SINGULAR SRB MEASURES FOR A NON 1–1 MAP OF THE UNIT SQUARE

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Abstract. We consider a map of the unit square which is not 1–1, such as the memory map studied in [8]. Memory maps are defined as follows: $x_{n+1} = M_\alpha(x_{n-1}, x_n) = \tau(\alpha \cdot x_n + (1 - \alpha) \cdot x_{n-1})$, where $\tau$ is a one-dimensional map on $I = [0, 1]$ and $0 < \alpha < 1$ determines how much memory is being used. In this paper we let $\tau$ to be the symmetric tent map. To study the dynamics of $M_\alpha$, we consider the two-dimensional map $G_\alpha : [x_{n-1}, x_n] \mapsto [x_n, \tau(\alpha \cdot x_n + (1 - \alpha) \cdot x_{n-1})]$. The map $G_\alpha$ for $\alpha \in (0, 3/4]$ was studied in [8]. In this paper we prove that for $\alpha \in (3/4, 1)$ the map $G_\alpha$ admits a singular Sinai-Ruelle-Bowen measure. We do this by applying Rychlik’s results for the Lozi map. However, unlike the Lozi map, the maps $G_\alpha$ are not invertible which creates complications that we are able to overcome.

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1. Introduction

Let $\tau$ be a piecewise, expanding map on $I = [0, 1]$. We consider a process $x_{n+1} = M_\alpha(x_n) \equiv \tau(\alpha \cdot x_n + (1 - \alpha) \cdot x_{n-1})$, $0 < \alpha < 1$, which we call a map with memory since the next state $x_{n+1}$ depends not only on current state $x_n$ but also on the past $x_{n-1}$. Note that $M_\alpha$ is a map from $[0, 1]^2$ to $[0, 1]$ and hence is not a dynamical system.

A natural method to study the long term behaviour of the process $M_\alpha$, is to study the invariant measures of the two dimensional transformation $G_\alpha : [x_{n-1}, x_n] \mapsto [x_n, M_\alpha(x_n)] = [x_n, \tau(\alpha \cdot x_n + (1 - \alpha) \cdot x_{n-1})]$.

In [8] we studied $G_\alpha$ with the tent map $\tau(x) = 1 - 2|x - 1/2|$, $x \in I$, for $\alpha \in (0, 3/4]$. For $0 < \alpha \leq 0.46$, we proved that $G_\alpha$ admits an absolutely continuous invariant measure (acim). We conjecture that acim exists also for $\alpha \in (0.46, 1/2)$. As $\alpha$ approaches 1/2 from below, that is, as we approach a balance between the
memory state and the present state, the support of the a.c.m.s become thinner until 
at $\alpha = 1/2$, all points have period 3 or eventually possess period 3. We proved that 
for $\alpha = 1/2$ all points (except two fixed points) are eventually periodic with period 
3. For $\alpha = 3/4$ we proved that all points of the line $x + y = 4/3$ (except the fixed 
point) are of period 2 and all other points (except $(0,0)$) are attracted to this line. 
For $1/2 < \alpha < 3/4$, we prove the existence of a global attractor: for all starting 
points in the square $[0,1]^2$ except $(0,0)$, the orbits are attracted to the fixed point 
$(2/3,2/3)$.

In this paper, we continue the study of transformation $G_\alpha$ for $\alpha \in (3/4,1)$ 
and prove the existence of a singular Sinai-Ruelle-Bowen measure $\mu_\alpha$. The invariant 
measure is singular with respect to Lebesgue measure since for $\alpha \in (3/4,1)$ the 
determinants of the derivative matrices of $G_\alpha$ are less than one, hence the support 
of the invariant measure is of Lebesgue measure 0. The invariant measure has two 
main properties: for Lebesgue almost every point $x \in [0,1]^2$ and any continuous 
function $g : [0,1]^2 \to \mathbb{R}$,
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(G_\alpha^k x) = \mu_\alpha(g),
\]
and the conditional measures induced by $\mu_\alpha$ on segments with expanding directions 
are one-dimensional absolutely continuous measures.

Our method follows closely the techniques in Rychlik [10] for the Lozi map. The 
most important difference between the Lozi map and the $G_\alpha$’s is the fact that our 
maps are not invertible. For maps that are invertible or locally invertible, there 
are results known [11 2 3 0 7 11 13 14 15 16]. However, to the best of our 
knowledge the existence of a singular SRB measure has, until now, not been proven 
for any non-invertible map.

2. Abstract Reduction Theorem

Similarly as Rychlik in [10], we will start with abstract considerations. Sections 
2 and 3 are taken from [10] almost without any changes. We present them here for 
completeness, to introduce the notation and the results we need in the following 
sections.

Let $(X, \Sigma, m)$ be a Lebesgue measure with a $\sigma$-algebra $\Sigma$ and a probability 
measure $m$. Let $T : X \to X$ be a measurable, nonsingular mapping, i.e., $T_* m \ll m$.

We define the Frobenius-Perron operator induced by $T$ as

\[
P_T f = \frac{d(T_*(fm))}{dm} \quad \text{(Radon – Nikodym derivative),}
\]

for $f \in L^1(X, \Sigma, m)$ and we have $P_T f \in L^1(X, \Sigma, m)$. Equivalently, we can define 
$P_T f$ as the unique element of $L^1(X, \Sigma, m)$ satisfying

\[
\int_X (h \circ T) \cdot f dm = \int_X h \cdot P_T f dm,
\]

for all $h \in L^\infty(X, \Sigma, m)$. This means that operator $P_T$ is the conjugate of the 
Koopman operator $K_T h = h \circ T$ acting on $L^\infty(X, \Sigma, m)$.

A measurable, countable partition $\beta$ of $X$ is called regular iff for every $A \in \beta$, 
$T(A)$ is $\Sigma$-measurable and $T|_A$ maps $(A, \Sigma|_A)$ onto $(T(A), \Sigma|_{T(A)})$ isomorphically.
For any regular partition $\beta$ we define $g_T : X \to \mathbb{R}^+$ as follows:

\begin{equation}
(2.1) \quad g_T(x) = \frac{d(T_*(\chi_A m))}{dm}(T x), \text{ for } x \in A \in \beta.
\end{equation}

We can write $g_T = \sum_{A \in \beta} K_T(P_T \chi_A) \cdot \chi_A$. The function $g_T$ is determined up to a set of measure 0 and does not depend on the choice of partition $\beta$. For piecewise differentiable map $T$ the function $g_T$ is the reciprocal of the Jacobian. Using $g_T$ we can express $P_T$ as follows

\begin{equation}
(2.2) \quad P_T f(x) = \sum_{y \in T^{-1}(x)} g_T(y) \cdot f(y), \text{ } x \in X.
\end{equation}

Equality (2.2) holds $m$ almost everywhere.

\[
\begin{array}{ccc}
Y & \xrightarrow{S} & Y \\
\downarrow \pi & & \downarrow \pi \\
X & \xrightarrow{T} & X
\end{array}
\]

Now, we consider the case when $T$ is a factor of another mapping $S : Y \to Y$, where $(Y, \Sigma_Y, \nu)$ is a Lebesgue space. We assume that $S$ is nonsingular. By $\xi$ we denote the measurable partition of $Y$ which is $S$-invariant, i.e., $S^{-1}\xi \leq \xi$. Let $X = Y/\xi$ and let $T = S_\xi$ be the factor map. We assume that $m = \pi_* \nu$ or $m = \nu \circ \pi$. We denote the natural projection by $\pi : Y \to X$. Let $C(x)$ denote the element $\pi^{-1}(x) \in \xi$. We have $S(C(x)) \subset C(Tx)$. We will find the relation between $P_T$ and $P_S$.

\[
\begin{array}{ccc}
L^1(Y, \Sigma_Y, \nu) & \xrightarrow{P_S} & L^1(Y, \Sigma_Y, \nu) \\
\downarrow E_\nu(\cdot | \xi) & & \downarrow E_\nu(\cdot | \xi) \\
L^1(X, \Sigma, m) & \xrightarrow{P_T} & L^1(X, \Sigma, m)
\end{array}
\]

**Proposition 1.** (Rychlik [10], Proposition 1) Let $E_\nu(\cdot | \xi) : L^1(Y, \Sigma_Y, \nu) \to L^1(X, \Sigma, m)$ be the operator of conditional expectation with respect to the $\sigma$-algebra generated by the partition $\xi$, see [4]. For any $f \in L^1(Y, \Sigma_Y, \nu)$ we have

\[P_T(E_\nu(f|\xi)) = E_\nu(P_S f|\xi).\]

**Proof.** Let $h \in L^\infty(X, \Sigma, m)$. Then:

\[
\int_X h \cdot P_T(E_\nu(f|\xi)) dm = \int_X (h \circ T) E_\nu(f|\xi) dm
\]

\[
= \int_Y (h \circ T \circ \pi)(E_\nu(f|\xi) \circ \pi) d\nu = \int_Y (h \circ \pi \circ S) f d\nu
\]

\[
= \int_Y (h \circ \pi) \cdot P_S f d\nu = \int_Y (h \circ \pi)(E_\nu(P_S f|\xi) \circ \pi) d\nu
\]

\[
= \int_X h \cdot E_\nu(P_S f|\xi) dm.
\]

We used two properties of the conditional expectation:
(a) If \( g \in L^\infty(Y, \Sigma_Y, \nu) \) is \( \xi \)-measurable, then \( E_\nu(gf|\xi) = gE_\nu(f|\xi) \).
(b) \( \int_Y E_\nu(f|\xi)d\nu = \int_Y f d\nu \).

We assume that \( S \) has a regular partition \( \mathcal{P} \) with the property
\[
S^{-1} \xi \vee \mathcal{P} = \xi.
\]
We will consider \( \xi = \mathcal{P}^- = \bigvee_{k=0}^\infty S^{-k} \mathcal{P} \) so this property will holds automatically.

**Lemma 1.** (Rychlik [10], Lemma 1) The family \( \beta = \{\pi(A)\}_{A \in \mathcal{P}} \) is a \( T \)-regular partition of \( X \).

**Proof.** By assumption (2.4) \( \mathcal{P} \leq \xi \) and thus \( \pi^{-1}(\pi A) = A \) for every \( A \in \mathcal{P} \). Hence, \( \beta \) is a partition. Also, \( \pi^{-1}(T(\pi A)) = S(A) \). Then, \( T(\pi A) \) is measurable, by the definition of the factor space and regularity of \( \mathcal{P} \). Then, \( T|_{\pi(A)} : \pi(A) \to T(\pi A) \) is the factor \( S|_{A} : A \to S(A) \). Moreover, by (2.4) \( T|_{\pi(A)} \) is 1-1 (since \( S|_{A} \) is almost everywhere 1-1) and \( (T|_{\pi(A)})^{-1} \) is the factor of \( (S|_{A})^{-1} \). So, \( T|_{\pi(A)} \) is an isomorphism of \( (\pi(A), \Sigma_{|\pi(A)}) \) and \( (T(\pi A), \Sigma_{|T(\pi A)}) \). \hfill \( \square \)

Let \( \{\nu_C\}_{C \in \xi} \) be the family of conditional measures of \( \nu \) with respect to \( \xi \). In the following proposition we relate \( g_S \), \( g_T \) and \( \{\nu_C\}_{C \in \xi} \).

**Proposition 2.** (Rychlik [10], Proposition 2) For almost every \( x \in X \) and \( \nu_{C(x)} \)-almost every \( y \in C(x) \), we have
\[
(2.5) \quad g_T(x) = g_S(y) \frac{d((S|_{A})^{-1}\nu_{C(Tx)})}{d\nu_{C(x)}}(y),
\]
where \( A \) is the element of \( \mathcal{P} \) which contains \( C(x) \). Note, that \( C(x) = S^{-1}(C(Tx)) \cap A \). In particular, \( (S|_{A})^{-1}\nu_{C(Tx)} \) is equivalent to \( \nu_{C(x)} \) for almost every \( x \in X \).

**Proof.** Let \( h \in L^\infty(Y, \Sigma_Y, \nu) \). Then,
\[
(2.6) \quad E_\nu(P_S h|\xi)(x) = \int_{C(x)} \sum_{A \in \mathcal{P}} (h \cdot g_S) \circ (S|_{A})^{-1} d\nu_{C(x)} = \int h d\sigma_{1,x},
\]
where
\[
\sigma_{1,x} = \sum_{z \in T^{-1}(x)} g_S \circ (S|_{C(z)})^{-1}\nu_{C(z)}.
\]
The first inequality in (2.6) follows by the definition of \( P_S h \) and the fact that \( E_\nu(P_S h|\xi) \) is almost surely constant on elements of \( \xi \). The second, by the definition of \( g_S \). Also, we have
\[
(2.7) \quad P_T E_\nu(h|\xi)(x) = \sum_{z \in T^{-1}(x)} \int_{C(z)} h d\nu_{C(z)} g_T(z) = \int h d\sigma_{2,x},
\]
where
\[
\sigma_{2,x} = \sum_{z \in T^{-1}(x)} g_T(z) \nu_{C(z)}.
\]
In view of Proposition [1] since \( h \) is arbitrary the measures \( \sigma_{1,x} \) and \( \sigma_{2,x} \) are equal for almost every \( x \). Since the measures \( \nu_{C(x)} \) have disjoint supports and since functions \( g_T \) and \( g_S \) are positive almost everywhere the equality (2.5) is proved. \hfill \( \square \)
We will consider situation when the elements of $\xi$ are endowed with some natural measure. Let $\{\ell_C\}_{C \in \xi}$ be a family of such measures such that for any $C \in \xi$ the measure $\ell_C$ is equivalent to $\nu_C$ and the Radon-Nikodym derivative
\[
\rho = \frac{d\nu_C}{d\ell_C},
\]
defined on $Y$ is $\Sigma_Y$-measurable. Then, for almost every $x \in X$, $(S|_{C(x)})^{-1}\ell_{C(Tx)}$ is also equivalent to $\ell_{C(x)}$. Also, the function
\[
\lambda(y) = \frac{d((S|_{C(x)})^{-1}\ell_{C(Tx)})}{d\ell_{C(x)}}(y), \quad y \in C(x),
\]
is $\Sigma_Y$-measurable.

Let us now consider the situation when $S : [0, 1]^2 \to [0, 1]^2$ preserves two families of cones, the cone of stable directions and the cone of unstable directions. Let $\xi = \mathcal{P}^-$ be the $S$-invariant partition which consists of pieces of lines with stable directions. On each element $C \in \xi$ we have Lebesgue measure $\ell_C$. If $\nu$ is the Lebesgue measure on $[0, 1]^2$, then we proved in Proposition 7 that for almost all $C$, the conditional measure $\nu_C$ is equivalent to $\ell_C$.

By Proposition 2 we have
\[
g_T \circ \pi = g_S \frac{\rho \circ S}{\rho} \lambda.
\]
For $y, y'$ belonging to the same $C$ it gives
\[
\frac{\rho(y)}{\rho(y')} = \frac{g_S(y)}{g_S(y')} \frac{\rho(Sy)}{\rho(Sy')} \frac{\lambda(y)}{\lambda(y')},
\]
and by induction
\[
(2.8) \quad \frac{\rho(y)}{\rho(y')} = \left( \prod_{k=0}^{n-1} \frac{g_S(S^k y)}{g_S(S^k y')} \frac{\lambda(S^k y)}{\lambda(S^k y')} \right) \frac{\rho(S^n y)}{\rho(S^n y')}.
\]
Formula (2.8) proves the following:

**Proposition 3.** (Rychlik’s Proposition 3) For almost every $x \in X$ and for $\nu_{C(x)}$-almost every $y, y' \in C(x)$, the following conditions are equivalent:

(a) \[
\lim_{n \to \infty} \frac{\rho(S^n y)}{\rho(S^n y')} = 1;
\]

(b) \[
\frac{\rho(y)}{\rho(y')} = \prod_{k=0}^{\infty} \frac{g_S(S^k y)}{g_S(S^k y')} \frac{\lambda(S^k y)}{\lambda(S^k y')}.
\]

**Remark 1.** If (a) holds almost everywhere, then (b) and the condition $\int \rho d\ell_C = 1$ determine $\rho$ completely.

3. **Existence of Absolutely Continuous Invariant Measures.**

As before, let $T$ be a nonsingular map of a Lebesgue space $(X, \Sigma, m)$. Let $\beta$ be a regular partition of $X$ such that $\beta^- = \bigvee_{k=0}^{\infty} T^{-k}(\beta)$ is a partition into points, i.e., $\beta$ is a generator for $T$. We will give conditions which prove that $T$ admits an invariant measure absolutely continuous with respect to $m$. We introduce the following notations: $g = g_T$, $g_n = g_{T^n}$, $P = P_T$, for any $A \in \Sigma$, $\beta(A) = \{B \in \beta :$
By supremum and infimum we understand the essential supremum or minimum.

(I) **Distortion condition:**
\[ \exists d > 0 \forall n \geq 1 \forall B \in \beta^n \sup_B g_n \leq d \cdot \inf_B g_n; \]

(II) **Localization condition:**
\[ \exists \varepsilon > 0 \exists 0 < r < 1 \forall n \geq 1 \forall B \in \beta^n m(T^n B) < \varepsilon \implies \sum_{B' \in \beta(T^n B)} \sup_{B'} g \leq r; \]

(III) **Bounded variation condition:**
\[ \sum_{B \in \beta} \sup_B g < +\infty. \]

**Remark 2.** If conditions (I) and (III) hold for \( T, \beta \) and \( g \), then they also hold for \( T^N, \beta^N \) and \( g_N \). Condition (I) holds with the same value of \( d \). Moreover,
\[ \sum_{B \in \beta^N} \sup_B g_N \leq \left( \sum_{B \in \beta} \sup_B g \right)^N. \]

**Theorem 1.** (Rychlik [10], Theorem 1) Let (I)–(III) be satisfied. Then, the sequence \((P_n)_{n \geq 1}\) is bounded in \( L^\infty(X, \Sigma, m) \) and the averages \( \frac{1}{n} \sum_{k=0}^{n-1} P^k 1 \) converge in \( L^1(X, \Sigma, m) \) to some \( \phi \in L^\infty(X, \Sigma, m) \) such that \( P\phi = \phi \).

**Proof.** For the proof we refer to [10] or to [5]. \( \square \)

Theorem 1 gives the existence of an absolutely continuous invariant measure. To improve on this result we introduce the following condition:

(IV) **Expanding condition:**
\[ \exists r \in (0, 1) \sup_X g \leq r. \]

We can assume that \( r \) is chosen to satisfy both (II) and (IV).

**Theorem 2.** (Rychlik [10], Theorem 2) Let (I)–(IV) be satisfied. Then, there exists a bounded, finite dimensional projection \( Q : L^1(X, \Sigma, m) \to L^\infty(X, \Sigma, m) \) such that
\[ (1) \] \( Q(L^1(X, \Sigma, m)) \subset L^\infty(X, \Sigma, m) \) and \( Q \) is bounded as an operator from \( L^1(X, \Sigma, m) \) to \( L^\infty(X, \Sigma, m) \);

\[ (2) \] for every \( f \in L^1(X, \Sigma, m) \) the averages \( \frac{1}{n} \sum_{k=0}^{n-1} P^k f \) converge in \( L^1(X, \Sigma, m) \) to \( Qf \).

\[ (3) \] The range \( \mathcal{R}(Q) \) of \( Q \) consists of all eigenvectors of \( P \) corresponding to the eigenvalue 1 and of 0 function.

\[ (4) \] There exist non-negative functions \( \phi_1, \phi_2, \ldots, \phi_s \in \mathcal{R}(Q) \) which span \( \mathcal{R}(Q) \) and \( \phi_i \wedge \phi_j = 0 \) as \( i \neq j \). Moreover, \( \int_X \phi_i dm = 1, i = 1, \ldots, s \) and if
\[ C_i = \bigcup_{n=0}^{\infty} T^{-n} \{ x : \phi_i(x) > 0 \} \]
is the basin of attraction of the measure \( \phi_i m \), then \( Q \) can be represented as
\[ Qf = \sum_{i=1}^{s} \int_{C_i} f dm \cdot \phi_i. \]
Moreover, $\bigcup_{i=1}^{n} C_{i} = X$ up to a set of measure 0 and $\{\phi_{i}\}_{i=1}^{n}$ are the only functions $\phi \in L^{1}(X, \Sigma, m)$ such that $\phi \cdot m$ is a $T$-invariant, ergodic, probabilistic measure.

Proof. For the proof we refer to [10].

4. Preliminary Results for Maps $G_{\alpha}$ when $\alpha \in (\frac{3}{4}, 1)$.

Recall the map $G = G_{\alpha}$:

$$G(x, y) = [y, \tau(\alpha y + (1 - \alpha)x)], \ (x, y) \in [0, 1]^2,$$

where $\tau$ is the tent map $x \mapsto 1 - 2|x - 1/2|$. $L$ denotes the line $\alpha y + (1 - \alpha)x = 1/2$ which divides $[0, 1]^2$ into two domains on which $G$ is 1–1. A more explicit formula for $G$ is

$$G(x, y) = \begin{cases} 
(y, 2(\alpha y + (1 - \alpha)x), & \text{if } y \text{ is below } L; \\
(y, 2 - 2(\alpha y + (1 - \alpha)x), & \text{if } y \text{ is above } L.
\end{cases}$$

We have two possibilities for the Jacobian matrices:

$$A_{\pm} = DG = \begin{bmatrix} 0 & 1 \\ (1 - \alpha) \tau' (\alpha y + (1 - \alpha)x) & \alpha \tau' (\alpha y + (1 - \alpha)x) \end{bmatrix} = \begin{bmatrix} \pm 2(1 - \alpha) & \pm 2\alpha \\ \pm 2(u(1 - \alpha) \pm 2\alpha \\
\end{bmatrix},$$

with $+$ sign for $(x, y) \in A_{1}$, the region below line $L$ and $-$ sign for $(x, y) \in A_{2}$, the region above line $L$. Similarly, when we consider the inverse branches $G_{1}^{-1}$ and $G_{2}^{-1}$, we have two Jacobian matrices:

$$B_{\pm} = DG^{-1} = \begin{bmatrix} -\alpha & 1 - \alpha \pm 1 \\ 1 & 0 \end{bmatrix}.$$

We now construct invariant cones of directions in the tangent spaces as in [10].

For $A_{\pm}$, we consider the direction vector in the form $(u, 1)$. Then,

$$\begin{bmatrix} 0 & 1 \\ \pm 2(1 - \alpha) & \pm 2\alpha \end{bmatrix} \begin{bmatrix} u \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \pm 2u (1 - \alpha) \pm 2\alpha \end{bmatrix} = \begin{bmatrix} \frac{1}{\pm 2u (1 - \alpha) \pm 2\alpha} \end{bmatrix}.$$

Let

$$S_{\pm}(u) = \frac{1}{\pm 2u (1 - \alpha) \pm 2\alpha}, \quad (4.1)$$

be the corresponding transformation on directions.

For $B_{\pm}$, we consider the direction vector in the form $(1, v)$. Then,

$$\begin{bmatrix} -\alpha & 1 - \alpha \pm 1 \\ 1 - \alpha & 0 \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} = \begin{bmatrix} \frac{-\alpha}{1 - \alpha} & \frac{2(1 - \alpha) - v}{2(1 - \alpha)} \\ 1 & 0 \end{bmatrix}.$$

Let

$$T_{\pm}(v) = \frac{1}{\frac{-\alpha}{1 - \alpha} \pm \frac{2(1 - \alpha) - v}{2(1 - \alpha)}} = \frac{2(1 - \alpha)}{2\alpha \pm v}, \quad (4.2)$$

be the corresponding transformation on directions.
Lemma 2. (Rychlik [10], Lemma 3) Let \( \theta_0 = \alpha - \sqrt{\alpha^2 + 2\alpha - 2} \), \( J_+ = \{ u \in \mathbb{R} | |2u(1-\alpha)| \leq \theta_0 \} \), \( J_- = \{ v \in \mathbb{R} | |v| \leq \theta_0 \} \). Then, \( J_+ \) and \( J_- \) are \( S_\pm \) and \( T_\pm \)-invariant, respectively.

Proof. First, note that \( \theta_0 < \alpha \). We will prove the case of \( S_+ \). The case for \( S_- \) is similar. It follows from \( |2u(1-\alpha)| \leq \theta_0 \) that

\[
2\alpha - \theta_0 \leq 2\alpha + 2u(1-\alpha) \leq 2\alpha + \theta_0.
\]

Thus,

\[
|2(1-\alpha)S_+(u)| \leq \frac{2(1-\alpha)}{2\alpha - \theta_0} \leq \theta_0,
\]

where the last inequality follows from the definition of \( \theta_0 \).

Now we prove the case of \( T_+ \). The case of \( T_- \) is similar. It follows from \( |v| \leq \theta_0 \) that

\[
-3\alpha + \sqrt{\alpha^2 + 2\alpha - 2} \leq -2\alpha + v \leq -\alpha - \sqrt{\alpha^2 + 2\alpha - 2}.
\]

Thus,

\[
|T_+(v)| = \frac{|2(1-\alpha)|}{|2\alpha - v|} = \frac{2(1-\alpha)}{|2\alpha + v|} \leq \frac{2(1-\alpha)}{\alpha + \sqrt{\alpha^2 + 2\alpha - 2}} = \theta_0.
\]

\( \Box \)

Remark 3. We also have \( S_\pm(J_+) \subseteq \{ u \in \mathbb{R} | |2u(1-\alpha)| \geq \frac{2(1-\alpha)}{2\alpha + \theta_0} \} = \{ u \in \mathbb{R} | |u| \geq \theta_1 \} \), \( T_\pm(J_-) \subseteq \{ v \in \mathbb{R} | |v| \geq 2(1-\alpha)\theta_1 \} \), where \( \theta_1 = \frac{1}{2\alpha + \theta_0} \).

Lemma 3. (Rychlik [10], Lemma 4) Let \( \kappa = \frac{\alpha - \sqrt{\alpha^2 + 2\alpha - 2}}{\alpha + \sqrt{\alpha^2 + 2\alpha - 2}} \), which is less than 1 (actually it is less than 0.5 and decreasing with respect to \( \alpha \)). Then, \( \sup_{J_+} |S'_\pm(u)| = \sup_{J_-} |T'_\pm(v)| = \kappa \).
Proof. It follows from (4.3) that
\[ |S'_\pm(u)| = 2(1 - \alpha)S^2_\pm(u) \leq 2(1 - \alpha) \left( \frac{\theta_0}{2(1 - \alpha)} \right)^2 = \kappa. \]

And, it follows from (4.4) that
\[ |T'_\pm(v)| = \frac{1}{2(1 - \alpha)} T^2_\pm(v) \leq \frac{1}{2(1 - \alpha)} \theta^2 = \kappa. \]

Using Lemma 2 and Lemma 3, we see that for any sequence \((\varepsilon_0, \varepsilon_1, \ldots) \in \{+,-\}^\infty\), we have \(|(T_{\varepsilon_{n-1}} \circ T_{\varepsilon_{n-2}} \circ \cdots \circ T_{\varepsilon_1} \circ T_{\varepsilon_0})(\mathcal{J}_-)\) \leq \kappa^n|\mathcal{J}_-| so the set
\[ \bigcap_{n=1}^{\infty} (T_{\varepsilon_{n-1}} \circ T_{\varepsilon_{n-2}} \circ \cdots \circ T_{\varepsilon_1} \circ T_{\varepsilon_0})(\mathcal{J}_-), \]
consists of exactly one point which can be expressed as a continued fraction:
\[ \frac{2(1 - \alpha)}{-2\alpha + \frac{\varepsilon_0 2(1 - \alpha)}{-2\alpha + \frac{\varepsilon_1 2(1 - \alpha)}{-2\alpha + \cdots}}}. \]

Similarly, for any sequence \((\eta_0, \eta_1, \ldots) \in \{+,-\}^\infty\), we have \(|(S_{\eta_{n-1}} \circ S_{\eta_{n-2}} \circ \cdots \circ S_{\eta_1} \circ S_{\eta_0})(\mathcal{J}_-)\) \leq \kappa^n|\mathcal{J}_-| so the set
\[ \bigcap_{n=1}^{\infty} (S_{\eta_{n-1}} \circ S_{\eta_{n-2}} \circ \cdots \circ S_{\eta_1} \circ S_{\eta_0})(\mathcal{J}_-), \]
consists of exactly one point which can be expressed as a continued fraction:
\[ \frac{\eta_0}{2\alpha + 2(1 - \alpha) \frac{\eta_1}{2\alpha + 2(1 - \alpha) \frac{\eta_2}{2\alpha + 2(1 - \alpha) \cdots}}}. \]

They are both convergent since \(\kappa < 1\).

Now, using the above construction we define invariant directions for \(G\). For points \(p \in U^* = [0,1]^2 \setminus \bigcup_{n=0}^{\infty} G^{-n}(L)\), setting \(\varepsilon_i = +\) or \(\varepsilon_i = -\), depending on whether \(G^n(p)\) is below or above the line \(L\), we obtain the invariant stable direction \(v(p) \in \mathcal{J}_-\),
\[ v(p) = \frac{2(1 - \alpha)}{-2\alpha + \frac{\varepsilon_0 2(1 - \alpha)}{-2\alpha + \frac{\varepsilon_1 2(1 - \alpha)}{-2\alpha + \cdots}}}. \]

To construct an invariant unstable direction for a point \(p\) we have to use \(G\)-preimages of \(p\). Since \(G\) is not invertible the “invariant” direction will depend on the chosen admissible past of the point \(p\). Some points have only one admissible past, for example, for the fixed point \((2/3,2/3)\) the only admissible past is \((\ldots,2,2,\ldots,2,2)\) and it has the unique well defined unstable direction. Other points have finite number or infinitely many admissible pasts. The richest case happens when the directions in the set of “invariant” directions form a Cantor set, namely the attractor of the Iterated Function System \(\{S_+, S_-\}\). For a point \(p \in U^n = [0,1]^2 \setminus \bigcup_{n=0}^{\infty} G^n(L)\) with specified past \((\ldots, k_{n-1}, \ldots, k_1, k_0) \in \{1,2\}^\infty\) we
choose $\eta_i = +$ when $k_i = 1$ or $\eta_i = -$ when $k_i = 2$, and obtain the invariant stable direction $u(p) \in J_+$,

$$u(p) := \frac{\eta_0}{2\alpha + 2(1 - \alpha) + 2 \alpha + 2(1 - \alpha) \eta_0^2 + \cdots}.$$

We now compute $\lambda^s(p)$ and $\lambda^u(p)$, which represent the rates of change on the length along directions of $E^s$ and $E^u$, respectively. For directions in $E^u$ the rate is independent of the chosen past of the point.

Lemma 4. (Rychlik [10], Lemma 5)

$$\lambda^s(p) = |v(p)| \frac{h_1(p)}{h_1(G(p))}, \quad \lambda^u(p) = |u(p)| \frac{h_2(p)}{h_2(G^{-1}(p))},$$

where,

$$h_1(p) = \frac{1}{\sqrt{(v(p))^2 + 1}}, \quad h_2(p) = \frac{1}{\sqrt{(u(p))^2 + 1}}.$$
Similarly, \( \lambda^s(\theta) = \frac{\|v(p)(1, v(G(p)))\|}{\|\theta, v(p)\|} = |v(p)| \frac{h_1(p)}{h_1(G(p))} \).

Similarly, \( DG^{-1}(p) \begin{bmatrix} u(p) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-\alpha}{1 - \alpha} \pm \frac{1}{2(1 - \alpha)} \\ 1 \end{bmatrix} \begin{bmatrix} u(p) \\ 1 \end{bmatrix} \) = \( u(p) \begin{bmatrix} \frac{-\alpha}{1 - \alpha} u(p) \pm \frac{1}{2(1 - \alpha)} u(p) \\ 1 \end{bmatrix} \) = \( u(p) \begin{bmatrix} S^{-1}_\pm(u(p)) \\ 1 \end{bmatrix} = u(p) \begin{bmatrix} u(G^{-1}(p)) \\ 1 \end{bmatrix} \),

and thus, \( \lambda^u(p) = \frac{||u(p)(u(G^{-1}(p)), 1)||}{||(u(p), 1)||} = |u(p)| \frac{h_2(p)}{h_2(G^{-1}(p))} \).

We need the conditions that both \( \theta_0 \) and \( \frac{\theta_0}{2(1 - \alpha)} \) are less than 1, which hold since \( \alpha \in \left(\frac{1}{2}, 1\right) \).

Now we present a proposition analogous to Proposition 5 in Rychlik [10].

**Proposition 4.** Let \( \lambda_+ = \frac{\theta_0}{2(1 - \alpha)}, \lambda_- = \theta_0 \). Then both \( \lambda_+ \), \( \lambda_- \in (0, 1) \). And there exists a constant \( C > 0 \) such that \( |\lambda^u_\pm(p)| \leq C \lambda^u \) if \( p \in U^s \), \( |\lambda^u_\pm(p)| \leq C \lambda^u \) if \( p \in U^u \).

**Proof.** Using Lemma 2 and the invariant sets \( J_+ \) and \( J_- \), it follows that \( h_1 \) and \( h_2 \) are bounded, i.e. there exists numbers \( c_1 \) and \( c_2 \) such that \( 0 < c_1 \leq h_i \leq c_2, i = 1, 2 \). Thus, by Lemma 3

\[
\lambda^s_\pm(p) = \lambda^s(G^{n-1}(p)) \cdot \lambda^s(G^{n-2}(p)) \cdots \lambda^s(G(p)) \lambda^s(p)
\]

\[
= |v(G^{n-1}(p))| \frac{h_1(G^{n-1}(p))}{h_1(G^n(p))} \cdot |v(G^{n-2}(p))| \frac{h_1(G^{n-2}(p))}{h_1(G^{n-1}(p))} \cdots \\
\cdot |v(G(p))| \frac{h_1(G(p))}{h_1(G^2(p))} \cdot |v(p)| \frac{h_1(p)}{h_1(G(p))}
\]

\[
= |v(G^{n-1}(p))| \cdot |v(G^{n-2}(p))| \cdots |v(p)| \frac{h_1(p)}{h_1(G^n(p))}
\]

\[
\leq \frac{c_2}{c_1} \theta_0^n.
\]
Similarly, we have

\[ \lambda_n^u(p) \leq \frac{c_2}{c_1} \left( \frac{\theta_0}{2(1-\alpha)} \right)^n. \]

\[ \square \]

Let \( P = P^{(1)} \) be the partition of the square \([0,1]^2\) into the regions of definition of the map \( G \), i.e., \( A_1 = \{(x,y): \alpha y + (1-\alpha)x \leq 1/2\} \) and \( A_2 = \{(x,y): \alpha y + (1-\alpha)x \geq 1/2\} \). These regions intersect, but the intersection is a negligible set both in a measure-theoretic and topological sense. We define \( P^{(n)} = P \cup G^{-1}(P) \cup G^{-2}(P) \cup \cdots \cup G^{n-1}(P) \). \( P^{(n)} \) is the defining partition for the map \( G^n \).

Let \( L \) denote the partition line

\[ L = \{p = (x,y): \alpha y + (1-\alpha)x = 1/2\}. \]

**Lemma 5.** (Rychlik [10], Lemma 8) For every \( N \geq 1 \) there is an open cover \( U_N \) of the unit square such that every element of \( U_N \) intersects no more than \( 2N \) elements of \( P^{(N)} \).

The proof is exactly the same as in [10].

**Proposition 5.** (Rychlik [10], Proposition 7) There exist constants \( F > 0 \) and \( 0 < r < 1 \) such that for any segment \( I \) with the direction from the unstable cone \( J_+ \) we have

\[ \Gamma_n(I) = \sum_{J \in P^{(n)}|I} \frac{|J|}{|G^n(J)|} \leq F(r^n + |I|), \]

where \( | \cdot | \) denotes the length of the segment.

**Proof.** The proof follows closely the proof from [10]. We choose \( N \) in such a way that

\[ r_0 = 2NC(\lambda_+)^N < 1, \]

where \( C \) and \( \lambda_+ \) are from Proposition [1]. Let \( \varepsilon_0 \) be the Lebesgue constant of the cover \( U_N \) from Lemma [5]. Let us define

\[ \gamma_n = \sum_{J \in P^{(nN)}|I} \frac{|J|}{|G^{nN}(J)|}, \quad n = 1, 2, \ldots \]

We will show that

\[ \gamma_{n+1} \leq r_0 \gamma_n + \frac{1}{\varepsilon_0} R_0 |I|, \]

where

\[ R_0 = \sup_I \gamma_1 = \sup_I \sum_{J \in P^{(N)}|I}} \frac{|J|}{|G^N(J)|}, \]

and \( I \) is any segment with the direction from the unstable cone \( J_+ \).

Let \( J \in P^{(nN)}|I \). Either \( |J| < \varepsilon_0 \) or \( |J| \geq \varepsilon_0 \). In the first case \( P^{((n+1)N)}|J \) consists of not more that \( 2N \) elements (Lemma [5]) as the partition \( P^{((n+1)N)} \) is...
obtained from $\mathcal{P}^{(nN)}$ in $N$ steps. Thus,
\begin{align}
\sum_{J' \in \mathcal{P}^{(nN)}(J)} \frac{|J'|}{G^{(n+1)N}(J')} &\leq 2N \max_{J'} \frac{|J'|}{G^{(n+1)N}(J')} \\
&= 2N \max_{J'} \left( \frac{|J'|}{G^{nN}(J')} \frac{|G^{nN}(J')|}{G^{(n+1)N}(J')} \right) \\
&= 2N \max_{J'} \frac{|J|}{G^{nN}(J')} \frac{|G^{nN}(J')|}{G^{(n+1)N}(J')} \\
&\leq 2N \frac{|J|}{G^{nN}(J')} \left( C(\lambda_+)^N \right) \frac{|G^{nN}(J')|}{G^{(n+1)N}(J')}.
\end{align} 
(4.8)

We have used the fact that $J' \subset J$ and $G^{nN}$ is a linear transformation on $J$, so the expansion rate is uniform on $J$.

In the second case we have
\begin{align}
\sum_{J' \in \mathcal{P}^{(nN)}(J)} \frac{|J'|}{G^{(n+1)N}(J')} &= \sum_{J' \in \mathcal{P}^{(nN)}(J)} \left( \frac{|J'|}{G^{nN}(J')} \frac{|G^{nN}(J')|}{G^{(n+1)N}(J')} \right) \\
&= \frac{|J|}{G^{nN}(J')} \sum_{J' \in \mathcal{P}^{(nN)}(J)} \frac{|G^{nN}(J')|}{G^{(n+1)N}(J')} \\
&\leq \frac{|J|}{G^{nN}(J')} R_0 \leq R_0 \frac{1}{\varepsilon_0} |J|.
\end{align} 
(4.9)

We used again the linearity of $G^{nN}$ on $J$. Moreover
\begin{align}
\sum_{J' \in \mathcal{P}^{(nN)}(J)} \frac{|G^{nN}(J')|}{G^{(n+1)N}(J')} &\leq \sum_{K \in \mathcal{P}(J)} \frac{|K|}{|G^{\lambda}(K)|} \leq R_0,
\end{align}
since intervals $G^{nN}(J')$ are elements of $\mathcal{P}(J)$, $G^{nN}(J)$ and $G^{nN}(J)$ has the direction from $J_+$. Also $|G^{nN}(J)| > |J| > \varepsilon_0$.

Summing up (4.8) and (4.9) over all $J \in \mathcal{P}^{(nN)}(I)$, we obtain
\begin{align}
\sum_{J' \in \mathcal{P}^{(nN)}(J)} \frac{|J'|}{G^{(n+1)N}(J')} &\leq \sum_{J \in \mathcal{P}(J)} \left( \gamma_n + R_0 \frac{1}{\varepsilon_0} |J| \right) \\
&= r_0 \gamma_n + R_0 \frac{1}{\varepsilon_0} |I|,
\end{align}
(4.10)
and (4.7) is proved. To obtain inequality (4.6) from (4.7) we proceed as follows.
Since $\gamma_1 \leq R_0$ by definition, the inequality (4.7) implies
\begin{align}
\gamma_n \leq r_0^n R_0 + \frac{R_0}{\varepsilon_0(1 - r_0)} |I|, \quad n = 1, 2, \ldots,
\end{align}
(4.11)
or, using capital gamma notation
\begin{align}
\Gamma_n(I) \leq r_0^n R_0 + \frac{R_0}{\varepsilon_0(1 - r_0)} |I|, \quad n = 1, 2, \ldots.
\end{align}

Let us define
\begin{align}
\bar{R}_i = \sup_I \Gamma_i(I) = \sup_I \sum_{J \in \mathcal{P}(I)} \frac{|J|}{G^{\lambda}(J)} \quad i = 1, 2, \ldots, N,
\end{align}
To make these estimates independent of $\ell$

Now, let $\xi$ be a segment with direction from the stable cone or points. Let $P_r$ be the direction from the stable cone or points. Let $\xi = (P_r, R)$ be an element of $\xi$. Consider $\xi = (P_r, R)$, where $\xi$ is independent of $\ell$

$$\Gamma_n(I) \leq r_0^N R + \frac{R}{\varepsilon_0(1 - r_0)}|I|.$$ 

To make these estimates independent of $\ell$ we can write

$$\Gamma_n(I) \leq r_0^N R + \frac{R}{\varepsilon_0(1 - r_0)}|I|.$$ 

Now, let $r = (r_0)^{1/N}$ and $F = \max\{\frac{R}{\varepsilon_0(1 - r_0)}, \frac{R}{\varepsilon_0(1 - r_0)}\}$. We obtain inequality (4.10).

We define $\mathcal{P}^- = V_{n=0}^\infty G_n^{-n}(\mathcal{P})$. Elements of $\mathcal{P}^-$ are either segments with direction from the stable cone or points. Let $\xi(p) \in \mathcal{P}^-$ denote an element of $\mathcal{P}^-$ containing point $p$.

**Lemma 6.** (corresponds to Lemma 9 of [10]) Let

$$D^*(\delta) = \{p \in [0,1]^2 : \text{dist}(G_n^p, L) \geq \delta \lambda_n^*(p), \text{for } n = 0, 1, 2, \ldots\}.$$ 

For every $p \in D^*(\delta)$ the distance from $p$ to the endpoints of $\xi(p)$ is not smaller than $\delta$. In particular, $|\xi(p)| \geq 2\delta$.

**Proof.** Assume that the distance from $p$ to one of the endpoints of $\xi(p)$ called $q$ is $\text{dist}(p, q) < \delta$. Since endpoints of elements $\xi$ belong to preimages $G_n^{-n}(L)$, there is an integer $k \geq 0$ such that $q \in G_n^{-k}(L)$. Then,

$$\text{dist}(G_n^k p, L) \leq \text{dist}(G_n^k p, G_n^k q) \leq \lambda_n^*(q) \text{dist}(p, q) < \delta \lambda_n^*(p),$$

which contradicts $p \in D^*(\delta)$.

**Lemma 7.** (corresponds to Lemma 10 of [10]) Let $(\lambda_n) = (\lambda_n)^\infty_{n=0}$ be a sequence of positive numbers such that $Z = \sum_{n=0}^{\infty} \lambda_n < +\infty$. Let

$$D^*(\delta, (\lambda_n)) = \{p \in [0,1]^2 : \text{dist}(G_n^p, L) \geq \delta \lambda_n, \text{for } n = 0, 1, 2, \ldots\}.$$ 

Let $I$ be a segment with direction from unstable cone. Then, there is a constant $A_1$ such that $|I \setminus D^*(\delta, (\lambda_n))| \leq A_1 \cdot Z \cdot \delta$.

**Proof.** We follow closely Rychlik [10]. Let

$$C(t) = \{q : \text{dist}(q, L) \leq t\},$$

where $t \geq 0$. Let $p \in I \setminus D^*(\delta, (\lambda_n))$. There exists $n \geq 0$ such that $\text{dist}(G_n^p, L) < \delta \lambda_n$. Let $J \in \mathcal{P}^n|I|$ be the subinterval containing point $p$. Then, $G_n^p$ belongs to the interval $G_n^J$ such that

$$|G_n^J \cap C(\delta \lambda_n)| \leq A_0 \cdot \delta \lambda_n,$$

for some constant $A_0$ independent of $\delta$ and $n$. As $G_n^J$ has a direction from the expanding cone and thus, the angle between $G_n^J$ and line $L$ is bounded away from 0. Thus, $p \in J \cap G_n^{-n}(C(\delta \lambda_n))$ and

$$|J \cap G_n^{-n}(C(\delta \lambda_n))| \leq \frac{A_0 \cdot \delta \lambda_n}{|G_n^J|} \cdot |J|.$$ 

By Proposition 5 this gives

$$|I \cap G_n^{-n}(C(\delta \lambda_n))| \leq A_0 \cdot \delta \lambda_n \cdot F(r^n + |I|) \leq A_0 F(1 + \text{diam}([0,1]^2)) \delta \lambda_n.$$
Summing up over all \( n \), we obtain
\[
|I \setminus D^*(\delta, (\lambda_n))| \leq A_0 F(1 + \sqrt{2})\delta Z.
\]
The Lemma is proved with \( A_1 = A_0 F(1 + \sqrt{2}) \).

**Corollary 1.** For any interval \( I \) with the direction from the expanding cone we have
\[
|I \setminus D^*(\delta)| \leq A_2 \cdot \delta,
\]
where \( A_2 = A_1 \sum_{n=0}^{\infty} C\lambda^- = A_1 C/(1 - \lambda^-) \).

**Proof.** Let \( \lambda_n = C\lambda^-_n \), \( n = 0, 1, 2, \ldots \). Since \( \lambda_n^* \leq C\lambda^-_n \) we have \( D^*(\delta) \supset D^*(\delta, (\lambda_n)) \). This proves the claim. \( \square \)

Let \( \nu \) denote the normalized Lebesgue measure on \([0, 1]^2\).

**Corollary 2.** The set \( \tilde{D}^* = \bigcup_{\delta > 0} D^*(\delta) \) is of full \( \nu \)-measure in \([0, 1]^2\). Moreover
\[
\nu([0, 1]^2 \setminus D^*(\delta)) \leq A_2 \cdot \delta.
\]

**Proof.** Follows by Corollary 1 and Fubini’s Theorem. \( \square \)

Let us consider the function \( 1/D(p) \) where \( D(p) = |\xi(p)| \). We will prove that it is integrable.
Proposition 6. (corresponds to Proposition 8 of [10]) There is a constant \( A_3 > 0 \) such that for an arbitrary \( \delta > 0 \),
\[
\nu(\{ p \in [0,1]^2 : D(p) < \delta \}) \leq A_3 \delta^2.
\]

Proof. If \( p \in \{ p \in [0,1]^2 : D(p) < \delta \} \), then \( \text{dist}(G^n p, L) < \delta \lambda_n^*(p) \) at least for two \( n_1 < n_2 \) since both ends of \( \xi(p) \) have to be trimmed (Lemma 6). This means that \( p \) is less that \( \delta \) close to preimage \( G^{-n_1}(L) \) and \( \text{dist}(G^{n_1} p, L) < \delta \lambda_{n_1}^*(p) = \eta \). Then, \( G^{n_1+2} p \in \Pi_{A_{4}\eta} = \{ (x,y) : 1-A_4\eta \leq x \leq 1, 0 \leq y \leq 1 \} \), for some constant \( A_4 > 0 \) independent of \( \delta \) and \( n_1 \). See Figure 3. Also, \( G^{n_1+2} p \in [0,1]^2 \setminus D^s(\eta) \) since \( \text{dist}(G^{n_2} p, L) < \delta \lambda_{n_2}^*(p) < \eta \), and \( G^{n_1+1} p \) is far from the line \( L \). Let \( n = n_1 + 2 \). We have \( G^n p \in \Pi_{A_{4}\eta} \setminus D^s(\eta) \). Since the vertical direction is in the expanding cone, by Corollary 1 and Fubini's Theorem, we have
\[
\nu(\Pi_{A_{4}\eta} \setminus D^s(\eta)) \leq A_2 \cdot A_4 \cdot \eta^2.
\]

Thus,
\[
\nu(\{ p \in [0,1]^2 : D(p) < \delta \}) \leq \sum_{n=0}^{\infty} \nu(G^{-n}(\Pi_{A_{4}\eta} \setminus D^s(\eta)) \leq \sum_{n=0}^{\infty} A_2 \cdot A_4 \cdot \eta^2 \cdot (2 \cdot \text{Jac}^{-1}(\alpha))^n,
\]
where \( \text{Jac}(\alpha) = 2(1-\alpha) \) is the Jacobian of both \( G_1 \) and \( G_2 \). We need the multiplier 2 because \( G \) is a 2 to 1 map (This is different from the Lozi map studied in [10].) By Lemma 4 and Proposition 4 we have
\[
\lambda_+^*(p) \leq C \lambda^a,
\]
where
\[
\lambda_- = \alpha - \sqrt{\alpha^2 + 2\alpha - 2}.
\]

We have
\[
\nu(\{ p \in [0,1]^2 : D(p) < \delta \}) \leq A_2 \cdot A_4 \cdot C^2 \cdot \delta^2 \cdot \sum_{n=0}^{\infty} \left( \frac{2(\alpha - \sqrt{\alpha^2 + 2\alpha - 2})^2}{2(1-\alpha)} \right)^n.
\]

It can be easily proved that for \( 3/4 < \alpha < 1 \) we have \( \frac{(\alpha - \sqrt{\alpha^2 + 2\alpha - 2})^2}{2(1-\alpha)} < 1 \). Thus, the series converges to some constant \( A(\alpha) \), and setting \( A_3 = A_2 \cdot A_4 \cdot C^2 \cdot A(\alpha) \) completes the proof of the proposition. \( \square \)

---

**Figure 4.** Partitions \( P^{(2)} \) and \( P^{(6)} \) for \( \alpha = 0.82 \).
Corollary 3. (corresponds to Corollary 3 of [10]) The function \( p \mapsto 1/D^\beta(p) \) is integrable on \([0, 1]^2\) for any \( \beta \in [1, 2) \).

Proof. We will use the following identity for positive random variables

\[
E(X) = \int_0^\infty P(X > t) \, dt
\]

which can be found, e.g., in [4], page 275. We have

\[
\int (1/D^\beta) \, d\nu = \int_0^\infty \nu(D^{-\beta} > \gamma) \, d\gamma \\
\leq 1 + \int_1^\infty \nu(D < \gamma^{-1/\beta}) \, d\gamma \\
\leq 1 + \int_1^\infty A_3 \gamma^{-2/\beta} \, d\gamma < +\infty
\]

□

In the following proposition we will discuss the family of conditional measures \( \{\nu_C\}_{C \in \mathcal{P}^-} \) of measure \( \nu \) on elements of the partition \( \mathcal{P}^- \). The theory of conditional measures can be reviewed by referring to [12] or [9]. Let \( \{\ell_C\}_{C \in \mathcal{P}^-} \) be the family of one-dimensional Lebesgue measures on the elements of \( \mathcal{P}^- \).

Proposition 7. (corresponds to Proposition 9 of [10]) For almost every \( C \in \mathcal{P}^- \), measure \( \nu_C \) is absolutely continuous with respect to \( \ell_C \) and the Radon-Nikodym derivative \( d\nu_C/d\ell_C \) is constant on \( C \), equal to \( 1/|C| \).

![Figure 5](image_url). A polygon \( A_n \) of the partition \( \mathcal{P}^n \) and the density \( \rho_n \).

Proof. We follow closely [10]. Let \( A_n \) be the polygon of the partition \( \mathcal{P}^n \), \( n \geq 1 \) containing \( C \in \mathcal{P}^- \). See Figure 5. Since \( A_n \) is convex, the projection of the measure \( \frac{1}{\nu(A_n)} \nu|A_n \) onto the \( x \)-axis is a measure absolutely continuous with respect to Lebesgue measure with density \( \rho_n \) which is positive on some interval \((a_n, b_n)\) and zero outside of this interval. Since \( \rho_n(t) \) is proportional to the length of the intersection of the vertical line \( x = t \) with the polygon \( A_n \), the density \( \rho_n \) is concave on \((a_n, b_n)\). We have \((a_{n+1}, b_{n+1}) \subset (a_n, b_n)\) and \( a_n \to a, b_n \to b \) where \( a \) and \( b \) are the end points of the projection of \( C \) onto the \( x \)-axis. Since \( \rho_n(a_n) = \rho_n(b_n) = 0, \int_{a_n}^{b_n} \rho_n = 1 \), and \( \rho_n \) are concave the family \( \{\rho_n\}_{n \geq 1} \) is uniformly bounded by \( 2/(b - a) \). Since they are concave their variations are also uniformly bounded by \( 4/(b - a) \). By Helly’s theorem ([3]), there exists a subsequence \( \rho_{n_k} \) convergent to some density \( \rho \) almost everywhere. \( \rho \) is concave as a limit of concave functions. Projecting \( \rho \) onto \( C \) we obtain \( \frac{d\nu_C}{d\ell_C} \) which is also concave. We will denote it again by \( \rho \).
Figure 6. Using the concavity of $\rho_{|G^nX|}$.

To use Proposition 3 we will prove that for almost every $x \in [0,1]^2$ and almost every $y, y' \in C(x) = C$, we have

$$\lim_{n \to \infty} \frac{\rho(G^n y)}{\rho(G^n y')} = 1.$$  

By Lemma 7 we assume that $\text{dist}(G^n C, L) > \delta \lambda^n$, where $\delta > 0$ and $\lambda \in (\lambda_-, 1)$. Now, we use the concavity of $\rho_{|G^nX|}$. See Figure 6. From the triangles in Figure 6 we have $w/v = 1 + u/z$. Thus,

$$\frac{\rho(G^n y)}{\rho(G^n y')} \leq \frac{w}{v} = 1 + \frac{u}{z} \leq 1 + \frac{\text{dist}(G^n y, G^n y')}{\text{dist}(G^n C, \partial(\xi(G^n x)))} \leq 1 + \frac{C \lambda^n}{\delta \lambda^n},$$

which goes to 1 as $n \to \infty$. Thus, $\limsup_{n \to \infty} \frac{\rho(G^n y)}{\rho(G^n y')} \leq 1$. By symmetry, we obtain (4.16). By Proposition 3 we have

$$\frac{\rho(y)}{\rho(y')} = \prod_{k=0}^{\infty} \frac{g_C(G^k y)}{g_C(G^k y')} \frac{\lambda(G^k y)}{\lambda(G^k y')}.$$  

Since the Jacobian of $G$ is constant and $G$ is piecewise linear and in particular linear on every $C \in \xi$, the right hand side of the above formula is constant.

5. Applying the Abstract Theorems to Transformations $G_\alpha$.

We will use the notation introduced in Section 2. Let $Y = [0,1]^2$, $S = G$ and $\nu$ be Lebesgue measure on $[0,1]^2$. The map $T$ is the factor map induced by $G$ on the space $X = [0,1]^2/P^\perp$.

By formula (2.5) we have

$$g_T(x) = g_C(y) \frac{d((G|A)x)^{-1} \nu_C(xz)}{d\nu_C(x)}(y),$$

for almost every $x$ and almost every $y$, where $A$ is an element of the partition $P$. 

Lemma 8. We can rewrite $g_T$ as follows:

\begin{equation}
5.1 \quad g_T = \frac{1}{\text{Jac}_G} \lambda^* \frac{D}{D \circ T},
\end{equation}

where $\lambda^*$ is defined in Lemma 4 and $D(x) = |C(x)|$.

Proof. We can write

\[
\frac{d((G|A)^{-1}\nu_{C(Tx)})}{d\nu_{C(x)}} = \frac{d((G|A)^{-1}\nu_{C(Tx)})}{d((G|A)^{-1}\nu_{C(Tx)})} \frac{d((G|A)^{-1}\nu_{C(Tx)})}{d\nu_{C(x)}}
\]

In view of Proposition 7 this gives the required formula for $g_T$.

Since $g_T$ given by formula (5.1) is very discontinuous we will replace it by considering instead of Lebesgue measure on $[0,1]^2$ an equivalent measure $\nu = \frac{1}{D}\bar{\nu}$, where $\bar{\nu}$ is the Lebesgue measure. Then, we define $m = \frac{1}{D}\bar{m}$, where $\bar{m}$ is the factor of the Lebesgue measure on $X = [0,1]^2/\mathcal{P}$.

Proposition 8. (Rychlik [10], Proposition 10) If we apply the results of Section 2 to the measure $\nu = \frac{1}{D}\bar{\nu}$, then

\begin{equation}
5.2 \quad g_T = \frac{1}{\text{Jac}_G} \lambda^*.
\end{equation}

Proof. Let $A \in \mathcal{P}^-$. By the definition of $g_T$ (2.1), we have

\[
\frac{d(T_*(\chi_A \cdot \bar{m}))}{dm} \frac{D}{D \circ T} = \frac{d(T_*(\chi_A \cdot \bar{m}))}{d(\frac{1}{D}\bar{m})} \frac{D}{D \circ T}
\]

(5.3)

\[
= \frac{1}{D} \circ (T_A)^{-1} \cdot dT_*(\chi_A \bar{m}) \frac{D}{D \circ T}
\]

\[
= \frac{1}{D} \circ T \cdot \frac{1}{\text{Jac}_G} \lambda^* \frac{D}{D \circ T} = \frac{1}{\text{Jac}_G} \lambda^*.
\]

We will now verify assumptions (I)-(IV).

Lemma 9. (Rychlik [10], Lemma 12) Condition (I) is satisfied.

Proof. Let $n \geq 1$ and $x_1, x_2 \in B \in \beta^{(n)}$. We can treat $x_1, x_2$ as elements of the Lebesgue space $X$ and also as points in $[0,1]^2$ or elements of $\mathcal{P}^-$. The points $G^kx_1$ and $G^kx_2$ are on the same side of the partition line $L$ for $k = 1, 2, \ldots, n - 1$. Since $\text{Jac}_G$ is constant, we need only to find a universal constant $d$ such that

\[
\frac{1}{d} \leq \frac{\lambda^*(x_1)}{\lambda^*(x_2)} \leq d.
\]

By Lemma 4 we have $\lambda^*(p) = |v(p)| \frac{h_1(p)}{h_1(G(p))}$, so

\[
\lambda^*_n(p) = |v(p)| \cdot |v(Gp)| \cdot \ldots \cdot |v(G^{n-1}p)| \cdot \frac{h_1(p)}{h_1(G^n(p))}.
\]

By Lemma 3 we have $|v(Gp) - v(Gp')| \leq \kappa |v(p) - v(p')|$, where $0 < \kappa < 1$. Thus, $|v(G^kx_1) - v(G^kx_2)| \leq \kappa^{n-k}|J_n|$, for $k = 1, 2, \ldots, n - 1$. Thus, there exists a constant $d_0$ such that

\[
\exp(-d_0\kappa^{n-k}) \leq \frac{\lambda^*(G^kx_1)}{\lambda^*(G^kx_2)} \leq \exp(d_0\kappa^{n-k}).
\]
Then, for some constant \( d_1 \) (we have to include the fractions \( h_1(x_1)/h_1(G^n(x_1)) \) and \( h_1(x_2)/h_1(G^n(x_2)) \)), we obtain

\[
\exp \left( -d_1 \sum_{k=0}^{n-1} \kappa^{n-k} \right) \leq \frac{\lambda_n^*(x_1)}{\lambda_n^*(x_2)} \leq \exp \left( d_1 \sum_{k=0}^{n-1} \kappa^{n-k} \right).
\]

Letting \( d = \exp(d_1/(1 - \kappa)) \), completes the proof. \(\square\)

**Lemma 10.** (Rychlik [10], Lemma 15) Conditions (II) and (IV) are satisfied for some iteration \( T^N, N \geq 1 \).

**Proof.** Condition (IV) is satisfied because \( g_n = (\text{Jac}_G)^{-n} \cdot \lambda_n^* \leq (\text{Jac}_G)^{-n} \cdot (C \theta_0^n) = C(\theta_0/\text{Jac}_G)^n \). For large \( n \), \( g_n < 1 \), since \( \theta_0/\text{Jac}_G < 1 \). We used Proposition 4 and \( \theta_0 = \alpha - \sqrt{\alpha^2 + 2\alpha - 2}, \text{Jac}_G = 2(1 - \alpha) \). For \( 3/4 < \alpha < 1 \), we have \( \theta_0/\text{Jac}_G < 1 \).

Now, we will prove that condition (II) is satisfied for some iterate of \( T \). The proof is similar to that of Proposition 4. Let \( N \) be such that \( r_0 = 2NC(\lambda_+)^N < 1 \) and let \( \varepsilon_0 \) be the Lebesgue constant of the cover \( U_N \) of Lemma 4. By Proposition 3, \( \lambda_+ = \theta_0/\text{Jac}_G \) so \( g_n \leq C \cdot \lambda_+ \) for \( n \geq 1 \).

Let \( B \in \beta^N \) and \( A = \pi^{-1}(B) \). Then, \( T^N(B) = \pi(G^NA) \). \( G^N A \) is a convex polygon such that

\[
\partial(G^N(A)) \subset \cup_{k=0}^{N} G^k(L).
\]

Except for \( L \) itself and the first image \( G(L) \subset \{ y = 1 \} \), all subsequent images \( G^k(L), k \geq 2 \), consist of segments with directions from the unstable cone \( J_+ \). Also, all images of the sides of \( [0, 1]^2 \) have this property. We can assume \( \varepsilon_0 \) is much smaller than the distance between \( L \) and \( G(L) \). There are two possibilities:

1. \( \text{diam}(G^N(A)) \geq \varepsilon_0 \). Then, \( G^N A \) contains a segment \( I \) with the unstable direction (from \( J_+ \)) of length \( A_5 \cdot \varepsilon_0 \), for some \( A_5 \leq 1 \). If all sides of \( G^N A \) belong to \( \cup_{k=2}^{N} G^k(L) \), i.e., they have directions from \( J_+ \), then obviously \( G^N A \) contains a segment \( I \) with the direction from \( J_+ \) of length \( \varepsilon_0 \). If one of the sides belongs to \( L \) and another to \( G(L) \), then \( G^N A \) also contains such segment since \( \varepsilon_0 \) is small. If only one side of \( G^N A \) belongs to \( L \) or \( G(L) \), then the worst case scenario is a triangle with two remaining sides with directions from \( J_+ \). Since the angle between directions from \( J_+ \) and \( L \) or \( G(L) \) is separated from zero, \( G^N A \) contains a segment \( I \) with the direction from \( J_+ \) of length \( A_5 \cdot \varepsilon_0 \), for some \( A_5 \leq 1 \).

By Corollary 11 for arbitrary \( \delta > 0 \), \( |I \cap D^+(\delta)| \leq A_2 \delta \). The set \( A = \pi^{-1} \nu(G^N A) = \pi^{-1}(T^N B) \) has measure larger than \( A_2^{-1} (1 - A_2 \delta) \cdot A_5 \cdot \varepsilon_0 \), where \( A_4 \) will be found in the following Lemma 14. So, we put \( \delta = \frac{1}{4} A_2^{-1} \) and \( \varepsilon = A_4^{-1} (1 - A_2 \delta) \cdot A_5 \cdot \varepsilon_0 = \frac{1}{2} A_4^{-1} . A_5 \cdot \varepsilon_0 \). Then, \( \nu(A) > \varepsilon \) and \( m(T^N B) = \nu(A) > \varepsilon \), since \( m = \pi_* \nu \).

2. \( \text{diam}(G^N(A)) < \varepsilon_0 \). Then, \( T^N(B) \) is contained in no more than \( 2N \) elements of \( \beta^N \) and

\[
\sum_{B' \in \beta^N(T^N B)} \sup_{B' \in B'} g_n \leq (2N)(C \lambda_+^N) = r_0.
\]

Thus, condition (II) is satisfied for \( T^N \) with \( r = r_0 \). \(\square\)

**Lemma 11.** (Rychlik [10], Lemma 14) Let \( I \) be a segment with the direction from \( J_+ \) and let \( \ell_I \) be the Lebesgue measure on \( I \). Then, the measure \( \pi_*(\ell_I) \) is absolutely continuous with respect to \( m \) and

\[
\frac{d\pi_*(\ell_I)}{dm}(x) = \frac{1}{\sin \angle(I, x)}.
\]
for $x \in X$, where $\angle(I, x)$ is the angle between segment $I$ and segment $C(x) \in P^-$. In particular, for some $A_4 > 0$ we have

\begin{equation}
\frac{1}{A_4} \leq \frac{d\pi_i(\ell_I)}{dm} \leq A_4.
\end{equation}

**Figure 7.** Strip $I_\delta$ for the proof of Lemma 11.

**Proof.** Fix some small $\delta > 0$. Let $I_\delta$ be a strip of width $\delta$ ($\delta/2$ on each side of $I$). We note that if $x \in D^i(\delta) \cap I$ and $\text{dist}(x, \partial I) > \delta$, then $\xi(x) \cap I_\delta$ is an interval of length $\delta/\sin \omega(x)$, where $\omega(x) = \angle(I, x)$. See Figure 7. So

$$
\nu_x(I_\delta) = \frac{\delta}{\sin \omega(x) \cdot D(x)}.
$$

Let $E$ be a subinterval of $I$. If $\tilde{\nu}$ is the Lebesgue measure on $[0, 1]^2$, we have

$$
\tilde{\nu}(\pi^{-1}(\pi E) \cap I_\delta) = \int_{\pi(E)} \nu_x(x \cap I_\delta) dm = \delta \int_{\pi(E)} \frac{1}{\sin \omega(x)} dm.
$$

On the other hand, by Corollary 1

$$
\tilde{\nu}(\pi^{-1}(\pi E) \cap I_\delta) = \ell_I(E) \cdot \delta + o(\delta).
$$
This proves (5.4). Since the angles between directions from $J_{-}$ and $J_{+}$ are separated from zero the inequality (5.5) is also proved.

**Remark 4.** Condition (III) holds since $\beta$ is finite.

Thus, we checked the assumptions of Theorems 1 and 2. Hence, we have

**Theorem 3.** The results of Theorems 1 and 2 apply to $G_{\alpha}$ maps for $\frac{3}{4} < \alpha < 1$.

6. Invariant measures for maps $G$.

We proved the existence of the invariant measures of the form $\phi \cdot m$ for the factor map $T$. Now, we will construct a $G$-invariant measure $\mu$ such that the projection $\pi_{*}(\mu)$ onto $X$ coincides with $\phi \cdot m$.

Let $f : [0, 1]^2 \to \mathbb{R}$ be a continuous function. We will define $\mu(f)$. Let

$$f^{<}(p) = \inf_{\xi(p)} f, \quad p \in [0, 1]^2,$$

and

$$f^{>}(p) = \sup_{\xi(p)} f, \quad p \in [0, 1]^2.$$

Both, $f^{<}$ and $f^{>}$ are $\Xi$-measurable ($\Xi$ is the $\sigma$-algebra generated by the partition $\xi = \mathcal{P}^{-}$). We define

$$\mu(f) = \lim_{n \to \infty} \tilde{\mu}((f \circ G^{n})^{<}),$$

where $\tilde{\mu} = \phi \cdot m$.

---

**Lemma 12.** The limits $\lim_{n \to \infty} \tilde{\mu}((f \circ G^{n})^{<})$ and $\lim_{n \to \infty} \tilde{\mu}((f \circ G^{n})^{>})$ exist and are equal.

---

**Figure 8.** Definition of the function $f^{<}_{n|j(n)}$. 

**Equation 6.**

$$\xi_{j(n)}(p) = G^{n}((\xi(G^{-j(n)}(p))))$$

$$G^{-j(n)}(p) \xleftarrow{} \xi(G^{-j(n)}(p))$$

---

$\xi(p)$

$\xi_{j(n)}(p) = G^{n}((\xi(G^{-j(n)}(p))))$

$G^{-j(n)}(p)$
Proof: This proof follows the proof of Lemma 15 in Rychlik [10], but we have to deal with the fact that $G$ is not invertible. This causes a need for more complicated notation. The map $G$ has two invertible “branches” $G_1 = G|_{A_1}$ and $G_2 = G|_{A_2}$. Corresponding inverses are $G_1^{-1}$ and $G_2^{-1}$. Let \( j(n) = (i_1, i_2, \ldots, i_n) \in \{1, 2\}^n \). Then, $G^{j(n)} = G_{i_1} \circ \cdots \circ G_{i_2} \circ G_{i_1}$ and $G^{-j(n)} = G_{i_1}^{-1} \circ G_{i_2}^{-1} \circ \cdots \circ G_{i_1}^{-1}$. Let us also introduce the notation \((j(n), n_{n+1}) = (i_1, i_2, \ldots, i_n, i_{n+1})\), for $i_{n+1} \in \{1, 2\}$.

We define $f_{n,j(n)}^< (p) = (f \circ G^n)^< \circ G^{-j(n)}(p)$, where $j(n)$ is such that $p \in G^{j(n)}([0, 1]^2)$. Note that

$$f_{n,j(n)}^< (p) = \inf_{q \in \xi^j(n)(p)} f(G^n(q)) = \inf_{s \in G^n(\xi^j(n)(p))} f(s)$$

is constant on $\xi(G^{-j(n)}(p))$, see Figure 8. Thus, $f_{n,j(n)}^<$ is $\Xi$-measurable. Also, $f_{n,j(n)}^< \leq f_{n+1, j(n)-1}^<$ since $G$ contracts segments $\xi \in \mathcal{P}^-$. We define

$$f_n^< = \min_{j(n)} f_{n,j(n)}^<.$$

Now, $f_n^<$ is $\Xi$ measurable and $f_n^< \leq f_{n+1}^<$. Similarly, we define $f_n^> = (f \circ G^n)^> \circ G^{-j(n)}$ and

$$f_n^> = \max_{j(n)} f_{n,j(n)}^>.$$

The functions $f_n^>$ are $\Xi$ measurable and $f_n^> \geq f_{n+1}^>$. We have $f \geq f_n^>$ for all $n \geq 1$ and $f_1^< \leq f_2^< \leq \cdots \leq f_n^< \leq \cdots$. Similarly, $f \leq f_n^>$ for all $n \geq 1$ and $f_1^> \geq f_2^> \geq \cdots \geq f_n^> \geq \cdots$. Also, if $\xi_n = G^n \xi$ is a partition of $G^n([0, 1]^2)$, then

$$f_n^>(p) - f_n^<(p) = \sup_{\xi_n(p)} f - \inf_{\xi_n(p)} f \leq \omega_\delta_n(f),$$

where $\delta_n = \sup_p \text{diam}(\xi_n(p))$ and

$$\omega_\delta(f) = \sup_{\text{dist}(x, y) < \delta} |f(x) - f(y)|,$$

is the modulus of continuity of $f$. Thus, $f_n^> - f_n^< \to 0$ as $n \to \infty$ and, consequently, $f_n^> \nearrow f$ and $f_n^< \searrow f$ uniformly as $n \to \infty$.

We have

$$f_n^< \circ G^n \leq (f \circ G^n)^< \leq (f \circ G^n)^> \leq f_n^> \circ G^n,$$

so $(f \circ G^n)^< \geq f_n^< \circ G^n$. Similarly, $(f \circ G^n)^> \leq f_n^> \circ G^n$. We have

$$|\tilde{\mu}(f \circ G^n)^> - \tilde{\mu}(f \circ G^n)^<| \leq \sup |(f \circ G^n)^> - (f \circ G^n)^<| \leq \sup |f_n^> - f_n^<| \leq \omega_\delta_n(f),$$

which goes to 0 as $n \to \infty$. Thus, both limits are the same if they exist. To show existence we write

$$f_n^< \circ G^n \leq (f \circ G^n)^< \leq (f \circ G^n)^> \leq f_n^> \circ G^n,$$

which implies

$$\tilde{\mu}(f_n^< \circ G^n) \leq \tilde{\mu}(f \circ G^n)^< \leq \tilde{\mu}(f \circ G^n)^> \leq \tilde{\mu}(f_n^> \circ G^n).$$
By the $T$-invariance of $\tilde{\mu}$ we have
$$\tilde{\mu}(f_n^\leq G^n) = \tilde{\mu}(f_n^\leq T^n) = \tilde{\mu}(f_n^\geq),$$
and similarly $\tilde{\mu}(f_n^\leq G^n) = \tilde{\mu}(f_n^\geq)$. Since both sequences $\{f_n^\leq\}$ and $\{f_n^\geq\}$ converge uniformly to the same limit we have $\lim_{n\to\infty} \tilde{\mu}(f_n^\leq) = \lim_{n\to\infty} \tilde{\mu}(f_n^\geq)$, which completes the proof. □

**Proposition 9.** (Rychlik [10], Proposition 11) Let $\bar{\mu}$ be an arbitrary measure on $X$ which is $T$-invariant and such that the sets of $\Sigma$ are $\bar{\mu}$ measurable. Then, there exists a unique measure on $Y$ such that $\mu$ is $S$-invariant and $\pi_*\mu = \bar{\mu}$.

**Proof.** Let $\mu$ be constructed as in Lemma 12 and let $\eta$ be some other $S$-invariant measure such that $\pi_*\eta = \bar{\mu}$. For every continuous function $f$ on $Y$ we have $\eta(f^\leq) \leq \eta(f) \leq \eta(f^\geq)$. Since $\eta(f^\leq) = (\pi_*\eta)(f^\leq) = \bar{\mu}(f^\leq)$ (and similarly for $f^\geq$) for any function $f$ and in particular for $f \circ S^n$, we get
$$\bar{\mu}((f \circ S^n)^\leq) \leq \eta(f) \leq \bar{\mu}((f \circ S^n)^\geq),$$
as $\eta(f \circ S^n) = \eta(f)$. Going to the limit completes the proof. □

**Corollary 4.** In view of Theorem 2 we can construct $G$-invariant measures $\mu_1, \mu_2, \ldots, \mu_s$ such that $\pi_*\mu_i = \phi_i \circ m$, $i = 1, 2, \ldots, s$.

**Theorem 4.** (Rychlik [10], Theorem 4) Let $\mu$ be a Borel, regular measure on $[0, 1]^2$ such that $\pi_\mu$ is absolutely continuous with respect to $m$. Then,
$$\frac{1}{n} \sum_{k=0}^{n-1} G^{k}\mu \xrightarrow{n \to \infty} \sum_{i=1}^{s} \mu(\bar{C}_i) \cdot \mu_i ,$$
where $\bar{C}_i = \pi^{-1}(C_i)$, $C_1, C_2, \ldots, C_s$ as in Theorem 2, $\mu_1, \mu_2, \ldots, \mu_s$ are as above, and the convergence is in $\ast$-weak topology of measures.

**Proof.** We refer to [10]. □

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