Five measurement bases determine pure quantum states on any dimension

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A long standing problem in quantum mechanics is the minimum number of projective measurements required for the characterisation of unknown pure quantum states. The solution to this problem will be crucial for the developing field of high-dimensional quantum information processing. In this work we demonstrate that any pure state is unambiguously reconstructed by projective measurements onto five orthonormal bases for any $d$-dimensional Hilbert space. Thus, in our method the number of measurements ($5d$) scales linearly with $d$. The reconstruction is robust against experimental errors and requires simple post-processing, regardless of $d$. We experimentally demonstrate the feasibility of our scheme through the reconstruction of 8-dimensional quantum states, encoded in the transverse momentum of single photons, obtaining fidelities greater than 0.96 ± 0.03.

With the development of high-dimensional quantum information processing techniques [1–4], the total dimension $d$ of quantum systems employed in experiments increases at a fast pace. Since the number of projective measurements employed by conventional quantum tomography methods scales with $d^4$ [5,17], it is of paramount importance the search for tomographic protocols specially adapted to higher dimensions. That is, schemes which require a lower number of measurements and a reduced complexity of the post-processing methods. The current approach to meet these two requirements consists of the use of a priori information about the set of states to be characterised. For instance, rank-$r$ quantum states can be reconstructed with high probability, from $O(rd\log d)$ projective measurements via compressed sensing techniques [18]. Nearly matrix product states can also be determined with a linear number of projective measurements with post-processing that is polynomial in the system size [19], and permutationally invariant quantum states of $n$ qubits can be reconstructed with $n+2$ mutually unbiased bases [20].

In this work we study the theoretical and experimental characterisation of unknown pure quantum states. In 1933 W. Pauli [21] considered the unambiguous characterisation of pure states from probability distributions generated by the measurement of a fixed set of observables. It has been shown that the number of observables must be larger than 3 for $d \geq 9$ [22]. Here, we show first that a set of probability distributions obtained by projective measurements onto a fixed set of 4 bases is informationally complete on the pure states up to a null measure subset of the Hilbert space. This number of bases is independent of the dimension of the underlying quantum system. The total number $4d$ of projections is an improvement compared with quantum tomography of pure states based on mutually unbiased bases [9], symmetric informationally complete positive-operator valued measure [10,11] and compressed sensing [18], which require $d(d+1)/2$, $d^2$ and $O(d\log^2 d)$ projective measurements respectively. Pure states can also be reconstructed via the expectation values of a fixed set of observables. In this case $4d-5$ observables are required and, consequently, the number of rank-1 projective measurements needed to obtain the expectation values is larger than in our proposal.

We also show that it is possible to achieve full informational completeness, that is, the unambiguous reconstruction of any pure state, with an additional basis, increasing the total to 5. This number is also independent of the dimension of the Hilbert space. In this case, the Hilbert space is decomposed into a finite number of disjoint sets of states such that each one of them is reconstructed with a particular set of five bases. These sets have in common the canonical basis whose measurement detects the particular family of states and allows us to construct the remaining 4 bases. Nevertheless, almost all pure states are reconstructed with a single family of 4 bases together with the canonical one. The remaining sets of 5 bases are necessary to reconstruct pure states in the null measure set. An interesting feature of the sets of 5 bases is that they allow us to certify the initial assumption on the purity of the state to be reconstructed. This class of certification can also be achieved by reconstructing pure states via expectation values [24]. However, here the number of observables is $5d-7$, which involves the realisation of a larger set of projective measurements than in our scheme based on 5 bases.

In order to demonstrate our tomographic scheme, we performed the experimental characterisation of eight-dimensional quantum states encoded on the transverse momentum of single-photons. In this case only 40 measurements are needed, and the obtained fidelities are higher than 0.96 ± 0.03. Our work shows that effective characterisation of unknown high-dimensional pure quantum states is feasible.

Quantum states of a $d$-dimensional system can be ex-
panded as linear combinations of the $d^2 - 1$ generators $T_j$ of the $su(d)$ algebra and the identity operator $I$, that is,

$$\rho = \frac{1}{d} I + \sqrt{\frac{d-1}{2d}} \sum_{j=1}^{d^2-1} r_j T_j,$$

(1)

where coefficients $r_j = Tr(\rho T_j)$ form the generalised Bloch vector. We consider a traceless, hermitian representation of the set of generators $\{T_j\}$ such as the generalised Gell-Mann basis $\tilde{B}$. This consists of $d-1$ diagonal operators $T_0$ and $d^2 - d$ non-diagonal operators $T_{k,m}$ and $\tilde{T}_{k,m}$. The latter are given by

$$T_{k,m} = |k\rangle\langle m| + |m\rangle\langle k|, \quad \tilde{T}_{k,m} = -i|k\rangle\langle m| + i|m\rangle\langle k|,$$

(2)

where $0 \leq k < m \leq d - 1$.

All pure states $|\Psi\rangle = \sum_{k=0}^{d-1} c_k |k\rangle$ satisfy the following set of $d(d-1)/2$ equations

$$2c_m c_k^* = Tr(|\Psi\rangle\langle \Psi| T_{m,k}) + i Tr(|\Psi\rangle\langle \Psi| \tilde{T}_{m,k}).$$

(3)

Interestingly, $d-1$ of the above equations characterise a certain set of pure states. We choose the set of equations with $m = k + 1$. This allows us to solve Eq. (3) recursively. In order to calculate the traces entering into Eq. (3) we consider $4d$ rank-one projective measurements associated to the eigenvectors of $T_{k,m}$ and $\tilde{T}_{k,m}$ with eigenvalues $\pm 1$. These $4d$ vectors can be sorted in four orthogonal bases for every dimension $d$. For $d \geq 3$ these bases are given by

$$B_1 = \{2|\nu\rangle \pm |2\nu+1\rangle\}_{\nu \in [0,(d-2)/2]},$$

$$B_2 = \{2|\nu\rangle \pm i|2\nu+1\rangle\}_{\nu \in [0,(d-2)/2]},$$

$$B_3 = \{2|\nu+1\rangle \pm |2\nu+2\rangle\}_{\nu \in [0,(d-2)/2]},$$

$$B_4 = \{2|\nu+1\rangle \pm i|2\nu+2\rangle\}_{\nu \in [0,(d-2)/2]},$$

(4)

where addition of labels is carried out modulo $d$. In the case of odd dimensions we consider the integer part of $(d-2)/2$ and every basis is completed with $|d\rangle$. Defining $p^+_{k,l}$ ($p^-_{k,l}$) as the probability of projecting the state $|\Psi\rangle$ onto the eigenvector of $T_{k,k+1}$ ($\tilde{T}_{k,k+1}$) with eigenvalue $\pm 1$, it is then possible to cast Eq. (3) as

$$2c_k c_{k+1}^* = \Lambda_k,$$

(5)

where $\Lambda_k = \sqrt{(d-1)/2d} [p^+_{k,l} - p^-_{k,l}] + i(p^+_{k,l} - p^-_{k,l})].$ Table I associates bases $B_k$ with the eigenvectors of $T_{k,k+1}$ and $\tilde{T}_{k,k+1}$ with eigenvalues $\pm 1$ and probabilities $p^+_{k,l}$ and $p^-_{k,l}$, respectively.

From Eq. (5) we can deduce an analytical expression for the probability amplitudes $c_k$ of the quantum state $|\Psi\rangle$ as functions of the probability distributions obtained by projecting onto the four bases $B_k$, that is,

$$c_k = \begin{cases} \frac{\Lambda}{2\sqrt{d}} & k = 1, \\ \cos \sum_{l=0}^{k/2-1} \frac{\Lambda_{l+1}}{\Lambda_{2l+2}} & k > 0 \text{ even}, \\ \frac{\Lambda}{2\sqrt{d}} \prod_{l=0}^{(k-3)/2} \frac{\Lambda_{l+2}}{\Lambda_{2l+2}} & k > 1 \text{ odd}, \end{cases}$$

(6)

| $B_k$ | $T_{k,k+1}$, $k$ even | $p^+_{k,l}$, $k$ even | $p^-_{k,l}$, $k$ even |
|---|---|---|---|
| $B_1$ | $T_{k,k+1}$, $k$ odd | $p^+_{k,l}$, $k$ odd | $p^-_{k,l}$, $k$ odd |

TABLE I. Association between $su(d)$ generators and transition probabilities.

where the coefficient $c_0$ is determined by the normalisation condition. If one of the coefficients $c_k$ vanishes then the system of equations cannot be recursively solved. However, the remaining equations and the normalisation condition are enough to reconstruct the state. This also holds in the case of two consecutive vanishing coefficients, that is, $c_l = c_{l+1} = 0$. In case of two non-consecutive vanishing coefficients the system of equations has infinite solutions. For instance, in $d = 4$ the set of equations is $2c_0 c_1^* = \Lambda_0$, $2c_1 c_2^* = \Lambda_1$, $2c_2 c_3^* = \Lambda_2$ and $2c_3 c_4^* = \Lambda_3$. If $c_0 = c_2 = 0$ or $c_1 = c_3 = 0$ then all left sides vanish. Thus, there are states that cannot be singled out via the bases $B_k$. These form a manifold $\Omega$ of dimension $d - 2$ and, consequently, are statistically unlikely. This means that, if pure states are randomly selected then with probability 1 they would be out of $\Omega$. Thus, bases $B_k$ are informationally complete on the set of pure states up to the null measure set $\Omega$.

The introduction of a fifth basis $B_0$, the canonical basis which is first to be measured, allows us to determine whether a state belongs to the manifold $\Omega$ or not depending on the number of vanishing coefficients detected. If the state is in $\Omega$ and has $m$ null coefficients, then we can reconstruct the state into the subspace associated to the $d - m$ non-vanishing coefficients. This is done with the bases $B_k$ ($k = 1, 2, 3, 4$) but defined for a subspace of dimension $d - m$. Thereby, any pure state can be reconstructed via measurements onto 5 orthonormal bases.

Measurements onto the set of bases allow us to test whether the assumed purity of the unknown state holds or not. A state $\rho$ is pure if and only if the equation $|\rho_{k,l}|^2 = |\rho_{k,k}||\rho_{l,l}|$ holds for every $k,l = 0, \ldots, d - 1$. Remarkably, these conditions for $l = k + 1$ are enough to ensure that $\rho$ determines a pure state. Indeed, given that $\rho$ is a quantum state then $\rho = A A^\dagger$ for a given operator $A$. So, every entry of $\rho$ satisfies $\rho_{k,l} = v_k \cdot v_l$, where $v_k$ is the $k$th column of $A$. If $l = k + 1$ the $d - 1$ equations $|\rho_{k,k+1}|^2 = |\rho_{k,k}||\rho_{k+1,k+1}|$ hold if and only if vectors $v_k$ and $v_{k+1}$ are parallel for every $k = 1, \ldots, d-1$. Consequently, $\rho$ is pure. This holds for any set of five bases.

Tomographic schemes reconstruct quantum states in a matrix space. Since quantum states form a null measure set in matrix space, noisy measurement results lead to matrices that do not represent quantum states. To overcome this problem experimental data is post-processed with maximum likelihood estimation [12]. An important feature of our method is that it delivers a vector of the un-
deriving Hilbert space for any set of noisy probabilities, being the normalisation of this vector the only procedure required to obtain a pure state from noisy probabilities.

The scheme here proposed is based on a priori information about the purity of the state to be reconstructed. This condition is difficult to realise in current experiments. However, it is possible to generate nearly pure states such as

\[
\rho = (1 - \lambda)|\Psi\rangle\langle\Psi| + \frac{\lambda}{d} I.
\]

Here, the target pure state is $|\Psi\rangle$ but the generation process adds a noise given by the maximally mixed state. The strength of the mixing process is given by $\lambda$. This model is in agreement with our experimental setup. In order to test the fidelity of the reconstruction by our method we generated a pure state according to the Haar measure and as a function of the dimension.

The state preparation stage has a continuous-wave laser operated at 690 nm and an acousto-optic modulator (AOM). This is used to generate controlled optical pulses. Optical attenuators (not shown for sake of clarity) decrease the mean photon number per pulse to the single-photon level. Thereafter, single photons are sent through two transmissive spatial light modulators (SLM1 and SLM2), each one formed by two polarisers, quarter wave plates (QWP) and a liquid crystal display (LCD). SLM1 and SLM2 work in amplitude-only and phase-only modulation configuration, respectively. SLM2 is placed on the image plane of SLM1, and the 8-dimensional quantum system is generated with 8 parallel slits addressed on the SLMs. After SLM2 the state of the transmitted photon is given by $\Psi_t$.

\[
|\Psi_t\rangle = \frac{1}{\sqrt{N}} \sum_{l=-7/2}^{7/2} \sqrt{n_l} e^{i \phi_l} |l\rangle,
\]

where $|l\rangle$ represents the state of the single photon crossing the $l$th-slit. Here, $t_l$ is the transmission for each slit controlled by SLM1; $\phi_l$ is the phase of each slit addressed by SLM2, and $N$ is a normalisation constant. The different values of $t_l$ and $\phi_l$ are configured by the grey level of the pixels in the SLMs.

We addressed in the SLMs slits with the width of 2 pixels, and 1 pixel of separation between them, where each pixel is a square of 32 $\mu$m of side length.

The state of Eq. (8) represents an 8-dimensional quantum state encoded in the linear transverse momentum of a single photon transmitted by the SLMs. However, since a three-qubit system is a 8-dimensional quantum system, this state can be used to simulate a 3-qubit system. We generated three different 3-qubit states: $|\Psi_U\rangle = \frac{1}{\sqrt{8}}[1, 1, 1, 1, 1, 1, 1, 1]$, $|\Psi_{GHZ}\rangle = \frac{1}{2}[1, 0, 0, -i, 0, 1, 1, 0]$ and $|\Psi_W\rangle = \frac{1}{\sqrt{2}}[1, 0, 1, 0, 1, 0, 0, 0]$. The first state has equal real probability amplitudes and the second and third states are analogous to the GHZ state up to local transformations. These two states can be reconstructed with the same set of five bases. The third state is analogous to a W state up to local transformations and requires a different set of five bases.
To guarantee the purity of the spatial qudit states, it is necessary to observe a high visibility in the interference patterns in the far-field plane of the SLMs. The value of $\lambda$ can be obtained from the relation $V = \frac{4-4\lambda}{4-3\lambda}$, where $V$ is the observed visibility in the far-field plane. In our experiment the visibilities are equal to $0.99 \pm 0.009$ leading to values of lambda equal to $0.037 \pm 0.033$. Thus the generated states have a purity equal to $0.93 \pm 0.05$.

In order to realise the projections onto the states of the 5 bases $B_i$, we used two additional modulators, SLM3 and SLM4, working in amplitude-only and phase-only, respectively, and a point-like avalanche photo-detector (APD). The SLMs in the projective measurement stage (see Fig. 1) are addressed with slits whose amplitudes and phases are defined to implement the projections required by our method. At the detection plane, the single-photon detection rate is proportional to the probability of projecting the initial state ($\Psi_{\text{GHZ}}$, $\Psi_U$ and $\Psi_W$) into the required basis states of Eq. (4).

From the experimental data we calculated the probability distributions associated to the 5 bases $B_i$ for the three initial states. With these probability distributions and using Eq. (3), for the appropriate set of 5 bases, we obtained a set of vectors in the Hilbert space, which were then normalised to obtain the final reconstructed states. The fidelity $F_i = |\langle \Psi_i | \Psi_{\text{theo}} \rangle|$ of the initial states with respect to the expected ones are $F_{\text{GHZ}} = 0.985 \pm 0.015$, $F_U = 0.96 \pm 0.03$ and $F_W = 0.96 \pm 0.03$. These were calculated considering the effect of the poisson noise in the detection process on the measured probability distributions and selecting the highest and smallest fidelity between the expected state and the estimated state for a particular noisy set of distributions. For comparison purposes we consider an experiment with a similar configuration where two pure states with non vanishing, real coefficients in $d = 8$ were reconstructed by means of measurements on mutually unbiased bases achieving fidelities of $0.91 \pm 0.03$ and $0.92 \pm 0.03$. Note that we achieved higher fidelities with a total of 40 projective measurements instead of 72, as in the compared case. The states reconstructed with our method are shown in Fig. 3. Fig. 3(a), 3(c) and 3(e) exhibits the real parts of each reconstructed density matrix, compared with the expected ones (insets). Fig. 3(b), 3(d) and 3(f) show the imaginary parts of the respective matrix, compared with the theoretical predictions (insets).

We have shown that projections onto the eigenstates of a fixed set of 4 rank-$d$ observables characterise all pure states of a $d$-dimensional quantum system up to a null measure set. These $4d$ measurements compares favourably with the typical $d^2$ scaling of measurements of known generic tomographic methods. The addition of a fifth observable allows us to detect whether a state
belongs to the null measure set or not. If this is the case, then it is always possible to construct a new set of 4 observables which determine unambiguously the state. All sets of five observables allows us to certify whether the initial assumption on the purity of the state holds or not. We also experimentally demonstrated the feasibility of our scheme achieving high fidelities in the characterisation of states in $d = 8$. An interesting feature of our proposal is that experimental errors can be taken into account by a simple procedure without resorting to complex optimisation algorithms.

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