Solution Transformations for GS String in $AdS_5 \times S^5$ by Conserved Quantities

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Abstract

For Light-cone gauge of Green-Schwarz superstring in $AdS_5 \times S^5$ background, we fix two bosonic variables $x^+ = \tau$ and $y^9 = \sigma$, and then perform the partial Legendre transformation of the remaining bosonic variables. We then obtain a Lagrangian which is linear in velocity after eliminating the metric of world sheet. For such a system, one can formulate its Poisson bracket and Hamiltonian. Since this system is free and without constraint, the hierarchy of infinite nonlocal conserved quantities given by Bena, Polchinski and Roiban, induce solution transformations due to Jacobi identity.

1 Introduction

Because of AdS/CFT correspondence $^{[1-3]}$, there has been much interest in the role of integrability in the world-sheet theory of type IIB strings in $AdS$ spaces. Metsaev and Tseytlin $^{[4]}$ gave the famous $^{PSU(2,2|4)}_{SO(2,4) \otimes SO(5)}$ coset model with Wess-Zumino term which describes string in $AdS_5 \times S^5$ background. Because of $\kappa$ symmetry, the model has same degree of freedom for bosonic and fermionic canonical variables. Its flat-space limit is the well known Green-Schwarz superstring $^{[4,5]}$. This model has attracted renewed interest and been studied in various aspects $^{[6,7]}$. The parametrization is a hot issue of them, which includes the work by Kallosh, Rajaraman, Rahmfeld(KRR) $^{[8]}$, Roiban and Siegel $^{[9]}$, and many other authors $^{[1,12,13,14]}$. A Light-cone gauge was given by Metsaev and Tseytlin using properly grouping $PSU(2,2|4)$ AdS base $^{[2,3]}$. Another Light-cone gauge was given in $Z_4$ grading matrix approach by Alday, Arutyunov and

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Tseytlin, et al. with Hamiltonian construction and quantization [15, 16, 17]. These
work simplifies the Lagrangian of the model and investigates its solution and symmetry
properties.

After the construction of the Metsaev and Tseytlin’s model, Bena, Polchinski and
Roiban [18] constructed one parameter flat currents which implies a hierarchy of infinite
nonlocal conserved quantities for the Green-Schwarz superstring in $AdS_5 \times S^5$ space-
time. Thus the world-sheet sigma model is probably completely integrable [19]. This
is a breakthrough which attracts much attention and many corresponding studies in
string theory [20, 21, 22, 23, 24, 25, 26, 27]. There are other approaches of the AdS strings,
including their quantization by Berkovits [28, 29, 30].

For integrable models in two dimension field theory, the solution transformation is
a traditional topic. However, Metsaev and Tseytlin’s model is different from ordi-
nary nonlinear $\sigma$-models in that it has a Wess-Zumino term and satisfies the virasoro
constraint. Consequently, we can’t use the usual methods such as Riemann-Hilbert
transformation directly.

The dynamic structure and Hamiltonian of the model which are important for the in-
tegrability were studied [15, 16, 17, 27]. Our work is motivated by these work, mostly by
the Legendre transformation of bosonic variables in the kinetic part of the Lagrangian
originally introduced by Alday, Arutyunov and Tseytlin, et al [15].

In this paper, we use the Light-cone $\kappa$ symmetry gauge by Metsaev and Tseytlin [12]
and the $S^5$ parametrization by Kallosh, Rahmfeld, Rajaraman and H. Lü, et al [7, 9].
We first fix $x^+ = \tau$ and $y^9 = \sigma$. After Legendre transformation of the remaining
classic variables in kinetic part of Lagrangian, the Lagrangian becomes linear in ‘velocity’ of
canonical variables and is degenerate. This system is actually a free Hamiltonian
system without any constraint. The poisson bracket can be induced from the final
Lagrangian [15, 16, 17]. We check that, the Jacobi identity of the poisson bracket is
satisfied for canonical variables, Hamiltonian and conserved quantities given by Bena,
et al [18]. Since the poisson bracket of the conserved quantities and Hamiltonian must
be identically zero, then due to Jacobi identity one may generate solutions from an
existing one by these conserved quantities. The degree of freedom of such solution
transformations is the same as the traditional Riemann-Hilbert transformations.

This paper is organized as follows. In section 2 we briefly review the Metsaev-
Tseytlin formulation of superstring on the $AdS_5 \times S^5$ background with $\kappa$ symmetry. In
section 3, after the Light-cone $\kappa$ symmetry gauge fixing, we perform partial Legendre
transform to obtain the final degenerate Lagrangian which is linear in velocities of
the dynamic variables. In section 4, we study the poisson structure and prove Jacobi
identity for system of finite degree of freedom whose Lagrangian is linear in velocities.
The formal extension to field theory is also given. In section 5, we firstly review the
flat currents and the infinite conserved quantities discovered by Bena, Polchinski and
Roiban [18]. Then we give the solution transformation from the conserved quantities,
we make some further discussions in the last section. Appendices include some detailed
calculations. In some parts of our paper, although the contents are known, we present the explicit derivation for self-containment and for the convenience to check the final results.

2 Coset model of $PSU(2,2|4)$

In this section, we first recall the superalgebra $psu(2,2|4)$ and the action of Green-Schwarz superstring in $AdS_5 \times S^5$ spacetime constructed by Metsaev and Tseytlin and its $\kappa$ symmetry \[3\] [111].

Superstring propagating in the $AdS_5 \times S^5$ spacetime can be described as the non-linear sigma-model whose target space is the coset superspace

$$PSU(2,2|4) \over SO(4,1) \otimes SO(5)$$

with the corresponding superalgebra $psu(2,2|4)$ in the $so(4|1) \oplus so(5)$ basis.

2.1 Superalgebra $psu(2,2|4)$

The generators of $psu(2,2|4)$ are $T_A = (P_a, J_{ab}, P_{a'}, J_{a'b'}, Q_{\alpha\alpha'})$, where the indices $a, b, c, d = 0, 1, 2, 3, 4; a', b', c', d' = 5, 6, 7, 8, 9; I, J = 1, 2; \alpha, \beta = 1, 2, 3, 4$. The commutation relations for the generators of the Lie superalgebra $psu(2,2|4)$ are

$$[P_a, P_b] = J_{ab}, \quad [P_{a'}, P_{b'}] = -J_{a'b'},$$

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \eta_{da}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac},$$

$$[J_{a'b'}, J_{c'd'}] = \eta_{b'c'}J_{a'd'} + \eta_{d'a'}J_{b'c'} - \eta_{a'c'}J_{b'd'} - \eta_{b'd'}J_{a'c'},$$

$$[P_a, J_{bc}] = \eta_{ab}P_c - \eta_{ac}P_b, \quad [P_{a'}, J_{b'c'}] = \eta_{a'b'}P_c - \eta_{a'c'}P_{b'},$$

$$[Q_{\alpha\alpha'}, P_a] = -\frac{1}{2} \epsilon_{I1J} Q_{\beta \alpha'} J(\gamma_a)_\beta \alpha, \quad [Q_{\alpha\alpha'}, P_{a'}] = \frac{1}{2} \epsilon_{I1J} Q_{\alpha\beta' J(\gamma_{a'})_\beta \alpha'},$$

$$[Q_{\alpha\alpha'}, J_{ab}] = -\frac{1}{2} Q_{\beta \alpha'} J(\gamma_{ab})_\beta \alpha, \quad [Q_{\alpha\alpha'}, J_{a'b'}] = -\frac{1}{2} Q_{\alpha\beta' J(\gamma_{a'b'})_\beta \alpha'},$$

$$\{Q_{\alpha\alpha'}, Q_{\beta \beta'} J\} = \delta_{IJ} \left[ -2i C_{\alpha' \beta} (C'_{\gamma a})_{\alpha \beta} P_a + 2 C_{\alpha \beta} (C'_{\gamma a'})_{\alpha' \beta'} P_{a'} \right] + \epsilon_{IJ} \left[ C_{\alpha' \beta} (C'_{\gamma a b'})_{\alpha \beta} J_{ab} - C_{\alpha \beta} (C'_{\gamma a' b'})_{\alpha' \beta'} J_{a'b'} \right],$$

where $\eta_{ab} = (---++)$, $\eta_{a'b'} = (+++++)$, $\{\gamma_a, \gamma^b\} = 2\eta^{ab}$, $\{\gamma_{a'}, \gamma^b\} = 2\eta^{a'b'}$, $\epsilon_{12} = -\epsilon_{21} = 1$. Here and after, the repeated indices are summed.
The gamma matrices satisfy
\[
\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \{\gamma^{a'}, \gamma^{b'}\} = 2\eta^{a'b'},
\]
\[
\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b], \gamma^{a'b'} = \frac{1}{2}[\gamma^a, \gamma^b],
\]
\[
(C_\gamma \dot{a})^t = C_\gamma \dot{a}, (C_\gamma \dot{a}'b')^t = -C_\gamma \dot{ab}, C^t = C = C \otimes C', \dot{a} = a, b \text{ or } a', b'.
\]

We may set
\[
\gamma_a = \begin{bmatrix} 0 & \sigma_a \\ \bar{\sigma}_a & 0 \end{bmatrix}, \sigma_a = (I, \sigma_1, \sigma_2, \sigma_3), \bar{\sigma}_a = (-I, \sigma_1, \sigma_2, \sigma_3),
\]
for \(a = 0, 1, 2, 3\), where \(I\) is the \(2 \times 2\) unit matrix,
\[
\gamma_4 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \gamma_5 = i\gamma_0, \gamma_6 = \gamma_1, \gamma_7 = \gamma_2, \gamma_8 = \gamma_3, \gamma_9 = \gamma_4,
\]
and
\[
C = C' = \begin{bmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{bmatrix}.
\]

The left-invariant Cartan 1-forms
\[
L^A = dX^M L^A_M, \quad X^M = (x, \theta),
\]
are given by
\[
\mathcal{J} = G^{-1} dG = L^A T_A \equiv L^a P_a + L^{a'} P_{a'} + \frac{1}{2} L^{ab} J_{ab} + \frac{1}{2} L^{a'b'} J_{a'b'} + L^{a1} Q_{a1},
\]
where \(G = G(x, \theta)\) is a coset representative in \(PSU(2,2|4)\).

The Cartan 1-form satisfies the zero-curvature equation \(d\mathcal{J} = -\mathcal{J} \wedge \mathcal{J}\). When decompose it according to the generators of the Lie algebra, we get Maurer-Cartan equations.

### 2.2 \(\kappa\) symmetry

The string theory action is the sum of the non-linear sigma-model action and a topological Wess-Zumino term to ensure \(\kappa\) symmetry. The Polyakov action given by Metsaev and Tseytlin \([4]\) is
\[
S = S_k + S_{WZ}
= -\frac{1}{2} \int_{\partial M_3} \sqrt{-g} g^{ij} (L_i^a L_j^a + L_i^{a'} L_j^{a'}) - \int_{\partial M_3} \sqrt{-g} \epsilon^{ij} (\bar{L}_i^1 L_j^2 + \bar{L}_i^2 L_j^1).
\]
Here $\sqrt{-g} = \sqrt{-\det g_{ij}}$, $g_{ij}g^{jk} = \delta_{ik}$, $i, j = 0, 1$ and $g_{ij}$ is the metric of the world-sheet.

$2 \times 2$ matrix $\epsilon^{ij} = -\epsilon^{ji}$, $\epsilon^{01} = 1$. Here and after denote $X^\alpha Y^\beta = X^\alpha Y^\beta \eta_{\alpha\beta}$. This action is invariant with respect to the local $\kappa$-transformations. Due to $L = \bar{L}^\gamma_{\gamma_0} = \bar{L}^\gamma_{CC}^\gamma$ for Majorana spinor $L$, the WZ term can be written as $S_{WZ} = -\int_{\partial M_3} d^2 \sigma \epsilon^{ij} (L^i_{1t} CC^\gamma L^2_j + L^2_i CC^\gamma L^1_j)$.

Let the variation of group element be $\delta G$ and $\rho = G^{-1} \delta G \equiv P_a \delta x^a + P_{a'} \delta x^{a'} + \frac{1}{2} J_{ab} \delta x^{ab} + \frac{1}{2} J_{a'b'} \delta x^{a'b'} + Q_{a'\alpha'}^l \delta \theta^{l\alpha\alpha'}$. The equation $\delta J = d\rho + [J, \rho]$ gives the variations of the Cartan 1-forms. The variation of action (5) with respect to $\delta x^a, \delta x^{a'}, \delta \theta^{l\alpha\alpha'}$ gives the equations of motion

$$
\partial_i (\gamma^{ij} L^a_i) + \gamma^{ij} L^a_{ib} L^b_j + i \epsilon^{ij} s^{IJ} L^a_i \gamma^a L^J_j = 0,
\partial_i (\gamma^{ij} L^{a'}_j) + \gamma^{ij} L^{a'b'} L^{b'}_j - \epsilon^{ij} s^{IJ} L^a_i \gamma^{a'} L^J_j = 0,
(L^a_i \gamma^a + i L^a_i \gamma^{a'})(\gamma^{ij} - \epsilon^{ij}) L^1_j = 0,
(L^a_i \gamma^a + i L^a_i \gamma^{a'})(\gamma^{ij} + \epsilon^{ij}) L^2_j = 0.
$$

(6)

while the variation of the metric $g_{ij}$ gives the virasoro constraint

$$
L^a_i L^a_j + L^{a'}_i L^{a'}_j = \frac{1}{2} g_{ij} g^{kl} \left( L^a_k L^a_i + L^{a'}_k L^{a'}_i \right),
$$

(7)

where $\gamma^{ij} = \sqrt{-g} \epsilon^{ij}$, $\det[\gamma^{ij}] = -1$. $s^{IJ} = (\sigma_3)^{IJ} = (-1)^{l+1} \delta_{IJ}$.

Substituting the virasoro constraint (4) into (5), one obtains the Nambu-Goto action

$$
S = -\int_{\partial M_3} (d^2 \sigma \sqrt{-G} + 2 \bar{L} L^2),
$$

(8)

where the induced metric $G_{ij} = L^a_i L^a_j + L^{a'}_i L^{a'}_j$ and $G = \det[G_{ij}]$.

We may check the $\kappa$ symmetry in the Nambu-Goto action, which can give the right degrees of freedom [31]. Consider the variation of $\delta_\kappa \tilde{\theta}^I$, one obtains

$$
\delta_\kappa S = 4i \int_{\partial M_3} d^2 \sigma \sqrt{-G} (\delta_\kappa \tilde{\theta}^I P^{ij}_+ L^+_j L^1_i + \delta_\kappa \tilde{\theta}^2 P^{ij}_2 L^+_j L^2_i),
$$

(9)

where

$$
L^+_i \equiv (L^a_i \gamma^a + i L^a_i \gamma^{a'}), \quad L^-_i \equiv (L^a_i \gamma^a - i L^a_i \gamma^{a'}).
$$

(10)

Define

$$
P^{ij}_\pm = \frac{1}{2} (G^{ij} \pm \frac{\epsilon^{ij}}{\sqrt{-G}}), \quad \gamma^a = -\frac{\epsilon^{ij} L^a_i L^-_j}{2 \sqrt{-G}}, \quad \gamma^a = 1, \quad \text{tr} \gamma = 0.
$$

(11)
One has
\[ \gamma P_{ij}^+ \mathcal{L}_j^+ = \pm P_{ij}^+ \mathcal{L}_j^+ , \quad (13) \]

where \( \kappa^1 \) and \( \kappa^2 \) are arbitrary, \( P_\pm = \frac{1 \pm \gamma}{2} \) are projector operators. For such variation, we have \( \delta \kappa S = 0 \).

This is a local symmetry of the model, thus this system is not definite, which has infinite solutions for given initial and boundary conditions. We must perform \( \kappa \) symmetry gauge fixing, only take half of the fermionic variables.

### 3 Light-cone gauge fixing and partial Legendre transformation

#### 3.1 parametrization

In this subsection, we follow the Light-cone \( \kappa \) symmetry gauge fixing by Metsaev and Tseytlin [12], while the \( S^5 \) part we use the parametrization of KRR [9], and H. Liu et al [7].

Define
\[
\begin{align*}
x^\pm &= \frac{1}{\sqrt{2}}(x^3 \pm x^0), \quad x = \frac{1}{\sqrt{2}}(x^1 + i x^2), \\
\bar{x} &= \frac{1}{\sqrt{2}}(x^1 - i x^2), \quad \phi = x^4, \\
x^a &= (x^+, x^-, x, \bar{x}, \phi), \quad \eta^+ = \eta^- = \eta^{xx} = \eta^{\bar{x}x} = 1.
\end{align*}
\]

The bosonic generators of AdS for Light-cone gauge are
\[
\begin{align*}
D &= P^4, \\
P^\pm &= \frac{1}{\sqrt{2}}(P^3 \pm P^0 + J^{43} \pm J^{40}), \quad \frac{1}{\sqrt{2}}(P^1 \pm i P^2 + J^{41} \pm i J^{42}); \\
K^\pm &= \frac{1}{2\sqrt{2}}(-P^3 \mp P^0 + J^{43} \mp J^{40}), \quad \frac{1}{2\sqrt{2}}(-P^1 \mp i P^2 + J^{41} \mp i J^{42}).
\end{align*}
\]
\[ J^{\pm x} = \pm \frac{1}{2} J^{01} \pm \frac{i}{2} J^{02} + \frac{1}{2} J^{31} + \frac{i}{2} J^{32}; \]
\[ J^{\pm x} = \pm \frac{1}{2} J^{01} \mp \frac{i}{2} J^{02} + \frac{1}{2} J^{31} - \frac{i}{2} J^{32}; \]
\[ J^{x\bar{x}} = -i J^{12}; \quad J^{+} = J^{03}. \quad (16) \]

Also define for fermionic generators
\[ Q_{\pm\alpha'} = Q_{1\alpha'} \pm i Q_{2\alpha'}, \quad q_{\alpha'\alpha} = Q_{-\alpha'\alpha}, \quad q_{\alpha'\alpha} = C_{\alpha\beta} C_{\alpha'\beta'} Q_{+\beta'}. \]

and
\[ q^{1i} = i 2\sqrt{2}^{-\frac{i}{2}} Q^{-i}, \quad q^{2i} = -i 2\sqrt{2}^{-\frac{i}{2}} Q^{+i}; \]
\[ q^{3i} = 2\sqrt{2}^{\frac{i}{2}} S^{+i}, \quad q^{4i} = 2\sqrt{2}^{\frac{i}{2}} S^{-i}; \]
\[ q_{1i} = 2\sqrt{2}^{\frac{i}{2}} S^{+i}, \quad q_{2i} = 2\sqrt{2}^{\frac{i}{2}} S^{-i}; \]
\[ q_{3i} = i 2\sqrt{2}^{-\frac{i}{2}} Q^{-i}, \quad q_{4i} = -i 2\sqrt{2}^{-\frac{i}{2}} Q^{+i}, \quad (17) \]

where the index \( i \) is the \( S^5 \) index \( \alpha' \).

We then take the parametrization following [12], and [9, 7] for \( S^5 \) part. The \( \kappa \)-symmetry gauge fixed representative group element is
\[ G(x, y, \theta, \eta, \phi) = g(x) g(\theta) g(\eta) g(y) g(\phi) \]
\[ g(x) = \exp(x^- P^+ + x^+ P^- + x^P + \bar{x} P \bar{x}); \]
\[ g(\theta) = \exp(\theta^i Q^i_+ + \theta^i Q^{+i}); \]
\[ g(\eta) = \exp(\eta^i s^+_i + \eta_i S^{+i}); \]
\[ g(y) = e^{y^i j_{56} e^{y^i} j_{78} e^{y^i} j_{89} e^{y^i} j_{90}} e^{y^i} p_0; \]
\[ g(\phi) = \exp(\phi D). \quad (18) \]

Define
\[ M = \exp\left(\frac{y^5}{2} \gamma^{56}\right) \exp\left(\frac{y^6}{2} \gamma^{67}\right) \exp\left(\frac{y^7}{2} \gamma^{78}\right) \exp\left(\frac{y^8}{2} \gamma^{89}\right) \exp\left(-\frac{i y^9}{2} \gamma^9\right), \]

and
\[ \tilde{\xi}_j = \xi_i M_{ij}, \quad \bar{\xi}^i = M^{-1}_{ij} \xi^j, \]

for fermionic variables.

The one form of \( G \) is
\[ J = G^{-1} (x, y, \theta, \eta, \phi) dG (x, y, \theta, \eta, \phi) \]
\[ = L_p^+ P^+ + L_p^0 P^0 + L_p^+ P^0 + L^+ D + L_K^+ K^+ + L_K^0 K^0 + L^+ K^+ + L^0 P^0 \]
\[ + L^+ J^{\pm x} + L^0 J^{x\bar{x}} + L^{\pm x} J^{x\bar{x}} + L^{x\bar{x}} J^{x\bar{x}} + L^+ J^{-} \]
\[ + \frac{1}{2} L_{a'b'} J_{a'b'} + L_{Q}^+ Q^+_i + L_{S}^+ S^+_i + L_{iQ} Q^{-i} + L_{iS} S^{-i}. \]
By the deliberately designed coset representative (18), one obtains the nonzero 1-forms

\[
L^{A'} = (\prod_{k'=A'+1}^{9} \sin y_{k'})dy_{A'} - \frac{1}{2}dx^+ \bar{\eta}_i (\gamma^{A'})_j^i \bar{\eta}^j \equiv u^{A'}dy_{A'} + v^{A'}dx^+, (u^0 = 1)
\]

\[
L^{A'B'} = -\frac{1}{2}dx^+ \bar{\eta}_i (\gamma^{A'B'})_j^i \bar{\eta}^j + \prod_{k'=A'+1}^{B'-1} \sin y_{k'} \cos y^{B'}dy^{A'}
\]

\[
L^+_p = e^\phi dx^+, \quad L^- = e^\phi [dx^+ - \frac{i}{2}(\theta_i d\theta^i + \theta^i d\theta_i)],
\]

\[
L^x_p = e^\phi d\bar{x}, \quad L^x = e^\phi dx, \quad L^D = d\phi,
\]

\[
L^i_K = e^{-\phi}(\frac{1}{4}dx^+ (\eta^i \eta_i)^2 + \frac{i}{2}(\eta_i d\eta^i + \eta^i d\eta_i)), \quad L^{ix} = \frac{i}{2}dx^+ (\eta^i \eta_i),
\]

\[
L^{-i} = \eta_i d\theta^i - \frac{1}{2}d\bar{x}(\eta^i \eta_i), \quad L^{-i} = -\eta_i d\theta_i + \frac{1}{2}d\bar{x}(\eta^i \eta_i),
\]

\[
L^{-i}_Q = e^{\frac{1}{2}\phi}(d\bar{x}^i + idx^i), \quad L^{+i}_Q = -ie^{\frac{1}{2}\phi}dx^+ \bar{\eta}^i,
\]

\[
L^{-i}_Q = e^{\frac{1}{2}\phi}(d\bar{x}^i - idx^i), \quad L^{+i}_Q = ie^{\frac{1}{2}\phi}dx^+ \bar{\eta}^i,
\]

\[
L^{-i}_S = e^{-\phi}(\frac{1}{2}dx^+ (\eta^i \eta_i) \bar{\eta}^i + d\eta^i), \quad L^{-i}_S = e^{-\frac{1}{2}\phi}(\frac{1}{2}dx^+ (\eta^i \eta_i) \bar{\eta}^i + d\eta^i).
\]

(19)

Using the formula \( L^a \equiv L^a_p - \frac{1}{2}L^a_k \) one gives

\[
L^a = e^\phi [dx^+ - \frac{i}{2}(\theta_i d\theta^i + \theta^i d\theta_i)] - \frac{1}{2}e^{-\phi}(\frac{1}{4}dx^+ (\eta^i \eta_i)^2 + \frac{i}{2}(\eta_i d\eta^i + \eta^i d\eta_i)), \quad (20)
\]

and has

\[
\mathcal{L}_k = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}(L^\mu_p L^\nu_p + L^\mu_p L^\nu_p + L^\mu_p L^\nu_p + L^\mu_p L^\nu_p + L^D_p L^D_p + \sum_{A'=5}^{9} L^A_p L^A_p)
\]

\[
= -\frac{1}{2}\gamma^{\mu\nu}(\gamma^{\delta\beta}G_{\delta\beta}x^\nu)
\]

\[
= -\frac{1}{2}\gamma^{\mu\nu} \{ e^\phi(\partial_\mu x^- - \frac{i}{2}(\theta^i \partial_\mu \theta_i + \theta_i \partial_\mu \theta^i))e^\phi \partial_\nu x^+
\]

\[+ e^2 \phi(\partial_\mu x^+ (\eta^i \eta_i) + \frac{i}{2}(\eta_i \partial_\mu \eta^i + \eta^i \partial_\mu \eta_i))e^\phi \partial_\nu x^+
\]

\[
+ e^2 \phi(\partial_\mu x^- (\eta^i \eta_i) + \frac{i}{2}(\eta_i \partial_\mu \eta^i + \eta^i \partial_\mu \eta_i))e^\phi \partial_\nu x^+
\]

+ \sum_{A'=5}^{9} (u^{A'} \partial_\nu y_{A'} + v^{A'} \partial_\nu x^+)(u^{A'} \partial_\nu y_{A'} + v^{A'} \partial_\nu x^+),
\]

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We next fix the gauge $x^+ = \tau$, $y^9 = \sigma$ and perform partial Legendre transformation for the remaining bosonic variables in $\mathcal{L}_k$.

Define $\frac{\partial \mathcal{L}_k}{\partial \dot{x}^i} = \pi_i$ for 8 bosonic variables. Perform Legendre transformation partially, we have (Appendix A)

$$\tilde{\mathcal{H}} = \pi_i \dot{z}_i - \mathcal{L}_k$$

$$= \frac{e^{2\phi}}{2\pi_+} \{2e^{2\phi} x' x'' + \phi^2 + \sum_{a'=5}^8 (u^a')^2(y^{a'})^2 + 1 + [2e^{-2\phi} \pi_x \pi_{\bar{x}} + \pi_D^2 + \sum_{a'=5}^8 (u^{a'})^{-2}(\pi_{a'})^2]
+\left[ (\pi_- (x'') - \frac{i}{2} (\theta^i \theta'_i + \theta_i \theta'_i) - \frac{i}{4} e^{-2\phi} (\eta^i \eta'_i + \eta_i \eta'_i) + \pi_x x' + \pi_{\bar{x}} \bar{x}' + \pi_D \phi' + 8 \pi_{a'} y^{a'} + v^9 \pi_- (x'') - \frac{i}{2} (\theta^i \theta'_i + \theta_i \theta'_i) - \frac{i}{4} e^{-2\phi} (\eta^i \eta'_i + \eta_i \eta'_i)) \right.
\left. + \frac{1}{8} e^{-2\phi} (\eta^i \eta'_i) \right]
- \sum_{a'=5}^8 \pi_{a'} y^{a'},\]
Adding $\pi_a$ and $\pi_{a'}$ as new variables, one can obtain a new Lagrangian density (Appendix A)

\[ \tilde{L} = \tilde{L}_k + L_{WZ} \]

\[ = \pi_{-} \dot{x} - \pi_{x} \dot{x} + \pi_{\bar{x}} \dot{\bar{x}} + \pi_{D} \dot{\phi} + \sum_{a'=5}^{8} \pi_{a'} \dot{y}_{a'} \]

\[ -\pi_{-} \left[ i \frac{1}{2} (\theta_{i} \dot{\theta}_{i} + \dot{\theta}_{i} \theta_{i}) + \frac{1}{4} e^{-2\phi} (\eta_{i} \dot{\eta}_{i} + \dot{\eta}_{i} \eta_{i}) \right] - \mathcal{H} \]

\[ = f_{i} \dot{z}_{i} + f_{a} \dot{z}_{a} - \mathcal{H}, \quad (25) \]

where $\mathcal{H}$ is the Hamiltonian density and define $H = \int d\sigma \mathcal{H}$. In appendix A, we have proved Lagrangian equations of $\tilde{L}$ include the equations of $L$ and the definition of $\pi_i$. We now see that the number of bosonic variables ($z_a, \pi_a, a = 1, 2, \cdots, 8$) and fermionic variables ($\theta^i, \theta_i, \eta^i, \eta_i; i = 1, 2, 3, 4$) are equal. This is an important reason for requiring the $\kappa$ symmetry.

Therefore all the coefficients $f$ are

\[ f_{-} = \pi_{-}, f_{x} = \pi_{x}, f_{\bar{x}} = \pi_{\bar{x}}, f_{D} = \pi_{D}, f_{a'} = \pi_{a'}, \]

\[ f_{\theta_{i}} = -\frac{i}{2} \pi_{-} \theta_{i}, f_{\bar{\theta}_{i}} = -\frac{i}{2} \pi_{-} \bar{\theta}_{i}, f_{\eta_{i}} = -\frac{i}{4} e^{-2\phi} \pi_{-} \eta_{i}, f_{\bar{\eta}_{i}} = -\frac{i}{4} e^{-2\phi} \pi_{-} \bar{\eta}_{i}, f_{\pi} = 0. \quad (26) \]

4 Poisson bracket and Jacobi identity

The Lagrangian (25) is degenerate in that it is linear in velocities $\dot{z}^a$. In this section, we study the Lagrangian linear in velocities. One sees that such system has a natural quasi-symplectic structure, we next derive the poisson bracket for such degenerate Lagrangian system.

4.1 Bosonic system

Assume the Lagrangian of bosonic system is

\[ L(x_{i}, \dot{x}_{i}) = \sum_{i} f_{i}(x) \dot{x}_{i} - g(x). \quad (27) \]

The Lagrangian equation is

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{i}} - \frac{\partial L}{\partial x_{i}} = 0, \]

giving

\[ \frac{\partial f_{i}}{\partial x_{j}} \dot{x}_{j} - \frac{\partial f_{j}}{\partial x_{i}} \dot{x}_{j} = -\frac{\partial g}{\partial x_{i}}. \]
Define
\[ \omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}, \]
we obtain
\[ \partial_i \omega_{jk} + \text{cyc}(i, j, k) = 0, \]
and
\[ \omega_{ij} \dot{x}_j = \frac{\partial g}{\partial x_i}. \]
If \( \omega \) has an inverse \( \Omega \)
\[ \Omega_{ij} \omega_{jl} = \delta_{il}, \]
we have
\[ \dot{x}_i = \Omega_{ij} \frac{\partial g}{\partial x_j}, \]
with
\[ \Omega_{ij} = -\Omega_{ji}. \]
Define poisson bracket
\[ \{ A, B \} = \frac{\partial A}{\partial x_i} \Omega_{ij} \frac{\partial B}{\partial x_j}. \]
Jacobi identity
\[ \{ A, \{ B, C \} \} + \{ B, \{ C, A \} \} + \{ C, \{ A, B \} \} = 0, \]
follows from
\[ \Omega_{il} \partial_l \Omega_{jk} + \Omega_{jl} \partial_l \Omega_{ki} + \Omega_{ki} \partial_l \Omega_{lj} = 0, \]
due to (28). One has
\[ \dot{A} = \{ A, g \}, \]
for \( A(x) \).

### 4.2 Bosonic and Fermionic system

For the system with both bosonic(\( x_i \) Grassmann even) and fermionic(\( \theta_\alpha \) Grassmann odd) variables, we have \( AB = (-1)^{ab} BA \), \( \partial_i \partial_j = (-1)^{ij} \partial_j \partial_i \), \( \partial_i (BC) = (\partial_i B) C + (-1)^{ib} B \partial_i C \) where \( \partial_i = \frac{\partial}{\partial z_i} \), and the Grassmann index of \( z_i \) is \( i \), while for \( A, B, C \), Grassmann indices are \( a, b, c \) respectively (they should be \( i, j, \hat{i}, \hat{j}, \hat{a}, \hat{b}, \hat{c} \), here we abuse the notation for simplicity).

Assume the Lagrangian is
\[ L(x_i, \dot{x}_i, \theta, \dot{\theta}) = \sum_i f_i(x, \theta) \dot{x}_i + \sum_\alpha \psi_\alpha(x, \theta) \dot{\theta}_\alpha - g(x, \theta). \]
The Lagrangian equation
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0,
\]
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_\alpha} - \frac{\partial L}{\partial \theta_\alpha} = 0,
\]
gives
\[
\omega_{ij} \dot{x}_j + \omega_{i\beta} \dot{\theta}_\beta = \frac{\partial g}{\partial x_i},
\]
\[
\omega_{\alpha j} \dot{x}_j + \omega_{\alpha \beta} \dot{\theta}_\beta = \frac{\partial g}{\partial \theta_\alpha},
\]
(31)

with
\[
\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = -\omega_{ji}, \quad \omega_{i\beta} = \frac{\partial f_i}{\partial \theta_\beta} + \frac{\partial \psi_\beta}{\partial x_i} = \omega_{i\beta} \omega_{\alpha \beta} = \frac{\partial \psi_\alpha}{\partial \theta_\beta} + \frac{\partial \psi_\beta}{\partial \theta_\alpha} = \omega_{\beta \alpha}.
\]

Denote
\[
\omega = \begin{pmatrix} \omega_{ij} & \omega_{i\beta} \\ \omega_{\alpha j} & \omega_{\alpha \beta} \end{pmatrix}, \quad \dot{z} = \begin{pmatrix} \dot{x}_j \\ \dot{\theta}_\beta \end{pmatrix}, \quad \dot{\theta} = \begin{pmatrix} \frac{\partial g}{\partial x_i} \\ \frac{\partial g}{\partial \theta_\alpha} \end{pmatrix}.
\]

The matrix elements
\[
\omega_{mn} = \partial_m f^n - (-1)^{m+n+1} \partial_n f^m
\]
satisfies
\[
\partial_l \omega_{mn} (-1)^{n^2+l+n} \text{cyc}(lmn) = 0.
\]

If \( \omega \) is invertible, one may find \( \Omega \), such that \( \Omega_{mn} \omega_{nl} = \delta_{ml} \) giving
\[
\partial_s \Omega_{lt} = -\Omega_{lm} \partial_s \omega_{mn} \Omega_{nt} (-1)^{s(t+m)}.
\]

We can further show
\[
\Omega_{mn} = (-1)^{m+n+1} \Omega_{mn},
\]
and
\[
\dot{z}_l = \Omega_{lm} \partial_m g.
\]

In the exponent of \(-1\), \( s, m, n, t \) stand for Grassmann indices, they can be even \((i, j)\) and odd \((\alpha, \beta)\).

Define poisson bracket
\[
\{A, B\} = A \frac{\partial}{\partial z_l} \Omega_{lm} \frac{\partial}{\partial z_m} B,
\]
(32)
one has
\[ \dot{A} = \{A, g\} = A \frac{\partial}{\partial z_l} \Omega_{lm} \frac{\partial}{\partial z_m}. \] (33)

Poisson bracket satisfy
\[ \{A, B\} = (-1)^{ab-1} \{B, A\}, \]
\[ \{A, BC\} = \{A, B\} C + (-1)^{ab} \beta \{A, C\}, \]
\[ \{A, \alpha B + \beta C\} = (-1)^{\alpha a} \alpha \{A, B\} + (-1)^{\beta a} \beta \{A, C\}, \]
for constants \( \alpha, \beta \). (34)

here the superscript \( a, b, \alpha \) and \( \beta \) are Grassmann indices for \( A, B, \alpha \) and \( \beta \) respectively.

Super-Jacobi identity is also satisfied
\[ (-1)^n \omega_{mn} \partial_l \omega_{lm} + (-1)^m \omega_{nl} \partial_m \omega_{ln} + (-1)^l \omega_{nm} \partial_n \omega_{ml} = 0, \] (35)
\[ \Rightarrow (-1)^n \Omega_{ki} \partial_l \Omega_{mn} + (-1)^m \Omega_{mi} \partial_k \Omega_{nk} + (-1)^l \Omega_{nj} \partial_m \Omega_{kj} = 0, \] (36)
\[ \Rightarrow (-1)^{ac} \{A, \{B, C\}\} + (-1)^{ab} \{B, \{C, A\}\} + (-1)^{bc} \{C, \{A, B\}\} = 0. \] (37)

4.3 Extension to field theory

In the following, integration of \( \sigma \) over one period is always assumed if no initial and end points.

The Lagrangian is
\[ L = \int d\sigma f_i(z(\sigma), z'(\sigma)) \dot{z}_i(\sigma) - \int d\sigma g(z(\sigma), z'(\sigma)), \] (38)
where the index \( i \) can be bosonic and fermionic.

The Lagrangian equation
\[ \frac{d}{dt} \left( \frac{\delta L}{\delta \dot{z}_i(\sigma)} \right) - \frac{\delta L}{\delta z_i(\sigma)} = 0, \]
gives
\[ \frac{d}{dt} \left[ (-1)^j \int d\sigma' \frac{\delta \dot{z}_j(\sigma')}{\delta \dot{z}_i(\sigma)} f_j(z(\sigma'), z'(\sigma')) \right] - \int d\sigma' \frac{\delta f_j(z(\sigma'), z'(\sigma'))}{\delta z_i(\sigma)} \dot{z}_j(\sigma') + \int d\sigma' \frac{\delta g(z(\sigma'), z'(\sigma'))}{\delta z_i(\sigma)} = 0. \]

Due to
\[ \frac{\delta \dot{z}_j(\sigma')}{\delta \dot{z}_i(\sigma)} = \delta_{ij} \delta(\sigma' - \sigma), \]
one has
\[ \int d\sigma' \omega_{ij}(\sigma, \sigma') \dot{z}_j(\sigma') = \int d\sigma' \frac{\delta g(z(\sigma'), z'(\sigma'))}{\delta z_i(\sigma)}. \]
which has the property

\[ \omega_{i(\sigma),j(\sigma')} = (-1)^{(i-1)(j-1)} \omega_{j(\sigma'),i(\sigma)}, \]

and

\[
(-1)^{j^2+ij} \frac{\delta}{\delta z_i(\sigma)} \omega_{i(\sigma),j(\sigma)} + (-1)^{l^2+il} \frac{\delta}{\delta z_i(\sigma)} \omega_{j(\sigma),l(\sigma)} + (-1)^{l^2+ij} \frac{\delta}{\delta z_j(\sigma)} \omega_{i(\sigma),i(\sigma)} = 0. \tag{39}
\]

Assume matrix \([\omega_{i(\sigma),j(\sigma')}]\) has an inverse \([\Omega_{j(\sigma'),i(\sigma)}]\)

\[
\int d\sigma' \omega_{i(\sigma),j(\sigma')} \Omega_{j(\sigma'),l(\sigma'')} = \delta_{ll} \delta(\sigma - \sigma''),
\]

\[
\int d\sigma \Omega_{i(\sigma'),j(\sigma)} \omega_{j(\sigma),l(\sigma'')} = \delta_{il} \delta(\sigma' - \sigma''),
\]

Then we have

\[ \Omega_{i(\sigma),j(\sigma')} = (-1)^{ij+1} \Omega_{j(\sigma'),i(\sigma)}. \]

The Lagrangian equation

\[
\int d\sigma' \omega_{i(\sigma),j(\sigma')} \dot{z}_j(\sigma') = \int d\sigma' \frac{\delta g(z(\sigma'), \dot{z}(\sigma'))}{\delta z_i(\sigma)} = \frac{\delta H}{\delta z_i(\sigma)}, \tag{40}
\]

gives

\[ \dot{z}_k(\sigma'') = \int d\sigma \Omega_{k(\sigma''),i(\sigma)} \frac{\delta H}{\delta z_i(\sigma)}, \]

where \( H = \int d\sigma g(z(\sigma), \dot{z}(\sigma)) \).

Define Poisson bracket

\[
\{A, B\} = \int d\sigma \int d\sigma' A \frac{\frac{\partial}{\partial z_i(\sigma)} \Omega_{i(\sigma),j(\sigma')} \delta B}{\partial z_j(\sigma')} = \int d\sigma \int d\sigma' (-1)^{(a-1)i} \frac{\delta A}{\delta z_i(\sigma)} \Omega_{i(\sigma),j(\sigma')} \frac{\delta B}{\delta z_j(\sigma')}, \tag{41}
\]

which implies

\[ \{A, B\} = (-1)^{ab+1} \{B, A\}. \]

We have

\[
\dot{z}_k(\sigma'') = \{z_k(\sigma''), H\} = \int d\sigma \int d\sigma' z_k(\sigma'') \frac{\frac{\partial}{\partial z_i(\sigma)} \Omega_{i(\sigma),j(\sigma')} \delta H}{\partial z_j(\sigma')}.
\]
\[ A = \{ A, H \}. \]  

The l.h.s. of Lagrangian equation (40) can be written as

\[
\int d\sigma \omega_{i(\sigma),j(\sigma')} \dot{z}_j(\sigma') = \int d\sigma' \{ \delta(\sigma' - \sigma) \frac{\partial f_j}{\partial z_i}(\sigma') - (-1)^{i+j+ij} \delta(\sigma - \sigma') \frac{\partial f_i}{\partial z_j}(\sigma) \}
\]

\[
+ \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma) \frac{\partial f_j}{\partial z_i}(\sigma') - (-1)^{i+j+ij} \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') \frac{\partial f_i}{\partial z_j}(\sigma) \}\dot{z}_j(\sigma').
\]

Using integration by parts and considering \( \delta(\sigma - \sigma') = \delta(\sigma' - \sigma) \) and \( \frac{\partial}{\partial \sigma} \delta(\sigma' - \sigma) = -\frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') \), we have

\[
\int d\sigma' \omega_{i(\sigma),j(\sigma')} \dot{z}_j(\sigma') = \tilde{\omega}_{ij}(\sigma) \dot{z}_j(\sigma) + A_{ij} \dot{z}_j(\sigma),
\]

where

\[
\tilde{\omega}_{ij}(\sigma) = \frac{\partial f_j}{\partial z_i}(\sigma) - (-1)^{i+j+ij} \frac{\partial f_i}{\partial z_j}(\sigma) - \left[ \frac{\partial f_j}{\partial z_i}(\sigma) - (-1)^{i+j+ij} \frac{\partial f_i}{\partial z_j}(\sigma) \right]
\]

\[
A_{ij}(\sigma) = -\left[ \frac{\partial f_j}{\partial z_i}(\sigma) + (-1)^{i+j+ij} \frac{\partial f_i}{\partial z_j}(\sigma) \right].
\]

Locality condition requires \( A_{ij} = 0 \). If this condition is satisfied, we may instead use

\[
\omega_{i(\sigma),j(\sigma')} = \tilde{\omega}_{ij}(\sigma) \delta(\sigma - \sigma'),
\]

and

\[
\Omega_{i(\sigma),j(\sigma')} = \tilde{\Omega}_{ij}(\sigma) \delta(\sigma - \sigma'),
\]

for the inverse of \( \omega_{i(\sigma),j(\sigma')} \) to get the correct equation of motion. We may also define poisson bracket as

\[
\{ A, B \} = \int d\sigma d\sigma' [ A_{ij} \frac{\delta}{\delta z_i(\sigma)} \Omega_{ij}(\sigma) \delta(\sigma - \sigma') \frac{\delta B}{\delta z_j(\sigma')} ]
\]

and show that the Jacobi identity is still valid for the case we are interested in in this paper (Appendix B).

On the contrary, for \( A_{ij} \neq 0 \), we find the locality of the field theory is broken. This is because the equation gives

\[
\dot{z}_i' = M_{ij} \dot{z}_j + N_i.
\]

Even though \( M_{ij} \) and \( N_i \) are local at each \( \sigma \). The quantity \( \dot{z}_j(\sigma) \) is determined by all data of \( M_{ij} \) and \( N_i \) at \( \sigma' = \sigma_0 \) to \( \sigma \), if \( \dot{z}_j(\sigma_0) \) is given at point \( \sigma_0 \). Thus the 2-D field theory is nonlocal.
For the system described by (25), we see that from (26)
\[ \frac{\partial f_a}{\partial z^b} = 0, \] (45)
for \( a = i, \alpha \), thus it does satisfies
\[ A_{ab} \equiv -\frac{\partial f_b}{\partial z^a} - (-1)^{a+b+ab} \frac{\partial f_a}{\partial z^b} = 0. \] (46)

The problem left is whether the supermatrix \( \tilde{\Omega}_{ij(\sigma)} \) is invertible. Further calculation of \( \omega_{ab} \) via
\[ \tilde{\omega}_{ab} = (-1)^{a+b+ab} \partial_b f_a + \partial_a f_b, \] (47)
for (25) gives:

(A) \[ \tilde{\omega}_{ij} = -\tilde{\omega}_{ji}, \]
\[ \tilde{\omega}_{x-\pi_-} = -1, \tilde{\omega}_{x\pi_+} = -1, \tilde{\omega}_{D\pi_D} = -1, \tilde{\omega}_{a'\pi_{a'}} = -1. \] (48)

(B) \[ \tilde{\omega}_{ia} = \tilde{\omega}_{ai}, \]
\[ \tilde{\omega}_{\pi_-\theta_i} = -\frac{i}{2} \theta^i, \tilde{\omega}_{\pi_-\theta^i} = -\frac{i}{2} \theta_i, \tilde{\omega}_{\pi_-\eta_i} = -\frac{i}{2} e^{-2\phi} \eta^i, \] (49)
\[ \tilde{\omega}_{\pi_-\eta_i} = -\frac{i}{2} e^{-2\phi} \eta^i, \tilde{\omega}_{\phi\eta^i} = \frac{i}{2} e^{-2\phi} \pi_- \eta_i. \] (50)

(C) \[ \tilde{\omega}_{\alpha\beta} = \tilde{\omega}_{\beta\alpha}, \]
\[ \tilde{\omega}_{\theta_i\theta^i} = -\frac{i}{2} \pi_- \tilde{\omega}_{\theta_i\eta^i} = -\frac{i}{2} e^{-2\phi} \pi_- \tilde{\omega}_{\eta^i\eta^i} = -\frac{i}{2} e^{-2\phi} \pi_- \tilde{\omega}_{\eta^i\eta^i}. \] (51)

We conclude that as long as the c number part of \( \pi_- \neq 0 \), \( \omega \) is invertible, and the Poisson bracket is well defined. On the other hand, we have from Appendix A,
\[ \pi_- = e^{\phi} J_- = e^{\phi} \frac{\partial L_k}{\partial x_0} = \frac{e^{2\phi}}{\sqrt{-G}} [e^{2\phi}((x^1)^2 + (x^2)^2) + \phi'^2 + \sum_{a'=5}^8 (u^{a'} y'^{a'})^2 + 1], \]
and
\[ \frac{1}{\pi_-} = e^{-2\phi} \frac{\sqrt{-G}}{e^{2\phi} (x_1^2 + x_2^2) + \phi'^2 + \sum_{a'=5}^8 (u^{a'} y'^{a'})^2 + 1}. \] (52)
Thus as long as \( G \) exits, we have a well defined Poisson bracket. This is a loose condition. This condition may break down, for example, when the "string tube" grows a new branch in the \( AdS_5 \times S_5 \) space. But in most cases, the Hamiltonian description is valid. The difficulty may appear in the quantum theory, where one has to take into account the whole space time. This needs further investigation.
5 Flat currents and solution transformation

5.1 Flat currents with one parameter

Bena, Polchinski and Roiban made an important discovery that the Metsaev and Tseytlin model has a one-parameter family of flat currents. This implies the model has infinite conserved nonlocal quantities. Here we review the equivalent form of their construction.

From \( J = G' - \frac{1}{dG'} \), one has

\[
dJ + J \wedge J = 0,
\]

giving the Maurer-Cartan equations

\[
dL_a = -L^b \wedge L^a b - i \gamma^a \wedge L^1,
\]

\[
dL_a' = -L' b \wedge L^b a' + L^1 \gamma^a' \wedge L^1,
\]

\[
dL_{ab} = L^d \wedge L^b - L^{ac} \wedge L^{b} + \epsilon_{ij} L^j \gamma^{ab} \wedge L^1,
\]

\[
dL_{a'b'} = L'^d \wedge L'^{b'} - L'^{a'c'} \wedge L'^{b'} - \epsilon_{ij} L^j \gamma^{a'b'} \wedge L^1,
\]

\[
dL^I = -\frac{1}{2} \gamma_a \epsilon_{ij} L^j \wedge L^a + \frac{1}{2} \gamma_{a'} \epsilon_{ij} L^j \wedge L^{a'}
\]

\[
+ \frac{1}{4} \gamma_{ab} L^j \wedge L^{ab} + \frac{1}{4} \gamma_{a'b'} L^j \wedge L^{a'b'}.
\]

(53)

We firstly introduce the world-sheet Hodge dual of the Maurer-Cartan 1-forms \( L^a \) and \( L^a' \). Let \( \sqrt{-g} g^{ij} = \gamma^{ij} \) and

\[
* L^a_k = -\epsilon^{ki} \gamma^{ij} L^i_j, \text{ and } \hat{a} = a, a', \epsilon^{01} = -\epsilon^{10} = 1.
\]

The equations of motion can be expressed as

\[
d* L^a + L^a \wedge * L^b + i s L^j \gamma^a \wedge L^j = 0, \quad (54)
\]

\[
d* L^a' + L^a' \wedge * L^b' - s' L^j \gamma^{a'} \wedge L^j = 0, \quad (55)
\]

\[
\delta^{ij}( * L^a \gamma^{a} + i* L^a \gamma^{a'} ) \wedge L^j + s^{ij} (L^a \gamma^{a} + iL^a' \gamma^{a'} ) \wedge L^j = 0. \quad (56)
\]

Introduce the forms with a parameter \( \lambda \),

\[
L^a (\lambda) = \frac{1}{2} \left( \lambda^2 - \lambda^{-2} \right) L^a + \frac{1}{2} \left( \lambda^2 - \lambda^{-2} \right) * L^a,
\]

\[
L^a' (\lambda) = \frac{1}{2} \left( \lambda^2 + \lambda^{-2} \right) L^a' + \frac{1}{2} \left( \lambda^2 - \lambda^{-2} \right) * L^a',
\]

\[
L^{ab} (\lambda) = L^{ab}, \quad L^{a'b'} (\lambda) = L^{a'b'},
\]

\[
L^1 (\lambda) = \lambda L^1, \quad L^2 (\lambda) = \lambda^{-1} L^2.
\]

(57)
When \( \det[\gamma_{ij}] = -1 \), we have
\[
\begin{align*}
^\star(\star A) &= A, \\
A \wedge ^\star B &= -^\star A \wedge B, \\
^\star A \wedge ^\star B &= -A \wedge B.
\end{align*}
\]
(58)

We can prove one forms (57) with a parameter also satisfy Maurer-Cartan equations (53) by (54) to (58). Thus the currents \( \mathcal{J}(\lambda) \) with spectral parameter \( \lambda \) expressed from the Cartan one forms (57),
\[
\mathcal{J}(\lambda) = L^a(\lambda) P_a + L^{a'}(\lambda) P_{a'} + \frac{1}{2} L^{ab}(\lambda) J_{ab} + \frac{1}{2} L^{a'b'}(\lambda) J_{a'b'} + L^\alpha I_{\alpha} Q^I_{\alpha},
\]
(59)
satisfies \( d\mathcal{J}(\lambda) + \mathcal{J}(\lambda) \wedge \mathcal{J}(\lambda) = 0 \). So the equation \( \mathcal{J}(\lambda) = G(\lambda)^{-1} dG(\lambda) \) is integrable and \( \mathcal{J}(\lambda) \) naturally leads to an infinite number of non-local conserved quantities.

From the above, we see that as long as equations of motion are satisfied, \( \mathcal{J}(\lambda) \) will be flat. However, after some gauge fixing, some equations of motion are missing. The \( \kappa \) symmetry gauge fixing cause half of equations (56) disappear and fixing \( x_0 = \tau, y_0 = \sigma \) cause two of (54,55) missing. Are they still satisfied? The answer is affirmative. The reason is that local symmetry cause \( \delta S = 0 \), under certain combination of canonical variables. Thus equations of motion are not independent. The number of redundancy of them exactly matches the number of missing equations in the gauge fixing. In references [16], the authors give a concise proof that the flat currents keeps flat under various symmetry transformations. This explains the origin of that the flat currents still exists after gauge fixing.

We then express the flat currents in terms of canonical variables for the system in section 3. One has
\[
^\star L^\hat{a} = -\epsilon^{ki} \gamma_{ij} L^\hat{j}.
\]
From (5) and (21), we have
\[
\delta S_k = -\frac{1}{2} \int d\tau d\sigma \gamma^{\mu\nu} x^\hat{a}_\mu G_{\hat{a}b} x^\hat{b}_\nu - \frac{1}{2} \int d\tau d\sigma \gamma^{\mu\nu} x^\hat{a}_\mu G_{\hat{a}b} x^\hat{b}_\nu - \frac{1}{2} \int d\tau d\sigma \gamma^{\mu\nu} x^\hat{a}_\mu G_{\hat{a}b} x^\hat{b}_\nu
\]
\[
= -\frac{1}{2} \int d\tau d\sigma \gamma^{\mu\nu} x^\hat{a}_\mu G_{\hat{a}b} x^\hat{b}_\nu,
\]
giving
\[
\frac{\partial L_k}{\partial x^\hat{a}_0} = -\gamma^{0\nu} G_{\hat{a}b} x^\hat{b}_\nu = J^\hat{a},
\]
and
\[
^\star L^\hat{a}_1 = \epsilon^{10} (G^{-1})^\hat{a}b J^\hat{b} = -(G^{-1})^\hat{a}b J^\hat{b}.
\]
From appendix A, \( J^\hat{a} \) is expressed in \( (\pi_1, z^i, z^i, z^{i\alpha}, z^{i\alpha}) \) for all ten \( \hat{a} \)'s. Further checking the remaining components of \( \mathcal{J}_1(\lambda) \) by (19), we find \( \mathcal{J}_1(\lambda) \) can be expressed by these variables too.
5.2 solution transformations

For the flat currents

$$\mathcal{J}(\lambda) = L^a(\lambda)P_a + L^a'(\lambda)P_a' + \frac{1}{2}L^{ab}(\lambda)J_{ab} + \frac{1}{2}L^{a'b'}(\lambda)J_{a'b'} + L^{\alpha\beta}(\lambda)Q_\alpha^\beta,$$

the solution $G(\lambda, \tau, \sigma)^{-1}\partial_\mu G(\lambda, \tau, \sigma) = \mathcal{J}_\mu(\lambda) \ (\mu = 0, 1)$ with given $G(\lambda, \tau_0, \sigma_0) = G_0$ is independent of the path of integration. It can be symbolically expressed as

$$G_0 P e^{L_C \mathcal{J}(\lambda)} = G(\lambda, \tau, \sigma),$$

where $C$ is any contour from $(\tau_0, \sigma_0)$ to $(\tau, \sigma)$, and $P$ denotes path ordering of the Lie algebra generators. Consider two paths ABC and ADC, where AB and DC are along $\sigma$ with length $L$ while BC and AD are along $\tau$. We have $G(\lambda, \tau_c, \sigma_c) = G_0 U_{AB} U_{BC} U_{AD} U_{DC}$, where

$$U_{AB} = Pe^{f^B_A \mathcal{J}(\lambda)}, \quad U_{BC} = Pe^{f^C_B \mathcal{J}(\lambda)},
\quad U_{AD} = Pe^{f^D_A \mathcal{J}(\lambda)}, \quad U_{DC} = Pe^{f^C_D \mathcal{J}(\lambda)}.$$

If the period of $\sigma$ is $L$, $U_{AD}$ and $U_{BC}$ are equal. We have

$$U_{DC} = U_{AD}^{-1} U_{AB} U_{BC} = U_{BC}^{-1} U_{AB} U_{BC}.$$

Thus in any representation of $PSU(2, 2|4)$, the matrices $\hat{U}_{DC}$ and $\hat{U}_{AB}$ are similar matrices and the supertraces of them are equal. That is, $F_1(\lambda) = \text{str} P e^{L \hat{J}_1(\lambda, \sigma, \tau)} = \text{str} \hat{U}(\tau)$ is a constant of motion [18] as well as their eigenvalues [18]. Let’s return to the system (24) in section 3, since $\mathcal{J}_1(\lambda)$ is a function of canonical variables and $F_1(\lambda)$ is always conserved, we have

$$\{F_1(\lambda), H\} = 0 \quad (61)$$

by (12). Notice $F_1(\lambda)$ and $H$ are not depending on $\tau = x^+$ when expressed by $\pi_i, z_i, z'_i, z_\alpha, z'_\alpha$ in section 3.

Due to Jacobi identity, we have

$$\{z_i, \{F_1(\lambda), H\}\} + \{F_1(\lambda), \{H, z_i\}\} + \{H, \{z_i, F_1(\lambda)\}\} = 0,$$

$$\{\{z_i, H\}, F_1(\lambda)\} = \{\{z_i, F_1(\lambda)\}, H\},$$

impling the action of Hamiltonian $H$ and the action of $F_1(\lambda)$ are commutable [32]. Assume a solution $z(\tau, \sigma)$ is given by $z(0, \sigma)$ and satisfies

$$\dot{z}_i = \{z_i, H\},$$

(62)

where $z_i$ can be bosonic and fermionic variables.
We may solve
\[
\frac{d}{dt} z_i = \{z_i, F_1(\lambda)\}, \tag{63}
\]
with \(z_i(\lambda, t = 0, \tau, \sigma) = z_i(\tau, \sigma)\). Then \(z_i(\lambda, t, \tau, \sigma)\) is a new solution of (62) for each fixed \(t\). This is a solution transformation. There are infinite generators of such transformations.

6 Discussions

In this paper, we construct the solution transformations by Jacobi identity of poisson bracket for one parameter flat currents with Hamiltonian, and the poisson bracket is constructed from Lagrangian which is linear in velocities with \(\kappa\) Light-cone gauge fixing. The relation of solution transformations for different \(\lambda, t\) is not clear, it seems that they form two parameter sets. Since the expression of \(F_1\) is complicated, the further investigation of examples is worth doing.

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A Hamiltonian Analysis

Let \(\mathcal{L} = \mathcal{L}_k + \mathcal{L}_{WZ}, z = z_i, z_\alpha\) and \(c\) number of \(\det(\frac{\partial \mathcal{L}_k}{\partial \dot{z}_i \dot{z}_j}) \neq 0\), define \(\pi_i = \frac{\partial \mathcal{L}_k}{\partial \dot{z}_i}\), \(\tilde{\mathcal{H}} = \pi_i \dot{z}_i - \mathcal{L}_k\). One can express \(\dot{z}_i\) and \(\tilde{\mathcal{H}}\) as the functions of \(\pi_i, z_i, z'_i, z_\alpha, z'_\alpha\). We have a new Lagrangian density
\[
\tilde{\mathcal{L}}_k = \pi_i \dot{z}_i - \tilde{\mathcal{H}}.
\]
The variation of \(\tilde{\mathcal{H}}\) is
\[
\delta \tilde{\mathcal{H}} = \delta \pi_i \dot{z}_i + \pi_i \delta \dot{z}_i - \frac{\partial \mathcal{L}_k}{\partial \dot{z}_i} \delta \dot{z}_i - \frac{\partial \mathcal{L}_k}{\partial z_i} \delta z_i - \frac{\partial \mathcal{L}_k}{\partial z'_i} \delta z'_i - \frac{\partial \mathcal{L}_k}{\partial z_\alpha} \delta z_\alpha - \frac{\partial \mathcal{L}_k}{\partial z'_\alpha} \delta z'_\alpha. \tag{64}
\]
Denote
\[
L_{u,f} \equiv \frac{\partial f}{\partial z_i} - \frac{\partial}{\partial \tau} \frac{\partial f}{\partial \dot{u}} - \frac{\partial}{\partial \sigma} \frac{\partial f}{\partial \dot{w}}.
\]
The equation $L_\pi, \tilde{\mathcal{L}}_k = 0$ gives

$$(\pi_j - \frac{\partial L_k}{\partial \dot{z}_j}) \frac{\partial \dot{z}_j}{\partial \pi_i} = 0,$$

implying

$$\pi_j - \frac{\partial L_k}{\partial \dot{z}_j} = 0,$$

(65)

when $c$ number of $\det(\frac{\partial^2 L}{\partial z_i \partial \dot{z}_j}) \neq 0$. We have

$$L_{z_i} \tilde{\mathcal{L}}_k \equiv \frac{\partial \tilde{\mathcal{L}}_k}{\partial z_i} - \frac{\partial}{\partial \dot{z}_i} \frac{\partial \tilde{\mathcal{L}}_k}{\partial z} + \left( \frac{\partial \mathcal{H}}{\partial z_i} - \frac{\partial}{\partial \dot{z}_i} \frac{\partial \mathcal{H}}{\partial z} \right) = L_{z_i} \mathcal{L}_k,$$

$$L_{z_{\alpha}} \tilde{\mathcal{L}}_k \equiv \frac{\partial \tilde{\mathcal{L}}_k}{\partial z_{\alpha}} - \frac{\partial}{\partial \dot{z}_{\alpha}} \frac{\partial \tilde{\mathcal{L}}_k}{\partial z_i} + \left( \frac{\partial \mathcal{H}}{\partial z_{\alpha}} - \frac{\partial}{\partial \dot{z}_{\alpha}} \frac{\partial \mathcal{H}}{\partial z_i} \right) = L_{z_{\alpha}} \mathcal{L}_k.$$  

The last step comes from (64) and (65).

For $\mathcal{L} = \mathcal{L}_k + \mathcal{L}_{WZ}$ and $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_k + \tilde{\mathcal{L}}_{WZ}$, we have

$$0 = L_{\pi_i} \tilde{\mathcal{L}} = L_{\pi_i} \tilde{\mathcal{L}}_k \Rightarrow \pi_i - \frac{\partial L_k}{\partial \dot{z}_i} = 0,$$

$$0 = L_{z_i} \tilde{\mathcal{L}}_k = L_{z_i} \tilde{\mathcal{L}}_k + L_{z_{\alpha}} \mathcal{L}_{WZ} = L_{z_i} \mathcal{L}_k + L_{z_{\alpha}} \mathcal{L}_{WZ} \Rightarrow L_{z_i} \mathcal{L} = 0,$$

$$0 = L_{z_{\alpha}} \tilde{\mathcal{L}}_k = L_{z_{\alpha}} \tilde{\mathcal{L}}_k + L_{z_{\alpha}} \mathcal{L}_{WZ} = L_{z_{\alpha}} \mathcal{L}_k + L_{z_{\alpha}} \mathcal{L}_{WZ} \Rightarrow L_{z_{\alpha}} \mathcal{L} = 0.$$  

Thus the Lagrangian equations of $\tilde{\mathcal{L}}$ with the variables $(\pi_i, \dot{z}_i, z_i, z_{\alpha}, \dot{z}_{\alpha}, z_{\alpha})$ give the definition of $\pi_i$ and the Lagrangian equation of $\mathcal{L}$ for variables $(\dot{z}_i, z_i, \dot{z}_{\alpha}, z_{\alpha})$.

This is a partial Legendre transformation for the part Lagrangian with arbitrary partition of $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$. This idea is firstly introduced by Arutyunov, Frolov et al. [10] in deriving the Hamiltonian of GS string.

For the Polyakov Lagrangian (21), the variation of $g^{\mu\nu}$ yields the well known Nambu-Goto Lagrangian,

$$\mathcal{L}_k = -\sqrt{(x_0^a G_{ab} x_1^b)^2 - (x_0^a G_{ab} x_1^b)(x_1^a G_{ab} x_1^b)} \equiv -\sqrt{-G},$$

where $a, b = +, -, x, \bar{x}, D$ and $A'(A' = 5, 6, 7, 8, 9)$. Then we fix the gauge $x^+ = \tau$, $y^0 = \sigma$, and write

$$L_\mu^a \equiv x_\mu^a = x_{\mu 0} + \alpha_i^a z_i^\mu,$$

where $z_i^\mu = \begin{cases} \dot{z}_i, & \mu = 0 \\ z_i^\mu, & \mu = 1 \end{cases}$, $i \neq x^+, y^0$, and $x_{\mu 0}$ is the rest of the $L_\mu^a$, including fermionic variables and some functions of coordinates.
We have for $z^i = x^a, x^b, D, y^5, y^6, y^7, y^8,$

$$\pi_i = \frac{\partial L_k}{\partial z^i} = -\frac{1}{\sqrt{-\mathcal{G}}} \left[ 2 \frac{\partial x_0^a}{\partial z^i} G_{ab} x_1^b (x_0^a G_{a_1 b_1} x_1^{b_1}) - 2 \frac{\partial x_0^a}{\partial z^i} G_{ab} x_0^b (x_1^a G_{a_1 b_1} x_1^{b_1}) \right]$$

$$= -\frac{1}{\sqrt{-\mathcal{G}}} \left[ \pi_i \partial x_0^a \right] \left[ \partial x_0^a \right] = \pi_i J_a,$$

$$J_a = \frac{1}{\sqrt{-\mathcal{G}}} \left[ G_{ab} x_1^b (x_0^a G_{a_1 b_1} x_1^{b_1}) - G_{ab} x_0^b (x_1^a G_{a_1 b_1} x_1^{b_1}) \right]$$

$$= \frac{1}{\sqrt{-\mathcal{G}}} \left[ \pi_i \partial x_0^a \right] \left[ \partial x_0^a \right] = \pi_i \partial x_0^a \partial x_0^a.$$
Another identity
\[
\frac{\partial L_k}{\partial x_0^a} x_0^a = -\frac{1}{\sqrt{-\mathcal{G}}}[x_0^a G_{ab} x_1^b(x_0^a G_{ab} x_1^b) - x_0^a G_{ab} x_0^b(x_1^a G_{ab} x_1^b)]
\]
\[
= -\frac{1}{\sqrt{-\mathcal{G}}}[x_0 x_1 x_0 x_1 - (x_0 x_0)(x_1 x_1)]
\]
\[
= -\sqrt{(x_0 x_1)^2 - (x_0 x_0)(x_1 x_1)} = -\sqrt{-\mathcal{G}} = \mathcal{L}_k,
\]
can be used in deriving (69).

The momenta $\pi_i$ are
\[
\pi_i = \frac{\partial x_0^a}{\partial \dot{z}_i} J_a,
\]
where $x_\mu = L_\mu^a$.

The nonzero $\frac{\partial x_0^a}{\partial \dot{z}_i}$ are
\[
\frac{\partial x_0^a}{\partial \dot{x}^-} = e^\phi, \frac{\partial x_0^a}{\partial \dot{x}^+} = e^\phi, \frac{\partial x_0^a}{\partial \dot{x}^0} = e^\phi, \frac{\partial x_0^a}{\partial \dot{x}^D} = \frac{\partial x_0^a}{\partial \dot{\phi}} = 1, \frac{\partial x_0^a}{\partial \dot{y}^a} = u^a, a' = 5, 6, 7, 8.
\]

One has
\[
\pi_+ = e^\phi J_-, \pi_+ = e^\phi J_x, \pi_+ = e^\phi J_x, \pi_D = J_D, \pi_{a'} = u^a J_{a'},
\]
and
\[
J_+ = e^{-\phi} \pi_-, J_x = e^{-\phi} \pi_x, J_x = e^{-\phi} \pi_x, J_D = \pi_D, J_{a'} = \frac{1}{u^a} \pi_{a'}.
\]

Noting from (69),
\[
L^+ = e^\phi dx^+, \quad L^- = e^\phi [dx^- - \frac{i}{2}(\theta^i d \theta_i + \theta_i d \theta^i)] - \frac{1}{2} e^{-\phi} \frac{1}{4}(\eta^2)^2 dx^+ + \frac{i}{2}(\eta^i d \eta_i + \eta_i d \eta^i),
\]
\[
L^\times = e^\phi dx^0, \quad L^\times = e^\phi dx^0,
\]
\[
L^D = d \phi, \quad L^D = d \phi,
\]
where $\eta^2 = \eta^i \eta_i$ and defining
\[
b_{00} = -\frac{i}{2}(\theta^i \dot{\theta}_i + \theta_i \dot{\theta}^i), \quad b_{00} = \frac{i}{2}(\eta^i \dot{\eta}_i + \eta_i \dot{\eta}^i),
\]
\[
b_{10} = -\frac{i}{2}(\theta^i \dot{\theta}'_i + \theta_i \dot{\theta}'^i), \quad b_{10} = \frac{i}{2}(\eta^i \dot{\eta}'_i + \eta_i \dot{\eta}'^i),
\]
from (72) and (73) one obtains
\[
x_0^+ = e^\phi \dot{x}^+, \quad x_0^- = e^\phi (\dot{x}^- + b_{00}) - \frac{1}{2} e^{-\phi} \frac{1}{4}(\eta^2)^2 \dot{x}^+ + b_{00},
\]
\[
x_0^x = e^\phi \dot{x}, \quad x_0^x = e^\phi \dot{x}, x_0^D = \dot{\phi}, x_0^A = u^A \dot{y}^A + v^A \dot{x}^+, \quad x_1^+ = e^\phi \dot{x}^+, \quad x_1^- = e^\phi (\dot{x}^- + b_{10}) - \frac{1}{2} e^{-\phi} \frac{1}{4}(\eta^2)^2 \dot{x}^+ + b_{10},
\]
\[
x_1^x = e^\phi \dot{x}, \quad x_1^x = e^\phi \dot{x}, x_1^D = \dot{\phi}', x_1^A = u^A y'^A + v^A \dot{x}^+.
\]
For gauge $x^+ = \tau$, $y^9 = \sigma$, we have

\[
\begin{align*}
\dot{x}^+ &= 1, \quad \dot{x}^t = 0, \quad \dot{y}^9 = 0, \quad \dot{y}^9 = 1, \\
x_0^{A'} &= u^{A'} y^{A'} + v^{A'}, \quad x_1^{A'} = u^{A'} y^{A'}, \\
x_0^0 &= e^\phi, \quad x_1^0 = 0, \\
x_0^9 &= v^9, \quad x_1^9 = u^9 = 1.
\end{align*}
\]

This implies

\[
\begin{align*}
\dot{x}_{00}^+ &= e^\phi, \quad \dot{x}_{00}^- = e^\phi b_{\theta} - \frac{1}{2} e^{-\phi} [b_{\theta} + \frac{1}{4} (\eta^2)^2], \quad x_{00}^{A'} = v^{A'}.
\end{align*}
\]

The equation $J_a x_1^a = 0$ gives

\[
J_+ x_1^+ + J_- x_1^- + \cdots + J_8 x_1^8 + J_9 x_1^9 = 0.
\]

So we can derive

\[
\begin{align*}
J_9 &= -(\frac{1}{x_1^9}) [J_+ x_1^+ + J_- x_1^- + \cdots + J_8 x_1^8] \\
&= -\{\pi_-(x' - b_{1\theta} - \frac{1}{2} e^{-2\phi} (x_{21})^2) + \pi_+ x' + \pi_\bar{x} \bar{x}' + \pi_D \phi' + \sum_{a' = 5}^8 \pi_{a'} y^{a'}\}. \quad (74)
\end{align*}
\]

We have $G^{-1}_{ab} = G_{ab}$. Equation $J_a (G^{-1}_{ab} J_b) = -x_1 G x_1$ gives

\[
\begin{align*}
2J_+ J_+ + 2J_+ J_2 + J_2^2 + \sum_{a' = 5}^8 J_{a'}^2 + J_9^2 \\
&= -[2x_1^+ x_1^- + 2x_1^x x_1^\bar{x} + (x_1^D)^2 + \sum_{a' = 5}^8 (x_1^{a'})^2 + (x_1^9)^2] \\
&= -2e^{2\phi} (x' x') + \phi^2 + \sum_{a' = 5}^8 (u_{a'} x^{a'})^2 + 1] \\
&= -[2e^{2\phi} (x' x') + \phi^2 + \sum_{a' = 5}^8 (u_{a'} x^{a'})^2 + 1]. \quad (75)
\end{align*}
\]

and

\[
\begin{align*}
J_+ &= \frac{1}{2J_-} \{-[2e^{2\phi} (x' x') + \phi^2 + (u_{a'})^2 (x^{a'})^2 + 1] - [2e^{-2\phi} \pi_x \pi_{\bar{x}} + \pi_D^2 + \sum_{a' = 5}^8 (u_{a'})^{-2} \pi_{a'}^2 \\
&\quad + (\pi_-(x' - b_{1\theta} - \frac{1}{2} e^{-2\phi} (x_{21})) + \pi_+ x' + \pi_\bar{x} \bar{x}' + \pi_D \phi' + \sum_{a' = 5}^8 \pi_{a'} x^{a'}])\}. \quad (77)
\end{align*}
\]
One has

\[ \tilde{H} = -x^a_{00}J_a, \]

with

\[ x^+_{00} = e^\phi, \quad x^-_{00} = e^\phi b_{0\theta} - \frac{1}{2} e^{-\phi} \left[ b_{0\eta} + \frac{1}{4} (\eta^2)^2 \right], \quad x^a_{00} = v^a, \quad x^9_{00} = v^9, \]

giving

\[
\tilde{H} = -J_x e^\phi - J_- [e^\phi b_{0\theta} - \frac{1}{2} e^{-\phi} [b_{0\eta} + \frac{1}{4} (\eta^2)^2]] - \sum_{a'=5}^8 J_{a'} v^{a'} - J_9 v^9 \\
= \frac{e^{2\phi}}{2\pi_-} \left\{ [2e^{2\phi} x'^a \bar{x}'^a + \phi'^2 + \sum_{a'=5}^8 (u^{a'})^2 (y'^a)^2 + 8] + [2e^{-2\phi} \pi_x \pi_x + \pi_D^2 + \sum_{a'=5}^8 (u'^a)^2 - 2 \pi_{a'a'}^2] \right\} \\
+ \left\{ (\pi_-(x'^- + b_{1\theta} - \frac{1}{2} e^{2\phi} b_{1\eta}) + \pi_x x' + \pi_x \bar{x}' + \pi_D \phi' + \sum_{a'=5}^8 \pi_{a'a'} y'^{a'} \right\} \\
- e^{-\phi} \pi_- [e^\phi b_{0\theta} - \frac{1}{2} e^{-\phi} [b_{0\eta} + \frac{1}{4} (\eta^2)^2]] - \sum_{a'=5}^8 \pi_{a'a'} \frac{v^{a'}}{u^{a'}} \\
+ v^9 \left\{ (\pi_-(x'^- + b_{1\theta} - \frac{1}{2} e^{2\phi} b_{1\eta}) + \pi_x x' + \pi_x \bar{x}' + \pi_D \phi' + \sum_{a'=5}^8 \pi_{a'a'} y'^{a'} \right\] \\
= \frac{e^{2\phi}}{2\pi_-} \left\{ [2e^{2\phi} x'^a \bar{x}'^a + \phi'^2 + \sum_{a'=5}^8 (u^{a'})^2 (y'^a)^2 + 8] + [2e^{-2\phi} \pi_x \pi_x + \pi_D^2 + \sum_{a'=5}^8 (u'^a)^2 - 2 \pi_{a'a'}^2] \right\} \\
+ \left\{ (\pi_-(x'^- - \frac{i}{2} (\theta^i \theta'^i + \theta_i \theta') - \frac{i}{4} e^{-2\phi} (\eta^i \eta'^i + \eta_i \eta'^i)) + \pi_x x' + \pi_x \bar{x}' + \pi_D \phi' \right\} \\
+ \sum_{a'=5}^8 \pi_{a'a'} y'^{a'} \right\} - \sum_{a'=5}^8 \pi_{a'a'} \frac{v^{a'}}{u^{a'}} + v^9 \left\{ (\pi_-(x'^- - \frac{i}{2} (\theta^i \theta'^i + \theta_i \theta') - \frac{i}{4} e^{-2\phi} (\eta^i \eta'^i + \eta_i \eta'^i)) \right\} \\
+ \pi_x x' + \pi_x \bar{x}' + \pi_D \phi' + \sum_{a'=5}^8 \pi_{a'a'} y'^{a'} \right\}. \tag{78}
\]

They are functions of coordinates for \( \tilde{L}_k \). Therefore the partial Legendre transformed Lagrangian density is

\[ \tilde{L}_k = \pi_- x'^- + \pi_x x + \pi_x \bar{x} + \pi_D \phi + \sum_{a'=5}^8 \pi_{a'a'} y'^{a'} - \tilde{H} \]
\[ \mathcal{L}_{WZ} = -\frac{e^\phi}{\sqrt{2}} \partial_\mu \partial_\nu x^+ [\bar{\eta} C'_{ij}(\partial_\nu \bar{\theta}^j + i \partial_\nu x \bar{\eta}^j) - \bar{\eta} C'_{ij}(\partial_\nu \bar{\theta}^j + i \partial_\nu x \bar{\eta}^j)] 
\]
\[ = -\frac{e^\phi}{\sqrt{2}} [\bar{\eta} C'_{ij}(\theta^j + i x^j \eta^j) - \bar{\eta} C'_{ij}(\theta^j + i x^j \eta^j)], \]

and the total Lagrangian density

\[ \tilde{\mathcal{L}} = \tilde{\mathcal{L}}_k + \mathcal{L}_{WZ} \]
\[ = \pi_- \dot{x}^- + \pi_x \dot{x} + \pi_\phi \dot{\phi} + \sum_{a'=5}^8 \pi_{a'} \dot{y}_{a'}^a 
- \pi_- \frac{i}{2} (\bar{\theta}^i \dot{\theta}_i + \theta_i \dot{\bar{\theta}}^i) + \frac{i}{4} e^{-2\phi}(\eta^i \dot{\eta}_i + \eta_i \dot{\eta}^i) - \mathcal{H} \]
\[ = f_i \dot{z}^i + f_\alpha \dot{z}^\alpha - \mathcal{H}. \]

It satisfies \( \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\pi}} = 0 \) and is linear in velocities. In addition to that, the locality condition \( \mathcal{A}_{ij} = 0 \) is satisfied.

The WZ term

\[ \mathcal{L}_{WZ} = -\frac{e^\phi}{\sqrt{2}} \partial_\mu \partial_\nu x^+ [\bar{\eta} C'_{ij}(\partial_\nu \bar{\theta}^j + i \partial_\nu x \bar{\eta}^j) - \bar{\eta} C'_{ij}(\partial_\nu \bar{\theta}^j + i \partial_\nu x \bar{\eta}^j)] \]

and the total Lagrangian density

\[ \tilde{\mathcal{L}} = \tilde{\mathcal{L}}_k + \mathcal{L}_{WZ} \]
\[ = \pi_- \dot{x}^- + \pi_x \dot{x} + \pi_\phi \dot{\phi} + \sum_{a'=5}^8 \pi_{a'} \dot{y}_{a'}^a 
- \pi_- \frac{i}{2} (\bar{\theta}^i \dot{\theta}_i + \theta_i \dot{\bar{\theta}}^i) + \frac{i}{4} e^{-2\phi}(\eta^i \dot{\eta}_i + \eta_i \dot{\eta}^i) - \mathcal{H} \]
\[ = f_i \dot{z}^i + f_\alpha \dot{z}^\alpha - \mathcal{H}. \]

We have the Hamiltonian density

\[ \mathcal{H} = \tilde{\mathcal{H}} - \pi_- [\frac{i}{2} (\bar{\theta}^i \dot{\theta}_i + \theta_i \dot{\bar{\theta}}^i) + \frac{i}{4} e^{-2\phi}(\eta^i \dot{\eta}_i + \eta_i \dot{\eta}^i)] - \mathcal{L}_{WZ} \]
\[ = -\dot{x}_{\alpha}^a J_a - \pi_- [\frac{i}{2} (\bar{\theta}^i \dot{\theta}_i + \theta_i \dot{\bar{\theta}}^i) + \frac{i}{4} e^{-2\phi}(\eta^i \dot{\eta}_i + \eta_i \dot{\eta}^i)] - \mathcal{L}_{WZ} \]
\[-J_+ e^\phi - J_- [e^\phi b_0 \theta - \frac{1}{2} e^{-\phi} b_0 \eta + \frac{1}{4} (\eta^2)^2] - \sum_{\alpha'=5}^8 J_{\alpha'} v^{\alpha'} - J_9 v^9 \]
\[-\pi_- [\frac{i}{2} (\theta^i \dot{\theta}^i + \theta_{\alpha} \dot{\theta}_{\alpha}) + \frac{i}{4} e^{-2\phi} (\eta^i \dot{\eta}^i + \eta_{\eta} \dot{\eta}_{\eta})] - L_{W Z} \]
\[= -\frac{\partial L_k}{\partial \dot{x}^k} - \frac{\partial L_{W Z}}{\partial \dot{x}^k} = -\pi_+. \quad (81)\]

**B Derivation for poisson bracket in field theory**

We first define \( \frac{\delta}{\delta z_i(\sigma)} L[z] \). Let \( z_i(\sigma) \) be a function of \( \sigma \), \( \tilde{z}_i(\sigma) \) be another function of \( \sigma \), where
\[
\Delta_\epsilon z_i(\sigma) = \tilde{z}_i(\sigma) - z_i(\sigma) = \delta_\epsilon (\sigma - \sigma') \eta_i(\sigma'), \quad (82)
\]
with \( \int d\sigma \delta_\epsilon (\sigma - \sigma') = 1 \), \( \delta_\epsilon (x) = \delta_\epsilon (-x) \), \( \delta_\epsilon (x + L) = \delta_\epsilon (x) \), \( \frac{\partial \eta_i}{\partial \sigma \eta_i} \delta_\epsilon (\sigma - \sigma') = \delta_\epsilon^{(n)} (\sigma - \sigma') \) exist. This is a smoothed change of \( z_i(\sigma) \) centered at \( \sigma' \).

One has
\[
\lim_{\epsilon \to 0} \int d\sigma f(\sigma) \delta_\epsilon (\sigma - \sigma') = f(\sigma'),
\]
for continuous \( f(\sigma) \).

In (82), \( \eta_i(\sigma') \) is an infinitesimal Grassmann number or an ordinary infinitesimal number. Define
\[
\frac{\delta}{\delta z_i(\sigma')} L = \frac{1}{\eta_i(\sigma')} \{ L[\tilde{z}] - L[z] \}, \quad \frac{\delta}{\delta z_i(\sigma')} L = \lim_{\epsilon \to 0} \frac{\delta}{\delta \epsilon z_i(\sigma')} L.
\]

We then have
\[
\frac{\delta}{\delta z_j(\sigma')} \frac{\delta}{\delta z_i(\sigma)} = (-1)^{ij} \frac{\delta}{\delta z_i(\sigma)} \frac{\delta}{\delta z_j(\sigma')}, \quad \frac{\delta}{\delta z_i(\sigma)} (AB) = (\frac{\delta A}{\delta z_i(\sigma)}) B + (-1)^{ia} A (\frac{\delta B}{\delta z_i(\sigma)}),
\]
\[
\frac{\delta}{\delta z_i(\sigma)} (A + B) = \frac{\delta A}{\delta z_i(\sigma)} + \frac{\delta B}{\delta z_i(\sigma)}. \quad (83)
\]

where the indices \( \hat{i}, \hat{j}, \hat{a} \) denote the Grassmann indices of \( z_i(\sigma), z_j(\sigma') \) and \( A \) respectively.

In text and in this appendix, we use \((-1)^{ia}\) instead of \((-1)^{i\hat{a}}\) if no confusion.
For an action

\[ S = \int d\sigma d\tau L(z, z', \dot{z}, \sigma, \tau), \]

the field equation is

\[ \delta S = 0 = \int d\sigma d\tau \delta z_i \left\{ \frac{\partial L}{\partial z_i} - \partial_\tau \left( \frac{\partial L}{\partial \dot{z}_i} \right) - \partial_\sigma \left( \frac{\partial L}{\partial z'_i} \right) \right\} + \text{surface terms}, \]

giving

\[ \frac{\partial L}{\partial z_i(\sigma)} - \partial_\tau \left( \frac{\partial L}{\partial \dot{z}_i(\sigma)} \right) - \partial_\sigma \left( \frac{\partial L}{\partial z'_i(\sigma)} \right) = 0. \]

We can alternatively write it as

\[ \frac{\delta L}{\delta z_i(\sigma)} - \partial_\tau \left( \frac{\delta L}{\delta \dot{z}_i(\sigma)} \right) = 0, \]

if no boundary term.

For

\[ L(z_i, z'_i, \dot{z}, \sigma, \tau) = \int d\sigma f_i(z(\sigma), z'(\sigma)) \dot{z}_i(\sigma) - \int d\sigma g(z(\sigma), z'(\sigma)), \]

the Lagrangian equation is

\[ \left[ \frac{\partial f_j}{\partial z_i} - (-1)^{i+j+ij} \frac{\partial f_i}{\partial z'_j} \right] \dot{z}_j + \left[ -\frac{\partial f_j}{\partial z'_i} - (-1)^{i+j+ij} \frac{\partial f_i}{\partial z'_j} \right] \dot{z}'_j = \frac{\partial g}{\partial z_i} + \left( \frac{\partial g}{\partial z'_i} \right)'. \]

With

\[ \tilde{\omega}_{ij} = \frac{\partial f_j}{\partial z_i} - (-1)^{i+j+ij} \frac{\partial f_i}{\partial z'_j}, \]
\[ A_{ij} = -\frac{\partial f_j}{\partial z'_i} - (-1)^{i+j+ij} \frac{\partial f_i}{\partial z'_j}, \]

we can write it as

\[ \tilde{\omega}_{ij} \dot{z}_j + A_{ij} \dot{z}'_j = \frac{\partial g}{\partial z_i} - \left( \frac{\partial g}{\partial z'_i} \right)'. \]

Define

\[ \tilde{\omega}_{i(\sigma), j(\sigma')} = \frac{\delta f_j(z(\sigma'), z'(\sigma')}{\delta z_i(\sigma)} - (-1)^{i+j+ij} \frac{\delta f_i(z(\sigma), z'(\sigma))}{\delta z'_j(\sigma')}, \]

one has

\[ J_0 = \frac{\delta}{\delta z_k(\sigma_k)} \tilde{\omega}_{i(\sigma), j(\sigma)} (-1)^{i+j+k} + \text{cyc}(i, j, k) = 0. \]

We have
\[
\int d\sigma' \dot{\omega}_{i(\sigma'),j(\sigma')} \dot{z}_j(\sigma') \\
= \int d\sigma' [\delta(\sigma' - \sigma) \frac{\partial f_i}{\partial z_j}(\sigma') + \delta'(\sigma' - \sigma) \frac{\partial f_j}{\partial z_i}(\sigma')] \dot{z}_j(\sigma') \\
- \int d\sigma'(-1)^{i+j+ij}[\delta(\sigma - \sigma') \frac{\partial f_i}{\partial z_j}(\sigma) + \delta'(\sigma - \sigma') \frac{\partial f_j}{\partial z_i}(\sigma)] \dot{z}_j(\sigma') \\
= \int d\sigma' [\delta(\sigma - \sigma) \frac{\partial f_i}{\partial z_j}(\sigma) - (-1)^{i+j+ij}\delta(\sigma - \sigma') \frac{\partial f_j}{\partial z_i}(\sigma)] \dot{z}_j(\sigma') \\
+ \int d\sigma' [-\delta(\sigma - \sigma) \frac{\partial f_i}{\partial z_j}(\sigma) - (-1)^{i+j+ij}\delta(\sigma - \sigma') \frac{\partial f_j}{\partial z_i}(\sigma)] \dot{z}_j(\sigma'),
\]
giving
\[
\lim_{\epsilon \to 0} \int d\sigma' \dot{\omega}_{i(\sigma),j(\sigma')} \dot{z}_j(\sigma') = \frac{\partial g}{\partial z_i} - (\frac{\partial g}{\partial z_i'})'.
\] (84)

When \(A_{ij} = 0\), let the inverse matrix of \(\tilde{\omega}\) be \(\tilde{\Omega}\),

\[
\tilde{\omega}_{ij}(\sigma) \tilde{\Omega}_{jk}(\sigma) = \delta_{ik}, \tilde{\Omega}_{ij}(\sigma) \tilde{\omega}_{jk}(\sigma) = \delta_{ik}.
\]

One can prove
\[
\frac{\delta}{\delta \epsilon \omega_i(\sigma)} \tilde{\Omega}_{ij}(\sigma) = \tilde{\Omega}_{il}(\sigma) \frac{\delta}{\delta \epsilon \omega_j(\sigma)} \tilde{\omega}_{lm}(\sigma) \tilde{\Omega}_{mj}(\sigma)(-1)^{(i+l)k}.
\] (85)

Field equation (84) can be written as
\[
\tilde{\omega}_{ij}(\sigma) \dot{z}_j(\sigma) = \frac{\partial}{\partial \epsilon \omega_i(\sigma)} \lim_{\epsilon \to 0} \int d\sigma' \frac{\delta}{\delta \epsilon \omega_j(\sigma')} g(\sigma'),
\]
and
\[
\dot{z}_l(\sigma) = \lim_{\epsilon \to 0} \int d\sigma' \dot{\Omega}_{li}(\sigma) \frac{\delta}{\delta \epsilon \omega_i(\sigma)} g(\sigma').
\] (86)

Denote
\[
\dot{\Omega}_{li(\sigma),j(\sigma)} = \dot{\Omega}_{ij}(\sigma) \delta_l(\sigma - \sigma'),
\]
and define the poisson bracket of functionals \(A\) and \(B\) as
\[
\{A, B\} = \lim_{\epsilon \to 0} \int d\sigma \int d\sigma' A \frac{\delta}{\delta \epsilon \omega_i(\sigma)} \dot{\Omega}_{li(\sigma),j(\sigma')} \frac{\delta}{\delta \epsilon \omega_j(\sigma')} B.
\]
Equations (86) is then
\[
\dot{z}_i(\sigma) = \{z_i(\sigma), H\},
\]
where $H = \int d\sigma g(z(\sigma), z'(\sigma))$.

We next study the key property of poisson bracket, the Jacobi identity. In contrast with $J_0$,

\[
J_1 \equiv \frac{\delta}{\delta z_k(\sigma_k)} \left[ (-1)^{i+j+k} \hat{\omega}_{ij}(\sigma_i) \delta_i(\sigma_i - \sigma_j) \right] + \text{cyc}(i,j,k),
\]

it is not identically zero however. The difference

\[
\frac{\delta}{\delta z_k(\sigma_k)} \hat{\omega}_{ij}(\sigma, j'(\sigma')) - \frac{\delta}{\delta z_k(\sigma_k)} \left[ \hat{\omega}_{ij}(\sigma_i) \delta_i(\sigma_i - \sigma_j) \right]
= -\delta_i(\sigma_i - \sigma_j) [g^j_k(\sigma_i) - g^j_k(\sigma_j)] \\
+ \delta_i(\sigma_i - \sigma_j) [G^i_{kj}(\sigma_i) - G^i_{kj}(\sigma_j)] \\
+ \delta_i(\sigma_i - \sigma_j) \frac{\partial}{\partial \sigma_i} G^i_{kj}(\sigma_i)
\]

\[
\equiv \Psi_{ijk}(\epsilon, \sigma_i, \sigma_j, \sigma_k),
\]

where

\[
g^j_k(\sigma) = \delta(\sigma - \sigma_k) \frac{\partial^2 f_j}{\partial z_k \partial z_i}(\sigma) + \delta(\sigma - \sigma_k) \frac{\partial^2 f_j}{\partial \sigma_i \partial z_i}(\sigma),
\]

\[
G^i_{kj}(\sigma) = \delta(\sigma - \sigma_k) \frac{\partial^2 f_j}{\partial z_k \partial z_i}(\sigma) + \delta(\sigma - \sigma_k) \frac{\partial^2 f_j}{\partial \sigma_i \partial z_i}(\sigma),
\]

has the property

\[
\lim_{\epsilon \to 0} \int d\sigma_i d\sigma_j d\sigma_k F_i(\epsilon, \sigma_i) F_j(\epsilon, \sigma_j) F_k(\epsilon, \sigma_k) \Psi_{ijk}(\epsilon, \sigma_i, \sigma_j, \sigma_k) = 0,
\]

for smooth periodic functions $F_i, F_j, F_k$ with

\[
\lim_{\epsilon \to 0} F_l^{(n)}(\epsilon, \sigma) = F_l^{(n)}(\sigma), l = i, j, k.
\]

one has

\[
\lim_{\epsilon \to 0} \int d\sigma_i d\sigma_j d\sigma_k \prod_{l=i,j,k} F_l(\epsilon, \sigma_l) J_1 = 0.
\]

Consider

\[
J(A, B, C)
= (-1)^{ac} \{ A, \{ B, C \} \} + (-1)^{ab} \{ B, \{ C, A \} \} + (-1)^{bc} \{ C, \{ A, B \} \}
= \lim_{\epsilon \to 0} \int d\sigma_j d\sigma_k (-1)^{ac} (-1)^{jb-1} \{ A, \frac{\delta B}{\delta z_j(\sigma)} \hat{\omega}_{jk}(\sigma_j) \delta_i(\sigma_j - \sigma_k) \frac{\delta C}{\delta z_k(\sigma_k)} \} + \text{cyc}(A, B, C).
\]

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\[
= \lim_{\epsilon' \to 0} \int d\sigma_i d\sigma_j \frac{\delta A}{\delta \epsilon' z_1(\sigma_i)} \bar{\Omega}_{il}(\sigma_i) \delta_{\epsilon'}(\sigma_i - \sigma_l) \frac{\delta}{\delta \epsilon' z_1(\sigma_l)} \\
\times \lim_{\epsilon \to 0} \int d\sigma_j d\sigma_k \frac{\delta B}{\delta \epsilon' z_2(\sigma_j)} \bar{\Omega}_{jk}(\sigma_j) \delta_{\epsilon'}(\sigma_j - \sigma_k) \frac{\delta C}{\delta \epsilon' z_2(\sigma_k)} \end{array} \right] (-1)^{\alpha + j(b-1) + i(a-1)} + \text{cyc}(A, B, C) \\
= \lim_{\epsilon' \to 0, \epsilon \to 0} \int d\sigma_j d\sigma_k \frac{\delta^2 B}{\delta \epsilon' z_2(\sigma_j) \delta \epsilon' z_2(\sigma_k)} \bar{\Omega}_{jk}(\sigma_j) \delta_{\epsilon'}(\sigma_j - \sigma_k) \frac{\delta C}{\delta \epsilon' z_2(\sigma_k)} \\
\left\{ (-1)^{\alpha c + j(b-1) + i(a-1)} \left[ \frac{\delta A}{\delta \epsilon' z_1(\sigma_i)} \bar{\Omega}_{il}(\sigma_i) \delta_{\epsilon'}(\sigma_i - \sigma_l) \\
\frac{\delta^2 B}{\delta \epsilon' z_1(\sigma_i) \delta \epsilon' z_1(\sigma_l)} \bar{\Omega}_{il}(\sigma_i) \delta_{\epsilon'}(\sigma_i - \sigma_l) \right] \right\} \\
\left\{ (-1)^{\alpha c + j(b-1) + i(a-1)} \frac{\delta A}{\delta \epsilon' z_1(\sigma_i)} \bar{\Omega}_{il}(\sigma_i) \delta_{\epsilon'}(\sigma_i - \sigma_l) \right\} \\
+ \text{cyc}(A, B, C) \equiv \alpha + \beta + \gamma. \tag{89} \]

This is a two fold limit. We first prove that the double limit \(\lim_{\epsilon' \to 0, \epsilon \to 0}\) exists and which does not depend on the manner of \(\epsilon, \epsilon' \to 0\).

For this purpose, we use integration by parts to change all \(\frac{\partial}{\partial \sigma} \delta_{\epsilon, \epsilon'}(\sigma - \sigma')\) to \(-\delta_{\epsilon, \epsilon'}(\sigma - \sigma') \frac{\partial}{\partial \sigma}\), in a properly chosen route. Eventually we have an expression with only \(\delta_{\epsilon, \epsilon'}\) functions and local functions in the integration. The boundary terms in integration by parts disappear because of the periodicity of local functions and the property of the conserved quantities.

Then one sees that the double limit \(\lim_{\epsilon' \to 0, \epsilon \to 0}\) is well-defined, it is irrelevant of the manner \(\epsilon, \epsilon' \to 0\). We have \(\lim_{\epsilon, \epsilon' \to 0} \lim\{\cdots\} = \lim_{\epsilon' \to 0} \lim_{\epsilon \to 0} \{\cdots\} = \lim_{\epsilon = \epsilon' \to 0} \{\cdots\},\) if the first limit \(\lim_{\epsilon \to 0} \{\cdots\}\) exists (which is easy to check), due to the multi limit theorem.

Since \(\delta_{\epsilon}\) function is regular and \(\frac{\partial}{\delta\epsilon}\) operation is also regular, thus before limit, we can check the validity of \([83]\) and treat these terms just as in the finite dimensional mechanics.

We then check \(\alpha + \beta + \gamma\) in \([83]\), and find when \(\epsilon = \epsilon'\), \(\alpha + \beta = 0\) because of the cyclic permutation. The term \(\lim_{\epsilon \to 0} \beta\) can be converted to an integration of the form as l.h.s. of \([88]\) by \([85]\), and it is also zero due to the equation \([88]\).
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