Flexural band gaps and vibration control of a periodic railway track

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Abstract
Periodic structures exhibit unique band gap characteristics by virtue of which they behave as vibro-acoustic filters thereby allowing only waves within a certain frequency range to pass through. In this paper, both lateral and vertical flexural wave propagation and vibration control of a periodic railway track are studied in depth. More precisely, a rail fastened on rigid sleeper blocks is modeled with an Euler-Bernoulli beam. The dispersion relations in both lateral and vertical directions are obtained using the Floquet-Bloch theorem and the resulting dispersion curves are verified using finite element (FE) models. Afterwards, tuned mass dampers (TMDs) with different mass ratios are designed to control vibrations of the examined rail along both lateral and vertical directions. Moreover, the influence of damping of rail and resonators on band structures is investigated. As a replacement to the conventional TMD, a novel possibility to control vibrations relies on using another rail as a lateral distributed resonator (LDR). Although the effectiveness of LDR is lower than that of localized resonators, the former represents a simple and promising way to control vibrations. Efficacy of the proposed control methods is finally verified using the results of transient simulation based on a random Gaussian white noise input.

Introduction
Nowadays, railway system is the most widely used medium of transportation between urban and rural areas. Due to traffic congestion issues in cities, the railway remains as a reliable alternative and is realized as the safest, on time, rapid and most convenient medium of transportation. However, due to the significant increase in speed and the operating frequency of trains, the interaction of wheel/rail is greatly enhanced and results in excessive noise and vibrations in tracks¹. This can cause fatigue damage and produce corrugations in rails, lead to loosening of fasteners and cause the sleepers to crack. A significant part of the railway infrastructure budget is thus always required to maintain the safety of such tracks. Also, such excessive vibrations affect both strength and serviceability requirements of buildings adjacent to tracks. At the same time, the generation of excessive noise results in noise pollution which is a major concern to the mental health of residents and also leads to hearing loss. Therefore, it is essential to protect rail track structures from undesired waves and large vibration amplitudes.
Propagation of elastic waves in periodic structures received much attention of researchers/scientists for decades\(^2,3\). Meanwhile, the concept of phononic crystals (PCs) introduced from solid-state physics opened a new direction to study the acoustic/elastic wave propagation in periodic structures. Differently from conventional periodic structures, PCs are a new class of materials/structures made by a periodic arrangement of artificial structural units. Such structures possess a unique wave filtering property and thereby exhibit band gaps in the frequency spectra. This is a result of either the Bragg scattering\(^2\) or a local resonance\(^4,5\). The frequency ranges wherein the freely propagating acoustic/elastic waves get attenuated are represented as band gaps or stop bands while waves of the remaining frequencies pass freely creating pass or propagation bands. Earlier studies conducted on band gaps in periodic structures are based on the Bragg scattering mechanism\(^2,3\). If the characteristic unit cell length \(l\) of a periodic structure is comparable to the wavelength \(\beta\) of the waves in the structure, Bragg band gaps are induced. The frequency of these gaps is determined by the Bragg condition \(l = n \left( \frac{\beta}{2} \right)\), where \(n = 1, 2, 3, \ldots\). To date, several studies have been carried out in the context of band gaps in periodic beams\(^2,6\), piping systems\(^7,8\), plates\(^3,9\) and railway tracks\(^10,11\). However, PCs with locally resonant units are classified as acoustic/elastic metamaterials because of their effective attenuation properties\(^12\). In addition to Bragg band gaps, locally resonant PCs entail additional band gaps induced by local resonances\(^4\). Recently, vibration control strategies based on such local resonances were used to filter the propagation of undesired waves in metamaterial beams\(^13,14\), rods\(^15\), shafts\(^16,17\) and piping systems\(^8\). Moreover, vibration transmission behavior in the vertical and lateral direction of railway tracks like the one depicted in Figure 1, were investigated by many researchers\(^18\)-\(^23\).

![Figure 1. Layout of a ballastless railway track fixed on sleeper blocks using fasteners.](image)

In the context of passive control, a system can be controlled in different ways. One of the commonly employed methods consists of attaching a secondary unit called tuned mass damper (TMD) to the main structure\(^24\). TMD consists of a mass that is attached to the main structure using a spring-damper element. The principle of TMD lies in transferring energy from the main system to the secondary system and dissipating it through this secondary system\(^25,26\). In order to harness an efficient energy dissipation, a TMD has to be optimally designed. For a given mass ratio, this is achieved by adjusting the stiffness and damping values so as to minimize any significant kinematic response quantity of the main system, typically displacements. Closed-form expressions exist for such damper parameters in case of an undamped single degree of freedom (SDoF) system subjected to harmonic excitations\(^25\). Similar optimal damper parameters for different types of responses and excitations are also available\(^27,28\). However, such methods are based on simplified assumptions and are applicable only to simple structures.
For more accurate designs and for complex structures, TMDs are designed by means of numerical optimization techniques.

In the context of railway tracks, different realizations of TMDs exist. Most of these systems employ attaching masses on either side of the rail which are allowed to deform in both the lateral and vertical directions. Damping in such cases is obtained by hysteresis in the attached damping layers. Also, magnetorheological elastomeric rail dampers endowed with variable stiffness obtained by means of magnetic fields exist. Energy can also be dissipated in TMDs using impacts/pounding of masses.

Figure 2. Simplified physical models of a periodic track structure subjected to lateral movement: (a) uncontrolled rail $R_1$; (b) rail $R_1$ endowed with lateral localized resonators (LLRs) in the middle of each unit cell, i.e., a controlled track; and (c) rails $R_1$ and $R_2$ diagonally connected using spring-damper systems.

In this study, flexural band gaps and vibration control of a ballastless periodic track structure is examined. Along this main vein, flexural wave propagation is both theoretically and numerically investigated. More precisely, two types of flexural waves are studied in the track: (i) lateral wave –Wave #A and (ii) vertical wave –Wave #B. The shear deformation in the rails is neglected, and tracks are modelled by means of Euler-Bernoulli beams. Torsional effects may occur in rails owing to the sectional asymmetry, but they are neglected as a first approximation. The dispersion relation that characterizes wave propagation in the rail is derived using the Floquet-Bloch theory of periodic structures and, is subsequently verified, by a FE model. In this respect, Figure 1 shows a simplified layout of the track consisting of the...
rail, \( R_1 \) and \( R_2 \), fixed on sleeper blocks using fasteners with some translational and rotational stiffnesses in both lateral and vertical directions. The sleeper blocks are assumed to be rigid and any flexibility of the parts underneath is neglected. Further, in order to tune band gap properties, localized resonators in both lateral (LLRs) and vertical (VLRs) directions are attached at the middle of each unit cell/span of the rail. The configurations of the track in lateral direction (Wave #A) without and with the LLRs are illustrated in Fig. 2a and 2b, respectively. Identical configurations are also used for the rail in the vertical direction (Wave #B). In this respect, coordinates \( x, y \) and \( z \), respectively represent the longitudinal, vertical and lateral directions.

A conventional TMD requires the installation of additional mass along with stiffness and damping components. However, any existing mass in the rail track system can be utilized as part of a control mechanism\(^8\). Therefore, in the context of Wave #A, a vibration control mechanism can be realized by connecting rails \( R_1 \) and \( R_2 \) using spring-damper units as shown in Fig. 2c. Thus, for controlling lateral vibration (Wave #A) in rail \( R_1 \), rail \( R_2 \) acts as a lateral distributed resonator (LDR) and vice versa. When a train passes over the track, elastic/acoustic waves propagate in both the rails \( R_1 \) and \( R_2 \) simultaneously. It is assumed herein that Wave #A propagates in both the rails \( R_1 \) and \( R_2 \) in the same phase. A simple way to couple both the rails is to connect their corresponding mid spans using a spring-damper system\(^8\). Here, this may not be effective as both the rails vibrate with the same amplitude, frequency and phase making the spring-damper system ineffective. Therefore, the two rails are attached diagonally as shown in Fig. 2c, i.e., the mid-span of one rail is connected to the middle of the adjacent span of the other rail. Derivation of dispersion relations is not straightforward for the case when LLRs/LDRs are attached to the rail or when damping is considered. Therefore, in these situations, a numerical model is established to conduct the relevant studies. The optimal values of spring and damper parameters of TMDs in both lateral and vertical directions are obtained by means of a genetic algorithm-based optimization.

The influence of damping of both rail \( R_1 \) and resonators on the band gaps is also studied herein. To show the effectiveness of the proposed devices, the uncontrolled response of \( R_1 \) is compared against the different controlled cases. In addition, to evaluate the performance of the optimized LLRs and LDRs solutions, dynamic analyses are carried out using a white noise loading. Results show that in the context of vibration control, LDRs work less efficiently than LLRs; nonetheless, as LDRs do not require any additional mass, they may lead to cost-saving solutions. In sum, the findings of this paper are promising in the context of flexural vibration control of periodic track structures and may be very helpful for the design of TMDs.

**Methods**

**Theoretical modelling and formulation of dispersion relationships.** An infinite periodic rail track of span \( l \) illustrated in Fig. 2a is adopted to investigate the flexural wave propagation characteristics. The study is conducted for the track in the lateral direction. Each unit cell is composed of a single span of the rail supported on both ends by rigid sleeper blocks connected using fasteners. Two such adjacent unit cells are depicted in Fig 3a. The governing equation of motion of the undamped rail considered as an Euler- Bernoulli beam is given as,

\[
\frac{\partial^2}{\partial x^2} \left[ EI_{yy} \frac{\partial^2 z(x,t)}{\partial x^2} \right] + \rho A \frac{\partial^2 z(x,t)}{\partial t^2} = 0
\]  

(1)
where $E$ and $I_{yy}$ are the modulus of elasticity and second moment of inertia about $y$ axis, respectively; $\rho$ and $A$ respectively denote the density and cross-sectional area and $z(x,t)$ represents the transverse displacement as a function of the spatial coordinate $x$ and time $t$.

A steady-state harmonic solution of the form $z(x,t) = Z(x)e^{i\omega t}$ is assumed, which when substituted in Eq. (1) yields,

$$EI_{yy}Z^{IV}(x) - \rho A \omega^2(x) = 0$$

where $\omega$ is the angular frequency. The solution of (2) provides the beam displacement amplitude $Z(x)$ which can be expressed as,

$$Z(x) = D_1 \cos(\alpha x) + D_2 \sin(\alpha x) + D_3 \cosh(\alpha x) + D_4 \sinh(\alpha x)$$

where $\alpha = \frac{\sqrt{-\frac{\rho A \omega^2}{EI_{yy}}}}{4}$ denotes the wave number of the flexural wave in the beam.

By applying the Floquet-Bloch periodic condition at each node of the two-unit cells shown in Fig. 3b, transverse displacements of the generic nodes $j+1$ and $j-1$ is related to that at node $j$ as

$$z_{j+1} = z_j e^{iql}, z_{j-1} = z_j e^{-iql}$$

where $l$ represents the length of the unit cell, $i$ is $\sqrt{-1}$ and $q$ signifies the Bloch parameter or the wavenumber, which is related to the wavelength $\lambda$ as $\lambda = 2\pi/q$. Similar relations are employed for shear forces, bending moments and rotations.

The constants $D_1, D_2, D_3$ and $D_4$ in (3) are obtained by means of the boundary conditions illustrated in Fig. 3c, which are used to calculate the bending moments $M$ and shear forces $S$. 

\[\text{Figure 3. Theoretical modelling of a track structure in the lateral direction. (a) two-unit cells; (b) periodic Floquet-Bloch condition imposed to the nodes for angular and transverse displacements; (c) single unit cell represented as a simple beam with rotation $\psi_0$ and transverse displacement $z_0$ at free end and clamped at the other end; (d) equilibrium of forces and moments at node $j$.}\]
on both sides of the node \( j \). The expressions for dynamic compliance coefficients\(^{38,39} \) at \( x = 0 \) and \( x = l \) for \( z_0 = 1 \) and \( \psi_0 = 0 \) read,

\[
S_0' = \frac{\alpha^3 E I_y y [\sin (\alpha l) + \sinh (\alpha l)]}{1 - \cos (\alpha l) \cosh (\alpha l)}
\]

\[
S_l' = \frac{\alpha^3 E I_y y [\cosh (\alpha l) \sin (\alpha l) + \cos (\alpha l) \sinh (\alpha l)]}{1 - \cos (\alpha l) \cosh (\alpha l)}
\]

\[
M_0' = \frac{\alpha^2 E I_y y [\cos (\alpha l) - \cosh (\alpha l)]}{1 - \cos (\alpha l) \cosh (\alpha l)}
\]

\[
M_l' = \frac{\alpha^2 E I_y y [\sinh (\alpha l) \sin (\alpha l)]}{1 - \cos (\alpha l) \cosh (\alpha l)}
\]

For \( z_0 = 0 \) and \( \psi_0 = 1 \), the dynamic compliance coefficients are given by

\[
S_0'' = \frac{-\alpha^2 E I_y y [\cosh (\alpha l) - \cos (\alpha l)]}{1 - \cos (\alpha l) \cosh (\alpha l)}
\]

\[
S_l'' = \frac{-\alpha^2 E I_y y [\sinh (\alpha l) \sin (\alpha l)]}{1 - \cos (\alpha l) \cosh (\alpha l)}
\]

\[
M_0'' = \frac{-\alpha E I_y y [\sin (\alpha l) - \sinh (\alpha l)]}{1 - \cos (\alpha l) \cosh (\alpha l)}
\]

\[
M_l'' = \frac{-\alpha E I_y y [\cosh (\alpha l) \sin (\alpha l) - \cos (\alpha l) \sinh (\alpha l)]}{1 - \cos (\alpha l) \cosh (\alpha l)}
\]

With reference to Fig. 3b, the shear forces and bending moments at node \( j \) are expressed as

\[
S^- = -S_0' z_j e^{-i\alpha l} + S_l' z_j + S_0'' \psi_j^+ e^{-i\alpha l} + S_l'' \psi_j^-
\]

\[
S^+ = S_0' z_j e^{i\alpha l} - S_l' z_j + S_0'' \psi_j^- e^{i\alpha l} + S_l'' \psi_j^+
\]

\[
M^- = M_0' z_j e^{-i\alpha l} + M_l' z_j - M_0'' \psi_j^+ e^{-i\alpha l} + M_l'' \psi_j^-
\]

\[
M^+ = M_0' z_j e^{i\alpha l} + M_l' z_j + M_0'' \psi_j^- e^{i\alpha l} - M_l'' \psi_j^+
\]

The equilibrium of forces and moments at node \( j \) in Fig. 3d entails,

\[
S^+ = S^- + k_z z_j
\]

\[
M^+ = M^- - k_{\psi y} \left( \psi_j^- + \psi_j^+ \right) \frac{2}{2}
\]

where, \( k_z \) and \( k_{\psi y} \) represent the translational and rotational stiffness of the fastening, respectively.

The kinematic compatibility condition at node \( j \) is given as,
\[ \psi_j^- = \psi_j^+ \] (9)

Eqs. (7)-(9) yield a set of linear homogeneous equations in term of \( \psi_j^- \), \( \psi_j^+ \) and \( z_j \) as,

\[ \psi_j^- - \psi_j^+ = 0 \] (10)
\[ (S''_0 e^{iql} - S''_l)\psi_j^- + (S''_l - S''_0 e^{-iql})\psi_j^+ + [2S'_0 \cos(ql) - 2S'_l - k_z]z_j = 0 \] (11)
\[ \left( M'''_0 e^{iql} - M'''_l + \frac{k_{\phi y}}{2}\right) \psi_j^- + \left( M''_0 e^{-iql} - M''_l + \frac{k_{\phi y}}{2}\right) \psi_j^+ + [2iM'_0 \sin(ql)]z_j = 0 \] (12)

Successively, Eqs. (10)-(12) can be written in a matrix form as,

\[
\begin{bmatrix}
1 & -1 & 0 \\
S''_0 e^{iql} - S''_l & S''_l - S''_0 e^{-iql} & 2S'_0 \cos(ql) - 2S'_l - k_z \\
M'''_0 e^{iql} - M'''_l + \frac{k_{\phi y}}{2} & M''_0 e^{-iql} - M''_l + \frac{k_{\phi y}}{2} & 2iM'_0 \sin(ql)
\end{bmatrix}
\begin{bmatrix}
\psi_j^- \\
\psi_j^+ \\
z_j
\end{bmatrix} = 0 \] (13)

A non-trivial solution of (13) entails,

\[
\begin{bmatrix}
1 & -1 & 0 \\
S''_0 e^{iql} - S''_l & S''_l - S''_0 e^{-iql} & 2S'_0 \cos(ql) - 2S'_l - k_z \\
M'''_0 e^{iql} - M'''_l + \frac{k_{\phi y}}{2} & M''_0 e^{-iql} - M''_l + \frac{k_{\phi y}}{2} & 2iM'_0 \sin(ql)
\end{bmatrix} = 0 \] (14)

The subsequent solution of (14) provides the dispersion relation of the periodic track structure as,

\[ [4M'_0 S''_0 \sin^2(ql)] + [2S'_0 \cos(ql) - 2S'_l - k_z][2M''_0 \cos(ql) - 2M''_l + k_{\phi y}] = 0 \] (15)

The dispersion relation is derived for the rail in the lateral direction, i.e. Wave #A; by replacing the corresponding stiffness values, the same relation holds for the track in the vertical direction, i.e. Wave #B.

**FE modeling.** In order to verify the wave propagation behavior obtained using the dispersion relation (15), a FE model of \( R_1 \) placed on rigid sleeper blocks using fasteners is made by means of the Euler-Bernoulli beam element BEAM4 available in the ANSYS 19.0 software. Since an infinite number of unit cells (spans) cannot be used in a simulation, a finite structure composed of 30 unit cells is used. A harmonic rotation of the form \( \psi_{i/p} e^{i2\pi ft} \) with \( f = \omega/2\pi \), is imposed at the left end of \( R_1 \) (in the first span) and the steady-state amplitude of the rotation \( \psi_{o/p}(f) \) is obtained at the right end (last span). The corresponding vibration transmittance \( T_\psi (dB) \) for this system is defined as

\[ T_\psi = 20 \log_{10} \left| \frac{\psi_{o/p}(f)}{\psi_{i/p}(f)} \right| \] (16)

**Design of vibration control mechanism.** For a LLR with a given mass ratio \( \sigma \), where \( \sigma = m_{t,\sigma}/(\rho A l) \), and \( m_{t,\sigma} \) is the mass of the resonator/TMD, its optimal stiffness \( k_{t,\sigma}^l \) and damping values \( c_{t,\sigma}^l \) determine the efficiency of control strategy. Let \( \| H_{\text{Control}} \|_{\infty} \) and \( \| H_{\text{Uncontrol}} \|_{\infty} \) refer to the peak value of \( \psi_{o/p}(f) \) with and without LLR, respectively. As a measure of the efficiency of LLR, a performance metric \( \eta = \| H_{\text{Control}} \|_{\infty} / \| H_{\text{Uncontrol}} \|_{\infty} \) is adopted, which is then minimized to obtain the optimal \( k_{t,\sigma}^l \) and \( c_{t,\sigma}^l \). A lower value of \( \eta \) denotes
better vibration suppression capabilities of the controlled structure. A genetic algorithm (GA)-based optimization is adopted for the design of a given mass ratio \( \sigma \) as follows,

\[
\{ k_\ell,\sigma \text{ and } c_\ell,\sigma \} = \arg \min(\eta)
\]  

subjected to,

\[
\{ LB \} \leq \{ k_\ell,\sigma \text{ and } c_\ell,\sigma \} \leq \{ UB \}
\]

\( \{ UB \} \) and \( \{ LB \} \) represent the upper and lower bound for \( k_\ell,\sigma \) and \( c_\ell,\sigma \), respectively. The values of these two bounds are selected such that the design variables do not adopt unrealistic values and results in faster optimization. \( \eta \) is calculated over the frequency range to be controlled. A similar GA-based optimization is used to determine the optimal spring-damper parameters \( k_\ell,\sigma \) and \( c_\ell,\sigma \) of the vertical localized resonator (VLR). Finally, the optimal parameters \( k_\ell \) and \( c_\ell \) for the LDR in the lateral direction are obtained using a similar optimization too.

In view of the design of TMDs, in the class of heuristic optimization algorithms, GA-based methods are commonly adopted\(^{29,30,32}\). The concept of GA which is a population-based stochastic search method is based on the principles of natural selection and genetics\(^ {40,41}\). GA-based optimization starts by selecting a random set of possible initial configurations \( X_0 \) which evolves towards the optimal solution in each generation. A simplified layout of the algorithm\(^ {42}\) is shown in Fig. 4. From any generation \( i \), the \( i + 1 \)th generation is obtained by means of selection, crossover and mutation. Selection involves finding a set of solutions from \( X_i \) which has the best fitness values and they are included directly in the next generation. While crossover involves finding new solutions by combining two best solutions from \( X_i \), mutation generates new solutions by applying random changes to the solutions in \( X_i \). This process is repeated until some desired convergence criterion is satisfied\(^ {33}\).

**Figure 4.** Genetic algorithm-based optimization for the design of tuned mass dampers.

**Results**

Based on the dispersion relation (15), the propagation characteristics of both Wave #A and Wave #B are initially studied. On the basis of these results, vibration control is designed for both the lateral and vertical cases. The influence of damping and the presence of LLR/VLR on the band gaps are also determined. Finally, the effectiveness of the optimized LLRs/LDRs is verified by imposing a random Gaussian white noise excitation.

**Propagation of Wave #A and Wave #B in \( R_1 \).** In order to determine the propagation characteristics of Wave #A and Wave #B, the track shown in Fig. 2a is considered. The dispersion relation provided in (15) corresponds to Wave #A in the undamped rail \( R_1 \). By substituting the compliance coefficients relations, the dispersion relation is obtained as,
\[
[(\cosh(\alpha l) - \cos(\alpha l))^2] \sin^2(q_l)
\]
\[
+ \left[\cos(q_l)\{\sin(\alpha l) + \sinh(\alpha l)\}\right]
\]
\[
- \{(\cosh(\alpha l)\sin(\alpha l) + \cos(\alpha l)\sinh(\alpha l)\}
\]
\[
- k_z \left\{\frac{1 - \cos(\alpha l)\cosh(\alpha l)}{2\alpha^3 E I_{yy}}\right\} \left[\cos(q_l)\{\sinh(\alpha l) - \sin(\alpha l)\}
\right]
\]
\[
- \{\cos(\alpha l)\sinh(\alpha l) - \cosh(\alpha l)\sin(\alpha l)\}
\]
\[
+ k_{\varphi y} \left\{\frac{1 - \cos(\alpha l)\cosh(\alpha l)}{2\alpha E I_{yy}}\right\} = 0
\]

The solution of Eq. (19) yields two pairs of \(q\) for each \(\omega: \pm q_1\) and \(\pm q_2\); the two signs indicate that the same waves propagating in opposite directions. The real part of \(q_l\) represents the phase difference between two adjacent cells while the imaginary part shows the decay rate of the amplitude.

| Component       | Property                  | Value       |
|-----------------|---------------------------|-------------|
| Rail            | Density (\(\rho\))       | 7850 kg/m\(^3\) |
|                 | Modulus of elasticity (\(E\)) | 2.1E11 N/m\(^2\) |
|                 | Area of cross-section (\(A\)) | 77.4E-4 m\(^2\) |
|                 | Second moment of inertia  |             |
|                 | \(I_{yy}\)                | 5.24E-6 m\(^4\) |
|                 | \(I_{zz}\)                | 32.17E-6 m\(^4\) |
| Fastening       | Material damping (\(\xi\)) | 2%          |
| Stiffness in lateral direction | Translational (\(k_x\)) | 10E6 N/m |
|                 | Rotational (\(k_{\varphi y}\)) | 7E4 Nm/rad |
| Stiffness in vertical direction | Translational (\(k_y\)) | 35E6 N/m |
|                 | Rotational (\(k_{\varphi z}\)) | 5E6 Nm/rad |
| Spacing (\(l\)) |                            | 0.625 m     |

Based on the characteristics of \(q\), three types of waves exist. For purely real \(q\) (\(|\text{Im}(q_l)| = 0\)), the waves of all frequencies travel freely through each unit cell thereby giving only pass bands in the dispersion relationships. Here adjacent cells vibrate with phase. Conversely, for purely imaginary \(q\) (\(|\text{Im}(q_l)| \neq 0\) and \(|\text{Re}(q_l)| = 0\) or \(\pi\)), the amplitude of wave reduces at each unit cell and they are referred to as evanescent waves. Now the adjacent unit cells vibrate either in or out of phase. For a complex \(q\), both \(|\text{Im}(q_l)|\) and \(|\text{Re}(q_l)|\) will be non-vanishing, (\(|\text{Im}(q_l)| > 0\) and \(0 < |\text{Re}(q_l)| < \pi\)) and the waves propagate and attenuate in the adjacent unit cells resulting in both pass and stop bands in the dispersion curves.

The properties of the laterally fastened track structure based on Table 1 are used in (19) to obtain the dispersion curves. Dispersion characteristics of only the first wave \((i.e., +q_1)\) travelling along the positive \(x\) direction in the track are investigated. The variation of \(|\text{Re}(q_l)|\) and \(|\text{Im}(q_l)|\) with the wave frequency \(f = 2\pi/\omega\) are plotted in Fig. 5a,b. In the frequency range \([0 \sim 1000]\) Hz, two band gaps are found with the frequency ranges of \([0 \sim 82]\) Hz and \([541 \sim 556]\) Hz, respectively, and are shown by shaded areas while remaining frequency regions indicate pass bands. To verify these results, an undamped FE model of the rail track consisting of 30 spans is considered. \(T_{\varphi}\) (dB) (from Eq. (16)) versus \(f\) is plotted in Fig. 5c.
which shows a perfect correspondence with the first band gap. However, FE model is not able to capture the second band gap. This may be due to its narrow bandwidth and low attenuation of waves. The first order band gap is emerged due to rail/fastening resonance while the second order band gap is caused by Bragg scattering.

**Figure 5.** Dispersion curves and transmittance $T_\psi$ (dB) of Wave #A: (a) real part of the dispersion relation, Re $(q_l)$; (b) imaginary part of the dispersion relation, Im $(q_l)$; and (c) $T_\psi$ (dB).

Similarly, the propagation characteristics of Wave #B is investigated. The properties of the vertically fastened track collected in Table 1 are used with the dispersion relation (19). The vertical direction being stiffer than the lateral entails a higher frequency range from $[0 – 2000]$ Hz, to be considered. Two band gaps $[0 – 156]$ Hz and $[1276 – 1359]$ Hz are found in this range and are represented by shaded areas in Fig. 6. Fig. 6c corresponds to the FE results and shows an excellent agreement with Fig. 6a,b, respectively.

**Figure 6.** Dispersion curves and transmittance $T_\psi$ (dB) of Wave #B: (a) real part of the dispersion relation, Re $(q_l)$; (b) imaginary part of the dispersion relation, Im $(q_l)$; and (c) $T_\psi$ (dB).

Vibration transmission characteristics of a controlled periodic track structure. From the results depicted in both Fig. 5 and 6, it is evident that the bandwidth of the first and second band gap is narrow and the latter also has very low attenuation. Consequently, for a wide frequency range waves can freely pass through the track, causing excessive noise and vibration. Thus, for both the lateral and vertical cases, it is aimed to control a certain frequency range of the first pass band. The full pass band is not considered as the frequency bounds are very large to be efficiently controlled using a SDoF TMD. Thus, the LLRs and VLRs are optimized in the frequency ranges $[300 – 500]$ Hz and $[500 – 1000]$ Hz, respectively.

Along this main vein, identical LLR and VLR are respectively attached to the center of each span of the rail. Figure 2b shows the rail $R_1$ endowed with LLRs. A similar configuration is adopted with VLRs. For a given mass ratio $\sigma$, the optimal damping coefficient and stiffness are obtained as $c^l_{\tau,\sigma}$ and $k^l_{\tau,\sigma}$ for the LLR and as $k^v_{\tau,\sigma}$ and $c^v_{\tau,\sigma}$ for the VLR using (17) and (18). The optimal parameters and the corresponding performance metric $\eta$ calculated for different values of $\sigma$ are listed in Table 2.
Table 2. Optimal TMD parameters for different mass ratios to control lateral and vertical flexural vibrations.

| Mass ratio (σ) | LLR | VLR | | | |
|---------------|-----|-----|-----|-----|-----|
|               | $k_{L,σ}$ (N/m) | $c_{L,σ}$ (Ns/m) | η | $k_{V,σ}$ (N/m) | $c_{V,σ}$ (Ns/m) | η |
| 0.10          | 1.89E7 | 5.15E3 | 0.121 | 5.24E7 | 1.10E4 | 0.199 |
| 0.15          | 2.93E7 | 7.41E3 | 0.057 | 7.56E7 | 1.68E4 | 0.106 |
| 0.20          | 4.00E7 | 9.90E3 | 0.027 | 1.03E8 | 2.07E4 | 0.058 |
| 0.25          | 5.13E7 | 1.15E4 | 0.013 | 1.24E8 | 2.61E4 | 0.034 |

Figure 7a,b show the vibration transmittance $T_ψ$ when damping in both the rail and the resonators are neglected for LLR and VLR, respectively. These plots correspond to a mass ratio $σ=0.20$ while the other mass ratios having identical variation of $T_ψ$ are not reported. In the absence of damping, a new band gap is opened around the natural frequency of the resonator and thus three band gaps are obtained in the frequency response of both the wave types. In the case of Wave #A, the band gap frequency ranges are $[0 − 75] \text{Hz}$, $[318 − 398] \text{Hz}$ and $[549 − 618] \text{Hz}$; conversely, for Wave #B, they are $[0 − 147] \text{Hz}$, $[556 − 646] \text{Hz}$ and $[1295 − 1416] \text{Hz}$. Both are represented by shaded areas in Fig. 7a,b.

In order to verify the influence of damping of $R_1$ and the resonators on the band gaps, four cases are examined for both types of wave; (i) uncontrolled $R_1$ with material damping ($ξ = 0.02$), (ii) both material damping in $R_1$ ($ξ = 0$) and damping in LLR/VLR is neglected ($k_{L,σ}^l = 4.0E7, c_{L,σ}^l = 0$ for Wave #A and $k_{L,σ}^v = 1.03E8, c_{L,σ}^v = 0$ for Wave #B), (iii) material damping ($ξ = 0.02$) is considered in $R_1$ while damping in LLR/VLR is neglected ($k_{L,σ}^l = 4.0E7, c_{L,σ}^l = 0$ for Wave #A and $k_{L,σ}^v = 1.03E8, c_{L,σ}^v = 0$ for Wave #B) and (iv) both material damping ($ξ = 0.02$) in $R_1$ and damping of LLR/VLR is considered ($k_{L,σ}^l = 4.0E7, c_{L,σ}^l = 9.90E3$ for Wave #A and $k_{L,σ}^v = 1.03E8, c_{L,σ}^v = 2.07E4$ for Wave #B). The transmittance $T_ψ$ of all the four cases for Wave #A is shown in Fig. 8a while Fig. 8b reveals the same trend for Wave #B. As damping of the structure is taken into account, the pass band peaks get lowered while the band gaps broaden and high damping causes the band gaps to vanish.
Figure 8. Effect of different damping values on the transmittance $T_{\psi}$ (dB) of $R_1$ when attached with LLRs and VLRs, respectively: (a) propagation of Wave #A; and (b) propagation of Wave #B.

The effectiveness of the designed LLRs and VLRs is also compared in Fig. 8a,b for both the uncontrolled $R_1$ (i.e., case i) against the case when $R_1$ is attached to LLRs/VLRs (i.e., case iv) with $\sigma = 0.20$. In both Fig. 8 and Table 2, it can be observed that a significant reduction in response is achieved for the rail $R_1$ in the considered frequency range.

Similar to the case of LLR, LDR is also designed in the same frequency range. The corresponding optimal stiffness and damping values are obtained as $1.44 N/m$ and $3.71E4 Ns/m$, respectively, with the performance metric $\eta = 0.147$. Vibration transmittance $T_{\psi}$ of $R_1$ with LDR is compared with LLR for different values of $\sigma$ in Fig. 9. The efficiency with LDR appears to be less when compared to the LLR. This may be due to low mass mobilization obtained with LDR. In contrast, for a LLR, the complete mass $m_{i\sigma}$ takes part in the tuned mass damper action; instead, when $R_2$ is connected to $R_1$, only a part of the mass of $R_2$ of a particular span is mobilized, leading to a low effective mass ratio.

Figure 9. Comparison between transmittance $T_{\psi}$ (dB) from the uncontrolled $R_1$ against LDRs and LLRs.

**Efficacy of control mechanism.** The designed vibration control mechanism is evaluated in the context of a general loading scenario in order to understand their efficiency. Only the lateral direction is considered here. A random Gaussian white noise rotation $RY_{i/p}$ about the $y$-axis is applied as an input to the left-most end of the rail and the output rotation at the right end is computed as $RY_{o/p}$. The input $RY_{i/p}$ is defined as a zero mean Gaussian process with unit standard deviation and is imposed for a duration of 5 s with a discretization time of $1e^{-3}$ s. Figure 10a shows the realization of the input rotation $RY_{i/p}$, whilst Fig.10b depicts the corresponding fast Fourier transform (FFT) and points out the white noise characteristics.
A full transient dynamic analysis is performed by means of the Newmark-β method with the assumption of linear variation of acceleration between two successive time instants\(^{38}\) (\(\gamma = 1/2\) and \(\beta = 1/6\)). Rayleigh damping was provided to the rail and the relevant coefficients were chosen to cover a frequency range of 0 to 500 Hz.

The response of the uncontrolled rail \(R_1\) is compared with the cases where LLRs and LDRs are used in Fig. 11. The LLR is considered with a mass ratio \(\sigma = 0.2\), and since the other mass ratios follow a similar pattern, they are not reported. A significant reduction in response is
observed for both cases. Figure 12 shows the relevant FFT of the responses which further illustrates the response attenuation when properly designed LLRs and LDRs are used.

**Discussion**

In order to study in depth, the propagation characteristics of the flexural waves, a phononic crystal theory-based metamaterial concept was utilized. The dispersion relation for the propagation of both Wave #A and Wave #B in an infinite periodic track was formulated by means of the Floquet-Bloch theorem, and the resulting dispersion characteristics were compared with FE models. Two band gaps were identified for both waves in the considered frequency range. The first band gap was due to the rail/fastening resonance while the second one was caused by the spatial periodicity in the track. By virtue of these band gaps, such spatially periodic structures act as filters and allow only waves of particular frequencies to pass through.

Further, to control vibrations within the respective first pass bands of the rail $R_1$, LLRs/VLRs were attached at the middle of each unit cell of the rail. The elimination of damping of both LLRs/VLRs and the rail $R_1$, entails a new band gap around the natural frequency of the resonator which can be observed in Fig. 7a,b. Thus, the wave filtering and attenuation capability of the track can be greatly enhanced with the LLRs/VLRs. At the same time, the introduction of resonators causes a small shift in the band gap frequencies as highlighted in Fig. 7. A reader can notice in Fig. 8 that when damping is taken into account in both $R_1$ and LLRs/VLRs, the vibration transmission peaks are lowered in the pass bands. Although the use of high damping results in vanishing of band gaps $^{13,14}$, Fig. 8 shows that a significant reduction in the amplitude of vibration is achieved.

Along these lines, it is apparent from Table 2 that with an increase of the mass ratio $\sigma$, the effectiveness of the control mechanism increases in both lateral and vertical directions. As expected, with an increase of $\sigma$, the efficiency of LLR/VLR improves; however, it also increases the optimal stiffness and damping values which may lead to a higher cost.

Furthermore, to control the vibration of $R_1$ for the propagation of Wave #A, a strategy based on the novel concept of LDRs was employed. The rail $R_2$ available in the full track structure was utilized as a LDR. When both the rails $R_1$ and $R_2$ are connected by spring-damper systems as in Fig. 2c, $R_2$ acts as LDR for $R_1$ and vice versa. Thus, the response of both rails can be equally reduced. From the results of Fig. 10 and the corresponding $\eta$ values of LLRs/LDRs, it is evident that LLRs perform better than the LDRs in terms of response reduction; but the adoption of LDRs may be a cost-saving solution. The relevant time history analysis further illustrates the effectiveness of the proposed vibration control mechanisms. Conversely, the corresponding FFT reported in Fig. 12, demonstrates the efficiency of the designed solution in the considered frequency range.

The control strategies of LLR/VLR are based on SDoF resonator systems. This design choice limits the usability of the control mechanism for a large frequency range. However, multiple-degrees-of-freedom (MDofFs) resonators may efficiently be employed in such situations. While in the context of LDRs, the center of rail $R_1$ was connected to the center of rail $R_2$ of the adjacent span, the study of more efficient configurations deserves further attention.

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Author Contributions

M.I. conceived the initial idea and designed the research together with A.K. M.I. developed the models, carried out the theoretical calculations, performed simulations and validated the results. The resonators optimization study was conducted by M.M.J and M.I. The authors
jointly discussed the results. M.I drafted the manuscript and prepared all the figures. OSB commented and revised the manuscript.