USING TWISTED ALEXANDER POLYNOMIALS TO DETECT FIBEREDNESS

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Abstract. In this paper we use twisted Alexander polynomials to prove that the exterior of a particular graph knot is not fibered. Then we build three 2-component graph links out of this knot, and use similar techniques to discuss their fiberedness.

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1. Introduction and Main Results

The main purpose of this paper is to find explicit applications of the relationship between twisted Alexander polynomials and fiberedness. For twisted Alexander polynomials, we will follow the definition given in [4]. In particular, we studied a graph knot $K$ that is included in a homology sphere $\Sigma$ (different from the 3-dimensional sphere $S^3$), and three 2-component links that have $K$ as one of their components. For graph links, D. Eisenbud and W. Neumann introduced splice diagrams and developed a method to use the combinatorial information included in splice diagrams to determine fiberedness and the Thurston norm, [2]. However, the technique they use only applies to graph links, whereas the method of this paper can theoretically be applied to any 3-manifold.

The following theorem of C. McMullen shows the ability of the (ordinary) Alexander polynomial to provide information on the Thurston norm and fiberedness for a general 3-manifold $N$. If $\phi = (m_1, \ldots, m_n) \in H^1(N; \mathbb{Z})$, then $\text{div}(\phi)$ is the greatest common divisor of $m_1, \ldots, m_n$.

Theorem 1.1. (McMullen, [9]) Let $N$ be a compact, connected, orientable 3-manifold whose boundary (if any) is a union of tori. Then for any $\phi \in H^1(N; \mathbb{Z})$

$$\deg(\Delta_{N,\phi}) \leq \|\phi\|_T + \begin{cases} 0, & b_1(N) \geq 2 \\ \text{div}(\phi) \cdot (1 + b_3(N)), & b_1(N) = 1 \end{cases}$$

Moreover, if $\phi$ is fibered, $\Delta_{N,\phi}$ is monic, and equality holds.

It is well-known that the converse of Theorem 1.1 is not true as we will show for the graph knot $K$, which has the splice diagram shown in Figure 1.
Proposition 1.2. The genus of the knot $K$ is 1, it has Alexander polynomial equal to $t^2 - t + 1$, and it is not fibered.

S. Friedl and T. Kim have generalized the result in Theorem 1.1 by considering the collection of twisted Alexander polynomials in the following theorem.

Theorem 1.3. (Friedl-Kim, [4]) Let $N$ be a 3-manifold different from $S^1 \times D^2$ and $S^1 \times S^2$. Let $\phi \in H^1(N; \mathbb{Z})$ be such that $(N, \phi)$ fibers over $S^1$. Then for every representation $\alpha : \pi_1(N) \to \text{Gl}_k(\mathbb{Z})$,

$$\Delta_{\alpha,\phi}^N \text{ is monic and } \text{deg}(\Delta_{\alpha,\phi}^N) = k||\phi||_T + \text{deg}(\Delta_{N,\phi,0}) + \text{deg}(\Delta_{N,\phi,2}). \quad (1.1)$$

Here, $\Delta_{N,\phi,0}$ and $\Delta_{N,\phi,2}$ are determined by the Alexander modules $H_0(N; \mathbb{Z}[F])$ and $H_2(N; \mathbb{Z}[F])$.

Theorem 1.3 leads one to believe that the collection of twisted Alexander polynomials gives stronger obstructions to fiberedness. This, in fact is confirmed by the following theorem of S. Friedl and S. Vidussi.

Theorem 1.4. (Friedl-Vidussi, [6]) Let $N$ be a compact, connected, orientable 3-manifold whose boundary (if any) is a union of tori. Let $\phi$ be non-trivial in $H^1(N; \mathbb{Z})$. Then if $\phi$ is not fibered, there is a representation $\alpha : \pi_1(N) \to \text{Gl}_k(\mathbb{Z})$ for which the conditions in (1.1) are not satisfied.

For knots of genus 1, this result has been enhanced to show that, there is some representation $\alpha$ for which the twisted Alexander polynomial vanishes, [5]. This result has been further generalized to any 3-manifold pair, $(N, \phi)$, where $\phi \in H^1(N; \mathbb{Z})$, [7].

The proof of Theorem 1.3 is not constructive. We have found explicit representations for the knot $K$ and one 2-component link containing $K$, for which (1.1) is violated.

Theorem 1.5. For the representation $\alpha : \pi_1(K) \to S_5 \to \text{GL}_5(\mathbb{Z})$ given in Theorem 3.2 $\Delta_{K,\phi}^\alpha$ is not monic.

In order to find the explicit representation, we will first calculate the fundamental group of the exterior of $K$, and then use the computer program Knottwister written by S. Friedl, [3].
2. Proof of Proposition 1.2

To prove the proposition, we use various results from [2]. (More details can be found in [10].)

Proof. As we can see in the diagram in Figure 2, there is one arrowhead vertex, we will call this vertex $v_1$. Considering the conventions in [2], this knot has 8 vertices. So $n = 1$, and $k = 8$. First, we will find $l_{ij}$ for $i = 1$, and $1 < j \leq 8$: $l_{12} = l_{13} = l_{14} = l_{15} = 0, l_{16} = 6, l_{17} = 3, l_{18} = 2$.

For boundary vertices and arrowhead vertices, $\delta_i = 1$. For this particular knot, each node has 3 arrowhead vertices and/or boundary vertices attached to it. So we have the following values for $\delta_i$ where $1 < i \leq 8$: $\delta_2 = \delta_4 = \delta_7 = \delta_8 = 1$ and $\delta_3 = \delta_5 = \delta_6 = 3$. Now we use Theorem 12.1 in [2] to compute the Alexander polynomial:

$$\Delta = (t - 1)(t^0 - 1)^{-1}(t^0 - 1)(t^0 - 1)^{-1}(t^0 - 1)(t^6 - 1)(t^3 - 1)^{-1}(t^2 - 1)^{-1}.$$  

Following the convention mentioned in [2] we cancel the terms $(t^0 - 1)$ and $(t^0 - 1)^{-1}$. Doing so we get

$$\Delta = \frac{(t - 1)(t^6 - 1)}{(t^3 - 1)(t^2 - 1)} = \frac{t^3 + 1}{t + 1} = t^2 - t + 1.$$  

To find the genus of the knot, we calculate the Thurston norm of the class $\phi = (1) \in H^1(\Sigma \setminus (\nu(K), \mathbb{Z}) \cong \mathbb{Z}$. By Theorem 11.1 in [2],

$$\|\phi\|_T = \| (1) \|_T = \sum_{j=2}^{8} (\delta_j - 2)|l_{1j}| = 1.$$  

So this knot has genus equal to 1 as claimed since $\|\phi\|_T = 2g - 1$. It remains to show it is not fibered. To show this, we use Theorem 11.2 in [2], which asserts that if some of the terms in the summation are zero, as in our case, then $K$ is not fibered.

3. Proof of the Main Theorem

3.1. The Fundamental Group. To find the explicit representation $\alpha$, we first need to calculate the fundamental group of its exterior. For a knot in $S^3$, 

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{knot_diagram.png}
\caption{Vertices of the Knot $K$}
\end{figure}
one can use the Wirtinger presentation of any blackboard projection of the knot to compute its fundamental group. Given that the knot $K$ is contained in a homology sphere $\Sigma$, this method is not directly available, because we do not have access to any blackboard presentation. The route we will follow uses instead the Seifert-Van Kampen theorem and the decomposition of the knot exterior into three components reflected by the splice diagram of $K$ given in Figure 1.

From now on, for the sake of simplicity, when we talk about the fundamental group of the exterior of a link or a knot $L$, we will call it the fundamental group of $L$. We will follow this convention in our notation as well. For example, we will denote the fundamental group of the exterior of the knot $K$ as $\pi_1(K)$ instead of $\pi_1(\Sigma \setminus (\nu(K)))$.

**Lemma 3.1.** The exterior of the knot $K$ has the following fundamental group:

\[\pi_1(K) = \langle x, y, s, t, b | xyx = yxy, stbst = bstb, xs = sx, xt = tx, s = x^{-1}yx^2yx^{-3}, x = (st)^{-1}b(st)^2b(st)^{-3} \rangle.\]

**Proof.** First, we will look at the three building blocks of the splice diagram. If we separate the middle node from the rest, we get the following splice diagram.

![splice diagram](image)

**Figure 3. Splice Diagram of the 3-Component Necklace**

The three-component necklace that this splice diagram represents is the one in Figure 4. The arrowhead vertex with weight 0 is the main loop, and the ones with weight 1 are the two hanging loops. We will call the main loop $N_0$, the loop hanging on the left $N_1$ and the one hanging on the right $N_2$. The following is its projection.

![3-component necklace](image)

**Figure 4. 3-Component Necklace**
To avoid making the diagrams busy, we put the names of the meridians on the arc and will not include the actual meridians in pictures. For this necklace, let $\mu(N_1) = s$, and $\mu(N_2) = t$ be the meridians of $N_1$ and $N_2$. Also since $N_0$ is made of two arcs $m$ and $n$, we can choose as meridian of this component either $m$ or $n$. Using the Wirtinger presentation for links, we see that the (simplified) fundamental group of this link is 

$$\pi_1(N) = \langle n, s, t | ns = sn, nt = tn \rangle.$$

The node on the left is the $(2,3)$ cable on the unknot, as we can read from its splice diagram (Proposition 7.3 in [2]). Hence it represents the right-handed trefoil knot with the canonical orientation. We will call it $T_L$.

The diagram in Figure 5 shows the node on the left separated from the rest.

Considering the projection of the right-handed trefoil shown in Figure 6, we can use the Wirtinger presentation for knots to calculate the fundamental group. Doing so will give us the following (simplified) fundamental group: 

$$\pi_1(T_L) = \langle x, y | xyx = yxy \rangle.$$ 

For this knot, we will choose the meridian to be $\mu(T_L) = x$. Then by the details discussed in Remark 3.13 of [1], the longitude will be $\lambda(T_L) = zxyx^{-3} = x^{-1}yx^2yx^{-3}$.

Splicing on the left, we identify the longitude of $T_L$ with the meridian of $N_1$ and the meridian of $T_L$ with the longitude of $N_1$. Doing so will yield the relations $s = x^{-1}yx^2yx^{-3}$ and $x = n$ respectively.

Since the node on the right is another copy of the right-handed trefoil knot, we will call it $T_R$. This knot has the fundamental group $\pi_1(T_R) = \langle a, b | aba = bab \rangle$. If we choose its meridian to be $a$, then the longitude is $caba^{-3} = a^{-1}ba^2ba^{-3}$. The splicing on the right happens along the $N_0$ component of the necklace, with meridian $\mu(N_0) = n$ and longitude $\lambda(N_0) = st$. Hence after splicing on the right, we will have the relations $st = a$ and
Given the fundamental groups of each of the building blocks, along with the
relations due to the splicing, the Seifert-Van Kampen Theorem states that
the fundamental group of the knot \( K \) is:

\[
\pi_1(K) = \langle x, y, n, s, t, a, b | xyx = yxy, aba = bab, ns = sn, nt = tn, x = n, s = x^{-1}xy^2yx^{-3}, st = a, n = a^{-1}ba^2ba^{-3} \rangle.
\]

Simplifying this group, we get:

\[
\pi_1(K) = \langle x, y, s, t, b | xyx = yxy, stbsta = bstb, sx = xs, xt = tx, s = x^{-1}xy^2yx^{-3}, x = (st)^{-1}b(st)^2b(st)^{-3} \rangle.
\]

\( \square \)

3.2. Finding an Explicit Representation \( \alpha \) that Shows \( K \) is not Fibered.

In this section, using the above presentation of \( \pi_1(K) \) we find an explicit representation of \( \pi_1(K) \to GL_5(\mathbb{Z}) \) for which the twisted Alexander polynomial is not monic. To do so, we use the computer program KnotTwister.

**Theorem 3.2.** For the representation \( \alpha : \pi_1(K) \to S_5 \to GL_5(\mathbb{Z}) \) given by

\[
\alpha(a) = (15234), \alpha(b) = (13524), \alpha(n) = (14523), \alpha(s) = (12345)
\]

\[
\alpha(t) = (15234), \alpha(x) = (14523), \alpha(y) = (34125),
\]

\( \Delta^K_{\alpha, \phi} \) is not monic. (Here, one-line permutation notation is used.)

**Proof.** KnotTwister takes the fundamental group of \( K \) along with a cohomology class \( \phi \) as the input data. For knots, \( \phi \) can be chosen to be the abelianization map \( \phi : \pi_1(K) \to \mathbb{Z} \). To identify explicitly the abelianization map \( \phi \) we add the commutator relations to the fundamental group found in Lemma [3.1]. Then the map \( \phi \) is given explicitly as:

\( \phi(x) = \phi(y) = \phi(s) = 0 \) and \( \phi(b) = \phi(t) = 1 \).

It can be easily checked that \( \alpha \) is a homomorphism, meaning that it respects the relations of the fundamental group. The ordinary Alexander polynomial is \( t^2 - t + 1 \), which is identical to that of the trefoil knot. However, KnotTwister gives the twisted Alexander polynomial \( \Delta^K_{\alpha, \phi} \) with coefficients modulo \( p \) for different prime numbers. The twisted Alexander polynomial given by this particular representation \( \alpha \) over \( \mathbb{F}_5[t^{\pm 1}], \mathbb{F}_7[t^{\pm 1}], \mathbb{F}_{11}[t^{\pm 1}], \mathbb{F}_{13}[t^{\pm 1}], \mathbb{F}_{17}[t^{\pm 1}], \mathbb{F}_{19}[t^{\pm 1}], \mathbb{F}_{23}[t^{\pm 1}], \) and \( \mathbb{F}_{29}[t^{\pm 1}] \) is equal to 0. Since the twisted Alexander polynomial associated with any one of these representations vanishes, it is not monic.

\( \square \)

We can conclude from the previous theorem and Theorem [1.3] that the knot \( K \) is not fibered. Clearly, having the polynomial vanish over any of the fields above would be sufficient to show it is not monic. However, the fact that it vanishes over all these fields is a strong evidence that it is indeed 0. Since the genus of \( K \) is 1 as we saw in Proposition [1.2], this observation is consistent with the enhanced version of Theorem [1.4] appearing in [5].
4. 2-Component Links Containing $K$

In this section, we discuss three 2-component links that contain the knot $K$ as a component. These links are the result of adding an arrowhead vertex to the three nodes of the splice diagram of $K$.

4.1. The Link $L_{\alpha}$. First, we put the second arrowhead vertex on the last node. The following is the splice diagram of this 2-component link. From now on, we call this link $L_{\alpha}$. Since this link contains the knot $K$ as a component, we can denote it as $L_{\alpha} = K_{\alpha} \cup K$, when $K_{\alpha}$ is the new component of the link.

![Figure 7. Splice Diagram of the Link $L_{\alpha}$](image)

Using the theorems in [2], we can easily prove the following proposition. The proof is similar to that of 1.2 and hence is omitted.

**Proposition 4.1.** The 2-component link $L_{\alpha}$ in Figure 7 has the following properties:

1. Its multivariable Alexander polynomial is:
   \[ \Delta_{L_{\alpha}}(t_1, t_2) = (t_1^{12} - t_1^6 + 1)(t_1^4 t_2^4 + t_1^2 t_2^2 + 1)(t_1^3 t_2^3 + 1). \]

2. For a general $\phi = (p, q)$, the Thurston norm is:
   \[ \| \phi \|_T = 7|p + q| + 12|p|. \]

3. If $N$ is the exterior of the link, the pairs $(N, (0, 1))$ and $(N, (1, -1))$ are not fibered.

**Remark 4.2.** From Proposition 4.1, we can observe that for the class $\phi = (1, -1)$ the single variable Alexander polynomial is

\[ \Delta_{L_{\alpha, \phi}} = 6(t - 1)(t^{12} - t^6 + 1). \]

Even though $\text{deg}(\Delta_{L_{\alpha, \phi}}) = \| \phi \|_T + 1$, the polynomial is not monic. So Theorem 1.1 states that this class is not fibered. However, for $\phi = (0, 1)$, we have the following ordinary Alexander polynomial:

\[ \Delta_{L_{\alpha, \phi}} = (t - 1)(t^4 + t^2 + 1)(t^3 + 1) = (t^6 - 1)(t^2 - t + 1). \]

In this case, the Alexander polynomial is monic, and $\text{deg}(\Delta_{L_{\alpha, \phi}}) = 8$. According to Theorem 1.1, this result is compatible with fiberedness, but we showed in Proposition 4.1 that it is not fibered.
4.2. Fundamenta! Group of the Exterior of $L_\alpha$. In order to use twisted Alexander polynomials to discuss the fiberedness of $L_\alpha$, we need to calculate the fundamental group of its exterior.

**Lemma 4.3.** The fundamental group of the exterior of $L_\alpha$ is:

$$\pi_1(L_\alpha) = \langle c, d, e, f, g, h, i, j, k, l, o, p, q, r, u, v, w, a, x, y, n, s, t | xyx = yxy, ns = sn, nt = tn, s = x^{-1}yx^2yx^{-3}, e = st, \rangle$$

$gd = cg, ve = dv, cf = ec, pg = fp, vh = gv, wi = hw, aj = ia, ek = je, rc = kr,$

$eo = le, rp = or, gq = pg, vr = qv, cu = rc, pv = up, hw = vh, ia = wi, jl = aj \rangle.$

**Proof.** Again, we need to decompose the link over its three nodes. For the node on the left and the one in the middle, the calculations are identical to those of the knot $K$. For $L_\alpha$, the node on the right before splicing is shown in Figure 8.

![Figure 8. Splice Diagram of the Link D on the Right](image)

The splice diagram in Figure 8 represents a 2-component link, as it has two arrowhead vertices. It is the $(2, 3)$ cable on the right-handed trefoil (see Proposition 7.3 in [2]). Hence each component is a copy of the right-handed trefoil knot, such that they have linking number 6. We call this 2-component link $D$. The blackboard projection of the link $D$ is shown in Figure 9. We only need to discuss the splicing relations on the right, as the ones on the left are identical to those of $K$. As for $K$, splicing on the right happens along the main loop of the necklace, $N_0$. If we choose to splice along the outer trefoil of $D$, and choose its meridian to be $\mu(D) = e$, the longitude will be $\lambda(D) = cpvwxergve^{-3}$. Hence the splicing relations are:

$$n = cpvwxergve^{-3}, \text{and } e = st.$$ 

Therefore, considering the fundamental groups of the three building blocks of $L_\alpha$ and the relations that result from splicing, we see that the fundamental group of the exterior of $L_\alpha$ is:

$$\pi_1(L_\alpha) = \langle c, d, e, f, g, h, i, j, k, l, o, p, q, r, u, v, w, a, x, y, n, s, t | xyx = yxy, ns = sn, nt = tn, s = x^{-1}yx^2yx^{-3}, e = st, \rangle$$

$gd = cg, ve = dv, cf = ec, pg = fp, vh = gv, wi = hw, aj = ia, ek = je, rc = kr,$

$eo = le, rp = or, gq = pg, vr = qv, cu = rc, pv = up, hw = vh, ia = wi, jl = aj \rangle.$

□
4.3. Finding Representations for $\pi_1(L_\alpha)$ in Two Cases. Since for all knots, the abelianization of their fundamental group is isomorphic to $\mathbb{Z}$, if one cohomology class is fibered, all are. However, it is possible that for the same link, some cohomology classes are fibered and others are not. Now we will show that two different cohomology classes for $L_\alpha$ are not fibered.

In the following theorem, we will find explicit representations for which $\Delta_{N,\phi}^\alpha$ is not monic, when $N$ is the exterior of $L_\alpha$ and $\phi$ is one of the classes $(0, 1)$ or $(1, -1)$. Consequently by Theorem 1.3, the pair $(N, \phi)$ is not fibered for either $\phi$.

**Theorem 4.4.** Let $N$ be the exterior of $L_\alpha$. For $\phi_1 = (0, 1)$, and $\phi_2 = (1, -1)$, there are corresponding representations $\alpha_1, \alpha_2 : \pi_1(N) \to S_5 \to GL(\mathbb{Z}, 5)$ such that $\Delta_{N,\phi_1}^\alpha$ and $\Delta_{N,\phi_2}^\alpha$ are not monic.

**Proof.** First, we need to understand what $\phi_1$ does as a map. We add all the commutator relations to the fundamental group in Lemma 4.3. This will
result in the following relations:
\[
c = d = e = f = g = h = i = j = k = t
\]
\[
o = l = p = q = r = u = v = w = a
\]
\[
s = 1, x = y = n = v^6.
\]

As expected for a 2-component link, the abelianization of \(\pi_1(L_\alpha)\) is isomorphic to \(\mathbb{Z} \oplus \mathbb{Z}\). We can see from the splice diagram of this link that the two components that survive are one of the hanging loops of the necklace, \(N_2\), and the trefoil knot inside the link \(D\). These are the arrowhead vertices in the splice diagram. Hence \(\phi_1\) is the homomorphism that sends \(v\) to 0, and \(t\) to 1. Again, Knottwister takes the fundamental group of \(L_\alpha\) from Lemma 4.3 along with the homomorphism \(\phi_1\) as an input. In multiplicative notation, \(\phi_1\) is the following map:
\[
\phi_1(c) = \phi_1(d) = \phi_1(e) = \phi_1(f) = \phi_1(g) = \phi_1(h) = \phi_1(i) = \phi_1(j) = \phi_1(k) = \phi_1(t) = 1
\]
\[
\phi_1(a) = \phi_1(l) = \phi_1(o) = \phi_1(p) = \phi_1(q) = \phi_1(r) = \phi_1(u) = \phi_1(v) = \phi_1(w) = \phi_1(1) = 0.
\]

Knottwister gives the following representation \(\alpha_1: \pi_1(N) \to S_5 \to GL(\mathbb{Z}, 5)\), when the elements in \(S_5\) are written in on-line permutation form:
\[
\begin{align*}
a & \mapsto (13245) & c & \mapsto (23415) & d & \mapsto (45321) & e & \mapsto (24351) \\
f & \mapsto (32514) & g & \mapsto (13524) & h & \mapsto (14532) & i & \mapsto (15234) \\
j & \mapsto (13524) & k & \mapsto (31425) & l & \mapsto (14325) & n & \mapsto (45312) \\
o & \mapsto (21345) & p & \mapsto (21345) & q & \mapsto (42315) & r & \mapsto (21345) \\
s & \mapsto (12345) & t & \mapsto (24351) & u & \mapsto (42315) & v & \mapsto (14325) \\
w & \mapsto (15342) & x & \mapsto (45312) & y & \mapsto (45213).
\end{align*}
\]

For this twist, the twisted Alexander polynomial, \(\Delta_{N,\phi_1}^{a_1}\), vanishes over the fields \(\mathbb{F}_1[t^\pm 1], \mathbb{F}_{11}[t^\pm 1], \mathbb{F}_{13}[t^\pm 1], \mathbb{F}_{17}[t^\pm 1], \mathbb{F}_{19}[t^\pm 1], \mathbb{F}_{23}[t^\pm 1]\), and \(\mathbb{F}_{29}[t^\pm 1]\). Since the twisted Alexander polynomial vanishes over these finite fields, it cannot be monic.

Now, we do the same for \(\phi_2 = (1, -1)\). Using multiplicative notation, \(\phi_2\) can be viewed as the map that acts as follows on the generators of \(\pi_1(L_\alpha)\):
\[
\phi_2(c) = \phi_2(d) = \phi_2(e) = \phi_2(f) = \phi_2(g) = \phi_2(h) = \phi_2(i) = \phi_2(j) = \phi_2(k) = \phi_2(t) = -1
\]
\[
\phi_2(a) = \phi_2(l) = \phi_2(o) = \phi_2(p) = \phi_2(q) = \phi_2(r) = \phi_2(u) = \phi_2(v) = \phi_2(w) = \phi_2(1) = 1
\]
\[
\phi_2(s) = 0, \phi_2(x) = \phi_2(y) = \phi_2(n) = 6.
\]

Given this information, Knottwister gives us the following representation \(\alpha_2\) (in one-line permutation form):
\[
\alpha_2: \pi_1(M) \to S_5 \to GL(\mathbb{Z}, 5)
\]
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For this representation, the twisted Alexander polynomial, $\Delta_{N,\varphi}^{a_2}$, vanishes over $\mathbb{F}_5[t^{\pm 1}]$ and all of the fields previously mentioned for $\Delta_{N,\varphi}^{a_1}$. Hence neither twisted Alexander polynomial is monic as claimed.

Again, by Theorem 1.3, the pairs $(N,(0,1))$ and $(N,(1,-1))$ are not fibered.

4.4. Links $L_\beta$ and $L_\gamma$. In this section, we briefly discuss the 2-component link that results from adding an arrowhead vertex to the middle node, $L_\beta$, and the one that results from adding it to the first node, $L_\gamma$. We use the theorems in [2] to conclude the following propositions.

**Proposition 4.5.** For the link $L_\beta$ the following are true:

1. The Alexander polynomial vanishes.
2. The Thurston norm of the class $\phi = (p, q)$ on $L_\beta$ is $|p + q|$.
3. No cohomology class $\phi$ on $L_\beta$ is fibered.

**Proposition 4.6.** The link $L_\gamma$ has the following properties.

1. The Alexander polynomial vanishes.
2. The Thurston norm for a class $\phi = (p, q)$ on this link is $7|p| + |6p + q|$.
3. No class $\phi$ on this link is fibered.

We can use similar techniques to find the fundamental groups of these links. We have discussed the three “building blocks” of $L_\gamma$ already. For the link $L_\beta$, notice that the middle node gives the splice diagram of a 4-component necklace. The following propositions give the fundamental groups of the exteriors of these two links.

**Proposition 4.7.** The fundamental group of $L_\beta$ is the following:

$$\pi_1(L_\beta) = \langle x, y, a, b, s, r, t, n | aba = bab, x = yx = yx, nr = rn, nt = tn, ns = sn, x = n, s = x^{-1}yx^2yx^{-3}, a = rst, n = a^{-1}ba^2ba^{-3} \rangle.$$

**Proposition 4.8.** The fundamental group of $L_\gamma$ is the following group:

$$\pi_1(L_\gamma) = \langle a, b, n, s, t, c, d, e, f, g, h, i, j, k, o, l, p, q, r, u, v, w | gd = cg, ve = dv, cf = ec, pg = fp, vh = gv, wi = hw, xj = ix, ek = je, rc = kr, eo = le, rp = or, gq = pg, vr = qv, cu = rc, pv = up, hw = vh, ix = wi, jl = xj, aba = bab, ns = sn, nt = tn, a = st, n = a^{-1}ba^2ba^{-3}, e = n, s = cpvwxergve^{-3} \rangle.$$
5. A “Secondary” Polynomial, $\tilde{\Delta}_1^\alpha(t)$

Since the ordinary Alexander polynomial is 0 for $L_\beta$ and $L_\gamma$, we may not use Theorem 1.1 to get a useful bound for the Thurston norm. From now on, we will only be concerned with the single-variable version of the twisted Alexander polynomial for simplicity. Also, since $\mathbb{F}[t^\pm 1]$ is a principal ideal domain, we replace $\mathbb{Z}[t^\pm 1]$ by $\mathbb{F}[t^\pm 1]$ in the definition of the Alexander module where $\mathbb{F} = F_p$ is a field. As a result, we have the following isomorphism:

$$H_1(N, \mathbb{F}^k[t^\pm 1]) \cong \mathbb{F}[t^\pm 1]^r \oplus \bigoplus_{j=1}^m \mathbb{F}[t^\pm 1]/(p_j(t))$$

for $p_1(t), ..., p_m(t) \in \mathbb{F}[t^\pm 1]$. The type of polynomials we will examine are defined by:

$$\tilde{\Delta}_{N, \phi}^\alpha := \prod_{j=1}^m p_j(t)$$

regardless of the rank $r$. Not much is known about these polynomials.

S. Friedl and T. Kim have proved the following theorem that relates these polynomials to the Thurston norm in [4].

**Theorem 5.1.** (Friedl-Kim, [4]) Let $L = L_1 \cup L_2 \cup ... \cup L_m$ be a link with $m$ components. Denote its meridian by $\mu_1, ..., \mu_m$. Let $\phi \in H^1(X(L); \mathbb{Z})$, be primitive and dual to a meridian $\mu_i$, when $X(L)$ denotes the exterior of $L$. Hence $\phi(\mu_i) = 1$ for some $i$ and $\phi(\mu_j) = 0$ for $j \neq i$. Then

$$\|\phi\|_T \geq \frac{1}{k} \text{deg}(\tilde{\Delta}_1^\alpha(t)) - 1.$$

Here, $k$ is the size of the representation $\alpha$.

Theorem 5.1 will help us improve the bound of the Thurston norm for the class $(0,1)$ for both $L_\beta$ and $L_\gamma$. Recall from Section 2.2 that for $L_\beta$, the Thurston norm of a general cohomology class $(p, q)$ is $|p + q|$. So for this link, $||(0,1)||_T = 1$. In this case, Knottwister computes the $\Delta_1^\alpha(t)$ to be $1 - t + t^2$ over $\mathbb{F}_{13}$ when $\alpha$ is trivial (so $k = 1$). Therefore, for the pair $(L_\beta, (0,1))$ we get

$$||(0,1)||_T \geq 2 - 1 = 1$$

which is a sharp bound.

Now, we consider the same cohomology classes on $L_\gamma$. We know from our calculations in section 2.2 that for this link, $||(p, q)||_T = |p + q|$. So for this link $||(0,1)||_T = 1$. Knottwister yields the $\Delta_1^\alpha(t) = 1 - t + t^2$ over $\mathbb{F}_{13}$ again, when $\alpha$ is trivial, which is again a sharp bound.

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