GEOMETRIC SCHOTTKY GROUPS AND NON COMPACT HYPERBOLIC SURFACES WITH INFINITY GENUS

JOHN A. ARREDONDO AND CAMILO RAMÍREZ MALUENDAS

Abstract. As the topological type of a non compact Riemann surface is determinated by its ends space and the ends having infinite genus. In this paper for a non compact Riemann surface $S$ with $n$ ends and infinite genus, we proved that given an explicitly geometric Schottky group $\Gamma$, which is an infinitely generated Fuchsian group, the quotient space by the hyperbolic plane $\mathbb{H}$ under the group $\Gamma$ is a hyperbolic surface homeomorphic to $S$ having infinite area.

1. Introduction

It is well known that for a manifold $M$ and a properly discontinuous action of a group $\Gamma$, the quotient $M/G$ correspond to a new manifold. In particular, every compact Riemann surface of genus $g$ grater than one $S_g$, is represented as the space of orbits of the hyperbolic plane $\mathbb{H}$ under the action of some subgroup $\Gamma_g$ of the automorphism group of $\mathbb{H}$. The fundamental group of this compact Riemann surface $\mathbb{H}/\Gamma_g$ is isomorphic to $\Gamma_g$. In this situation, some properties of the Riemann surface $\mathbb{H}/\Gamma_g$ coming from the properties of the group $\Gamma_g$, which immediately put in the map Fuchsian groups, and in particular, classical Schottky groups, in fact, every such compact orientable surface is the quotient of the Riemann sphere by a Schottky group. The situation is still the same, if our interest is the world of the non-compact Riemann surfaces with infinite genus, the groups $\Gamma_g$ are still Fuchsian, but now the interest is a class of Fuchsian groups called geometric Schottky groups, which was recently introduced by Anna Zielicz [Zie15]. This groups can be constructed in a similar way to a classical Schottky group but their class also contains many infinitely generated groups, a fact related with the non-compact nature of a surface.

Non-compact Riemann surfaces are mainly distinguished by the ends space and the genus associated in each end. Between this surfaces stand out the Infinite Loch Ness monster (the only surface having infinitely many handles and only one way to go to infinity) and the Jacob’s ladder (the only orientable surface having two ways to go to infinity and infinitely many handles in each) see [PS81] and [Ghy95], which are some of the usual examples in this field, in fact, in [ARM17] the authors give an explicit Fuchsian group $\Gamma$ to generated a Loch Ness monster with hyperbolic structure as quotient $\mathbb{H}/\Gamma$. Motivated by this particularity, the various investigations on non-compact Riemann surfaces (see e.g., [AMV17], [LT16], [Mat18], [RMV17], and others) and the characterization given by the uniformization theorem in terms of universal covers for Riemann surfaces (see e.g. [Abi81], [FK92]), naturally arises the following inquiry:

Question 1.1. Given natural numbers $1 < m \leq s$, which is a Fuchsian group $\Gamma_{m,s}$ such that

$$\mathbb{H}/\Gamma_{m,s} = S_{m,s}$$

is a Riemann surface having $s$ ends and $m$ ends with infinity genus? See Figure 1.

The rest of this paper is dedicated to answer this question, we give an explicit Fuchsian group for any non-compact surface characterized by their ends and the genus in each end. More precisely proving the following result:
Figure 1. Topological orientable surface having $s$ ends and $m$ ends with infinite genus.

**Theorem 1.** Let $m$ and $s$ be two positive integers satisfying $1 < m \leq s$ and let $\Gamma_{m,s}$ be the Fuchsian group generated by the Möbius transformations

$$f_t(z) = \frac{-5tz + (25t^2 - 1)}{z - 5t}, \quad f_t^{-1}(z) = \frac{-5tz - (25t^2 - 1)}{-z - 5t},$$

$$g_{k,n}(z) = \frac{-(17 + (5k - 3) \cdot 2^n \cdot 10)z + (17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{2^n \cdot 10z - (13 + (5k - 3) \cdot 2^n \cdot 10)},$$

$$g_{k,n}^{-1}(z) = \frac{-(13 + (5k - 3) \cdot 2^n \cdot 10)z - (17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{-2^n \cdot 10z - (17 + (5k - 3) \cdot 2^n \cdot 10)},$$

$$h_{k,n}(z) = \frac{-(13 + (5k - 3) \cdot 2^n \cdot 10)z + (17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{2^n \cdot 10z - (17 + (5k - 3) \cdot 2^n \cdot 10)},$$

$$h_{k,n}^{-1}(z) = \frac{-(17 + (5k - 3) \cdot 2^n \cdot 10)z - (17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{-2^n \cdot 10z - (13 + (5k - 3) \cdot 2^n \cdot 10)},$$

where $t \in \{1, \ldots, s - 1\}$, $k \in \{1, \ldots, m\}$ and $n \in \mathbb{N}$. Then the quotient space $S_{m,s} := \mathbb{H}/\Gamma_{m,s}$ is a geodesic complete Riemann surface having $s$ ends and $m$ ends with infinite genus.

It follows from The Uniformization Theorem, see [Bea84, p.174].

**Corollary 2.** The fundamental group of the Riemann surface $S_{m,s}$ is isomorphic to the Fuchsian group $\Gamma_{m,s}$.

The paper is organized as follows: In section 2 we collect the principal tools used through the paper and section 3 is dedicated to the proof of our main result, which is divided into five steps: In step 1 we define a suitable family of half circles of the hyperbolic plane, called $\mathcal{C}$, then we define by $J$, a family of Möbius transformation having as isometric circles the elements of $\mathcal{C}$. Hence, the elements of $J$ generate the group $\Gamma_{s,m}$. In step 2 we prove that $\Gamma_{s,m}$ is a Fuchsian group with a Schottky structure. In step 3 is proved that $\mathbb{H}/\Gamma_{s,m}$ is a complete geodesic Riemann surface. Finally in In step 4 and step 5 we characterize the ends space and the genus of each end for the Riemann surface.
2. PRELIMINARIES

2.1. Ends spaces. By a surface \( S \) we mean a connected and orientable 2-dimensional manifold.

**Definition 2.1.** ([Fre31]) Let \( U_1 \supset U_2 \supset \cdots \) be an infinite nested sequence of non-empty connected open subsets of \( S \), so that

* The boundary of \( U_n \) in \( S \) is compact for every \( n \in \mathbb{N} \).
* For any compact subset \( K \) of \( S \) there is \( l \in \mathbb{N} \) such that \( U_l \cap K = \emptyset \). We shall denote the sequence \( U_1 \supset U_2 \supset \cdots \) as \( (U_n)_{n \in \mathbb{N}} \).

Two sequences \( (U_n)_{n \in \mathbb{N}} \) and \( (U'_n)_{n \in \mathbb{N}} \) are equivalent if for any \( l \in \mathbb{N} \) it exists \( k \in \mathbb{N} \) such that \( U_l \supset U'_k \) and \( n \in \mathbb{N} \) it exists \( m \in \mathbb{N} \) such that \( U'_n \supset U_m \). We will denote the set of ends by \( \text{Ends}(S) \) and each equivalence class \( [U_n]_{n \in \mathbb{N}} \in \text{Ends}(S) \) is called an end of \( S \).

For every non-empty open subset \( U \) of \( S \) in which its boundary \( \partial U \) is compact, we define the set

\[
U^* := \{ \{U_n\}_{n \in \mathbb{N}} \in \text{Ends}(S) : U_j \subset U \text{ for some } j \in \mathbb{N} \}.
\]

The collection of all sets of the form \( U^* \), with \( U \) open with compact boundary of \( S \), forms a base for the topology of \( \text{Ends}(S) \).

For some surfaces their ends space carries extra information, namely, those ends that carry infinite genus. This data, together with the space of ends and the orientability class, determines the topology of \( S \). The details of this fact are discussed in the following paragraphs. Given that, this article only deals with orientable surfaces; from now on, we dismiss the non-orientable case.

A surface is said to be planar if all of its compact subsurfaces are of genus zero. An end \( [U_n]_{n \in \mathbb{N}} \) is called planar if there is \( l \in \mathbb{N} \) such that \( U_l \) is planar. The genus of a surface \( S \) is the maximum of the genera of its compact subsurfaces. Remark that, if a surface \( S \) has infinite genus, there is no finite set \( \mathcal{C} \) of mutually non-intersecting simple closed curves with the property that \( S \setminus \mathcal{C} \) is connected and planar. We define \( \text{Ends}_\infty(S) \subset \text{Ends}(S) \) as the set of all ends of \( S \) which are not planar or with infinite genus. It comes from the definition that \( \text{Ends}_\infty(S) \) forms a closed subspace of \( \text{Ends}(S) \).

**Theorem 2.2** (Classification of non-compact and orientable surfaces, [Ker23], [Ric63]). Two non-compact and orientable surfaces \( S \) and \( S' \) having the same genus are homeomorphic if and only if there is a homeomorphism \( f : \text{Ends}(S) \to \text{Ends}(S') \) such that \( f(\text{Ends}_\infty(S)) = \text{Ends}_\infty(S') \).

**Proposition 2.3.** ([Ric63] Proposition 3). The space of ends of a connected surface \( S \) is totally disconnected, compact, and Hausdorff. In particular, \( \text{Ends}(S) \) is homeomorphic to a closed subspace of the Cantor set.

**Remark 2.4.** ([Spe49]) A surface \( S \) has exactly \( n \) ends if and only if for all compact subset \( K \subset S \) there is a compact \( K' \subset S \) such that \( K \subset K' \) and \( S \setminus K' \) are \( n \) connected component.

2.2. Hyperbolic plane. Let \( \mathbb{C} \) be the complex plane. Namely the upper half-plane \( \mathbb{H} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) equipped with the riemannian metric \( ds = \sqrt{dz^2 + dy^2} \) is well known as either the hyperbolic or Lobachevski plane. It comes with a group of transformation called the isometries of \( \mathbb{H} \) denoted by \( \text{Isom}(\mathbb{H}) \), which preserves the hyperbolic distance on \( \mathbb{H} \) defined by \( ds \). Strictly, the group \( \text{PSL}(2, \mathbb{R}) \) is a subgroup \( \text{Isom}(\mathbb{H}) \) of index 2, where \( \text{PSL}(2, \mathbb{R}) \) is composed by all
fractional linear transformations or Möbius transformations

(2) \[ f : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \frac{az + b}{cz + d}, \]

where \( a, b, c \) and \( b \) are real numbers satisfying \( ad - bc = 1 \). The group \( PSL(2, \mathbb{R}) \) could also be thought as the set of all real matrices having determinant one. Throughout this paper, unless specified in a different way, we shall always write the elements of \( PSL(2, \mathbb{R}) \) as Möbius transformations.

**Half-circles.** Given the Möbius transformation \( f \in PSL(2, \mathbb{R}) \) as in equation (2) with \( c \neq 0 \), then the half-circle

(3) \[ C(f) := \{ z \in \mathbb{H} : |cz + d|^{-2} = 1 \} \]

will be called the **isometric circle of** \( f \) (see e.g., [Mas88, p. 9]). We note that \( \frac{-d}{c} \in \mathbb{R} \) the center of \( C(f) \) is mapped by \( f \) onto the infinity point \( \infty \). Further, \( f \) sends the half-circle \( C(f) \) onto \( C(f^{-1}) \) the isometric circle of the Möbius transformation \( f^{-1} \), as such:

(4) \[ C(f^{-1}) = \{ z \in \mathbb{H} : |-cz + a|^{-2} = 1 \}. \]

Given a half-circle \( C \) which center and radius are \( \alpha \in \mathbb{R} \) and \( r > 0 \) respectively, then the set \( \hat{C} := \{ z \in \mathbb{H} : |z - \alpha| > r \} \) is called the **outside** of \( C \).

**Remark 2.5.** The isometric circles \( C(f) \) and \( C^{-1}(f) \) have the same radius \( r = |c|^{-1} \), and their respective centers are \( \alpha = \frac{-d}{c} \) and \( \alpha^{-1} = \frac{a}{c} \).

**Remark 2.6.** We let \( L_{\alpha-2r} \) and \( L_{\alpha+2r} \) be the two straight, orthogonal lines to the real axis \( \mathbb{R} \) through the points \( \alpha - 2r \) and \( \alpha + 2r \), respectively. Then the reflection \( f_C \) sends \( L_{\alpha-2r} \) (analogously, \( L_{\alpha+2r} \)) onto the half-circle whose ends points are \( \alpha + \frac{r}{2} \) and \( \alpha \) (respectively, \( \alpha - \frac{r}{2} \) and \( \alpha \)). See the Figure 2.

Given that \( f_C \) is an element of \( PSL(2, \mathbb{R}) \), then for every \( \epsilon < \frac{r}{2} \) the closed hyperbolic \( \epsilon \)-neighborhood of the half-circle \( C \) does not intersect any of the hyperbolic geodesics \( L_{\alpha-2r}, f_C(L_{\alpha-2r}), L_{\alpha+2r}, \) and \( f_C(L_{\alpha+2r}) \).

**Lemma 2.7.** Let \( C_1 \) and \( C_2 \) be two disjoint half-circles having centers and radius \( \alpha_1, \alpha_2 \in \mathbb{R} \), and \( r_1, r_2 > 0 \), respectively. Suppose that the complex norm \( |\alpha_1 - \alpha_2| > (r_1 + r_2) \) then for every \( \epsilon < \frac{\max\{r_1, r_2\}}{2} \) the closed hyperbolic \( \epsilon \)-neighborhoods of the half-circles \( C_1 \) and \( C_2 \) are disjoint.
Proof. By hypothesis $|\alpha_1 - \alpha_2| > (r_1 + r_2)$, then the open strips $S_1$ and $S_2$ are disjoint, where

$$S_1 := \{ z \in \mathbb{H} : \alpha_1 - 2r_1 < \text{Re}(z) < \alpha_1 + 2r_1 \} \quad \text{and} \quad S_2 := \{ z \in \mathbb{H} : \alpha_2 - 2r_2 < \text{Re}(z) < \alpha_2 + 2r_2 \}.$$  

We remark that the half-circle $C_i$ belongs to the open strip $S_i$, for every $i \in \{1, 2\}$. Further, the transformation

$$f : \mathbb{H} \rightarrow \mathbb{H}, \quad z \mapsto \frac{r_2z}{\sqrt{r_1r_2}} + \frac{(r_1c_2 - r_2c_1)}{\sqrt{r_1r_2}}$$

of $PSL(2, \mathbb{R})$ sends the open strip $S_1$ onto the open strip $S_2$ and vice versa. We must suppose without generality that $r_1 = \max\{r_1, r_2\}$. Then, using the remark 2.6, we have that for every $\epsilon < \frac{r_1}{2}$ the closed hyperbolic $\epsilon$-neighborhood of the half-circle $C_1$ is contained in the open strip $S_1$. Now, given that $f$ is an element of $PSL(2, \mathbb{R})$, then the closed hyperbolic $\epsilon$-neighborhood of the half-circle $C_2$ is contained in the open strip $S_2$. Since $S_1 \cap S_2 = \emptyset$ that implies that for every $\epsilon < \frac{\max\{r_1, r_2\}}{2}$ the closed hyperbolic $\epsilon$-neighborhoods of the half-circles $C_1$ and $C_2$ are disjoint.

2.3. Fuchsian groups and Fundamental region. The group $PSL(2, \mathbb{R})$ comes with a topological structure from the quotient space between a group composed by all the real matrices $g := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant exactly $\text{det}(g) = 1$ under $\{ \pm \text{Id} \}$. A subgroup $\Gamma$ of $PSL(2, \mathbb{R})$ is called Fuchsian if $\Gamma$ is discrete. We shall denote as $PSL(2, \mathbb{Z})$ the Fuchsian group of all M"obius transformation with entries in the integers numbers.

Definition 2.8. [Kat92] Given a Fuchsian group $\Gamma < PSL(2, \mathbb{R})$. A closed region $R$ of the hyperbolic plane $\mathbb{H}$ is said to be a fundamental region for $\Gamma$ if it satisfies the following facts:

(i) The union $\bigcup_{f \in \Gamma} f(R) = \mathbb{H}.$

(ii) The intersection of the interior sets $\text{Int}(R) \cap f(\text{Int}(R)) = \emptyset$ for each $f \in \Gamma \setminus \{ \text{Id} \}$.

The different set $\partial R = R \setminus \text{Int}(R)$ is called the boundary of $R$ and the family $\Sigma := \{ f(R) : f \in \Gamma \}$ is called the tessellation of $\mathbb{H}$.

If $\Gamma$ is a Fuchsian group and each one of its elements are described as the equation $z$ such that $c \neq 0$, then the subset $R_0$ of $\mathbb{H}$ defined as follows

$$R_0 := \bigcap_{f \in \Gamma} \overline{C(f)} \subseteq \mathbb{H},$$

is a fundamental domain for the group $\Gamma$ (see e.g., [For25], [Mas88] Theorem H.3 p. 32, [Kat92] Theorem 3.3.5]). The fundamental domain $R_0$ is well-known as the Ford region for $\Gamma$.

On the other hand, we can get a Riemann surface from any Fuchsian group $\Gamma$. It is only necessary to define the action as follows

$$\alpha : \Gamma \times \mathbb{H} \rightarrow \mathbb{H}, \quad (f, z) \mapsto f(z),$$

which is proper and discontinuous (see [Kat10] Theorem 8. 6]). Now, we define the subset

$$W := \{ w \in \mathbb{H} : f(w) = w \text{ for any } f \in \Gamma \setminus \{ \text{Id} \} \} \subseteq \mathbb{H}.$$  

We note that
The subset $W$ is countable and discrete. Given that each element contained in $\Gamma$ fixes finitely many points of $\mathbb{H}$ and every Fuchsian group is countable. Thus we conclude that the set $W$ is countable. Contrarily, if $W$ is not discrete, then there is a point $w \in W$ and $\varepsilon > 0$ such that the ball $B_\varepsilon(w)$ contains infinitely many points of $W$. In other words, the set $\{f \in \Gamma : f(B_\varepsilon(w)) \cap B_\varepsilon(w) \neq \emptyset\}$ is infinite. Clearly, it is a contradiction to proper discontinuity of the group $\Gamma$ on $\mathbb{H}$.

The action $\alpha$ leaves invariant the subset $W$. If $w$ is a point contained in $W$, which is fixed by an element $f \in \Gamma \setminus \{Id\}$, then for any $g \in \Gamma$ the point $f(w)$ is fixed by the composition of isometries $f \circ g \circ f^{-1}(z)$.

Then the action $\alpha$ restricted to the hyperbolic plane $\mathbb{H}$ removing the subset $W$ is free, proper and discontinuous. Therefore, the quotient space (also called the space of the $\Gamma$-orbits)

$$S := (\mathbb{H} \setminus W)/\Gamma$$

is well-defined and via the projection map

$$\pi : (\mathbb{H} \setminus W) \to S, \ z \mapsto [z],$$

it comes with a hyperbolic structure, it means, $S$ is a Riemann surface (see e.g., [Lee00]).

**Remark 2.9.** If $R$ is a locally finite fundamental domain for the Fuchsian group $\Gamma$, then the quotient space $(\mathbb{H} \setminus W)/\Gamma$ is homeomorphic to $R/\Gamma$ (see Theorem 9.2.4 on [Bea83]).

### 2.4. Classical Schottky groups and Geometric Schottky groups

In general, by a classical Schottky group it is understood as a finitely generated subgroup of $PSL(2, \mathbb{C})$, generated by isometries sending the exterior of one circular disc to the exterior of a different circular disc, both of them disjoint. From the various definitions, we consider the one given in [Mas88], but there are alternative and similar definitions that can be found in [But98, Car12, MT13].

Let $C_1, C'_1, \ldots, C_n, C'_n$ be a set of disjoint countably of circles in the extended complex plane $\hat{\mathbb{C}}$, for any $n \in \mathbb{N}$, bounding a common region $D$. For every $j \in \{1, \ldots, n\}$, we consider the M"{o}bius transformation $f_j$, which sends the circle $C_j$ onto the circle $C'_j$, i.e., $f_j(C_j) = C'_j$ and $f_j(D) \cap D = \emptyset$.

The group $\Gamma$ generated by the set $\{f_j, f_j^{-1} : j \in \{1, \ldots, n\}\}$ is called a classical Schottky group.

The Geometric Schottky groups can be acknowledged as a nice generalization of the Classical Schottky groups because the definition of the first group is extended to the second one, in the sense that the Classical Schottky groups are finitely generated by definition, while the Geometric Schottky group can be infinitely generated. This geometric groups were done thanks to Anna Zięlicz (see [Zie15, Section 3]) and they are the backbone of our main result Theorem 1.

**Definition 2.10.** [Zie15, Definition 2. p. 28] Let $\{A_k : k \in I\}$ be a family of straight segments in the real line $\mathbb{R}$, where $I$ is a symmetric subset$^2$ of $\mathbb{Z}$ and let $\{f_k : k \in I\}$ be a subset of $PSL(2, \mathbb{R})$. The pair

$$\mathcal{U}(A_k, f_k, I) := (\{A_k\}, \{f_k\})_{k \in I}$$

is called a Schottky description$^3$ if it satisfies the following conditions:

1. The closures subsets $\overline{A_k}$ in $\mathbb{C}$ are mutually disjoint.
2. None of the $\overline{A_k}$ contains a closed half-circle.
3. For every $k \in I$, we denote as $C_k$ the half-circle whose ends points coincide to the ends points of $\overline{A_k}$, which is the isometric circle of $f_k$. Analogously, the half-circle $C_{-k}$ is the isometric circle of $f_{-k} := f_k^{-1}$.

$^1$In the sense due to Beardon on [Bea83, Definition 9.2.3].

$^2$A subset $I \subseteq \mathbb{Z}$ is called symmetric if it satisfies that $0 \notin I$ and for every $k \in I$ implies $-k \in I$.

$^3$The writer gives this definition to the Poincaré disc and we use its equivalent to the half plane.
(4) For each \( k \in I \), the Möbius transformation \( f_k \) is hyperbolic.

(5) There is an \( \epsilon > 0 \) such that the closed hyperbolic \( \epsilon \)-neighborhood of the half-circles \( C_k, k \in I \) are pairwise disjoint.

Definition 2.11. [Zie15, Definition 3. p. 29] A subgroup \( \Gamma \) of \( \text{PSL}(2, \mathbb{R}) \) is called Schottky type if there exists a Schottky description \( \mathcal{U}(A_k, f_k, I) \) such that the generated group by the set \( \{f_k : k \in I\} \) is equal to \( \Gamma \), i.e. we have \( \Gamma = \langle f_k : k \in I \rangle \).

We note that any Schottky description \( \mathcal{U}(A_k, f_k, I) \) defines a Geometric Schottky group.

Proposition 2.12. [Zie15, Proposition 4. p.34] Every Geometric Schottky group \( \Gamma \) is a Fucshian group.

The standard fundamental domain for the Geometric Schottky group \( \Gamma \) having Schottky description \( \mathcal{U}(A_k, f_k, I) \) is the intersection of all outside of the half-circle associated to the transformations \( f_k \), i.e.,

\[
F(\Gamma) := \bigcap_{k \in I} \overline{C_k} \subset \mathbb{H}.
\]

Proposition 2.13. [Zie15, Proposition 2. p. 33] The standard fundamental domain \( F(\Gamma) \) is a fundamental domain for the Geometric Schottky group \( \Gamma \).

3. Proof of main result

The strategy to prove the theorem is as follows. First the group \( \Gamma \) is defined. Next it is proved that this group is Fuchsian. After this, we argue that the quotient \( \mathbb{H}/\Gamma_{m,s} \) is a geodesically complete Riemann surface. Finally, we will see that this quotient is the adequate surface.

Step 1. Building the group \( \Gamma \). Given that we shall construct a Riemann Surface with \( s \) ends for \( s \in \mathbb{N} \), then for each \( t \in \{1, \ldots, s-1\} \) we denote as \( C(f_t) := \{z \in \mathbb{H} : |z - 5t| = 1\} \) and \( C(f_t^{-1}) := \{z \in \mathbb{H} : |z + 5t| = 1\} \) the half-circle having center \( 5t \) and \( -5t \) in the real line, respectively, and the same radius equality to 1 (see Figure 3). Let \( f_t \) and \( f_t^{-1} \) be the Möbius transformations having as isometric circle the half-circle \( C(f_t) \) and \( C(f_t^{-1}) \) respectively. Using Remark 2.5 we compute and get

\[
f_t(z) = \frac{-5tz + (25t^2 - 1)}{z - 5t}, \quad f_t^{-1}(z) = \frac{-5tz - (25t^2 - 1)}{-z - 5t}.
\]

Remark 3.1. For each \( t \in \{1, \ldots, s-1\} \), we note that the Möbius transformations \( f_t \) and \( f_t^{-1} \) satisfy the following properties:

1. The function \( f_t^{-1} \) is the inverse function of \( f_t \).
2. \( f_t \) and \( f_t^{-1} \) are both hyperbolic.
The map $f_t$ sends $\hat{C}(f_t) := \{z \in \mathbb{H} : |z - 5t| < 1\}$ the inside of the half circle $C(f_t)$ onto $\hat{C}(f_t^{-1}) := \{z \in \mathbb{H} : |z + 5t| > 1\}$ the outside of the half circle $C(f_t^{-1})$.

Analogously, the Möbius transformations $f_t$ sends $\hat{C}(f_t) := \{z \in \mathbb{H} : |z - 5t| > 1\}$ the outside of the half circle $C(f_t)$ onto $\hat{C}(f_t^{-1}) := \{z \in \mathbb{H} : |z + 5t| < 1\}$ the inside of the half circle $C(f_t^{-1})$.

Moreover, the half-circle $C(f_t)$ is mapped onto the half-circle $C(f_t^{-1})$ under $f_t$.

In order to hold the Riemann surface having $m \leq s$ ends with infinite genus, it is necessary to introduce the following infinite families of half-circles and Möbius transformations.

For each $k \in \{1, \ldots, m\}$ we consider the closed intervals $I_k := [5k - 3, 5k - 2]$ and $\hat{I}_k := [-(5k - 2), -(5k - 3)]$ on the real line, which can be written as the following union of closed subintervals

$$I_k = \bigcup_{n \in \mathbb{N}} \left[ 5k - 3 + \frac{1}{2n-1}, 5k - 3 + \frac{1}{2n} \right],$$

$$\hat{I}_k = \bigcup_{n \in \mathbb{N}} \left[ -(5k - 3 + \frac{1}{2n}), -(5k - 3 + \frac{1}{2n-1}) \right].$$

We shall introduce the following notation for the closed subintervals in equation (14) (see Figure 4).

$$I_{k,n} := \left[ 5k - 3 + \frac{1}{2n-1}, 5k - 3 + \frac{1}{2n} \right],$$

$$\hat{I}_{k,n} := \left[ -(5k - 3 + \frac{1}{2n}), -(5k - 3 + \frac{1}{2n-1}) \right].$$

**Figure 4.** Closed intervals $I_k$ and $\hat{I}_k$ as the union of closed subintervals.

**Remark 3.2.** By construction, the intervals $I_{k,n}$ and $\hat{I}_{k,n}$ have length $\frac{1}{2n}$, for each $n \in \mathbb{N}$. Moreover, they are symmetric with respect to the imaginary axis.

Now, we divide the closed interval $I_{k,n}$ into ten closed subintervals having length $\frac{1}{2n \cdot 10}$, and we consider the following two points in $I_{k,n}$

$$\alpha_{k,n} := 5k - 3 + \frac{1}{2n} + \frac{3}{2^n \cdot 10} = \frac{13 + (5k - 3) \cdot 2^n \cdot 10}{2^n \cdot 10},$$

$$\beta_{k,n} := 5k - 3 + \frac{1}{2n} + \frac{7}{2^n \cdot 10} = \frac{17 + (5k - 3) \cdot 2^n \cdot 10}{2^n \cdot 10},$$
then we denote as $C(g_{k,n})$ and $C(h_{k,n})$ the half-circles with center in $\alpha_{k,n}$ and $\beta_{k,n}$ respectively, and radius $r_n := \frac{1}{2^n \cdot 10}$ (see Figure 5).

Analogously, we divide the closed interval $\hat{I}_{k,n}$ into ten closed subintervals having length $\frac{1}{2^n \cdot 10}$, and we consider the following two points in $\hat{I}_{k,n}$

\[
\alpha_{k,n}^{-1} := -\frac{17 + (5k - 3) \cdot 2^n \cdot 10}{2^n \cdot 10},
\]
\[
\beta_{k,n}^{-1} := -\frac{13 + (5k - 3) \cdot 2^n \cdot 10}{2^n \cdot 10},
\]

then we denote as $C(g_{k,n}^{-1})$ and $C(h_{k,n}^{-1})$ the half-circles with center in $\alpha_{k,n}^{-1}$ and $\beta_{k,n}^{-1}$ respectively, and radius $r_n := \frac{1}{2^n \cdot 10}$ (see Figure 6).

Remark 3.3. The points $\alpha_{k,n}$, $\beta_{k,n}^{-1}$, and $\beta_{k,n}$, $\alpha_{k,n}^{-1}$ are symmetric with respect to the imaginary axis, in other words, we have the relations $-\alpha_{k,n} = \beta_{k,n}^{-1}$ and $-\beta_{k,n} = \alpha_{k,n}^{-1}$.
Now we shall calculate the Möbius transformation \( g_{k,n}(z) = \frac{a_{k,n}z + b_{k,n}}{c_{k,n}z + d_{k,n}} \) and its respective inverse \( g_{k,n}^{-1}(z) = \frac{d_{k,n}z - b_{k,n}}{-c_{k,n}z + a_{k,n}} \), which have as isometric circles the half-circles \( C(g_{k,n}) \) and \( C(g_{k,n}^{-1}) \), respectively. By remark 2.5 we get

\[
\begin{align*}
c_{k,n} &= 2^n \cdot 10, \\
a_{k,n} &= -\frac{2^n \cdot 10(17 + (5k - 3) \cdot 2^n \cdot 10)}{2^n \cdot 10} = -(17 + (5k - 3) \cdot 2^n \cdot 10), \\
d_{k,n} &= -\frac{2^n \cdot 10(13 + (5k - 3) \cdot 2^n \cdot 10)}{2^n \cdot 10} = -(13 + (5k - 3) \cdot 2^n \cdot 10).
\end{align*}
\]

Now we substitute these values in the determinant \( a_{1,n}d_{1,n} - b_{1,n}c_{1,n} = 1 \), and computing we hold

\[
b_{k,n} = \frac{(17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{2^n \cdot 10}.
\]

Hence, from the equations (18) and (19) we get explicitly the Möbius transformations (20)

\[
\begin{align*}
g_{k,n}(z) &= \frac{-(17 + (5k - 3) \cdot 2^n \cdot 10)z + (17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{2^n \cdot 10z - (13 + (5k - 3) \cdot 2^n \cdot 10)}, \\
g_{k,n}^{-1}(z) &= \frac{-(13 + (5k - 3) \cdot 2^n \cdot 10)z - (17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{-2^n \cdot 10z - (17 + (5k - 3) \cdot 2^n \cdot 10)}.
\end{align*}
\]

**Remark 3.4.** The Möbius transformations \( g_{k,n} \) and \( g_{k,n}^{-1} \) satisfy the following properties.

1. They are hyperbolic.
2. The map \( g_{k,n} \) sends \( \hat{C}(g_{k,n}) \) the inside of the half circle \( C(g_{k,n}) \) onto \( \hat{C}(g_{k,n}^{-1}) \) the outside of the half circle \( C(g_{k,n}^{-1}) \).
3. Analogously, the Möbius transformations \( g_{k,n} \) sends \( \hat{C}(g_{k,n}) \) the outside of the half circle \( C(g_{k,n}) \) onto \( \hat{C}(g_{k,n}^{-1}) \) the inside of the half circle \( C(g_{k,n}^{-1}) \).
4. Moreover, the half-circle \( C(g_{k,n}) \) is mapped onto the half-circle \( C(g_{k,n}^{-1}) \) under \( g_{k,n} \).

Likewise, we shall calculate the Möbius transformation \( h_{k,n}(z) = \frac{a_{k,n}z + b_{k,n}}{c_{k,n}z + d_{k,n}} \) and its respective inverse \( h_{k,n}^{-1}(z) = \frac{d_{k,n}z - b_{k,n}}{-c_{k,n}z + a_{k,n}} \), which have as isometric circles the half-circles \( C(h_{k,n}) \) and \( C(h_{k,n}^{-1}) \), respectively. By remark 2.5 we get

\[
\begin{align*}
c_{k,n} &= 2^n \cdot 10, \\
a_{k,n} &= -(13 + (5k - 3) \cdot 2^n \cdot 10), \\
d_{k,n} &= -(17 + (5k - 3) \cdot 2^n \cdot 10).
\end{align*}
\]

Now, we substitute these values in the determinant \( a_{k,n}d_{k,n} - b_{k,n}c_{k,n} = 1 \), and computing we hold

\[
b_{k,n} = \frac{(17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{2^n \cdot 10}.
\]
Hence, from the equations (21) and (22) we get the Möbius transformations (23)

\[
h_{k,n}(z) = \frac{-(13 + (5k - 3) \cdot 2^n \cdot 10)z + (17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{2^n \cdot 10z - (17 + (5k - 3) \cdot 2^n \cdot 10)},
\]

\[
h_{k,n}^{-1}(z) = \frac{-(17 + (5k - 3) \cdot 2^n \cdot 10)z - (17 + (5k - 3) \cdot 2^n \cdot 10)(13 + (5k - 3) \cdot 2^n \cdot 10) - 1}{-2^n \cdot 10z - (13 + (5k - 3) \cdot 2^n \cdot 10)}.
\]

**Remark 3.5.** The Möbius transformations \(h_{k,n}\) and \(h_{k,n}^{-1}\) satisfy the following properties.

1. They are hyperbolic.
2. The map \(h_{k,n}\) sends \(\hat{C}(h_{k,n})\) the inside of the half circle \(C(h_{k,n})\) onto \(\hat{C}(h_{k,n}^{-1})\) the outside of the half circle \(C(h_{k,n})\).
3. Analogously, the Möbius transformations \(h_{k,n}\) sends \(\hat{C}(h_{k,n})\) the outside of the half circle \(C(h_{k,n})\) onto \(\hat{C}(h_{k,n}^{-1})\) the inside of the half circle \(C(h_{k,n})\).
4. Moreover, the half-circle \(C(h_{k,n})\) is mapped onto the half-circle \(C(h_{k,n}^{-1})\) under \(h_{k,n}\).

From the previous construction of Möbius transformations and their respectively half-circles (see equations (13), (20), and (23)) we define the sets (24)

\[
\mathcal{J} := \{f_t, f_t^{-1}, g_{k,n}, g_{k,n}^{-1}, h_{k,n}, h_{k,n}^{-1} : t \in \{1, \ldots, s - 1\}, k \in \{1, \ldots, m\}, n \in \mathbb{N}\},
\]

\[
\mathcal{C} := \{C(f_t), C(f_t^{-1}), C(g_{k,n}), C(g_{k,n}^{-1}), C(h_{k,n}), C(h_{k,n}^{-1}) : t \in \{1, \ldots, s - 1\}, k \in \{1, \ldots, m\}, n \in \mathbb{N}\}.
\]

Then let \(\Gamma_{m,s}\) be the subgroup of \(PSL(2, \mathbb{R})\) generated by \(\mathcal{J}\). By construction the elements of \(\Gamma_{m,s}\) are hyperbolic and the half-circles of \(\mathcal{C}\) are pairwise disjoint.

**Step 2.** **The group \(\Gamma_{m,s}\) is a Fuchsian group.** We will prove that \(\Gamma_{m,s}\) is a Geometric Schottky group i.e., we shall define a Schottky description for \(\Gamma_{m,s}\). Hence, by proposition 2.12 we will conclude that \(\Gamma_{m,s}\) is Fuchsian.

We consider \(P = \{p_n\}_{n \in \mathbb{N}}\) the set conformed by all primes numbers. Hence, the map \(\psi : \mathcal{J} \rightarrow \mathcal{Z}\) defined by

\[
f_t \mapsto p_t^k, \quad f_t^{-1} \mapsto -p_t^k, \quad g_{k,n} \mapsto p_2 \cdot p_{k+n}, \quad g_{k,n}^{-1} \mapsto -p_2 \cdot p_{k+n}, \quad h_{k,n} \mapsto p_3 \cdot p_{k+n}, \quad h_{k,n}^{-1} \mapsto -p_3 \cdot p_{k+n},
\]

is injective, for every \(t, k, n \in \mathbb{N}\). We remark that \(I := \psi(\mathcal{J})\) the image of \(\mathcal{J}\) under \(\psi\) is a symmetric subset of the integers numbers \(\mathbb{Z}\), then for each element \(k\) of \(I\) there is a unique Möbius transformation \(f\) belonged to \(\mathcal{J}\) such that \(\psi(f) = k\), then we label the transformation \(f\) as \(f_k\). Therefore, we re-write the sets in the equation (24) as

\[
\mathcal{J} := \{f_k : k \in I\},
\]

\[
\mathcal{C} := \{C(f_k) : k \in I\}.
\]

Similarly, the center and radius of each half circle \(C(f_k)\) is re-written as \(\alpha_k\) and \(r_k\), respectively. Now, we define the set \(\{A_k : k \in I\}\) being \(A_k\) the straight segment in the real line \(\mathbb{R}\) having as ends point the same ones of the half circle \(C(f_k)\). So, we shall prove that the pair

\[
U(A_k, f_k, I) := (\{A_k\}, \{f_k\})_{k \in I}
\]

is a Schottky’s description. We note that by the inductive construction of the family \(\mathcal{J}\) described above (see equation (25)), the object \(U(A_k, f_k, I)\) satisfies conditions 1 to 4 in definition 2.10. Thus, we must only prove that the requirement 5 of the definition 2.10 is done.
Given the Möbius transformation \( f_k \in \mathcal{J} \) and its respective half circle \( C(f_k) \in \mathcal{C} \) we construct the open vertical strip

\[
M_k := \{ z \in \mathbb{H} : \alpha_k - 2r_k < \text{Re}(z) < \alpha_k + 2r_k \},
\]

being \( \alpha_k \) and \( r_k \) the center and radius of \( C(f_k) \), respectively.

**Remark 3.6.** We have the following facts.

1. The half circle \( C(f_k) \) is contained into the open strips \( M_k \) i.e., \( C(f_k) \subset M_k \).
2. The \( \epsilon \)-neighborhood \( B^k_\epsilon \) of the half circle \( C(f_k) \) is contained into the open strips \( M_k \), choosing \( \epsilon < \frac{r_k}{2} \).
3. For any two different transformations \( f_k \neq f_j \in \mathcal{J} \) the intersection of their open strips associated is empty, \( M_k \cap M_j = \emptyset \).

On the other hand, there is an element \( i \in I \) such that the radius \( r_i \) of the half circle \( C(f_i) \) is equal to 1 i.e., \( r_i = 1 \) (see isometric circle of the Möbius transformation of the equation (13)). So, \( r_k \leq r_i = 1 \) for all \( k \in I \). Then, by construction we have the relation

\[
|\alpha_k - \alpha_i| > r_k + r_i = r_k + 1,
\]

for all \( k \in I \), applying Lemma 2.7 we have that for every \( \epsilon < \frac{\max\{r_k, r_i\}}{2} = \frac{r_i}{2} = \frac{1}{2} \) the closure of the \( \epsilon \)-neighborhoods \( B^k_\epsilon \) and \( B^i_\epsilon \) of the half circles \( C(f_k) \) and \( C(f_i) \) are disjoint. Moreover, this lemma assures that the closure of the \( \epsilon \)-neighborhood \( B^k_\epsilon \) is contained in the open strip \( M_k \). This implies that the closed hyperbolic \( \epsilon \)-neighborhood \( B^k_\epsilon, k \in I \), are pairwise disjoint, fixing \( \epsilon < \frac{1}{2} \). Therefore, requirement 5 of definition 2.10 is done, then we can conclude that \( U(A_k, f_k, I) \) is a Schottky description.

Finally, we consider the set \( \mathcal{J} \) described in equation (25). Then from proposition 2.12 we hold that the Geometric Schottky group

\[(27) \quad \Gamma_{m,s} := \langle f_k : k \in I \rangle \]

is a Fuchsian group.

**Step 3.** **Holding the suitable surface.** We remember that the Fuchsian group \( \Gamma_{m,s} \) defined above acts freely and properly discontinuously on the subset \( \mathbb{H} \setminus W \), being \( W := \{ w \in \mathbb{H} : f(w) = w \text{ for any } \Gamma_{m,s} - \{ \text{Id} \} \} \) (see equation (8)). Nevertheless, in this case the set \( W \) is the empty set, because the half circles belonging to \( \mathcal{C} \) are pairwise disjoint. Then, the quotient space

\[(28) \quad S_{m,s} := \mathbb{H}/\Gamma_{m,s} \]

is a geodesic complete Riemann surface via the projection map \( \pi : \mathbb{H} \to S_{m,s}, \) such as \( z \mapsto [z] \) [Sch11].

**Step 4.** **The surface \( S_{m,s} \) has \( s \) ends.** We claim that the surface \( S_{m,s} \) has exactly \( m \) ends and \( s \) ends with infinite genus. To prove this, it is necessary to introduce the following construction.

**Construction 3.7.** For the hyperbolic plane \( \mathbb{H} \) there is an increasing sequence of connected compact sets \( K_1 \subset K_2 \subset \ldots \) such that \( \mathbb{H} = \bigcup_{l \in \mathbb{N}} K_l \), and whose complements define the only one end of the hyperbolic plane. In other words, the infinite nested sequence of non-empty connected subsets of \( \mathbb{H} \),

\[ \mathbb{H} \setminus K_1 \supset \mathbb{H} \setminus K_2 \supset \ldots \]
defined the equivalent class \([\mathbb{H} \setminus K_l]_{l \in \mathbb{N}}\), which is the only one end of \(\mathbb{H}\) (see Definition 2.1). More precisely, for each \(l \in \mathbb{N}\), we define the compact subset

\[
K_l := \left\{ z \in \mathbb{H} : -5(s - 1) - l \leq \text{Re}(z) \leq 5(s - 1) + l \text{ and } \frac{1}{l} \leq \text{Im}(z) \leq l + 1 \right\}.
\]

We remark that if \(K\) is a compact subset of \(\mathbb{H}\), then there is \(l \in \mathbb{N}\) such that \(K \subset K_l\) (see Figure 7).

\[
\text{Figure 7. Geometric representation of the compact subset } K_l \text{ on the hyperbolic plane.}
\]

Now, for each \(t \in \{1, \ldots, s\}\) we define the open subset \(U_t\) of the hyperbolic plane \(\mathbb{H}\) as follows

\[
U_t := \begin{cases} 
\{ z \in \mathbb{H} : -5t \leq \text{Re}(z) \leq -5(t - 1) \text{ and } 0 < \text{Im}(z) < 1 \} \cup \\
\{ z \in \mathbb{H} : 5(t - 1) \leq \text{Re}(z) \leq 5t \text{ and } 0 < \text{Im}(z) < 1 \}.
\end{cases}
\]

* Otherwise, if \(t = s\), then \(U_t = U_s\) is the complement

\[
U_t = U_s := \mathbb{H} \setminus \{ z \in \mathbb{H} : -5(t - 1) \leq \text{Re}(z) \leq 5(t - 1) \text{ and } 0 < \text{Im}(z) \leq 1 \}.
\]

We remark that the open subsets \(U_1, \ldots, U_s\) are pair disjoint.

On the other hand, following Proposition 2.13 the standard fundamental domain

\[
F(\Gamma_{m,s}) = \bigcap_{k \in \mathcal{I}} \mathcal{C}(f_k) \subset \mathbb{H},
\]

is a fundamental domain for the Fuchsian group \(\Gamma_{m,s}\). We note that the fundamental domain \(F(\Gamma_{m,s})\) is connected, locally connected and its boundary is the set of half circles \(\mathcal{C}\) (see equation (24)). It is easy to check that the hyperbolic area of \(F(\Gamma_{m,s})\) is infinite.

By construction, if \(\gamma\) is a horizontal line belonged to the horizontal strip

\[
M := \{ z \in \mathbb{H} : 0 < \text{Im}(z) < 1 \},
\]

then the image of \(\gamma \cap F(\Gamma_{m,s})\) under \(\pi\) are \(s\) disjoint curves. We shall abuse of the language and the word horocycle would mean horizontal line. Contrary, if \(\gamma\) is a horocycle belonged to \(\mathbb{H} \setminus M\), then the the image of \(\gamma \cap F(\Gamma_{m,s}) = \gamma\) under the projection \(\pi\) is a curve. Considering the objects described in the Construction 3.7, it implies that for each \(t \in \{1, \ldots, s\}\) and each \(l \in \mathbb{N}\) the image of \((U_t \setminus K_l) \cap F(\Gamma_{m,s})\) under \(\pi\) is a path-wise connected subset of \(S_{m,s}\), then the open subset

\[
C_{t,l} := \pi((U_t \setminus K_l) \cap F(\Gamma_{m,s})) \subset S_{m,s}
\]

is connected and its boundary in \(S_{m,s}\) is compact for each \(l \in \mathbb{N}\) and \(t \in \{1, \ldots, s\}\). By definition of \(K_l\) we hold that

\[
S_{m,s} \setminus \pi(K_l) = S_{m,s} \setminus \pi(K_l \cap F(\Gamma_{m,s})) = \bigcup_{t=1}^{s} C_{t,l}.
\]

\[\text{An horizontal line or a circle tangent at the real line (without its points of tangency) is called and horocycle.}\]
Let $\tilde{K}$ be a compact subset of $S_{m,s}$, by Remark 2.4 we must prove that in there exist a compact subset $K'$ of $S_{m,s}$ such that $\tilde{K} \subset K'$ and $S_{m,s} \setminus K'$ are exactly $s$ connected component. Let $K$ denote the compact subset of $\mathbb{H}$ such that $\pi(K) = \tilde{K}$. From the Construction 3.7 there is a compact subset $K_l$ of the hyperbolic plane $\mathbb{H}$ such that $\tilde{K} \subset K_l$, for any $l \in \mathbb{N}$. Let $K'$ be the image of $K_l \cap F(\Gamma_{m,s})$ under the projection $\pi$, $K' := \pi(K_l \cap F(\Gamma_{m,s}))$. Then we shall prove that $S_{m,s} \setminus K'$ are exactly $s$ connected components, it means that $S_{m,s} \setminus K'$ is the disjoint union

$$S_{m,s} \setminus K' = \bigcup_{t=1}^{s} C_{t,l}.$$  

We just should prove that the subsets $C_{t,l}$ are disjoint. We will proceed by contradiction and will suppose that there is $u \neq t \in \{1, \ldots, s\}$ such that $C_{u,t} \cap C_{t,l} \neq \emptyset$. Let $z$ be an element in the fundamental domain $F(\Gamma_{m,s})$ such that $\pi(z) \in C_{u,t} \cap C_{t,l}$, then $z \in (U_u \setminus K_l) \cap F(\Gamma_{m,s})$ and $z \in (U_t \setminus K_l) \cap F(\Gamma_{m,s})$, which implies that $z$ is an element on the intersection of $U_u \cap U_t$. Clearly, this is a contradiction because the open subsets $U_z$ are disjoint (see Construction 3.7). This proves that the surface $S_{m,s}$ has $s$ ends.

**Step 5. The surface $S_{m,s}$ has $m$ ends with infinite genus.** Given the exhaustion of $\mathbb{H} = \bigcup_{t \in \mathbb{N}} K_t$ by compact subsets in the Construction 3.7, the image of each element of the family 

$$\{K_l \cap F(\Gamma_{m,s})\}_{l \in \mathbb{N}}$$

under the projection $\pi$ is a compact subset of $S_{m,s}$, which we denote as $K_l$. Hence, the family $\{K_l\}_{l \in \mathbb{N}}$ is also an exhaustion of $S_{m,s} = \bigcup_{t \in \mathbb{N}} K_t$ by compact subsets. Using the ideas above, we can write the set

$$S_{m,s} \setminus K_l = \bigcup_{t=1}^{s} C_{t,l},$$

where each $C_{t,l}$ with $t \in \{1, \ldots, s\}$ and $l \in \mathbb{N}$ is a connected component whose boundary is compact in $S_{m,s}$ and, $C_{t,l} \supset C_{t,l+1}$. In other words, the ends space $\text{Ends}(S_{m,s})$ are all the nested sequences $(C_{t,l})_{l \in \mathbb{N}}$ i.e., $\text{Ends}(S_{m,s}) = \{[C_{t,l}]_{l \in \mathbb{N}} : t \in \{1, \ldots, s\}\}$. By construction follows that the ends $[C_{t,l}]_{l \in \mathbb{N}}$ with $t \in \{m+1, \ldots, s\}$ is planar because of the subsurface $C_{t,l}$ is homeomorphic to the cylinder for all $l \in \mathbb{N}$ and all $t \in \{m+1, \ldots, s\}$.

Now, we shall prove that the end $[C_{k,l}]_{l \in \mathbb{N}}$ with $k \in \{1, \ldots, m\}$ have infinite genus. In other words, we will prove that each subsurface $C_{k,l}$ of $S_{m,s}$ has infinite genus with $k \in \{1, \ldots, m\}$ and $l \in \mathbb{N}$.

Given the end $[C_{k,l}]_{l \in \mathbb{N}}$ and the closed intervals $I_k = \bigcup_{k \in \mathbb{N}} I_{k,n}$ and $\hat{I}_k = \bigcup_{k \in \mathbb{N}} \hat{I}_{k,n}$ (see Equations (14) and (15)) with $k \in \{1, \ldots, m\}$, we define the two vertical strips

$$M_{k,n} := \{z \in \mathbb{H} : \text{Re}(z) \in \text{Int}(I_{k,2n-1} \cup I_{2k,n}) \text{ and } 0 < \text{Im}(z) < 1\} \subset F(\Gamma_{m,s}),$$

$$\hat{M}_{k,n} := \{z \in \mathbb{H} : \text{Re}(z) \in \text{Int}(\hat{I}_{k,2n-1} \cup \hat{I}_{2k,n}) \text{ and } 0 < \text{Im}(z) < 1\} \subset F(\Gamma_{m,s}).$$

For $l = 1$, the image $(M_{k,n} \cup \hat{M}_{k,n}) \setminus K_1$ under $\pi$ is a subsurface of $C_{1,1}$, denote as $S_k := \pi((M_{k,n} \cup \hat{M}_{k,n}) \setminus K_1)$, which is homeomorphic to the torus punctured by 2 points, for each $k \in \mathbb{N}$ (see Figure 8). We remark that if $u \neq k \in \mathbb{N}$ then the subsurface $S_u$ and $S_k$ are disjoint, $S_u \cap S_k = \emptyset$, this implies that the subsurface $C_{1,1}$ of $S_{m,s}$ has infinite genus.

For $l = 2$, the image $(M_{k,n} \cup \hat{M}_{k,n}) \setminus K_2$ under $\pi$ is a subsurface of $C_{1,2}$, denote as $S_k := \pi((M_{k,n} \cup \hat{M}_{k,n}) \setminus K_2)$, is homeomorphic to the torus punctured by 2 points, for each $k > 1$. We remark that if $u \neq k \in \mathbb{N}$ then the subsurface $S_u$ and $S_k$ are disjoint, $S_u \cap S_k = \emptyset$, this implies that the subsurface $C_{1,2}$ of $S_{m,s}$ has infinite genus.

For each $l \in \mathbb{N}$, the image $(M_{k,n} \cup \hat{M}_{k,n}) \setminus K_l$ under $\pi$ is a subsurface of $C_{t,l}$, denote as $S_k := \pi((M_{k,n} \cup \hat{M}_{k,n}) \setminus K_l)$, is homeomorphic to the torus punctured by 2 points, for each $k > l$. We
a. The image of \((M_{k,n} \cup \hat{M}_{k,n}) \setminus K_1\) under \(\pi\) is the subsurface \(S_k\).

b. The subsurface \(S_k\) is topologically equivalent to two squares with identifications.

c. Identifying the sides \(C(g_{k,2n}^{-1})\) and \(C(h_{k,2n})\).

d. Cutting we get three triangles.

Figure 8. Graphic description of the process of taking a fundamental region on the Hyperbolic plane and through the quotient with a geometric Schottky group arise a non-compact Riemann surface.

e. Identifying the sides \(C(g_{k,2n})\) and \(C(g_{k,2n}^{-1})\); \(C(h_{k,2n-1})\) and \(C(h_{k,2n-1}^{-1})\); \(C(h_{k,2n})\) and \(C(h_{k,2n}^{-1})\), respectively.

f. Finally is obtained a torus with two punctured.

remark that if \(u \neq k \in \mathbb{N}\) then the subsurface \(S_u\) and \(S_k\) are disjoint, \(S_u \cap S_k = \emptyset\), this implies that the subsurface \(C_{t,l}\) of \(S_{m,s}\) has infinite genus. Hence we conclude that the end \([C_k]_{k \in \mathbb{N}}\) has infinite genus, for each \(k \in \{1, \ldots, m\}\).

Corollary 3. The fundamental group of the Riemann surface \(S_{s,m}\) is isomorphic to \(\Gamma_{s,m}\).

Corollary 4. Let \(\Gamma_s\) be the subgroup of \(PSL(2,\mathbb{R})\) generated by the set \(\{f_1, f_1^{-1}, \ldots, f_s, f_s^{-1}\}\) (see equation [13]) then the quotient \(S_s := \mathbb{H}/\Gamma_s\) is complete Riemann surface having exactly \(s\) ends and genus zero. Moreover, the fundamental group of the Riemann surface \(S_s\) is isomorphic to \(\Gamma_s\).

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FUNDACIÓN UNIVERSITARIA KONRAD LORENZ. CP. 110231, BOGOTÁ, COLOMBIA.

E-mail address: alexander.arredondo@konradlorenz.edu.co

UNIVERSIDAD NACIONAL DE COLOMBIA, SEDE MANIZALES. MANIZALES, COLOMBIA.

E-mail address: camramirezma@unal.edu.co