CLASSIFYING BICROSSED PRODUCTS OF HOPF ALGEBRAS

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Abstract. Let $A$ and $H$ be two Hopf algebras. We shall classify up to an isomorphism that stabilizes $A$ all Hopf algebras $E$ that factorize through $A$ and $H$ by a cohomological type object $H^2(A, H)$. Equivalently, we classify up to a left $A$-linear Hopf algebra isomorphism, the set of all bicrossed products $A owtie H$ associated to all possible matched pairs of Hopf algebras $(A, H, \triangleleft, \triangleright)$ that can be defined between $A$ and $H$. In the construction of $H^2(A, H)$ the key role is played by special elements of $CoZ^1(H, A) \times \text{Aut}_{CoAlg}(H)$, where $CoZ^1(H, A)$ is the group of unitary cocentral maps and $\text{Aut}_{CoAlg}(H)$ is the group of unitary automorphisms of the coalgebra $H$. Among several applications and examples, all bicrossed products $H_4 \bowtie k[C_n]$ are described by generators and relations and classified: they are quantum groups at roots of unity $H_{4n, \omega}$ which are classified by pure arithmetic properties of the ring $\mathbb{Z}_n$. The Dirichlet’s theorem on primes is used to count the number of types of isomorphisms of this family of $4n$-dimensional quantum groups. As a consequence of our approach the group $\text{Aut}_{Hopf}(H_{4n, \omega})$ of Hopf algebra automorphisms is fully described.

Introduction

The aim of this paper is to prove that there exists a very rich theory behind the so-called bicrossed product (double cross product in Majid’s terminology) of two objects arising from the factorization problem. This theory deserves to be developed further mainly because of its major impact in at least three different problems: the classification of objects of a given dimension, the development of a general descent type theory for a given extension $A \subseteq E$ recently introduced in [3] and the development of some new types of cohomologies that will be the key players for both problems. All results presented below provide an answer at the level of Hopf algebras for the classification problem and offer an argument for the role that bicrossed products can play. In particular, we describe a tempting way of approaching the classification problem for finite quantum groups. The pioneers of this approach, at the level of groups, were Douglas [20], Rédei [43] and Cohn [18] related to the problem of classifying all groups that factorize through two cyclic groups.

In order to maintain a general frame for our discussion, considering that bicrossed products of two objects were introduced and studied in various areas of mathematics, we will consider $\mathcal{C}$ a category whose objects are sets endowed with various algebraic, topological...
or differential structures. To illustrate, we can think of \( \mathcal{C} \) as the category of groups, groupoids or quantum groupoids, algebras, Hopf algebras, local compact groups or local compact quantum groups, Lie groups, Lie algebras and so on. Let \( A \) and \( H \) be two given objects of \( \mathcal{C} \). We say that an object \( E \in \mathcal{C} \) factorizes through \( A \) and \( H \) if \( E \) can be written as a 'product' of \( A \) and \( H \), where \( A \) and \( H \) are subobjects of \( E \) having minimal intersection. Here, the 'product' depends on the nature of the category. For instance, if \( \mathcal{C} \) is the category of groups, then a group \( E \) factorizes through two subgroups \( A \) and \( H \) if and only if \( E \cong A \oplus H \), that is \( E \) is the coproduct of \( A \) and \( H \). Now we take a step forward and consider \( \mathcal{C} = \mathcal{Gr} \), the category of groups. Here things change radically: the factorization problem becomes very difficult and it was formulated by O. Ore [38] but its roots are much older and descend to E. Maillet's 1900 paper [28]. It can be seen as the dual of the more famous extension problem of O. L. Hölder. Moreover, little progress has been made on this problem so far. For instance, in the case of two cyclic groups \( A \) and \( H \), not both finite, the problem was opened by Rédei in [43] and closed by Cohn in [18], without the classification part. If \( A \) and \( H \) are both finite cyclic groups the problem is more difficult and seems to be still an open question, even though J. Douglas [20] has devoted four papers to the subject. In [19] all groups that factorize through an alternating group or a symmetric group are completely described. One of the famous results in this direction remains Ito's theorem [25] proved in the 50's: any product of two abelian groups is a meta-abelian group. An important step in dealing with the factorization problem was the construction of the bicrossed product \( A \bowtie H \) associated to a matched pair \( (A, H, \triangleleft, \triangleright) \) of groups, given by Takeuchi [45]. A group \( E \) factorizes through two subgroups \( A \) and \( H \) if and only if there exists a matched pair of groups \( (A, H, \triangleleft, \triangleright) \) such that \( E \cong A \bowtie H \). Thus, at the level of groups, the factorization problem can be restated in a purely computational manner: Let \( A \) and \( H \) be two given groups. Describe the set of all matched pairs \( (A, H, \triangleleft, \triangleright) \) and classify up to an isomorphism all bicrossed products \( A \bowtie H \).

This is a strategy that has to be followed for the factorization problem in any category \( \mathcal{C} \). In fact, the bicrossed product construction at the level of groups served as a model for similar constructions in other fields of mathematics. The first step was made by Majid in [31, Proposition 3.12] where a twisted version of the bicrossed product of two Hopf algebras associated to a matched pair was introduced, under the name of double cross...
product. Its purpose was not to solve the factorization problem, but to give an elegant
description for the Drinfel’d double of a finite dimensional Hopf algebra. Then the
construction was performed for Lie Algebras [30], [35] and Lie groups [30], [27], algebras
[15] or more generally in [17], coalgebras [14], groupoids [1], [29], locally compact groups
[7] or locally compact quantum groups [46], and so on. At the level of algebras the bicrossed
product of two algebras is usually called the twisted tensor product algebra
and its construction plays an important role in noncommutative geometry in the sense
of Brezinski and Majid (see [10], [11], [12]).

To conclude, if we are only looking for the description part of the factorization problem,
formulated in an arbitrary category $C$, the following general principle should work: an
object $E \in C$ factorizes through $A$ and $H$ if and only if $E \cong A \rtimes H$, where $A \rtimes H$ is a
‘bicrossed product’ in the category $C$ associated to a ‘matched pair’ between the objects
$A$ and $H$. The classification part of the factorization problem is now clear: it consists
of classifying the bicrossed products $A \rtimes H$ associated to all matched pairs between $A$
and $H$. This is the strategy that we follow for the category of Hopf algebras. For other
categories, the steps taken in this direction are still shy, including the group case as
well as the algebra case. The problem of classifying bicrossed products of two algebras
started with [14, Examples 2.11] where all bicrossed products between two group algebras
of dimension two are completely described and classified. For recent results related to
the classification of bicrossed product of two algebras we refer to [40], [41], [22].

Looking at the classification part of the factorization problem for Hopf algebras we
emphasize that the theory of bicrossed products can play an important role, not exploited
until now, in the classification of finite quantum groups. The problem of classifying up
to isomorphism all Hopf algebras of a given dimension is one of the central themes
in Hopf algebra theory. There are complete classifications for Hopf algebras of given
small dimensions: see for instance, [5], [6], [8], [23], [34], [37], [39], [47] and their list of
references. However, there are no general methods of tackling this problem. Following
Hölder’s model from group extensions, the only method for classifying a certain class
(pointed, semisimple, etc) of Hopf algebras of a given dimension relies on the theory of
Hopf algebra extensions [5], [33] and the Radford biproduct. Unfortunately, for Hopf
algebras the method has its limitations and is not an exhaustive one. We shall highlight
a new way of classifying Hopf algebras of a given dimension using bicrossed products
instead of extension theory for Hopf algebras.

The paper is organized as follows. In Section 1 we recall the construction of the bicrossed
product associated to a matched pair of Hopf algebras $(A, H, \triangleleft, \triangleright)$. Majid’s Theorem 1.2
proves that the factorization problem for Hopf algebras can be restated in a computa-
tional manner exactly as in the group case: \textit{Let $A$ and $H$ be two given Hopf algebras.}
\textit{Describe the set of all matched pairs $(A, H, \triangleleft, \triangleright)$ and classify up to an isomorphism all
bicrossed products $A \triangleleft H$.}

Section 2 is devoted to proving some purely technical results which will be intensively
used throughout the paper. Theorem 2.2 describes completely the set of all morphisms of
Hopf algebras between two arbitrary bicrossed products. In particular, the set of all Hopf
algebra maps between two semi-direct (or smash) products of Hopf algebras is described
in Corollary 2.3. This result gives the first application: the parametrization of all Hopf algebra morphisms between two Drinfel’d doubles associated to two finite groups $G$ and $H$ is given in Corollary 2.4. In particular, if $G = H$, the space $\text{End}_{\text{Hopf}}(D(k[G]))$ of all Hopf algebra endomorphisms is fully described.

Section 3 deals with the classification part of the factorization problem. Let $A$ and $H$ be two given Hopf algebras. Theorem 3.8 is the classification theorem for bicrossed products: all Hopf algebras $E$ that factorize through $A$ and $H$ are classified up to an isomorphism that stabilizes $A$ by a cohomological type object $\mathcal{H}^2(A, H)$ in the construction of which the key role is played by pairs $(r, v) \in C(Z^1(H, A) \times \text{Aut}_{\text{CoAlg}}^1(H))$, consisting of a unitary cocommutative map $r : H \to A$ and a unitary automorphism of coalgebras $v : H \to H$ related by a certain compatibility condition. The classification of bicrossed products up to an isomorphism that stabilizes one of the terms has at least two strong motivations: the first one is the cohomological point of view which descends from the classification theory of Holder’s group extensions [41, Theorem 7.34] and the second one is the problem of describing and classifying the factorization $A$-forms of a Hopf algebra extension from descent theory [3]. We indicate only two of the several applications of Theorem 3.8: Corollary 3.9 gives necessary and sufficient conditions for two generalized quantum doubles $D_A(A, H)$ and $D_B(A, H')$ to be isomorphic as Hopf algebras and left $A$-modules. Corollary 3.10 is a Schur-Zassenhaus type theorem for the bicrossed product of Hopf algebras: it provides necessary and sufficient conditions for a bicrossed product $A \bowtie H$ to be isomorphic to a semi-direct product $A \# H'$ of Hopf algebras.

In Section 4 we provide some explicit examples for the factorization problem: for two given Hopf algebras $A$ and $H$ we describe by generators and relations and classify up to an isomorphism all Hopf algebras $E$ that factorize through $A$ and $H$. We go through the following three steps: first of all, we compute the set of all matched pairs between $A$ and $H$. This is the computational part of our schedule and can not be avoided. Then we describe by generators and relations the bicrossed products $A \bowtie H$ associated to all these matched pairs. Finally, using Theorem 2.2 we classify up to an isomorphism these bicrossed products $A \bowtie H$. As an application, the group $\text{Aut}_{\text{Hopf}}(A \bowtie H)$ of all Hopf algebra automorphisms of a given bicrossed product is computed.

Let $k$ be a field of characteristic $\neq 2$ and $H_4$ the Sweedler’s four dimensional Hopf algebra. For a positive integer $n$, let $C_n$ be the cyclic group of order $n$ generated by $c$ and $U_n(k) = \{\omega \in k \mid \omega^n = 1\}$ the cyclic group of $n$th roots of unity in $k$ of order $\nu(n) = |U_n(k)|$. The group $U_n(k)$ depends heavily on the base field $k$. Proposition 4.3 proves that $U_n(k)$ parameterizes the set of all matched pairs $(H_4, k[C_n], \langle c, \varphi \rangle)$, i.e. there exists a bijective correspondence between the set of all matched pairs $(H_4, k[C_n], \langle c, \varphi \rangle)$ and the group $U_n(k)$. Corollary 4.4 shows that a Hopf algebra $E$ factorizes through $H_4$ and $k[C_n]$ if and only if $E \cong H_{4n, \omega}$ for some $\omega \in U_n(k)$; this quantum group $H_{4n, \omega}$ at root of unity is described explicitly by generators and relations. It is a non-commutative non-cocommutative, pointed and non-semisimple $4n$-dimensional Hopf algebra. Theorem 4.7 and Theorem 4.10 describe precisely the number of types of isomorphisms of this family of Hopf algebras $\{H_{4n, \omega} \mid \omega \in U_n(k)\}$. The beauty of this classification result is given by Dirichlet’s theorem on primes in an arithmetical progression which was used in a key step in proving Theorem 4.10. Let $\nu(n) = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime decomposition of $\nu(n)$. If
Throughout this paper, \(\nu(n)\) is odd, then the number of types of such Hopf algebras is \((\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)\). On the other hand, if \(\nu(n) = 2^{\alpha_1}p_2^{\alpha_2} \cdots p_r^{\alpha_r}\) is even, then the number of types of such Hopf algebras is \(\alpha_1(\alpha_2 + 1) \cdots (\alpha_r + 1)\). As an application, the infinite abelian group \(\text{Aut}_{\text{Hopf}}(H_{4n}, \omega)\) of Hopf algebra automorphisms of \(H_{4n}, \omega\) is described in Corollary 1.12.

The results proven in this paper at the level of Hopf algebras can serve as a model for obtaining similar results for all the fields of mathematics where the bicrossed product is constructed.

1. Preliminaries

Throughout this paper, \(k\) will be a field. For a positive integer \(n\) we denote by \(U_n(k) = \{\omega \in k \mid \omega^n = 1\}\) the cyclic group of \(n\)-th roots of unity in \(k\), and by \(\nu(n) = |U_n(k)|\) the order of the group \(U_n(k)\). Of course, \(\nu(n)\) is a divisor of \(n\); if \(\nu(n) = n\), then any generator of \(U_n(k)\) is called a primitive \(n\)-th root of unity \([24]\). The group \(U_n(k)\) depends heavily on the base field \(k\). For instance, \(U_n(k) = \{1\}\), for any positive integer \(t\), if \(k\) is a field of characteristic \(p > 0\). If \(\text{Char}(k) = p\) and \(p|n\) then there is no primitive \(n\)-th root of unity in \(k\) \([24\text{ Pag. } 295]\). Furthermore, if \(k\) is a finite field with \(p^n\) elements, then \(\nu(n) = \gcd(n, p^n - 1)\). If \(\text{Char}(k) \nmid n\) and \(k\) is algebraically closed then \(\nu(n) = n\).

Unless specified otherwise, all algebras, coalgebras, bialgebras, Hopf algebras, tensor products and homomorphisms are over \(k\). For a coalgebra \(C\), we use Sweedler’s \(\Sigma\)-notation: \(\Delta(c) = c_{(1)} \otimes c_{(2)}\), \((I \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}\), etc (summation understood).

Let \(A\) and \(H\) be two Hopf algebras. \(H\) is called a right \(A\)-module coalgebra if \(H\) is a coalgebra in the monoidal category \(\mathcal{M}_A\) of right \(A\)-modules, i.e. there exists \(\triangleleft : H \otimes A \rightarrow H\) a morphism of coalgebras such that \((H, \triangleleft)\) is a right \(A\)-module. A morphism between two right \(A\)-module coalgebras \((H, \triangleleft)\) and \((H', \triangleleft')\) is a morphism of coalgebras \(\psi : H \rightarrow H'\) that is also right \(A\)-linear. Furthermore, \(\psi\) is called unitary if \(\psi(1_H) = 1_{H'}\).

Similarly, \(A\) is a left \(H\)-module coalgebra if \(A\) is a coalgebra in the monoidal category of left \(H\)-modules, that is there exists \(\triangleright : H \otimes A \rightarrow A\) a morphism of coalgebras such that \((A, \triangleright)\) is also a left \(H\)-module. For further computations, the fact that two \(k\)-linear maps \(\triangleleft : H \otimes A \rightarrow H\), \(\triangleright : H \otimes A \rightarrow A\) are coalgebra maps can be written explicitly as follows:

\[
\Delta_H(h \triangleleft a) = h_{(1)} \triangleleft a_{(1)} \otimes h_{(2)} \triangleleft a_{(2)}, \quad \varepsilon_A(h \triangleleft a) = \varepsilon_H(h)\varepsilon_A(a) \tag{1}
\]

\[
\Delta_A(h \triangleright a) = h_{(1)} \triangleright a_{(1)} \otimes h_{(2)} \triangleright a_{(2)}, \quad \varepsilon_A(h \triangleright a) = \varepsilon_H(h)\varepsilon_A(a) \tag{2}
\]

for all \(h \in H\), \(a \in A\). The actions \(\triangleleft : H \otimes A \rightarrow H\), \(\triangleright : H \otimes A \rightarrow A\) are called trivial if \(h \triangleleft a = \varepsilon_A(a)h\) and respectively \(h \triangleright a = \varepsilon_H(h)a\), for all \(a \in A\) and \(h \in H\).

Bicrossed product of Hopf algebras. The bicrossed product of two Hopf algebras was introduced by Majid in \([31\text{ Proposition 3.12}]\) under the name of double cross product. We shall adopt the name of bicrossed product from \([26\text{ Theorem IX 2.3}]\) that is also used for the similar construction at the level of groups \([15]\), groupoids \([4]\) etc. A matched pair of Hopf algebras is a system \((A, H, \triangleleft, \triangleright)\), where \(A\) and \(H\) are Hopf algebras, \(\triangleleft : H \otimes A \rightarrow H\), \(\triangleright : H \otimes A \rightarrow A\) are coalgebra maps such that \((A, \triangleright)\) is a left \(H\)-module.
coalgebra, \((H, □)\) is a right \(A\)-module coalgebra and the following compatibilities hold for any \(a, b \in A, g, h \in H\).

\[
G \triangleright 1_A = \varepsilon_H(h)1_A, \quad 1_H \triangleleft a = \varepsilon_A(a)1_H
\]

\[
g \triangleright (ab) = ((g_1) \triangleright a_1)(g_2 \triangleleft a_2) \triangleright b
\]

\[
(gh) \triangleleft a = (g \triangleleft (h_1 \triangleright a_1))(h_2 \triangleleft a_2)
\]

\[
g_1 \triangleleft a_1 \otimes g_2 \triangleright a_2 = g(2) \triangleleft a_2 \otimes g_1 \triangleright a_1
\]

Let \((A, H, □, △)\) be a matched pair of Hopf algebras; the bicrossed product \(A \bowtie H\) of \(A\) with \(H\) is the \(k\)-module \(A \otimes H\) with the multiplication given by

\[
(a \bowtie h) \cdot (c \bowtie g) := a(h_1 \triangleright c_1) \bowtie (h_2 \triangleleft c_2)g
\]

for all \(a, c \in A, h, g \in H\), where we denoted \(a \otimes h\) by \(a \bowtie h\). \(A \bowtie H\) is a Hopf algebra with the antipode

\[
S_{A \bowtie H}(a \bowtie h) = S_H(h_2) \triangleright S_A(a_2) \bowtie S_H(h_1) \triangleleft S_A(a_1)
\]

for all \(a \in A\) and \(h \in H\).

**Examples 1.1.** 1. Let \((A, □)\) be a left \(H\)-module coalgebra and consider \(H\) as a right \(A\)-module coalgebra via the trivial action, i.e. \(h \triangleleft a = \varepsilon_A(a)h\). Then \((A, H, □, △)\) is a matched pair of Hopf algebras if and only if \((A, □)\) is also a left \(H\)-module algebra and the following compatibility condition holds

\[
g_1 \otimes g_2 \triangleright a = g(2) \otimes g_1 \triangleright a
\]

for all \(g \in H\) and \(a \in A\). In this case, the associated bicrossed product \(A \bowtie H = A\#H\) is the left version of the semi-direct (smash) product of Hopf algebras as defined by Molnar [36] in the cocommutative case, for which the compatibility condition (9) holds automatically. Thus, \(A\#H\) is the \(k\)-module \(A \otimes H\), where the multiplication (7) takes the form:

\[
(a\#h) \cdot (c\#g) := a(h_1 \triangleright c_1)\# h_2 g
\]

for all \(a, c \in A, h, g \in H\), where we denoted \(a \otimes h\) by \(a\#h\). \(A\#H\) is a Hopf algebra with the coalgebra structure given by the tensor product of coalgebras and the antipode

\[
S_{A\#H}(a\#h) = S_H(h_2) \triangleright S_A(a_2) \# S_H(h_1)
\]

for all \(a \in A\) and \(h \in H\).

Similarly, let \((H, □)\) be a right \(A\)-module coalgebra and consider \(A\) as left \(H\)-module coalgebra via the trivial action, i.e \(h \triangleright a := \varepsilon_H(h)a\). Then \((A, H, □, △)\) is a matched pair of Hopf algebras if and only if \((H, □)\) is also a right \(A\)-module algebra and

\[
g \triangleleft a_1 \otimes a_2 = g \triangleleft a_2 \otimes a_1
\]

for all \(g \in H\) and \(a \in A\). In this case, the associated bicrossed product \(A \bowtie H = A\#^r H\) is the right version of the smash product of Hopf algebras. Thus, \(A\#^r H\) is the \(k\)-module \(A \otimes H\), where the multiplication (7) takes the form:

\[
(a\#^r h) \cdot (c\#^r g) := a c_1 \# (h \triangleleft c_2) g
\]
for all \(a, c \in A, h, g \in H\), where we denoted \(a \otimes h\) by \(a\#h\). \(A\#^r H\) is a Hopf algebra with the coalgebra structure given by the tensor product of coalgebras and the antipode

\[ S_{A\#^r H}(a\#h) = S_A(a_{(2)}) \# S_H(h) \triangleleft S_A(a_{(1)}) \]

for all \(a \in A\) and \(h \in H\).

2. Let \(G\) and \(K\) be two groups, \(A = k[G]\) and \(H = k[K]\) the group algebras. There exists a bijection between the set of all matched pairs of Hopf algebras \((k[G], k[K], \triangleleft, \triangleright)\) and the set of all matched pairs of groups \((G, K, \triangleleft, \triangleright)\) in the sense of Takeuchi [45]. The bijection is given such that there exists an isomorphism of Hopf algebras \(k[G] \triangleright k[K] \cong k[G \bowtie H]\), where \(G \bowtie H\) is the Takeuchi’s bicrossed product of groups ([26], pg. 207).

3. The fundamental example of a bicrossed product is the Drinfel’d double \(D(H)\). Let \(H\) be a finite dimensional Hopf algebra. Then we have a matched pair of Hopf algebras \(((H^*)_{\text{cop}}, H, \triangleleft, \triangleright)\), where the actions \(\triangleleft\) and \(\triangleright\) are defined by:

\[ h \triangleleft h^* := \langle h^*, S_{H}^{-1}(h_{(3)})h_{(1)} \rangle h_{(2)}, \quad h \triangleright h^* := \langle h^*, S_{H}^{-1}(h_{(2)}) \triangleright h_{(1)} \rangle \]

for all \(h \in H\) and \(h^* \in H^*\) ([26] Theorem IX.3.5]). The Drinfel’d double of \(H\) is the bicrossed product associated to this matched pair, i.e. \(D(H) = (H^*)_{\text{cop}} \bowtie H\).

4. Majid generalized the construction of the Drinfel’d double to the infinite dimensional case. Let \(A\) and \(H\) be two Hopf algebras and \(\lambda : H \otimes A \to k\) a skew pairing. Then there exists a matched pair of Hopf algebras \((A, H, \triangleleft = \triangleleft\lambda, \triangleright = \triangleright\lambda)\), where the actions \(\triangleleft\) and \(\triangleright\) arise from \(\lambda\) via

\[ h \triangleleft a = h_{(2)} \lambda^{-1}(h_{(1)}, a_{(1)}) \lambda(h_{(3)}, a_{(2)}) = h_{(2)} \lambda(S(h_{(1)})h_{(3)}, a) \]  

\[ h \triangleright a = a_{(2)} \lambda^{-1}(h_{(1)}, a_{(1)}) \lambda(h_{(2)}, a_{(3)}) = a_{(2)} \lambda(S(h), a_{(1)}S(a_{(3)})) \]

for all \(h \in H\) and \(a \in A\) ([31], Example 7.2.6]). The corresponding bicrossed product \(A \bowtie\lambda H\) associated to this matched pair is called a generalized quantum double and it will be denoted by \(D_\lambda(A, H)\).

A Hopf algebra \(E\) factorizes through two Hopf algebras \(A\) and \(H\) if there exists injective Hopf algebra maps \(i : A \to E\) and \(j : H \to E\) such that the map

\[ A \otimes H \to E, \quad a \otimes h \mapsto i(a)j(h) \]

is bijective. The main theorem which characterizes the bicrossed product is the next theorem due to Majid [32] Theorem 7.2.3]. The normal version of it was recently proven in [13, Proposition 2.2] and more generally in [11, Theorem 2.1]. First we recall that a Hopf subalgebra \(A\) of a Hopf algebra \(E\) is called normal if \(x_{(1)}aS(x_{(2)}) \in A\) and \(S(x_{(1)})ax_{(2)} \in A\), for all \(x \in E\) and \(a \in A\).

**Theorem 1.2.** Let \(A, H\) be two Hopf algebras. A Hopf algebra \(E\) factorizes through \(A\) and \(H\) if and only if there exists a matched pair of Hopf algebras \((A, H, \triangleleft, \triangleright)\) such that \(E \cong A \bowtie H\), an isomorphism of Hopf algebras.

Furthermore, a Hopf algebra \(E\) factorizes through a normal Hopf subalgebra \(A\) and a Hopf subalgebra \(H\) if and only if \(E\) is isomorphic as a Hopf algebra to a semidirect product \(A\# H\).
Proof. The bicrossed product $A \bowtie H$ factorizes through $A$ and $H$ via the canonical maps $i_A : A \to A \bowtie H$, $i_A(a) = a \bowtie 1_H$ and $i_H : H \to A \bowtie H$, $i_H(h) = 1_A \bowtie h$. Conversely, assume that $E$ factorizes through $A$ and $H$ and we view $A$ and $H$ as Hopf subalgebras of $E$ via the identification $A \cong i(A)$ and $H \cong j(H)$. For any $a \in A$ and $h \in H$ we view $ha \in E$; as $E$ factorizes through $A$ and $H$, we can find an unique element $\sum_{i=1}^{t} a_i^{(a,h)} \otimes h_i^{(a,h)} \in A \otimes H$ such that

$$ha = \sum_{i=1}^{t} a_i^{(a,h)} h_i^{(a,h)}$$

We define the maps $\triangleleft : H \otimes A \to H$ and $\triangleright : H \otimes A \to A$ via:

$$h \triangleleft a := \sum_{i=1}^{t} \varepsilon_A(a_i^{(a,h)}) h_i^{(a,h)}, \quad h \triangleright a := \sum_{i=1}^{t} \varepsilon_H(h_i^{(a,h)}) a_i^{(a,h)} \quad (16)$$

Then $(A, H, \triangleleft, \triangleright)$ is a matched pair of Hopf algebras and the multiplication map $A \bowtie H \to E$, $a \bowtie h \mapsto ah$ is an isomorphism of Hopf algebras. The details are proven in [22, Theorem 7.2.3]. The final statement is [13, Proposition 2.2].

Theorem 1.2 proves that the factorization problem for Hopf algebras can be restated in a computational manner: *Let $A$ and $H$ be two given Hopf algebras. Describe the set of all matched pairs $(A, H, \triangleright, \triangleleft)$ and classify up to an isomorphism all bicrossed products $A \bowtie H$.\*

The Hopf algebras $A$ and $H$ play a symmetric role in a bicrossed product $A \bowtie H$. In other words, finding all matched pairs $(H, A, \triangleright', \triangleleft')$ reduces in fact to finding all matched pairs $(A, H, \triangleright, \triangleleft)$. In the case where both Hopf algebras $A$ and $H$ have bijective antipode we indicate an explicit way of constructing a matched pair $(H, A, \triangleright', \triangleleft')$ out of a given matched pair $(A, H, \triangleright, \triangleleft)$ such that the associated Hopf algebras $A \bowtie H$ and $H \bowtie A$ are isomorphic. It is the Hopf algebra version of [2 Proposition 2.5] proved for matched pairs of groups.

**Proposition 1.3.** *Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras with bijective antipodes. We define the actions $\triangleright' : A \otimes H \to H$ and $\triangleleft' : A \otimes H \to A$ by

$$a \triangleright' h := S_H \left( S_H^{-1}(h(1)) \triangleleft S_A^{-1}(a(h(1))) \right) \triangleleft S_A \left( S_H^{-1}(h(2)) \triangleright S_A^{-1}(a(h(2))) \right)$$

$$a \triangleleft' h := S_H \left( S_H^{-1}(h(1)) \triangleleft S_A^{-1}(a(h(1))) \right) \triangleright S_A \left( S_H^{-1}(h(2)) \triangleright S_A^{-1}(a(h(2))) \right)$$

for all $a \in A$ and $h \in H$. Then $(H, A, \triangleright', \triangleleft')$ is a matched pair of Hopf algebras and there exists an isomorphism of Hopf algebras $A \bowtie H \cong H \bowtie A$, where $H \bowtie A$ is the bicrossed product associated to $(H, A, \triangleright', \triangleleft')$.\*

**Proof.** The proof is a straightforward verification based on Theorem 1.2 as follows: let $E := A \bowtie H$ be the bicrossed product associated to the matched pair $(A, H, \triangleright, \triangleleft)$. Then $E$ factorizes through $A$ and $H$; hence $E$ also factorizes through $H$ and $A$. Next we write down the cross relation in the bicrossed product $A \bowtie H$

$$(1_A \bowtie h) \cdot (a \bowtie 1_H) = h(1) \triangleright a(1) \bowtie h(2) \triangleleft a(2)$$
for \( a = S_A^{-1}(a') \) and \( h = S_H^{-1}(h') \), with \( a' \in A \) and \( h' \in H \) and then we apply \( S_{A \triangleleft \triangleright H} \) to it. We shall obtain an expression for \( a' \triangleright h' \) in terms of the actions \( \triangleright \) and \( \triangleleft \) as follows:

\[
a' \triangleright h' = S_H \left( S_H^{-1}(h'_{(1)}) \triangleright S_A^{-1}(a'_{(1)}) \right) \triangleright S_H \left( S_H^{-1}(h'_{(2)}) \triangleleft S_A^{-1}(a'_{(2)}) \right) \triangleright S_H \left( S_H^{-1}(h'_{(3)}) \triangleright S_A^{-1}(a'_{(3)}) \right) \triangleright S_H \left( S_H^{-1}(h'_{(4)}) \triangleleft S_A^{-1}(a'_{(4)}) \right)
\]

Now, the actions \( \triangleright' \) and \( \triangleleft' \) of the new matched pair \((H, A, \triangleright', \triangleleft')\) are obtained using (16), i.e. we first apply \( \varepsilon_A \otimes \text{Id} \) and then \( \text{Id} \otimes \varepsilon_H \) to this formula. Finally, the k-linear map \( \varphi: A \otimes H \to H \otimes A, \varphi(a \otimes h) := S_H(h) \otimes S_A(a) \) is bijective and we can easily prove that \( \psi: A \triangleright H \to H \triangleright' A, \psi := \varphi \circ S_{A \triangleleft \triangleright H} \) is an isomorphism of Hopf algebras.

\[\square\]

2. The morphisms between two bicrossed products

In order to describe the morphisms between two arbitrary bicrossed products \( A \triangleright H \) and \( A' \triangleright H' \) we need the following:

**Lemma 2.1.** Let \( C, D \) and \( E \) be three coalgebras. Then there exists a bijective correspondence between the set of all morphisms of coalgebras \( \alpha: C \to D \otimes E \) and the set of all pairs \((u, p)\), where \( u: C \to D \) and \( p: C \to E \) are morphisms of coalgebras satisfying the symmetry condition

\[
p(c_{(1)}) \otimes u(c_{(2)}) = p(c_{(2)}) \otimes u(c_{(1)})
\]

for all \( c \in C \). Under the above correspondence the morphism of coalgebras \( \alpha: C \to D \otimes E \) corresponding to the pair \((u, p)\) is given by:

\[
\alpha(c) = u(c_{(1)}) \otimes p(c_{(2)})
\]

for all \( c \in C \).

**Proof.** Let \( \alpha: C \to D \otimes E \) be a morphism of coalgebras. We adopt the temporary notation \( \alpha(c) = c_{[0]} \otimes c_{[1]} \in D \otimes E \) (summation understood). Define the following two linear maps:

\[
u: C \to D, \quad u := (\text{Id} \otimes \varepsilon_E) \circ \alpha, \quad \text{i.e.} \quad u(c) = \varepsilon_E(c_{[1]})c_{[0]}
\]

\[
p: C \to E, \quad p := (\varepsilon_D \otimes \text{Id}) \circ \alpha, \quad \text{i.e.} \quad p(c) = \varepsilon_D(c_{[0]})c_{[1]}
\]

for all \( c \in C \). Then \( u \) and \( p \) are coalgebra maps as compositions of coalgebra maps. Since \( \alpha \) is a coalgebra map we have \( \Delta_{D \otimes E} \circ \alpha(c) = (\alpha \otimes \alpha) \circ \Delta_C(c) \) for any \( c \in C \), which is equivalent to:

\[
c_{[0]}c_{[1]} \otimes c_{[1]}c_{[2]} \otimes c_{[2]}c_{[3]} = c_{[2]}c_{[1]}c_{[1]}c_{[2]}c_{[0]}c_{[1]}c_{[2]}c_{[0]}c_{[2]}c_{[1]}
\]

(19)

If we apply \( \text{Id} \otimes \varepsilon_E \otimes \varepsilon_D \otimes \text{Id} \) to (19) we obtain \( c_{[0]}c_{[1]}c_{[1]} = u(c_{[1]}) \otimes p(c_{[2]}) \) that is (18) holds. With this form for \( \alpha \) and having in mind that \( u \) and \( p \) are coalgebra maps the equation (19) takes the form:

\[
u(c_{[1]}) \otimes p(c_{[3]}) \otimes u(c_{[2]}) \otimes p(c_{[4]}) = u(c_{[1]}) \otimes p(c_{[2]}) \otimes u(c_{[3]}) \otimes p(c_{[4]})
\]

Applying \( \varepsilon_D \otimes \text{Id} \otimes \text{Id} \otimes \varepsilon_E \to \) to the above identity we obtain the symmetry condition (17). Conversely, it is easy to see that any map \( \alpha \) given by (18), for some coalgebra
maps $u$ and $p$ satisfying (17), is a coalgebra map and it follows from the proof that the correspondence is bijective.

Now we can describe all Hopf algebra morphisms between two bicrossed products.

**Theorem 2.2.** Let $(A, H, \triangleright, \triangleleft)$ and $(A', H', \triangleright', \triangleleft')$ be two matched pairs of Hopf algebras.

Then there exists a bijective correspondence between the set of all morphisms of Hopf algebras $\psi : A \triangleright H \to A' \triangleright' H'$ and the set of all quadruples $(u, p, r, v)$, where $u : A \to A'$, $p : A \to H'$, $r : H \to A'$, $v : H \to H'$ are unitary coalgebra maps satisfying the following compatibility conditions:

\[ u(a_{(1)}) \otimes p(a_{(2)}) = u(a_{(2)}) \otimes p(a_{(1)}) \quad (20) \]
\[ r(h_{(1)}) \otimes v(h_{(2)}) = r(h_{(2)}) \otimes v(h_{(1)}) \quad (21) \]
\[ u(ab) = u(a_{(1)}) (p(a_{(2)}) \triangleright' u(b)) \quad (22) \]
\[ p(ab) = (p(a) \triangleright u(b_{(1)})) p(b_{(2)}) \quad (23) \]
\[ r(hg) = r(h_{(1)}) (v(h_{(2)}) \triangleright' r(g)) \quad (24) \]
\[ v(hg) = (v(h) \triangleright' r(g_{(1)})) v(g_{(2)}) \quad (25) \]
\[ r(h_{(1)}) (v(h_{(2)}) \triangleright' u(b)) = u(h_{(1)} \triangleright b_{(1)}) \left( p(h_{(2)} \triangleright b_{(2)}) \triangleright' r(h_{(3)} \triangleleft b_{(3)}) \right) \quad (26) \]
\[ (v(h) \triangleright' u(b_{(1)})) p(b_{(2)}) = \left( p(h_{(1)} \triangleright b_{(1)}) \triangleright' r(h_{(2)} \triangleleft b_{(2)}) \right) v(h_{(3)} \triangleleft b_{(3)}) \quad (27) \]

for all $a, b \in A$, $g, h \in H$.

Under the above correspondence the morphism of Hopf algebras $\psi : A \triangleright H \to A' \triangleright' H'$ corresponding to $(u, p, r, v)$ is given by:

\[ \psi(a \triangleright h) = u(a_{(1)}) (p(a_{(2)}) \triangleright' r(h_{(1)})) \triangleright' p(a_{(3)}) \triangleright' r(h_{(2)}) v(h_{(3)}) \quad (28) \]

for all $a \in A$ and $h \in H$.

**Proof.** Let $\psi : A \triangleright H \to A' \triangleright' H'$ be a morphism of Hopf algebras. We define

\[ \alpha : A \to A' \triangleright' H', \quad \alpha(a) := \psi(a \triangleright 1_H) \]
\[ \beta : H \to A' \triangleright' H', \quad \beta(h) := \psi(1_A \triangleright h) \]

Then $\alpha : A \to A' \otimes H'$ and $\beta : H \to A' \otimes H'$ are unitary morphisms of coalgebras as compositions of such maps (we recall that the coalgebra structure on $A' \triangleright' H'$ is the tensor product of coalgebras $A' \otimes H'$) and

\[ \psi(a \triangleright h) = \psi((a \triangleright 1_H)(1_A \triangleright h)) = \psi(a \triangleright 1_H) \psi(1_A \triangleright h) = \alpha(a) \beta(h) \quad (29) \]

for all $a \in A$ and $h \in H$. It follows from Lemma 2.1 applied to $\alpha$ and $\beta$ that there exist four coalgebra maps $u : A \to A'$, $p : A \to H'$, $r : H \to A'$, $v : H \to H'$ such that

\[ \alpha(a) = u(a_{(1)}) \otimes p(a_{(2)}), \quad \beta(h) = r(h_{(1)}) \otimes v(h_{(2)}) \quad (30) \]

and the pairs $(u, p)$ and $(r, v)$ satisfy the symmetry conditions (20) and (21). Explicitly $u$, $p$, $r$ and $v$ are defined by

\[ u(a) = ((\text{Id} \otimes \varepsilon_{H'}) \circ \psi)(a \triangleright 1_H), \quad p(a) = ((\varepsilon_{A'} \otimes \text{Id}) \circ \psi)(a \triangleright 1_H) \]
\[ r(h) = ((\text{Id} \otimes \varepsilon_{H'}) \circ \psi)(1_A \triangleright h), \quad v(h) = ((\varepsilon_{A'} \otimes \text{Id}) \circ \psi)(1_A \triangleright h) \]
for all $a \in A$ and $h \in H$. All these maps are unitary coalgebra maps. Now, for any $a \in A$ and $h \in H$ we have:

$$\psi(a \bowtie h) = \alpha(a)\beta(h)$$

$$= (u(a_{(1)}) \triangleright p(a_{(2)})) \cdot (r(h_{(1)}) \triangleright v(h_{(2)}))$$

$$= u(a_{(1)}) (p(a_{(2)}) \triangleright r(h_{(1)})) \triangleright (p(a_{(3)}) \triangleright r(h_{(2)})) v(h_{(3)})$$

\text{i.e. (28) also holds. Thus any bialgebra map } \psi : A \bowtie H \rightarrow A' \bowtie H' \text{ is uniquely determined by the formula (29) for some unitary coalgebra maps } \alpha : A \rightarrow A' \otimes H' \text{ and } \beta : H \rightarrow A' \otimes H' \text{ or, equivalently, in the more explicit form given by (28), for some unique quadruple of unitary coalgebra maps } (u, p, r, v).$

Now, a map } \psi \text{ given by (29) is a morphism of algebras if and only if } \alpha : A \rightarrow A' \triangleright H' \text{ and } \beta : H \rightarrow A' \triangleright H' \text{ are algebra maps and the following commutativity relation holds}

$$\beta(h)\alpha(b) = \alpha(h_{(1)} \triangleright b_{(1)}) \beta(h_{(2)} \triangleleft b_{(2)}) \quad (31)$$

for all $h \in H$ and $b \in A$. Indeed, if } \psi \text{ is an algebra map then } \alpha \text{ and } \beta \text{ are algebra maps as compositions of algebra maps. On the other hand:}

$$\psi(a \bowtie h)\psi(b \bowtie g) = \alpha(a)\beta(h)\alpha(b)\beta(g)$$

and

$$\psi((a \bowtie h)(b \bowtie g)) = \alpha(a)\alpha(h_{(1)} \triangleright b_{(1)})\beta(h_{(2)} \triangleleft b_{(2)})\beta(g)$$

Hence, the relation (31) follows by taking $a = 1_A$ and $g = 1_H$ in the identity above. Conversely is obvious. Now, we write down the explicit conditions for $\alpha$ and $\beta$ to be algebra maps. First, we prove that $\alpha : A \rightarrow A' \triangleright H'$, $\alpha(a) = u(a_{(1)}) \triangleright p(a_{(2)})$ is an algebra map if and only if (22) and (23) hold. Indeed, $\alpha(ab) = \alpha(a)\alpha(b)$ is equivalent to:

$$u(a_{(1)}b_{(1)}) \triangleright p(a_{(2)}b_{(2)}) = u(a_{(1)}) (p(a_{(2)}) \triangleright u(b_{(1)})) \triangleright (p(a_{(3)}) \triangleright u(b_{(2)})) p(b_{(3)}) \quad (32)$$

If we apply $\text{Id} \otimes \varepsilon_{H'}$ to (32) we obtain (22) while if we apply $\varepsilon_A \otimes \text{Id}$ to (32) we obtain (23). Conversely is obvious. In a similar way we can show that $\beta : H \rightarrow A' \triangleright H'$, $\beta(h) = r(h_{(1)}) \triangleright v(h_{(2)})$ is an algebra map if and only if (24) and (25) hold. Finally, we prove that the commutativity relation (31) is equivalent to (26) and (27). Indeed, using the expressions of $\alpha$ and $\beta$ in terms of $(u, p)$ and respectively $(r, v)$, the equation (31) is equivalent to:

$$r(h_{(1)})(v(h_{(2)}) \triangleright u(b_{(1)})) \triangleright (v(h_{(3)}) \triangleright u(b_{(2)})) p(b_{(3)}) =$$

$$u(h_{(1)} \triangleright b_{(1)}) (p(h_{(2)} \triangleright b_{(2)}) \triangleright r(h_{(3)} \triangleleft b_{(3)})) \triangleright$$

$$\left(p(h_{(4)} \triangleright b_{(4)}) \triangleleft r(h_{(5)} \triangleleft b_{(5)})\right) v(h_{(6)} \triangleleft b_{(6)})$$

If we apply $\text{Id} \otimes \varepsilon_{H'}$ to the above identity we obtain (26) while if we apply $\varepsilon_A \otimes \text{Id}$ to it we get (27). Conversely, the commutativity condition (31) follows straightforward from (26) and (27).

To conclude, we have proved that a bialgebra map } \psi : A \bowtie H \rightarrow A' \bowtie H' \text{ is uniquely determined by a quadruple } (u, p, r, v) \text{ of unitary coalgebra maps satisfying the compatibility conditions (20)-(27) such that } \psi : A \bowtie H \rightarrow A' \bowtie H' \text{ is given by (28).}
Conversely, the fact that \( \psi \) given by (28) is a morphism of bialgebras, for some \((u, p, r, v)\) satisfying (20)-(27) is straightforward and follows directly from the proof. The fact that \( \psi \) is an algebra map is proven above. \( \psi \) is also a coalgebra map as a composition of coalgebra maps. More precisely, the equation (28) can be written in the equivalent form (29) namely, \( \psi = m_{A'\triangleright H'} \circ (\alpha \otimes \beta) \), where \( m_{A'\triangleright H'} \) is the multiplication map on the bicrossed product \( A' \triangleright \triangleleft H' \) and \( \alpha, \beta \) are the unitary coalgebra maps obtained from \((u, p, r, v)\) via the formulas (30).

In the next corollary \( A\#H \) will be a semi-direct product of Hopf algebras constructed in (1) of Example 1.1 as a special case of a bicrossed product.

**Corollary 2.3.** Let \( A\#H \) and \( A'\#'H' \) be two semi-direct products of Hopf algebras associated to two left actions \( \triangleright : H \otimes A \to A \) and \( \triangleright : H' \otimes A' \to A' \). Then there exists a bijection between the set of all morphisms of Hopf algebras \( \psi : A\#H \to A'\#'H' \) and the set of all quadruples \((u, p, r, v)\), where \( u : A \to A' \), \( r : H \to A' \) are unitary coalgebra maps, \( p : A \to H' \), \( v : H \to H' \) are morphism of Hopf algebras satisfying the following compatibility conditions:

\[
\begin{align*}
    u(a_{(1)}) \otimes p(a_{(2)}) &= u(a_{(2)}) \otimes p(a_{(1)}) \\
    r(h_{(1)}) \otimes v(h_{(2)}) &= r(h_{(2)}) \otimes v(h_{(1)}) \\
    u(ab) &= u(a_{(1)}) (p(a_{(2)}) \triangleright u(b)) \\
    r(hg) &= r(h_{(1)}) (v(h_{(2)}) \triangleright r(g)) \\
    r(h_{(1)}) (v(h_{(2)}) \triangleright u(b)) &= u(h_{(1)} \triangleright b_{(1)}) \left(p(h_{(2)} \triangleright b_{(2)}) \triangleright r(h_{(3)})\right) \\
    v(h)p(b) &= p(h_{(1)} \triangleright b) v(h_{(2)})
\end{align*}
\]

for all \( a, b \in A, g, h \in H \).

Under the above bijection the morphism of Hopf algebras \( \psi : A\#H \to A'\#'H' \) corresponding to \((u, p, r, v)\) is given by:

\[
\psi(a\#h) = u(a_{(1)}) \left(p(a_{(2)}) \triangleright r(h_{(1)})\right) \# p(a_{(3)}) v(h_{(2)})
\]

for all \( a \in A \) and \( h \in H \).

**Proof.** We apply Theorem 2.2 in the case that the two right actions \( \triangleright, \triangleright \) are the trivial actions. In this case, (23) is equivalent to \( p : A \to H' \) being an algebra map, i.e. \( p \) is a Hopf algebra map while (25) is equivalent to \( v : H \to H' \) being an algebra map, i.e. \( v \) is a Hopf algebra map. Finally, (24), (26), (27) and (28) take the simplified forms (33)-(35), (36), (37) and (38) respectively. \( \square \)

Using Theorem 2.2 we can completely describe \( \text{End}(A \bowtie H) \), the space of all Hopf algebra endomorphisms of an arbitrary bicrossed product. In particular, we obtain a description of \( \text{End}(D(H)) \), the group of all Hopf algebra endomorphisms of a Drinfel’d double \( D(H) \). If \( H = k[G] \), for a finite group \( G \), the group \( \text{End}(D(k[G])) \) of all Hopf algebra endomorphisms is described below in full details.

Let \( G \) be a finite group. The Drinfel’d double of \( k[G] \) is described as follows. Take \( \{e_g\}_{g \in G} \) to be a dual basis for the basis \( \{g\}_{g \in G} \) of \( k[G] \). \( A := (k[G]^*)^{\text{op}} \) is a Hopf
algebra with the multiplication and the comultiplication given by
\[ e_g \cdot e_h := \delta_{g,h} e_g, \quad \Delta_A(e_g) := \sum_{x \in G} e_x \otimes e_{gx^{-1}}, \quad 1_A := \sum_{g \in G} e_g, \quad \varepsilon_A(e_g) := \delta_{g,1_G} \]
for all \( g, h \in G \), where \( \delta(-,-) \) is the Kronecker delta. The left action of \( H = k[G] \) on \( A = (k[G]^*)^\text{cop} \) given by \(^{[13]}\) takes the form
\[ g \triangleright e_h := e_{gh^{-1}} \quad \text{or} \quad (g \triangleright f)(z) := f(g^{-1}zg) \quad (40) \]
for all \( g, h, z \in G \) and \( f \in (k[G]^*)^\text{cop} \). Using the action \(^{[13]}\), we have that \( D(k[G]) = (k[G]^*)^\text{cop} \otimes_k k[G] \). Thus, \( D(k[G]) \) is the Hopf algebra having the basis \( \{ e_h \# g \}_{g, h \in G} \) with the multiplication and the comultiplication given by
\[ (e_h \# g) \cdot (e_x \# y) = \delta_{h,gy^{-1}} e_h \# gy, \quad \Delta(e_h \# g) = \sum_{x \in G} (e_x \# g) \otimes (e_{hx^{-1}} \# g) \]
for all \( h, g, x \) and \( y \in G \).

The following gives the parametrization of all Hopf algebra morphisms between two Drinfel’d doubles associated to two finite groups \( G \) and \( H \). In particular, if \( G = H \), the space \( \text{End}(D(k[G])) \) of all Hopf algebra endomorphisms is described.

**Corollary 2.4.** Let \( G, H \) be two finite groups. Then any Hopf algebra morphism \( \psi : D(k[G]) \to D(k[H]) \) between the Drinfel’d doubles has the form\(^{[14]}\)
\[ \psi(e_g \# g') = \sum_{x \in G, y \in H} \lambda(gx^{-1}, z) \omega(g', y) \theta(? , x) \delta(? , zy^{-1}) \# z v(g') \quad (41) \]
for all \( g, g' \in G \), where \( (\lambda, \omega, \theta, v) \) is a quadruple such that \( v : G \to H \) is a morphism of groups, \( \theta : H \times G \to k \), \( \lambda : G \times H \to k \), \( \omega : G \times H \to k \) are three maps satisfying the following compatibilities:
\[
\begin{align*}
\theta(1,g) &= \delta_{g,1_G} \quad (42) \\
\sum_{x \in G} \theta(h,x) &= 1 \quad (43) \\
\theta(hh',g) &= \sum_{x \in G} \theta(h,x) \theta(h',x^{-1}g) \quad (44) \\
\omega(1,g) &= 1 \quad (45) \\
\omega(g,hh') &= \omega(g,h) \omega(g,h') \quad (46) \\
\sum_{y \in H} \lambda(g,y) &= \delta_{g,1} \quad (47) \\
\sum_{x \in G} \lambda(x,h) &= \delta_{1,h} \quad (48)
\end{align*}
\]

\(^{1}\)We denote by \( \theta(? , y)\delta_{(? , aba^{-1})} \in k[H]^* \) the \( k \)-linear map sending any \( z \in H \) to \( \theta(z,y)\delta_{(z,aba^{-1})} \).
as augmented algebras if and only if the compatibility conditions (42)-(44) hold. Now, for all \( g \) such that \( \delta_{g,g'} \lambda(g,h) \) for any \( \theta \):

\[
\sum_{g \in G, y \in H} \lambda(g^{-1}, y) \theta(h, x) \theta(y^{-1}hy, g') = \delta_{g,g'} \theta(h, g)
\]

(52)

(53)

\[
\sum_{x \in G, y \in H} \lambda(g^{-1}, y) \theta(h, x) \theta(y^{-1}hy, g') = \delta_{g,g'} \theta(h, g)
\]

(54)

\[
\lambda(gg^{-1}, v(g) h v(g)^{-1}) = \lambda(g', h)
\]

(55)

for all \( g, g' \in G, h, h' \in H \).

The correspondence \( \psi \leftrightarrow (\lambda, \omega, \theta, v) \) between the set of all Hopf algebra morphisms \( \psi : D(k[G]) \to D(k[H]) \) and the set of all maps \( (\lambda, \omega, \theta, v) \) satisfying the compatibility conditions (42) - (55) is bijective.

**Proof.** It follows by a direct computation from Corollary 2.3 applied for \( A = (k[G])^{\mathrm{cop}}, H = k[H] \) and the left action \( \rightharpoonup \) given by (40). We shall indicate only a sketch of the proof, the details being left to the reader. First we should notice that any Hopf algebra map \( v : k[G] \to k[H] \) is uniquely determined by a morphism of the groups which will be denoted also by \( v : G \to H \). We shall prove now that any unitary coalgebra map \( u : (k[G])^{\mathrm{cop}} \to (k[H])^{\mathrm{cop}} \) is given by the formula

\[
u(e_y)(h) = \theta(h, g)
\]

for all \( g \in G, h \in H \) and for a unique map \( \theta : H \times G \to k \) satisfying (42)-(44). Indeed, first we notice that a unitary coalgebra map \( u : (k[G])^{\mathrm{cop}} \to (k[H])^{\mathrm{cop}} \) is in fact the same as a unitary coalgebra map \( u : k[G]^* \to k[H]^* \). From the duality algebras/coalgebras any such map \( u \) is uniquely implemented by a map of augmented algebras \( f : k[H] \to k[G] \) (i.e. \( f \) is an algebra map and \( \varepsilon_{k[G]} \circ f = \varepsilon_{k[H]} \)) by the formula \( u = f^* \), i.e. \( u(e_h) = e_h \circ f \), for all \( h \in G \). Now, any \( k \)-linear map \( f : k[H] \to k[G] \) is uniquely defined by a map \( \theta : H \times G \to k \) as follows:

\[
f(h) = \sum_{x \in G} \theta(h, x) x
\]

for any \( g \in H \). We can easily prove that such a linear map is an endomorphism of \( k[G] \) as augmented algebras if and only if the compatibility conditions (42)-(44) hold. Now, any \( k \)-linear map \( r : k[G] \to (k[H])^* \) is implemented by a unique map \( \omega : G \times H \to k \) such that

\[
r(g) = \sum_{y \in H} \omega(g, y) e_y
\]

for all \( g \in G \). Such a map \( r = r_{\omega} \) is an unitary morphism of coalgebras if and only if \( \omega \) satisfies the compatibility conditions (45)-(46). Finally, any \( k \)-linear map

\[
\sum_{y \in H} \lambda(g, y) \lambda(g', y^{-1}h) = \delta_{g,g'} \lambda(g,h)
\]

(49)

\[
\sum_{x \in G} \lambda(x, h) \lambda(gx^{-1}, h') = \delta_{h,h'} \lambda(g,h)
\]

(50)

\[
\sum_{x \in G} \theta(h, x) \lambda(gx^{-1}, h') = \sum_{x \in G} \theta(h, x) \lambda(x^{-1}g, h')
\]

(51)
$p : (k[G]^\ast)^{\text{cop}} \to k[H]$ is given in a unique way by a map $\lambda : G \times H \to k$ such that
\[ p(e_g) = \sum_{y \in H} \lambda(g, y)y \]
for all $g \in G$. Such a map $p = p_\lambda$ is a morphism of Hopf algebras if and only if $\lambda$ satisfies the compatibility conditions (47)-(50).

Hence we have described the set of data $(u, p, r, v)$ of Corollary 2.3. As $H = k[H]$ is cocommutative the compatibility condition (34) is trivially fulfilled. Moreover, by a straightforward computation the compatibility conditions (33) and (35)-(38) take the form (51)-(55). Now, by a direct computation we can show that the expression of the morphism $\psi : D(k[G]) \to D(k[H])$ given by (39) takes the following simplified form:
\[ \psi(e_g \# g') = \sum_{a,b,c \in H, x,y \in G} \lambda(xy^{-1}, a) \lambda(gx^{-1}, c) \omega(g', b) \theta(?, y)\delta(?, aba^{-1}) \# c v(g') \]
\[ = \sum_{a,b,c \in H, x,y \in G} (\sum_{x \in G} \lambda(xy^{-1}, a) \lambda(gx^{-1}, c) \omega(g', b) \theta(?, y)\delta(?, aba^{-1}) \# c v(g')) \]
\[ = \sum_{a,b,c \in H, x,y \in G} (\sum_{x \in G} \lambda(xy^{-1}, a) \lambda(gx^{-1}, c) \omega(g', b) \theta(?, y)\delta(?, aba^{-1}) \# c v(g')) \]
i.e., by interchanging the summation indices, we proved that (41) holds.

3. The classification of bicrossed products

Theorem 2.2 can be used to indicate when two arbitrary bicrossed products $A \bowtie H$ and $A' \bowtie H'$ are isomorphic. Hence it gives the answer to the classification part of the factorization problem for Hopf algebras and will be used in its full generality for explicit examples in Section 4. In general, since the result is very technical and not so transparent, we restrict ourselves to a special kind of classification, namely the one that stabilizes one of the terms of the bicrossed product. As explained in the introduction the classification of bicrossed products up to an isomorphism that stabilizes one of the terms has two motivations: the first one is the cohomological point of view which descends to the classification theory of group extensions and the second one is the problem of describing and classifying the $A$-forms of a Hopf algebra extensions from descent theory [3].

Let $A$ and $H$ be two Hopf algebras. We shall classify up to an isomorphism that stabilizes $A$ the set of all Hopf algebras $E$ that factorize through $A$ and $H$. It follows from Theorem 1.2 that we have to classify all bicrossed products $A \bowtie H$ associated to all possible matched pairs of Hopf algebras $(A, H, \triangleright, \triangleleft)$. First we need to recall from [5] the following concept that will play a crucial role in the paper.
Definition 3.1. Let $A$ and $H$ be two Hopf algebras. A coalgebra map $r : H \to A$ is called cocentral if the following compatibility holds:

$$r(h_{(1)}) \otimes h_{(2)} = r(h_{(2)}) \otimes h_{(1)}$$

(56)

for all $h \in H$. We denote by $CoZ^1(H, A)$ the group with respect to the convolution product of all unitary cocentral maps from $H$ to $A$.

Remark 3.2. If $r \in CoZ^1(H, A)$ then $S^2_A \circ r = r$. Indeed, since $r$ is a coalgebra map, the inverse of $r$ in the group $CoZ^1(H, A)$ is $r^{-1} = S_A \circ r$, which is still a unitary cocentral map, i.e. in particular a coalgebra map. Hence, the inverse of $S_A \circ r = r^{-1}$ in the group $CoZ^1(H, A)$ is $S_A \circ (S_A \circ r) = S^2_A \circ r$. Thus $S^2_A \circ r = r$.

Examples 3.3. 1. If $H$ is cocommutative, then the group $CoZ^1(H, A)$ coincides with the group of all unitary coalgebra maps $r : H \to A$ with the convolution product.

In particular, let $G$ and $G'$ be two groups, $H = k[G]$ and $A = k[G']$ the corresponding group algebras. Then the group $CoZ^1(k[G], k[G'])$ is isomorphic to the group of all unitary maps $r : G \to G'$.

2. Let $H = A := H_4$ be the Sweedler’s four dimensional Hopf algebra. Then, by a routine computation proved in Lemma we can show that $CoZ^1(H_4, H_4)$ is the trivial group with only one element, namely the trivial unitary cocentral map $r : H_4 \to H_4$, $r(h) = \varepsilon(h)1_H$, for all $h \in H_4$.

3. Consider now $A = H_4$ and $H = k[C_n]$, where $C_n$ is the cyclic group of order $n$ generated by $c$. Let $r : k[C_n] \to H_4$ be a unitary coalgebra map. Then (56) is trivially fulfilled as $H$ is cocomutative. Moreover, since $r$ is a coalgebra map we have $r(c^i) \in \{1, g\}$ for all $i \in \{1, 2, ..., n-1\}$ and $r(1) = 1$. Thus $CoZ^1(k[C_n], H_4)$ is the abelian group $C_2 \times C_2 \times \cdots \times C_2$ of order $2^{n-1}$.

On the other hand, we can prove that $CoZ^1(H_4, k[C_n])$ is the group with only one element, namely the trivial unitary cocentral map $r : H_4 \to k[C_n]$, $r(h) = \varepsilon(h)1_{C_n}$, for all $h \in H_4$.

4. A general method of constructing unitary cocentral maps is the following. Let $\psi : A \otimes H \to A \otimes H$ be a left $A$-linear Hopf algebra isomorphism. Then

$$r = r_\psi : H \to A, \quad r(h) = (\text{Id} \otimes \varepsilon_H) \circ \psi(1_A \bowtie h)$$

for all $h \in H$ is a unitary cocentral map. Furthermore, $ar_\psi(h) = r_\psi(h) a$, for all $h \in H$ and $a \in A$. Conversely, if $r : H \to A$ is a unitary cocentral map such that Im$(r) \subseteq Z(A)$, then

$$\psi = \psi_r : A \otimes H \to A \otimes H, \quad \psi(a \bowtie h) := a r(h_{(1)}) \otimes h_{(2)}$$

is a left $A$-linear Hopf algebra isomorphism. For further details we refer to Corollary 3.11

Definition 3.4. Let $A$ be a Hopf algebra, $A \bowtie H$ and $A \bowtie' H'$ two bicrossed products associated to two matched pairs of Hopf algebras $(A, H, \triangleright, \triangleleft)$ and $(A, H', \triangleright', \triangleleft')$. We say that a morphism of Hopf algebras $\psi : A \bowtie H \to A \bowtie' H'$ stabilizes $A$ if the following
Theorem 2.2 takes the form
\[ \alpha(57) \] commutative if and only if the map \( \alpha \) and satisfying (20)-(27). Now, such a morphism \( \psi : A \to A \bowtie H \). This (iso)morphisms are fully described in the following:

\[ (20)-(27) \]

There is a one-to-one correspondence between the set of all Hopf algebra morphisms \( \psi : A \bowtie H \to A \bowtie H' \) that stabilize \( A \) and the set of all pairs \( (r,v) \), where \( r : H \to A \), \( v : H \to H' \) are unitary coalgebra maps satisfying the following compatibility conditions for any \( a \in A \), \( g \in H \):

\[ r(h_{1}) \otimes v(h_{2}) = r(h_{2}) \otimes v(h_{1}) \]
\[ r(h g) = r(h_{1})(v(h_{2}) \triangleright r(g)) \]
\[ v(h g) = (v(h) \triangleleft r(g))(v(g_{2})) \]
\[ h \triangleright a = r(h_{1})(v(h_{2}) \triangleright a(1)) (S_{A} \circ r)(h_{3} \triangleleft a(2)) \]
\[ v(h \triangleleft a) = v(h) \triangleleft' a \]

Under the above bijection the morphism \( \psi : A \bowtie H \to A \bowtie H' \) corresponding to \( (r,v) \) is given by:

\[ \psi(a \bowtie h) = a r(h_{1}) \triangleright' v(h_{2}) \]

for all \( a \in A \) and \( h \in H \).

(2) The left \( A \)-linear Hopf algebra morphism \( \psi : A \bowtie H \to A \bowtie H' \) given by \( (63) \) is an isomorphism if and only if \( v : H \to H' \) is bijective\(^2\).

**Proof.** (1) We shall apply Theorem 2.2 for \( A' = A \). Any morphism of Hopf algebras \( \psi : A \bowtie H \to A \bowtie H' \) is given by (28) for some unitary coalgebra maps \( (u,p,r,v) \) satisfying (20)-(27). Now, such a morphism \( \psi : A \bowtie H \to A \bowtie H' \) makes the diagram

\[ \begin{array}{ccc}
A & \xrightarrow{i_{A}} & A \bowtie H \\
\downarrow{\text{Id}_{A}} & & \downarrow{\psi} \\
A & \xrightarrow{i_{A}} & A \bowtie H'
\end{array} \]

commutative if and only if \( \alpha(a) = a \bowtie H' \) constructed in the proof of Theorem 2.2 takes the form \( \alpha(a) = a \otimes 1_{H'} \). This is equivalent to the fact that \( u : A \to A \) and \( p : A \to H' \), constructed in the same proof, are precisely the following: \( u(a) = a \) and \( p(a) = 1_{A}(a)1_{H'} \), for any \( a \in A \). With these maps \( u \) and \( p \), the compatibility relations (20)-(27) are reduced to (58)-(62). For instance, (26) takes the form

\[ (h_{1} \triangleright a(1))r(h_{2} \triangleleft a(2)) = r(h_{1})(v(h_{2}) \triangleright' a) \]

That is \( v : (H,\triangleleft) \to (H',\triangleleft') \) is an unitary isomorphism of right \( A \)-module coalgebras.

\(^2\)That is \( v : (H,\triangleleft) \to (H',\triangleleft') \) is an unitary isomorphism of right \( A \)-module coalgebras.
which is equivalent to \( [61] \), as \( r \) is invertible in the convolution algebra \( \text{Hom}(H, A) \) with the inverse \( S_A \circ r \). We also note that \( [27] \) takes the easier form given by \( [62] \) which means precisely the fact that the unitary coalgebra map \( v : H \to H' \) is also a morphism of right \( A \)-modules. Finally, the formula of \( \psi \) given by \( [28] \) takes the simplified form \( [63] \).

(2) Assume first that \( v : H \to H' \) is bijective with the inverse \( v^{-1} \). Applying \( \text{Id} \otimes v^{-1} \) in \( [58] \) we obtain that

\[
r(h_{(1)}) \otimes h_{(2)} = r(h_{(2)}) \otimes h_{(1)}
\]

for all \( h \in H \). Thus, \( r \) is a unitary cocentral map. In particular, it follows from Remark \( [3.2] \) that \( S_A^2 \circ r = r \). Using this observation we can easily prove that the map

\[
\psi^{-1} : A \bowtie H' \to A \bowtie H, \quad \psi^{-1}(a \bowtie h') = a (S_A \circ r \circ v^{-1})(h'_{(1)}) \bowtie v^{-1}(h'_{(2)})
\]

for all \( a \in A \) and \( h' \in H' \) is the inverse of \( \psi \).

Conversely, assume that \( \psi : A \bowtie H \to A \bowtie H' \) given by \( [63] \) is bijective, that is an isomorphism of Hopf algebras and left \( A \)-modules. Then, there exists a left \( A \)-module Hopf algebra morphism \( \varphi : A \bowtie H' \to A \bowtie H \) such that \( \psi \circ \varphi = \text{Id}_{A \bowtie H'} \) and \( \varphi \circ \psi = \text{Id}_{A \bowtie H} \). It follows from the first part of the theorem that there exist two unitary coalgebra maps \( q : H' \to A \) and \( t : H' \to H \) satisfying the compatibility conditions \( [58] \)-\( [62] \), where the pair \( (<a, b>) \) is interchanged with \( (<a', b'>) \), such that \( \varphi \) is given by

\[
\varphi (a \bowtie h') = a q(h'_{(1)}) \bowtie t(h'_{(2)})
\]

for all \( a \in A \) and \( h' \in H' \). Let \( h \in H \). From \( (\varphi \circ \psi)(1_A \bowtie h) = 1_A \bowtie h \) we obtain

\[
1_A \bowtie h = r(h_{(1)})(q \circ v)(h_{(2)}) \bowtie (t \circ v)(h_{(3)})
\]

If we apply \( \varepsilon_A \) on the first position in the above equality we obtain \( t \circ v = \text{Id}_H \). On the other hand, let \( h' \in H' \). From \( (\psi \circ \varphi)(1_A \bowtie h') = 1_A \bowtie h' \) we obtain

\[
1_A \bowtie h' = q(h'_{(1)})(r \circ t)(h'_{(2)}) \bowtie (v \circ t)(h'_{(3)})
\]

If we apply \( \varepsilon_A \) on the first position we obtain \( v \circ t = \text{Id}_{H'} \). Hence, it follows that \( v \) is bijective and \( v^{-1} = t \), as needed.

Now, we shall fix two Hopf algebras \( A \) and \( H \). We define the small category \( \mathcal{MP}(A, H) \) of all matched pairs as follows: the objects of \( \mathcal{MP}(A, H) \) are the set of all pairs \( (<a, b>) \), such that \( (A, H, <a, b>) \) is a matched pair of Hopf algebras. A morphism \( \psi : (<a, b>) \to (<a', b'>) \) in the category \( \mathcal{MP}(A, H) \) is a Hopf algebra map \( \psi : A \bowtie H \to A \bowtie H' \) that stabilizes \( A \). In order to classify up to an isomorphism that stabilizes \( A \) all Hopf algebras \( E \) that factorize through \( A \) and \( H \) we have to describe the skeleton of the category \( \mathcal{MP}(A, H) \). This will be done next.

**Definition 3.6.** Let \( A \) and \( H \) be two Hopf algebras. Two objects \( (<a, b>) \) and \( (<a', b'>) \) of the category \( \mathcal{MP}(A, H) \) are called **cohomologous** and we denote this by \( (<a, b>) \approx (<a', b'>) \) if there exists a pair of maps \( (r, v) \) such that:
Example 1.1. We shall prove a necessary and sufficient condition for the bicrossed products associated to the matched pairs $(A,H,\triangleright,\triangleleft)$. It follows from Theorem 3.5 that $(A,H,\triangleright,\triangleleft)$ is an equivalence relation on the set $\mathcal{MP}(A,H)$, where $\triangleright,\triangleleft$ are the left action

$$r(hg) = r(h_{(1)}) (v(h_{(2)})) \triangleright r(g))$$

$$v(hg) = (v(h) \triangleleft r(g_{(1)})) v(g_{(2)})$$

(2) The actions $(\triangleright,\triangleleft)$ are implemented from $(\triangleright',\triangleleft')$ via $(r,v)$ as follows:

$$h \triangleleft a = v^{-1}(v(h) \triangleleft' a)$$

$$h \triangleright a = r(h_{(1)}) (v(h_{(2)})) \triangleright' a_{(1)} (S_{A} \circ r \circ v^{-1})(v(h_{(3)})) \triangleleft' a_{(2)}$$

for all $a \in A$ and $h \in H$, where $v^{-1}$ is the usual inverse of the bijective map $v$.

**Remark 3.7.** The condition (66) is equivalent to saying that $v : (A,\triangleleft) \rightarrow (H,\triangleleft')$ is an isomorphism of right $A$-module coalgebras. There exists a trivial object in $\mathcal{MP}(A,H)$, namely $(\triangleleft,\triangleright')$, where $\triangleright,\triangleright'$ are both the trivial actions. An object $(\triangleright,\triangleleft)$ of $\mathcal{MP}(A,H)$ is called a coboundary if $(\triangleright,\triangleleft)$ is cohomologous with the trivial object $(\triangleleft,\triangleright')$. Thus, if we write down the conditions from Definition 3.6 we can easily prove that an object $(\triangleright,\triangleleft)$ of $\mathcal{MP}(A,H)$ is a coboundary if and only if the right action $\triangleleft$ is the trivial action and the left action $\triangleright$ is implemented by

$$h \triangleright a = r(h_{(1)}) a S_{A}(r(h_{(2)}))$$

for some unitary cocentral map $r : H \rightarrow A$ that is also a morphism of Hopf algebras. This is in fact the necessary and sufficient condition for a bicrossed product $A \bowtie H$ to be isomorphic as left $A$-modules and Hopf algebras to the usual tensor product $A \otimes H$.

The classification theorem now follows: the set of all isomorphism types of bicrossed products $A \bowtie H$ which stabilize $A$ (i.e. the skeleton of the category $\mathcal{MP}(A,H)$) is in bijection with a cohomologically type pointed set $\mathcal{H}^{2}(A,H)$.

**Theorem 3.8. (The classification of bicrossed products)** Let $A$ and $H$ be two Hopf algebras. Then $\approx$ is an equivalence relation on the set $\mathcal{MP}(A,H)$ and there exists an one-to-one correspondence between the set of objects of the skeleton of the category $\mathcal{MP}(A,H)$ and the pointed quotient set $\mathcal{H}^{2}(A,H) := \mathcal{MP}(A,H)/\approx$.

**Proof.** It follows from Theorem 3.5 that $(\triangleright,\triangleleft) \approx (\triangleright',\triangleleft')$ if and only if there exists a left $A$-linear Hopf algebra isomorphism $\psi : A \bowtie H \rightarrow A \bowtie' H$, where $A \bowtie H$ and $A \bowtie' H$ are the bicrossed products associated to the matched pairs $(A,H,\triangleright,\triangleleft)$ and respectively $(A,H,\triangleright',\triangleleft')$. The compatibility condition (67) is exactly (61) taking into account (66). Thus, $\approx$ is an equivalence relation on the set $\mathcal{MP}(A,H)$ and we are done. \qed

Theorem 3.5 has several applications. Three of them are given bellow. First we shall apply it for infinite dimensional quantum doubles. Let $D_{\lambda}(A,H)$ and $D_{\lambda'}(A,H')$ be two generalized quantum doubles associated to two skew pairings $\lambda, \lambda'$ as constructed in (4) Example 1.1. We shall prove a necessary and sufficient condition for $D_{\lambda}(A,H) \cong D_{\lambda'}(A,H')$, an isomorphism of Hopf algebras and left $A$-modules.
Corollary 3.9. Let $\lambda : H \otimes A \to k$, $\lambda' : H' \otimes A \to k$ be two skew pairings of Hopf algebras, $D_\lambda(A, H)$ and $D_{\lambda'}(A, H')$ the generalized quantum doubles. The following are equivalent:

1. There exists a left $A$-linear Hopf algebra isomorphism $D_\lambda(A, H) \cong D_{\lambda'}(A, H')$;
2. There exists a pair of maps $(r, v)$, where $r : H \to A$ is a unitary cocentral map, $v : H \to H'$ is an unitary isomorphism of coalgebras satisfying the following four compatibility conditions:

$$r(hg) = r(h(1))r(g(2))\lambda'(v(h(2)), r(g(3))S_A(r(g(1))))$$
$$v(hg) = v(h(2))v(g(2))\lambda'(S_{H'}(v(h(1)))v(h(3)), r(g(1)))$$
$$v(h(2))\lambda(S_H(h(1))h(3), a) = v(h(2))\lambda'(S_{H'}(v(h(1)))v(h(3)), a)$$
$$a(2)r(h(2))\lambda(S_H(h(1)), a(1))\lambda(h(3), a(3)) =$$
$$= r(h(1))a(2)\lambda'(v(h(3)), a(3))\lambda'(S_{H'}(v(h(2))), a(1))$$

for all $a \in A$, $g, h \in H$.

Proof. First we note that $D_\lambda(A, H) = A \bowtie_\lambda H$, where the matched pair $(A, H, \langle \lambda, \rangle_\lambda)$ is given in (14) and (15). Now, we are in position to apply Theorem 3.5. The four compatibility conditions from the second statement are precisely (59)-(62) applied to the matched pairs associated to $\lambda$ and $\lambda'$. In order to simplify these compatibilities, the axioms of the skew pairings $\lambda$ and $\lambda'$ are used as well as the fact that $\lambda(h, a) = \lambda(S_H(h), S_A(a))$, for all $h \in H$ and $a \in A$ [21, Lemma 1.4]. The third compatibility above is (62), which means exactly the fact that $v : H \to H'$ is also a right $A$-module map. 

The next corollary provides necessary and sufficient conditions for a bicrossed product $A \bowtie H$ to be isomorphic to a smash product $A \# H'$ such that the isomorphism stabilizes $A$.

Corollary 3.10. Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras, $H'$ a Hopf algebra and $(A, \trianglerighthook)$ a left $H'$-module algebra and coalgebra satisfying the compatibility condition (3). The following are equivalent:

1. There exists a left $A$-linear Hopf algebra isomorphism $A \bowtie H \cong A \# H'$;
2. The right action $\triangleright$ is the trivial action and there exists a pair $(r, v)$, where $r : H \to A$ is a unitary cocentral map, $v : H \to H'$ is an isomorphism of Hopf algebras such that the left action $\trianglerighthook$ is implemented by the formula:

$$h \trianglerighthook a = r(h(1)) (v(h(2)) \trianglerighthook a) (S_A \circ r)(h(3))$$

(68)

and the following compatibility condition holds:

$$r(hg) = r(h(1))(v(h(2)) \trianglerighthook r(g))$$

(69)

for all $h, g \in H$ and $a \in A$.

Proof. We apply Theorem 3.5 for the matched pair $(A, H', \trianglerighthook', \triangleleft')$, where $\triangleleft'$ is the trivial action. Using the fact that $v$ is bijective, we obtain from the compatibility condition
that the right action $\triangleright$ is also the trivial action. On the other hand (61) takes the equivalent form (68) using that $r$ is invertible in the convolution with the inverse $S_A \circ r$.

As a special case of Theorem 3.5 we have the following interesting result:

**Corollary 3.11.** Let $A$, $H$, $H'$ be three Hopf algebras. Then there exists a bijection between the set of all left $A$-linear Hopf algebra isomorphisms $\psi : A \otimes H \to A \otimes H'$ and the set of all pairs $(r, v)$, where $v : H \to H'$ is an isomorphism of Hopf algebras, $r : H \to A$ is a unitary cocentral map and a morphism of Hopf algebras with $\text{Im}(r) \subset Z(A)$, the center of $A$.

Under the above bijection the left $A$-linear Hopf algebra isomorphism $\psi : A \otimes H \to A \otimes H'$ corresponding to $(r, v)$ is given by:

$$\psi(a \otimes h) = ar(h_{(1)}) \otimes v(h_{(2)})$$

(70)

for all $a \in A$ and $h \in H$.

**Proof.** We consider $\triangleleft$, $\triangleleft'$, $\triangleright$ and $\triangleright'$ to be all the trivial actions in Theorem 3.5. The compatibility conditions (59)-(61) implies in this case that $r$ and $v$ are also algebra maps, while (61) becomes

$$r(h_{(1)}) a (S_A \circ r)(h_{(2)}) = \varepsilon_H(h)a$$

for all $a \in A$, $h \in H$ which is equivalent to the centralizing condition $r(h)a = ar(h)$, for all $a \in A$ and $h \in H$. □

**Corollary 3.12.** Let $A$, $H$ and $H'$ be three Hopf algebras. Then $H$ and $H'$ are isomorphic as Hopf algebras if and only if there exists a left $A$-linear Hopf algebra isomorphism $A \otimes H \cong A \otimes H'$.

**Proof.** If $f : H \to H'$ is an isomorphism of Hopf algebras, then $\text{Id}_A \otimes f : A \otimes H \to A \otimes H'$ is a left $A$-linear Hopf algebra isomorphism. Conversely, if $\psi : A \otimes H \to A \otimes H'$ is a left $A$-linear Hopf algebra isomorphism then the map $v : H \to H'$ constructed in the proof of Corollary 3.11 is in fact an isomorphism of Hopf algebras. □

4. Examples

This section is devoted to the construction of some explicit examples: for two given Hopf algebras $A$ and $H$ we will describe and classify all Hopf algebras $E$ that factorize through $A$ and $H$. There are three steps that we have to go through: first of all we have to compute the set of all matched pairs between $A$ and $H$. Then we have to describe by generators and relations all bicrossed products $A \ltimes H$ associated to these matched pairs. Finally, using Theorem 2.2, we shall classify up to an isomorphism the bicrossed products $A \ltimes H$. As an application, the group $\text{Aut}_{\text{Hopf}}(A \ltimes H)$ of all Hopf algebra automorphisms of a given bicrossed product is computed.
For a Hopf algebra \( H, G(H) \) is the set of group-like elements of \( H \) and for \( g, h \in G(H) \) we denote by \( P_{g,h}(H) \) the set of all \((g,h)\)-primitive elements, that is
\[
P_{g,h}(H) = \{ x \in H \mid \Delta_H(x) = x \otimes g + h \otimes x \}
\]

The following result is very useful in computing all matched pairs between \( A \) and \( H \):

**Lemma 4.1.** Let \((A,H,\prec,\succ)\) be a matched pair of Hopf algebras, \( a, b \in G(A) \) and \( g, h \in G(H) \). Then:

1. \( g \succ a \in G(A) \) and \( g \prec a \in G(H) \);
2. If \( x \in P_{a,b}(A) \), then \( g \prec x \in P_{g \alpha a, g \alpha b}(H) \) and \( g \succ x \in P_{g \alpha a, g \alpha b}(A) \);
3. If \( y \in P_{g,h}(H) \), then \( y \prec a \in P_{g \alpha a, h \alpha (H)} \) and \( y \succ a \in P_{g \alpha a, h \alpha a}(A) \).

In particular, if \( x \) is an \((1_A, b)\)-primitive element of \( A \), then \( g \prec x \) is an \((g, g \alpha b)\)-primitive element of \( A \) and \( g \prec x \) is an \((g, g \alpha b)\)-primitive element of \( H \).

**Proof.** Straightforward: in fact for (1)-(3) we only use the fact that \( \prec \) and \( \succ \) are coalgebra maps, i.e. (1) and (2) hold. For the final statement we use the normalizing conditions (3) for \((\prec, \succ)\). \( \square \)

From now on, \( C_n \) will be the cyclic group of order \( n \) generated by \( c \) and \( k \) will be a field of characteristic \( \neq 2 \). Let \( A := H_4 \) be the Sweedler’s 4-dimensional Hopf algebra having \( \{1, g, x, gx\} \) as a basis subject to the relations:
\[
g^2 = 1, \quad x^2 = 0, \quad xg = -gx
\]
with the coalgebra structure and antipode given by:
\[
\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \Delta(gx) = gx \otimes g + 1 \otimes gx
\]
\[
\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g, \quad S(x) = -gx
\]

In order to compute the set of all matched pairs between \( H_4 \) and \( k[C_n] \) we need the following elementary result.

**Lemma 4.2.** Let \( k \) be a field of characteristic \( \neq 2 \), \( H_4 \) the Sweedler’s 4-dimensional Hopf algebra and \( k[C_n] \) the group algebra of \( C_n \). Then:
\[
P_{c^i, c^j}(k[C_n]) = \{ \lambda c^i - \lambda c^j \mid \lambda \in k \}
\]
\[
G(H_4) = \{1, g\}, \quad P_{1,1}(H_4) = \{0\}, \quad P_{g,g}(H_4) = \{0\}
\]
\[
P_{1,g}(H_4) = \{ \alpha - \alpha g + \beta x \mid \alpha, \beta \in k \}, \quad P_{g,1}(H_4) = \{ \alpha - \alpha g + \beta gx \mid \alpha, \beta \in k \}
\]
for all \( i, j = 0, 1, \cdots, n - 1 \).

**Proof.** Everything is just a straightforward computation. For example, let \( z = \lambda_0 + \lambda_1 c + \lambda_2 c^2 + \cdots + \lambda_{n-1} c^{n-1} \in P_{c^i, c^j}(k[C_n]) \), for some \( \lambda_j \in k \). Let \( t \neq i, t \neq j \) and apply \( \text{Id} \otimes (c^t)^* \) in \( \Delta(z) = z \otimes c^i + c^j \otimes z \) (where \((c^t)^*\)_{t=0, \cdots, n-1} is the dual basis of \((c^t)_{t=0, \cdots, n-1} \) we obtain \( \lambda_t c^i = \lambda_t c^j \), thus \( \lambda_t = 0 \). Hence, \( z = \lambda_i c^i + \lambda_j c^j \). Now using \( \Delta(z) = z \otimes c^i + c^j \otimes z \) we obtain that \( \lambda_i + \lambda_j = 0 \), and thus \( z = \lambda c^i - \lambda c^j \), for some \( \lambda \in k \). \( \square \)
For a positive integer \( n \) let \( U_n(k) = \{ \omega \in k \mid \omega^n = 1 \} \) be the cyclic group of \( n \)-th roots of unity in \( k \). The group \( U_n(k) \) parameterizes the set of all matched pairs \((H_4, k[C_n], \langle, \rangle)\).

**Proposition 4.3.** Let \( k \) be a field of characteristic \( \neq 2 \), \( n \) a positive integer and \( C_n \) the cyclic group of order \( n \). Then there exists a bijective correspondence between the set of all matched pairs \((H_4, k[C_n], \langle, \rangle)\) and \( U_n(k) \) such that the matched pair \((\langle, \rangle)\) corresponding to an \( n \)-th root of unity \( \omega \in U_n(k) \) is given by:

\[
\begin{array}{c|cccc}
\langle & 1 & g & x & gx \\
\hline
1 & 1 & 1 & 0 & 0 \\
c & c & c & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c^k & c^k & c^k & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c^{n-1} & c^{n-1} & c^{n-1} & 0 & 0 \\
\end{array}
\qquad
\begin{array}{c|cccc}
\rangle & 1 & g & x & gx \\
\hline
1 & 1 & 1 & 0 & 0 \\
c & 1 & g & \omega x & \omega gx \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c^k & 1 & g & \omega^k x & \omega^k gx \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c^{n-1} & 1 & g & \omega^{n-1} x & \omega^{n-1} gx \\
\end{array}
\]

**Proof.** Let \((H_4, k[C_n], \langle, \rangle)\) be a matched pair. It follows from the normalizing conditions \( \text{(4)} \) that

\[c^i \triangleright 1 = 1, \quad 1 \triangleright z = z, \quad 1 \triangleleft z = \varepsilon(z)1, \quad c^i \triangleleft 1 = c^i\]

for all \( i = 0, \ldots, n - 1 \) and \( z \in H_4 \). We prove now that \( c \triangleright g = g \). Indeed, it follows from Lemma \( \text{[4.1]} \) that \( c \triangleright g \) is a group-like element in \( H_4 \). Thus, using Lemma \( \text{[4.2]} \) we obtain that \( c \triangleright g \in \{1, g\} \). If \( c \triangleright g = 1 \) using the fact that \( \triangleright \) is a left action we obtain \( c^i \triangleright g = 1 \), for any positive integer \( i \). In particular, \( g = 1 \triangleright g = c^n \triangleright g = 1 \), contradiction. Thus, \( c \triangleright g = g \) and hence \( c^i \triangleright g = g \), for all \( i = 1, \ldots, n - 1 \).

Now, \( x \) is an \((1,g)\)-primitive element of \( H_4 \). As \( c \triangleright g = g \) we obtain using Lemma \( \text{[4.1]} \) that \( c \triangleright x \) is an \((1,g)\)-primitive element of \( H_4 \). It follows from Lemma \( \text{[4.2]} \) that \( c \triangleright x = \beta - \beta g + \omega x \), for some \( \beta, \omega \in k \). We will first prove that \( \omega^n = 1 \) and later on that \( \beta = 0 \). Indeed, by induction, having in mind that \( \triangleright \) is a left action we obtain:

\[c^i \triangleright x = \beta \sum_{j=0}^{i-1} \omega^j - \beta (\sum_{j=0}^{i-1} \omega^j) g + \omega^i x\]

for any positive integer \( i \). In particular,

\[x = 1 \triangleright x = c^n \triangleright x = \beta \sum_{j=0}^{n-1} \omega^j - \beta (\sum_{j=0}^{n-1} \omega^j) g + \omega^n x\]

thus \( \omega^n = 1 \) and \( \beta \sum_{j=0}^{n-1} \omega^j = 0 \). Similarly, we obtain that for any \( i = 1, \ldots, n - 1 \) we have:

\[c^i \triangleright (gx) = \gamma \sum_{j=0}^{i-1} \zeta^j - \gamma (\sum_{j=0}^{i-1} \zeta^j) g + \zeta^i gx\]

for some \( \gamma, \zeta \in k \), such that \( \zeta^n = 1 \) and \( \gamma \sum_{j=0}^{n-1} \zeta^j = 0 \). At the end of the proof we will see that \( \gamma = \beta = 0 \) and \( \zeta = \omega \).
Meanwhile, using what we already know about the left action $\triangleright$, we shall prove that the other action $\triangleleft$ of the matched pair $(H_4, k[C_n], \triangleright, \triangleright)$ is necessarily the trivial action of $H_4$ on $k[C_n]$. First of all, using Lemma 4.11 we obtain that $c^i \triangleleft g$ is a group-like element in $k[C_n]$, hence $c^i \triangleleft g \in \{1, c, \cdots, c^{n-1}\}$. We claim now that $c^i \triangleleft g = c^i$, for all $i = 1, \cdots, n-1$. Indeed, suppose that there exist $i \neq j$ such that $c^i \triangleleft g = c^j$. Then, the compatibility condition (6) from the definition of a matched pair applied for $(c^i, x)$ takes the form:

$$c^i \triangleleft x \otimes c^i \triangleright 1 + c^i \triangleleft g \otimes c^i \triangleright x = c^i \triangleleft 1 \otimes c^i \triangleright x + c^i \triangleleft x \otimes c^i \triangleright g$$

which is equivalent to:

$$c^i \triangleleft x \otimes 1 + \beta \left( \sum_{t=0}^{i-1} \omega^t \right) c^i \otimes 1 - \beta \left( \sum_{t=0}^{i-1} \omega^t \right) c^i \otimes g + \omega^i c^i \otimes x = c^i \triangleleft x \otimes g$$

Now observe that the coefficient of $c^i \otimes x$ in the left hand side of the equality is $\omega^i$ while the coefficient of the same element in the right hand side of the equality is 0 if $i \neq j$. Since $\omega^n = 1$ we reached a contradiction. Thus $c^i \triangleleft g = c^i$, for all $i = 1, \cdots, n-1$. Now, $c^i$ is a group like element of $k[C_n]$ and $x \in P_{\triangleright g}(H_4)$. Thus, using Lemma 4.11 we obtain that $c^i \triangleleft x \in P_{\triangleright, \triangleleft g}(k[C_n]) = P_{\triangleright, \triangleleft}(k[C_n])$. Hence, it follows from Lemma 4.2 that $c^i \triangleleft x = 0$. A similar argument shows that $c^i \triangleleft (gx) = 0$, for all $i = 1, \cdots, n-1$.

Thus we have proved that $\triangleleft$ is the trivial action. In this case the compatibility conditions from the definition of a matched pair become simpler. More precisely the compatibility (1) becomes $h \triangleright (ab) = (h_{(1)} \triangleright a) (h_{(2)} \triangleright b)$, for all $h \in k[C_n]$, $a$ and $b \in H_4$, the compatibility (5) becomes trivially fulfilled while the compatibility (6) becomes $h_{(1)} \otimes h_{(2)} \triangleright a = h_{(2)} \otimes h_{(1)} \triangleright a$, for all $h \in k[C_n]$, $a \in H_4$ and is also automatically satisfied since $k[C_4]$ is cocommutative.

Now, using the compatibility condition (11) in its simplified form $c \triangleright (gx) = (c \triangleright g)(c \triangleright x)$, we obtain:

$$\gamma - \gamma g + \zeta \cdot gx = g(\beta - \beta g + \omega x) = -\beta + \beta g + \omega gx$$

and therefore, $\gamma = -\beta$ and $\zeta = \omega$. Moreover, since $0 = c \triangleright x^2 = (c \triangleright x)(c \triangleright x)$, we find that

$$0 = (\beta - \beta g + \omega x)^2 = 2\beta^2 - 2\beta^2 g + 2\beta \omega x$$

As $\text{char}(k) \neq 2$ we obtain that $\beta = 0$ and, as a consequence, that $c^i \triangleright x = \omega^i x$ and $c^i \triangleright (gx) = \omega^i (gx)$, for all $i = 1, \cdots, n-1$ and hence the left action $\triangleright$ is also fully described. We are left to verify the rest of the compatibilities, and, since this is a routinely check, we omit it. \hfill $\square$

We prove now that a Hopf algebra $E$ factorizes through $H_4$ and $k[C_n]$ if and only if $E \cong H_{4n, \omega}$, where $H_{4n, \omega}$ is a quantum group associated to any $\omega$ an $n$-th root of unity as described bellow.

**Corollary 4.4.** Let $k$ be a field of characteristic $\neq 2$ and $n$ a positive integer. Then a Hopf algebra $E$ factorizes through $H_4$ and $k[C_n]$ if and only if $E \cong H_{4n, \omega}$, for some
ω ∈ U_n(k), where we denote by H_{4n,ω} the Hopf algebra generated by g, x and c subject to the relations:

\[ g^2 = c^n = 1, \quad x^2 = 0, \quad xg = gx, \quad cg = gc, \quad cx = \omega xc \]

with the coalgebra structure and antipode given by:

\[ \Delta(g) = g \otimes g, \quad \Delta(c) = c \otimes c, \quad \Delta(x) = x \otimes 1 + g \otimes x \]

\[ \varepsilon(c) = \varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(c) = c^{n-1}, \quad S(g) = g, \quad S(x) = -gx \]

H_{4n,ω} is a pointed non-semisimple 4n-dimensional Hopf algebra.

**Proof.** It follows from Proposition 4.3 and Theorem 1.2. The Hopf algebra H_{4n,ω} is the bicrossed product H_4 ≀ k[C_n] associated to the matched pair given in Proposition 4.3. Indeed, up to canonical identification, H_4 ≀ k[C_n] is generated as an algebra by \( g = g \bowtie 1, \quad x = x \bowtie 1 \) and \( c = 1 \bowtie c \). Hence,

\[ cx = (1 \bowtie c)(x \bowtie 1) = c \bowtie x(1) \bowtie c \bowtie x(2) = \omega x \bowtie 1 \bowtie c = \omega (x \bowtie 1)(1 \bowtie c) = \omega xc \]

H_4 ≀ k[C_n] is pointed by being generated by two group-likes and a primitive element [12] Lemma 1] and non-semisimple as \( H_4 \) is non-semisimple.

**Remark 4.5.** A k-basis in H_{4n,ω} is given by \{c^i, gc^i, xc^i, gx c^i | i = 0, \ldots, n - 1\}. We note that \( H_{4n,ω} \) is the bicrossed product \( H_4 \bowtie k[C_n] \) associated to the matched pair in which the right action \( \bowtie : k[C_n] \otimes H_4 \to k[C_n] \) is the trivial action. Thus, \( H_{4n,ω} \) is in fact the semi-direct product \( H_4 \# k[C_n] \) associated to the left action \( \bowtie : k[C_n] \otimes H_4 \to H_4 \) given in Proposition 4.3. In particular, \( H_4 \) is a normal Hopf subalgebra of \( H_{4n,ω} \).

In order to classify these Hopf algebras \( H_{4n,ω} \) we still need one more elementary lemma:

**Lemma 4.6.** Let k be a field of characteristic \( \neq 2 \), \( H_4 \) the Sweedler’s 4-dimensional Hopf algebra and \( k[C_n] \) the group algebra of \( C_n \). Then:

1. \( u : H_4 \to H_4 \) is a unitary coalgebra map if and only if \( u \) is the trivial morphism \( u(h) = \varepsilon(h)1, \) for all \( h \in H_4 \), or there exist \( \alpha, \beta, \gamma, \delta \in k \) such that \( u(1) = 1, \) \( u(g) = g, \)
\( u(x) = \alpha - \alpha g + \beta x, \) and \( u(gx) = \gamma - \gamma g + \delta gx \). In particular, \( \text{CoZ}^1(H_4, H_4) \) is the trivial group with only one element.

2. \( u : H_4 \to H_4 \) is a Hopf algebra morphism if and only if \( u \) is the trivial morphism \( u(h) = \varepsilon(h)1, \) for all \( h \in H_4 \), or there exists \( \beta \in k \) such that \( u(1) = 1, \) \( u(g) = g, \)
\( u(x) = \beta x, \) and \( u(gx) = \beta gx \). In particular, \( \text{Aut}_{\text{Hopf}}(H_4) \cong k^* \).

3. Assume that \( n \) is odd. Then:

3a. \( p : H_4 \to k[C_n] \) is a morphism of Hopf algebras if and only if \( p \) is the trivial morphism: \( p(h) = \varepsilon(h)1, \) for all \( h \in H_4 \).

3b. \( r : k[C_n] \to H_4 \) is a morphism of Hopf algebras if and only if \( r \) is the trivial morphism: \( r(c^i) = 1, \) for all \( i = 0, \ldots, n - 1 \).

4. Assume that \( n = 2m \) is even. Then:

4a. \( p : H_4 \to k[C_n] \) is a morphism of Hopf algebras if and only if \( p \) is the trivial morphism or \( p \) is given by: \( p(1) = 1, \) \( p(g) = c^m, \) \( p(x) = p(gx) = 0. \)
(4b) \( r : k[C_n] \to H_4 \) is a morphism of Hopf algebras if and only if \( r \) is the trivial morphism or \( r \) is given by \( r(c^i) = g^i \), for all \( i = 0, \ldots, n-1 \).

**Proof.** (1) Let \( u : H_4 \to H_4 \) be a unitary coalgebra map. Then \( u(g) \in G(H_4) = \{1, g\} \) and \( u(x) \in P_1, u(g)(H_4) \) respectively \( u(gx) \in P_{u(g),1}(H_4) \). By applying Lemma 4.2 we arrive at the desired conclusion: if \( u(g) = 1 \), then \( u \) is the trivial morphism, while if \( u(g) = g \), then \( u \) has the second form. A little computation shows that among all these morphisms only the trivial one satisfies the cocycle condition (3).

(2) It follows from (1). If \( u : H_4 \to H_4 \) is the non-trivial morphism, then considering the relations on the generators of \( H_4 \) we obtain that \( u \) is also an algebra map if and only if \( \alpha = \gamma = 0 \) and \( \delta = \beta \). The final assertion is obvious.

(3) and (4) Let now \( p : H_4 \to k[C_n] \) be a Hopf algebra map. Then, \( p(g) \) is a group-like element in \( k[C_n] \), i.e. \( p(g) = c^i \), for some \( i \in \{0,1,\ldots,n-1\} \). It follows from

\[
1 = p(g^2) = p(g)^2 = c^{2i}
\]

that \( n \mid 2i \). If \( n \) is odd, then \( n \mid i \), hence \( i = 0 \) i.e. \( p(g) = 1 \). If \( n = 2m \) is even, then \( m \mid i \), hence \( i = 0 \) or \( i = m \). Thus, \( p(g) = 1 \) or \( p(g) = c^m \). On the other hand, \( x \) is an \( (1,g) \)-primitive element of \( H_4 \), hence \( p(x) \in P_1, p(g)(k[C_n]) \). Using Lemma 4.2 we obtain that \( p(x) = 0 \) if \( p(g) = 1 \), and \( p(x) = \lambda - \lambda c^m \) if \( p(g) = c^m \). In the last case we have:

\[
0 = p(x^2) = p(x)^2 = (\lambda - \lambda c^m)^2 = 2\lambda^2 - 2\lambda^2 c^m
\]

therefore \( \lambda = 0 \) i.e. \( p(x) = 0 \). A similar discussion describes Hopf algebra maps \( r : k[C_n] \to H_4 \) and we are done.

We shall give now the necessary and sufficient conditions for two Hopf algebras \( H_{4n,\omega} \) and \( H_{4n,\omega'} \) to be isomorphic. The classification Theorem 4.7 below depends on the structure of \( U_n(k) \). We denote by \( \nu(n) = [U_n(k)] \), the order of the cyclic group \( U_n(k) \) and we shall fix \( \xi \) a generator of \( U_n(k) \). Let \( \omega = \xi^t \), \( \omega' = \xi^s \) be two arbitrary \( n \)-th roots of unity, for some positive integers \( t, s \in \{0, \ldots, \nu(n) - 1\} \).

**Theorem 4.7.** Let \( k \) be a field of characteristic \( \neq 2 \), \( n \) a positive integer, \( \xi \) a generator of the group \( U_n(k) \) of order \( \nu(n) \) and \( l, t \in \{0, \ldots, \nu(n) - 1\} \). Then:

1. Assume that one of the positive integers \( n \) or \( \nu(n) \) is odd. Then the Hopf algebras \( H_{4n,\xi^t} \) and \( H_{4n,\xi^s} \) are isomorphic if and only if there exists \( s \in \{0,1,\ldots,n-1\} \) such that \( \gcd(s, n) = 1 \) and \( \nu(n) \mid l - ts \).

2. Assume that \( n \) and \( \nu(n) \) are both even. Then the Hopf algebras \( H_{4n,\xi^t} \) and \( H_{4n,\xi^s} \) are isomorphic if and only if there exists \( s \in \{0,1,\ldots,n-1\} \) such that \( \gcd(s, n) = 1 \) and one of the following conditions hold: \( \nu(n) \mid l - ts \) or \( 2(l - ts) = \nu(n)q \), for some odd integer \( q \).

**Proof.** We shall prove more. In fact we will describe the set of all possible Hopf algebra isomorphisms between \( H_{4n,\xi^t} \) and \( H_{4n,\xi^s} \). We denote by \( H_{4n,\xi^t} := H_4 \# k[C_n] \) (resp. \( H_{4n,\xi^s} := H_4 \# k[C_n] \)) the semidirect product implemented by the left action \( b : k[C_n] \otimes H_4 \to H_4 \), \( c \triangleright x = \xi^t x \) (resp. \( c \triangleright x = \xi^s x \)) from Proposition 4.3. We recall from Corollary 2.3 that \( \psi : H_{4n,\xi^t} \to H_{4n,\xi^s} \) is an isomorphism of Hopf algebras if and only
if there exists two unitary coalgebra maps $u : H_4 \to H_4$, $r : k[C_n] \to H_4$ and two morphisms of Hopf algebras $p : H_4 \to k[C_n]$, $v : k[C_n] \to k[C_n]$ such that for any $a, b \in H_4$, $g, h \in k[C_n]$ we have

$$u(a_{(1)}) \otimes p(a_{(2)}) = u(a_{(2)}) \otimes p(a_{(1)})$$

$$r(h_{(1)}) \otimes v(h_{(2)}) = r(h_{(2)}) \otimes v(h_{(1)})$$

$$u(ab) = u(a_{(1)}) (p(a_{(2)}) b' u(b))$$

$$r(hg) = r(h_{(1)}) (v(h_{(2)}) b' r(g))$$

$$r(h_{(1)}) \left( v(h_{(2)}) b' u(b) \right) = u(h_{(1)} \triangleright b_{(1)}) \left( p(h_{(2)} \triangleright b_{(2)}) b' r(h_{(3)}) \right)$$

$$v(h) p(b) = p(h_{(1)} \triangleright b) v(h_{(2)})$$

and $\psi = \psi_{(u,r,p,v)} : H_{4n,ξ^i} \to H_{4n,ξ^i}$ is given by:

$$\psi(a \# h) = u(a_{(1)}) (p(a_{(2)}) b' r(h_{(1)})) \# p(a_{(3)}) v(h_{(2)})$$

for all $a \in H_4$ and $h \in k[C_n]$. In what follows we describe completely all quadruples $(u, r, p, v)$ which satisfy the compatibility conditions (71)-(76) such that $\psi = \psi_{(u,r,p,v)}$ given by (77) is bijective. First, we should notice that (72) holds trivially as $p(a_{(1)}) \otimes p(a_{(2)})$ is bijective. Next we shall prove simultaneously that any Hopf algebra map $v : k[C_n] \to k[C_n]$ is cocommutative. Also any morphism of Hopf algebras $v : k[C_n] \to k[C_n]$ is given by

$$v : k[C_n] \to k[C_n], \quad v(c) = e^s$$

for some $s = 0, \ldots, n - 1$. For future use, we note that such a morphism $v$ is bijective if and only if $(s, n) = 1$.

Next we shall prove simultaneously that any Hopf algebra map $p : H_4 \to k[C_n]$ of a such quadruple $(u, r, p, v)$ is the trivial morphism

$$p : H_4 \to k[C_n], \quad p(z) = \varepsilon(z) 1$$

for any $z \in H_4$ and any unitary coalgebra morphism $u : H_4 \to H_4$ of a such quadruple $(u, r, p, v)$ is given by

$$u : H_4 \to H_4, \quad u(1) = 1, \quad u(g) = g, \quad u(x) = \gamma x, \quad u(gx) = \gamma gx$$

for some non-zero scalar $\gamma \in k^*$. Indeed, it follows from (3a) and (4a) of Lemma 4.6 that $p(x) = 0$ and $p(gx) = 0$. Now, using the normalizing conditions (83), the fact that $r, v$ are unitary maps and $p$ is a coalgebra map we obtain from (77) that

$$\psi(x \# 1) = u(x_{(1)}) \# p(x_{(2)}) = u(x) \# 1 + u(g) \# p(x) = u(x) \# 1$$

As $\psi$ has to be an isomorphism and $x$ is an element of the basis, we obtain that $u(x) \neq 0$. It follows from (1) of Lemma 4.6 that $u(x) = \alpha - \alpha g + \gamma x$, for some $\alpha, \gamma \in k$. Now, by applying (71) for $a = x$, we obtain:

$$(\alpha - \alpha g + \gamma x) \otimes 1 = (\alpha - \alpha g + \gamma x) \otimes p(g)$$

As $u(x) \neq 0$, we must have $p(g) = 1$ and therefore $p : H_4 \to k[C_n]$ is the trivial morphism. Condition (83) is then equivalent to $u : H_4 \to H_4$ being an algebra map, i.e. $u$ is a morphism of Hopf algebras. Using (2) of Lemma 4.6 we obtain that $u$ is given by (80), where $\gamma \in k^*$, since $u(x) \neq 0$. Thus, we fully described the maps $u, v$ and $p$ in
the quadruple \((u, r, p, v)\). Furthermore, since \(p\) is the trivial morphism the compatibility conditions (71) and (76) are trivially fulfilled.

We focus now on the unitary morphism of coalgebras \(r : k[C_n] \to H_4\) in a such quadruple. As \(r\) is a coalgebra map we have that \(r(c^i) \in G(H_4) = \{1, g\}\), for all \(i = 0, \ldots, n - 1\). Since \(c^i v' z = z\), for all \(i = 0, \ldots, n - 1\) and \(z \in \{1, g\}\), it follows that (73) is equivalent to \(r : k[C_n] \to H_4\) being an algebra map, that is \(r : k[C_n] \to H_4\) is a Hopf algebra map.

In conclusion, taking into account that \(p\) is the trivial morphism, we proved so far that any potential isomorphism \(\psi : H_{4n, \xi} \to H_{4n, \xi'}\) is defined by the formula

\[
\psi : H_4 \# h k[C_n] \to H_4 \# h k[C_n], \quad \psi(a \# h) = u(a) r(h(1)) \# v(h(2))
\]

(81) for all \(a \in H_4\) and \(h \in k[C_n]\), where \(u : H_4 \to H_4\) is the isomorphism of Hopf algebras given by (80), \(v : k[C_n] \to k[C_n]\) is the Hopf algebra map given by (75) and \(r : k[C_n] \to H_4\) is a morphism of Hopf algebras satisfying the compatibility condition (75) in its simplified form, namely

\[
r(h(1))(v(h(2)) \triangleright u(b)) = u(h(1) \triangleright b) r(h(2))
\]

(82) for all \(h \in k[C_n]\) and \(b \in H_4\).

Finally, it remains to describe when there exists a Hopf algebra map \(r : k[C_n] \to H_4\) satisfying the compatibility condition (82) such that \(\psi\) given by (81) is bijective.

According to Lemma 4.6 we distinguish two cases. Suppose first that \(n\) is odd. Then using (3b) of Lemma 4.6 we obtain that \(r : k[C_n] \to H_4\) is also the trivial morphism namely, \(r(c^i) = \varepsilon(c^i) 1 = 1\), for all \(i = 0, 1, \ldots, n - 1\). Hence, \(\psi\) given by (81) takes the simplified form \(\psi(a \# h) = u(a) \# v(h)\) and \(\psi\) is an isomorphism if and only if \(v\) is bijective, i.e. if and only if \((s, n) = 1\). Moreover, the compatibility condition (82) becomes:

\[
v(h) \triangleright u(a) = u(h \triangleright a)
\]

(83) for all \(a \in H_4\) and \(h \in k[C_n]\). Applying the above compatibility for \(a = x\) and \(h = c\) we obtain \(\xi^{-st} = 1\), thus \(\nu(n) \mid (l - st)\). Moreover, it is easy to see that if \(\nu(n) \mid (l - st)\), then (83) holds for any \(a \in H_4\) and \(h \in k[C_n]\).

Finally, suppose now that \(n\) is even and use (4b) of Lemma 4.6. If \(r\) is the trivial morphism then the proof follows exactly as in the above odd case. Assume now that \(r(c) = g\). Hence, \(\psi\) is given by \(\psi(a \# h) = u(a) r(h) \# v(h)\), for all \(a \in H_4\) and \(h \in \{1, c, \ldots, c^{n-1}\}\).

It is easy to see that \(\psi\) is an isomorphism if and only if \(v\) is an isomorphism, i.e. if and only if \((s, n) = 1\). Now, we observe that the compatibility condition (82) is equivalent to \(\xi^{l-ts} = -1\). Indeed, (82) applied for \(a = x\) and \(h = c\) gives precisely \(\xi^{l-ts} = -1\). Conversely, if (82) holds for \(a = x\) and \(h = c\), then it is straightforward to see that it is fulfilled for any \(a \in H_4\) and any \(h \in k[C_n]\).

We shall prove now that \(\xi^{l-ts} = -1\) if and only if \(\nu(n)\) is even and \(2(l - ts) = \nu(n) q\), for some odd integer \(q\) and this finishes the proof.

Indeed, since \(\text{Char}(k) \neq 2\) and \(\varepsilon(n) = 1\), the equation \(\xi^{l-ts} = -1\) is possible only if \(\nu(n)\) is even. Consider \(\nu(n) = 2m\), for a positive integer \(m\). As \(\xi^{2m} = 1\), we obtain \(\xi^{m} = -1\), otherwise using that \(2m\) is the order of \(\xi\) we will obtain \(2m\mid m\), contradiction. Assume first that \(\xi^{l-ts} = -1\). Then, \(\xi^{2(l-ts)} = 1\) and hence \(\nu(n) \mid 2(l - ts)\), that is \(l - ts = mq\),
for some integer $q$. Thus, $-1 = \xi^{l-ts} = (\xi^m)^q = (-1)^q$, hence $q$ is odd. Conversely, is straightforward.

**Corollary 4.8.** Let $k$ be a field of characteristic $\neq 2$, $n$ a positive integer and $\xi$ a generator of the group $U_n(k)$. Then, $H_{4n,\xi}$ is isomorphic to $H_{4n,\xi'}$, for any positive integer $l \in \{1, \ldots, \nu(n) - 1\}$ such that $\gcd(l, n) = 1$.

In particular, if $n = p$ is a prime odd number, then a Hopf algebra factorizes through $H_4$ and $k[C_p]$ if and only if it is isomorphic to $H_{4n,1}$ or $H_{4n,\xi}$, where $\xi$ is a generator of $U_p(k)$.

**Proof.** We apply Theorem 4.7 for $t = 1$ by taking $s := l$ in its statement. □

In what follows we continue our investigation in order to indicate precisely the number of all types of isomorphisms of Hopf algebras which factorize through $H_4$ and $k[C_n]$. For this we need the following lemma in the proof of which we use Dirichlet’s theorem on primes in an arithmetical progression.

**Lemma 4.9.** Let $n$ and $m$ be two positive integers such that $m \mid n$ and $\varphi : \mathbb{Z}_n \to \mathbb{Z}_m$ the canonical projection $\varphi(a + n\mathbb{Z}) = a + m\mathbb{Z}$, for all $a \in \mathbb{Z}$. Then $\varphi(U(\mathbb{Z}_n)) = U(\mathbb{Z}_m)$.

**Proof.** Consider $a + m\mathbb{Z} \in U(\mathbb{Z}_m)$. Thus, $\gcd(a, m) = 1$. The Dirichlet’s theorem [16] ensure the fact that in the set $\{a + km | k \in \mathbb{Z}\}$ there exists infinitely many primes. In particular, there exists a prime number $p \in \{a + km | k \in \mathbb{Z}\}$ such that $p \nmid n$. As $\varphi(p + n\mathbb{Z}) = a + m\mathbb{Z}$, and $p + n\mathbb{Z} \in U(\mathbb{Z}_n)$, we deduce that $U(\mathbb{Z}_m) \subseteq \varphi(U(\mathbb{Z}_n))$. Obviously, $\varphi(U(\mathbb{Z}_n)) \subseteq U(\mathbb{Z}_m)$, hence our claim. □

We shall describe and count the set of types of Hopf algebras that factorize through $H_4$ and $k[C_n]$.

**Theorem 4.10.** Let $k$ be a field of characteristic $\neq 2$, $n$ a positive integer, $\xi$ a generator of $U_n(k)$ and $\nu(n) = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime decomposition of $\nu(n)$. Then:

1. There exists an isomorphism of Hopf algebras $H_{4n,\xi} \simeq H_{4n,\xi'\nu(n)}$, for all $t = 0, \ldots, \nu(n) - 1$. In particular, $H_{4n,\xi} \simeq H_{4n,\xi^{-i}}$, for any $i = 0, \ldots, \nu(n) - 1$.

2. Assume that $\nu(n)$ is odd. Then the set of types of Hopf algebras that factorize through $H_4$ and $k[C_n]$ is in bijection with the set of all Hopf algebras $H_{4n,\xi^d}$, where $d$ running over all positive divisors of $\nu(n)$. In particular, the number of types of such Hopf algebras is

   $$(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)$$

3. Assume that $\nu(n) = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is even. Then the set of types of Hopf algebras that factorize through $H_4$ and $k[C_n]$ is in bijection with the set of all Hopf algebras $H_{4n,\xi^d}$, where $d$ is running over all positive divisors of $\frac{\nu(n)}{2}$. In particular, the number of types of such Hopf algebras is

   $$\alpha_1(\alpha_2 + 1) \cdots (\alpha_r + 1)$$
Proof. (1) Let \( d = \text{gcd}(t, \nu(n)) \). Consider two positive integers \( a \) and \( m \) such that \( t = da \) and \( \nu(n) = dm \). Since \( \text{gcd}(a, m) = 1 \) and \( m \mid \nu(n) \mid n \), we obtain from Lemma \[ \text{[4.9]} \] that there exists \( s \in \{0, 1, ..., n - 1\} \) such that \( \text{gcd}(s, n) = 1 \) and \( m \mid a - s \). Multiplying the last relation by \( d \), we obtain \( \nu(n) \mid t - ds \). This, together with Theorem \[ \text{[4.7]} \] proves that \( H_{4n, \xi^d} \simeq H_{4n, \xi^s} \). Since \( \nu(n) \mid n \), the last statement follows from the first part using that \( \text{gcd}(i, \nu(n)) = \text{gcd}(n - i, \nu(n)) \), for any \( i = 1, \ldots, \nu(n) - 1 \).

(2) In the first part we have proved that any Hopf algebra \( H_{4n, \xi^d} \) is isomorphic to \( H_{4n, \xi^s} \), for some divisor \( d \) of \( \nu(n) \). We will show now that if \( d_1 \) and \( d_2 \) are two distinct positive divisors of \( \nu(n) \) then \( H_{4n, \xi^{d_1}} \) and \( H_{4n, \xi^{d_2}} \) are not isomorphic. This will prove our claim.

Let therefore \( d_1 \) and \( d_2 \) be two distinct positive divisors of \( \nu(n) \). Suppose that \( H_{4n, \xi^{d_1}} \simeq H_{4n, \xi^{d_2}} \). By Theorem \[ \text{[4.7]} \] there exists then \( s \in \{0, 1, ..., n - 1\} \) such that \( \text{gcd}(s, n) = 1 \) and \( \nu(n) \mid d_1 - sd_2 \). Since \( d_1 \) and \( d_2 \) are distinct, they differ by the exponent of a prime number, say \( p \). Let \( \alpha \) and \( \beta \) be the respective exponents of \( d_1 \) and \( d_2 \). We do not restrict the generality by assuming that \( \alpha > \beta \). Since

\[
\frac{\nu(n)}{p^\alpha} \mid \frac{d_1}{p^\beta} - \frac{d_2}{p^\beta},
\]

\( p \mid \frac{\nu(n)}{p^\alpha} \), \( p \mid \frac{d_1}{p^\beta} \), and \( p \nmid \frac{d_2}{p^\beta} \), we have arrived at the desired contradiction.

(3) We first prove that if \( d_1 \) and \( d_2 \) are two distinct positive divisors of \( \frac{\nu(n)}{2} \) then \( H_{4n, \xi^{d_1}} \not\simeq H_{4n, \xi^{d_2}} \). Then we prove that if \( d_1 \) is a positive divisor of \( \nu(n) \) then there exists \( d_2 \) a positive divisor of \( \frac{\nu(n)}{2} \) such that \( H_{4n, \xi^{d_1}} \simeq H_{4n, \xi^{d_2}} \). This, together with assertion (1) will finish the proof.

Suppose \( d_1 \) and \( d_2 \) are two distinct positive divisors of \( \frac{\nu(n)}{2} \). If \( s \in \{0, 1, ..., n - 1\} \) and \( \text{gcd}(s, n) = 1 \), then \( \frac{\nu(n)}{2} \nmid d_1 - sd_2 \). Indeed, \( d_1 \) and \( d_2 \) being distinct they differ by the exponent of a prime number, \( p \). Let \( \alpha \) and \( \beta \) be the respective exponents of \( d_1 \) and \( d_2 \). We may assume, without loss of generality that \( \alpha > \beta \). Since \( p \mid \frac{\nu(n)}{2p^\beta} \), \( p \mid \frac{d_1}{p^\beta} \), and \( p \nmid \frac{d_2}{p^\beta} \) (recall that \( \nu(n) \mid n \) and \( \text{gcd}(s, n) = 1 \)) we cannot have

\[
\frac{\nu(n)}{2p^\beta} \mid \frac{d_1}{p^\beta} - \frac{s}{p^\beta} \frac{d_2}{p^\beta},
\]

hence, neither \( \frac{\nu(n)}{2} \mid d_1 - sd_2 \). This being the case, we deduce from Theorem \[ \text{[4.7]} \] that \( H_{4n, \xi^{d_1}} \) and \( H_{4n, \xi^{d_2}} \) are not isomorphic.

For the second claim, consider \( d_1 \) a positive divisor of \( \nu(n) \). If \( d_1 = \nu(n) \), then \( H_{4n, \xi^{d_1}} = H_{4n, \xi^{\frac{\nu(n)}{2}}} \simeq H_{4n, \xi^{\frac{\nu(n)}{2}}} \), as it results from Theorem \[ \text{[4.7]} \] observing that:

\[
2 \left( \frac{\nu(n)}{2} - 1 \cdot 0 \right) = \nu(n) \cdot 1
\]
Thus $d_2 = \frac{\nu(n)}{2}$ in this case. If $d_1 | \frac{\nu(n)}{2}$, we take $d_2 = d_1$. If $d_1 \neq \nu(n)$ and $d_1 \nmid \frac{\nu(n)}{2}$, we take $d_2 = \frac{d_1}{2}$. Indeed, $2 \nmid \frac{\nu(n)}{d_1}$, hence $\nu(n) = 2^\alpha u$ and $d_1 = 2^\alpha v$, for some positive integer $\alpha$ and odd integers $u$ and $v$ such that $v \mid u$. Let $-q$ be the product of all prime divisors of $n$ that do not divide $2^u v$, and $s = 2 - \frac{u}{v}q$. Then $q$ is an odd integer, $-\frac{u}{v}q \leq \frac{n}{2}$, and $gcd(s,n) = 1$. Also,

$$s = 2 - \frac{u}{v}q \leq 2 + \frac{n}{2} \leq n - 1$$

as soon as $n \geq 6$. Multiplying $s = 2 - \frac{u}{v}q$ by $d_2 = \frac{d_1}{2}$, we find that $2(d_1 - sd_2) = \nu(n)q$. Therefore, when $n \geq 6$, we have $H_{4n,\xi^4} \cong H_{4n,\xi^2}$, by virtue of Theorem 4.10. If $n < 6$ then $\nu(n) = 2$ or $\nu(n) = 4$, cases in which there is nothing more to prove.

**Example 4.11.** A straightforward computation based on Theorem 4.10 shows the following: if $\nu(n) \in \{3, 4, 5, 6, 7\}$ there are two types of isomorphisms of bicrossed products between $H_4$ and $k[C_n]$ as follows: $H_{4n,1}$ and $H_{4n,\xi}$. If $\nu(n) = 8$ or $\nu(n) = 9$ then there are three types of isomorphisms of bicrossed products between $H_4$ and $k[C_n]$ namely $H_{4n,1}$, $H_{4n,\xi}$ and $H_{4n,\xi^2}$ if $\nu(n) = 8$ and respectively $H_{4n,1}$, $H_{4n,\xi}$ and $H_{4n,\xi^3}$ for $\nu(n) = 9$.

We conclude the discussion on these family of quantum groups by describing the group of Hopf algebra automorphisms of $H_{4n,\xi^t}$. For any $t = 0, 1, \ldots, \nu(n) - 1$ we define:

$$U_t(Z_n) := \{ \hat{s} \in U(Z_n) ; \nu(n) \mid t(s - 1) \}$$

$$V_t(Z_n) := \{ \hat{l} \in U(Z_n) ; 2t(l - 1) = \nu(n)q, \text{ for some odd integer } q \}$$

$$\tilde{U}_t(Z_n) := U_t(Z_n) \cup V_t(Z_n)$$

**Corollary 4.12.** Let $k$ be a field of characteristic $\neq 2$, $n$ a positive integer, $\xi$ a generator of the group $U_n(k)$ of order $\nu(n)$ and $t = 0, 1, \ldots, \nu(n) - 1$. Then $U_t(Z_n)$, $\tilde{U}_t(Z_n)$ are subgroups of the group of units $U(Z_n)$ and there exists an isomorphism of groups

$$\text{Aut}_{\text{Hopf}}(H_{4n,\xi^t}) \simeq k^* \times U_t(Z_n) \quad \text{or} \quad \text{Aut}_{\text{Hopf}}(H_{4n,\xi^t}) \simeq k^* \times \tilde{U}_t(Z_n)$$

the latter case holds if and only if $n$ and $\nu(n)$ are both even.

**Proof.** To start with we prove that $U_t(Z_n)$ and $\tilde{U}_t(Z_n)$ are subgroups of $U(Z_n)$. Indeed, take $s, p \in U_t(Z_n)$. Then $\nu(n) \mid t(s - 1)$, $\nu(n) \mid t(p - 1)$. It follows that $\nu(n) \mid t(s - 1)(p - 1)$ which is equivalent to $\nu(n) \mid t(sp - p - s + 1)$. Moreover, we also have $\nu(n) \mid t(s + p - 2)$. Thus, we get $\nu(n) \mid t(sp - p)$. Hence, we obtain $sp \in U_t(Z_n)$ as desired.

Take now $s \in U_t(Z_n)$ and $l \in V_t(Z_n)$. Then $2t(l - 1) = \nu(n)q$, for an odd integer $q$ and $t(s - 1) = \nu(n)w$, for some integer $w$. It follows that $t(s - 1)(l - 1) = \nu(n)w(l - 1)$ which is equivalent to $t(sl - s - l + 1) = \nu(n)w(l - 1)$. Moreover, we also have $2t(s + l - 2) = \nu(n)(2wl + q)$. Hence we get $2t(sl - 1) = \nu(n)(2wl + q)$ and since $2wl + q$ is odd it follows that $sp \in V_t(Z_n) \subseteq \tilde{U}_t(Z_n)$ as desired. Finally, if $l, r \in V_t(Z_n)$ then $2t(l - 1) = \nu(n)q_1$ and $2t(r - 1) = \nu(n)q_2$, for some odd integers $q_1, q_2$. It is straightforward to see that
$2t(lr - 1) = \nu(n)(q_1(r - 1) + (q_1 + q_2))$ and since $q_1(r - 1) + (q_1 + q_2)$ is an even integer we get $\nu(n) | (lr - 1)$ and thus $lr \in U_l(\mathbb{Z}_n) \subset \widetilde{U}_l(\mathbb{Z}_n)$.

Suppose first that $n$ is odd. Then according to the proof of Theorem 4.7 any Hopf algebra automorphism of $H_{4n, \xi^t}$ has the following form:

$$\psi_{\gamma, s} : H_{4n, \xi^t} \to H_{4n, \xi^t}, \quad \psi_{\gamma, s}(a \# h) = u(a) \# v(h)$$

(84)

where the Hopf algebra maps $u : H_4 \to H_4$ and $v : k[C_n] \to k[C_n]$ are defined as follows:

$$u(1) = 1, \quad u(g) = g, \quad u(x) = \gamma x, \quad v(c) = \epsilon^s$$

for some non-zero scalar $\gamma \in k^*$ and $s \in U_l(\mathbb{Z}_n)$. By a straightforward computation it can be seen that $\psi_{\gamma, s} \circ \psi_{\zeta, s'} = \psi_{\gamma \zeta, ss'}$. Then, the following map is an isomorphism of groups:

$$\Gamma : \text{Aut}_{\text{Hopf}}(H_{4n, \xi^t}) \to k^* \times U_l(\mathbb{Z}_n), \quad \Gamma(\psi_{\gamma, s}) = (\gamma, s)$$

Assume now that $n$ is even. Then, again by the proof of Theorem 4.7 we have two types of Hopf algebra automorphism for $H_{4n, \xi^t}$. The first such type of automorphisms is given by the maps $\psi_{\gamma, s}$ defined in (84) while the second one is given by:

$$\varphi_{\sigma, l} : H_{4n, \xi^t} \to H_{4n, \xi^t}, \quad \varphi_{\sigma, l}(a \# h) = u(a)r(h(1)) \# v(h(2))$$

for some $\sigma \in k^*$ and $l \in V_l(\mathbb{Z}_n)$, where the Hopf algebra maps $u : H_4 \to H_4$, $r : k[C_n] \to H_4$ and $v : k[C_n] \to k[C_n]$ are defined as follows:

$$u(1) = 1, \quad u(g) = g, \quad u(x) = \sigma x, \quad r(c) = g, \quad v(c) = \epsilon^l$$

Now define $\widetilde{\Gamma} : \text{Aut}_{\text{Hopf}}(H_{4n, \xi^t}) \to k^* \times \widetilde{U}_l(\mathbb{Z}_n)$ as follows:

$$\widetilde{\Gamma}(\psi_{\gamma, s}) = (\gamma, s) \in k^* \times U_l(\mathbb{Z}_n), \quad \widetilde{\Gamma}(\varphi_{\sigma, l}) = (\sigma, l) \in k^* \times V_l(\mathbb{Z}_n)$$

for all $\gamma, \sigma \in k^*$ and $s \in U_l(\mathbb{Z}_n), l \in V_l(\mathbb{Z}_n)$. As $U_l(\mathbb{Z}_n) \cap V_l(\mathbb{Z}_n) = \emptyset$ it can be easily seen that $\widetilde{\Gamma}$ is well defined and bijective. Moreover:

$$\widetilde{\Gamma}(\psi_{\gamma, s} \circ \psi_{\zeta, s'}) = \widetilde{\Gamma}(\psi_{\gamma \zeta, ss'}) = (\gamma \zeta, ss') = (\gamma, s)(\zeta, s') = \widetilde{\Gamma}(\psi_{\gamma, s}) \widetilde{\Gamma}(\psi_{\zeta, s'})$$

$$\widetilde{\Gamma}(\psi_{\gamma, s} \circ \varphi_{\sigma, l}) = \widetilde{\Gamma}((\gamma \sigma, s)) = (\gamma \sigma, s) = \widetilde{\Gamma}(\psi_{\gamma, s}) \widetilde{\Gamma}(\varphi_{\sigma, l})$$

$$\widetilde{\Gamma}(\varphi_{\sigma, l} \circ \varphi_{\tau, l'}) = \widetilde{\Gamma}((\sigma \tau, ll')) = (\sigma \tau, ll') = \widetilde{\Gamma}(\varphi_{\sigma, l}) \widetilde{\Gamma}(\varphi_{\tau, l'})$$

i.e. $\widetilde{\Gamma}$ is a morphism of groups. \(\Box\)

**Examples 4.13.** 1. Let $K = C_2 \times C_2 = \langle a, b \mid a^2 = b^2 = 1, ab = ba \rangle$ be the Klein’s group and $k$ a field of characteristic $\neq 2$. By a straightforward computation, similar to the one performed in Proposition 4.3 we can prove that $(H_4, k[C_2 \times C_2], \langle \cdot, \cdot \rangle)$ is a matched pair if and only if both actions $\cdot, \cdot$ are trivial or the right action $\cdot$ is trivial and the left action $\cdot = \cdot^i$, for $i = 1, 2, 3$, where $\cdot^i$ are given as follows:

| \cdot^1 | 1 \ g \ x \ gx | \cdot^2 | 1 \ g \ x \ gx | \cdot^3 | 1 \ g \ x \ gx |
|---|---|---|---|---|
| 1 | 1 \ g \ x \ gx | 1 | 1 \ g \ x \ gx | 1 | 1 \ g \ x \ gx |
| a | 1 \ g \ x \ gx | a | 1 \ g \ -x \ -gx | a | 1 \ g \ -x \ -gx |
| b | 1 \ g \ -x \ -gx | b | 1 \ g \ x \ gx | b | 1 \ g \ -x \ -gx |
| ab | 1 \ g \ -x \ -gx | ab | 1 \ g \ -x \ -gx | ab | 1 \ g \ x \ gx |
However, using Theorem 2.2 we can prove after a rather long but straightforward computation that all the associated bicrossed products \( H_4 \bowtie k[C_2 \times C_2] \) are isomorphic to the trivial one, namely the tensor product \( H_4 \otimes k[C_2 \times C_2] \) of Hopf algebras.

2. The examples we have chosen in this section are relevant for showing that there might be an unexplored theory in finding some new families of finite quantum groups which are the bicrossed product of two given finite dimensional Hopf algebras \( A \) and \( H \). They can serve as a model for proving similar results in the future. In [9] the problem was solved for the case \( A = H = H_4 \) by showing that a Hopf algebra \( E \) factorizes through \( H_4 \) and \( H_4 \) if and only if \( E \cong H_4 \otimes H_4 \) or \( E \cong H_{16,\lambda} \), for some \( \lambda \in k \), where \( H_{16,\lambda} \) is the Hopf algebra generated by \( g, x, G, X \) subject to the relations:

\[
\begin{align*}
g^2 &= G^2 = 1, \quad x^2 = X^2 = 0, \quad gx = -xg, \quad GX = -XG, \\
gG &= Gg, \quad gX = -Xg, \quad xG = -Gx, \quad xX + Xx = \lambda (1 - Gg)
\end{align*}
\]

with the coalgebra structure given by

\[
\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \Delta(G) = G \otimes G, \quad \Delta(X) = X \otimes 1 + G \otimes X,
\]

\[
\varepsilon(g) = \varepsilon(G) = 1, \quad \varepsilon(x) = \varepsilon(X) = 0
\]

However, up to an isomorphism of Hopf algebras there are only three Hopf algebras that factorize through \( H_4 \) and \( H_4 \), namely

\( H_4 \bowtie H_4 \), \( H_{16,0} \) and \( H_{16,1} \cong D(H_4) \)

where \( D(H_4) \) is the Drinfel’d double of \( H_4 \). For the proofs of these theorems and related results we refer to [9].

5. Conclusions, outlooks and open problems

The bicrossed product for groups or Hopf algebras has served as a model for similar constructions in other fields of study: algebras, coalgebras, Lie groups and Lie algebras, locally compact groups, multiplier Hopf algebras, locally compact quantum groups, groupoids, von Neumann algebras, etc. Thus, all the results proven in this paper at the level of Hopf algebras can serve as a model for obtaining similar results for all the fields mentioned above. This is the first direction for further study.

The second one is a continuation of the classification problem for Hopf algebras that factorize through two given Hopf algebras. The first question in this direction is the following:

**Question 1:** Let \( G \) and \( H \) be two finite cyclic groups. Describe by generators and relations and classify all bicrossed products \( k[G]^* \bowtie k[H] \) and \( k[G]^* \bowtie k[H]^* \)?

The third direction for further study is given by some open questions that are directly related to the results of this paper. For instance, having as a starting point Corollary 2.3 we can ask:

**Question 2:** Let \( G \) and \( H \) be two finite groups such that there exists an isomorphism of Hopf algebras \( D(k[G]) \cong D(k[H]) \). What is the relation between the groups \( G \) and \( H \)?

---

\(^3\)A detailed proof of these results can be provided upon request.
We believe that there exist two non-isomorphic groups such that their corresponding
Drinfel’d doubles are isomorphic but unfortunately we could not find such an example.

As we already mentioned in the introduction, one of the most important results on the
structure of products of groups is Ito’s theorem [25]: any product of two abelian groups
is a meta-abelian group. Starting from here we can ask if an Ito type theorem holds for
Hopf algebras, that is:

**Question 3 (Ito theorem for Hopf algebras):** Let \((A,H,\triangleleft,\triangleright)\) be a matched pair
between two commutative Hopf algebras. Is the bicrossed product \(A\bowtie H\) a meta-abelian
Hopf algebra?

What should a meta-abelian Hopf algebra be is part of the problem. Having in mind the
group case, one of the possible definitions is the following: a Hopf algebra is called meta-
abelian if it is isomorphic as a Hopf algebra to a crossed product of two commutative
Hopf algebras in the sense of [1].

Kaplansky’s tenth conjecture was invalidated at the end of the 90’s. However, we believe
that the following bicrossed version of it might be true:

**Question 4 (Bicrossed Kaplansky’s tenth conjecture):** Let \(A\) and \(H\) be two finite
dimensional Hopf algebras. Is the set of types of isomorphisms of all bicrossed products
\(A\bowtie H\) finite?

We can iterate the bicrossed product by constructing new examples of Hopf algebras
obtained as a sequence of the form \((A\bowtie^1 H_1)\bowtie^2 H_2)\cdots\). The following is a natural
question in the context:

**Question 5:** Let \(A, H_1, H_2\) be Hopf algebras and \((A,H_1,\triangleleft^1,\triangleright^1),\ (A\bowtie^1 H_1, H_2, \triangleleft^2,\triangleright^2)\)
matched pairs of Hopf algebras where \(A\bowtie^1 H_1\) is the bicrossed product associated to the
first matched pair. Do there exist two matched pairs of Hopf algebras \((H_1,H_2,\triangleleft^2,\triangleright^2)\)
and \((A,H_1 \bowtie^2 H_2,\triangleleft^1,\triangleright^1)\) such that the canonical map

\[
(A\bowtie^1 H_1)\bowtie^2 H_2 \to A\bowtie^1'(H_1\bowtie^2' H_2), \quad (a\bowtie^1 g)\bowtie^2 h \mapsto a\bowtie^1'(g\bowtie^2' h)
\]

is an isomorphism of Hopf algebras?

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