The Brown measure of the sum of a self-adjoint element and an elliptic element

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Abstract

We completely determine the Brown measure of the sum of a self-adjoint element and an elliptic element, which is the limiting eigenvalue distribution of the random matrix

\[
Y_N + \sqrt{s - \frac{t^2}{2}} X_N + i\sqrt{\frac{t}{2}} X_N'
\]

where \(Y_N\) is an \(N \times N\) deterministic Hermitian matrix whose eigenvalue distribution converges as \(N \to \infty\) and \(X_N\) and \(X_N'\) are independent Gaussian unitary ensembles. We also study various asymptotic behaviors of this Brown measure as the variance of the elliptic element approaches infinity.

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1 Introduction

1.1 The sum of a self-adjoint element and an elliptic element

An elliptic element is an element in a $W^*$-probability space of the form $z = x + iy$ where $x$ and $y$ are freely independent semicircular elements, possibly with different variances. By substracting the mean $\tau(z)$ if necessary, we only consider the case $\tau(z) = 0$ in this paper. The variance of such an element is given by

$$\tau(z^*z) = \tau(x^*x) + \tau(y^*y).$$

Once the variance of $z$ is given, say $s$, there are several possibilities for the variances of $x$ and $y$. We use the parameters $t = 2\tau(y^*y)$, and $\tau(x^*x) = s - \frac{t}{2}$. Under the parameters $s, t$, the elliptic element $z$ then has the form

$$\tilde{s}_{s-t/2} + i\sigma_z,$$

where $\tilde{s}_{s-t/2}$ and $\sigma_z$ are freely independent centered semicircular elements with variances $s - \frac{t}{2}$ and $\frac{t}{2}$ respectively in a certain $W^*$-probability space.

Suppose that $y_0$ is a bounded self-adjoint element in the $W^*$-probability space containing $\tilde{s}_{s-t/2}$ and $\sigma_z$; suppose also that all the three elements are freely independent. In this paper, we compute the Brown measure of the element

$$y_0 + \tilde{s}_{s-t/2} + i\sigma_z.$$

We show that the Brown measure of $y_0 + \tilde{s}_{s-t/2} + i\sigma_z$ is a push-forward of the Brown measure of $y_0 + c_s$ where $c_s = \tilde{s}_{s-t/2} + i\sigma_z$ is the Voiculescu’s circular element. The Brown measure of $y_0 + c_s$ was computed and analyzed by Zhong and the author [25]. We also study the asymptotic behavior of the Brown measure of $y_0 + \tilde{s}_{s-t/2} + i\sigma_z$ as

1. $s, t \to \infty$ such that the ratio $s/t$ remains as a constant $> \frac{1}{2}$;
2. $s \to \infty$ and $t$ is kept fixed; and
3. $s, t \to \infty$ such that the ratio $s/t = \frac{1}{2}$.

If $s \geq t$, our results can be computed by the results of Zhong and the author [25] in which the Brown measure of $x_0 + c_t$ is computed, with $x_0 = y_0 + \tilde{s}_{s-t}$, where $c_t$ is a circular element, freely independent of $x_0$. If $s < t$, $y_0 + \tilde{s}_{s-t/2} + i\sigma_z$ is not a sum of a self-adjoint element and a circular element. We need a more general method.

We use the result introduced in [21] to compute the Brown measure of $y_0 + \tilde{s}_{s-t/2} + i\sigma_z$ in terms of the Hermitian part $y_0 + \tilde{s}_{s-t/2}$ and $t$ (the parameter of the semicircular element in the skew-Hermitian part). We combine this method with techniques in free probability to determine the Brown measure of $y_0 + \tilde{s}_{s-t/2} + i\sigma_z$ in terms of $y_0$, and $s$ and $t$. The results in [21] used a PDE method introduced in the work of Driver, Hall and Kemp [12]; this method has been used in subsequent work by other authors [10][21][25]. See also the expository article [19] by Hall for an introduction to the PDE method.

Our results have direct connections to random matrix theory. If $X_N$ and $X'_N$ are independent Gaussian unitary ensembles (GUEs), and $Y_N$ is a sequence of $N \times N$ self-adjoint deterministic matrices whose empirical eigenvalue distributions converge weakly to the law of $y_0$, then $Y_N$, $X_N$ and $X'_N$ are asymptotically free in the sense of Voiculescu [33]. If $s > \frac{1}{2}$, by [29] Theorem 6], the empirical eigenvalue distribution of the (almost surely non-normal) random matrix

$$Y_N + \sqrt{s - \frac{t}{2}}X_N + i\sqrt{\frac{t}{2}}X'_N,$$
converges to the Brown measure of \( y_0 + \sigma_{\frac{1}{2}} + i \sigma_{\frac{1}{2}} \) as \( N \to \infty \). The Brown measure of the case \( s = \frac{1}{2} \) is studied in [21], and it is a special case of the results in this paper. In this \( s = \frac{1}{2} \) special case, the random matrix model is not a sum of a random matrix and a Ginibre ensemble. We cannot apply \([29]\) to conclude that the empirical eigenvalue distribution converges to the Brown measure; it is still an open problem to give a mathematical proof of the convergence. Nevertheless, numerical simulations in \([21]\) suggest that the Brown measure of \( y_0 + i \sigma_{\frac{1}{2}} \) is indeed the limiting eigenvalue distribution of \( Y_N + i \sqrt{t/2}X_N \), where \( Y_N \) and \( X_N \) are the same matrices as above.

The Brown measure computed in the case where \( y_0 = 0 \) is the elliptic law \([8]\) (see also \([15]\)); its name is due to the fact that its support is a region bounded by an ellipse centered at the origin. In the even more special case \( s = t \), the Brown measure is called the circular law since its support is a disk centered at the origin. The circular law was first discovered by Ginibre \([14]\) in the case when the entries come with more relaxed assumptions. The assumptions

**Assumption 1.1.**

\[ a \] is a variable unique compactly supported probability measure on \( R \), however, in this case, the element \( a \) is not a Dirac measure. However, in the particular case, the elliptic law was first computed by Girko \([15]\) as a limiting eigenvalue distribution of a certain random matrix model. The Brown measure, in the operator framework, was computed by Biane and Lehner \([8]\) and various later work of others.

The Brown measure of operators of the form \( X + iY \) where \( X \) and \( Y \) are freely independent has been analyzed at a nonrigorous level in the physics literature. Stephanov \([30]\) used the case when \( X \) is Bernoulli distributed and \( Y \) is a GUE to provide a model of QCD. Janik et al. \([26]\) identified the domain where the eigenvalues cluster in the large-\( N \) limit when \( X \) is an arbitrary self-adjoint random matrix and \( Y \) is a GUE. Jarosz and Nowak \([27, 28]\) computed the limiting eigenvalue distribution for general self-adjoint \( X \) and \( Y \). Belinschi et al. \([3, 4]\) put the results in \([27, 28]\) on a more rigorous basis; however, there have not been analytic results about the Brown measure of \( X + iY \) obtained under this framework.

Since this article was posted on the arXiv, the results of this article have been extended by several papers. In \([24]\), Theorem \([12]\) is extended to the case when \( y_0 \) is an unbounded self-adjoint element. Zhong \([35]\) computes the Brown measure of \( y_0 + \sigma_{\frac{1}{2}} + i \sigma_{\frac{1}{2}} \) for arbitrary bounded operator \( y_0 \). Hall and the author \([20]\) compute the Brown measure of the multiplicative analogue of the operator considered in this paper.

### 1.2 Statements of results

Let \( y_0 \) be a bounded self-adjoint element, \( \sigma_{\frac{1}{2}} \) and \( \sigma_{\frac{1}{2}} \) be semicircular elements with variances \( s - t/2 \) and \( t/2 \) in a \( W^* \)-probability space \((\mathcal{A}, \tau)\), which is a finite von Neumann algebra \( \mathcal{A} \) with a faithful, normal, tracial state \( \tau \). Suppose also that all three of them are freely independent. Throughout the paper, we let \( \nu \) be the law (or distribution) of \( y_0 \), which is the unique compactly supported probability measure on \( \mathbb{R} \) such that

\[
\int x^n \, d\nu(x) = \tau(y_0^n), \quad \text{for all } n \in \mathbb{N}.
\]

Recall that, in this paper, we compute the Brown measure of the element

\[ y_0 + \sigma_{\frac{1}{2}} + i \sigma_{\frac{1}{2}} \in \mathcal{A}. \]

Background information of free probability and Brown measure is reviewed in Section \([2]\) The choice of the parameters \( s, t \) comes from the context of the two-parameter Segal–Bargmann transform \([11, 18, 23]\). It is an interpolation between the self-adjoint element \( y_0 + \sigma_s \) and the element \( y_0 + i \sigma_s \) studied in \([21]\).

We make the following standing assumption about the element \( y_0 + \sigma_{\frac{1}{2}} + i \sigma_{\frac{1}{2}} \). We use \( \text{Law}(a) \) to denote the law of any self-adjoint random variable \( a \in \mathcal{A} \) and \( \text{Brown}(a) \) to denote the Brown measure of any non-self-adjoint random variable \( a \in \mathcal{A} \).

**Assumption 1.1.** Throughout the paper, we assume either \( s > \frac{1}{2} \) or \( \nu \) is not a Dirac measure, so that \( \text{Law}(y_0 + \sigma_{\frac{1}{2}}) \) is not a Dirac measure.

When this assumption does not hold, that is, if \( \text{Law}(y_0 + \sigma_{\frac{1}{2}}) \) is a Dirac measure, then one cannot apply the results from \([21]\). However, in this case, the element \( y_0 + \sigma_{\frac{1}{2}} + i \sigma_{\frac{1}{2}} \) has the form \( u 1 + i \sigma_{\frac{1}{2}} \) for some constant \( u \in \mathbb{R} \) (where \( 1 \) is the identity element in \( \mathcal{A} \)). The Brown measure is then a semicircular distribution centered at \( u \) with variance \( t/2 \) on the vertical line through the point \( u \). Under Assumption \([1.1]\) by the results in \([21]\), the Brown measure is absolutely continuous with respect to the Lebesgue measure on the plane.

The following theorem summarizes Theorems \([3.3] \) and \([3.7] \) the proofs can be found in Sections \([3.2] \) and \([3.3] \) The results in \([25] \) and \([21] \) show that both \( \text{Brown}(y_0 + \sigma_{\frac{1}{2}} + i \sigma_{\frac{1}{2}}) \) and \( \text{Brown}(y_0 + \sigma_s) \) can be pushed forward to \( \text{Law}(y_0 + \sigma_s) \). Points
2 and 3 of the following theorem are proved by comparing these two push-forward maps. We then use the push-forward result to compute the density of Brown\((y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2})\) given in Point 1 of the following theorem.

**Theorem 1.2.** 1. For each \(s \geq \frac{1}{2} > 0\), there is a continuous function \(b_{s,t} : \mathbb{R} \to [0, \infty)\) such that the Brown measure of \(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2}\) is supported in the closure of the set

\[
\Omega_{s,t} = \{a + ib \in \mathbb{C} \mid |b| < b_{s,t}(a)\}.
\]

The boundary of \(\Omega_{s,t}\) is of measure zero with respect to the Brown measure. The Brown measure is absolutely continuous with respect to the Lebesgue area measure on \(\mathbb{C}\), with density

\[
w_{y_0,s,t}(a + ib) = \frac{1}{2\pi t} \left(1 + \frac{t}{2s} \int_{\mathbb{R}} \frac{d\nu(x)}{(a_{s,t}(a) - x)^2 + v_{y_0,s}(a_{s,t}(a))^2}\right),
\]

for \(|b| < b_{s,t}(a)\), where \(a_{s,t}\) is a certain homeomorphism on \(\mathbb{R}\) and \(v_{y_0,s}\) is a certain nonnegative continuous function on \(\mathbb{R}\) such that \(a_{s,t}\) and \(v_{y_0,s} \circ a_{s,t}\) are differentiable in \(\Omega_{s,t} \cap \mathbb{R}\). In particular, the density is constant in the vertical direction.

2. The Brown measure of \(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2}\) is the push-forward measure of the Brown measure of \(y_0 + c_s\) by the homeomorphism \(U_{s,t} : \mathbb{C} \to \mathbb{C}\),

\[U_{s,t}(\alpha + i\beta) = a_{s,t}(\alpha) + \frac{t}{s}\beta\]

where \(a_{s,t}\) is the inverse function of \(a_{s,t}\).

3. The push-forward measure of the Brown measure of \(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2}\) by the map, constant in the vertical directions,

\[Q_{s,t}(a + ib) := \frac{1}{s - t}[sa - t\alpha_{s,t}(a)]\]

is the law of the self-adjoint element \(y_0 + \sigma_s\).

We now describe briefly how to compute the functions \(a_{s,t}\), \(b_{s,t}\), and \(v_{y_0,s} \circ a_{s,t}\) from the above theorem in \(\Omega_{s,t} \cap \mathbb{R}\). Given \(a \in \mathbb{R}\), we try to solve for \(\alpha \in \mathbb{R}\) and \(v > 0\) the equations

\[
\int \frac{d\nu(x)}{(\alpha - x)^2 + v^2} = \frac{1}{s}
\]

\[
\frac{(2s - t)\alpha}{s} - (s - t) \int \frac{x d\nu(x)}{(\alpha - x)^2 + v^2} = a.
\]  \( (1.1) \)

The following proposition shows that \(a \in \Omega_{s,t} \cap \mathbb{R}\) is precisely when \((1.1)\) has a unique pair of solution. It also shows how the functions \(a_{s,t}\), \(v_{y_0,s} \circ a_{s,t}\) and \(b_{s,t}\) in Theorem 1.2 are computed using the solution. This proposition is proved in Corollary 3.3.

**Proposition 1.3.** Given any \(a \in \mathbb{R}\), \((1.1)\) has a pair of solution \(\alpha \in \mathbb{R}\) and \(v > 0\) if and only if \(a \in \Omega_{s,t} \cap \mathbb{R}\). In this case, the solution is unique, and \(a_{s,t}(a) = \alpha\), \(v_{y_0,s}(a_{s,t}(a)) = v\) and \(b_{s,t}(a) = \frac{\alpha}{s}v\).

In the special case \(s = t\), we obtain \(a_{s,t}(a) = a\) and, by Theorem 1.2,

\[w_{y_0,s,s}(a + ib) = \frac{1}{\pi s} \left(1 - \frac{t}{2s} \frac{d}{da} \int \frac{x d\nu(x)}{(\alpha - x)^2 + v_{y_0,s}(a)^2}\right)\]

which reduces to the results in \([25]\). In another special case \(t = 2s\), the equations in \((1.1)\) reduces to \((1.4)\) and \((1.5)\) in \([21]\); the function \(a_{s,t}\) is the function \(a_0^s\) in \([21]\) and the density is given by

\[\frac{1}{2\pi s} \left(\frac{da_0^s}{da} - \frac{1}{2}\right)\].

Thus, in the case, Theorem 1.2 reduces to the results in \([21]\).

In Sections 4 and 5, we also investigate the asymptotic behaviors of the Brown measure of \(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2}\), which are summarized in the following theorem; roughly speaking, the Brown measure of \(y_0 + \tilde{\sigma}_{s-t/2} + i\sigma_{t/2}\) behaves like the Brown measure of \(\tilde{\sigma}_{s-t/2} + i\sigma_{t/2}\). Point 1 of the following theorem is proved in Theorems 5.1 and 5.2, Point 2 is proved in Theorem 5.3 and 5.4 and Point 3 is proved in Theorem 5.5. See these theorems for the precise statements.
Theorem 1.4. In all of the following three limiting regimes, the function $b_{s,t}$ is unimodal for all large enough $s$.

1. As $s, t \to \infty$ such that the ratio $s/t$ remains as a constant $> \frac{1}{2}$, the domain $\Omega_{s,t}$ is asymptotically equivalent to a region bounded an ellipse centered at $(\tau(y_0), 0)$ with horizontal semi-axis of length $\frac{2s-t}{\sqrt{s}}$ and vertical semi-axis of length $\frac{t}{\sqrt{s}}$. The density $w_{y_0,s,t}$ converges to the constant

$$\frac{1}{\pi (2s-t)t}.$$

Both convergences are uniform outside any neighborhood of the endpoints of $\Omega_{s,t} \cap \mathbb{R}$.

2. As $s \to \infty$ and $t$ is kept fixed: the domain $\Omega_{s,t}$ is asymptotically equivalent to a region bounded by a long and thin ellipse centered at $(\tau(y_0), 0)$, with horizontal semi-axis of length $2\sqrt{s}$ and vertical semi-axis of length $\frac{t}{\sqrt{s}}$. The density converges to the constant

$$\frac{1}{2\pi t}.$$

Both convergences are uniform outside any neighborhood of the endpoints of $\Omega_{s,t} \cap \mathbb{R}$.

3. As $s, t \to \infty$ such that the ratio $s/t = \frac{1}{2}$, the domain $\Omega_{s,t}$ is asymptotically equivalent to a region bounded a narrow and tall ellipse centered at $(\tau(y_0), 0)$, with vertical semi-axis of length $2\sqrt{s}$. The set $\Omega_{s,t} \cap \mathbb{R}$ concentrates around $\tau(y_0)$; more precisely, given any $c > 1$, we have

$$-\frac{4c\tau(y_0^2)}{\sqrt{s}} < \inf(\Omega_{s,t} \cap \mathbb{R}) - \tau(y_0) < 0 < \sup(\Omega_{s,t} \cap \mathbb{R}) - \tau(y_0) < \frac{4c\tau(y_0^2)}{\sqrt{s}},$$

for all large enough $s$.

We do not have a density estimate for the last case.

2. Background and previous results

2.1 Free random variables

**Definition 2.1.** 1. We call $(\mathcal{A}, \tau)$ a $W^*$-probability space if $\mathcal{A}$ is a von Neumann algebra and $\tau$ is a normal, faithful tracial state on $\mathcal{A}$. The elements in $\mathcal{A}$ are called non-commutative random variables, or simply random variables.

2. The $*$-subalgebras $A_1, \ldots, A_n \subset \mathcal{A}$ are said to be freely independent if given an $i_1, i_2, \ldots, i_m \in \{1, \ldots, n\}$ with $i_k \neq i_{k+1}$, $a_{i_k} \in \mathcal{A}_{i_k}$ are centered, then we also have $\tau(a_{i_1}a_{i_2} \cdots a_{i_m}) = 0$. The random variables $a_1, \ldots, a_m$ are freely independent if the $*$-algebras they generate are freely independent.

3. For a self-adjoint element $a \in \mathcal{A}$, the distribution, or the law, of $a$ is a compactly supported measure $\mu$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} f \, d\mu = \tau(f(a))$$

for all continuous function $f$. We denote by $\text{Law}(a)$ the law of $a$.

We now introduce the random variables that are key to this paper. The semicircular element $\sigma_t$ has the semicircular distribution, or the semicircle law of variance $t$, supported on $[-2\sqrt{t}, 2\sqrt{t}]$ with density

$$\frac{\sqrt{4t-x^2}}{2\pi t} \, dx.$$

The circular element $\xi$ has the form $\tilde{\sigma}_z + i\sigma_z$ where $\tilde{\sigma}_z$ and $\sigma_z$ are freely independent semicircular elements. The elliptic element has the form $\tilde{\sigma}_z + i\sigma_z$ where $\tilde{\sigma}_z$ and $\sigma_z$ are freely independent semicircular elements.
2.1.1 The $R$-transform

Let $a \in \mathcal{A}$ be a self-adjoint element with law $\mu$. Then we consider the Cauchy transform

$$G_a(z) = \int \frac{1}{z-x} d\mu(x)$$

defined outside the spectrum of $a$. The Cauchy transform $G_a$ is univalent around $\infty$. Denote by $K_a$ the inverse of $G_a$ at $\infty$, and let

$$R_a(z) = K_a(z) - \frac{1}{z}.$$ 

We call $K_a$ the $K$-transform of $a$ and $R_a$ the $R$-transform of $a$.

**Theorem 2.2** (E2). If $a_1, a_2 \in \mathcal{A}$ are freely independent self-adjoint random variables, then the $R$-transform of the random variable $a = a_1 + a_2$ is given by

$$R_a = R_{a_1} + R_{a_2}.$$ 

Using the notations in the theorem, the distribution of $a$ is called the free convolution of $a_1 + a_2$.

2.2 The Brown measure

In this section, we review the definition of the Brown measure, which was introduced by Brown [9]. Let $a \in \mathcal{A}$. We define a function $S$ by

$$S(\lambda, \varepsilon) = \tau[\log(|a - \lambda|^2 + \varepsilon)], \quad \lambda \in \mathbb{C}, \varepsilon > 0.$$ 

Then

$$S(\lambda, 0) = \lim_{\varepsilon \to 0^+} S(\lambda, \varepsilon)$$

exists as a subharmonic function on $\mathbb{C}$, with value in $\mathbb{R} \cup \{-\infty\}$. The Brown measure of $a$, denoted by $\text{Brown}(a)$, is defined to be

$$\text{Brown}(a) = \frac{1}{4\pi} \Delta S(\lambda, 0)$$

where the Laplacian is in distributional sense.

One can see that $S(\lambda, 0)$ does define a harmonic function outside the spectrum of $a$; the Brown measure of $a$ is a probability measure supported on the spectrum of $a$. The support of $\text{Brown}(a)$, however, can be a proper subset of the spectrum of $a$.

The Brown measure of an $N \times N$ matrix is the empirical eigenvalue distribution of the matrix. If a sequence of random matrices $A_N$ converges in $\ast$-distribution to an element $a$ in a non-commutative probability space, one generally expects that the empirical eigenvalue distribution of $A_N$ converges to the Brown measure of $a$; this, however, is not always the case. A counter-example is the nilpotent matrix

$$\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}$$

this sequence of matrices converges to the Haar unitary element in $\ast$-distribution but the empirical eigenvalue distribution is always the Dirac measure at 0.

The Brown measure of the circular element $c_s = \sigma_s + i\sigma_s$ is called the circular law and is supported in the disk of radius $\sqrt{s}$ centered at the origin. The density is the constant

$$\frac{1}{\pi s}$$

in the support. The circular element is an $R$-diagonal element. The Brown measure of the circular element can be computed by the method developed by Haagerup and Larsen [16] and Haagerup and Schultz [17].
The Brown measure of the elliptic element $\sigma_{\frac{1}{2}} + i\sigma_{\frac{1}{2}}$ is called the elliptic law and is supported in an ellipse with semi-axes on the real and imaginary axes of length $\frac{2s}{\sqrt{\pi}}$ and $\frac{t}{\sqrt{\pi}}$ respectively. The density is the constant
\[
\frac{1}{\pi \sqrt{2s - t}}
\]
in the support. The elliptic law was computed by Biane and Lehner [8].

2.3 Biane’s free convolution formula

In this section, we review the results of the distribution of the free convolution of a self-adjoint element and a semicircular element established by Biane [7]; several functions and a domain also come up in our study of Brown measure. Given a self-adjoint random variable $x_0$ with law $\mu$, we consider the function
\[
v_{x_0,t}(u) = \inf \left\{ v > 0 \left| \int_{\mathbb{R}} \frac{d\mu(x)}{(x-u)^2 + v^2} > \frac{1}{t} \right. \right\}.
\]
That is, if
\[
\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v_{x_0,t}(u)^2} > \frac{1}{t},
\]
then $v_{x_0,t}(u)$ is defined to be the unique positive number such that
\[
\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v_{x_0,t}(u)^2} = \frac{1}{t},
\]
otherwise, if
\[
\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v_{x_0,t}(u)^2} \leq \frac{1}{t},
\]
then we set $v_{x_0,t}(u) = 0$. It is noted in [7] that the function $v_{x_0,t}$ is continuous on $\mathbb{R}$ and is differentiable at the points $u$ where $v_{x_0,t}(u) > 0$.

Definition 2.3. We introduce the following notations.

1. $\Delta_{x_0,t} = \{ u + iv \in \mathbb{C} | v > v_{x_0,t}(u) \}$ is the region above the graph of $v_{x_0,t}$ in the upper half plane.

2. $H_{x_0,t}(z) = \frac{z + tG_{x_0}(z)}{2}$, $z \in \Delta_{x_0,t}$.

Theorem 2.4 (7). 1. The function $H_{x_0,t}$ is an injective conformal map, from $\Delta_{x_0,t}$ onto the upper half plane $\mathbb{C}^+$; the function $H_{x_0,t}$ extends to a homeomorphism from the closure $\overline{\Delta_{x_0,t}}$ of $\Delta_{x_0,t}$ onto $\mathbb{C}^+ \cup \mathbb{R}$. In particular, $H_{x_0,t}(u + iv_{x_0,t}(u))$ is real.

2. The function $H_{x_0,t}$ satisfies
\[
G_{x_0+\sigma_1}(H_{x_0,t}(z)) = G_{x_0}(z).
\]

3. The measure $\text{Law}(x_0 + \sigma_1)$ is absolutely continuous with respect to the Lebesgue measure; its density $p_{x_0,t}$ can be computed by the function $\psi_{x_0,t}(u) := H_{x_0,t}(u + iv_{x_0,t}(u))$. The function $\psi_{x_0,t} : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, and
\[
p_{x_0,t}(\psi_{x_0,t}(u)) = \frac{\psi_{x_0,t}(u)}{\pi t}.
\]

4. As a consequence, the support of $\text{Law}(x_0 + \sigma_1)$ is the closure of the open set $\{ \psi_{x_0,t}(u) | v_{x_0,t}(u) > 0 \}$.

Remark 2.5. Let $\Lambda_{x_0,t} = \{ u + iv \in \mathbb{C} | v < v_{x_0,t}(u) \}$. The map $H_{x_0,t}$ can be extended to an injective conformal map on $(\Lambda_{x_0,t})^c$ by Schwarz reflection with a continuous extension to $\Lambda_{x_0,t}^c$. From now on, $H_{x_0,t}$ means the extension defined on $\Lambda_{x_0,t}^c$. If $v_{x_0,t}(u) > 0$, $H_{x_0,t}$ maps both boundary points $u \pm iv_{x_0,t}(u)$ of $\Lambda_{x_0,t}$ to the same point in the support of $\text{Law}(x_0 + \sigma_1)$.

We then define the right inverse $H_{x_0,t}^{-1}$ of $H_{x_0,t}$ as follows. Outside the interior of the support of $\text{Law}(x_0 + \sigma_1)$, which is the closure of an open set by Theorem 2.4, $H_{x_0,t}^{-1}$ is defined to be the inverse of $H_{x_0,t}$. Given any $q$ in the interior of the support of $\text{Law}(x_0 + \sigma_1)$, we define
\[
H_{x_0,t}^{-1}(q) = u + iv_{x_0,t}(u)
\]
where $u$ is chosen such that $H_{x_0,t}(u + iv_{x_0,t}(u)) = q$. Thus, the restriction of $H_{x_0,t}^{-1}(q)$ to $\mathbb{C}^+ \cup \mathbb{R}$ is the inverse of $H_{x_0,t}$ on $\Delta_{x_0,t}$.
2.4 Sum of a self-adjoint and a circular elements

In [25], the author and Zhong computed the Brown measure of \( x_0 + c_t \), where \( x_0 \) is a self-adjoint element freely independent of the circular element \( c_t \), using the method introduced by Driver, Hall and Kemp [12]. Interestingly, the support of the Brown measure is bounded by the graph of Biane’s function \( v_{x_0,t} \) introduced in Section 2.3 and the density is closely related to the law of the self-adjoint element \( x_0 + \sigma_t \). In this section, we review the results established in [25].

**Theorem 2.6.** Let
\[
\Lambda_{x_0,t} = \{ u + iv \in \mathbb{C} | |v| < v_{x_0,t}(u) \}.
\]
Then \( \Lambda_{x_0,t} \) is a set of full measure with respect to \( \text{Brown}(x_0 + c_t) \), and its density \( w_{x_0,t} \) has the form
\[
w_{x_0,t}(u + iv) = \frac{1}{2\pi t} \frac{d\psi_{x_0,t}(u)}{du}, \quad u + iv \in \Lambda_{x_0,t}
\]
where \( \psi_{x_0,t} \) is defined in Theorem 2.2. The density is constant along the vertical segments.

Furthermore, the push-forward of \( \text{Brown}(x_0 + c_t) \) by
\[
\Psi_{x_0,t}(u + iv) = H_{x_0,t}(u + iv_{x_0,t}(u)), \quad u + iv \in \Lambda_{x_0,t}
\]
which is independent of \( v \), is the law of \( x_0 + \sigma_t \).

2.5 Sum of a self-adjoint and an imaginary multiple of semicircular elements

Hall and the author computed in [21] the Brown measure of \( x_0 + i\sigma_t \), a sum of a self-adjoint element and an imaginary multiple of semicircular element. The computation of the Brown measure of elements of the form \( x_0 + i\sigma_t \) covers the case \( x_0 + c_t \) which has the same \(*\)-moments as \( x_0 + \sigma_{t/2} + i\delta_{t/2} \) where \( \delta_{t/2} \) is freely independent semicircular elements, both freely independent of \( x_0 \). The results in [21] show that there is a connection between the Brown measure of \( x_0 + i\sigma_t \), that of \( x_0 + c_t \) as well as the law of \( x_0 + \sigma_t \), for the same self-adjoint element \( x_0 \).

We need the following notations to describe the results in [21].

**Definition 2.7.** Let \( x_0 \) be a self-adjoint element.

1. Given any \( r \in \mathbb{R} \), let \( H_{x_0,r}(z) = z + rG_{x_0}(z) \), \( z \in \Delta_{x_0,|r|} \). Compared to the holomorphic function \( H \) in Definition 2.3 we allow \( r \) negative in this notation. By the results in [21], for \( t > 0 \), the map \( H_{x_0,-t}(z) \) is an injective conformal map on \( \Delta_{x_0,t} \) (see Definition 2.3 using \( x_0 \) and the positive \( t \), not \(-t\)). In [21], the authors use the notation \( J_t \) instead of \( H_{x_0,-t} \). Furthermore, \( H_{x_0,r} \) can be extended on \( \mathbb{C} \) by Schwarz reflection.

2. Define \( h_{x_0,t}(u) = \text{Re}[H_{x_0,-t}(u + iv_{x_0,t}(u))] \) on \( \mathbb{R} \). This function \( h_{x_0,t} \) is a homeomorphism from \( \mathbb{R} \) to \( \mathbb{R} \); it is a strictly increasing function. If \( v_{x_0,t}(u) > 0 \), we have \( h_{x_0,t}(u) > 0 \).

3. Denote by \( h_{x_0,t}^{-1} \) the inverse of \( h_{x_0,t} \).

The following theorem established in [21] computes the Brown measure of \( x_0 + i\sigma_t \).

**Theorem 2.8.** Let
\[
\Omega_{x_0,t} = [H_{x_0,-t}(\Lambda_{x_0,t}^c)]^c.
\]
Then we can write \( \Omega_{x_0,t} \) as
\[
\Omega_{x_0,t} = \{ a + ib \in \mathbb{C} | |b| < b_{x_0,t}(a) \}
\]
where \( b_{x_0,t}(a) = 2v_{x_0,t}(h_{x_0,t}^{-1}(a)) \) is a nonnegative function on \( \mathbb{R} \). The set \( \Omega_{x_0,t} \) itself is a set of full measure with respect to \( \text{Brown}(x_0 + i\sigma_t) \).

Inside \( \Omega_{x_0,t} \), \( \text{Brown}(x_0 + i\sigma_t) \) is absolutely continuous with respect to the Lebesgue measure on the plane with a strictly positive density; the density has the form
\[
\frac{1}{2\pi t} \left( \frac{db_{x_0,t}^{-1}(a)}{da} - \frac{1}{2} \right), \quad a + ib \in \Omega_{x_0,t}.
\]
In particular, the density is independent of \( b \) and is constant along the vertical segments.
We now describe the connections of Brown$(x_0 + c_t)$, Brown$(x_0 + i\sigma_t)$, and Law$(x_0 + \sigma_t)$. Let $U_{x_0,t} : \overline{\Lambda_{x_0,t}} \to \overline{\Omega_{x_0,t}}$ be a homeomorphism defined by
\[U_{x_0,t}(u + iv) = h_{x_0,t}(u) + 2iv.\]
Note that the map $U_{x_0,t}$ takes the vertical line segments in $\Lambda_{x_0,t}$ linearly to vertical line segments in $\Omega_{x_0,t}$. Also, recall that $\Lambda_{x_0,t}$ defined in [2.2] is an open set of full measure of Brown$(x_0 + c_t)$. The following theorem establishes the push-forward relations between Brown$(x_0 + c_t)$, Brown$(x_0 + i\sigma_t)$ and Law$(x_0 + \sigma_t)$. It is proved in [21].

**Theorem 2.9.**

1. The push-forward measure of Brown$(x_0 + c_t)$ under $U_{x_0,t}$ is the Brown measure Brown$(x_0 + i\sigma_t)$.

2. The push-forward of Brown$(x_0 + i\sigma_t)$ under the map
\[Q_{x_0,t}(a + ib) := 2h_{x_0,t}^{-1}(a) - a\] is the law of $x_0 + \sigma_t$. The map $Q_{x_0,t}$ agrees with $\Psi_{x_0,t} \circ U_{x_0,t}^{-1}$ where $\Psi_{x_0,t}$ is defined in Theorem 2.8. Alternatively, by Definition 8.1 of [21], we can write
\[Q_{x_0,t}(a + ib) = H_{x_0,t} \circ H_{x_0,-t}^{-1}(a + ib_{x_0,t}(a)), \quad a \in \Omega_{x_0,t}.\]

Moreover, $Q_{x_0,t}$ is a diffeomorphism on $\Omega_{x_0,t} \cap \mathbb{R}$.

Although $Q_{x_0,t}$ is not an invertible map, Point 2 of Theorem 2.9 characterizes the probability measure on $\Omega_{x_0,t}$ whose density is constant along vertical segments. Similar results of the following proposition for the Brown measures of different random variables can be found in [12, 23].

**Proposition 2.10.** The Brown measure of $x_0 + i\sigma_t$ is the unique measure $m$ on $\Omega_{x_0,t}$ that is absolutely continuous with respect to the Lebesgue measure such that the density is constant along vertical segments and the push-forward of $m$ by $Q_{x_0,t}$ is Law$(x_0 + \sigma_t)$.

**Proof.** Suppose that $dm(a + ib) = g(a) \, da \, db$ on $\Omega_{x_0,t}$. Write $u = Q_{x_0,t}(a)$. Since $\Omega_{x_0,t}$ has the form described in Theorem 2.8, the push-forward of $m$ by $Q_{x_0,t}$ has the form
\[4\pi t(h_{x_0,t}^{-1}(a))g(a) \, da = 4\pi t(h_{x_0,t}^{-1}(a))g(a) \frac{da}{du}, \quad u \in Q_{x_0,t}(\Omega_{x_0,t} \cap \mathbb{R}).\]

By the definition (2.3) of $Q_{x_0,t}$ and Theorem 2.8 the density of Brown$(x_0 + i\sigma_t)$ has the form $(1/4\pi t)(du/da)$ that is strictly positive.

By Point 2 of Theorem 2.9, taking $g(a) = (1/4\pi t)(du/da)$ to be the density of Brown$(x_0 + i\sigma_t)$ gives Law$(x_0 + \sigma_t)$; that is, Law$(x_0 + \sigma_t)$ has the form
\[\frac{1}{\pi t} \frac{1}{h_{x_0,t}^{-1}(a)} \, du, \quad u \in Q_{x_0,t}(\Omega_{x_0,t}).\]

Since $du/da$ is positive, the only $g(a)$ that makes the measure in (2.4) equal to Law$(x_0 + i\sigma_t)$ is $(1/4\pi t)(du/da)$. This shows that Brown$(x_0 + i\sigma_t)$ is the only measure on $\Omega_{x_0,t}$ that is absolutely continuous with respect to the Lebesgue measure such that the density is constant along vertical segments and the push-forward of $m$ by $Q_{x_0,t}$ is Law$(x_0 + \sigma_t)$.

### 3 The Brown measure computation

Let $y_0$ be a self-adjoint element, $\sigma_{s-\frac{i}{2}}$ and $\sigma_{\frac{i}{2}}$ be two semicircular elements, all freely independent. Denote the law of $y_0$ by $\nu$. We study the Brown measure of
\[y_0 + \sigma_{s-\frac{i}{2}} + i\sigma_{\frac{i}{2}}\]
with $0 < \frac{i}{2} \leq s$.

If the law of $y_0 + \sigma_{s-\frac{i}{2}}$ is a Dirac mass at one point, then the Brown measure of $y_0 + \sigma_{s-\frac{i}{2}} + i\sigma_{\frac{i}{2}}$ is singular with respect to the Lebesgue measure on the plane, and is a semicircular distribution along a vertical segment. Thus, we recall our standing assumption (Assumption 1.1) that either $s > \frac{i}{2}$ or $\nu$ is not a Dirac mass, so that Law$(y_0 + \sigma_{s-\frac{i}{2}})$ is not a Dirac mass.

For convenience, we define
\[x_0 = y_0 + \sigma_{s-\frac{i}{2}}.\]
In particular, Proposition 3.2.

The function $\Lambda_{y_0,s}$ is an injective conformal mapping from $\Omega_{s,t}$ to the complement of the support of $\text{Law}(y_0 + \sigma_s)$. We also write the boundary of $\Omega_{s,t}$ as $\Omega_{s,t}$. This shows that, when $x_0 = 0$, $F_{s,t}$ is the additive analogue of the function $f_{s,t}$ introduced in (2.3) in the context of free Segal–Bargmann–Hall transform.

**Proposition 3.2.** The inverse $F_{s,t}^{-1}$ of $F_{s,t}$ can be written as

$$F_{s,t}^{-1}(z) = (\Lambda_{y_0,s})^{-1}(z)$$

for all $z$ outside the support of $\text{Law}(y_0 + \sigma_s)$.

This shows that, when $y_0 = 0$, $F_{s,t}$ is the additive analogue of the function $f_{s,t}$ introduced in (2.3) in the context of free Segal–Bargmann–Hall transform.

**Proof.** Recall that we denote $y_0 + \sigma_s$ by $x_0$. By Theorem 2.4,

$$G_{y_0 + \sigma_s} (H_{x_0,t/2}(z)) = G_{x_0} (H_{x_0,t/2}(z)) = G_{x_0}(z) = G_{y_0 + \sigma_s}(z)$$

because $\sigma_{s-t/2}$ has the same distribution as $\sigma_s$. When $z$ is large, (3.4) becomes

$$H_{x_0,t/2}^{-1}(z) = K_{y_0 + \sigma_{s-t/2}}(G_{y_0 + \sigma_s}(z)).$$
Since the $R$-transform of the sum of two freely independent variables is the sum of the $R$-transforms of each variable (See Section 2.1.1),

$$R_{y_0 + \alpha, s - t/2}(z) = R_{y_0}(z) + R_{\alpha, s - t/2}(z) = R_{y_0}(z) + \left( s - \frac{t}{2} \right) z.$$ 

Subtracting by $\frac{1}{z}$ gives us

$$K_{y_0 + \alpha, s - t/2}(z) = K_{y_0}(z) + \left( s - \frac{t}{2} \right) z. \quad (3.6)$$

Therefore,

$$K_{y_0 + \alpha, s - t/2}(G_{y_0 + \alpha}(z)) = K_{y_0}(G_{y_0 + \alpha}(z)) + \left( s - \frac{t}{2} \right) G_{y_0 + \alpha}(z). \quad (3.7)$$

By the definition of $F_{s,t}^{-1}$ in (3.1),

$$F_{s,t}^{-1}(z) = H_{x_0, -t/2}(H_{x_0, t/2}^{-1}(z)) = H_{x_0, t/2}^{-1}(z) - \frac{t}{2}G_{x_0, s - t/2}(H_{x_0, t/2}^{-1}(z)) \quad (3.8)$$

Using (3.5) and (3.7), the above becomes

$$F_{s,t}^{-1}(z) = K_{y_0 + \alpha, s - t/2}(G_{y_0 + \alpha}(z)) - \frac{t}{2}G_{y_0 + \alpha}(z)$$

$$= K_{y_0}(G_{y_0 + \alpha}(z)) + \left( s - \frac{t}{2} \right) G_{y_0 + \alpha}(z) - \frac{t}{2}G_{y_0 + \alpha}(z) \quad (3.9)$$

Now, since $H_{y_0, s}$ satisfies $G_{y_0 + \alpha, s}(H_{y_0, s}(z)) = G_{y_0}(z)$, we have

$$H_{y_0, s}^{-1}(z) = K_{y_0}(G_{y_0 + \alpha}(z))$$

for all large enough $|z|$. It follows from (3.9) that $F_{s,t}^{-1}$ can be written as

$$F_{s,t}^{-1}(z) = H_{y_0, s}^{-1}(z) + (s - t)G_{y_0}(H_{y_0, s}^{-1}(z)) = (H_{y_0, s - t} \circ H_{y_0, s}^{-1})(z)$$

for all large enough $z$. Since both sides of the above expression are defined on the complement of the support of $\text{Law}(y_0 + \alpha, s)$, (3.3) holds for all $z$ in the complement of the support of $\text{Law}(y_0 + \alpha, s)$ by analytic continuation. \hfill \Box

**Proof of Theorem 3.1** The function $F_{s,t}^{-1}$ is an injective conformal map on the complement of the support of $\text{Law}(y_0 + \alpha, s)$. Thus, by Proposition 3.3

$$H_{y_0, s - t}(z) = F_{s,t}^{-1} \circ H_{y_0, s}(z), \quad z \in \Delta_{y_0, s}$$

is an injective conformal map onto

$$\{a + ib \in \mathbb{C} \mid |b| > b_{s,t}(a)\}.$$ 

Now, that the function $H_{y_0, s - t}$ extends to a homeomorphism on $\Delta_{y_0, s}$ follows from an elementary topological argument by regarding $\Delta_{y_0, s} \cup \{\infty\}$ and $\{a + ib \in \mathbb{C} \mid |b| > b_{x_0, t}(a)\} \cup \{\infty\}$ as two disks in the Riemann sphere. Thus, $H_{y_0, s - t}$ is an injective conformal map on $(\Lambda_{y_0, s})^c$ and extends to a homeomorphism on $\Lambda_{y_0, s}$ by Schwarz reflection about the real axis.

Equation (3.2) is a restatement of Proposition 3.3. If $s = t$, the holomorphic function $H_{y_0, s - t}$ is the identity map; therefore, $\Omega_{s,t} = \Lambda_{y_0, s}$ by (3.2). \hfill \Box

### 3.2 Two push-forward properties

In Section 2.1.1 we establish the connection between $\Lambda_{y_0, s}$ and $\Omega_{s,t}$ through the map $H_{y_0, s - t}$. In this section, we prove that the push-forward measure of $\text{Brown}(y_0 + \alpha, s)$ by a canonical map constructed using $H_{y_0, s - t}$ is $\text{Brown}(y_0 + \alpha, s - t/2 + i\sigma_{t/2})$. The main observation is that both $\text{Brown}(y_0 + \alpha, s)$ and $\text{Brown}(y_0 + \alpha, s - t/2 + i\sigma_{t/2})$ can be pushed forward to $\text{Law}(y_0 + \alpha, s)$, by Theorems 2.6 and 2.9. These push-forward maps are not injective; nevertheless, Proposition 2.10 shows that they characterize $\text{Brown}(y_0 + \alpha, s)$ and $\text{Brown}(y_0 + \alpha, s - t/2 + i\sigma_{t/2})$.

For convenience, we use the notations $a + ib$ for the points in $\Omega_{s,t}$, $\alpha + i\beta$ for the points in $\Lambda_{y_0, s}$, and $u$ for the points in the support of $\text{Law}(y_0 + \alpha, s)$.  

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Figure 2: Push-forward relations between the probability measures $\text{Brown}(y_0 + c_s)$, $\text{Brown}(y_0 + \sigma_{t/2} + i\sigma_{t/2})$, and $\text{Law}(y_0 + \sigma_s)$, where $x_0 = y_0 + \sigma_{t/2}$.

Define the function $a_{s,t} : \mathbb{R} \to \mathbb{R}$ by

\[ a_{s,t}(\alpha) = \Re[H_{y_0,s-t}(\alpha + iv_{y_0,s}(\alpha))], \quad \alpha \in \mathbb{R}. \]

Let $U_{s,t} : \Lambda_{y_0,s} \to \Omega_{s,t}$ be defined by

\[
\begin{align*}
\Re U_{s,t}(\alpha + i\beta) &= a_{s,t}(\alpha) \\
\Im U_{s,t}(\alpha + i\beta) &= \frac{t\beta}{s}.
\end{align*}
\]

We will prove that $a_{s,t}$ is a homeomorphism on $\mathbb{R}$ in Proposition 3.4. We can then immediately see that $U_{s,t}$ is indeed a homeomorphism on the complex plane $\mathbb{C}$. In this section, we prove the following two push-forward properties that are introduced in Points 2 and 3 of Theorem 1.2.

**Theorem 3.3.** We have the following results about push-forward measures.

1. The push-forward of $\text{Brown}(y_0 + c_s)$ under the map $U_{s,t}$ is $\text{Brown}(y_0 + \sigma_{t/2} + i\sigma_{t/2})$.

2. The push-forward of $\text{Brown}(y_0 + \sigma_{t/2} + i\sigma_{t/2})$ by the map

\[ Q_{s,t}(a + ib) = \frac{1}{s - t}[sa - t\alpha_{s,t}(a)] \]

is $\text{Law}(y_0 + \sigma_s)$.

Recall that the function $F_{s,t}$ is defined in (3.1). By Theorems 2.7 and 2.9 the push-forward of $\text{Brown}(y_0 + c_s)$ by $\Psi_{y_0,s}$ defined by

\[ \Psi_{y_0,s}(\alpha + i\beta) = H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)), \quad \alpha + i\beta \in \Lambda_{y_0,s} \]

and the push-forward of $\text{Brown}(y_0 + \sigma_{t/2} + i\sigma_{t/2})$ by $Q_{x_0,t/2}$ (where $x_0 = y_0 + \sigma_{t/2}$) defined by

\[ Q_{x_0,t/2}(a + ib) = F_{s,t}(a + ib_{s,t}(a)) \]

are both $\text{Law}(y_0 + \sigma_{t/2} + i\sigma_{t/2})$. In the proof of Theorem 3.3 we actually can see that $Q_{s,t} = Q_{x_0,t/2}$. Figure 2 illustrates the push-forward relations between all of these measures.

Before we prove this theorem, we first study the function $a_{s,t}$ in the definition of $U_{s,t}$.

**Proposition 3.4.** The function $a_{s,t}$ is strictly increasing. It is a homeomorphism onto $\mathbb{R}$. In particular, $a_{s,t}$ has an inverse on $\mathbb{R}$ that is also strictly increasing. Furthermore, $a'_{s,t}(\alpha) > 0$ for all $\alpha \in \Lambda_{y_0,s} \cap \mathbb{R}$.

The upper boundary curve $\alpha + ib_{s,t}(\alpha)$ of $\Omega_{s,t}$ can be parametrized by $\alpha \in \Lambda_{y_0,s} \cap \mathbb{R}$. The parameterization is

\[ a + ib_{s,t}(\alpha) = a_{s,t}(\alpha) + \frac{it}{s}v_{y_0,s}(\alpha). \]

**Proof.** By a direct computation,

\[ a_{s,t}(\alpha) = \frac{s - t}{s} \left( \frac{t\alpha}{s - t} + \Re[H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha))] \right). \]
If \( s > t \), then \( a_{s,t} \) is strictly increasing because \( \text{Re}[H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha))] \) is strictly increasing in \( \alpha \in \mathbb{R} \) by Theorem 2.4. If \( s < t \), then we write

\[
a_{s,t}(\alpha) = \frac{t - s}{s} \left( \frac{(2s - t)\alpha}{t - s} + \text{Re}[H_{y_0,-s}(\alpha + iv_{y_0,s}(\alpha))] \right)
\]

which is a strictly increasing function since \( \text{Re}[H_{y_0,-s}(\alpha + iv_{y_0,s}(\alpha))] \) is strictly increasing in \( \alpha \in \mathbb{R} \), by Point 2 of Definition 2.7. If \( s = t \), \( a_{s,t} \) is just the identity function. In any case, if \( v_{y_0,s}(\alpha) > 0 \), \( a_{s,t} \) is differentiable at \( \alpha \) and \( a'_{s,t}(\alpha) > 0 \) by Point 2 of Definition 2.7.

By Theorem 3.1, \( a + iv_{y_0,s}(\alpha) = H_{s-t}(\alpha + iv_{y_0,s}(\alpha)) \) for a unique \( \alpha \in \Lambda_{y_0,s} \cap \mathbb{R} \). The imaginary part of \( H_{s-t}(\alpha + iv_{y_0,s}(\alpha)) \) is given by

\[
v_{y_0,s}(\alpha) \left( 1 - (s - t) \int \frac{1}{(\alpha - x)^2 + v_{y_0,s}(\alpha)^2} \, d\nu(x) \right) = \frac{t}{s} v_{y_0,s}(\alpha)
\]

by (2.1). This proves the parametrization (3.11).

**Proposition 3.5.** The function \( U_{s,t} : \Lambda_{y_0,s} \to \Omega_{s,t} \) defined by (3.10) is a diffeomorphism; it extends to a homeomorphism from \( \Lambda_{y_0,s} \) to \( \Omega_{s,t} \). Moreover, it agrees with \( H_{y_0,-s-t} \) on the boundary of \( \Lambda_{y_0,s} \).

**Proof.** By Point 1 of Theorem 3.7, \( a_{s,t} \) is injective, strictly increasing and differentiable in \( \Lambda_{y_0,s} \cap \mathbb{R} \) with nonzero derivative; therefore, \( U_{s,t} \) is a diffeomorphism from \( \Lambda_{y_0,s} \) onto \( \Omega_{s,t} \). Since \( a_{s,t} \) is a homeomorphism defined on \( \mathbb{R} \), the map \( U_{s,t} \) can be extended to a homeomorphism in \( \mathbb{C} \); in particular, it is a homeomorphism from \( \Lambda_{y_0,t} \) to \( \Omega_{s,t} \).

It is clear from (3.11) that \( U_{s,t} \) agrees with \( H_{y_0,-s-t} \) on the boundary of \( \Lambda_{y_0,s} \). \( \square \)

Before we prove Theorem 3.3, we write the function \( \alpha_{s,t} \) in Theorem 3.7 as the solution of the following integral equation

\[
a = \alpha_{s,t}(\alpha) + (s - t) \int \frac{(\alpha_{s,t}(\alpha) - x) \, d\nu(x)}{(\alpha_{s,t}(\alpha) - x)^2 + v_{y_0,s}(\alpha_{s,t}(\alpha))^2}.
\]

**Proof of Theorem 3.3** Recall that the density of \( \text{Brown}(y_0 + c_s) \) is constant along vertical segments in \( \Lambda_{y_0,s} \). By (3.10), the Jacobian matrix of \( U_{s,t} \) on \( \Lambda_{y_0,s} \) is diagonal and \( \text{Im}(U_{s,t}(\alpha + i\beta)) \) depends linearly in \( \beta \). Thus, the density of the push-forward measure of \( \text{Brown}(y_0 + c_s) \) by \( U_{s,t} \) is again constant along vertical segments in \( \Omega_{s,t} \).

We apply Proposition 2.10 to show that the push-forward of \( \text{Brown}(y_0 + c_s) \) by \( U_{s,t} \) is \( \text{Brown}(y_0 + \tilde{s}_s-t/2 + i\tilde{t}_t/2) \). By Proposition 3.2, for any \( \alpha + i\beta \in \Lambda_{y_0,s} \),

\[
Q_{x_0,t} \circ U_{s,t}(\alpha + i\beta) = F_{s,t}(a_{s,t}(\alpha) + ib_{s,t}(a_{s,t}(\alpha))) = H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) = \Psi_{y_0,s}(\alpha + i\beta).
\]

This shows that if we further push forward by \( Q_{x_0,t} \) the push-forward of \( \text{Brown}(y_0 + c_s) \) by \( U_{s,t} \), we get the push-forward of \( \text{Brown}(y_0 + c_s) \) by \( \Psi_{y_0,s} \), which is \( \text{Law}(y_0 + \sigma_s) \) by Theorem 2.6. This completes the proof of Point 1 of the theorem.

We now prove Point 2. By Point 1, \( \text{Brown}(y_0 + \tilde{s}_s-t/2 + i\tilde{t}_t/2) \) is the push-forward measure of \( \text{Brown}(y_0 + c_s) \). Since \( U_{s,t} \) is a diffeomorphism on \( \Lambda_{y_0,s} \), the push-forward of \( \text{Brown}(y_0 + \tilde{s}_s-t/2 + i\tilde{t}_t/2) \) by \( \Psi_{y_0,s} \circ U_{s,t}^{-1} \) is \( \text{Law}(y_0 + \sigma_s) \). (In fact, by the proof of Point 1, \( \Psi_{y_0,s} \circ U_{s,t}^{-1} = Q_{x_0,t} \)) We then compute

\[
\Psi_{y_0,s} \circ U_{s,t}^{-1}(a + ib) = \Psi_{y_0,s}(\alpha_{s,t}(a) + \frac{s}{t}b) = \alpha_{s,t}(a) + s \int \frac{\alpha_{s,t}(a) - x}{(\alpha_{s,t}(a) - x)^2 + v_{y_0,s}(\alpha_{s,t}(a))^2} \, d\nu(x)
\]

which we use (3.12) in the last equality. The above equation simplifies to the definition of \( Q_{x_0,t} \), completing the proof. \( \square \)

The density \( w_{y_0,s} \) of \( \text{Brown}(y_0 + \tilde{s}_s-t/2 + i\tilde{t}_t/2) \) can be computed in terms of the density \( w_{y_0,s} \) of \( \text{Brown}(y_0 + c_s) \). We will give an alternative formula in the next section.
Corollary 3.6. Let \( r = t/s \) and write \( a + ib = U_{s,t}(\alpha + i\beta) \) for all \( \alpha + i\beta \in \Lambda_{y_0,s} \). Then we have
\[
w_{y_0,s,t}(a + ib) = \frac{w_{y_0,s}(\alpha + i\beta)}{r r + 2\pi(1 - r)s \cdot w_{y_0,s}(\alpha + i\beta)}
\]
for all \( a + ib \in \Omega_{s,t} \).

Proof. Denote \( r = t/s \). We can write the function \( a_{s,t}(\alpha) \) defined in Proposition 3.4 as
\[
a_{s,t}(\alpha) = \alpha + (1 - r)s \text{Re} \left[ \int \frac{dv(y)}{\alpha + iv_{y_0,s}(\alpha) - y} \right] = \alpha + (1 - r)[H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) - \alpha] = (1 - r)\psi_{y_0,s}(\alpha) + r\alpha.
\]
So, we have
\[
\frac{da_{s,t}(\alpha)}{d\alpha} = r + 2\pi(1 - r)s \cdot w_{y_0,s}(\alpha + i\beta).
\]
By Theorem 3.3 we can compute the density \( w_{y_0,s,t}(a + ib) \) \( da \) \( db \) in terms of \( w_{y_0,s} \) as
\[
\begin{align*}
\frac{w_{y_0,s,t}(a + ib) \cdot da \cdot db}{w_{y_0,s}(\alpha + i\beta) \cdot da \cdot db} &= \frac{w_{y_0,s}(\alpha + i\beta) \cdot \frac{da}{da} \cdot db}{\frac{da}{da} \cdot db} = \frac{1}{r} \frac{w_{y_0,s}(\alpha + i\beta)}{r r + 2\pi(1 - r)s \cdot w_{y_0,s}(\alpha + i\beta)} \cdot da \cdot db,
\end{align*}
\]
completing the proof.

3.3 The density of the Brown measure

The main theorem of this section is to compute the density of \( \text{Brown}(y_0 + \hat{\sigma}_{s-t/2} + i\sigma_{t/2}) \) stated in Point 1 of Theorem 1.2

Theorem 3.7. The Brown measure of \( y_0 + \hat{\sigma}_{s-t/2} + i\sigma_{t/2} \) is absolutely continuous with respect to the Lebesgue measure on the plane and is supported on \( \Omega_{s,t} \). The open set \( \Omega_{s,t} \) is a set of full measure of the Brown measure. The density of the Brown measure is given by
\[
w_{y_0,s,t}(a + ib) = \frac{1}{2\pi t} \left( 1 + t \frac{d}{da} \int \frac{\alpha_{s,t}(\alpha) - x}{(\alpha_{s,t}(\alpha) - x)^2 + v_{y_0,s}(\alpha_{s,t}(\alpha))^2} dv(y) \right)
\]
on the set \( \Omega_{s,t} \). In particular, the density is constant along the vertical segments.

Proof. We only need to compute the density. The proof uses the first push-forward property stated in Theorem 3.3. By Theorem 2.6 Brown\((y_0 + c_s)\) is given by
\[
\begin{align*}
\frac{1}{2\pi s} \frac{d}{da} H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) &\cdot da \cdot db = \frac{1}{2\pi s} \frac{d}{da} \left( \alpha_{s,t}(\alpha) + t \int \frac{(\alpha - x) dv(y)}{(\alpha - x)^2 + v_{y_0,s}(\alpha)^2} \right) \cdot da \cdot db
\end{align*}
\]
for \( \alpha + i\beta \in \Lambda_{y_0,s} \). The determinant of the Jacobian matrix of \( U_{s,t} \) defined in (3.10) is \( (t/s)(da_{s,t}/da) \). By the push-forward property in Point 1 of Theorem 3.3, we compute \( \text{Brown}(y_0 + \hat{\sigma}_{s-t/2} + i\sigma_{t/2}) \) by doing a change of variable \( a + ib = a_{s,t}(\alpha) + i(t/s)\beta \) to the above formula of Brown\((y_0 + c_s)\) and get
\[
\text{Brown}(y_0 + \hat{\sigma}_{s-t/2} + i\sigma_{t/2}) = \frac{1}{2\pi t} \frac{d}{da} \left( a + t \int \frac{(\alpha_{s,t}(\alpha) - x) dv(y)}{(\alpha_{s,t}(\alpha) - x)^2 + v_{y_0,s}(\alpha_{s,t}(\alpha))^2} \right) \cdot da \cdot db
\]
on \( \Omega_{s,t} \). We have completed the proof.
Before we end this section, we prove Proposition 1.3 in the following corollary.

**Corollary 3.8.** Given any $a \in \mathbb{R}$, (1.1) has a pair of solution $\alpha \in \mathbb{R}$ and $v > 0$ if and only if $a \in \Omega_{s,t} \cap \mathbb{R}$. In this case, the solution is unique; moreover, we have $\alpha_{s,t}(a) = \alpha, v_{y_0,s}(\alpha_{s,t}(a)) = v$ and $b_{s,t}(a) = \frac{v}{s}v$.

**Proof.** Let $a \in \Omega_{s,t} \cap \mathbb{R}$. Then, by (2.1) and (3.12), $\alpha = \alpha_{s,t}(a)$ and $v = v_{y_0,s}(\alpha_{s,t}(a))$ is a pair of solution of (1.1). This shows existence of the equation. We now show the solution is indeed unique. Suppose that $\alpha \in \mathbb{R}$ and $v > 0$ is a pair of solution. We must show that $\alpha = \alpha_{s,t}(a)$ and so $\alpha = \alpha_{s,t}(a)$.

Conversely, suppose that (1.1) has a pair of solution $\alpha \in \mathbb{R}$ and $v > 0$. Then the argument that shows uniqueness of solution in the preceding paragraph proves that $v = v_{y_0,s}(\alpha_{s,t}(a))$ and so $\alpha = \alpha_{s,t}(a)$. Thus, (3.11) shows $b_{s,t}(a) = tv/s > 0$, and so $a \in \Omega_{s,t} \cap \mathbb{R}$.

\[\square\]

\section{Asymptotic behaviors of adding a circular element}

\subsection{The graph of $v_{y_0,s}$ as $s \to \infty$}

In this section, we study the asymptotic behavior of $v_{y_0,s}$ and $\Lambda_{y_0,s}$ as $s \to \infty$. Below is the main theorem of this section.

**Theorem 4.1.** The following asymptotic behaviors of the graph of $v_{y_0,s}$ hold.

1. Let $D_{\nu} = \sup \{|x - y| \mid x, y \in \text{supp } \mu\}$. When $s \geq 4D_{\nu}^2$, the function $v_{y_0,s}$ is unimodal. In particular, $\Lambda_{y_0,s} \cap \mathbb{R}$ is an interval.

2. Given any $c > 1$, we have

\[|\sup \Lambda_{y_0,s} \cap \mathbb{R} - (\tau(y_0) + \sqrt{s})| < \frac{3c\tau(y_0^2)}{2\sqrt{s}}\]

and

\[|\inf \Lambda_{y_0,s} \cap \mathbb{R} - (\tau(y_0) - \sqrt{s})| < \frac{3c\tau(y_0^2)}{2\sqrt{s}}\]

for all large enough $s$. In particular,

\[\Lambda_{y_0,s} \cap \mathbb{R} \subset \left(\tau(y_0) - \sqrt{s} - \frac{3c\tau(y_0^2)}{2\sqrt{s}}, \tau(y_0) + \sqrt{s} + \frac{3c\tau(y_0^2)}{2\sqrt{s}}\right)\]

for all large enough $s$.

3. Given $\phi_0 \in (0, \pi/2)$, then for all large enough $s$, for all $|\cos \phi| \leq \cos \phi_0$, the unique $\alpha \in \mathbb{R}$ such that

\[H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) = 2\sqrt{s} \cos \phi\]

satisfies

\[|\alpha + iv_{y_0,s}(\alpha) - \sqrt{s}e^{i\phi}| < \frac{1}{(\sin \phi_0)\sqrt{s}}\]

Point 1 of Theorem 4.1 is a known result in [22 Theorem 3.2]. We state it here for completeness; it is also useful for us to understand the asymptotic behaviors of $\Lambda_{y_0,s}$.
We study the asymptotic behaviors of \( v_{y_0,s} \) by looking at \( v_{y_0,1} \), whose graph is scaled by \( \sqrt{s} \) the graph of \( v_{y_0,s} \). We look at

\[
H_{y_0,1}(z) = z + G_{y_0}(z).
\]

If \( s \) is large enough, \( H_{y_0,1} \) is defined for all \( |z| > \frac{1}{2} \) since \( y_0 \) is assumed to be bounded.

We assume \( y_0 \) is centered and has unit variance until the proof of Theorem 4.1 for simplicity. The function \( H_{y_0,1} \) is the inverse subordination function of the free convolution \( \frac{\tau(y_0^0)}{\sqrt{s}} + \sigma_1 \). When \( s \) is large, \( \frac{\tau(y_0^0)}{\sqrt{s}} + \sigma_1 \) behaves like \( \sigma_1 \); our strategy is to compare \( \frac{\tau(y_0^0)}{\sqrt{s}} + \sigma_1 \) with \( \sigma_1 \). Denote by \( k(z) \) the function \( H_{0,1}(z) \); that is

\[
k(z) = z + \frac{1}{z}.
\]

The techniques in this section are similar to techniques in proving the supercoverage results in [5, 6, 34].

**Lemma 4.2.** Assume \( y_0 \) is a bounded random variable with \( \tau(y_0) = 0 \) and \( \tau(y_0^1) = 1 \). Then given any \( c > 1 \), there exists \( s_0 > 0 \) such that

\[
|H_{y_0,1}(z) - k(z)| < \frac{c}{s|z|^3}, \quad |z| > \frac{1}{2}
\]

for all \( s \geq s_0 \).

**Proof.** When \( s \) is large enough, we can write

\[
H_{y_0,1}(z) = k(z) + \frac{1}{s} \sum_{n=2}^{\infty} \frac{\tau(y_0^0)}{s^{\frac{n}{2} - 1} z^n + 1}
\]

for all \( |z| > \frac{1}{2} \). Observe that

\[
\left| \sum_{n=2}^{\infty} \frac{\tau(y_0^0)}{s^{\frac{n}{2} - 1} z^n + 1} \right| \leq \frac{\tau(y_0^0)}{|z|^3} + \frac{1}{|z|^3} \sum_{n=3}^{\infty} \frac{\tau(y_0^0)}{s^{\frac{n}{2} - 1}(1/2)^{n-2}}
\]

for all \( |z| > \frac{1}{2} \). Since we assume \( \tau(y_0^0) = 1 \) and

\[
\lim_{s \to \infty} \sum_{n=3}^{\infty} \frac{\tau(y_0^0)}{s^{\frac{n}{2} - 1}(1/2)^{n-2}} = 0,
\]

the result follows.

We compute that \( k'(z) = 1 - \frac{1}{z^2} \); the double zeros of \( k \) are 1 and \(-1\). The next lemma shows that \( H_{y_0,1} \) also has double zeros at a point close to 1 and a point close to \(-1\). Since \( v_{y_0,1} \) is unimodal for large \( s \), these two points are the only double zeros of \( H_{y_0,1} \). Since \( H_{y_0,1} \) is symmetric about the real axis, these two double zeros must be real numbers. Again since \( v_{y_0,1} \) is unimodal for large \( s \), \( \Lambda_{y_0,1} \cap \mathbb{R} \) is an open interval and the two double zeros of \( H_{y_0,1} \) are the endpoints of \( \Lambda_{y_0,1} \cap \mathbb{R} \).

**Lemma 4.3.** Given any \( c > 1 \), there exists \( s_0 \) such that

\[
|H'_{y_0,1}(\pm 1 + re^{i\theta}) - k'(\pm 1 + re^{i\theta})| < \frac{3c}{s(1-r)^3}
\]

for all \( s \geq s_0 \) and \( r < \frac{1}{2} \).

**Proof.** Recall that

\[
H_{y_0,1}(z) = k(z) + \frac{1}{s} \sum_{n=2}^{\infty} \frac{\tau(y_0^0)}{s^{\frac{n}{2} - 1} z^n + 1};
\]

we compute

\[
H'_{y_0,1}(z) = 1 - \frac{1}{z^2} - \frac{1}{s} \left( \frac{3\tau(y_0^0)}{z^4} + \frac{1}{z^4} \sum_{n=3}^{\infty} \frac{(n+1)\tau(y_0^0)}{s^{\frac{n}{2} - 1} z^{n-2}} \right)
\]

(4.1)
Let $c > 1$ be given. If $z = 1 + re^{i\theta}$ with $r < 1/2$, then for all large enough $s$,

$$
\left| \frac{3\tau(y_0^2)}{z^4} + \frac{1}{z^4} \sum_{n=3}^{\infty} \frac{\alpha(n+1)\tau(y_0^2)}{s^{n-1}z^{n-2}} \right| < \frac{3c}{(1-r)^2}
$$

since $|z| > 1 - r > 1/2$ and $\tau(y_0^2) = 1$. The case for $z = 1 - re^{i\theta}$ is similar.

\[ \square \]

**Proposition 4.4.** We have

$$1 - \frac{3c}{2s} < \sup_{v_{\psi,1}} \Lambda_{v_{\psi,1}} \cap \mathbb{R} < 1 + \frac{3c}{2s}$$

and

$$-1 - \frac{3c}{2s} < \inf_{v_{\psi,1}} \Lambda_{v_{\psi,1}} \cap \mathbb{R} < -1 + \frac{3c}{2s}$$

for all large enough $s$. In particular,

$$\Lambda_{v_{\psi,1}} \cap \mathbb{R} \subset \left(-1 - \frac{3c}{2s}, 1 + \frac{3c}{2s}\right)$$

for all large enough $s$.

**Proof.** Recall that $\Lambda_{v_{\psi,1}} \cap \mathbb{R}$ and $\Lambda_{v_{\psi,1}} \cap \mathbb{R}$ are the only double zeros for $H_{v_{\psi,1}}$ when $s$ is large enough so that $v_{y_0,z}$ is unimodal.

Let $c > 1$. We compute, with $z = 1 + re^{i\theta}$,

$$\left| 1 - \frac{1}{z^2} \right| = \left| \frac{r(2e^{i\theta} + re^{2i\theta})}{(1 + re^{i\theta})^2} \right| < \frac{r(2 - r)}{(1 + r)^2}.$$

Then, by choosing any $1 < c' < c$ in Lemma 4.3, $r = \frac{c'}{c}$ satisfies

$$\left| H_{v_{\psi,1}}(1 + re^{i\theta}) - k'(1 + re^{i\theta}) \right| < \frac{3c'}{s(1-r)^2} < \frac{r(2 - r)}{(1 + r)^2} \leq \left| 1 - \frac{1}{z^2} \right|$$

for all large enough $s$, because, if $s$ is large enough

$$\frac{3c'(1+r)^2}{r(2-r)(1-r)^2} = \frac{3c'(1+r)^2}{3c(2-r)(1-r)^4} < s.$$

By Rouché’s theorem, we have

$$1 - \frac{3c}{2s} < \sup_{v_{\psi,1}} \Lambda_{v_{\psi,1}} \cap \mathbb{R} < 1 + \frac{3c}{2s}.$$

The proof of

$$-1 - \frac{3c}{2s} < \inf_{v_{\psi,1}} \Lambda_{v_{\psi,1}} \cap \mathbb{R} < -1 + \frac{3c}{2s}$$

is similar.

\[ \square \]

**Proposition 4.5.** Given any $\varphi_0 \in (0, \pi/2)$, then for all large enough $s$, for all $|\cos \varphi| \leq \cos \varphi_0$, the unique $\alpha \in \mathbb{R}$ such that

$$H_{\psi,1}(\alpha + iv_{\psi,1}(\alpha)) = 2 \cos \varphi.$$

satisfies

$$|\alpha + iv_{\psi,1}(\alpha) - e^{i\varphi}| < \frac{1}{(\sin \varphi_0)s}.$$

**Proof.** Fix $\varphi_0 \in (0, \pi/2)$ and let $r = \frac{1}{(\sin \varphi_0)s}$. Then, given any $\varphi \in (0, \pi)$ such that $\sin \varphi \geq \sin \varphi_0$, we have, for large $s$,

$$|k(e^{i\varphi} + re^{i\theta}) - k(e^{i\varphi})| = \left| re^{i\theta} \left( \frac{e^{i\varphi} + re^{i\theta}}{e^{i\varphi} + re^{i\theta}} \right) \right| \geq \frac{1}{\sin \varphi_0} \frac{2 \sin \varphi_0 - r}{1 + r}. \quad (4.2)$$
Fix any $1 < c < 2$. The lower bound in (4.2) of $s \left| k(e^{i\varphi} + re^{i\theta}) - k(e^{i\varphi}) \right|$ converges to 2 as $s \to \infty$. It follows from Lemma 4.2 that, for all large enough $s$,

$$\left| H_{\frac{y_0}{\sqrt{s}}} \left( e^{i\varphi} + re^{i\theta} \right) - k(e^{i\varphi}) \right| < \frac{c}{s(1-r)^3} \left| k(e^{i\varphi} + re^{i\theta}) - k(e^{i\varphi}) \right|$$

by Rouche’s theorem, there exists a point $p_{\cos \varphi}$ such that $\left| p_{\cos \varphi} - e^{i\varphi} \right| < \frac{1}{(\sin \varphi_0)s}$ and

$$H_{\frac{y_0}{\sqrt{s}}} (p_{\cos \varphi}) = 2 \cos \varphi.$$

In particular, $H_{\frac{y_0}{\sqrt{s}}} (p_{\cos \varphi}) \in \mathbb{R}$. The proposition now follows from the fact that $v_{\frac{y_0}{\sqrt{s}},1}(\alpha)$ is the unique positive number (if exists) such that

$$H_{\frac{y_0}{\sqrt{s}}} (\alpha + iv_{\frac{y_0}{\sqrt{s}},1}(\alpha)) \in \mathbb{R}.$$

This completes the proof.

**Proof of Theorem 4.1.** Point 1 is a result in [22, Theorem 3.2] which states that $v_s$ is unimodal for $s \geq 4D_0^2$. This implies $\Lambda_{y_0,s} \cap \mathbb{R} = (\inf \Lambda_{y_0,s}, \sup \Lambda_{y_0,s})$ is an interval.

Let

$$Y = \frac{y_0 - \tau(y_0)}{\sqrt{\tau(y_0^2)}}$$

and write $t = s/\tau(y_0^2)$. By Theorem 2.6, $\Lambda_{y_0,s}$ is the domain of full measure of $\text{Brown}(y_0 + c_s)$. Since $\text{Brown}(y_0 + c_s)$ is the push-forward of $\text{Brown} \left( \frac{1}{\sqrt{t}} + c_1 \right)$ by the function

$$z \mapsto \tau(y_0) + z\sqrt{t\tau(y_0^2)} = \tau(y_0) + z\sqrt{s}$$

by [17] Proposition 2.14. Thus,

$$\Lambda_{y_0,s} = \left\{ \tau(y_0) + z\sqrt{s} \in \mathbb{C} \mid z \in \Lambda_{\frac{y_0}{\sqrt{s}},1} \right\}.$$

Points 2 and 3 then follow from applying Proposition 4.4 and Proposition 4.5 with $t = s/\tau(y_0^2)$ in place of $s$ respectively; $\Lambda_{y_0,s}$ is obtained by scaling $\Lambda_{\frac{y_0}{\sqrt{s}},1}$ by $\sqrt{s}$ and translating by $\tau(y_0)$.

### 4.2 The density as $s \to \infty$

In this section, we estimate the density of $\text{Brown}(y_0 + c_s)$ for large $s$. The Brown measure of $c_s$ is the uniform measure on the disk of radius $\sqrt{s}$; that is, the density is the constant

$$\frac{1}{\pi s}$$

inside the unit disk. The following theorem states that for a fixed $y_0$, as $s \to \infty$, the density $w_{y_0,s}$ of $\text{Brown}(y_0 + c_s)$ is approximately the same constant in (4.3).

**Theorem 4.6.** Denote by $w_{y_0,s}$ the density of $\text{Brown}(y_0 + c_s)$. Then, for any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, we have

$$\left| w_{y_0,s}(\alpha + i\beta) - \frac{1}{\pi s} \right| < \frac{c\tau(y_0^2)}{2\pi s^2 \sin^2 \varphi_0} \left( 3 + \frac{2}{\sin \varphi_0} \right), \quad |\psi_{y_0,s}(\alpha)| < 2\sqrt{s} \cos \varphi_0$$

for all large enough $s$.

To simplify the computation, we assume $\tau(y_0) = 0$ and $\tau(y_0^2) = 1$ until the proof of the theorem. The key is to estimate the difference between the complex derivatives $H_{\frac{y_0}{\sqrt{s}}}'$ and $1'$; indeed the density is directly related to the real part of the complex derivative of the subordination function $H_{\frac{y_0}{\sqrt{s}}}'$. 
Lemma 4.7. Given any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, for all sufficient large $s$, the unique $\alpha$ such that

$$H_{\varphi_0,1}(\alpha + iv_{\varphi_0,1}(\alpha)) = 2\cos \varphi, \quad \sin \varphi > \sin \varphi_0$$

satisfies

$$\left| \frac{1}{\text{Re}(1/k'(\alpha + iv_{\varphi_0,1}(\alpha)))} - \frac{1}{\text{Re}(1/k'(e^{i\varphi})]} \right| < \frac{2c}{s \sin^3 \varphi_0}.$$  \hspace{1cm} (4.4)

Proof. Fix any $\varphi_0 \in (0, \pi/2)$ and $c > 1$. By Proposition 4.5, for any $\varphi \in (0, \pi)$ such that $\sin \varphi > \sin \varphi_0$, the unique $\alpha \in \mathbb{R}$ such that

$$H_{\varphi_0,1}(\alpha + iv_{\varphi_0,1}(\alpha)) = 2\cos \varphi.$$ 

satisfies

$$\left| \alpha + iv_{\varphi_0,1}(\alpha) - e^{i\varphi} \right| < \frac{1}{(\sin \varphi_0)s}.$$  \hspace{1cm} (4.4)

for all large enough $s$. We know that $\frac{1}{\text{Re}(1/k'(z))} = 2$ because

$$\frac{1}{k'} = \frac{e^{i\varphi}}{e^{i\varphi} - e^{-i\varphi}} = \frac{1}{2}(1 - i \cot \varphi).$$  \hspace{1cm} (4.5)

Using (4.4) and (4.5), we have

$$\frac{1}{(1/2 - |\text{Re}(1/k'(w)) - \text{Re}(1/k'(e^{i\varphi}))|)^2} < 4\sqrt{c}$$  \hspace{1cm} (4.6)

for all large enough $s$.

Write $z = e^{i\varphi}$ and $w = \alpha + iv_{\varphi_0,1}(\alpha)$. Observe that

$$\frac{1}{k'(w)} - \frac{1}{k'(z)} = \frac{w^2}{w^2 - 1} - \frac{z^2}{z^2 - 1} = \frac{(z - w)(z + w)}{(w^2 - 1)(z^2 - 1)}.$$  \hspace{1cm} (4.7)

Also, it is straightforward to check that $|z^2 - 1| = |e^{2i\varphi} - 1| = 2\sin \varphi$, and, by (4.4),

$$|w^2 - z^2| = |w - z| |w + z| < \frac{1}{(\sin \varphi_0)s} \left(2 + \frac{1}{(\sin \varphi_0)s}\right).$$

We have, for all large enough $s$,

$$\left| \frac{1}{k'(w)} - \frac{1}{k'(z)} \right| < \frac{1}{4 \sin^2 \varphi_0 s (\sin \varphi_0)}.$$  \hspace{1cm} (4.7)

Thus, by the mean value theorem (applied to the function $1/(\frac{1}{2} + x)$), and (4.4)-(4.7),

$$\left| \frac{1}{\text{Re}(1/k'(w))} - \frac{1}{\text{Re}(1/k'(e^{i\varphi})]} \right| \leq \frac{|\text{Re}(1/k'(w)) - \text{Re}(1/k'(e^{i\varphi}))|}{(1/2 - |\text{Re}(1/k'(w)) - \text{Re}(1/k'(e^{i\varphi}))|)^2}$$

$$< 4\sqrt{c} \frac{\sqrt{c}}{2s \sin^3 \varphi_0} = \frac{2c}{s \sin^3 \varphi_0}.$$  \hspace{1cm} (4.7)

for all large enough $s$, completing the proof. \hfill \Box

Lemma 4.8. For any $c > 1$, we have

$$\left| \frac{1}{\text{Re}(1/H_{\varphi_0,1}(z))} - \frac{1}{\text{Re}(1/k'(z))} \right| < \frac{3c}{s |z|^2} \frac{1}{[\text{Re}(1/k'(z))]^2} \frac{1}{|k'(z)|^2},$$

for all large enough $s$. 






















When $|z| = 1$ but $z \neq 1, -1$, the right hand side of the inequality does not divide by zero. More explicitly, if $z = e^{i\varphi}$, we have

$$|k'(z)| = |z^2 - 1| = 2 \sin \varphi.$$  \hspace{1cm} (4.8)

**Proof.** Let $c > 1$. By (4.1), for all $|z| > \frac{1}{2}$,

$$|H'_{\frac{\varphi}{\sqrt{s}}, 1}(z) - k'(z)| \leq \frac{1}{s|z|^3} \left(3\tau(y_0^2) + \sum_{n=3}^{\infty} \frac{(n + 1) |\tau(y^n)|}{s^{\frac{3}{2}}(1/2)^{n-2}} \right) < \frac{3c^{1/3} \tau(y_0^2)}{s|z|^4}$$  \hspace{1cm} (4.9)

for all large enough $s$. We then must have

$$\left| \frac{1}{\text{Re}(1/H'_{\frac{\varphi}{\sqrt{s}}, 1}(z))} \right| < \frac{c^{1/3}}{\text{Re}(1/k'(z))} \quad \text{and} \quad \left| \frac{1}{H'_{\frac{\varphi}{\sqrt{s}}, 1}(1)(z)} \right| < \frac{c^{1/3}}{|k'(z)|}$$

for all large enough $s$. Therefore, we have

$$\left| \frac{1}{\text{Re}(1/H'_{\frac{\varphi}{\sqrt{s}}, 1}(z))} - \frac{1}{\text{Re}(1/k'(z))} \right| = \left| \frac{\text{Re}(1/k'(z)) - \text{Re}(1/H'_{\frac{\varphi}{\sqrt{s}}, 1}(z))}{\text{Re}(1/H'_{\frac{\varphi}{\sqrt{s}}, 1}(z))\text{Re}(1/k'(z))} \right|$$

$$\leq \frac{c^{1/3}}{|\text{Re}(1/k'(z))|^2 |k'(z)|^2} \leq \frac{3c\tau(y_0^2)}{s|z|^4} \frac{1}{|\text{Re}(1/k'(z))|^2} \frac{1}{|k'(z)|^2},$$

which is the desired inequality since we assume $\tau(y_0^2) = 1$ until the proof of Theorem 4.6. $\square$

**Lemma 4.9.** Given any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, for all sufficient large $s$, the unique $\alpha$ such that

$$H_{\frac{\varphi}{\sqrt{s}}, 1}(\alpha + iv_{\frac{\varphi}{\sqrt{s}}, 1}(\alpha)) = 2 \cos \varphi, \quad \sin \varphi > \sin \varphi_0$$

satisfies

$$\left| \frac{1}{\text{Re}(1/H'_{\frac{\varphi}{\sqrt{s}}, 1}(w))} - 2 \right| < \frac{c}{s\sin^2 \varphi_0} \left(3 + \frac{2}{\sin \varphi_0} \right)$$

where $w = \alpha + iv_{\frac{\varphi}{\sqrt{s}}, 1}(\alpha)$.

**Proof.** Let $c > 1$. Write $z = e^{i\varphi}$ and $w = \alpha + iv_{\frac{\varphi}{\sqrt{s}}, 1}(\alpha)$. Recall that $\frac{1}{\text{Re}(1/k'(z))} = 2$ by (4.5). We estimate

$$\left| \frac{1}{\text{Re}(1/H'_{\frac{\varphi}{\sqrt{s}}, 1}(w))} - 2 \right| \leq \left| \frac{1}{\text{Re}(1/H'_{\frac{\varphi}{\sqrt{s}}, 1}(w))} - \frac{1}{\text{Re}(1/k'(w))} \right| + \left| \frac{1}{\text{Re}(1/k'(w))} - \frac{1}{\text{Re}(1/k'(z))} \right|. \hspace{1cm} (4.10)$$

We estimate the first term in (4.10) using Proposition 4.5 and Lemmas 4.7 and 4.8. Fix any $1 < c' < c$. For all large enough $s$, the first term is bounded by

$$\frac{3c'}{s|w|^2} \frac{1}{|\text{Re}(1/k'(w))|^2} \frac{1}{|k'(w)|^2} \leq \frac{3c'}{s[1 - 1/(\sin \varphi_0 s)^4]} \left( \frac{1}{\text{Re}(1/k'(e^{i\varphi}))} + \frac{2c'}{s\sin^3 \varphi_0} \right)^2 \frac{1}{|k'(w)|^2}$$

$$< \frac{12c}{s^4 \sin^2 \varphi} \leq \frac{3c}{s \sin^2 \varphi_0}$$
by (4.8) and Lemmas 4.7 and 4.8.

By Lemma 4.7, the second term in (4.10) is bounded by

\[ \left| \frac{1}{\Re(1/k'(w))} - \frac{1}{\Re(1/k'(z))} \right| < \frac{2c}{s \sin^2 \varphi_0}. \]

The result then follows from adding these estimates.

**Proposition 4.10.** Denote by \( w_{\varphi_0, 1} \) the density of \( \text{Brown}(\sqrt{y_0} + c_1) \). Then, for any \( c > 1 \) and \( \varphi_0 \in (0, \pi/2) \), we have

\[ \left| w_{\varphi_0, 1}(\alpha + i\beta) - \frac{1}{\pi} \right| < \frac{c}{2 \pi s \sin^2 \varphi_0} \left( 3 + \frac{2}{\sin \varphi_0} \right), \quad \left| \psi_{\varphi_0,1}(\alpha) \right| < 2 \cos \varphi_0 \]

for all large enough \( s \).

**Proof.** By Equation (3.31) of [25],

\[ \operatorname{Re} \left( \frac{1}{H_{\varphi_0,1}'(w)} \right) \frac{d\psi_{\varphi_0,1}(\alpha)}{d\alpha} = 1 \]

where \( w = \alpha + iv_{\varphi_0,1}(\alpha) \). (This formula appeals to the subordination function \( H_{\varphi_0,1}^{-1} \) of the free convolution \( \sqrt{y_0} + \sigma_1 \) has an analytic continuation in a neighborhood of any point \( \psi_{\varphi_0,1}(\alpha + iv_{\varphi_0,1}(\alpha)) \) if \( v_{\varphi_0,1}(\alpha) > 0 \); see [2] Theorem 3.3(1).) Thus, we can express the real derivative through complex derivative

\[ \frac{d\psi_{\varphi_0,1}(\alpha)}{d\alpha} = \frac{1}{\Re(1/H_{\varphi_0,1}'(w))}. \]

By Lemma 4.9, given any \( c > 1 \) and \( \varphi_0 \in (0, \pi/2) \), for all sufficiently large \( s \), the unique \( \alpha \) such that

\[ \psi_{\varphi_0,1}(\alpha) = 2 \cos \varphi, \quad \sin \varphi > \sin \varphi_0 \]

satisfies

\[ \left| \frac{d\psi_{\varphi_0,1}(\alpha)}{d\alpha} - 2 \right| < \frac{c}{2 \pi s \sin^2 \varphi_0} \left( 3 + \frac{2}{\sin \varphi_0} \right). \]

The proposition now follows from Theorem 2.6.

All the estimates in this section that we have done are under the assumption \( \tau(y_0) = 0 \) and \( \tau(y_0^2) \). We are now ready to prove the estimate of the density of \( \text{Brown}(y_0 + c_s) \) for arbitrary \( \tau(y_0) \) and \( \tau(y_0^2) \).

**Proof of Theorem 4.6.** Without loss of generality, we assume \( \tau(y_0) = 0 \), since otherwise we translate the density by \( \tau(y_0) \).

We first assume \( \tau(y_0^2) = 1 \). Let \( w = \alpha + iv_{y_0, s}(\alpha) \) and \( z = \frac{w}{\sqrt{s}} \). Then

\[ z = \frac{\alpha}{\sqrt{s}} + iv_{y_0, s} \left( \frac{\alpha}{\sqrt{s}} \right). \]

Since \( \text{Brown}(y_0 + c_s) \) is the push-forward measure of \( \text{Brown} \left( \frac{y_0}{\sqrt{s}} + c_1 \right) \) by \( z \mapsto \sqrt{s}z \),

\[ w_{y_0, s}(\alpha + i\beta) = \frac{1}{s} \cdot w_{\varphi_0, 1} \left( \frac{1}{\sqrt{s}}(\alpha + i\beta) \right), \quad z \in \Lambda_{y_0, s}. \]

By Proposition 4.10 for any \( c > 1 \) and \( \varphi_0 \in (0, \pi/2) \), we have

\[ \left| w_{y_0, s}(\alpha + i\beta) - \frac{1}{\pi s} \right| < \frac{c}{2 \pi s^2 \sin^2 \varphi_0} \left( 3 + \frac{2}{\sin \varphi_0} \right), \quad \left| \psi_{y_0, s}(\alpha) \right| < 2 \sqrt{s} \cos \varphi_0 \]

for all large enough \( s \). This establishes the result with \( \tau(y_0^2) = 1 \).
For arbitrary $\tau(y_0^2)$, let $Y = \frac{y_0}{\sqrt{\tau(y_0^2)}}$. We consider the random variable $\frac{1}{\sqrt{\tau(y_0^2)}} (y_0 + c s)$ which has the same $s$-moments, hence the same Brown measure, as $Y + c_s$, where $t = s/\tau(y_0^2)$.

By the result for $\tau(y_0^2) = 1$, given any $c > 1$ and $\varphi_0 \in (0, \pi/2)$, we have

$$\left| w_{Y,t}(\alpha + i\beta) - \frac{1}{\pi t} \right| < \frac{c}{2\pi^2 \sin^2 \varphi_0} \left( 3 + \frac{2}{\sin \varphi_0} \right), \quad \left| \psi_{Y,t}(\alpha) \right| < 2\sqrt{t} \cos \varphi_0$$

(4.11)

for all large enough $t$. Now, since $\text{Brown}(y_0 + c_s)$ is the push-forward measure of $\text{Brown}(Y + c_t)$ by $z \mapsto \sqrt{\tau(y_0^2)} z$, by (4.11), we must have

$$\left| w_{y_0,s}(\alpha + i\beta) - \frac{1}{\pi s} \right| < \frac{c\tau(y_0^2)}{2\pi s^2 \sin^2 \varphi_0} \left( 3 + \frac{2}{\sin \varphi_0} \right), \quad \left| \psi_{y_0,s}(\alpha) \right| < 2\sqrt{s} \cos \varphi_0$$

for all large enough $s$. 

\[ \square \]

## 5 Asymptotic behaviors of adding an elliptic element

In this section, we study three limiting behaviors of $\text{Brown}(y_0 + \sigma_{s,t/2} + i\sigma_{s,t/2})$ as $s \to \infty$. The first regime is to keep $s$ and $t$ at the same ratio $r = t/s$; the second regime is to keep $t$ fixed; the last regime is to fix $s = t/2$.

### 5.1 Fix $s/t$ and let $s, t \to \infty$

#### 5.1.1 Domain behavior

In this section, we discuss the asymptotic behavior of the domain of $\text{Brown}(y_0 + \sigma_{s-t/2} + i\sigma_{s-t/2})$ for a fixed $r = t/s$. When $y_0 = 0$, the domain of $\sigma_{s-t/2} + i\sigma_{s-t/2}$ has the shape of an ellipse with boundary

$$\frac{2s-t}{\sqrt{s}} \cos \varphi + i \frac{s}{t} \sin \varphi, \quad \varphi \in [0, 2\pi]$$

(5.1)

(See [8 Example 5.3]). As $s \to \infty$ with $r = t/s$ fixed, the random variable $y_0 + \sigma_{s-t/2} + i\sigma_{s-t/2}$ behaves like the elliptic element $\tau(y_0) + \sigma_{s-t/2} + i\sigma_{s-t/2}$. Roughly speaking, the domain $\Omega_{s,t}$ of $\text{Brown}(y_0 + \sigma_{s-t/2} + i\sigma_{s-t/2})$ is asymptotically an ellipse with boundary as in (5.1) translated by $\tau(y_0)$. The following theorem states precisely the asymptotic behavior of the domain $\Omega_{s,t}$ of $\text{Brown}(y_0 + \sigma_{s-t/2} + i\sigma_{s-t/2})$; the main tool is Theorem 4.1.

**Theorem 5.1.** Fix the ratio $r = t/s$. The following asymptotic behaviors of the graph of $\Omega_{s,t}$ hold.

1. Let $D_{\nu} = \sup \{|x - y| \mid x, y \in \text{supp } \mu\}$. When $s \geq 4D_{\nu}^2$, the function $b_{s,t}$ is unimodal. In particular, $\Omega_{s,t} \cap \mathbb{R}$ is an interval.

2. Given any $c > 1$, we have

$$\left| \sup_{\Omega_{s,t} \cap \mathbb{R}} - \left( \tau(y_0) + \frac{2s-t}{\sqrt{s}} \right) \right| < \frac{c(3r + 2|1-r|)\tau(y_0^2)}{2\sqrt{s}}$$

and

$$\left| \inf_{\Omega_{s,t} \cap \mathbb{R}} - \left( \tau(y_0) - \frac{2s-t}{\sqrt{s}} \right) \right| < \frac{c(3r + 2|1-r|)\tau(y_0^2)}{2\sqrt{s}}$$

for all sufficiently large $s$. In particular, $\Lambda_{y_0,s} \cap \mathbb{R}$ is contained in

$$\left( \tau(y_0) - \frac{2s-t}{\sqrt{s}} \right) - \frac{c(3r + 2|1-r|)\tau(y_0^2)}{2\sqrt{s}}, \tau(y_0) + \frac{2s-t}{\sqrt{s}} + \frac{c(3r + 2|1-r|)\tau(y_0^2)}{2\sqrt{s}}$$

for all large enough $s$. 

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3. Given any \( \varphi_0 \in (0, \pi/2) \), then for all large enough \( s \), for all \( |\cos \varphi| \leq \cos \varphi_0 \), the unique \( \alpha \in \mathbb{R} \) such that
\[
H_{y_0,s}(\alpha + iv_{y_0,s}(\alpha)) = 2\sqrt{s} \cos \varphi.
\]
satisfies
\[
|U_{s,t}(\alpha + iv_{y_0,s}(\alpha)) - \left( \frac{2s - t}{\sqrt{s}} \cos \varphi + i \frac{t}{\sqrt{s}} \sin \varphi \right) | < \frac{r}{(\sin \varphi_0)\sqrt{s}}.
\]

**Proof.** Point 1 follows directly from \([22\text{, Theorem 3.2}]\) which states that \( v_{y_0,s} \) is unimodal for \( s \geq 4D_\nu^2 \), because, by Proposition 3.4 we have
\[
b_{s,t} = \frac{t}{s} v_{y_0,s}.
\]

Fix \( r = t/s \) throughout this proof. We now prove Point 2. Without loss of generality, we assume \( \tau(y_0) = 0 \). We first estimate \( a_{1,r}(\alpha^*) \) where
\[
\alpha^* = \sup \Lambda_{y_0/\sqrt{\pi}1} \cap \mathbb{R}.
\]
We compute
\[
a_{1,r}(\alpha^*) - (2 - r) = (\alpha^* - 1) \left( 1 - \frac{1 - r}{\alpha^*} \right) + \frac{(1 - r)\tau(y_0^2)}{s(\alpha^*)^3} + \frac{(1 - r)}{s^{3/2}} \sum_{n=3}^{\infty} \frac{\tau(y_0^n)}{s^{(n-3)/2}(\alpha^*)^{n+1}}. \tag{5.2}
\]
By Proposition 4.4 (with \( s \) replaced by \( s/\tau(y_0^2) \)), given any \( c > 1 \), for all large enough \( s \), we have
\[
|a_{1,r}(\alpha^*) - (2 - r)| < \frac{c(3r + 2|1 - r|)\tau(y_0^2)}{2s}.
\]
Since
\[
\sup \Omega_{s,t} \cap \mathbb{R} = \sqrt{s} a_{1,r}(\alpha^*)
\]
we have
\[
\left| \sup \Omega_{s,t} \cap \mathbb{R} - (\tau(y_0) + \frac{2s - t}{\sqrt{s}}) \right| < \frac{c(3r + 2|1 - r|)\tau(y_0^2)}{2s}
\]
for all sufficiently large \( s \). The estimate for \( \inf \Omega_{s,t} \cap \mathbb{R} \) is similar.

We prove Point 3 now. By Theorem 3.3, we know that
\[
\Omega_{s,t} = U_{s,t}(\Lambda_{y_0,s}).
\]
Suppose \( \alpha \) is chosen such that \( \psi_{y_0,s}(\alpha) = 2\sqrt{s} \cos \varphi \). We compute the upper boundary curve \( a + ib_{s,t}(a) = U_{s,t}(\alpha + iv_{y_0,s}(\alpha)) \) as
\[
a_{s,t}(\alpha) = (1 - r)\psi_{y_0,s}(\alpha) + r\alpha = 2(1 - r)\sqrt{s} \cos \varphi + r\alpha;
\]
\[
b_{s,t}(a) = b_{s,t}(a_{s,t}(\alpha)) = rv_{y_0,s}(\alpha).
\]
So, we have
\[
|a + ib_{s,t}(a) - \sqrt{s}[(2 - r) \cos \varphi + ir \sin \varphi]| = r |\alpha + iv_{y_0,s}(\alpha) - \sqrt{s}e^{i\varphi}|. \tag{5.3}
\]
Therefore, by Theorem 4.4 for any \( \varphi_0 \in (0, \pi/2) \),
\[
|a + ib_{s,t}(a) - \sqrt{s}[(2 - r) \cos \varphi + ir \sin \varphi]| = r |\alpha + iv_{y_0,s}(\alpha) - \sqrt{s}e^{i\varphi}| < \frac{r}{(\sin \varphi_0)\sqrt{s}}
\]
for all sufficiently large \( s \). This proves Point 3.

\[\square\]
5.1.2 Density behavior

In this section, we investigate the asymptotic behavior of the density of \( \text{Brown}(y_0 + \tilde{s}_{-t/2} + i\sigma_{t/2}) \) for a fixed \( r = t/s \). In the case \( y_0 = 0 \), \( \text{Brown}(y_0 + \tilde{s}_{-t/2} + i\sigma_{t/2}) \) is the elliptic law, with constant density

\[
\frac{1}{\pi} \frac{s}{t} (2s - t)t
\]

in domain \( \Omega_{s,t} \), which is a region bounded by an ellipse in this case (See Example 5.3).

Denote by \( w_{y_0,s,t} \) the density of \( \text{Brown}(y_0 + \tilde{s}_{-t/2} + i\sigma_{t/2}) \). We will prove that as \( s \) large and \( r = t/s \) fixed, the density \( w_{y_0,s,t} \) is approximately the same constant in (5.4). The main tool is the estimate of the density of \( \text{Brown}(y_0 + c_a) \) in Theorem 4.6.

**Theorem 5.2.** Fix \( r = t/s \). Given any \( c > 1 \) and \( \varphi_0 \in (0, \pi/2) \), we have

\[
\left| w_{y_0,s,t}(a + ib) - \frac{1}{\pi} \frac{s}{t} \left( \frac{s}{2s - t} \right) \right| < \frac{c \tau(y_0^2)}{2\pi\sin^2\varphi_0}\left( \frac{1}{2s - t} \right)^2 \left( 3 + \frac{2}{\sin\varphi_0} \right)
\]

whenever \( w_{y_0,s}(\alpha + i\beta) < 2\sqrt{s} \cos \varphi_0 \), for all large enough \( s \).

**Proof.** Let \( c > 1 \) be given. By Corollary 3.6, if we write \( a + ib = U_{s,t}(\alpha + i\beta) \) for all \( \alpha + i\beta \in \Lambda_{y_0,s} \). Then we have

\[
w_{y_0,s,t}(a + ib) = \frac{1}{r} \frac{w_{y_0,s}(\alpha + i\beta)}{r + 2\pi(1 - r)s \cdot w_{y_0,s}(\alpha + i\beta)}
\]

for all \( a + ib \in \Omega_{s,t} \).

Now, by the formula

\[
\frac{1}{\pi s} \frac{1}{2 - r} = \frac{1}{\pi s} \frac{1}{r + 2\pi(1 - r)s \cdot (1/\pi s)}
\]

and Theorem 4.6 for any \( 1 < c' < c \), if \( w_{y_0,s}(\alpha) < 2\sqrt{s} \cos \varphi_0 \), then we have \( \pi sw_{y_0,s}(\alpha + i\beta) \rightarrow 1 \), and

\[
\left| \frac{w_{y_0,s}(\alpha + i\beta)}{r + 2\pi(1 - r)s \cdot w_{y_0,s}(\alpha + i\beta)} - \frac{1/(\pi s)}{2 - r} \right| \leq \frac{r |w_{y_0,s}(\alpha + i\beta) - 1/(\pi s)|}{|r + 2\pi(1 - r)s \cdot w_{y_0,s}(\alpha + i\beta)|[2 - r]} \left| \frac{c \tau(y_0^2)}{2\pi s^2\sin^2\varphi_0} \left( 3 + \frac{2}{\sin\varphi_0} \right) \frac{1}{(2 - r)^2} \right|
\]

for all large enough \( s \). The proof follows from dividing the above estimate by \( r \).

\[\square\]

5.2 Fix \( t \) and let \( s \rightarrow \infty \)

In this section, we investigate the asymptotic behavior of \( \text{Brown}(y_0 + \tilde{s}_{-t/2} + i\sigma_{t/2}) \) with \( t \) fixed and \( s \rightarrow \infty \).

5.2.1 Domain behavior

The following theorem states that \( \Omega_{s,t} \) has the shape of an ellipse in the limit with fixed \( t \) as \( s \rightarrow \infty \), except points close to the endpoints of \( \Omega_{s,t} \cap \mathbb{R} \). The limiting ellipse has a very short minor axis; it is a long and thin ellipse.

**Theorem 5.3.** Fix \( t > 0 \). The following asymptotic behaviors of the graph of \( \Omega_{s,t} \) hold.

1. Let \( D_\nu = \sup\{|x - y| \mid x, y \in \text{supp } \mu \} \). When \( s \geq 4D_\nu^2 \), the function \( b_{s,t} \) is unimodal. In particular, \( \Omega_{s,t} \cap \mathbb{R} \) is an interval.

2. Given any \( c > 1 \), we have

\[
\left| \sup \Omega_{s,t} \cap \mathbb{R} - (\tau(y_0) + 2\sqrt{s}) \right| < \frac{c |\tau(y_0^2) - t|}{\sqrt{s}}
\]

and

\[
\left| \inf \Omega_{s,t} \cap \mathbb{R} - (\tau(y_0) - 2\sqrt{s}) \right| < \frac{c |\tau(y_0^2) - t|}{\sqrt{s}}
\]
It follows that
\[ \Lambda_{y_0, s} \cap \mathbb{R} \subset \left( \tau(y_0) - 2\sqrt{s} - \frac{c{|\tau(y_0^2) - t|}}{\sqrt{s}}, \tau(y_0) + 2\sqrt{s} + \frac{c{|\tau(y_0^2) - t|}}{\sqrt{s}} \right) \]
for all large enough \( s \).

3. Given any \( \varphi_0 \in (0, \pi/2) \), then for all large enough \( s \), for all \( |\cos \varphi| \leq \cos \varphi_0 \), the unique \( \alpha \in \mathbb{R} \) such that
\[ H_{y_0, s}(\alpha + iv_{y_0, s}(\alpha)) = 2\sqrt{s} \cos \varphi, \]
satisfies
\[ \left| U_{s, t}(\alpha + iv_{y_0, s}(\alpha)) - \left[ \frac{2s - t}{\sqrt{s}} \cos \varphi + i \frac{t}{\sqrt{s}} \sin \varphi \right] \right| < \frac{t}{(\sin \varphi_0)s^{3/2}}. \]
Furthermore, we have
\[ \lim_{s \to \infty} \sup \{ \text{Im} z | z \in \Omega_{s, \tau} \} = 0. \]

**Proof.** Point 1 follows directly from Theorem 3.3 and [22, Theorem 3.2] which states that \( v_{y_0, s} \) is unimodal for \( s \geq 4D_\alpha^2 \), because, by (3.11), we have
\[ b_{s, t} = \frac{t}{s} v_{y_0, s}. \]

Fix \( t > 0 \). We now prove Point 2. Without loss of generality, we assume \( \tau(y_0) = 0 \). We first estimate \( a_{1, r}(\alpha^*) \) where
\[ \alpha^* = \sup \Lambda_{y_0/\sqrt{\tau_1}, s} \cap \mathbb{R}. \]
We calculate
\[ a_{1, r}(\alpha^*) - 2 = \alpha^* - 2 + (1 - r) \sum_{n=0}^{\infty} \frac{\tau(y_0^n)}{s^2(\alpha^*)^{n+1}} \]
\[ = \alpha^* - 1 + \frac{1 - \alpha^*}{\alpha^*} - \frac{t}{s\alpha^*} + \frac{\tau(y_0^2)}{s^2(\alpha^*)^3} + \sum_{n=3}^{\infty} \frac{\tau(y_0^n)}{s^2(\alpha^*)^{n+1}} \]
\[ = (\alpha^* - 1)\frac{\alpha^* - 1}{\alpha^*} + \frac{\tau(y_0^2) - t(\alpha^*)^2}{s(\alpha^*)^3} + \sum_{n=3}^{\infty} \frac{\tau(y_0^n)}{s^2(\alpha^*)^{n+1}}. \]
By Proposition 4.4 (with \( s \) replaced by \( s/\tau(y_0^2) \)), given any \( c > 1 \), for all large enough \( s \), we have (by keeping the only order \( 1/s \) term)
\[ |a_{1, r}(\alpha^*) - 2| < \frac{c{|\tau(y_0^2) - t|}}{s}. \]
It follows that
\[ \left| \sup \Omega_{s, \tau} \cap \mathbb{R} - (\tau(y_0) + 2\sqrt{s}) \right| < \frac{c{|\tau(y_0^2) - t|}}{\sqrt{s}} \]
for all sufficiently large \( s \). The estimate for \( \inf \Omega_{s, \tau} \cap \mathbb{R} \) is similar.

We now prove Point 3. By (5.3),
\[ |a + ib_{s, t}(a) - \sqrt{s}[(2 - r) \cos \varphi + ir \sin \varphi]| = r |a + iv_{y_0, s}(\alpha) - \sqrt{s} e^{i\varphi}|. \]
Therefore, by Theorem 4.4 for any \( \varphi_0 \in (0, \pi/2) \),
\[ |a + ib_{s, t}(a) - \sqrt{s}[(2 - r) \cos \varphi + ir \sin \varphi]| < \frac{t}{(\sin \varphi_0)s^{3/2}} \]
(5.5)
for all sufficiently large \( s \).
Let \( \varphi_0 = \frac{\pi}{6} \) so that \( \sin \varphi > 1/2 \) for all \( \varphi \) such that \( |\cos \varphi| < \cos \varphi_0 \). We label by \( \alpha_{\varphi} \) the unique \( \alpha \in \mathbb{R} \) such that
\[ H_{y_0, s}(\alpha + iv_{y_0, s}(\alpha)) = 2\sqrt{s} \cos \varphi, \quad |\cos \varphi| \leq \cos \varphi_0. \]

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By (5.3), we have
\[ \sup\{b_{s,t}(a_{s,t}(\alpha))|\alpha_\pi - \varphi_0 < \alpha < \alpha_\varphi_0\} > \frac{t}{\sqrt{s}} - \frac{2t}{s^{3/2}}. \]

Since
\[ b_{s,t}(a_{s,t}(\alpha_\varphi_0)) < \frac{t}{2\sqrt{s}} + \frac{2t}{s^{3/2}}, \]
and, by Point 1, the function \( b_{s,t} \) is unimodal,
\[ b_{s,t}(a_{s,t}(\alpha)) < \frac{t}{2\sqrt{s}} + \frac{2t}{s^{3/2}}, \quad \alpha \geq \alpha_\varphi_0 \text{ or } \alpha \leq \alpha_\pi - \varphi_0. \quad (5.6) \]
For all \( \alpha_\pi - \varphi_0 < \alpha < \alpha_\varphi_0 \),
\[ \sup\{b_{s,t}(a_{s,t}(\alpha))|\alpha_\pi - \varphi_0 < \alpha < \alpha_\varphi_0\} < \frac{t}{\sqrt{s}} + \frac{2t}{s^{3/2}}. \quad (5.7) \]
Therefore, we conclude
\[ \lim_{s \to \infty} \sup\{\Im z \mid z \in \Omega_{s,t}\} = 0 \]
by (5.6) and (5.7).

**5.2.2 Density behavior**

If we consider the special case of \( y_0 = 0 \), Brown(\( y_0 + \tilde{s}_{s-t/2} + i\sigma_{t/2} \)) is just the elliptic law; as mentioned in (5.4), it has a constant density
\[ \frac{1}{\pi} \frac{s}{(2s-t)t}. \]
If we fixed \( t \) and let \( s \to \infty \), this density converges to the constant \( 1/(2\pi t) \).

The following theorem states that if we consider an arbitrary self-adjoint initial condition \( y_0 \), the density of Brown(\( y_0 + \tilde{s}_{s-t/2} + i\sigma_{t/2} \)) also converges to \( 1/(2\pi t) \); the convergence is uniform away the endpoints of \( \Omega_{s,t} \cap \mathbb{R} \).

**Theorem 5.4.** Denote by \( w_{y_0,s,t} \) the density of Brown(\( y_0 + \tilde{s}_{s-t/2} + i\sigma_{t/2} \)). Then given any \( c > 1 \) and \( \varphi_0 \in (0, \pi/2) \), there is an \( s_0 > 0 \) such that
\[ \left| w_{y_0,s,t}(a + ib) - \frac{1}{2\pi} \right| < \frac{c}{4\pi s}, \quad |\psi_{y_0,s}(a_{s,t}(\alpha))| < 2\sqrt{s} \cos \varphi_0 \]
for all \( s > s_0 \).

**Proof.** Let \( c > 1 \) and \( \varphi_0 \in (0, \pi/2) \) be given. By Corollary 3.6 if we write \( (a, b) = U_{s,t}(\alpha, \beta) \) for all \( \alpha + i\beta \in \Lambda_{y_0,s} \). Then we have
\[ w_{y_0,s,t}(a + ib) = \frac{s\pi w_{y_0,s}(\alpha + i\beta)}{t/(2s) + (1 - t/s)\pi s \cdot w_{y_0,s}(\alpha + i\beta)} \quad (5.8) \]
for all \( a + ib \in \Omega_{s,t} \).

By Theorem 4.6 given any \( 1 < c' < c \), we have
\[ |\pi s \cdot w_{y_0,s}(\alpha + i\beta) - 1| < \frac{c'(y_0^2)}{2s \sin^2 \varphi_0} \left( 3 + \frac{2}{\sin \varphi_0} \right), \quad |\psi_{y_0,s}(\alpha)| < 2\sqrt{s} \cos \varphi_0 \]
for all large enough \( s \). Then, we compute
\[ \left| \frac{s\pi w_{y_0,s}(\alpha + i\beta)}{t/(2s) + (1 - t/s)\pi s \cdot w_{y_0,s}(\alpha + i\beta)} - 1 \right| = \frac{t}{s} \left| \frac{s\pi w_{y_0,s}(\alpha + i\beta)}{t/(2s) + (1 - t/s)\pi s \cdot w_{y_0,s}(\alpha + i\beta)} - 1 \right| \]
\[ < \frac{c't}{s} \left[ \frac{1}{2} + \frac{c'(y_0^2)}{2s \sin^2 \varphi_0} \left( 3 + \frac{2}{\sin \varphi_0} \right) \right] \]
\[ < \frac{ct}{2s}. \]
for all large enough \( s \), since \( \frac{t}{(2s)} + (1 - t/s)\pi s \cdot w_{y_0,s}(\alpha + i\beta) \) converges to 1. Thus, using (5.8), we have the estimate (uniform for all \( |\psi_{y_0}(\alpha_{s,t}(a))| < 2\sqrt{s} \cos \varphi_0 \))

\[
w_{y_0,s}(a + ib) - \frac{1}{2\pi t} = \frac{1}{2\pi t} \left| \frac{s\pi w_{y_0,s}(\alpha + i\beta)}{t/2s + (1 - t/s)\pi s \cdot w_{y_0,s}(\alpha + i\beta)} - 1 \right| < \frac{c}{4\pi s}
\]

for all sufficiently large \( s \).

\[
\square
\]

### 5.3 Set \( s = t/2 \) and let \( s \to \infty \)

In this section, we investigate the asymptotic behavior of Brown \((y_0 + \sigma_{s-t/2} + i\sigma_{t/2})\) with \( s = t/2 \) and \( s \to \infty \). Note that, when \( s = t/2 \), the random variable \( y_0 + \sigma_{s-t/2} + i\sigma_{t/2} \) is \( y_0 + i\sigma_s \).

**Theorem 5.5.**

1. Let \( D_v = \sup\{|x - y| | x, y \in \text{supp}\mu\} \). When \( s \geq 4D_v^2 \), the function \( b_{s,t} \) is unimodal. In particular, \( \Omega_{s,t} \cap \mathbb{R} \) is an interval.

2. We have

\[
-\frac{4c\tau(y_0^2)}{\sqrt{s}} < \inf(\Omega_{s,t} \cap \mathbb{R}) - \tau(y_0) < 0 < \sup(\Omega_{s,t} \cap \mathbb{R}) - \tau(y_0) < \frac{4c\tau(y_0^2)}{\sqrt{s}}
\]

for all \( s \) large enough. In particular,

\[
\Omega_{s,t} \cap \mathbb{R} \subset \left( \tau(y_0) - \frac{4c\tau(y_0^2)}{\sqrt{s}}, \tau(y_0) + \frac{4c\tau(y_0^2)}{\sqrt{s}} \right)
\]

for all \( s \) large enough.

3. We also have

\[
|\sup\{|\text{Im} z| | z \in \Omega_{s,t}\} - 2\sqrt{s}| < \frac{2c}{\sqrt{s}}
\]

for all large enough \( s \).

**Proof.** Point 1 follows directly from [22, Theorem 3.2] which states that \( v_{y_0,s} \) is unimodal for \( s \geq 4D_v^2 \), because, (4.1), we have

\[
b_{s,t} = 2v_{y_0,s}.
\]

We now prove Point 2. Let \( c > 1 \) be given. Without loss of generality, we assume \( \tau(y_0) = 0 \). Denote

\[
M_s = \sup(\Lambda_s \cap \mathbb{R}) \quad \text{and} \quad m_s = \inf(\Lambda_s \cap \mathbb{R}).
\]

Then \( \sup(\Omega_{y_0,s} \cap \mathbb{R}) = a_{y_0,s}(M_s) \) and \( \inf(\Omega_{y_0,s} \cap \mathbb{R}) = a_{y_0,s}(m_s) \). First, \( M_s > \sup(\text{supp}\nu) \) by Point 1 of Theorem 4.1. Recall from Definition [22] that (since \( M \) is real)

\[
a_{y_0,s}(M_s) = H_{y_0,-s}(M_s)
\]

\[
= M_s - s \int \frac{d\nu(x)}{M_s - x}
\]

\[
= \frac{1}{M_s}(M_s^2 - s) - \frac{s}{M_s^2} \sum_{n=2}^{\infty} \frac{\tau(y_0^n)}{M_s^{n-2}}.
\]

(5.9)

Now, by Theorem 4.1, we have

\[
\sqrt{s} - \frac{3c\tau(y_0^2)}{2\sqrt{s}} < M_s < \sqrt{s} + \frac{3c\tau(y_0^2)}{2\sqrt{s}}
\]

for all large enough \( s \). Thus we can estimate \( |a_{y_0,s}(M_s)| \) by (5.9)

\[
|a_{y_0,s}(M_s)| = \left| (M_s - \sqrt{s}) \left( 1 + \frac{\sqrt{s}}{M_s} \right) - \frac{s}{M_s^2} \sum_{n=2}^{\infty} \frac{\tau(y_0^n)}{M_s^{n-2}} \right|
\]

\[
< \frac{3c\tau(y_0^2)}{\sqrt{s}} + \frac{c\tau(y_0^2)}{\sqrt{s}}
\]

\[
= \frac{4c\tau(y_0^2)}{\sqrt{s}}.
\]

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By that $\text{Brown}(y_0 + i\sigma_s)$ is symmetric about the real axis and the holomorphic moments of $\text{Brown}(y_0 + i\sigma_s)$ agree with the corresponding holomorphic moments of $y_0 + i\sigma_s$ [9],

$$
\int a \, d\text{Brown}(y_0 + i\sigma_s)(a + ib) = \int (a + ib) \, d\text{Brown}(y_0 + i\sigma_s)(a + ib) = \tau(y_0 + i\sigma_s) = 0. 
$$

(5.10)

It is impossible that $a_{y_0,s}(M_s) \leq 0$; otherwise, since $\Omega_{y_0,s}$ is not a subset of the imaginary axis, the integral in (5.10) is negative, contradicting that the integral is 0.

The estimate for $a_{y_0,s}(m_s)$ is similar.

To prove Point 3, we let $\varphi_0 \in (0, \pi/2)$ such that $1/(\sin \varphi_0) < c$. By Theorem 4.1, if we write $\alpha$ the unique real number such that $H_{y_0,s}(\alpha \varphi + iv_{y_0,s}(\alpha \varphi)) = 2\sqrt{s} \cos \varphi$, $|\cos \varphi| \leq \cos \varphi_0$,

$$
|\alpha \varphi + iv_{y_0,s}(\alpha \varphi) - \sqrt{s} e^{i\varphi}| < \frac{1}{(\sin \varphi_0)\sqrt{s}}.
$$

Thus, we have

$$
\sqrt{s} - \frac{1}{(\sin \varphi_0)\sqrt{s}} < \sup \{v_{y_0,s}(\alpha \varphi) \mid |\cos \varphi| < \cos \varphi_0\} < \sqrt{s} + \frac{1}{(\sin \varphi_0)\sqrt{s}}.
$$

Also, for all $\alpha \geq \alpha \varphi_0$ or $\alpha \leq \alpha \pi - \varphi_0$, we have, by unimodality of $v_{y_0,s},$

$$
v_{y_0,s}(\alpha) < \sqrt{s} \sin \varphi_0 + \frac{1}{(\sin \varphi_0)\sqrt{s}} < \sqrt{s} - \frac{1}{\sqrt{s} \sin \varphi_0} < \sup \{v_{y_0,s}(\alpha \varphi) \mid |\cos \varphi| < \cos \varphi_0\}
$$

for all large enough $s$. It follows that

$$
\left|\sup_{\alpha \in \mathbb{R}} v_{y_0,s}(\alpha) - \sqrt{s}\right| < \frac{1/(\sin \varphi_0)}{\sqrt{s}} < \frac{c}{\sqrt{s}}
$$

for all sufficiently large $s$. Because $b_{s,t} = 2v_{y_0,s}$, Point 3 of this theorem is established.

\begin{proof}
\end{proof}

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References

[1] Bai, Z. D. Circular law. Ann. Probab. 25, 1 (1997), 494–529.

[2] Belinschi, S. T. The Lebesgue decomposition of the free additive convolution of two probability distributions. Probab. Theory Related Fields 142, 1-2 (2008), 125–150.

[3] Belinschi, S. T., Mai, T., and Speicher, R. Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem. J. Reine Angew. Math. 732 (2017), 21–53.

[4] Belinschi, S. T., Śniady, P., and Speicher, R. Eigenvalues of non-Hermitian random matrices and Brown measure of non-normal operators: Hermitian reduction and linearization method. Linear Algebra Appl. 537 (2018), 48–83.

[5] Bercovici, H., and Voiculescu, D. Superconvergence to the central limit and failure of the Cramér theorem for free random variables. Probab. Theory Related Fields 103, 2 (1995), 215–222.
[6] Bercovici, H., Wang, J.-C., and Zhong, P. Superconvergence to freely infinitely divisible distributions. Pacific J. Math. 292, 2 (2018), 273–290.

[7] Biane, P. On the free convolution with a semi-circular distribution. Indiana Univ. Math. J. 46, 3 (1997), 705–718.

[8] Biane, P., and Lehner, F. Computation of some examples of Brown’s spectral measure in free probability. Colloq. Math. 90, 2 (2001), 181–211.

[9] Brown, L. G. Lidskii’s theorem in the type II case. In Geometric methods in operator algebras (Kyoto, 1983), vol. 123 of Pitman Res. Notes Math. Ser. Longman Sci. Tech., Harlow, 1986, pp. 1–35.

[10] Demni, N., and Hamdi, T. Support of the Brown measure of the product of a free unitary Brownian motion by a free self-adjoint projection. J. Funct. Anal. 282, 6 (2022), Paper No. 109362, 36.

[11] Driver, B. K., and Hall, B. C. Yang-Mills theory and the Segal–Bargmann transform. Comm. Math. Phys. 201, 2 (1999), 249–290.

[12] Driver, B. K., Hall, B. C., and Kemp, T. The Brown measure of the free multiplicative Brownian motion. Probab. Theory Related Fields (To appear).

[13] Ginibre, J. Statistical Ensembles of Complex, Quaternion and Real Matrices. J. Math. Phys. 6 (1965), 440–449.

[14] Girko, V. L. The circular law. Teor. Veroiatnost. i Primenen. 29, 4 (1984), 669–679.

[15] Girko, V. L. The elliptic law. Teor. Veroiatnost. i Primenen. 30, 4 (1985), 640–651.

[16] Haagerup, U., and Larsen, F. Brown’s spectral distribution measure for $R$-diagonal elements in finite von Neumann algebras. J. Funct. Anal. 176, 2 (2000), 331–367.

[17] Haagerup, U., and Schultz, H. Brown measures of unbounded operators affiliated with a finite von Neumann algebra. Math. Scand. 100, 2 (2007), 209–263.

[18] Hall, B. C. A new form of the Segal–Bargmann transform for Lie groups of compact type. Canad. J. Math. 51, 4 (1999), 816–834.

[19] Hall, B. C. PDE methods in random matrix theory. In Harmonic analysis and applications, vol. 168 of Springer Optim. Appl. Springer, Cham, [2021] ©2021, pp. 77–124.

[20] Hall, B. C., and Ho, C.-W. The Brown measure of a family of free multiplicative Brownian motions. preprint arXiv:2104.07859 (2021).

[21] Hall, B. C., and Ho, C.-W. The Brown measure of the sum of a self-adjoint element and an imaginary multiple of a semicircular element. Lett. Math. Phys. 112, 2 (2022), Paper No. 19, 61.

[22] Hasebe, T., and Ueda, Y. Large time unimodality for classical and free Brownian motions with initial distributions. ALEA Lat. Am. J. Probab. Math. Stat. 15, 1 (2018), 353–374.

[23] Ho, C.-W. The two-parameter free unitary Segal-Bargmann transform and its Biane-Gross-Malliavin identification. J. Funct. Anal. 271, 12 (2016), 3765–3817.

[24] Ho, C.-W. The Brown measure of unbounded variables with free semicircular imaginary part. preprint arXiv:2011.14222 (2020).

[25] Ho, C.-W., and Zhong, P. Brown measures of free circular and multiplicative Brownian motions with self-adjoint and unitary initial conditions. J. Eur. Math. Soc. (JEMS) (To appear).

[26] Janik, R. A., Nowak, M. A., Papp, G., Wambach, J., and Zahed, I. Non-Hermitian random matrix models: Free random variable approach. Phys. Rev. E 55 (Apr 1997), 4100–4106.

[27] Jarosz, A., and Nowak, M. A. A novel approach to non-Hermitian random matrix models. preprint arXiv:math-ph/0402057 (2004).
[28] Jarosz, A., and Nowak, M. A. Random Hermitian versus random non-Hermitian operators—unexpected links. *Journal of Physics A: Mathematical and General* 39, 32 (Jul 2006), 10107–10122.

[29] Śniady, P. Random regularization of Brown spectral measure. *J. Funct. Anal.* 193, 2 (2002), 291–313.

[30] Stephanov, M. A. Random matrix model of QCD at finite density and the nature of the quenched limit. *Phys. Rev. Lett.* 76 (1996), 4472–4475.

[31] Tao, T., and Vu, V. Random matrices: universality of ESDs and the circular law. *Ann. Probab.* 38, 5 (2010), 2023–2065. With an appendix by Manjunath Krishnapur.

[32] Voiculescu, D. Addition of certain non-commuting random variables. *J. Funct. Anal.* 66, 3 (1986), 323–346.

[33] Voiculescu, D. Limit laws for random matrices and free products. *Invent. Math.* 104, 1 (1991), 201–220.

[34] Wang, J.-C. Local limit theorems in free probability theory. *Ann. Probab.* 38, 4 (2010), 1492–1506.

[35] Zhong, P. Brown measure of the sum of an elliptic operator and a free random variable in a finite von neumann algebra. *preprint arXiv:2108.09844* (2021).