The homotopy operator method for symbolic integration by parts and inversion of divergences with applications

Douglas Poole and Willy Hereman

Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887, USA

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Using standard calculus, explicit formulas for one-, two- and three-dimensional homotopy operators are presented. A derivation of the one-dimensional homotopy operator is given. A similar methodology can be used to derive the multi-dimensional versions. The calculus-based formulas for the homotopy operators are easy to implement in computer algebra systems such as Mathematica, Maple, and REDUCE. Several examples illustrate the use, scope, and limitations of the homotopy operators. The homotopy operator can be applied to the symbolic computation of conservation laws of nonlinear partial differential equations (PDEs). Conservation laws provide insight into the physical and mathematical properties of the PDE. For instance, the existence of infinitely many conservation laws establishes the complete integrability of a nonlinear PDE.

Keywords: homotopy operator, exact differential function, Euler operator, total divergence, conservation law, complete integrability.

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1. Introduction

In many applications one needs to integrate exact expressions involving several unspecified functions of multiple independent variables. The “homotopy operator” presented in this paper performs that task. Indeed, we give a formula based on a homotopy that integrates one-dimensional (1D) expressions, or inverts a total divergence in two dimensions (2D) or three-dimensions (3D). The formulas are derived using a principle of the calculus of variations and are written in the language of standard calculus.

The concept of homotopy, contributed to Poincaré, is certainly familiar to differential geometers and algebraic topologists. Briefly, a homotopy is a continuous map, \( T: X \times [0, 1] \rightarrow Y \) between two functions \( u \) and \( u_0 \) (both are maps from a space \( X \) to a space \( Y \)), such that \( T(x, 0) = u_0(x) \) and \( T(x, 1) = u(x) \). For example, \( T(x, \lambda) = \lambda u(x) + (1-\lambda)u_0(x) = u_0(x) + \lambda(u(x) - u_0(x)) \) is a homotopy. If \( u_0 = 0 \) then \( T(x, \lambda) = \lambda u(x) \) expresses a scaling of \( u \) with a parameter \( \lambda \). The homotopy operator used in this paper is more elaborate for it integrates an expression with respect to an auxiliary variable \( \lambda \) from 0 to 1.

Homotopy operators presumably first appeared in the inverse problem of the calculus of variations in works by Volterra. They also appear in the proof of the converse of the Poincaré lemma, which states that closed differential \( k \)-forms are exact, but only on domains with certain topological assumptions such as

*Corresponding author. Email: whereman@mines.edu
D. Poole and W. Hereman

a star-shaped domain. Once the domain is appropriately restricted, the proof of the converse of Poincaré’s lemma requires the construction of a homotopy operator. More generally, the construction of suitable homotopy operators is the key to exactness proofs for many complexes such as the de Rham complex and the bi-variational complex.

One method to investigate the complete integrability of a system of partial differential equations (PDEs) is to determine whether the system has infinitely many conservation laws. While developing a method for computing conservation laws of nonlinear PDEs in \((1+1)\) dimensions, the authors needed a technique to symbolically integrate expressions involving unspecified functions. Mathematica’s \texttt{Integrate} function often failed to integrate such integrands, in particular, when transcendental functions were present. The computation of conservation laws of nonlinear PDEs in multiple space variables, requires a tool to invert total divergences. The homotopy operator used in the proof of the exactness of the (b-)variational complex can do the required integrations. The homotopy operator in [5] has since been translated into the language of standard calculus and written into a more efficient algorithmic form. Although the homotopy operator for one variable does not appear in work by Kruskal et al. [12], it takes only one extra step to derive it. Taking that step, it also became clear how to generalize the formula to cover multiple variables.

The calculus-based form given in [8, 9] makes the homotopy operator easy to apply by researchers not familiar with differential forms. However, two issues must be addressed before the homotopy operator becomes a ready-to-use tool. The first issue relates to the inversion of a total divergence, \(\text{Div}^{-1}\), which does not have a unique answer. In analogy with standard integration with \(\int x\), where the result is up to an arbitrary constant, \(\text{Div}^{-1}\) is defined up to curl terms. The homotopy operator will produce a particular choice for the curl terms, often creating very large vectors. Hence, we discuss an algorithm to remove the unwanted curl terms. Secondly, the homotopy operator fails to work on expressions involving terms of degree zero. A solution to this problem is also presented in this paper, thereby extending the applicability of the homotopy operator.

With applications in mind far beyond the computation of conservation laws, the homotopy operator has its own \textit{Mathematica} code, \texttt{HomotopyIntegrator.m}. To our knowledge, \texttt{HomotopyIntegrator.m} is currently the only full-fledged implementation of the homotopy operator in \textit{Mathematica}. In collaboration with Hereman [9], Deconinck and Nivala [14, 15] have recently developed similar code in \textit{Maple}. Although applicable to exact as well as non-exact expressions, their code is restricted to the 1D case. Starting from version 9 of \textit{Maple}, the homotopy operator is now part of the kernel of \textit{Maple} to broaden the scope of its \texttt{Integrate} function. Anderson [16] and Cheviakov [17, 18] offer implementations of the homotopy operator in \textit{Maple}, not as stand-alone tools but as a component of broader software packages.

The multi-dimensional homotopy operator is an essential tool for the symbolic computation of fluxes of conservation laws of nonlinear PDEs involving multiple space variables. Most of such systems are not completely integrable for they only have a finite number of conservation laws [11]. As an application, we compute a conservation law of the \((2+1)\)-dimensional Zakharov–Kuznetsov equation [19], which describes ion-acoustic solitons in magnetized plasmas. Recently, we developed the \textit{Mathematica} package \texttt{ConservationLawsMD.m} [20] that automates the computation of polynomial conservation laws of nonlinear polynomial PDEs involving multiple space variables and time. The multi-dimensional homotopy operator is a key tool in that package.
2. Preliminary Definitions

Operations are carried out on differential functions, $f(x, u^{(M)}(x))$, where $u^{(M)}(x)$ denotes the dependent variable $u = (u^1, ..., u^j, ..., u^N)$ and its partial derivatives (up to order $M$) with respect to independent variable $x$. Although we allow variable coefficients, the differential functions should not contain terms that are functions of $x$ only. For simplicity of notation in the examples, we will denote $u^1, u^2, u^3$, etc. by $u, v, w$, etc.

We consider only 1 D, 2 D, and 3 D cases. Therefore, the independent space variable, $x$, will have at most three components. Thus, $x$ represents $x$, $(x, y)$, and $(x, y, z)$ in 1 D, 2 D, and 3 D problems, respectively. Partial derivatives are denoted by $u_{k1x}k2y k3z = \frac{\partial^k u^{(M)}}{\partial x^{k1} \partial y^{k2} \partial z^{k3}}$. For example, $\frac{\partial^3 u}{\partial x^2 \partial y}$ is written as $u_{3x2y}$.

For simplicity, we often abbreviate $f(x, u^{(M)}(x))$ by $f$. The notation $f[\lambda u]$ means that in $f$ one replaces $u$ by $\lambda u$, $u_x$ by $\lambda u_x$, and so on for all derivatives of $u$, where $\lambda$ is an auxiliary parameter. For example, if $f = xu^2v_x + u_xv_x \sin w$ then $f[\lambda u] = \lambda^2 xu^2v_x + \lambda u_xv_x \sin \lambda w$.

Two additional operators are needed before the homotopy operator can be defined. The total derivative operator allows one to algorithmically compute derivatives of differential functions based on the chain rule of differentiation.

**Definition 2.1**: With $x = x$, the 1 D total derivative operator $D_x$ acting on $f = f(x, u^{(M)}(x))$ is defined as

$$D_x f = \frac{\partial f}{\partial x} + \sum_{j=1}^{N} \sum_{k=0}^{M_j^1} u^j_{(k+1)x} \frac{\partial f}{\partial u^j_{kx}},$$

where $M^j_1$ is the order of $f$ in component $u^j$ and $M = \max_{j=1, ..., N} M^j_1$. The partial derivative, $\frac{\partial}{\partial x}$, acts on any $x$ that appears explicitly in $f$, but not on $u^j$ or any partial derivatives of $u^j$. The total derivative operators in 2 D and 3 D are defined analogously. For example, the 3 D total derivative operator with respect to $x$ is

$$D_x f = \frac{\partial f}{\partial x} + \sum_{j=1}^{N} \sum_{k=1}^{M_j^1} \sum_{l=0}^{M_l^1} \sum_{m=0}^{M_m^1} \sum_{k=0}^{M_k^1} u^j_{(k+1)x}k2y k3z \frac{\partial f}{\partial u^j_{k1x}k2y k3z},$$

where $M^j_1, M^j_2$ and $M^j_3$ are the orders of $f$ for component $u^j$ with respect to $x, y$, and $z$, respectively.

Obviously, $D_y$ and $D_z$ can be defined analogously. Note that for the 3 D case, $M$ is the maximum order for derivatives on $u^j$, $j = 1, ..., N$.

The Euler operator, also known as the variational derivative, is one of the most important tools in the calculus of variations.

**Definition 2.2**: The Euler operator for the 1 D case where $f = f(x, u^{(M)}(x))$ is

$$\mathcal{L}_{u^j(x)} f = \sum_{k=0}^{M^j_1} (-D_x)^k \frac{\partial f}{\partial u^j_{kx}}$$

$$= \frac{\partial f}{\partial u^j} - D_x \frac{\partial f}{\partial u^j_x} + D_x^2 \frac{\partial f}{\partial u^j_{2x}} - D_x^3 \frac{\partial f}{\partial u^j_{3x}} + \cdots + (-D_x)^{M^j_1} \frac{\partial f}{\partial u^j_{M^j_1 x}},$$

$j = 1, ..., N$. The 2 D and 3 D Euler operators (variational derivatives) are defined
analogously. For example, in the 2D case where \( x = (x, y) \) and \( f = f(x, u^{(M)}(x)) \), the Euler operator is defined as

\[
\mathcal{L}_w(x,y)f = \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} (-D_x)^{k_1}(-D_y)^{k_2} \frac{\partial f}{\partial u_{k_1x k_2y}}, \quad j = 1, \ldots, N. \tag{2}
\]

In this paper we use the Euler operators to test if differential functions are exact.

**Definition 2.3:** Let \( f = f(x, u^{(M)}(x)) \) be a differential function of order \( M \). When \( x = x \), \( f \) is called *exact* if there exists a differential function \( F(x, u^{(M-1)}(x)) \) such that \( f = D_x F \). When \( x = (x, y) \) or \( x = (x, y, z) \), \( f \) is *exact* if there exists a differential vector function \( F(x, u^{(M-1)}(x)) \) such that \( f = \text{Div} \, F \). In the 2D case, \( F = (F^1, F^2) \) and \( \text{Div} \, F = D_x F^1 + D_y F^2 \). The definition of \( \text{Div} \) in 3D is analogous.

**Theorem 2.4:** A differential function \( f = f(x, u^{(M)}(x)) \) is exact if and only if \( \mathcal{L}_u(x)f \equiv 0 \). Here, \( \mathbf{0} \) is the vector \((0,0,\ldots,0)\) which has \( N \) components matching the number of components of \( u \).

A proof for Theorem 2.4 is given in [11].

### 3. Integration by Parts Using the One-Dimensional Homotopy Operator

In this section we show how to compute \( F = D_x^{-1} f = \int f \, dx \) when \( f \) is exact. Direct integration by parts is cumbersome and prone to errors if \( f \) contains high-order derivatives of several dependent variables. The 1D homotopy operator provides an alternate method for integrating exact differential functions (with one independent variable) and multiple dependent variables, often eliminating the need for integration by parts.

**Definition 3.1:** Let \( x = x \) be the independent variable and \( f = f(x, u^{(M)}(x)) \) be an exact differential function, i.e. there exist a function \( F \) such that \( F = D_x^{-1} f \). Thus, \( F \) is the integral of \( f \). The 1D homotopy operator is defined as

\[
\mathcal{H}_{u(x)} f = \int_{\lambda_0}^{\lambda} \left( \sum_{j=1}^{N} \mathcal{I}_{u^j(x)} f \right) \left[ \lambda u \right] \frac{d\lambda}{\lambda}, \tag{3}
\]

where \( u = (u^1, \ldots, u^j, \ldots, u^N) \). The integrand, \( \mathcal{I}_{u^j(x)} f \), is defined as

\[
\mathcal{I}_{u^j(x)} f = \sum_{k=1}^{M_1^j} \left( \sum_{i=0}^{k-1} u^{(i)}_{kx} (-D_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u^j_{kx}}, \tag{4}
\]

where \( M_1^j \) is the order of \( f \) in dependent variable \( u^j \) with respect to \( x \).

The usual homotopy operator with \( \lambda_0 = 0 \) applies when \( \{3\} \) converges. However, due to a possible singularity at \( \lambda = 0 \), \( \{3\} \) might diverge for \( \lambda_0 \to 0 \). This can occur with rational as well as irrational integrands. In such cases, one can take \( \lambda_0 \to \infty \) or, alternatively, evaluate the indefinite integral and let \( \lambda \to 1 \). As shown in [15], the homotopy operator with appropriately selected limits remains a valid tool for integrating non-polynomial functions.

In the next section we prove that \( F = \mathcal{H}_{u(x)} f \). For now, we illustrate the application of the homotopy operator formulas \( \{3\} \) and \( \{4\} \) with an example.
Example 3.2 Let \((u^1, u^2) = (u, v)\) and take \(f(x, u^{(3)}(x)) = u^2 + 2xuv_x + uxv_{3x} + u_{2x}v_{2x} - 3v_x^2v_{2x}\). Applying the Euler operator \((\frac{\partial}{\partial u_x})\) separately for \(u\) and \(v\), with \(M^1 = 2\) and \(M^2 = 3\), one readily verifies that \(\mathcal{L}_{u(x)}f \equiv 0\) and \(\mathcal{L}_{v(x)}f \equiv 0\). Thus, \(f\) is exact. To find \(F = \int f \, dx\), using \((\ref{eq:3})\), first compute the integrands,

\[
\mathcal{I}_{u(x)}f = (u\mathcal{I})(\frac{\partial f}{\partial u_x}) + (u_x\mathcal{I} - u\mathcal{D}_x)(\frac{\partial f}{\partial u_{2x}}) = u(2xu + v_{3x}) + (u_x\mathcal{I} - u\mathcal{D}_x)(v_{2x})
\]

\[
= 2xu^2 + uxv_{2x},
\]

\[
\mathcal{I}_{v(x)}f = (v\mathcal{I})(\frac{\partial f}{\partial v_x}) + (v_x\mathcal{I} - v\mathcal{D}_x)(\frac{\partial f}{\partial v_{2x}}) = (v_x\mathcal{I} - v\mathcal{D}_x)(u_{2x} - 3v_x^2) + (v_x\mathcal{I} - v\mathcal{D}_x + v\mathcal{D}_x^2)(u_x)
\]

\[
= u_xv_{2x} - 3v_x^3,
\]

where \(\mathcal{I}\) is the identity operator. Next, sum the integrands, replace \(u, u_x, v_x\) and \(v_{2x}\) with \(\lambda u, \lambda u_x, \lambda v_x,\) and \(\lambda v_{2x}\), respectively, divide by \(\lambda\), and integrate with respect to \(\lambda\). In detail,

\[
F = \mathcal{H}_{u(x)}f = \int_0^1 (\mathcal{I}_{u(x)}f + \mathcal{I}_{v(x)}f) \left[ \frac{\lambda u}{\lambda} \right] d\lambda
\]

\[
= \int_0^1 (2xu^2 + 2\lambda u_xv_{2x} - 3\lambda^2 v_x^3) \lambda u_x d\lambda
\]

\[
= (\lambda^2 xu^2 + \lambda^2 u_xv_{2x} - \lambda^3 v_x^3)\bigg|_0^1
\]

\[
= xu^2 + u_xv_{2x} - v_x^3.
\]

Clearly, \(\mathcal{D}_xF = f\). Here \(\lambda_0 = 0\) was used since \((\ref{eq:3})\) converged for \(\lambda_0 \to 0\).

For polynomial differential expressions, \((\ref{eq:3})\) replaces integration by parts \((in x)\) with a few differentiations following by a standard integration \((of a polynomial)\) with respect to an auxiliary parameter \(\lambda\). The second example illustrates the integration of a rational differential function where \((\ref{eq:3})\) diverges for \(\lambda_0 \to 0\).

Example 3.3 Let \((u^1, u^2) = (u, v)\) and take \(f(x, u^{(1)}(x)) = (uv_x + uv_x)/(uv)^2\). Using \((\ref{eq:4})\), \(\mathcal{I}_{u(x)}f = \frac{1}{uv}\) and \(\mathcal{I}_{v(x)}f = \frac{1}{uv}\). Evaluation of \((\ref{eq:3})\) gives

\[
F = \mathcal{H}_{u(x)}f = \int_{\lambda_0}^1 \frac{2}{\lambda^2 uv} \, d\lambda = - \frac{1}{\lambda^2 uv} \bigg|_{\lambda_0}^1.
\]

Obviously, \(\lambda_0 \to 0\) would cause the integral to diverge. Instead, take \(\lambda_0 \to \infty\) so that \(F = \mathcal{H}_{u(x)}f = -\frac{1}{uv}\). Alternatively, when a singularity occurs at \(\lambda = 0\), one could compute the indefinite integral and let \(\lambda \to 1\). In either case, \(\mathcal{D}_xF = f\).

4. The One-Dimensional Homotopy Operator

The 1D homotopy operator in Definition \((\ref{eq:4})\) is an expanded version of the one \((in terms of higher-Euler operators)\) given in \([9, 14, 15, 18]\). Although they give the same result \([11, 21]\), the expanded homotopy operator formula is computationally more efficient for two reasons: the number of times that the total derivative is applied is significantly smaller and the combinatorial coefficients have been eliminated.
To prove that the homotopy operator [3] does indeed integrate an exact differential expression, some extra definitions and theorems are needed. We first define a degree operator, \( \mathcal{M} \). Application of \( \mathcal{M} \) to a differential function yields that function multiplied by its degree; hence its name.

**Definition 4.1:** Let \( x = x \) be the independent variable. The degree operator \( \mathcal{M} \) acting on a differential function \( f = f(x, u^{(M)}(x)) \) is defined [12] as

\[
\mathcal{M}f = \sum_{j=1}^{N} \sum_{i=0}^{M_j} u_{ix}^i \partial \frac{f}{\partial u_{ix}},
\]

where \( f \) has order \( M_j^i \) in \( u^j \) with respect to \( x \).

The next example shows how the degree operator works.

**Example 4.2** Let \( u = (u^1, u^2) = (u, v) \) and \( f(x, u^{(5)}(x)) = (u)^p (v_2x)^q (u_5x)^r \), where \( p, q, \) and \( r \) are nonzero rational numbers. Using (5),

\[
\mathcal{M}f = u \partial \frac{(u)^p (v_2x)^q (u_5x)^r}{\partial u} + u_{5x} \partial \frac{(u)^p (v_2x)^q (u_5x)^r}{\partial u_{5x}} + v_{2x} \partial \frac{(u)^p (v_2x)^q (u_5x)^r}{\partial v_{2x}}
\]

\[
= pu (u)^{p-1} (v_2x)^q (u_5x)^r + ru_{5x} (u)^p (v_2x)^q (u_5x)^{r-1} + qv_{2x} (u)^p (v_2x)^q-1 (u_5x)^r.
\]

Note that the total degree of \( f \) has become a factor. Compare this with \( x(x^n)' = x(nx^{n-1}) = nx^n (n \) is a nonzero rational) in 1 D calculus.

It is trivial to prove that \( \mathcal{M} \) is a linear operator. Less straightforward is finding the kernel (\( \text{ker} \)) of \( \mathcal{M} \). For what follows, we need the notion of homogeneity.

**Definition 4.3:** A differential function \( f = f(x, u^{(M)}(x)) \) is called homogeneous of degree \( p \) in \( u \) (and its derivatives) if \( f[\lambda u] = \lambda^p f \).

Note the analogy with the definition of a homogeneous function of several variables. For example for the two-variables case, \( f(x, y) \) is called homogeneous of degree \( p \) if \( f(\lambda x, \lambda y) = \lambda^p f(x, y) \). Examples are \( f(x, y) = ax^2 + by + cy^2 \) of degree 2 and \( f(x, y) = 1/\sqrt{ax^2 + by + cy^2} \) of degree -1 \((a, b, \) and \( c \) are constants).

**Theorem 4.4:** If \( f = \frac{k}{\ell} \) where \( k = k(x, u^{(M)}(x)) \) and \( \ell = \ell(x, u^{(M)}(x)) \) are homogeneous differential functions of the same degree (so, \( f \) is of degree 0) then \( \mathcal{M}f = 0 \).

**Proof:** Suppose that \( k \) and \( \ell \) are homogeneous differential functions of degree \( p \). Obviously, \( \mathcal{M}k = pk \) and \( \mathcal{M}\ell = p\ell \). Hence, \( \mathcal{M} \left( \frac{k}{\ell} \right) = \frac{\ell \mathcal{M}k - k \mathcal{M}\ell}{\ell^2} = \frac{plk - pk\ell}{\ell^2} = 0 \). \( \square \)

Conversely, one could ask “Which differential functions are in the kernel of the degree operator?” For simplicity, consider the case where \( f(u, v, u_x, v_x) \). Any \( f \in \text{ker} \mathcal{M} \) must satisfy \( \mathcal{M}f = 0 \), or explicitly,

\[
u \frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} + v \frac{\partial f}{\partial v} + v_x \frac{\partial f}{\partial v_x} = 0.
\]

The jet variables \( u, v, u_x, \) and \( v_x \) are independent. Hence, [6] is a linear first-order PDE which can be solved with the method of the characteristics[1]. Solving the

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[1] We are indebted to Mark Hickman for this argument and subsequent derivation.
first-order system of ODEs,

\[
\frac{du}{d\tau} = u, \quad \frac{du_x}{d\tau} = u_x, \quad \frac{dv}{d\tau} = v, \quad \frac{dv_x}{d\tau} = v_x,
\]

where \( \tau \) parameterizes the characteristic curves, yields

\[
u = c_1 e^{\tau}, \quad u_x = c_2 e^{\tau}, \quad v = c_3 e^{\tau}, \quad v_x = c_4 e^{\tau}, \quad (7)
\]

where the \( c_i \) are arbitrary constants. The first equation in (7) determines \( e^{\tau} = \frac{u}{c_1} \). Using it to eliminate \( \tau \) from the remaining equations of (7), yields

\[
u_x = \frac{c_2}{c_1} u = C_1 u, \quad v = \frac{c_3}{c_1} u = C_2 u, \quad v_x = \frac{c_4}{c_1} u = C_3 u,
\]

where \( C_1, C_2, \) and \( C_3 \) are arbitrary constants. Thus, the general solution of (6) is \( f(C_1, C_2, C_3) = f(\frac{u_x}{u}, \frac{v}{u}, \frac{v_x}{u}) \), where the functional form of \( f \) is arbitrary.

**Example 4.5** By construction,

\[
f = D_x \left( \frac{u + v}{u - v} \right) = \frac{2(uv_x - u_x v)}{(u - v)^2}, \quad (8)
\]

is exact. Furthermore, \( Mf = 0 \) by Theorem 4.4. Given \( f(u, u_x, v, v_x) \), not necessarily rational, one can a priori test whether or not \( f \in \text{Ker} \ M \). Indeed, applying the replacement rules, \( u \to u, \ u_x \to \mu u, \ v \to \mu u, \ v_x \to \mu u \), to \( f \) should yield an expression which is independent of \( u, u_x, v, \) and \( v_x \). Applying the replacement rules to \( f \) in (8) gives \( \frac{2}{\mu} \), which only depends on \( \mu \).

The argument above can be extended to any differential function \( f \) no matter which jet variables are present. The above “kernel test” can be performed on exact as well as non-exact differential functions.

Next, the inverse of the degree operator will be derived, which requires the use of a homotopy [21]. To avoid a potentially diverging integral at 0, the homotopy is defined on \([\lambda_0, 1]\). To begin with, consider only differential functions with one term, e.g., \( u u_x, x u_x, u_x/u, u_x^2/(\sqrt{u_x^2 + v_x^2}) \), or take \( f \) in Example 4.2.

**Theorem 4.6:** Let \( f(x, u^{(M)}(x)) \) be any single term differential function in 1D and let \( g(x, u^{(M)}(x)) = Mf(x, u^{(M)}(x)) \), where \( Mf(x, u^{(M)}(x)) \neq 0 \). Then,

\[
M^{-1} g(x, u^{(M)}(x)) = \int_{\lambda_0}^{1} g[\lambda u] \frac{d\lambda}{\lambda}, \quad (9)
\]

**Proof:** If \( g(x, u^{(M)}(x)) \) has order \( M_1^2 \) in \( u^j \) with respect to \( x \), then \( g[\lambda u] \) also has order \( M_1^2 \) in \( u^j \). Furthermore, using the chain rule,

\[
\frac{d}{d\lambda} g[\lambda u] = \sum_{j=1}^{N} \sum_{i=0}^{M_1^2} \frac{\partial g[\lambda u]}{\partial \lambda} \frac{d\lambda u^j_x}{d\lambda} = \frac{1}{\lambda} \sum_{j=1}^{N} \sum_{i=0}^{M_1^2} u^j_x \frac{\partial g[\lambda u]}{\partial u^j_x} = \frac{1}{\lambda} M g[\lambda u], \quad (10)
\]
by the definition of $\mathcal{M}$. Integrate both sides of (10) with respect to $\lambda$ yields

$$
\int_{\lambda_0}^{1} \frac{d}{d\lambda} g[\lambda u] \, d\lambda = \int_{\lambda_0}^{1} \mathcal{M} \frac{g[\lambda u]}{\lambda} \, d\lambda,
$$

$$
g[\lambda u] \bigg|_{\lambda_0}^{1} = \mathcal{M} \int_{\lambda_0}^{1} g[\lambda u] \, d\lambda / \lambda,
$$

$$
g(x, u^{(M)}(x)) - g[\lambda_0 u] = \mathcal{M} \int_{\lambda_0}^{1} g[\lambda u] \, d\lambda / \lambda. \quad (11)
$$

To get (9), $g[\lambda_0 u]$ must be 0. This will affect the choice of $\lambda_0$. Indeed, using (9),

$$
g[\lambda_0 u] = \sum_{j=1}^{N} \sum_{i=0}^{M_{ij}} \lambda_0^j u^{i}_{ix} \frac{\partial f[\lambda_0 u]}{\partial (\lambda_0 u^{i}_{ix})} = \lambda_0 \sum_{j=1}^{N} \sum_{i=0}^{M_{ij}} u^{i}_{ix} \frac{\partial f[\lambda_0 u]}{\partial (\lambda_0 u^{i}_{ix})}. \quad (12)
$$

Depending on the form of $f$, there are two choices for $\lambda_0$ that make $g[\lambda_0 u] = 0$.

i. If $g[\lambda_0 u]$ in (12) is an expression in fractional form where $\lambda_0$ is a factor in the denominator, then let $\lambda_0 \to \infty$ to get $g[\lambda_0 u] = 0$. The integral on the right-hand side of (11) will then converge for $\lambda_0 \to \infty$.

ii. For all other cases, provided $\lambda_0$ does not drop out of (12) after simplification, set $\lambda_0 = 0$ to get $g[\lambda_0 u] = 0$. In the simplest case, when $f$ (and therefore also $g$) is a homogeneous monomial of degree $p$, then $g[\lambda u]$ has as factor $\lambda^p$ and, consequently, $g[\lambda_0 u] = 0$ for $\lambda_0 = 0$.

With $g[\lambda_0 u] = 0$, apply $\mathcal{M}^{-1}$ to both sides of (11) to get (9). \hfill \Box

Note that Theorem 4.6 excludes the case where $\mathcal{M} f(x, u^{(M)}(x)) = 0$. If $f \in \text{Ker} \, \mathcal{M}$, then $g(x, u^{(M)}(x)) = 0$ and Theorem 4.6 becomes trivial. In view of Theorem 4.4, the homotopy operator cannot be applied to expressions of degree 0. In particular, the homotopy operator would incorrectly handle integrands involving ratios of polynomial or irrational differential functions of like degree. In Section 7 we provide a method to overcome this shortcoming of the homotopy operator.

To show that (9) does indeed invert the $\mathcal{M}$ operator, in the next example we apply $\mathcal{M}^{-1}$ to the result of Example 4.2.

**Example 4.7** Let $u = (u^1, u^2) = (u, v)$ and $g(x, u^{(M)}(x)) = (p + q + r) (u^p) (v^q) (u^5 v^r)^r$. Then,

$$
\mathcal{M}^{-1} g(x, u^{(M)}(x)) = \int_{0}^{1} (p + q + r) (\lambda u)^p (\lambda v^q) (\lambda u^5 v^r) \, \frac{d\lambda}{\lambda}
$$

$$
= (p + q + r) (u^p) (v^q) (u^5 v^r) \int_{0}^{1} \lambda^{p+q+r-1} \, d\lambda
$$

$$
= (p + q + r) (u^p) (v^q) (u^5 v^r) \left[ \frac{\lambda^{p+q+r}}{p+q+r} \right]_{0}^{1}
$$

$$
= (u^p) (v^q) (u^5 v^r)^r. \quad (13)
$$

As with all polynomial integrands, we took $\lambda_0 = 0$ as the lower limit of the homotopy integral. Note that (13) is identical to $f$ given in Example 4.2.

**Remark 1:** Let $f(x, u^{(M)}(x)) = f_1 + f_2 + \cdots + f_i + \cdots + f_q$, where each $f_i = \cdots$
\( f_i(x, u^{(M)}(x)) \) is a single-term differential function (not necessarily a monomial). Theorem 4.6 holds for \( f \) provided that \( Mf_i \neq 0 \) for \( i = 1, \ldots, q \).

Next, we establish the commutation of the degree operator (and its inverse) with the total derivative operator.

**Lemma 4.8:** \( M D_x = D_x M \) and \( M^{-1} D_x = D_x M^{-1} \).

**Proof:** A proof is given in [11].

Now follows the key theorem for this Section which states that the 1D homotopy operator does indeed integrate an exact differential function.

**Theorem 4.9:** Let \( f = f(x, u^{(M)}(x)) \) be exact, i.e. \( D_x F = f \) for some differential function \( F(x, u^{(M-1)}(x)) \). Then \( F = D_x^{-1} f = H_{u(x)} f \).

**Proof:** The proof for the scalar case \( (u = u) \) in [21] can be generalized to the vector case. Indeed, first consider a fixed component \( u^j \) of \( u \) and multiply \( L_{u^j(x)f} \) by \( u^j \) to restore the degree. Subsequently, split off \( u^j \frac{\partial f}{\partial u^j} \). This yields

\[
u^j L_{u^j(x)f} = u^j \sum_{k=0}^{M^j_x} (-D_x)^k \frac{\partial f}{\partial u^j_{kx}} = u^j \frac{\partial f}{\partial u^j} + u^j \sum_{k=1}^{M^j_x} (-D_x)^{k-1} \frac{\partial f}{\partial u^j_{kx}}.
\]

(14)

Next, integrate the last term by parts and split off \( \frac{\partial f}{\partial u^j_{M^j_x}} \) is split off. Continuing with (14), the computations proceed as follows:

\[
u^j L_{u^j(x)f} = u^j \frac{\partial f}{\partial u^j} - D_x \left( u^j \sum_{k=1}^{M^j_x} (-D_x)^{k-1} \frac{\partial f}{\partial u^j_{kx}} \right) + u^j \sum_{k=1}^{M^j_x} (-D_x)^{k-1} \frac{\partial f}{\partial u^j_{kx}}
\]

\[
= u^j \frac{\partial f}{\partial u^j} + u^j \frac{\partial f}{\partial u^j} - D_x \left( u^j \sum_{k=1}^{M^j_x} (-D_x)^{k-1} \frac{\partial f}{\partial u^j_{kx}} \right) + u^j \sum_{k=2}^{M^j_x} (-D_x)^{k-2} \frac{\partial f}{\partial u^j_{kx}}
\]

\[
= u^j \frac{\partial f}{\partial u^j} + u^j \frac{\partial f}{\partial u^j} - D_x \left( u^j \sum_{k=1}^{M^j_x} (-D_x)^{k-1} \frac{\partial f}{\partial u^j_{kx}} + u^j \sum_{k=2}^{M^j_x} (-D_x)^{k-2} \frac{\partial f}{\partial u^j_{kx}} \right)
\]

\[
+ u^j \sum_{k=2}^{M^j_x} (-D_x)^{k-2} \frac{\partial f}{\partial u^j_{kx}}
\]

\[
= \ldots
\]

\[
= u^j \frac{\partial f}{\partial u^j} + u^j \frac{\partial f}{\partial u^j} + \ldots + u^j \frac{\partial f}{\partial u^j_{M^j_x}} - D_x \left( u^j \sum_{k=1}^{M^j_x} (-D_x)^{k-1} \frac{\partial f}{\partial u^j_{kx}}
\]

\[
+ u^j \sum_{k=2}^{M^j_x} (-D_x)^{k-2} \frac{\partial f}{\partial u^j_{kx}} + \ldots + u^j \sum_{k=1}^{M^j_x} (-D_x)^{k-M^j_x} \frac{\partial f}{\partial u^j_{kx}} \right)
\]
\begin{align*}
&= \sum_{i=0}^{M_i} u_{ix}^j \frac{\partial f}{\partial u_{ix}} - D_x \left( \sum_{i=0}^{M_i-1} u_{ix}^j \sum_{k=i+1}^{M_i} (-D_x)^{k-\ell(i+1)} \frac{\partial f}{\partial u_{ix}^j} \right) \\
&= \sum_{i=0}^{M_i} u_{ix}^j \frac{\partial f}{\partial u_{ix}^j} - D_x \left( \sum_{k=1}^{M_i} \left( \sum_{i=0}^{k-1} u_{ix}^j (-D_x)^{k-\ell(i+1)} \right) \frac{\partial f}{\partial u_{ix}^j} \right). \quad (15)
\end{align*}

Now, sum (15) over all components \( u^j \) to get
\begin{align*}
\sum_{j=1}^{N} u^j \mathcal{L}_{w^j(x)} f &= \sum_{j=1}^{N} \sum_{i=0}^{M_i} u_{ix}^j \frac{\partial f}{\partial u_{ix}^j} - \sum_{j=1}^{N} D_x \left( \sum_{k=1}^{M_i} \left( \sum_{i=0}^{k-1} u_{ix}^j (-D_x)^{k-\ell(i+1)} \right) \frac{\partial f}{\partial u_{ix}^j} \right) \\
&= \mathcal{M} f - D_x \left( \sum_{j=1}^{N} \sum_{k=1}^{M_i} \left( \sum_{i=0}^{k-1} u_{ix}^j (-D_x)^{k-\ell(i+1)} \right) \frac{\partial f}{\partial u_{ix}^j} \right). \quad (16)
\end{align*}

Since \( f \) is exact, by Theorem 2.4, \( \mathcal{L}_{w^j(x)} f \equiv 0 \) for \( j = 1, \ldots, N \), which implies that \( \sum_{j=1}^{N} \mathcal{L}_{w^j(x)} f \equiv 0 \). Hence,
\begin{align*}
\mathcal{M} f &= D_x \left( \sum_{j=1}^{N} \sum_{k=1}^{M_i} \left( \sum_{i=0}^{k-1} u_{ix}^j (-D_x)^{k-\ell(i+1)} \right) \frac{\partial f}{\partial u_{ix}^j} \right). \quad (17)
\end{align*}

Apply \( \mathcal{M}^{-1} \) to both sides and replace \( \mathcal{M}^{-1} D_x \) by \( D_x \mathcal{M}^{-1} \) using Lemma 4.8. Thus,
\begin{align*}
f &= \mathcal{M}^{-1} f = D_x \left( \sum_{j=1}^{N} \sum_{k=1}^{M_i} \left( \sum_{i=0}^{k-1} u_{ix}^j (-D_x)^{k-\ell(i+1)} \right) \frac{\partial f}{\partial u_{ix}^j} \right). \quad (17)
\end{align*}

Apply \( D_x^{-1} \) to both sides of (17) and use (9) to obtain
\begin{align*}
D_x^{-1} f &= \int_{\lambda_0}^{1} \left( \sum_{j=1}^{N} \sum_{k=1}^{M_i} \left( \sum_{i=0}^{k-1} u_{ix}^j (-D_x)^{k-\ell(i+1)} \right) \frac{\partial f}{\partial u_{ix}^j} \right) \lambda u \, d\lambda. \quad (18)
\end{align*}

The right hand side of (18) is identical to (9) with (1).

The homotopy operator [3] has been coded in Mathematica syntax and is part of the package HomotopyIntegrator.m [13]. In extensive testing [11], the expanded formula (3) out performed the implementation of the homotopy operator in [4, 10], dramatically reducing CPU time on complicated expressions. The code HomotopyOperator.m also integrates differential expressions that Mathematica’s Integrate function fails to integrate. For example, Integrate cannot integrate the (exact) expression
\[ f = u_x v_{2x} \cos u + v_{3x} \sin u - v_{4x}. \quad (19) \]

The HomotopyIntegrator.m code returns \( F = D_x^{-1} f = v_{2x} \sin u - v_{3x} \). It is easy to verify that \( D_x F = f \). Needless to say, one does not need the homotopy operator to integrate simple expressions like (19) for it can easily be done with pen on paper.
The homotopy operator is a tool for integrating long and more complicated differential functions, in particular, those where Mathematica’s Integrate function fails and integration by hand becomes intractable or prone to errors.

5. Multi-Dimensional Homotopy Operators

The 2D and 3D homotopy operators invert a divergence in 2D and 3D, respectively. Such inversions are considerably more difficult to do by hand. The 2D and 3D homotopy operators given in this paper were developed using the technique of proof in Theorem 4.9 Therefore, the homotopy operators presented below are different from the ones in [9, 10, 14, 18], where they were defined in terms of higher-order Euler operators. Like in the 1D case, the expanded versions of multidimensional homotopy operators require considerably less computations [11].

Definition 5.1: Let \( f(x, y, u^{(M)}(x, y)) \) be an exact differential function involving two independent variables \( x = (x, y) \). The 2D homotopy operator is a “vector” operator with two components,

\[
\left( \mathcal{H}_{u(x,y)}^{(x)} f, \mathcal{H}_{u(x,y)}^{(y)} f \right),
\]

where

\[
\mathcal{H}_{u(x,y)}^{(x)} f = \int_{\lambda_0}^{1} \left( \sum_{j=1}^{N} \mathcal{I}_{u(x,y)}^{(x)} f \right) \left[ \lambda u \right] \frac{d\lambda}{\lambda} \quad \text{and} \quad \mathcal{H}_{u(x,y)}^{(y)} f = \int_{\lambda_0}^{1} \left( \sum_{j=1}^{N} \mathcal{I}_{u(x,y)}^{(y)} f \right) \left[ \lambda u \right] \frac{d\lambda}{\lambda},
\]

The 2D homotopy operator is denoted as

\[
\mathcal{I}_{u(x,y)}^{(x)} f = \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} \left( \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} B(x) u_{i_1, i_2} \left( -D_x \right)^{k_1 - i_1 - 1} \left( -D_y \right)^{k_2 - i_2} \right) \frac{\partial f}{\partial u_{k_1 x k_2 y}},
\]

with combinatorial coefficient \( B(x) = B(i_1, i_2, k_1, k_2) \) defined as

\[
B(i_1, i_2, k_1, k_2) = \frac{(i_1 + i_2)(k_1 + k_2 - i_1 - i_2 - 1)}{(i_1 + i_2)(k_1 - i_1 - 1) k_1 k_2}.
\]

Similarly, the y-integrand, \( \mathcal{I}_{u(x,y)}^{(y)} f \), is defined as

\[
\mathcal{I}_{u(x,y)}^{(y)} f = \sum_{k_1=0}^{M_1} \sum_{k_2=1}^{M_2} \left( \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2-1} B(y) u_{i_1, i_2} \left( -D_x \right)^{k_1 - i_1} \left( -D_y \right)^{k_2 - i_2 - 1} \right) \frac{\partial f}{\partial u_{k_1 x k_2 y}},
\]

where \( B(y) = B(i_2, i_1, k_2, k_1) \).

Definition 5.2: Let \( f(x, u^{(M)}(x)) \) be a differential function of three independent variables where \( x = (x, y, z) \). The homotopy operator in 3D is a three-component (vector) operator,

\[
\left( \mathcal{H}_{u(x,y,z)}^{(x)} f, \mathcal{H}_{u(x,y,z)}^{(y)} f, \mathcal{H}_{u(x,y,z)}^{(z)} f \right),
\]
where the \( x \)-component is given by

\[
H_{u(x,y,z)}^{(x)} f = \int_{0}^{1} \left( \sum_{j=1}^{N} I_{u(x,y,z)}^{(x)}(j) \right) [\lambda u] \frac{d\lambda}{\lambda}.
\]

The \( y \)- and \( z \)-components are defined analogously. The \( x \)-integrand is given by

\[
I_{u(x,y,z)}^{(x)} f = \sum_{k_1=1}^{M_1} \sum_{k_2=0}^{M_2} \sum_{k_3=0}^{M_3} \sum_{i=0}^{k_1-1} \sum_{j=0}^{k_2-1} \sum_{k=0}^{k_3-1} \left( B^{(x)}(u) \right) u_{i_1 x i_2 y i_3 z}^{(x)} \frac{\partial f}{\partial u_{i_1 x i_2 y i_3 z}}.
\]

with combinatorial coefficient \( B^{(x)} = B(i_1, i_2, i_3, k_1, k_2, k_3) \) defined as

\[
B(i_1, i_2, i_3, k_1, k_2, k_3) = \frac{(i_1 + i_2 + i_3) \cdot (i_2 + i_3) \cdot (k_1 + k_2 + k_3 - i_1 - i_2 - i_3) \cdot (k_2 + k_3 - i_2)}{(k_1 + k_2 + k_3) \cdot (k_2 + k_3) \cdot (k_2 + k_3)}.
\]

As one might expect, the integrands \( I_{u(x,y,z)}^{(y)} f \) and \( I_{u(x,y,z)}^{(z)} f \) are defined analogously. Based on cyclic permutations, they have combinatorial coefficients \( B^{(y)} = B(i_2, i_3, i_1, k_1, k_2, k_3) \) and \( B^{(z)} = B(i_3, i_1, i_2, k_3, k_1, k_2) \), respectively.

The homotopy with \( \lambda_0 = 0 \) is used, except when singularities at \( \lambda = 0 \) occur.

**Theorem 5.3:** Let \( f = f(x, u^{(M)}(x)) \) be exact, i.e. \( f = \text{Div} \ F \) for some \( F(x, u^{(M-1)}(x)) \). Then, in the 2D case, \( F = \text{Div}^{-1} f = \left( H^{(x)}_{u(x,y)} f, H^{(y)}_{u(x,y)} f \right) \). Analogously, in 3D, \( F = \text{Div}^{-1} f = \left( H^{(x)}_{u(x,y,z)} f, H^{(y)}_{u(x,y,z)} f, H^{(z)}_{u(x,y,z)} f \right) \).

**Proof:** The very lengthy proof is similar to that of Theorem 4.9. It does not add much to the understanding of the theory. Nevertheless, details can be found in [11, Appendix A].

The following 2D example demonstrates how (20) works.

**Example 5.4** With \( u = (u, v) \) and \( x = (x, y) \), let

\[
f(x, u^{(3)}(x)) = u^2 x_2 y + 2 u x_2 y_2 + 3 u x v_x \cos v - 4 u_y v_x v_y - 2 u_2 y v_x^2
\]

\[
+ v_2 y \cos u + 3 u_2 x \sin v - u_y v_y \sin u.
\]

Using (21), compute

\[
I_{u(x,y)}^{(x)} = u \frac{\partial f}{\partial u_x} + (u_x I - u D_x) \frac{\partial f}{\partial u_{2x}} + \frac{1}{3} \left( u_{2y} I - u_y D_y + u D_2 y \right) \frac{\partial f}{\partial u_{x2y}}
\]

\[
= 3u^2 w_2 y + 3 u_x \sin v,
\]

\[
I_{v(x,y)}^{(x)} = v \frac{\partial f}{\partial v_x} + \frac{1}{2} (v_y I - v D_y) \frac{\partial f}{\partial v_{xy}}
\]

\[
= 3 u x v \cos v - 2 u_y v v_x - 2 u_y v_x v_y - 2 u_{2y} v v_x.
\]
Likewise, using (22), compute

\[
I^{(y)}_{u(x,y)} = u \frac{\partial f}{\partial u_y} + (u_y I - u D_y) \frac{\partial f}{\partial u_{2y}} + \frac{1}{3} (2u_{xy} I - u_y D_x - u_x D_y + 2u D_x D_y) \frac{\partial f}{\partial u_{2y}}
\]

\[
= -u v_y \sin u - 2u_y v_x^2,
\]

\[
I^{(y)}_{v(x,y)} = v \frac{\partial f}{\partial v_y} + (v_y I - v D_y) \frac{\partial f}{\partial v_{2y}} + \frac{1}{2} (v_x I - v D_x) \frac{\partial f}{\partial v_{xy}}
\]

\[
= 2u_y v v_{2x} - 2u_y v_x^2 + 2u_{xy} v v_x + v_y \cos u.
\]

Then, using (20), compute

\[
\mathcal{H}^{(x)}_{u(x,y)} f = \int_0^1 \left( \mathcal{I}^{(x)}_{u(x,y)} f + \mathcal{I}^{(x)}_{v(x,y)} f \right) [\lambda u] \frac{d\lambda}{\lambda}
\]

\[
= \int_0^1 \left( 3\lambda^2 u^2 v_{2y} + 3u_x \sin \lambda v + 3\lambda u_x v \cos \lambda v - 2\lambda^2 u v v_{xy} \right.
\]

\[
-2\lambda^2 u_y v_x v_y - 2\lambda^2 u_{2y} v v_x^2 \big) \, d\lambda
\]

\[
= u^2 v_{2y} + 3u_x \sin v - \frac{2}{3} u_y v v_{xy} - \frac{2}{3} u_y v_x v_y - \frac{2}{3} u_{2y} v v_x,
\]

\[
\mathcal{H}^{(y)}_{u(x,y)} f = \int_0^1 \left( \mathcal{I}^{(y)}_{u(x,y)} f + \mathcal{I}^{(y)}_{v(x,y)} f \right) [\lambda u] \frac{d\lambda}{\lambda}
\]

\[
= \int_0^1 \left( v_y \cos \lambda u - \lambda u v_y \sin \lambda u - 4\lambda^2 u_y v_x^2 + 2\lambda^2 u_y v v_{2x} + 2\lambda^2 u_{xy} v v_x \right) \, d\lambda
\]

\[
= \frac{2}{3} u_y v v_{2x} + \frac{2}{3} u_{xy} v v_x - \frac{4}{3} u y v_x^2 + v_y \cos u.
\]

Thus, the homotopy operator gives the vector

\[
F = \text{Div}^{-1} f = \left( u^2 v_{2y} + 3u_x \sin v - \frac{2}{3} u_y v v_{xy} - \frac{2}{3} u_y v_x v_y - \frac{2}{3} u_{2y} v v_x \right)
\]

\[
\left( \frac{2}{3} u_y v v_{2x} + \frac{2}{3} u_{xy} v v_x - \frac{4}{3} u y v_x^2 + v_y \cos u \right).
\] (24)

Obviously, there are infinitely many choices for \(F\) because the addition to \(F\) of a 2D “curl vector,” \(K = (D_y \theta - D_x \theta)\), where \(\theta\) is an arbitrary differential expression, will produce an identical divergence. Indeed, \(\text{Div} G = \text{Div} (F + K) = \text{Div} F + D_x (D_y \theta) - D_y (D_x \theta) = \text{Div} F = f\). The same happens in 3D where a curl vector, \(K = (D_y \eta - D_z \xi, D_z \theta - D_x \eta, D_x \xi - D_y \theta)\) involves three arbitrary differential functions \(\theta, \eta\) and \(\xi\). Often a concise result can be obtained for (24) by removing undesirable curl terms. This is the subject of the next section.

6. Removing Curl Terms

The homotopy operator acting on an exact multi-dimensional differential function returns an \(F\) which includes a curl vector. Although the homotopy operator consistently leads to a particular choice for the curl vector, \(F\) can be of unmanageable size due to the presence of dozens, if not hundreds, of unwanted terms. Removing the curl terms makes \(F\) shorter.

There are several methods for removing curl vectors. Doing so by hand requires educated guess work. Ideally, one could modify the integrand of the homotopy operator so that the least number of curl terms would be generated. We are currently
investigating this strategy. For now, we propose a simple, yet effective algorithm that uses only linear algebra.

To remove the curl terms from (24), first attach undetermined coefficients, \(k_1, \ldots, k_p\), to the terms of (24) and call the new vector \(\tilde{F}\). Doing so,

\[
\tilde{F} = \left( k_1 u^2 u_{2y} + k_2 u_x \sin v + k_3 u_y v v_{xy} + k_4 u_y v_{vx} v_y + k_5 u_{2y} v v_x \\
k_6 u_y v v_{2x} + k_7 u_{xy} v v_x + k_8 u_y v_{2x}^2 + k_9 v_y \cos u \right).
\] (25)

Next, compute

\[
\text{Div} \tilde{F} = 2k_1 u u_x u_y + k_1 u^2 u_{2y} + k_2 u_x v v_x \cos v + k_2 u_{2y} \sin v + (k_3 + k_4 + 2k_8) u_y v v_{xy} + (k_3 + k_6) u_y v v_{2xy} + (k_3 + k_7) u_{xy} v v_y \\
+ (k_4 + k_6) u_y v_{2x} v_y + (k_4 + k_7) u_{xy} v_x v_y + (k_5 + k_6) u_{2y} v v_{2x} + (k_5 + k_7) u_x v v_{2x} \\
+ (k_5 + k_8) u_{xy} v v_x + (k_5 + k_8) u_{2y} v_{2x}^2 + k_9 v_y \cos u - k_9 u_y v_y \sin u.
\] (26)

Since \(F\) in (24) and \(\tilde{F}\) in (25) should differ only by a curl vector, their divergences must be identical. Thus, \(\text{Div} \tilde{F} = \text{Div} F \equiv f\) in (23). Gathering like terms in (23) and (26) leads to the linear system

\[
k_1 = 1, \quad k_2 = 3, \quad k_3 + k_4 + 2k_8 = -4, \quad k_3 + k_6 = 0, \\
k_3 + k_7 = 0, \quad k_4 + k_6 = 0, \quad k_4 + k_7 = 0, \quad k_5 + k_6 = 0, \\
k_5 + k_7 = 0, \quad k_5 + k_8 = -2, \quad k_9 = 1.
\] (27)

To find the undetermined coefficients that will produce a \(\tilde{F}\) without curl terms, first solve for the \(k_i\) that appear only in equations with non-zero right-hand sides. In (27), clearly \(k_1 = 1, k_2 = 3,\) and \(k_9 = 1.\) The next variable to solve is \(k_8\) since it appears in equations with non-zero right hand sides. Coefficients \(k_3\) through \(k_7\) appear in equations with zero right-hand sides; these equations are solved last. Solving (27) in this order yields

\[
k_1 = 1, \quad k_2 = 3, \quad k_3 = -k_7, \quad k_4 = -k_7, \\
k_5 = -k_7, \quad k_6 = k_7, \quad k_8 = k_7 - 2, \quad k_9 = 1.
\] (28)

where \(k_7\) is arbitrary. Set \(k_7 = 0\) to eliminate as many terms as possible in (25). Substitute (28) with \(k_7 = 0\) into (25), to get

\[
\tilde{F} = \left( u^2 u_{2y} + 3u_x \sin v \\
u_y \cos u - 2u_y v_{x}^2 \right).
\] (29)

By construction, \(\text{Div} \tilde{F} = \text{Div} F = f\), but (29) is free of curl terms.

7. Extending the Applicability of the Homotopy Operator

It follows from (16), that the 1D homotopy operator in Definition 3.1, will not work when \(f(x, \mathbf{u}^{(M)}(x))\) (or for that matter any term of it) belongs to \(\text{Ker} M\). Indeed, if \(M f = 0\), in particular when \(f\) satisfies the conditions of Theorem 4.4, then \(\sum_{j=1}^N I_{w_j}(x) f = C\), where \(C\) is an arbitrary constant. Thus, the homotopy operator would return \(C\), in fact 0 for it ignores constants of integration.

The next two examples (in 1D) demonstrate a problem that occurs when some terms of \(f\) belong to \(\text{Ker} M\) while others do not.
Example 7.1 Let \( \mathbf{u} = (u^1, u^2) = (u, v) \) and \( F = \frac{u^2}{v} \). Then,

\[
f = D_x F = \frac{u_{2x} v - u_x v_x}{v^2} = \frac{u_{2x}}{v} - \frac{u_x v_x}{v^2}.
\]

Obviously, \( f \) is of degree zero. Hence, \( Mf = 0 \) by Theorem 4.4. Therefore, trying to integrate \( f \) with the homotopy operator will fail. Indeed, using (4) for \( u(x) \) and \( v(x) \), respectively, yields

\[
I_{u(x)} f = u \frac{\partial f}{\partial u_x} + (u_x I - u D_x) \frac{\partial f}{\partial u_{2x}} = -u \frac{v_x}{v^2} + u \frac{v_x}{v^2} + \frac{u_x v_x}{v} = \frac{u_x}{v},
\]

\[
I_{v(x)} f = v \frac{\partial f}{\partial v_x} = -v \frac{u_x}{v^2} = -\frac{u_x}{v}.
\]

Clearly, \( I_{u(x)} f + I_{v(x)} f = 0 \) and (3) gives 0 instead of \( F \).

Furthermore, if \( f \) is the sum of monomial, rational, or irrational terms (and fractions thereof) where any of these terms are in \( \text{Ker} \ M \), then the homotopy operator will return an incorrect result. Indeed, the homotopy operator would correctly integrate the terms not in \( \text{Ker} \ M \), but annihilate the terms in \( \text{Ker} \ M \). The following example highlights this situation.

Example 7.2 Let \( \mathbf{u} = (u^1, u^2) = (u, v) \) and \( F = \frac{u^2 + v}{u - v} \). Then,

\[
f = D_x F = \frac{u^2 u_x + u^2 v_x - 2 u u_x v + u v_x - u x v}{(u - v)^2}
\]

\[
= \frac{u^2 u_x}{(u - v)^2} + \frac{u^2 v_x}{(u - v)^2} - 2 \frac{u u_x v}{(u - v)^2} + \frac{u v_x}{(u - v)^2} - \frac{u x v}{(u - v)^2}
\]

is exact by construction. Applying the degree operator to \( f \), term by term, gives

\[
Mf = \frac{u^2 u_x}{(u - v)^2} + \frac{u^2 v_x}{(u - v)^2} - 2 \frac{u u_x v}{(u - v)^2} + 0 + 0.
\]

Because the last two terms of (30) are annihilated by \( M \), the homotopy operator will incorrectly integrate \( f \). One would obtain \( F = \frac{u^2 + v}{u - v} \) instead of \( F = \frac{u^2 + v}{u - v} \).

A method to overcome these shortcomings will now be proposed. A coordinate for the “jet” space where \( f(\mathbf{x}, \mathbf{u}^{(M)}(\mathbf{x})) \) resides is given by

\[
\mathbf{u}^{(M)}(\mathbf{x}) = (u^1, u^1_x, u^1_y, u^1_z, u^2, u^2_x, u^2_y, u^2_z, \ldots, u^1, u^2, \ldots, u^N, u^N_x, u^N_y, u^N_z),
\]

which has origin \( 0 = (0, \ldots, 0) \). By shifting the coordinate away from 0, \( f \) can be taken out of \( \text{Ker} \ M \). It suffices to shift one of the variables that appear in the denominator of the term in \( \text{Ker} \ M \). After integrating, the shift must be undone to put the integral at the origin. The following examples illustrate the procedure.

Example 7.3 Returning to Example 7.1 where \( f(\mathbf{x}, \mathbf{u}^{(1)}(\mathbf{x})) = \frac{u_{2x} v - u_x v_x}{v^2} \). Since \( v \) appears in the denominator, replace \( v \) with \( v - v_0 \) to get \( f_0 = \frac{u_{2x} (v - v_0) - u_x v_x}{(v - v_0)^2} \). Then use the homotopy operator to integrate \( f_0 \). Using (4), first compute

\[
\mathcal{I}_{u(x)} f_0 = \frac{u_x}{v - v_0} \quad \text{and} \quad \mathcal{I}_{v(x)} f_0 = -\frac{u_x v}{(v - v_0)^2}.
\]
Next, using (3), compute
\[ \mathcal{H}_{u(x)} f_0 = \int_0^1 \left( \frac{u_x}{v - v_0} - \frac{u_x v}{(v - v_0)^2} \right) \frac{d\lambda}{\lambda} = -\int_0^1 \frac{u_x v_0}{(\lambda v - v_0)^2} d\lambda \]
\[ = \frac{u_x v_0}{v(\lambda v - v_0)} \bigg|_0^1 = \frac{u_x}{v - v_0}. \]

Finally, remove the shift by replacing \( v \) with \( v + v_0 \) (or simply set \( v_0 = 0 \)) to get
\[ F = \mathcal{H}_{u(x)} f = \frac{u_x}{v}. \]

**Example 7.4** To integrate \( f \) from Example 7.2, the shift only needs to be applied to the last two terms of (30) because these are in \( \text{Ker} \mathcal{M} \). Therefore, we will apply the homotopy operator to \( f_0 = u^2 (u - 2v)/(u - v)^2 + uv/(u - u_0 - v)/(u - v)^2 - \frac{u_x v}{(u - u_0 - v)^2}, \)

where \( u \) has been replaced by \( u - u_0 \) in the last two terms. Note that it suffices to shift just one of the two variables in the denominator. Keeping the number of shifts to a minimum leads to a simpler integrand for the integration over \( \lambda \). The integrands for the homotopy operator are
\[ I_{u(x)} f_0 = \frac{u^2(u - 2v)}{(u - v)^2} - \frac{uv}{(u - u_0 - v)^2} \quad \text{and} \quad I_{v(x)} f_0 = \frac{u^2 v}{(u - v)^2} + \frac{v(u - u_0)}{(u - u_0 - v)^2}. \]

The first terms in each integrand have not been shifted. However, the second terms have been shifted. Applying (3),
\[ \mathcal{H}_{u(x)} f_0 = \int_0^1 \left( \frac{u^2}{u - v} - \frac{u_0 v}{(u - u_0 - \lambda v)^2} \right) d\lambda \]
\[ = \left( \frac{\lambda u^2}{u - v} + \frac{u_0 v}{(u - v)(\lambda u - u_0 - \lambda v)} \right) \bigg|_0^1 \]
\[ = \frac{u^2}{u - v} + \left( \frac{u_0 v}{(u - v)(u - u_0 - v)} + \frac{v}{u - v} \right) \]
\[ = \frac{u^2}{u - v} + \frac{v}{u - u_0 - v}. \]

Finally, setting \( u_0 = 0 \) yields the desired result, \( F = \frac{u^2 + v}{u - v}. \)

Similar difficulties might occur with functions in fractional form involving multiple independent variables. Again, a shift of a coordinate in the denominator will solve the problem. However, the complexity of the integration over \( \lambda \) increases rapidly when there are multiple independent variables (with shifts). Implementation of the 2D and 3D homotopy operators (in Definitions 5.1 and 5.2) requires the same amount of care. A discussion of the kernel of the homotopy operator, including an alternate strategy for dealing with the integration of homogeneous expressions can be found in [15].
8. Application: Computation of Conservation Laws of Nonlinear PDEs

This Section covers the application of the homotopy operator to the computation of conservation laws of nonlinear PDEs. We developed a Mathematica package, ConservationLawsMD.m\[^{[20]}\], that automates these computations for nonlinear polynomial PDEs with a maximum of three space variables in addition to time.

**Definition 8.1:** A conservation law for a given PDE with independent variables \(t\) and \(x\) is defined as

\[
D_t \rho + \text{Div } J = 0, \tag{31}
\]

where \(\rho = \rho(t, x, u^{(M)}(t, x))\) is the conserved density, \(J = J(t, x, u^{(P)}(t, x))\) is the associated flux, \(D_t\) is the total derivative with respect to \(t\) and \(\text{Div}\) is the total divergence with respect to \(x\). (31) is satisfied on solutions of the given PDE\[^{[6]}\].

A conservation law is found by first computing the density, \(\rho\), followed by the computation of the flux \(J\). The latter will require the use of the homotopy operator. Following the approach by Hereman et al.\[^{[8, 9, 10]}\], a candidate density is built as a linear combination (with undetermined coefficients) of differential terms which are invariant under the scaling symmetry of the given PDE. Once the form of \(\rho\) is determined, one computes \(D_t \rho\) and removes all time-derivatives using the PDE. By (31), \(D_t \rho\) must be a divergence. Thus, using Theorem 2.4, one requires that

\[
\mathcal{L}_{w_i(x)}(D_t \rho) \equiv 0, \quad j = 1, \ldots, N,
\]

where \(\mathcal{L}_{w_i(x)}\) is the Euler operator. This leads to a linear system for the undetermined coefficients. Substituting its solution into the candidate for \(\rho\) gives the actual density. Finally, the flux \(J = -\text{Div}^{-1}(D_t \rho)\) is computed with the homotopy operator.

To illustrate the method, we compute a conservation law for the \((2 + 1)\)-dimensional Zakharov-Kuznetsov (ZK) equation\[^{[19]}\] which describes ion-acoustic solitons in magnetized plasmas. After re-scaling, the PDE takes the form

\[
u_t + \alpha uu_x + \beta(\Delta u)_x = 0, \tag{32}
\]

where \(\alpha\) and \(\beta\) are parameters and \(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\) is the Laplacian.

8.1. Computation of the Density

It is easy to verify that (32) is invariant under the scaling (dilation) symmetry,

\[(t, x, y, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^{-1}y, \lambda^2u). \tag{33}\]

This scaling symmetry can be computed as follows. Assume that (32) scales uniformly under

\[(t, x, y, u) \rightarrow (T, X, Y, U) = (\lambda^a t, \lambda^b x, \lambda^c y, \lambda^d u), \tag{34}\]

with undetermined (rational) exponents \(a, b, c,\) and \(d\). Applying the chain rule to (32) yields

\[
\lambda^{a-d} U_T + \alpha \lambda^{b-2d} U U_X + \beta \lambda^{3b-d} U_3X + \beta \lambda^{b+2c-d} U_{X2Y} = \lambda^{a-d}(U_T + \alpha \lambda^{b-d-a} U U_X + \beta \lambda^{3b-a} U_3X + \beta \lambda^{b+2c-a} U_{X2Y}) = 0. \tag{35}
\]
One obtains (32) in the new variables \((T, X, Y, U)\), up to the common factor \(\lambda^{n-d}\), if \(b - d - a = 3b - a = b + 2c - a = 0\). Setting \(b = -1\), one gets \(a = -3, c = -1,\) and \(d = 2\), which determines (33). A more algorithmic method for computing scaling symmetries can be found in [8, 9, 10].

The scaling symmetry carries over to conservation laws and therefore can be used to construct densities \((\rho, J)\). Indeed, the candidate density may be trivial or a linear combination of two or more of the terms is a divergence. For example, \(u_1^{(1)} = u, J^{(1)} = \left(\frac{1}{2}\alpha u^2 + \beta u_{2x} + \beta u_{xy}\right)\), expressing conservation of mass. It is straightforward to verify that

\[
\rho^{(2)} = u^2, \quad J^{(2)} = \left(\frac{2}{3}\alpha u^3 - \beta (u_x^2 - u_y^2) + 2\beta u (u_{2x} + u_{2y}) - 2\beta u_x u_y\right).
\]

also are a density-flux pair of (32); the corresponding conservation law expresses conservation of momentum. Note that the densities in (36) and (37) scale with \(\lambda^2\) and \(\lambda^4\), respectively. For brevity, we say that \(\rho^{(1)}\) has rank 2; \(\rho^{(2)}\) has rank 4. The fluxes \(J^{(1)}\) and \(J^{(2)}\) have ranks 4 and 6. As a whole, the conservation laws (31) with \((\rho^{(1)}, J^{(1)})\) and \((\rho^{(2)}, J^{(2)})\) have ranks 5 and 7, respectively.

We will construct a third density, \(\rho^{(3)}\), which has rank 6, i.e. each monomial in \(\rho^{(3)}\) depends on \(u\) and its derivatives so that every term scales with \(\lambda^6\). First, construct the list \(\mathcal{P} = \{u^3, u^2, u\}\) containing powers of the dependent variable of rank 6 or less. Second, bring all terms in \(\mathcal{P}\) with a rank less that 6 up to rank 6 by applying \(D_x\) and \(D_y\). Considering all possible combinations of derivations yields

\[
\mathcal{Q} = \{u^3, u_x^2, uu_{2x}, u_y^2, uu_{2y}, u_x u_y, uu_{xy}, u_{4x}, u_{3xy}, u_{2x2y}, u_{x3y}, u_{4y}\}.
\]

Third, remove all terms from \(\mathcal{Q}\) that are divergences (because they belong to the flux). For example, \(u_{4x} = \text{Div} (u_{3x}, 0)\). This leaves

\[
\mathcal{Q} = \{u^3, u_x^2, uu_{2x}, u_y^2, uu_{2y}, u_x u_y, uu_{xy}\}.
\]

Fourth, find terms that are divergence-equivalent and remove all but the lowest order term. Two or more terms are divergence-equivalent when a linear combination of the terms is a divergence. For example, \(u_x^2\) and \(uu_{2x}\) are divergence-equivalent since \(u_x^2 + uu_{2x} = \text{Div} (u_{2x}, 0)\). Consequently, \(uu_{2x}\) can be removed from \(\mathcal{Q}\). Doing so, \(\mathcal{Q} = \{u^3, u_x^2, u_y^2, uu_{xy}\}\). If divergences and divergence-equivalent terms were not removed, they would lead to trivial and equivalent conservation laws, respectively.

Now, form a candidate density by linearly combining the terms in \(\mathcal{Q}\) with undetermined coefficients \(c_i\),

\[
\rho^{(3)} = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y.
\]

All, part, or none of the candidate density may be an actual density for (32). Indeed, the candidate density may be trivial or a linear combination of two or more independent densities. Computation of the undetermined coefficients will reveal the nature of the candidate density.

With (38), compute

\[
D_t \rho^{(3)} = 3c_1 u^2 u_t + 2c_2 u_x u_{tx} + 2c_3 u_y u_{ty} + c_4 (u_y u_{tx} + u_x u_{ty}).
\]
Using (32), replace \( u_t \) with \(- (\alpha u u_x + \beta u_{3x} + \beta u_{2x} y)\). Set \( E = - D_t p^{(3)} \), to get

\[
E = 3c_1 u^2 (\alpha u u_x + \beta u_{3x} + \beta u_{2x} y) + 2c_2 u_x (\alpha u u_x + \beta u_{3x} + \beta u_{2x} y)_x \\
+ 2c_3 u_y (\alpha u u_x + \beta u_{3x} + \beta u_{2x} y)_y + c_1 u_y (\alpha u u_x + \beta u_{3x} + \beta u_{2x} y)_x \\
+ c_4 u_x (\alpha u u_x + \beta u_{3x} + \beta u_{2x} y)_y.
\] (40)

Since (40) must be a divergence, use (2) and require

\[
\mathcal{L}_{u(x,y)} E = -2(3c_1 \beta + c_3 \alpha) u_x u_{2y} + 2(3c_1 \beta + c_3 \alpha) u_y u_{xy} + 3(3c_1 \beta + c_2 \alpha) u_x u_{2x} \\
+ 2c_4 \alpha u_x u_{xy} + c_4 \alpha u_y u_{2x} = 0.
\] (41)

It follows from (41) that

\[
3c_1 \beta + c_2 \alpha = 0, \quad 3c_1 \beta + c_3 \alpha = 0, \quad \alpha c_4 = 0.
\] (42)

Before solving the system it is necessary to check for potential compatibility conditions on the parameters \( \alpha \) and \( \beta \). This is done by setting each \( c_i = 1 \), one at a time, and algebraically eliminating the other undetermined coefficients. See, e.g., [11, 22] for details about computing compatibility conditions. It turns out that there are no compatibility conditions for (42) and the solution is

\[
c_2 = -\frac{3\beta}{\alpha} c_1, \quad c_3 = -\frac{3\beta}{\alpha} c_1, \quad c_4 = 0,
\] (43)

where \( c_1 \) is arbitrary. Substitute (43) into (38) and set \( c_1 = 1 \) to get

\[
\rho^{(3)} = u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2).
\] (44)

### 8.2. Computation of the Flux

After substitution of (43) and \( c_1 = 1 \) into (40),

\[
E = 3u^2 (\alpha u u_x + \beta u_{3x} + \beta u_{2x} y) - \frac{6\beta}{\alpha} u_x (\alpha u u_x + \beta u_{3x} + \beta u_{2x} y)_x \\
- \frac{6\beta}{\alpha} u_y (\alpha u u_x + \beta u_{3x} + \beta u_{2x} y)_y.
\]

Since \( E = \text{Div} J^{(3)} \), the flux \( J^{(3)} \) can be computed with the 2D homotopy operator which inverts divergences. Using (20), the integrands are

\[
I^{(x)}_{u(x,y)} = 3 \alpha u^4 + \beta \left( 9 u^2 (u_{2x} + \frac{2}{5} u_{2y}) - 6 u (3 u_x^2 + u_y^2) \right) + \frac{\beta^2}{\alpha} \left( 6 u_{2x}^2 + 5 u_{xy}^2 + \frac{3}{2} u_{2y}^2 \right) \\
+ \frac{3}{2} u (u_{4y} + u_{2x} u_{2y}) - u_x (12 u_{3x} + 7 u_{2y}) - u_y (3 u_{3y} + 8 u_{2x} y) + \frac{5}{2} u_{2x} u_{2y},
\]

and

\[
I^{(y)}_{u(x,y)} = 3 \beta u (u u_{xy} - 4 u_x u_y) - \frac{\beta^2}{\alpha} (3 u (u_{3xy} + u_{x3y}) + u_x (13 u_{2xy} + 3 u_{3y}) \\
+ 5 u_y (u_{3x} + 3 u_{2x} y) - 9 u_{xy} (u_{2x} + u_{2y})).
\]
The 2D homotopy operator formulas in (20) yield

\[ H_{u(x,y)}^{(x)} E = \int_0^1 \left( \mathcal{T}_{u(x,y),E}^{(x)} \right) [\lambda u] \frac{d\lambda}{\lambda} \]

\[ = \int_0^1 (3\alpha \lambda^2 u^4 + \beta \lambda^2 (9u^2(u_{2x} + \frac{2}{3}u_{2y}) - 6u(3u_x^2 + u_y^2)) + \frac{\beta^2}{\alpha} \lambda \left( 6u_{2x}^2 + 5u_{xy}^2 + \frac{3}{2}u_{2y}^2 + \frac{3}{2}u(u_{2x} + u_y) - u_x(12u_{3x} + 7u_{x2y}) \right) - u_y(3u_{3y} + 8u_{2xy}) + \frac{2}{3}u_{2x}u_{2y}) \) d\lambda \]

\[ = \frac{3}{\alpha} \alpha u + \beta (3u^2(u_{2x} + \frac{2}{3}u_{2y}) - 2u(3u_x^2 + u_y^2)) + \frac{\beta^2}{\alpha} \lambda \left( 3u_{2x}^2 + \frac{3}{2}u_{2y}^2 + \frac{3}{4}u(u_{2x} + u_y) - u_x(6u_{3x} + \frac{7}{2}u_{x2y}) \right) - u_y(3u_{3y} + 4u_{2xy}) + \frac{2}{3}u_{2x}u_{2y}, \]

\[ H_{u(x,y)}^{(y)} E = \int_0^1 \left( \mathcal{T}_{u(x,y),E}^{(y)} \right) [\lambda u] \frac{d\lambda}{\lambda} \]

\[ = \int_0^1 \left( 3\lambda^2 \beta u(u_{ux} - 4u_{xu}) - \frac{\beta^2}{\alpha} \lambda (3u(u_{3xy} + u_{x3y}) + u_x(3u_{3y} + 13u_{2xy})) + u_y(5u_{3x} + 15u_{x2y}) - \frac{9}{2}u_{2y}(u_{2x} + u_{xy})) \) d\lambda \]

\[ = \beta u(u_{ux} - 4u_{xu}) - \frac{\beta^2}{\alpha} \lambda (3u(u_{3xy} + u_{x3y}) + u_x(13u_{2xy} + 3u_{3y})) + 5u_y(u_{3x} + 3u_{x2y}) - 9u_{xy}(u_{2x} + u_{2y}). \]

The flux \( J = (H_{u(x,y)}^{(x)} E, H_{u(x,y)}^{(y)} E) \) has a curl term, \( K = (D_y\theta, -D_x\theta), \) with

\[ \theta = 2\beta u^2 u_y + \frac{\beta^2}{\alpha} \left( 3u(u_{2xy} + u_{3y}) + 5(2u_x u_{xy} + 3u_y u_{2y} + u_{2xy}) \right). \]

After removing \( K \) with the technique in Section 6, the density-flux pair reads

\[ \rho^{(3)} = u^3 - \frac{3\beta}{\alpha}(u_{2x}^2 + u_{2y}^2), \]

\[ J^{(3)} = \left( \frac{3\alpha}{2}u^4 + 3\beta u^2 u_{2x} - 6\beta u(u_{2x}^2 + u_{2y}^2) + \frac{3\beta^2}{\alpha}(u_{2x}^2 - u_{2y}^2) - \frac{6\beta^2}{\alpha}(u_x(u_{3x} + u_{x2y}) + u_y(u_{2xy} + u_{3y})) \right), 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{2x} + u_{2y}). \]

(45)

There is one additional density-flux pair [11] with explicit dependence on \( x \) and \( t \),

\[ \rho^{(4)} = tu^2 - \frac{2}{\alpha} x u, \]

\[ J^{(4)} = \left( t \frac{3\alpha}{2} u^3 - \beta(u_{x}^2 - u_y^2) + 2\beta u(u_{2x} + u_{2y})) - \frac{2}{\alpha} x(u_{2x}^2 + \beta u_{2x}) + \frac{2\beta}{\alpha} u_x, \right. \]

\[ - 2\beta(tu_x u_y + \frac{1}{\alpha} x u_{xy}) \), \]

(46)

which expresses the conservation of center of mass. No other density-flux pairs than the four reported above could be found [11] with ConservationLawsMD.m. Zakharov and Kuznetsov [19] found three “integrals of motion” which are identical to (36), (37), and (46). Without mention of fluxes, Infeld [23] reports the densities in (37) and (45) and states that another constant of motion exists, but only for the ZK equation in 3D. Shivamoggi et al. [24] claimed that there are only four conservation laws for the ZK equation, but gave different results. Shivamoggi computed
conservation laws for the potential ZK equation, \(v_t + \frac{1}{2}v_x^2 + v_{3x} + v_{x2y} = 0\), derived from (32) with \(\alpha = \beta = 1\) by setting \(u = v_x\) followed by an integration with respect to \(x\). Doing so, he produced four nonlocal conservation laws for (32). The densities for the potential ZK equation reported in [24] are correct but the fluxes for three of the four conservation laws are incorrect due to typographical errors [25].

In summary, the existence of only four conservation laws confirms that (32) is not completely integrable.

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