Spin mixture AKLT states for universal quantum computation

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The spin-3/2 Affleck-Kennedy-Lieb-Tasaki (AKLT) valence-bond states on the hexagonal and
other trivalent Archimedean lattices were shown to be universal resource states for measurement-
based quantum computation (MBQC). It is still unclear whether AKLT states of higher spin mag-
nitude can also support universal MBQC. We demonstrate that several 2D AKLT states involving
mixture of spin-2 and other lower-spin entities are also universal for MBQC. This includes a spin-2
spin-3/2 mixture and two other spin-2 spin-1 mixtures. In addition, we examine the universality for
the spin-2 AKLT state on the Kagomé lattice and provide evidence and argument that, however, it
is likely not universal.

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I. INTRODUCTION

Quantum computation by local measurements (MBQC) is shown to provide the same power of
computation as the standard circuit model [1–3]. Studying various aspects of this model can also offer
new perspectives and insights, such as dramatic improvement of the resource overhead in linear-optics
quantum computation [4], the utilization of topological protection [5], renormalization [6] and holographic
principles [7], and the exploration of quantum computational universality in phases of matter [8–11]
and the relation to symmetries [12] and contextuality [13], just to name a few. There are still several aspects in MQBC
to be understood, such as complete characterization of universal resource states and whether they can
arise as unique ground states of two-body interacting Hamiltonians [14, 15].

An interesting family is the so-called Affleck-Kennedy-Lieb-Tasaki (AKLT) states [16], which can be defined
on any graph and are unique ground states of two-body interacting Hamiltonians with suitable boundary conditions.
The insight why such family might be useful for MBQC originated from the result that the 1D spin-1
AKLT state can be used for simulating one-qubit gate operation on a single qubit [17, 18]. But the full quantum
computational universality was only established recently in the spin-3/2 AKLT state on the honeycomb lattice [19–
21]. Other trivalent lattices, such as the square-octagon and the cross lattices, have also been shown to host spin-
3/2 AKLT states that are universal [22]. However, no AKLT states with higher spins (than 3/2) are known to
be universal and this problem has remained open. In particular, it is not clear whether the spin-2 AKLT states
on the square and Kagomé lattices can provide quantum computational universality.

The purpose of this paper is to show that there do exist universal AKLT states that involve spin-2 entities mixed
with other lower spin ones such as spin-3/2 and spin-1. Additionally, we provide evidence that the AKLT state
on the Kagomé lattice is not universal due to frustration. The emerging picture from our study on the quantum
computational universality is this: AKLT states involving spin-2 and other lower spin entities are universal if
they reside on a two-dimensional frustration-free regular lattice with any combination of spin-2, spin-3/2, spin-1
and spin-1/2 (consistent with the lattice), provided that spin-2 sites are not neighbors. (We do not know how to
lift the restriction.)

The task of proving universality of higher-spin AKLT states poses a greater challenge than for the spin-3/2
ones, as a straightforward extension of the POVM for the trivalent case does not work. Additional elements must
be introduced to complete the spin-2 POVM. An appropriate POVM that reduces the local spins to qubit-like
structure is one crucial step. For example, a POVM for the spin-2 case has been constructed via the inclusion of
four more directions (in addition to the three x,y,z axes) that are the diagonals of a cube [9]. However, introducing
these additional directions spoils the connection of the post-POVM state to graph states, a nice feature which
spin-1 and spin-3/2 AKLT states have [19, 21]. Moreover, if the POVM outcome at a given site is along the diagonal
directions, we may need to treat this as an error outcome and to remove such error sites, e.g. by measuring the sur-
rounding sites appropriately, thereby removing the error sites and their neighboring sites. Such local corrections
can restore the post-POVM states to graph states at the expense of some logical qubits. If the error rate is small
enough and if the resulting graph is in the supercritical phase, then the state is a universal resource.

We shall first consider the lattice, shown in Fig. 1, which can be regarded as being modified from the square
lattice by replacing in the checker-board pattern spin-2...
The study of the mixture, spin-2 approaches is relevant in advancing our understanding of whether the error rate (which depends on the particular POVM used) $\epsilon$ is small enough, then the probability for removing a spin-2 site and its neighbors is small, we may still extract a universal resource state. For the spin-2 POVM constructed in Ref. \cite{9} and described above, $p_{\text{err}} = 3/5$, which seems not small. We shall provide another construction with $p_{\text{err}} = 1/5$ below. In addition to the lattice in Fig. 1a, we shall also examine two other lattices (Figs. 1b,c) which support AKLT states with mixture of spin-2 and spin-1 entities. Moreover, we shall investigate the universality for the spin-2 AKLT state on the Kagomé lattice (Fig. 1d).

As we shall demonstrate below that the three AKLT states consisting of spin-2 and other lower-spin entities are indeed all universal resource states. We believe our approach is relevant in advancing our understanding of the spin-2 only case. The study of the mixture, spin-2 and other higher-spin states may help to shed light on how magnetic properties, spin magnitudes, and lattice geometry conspire to quantum computational universality.

**II. THE GENERALIZED MEASUREMENT FOR REDUCTION TO QUBITS**

One approach for universality is to show that the state in question can be converted, via local measurements, to a known resource state. We construct a POVM for this approach. (Another possible approach is to directly construct universal gates, but this still relies on the first-step POVM.) The POVM for spin-3/2 sites consists of three rank-two elements \cite{19} and these are expressed in terms of the virtual qubits as follows,

\begin{align}
\tilde{F}_z &= \sqrt{\frac{2}{3}}(|000\rangle\langle 000| + |111\rangle\langle 111|) \\
\tilde{F}_x &= \sqrt{\frac{2}{3}}(|++\rangle\langle ++| + |--\rangle\langle --|) \\
\tilde{F}_y &= \sqrt{\frac{2}{3}}(|i\rangle\langle i| + |(-)\rangle\langle (-)|)
\end{align}

The $|0/1\rangle$, $|+/-\rangle$, and $|i/-i\rangle$ are eigenstates of the Pauli $X$, $Y$, and $Z$ matrices, respectively. The POVM gives rise to three possible outcomes, and any of them is regarded as a good outcome. Note that the POVM for spin-1 sites is almost identical except the proportional constants and the number of virtual qubits \cite{21}.

A POVM for spin-2 particles was given in Ref. \cite{9}. But for the purpose of establishing quantum computational universality, that POVM yield outcomes that are not eigenstates of Pauli operators and the $p_{\text{err}} = 3/5$ is not small. A key point here is that instead we propose to use a different POVM consisting of three rank-two elements and three other rank-one elements, expressed in terms of the four virtual qubits representing a spin-2 particle,

\begin{align}
\tilde{K}_z &= \frac{1}{\sqrt{3}}(|\text{GHZ}_x\rangle\langle \text{GHZ}_x|) \\
\tilde{K}_x &= \frac{1}{\sqrt{3}}(|\text{GHZ}_y\rangle\langle \text{GHZ}_y|) \\
\tilde{K}_y &= \frac{1}{\sqrt{3}}(|\text{GHZ}_z\rangle\langle \text{GHZ}_z|)
\end{align}

where $|\psi^{\otimes 4}\rangle$ is a short-hand notation for $|\psi,\psi,\psi,\psi\rangle$, equivalent to an eigenstate $|S = 2, S_\alpha\rangle$ of the spin-2 operator $S_\alpha$ with an eigenvalue $S_\alpha = \pm 2$ in either $\alpha = x, y, or z$ direction. The first three elements are similar to those in spin-3/2 sites, except the number of virtual qubits being four, and correspond to good outcomes of type $x$, $y$, and $z$, respectively. Associated with the last three elements, $|\text{GHZ}_x\rangle \equiv (|0000\rangle - |1111\rangle)/\sqrt{2}$, $|\text{GHZ}_y\rangle \equiv (|+++\rangle - |---\rangle)/\sqrt{2}$, and $|\text{GHZ}_z\rangle \equiv (|iiii\rangle - |---\rangle)/\sqrt{2}$ are the corresponding states and they will be regarded as error outcomes of type $x$, $y$, and $z$, respectively. But these GHZ states are at least eigenstates for certain product combination of Pauli operators. It can be verified that $\sum_\alpha \tilde{F}_\alpha^\dagger \tilde{F}_\alpha + \sum_\alpha \tilde{K}_\alpha^\dagger \tilde{K}_\alpha = \Pi_S$, where $\Pi_S$ is the projection onto the symmetric subspace of four qubits, i.e., identity in the spin-2 Hilbert space. The reduced density matrix for a single site of the AKLT state is a completely mixed state, and therefore.
III. ALTERNATIVE PERSPECTIVE ON THE ERRORS: LOGICAL PAULI MEASUREMENTS ON GRAPH STATES

The $\tilde{K}$’s operators can be rewritten as

$$\tilde{K}_\alpha = \sqrt{1/2} |\text{GHZ}_\alpha^+\rangle \langle \text{GHZ}_\alpha^+| \tilde{F}_\alpha.$$  (3)

This suggests that we can think of the error outcome associated with $\tilde{K}_\alpha$ as arising from a two-step process: (1) first a result in the outcome associated with $\tilde{F}_\alpha$ is obtained; (2) then a further measurement is done in the basis $|\text{GHZ}_\alpha^-\rangle$ and the result is $|\text{GHZ}_\alpha^-\rangle$. We have previously shown that for a state being obtained via $|\psi\rangle$ via $\otimes_{v \in V} |\tilde{F}_\alpha|_{\text{AKLT}}$ it is an encoded graph state [19][21]. A measurement in the basis $|\text{GHZ}_\alpha^\pm\rangle$ corresponds to a measurement in the effective logical X or Y basis. Thus the consequence of $\tilde{K}$’s is to transform the graph state $|\psi\rangle$ to another graph state by Pauli measurements.

The errors only occur on spin-2 sites. For each spin-2 site that is contained in a multi-site domain, the effect of the measurement in the basis $|\text{GHZ}_\alpha^\pm\rangle$ correspond to shrinking the domain by one site without affecting the quantum correlations of the present domain with others (i.e., the graph remains the same). Therefore, it essentially has no effect on the graph state, except a possible Pauli Z unitary on neighboring domains. However, for a single-site spin-2 domain, the measurement outcome $\tilde{K}_\alpha$ amounts to either a logical X or Y measurement on this logical qubit of the graph state. (Whether it is an X or Y error depends on the POVM outcome of its neighbors, and can be easily deduced [19][21].) Pauli measurements on a graph state simply results in another graph state, whose graph can be easily deduced from simple rules [22]. However, they can make the original planar graph become nonplanar, and the percolation argument on planar graphs cannot directly apply.

![Graph transformation rules on Y errors](image)

**FIG. 2.** Graph transformation rules on Y errors (or measurements) on a single-site spin-2 domain. (a) and (b) illustrate the case where four distinct domains are connected to center spin-2 site; (b) is different from (a) in that there is an additional edge between domains 4 and 5. To further make the graph remain planar, a Pauli Z measurement is done on any of the neighboring domain, say, 2. (c) shows the case where three distinct domains are connected to the spin-2 site. (This case can arise, e.g., as one of the neighboring domains was deleted in the second step of (a) or (b) associated with other spin-2 site.) (d) and (e) exemplify the cases of, respectively, two and one domain connected to a spin-2 site. Note that this does not exhaust all possibilities (due to other possible connections between neighbors, albeit planar) but just serves to illustrate that Y errors can be treated to maintain planarity. For X errors, there can be long-range edges created. We take the simplest approach to simply remove them by measuring their neighboring domains in the logical Pauli Z basis.

The above discussions suggest that one approach to proceed for establishing universality is to employ certain procedure to recover the planarity of the graphs, caused by the $\tilde{K}_\alpha$ outcomes on single-site spin-2 domains. As the single-site spin-2 domains are separated from one another, the Y errors can be easily dealt with. Such a single-site spin-2 domain can connect to at most four other domains (in general, zero, two or four, due to the modulo-2 operation in determining the graph [19]). If the spin-2 domain is connected to fewer than four other domains, then the Y error does not change planarity. For the spin-2 domain that is connected to four other domains, we can simply apply an additional Pauli Z measurement on any neighboring domain so as to recover planarity. See Fig. [2] for illustration of these measurements. The X errors are more troublesome, however, as they can result in long-range connectivity (edges), and the planarity is harder to restore (it also depends on which reference neighbor...
to use for applying the graph rule [23]). Without treating X errors case by case that depends on the graphical properties of the error vertex, its neighbors, and the neighbors of its neighbors, we shall take the minimalist approach to remove all X errors by measuring logical Z on all their neighbors, therefore removing not only the error domains but also their neighboring domains. (This can be costly, but simplifies the simulations. We note that it is not necessary to measuring Z on all neighbors and fewer neighbors will do; see the Appendix.) In summary, we shall first treat all the Y errors to recover planarity locally and then deal with X errors with described above, thereby recovering the full planarity (note the order is not essential).

IV. WORST CASE SCENARIO AMONG TYPICAL F POVM OUTCOMES

We have just described a procedure to deal with all POVM error outcomes. However, in order to perform simulations to determine the quantum computational universality for the AKLT states, we need a method to sample ˜F’s and ˜K from the exact distribution. The question of exact sampling amounts to sampling outcomes as a result of measuring Pauli X/Y on some subset of qubits in a graph state. It is a linear algebra problem and one can formulate the solutions in principle. However, it turns out that we can actually avoid the exact sampling and consider the ‘worst scenario’ and still demonstrate universality.

Because of the 2-step viewpoint of ˜K’s as in Eq. (3), we first sample POVM outcomes according to ˜F’s, i.e. assuming no error POVM. The exact sampling could be obtained by assigning some spin-2 sites to flip from ˜Fα to ˜Kα according the correct probability distribution, if one had solved this problem of sampling Pauli X/Y measurements on a subset of qubits for a graph state. Since measurement in the basis +/− on a spin-2 site within a domain of multiple sites is to shrink the domain by one site, the graph for the (encoded) graph state remains unchanged. These spin-2’s residing in multiple-site domains do not cause a problem. As discussed earlier, the problem arises from Pauli X/Y measurement on those spin-2 sites forming single-site domains. Thus, instead of the second step to sample further ˜K’s for the complete exact distribution, we shall consider what we call ‘the worst case scenario among the typical samples’: all the single-site spin-2 domains with ˜Fα will be flipped to ˜Kα. Then we check whether the resulting graphs are in the supercritical phase or not. This is done by checking the existence of percolated paths as one gradually deleting vertices (site percolation) or edges (bond percolation) [19]. If the graphs (before deletion) are in the supercritical phase, as the probability of deletion increases, there will be a clear signature of phase transition. We have implemented such a worst case scenario on the lattice of Fig. 1a and performed site percolation simulation on the resultant graphs, shown in Fig. 3. It is seen that the graphs, even in the worst case scenario, reside in the supercritical phase. This shows that the AKLT state on Fig. 1a, which is a mixture of spin-3/2 and spin-2 parti-
cles, is a universal resource for MBQC.

The techniques developed above can be applied to other spin-1/2, spin-1, spin-3/2 and spin-2 mixtures. We have also performed simulations for the AKLT states on the lattices in Figs. 1b, c, and confirmed that indeed they are also universal for MQBC; see Figs. 2a-c.

V. THE SPIN-2 AKLT STATE ON THE KAGOMÉ LATTICE

For the spin-2 only AKLT states, the worst-case scenario cannot be directly applied. Multi-site domains can be flipped to an error domain, because all the sites in each domain have some probability to be flipped from $\tilde{F}$ to $K$. However, even assume the best case scenario, we demonstrate that the spin-2 AKLT state on the Kagomé lattice is likely not universal. What we have done is to sample only the $\tilde{F}$’s POVM outcomes, thereby assuming no errors, i.e., the best scenario. We then check whether the resulting graphs possess traversing paths according to percolation. What we have found is that the random graphs generated from sampling only $\tilde{F}$’s are already in the subcritical phase, i.e., no traversing paths exist for large enough lattice sizes. Thus, the spin-2 AKLT state on the Kagomé is not likely universal. (What we have not ruled out is the possibility of other POVMs that might have enabled universal quantum computation, even though we believe that it is unlikely.)

The main reason for the lack of universality is due to geometric frustration in the lattice. The result can be understood intuitively as follows, similar to the case of the star lattice. Consider a triangle in the Kagomé lattice. Due to geometric frustration, the POVM outcome with $(x, x, x)$, $(y, y, y)$ and $(z, z, z)$ cannot appear. The remaining 24 instances are divided in the follow two cases. Case 1: There are 6 outcomes with all three labels be different; Case 2: 18 outcomes with only two labels being the same and the third being distinct. In Case 1 the three edges remain, whereas in Case 2 two of the three edges are removed and the remaining one becomes an internal edge. This gives rise to an average probability of $p_{\text{delete}}^{\text{bond}} = 1/2$ to remove an edge from an triangle. However, the probability that an edge is occupied $1 - p_{\text{delete}}^{\text{bond}} = 1/2$ is lower than the bond percolation threshold for the Kagomé lattice $p_{\text{th}}^{\text{bond}} \approx 0.5244$, and therefore, the random graphs are not in the supercritical phase, implying non-universality for the original AKLT state. This intuitive argument does not take into account of correlated errors, but that is exactly what our simulations have dealt with.

VI. CONCLUDING REMARKS

We have shown that three particular 2D AKLT states (see Fig. 1) involving mixture of spin-2 and other lower-spin entities are also universal for MBQC. The results are nontrivial as they demonstrate that AKLT states with higher spins can still be universal. We have also demonstrated that the spin-2 AKLT state on the Kagomé lattice (Fig. 1c) is not likely universal. What we have learned from the results in this work is an emerging picture on the universality of AKLT states involving spin-2 and other lower spin entities: any 2D frustration-free lattice with any combination of spin-2, spin-3/2, spin-1 and spin-1/2, except that spin-2 sites are not neighbors, will host a universal AKLT state.

With the spin-2 POVM constructed, it is also possible to use tensor network description or the Gross-Eisert correlation-space MBQC approach to construct universal one-qubit and two-qubit gates. However, the universality relies on the existence of a computational ‘backbone’ (which lives on the graph for the domains instead of the original lattice), whose existence is exactly what the percolation simulations are about. We conclude by leaving open the question of the quantum computational universality regarding the spin-2 AKLT state on the square lattice.

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Appendix A: Other approaches for treating errors

The sites with POVM outcomes $K$’s are regarded as error sites. As illustrated in Fig. 4, the four virtual qubits are post-selected to have GHZ entanglement, and such quantum correlation is teleported via singlet correlations to the outer four virtual qubits. Naively, one can remove all domains (by measuring in any site in the domain in its canonical basis) that surround such an error, removing the GHZ correlation. But this may be too costly in losing useful sites. Below we describe two similar ‘surgery’ procedures, and they can be combined with the one described in the main text.

1. Procedure 1

For each error site, if there exists at least one surrounding domain that has the same type (x, y, or z), remove these domains (by measuring in their canonical basis) but not others with different types. If no domains have the same type with the center error sites, then remove all these domains surrounding the error site. (The error site and those removed are thus decoupled from the remaining of the system.)

Justification. Suppose the center spin-2 site has error GHZ and the spin-3/2 to its left has the POVM outcome $F_z$. Then the virtual qubit that connects to the center site can have values $|0\rangle$ or $|1\rangle$. But if, as a
first step, we measure that spin-3/2 site in its canonical basis (i.e., the two components in \( F \)), to project the 3 virtual qubits to either \( |000 \rangle \) or \( |111 \rangle \). If we get \( |000 \rangle \), this collapses the virtual GHZ state of the four virtual particles down to \( |0000 \rangle \). Hence, all the virtual qubits connecting to the center site are collapsed to \( |0 \rangle \). If any other neighboring spin-3/2 site has the same POVM outcome, \( z \), then the three virtual qubits on this site will be collapsed to \( |000 \rangle \) as well (and those that belong to the same domain will be collapsed to either \( |000 \rangle \) or \( |111 \rangle \) depending on the location in the domain). The case of \( |111 \rangle \) can be similarly dealt with. Therefore, the center site and the neighboring spin-3/2 sites having the same type of POVM outcome (and their domain) are essentially removed. The remaining spin-3/2 sites that do not share the same POVM outcome essentially are left with two virtual qubits, as the third one is contracted with \( |0 \rangle \). It becomes as if this were a spin-1 AKLT site with the same POVM outcome. But there are some spin-2 sites that have neighboring spin-3/2 sites all of different POVM outcome types. One way to proceed, as a second step in the procedure, is to measure all these sites further depending on the location in the domain). The case of \( |111 \rangle \) can be similarly dealt with. Therefore, the center site and the neighboring spin-3/2 sites having the same POVM outcome. But there are some spin-2 sites that have neighboring spin-3/2 sites all of different POVM outcome types. One way to proceed, as a second step in the procedure, is to measure all these sites further in their canonical basis to project down to a single component, thereby removing them. It is clear that the state after Procedure 1 is still an encoded graph state (which we simply refer to as a graph state).

**FIG. 4.** Teleportation of virtual qubits. When obtaining the POVM outcome \( \text{GHZ}^- \), this state of the center four virtual qubits are teleported to the outer four. The outer four virtual qubits are now in \( \text{GHZ}^- \).

Suppose we measure A and B and obtain \( |0 \rangle \) and \( |1 \rangle \) respectively. Then the remaining tensors for A, B, C, and D are

\[
|0\rangle \langle \gamma_A A | 1 \rangle \langle \gamma_A A | \otimes |1\rangle \langle \gamma_B B | 2 \rangle \langle \gamma_B B | \otimes F_C \otimes F_D \left[ |0_1 0_2 0_3 0_4 \rangle - |1_1 1_2 1_3 1_4 \rangle \right]
\]

\[
= |0\rangle \langle \gamma_A A | 1 \rangle \langle \gamma_B B | 1 \rangle \otimes F_C \otimes F_D \left[ \langle \gamma_A A | 0 \rangle \langle \gamma_B B | 2 \rangle |0_3 0_4 \rangle - \langle \gamma_A A | 1 \rangle \langle \gamma_B B | 2 \rangle |1_3 1_4 \rangle \right].
\]

As \( \langle \gamma_A A | 0 \rangle \), \( \langle \gamma_A A | 1 \rangle \), \( \langle \gamma_B B | 0 \rangle \) and \( \langle \gamma_B B | 1 \rangle \) all have the same magnitude (due to the unbiased basis for \( x, y, \) and \( z \)), the state inside the square bracket on the second line is a Bell state, which differs from the singlet by a Pauli unitary.

**3. Modified procedures**

The alternative perspective in understanding the spin-2 POVM \( \tilde{K} \)'s as Pauli measurements leads to the modified Procedures 1’ and 2’.

**Procedure 1’**. For those spin-2 sites with error sitting on multi-site domains, we need not do anything about...
them. But for those single-site domains with error (i.e., are themselves domains and neighboring sites having different types), remove all surrounding domains, as described in Procedure 1. (The error site and those removed are thus decoupled from the remaining of the system.)

**Procedure 2'.** For those spin-2 sites with error sitting on multi-site domains, we need not do anything about them. If no domains have the same type with the center error sites (i.e. single-site spin-2 domains), one can concentrate the virtual GHZ correlation to a two-site Bell correlation between two domains by measuring all but two surrounding domains in their respective canonical basis. The two remaining domains are thus linked with an edge.

We have performed some tests with these two procedures. Instead of flipping all single-site spin-2 domains with probability one, we flipped them with a probability of 1/5, i.e., the corresponding $p_{\text{err}}$ (without taking into correlated errors), and used Procedure 1' and 2' to check the percolation properties of the resultant graphs. We found that they are all in the supercritical phase. In fact, even if the probability of flipping was increased to 1/2 or even higher, the resultant graphs were still in the supercritical phase.

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