Regularity and strict positivity of densities for the stochastic heat equation on $\mathbb{R}^d$

Le Chen and Jingyu Huang

Friday 8th February, 2019

Abstract

In this paper, we study the stochastic heat equation with a general multiplicative Gaussian noise that is white in time and colored in space. Both regularity and strict positivity of the densities of the solution have been established. The difficulty, and hence the contribution, of the paper lie in three aspects, which include rough initial conditions, degenerate diffusion coefficient, and weakest possible assumptions on the correlation function of the noise. In particular, our results cover the parabolic Anderson model starting from a Dirac delta initial measure.

Keywords: Stochastic heat equation, parabolic Anderson model, Malliavin calculus, negative moments, regularity of density, strict positivity of density, measure-valued initial conditions, spatially colored noise.

AMS 2010 subject classification. Primary 60H15; Secondary 35R60, 60G60.

Contents

1 Introduction 2

1.1 Regularity of density ........................................ 4
1.2 Strict positivity of density ................................. 7
1.3 Outline of the paper and some notation .................... 9

2 Some preliminaries 10

2.1 Definition and existence of a solution .................... 10
2.2 Some auxiliary functions ................................. 12
2.3 Malliavin calculus ........................................ 13
2.4 Sufficient conditions for strict positivity of density .... 15

3 Regularity of densities 16

3.1 Nonnegative moments (Proof of Theorem 1.8) ............ 16
3.2 Malliavin derivatives of $u(t, x)$ ........................ 20
3.3 Density at a single point (Proof of Theorem 3.3) ......... 27
3.4 Assumption 1.10 (Properties of $k(t)$) .................... 33
3.5 Density at multiple points (Proof of Theorem 1.4) ......... 37
4 Strict positivity of density

4.1 Proof of Theorem 1.12 .................................................. 44
4.2 Properties of the function Ψ_n(t, x; ℓ) .............................. 47
4.2.1 Proof of part (1) of Proposition 4.1 .............................. 49
4.2.2 Proof of part (2) of Proposition 4.1 .............................. 54
4.2.3 Proof of part (3) of Proposition 4.1 .............................. 58
4.2.4 Proof of Lemma 4.2 .................................................. 61
4.3 Moments of ̂u_n^z(t, x) and its first two derivatives ........... 63
4.3.1 Moments of ̂u_n^z(t, x) .............................................. 64
4.3.2 Moments of ̂u_n,i^z(t, x) .............................................. 67
4.3.3 Moments of ̂u_n,i,k^z(t, x) .......................................... 68
4.3.4 Moment increments in z ............................................. 70
4.4 Almost convergence of ̂u_n^0(T, x_i) to ρ(u(T, x_i)) ............. 76
4.5 Conditional boundedness .............................................. 80
4.5.1 Spatial Hölder continuity of ̂u_n^z(t, x) and its two derivatives 80
4.5.2 Some Grownwall-type inequalities .............................. 83
4.5.3 Proof of the conditionally boundedness (Proposition 4.14) .... 85

Bibliography 92

1 Introduction

In this paper, we study both the regularity and strict positivity of the density to the following stochastic heat equation (SHE) with rough initial conditions:

\begin{align}
\begin{cases}
\left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right) u(t, x) = \rho(t, x, u(t, x)) \dot{W}(t, x), & x \in \mathbb{R}^d, \ t > 0, \\
u(0, \cdot) = \mu(\cdot),
\end{cases}
\end{align}

where d ≥ 1 and \( \Delta = \sum_{i=1}^{d} \partial^2 / \partial x_i^2 \) is the Laplacian operator. The noise \( \dot{W} \) is a centered Gaussian noise that is white in time and homogeneously colored/correlated in space. Informally, \[ \mathbb{E} \left[ \dot{W}(t, x) \dot{W}(s, y) \right] = \delta_0(t - s) f(x - y), \] where \( \delta_0 \) is the Dirac delta measure with unit mass at zero and \( f \) is a “correlation function”, that is, a nonnegative and nonnegative definite function that is not identically equal to zero. The Fourier transform of \( f \) is denoted by \( \hat{f} \)

\[ \hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}^d} \exp(-i \xi \cdot x) f(x) dx, \]

which is again a nonnegative and nonnegative definite measure. Here, \( \hat{f} \) is usually called the spectral measure. It is well known that the minimum assumption on the correlation function \( f \) is Dalang’s condition \[ \[12\], namely,

\[ \Upsilon(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{\beta + |\xi|^2} < +\infty \quad \text{for some and hence for all } \beta > 0. \]
Since we will need some regularity of the solution, we will work under the following slightly stronger condition, which will also be called Dalang’s condition, than condition (1.2) as in [10]:

\[ \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{(1 + |\xi|^2)^{1-\alpha}} < +\infty, \quad \text{for some } \alpha \in (0, 1]. \]  

(1.3)

One aim of this paper is to work under the weakest possible assumptions on \( f \) but still under Dalang’s condition (1.3). In particular, we don’t assume any scaling properties on \( f \).

The nonlinear coefficient \( \rho(t, x, z) \) is assumed to be a continuous function which is differentiable in the third argument with a bounded derivative. In particular, our results below will cover the important linear case \( \rho(t, x, u) = \lambda u \), which is called the parabolic Anderson model (PAM) [7].

The precise meaning of the “rough initial conditions/data” are specified as follows. We first note that by the Jordan decomposition, any signed Borel measure \( \mu \) can be decomposed as \( \mu = \mu_+ - \mu_- \) where \( \mu_\pm \) are two non-negative Borel measures with disjoint support. Denote \( |\mu| := \mu_+ + \mu_- \). The rough initial data refers to any signed (Borel) measure \( \mu \) such that

\[ \int_{\mathbb{R}^d} e^{-a|x|^2} |\mu|(dx) < +\infty, \quad \text{for all } a > 0. \]  

(1.4)

Let \( J_0(t, x) \) be the solution to the homogeneous equation, that is,

\[ J_0(t, x) = (\mu \ast G(t, \cdot))(x) := \int_{\mathbb{R}^d} G(t, x - y)\mu(dy), \]  

(1.5)

where \( G(t, x) \) is the heat kernel

\[ G(t, x) := (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right). \]

It is easy to see that condition (1.4) is equivalent to, in case \( \mu \geq 0 \), the condition that the solution to the homogeneous equation \( J_0(t, x) \) exists for all \( t > 0 \) and \( x \in \mathbb{R}^d \). It is better to keep in mind the following examples of rough initial conditions:

\[ \mu(dx) = \exp\left(|x|^{7/4}\right) dx, \quad \mu = \delta_0 \quad \text{and} \quad \mu = \sum_{n \in \mathbb{Z}^d} \exp\left(\sqrt{|n|}\right) \delta_n, \]  

(1.6)

where \( \delta_{x_0} \) is the Dirac delta measure with unit mass at \( x = x_0 \). Such rough initial conditions are important; see, e.g., Amir, Corwin and Quastel [1] and Bertini and Giacomin [4] where the Dirac delta initial data and the exponential of two sided Brownian motion, respectively, are their crucial choices for the initial conditions.

The aim of this paper is to establish the regularity and strict positivity of the joint density of \((u(t, x_1), \cdots, u(t, x_m))\) for degenerate diffusion coefficient \( \rho \), starting from rough initial data, and under the weakest possible assumptions on the correlation function \( f \). In particular, due to the importance of the PAM, especially its relation with the stochastic Burgers and the Kardar-Parisi-Zhang (KPZ) equation through the Hopf-Cole transform, all our main results in this paper, namely, Theorems 1.1, 1.4, 1.8 and 1.12, apply to the PAM with rough initial data. The space time white noise case has been recently studied in [9]. We should emphasize that working on \( \mathbb{R}^d \) with noise that is white in time and colored in space brings many more challenges that one does not have in the space-time white noise case. We will elaborate more on these difficulties in the next two subsections.

Now we are ready to state our main results.
1.1 Regularity of density

The first set of results of this paper concern the regularity of the density. Theorem 1.1 below gives necessary and sufficient conditions for $u(t, x)$ to have a smooth density, but this result is only for a single space-time point. By one additional assumption on $f$ (Assumption 1.3) and by imposing a mild cone condition on the diffusion coefficient $\rho$, Theorem 1.4 below gives sufficient conditions for the existence of smooth density at multiple points. These results generalize corresponding results in [9] for space-time white noise case (hence $d = 1$).

**Theorem 1.1.** Suppose that $\rho : [0, \infty) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is continuous and $f$ satisfies condition (1.3) for some $\alpha \in (0, 1]$. Let $u(t, x)$ be the solution to (1.1) starting from an initial measure $\mu$ that satisfies (1.4). Then the following two statements are true:

(a) If $\rho$ is differentiable in the third argument with a bounded Lipschitz continuous derivative, then for all $t > 0$ and $x \in \mathbb{R}^d$, $u(t, x)$ has an absolutely continuous law with respect to the Lebesgue measure on $\mathbb{R}$ if and only if

$$t > t_0 := \inf \left\{ s > 0, \sup_{y \in \mathbb{R}^d} |\rho(s, y, (G(s, \cdot) * \mu)(y))| \neq 0 \right\}. \quad (1.7)$$

(b) If $\rho$ is infinitely differentiable in the third argument with bounded derivatives of all orders, then for all $t > 0$ and $x \in \mathbb{R}^d$, $u(t, x)$ has a smooth density if and only if condition (1.7) holds.

This theorem will be proved in Section 3.3.

**Example 1.2.** Here we would like to point out several examples on $\rho$:

(1) Under the degenerate condition (see (1.11) below), it is clear from (1.7) that $t_0 = 0$.

(2) For the PAM, that is, $\rho(t, x, z) = \lambda z$, $\lambda \neq 0$, with delta initial data $\mu = \delta_0$, then $t_0 = 0$ because

$$\sup_{y \in \mathbb{R}^d} |\rho(s, y, (G(s, \cdot) * \delta_0)(y))| = |\lambda| \sup_{y \in \mathbb{R}^d} |G(s, y)| = \frac{|\lambda|}{(2\pi s)^{d/2}} > 0.$$

(3) Let $\rho(t, x, z) = \exp \left(\left((4z - 1)(4z - 3)\right)^{-1}\right) \mathbb{I}_{[1/4,3/4]}(z)$ and $\mu(dx) = \mathbb{I}_{[-1,1]}(x)dx$. Clearly, $\rho$ is a smooth function in the third argument with compact support. Moreover, one can check from (1.7) that $t_0 = 0$. Therefore, there is a smooth density for all time $t > 0$. However, Mueller and Nualart’s result [20] fails in this example since $\rho(\mathbb{I}_{[-1,1]}(x)) = 0$ for all $x \in \mathbb{R}^d$.

(4) If $\rho(t, x, z) = \mathbb{I}_{[\tau, \tau]}(t)z$ and $\mu(dx) = dx$ with two deterministic constants $\tau' > \tau > 0$, then $u(t, x)$ has a smooth density if only if $t > t_0 = \tau$.

For the regularity of density at multiple points and also for the strict positivity of density (Theorem 1.12 below), we need some assumptions on the correlation function $f$ as follows, which are satisfied by most common-seen examples (see Example 1.6 below).

**Assumption 1.3.** Assume that for some $A > 1$ large enough, the correlation function satisfies that

$$\sup_{|z| \geq A} f(z) < \infty \quad \text{and} \quad \inf_{|z| < 1/A} f(z) > 0.$$
Theorem 1.4. Suppose that the condition (1.3) is satisfied for some $\alpha \in (0, 1]$. Let $u(t, x)$ be the solution to (1.1) starting from a nonnegative measure $\mu > 0$ that satisfies (1.4). Suppose that for some constants $\beta > 0$, $\gamma \in (0, 1 + \alpha)$ and $l_\rho > 0$,

$$|\rho(t, x, z)| \geq l_\rho \exp \left\{ -\beta \left[ \log \frac{1}{|z|^{\gamma}} \right] \right\}, \quad \text{for all } (t, x, z) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}. \quad (1.8)$$

Then for any $m$ distinct points $\{x_1, \ldots, x_m\} \subseteq \mathbb{R}^d$ and $t > 0$, under Assumption 1.3, the following two statements are true:

(a) If $\rho$ is differentiable in the third argument with a bounded Lipschitz continuous derivative, then the law of the random vector $(u(t, x_1), \ldots, u(t, x_m))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$.

(b) If $\rho$ is infinitely differentiable in the third argument with bounded derivatives of all orders, then the random vector $(u(t, x_1), \ldots, u(t, x_m))$ has a smooth density on $\mathbb{R}^m$.

This theorem will be proved in Section 3.5.

Example 1.5. Let us see several examples on the cone condition (1.8):

1. The degenerate condition (see (1.11) below) trivially ensures the cone condition (1.8).
2. For the PAM, that is, $\rho(t, x, z) = \lambda z$, $\lambda \neq 0$, then the cone condition (1.8) is satisfied with $\gamma = \beta = 1$ and $l_\rho = |\lambda|$.
3. Let $\rho(t, x, z) = z^{2n+1}/(1+z^{2n})$, $n \in \mathbb{N}$. This $\rho$ has linear growth for large $x$ and approaches zero as $x$ tends to zero in a polynomial rate. Hence, the cone condition (1.8) is also satisfied.

Example 1.6. We remark that all the commonly-seen correlation functions $f$ satisfy Assumption 1.3. Here are some examples:

(1) For the Riesz kernel $f(x) = |x|^{-\beta'}$ with $\beta' \in (0, 2 \wedge d)$.
(2) Let $f(x)$ be the Bessel kernel of order $\alpha' > 0$ (see, e.g., Section 6.1 of [16])

$$f(x) = \int_0^\infty w^{\alpha'-\frac{d-2}{2}} e^{-w} e^{-\frac{|x|^2}{4w}} \, dw. \quad (1.9)$$

Note that $f$ does not satisfy any scaling property. Dalang’s condition (1.2)/(1.3) requires that $\alpha' > d - 2$; see Proposition 3.7. Assumption 1.3 is trivially satisfied.

(3) For the fractional noise $f(x) = \prod_{j=1}^d |x_j|^{2H_j-2}$ with $H_j \in (1/2, 1)$ and $d - \sum_{j=1}^d H_j < 1$, Assumption 1.3 is not satisfied (the supremum outside a ball can be infinity). One can see that coordinate-wisely Assumption 1.3 is still true and Lemma 3.9 can adapted to the coordinate-wise form easily. Hence, Theorem 1.4 still holds in this case.

Remark 1.7. For the space-time white noise (hence $d = 1$) case, while Assumption 1.3 is not satisfied. Nevertheless, Theorem 1.4 for the space-time white noise case has been proved in [9].
Que-Sardanyons and Sanz-Solé [26] for the stochastic wave equation on $\mathbb{R}^3$ (see Nualart and Que-Sardanyons [22] as well). All these results rely on the property that

$$
\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E} [ |u(t,x)|^p ] < \infty, \quad \text{for any } T > 0,
$$

(1.10)

which doesn’t hold any more for rough initial data. Hence, this quite standard fact — $u(t,x) \in \mathbb{D}^\infty$ — requires a proof in our setting. To prove this property, we need to introduce the Malliavin calculus localized to a space-time subdomain as in [9] in order to avoid possible singularities at $t = 0$ and $x = \infty$; see Section 2.3 and Section 3.2 for more detail.

The second and also the major obstacle is to establish the negative moments of the determinant of the corresponding Malliavin matrix (see Section 2.3). This obstacle can be circumvented by imposing the following nondegenerate condition

$$
\inf_{x \in \mathbb{R}} |\rho(x)| \geq c \quad \text{for some constant } c > 0.
$$

(1.11)

or similar conditions as in [22] and [19]. However, this condition excludes the important case — PAM. An alternative compromise to avoid this issue is to prove a “local” result as those in [3], where the smooth joint density of $(u(t,x_1), \ldots, u(t,x_d))$ is proved over the domain $\{\rho \neq 0\}^d$ instead of $\mathbb{R}^d$. As for the degenerate case, that is, the case that includes the PAM, Pardoux and Zhang [25] showed that the Malliavin matrix is invertible a.s., which enabled them to establish the existence of the density. Much later Mueller and Nualart [20] succeeded in establishing the smooth density. Both [25] and [20] handle the one-dimensional SHE over an interval with space-time white noise. Recently, Chen, Hu and Nualart [9] extended the above results to the one-dimensional SHE over the whole $\mathbb{R}$. In this paper, we carry out this program to extend these results further to SHE on $\mathbb{R}^d$. Indeed, following similar ideas as in [9, 20], we can transform the arguments of the proof of the strict positivity of the solution in [10] into a stopping-time argument in order to obtain a better convergence rate (see (1.14)), which will in turn guarantee the existence of negative moments of all orders for a related SHE over $\mathbb{R}^d$. More precisely, we will establish Theorem 1.8 for the following SHE:

$$
\begin{cases}
\left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right) u(t,x) = H(t,x)\sigma(t,x,u(t,x))\dot{W}(t,x), & t > 0, \ x \in \mathbb{R}^d, \\
u(0,\cdot) = \mu(\cdot),
\end{cases}
$$

(1.12)

where $H(t,x)$ is a bounded and adapted process and $\sigma(t,x,z)$ is a measurable and locally bounded function which is Lipschitz continuous in $z$, uniformly in both $t$ and $x$, satisfying that $\sigma(t,x,0) = 0$.

**Theorem 1.8.** Suppose that condition (1.3) is satisfied for some $\alpha \in (0,1]$. Let $u(t,x)$ be the solution to (1.12) starting from a deterministic and nonnegative measure $\mu > 0$ that satisfies (1.4). Let $\Lambda > 0$ be a constant such that

$$
|H(t,x,\omega)\sigma(t,x,z)| \leq \Lambda |z| \quad \text{for all } (t,x,z,\omega) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \times \Omega.
$$

(1.13)

Then for any compact set $K \subseteq \mathbb{R}^d$ and $t > 0$, there exists a finite constant $B > 0$ which only depends on $\Lambda$, $\delta$, $K$ and $t > 0$ such that for any $\delta \in (0,1)$ and for all small enough $\epsilon > 0$,

$$
\mathbb{P} \left( \inf_{x \in K} u(t,x) < \epsilon \right) \leq \exp \left( -B \left\{ |\log(\epsilon)| \cdot |\log (|\log(\epsilon)|)\right\}^{1+\alpha} \right).
$$

(1.14)
Consequently, for all \( p > 0 \),
\[
\mathbb{E} \left( \inf_{x \in K} u(t, x) \right)^{-p} < +\infty. \tag{1.15}
\]

This theorem will be proved in Section 3.1.

**Remark 1.9.** Finally, we would like to mention a recent work that is partially related to this paper. In [6], Cannizzaro, Friz and Gassiat studied the following generalized PAM (gPAM),
\[
(\partial_t - \Delta) u(t, x) = g(u(t, x))\xi(x), \quad u(0, \cdot) = u_0(\cdot),
\]
where \( \xi = \xi(x, \omega) \) is space (not space-time) white noise, \( x \in \mathbb{T}^2 \) (two-dimensional torus) and the initial data is Hölder continuous. The diffusion coefficient \( g(\cdot) \) is assumed to be sufficiently smooth and to be compactly supported (see Proposition 3.28 [ibid]). This last property excludes the linear PAM. The space dimension 2 is the critical case when one needs to use Hairer’s theory of the regularity structures [17, 18] to handle properly the renormalization procedure. For this path-wise approach, it is natural to obtain some almost-sure results such as the strict positivity of the solution; see Theorem 5.1 [ibid]. Hence by the first part of Bouleau-Hirsch’s criterion (see part (1) of Theorem 2.10 below), they obtain the existence of the density. However, it seems very hard to establish the smooth density in their framework since it requires the \( L^p(\Omega) \)-moments of the solution. Comparing to our results here, we work under the cases when the noises are good enough that there is no need of renormalization in order to make sense of the solution. The difficulties/contributions of this paper lie in degenerate diffusion coefficients \( \rho \), rough initial data \( \mu \), and weakest possible assumptions on the correlation function \( f \) (but still under Dalang’s condition (1.3)). Contrary to [ibid], the moments formula/bounds are the basic tools for us.

### 1.2 Strict positivity of density

As for the strict positivity of the density, most known results assume the boundedness of the diffusion coefficient \( \rho \); see, e.g., Theorem 2.2 of Bally and Pardoux [3], Theorem 4.1 of Hu et al [19] and Theorem 5.1 of E. Nualart [23]. This nondegenerate condition, which excludes the important case: the parabolic Anderson model \( \rho(u) = \lambda u \), has been removed in a recent work by Chen et al [9, Theorem 1.4]. Moreover, the results in [9] allow the rough initial conditions. However, the results in [9] cover only the case of space-time white noise (hence \( d = 1 \)). The goal of the next theorem is to extend Theorem 1.4 of [9] to higher spatial dimensions with more general noises. Recall that Theorem 1.4 is proved under Assumption 1.3 on \( f \). Here we need the following two assumptions on \( f \). Denote
\[
k(t) := \int_{\mathbb{R}^d} f(z) G(t, z) dz = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( -\frac{t|\xi|^2}{2} \right), \tag{1.16}
\]
where the second equality is due to the Plancherel theorem.

**Assumption 1.10.** Assume that for some \( \beta \in (0, 1) \), the limit \( \lim_{t \downarrow 0} t^\beta k(t) \) exists and belongs to \((0, \infty)\), or equivalently, \( k(t) \asymp t^{-\beta} \) as \( t \to 0_+ \) (see the notation at the end of this section).

**Assumption 1.11.** Assume that the correlation function \( f \) is a locally bounded function on \( \mathbb{R}^d \setminus \{0\} \).
Theorem 1.12. Suppose \( \rho(t, x, z) = \rho(z) \) and \( \rho \in C^\infty(\mathbb{R}) \) such that all derivatives of \( \rho \) are bounded. Let \( \{x_1, \ldots, x_m\} \subseteq \mathbb{R}^d \) be \( m \) distinct points. Then under Assumptions 1.10 and 1.11, for any \( t > 0 \), the joint law of the random vector \((u(t, x_1), \ldots, u(t, x_m))\) admits a smooth density \( p(y) \), and \( p(y) > 0 \) if \( y \) belongs both to \( \{\rho \neq 0\}^m \) and to the interior of the support of the law of \((u(t, x_1), \ldots, u(t, x_m))\).

This theorem will be proved in Section 4.

Example 1.13. We remark that all the commonly-seen correlation functions \( f \) satisfy Assumption 1.10 (see Proposition 3.7 below for the proof). Here are some examples:

1. For the space-time white noise case (hence \( d = 1 \), Assumption 1.10 is satisfied with \( \beta = 1/2 \).
2. For the Riesz kernel \( f(x) = |x|^{-\beta'} \) with \( \beta' \in (0, 2 \wedge d) \), Assumption 1.10 is satisfied with \( \beta = \beta'/2 \).
3. For the fractional noise \( f(x) = \prod_{j=1}^d |x_j|^{2H_j-2} \) with \( H_j \in (1/2, 1) \) and \( d - \sum_{j=1}^d H_i < 1 \), Assumption 1.10 is satisfied with \( \beta = d - \sum_{j=1}^d H_j \).
4. When \( f(x) \) is the Bessel kernel (1.9) of order \( \alpha' \in (0 \lor (d - 2), d) \), Assumption 1.10 is satisfied with \( \beta = (d - \alpha')/2 \).

Remark 1.14. Examples (1), (2) and (4) in Example 1.13 clearly satisfy Assumption 1.11. In general, let \( H \subseteq \mathbb{R}^d \) be the set of points where \( f \) fails to be locally bounded. In this case, Theorem 1.12 is still true provided that the \( m \) points \( \{x_1, \ldots, x_m\} \subseteq \mathbb{R}^d \) satisfy the condition that

\[
\bigcup_{i,j=1, \ldots, m, i \neq j} \{x_i - x_j\} \cap H = \emptyset.
\]

In case of examples (1), (2) and (4) in Example 1.13, \( H = \{0\} \). Hence, we only need to require that all \( m \) points are distinct. In case of example (3) in Example 1.13, we have that \( H = \{z \in \mathbb{R}^d, z_i = 0 \text{ for some } i = 1, \ldots, d\} \). Hence, we need to instead require that the projections of the \( m \) points on each coordinate are distinct.

We should emphasize that moving from the space-time white noise to the spatially colored noises of the current paper makes a significant difference in the perturbation strategy. For the space-time white noise case as in [3, 9], the perturbation function takes the following form (see, e.g., Eq. (8.8) of [9])

\[
h^i_n(t, x) = c_n \mathbb{I}_{[T-2^{-n}, T]}/(t) \mathbb{I}_{[x_i-2^{-n}, x_i+2^{-n}]}(x),
\]

where \( c_n \) is some normalization constant. Here, \( x \) lives in a compact set, which could be used to simplify many arguments. However, for the spatially colored noise, one needs to take the following perturbation function (see (4.1.1) below) as in [23],

\[
h^i_n(t, x) = c_n \mathbb{I}_{[T-2^{-n}, T]}(t) G(T - s, x_i - x).
\]

Here, \( x \) no longer lives in a compact set, which makes arguments much more involved than those in [9].

Another complication/difficulty comes from the rough initial data (see examples in (1.6)). Since we need to prove some almost-sure results for some supremums (see (2.4.4)), we will need to show many sharp moments estimates and their increments in order to first apply the Kolmogorov continuity theorem to take care of the supremums and then apply Borel-Cantelli lemma to obtain path-wise results. If one assumes that initial data to be bounded, then property (1.10) holds, with which all these arguments can be significantly simplified.
Remark 1.15. Finally, let us point out some subtleties on the assumptions on the correlation functions $f$ among the three main theorems of this paper, namely, Theorems 1.1, 1.4 and 1.12. Theorem 1.1 makes the weakest assumption on the correlation function $f$, namely, Dalang’s condition (1.3). All examples on $f$ in Examples 1.6 and 1.13 work for Theorem 1.1. On the other hand, one can construct examples as follows that work for Theorem 1.1 but not for either Theorem 1.4 or Theorem 1.12:

(1) For $d = 1$ and for any $a > 0$ and $c \in [0, 1/2]$, one can show that

$$f(x) = \delta_0(x) + c(\delta_{a}(x) + \delta_{-a}(x))$$

is nonnegative and nonnegative definite since $\hat{f}(\xi) = 1 + 2c \cos(a \xi) \geq 0$. More generally, one can have

$$f(x) = \delta_0(x) + \sum_{i=1}^{\infty} c_i(\delta_{a_i}(x) + \delta_{-a_i}(x))$$

with $a_i > 0$ and $c_i > 0$ such that $\sum_{i=1}^{\infty} c_i \leq 1/2$. In this case, $\hat{f}(\xi) = 1 + 2\sum_{i=1}^{\infty} c_i \cos(a_i \xi)$.

(2) Similarly, in any spatial dimensions $d \geq 1$, one can construct the following example: Let $\beta \in (0, d \land 2)$ and

$$f(x) = |x|^{-\beta} + \sum_{i=1}^{\infty} c_i(|x - a_i|^{-\beta} + |x + a_i|^{-\beta})$$

where $a_i \in \mathbb{R}^d \setminus \{0\}$ and $c_i > 0$ such that $\sum_{i=1}^{\infty} c_i \leq 1/2$. In this case,

$$\hat{f}(\xi) = C_{\beta, d} |\xi|^{\beta - d} \left(1 + 2\sum_{i=1}^{\infty} c_i \cos(a_i \cdot \xi)\right) \geq 0.$$

1.3 Outline of the paper and some notation

The paper is organized as follows. In Section 2, we give some preliminaries over the definition of the solution and some known results about the solution that will be used in this paper (Section 2.1), some auxiliary functions (Section 2.2), Malliavin calculus and its localized version (Section 2.3), and a criterion for the strict positivity of density (Section 2.4).

The two regularity results (Theorems 1.1 and 1.4) are proved in Section 3. In particular, we first prove the existence of negative moments of all orders (Theorem 1.8) in Section 3.1. In Section 3.2, we study the Malliavin derivatives of $u(t, x)$ in the localized Sobolev spaces. The existence and smoothness of density at a single space-time point (Theorem 1.1) and at multiple points (Theorem 1.4) are proved in Sections 3.3 and 3.5, respectively.

The strict positivity of the density (Theorem 1.12) is proved in Section 4. The proof of Theorem 1.12 is outlined in Section 4.1, which is essentially an application of Theorem 2.13 in Section 2.4. The rest parts, namely, subsections from 4.2 to 4.5, are devoted to prove all properties that are needed in the proof of Theorem 1.12 in Section 4.1.

Throughout this paper, we use $C$ to denote a generic constant whose value may vary at different occurrences. The function $\rho(u(t, x))$ should be understood as $\rho(t, x, u(t, x))$. Let $\|\cdot\|_p$ denote the $L^p(\Omega)$-norm. For $x \in \mathbb{R}^d$ and $A \in (\mathbb{R}^d)^{\otimes m}$,

$$|x| := \sqrt{x_1^2 + \cdots + x_d^2} \quad \text{and} \quad \|A\| := \left(\sum_{i_1, \ldots, i_m=1}^{d} A_{i_1, \ldots, i_m}^2\right)^{1/2}.$$
For two functions $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}$, we write $g_1(t) \asymp g_2(t)$ as $t \to 0_+$ if for some constants $C_1, C_2 > 0$, both $g_1(t) \leq C_1 g_2(t)$ and $g_2(t) \leq C_2 g_1(t)$ hold as $t \to 0_+$.

## 2 Some preliminaries

### 2.1 Definition and existence of a solution

Let $\mathcal{H}$ be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing smooth functions, endowed with the inner product

$$
\langle \phi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x)f(x-y)\psi(y)dxdy = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\mathcal{F}\overline{\psi}(\xi)\widehat{f}(d\xi), \quad \text{for } \phi, \psi \in \mathcal{S}(\mathbb{R}^d).
$$

For any measurable sets $T \subseteq [0, \infty)$ and $S \subseteq \mathbb{R}^d$, let $\mathcal{H}_{T,S}$ be the completion of $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)$ with respect to the inner product

$$
\langle \phi, \psi \rangle_{\mathcal{H}_{T,S}} = \int_T \int_S \int_S \phi(s,x)\psi(s,y)f(x-y)dxdyds, \quad \text{for } \phi, \psi \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d). \quad (2.1.1)
$$

It is known that both $\mathcal{H}$ and $\mathcal{H}_{T,S}$ may contain distributions.

Recall that a spatially homogeneous Gaussian noise that is white in time is an $L^2(\Omega)$-valued mean zero Gaussian process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$
\{ W(\psi) : \psi \in C_0^\infty([0, \infty) \times \mathbb{R}^d) \},
$$

such that

$$
\mathbb{E} [W(\psi)W(\phi)] = \int_0^\infty ds \int_{\mathbb{R}^{2d}} \psi(s,x)\phi(s,y)f(x-y)dxdy. \quad (2.1.2)
$$

Let $\mathcal{B}_b(\mathbb{R}^d)$ be the collection of Borel measurable sets with finite Lebesgue measure. As in Dalang-Walsh theory [12, 13, 14, 15, 27], one can extend $F$ to a $\sigma$-finite $L^2(\Omega)$-valued martingale measure $B \mapsto F(B)$ defined for $B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$. Then define

$$
W_t(B) := F([0, t] \times B), \quad B \in \mathcal{B}_b(\mathbb{R}^d).
$$

Let $(\mathcal{F}, t \geq 0)$ be the natural filtration generated by $M(\cdot)$ and augmented by all $\mathbb{P}$-null sets $\mathcal{N}$ in $\mathcal{F}$, that is,

$$
\mathcal{F}_t := \sigma \left( W_s(A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d) \right) \vee \mathcal{N}, \quad t \geq 0,
$$

Then for any adapted, jointly measurable (with respect to $\mathcal{B}((0, \infty) \times \mathbb{R}^d) \times \mathcal{F}$) random field $\{ X(t,x) : t > 0, x \in \mathbb{R}^d \}$ such that

$$
\int_0^\infty ds \int_{\mathbb{R}^{2d}} dxdy \| X(s,y)X(s,x) \|_{p/2} f(x-y) < \infty,
$$

the stochastic integral

$$
\int_0^\infty \int_{\mathbb{R}^d} X(s,y)W(dsdy)
$$
is well-defined in the sense of Dalang-Walsh. Here we only require the joint-measurability instead of predictability; see Proposition 2.2 in [11] for this case or Proposition 3.1 in [8] for the space-time white noise case. Throughout this paper, \( \| \cdot \|_p \) denotes the \( L^p(\Omega) \)-norm.

We formally write the SPDE (1.1) in the integral form

\[
  u(t, x) = J_0(t, x) + I(t, x)
\]

where

\[
  I(t, x) := \int_{[0,t] \times \mathbb{R}^d} G(t - s, x - y) \rho(u(s, y)) W(dsdy).
\]

The above stochastic integral is understood in the sense of Walsh [12, 27].

**Definition 2.1.** A process \( u = (u(t, x), (t, x) \in (0, \infty) \times \mathbb{R}^d) \) is called a random field solution to (1.1) if

1. \( u \) is adapted, that is, for all \((t, x) \in (0, \infty) \times \mathbb{R}^d\), \( u(t, x) \) is \( \mathcal{F}_t \)-measurable;
2. \( u \) is jointly measurable with respect to \( \mathcal{B}((0, \infty) \times \mathbb{R}^d) \times \mathcal{F} \);
3. \( \| I(t, x) \|_2 < +\infty \) for all \((t, x) \in (0, \infty) \times \mathbb{R}^d \);
4. \( I \) is \( L^2(\Omega) \)-continuous, that is, the function \((t, x) \mapsto I(t, x)\) mapping \((0, \infty) \times \mathbb{R}^d \) into \( L^2(\Omega) \) is continuous;
5. \( u \) satisfies (2.1.3) a.s., for all \((t, x) \in (0, \infty) \times \mathbb{R}^d \).

The following results are from [11] and [10], where \( \rho \) is assumed to be a function of one variable. The extension to the current setting, that is, \( \rho \) is a function of three variables, is routine. In particular, Theorem 2.2 gives the existence and uniqueness of a random field solution. Theorem 2.3 supplies us with some useful moment bounds. Theorem 2.4 gives the Hölder regularity of the solution and finally, the comparison principle is stated in Theorem 2.5.

**Theorem 2.2 (Theorem 2.4 in [11]).** If the initial data \( \mu \) satisfies (1.4), then under Dalang’s condition (1.2), SPDE (1.1) has a unique (in the sense of versions) random field solution \( \{ u(t, x) : t > 0, x \in \mathbb{R}^d \} \) starting from \( \mu \). This solution is \( L^2(\Omega) \)-continuous.

**Theorem 2.3 (Theorem 1.5 in [10]).** Under Dalang’s condition (1.2), if the initial data \( \mu \) is a signed measure that satisfies (1.4), then the solution \( u \) to (1.1) for any given \( t > 0 \) and \( x \in \mathbb{R}^d \) is in \( L^p(\Omega) \), \( p \geq 2 \), and

\[
  \| u(t, x) \|_p \leq \sqrt{2} \left[ \| \mu \| \ast G(t, \cdot) \right] H(t; \gamma_p)^{1/2},
\]

where \( \bar{\zeta} = |\rho(0)| / \text{Lip}_p \) and \( \gamma_p = 32p \text{Lip}_p^2 \) and \( H(t; \gamma_p) \) is defined in (2.2.2) below. Moreover, if for some \( \alpha \in (0, 1] \) condition (1.3) is satisfied, then when \( p \geq 2 \) is large enough, there exists some constant \( C > 0 \) such that

\[
  \| u(t, x) \|_p \leq C \left[ \bar{\zeta} + (\| \mu \| \ast G(t, \cdot)) (x) \right] \exp \left( Cp^{1/\alpha} t \right).
\]
Theorem 2.4 (Theorem 1.6 of [10]). Suppose that $\mu$ is any measure that satisfies (1.4) and $f$ satisfies (1.3) for some $\alpha \in (0,1]$. Then the solution to (1.1) starting from $\mu$ is a.s. $\beta_1$-Hölder continuous in time and $\beta_2$-Hölder continuous in space on $(0,\infty) \times \mathbb{R}^d$ with

$$\beta_1 \in (0,\alpha/2) \quad \text{and} \quad \beta_2 \in (0,\alpha).$$

Theorem 2.5 (Comparison principle [10]). Assume that $f$ satisfies Dalang’s condition (1.2). Let $u_1(t,x)$ and $u_2(t,x)$ be two solutions to (1.1) with the initial measures $\mu_1$ and $\mu_2$ that satisfy (1.4), respectively. Then

(a) (Weak comparison principle) If $\mu_1 \leq \mu_2$, then

$$\mathbb{P}(u_1(t,x) \leq u_2(t,x)) = 1 \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}^d. \quad (2.1.6)$$

(b) (Strong comparison principle) If, in addition, $f$ satisfies (1.3) for some $\alpha \in (0,1]$, then $\mu_1 < \mu_2$ implies that

$$\mathbb{P}(u_1(t,x) < u_2(t,x) \text{ for all } t \geq 0 \text{ and } x \in \mathbb{R}^d) = 1. \quad (2.1.7)$$

2.2 Some auxiliary functions

Recall that $k(t)$ is defined in (1.16). Define

$$h_0(t) := 1 \quad \text{and} \quad h_n(t) = \int_0^t ds h_{n-1}(s)k(t-s) \quad \text{if } n \geq 1. \quad (2.2.1)$$

Let

$$H(t;\gamma) := \sum_{n=0}^{\infty} \gamma^n h_n(t). \quad (2.2.2)$$

This function is defined through the correlation function $f$. The following lemma tells us that this function has an exponential asymptotic bound.

Lemma 2.6 (Lemma 2.5 in [11] or Lemma 3.8 in [2]). For all $t \geq 0$ and $\gamma \geq 0$,

$$\limsup_{t \to \infty} \frac{1}{t} \log H(t;\gamma) \leq \inf \left\{ \beta > 0 : \Upsilon(\beta) < \frac{1}{\gamma} \right\}. \quad (2.2.3)$$

Lemma 2.7 (Lemma 3.1 in [10]). Suppose that $\mu$ is a signed measure that satisfies condition (1.4) and let $J_0(t,x)$ be the solution to the homogeneous equation (see (1.5)). If a nonnegative function $g: \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}_+$ satisfies the following integral inequality

$$g(t,x)^2 \leq J_0^2(t,x) + \lambda^2 \int_0^t ds \int_{\mathbb{R}^d} G(t-s,x-y_1)G(t-s,x-y_2)$$

$$\times f(y_1 - y_2)g(s,y_1)g(s,y_2)dy_1dy_2, \quad (2.2.4)$$

for all $t > 0$ and $x \in \mathbb{R}^d$, then

$$g(t,x) \leq (|\mu| * G(t,\cdot))(x) H(t;2\lambda^2)^{1/2}. \quad (2.2.5)$$
In the proof of Lemma 3.1 of [10], the authors actually prove the following result, which will be useful in this paper.

**Lemma 2.8.** Suppose that \( \mu \) is a signed measure that satisfies condition (1.4). For \( \lambda \geq 0 \), define

\[
g_0(t, x) := (|\mu| * G(t, \cdot))(x) \quad \text{and for } n \geq 1
\]

\[
g_n(t, x) = J_0^2(t, x) + \chi \int_0^t ds \int_{\mathbb{R}^d} G(t-s, x-y_1)G(t-s, x-y_2)f(y_1 - y_2) \times g_{n-1}(s, y_1)g_{n-1}(s, y_2) dy_1 dy_2,
\]

with \( t > 0 \) and \( x \in \mathbb{R}^d \). Then for all \( n \geq 0, t > 0 \) and \( x \in \mathbb{R}^d \), it holds that

\[
g_n(t, x) \leq g_0(t, x) \left( \sum_{i=0}^{n} (2\chi)^i h_i(t) \right)^{1/2}.
\]

**Lemma 2.9.** Suppose that \( \mu \) is a signed measure that satisfies condition (1.4). For \( \lambda > 0 \), define

\[
g(t, x) := (|\mu| * G(t, \cdot))(x)H(t; 2\lambda^{2})^{1/2}
\]

\[
h(t, x) := \int_0^t ds \int_{\mathbb{R}^d} dy dy' G(t-s, x-y)g(s, y)f(y-y')g(s, y')G(t-s, x-y').
\]

Then

\[
h(t, x) \leq \lambda^{-2} g_0^2(t, x).
\]

**Proof.** Set \( \gamma = 2\lambda^2 \) and \( g_0(t, x) := (|\mu| * G(t, \cdot))(x) \). By the same arguments as the proof of Lemma 3.1 of [10] and the definition of \( H(t; \gamma) \) in (2.2.2), we see that

\[
h(t, x) \leq 2g_0^2(t, x) \int_0^t ds \left( \sum_{i=0}^{\infty} \gamma^i h_i(s) \right) k(t-s)
\]

\[
= 2g_0^2(t, x) \sum_{i=0}^{\infty} \gamma^i h_{i+1}(s)
\]

\[
= \frac{1}{\chi^2} g_0^2(t, x) (H(t; \gamma) - h_0(t)) \leq \frac{1}{\chi^2} g_0^2(t, x) H(t; \gamma).
\]

The following identity will be used many times in this paper:

\[
G(s, x)G(t, y) = G(s + t, x + y)G \left( \frac{st}{s+t}, \frac{tx - sy}{s+t} \right).
\]

### 2.3 Malliavin calculus

Now we recall some basic facts on Malliavin calculus associated with \( W \). Denote by \( C_p^\infty(\mathbb{R}^n) \) the space of smooth functions with all their partial derivatives having at most polynomial growth at infinity. Let \( S \) be the space of simple functionals of the form

\[
F = f(W(A_1), \ldots, W(A_n)),
\]

(2.3.1)
Theorem 2.10 \[ \text{the density are the following:} \]

\[ D = \text{Malliavin matrix} \]

In a similar way we define the iterated derivative \( D^k \). The derivative operator \( D^k \) for positive integers \( k \geq 1 \) is a closable operator from \( L^p(\Omega) \) into \( L^p(\Omega; L^2([0, \infty); \mathcal{H})^k) \) for any \( p \geq 1 \). Let \( k \) be some positive integer. For any \( p > 1 \), let \( \mathbb{D}^{k,p} \) be the completion of \( \mathcal{S} \) with respect to the norm

\[
\| F \|_{k,p}^p := \mathbb{E}(|F|^p) + \sum_{j=1}^k \mathbb{E} \left[ \left( \int_{([0, \infty) \times \mathbb{R}^d)^p} (D_{\theta_1, z_1} \cdots D_{\theta_j, z_j} F) \left( D_{\theta_1, z_1} \cdots D_{\theta_j, z_j} F \right) \right. \right.
\]

\[
\times \left. \left. \prod_{i=1}^j f(z_i - z_i') d\theta_i dz_i d\theta_i' \right)^{p/2} \right].
\] (2.3.2)

Denote \( \mathbb{D}^\infty := \cap_{k,p} \mathbb{D}^{k,p} \).

Suppose that \( F = (F^1, \ldots, F^d) \) is a \( d \)-dimensional random vector whose components are in \( \mathbb{D}^{1,2} \). The following random symmetric nonnegative definite matrix

\[
\sigma_F = \left( \langle DF^i, DF^j \rangle_{L^2([0, \infty); \mathcal{H})} \right)_{1 \leq i, j \leq d}
\] (2.3.3)

is called the Malliavin matrix of \( F \). The classical criteria for the existence and regularity of the density are the following:

**Theorem 2.10** (Bouleau and Hirsch [5]). Suppose that \( F = (F^1, \ldots, F^d) \) is a \( d \)-dimensional random vector whose components are in \( \mathbb{D}^{1,2} \). Then

1. If \( \det(\sigma_F) > 0 \) almost surely, the law of \( F \) is absolutely continuous with respect to the Lebesgue measure.

2. If \( F^i \in \mathbb{D}^\infty \) for each \( i = 1, \ldots, d \) and \( \mathbb{E} \left[ (\det \sigma_F)^{-p} \right] < \infty \) for all \( p \geq 1 \), then \( F \) has a smooth density.

As in [9], we need to introduce a localized version of the above theorem in order to deal with the rough initial data. For any measurable sets \( T \subset [0, \infty) \) and \( S \subset \mathbb{R}^d \), and for any \( p \geq 1 \), let \( \mathbb{D}^{k,p}_{T,S} \) be the completion of \( \mathcal{S} \) with respect to the norm

\[
\| F \|_{k,p,T,S}^p := \mathbb{E}(|F|^p) + \sum_{j=1}^k \mathbb{E} \left[ \left( \int_{(T \times S)^2} (D_{\theta_1, z_1} \cdots D_{\theta_j, z_j} F) \left( D_{\theta_1, z_1} \cdots D_{\theta_j, z_j} F \right) \right. \right.
\]

\[
\times \left. \left. \prod_{i=1}^j f(z_i - z_i') d\theta_i dz_i d\theta_i' \right)^{p/2} \right].
\] (2.3.4)

Similarly, denote \( \mathbb{D}^{\infty}_{T,S} := \cap_{k,p} \mathbb{D}^{k,p}_{T,S} \). Let \( \mathcal{H}_{T,S} \) be the Hilbert space completed from the Schwartz space \( \mathcal{S}(\mathbb{R}^{1+d}) \) with respect to the following inner product:

\[
( g, h )_{\mathcal{H}_{T,S}} = \int_{T \times S^2} g(s, y) h(s, z) f(y - z) dy dz ds.
\] (2.3.5)

Note that \( \mathcal{H}_{T,S} \) may contain distributions.
Theorem 2.11 (Chen et al [9]). Suppose that $F = (F^1, \ldots, F^d)$ is a $d$-dimensional random vector whose components are in $\mathbb{D}^{1,2}_{T,S}$. Let

$$
\sigma_{F,T,S} := \left( \langle DF^i, DF^j \rangle_{\mathcal{H},T,S} \right)_{1 \leq i, j \leq d}.
$$

Then

1. If $\det(\sigma_{F,T,S}) > 0$ almost surely, the law of $F$ is absolutely continuous with respect to the Lebesgue measure.

2. If $F^i \in \mathbb{D}^{\infty}_{T,S}$ for each $i = 1, \ldots, d$ and $\mathbb{E}[\det(\sigma_{F,T,S})^{-p}] < \infty$ for all $p \geq 1$, then $F$ has a smooth density.

Lemma 2.12 (Chen et al [9]). Let $\{F_m, m \geq 1\}$ be a sequence of random variables converging to $F$ in $L^p(\Omega)$ for some $p > 1$. Suppose that $\sup_m \|F_m\|_{n,p,T,S} < \infty$ for some integer $n \geq 1$. Then $F \in \mathbb{D}^{n,p}_{T,S}$.

### 2.4 Sufficient conditions for strict positivity of density

In this part, we introduce a criterion for the strict positivity of density. Recall that $W = \{W_t, t \geq 0\}$ can be viewed as a cylindrical Wiener process in the Hilbert space $\mathcal{H}$ with the covariance given by (2.1.2). Let $h = (h^1, \ldots, h^m) \in L^2(\mathbb{R}_+; \mathcal{H})^m$ and $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$. Define a translation of $W_t$, denoted by $\hat{W}_t$, as follows:

$$
\hat{W}_t(g) := W_t(g) + \sum_{i=1}^m z_i \int_0^t ds \int_{\mathbb{R}^d} dy dy' h^i(s, y)g(y)f(y-y'), \text{ for any } g \in \mathcal{H}. \quad (2.4.1)
$$

Then $\{\hat{W}_t, t \geq 0\}$ is a cylindrical Wiener process in $\mathcal{H}$ on the probability space $(\Omega, \mathcal{F}, \hat{P})$, where

$$
\frac{d\hat{P}}{dP} = \exp\left( -\sum_{i=1}^m z_i \int_0^\infty ds \int_{\mathbb{R}^d} h^i(s, y)W(dsdy) - \frac{1}{2} \sum_{i=1}^m z_i^2 \int_0^\infty ds \int_{\mathbb{R}^{2d}} dy dy' h^i(s, y)h^i(s, y')f(y-y') \right).
$$

For any predictable process $Z \in L^2(\Omega \times \mathbb{R}_+; \mathcal{H})$, we have that

$$
\int_0^\infty \int_{\mathbb{R}^d} Z(s, y)\hat{W}(dsdy) = \int_0^\infty \int_{\mathbb{R}^d} Z(s, y)W(dsdy) + \sum_{i=1}^m z_i \int_0^\infty \int_{\mathbb{R}^{2d}} Z(s, y)h^i(s, y)f(y-y')dsdydy'.
$$

In the following, we write $\sum_{i=1}^m z_i h^i(s, y) =: (z, h(s, y))$. Let $\hat{u}_z^m(t, x)$ be the solution to (1.1) with respect to $\hat{W}$, that is,

$$
\hat{u}_z(t, x) = J_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\rho(\hat{u}_z(s, y))W(dsdy)
+ \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\rho(\hat{u}_z(s, y)) (z, h(s, y')) f(y-y')dsdydy'. \quad (2.4.2)
$$
Then, the law of $u(t, x)$ under $\mathbb{P}$ coincides with the law of $\hat{\alpha}_z(t, x)$ under $\hat{\mathbb{P}}$.

The following theorem is an extension of Theorem 3.3 of Bally and Pardoux [3], which allows one to consider the case with unbounded diffusion parameter such as the parabolic Anderson model.

**Theorem 2.13** (Chen et al [9]). Let $F$ be an $m$-dimensional random vector measurable with respect to $W$, such that each component of $F$ is in $D^{3,2}$. Assume that for some $f \in C(\mathbb{R}^m)$ and for some open subset $\Gamma$ of $\mathbb{R}^m$, it holds that

$$\mathbb{I}_\Gamma(y) \left[ \mathbb{P} \circ F^{-1} \right](dy) = \mathbb{I}_\Gamma(y) f(y)dy.$$  

Fix a point $y_* \in \Gamma$. Suppose that there exists a sequence $\{h_n\}_{n \in \mathbb{N}} \subseteq L^2(\mathbb{R}_+; \mathcal{H})^m$ such that the associated random field $\phi_n(z) = F(\hat{W}^z, n)$ satisfies the following two conditions.

(i) There are constants $c_0 > 0$ and $r_0 > 0$ such that for all $r \in (0, r_0]$, the following limit holds true:

$$\liminf_{n \to \infty} \mathbb{P} \left( |F - y_*| \leq r \text{ and } |\det \partial_z \phi_n(0)| \geq \frac{1}{c_0} \right) > 0. \quad (2.4.3)$$

(ii) There are some constants $\kappa > 0$ and $K > 0$ such that

$$\lim_{n \to \infty} \mathbb{P} \left( \sup_{|z| \leq \kappa} \|\phi_n(z)\|_{C^2} \leq K \left| |F - y_*| \leq r_0 \right) = 1, \quad (2.4.4)$$

where

$$\|\phi_n(z)\|_{C^2} := |\phi_n(z)| + \|\partial_x \phi_n(z)\| + \|\partial^2_x \phi_n(z)\|.$$

Then $f(y_*) > 0$.

### 3 Regularity of densities

In this section, we will prove Theorems 1.1 and 1.4.

#### 3.1 Nonnegative moments (Proof of Theorem 1.8)

In this section, we will prove Theorem 1.8. We will need the following lemma.

**Lemma 3.1** (Lemma 3.4 in [9]). For any $a, b \in \mathbb{R}$, $\gamma \geq 0$ and $T > 0$ such that $b - a > \gamma T$, it holds that

$$0 < \inf_{0 \leq t + s \leq T} \inf_{a - \gamma(t+s) \leq x \leq b + \gamma(t+s)} (\mathbb{I}_{[a-\gamma s, b+\gamma s]} \ast G_1(t, \cdot))(x) \leq 1,$$

where $G_1(t, x)$ is the heat kernel function on $\mathbb{R}$.

In the following proof, we will use the notation: For any $a, b \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$,

$$[a + \alpha, b + \beta] := [a_1 + \alpha, b_1 + \beta] \times \cdots \times [a_d + \alpha, b_d + \beta].$$
Proof of Theorem 1.8. Recall that $F_t$ is the natural filtration generated by the noise $\hat{W}$. Fix an arbitrary compact set $K \subset \mathbb{R}^d$ and let $T > 0$. We are going to prove Theorem 1.8 for $\inf_{x \in K} u(T, x)$ in two steps.

Case I. In this case, we assume that

(H) For some cube $[a, b] = [a_1, b_1] \times \cdots \times [a_d, b_d]$ and some nonnegative function $g$ the initial measure $\mu$ satisfies that $\mathbb{I}_{[a,b]}(x)\mu(dx) = g(x)dx$. Moreover, for some $c > 0$, $g(x) \geq c\mathbb{I}_{[a,b]}(x)$ for all $x \in \mathbb{R}^d$.

Thanks to the weak comparison principle (see (2.1.6)), we may assume that $g(x) = \mathbb{I}_{[a,b]}(x)$. For any $t > 0$, we denote

$$I_t = [a - \gamma t, b + \gamma t].$$

Choose and fix $\gamma$ such that $K \subseteq \mathcal{I}_T$. Take

$$\beta = \frac{1}{2} \inf_{0 \leq t + s \leq T} \inf_{x \in \mathcal{I}_t} (\mathbb{I}_{[a-\gamma s,b+\gamma s]} * G(t, \cdot))(x)$$

$$= \frac{1}{2} \prod_{i=1}^d \inf_{0 \leq t + s \leq T} \inf_{x_i \in [a_i-\gamma (t+s), b_i + \gamma (t+s)]} (\mathbb{I}_{[a_i-\gamma s,b_i+\gamma s]} * G_1(t, \cdot))(x_i).$$

Thanks to Lemma 3.1, we have $0 < \beta < 1/2$. Define

$$T_0 := 0, \quad \text{and} \quad T_k := \inf \left\{ t > T_{k-1} : \inf_{x \in \mathcal{I}_t} u(t, x) \leq \beta^k \right\}, \quad k \geq 1.$$ 

Let $\hat{W}_k(t, x) = \hat{W}(t + T_{k-1}, x)$, and $H_k(t, x) = H(t + T_{k-1}, x)$. For each $k$, let $u_k(t, x)$ be the unique solution to (1.12) with initial data $u_k(0, x) = \beta^{k-1} \mathbb{I}_{[a-\gamma T_{k-1}, b+\gamma T_{k-1}]}(x)$. So we see that the random field $w_k(t, x) = \beta^{1-k} u_k(t, x)$ solves the equation

$$\begin{aligned}
\left\{ \left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right) w_k(t, x) = H_k(t, x)\sigma_k(t, x, w_k(t, x))\hat{W}_k(t, x), \quad t > 0, \; x \in \mathbb{R}^d,
\right.

w_k(0, x) = I_{[a-\gamma T_{k-1}, b+\gamma T_{k-1}]}(x),
\end{aligned}$$

where

$$\sigma_k(t, x, z) = \beta^{1-k} \sigma(t, x, \beta^{k-1} z).$$

Set $\tau_n := \frac{2\pi}{n}$ and

$$S(t_1, t_2) = \left\{ (t, x) : t \in [0, t_2 - t_1], \; x \in [a - \gamma (t_1 + t_2), b + \gamma (t_1 + t_2)] \right\}.$$ 

Define the following events

$$\mathcal{D}_{k,n} := \{ T_k - T_{k-1} \leq \tau_n \}, \quad \text{for} \; 1 \leq k \leq n.$$ 

Then following exactly the same arguments as those in the proof of Theorem 1.5 in [9] we see that for $k \leq n$,
\[ \mathbb{P}\left( \mathcal{D}_{k,n} \cap \left\{ \sup_{(t,x) \in S(T_{k-1}, T_k)} |w_k(t, x) - w_k(0, x)| \geq 1 - \beta \right\} \mid \mathcal{F}_{T_{k-1}} \right) \leq \beta^{-p} \mathbb{E}\left[ \sup_{(t,y) \in [0, \tau_n] \times \mathcal{I}_T} |I_k(t, y)|^p \right]. \] (3.1.8)

Next we need to find a deterministic upper bound for the conditional probability in (3.1.8).

\[ \mathbb{E}\left[ |I_k(s, x) - I_k(s', x)|^p \right] \leq C |s - s'|^{\alpha} \sup_{(t,y) \in [0, \tau_n] \times \mathcal{I}_T} \|w_k(t,y)\|^p, \]

for all \((s, s', x) \in [0, \tau_n]^2 \times \mathcal{I}_T\). By Theorem 2.3, we see that for some constant \(Q > 0\),

\[ \sup_{(t,y) \in [0, \tau_n] \times \mathbb{R}^d} \|w_k(t,y)\|^p \leq Q^p \exp\left( Q \frac{1 - \eta}{\alpha} \right) =: C_{p, \tau_n}. \] (3.1.9)

Choose \(p\) large enough such that \(1 - \frac{2}{p} \left( \frac{2}{\alpha} - 1 \right) > 0\). Then by the Kolmogorov continuity theorem, for some constant \(C > 0\) and for all \(0 < \eta < 1 - \frac{2}{p} \left( \frac{2}{\alpha} - 1 \right)\), we have that

\[ \mathbb{E}\left[ \sup_{(t,x) \in [0, \tau_n] \times \mathcal{I}_T} \frac{|I_k(t, x)|}{\tau_n^{\alpha/2}} \right]^p \leq \mathbb{E}\left[ \sup_{(s,s',x') \in [0, \tau_n]^2 \times \mathcal{I}_T} \frac{|I_k(s, x) - I_k(s', x)|}{|s - s'|^{\alpha/2}} \right]^p \leq C^p C_{p, \tau_n}, \] (3.1.10)

which implies that for some constant \(Q' > 0\),

\[ \beta^{-p} \mathbb{E}\left[ \sup_{(s,x) \in [0, \tau_n] \times \mathcal{I}_T} |I_k(s, x)|^p \right] \leq \tau_n^{\alpha p/2} \exp\left( Q' \frac{1 - \eta}{\alpha} \tau_n \right) = \exp\left( Q' \frac{1 - \eta}{\alpha} \tau_n + \frac{1}{2} \alpha \eta \log(\tau_n) p \right). \]

Take \(\eta = \theta \left( 1 - \frac{2}{p} \left( \frac{2}{\alpha} - 1 \right) \right)\) with some \(\theta \in (0, 1)\). Then the above exponent becomes

\[ f(p) := Q' p \frac{1 - \eta}{\alpha} \tau_n + \frac{1}{2} \theta \left( \alpha - \frac{2}{p} [2 - \alpha] \right) \log(\tau_n) p. \]

Optimizing in \(p\) shows that \(f(p)\) achieves its global minimum at

\[ p' = \left( \frac{\alpha^2 \theta n \log \left( \frac{n}{2\tau_n} \right)}{4(\alpha + 1)Q'T} \right)^\alpha \]

and for some constant \(Q'' > 0\),

\[ \min_{p \geq 2} f(p) \leq -Q'' n^\alpha (\log n)^{1+\alpha}. \] (3.1.11)

Thus, for some finite constant \(C > 0\),

\[ P\left( \mathcal{D}_{k,n} \mid \mathcal{F}_{T_{k-1}} \right) \leq C \exp\left( -C n^\alpha (\log n)^{1+\alpha} \right). \] (3.1.12)

Then following exactly the same arguments as those leading to (3.8) in [9], we see that for some constant \(B = B(\Lambda, T) > 0\) such that for \(\epsilon > 0\) small enough,

\[ \mathbb{P}\left( \inf_{x \in K} u(T, x) < \epsilon \right) \leq \exp\left( -B \left\{ |\log(\epsilon)| \log (|\log(\epsilon)|) \right\}^{1+\alpha} \right). \] (3.1.13)
Case II. Now we consider the general initial data. Set $Q = (a, b)^d$ for some arbitrary constants $a < b$. Choose and fix an arbitrary $\theta \in (0, T \wedge 1)$. Set

$$ \Theta(\omega) := 1 \land \inf_{x \in Q} u(\theta, x, \omega). $$

Since $u(\theta, x) > 0$ for all $x \in \mathbb{R}$ a.s. (see Theorem 2.5) and $u(\theta, x, \omega)$ is continuous in $x$, we see that $\Theta > 0$ a.s. Hence, $u(\theta, x, \omega) \geq \Theta(\omega) \mathbb{I}_Q(x)$ for all $x \in \mathbb{R}$. Denote $V(t, x, \omega) := \Theta(\omega)^{-1} u(t+\theta, x, \omega)$. By the Markov property, $V(t, x)$ solves the following time-shifted SPDE

$$ \begin{cases} 
\left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right) V(t, x) = \tilde{H}(t, x) \tilde{\sigma}(t, x, u(t, x)) \tilde{W}_\theta(t, x), & t > 0, \ x \in \mathbb{R}^d, \\
V(0, x) = \Theta^{-1} u(\theta, x),
\end{cases} 
$$

(3.1.14)

where $\tilde{W}_\theta(t, x) = \tilde{W}(t+\theta, x)$, $\tilde{H}(t, x) = H(t+\theta, x)$ and $\tilde{\sigma}(t, x, z, \omega) = \Theta(\omega)^{-1} \sigma(t + \theta, x, \Theta(\omega) z)$. Notice that condition (1.13) is satisfied by $\tilde{H}$ and $\tilde{\sigma}$ with the same constant $\Lambda$, that is,

$$ \left| \tilde{H}(t, x, \omega) \tilde{\sigma}(t, x, z, \omega) \right| \leq \Lambda |z| \quad \text{for all} \ (t, x, z, \omega) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega. $$

The initial data $V(0, x)$ satisfies assumption (H) in Case I. Hence, we can conclude from (3.1.13) that for $\epsilon > 0$ small enough,

$$ \mathbb{P} \left( \inf_{x \in K} u(T + \theta, x) < \epsilon \right) \leq \mathbb{P} \left( \inf_{x \in K} V(T, x) < \Theta^{-1} \epsilon \right) $$

$$ \leq \int_0^1 1 \wedge \exp \left( -B \left\{ |\log(\zeta^{-1}|) | \log |\log(\zeta^{-1}|) \right\}^{1+\alpha} \right) \mu_\Theta(\mathrm{d}\zeta), $$

where $\mu_\Theta$ denotes the law of the random variable $\Theta$, which is supported over $[0, 1]$. For any $\delta \in (0, 1),$

$$ \mathbb{P} \left( \inf_{x \in K} u(T + \theta, x) < \epsilon \right) $$

$$ \leq \int_0^1 1 \wedge \exp \left( -B \left\{ |\log(\zeta^{-1}|) | \log |\log(\zeta^{-1}|) \right\}^{1+\alpha} \right) \mu_\Theta(\mathrm{d}\zeta). $$

The integrand of the right hand side of the above inequality is bounded by one and goes to zero as $\epsilon$ goes to zero because $\delta \in (0, 1)$. Hence, the dominated convergence theorem shows that

$$ \lim_{\epsilon \to 0^+} \frac{\mathbb{P} \left( \inf_{x \in K} u(T + \theta, x) < \epsilon \right)}{\exp \left( -B \left\{ |\log(\epsilon)| \cdot |\log (|\log(\epsilon)|) \right\}^{1+\alpha} \right)} = 0, $$

which implies that

$$ \mathbb{P} \left( \inf_{x \in K} u(T + \theta, x) < \epsilon \right) \leq \exp \left( -B' \left\{ |\log(\epsilon)| \cdot |\log (|\log(\epsilon)|) \right\}^{1+\alpha} \right) $$

19
for $\epsilon$ small enough. Then using the fact that $(t, x) \mapsto u(t, x)$ is continuous a.s., by letting $\theta$ go to zero, we can conclude that (1.14) holds for $\epsilon > 0$ small enough.

Finally, the existence of the negative moments is a direct consequence of (1.14). This completes the proof of Theorem 1.8.

### 3.2 Malliavin derivatives of $u(t, x)$

In this section, we study the Malliavin derivatives of $u(t, x)$. Due to the difficulties caused by the initial data, we need to show that $u(t, x)$ lives in some Sobolev spaces restricted to some measurable sets $T \times S \subset [0, \infty) \times \mathbb{R}^d$.

**Proposition 3.2.** Suppose that $\rho$ is a $C^1$ function with bounded Lipschitz continuous derivative. Suppose that the initial data $\mu$ satisfies condition (1.4). Then

1. For any $(t, x) \in [0, T] \times \mathbb{R}^d$, $u(t, x)$ belongs to $D^{1,p}$ for all $p \geq 1$.
2. The Malliavin derivative $Du(t, x)$ defines an $L^2([0, T]; H)$-valued process that satisfies the following linear stochastic differential equation

$$D_{\theta, \xi}u(t, x) = \rho(u(\theta, \xi))G(t - \theta, x - \xi) + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)\rho'(u(s, y))D_{\theta, \xi}u(s, y)W(ds, dy),$$

for all $(\theta, \xi) \in [0, T] \times \mathbb{R}^d$.
3. If $\rho \in C^\infty(\mathbb{R}^d)$ and it has bounded derivatives of all orders, and if for some measurable sets $T \subset [0, t]$ and $S \subset \mathbb{R}^d$, the initial data satisfies the following condition

$$\sup_{(s, y) \in T \times S} J_0^2(s, y) < \infty,$$

then $u(t, x) \in \mathbb{D}^{\infty}_{T, S}$.

Throughout this section, let $\mathcal{H}_T$ denote the space $L^2([0, T]; H)$, that is,

$$\langle h, g \rangle_{\mathcal{H}_T} := \int_0^T \langle h(s, \cdot), g(s, \cdot) \rangle_H ds, \quad \text{for } h, g \in \mathcal{H}_T.$$

Recall the space $\mathcal{H}_{T, S}$ defined by the norm in (2.3.5).

**Proof of part (1) of Proposition 3.2.** Fix $p \geq 2$ and $T > 0$. Consider the Picard approximations $u_m(t, x)$ in the proof of the existence of the random field solution in [10], that is, $u_0(t, x) = J_0(t, x)$, and for $m \geq 1$,

$$u_m(t, x) = J_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)\rho(u_{m-1}(s, y))W(dsdy).$$

It is proved in [10] that $u_m(t, x)$ converges to $u(t, x)$ in $L^p(\Omega)$ as $m \to \infty$ and

$$\sup_{m \in \mathbb{N}} \| u_m(t, x) \|_p \leq C_1((1 + |\mu|) * G(t, \cdot))(x) \quad \text{for all } t \in (0, T] \text{ and } x \in \mathbb{R}^d.$$  

(3.2.3)
Now we claim that for some constants $C_2 > 0$ and $\gamma > 0$, it holds that
\[
\sup_{m \in \mathbb{N}} \mathbb{E} \left( \| Du_m(t, x) \|_{H_T}^p \right) < \left[ C_2((1 + |\mu|) * G(t, \cdot))(x)H(t; \gamma)^{1/2} \right]^p
\] (3.2.4)
for all $t > 0$ and $x \in \mathbb{R}^d$. It is clear that $0 \equiv Du_0(t, x)$ satisfies (3.2.4). Assume that $Du_k(t, x)$ satisfies (3.2.4) for all $k < m$. Now we shall show that $Du_m(t, x)$ satisfies (3.2.4) as well. Notice that
\[
D_{\theta, \xi} u_m(t, x) = G(t - \theta, x - \xi) \rho(\mu_{m-1}(\theta, \xi))
\]
\[
+ \int_{\theta}^{t} \int_{\mathbb{R}^d} G(t - s, x - y) \rho'(\mu_{m-1}(s, y)) D_{\theta, \xi} u_{m-1}(s, y) W(dsdy)
\] (3.2.5)
Then by Minkowski’s inequality and (3.2.3), we see that
\[
\mathbb{E} \left( \| A_m \|_{H_T}^p \right) = \mathbb{E} \left( \int_0^t d\theta \int_{\mathbb{R}^{2d}} d\xi d\xi' G(t - \theta, x - \xi) \rho(\mu_{m-1}(\theta, \xi)) f(\xi - \xi')
\]
\[
\times G(t - \theta, x - \xi') \rho(\mu_{m-1}(\theta, \xi') \right)^{p/2} \right)
\]
\[
\leq C \left( \int_0^t d\theta \int_{\mathbb{R}^{2d}} d\xi d\xi' G(t - \theta, x - \xi) \left( 1 + \| \mu_{m-1}(\theta, \xi) \|_p \right) f(\xi - \xi')
\]
\[
\times G(t - \theta, x - \xi') \left( 1 + \| \mu_{m-1}(\theta, \xi') \|_p \right)^{p/2} \right)
\]
\[
\leq C' \left( \int_0^t d\theta \int_{\mathbb{R}^{2d}} d\xi d\xi' G(t - \theta, x - \xi) \left( 1 + (|\mu| * G(\theta, \cdot))(\xi) \right) f(\xi - \xi')
\]
\[
\times G(t - \theta, x - \xi') \left( 1 + (|\mu| * G(\theta, \cdot))(\xi') \right)^{p/2} \right)
\]
where the constant $C'$ does not depend on $m$. Therefore, by Lemma 2.8 with $n = 1$, we see that for some $C_3 > 0$ which depends on the Lipschitz constant of $\rho$ and $C_1$ such that
\[
\sup_{m \in \mathbb{N}} \mathbb{E} \left( \| A_m \|_{H_T}^p \right) < \left[ C_3((1 + |\mu|) * G(\theta, \cdot))(x) \right]^p
\]
for all $t > 0$ and $x \in \mathbb{R}^d$.

As for $B_m$, by the Burkholder-Davis-Gundy inequality and by the boundedness of $\rho'$,
\[
\mathbb{E} \left( \| B_m \|_{H_T}^p \right) \leq C \mathbb{E} \left( \left( \int_0^t d\theta \int_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) \| Du_{m-1}(s, y) \|_{H_T} f(y - y')
\]
\[
\times G(t - s, x - y') \| Du_{m-1}(s, y') \|_{H_T} \right)^{p/2} \right)
\]
\[
\leq C \left( \int_0^t d\theta \int_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) \| Du_{m-1}(s, y) \|_{H_T} \right)^p f(y - y')
\]
\[
\times G(t-s, x-y') \|Du_{m-1}(s, y')\|_{\mathcal{H}_T} \right)^{p/2}.
\]

Then by the induction assumption and Lemma 2.9,
\[
\mathbb{E}\left(\|B_m\|_{\mathcal{H}_T}^p\right) \leq \left[ C\gamma^{-1/2}((1 + |\mu|) * G(t, \cdot))(x) H(t; \gamma)^{1/2}\right]^p.
\]

Since the function \( \gamma \mapsto H(t; \gamma) \) is nonincreasing, by increasing the value of \( \gamma \), one can make sure that the above mapping is a contraction. Therefore, (3.2.4) is true. Finally, by Lemma 2.12, we can conclude that \( u(t, x) \in \mathbb{D}^{1, p} \). This proves part (1) of Proposition 3.2.

**Proof of part (2) of Proposition 3.2.** The proof is similar to the proof of part (2) of Proposition 5.1 in [9]. Fix \( t \in (0, T] \) and \( x \in \mathbb{R}^d \). By Lemma 1.2.3 of [21], \( D_{\theta, \xi}u_m(t, x) \) converges to \( D_{\theta, \xi}u(t, x) \) in the weak topology of \( L^2(\Omega; \mathcal{H}_T) \), that is, for any \( h \in \mathcal{H}_T \) and any square integrable random variable \( F \in \mathcal{F}_t \),
\[
\lim_{n \to \infty} \mathbb{E}\left(\langle D_{\theta, \xi}u_m(t, x) - D_{\theta, \xi}u(t, x), h \rangle_{\mathcal{H}_T} F\right) = 0.
\]

Hence, we need to prove that the right-hand of (3.2.5) converges to the right-hand side of (3.2.1) in this weak topology of \( L^2(\Omega; \mathcal{H}_T) \). By the Cauchy-Schwartz inequality,
\[
\mathbb{E}\left(\|\langle (\rho(u) - \rho(u_m) \rangle G(t-\cdot, x-\cdot) F\|_{\mathcal{H}_T}\|F\|_2\right) \leq \text{Lip}_p \|h\|_{\mathcal{H}_T} \|F\|_2 \times \left(\int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \|u(s, y) - u_m(s, y)\|_2^2 \times \int_{\mathbb{R}^d} G(t-s, x-y') \|u(s, y') - u_m(s, y')\|_2^2 f(y-y') ds dy dy'\right)^{1/2}.
\]

Denote
\[
F_m(t, x) := \begin{cases} \|u(t, x)\|_2 & \text{if } m = -1, \\ \|u(t, x) - u_m(t, x)\|_2 & \text{if } m \geq 0. \end{cases}
\]

For some positive constant \( \lambda \) large enough, we have that
\[
F_m^2(t, x) \leq \lambda \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t-s, x-y) G(t-s, x-y') f(y-y') F_m-1(s, y) F_m-1(s, y')
\]
for all \( m \geq 0 \). We claim that
\[
F_m(t, x) \leq C(|\mu| * G(t, \cdot))(x) \left[ \sum_{i=m+1}^{\infty} \lambda^i h_i(t) \right]^{1/2} \text{ for all } m \geq -1.
\]

It is true for \( m = -1 \) by Theorem 2.3. Suppose that it is true for \( m > -1 \). We now prove that it holds for \( m + 1 \). By the induction assumption,
\[
F_{m+1}^2(t, x) \leq C^2 \lambda \int_0^t ds \left( \sum_{i=m+1}^{\infty} \lambda^i h_i(s) \right) \int_{\mathbb{R}^{2d}} dy dy' G(t-s, x-y) G(t-s, x-y') f(y-y')
\]
for all
Hence, it holds that
\[
\mu \ast G(s, \cdot)(y) (\mu \ast G(s, \cdot))(y').
\]
Then by the same arguments as the proof of Lemma 2.2 of [10], we see that
\[
F_{m+1}(t, x) \leq C(|\mu| \ast G(t, \cdot))(x) \left( \lambda \int_0^t \left[ \sum_{i=m+1}^{\infty} \lambda_i h_i(s) \right] k(t - s)ds \right)^{1/2}
= C(|\mu| \ast G(t, \cdot))(x) \left( \sum_{i=m+1}^{\infty} \lambda_i \gamma h_i(t) \right)^{1/2}.
\]
Hence, it holds for \( m + 1 \). This proves (3.2.7). By the same argument as above, we see that
\[
F_m^2(t, x) \leq C (|\mu| \ast G(t, \cdot))(x)^2 \sum_{i=m+2}^{\infty} \lambda_i \gamma h_i(t) \to 0 \quad \text{as} \quad m \to \infty,
\]
because \( \sum_{i=0}^{\infty} \gamma^i h_i(s) = H(t; \gamma) < \infty \) for any \( \gamma > 0 \). Therefore,
\[
\lim_{m \to \infty} \mathbb{E} \left( \langle (\rho(u) - \rho(u_m)) G(t - \cdot, x - \cdot), h \rangle_{H_T} F \right) = 0.
\]
Now denote the second term on the right-hand side of (3.2.1) by \( B(\theta, \xi) \). Recall that \( B_m(\theta, \xi) \) is defined in (3.2.5). It remains to prove that
\[
\lim_{m \to \infty} \mathbb{E} \left( \langle B - B_m, h \rangle_{H_T} F \right) = 0.
\]
Notice that
\[
B(\theta, \xi) - B_m(\theta, \xi)
= \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) (\rho'(u(s, y)) - \rho'(u_{m-1}(s, y))) D_{\theta, \xi} u_{m-1}(s, y) W(dsdy)
+ \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) \rho'(u(s, y)) (D_{\theta, \xi} u(s, y) - D_{\theta, \xi} u_{m-1}(s, y)) W(dsdy)
=: B_{m,1}(\theta, \xi) + B_{m,2}(\theta, \xi).
\]
Because \( F \) is square integrable, for some adapted random field \( \{\Phi(s, y), s \in [0, T], y \in \mathbb{R} \} \) with
\[
\int_0^t \int_{\mathbb{R}^d} \mathbb{E} [\Phi(s, y) \Phi(s, y')] f(y - y') dsdydy' < \infty,
\]
it holds that
\[
F = \mathbb{E}[F] + \int_0^t \int_{\mathbb{R}^d} \Phi(s, y) W(dsdy).
\]
Hence,
\[
\mathbb{E} \left( \langle B_{m,1}(t, x), h \rangle_{H_T} F \right) = \mathbb{E} \left[ \int_0^t ds \int_{\mathbb{R}^d} dydy' G(t - s, x - y) f(y - y') \times \Phi(s, y') (\rho'(u(s, y)) - \rho'(u_{m-1}(s, y))) \langle Du_{m-1}(s, y), h \rangle_{H_T} \right]
\]

23
where we have applied the Cauchy-Schwartz inequality. It is clear that
\[ \lim_{m \to \infty} I_2(m) = 0. \]
On the one hand, since \( \rho \) is bounded,
\[
I_{2,m} \leq C \int_0^t ds \int_{\mathbb{R}^d} dy \ G(t - s, x - y) f(y - y') G(t - s, x - y')
\]
\[ \times \mathbb{E}\left[ \left| \rho'(u(s, y)) - \rho'(u_{m-1}(s, y)) \right| \left| \rho'(u(s, y')) - \rho'(u_{m-1}(s, y')) \right| \right] \]
\[ \times \| D u_{m-1}(s, y), h \|_{\mathcal{H}_T} \| D u_{m-1}(s, y'), h \|_{\mathcal{H}_T} \right]^{1/2} = I_1^{1/2} I_{2,m}^{1/2}, \]
Then by (3.2.4) and Lemma 2.9, we find an integrable upper bound of the integrand that
does not depend on \( m \). On the other hand, the expectation in \( I_{2,m} \) is bounded by
\[
\text{Lip}_p^2 \| u(s, y) - u_{m-1}(s, y) \|_4 \| u(s, y') - u_{m-1}(s, y') \|_4
\]
\[ \times \| D u_{m-1}(s, y), h \|_{\mathcal{H}_T} \| D u_{m-1}(s, y'), h \|_{\mathcal{H}_T} \leq \Theta_T. \]

**Proof of part (3) of Proposition 3.2.** Fix \( S \subset \mathbb{R}^d \) and \( T \subset [0, T] \). Recall that \( \mathcal{H}_{T,S} \) is defined in (2.1.1). We will prove the following property by induction:
\[
\sup_{m \in \mathbb{N}} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \| D^n u_m(t, x) \|_{\mathcal{H}_{T,S}^n}^p < \infty \quad \text{for all } n \geq 1 \text{ and all } p \geq 2. \tag{3.2.11}
\]

**Step 1.** Consider the case \( n = 1 \). We will prove by induction that for some finite constant \( \Theta_T > 0 \) independent of \( m \),
\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \| D u_m(t, x) \|_{\mathcal{H}_{T,S}}^p < \Theta_T. \tag{3.2.12}
\]
Since \( u_0(t, x) \) is deterministic, (3.2.12) is true for \( m = 0 \). Now suppose (3.2.12) holds for all \( k \leq m \) and we will consider \( k = m + 1 \). Notice that
\[
D_{\theta_1, \xi_1} u_{m+1}(t, x) = \rho(u_m(\theta_1, \xi_1)) G(t - \theta_1, x - \xi_1)
\]
\[ + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) \rho'(u_m(s, y)) D_{\theta_1, \xi_1} u_m(s, y) W(dsdy). \]
Thus,
\[
\|Du_{m+1}(t, x)\|_{\mathcal{H}_T, s}^2 \leq 2 \|\rho(u_m(\cdot , *))G(t - \cdot , x - *)\|_{\mathcal{H}_T, s}^2 + 2 \left\| \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)\rho'(u_m(s, y))Du_m(s, y)W(dsdy) \right\|_{\mathcal{H}_T, s}^2 = 2I_1 + 2I_2.
\]

Using Minkowski’s inequality, we obtain
\[
I_1 = \mathbb{E}\left\{ \left( \int_T \int_{S^2} G(t - \theta_1, x - \xi_1)G(t - \theta_1, x - \xi'_1)f(\xi_1 - \xi'_1)\right.\left. \times \rho(u_m(\theta_1, \xi_1))\rho(u_m(\theta_1, \xi'_1))d\xi_1d\xi'_1d\theta_1 \right) \right\}^{\frac{2}{p}} \leq \int_T \int_{S^2} \|\rho(u_m(\theta_1, \xi_1))\rho(u_m(\theta_1, \xi'_1))\|_2 G(t - \theta_1, x - \xi_1)G(t - \theta_1, x - \xi'_1)f(\xi_1 - \xi'_1)d\xi_1d\xi'_1d\theta_1 \leq C \sup_{m \in \mathbb{N}} \sup_{s \in T, y \in S} (1 + \|u_m(s, y)\|_p^2) \int_0^T \int_{\mathbb{R}^d} G(s, \xi_1)G(s, \xi'_1)f(\xi_1 - \xi'_1)d\xi_1d\xi'_1ds \leq C \sup_{s \in T, y \in S} (1 + (|\mu| * G(s, \cdot))(y)) < \infty,
\]
where we have used the fact that \(\|u_m(s, y)\|_p \leq \|u(s, y)\|_p\) and the moment bound (2.1.4). As for \(I_2\), using the Burkholder-Davis-Gundy inequality, by the boundedness of \(\rho'\), we see that
\[
I_2 \leq \|\rho'\|^2_{L^\infty(\mathbb{R})} \int_0^t ds \int_{\mathbb{R}^d} dy dy' G(t - s, x - y)G(t - s, x - y')f(y - y') \times \left\| \|Du_m(s, y)\|_{\mathcal{H}_T, s} \right\|_p \cdot \left\| \|Du_m(s, y')\|_{\mathcal{H}_T, s} \right\|_p.
\]
Then one can apply Lemma 2.8 with constant initial data to conclude that (3.2.12) holds.

**Step 2.** We have proved (3.2.11) for \(n = 1\) in the previous step. We assume that (3.2.11) holds for \(n - 1\) with \(n \geq 1\). Now we will prove by induction on \(m\) that for some finite constant \(\Theta_T > 0\) independent of \(m\),
\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \left\| \|Du_m(t, x)\|_{\mathcal{H}_T, s} \right\|_p < \Theta_T. \tag{3.2.13}
\]

Denote
\[
\alpha := ((\theta_1, \xi_1), \ldots, (\theta_n, \xi_n)) \quad \text{and} \quad \dot{\alpha}_k := ((\theta_1, \xi_1), \ldots, (\theta_{k-1}, \xi_{k-1}), (\theta_{k+1}, \xi_{k+1}), \ldots, (\theta_n, \xi_n)). \tag{3.2.14}
\]
Clearly, the case \(m = 0\) is true since \(D^n_{\alpha}u_0(t, x) \equiv 0\). By Lemma 5.6 in [9],
\[
D^n_{\alpha}u_{m+1}(t, x) = \sum_{k=1}^n D^{n-1}_{\dot{\alpha}_k} \rho(u_m(\theta_k, \xi_k))G(t - \theta_k, x - \xi_k)
\]
Thus,

$$\|D^n u_{m+1}(t, x)\|_{\mathcal{H}^{\alpha_n}_{T, S}} \leq \sum_{k=1}^n \left( \int_T \int_{S^2} \langle D^{n-1}_\alpha \rho (u_m(\theta_k, \xi_k)) , D^{n-1}_\alpha \rho (u_m(\theta_k, \xi'_k)) \rangle_{\mathcal{H}^{\alpha(n-1)}_{T, S}} \right)$$

\[ \times \left( G(t - \theta_k, x - \xi_k) , G(t - \theta_k, x - \xi'_k) \right) ]^{\frac{1}{p}} \]

\[ + \left\| \int_0^t \int_{\mathbb{R}_d} D^n \rho (u_m(s, y)) G(t - s, x - y) W(dsdy) \right\|_{\mathcal{H}^{\alpha_n}_{T, S}} \]

\[ =: \left( \sum_{k=1}^n J_{1,k} \right) + J_2. \]

For $J_{1,k}$, by Minkowski’s inequality, we see that

$$\|J_{1,k}\|^2 \leq \int_T d\theta_k \int_{S^2} d\xi_k d\xi'_k \left( \left\| D^{n-1}_\alpha \rho (u_m(\theta_k, \xi_k)) \right\|_{\mathcal{H}^{\alpha(n-1)}_{T, S}} \right) \left( \left\| D^{n-1}_\alpha \rho (u_m(\theta_k, \xi'_k)) \right\|_{\mathcal{H}^{\alpha(n-1)}_{T, S}} \right)$$

\[ \times \left( G(t - \theta_k, x - \xi_k) , G(t - \theta_k, x - \xi'_k) \right) ]^{\frac{1}{p}} \]

\[ =: J^*_{1,k}. \]

As for $J_2$, by the Burkholder-Davis-Gundy inequality and Minkowski’s inequality,

$$\|J_2\|^2 \leq 4p \int_T \left[ \left( \int_0^t \int_{\mathbb{R}_d} \langle D^n \rho (u_m(s, y)) , D^n \rho (u_m(s, y)) \rangle_{\mathcal{H}^{\alpha_n}_{T, S}} \right) \times \left( G(t - s, x - y) G(t - s, x - y') \right) \right]^{\frac{1}{2}} dydy'ds \]

\[ \leq 4p \int_0^t \int_{\mathbb{R}_d} \left( \left\| D^n \rho (u_m(s, y)) \right\|_{\mathcal{H}^{\alpha_n}_{T, S}} \right) \left( \left\| D^n \rho (u_m(s, y')) \right\|_{\mathcal{H}^{\alpha_n}_{T, S}} \right) \]

\[ \times \left( G(t - s, x - y) G(t - s, x - y') \right) dydy'ds \]

\[ =: 4pJ^*_{2}. \]

Therefore, by $(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$, we see that

$$\left\| D^n u_m(t, x) \right\|^2_{\mathcal{H}^{\alpha_n}_{T, S}} \leq C_n \sum_{k=1}^n J^*_{1,k} + C_n 4pJ^*_{2}. $$

where $C_n = n + 1$. By Lemma 5.5 of [9] and the induction assumption, we have that

$$J^*_{1,k} \leq C \sup_{m \in \mathbb{N}, (s, y) \in [0,T] \times \mathbb{R}} \left\| D^{n-1} \rho (u_m(s, y)) \right\|_{\mathcal{H}^{\alpha(n-1)}_{T, S}}^2 < \infty. $$

Let

$$\Delta^n (\rho, u) := D^n \rho (u) - \rho' (u) D^n u, \quad (3.2.15)$$

26
be all terms in the summation of $D^n\rho(u)$ that have Malliavin derivatives of order less than or equal to $n - 1$. Then

$$J^*_2 \leq 2\|\rho\|^2_{L^\infty(\mathbb{R})} C_n \int_0^t \int_{\mathbb{R}^d} \left\|D^n u_m(s,y)\right\|_{\mathcal{H}_{T,S}^n} \left\|D^n u_m(s,y')\right\|_{\mathcal{H}_{T,S}^n} \times G(t-s,x-y)G(t-s,x-y')f(y-y')dy'ds$$

$$+ 2C_n \int_0^t \int_{\mathbb{R}^d} \left\|\Delta^n (\rho, u_m(s,y))\right\|_{\mathcal{H}_{T,S}^{\otimes n}} \left\|\Delta^n (\rho, u_m(s,y'))\right\|_{\mathcal{H}_{T,S}^{\otimes n}} \times G(t-s,x-y)G(t-s,x-y')f(y-y')dy'ds$$

$$=: 2\|\rho\|^2_{L^\infty(\mathbb{R})} C_n J^*_{2,1} + 2C_n J^*_{2,2}.$$

By the induction assumption, we have that

$$J^*_{2,2} \leq C \sup_{m \in \mathbb{N}} \sup_{(s,y) \in [0,T] \times \mathbb{R}} \left\|\Delta^n (\rho, u_m(s,y))\right\|_{\mathcal{H}_{T,S}^{\otimes (n-1)}}^2 < \infty.$$

Therefore, for some $C'_n > 0$,

$$\left\|D^n u_m(t,x)\right\|_{\mathcal{H}_{T,S}^n}^2 \leq C'_n + C'_n \int_0^t \int_{\mathbb{R}^d} \left\|D^n u_m(s,y)\right\|_{\mathcal{H}_{T,S}^{\otimes n}} \left\|D^n u_m(s,y')\right\|_{\mathcal{H}_{T,S}^{\otimes n}} \times G(t-s,x-y)G(t-s,x-y')f(y-y')dy'ds.$$

Finally, an application of Lemma 2.9 with $\mu(dx) \equiv \sqrt{C'_n} dx$ and $g(t,x) = \left\|D^n u_m(t,x)\right\|_{\mathcal{H}_{T,S}^n}$ proves (3.2.13). Therefore, (3.2.11) holds. This completes the proof of part (3) of Proposition 3.2. \qed

### 3.3 Density at a single point (Proof of Theorem 3.3)

In this section, we will prove Theorem 1.1. We need to first prove two lemmas and one special case (Theorem 3.5 below).

**Lemma 3.3.** If for some cube $Q' = (a'_1, b'_1) \times \cdots \times (a'_d, b'_d) \subset \mathbb{R}^d$, $a'_i < b'_i$, the measure $\mu$ restricted to this $Q' = [a'_1, b'_1] \times \cdots \times [a'_d, b'_d]$ has a density $f(x)$ with $f(x)$ being $\beta$-Hölder continuous for some $\beta \in (0,1)$, then for any compact set $Q \subset Q'$ and $T > 0$, there exists some finite constant $C := C(Q, Q', T, \beta, \mu) > 0$ such that

$$\left| \int_{\mathbb{R}^d} G(t,x-y)\mu(dy) - f(x) \right| \leq Ct^{\beta/2} \quad \text{for all } (t, x) \in (0,T] \times Q.$$

**Proof.** Fix $x \in Q$. Notice that

$$\left| \int_{\mathbb{R}^d} G(t,x-y)\mu(dy) - f(x) \right| \leq \int_{Q'} G(t,x-y) |f(y) - f(x)| dy$$

$$+ \int_{Q'} G(t,x-y)|\mu(dy)| + |f(x)| \int_{Q''} G(t,x-y)dy$$

$$=: I_1 + I_2 + I_3.$$
By the H"older continuity of $f$,

$$I_1 \leq C \int_{Q'} G(t, x - y) |y - x|^\beta dy$$

$$\leq C \int_{\mathbb{R}^d} t^{-d/2} G \left(1, \frac{y}{t^{1/2}}\right) |y|^\beta dy$$

$$= Ct^{\beta/2} \int_{\mathbb{R}^d} G(1, z) |z|^\beta dz = Ct^{\beta/2}.$$

Denote $\text{dist}(y, Q) = \min (|y - z| : z \in Q)$ and $\text{dist}(Q_1, Q_2) = \min (|y - z| : y \in Q_1, z \in Q_2)$. It is clear that $\text{dist}(y, x) \geq \text{dist}(y, Q) \geq \text{dist}(Q^c, Q) > 0$, for all $x \in Q$ and $y \in Q^c$.

Hence,

$$e^{-\frac{|y - x|^2}{4t}} \leq e^{-\frac{\text{dist}(Q^c, Q)^2}{4t}} e^{-\frac{\text{dist}(y, Q)^2}{4t}}, \quad \text{for all } x \in Q, y \in Q^c \text{ and } t \in (0, T],$$

which implies that

$$I_3 \leq Ct^{-d/2} e^{-\frac{\text{dist}(Q^c, Q)^2}{4t}} \left(\sup_{x \in Q'} |f(x)|\right) \int_{Q^c} e^{-\frac{\text{dist}(y, Q)^2}{4t}} dy \leq Ct.$$

Similarly,

$$I_2 \leq Ct^{-d/2} e^{-\frac{\text{dist}(Q^c, Q)^2}{4t}} \int_{Q^c} e^{-\frac{\text{dist}(y, Q)^2}{4t}} |\mu|(dy) \leq Ct,$$

which completes the proof of Lemma 3.3.

\[\square\]

**Lemma 3.4.** Let $u$ be the solution with the initial data $\mu$ that satisfies condition (1.4). Suppose there exists a cube $Q' = [a_1, b_1] \times \cdots \times [a_d, b_d]$ with $a_i < b_i$ such that the measure $\mu$ restricted to $Q'$ has a bounded density $g(x)$. Then for any $Q \subset Q'$, the following properties hold:

1. For all $T > 0$, $\sup_{t \in [0, T]} \|u(t, x)\| < +\infty$;

2. If $g$ is $\beta$-H"older continuous on $Q'$ for some $\beta \in (0, 1)$, then for all $x \in Q$,

$$\|u(t, x) - u(s, x)\| \leq Ct |t - s|^{\alpha/\beta} \quad \text{for all } 0 \leq s \leq t \leq T \text{ and } p \geq 2.$$

**Proof.** By Theorem 2.3, for all $x \in Q$ and $p \geq 2$,

$$\limsup_{t \to 0} \|u(t, x)\| \leq C \limsup_{t \to 0} \left(|\mu * G(t, \cdot)(x)\right) \leq C \sup_{y \in Q'} |g(y)|,$$

which implies part (1). As for part (2), notice that $u(t, x)$ consists of two parts as in (2.1.3). This property for $J_0(t, x)$ is guaranteed by Lemma 3.3 and that for $I(t, x)$ is proved in step 3 of the proof of Theorem 1.6 in [10].

\[\square\]

Next we will prove a sufficient condition for the existence and smoothness of density at a single point.
**Theorem 3.5.** Let $u(t,x)$ be the solution to equation (1.1) starting from an initial measure $\mu$ that satisfies (1.4). Assume that $\mu$ is proper at some point $x_0 \in \mathbb{R}^d$ with a density function $g$ over a neighborhood $Q$ of $x_0$. Suppose that $\rho(0,x_0,g(x_0)) \neq 0$ and that $g$ is $\beta$-Hölder continuous on $Q$ for some $\beta \in (0,1)$. We also assume that (1.3) holds for some $0 < \alpha \leq 1$. Then we have the following two statements:

(a) If $\rho$ is differentiable in the third argument with bounded Lipschitz continuous derivative, then for all $t > 0$ and $x \in \mathbb{R}^d$, $u(t,x)$ has an absolutely continuous law with respect to the Lebesgue measure.

(b) If $\rho$ is infinitely differentiable in the third argument with bounded derivatives, then for all $t > 0$ and $x \in \mathbb{R}^d$, $u(t,x)$ has a smooth density.

**Proof.** By Proposition 3.2, we know that

$$D_{\theta,\xi}u(t,x) = \rho(u(\theta,\xi))G(t - \theta, x - \xi) + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)\rho'(u(s,y))D_{\theta,\xi}u(s,y)W(dy)ds.$$

(3.3.2)

By part (3) of Proposition 3.2, we know that $u(t,x) \in D_{T,S}$. Denote by

$$C(t,x) = \int_0^t \|D_{\theta,u}(t,x)\|_H^2 d\theta.$$

(3.3.3)

Then both parts (a) and (b) will be proved once we can show that $\mathbb{E}[C(t,x)^{-p}] < \infty$ for all $p \geq 2$. By the assumption on $g$, we may find a cube $Q$ such that $|\rho(0,x,g(x))| \geq \delta > 0$ for all $x \in Q$. Let $\psi$ be a smooth function supported in $Q$ such that $0 \leq \psi(\xi) \leq 1$ and $\langle D_{\theta,u}(t,x), \psi \rangle_H \leq 1$.

Set

$$Y_\theta(t,x) = \int_{\mathbb{R}^d} D_{\theta,\xi}u(t,x) f(\xi - \xi')\psi(\xi')d\xi d\xi'.$$

(3.3.4)

Then, choose $0 < r < 1$ and $\epsilon < t$. By the Cauchy-Schwarz inequality,

$$C(t,x) \geq \int_0^t \langle D_{\theta,u}(t,x), \psi \rangle_H^2 d\theta \geq \int_0^t Y_\theta^2(t,x)d\theta \geq \epsilon^r Y_\theta^2(t,x) - \int_0^t |Y_\theta^2(t,x) - Y_\theta^2(t,x)| d\theta.
$$

Hence

$$\mathbb{P}(C(t,x) < \epsilon) \leq \mathbb{P}\left(\left|Y_0(t,x)\right| < \sqrt{2t \epsilon}\right) + \mathbb{P}\left(\int_0^t \left|Y_\theta^2(t,x) - Y_0^2(t,x)\right| d\theta > \epsilon\right) =: \mathbb{P}(A_1) + \mathbb{P}(A_2).$$
We will estimate $\mathbb{P}(A_1)$ and $\mathbb{P}(A_2)$. First, for $\mathbb{P}(A_1)$, in both sides of (3.2.1), take the inner product with $\psi$ in $\mathcal{H}$, we see that $Y_\theta(t, x)$ solves the following integral equation

$$
Y_\theta(t, x) = \int_{\mathbb{R}^d} \psi(\xi) \rho(u(\theta, \xi)) G(t - \theta, x - \xi) d\xi + \int_\theta^t \int_{\mathbb{R}^d} G(t - s, x - y) \rho'(u(s, y)) Y_\theta(s, y) W(ds dy). \tag{3.3.5}
$$

In particular, $Y_0(t, x)$ solves the following SPDE

$$
\begin{cases}
\left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right) Y_0(t, x) = \rho'(u(t, x)) Y_0(t, x) \dot{W}(t, x) \\
Y_0(0, x) = \psi(x) \rho(0, x, u_0(x)).
\end{cases} \tag{3.3.6}
$$

Hence Theorem 1.8 implies that for all $p \geq 1$, $t > 0$ and $x \in \mathbb{R}^d$,

$$
\mathbb{P}(A_1) \leq C_{t, x, \rho, \epsilon}^p, \text{ for } \epsilon \text{ small enough.} \tag{3.3.7}
$$

As for $\mathbb{P}(A_2)$, for all $q \geq 1$, by Minkowski’s inequality, we see that

$$
\mathbb{P}(A_2) \leq \frac{1}{c^q} \mathbb{E} \left[ \left( \int_0^t \left| Y_\theta^2(t, x) - Y_0^2(t, x) \right| d\theta \right)^q \right] \\
\leq \frac{1}{c^q} \left( \int_0^t \left\| Y_\theta^2(t, x) - Y_0^2(t, x) \right\|_q d\theta \right)^q \\
\leq \epsilon^{q(r-1)} \sup_{(\theta, x) \in [0, t] \times \mathbb{R}^d} \left( \| Y_\theta(t, x) - Y_0(t, x) \|_{2q}^2 \| Y_\theta(t, x) + Y_0(t, x) \|_{2q}^2 \right). 
$$

By the same arguments as the proof of Theorem 6.1 of [9], we have

$$
\sup_{(\theta, x) \in [0, t] \times \mathbb{R}^d} \mathbb{E} \left[ \left| Y_\theta(t, x) \right|^{2q} \right] < \infty. \tag{3.3.8}
$$

Now we write

$$
Y_\theta(t, x) - Y_0(t, x) = \Psi_1 - \Psi_2 + \Psi_3, \tag{3.3.9}
$$

where

$$
\Psi_1 = \int_{\mathbb{R}^d} \psi(\xi) \left[ \rho(u(\theta, \xi)) G(t - \theta, x - \xi) - \rho(u(0, \xi)) G(t, x - \xi) \right] d\xi, \\
\Psi_2 = \int_\theta^t \int_{\mathbb{R}^d} G(t - s, x - y) \rho'(u(s, y)) Y_\theta(s, y) W(ds dy), \\
\Psi_3 = \int_\theta^t \int_{\mathbb{R}^d} G(t - s, x - y) \rho'(u(s, y)) (Y_\theta(s, y) - Y_0(s, y)) W(ds dy).
$$

By the Lipschitz continuity of $\rho$, we have that

$$
\mathbb{E} \left[ |\Psi_1|^{2q} \right] \leq C \mathbb{E} \left[ \left( \int_{\mathbb{R}^d} \psi(\xi) |u(\theta, \xi) - u(0, \xi)| G(t - \theta, x - \xi) d\xi \right)^{2q} \right] \\
+ C \left( \int_{\mathbb{R}^d} \psi(\xi) |G(t - \theta, x - \xi) - G(t, x - \xi)| (1 + |u(0, \xi)|) d\xi \right)^{2q}
$$

30
is proved by applying Lemma 3.3.9. As for $\Psi_{12}$, by the mean-value theorem, we see that for some $r \in [t - \theta, t]$, 

$$\Psi_{12} \leq C \sup_{x \in Q} (1 + |g(x)|)^{2q} \left( \int_{\mathbb{R}^d} \psi(\xi) \left| \frac{\partial}{\partial t} G(s, x - \xi) \right| d\xi \right)^2 \leq C \theta^q.$$ 

For $\Psi_2$, 

$$\|\Psi_2\|_{2q}^2 \leq \int_0^\theta ds \int_{\mathbb{R}^d} dy dy' G(t-s, x-y) G(t-s, x-y') \|Y_0(s, y) Y_0(s, y')\|_{2q} f(y - y')$$ 

$$\leq C \sup_{(s, y) \in [0, \theta] \times \mathbb{R}^d} \|Y_0(s, y)\|_{2q} \int_0^\theta ds \int_{\mathbb{R}^d} e^{-(t-s)\|\xi\|^2} \widehat{f}(d\xi).$$ 

Because 

$$e^{-t|x|^2} \leq \frac{C}{(1 + t|\xi|^2)^{1-\alpha}} \leq \frac{C}{t^{1-\alpha}((1/T) + |\xi|^2)^{1-\alpha}}$$ 

for all $(t, x) \in [0, T] \times \mathbb{R}^d$, by (1.3), we see that 

$$\|\Psi_2\|_{2q}^2 \leq C \int_0^\theta \frac{1}{(t-s)^{1-\alpha}} ds \leq C \theta^\alpha,$$ 

for all $\theta \in [0, t]$. Applying the Burkholder-Davis-Gundy inequality to $\Psi_3$ in (3.3.9) yields 

$$\|Y_\theta(t, x) - Y_0(t, x)\|_{2q}^2 \leq C \left( \|\Psi_1\|_{2q}^2 + \|\Psi_2\|_{2q}^2 \right)$$ 

$$+ C \int_0^t ds \int_{\mathbb{R}^d} dy dy' G(t-s, x-y) G(t-s, x-y') f(y - y')$$ 

$$\times \sup_{z \in \mathbb{R}^d} \|Y_\theta(s, z) - Y_0(s, z)\|_{2q}^2 \leq C \theta^{\alpha + \beta} + C \int_0^t (t-s)^{-\alpha-1} \sup_{z \in \mathbb{R}^d} \|Y_\theta(s, z) - Y_0(s, z)\|_{2q}^2 ds.$$ 

Then Gronwall’s Lemma (see, e.g., Lemma A.2 in [9]) implies that 

$$\sup_{z \in \mathbb{R}^d} \|Y_\theta(s, z) - Y_0(s, z)\|_{2q}^2 \leq C \theta^{\alpha + \beta}.$$ 

Therefore, 

$$\sup_{0 \leq \theta \leq t', x \in \mathbb{R}^d} \mathbb{E} \left[ (Y_\theta(t, x) - Y_0(t, x))^2 \right] \leq C \epsilon^{r(\alpha + \beta)q},$$ 

(3.3.11) which implies that 

$$\mathbb{P}(\mathcal{A}_2) \leq C \epsilon^{q(1-t')^2} \epsilon^{r(\alpha + \beta)q} = C \epsilon^{q(1-t'+\frac{\alpha+\beta}{2})}.$$ 

(3.3.12) Thus we can choose any $r \in (1 - \frac{\alpha + \beta}{2}, 1) \subseteq (0, 1)$. Theorem 3.5 is proved by applying Lemma A.1 of [9].
Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Recall the mild solution \( u(t, x) = J_0(t, x) + I(t, x) \) in (2.1.3) and the critical time \( t_0 \) is defined in (1.7), that is,

\[
t_0 := \inf \left\{ s > 0, \sup_{y \in \mathbb{R}^d} |\rho(s, y, (G(s, \cdot) \ast \mu)(y))| \neq 0 \right\}.
\]

If condition (1.7) is not satisfied, that is, \( t \leq t_0 \), then \( I(t, x) \equiv 0 \). In this case, \( u(t, x) \equiv J_0(t, x) \) is deterministic. Hence, \( u(t, x) \) doesn’t have a density. This proves one direction for both parts (a) and (b).

On the other hand, if condition (1.7) is satisfied, that is, \( t > t_0 \), then by the continuity of the function

\[
(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \ni (t, x, z) \mapsto (t, x, \rho(t, x, (G(t, \cdot) \ast \mu)(x) + z) \in \mathbb{R},
\]
we know that for some \( \epsilon_0 \in (0, t - t_0) \) and some \( x_0 \in \mathbb{R}^d \), it holds that

\[
\rho(t_0 + \epsilon, y, (G(t_0 + \epsilon, \cdot) \ast \mu)(y) + z) \neq 0 \tag{3.3.13}
\]

for all \( (\epsilon, y, z) \in (0, \epsilon_0) \times B(x_0, \epsilon_0) \times [-\epsilon_0, \epsilon_0] \), where \( B(x, r) := \{ y \in \mathbb{R}^d : \|x - y\| \leq r \} \). Let

\[
\tau := (t_0 + \epsilon_0) \wedge \inf \left\{ t > t_0, \sup_{y \in B(x_0, \epsilon_0)} |I(t, y)| \geq \epsilon_0 \right\}.
\]

Denote \( \tilde{W}_\ast(t, x) := \tilde{W}(t + \tau, x) \) and \( \rho_\ast(t, x, z) := \rho(t + \tau, x, z) \). Let \( u_\ast(t, x) \) be the solution to the following stochastic heat equation

\[
\begin{cases}
\left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right) u_\ast(t, x) = \rho_\ast(t, x, u_\ast(t, x)) \tilde{W}_\ast(t, x), & t > 0, x \in \mathbb{R}^d, \\
u_\ast(0, x) = u(\tau, x), & x \in \mathbb{R}^d.
\end{cases}
\tag{3.3.14}
\]

By the construction, \( \sup_{y \in B(x_0, \epsilon_0)} |I(\tau, y)| \leq \epsilon_0 \) and thus, property (3.3.13) implies that

\[
\rho_\ast(0, y, u(\tau, y)) = \rho(\tau, y, J_0(\tau, y) + I(\tau, y)) \neq 0, \quad \text{for all } y \in B(x_0, \epsilon_0).
\]

Because \( y \mapsto u(\tau, y) \) is \( \beta \)-Hölder continuous a.s. for any \( \beta \in (0, \alpha) \), we can apply Theorem 3.5 to SPDE (3.3.14) to see that if \( \rho \) is differentiable in the third argument with bounded Lipschitz continuous derivative, then \( u_\ast(t, x) \) has a conditional density, denoted as \( \psi_t(x) \), that is absolutely continuous with respect to the Lebesgue measure. Moreover, if \( \rho \) is infinitely differentiable in the third argument with bounded derivatives, then this conditional density \( x \mapsto \psi_t(x) \) is smooth a.s. For any nonnegative continuous function \( g \) on \( \mathbb{R} \) with compact support,

\[
\mathbb{E} [g(u(t, x))] = \mathbb{E} [\mathbb{E} [g(u(t, x))|\mathcal{F}_\tau]]
\]

\[
= \mathbb{E} [\mathbb{E} [g(u_\ast(t - \tau, x))|\mathcal{F}_\tau]]
\]

\[
= \mathbb{E} \left[ \int_{\mathbb{R}} g(x) \psi_{t-\tau} (x) dx \right]
\]

32
Therefore, if \( \rho \) is differentiable in the third argument with bounded Lipschitz continuous derivative, then \( u(t, x) \) has a density, which is equal to \( E[\psi_{t-\tau}(x)] \). It is clear that this density is absolutely continuous with respect to the Lebesgue measure. Moreover, if \( \rho \) is infinitely differentiable in the third argument with bounded derivatives, this density is smooth. This completes the proof of Theorem 1.1.

### 3.4 Assumption 1.10 (Properties of \( k(t) \))

In this part, we study properties of \( k(t) \) defined in (1.16), which is closely related to the following function

\[
V_d(t) := \int_0^t ds \int_{\mathbb{R}^d} dy dy' \ G(s, y)G(s, y')f(y - y').
\]  

(3.4.1)

By Fourier transform, we see that

\[
V_d(t) = (2\pi)^{-d} \int_0^t ds \int_{\mathbb{R}^d} e^{-s|\xi|^2} \hat{f}(d\xi) = \int_0^t k(2s)ds.
\]  

(3.4.2)

**Lemma 3.6.** If

\[
\bar{B} := \left\{ \beta \in [0, 1) : \limsup_{t \downarrow 0} t^\beta k(t) < \infty \right\},
\]

\[
\underline{B} := \left\{ \beta \in [0, 1) : \liminf_{t \downarrow 0} t^\beta k(t) > 0 \right\},
\]

then the following properties hold:

(1) If \( \bar{B} \neq \emptyset \), then \( \inf \bar{B} \geq \sup \underline{B} \).

(2) If, for some \( \beta \in [0, 1) \), \( \lim_{t \downarrow 0} t^\beta k(t) \) exists and belongs to \( (0, \infty) \), then \( \inf \bar{B} = \sup \underline{B} = \beta \).

(3) If \( \bar{B} \neq \emptyset \), then for any \( \beta \in \bar{B} \),

\[
\sup_{t \in [0,T]} t^\beta k(t) < \infty \quad \text{and} \quad V_d(t) \leq C t^{1-\beta}.
\]

(4) \( \underline{B} \) is never an empty set since \( 0 \in \underline{B} \). Moreover, for any \( \beta \in \underline{B} \),

\[
\inf_{t \in [0,T]} t^\beta k(t) > 0 \quad \text{and} \quad V_d(t) \geq C t^{1-\beta}.
\]

**Proof.** Notice that \( g(t) \) is a strictly positive and nonincreasing function on \((0, \infty)\), and \( k(t) \) may blow up at \( t = 0 \), from which part (1) is clear.

As for (2), for any \( \beta' < \beta \), we have that \( \limsup_{t \downarrow 0} t^{\beta'} k(t) = \infty \), which implies that \( \beta = \inf \bar{B} \). Similarly, for any \( \beta'' > \beta \), we have that \( \liminf_{t \downarrow 0} t^{\beta''} k(t) = 0 \), which implies that \( \beta = \sup \underline{B} \).
(3) Fix \( t \in [0, T] \). Denote \( h(t) = \int_{\mathbb{R}^d} e^{-s(1+|\xi|^2)/2} \widehat{f}(d\xi) \). It is clear that
\[
h(t) \leq k(t) \leq e^T h(t).
\]
The function \( h(t) \) is a smooth function for \( t \in (0, T] \) because, by Dalang’s condition (1.2),
\[
h^{(n)}(t) = \int_{\mathbb{R}^d} \frac{e^{-s(1+|\xi|^2)/2}}{(1+|\xi|^2)^n} \widehat{f}(d\xi) < \infty, \quad \text{for all } n \geq 1.
\]
Hence, for some \( \beta > 0 \),
\[
\sup_{t \in [0, T]} t^\beta k(t) < \infty \iff \sup_{t \in [0, T]} t^\beta h(t) < \infty \iff \limsup_{t \downarrow 0} t^\beta h(t) < \infty \iff \limsup_{t \downarrow 0} t^\beta k(t) < \infty.
\]
Finally, notice that
\[
V_d(t) = (2\pi)^{-d} \int_0^t k(2s)ds \leq C \left[ \sup_{s \in [0, T]} s^\beta k(s) \right] \int_0^t s^{-\beta}ds = Ct^{1-\beta}.
\]
This proves (3).

(4) Since \( k(t) \) is a strictly positive and nonincreasing function, we see that \( 0 \in B \). Since \( h(t) > 0 \). We have that
\[
e^{-T} h(t)^{-1} \leq k(t)^{-1} \leq h(t)^{-1}.
\]
By the same arguments as in part (3), noticing that \( h^{-1}(t) \) is a smooth function on \((0, \infty)\), we see that
\[
\inf_{t \in [0, T]} t^\beta k(t) > 0 \iff \sup_{t \in [0, T]} t^{-\beta} k(t)^{-1} < \infty \iff \sup_{t \in [0, T]} t^{-\beta} h(t)^{-1} < \infty \iff \limsup_{t \downarrow 0} t^{-\beta} h(t)^{-1} < \infty \iff \limsup_{t \downarrow 0} t^{-\beta} k(t)^{-1} < \infty \iff \liminf_{t \downarrow 0} t^\beta k(t) > 0,
\]
and
\[
V_d(t) = (2\pi)^{-d} \int_0^t k(2s)ds \geq C \left[ \inf_{s \in [0, T]} s^\beta k(s) \right] \int_0^t s^{-\beta}ds = Ct^{1-\beta},
\]
which proves (4). This completes the proof of Lemma 3.6.

The following proposition shows that many common correlation functions satisfy Assumption 1.10.

**Proposition 3.7.** We have that
(1) For the space-time white noise case, that is, \( f(x) = \delta_0(x), \) \( \lim_{t \downarrow 0} t^{d/2} k(t) = (2\pi)^{-d/2}. \) In particular, when \( d = 1, \text{inf} \mathcal{B} = \text{sup} \mathcal{B} = 1/2, \) and when \( d \geq 2, \mathcal{B} = \mathcal{B} = \phi. \)

(2) For the Riesz kernel \( f(x) = |x|^{-\beta} \) with \( \beta' \in (0, 2 \wedge d), \) \( \lim_{t \downarrow 0} t^{\beta'/2} g(t) = C \in (0, \infty) \) and hence, \( \text{inf} \mathcal{B} = \text{sup} \mathcal{B} = \beta'/2. \)

(3) For the fractional noise case, that is, \( f(x) = \prod_{j=1}^d |x_j|^{2H_j-2} \) with \( H_j \in (1/2, 1) \) and \( d - \sum_{j=1}^d H_j < 1, \) we have that \( \text{inf} \mathcal{B} = \text{sup} \mathcal{B} = d - H, \) where \( H := \sum_{j=1}^d H_j. \)

(4) If \( f(0) < \infty \) (or equivalently \( \hat{f} \in L^1(\mathbb{R}^d) \)), then \( \text{inf} \mathcal{B} = \text{sup} \mathcal{B} = 0. \)

(5) If \( f \) is the Bessel kernel of order \( \alpha' > 0 \) (see (1.9)), then we have that

(a) If \( \alpha' > d - 2, \) \( f \) satisfies Dalang’s condition (1.3) for any \( \alpha \in (0, (\alpha' + 2 - d)/2). \)

(b) As \( t \to 0, \) the function \( k(t) \) defined in (1.16) satisfies the following property:

\[
k(t) \simeq \begin{cases} t^{\alpha'-d} & \text{if } \alpha' \in (0, d), \\ \log(1/t) & \text{if } \alpha' = d, \\ 1 & \text{if } \alpha' > d. \\
\end{cases}
\]

Proof.

(1) For the space-time white noise, \( k(t) = G(t, 0) = (2\pi t)^{-d/2}. \) It is clear that \( d/2 \in [0, 1) \) if and only if \( d = 1. \) This proves (1).

(2) For the Riesz kernel case, \( \hat{f}(\xi) = C|\xi|^{\beta'-d} \) and hence,

\[
k(t) = C \int_{\mathbb{R}^d} e^{-t|\xi|^2/2} |\xi|^{\beta'-d} d\xi = C \int_0^\infty e^{-tr} r^{\beta'-d} r^{d-1} dr = Ct^{-\beta'/2},
\]

This case is proved by an application of part (2) of Lemma 3.6.

(3) In this case,

\[
\hat{f}(d\xi) = \prod_{j=1}^d C_j |\xi_j|^{1-2H_j} d\xi_j \quad \text{with} \quad C_j = \frac{\Gamma(2H_j + 1) \sin(\pi H_j)}{2\pi} > 0.
\]

Therefore,

\[
k(t) = \prod_{j=1}^d C_j \int_{\mathbb{R}} e^{-t\xi_j^2/2} |\xi_j|^{1-2H_j} d\xi_j = \prod_{j=1}^d C_j \Gamma(1 - H_j)t^{H_j-1} = Ct^{\sum_{j=1}^d H_j-1}.
\]

Then an application of part (2) of Lemma 3.6 proves part (3).

(4) If \( f(0) < \infty, \) then \( k(t) \leq f(0) < \infty, \) where the first inequality is due to the fact that both \( f(\cdot) \) and \( G(t, \cdot) \) are nonnegative definite. Hence, \( \text{inf} \mathcal{B} = 0. \)

(5) Now we study the Bessel kernel (1.9). Notice that

\[
f(x) = (4\pi)^{d/2} \int_0^\infty w^{\alpha'-2} e^{-w} G(2w, x) dw,
\]
where $G(w, x)$ is the heat kernel. Hence, the Fourier transform of $f$ is equal to

$$\hat{f}(x) = (4\pi)^{d/2} \int_0^\infty w^{\alpha'/2} e^{-w |\xi|^2} dw$$

$$= (4\pi)^{d/2} (1 + |\xi|^2)^{-\alpha'/2} \int_0^\infty e^{-w \alpha'/2} dw$$

$$= (4\pi)^{d/2} \Gamma(\alpha'/2)(1 + |\xi|^2)^{-\alpha'/2}.$$

Hence, Dalang’s condition (1.3) becomes

$$\int_{\mathbb{R}^d} \frac{d\xi}{(1 + |\xi|^2)^{\alpha'/2 + 1}} < \infty,$$

which implies that $\alpha' + 2 - 2\alpha > d$. Therefore, if $\alpha' > d - 2$, Dalang’s condition (1.3) is satisfied for any $\alpha \in (0, (\alpha' + 2 - d)/2)$. This proves part (a).

As for (b), notice that from (1.16), by the spherical coordinates,

$$k(t) = C \int_{\mathbb{R}^d} \frac{e^{-t|\xi|^2/2}}{(1 + |\xi|^2)^{\alpha'/2}} d\xi = C \int_0^\infty \frac{e^{-t r^2/2}}{(1 + r^2)^{\alpha'/2}} r^{d-1} dr = CU \left(\frac{d}{2}, \frac{2 + d - \alpha'}{2}, \frac{t}{2}\right),$$

where $U(a, b, z)$ is the confluent hypergeometric function (see [24, Chapter 13]) and the last equation is due to [24, Eq. 13.4.4 on p. 326]. Therefore, by the seven cases from 13.2.16 to 13.3.22 of [24], we see that as $t \to 0$,

$$k(t) = \begin{cases} 
C t^{\alpha'/2} + O(t^{1 + \alpha'/2}) & \alpha' \in (0, d - 2), \\
C t^{-1} + O(\log t) & \alpha' = d - 2, \\
C t^{\alpha'/2} + C + O(t^{1 + \alpha'/2}) & \alpha' \in (d - 2, d), \\
C \log(1/t) + C' + O(t \log t) & \alpha' = d, \\
C + O(t^{\alpha'/2}) & \alpha' \in (d, d + 2), \\
C + O(t \log t) & \alpha' = d + 2, \\
C + O(t) & \alpha' > d + 2.
\end{cases}$$

Combining the above cases proves part (b). This completes the proof of Proposition 3.7. □

We will need the following lemma.

**Lemma 3.8.** For all $\alpha > 0$, it holds that

$$\sup_{t > 0} \frac{V_d(\alpha t)}{V_d(t)} < \infty.$$

**Proof.** Simple calculations show that

$$1 = \inf_{x \in \mathbb{R}} \frac{1 + x^2}{x^2} (1 - e^{-x^2}) \leq \sup_{x \in \mathbb{R}} \frac{1 + x^2}{x^2} (1 - e^{-x^2}) := C' < \infty.$$
Noticing that by (3.4.2),
\[ V_d(t) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{1 - e^{-t||\xi||^2}}{||\xi||^2} \hat{f}(d\xi). \]
we see that for all \( t > 0 \),
\[ (2\pi)^{-d} t \int_{\mathbb{R}^d} \frac{1}{1 + t||\xi||^2} \hat{f}(d\xi) \leq V_d(t) \leq C'(2\pi)^{-d} t \int_{\mathbb{R}^d} \frac{1}{1 + t||\xi||^2} \hat{f}(d\xi). \]
In case of \( \alpha < 1 \),
\[ \frac{V_d(\alpha t)}{V_d(t)} \leq \frac{C'\alpha t}{t} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{1 + \alpha t||\xi||^2} = C' \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{1 + \alpha t||\xi||^2} \leq C', \]
and in case of \( \alpha \geq 1 \),
\[ \frac{V_d(\alpha t)}{V_d(t)} \leq \frac{\alpha C' t}{t} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{1 + t||\xi||^2} = C' \alpha \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{1 + t||\xi||^2} \leq C'\alpha. \]
This proves the lemma. \( \square \)

### 3.5 Density at multiple points (Proof of Theorem 1.4)

We start the proof of Theorem 1.4 by denoting
\[ f_{\gamma,\beta}(x) := \exp \left\{ -2\beta \left[ \log \frac{1}{|x|} \wedge 1 \right]^\gamma \right\}, \quad \text{for } x \in \mathbb{R}, \]
where \( \gamma \in (0, 1 + \alpha) \) and \( \beta > 0 \) are the constants given in the condition (1.8). Fix \( t > 0 \) and \( m \) distinct points \( \{x_1, \ldots, x_m\} \subseteq \mathbb{R}^d \). Let \( \epsilon_0 \) be any positive constant such that
\[ \epsilon_0 \leq \min(t/2, 1) \quad \text{and} \quad 2(2\epsilon_0)^{1/2} \leq \min_{i \neq j} |x_i - x_j|. \]

We begin by writing
\[ D_{r,z}u(t, x) = \rho(u(r, z))G(t - r, x - z) + Q_{r,z}(t, x), \]
where \( r \in (0, t], z \in \mathbb{R}^d, \) and
\[ Q_{r,z}(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)\rho'(u(s, y))D_{r,z}u(s, y)W(dsdy). \]

Denote
\[ \psi_{r,z}(t, x) := D_{r,z}u(t, x). \]
Let $S_{r,z}(t, x)$ be the solution to the equation

\[ S_{r,z}(t, x) = G(t - r, x - z) + \int_r^t \int_{\mathbb{R}^d} G(t - s, x - y) \rho'(u(s, y)) S_{r,z}(s, y) W(dsdy). \quad (3.5.4) \]

Using the uniqueness of the solution to the SPDE, we can write

\[ \psi_{r,z}(t, x) = S_{r,z}(t, x) \rho(u(r, z)) . \quad (3.5.5) \]

Note that this also shows that the Malliavin derivative of the solution $(r, z) \mapsto D_{r,z}u(t, x)$ is a function in $L^2(\mathbb{R}^d; \mathcal{H})$. Set $\mathcal{T} := [t/2, t]$. For some $R \geq 2A$ large enough where $A$ is the constant in Assumption 1.3, set $S = \{ x \in \mathbb{R}^d, |x| \leq R \}$ such that $x_i \in S$ for all $i = 1, \cdots, m$. Define

\[ \tilde{V}_d(\epsilon) := \int_0^t \int_{\mathcal{S}^2} dzd' G(r, z)G(r, z') f(z - z') . \quad (3.5.6) \]

Recall that $V_d(\cdot)$ is defined in (3.4.1). The following lemma shows that both $V_d(\epsilon)$ and $\tilde{V}_d(\epsilon)$ has the same order as $\epsilon$ goes to zero.

**Lemma 3.9.** Under Assumption 1.3, it holds that

\[ \lim_{\epsilon \to 0^+} \frac{V_d(\epsilon)}{\tilde{V}_d(\epsilon)} = 1. \]

**Remark 3.10.** We first remark that if $f$ satisfies some scaling property, such as the Riesz kernel, then this property can be easily proved. Let us see this through Riesz kernel case. By the l’Hospital rule,

\[
\lim_{\epsilon \to 0^+} \frac{V_d(\epsilon)}{\tilde{V}_d(\epsilon)} = \lim_{\epsilon \to 0^+} \frac{\int_{|x| \leq R} \int_{|z| \leq R} G(\epsilon, z)G(\epsilon, z') f(z - z') dzd' \int_{|z| \leq R} \int_{|z'| \leq R} G(\epsilon, z)G(\epsilon, z') f(z - z') dzd'}{\int_{|x| \leq R} \int_{|z| \leq R} G(1, z)G(1, z') f(\sqrt{\epsilon}(z - z')) dzd' \int_{|z| \leq R} \int_{|z'| \leq R} G(1, z)G(1, z') f(\sqrt{\epsilon}(z - z')) dzd'} = 1,
\]

where the last step is due to the dominated convergence theorem. However, for general $f$, to prove this property is much less straightforward. Indeed, we need to impose some conditions on $f$, namely, Assumption 1.3.

**Proof of Lemma 3.9.** Let $A > 1$ be the constant in Assumption 1.3. Throughout the proof, we assume that $\epsilon \in (0, 1/(A^2R^2))$. For any $\epsilon > 0$ and $H \subseteq \mathbb{R}^{2d}$, denote

\[ I_H(\epsilon) := \int_H dzd' G(\epsilon, z)G(\epsilon, z') f(z - z'). \]

By the l’Hospital rule,

\[ \lim_{\epsilon \to 0^+} \frac{V_d(\epsilon)}{\tilde{V}_d(\epsilon)} = \lim_{\epsilon \to 0^+} \frac{I_{\mathcal{S}^2}(\epsilon)}{I_{\mathcal{S}^2}(\epsilon)}. \]
Notice that by (2.2.8), \( G(\epsilon, z)G(\epsilon, z') = G(2\epsilon, z - z')G(\epsilon/2, (z + z')/2) \). Hence, by change of variables \( y = z - z' \) and \( y' = (z + z')/2 \), we see that

\[
I_{R^{2d}}(\epsilon) = \int \int_{\mathbb{R}^{2d}} dy \, dy' G(2\epsilon, y)G(\epsilon/2, y') f(y)
\]
\[
= \int_{\mathbb{R}^{d}} G(2\epsilon, y) f(y) dy
\]
\[
= \int_{|y| \leq \frac{R}{\sqrt{\epsilon}}} G(2, y) f(\sqrt{\epsilon}y) dy + \int_{|y| > \frac{R}{\sqrt{\epsilon}}} G(2, y) f(\sqrt{\epsilon}y) dy.
\]

Recall that \( R \geq 2A \). By Assumption 1.3,

\[
I_{R^{2d}}(\epsilon) \leq \int_{|y| \leq \frac{R}{\sqrt{\epsilon}}} G(2, y) f(\sqrt{\epsilon}y) dy + C_R \Theta(\epsilon),
\]

where

\[
C_R := \sup_{|y| \geq R/2} f(y) \quad \text{and} \quad \Theta(\epsilon) := \int_{|y| > \frac{R}{\sqrt{\epsilon}}} G(2, y) dy.
\]

Similarly,

\[
I_{S^2}(\epsilon) \geq \int_{|y| \leq R} dy \int_{|y'| \leq R/2} dy' G(\epsilon/2, y') \int_{|y| \leq R} dy G(2\epsilon, y)
\]
\[
= \int_{|y| \leq \frac{R}{\sqrt{\epsilon}}} dy \int_{|y'| \leq \frac{R}{2\sqrt{\epsilon}}} dy' G(2, y) f(\sqrt{\epsilon}y) dy + \int_{|y| > \frac{R}{\sqrt{\epsilon}}} dy \int_{|y'| > \frac{R}{2\sqrt{\epsilon}}} dy' G(1/2, y').
\]

For any \( \delta \in (0, 1) \), as \( \epsilon \) is small enough, we can always ensure that

\[
\int_{|y'| \leq \frac{R}{2\sqrt{\epsilon}}} dy' G(1/2, y') \geq \delta,
\]

which implies that

\[
I_{S^2}(\epsilon) \geq \delta \int_{|y| \leq \frac{R}{\sqrt{\epsilon}}} f(\sqrt{\epsilon}y) G(2, y) dy.
\]

Therefore, for \( \epsilon \) small enough,

\[
\frac{I_{R^{2d}}(\epsilon)}{I_{S^2}(\epsilon)} \leq \delta^{-1} \frac{\int_{|y| \leq \frac{R}{\sqrt{\epsilon}}} G(2, y) f(\sqrt{\epsilon}y) dy + C_R \Theta(\epsilon)}{\int_{|y| \leq \frac{R}{\sqrt{\epsilon}}} G(2, y) f(\sqrt{\epsilon}y) dy}
\]
\[
\leq \delta^{-1} \left( 1 + \frac{C_R \Theta(\epsilon)}{\int_{|y| \leq \frac{R}{\sqrt{\epsilon}}} G(2, y) f(\sqrt{\epsilon}y) dy} \right).
\]

Because \( \epsilon < 1/(A^2R^2) \), that is \( \sqrt{\epsilon}R \leq \sqrt{\epsilon}A < 1/A \), by Assumption 1.3,

\[
\int_{|y| \leq \frac{R}{\sqrt{\epsilon}}} G(2, y) f(\sqrt{\epsilon}y) dy \geq \int_{|y| \leq R} G(2, y) f(\sqrt{\epsilon}y) dy \geq C_{R,f},
\]

where

\[
C_{R,f} := \inf_{|z| < 1/A} f(z) \int_{|y| \leq R} G(2, y) dy > 0.
\]
Therefore,
\[
\frac{I_{R^2}(\epsilon)}{I_{S^2}(\epsilon)} \leq \delta^{-1} \left( 1 + C^{-1}_{R,1} C_{R}(\epsilon) \right) \rightarrow \delta^{-1}, \quad \text{as } \epsilon \downarrow 0.
\]

Finally, since \( \delta \) can be arbitrarily close to 1, this proves the lemma. \( \square \)

Let us continue our proof of Theorem 1.4. Define
\[
\sigma_{i,j} := \langle Du(t, x_i), Du(t, x_j) \rangle_{\mathcal{H}_{r,s}} \nonumber
= \int_T dr \iint_{S^2} dzd{z'} \psi_{r,z}(t, x_i) \psi_{r,z'}(t, x_j) f(z - z').
\]
Let \( \sigma \) be the matrix with entries \( \sigma_{i,j}, 1 \leq i, j \leq m \). For any \( \xi \in \mathbb{R}^m \) and any \( \epsilon \in (0, \epsilon_0) \), consider the inner product in the Euclidean space
\[
\langle \sigma \xi, \xi \rangle = \sum_{i,j=1}^m \xi_i \xi_j \int_T \iint_{S^2} dzd{z'} \psi_{r,z}(t, x_i) \psi_{r,z'}(t, x_j) f(z - z') \nonumber
\geq \int_{t-\epsilon}^t dr \iint_{S^2} dzd{z'} \left( \sum_{i=1}^m \psi_{r,z}(t, x_i) \xi_i \right) \left( \sum_{i=1}^m \psi_{r,z'}(t, x_j) \xi_j \right) f(z - z') \nonumber
\geq \sum_{j=1}^m \xi_j^2 \int_{t-\epsilon}^t dr \iint_{S^2} dzd{z'} \psi_{r,z}(t, x_j) \psi_{r,z'}(t, x_j) f(z - z') \nonumber
+ \sum_{j=1}^m \sum_{i \neq j} \xi_i \xi_j \int_{t-\epsilon}^t dr \iint_{S^2} dzd{z'} \psi_{r,z}(t, x_i) \psi_{r,z'}(t, x_j) f(z - z') \nonumber
= I_\epsilon^* + I^{(1)}_\epsilon(\xi).
\]
Apply twice the following inequality
\[
\|a + b\|^2 \geq (\|a\| - \|b\|)^2 \geq \frac{2}{3} \|a\|^2 - 2 \|b\|^2 \tag{3.5.7}
\]
to see that
\[
I_\epsilon^* \geq \frac{2}{3} \sum_{j=1}^m \xi_j^2 \int_{t-\epsilon}^t dr \iint_{S^2} dzd{z'} \rho(u(r, z)) \rho(u(r, z')) \nonumber
\times G(t - r, x_j - z) G(t - r, x_j - z') f(z - z') \nonumber
- 2 \sum_{j=1}^m \xi_j^2 \int_{t-\epsilon}^t dr \iint_{S^2} dzd{z'} Q_{r,z}(t, x_j) Q_{r,z'}(t, x_j) f(z - z') \nonumber
\geq \frac{4}{9} \sum_{j=1}^m \xi_j^2 \int_{t-\epsilon}^t dr \iint_{S^2} dzd{z'} \rho(u(t, x_j))^2 G(t - r, x_j - z) G(t - r, x_j - z') f(z - z') \nonumber
- \frac{4}{3} \sum_{j=1}^m \xi_j^2 \int_{t-\epsilon}^t dr \iint_{S^2} dzd{z'} \left[ \rho(u(t, x_j)) - \rho(u(r, z)) \right] \left[ \rho(u(t, x_j)) - \rho(u(r, z')) \right] \nonumber
\times G(t - r, x_j - z) G(t - r, x_j - z') f(z - z') \nonumber
- 2 \sum_{j=1}^m \xi_j^2 \int_{t-\epsilon}^t dr \iint_{S^2} dzd{z'} Q_{r,z}(t, x_j) Q_{r,z'}(t, x_j) f(z - z')
\]
\[ \frac{4}{9} I^{(0)}_\epsilon(\xi) - \frac{4}{3} I^{(2)}_\epsilon(\xi) - 2 I^{(3)}_\epsilon(\xi). \]

So
\[ (\det \sigma)^{1/d} \geq \inf_{|\xi|=1} \langle \sigma \xi, \xi \rangle \geq \inf_{|\xi|=1} I^{(0)}_\epsilon(\xi) - 2 \sum_{i=1}^{3} \sup_{|\xi|=1} |I^{(i)}_\epsilon(\xi)|. \] (3.5.8)

By the same arguments as those in the proof of Theorem 1.2 of [9], we see that
\[ I^{(0)}_\epsilon(\xi) \geq \inf_{x \in K} \rho(u(t, x))^2 \int_0^\epsilon \int \int_{S^2} dz' G(r, z) G(r, z') f(z - z'). \]
\[ = \sqrt{V}(\epsilon) \inf_{x \in K} \rho(u(t, x))^2. \]

For \( I^{(i)}_\epsilon, i = 1, 2, 3 \), we will estimate their upper bound of the \( L^p \) norm. By Minkowski’s inequality and the Cauchy-Schwarz inequality, we see that
\[
\left\| \sup_{|\xi|=1} I^{(1)}_\epsilon \right\|_p \leq \sum_{j=1}^m \sum_{i \neq j} \int_{t-\epsilon}^t dr \int \int_{S^2} d\phi d\rho \|\psi_{r,z}(r, \phi, \rho, t, x, j)\|_p \phi(z - z')
\leq \sum_{j=1}^m \sum_{i \neq j} \int_{t-\epsilon}^t dr \int \int_{S^2} d\phi d\rho \|\psi_{r,z}(r, \phi, \rho, t, x, j)\|_{2p} \phi(z - z')
\leq C_{t,K} \sum_{j=1}^m \sum_{i \neq j} \int_{t-\epsilon}^t dr \int \int_{S^2} d\phi d\rho \|S_{r,z}(r, \phi, \rho, t, x, j)\|_{4p} \phi(z - z'),
\]
where
\[ C_{t,K} := \sup_{(s,y) \in [t/2,t] \times K} \|\rho(u(s, y))\|_{2p}. \]

The next lemma gives a moment bound for \( S_{r,z}(t, x) \).

**Lemma 3.11.** For \( t \in (0, T], x \in \mathbb{R}^d \) and \( p \geq 2 \), there exists a constant \( C = C(T, p) > 0 \) such that
\[ \|S_{r,z}(t, x)\|_p \leq CG(t - r, x - z), \quad \text{for all } r \in (0, t] \text{ and } x, z \in \mathbb{R}^d. \]

**Proof.** Because \( \rho' \) is bounded, by the Burkholder-Davis-Gundy inequality,
\[
\|S_{r,z}(t, x)\|_p^2 \leq C_p G(t - r, x - z)^2 + C_p \int_r^t ds \int \int_{\mathbb{R}^{2d}} dy dy' \phi(t - s, x - y) G(t - s, x - y'^') \times \|S_{r,z}(s, y)\|_p \|S_{r,z}(s, y')\|_p \phi(y - y').
\]
By setting \( \theta = t - r, \eta = x - z \) and \( g(\theta, \eta) = \|S_{r,z}(\theta + r, \eta + z)\|_p \), we see that
\[
g(\theta, \eta)^2 \leq C_p G(\theta, \eta)^2 + C_p \int_0^\theta \int \int_{\mathbb{R}^{2d}} dy dy' \phi(t - s, \eta - y) G(t - s, \eta - y'^') \times g(s, y)g(s, y') \phi(y - y').
\]
Therefore, this lemma is proved by an application of Lemma 2.7 with \( \mu = \sqrt{C_p} \delta_0. \)
By Lemma 3.11 and the property of $\Psi_n(T, x; 0)$ in Proposition 4.1, we have that

$$\left\| \sup_{|\xi|=1} I^{(1)}_\xi \right\|_p \leq C \sum_{j=1}^m \sum_{i \neq j} \int_{t-\epsilon}^t \int_{S^2} d\xi d\eta' \, G(t-r, x_i - z)G(t-r, x_j - z') f(z - z')$$

$$\leq C \sum_{j=1}^m \sum_{i \neq j} C \xi_i - x_j \, \bar{V}_d(\epsilon) \epsilon^\beta$$

where the constant $C$ in the last expression depends on $x_i, i = 1, \ldots, m$.

To estimate $I^{(2)}_\epsilon(\xi)$, we note that by Theorem 2.4,

$$\| u(r, z) - u(s, y) \|_p \leq C_p (|r-s|^2 + |y-z|^\alpha) \quad \text{for all } r, s \in [0, T], y, z \in K \; , \quad (3.5.9)$$

for some constant $C_p$ which depends on $T$ and $K$. Thus we have

$$\left\| \sup_{|\xi|=1} I^{(2)}_\epsilon(\xi) \right\|_p \leq \sum_{j=1}^m \int_{t-\epsilon}^t \int_{S^2} d\xi d\eta' \, \| u(r, z) - u(t, x_j) \|_{2p} \| u(r, z') - u(t, x_j) \|_{2p}$$

$$\times G(t-r, x_j - z)G(t-r, x_j - z') f(z - z')$$

$$\leq C \int_{t-\epsilon}^t \int_{S^2} d\xi d\eta' \, (|r|^2 + |y|^\alpha) (|r|^2 + |z'|^\alpha) G(r, z)G(r, z') f(z - z')$$

$$\leq C \left( \xi^2 + \sqrt{\epsilon} \right)^2 \int_{t-\epsilon}^t \int_{S^2} d\eta d\eta' \, G(r, z)G(r, z') f(z - z')$$

$$= C \epsilon \alpha \bar{V}_d(\epsilon).$$

To estimate $I^{(3)}_\epsilon(\xi)$, we first claim that

$$\| Q_{r,z}(t, x) \|^2_{2p} \leq C G(t-r, x-z)^2(t-r)^\alpha, \quad \text{for all } z \in K \; . \quad (3.5.10)$$

Indeed, by the Burkholder-Davis-Gundy inequality, we see that

$$\| Q_{r,z}(t, x) \|^2_{2p} \leq 4p \int_r^t ds \int_{\mathbb{R}^{2d}} dy dy' \, G(t-s, x-y)G(t-s, x-y')$$

$$\times \| D_{r,z} u(s, y) \|_{2p} \| D_{r,z} u(s, y') \|_{2p} f(y - y')$$

$$\leq C_{t,K} p \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' \, G(t-s, x-y)G(t-s, x-y')$$

$$\times \| S_{r,z} u(s, y) \|_{4p} \| S_{r,z} u(s, y') \|_{4p} f(y - y')$$

$$\leq C \int_r^t ds \int_{\mathbb{R}^{2d}} dy dy' \, G(t-s, x-y)G(t-s, x-y')$$

$$\times G(s-r, y-z)G(s-r, y'-z) f(y - y') ,$$

where we have applied Lemma 3.11 and have used the inequality that for $z \in K$ and $r \in (t-\epsilon, t)$,

$$\| D_{r,z} u(s, y) \|_{2p} \leq \| S_{r,z} u(s, y) \|_{4p} \| \rho(u(r, z)) \|_{4p} \leq C_{t,K} \| S_{r,z} u(s, y) \|_{4p} .$$
Next we use identity (2.2.8) to see that that
\[
\|Q_{r,z}(t,x)\|_{2^p}^2 \leq C \int_r^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t-r,x-z)G(t-r,x-z) \\
\times G\left(\frac{(t-s)(s-r)}{t-r},y-\frac{t-s}{t-r}z-\frac{s-r}{t-r}x\right) \\
\times G\left(\frac{(t-s)(s-r)}{t-r},y'-\frac{t-s}{t-r}z-\frac{s-r}{t-r}x\right) f(y-y') \\
\leq C G(t-r,x-z)^2 \int_r^t ds \int_{\mathbb{R}^d} e^{-\frac{2(t-s)(s-r)}{t-r} |\xi|^2} \hat{f}(d\xi).
\]
Notice that for \( \beta > 0, x \geq 0 \) and \( \theta \in (0, \Theta) \),
\[
(1 + x^\beta) e^{-\theta x^2} = \theta^{-\beta/2} e^{-((\sqrt{\beta}x)^2)} \leq \theta^{-\beta/2} \max_{y>0} e^{-\theta y^2} (\Theta^{\beta/2} + y^\beta). \tag{3.5.11}
\]
Apply the above inequality with \( \beta = 2(1-\alpha) \) and
\[
\theta := \frac{2(t-s)(s-r)}{t-r} \leq t-r < T =: \Theta
\]
to see that
\[
(1 + |\xi|^2)^{1-\alpha} e^{-\frac{2(t-r)(s-r)}{t-r} |\xi|^2} \leq C \left[ \frac{t-r}{(t-s)(s-r)} \right]^{1-\alpha}. \tag{3.5.12}
\]
Hence, by the Beta integral and condition (1.3),
\[
\|Q_{r,z}(t,x)\|_{2^p}^2 \leq C G(t-r,x-z)^2 \int_r^t ds \left[ \frac{t-r}{(t-s)(s-r)} \right]^{1-\alpha} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{(1 + |\xi|^2)^{1-\alpha}}
\]
which proves (3.5.10). Therefore, by Minkowski’s inequality, we have that
\[
\left\| \sup_{|\xi|=1} I^{(3)}(\xi) \right\| \leq C \int_{t-r}^t dr \int_{\mathbb{R}^d} dz dz' \|Q_{r,z}(t,x)\|_{2p} \|Q_{r,z'}(t,x)\|_{2p} f(z-z')
\]
\[
\leq C \int_{t-r}^t dr (t-r)^\alpha \int_{\mathbb{R}^d} dz dz' G(t-r,z)G(t-r,z') f(z-z')
\]
\[
\leq C e^\alpha \overline{V}_d(\epsilon).
\]
Finally, by choosing \( \eta \in (0, \alpha \wedge \beta) \), we have that
\[
P \left( (\det \sigma)^{1/d} < \overline{V}_d(\epsilon) e^\eta \right) \leq P \left( \frac{4}{9} \inf_{|\xi|=1} |I^{(0)}(\xi)| < 2 \overline{V}_d(\epsilon) e^\eta \right) + \sum_{i=1}^3 P \left( 2 \sup_{|\xi|=1} |I^{(i)}(\xi)| > \frac{1}{3} \overline{V}_d(\epsilon) e^\eta \right)
\]
\[
\leq P \left( \inf_{x \in K} f_{\beta,\gamma}(u(t,x)) < 5 e^\eta \right) + C e^{\beta(\alpha-\eta)},
\]
where \( f_{\beta,\gamma} \) is defined in (3.5.1). Notice that for any \( \theta \) and \( x \in (0,1) \),
\[
\exp \left\{ -2\beta \left[ \log \left( \frac{1}{x} \right) \right]^\gamma \right\} < \theta \iff x < \exp \left\{ -(2\beta)^{-1/\gamma} \left[ \log \left( \frac{1}{\theta} \right) \right]^{1/\gamma} \right\}.
\]
43
Hence, as $\epsilon$ is small enough, for some constant $C_0 > 0$,

$$
P \left( \inf_{x \in K} f_{\beta,\gamma}(u(t, x)) < 5 \epsilon^\eta \right) = P \left( \inf_{x \in K} u(t, x) \wedge 1 < \exp \left\{ -C_0 \left( \frac{\eta}{2\beta} \right)^{\frac{1}{\gamma}} \log \frac{1}{\epsilon}^{\frac{1}{\gamma}} \right\} \right)
$$

$$
= P \left( \inf_{x \in K} u(t, x) < \exp \left\{ -C_0 \left( \frac{\eta}{2\beta} \right)^{\frac{1}{\gamma}} \log \frac{1}{\epsilon}^{\frac{1}{\gamma}} \right\} \right)
$$

$$
\leq \exp \left( -C \left[ \log(1/\epsilon) \right]^{\frac{1 + \alpha}{\gamma}} \right),
$$

where in the last step we have applied Theorem 1.8. Therefore,

$$
P \left( (\det \sigma)^{1/d} < \tilde{V}_d(\epsilon^{\eta}) \right) \leq \exp \left( -C \left[ \log(1/\epsilon) \right]^{\frac{1 + \alpha}{\gamma}} \right) + C \epsilon^{p(\alpha - \eta)}. \quad (3.5.13)
$$

Finally, by Lemma 3.9 and part (4) of Lemma 3.6, $\tilde{V}_d \geq CV_d(\epsilon) \geq C'\epsilon^{1-\beta}$. From (3.5.13) we see that as $\epsilon$ small enough,

$$
P \left( (\det \sigma)^{1/d} < \epsilon^{1+\eta-\beta} \right) \leq P \left( (\det \sigma)^{1/d} < R(\epsilon)^{\eta} \right) \leq \exp \left( -C \left[ \log(1/\epsilon) \right]^{\frac{1 + \alpha}{\gamma}} \right) + C \epsilon^{p(\alpha - \eta)}.
$$

Because $\gamma < 1 + \alpha$, an application of Lemma A.1 of [9] shows that $E[(\det \sigma)^{-p}] < \infty$ for all $p > 0$. Hence, Theorem 2.11, together with Proposition 3.2, implies that for the choice of $T$ and $S$, both parts (a) and (b) of Theorem 1.4 hold. This completes the whole proof of Theorem 1.4.

4 Strict positivity of density

The aim of this section is to prove the positivity of the joint density as stated in Theorem 1.12. Throughout this section, we will fix a set of arbitrary $m$ disjoint points $\{x_1, \ldots, x_m\} \subseteq \mathbb{R}^d$. We will assume that the initial data $\mu$ is nonnegative, and if not, one may simply replace $\mu$ by $|\mu|$. All arguments go through.

The outline of the proof of Theorem 1.12 is given in Section 4.1. All technical details are given in the subsequent sections.

4.1 Proof of Theorem 1.12

The proof of Theorem 1.12 follows the same arguments as those in the proof of Theorem 1.4 of [9]. For completeness, we present this proof below.

Proof of Theorem 1.12. Choose and fix an arbitrary final time $T$. We will prove Theorem 1.12 for $t = T$. Throughout the proof, we fix $\kappa > 0$ and assume that $|z| \leq \kappa$ where $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ and $t \in (0, T]$. Without loss of generality, one may assume that $T > 1$ in order that $T - 2^{-n} > 0$ for all $n \geq 1$. Otherwise we simply replace all “$n \geq 1$” in the proof below by “$n \geq N$” for some large $N > 0$. The proof consists of three steps.

Step 1. For $n \geq 1$, define $h_n$ as follows:

$$
h_n(s, y) := c_n \mathbb{I}_{[T - 2^{-n}, T]}(s)G(T - s, x_i - y), \quad \text{for } 1 \leq i \leq m, \quad (4.1.1)
$$
where
\[ c_n^{-1} := \int_{T-2^{-n}}^T ds \int_{\mathbb{R}^d} dy' G(T-s, x_i - y) G(T-s, x_i - y') f(y - y'). \] (4.1.2)

Under Assumption 1.10,
\[ c_n = V(2^{-n})^{-1} \asymp 2^{(1-\beta)n}, \] (4.1.3)
or equivalently,
\[ c_n 2^{-n} \leq C 2^{-\beta n} \quad \text{and} \quad V_d(2^{-n}) \leq C 2^{-(1-\beta)n} \quad \text{for all } n \in \mathbb{N}, \] (4.1.4)
see parts (2) and (3) of Lemma 3.6.

Let \( \hat{W}_z^n \) be the cylindrical Wiener process translated by \( h_n \) and \( z \). Let \( \{ \hat{u}_z^n(t, x), (t, x) \in (0, T] \times \mathbb{R}^d \} \) be the random field shifted with respect to \( \hat{W}_z^n \), that is, \( \hat{u}_z^n(t, x) \) satisfies the following equation:
\[
\hat{u}_z^n(t, x) = J_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \rho(\hat{u}_z^n(s, y)) W(ds, dy) \nonumber
\]
\[ + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \rho(\hat{u}_z^n(s, y)) \langle z, h_n(s, y') \rangle f(y-y') ds dy dy'. \] (4.1.5)

For \( x \in \mathbb{R}^d \), denote the gradient vector and the Hessian matrix of \( \hat{u}_z^n(t, x) \) by
\[
\hat{u}_z^{n,i}(t, x) := \partial_{z_i} \hat{u}_z^n(t, x) \quad \text{and} \quad \hat{u}_z^{n,i,k}(t, x) := \partial_{z_i z_k} \hat{u}_z^n(t, x), \] (4.1.6)
respectively. From (4.1.5), we see that
\[
\hat{u}_z^n(s, y) = u(s, y) \quad \text{for } s \leq T - 2^{-n} \text{ and } y \in \mathbb{R}^d. \]

Hence, \( \{ \hat{u}_z^{n,i}(t, x), (t, x) \in (0, T] \times \mathbb{R}^d \} \) satisfies
\[
\hat{u}_z^{n,i}(t, x) = \theta_z^{n,i}(t, x) \] (4.1.7)
\[ + \mathbb{I}(t > T - 2^{-n}) \int_{T-2^{-n}}^t \int_{\mathbb{R}^d} G(t-s, x-y) \rho'(\hat{u}_z^n(s, y)) \hat{u}_z^{n,i}(s, y) W(ds, dy) \nonumber
\]
\[ + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \rho'(\hat{u}_z^n(s, y)) \hat{u}_z^{n,i}(s, y) \langle z, h_n(s, y') \rangle f(y-y') ds dy dy', \]
where
\[
\theta_z^{n,i}(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \rho(\hat{u}_z^n(s, y)) h_n^i(s, y') f(y-y') ds dy dy'. \] (4.1.8)

Similarly, \( \{ \hat{u}_z^{n,i,k}(t, x), (t, x) \in (0, T] \times \mathbb{R}^d \} \) satisfies
\[
\hat{u}_z^{n,i,k}(t, x) = \theta_z^{n,i,k}(t, x) \] (4.1.9)
\[
+ \int_0^t \int_{\mathbb{R}^{2d}} G(t - s, x - y) \rho'(\hat{\mathbf{u}}(s, y)) \hat{\mathbf{u}}_s(s, y) h_n(s, y') f(y - y') ds dy dy' \\
+ \mathbb{I}_{\{t > T - 2^n\}} \int_{T - 2^n}^t \int_{\mathbb{R}^d} G(t - s, x - y) \rho''(\hat{\mathbf{u}}(s, y)) \hat{\mathbf{u}}_{ss}(s, y) \hat{\mathbf{u}}_s(s, y) W(ds dy) \\
+ \int_0^t \int_{\mathbb{R}^{2d}} G(t - s, x - y) \rho'(\hat{\mathbf{u}}(s, y)) \hat{\mathbf{u}}_{s,k}(s, y) \langle \mathbf{z}, \mathbf{u}_n(s, y') \rangle f(y - y') ds dy dy'.
\]

where

\[
\theta^{n,i,k}(t, x) := \partial_{z_i} \theta^{n,i}(t, x) \\
= \int_0^t \int_{\mathbb{R}^{2d}} G(t - s, x - y) \rho'(\hat{\mathbf{u}}(s, y)) \hat{\mathbf{u}}_{s,i,k}(s, y) h_n(s, y') f(y - y') ds dy dy',
\]

Note that the second term on the right-hand side of (4.1.9) is equal to \( \theta^{n,k,i}(t, x) \).

Denote

\[
\| \{ \hat{\mathbf{u}}^n(t, x_i) \}_{1 \leq i \leq m} \|_{C^2} \\
= \| \{ \hat{\mathbf{u}}^n(t, x_i) \}_{1 \leq i \leq m} \| + \| \{ \hat{\mathbf{u}}^{n,i}(t, x_j) \}_{1 \leq i, j \leq m} \| + \| \{ \hat{\mathbf{u}}^{n,i,k}(t, x_j) \}_{1 \leq i, j, k \leq m} \|.
\]

Suppose that \( y \in \mathbb{R}^m \) belongs to the interior of the support of the joint law of

\[
(u(T, x_1), \ldots, u(T, x_m))
\]

and \( \rho(y_i) \neq 0 \) for all \( i = 1, \ldots, m \). By Theorem 2.13, Theorem 1.12 is proved once we show that there exist some positive constants \( c_1, c_2, r_0 \), and \( \kappa \) such that the following two conditions are satisfied:

\[
\liminf_{n \to \infty} \mathbb{P} \left( \| \{ u(T, x_i) - y_i \}_{1 \leq i \leq m} \| \leq r \text{ and } \det \left[ \{ \hat{\mathbf{u}}^n_0(T, x_j) \}_{1 \leq i, j \leq m} \right] \geq c_1 \right) > 0, \quad (4.1.12)
\]

for all \( r \in (0, r_0) \), and

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{|x| \leq \kappa} \| \{ \hat{\mathbf{u}}^n(T, x_i) \}_{1 \leq i \leq m} \|_{C^2} \leq c_2 \left| \{ u(T, x_i) - y_i \}_{1 \leq i \leq m} \| \leq r_0 \right) = 1. \quad (4.1.13)
\]

These two conditions are verified in the following two steps.

Step 2. Let \( y \) be a point in the intersection of \( \{ \rho \neq 0 \}^m \) and the interior of the support of the joint law of \( (u(T, x_1), \ldots, u(T, x_m)) \). Then there exists \( r_0 \in (0, 1) \) such that for all \( 0 < r \leq r_0 \),

\[
\mathbb{P} \left( \{ u(T, x_1), \ldots, u(T, x_m) \} \in B(y, r) \right) \cap \left\{ \prod_{i=1}^m |\rho(u(T, x_i))| \geq 2c_0 \right\} > 0,
\]

46
where
\[ c_0 = \frac{1}{2} \inf_{(z_1, \ldots, z_d) \in B(y, r_0)} \prod_{i=1}^{m} \rho(z_i). \]

Due to (4.4.7) below, it holds that
\[
\lim_{n \to \infty} \det \left[ \left\{ \hat{u}_0^{n,i}(T, x_j) \right\}_{1 \leq i, j \leq m} \right] = \prod_{i=1}^{m} \rho(u(T, x_i)) \quad \text{a.s.}
\]

Hence, by denoting
\[
A := \left\{ (u(T, x_1), \ldots, u(T, x_m)) \in B(y, r) \right\},
\]
\[
D := \left\{ \prod_{i=1}^{m} |\rho(u(T, x_i))| \geq 2c_0 \right\},
\]
\[
E_n := \left\{ \left| \det \left[ \left\{ \hat{u}_0^{n,i}(T, x_j) \right\}_{1 \leq i, j \leq m} \right] - \prod_{i=1}^{m} \rho(u(T, x_i)) \right| < c_0 \right\},
\]
\[
G_n := \left\{ \left| \det \left[ \left\{ \hat{u}_0^{n,i}(T, x_j) \right\}_{1 \leq i, j \leq m} \right] \right| \geq c_0 \right\},
\]
we see that
\[
\mathbb{P}(A \cap G_n) \geq \mathbb{P}(A \cap D \cap E_n) \to \mathbb{P}(A \cap D) > 0, \quad \text{as } n \to \infty.
\]

Therefore,
\[
\liminf_{n \to \infty} \mathbb{P} \left( \left\{ (u(T, x_1), \ldots, u(T, x_m)) \in B(y, r) \right\} \cap \left\{ \left| \det \left[ \left\{ \hat{u}_0^{n,i}(T, x_j) \right\}_{1 \leq i, j \leq m} \right] \right| \geq c_0 \right\} \right) > 0,
\]
which proves condition (4.1.12).

**Step 3.** From (4.1.11), we see that
\[
\left\| \left\{ \hat{u}_n^{n}(T, x_i) \right\}_{1 \leq i \leq m} \right\|_{C^2} \leq \sum_{i=1}^{m} \left| \hat{u}_n^{n}(T, x_i) \right| + \sum_{i,j=1}^{m} \left| \hat{u}_n^{n,i}(T, x_j) \right| + \sum_{i,j,k=1}^{m} \left| \hat{u}_n^{n,i,k}(T, x_j) \right|.
\]

By Proposition 4.14 below, there exists some constant \( K_{r_0} \) independent of \( n \) such that
\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{|x| \leq \kappa} \left\| \left\{ \hat{u}_n^{n}(T, x_i) \right\}_{1 \leq i \leq m} \right\|_{C^2} \leq K_{r_0} \left\| \left\{ u(T, x_i) \right\}_{1 \leq i \leq m} \right\|_{C^2} \leq K_{r_0} \right) = 1,
\]
where \( \kappa \) is fixed as at the beginning of the proof. Therefore, condition (4.1.13) is also satisfied. This completes the proof of Theorem 1.12.

\[ \square \]

### 4.2 Properties of the function \( \Psi_n(t, x; \ell) \)

For \( t \in [0, T], x \in \mathbb{R}^d \) and \( k \in \mathbb{N} \) define
\[
\Psi_n^i(t, x; k) := \int_0^t \int_{\mathbb{R}^{2d}} G(t-s, x-y, \ell) J_0^i(s, y) h_n(s, y') f(y-y') ds dy dy',
\]
\[
\Psi_n(t, x; k) := \int_0^t \int_{\mathbb{R}^{2d}} G(t-s, x-y, \ell) J_0^i(s, y) \left\{ 1, h_n(s, y') \right\} f(y-y') ds dy dy',
\]
\[ (4.2.1) \]
where \( 1 = (1, \ldots, 1) \in \mathbb{R}^m \). We will use the convention that
\[
Ψ_n^i(t, x) := Ψ_n^i(t, x; 0) \quad \text{and} \quad Ψ_n(t, x) := Ψ_n(t, x; 0).
\]

The aim of this subsection is to prove the following proposition for the properties of \( Ψ_n(t, x; k) \) and we will use the convention that
\[
J_0^k(kt, x) \equiv 1 \quad \text{if} \ k = 0.
\]

**Proposition 4.1.** Under Assumption 1.11, for all \((t, x) \in (0, T] \times \mathbb{R}^d, \ n \in \mathbb{N}, k \in \{0, 1, 2, 3\} \) and \( i \in \{1, \ldots, m\} \), the following properties hold:

1. \( Ψ_n^i(t, x; k) \) are nonnegative. When \( k = 0 \), it holds that
\[
Ψ_n^i(t, x) \leq Ψ_n^i(T, x_i) \mathbb{I}_{\{t > T - 2^{-n}\}} = \mathbb{I}_{\{t > T - 2^{-n}\}}.
\]
   For \( k \in \{1, 2, 3\} \), there exists some constant \( C > 0 \) independent of \( n \) such that
\[
Ψ_n(t, x; k) \leq C J_0^k(kt, x) \mathbb{I}_{\{t > T - 2^{-n}\}}.
\]

2. Under Assumption 1.11, for any \( x \neq x_1 \), there exists some finite constant \( C_x > 0 \) such that
\[
Ψ_n^i(T, x; k) \leq C_x 2^{-n}.
\]
   Moreover, under Assumption 1.10, \( Ψ_n^i(T, x; k) \leq C_x 2^{-\beta n} \to 0 \) as \( n \to \infty \).

3. For all \( n \in \mathbb{N}, (t, x) \in [T - 2^{-n}, T] \times \mathbb{R}^d \) and \( k \in \{0, 1, 2, 3\} \), there exists some constant \( C > 0 \) independent of \( n \) such that
\[
∫_t^{T - 2^{-n}} ds ∫_{\mathbb{R}^d} dy G(t - s, x - y)G(t - s, x - y')f(y - y')
× J_0^k(s, y)J_0^k(s, y') \leq C J_0^{2k}(kt, x)V_d(2^{-n}).
\]

The following lemma will also be used in order to apply our Picard iterations as those in the proof of Lemma 4.7 below.

**Lemma 4.2.** For all \( n \in \mathbb{N}, T > 1, t > 0, x \in \mathbb{R}^d \) and \( k \in \{2, 3\} \), it holds that
\[
J_0^k(kt, x) \mathbb{I}_{\{T - 2^{-n}, T\}}(t) \leq C \mathbb{I}_{\{t > T - 2^{-n}\}} ∫_{\mathbb{R}^d} G(t, x - y)J_0^k(kt, y)dy < \infty.
\]

We need to define the augmented initial measure as follows:

**Definition 4.3.** Let \( \mu \) be a nonnegative measure that satisfies (1.4). The augmented initial measure, or the star version of \( \mu \) is a nonnegative measure defined as
\[
μ^*(dx) := μ(dx) + [1 + J_0^2(2T, x) + J_0^3(3T, x)] dx,
\]
where \( J_0(t, x) \) is the solution to the homogeneous equation (see (1.5)). Let \( J_0^*(t, x) \) and \( Ψ_n^*(t, x; k) \) denote the corresponding star versions of \( J_0(t, x) \) and \( Ψ_n(t, x; k) \), respectively:

\[
J_0^*(t, x) := ∫_{\mathbb{R}^d} G(t, x - y)μ^*(dy),
\]
\[
Ψ_n^*(t, x; k) := ∫_0^t ∫_{\mathbb{R}^d} G(t - s, x - y)J_0^*(s, y)J_0^k(s, y) \left(1, h_n^i(s, y')\right) f(y - y')dysdy'.
\]
By Lemma 4.2, \( \mu^* \) is a legal initial measure (that is, it satisfies (1.4)). For a given initial measure, we may augment it twice, namely, \( \mu^{**} \). The following facts will be often used, the proofs of which are apparent and left for the interested reader.

**Lemma 4.4.** The following properties hold:

1. Clearly, \( \Psi_n^*(t, x; 0) \equiv \Psi_n(t, x; 0); \)
2. \( 1 + J_0(t, x) \leq J_0^*(t, x); \)
3. \( \Psi_n(t, x; k) \leq \Psi_n^*(t, x; k), \text{ for all } k \in \{1, 2, 3\}; \)
4. \( \sum_{k=0}^{3} \left( \Psi_n(t, x; k) + J_0^*(kt, x)2^{-(1-\beta)n/2} \mathbb{I}_{(t>T-2^{-n})} \right) \leq CJ_0^*(t, x)\mathbb{I}_{(t>T-2^{-n})} \text{ (due to (4.2.3), (4.2.4) and Lemma 4.2)}; \)
5. \( \Psi_n(t, x; k) \leq C\Psi_n^*(t, x; 1) \text{ for all } k \in \{0, 1, 2, 3\} \text{ (due to part (4)}. \)

In the rest part of this subsection, we will prove Proposition 4.1 and Lemma 4.2.

### 4.2.1 Proof of part (1) of Proposition 4.1

The proof consists of the following four steps for \( k = 0, \ldots, 3 \). Set \( \epsilon := 2^{-n} \).

**Step 0.** In this step, we study the case when \( k = 0 \). The nonnegativity of \( \Psi_n^i \) is clear. It is clear that \( \Psi_n^i(t, x) \equiv 0 \text{ for } t \leq T - \epsilon \). When \( t \in [T - \epsilon, T] \), we have that

\[
\Psi_n^i(t, x) = c_n \int_0^{T-t} ds \int_{\mathbb{R}^d} dy dy' G(s, x - y)G(s + T - t, x_i - y')f(y - y')
\]

\[
= c_n (2\pi)^{-d} \int_0^{T-t} ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( -\left(s + \frac{T-t}{2}\right) |\xi|^2 - i(x - x_i) \cdot \xi \right)
\]

\[
\leq c_n (2\pi)^{-d} \int_0^{T-t} ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left(-\frac{s}{2} |\xi|^2 \right)
\]

\[
\leq c_n (2\pi)^{-d} \int_0^{T-t} ds \int_{\mathbb{R}^d} \exp (-s|\xi|^2) \hat{f}(d\xi)
\]

\[
= c_n \int_0^{T-t} ds \int_{\mathbb{R}^d} dy dy' G(s, y)G(s, y')f(y - y').
\]

Therefore,

\[
0 \leq \Psi_n^i(t, x) \leq V_d(\epsilon)^{-1} V_d(\max (\epsilon - (T - t), 0)). \tag{4.2.11}
\]

In particular, the above two inequalities become equalities when \( x = x_i \) and \( t = T \), respectively. This proves (4.2.3).

**Step 1.** In this step, we prove (4.2.4) for \( k = 1 \). We need only prove the case when \( t \geq T - \epsilon \). In this case, using (2.2.8) in the following form

\[
G(t - s, x - y)G(s, y - z) = G(t, x - z)G\left(\frac{(t - s)s}{t}, y - z - \frac{s}{t}(x - z)\right), \tag{4.2.12}
\]
we can apply similar arguments as above to see that

\[
\Psi_n^i(t, x; 1) = \int_0^t ds \int_{\mathbb{R}^d} dy dy' \mu(dz) \mu(dz) h_n^i(s, y') f(y - y') \\
\times G(t, x - z) G \left( \frac{s(t - s)}{t}, y - z - \frac{s}{t}(x - z) \right)
\]

\[= c_n (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(dz) G(t, x - z) \int_0^\epsilon ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \]

\[\times \exp \left( -\frac{1}{2} \left( \frac{s(t - s)}{t} + T - t + s \right) |\xi|^2 \right) \]

\[\leq c_n (2\pi)^{-d} J_0(t, x) \int_0^\epsilon ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( -\frac{1}{2} \left( \frac{s(t - s)}{t} + s \right) |\xi|^2 \right). \tag{4.2.13}\]

Hence,

\[
\Psi_n^i(t, x; 1) \leq c_n (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(dz) G(t, x - z) \int_0^\epsilon ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \]

\[\times \exp \left( -\frac{1}{2} \left( \frac{s(t - s)}{t} + T - t + s \right) |\xi|^2 \right) \]

\[\leq c_n (2\pi)^{-d} J_0(t, x) \int_0^\epsilon ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( -\frac{1}{2} \left( \frac{s(t - s)}{t} + s \right) |\xi|^2 \right). \tag{4.2.14}\]

Because \( T > 1 \), when \( n \) is sufficiently large, say \( n \geq 2 \) (which implies \( \epsilon \in (0, 1/4] \)), the \( ds \)-integral satisfies that

\[s \leq \epsilon < \frac{T - \epsilon}{2} \leq \frac{t}{2} \implies \frac{s(t - s)}{t} \geq \frac{s}{2}. \tag{4.2.15}\]

Hence,

\[
\Psi_n^i(t, x; 1) \leq c_n (2\pi)^{-d} J_0(t, x) \int_0^\epsilon ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( -\frac{3}{4} s|\xi|^2 \right) = \frac{4}{3} c_n J_0(t, x) V_d(3\epsilon/4),
\]

where the last equality is due to (3.4.2). Finally, an application of Lemma 3.8 proves (4.2.14) for \( k = 1 \).

**Step 2.** Now we study the case \( k = 2 \). In this case,

\[
\Psi_n^i(t, x; 2) = \int_0^t ds \int_{\mathbb{R}^{2d}} \mu(dz_1) \mu(dz_2) \int_{\mathbb{R}^{2d}} dy dy' h_n^i(s, y') f(y - y') \\
\times G(t - s, y - z_1) G(s, y - z_2).
\]

Then we apply the following bounds

\[
G(s, y - z) \leq 2^{d/2} G(2s, y - z), \quad \text{and} \quad G(t - s, x - y) \leq 2^{3d/2} |\pi(t - s)|^{d/2} G(2(t - s), x - y)^2.
\]

(4.2.15)

to turn the three \( G \)'s into two pairs of \( G \)'s:

\[
\Psi_n^i(t, x; 2) \leq C \int_0^t ds \int_{\mathbb{R}^{2d}} \mu(dz_1) \mu(dz_2) \int_{\mathbb{R}^{2d}} dy dy' h_n^i(s, y') f(y - y')
\]

50
\[
\times G(2(t-s), x-y)G(2s, y-z_1) \cdot G(2(t-s), x-y)G(2s, y-z_2).
\]

Then apply (4.2.12) for these two pairs of G’s to see that

\[
\Psi_i^n(t, x; 2) \leq C \int_0^t ds(t-s)^{d/2} \int_{\mathbb{R}^{2d}} \mu(dz_1)\mu(dz_2) \int_{\mathbb{R}^{2d}} dy dy' h^n_i(s, y') f(y-y') \\
\times G(2t, x-z_1)G\left(\frac{2s(t-s)}{t}, y-z_1 - \frac{s}{t}(x-z_1)\right) \\
\times G(2t, x-z_2)G\left(\frac{2s(t-s)}{t}, y-z_2 - \frac{s}{t}(x-z_2)\right).
\]

Then apply (2.2.8) and the definition of \(h^n_i\) in (4.1.1) to see that

\[
\Psi_i^n(t, x; 2) \leq C c_n \int_{T-e}^{t} ds(t-s)^{d/2} \int_{\mathbb{R}^{2d}} \mu(dz_1)\mu(dz_2) G(2t, x-z_1)G(2t, x-z_2) \\
\times \int_{\mathbb{R}^{2d}} dy dy' f(y-y') G(T-s, x_i-y') \\
\times G\left(\frac{s(t-s)}{t}, y - \frac{z_1 + z_2}{2} - \frac{s}{2t}(2x-z_1-z_2)\right) \\
\times G\left(\frac{4s(t-s)}{t}, z_1 - z_2 + \frac{s}{t}(z_1-z_2)\right).
\]

Then bound the last \(G(t, x)\) by \(Ct^{-d/2}\) to see that

\[
\Psi_i^n(t, x; 2) \leq C c_n \int_{T-e}^{t} ds(t/s)^{d/2} \int_{\mathbb{R}^{2d}} \mu(dz_1)\mu(dz_2) G(2t, x-z_1)G(2t, x-z_2) \\
\times \int_{\mathbb{R}^{2d}} dy dy' f(y-y') G(T-s, x_i-y') \\
\times G\left(\frac{s(t-s)}{t}, y - \frac{z_1 + z_2}{2} - \frac{s}{2t}(2x-z_1-z_2)\right).
\]

For \(T > 1\) and \(n\) large enough, we have that \(T - 2^{-n} \geq 1/2\) and hence, we can bound \(s^{-d/2}\) from above by \(2^{d/2}\), so that we can get rid of the factor \((t/s)^{d/2}\). Then by Fourier transform, we see that

\[
\Psi_i^n(t, x; 2) \leq C c_n \int_{T-e}^{t} ds \int_{\mathbb{R}^{2d}} \mu(dz_1)\mu(dz_2) G(2t, x-z_1)G(2t, x-z_2) \int_{\mathbb{R}^d} \hat{f}(d\xi) \\
\times \exp\left(-\frac{1}{2} \left[\frac{s(t-s)}{t}T - s \right] |\xi|^2 - i \left(\frac{z_1 + z_2}{2} + \frac{s}{2t}(2x-z_1-z_2)\right) \cdot \xi\right) \\
= C c_n \int_{0}^{t+T} ds \int_{\mathbb{R}^{2d}} \mu(dz_1)\mu(dz_2) G(2t, x-z_1)G(2t, x-z_2) \int_{\mathbb{R}^d} \hat{f}(d\xi) \\
\times \exp\left(-\frac{1}{2} \left[\frac{s(t-s)}{t} + T - t + s \right] |\xi|^2 \\
- i \left(x_i - \frac{z_1 + z_2}{2} - \frac{t-s}{2t}(2x-z_1-z_2)\right) \cdot \xi\right). \tag{4.2.16}
\]
By (4.2.14),

$$
\Psi^i_n(t, x; 2) \leq Cc_n \int_0^{t-T} ds \int_{\mathbb{R}^d} \mu(dz_1) \mu(dz_2) G(2t, x - z_1)G(2t, x - z_2)
\times \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( -\frac{3}{4}s|\xi|^2 \right)
\leq Cc_n J^i_n(2t, x)V_d(3\epsilon/4).
$$

Finally, this case (that is, \( k = 2 \)) is proved by an application of Lemma 3.8.

**Step 3.** Now we prove the case \( k = 3 \). In this case,

$$
\Psi^i_n(t, x; 3) = \int_0^t ds \int_{\mathbb{R}^d} \mu(dz_1) \mu(dz_2) \mu(dz_3) \int_{\mathbb{R}^d} dydy'h^i_n(s, y')f(y - y')
\times G(t - s, x - y)G(s, y - z_1)G(s, y - z_2)G(s, y - z_3).
$$

Then we apply the following bounds

$$
G(s, y - z) \leq 3^{d/2}G(3s, y - z), \quad \text{and}
G(t - s, x - y) = 3^{3d/2}[2\pi(t - s)]^dG(3(t - s), x - y)^3
$$

(4.2.17)

to turn the four \( G \)'s into three pairs of \( G \)'s:

$$
\Psi^i_n(t, x; 3) \leq C \int_0^t ds(t - s)^d \int_{\mathbb{R}^d} \mu(dz_1) \mu(dz_2) \mu(dz_3) \int_{\mathbb{R}^d} dydy'h^i_n(s, y')f(y - y')
\times \prod_{i=1}^3 G(3(t - s), x - y)G(3s, y - z_i).
$$

Then apply (4.2.12) for these three pairs of \( G \)'s to see that

$$
\Psi^i_n(t, x; 3) \leq C \int_0^t ds(t - s)^d \int_{\mathbb{R}^d} \mu(dz_1) \mu(dz_2) \mu(dz_3) \int_{\mathbb{R}^d} dydy'h^i_n(s, y')f(y - y')
\times \prod_{i=1}^3 G(3t, x - z_i)G\left(\frac{3s(t - s)}{t}, y - z_i - \frac{s}{t}(x - z_i)\right).
$$

Then apply (2.2.8) and the definition of \( h^i_n \) in (4.1.1) to see that

$$
\Psi^i_n(t, x; 3) \leq Cc_n \int_{T-\epsilon}^t ds(t - s)^d \int_{\mathbb{R}^d} \mu(dz_1) \mu(dz_2) \mu(dz_3) \prod_{i=1}^3 G(3t, x - z_i)
\times \int_{\mathbb{R}^d} dydy' f(y - y')G(T - s, x_i - y')
\times \prod_{i=1}^3 G\left(\frac{3s(t - s)}{t}, y - z_i - \frac{s}{t}(x - z_i)\right).
$$

Notice by (4.2.15) that

$$
\prod_{i=1}^3 G\left(\frac{3s(t - s)}{t}, y - z_i - \frac{s}{t}(x - z_i)\right)
$$

52
By the same reasoning as before, we can get rid of the factor \((t/s)^d\) and then by Fourier transform we see that

\[
\Psi_i^t(x;3) \leq Cc_n \int_{T-t}^t ds (t/s)^d \int \int_{\mathbb{R}^{2d}} \mu(dz_1) \mu(dz_2) \mu(dz_3) \prod_{i=1}^d G(3t, x - z_i) \times \int \int_{\mathbb{R}^{2d}} dy dy' f(y - y') G(T - s, x_i - y') \times G\left(\frac{3s(t-s)}{2t}, y - \frac{z_1 + z_2 + 2z_3}{4} - \frac{s}{4t}(4x - z_1 - z_2 - 2z_3)\right) \cdot \xi. \tag{4.2.19}
\]

By (4.2.14),

\[
\Psi_i^t(x;3) \leq Cc_n \int_{0}^{t+t-T} ds \int \int_{\mathbb{R}^{3d}} \mu(dz_1) \mu(dz_2) \mu(dz_3) \prod_{i=1}^d G(3t, x - z_i) \int_{\mathbb{R}^d} \hat{f}(d\xi) \times \exp\left(-\frac{1}{2}\left[\frac{3s(t-s)}{2t} + T - s\right] |\xi|^2 \right.
\]
\[
\left. - i \left(\frac{z_1 + z_2 + 2z_3}{4} + \frac{s}{4t}(4x - z_1 - z_2 - 2z_3)\right) \cdot \xi \right). \tag{4.2.19}
\]

Hence,

\[
\Psi_i^n(x;3) \leq Cc_n \int_{T-t}^t ds (t/s)^d \int \int_{\mathbb{R}^{2d}} \mu(dz_1) \mu(dz_2) \mu(dz_3) \prod_{i=1}^d G(3t, x - z_i) \times \exp\left(-\frac{1}{2}\left[\frac{3s(t-s)}{2t} + T - s\right] |\xi|^2 \right.
\]
\[
\left. - i \left(\frac{z_1 + z_2 + 2z_3}{4} + \frac{s}{4t}(4x - z_1 - z_2 - 2z_3)\right) \cdot \xi \right). \tag{4.2.19}
\]

By (4.2.18),

\[
\Psi_i^n(x;3) \leq Cc_n \int_{0}^{t+t-T} ds \int \int_{\mathbb{R}^{3d}} \mu(dz_1) \mu(dz_2) \mu(dz_3) \prod_{i=1}^d G(3t, x - z_i) \times \int_{\mathbb{R}^d} \hat{f}(d\xi) \times \exp\left(-\frac{7}{8} s|\xi|^2 \right). \tag{4.2.19}
\]
Finally, this case (that is, $k = 3$) is proved by an application of Lemma 3.8.

\[ \leq C_{\epsilon} f_0^3(3t, x)V_d(\gamma/8). \]

4.2.2 Proof of part (2) of Proposition 4.1

The proof relies on the following lemma.

**Lemma 4.5.** Suppose $f$ satisfies Assumption 1.11, that is, $f$ is locally bounded on $\mathbb{R}^d \setminus \{0\}$. Then for any $x \in \mathbb{R}^d \setminus \{0\}$, there exists some constant $C_x > 0$,

\[
\limsup_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} G(\epsilon, y) f(y + x) < C_x. \tag{4.2.20}
\]

**Proof.** Notice that

\[
\int_{\mathbb{R}^d} G(\epsilon, y) f(y + x) dy = \int_{|y| \leq |\Delta x|/2} G(\epsilon, y) f(y + x) dy + \int_{|y| > |\Delta x|/2} G(\epsilon, y) f(y + x) dy =: I_{0,1}(\epsilon) + I_{0,2}(\epsilon).
\]

Since $f$ is a locally bounded function on $\mathbb{R}^d \setminus \{0\}$, we see that

\[
\int_{\mathbb{R}^d} G(1, z) \mathbb{I}_{\{|z| \leq |x|/(2\sqrt{\pi})\}} f(\sqrt{\epsilon} z + x) dz \leq \left( \sup_{\frac{|z|}{2} \leq |z| \leq \frac{3|z|}{2}} f(z) \right) \int_{\mathbb{R}^d} G(1, z) dz =: C_x.
\]

Hence,

\[
\limsup_{\epsilon \downarrow 0} I_{0,1}(\epsilon) = \limsup_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} G(1, z) \mathbb{I}_{\{|z| \leq |x|/(2\sqrt{\pi})\}} f(\sqrt{\epsilon} z + x) dz \leq C_x.
\]

As for $I_{0,2}(\epsilon)$, notice that

\[
I_{0,2}(\epsilon) = \int_{|y| > |\Delta x|/2} (2\pi \epsilon)^{-d/2} \exp \left( -\frac{|y|^2}{4\epsilon} - \frac{|y|^2}{4\epsilon} \right) f(y + \Delta x) dy
\]

\[
\leq \int_{\mathbb{R}^d} (2\pi \epsilon)^{-d/2} \exp \left( -\frac{|\Delta x|^2}{16\epsilon} - \frac{|y|^2}{8\epsilon} \right) f(y + \Delta x) dy
\]

\[
= C \exp \left( -\frac{|\Delta x|^2}{32\epsilon} \right) \int_{\mathbb{R}^d} \exp \left( -2\epsilon |\xi|^2 - i\Delta x \cdot \xi \right) \hat{f}(d\xi)
\]

\[
\leq C \exp \left( -\frac{|\Delta x|^2}{32\epsilon} \right) \int_{\mathbb{R}^d} \exp \left( -2\epsilon |\xi|^2 - \frac{|\Delta x|^2}{32\epsilon} \right) \hat{f}(d\xi).
\]

Because $g(\epsilon) := 2\epsilon |\xi|^2 + |\Delta x|^2/(32\epsilon)$ achieves its global minimum at $\epsilon = \frac{|\Delta x|}{8|\xi|}$ with

\[
\min_{\epsilon > 0} g(\epsilon) = g \left( \frac{|\Delta x|}{8|\xi|} \right) = \frac{|\Delta x||\xi|}{2}, \tag{4.2.21}
\]

we see that

\[
I_{0,2}(\epsilon) \leq C \exp \left( -\frac{|\Delta x|^2}{32\epsilon} \right) \int_{\mathbb{R}^d} \exp \left( -\frac{|\Delta x|^2}{2|\xi|} \right) \hat{f}(d\xi) \to 0, \text{ as } \epsilon \to 0,
\]

where the integral is finite thanks to Dalang’s condition (1.2). Combining the above two terms proves the lemma.
We will prove (4.2.5) in four steps. Throughout the proof, \( x \neq x_i \) and denote \( \Delta x := x - x_i \).

**Step 0.** We first study the case when \( k = 0 \). Denote

\[
I_0(\epsilon) := \int_0^\epsilon ds \iint_{\mathbb{R}^d} G(s, x - y)G(s, x_i - y')f(y - y')dydy'.
\]

Then \( \Psi_\beta^j(T, x; 0) = c_n I_0(2^{-n}) \). By Fourier transform, we see that

\[
I_0(\epsilon) = (2\pi)^{-d} \int_0^\epsilon ds \int_{\mathbb{R}^d} e^{-s|\xi|^2 - i(x - x_i) \cdot \xi} \hat{f}(d\xi)
= \int_0^\epsilon ds \int_{\mathbb{R}^d} G(2s, y)f(y + \Delta x)dy.
\]

By l'Hôpital's rule,

\[
\limsup_{\epsilon \downarrow 0} \frac{I_0(\epsilon)}{\epsilon} = \limsup_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} G(2\epsilon, y)f(y + \Delta x)dy \leq C_{\Delta x},
\]

where the last step is due to Lemma 4.5. Therefore,

\[
\limsup_{\epsilon \downarrow 0} \frac{I_0(\epsilon)}{\epsilon} \leq C_{\Delta x} \tag{4.2.22}
\]

In particular, for some constant \( C_x > 0 \), \( I_0(\epsilon) \leq C_x \epsilon \) for all \( \epsilon \in (0, 1) \). Clearly, from the above limit we can see that if \( f \) blows up at \( x = 0 \), this constant \( C_x \) will blow up at \( x = x_i \).

**Finally, \( 4.2.5 \) is proved by setting \( \epsilon = 2^{-n} \).**

**Step 1.** In this step, we will prove (4.2.5) for \( k = 1 \). Denote

\[
\tilde{s} := \frac{s(T - s)}{T} + s \quad \text{and} \quad \tilde{\epsilon} := \epsilon(T - \epsilon)/T + \epsilon.
\]

We first prove the case \( k = 1 \). From (4.2.13) with \( t = T \), we denote

\[
I_1(\epsilon) := (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(dz)G(T, x - z) \int_0^\epsilon ds \int_{\mathbb{R}^d} \hat{\mu}(d\xi) \exp \left(-\frac{1}{2}\tilde{s}|\xi|^2 - i\Delta^s x \cdot \xi \right)
\]

with

\[
\Delta^s x := x_i - z - \frac{T - s}{T}(x - z),
\]

so that \( \Psi_\beta^j(T, x; 1) = c_n I_1(2^{-n}) \). By Fourier transform, we see that

\[
I_1(\epsilon) = \int_{\mathbb{R}^d} \mu(dz)G(T, x - z) \int_0^\epsilon ds \int_{\mathbb{R}^d} dy G(\tilde{s}, y) f(y + \Delta^s x)
\]

By l'Hôpital’s rule,

\[
\limsup_{\epsilon \downarrow 0} \frac{I_1(\epsilon)}{\epsilon} = \limsup_{\epsilon \downarrow 0} I'_1(\epsilon)
= \limsup_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} \mu(dz)G(T, x - z) \int_{\mathbb{R}^d} dy G(\tilde{\epsilon}, y) f(y + \Delta^\epsilon x)
\]

55
which implies that with change of variable \( y \leftrightarrow y' = y - \frac{\epsilon}{T}(z - x) \), we see that
\[
I_1^*(\epsilon) = \int_{\mathbb{R}^d} \mu(dz)G(T, x - z) \int_{\mathbb{R}^d} dy' G\left(\bar{\epsilon}, y' + \frac{\epsilon}{T}(z - x)\right) f(y' + x_i - x).
\]
Because
\[
G\left(\bar{\epsilon}, y' + \frac{\epsilon}{T}(z - x)\right) \leq CG(2\bar{\epsilon}, y') \exp\left(\frac{\epsilon^2|x - z|^2}{2\epsilon T^2}\right),
\]
we see that
\[
I_1^*(\epsilon) \leq \int_{\mathbb{R}^d} \mu(dz)G(T, x - z) \exp\left(\frac{\epsilon^2|x - z|^2}{2\epsilon T^2}\right) \int_{\mathbb{R}^d} dy' G\left(\bar{\epsilon}, y'\right) f(y' + x_i - x).
\]
Because \( \frac{\epsilon^2}{T} \to 0 \) as \( \epsilon \to 0 \), when \( \epsilon \) is sufficiently small, we have that
\[
G(T, x - z) \exp\left(\frac{\epsilon^2|x - z|^2}{2\epsilon T^2}\right) \leq CG(2T, x - z),
\]
which implies that
\[
\limsup_{\epsilon \to 0} I_1^*(\epsilon) \leq \limsup_{\epsilon \to 0} CJ_0(2T, x) \int_{\mathbb{R}^d} dy' G\left(\bar{\epsilon}, y'\right) f(y' + x_i - x)
\]
\[
\leq C_{\Delta x} J_0(2T, x),
\]
where the last step is due to Lemma 4.5. Therefore, for some constant \( C_x > 0 \), \( I_1(\epsilon) \leq C_x \epsilon \). This proves (4.2.5) for \( k = 1 \).

**Step 2.** Now we study the case \( k = 2 \). From (4.2.16) with \( t = T \), we denote
\[
I_2(\epsilon) := (2\pi)^{-d} \iint_{\mathbb{R}^{2d}} \mu(dz_1)\mu(dz_2)G(2T, x - z_1)G(2T, x - z_2)
\]
\[
\times \int_0^\epsilon ds \iint_{\mathbb{R}^d} \hat{f}(d\xi) \exp\left(-\frac{1}{2}\bar{s}^2|\xi|^2 - i\Delta_{z_1, z_2}\epsilon x \cdot \xi\right)
\]
with
\[
\Delta_{z_1, z_2}\epsilon x := x_i - \frac{z_1 + z_2}{2} - \frac{T - s}{2T}(2x - z_1 - z_2),
\]
so that \( \Psi_2^x(T, x; 2) = c_n I_2(2^{-n}) \). Let \( \bar{z} = (z_1 + z_2)/2 \). Then we can apply the same arguments for \( I_1 \) with \( z \) replaced by \( \bar{z} \). Indeed, by L'Hôpital's rule,
\[
\limsup_{\epsilon \to 0} \frac{I_2(\epsilon)}{\epsilon} = \limsup_{\epsilon \to 0} I_2'(\epsilon)
\]
\[
= \limsup_{\epsilon \to 0} \iint_{\mathbb{R}^{2d}} \mu(dz_1)\mu(dz_2)G(2T, x - z_1)G(2T, x - z_2)
\]
\[
\times \int_{\mathbb{R}^d} dy G\left(\bar{\epsilon}, y\right) f\left(y + \Delta_{z_1, z_2}\epsilon x\right)
\]
\[ I_2^*(\epsilon) = \limsup_{\epsilon \downarrow 0} I_2^*(\epsilon). \]

Noticing that \( \Delta_{z_1, z_2} x = (x_i - x) - \frac{\epsilon}{T} (\tilde{z} - x) \), by change of variable \( y \leftrightarrow y' = y - \frac{\epsilon}{T} (\tilde{z} - x) \), we see that

\[
I_2^*(\epsilon) = \int_{\mathbb{R}^d} \mu(dz_1) \mu(dz_2) G(2T, x - z_1) G(2T, x - z_2) \times \int_{\mathbb{R}^d} dy' G \left( \bar{\epsilon}, y' + \frac{\epsilon}{T}(\tilde{z} - x) \right) f(y' + x_i - x).
\]

Then by (4.2.23), we see that

\[
I_2^*(\epsilon) \leq \int_{\mathbb{R}^d} \mu(dz_1) \mu(dz_2) G(2T, x - z_1) G(2T, x - z_2) \exp \left( \frac{\epsilon^2 |x - \tilde{z}|^2}{2 \epsilon T^2} \right) \times \int_{\mathbb{R}^d} dy' G \left( \bar{\epsilon}, y' \right) f(y' + x_i - x).
\]

Notice that

\[
\exp \left( \frac{\epsilon^2 |x - \tilde{z}|^2}{2 \epsilon T^2} \right) \leq \exp \left( \frac{\epsilon^2 |x - z_1|^2}{4 \epsilon T^2} \right) \exp \left( \frac{\epsilon^2 |x - z_2|^2}{4 \epsilon T^2} \right).
\]

Then by the same arguments as for \( I_1 \), when \( \epsilon \) is sufficiently small, we have that

\[
G(2T, x - z_1) G(2T, x - z_2) \exp \left( \frac{\epsilon^2 |x - \tilde{z}|^2}{2 \epsilon T^2} \right) \leq CG(4T, x - z_1) G(4T, x - z_2),
\]

which implies that

\[
\limsup_{\epsilon \downarrow 0} I_2^*(\epsilon) \leq \limsup_{\epsilon \downarrow 0} C J_0^2(4T, x) \int_{\mathbb{R}^d} dy' G \left( \bar{\epsilon}, y' \right) f(y' + x_i - x)
\]

\[
\leq CJ_0^2(4T, x),
\]

where the last step is due to Lemma 4.5. Therefore, for some constant \( C_x > 0 \), \( I_2(\epsilon) \leq C_x \epsilon \).

This proves (4.2.5) for \( k = 2 \).

**Step 3.** Now we study the case \( k = 3 \). From (4.2.19) with \( t = T \), we denote

\[
I_3(\epsilon) := (2\pi)^{-d} \int \int_{\mathbb{R}^d} \prod_{i=1}^3 \mu(dz_i) G(3T, x - z_i) \int_0^\epsilon ds \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( \frac{-1}{2} |\xi|^2 - i \Delta_{s,x}^3 x \cdot \xi \right)
\]

with

\[
\Delta_{s,x}^3 x := x_i - \frac{z_1 + z_2 + 2z_3}{4} - \frac{T - s}{4T} (4x - z_1 - z_2 - 2z_3),
\]

so that \( \Psi_n^t(T; x; 3) = c_n I_3(2^{-n}) \). Let \( \bar{z} = (z_1 + z_2 + 2z_3)/4 \). Then we can apply the same arguments for \( I_1 \) with \( z \) replaced by \( \bar{z} \). Indeed, by L'Hôpital's rule,

\[
\limsup_{\epsilon \downarrow 0} \frac{I_3(\epsilon)}{\epsilon} = \limsup_{\epsilon \downarrow 0} I_3^*(\epsilon)
\]

\[
= \limsup_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} \int \prod_{i=1}^3 \mu(dz_i) G(3T, x - z_i) \int_{\mathbb{R}^d} dy G(\bar{\epsilon}, y) f(y + \Delta_{s,x}^3 x)
\]

\[= \frac{1}{2}\limsup_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} \int \mu(dz_1) \mu(dz_2) G(2T, x - z_1) G(2T, x - z_2) \exp \left( \frac{\epsilon^2 |x - \bar{z}|^2}{2 \epsilon T^2} \right) \times \int_{\mathbb{R}^d} dy G(\bar{\epsilon}, y) f(y + \Delta_{s,x}^3 x).
\]
Then by (4.2.23), we see that

\[ I_3^*(\epsilon) \leq \int \int \int_{\mathbb{R}^{2d}} \exp \left( \frac{\epsilon^2 |x - \bar{z}|^2}{2T^2} \right) \prod_{i=1}^{3} \mu(\text{d}z_i) G(2T, x - z_i) \int_{\mathbb{R}^d} \text{d}y' G(\bar{\epsilon}, y') f(y' + x_i - x). \]

Notice that

\[ \left| x - \frac{z_1 + z_2 + 2z_3}{4} \right|^2 = \left| \frac{x - z_1}{4} + \frac{x - z_2}{4} + \frac{x - z_3}{2} \right|^2 \leq \frac{1}{4} \left| x - z_1 \right|^2 + \frac{1}{4} \left| x - z_2 \right|^2 + \frac{1}{2} \left| x - z_3 \right|^2, \]

which implies that

\[ \exp \left( \frac{\epsilon^2 |x - \bar{z}|^2}{2T^2} \right) \leq \prod_{i=1}^{3} \exp \left( \frac{\epsilon^2 |x - z_i|^2}{2T^2} \right). \]

Then by the same arguments as for \( I_1 \), when \( \epsilon \) is sufficiently small, we have that

\[ \prod_{i=1}^{3} G(3T, x - z_i) \exp \left( \frac{\epsilon^2 |x - z_i|^2}{2T^2} \right) \leq C \prod_{i=1}^{3} G(6T, x - z_i), \]

which implies that

\[ \limsup_{\epsilon \downarrow 0} I_3^*(\epsilon) \leq \limsup_{\epsilon \downarrow 0} C J_0^3(6T, x) \int_{\mathbb{R}^d} \text{d}y' G(\bar{\epsilon}, y') f(y' + x_i - x) \]

\[ \leq C_{\Delta_x} J_0^3(6T, x), \]

where the last step is due to Lemma 4.5. Therefore, for some constant \( C_x > 0 \), \( I_3^*(\epsilon) \leq C_x \epsilon \).

This proves (4.2.5) for \( k = 3 \).

4.2.3 Proof of part (3) of Proposition 4.1

In this part, we will prove (4.2.6). It is clear that the case of \( k = 0 \) is a direct consequence of the definition of \( c_n^{-1} \) in (4.1.2). In the following, we need only to prove the cases for \( k = 1, 2, 3 \). Denote the triple integral in (4.2.6) by \( I \).

**Step 1.** We first prove the case when \( k = 1 \). By (4.2.12), we see that

\[
I = \int_{T-2}^{t} ds \int_{\mathbb{R}^{2d}} \mu(\text{d}z) \mu(\text{d}z') G(t, x - z) G(t, x - z') \int_{\mathbb{R}^{2d}} \text{d}y \text{d}y' f(y - y') \\
\times G\left( \frac{(t-s)s}{t}, y - z - \frac{s}{t}(x - z) \right) G\left( \frac{(t-s)s}{t}, y' - z' - \frac{s}{t}(x - z') \right)
\]
\begin{align*}
\leq C \int_{T-2^{-n}}^{t} ds \int_{\mathbb{R}^{d}} \mu(dz) \mu(dz') G(t, x - z) G(t, x - z') \int_{\mathbb{R}^{d}} \hat{f}(d\xi) \exp \left(-\frac{(t - s)s}{t} |\xi|^2 \right) \\
\leq C J_0^2(t, x) \int_{0}^{2^{-n} + t - T} ds \int_{\mathbb{R}^{d}} \hat{f}(d\xi) \exp \left(-\frac{(t - s)s}{t} |\xi|^2 \right).
\end{align*}

Then by (4.2.14) and Lemma 3.8, we have that

\begin{align*}
I \leq C J_0^2(t, x) \int_{0}^{2^{-n}} ds \int_{\mathbb{R}^{d}} \hat{f}(d\xi) \exp \left(-\frac{s}{2} |\xi|^2 \right) \\
= C J_0^2(t, x) V_d(2 \times 2^{-n}) \\
\leq C J_0^2(t, x) V_d(2^{-n}),
\end{align*}

Step 2. The case \( k = 2 \) is more involved. First we see that

\begin{align*}
I = \int_{T-2^{-n}}^{t} ds \int_{\mathbb{R}^{d}} \cdots \int_{\mathbb{R}^{d}} \mu(dz_1) \mu(dz_2) \mu(dz'_1) \mu(dz'_2) \int_{\mathbb{R}^{d}} dy dy' f(y - y') \\
\times G(t - s, x - y) G(s, y - z_1) G(s, y - z_2) \\
\times G(t - s, x - y') G(s, y' - z'_1) G(s, y' - z'_2).
\end{align*}

Then we use (4.2.15) to turn the above six \( G \)'s to four pairs of \( G \)'s:

\begin{align*}
I \leq C \int_{T-2^{-n}}^{t} ds (t - s)^d \int_{\mathbb{R}^{d}} \cdots \int_{\mathbb{R}^{d}} \mu(dz_1) \mu(dz_2) \mu(dz'_1) \mu(dz'_2) \int_{\mathbb{R}^{d}} dy dy' f(y - y') \\
\times G(2(t - s), x - y) G(2s, y - z_1) G(2(t - s), x - y') G(2s, y - z'_1) \\
\times G(2(t - s), x - y) G(2s, y - z_2) G(2(t - s), x - y') G(2s, y - z'_2).
\end{align*}

Then we apply the relation (2.2.8) several times to see that the right-hand side of the above inequality is equal to

\begin{align*}
= \int_{T-2^{-n}}^{t} ds (t - s)^d \int_{\mathbb{R}^{d}} \cdots \int_{\mathbb{R}^{d}} \mu(dz_1) \mu(dz_2) \mu(dz'_1) \mu(dz'_2) \int_{\mathbb{R}^{d}} dy dy' f(y - y') \\
\times G(2t, x - z_1) G(2t, x - z_2) G(2t, x - z'_1) G(2t, x - z'_2) \\
\times G \left( \frac{2s(t - s)}{t}, y - z_1 - \frac{s}{t}(x - z_1) \right) G \left( \frac{2s(t - s)}{t}, y - z_2 - \frac{s}{t}(x - z_2) \right) \\
\times G \left( \frac{2s(t - s)}{t}, y' - z'_1 - \frac{s}{t}(x - z'_1) \right) G \left( \frac{2s(t - s)}{t}, y' - z'_2 - \frac{s}{t}(x - z'_2) \right) \\
= \int_{T-2^{-n}}^{t} ds (t - s)^d \int_{\mathbb{R}^{d}} \cdots \int_{\mathbb{R}^{d}} \mu(dz_1) \mu(dz_2) \mu(dz'_1) \mu(dz'_2) \int_{\mathbb{R}^{d}} dy dy' f(y - y') \\
\times G(2t, x - z_1) G(2t, x - z_2) G(2t, x - z'_1) G(2t, x - z'_2) \\
\times G \left( \frac{s(t - s)}{t}, y - \frac{z_1 + z_2}{2} - \frac{s}{2t}(2x - z_1 - z_2) \right) G \left( \frac{4s(t - s)}{t}, z_1 - z_2 + \frac{s}{t}(z_1 - z_2) \right) \\
\times G \left( \frac{s(t - s)}{t}, y' - \frac{z'_1 + z'_2}{2} - \frac{s}{2t}(2x - z'_1 - z'_2) \right) G \left( \frac{4s(t - s)}{t}, z'_1 - z'_2 + \frac{s}{t}(z'_1 - z'_2) \right).
\end{align*}
Then by bounding $G(t, x)$ by $(2\pi t)^{-d/2}$, we can get rid of two $G$’s:

$$I \leq C \int_{T^{-2-n}}^tds \int_{\mathbb{R}^{4d}} \prod_{i=1}^{3} \mu(dz_i) \mu(dz_i') \int_{\mathbb{R}^{2d}} dydy' f(y - y')$$

$$\times G(t, x - z_1)G(t, x - z_2)G(t, x - z_3)G(t, x - z_3')$$

$$\times G(t, x - z_1')G(t, x - z_2')G(t, x - z_2').$$

For $T > 1$ and $n$ large enough, we have that $T - 2^{-n} \geq 1/2$ and hence, we can bound $t^d/s^d$ from above by $2^d$, so that we can get rid of the factor $t^d/s^d$. Then by Fourier transform, we see that

$$I \leq C J_0^3(2t, x) \int_{T^{-2-n}}^t ds \int_{\mathbb{R}^{4d}} d\xi \hat{f}(d\xi) \exp \left( -\frac{2s(t-s)}{t} |\xi|^2 \right)$$

$$= C J_0^3(2t, x) \int_{T^{-2-n}}^t ds \int_{\mathbb{R}^{4d}} d\xi \hat{f}(d\xi) \exp \left( -\frac{2s(t-s)}{t} |\xi|^2 \right).$$

Then by (4.2.14),

$$I \leq C J_0^3(2t, x) \int_{0}^{2^{-n}} ds \int_{\mathbb{R}^{4d}} d\xi \hat{f}(d\xi) \exp (-s|\xi|^2)$$

$$= C J_0^3(2t, x) V_d(2^{-n}),$$

which proves (4.2.6) for $k = 2$.

**Step 3.** The case when $k = 3$ can be proved in a similar way. First we see that

$$I = \int_{T^{-2-n}}^t ds \int_{\mathbb{R}^{6d}} \prod_{i=1}^{3} \mu(dz_i) \mu(dz_i') \int_{\mathbb{R}^{2d}} dydy' f(y - y')$$

$$\times G(t - s, x - y)G(s, y - z_1)G(s, y - z_2)G(s, y - z_3)$$

$$\times G(t - s, x - y)G(s, y' - z_1')G(s, y' - z_2')G(s, y' - z_3').$$

Then we use (4.2.17) to turn the above eight $G$’s to six pairs of $G$’s:

$$I \leq C \int_{T^{-2-n}}^t ds (s-t)^{2d} \int_{\mathbb{R}^{6d}} \prod_{i=1}^{3} \mu(dz_i) \mu(dz_i') \int_{\mathbb{R}^{2d}} dydy' f(y - y')$$

$$\times \prod_{i=1}^{3} [G(3(t-s), x - y)G(3s, y - z_i)]$$
\[ \times \prod_{i=1}^{3} [G(3(t-s), x-y')G(3s, y'-z_i')] \].

Then we apply the relation (2.2.8) several times to see that the right-hand side of the above inequality is equal to

\[ C \int_{T-2^{-n}}^{t} ds \left( t-s \right)^{2d} \prod_{i=1}^{3} \mu(dz_i) \mu(dz'_i) \left( \int_{\mathbb{R}^{2d}} dy dy' f(y-y') \right) \]

\[ \times \prod_{i=1}^{3} G(3t, x-z_i)G(3t, x-z'_i) \]

\[ \times \prod_{i=1}^{3} G \left( \frac{3s(t-s)}{t}, y-z_i - \frac{s}{t}(x-z_i) \right) \]

\[ \prod_{i=1}^{3} G \left( \frac{3s(t-s)}{t}, y'-z'_i - \frac{s}{t}(x-z'_i) \right) . \]

By (4.2.18), we see that the product of the last six \( G \)'s is bounded by

\[ \leq C \left( \frac{s(t-s)}{t} \right)^{-2d} G \left( \frac{3s(t-s)}{2t}, y - \frac{z_1 + z_2 + 2z_3}{4} - \frac{s}{4t}(4x - z_1 - z_2 - 2z_3) \right) \]

\[ \times G \left( \frac{3s(t-s)}{2t}, y' - \frac{z'_1 + z'_2 + 2z'_3}{4} - \frac{s}{4t}(4x - z'_1 - z'_2 - 2z'_3) \right) . \]

By a similar argument as above we can remove the factor \( (t/s)^{2d} \), hence, by Fourier transform, we see that

\[ I \leq C \int_{T-2^{-n}}^{t} ds \int_{\mathbb{R}^{d}} \prod_{i=1}^{3} \left\{ \mu(dz_i) \mu(dz'_i)G(3t, x-z_i)G(3t, x-z'_i) \right\} \]

\[ \times \int_{\mathbb{R}^{d}} d\xi \hat{f}(d\xi) \exp \left( -\frac{3s(t-s)}{2t} |\xi|^2 \right) \]

\[ = C J_0^b(3t, x) \int_{T-2^{-n}}^{t} ds \int_{\mathbb{R}^{d}} d\xi \hat{f}(d\xi) \exp \left( -\frac{3s(t-s)}{2t} |\xi|^2 \right) \]

\[ \leq C J_0^b(3t, x) \int_{0}^{2^{-n}} ds \int_{\mathbb{R}^{d}} d\xi \hat{f}(d\xi) \exp \left( -\frac{3s(t-s)}{2t} |\xi|^2 \right) . \]

Then by (4.2.14) and Lemma 3.8,

\[ I \leq C J_0^b(3t, x) \int_{0}^{2^{-n}} ds \int_{\mathbb{R}^{d}} d\xi \hat{f}(d\xi) \exp \left( -\frac{3}{4} s |\xi|^2 \right) \]

\[ \leq C J_0^b(3t, x)V_d(2^{-n}), \]

which proves (4.2.6) for \( k = 3 \). With this we have completed the whole proof of Proposition 4.1. \( \square \)

### 4.2.4 Proof of Lemma 4.2

In this part, we will prove Lemma 4.2 in two steps. Denote the integral in (4.2.7) by \( I_k \).
Step 1 \((k = 2)\).  We first prove that \(I_2\) is finite so that \(J_0^2(2T, x)\) can be viewed as a legal initial data (see (1.4)).  Fix \(t \in [T - 2^{-n}, T]\).  Denote \(\tilde{z} = (z + z')/2\) below.  Notice that by (2.2.8),

\[
I_2 = \iiint_{\mathbb{R}^3d} G(t, x - y)G(2T, y - z)G(2T, y - z')\mu(dz)\mu(dz')dy
= \iiint_{\mathbb{R}^3d} G(t, x - y)G(4T, z - z')G(T, y - \tilde{z})\mu(dz)\mu(dz')dy.
\]

Then by semigroup property, we see that

\[
I_2 = \int_{\mathbb{R}^2d} G(4T, z - z')G(t + T, x - \tilde{z})\mu(dz)\mu(dz'). \tag{4.2.24}
\]

Now, since \(t \in [T - 2^{-n}, T]\), we can get rid of \(t\) to see that

\[
I_2 \leq C\int_{\mathbb{R}^{2d}} G(8T, z - z')G(2T, x - \tilde{z})\mu(dz)\mu(dz')
= C\int_{\mathbb{R}^{2d}} G(4T, x - z)G(4T, x - z')\mu(dz)\mu(dz')
= CJ_0^2(4T, x) < \infty,
\]

where we have applied (2.2.8).

It remains to prove the first inequality in (4.2.7).  Actually, because \(t \in [T - 2^{-n}, T]\),

\[
J_0^2(2t, x) = \iiint_{\mathbb{R}^3d} G(2t, x - z)G(2t, x - z')\mu(dz)\mu(dz')
= \iiint_{\mathbb{R}^3d} G(4t, z - z')G(t, x - \tilde{z})\mu(dz)\mu(dz')
\leq C\iiint_{\mathbb{R}^3d} G(4T, x - z')G(t + T, x - \tilde{z})\mu(dz)\mu(dz') = CJ_2,
\]

where the last equality is due to (4.2.24).

Step 2 \((k = 3)\).  As the previous case we first prove that \(I_3\) is finite so that \(J_0^3(3T, x)\) can be viewed as a legal initial data (see (1.4)).  Fix \(t \in [T - 2^{-n}, T]\).  Denote \(\tilde{z} = (z_1 + z_2 + z_3)/3\) below.  Notice that

\[
I_3 = \int_{\mathbb{R}^{4d}} G(t, x - y) \left[ \prod_{i=1}^{3} G(3T, y - z_i)\mu(dz_i) \right] dy.
\]

Applying 2.2.8 twice gives that

\[
\prod_{i=1}^{3} G(3T, y - z_i) = G(6T, z_1 - z_2)G\left(\frac{9T}{2}, \frac{z_3 - (z_1 + z_2)}{2}\right) G\left(\frac{9T}{2}, y - \frac{z_1 + z_2 + z_3}{3}\right). \tag{4.2.25}
\]

Hence, by semigroup property to integrate over \(dy\), we see that

\[
I_3 = \iiint_{\mathbb{R}^{3d}} G(6T, z_1 - z_2)G\left(\frac{9T}{2}, \frac{z_3 - (z_1 + z_2)}{2}\right) G(t + T, x - \tilde{z}) \prod_{i=1}^{3} \mu(dz_i). \tag{4.2.26}
\]
Now, since \( t \in [T - 2^{-n}, T] \), we see that
\[
I_3 \leq \iiint_{\mathbb{R}^d} G(12T, z_1 - z_2)G\left(9T, z_3 - \frac{z_1 + z_2}{2}\right) G(2T, x - \tilde{z}) \prod_{i=1}^3 \mu(dz_i)
\]
\[
= \iiint_{\mathbb{R}^d} \prod_{i=1}^3 G(6T, y - z_i) \mu(dz_i)
\]
\[
= C J_0^3(6T, x) < \infty,
\]
where we have applied (4.2.25) on the second line.

It remains to prove the first inequality in (4.2.7). Actually, because \( t \in [T - 2^{-n}, T] \), by (4.2.25),
\[
J_0^3(3t, x) = \iiint_{\mathbb{R}^d} G(6t, z_1 - z_2)G\left(\frac{9t}{2}, z_3 - \frac{z_1 + z_2}{2}\right) G(t, x - \tilde{z}) \prod_{i=1}^3 \mu(dz_i)
\]
\[
\leq C \iiint_{\mathbb{R}^d} G(6T, z_1 - z_2)G\left(\frac{9T}{2}, z_3 - \frac{z_1 + z_2}{2}\right) G(t + T, x - \tilde{z}) \prod_{i=1}^3 \mu(dz_i)
\]
\[
= C I_3,
\]
where the last equality is due to (4.2.26). This proves Lemma 4.2. \( \Box \)

### 4.3 Moments of \( \widehat{u}_2^n(t, x) \) and its first two derivatives

The aim of this subsection is to prove the following proposition.

**Proposition 4.6.** For all \( \kappa > 0, 1 \leq i, k \leq m, n \in \mathbb{N}, p \geq 2, t \in [0, T] \) and \( x \in \mathbb{R}^d \), we have that

\[
\left\| \sup_{|x| \leq \kappa} \left| \widehat{u}_2^n(t, x) \right| \right\|_p \leq C(1 + J_0(t, x)), \quad (4.3.1)
\]
\[
\left\| \sup_{|x| \leq \kappa} \left| \widehat{u}_2^{n,i}(t, x) \right| \right\|_p \leq C \left( \Psi_n^*(t, x; 1) + 2^{-(1-\beta)n/2} J_0^*(t, x) \right), \quad (4.3.2)
\]
\[
\left\| \sup_{|x| \leq \kappa} \left| \widehat{u}_2^{n,i,k}(t, x) \right| \right\|_p \leq C \left( \Psi_n^*(t, x; 1) + 2^{-(1-\beta)n/2} J_0^*(t, x) \right), \quad (4.3.3)
\]
\[
\left\| \sup_{|x| \leq \kappa} \left| \theta_2^{n,i}(t, x) \right| \right\|_p \leq C \left( \Psi_n(t, x) + \Psi_n(t, x; 1) \right), \quad (4.3.4)
\]
\[
\left\| \sup_{|x| \leq \kappa} \left| \theta_2^{n,i,k}(t, x) \right| \right\|_p \leq C \Psi_n^*(t, x; 1). \quad (4.3.5)
\]

**Proof.** Notice that
\[
\left\| \sup_{|x| \leq \kappa} \left| \widehat{u}_2^n(t, x) \right| \right\|_p \leq \left\| \sup_{|x| \leq \kappa} \left| \widehat{u}_2^n(t, x) - \widehat{u}_2^n(t, x) \right| \right\|_p + \left\| \widehat{u}_2^n(t, x) \right\|_p.
\]
Then we apply the Kolmogorov continuity theorem and (4.3.21) to the first term and apply (4.3.7) to the second term to see that
\[
\left\| \sup_{|z| \leq \kappa} |\hat{\eta}_z^n(t, x)| \right\|_p \leq C \sum_{\ell=0,1} \left[ \Psi_n(t, x; \ell) + J_0^\ell(\ell t, x) 2^{-(1-\beta)n/2} \right] + C(1 + J_0(t, x)),
\]
which proves (4.3.1). Similarly, (4.3.14) and (4.3.22) imply (4.3.2); (4.3.17) and (4.3.23) imply (4.3.3).

As for (4.3.4), from (4.1.8), we see that
\[
\left\| \sup_{|z| \leq \kappa} |\hat{\eta}_z^n(t, x)| \right\|_p \leq C \int_0^t \int_{\mathbb{R}^{2d}} G(t-s, x-y) \sup_{|z| \leq \kappa} |\hat{\eta}_z^n(s, y)| \left\| h_n^i(s, y') f(y-y') ds dy' \right\|.
\]

Then we can apply (4.3.1) to obtain (4.3.4). Similarly, from (4.1.10), since \( \rho' \) is bounded,
\[
\left\| \sup_{|z| \leq \kappa} |\hat{\eta}_z^n(t, x)| \right\|_p \leq C \int_0^t \int_{\mathbb{R}^{2d}} G(t-s, x-y) \sup_{|z| \leq \kappa} |\hat{\eta}_z^n(s, y)| \left\| h_n^i(s, y') f(y-y') ds dy' \right\|.
\]

Then use the following bound,
\[
\left\| \sup_{|z| \leq \kappa} |\hat{\eta}_z^n(s, y)| \right\|_p \leq C J_0^\kappa(t, x), \quad (4.3.6)
\]
which is a consequence of (4.2.4) and (4.3.2), to obtain (4.3.5). Recall that \( J_0^\kappa(t, x) \) is defined in (4.2.9). This completes the proof of Proposition 4.6.

### 4.3.1 Moments of \( \hat{\eta}_z^n(t, x) \)

In the next lemma, we study the moments of \( \hat{\eta}_z^n(t, x) \).

**Lemma 4.7.** For any \( p \geq 2, T > 1, \ (t, x) \in [0, T] \times \mathbb{R}^d \) and \( \kappa > 0 \), there exists some constant \( \beta > 0 \) independent of \( n \) such that
\[
\sup_{n \in \mathbb{N}} \sup_{|z| \leq \kappa} \| \hat{\eta}_z^n(t, x) \|_p \leq C(1 + J_0(t, x)), \quad (4.3.7)
\]
where \( \tau := \rho(0) / \text{Lip}_p \). As a consequence,
\[
\sup_{|z| \leq \kappa} \| \hat{\eta}_z^n(t, x) \|_p \leq C \left( \Psi_n^i(t, x) + \Psi_n^i(t, x; 1) \right), \quad (4.3.8)
\]
\[
\max_{1 \leq i \leq m} \sup_{|z| \leq \kappa} \| \hat{\eta}_z^n(t, x) \|_p < C(1 + J_0(t, x)) \mathbb{I}_{\{ t > T-2^{-n} \}}, \quad (4.3.9)
\]
and under Assumption 1.10, for \( x \neq x_i \),
\[
\lim_{n \to \infty} \hat{\eta}_z^n(t, x) = 0 \ a.s. \ for \ all \ t \in [0, T] \ and \ |z| \leq \kappa. \quad (4.3.10)
\]


Proof. We prove this Lemma in two steps.

**Step 1.** We first prove (4.3.7). Recall that \( \hat{u}_m^n(t, x) \) satisfies (4.1.5). We will use Picard iteration to show its existence and uniqueness and moment bounds. Since \( m \) is used to denote the number of preselected points \( x_i \), we will use \( m' \) for the Picard iteration. Define

\[
\hat{u}_{m',0}(t, x) = J_0(t, x).
\]

and

\[
\hat{u}_{m',m+1}(t, x) = J_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \rho(\hat{u}_{m',m}(s, y)) W(dsdy)
\]

\[
+ \int_0^t \int_{\mathbb{R}^{2d}} G(t-s, x-y) \rho(\hat{u}_{m',m}(s, y)) \langle z, h_n(s, y') \rangle f(y-y') dydy'ds
\]

for \( m' \geq 1 \). Recall that \( I_0 := \sup_{0 \leq s \leq T} \| W(s) \| \). Then we have that

\[
\frac{\tau + |\hat{u}_{m',m+1}(t, x)|}{\tau + J_0(t, x)} \leq 1 + \left| \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \frac{\rho(\hat{u}_{m',m}(s, y))}{\rho(\hat{u}_m^n(s, y))} W(dsdy) \right|
\]

\[
+ \left| \int_0^t \int_{\mathbb{R}^{2d}} G(t-s, x-y) \frac{\rho(\hat{u}_{m',m}(s, y))}{\rho(\hat{u}_m^n(s, y))} \langle z, h_n(s, y') \rangle f(y-y') dydy'ds \right|.
\]

Set

\[
\Theta_{T, m', n} := \sup_{0 \leq s \leq T} e^{-\beta s} \left\| \frac{\tau + |\hat{u}_{m',m}(s, y)|}{\tau + J_0(s, y)} \right\|_p.
\]

Taking \( L^p(\Omega) \)-norm on both sides and multiplying by \( e^{-\beta t} \), we have that

\[
e^{-\beta t} \left\| \frac{\tau + |\hat{u}_{m',m+1}(t, x)|}{\tau + J_0(t, x)} \right\|_p
\]

\[
\leq 1 + e^{-\beta t} \left| \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \frac{\rho(\hat{u}_{m',m}(s, y))}{\rho(\hat{u}_m^n(s, y))} W(dsdy) \right|_p
\]

\[
+ e^{-\beta t} \left| \int_0^t \int_{\mathbb{R}^{2d}} G(t-s, x-y) \frac{\rho(\hat{u}_{m',m}(s, y))}{\rho(\hat{u}_m^n(s, y))} \langle z, h_n(s, y') \rangle f(y-y') dydy'ds \right|_p
\]

\[
\leq 1 + \Theta_{T, m', n} C_p \liprho I_1 + \Theta_{T, m', n} \liprho I_2,
\]

where

\[
I_1 := \left( \int_0^t ds e^{-2\beta(s-t)} \int_{\mathbb{R}^d} dydy' \frac{G(t-s, x-y) \tau + J_0(s, y)}{\tau + J_0(t, x)} \right)^{1/2}
\]

\[
\times \frac{G(t-s, x-y') \tau + J_0(s, y')}{\tau + J_0(t, x)} f(y-y')
\]

and

\[
I_2 := \int_0^t \int_{\mathbb{R}^{2d}} e^{-\beta(t-s)} \frac{G(t-s, x-y) \tau + J_0(s, y)}{\tau + J_0(t, x)} \langle z, h_n(s, y') \rangle f(y-y') dydy'ds.
\]
For $I_1$, using Minkowski’s inequality we obtain that
\[
I_1 \leq \left( \int_0^t \int_{\mathbb{R}^d} e^{-2\beta(t-s)} G(t-s, x-y) G(t-s, x-y') \left( \frac{\tau}{\tau + J_0(t, x)} \right)^2 f(y-y') dy dy' ds \right)^{1/2} 
+ \left( \int_0^t \int_{\mathbb{R}^d} \frac{G(t-s, x-y) J_0(s, y) G(t-s, x-y') J_0(s, y') e^{-2\beta(t-s)} f(y-y') dy dy' ds}{\tau + J_0(t, x)} \right)^{1/2} 
\leq \left( \int_0^t \int_{\mathbb{R}^d} e^{-2\beta(t-s)} G(t-s, x-y) G(t-s, x-y') f(y-y') dy dy' ds \right)^{1/2} 
+ \left( \int_0^t \int_{\mathbb{R}^d} \frac{G(t-s, x-y) J_0(s, y) G(t-s, x-y') J_0(s, y') e^{-2\beta(t-s)} f(y-y') dy dy' ds}{J_0(t, x)} \right)^{1/2}.
\]

In the second summand of $I_1$, using the identity (4.2.12) and the Fourier transform we obtain that
\[
I_1 \leq (2\pi)^{-d/2} \left( \int_0^t \int_{\mathbb{R}^d} e^{-2\beta(t-s)} e^{-2(t-s)|\xi|^2} \hat{f}(\xi) d\xi ds \right)^{1/2} 
+ (2\pi)^{-d/2} \left( \int_0^t \int_{\mathbb{R}^d} e^{-2\beta(t-s)} e^{-2(t-s)|\xi|^2} \hat{f}(\xi) d\xi ds \right)^{1/2}.
\]

The dominated convergence theorem shows that $I_1$ can be arbitrarily small if $\beta$ is sufficiently large.

Similarly, $I_2$ can be bounded as the following
\[
I_2 \leq \int_0^t \int_{\mathbb{R}^d} e^{-\beta(t-s)} G(t-s, x-y) \langle z, h_n(s, y') \rangle f(y-y') dy dy' ds 
+ \int_0^t \int_{\mathbb{R}^d} e^{-\beta(t-s)} G(t-s, x-y) J_0(s, y) \langle z, h_n(s, y') \rangle f(y-y') dy dy' ds 
=: I_{21}(\beta) + I_{22}(\beta).
\]

By Proposition 4.1, we see that
\[
I_{21}(\beta) \leq I_{21}(0) \leq \kappa \Psi_m(t, x) \leq \kappa m,
\]
and
\[
I_{22}(\beta) \leq I_{22}(0) \leq \kappa \Psi_m(t, x) \leq \kappa \max_{i=1,...,m} \sup_{(t, x) \in (0, T] \times \mathbb{R}^d, n \in \mathbb{N}} \psi_n^i(t, x) < \infty.
\]

Hence, we can again apply the dominated convergence theorem to show that $I_2$ can be arbitrarily small if $\beta$ is large enough.

Therefore, the above arguments show that
\[
\Theta_{T, m'+1, n} \leq 1 + \Theta_{T, m', n} C_\beta C_p \text{Lip}_p,
\]
where $C_\beta$ can be arbitrarily small if $\beta$ is large enough. Then the induction on $m'$ shows that for some $\beta$ sufficiently large it holds that
\[
\sup_{(t, x) \in (0, T] \times \mathbb{R}^d} \sup_{n \in \mathbb{N}} \sup_{|x| \leq \kappa} e^{-\beta t} \left\| \frac{\tau + |\hat{u}_{m'}(t, x)|}{\tau + J_0(t, x)} \right\|_p \leq C, \quad \text{for all } m' \in \mathbb{N}. \quad (4.3.12)
\]
Next, by considering the difference
\[
\left| \hat{u}_{z,m+1}(t, x) - \hat{u}_{z,m'}(t, x) \right|, \tag{4.3.13}
\]
it is easy to show the existence and uniqueness of the solution. We also have the moment bound
\[
\sup_{|x| \leq \kappa} \sup_{n \in \mathbb{N}} \| \hat{u}_{z}(t, x) \|_p \leq \tau + (\tau + J_0(t, x))e^\beta t,
\]
for some \(\beta\) sufficiently large. Finally, because \(t \in [0, T]\), the above inequality is equivalent to (4.3.7).

**Step 2.** Now we study the rest properties that are related to \(\theta_{z}^{n,i}(t, x)\). By Minkowski’s inequality and the Lipschitz continuity of \(\rho\), we have that
\[
\| \theta_{z}^{n,i}(t, x) \|_p \leq \int_{0}^{t} ds \int_{\mathbb{R}^d} dy dy' G(t - s, x - y) \| \rho(\hat{u}_{z}(s, y)) \|_p h_{n}(s, y') f(y - y')
\]
\[
\leq \text{Lip}_\rho \int_{0}^{t} ds \int_{\mathbb{R}^d} dy dy' G(t - s, x - y) \left( \tau + \| \hat{u}_{z}(s, y) \|_p \right) h_{n}(s, y') f(y - y'),
\]
where we recall that \(\tau = \rho(0)/\text{Lip}_\rho\). Then by (4.3.7), for some constant \(C_T > 0\),
\[
\| \theta_{z}^{n,i}(t, x) \|_p \leq C_T \text{Lip}_\rho \int_{0}^{t} ds \int_{\mathbb{R}^d} dy dy' G(t - s, x - y) \left( 1 + J_0(s, y) \right) h_{n}(s, y') f(y - y')
\]
\[
= C_T \text{Lip}_\rho \left[ \Psi_{n}(t, x) + \Psi_{n}(t, x; 1) \right],
\]
which proves (4.3.8). Property (4.3.9) is a direct consequence of (4.3.8), (4.2.3) and (4.2.4). Finally, By (4.3.8) and the bounds in parts (3) and (4) of Proposition 4.1, one can apply the Borel-Cantelli lemma to obtain (4.3.10). This completes the proof of Lemma 4.7.

### 4.3.2 Moments of \(\hat{u}_{z}^{n,i}(t, x)\)

In the next lemma, we study the moments of \(\hat{u}_{z}^{n,i}(t, x)\).

**Lemma 4.8.** For any \(p \geq 2\), \(n \in \mathbb{N}\), \(i = 1, \ldots, d\), and \(\kappa > 0\), we have that
\[
\sup_{|x| \leq \kappa} \| \hat{u}_{z}^{n,i}(t, x) \|_p \leq C \sum_{\ell=0,1} \left[ \Psi_{n}(t, x; \ell) + J_0(\ell t, x)2^{-|\beta|} \Pi_{|t|>T-2-\kappa} \right], \tag{4.3.14}
\]
and as a consequence,
\[
\max_{1 \leq i, k \leq d} \sup_{|x| \leq \kappa} \| \theta_{z}^{n,i,k}(t, x) \|_p \leq C \left( \Psi_{n}(t, x; 0) + \Psi_{n}(t, x; 1) \right). \tag{4.3.15}
\]

**Proof.** Recall that \(\hat{u}_{z}^{n,i}(t, x)\) satisfies (4.1.7). We claim that
\[
\sup_{|x| \leq \kappa} \| \hat{u}_{z}^{n,i}(t, x) \|_p \leq \sum_{i=1}^{3} I_i,
\]

where the $I_i$ are defined and bounded as follows: From (4.1.7), By (4.3.8),
\[ I_1 := \sup_{|z| \leq \kappa} \| \theta_n^{i,j}(t, x) \|_p \leq C \left( \Psi_n(t, x; 0) + \Psi_n(t, x; 1) \right). \]

By the boundedness of $\rho'$, (4.3.16) and (4.2.6) (together with Assumption 1.10), we see that
\[ I_2 := \| \rho' \|_{L^\infty} \mathbb{I}_{(t > T_2^{-n})} \left( \int_{T_2^{-n}}^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y)G(t - s, x - y') f(y - y') \times \sup_{|z| \leq \kappa} \| \hat{u}_{n,i}^j(s, y) \|_p \sup_{|z| \leq \kappa} \| \hat{u}_{n,i}^j(s, y') \|_p \right)^{1/2}. \]

By the boundedness of $\rho'$, (4.3.16) and (4.2.1), we see that
\[ I_3 := \| \rho' \|_{L^\infty} \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) \langle 1, h_n(s, y') \rangle f(y - y') \sup_{|z| \leq \kappa} \| \hat{u}_{n,i}^j(s, y) \|_p. \]

Thanks (4.2.3) and (4.2.4), when evaluating the upper moment bounds of $\hat{u}_{n,i}^j(t, x)$, we can treat $\hat{u}_{n,i}^j(t, x)$ as if it starts from the initial data $C(1 + \mu)$. Hence, we can apply the same Picard iteration scheme as in the proof of Lemma 4.7 to see that
\[ \max_{1 \leq i \leq d} \sup_{|z| \leq \kappa} \| \hat{u}_{n,i}^j(t, x) \|_p < C(1 + J_0(t, x)) \mathbb{I}_{(t > T_2^{-n})}. \]

Then plugging the bound (4.3.16) back to $I_2$ and $I_3$ shows that
\[ I_2 \leq C \mathbb{I}_{(t > T_2^{-n})} (1 + J_0(t, x)) 2^{-(1-\beta)n/2}, \]
\[ I_3 \leq C \left( \Psi_n(t, x; 0) + \Psi_n(t, x; 1) \right), \]

which proves (4.3.14). Finally, the proof for (4.3.15) is the same as those for $I_1$ above. This completes the whole proof of Lemma 4.8. \qed

### 4.3.3 Moments of $\hat{u}_{n,i,k}^j(t, x)$

In the next lemma, we study the moments of $\hat{u}_{n,i,k}^j(t, x)$.

**Lemma 4.9.** For any $p \geq 2$, $n \in \mathbb{N}$, $1 \leq i, k \leq d$, and $\kappa > 0$, we have that
\[ \sup_{|z| \leq \kappa} \| \hat{u}_{n,i,k}^j(t, x) \|_p \leq C \left[ \Psi_n^*(t, x; 1) + J_0^*(t, x) 2^{-(1-\beta)n/2} \mathbb{I}_{(t > T_2^{-n})} \right]. \] (4.17)

**Proof.** We can write the six parts of $\hat{u}_{n,i,k}^j(t, x)$ in (4.1.9) as
\[ \hat{u}_{n,i,k}^j(t, x) = \theta_{n,i,k}^j(t, x) + \theta_{n,k,j}^i(t, x) + \sum_{\ell=1}^{4} U_{\ell}^n(t, x). \] (4.18)

Hence, we have that
\[ \sup_{|z| \leq \kappa} \| \hat{u}_{n,i,k}^j(t, x) \|_p \leq \sum_{i=0}^{4} I_i, \]

68
where the $I_i$ are defined and bounded as follows: By (4.3.15),

$$I_0 := \sup_{|x| \leq \kappa} \|\theta^{n,i,k}(t, x)\|_p + \sup_{|x| \leq \kappa} \|\theta^{n,k,i}(t, x)\|_p \leq C \left(\Psi_n(t, x) + \Psi_n(t, x; 1)\right).$$

By the boundedness of $\rho''$, the moments bound for $\hat{u}^{n,i}(t, x)$ in (4.3.16) and (4.2.6) (together with Assumption 1.10), we see that

$$I_1 := \sup_{|x| \leq \kappa} \|U^n_1(t, x)\|_p \leq C \left\|\rho''\right\|_{L^\infty} \mathbb{I}_{\{t > T - 2^{-n}\}} \left(\int_{T-2^{-n}}^t ds \iint_{\mathbb{R}^{2d}} G(t - s, x - y)G(t - s, x - y') f(y - y') \times \sup_{|x| \leq \kappa} \left(\left\|\hat{u}^{n,i}(s, y)\right\|_2 \left\|\hat{u}^{n,i}(s, y')\right\|_2 \left\|\hat{u}^{n,k}(s, y')\right\|_2\right) \right)^{1/2} \leq C \left\|\rho''\right\|_{L^\infty} \mathbb{I}_{\{t > T - 2^{-n}\}} \left(\int_{T-2^{-n}}^t ds \iint_{\mathbb{R}^{2d}} G(t - s, x - y)G(t - s, x - y') f(y - y') \times (1 + J_0(s, y)) (1 + J_0(s, y')) \right)^{1/2} \leq C 2^{-(1 - \beta) / 2}\left(1 + \tilde{J}_0^2(2t, x)\right) \mathbb{I}_{\{t > T - 2^{-n}\}}. \tag{4.3.19}$$

By the boundedness of $\rho''$, (4.3.16) and (4.2.1), we see that

$$I_2 := \sup_{|x| \leq \kappa} \|U^n_2(t, x)\|_p \leq \left\|\rho''\right\|_{L^\infty} \int_0^t ds \iint_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) \langle 1, h_n(s, y') \rangle f(y - y') \times \sup_{|x| \leq \kappa} \left(\left\|\hat{u}^{n,i}(s, y)\right\|_2 \left\|\hat{u}^{n,k}(s, y)\right\|_2\right) \leq C \left(\Psi_n(t, x) + \Psi_n(t, x; 2)\right).$$

Similarly,

$$I_3 := \sup_{|x| \leq \kappa} \|U^n_3(t, x)\|_p \leq C \left\|\rho''\right\|_{L^\infty} \mathbb{I}_{\{t > T - 2^{-n}\}} \left(\int_{T-2^{-n}}^t ds \iint_{\mathbb{R}^{2d}} G(t - s, x - y)G(t - s, x - y') f(y - y') \times \sup_{|x| \leq \kappa} \left(\left\|\hat{u}^{n,i,k}(s, y)\right\|_p \left\|\hat{u}^{n,i,k}(s, y')\right\|_p\right) \right)^{1/2} \leq C \left(\Psi_n(t, x) + \Psi_n(t, x; 2)\right).$$

and

$$I_4 := \sup_{|x| \leq \kappa} \|U^n_4(t, x)\|_p \leq \left\|\rho''\right\|_{L^\infty} \int_0^t ds \iint_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) \langle 1, h_n(s, y') \rangle f(y - y') \times \sup_{|x| \leq \kappa} \left\|\hat{u}^{n,i,k}(s, y)\right\|_p .$$
Notice that
\[
I_0 + I_1 + I_2 \leq C (\Psi_n(t, x) + \Psi_n(t, x; 1) + \Psi_n(t, x; 2)) + C 2^{-(1-\beta)n/2} (1 + J_0^2(t, x)) \mathbb{I}_{\{t>T-2^n\}}
\]
\[
\leq C \left[ 1 + J_0(t, x) + J_0^2(t, x) \right] \mathbb{I}_{\{t>T-2^n\}}
\]
\[
\leq C J_0^*(t, x) \mathbb{I}_{\{t>T-2^n\}},
\]
where in the second inequality we have applied (4.2.3) and (4.2.4), and the last inequality is due to Lemma 4.2. Therefore, \( \|\tilde{u}_{z,i}^{n,i,k}(t, x)\|_p \) satisfies a similar integral inequality as that for \( \|\tilde{u}_z^n(t, x)\|_p \). Hence, we can carry out the same Picard iteration scheme as that in the proof of Lemma 4.7 to conclude that
\[
\max_{1 \leq l \leq d} \sup_{|z_n| \leq \kappa} \|\tilde{u}_{z,i}^{n,i,k}(t, x)\|_p \leq C (1 + J_0^*(t, x)) \mathbb{I}_{\{t>T-2^n\}} \leq C J_0^*(t, x) \mathbb{I}_{\{t>T-2^n\}}. \tag{4.3.20}
\]

Then by plugging the above bounds back to the upper bounds for \( I_3 \) and \( I_4 \), we see that
\[
I_3 \leq C 2^{-(1-\beta)n/2} J_0^*(t, x) \mathbb{I}_{\{t>T-2^n\}} \quad \text{and} \quad I_4 \leq C \Psi_n^*(t, x; 1).
\]

Finally, we can use Lemma 4.4 to upgrade the bounds for \( I_0, I_1 \) and \( I_2 \) into either \( C \Psi_n^*(t, x; 1) \) or \( C 2^{-(1-\beta)n/2} J_0^*(t, x) \mathbb{I}_{\{t>T-2^n\}} \). This completes the proof of Lemma 4.9. \( \square \)

### 4.3.4 Moment increments in z

Since we want to bring the “\( \sup_{|z_n| \leq \kappa} \)” inside the expectation, we need to study the moment increments in \( z \).

**Lemma 4.10.** For all \( \kappa > 0, 1 \leq i, k \leq d, n \in \mathbb{N}, p \geq 2, t \in [0, T] \) and \( x \in \mathbb{R}^d \), we have
\[
\sup_{|z_n| \leq \kappa} \|\tilde{u}_z^n(t, x) - \tilde{u}_z^n(t, x')\|_p \leq C |z - z'| \sum_{\ell=0,1} (\Psi_n(t, x; \ell) + 2^{-(1-\beta)n/2} J_0^*(t, x)), \tag{4.3.21}
\]
\[
\sup_{|z_n| \leq \kappa} \|\tilde{u}_{z,i}^n(t, x) - \tilde{u}_{z,i}^n(t, x)\|_p \leq C |z - z'| (\Psi_n^*(t, x; 1) + 2^{-(1-\beta)n/2} J_0^*(t, x)) \tag{4.3.22}
\]
\[
\sup_{|z_n| \leq \kappa} \|\tilde{u}_{z,i}^{n,i,k}(t, x) - \tilde{u}_{z,i}^{n,i,k}(t, x)\|_p \leq C |z - z'| (\Psi_n^{**}(t, x; 1) + 2^{-(1-\beta)n/2} J_0^{**}(t, x)) \tag{4.3.23}
\]

**Proof.** We will prove these three inequalities in this lemma in three steps.

**Step 1.** In this step we prove (4.3.21). Notice that
\[
\tilde{u}_z^n(t, x) - \tilde{u}_z^n(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \left[ \rho(\tilde{u}_z^n(s, y)) - \rho(\tilde{u}_z^n(s, y)) \right] \mathbb{W}(dsdy)
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \left[ \rho(\tilde{u}_z^n(s, y)) - \rho(\tilde{u}_z^n(s, y)) \right] \langle z, h_n(s, y') \rangle f(y - y') dydy'ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \rho(\tilde{u}_z^n(s, y)) \langle z - z', h_n(s, y') \rangle f(y - y') dydy'ds.
\]

Hence, we have that
\[
\sup_{|z_n| \leq \kappa} \|\tilde{u}_z^n(t, x) - \tilde{u}_z^n(t, x)\|_p \leq C \sum_{i=1}^3 I_i,
\]

70
where $I_i$ are defined and bounded as follows: By the Lipschitz continuity of $\rho$,

\[ I_1^2 := \text{Lip}_\rho^2 \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t-s, x-y) G(t-s, x-y') f(y-y') \times \sup_{|\xi| \leq \kappa} \left( \|\hat{u}_n^n(s, y) - \hat{u}_n^n(s, y)\|_p \|\hat{u}_n^n(s, y') - \hat{u}_n^n(s, y')\|_p \right), \]

\[ I_2 := \text{Lip}_\rho \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t-s, x-y) \langle 1, h_n(s, y') \rangle f(y-y') \times \sup_{|\xi| \leq \kappa} \left( \|\hat{u}_n^n(s, y) - \hat{u}_n^n(s, y)\|_p \right). \]

By the linear growth of $\rho$, (4.3.7) and (4.2.1),

\[ I_3 := |z - z'| \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t-s, x-y) \langle 1, h_n(s, y') \rangle f(y-y') \times \left( 1 + \sup_{|\xi| \leq \kappa} \|\hat{u}_n^n(s, y)\|_p \right) \leq C|z - z'| \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t-s, x-y) \langle 1, h_n(s, y') \rangle f(y-y') \times (1 + J_0(s, y)) \leq C|z - z'| \langle \Psi_n(t, x) + \Psi_n(t, x; 1) \rangle. \] (4.3.24)

By (4.2.3) and (4.2.4), we see that

\[ I_3 \leq C |z - z'| (1 + J_0(t, x)) \mathbb{1}_{\{t > T-2^{-n}\}}. \]

Hence, we can apply the same Picard iterate scheme as in the proof of Lemma 4.7 to see that

\[ \sup_{|\xi| \leq \kappa} \|\hat{u}_n^n(t, x) - \hat{u}_n^n(t, x)\|_p \leq C |z - z'| (1 + J_0(t, x)) \mathbb{1}_{\{t > T-2^{-n}\}}. \] (4.3.25)

Then plugging the moment bound (4.3.25) back to the upper bounds for $I_1$ and $I_2$ shows that

\[ I_1 \leq C |z - z'| \mathbb{1}_{\{t > T-2^{-n}\}} (1 + J_0(t, x)) 2^{-1/2}, \]

\[ I_2 \leq C |z - z'| \langle \Psi_n(t, x) + \Psi_n(t, x; 1) \rangle, \]

which proves (4.3.21).

**Step 2.** Now we will prove (4.3.22). Similar to the previous case, we have

\[ \sup_{|\xi| \leq \kappa} \|\hat{u}_{n,i}^n(t, x) - \hat{u}_{n,i}^n(t, x)\|_p \leq C \sum_{i=1}^6 I_i, \]

with $I_i$ being defined and bounded as follows. By the Lipschitz continuity of $\rho$ and (4.3.25),

\[ I_1 := \sup_{|\xi| \leq \kappa} \|\theta_{i}^{n,i}(t, x) - \theta_{i}^{n,i}(t, x)\|_p \]
\[
\leq \text{Lip}_\rho \int_0^t ds \iint_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) f(y - y') h_n^i(s, y') \\
\times \sup_{|z| \leq \kappa} \|\hat{\nu}_z^n(s, y) - \hat{\nu}_z^n(s, y')\|_p \\
\leq C \|z - z'\| (\Psi_n(t, x) + \Psi_n(t, x; 1)).
\]

By the Lipschitz continuity of \(\rho\) and the Schwartz inequality,
\[
I_2 := \text{Lip}^2_\rho \mathbb{I}_{\{t > T - 2^{-n}\}} \int_0^t ds \iint_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) G(t - s, x - y') f(y - y') \\
\times \sup_{|z| \leq \kappa} \left[ \|\hat{\nu}_z^n(s, y) - \hat{\nu}_z^n(s, y')\|_p^2 \right] \\
\times \sup_{|z| \leq \kappa} \left[ \|\hat{\nu}_z^n(s, y) - \hat{\nu}_z^n(s, y')\|_p^2 \right].
\]

Then by (4.3.25), (4.3.16) and (4.2.6),
\[
I_2 \leq C \mathbb{I}_{\{t > T - 2^{-n}\}} \left( \int_0^t ds \iint_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) G(t - s, x - y') f(y - y') \\
\times (1 + J_0(s, y))^2 (1 + J_0(s, y'))^2 \right)^{1/2} |z - z'| \\
\leq C |z - z'| \mathbb{I}_{\{t > T - 2^{-n}\}} (1 + J_0^2(2t, x)) 2^{-(1-\beta)n/2}.
\]

By the boundedness of \(\rho\),
\[
I_3 := \|\rho''\|_{L^\infty} \mathbb{I}_{\{t > T - 2^{-n}\}} \left( \int_0^t ds \iint_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) G(t - s, x - y') f(y - y') \\
\times \sup_{|z| \leq \kappa} \|\hat{\nu}_z^n(s, y) - \hat{\nu}_z^n(s, y')\|_p \\
\times \sup_{|z| \leq \kappa} \|\hat{\nu}_z^n(s, y') - \hat{\nu}_z^n(s, y')\|_p \right)^{1/2}.
\]

Similarly, we have that
\[
I_4 := \text{Lip}_\rho \int_0^t ds \iint_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) \langle 1, h_n(s, y') \rangle f(y - y') \\
\times \sup_{|z| \leq \kappa} \|\hat{\nu}_z^n(s, y) - \hat{\nu}_z^n(s, y')\|_{2p} \|\hat{\nu}_z^n(s, y')\|_{2p} \\
\leq C |z - z'| \mathbb{I}_{\{t > T - 2^{-n}\}} (\Psi_n(t, x; 0) + \Psi_n(t, x; 2)),
\]

and
\[
I_5 := \|\rho''\|_{L^\infty} \int_0^t ds \iint_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) \langle 1, h_n(s, y') \rangle f(y - y') \\
\times \sup_{|z| \leq \kappa} \|\hat{\nu}_z^n(s, y) - \hat{\nu}_z^n(s, y')\|_p,
\]

and
\[
I_6 := \|\rho''\|_{L^\infty} \int_0^t ds \iint_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) f(y - y')
\]
\[
\times \sup_{|z|\leq \kappa} \left( \left\| \hat{u}_{n,x}^{n,i}(s, y) \right\|_p (z - z', h_n(s, y')) \right) \\
\leq C |z - z'| \mathbb{I}_{\{t > T - 2^{-n}\}} (\Psi_n(t, x; 0) + \Psi_n(t, x; 1)).
\]

Now we group terms in order to apply the Picard iteration. By Lemma 4.2, (4.2.8), (4.2.3) and (4.2.4), we see that

\[
\sum_{i=1,2,4,6} I_i \leq C J_6^0(t, x) |z - z'|.
\]

Hence, by the same Picard iteration as in the proof of Lemma 4.7 to see that

\[
\sup_{|z|\leq \kappa} \left\| \hat{u}_{n,x}^{n,i}(t, x) - \hat{u}_{n,x}^{n,i}(t, x) \right\|_p \leq C |z - z'| J_6^0(t, x).
\]

Finally, plugging this upper bound back to the upper bounds for \(I_3\) and \(I_5\) proves (4.3.22).

**Step 3.** The proof for (4.3.23) is similar to Step 2. We have, instead of six, fourteen terms:

\[
\sup_{|z|\leq \kappa} \left\| \hat{u}_{n,x}^{n,i,k}(t, x) - \hat{u}_{n,x}^{n,i,k}(t, x) \right\|_p \leq C \sum_{j=1}^{14} I_j.
\]

In the following, we will specify each of these \(I_j\) and give estimates on them. Recall that we can write the six parts of \(\hat{u}_{n,x}^{n,i,k}(t, x)\) in (4.1.9) as

\[
\hat{u}_{n,x}^{n,i,k}(t, x) = \theta_{n,i,k}^{n,i,k}(t, x) + \theta_{n,x}^{n,i,k}(t, x) + \sum_{\ell=1}^{4} U^n_{\ell}(t, x).
\]

(1-2) By the Lipschitz continuity and the boundedness of \(\rho'\),

\[
I_1 := \sup_{|z|\leq \kappa} \left\| \theta_{n,i,k}^{n,i,k}(t, x) - \theta_{n,x}^{n,i,k}(t, x) \right\|_p \\
\leq \text{Lip}_{\rho'} \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) f(y - y') h_n^i(s, y') \\
\times \sup_{|z|\leq \kappa} \left( \left\| \hat{u}_{n,x}^{n,i,k}(s, y) - \hat{u}_{n,x}^{n,i,k}(s, y) \right\|_p \left\| \hat{u}_{n,x}^{n,i,k}(s, y) \right\|_p \right) \\
+ \|\rho\|_{L^\infty} \int_0^t ds \int_{\mathbb{R}^{2d}} dy dy' G(t - s, x - y) f(y - y') h_n^i(s, y') \\
\times \sup_{|z|\leq \kappa} \left( \left\| \hat{u}_{n,x}^{n,i,k}(s, y) - \hat{u}_{n,x}^{n,i,k}(s, y) \right\|_p \right) \\
\leq C |z - z'| (\Psi_n(t, x; 0) + \Psi_n(t, x; 2) + \Psi_n(t, x; 1)) \\
\leq C |z - z'| \Psi_n^*(t, x; 1),
\]

where we have applied (4.3.27), (4.3.16) and (4.3.25) in the second inequality and Lemma 4.4 in the last inequality. Similarly,

\[
I_2 := \sup_{|z|\leq \kappa} \left\| \theta_{n,x}^{n,i,k}(t, x) - \theta_{n,x}^{n,i,k}(t, x) \right\|_p \leq C |z - z'| \Psi_n^*(t, x; 1).
\]
(3-5) Terms from $I_3$ to $I_5$ come from $U_1$. By the Lipschitz continuity of $\rho''$,

\[
I_3^2 := \text{Lip}_{\rho''} \| \theta_{t,T-2-n} \|_t \int_{T-2-n}^t \int_{\mathbb{R}^d} dydy' \ G(t-s,x-y)G(t-s,x-y')f(y-y')
\times \sup_{|z| \leq |z'|} \left( \| \hat{u}_z^{n,i}(s,y) - \hat{u}_z^{n,i}(s,y') \|_{L_p} \right)
\times \sup_{|z| \leq |z'|} \left( \| \hat{u}_z^{n,k}(s,y) - \hat{u}_z^{n,k}(s,y') \|_{L_p} \right)
\leq C|z - z'| \| \theta_{t,T-2-n} \|_t J_\sigma^*(3t, x)^{2(1-\beta)/n},
\]

where the last inequality is due to (4.2.6) applied to $\mu^*$ and Assumption 1.10. Similarly,

\[
I_4 := \| \rho'' \|_L^2 \| \theta_{t,T-2-n} \|_t \int_{T-2-n}^t \int_{\mathbb{R}^d} dydy' \ G(t-s,x-y)G(t-s,x-y')f(y-y')
\times \sup_{|z| \leq |z'|} \left( \| \hat{u}_z^{n,i}(s,y) - \hat{u}_z^{n,i}(s,y') \|_{L_p} \right)
\times \sup_{|z| \leq |z'|} \left( \| \hat{u}_z^{n,k}(s,y) - \hat{u}_z^{n,k}(s,y') \|_{L_p} \right)
\leq C|z - z'| \| \theta_{t,T-2-n} \|_t J_\sigma^*(2t, x)^{2(1-\beta)/n},
\]

and

\[
I_5 := \| \rho'' \|_L^2 \| \theta_{t,T-2-n} \|_t \int_{T-2-n}^t \int_{\mathbb{R}^d} dydy' \ G(t-s,x-y)G(t-s,x-y')f(y-y')
\times \sup_{|z| \leq |z'|} \left( \| \hat{u}_z^{n,i}(s,y) - \hat{u}_z^{n,i}(s,y') \|_{L_p} \right)
\times \sup_{|z| \leq |z'|} \left( \| \hat{u}_z^{n,k}(s,y) - \hat{u}_z^{n,k}(s,y') \|_{L_p} \right)
\leq C|z - z'| \| \theta_{t,T-2-n} \|_t J_\sigma^*(2t, x)^{2(1-\beta)/n}.
\]

(6-9) Terms from $I_6$ to $I_9$ come from $U_2$. By the Lipschitz continuity of $\rho''$,

\[
I_6 := \text{Lip}_{\rho''} \int_0^t \int_{\mathbb{R}^d} dydy' \ G(t-s,x-y)G(t-s,x-y') \langle 1, h_n(s,y') \rangle f(y-y')
\times \sup_{|z| \leq |z'|} \left( \| \hat{u}_z^{n,i}(s,y) - \hat{u}_z^{n,i}(s,y') \|_{L_p} \right)
\times \sup_{|z| \leq |z'|} \left( \| \hat{u}_z^{n,k}(s,y) - \hat{u}_z^{n,k}(s,y') \|_{L_p} \right)
\leq C|z - z'| T \Psi_\sigma^*(t, x; 3),
\]

74
and by the boundedness of $\rho''$,

\[
I_7 := \|\rho''\|_{L^\infty} \int_0^t \int_{\mathbb{R}^{2d}} dydy' G(t-s, x-y) G(t-s, x-y') \langle 1, \mathbf{h}_n(s, y') \rangle f(y-y')
\times \sup_{\|z\| \leq \kappa} \left( \|\tilde{u}_{n,i}^z(s, y) - \tilde{u}_{z'}^n(s, y)\|_{2p} \|\tilde{u}_{z'}^{n,k}(s, y)\|_{2p} \right)
\leq C|z - z'|\Psi_n^*(t, x; 2),
\]

and

\[
I_8 := \|\rho''\|_{L^\infty} \int_0^t \int_{\mathbb{R}^{2d}} dydy' G(t-s, x-y) G(t-s, x-y') \langle 1, \mathbf{h}_n(s, y') \rangle f(y-y')
\times \sup_{\|z\| \leq \kappa} \left( \|\tilde{u}_{n,i}^z(s, y)\|_{2p} \|\tilde{u}_{z'}^{n,k}(s, y) - \tilde{u}_{z'}^{n,k}(s, y)\|_{2p} \right)
\leq C|z - z'|\Psi_n^*(t, x; 2),
\]

and

\[
I_9 := \|\rho''\|_{L^\infty} \int_0^t \int_{\mathbb{R}^{2d}} dydy' G(t-s, x-y) G(t-s, x-y') f(y-y')
\times \sup_{\|z\| \leq \kappa} \left( \|\tilde{u}_{n,i}^z(s, y)\|_{2p} \|\tilde{u}_{z'}^{n,k}(s, y)\|_{2p} \right)
\leq C|z - z'|\Psi_n^*(t, x; 2).
\]

(10-11) Terms for $I_{10}$ and $I_{11}$ come from $U_3$. By the Lipschitz continuity of $\rho'$,

\[
I_{10} := \text{Lip}_\rho \mathbb{I}_{\{t > T-2^n\}} \left( \int_{T-2^n}^t \int_{\mathbb{R}^{2d}} dydy' G(t-s, x-y) G(t-s, x-y') f(y-y') \times \sup_{\|z\| \leq \kappa} \left( \|\tilde{u}_{n,i}^z(s, y) - \tilde{u}_{z'}^n(s, y)\|_{2p} \|\tilde{u}_{z'}^{n,k}(s, y)\|_{2p} \right) \times \sup_{\|z\| \leq \kappa} \left( \|\tilde{u}_{n,i}^z(s, y') - \tilde{u}_{z'}^n(s, y')\|_{2p} \|\tilde{u}_{z'}^{n,k}(s, y')\|_{2p} \right) \right)^{1/2}
\leq C|z - z'|\mathbb{I}_{\{t > T-2^n\}} J(t, x)^{2-\frac{1}{2}},
\]

and by the boundedness of $\rho'$,

\[
I_{11} := \|\rho''\|_{L^\infty} \mathbb{I}_{\{t > T-2^n\}} \left( \int_{T-2^n}^t \int_{\mathbb{R}^{2d}} dydy' G(t-s, x-y) G(t-s, x-y') f(y-y') \times \sup_{\|z\| \leq \kappa} \left\{ \left\| \tilde{u}_{n,i,k}^z(s, y) - \tilde{u}_{z'}^{n,i,k}(s, y) \right\| \|\tilde{u}_{z'}^{n,i,k}(s, y')\|_{p} \right\} \right)^{1/2}.
\]

(12-14) Terms from $I_{12}$ to $I_{14}$ come from $U_4$. In particular,

\[
I_{12} := \text{Lip}_\rho \int_0^t \int_{\mathbb{R}^{2d}} dydy' G(t-s, x-y) \langle 1, \mathbf{h}_n(s, y') \rangle f(y-y')
\times \sup_{\|z\| \leq \kappa} \left( \|\tilde{u}_{n,i}^z(s, y) - \tilde{u}_{z'}^n(s, y)\|_{2p} \|\tilde{u}_{z'}^{n,k}(s, y)\|_{2p} \right).
\]
\[ \leq C|z - z'|\Psi_n^*(t,x;2), \]
and by the boundedness of $\rho'$,
\[
I_{13} := \|\rho'\|_{L^\infty} \int_0^t ds \int_{\mathbb{R}^{2d}} dy' G(t-s, x-y) \langle 1, h_n(s, y') \rangle f(y-y') \times \sup_{|z| \vee |z'| \leq \kappa} \left\| \hat{u}^{n,i,k}_z(s,y) - \hat{u}^{n,i,k}_z(s,y) \right\|_p,
\]
and
\[
I_{14} := \|\rho'\|_{L^\infty} \int_0^t ds \int_{\mathbb{R}^{2d}} dy' G(t-s, x-y) f(y-y') \times \sup_{|z| \vee |z'| \leq \kappa} \left( \left\| \hat{u}^{n,i,k}_z(s,y) \right\|_p \langle z - z', h_n(s,y') \rangle \right) \leq C|z - z'|\Psi_n^*(t,x;1).
\]
Therefore, by Lemma 4.4, we see that
\[
\sum_{1 \leq i \leq 14} I_i \leq \sum_{i=1}^3 \left( \Psi_n^*(t,x;\ell) + 2^{-(1-\beta)n/2} \mathbb{I}_{(t>T-2^{-n})} J_0^* (\ell t,x) \right)
\leq \mathbb{I}_{(t>T-2^{-n})} |z - z'| J_0^* (t,x).
\]
Together with $I_{11}$ and $I_{13}$, we can apply the same Picard iteration scheme as that in the proof of Lemma 4.7 to see that
\[
\sup_{|z| \vee |z'| \leq \kappa} \left\| \hat{u}^{n,i,k}_z(t,x) - \hat{u}^{n,i,k}_z(t,x) \right\|_p \leq C|z - z'| J_0^* (t,x).
\]
Then plugging this bounds back to $I_{11}$ and $I_{13}$ gives that
\[
I_{11} \leq C|z - z'| J_0^* (t,x) 2^{-(1-\beta)n/2} \quad \text{and} \quad I_{13} \leq C|z - z'| \Psi_n^*(t,x;1).
\]
Finally, we can use Lemma 4.4 to upgrade the moment bounds for $I_i$, $i \notin \{11,13\}$, to those with double augmented initial measure $\mu^{**}$. With this, we complete the proof of Lemma 4.10. \qed

4.4 Almost convergence of $\hat{u}^{n,i}_0(T,x_i)$ to $\rho(u(T,x_i))$

The aim of this part is to prove the following convergence
\[
\lim_{n \to \infty} \hat{u}^{n,i}_0(T,x_i) = \rho(u(T,x_i)) \quad \text{a.s.,}
\]
which is used in step 2 of the proof of Theorem 1.12. This result is proved through the following three lemmas (see (4.4.7)):

Lemma 4.11. For any $\kappa > 0$, $p \geq 2$, $n \in \mathbb{N}$ and $t \in [0,T]$ and $x \in \mathbb{R}^d$ we have
\[
\left\| \sup_{|z| \leq \kappa} |\hat{u}^{n,i}_z(t,x) - u(t,x)| \right\|_p \leq C\kappa \sum_{\ell=0,1} (\Psi_n(t,x;\ell) + J_0^*(\ell t,x) 2^{-(1-\beta)n/2}). \tag{4.4.1}
\]
Proof. We note that
\[
\begin{align*}
\hat{u}_z^n(t, x) - u(t, x) &= \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) \rho(\hat{u}_z^n(s, y)) \langle z, h_n(s, y') \rangle f(y - y') ds dy' \\
&\quad + \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) [\rho(\hat{u}_z^n(s, y)) - \rho(u(s, y))] W(ds dy).
\end{align*}
\]
Hence, by the moment bounds for \( \sup_{|z| \leq \kappa} \hat{u}_z^n(t, x) \) in (4.3.1), we see that
\[
\begin{align*}
\sup_{|z| \leq \kappa} \|\hat{u}_z^n(t, x) - u(t, x)\|_p \\
&\leq C\kappa \int_0^t ds \int_{\mathbb{R}^d} dy' G(t - s, x - y) (1 + J_0(s, y)) \langle 1, h_n(s, y') \rangle f(y - y') \\
&\quad + C \left( \int_0^t ds \int_{\mathbb{R}^d} dy' G(t - s, x - y) G(t - s, x - y') \times \left( \sup_{|z| \leq \kappa} \|\hat{u}_z^n(s, y) - u(s, y)\|_p \right) \left( \sup_{|z| \leq \kappa} \|\hat{u}_z^n(s, y') - u(s, y')\|_p \right) f(y - y') \right)^{1/2} \\
&=: I_1 + I_2.
\end{align*}
\]
Notice that
\[
I_1 = C\kappa (\Psi_n(t, x) + \Psi_n(t, x; 1)) \leq C\kappa (1 + J_0(t, x)).
\]
By the same Picard iteration as in the proof of Lemma 4.7, we see that
\[
\sup_{|z| \leq \kappa} \|\hat{u}_z^n(t, x) - u(t, x)\|_p \leq C\kappa (1 + J_0(t, x)). \tag{4.4.2}
\]
Plug this upper bound back to \( I_2 \) to see that
\[
I_2 \leq C\kappa (1 + J_0(t, x)) 2^{-2(1 - \beta)n},
\]
which proves (4.4.1) with the supremum outside of the \( L^p(\Omega) \)-norm. Finally, thanks to (4.3.21), one can apply the Kolmogorov continuity theorem to move the supremum inside the norm. This completes the proof. \( \square \)

**Lemma 4.12.** For any \( \kappa > 0, 1 \leq i \leq d \) and \( n \in \mathbb{N} \), it holds that
\[
\left\| \sup_{|z| \leq \kappa} |\theta_{z}^{n,i}(T, x_i) - \rho(u(T, x_i))| \right\|_p \leq C2^{-n\alpha/2} + C\kappa (\Psi_n(t, x_i) + \Psi_n(t, x_i; 1)), \tag{4.4.3}
\]
where \( \alpha \in (0, 1] \) is the parameter in condition (1.3). As a consequence (together with (4.3.4)), for all \( x \in \mathbb{R}^d \), with probability one,
\[
\lim_{n \to \infty} \theta_{z}^{n,i}(T, x) = \rho(u(T, x_i)) \mathbb{1}_{\{x = x_i\}}. \tag{4.4.4}
\]

**Proof.** By (4.1.8), we see that
\[
\theta_{z}^{n,i}(T, x_i) - \rho(u(T, x_i))
\]

Applying the Hölder continuity of \( u \)

By \((4.4.2)\) in the proof of Lemma 4.11, we have

\[
I_1 \leq C \kappa \int_0^T ds \int_{\mathbb{R}^{2d}} dy dy' G(T - s, x_i - y)(1 + J_0(s, y)) h_n^i(s, y') f(y - y')
\]
\[
\leq C \kappa (\Psi_n(T, x_i) + \Psi_n(T, x_i; 1)) \leq C \kappa.
\]

Applying the Hölder continuity of \( u(s, y) \) (see Theorem 2.4), we have that

\[
I_2 \leq C \kappa \int_0^T ds \int_{\mathbb{R}^{2d}} dy dy' G(T - s, x_i - y) G(T - s, x_i - y')(|T - s|^{\alpha/2} + |x_i - y|^{\alpha})
\]
\[
=: I_{2,1} + I_{2,2}.
\]

By the Plancherel theorem and recalling that \( k(\cdot) \) is defined in (1.16), we have that

\[
I_{2,1} = C \kappa \int_0^{2^{-n}} s^{\alpha/2} k(2s) ds \leq C \kappa 2^{-\alpha n/2} V(2^{-n}) \leq C 2^{-\alpha n/2}.
\]

As for \( I_{2,2} \), notice that

\[
G(t, x)|x|^{\alpha} \leq C t^{\alpha/2} G(2t, x) \exp \left(-\frac{|x|^2}{4t}\right) \left|\frac{x}{\sqrt{t}}\right|^{\alpha}
\]
\[
\leq C t^{\alpha/2} G(2t, x) \left(\sup_{z \in \mathbb{R}} e^{-\frac{z^2}{4t}} |z|^{\alpha}\right)
\]
\[
\leq C t^{\alpha/2} G(2t, x).
\]

We can apply the above inequality to see that

\[
I_{2,2} \leq C \kappa \int_0^{2^{-n}} ds s^{\alpha/2} \int_{\mathbb{R}^{2d}} dy dy' G(2s, x_i - y) G(s, x_i - y')
\]
\[
= C \kappa \int_0^{2^{-n}} ds s^{\alpha/2} \int_{\mathbb{R}^{d}} \hat{f}(d\xi) \exp \left(-\frac{3s}{2} |\xi|^2\right)
\]

78
shows that

\[ 4.4.4 \]

\[ \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp(-s|\xi|^2) \]

\[ = C \kappa 2^{-na/2} V(2^{-n}) \leq C 2^{-na/2}. \]

Combining the estimates of \( I_1 \) and \( I_2 \) proves (4.4.3). Finally, (4.4.4) can be obtained by an application of the Borel-Cantelli lemma thanks to (4.4.3) when \( x = x_i \) and (4.3.4) when \( x \neq x_i \). This completes the proof of Lemma 4.12.

\[ \square \]

**Lemma 4.13.** For any \( \kappa > 0 \), \( 1 \leq i \leq m \) and \( n \in \mathbb{N} \), it holds that

\[ \left\| \sup_{|x| \leq \kappa} |\hat{u}^{n,i}(T, x_i) - \rho(u(T, x_i))| \right\|_p \leq C \left( 2^{-an/2} + 2^{-(1-\beta)n/2} + \kappa \right), \]

(4.4.6)

where \( \alpha \in (0, 1] \) is the parameter in condition (1.3). As a consequence, for all \( x \in \mathbb{R}^d \),

\[ \lim_{n \to \infty} \hat{u}^{n,i}(T, x) = \rho(u(T, x_i)) \mathbb{I}_{ \{ x = x_i \} } \text{ a.s.} \]

(4.4.7)

**Proof.** We begin by writing

\[ \hat{u}^{n,i}(T, x) - \rho(u(T, x_i)) \]

\[ = \theta^{n,i}(T, x_i) - \rho(u(T, x_i)) \]

\[ + \int_{T-2^{-n}}^T \int_{\mathbb{R}^d} \hat{G}(T - s, x_i - y) \rho(\hat{u}^{n}(s, y)) \hat{u}^{n,i}(s, y) W(dsdy) \]

\[ + \int_0^T \int_{\mathbb{R}^d} dy' G(T - s, x_i - y) \rho(\hat{u}^{n}(s, y)) \hat{u}^{n,i}(s, y) (\mathbf{z}, h_n(s, y')) f(y - y') \]

\[ := I_1 + I_2 + I_3. \]

Lemma 4.12 shows that

\[ \left\| \sup_{|x| \leq \kappa} |I_1| \right\|_p \leq C 2^{-na/2} + C \kappa (\Psi_n(T, x_i) + \Psi_n(T, x_i; 1)) \]

\[ \leq C 2^{-na/2} + C \kappa. \]

As for \( I_2 \), by (4.3.6)) and the boundedness of \( \rho' \), we see that

\[ I_2(\mathbf{z}) - I_2(\mathbf{z}') \]

\[ = \int_{T-2^{-n}}^T \int_{\mathbb{R}^d} \hat{G}(T - s, x_i - y) \left[ \rho'(\hat{u}^{n}(s, y)) \hat{u}^{n,i}(s, y) - \rho'(\hat{u}^{n}(s, y)) \hat{u}^{n,i}(s, y) \right] W(dsdy) \]

\[ = \int_{T-2^{-n}}^T \int_{\mathbb{R}^d} \hat{G}(T - s, x_i - y) \left[ \rho'(\hat{u}^{n}(s, y)) - \rho'(\hat{u}^{n}(s, y)) \right] \hat{u}^{n,i}(s, y) W(dsdy) \]

\[ + \int_{T-2^{-n}}^T \int_{\mathbb{R}^d} \hat{G}(T - s, x_i - y) \left[ \hat{u}^{n,i}(s, y) - \hat{u}^{n,i}(s, y) \right] W(dsdy) \]

\[ := I_{21}(\mathbf{z}) + I_{21}(\mathbf{z}'). \]

For \( I_{21}(\mathbf{z}) \), we note that

\[ \|I_{21}\|_p \leq \left( \int_{T-2^{-n}}^T ds \int_{\mathbb{R}^d} dy' \hat{G}(T - s, x_i - y) G(T - s, x_i - y') \right) \]

\[ \leq C \kappa 2^{-na/2} V(2^{-n}) \leq C 2^{-na/2}. \]
\[
\begin{align*}
\times \| \hat{u}_x^n(s, y) - \hat{u}_x^n(s, y) \|_{2p}^2 \\
\times \| \hat{u}_x^n(s, y - y') - \hat{u}_x^n(s, y') \|_{2p}^2 \| \hat{u}_x^n(s, y') \|_{2p} f(y - y') \right) ^{1/2}.
\end{align*}
\]

From (4.3.27) and (4.3.16) we see that
\[
\| I_{21} \|_p \leq |z - z'| \left( \int_{T - 2^{-n}}^T ds \int_{\mathbb{R}^d} dy dy' G(T - s, x_i - y)G(T - s, x_i - y')J_0^*(s, y)J_0^*(s, y')^2 \right)^{1/2}
\]
\[
\leq C |z - z'| 2^{- (1 - \beta)n/2},
\]
where the last inequality follows from (4.2.6) applied to \( \mu^* \). Similarly we have
\[
\| I_{22} \|_p \leq C |z - z'| 2^{- (1 - \beta)n/2}.
\]

Thus, an application of Kolmogorov continuity theorem shows that
\[
\left\| \sup_{|z| \leq \kappa} |I_3| \right\|_p \leq 2^{ - (1 - \beta)n/2}.
\]

The case for \( I_3 \) can be proved in a similar way:
\[
\left\| \sup_{|z| \leq \kappa} |I_3| \right\|_p \leq C \kappa \int_0^T ds \int_{\mathbb{R}^d} dy dy' G(T - s, x_i - y)J_0^*(s, y) (1, h_\kappa(s, y')) f(y - y')
\]
\[
= C \kappa \Psi_n^*(T, x_i; 1) \leq C \kappa J_0^*(T, x_i) = C' \kappa.
\]

This proves (4.4.6). Finally, when \( \kappa = 0 \), (4.4.7) is proved by an application of the Borel-Cantelli lemma thanks to (4.4.6) when \( x = x_i \) and (4.3.2). This completes the proof of Lemma 4.13.

\[\square\]

### 4.5 Conditional boundedness

The aim of this subsection is to prove the following proposition.

**Proposition 4.14.** Let \( y \in \mathbb{R}^d \) be a point chosen as in Theorem 1.12. For all \( \kappa > 0 \) and \( r > 0 \), there exists constant \( K > 0 \) such that for all \( 1 \leq i, k \leq m \),

\[
\lim_{n \to \infty} \mathbb{P} \left( \left\| \sup_{|z| \leq \kappa} |\hat{u}_x^n(T, x_i)| \vee \sup_{|z| \leq \kappa} |\hat{u}_x^n(T, x_i)| \vee \sup_{|z| \leq \kappa} |\hat{u}_x^n(i, k)(T, x_i)| \right\| \leq K \bigg| Q_r \right) = 1,
\]

where
\[
Q_r := \left\{ \left| \{u(T, x_i) - y_i\}_1^{i \leq m} \right| \leq r \right\}.
\]

#### 4.5.1 Spatial Hölder continuity of \( \hat{u}_x^n(t, x) \) and its two derivatives

We need first prove several lemmas.
Lemma 4.15. For $T > 0$, there exists $C = C(T)$ such that for all $n \in \mathbb{N}$, $p \geq 2$, $1 \leq i, k \leq m$, $\kappa > 0$, $t \in [T - 2^{-n}, T]$ and $x, y \in \mathbb{R}^d$,

$$\left\| \sup_{|x| \leq \kappa} |\hat{u}^n_x(t, x) - \hat{u}^n_x(t, y)| \right\|_p \leq C \left( 1 + J_0(2t, x) + J_0(2t, y) \right) |x - y|^{\alpha}, \quad (4.5.2)$$

$$\left\| \sup_{|x| \leq \kappa} |\hat{u}^{n,i}_x(t, x) - \hat{u}^{n,i}_x(t, y)| \right\|_p \leq C \left( J_0^*(2t, x) + J_0^*(2t, y) \right) |x - y|^{\alpha}, \quad (4.5.3)$$

$$\left\| \sup_{|x| \leq \kappa} |\hat{u}^{n,i,k}_x(t, x) - \hat{u}^{n,i,k}_x(t, y)| \right\|_p \leq C \left( J_0^{**}(2t, x) + J_0^{**}(2t, y) \right) |x - y|^{\alpha}, \quad (4.5.4)$$

where $\alpha \in (0, 1]$ is the parameter that is given in (1.3).

Proof. We prove these three inequalities in three steps. The workhorse is the following two inequalities (see Lemma 3.1 of [10]): For all $\alpha \in (0, 1]$, $t' \geq t > 0$ and $x, y \in \mathbb{R}^d$, it holds that

$$|G(t, x) - G(t, y)| \leq \frac{C}{t^{3/2}} [G(2t, x) + G(2t, y)] |x - y|^{\alpha}, \quad (4.5.5)$$

$$|G(t', x) - G(t', y)| \leq \frac{C}{t^{3/2}} G(4t', x) |t' - t|^{\alpha/2}. \quad (4.5.6)$$

In the following, we will apply the above two inequalities with $\alpha$ that is given in (1.3).

Step 1. We first prove (4.5.2). Notice that

$$\hat{u}^n_x(t, x) - \hat{u}^n_x(t, y) = J_0(t, x) - J_0(t, y)$$

$$+ \int_0^t \int_{\mathbb{R}^d} \left[ G(t - s, x - z) - G(t - s, y - z) \right] \rho (\hat{u}^n_x(s, z)) W(dsdz)$$

$$+ \sum_{i=1}^m z_i \int_0^t ds \int_{\mathbb{R}^{2d}} dz dz' \left[ G(t - s, x - z) - G(t - s, y - z) \right]$$

$$\times \rho (\hat{u}^n_x(s, z)) h^i_n(s, z') f(z - z')$$

$$=: I_1 + I_2 + I_3.$$

By (4.5.5)

$$|I_1| \leq \frac{C}{t^{3/2}} (J_0(2t, x) + J_0(2t, y)) |x - y|^{\alpha}.$$

Then use the fact that $t > T/2$ to absorb the factor $t^{-\alpha/2}$ into the constant.

Denote

$$\tilde{\mu}(dx) := \mu(dx) + dx \quad \text{and} \quad \tilde{J}_0(t, x) := (G(t, \cdot) \ast \tilde{\mu})(x) = 1 + J_0(t, x). \quad (4.5.7)$$

By (4.3.1), we see that

$$\sup_{|x| \leq \kappa} \left\| I_2 \right\|_p^2 \leq \int_0^t ds \int_{\mathbb{R}^{2d}} dz dz' f(z - z')$$

$$\times (G(t - s, x - z) - G(t - s, y - z)) \tilde{J}_0(s, z)$$

$$\times (G(t - s, x - z') - G(t - s, y - z')) \tilde{J}_0(s, z')$$

81
\[
\leq C \left( J_0(2t, x) + J_0(2t, y) \right)^2 |x - y|^{2\alpha},
\]
where the last inequality is proved in Step 1 of the proof of Theorem 1.8 in [10, Section 3].

then an application of Kolmogorov continuity theorem shows that

\[
\left\| \sup_{|x| \leq \kappa} |I_2| \right\|_p \leq C \left( J_0(2t, x) + J_0(2t, y) \right) |x - y|^{\alpha}.
\]

As for \( I_3 \), by (4.3.1), we see that

\[
\left\| \sup_{|x| \leq \kappa} |I_3| \right\|_p \leq C \kappa \int_0^t ds \int_{\mathbb{R}^{2d}} \, dz \, dz' \ |G(t - s, x - z) - G(t - s, y - z)|
\]

\[
\times \tilde{J}_0(s, z) \langle 1, h_n(s, z') \rangle f(z - z').
\]

By the fact that \( G(t, x) \leq 2^{d/2} G(2t, x) \) and

we see that

\[
\left\| \sup_{|x| \leq \kappa} |I_3| \right\|_p \leq C \kappa |x - y|^{\alpha} (\Theta(x) + \Theta(y)),
\]

where

\[
\Theta(x) := \sum_{i=1}^m \mathbf{1}_{\{t > T - 2^n\}} \int_{T - 2^n}^t \frac{ds}{(t - s)^{\alpha/2}} \int_{\mathbb{R}^d} \tilde{\mu}(\tilde{z}) \int_{\mathbb{R}^{2d}} \, dz \, dz'
\]

\[
\times G(2(t - s), x - z) G(2s, z - \tilde{z}) G(2(T - s), x_i - z') f(z - z').
\]

Then apply (4.2.12) for the first two \( G \)'s and use the Fourier transform to see that

\[
\Theta(x) \leq \sum_{i=1}^m \mathbf{1}_{\{t > T - 2^n\}} \int_{T - 2^n}^t \frac{ds}{(t - s)^{\alpha/2}} \int_{\mathbb{R}^d} \tilde{\mu}(\tilde{z}) G(2t, x - \tilde{z}) \int_{\mathbb{R}^{2d}} \, dz \, dz'
\]

\[
\times G \left( \frac{2(t - s)}{t}, z - \tilde{z} - \frac{s}{t}(x - \tilde{z}) \right) G(2(T - s), x_i - z') f(z - z')
\]

\[
\leq C \sum_{i=1}^m \mathbf{1}_{\{t > T - 2^n\}} \tilde{J}_0(2t, x) \int_0^{2^{n+t-T}} \frac{ds}{s^{\alpha/2}} \int_{\mathbb{R}^d} \hat{f}(d\xi)
\]

\[
\times \exp \left( - \left[ \frac{s(t - s)}{t} + T - t + s \right] |\xi|^2 \right)
\]

\[
\leq C \mathbf{1}_{\{t > T - 2^n\}} \tilde{J}_0(2t, x) \int_0^{2^{n+t-T}} \frac{ds}{s^{\alpha/2}} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( - \frac{3}{2} s |\xi|^2 \right),
\]

where in the last inequality we have used (4.2.14) and the fact that \( t \leq T \). Notice that

\[
\int_0^{2^n} \frac{ds}{s^{\alpha/2}} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( - \frac{3}{2} s |\xi|^2 \right) \leq e^{2-n} \int_0^\infty \frac{ds}{s^{\alpha/2}} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( -s(1 + |\xi|^2) \right)
\]

\[
= C \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{(1 + |\xi|^2)^{1-\alpha/2}} < \infty,
\]

82
which implies that \( \Theta(x) \leq \mathbb{I}_{(t>T-2^{-n})} \tilde{J}_0(2t, x) \). Therefore,
\[
\left\| \sup_{|\alpha| \leq \kappa} |I_3| \right\|_p \leq C\kappa|x - y|^\alpha \mathbb{I}_{(t>T-2^{-n})} \left( 1 + J_0(2t, x) + J_0(2t, y) \right).
\]
Combining these bounds prove (4.5.2).

**Step 2.** Now we prove (4.5.3). From (4.1.7), write the difference \( \hat{u}_{n,i}^z(t, x) - \hat{u}_{n,i}^z(t, y) \) in three parts as above. By the moment bound (4.3.1), we see that the difference for \( \theta_{n,i}^z \) reduces to
\[
\left\| \sup_{|\alpha| \leq \kappa} |\theta_{n,i}^z(t, x) - \theta_{n,i}^z(t, y)| \right\|_p \leq C \int_0^t ds \int_{\mathbb{R}^2} dz dz' \tilde{J}_0(s, z) h^z_n(s, z') f(z - z') \times |G(t - s, x - z) - G(t - s, y - z)|.
\]
Comparing the right-hand side of the above inequality with that of (4.5.8), we see that
\[
\left\| \sup_{|\alpha| \leq \kappa} |\theta_{n,i}^z(t, x) - \theta_{n,i}^z(t, y)| \right\|_p \leq C\kappa|x - y|^\alpha \mathbb{I}_{(t>T-2^{-n})} \left( 1 + J_0(2t, x) + J_0(2t, y) \right).
\]
Thanks to the boundedness of \( \rho' \), by the same arguments using moment bound (4.3.2) in the form of
\[
\left\| \sup_{|\alpha| \leq \kappa} |\hat{u}_{n,i}^z(t, x)| \right\|_p \leq C J_0^*(t, x),
\]
both the second and third part can be proved in exactly the same way as Step 1, except that \( \tilde{J}_0(t, x) \) should be replaced by \( J_0^*(t, x) \). This proves (4.5.3).

**Step 3.** The result (4.5.4) can be proved in the same way. We leave the details for interested readers. This completes the proof of Lemma 4.15.

### 4.5.2 Some Grownwall-type inequalities

We will need the following lemma, which is Lemma A.3 in [9] with \( \alpha \in (1, 2] \) replaced by \( \beta = 1/\alpha \). Note that the range of \( \alpha \) for Lemma A.3 ibid. could be any \( \alpha > 1 \) just as Lemma A.2 ibid. Recall that the two-parameter Mittag-Leffler function is defined as
\[
E_{\alpha, \beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0, z \in \mathbb{C}.
\]

**Lemma 4.16.** (Lemma A.3 of [9]) Suppose that \( \beta \in [0, 1) \), \( \lambda > 0 \), \( T > \epsilon > 0 \), and \( \theta_{\epsilon} : \mathbb{R}_+ \to \mathbb{R} \) is a locally integrable function.

1. If \( f \) satisfies that
\[
f(t) = \theta_{\epsilon}(t) + \lambda \epsilon^{(1-\beta)} \mathbb{I}_{(t>T-\epsilon)} \int_{T-\epsilon}^t (t - s)^{-\beta} f(s) ds
\]
for all \( t \in (0, T] \), then
\[
f(t) = \theta_{\epsilon}(t) + \mathbb{I}_{\{t > T - \epsilon\}} \int_{T - \epsilon}^{t} K_{\lambda^{(1 - \beta)}}(t - s)\theta_{\epsilon}(s)ds,
\]
(4.5.11)
for all \( t \in (0, T] \), where \( K_{\lambda^{(1 - \beta)}}(t) \) is defined as
\[
K_{\lambda}(t) := t^{-\beta} \lambda \Gamma(1 - \beta) E_{1 - \beta, 1 - \beta}(t^{1 - \beta}\lambda \Gamma(1 - \beta)).
\]
Moreover, when \( \theta_{\epsilon}(t) \geq 0 \), if the equality in (4.5.10) is replaced by \( \leq \) (resp. \( \geq \)), then the equality in the conclusion (4.5.11) should be replaced by \( \leq \) (resp. \( \geq \)) accordingly.

(2) When \( \theta_{\epsilon}(t) \geq 0 \), for some nonnegative constant \( C \) that depends on \( \alpha, \lambda \) and \( T \) (not on \( \epsilon \)), we have
\[
f(t) \leq \theta_{\epsilon}(t) + C \mathbb{I}_{\{t > T - \epsilon\}} \epsilon^{-(1 - 1/\alpha)} \int_{T - \epsilon}^{t} (t - s)^{-1/\alpha}\theta_{\epsilon}(s)ds,
\]
(4.5.12)
for all \( t \in [0, T] \). Moreover, if \( \theta_{\epsilon}(t) \equiv \theta_{\epsilon} \) is a nonnegative constant, then for the same constant \( C \), it holds that
\[
f(t) \leq C\theta_{\epsilon}, \quad \text{for all } t \in [0, T].
\]
(4.5.13)

Similar to [9], we still need to introduce the some functionals: for any function \( \theta : \mathbb{R}_{+} \rightarrow \mathbb{R} \), \( n \in \mathbb{N} \), \( t \in (0, T] \) and \( \beta \in [0, 1) \), define
\[
F_{n}[\theta](t) := \theta(t) + (1 - \beta) \mathbb{I}_{\{t > T - 2^{-n}\}} 2^{n(1 - \beta)} \int_{T - 2^{-n}}^{t} (t - s)^{-\beta}\theta(s)ds,
\]
(4.5.14)
and for \( m \geq 1 \),
\[
F_{n}^{m}[\theta](t) := m (1 - \beta) \mathbb{I}_{\{t > T - 2^{-n}\}} 2^{mn(1 - \beta)} \int_{T - 2^{-n}}^{t} (t - s)^{m(1 - \beta) - 1}\theta(s)ds.
\]
(4.5.15)

**Lemma 4.17.** (Lemma 8.17 of [9]) For all \( n, m, m' \geq 1 \) and \( T > 0 \), the following properties hold for all \( t \in (0, T] \):
\[
F_{n}[\theta](t) = \theta(t) + F_{n}^{1}[\theta](t),
\]
(4.5.16)
\[
\lim_{n \to \infty} F_{n}^{m}[\theta](t)\mathbb{I}_{\{t < T\}} = 0,
\]
(4.5.17)
\[
F_{n}^{m}[\theta](t) \leq \left( F_{n}^{m}[\theta^{m'}](t) \right)^{1/m'},
\]
(4.5.18)
\[
F_{n}^{m}[F_{n}^{m'}[\theta]](t) = C_{m, m', \alpha} F_{n}^{m + m'}[\theta](t).
\]
(4.5.19)

If \( \theta \) is left-continuous at \( T \), then
\[
\lim_{n \to \infty} F_{n}[\theta](T) = \theta(T).
\]
(4.5.20)
4.5.3 Proof of the conditionally boundedness (Proposition 4.14)

Now we are ready to prove our last result, Proposition 4.14, in order to complete the proof of Theorem 1.12.

Proof of Proposition 4.14. In this proof we assume \( t \in (0, T] \). Throughout the proof, \( p \) is an arbitrary number that is greater than or equal to 2. The proof consists of the following four steps.

**Step 1.** We first prove (4.5.1) for \( \hat{u}_n^p(T, x_i) \). Notice that

\[
\begin{align*}
\hat{u}_n^p(t, x_i) - u(t, x_i) &= \int_0^t \int_{\mathbb{R}^d} G(t - s, x_i - y) [\rho(\hat{u}_n^p(s, y)) - \rho(u(s, y))] W(dy)ds + \sum_{j \neq i} z_j \int_0^t \int_{\mathbb{R}^d} G(t - s, x_i - y) \rho(\hat{u}_n^p(s, y)) h_i^n(s, y')f(y - y')dsdydy' \\
&+ z_i \int_0^t \int_{\mathbb{R}^d} G(t - s, x_i - y) [\rho(\hat{u}_n^p(s, y)) - \rho(\hat{u}_n^p(s, x_i))] h_i^n(s, y')f(y - y')dsdydy' \\
&+ z_i \int_0^t \int_{\mathbb{R}^d} G(t - s, x_i - y) [\rho(u(s, x_i)) - \rho(u(t, x_i))] h_i^n(s, y')f(y - y')dsdydy' \\
&+ z_i \psi_i(t, x_i) \rho(u(t, x_i)) \\
&=: \sum_{t=1}^{6} I_t(t).
\end{align*}
\]

Then by the Lipschitz continuity of \( \rho \),

\[
|I_6| \leq |z_i|c_n \text{ Lip}_\rho \mathbb{I}_{\{t > T-2^{-n}\}} \int_{T-2^{-n}}^t ds |\hat{u}_n^p(s, x_i) - u(s, x_i)| \\
\times \int_{\mathbb{R}^d} dydy' G(t - s, x_i - y)G(T - s, x_i - y')f(y - y') \\
= &C_c_n \mathbb{I}_{\{t > T-2^{-n}\}} \int_{T-2^{-n}}^t ds |\hat{u}_n^p(s, x_i) - u(s, x_i)| \\
\times \int_{\mathbb{R}^d} f(d\xi) \exp \left( -\frac{1}{2} |t - s + T - s| |\xi|^2 \right) \\
\leq &C_c_n \mathbb{I}_{\{t > T-2^{-n}\}} \int_{T-2^{-n}}^t ds |\hat{u}_n^p(s, x_i) - u(s, x_i)| \int_{\mathbb{R}^d} f(d\xi) \exp \left( -(t - s)|\xi|^2 \right) \\
= &C_c_n \mathbb{I}_{\{t > T-2^{-n}\}} \int_{T-2^{-n}}^t ds |\hat{u}_n^p(s, x_i) - u(s, x_i)| k(2(t - s)),
\]

where \( k(t) \) is defined in (1.16). Then by Assumption 1.10 and Lemma 3.6, we have that \( c_n \leq C2^{(1-\beta)n} \) and

\[
\sup_{|x| \leq \kappa} |\hat{u}_n^p(t, x_i) - u(t, x_i)|
\]
\[
\leq \sum_{t=1}^{5} \sup_{|z| \leq \kappa} |I_t(t)| + C2^{n(1-\beta)} \mathbb{I}_{(t>T-2^{-n})} \int_{T-2^{-n}}^{t} (t-s)^{-\beta} \sup_{|z| \leq \kappa} |\hat{u}_z^n(s, x_i) - u(s, x_i)| \, ds.
\]

Hence, by Lemma 4.16 (see (4.5.12)) and by (4.5.14), with probability one,

\[
\sup_{|z| \leq \kappa} |\hat{u}_z^n(t, x_i) - u(t, x_i)| \leq C \sum_{t=1}^{5} F_n \left[ \sup_{|z| \leq \kappa} |I_t(\cdot)| \right](t) =: CM_{n}(t) + CF_n \left[ \sup_{|z| \leq \kappa} |I_5(\cdot)| \right](t). \tag{4.5.21}
\]

We estimate each term in the above sum separately. Using the same method as in the proof of Lemma 4.11 or the proof of Lemma 4.10 for \( I_1 \) in proving (4.3.21), an application of Kolmogorov continuity theorem shows that

\[
\left\| \sup_{t \in [0, T]} \left\| \sup_{|z| \leq \kappa} |I_1(t)| \right\| \right\| \leq C2^{-n(1-\beta)/2}.
\]

By (4.3.1) and (4.2.5), and since \( i \neq j \), we see that

\[
\left\| \sup_{t \in [0, T]} \left\| \sup_{|z| \leq \kappa} |I_2(t)| \right\| \right\| \leq C2^{-n\beta}.
\]

As for \( I_3(t) \) and \( I_4(t) \), we claim that

\[
\left\| \sup_{t \in [0, T]} \left\| \sup_{|z| \leq \kappa} |I_\ell(t)| \right\| \right\| \leq C2^{-n\alpha/2}, \quad \text{for } \ell = 3, 4. \tag{4.5.22}
\]

We first prove the case for \( I_3 \). By the Minkowski inequality,

\[
\left\| \sup_{|z| \leq \kappa} |I_3(t)| \right\| \leq C2^{n(1-\beta)} \mathbb{I}_{(t>T-2^{-n})} \int_{T-2^{-n}}^{t} ds \int_{\mathbb{R}^{2d}} dy dy' f(y - y') \times G(t-s, x_i-y)G(T-s, x_i-y') \left\| \sup_{|z| \leq \kappa} |\hat{u}_z^n(s, y) - \hat{u}_z^n(s, x_i)| \right\|.
\]

Then by Lemma 4.15, for \( s \in [T - 2^{-n}, t] \),

\[
\left\| \sup_{|z| \leq \kappa} |\hat{u}_z^n(s, y) - \hat{u}_z^n(s, x_i)| \right\| \leq C(1 + J_0(2s, y))|y - x_i|^\alpha.
\]

Now we use the notation \( \tilde{\mu} \) and \( \tilde{J}_0(t, x) \) as in (4.5.7). Thus,

\[
\left\| \sup_{|z| \leq \kappa} |I_3(t)| \right\| \leq C2^{n(1-\beta)} \mathbb{I}_{(t>T-2^{-n})} \int_{T-2^{-n}}^{t} ds \int_{\mathbb{R}^{d}} \tilde{\mu}(dz) \int_{\mathbb{R}^{2d}} dy dy' f(y - y') \times G(t-s, x_i-y)G(T-s, x_i-y')G(2s, y-z) |y - x_i|^\alpha.
\]

86
Now we apply the inequality (4.4.5) to see that
\[
\left\| \sup_{|x| \leq \kappa} |I_3(t)| \right\|_p \leq C^2 n^{(1-\beta)} \|I_{(t>T-2^{-n})} \|_p \int_{T-2^{-n}}^t ds \left( \frac{t - s}{s} \right)^{\alpha/2} \int_{\mathbb{R}^d} d\bar{\mu}(dz) \int_{\mathbb{R}^{2d}} dy dy' \int_0^s dy dy' G(y - y')
\]
\[
\times G(2(t - s), x_i - y) G(T - s, x_i - y') G(2s, y - z)
\]
\[
= C^2 n^{(1-\beta)} \|I_{(t>T-2^{-n})} \|_p \int_{T-2^{-n}}^t ds \left( \frac{t - s}{s} \right)^{\alpha/2} \int_{\mathbb{R}^d} d\bar{\mu}(dz) G(2t, x_i - z)
\]
\[
\times \int_{\mathbb{R}^{2d}} dy dy' f(y - y') G\left( \frac{2s(t - s)}{t}, y - z - \frac{s}{t} (x_i - y) \right) G(t - s, x_i - y')
\]
\[
\leq C^2 n^{(1-\beta)} \tilde{J}_0(2t, x_i) \|I_{(t>T-2^{-n})} \|_p \int_{T-2^{-n}}^t s^{\alpha/2} \int_{\mathbb{R}^d} f(d\xi)
\]
\[
\times \exp \left( -\frac{1}{2} \left( \frac{2s(t - s)}{t} + T - t + s \right) |\xi|^2 \right),
\]
where we have applied (4.2.12). Then by (4.2.14) and Assumption 1.10, we see that
\[
\left\| \sup_{|x| \leq \kappa} |I_3(t)| \right\|_p \leq C^2 n^{(1-\beta)} \tilde{J}_0(2t, x_i) \|I_{(t>T-2^{-n})} \|_p \int_{T-2^{-n}}^t s^{\alpha/2} \int_{\mathbb{R}^d} f(d\xi) \exp \left( -s |\xi|^2 \right)
\]
\[
\leq C^2 n^{(1-\beta)} \tilde{J}_0(2t, x_i) \|I_{(t>T-2^{-n})} \|_p 2^{-n\alpha/2} V_d(2^{-n})
\]
\[
\leq C^2 n^{-\alpha/2},
\]
which proves (4.5.22) for \( \ell = 3 \).

As for \( I_4 \), by the Hölder continuity of \( u(t, x) \) (see Steps 2 & 3 of the proof of Theorem 1.8 in [10]), we see that
\[
\left\| \sup_{|x| \leq \kappa} |I_4(t)| \right\|_p \leq C^2 n^{(1-\beta)} \|I_{(t>T-2^{-n})} \|_p \int_{T-2^{-n}}^t ds \|u(s, x_i) - u(t, x_i)\|_p
\]
\[
\times \int_{\mathbb{R}^{2d}} dy dy' G(t - s, x_i - y) G(T - s, x_i - y') f(y - y')
\]
\[
\leq C^2 n^{(1-\beta)} \|I_{(t>T-2^{-n})} \|_p \int_{T-2^{-n}}^t (t - s)^{-\beta} \|u(s, x_i) - u(t, x_i)\|_p ds
\]
\[
\leq C^2 n^{(1-\beta)} \|I_{(t>T-2^{-n})} \|_p \int_{T-2^{-n}}^t (t - s)^{-\beta} (t - s)^{\alpha/2} ds
\]
\[
\leq C^2 n^{-\alpha/2},
\]
which proves (4.5.22) for \( \ell = 4 \).

Hence, the above computations show that
\[
\sup_{t \in [0, T]} \left\| M_n^*(t) \right\|_p = \sup_{t \in [0, T]} \left\| \sum_{\ell=1}^4 F_n \left[ \sup_{|x| \leq \kappa} |I_{\ell}(\cdot)| \right] (t) \right\|_p \leq C^2 2^{-\frac{\min(2\beta, 1-\beta, \alpha)}}{2}
\]
By the Borel-Cantelli lemma, we see that, for all \( t \in [0, T] \),
\[
\lim_{n \to \infty} M_n^*(t) = \lim_{n \to \infty} \sum_{\ell=1}^4 F_n \left[ \sup_{|x| \leq \kappa} |I_{\ell}(\cdot)| \right] (t) = 0, \text{ a.s.}
\]
(4.5.23)
As for $I_5$, since $\Psi_n(t, x_i)$ is bounded by one (see (4.2.3)), we have that
\[
|\Psi_n(t, x_i)\rho(u(t, x_i))| \leq |\rho(u(t, x_i))|, \quad \text{a.s.} \tag{4.5.24}
\]
Hence, for all $n \in \mathbb{N}$,
\[
F_n[|I_5(\cdot)|](t) \leq F_n[|\rho(u(\cdot, x_i))|](t). \tag{4.5.25}
\]
Therefore, by combining (4.5.21), (4.5.23) and (4.5.25) we have that
\[
\sup_{|x| \leq \kappa}|\hat{u}_n^n(t, x_i) - u(t, x_i)| \leq M_n^*(t) + C|\rho(u(t, x_i))| + CF_n^1[|\rho(u(\cdot, x_i))|](t) \quad \text{a.s.} \tag{4.5.26}
\]
for all $n \in \mathbb{N}$. Finally, by letting $t = T$ and sending $n \to \infty$, and by the Hölder continuity of $s \mapsto \rho(u(s, x_i))$, we have that
\[
\limsup_{n \to \infty} \sup_{|x| \leq \kappa}|\hat{u}_n^n(T, x_i)| \leq \limsup_{n \to \infty} \sup_{|x| \leq \kappa}|\hat{u}_n^n(T, x_i) - u(T, x_i)| + |u(T, x_i)|
\]
\[
\leq C|\rho(u(T, x_i))| + |u(T, x_i)|, \quad \text{a.s.}
\]
This proves Proposition 4.14 for $\hat{u}_n^n(T, x)$.

**Step 2.** Now we prove Proposition 4.14 for $\hat{u}_n^n(t, x)$. Notice that
\[
\hat{u}_n^n(t, x_i) - \rho(u(t, x_i))
\]
\[
= \theta_n^n(t, x_i)
\]
\[
+ \chi_{\{t > T - 2^{-n}\}} \int_t^T \int_{T - 2^{-n}}^{T} G(t - s, x_i - y)\rho' \left(\hat{u}_n^n(s, y) - \hat{u}_n^n(s, x_i)\right) \hat{u}_n^n(s, y) W(dsdy)
\]
\[
+ \sum_{j \neq i} \sum_{k} \int_0^t \int_{\mathbb{R}^{2d}} G(t - s, x_i - y)\rho' \left(\hat{u}_n^n(s, y) - \hat{u}_n^n(s, x_i)\right) h_{n, s}^i(s, y') f(y - y') dsdyd'y'
\]
\[
+ z_i \int_0^t \int_{\mathbb{R}^{2d}} G(t - s, x_i - y)\rho' \left(\hat{u}_n^n(s, y) - \hat{u}_n^n(s, x_i)\right) h_{n, s}^i(s, y') f(y - y') dsdyd'y'
\]
\[
+ z_i \int_0^t \int_{\mathbb{R}^{2d}} G(t - s, x_i - y)\rho' \left(\hat{u}_n^n(s, y) - \hat{u}_n^n(s, x_i)\right) h_{n, s}^i(s, y') f(y - y') dsdyd'y'
\]
\[
+ z_i \int_0^t \int_{\mathbb{R}^{2d}} G(t - s, x_i - y)\rho' \left(\hat{u}_n^n(s, y) - \hat{u}_n^n(s, x_i)\right) h_{n, s}^i(s, y') f(y - y') dsdyd'y'
\]
\[
= \sum_{\ell = 0}^6 I_{\ell}(t).
\]
By similar arguments as those in Step 1, we see that
\[
\sup_{|x| \leq \kappa}|\hat{u}_n^n(t, x_i) - \rho(u(t, x_i))| \leq \sum_{\ell = 0}^5 \sup_{|x| \leq \kappa}|I_{\ell}(t)| + C \chi_{\{t > T - 2^{-n}\}} 2^{n(1 - \beta)}
\]
\[
\times \int_t^T (t - s)^{-\beta} \sup_{|x| \leq \kappa}|\hat{u}_n^n(s, x_i) - \rho(u(s, x_i))| ds, \tag{4.5.27}
\]
88
The term and (by Lemma 4.16), with probability one,
\[
\sup_{|z| \leq \kappa} |\hat{u}_{z}^{n,i}(t, x_i) - \rho(u(t, x_i))| \leq C \sum_{\ell=0}^{5} F_n \left( \sup_{|z| \leq \kappa} |I_\ell(\cdot)| \right)(t).
\] (4.5.28)

We first consider \(I_0(t)\). Decompose \(z_i \theta_i^{n,i}(t, x_i)\) into three parts
\[
z_i \theta_i^{n,i}(t, x_i) = \hat{u}_{z_i}^n(t, x_i) - u(t, x_i)
+ \int_0^t \int_{\mathbb{R}^d} G(t - s, x_i - y) [\rho(u(s, y)) - \rho(\hat{u}_{z_i}^n(s, y))] W(dsdy)
- \sum_{j \neq i} z_j \theta_j^{n,j}(t, x_i)
=: I_{0,1}(t) - I_{0,2}(t) - I_{0,3}(t).
\]

Notice that \(I_{0,2}(t)\) is equal to \(I_1(t)\) in Step 1. From (4.3.4) and Proposition 4.1, we see that
\[
\sup_{t \in [0, T]} \left\| F_n \left( \sup_{|z| \leq \kappa} |I_{0,3}(\cdot)| \right)(t) \right\|_p \leq C 2^{-\beta n}.
\]

As for \(I_{0,1}(t)\), by (4.5.26) and (4.5.19), we see that with probability one,
\[
F_n \left( \sup_{|z| \leq \kappa} |I_{0,1}(\cdot)| \right)(t) \leq F_n[M^*_\kappa](t) + C|\rho(u(t, x_i))| + C \sum_{\ell=1}^{2} F_n \left[ |\rho(u(\cdot, x_i))| \right](t).
\] (4.5.29)

The terms \(I_1\) to \(I_4\) are similar to those in Step 1 and by the same arguments as those in Step 1 and using the fact that \(\rho'\) is bounded, we see that
\[
\sup_{t \in [0, T]} \left\| \sum_{\ell=1}^{4} F_n \left( \sup_{|z| \leq \kappa} |I_\ell(\cdot)| \right)(t) \right\|_p \leq C 2^{-\frac{\beta}{2} \min(2, 3 - \beta, 1 - \alpha)}.
\]

Set
\[
M^{**}_n(t) := \sum_{\ell=0}^{1} F_n \left( \sup_{|z| \leq \kappa} |I_{0,\ell}(\cdot)| \right)(t) + \sum_{\ell=1}^{4} F_n \left( \sup_{|z| \leq \kappa} |I_\ell(\cdot)| \right)(t) + F_n[M^*_\kappa](t).
\]

Hence,
\[
\sup_{t \in [0, T]} \left\| M^{**}_n(t) \right\|_p \leq C 2^{-\frac{\beta}{2} \min(2, 3 - \beta, 1 - \alpha)}
\]
and by the Borel-Cantelli lemma,
\[
\lim_{n \to \infty} M^{**}_n(t) = 0, \quad \text{a.s. for all } t \in [0, T].
\] (4.5.30)

The term \(I_5\) part is identical to the term \(I_5\) in Step 1, so that we have the bound (4.5.25). Combining (4.5.28), (4.5.29), (4.5.30) and (4.5.25) shows that for all \(t \in (0, T]\) and \(n \in \mathbb{N}\),
\[
\sup_{|z| \leq \kappa} |\hat{u}_{z}^{n,i}(t, x_i) - \rho(u(t, x_i))| \leq C M^{**}_n(t) + C |\rho(u(t, x_i))| + C \sum_{\ell=1}^{2} F_n \left[ |\rho(u(\cdot, x_i))| \right](t) \quad \text{a.s.}
\] (4.5.31)
Finally, by letting $t = T$ and sending $n \to \infty$, and by the Hölder continuity of $s \mapsto \rho(u(s, x_i))$, we have that

$$\limsup_{n \to \infty} \sup_{|x| \leq \kappa} \left| \hat{u}^{n,i}_z(T, x_i) \right| \leq C |\rho(u(T, x_i))| \text{ a.s.}$$

This proves Proposition 4.14 for $\hat{u}^{n,i}_z(T, x_i)$.

**Step 3.** In this step, we will prove for all $t \in [0, T]$, with probability one,

$$F_n \left[ \sup_{|x| \leq \kappa} \left| \hat{\theta}^{n,i,k}_z(\cdot, x_i) \right| (t) \right] \leq C M^t_n(t) + C |\rho(u(t, x_i))| + C \sum_{\ell=1}^3 F^n_\ell \left[ |\rho(u(\cdot, x_i))| \right] (t) \quad (4.5.32)$$

with

$$M^t_n(t) := F_n \left[ \sum_{j \neq i} \sup_{|x| \leq \kappa} \left| z_j \hat{\theta}^{n,i,k}_z(\cdot, x_i) \right| (t) \right] + F_n[M^*_n](t)$$

satisfying that

$$\sup_{t \in [0, T]} \left\| M^t_n(t) \right\|_p \leq 2^{-\frac{n}{2}} \min(2\beta,1-\beta,\alpha) \text{ and } \lim_{n \to \infty} M^t_n(t) = 0, \text{ a.s. for all } t \in [0, T].$$

Indeed, by (4.3.15), we need only to consider the case when $k = i$ (see, e.g., the arguments leading to (4.5.33) below). Notice from (4.1.7) that

$$z_i \hat{\theta}^{n,i}_z(t, x_i) = \hat{\theta}^{n,i}_z(t, x_i) - \theta^{n,i}_z(t, x_i) - \sum_{j \neq i} z_j \theta^{n,i,k}_z(t, x_i) - I(t, x_i),$$

where

$$I(t, x_i) := \mathbb{I}_{\{t > T-2^{-n}\}} \int_{T-2^{-n}}^t \int_{\mathbb{R}^d} G(t-s, x_i-y) \rho'(\hat{u}^{n}_z(s, y)) \hat{u}^{n,i}_z(s, y) W(dsdy).$$

By (4.5.31), we see that

$$F_n \left[ \sup_{|x| \leq \kappa} \left| \hat{u}^{n,i}_z(\cdot, x_i) \right| \right] (t) \leq F_n \left[ \sup_{|x| \leq \kappa} \left| \hat{u}^{n,i}_z(\cdot, x_i) - \rho(u(\cdot, x_i)) \right| \right] (t) + F_n \left[ |\rho(u(\cdot, x_i))| \right] (t)$$

$$\leq C F_n[M^*_n](t) + C |\rho(u(t, x_i))| + C \sum_{\ell=1}^3 F^n_\ell \left[ |\rho(u(\cdot, x_i))| \right] (t) \text{ a.s.}$$

Notice that $\hat{\theta}^{n,i}_z(t, x_i)$ is the $I_0$ term in Step 2, hence by (4.5.29)

$$F_n \left[ \sup_{|x| \leq \kappa} \left| \hat{\theta}^{n,i}_z(\cdot) \right| \right] (t) \leq C M^*_n(t) + C |\rho(u(t, x_i))| + C \sum_{\ell=1}^2 F^n_\ell \left[ |\rho(u(\cdot, x_i))| \right] (t) \text{ a.s.}$$

For $i \neq j$, from (4.3.5) and Proposition 4.1, we see that

$$\sup_{t \in [0, T]} \left\| \sup_{j \neq i} \sum_{j \neq i} z_j \hat{\theta}^{n,i,k}_z(t, x_i) \right\|_p \leq C 2^{-n\beta}. \quad (4.5.33)$$
The $I$ term coincides with the $I_1$ term in Step 2. Combining the above four terms proves (4.5.32).

**Step 4.** In this last step, we will prove Proposition 4.14 for $\hat{u}_{z,i}^{n,k}(T, x_i)$. Write the six parts of $\hat{u}_{z,i}^{n,k}(t, x_i)$ in (4.1.9) as in (4.3.18), that is,

$$\hat{u}_{z,i}^{n,k}(t, x_i) = \theta_{z,i}^{n,k}(t, x_i) + \theta_{z,k}^{n,i}(t, x_i) + \sum_{\ell=1}^{4} U_{4,\ell}^{n}(t, x_i). \quad (4.5.34)$$

We first consider the term $U_{4}^{n}(t, x_i)$ which contributes to the recursion. Write $U_{4}^{n}(t, x_i)$ in three parts

$$U_{4}^{n}(t, x_i) = \sum_{j \neq i} z_j \int_{0}^{t} \int_{\mathbb{R}^d} G(t - s, x_i - y) \rho'(\hat{u}_{z}^{n}(s, y)) \hat{u}_{z,i}^{n,k}(s, y) h_{n}^i(s, y') f(y - y') ds dy dy'$$

$$+ z_i \int_{0}^{t} \int_{\mathbb{R}^d} G(t - s, x_i - y) \rho' \left( \hat{u}_{z}^{n}(s, y) - \hat{u}_{z,i}^{n,k}(s, x_i) \right) h_{n}^i(s, y') f(y - y') ds dy dy'$$

$$+ z_i \int_{0}^{t} \int_{\mathbb{R}^d} G(t - s, x_i - y) \rho'(\hat{u}_{z}^{n}(s, y)) \hat{u}_{z,i}^{n,k}(s, x_i) h_{n}^i(s, y') f(y - y') ds dy dy'$$

$$=: U_{4,1}^{n}(t, x_i) + U_{4,2}^{n}(t, x_i) + U_{4,3}^{n}(t, x_i).$$

Notice that

$$\sup_{|x| \leq \kappa} |U_{4,3}^{n}(t, x_i)| \leq C \mathbb{I}_{\{T > T - 2^{-n}\}} 2^{n(1-\beta)} \int_{T - 2^{-n}}^{t} (t - s)^{-\beta} \sup_{|x| \leq \kappa} |\hat{u}_{z,i}^{n,k}(s, x_i)| ds, \text{ a.s.}$$

Therefore, by Lemma 4.16,

$$\sup_{|x| \leq \kappa} |\hat{u}_{z,i}^{n,k}(t, x_i)| \leq C \sum_{\kappa} \left[ \sup_{|x| \leq \kappa} I_{4,1}(\cdot, x_i) \right] (t) \text{ a.s.,} \quad (4.5.35)$$

where the summation is over all terms on the right-hand side of (4.5.34) except $U_{4,3}^{n}(t, x_i)$ and $I_{4,1}^{n}$ stands for such a generic term.

By the similar arguments as in the previous steps, using (4.5.4), one can show that

$$\lim_{n \to \infty} F_n \left[ \sup_{|x| \leq \kappa} |U_{4,\ell}^{n}(\cdot, x_i)| \right] (t) = 0, \text{ a.s. for } \ell = 1, 2 \text{ and } t \in [0, T]. \quad (4.5.36)$$

By (4.3.2), (4.3.3), the boundedness of both $\rho'$ and $\rho''$, and (4.2.4), we can obtain moment bounds for both $\sup_{|x| \leq \kappa} |U_{4}^{n}(t, x_i)|$ and $\sup_{|x| \leq \kappa} |U_{3}^{n}(t, x_i)|$ and then argue using the Borel-Cantelli lemma as above to conclude that

$$\lim_{n \to \infty} F_n \left[ \sup_{|x| \leq \kappa} |U_{3}^{n}(\cdot, x_i)| \right] (t) = 0, \text{ a.s. for } \ell = 1, 3 \text{ and } t \in [0, T]. \quad (4.5.37)$$

As for the term $U_{2}^{n}$, because

$$|\hat{u}_{z,i}^{n,k}(s, y)| \leq \frac{1}{2} (\hat{u}_{z,i}^{n,k}(s, y)^2 + \hat{u}_{z}^{n}(s, y)^2),$$
we only need to consider the case when \( i = k \). By the boundedness of \( \rho'' \), we see that

\[
U_2^n(t,x_i) \leq C \int_0^t \int_{\mathbb{R}^{2d}} G(t-s,x_i-y) \hat{u}^{n,i}_z(s,y)^2 \langle z,h_n(s,y') \rangle f(y-y') dsdydy' \\
\leq C \int_0^t \int_{\mathbb{R}^{2d}} G(t-s,x_i-y) \left[ \hat{u}^{n,i}_z(s,y) - \hat{u}^{n,i}_z(s,x_i) \right]^2 \langle z,h_n(s,y') \rangle f(y-y') dsdydy' \\
+ C \int_0^t \int_{\mathbb{R}^{2d}} G(t-s,x_i-y) \left[ \hat{u}^{n,i}_z(s,x_i) - \rho(u(s,x_i)) \right]^2 \langle z,h_n(s,y') \rangle f(y-y') dsdydy' \\
+ C \int_0^t \int_{\mathbb{R}^{2d}} G(t-s,x_i-y) \left[ \rho(u(s,x_i)) - \rho(u(t,x_i)) \right] \langle z,h_n(s,y') \rangle f(y-y') dsdydy' \\
+ C \rho(u(t,x_i))^2 \\=: U_{2,1}^n(t,x_i) + U_{2,2}^n(t,x_i) + U_{2,3}^n(t,x_i) + C \rho(u(t,x_i))^2.
\]

By (4.5.3) and the Hölder continuity of \( s \mapsto \rho(u(s,x_i)) \) one can prove in the same way as before that

\[
\lim_{n \to \infty} F_n \left[ \sup_{|x| \leq \kappa} U_{2,\ell}^n(\cdot,x_i) \right](t) = 0, \quad \text{a.s. for } \ell = 1, 3 \text{ and } t \in [0,T]. \tag{4.5.38}
\]

Notice that

\[
\sup_{|x| \leq \kappa} U_{2,2}^n(t,x_i) \leq CF_n^1 \left[ \sup_{|x| \leq \kappa} \left( \hat{u}^{n,i}_z(\cdot,x_i) - \rho(u(\cdot,x_i)) \right)^2 \right](t).
\]

By applying (4.5.18) on (4.5.31) with \( m' = 2 \), we see that with probability one,

\[
\sup_{|x| \leq \kappa} \left( \hat{u}^{n,i}_z(s,x_i) - \rho(u(s,x_i)) \right)^2 \leq C[M^{**}_n(t)]^2 + C \rho^2(u(t,x_i)) + C \sum_{\ell=1}^2 F_n^\ell \left[ \rho^2(u(\cdot,x_i)) \right](t).
\]

Hence, by (4.5.19), for all \( n \in \mathbb{N} \),

\[
\sup_{|x| \leq \kappa} U_{2,2}^n(t,x_i) \leq CF_n^1[M^{**}_n](t) + C \sum_{\ell=1}^3 F_n^\ell \left[ \rho^2(u(\cdot,x_i)) \right](t) \quad \text{a.s.}
\]

Then another application of (4.5.19) shows that

\[
F_n \left[ \sup_{|x| \leq \kappa} U_{2,2}^n(\cdot,x_i) \right](t) \leq C \sum_{\ell=1}^2 F_n^\ell[M^{**}_n](t) + C \sum_{\ell=1}^4 F_n^\ell \left[ \rho^2(u(\cdot,x_i)) \right](t) \quad \text{a.s.,}
\]

for all \( t \in [0,T] \). Therefore,

\[
\limsup_{n \to \infty} F_n \left[ \sup_{|x| \leq \kappa} U_{2,2}^n(\cdot,x_i) \right](t) \leq C \rho^2(u(t,x_i)) \quad \text{a.s. for all } t \in [0,T]. \tag{4.5.39}
\]

Finally, by combining (4.5.32), (4.5.36), (4.5.37), (4.5.38) and (4.5.39), setting \( t = T \), and sending \( n \) to infinity, we can conclude that

\[
\limsup_{n \to \infty} \sup_{|x| \leq \kappa} |\hat{u}^{n,i,k}_z(T,x_i)| \leq C |\rho(u(T,x_i))| + C \rho^2(u(T,x_i)).
\]

This proves the case for \( \hat{u}^{n,i,k}_z(T,x) \). With this, we have completed the whole proof of Proposition 4.14. \( \square \)
References

[1] Amir, Gideon, Ivan Corwin and Jeremy Quastel. Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions. *Comm. Pure Appl. Math.* 64 (2011), no. 4, 466–537.

[2] Balan, Raluca M. and Le Chen. Parabolic Anderson model with space-time homogeneous Gaussian noise and rough initial condition *J. Theoret. Probab.*, to appear 2017.

[3] V. Bally and E. Pardoux. Malliavin calculus for white noise driven parabolic SPDEs. *Potential Anal.* 9 (1998), no. 1, 27–64.

[4] Bertini, Lorenzo and Giambattista Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.* 183 (1997), no. 3, 571–607.

[5] Bouleau, Nicolas and Francis Hirsch. *Dirichlet forms and analysis on Wiener space.* De Gruyter Studies in Mathematics, 14. Walter de Gruyter & Co., Berlin, 1991. x+325 pp.

[6] Cannizzaro, Giuseppe, Peter K. Friz and Paul Gassiat. Malliavin Calculus for regularity structures: the case of $g$PAM. *J. Funct. Anal.*, 272 (2017), no. 1, 363–419.

[7] Carmona, René A. and Stanislav A. Molchanov. Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.*, 108 (1994), no. 518, viii+125 pp.

[8] Chen, Le and Robert C. Dalang. Moments and growth indices for nonlinear stochastic heat equation with rough initial conditions. *Ann. Probab.* 43 (2015), no. 6, 3006–3051.

[9] Chen, Le, Yaozhong Hu and David Nualart. Regularity and strict positivity of densities for the nonlinear stochastic heat equation. *Mem. Amer. Math. Soc.*, 2019, to appear.

[10] Chen, Le and Jingyu Huang. Comparison principle for stochastic heat equation on $\mathbb{R}^d$. *Ann. Probab.* 2019, to appear.

[11] Chen, Le and Kunwoo Kim. Nonlinear stochastic heat equation driven by spatially colored noise: moments and intermittency. *Acta Math. Sci. Ser. B*, 2019, to appear.

[12] Dalang, Robert C. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.’s. *Electron. J. Probab.* 4 (1999), no. 6, 29 pp.

[13] Dalang, Robert, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. *A minicourse on stochastic partial differential equations.* Held at the University of Utah, Salt Lake City, UT, May 8-19, 2006. Edited by Khoshnevisan and Firas Rassoul-Agha. Lecture Notes in Mathematics, 1962. Springer-Verlag, Berlin, 2009. xii+216 pp.

[14] Dalang, Robert C. and Lluís Quer-Sardanyons. Stochastic integrals for spde’s: a comparison. *Expo. Math.* 29 (2011), no. 1, 67–109.

[15] Foondun, Mohammud and Davar Khoshnevisan. On the stochastic heat equation with spatially-colored random forcing. *Trans. Amer. Math. Soc.* 365 (2013), no. 1, 409–458.

[16] Grafakos, Loukas *Modern Fourier analysis.* Third edition. Graduate Texts in Mathematics, 250. Springer, New York, 2014. xvi+624 pp.

[17] Hairer, Martin. A theory of regularity structures. *Invent. Math.*, 198 (2) (2014), 1–236.
[18] Hairer, Martin. Solving the KPZ equation. *Ann. of Math.*, (2) 178 (2013), no. 2, 559–664.

[19] Y. Hu, J. Huang, D. Nualart, X. Sun. Smoothness of the joint density for spatially homogeneous SPDEs. *J. Math. Soc. Japan*, Vol. 67, No. 4 (2015), 1605–1630.

[20] Mueller, Carl and David Nualart. Regularity of the density for the stochastic heat equation. *Electron. J. Probab.*, 13 (2008), no. 74, 2248–2258.

[21] D. Nualart. *Malliavin calculus and its applications*. CBMS Regional Conference Series in Mathematics, 110. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2009.

[22] D. Nualart and L. Quer-Sardanyons. Existence and smoothness of the density for spatially homogeneous SPDEs. *Potential Anal.* 27 (2007), no. 3, 281–299.

[23] E. Nualart. On the density of systems of non-linear spatially homogeneous SPDEs. *Stochastics*, 85 (2013), no. 1, 48–70.

[24] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC. Cambridge Univ. Press, Cambridge, 2010.

[25] Pardoux, Etienne and Tusheng Zhang. Absolute continuity of the law of the solution of a parabolic SPDE. *J. Funct. Anal.* 112 (1993), no. 2, 447–458.

[26] Quer-Sardanyons, Lluís and Marta Sanz-Solé. A stochastic wave equation in dimension 3: smoothness of the law. *Bernoulli*, 10 (2004), no. 1, 165–186.

[27] Walsh, John B. *An Introduction to Stochastic Partial Differential Equations*. In: École d’été de probabilités de Saint-Flour, XIV—1984, 265–439. Lecture Notes in Math. 1180, Springer, Berlin, 1986.

**Le CHEN**
Department of Mathematical Sciences
University of Nevada, Las Vegas
Box 454020,
4505 S. Maryland Pkwy.
Las Vegas, NV 89154-4020, USA.
E-mails: le.chen@unlv.edu
chenle02@gmail.com

**Jingyu Huang**
School of Mathematics
University of Birmingham
Edgbaston, Birmingham,
B15 2TT, UK
Email: j.huang.4@bham.ac.uk