Constrained Restless Bandits for Dynamic Scheduling in Cyber-Physical Systems

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ABSTRACT This paper develops a sequential decision-making framework called constrained restless multi-armed bandits (CRMABs) to model problems of resource allocation under uncertainty and dynamic availability constraints. The decision-maker’s objective is to maximize the long-term cumulative reward. This can only be achieved by considering the impact of current actions on the future evolution of states. The uncertainty about the future availability of arms and partial state-information makes this objective challenging. CRMABs can be applied to resource allocation problems in cyber-physical systems, including sensor/relay scheduling. Whittle’s index policy, online rollout, and myopic policies are studied as solutions for CRMABs. First, the conditions for the applicability of Whittle’s index policy are studied, and the indexability result is claimed under some structural assumptions. An algorithm for index computation is presented. The online rollout policy for partially observable CRMABs is proposed as a low-complexity alternative to the index policy, and the complexity of these schemes is analyzed. An upper bound on the optimal value function is derived, which helps assess the sub-optimality of various solutions. The simulation study compares the performance of these policies and shows that the rollout policy is the better performing solution. In some settings it shows about 14% gain relative to Whittle’s index and myopic policies. Finally, an application to the problem of wildfire management is presented. Decision-making using CRMABs is analyzed from the perspective of a central agency tasked with fighting wildfires in multiple regions under logistic constraints.

INDEX TERMS Markov decision processes, restless multi-armed bandits, dynamic scheduling, cyber-physical systems, reinforcement learning, sequential decision models, stochastic control.

I. INTRODUCTION

Restless multi-armed bandits (RMABs) are a class of discrete-time stochastic control problems which involve sequential decision making with a finite set of actions (called arms). RMABs are used in applications involving decision making under uncertainty in evolving environments. They have been extensively studied for scheduling applications in opportunistic communication systems, dynamic relay selection in wireless relay networks, queuing systems, multi-agent systems, recommendation systems, unmanned aerial vehicle routing, internet of things, scheduling machine maintenance, and cyber-physical systems [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. Reference [15] presented a field study that utilized an RMAB based system for service quality improvement in a public health setting.

Let us first look at some application scenarios involving dynamic scheduling in uncertain environments under resource constraints. Cyber-physical systems (CPS) have recently received much attention in view of their potential applications relating to environment, health care, security, etc. [16], [17]. A simple representation of the interaction
In CPS, there are constraints, such as limited communication channels. During this time, no input is provided to or feedback is received from the other local controllers, which may still interact with their respective sub-processes, depending on the application. Local controllers are occasionally unavailable temporarily, either due to maintenance or due to resource constraints such as limited energy reserves. The solid lines indicate the control signals from the global controller, while the dotted lines indicate information signals about the environment through the local controllers.

Between various elements of a layered CPS is shown in Fig. 1. In highly complex cyber-physical systems, there may be layers of decision making involving local controllers. Here, the global controller’s goal is to control a complex process with many independent sub-processes, to maximize its rewards. For this purpose a local controller is assigned to interact with each sub-process. The global controller aims for macro-optimization, while the local agents/controllers are tasked with micro-optimization.

In these systems, the problem of scheduling and resource allocation occurs at multiple places, such as sensor scheduling for monitoring dynamic processes and task scheduling for controlling sub-processes, etc. Further, these scheduling problems often come with resource availability and latency constraints. For instance, some of the local controllers may not be available in certain time slots on account of their being engaged in maintenance. This availability may be time-varying and depend on factors such as maintenance time. Similarly, communication channels might become unavailable due to signal outages or because they are already in use. The objective is for the source to select $M$ out of these $N$ paths during each time slots to maximize long-term data transmission quality/rate. However, the source lacks direct information about the quality and availability of each path, complicating the relay selection process. This challenge is addressed through a feedback mechanism where, at the end of each interval, the source receives feedback in the form of success or failure for the paths used. Based on this feedback, the source updates its selection for the subsequent interval, taking into account both path quality and the factors contributing to path unavailability.

In this paper we formulate the problem of constrained restless multi-armed bandits which can be used to represent various problems involving adaptive scheduling under dynamic resource constraints. In the following discussion we provide a brief overview of RMABs.

### A. RESTLESS MULTI-ARMED BANDITS

An RMAB is described as follows. There is a decision maker or source that has $N$ independent arms. Each arm can be in one of a finite set of states, and the state evolves according to Markovian law. The play of an arm yields a state-dependent reward. It is assumed that the decision maker knows the statistical characteristics of state evolution for each arm. The system is time-slotted. The decision maker plays $M$ out of $N$ arms in each slot. The goal is to determine the sequence of plays of the arms that maximize the long-term cumulative reward. These planning problems are non-trivial because there is a trade-off between the immediate and future rewards. The choice that yields low immediate reward may yield better future reward.
TABLE 1. Major differences in the relevant literature on restless multi-armed bandit (RMAB) models.

| Paper                  | Arm Model                        | Constraints (Arm availability, budget) | Policies analysed | Upper bounds |
|------------------------|----------------------------------|----------------------------------------|-------------------|--------------|
| Whittle 1988 [18]      | RMAB (MDP) (State is observable) | Arms always available (No constraints on arms) | Index policy     | Yes          |
| Dayanik 2002 [19]      | CRMB (MDP) (State is observable) | Dynamic availability (stochastic)      | Index policy     | No           |
| Liu and Zhao 2010 [2]  | RMAB (POMDP) (Observable only for played arms) | Arms always available (No constraints on arms) | Index policy     | Yes          |
| Meslem et al. 2018 [20]| RMAB (POMDP) (State not observable when played or not played) | Arms always available (No constraints on arms) | Index policy     | No           |
| Mehta et al. 2018 [21] | CRMB (POMDP) (State not observable when played or not played) | Arms always available (sometimes with extra cost) | Index policy     | No           |
| Kaza et al. 2019 [3]   | RMAB (POMDP) (State not observable when played or not played) | Cumulative feedback after multiple transitions (No additional constraints on arms) | Index policy     | Yes          |
| Meslem et al. 2020 [22]| Multi-action RMAB (MDP) (State is observable) | Arms always available (No constraints on arms) | Rollout policy   | No           |
| Akbarzadeh 2022 [23]   | RMAB (POMDP) (Observable only for played arms) | Arms always available (No constraints on arms) | Index policy     | No           |
| Rodriguez-Diaz 2023 [24]| RMAB (MDP) (State is observable) | Arms always available (Flexible budget constraint) | Heuristic policy using Primal dual linear program | No |
| Liu 2024 [25]          | RMAB (POMDP) (Imperfect/error prone state Observations) | Arms always available (No additional constraints on arms) | Low complexity algorithm for Index computation | Yes |
| This paper             | CRMB (POMDP) (State not observable when played or not played) | Dynamic Availability (stochastic, semi-deterministic) | Index policies, Rollout policy | Yes |

A typical RMAB model assumes that the arms are always available, and the objective is to determine the optimal subset of arms to be played in a given state (in each time slot). In RMAB applications to real world systems, arms might represent machines, communication channels, sensors, and various other subsystems. In conventional RMAB literature all the arms are always assumed to be available, and the problem is to find the optimal subset of arms each time slot. The only constraint is on the number of arms that can be played in each slot. This is called a weak coupling constraint. However, these arms (machines, communications channels, sensors) can go unavailable temporarily due to maintenance, repair, energy constraints, etc. Thus, the availability of arms is intermittent and time-varying, which is an additional constraint on the arms and the decision maker. We refer to such RMABs as constrained restless multi-armed bandits (CRMAB). This is a generalization of restless multi-armed bandits. Models for the availability of arms may vary across applications. We study CRMABs with two different availability models - stochastic and semi-deterministic.

Apart from scheduling in CPS, another potential application for CRMABs is the problem of dynamic relay selection in wireless networks with evolving channel conditions and intermittent availability of relays due to energy constraints or other application interrupts.

B. RELATED WORK

The literature on restless bandits is vast and includes different variations on bandits and their applications. We mention a few that are relevant to our work. The major differences in the relevant literature are summarized in Table 1. In the following, we discuss the important developments in the theory of restless bandits.

The restless multi-armed bandit problem was first proposed in [18]. It was inspired by the work on rested bandits [27]. In [27], index policies were introduced for rested multi-armed bandits, where states of arms are frozen when they are not played. This index policy is now known as Gittins index policy. Later, [18] studied restless bandits and introduced a heuristic index policy, which is now referred to as Whittle’s index policy. The popularity of Whittle’s index policy is due to its asymptotic optimality in some examples and its near-optimal performance in others (see [18], [28], [29]). Whittle’s index policy for other applications such as machine maintenance and repair problems are analysed in [5] and [13]. Recently, [14] applied the RMAB framework to the problem of risk-sensitive scheduling in CPS. Here, an exponential cost function is defined instead of a linear function. This variant is termed a risk-sensitive RMAB, and the corresponding index policy is a risk-sensitive index policy. In classical restless bandit literature, current states of all the arms are observable in every time slot [8], [9], [18]. Later, this assumption was relaxed, and restless bandit models with partially observable states were studied, where states are observable only for those arms that are played [2], [29]. More recent work on restless bandits further generalized this model to the case where states of all arms are partially observable. This is referred as the hidden restless bandit [20], [30]. In [3], further generalization is considered where multiple state transitions are allowed in a single decision interval. In [23], a RMAB model where states of the active arms are reset according to known distribution is considered. References [13] and [31] consider some variations of multi-state RMAB models; optimal threshold policy and indexability results under some structural assumptions. Whittle’s index policy for RMABs was studied for job scheduling and dynamic routing on...
servers in [32] and [33], where authors considered scenario of servers being available intermittently.

An RMAB variant with availability constraints on arms was first proposed in [19]. It was applied to the machine repair problem where machine availability is time-varying, and machine state is observable. This paper introduced two models—1) Unavailable arms cannot be played, and 2) Unavailable arms can be played with some additional penalty. Whittle’s index policy is studied. Since states are exactly observable, a closed-form expression for the index is easily obtained. The second model of [19] was generalized for observable, a closed-form expression for the index is easily obtained. Further, we also consider a semi-deterministic availability model. The results claimed in [21] do not necessarily apply to the current model because of differences in belief update rules and assumptions on reward structure. Additionally, we study the rollout policy, provide upper bounds on optimal value function, and analyze the sample complexity of the index computation algorithms.

The literature on POMDPs, RMABs makes use of certain common techniques and procedures. These include defining action value functions and using induction principle to derive their structural properties. Another common aspect is proving sub-modularity of the value function, which will lead to a threshold structure of optimal policy (see [34], [35]). The threshold structure of the optimal policy is used to prove indexability of restless single armed bandits. This allows one to apply Whittle’s index policy. One must note the differences in modeling that require redoing or following similar procedures as it is not obvious that the same results hold.

C. CONTRIBUTIONS

1) We consider the problem of restless multi-armed bandit with dynamic resource constraints and partially observable states. It is referred to as partially observable CRMAB. We study two availability models—stochastic and semi-deterministic.

2) We analyze the constrained restless single-armed bandit (CRSAB) problem under some structural assumptions, show the threshold structure of optimal policy and thereby indexability.

3) We utilize the indexability result to define an index policy and present an algorithm for index computation based on two-timescale stochastic approximation (TTSA).

4) We propose an online rollout policy for partially observable CRMABs, as an alternative to the index policy. This does not require any structural assumptions on the model, unlike Whittle’s index policy.

5) We present a detailed analysis on the sample and computational complexity of the online rollout policy. We also present a conjecture on the sample complexity of the index policy by drawing parallels between TTSA and the primal-dual schemes used to solve constrained MDP problems.

6) We derive an upper bound on the optimal value function of the CRMAB problem. This bound can be used to estimate the degree of sub-optimality of various policies.

7) We present a simulation study which compares the performance of various policies in different parametric settings. We observe that the online rollout policy performs better than Whittle’s index policy which is followed closely by the myopic policy.

8) We present a case study on the application of CRMAB models to wildfire management. We also discuss the limitations of the proposed model in the context of this application, along with possibilities for further improvement in the future.

The rest of this document is organized as follows. The system model is explained in Section II and the constrained restless single armed bandit is analyzed in Section III. Sample complexity results are discussed in Section IV. Online rollout policy is presented in Section V. Bounds on value functions are derived in Section VI. Numerical simulations are presented in Section VII. A case study on wildfire management is presented in Section VIII, and concluding remarks are given in Section IX.

II. MODEL DESCRIPTION AND PRELIMINARIES

Consider a restless multi-armed bandit with N independent arms. Each arm can be one of two states, state 0 or state 1. The state of each arm evolves according to a discrete time Markov chain. Sometimes the arms might become unavailable. The evolution of states also depends on the availability of arms. The availability of arms is time varying. Let us introduce some notation to formalize the model. The system is time-slotted and time is indexed by t. Let $X_n(t) \in \{0, 1\}$ denote the state of arm n at the beginning of time slot t.

Let $Y_n(t) \in \{0, 1\}$ denote the availability of arm n at the beginning of time slot t and

$$Y_n(t) = \begin{cases} 1 & \text{if arm } n \text{ is available}, \\ 0 & \text{if arm } n \text{ is not available}. \end{cases}$$

Each arm has two actions associated with it when it is available, either play or don’t play. When it is unavailable it cannot be played. However, it’s state continues to evolve. Let $a_n^i(t) \in \{0, 1\}$ denote the action corresponding to arm n when it is available, it is described as follows.

$$a_n^1(t) = \begin{cases} 1 & \text{if arm } n \text{ is available and played}, \\ 0 & \text{if arm } n \text{ is available and not played}. \end{cases}$$
Let $\rho_n(t)$ be the action corresponding to arm $n$ when it is not available. As it cannot be played, $\rho_n(t) := 0$.

The state of arm $n$ changes at beginning of time slot $(t + 1)$ from state $i$ to $j$ according to transition probabilities $p^n_{ij}$. These are defined as follows.

$$p^n_{ij} := \Pr[X_n(t + 1) = j \mid X_n(t) = i].$$

When arm $n$ is played, the result is either success or failure. A binary signal is observed at the end of each slot that describes the event of success or failure (ACK or NACK in communication parlance). Let $Z_n(t)$ be the binary signal that is received by the source (decision maker) at the end of slot $t$. It is given as

$$Z_n(t) = \begin{cases} 1 & \text{if arm } n \text{ is played and that resulted success,} \\ 0 & \text{If arm } n \text{ is played and no success.} \end{cases}$$

When arm $n$ is not played, no signal is observed from that arm. Let $\rho_n(i)$ be the probability of success from playing arm $n$.

$$\rho_n(i) = \Pr(Z_n(t) = 1 \mid X_n(t) = i, a_n(t) = 1), \text{ for } i = 0, 1. \text{ It is the probability that signal } Z_n(t) = 1 \text{ is observed given that arm } n \text{ is in state } i \text{ and action } a_n(t) = 1. \text{ We will assume } \rho_{n,0} < \rho_{n,1}, \text{ i.e., the probability of success is higher from state } 1 \text{ than from state } 0.$$

The play of arm $n$ yields a state dependent reward. Let $\eta_{n,i}$ be the reward obtained by playing arm $n$ given that $X_n(t) = i$. When arm $n$ is not played, no reward is obtained. Further, we suppose that $0 \leq \eta_{n,0} < \eta_{n,1} \leq 1$ for all $n$.

The decision maker or source cannot observe the exact state vector at any arbitrary time $t$. However, it can exactly observe the current availability vector at the beginning of each time slot. That is, $Y(t) = [Y_1(t), \ldots, Y_N(t)]$ is known at beginning of slot $t$. Since the source does not know the exact states of arms, it maintains a ‘belief’ about each of them. Let $\pi_n(t)$ be the belief about arm $n$. It is the probability of being in state $0$, given the history $H_t$ up to time $t$.

The history up to time $t$ is given as

$$H_t := (Y_n(s), a_n(s), Z_n(s))_{0 \leq s \leq t; n \leq t}.$$

The belief vector is given as $\pi(t) = [\pi_1(t), \ldots, \pi_n(t)]$, with $\pi_n(t) = \Pr(X_n(t) = 0 \mid H_t)$.

### A. AVAILABILITY MODELS

We consider two availability models, namely, stochastic and semi-deterministic. In the stochastic model, future availability depends on a probability value conditioned on current availability. In the semi-deterministic model, future availability is deterministic when an arm goes unavailable. This model is useful in applications in which some sub-systems are occasionally down for a fixed period of time for maintenance.

1) STOCHASTIC

The future availability of arm $n$, $Y_n(t + 1)$, is dependent on current availability $Y_n(t) = y$, action $a_n(t) = a$ and current state of arm $X_n(t) = i$. We define

$$\theta^a_n(i, y) := \Pr(Y_n(t + 1) = 1 \mid X_n(t) = i, Y_n(t) = y, a_n(t) = a).$$

We replace knowledge of state $X_n(t) = i$ with belief $\pi_n(t) = \pi$, and we rewrite $\theta^a_n(i, y)$ as $\theta^a_n(\pi, y)$. The availability model is described as follows.

$$Y_n(t + 1) = \begin{cases} 1, & \text{w.p. } \theta^a_n(\pi, y), \\ 0, & \text{w.p. } 1 - \theta^a_n(\pi, y). \end{cases}$$

The decision maker knows the probability of availability $\theta_n(\pi, y)$. Notice that this model satisfies Markov property.

In general, $\theta^a_n(i, y)$ depends on the state of arm $n$, current availability $y$ and action of that arm $a$.

The model for $\theta^a_n(\pi, y)$ might depend on application. For simplicity, we assume that $\theta^a_n(\pi, y)$ is to be linearly dependent on $\pi$ and $y$ for each $a \in [0, 1]$.

2) SEMI-DETERMINISTIC

The future availability for unavailable arms has a deterministic model. When available arms turn unavailable, they remain unavailable for exactly $T_0$ slots and then become available. That is, if arm $n$ is unavailable, i.e., $Y_n(t) = 0$, then

$$Y_n(t + t') = \begin{cases} 0, & \text{for } t' = 1, \ldots, T_0 - 1, \\ 1, & \text{for } t' = T_0. \end{cases}$$

If arm $n$ is available, i.e., $Y_n(t) = 1$, then

$$Y_n(t + 1) = \begin{cases} 1, & \text{w.p. } \theta^a_n(\pi, 1), \\ 0, & \text{w.p. } 1 - \theta^a_n(\pi, 1). \end{cases}$$

### B. PROBLEM FORMULATION

As the exact state is not observable the state is redefined in terms of belief and availability. Consider the perceived state $S_n(t) = (\pi_n(t), Y_n(t)) \in [0, 1] \times [0, 1]$ in beginning of time slot $t$. Using the belief $\pi_n(t)$, we compute the expected reward from play of arm $n$ at time $t$ as follows.

$$\eta(\pi_n(t), y = 1) := \pi_n(t)\eta_{n,0} + (1 - \pi_n(t))\eta_{n,1}$$

and $\eta(\pi_n(t), y = 0) := 0$.

We next define the optimization problem as reward maximization. Let $\phi(t)$ be the policy of the source such that $\phi(t) : H_t \rightarrow \{1, \ldots, N\}$ maps the history to $M$ arms in slot $t$. Let

$$a^\phi_n(t) = \begin{cases} 1 & \text{if } n \in \phi(t), \\ 0 & \text{if } n \notin \phi(t). \end{cases}$$
The infinite horizon discounted cumulative reward under strategy \( \phi \) for initial state information \((\pi, y)\), \( \pi = (\pi_1(1), \cdots, \pi_N(1)) \) and \( y = (y_1(1), \cdots, y_N(1)) \) is given by

\[
V_\phi(\pi, y) = E^\phi \left( \sum_{t=0}^{\infty} \beta^{t-1} \left[ \sum_{n=1}^{N} \delta_n^0(t) q(\pi_n(t), Y_n(t)) \right] \right).
\]

In each time slot \( M \) arms are played; hence the constraint \( \sum_{n=1}^{N} \delta_n^0(t) = M \). Here, \( \beta \in (0, 1) \) is the discount parameter. The objective is to find a policy \( \phi \) that maximizes \( V_\phi(\pi, y) \) for all \( \pi \in [0, 1]^N \), \( y \in [0, 1]^N \). The problem (1) is a constrained hidden Markov restless multi-armed bandit. The optimal solution for problem (1) is computationally intractable; it is known to be PSPACE-hard [36]. The major difficulty here is due to the integer constraint, \( \sum_{n=1}^{N} \delta_n^0(t) = M \), \( \delta_n^0(t) \in \{0, 1\} \). The key idea is to introduce a relaxed version of problem (1). This is done by replacing the exact integer constraint with the following expectation constraint.

\[
E^\phi \left( \sum_{t=0}^{\infty} \beta^{t-1} \left[ \sum_{n=1}^{N} \delta_n^0(t) \right] \right) = \frac{M}{1 - \beta}.
\]

Now, using Lagrangian relaxation of the problem, we can reduce the dimension of the relaxed RMAB problem into \( N \) restless single-armed bandits. In the next section we study the constrained restless single-armed bandit problem and define an index policy.

Note: The above formulation assumes that at least \( M \) are available in each slot. When the number of available arms (say \( m \)) in a slot is less than \( M \), then \( M - m \) dummy arms with minuscule rewards (say \( \epsilon_i > 0 \) for state \( i \)) are played.

### III. CONSTRAINED RESTLESS SINGLE ARMED BANDIT

As there is only one arm, the problem of the decision maker here is to decide in each time slot whether or not to play the arm. We drop the subscript \( n \), the sequence number of the arms; so, \( \rho_{n,i} \equiv \rho_i \), \( \eta_{n,i} \equiv \eta_i \), \( \theta_i^y(y) \equiv \theta_i^y \). The analysis of the single arm problem proceeds by assigning a subsidy \( w \) for not playing the arm.

Recall that the source maintains and updates its belief about state of the arm at the end of every time slot. The update rules are based on previous actions, availability and observations of the arms and it is given as follows.

1) If the arm is available, played and a success is observed, i.e., \( a(t) = 1 \), \( Y(t) = 1 \), \( Z(t) = 1 \). Then the new belief \( \pi(t + 1) = \Gamma_1(\pi(t)) \), and it is

\[
\Gamma_1(\pi) = \frac{\pi p_0 p_{00} + (1 - \pi) \rho_1 p_{10}}{\pi p_0 + (1 - \pi) \rho_1}.
\]

This update is according to the Bayes rule.

2) If the arm is available, played and success is not observed, i.e., \( a(t) = 1, Y(t) = 1, Z(t) = 0 \), then the belief \( \pi(t + 1) = \Gamma_0(\pi(t)) \), and it is

\[
\Gamma_0(\pi) = \frac{\pi (1 - \rho_0) p_{00} + (1 - \pi)(1 - \rho_1)p_{10}}{\pi(1 - \rho_0) + (1 - \pi)(1 - \rho_1)}.
\]

3) If the arm is available but not played, there is no observation, i.e., \( a(t) = 0, Y(t) = 1 \). Then the belief \( \pi(t + 1) = \gamma_1^0(\pi(t)) \) and it is given by

\[
\gamma_1^0(\pi) = \pi p_{00} + (1 - \pi)p_{10}.
\]

4) If the arm is not available, then it cannot be played and no observation is available, i.e., \( a(t) = 0, Y(t) = 0 \). We consider the belief \( \pi(t + 1) = \gamma_0^0(\pi(t)) \) and it is updated according to following rule.

\[
\gamma_0^0(\pi) = \frac{p_{10}}{p_{01} + p_{10}} \text{ or } \pi p_{00} + (1 - \pi)p_{10}.
\]

In this case, belief is either taken to be the stationary probability or the value obtained by natural evolution of the Markov chain.

### A. VALUE FUNCTIONS

Given a state \((\pi, y)\), let \( V(\pi, y) \) denote the expected cumulative discounted reward achieved by the optimal policy. \( V \) is called the optimal value function. Let us now define the values of different actions depending on the belief and availability, in terms of \( V \). The value for action \( a \), given belief \( \pi \) and availability \( y \) is denoted as \( \mathcal{L}^a V(\pi, y) \) for \( a \in \mathcal{A}_y \), \( y \in \{0, 1\} \). Here, \( \mathcal{L}^a V \) is called action value function. \( \mathcal{A}_y \) is the set of possible actions for availability \( y \). For our model, we have \( \mathcal{A}_1 = \{0, 1\} \) and \( \mathcal{A}_0 = \{0\} \). When the arm is unavailable (\( y = 0 \)), it cannot be played.

The value functions for stochastic availability model are given as follows.

a) For action \( a = 1 \), and availability \( y = 1 \) :

\[
\mathcal{L}^1 V(\pi, 1) = \eta(\pi) + \beta \rho(\pi) \times \left[ \theta_1^1(\pi)V(\Gamma_1(\pi), 1) + (1 - \theta_1^1(\pi))V(\Gamma_1(\pi), 0) \right] + \beta (1 - \rho(\pi)) \times \left[ \theta_0^1(\pi) V(\Gamma_0(\pi), 1) + (1 - \theta_0^1(\pi))V(\Gamma_0(\pi), 0) \right]
\]

Here, \( \eta(\pi) = \eta_0 \pi + \eta_1 (1 - \pi) \).

The value function consists of immediate expected reward and discounted future value. So, the first term is immediate reward, \( \eta(\pi) \). The second term and third terms depend on probability of observing success or failure. These terms also include the future value function and expectation w.r.t. availability probability.

b) For action \( a = 0 \), and availability \( y = 1 \) :

\[
\mathcal{L}^0 V(\pi, 1) = w + \beta \times \left[ \theta_0^0(\pi) V(\gamma_0^0(\pi), 1) + (1 - \theta_0^0(\pi))V(\gamma_0^0(\pi), 0) \right]
\]

If the arm is available and is not played, the immediate reward is a subsidy \( w \). The second term of the value
function includes the expectation of future value which depends on availability probability and updated belief.

c) Action \( a = 0 \), availability \( y = 0 \),

\[
\mathcal{L}^0 V(\pi, 0) = w + \beta \\
\times \left[ \theta_0^0(\pi)V(\gamma_0^0(\pi), 1) + (1 - \theta_0^0(\pi))V(\gamma_0^0(\pi), 0) \right]
\]

This value function is very similar to preceding case. If the arm is unavailable, it cannot be played. The value function consists of immediate reward as subsidy \( w \) and the expected future value which depends on availability probability and updated belief. This updated belief could stationary probability or the value obtained by natural evolution of the Markov chain.

We will now write down action value function expressions for the semi-deterministic availability model. Recall that when arm is not available then it cannot be played for a fixed amount of time \( T_0 \). Thus, the value function differs from earlier stochastic model for availability \( y = 0 \) and action \( a = 0 \). The value function for availability \( y = 1 \) and action \( a = 0 \) or \( a = 1 \) is similar to that of stochastic availability model. The value functions are given as follows.

a) Action \( a = 1 \), and availability \( y = 1 \):

\[
\mathcal{L}^1 V(\pi, 1) = \eta(\pi) + \beta p(\pi) \\
\times \left[ \theta_1^1(\pi)V(\Gamma_1(\pi), 1) + (1 - \theta_1^1(\pi))V(\Gamma_1(\pi), 0) \right] \\
+ \beta(1 - \rho(\pi)) \\
\times \left[ \theta_1^1(\pi)V(\Gamma_0(\pi), 1) + (1 - \theta_1^1(\pi))V(\Gamma_0(\pi), 0) \right]
\]

b) Action \( a = 0 \), and availability \( y = 1 \):

\[
\mathcal{L}^0 V(\pi, 1) = w + \beta \\
\times \left[ \theta_0^0(\pi)V(\gamma_0^0(\pi), 1) + (1 - \theta_0^0(\pi))V(\gamma_0^0(\pi), 0) \right]
\]

c) Action \( a = 0 \), and availability \( y = 0 \):

\[
\mathcal{L}^0 V(\pi, 0) = w \frac{(1 - \beta T_0)}{(1 - \beta)} + \beta T_0 V \left( (\gamma_0^0 T_0(\pi), 1) \right)
\]

Since the arm is unavailable for \( T_0 \) number of slots, the discounted reward obtained in this period is \( w \frac{(1 - \beta T_0)}{(1 - \beta)} \). The second term is future discounted value after \( T_0 \) slots when the arm becomes available.

Observe that there is no available choice of actions for \( y = 0 \). However, the value function \( \mathcal{L}^0 V(\pi, 0) \) is important as it impacts other value functions.

The optimal value function \( V(\pi, y) \) satisfies the following dynamic programming optimality equations.

\[
V(\pi, y) = \max_{a \in \mathcal{A}_y} \mathcal{L}^a V(\pi, y), \\
\forall \pi \in [0, 1] \text{ and } y \in \{0, 1\}. \tag{3}
\]

Note that we sometimes use the notation \( \mathcal{L}^a V(\pi, y) \) in place of \( \mathcal{L}^0 V(\pi, y) \) to emphasize the dependence on \( w \). We obtain all results assuming \( \eta_0 = \rho_0 \) and \( \eta_1 = \rho_1 \).

### B. STRUCTURAL RESULTS

In the following, we derive structural results for value functions in the case of stochastic availability. These results also hold true for the semi-deterministic availability model because the value functions are similar except at availability \( y = 0 \). We first define a threshold type policy.

**Definition 1 (Threshold Type Policy):** A policy is said to be of threshold type if one of the following is true.

1. For \( y = 1 \) and \( \forall \pi \in [0, 1] \), \( \mathcal{L}^1 V(\pi, 1) > \mathcal{L}^0 V(\pi, 1) \).

   In this case the optimal action is to play the arm.

2. For \( y = 1 \) and \( \forall \pi \in [0, 1] \), \( \mathcal{L}^1 V(\pi, 1) < \mathcal{L}^0 V(\pi, 1) \).

   In this case not playing the arm is always optimal.

3. There exists a \( \pi_T \in (0, 1) \), such that, \( \mathcal{L}^1 V(\pi, 1) > \mathcal{L}^0 V(\pi, 1) \) for all \( \pi < \pi_T \), and \( \mathcal{L}^1 V(\pi, 1) < \mathcal{L}^0 V(\pi, 1) \) for all \( \pi > \pi_T \). Here, \( \pi_T \) is a threshold at which both actions are optimal and obtain same value from both the actions.

To claim the existence of threshold type policy result we prove following structural properties of the value functions. Using these properties, we will show that the optimal value function is submodular. The optimal threshold policy result follows from submodularity.

**Lemma 1:** For both stochastic and semi-deterministic availability

1. value functions \( V(\pi, y) \) and \( \mathcal{L}^a W(\pi, y) \) are convex in \( \pi \) for \( a \in \mathcal{A}_y, y \in [0, 1] \).

2. value functions \( V(\pi, y) \) and \( \mathcal{L}^a W(\pi, y) \) are convex in \( w \) for all \( \pi \in [0, 1] \), \( a \in \mathcal{A}_y \), \( y \in [0, 1] \).

The proof is given in Appendix A.

We note that convexity of value function is not enough to show threshold policy. This is because the belief update for played arm is non-linear in current belief. Even with some structural assumptions on transition probabilities, it is difficult prove submodularity. This is more clear from [34, Lemma 2.1 and Eqn.(4)], where submodularity and threshold behavior is proved when either playing or not playing action provides perfect state information. This is not true in our model. Hence we require an alternative proof technique. This is given in the following.

**Remark 1:**

- We know about the continuity and convexity of value functions in \( \pi \). We also know that value functions are absolutely continuous in \( \pi \). Further, value functions are Lipschitz in \( \pi \), this is because rewards are bounded and discounted with parameter \( 0 < \beta < 1 \). Hence, partial derivative of value function w.r.t. \( \pi \) is bounded. Next, in Lemma 2, we derive a tight Lipschitz constant. This constant will be used in subsequent lemmas to prove submodularity and threshold policy result.

- A tighter Lipschitz constant allows a wider range of transition probabilities for which threshold policy result
can be proved analytically. A more relaxed Lipschitz constant gives a smaller range of transition probabilities for which the result is provable. We believe this is a technical limitation that does not allow us to leverage the structure of the problem to find out the smallest Lipschitz constant.

Now, we show that the partial derivative of the value function w.r.t. \( \pi \) is bounded. A tighter bound is derived under some conditions on state transition probabilities. The bound is obtained under the assumption that \( \theta \) is independent of \( \pi \). When \( \theta \) is dependent on \( \pi \), we need additional properties on the value function which are mentioned after the next Lemma.

Lemma 2: Given that \( \theta^a_{\pi}, y \in \{0, 1\}, a \in \mathcal{A}_y \) is independent of \( \pi \) and for any of the following conditions,

1) \( 0 < p_{00} - p_{10} < \frac{1 - b}{1 + b} \),
2) \( 0 < p_{10} - p_{00} < \frac{1 - b}{1 + b} \),

absolute values the derivatives of the action value functions are bounded, i.e., \( \left| \frac{\partial C(V(\pi, \theta))}{\partial \pi} \right|, \left| \frac{\partial C(V(\pi, \theta))}{\partial \pi} \right| \) and \( \left| \frac{\partial \pi}{\partial \pi} \right| \) are bounded by \( k_\pi \rho_1 - \rho_0 \), where \( k = \frac{1}{1 - p_{00} - p_{10}} > 1, b = \frac{a_{1} - a_{0}}{\rho_1 - \rho_0}, c = \max \{1, \frac{a_{1} - a_{0}}{\rho_1 - \rho_0}\} \).

The proof is given in Appendix B.

Remark 2:

- When \( \rho_0 = \eta_0 \) and \( \rho_1 = \eta_1 \), the bound on the derivatives in Lemma 2 becomes \( k \rho_1 - \rho_0 \) under the conditions \( 0 < p_{00} - p_{10} < 1/2 \), or \( 0 < p_{10} - p_{00} < 1 \).
- Also notice that when \( \rho_0 = \eta_0 \) and \( \rho_1 = \eta_1 \), we can have a better Lipschitz constant with less restrictive conditions on transition probabilities. For example, let the ratio of ratios \( b, c \) be \( b = c = 1 \). The conditions become \( 0 < p_{00} - p_{10} < 0.477 \) or \( 0 < p_{10} - p_{00} < 0.954 \).

Remark 3: In Lemma 2 we had assumed that \( \theta^a_{\pi} \) is independent of \( \pi \). Instead, suppose \( \theta^a_{\pi}(\pi) \) is a linear function of \( \pi \) for given \( a \) and \( y \). To obtain a bound on the partial derivative of the value functions w.r.t. \( \pi \), we need to bound \( V(\pi, 1) - V(\pi, 0) \) in terms of \( \rho_0 - \rho_1 \) (see eqn. (23)). It is difficult to derive a tight bound because of the additional term \( V(\pi, 1) - V(\pi, 0) \). We believe that one can have loose bound on \( V(\pi, 1) - V(\pi, 0) \) which may introduce a more stringent condition on the difference of transition probabilities and in turn a loose Lipschitz constant. Hence, we do not analyze this scenario here.

Let \( D(\pi) := L^1 V(\pi, 1) - L^0 V(\pi, 1) \). It gives the advantage of playing the arm in belief state \( \pi \) when it is available. The following lemma states that this advantage decreases as the belief increases.

Lemma 3: Given that \( \theta^a_{\pi}, y \in \{0, 1\}, a \in \mathcal{A}_y \) is independent of \( \pi \) and for any of the following conditions,

1) \( 0 < p_{00} - p_{10} < \frac{b}{b + c + 1} \),
2) \( 0 < p_{10} - p_{00} < \frac{b}{b + c + 1} \),

the function \( D(\pi) \) is decreasing in \( \pi \).

The proof is given in Appendix C.

Remark 4: Note that \( L^1 V(\pi, 1) \) and \( L^0 V(\pi, 1) \) are convex in \( \pi \). Convexity of value functions and the preceding Lemma 3 suggest that \( D(\pi) \) has at most one root in \( \pi \in (0, 1) \).

We now state our main result, the optimal threshold policy result.

Theorem 1: Constrained restless single armed bandits of stochastic and semi-deterministic availability types satisfying either

1) \( 0 < p_{00} - p_{10} \leq \frac{b}{b + c + 1} \) or
2) \( 0 < p_{10} - p_{00} \leq \frac{b}{b + c + 1} \),

admit an optimal policy of threshold type.

Proof: Suppose \( L^1 V(\pi, 1) > L^0 V(\pi, 1) \), at \( \pi = 0 \). That is, playing the arm is advantageous than not playing it. From Lemma 3, \( D(\pi) \) can have at most one root in \( [0, 1] \).

Case 1: \( D(\pi) \) has a root in \( [0, 1] \) : From Lemma 3, we know that this advantage decreases as \( \pi \) increases. So, there exists a \( \pi_T \in (0, 1) \) : \( D(\pi_T) = L^1 V(\pi, 1) - L^0 V(\pi, 1) < 0, \forall \pi_T \). Hence the policy is of threshold type by definition 1.

Case 2) \( D(\pi) \) has no root in \( (0, 1) \) : This means \( D(\pi) > 0, \pi \in (0, 1) \). Hence the optimal policy always choose to play the arm and is threshold type by definition. Similar arguments can be made when \( L^1 V(\pi, 1) < L^0 V(\pi, 1) \), to claim the result.

C. INDEXABILITY

Index for a constrained restless single armed bandit in a given state is defined as the minimum amount of subsidy for which the value of not playing the arm becomes greater than or equal to the value of playing the arm. As subsidy is provided for not playing the arm, a higher value of the index indicates greater gains from playing the arm. To use these indices in decision making they need to be well defined. For this we need to first prove indexability of CRSABs. In this section, we define indexability for an arm (CRSAB) and prove that it is indexable. To claim this we make use of the optimal threshold policy result. Hence, we assume same conditions on transition probabilities similar to those in Theorem 1.

We now define indexability for a CRSAB and provide sufficient conditions.

For a given subsidy \( w \), let \( G(w) \) be a set formed by members \( (\pi, y) \) of perceived state space \( S = \{0, 1\} \times \{0, 1\} \) for which not playing the arm when available is optimal.

\[
G(w) := \{[0, 1] \times \mathcal{A}_0 \} \cup \{[0, 1] \times \mathcal{A}_1 : L^1 V(\pi, 1) \leq L^0 V(\pi, 1)\}
\]

Definition 2 (Indexability): The arm is indexable if the set \( G(w) \) is increasing in \( w \in \mathbb{R} \).

Intuitively, indexability suggests that, if not-playing is the optimal choice for a given subsidy \( w \), then it is also the optimal choice at higher values of subsidy \( w' > w \).

Remark 5:

- Action value functions \( L^1 V(\pi, 1) \) and \( L^0 V(\pi, 1) \) are non-decreasing and strictly increasing in subsidy \( w \),
respectively. The proof of this straightforward, it uses the principle of mathematical induction.

- For a CRSAB, Theorem 1 shows that there exists a threshold belief $\pi_T \in (0, 1)$ at the optimal action switches from playing the arm to not playing as we cross over to its right from the left. This threshold is a function of subsidy $w$. In the following, we will see that the threshold moves left the segment as subsidy increases.

To prove indexability of the arm, we use the following Lemma from [20] and provide a sketch of the proof.

**Lemma 4:** Let

$$\pi_T(w)$$

$$= \inf \{0 \leq p \leq 1 : L^1_w V(p, 1) \leq L^0_w V(p, 1) \} \in [0, 1].$$

If $\frac{\partial L^1_w V(p, 1)}{\partial w} \mid _{p=\pi_T(w)} < \frac{\partial L^0_w V(p, 1)}{\partial w} \mid _{p=\pi_T(w)}$, then $\pi_T(w)$ is a monotonically decreasing function of $w$.

**Proof sketch:** This proof is by contradiction. Assume that thresholds $\pi_T(w) < \pi_T(w')$ for $w < w'$, under given ‘if’ condition. By definition of threshold, $L^0_w V(\pi_T(w), 1) = L^0_w V(\pi_T(w), 1)$. For some $w' = w + \epsilon$, $\epsilon \in (0, c), c < 1$, we have $L^1_w V(\pi_T(w'), 1) \geq L^0_w V(\pi_T(w'), 1)$. This means $\frac{\partial L^1_w V(p, 1)}{\partial w} \mid _{p=\pi_T(w)} > \frac{\partial L^0_w V(p, 1)}{\partial w} \mid _{p=\pi_T(w)}$, which contradicts our assumption.

Define $D(\pi, w) := L^1_w V(p, 1) - L^0_w V(p, 1)$.

**Theorem 2:** A CRSAB with bounded subsidy $w \in [w_l, w_h]$ and discount parameter $\beta \in (0, 1)$, is indexable under either of the following conditions:

1. $0 < p_{00} - p_{10} < b \beta^{k+1}$ or
2. $0 < p_{10} - p_{00} < \frac{b}{\beta}$.

**Proof:** The proof proceeds in the following steps. (1) From Lemma 1, it can be seen that the functions $L^0_w V(\pi, y)$ $a \in A_\pi$ are convex and Lipschitz in $w$. This implies that they are absolutely continuous. (2) It means $D(\pi, w)$ is absolutely continuous, which implies that it is differentiable w.r.t $w$ almost everywhere in the interval $[w_l, w_h]$, for all $\pi \in [0, 1]$. (3) This implies that the threshold $\pi_T(w) := \{\pi \in [0, 1] \mid D(\pi, w) = 0\}$ is absolutely continuous on $[w_l, w_h]$; hence, $\pi_T(w)$ is differentiable w.r.t $w$ almost everywhere. (4) From Remark 2, $D(\pi, w)$ is decreasing in $w$; hence, $\frac{\partial D(\pi, w)}{\partial w} \leq 0$ almost everywhere in $[w_l, w_h]$. This implies $\frac{\partial \pi_T(w)}{\partial w} \leq 0$. Now, using Lemma 4 we can say that $\pi_T(w)$ decreases with $w$. This means as subsidy $w$ increases, the set $G(w)$ also increases. Hence, the arm is indexable.

Indexability ensures a well defined index for an arm. Now we will be able use this index to define heuristic index based policies for solving the constrained restless multi-armed bandit problem. One such policy is the Whittle’s index policy in which the arm with the highest index is played in each slot. Before proceeding to apply this policy to the CRMAB problem, we need an algorithm to compute Whittle’s index.

**D. COMPUTING WHITTLE’S INDEX**

The index of a CRSAB is the minimum subsidy required to make not-playing the optimal action; it is defined below. Note that closed form expressions for value functions are not available. It is difficult to obtain a closed form expression for the index. We devise an algorithm for Whittle’s index computation for CRSABs. The argument for convergence of this algorithm is based on stochastic approximation schemes.

**Definition 3 (Whittle’s Index):** For a given belief $\pi \in [0, 1]$, Whittle’s index $W(\pi)$ is the minimum subsidy for which, not playing the arm will be the optimal action.

$$W(\pi) = \inf \{w \in \mathbb{R} : L^0_w V(\pi, 1) \geq L^1_w V(\pi, 1)\}$$

An algorithm for computing Whittle’s index. Algorithm 1 is based on two timescale stochastic approximation.

**Algorithm 1** W1 Computes Whittle’s Index for CRSAB

**Input:** Reward values $\eta_0$, $\eta_1$, initial subsidy $w_0$, tolerance $h$, grid over belief space $G([0, 1])$.

**Output:** Whittle’s index $W(\pi)$ for $\pi \in G([0, 1])$.

1. $w_t \leftarrow w_0$;
2. While $|L^1_{w_t} V(\pi, 1) - L^0_{w_t} V(\pi, 1)| > h$ do
   a. $w_{t+1} = w_t + \alpha_t (L^1_{w_t} V(\pi, 1) - L^0_{w_t} V(\pi, 1))$.
   b. $t = t + 1$; compute $L^0_{w_t} V(\pi, 1), L^1_{w_t} V(\pi, 1)$.
3. Return $W(\pi, 1) \leftarrow w_t$.

The algorithm runs on two timescales; value iteration algorithm runs on the faster timescale, while subsidy $w$ is updated on the slower timescale. That is, the value function $L^0_w V(\pi, 1), L^1_w V(\pi, 1)$ are updated on faster timescale, while the value of $w_t$ is updated along the slower one. In this algorithm, the algorithm on the faster timescale views $w_t$ as quasi-static, and runs value iteration till convergence. Whenever $|L^1_{w_t} V(\pi, 1) - L^0_{w_t} V(\pi, 1)| > h$, then $w_t$ is updated according to equation (4). Otherwise, subsidy or index $W(\pi, 1) = w_t$.

In stochastic approximation, two-timescale algorithms converge if the sequence $\alpha_t$ is decreasing, $\sum \alpha_t = \infty$ and $\sum \alpha_t^2 < \infty$. This convergence is almost sure as shown in Theorem 2, [37, Chapter 6]. If $\alpha_t$ is replaced with a tiny constant value $\alpha$, there is convergence with high probability; see [37, Section 9.3].

A closed form index formula is not feasible for our model and the stochastic approximation algorithm is computation-ally expensive. This motivates us look for alternate heuristic policy without compromising on performance.

**IV. SAMPLE COMPLEXITY RESULTS**

In the following we discuss sample complexity results for Whittle’s index policy. The line of thought is as follows.
1) We used two-timescale stochastic approximation (TTSA) for Whittle’s index computation in Algorithm 1. The subsidy \( w \) is updated along the slower timescale and value iteration is performed along the faster timescale.

2) One way to estimate the complexity of Algorithm 1 is by using available sample complexity results for one-timescale stochastic approximation (OTSA) (along the slower timescale) and value iteration (along the faster timescale). So the overall complexity would be the number of iterations for convergence of OTSA multiplied by number of iterations for convergence of value iteration.

3) However, the above approach might only give a loose estimate of the complexity of Algorithm 1.

4) Hence, we discuss other possible approaches to estimate complexity such as the primal dual algorithm of discounted MDPs.

We now discuss some known results regarding the sample complexity of SA and value iteration. Recently, there has been surge of interest in finite time analysis of two-timescale stochastic approximations [38, 39, 40, 41]. This analysis claims the existence of finite time \( T^* \) after which the iterate (algorithm) remains in the \( \epsilon \)-neighbourhood of the optimal solution, with high probability. Concentration bounds are utilized to show this.

There has been work on sample complexity of solving discounted MDPs with value iteration and Q-learning algorithms, [42], [43]. Sample complexity results provide explicit value (expressions) of \( T^* \) for which the iterate converges to \( \epsilon \)-optimal solution. These are stronger claims than finite time analysis, and are often difficult to obtain. No direct results are available on the sample complexity of TTSA algorithms. Even for OTSA in a general setting, few results are available till date [37, Section 4.2], [44, Section VI]. Using the available literature, we comment on sample complexity of Whittle’s index computation algorithm.

We state all results for the constrained restless single-armed bandit model.

Sample complexity of value iteration for MDP problems was given by [45, Section 4.3]. We note that our state space is the belief space \([0, 1]\). We discretize this state space using a grid. Let \( G([0, 1]) \) denote a grid over the interval \([0, 1]\), and \(|G|\) denotes the number of grid points. In our value iteration algorithm, the number of computations required in each iteration is \( O(|G|^2) \). The maximum number of iterations required to find \( \epsilon \)-optimal policy is

\[
T^* \leq \frac{B + \log(1/\epsilon) + \log(1/(1 - \beta)) + 1}{1 - \beta}.
\]

The derivation can be found in [45]. Here, \( B \) is in bits, which is the memory size in a linear program. The number of iterations is polynomial in \(|G|\), \( B \) and \( 1/(1 - \beta) \). The lower bound on \( T^* \) is given as follows,

\[
T^* \geq \frac{1 - \epsilon}{2} \log \left( \frac{1}{\epsilon(1 - \beta)} \right)
\]

The lower bound is summarized in the following Lemma.

**Lemma 5 (Lower Bound):** The worst case complexity of computing value functions \( L_w^\pi(V(\pi), 1) \) and \( L_1^\pi(V(\pi), 1) \) \( \forall \pi \in G([0, 1]) \) with an error tolerance \( \epsilon \) is \( \Omega(|G|^2T^*) \) with \( T^* = \frac{1}{1 - \beta} \log \left( \frac{1}{\epsilon(1 - \beta)} \right) \).

Here, the minimum number of iterations required for convergence to an \( \epsilon \)-optimal answer is given by \( T^* \). The number of computations needed for each iteration is at most 2\(|G|^2\) as there are two possible actions.

More recently, faster variants of value iteration algorithm have been proposed in [43]. For example, the high precision randomized value iteration, which provides bounds on sample complexity with high probability. This bound depends on the number of states, actions, discount parameter, \( \epsilon \) (near optimal parameter) and \( \delta \) (probability confidence). In the derivation of this bound, concentration inequalities are used. We state the result [43, Lemma 4.9] for our single armed bandit without proof. For our case, this sample complexity for obtaining \( \epsilon \)-optimal value function with probability \( (1 - \delta) \) can be given as

\[
\tilde{O} \left( \left( |G|^2 + \frac{|G|}{(1 - \beta)^2} \right) \log \left( \frac{R_{\max}}{\epsilon} \right) \log \left( \frac{1}{\delta} \right) \right),
\]

where \( R_{\max} \) is the maximum possible reward over all states and actions and \( \tilde{O} \) is used to hide a polylogarithmic factor in the input, i.e., \( \tilde{O}(f(\pi)) = O(f(\pi)(\log f(\pi))^{O(1)}) \). It is important to note that the sample complexity has polynomial dependence on \( \frac{1}{\epsilon(1 - \beta)} \).

For Whittle’s index computation algorithm, sample complexity results are unknown and very challenging to determine from finite time analysis of TTSA. Sample complexity results known for SA are not very informative (see [37, Chapter 4.2] and [44]) in the sense that they claim convergence to the optimal neighbourhood after a finite ‘large enough’ number of iterations. This ‘large enough’ number is however not known. For estimating the complexity of Algorithm 1, we can utilize sample complexity results derived for constrained discounted Markov decision processes with primal-dual methods, [46].

Primal dual algorithm is used in solving constrained optimization problem, [47]. Basically in this one would like to find a solution to Lagrangian relaxed problem which is unconstrained. It is required to find optimal solution to both primal and dual variables; this boils down to finding the saddle point condition. In primal dual algorithm, the idea is to fix dual variables, i.e., Lagrangian multipliers, and solve the problem for primal variables. The optimal solution obtained for the primal is dependent on the dual variables. If we change the dual variable according to gradient ascent/descent method, then we get a new optimal primal solution. Thus, these primal-dual solutions are coupled iterations and can be analyzed using ideas of two timescale approach, where dual variables are updated on slower timescale (assumed as quasi-static) and the primal variables are updated on the faster timescale. Such two timescale approach can be analyzed using TTSA algorithms, [37, Chapter 5]. Similar ideas are
employed for constrained discounted MDPs in [48] in case of actor-critic methods in constrained MDPs. For the sake of clarity we describe TTSA algorithm from [37, Chapter 6] below:

\[ \xi_{t+1} = \xi_t + \alpha_t \left[ f(\xi_t, w_t) + M^1_{t+1} \right], \]
\[ w_{t+1} = w_t + \alpha_t \left[ g(\xi_t, w_t) + M^2_{t+1} \right], \]

where, \( M^1_{t+1} \) and \( M^2_{t+1} \) are martingale difference noise terms. The \( \{\alpha_t\} \) are step sizes \( t \geq 1 \) and satisfy the following condition:

\[ \sum_{t=1}^{\infty} \alpha_t = \infty, \]
\[ \sum_{t=1}^{\infty} (\alpha_t^2 + \kappa_t^2) < \infty, \]
\[ \alpha_t / \kappa_t \to 0, \text{ as } t \to \infty. \]

The last condition on step sizes implies that \( w_t \) moves on slower timescale than \( \xi_t \). The convergence analysis is given in [37, Chapter 6]. This analysis is also valid when \( \alpha_t / \kappa_t = \alpha / \beta = \epsilon_1 < 1, \alpha_t = \alpha < 1, \kappa_t = \kappa < 1 \) for \( t \geq 1 \) but convergence guarantees are in probabilistic sense instead of almost sure convergence.

Now coming back to sample complexity results, we would like to state a result similar to [46], which uses two-timescale approach in discounted MDP and make use of primal-dual stochastic algorithm. In [46, Theorem 1], the duality gap is expressed as function of \( T \), state \( S \), action \( A \) and discount parameter and this in turn describes the convergence rate. In their algorithm, the step sizes are chosen such that two timescale behavior holds in their setting. Using these concepts, they derive sample complexity result in terms of number of states, number of actions, discount parameter and desired accuracy. We state their result in our framework using similar step sizes and conjecture that this is the optimal sample complexity for Whittle index computation with a single-arm restless bandit. Suppose the step size \( \alpha_t = \alpha \) for update rule \( w_t \) is set to \( \frac{1-\beta}{\sqrt{\log \frac{\beta G}{2\epsilon}}} \), and value iteration algorithm in index computation algorithm is updated at timescale \( \sqrt{\frac{|G|}{T}} \tilde{C} \) for all \( t \). Note that this choice of step sizes satisfy two-timescale conditions discussed before. In fact the ratio of these is \( \frac{1-\beta}{\sqrt{\log \frac{\beta G}{2\epsilon}}} \ll 1 \), where \( \tilde{C} > 1 \). Then, we expect the following result.

Result 1 (Conjecture on sample complexity of index computation): For Whittle index computation algorithm 1, stepsize of \( w_t \) update rule is \( \frac{1-\beta}{\sqrt{|G|T}} \), and value iteration is updated at faster timescale with stepsize \( \sqrt{\frac{|G|}{T}} \). Let \( \tilde{w} \) be the output of the Algorithm 1, and \( w^* \) be the true value of index at which the value functions for both actions are equal. Then, we have

\[ |V_{\tilde{w}}(\pi, y) - V_{w^*}(\pi, y)| \leq O \left( \frac{2|G| \log(2|G|)}{T} \tilde{C} \sqrt{\frac{1}{1-\beta^2}} \right). \]

To guarantee \( |V_{\tilde{w}}(\pi, y) - V_{w^*}(\pi, y)| \leq \epsilon \), the optimal number of sample required is

\[ T = O \left( \frac{2|G| \log(2|G|)}{(1-\beta^2)|C|} \right). \]

Here, \( V_{w}(\pi, y) \) denotes the optimal value function (defined in (3)) with subsidy \( w \). We do not provide a proof here; we believe this result will hold true from the analysis of [46] and our result will require similar type of analysis.

Remark 6: From comparison of sample complexity result in Eqn. (11) with sample complexity of randomized value iteration algorithm in Eqn. (5), we observe that there is increase in number of samples needed by factor of \( \frac{1}{1-\beta^2} \).

V. ONLINE ROLLOUT POLICY

We now present a simulation based approach referred to as online rollout policy. In the Whittle’s approach the constraint

\[ \sum_{i=1}^{\infty} \beta^{i-1} a_i(t) = M \]

is first relaxed to a discounted form

\[ \sum_{i=1}^{\infty} \beta^{i-1} a_i(t) = M \]

and then Lagrangian relaxation method is applied. Instead, we directly employ a simulation based look-ahead approach. We call this the rollout policy as many trajectories are ‘rolled out’ using a simulator and the value of each action is estimated based on the cumulative information obtained from a trajectory.

We now describe the rollout policy for \( M = 1 \). We compute the value estimate for trajectory \( l \) with starting belief \( \pi = (\pi_1, \cdots, \pi_N) \), availability \( y = (y_1, y_2, \cdots, y_N) \), and initial action \( \xi \in \{1, 2, \cdots, N\} \). Here, \( \xi(h, l) = n \) means arm \( n \) is played at step \( h \) in trajectory \( l \), \( \beta^h \) denotes the belief about arm \( n \) after \( h \) steps and \( \beta^h \) denotes the availability of arm \( n \) is denoted by \( y_n(h, l) \in \{0, 1\} \). Reward obtained from arm \( n \) is \( R_n^h(h, l) \) and the availability of arm \( n \) is denoted by \( y_n(h, l) \in \{0, 1\} \). Recall that the play of an arm depends on availability of that arm.

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We now describe the rollout policy for \( M = 1 \). We compute the value estimate for trajectory \( l \) with starting belief \( \pi = (\pi_1, \cdots, \pi_N) \), availability \( y = (y_1, y_2, \cdots, y_N) \), and initial action \( \xi \in \{1, 2, \cdots, N\} \). Here, \( \xi(h, l) = n \) means arm \( n \) is played at step \( h \) in trajectory \( l \), so \( \beta^h_0(h, l) = 1 \), and \( \beta^h_0(h, l) = 0 \) for \( \forall i \neq n \). The value estimate for initial action \( \xi \) along trajectory \( l \) is given by

\[ Q^\phi_0(\pi, y, \xi) = \sum_{h=1}^{H} \beta^{h-1} \sum_{n=1}^{N} R_n^h(h, l) \]

\[ = \sum_{h=1}^{H} \beta^{h-1} \sum_{n=1}^{N} r_n(\pi(h, l), y_n(\pi(h, l), b_n^h(\pi(h, l))). \]

Then, averaging over \( L \) trajectories the value estimate for action \( \xi \) in state \( \pi \) under policy \( \phi \) is

\[ Q^\phi_{H, L}(\pi, y, \xi) = \frac{1}{L} \sum_{l=1}^{L} Q^\phi_l(\pi, y, \xi). \]
Here, the base policy $\phi$ is myopic (greedy), it chooses the arm with the highest immediate reward, along each trajectory. Now we perform one step policy improvement, and the optimal action is selected as

$$j^*(\pi, y) = \arg \max_{1 \leq j \leq N} \left[ \tilde{r}(\pi, y, \xi = j) + \beta \tilde{Q}^j_{H,L}(\pi, y, \xi) \right].$$

(13)

Here, $\tilde{r}(\pi, y, \xi) = \sum_{n=1}^{N} r_n(\pi_n, y_n, b_n)$.

In each time slot $t$ with belief $\pi(t)$ and availability $y(t)$, online rollout policy plays the arm $j^*(\pi(t), y(t))$ obtained according to (13).

The detailed discussion on rollout policy for MDP and restless bandits with complex action space is given in [22]. In [49], [50], roll-out policy is extended to partially observable restless multi-state restless bandits.

A. PLAYING MULTIPLE ARMS USING ONLINE ROLL-OUT POLICY

Our discussion above consider the case $M = 1$, that is, only one arm is played in each time slot. In particular this is assumed while employing the base policy $\phi$. When a decision maker plays more than one arm per slot, employing a base policy with future look-ahead is non-trivial. This is due to the large number of possible combinations of $M$ out of $N$ available arms, i.e., $\binom{N}{M}$. Since the rollout policy depends on future look-ahead actions, it can be computationally expensive to implement as each time step we need to choose from $\binom{N}{M}$. We reduce these computations for base policy $\phi$ by employing a myopic rule in look-ahead approach, where we select $M$ arms with highest immediate rewards while computing value estimates of trajectories.

In this case, $\sum_{n=1}^{N} b^n(h, l) = M$, with $M > 1$. The set of arms played at step $h$ in trajectory $l$ is $\xi(h, l) \subseteq \mathcal{N} = \{1, 2, \cdots, N\}$, with $|\xi(h, l)| = M$. Here, $b^n(h, l) = 1$ if $n \in \xi(h, l)$. The base policy $\phi$ uses myopic decision rule and the one step policy improvement is given by

$$j^*(\pi, y) = \arg \max_{\xi \subseteq \mathcal{N}} \left[ \tilde{r}(\pi, y, \xi) + \beta \tilde{Q}^j_{H,L}(\pi, y, \xi) \right].$$

(14)

Here, $\tilde{r}(\pi, y, \xi) = \sum_{n=1}^{N} r_n(\pi_n, y_n, b_n)$. The computation of $\tilde{Q}^j_{H,L}(\pi, y, \xi)$ is similar to the preceding discussion. At time $t$ with belief $\pi(t)$ and availability $y(t)$, rollout policy plays the subset of arms $j^*$ obtained according to (14). A more detailed discussion on rollout policy can be found in [22].

B. COMPUTATIONAL COMPLEXITY

We now present the computational complexity of online rollout policy. As rollout policy is a heuristic (lookahead) policy which does not require convergence analysis, and we only present its computational complexity.

WI computation is done offline where we compute and store the index values for each element on the grid $G$, for all arms $(N)$. During online implementation, when a belief state $[\pi_1, \ldots, \pi_N]$ is observed, the corresponding index values are drawn from the stored data. On the other hand, online rollout policy is implemented online and its computational complexity is stated in the following Lemma.

Lemma 6: The online rollout policy has a worst case complexity of $\mathcal{O}(A(\mathcal{H}L + 2)T)$ for number of iterations $T$, when the base policy is myopic. Here $|A|$ is the number of possible actions in each iteration.

Proof:

- Case $M = 1$ (Only one arm is played): For each iteration, we need to compute the value estimates $\{\tilde{Q}^j_{H,L}(\pi, y, \xi)\}$ for $N$ possible initial actions (arms). This takes $\mathcal{O}(\mathcal{H}L)$ computations as there are $L$ trajectories of horizon (look ahead) length $H$ for each of the $N$ initial actions. For policy improvement step in Eqn 13, it takes another $\mathcal{O}(2N)$ computations. Thus total computation complexity in each iteration is $\mathcal{O}(2N + \mathcal{H}L) = \mathcal{O}(N(\mathcal{H}L + 2))$. For $T$ time steps, the computational complexity is $\mathcal{O}(N(\mathcal{H}L + 2)T)$.

- Case $M > 1$ (Multiple arms are played): Here the number of possible actions is $\binom{N}{M}$ and $\binom{N}{M} \approx O(N^M)$, which is a polynomial in $N$ for fixed $M$. Thus the complexity would be very high if all the possible actions are considered. The value estimates are computed only for some $|A|$ initial actions, $N \leq |A| < \binom{N}{M}$. The computations required for value estimate $\tilde{Q}^j_{H,L}(\pi, y, \xi)$ per iteration is $\mathcal{A}HL$. As there are $A$ number of subsets considered in Eqn. 14, the computations needed for policy improvement steps are at most $2|A|$. So, the per-iteration computation complexity is $\mathcal{O}(|A||(\mathcal{H}L + 2)|)$. Hence for $T$ time steps the computational complexity would be $\mathcal{O}(|A||(\mathcal{H}L + 2)T)$.

Remark 7: Note that computation complexity of online rollout policy in each iteration depends linearly on lookahead horizon length $H$, number of trajectories $L$ and number of arms $N$, in the case of where one arm is played. The offline computation of index (sample) complexity depends on the polynomial $\frac{1}{1 - \beta \pi^2}$, and is linear in number of arms $N$.

VI. BOUNDS ON OPTIMAL VALUE FUNCTIONS

In the previous sections we studied Whittle’s index and online rollout policies as solutions to the CRMAB problem. Both these are heuristic policies which are not necessarily optimal. The difference between the optimal value function and the value generated by a policy would be the absolute measure of goodness of a policy. However, it is hard to compute the optimal value function of the original CRMAB problem over a polymatroid belief space. Hence, an upper bound on optimal value functions is computed to provide an estimate of the difference.

In this section we shall derive upper bounds on the optimal value function of a CRMAB. First, we shall compare its value function to that of a RMAB (unconstrained). In the following discussion we use the terms ‘unconstrained restless bandits’ and ‘restless bandits’ interchangeably.
A. RELATION BETWEEN VALUE FUNCTIONS OF RMAB AND CRMAB

Let $U(\pi)$ be the value function of an restless single armed bandit which is always available. $U(\pi)$ is the solution of the following dynamic program.

$$U(\pi) = \max\{U_S(\pi), U_{NS}(\pi)\},$$

$$U_S(\pi) = \eta(\pi) + \beta\left[ \rho(\pi)U(\Gamma_1(\pi)) + (1-\rho(\pi))U(\Gamma_0(\pi)) \right],$$

$$U_{NS}(\pi) = w + \beta U(\gamma^0_1(\pi)).$$

(15)

The following Lemma states that for the same Markov chain parameters, the optimal value of the restless single armed bandit is greater than that of constrained restless single armed bandit.

**Lemma 7:** For any given set of parameters $p_{00}, p_{10}, p_0, p_1, \eta_0, \eta_1, w$, each of the following statements is true.

1) For belief update rules $\gamma^0_1(\pi) = \pi p_{00} + (1-\pi)p_{10}$ and $\gamma^0(\pi) = q$, the inequality $U(\pi) \geq V(\pi, y)$, holds $\forall \pi \in \Pi_{r}, y \in \{0, 1\}$, where

$$\Pi_r = \{\pi \in [0, 1]| U_{NS}(\pi) \leq U_{NS}(\Gamma_1(\pi)), U_{NS}(\pi), U_{NS}(\Gamma_0(\pi))\}$$

(16)

2) If belief update rule $\gamma^0_1(\pi) = \gamma^0_1(\pi) = \pi p_{00} + (1-\pi)p_{10}$, the inequality $U(\pi) \geq V(\pi, y)$, holds $\forall \pi \in [0, 1], y \in \{0, 1\}$.

The proof is straightforward, through induction.

B. BOUNDS ON VALUE FUNCTIONS

We shall now derive an upper bound on the value function of the constrained bandit. The Lagrangian relaxation provides an upper bound on the value function of the original problem. This has been studied for weakly couple Markov decision processes by [51] and [52]. However, the applicability of this result for constrained availability case is not obvious. We extend this idea for CRMABs and provide proof for the generalized case of partial observability and constrained availability.

Let us now look at the constrained multi-armed bandit problem as a set of N single armed bandits. We will be slightly abusing the notation in order to keep the mathematical expressions simpler; any change in notation is mentioned.

The CRMAB problem can be described as the following dynamic program. Given belief vector $\pi \in [0, 1]^N$ and availability vector $y \in [0, 1]^N$, find $J(\pi, y)$ satisfying

$$J(\pi, y) = \max_{a_0 \in A_0} \left\{ R(\pi, y, a) + \beta \sum_{o_0 \in S_0, y' \in S_y} \Pr(o, y'|\pi, y, a) J(\Gamma^0(\pi), y') \right\}$$

s.t. $\|a\|_1 = M, A_0 := A_{y_1} \times A_{y_2} \times \ldots \times A_{y_n},$  

(17)

Here, $\Gamma^0$ is the belief (vector) update rule for observation vector $o$. So, $\Gamma^{o_n}$ is the belief update rule based on observation $o_n$ for arm $n$. And $\Gamma^0 : S_o \rightarrow \{0, 1\}^N$. Here, $S_o$ is the observation set with the set of all possible observation vectors. $S_{o_m}$ is the set of possible observations for arm $n$. An observation vector $o$ also contains some ‘no observation’ elements corresponding to the unplayed arms.

The Lagrangian relaxed dynamic program of the above optimization problem is written as

$$J^\lambda(\pi, y) = \max_{a \in A_0} \left\{ R(\pi, y, a) + \lambda[M - \|a\|_1] \right\}$$

$$+ \beta \sum_{o \in S_o, y' \in S_y} \Pr(o, y'|\pi, y, a) J^\lambda(\Gamma^o(\pi), y'),$$

$$\lambda \geq 0.$$

(18)

The following Lemma states that the Lagrange relaxed value function of CRMAB can be written as a linear combination of value functions of $N$ constrained single armed bandits.

**Lemma 8:**

$$J^\lambda(\pi, y) = \frac{M\lambda}{1 - \beta} + \sum_{n=1}^{N} J^\lambda(\pi_n, y_n),$$

(19)

where,

$$J^\lambda(\pi_n, y_n) = \max_{o_n \in A_{y_n}} \left\{ r_n(\pi_n, y_n, a_n) - \lambda a_n + \beta \sum_{o_n \in S_{o_n}} \Pr(o_n|\pi_n, y_n, a_n) \right\} \times \left[ \frac{\lambda}{1 - \beta} \right].$$

The proof is given in Appendix D. The Lagrangian relaxed value function $U^\lambda(\pi)$ for an RMAB is given as follows (in [3]). This provides an upper bound on Whittle’s index policy for RMAB.

**Lemma 9:**

$$U^\lambda(\pi) = \frac{M\lambda}{1 - \beta} + \sum_{n=1}^{N} U^\lambda(\pi_n),$$

(20)

$$U^\lambda(\pi_n) = \max_{o_n \in [0, 1]} \left\{ r_n(\pi_n, a_n) - \lambda a_n \right\}$$

$$+ \beta \sum_{o_n \in S_{o_n}} \Pr(o_n|\pi_n, a_n) U^\lambda(\Gamma^{o_n}(\pi_n)).$$

(21)

The following corollary based on Lemma 7 and Lemma 8 states that for the same set of state transition probabilities and rewards, the Lagrange relaxed value function of RMAB is greater than that of CRMAB. This means, an upper bound on value can be computed using either of the functions.

**Corollary 1:** The inequality $J^\lambda(\pi, y) \leq U^\lambda(\pi)$ holds for each of the following cases.

1) $\gamma^0_1(\pi) = q, \gamma^0_1(\pi) = \pi p_{00} + (1-\pi)p_{10}, \forall \pi \in \Pi_r, y \in \{0, 1\},$
2) \( \gamma_0^y(\pi) = \gamma_1^y(\pi) = \pi p_{00} + (1 - \pi) p_{10}, \forall \pi \in [0, 1], y \in \{0, 1\}, \)

\[ \Pi_F = \{ \pi \in [0, 1] \mid U_{NS}(\pi) \leq U_{NS}(\Gamma_1(\pi)), U_{NS}(\pi), U_{NS}(\Gamma_0(\pi)) \}. \]

**Proof:** From Lemma 7 we know that the value functions of constrained restless single armed bandits are upper bounded by those of restless single armed bandits. It follows that their summation as given in Lemma 8 is also similarly bounded.

**VII. SIMULATION STUDY**

**A. COMPARISON OF POLICY PERFORMANCE**

We consider different parametric scenarios and evaluate the performance of Whittle’s index policy (WI), online rollout policy, and myopic policy (MP) in terms of their value (discounted cumulative reward).

Whittle’s index policy plays the arms with highest indices \( Y_n(t)W_n(\pi_n(t)) \). Myopic policy chooses the arms with highest expected immediate rewards, i.e., it considers the expression \( Y_n(t)(\pi_n(t))\eta_{n,0} + (1 - \pi_n(t))\eta_{n,1} \) as index for arm \( n \). Online rollout policy estimates the values of various actions in a state and chooses those actions with the highest estimated values.

The impact of parameters such as the number of arms \( N \), number of played arms per slot \( M \), number of always available arms \( K \), and the reward structure is illustrated using numerical examples. In the following numerical examples, policies are evaluated for different bandit instances (a parameter set is called an instance). A bandit instance is specified by giving the values of 1) number of arms \( N \), 2) state transition probabilities of arms \( p_{00}(y, a) \), 3) availability probabilities of arms \( \eta_n(y) \), 4) reward structure \( \eta_{n,i} \), 5) success probabilities \( \rho_n(i) \). For each bandit instance, the value function of each policy is computed, and averaged over numerous sample sequences of states and arm availability.

1) EXPERIMENT 0 - PERFECT OBSERVABILITY FOR PLAYED ARMS

We consider 10-armed restless bandit instances, i.e. \( N = 10 \). In this experiment we assume that exact state is observed for played arms, i.e. \( \rho_0 = 0, \rho_1 = 1 \). The parameter set for stochastic availability model is given in Table 2. We also present simulations for the semi-deterministic model. We use same parameters as in previous model where ever applicable, and chose \( T_0 = 3 \) or \( T_0 = 5 \). For rollout policy we use \( H = 3 \) and \( L = 100 \). A comparison of discounted rewards generated by Whittle’s index policy and myopic policy for stochastic and semi-deterministic availability models is given in Table 3 and Tables 4.5, respectively. A discount factor of \( \beta = 0.99 \) was used. We observe that the rollout policy performs the best among the three policies. Whittle’s index policy is only marginally better than the myopic policy. Also notice the closeness of various performances to the upper bound for \( M = 1 \) in Table 3. For \( M = 1 \), both rollout and Whittle index policies can be practically considered optimal. As \( M \) increases they move away from the bound. Also notice that the performance of myopic policy gets closer to rollout and WI as \( M \) increases. This might be due to the inherent sub-optimality of assigning an index value to each action using heuristic approaches (such as myopic, Whittle’s index or rollout). Increase in \( M \) increases the number of possible actions which accentuates the sub-optimality of these heuristics.

Table 6 summarizes the comparison of performances by using the metric \( >_{MP} \) which denotes % improvement over myopic policy. It shows that the rollout policy is the closest to optimal (given that it has the least distance from \( L_{b,\beta} \)) in this case, with \( \beta = 0.99 \), the rollout policy shows about 11% improvement over the myopic policy.

We also experimented with \( \beta = 0.95 \). Figure 3 summarizes the comparison of polices for two bandit instances with \( N = 5 \) and \( N = 10 \) for this case. The rollout policy still outperforms by around 14%, while Whittle’s index policy and Myopic policy perform the same.

**TABLE 2.** Experiment 0: Parameter set \( (N = 10, K = 5) \).

| Arm | \( \theta^1_n \) | \( \theta^0_n \) | \( \rho_0 \) | \( \rho_1 \) | \( \eta_n \) | \( \eta_{00} \) | \( \rho_{10} \) |
|-----|----------------|----------------|----------|----------|----------|----------|----------|
| 1   | \([1, 1, 1]\) | 0   | 1   | 0   | 0.9 | 0.5 | 0.41 |
| 2   | \([0.3, 0.75, 0.8]\) | 0   | 1   | 0   | 0.97 | 0.45 | 0.4 |
| 3   | \([1, 1, 1]\) | 0   | 1   | 0   | 0.82 | 0.45 | 0.45 |
| 4   | \([0.95, 0.9, 0.85]\) | 0   | 1   | 0   | 0.85 | 0.78 | 0.15 |
| 5   | \([1, 1, 1]\) | 0   | 1   | 0   | 0.66 | 0.6 | 0.5 |
| 6   | \([1, 1, 1]\) | 0   | 1   | 0   | 0.72 | 0.6 | 0.5 |
| 7   | \([1, 1, 1]\) | 0   | 1   | 0   | 0.75 | 0.7 | 0.5 |
| 8   | \([0.8, 0.7, 0.6]\) | 0   | 1   | 0   | 0.45 | 0.7 | 0.6 |
| 9   | \([0.9, 0.85, 0.95]\) | 0   | 1   | 0   | 0.75 | 0.4 | 0.3 |
| 10  | \([0.95, 0.9, 0.9]\) | 0   | 1   | 0   | 0.72 | 0.45 | 0.25 |

2) EXPERIMENT 1 (STOCHASTIC AVAILABILITY) - PARTIALLY OBSERVABLE STATES

We consider a 15-armed bandit instance with stochastic availability model. The parameter set is given in Table 7. The first five arms are always available while remaining are available according to action dependent probabilities. Table 8 shows the discounted cumulative rewards achieved by various policies. While rollout policy is still the best, WI and myopic policies follow very closely.

3) EXPERIMENT 2 (SEMI-DETERMINISTIC AVAILABILITY) - PARTIALLY OBSERVABLE STATES

We again consider a 15-armed bandit instance with semi-deterministic availability model. The parameters used for this experiment are same as in Experiment 1, except for the availability parameters. Recall that semi-deterministic availability is characterized by parameters \( \theta^1_n, \theta^0_n, T_0 \). Here, \( \theta^1_n, \theta^0_n \) are same as in Experiment 1, and \( T_0 \) is chosen to be 3 slots. The discounted cumulative rewards achieved by various policies are shown in tables 9. Again, the ordering on policy performance remains the same (as in Experiment 1), for \( H = 3 \).

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FIGURE 3. Comparison of cumulative rewards achieved by different policies as a function of time for two problem instances with $N = 5$, $N = 10$ (parameters from Table 2), and discount factor $\beta = 0.95$. The online rollout policy outperforms both the Myopic and Whittle index policies by around 14%, with the latter two exhibiting similar performance.

| Availability | Arms played (M) | $L_b$ | WI (Mean) | Myopic (MP) | Rollout policy |
|--------------|-----------------|-------|-----------|-------------|----------------|
| all          | 1               | 62.55 | 61.1      | 56.2        | 62.5           |
| stochastic   | 1               | 61.8  | 59.11     | 55          | 61.46          |
| stochastic   | 2               | 117.8 | 111.6     | 108         | 113.9          |
| stochastic   | 3               | 173.8 | 162.9     | 160.5       | 161.1          |
| stochastic   | 4               | 229.8 | 211.3     | 210.2       | 214.3          |

TABLE 4. Experiment 0 - Semi-deterministic availability ($T_0 = 3$): Discounted cumulative rewards achieved by various policies, with random initial belief, and the upper bound $L_b$, $N = 10$, $T = 600$, $\beta = 0.99$.

| Availability | Arms played (M) | $L_b$ | WI (Mean) | Myopic (MP) | Rollout policy |
|--------------|-----------------|-------|-----------|-------------|----------------|
| semi-deterministic | 1           | 61.6  | 58.28     | 54.65       | 60.37          |
| semi-deterministic | 2           | 114.6 | 110.5     | 107.5       | 113.10         |
| semi-deterministic | 3           | 167.6 | 160.8     | 159         | 163.1          |
| semi-deterministic | 4           | 220.6 | 206.9     | 206         | 208.1          |

Table 8 and Table 9 correspond to $N = 15$, $M = 1$. Here, all the policies are close to the upper bound. This suggests that for systems with higher number of arms, the difference between these policies reduces and the simplest policy (myopic) is enough to give near-optimal results.

Overall, the simulation study suggests that the rollout policy is a better performing, low complexity alternative to Whittle’s index policy.

VIII. APPLICATION: WILDFIRE MANAGEMENT

In this section we present application of wildfire management. Forest fires are a frequent occurrence in some parts of the world and often have devastating effect on the livability of the areas through their impact on temperature, water and air quality, etc. Consider a situation where a central
agency is tasked with managing forest fires in multiple regions. Each region also has local incident controllers with limited resources who are actively engaged in dealing with the situation. The central agency can actively monitor and provide additional resources to only a limited number of regions, simultaneously, due to logistic constraints.

Theories of fire dynamics have been developed for modelling, prediction and simulation of fire behaviour. These have been used to develop theories of firefighting based on institutional firefighting experience and field data, across the world [53], [54], [55], [56]. Various index systems have been developed across the world to quantify the possibility of fire starting in a given region. These are used for forest fire management. Examples include the Canadian Forest Fire Weather Index (FWI) system and the Australian Forest Fire Danger Index (FFDI) [57], [58], [59]. These indices take into account various parameters such as air temperature, relative humidity, wind speed, fuel moisture, fuel load, etc. Local incident controllers utilize parameters such as rate of spread (RoS) and flame length values below a threshold, which means that the fires in the region are sufficiently suppressed and have low probability of re-acceleration. They will eventually die down with the continued effort of local controllers. High-danger state represents high RoS, flame lengths and depths, with high probability of fire acceleration, and requires additional resources. Effectively, states can be defined as using relative flow rate $RF = \text{available flow rate/critical flow rate}$. The state of a region $n$ in time slot $t$ is defined as follows.

$$X_n(t) = \begin{cases} 0(\text{high – danger/bad}) & \text{if } RF_n \leq \beta_f, \\ 1(\text{low – danger/not bad}) & \text{if } RF_n > \beta_f, \end{cases}$$

(22)

where $\beta_f$ is a threshold predetermined by the central agency’s decision makers.

- **Actions:** In a given time slot the central agency can actively engage only $M$ of the $N$ locations. $a_n(t) = 1$ if the central agency is actively engaged with region $n$, and $a_n(t) = 0$ otherwise. Active engagement brings in additional flow rate for fire suppression.

- **Transition probabilities between states:** Given the estimated rate of spread and flame depth in a region, the state transition probabilities can be computed as follows. The relation between CFI, RoS, and $F_{FDI}$ takes the form $CFI(L_d = l) = \alpha_1(l)RoS^{\alpha_2(l)}$, the values of $\alpha_1(l), \alpha_2(l)$ are given in [60, Table 2]. This relation is used to compute the critical flow rate. Then, using the locally available flow rate and the flow rates added by the central agency, the probability of moving from high-danger state to a low-danger state is estimated.

- **Observability of states:** The state of a region is fully observable by the central agency only when it actively engages with that region. This scenario is represented by choosing $\rho_0 = 0$, $\rho_1 = 1$.

- **Availability Constraints:** There is a possibility of delay or temporary unavailability of resources such as water tankers, aircraft, personnel, etc., in a given time slot due to various transport or logistic reasons. The semi-deterministic availability model is applied to this scenario assuming that these constraints are surely overcome by the central agency, although with some delay. Once resources are unavailable for region $n$, they remain unavailable for $T_0 = 2$ time slots. Given current availability the probability of unavailability in the next time slot is $1 - \theta_n^a(\pi_n(t), y)$

$$\theta_n^a(\pi_n(t), y) = E_i[\theta_n^a(i, y = 1)]$$

when $\theta_n^a(i, y = 1)$ can take values $0.5$ for $i = 0$ and $0.95$ for $i = 1$. That is, in high – danger state there is high uncertainty.

- **Cost:** Let $C_0$ be the cost of engagement when state is 0 (high danger) and $C_1$ be the cost of engagement when state is 1 (low danger). Further resource investment is higher in state 0 than state 1, $C_0 \geq C_1$. Without

| Availability | $L_0$ | WI | Myopic | Rollout ($H = 3$) | Rollout ($H = 5$) |
|--------------|------|----|--------|-------------------|-------------------|
| Semi-det.    | 65.7 | 64.3 | 64.7   | 64.7              | 63.7              |

| Availability | $L_0$ | WI | Myopic | Rollout ($H = 3$) | Rollout ($H = 5$) |
|--------------|------|----|--------|-------------------|-------------------|
| Stochastic   | 65.7 | 64.7 | 64.3   | 65                | 65.4              |

Table 8. Experiment 1 - Stochastic availability: Discounted Cumulative Rewards from various policies, with random initial belief. $N = 15$, $M = 1$, $T = 600$, $\beta = 0.99$.

Table 9. Experiment 2 - Semi-deterministic availability: Discounted Cumulative Rewards from various policies, with random initial belief. $N = 15$, $M = 1$, $T = 600$, $\beta = 0.99$. 

Table 10. Rewards from various policies, with random initial belief.
engagement the costs are \( C_0 \) and \( C_1 \). Thus total cost is \( C_0 + \tilde{C}_0 \) and \( C_1 + \tilde{C}_1 \). The values of these costs depend on the critical flow rates required to suppress the fires, human worker costs, transportation and fuel costs, and might include loss of life and property.

In summary, our purpose here was to present a broad framework for planning and resource management in such scenarios. Once the model parameters are computed, simulation of the decision making is similar to that presented in Section VII.

The limitations and possible extensions of this model for wildfire management include the following:

- Realistic estimation of the model parameters needs access to field data, and expertise in multiple domains. It is beyond the scope of this work. Hence, more domain specific simulation studies and field work are needed to beneficially apply the model to real world applications.
- Also, sensitivity of the policies to parameter estimation errors and cost definitions needs to be studied to better understand their practical applicability.
- The two state model of high or low danger can be extended to a multi-state model. This might allow for a more precise control of the situation provided the estimation errors are contained.
- Further, to model situations involving severe uncertainty, distributionally robust and also more risk-aware formulations of CRMABs can be pursued.

IX. DISCUSSION AND CONCLUSION

In this paper, the problem of constrained restless multi-armed bandits is studied. These constraints are in the form of time varying availability of arms. The solution methods studied include Whittle’s index policy, online rollout policy, and myopic policy.

For Whittle’s index policy, index computation is done offline, and the indices are used for online decision making. In contrast, the implementation of the online rollout policy is entirely online. Complexity analysis shows that rollout policy with a short look ahead is less complex than Whittle’s index policy. However, there is a trade-off between offline computation and online decision time. Numerical experiments show that when only one arm is played per time slot, the online rollout policy is almost optimal and is followed closely by the index policy. This suggests that the rollout policy with a short look ahead can be used as an alternative to Whittle’s index policy under computational scarcity. Further, as the arms played per slot increases the performance of these policies seem to get closer to each other.

For applications with a very large number of subsystems or arms, we believe that the online rollout policy is a better alternative to Whittle’s index policy, as the performance of the rollout policy has been shown to be superior in our simulation study. To employ the index policy, the indexability of arms needs to be verified using the theoretical results provided in the earlier sections. These indexability results require heavy structural assumptions on the model for partially observable CRMABs, as seen in Section III, Theorem 2 of the paper. In contrast, the online rollout policy does not require any structural assumptions on the model. Hence, we believe that the online rollout policy has potential to become the main solution for CRMABs with more general additional constraints, particularly with multi-state, partially observable models. Furthermore, we discussed the results on sample complexity for Whittle’s index and rollout policies in Section IV. Whittle’s index can be computed offline, and the sample complexity scales with the discount parameter \( \beta \), being proportional to \( \mathcal{O}(1/(1 - \beta^4)) \). For \( \beta \) closer to 1, index computation can become very costly due to higher sample complexity, whereas the rollout policy may have lower sample complexity in such cases. Recently, rollout policies have been studied for other applications like MDP and POMDP in [61], but there has been no work on rollout policies for CRMABs with partially observable models.

A useful research direction would be to study variations in the online rollout policy, such as using different base policies.

Another future direction could be developing learning algorithms for scenarios where the systems parameters are unknown and the state space is large. Further, finding tractable solutions to constrained bandits for the multi-state, multi-action case under partial observability will increase the scope of application.

APPENDIX

A. PROOF OF LEMMA 1

Proof: The proof uses the principle of mathematical induction.

1) First prove convexity of value functions in \( \pi \) for fixed \( w \).

Let \( V_1(\pi, 1) := \max\{\pi \eta_0 + (1 - \pi) \eta_1, w\} \), and \( V_1(\pi, 0) := w \). Observe that \( V_1(\pi, 1) \) is convex function in \( \pi \) and \( V_1(\pi, 0) \) is constant function in \( \pi \); hence convex in \( \pi \).

We next assume \( V_0(\pi, 1) \), and \( V_0(\pi, 0) \) are convex in \( \pi \). The action value functions are given by

\[
\mathcal{L}^1 V_0(\pi, 1) = \eta(\pi) + \beta \rho(\pi) \left[ \theta^1(\pi) V_0(\Gamma_1(\pi), 1) + (1 - \theta^1(\pi)) \right]
\]

\[
\times V_0(\Gamma_1(\pi), 0) + \beta (1 - \rho(\pi)) \left[ \theta^1(\pi) V_0(\Gamma_0(\pi), 1) + (1 - \theta^1(\pi)) V_0(\Gamma_0(\pi), 0) \right],
\]

\[
\mathcal{L}^0 V_0(\pi, 1) = w + \beta \left[ \theta^0(\pi) V_0(\gamma^0(\pi), 1) + (1 - \theta^0(\pi)) \right]
\]

\[
\times V_0(\gamma^0(\pi), 0), \] \[
\mathcal{L}^0 V_0(\pi, 0) = w + \beta \left[ \theta^0(\pi) V_0(\gamma^0(\pi), 1) + (1 - \theta^0(\pi)) \right]
\]

\[
\times V_0(\gamma^0(\pi), 0). \]

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Define $V := \gamma_0/(1-\gamma_0)$, $w := \gamma_0/(1-\gamma_0)$, and $\eta := \gamma_0/(1-\gamma_0)$. Thus, convexity is not obvious from preceding equations, so we will rearrange the terms. Define
\[
\begin{align*}
 b_0 := [\pi p_{00}(1-\rho_0) + (1-\pi)p_{10}(1-\rho_1), \pi(1-p_{00})] \\
 b_1 := [\pi p_{00}\rho_0 + (1-\pi)p_{10}\rho_1, \pi(1-p_{00})\rho_0] \\
 + (1-\pi)(1-p_{10})\rho_1], \\
c_0 := [\pi p_{00} + (1-\pi)p_{10}, \pi(1-p_{00})] \\
+ (1-\pi)(1-p_{10})]. \\
\end{align*}
\]
Also, observe that the absolute value of slope of $V_{\pi}(\gamma_0, 1)$ w.r.t. $\eta = 0$ is 0, thus bounded by $\kappa c(\rho_1 - \rho_0)$. Next we assume that $|\partial V_{\pi}(\gamma_0, 1)/\partial \eta| \leq \kappa c(\rho_1 - \rho_0)$, and compute the partial derivatives of $L^1V_{\pi}(\gamma_0, 1), L^0V_{\pi}(\gamma_0, 1)$ and $V_{n+1}(\pi, 1)$ w.r.t. $\pi$.

\[
L^1V_{\pi}(\gamma_0, 1) = w + \beta(\theta^0(\pi)V_n(\gamma_0, 1) + (1-\theta^0(\pi))V_n(\gamma_0, 1) + (1-\theta^0(\pi))V_n(\gamma_0, 1) + (1-\theta^0(\pi))V_n(\gamma_0, 1)).
\]

Taking partial derivative w.r.t. $\pi$, we have
\[
\frac{\partial L^1V_{\pi}(\gamma_0, 1)}{\partial \pi} = \frac{\partial L^0V_{\pi}(\gamma_0, 1)}{\partial \pi} = \frac{\partial V_{\pi}(\gamma_0, 1)}{\partial \pi}.
\]

From [62, Lemma 2], given a convex function $g(x)$, the function $\|x\|_{L^1}(x/\|x\|_{L^1})$ is also convex. Hence $L^1V_{\pi}(\gamma_0, 1), L^0V_{\pi}(\gamma_0, 1)$ and $L^0V_{\pi}(\gamma_0, 0)$ are convex functions in $\pi$. As $V_{n+1}(\pi, 1) = \max(L^1V_{\pi}(\gamma_0, 1), L^0V_{\pi}(\gamma_0, 1))$, it is also convex in $\pi$. Similarly, $V_{n+1}(\pi, 0)$ is convex function in $\pi$. As $n \to \infty$, $V_{n}(\pi, 1)$ and $V_{n}(\pi, 0)$ converges uniformly to $V(\pi, 1)$ and $V(\pi, 0)$, respectively. Thus $V(\pi, 1)$ and $V(\pi, 0)$ are convex in $\pi$. Also, $L^2V_{\pi}(\pi, y)$ is convex in $\pi$ for fixed $w$.

2) Analogously, the value functions are convex in $w$ for fixed $\pi$. The proof technique is similar to that presented above.

\[\Box\]

**B. PROOF OF LEMMA 2**

**Proof:** The proof makes use of the principle of induction. We write down the steps for the case $p_{00} > p_{10}$, similar steps will lead to the result.

In the first step, let $V_{1}(\pi, 1) = \max(\eta(\pi), w)$, where $\eta(\pi) = \pi(\gamma_0 - \eta_1) + \eta_1$ and $\eta_0 < \eta_1$. Hence, the absolute value of slope of $V_{1}(\pi, 1)$ w.r.t. $\pi$ is bounded by $\kappa c(\rho_1 - \rho_0)$. Also, observe that the absolute value of slope of $V_{1}(\pi, 0)$ w.r.t. $\pi$ is 0, thus bounded by $\kappa c(\rho_1 - \rho_0)$.

Taking partial derivative w.r.t. $\pi$, we have
\[
\frac{\partial V_{1}(\pi, 0)}{\partial \pi} = \frac{\partial V_{0}(\pi, 0)}{\partial \pi}.
\]

Assuming $\theta^a(\pi)$ is independent of $\pi$, i.e., $\theta^a(\pi) = \theta^a_\gamma$, and using
\[
\frac{\partial V_{\pi}(\gamma, 1)}{\partial \pi} = \frac{\partial V_{\pi}(\gamma, 0)}{\partial \pi};
\]

}\[23\]
\[
\begin{align*}
\frac{\partial L^1 V_\pi(\pi, 1)}{\partial \pi} & \leq (\rho_1 - \rho_0) \\
& \times \left( -b + \beta_1 \right) [V_\pi(\Gamma_0(\pi), 1) - V_\pi(\Gamma_1(\pi), 1)] \\
& + \beta \left[ V_\pi(\Gamma_0(\pi), 0) - V_\pi(\Gamma_1(\pi), 0) \right] \\
& + \beta \rho_0 \rho_0 - p_{10} \\
& \times \left( \left[ V_\pi(\Gamma_0(\pi), 1) - V_\pi(\Gamma_1(\pi), 1) \right] \\
& + \left[ V_\pi(\Gamma_0(\pi), 0) - V_\pi(\Gamma_1(\pi), 0) \right] \right) \\
& + \left[ V_\pi(\Gamma_0(\pi), 1) - V_\pi(\Gamma_1(\pi), 1) \right] \right) \\
& + \left[ V_\pi(\Gamma_0(\pi), 0) - V_\pi(\Gamma_1(\pi), 0) \right] \right) \right) \right) \\
& \leq \kappa\mathcal{C}(\rho_1 - \rho_0) \{-b + \beta(b + 3c)(\rho_0 - p_{10})\}.
\end{align*}
\]

Using \( \frac{\partial V_\pi(\pi, 1)}{\partial \pi} \geq -\kappa\mathcal{C}(\rho_1 - \rho_0) \) and \( V_\pi(\Gamma_1(\pi), 1) - V_\pi(\Gamma_0(\pi)) \geq -\kappa\mathcal{C}(\rho_1 - \rho_0) p_{00} - p_{10} \) in (25), we have

\[
\begin{align*}
\frac{\partial L^1 V_\pi(\pi, 1)}{\partial \pi} & \geq (\rho_1 - \rho_0) \\
& \times \left( -b + \beta\kappa\mathcal{C}(\rho_1 - \rho_0)(\rho_0 - p_{10}) - \beta\kappa\mathcal{C}\rho_1(p_00 - p_{10}) \\
& - \beta\kappa\mathcal{C}(1 - \rho_0)(\rho_0 - p_{10}) \\
& \geq -\kappa\mathcal{C}(\rho_1 - \rho_0)(b + \beta(\rho_0 - p_{10})(c - b)) \\
& \geq -\kappa\mathcal{C}(\rho_1 - \rho_0).
\end{align*}
\]

Under the condition \( 0 < p_{00} - p_{10} < \frac{b + 1}{\mathcal{C}} \), the expression \( \{-b + \beta(b + 3c)(\rho_0 - p_{10})\} < 1 \) and the value of \( \left| \frac{\partial L^1 V_\pi}{\partial \pi} \right| \) is bounded by \( \kappa\mathcal{C}(\rho_1 - \rho_0) \). By induction \( \left| \frac{\partial L^1 V_\pi}{\partial \pi} \right| \) is also bounded by the same. Similarly, for the case \( p_{00} < p_{10} \), we have

\[
\begin{align*}
\frac{\partial L^1 V_\pi(\pi, 1)}{\partial \pi} & \leq (\rho_1 - \rho_0) \\
& \times \left( -b + \beta\kappa\mathcal{C}(\rho_1 - \rho_0)(\rho_0 - p_{10}) \\
& + \beta\kappa\mathcal{C}(1 - \rho_0)(\rho_0 - p_{10}) \\
& \leq \kappa\mathcal{C}(\rho_1 - \rho_0)\{-b + \beta(b + 3c)(\rho_0 - p_{10})\}.
\end{align*}
\]

For \( p_{00} > p_{10} \) we have \( p_{10} \leq \Gamma_1(\pi), \Gamma_0(\pi) \leq p_{00} \). From our assumption on \( V_\pi \) we obtain

\[
\begin{align*}
\left| V_\pi(\Gamma_1(\pi), 1) - V_\pi(\Gamma_0(\pi)) \right| & \leq \kappa\mathcal{C}(\rho_1 - \rho_0)\Gamma_1(\pi) - \Gamma_0(\pi) \right| \\
& \leq \kappa\mathcal{C}(\rho_1 - \rho_0)p_{00} - p_{10}.
\end{align*}
\]

Substituting in (24), we get

\[
\begin{align*}
\frac{\partial L^1 V_\pi(\pi, 1)}{\partial \pi} & \leq (\rho_1 - \rho_0) \\
& \times \left( -b + \beta\kappa\mathcal{C}(\rho_1 - \rho_0)(\rho_0 - p_{10}) + \beta\kappa\mathcal{C}\rho_1(p_00 - p_{10}) \\
& + \beta\kappa\mathcal{C}(1 - \rho_0)(\rho_0 - p_{10}) \\
& \leq (\rho_1 - \rho_0)(-b + 3\beta\kappa\mathcal{C}(p_00 - p_{10}))
\end{align*}
\]

Thus the solution is obtained.
By induction \( \frac{\partial L^0V_{\pi}}{\partial \pi} \) is bounded by \( \kappa c(\rho_1 - \rho_0) \). As \( V_{n+1}(\pi, 1) = \max\{L^1V_n(\pi, 1), L^0V_n(\pi, 1)\} \), the derivative of \( V_{n+1}(\pi, 1) \) is also bounded by \( \kappa c(\rho_1 - \rho_0) \). As \( n \to \infty \), we get \( V_\pi(\pi, 1) \to V(\pi, 1) \) and \( V_\pi(\pi, 0) \to V(\pi, 0) \) uniformly. Hence we have desired bound on partial derivative of value functions.

C. PROOF OF LEMMA 3

Proof: To show that \( D(\pi) \) is decreasing, it is enough to show that \( \frac{\partial D(\pi)}{\partial \pi} < 0 \).

\[
\frac{\partial D(\pi)}{\partial \pi} = \frac{\partial L^1V(\pi, 1)}{\partial \pi} - \frac{\partial L^0V(\pi, 1)}{\partial \pi}.
\]

Using Lemma 2, for the case \( p_{00} > p_{10} \),

\[
\frac{\partial D(\pi)}{\partial \pi} \leq \kappa c(\rho_1 - \rho_0) \{ -b + \beta(b + 3c)(p_{00} - p_{10}) \}
+ \beta \kappa c(\rho_1 - \rho_0)(p_{00} - p_{10})
\leq \kappa c(\rho_1 - \rho_0)
\times \{-b + \beta(b + 3c + 1)(p_{00} - p_{10})\}.
\]

Under the condition \( 0 < p_{00} - p_{10} < \frac{b}{\beta + 3c + 1} \), the coefficient \( -b + \beta(b + 3c + 1)(p_{00} - p_{10}) \) is negative; hence, the derivative \( \frac{\partial D(\pi)}{\partial \pi} \) is negative.

Similarly, for the case \( p_{00} < p_{10} \),

\[
\frac{\partial D(\pi)}{\partial \pi} \leq \kappa c(\rho_1 - \rho_0) \{ -b + \beta(b + c)(p_{10} - p_{00}) \}
+ \beta \kappa c(\rho_1 - \rho_0)(p_{10} - p_{00})
\leq \kappa c(\rho_1 - \rho_0)
\times \{-b + \beta(b + c + 1)(p_{10} - p_{00})\}.
\]

Under the condition \( 0 < p_{10} - p_{00} < \frac{b}{\beta + c + 1} \), the coefficient \( -b + \beta(b + c + 1)(p_{10} - p_{00}) \) is negative; hence, the derivative \( \frac{\partial D(\pi)}{\partial \pi} \) is negative.

D. PROOF OF LEMMA 8

Proof: Denote the expressions on the right hand side of (18) and (19) as \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) respectively. We need to show that substituting \( \mathcal{E}_2 \) in (18) gives (19), i.e. \( \mathcal{E}_1(\mathcal{E}_2) = \mathcal{E}_2 \). That means, it suffices to show that the following expression \( \mathcal{E}_1(\mathcal{E}_2) - \mathcal{E}_2 \) equals 0.

\[
\max_{a \in A_0} \left\{ \sum_{n=1}^N [r_n(\pi_n, y_n, a_n) - \lambda a_n] + \lambda M \right\}
+ \beta \sum_{a \in A_0} \sum_{y' \in S_y} \Pr(a|\pi, y, a) \Pr(y'|y, a)
\times \left\{ \frac{M \lambda}{1 - \beta} + \sum_{n=1}^N J^k(\Gamma^{a_n}(\pi_n), y_n) \right\}
- \frac{M \lambda}{1 - \beta} - \sum_{n=1}^N J^k(\pi_n, y_n),
\]

using \( \sum_{a \in A_0} \sum_{y' \in S_y} \Pr(a|\pi, y, a) \Pr(y'|y, a) = 1 \), and rearranging the terms,

\[
= - \sum_{n=1}^N J^k(\pi_n, y_n)
+ \max_{a \in A_0} \left\{ \sum_{n=1}^N [r_n(\pi_n, y_n, a_n) - \lambda a_n] + \beta \sum_{a \in A_0} \sum_{y' \in S_y} \Pr(a|\pi, y, a) \Pr(y'|y, a) J^k(\Gamma^{a_n}(\pi_n), y_n') \right\},
\]

reordering the summations and suitably expanding,

\[
= - \sum_{n=1}^N J^k(\pi_n, y_n)
+ \max_{a \in A_0} \left\{ \sum_{n=1}^N [r_n(\pi_n, y_n, a_n) - \lambda a_n] \right\}
+ \beta \sum_{a \in A_0} \sum_{y' \in S_y} \Pr(y'|y, a) J^k(\Gamma^{a_n}(\pi_n), y_n') \right\},
\]

where, \( y'_{-n} \) is the observation vector \( o \) omitting the \( n^{th} \) element. So is the case with \( y'_{-n} \) and so on.

\[
= - \sum_{n=1}^N J^k(\pi_n, y_n)
+ \max_{a \in A_0} \left\{ \sum_{n=1}^N [r_n(\pi_n, y_n, a_n) - \lambda a_n] \right\}
+ \beta \sum_{a \in A_0} \sum_{y' \in S_y} \Pr(y'|y, a) J^k(\Gamma^{a_n}(\pi_n), y_n') \right\},
\]

\[
= \sum_{n=1}^N \left( - J^k(\pi_n, y_n) + \frac{\max_{a \in A_0} \left\{ [r_n(\pi_n, y_n, a_n) - \lambda a_n] \right\}}{1 - \beta} \right)
+ \beta \sum_{a \in A_0} \sum_{y' \in S_y} \Pr(y'|y, a) J^k(\Gamma^{a_n}(\pi_n), y_n') \right\},
\]

\[
= 0.
\]

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