On the symmetric rearrangement of the gradient of a Sobolev function

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Abstract

In this paper, we generalize a classical comparison result for solutions to Hamilton-Jacobi equations with Dirichlet boundary conditions, to solutions to Hamilton-Jacobi equations with non-zero boundary trace.

As a consequence, we prove the isoperimetric inequality for the torsional rigidity (with Robin boundary conditions) and for other functionals involving such boundary conditions.

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1 Introduction

Let $\Omega$ be a bounded, open and Lipschitz set and let $u \in W^{1,p}(\Omega)$, for some $p \geq 1$, be a non-negative function.

In this paper, we deal with the problem of comparing a function $u \in W^{1,p}(\Omega)$ with a radial function having the modulus of the gradient equi-rearranged with $|\nabla u|$. Hence, we aim to extend the results contains in and Nunziante [GN84] to a more general setting.

Throughout this article, $|\cdot|$ will denote both the $n$-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure, the meaning will be clear by the context.

If $A$ is a bounded and open set with the same measure as $\Omega$, we say that a function $f^* \in L^p(A)$ is equi-rearranged to $f \in L^p(\Omega)$ if they have the same distribution function, i.e.

\textbf{Definition 1.1.} Let $f : \Omega \to \mathbb{R}$ be a measurable function, the distribution function of $f$ is the function $\mu_f : [0, +\infty[ \to [0, +\infty[$ defined by

$$\mu_f(t) = |\{ x \in \Omega : |f(x)| > t \}|.$$

In order to state our results, we recall some definitions

\textbf{Definition 1.2.} Let $f : \Omega \to \mathbb{R}$ be a measurable function:
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- the *decreasing rearrangement* of \( f \), denoted by \( f^* \), is the distribution function of \( \mu_f \). Moreover, we can write
  \[
  f^*(s) = \inf \{ t \geq 0 \mid \mu_f(t) < s \};
  \]

- the *increasing rearrangement* of \( f \) is defined as
  \[
  f_*(s) = f^*(|\Omega| - s);
  \]

- the *spherically symmetric decreasing rearrangement* of \( f \), defined in \( \Omega^\sharp \) i.e. the ball centered at the origin with the same measure as \( \Omega \), is the function
  \[
  f^\sharp(x) = f^*(\omega_n|x|^n),
  \]
  where \( \omega_n \) is the measure of the \( n \)-dimensional unit-ball of \( \mathbb{R}^n \);

- the *spherically symmetric increasing rearrangement* of \( f \), defined in \( \Omega^\sharp \), is
  \[
  f^\sharp_*(x) = f_*(\omega_n|x|^n).
  \]

Clearly, we can construct several rearrangements of a given function \( f \), but the one we will refer to is the spherically symmetric increasing rearrangement defined in \( \Omega^\sharp \).

The starting point of our work, and many others, is [GN84, Theorem 2.2]

Theorem 1.1. Let \( p \geq 1 \), \( f : \Omega \to \mathbb{R}, \ H : \mathbb{R}^n \to \mathbb{R} \) be measurable non-negative functions and let \( K : [0, +\infty) \to [0, +\infty) \) be a strictly increasing real-valued function such that

\[
0 \leq K(|y|) \leq H(y) \quad \forall y \in \mathbb{R}^n \quad \text{and} \quad K^{-1}(f) \in L^p(\Omega)
\]

Let \( v \in W_0^{1,p}(\Omega) \) be a function that satisfy

\[
\begin{cases}
  H(\nabla v) = f(x) & \text{a.e. in } \Omega \\
  v = 0 & \text{on } \partial \Omega
\end{cases}
\]

then, denoting with \( \overline{v} \) the unique decreasing spherically symmetric solution to

\[
\begin{cases}
  K(|\nabla \overline{v}|) = f_\sharp(x) & \text{a.e. in } \Omega^\sharp \\
  \overline{v} = 0 & \text{on } \partial \Omega^\sharp
\end{cases}
\]

it holds

\[
\|v\|_{L^1(\Omega)} \leq \|\overline{v}\|_{L^1(\Omega^\sharp)} \tag{1.1}
\]

They give also a similar result for the spherically symmetric decreasing rearrangement of the gradient, with an \( L^\infty \) comparison.

In recent decades, many authors studied this kind of problems, in particular in [ALT89] Alvino, Lions and Trombetti proved the existence of a spherically symmetric rearrangement of the gradient of \( v \) which gives a \( L^q \) comparison as in (1.1) for a fixed \( q \).
Moreover, Cianchi in [Cia96] gives a characterization of such rearrangement; clearly, the rearrangement found by Cianchi is different both from the spherically symmetric increasing and decreasing rearrangement if \( q \in (1, \infty) \).

Furthermore, in [FP91] and [FPV93] the authors studied the optimization of the norm of a Sobolev function in the class of functions with fixed rearrangement of the gradient.

Incidentally, let us mention that the case where the \( L_{q,1}^{1} \) Lorentz norm, see Section 2 for its definition, takes the place of the \( L_{q}^{1} \) norm in (1.1) has been studied in [Tal]. In particular, he stated the following

**Theorem 1.2.** Let \( u \) be a real-valued function defined in \( \mathbb{R}^{n} \). Suppose \( u \) is nice enough - e.g. Lipschitz continuous - and the support of \( u \) has finite measure. Let \( M \) and \( V \) denote the distribution function of \( |\nabla u| \) and the measure of the support of \( u \), respectively.

Let \( v \) the real-valued function defined in \( \mathbb{R}^{n} \) that satisfies the following conditions:

1. \( |\nabla v| \) is a rearrangement of \( |\nabla u| \);
2. the support of \( v \) has the same measure of the support of \( u \);
3. \( v \) is radially decreasing and \( |\nabla v| \) is radially increasing.

Then
\[
\|u\|_{L_{p,1}(\Omega)} \leq \|v\|_{L_{p,1}(\Omega^\#)} \quad \text{if } n = 1 \text{ or } 0 < p \leq \frac{n}{n-1},
\]

furthermore
\[
\|v\|_{L_{p,1}(\Omega^\#)} = \frac{p^2}{\omega_n (n+p)} \int_0^\infty \left[ V^{\frac{1}{p}+\frac{1}{n}} - (V-M(t))^{\frac{1}{p}+\frac{1}{n}} \right] dt.
\]

On the other hand, the problem of studying the rearrangement of the Laplacian has been widely studied by several authors. The bibliography is extensive; for the sake of completeness, let us recall some of the works: [Tal76] for the Dirichlet boundary conditions, [ACNT21; ANT20; AGM22] for the Robin conditions.

As we already said, we focus on the case in which the functions do not vanish on the boundary. Our main theorem is the following:

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^{n} \) be a bounded, open and Lipschitz set and let \( u \in W^{1,p}(\Omega) \) be a non-negative function. If we denote with \( \Omega^\# \) the ball centered at the origin with same measure as \( \Omega \), then there exists a non-negative function \( u^* \in W^{1,p}(\Omega^\#) \) that satisfies

\[
\begin{cases}
|\nabla u^*| = |\nabla u|_p(x) & \text{a.e. in } \Omega^\# \\
\int_{\partial \Omega} u d\mathcal{H}^{n-1} & \text{on } \partial \Omega^\#
\end{cases}
\]

and such that
\[
\|u\|_{L^1(\Omega)} \leq \|u^*\|_{L^1(\Omega^\#)},
\]
Remark 1.4. By the explicit expression of $u^*$ on the boundary and the Hölder inequality, we can estimate the $L^p$ norm of the trace:

$$\left| \partial \Omega \right|^{p-1} \int_{\partial \Omega} (u^*)^p \, d\mathcal{H}^{n-1} = \left( \int_{\partial \Omega} u \, d\mathcal{H}^{n-1} \right)^p \leq \left| \partial \Omega \right|^{p-1} \int_{\partial \Omega} u^p \, d\mathcal{H}^{n-1} \quad \forall p \geq 1. \quad (1.4)$$

This result allows us to compare solutions to PDE with Robin boundary conditions with solutions to their symmetrized.

Precisely we are able to compare solutions to

$$\begin{cases}
-\Delta u = 1 & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \beta |\partial \Omega| \, u = 0 & \text{on } \partial \Omega
\end{cases} \quad (1.5)$$

with the solution to

$$\begin{cases}
-\Delta v = 1 & \text{in } \Omega^\sharp \\
\frac{\partial v}{\partial \nu} + \beta |\partial \Omega^\sharp| \, v = 0 & \text{on } \partial \Omega^\sharp
\end{cases} \quad (1.6)$$

In particular we get

**Corollary 1.5.** Let $\beta > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded, open and Lipschitz set. If we denote with $\Omega^\sharp$ the ball centered at the origin with same measure as $\Omega$, it holds

$$T(\Omega, \beta) \geq T(\Omega^\sharp, \beta) \quad (1.7)$$

where

$$T(\Omega, \beta) = \inf_{w \in W^{1,2}(\Omega)} \frac{\int_{\Omega} |\nabla w|^2 \, dx + \beta |\partial \Omega| \int_{\partial \Omega} w^2 \, d\mathcal{H}^{n-1}}{\left( \int_{\Omega} w \, dx \right)^2} \quad \text{for } w \in W^{1,2}(\Omega). \quad (1.8)$$

The paper is organized as follows. In Section 2 we recall some basic notions, definitions and classical results and we prove Theorem 1.3. Eventually, Section 3 is dedicated to the application to the Robin torsional rigidity and in Section 4 we get a comparison between Lorentz norm of $u$ and $u^*$.

## 2 Notations, Preliminaries and proof of the main result

Observe that obviously $\forall p \geq 1$

$$\|f\|_{L^p(\Omega)} = \|f^*\|_{L^p([0,|\Omega|])} = \|f^\sharp\|_{L^p(\Omega^\sharp)} = \|f_*\|_{L^p([0,|\Omega|])} = \|f_*\|_{L^p(\Omega^\sharp)},$$

moreover, the Hardy-Littlewood inequalities hold true:

$$\int_{\Omega} |f(x)g(x)| \, dx \leq \int_0^{|\Omega|} f^*(s)g^*(s) \, ds = \int_{\Omega^\sharp} f^\sharp(x)g^\sharp(x) \, dx,$$
\[ \int_{\Omega} f^*(x) g_\ast(x) \, dx = \int_0^{\|\Omega\|} f^*(s) g_\ast(s) \, ds \leq \int_{\Omega} |f(x)g(x)| \, dx. \]

Finally, the operator which assigns to a function its symmetric decreasing rearrangement is a contraction in $L^p$, see ([Chi79]), i.e.

\[ \|f^* - g^*\|_{L^p([0,\|\Omega\|])} \leq \|f - g\|_{L^p(\Omega)} \tag{2.1} \]

One can find more results and details about rearrangements for instance in [HLP88] and in [Tal].

Other powerful tools are the pseudo-rearrangements. Let $u \in W^{1,p}(\Omega)$ and let $f \in L^1(\Omega)$, as in [AT78] for $s \in [0,|\Omega|]$, there exists a subset $D(s) \subseteq \Omega$ such that

1. $|D(s)| = s$;
2. $D(s_1) \subseteq D(s_2)$ if $s_1 < s_2$;
3. $D(s) = \{ x \in \Omega \mid |u(x)| > t \}$ if $s = \mu(t)$.

So the function

\[ \int_{D(s)} f(x) \, dx \]

is absolutely continuous, therefore it exists a function $F$ such that

\[ \int_0^s F(t) \, dt = \int_{D(s)} f(x) \, dx \tag{2.2} \]

We will use the following property ([AT78, Lemma 2.2])

**Lemma 2.1.** Let $f \in L^p$ for $p > 1$ and let $D(s)$ be a family described above. If $F$ is defined as in (2.2), then there exists a sequence $\{ F_k \}$ such that $F_k$ has the same rearrangement as $f$ and

\[ F_k \rightharpoonup F \quad \text{in } L^p([0,|\Omega|]) \]

If $f \in L^1$ it follows that

\[ \lim_k \int_0^{[0,|\Omega|]} F_k(s) g(s) \, ds = \int_0^{[0,|\Omega|]} F(s) g(s) \, ds \]

for each function $g \in BV([0,|\Omega|])$.

Moreover, for sake of completeness, we will recall the definition of the Lorentz norm.

**Definition 2.1.** Let $\Omega \subseteq \mathbb{R}^n$ a measurable set, $0 < p < +\infty$ and $0 < q < +\infty$. Then a function $g$ belongs to the Lorentz space $L^{p,q}(\Omega)$ if

\[ \|g\|_{L^{p,q}(\Omega)} = \left( \int_0^{+\infty} \left[ \frac{1}{t} g^*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty. \tag{2.3} \]
Let us notice that for $p = q$ the Lorentz space $L^{p,p}(\Omega)$ coincides with the Lebesgue space $L^p(\Omega)$ by the Cavalieri’s principle.

Let us now prove the main Theorem.

**Proof of Theorem 1.3.** Let us consider $\varepsilon$ and $\delta := \delta_\varepsilon$ and the sets
\begin{align*}
\Omega_\varepsilon &= \{ x \in \mathbb{R}^n \mid d(x, \Omega) < \varepsilon \} \quad \Sigma_\varepsilon = \Omega_\varepsilon \setminus \Omega, \\
\Omega_\varepsilon^\delta &= \{ x \in \mathbb{R}^n \mid d(x, \Omega^\delta) < \delta \} \quad \Sigma_\varepsilon^\delta = \Omega_\varepsilon^\delta \setminus \Omega^\delta,
\end{align*}
where, since $|\Sigma_\varepsilon|/\varepsilon \to |\partial \Omega|$ and $|\Sigma_\varepsilon^\delta|/\delta \to |\partial \Omega^\delta|$ as $\varepsilon \to 0$, we have
\begin{equation}
\lim_{\varepsilon \to 0} \frac{\delta}{\varepsilon} = \frac{|\partial \Omega|}{|\partial \Omega^\delta|}.
\end{equation}
Let $d(\cdot, \Omega)$ defined as follows:
\begin{equation*}
d(x, \Omega) := \inf_{y \in \Omega} |x - y|.
\end{equation*}

Then we divide the proof into four steps.

**Step 1** First of all we assume $\Omega$ with $C^{1,\alpha}$ boundary, $u \in W^{1,\infty}(\Omega)$ and $u \geq \sigma > 0$ in $\Omega$.

So we can consider the following 'linear' extension of $u$, $u_\varepsilon$ in $\Omega_\varepsilon$
\begin{equation*}
u_\varepsilon(x) = u(p(x)) \left(1 - \frac{d(x, \partial \Omega)}{\varepsilon}\right) \quad \forall x \in \Omega_\varepsilon \setminus \Omega,
\end{equation*}
where $p(x)$ is the projection of $x$ on $\partial \Omega$ (for $\varepsilon$ sufficiently small, this definition is well posed since $\Omega$ is smooth, see [GT]). The function $u_\varepsilon$, has the following properties:

(a) $u_\varepsilon|_\Omega = u$,
(b) $u_\varepsilon = 0$ on $\partial \Omega_\varepsilon$,
(c) $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq |\nabla u_\varepsilon|(y) \forall y \in \Sigma_\varepsilon$ for $\varepsilon$ sufficiently small,
(d) $\lim_{\varepsilon \to 0^+} \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|\,dx = \int_{\partial \Omega} u\,d\mathcal{H}^{n-1}$.

Properties (a) and (b) follow immediately by the definition of $u_\varepsilon$, while (c) is a consequence of the regularity of $u$. Property (d) can be obtained by an easy calculation, indeed
\begin{equation*}
\nabla u_\varepsilon(x) = \nabla \left( u(p(x)) \left[1 - \frac{d(x, \partial \Omega)}{\varepsilon}\right] - u(p(x))\frac{\nabla d(x, \partial \Omega)}{\varepsilon}\right)
\end{equation*}
For the first term, we can notice
\begin{equation*}
\int_{\Sigma_\varepsilon} \left|\nabla \left(u(p(x))\right)\right| \left[1 - \frac{d(x, \partial \Omega)}{\varepsilon}\right] dx \leq L \int_{\Sigma_\varepsilon} dx = L|\Sigma_\varepsilon|
\end{equation*}
where $L$ is the $L^\infty$ norm of $\nabla u(p(x))$. Now we deal with the second term and, keeping in mind that $|\nabla d| = 1$ and using coarea formula, we have

$$\lim_{\varepsilon \to 0^+} \int_{\Sigma_\varepsilon} |\nabla u| \, dx = \lim_{\varepsilon \to 0^+} \int_{\Sigma_\varepsilon} u(p) \, dx = \lim_{\varepsilon \to 0^+} \int_0^\varepsilon \int_{\Gamma_t} (u \circ p) \, d\mathcal{H}^{n-1}$$

where $\Gamma_t = \{ x \in \Sigma_\varepsilon : d(x, \partial \Omega) = \varepsilon \}$. By continuity of $u$ and Lebesgue differentiation theorem we get

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\Gamma_t} u \circ p \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} u \, d\mathcal{H}^{n-1}$$

that proves property $(d)$.

For every $\varepsilon > 0$, we consider the following problem

$$\begin{cases}
|\nabla v_\varepsilon|(x) = |\nabla u_\varepsilon|(x) & \text{in } \Omega^\varepsilon_t \\
v_\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon_t
\end{cases} \quad (2.5)$$

and by Theorem 1.1 it holds

$$\|u_\varepsilon\|_{L^1(\Omega^\varepsilon_t)} \leq \|v_\varepsilon\|_{L^1(\Omega^\varepsilon_t)}. \quad (2.6)$$

Moreover it exists $\bar{\varepsilon}$ such that for every $\varepsilon \leq \bar{\varepsilon}$

$$|\nabla v_\varepsilon|(x) = |\nabla u_\varepsilon|(x) = |\nabla u|(x) \quad \forall x \in \Omega^\varepsilon. \quad (2.7)$$

We can see $u_\varepsilon$ as a $W^{1,1}(\Omega^\varepsilon)$ function and we have

$$\int_{\Omega^\varepsilon_t} |\nabla v_\varepsilon| = \int_{\Omega^\varepsilon_t} |\nabla u_\varepsilon| = \int_{\Omega} |\nabla u| + \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon| \leq \|\nabla u\|_{L^1(\Omega)} + 2\|u\|_{L^1(\partial\Omega)}. \quad (2.8)$$

by property $(d)$.

Finally, by Poincarè and (2.8), there exists a constant $0 < C = C(n, \Omega)$ such that

$$\|v_\varepsilon\|_{W^{1,1}(\Omega^\varepsilon_t)} \leq C\|\nabla v_\varepsilon\|_{L^1(\Omega^\varepsilon_t)} \leq C(n, \Omega)\|u\|_{W^{1,1}(\Omega)}.$$ 

Therefore, up to a subsequence, there exists a limit function $u^* \in BV(\Omega^1_t)$ such that ([AFP00, Proposition 3.13])

$$v_\varepsilon \to u^* \text{ in } L^1(\Omega^1_t) \quad \nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla u^* \text{ in } \Omega$$

namely

$$\lim_{\varepsilon \to 0} \int_{\Omega_t^\varepsilon} \varphi \, d\nabla v_\varepsilon = \int_{\Omega_t^1} \varphi \, d\nabla u^* \quad \forall \varphi \in C_0(\Omega, \mathbb{R}^n).$$

Our aim is to show that $u^*$ satisfies properties (1.2), (1.3) and (1.4).
Concerning (1.2) then \(|\nabla u^*| = |\nabla u|_z^2\) follows from (2.7).

To find the value of \(u^*\) at the boundary, we observe that, from (2.5) and (2.7), we have

\[ \int_{\Sigma_{\varepsilon}} |\nabla u_{\varepsilon}| = \int_{\Sigma_{\varepsilon}^\natural} |\nabla v_{\varepsilon}|. \]

Now, for \(t > 0\) setting \(\Gamma_t = \{ d(x, \Omega) = t \}, \Gamma_t^\natural = \{ d(x, \Omega^\natural) = t \}, r = \left( \frac{\|\Omega\|}{\omega_n} \right)^{\frac{1}{n}}\) and recalling that \(v_{\varepsilon}\) is radially symmetric we have

\[ \int_{\Sigma_{\varepsilon}^\natural} |\nabla v_{\varepsilon}| = \int_{r}^{r+\delta} \int_{\Gamma_{t}^\natural} |\nabla v_{\varepsilon}| d\mathcal{H}^{n-1} dt = |\Gamma_{t}^\natural| \int_{r}^{r+\delta} -v'_{\varepsilon} |\Gamma_{t}^\natural| dt = |\Gamma_{t}^\natural| v_{\varepsilon}(r). \]

Therefore by monotonicity of \(|\Gamma_{t}^\natural|\) we have

\[ |\Gamma_{t}^\natural| v_{\varepsilon}(r) \leq \int_{r}^{r+\delta} (v'_{\varepsilon}(t)|\Gamma_{t}^\natural|) dt \leq |\Gamma_{r+\delta}^\natural| v_{\varepsilon}(r) \]

and since

\[ |\Gamma_{t}^\natural| v_{\varepsilon}(r) = \int_{\partial\Omega} v_{\varepsilon} d\mathcal{H}^{n-1} \]

using the fact that \(v_{\varepsilon} \to v\) in \(L^1(\Omega), \nabla v_{\varepsilon} = \nabla u\) in \(\Omega\) and the continuity embedding of \(W^{1,1}(\Omega)\) in \(L^1(\Omega)\), in the end we have

\[ \int_{\Sigma_{\varepsilon}^\natural} |\nabla v_{\varepsilon}| \to \int_{\partial\Omega} u^* d\mathcal{H}^{n-1}. \]

Using property (d) we obtain

\[ \int_{\partial\Omega} u d\mathcal{H}^{n-1} = \int_{\partial\Omega} u^* d\mathcal{H}^{n-1}. \]

In the end we have that for \(u^*\) it holds

\[ \begin{cases} |\nabla u^*| = |\nabla u|_z^2 & \text{in } \Omega^\natural \\ u^* = \frac{\int_{\partial\Omega} u d\mathcal{H}^{n-1}}{|\partial\Omega^\natural|} & \text{on } \partial\Omega^\natural. \end{cases} \tag{2.9} \]

that proves (1.2).

Furthermore by

\[ \|u_{\varepsilon}\|_{L^1(D)} \to \|u\|_{L^1(D)} \quad \text{and} \quad \|v_{\varepsilon}\|_{L^1(D^\natural)} \to \|u^*\|_{L^1(D^\natural)}, \]

we can pass to the limit \(\varepsilon \to 0\) in (2.6) and we get

\[ \|u\|_{L^1(\Omega)} \leq \|u^*\|_{L^1(\Omega^\natural)}. \]

that proves (1.3).
Step 2 Now we remove the extra-assumption $u \geq \delta > 0$ defining

$$u_\sigma := u + \sigma.$$ 

Then $u_\sigma$ is strictly positive in $\Omega$ and we can apply the previous result: there exists a function $v_\sigma$ in $\Omega^2$ such that

$$\begin{cases}
|\nabla v_\sigma| = |\nabla u_\sigma| = |\nabla u| \\
v_\sigma = \int_{\partial \Omega} u_\sigma \, d\mathcal{H}^{n-1} / |\partial \Omega^2| = \int_{\partial \Omega} u \, d\mathcal{H}^{n-1} / |\partial \Omega^2| + \sigma |\partial \Omega| / |\partial \Omega^2| \quad \text{on } \partial \Omega^2,
\end{cases}$$

and

$$\|u_\sigma\|_{L^1(\Omega)} \leq \|v_\sigma\|_{L^1(\Omega^2)}, \quad (2.10)$$

If we define

$$u^* := v_\sigma - \sigma |\partial \Omega| / |\partial \Omega^2|,$$

then $u^*$ solves

$$\begin{cases}
|\nabla u^*| = |\nabla u| & \text{in } \Omega^2 \\
u^* = \int_{\partial \Omega} u \, d\mathcal{H}^{n-1} / |\Omega^2| & \text{on } \partial \Omega^2,
\end{cases} \quad (2.11)$$

Sending $\sigma \to 0$ in (2.10) we have

$$\|u\|_{L^1(\Omega)} \leq \|u^*\|_{L^1(\Omega^2)}.$$ 

Step 3 Now we remove the assumption on the regularity of $\Omega$.

Let $\Omega$ be a bounded, open and Lipschitz set, and $u \in W^{1,\infty}(\Omega)$. Then there exists a sequence $\{\Omega_k\} \subset \mathbb{R}^n$ of open set with $C^2$ boundary such that $\Omega \subset \Omega_k$, $\forall k \in \mathbb{N}$ (for instance you can mollify $\chi_\Omega$ and take a suitable superlevel set) and

$$|\Omega_k \triangle \Omega| \to 0 \quad \mathcal{H}^{n-1}(\partial \Omega_k) \to \mathcal{H}^{n-1}(\partial \Omega) \quad \text{for } k \to +\infty.$$ 

Let $\tilde{u}$ be an extension of $u$ in $\mathbb{R}^n$ such that

$$\tilde{u}|_{\Omega} \equiv u, \quad \|\tilde{u}\|_{W^{1,\infty}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,\infty}(\Omega)}.$$ 

We define

$$u_k = \tilde{u}\chi_{\Omega_k},$$

and clearly $u_k = u$ in $\Omega$. By the previous step, we can construct $u^*_k \in W^{1,\infty}(\Omega_k^2)$ such that it is radial, $|\nabla u_k|_* = |\nabla u^*_k|_*$ and

$$\begin{align*}
\|u_k\|_{L^1(\Omega_k)} & \leq \|u_k^*\|_{L^1(\Omega_k^2)} \quad (2.12) \\
\int_{\partial \Omega_k} u_k \, d\mathcal{H}^{n-1} & = \int_{\partial \Omega_k^2} u_k^* \, d\mathcal{H}^{n-1} \quad (2.13)
\end{align*}$$
Therefore, since \( \| u_k \|_{W^{1,p}(\Omega)} \leq M \), for all \( p \), the sequence \( \{ u_k^* \} \) is equibounded in \( W^{1,p}(\Omega^\#) \) and it has a subsequence which converges strongly in \( L^p \) and weakly in \( W^{1,p} \) to a function \( w \).

Let us prove that \( \| \nabla u \| \) and \( \| \nabla w \| \) has the same rearrangement.

\[
\limsup_k \| \nabla u_k^* \| - \| \nabla u \| \leq \lim_k \| (f_k)_z - f_z \|_{L^p(\mathbb{R}^n)}
\]

where

\[
f(x) = \begin{cases} \| \nabla \hat{u} \|_{L^\infty(\mathbb{R}^n)} & \text{in } \Omega \setminus \Omega_k \\ \| \nabla \hat{u} \|_{L^\infty(\mathbb{R}^n)} & \text{in } \mathbb{R}^n \setminus \Omega_k \end{cases}
\]

and

\[
f_k = \begin{cases} \| \nabla u_k \|_{L^\infty(\mathbb{R}^n)} & \text{in } \Omega_k \\ \| \nabla \hat{u} \|_{L^\infty(\mathbb{R}^n)} & \text{in } \mathbb{R}^n \setminus \Omega_k \end{cases}
\]

So using (2.1) we have

\[
\| (f_k)_z - f_z \|_{L^p(\mathbb{R}^n)} \leq \| f_k - f \|_{L^p(\mathbb{R}^n)} = \| f_k - f \|_{L^p(\Omega, \mathbb{R}^n)} \leq 2\| \nabla \hat{u} \|_{L^\infty(\mathbb{R}^n)}|\Omega_k \setminus \Omega|
\]

that tends to 0 as \( k \to +\infty \) by the fact that \( |\Omega_k \setminus \Omega| \to 0 \).

Hence, the functions \( \nabla w \) and \( \nabla u \) has the same rearrangement, by the uniqueness of the weak limit in \( \Omega^\# \).

In the end, passing to limit \( k \to +\infty \) in (2.12) and (2.13), we have

\[
\| u \|_{L^1(\Omega)} \leq \| w \|_{L^1(\Omega)}
\]

\[
\int_{\partial \Omega} u \, d\mathcal{H}^{n-1} = \int_{\partial \Omega} w \, d\mathcal{H}^{n-1}.
\]

Hence \( w = u^* \).

**Step 4** Finally, we proceed by removing the assumption \( u \in W^{1,\infty}(\Omega) \).

If \( u \in W^{1,p}(\Omega) \), by Meyers-Serrin Theorem, there exists a sequence \( \{ u_k \} \subset C^\infty(\Omega) \cap W^{1,p}(\Omega) \) such that \( u_k \to u \) in \( W^{1,p}(\Omega) \). We can apply previous step to obtain \( u_k^* \in W^{1,\infty}(\Omega^\#) \) such that \( |\nabla u_k| \) and \( |\nabla u_k^*| \) are equally distributed and

\[
\| u_k \|_{L^1(\Omega)} \leq \| u_k^* \|_{L^1(\Omega^\#)} \quad \forall k \in \mathbb{N} \quad (2.14)
\]

\[
\int_{\partial \Omega} u_k \, d\mathcal{H}^{n-1} = \int_{\partial \Omega} u_k^* \, d\mathcal{H}^{n-1} \quad \forall k \in \mathbb{N}. \quad (2.15)
\]

Arguing as the previous step, there exists a function \( w \) such that, up to a subsequence \( u_k^* \to w \) in \( L^p(\Omega) \) and \( \nabla u_k^* \to \nabla w \) in \( L^p(\Omega; \mathbb{R}^n) \)

and \( |\nabla w| \) has the same rearrangement as \( |\nabla u| \).

Finally, sending \( k \to +\infty \) in (2.14) and (2.15), we have

\[
\| u \|_{L^1(\Omega)} \leq \| w \|_{L^1(\Omega^\#)}
\]

\[
\int_{\partial \Omega} u \, d\mathcal{H}^{n-1} = \int_{\partial \Omega} w \, d\mathcal{H}^{n-1}.
\]

Hence \( w = u^* \).
3 An application to torsional rigidity

Let $\beta > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded and open set with Lipschitz boundary and let us consider the functional

$$F_{\beta}(\Omega, w) = \frac{\int_{\Omega} |\nabla w|^2 \, dx + \beta |\partial \Omega| \int_{\partial \Omega} w^2 \, d\mathcal{H}^{n-1}}{\left( \int_{\Omega} w \, dx \right)^2} \quad \text{if } w \in W^{1,2}(\Omega) \quad (3.1)$$

and the associate minimum problem

$$T(\Omega, \beta) = \min_{w \in W^{1,2}(\Omega)} F_{\beta}(w) \quad (3.2)$$

The minimum $u$ is a weak solution to

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta |\partial \Omega| u = 0 & \text{on } \partial \Omega \end{cases} \quad (3.3)$$

Our aim is to compare $T(\Omega, \beta)$ with

$$T(\Omega^\sharp, \beta) := \min_{v \in W^{1,2}(\Omega)} F_{\beta,\sharp}(v) = \min_{v \in W^{1,2}(\Omega)} \frac{\int_{\Omega^\sharp} |\nabla v|^2 \, dx + \beta |\partial \Omega^\sharp|^2 \int_{\partial \Omega^\sharp} v^2 \, d\mathcal{H}^{n-1}}{\left( \int_{\Omega^\sharp} v \, dx \right)^2}$$

where the minimum is a weak solution to

$$\begin{cases} -\Delta z = 1 & \text{in } \Omega^\sharp \\ \frac{\partial z}{\partial \nu} + \beta |\partial \Omega^\sharp| z = 0 & \text{on } \partial \Omega^\sharp \end{cases} \quad (3.4)$$

**Proof of Corollary 1.5.** Let $w \in W^{1,p}(\Omega)$, by Theorem 1.3 and Remark 1.4 there exists $w^* \in W^{1,\infty}(\Omega^\sharp)$ radial such that

$$\int_{\Omega} |\nabla w|^2 \, dx = \int_{\Omega^\sharp} |\nabla w^*|^2 \, dx \quad \int_{\Omega} |w| \, dx \leq \int_{\Omega^\sharp} |w^*| \, dx \quad |\partial \Omega|^2 \int_{\partial \Omega} (w^*)^2 \leq |\partial \Omega| \int_{\partial \Omega} w^2$$

Therefore

$$F_{\beta}(w) \geq F_{\beta}(w^*)$$

Passing to the infimum on right-hand side and successively to the left-hand side, we obtain

$$T(\Omega, \beta) \geq T(\Omega^\sharp, \beta)$$

$\square$
**Remark 3.1.** We highlight that all the arguments work also in the non-linear case, where the functional

$$
F_{\beta,p}(w) = \int_{\Omega} |\nabla w|^p \, dx + \beta |\partial \Omega|^{p-1} \int_{\partial \Omega} w^p \, d\mathcal{H}^{n-1} \quad \text{for } w \in W^{1,p}(\Omega). \tag{3.5}
$$

is considered.

### 4 A weighted $L^1$ comparison

Let us check how extend the result by [Tal] to the case of function non vanishing on the boundary.

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and Lipschitz set. Let $f \in L^\infty(\Omega)$ be a function such that

$$
f^*(t) \geq \left(1 - \frac{1}{n}\right) \frac{1}{t} \int_0^t f^*(s) \, ds \quad \forall t \in [0, |\Omega|]. \tag{4.1}
$$

If $u \in W^{1,p}(\Omega)$ and $u^*$ is the function given by Theorem 1.3, then

$$
\int_{\Omega} f(x) u(x) \, dx \leq \int_{\Omega} f^*(x) u^*(x) \, dx. \tag{4.2}
$$

**Proof.** If $u \in W_0^{1,p}(\Omega)$, the result is contained in [Tal]. We recall it, for sake of completeness. By [GN84, eq. 2.7] it is known

$$
u^*(s) \leq \frac{1}{n \omega_n} \int_s^{|\Omega|} \frac{F(t)}{t^{1-\frac{1}{n}}} \, dt \tag{4.3}
$$

where $F$ is a function such that

$$
\int_0^s F(t) \, dt = \int_{D(s)} |\nabla u^*|_*(s) \, ds
$$

with $D(s)$ defined in Section 2.

Setting $g(t) := \frac{1}{t^{1-\frac{1}{n}}} \int_0^t f^*(s) \, ds$, multiplying both terms of (4.3) for $f^*(s)$, integrating from 0 to $|\Omega|$ and using Fubini’s Theorem we get

$$
\int_0^{[\Omega]} f^*(s) u^*(s) \, ds \leq \frac{1}{n \omega_n} \int_0^{[\Omega]} f^*(s) \left( \int_s^{[\Omega]} \frac{F(t)}{t^{1-\frac{1}{n}}} \, dt \right) \, ds = \frac{1}{n \omega_n} \int_0^{[\Omega]} F(t) g(t) \, dt \tag{4.4}
$$
Let us suppose that \( g(t) \) is non-decreasing, so \( g_*(s) = g(s) \) and by Lemma 2.1 there exists a sequence \( \{F_k\} \) such that \((F_k)_* = (\nabla u)_* \) and \( F_k \rightarrow F \) in \( BV \). Therefore

\[
\int_0^{[\Omega]} F(t)g(t) \, dt = \lim_k \int_0^{[\Omega]} F_k(t)g(t) \, dt
\]

Using Hardy-Littlewood’s inequality we have

\[
\lim_k \int_0^{[\Omega]} F_k(t)g(t) \, dt \leq \int_0^{[\Omega]} \left| \nabla u_*(t)g_*(t) \right| \, dt = \int_0^{[\Omega]} \left| \nabla u_*(t)g(t) \right| \, dt
\]

Hence, by (4.4) and Fubini’s Theorem, we obtain

\[
\int_0^{[\Omega]} f^*(t)u^*(t) \, dt \leq \frac{1}{n\omega_1^n} \int_0^{[\Omega]} \left| \nabla u_*(t) \right| g(t) \, dt
\]

\[
= \frac{1}{n\omega_1^n} \int_0^{[\Omega]} \left| \nabla u_*(t) \right| \left( \frac{1}{t^{1-\frac{n}{2}}} \int_0^t f^*(s) \, ds \right) \, dt
\]

\[
= \int_0^{[\Omega]} f^*(s) \left( \frac{1}{n\omega_1^n} \int_s^{[\Omega]} \left| \nabla u_*(t) \right| \, dt \right) \, ds
\]

\[
= \int_0^{[\Omega]} f^*(s)(u^*)^*(s) \, ds
\]

Therefore, by Hardy-Littlewood inequality, we have

\[
\int_{\Omega} f(x)u(x) \, dx \leq \int_0^{[\Omega]} f^*(t)u^*(t) \leq \int_0^{[\Omega]} f^*(s)(u^*)^*(s) \, ds = \int_{\Omega^\prime} f^*(x)u^*(x) \, dx \quad (4.5)
\]

But we have to deal with the assumption that \( g \) is non-decreasing, that is

\[
g'(t) \geq 0 \iff \frac{d}{dt} \left( \frac{1}{t^{1-\frac{n}{2}}} \int_0^t f^*(s) \, ds \right) = -\frac{n-1}{t^{2-\frac{n}{2}}} \left( \int_0^t f^*(s) \, ds \right) + \frac{1}{t^{1-\frac{n}{2}}} f^*(t) \geq 0,
\]

hence, if and only if

\[
f^*(t) \geq \left(1 - \frac{1}{n}\right) \frac{1}{t} \int_0^t f^*(s) \, ds.
\]

Now let us deal with \( u \notin W_0^{1,p}(\Omega) \). Suppose that \( u \in C^2(\Omega) \) is a non-negative function, that \( \Omega \) has \( C^2 \) boundary and that \( f \) satisfies (4.1). Proceeding as in Step 1 of Theorem 1.3, for every \( \varepsilon > 0 \) we can construct \( u_\varepsilon \) that coincides with \( u \) in \( \Omega \) and is zero on \( \partial \Omega \). Moreover we can extend \( f \) to \( \Omega_\varepsilon \) simply defining

\[
f_\varepsilon(t) = \begin{cases} f(x) & \text{in } \Omega \\ f^*(|\Omega|) & \text{in } \Omega_\varepsilon \setminus \Omega \end{cases}
\]
The rearrangement, for every \( \varepsilon > 0 \), is

\[
f^*_\varepsilon(t) = \begin{cases} 
  f^*(t) & \text{in } [0,|\Omega|] \\
  f^*(|\Omega|) & \text{in } [|\Omega|,|\Omega_\varepsilon|],
\end{cases}
\]

so we just have to check \((4.1)\) for \( t \in [|\Omega|,|\Omega_\varepsilon|] \), namely

\[
f^*_\varepsilon(t) \geq \left(\frac{n-1}{n}\right) \frac{1}{t} \int_0^t f^*_\varepsilon(s) \, ds.
\]

(4.6)

Keeping in mind that \( f \) verifies \((4.1)\), we have

\[
f^*_\varepsilon(t) = f^*(|\Omega|) \geq \left(\frac{n-1}{n}\right) \frac{1}{|\Omega|} \int_0^{|\Omega|} f^*(s) \, ds.
\]

If we show that

\[
\frac{1}{|\Omega|} \int_0^{|\Omega|} f^*(s) \, ds \geq \left[ \frac{1}{t} \int_0^{|\Omega|} f^*(s) \, ds + \frac{t - |\Omega|}{t} f^*(|\Omega|) \right] = \frac{1}{t} \int_0^t f^*_\varepsilon(s) \, ds
\]

then \((4.6)\) is true. By direct calculations

\[
\frac{t - |\Omega|}{t|\Omega|} \int_0^{|\Omega|} f^*(s) \, ds \geq \frac{t - |\Omega|}{t} f^*(|\Omega|) \iff \frac{1}{|\Omega|} \int_0^{|\Omega|} f^*(s) \, ds \geq f^*(|\Omega|).
\]

that is true of the fact that \( f^* \) is decreasing.

So, \( \forall \varepsilon > 0 \) we can apply the first part of the Theorem obtaining

\[
\int_{\Omega_\varepsilon} u_\varepsilon f_\varepsilon \, dx \leq \int_{\Omega_\varepsilon} v_\varepsilon f^*_\varepsilon \, dx
\]

Sending \( \varepsilon \to 0 \) we get

\[
\int_{\Omega} u f \, dx \leq \int_{\Omega_\varepsilon} u^* f^*_\varepsilon \, dx.
\]

Arguing as in Theorem 1.3, we get \((4.2)\).

**Remark 4.2.** Condition \((4.1)\) implies the \( f \) is strictly positive. Moreover, if the essential oscillation of \( f \) is bounded

\[
\text{ess osc } |f| := \frac{\text{ess sup}_{x \in \Omega} |f(x)|}{\text{ess inf}_{x \in \Omega} |f(x)|} \leq \frac{n}{n-1}
\]

then \((4.1)\) is satisfied.
Theorem 4.1 allows us to compare the minimum of
\[
T_{\beta,f}(\Omega) := \min_{w \in W^{1,2}(\Omega)} \left\{ \frac{1}{2} \int_{\Omega} |\nabla w|^2\,dx + \frac{\beta |\partial \Omega|}{2} \int_{\partial \Omega} w^2\,dH^{n-1} - \int_{\Omega} wf\,dx \right\}
\]
with the one of
\[
T_{\beta,f}(\Omega^\sharp) := \min_{v \in W^{1,2}(\Omega^\sharp)} \left\{ \frac{1}{2} \int_{\Omega^\sharp} |\nabla v|^2\,dx + \frac{\beta |\partial \Omega^\sharp|}{2} \int_{\partial \Omega^\sharp} v^2\,dH^{n-1} - \int_{\Omega^\sharp} vf\,dx \right\}.
\]

**Corollary 4.3.** Let \( \beta > 0 \), let \( \Omega \subset \mathbb{R}^n \) be a bounded, open and Lipschitz set. If \( f \) satisfies (4.1), then denoting with \( \Omega^\sharp \) the ball centered at the origin with same measure as \( \Omega \), it holds
\[
T_{\beta,f}(\Omega) \geq T_{\beta,f}(\Omega^\sharp)
\]
Moreover we can use Theorem 4.1 to get a comparison between Lorentz norm of \( u \) and \( u^\star \).

**Corollary 4.4.** Let \( 1 \leq p \leq \frac{n}{n-1} \), under the assumption of Theorem 1.3 it holds
\[
\|u\|_{L^p,1}(\Omega) \leq \|u^\star\|_{L^p,1}(\Omega^\sharp)
\]
where \( u^\star \) is the function given by Theorem 1.3

**Proof.** Let us explicit the \( L^{p,1} \) norm of \( u \)
\[
\|u\|_{L^{p,1}(\Omega)} = \int_0^{+\infty} t^{\frac{1}{p}-1} u^\star(t)\,dt = \int_0^{+\infty} t^{\frac{1}{p}} u^\star(t)\,dt
\]
Hence by Theorem 4.1, it is sufficient that
\[
t^{-\frac{1}{p}} - \frac{n-1}{n} \int_0^t s^{-\frac{1}{p}}\,ds \geq 0.
\]
If we compute
\[
\frac{1}{t} \int_0^t s^{-\frac{1}{p}}\,ds = \frac{1}{t} p t^{-\frac{1}{p}+1} = p t^{-\frac{1}{p}},
\]
then we have
\[
t^{-\frac{1}{p}} - \frac{n-1}{n} t \int_0^t s^{-\frac{1}{p}}\,ds = t^{-\frac{1}{p}} \left( 1 - \frac{n-1}{n} p \right) \geq 0 \iff p \leq \frac{n}{n-1}
\]
so (4.8) is true and we can apply Theorem 4.1 obtaining
\[
\int_0^{+\infty} t^{-\frac{1}{p}} u^\star(t)\,dt \leq \int_0^{+\infty} t^{-\frac{1}{p}} u^\star(t)\,dt
\]
that is (4.7).

**Remark 4.5.** We emphasize that the bound \( p \leq \frac{n}{n-1} \) is the best we can hope for Lorentz norm \( L^{p,1} \). Indeed, if by absurd (4.7) holds for \( p > \frac{n}{n-1} \), by the embedding of \( L^{p,q} \) spaces, \( L^{p,1}(\Omega) \subseteq L^{q,q}(\Omega) = L^q(\Omega) \), which gives a contradiction.
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Abstract

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