A Note on the Number of Permutations whose 
Cycle Lengths Are Prime Numbers

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Abstract
Let $A$ be a set of natural numbers and let $S_{n,A}$ be the set of all permutations of $[n] = \{1, 2, \ldots, n\}$ with cycle lengths belonging to $A$. For $A(n) = A \cap [n]$, the limit $\rho = \lim_{n \to \infty} |A(n)|/n$ (if it exists) is usually called the density of set $A$. (Here $|B|$ stands for the cardinality of the set $B$.) Several studies show that the asymptotic behavior of the cardinality $|S_{n,A}|$, as $n \to \infty$, depends on the density $\rho$. It turns out that the assumption $\rho > 0$ plays an essential role in the asymptotic analysis of $|S_{n,A}|$. Kolchin (1999) noticed that there is a lack of studies on classes of permutations satisfying $\rho = 0$ and proposed investigations of certain particular cases. In this note, we consider the permutations whose cycle lengths are prime numbers, that is, we assume that $A = \mathcal{P}$, where $\mathcal{P}$ denotes the set of all primes. From the Prime Number Theorem it follows that $\rho = 0$ for this class of permutations. We deduce an asymptotic formula for the summatory function $\sum_{k \leq n} |S_k,\mathcal{P}|/k!$ as $n \to \infty$. In our proof we employ the classical Hardy-Littlewood-Karamata Tauberian theorem.

Key words: permutation, cycle length, prime number, Hardy-Littlewood-Karamata Tauberian theorem

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1 Introduction, Motivation and Statement of the Main Result

We start with some notation and conventions that will be used freely in the text of the paper.

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The letter \( p \) without subscript denotes a prime number. We write \( \mathcal{P} \) for the set of all primes. If the primes in \( \mathcal{P} \) are arranged in increasing order, then \( p_k \) denotes the \( k \)th smallest prime, \( k = 1, 2, \ldots \). For a function \( \psi(y) \) of a real variable \( y \) which assumes real or complex values, we write \( \sum_{p} \psi(p) \) instead of \( \sum_{p \in \mathcal{P}} \psi(p) \). We also use the usual notation \( \pi(y) \) for the number of primes not exceeding \( y > 0 \).

Further on, \( A \) always denotes a set of positive integers. Let \( [n] = \{1, 2, \ldots, n\} \). Then, we define the set \( A(n) = A \cap [n] \). We denote the cardinality of any set \( B \) by \( |B| \). The limit

\[
\rho = \lim_{n \to \infty} \frac{|A(n)|}{n},
\]

if it exists, is called the density of the set \( A \). Clearly, \( 0 \leq \rho \leq 1 \).

We write \( S_{n,A} \) for the set of permutations of \( n \) letters whose cycle lengths belong to \( A \). We denote the cardinality of \( S_{n,A} \) by \( P_{n,A} = |S_{n,A}| \). If \( A = \mathcal{P} \), we write \( P_n = |S_{n,p}| \).

For two sequences of real numbers \( \{u_n\}_{n \geq 1} \) and \( \{v_n\}_{n \geq 1} \), we write \( u_n \sim v_n \) if \( \lim_{n \to \infty} u_n/v_n = 1 \). In a similar and traditional manner, we shall also explore the commonly used symbol \( O(.) \); for its definition, we refer the reader, e.g., to \[1\], \[7\] and \[10\].

Finally, we show the approximations of two constants that will be used further, namely: the Euler’s constant

\[
\gamma = 0.577215
\]

and the constant

\[
c = \gamma - \sum_{p} \left( \log \left( \frac{1}{1 - 1/p} - \frac{1}{p} \right) \right) = 0.261497,
\]

which appears in an asymptotic estimate of the partial sum \( \sum_{p \leq y} 1/p \) and in Mertens’ formula for the partial product \( \prod_{p \leq y} (1 - 1/p) \) as \( y \to \infty \). More details may be found in \[10\] Theorems 1.10, 1.12.

The problem on the asymptotic enumeration of permutations whose cycle lengths are constrained has attracted the interest of several authors in the middle of the last century. The last two chapters of Kolchin’s monograph \[4\] are devoted to certain aspects from this topic. Kolchin surveyed various important results and discussed in detail the existing approaches and methods applied in this area. It seems that one of the main reasons for the interest in asymptotic enumeration problems of this type is the relationship between the cardinalities of sets of permutations \( S_{n,A} \), for certain particular sets \( A \), and the theory of equations containing an unknown permutation of \( n \) letters. For more details on asymptotic enumeration problems related to \( S_{n,A} \), we refer the reader, e.g., to \[4\] Section 4.4 and Chapter 5, \[7\] Section 8.2 and \[12\] Chapter 3.

The enumeration of permutations with cyclic structure constrained by a certain set \( A \) is based on the following generating function:

\[
f_A(z) := \sum_{n=0}^{\infty} \frac{P_{n,A} z^n}{n!} = \exp \left( \sum_{k \in A} \frac{z^k}{k} \right).
\]
A proof of this identity may be found, e.g., in [4, Theorem 5.1.2]. It turns out that, for several $A$’s, the asymptotic behavior of $P_{n,A}$ depends on the density $\rho$, given by (1). The case $\rho > 0$ was studied under several additional conditions on $A$ by many authors. As an illustration, below we give a typical asymptotic result obtained by Yakimiv in [11] (see also [12, Theorem 3.3.1]).

**Theorem (Yakymiv (1991)).** Suppose that, for a certain $A$, its density is $\rho > 0$. Moreover, for $m \geq n$ and $m = O(n)$, we assume that

$$\left|\{k : k \leq m, k \in A, m - k \in A\}\right|/n \to \rho^2$$

as $n \to \infty$. Then, we have

$$P_{n,A} \sim n! n^{\rho - 1} e^{L(n)} - \gamma \rho / \Gamma(\rho),$$

where $L(n) = \sum_{k \in A(n)} 1/k - \rho \log n$ and $\Gamma(.)$ is the Euler’s gamma function.

**Remark 1.** Condition (4) shows that the set $A$ may be considered as a realization of a random set containing each integer with probability $\rho$ independently from the other integers. It is also interesting to note that, under entirely different conditions on $A$, Kolchin [3] (see also [4, Theorem 4.4.10]) obtained the same asymptotic equivalence (5).

There are many examples of permutations with constraints on their cycle structure whose basic generating function (3) has a single dominant singularity at $z = 1$. In Sections 11 and 12 of his excellent survey [7], Odlyzko classified and discussed the methods used to extract asymptotic information about coefficients of analytic generating functions. He suggested two main classes of generating functions, depending on whether the main singularity is large or small. Functions with large singularities (i.e., ones that grow rapidly as the argument approaches the circle of convergence) are usually analyzed using the saddle point method; see [7, Sections 12.1 and 12.2]. In [4, Chapters 4 and 5] Kolchin demonstrated this method several times in the context of permutations with constraints on their cycles. On the other hand, some generating functions of the form (3) have small singularities on the circle of convergence and admit applications of other methods: Tauberian theory and transfer theorems (including those due to Darboux and Jungen); see [7, Sections 8.2, 11.1 and 11.2] and [12, Chapter 3].

In [4, Section 4.4], Kolchin surveyed asymptotic results on constrained permutations whose density $\rho$ is positive and noticed that there is a lack of studies in the case $\rho = 0$. Kolchin concluded his discussion with an open problem: he proposed to handle particular cases of $A$’s whose density $\rho = 0$. This, in fact, motivates us to study the case $A = \mathcal{P}$ and $P_{n,A} = P_n$. In this case the Prime Number Theorem (PNT) implies that $\rho = 0$. (For more details, historical remarks and stronger results on the remainder term in the PNT, we refer the reader to [10, p. 12 and Chapter II.4].) We also note that $P_n/n!$ is the probability that a permutation of $n$ letters chosen uniformly at random has cycle lengths which are prime numbers. The sequence $\{P_n\}_{n \geq 1}$ is A218002 in the On-Line Encyclopedia of Integer Sequences [9].

Below we state our main result.
Theorem 1. We have
\[ \sum_{k=0}^{n} \frac{P_k}{k!} \sim e^c \log n \]
as \( n \to \infty \), where \( c \) is the constant, given by \( c = 1.298873 \).

Remark 2. Theorem 1 has a probabilistic interpretation. Suppose that
the set \([k], k = 1, 2, ..., n,\) is chosen uniformly at random (i.e., with probability
\( 1/n \)) and then a permutation of \([k]\) is selected uniformly at random (i.e. with
probability \( 1/k! \)). Combining Theorem 1 with the total probability formula, one
can observe that the probability that this permutation has cycle lengths, which
are prime numbers, is asymptotically equivalent to
\[ \frac{e^c \log n}{n}. \]

The proof of Theorem 1 relies on a classical Tauberian theorem of Hardy,
Littlewood and Karamata (see [2] and [7, Theorem 8.7]).

Our paper is organized as follows. The next Section 2 contains some r emarks
on our method of proof. We discuss there the singularities of the underlying
generating function and answer the question: why we have chosen Tauberian
theory as a main tool of our approach? The proof of Theorem 1 is given in
Section 3.

2 Some Remarks on the Method of Proof

We first note that, for \( A = \mathcal{P} \) and \( P_{n,A} = P_n \), (6) implies
\[ f(z) := \sum_{n=0}^{\infty} \frac{P_n z^n}{n!} = \exp \left( \sum_{p} \frac{z^p}{p} \right). \]

For the sake of convenience, we also set
\[ \varphi(z) := \sum_{p} \frac{z^p}{p} = \sum_{n=1}^{\infty} \frac{z^{p_n}}{p_n}, \]
where in the last equality we have used that the primes in \( \mathcal{P} \) are arranged in
increasing order. As first observation, we will show that the unit circle is a
natural boundary for \( \varphi(z) \). The proof of this fact relies essentially on the Fabry
gap theorem, which we state below. For more details, we refer the reader to [5,
Section 5.3].

**Fabry Gap Theorem.** Suppose that the power series
\[ g(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n} \]
has radius of convergence 1, where \( 0 \leq \lambda_1 < \lambda_2 < ... \). If \( \lim_{n \to \infty} \lambda_n / n = \infty \),
then the circle of convergence \(|z| = 1\) is a natural boundary of \( g \).
It is well known that from the PNT in its minimal form, namely \(\pi(y) \sim y / \log y\) as \(y \to \infty\), it follows that the \(n\)th smallest prime \(p_n\) satisfies
\[
p_n \sim n \log n \quad n \to \infty; \tag{8}
\]
see, e.g., [10, Part I, Chapter 1.0, Exercise 14]. The gaps between the successive primes in \(\mathcal{P}\) and [8] show that \(\phi\) satisfies the conditions of the Fabry gap theorem. Thus we observe that the circle \(|z| = 1\) is a natural boundary for \(\phi\) and \(f\). Hence any application of transfer theorems (see [7, Section 11.1]) fails since an analytic continuation of the underlying generating function beyond its circle of convergence is not possible. We recall that the general transfer theorems (see [7, Theorem 11.4]) require that the function \(f(z)\), defined by (6), has to be analytic in the domain \(\Delta(1, \phi, \eta) = \{z : |z| \leq 1 + \eta, |\arg(z - 1)| \geq \phi\} \setminus \{1\}\), where \(\eta > 0\) and \(0 < \phi < \pi/2\). The dominant singularity of the function \(f(z)\) does not satisfy also the algebraic-logarithmic conditions of the Darboux and Jungen theorems [7, Section 11.2] and thus their application is also impossible. Both methods discussed above are usually used to extract asymptotic information for the coefficients of analytic generating functions with small dominant singularities. As it was expected, our attempt to imitate a saddle point approximation failed since the function \(f(z)\) does not have large enough singularity at \(z = 1\). The computational details for showing this are based on the PNT. In fact, in [6] was shown that a weak form of the PNT (namely, \(\pi(y) = y / \log y + O(y / \log^2 y), y \to \infty\)) yields the asymptotic of the first derivative of the function \(\phi(z)\) (see [7]), as \(z \to 1^-\). We have
\[
\phi'(z) \sim \frac{1}{(1 - z) \log \frac{1}{1 - z}}. 
\]
(In addition, De Bruijn [11] Section 3.14, Exercise 5] proposed an asymptotic series in the sense of Poincaré for \(\phi'(z)\).) In a way similar to that given in [6], one can also obtain
\[
\phi''(z) \sim \frac{1}{(1 - z)^2 \log \frac{1}{1 - z}}
\]
and
\[
\phi'''(z) \sim \frac{2}{(1 - z)^3 \log \frac{1}{1 - z}}.
\]
These computations show that the third-order term in the Taylor expansion of \(\phi(Re^{i\theta})\) around the point \(\theta = 0\) is too large as \(R \to 1^-\), and so, we are in the situation of an invalid application of the saddle point method, described in [7, Example 12.3].

The disadvantages discussed above show that these methods are not adequate for asymptotic analysis of the coefficients of the function \(f(z)\), given by [6]. Clearly, its singularity at \(z = 1\) is small. Hence we prefer to apply a classical Tauberian theorem by Hardy-Littlewood-Karamata. Its proof may be found in [2] Chapter 7. We use it in the form given in [7, Section 8.2].
Hardy-Littlewood-Karamata Theorem. (See [7; Theorem 8.7, p. 1225].) Suppose that $h_k \geq 0$ for all $k$, and that

$$h(z) = \sum_{k=0}^{\infty} h_k z^k$$

converges for $0 \leq z < r$. If there is an $\alpha \geq 0$ and a function $L(t)$ that varies slowly at infinity such that

$$h(z) \sim (r - z)^{-\alpha} L\left(\frac{1}{r - z}\right), \quad z \to r^-,$$

then

$$\sum_{k=0}^{n} h_k r^k \sim \left(\frac{n}{r}\right)^\alpha \frac{L(n)}{\Gamma(\alpha + 1)}, \quad n \to \infty.$$

Remark 3. A function $L(t)$ varies slowly at infinity if, for every $u > 0$, $L(ut) \sim L(t)$ as $t \to \infty$.

Remark 4. Odlyzko [7, Example 8.8] applied the Hardy-Littlewood-Karmata Tauberian theorem to obtain a general formula for the $n$th partial sum of the coefficients (probabilities) $P_{k,A}/k!$, where the set $A$ has density $\rho \in [0, 1]$. He showed that

$$\sum_{k=0}^{n} \frac{P_{k,A}}{k!} \sim f_A(1 - 1/n) \frac{\Gamma(\rho + 1)}{\Gamma(\alpha + 1)},$$

where $f_A$ is the function, defined by (3). Hence, an essential step in the proof of Theorem 1 is to determine the asymptotic behavior of $f(z)$ as $z \to 1^-$ (see (6)). Additional results and references on studies in this direction may be also found in [8].

3 Proof of Theorem 1

For convenience, in (6) and (7) we set $z = e^{-t}$. Then, we deal first with $\varphi(e^{-t})$. We need to study the behavior of $\varphi(e^{-t})$ as $t \to 0^+$, since by (7) $z \to 1^-$ and $t \to 0^+$ are equivalent. We start our computations with the following representation:

$$\varphi(e^{-t}) = \sum_p \frac{1 - (1 - e^{-pt})}{p} = \varphi_1(e^{-t}) + \varphi_2(e^{-t}) + \varphi_3(e^{-t}),$$

where

$$\varphi_1(e^{-t}) = \sum_{p \leq \left(\frac{1}{\log \frac{1}{t}}\right)/\log \log \frac{1}{t}} \frac{1}{p},$$

$$\varphi_2(e^{-t}) = \sum_{p \leq \left(\frac{1}{\log \frac{1}{t}}\right)/\log \log \frac{1}{t}} \frac{1 - e^{-pt}}{p},$$

$$\varphi_3(e^{-t}) = \sum_{p \in \mathbb{P}} \frac{1 - e^{-pt}}{p}.$$
\[ \varphi_3(e^{-t}) = \sum_{p > \left( \frac{t \log \log \frac{1}{t}}{\log \log \frac{1}{t}} \right)} \frac{e^{-pt}}{p}. \]

We notice that \( \varphi_j(e^{-t}), j = 1, 2, 3, \) are well defined for sufficiently small \( t. \) (An upper bound for \( t \) could be \( e^{-e} \) since, for \( t < e^{-e} \), the function \( \log \log \frac{1}{t} \), which appears in our further computations, is well defined.)

The asymptotic estimate for \( \varphi_1(e^{-t}) \) is obtained by means of a classical result from Number Theory, given in [10, Theorems 1.10 and 1.12] and related to the asymptotic growth of the partial sum \( \sum_{p \leq y} 1/p \) as \( y \to \infty \). An application of this result immediately yields

\[
\varphi_1(e^{-t}) = \log \log \left( \frac{\log \frac{1}{t}}{\log \log \frac{1}{t}} \right) + c + O \left( \frac{1}{\log \log \frac{1}{t}} \right), \tag{12}
\]

where the constant \( c \) is defined by \( (2) \).

The asymptotic estimate for \( \varphi_2(e^{-t}) \) will be based again on the PNT in its minimal form. We apply first the inequality \( (1 - e^{-pt})/p \leq t \) to each term in \( \varphi_2(e^{-t}) \). Then, by the PNT, we have

\[
|\varphi_2(e^{-t})| \leq t \sum_{p \leq \left( \frac{t \log \log \frac{1}{t}}{\log \log \frac{1}{t}} \right)} 1 \sim \frac{t \left( \log \log \frac{1}{t} \right)}{(\log \log \frac{1}{t}) \log \log \frac{1}{t}} = O \left( \frac{1}{\log \log \frac{1}{t}} \right) \tag{13}
\]
as \( t \to 0^+ \). Finally, for \( \varphi_3(e^{-t}) \), we obtain

\[
\varphi_3(e^{-t}) \leq \frac{t \log \log \frac{1}{t}}{\log 1/t} \sum_{p > \left( \frac{t \log \log \frac{1}{t}}{\log \log \frac{1}{t}} \right)} e^{-pt} < \frac{t \log \log \frac{1}{t}}{\log 1/t} \sum_{m > \left( \frac{t \log \log \frac{1}{t}}{\log \log \frac{1}{t}} \right)} e^{-mt}.
\]

where in the last sum we have also included all terms \( e^{-mt}, m \notin \mathcal{P} \). The last sum is a tail of a geometric progression and hence we easily obtain the following asymptotic estimate:

\[
\varphi_3(e^{-t}) = O \left( \frac{\log \log \frac{1}{t}}{\log 1/t} \cdot \exp\left( -\left( \frac{\log \log \frac{1}{t}}{\log \log \frac{1}{t}} \right) \right) \right). \tag{14}
\]
Combining (11) - (14) all together, we deduce that
\[ \varphi(e^{-t}) = \log \log \frac{1}{t} + c + O\left(\frac{1}{\log \log \frac{1}{t}}\right), \quad t \to 0^+. \]

Then (6) and (7) imply
\[ f(e^{-t}) = e^c \left( \log \frac{1}{t} \right) \left( 1 + O\left( \frac{1}{\log \log \frac{1}{t}} \right) \right), \quad t \to 0^+. \quad (15) \]

To complete the proof of Theorem 1, we recall that \( t = \log \frac{1}{z} \). Since
\[ \log \frac{1}{z} = -\log z = -\log (1 - (1 - z)) \sim 1 - z, \quad z \to 1^- , \]
we can rewrite (15) as follows
\[ f(z) \sim e^c \log \frac{1}{1 - z}, \quad z \to 1^- . \]

Therefore the series \( f(z) \), defined by (6), satisfies condition (9) of Hardy-Littlewood-Karamata Tauberian theorem with \( h(z) = f(z), r = 1, \alpha = 0 \) and \( L(t) = \log t \).
Now, the asymptotic equivalence of Theorem 1 follows immediately from (10).

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