Resolution Graphs of Some Surface Singularities, I.
(Cyclic Coverings)

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Abstract. The article starts with some introductory material about resolution graphs of normal surface singularities (definitions, topological/homological properties, etc). Then we discuss the problem of N-cyclic coverings \((X_{f,N}, 0) \to (X, 0)\) of \((X, 0)\), branched along \(\{f = 0\}, 0\), where \(f : (X, 0) \to (\mathbb{C}, 0)\) is the germ of an analytic function. We present non-trivial examples in order to show that from the embedded resolution graph \(\Gamma(X, f)\) of \(f\) it is not possible to recover the resolution graph of \((X_{f,N}, 0)\). The main results are the construction of a “universal covering graph” of \(\Gamma(X, f)\) from the topology of the germ \(f\), and the completely combinatorial construction of the resolution graph of \((X_{f,N}, 0)\) from this universal graph of \(f\) and the integer \(N\). For this we also prove some classification theorems of “graph coverings”, results which are purely graph-theoretical. In the last part, we connect the properties of the universal covering graph with the topological invariants of \(f\), e.g. with the nilpotent part of its algebraic monodromy.

Introduction.

The present article has several goals. First of all, it is an article with some expository character, presenting several aspects of the resolution graphs of normal surface singularities. Moreover, it creates the good language and right point of view in the graph-codifications of important geometric constructions, such as cyclic coverings of surface singularities, series of singularities, degeneration of curves, etc. This is realized by the introduction of a new combinatorial object: the “covering of graphs” (Section 1). We will present two main applications: the case of cyclic coverings of normal surface singularities (the present article, in the sequel: Part I), and the case of some series of singularities (a joint work with Ágnes Szilárd [30], in the sequel: Part II).

Consider a normal surface singularity \((X, 0)\) and a germ of an analytic function \(f : (X, 0) \to (\mathbb{C}, 0)\). Let \(X_{f,N}\) be the (normalized) cyclic \(N\)-covering of \((X, 0)\) branched along \(\{f = 0\}, 0\). The final goal of Part I is to recover the resolution graph \(\Gamma(X_{f,N})\) of \(X_{f,N}\) from some invariants of the pair \((X, f)\) and the integer \(N\).

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Particular cases suggest that $\Gamma(X, f, N)$ might be computable from the embedded resolution graph $\Gamma(X, f)$ of $f$ and the integer $N$, but this is not true in general. In fact, $\Gamma(X, f)$ and $N$ codify all the local data of the covering, but they miss some global information. This global data is in a close relationship with the monodromy representation $arg_*$ of the Milnor fibration associated with $f$, which plays an important role in the global geometry of all the cyclic coverings.

In the second section we present the basic definitions and some of the (elementary) properties of embedded resolution graphs. This section also serves as an introduction to Part II. Then we discuss “local/global” properties, showing e.g. that the embedded resolution graph does not determine the representation $arg_*$. The additional global information of $arg_*$ will be codified in another graph $G(X, f)$, called the “universal covering graph” of $\Gamma(X, f)$.

This already motivates the material of Section 1, where a detailed presentation of the “cyclic covering of graphs” can be found. The section culminates in the classification theorems, which are indispensable in the applications.

In the third section we present an algorithm which provides the resolution graph $\Gamma(X, f, N)$, in a purely combinatorial way, from the universal covering graph $G(X, f)$ of $\Gamma(X, f)$ and the integer $N$. In fact, each graph $\Gamma(X, f, N)$ appears as a graph covering $\Gamma(X, f) \to \Gamma(X, f)$ modified by Hirzebruch–Jung strings. The family of graphs $\{\Gamma(X, f, N)\}_N$ behaves like a complicated series of graphs, which is coordinated by a unique graph, the universal covering of $\Gamma(X, f)$. The global nature of the construction is emphasized by non–trivial examples. On the other hand, we always stress the particular cases when the global information can be recovered from the local (i.e. when $G(X, f)$ is determined from $\Gamma(X, f)$), e.g. when the link of $(X, 0)$ is a rational homology sphere.

Each section and subsection has its own introduction, where the reader can find more guiding information (see also the introduction of Part II).

1. $\mathbb{Z}$–coverings of graphs

1.1. In this section we present some graph–theoretical constructions. The construction of “cyclic covering of graphs” is motivated by the structure of the resolution graphs of cyclic coverings of the normal surface singularities $(X, 0)$ with branch locus $(f^{-1}(0), 0) \subset (X, 0)$ where $f : (X, 0) \to (\mathbb{C}, 0)$ is a germ of an analytic function (cf. 2.7). All these resolution graphs are controlled by a single “universal covering graph” which is a $\mathbb{Z}$–covering of the embedded resolution graph of the pair $(f^{-1}(0), 0) \subset (X, 0)$.

Graph coverings can be defined for an arbitrary discrete group $G$, and they have important applications even for $G \neq \mathbb{Z}$. The general description, together with some applications, will be published elsewhere. In this article we only discuss the case $G = \mathbb{Z}$, and we call the corresponding coverings “cyclic” or “$\mathbb{Z}$–coverings”.

1.2. Notations. For any graph $\Gamma$, we denote the set of vertices by $\mathcal{V}(\Gamma)$ and the set of edges by $\mathcal{E}(\Gamma) \subset \mathcal{V}(\Gamma) \times \mathcal{V}(\Gamma)$. If there is no danger of confusion, we denote them simply by $\mathcal{V}$ and $\mathcal{E}$.

For simplicity, we will assume that our graphs have no loops. The interested reader can easily extend all our results for graphs with loops. If $v \in \mathcal{V}$ is a fixed vertex, let $\mathcal{E}_v$ be the set of edges which have $v$ as one of their endpoints. If $|\Gamma|$ is
the topological realization of the graph \( \Gamma \), then we denote the rank of \( H_1(\|\Gamma\|, \mathbb{Z}) \) by \( c_\Gamma \), i.e. \( c_\Gamma \) is the number of independent cycles of the graph \( \Gamma \).

1.3. Definitions. A morphism of graphs \( p : \Gamma_1 \to \Gamma_2 \) consists of two maps \( p_V : V(\Gamma_1) \to V(\Gamma_2) \) and \( p_E : E(\Gamma_1) \to E(\Gamma_2) \), such that if \( e \in E(\Gamma_1) \) has endpoints \( v_1 \) and \( v_2 \), then \( p_E(e) \) is an edge in \( \Gamma_2 \) with endpoints \( p_V(v_1) \) and \( p_V(v_2) \). If \( p_V \) and \( p_E \) are isomorphisms of sets, then we say that \( p \) is an isomorphism of graphs.

If \( \Gamma \) is a graph, we say that \( \mathbb{Z} \) acts on \( \Gamma \), if there are group–actions \( a_V : \mathbb{Z} \times V \to V \) and \( a_E : \mathbb{Z} \times E \to E \) of \( \mathbb{Z} \) with the following compatibility property: if \( e \in E \) has endpoints \( v_1 \) and \( v_2 \), then \( a_E(h, e) \) has endpoints \( a_V(h, v_1) \) and \( a_V(h, v_2) \). The action is trivial if \( a_V \) and \( a_E \) are trivial actions.

If \( \mathbb{Z} \) acts on both \( \Gamma_1 \) and \( \Gamma_2 \), then a morphism \( p : \Gamma_1 \to \Gamma_2 \) is equivariant if the maps \( p_V \) and \( p_E \) are equivariant with respect to the actions of \( \mathbb{Z} \). If additionally \( p \) is an isomorphism then it is called an equivariant isomorphism of graphs.

1.4. – The “segment graph”. The simplest graph is the “segment graph” \( S \) which has two vertices \( V = \{ v_1, v_2 \} \) and one edge \( E = \{ e = (v_1, v_2) \} \). In the sequel \( S \) will always carry the trivial action of the structure group \( \mathbb{Z} \).

1.5. – The “standard blocks” of \( \mathbb{Z} \)-coverings. A “standard block” \( B \), which covers the segment graph \( S \), can be constructed as follows.

Fix three strictly positive integers \( n_1, n_2 \) and \( d \), and set \( [n, m] = \text{l.c.m.}\{n, m\} \). The standard block \( B(n_1, n_2, d) \) is a graph which consists of \( n_1 + n_2 \) vertices \( V = \{ P_1, ..., P_{n_1}, P_1', ..., P_{n_2}' \} \) and \( d \cdot [n_1, n_2] \) edges \( E = \{ e_1, ..., e_d[n_1, n_2] \} \) and has the following structure: for each \( k \in \{ 1, ..., d[n_1, n_2] \} \), the endpoints of the edge \( e_k \) are \( P_{i(k)} \) and \( P_{j(k)} \), where \( i(k) \equiv k \) (mod \( n_1 \)) and \( j(k) \equiv k \) (mod \( n_2 \)). Notice that \( P_1 \) and \( P_1' \) are connected by exactly \( d \) edges. The graph \( B(n_1, n_2, d) \) has a natural \( \mathbb{Z} \)-action given by \( a_E(n, e_k) = e_l \), where \( l \equiv k + n \) (mod \( d[n_1, n_2] \)) and \( a_A(1, P_{i(k)}') = P_{i(k)}' \), where \( i \equiv k + n \) (mod \( n_1 \)) (i = 1, 2).

Consider the “segment graph” \( S \) and the trivial \( \mathbb{Z} \)-action on it. Then \( p : B(n_1, n_2, d) \to S \) defined by \( p(P_i) = v_i \) (i = 1, 2) and \( p(e_k) = (v_1, v_2) \) is an equivariant morphism.

Actually, almost all the finite coverings of \( S \) are “standard blocks”. Indeed, consider an arbitrary equivariant morphism \( p : B \to S \) of graphs, where \( B \) is a finite connected graph with \( E \neq \emptyset \) and with a \( \mathbb{Z} \)-action, \( p \) is an equivariant morphism such that the restriction of the action of \( \mathbb{Z} \) on \( p^{-1}(v_1) \) (i = 1, 2), respectively on \( p^{-1}((v_1, v_2)) \), is transitive. Then, it is not difficult to show that \( p : B \to S \) is equivalent with the standard block \( B(n_1, n_2, d) \), where \( n_i \mathbb{Z} \) (respectively \( d[n_1, n_2] \) \( \mathbb{Z} \)) is the maximal subgroup of \( \mathbb{Z} \) which acts trivially on \( p^{-1}(v_1) \) (i = 1, 2) (respectively on \( p^{-1}((v_1, v_2)) \)).

1.6. – Cyclic (or \( \mathbb{Z} \)-) coverings of graphs. Now, consider an arbitrary graph \( \Gamma \). We assume that \( \mathbb{Z} \) acts on \( \Gamma \) in a trivial way. Any edge \( e \) with endpoints \( v_1 \) and \( v_2 \) determines a natural “segment subgraph” \( S_e = \{ \{ v_1, v_2 \}, \{ e \} \} \) of \( \Gamma \).

1.7. Definition. A \( \mathbb{Z} \)-covering, or cyclic covering, of the finite graph \( \Gamma \) consists of a finite graph \( G \), that carries a \( \mathbb{Z} \)-action, together with an equivariant morphism
p : G \to \Gamma$ such that the restriction of the $\mathbb{Z}$-action on any set of type $p^{-1}(v)$ ($v \in \mathcal{V}(\Gamma)$, respectively $p^{-1}(e)$ ($e \in \mathcal{E}(\Gamma)$), is transitive.

This definition can be reformulated in terms of “standard blocks” as follows. For any $v \in \mathcal{V}(\Gamma)$ (respectively edge $e \in \mathcal{E}(\Gamma)$ with endpoints $\{v_1, v_2\}$), let $n_v \mathbb{Z}$ (respectively $d_e [n_{v_1}, n_{v_2}] \mathbb{Z}$) be the maximal subgroup of $\mathbb{Z}$ which acts trivially on $p^{-1}(v_i)$ ($i = 1, 2$) (respectively on $p^{-1}(e)$). This defines a system of strictly positive integers $(n, d) = \{(n_v)_{v \in \mathcal{V}(\Gamma)}; \{d_e\}_{e \in \mathcal{E}(\Gamma)}\}$. This system will be called covering data.

1.8. Definition. Fix a graph $\Gamma$ with a trivial $\mathbb{Z}$-action, and a system of integers (covering data) $(n, d) = \{(n_v)_{v \in \mathcal{V}(\Gamma)}; \{d_e\}_{e \in \mathcal{E}(\Gamma)}\}$. A $\mathbb{Z}$-covering (or cyclic covering) of type $(n, d)$ of the graph $\Gamma$ consists of a graph $G$, that carries a $\mathbb{Z}$-action, together with an equivariant morphism $p : G \to \Gamma$ such that for any $v \in \mathcal{V}(\Gamma)$ (resp. $e \in \mathcal{E}(\Gamma)$) the set $p^{-1}(v)$ (resp. $p^{-1}(e)$) consists of $n_v$ vertices (resp. $n_e = d_e \cdot |n_{v_1}, n_{v_2}|$ edges). Moreover, we assume that for any edge $e$, the subgraph $p^{-1}(S_e)$ is a “standard block” $B = B(n_{v_1}, n_{v_2}, d_e)$ such that the restriction of the $\mathbb{Z}$-action of $\mathbb{Z}$ to $p^{-1}(S_e)$ coincides with the natural $\mathbb{Z}$-action of $B$.

The above definition shows that any cyclic covering $G \to \Gamma$ is constructed from standard blocks which are glued equivariantly along the vertices $\{p^{-1}(v)\}$, where $v$ runs over the vertices of $\Gamma$ with degree $\geq 2$.

1.9. Definition. Two cyclic coverings $p_i : G_i \to \Gamma$ ($i = 1, 2$) are equivalent $(G_1 \sim G_2)$ if there is an equivariant isomorphism $q : G_1 \to G_2$ such that $p_2 \circ q = p_1$.

The set of equivalence classes of cyclic coverings of $\Gamma$, associated with a system of integers $(n, d)$, is denoted by $\mathcal{G}(\Gamma, (n, d))$.

1.10. Examples.

a.) If $\Gamma = S$, then $\mathcal{G}(\Gamma, (n, d))$ has exactly one element for any $(n, d)$.

b.) Let $\Gamma$ be the cyclic graph with two vertices and two edges: $\mathcal{V}(\Gamma) = \{v_1, v_2\}$, $\mathcal{E}(\Gamma) = \{e_1, e_2\}$ with both $e_i$ ($i = 1, 2$) having $v_1$ and $v_2$ as endpoints.

Set $n_1 = n_2 = n$. Then $\mathcal{G}(\Gamma, (n, d))$ has exactly $n$ elements. Notice that we can have graphs that are not equivalent as cyclic coverings over $\Gamma$, but they are isomorphic as graphs. (Take e.g. the case $n = 3$.)

The fact that in this case $\mathcal{G}(\Gamma, (n, d))$ is independent of the choice of $d$, is not a particularity of this example: it is true in the most general situation, cf. (1.15).

1.11. – The trivial covering of $(\Gamma, (n, d))$. There is a special element in $\mathcal{G}(\Gamma, (n, d))$ which can be constructed as follows. Fix a distinguished edge in each standard block $B(n_{v_1}, n_{v_2}, d_e)$. Then construct $G$ in such a way that whenever the vertex $v \in \mathcal{V}(\Gamma)$ of $\Gamma$ is adjacent to the edges $\{e\}_{e \in \mathcal{E}_v}$ of $\Gamma$, then the distinguished edges of all the blocks $\{p^{-1}(e)\}_{e \in \mathcal{E}_v}$ have a common endpoint (which is one of the vertices of $G$ in $p^{-1}(e)$). Notice that this condition together with the existence of the $\mathbb{Z}$-action determines all the other adjacency relations.

The equivalence class of $G$ constructed in this way does not depend on the choice of the distinguished edges in the standard blocks. Indeed, if we have two different choices of the distinguished edges in the standard blocks which provide two graphs $G_1$ and $G_2$ by the above construction (both covering $\Gamma$), then for any edge $e$ of $\Gamma$ denote the distinguished edges in the blocks $p_i^{-1}(S_e)$ by $e_i$ ($i = 1, 2$).
Then the map $q(e_1) = e_2$ (for any $e$) can be extended to an equivariant isomorphism $q : G_1 \rightarrow G_2$ with $p_2 \circ q = p_1$.

The covering just constructed (or its class) is denoted by $p : T \rightarrow \Gamma$ and it is called the trivial cyclic covering of $(\Gamma, (n, d))$.

The trivial covering is characterized by the existence of a (non-equivariant!) morphism of graphs $s : \Gamma \rightarrow T$ with $p \circ s = \text{id}_\Gamma$ (i.e. $s$ is a section with $s(e) =$ distinguished edge above $e$).

1.12. – $G(\Gamma, (n, d))$ as a homogeneous space. Classification.

If $p : G \rightarrow \Gamma$ is a cyclic covering of $\Gamma$, then $G$ is obtained by an equivariant gluing of $\# \mathcal{E}(\Gamma)$ standard blocks. Above any vertex $v \in \mathcal{V}(\Gamma)$ we have to glue together $\# \mathcal{E}_v$ standard blocks.

Regard $\Gamma$ as a union of segments, with some endpoints glued together. Then it is useful to introduce an index set of all the endpoints of the segments. This is: $I = \bigcap_v \mathcal{E}_v = \bigcap_v \{ e_v \}_{e_v \in \mathcal{E}_v}$.

Now consider the following group indexed exactly over this set: $B(\Gamma, n) = \prod_v \prod_{e_v \in \mathcal{E}_v} \mathbb{Z}_{n_v} = \prod_{e_v \in I} \mathbb{Z}_{n_v}$. Then $B(\Gamma, n)$ describes the equivariant gluings in $G$. A typical element in $B(\Gamma, n)$ is $\{ b_{e_v} \}_{e_v \in I}$, where $b_{e_v} \in \mathbb{Z}_{n_v}$. A generator set $\{ g_{e_v} \}_{e_v \in I}$ has the form $g_{e_v} = (0, \ldots, 1, \ldots, 0)$, where all the entries are zero except the place $e_v$, where we put the generator 1 of $\mathbb{Z}_{n_v}$.

$B(\Gamma, n)$ acts in a natural way on the set $G(\Gamma, (n, d))$. We give the action of $g_{e_v}$ for each $e_v \in I$. For this, fix $v \in \mathcal{V}(\Gamma)$ and an edge $e = e_v$ with endpoints $v$ and $w$. If $G \rightarrow \Gamma$ is a covering, then above the segment $S_e$ we have the block $B(n_v, n_w, d_v)$. In particular, above the vertex $v$ we have $n_v$ vertices cyclically permuted by the $\mathbb{Z}$-action. Call them $P_1, \ldots, P_{n_v}$ so that the action is $P_1 \rightarrow P_2 \rightarrow \ldots \rightarrow P_{n_v} \rightarrow P_1$.

We can detach the endpoint $v$ of the segment $e$ from the graph $\Gamma$. In the graph $G$ this means that we separate the endpoints of the block $B(n_v, n_w, d_v)$, that stay above $v$, from $G$. In this way we obtain the graph $\bar{G}_v$. It is represented in the right hand side of the next diagram.
Hence, we can re-obtain $G$ from $\tilde{G}_v$ if we re-glue $P_i$ and $Q_i$ ($i = 1, \ldots, n_v$). We emphasize that the action is $P_i \to P_{i+1}$ and similarly $Q_i \to Q_{i+1}$. Now, by definition, $g_{e_v} \ast G$ is obtained by the equivariant gluing $P_1 = Q_2$, $P_2 = Q_3, \ldots, P_{n_v} = Q_1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Diagram of graphs with identified edges.}
\end{figure}

Notice that there is a canonical equivariant identification of the edges of $G$ and $g_{e_v} \ast G$ (which, in general, cannot be extended to a morphism of graphs). Indeed, there are natural morphisms $q_1 : \tilde{G}_v \to G$ and $q_2 : \tilde{G}_v \to g_{e_v} \ast G$ which correspond to the gluings described above. They induce isomorphisms at the level of edges, hence an isomorphism of sets:

$$i_{g_{e_v}}(G) : \mathcal{E}(G) \to \mathcal{E}(g_{e_v} \ast G).$$

On the above diagram of graphs, we decorated by $\tilde{e}_1$ and $\tilde{e}_2$ two identified pairs of edges; this partial identification can be extended to a unique equivariant identification of edges.

It is not difficult to see, that this construction $(g_{e_v}, G) \mapsto g_{e_v} \ast G$ can be extended to a group action of $B(\Gamma, \mathbf{n})$. Moreover, for any $g \in B(\Gamma, \mathbf{n})$, one can define inductively an identification isomorphism:

$$i_g(G) : \mathcal{E}(G) \to \mathcal{E}(g \ast G).$$

The verification of the following easy facts are left to the reader.

1.13. Lemma. \hspace{1em} a) If $G_1 \sim G_2$ then $g \ast G_1 \sim g \ast G_2$ for any $g$. In particular, we obtain a group action $B(\Gamma, \mathbf{n}) \times G(\Gamma, (\mathbf{n}, \mathbf{d})) \to G(\Gamma, (\mathbf{n}, \mathbf{d}))$.

b) The above action on $G(\Gamma, (\mathbf{n}, \mathbf{d}))$ is transitive.

For a fixed $G$ let $Iso(G)$ be the isotropy subgroup $\{g \in B(\Gamma, \mathbf{n}) : g \ast G = G\}$ of $G$. Since $B(\Gamma, \mathbf{n})$ is abelian and the action is transitive, all the isotropy subgroups coincide. We denote this subgroup by $Iso \subset B(\Gamma, \mathbf{n})$. An immediate consequence of the previous lemma is that $G$ can be represented as a homogeneous set $B(\Gamma, \mathbf{n}) / Iso$.

In the next lemma, we will give a (possible) generator set of $Iso$. Fix a covering $p : G \to \Gamma$. If $e = (v, w)$ is an edge of $\Gamma$, then we can detach both endpoints of $e$ from $\Gamma$. We make the same twist above the separated endpoints as before (but now above both endpoints). This operation does not change the equivalence class of $G$, since we just rotate the block $p^{-1}(e)$, which is equivariant. In other words, for any edge $e = (v, w)$, the equivariance of block $p^{-1}(e)$ gives that $\tilde{g}_e \ast G \sim G$, where $\tilde{g}_e = g_{e_v} \cdot g_{e_w}$. 
Similarly, if \( v \) is a vertex, then we take the product \( \hat{g}_v = \prod_{e_v \in E_v} g_{e_v} \) and equivariance of the fiber above \( v \) gives \( \hat{g}_v \ast G \sim G \).

1.14. Lemma. \quad Iso is generated by the elements \( \{\hat{g}_e\}_{e \in \mathcal{E}(\Gamma)} \) and \( \{\hat{g}_v\}_{v \in V(\Gamma)} \).

Proof. We will prove that \( Iso(T) \) is generated by the elements \( \{\hat{g}_e\}_{e \in \mathcal{E}(\Gamma)} \) and \( \{\hat{g}_v\}_{v \in V(\Gamma)} \).

Fix a section \( s : \Gamma \to T \) of \( T \), which determines a set of distinguished edges \( \{s(e)\}_{e \in \mathcal{E}(\Gamma)} \) of \( T \) (cf. 1.11). Recall that for any \( g \in B(\Gamma, \mathbf{n}) \), as we already mentioned, there is a canonical equivariant identification \( i_g = i_g(T) : \mathcal{E}(T) \to \mathcal{E}(g \ast T) \).

Fix an element \( g \in Iso(T) \), and set \( s_g(e) := i_g(s(e)) \). The correspondence \( e \mapsto s_g(e) \), in general, cannot be extended to a section of \( g \ast T \to \Gamma \), (because, for a fixed \( v \in V(\Gamma) \), the set of endpoints above \( v \) of the edges \( \{s_g(e_v)\}_{e_v \in E_v} \) might contain more than one vertex).

On the other hand, since \( g \ast T \) is trivial, there is a section \( s' : \Gamma \to g \ast T \), which again determines a set of distinguished edges \( \{s'(e)\}_{e \in \mathcal{E}(\Gamma)} \).

For any block of \( g \ast T \) above the segment \( S_v \) of \( \Gamma \), we can find an integer \( k_v \in \mathbb{Z} \) such that \( e \mathcal{E}(k_v, s_g(e)) = s'(e) \). Set \( h = \prod_v \hat{g}_v^{-k_v} \); then the endpoints of the edges \( \{i_{\hat{g}_v}(s(e))\} \) above any vertex \( v \) consists of exactly one vertex, i.e. they form a section of \( (h \circ g) \ast T \). But, the elements \( g' \) of \( B(\Gamma, \mathbf{n}) \), which have the property that for a section \( s \) of \( T \), the set of edges \( \{i_{\hat{g}_v}(s(e))\} \) can be extended to a section, have the form \( g' = \prod_v \hat{g}_v^{-k_v} \) (for some integers \( k_v \)). Hence \( g = \prod_v \hat{g}_v^{k_v} \).

Now, for any edge \( e \) with endpoints \( v_1 \) and \( v_2 \) define \( n_e = d_e[n_{v_1}, n_{v_2}] \). Let \( A(\Gamma, \mathbf{n}) \) be the group \( \prod_{e \in \mathcal{E}(\Gamma)} \mathbb{Z}_{n_e} \times \prod_{v \in V(\Gamma)} \mathbb{Z}_{n_v} \). The generators of \( A(\Gamma, \mathbf{n}) \) are \( \hat{g}_e = (0, ..., 1, ..., 0) \) (resp. \( \hat{g}_v \)), whose all entries are zero except at the place \( e \) (resp. \( v \)), where we put the generator \( 1 \) of \( \mathbb{Z}_{n_e} \) (resp. of \( \mathbb{Z}_{n_v} \)). Consider the map \( \theta : A(\Gamma, \mathbf{n}) \to B(\Gamma, \mathbf{n}) \) given by \( \theta(\hat{g}_e) = \hat{g}_e = g_{e_{v_1}} \cdot g_{e_{v_2}} \) and \( \theta(\hat{g}_v) = \hat{g}_v = \prod_{e_v \in E_v} g_{e_v} \).

Then by the above discussion \( G(\Gamma, (\mathbf{n}, \mathbf{d})) \) can be identified with the factor group \( \text{coker } \theta \). Indeed, just notice that we have a distinguished element of \( G \), namely the trivial covering \( T \). The action \( g \in B(\Gamma, \mathbf{n}) \to g \ast T \in G \) identifies \( G(\Gamma, (\mathbf{n}, \mathbf{d})) \) with the factor group \( \text{coker } \theta = B(\Gamma, \mathbf{n})/Iso \).

The above discussions are summarized in the following theorem:

1.15. Theorem. \quad a.) One has the following exact sequence of abelian groups:

\[
A(\Gamma, \mathbf{n}) \xrightarrow{\partial} B(\Gamma, \mathbf{n}) \xrightarrow{\ast T} G(\Gamma, (\mathbf{n}, \mathbf{d})) \to 0
\]

In particular, \( G(\Gamma, (\mathbf{n}, \mathbf{d})) \) is independent of \( \mathbf{d} \) (therefore, sometimes we will use the more natural \( G(\Gamma, \mathbf{n}) \) for it).

b.) If \( \Gamma = \coprod_{i=1}^k \Gamma_i \) has \( k \) connected components \( \{\Gamma_i\}_{i=1}^k \), then

\[
G(\Gamma, (\mathbf{n}, \mathbf{d})) = \bigoplus_{i=1}^k G(\Gamma_i, (\mathbf{n}, \mathbf{d})).
\]

1.16. \quad For a subgraph \( \Gamma' \subseteq \Gamma \). Fix a graph \( \Gamma \) and integers \( (\mathbf{n}, \mathbf{d}) = \{(n_v)_{v \in V}; (d_e)_{e \in E}\} \) as above. Then any subgraph \( \Gamma' \) (i.e. graph \( \Gamma' \) with \( V(\Gamma') \subseteq V(\Gamma) \) and \( E(\Gamma') \subseteq E(\Gamma) \)) inherits a system of integers \( \{(n_v)_{v \in V(\Gamma')}; (d_e)_{e \in E(\Gamma')}\} \).
1.18. Corollary. For any subgroup \( \Gamma' \subset \Gamma \), \( pr : G(\Gamma, (n, d)) \to G(\Gamma', (n, d)) \) is onto.

The construction of the coverings \( G \to \Gamma \) suggests that the group \( G(\Gamma) \) describes the possible twists above \( \Gamma \). Intuitively it is clear that a non-trivial twist can be realized only above cycles of \( \Gamma \). The next theorems state this fact rigorously.

1.19. Theorem. Assume that \( \Gamma \) is a tree. Then \( G(\Gamma, (n, d)) = 0 \) for any \( (n, d) \).

Proof. We can assume that \( \Gamma \) is connected (cf. 1.15). We prove the vanishing of \( G \) by induction on the number of edges of \( \Gamma \). If \( \#E = 1 \), then \( G = 0 \) by (1.10a) or by (1.15). If \( \#E \geq 2 \), fix a connected subgroup \( \Gamma' \subset \Gamma \) such that \( V(\Gamma') = V(\Gamma) \setminus \{v\} \), and \( E(\Gamma') = E(\Gamma) \setminus \{(v, w)\} \) (i.e. \( \Gamma' \) is obtained from \( \Gamma \) by deleting a vertex \( v \) with \( \#E_v = 1 \)). Then, in (1.17), \( ker(pr_A) = \mathbb{Z}_{n_v} \times \mathbb{Z}_{d(v,w)}[n_v,n_w] \), \( ker(pr_B) = \mathbb{Z}_{n_v} \times \mathbb{Z}_{n_w} \), and \( \theta : ker(pr_A) \to ker(pr_B) \) is given by \( (\tilde{a},\tilde{b}) \mapsto (\tilde{a}+\tilde{b},\tilde{b}) \), hence \( \theta \) is onto. Therefore \( pr : G(\Gamma) \to G(\Gamma') \) is an isomorphism. Hence the theorem follows.

In Part II (5.53), we need the following generalization of (1.19).

1.20. Theorem. Fix a graph \( \Gamma \) and \( (n, d) \) as above. Denote by \( V^1 \) the set of vertices \( \{v \in V(\Gamma) : n_v = 1\} \). Let \( \Gamma_1 \) be the subgraph of \( \Gamma \) obtained from \( \Gamma \) by deleting the vertices \( V^1 \) and all those edges which have at least one endpoint in \( V^1 \). If each connected component of \( \Gamma_1 \) is a tree, then \( G(\Gamma, (n, d)) = 0 \).

The proof is similar to the proof of 1.19 (using the diagram 1.17), and it is left to the reader.

Now, let \( \Gamma \) be a cyclic graph, i.e. \( V(\Gamma) = \{v_1,v_2,\ldots,v_k\} \), \( E(\Gamma) = \{(v_1,v_2),(v_2,v_3),\ldots,(v_k,v_1)\} \).
1.21. Theorem. If \( \Gamma \) is a cyclic graph as above, and \( d = \text{g.c.d.}\{n_v : v \in \mathcal{V}(\Gamma)\} \), then \( \mathcal{G}(\Gamma, (n, d)) = \mathbb{Z}_d \).

Proof. Let \( \Gamma' \) be the subgraph of \( \Gamma \) with \( \mathcal{V}(\Gamma') = \mathcal{V}(\Gamma) \) and \( \mathcal{E}(\Gamma') = \mathcal{E}(\Gamma) \setminus \{(v_k, v_{k+1})\} \). Then in (1.17) \( \ker(p_G) = \mathbb{Z}_{d_k}[n_{v_k}, n_{v_{k+1}}] \), \( \ker(p_B) = \mathbb{Z}_{n_{v_k}} \times \mathbb{Z}_{n_{v_{k+1}}} \), and \( \mathcal{G}(\Gamma', n) = 0 \) (because \( \Gamma' \) is a tree). Hence \( \theta = \mathbb{Z}_{[n_{v_k}, n_{v_{k+1}}]} \rightarrow \mathcal{G}(\Gamma, n) \) is onto for any \( k \). Therefore, \( \mathcal{G}(\Gamma, n) \) is cyclic of order \( d' \) with \( d'|d \). Now, consider the map \( B(\Gamma, n) = \prod_k \mathbb{Z}_{n_{v_k}}^2 \rightarrow \mathbb{Z}_d \) given by \( (a_1, b_1), (a_2, b_2), \ldots \) \( \mapsto \sum a_i - \sum b_i \).

This is obviously onto. The subgroup \( \text{Is}o \) is generated by \( (\ldots, (0, 0), (1, 1), (0, 0), \ldots) \) and \( (\ldots, (0, 1), (1, 0), \ldots) \). Therefore, there is a well-defined map \( \mathcal{G}(\Gamma, n) \rightarrow \mathbb{Z}_d \), which is onto. Hence \( d = d' \). \( \diamond \)

The proofs of (1.19) and (1.21) combined give:

1.22. Corollary. Let \( \Gamma \) be a graph with \( c_\Gamma = 1 \) and fix the system of integers \( n = \{n_v\}_v \). Let \( \Gamma' \) be the (unique) minimal cyclic subgraph of \( \Gamma \), and set \( d := \text{g.c.d.}\{n_v : v \in \mathcal{V}(\Gamma')\} \). Then \( \mathcal{G}(\Gamma, n) = \mathbb{Z}_d \).

We end this sphere of thought with the following fact:

1.23. Lemma. If \( p : G \rightarrow \Gamma \) is a cyclic covering then \( c_G \geq c_\Gamma \).

Proof. If \( [G] \) and \( [\Gamma] \) denote the topological realizations of the corresponding graphs, then \( H_1([G], Q)^\vee = H_1([\Gamma], Q) \), hence the result follows. Cf. also with (3.11), and also with (2.14), (3.21) and (3.25). \( \diamond \)

1.24. The \textit{mod} \( N \)-construction. (This will be crucial in Section 3.)

Fix an arbitrary graph \( \Gamma \) and system of integers \( n = \{n_v\}_v \) as above. For any strict positive integer \( N \), we introduce a new set of integers \( (N, n) := \{(N, n_v)\}_v \) and \( d', \) where \( d' := (d, N/(N, [n_{v_1}, n_{v_2}])) \). In other words: if \( n_e = d_e \cdot [n_{v_1}, n_{v_2}] \), then \( (n_e, N) = d'_e \cdot [n_{v_1}, N, (n_{v_2}, N)] \).

We define a natural map:

\[ \text{mod}_N : \mathcal{G}(\Gamma, (n, d)) \rightarrow \mathcal{G}(\Gamma, ((N, n), d')) \]

as follows. Let \( G \) be a representative of an element of \( \mathcal{G}(\Gamma, (n, d)) \). Then \( \text{mod}_N(G) \) is the “orbit graph” of the induced action of \( N \mathbb{Z} \). More precisely, we introduce the equivalence relation \( \sim_N \) on \( \mathcal{V}(G) \), respectively on \( \mathcal{E}(G) \): \( v_1 \sim_N v_2 \) (resp. \( e_1 \sim_N e_2 \)) if there is an integer \( k \) such that \( a_G(kN, v_1) = v_2 \) (resp. \( a_E(kN, e_1) = e_2 \)). Then \( \text{mod}_N(G) = G/\sim_N \).

1.25. Variations on the theme of coverings. We can extend our set of coverings if we change the definition of the “standard block”. Actually, what is really important in the definition of a block \( B \) is summarized in the following two principles:

(a) \( B \) must be equivariant (i.e. it has a \( \mathbb{Z} \)-action);
(b) \( B \) must be rigid in the following sense: if \( a_E(h, e) = e \) for some \( h \in \mathbb{Z} \) and \( e \in \mathcal{E}(B) \), then all the edges and vertices of \( B \) are left invariant by the action of \( h \).

Otherwise, the block can be as complicated as we want.

(1) **First variation.** Assume that the vertices of all our graphs have two types: arrowhead vertices \( A \) and non-arrowhead vertices \( W \), i.e. \( \mathcal{V} = A \bigsqcup W \). Then in
the definition of the coverings \( p : G \to \Gamma \) we add the following axiom: \( \mathcal{A}(G) = p^{-1}(\mathcal{A}(\Gamma)) \).

(2) Second variation. Assume that our graphs have some decorations. Then for coverings \( p : G \to \Gamma \) we require additionally, that the decoration of \( G \) must be equivariant. (At this moment, we require no connection between the decorations of \( G \) and \( \Gamma \).)

(3) Third variation. The following construction is used extensively in Section 3: we change (in an equivariant way) each edge of \( G \) into a string. More precisely, we start with the following data:

(a) a graph \( \Gamma \) and a system \((n, d)\) as above;
(b) a covering \( p : G \to \Gamma \), as an element of \( \mathcal{G}(n, d) \);
(c) for each edge \( e \) of \( \Gamma \), we fix a string \( Str(e) \) (which may have some decorations):

\[
Str(e) : \quad \begin{array}{c}
- k_1 \\
- k_2 \\
\vdots \\
- k_s
\end{array}
\]

Denote the collection \( \{Str(e)\}_{e \in E(\Gamma)} \) of strings by \( Str \).

Then the new graph \( G(\text{Str}) \) is constructed as follows: we replace each edge \( e \in p^{-1}(e) \) (with endpoints \( \tilde{v}_1 \) and \( \tilde{v}_2 \)) of \( G \) with the decorated string \( \overline{Str(e)} \), as it is shown below:

\[
\begin{array}{cccc}
\tilde{v}_1 & \tilde{v}_2 \\
\text{replaced} & \quad \begin{array}{ccc}
- k_1 \\
- k_2 \\
\vdots \\
- k_s
\end{array} & \quad \begin{array}{ccc}
\tilde{v}_1 \\
\tilde{v}_2
\end{array}
\end{array}
\]

2. Graphs associated with analytic germs

Let \((X, x)\) be a normal surface singularity and fix the germ \( f : (X, x) \to (\mathbb{C}, 0)\) of an analytic function. In the first subsection we review the definition of the embedded resolution graph \( \Gamma(X, f) \) of \( f \), and we introduce our basic notations. For more details the reader is invited to consult the books of H. Laufer [19] and Eisenbud–Neumann [10], as well as the survey article of J. Lipman [22]. In the next subsection, we discuss Jung’s method of desingularization of normal surface singularities. Here we also recall the basic arithmetical properties of the resolution graph of Hirzebruch–Jung singularities. The third subsection deals with the topology of the link \( L_X \) of \((X, x)\) and of the pair \((X, f^{-1}(x))\). We discuss in details the representation \( \text{arg}_*(f) \) provided by the Milnor fibration associated with \( f \). This representation will be crucial in the next subsection, and in Section 3. In the last subsection we introduce the “universal cyclic \( \mathbb{Z} \)-covering” \( G(X, f) \) of the embedded resolution \( \Gamma(X, f) \). This covering graph coordinates all the resolution graphs of the cyclic coverings of \((X, x)\) branched along \( f^{-1}(0) \) (see the next section).

In the body of this section we present many examples in order to clarify the relationship between the objects \( \Gamma(X, f) \), \( \text{arg}_*(f) \), \( L_X \) and \( G(X, f) \). The verification of these examples sometimes is not absolutely trivial, but basically all of them are based on the main result (3.7). Even if that theorem is presented only in the next section, we prefer to give all these examples already here. The reader can skip the verification of them at the first reading.
The embedded resolution graph $\Gamma(X, f)$.

2.1. The embedded resolution. Let $(X, x)$ be a normal surface singularity and let $f : (X, x) \rightarrow (\mathbb{C}, 0)$ be the germ of an analytic function.

An embedded resolution $\phi : (Y, D) \rightarrow (U, f^{-1}(0))$ of $(f^{-1}(0), x) \subset (X, x)$ is characterized by the following properties. There is a sufficiently small neighborhood $U$ of $x$ in $X$, smooth analytic manifold $Y$, and an analytic proper map $\phi : Y \rightarrow U$ such that:

1) if $E = \phi^{-1}(x)$, then the restriction $\phi|_{Y\setminus E} : Y\setminus E \rightarrow U \setminus \{x\}$ is biholomorphic, and $Y\setminus E$ is dense in $Y$;

2) $D = \phi^{-1}(f^{-1}(0))$ is a divisor with only normal crossing singularities, i.e. at any point $P$ of $E$, there are local coordinates $(u, v)$ in some small neighbourhood of $P$, such that in these coordinates $f \circ \phi = u^a v^b$ for some non-negative integers $a$ and $b$.

If such an embedded resolution $\phi$ is fixed, then $E = \phi^{-1}(x)$ is called the exceptional divisor associated with $\phi$. Let $E = \cup_{w \in W} E_w$ be its decomposition in irreducible divisors. The closure $S$ of $\phi^{-1}(f^{-1}(0))\setminus \{0\}$ is called the strict transform of $f^{-1}(0)$. Let $\cup_{a \in A} S_a$ be its irreducible decomposition. Obviously, $D = E \cup S$.

In this section, for simplicity, we will assume that $W \neq \emptyset$, any two irreducible components of $E$ have at most one intersection point, and no irreducible exceptional divisor has a self-intersection. This can always be realized by some additional blow-ups.

2.2. The embedded resolution graph. We construct the embedded resolution graph $\Gamma(X, f)$ of the pair $(X, f)$, associated with a fixed resolution $\phi$, as follows. Its vertices $V = W \bigsqcup A$ consist of the nonarrowhead vertices $W$ corresponding to the irreducible exceptional divisors, and arrowhead vertices $A$ corresponding to the irreducible divisors of the strict transform $S$. If two irreducible divisors corresponding to $v_1, v_2 \in V$ have an intersection point then $(v_1, v_2) = (v_2, v_1)$ is an edge of $\Gamma(X, f)$. The set of edges is denoted by $E$.

For any $w \in W$, we denote by $V_w$ the set of vertices $v \in V$ adjacent to $w$. Its cardinality $\#V_w$ is called the degree $\delta_w$ of $w$. In Section 1 we introduced for any graph the set $E_w$ of edges adjacent to $w$. Since, by our assumption, any two vertices are connected by at most one edge, one has the identification $E_w = V_w$.

The graph $\Gamma(X, f)$ is decorated as follows. Any vertex $w \in W$ is decorated with three numbers. The first is the self-intersection $e_w := E_w \cdot E_w$. Equivalently, $e_w$ is the Euler-number of the normal bundle of $E_w$ in $Y$. The second is the genus $g_w$ of $E_w$. The third decoration is given by the multiplicity of $f$. More precisely, for any $v \in V$, let $m_v$ be the vanishing order of $f \circ \phi$ along the irreducible divisor corresponding to $v$. For example, if $f$ defines an isolated singularity, then for any $a \in A$ one has $m_a = 1$.

In all our graphs, we put the multiplicities in parentheses (e.g.: $(3)$) and the genera in brackets (e.g.: $[3]$). In order to simplify the graphs, if $g_w = 0$ for some $w$, then we omit $[0]$ from the graph.

2.3. First properties of the graph.

(1) Notice that the combinatorics of the graph and the self-intersection numbers codify completely the intersection matrix $(E_w \cdot E_v)_{(w, v) \in W \times W}$ of the irreducible
components of $E$. Moreover, this matrix is negative definite, see e.g. [25] page 230; [19] page 49, or [11].

(2) The following compatibility property holds between self–intersections and multiplicities. For any $w \in \mathcal{W}$ one has:

$$e_w m_w + \sum_{v \in \mathcal{V}_w} m_v = 0.$$  

Obviously, $m_v > 0$ for any $v \in \mathcal{V}$, hence the set of multiplicities determine the self–intersection numbers completely. The advantage of this fact is the following: a multiplicity can always be determined by a local computation, on the other hand the Euler–number $e_w$ is a global characteristic class.

Similarly, since the intersection matrix $(E_w \cdot E_v)_{(w,v) \in \mathcal{W} \times \mathcal{W}}$ is negative definite, these relations determine the multiplicities $\{m_w\}_{w \in \mathcal{W}}$ in terms of the self–intersection numbers $\{e_w\}_{w \in \mathcal{W}}$ and the multiplicities $\{m_a\}_{a \in \mathcal{A}}$.

\textbf{2.4. – The resolution of $(X, x)$}. We say that $\phi : Y \to U$ is a resolution of $(X, x)$ if $Y$ is a smooth analytic manifold, $U$ a neighbourhood of $x$ in $X$, $\phi$ is a proper analytic map, such that $Y \setminus E$ (where $E = \phi^{-1}(x)$) is dense in $Y$ and the restriction $\phi|_{Y \setminus E} : Y \setminus E \to U \setminus \{x\}$ is biholomorphic.

The topology of the resolution and the combinatorics of the irreducible exceptional divisors $\bigcup_w E_w$ can be codified in the graph $\Gamma(X)$, which is called the dual resolution graph of $(X, x)$ associated with $\phi$. If the divisor $E$ is not a normal crossing divisor, then this codification can be slightly complicated, so in the sequel we will assume that the irreducible components of $E$ are smooth and intersect each other transversally, the irreducible components have no self–intersections, and there is no intersection point which is contained in more than two components. In this case, similarly as in the situation of the embedded resolution, the vertices of the dual graph correspond to the irreducible components of $E$, the edges to the intersections of these components, and each vertex $w$ carry two decorations: the genus $g_w$ of $E_w$, and the self–intersection $E_w^2$.

Actually, one can obtain a possible graph $\Gamma(X)$ from any $\Gamma(X, f)$ by deleting all the arrows and multiplicities of the graph $\Gamma(X, f)$.

The graphs $\Gamma(X, f)$ and $\Gamma(X)$ are connected (see [19], or Zariski’s Main Theorem, e.g. in [12]).

\textbf{2.5. Example. – Plane curve singularities}. (see e.g. [7]) Assume that $(X, x)$ is smooth. Then the singular germ $(f^{-1}(x), x) \subset (X, x)$ can be resolved only by quadratic modifications. In this case, the graph $\Gamma(X, f)$ is a tree, and $g_w = 0$ for any $w \in \mathcal{W}$.

\textbf{2.6. Example. – The normalization of an arbitrary surface singularity}. If $(X, x)$ is a surface singularity, but it is not normal, then there is a canonical way to construct its normalization $\tilde{X}$ (see e.g. [19]). If $(X, x)$ has $k$ local irreducible components, then its normalization will split into $k$ normal singular space–germs. The corresponding resolution graph of $(\tilde{X}, \{x_1, \ldots, x_k\})$ will have $k$ connected components, each corresponding exactly to a resolution graph of the irreducible components $(\tilde{X}, x_i)$. 

2.7. Example. – Cyclic coverings. Start with a normal surface singularity \((X, x)\) and a germ \(f : (X, x) \to (\mathbb{C}, 0)\). Consider the covering \(b : (\mathbb{C}, 0) \to (\mathbb{C}, 0)\) given by \(z \mapsto z^N\), and construct the fiber product:

\[(X, x) \prod_{f, b} (\mathbb{C}, 0) = \{(x', z) \in (X \times \mathbb{C}, x \times 0) : f(x') = z^N\}.

By definition, \(X_{f,N}\) is the normalization of \((X, x) \prod_{f, b} (\mathbb{C}, 0)\). There is a natural ramified covering \(X_{f,N} \to X\) branched along \(f^{-1}(0)\).

2.8. Example. – Hirzebruch–Jung singularities. (See [14, 19, 5]). For a normal surface singularity, the following conditions are equivalent. If \((X, x)\) satisfies (one of) them, then it is called Hirzebruch–Jung singularity.

a) The resolution graph \(\Gamma(X)\) is a string, and \(g_w = 0\) for any \(w \in W\). (If the graph is minimal then \(e_w \leq -2\) for any \(w\).)

b) There is a finite proper map \(\pi : (X, x) \to (\mathbb{C}^2, 0)\) such that the reduced discriminant locus of \(\pi\), in some local coordinates \((u, v)\) of \((\mathbb{C}^2, 0)\), is \(\{uv = 0\}\).

c) \((X, x)\) is isomorphic with exactly one of the “model spaces” \(\{A_{n,q}\}_{n,q}\), where \(A_{n,q}\) is the normalization of \(\{xy^n-q + z^n = 0\}, 0\), with \(0 < q < n, (n, q) = 1\).

If there is a map \(\pi\) as in (b) with smooth reduced discriminant locus, then \((X, x)\) is automatically smooth.

Jung’s method. Hirzebruch–Jung singularities.

2.9. Jung’s method [17] gives not only a qualitative proof of the existence of the resolution \(\phi : (\mathcal{Y}, E) \to (X, x)\) of a normal surface singularity \((X, x)\) (see, e.g. [19] Theorem 2.1; or [5] Theorem 6.1), but also a rather clear recipe how this resolution can be constructed in concrete cases (see, e.g. [19] chapter III, or [22]).

The Jungian strategy (presented for the case of a normal surface singularity \((X, x)\)) can be summarized in the following diagram:

\[
\begin{array}{ccc}
X^{res} & \xrightarrow{\rho} & \hat{X}' & \xrightarrow{\phi'_{\Delta}} & (X, x) \\
\downarrow \pi' & & \downarrow \pi & & \downarrow \pi \\
(\mathcal{Y}, D) & \xrightarrow{\phi_{\Delta}} & (\mathbb{C}^2, 0) \supset (\Delta, 0)
\end{array}
\]

where:

a) \(\pi : (X, x) \to (\mathbb{C}^2, 0)\) is a proper finite map with (reduced) discriminant locus \((\Delta, 0) \subset (\mathbb{C}^2, 0)\).

b) \(\phi_{\Delta}\) is an embedded resolution of \((\Delta, 0) \subset (\mathbb{C}^2, 0)\). In particular, \(D = \phi_{\Delta}^{-1}(\Delta)\) is a divisor with normal crossing singularities.

c) \(\pi' : X' \to \mathcal{Y}\) is the pullback of \(\pi\) via \(\phi_{\Delta}\), and \(\hat{X}'\) is the normalization of \(X'\). Then \(\hat{X}'\) has only normal singularities, and the discriminant of the projection \(\hat{\pi}'\) has normal crossing singularities only. This property characterizes exactly the Hirzebruch–Jung singularities (cf. 2.8).

d) \(X^{res} \to \hat{X}'\) is the resolution of the (Hirzebruch–Jung) singularities of \(\hat{X}'\).
2.10. Example. In general, the computation of the discriminant locus $\Delta$ of some projection $\pi$, and the whole process, can be rather complicated. But in some cases, the above recipe is really nice. For example, if $(X, x) = (\{ f(x, y, z) + z^N = 0 \}, 0) \subset (\mathbb{C}^3, 0)$, and $\pi$ is induced by $(x, y, z) \mapsto (x, y)$, then $(\Delta, 0) = (\{ f = 0 \}, 0) \subset (\mathbb{C}^2, 0)$. It turns out, that the dual resolution graph of $(X, x)$ can be recovered from the embedded resolution graph of $f$ and the integer $N$ (see 3.12).

The above strategy shows that, in order to resolve $(X, x)$, we have to know two things: the embedded resolution of plane curve singularities, and the resolution of Hirzebruch–Jung singularities.

2.11. – Hirzebruch–Jung singularities. [14, 19, 5] (cf. also 2.8).

From our point of view, (and also from the point of view of the strategy 2.9), it is more convenient to consider a bigger class of “models” instead of $\{ A_{n,q} \}$. For any three strictly positive integers $a, b$ and $N$, with $\text{g.c.d.}(a, b, N) = 1$, we define $(X, x) = (X(a, b, N), x)$ as the unique singularity lying over the origin in the normalization of $(\{ \alpha a + \beta b + \gamma N = 0 \}, 0)$. Let the germ $\gamma : (X(a, b, N), x) \rightarrow (\mathbb{C}, 0)$ be induced by $(\alpha, \beta, \gamma) \mapsto \gamma$. In the sequel, we give the embedded resolution graph $\Gamma(X, \gamma)$ of the germ $\gamma$. Obviously, if we delete the arrows and multiplicities of $\Gamma(X, \gamma)$ we obtain the resolution graph $\Gamma(X)$ of $(X(a, b, N), x)$.

First, consider the unique $0 \leq \lambda < N/(a, N)$ and $m_1 \in \mathbb{N}$ with:

$$b + \lambda \cdot \frac{a}{(a, N)} = m_1 \cdot \frac{N}{(a, N)}.$$

If $\lambda \neq 0$, then consider the continuous fraction:

$$\frac{N/(a, N)}{\lambda} = k_1 - \frac{1}{k_2 - \frac{1}{\ddots - \frac{1}{k_s}}}, \quad k_1, \ldots, k_s \geq 2.$$

Then the following string, denoted by $Str(a, b; N)$, is the embedded resolution graph $\Gamma(X, \gamma)$ of $\gamma$:

$$\begin{array}{cccc}
(a, N) & -k_1 & -k_2 & -k_s \\
(m_1) & (m_2) & (m_s) & (b, N_1)
\end{array}$$

The arrow on the left (resp. right) hand side codifies the strict transform of $\{ \alpha = 0 \}$ (resp. of $\{ \beta = 0 \}$). All vertices have genus $g_w = 0$, i.e. they represent rational irreducible exceptional divisors. The first vertex has multiplicity $m_1$ given by the above congruence. Hence $m_2, \ldots, m_s$ can easily be computed using (2.3), namely:

$$-k_1 m_1 + \frac{a}{(a, N)} + m_2 = 0, \quad \text{and} \quad -k_i m_i + m_{i-1} + m_{i+1} = 0 \text{ for } i \geq 2.$$

This resolution resolves also the germ $\alpha$ (induced by the projection $(\alpha, \beta, \gamma) \mapsto \alpha$). The multiplicities of $\alpha$ along the (same!) divisors (exceptional divisors and strict transforms of $\{ \alpha = 0 \}$ and $\{ \beta = 0 \}$) are given in the next graph. (This is, in fact, the graph $\Gamma(X, \alpha)$, if we delete the arrow with zero multiplicity):
The other multiplicities can again be computed by (2.3). We emphasize again: the arrows codify the same strict transforms as the arrows of $\Gamma(X, \gamma)$.

The multiplicity of $\beta$ along the corresponding irreducible divisors can be determined symmetrically:

\[
\begin{array}{ccccccc}
(0) & -k_1 & -k_2 & \cdots & -k_s & (\lambda) & (b, N) \\
((a, N)) & \lambda & \lambda & \cdots & \lambda & (b, N) \\
\end{array}
\]

where $0 \leq \tilde{\lambda} < \frac{N}{(N, b)}$ and

\[
a + \tilde{\lambda} \cdot \frac{b}{(b, N)} = m_s \cdot \frac{N}{(b, N)}.
\]

Obviously, the embedded resolution graph $\Gamma(X, \alpha^i \beta^j \gamma^k)$ of the function $\alpha^i \beta^j \gamma^k$ defined on $X$ can be deduced easily from the above resolution graphs. It has the same shape, the same self–intersections and genera, and the multiplicity $m_v$ (for any vertex $v$) satisfies:

\[
m_v(\alpha^i \beta^j \gamma^k) = i \cdot m_v(\alpha) + j \cdot m_v(\beta) + k \cdot m_v(\gamma).
\]

**Notation.** The decorated string $\Gamma(X, \alpha^i \beta^j \gamma^k)$ will sometimes be denoted by:

\[
\text{Str}(a, b; N | i, j; k).
\]

Form the point of view of the classification theorem (2.8c), $X(a, b, N)$ is an $A_{n,q}$–singularity, where $n = \frac{N}{(a, N)(b, N)}$ and $q = \frac{\lambda}{(b, N)}$ (cf. e.g. [5], page 83-84).

If $\lambda = 0$, then the string has no vertices, in particular $(X(a, b, N), x)$ is smooth. Moreover, in this case, the zero set of $\gamma$ (on $X$) has only a normal crossing singularity: in some local coordinates $(u, v)$ of $(X, x)$, it can be represented as $\gamma = u^{a/(a, N)} v^{b/(b, N)}$. (In this case the above string becomes a double arrow, without any non–arrowhead vertices.)

**The topology of the link of $f$.**

2.12. – The link of $(X, x)$. Let $(X, x)$ be a normal surface singularity, and fix an embedding $(X, x) \subset (\mathbb{C}^N, 0)$ for some $N$. Then, for sufficiently small $\epsilon_0 > 0$, all the spheres $S_\epsilon = \{ z \in \mathbb{C}^N : ||z|| = \epsilon \}$ $(0 < \epsilon \leq \epsilon_0)$ intersect $(X, x)$ transversally (see, e.g. [24, 23]), and the differentiable manifold $S_\epsilon \cap X$ does not depend on the choice of $\epsilon$ and of the embedding $(X, x) \subset \mathbb{C}^N, 0)$. It inherits a natural orientation and it is always connected. It is called the link of $(X, x)$, and denoted by $L_X$

From a topological point of view, $L_X$ characterizes $(X, x)$ completely. If $B_\epsilon$ denotes the ball $\{ z \in \mathbb{C}^N : ||z|| \leq \epsilon \}$, then for $\epsilon$ sufficiently small ($B_\epsilon \cap X, x$ is homeomorphic to $(\text{Cone}(L_X), \text{vertex of the cone})$. We will write $U = B_\epsilon \cap X$ for a small $\epsilon$. For such a $U$, consider an embedded resolution $\phi : \mathcal{Y} \to U$. Then the inclusion $\phi^{-1}(0) = E \hookrightarrow \mathcal{Y}$ admits a strong deformation retract $r : \mathcal{Y} \to E$, and $\mathcal{Y}$ is a manifold with smooth boundary. Moreover, the restriction of $\phi$ to $\partial \mathcal{Y}$ identifies
∂V with \( L_X = \partial U \). This shows that \( L_X \) is the \textit{plumbed manifold} \( M(L_X) \) associated with the graph \( \Gamma(X) \) (for details, see [31]), i.e. \( \Gamma(X) \) determines completely the 3–manifold \( L_X \). The converse is also true: W. Neumann in [31] proved that the topology of the (minimal) resolution of the singularity \( (X, x) \) is determined by the oriented homeomorphism type of the link \( L_X \).

\[ H_1(L_X, \mathbb{Z}) \approx \text{coker } A \oplus \mathbb{Z}^{2g + c_r} \]

In particular, \( L_X \) is an integer (resp. rational) homology sphere if and only if \( g = c_r = 0 \) and \( \det A = \pm 1 \) (resp. \( g = c_1 = 0 \); i.e. \( \Gamma \) is a tree with \( g_w = 0 \) for all \( w \)).

\[ \text{The homology of } L_X. \] Consider the intersection matrix \( A \) given by \( A_{v, w} = E_v \cdot E_w \). This defines a bilinear form \( (\mathbb{Z}^W)^{\otimes 2} \to \mathbb{Z} \), or equivalently a \( \mathbb{Z} \)-linear map \( A : \mathbb{Z}^W \to (\mathbb{Z}^W)^* \) (where for a \( \mathbb{Z} \)-module \( M \), \( M^* \) denotes its dual \( \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \)). Since \( A \) is non–degenerate, \( \text{coker } A \) is a torsion group with \( |\text{coker } A| = |\det A| \). Then the following holds:

\[ \text{The links of germs } f : (X, x) \to (\mathbb{C}, 0). \] By similar arguments (and notations) as above, for sufficiently small \( \epsilon \) the intersection \( S_\epsilon \cap f^{-1}(0) \subset S_\epsilon \cap X = L_X \) defines a 1–dimensional compact (in general non–connected) orientable submanifold \( L_f \) of \( L_X \), called the link of \( f \). The link–components (=connected components of \( L_f \)) are indexed by \( \mathcal{A} \) in \( \Gamma(X, f) \). Similarly as above, one can recover the link \( L_f \subset L_X \) from the graph \( \Gamma(X, f) \) by a plumbing construction.

\[ \text{The homology group } H_1(L_X \setminus L_f, \mathbb{Z}). \] Let \( \mathbb{Z}^V \) be the free abelian group generated by \( \{[v]\}_{v \in V} \) (recall \( V = W \coprod \mathcal{A} \)). Define the group \( H_1 \) as the quotient of \( \mathbb{Z}^V \) factorized by the subgroup generated by:

\[ e_w[w] + \sum_{v \in V_w} [v] \quad (\text{for all } w \in W). \]

Let \( i \) be the composed map \( \mathbb{Z}^A \hookrightarrow \mathbb{Z}^V \to H_1 \). Then one has the following exact sequence:

\[ 0 \to \mathbb{Z}^A \xrightarrow{i} H_1 \to \text{coker } A \to 0. \]

\[ \text{Proposition. } ([35, 15, 31]) \] \( \text{There is a natural exact sequence:} \)

\[ 0 \to H_1 \xrightarrow{j} H_1(L_X \setminus L_f, \mathbb{Z}) \xrightarrow{\delta} H_1(E, \mathbb{Z}) \to 0. \]

Geometrically, the map \( j \) can be described as follows. Identify \( L_X \) with \( \partial V \) and \( L_f \) with the intersection \( \cup_w S_a \cap L_X \), of \( L_X \) with the strict transform \( S \). For each \( v \in V \), define \( M_v \subset L_X \setminus L_f \) as a naturally oriented circle in the transversal slice of the corresponding irreducible divisor of \( D = (\cup_w E_w) \cup (\cup_a S_a) \). Equivalently, for \( w \in W \) we can take \( M_w \) as a generic fiber of \( T_w \to B_w \), and for \( a \in \mathcal{A} \), \( M_a \) is a topological standard meridian (see e.g. [10]) of \( S_a \cap L_X \subset L_X \). Then \( j([v]) = [M_v] \) for any \( v \in V \).

If \( S \) is the strict transform, then the strong deformation retract \( r : \mathcal{Y} \to E \) can be chosen such that it preserves \( S \), hence induces a map \( r' : \mathcal{Y} \setminus D \to E \setminus S \). Then \( q \) is the composed map:
\[ H_1(L_X \setminus L_f, \mathbb{Z}) \overset{\delta^{-1}}{\to} H_1(\mathbb{V} \setminus D, \mathbb{Z}) \overset{\gamma}{\to} H_1(E \setminus S, \mathbb{Z}) \to H_1(E, \mathbb{Z}). \]

2.19. – The Milnor fibration. Fix a map \( f : (X, x) \to (\mathbb{C}, 0) \). Then \( \arg = f/|f| : L_X \setminus L_f \to S^1 \) is a \( \mathcal{C}^\infty \) fibration ([24, 23]). This induces \( \pi_1(\arg) : \pi_1(L_X \setminus L_f) \to \mathbb{Z} \) at the fundamental group level, and \( \arg_* : H_1(L_X \setminus L_f, \mathbb{Z}) \to \mathbb{Z} \) at homology level. Obviously, if \( ab : \pi_1 \to \pi_1/[\pi_1, \pi_1] = H_1 \) is the abelianization map, then \( \pi_1(\arg) = ab \circ \arg_* \).

The connection with the exact sequence (2.18) is the following. Let \( m : \mathbb{Z} \mathcal{V} \to \mathbb{Z} \) be defined by \( m([v]) = m_v \) (where \( m_v \) is the multiplicity of \( f \) along \( E_v \) as in 2.2). Then by (2.3), this induces a well-defined map \( m : H_f \to \mathbb{Z} \). Then \( \arg_* \circ j = m \) (cf. [10]):

\[ \begin{array}{ccc}
0 & \to & H_f \overset{j}{\to} H_1(L_X \setminus L_f, \mathbb{Z}) \overset{\gamma}{\to} H_1(E, \mathbb{Z}) \to 0.
\end{array} \]

\( \xymatrix{ & \mathbb{Z} \\
m \ar[ur] & & \arg_* \ar[u]}
\]

In particular, \( \arg_* \) contains all the information about the multiplicities, but as we will see later, from the multiplicities we cannot recover the representation \( \arg_* \) (cf. Examples 2.22–2.26). On the other hand, in the study of cyclic coverings of \((X, x)\) branched along \( f^{-1}(0) \), the representation \( \arg_* \) plays a crucial role. In 2.27, the additional information about \( \arg_* \), which is not contained in \( \Gamma(X, f) \) will be codified in a \( \mathbb{Z} \)-covering graph of \( \Gamma(X, f) \). Moreover, one can verify the following:

2.20. Proposition. The fibration \( \arg : L_X \setminus L_f \to S^1 \) is completely determined (up to isotopy) by the induced representation \( \arg_* : H_1(L_X \setminus L_f, \mathbb{Z}) \to \mathbb{Z} \). Moreover, if \( \mathbb{Z}_d := \text{coker}(\arg_* : H_1(L_X \setminus L_f, \mathbb{Z}) \to \mathbb{Z}) \), then the (Milnor) fiber of \( \arg \) has \( d \) connected components which are cyclically permuted by the monodromy.

In general, one has the following divisibility conditions: \( d | \text{g.c.d.}\{m_v : v \in \mathcal{V}\} \) (g.c.d.\( \{m_a : a \in \mathcal{A}\} \), but it is possible that \( d \neq \text{g.c.d.}\{m_v : v \in \mathcal{V}\} \). This means that \( d \) cannot be determined from \( \mathbb{m} \).

Since \( \mathbb{m} \), in general, does not determine the representation \( \arg_* \) (cf. 2.19), the graph \( \Gamma(X, f) \) alone does not determine the Milnor fibration associated with \( f \).

2.21. – The particular case when \( L_X \) is a rational homology sphere. Set \( f : (X, x) \to (\mathbb{C}, 0) \) as above, and assume that \( L_X \) is a rational homology sphere. This is equivalent to the vanishing of \( H_1(E, \mathbb{Z}) \). Then by (2.18 – 2.19), \( \pi_1(\arg) : \pi_1(L_X \setminus L_f) \to \mathbb{Z} \) is completely determined by \( \mathbb{m} : H_f \to \mathbb{Z} \), and \( \mathbb{m} \) is determined by \( \{m_a\}_{a \in \mathcal{A}} \) via (2.3). Hence, the set \( \{m_a\}_{a \in \mathcal{A}} \) determines completely the Milnor fibration up to an isotopy. Moreover, if \( d = \text{g.c.d.}\{m_v : v \in \mathcal{V}\} \), then the fiber has \( d \) connected components, and they are cyclically permuted by the monodromy.

The next examples show that these properties are not true if \( L_X \) is not a rational homology sphere, i.e. even with the same multilink, different representations \( H_1(L_X \setminus L_f, \mathbb{Z}) \to \mathbb{Z} \) do occur.
2.22. Example. Set \((X, x) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (\mathbb{C}^3, 0)\) and take \(f_1(x, y, z) = z^2\) and \(f_2(x, y, z) = z^2 - y\). Then \(\Gamma(X, f_1) = \Gamma(X, f_2)\) is:

\[
\begin{array}{c}
\text{[3]} \\
-1 \\
\text{(2)}
\end{array}
\]

Since \(\text{coker}(m) = \mathbb{Z}_2\), in both cases \(\text{coker}(\text{arg}^*)\) is a factor group of \(\mathbb{Z}_2\) (cf. 2.19). We will show that in the first case \(\text{coker}(\text{arg}^*) = \mathbb{Z}_2\) and in the second case \(\text{arg}^*\) is onto. Indeed, the Milnor fibration of \(z^2\) is the pullback by \(z \mapsto z^2\) of the Milnor fibration of \(z\), hence \(\text{coker}(\text{arg}^*) = \mathbb{Z}_2\). In order to prove the second statement, it is enough to verify that the double covering \(\{x^2 + y^7 - z^{14} = w^2 + y - z^2 = 0\} \subset \mathbb{C}^4\) is irreducible (notice that our equations are quasi–homogeneous, so we can replace a small ball centered at the origin with the whole affine space). But this is true if its intersection with \(y = 1\), i.e. \(C := \{x^2 = z^{14} - 1; w^2 = z^2 - 1\} \subset \mathbb{C}^3\), is irreducible.

The covering \(C \rightarrow \mathbb{C} ((x, w, z) \mapsto z)\) is a \(\mathbb{Z}_2 \times \mathbb{Z}_2\) covering. The monodromy around \(\pm 1\) is \((-1, -1)\), and around any \(\alpha\) with \(\alpha^{14} = 1\) and \(\alpha^2 \neq 1\) is \((-1, +1)\) (here \(\mathbb{Z}_2 = \{+1, -1\}\)). Hence the global monodromy group is the whole group \(\mathbb{Z}_2 \times \mathbb{Z}_2\). In particular \(C\) is irreducible.

Notice also that the multiplicities of \(f_2\) are all even numbers, but there is no germ \(g : (X, x) \rightarrow (\mathbb{C}, 0)\) with \(f_2 = g^2\). (Actually, there is no homotopy between \(f_2\) and \(g^2\) for any \(g\).)

2.23. Example. Set \((X, x) = (\{z^2 + y(x^{12} - y^{18}) = 0\}, 0)\) and \(f_1 = x^2\) and \(f_2 = x^2 - y^3\). Then \(\Gamma(X, f_1) = \Gamma(X, f_2)\) is the graph:

\[
\begin{array}{c}
[3] \\
\text{(2)} \\
-1 \\
\text{(6)} \\
-2 \\
\text{(4)} \\
-2 \\
\text{(2)}
\end{array}
\]

By a similar argument as in (2.22) one has that \(\text{arg}^*(f_1)\) has cokernel \(\mathbb{Z}_2\), and \(\text{arg}^*(f_2)\) is onto.

2.24. Example. Set \((X, x) = (\{z^2 + (x^2 - y^3)(x^3 - y^2) = 0\}, 0)\), \(f_1 = x^2\) and \(f_2 = x^2 - y^3\). Then \(\Gamma(X, f_1) = \Gamma(X, f_2)\) is the following graph:

\[
\begin{array}{c}
\text{(2)} \\
-1 \\
\text{(6)} \\
\text{(2)} \\
-4
\end{array}
\]

Then again: \(\text{arg}^*(f_1)\) has cokernel \(\mathbb{Z}_2\), and \(\text{arg}^*(f_2)\) is onto.

Notice that in the above examples, \((X, x) = (\{z^2 + h(x, y) = 0\}, 0)\), and \(f_2\) divides \(h\) but it is not equal to \(h\). For all such cases the monodromy argument
given in (2.22) is valid. (So the interested reader can construct many—many similar examples, with even more additional properties.) But all these examples define non–isolated singularities. In order to construct examples of germs which define isolated singularities, we will use the well–known construction of series of singularities. Namely, assume that \( f_1 \) and \( f_2 \) have the same graph but have different representations \( \text{arg}_* \), and their zero sets have non–isolated singularities (e.g. they are constructed by the above method). Next, we find a germ \( g \) such that the zero set of \( f_i \) and \( g \) have no common components (for \( i = 1, 2 \)). Then, for a sufficiently large \( k \) the germs \( f_1 + g^k \) and \( f_2 + g^k \) define isolated singularities whose embedded resolution graphs are identical, but the representations \( \text{arg}_* \) are different.

2.25. Example. Set \( (X, x) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (\mathbb{C}^3, 0) \) and take \( f_1(x, y, z) = z^2 \) and \( f_2(x, y, z) = z^2 - y \) as in (2.22). Let \( P \) be the intersection point of the strict transform \( S_0 \) of \( \{f_i = 0\} \) with the exceptional divisor \( E \). Then, in some local coordinate system \((u, v)\) of \( P, \{u = 0\}\) represents \( E \) (in a neighborhood of \( P \)), \( \{v = 0\}\) represents \( S_0 \), and \( f_i = u^2v^2 \). Consider \( g = y \). Since \( y \) in the neighborhood of \( P \) can be represented as \( y = u^2 \) (modulo a local invertible germ), \( f_1 + g^k \) near \( P \) has the form \( u^2v^2 + u^{2k} \). For example, if \( k = 2 \), then one needs one more blowing up in order to resolve \( f_1 + g^k \).

Therefore, \( \Gamma(X, z^2 + y^k) = \Gamma(X, z^2 - y + y^k) \) for any \( k \geq 2 \); and for \( k = 2 \), the graphs have the following form:

\[
\begin{array}{c}
\text{[3]} \\
-2 \\
(2) \\
(4) \\
-1 \\
(1)
\end{array}
\]

(1)

Notice that now \( m \) is onto, hence for both \( i = 1, 2 \), \( \text{arg}_*(f_i) \) is onto. Nevertheless, \( \text{arg}_*(f_1) \neq \text{arg}_*(f_2) \) because their restrictions to a subgroup of \( H_1(L_X \setminus L_f) \) are different.

Indeed, let \( \mathcal{Y}' \) be a tubular neighborhood of the irreducible exceptional divisor \( E \) of genus 3. This curve \( E \) can be contracted by Grauert theorem [11]. Then \( E \) contracted in \( \mathcal{Y}' \) gives birth to a singularity \((X', x)\) with the same resolution graph as the surface singularity in (2.22). Moreover, the germs \( f_i \) \((i = 1, 2)\) induce germs \( f_i': (X', x) \to (\mathbb{C}, 0) \), such that they have the same embedded resolution graphs as the germs in (2.22). In particular, \( \text{coker}(\text{arg}_*(f_1')) = \mathbb{Z}_2 \) and \( \text{arg}_*(f_2') \) is onto. But \( \text{arg}_*(f_1') \) is the composed map

\[
H_1(\mathcal{Y}' \setminus D, \mathbb{Z}) \to H_1(\mathcal{Y} \setminus D, \mathbb{Z}) \xrightarrow{\text{arg}_*(f_i)} \mathbb{Z},
\]

hence \( \text{arg}_*(f_1) \neq \text{arg}_*(f_2) \).

2.26. Example. Set \( (X, x) = (\{z^2 + (x^2 - y^3)(x^3 - y^2) = 0\}, 0) \) and \( f_1 = x^2 + y^k \) and \( f_2 = x^2 - y^3 + y^k \), where \( k \geq 4 \). Then by similar argument as before, \( \Gamma(X, f_1) = \Gamma(X, f_2) \). This graph for \( k = 4 \) is:
Let $E_i$ ($i = 1, 2, 3$) be the irreducible exceptional divisors with self intersection numbers $-2, -4, -4$ respectively, and $\mathcal{Y}'$ be the union of small tubular neighborhoods of them. Then collapsing the curve $\cup_i E_i$ in $\mathcal{Y}'$ creates a singularity $(X', x)$ with the same graph as in (2.24). Repeating the arguments of (2.25) (but using (2.24) instead of (2.22)) one has $\text{arg}_*(f_1) \neq \text{arg}_*(f_2)$.

**The universal cyclic covering of $\Gamma(X, f)$**.

**2.27.** The covering $p : G(X, f) \to \Gamma(X, f)$. As we already noticed, the embedded resolution graph $\Gamma(X, f)$ does not codify all the information about $\text{arg}_*$. On the other hand, this information is needed in the study of cyclic coverings of $(X, x)$. In this section, we define a cyclic covering of $\Gamma(X, f)$ (cf. Section 1) which will control the behavior of the resolution graphs of all the cyclic coverings $\{X_{f, N}\}_N$ (cf. 2.7). The graph $G(X, f)$ was already considered in the literature by Ph. Du Bois and F. Michel from a completely different point of view, see [8].

Let $(X, x)$ be a normal surface singularity and $f : (X, x) \to (\mathbb{C}, 0)$ the germ of an analytic function. Fix an embedded resolution $\phi : (\mathcal{Y}, D) \to (X, f^{-1}(0))$ of $(f^{-1}(0), x) \subset (X, x)$ (as in 2.1) with embedded resolution graph $\Gamma(X, f)$.

Let $T(E_w)$ ($w \in W$) be a small tubular neighborhood of the irreducible divisor $E_w$. By our assumption that any two irreducible exceptional divisor has at most one intersection point (see the first subsection of this chapter), for any $e = (v, w) \in E$, the intersection $T(E_w) \cap T(E_v)$ ($(v, w) \in W \times W$) is homeomorphic to a multidisc $D \times D$. This will be denoted by $T_e$. If $T(S_a)$ ($a \in A$) is a small tubular neighborhood of the irreducible component $S_a$ of the strict transform $S$ (cf. 2.1), and $a$ is adjacent to $w_a \in W$, then corresponding to the edge $e = (a, w_a)$ we introduce the multidisc $T_e = T(S_a) \cap T(E_{w_a})$. Set $T = (\cup_w T(E_w)) \cup (\cup_a T(S_a))$.

Now, consider the smooth nearby fiber $f^{-1}(\delta) \subset X$ lifted via $\phi$. For sufficiently small $\delta > 0$, the fiber $F := (f \circ \phi)^{-1}(\delta) \subset \mathcal{Y}$ is in $T$. Set $F_w = F \cap T(E_w)$ for any $w \in W$, $F_a = F \cap T(S_a)$ for any $a \in A$, and $F_e = F \cap T_e$ for any $e \in E$.

It is not very difficult to construct a geometric monodromy $h_g : F \to F$ of the fibration $f^{-1}(S^1_{\delta}) \to S^1_{\delta}$ (where $S^1_{\delta} = \{z \in \mathbb{C} : |z| = \delta\}$ and $\delta$ is sufficiently small) which preserves the subspaces $\{F_v\}_{v \in V}$ and $\{F_e\}_{e \in E}$. Then the connected components of $F_v$ (resp. of $F_e$) are cyclically permuted by the geometric monodromy. Let $n_v$ (resp. $n_e$) be the number of connected components of $F_v$ (resp. $F_e$). Then, for any $e = (v_1, v_2)$, $n_e = d_e \cdot [n_{v_1}, n_{v_2}]$ for some $d_e \geq 1$.

Now, we are able to construct the covering $p : G(X, f) \to \Gamma(X, f)$ associated with the resolution $\phi$. Above a vertex $v \in \mathcal{V}(\Gamma(X, f))$ there are exactly $n_v$ vertices of $G(X, f)$, they correspond to the connected components of $F_v$. The $\mathbb{Z}$-action is induced by the monodromy (by the identification $I_z = (h_g)_z$). If $v$ is an arrowhead in $\Gamma$ then by convention, all the vertices in $G$ above $v$ are arrowheads (cf. 1.25 (1)). Above an edge $e$ of $\Gamma$, there are $n_e$ edges of $G$. They corresponds to the connected components of $F_e$. The $\mathbb{Z}$-action is again generated by the monodromy.
Fix an edge $\tilde{e}$ of $G$ (above the edge $e$ of $\Gamma$) which corresponds to the connected component $F_{\tilde{e}}$ of $F_e$. Similarly, take a vertex $\tilde{v}$ of $G$ (above the vertex $v$ of $\Gamma$) which corresponds to the connected component $F_{\tilde{v}}$ of $F_v$. Then $\tilde{e}$ has as an endpoint the vertex $\tilde{v}$ (in $G$) if and only if $F_{\tilde{e}} \subset F_{\tilde{v}}$. In particular, with the same notations as above, $\tilde{v}_1$ and $\tilde{v}_2$ are connected in $G$ by (at least) one edge if and only if $F_{\tilde{v}_1} \cap F_{\tilde{v}_2} \neq \emptyset$. Set $e = (\tilde{v}_1, \tilde{v}_2)$. If $F_{\tilde{v}_1} \cap F_{\tilde{v}_2} \neq \emptyset$, then $F_{\tilde{v}_1} \cap F_{\tilde{v}_2}$ has exactly $d_e = [n_{v_1}, n_{v_2}]/n_e$ connected components, hence $\tilde{v}_1$ and $\tilde{v}_2$ are connected exactly by $d_e$ edges. Therefore, above the segments of $\Gamma$ we have exactly the “standard blocks” of (1.5) in $G$.

In the sequel, we will use the notation $(m, d) = \{\{n_v\}_{v \in V}; \{d_e\}_{e \in E}\}$, where $n_e = d_e [n_{v_1}, n_{v_2}]$ for any edge $e = (v_1, v_2)$.

The next lemma establishes the number of connected components of $G(X, f)$.

2.28. Lemma. The number of connected components of the graph $G(X, f)$ is equal to the number of connected components of the Milnor fiber $F$ of the germ $f$.

Proof. Let $|G|$ be the topological realization of the graph $G$ considered as a 1–dimensional simplicial complex. Then it is not difficult to construct a continuous map $\alpha: F \to |G|$ which maps $F_{\tilde{e}}$ to the zero-cell (vertex) $\tilde{v}$, maps $F_{\tilde{v}}$ to the one–cell (edge) $\tilde{e}$, and for any $P \in |G|$, the space $\alpha^{-1}(P)$ is connected. (Notice that $F_{\tilde{e}} \approx S^1 \times \tilde{e}$, then $\alpha$ restricted to $F_{\tilde{e}}$ can be identified with the second projection $S^1 \times \tilde{e} \to \tilde{e}$.) Hence $\pi_0(\alpha): \pi_0(F) \to \pi_0(|G|)$ is an isomorphism. $\diamond$

Now, it is well–known, that the fibrations $f: f^{-1}(S^1_1) \to S^1_1$ and $\arg = f/|f|: L_X \setminus L_f \to S^1$ are equivalent. Hence by the long homotopy exact sequence:

$$\pi_1(L_X \setminus L_f) \xrightarrow{\pi_1(\arg)} \pi_1(S^1) \xrightarrow{\pi_1(\arg)} Z \to \pi_0(F) \to 0$$

(where $\pi_1(\arg) = ab \circ \arg_*$, cf. 2.19) we obtain that $|\pi_0(F)| = |\coker(\arg_*)|$. Therefore:

2.29. Corollary. $G(X, f)$ has $|\coker(\arg_*(f))|$ connected components. In particular, if $f$ defines an isolated singularity, then $G(X, f)$ is a connected graph. (But $G(X, f)$ may be connected even for germs $f$ with g.c.d.$\{m_v: v \in V\} \neq 1$.)

Before we state the second part of this corollary, we make the following discussion.

Fix a connected subgraph $\Gamma'$ of $G(X, f)$ with non–arrowhead vertices $W'$. Since the intersection form associated with the exceptional divisors $E':= \bigcup_{w \in W'} E_w$ is negative definite, by Grauert theorem [11] $E' \subset Y$ can be contracted to a singular point. Let $(X', x')$ be this singular point. Moreover, since $f \circ \phi$ is zero along $E'$, it gives rise to a germ $f'$ defined on $(X', x')$. It is obvious, that $\Gamma'$ is the embedded resolution graph of a singularity $((f')^{-1}(0), x') \subset (X', x')$. Let the corresponding representation be denoted by $\arg_*(f')$.

2.30. Corollary. Consider the universal cyclic covering $p: G(X, f) \to \Gamma(X, f)$ associated with $f$ and the resolution $\phi$. Fix a connected subgraph $\Gamma'$ of $\Gamma(X, f)$. Then $p^{-1}(\Gamma')$ has $|\coker(\arg_*(f'))|$ connected components.

2.31. – The case when $L_X$ is a rational homology sphere. If $L_X$ is a rational homology sphere, then the Milnor fibration is completely determined by
for any system of integers \((n)\). Hence, \(G(X, f)\) contains the same amount of information as \(\Gamma(X, f)\), and it can always be recovered from \(\Gamma(X, f)\).

2.32. Lemma. Assume that \(L_X\) is a rational homology sphere. Then \(n_v := \text{g.c.d.}\{m_w : w \in \mathcal{V}_v \cup \{v\}\}\) for any \(v \in \mathcal{V}(\Gamma)\); and \(n_e := \text{g.c.d.}(m_{v_1}, m_{v_2})\) for any \(e = (v_1, v_2) \in \mathcal{E}(\Gamma)\). (In particular, for \(a \in \mathcal{A}(\Gamma)\) one has \(n_a = n_e\), where \(e = (a, w_a) \in \mathcal{E}(\Gamma)\).) Moreover, the number of connected components of \(G(X, f)\) is exactly \(\text{g.c.d.}\{m_v : v \in \mathcal{V}(\Gamma)\}\).

Proof. If \(T_v\) is a tubular neighborhood of \(E_v\) as in (2.27), then \(\text{coker}(\pi_1(T_v \setminus \phi^{-1}(f^{-1}(0)))) \to \mathbb{Z} = \text{m}_v : w \in \mathcal{V}_v \cup \{v\}\mathbb{Z}\), hence the statement about \(n_v\) follows (by 2.30). Similarly, for \(e = (v, w)\), the cokernel of \(\pi_1(T_v \cap T_w \setminus \phi^{-1}(f^{-1}(0))) \to \mathbb{Z}\) is \(\mathbb{Z}/(m_v\mathbb{Z} + m_w\mathbb{Z})\). The number of connected components of \(G\) is \(\text{card}(\text{coker}(\arg_g) = \text{card}(\text{coker}(\mathbf{m} : H_\Gamma \to \mathbb{Z}))\).

Notice also that in this case \(G(\Gamma(X, f), (n, d))\) contains only one element (because \(\Gamma\) is a tree, cf. 1.19). This class is represented by \(G(X, f)\).

For example, if \((X, x)\) is smooth, then \(L_X = S^3\), hence for any plane curve singularity, the universal cyclic covering \(G(X, f) \to \Gamma(X, f)\) can completely be determined from the embedded resolution graph \(\Gamma(X, f)\) of \(f\).

2.33. Corollary. Consider an arbitrary germ \(f : (X, x) \to (\mathbb{C}, 0)\) (i.e. without any restriction about \(L_X\)), and consider the universal cyclic covering \(G(X, f) \to \Gamma(X, f)\). Then for any \(w \in \mathcal{W}(\Gamma)\) with \(g_w = 0\), \(p^{-1}(w)\) consists of exactly \(n_w = \text{g.c.d.}\{m_v : v \in \mathcal{V}_w \cup \{w\}\}\) vertices of \(G\). Similarly, for any \(a \in \mathcal{A}(\Gamma)\), \(\#p^{-1}(a) = n_a = \text{g.c.d.}(m_a, m_{w_a})\), where \((a, w_a) \in \mathcal{E}(\Gamma)\). Moreover, the number of edges \(n_e\) above \(e = (v_1, v_2)\) is \(\text{g.c.d.}(m_{v_1}, m_{v_2})\).

For \(w \in \mathcal{W}(\Gamma)\) with \(g_w > 0\) the following divisibility holds: \(n_w | \text{g.c.d.}\{m_v : v \in \mathcal{V}_w \cup \{w\}\}\).

Proof. Notice that if an irreducible rational exceptional divisor is contracted to a singular point then the link of this singular point is a rational homology sphere. Then use (2.29–2.32).

2.34. — The case when \(\Gamma(X, f)\) is a tree. Recall that if \(\Gamma(X, f)\) is a tree, then for any system of integers \((n, d)\), by (1.19) the class \(G(\Gamma(X, f), (n, d)) = 0\), hence all the coverings are equivalent. One the other hand, if \(L_X\) is not rational homology sphere, even if \(\Gamma(X, f)\) is a tree, the covering data \(\{n_x\}_{x \in V \cup E}\) of the universal covering \(G \to \Gamma\) (more precisely, the integers \(n_w\) with \(g_w > 0\) are not determined by \(\Gamma(X, f)\) (see the next examples). Hence already in this case, the covering \(G(X, f) \to \Gamma(X, f)\) contains some additional information about the representation \(\arg_g\).

2.35. Example. Set \((X, x) = \{(x^2 + y^7 - z^{14} = 0)\} \subset (\mathbb{C}^3, 0)\) and take \(f_1(x, y, z) = z^2\) and \(f_2(x, y, z) = z^2 - y\) (cf. 2.22). Then the coverings \(p : G(X, f_1) \to \Gamma(X, f_1)\) for \(i = 1, 2\) are:
In order to count the number of vertices above the irreducible divisor $E$ (with $g = 3$), we have to consider the representations $\text{arg}_*(f_i)$ (cf. 2.29). In the first case, the cokernel of this representation is $\mathbb{Z}_2$, in the second case it is trivial. Hence, above $E$, in the first case one has two vertices, and in the second case only one.

Notice, that the number of connected components (and even the Euler–characteristic) of the graphs $G(X, f_i)$ are different.

2.36. Example. Set $(X, x) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (\mathbb{C}^3, 0)$ and take $f_1(x, y, z) = z^2 + y^2$ and $f_2(x, y, z) = z^2 - y + y^2$ (cf. 2.25). By a similar argument as above, the coverings $p : G(X, f_i) \to \Gamma(X, f_i)$ (for $i = 1, 2$) are:

In this case the number of independent cycles of the graphs $G(X, f_i)$ is different.

2.37. If all the irreducible exceptional divisors of $\phi$ are rational (i.e. $g_w = 0$ for all $w$), then the type $(\mathbf{n}, \mathbf{d})$ of the covering is completely determined by $\Gamma(X, f)$ (cf. 2.33). But if $\Gamma(X, f)$ is not a tree, then $\mathcal{G}(\Gamma(X, f), (\mathbf{n}, \mathbf{d}))$ can be non–trivial. Hence again, $G(X, f)$ carries some additional information about $\text{arg}_*$. 
The monodromy representation of the regular covering consists of more than one point (cf. 2.6). A singular point of along \( f \) is given by two non-equivalent representatives of \( G \). The germ of an analytic function. For any integer \( n \in (\mathbb{Z})_\rightarrow \): \( G(=2) \). This is a general fact: for a connected graph \( G \) is abelian, hence the monodromy representation over \( f, N \) is denoted by \( \Gamma \) (provided that \( g_w = 0 \) for any \( w \)).

**3. The resolution graph of cyclic coverings**

The monodromy representation of cyclic coverings.

**3.1.** Let \( (X, x) \) be a normal surface singularity and \( f : (X, x) \to (\mathbb{C}, 0) \) the germ of an analytic function. For any integer \( N \geq 1 \), take \( b : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) given by \( z \mapsto z^N \), and let \( X_{f, N} \) be the normalization of the fiber product \( \{ (x', z) \in (X \times \mathbb{C}, x \times 0) : f(x') = z^N \} \) (cf. 2.7). The second projection \( (x', z) \to X \times \mathbb{C} \to \mathbb{C} \) induces an analytic map \( X_{f, N} \to \mathbb{C} \), still denoted by \( z \). The first projection \( (x', z) \to x' \) gives rise to a ramified cyclic \( N \)-covering \( pr : X_{f, N} \to X \), branched along \( f^{-1}(0) \). If \( f^{-1}(0) \) has an isolated singular point at \( x \), then there is only one (singular) point of \( X_{f, N} \) lying above \( x \in X \). But in general \( pr^{-1}(x) \) contains more than one point (cf. 2.6).

We regard \( \mathbb{Z}_N \) as the group of \( N \)-th roots of unity \( \{ \xi_k = e^{2\pi i k/N} : 0 \leq k \leq N-1 \} \), then \( (x', z) \to (x', \xi_k z) \) induces a \( \mathbb{Z}_N \)-Galois action of \( X_{f, N} \) over \( X \).

If \( P \in X \setminus f^{-1}(0) \) is a point in the complement of the branch locus, then \( pr^{-1}(P) \) consists of \( N \) points and they are cyclically permuted by the Galois action of \( \mathbb{Z}_N \).

The monodromy representation of the regular covering \( pr|_{X_{f, N} \setminus \{ z=0 \} : X_{f, N} \setminus \{ z=0 \} \to X \setminus \{ f = 0 \}} \) is denoted by \( \varphi_N : \pi_1(X \setminus \{ f = 0 \}) \to \mathbb{Z}_N \). Notice that \( X \setminus \{ f = 0 \} \) is connected, and \( \mathbb{Z}_N \) is abelian, hence the monodromy representation

**2.38. Example.** Set \( (X, x) = (\{ z^2 + (x^2 - y^3)(x^3 - y^2) = 0 \}, 0) \) and \( f_1 = x^2 + y^4 \) and \( f_2 = x^2 - y^3 + y^4 \) (cf. 2.26). Then the coverings \( p : G(X, f_i) \to \Gamma(X, f_i) \) (for \( i = 1, 2 \)) are:

In this case, \( G(\Gamma(X, f), (\mathbf{n}, d)) = \mathbb{Z}_2 \) (cf. 1.22). The above examples provide the two non-equivalent representatives of \( G = \mathbb{Z}_2 \).

The number of independent cycles \( c_G \) for both graphs \( G(X, f_i) \) are the same (= 2). This is a general fact: for a connected graph \( G \) one has: \( 1 - c_G = \#\mathcal{V}(G) - \#\mathcal{E}(G) \), but \( \#\mathcal{V}(G(X, f_i)) \) and \( \#\mathcal{E}(G(X, f_i)) \) are determined by \( \Gamma \) (provided that \( g_w = 0 \) for any \( w \)).
does not depend on the choice of the basepoint. So, we will omit the basepoint of the fundamental group.

By the local cone structure of \((X, \{f = 0\})\), we can replace the group \(\pi_1(X \setminus \{f = 0\})\) by \(\pi_1(L_X \setminus L_f)\). The following property of the cyclic coverings is well-known:

### 3.2. Lemma

Let \(\pi_1(\text{arg}) : \pi_1(L_X \setminus L_f) \to \pi_1(S^1) = \mathbb{Z}\) be the morphism induced by the Milnor fibration of \(f\) (at the fundamental group level) (cf. 2.19); and let \(\text{mod}_N : \mathbb{Z} \to \mathbb{Z}_N\) be the natural projection \(1 \mapsto 1\). Then \(\varphi_N = \text{mod}_N \circ \pi_1(\text{arg})\).

Since \(\mathbb{Z}_N\) is abelian, \(\varphi_N = r_N \circ ab\) for some \(r_N : H_1(L_X \setminus L_f, \mathbb{Z}) \to \mathbb{Z}_N\) (where \(ab : \pi_1 \to \pi_1/\{\pi_1, \pi_1\} = H_1\) is the abelianization map). Hence, \(r_N = \text{mod}_N \circ \arg_*\), where \(\arg_* : H_1(L_X \setminus L_f, \mathbb{Z}) \to \mathbb{Z}\) is induced by the Milnor fibration \(\text{arg} = f/|f|\). This implies the following:

### 3.3. Lemma

If the Milnor fiber \(F\) of \(f\) has \(k\) connected components, then \(X_{f,N}\) has \((k, N)\) connected components.

*Proof.* By the long homotopy exact sequence of the Milnor fibration \(k = |\text{coker} \arg_*|\), hence the range of \(\arg_*\) is \(k\mathbb{Z} \subseteq \mathbb{Z}\). But \(\text{mod}_N(k\mathbb{Z}) = k\mathbb{Z}_N\) has index \((k, N)\) in \(\mathbb{Z}_N\), hence \(\text{coker} r_N = \text{coker} \varphi_N \approx \mathbb{Z}_{(k, N)}\). Again, by the long homotopy exact sequence of the regular covering \(pr\) over \(X \setminus \{f = 0\}\), the integer \(|\text{coker} \varphi_N|\) is the number of connected components of \(X_{f,N} \setminus \{z = 0\}\). But this is exactly the number of connected (or irreducible) components of \(X_{f,N}\).

The number \((N, k)\) is exactly the number of points of \(X_{f,N}\) lying above \(x \in X\) (i.e. \(\#pr^{-1}(x)\)). In the sequel, we will use the germ–notation \((X_{f,N}, \{x_1, \ldots, x_{(N,k)}\})\), which means that \(X_{f,N}\) consists of \((N, k)\) disjoint space germs \((X_{f,N}, x_i)^{(N,k)}\). Obviously, they are all isomorphic with each other – an isomorphism is given by the Galois action (which permutes the points \(\{x_i\}_{i=1}^{(N,k)}\)).

**3.4. Remark.** The number of (singular) points of \(X_{f,N}\) lying above \(x \in X\) cannot be determined from the embedded resolution graph of \(f\) and the integer \(N\).

Indeed, consider the situation described in (2.22) and (2.35). Then \(\#pr^{-1}(x) = 2\) in the first case, and \(\#pr^{-1}(x) = 1\) in the second case.

The above remark already suggests (and we will see a lot of other examples later) that the resolution graph of \(X_{f,N}\) cannot be reconstructed from the graph \(\Gamma(X, f)\) and \(N\). In fact, this was the very reason why we constructed the universal cyclic covering \(G(X, f) \to \Gamma(X, f)\). For example, related to the above discussion: the number of connected components of \(G(X, f)\) is exactly \(k\) (cf. 2.28), hence \((k, N) = \#pr^{-1}(x)\) is determined by \(G(X, f)\) and the integer \(N\).

### 3.5. Definitions

**a.)** The resolution graph \(\Gamma(X_{f,N})\), by definition, is the union of the resolution graphs of \((X_{f,N}, x_i)^{(k,N)}\).

**b.)** The embedded resolution graph \(\Gamma(X_{f,N}, z)\) of \(z : (X_{f,N}, \{x_1, \ldots, x_{(k,N)}\}) \to (\mathbb{C}, 0)\) is the union of the embedded resolution graphs of \(z : (X_{f,N}, x_i) \to (\mathbb{C}, 0)\) \((1 \leq i \leq (k, N))\).

In both cases \(\Gamma\) has \((k, N)\) identical connected components.
The embedded resolution graph of $\Gamma(X, f, N, z)$.

3.6. – The main construction. Let $p : G(X, f) \to \Gamma(X, f)$ be the universal cyclic covering of the embedded resolution graph $\Gamma(X, f)$ associated with the germ $f : (X, x) \to (\mathbb{C}, 0)$ (cf. 2.27). For brevity we will use $\Gamma = \Gamma(X, f)$ and $G = G(X, f)$ throughout this subsection. The covering $G$ is an element of $G(\Gamma, (n, d))$, where $n_v = \#p^{-1}(v) \subset \mathcal{V}(G)$ for any vertex $v \in \mathcal{V}(\Gamma)$, and $n_v = d_v [n_{v_1}, n_{v_2}] = \#p^{-1}(e) \subset \mathcal{E}(G)$ for any $e \in \mathcal{E}(\Gamma)$. Recall that the graph $\Gamma$ has the following decorations: multiplicities $\{m_v\}_{v \in \mathcal{V}(\Gamma)}$, genera $\{g_w\}_{w \in \mathcal{W}(\Gamma)}$ (and self intersection numbers, which are less important in this construction). By our convention: $A(G) = p^{-1}(A(\Gamma))$.

Now, for any fixed integer $N \geq 1$, we construct a new graph in four steps.

Step 1. The graph $G$ has a $\mathbb{Z}$–action. The “orbit graph” of the subgroup $N\mathbb{Z} \subset \mathbb{Z}$ is denoted by $mod_N(G)$ (for details, see 1.24).

The new covering $mod_N(p) : mod_N(G) \to \Gamma$ is an element of $G(\Gamma, ((N, n, d'), (N, n, d') \cap \frac{1}{N}G_N))$, where $(N, n) = (g.c.d.(N, n_v), \{e\})_{v \in \mathcal{V}(\Gamma)}$, and $d_v = (d_v, N/(N, [n_{v_1}, n_{v_2}]))$ for any edge $e = (v_1, v_2) \in \mathcal{E}(\Gamma)$ (i.e. $\#mod_N(p)^{-1}(e) = g.c.d.(N, \#p^{-1}(e))$ for any $e$).

Step 2. We put decorations (multiplicities and genera) on $mod_N(G)$ as follows.

(a) The multiplicity $(m_v)$ of any vertex $\tilde{v} \in \mathcal{V}(mod_N(G))$, which lies above $v \in \mathcal{V}(\Gamma)$ is $m_v = (m_v, N)$.

(b) The genus $[\tilde{g_w}]$ of any vertex $\tilde{w} \in \mathcal{W}(mod_N(G))$, which lies above $w \in \mathcal{W}(\Gamma)$ with genus $g_w$, is given by:

$$2 - 2\tilde{g_w} = \frac{(2 - 2g_w - \delta_w) \cdot (m_w, N) + \sum_{v \in \mathcal{V}_w(\Gamma)} g.c.d.(m_v, m_w, N)}{(N, n_w)},$$

where $\delta_w = \#\mathcal{V}_w(\Gamma)$.

Step 3. Any edge $\tilde{e}$ of $mod_N(G)$, with endpoints $\tilde{v}_1$ and $\tilde{v}_2$, lying above $e$ with endpoints $v_1$ and $v_2 \in \mathcal{E}(\Gamma)$, will be replaced by a string $Str(e)$ as follows (cf. 2.11 and 1.25 (3)).

If $t_e = g.c.d.(m_{v_1}, m_{v_2}, N)$, then set $N' = n/t_e$ and $m'_{v_i} = m_{v_i}/t_e$ for $i = 1, 2$.

Consider the unique $0 < \lambda < N'/(m'_{v_1}, N')$ and $m_1 \in \mathbb{N}$ with:

$$m'_{v_2} + \lambda \cdot \frac{m'_{v_1}}{m'_{v_1}, N'} = m_1 \cdot \frac{N'}{m'_{v_1}, N'}. $$

If $\lambda = 0$, then the edge $\tilde{e}$ remains unchanged.

If $\lambda \neq 0$, then take the continuous fraction:

$$\frac{N'/(m'_{v_1}, N')}{\lambda} = k_1 - \frac{1}{1 - \frac{1}{k_2 - \frac{1}{\ddots - \frac{1}{k_s}}}}$$

$k_1, \ldots, k_s \geq 2$.

If both $v_1$ and $v_2$ are non–arrowhead vertices, then $Str(e)$ denotes the following decorated string:
RESOLUTION GRAPHS OF CYCLIC COVERINGS

\[
\begin{array}{cccc}
[\tilde{g}_{v_1}] & [0] & [0] & [0] \\
- k_1 & - k_2 & \cdots & - k_s \\
(m_{v_1}) & (m_1) & (m_2) & (m_s) \\
\end{array}
\]

with genera \([\tilde{g}_{v_1}], [0], [0], [\tilde{g}_{v_2}]\), self intersection numbers \(-k_1, \ldots, -k_s\), and multiplicities \(m_{v_1}, m_1, \ldots, m_s, m_{v_2}\).

The multiplicities \(m_{v_1}\) and \(m_{v_2}\) were already determined in step 2, namely \(m_{v_1} = \frac{m_{v_1}}{N}\); and \(m_1\) is the number given by the above congruence. Moreover, the multiplicities \(m_2, \ldots, m_s\) can easily be determined using (2.3 (2)), namely

\[m_2 = k_1m_1 - m_{v_1};\quad \text{and} \quad m_{i+1} = k_1m_i - m_{i-1} \quad \text{for} \quad i \geq 2.\]

Then each edge \(\tilde{e}\):

\[
\begin{array}{cccc}
[\tilde{g}_{v_1}] & [\tilde{g}_{v_2}] \\
(m_{v_1}) & (m_{v_2}) \\
\end{array}
\]

of \(\text{mod}_N(G)\), lying above \(e\), is replaced by the string \(\overline{\text{Str}}(e)\).

Moreover, if \(v_2 \in A(\Gamma)\), then the edge \(\tilde{e}\):

\[
\begin{array}{cccc}
[\tilde{g}_{v_1}] \\
(m_{v_1}) \\
\end{array}
\]

of \(\text{mod}_N(G)\), lying above \(e\), is replaced by the “modified” string \(\overline{\text{Str}}(e)\) (which has the same decorations as the “original \(\overline{\text{Str}}(e)\)”):

\[
\begin{array}{cccc}
[\tilde{g}_{v_1}] & [0] & [0] & [0] \\
- k_1 & - k_2 & \cdots & - k_s \\
(m_{v_1}) & (m_1) & (m_2) & (m_s) \\
\end{array}
\]

The new graph resulting from inserting all the necessary strings into \(\text{mod}_N(G)\) is denoted by \(\text{mod}_N(G)(\text{Str})\).

**Step 4.** The decoration of \(\text{mod}_N(G)(\text{Str})\) is not complete. All the vertices have multiplicities, all the non–arrowhead vertices have genera, but some of the self intersection numbers are missing (corresponding exactly to the vertices \(\tilde{v}\) of \(\text{mod}_N(G)\)). Now, we add these numbers using the relation (2.3 (2)) applied for \(\text{mod}_N(G)(\text{Str})\) (namely, for any \(w \in W\), the relation \(e_wm_w + \sum_{v \in V_w} m_v = 0\) provides \(e_w\)).

3.7. **Theorem.** If \(\phi : (Y, D) \to (X, x)\) is an embedded resolution of \((f^{-1}(0), x) \subset (X, x)\), and \(p : G(X, f) \to \Gamma(X, f)\) the universal covering graph associated with \(\phi\). Then the graphs \(\Gamma(X_{f,N}, z)\) and \(\Gamma(X_{f,N})\) can be determined from \(p : G(X, f) \to \Gamma(X, f)\) and from the integer \(N\). Namely:
a) The decorated graph \( \text{mod}_N(G)(\text{Str}) \) constructed in (3.6) is (a possible) embedded resolution graph \( \Gamma(X_{f,N}, z) \).

b) If we delete all the arrows and multiplicities of \( \text{mod}_N(G)(\text{Str}) \), then we obtain a resolution graph \( \Gamma(X_{f,N}) \) of \( X_{f,N} \).

3.8. Example. Set \((X, x) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (C^3, 0)\). Take \( f_1(x, y, z) = z^2 + y^2 \) and \( f_2(x, y, z) = z^2 - y + y^2 \) (cf. 2.25 and 2.36). Then \( \Gamma(X, f_1) = \Gamma(X, f_2) \), but in general, the graphs \( \Gamma(X_{f_i,N}, z) \ (i = 1, 2) \) are not the same. For example, these graphs for \( N = 2 \) are:

For \( N = 4 \), the graphs \( \Gamma(X_{f_i,N}, z) \) are:

For any odd \( N \) the orbit graphs \( \text{mod}_N(G) \), for \( i = 1, 2 \), are the same. Hence in this case \( \Gamma(X_{f_1,N}, z) = \Gamma(X_{f_2,N}, z) \). For \( N = 3 \) this graph is:
3.9. Example. Set \((X, x) = \{(z^2 + (x^2 - y^3)(x^3 - y^2) = 0\}, 0\) and \(f_1 = x^2 + y^4\) and \(f_2 = x^2 - y^3 + y^4\) (cf. 2.26 and 2.38). The graphs \(\Gamma(f_i, N, z)\) for \(N = 2\) are:

\[
\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
-1 & -3 & -2 & -2 & -2 & -2 & -2 \\
(2) & (2) & (4) & (3) & (2) & (1) & (1) \\
\end{array}
\]

The decorations are the following. The arrows have multiplicity \(1\), \(m_a = 4\), \(m_b = 3\), and the multiplicities of the unmarked nodes are \(m = 1\): \(e_a = e_b = -2\), and the self intersection of the unmarked nodes are \(e = -4\). All the genera are zero.

3.10. Remark. The construction (3.6) gives non–minimal resolution graphs, in general. They can be simplified by blowing–down the \((-1)–\)rational exceptional divisors \(E_w\) with \(\delta_w \leq 2\).

3.11. Remark. Using the formula of (Step 2.), it is easy to prove that \(\tilde{g}_\omega \geq g_w\). Moreover, \(c_{mod_x(G)} \geq c_F\) (cf. also with (1.23) and (3.21)). Therefore:
\[
\text{rank } H_1(L_{X, f, N}) \geq \text{rank } H_1(L_X).
\]
3.12. – Modification of (3.6) for the case when $L_X$ is a rational homology sphere. If $L_X$ is a rational homology sphere, then the above algorithm can be simplified. In this case the universal covering $G(X, f) \to \Gamma(X, f)$ can completely be reconstructed from $\Gamma(X, f)$ (see 2.31–2.32). This means that the graph $\Gamma(X_{f,N}, z)$ is completely determined by the embedded resolution graph $\Gamma(X, f)$ of $f$ and the integer $N$, and the reader can easily reconstruct this new algorithm.

In particular, if $(X, x) = (\mathbb{C}^2, 0)$, and $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is an arbitrary isolated plane curve singularity, then $(X_{f,N}, 0) = \{(f(x, y) = z^N), 0\} \subset (\mathbb{C}^3, 0)$, and $z : (X_{f,N}, 0) \to (\mathbb{C}, 0)$ is induced by the projection $(x, y, z) \mapsto z$. Therefore, our algorithm also provides the resolution graph of $(Y, 0) = \left\{(g = 0), 0\right\}$, where $g = f(x, y) + z^N$. For the algorithm in this case see [28]. The idea of the construction in the case of $f + z^N$ can already be found in the book of Laufer [19]. For other particular cases, see [4, 33, 34].

The general algorithm can be compared with some results of Eriko Hironaka, who considers the global case of cyclic coverings.

3.13. Proof of the Theorem (3.7). Consider the “Jungian diagram” (cf. 2.9):

\[ \xymatrix{ X^{res} \ar[r]^-n & \tilde{X}' \ar[r]^-{\phi'} & X' \ar[r]^-{\phi} & (X_{f,N}, x) \ar@/_/[d]_{\pi'} \ar@/^/[d]_{\pi} \ar@{<-}[dl]^-{\tilde{\pi}'} \ar@{=}^{\phi}(\mathcal{Y}, D) \ar[r]^-{\phi} & (X, x) \ar@/^/[u]_{\rho} } \]

where:

a) the proper map $\pi : (X_{f,N}, x) \to (X, x)$ is induced by the projection $(x', z) \mapsto x'$, and it is an $N$–covering with branch locus $f^{-1}(0)$ (cf. 3.1).

b) $\phi$ is a fixed embedded resolution of $(f^{-1}(0), x) \subset (X, x)$, and $D$ is the divisor $\phi^{-1}(f^{-1}(0))$. The dual graph of $D$ is exactly $\Gamma(X, f)$.

c) $\pi' : X' \to \mathcal{Y}$ is the pullback of $\pi$ via $\phi$, and $\tilde{X}'$ is the normalization of $X'$.

Then $\tilde{X}'$ has only Hirzebruch–Jung singularities (cf. 2.8 and 2.9).

d) $X^{res} \to \tilde{X}'$ is the resolution of the Hirzebruch–Jung singularities of $\tilde{X}'$.

If $P$ is a generic point of the irreducible exceptional divisor $E_w \subset E = \phi^{-1}(x)$ ($w \in W(\Gamma(X, f))$), then we fix local coordinates $(u, v)$ in a neighbourhood $U$ of $P$ such that $\{u = 0\} = E_w \cap U$, and $f \circ \phi|_U = u^{m_w}$. Let $P'$ be the unique point in $X'$ above $P$ and consider its neighbourhood $U' := (\pi')^{-1}(U)$. Then $U' = \{(u, v, z) \in (U \times \mathbb{C}, P \times 0) : u^{m_w} = z^N\}$. This shows that $n^{-1}(P')$ contains exactly $(N, m_w)$ points, which correspond to the irreducible components of $U'$. Indeed, $U' = \bigcup_i U'_i$, where $U'_i = \{(u, v, z) : u^{m_w/(N, m_w)} = \epsilon_i \cdot z^{N/(N, m_w)}\}$, where $\epsilon_i$, for $i = 0, 1, \ldots, (N, m_w) - 1$, are the $(N, m_w)$-roots of unity. Moreover, the normalization of any $U'_i$ is smooth. For example, for $i = 0$, a possible normalization map of $U'_i$ is $\left(\mathbb{C}^2, 0\right) \to U'_0$ given by $(t, v) \mapsto (u(t), v, z(t))$, where

\[ u(t) = t^{N/(N, m_w)}, \quad z(t) = t^{m_w/(N, m_w)}. \]

Therefore, $(\pi')^{-1}(E_w) \to E_w$ is a regular covering of degree $(N, m_w)$ above $E_w^0 := E_w \setminus D \setminus E_w$. The covering $U' \to U$ is a Galois covering with Galois group action
of $\mathbb{Z}_N = \{\xi : \xi^N = 1\}$ given by $(u, v, z) \mapsto (u, v, \xi z)$. Notice that $\xi^{m_w}$ preserves the components $U'_i$, therefore the covering $(\tilde{\pi}')^{-1}(E^0_w) \to E^0_w$ is a Galois covering with Galois group $\mathbb{Z}_N/m_w\mathbb{Z}_N = \mathbb{Z}_{(N,m_w)}$. This regular covering extends to a branched covering $(\tilde{\pi}')^{-1}(E_w) \to E_w$, with branch locus $E_w \setminus E^0_w$. The curve $(\tilde{\pi}')^{-1}(E_w)$ contains exactly the irreducible exceptional divisors of $X'$ lying above $E_w$. Then the number $i_w$ of these irreducible exceptional components is exactly the number of irreducible components of $(\tilde{\pi}')^{-1}(E^0_w)$, and this number coincides with the number of connected components of $(\tilde{\pi}')^{-1}(E^0_w)$. The representation associated with the covering $(\tilde{\pi}')^{-1}(E^0_w) \to E^0_w$ is denoted by:

$$\rho_w : \pi_1(E^0_w) \to \mathbb{Z}_{(N,m_w)},$$

and the number $i_w$ of connected components of $(\tilde{\pi}')^{-1}(E^0_w)$ is $|\ker(\rho_w)|$.

Now, let $T_w$ be a tubular neighborhood of $E_w$ in $Y$ (cf. 2.27). Then $(\tilde{\pi}')^{-1}(T_w \setminus D) \to T_w \setminus D$ is a regular $N$–covering. The corresponding representation is denoted by:

$$r_w : \pi_1(T_w \setminus D) \to \mathbb{Z}_N.$$

This representation is induced by $arg_*$, as it is explained in (3.1–3.2). Hence, the following composed map:

$$\pi_1(T_w \setminus D) \xrightarrow{j} \pi_1(L_X \setminus L_f) \xrightarrow{arg_*} \mathbb{Z} \xrightarrow{pr_N} \mathbb{Z}_N$$

is exactly the map $r_w$ (above, $j$ is induced by the natural inclusion).

Now, notice that $T_w \setminus D = E^0_w \times S^1$, where $S^1$ can be represented by an oriented circle in a transversal slice of $E^0_w$ in $T_w$. In particular, $\pi_1(T_w \setminus D) = \pi_1(E^0_w) \times \mathbb{Z}$, and $arg_*((0, 1z)) = m_w \in \mathbb{Z}$. This shows that $r_w((0, 1z))$ is the class of $m_w$ in $\mathbb{Z}_N$.

The above discussion provides the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(T_w \setminus D) & \xrightarrow{j} & \pi_1(L_X \setminus L_f) \\
\uparrow & & \xrightarrow{arg_*} \\
\pi_1(E^0_w) & \xrightarrow{pr_N} & \mathbb{Z}_N \\
\downarrow & & \downarrow pr_{(N,m_w)} \\
\pi_1(E^0_w) & \xrightarrow{pr_w} & \mathbb{Z}_{(N,m_w)}
\end{array}$$

Since the class of $m_w$ in $\mathbb{Z}_{(N,m_w)}$ is zero, one has: $|\ker(pr_{(N,m_w)} \circ r_w)| = |\ker(\rho_w)|$.

Therefore, $i_w = |\ker(\rho_w)|$ is the cardinality of the cokernel of the following composed map:

$$p_w : \pi_1(T_w \setminus D) \xrightarrow{arg_* \circ j} \mathbb{Z} \xrightarrow{pr_N} \mathbb{Z}_N \xrightarrow{pr_{(N,m_w)}} \mathbb{Z}_{(N,m_w)}.$$
(m_w, m_v) is the number of edges of G above the edge (v, w) \in E(\Gamma), (N, n_e) = (N, m_w, m_v) is the number of edges in mod_N(G) above (v, w). Hence it coincides with \#(\tilde{\pi}')^{-1}(P). Moreover, the following diagram provides the adjacency relations in G:

$$
\pi_1(T_w \setminus D) \xrightarrow{\arg_{\circ,j}} \mathbb{Z} \rightarrow \mathbb{Z}_{n_w} \rightarrow 0
$$

$$
\pi_1(T_w \cap T_v \setminus D) \xrightarrow{\arg_{\circ,j}} \mathbb{Z} \rightarrow \mathbb{Z}_{n_e} \rightarrow 0
$$

The group \mathbb{Z}_{n_w} (resp. \mathbb{Z}_{n_e}) is the index set of the vertices (resp. edges) of G above w (resp. above e = (w, v)). The edge in G indexed by \ell \in \mathbb{Z}_{n_e} has as one of its endpoints the vertex indexed by a(\ell).

Now, in the commutative diagram:

$$
\begin{array}{ccc}
\mathbb{Z}_{n_w} & \rightarrow & \mathbb{Z}_{(N, n_w)} \\
\uparrow a & & \uparrow a \\
\mathbb{Z}_{n_e} & \rightarrow & \mathbb{Z}_{(N, n_e)}
\end{array}
$$

the left column codifies the adjacency relations in G, while the right column the adjacency relations in mod_N(G) (in a similar way). Therefore, by the above discussion, it is clear that mod_N(G) is the dual graph of the exceptional divisors of \tilde{X}'.

The space \tilde{X}' has only Hirzebruch–Jung singularities. The resolution of these singularities does not change the multiplicities of z along the irreducible exceptional divisors of \tilde{X}', nor the topology of them. So, already at the level of \tilde{X}' we can compute these invariants. First notice that the irreducible components above E_w are cyclically permuted by the Galois action, so their multiplicities and genera are the same. The formula (*), (of this proof) shows that at a generic point of (\tilde{\pi}')^{-1}(E_w), the multiplicity of z is m_w/(N, m_w), proving (Step 2, a).

Above the branch point E_v \cap E_w of the covering (\tilde{\pi}')^{-1}(E_w) \rightarrow E_w, there are (N, m_w, m_v) points, therefore, by an Euler–characteristic argument:

$$
\chi((\tilde{\pi}')^{-1}(E_w)) = (2 - 2g_w - \delta_w)(N, m_w) + \sum_{v \in v_w} (N, m_w, m_v).
$$

But this is (N, n_w) \cdot (2 - 2g), proving (Step 2, b).

Now, we have to resolve the Hirzebruch–Jung singularities of \tilde{X}'. The singular points \{P_c\} of \tilde{X}' are codified by the edges \hat{c} of mod_N(G), and their local equations are \{y_{\text{mod}}^m/m_v \cdot t_z = z^{n/c}\}, where \xi = (N, m_w, m_v), c = (v, w). Therefore, any edge \hat{c} of mod_N(G) must be replaced by the string of the corresponding Hirzebruch–Jung singularity. Hence the last part follows from subsection (2.11). ◊

**Cyclic coverings and monodromy.**

**3.14.** The main goal of the present section is to compare the covering p : G(X, f) \rightarrow \Gamma(X, f) with the algebraic monodromy operator associated with the Milnor fibration arg : LX \setminus L_f \rightarrow S^1.

Start again with a normal surface singularity (X, x) and a germ f : (X, x) \rightarrow (\mathbb{C}, 0) of an analytic function. The characteristic map of the Milnor fibration F \rightarrow
$L_X \setminus L_f \xrightarrow{\text{arq}} S^1$ is called the geometric monodromy $h_{geom} : F \to F$ (where $F$ denotes the Milnor fiber).

At homology level $h_{geom}$ induces the algebraic monodromies $h_q : H^q(F,\mathbb{Z}) \to H^q(F,\mathbb{Z})$ ($q = 0, 1$). If $F$ has $k$ connected components, then $H^0(F,\mathbb{Z}) \approx \mathbb{Z}^k$ and $h_0(x_1, ..., x_k) = (x_2, ..., x_k, x_1)$ (cf. 2.20). In particular, $h_0$ is finite.

The monodromy $h_1$ is more complicated – in general it is not finite. But, by the Monodromy Theorems (see, e.g. [6, 18, 21, 1, 2], all the eigenvalues of $h_1$ are roots of unity, and its Jordan decomposition contains blocks only of size one and two.

In this section, we connect the Jordan decomposition of $h_1$ with the number of independent cycles in the cyclic coverings with branch locus $\{f = 0\}$. Finally, we show that the number of Jordan blocks of $h_1$ is completely determined by the universal cyclic covering graph $G(X, f) \to \Gamma(X, f)$. Some results of this subsection can be compared with some results of Ph. Du Bois and F. Michel [8].

3.15. – The characteristic polynomial. For $q = 0, 1$ we define $\Delta_q(t) = \det((tI - h_q))$. Then, for $q = 0$, $\Delta_0(t) = t^k - 1$. The characteristic polynomial $\Delta_1(t)$ is determined by $\Gamma(X, f)$, via A’Campo’s formula [1, 2]:

\[
\Delta_0(t)/\Delta_1(t) = \prod_{w \in W(\Gamma)} (t^{m_w} - 1)^{2 g_w - \delta_w}.
\]

It is convenient to use the notation $H = H^1(F,\mathbb{C})$; and $H_\lambda$ for the generalized eigenspace corresponding to the eigenvalue $\lambda$ of $h_1$ (i.e. $H_\lambda = \{x \in H | (h_1 - \lambda I)^n x = 0 \text{ for some sufficiently large } n\}$.) The dimension $\dim_{\mathbb{C}} H_\lambda$ is exactly the order of $t - \lambda$ in $\Delta_1(t)$.

3.17. – The Jordan blocks of $h_1$. Let $\#_\lambda^1(f)$ be the number of Jordan blocks of size 1 of the restriction $h_1|_{H_\lambda}$. Obviously, $\#_\lambda^1(f) + \#_\lambda^2(f) = \dim H_\lambda$ and $\dim \ker(h_1 - \lambda I) = \#_\lambda^1(f) + \#_\lambda^2(f)$.

First consider $\lambda = 1$.

3.18. Proposition. (cf. [29]) Let $c_{\Gamma(X)}$ be the number of independent cycles in $\Gamma(X)$, and $g = \sum_{w \in W(\Gamma)} g_w$. Then:

a) $\dim H_1 = 2g + 2c_{\Gamma(X)} + \# A(\Gamma) - 1$;

b) $\dim \ker(h_1 - 1) = 2g + c_{\Gamma(X)} + \# A(\Gamma) - 1$.

Therefore, $\#_1^2(f) = c_{\Gamma(X)}$. In particular, $\#_1^2(f)$ is independent of the germ $f$, and depends only on the topology of the link $L_X$.

Proof. The order of $t - 1$ in $\Delta_0/\Delta_1$ is $1 - \dim H_1$. Via A’Campo’s formula this is $\sum_w (2 - 2g_w - \delta_w)$. Now, use $\sum_w \delta_w = \# A + 2\# \mathcal{E}$, and the “Euler-characteristic” identity $1 - c_{\Gamma(X)} = \# W - \# \mathcal{E}$, and (a) follows. By the Wang exact sequence associated with the Milnor fibration

\[
H^0(F) \xrightarrow{h_0} H^0(F) \to H^1(L_X \setminus L_f) \to H^1(F) \xrightarrow{h_1} H^1(F)
\]

one has that $\coker(h_0 - I) \approx \mathbb{C}$, hence $\dim \ker(h_1 - 1) = \dim H^1(L_X \setminus L_f) - 1$. But using (2.17) and (2.18) $\dim H^1(L_X \setminus L_f) = \dim H_1(E) + \# A = 2g + c_{\Gamma} + \# A$. \(\Diamond\)
As a corollary of (3.18), we obtain that if $L_X$ has no cycles (e.g. $(X,x)$ is smooth as in the case of plane curve singularities), then $f$ has no Jordan block of size 2 with eigenvalue $\lambda = 1$.

Now, assume that $f : (X,x) \to (\mathbb{C},0)$ is the smoothing of an isolated singularity at $x$. Then $\text{arg}_x(f)$ is onto, hence for any $N$, $X_{f,N}$ is connected. Moreover, there is natural identification of the Milnor fibers of $f$ and $z : (X_{f,N},x) \to (\mathbb{C},0)$ such that the monodromy of $z$ is the $N^{th}$-power of the monodromy of $f$. Therefore, by (3.18) $c_{\Gamma(X_{f,N})} = \#_2^\lambda(z) = \sum_{\lambda^N=1} \#_2^\lambda(f)$. Thus we have the following:

3.19. Corollary. If $f$ defines an isolated singularity, then:

$$
\sum_{\lambda^N=1} \#_2^\lambda(f) = c_{\Gamma(X_{f,N})}.
$$

Since any eigenvalue of $h_1$ is a root of unity, $h_1$ is of finite order if and only if it has no Jordan blocks of size 2. Therefore, the above corollary generalizes a theorem of A. Durfee, which says that for plane curve singularities $f : (\mathbb{C}^2,0) \to (\mathbb{C},0)$, $h_1$ has finite order if and only if the graphs of the cyclic coverings have no cycles [9].

3.20. Remark. In some particular cases $c_{\Gamma(X_{f,N})}$ can be computed from the multiplicity system of $\Gamma(X,f)$ alone. For example, if $L_X$ is a rational homology sphere, then by (2.31):

$$
1 - c_{\Gamma(X_{f,N})} = \#W(\Gamma(X_{f,N})) - \#E(\Gamma(X_{f,N}))
$$

where $M_w = \text{g.c.d.}\{m_v\}_{v \in \gamma_w, m_w}$ and $m_e = \text{g.c.d.}(m_v_1, m_v_2)$ with $e = (v_1, v_2)$, and $E_n$ denotes the set of edges connecting two non–arrowheads. Then by (3.19) and by an easy computation using $\#W - \#E_n = 1$ one has:

$$
\sum_{\lambda^N=1} \#_2^\lambda(f) = \sum_{e \in E_n(\Gamma(X,f))} ((m_e, N) - 1) - \sum_{w \in W(\Gamma(X,f))} ((M_w, N) - 1)
$$

This is nothing else than W. Neumann’s formula for $\#_2^\lambda(f)$ [10, 32].

Now, we will show that, in order to determine the numbers $\#_2^\lambda(f)$, we don’t have to consider all the cyclic coverings, but only $G(X,f)$, the universal cyclic covering graph of $\Gamma(X,f)$. First notice that the identity (3.19) is valid even if $f^{-1}(0)$ has non-isolated singularities, but $G(X,f)$ is connected. Then $X_{f,N}$ is connected for any $N$.

3.21. Corollary. Assume that the universal covering graph $G(X,f)$ is connected (e.g. $f$ defines an isolated singularity). Then:

(a)$$
\sum_{\lambda^N=1} \#_2^\lambda(f) = c_{\text{mod}_N(G(X,f))}.
$$

(b)$$
\sum_{\lambda} \#_2^\lambda(f) = c_{G(X,f)};
$$

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i.e. the number of independent cycles of $G(X, f)$ is exactly the total number of Jordan blocks of size 2 of the algebraic monodromy $h_1$ of $f$.

(c) Let $\xi_n$ be a primitive $n^{th}$-root of unity. Then:

$$\phi(n) \cdot \#^2_{\xi_n} = \sum_{k|n} \mu(n/k) \cdot c_{mod}(G(X, f))$$

where $\phi$ is the Euler function and $\mu$ is the Möbius function, namely: $\phi(n) = \#\{1 \leq k \leq n : (k, n) = 1\}$, and $\mu(k) = 1$ for $k = 1$, $\mu(k) = (-1)^t$ if $k$ is a product of $t$ different primes, and $\mu(k) = 0$ if $p^2|k$ for some prime $p$.

Proof. (a) follows from a similar argument as (3.19) and the main theorem (3.7). For (b), take an $N$ which is multiple of all the integers $\{n_v\}_{v \in V}$ and $\{n_e\}_{e \in E}$ (the integers which characterize the type of the universal covering $G(X, f) \rightarrow \Gamma(X, f)$), and also $\lambda^N = 1$ for any eigenvalue $\lambda$ of $h_1$. For (c), first notice that $\#^2_{\xi_n}$ does not depend on the choice of $\xi_n$ (because $h_1$ is defined over $\mathbb{Z}$). Then (c) follows from the Möbius Inversion Formula (see, e.g. [16], page 107).  

3.22. Example. Set $(X, x)$ as in (2.36) (cf. also (2.25)). Then $h_1(f_1)$ is finite, but $h_1(f_2)$ has a Jordan block of size 2, with eigenvalue $\lambda = -1$. In this case $\Gamma(X, f_1) = \Gamma(X, f_2)$, but $\arg_*(f_1) \neq \arg_*(f_2)$. This shows a subtle connection between the Jordan block structure of $h_1(f)$ and the representation $\arg_*(f)$.

3.23. Example. Set $(X, x)$ as in (2.38) (or 2.26). Then for both $i = 1, 2$, $h_1(f_i)$ has a Jordan block of size 2 with eigenvalue $\lambda = 1$, because $\Gamma(X, f)$ has a cycle. Since $G(X, f)$ has two independent cycles, there is one more Jordan block of size 2 and this one has eigenvalue $\lambda = -1$.

3.24. Actually, an even stronger connection can be established between the monodromy $h_1$ and the graph $G(X, f)$. Let $|G|$ be the topological realization of $G(X, f)$. The $\mathbb{Z}$-action of $G(X, f)$ induces a “geometric action” $h_{G, \text{geom}}$ on $|G|$ (by the identification $1_g = h_{G, \text{geom}}$). At homological level, this induces a finite morphism $h_{|G|} : H^i(|G|) \rightarrow H^i(|G|)$.

In the sequel, $H^*(Y)$ denotes the cohomology with complex coefficients. We invite the reader to review the notations and the results of subsection (2.27). Recall that $\mathcal{V}(\Gamma)$ is the index set of the irreducible components of $D = \phi^{-1}(f^{-1}(0))$. Let $T_v$ be a small tubular neighborhood of the irreducible component corresponding to $v \in \mathcal{V}(\Gamma)$. For any edge $e = (v_1, v_2)$ of $\Gamma$ set $T_e = T_{v_1} \cap T_{v_2}$. If $F = f^{-1}(\delta)$ is the Milnor fiber, then for $\delta$ sufficiently small $F \subset \cup_v T_v$. Put $F_v = F \cap T_v$ and $F_e = F \cap T_e$ for any $v \in \mathcal{V}(\Gamma)$ and $e \in \mathcal{E}(\Gamma)$. Then by Mayer–Vietoris argument one has the following exact sequence:

$$0 \rightarrow H^0(F) \rightarrow \oplus_v H^0(F_v) \rightarrow \oplus_e H^0(F_e) \rightarrow H^1(F) \rightarrow \oplus_v H^1(F_v) \rightarrow \oplus_e H^1(F_e) \rightarrow 0.$$
This provides an exact sequence:

\[ 0 \to H^1(|G|) \xrightarrow{\alpha} H^1(F) \xrightarrow{\delta^0} \oplus_e H^1(F_e) \xrightarrow{\delta^1} \oplus_e H^1(F_e) \to 0. \]

The monodromy acts on this exact sequence, the operators on the corresponding groups are: \( h_{|G|} \), \( h_1 \), \( \oplus_w h_w \) and \( \oplus_e h_e \). Notice that for any \( w \in W(\Gamma) \) the natural projection of \( T_w \) to \( E_w \) induces a \( m_w \)-covering of \( F_w \to E_w \) with Galois group \( \mathbb{Z}_{m_w} \). Moreover, the monodromy action \( h_w \) can be identified with the action of the generator 1 of this Galois group. In particular, \( h_w \) has finite order. Now, it is elementary to verify that the other monodromy operators \( h_a \) \( (a \in A) \) and \( h_e \) \( (e \in E) \) are also of finite order. On the other hand, \( h_{|G|}^* \) is finite by its construction.

Let \( N_0 \) be an integer such that \( \lambda^{N_0} = 1 \) for any eigenvalue \( \lambda \) of \( h_1 \) and \( h_e \). Then \( \delta^0(\text{im}(h_1^{N_0} - 1)) = 0 \), hence \( H^1(|G|) \subset \text{im}(h_1^{N_0} - 1) \). But, by the above corollary (3.21) (part b), these spaces have the same dimension. Therefore, we have:

3.25. Corollary. Fix an \( N_0 \) such that \( \lambda^{N_0} = 1 \) for any eigenvalue \( \lambda \) of \( h_1(f) \). Then the pairs \( (H^1(|G|), h_{|G|}^*) \) and \( (\text{im}(h_1^{N_0} - 1), h_1) \) are isomorphic. This identification is compatible with the generalized eigenspace decomposition, in particular, for any eigenvalue \( \lambda \) there exists the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & H^1(|G|)_\lambda \\
\downarrow & & \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
0 & \to & H^1(F)_\lambda \\
\end{array}
\]

This shows that for any \( \lambda \):

\[ \#^2 \lambda(f) = H^1(|G|)_\lambda. \]

Actually, in this diagram we can also see all the cohomology groups \( H^1(\text{mod}_N(G)) \) for arbitrary integer \( N \). Indeed, the “orbit projection” \( \sigma_N : |G| \to |\text{mod}_N(G)| \) induces an injective morphism \( \sigma_N^* : H^1(|\text{mod}_N(G)|) \to H^1(|G|) \) (where \( \text{im}(\sigma_N^*) \) are the invariant cocycles), therefore \( \sigma_N^* \) can be identified with the inclusion

\[ H^1(\text{mod}_N(G)) = \ker(h_{|G|}^N - 1) \hookrightarrow H^1(|G|). \]

3.26. The fact that the \( 2 \times 2 \)-Jordan blocks can be characterized by a graph with a \( \mathbb{Z} \)-action, has the following interesting consequence (cf. [29]):

3.27. Proposition. Assume that \( c_\Gamma = 0 \) and \( G(X, f) \) is connected. Assume that \( h_1 \) has a Jordan block of size 2 with an eigenvalue \( \beta \) with \( \beta^k = 1 \). Write \( k = pq \) with \( (p, q) = 1 \). Then \( h_1 \) has a Jordan block of size 2 with some eigenvalue, say \( \lambda \), such that either \( \lambda^p = 1 \) or \( \lambda^q = 1 \).

Proof. Fix a vertex \( w^* \in W(\Gamma) \) such that there is an arrow \( a \in A(\Gamma) \) with \( (a, w^*) \in \mathcal{E}(\Gamma) \). Let \( \mathcal{E}_n \) be the set of edges of \( \Gamma \) connecting two non–arrowhead vertices. For any \( e \in \mathcal{E}_n \) let \( w(e) \) be the vertex of \( e \) which has larger distance in \( \Gamma \) from \( w^* \) (i.e. \( w(e) \) and \( w^* \) are in two different components of \( \Gamma \setminus \{e\} \)). Then \( e \mapsto w(e) \) defines a bijection \( \mathcal{E}_w \to W \setminus \{w^*\} \). By Euler–characteristic argument:

\[ 1 - c_{\text{mod}_N(G)} = \sum_{w \in W} \text{g.c.d.}(k, n_w) - \sum_{e \in \mathcal{E}_n} \text{g.c.d.}(k, n_e). \]
But \( n_{w} = 1 \) (cf. 2.33), hence:
\[
c_{\text{mod}_k(G)} = \sum_{\nu \in \mathcal{E}_n} \text{g.c.d.}(k,n_{\nu}) - \text{g.c.d.}(k,n_{w(\nu)}),
\]
where \( n_{w(\nu)} | n_{\nu} \).

If \( \#_2^{p}(f) \neq 0 \) for some \( \beta \) such that \( \beta^k = 1 \), then \( c_{\text{mod}_k(G)} \neq 0 \), hence there exists \( \nu \in \mathcal{E}_n \) with \( (k,n_{\nu}) > (k,n_{w(\nu)}) \). But then, for the same \( \nu \), a similar strict inequality holds either for \( p \) or for \( q \) (instead of \( k \)). This ends the proof. \( \diamond \)

3.28. Remark. The above proof contains a generalization of the result (3.20) of Neumann. Namely, even if \( g \neq 0 \) (but with \( c_{\Gamma(X)} = 0 \)), there exist positive integers \( n_i \) and \( m_i \) (\( i \in I \)), with \( m_i | n_i \), such that:
\[
\sum_{\lambda^3 = 1} \#_3^{\lambda}(f) = \sum_i \text{g.c.d.}(N,n_i) - \text{g.c.d.}(N,m_i).
\]

3.29. The finiteness of \( h_1 \) revisited. Above we proved the characterization \( \sum_{\lambda} \#_2^{\lambda}(f) = c_{\text{G}(X,f)} \), but we can still ask: when is this number zero?

For example, Lê D. T. proved that the monodromy of an irreducible germ \( f: (\mathbb{C}^2,0) \to (\mathbb{C},0) \) is always finite [20].

In Lê’s result the irreducibility assumption is really important. Indeed, if we take \( f: (\mathbb{C}^2,0) \to (\mathbb{C},0) \) given by \( f(x,y) = (x^2 + y^3)(x^3 + y^2) \), then by (2.38), the resolution of the double covering \( \mathbb{C}^2_{f,2} \) has a cycle, hence \( h_1(f) \) is not finite (historically, this is the first germ with non–finite monodromy, found by A’Campo [1]).

On the other hand, if \( f: (X,x) \to (\mathbb{C},0) \) has a finite monodromy \( h_1(f) \), then \( c_{\Gamma(X)} \) must be zero (cf. 3.18).

So, this motivates to investigate the case when \( c_{\Gamma(X)} = 0 \) and \( \#\mathcal{A}(\Gamma) = 1 \).

Under a more restrictive condition, when \( L_X \) is an integer homology sphere, W. Neumann (using a stronger version of the identity (3.20)) proved that \( h_1(f) \) is finite, provided that \( \#\mathcal{A}(\Gamma) = 1 \) (see [10] page 111, or [32]). The next result is a generalization of the results of Lê and Neumann.

3.30. Theorem. (cf. also with [29]) Assume that \( L_X \) is a rational homology sphere and \( f: (X,x) \to (\mathbb{C},0) \) defines an isolated singularity with \( \#\mathcal{A}(\Gamma) = 1 \). Then if \( \#_2^{\lambda} \neq 0 \) for some \( \lambda \) of order \( k \), then there exists a prime number \( p \geq 2 \) such that \( p|k \) and \( p^2 \) divides \( |H_1(L_X,\mathbb{Z})| \).

In particular, if \( |H_1(L_X,\mathbb{Z})| \) is square free, then \( h_1(f) \) is finite for any \( f \).

(Recall that the order \( |H_1(L_X,\mathbb{Z})| \) of the torsion group \( H_1(L_X,\mathbb{Z}) \) is exactly \( |\text{det}(E_w \cdot E_v)| \).)

Proof. Assume that \( \#_2^{\lambda} \neq 0 \) with \( \lambda^k = 1 \). Then by (3.27), there is a prime number \( p \) and eigenvalue \( \eta \) such that \( p|k \), \( \eta^{p^2} = 1 \) for some \( \alpha \geq 1 \), and \( \#_2^{\alpha} \neq 0 \).

Write the characteristic polynomial \( \Delta(t) = \det(th_1 - 1) \) as a product of cyclotomic polynomials \( \prod_i \phi_i(t)^{n_i} \). By Wang’s exact sequence and Alexander duality, coker \( (h_1 - 1) = H_1(L_X,\mathbb{Z}) \). Then \( \Delta(1) = \prod_i \phi_i(1)^{n_i} = |H_1(L_X,\mathbb{Z})| \).

Now, if \( l \) is not a prime power, then \( \phi_l(1) = 1 \), whereas \( \phi_l(1) = q \) if \( l \) is a power of the prime \( q \). Since \( h_1 \) has a double eigenvalue of order a power of the prime \( p \), we obtain the divisibility \( p^2|\Delta(1) \). \( \diamond \)
3.31. Example. [29] Consider the embedded resolution graph of \( f : (X, x) \to (\mathbb{C}, 0) \), where \( (X, x) = (\{ x^{10} + x^3y^2 + y^3 + z^2 = 0 \}, 0) \subseteq (\mathbb{C}^3, 0) \) and \( f(x, y, z) = x \):

![Graph Image]

Then \( |\det(E_v \cdot E_w)| = |H_1(L_X, \mathbb{Z})| = 4 \). Then \( G \) can be computed from 2.33, hence the monodromy \( h_1 \) contains exactly one Jordan block of size 2 and the eigenvalue of this block is \(-1\).

Finally, we ask: in the above result (3.30), is it really important for \( L_X \) to be a rational homology sphere? Is the monodromy finite, for example, if \( c_1(\Gamma) = 0 \), \( \#A(\Gamma(X, f)) = 1 \) and \( \text{Tors}(H_1(L_X, \mathbb{Z})) = 0 \) (i.e. the intersection matrix \((E_w \cdot E_v)\) is unimodular)? The answer is negative! A possible counterexample is the following:

3.32. Example. In theorem (3.30) not only \( c_1 = 0 \) is important but also \( g = 0 \). To see this, take \( (X, x) = (\{ x^2 + y^7 - z^{14} = 0 \}, 0) \subseteq (\mathbb{C}^3, 0) \) and the function \( f_2(x, y, z) = z^2 - y \) as in (2.25) and (2.36). Let \( P \) be the intersection point of the strict transform \( S_0 \) of \( \{ f_2 = 0 \} \) with the exceptional divisor \( E \). Then, in some local coordinate system \((u, v)\) of \( P \), \( \{ u = 0 \} \) represents \( E \) (in a neighborhood of \( P \)), \( \{ v = 0 \} \) represents \( S_0 \), and \( f_2 = u^2v^2 \) (cf. 2.25). Consider \( g = z \). Since \( z \) in the neighborhood of \( P \) can be represented as \( z = u \) (modulo a local invertible germ), \( f_2 + g^k \) near \( P \) has the form \( u^2v^2 + u^k \). For example, if \( k = 3 \), then one needs two more blowing ups in order to resolve \( f_2 + g^k \). In this case, the graph \( \Gamma(X, -y + z^2 + z^3) \) and its covering \( G(X, -y + z^2 + z^3) \) are the following:

![Graph Image]

So, in this case, the intersection matrix of \( \Gamma \) is unimodular, hence \( H_1(L_X, \mathbb{Z}) \) is torsion free, \( c_1 = 0 \), and \( \#A = 1 \), but \( \#_{-1}^2 = 1 \), hence \( h_1 \) is not finite. The reason is that \( g \neq 0 \).
This also shows that part b) of Theorem B in [29] is true only with the additional assumption \( g = 0 \). (The other statements of [29] are correct; the error in (B,b) comes from the identity (1) on page 591.)

References

[1] N. A’Campo, Sur la monodromie des singularités isolées d’hypersurfaces complexes, Inventiones math., **20** (1973), 147-169.
[2] N. A’Campo, La fonction zêta d’une monodromie, Comment. Math. Helv., **50** (1975), 233-248.
[3] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable maps, Volume 1 and 2, Monographs Math., **82-83**, Birkhäuser, Boston, 1988.
[4] E. Artal–Bartolo, Forme de Seifert des singularités de surface, C. R. Acad. Sci. Paris, t. **313**, Série I (1991), 689-692.
[5] W. Barth, C. Peters and A. Van de Ven, Compact Complex Surfaces, Springer-Verlag, 1984.
[6] E. Brieskorn, Die Monodromie der isolierten Singularitäten von Hyperflächen, Man. Math. **2**(1970), 103-161.
[7] E. Brieskorn and H. Knörrer, Plane Algebraic Curves, Birkhäuser, Boston, 1986.
[8] Ph. Du Bois and F. Michel, The integral Seifert form does not determine the topology of plane curve germs, J. of Algebraic Geometry **3** (1994), 1-38.
[9] A. H. Durfee, The Monodromy of a Degenerating Family of Curves, Inventiones Math., **28** (1975), 231-241.
[10] D. Eisenbud and W. Neumann, Three-Dimensional Link Theory and Invariants of Plane Curve Singularities, Ann. of Math. Studies **110**, Princeton University Press, 1985.
[11] H. Grauert, Über Modifikationen und exceptionelle analytische Mengen, Math. Ann., **146** (1962), 331-368.
[12] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics 52, Springer-Verlag, 1977.
[13] J. Harer, A. Kas and R. Kirby, Handlebody decomposition of complex surfaces, Mem. Amer. Math. Soc., **62** (1986) no 350.
[14] F. Hirzebruch, Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei complexen Veränderlichen, Math. Ann., **126** (1953), 1-22.
[15] F. Hirzebruch, W. D. Neumann and S. S. Koh, Differentiable manifolds and quadratic forms, Math. Lectures Notes, 4, Dekker, New York, 1972.
[16] Hua Loo Keng: Introduction to Number Theory, Springer-Verlag, 1982.
[17] H. W. E. Jung, Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränderlichen \( x, y \) in der Umgebung einen stelle \( x = a, y = b \), J. Reine Angew. Math., **133** (1908), 289-314.
[18] A. Landman, On the Picard–Lefschetz formula for algebraic manifolds acquiring general singularities, Thesis, Berkeley 1967.
[19] H. B. Laufer, Normal two-dimensional singularities, Annals of Math. Studies **71**, Princeton University Press 1971.
[20] Lê Dũng Tráng, Sur les noeuds algébriques, Compositio Math., **25** (1972), 281-321.
[21] Lê Dũng Tráng, Some remarks on relative monodromy, In: Real and Complex Singularities, Oslo 1976, P. Holm ed., Sijthoff & Noordhoff, Alphen a/d Rijn 1977, 397-403.
[22] J. Lipman, Introduction to resolution of singularities, Proc. Symp. Pure Math., **29** (1875), 187-230.
[23] E. J. N. Looijenga, Isolated Singular Points on Complete Intersections, London Math. Soc. Lecture Note Series **77**, Cambridge University Press, Cambridge, 1984.
[24] J. Milnor, Singular Points of Complex Hypersurfaces, Annals of Math. Studies, Vol. **61**, Princeton University Press, 1868.
[25] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion of simplicity, IHES Publications Math., **9** (1961), 229-246.
[26] A. Némethi, Dedekind sums and the signature of \( f(x, y) + z^N \), Selecta Mathematica, New series, **4** (1998), 361-376.
[27] A. Némethi, Dedekind sums and the signature of \( f(x, y) + z^N \), II., Selecta Mathematica, New series, **5**, 161-179 (1999).
[28] A. Némethi, The signature of $f(x,y) + z^n$, Proceedings of Singularity Conference, (C.T.C Wall’s 60th birthday meeting), Liverpool, August 1996; London Math. Soc. lecture Notes, 263 (1999), 131-149.
[29] A. Némethi and J. Steenbrink, On the monodromy of curve singularities, Math. Zeitschrift, 223 (1996), 587-593.
[30] A. Némethi and Á. Szilárd, Resolution graphs of some surface singularities, II. (generalized Iomdin series), this Proceedings.
[31] W. D. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Transaction of the AMS, 268, 2 (1981), 299-344.
[32] W. D. Neumann, Splicing Algebraic Links, Advanced Studies in Pure Math., 8 (1986), Complex Analytic singularities, 349-361.
[33] I. Ono and K. Watanabe, On the singularity of $z^p + y^q + x^{pq} = 0$, Sci. Rep. Tokyo Kyoika Daigaku Sect. A, 12 (1974), 123-128.
[34] P. Orlik and Ph. Wagreich, Isolated singularities of algebraic surfaces with $\mathbb{C}^*$ action, Ann. of Math., (2) 93 (1971), 205-228.
[35] A. Scharf, Faserungen von Graphenmannigfaltigkeiten, Dissertation, Bonn, 1973; summarized in Math. Ann., 215 (1975), 35-45.

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