Research article

On the solution of fractional modified Boussinesq and approximate long wave equations with non-singular kernel operators

Thongchai Botmart\(^1\), Ravi P. Agarwal\(^2\), Muhammed Naeem\(^3,\ast\), Adnan Khan\(^4\) and Rasool Shah\(^4,\ast\)

\(^1\) Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
\(^2\) Department of Mathematics, Texas A & M University-Kingsville, Kingsville, TX 78363, USA
\(^3\) Deanship of Joint First Year Umm Al-Qura University Makkah, Saudi Arabia
\(^4\) Department of Mathematics, Abdul Wali khan university Mardan 23200, Pakistan

\ast Correspondence: Email: mfaridoon@uqu.edu.sa, rasoolshahawkum@gmail.com.

Abstract: In this paper, we used the Natural decomposition approach with nonsingular kernel derivatives to explore the modified Boussinesq and approximate long wave equations. These equations are crucial in defining the features of shallow water waves using a specific dispersion relationship. In this research, the convergence analysis and error analysis have been provided. The fractional derivatives Atangana-Baleanu and Caputo-Fabrizio are utilised throughout the paper. To obtain the equations results, we used Natural transform on fractional-order modified Boussinesq and approximate long wave equations, followed by inverse Natural transform. To verify the approach, we focused on two systems and compared them to the exact solutions. We compare exact and analytical results with the use of graphs and tables, which are in strong agreement with each other, to demonstrate the effectiveness of the suggested approaches. Also compared are the results achieved by implementing the suggested approaches at various fractional orders, confirming that the result comes closer to the exact solution as the value moves from fractional to integer order. The numerical and graphical results show that the suggested scheme is computationally very accurate and simple to investigate and solve fractional coupled nonlinear complicated phenomena that exist in science and technology.

Keywords: Caputo-Fabrizio and Atangana-Baleanu operators; fractional approximate long wave equations; fractional modified Boussinesq equations; Adomian decomposition method; natural transform

Mathematics Subject Classification: 26A33, 34A25, 35M13, 35R11
1. Introduction

In a letter to L’Hospital in 1695, the famous mathematician Leibniz introduced the concept of fractional derivative for the first time. Fractional calculus is concerned with non-integral differential and integral operators. The integer-order differential operator is a local operator, but the fractional-order differential operator is non-local, indicating that a system’s next state is determined not only by its current state but also by all of its past states. It is more realistic, which is one of the main reasons for the popularity of fractional calculus. Fractional calculus was found to be more suitable for modelling real-world problems than classical calculus. The theory of fractional calculus provides an effective and systematic interpretation of nature’s reality. The following are some basic works on fractional calculus on various topics, such as Podlubny [1], Caputo [2], Kiryakova [3], Jafari and Seifi [4, 5], Momani and Shawagfeh [6], Oldham and Spanier [7], Diethelm et al. [8], Miller and Ross [9], Kemple and Beyer [10], Kilbas and Trujillo [11].

Fractional differential equations (FDEs) are widely widely used in various fields of science. FDEs have attracted a lot of attention in the last few years because of their diverse uses in physics and engineering. Due to its proven usefulness in a wide range of very diverse disciplines of science and engineering, fractional partial differential equations (PDEs) have gained importance and reputation among FDEs in recent years. For example, fractional derivatives in a fluid-dynamic traffic model can be utilised to overcome the insufficiency produced by the assumption of a continuous flow of traffic. Nonlinear FPDE solutions are of tremendous interest in both mathematics and practical applications [12–21]. The world’s most important processes are represented by nonlinear equations. Manipulation of nonlinear processes is essential in physics, applied mathematics, and engineering problems. The significance of finding the exact solution to nonlinear partial differential equations is still a prominent issue in physics and applied mathematics, requiring implemented using different techniques to determine new approximate or exact solutions [22–25].

In order to obtain explicit solutions for nonlinear equations of integer order, various techniques have been used. To solve FPDEs, however, only a less numerous approaches are used, such as Laplace variational iteration method (LVIM) [26], iterative Laplace transform method (ILTM) [27], optimal Homotopy asymptotic method (OHAM) [28], approximate-analytical method (AAM) [29], reduced differential transform method (RDTM) [30], Laplace-Adomian decomposition method (LADM) [31], natural transform decomposition method (NTDM) [32], Elzaki transform decomposition method (ETDM) [33–35], new meshfree technique (NMT) [36–40], Homotopy analysis method (HAM) [41], generalized exponential rational function method (GERFM) [42, 43] and much more [44–51]. The goal of this paper is to implement the natural decomposition approach to solve modified Boussinesq equations and approximate long wave equations having fractional-order. Natural decomposition methods avoid roundoff errors by not requiring prescriptive assumptions, linearization, discretization, or perturbation.

Whitham [52], Broer [53], and Kaup [54] discovered the Whitham-Broer-Kaup (WBK) equations in the twentieth century, which describe the propagation of shallow water waves with various dispersion
relations. Consider the fractional order coupled WBK equations [55]:

\[
\begin{align*}
\frac{\partial^\wp u}{\partial \xi^\wp} + u \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \xi} + b \frac{\partial^2 u}{\partial \xi^2} &= 0, \\
\frac{\partial^\wp v}{\partial \xi^\wp} + u \frac{\partial v}{\partial \xi} + v \frac{\partial u}{\partial \xi} + a \frac{\partial^3 u}{\partial \xi^3} - b \frac{\partial^2 v}{\partial \xi^2} &= 0, \quad 0 < \wp \leq 1.
\end{align*}
\] (1.1)

where \( v = v(\xi, \tau) \) is the height that deviates from the liquid’s equilibrium position and \( u = u(\xi, \tau) \) is the horizontal velocity. The order of the time-fractional derivative is in this case. Furthermore, \( a \) and \( b \) are constants that represent different diffusion strengths, therefore Eq (1.1) becomes a modified Boussinesq equation if \( a = 1 \) and \( b = 0 \). Similarly, the system denotes the standard long wave equation for \( a = 0 \) and \( b = 1 \). These equations are used in ocean and coastal engineering to explain the propagation of waves in dissipative and nonlinear media. They are recommended for situations involving water leakage in porous subsurface stratum and are frequently used in hydrodynamics. Furthermore, Eq (1.1) serves as the foundation for a number of models that represent the unconfined subsurface, such as drainage and groundwater problems [56, 57].

The following is a summary of the paper’s structure. The essential definitions related to fractional calculus are briefly discussed in Section 2. The NTDM is a method implemented to solve coupled WBK equations having fractional order with non-singular definitions are given in Section 3. In Section 4, we discussed the uniqueness and convergence results. Two examples of fractional-order modified Boussinesq and approximate long wave equations are given in Section 5 to validate the approaches. The article’s brief conclusions are offered in Section 6.

2. Basic preliminaries

In the literature, there are various fractional derivative definitions; for additional details, see [58–60]. This section includes Caputo, Riemann-Liouville, Atangana-Baleanu and Caputo-Fabrizio definitions for the benefit of the readers.

**Definition 2.1.** For a function \( h \in C_v, v \geq -1 \), the Riemann-Liouville fractional integral operator is given as [61]

\[
I^\wp h(\eta) = \frac{1}{\Gamma(\wp)} \int_0^\eta (\eta - \zeta)^{\wp-1} h(\zeta) d\zeta, \quad \wp > 0, \quad \eta > 0.
\] (2.1)

**Definition 2.2.** The fractional derivative \( h(\eta) \) in Caputo manner is given as [61]

\[
D^\wp h(\eta) = I^{\wp-m} D^m h(\eta) = \frac{1}{m-\wp} \int_0^\eta (\eta - \zeta)^{m-\wp-1} h^{(m)}(\zeta) d\zeta,
\] (2.2)

for \( m - 1 < \wp \leq m, \quad m \in N, \quad \eta > 0, h \in C_v^m, v \geq -1 \).

**Definition 2.3.** Caputo-Fabrizio fractional derivative for \( h(\eta) \) is given as [61]

\[
D^\wp h(\eta) = \frac{F(\wp)}{1-\wp} \int_0^\eta \exp \left( \frac{-\wp(\eta - \zeta)}{1-\wp} \right) D(h(\zeta)) d\zeta,
\] (2.3)

where \( 0 < \wp < 1 \) and \( F(\wp) \) is a normalization function with \( F(0) = F(1) = 1 \).
Definition 2.4. The Atangana-Baleanu Caputo derivative of fractional-order for \( h(\eta) \) is defined as [61]

\[
D_\zeta^\alpha h(\eta) = \frac{B(\varphi)}{1 - \varphi} \int_0^\eta E_\varphi \left( -\varphi(\eta - \zeta) \right) D(h(\zeta))d\zeta,
\]  
(2.4)

where \( 0 < \varphi < 1 \), where \( B(\varphi) \) is Normalization function and \( E_\varphi(z) = \sum_{m=0}^\infty \frac{z^m}{\Gamma(\varphi + m + 1)} \) is the Mittag-Leffler function.

Definition 2.5. For a function \( u(\mathcal{H}) \), the Natural transformation is defined as

\[
\mathcal{N}(u(\mathcal{H})) = \mathcal{U}(s, \vartheta) = \int_{-\infty}^{\infty} e^{-s\vartheta} u(\vartheta \mathcal{H})d\vartheta, \quad s, \vartheta \in (-\infty, \infty).
\]  
(2.5)

Natural transformation of \( u(\mathcal{H}) \) for \( \vartheta \in (0, \infty) \) is given as

\[
\mathcal{N}(u(\mathcal{H}))H(\mathcal{H}) = \mathcal{N}^+ = \mathcal{U}^+(s, \vartheta) = \int_{-\infty}^{\infty} e^{-s\vartheta} u(\vartheta \mathcal{H})d\vartheta, \quad s, \vartheta \in (0, \infty).
\]  
(2.6)

here \( H(\mathcal{H}) \) is the Heaviside function.

Definition 2.6. For a function \( \mathcal{U}(s, \vartheta) \), the Natural inverse transformation is given as

\[
\mathcal{N}^{-1}[\mathcal{U}(s, \vartheta)] = u(\mathcal{H}), \quad \forall \mathcal{H} \geq 0.
\]  
(2.7)

Lemma 2.1. If linearity property having Natural transformation for \( u_1(\mathcal{H}) \) is \( u_1(s, \vartheta) \) and \( u_2(\mathcal{H}) \) is \( u_2(s, \vartheta) \), then

\[
\mathcal{N}[c_1u_1(\mathcal{H}) + c_2u_2(\mathcal{H})] = c_1\mathcal{N}[u_1(\mathcal{H})] + c_2\mathcal{N}[u_2(\mathcal{H})] = c_1\mathcal{U}_1(s, \vartheta) + c_2\mathcal{U}_2(s, \vartheta),
\]  
(2.8)

where \( c_1 \) and \( c_2 \) are constants.

Lemma 2.2. If Natural inverse transformation of \( \mathcal{U}_1(s, \vartheta) \) and \( \mathcal{U}_2(s, \vartheta) \) are \( u_1(\mathcal{H}) \) and \( u_2(\mathcal{H}) \) respectively then

\[
\mathcal{N}^{-1}[c_1\mathcal{U}_1(s, \vartheta) + c_2\mathcal{U}_2(s, \vartheta)] = c_1\mathcal{N}^{-1}[\mathcal{U}_1(s, \vartheta)] + c_2\mathcal{N}^{-1}[\mathcal{U}_2(s, \vartheta)] = c_1u_1(\mathcal{H}) + c_2u_2(\mathcal{H}),
\]  
(2.9)

where \( c_1 \) and \( c_2 \) are constants.

Definition 2.7. The Natural transformation of Caputo operator \( D_\zeta^\alpha u(\mathcal{H}) \) is define as [61]

\[
\mathcal{N}[D_\zeta^\alpha u(\mathcal{H})] = \left( \frac{s}{\vartheta} \right)^\alpha \left( \mathcal{N}[u(\mathcal{H})] - \left( \frac{1}{s} \right) u(0) \right).
\]  
(2.10)

Definition 2.8. In terms of Caputo-Fabrizio, the Natural transformation of \( D_\zeta^\alpha u(\mathcal{H}) \) is given as [61]

\[
\mathcal{N}[D_\zeta^\alpha u(\mathcal{H})] = \frac{1}{1 - \varphi + \varphi(\vartheta)^\alpha} \left( \mathcal{N}[u(\mathcal{H})] - \left( \frac{1}{s} \right) u(0) \right).
\]  
(2.11)

Definition 2.9. In terms of Atangana-Baleanu Caputo derivative, the Natural transformation of \( D_\zeta^\alpha u(\mathcal{H}) \) is defined as [61]

\[
\mathcal{N}[D_\zeta^\alpha u(\mathcal{H})] = \frac{M[\varphi]}{1 - \varphi + \varphi(\vartheta)^\alpha} \left( \mathcal{N}[u(\mathcal{H})] - \left( \frac{1}{s} \right) u(0) \right).
\]  
(2.12)
3. Methodology

This section introduces a general numerical methodology based on the Natural transform for the following equation:

\[ D_\alpha^\nu(\xi, \mathfrak{I}) = L(\nu(\xi, \mathfrak{I})) + N(\nu(\xi, \mathfrak{I})) + h(\xi, \mathfrak{I}), \tag{3.1} \]

with initial source

\[ \nu(\xi, 0) = \phi(\xi), \tag{3.2} \]

here $L$ is linear, $N$ is nonlinear and $h(\xi, \mathfrak{I})$ represents source term.

3.1. Case I ($NTDM_{CF}$):

We get this by applying Caputo-Fabrizio fractional derivative with the aid of Natural transformation of Eq (3.1),

\[ \frac{1}{p(\varphi, \vartheta, s)} \left( N[\nu(\xi, \mathfrak{I})] - \frac{\phi(\xi)}{s} \right) = N[M(\xi, \mathfrak{I})], \tag{3.3} \]

where

\[ p(\varphi, \vartheta, s) = 1 - \varphi + \varphi(\frac{s}{\vartheta}). \tag{3.4} \]

We may express Eq (3.3) as, using the inverse Natural transformation

\[ \nu(\xi, \mathfrak{I}) = N^{-1} \left( \frac{\phi(\xi)}{s} + p(\varphi, \vartheta, s)N[M(\xi, \mathfrak{I})] \right), \tag{3.5} \]

$N(\nu(\xi, \mathfrak{I}))$ can be decomposed into

\[ N(\nu(\xi, \mathfrak{I})) = \sum_{i=0}^{\infty} A_i, \tag{3.6} \]

where the Adomian polynomials is represented by $A_5$. We assume that the numerical solution of Eq (3.1) exists

\[ \nu(\xi, \mathfrak{I}) = \sum_{i=0}^{\infty} \nu_i(\xi, \mathfrak{I}). \tag{3.7} \]

By putting Eqs (3.6) and (3.7) into (3.5), we get

\[ \sum_{i=0}^{\infty} \nu_i(\xi, \mathfrak{I}) = N^{-1} \left( \frac{\phi(\xi)}{s} + p(\varphi, \vartheta, s)N[h(\xi, \mathfrak{I})] \right) \]

\[ + N^{-1} \left( p(\varphi, \vartheta, s)N \left[ \sum_{i=0}^{\infty} L(\nu_i(\xi, \mathfrak{I})) + A_3 \right] \right), \tag{3.8} \]
From (3.8), we obtain
\[
\nu_{CF}^{0}(\xi, \Im) = N^{-1} \left( \frac{\phi(\xi)}{s} + p(\varphi, \vartheta, s)N[h(\xi, \Im)] \right),
\]
\[
\nu_{CF}^{1}(\xi, \Im) = N^{-1} \left( p(\varphi, \vartheta, s)N \left[ \mathcal{L}(\nu_{0}(\xi, \Im)) + A_{0} \right] \right),
\]
\[
\vdots
\]
\[
\nu_{CF}^{l+1}(\xi, \Im) = N^{-1} \left( p(\varphi, \vartheta, s)N \left[ \mathcal{L}(\nu_{l}(\xi, \Im)) + A_{l} \right] \right), \quad l = 1, 2, 3, \ldots
\]
we obtain the NTDM solution of (3.1) by putting (3.9) into (3.7),
\[
\nu_{CF}(\xi, \Im) = \nu_{CF}^{0}(\xi, \Im) + \nu_{CF}^{1}(\xi, \Im) + \nu_{CF}^{2}(\xi, \Im) + \cdots. \tag{3.10}
\]

3.2. Case II (NTDM\textsubscript{ABC})

We get this by applying Atangana-Baleanu derivative with the aid of Natural transformation of Eq (3.1),
\[
\frac{1}{q(\varphi, \vartheta, s)} \left( N[\nu(\xi, \Im)] - \frac{\phi(\xi)}{s} \right) = N[M(\xi, \Im)], \tag{3.11}
\]
where
\[
q(\varphi, \vartheta, s) = \frac{1 - \varphi + \varphi(\xi)^{\varphi}}{B(\varphi)}. \tag{3.12}
\]
On applying inverse Natural transformation (2.7), we express (3.11) as,
\[
\nu(\xi, \Im) = N^{-1} \left( \frac{\phi(\xi)}{s} + q(\varphi, \vartheta, s)N[M(\xi, \Im)] \right), \tag{3.13}
\]
\(\mathbb{N}(\nu(\xi, \Im))\) can be decomposed into
\[
\mathbb{N}(\nu(\xi, \Im)) = \sum_{i=0}^{\infty} A_{i}, \tag{3.14}
\]
where the Adomian polynomials [62, 63] is represented by \(A_{5}\). We assume that the numerical solution of Eq (3.1) exists
\[
\nu(\xi, \Im) = \sum_{i=0}^{\infty} \nu_{i}(\xi, \Im). \tag{3.15}
\]
By putting Eqs (3.14) and (3.15) into (3.13), we get
\[
\sum_{i=0}^{\infty} \nu_{i}(\xi, \Im) = N^{-1} \left( \frac{\phi(\xi)}{s} + q(\varphi, \vartheta, s)N[h(\xi, \Im)] \right)
\]
\[
+ N^{-1} \left( q(\varphi, \vartheta, s)N \left[ \sum_{i=0}^{\infty} \mathcal{L}(\nu_{i}(\xi, \Im)) + A_{3} \right] \right). \tag{3.16}
\]
From (3.8), we get

\[ v_0^{ABC}(\xi, \mathfrak{G}) = \mathcal{N}^{-1} \left( \frac{\phi(\xi)}{s} + q(\varphi, \theta, s)\mathcal{H}(h(\xi, \mathfrak{G})) \right), \]

\[ v_1^{ABC}(\xi, \mathfrak{G}) = \mathcal{N}^{-1} \left( q(\varphi, \theta, s)\mathcal{H}(v_0(\xi, \mathfrak{G}) + A_0) \right), \]

\[ \vdots \]

\[ v_{l+1}^{ABC}(\xi, \mathfrak{G}) = \mathcal{N}^{-1} \left( q(\varphi, \theta, s)\mathcal{H}(v_l(\xi, \mathfrak{G}) + A_l) \right), \quad l = 1, 2, 3, \ldots, \]

we obtain the NTDM_{ABC} solution of (3.1) by putting (3.17) into (3.15),

\[ v^{ABC}(\xi, \mathfrak{G}) = v_0^{ABC}(\xi, \mathfrak{G}) + v_1^{ABC}(\xi, \mathfrak{G}) + v_2^{ABC}(\xi, \mathfrak{G}) + \ldots. \] (3.18)

4. Convergence analysis

Here we discuss uniqueness and convergence and of the NTDM_{CF} and NTDM_{ABC}.

**Theorem 4.1.** The result of (3.1) is unique for NTDM_{CF} when \( 0 < (\lambda_1 + \lambda_2)(1 - \varphi + \varphi \mathfrak{G}) < 1 \).

**Proof.** Let \( H = (C[J], ||.||) \) with the norm \( ||\phi(\mathfrak{G})|| = \max_{\mathfrak{G}\in J} |\phi(\mathfrak{G})| \) is Banach space, \( \mathcal{V} \) continuous function on \( J \). Let \( I : H \to H \) is a non-linear mapping, where

\[ \nu^C_{l+1} = \nu^C_0 + \mathcal{N}^{-1}[p(\varphi, \theta, s)\mathcal{H}(|\mathcal{V}(\nu_l(\xi, \mathfrak{G})| + \mathcal{N}(\nu_l(\xi, \mathfrak{G})))], \quad l \geq 0. \]

Suppose that \( |\mathcal{H}(\nu) - \mathcal{H}(\nu^*)| < \lambda_1|\nu - \nu^*| \) and \( |\mathcal{N}(\nu) - \mathcal{N}(\nu^*)| < \lambda_2|\nu - \nu^*| \), where \( \nu := \nu(\xi, \mathfrak{G}) \) and \( \nu^* := \nu^*(\xi, \mathfrak{G}) \) are distinct function values and \( \lambda_1, \lambda_2 \) are Lipschitz constants.

\[ ||I\nu - I\nu^*|| \leq \max_{\mathfrak{G}\in J}|\mathcal{N}^{-1}[p(\varphi, \theta, s)\mathcal{H}(|\mathcal{V}(\nu) - \mathcal{V}(\nu^*)|)]
\]

\[ + p(\varphi, \theta, s)\mathcal{N}(|\mathcal{N}(\nu) - \mathcal{N}(\nu^*)|)]\]

\[ \leq \max_{\mathfrak{G}\in J}[\lambda_1|\mathcal{N}^{-1}[p(\varphi, \theta, s)\mathcal{H}(|\nu - \nu^*|)]
\]

\[ + \lambda_2|\mathcal{N}^{-1}[p(\varphi, \theta, s)\mathcal{N}(|\nu - \nu^*|)]]\]

\[ \leq \max_{\mathfrak{G}\in J}[(\lambda_1 + \lambda_2)|\mathcal{N}^{-1}[p(\varphi, \theta, s)\mathcal{H}(|\nu - \nu^*|)]
\]

\[ = (\lambda_1 + \lambda_2)(1 - \varphi + \varphi \mathfrak{G})||\nu - \nu^*||. \] (4.1)

\( I \) is contraction as \( 0 < (\lambda_1 + \lambda_2)(1 - \varphi + \varphi \mathfrak{G}) < 1 \). From Banach fixed point theorem the result of (3.1) is unique. \( \square \)

**Theorem 4.2.** The result of (3.1) is unique for NTDM_{ABC} when \( 0 < (\lambda_1 + \lambda_2)(1 - \varphi + \varphi \frac{\mathfrak{G}_{\text{min}}}{\Gamma(\mu+1)}) < 1 \).

**Proof.** Let \( H = (C[J], ||.||) \) with the norm \( ||\phi(\mathfrak{G})|| = \max_{\mathfrak{G}\in J} |\phi(\mathfrak{G})| \) be the Banach space, \( \mathcal{V} \) continuous function on \( J \). Let \( I : H \to H \) is a non-linear mapping, where

\[ \nu^C_{l+1} = \nu^C_0 + \mathcal{N}^{-1}[p(\varphi, \theta, s)\mathcal{H}(|\mathcal{V}(\nu_l(\xi, \mathfrak{G})| + \mathcal{N}(\nu_l(\xi, \mathfrak{G})))], \quad l \geq 0. \]
Suppose that \(|L(\nu) - L(\nu')| < \lambda_1|\nu - \nu'|\) and \(|N(\nu) - N(\nu')| < \lambda_2|\nu - \nu'|\), where \(\nu := \nu(\zeta, \mathfrak{F})\) and \(\nu' := \nu'(\zeta, \mathfrak{F})\) are two different function values and \(\lambda_1, \lambda_2\) are Lipschitz constants.

\[
\|I\nu - I\nu'\| \leq \max_{\nu \in J} |N^{-1}[q(\varphi, \vartheta, s)N[L(\nu) - L(\nu')]]
+ q(\varphi, \vartheta, s)N[N(\nu) - N(\nu')]| \\
\leq \max_{\nu \in J} [\lambda_1 N^{-1}[q(\varphi, \vartheta, s)N[|\nu - \nu'|]]
+ \lambda_2 N^{-1}[q(\varphi, \vartheta, s)N[|\nu - \nu'|]]] \\
\leq \max_{\nu \in J} (\lambda_1 + \lambda_2)N^{-1}[q(\varphi, \vartheta, s)N[|\nu - \nu'|]] \\
= (\lambda_1 + \lambda_2)(1 - \varphi + \varphi \frac{\mathfrak{F}^0}{\Gamma^0 + 1})|\nu - \nu'|.
\]

\(I\) is contraction as \(\lambda_1 + \lambda_2)(1 - \varphi + \varphi \frac{\mathfrak{F}^0}{\Gamma^0 + 1} < 1\). From Banach fixed point theorem the result of (3.1) is unique. \(\Box\)

**Theorem 4.3.** The NTDM\(_{CF}\) result of (3.1) is convergent.

**Proof.** Let \(\nu_m = \sum_{r=0}^{m} \nu_r(\zeta, \mathfrak{F})\). To show that \(\nu_m\) is a Cauchy sequence in H. Let,

\[
\|\nu_m - \nu_n\| = \max_{\nu' \in J} \left| \sum_{r=n+1}^{m} \nu_r \right|, \quad n = 1, 2, 3, \cdots \\
\leq \max_{\nu' \in J} \left| N^{-1} \left[ p(\varphi, \vartheta, s)N \left[ \sum_{r=n+1}^{m} (L(\nu_{r-1}) + N(\nu_{r-1})) \right] \right] \right| \\
= \max_{\nu' \in J} \left| N^{-1} \left[ p(\varphi, \vartheta, s)N \left[ \sum_{r=n+1}^{m} (L(\nu_r) + N(\nu_r)) \right] \right] \right| \\
\leq \max_{\nu' \in J} \left| N^{-1} \left[ p(\varphi, \vartheta, s)N \left[ (L(\nu_{m-1}) - L(\nu_{n-1})) + N(\nu_{m-1}) - N(\nu_{n-1}) \right] \right] \right| \\
\leq \lambda_1 \max_{\nu' \in J} \left| N^{-1} \left[ p(\varphi, \vartheta, s)N \left[ (L(\nu_{m-1}) - L(\nu_{n-1})) \right] \right] \right| \\
+ \lambda_2 \max_{\nu' \in J} \left| N^{-1} \left[ p(\varphi, \vartheta, s)N \left[ (N(\nu_{m-1}) - N(\nu_{n-1})) \right] \right] \right| \\
= (\lambda_1 + \lambda_2)(1 - \varphi + \varphi \mathfrak{F})|\nu_{m-1} - \nu_{n-1}|.
\]

Let \(m = n + 1\), then

\[
|\nu_{n+1} - \nu_n| \leq \lambda |\nu_n - \nu_{n-1}| \leq \lambda^2 |\nu_{n-1} - \nu_{n-2}| \leq \cdots \leq \lambda^n |\nu_1 - \nu_0|,
\]

where \(\lambda = (\lambda_1 + \lambda_2)(1 - \varphi + \varphi \mathfrak{F})\). Similarly, we have

\[
|\nu_m - \nu_n| \leq |\nu_{n+1} - \nu_n| + |\nu_{n+2} - \nu_{n+1}| + \cdots + |\nu_m - \nu_{m-1}|, \\
(\lambda^{n+1} + \cdots + \lambda^{m-1})|\nu_1 - \nu_0| \\
\leq \lambda^n \left( \frac{1 - \lambda^{m-n}}{1 - \lambda} \right) |\nu_1|,
\]

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As $0 < \lambda < 1$, we get $1 - \lambda^{m-n} < 1$. Therefore,
\[
\|v_m - v_n\| \leq \frac{\lambda^n}{1 - \lambda} \text{max}_{\mathcal{E} \in J} \|v_1\|. \tag{4.6}
\]

Since $\|v_1\| < \infty$, $\|v_m - v_n\| \to 0$ when $n \to \infty$. As a result, $v_m$ is a Cauchy sequence in $H$, implying that the series $v_m$ is convergent.

\textbf{Theorem 4.4.} The NTDM$_{ABC}$ result of (3.1) is convergent.

\textit{Proof.} Let $v_m = \sum_{r=0}^{m} v_r(\zeta, \mathcal{J})$. To show that $v_m$ is a Cauchy sequence in $H$. Let,
\[
\|v_m - v_n\| = \text{max}_{\mathcal{E} \in J} \sum_{r=n+1}^{m} v_r, \quad n = 1, 2, 3, \ldots
\]
\[
\leq \text{max}_{\mathcal{E} \in J} \left| N^{-1} \left[ q(\varphi, \partial, s)N \left( \sum_{r=n+1}^{m} (L(v_{r-1}) + N(v_{r-1})) \right) \right] \right|
\]
\[
= \text{max}_{\mathcal{E} \in J} \left| N^{-1} \left[ q(\varphi, \partial, s)N \left( \sum_{r=n+1}^{m-1} (L(u_r) + N(u_r)) \right) \right] \right|
\]
\[
\leq \lambda_1 \text{max}_{\mathcal{E} \in J} |N^{-1} [q(\varphi, \partial, s)N [(L(v_{m-1}) - L(v_{n-1}) + N(v_{m-1}) - N(v_{n-1}))]]]
\]
\[
+ \lambda_2 \text{max}_{\mathcal{E} \in J} |N^{-1} [p(\varphi, \partial, s)N [(N(v_{m-1}) - N(v_{n-1}))]]]
\]
\[
= (\lambda_1 + \lambda_2)(1 - \varphi + \varphi \frac{\zeta^q}{\Gamma(\varphi + 1)}) \|v_{m-1} - v_{n-1}\|
\]

Let $m = n + 1$, then
\[
\|v_{n+1} - v_n\| \leq \lambda \|v_n - v_{n-1}\| \leq \lambda^2 \|v_{n-1}v_{n-2}\| \leq \cdots \leq \lambda^n \|v_1 - v_0\|, \tag{4.8}
\]
where $\lambda = (\lambda_1 + \lambda_2)(1 - \varphi + \varphi \frac{\zeta^q}{\Gamma(\varphi + 1)})$. Similarly, we have
\[
\|v_m - v_n\| \leq \|v_{m+1} - v_n\| + \|v_{n+2}v_{n+1}\| + \cdots + \|v_m - v_{m-1}\|
\]
\[
(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) \|v_1 - v_0\|
\]
\[
\leq \lambda^n \left( \frac{1 - \lambda^{m-n}}{1 - \lambda} \right) \|v_1\|, \tag{4.9}
\]

As $0 < \lambda < 1$, we get $1 - \lambda^{m-n} < 1$. Therefore,
\[
\|v_m - v_n\| \leq \frac{\lambda^n}{1 - \lambda} \text{max}_{\mathcal{E} \in J} \|v_1\|. \tag{4.10}
\]

Since $\|v_1\| < \infty$, $\|v_m - v_n\| \to 0$ when $n \to \infty$. As a result, $v_m$ is a Cauchy sequence in $H$, implying that the series $v_m$ is convergent.

\section{5. Numerical examples}

In this section we investigate the analytical solution for a few problems of fractional order coupled WBK equations. We chose these equations because they contain closed form solutions and are well-known methods for analysing results in the literature.
5.1. Example 1

Consider the modified Boussinesq (MB) equations of fractional order

\[
D_\alpha^\rho u = -u \frac{\partial u}{\partial \xi} - \frac{\partial v}{\partial \xi},
\]
\[
D_\alpha^\rho v = -u \frac{\partial v}{\partial \xi} - v \frac{\partial u}{\partial \xi} - \frac{\partial^3 u}{\partial \xi^3}, \quad 0 < \rho \leq 1,
\]
(5.1)

with initial source

\[
u(\xi, 0) = -2\ell^2 \coth^2[\ell(\xi + c)].
\]
(5.2)

By applying the Natural transform to Eq (5.1), we have

\[
N[D_\alpha^\rho u(\xi, \mathcal{J})] = -N\left\{ u \frac{\partial u}{\partial \xi} - \frac{\partial v}{\partial \xi} \right\},
\]
\[
N[D_\alpha^\rho v(\xi, \mathcal{J})] = -N\left\{ u \frac{\partial v}{\partial \xi} - v \frac{\partial u}{\partial \xi} - \frac{\partial^3 u}{\partial \xi^3} \right\}.
\]
(5.3)

Define the non-linear operator as

\[
\frac{1}{s^\rho} N[u(\xi, \mathcal{J})] - s^{2-\rho} u(\xi, 0) = N\left[-u \frac{\partial u}{\partial \xi} - \frac{\partial v}{\partial \xi}\right],
\]
\[
\frac{1}{s^\rho} N[v(\xi, \mathcal{J})] - s^{2-\rho} v(\xi, 0) = N\left[-u \frac{\partial v}{\partial \xi} - v \frac{\partial u}{\partial \xi} - \frac{\partial^3 u}{\partial \xi^3}\right].
\]
(5.4)

The above equation is reduced to by simplifying it

\[
N[u(\xi, \mathcal{J})] = s^2 \left[ -2\ell \coth[\ell(\xi + c)] + \frac{\varphi(s - \varphi(s - \varphi))}{s^2} \right] N\left[-u \frac{\partial u}{\partial \xi} - \frac{\partial v}{\partial \xi}\right],
\]
\[
N[v(\xi, \mathcal{J})] = s^2 \left[ -2\ell^2 \coth^2[\ell(\xi + c)] + \frac{\varphi(s - \varphi(s - \varphi))}{s^2} \right] N\left[-u \frac{\partial v}{\partial \xi} - v \frac{\partial u}{\partial \xi} - \frac{\partial^3 u}{\partial \xi^3}\right].
\]
(5.5)

We get by applying inverse \( NT \) to Eq (5.5)

\[
u(\xi, \mathcal{J}) = \left[ -2\ell^2 \coth^2[\ell(\xi + c)] \right] + N^{-1}\left[ -u \frac{\partial v}{\partial \xi} - v \frac{\partial u}{\partial \xi} - \frac{\partial^3 u}{\partial \xi^3}\right].
\]
(5.6)
5.1.1. Now we implementing $NTDM_{CF}$

Assume that the unknown functions $u(\xi, \mathfrak{F})$ and $v(\xi, \mathfrak{F})$ yield the infinite series result as follows

$$u(\xi, \mathfrak{F}) = \sum_{l=0}^{\infty} u_l(\xi, \mathfrak{F}) \quad \text{and} \quad v(\xi, \mathfrak{F}) = \sum_{l=0}^{\infty} v_l(\xi, \mathfrak{F})$$  \hspace{1cm} (5.7)

The Adomian polynomials represent the non-linear terms and are denoted by $uu_{\xi} = \sum_{l=0}^{\infty} A_l$, $uv_{\xi} = \sum_{l=0}^{\infty} B_l$ and $vu_{\xi} = \sum_{l=0}^{\infty} C_l$. We can write the Eq (5.6) as a result of applying these terms,

$$\begin{align*}
\sum_{l=0}^{\infty} u_{l+1}(\xi, \mathfrak{F}) &= \omega - 2\ell \coth[\ell(\xi + c)] \\
&\quad + N^{-1} \left[ \frac{\varphi(s - \varphi(s - \varphi))}{s^2} \mathcal{N} \left( - \sum_{l=0}^{\infty} A_l - \sum_{l=0}^{\infty} v_l \right) \right], \\
\sum_{l=0}^{\infty} v_{l+1}(\xi, \mathfrak{F}) &= -2\ell^2 \csch^2[\ell(\xi + c)] \\
&\quad + N^{-1} \left[ \frac{\varphi(s - \varphi(s - \varphi))}{s^2} \mathcal{N} \left( - \sum_{l=0}^{\infty} B_l - \sum_{l=0}^{\infty} C_l - \sum_{l=0}^{\infty} u_{l\xi \xi} \right) \right].
\end{align*}$$  \hspace{1cm} (5.8)

We can write as follows by comparing both sides of Eq (5.8)

$$\begin{align*}
u_0(\xi, \mathfrak{F}) &= \omega - 2\ell \coth[\ell(\xi + c)], \\
u_0(\xi, \mathfrak{F}) &= -2\ell^2 \csch^2[\ell(\xi + c)].
\end{align*}$$  \hspace{1cm} (5.9)

$$\begin{align*}
u_0(\xi, \mathfrak{F}) &= -2\ell^2 \omega \csch^2[\ell(\xi + c)] (\varphi(\mathfrak{F} - 1) + 1), \\
u_0(\xi, \mathfrak{F}) &= -4\ell^3 \omega \coth[\ell(\xi + c)] \csch^2[\ell(\xi + c)] (\varphi(\mathfrak{F} - 1) + 1) \\
u_0(\xi, \mathfrak{F}) &= 2\ell^2 \omega \csch^2[\ell(\xi + c)] (\varphi(\mathfrak{F} - 1) + 1) - 4\ell^3 \omega^2 \coth[\ell(\xi + c)] \\
&\quad \csch[\ell(\xi + c)] [(1 - \varphi)^2 + 2\varphi(1 - \varphi) \mathfrak{F} + \frac{\varphi^2 \mathfrak{G}^2}{2}], \\
u_0(\xi, \mathfrak{F}) &= 4\ell^3 \omega \coth[\ell(\xi + c)] \csch^2[\ell(\xi + c)] (\varphi(\mathfrak{F} - 1) + 1) - 4\ell^4 \omega^2 (2 + \cosh[2\ell(\xi + c)]) \\
&\quad \csch^4[\ell(\xi + c)] [(1 - \varphi)^2 + 2\varphi(1 - \varphi) \mathfrak{F} + \frac{\varphi^2 \mathfrak{G}^2}{2}].
\end{align*}$$  \hspace{1cm} (5.10)

The remaining components of the Natural decomposition technique result $u_l$ and $v_l$ for $l \geq 3$ can be performed easily. The series form result is calculated as follows:

$$\begin{align*}
u(\xi, \mathfrak{F}) &= \sum_{l=0}^{\infty} u_l(\xi, \mathfrak{F}) = 0 + u_1(\xi, \mathfrak{F}) + u_2(\xi, \mathfrak{F}) + \cdots, \\
u(\xi, \mathfrak{F}) &= \omega - 2\ell \coth[\ell(\xi + c)] - 2\ell^2 \omega \csch^2[\ell(\xi + c)] (\varphi(\mathfrak{F} - 1) + 1) + 2\ell^3 \omega \csch^2[\ell(\xi + c)] \\
&\quad (\varphi(\mathfrak{F} - 1) + 1) - 4\ell^3 \omega^2 \coth[\ell(\xi + c)] \csch^2[\ell(\xi + c)] [(1 - \varphi)^2 + 2\varphi(1 - \varphi) \mathfrak{F} + \frac{\varphi^2 \mathfrak{G}^2}{2}] + \cdots. \\
\end{align*}$$  \hspace{1cm} (5.11)
v(ξ, Ω) = ∑l=0∞ v_l(ξ, Ω) = v_0(ξ, Ω) + v_1(ξ, Ω) + v_2(ξ, Ω) + ⋅⋅⋅,

v(ξ, Ω) = -2ξ² csch²[ξ(ξ + c)] - 4ξ³ ω coth[ξ(ξ + c)] csch²[ξ(ξ + c)](φ(Ω - 1) + 1) + 4ξ³ ω coth[ξ(ξ + c)] csch²[ξ(ξ + c)](φ(Ω - 1) + 1) - 4ξ⁴ ω²(2 + cosh[2ξ(ξ + c)])

csch⁴[ξ(ξ + c)]((1 - φ)² + 2φ(1 - φ)Ω + φ²Ω²) + ⋅⋅⋅. (5.12)

5.1.2. Now we implementing NTDM_{ABC}

Assume that the unknown functions u(ξ, Ω) and v(ξ, Ω) yield the infinite series result as follows

u(ξ, Ω) = ∑l=0∞ u_l(ξ, Ω) and v(ξ, Ω) = ∑l=0∞ v_l(ξ, Ω) (5.13)

The Adomian polynomials represent the non-linear terms and are denoted by uu_ξ = ∑l=0∞ A_l, uv_ξ = ∑l=0∞ B_l and vu_ξ = ∑l=0∞ C_l. We can write the Eq (5.6) as a result of applying these terms,

\[ \sum_{l=0}^{∞} u_{l+1}(ξ, Ω) = \omega - 2ξ \coth[ξ(ξ + c)] \]
\[ + \mathcal{N}^{-1} \left[ \frac{\partial^2(\varphi^2 - \varphi(\partial^2 - s^2))}{s^2} \right] \mathcal{N}\left\{ - \sum_{l=0}^{∞} A_l - \sum_{l=0}^{∞} v_{l+1}(ξ, Ω) \right\}. \] (5.14)

\[ \sum_{l=0}^{∞} v_{l+1}(ξ, Ω) = -2ξ² \csch²[ξ(ξ + c)] \]
\[ + \mathcal{N}^{-1} \left[ \frac{\partial^2(\varphi^2 - \varphi(\partial^2 - s^2))}{s^2} \right] \mathcal{N}\left\{ - \sum_{l=0}^{∞} B_l - \sum_{l=0}^{∞} C_l - \sum_{l=0}^{∞} u_{l+1}(ξ, Ω) \right\}. \]

We can write as follows by comparing both sides of Eq (5.14)

\[ u_0(ξ, Ω) = \omega - 2ξ \coth[ξ(ξ + c)], \]
\[ v_0(ξ, Ω) = -2ξ² \csch²[ξ(ξ + c)]. \]
\[ u_1(ξ, Ω) = -2ξ² \omega \csch²[ξ(ξ + c)] \left( 1 - \varphi + \frac{\varphi \bar{\gamma}^\varphi}{\Gamma(\varphi + 1)} \right), \]
\[ v_1(ξ, Ω) = -4ξ³ \omega \coth[ξ(ξ + c)] \csch²[ξ(ξ + c)] \left( 1 - \varphi + \frac{\varphi \bar{\gamma}^\varphi}{\Gamma(\varphi + 1)} \right). \] (5.15)

\[ u_2(ξ, Ω) = 2ξ² \omega \csch²[ξ(ξ + c)] \left( 1 - \varphi + \frac{\varphi \bar{\gamma}^\varphi}{\Gamma(\varphi + 1)} \right) - 4ξ³ \omega² \coth[ξ(ξ + c)] \csch²[ξ(ξ + c)] \]
\[ \left[ \frac{\varphi² \bar{\gamma}²}{\Gamma(2\varphi + 1)} + 2\varphi(1 - \varphi) \frac{\bar{\gamma}^\varphi}{\Gamma(\varphi + 1)} + (1 - \varphi)² \right], \]
\[ v_2(ξ, Ω) = 4ξ³ \omega \coth[ξ(ξ + c)] \csch²[ξ(ξ + c)] \left( 1 - \varphi + \frac{\varphi \bar{\gamma}^\varphi}{\Gamma(\varphi + 1)} \right) - 4ξ⁴ \omega²(2 + \cosh[2ξ(ξ + c)]) \]
\[ \csch⁴[ξ(ξ + c)] \left[ \frac{\varphi² \bar{\gamma}²}{\Gamma(2\varphi + 1)} + 2\varphi(1 - \varphi) \frac{\bar{\gamma}^\varphi}{\Gamma(\varphi + 1)} + (1 - \varphi)² \right]. \] (5.16)
The remaining components of the natural decomposition technique result $u_l$ and $v_l$ for $l \geq 3$ can be performed easily. The series form result is calculated as follows:

$$u(\xi, \mathfrak{I}) = \sum_{l=0}^{\infty} u_l(\xi, \mathfrak{I}) = u_0(\xi, \mathfrak{I}) + u_1(\xi, \mathfrak{I}) + u_2(\xi, \mathfrak{I}) + \cdots,$$

$$u(\xi, \mathfrak{I}) = \omega - 2\ell \coth[\ell(\xi + c)] - 2\ell^2 \omega \csch^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{\varphi \mathfrak{I}^2}{\Gamma(\varphi + 1)} \right)$$
$$+ 2\ell^2 \omega \csch^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{\varphi \mathfrak{I}^2}{\Gamma(\varphi + 1)} \right) - 4\ell^3 \omega^2 \coth[\ell(\xi + c)]$$
$$\csch^2[\ell(\xi + c)] \left[ \frac{\varphi^2 \mathfrak{I}^2}{\Gamma(2\varphi + 1)} + 2\varphi(1 - \varphi) \frac{\mathfrak{I}^2}{\Gamma(\varphi + 1)} + (1 - \varphi)^2 \right] + \cdots. \quad (5.17)$$

$$v(\xi, \mathfrak{I}) = \sum_{l=0}^{\infty} v_l(\xi, \mathfrak{I}) = v_0(\xi, \mathfrak{I}) + v_1(\xi, \mathfrak{I}) + v_2(\xi, \mathfrak{I}) + \cdots,$$

$$v(\xi, \mathfrak{I}) = -2\ell^2 \csch^2[\ell(\xi + c)] - 4\ell^3 \omega \coth[\ell(\xi + c)] \csch^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{\varphi \mathfrak{I}^2}{\Gamma(\varphi + 1)} \right)$$
$$+ 4\ell^3 \omega \coth[\ell(\xi + c)] \csch^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{\varphi \mathfrak{I}^2}{\Gamma(\varphi + 1)} \right) - 4\ell^4 \omega^2 (2 + \cosh[2\ell(\xi + c)])$$
$$\csch^4[\ell(\xi + c)] \left[ \frac{\varphi^2 \mathfrak{I}^2}{\Gamma(2\varphi + 1)} + 2\varphi(1 - \varphi) \frac{\mathfrak{I}^2}{\Gamma(\varphi + 1)} + (1 - \varphi)^2 \right] + \cdots. \quad (5.18)$$

For Eq (5.1), the exact result is obtained at $\varphi = 1$,

$$u(\xi, \mathfrak{I}) = \omega - 2\ell \coth[\ell(\xi + c - \omega \mathfrak{I})],$$
$$v(\xi, \mathfrak{I}) = -2\ell^2 \csch^2[\ell(\xi + c - \omega \mathfrak{I})]. \quad (5.19)$$

5.2. Example 2

Consider the approximate long wave (ALW) equations with arbitrary order

$$D^\varphi_u = -u_{\xi} - \frac{\partial u}{\partial \xi} - \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2},$$
$$D^\varphi_v = -u_{\xi} - \frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2}, \quad 0 < \varphi \leq 1, \quad (5.20)$$

with initial source

$$u(\xi, 0) = \omega - \ell \coth[\ell(\xi + c)],$$
$$v(\xi, 0) = -\ell^2 \csch^2[\ell(\xi + c)]. \quad (5.21)$$

By applying the Natural transform to Eq (5.20), we have

$$\mathcal{N}[D^\varphi_u(\xi, \mathfrak{I})] = -\mathcal{N} \left[ u_{\xi} - \frac{\partial u}{\partial \xi} - \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} \right],$$
$$\mathcal{N}[D^\varphi_v(\xi, \mathfrak{I})] = -\mathcal{N} \left[ \frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} \right]. \quad (5.22)$$
Define the non-linear operator as

\[
\frac{1}{s^\nu} \mathcal{N}[u(\xi, \mathcal{J})] - s^{2-\nu} u(\xi, 0) = \mathcal{N} \left[ -u \frac{\partial u}{\partial \xi} - v \frac{\partial v}{\partial \xi} - \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} \right],
\]

\[
\frac{1}{s^\nu} \mathcal{N}[v(\xi, \mathcal{J})] - s^{2-\nu} v(\xi, 0) = \mathcal{N} \left[ -u \frac{\partial v}{\partial \xi} - v \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} \right].
\]

(5.23)

The above equation is reduced to by simplifying it

\[
\mathcal{N}[u(\xi, \mathcal{J})] = s^2 \left[ \omega - \ell \coth[\ell(\xi + c)] \right] + \frac{\varphi(s - \varphi(s - \varphi))}{s^2} \mathcal{N} \left[ -u \frac{\partial u}{\partial \xi} - \frac{\partial v}{\partial \xi} - \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} \right],
\]

\[
\mathcal{N}[v(\xi, \mathcal{J})] = s^2 \left[ -\ell^2 \csch^2[\ell(\xi + c)] \right] + \frac{\varphi(s - \varphi(s - \varphi))}{s^2} \mathcal{N} \left[ -u \frac{\partial v}{\partial \xi} - v \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} \right].
\]

(5.24)

We get by applying inverse \( NT \) to Eq (5.24)

\[
u(\xi, \mathcal{J}) = -\ell^2 \csch^2[\ell(\xi + c)] + \mathcal{N}^{-1} \left[ \frac{\varphi(s - \varphi(s - \varphi))}{s^2} \mathcal{N} \left[ -u \frac{\partial v}{\partial \xi} - v \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} \right] \right].
\]

(5.25)

5.2.1. Now we apply \( NTDM_{CF} \)

Assume that the unknown functions \( u(\xi, \mathcal{J}) \) and \( v(\xi, \mathcal{J}) \) yield the infinite series result as follows

\[
u(\xi, \mathcal{J}) = \sum_{l=0}^{\infty} v_i(\xi, \mathcal{J}) \quad \text{and} \quad v(\xi, \mathcal{J}) = \sum_{l=0}^{\infty} v_i(\xi, \mathcal{J}).
\]

(5.26)

The Adomian polynomials represent the non-linear terms and are denoted by \( uu_\xi = \sum_{l=0}^{\infty} A_l, \quad uv_\xi = \sum_{l=0}^{\infty} B_l \) and \( vu_\xi = \sum_{l=0}^{\infty} C_l \). We can write the Eq (5.25) as a result of applying these terms,

\[
\sum_{l=0}^{\infty} u_l(\xi, \mathcal{J}) = \omega - \ell \coth[\ell(\xi + c)] + \mathcal{N}^{-1} \left[ \varphi(s - \varphi(s - \varphi)) \mathcal{N} \left\{ -\sum_{l=0}^{\infty} A_l - \sum_{l=0}^{\infty} v_l \xi - \frac{1}{2} \sum_{l=0}^{\infty} v_l \xi \right\} \right].
\]

\[
\sum_{l=0}^{\infty} v_l(\xi, \mathcal{J}) = -\ell^2 \csch^2[\ell(\xi + c)] + \mathcal{N}^{-1} \left[ \varphi(s - \varphi(s - \varphi)) \mathcal{N} \left\{ -\sum_{l=0}^{\infty} B_l - \sum_{l=0}^{\infty} C_l + \frac{1}{2} \sum_{l=0}^{\infty} v_l \xi \right\} \right].
\]

(5.27)

We can write as follows by comparing both sides of Eq (5.27)

\[
u_0(\xi, \mathcal{J}) = \omega - \ell \coth[\ell(\xi + c)],
\]

\[
u_0(\xi, \mathcal{J}) = -\ell^2 \csch^2[\ell(\xi + c)].
\]

\[
u_1(\xi, \mathcal{J}) = -\ell^2 \omega \csch^2[\ell(\xi + c)] (\varphi(\mathcal{J} - 1) + 1),
\]

\[
u_1(\xi, \mathcal{J}) = -2\ell^3 \omega \coth[\ell(\xi + c)] \csch^2[\ell(\xi + c)] (\varphi(\mathcal{J} - 1) + 1).
\]
The Adomian polynomials represent the non-linear terms and are denoted by $u_\ell$ and $v_\ell$ for $(l \geq 3)$ can be performed easily. The series form result is calculated as follows:

$$
u_2(\xi, \xi) = - \ell^2 \omega \text{csch}^2[\ell(\xi + c)](\varphi(\xi - 1) + 1) - 2 \ell^2 \omega^2 \coth[\ell(\xi + c)]$$

$$\text{csch}^2[\ell(\xi + c)]((1 - \varphi)^2 + 2\varphi(1 - \varphi)\xi + \frac{\varphi^2 \xi^2}{2}),$$

$$v_2(\xi, \xi) = -2 \ell^3 \omega \coth[\ell(\xi + c)] \text{csch}^2[\ell(\xi + c)]((\varphi(\xi - 1) + 1) - 2 \ell^4 \omega^2(2 + \cosh[2\ell(\xi + c)])$$

$$\text{csch}^4[\ell(\xi + c)]((1 - \varphi)^2 + 2\varphi(1 - \varphi)\xi + \frac{\varphi^2 \xi^2}{2}).$$

The remaining components of the Natural decomposition technique result $u_l$ and $v_l$ for $(l \geq 3)$ can be performed easily. The series form result is calculated as follows:

$$u(\xi, \xi) = \sum_{l=0}^{\infty} u_l(\xi, \xi) = u_0(\xi, \xi) + u_1(\xi, \xi) + u_2(\xi, \xi) + \cdots,$$

$$u(\xi, \xi) = \omega - \ell \coth[\ell(\xi + c)] - \ell^2 \omega \text{csch}^2[\ell(\xi + c)]((\varphi(\xi - 1) + 1) - \ell^2 \omega \text{csch}^2[\ell(\xi + c)]((\varphi(\xi - 1) + 1)$$

$$- 2 \ell^2 \omega^2 \coth[\ell(\xi + c)] \text{csch}^2[\ell(\xi + c)]((1 - \varphi)^2 + 2\varphi(1 - \varphi)\xi + \frac{\varphi^2 \xi^2}{2}) + \cdots,$$

$$v(\xi, \xi) = \sum_{l=0}^{\infty} v_l(\xi, \xi) = v_0(\xi, \xi) + v_1(\xi, \xi) + v_2(\xi, \xi) + \cdots,$$

$$v(\xi, \xi) = -\ell^2 \text{csch}^2[\ell(\xi + c)] - 2 \ell^3 \omega \coth[\ell(\xi + c)] \text{csch}^2[\ell(\xi + c)]((\varphi(\xi - 1) + 1)$$

$$- 2 \ell^2 \omega \coth[\ell(\xi + c)] \text{csch}^2[\ell(\xi + c)]((\varphi(\xi - 1) + 1) - 2 \ell^4 \omega^2(2 + \cosh[2\ell(\xi + c)])$$

$$\text{csch}^4[\ell(\xi + c)]((1 - \varphi)^2 + 2\varphi(1 - \varphi)\xi + \frac{\varphi^2 \xi^2}{2}) \cdots.$$  

(5.28)

5.2.2. Now we apply $NTDM_{ABC}$

Assume that the unknown functions $u(\xi, \xi)$ and $v(\xi, \xi)$ yield the infinite series result as follows

$$u(\xi, \xi) = \sum_{l=0}^{\infty} u_l(\xi, \xi) \quad \text{and} \quad v(\xi, \xi) = \sum_{l=0}^{\infty} v_l(\xi, \xi)$$  

(5.29)

The Adomian polynomials represent the non-linear terms and are denoted by $uu_\xi = \sum_{l=0}^{\infty} A_l, uv_\xi = \sum_{l=0}^{\infty} B_l$ and $vu_\xi = \sum_{l=0}^{\infty} C_l$. We can write the Eq (5.25) as a result of applying these terms,

$$\sum_{l=0}^{\infty} u_l(\xi, \xi) = \left[ \omega - \ell \coth[\ell(\xi + c)] \right]$$

$$+ \mathcal{N}^{-1} \left[ \frac{\partial^\mu (s^\varphi + \varphi(\partial^\varphi - s^\varphi))}{s^{2\varphi}} \right] \mathcal{N} \left\{ - \sum_{l=0}^{\infty} A_l - \sum_{l=0}^{\infty} v_\xi - \frac{1}{2} \sum_{l=0}^{\infty} v_{\xi \xi} \right\},$$

(5.30)

$$\sum_{l=0}^{\infty} v_l(\xi, \xi) = \left[ - \ell^2 \text{csch}^2[\ell(\xi + c)] \right]$$

$$+ \mathcal{N}^{-1} \left[ \frac{\partial^\mu (s^\varphi + \varphi(\partial^\varphi - s^\varphi))}{s^{2\varphi}} \right] \mathcal{N} \left\{ - \sum_{l=0}^{\infty} B_l - \sum_{l=0}^{\infty} C_l - \frac{1}{2} \sum_{l=0}^{\infty} v_{\xi \xi} \right\}.$$
We can write as follows by comparing both sides of Eq (5.30)

\[ u_0(\xi, \mathfrak{J}) = \omega - \ell \coth[\ell(\xi + c)], \]
\[ v_0(\xi, \mathfrak{J}) = -\ell^2 \text{csch}^2[\ell(\xi + c)]. \]

\[ u_1(\xi, \mathfrak{J}) = -\ell^2 \omega \text{csch}^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{3\varphi^3}{(\varphi + 1)} \right), \]
\[ v_1(\xi, \mathfrak{J}) = -2\ell^3 \omega \coth[\ell(\xi + c)] \text{csch}^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{3\varphi^3}{(\varphi + 1)} \right). \]

\[ u_2(\xi, \mathfrak{J}) = -\ell^2 \omega \text{csch}^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{3\varphi^3}{(\varphi + 1)} \right) - 2\ell^3 \omega^2 \coth[\ell(\xi + c)] \]
\[ \text{csch}^2[\ell(\xi + c)] \left[ \frac{3\varphi^3}{(2\varphi + 1)} + 2\varphi(1 - \varphi) \frac{3\varphi}{(\varphi + 1)} + (1 - \varphi)^2 \right] \]
\[ v_2(\xi, \mathfrak{J}) = -2\ell^3 \omega \coth[\ell(\xi + c)] \text{csch}^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{3\varphi^3}{(\varphi + 1)} \right) - 2\ell^4 \omega^2(2 + \cosh[2\ell(\xi + c)]) \]
\[ \text{csch}^4[\ell(\xi + c)] \left[ \frac{3\varphi^3}{(2\varphi + 1)} + 2\varphi(1 - \varphi) \frac{3\varphi}{(\varphi + 1)} + (1 - \varphi)^2 \right] \cdot \]

The remaining components of the Natural decomposition technique result \(u_l\) and \(v_l(l \geq 3)\) can be performed easily. The series form result is calculated as follows:

\[ u(\xi, \mathfrak{J}) = \sum_{l=0}^{\infty} u_l(\xi, \mathfrak{J}) = u_0(\xi, \mathfrak{J}) + u_1(\xi, \mathfrak{J}) + u_2(\xi, \mathfrak{J}) + \cdots, \]

\[ u(\xi, \mathfrak{J}) = \omega - \ell \coth[\ell(\xi + c)] - 2\ell^3 \omega \coth[\ell(\xi + c)] \text{csch}^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{3\varphi^3}{(\varphi + 1)} \right) \]
\[ - \ell^2 \omega \text{csch}^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{3\varphi^3}{(\varphi + 1)} \right) - 2\ell^3 \omega^2 \coth[\ell(\xi + c)] \text{csch}^2[\ell(\xi + c)] \left[ \frac{3\varphi^3}{(2\varphi + 1)} \right] \]
\[ + 2\varphi(1 - \varphi) \frac{3\varphi}{(\varphi + 1)} + (1 - \varphi)^2 \right] \cdots, \]

\[ v(\xi, \mathfrak{J}) = \sum_{l=0}^{\infty} v_l(\xi, \mathfrak{J}) = v_0(\xi, \mathfrak{J}) + v_1(\xi, \mathfrak{J}) + v_2(\xi, \mathfrak{J}) + \cdots, \]

\[ v(\xi, \mathfrak{J}) = -\ell^2 \text{csch}^2[\ell(\xi + c)] - 2\ell^3 \omega \coth[\ell(\xi + c)] \text{csch}^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{3\varphi^3}{(\varphi + 1)} \right) - 2\ell^3 \omega \coth[\ell(\xi + c)] \]
\[ \text{csch}^2[\ell(\xi + c)] \left( 1 - \varphi + \frac{3\varphi^3}{(\varphi + 1)} \right) - 2\ell^4 \omega^2(2 + \cosh[2\ell(\xi + c)]) \text{csch}^4[\ell(\xi + c)] \left[ \frac{3\varphi^3}{(2\varphi + 1)} \right] \]
\[ + 2\varphi(1 - \varphi) \frac{3\varphi}{(\varphi + 1)} + (1 - \varphi)^2 \right] \cdots. \]

(5.31)

For Eq (5.20), the exact result is obtained at \( \varphi = 1 \)

\[ u(\xi, \mathfrak{J}) = \omega - \ell \coth[\ell(\xi + c - \omega \mathfrak{J})], \]
\[ v(\xi, \mathfrak{J}) = -\ell^2 \text{csch}^2[\ell(\xi + c - \omega \mathfrak{J})]. \]
5.3. Numerical results and discussion

We used two unique methods to investigate the numerical solution of systems of coupled modified Boussinesq and approximate long wave equations with fractional order in this work. Through Maple, you can find numerical data for the system of coupled modified Boussinesq and approximate long wave equations for any order at various values of space and time variables. For the system in Problem 1, we construct numerical simulations at various values of $\xi$ and $\Im$ in Tables 1 and 2. Tables 3 and 4 show a numerical comparison of the variational iteration method, Adomian decomposition method and Natural decomposition method in terms of absolute error for Eq (5.1).

| $\Im$ | $\xi$ | $\varphi = 0.4$ | $\varphi = 0.6$ | $\varphi = 0.8$ | $\varphi = 1$ (approx) | $\varphi = 1$ (exact) |
|------|------|----------------|----------------|----------------|----------------------|----------------------|
| 0.2  | 0.2  | -0.254859      | -0.254834      | -0.254820      | -0.254801            | -0.254799            |
| 0.4  | 0.6  | -0.249632      | -0.249608      | -0.249595      | -0.249577            | -0.249574            |
|      | 0.8  | -0.247212      | -0.247188      | -0.247176      | -0.247158            | -0.247155            |
|      | 1    | -0.244911      | -0.244888      | -0.244876      | -0.244859            | -0.244855            |
| 0.4  | 0.2  | -0.254864      | -0.254850      | -0.254838      | -0.254818            | -0.254812            |
|      | 0.6  | -0.249637      | -0.249623      | -0.249612      | -0.249593            | -0.249586            |
|      | 0.8  | -0.247217      | -0.247203      | -0.247193      | -0.247174            | -0.247167            |
|      | 1    | -0.244915      | -0.244902      | -0.244892      | -0.244874            | -0.244866            |
| 0.6  | 0.2  | -0.254868      | -0.254861      | -0.254853      | -0.254835            | -0.254826            |
|      | 0.4  | -0.252187      | -0.252169      | -0.252158      | -0.252138            | -0.252132            |
|      | 0.6  | -0.249640      | -0.249634      | -0.249626      | -0.249609            | -0.249599            |
|      | 0.8  | -0.247220      | -0.247214      | -0.247206      | -0.247189            | -0.247178            |
|      | 1    | -0.244918      | -0.244912      | -0.244905      | -0.244889            | -0.244877            |
| 0.8  | 0.2  | -0.254874      | -0.254870      | -0.254866      | -0.254851            | -0.254840            |
|      | 0.4  | -0.252192      | -0.252189      | -0.252185      | -0.252171            | -0.252158            |
|      | 0.6  | -0.249644      | -0.249642      | -0.249638      | -0.249624            | -0.249611            |
|      | 0.8  | -0.247226      | -0.247222      | -0.247218      | -0.247204            | -0.247190            |
|      | 1    | -0.244923      | -0.244921      | -0.244917      | -0.244904            | -0.244889            |
| 1    | 0.2  | -0.254882      | -0.254879      | -0.254878      | -0.254868            | -0.254854            |
|      | 0.4  | -0.252199      | -0.252197      | -0.252196      | -0.252187            | -0.252171            |
|      | 0.6  | -0.249654      | -0.249650      | -0.249649      | -0.249640            | -0.249623            |
|      | 0.8  | -0.247234      | -0.247230      | -0.247229      | -0.247220            | -0.247202            |
|      | 1    | -0.244932      | -0.244928      | -0.244927      | -0.244919            | -0.244900            |
Table 2. The exact, $NTDM_{CF}$ and $NTDM_{ABC}$ results of $v(\xi, \Im)$ for Problem 1 at various fractional-order of $\varphi$ having different values of $\xi$ and $\Im$.

| $\Im$ | $\xi$ | $\varphi = 0.4$ | $\varphi = 0.6$ | $\varphi = 0.8$ | $\varphi = 1$ (approx) | $\varphi = 1$ (exact) |
|-------|-------|-----------------|-----------------|-----------------|----------------------|----------------------|
| 0.2   | 0.2   | -0.013760       | -0.013754       | -0.013751       | -0.013747            | -0.013747            |
| 0.4   | 0.6   | -0.013066       | -0.013061       | -0.013058       | -0.013055            | -0.013055            |
| 0.8   | 0.8   | -0.0112414      | -0.0112410      | -0.0112407      | -0.0112404           | -0.0112404           |
|       | 0.4   | -0.012415       | -0.012413       | -0.012411       | -0.012407            | -0.012407            |
|       | 0.8   | -0.011802       | -0.011799       | -0.011797       | -0.011794            | -0.011794            |
|       | 0.2   | -0.011223       | -0.011219       | -0.011217       | -0.011213            | -0.011213            |
|       | 0.4   | -0.013067       | -0.013064       | -0.013062       | -0.013058            | -0.013058            |
|       | 0.8   | -0.011803       | -0.011801       | -0.011799       | -0.011797            | -0.011797            |
|       | 0.4   | -0.013068       | -0.013067       | -0.013065       | -0.013061            | -0.013061            |
|       | 0.8   | -0.011803       | -0.011801       | -0.011799       | -0.011797            | -0.011797            |
|       | 0.4   | -0.013068       | -0.013067       | -0.013065       | -0.013061            | -0.013061            |
|       | 0.8   | -0.011803       | -0.011801       | -0.011799       | -0.011797            | -0.011797            |
|       | 0.2   | -0.011223       | -0.011219       | -0.011217       | -0.011213            | -0.011213            |
|       | 0.4   | -0.013067       | -0.013064       | -0.013062       | -0.013058            | -0.013058            |
|       | 0.8   | -0.011803       | -0.011801       | -0.011799       | -0.011797            | -0.011797            |
|       | 0.2   | -0.013764       | -0.013762       | -0.013761       | -0.013758            | -0.013758            |
|       | 0.4   | -0.013071       | -0.013069       | -0.013068       | -0.013065            | -0.013065            |
|       | 0.8   | -0.011805       | -0.011803       | -0.011802       | -0.011800            | -0.011800            |
|       | 0.2   | -0.013768       | -0.013766       | -0.013764       | -0.013762            | -0.013762            |
|       | 0.4   | -0.013074       | -0.013071       | -0.013070       | -0.013068            | -0.013068            |
|       | 0.8   | -0.011808       | -0.011806       | -0.011804       | -0.011803            | -0.011803            |
|       | 0.2   | -0.013768       | -0.013766       | -0.013764       | -0.013762            | -0.013762            |
|       | 0.4   | -0.013074       | -0.013071       | -0.013070       | -0.013068            | -0.013068            |
|       | 0.8   | -0.011808       | -0.011806       | -0.011804       | -0.011803            | -0.011803            |
|       | 0.2   | -0.013768       | -0.013766       | -0.013764       | -0.013762            | -0.013762            |
|       | 0.4   | -0.013074       | -0.013071       | -0.013070       | -0.013068            | -0.013068            |
|       | 0.8   | -0.011808       | -0.011806       | -0.011804       | -0.011803            | -0.011803            |
Table 3. Error comparison between ADM [64], VIM [65], NTDM$_{CF}$ and NTDM$_{ABC}$ for $u(\xi, \zeta)$ of problem 1 at $\varphi = 1$.

| $\zeta$ | $\xi$ | $|u_{Exact} - u_{ADM}|$ | $|u_{Exact} - u_{VIM}|$ | $|u_{Exact} - u_{NTDM_{CF}}|$ | $|u_{Exact} - u_{NTDM_{ABC}}|$ |
|---------|-------|---------------------|---------------------|---------------------|---------------------|
| 0.1     | 0.1   | 8.16297E-7          | 6.35269E-5          | 9.0000E-10          | 9.0000E-10          |
| 0.1     | 0.3   | 7.64245E-7          | 1.90854E-4          | 9.0000E-10          | 9.0000E-10          |
| 0.1     | 0.5   | 7.16083E-7          | 3.18549E-4          | 9.0000E-10          | 9.0000E-10          |
| 0.2     | 0.1   | 3.26243E-6          | 6.18930E-5          | 1.9000E-9           | 1.9000E-9           |
| 0.2     | 0.3   | 3.05458E-6          | 1.85945E-4          | 1.9000E-9           | 1.9000E-9           |
| 0.2     | 0.5   | 2.86226E-6          | 3.10352E-4          | 1.8000E-9           | 1.9000E-9           |
| 0.3     | 0.1   | 7.33445E-7          | 6.03095E-5          | 2.9000E-9           | 2.9000E-9           |
| 0.3     | 0.3   | 6.86758E-6          | 1.81187E-4          | 2.8000E-9           | 2.8000E-9           |
| 0.3     | 0.5   | 6.43557E-6          | 3.02408E-4          | 2.7000E-9           | 2.7000E-9           |
| 0.1     | 0.1   | 1.30286E-5          | 5.87746E-5          | 3.9000E-9           | 3.9000E-9           |
| 0.1     | 0.3   | 1.22000E-5          | 1.76574E-4          | 3.7000E-9           | 3.7000E-9           |
| 0.1     | 0.5   | 1.14333E-5          | 2.94707E-4          | 3.6000E-9           | 3.6000E-9           |
| 0.3     | 0.1   | 2.03415E-5          | 5.72867E-5          | 4.9000E-9           | 4.9000E-9           |
| 0.3     | 0.3   | 1.90489E-5          | 1.72102E-4          | 4.7000E-9           | 4.7000E-9           |
| 0.3     | 0.5   | 1.78528E-5          | 2.87241E-4          | 4.5000E-9           | 4.5000E-9           |

Table 4. Error comparison between ADM [64], VIM [65], NTDM$_{CF}$ and NTDM$_{ABC}$ for $v(\xi, \zeta)$ of problem 1 at $\varphi = 1$.

| $\zeta$ | $\xi$ | $|v_{Exact} - v_{ADM}|$ | $|v_{Exact} - v_{VIM}|$ | $|v_{Exact} - v_{NTDM_{CF}}|$ | $|v_{Exact} - v_{NTDM_{ABC}}|$ |
|---------|-------|---------------------|---------------------|---------------------|---------------------|
| 0.1     | 0.1   | 5.88676E-5          | 1.65942E-5          | 5.8000E-10          | 5.8000E-10          |
| 0.1     | 0.3   | 5.56914E-5          | 4.98691E-5          | 5.4000E-10          | 5.4000E-10          |
| 0.1     | 0.5   | 5.27169E-5          | 8.32598E-5          | 4.9000E-10          | 4.9000E-10          |
| 0.2     | 0.1   | 1.18213E-4          | 1.06813E-5          | 2.3200E-9           | 2.3200E-9           |
| 0.2     | 0.3   | 1.11833E-4          | 4.83269E-5          | 2.1400E-9           | 2.1400E-9           |
| 0.2     | 0.5   | 1.05858E-4          | 8.06837E-5          | 1.9900E-9           | 1.9900E-9           |
| 0.3     | 0.1   | 1.78041E-4          | 1.55880E-5          | 5.2300E-9           | 5.2300E-9           |
| 0.3     | 0.3   | 1.68429E-4          | 4.68440E-5          | 4.8300E-9           | 4.8300E-9           |
| 0.3     | 0.5   | 1.59428E-4          | 7.82068E-5          | 4.4700E-9           | 4.4700E-9           |
| 0.4     | 0.1   | 2.38356E-4          | 1.51135E-5          | 9.3000E-9           | 9.3000E-9           |
| 0.4     | 0.3   | 2.25483E-4          | 4.54174E-5          | 8.5800E-9           | 8.5800E-9           |
| 0.4     | 0.5   | 2.13430E-4          | 7.58243E-5          | 7.9600E-9           | 7.9600E-9           |
| 0.5     | 0.1   | 2.99162E-4          | 1.46569E-5          | 1.4530E-8           | 1.4530E-8           |
| 0.5     | 0.3   | 2.83001E-4          | 4.40448E-5          | 1.3430E-8           | 1.3430E-8           |
| 0.5     | 0.5   | 2.67868E-4          | 7.35317E-5          | 1.2430E-8           | 1.2430E-8           |

The outcomes of a calculations for the coupled system considered in Problem 2 are shown in Tables 5 and 6. Similarly, in Tables 7 and 8, we compare the Adomian decomposition method, natural decomposition method and variational iteration method solutions to Eq (5.20).
Table 5. The exact, $NTDM_{CF}$ and $NTDM_{ABC}$ results of $u(\xi, \Im)$ for Problem 2 at various fractional-order of $\varphi$ having different values of $\xi$ and $\Im$.

| $\xi$ | $\Im$ | $\varphi = 0.4$ | $\varphi = 0.6$ | $\varphi = 0.8$ | $\varphi = 1(approx)$ | $\varphi = 1(exact)$ |
|-------|-------|----------------|----------------|----------------|---------------------|-------------------|
| 0.2   |       | -0.124923      | -0.124912      | -0.124907      | -0.124899           | -0.124899         |
| 0.4   |       | -0.123582      | -0.123572      | -0.123567      | -0.123559           | -0.123559         |
| 0.6   |       | -0.122308      | -0.122299      | -0.122294      | -0.122287           | -0.122287         |
| 0.8   |       | -0.121098      | -0.121089      | -0.121084      | -0.121077           | -0.121077         |
| 1     |       | -0.119947      | -0.119938      | -0.119934      | -0.119927           | -0.119927         |
| 0.2   |       | -0.124925      | -0.124919      | -0.124914      | -0.124906           | -0.124906         |
| 0.4   |       | -0.123584      | -0.123578      | -0.123574      | -0.123566           | -0.123566         |
| 0.6   |       | -0.122310      | -0.122305      | -0.122300      | -0.122293           | -0.122293         |
| 0.8   |       | -0.121100      | -0.121094      | -0.121090      | -0.121083           | -0.121083         |
| 1     |       | -0.119948      | -0.119943      | -0.119940      | -0.119933           | -0.119933         |
| 0.2   |       | -0.124926      | -0.124924      | -0.124920      | -0.124913           | -0.124913         |
| 0.4   |       | -0.123585      | -0.123583      | -0.123580      | -0.123572           | -0.123572         |
| 0.6   |       | -0.122311      | -0.122309      | -0.122306      | -0.122299           | -0.122299         |
| 0.8   |       | -0.121101      | -0.121098      | -0.121096      | -0.121089           | -0.121089         |
| 1     |       | -0.119950      | -0.119947      | -0.119945      | -0.119938           | -0.119938         |
| 0.2   |       | -0.124927      | -0.124928      | -0.124926      | -0.124920           | -0.124920         |
| 0.4   |       | -0.123588      | -0.123586      | -0.123585      | -0.123579           | -0.123579         |
| 0.6   |       | -0.122315      | -0.122312      | -0.122311      | -0.122305           | -0.122305         |
| 0.8   |       | -0.121104      | -0.121102      | -0.121100      | -0.121095           | -0.121095         |
| 1     |       | -0.119953      | -0.119950      | -0.119949      | -0.119944           | -0.119944         |
| 0.2   |       | -0.124935      | -0.124933      | -0.124931      | -0.124927           | -0.124927         |
| 0.4   |       | -0.123592      | -0.123590      | -0.123589      | -0.123585           | -0.123585         |
| 0.6   |       | -0.122319      | -0.122316      | -0.122315      | -0.122311           | -0.122311         |
| 0.8   |       | -0.121107      | -0.121106      | -0.121104      | -0.121101           | -0.121101         |
| 1     |       | -0.119958      | -0.119955      | -0.119953      | -0.119950           | -0.119950         |
Table 6. The exact, $NTDM_{CF}$ and $NTDM_{ABC}$ results of $u(\xi, \mathcal{I})$ for Problem 2 at various fractional-order of $\varphi$ having different values of $\xi$ and $\mathcal{I}$.

| $\xi$ | $\mathcal{I}$ | $\varphi = 0.4$ | $\varphi = 0.6$ | $\varphi = 0.8$ | $\varphi = 1(approx)$ | $\varphi = 1(exact)$ |
|-------|---------------|----------------|----------------|----------------|----------------------|----------------------|
| 0.2   | 0.4           | -0.006880      | -0.006877      | -0.006875      | -0.006873            | -0.006873            |
|       | 0.6           | -0.006533      | -0.006530      | -0.006529      | -0.006527            | -0.006527            |
| 0.2   | 0.6           | -0.006207      | -0.006205      | -0.006203      | -0.006202            | -0.006202            |
| 0.8   | 0.6           | -0.005900      | -0.005898      | -0.005897      | -0.005895            | -0.005895            |
| 1     | 0.6           | -0.005611      | -0.005609      | -0.005608      | -0.005606            | -0.005606            |
| 0.2   | 0.6           | -0.006880      | -0.006879      | -0.006877      | -0.006875            | -0.006875            |
| 0.4   | 0.6           | -0.006533      | -0.006532      | -0.006531      | -0.006529            | -0.006529            |
| 0.4   | 0.6           | -0.006207      | -0.006206      | -0.006205      | -0.006203            | -0.006203            |
| 0.8   | 0.6           | -0.005901      | -0.005899      | -0.005898      | -0.005897            | -0.005897            |
| 1     | 0.6           | -0.005612      | -0.005610      | -0.005608      | -0.005606            | -0.005606            |
| 0.2   | 0.6           | -0.006881      | -0.006880      | -0.006879      | -0.006877            | -0.006877            |
| 0.4   | 0.6           | -0.006534      | -0.006533      | -0.006532      | -0.006530            | -0.006530            |
| 0.6   | 0.6           | -0.006208      | -0.006207      | -0.006206      | -0.006205            | -0.006205            |
| 0.8   | 0.6           | -0.005901      | -0.005900      | -0.005900      | -0.005898            | -0.005898            |
| 1     | 0.6           | -0.005612      | -0.005611      | -0.005610      | -0.005608            | -0.005608            |
| 0.2   | 0.6           | -0.006881      | -0.006880      | -0.006880      | -0.006879            | -0.006879            |
| 0.4   | 0.6           | -0.006534      | -0.006534      | -0.006534      | -0.006532            | -0.006532            |
| 0.6   | 0.6           | -0.006208      | -0.006208      | -0.006208      | -0.006206            | -0.006206            |
| 0.8   | 0.6           | -0.005901      | -0.005901      | -0.005901      | -0.005900            | -0.005900            |
| 1     | 0.6           | -0.005612      | -0.005612      | -0.005612      | -0.005611            | -0.005611            |
| 0.2   | 0.6           | -0.006881      | -0.006882      | -0.006882      | -0.006881            | -0.006881            |
| 0.4   | 0.6           | -0.006534      | -0.006535      | -0.006535      | -0.006534            | -0.006534            |
| 0.6   | 0.6           | -0.006208      | -0.006209      | -0.006209      | -0.006208            | -0.006208            |
| 0.8   | 0.6           | -0.005901      | -0.005902      | -0.005902      | -0.005901            | -0.005901            |
| 1     | 0.6           | -0.005612      | -0.005613      | -0.005613      | -0.005612            | -0.005612            |
Table 7. Error comparison between ADM [64], VIM [65], NTD\textsubscript{CF} and NTD\textsubscript{ABC} for $u(\xi, \mathfrak{I})$ of Problem 2 at $\phi = 1$.

| $\mathfrak{I}$ | $\xi$ | $|u_{\text{Exact}} - u_{\text{ADM}}|$ | $|u_{\text{Exact}} - u_{\text{VIM}}|$ | $|u_{\text{Exact}} - u_{\text{NTD}_{\text{CF}}}|$ | $|u_{\text{Exact}} - u_{\text{NTD}_{\text{ABC}}}|$ |
|---|---|---|---|---|---|
| 0.1 | 0.1 | 8.02989E-6 | 1.23033E-4 | 8.0000E-10 | 8.0000E-10 |
| 0.5 | 0.3 | 7.38281E-6 | 3.69597E-4 | 9.0000E-10 | 9.0000E-10 |
| 0.5 | 0.5 | 6.79923E-6 | 4.92780E-4 | 8.0000E-10 | 8.0000E-10 |
| 0.1 | 0.1 | 3.23228E-5 | 1.69274E-5 | 3.7000E-9 | 3.7000E-9 |
| 0.5 | 0.3 | 7.3051E-5 | 6.55176E-5 | 7.7000E-9 | 7.7000E-9 |
| 0.5 | 0.5 | 6.19760E-5 | 2.12346E-5 | 7.3000E-9 | 7.3000E-9 |
| 0.1 | 0.1 | 7.32051E-5 | 1.12345E-5 | 8.2000E-9 | 8.2000E-9 |
| 0.5 | 0.3 | 6.73006E-5 | 6.55176E-5 | 7.7000E-9 | 7.7000E-9 |
| 0.5 | 0.5 | 6.19760E-5 | 2.12346E-5 | 7.3000E-9 | 7.3000E-9 |

Table 8. Error comparison between ADM [64], VIM [65], NTD\textsubscript{CF} and NTD\textsubscript{ABC} for $v(\xi, \mathfrak{I})$ of Problem 2 at $\phi = 1$.

| $\mathfrak{I}$ | $\xi$ | $|v_{\text{Exact}} - v_{\text{ADM}}|$ | $|v_{\text{Exact}} - v_{\text{VIM}}|$ | $|v_{\text{Exact}} - v_{\text{NTD}_{\text{CF}}}|$ | $|v_{\text{Exact}} - v_{\text{NTD}_{\text{ABC}}}|$ |
|---|---|---|---|---|---|
| 0.1 | 0.1 | 4.81902E-4 | 1.23033E-4 | 2.9100E-10 | 2.9100E-10 |
| 0.5 | 0.3 | 4.50818E-4 | 1.76000E-4 | 2.6800E-10 | 2.6800E-10 |
| 0.5 | 0.5 | 4.22221E-4 | 2.69597E-4 | 2.4900E-10 | 2.4900E-10 |
| 0.1 | 0.1 | 9.76644E-4 | 2.69597E-4 | 1.1620E-9 | 1.1620E-9 |
| 0.5 | 0.3 | 9.13502E-4 | 2.69597E-4 | 1.0730E-9 | 1.0730E-9 |
| 0.5 | 0.5 | 8.55426E-4 | 2.69597E-4 | 9.9400E-10 | 9.9400E-10 |
| 0.1 | 0.1 | 1.48482E-3 | 2.69597E-4 | 2.6150E-9 | 2.6150E-9 |
| 0.5 | 0.3 | 1.38858E-3 | 2.69597E-4 | 2.4170E-9 | 2.4170E-9 |
| 0.5 | 0.5 | 1.30009E-3 | 2.69597E-4 | 2.2380E-9 | 2.2380E-9 |
| 0.1 | 0.1 | 2.00070E-3 | 2.69597E-4 | 4.6480E-9 | 4.6480E-9 |
| 0.5 | 0.3 | 1.87661E-3 | 2.69597E-4 | 4.2960E-9 | 4.2960E-9 |
| 0.5 | 0.5 | 1.75670E-3 | 2.69597E-4 | 3.9800E-9 | 3.9800E-9 |
| 0.1 | 0.1 | 2.54396E-3 | 2.69597E-4 | 7.2650E-9 | 7.2650E-9 |
| 0.5 | 0.3 | 2.37815E-3 | 2.69597E-4 | 6.7150E-9 | 6.7150E-9 |
| 0.5 | 0.5 | 2.22578E-3 | 2.69597E-4 | 6.2190E-9 | 6.2190E-9 |

On the basis of the information in the above tables, we may say that the Natural decomposition approach produces more accurate results.

Figure 1 illustrates the behaviour of the exact and Natural decomposition technique result for $u(\xi, \mathfrak{I})$ of Problem 1, while Figure 2 displays the behavior of the analytical result at different fractional-orders.
of \( \varphi \). Figure 3 illustrates the solution for \( u(\xi, \Im) \) in various fractional-orders. Figure 4 demonstrates the behaviour of the exact and analytical solutions for \( v(\xi, \Im) \) and Figure 5 shows the absolute error for Example 1.

**Figure 1.** Example 1 exact and numerical solutions for \( u(\xi, \Im) \) at \( \varphi = 1 \).

**Figure 2.** Example 1 analytical solution graph for \( u(\xi, \Im) \) at \( \varphi = 0.8 \) and 0.6.

**Figure 3.** Example 1 analytical solution graph for \( u(\xi, \Im) \) at various values of \( \varphi \).
Figure 4. Example 1 exact and analytical solution of $v(\xi, \Im)$ at $\varphi = 1$.

Figure 5. Example 1 analytical solution graph for $v(\xi, \Im)$ at various values of $\varphi$.

Figure 6 shows the behaviour of the exact and Natural decomposition technique results for $u(\xi, \Im)$ of Problem 2, whereas Figure 7 displays the nature of the analytical solution at various fractional-orders of $\varphi$. Figure 8 shows the solution graph for Problem 2 at various fractional-orders of $u(\xi, \Im)$. Figure 9 shows the behavior of the exact and analytical solution for $v(\xi, \Im)$ whereas Figure 10 shows the absolute error of Problem 2. We draw the given graphs within domain $-100 \leq \xi \leq 100$ having $c = 10$, $\omega = 0.005$ and $\ell = 0.15$.

Figure 6. Example 2 exact and numerical solutions for $u(\xi, \Im)$ at $\varphi = 1$. 

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Figure 7. Example 2 analytical solution graph for $u(\xi, \Im)$ at $\varphi = 0.8$ and 0.6.

Figure 8. Example 2 analytical solution graph for $u(\xi, \Im)$ at various values of $\varphi$.

Figure 9. Example 2 exact and analytical solution of $v(\xi, \Im)$ at $\varphi = 1$. 
6. Conclusions

The natural decomposition technique is used to solve the coupled modified Boussinesq and approximate long wave equations in this paper. Two examples are solved to demonstrate and validate the effectiveness of the suggested technique. In comparison to existing analytical approaches for determining approximate solutions of nonlinear coupled fractional partial differential equations, the present method is efficient and straightforward. The derived results have been shown in graphical and tabular form. The proposed method generates a sequence of results in the form of a recurrence relation having greater accuracy and less calculations. In terms of absolute error, calculations were performed for both fractional coupled systems. At \( \psi = 1 \), a number of computational solutions are compared to well-known analytical techniques and the exact results. The benefits of the current methods include less calculations and greater accuracy. Moreover, the proposed method is shown to be simple and effective, and it may be applied to solve various fractional-order differential equation systems.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, New York: Academic Press, 1999. http://dx.doi.org/10.1016/s0076-5392(99)x8001-5
2. M. Caputo, Linear models of dissipation whose Q is almost frequency independent, *Geophys. J. Int.*, 13 (1967), 529–539. http://dx.doi.org/https://doi.org/10.1111/j.1365-246X.1967.tb02303.x
3. V. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus, *J. Comput. Appl. Math.*, **118** (2000), 241–259. http://dx.doi.org/10.1016/S0377-0427(00)00292-2

4. H. Jafari, S. Seifi, Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation, *Commun. Nonlinear Sci.*, **14** (2009), 2006–2012. http://dx.doi.org/10.1016/j.cnsns.2008.05.008

5. H. Jafari, S. Seifi, Solving a system of nonlinear fractional partial differential equations using homotopy analysis method, *Commun. Nonlinear Sci.*, **14** (2009), 1962–1969. http://dx.doi.org/10.1016/j.cnsns.2008.06.019

6. S. Momani, N. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, *Appl. Math. Comput.*, **182** (2006), 1083–1092. http://dx.doi.org/10.1016/j.amc.2006.05.008

7. K. Oldham, J. Spanier, *The fractional calculus: theory and applications of differentiation and integration to arbitrary order*, New York: Academic Press, 1974.

8. K. Diethelm, N. Ford, A. Freed, A predictor-corrector approach for the numerical solution of fractional differential equation, *Nonlinear Dynam.*, **29** (2002), 3–22. http://dx.doi.org/10.1023/A:1016592219341

9. K. Millerand, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.

10. S. Kemple, H. Beyer, Global and causal solutions of fractional differential equations, *Proceedings of 2nd international workshop*, 1997, 210–216.

11. A. Kilbas, J. Trujillo, Differential equations of fractional order: methods, results and problem, *Appl. Anal.*, **78** (2001), 153–192. http://dx.doi.org/10.1080/00036810108840931

12. R. Hilfer, Fractional calculus and regular variation in thermodynamics, In: *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000, 429–463. http://dx.doi.org/10.1142/9789812817747_0009

13. S. Saha Ray, B. Poddar, R. Bera, Analytical solution of a dynamic system containing fractional derivative of order one-half by Adomian decomposition method, *J. Appl. Mech.*, **72** (2005), 290–295. http://dx.doi.org/10.1115/1.1839184

14. S. Saha Ray, R. Bera, An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method, *Appl. Math. Comput.*, **167** (2005), 561–571. http://dx.doi.org/10.1016/j.amc.2004.07.020

15. S. Saha Ray, R. Bera, Analytical solution of a fractional diffusion equation by Adomian decomposition method, *Appl. Math. Comput.*, **174** (2006), 329–336. http://dx.doi.org/10.1016/j.amc.2005.04.082

16. S. Saha Ray, Exact solutions for time-fractional diffusion-wave equations by decomposition method, *Phys. Scr.*, **75** (2007), 53–61. http://dx.doi.org/10.1088/0031-8949/75/1/008

17. S. Saha Ray, A new approach for the application of Adomian decomposition method for the solution of fractional space diffusion equation with insulated ends, *Appl. Math. Comput.*, **202** (2008), 544–549. http://dx.doi.org/10.1016/j.amc.2008.02.043
18. S. Saha Ray, R. Bera, Analytical solution of the Bagley Torvik equation by Adomian decomposition method, *Appl. Math. Comput.*, **168** (2005), 398–410. http://dx.doi.org/10.1016/j.amc.2004.09.006

19. K. Nisar, K. Ali, M. Inc, M. Mehanna, H. Rezazadeh, L. Akinyemi, New solutions for the generalized resonant nonlinear Schrodinger equation, *Results Phys.*, **53** (2002), 105153. http://dx.doi.org/10.1016/j.rinp.2021.105153

20. M. Alesemi, N. Iqbal, A. Hamoud, The analysis of fractional-order proportional delay physical models via a novel transform, *Complexity*, **2022** (2022), 2431533. http://dx.doi.org/10.1155/2022/2431533

21. H. Yepez-Martinez, M. Khater, H. Rezazadeh, M. Inc, Analytical novel solutions to the fractional optical dynamics in a medium with polynomial law nonlinearity and higher order dispersion with a new local fractional derivative, *Phy, Lett, A*, **420** (2021), 127744. http://dx.doi.org/10.1016/j.physleta.2021.127744

22. P. Sunthrayuth, A. Zidan, S. Yao, R. Shah, M. Inc, The comparative study for solving fractional-order Fornberg-Whitham equation via ρ-Laplace transform, *Symmetry*, **13** (2021), 784. http://dx.doi.org/10.3390/sym13050784

23. K. Nonlaopon, A. Alsharif, A. Zidan, A. Khan, Y. Hamed, R. Shah, Numerical investigation of fractional-order Swift-Hohenberg equations via a novel transform, *Symmetry*, **13** (2021), 1263. http://dx.doi.org/10.3390/sym13071263

24. M. Naeem, A. Zidan, K. Nonlaopon, M. Syam, Z. Al-Zhour, R. Shah, A new analysis of fractional-order equal-width equations via novel techniques, *Symmetry*, **13** (2021), 886. http://dx.doi.org/10.3390/sym13050886

25. R. Agarwal, F. Mofarreh, R. Shah, W. Luangboon, K. Nonlaopon, An analytical technique, based on natural transform to solve fractional-order parabolic equations, *Entropy*, **23** (2021), 1086. http://dx.doi.org/10.3390/e23081086

26. M. Areshi, A. Khan, R. Shah, K. Nonlaopon, Analytical investigation of fractional-order Newell-Whitehead-Segel equations via a novel transform, *AIMS Mathematics*, **7** (2022), 6936–6958. http://dx.doi.org/10.3934/math.2022385

27. H. Khan, A. Khan, M. Al-Qurashi, R. Shah, D. Baleanu, Modified modelling for heat like equations within Caputo operator, *Energies*, **13** (2020), 2002. http://dx.doi.org/10.3390/en13082002

28. M. Alesemi, N. Iqbal, M. Abdo, Novel investigation of fractional-order Cauchy-reaction diffusion equation involving Caputo-Fabrizio operator, *J. Funct. Space.*, **2022** (2022), 4284060. http://dx.doi.org/10.1155/2022/4284060

29. H. Thabet, S. Kendre, J. Peters, Travelling wave solutions for fractional Korteweg-de Vries equations via an approximate-analytical method, *AIMS Mathematics*, **4** (2019), 1203–1222. http://dx.doi.org/10.3934/math.2019.4.1203

30. A. Iqbal, A. Akgul, R. Shah, A. Bariq, M. Mossa Al-Sawalha, A. Ali, On solutions of fractional-order gas dynamics equation by effective techniques, *J. Funct. Space.*, **2022** (2022), 3341754. http://dx.doi.org/10.1155/2022/3341754
31. W. Mohammed, N. Iqbal, Impact of the same degenerate additive noise on a coupled system of fractional space diffusion equations, *Fractals*, **30** (2022), 22400333. http://dx.doi.org/10.1142/S0218348X22400333

32. H. Eltayeb, Y. Abdalla, I. Bachar, M. Khabir, Fractional telegraph equation and its solution by natural transform decomposition method, *Symmetry*, **11** (2019), 334. http://dx.doi.org/10.3390/sym11030334

33. Hajira, H. Khan, A. Khan, P. Kumam, D. Baleanu, M. Arif, An approximate analytical solution of the Navier-Stokes equations within Caputo operator and Elzaki transform decomposition method, *Adv. Differ. Equ.*, **2020** (2020), 622. http://dx.doi.org/10.1186/s13662-020-03058-1

34. P. Sunthrayuth, F. Ali, A. Alderremy, R. Shah, S. Aly, Y. Hamed, J. Katle, The numerical investigation of fractional-order Zakharov-Kuznetsov equations, *Symmetry*, **11** (2019), 334. http://dx.doi.org/10.3390/sym11030334

35. Hajira, H. Khan, A. Khan, P. Kumam, D. Baleanu, M. Arif, An approximate analytical solution of the Navier-Stokes equations within Caputo operator and Elzaki transform decomposition method, *Adv. Differ. Equ.*, **2020** (2020), 622. http://dx.doi.org/10.1186/s13662-020-03058-1

36. F. Mirzaee, S. Rezaei, N. Samadyar, Application of combination schemes based on radial basis functions and finite difference to solve stochastic coupled nonlinear time fractional sine-Gordon equations, *Comp. Appl. Math.*, **41** (2022), 10. http://dx.doi.org/10.1007/s40314-021-01725-x

37. F. Mirzaee, S. Rezaei, N. Samadyar, Solution of time-fractional stochastic nonlinear sine-Gordon equation via finite difference and meshfree techniques, *Math. Method. Appl. Sci.*, **45** (2022), 3426–3438. http://dx.doi.org/10.1002/mma.7988

38. F. Mirzaee, S. Rezaei, N. Samadyar, Solving one-dimensional nonlinear stochastic Sine-Gordon equation with a new meshfree technique, *Int. J. Numer. Model. El.*, **34** (2021), 2856. http://dx.doi.org/10.1002/jnm.2856

39. F. Mirzaee, N. Samadyar, Combination of finite difference method and meshless method based on radial basis functions to solve fractional stochastic advection-diffusion equations, *Eng. Comput.*, **36** (2020), 1673–1686. http://dx.doi.org/10.1007/s00366-019-00789-y

40. F. Mirzaee, N. Samadyar, Numerical solution of time fractional stochastic Korteweg-de Vries equation via implicit meshless approach, *Iran. J. Sci. Technol. Trans. Sci.*, **43** (2019), 2905–2912. http://dx.doi.org/10.1007/s40495-019-00763-9

41. S. Abbasbandy, The application of homotopy analysis method to nonlinear equations arising in heat transfer, *Phys. Lett. A*, **360** (2006), 109–113. http://dx.doi.org/10.1016/j.physleta.2006.07.065

42. M. Khater, A. Jhangeer, H. Rezazadeh, L. Akinyemi, M. Ali Akbar, M. Inc, et al., New kinds of analytical solitary wave solutions for ionic currents on microtubules equation via two different techniques, *Opt. Quant. Electron.*, **53** (2021), 509. http://dx.doi.org/10.1007/s11082-021-03267-2

43. F. Samsami Khodadad, S. Mirhosseini-Alizamini, B. Günay, L. Akinyemi, H. Rezazadeh, M. Inc, Abundant optical solitons to the Sasa-Satsuma higher-order nonlinear Schrodinger equation, *Opt. Quant. Electron.*, **53** (2021), 702. http://dx.doi.org/10.1007/s11082-021-03338-4

44. A. Kanwal, C. Phang, J. Loh, New collocation scheme for solving fractional partial differential equations, *Hacet. J. Math. Stat.*, **49** (2020), 1107–1125. http://dx.doi.org/10.15672/hujms.459621
45. Y. Ng, C. Phang, J. Loh, A. Isah, Analytical solutions of incommensurate fractional differential equation systems with fractional order $1 < \alpha, \beta < 2$ via bivariate Mittag-Leffler functions, *AIMS Mathematics*, 7 (2022), 2281–2317. http://dx.doi.org/10.3934/math.2022130

46. N. Samadyara, Y. Ordokhania, F. Mirzaee, The couple of Hermite-based approach and Crank-Nicolson scheme to approximate the solution of two dimensional stochastic diffusion-wave equation of fractional order, *Eng. Anal. Bound. Elem.*, 118 (2020), 285–294. http://dx.doi.org/10.1016/j.enganabound.2020.05.010

47. N. Samadyara, Y. Ordokhania, F. Mirzaee, Hybrid Taylor and block-pulse functions operational matrix algorithm and its application to obtain the approximate solution of stochastic evolution equation driven by fractional Brownian motion, *Commun. Nonlinear Sci.*, 90 (2020), 105346. http://dx.doi.org/10.1016/j.cnsns.2020.105346

48. N. Samadyar, F. Mirzaee, Orthonormal Bernoulli polynomials collocation approach for solving stochastic Volterra integral equations of Abel type, *Int. J. Numer. Model. El.*, 33 (2020), 2688. http://dx.doi.org/10.1002/jnm.2688

49. F. Mirzaee, K. Sayevand, S. Rezaei, N. Samadyar, Finite difference and spline approximation for solving fractional stochastic advection-diffusion equation, *Iran. J. Sci. Technol. Trans. Sci.*, 45 (2021), 607–617. http://dx.doi.org/10.1007/s40995-020-01036-6

50. F. Mirzaee, S. Rezaei, N. Samadyar, Numerical solution of two-dimensional stochastic time-fractional Sine-Gordon equation on non-rectangular domains using finite difference and meshfree methods, *Eng. Anal. Bound. Elem.*, 127 (2021), 53–63. http://dx.doi.org/10.1016/j.enganabound.2021.03.009

51. H. Halidoua, S. Abbagariab, A. Houwec, M. Incdef, B. Thomasg, Rational W-shape solitons on a nonlinear electrical transmission line with Josephson junction, *Phys. Lett. A*, 430 (2022), 127951. http://dx.doi.org/10.1016/j.physleta.2022.127951

52. G. Whitham, Variational methods and applications to water waves, In: *Hyperbolic equations and waves*, Berlin: Springer, 1970. http://dx.doi.org/10.1007/978-3-642-87025-5_16

53. L. Broer, Approximate equations for long water waves, *Appl. sci. Res.*, 31 (1975), 377–395. http://dx.doi.org/10.1007/BF00418048

54. D. Kaup, A higher-order water-wave equation and the method for solving it, *Prog. Theor. Phys.*, 54 (1975), 396–408. http://dx.doi.org/10.1143/PTP.54.396

55. S. Saha Ray, A novel method for travelling wave solutions of fractional Whitham-Broer-Kaup, fractional modified Boussinesq and fractional approximate long wave equations in shallow water, *Math. Method. Appl. Sci.*, 38 (2015), 1352–1368. http://dx.doi.org/10.1002/mma.3151

56. K. Nonlaopon, M. Naeem, A. Zidan, R. Shah, A. Alsanad, A. Gumaei, Numerical investigation of the time-fractional Whitham-Broer-Kaup equation involving without singular kernel operators, *Complexity*, 2021 (2021), 7979365. http://dx.doi.org/10.1155/2021/7979365

57. R. Shah, H. Khan, D. Baleanu, Fractional Whitham-Broer-Kaup equations within modified analytical approaches, *Axioms*, 8 (2019), 125. http://dx.doi.org/10.3390/axioms8040125

58. K. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.
59. I. Podlubny, Fractional differential equations. In: Mathematics in science and engineering, San Diego: Academic Press, 1999, 1–340.

60. K. Diethelm, The analysis of fractional differential equations, Berlin: Springer-Verlag, 2010. http://dx.doi.org/10.1007/978-3-642-14574-2

61. M. Zhou, A. Ravi Kanth, K. Aruna, K. Raghavendar, H. Rezazadeh, M. Inc, et al., Numerical solutions of time fractional Zakharov-Kuznetsov equation via natural transform decomposition method with nonsingular kernel derivatives, J. Funct. Space., 2021 (2021), 9884027. http://dx.doi.org/10.1155/2021/9884027

62. G. Adomian, A new approach to nonlinear partial differential equations, J. Math. Anal. Appl., 102 (1984), 420–434. http://dx.doi.org/10.1016/0022-247X(84)90182-3

63. G. Adomian, Solving frontier problems of physics: the decomposition method, Dordrecht: Springer, 1994. http://dx.doi.org/10.1007/978-94-015-8289-6

64. S. El-Sayed, D. Kaya, Exact and numerical travelling wave solutions of Whitham-Broer-Kaup equations, Appl. Math. Comput., 167 (2005), 1339–1349. http://dx.doi.org/10.1016/j.amc.2004.08.012

65. M. Rafei, H. Daniali, Application of the variational iteration method to the Whitham-Broer-Kaup equations, Comput. Math. Appl., 54 (2007), 1079–1085. http://dx.doi.org/10.1016/j.camwa.2006.12.054

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