A Polynomial Decision for 3-sat

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Abstract

We propose a polynomially bounded, in time and space, method to decide whether a given 3-sat formula is satisfiable or not.

The tools we use here are, in fact, very simple. We first decide satisfiability for a particular 3-sat formula, called pivoted 3-sat and, after a plain transformation, still keeping the polynomial boundaries, it is shown that 3-sat formula is unsatisfiable if and only if its corresponding pivoted formula is unsatisfiable.

1 Introduction

After the preliminary definitions that establish a common language, we define a Pivoted 3-sat formula, the main subject we deal with. After that, we show that any 3-sat formula has a stronger version in the format of a Pivoted 3-sat in the sense that a Pivoted 3-sat formula is unsatisfiable if and only if the original 3-sat formula is unsatisfiable and, if the original 3-sat formula is a theorem (true under any valuation), its pivoted version is not a theorem.

Secondly, we show that we can solve, that is, we can decide if the pivoted formula is Satisfiable or Unsatisfiable, in a polynomial use of space and time. Our resolution methods will require no more than basic knowledge of Boolean Logic and some basic skills.

An excellent way to start on the Complexity topic is by reading Scott Aaronson breathtaking survey,
https://www.scottaaronson.com/papers/pnp.pdf

The question is, “What is computing?” What kind of reasoning, operation, can be performed using an algorithm” and how fast the operation can be performed. See [2] for a general overview, or consult the roots, [12], [10], [11], [5], [1], [7], [5], [9].
A precise definition of *algorithm* was given by Alan Turing in 1937 (see [13]). The natural question that arises is: *What is the computational difficulty of performing such an algorithm?* See, in chronological order, [11], [10], [7], [5], [2]. Classification of complexity is found in [6], [4] and [2]. We aim to open a new trail to understand complexity questions on polynomially solving our problem that obviously can be solved in exponential time and space and draw new lines on the complexity classification.

In Section 2, we provide classical definitions of valuation for Boolean Logic, define SAT, pose some more standard definitions, and introduce Pivoted 3-sat the special kind of 3-sat we focus on. To solve the central problem, decide satisfiability or unsatisfiability, of Pivoted 3-sat we write a digraph, the cylindrical digraph, a digraph where disjunctions of formulas play the role of edges endowed with labels. We show that, given a 3-sat formula \( \Psi \), there is a pivoted 3-sat formula \( \Psi_T \) so that \( \Psi \) is unsatisfiable if and only if \( \Psi_T \) is unsatisfiable.

After building the cylindrical digraph, we detach parts of the cylindrical digraph, the closed digraphs that mark unsatisfiable combinations of formulas in a pivoted 3-sat.

In Section 3, we fully describe the algorithms we use in the process of deciding whether the closed digraph represents all \( 2^m \) combinations of valuation over the conjugated literals of a Pivoted 3-sat. We will not write down all possible combinations to have a polynomial bound in our solution. Rather, we analyze the question there is a satisfiable combination by generating a digraph.

Section 4 is the connecting link between a 3-sat formula and a pivoted 3-sat formula. We demonstrate that given a 3-sat formula \( \Psi \), then there exists a strong version of \( \Psi \), \( \Psi_T \), so that \( \Psi \) is unsatisfiable if and only if \( \Psi_T \) is unsatisfiable.

Section 5 is quite a technical section and, as usual, a profound and critical reading follows after unraveling the basic Section, 2 and the foundation part, Section 3.

Finally, a very long part of this paper is devoted to the examples, Section 6. This section can be regarded as a companion section.

## 2 Basic Definitions

We work with a Boolean Language whose basic symbols are \( \lor, \land, \neg \) endowed with a finite set of atoms \( \mathcal{A} \). The set of literals, \( \mathcal{L} \), is the set \( \mathcal{A} \cup \{\neg p | p \in \mathcal{A}\} \). A pair formed by a literal together with its negation is called a conjugated
pair. Besides a finite set of atoms, we have the symbols $\top$ and $\bot$.

Here we give the (classical) valuation definition over a Boolean formula. That is the definition we will use henceforth.

**Definition 2.1.** Given a finite set of atoms, $\mathcal{A}$, $\psi$ belongs to the set of Boolean formulas $\mathcal{F}$ if

1. $\psi \in \mathcal{A}$;
2. $\psi = \chi \land \phi$ and both $\chi$ and $\phi$ belong to $\mathcal{F}$;
3. $\psi = \chi \lor \phi$ and both $\chi$ and $\phi$ belong to $\mathcal{F}$;
4. $\psi = \neg \chi$ and $\chi$ belongs to $\mathcal{F}$;
5. $\psi$ is either $\top$ or $\bot$.

We use the symbols

1. $\psi \rightarrow \chi$ to denote $\neg \psi \lor \chi$;
2. $\psi \equiv \chi$ to denote $(\psi \rightarrow \chi) \land (\chi \rightarrow \psi)$.

**Definition 2.2.** A valuation over a set of atoms $\mathcal{A}$ is a mapping $v$ from $\mathcal{A}$ to the set $\{\text{True}, \text{False}\}$ and, as usual, we extend the mapping $v$ from a formula $\psi$ to by reducing the complexity of writing a formula. Define the extension mapping $v$ to a mapping $v'$ from the formulas into the set $\{\text{True}, \text{False}\}$ as above,

1. If $\psi \in \mathcal{A}$, $v'(\psi) = \text{True}$ if $v(\psi) = \text{True}$;
2. $\psi = \chi \land \phi$, $v'(\psi) = \text{True}$ if $v'(\chi) = \text{True}$ and $v'(\phi) = \text{True}$;
3. $\psi = \chi \lor \phi$, $v'(\psi) = \text{True}$ if $v'(\chi) = \text{True}$ or $v'(\phi) = \text{True}$;
4. $\psi = \neg \chi$, $v'(\psi) = \text{False}$ if and only if $v'(\chi) = \text{True}$;
5. $v'(\bot) = \text{False}$ and $v'(\top) = \text{True}$.

If we do not have $v'(\psi) = \text{True}$, then $v'(\psi) = \text{False}$.

**Definition 2.3** (SAT). A formula $\chi$ in Boolean Logic is said satisfiable if there is a valuation $v$ from the set $\{\text{True}, \text{False}\}$ onto the set of atoms of $\chi$ so that $v(\chi)$ is $\text{True}$. If no such valuation exists, that is $v(\chi)$ is $\text{False}$ for any valuation, then we say that $\chi$ is unsatisfiable.
Any formula in the Boolean Logic is **decidable**, that is, given a formula \( \chi \), we can decide if \( \chi \) is **satisfiable** (True for some valuation) or **unsatisfiable** (False for any valuation). The complexity problem consists of deciding between the values, **True** or **False** in the most optimal way, see \[.\]

We use the symbols: **AND**, **OR**, **NOT** and **IMPLIES** to denote external logical symbols.

**Definition 2.4.** A **2-sat** formula \( \Psi \) is a conjunction of a finite number of a disjunction of at most two literals. We write

\[
\Psi \equiv (l_1^1 \lor l_2^1) \land \cdots \land (l_s^1 \lor l_2^1) \equiv C_1 \land \cdots \land C_s
\]

A subformula \( C_k \equiv l_1^k \lor l_2^k \), \( 1 \leq k \leq s \) of \( \Psi \) is called a clause.

A **3-sat** formula \( \Psi \) is a conjunction of a finite number of a disjunction of at most three literals. We write

\[
\Psi \equiv (l_1^1 \lor l_2^1 \lor l_3^1) \land \cdots \land (l_s^1 \lor l_2^1 \lor l_3^1) \equiv C_1 \land \cdots \land C_s
\]

A subformula \( C_k \equiv l_1^k \lor l_2^k \lor l_3^k \), \( 1 \leq k \leq s \) of \( \Psi \) is called a clause.

Define pivoted **3-sat** formulas,

**Definition 2.5.** A **Pivoted 3-sat** formula is a **3-sat** formula of the form,

\[
(a_1 \lor p_1^1 \lor q_1^1) \land \cdots \land (a_1 \lor p_1^{n_1} \lor q_1^{n_1}) \land
\]

\[
(-a_1 \lor r_1^1 \lor s_1^1) \land \cdots \land (-a_1 \lor r_1^{n_2} \lor s_1^{n_2}) \land \cdots \land
\]

\[
(a_m \lor p_m^1 \lor q_m^1) \land \cdots \land (a_m \lor p_m^{n_m} \lor q_m^{n_m}) \land
\]

\[
(-a_m \lor r_m^1 \lor s_m^1) \land \cdots \land (-a_m \lor r_m^{n_m} \lor s_m^{n_m})
\]

or, in factored form,

\[
(a_1 \lor (p_1^1 \lor q_1^1) \land \cdots \land (p_1^{n_1} \lor q_1^{n_1})) \land
\]

\[
(-a_1 \lor (r_1^1 \lor s_1^1) \land \cdots \land (r_1^{n_2} \lor s_1^{n_2})) \land \cdots \land
\]

\[
(a_m \lor (p_m^1 \lor q_m^1) \land \cdots \land (p_m^{n_m} \lor q_m^{n_m})) \land
\]

\[
(-a_m \lor (r_m^1 \lor s_m^1) \land \cdots \land (r_m^{n_m} \lor s_m^{n_m}))
\]

where the set of pivots, \( \{a_1, -a_1, \ldots, a_m, -a_m\} \) has no intersection with the set of entries,

\[
\{p_1^1, q_1^1, \ldots, p_1^{n_1}, q_1^{n_1}, r_1^1, s_1^1, \ldots, r_1^{n_2}, s_1^{n_2}, \ldots, p_m^1, q_m^1, r_m^1, s_m^1, \ldots, r_m^{n_m}, s_m^{n_m}\}
\]

Denote the pivot \( a_i \) by \( 1i \) and its conjugated pair \( -a_i \) by \( 2i \). 

4
Call the set of entries of $a_i$ the set of conjunctions,

$$ (p_1^i \lor q_1^i) \land \cdots \land (p_{ka_i}^i \lor q_{ka_i}^i) $$

and the set of entries of $\neg a_i$ the set of conjunctions,

$$ (r_1^i \lor s_1^i) \land \cdots \land (r_{ka_i}^i \lor s_{ka_i}^i) $$

**Definition 2.6.** Given a Pivoted formula $\Psi$, replace any clause of the form $a_i \lor p$, where $a_i$ is a pivot, by the clauses $(a_i \lor p \lor q_p) \land (a_i \lor p \lor \neg q_p)$, where the pairs of conjugated, $q_p$ and $\neg q_p$ are fresh new literals.

A pivoted **3-sat** formula is said complete if all of its clauses contain three literals.

**Lemma 2.7.** Any pivoted **2-sat** formula is logically equivalent to its complete version.

From now on, we work with complete **3-sat** pivoted formulas.

It is well known that **2-sat** formulas can be polynomially solved, a result originally shown in [3]. See [4]. We will store unsatisfiable groups of pivoted **3-sat** formulas in groups of **2-sat**.

**Proposition 2.8.** Given **2-sat** formula $\Psi$,

$$ (p_1 \lor q_1) \land \cdots \land (p_k \lor q_k) $$

$\Psi$ is unsatisfiable if and only if there is a pair of conjugated literals, $\{a, \neg a\}$ and a set of conjugated literals

$$ \{b_1, \neg b_1, \ldots, b_r, \neg b_r, c_1, \neg c_1, \ldots, c_s, \neg c_s\} $$

so that

$$ (a \lor b_1) \land (\neg b_1 \lor b_2) \land \cdots \land (\neg b_r \lor a) $$

and

$$ (\neg a \lor c_1) \land (\neg c_1 \lor c_2) \land \cdots \land (\neg c_s \lor \neg a) $$

are subformulas of $\Psi$.

Explore the above result as a support tool to achieve our claim. We store the data contained in a pivoted **3-sat** formula in a cylindrical digraph.

**Definition 2.9.** Let $\Psi$ be a pivoted (complete) **3-sat** formula. The cylindrical digraph associated to $\mathcal{P}$, $\mathcal{C}_{\text{indr}} = (V, E, \text{label})$, where $V$ is a set of vertices, $E$ is a set of edges and label is a mapping from $E$ to the set of parts of the set of entries is,
1. The set of vertices is the set of literals together with their negations;

2. If \( a \lor b \) belongs to the set of entries \( \{i_1 j_1, i_2 j_2, \ldots, i_k j_k\} \), then,

   (a) \( \neg a \Rightarrow b \) and \( \neg b \Rightarrow a \) belong to the set edges, \( E \);

   (b) \( i_1 j_1, i_2 j_2, \ldots, i_k j_k \) is the label of both edges \( \neg a \Rightarrow b \) and \( \neg b \Rightarrow a \).

Sequences of vertices connected by \( \Rightarrow \) are a path, as we define below,

**Definition 2.10.** Let \( a \) and \( b \) be two literals, there is a path from \( a \) to \( b \) if there is a subdigraph of \( C_{lmdr} \) the form

\[
a \xRightarrow{t_1} c^1 \xRightarrow{t_2} c^2 \xRightarrow{t_3} \ldots \xRightarrow{t_{k-1}} c^{k-1} \xRightarrow{t_k} b
\]

(1)

Some subgraphs of the cylindrical digraph will be associated to combinations of unsatisfiable formulas and we keep track of these subgraphs and, thus, we can decide, in an optimal way whether they represent or not all possible unsatisfiable combinations.

We search for sequences of the form,

\[
\neg a \Rightarrow b^1 \Rightarrow b^2 \Rightarrow \ldots \Rightarrow b^{t_1} \Rightarrow a
\]

and

\[
a \Rightarrow c^1 \Rightarrow c^2 \Rightarrow \ldots \Rightarrow c^{t_2} \Rightarrow \neg a
\]

(2)

We conclude that a Pivoted 3-sat formula \( \Psi \) has an unsatisfiable combination, with the entries chosen among the set of labels over sequences depicted in Equation 2.

**Notation 2.11.** Let \( 2^m \), the set of all mappings from \( m \) to \( 2 = \{1, 2\} \). Write an element of \( 2^m \) as \( \{1j_1, 2j_2, \ldots, mj_m\} \), that is, the above set represents the graphic of a mapping \( f : m \mapsto 2 \) and an entry, \( ij \) has the meaning that \( f(i) = j \).

We identify an entry \( ap \) with \( p^1 \) and an entry \( \neg ap \) with \( p^2 \), \( 1 \leq p \leq m \).

**Definition 2.12.** Given a Pivoted 3-sat formula, \( \mathcal{C} \), our task is to verify, in the most efficient way, regarding the use of time and space, whether for all all combinations \( \sigma \) in \( 2^m \), for all literals in \( \sigma \) regarded as False whether the resulting 2-sat formula is unsatisfiable, or, in a specular rephrasing, we verify whether there is an element \( \sigma = \{1j_1, 2j_2, \ldots, mj_m\} \) in \( 2^m \), in which the combination,

\[
\bigwedge_{1 \leq r \leq m} \{\land(p^{r j_r} \lor q^{r j_r})\mid (p^{r j_r} \lor q^{r j_r}) \in \text{ contents } r j_r\}
\]

(3)

is satisfiable.
In short, we ask if the Pivoted $3$-sat formula is unsatisfiable or satisfiable, by first making valuations ranging over the set of pivots. Using Proposition 2.8, we search the sequences that lead to unsatisfiable formulas and, thereafter, we decide whether we wrote all possible combinations or not.

We set the keystones by defining the cylindrical digraph associated with a given pivoted sat formula. Any disjunction contained in an entry plays the role of a vertex in a cylindrical digraph.

**Definition 2.13.** Given a Pivoted Formula $\Psi$, an interval $[p, q]$ is the sub-digraph that contains all sequences in between $p$ and $q$ and contains no loops, that is there is not a vertex $r$ so that a sequence

$$r \Rightarrow \cdots \Rightarrow r$$

belongs to $[p, q]$ and the set of necessarily true literals, $\mathcal{NEC}$, is the set of all pair of conjugated literals so that the intervals $[-a, a]$ and $[a, -a]$ are non-empty.

**Definition 2.14.** A pair of necessarily true literals is a conjugated pair of literals, $\{a, -a\}$ so that there are paths from $a$ to $-a$ and from $-a$ to $a$.

The set of necessarily true literals is denoted by $\mathcal{NEC}$.

For each pair of conjugated literals, $\{a, -a\} \in \mathcal{NEC}$, we write distinct copies of $[-a, a]$ and $[a, -a]$. As usual, distinct copies are fabricated using Cartesian Products to differentiate intervals.

**Definition 2.15.** Given a cylindrical digraph, suppose that the cardinality of $\mathcal{NEC}$ is an even number $r \geq 2$ and the set of necessarily true literals is $\{p_1, -p_1, \ldots, p_r/2, -p_r/2\}$. Define the set of closed digraph, $\mathcal{CSD}$ as the union of all digraphs of the form $[-p_l, p_l] \oplus [p_l, -p_l]$, 1.

$$[p_l, -p_l] \times \{2l\}, \quad 1 \leq l \leq r/2;$$

2. $[-p_l, p_l] \times \{2l - 1\}, \quad 1 \leq l \leq r/2$.

Define the edges and labels as the edges and labels inherited by the projection to the interval and endowed with the identification $(p_{2l-1}, 2l - 1) \sim (p_{2l}, 2l)$.

**Definition 2.16.** A chain is a sequence contained in the set of closed digraphs,

$$(-a, 2l - 1) \overset{l_0}{\Rightarrow} (c^1, 2l - 1) \overset{l_1}{\Rightarrow} (c^2, 2l - 1) \overset{l_2}{\Rightarrow} \cdots \overset{l_{k-1}}{\Rightarrow} (c^k, 2l - 1) \overset{l_k}{\Rightarrow} (a, 2l - 1) \sim (a, 2l)$$

$$\overset{l_{k+1}}{\Rightarrow} (c^{k+1}, 2l) \overset{l_{k+2}}{\Rightarrow} (c^{k+2}, 2l) \overset{l_{k+3}}{\Rightarrow} \cdots \overset{l_{t-1}}{\Rightarrow} (c^{t+1}, 2l) \overset{l_t}{\Rightarrow} (-a, 2l)$$
The set of all chains is denoted by \( \mathcal{P} \) and called Expansion of the Closed Digraph.

Consider the set of all chains in a closed digraph. Enumerate them by \( \{1, \ldots, s\} \). To each chain \( Ch_t \) enumerated by \( t \), consider the Cartesian Product \( Ch_t \times \{t\} \), labels, edges inherited by the labels and edges in \( Ch_t \).

**Definition 2.17.** The mapping

\[
Ch : \mathcal{P} \mapsto \mathcal{CSD} \\
(v, i) \mapsto v
\]

is the projection of vertices of \( \mathcal{P} \) onto \( \mathcal{CSD} \).

The expansion to \( \mathcal{P} \) is the fixed (exponential) writing of all maximal chains contained in a closed digraph. The enumeration of the chains is arbitrary and, from now on, remains fixed.

Given the set of chains, if \( (v, t) \) belongs to a chain in \( \mathcal{P} \), then \( v \) is a vertex in the cylindrical digraph and \( (u, v) \) is an edge of the cylindrical digraph if there is a chain \( Br_t \) and two arrows \( (u, t) \Rightarrow (v, t) \) in \( Br_t \), in \( Br_t \).

**Definition 2.18.** Consider an edge \( e = ((u, t) \Rightarrow (v, t)) \) in \( \mathcal{P} \). Define \( Ch(e) \) as

\[
Ch(u, t) \Rightarrow Ch(v, t)
\]

**Definition 2.19.** A compatible set of entries is a set \( U = \{i_1j_1, \ldots, i_nj_n\} \) so that for all pair \( i_r \) and \( i_s \), if \( i_r = i_s \) then \( j_r = j_s \). If a set of entries is not compatible, it is said an incompatible set.

A set of edges \( E = \{e_1, e_2, \ldots, e_k\} \) is said incompatible if the union of it set of labels, \( U = \{i_1j_1, \ldots, i_nj_n\} \) is an incompatible set of entries. If a set of edges is not incompatible, we say that \( \forall \) is compatible.

We wrote digraphs that encode all unsatisfiable choices. We must decide whether we have, in fact, the set of all unsatisfiable choices. Again, we will not spell all the compatible, if any, choices. It is an expsize task.

Any chain \( Br \) and a compatible choice in \( Br \) is an unsatisfiable combination. After writing all unsatisfiable combinations we must decide whether we wrote all possible combinations (we have an unsatisfiable formula) or not (the pivoted formula is satisfiable).

**Definition 2.20.** Given a chain in \( \mathcal{P} \),

\[
p = c^0 \Rightarrow c^1 \Rightarrow c^2 \Rightarrow \ldots \Rightarrow c^{k-1} \Rightarrow c^k
\]

a compatible choice in \( p \) is a set of compatible entries \( E = \{j_1i_1, \ldots, j_ri_i\} \), so that for each edge \( e_i = c^i \Rightarrow c^{i+1} \), \( 0 \leq i < k \), there is an entry \( ij \in E \) so that \( ij \in \text{label}(e_i) \).
If one casts a tableau where all combinations of pivots, \( \sigma \in 2^m \) labeled as False and there is a sequence
\[
\neg a \Rightarrow c^1 \Rightarrow c^2 \Rightarrow \ldots \Rightarrow c^{k-1} \Rightarrow c^k \\
a \Rightarrow c^k+1 \Rightarrow c^k+2 \Rightarrow \ldots \Rightarrow c^{t-1} \Rightarrow \neg a = c^t
\]
where \( \{q_0, q_1, q_2, \ldots, q_{k-1}, q_k, q_{k+1}, \ldots, q_{t-1}, q_t\} \subseteq \sigma \) and each \( q_j \) belongs to the label of the label \( l_j \), \( 0 \leq j \leq t-1 \), then the branch that falsifies the elements of \( \sigma \) closes. So, the next question is whether all possible sequences appear in \( \mathcal{P} \) and, therefore the pivoted formula is unsatisfiable or not, that is, the pivoted formula is satisfiable.

Given a Pivoted 3-sat formula \( \Psi \), the set of all compatible choices in \( \mathcal{P} \) is the set of all unsatisfiable combinations we will find in \( \Psi \). We now ask how one can decide whether the set of all compatible choices entails all possible combinations, thus, the Pivoted 3-sat is unsatisfiable.

Define the set of vertices orthogonal to the set of chains.

**Definition 2.21.** A set of edges \( \tilde{\text{seq}} = \{e_0, \ldots, e_n\} \) is called an antichain if for all \( \text{seq} \in \mathcal{P} \), there is an element \( e \in \tilde{\text{seq}} \) so that \( e \) is an edge of \( \text{seq} \).

We do not focus on the set of vertices any more. Instead, focus on the set of edges, or, more specifically, on all the compatible choices of labels in \( \mathcal{P} \). If all possible combinations were written, the formula would be unsatisfiable. Otherwise, the formula is satisfiable. Do all possible choices encompass all the possible combinations?

**Definition 2.22 (ENTAILS).** Let \( T(2^m) \) be a set of partial mappings of \( m \) into \( \{1, 2\} \). We say that \( T(2^m) \) entails \( 2^m \) if for all \( \gamma \in 2^m \), there is an \( \eta \in T(2^m) \) so that \( \eta \) is a restriction of \( \gamma \), that is, \( \eta \) is a subset of some \( \gamma = \{i_1, i_2, \ldots, i_m m\} \).

We search for chains in \( \mathcal{CSD} \). Writing all the paths is an expensive task. Recall that in Definition 2.15, \( \mathcal{P} \) denotes the set of all chains in \( \mathcal{CSD} \).

**Definition 2.23.** Let \( \tau \) be the set of all entries \( \{1j_1, \ldots, mj_m\} \) contained in \( 2^m \) so that
\[
\exists \text{seq} \in \mathcal{P} \quad \forall e \in \text{seq} \quad \exists 1 \leq l \leq m(i_l j_l \in \text{label}(e))
\]

Let \( \tilde{\tau} \) be the complementary of \( \tau \), that is, the set of all entries \( \tilde{\eta} = \{1j_1, \ldots, mj_m\} \) so that
\[
\forall \text{seq} \in \mathcal{P} \quad \exists e \in \text{seq} \quad \forall 1 \leq l \leq m(i_l j_l \notin \text{label}(e))
\]
Proposition 2.24 below marks a decisive step to build our theoremhood. After building the closed digraphs, and applying Proposition 2.24, our search is focused on looking for compatible antichains. Again, we emphasize that our answer is a plain output, YES or NO, without naming the compatible antichains if any. In other words, we have a plain program and we expect an answer YES or NO regarding the possibility of the existence of some kind of input.

Proposition 2.24. The following assertions are equivalent,

1. There is no compatible set of entries in $\tilde{\tau}$;
2. $\tau$ entails $2^m$;
3. There is no compatible antichain in $P$.

Proof. Clearly 1 and 2 are equivalent. If seq is a compatible set of entries in $\tilde{\tau}$, then, seq $\notin \tau$ for the sets are complementary.

1 and 2 imply 3. Suppose that there is a compatible antichain in $P$, $Ach = \{e_1, \ldots, e_k\}$. Let $lb(Ach)$ be the union of the set of labels of each vertex of Ach. As Ach is compatible, $lb(Ach)$ has the form

$$\{t_1r_1, t_2r_2, \ldots, t_lr_l\}$$

where $1 \leq t_1 < t_2 < \ldots < t_l \leq m$ and $r_i$ is either 1 or 2. That is, for all $1 \leq n \leq l$ we have only one choice, either $r_n = 1$ or $r_n = 2$. So, for all $1 \leq s \leq m$, we can choose a $\tilde{t}_s$ so that $\tilde{t}_s \notin ach$.

Therefore, there is a non-empty set

$$Comp = \{\tilde{t}_11, \tilde{t}_22, \ldots, \tilde{t}_m m\}$$

that is compatible and $Comp \cap Ach = \emptyset$. Conclude that $Comp$ belongs to $\tilde{\tau}$ and, thus, there is a compatible array in $\tilde{\tau}$.

Suppose that 1 is false and $s = \{t_1r_1, t_2r_2, \ldots, t_mr_m\}$ is a compatible set of entries in $\tilde{\tau}$ and, then, for all sequence $s_r \in P$, there is an edge $e_r$ so that $label(e_r) \cap s = \emptyset$. Let $A = \{e_r | e_r \in s_r \ AND \ (label(e_r) \cap s = \emptyset)\}$. As $s$ is compatible, $A$ is compatible, so, $A$ is compatible antichain in $P$. \qed

Due to Theorem 2.24 we can solve our game by deciding whether all antichains are incompatible or if there is a compatible antichain. Bear in mind that the word “entails $2^m$” encodes the fact that all the combinations in the pivoted 3-sat formula are unsatisfiable.
3 The Search For Compatible Antichains

In this section, we develop the tools we use for deciding whether a closed digraph entails $2^m$. Using Proposition 2.24, we search for compatible antichains in the closed digraph.

There is no need to search for unsatisfiable combinations. Using Proposition 2.24, we search compatible antichains, that is, we deal with pivots. Sequences of vertices were used to build the closed digraph. By using Proposition 2.24, our attention goes to the question of whether there is a compatible antichain in the closed digraphs and, thus, the set of closed digraphs do not entail $2^m$. Again, we stress that we do not develop tools to write out all the antichains and, therefore, spell out all the possible compatible combinations which is an expsize problem. The polysize problem consists of the search for a plain output YES, there are antichains or NO, there is no antichain.

The set of all chains stores all possible unsatisfiable combinations one can perform. We already wrote the closed digraph and, now, our attention is devoted to the search for compatible antichains, a search we perform in the set of labels. Each edge of a digraph has a label that will be used to point out the relevant parts that weigh in our search for antichains.

Given a closed digraph, we first modify the shape of the digraph to avoid exponential branching, the subject of the first Subsection 3.1. Lastly, we show that the reshaping, the Linearized Digraph does not change the set of compatible antichains.

In the second step, Subsection 3.2, after modifying the closed digraph we write a digraph that marks the compatible antichains orthogonal to the chains of the Linearized Digraph. A result of an empty digraph signalizes no compatible antichains in the Linearized Digraphs and, as we show in this Section, no compatible antichain in $\mathcal{P}$. Proposition 2.24 below marks a decisive step to build our theoremhood. After building the closed digraphs, and applying Proposition 2.24 our search is focused on looking for compatible antichains. Again, we emphasize that our answer is a plain output, YES or NO, without naming the compatible antichains, if any. In other words, we have a plain program and we expect an answer YES or NO regarding the possibility of the existence of some kind of input.

3.1 A Simpler Digraph

Here, we model a new shape to the set of closed digraphs. To avoid an exponential search for the antichains, we build a digraph, the Linearized
Digraph less complex than the closed digraph that is a mirror of the antichains in the Expansion of the Closed Digraph. The search for compatible antichains is unchanged because we show there are compatible antichains in the Linearized Digraph if and only if there are compatible antichains in the closed digraph. We show that the search for compatible antichains over the Linearized Digraph is polynomial in time and space.

In this section, we develop the tools for deciding whether a closed digraph entails $2^m$. Using Proposition 2.24, we search for compatible antichains in the closed digraph instead of searching for unsatisfiable combinations. The set of sequences of vertices, played the main role in building closed digraphs. Guided by Proposition 2.24, our attention goes to the question of whether the labels contain a compatible antichain and, thus, the set of closed digraphs do not entail $2^m$. Again, we stress that we do not develop tools to write out all the antichains and, therefore, spell out all the possible compatible combinations, which is an expsize problem. The polysize problem consists of the search for a plain output YES, there are antichains or NO, there is no antichain.

The set of all chains stores all possible unsatisfiable combinations.

Given a closed digraph, we first modify the shape of the digraph to avoid exponential branching, the subject of the first Subsection 3.1. We must show that the reshaping, the Linearized Digraph does not change the set of compatible antichains.

Our proof strategy, schematized to avoid exponential multiplication on writing the set of chains, is the transformation of closed digraphs into linear digraphs.

**Definition 3.1.** Given a closed digraph, define,

1. $R$, the set of roots is $\{a \in V | \exists b \in V ((a \Rightarrow b) \in E)\}$;
2. $T$, the set of tops is $\{a \in V | \exists b \in V ((b \Rightarrow a) \in E)\}$;
3. If there are at least two edges
   
   $a \Rightarrow b_1$
   $a \Rightarrow b_2$

   or
   
   $b_1 \Rightarrow a$
   $b_2 \Rightarrow a$

   where $b_1 \neq b_2$, then $a$ is a branching.
**Definition 3.2.** A sequence of labeled edges in a closed digraph is called a maximal branch if seq is of the form

\[ c^0 l_1 \Rightarrow c^1 l_2 \Rightarrow c^2 l_3 \cdots c^{k-1} l_k \Rightarrow c^k \]  

(4)

where \( c^k \) is either a root or a branching and \( c^0 \) is either a top or a branching and no other branching belongs to the sequence.

If an interval \([a,b]\) contains no branching besides \( a \) and \( b \), then it is called a linear interval.

A linear interval \([a,b]\) where \( b \) is a root, is called a root interval.

**Definition 3.3.** Use the term branch to design a linear sequence

\[ c^0 \Rightarrow c^1 \Rightarrow \cdots \Rightarrow c^{k-1} \Rightarrow c^k \]

where except by, perhaps, \( c^0 \) and \( c^k \), no other vertex is a branching.

We will define the operations over the closed digraphs, Lifting, Multiplication of Branches and the Addition of Labels, the operations 3.4, 3.5 and 3.7 and, without any loss in our search for compatible antichains, the given closed digraph is reshaped into a simpler digraph, the Linearized Digraph. Of course, we have to show that we do not lose or gain information about compatible antichains. Examples of the operations and more explanations can be found in the institutional page,

www.ime.usp.br/~weiss

In addition, Example 6.1 illustrates the use of all operations defined next.

**Definition 3.4 (Lifting).** Given a closed digraph \( Gr \),

- If \( B \) and \( R \) are two consecutive linear intervals, that is, they share a common branching, \( v \),

\[ br_{11} = v_{11} \Rightarrow \cdots \Rightarrow v_{1t_{11}} \Rightarrow v \]

\[ \cdots \]

\[ br_{1k} = v_{1k} \Rightarrow \cdots \Rightarrow v_{kt_{1k}} \Rightarrow v \]

and

\[ br_{21} = v \Rightarrow v_{21} \Rightarrow \cdots \Rightarrow v_{2t_{21}} \]

\[ \cdots \]

\[ br_{2k} = v \Rightarrow v_{2k} \Rightarrow \cdots \Rightarrow v_{2t_{2k}} \]
the Lifting of $Gr$ by $B$ and $R$ is $Gr(B\ast R) = \langle V_{B\ast R}, E_{B\ast R}, \text{label} \rangle$,

\[
\begin{align*}
V_{B\ast R} &= (V_G \setminus \{v\}) \cup (\{v\} \times \{1, \ldots, k\}) \\
E_{B\ast R} &= (E_G \setminus \{v_{t_1} \Rightarrow v, \ldots, v_{k_t} \Rightarrow v, \ldots, v \Rightarrow v_2, \ldots, v \Rightarrow v_2, \ldots\}) \\
&\quad \cup \{v_{t_1} \Rightarrow (v, 1), \ldots, v_{k_t} \Rightarrow (v, k), \\
&\quad \ldots, (v, 1) \Rightarrow v_2, \ldots, (v, k) \Rightarrow v_2\} \\
\end{align*}
\]

The labels of each $w \Rightarrow (v, j)$ and $(v, j) \Rightarrow w$ are the same labels associated with $w \Rightarrow v$ and $v \Rightarrow w$, respectively.

- If $R$ is of the form

\[
\begin{align*}
br_1 &= v_1 \Rightarrow \cdots \Rightarrow v_{t_1} \Rightarrow r \\
\vdots \\
br_n &= v_n \Rightarrow \cdots \Rightarrow v_{t_n} \Rightarrow r
\end{align*}
\]

where $r$ is a root, the Lifting of $Gr$ by $R$ is $Gr(R) = \langle V_R, E_R, \text{label} \rangle$ given by,

\[
\begin{align*}
V_R &= (V \setminus \{r\}) \cup (\{r\} \times \{1, \ldots, n\}) \\
E_R &= (E \setminus \{v_{t_1} \Rightarrow r, \ldots, v_{t_n} \Rightarrow r\}) \cup \\
&\quad \{v_{t_1} \Rightarrow (r, 1), \ldots, v_{t_n} \Rightarrow (r, n)\}
\end{align*}
\]

The labels of each $v_{t_j} \Rightarrow (r, j)$ are the same labels of $v_{t_j} \Rightarrow r$.

- If $B$ is of the form

\[
\begin{align*}
br_{11} &= a \Rightarrow v_1 \Rightarrow \cdots \Rightarrow v_{t_1} \\
\vdots \\
br_{1k} &= a \Rightarrow v_m \Rightarrow \cdots \Rightarrow v_{t_m}
\end{align*}
\]

where $a$ is a top, the Lifting of $Gr$ by $B$ is $Gr(B) = \langle V_B, E_B, \text{label} \rangle$ given by,

\[
\begin{align*}
V_B &= (V \setminus \{a\}) \cup (\{a\} \times \{1, \ldots, m\}) \\
E_B &= (E \setminus \{a \Rightarrow v_1, \ldots, a \Rightarrow v_m\}) \cup \\
&\quad \{(a, 1) \Rightarrow v_1, \ldots, (a, m) \Rightarrow v_m\}
\end{align*}
\]

The labels of each $(a, j) \Rightarrow (v_j)$ are the labels of $a \Rightarrow v_j$

**Definition 3.5** (Multiplication of a branch). Let

\[
br = v \Rightarrow v_1 \Rightarrow \cdots \Rightarrow v_j \Rightarrow u
\]

be a maximal branch contained in a linear interval $[v, u] = \langle V, E, \text{label} \rangle$, $m = \{1, \ldots, m\}$ a finite set and $M = \{v_1, \ldots, v_j\} \times \{1, \ldots, m\}$. 

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The Multiplication of the branch \( br \) is the the interval \([v,u]'\), where the vertices are given by the set \((V \setminus \{v_1,\ldots,v_j\}) \cup M\) and the edges of \(V \setminus \{v_1,\ldots,v_j\}\) plus the edges in the new sequences,

\[
v \Rightarrow (v_1,i) \Rightarrow \cdots \Rightarrow (v_j,i) \Rightarrow u\]

\(1 \leq i \leq m\).

As one cannot form edges among vertices in distinct linear sequences, the definition is consistent.

**Definition 3.6.** Given a closed digraph and a vertex \(a\), define \(U_p a\) and \(Dow n a\), respectively as sets of vertices,

\[
\{v \in V \mid \exists n \in \mathbb{N} \{v_1,v_2,\ldots,v_n\} \subseteq V \mid (v \Rightarrow v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_n \Rightarrow a)\}
\]

\[
\{w \in V \mid \exists n \in \mathbb{N} \{w_1,w_2,\ldots,w_n\} \subseteq V \mid (a \Rightarrow w_n \Rightarrow \cdots \Rightarrow w_2 \Rightarrow w_1 \Rightarrow w)\}
\]

An edge \(v_1 \Rightarrow v_2\) belongs to \(U_p a\) if \(v_1\) belongs to \(U_p a\). An analogous definition is given to \(Dow n a\), that is, \(v_2\) belongs to \(Dow n a\).

Next, we present a tool to avoid the creation of new compatible antichains in the linearized digraph we will build next as a simpler, easier to manage compared to the closed digraph. Suppose there is no addition of new labels. In that case, the Linearized Digraph we build next as a simpler digraph will present new compatible antichains with no corresponding antichain under a suitable mapping in \(\mathcal{P}\).

**Definition 3.7** (Adding a Label). Let the set of the branching in a closed digraph be \(B = \{a_1,\ldots,a_t\}\). Consider a set of conjugated pairs of literals, \(B = \{p_{a_1},p_{a_1},\ldots,p_{a_t},p_{a_t}\}\) that are all distinct from the literals we have been using. The addition of the new pair of literals to the set of branching is the new set of labels, \(label'\) defined by,

\[
label'(e) = label(e) \cup \{p_{a_i} \mid e \in Dow n(a_i)\} \cup \{\neg p_{a_i} \mid e \in Up(a_i)\}
\]

We do not overload our manuscript, so we drop the prime symbol, \(label'\) and write only \(label\).

In Example 6.1, we emphasize the role of adding a pair of conjugated literals that work to prevent the creation of a new antichain that never existed in the originally set \(\mathcal{P}\).

To perform Lifting, Multiplication of Branches and Addition of a Label to obtain a simpler digraph, we must show that there will be no prejudice in counting antichains. We cannot create or erase antichains.
Reshape a closed digraph into a set of linear intervals on Procedure 3.8. The mainstay to sustain our construction is that, after writing Procedure 3.8, we obtain a linear digraph that is equivalent to the originally closed digraph. We show that our remodeling will not impair the preexisting antichains or create new antichains.

Lifting, Multiplying Branches and Adding labels in a closed digraph are operations used to generate linear digraphs with no branching and, mainly, no loss of information about compatible antichains stored in $\mathcal{P}$ with no exponential multiplication of branches. The operations are illustrated in Example 6.1.

**Procedure 3.8** (Changing a Closed Digraph into a Linear Digraph). Given a closed digraph $CSD$, and a set of conjugated literals distinct of any literals from the set of labels of $CSD$, $S = \{p_1, \neg p_1, \ldots, p_r, \neg p_r\}$, where $V$ is the cardinality of literals (vertices). First, we transform the closed digraph, by Multiplying branches, into a digraph whose intervals, from branching to a root, are maximal linear sequences. After we obtain a digraph whose intervals, from branching to a root, are maximal linear sequences, we multiply branches, add suitable labels, and perform lifting.

1. Let the roots of the closed digraph be $\{e_1, \ldots, e_n\}$ and let the set of all root intervals be

   $r_{11} \Rightarrow \cdots \Rightarrow e_1$
   ..
   $r_{l1} \Rightarrow \cdots \Rightarrow e_1$
   ..
   $r_{1s} \Rightarrow \cdots \Rightarrow e_n$
   ..
   $r_{ls} \Rightarrow \cdots \Rightarrow e_n$

   Lift each maximal branch. Obtain the digraph $Modf$,

   $r_{11} \Rightarrow \cdots \Rightarrow (e_1, 1)$
   ..
   $r_{l1} \Rightarrow \cdots \Rightarrow (e_1, l_1)$
   ..
   $r_{1s} \Rightarrow \cdots \Rightarrow (e_r, 1)$
   ..
   $r_{ls} \Rightarrow \cdots \Rightarrow (e_r, l_r)$

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2. Suppose we obtain sets of root intervals of the form,

\[ u \Rightarrow \cdots \Rightarrow v_1 \\
\vdots \\
\Rightarrow \cdots \Rightarrow v_n \]

where \( \{v_1, \ldots, v_n\} \) are roots in the digraph.

Let all the maximal branches ending on \( u \) be,

\[ w_1 \Rightarrow \cdots \Rightarrow u \\
\vdots \\
w_t \Rightarrow \cdots \Rightarrow u \]

Multiply the branches by \( m = \max\{n, t\} \) and lift. Add to the set of labels a pair of conjugated literals \( p_u \) and \( \neg p_u \) never used in our process, to, respectively the edges in \( \text{Down}_u \) and \( \text{Up}_u \). Rewrite \( \text{Modf} \) as,

\[ w_1 \Rightarrow \cdots \Rightarrow (u, 1) \Rightarrow \cdots \Rightarrow v_1 \\
\vdots \\
w_m \Rightarrow \cdots \Rightarrow (u, m) \Rightarrow \cdots \Rightarrow v_m \]

Proceed until all intervals are lifted.

3. Finally, we have sets of maximal branches of the form,

\[ v \Rightarrow \cdots \Rightarrow u \Rightarrow \cdots \Rightarrow t_1 \\
\vdots \\
v \Rightarrow \cdots \Rightarrow u \Rightarrow \cdots \Rightarrow t_o \]

where \( v \) is a top and \( \{t_1, \ldots, t_o\} \) are roots. Lift all the branches and obtain

\[ (v, 1) \Rightarrow \cdots \Rightarrow u \Rightarrow \cdots \Rightarrow t_1 \\
\vdots \\
(v, o) \Rightarrow \cdots \Rightarrow u \Rightarrow \cdots \Rightarrow t_o \]

Call the Linearized Closed Digraph to the linear digraphs obtained after the successive applications of Lifting, Multiplication and Addition of a label. Denote the Linearized Closed Digraph by \( \text{Linclsd} \).

Suppose that each linear branch has the cardinality \( k \) and write each linear branch in \( \text{Linclsd} \) as \( \text{Br}_i \) as a short for \( \text{Br}_i \times \{i\} \). Define the edges whose arrow is inherited by \( v \Rightarrow u \).
We show that a Linearized Digraph, despite being simpler than the set of chains, preserves compatible antichains in the sense that we do not create nor erase information about compatible antichains in Lemma 3.16. Recall that the vertices of either \( \mathcal{P} \) and \( \text{Linclsd} \) are pairs, a vertex in the closed digraph and a number.

**Definition 3.9.** The mapping

\[
\text{Proj} : \text{Linclsd} \to \mathcal{CSD} \\
(v, j) \mapsto v
\]

is the projection of vertices of \( \text{Linclsd} \) onto \( \mathcal{CSD} \).

**Definition 3.10.** Consider the edge \( e' = (u' \Rightarrow v') \) in \( \text{Linclsd} \). Define \( \text{Proj}(e') \) as the edge,

\[
\text{Proj}(u') \Rightarrow \text{Proj}(v')
\]

Definitions 3.10 and 2.18 are well posed for if \( u' \Rightarrow v' \) and \( u'' \Rightarrow v'' \) are two edges, respectively in \( \text{Linclsd} \) and in \( \mathcal{P} \), then \( \text{Proj}(u') = \text{Proj}(v') \) and \( \text{Ch}(u'') = \text{Ch}(v'') \) are, equally, edges in \( \mathcal{CSD} \).

**Lemma 3.11.** Let \( \text{ach} \) be an antichain in the Linearized Digraph. Then, for all branching \( a \), either \( \text{Proj}(\text{Ch}(\text{ach})) \cap \text{Up}(a) \) or \( \text{Proj}(\text{Ch}(\text{ach})) \cap \text{Down}(a) \) is empty.

**Proof.** Due to the addition of labels, there is no branching \( a \in \mathcal{CSD} \) so that two edges \( e_1 \in \text{Up}(a) \) and \( e_2 \in \text{Down}(a) \) belong to a compatible antichain \( \text{ach} \) in the Linearized Digraph because both are incompatible.

**Definition 3.12.** The set of all antichains in \( \mathcal{P} \) is denoted by \( \Sigma \) and the set of all antichains in \( \text{Linclsd} \) is denoted by \( \Theta \).

Have we created or erased antichains? As the relevant antichains are kept, we search for compatible antichains in \( \text{Linclsd} \). We show that there are compatible antichains in \( \text{Linclsd} \) if and only if there are compatible antichains in \( \mathcal{P} \).

Given a branching \( a \), recall the definition of a branch in \( \text{Up}(a) \) is a linear sequence, in \( \mathcal{P} \) of the form,

\[
v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow a
\]

**Definition 3.13.** For all set of edges \( Z \) in \( \mathcal{P} \) and branching \( a \), we say that \( Z \) represents \( \text{Up}(a) \) if for all branching \( \text{Br} \) in \( \text{Up}(a) \), there is a \( z \in \text{Ch}(Z) \)
so that \( z \) belongs to \( \text{Br} \). A symmetric definition applies to \( \text{Down}(a) \), that is, for all branching \( \text{Br} \) in \( \text{Down}(a) \), there is a \( z \in \text{Ch}(Z) \) so that \( z \) belongs to \( \text{Br} \).

For all set of edges \( Z \) in \( \mathcal{P} \), \( Z \) involves a set of branching \( \mu \) if for all branching \( a \in \mu \), either \( Z \) represents \( \text{Up}(a) \) or \( Z \) represents \( \text{Down}(a) \).

**Proposition 3.14.** Given an antichain \( \sigma \) in \( \mathcal{P} \), for all branching \( a \), \( \sigma \) represents \( \text{Up}(a) \) or \( \text{Down}(a) \).

**Proof.** Let \( \sigma \) be an antichain in \( \mathcal{P} \). The antichain \( \text{Up}(a) \) or \( \text{Down}(a) \) where “or” is not exclusive.

Suppose otherwise. Let the branches of \( \text{Up}(a) \) and \( \text{Down}(a) \) as, respectively,

\[
\begin{align*}
  u_{11} & \ldots u_{r1} \\
  \vdots & \vdots \\
  u_{1j_1} & \ldots u_{rj_r} \\
  a & \ldots a
\end{align*}
\]

and

\[
\begin{align*}
  a & \ldots a \\
  v_{11} & \ldots v_{t1} \\
  \vdots & \vdots \\
  v_{1j_1} & \ldots v_{tj_t}
\end{align*}
\]

Combine the above chains and obtain the chains in \( \mathcal{P} \),

\[
\begin{align*}
  u_{11} & \ldots u_{11} \ldots u_{r1} \ldots u_{r1} \\
  \vdots & \vdots \vdots \vdots \vdots \vdots \\
  u_{1j_1} & \ldots u_{1j_1} \ldots u_{rj_r} \ldots u_{rj_r} \\
  (a, 1, 1) & \ldots (a, 1, t) \ldots (a, r, 1) \ldots (a, r, 1) \\
  v_{11} & \ldots v_{t1} \ldots v_{11} \ldots v_{t1} \\
  \vdots & \vdots \vdots \vdots \vdots \\
  v_{1j_1} & \ldots v_{tj_t} \ldots v_{1j_1} \ldots v_{tj_t}
\end{align*}
\]

If the antichain \( \sigma \) has no edges in columns, say \( r \) and \( s \) in, respectively \( \text{Up}(a) \) and \( \text{Down}(a) \), then the combination of the two columns \( r \times s \), does not contain any edge of the antichain.

In conclusion, for any antichain \( \sigma \), for any branching \( a \), either all edges of \( \text{Ch}(\sigma) \) are a member of one edge in \( \text{Up}(a) \) or all edges of \( \text{Ch}(\sigma) \) belong to \( \text{Down}(a) \).

\[\square\]

**Proposition 3.15.** Given an antichain \( \sigma \) in \( \mathcal{P} \), \( \sigma \) involves the set of branching of \( \mathcal{P} \).
Proof. Let \( a_1 < \cdots < a_s \) be an arbitrary ordering of all branches in the closed digraph.

Use Proposition 3.14 to \( a_1 \). If \( \sigma \) represents \( Up(a_1) \), obtain a new antichain \( \sigma_1 \) by defining,

1. Let \( Br_{t1}, \ldots, Br_{ts} \) be the set of branches in \( Up(a_1) \), still in the closed digraph. For all \( 1 \leq i \leq s \), choose a \( x_i \rightarrow y_i \in Br_{ti} \cap Ch(\sigma) \). Erase any edge whose projection via \( Ch \) in \( Br_{ti} \) is different from \( x_i \rightarrow y_i \) and add to \( \sigma_1 \) all \( (x_i, t) \rightarrow (y_i, t) \) whose projection is \( x_i \rightarrow y_i \).

Otherwise, \( \sigma \) represents \( Down(a_1) \) and proceed similarly.

Obtain a new antichain \( \sigma_1 \). Notice that \( Ach(\sigma_1) \subseteq Ach(\sigma) \). Suppose that we obtained \( \sigma_i, 1 \leq i < t < r \) and that \( Ach(\sigma_t) \subseteq \cdots \subseteq Ach(\sigma_1) \subseteq Ach(\sigma) \) and that either,

1. \( Ach(\sigma_t) \cap Up(a_1) = \emptyset \) and \( Ach(\sigma_t) \cap Down(a_1) \) has a singleton edge at each branch in \( Down(a_1) \) or,

2. \( Ach(\sigma_t) \cap Up(a_1) \) has a singleton edge at each branch in \( Up(a_1) \) and \( Ach(\sigma_t) \cap Down(a_1) = \emptyset \).

Suppose that \( \sigma_t \) represents \( Up(a_{t+1}) \).

1. Let \( Br_{t+1}, \ldots, Br_{t+s_{t+1}} \) be the branches in \( Up(a_{t+1}) \). For all \( 1 \leq i \leq s_{t+1} \), choose a \( x_i \rightarrow y_i \in Br_{ti+1} \cap Ch(\sigma_t) \). Erase any edge whose projection via \( Ch \) in \( Br_{ti+1} \) is different from \( x_i \rightarrow y_i \) and add to \( \sigma_{t+1} \) all \( (x_i, t) \rightarrow (y_i, t) \) whose projection is \( x_i \rightarrow y_i \).

Otherwise, \( \sigma_t \) represents \( Down(a_{t+1}) \) and a mirror reasoning applies.

As a result, obtain a series of nested antichain that we build (can build in several random ways), \( Ach(\sigma) \supseteq \cdots \supseteq Ach(\sigma_t) \), where the cardinality of \( t \) does not exceed the cardinality of the set of branching. Given the already chosen ordering, for all branching \( a_i \), we choose preferentially \( Up(a_i) \), if possible, and, if we do not have this choice, we have the choice \( Down(a_i) \).

We have that \( Ach(\sigma_t) \) cannot be reduced and, using Proposition 3.14, represents any branching in the set of branches, thus, \( \sigma \), as well as its reduction, involves the set of branching.

Lemma 3.16. There is an onto mapping \( Eq \) from \( \Sigma \) to \( \Theta \).

Proof. The operations of Lifting, Multiplying and Adding Labels are fixed, so, after reducing an antichain \( \sigma \) to its minimal version \( \sigma_t \), we obtain, in the Linearized Digraph, an antichain \( \overline{\sigma_t} \), the result of the fixed operation over \( \sigma_t \) in the Linearized Digraph. The onto mapping is given by \( Eq(\sigma) \) is the expansion of \( Ch(\sigma_t) \).
In short, our quest is whether there is a compatible antichain in a closed
digraph is equivalent to the quest whether there is a compatible antichain
in a linearized digraph.

3.2 Digraph of Compatible Antichains

Now, our attention is devoted to the search for compatible antichains, which
we perform in the set of labels. Each edge of a digraph has a label that will be
used to point out the relevant parts that weigh in our search for antichains.

In outline, once we have linear digraphs, $k$ linear digraphs, each linear
digraph in the column $r$ endowed $l_r$ vertices. To decide the existence of
compatible antichains in the Linearized Digraph, we filter series of digraphs
associated to each vertex $v$. These digraphs, called nested digraph will mark
compatible antichains. In the world of Nested Digraphs, the set of labels
plays the role of vertices. To avoid a heavy load, we discard unnecessary ele-
ments. Labels are the fundamental elements and, as we advance our search,
labels play the role of vertices and the connection if given by compatibility.

Up to now, we no longer need the set of literals, that played the role of
vertices in the Linearized Digraph and whose sequences encode unsatisfiable
combinations. Once we cleaned new paths, we pursued for new strategy
for solving a card. The maximal chains in the Closed Digraph were used
to mark the unsatisfiable combinations and our search is centered on the
search for a compatible antichain. So, we are dealing only with the set of
labels.

It is time to discharge vertices and edges in the Linearized Digraph be-
cause these elements were used to mark unsatisfiable choices and we no
longer use them. We use lighter gadgets ruling out the set of vertices, the
set of literals.

A sequence of linear digraphs of the form,

$$a \Rightarrow b_1 \Rightarrow l_2 \Rightarrow \ldots \Rightarrow b_t \Rightarrow l_{t+1} \Rightarrow c$$

corresponds to the sequence

$$l_1 \rightarrow l_2 \rightarrow \cdots \rightarrow l_t \rightarrow l_{t+1}$$

After modifying the closed digraph we write a digraph that marks the
compatible antichains orthogonal to the chains of the Linearized Digraph.
A result of an empty digraph signalizes no compatible antichains in the
Linearized Digraphs and, as we show in this Section, no compatible antichain
in $P$. 

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Let \((r, s)\) be a enumeration of the pair \textit{column} and \textit{location} in the array of \(k\) columns each with \(m_k\) elements. Write the pair edge together with the enumeration. An edge \((p, i, j) \Rightarrow (q, i, j + 1) \Rightarrow (r, i, j + 2)\) belongs to column \(i\) and the vertices belong to, respectively, the \(j, j+1\) and \(j+2\) places.

**Definition 3.17.** Given a Linearized Digraph, define its Companion Digraph as the digraph whose vertices are the set of indexes associated with each edge of the linearized digraph. Define an edge \((l_1, i, j) \Rightarrow (l_2, i, j + 1)\) if there is a sequence \((p, i, j) \Rightarrow (q, i, j + 1) \Rightarrow (r, i, j + 2)\) in the linearized digraph.

If \((p, i, j) \Rightarrow (q, i, j + 1) \Rightarrow (r, i, j + 2)\) and \((p', i', j') \Rightarrow (q', i', j' + 1) \Rightarrow (r', i', j' + 2)\) we have two distinct edges \((l_1, i, j) \Rightarrow (l_2, i, j + 1)\) and \((l_1, i', j') \Rightarrow (l_2, i', j' + 1)\), spite labels are the same.

We will not overload our manuscript. Distinct edges are originated from distinct columns and places. From now on we omit all extra notation and drop the pair column location.

Now, set the scenario for building nested digraphs associated with each vertex (a label). We expect that a \(k^{th}\)-interaction to generate an output \textit{satisfiable}. We set the necessary definitions.

**Definition 3.18.** Let \(Lab\) be a Digraph. We say that \(Lab\) has dept \(k\) if the length of all maximal sequences of \(Lab\) is \(k\).

**Definition 3.19.** Let \(Lab\) be a Digraph and \(v\) a vertex of \(Lab\). An edge-labeled digraph for \(v\) is a closed rooted digraph whose edges are labeled and whose root is \(v\). We label the edges with a set of vertices, that is, there is a mapping from the set of edges of \(Gr(v)\) into sets of vertices.

\[
G(v) = \langle V_v, E_v, \text{labels}_v : E_v \mapsto \mathcal{P}(V_v) \rangle
\]

The mapping \(\text{labels}_v\) associate to each edge a set of vertices and \(\text{labels}(v)\) is called the label of the vertex \(v\).

We require, moreover, that all maximal sequences in \(G\) have length \(p\).

Define the basic operations over \textit{labeled digraphs} we perform to obtain a simpler digraph and, without altering the the state of the closed digraph concerning the question there are compatible antichains or not.

**Definition 3.20.** Given two labeled digraphs whose root is \(v\),

\[
Gr_v = \langle V_v, E_v, \text{labels}_v : V_v \mapsto \mathcal{P}(E_v) \rangle
\]
and
\[ Gr'_v = (V'_v, E'_v, labels'_v : E'_v \mapsto \mathcal{P}(E'_v)) \]
their union, \( Gr_v \cup Gr'_v \) is given by
\[ (V \cup V', E \cup E', labels''_v : E \cup E' \mapsto \mathcal{P}(V \cup V')) \]
where \( labels''_v : E \cup E' \mapsto \mathcal{P}(V \cup V') \) is given by
\[
\begin{align*}
labels''_v(v) &= label(v), \text{ if } v \in E_v \setminus E'_v \\
labels''_v(v) &= label'(v), \text{ if } v \in E'_v \setminus E_v \\
labels''_v(v) &= label(v) \cup label'(v), \text{ if } v \in E_v \cap E'_v
\end{align*}
\]
Their intersection is given by,
\[ (E_v \cap E'_v, E_v \cap E'_v, labels''_v : E_v \cap E'_v \mapsto \mathcal{P}(E_v \cap E'_v)) \]
where \( labels''_v(v) = label(v) \cap label'(v) \), if \( v \in E_v \cap E'_v \).

**Definition 3.21 (Nested Digraph)**. A rooted edge-labeled digraph, \( Gr_v = (V_v, E_v, labels_v : E_v \mapsto \mathcal{P}(E_v)) \) is a nested digraph of length \( p \) if \( Gr_v \) is the union of a number \( s \) of linear digraphs of a fixed depth \( p \geq 1 \),
\[ v \leftarrow a_1^2 \leftarrow a_1^3 \leftarrow \cdots \leftarrow a_1^p \]
so that for each linear digraph, we have \( label(a_1^{i-1} \leftarrow a_1^i) = \{a_1^p, a_1^{p-1}, \ldots, a_1^i\} \), \( 1 \leq i \leq s \).

Notice that not necessarily vertices at each level \( j, 1 \leq j \leq p \) are distinct but, at each distinct level, the vertex are pairwise distinct.

A compatible nested digraph of length \( p \) is the union of linear digraphs of length \( p \) whose labels are compatible sets.

**Lemma 3.22.** The union of two nested digraphs of depth \( p \), \( G(v) \) and \( G(v)' \) is a nested digraph.

The intersection of two nested digraphs of depth \( p \) is not necessarily a nested digraph but we can select the maximal nested digraph contained in the intersection. Uniqueness follows from the fact that, given two nested maximal digraphs, \( G(v) \) and \( G(v)' \), then the union of a nested digraph is nested and, therefore, we cannot have two distinct maximal nested digraphs contained in \( G(v) \cap G(v)' \).

The maximal nested digraph contained in the intersection of two nested digraphs, \( G(v) \) and \( G(v)' \), contains the set of all compatible antichains contained in both \( G(v) \) and \( G(v)' \).
**Definition 3.23.** Let $G(v)$ and $G(v)'$ be two nested digraphs of depth $p$. Denote by $\text{max}G(v)$ the Maximal Nested Subdigraph contained in $G(v) \cap G(v)'$.

We will write an algorithm, Pseudocode 3.24, to obtain $\text{max}G(v)$, the maximal nested digraph contained in the intersection of two nested digraphs.

**Pseudocode 3.24.** Given a digraph $G(v) = \langle V,E,\text{label} \rangle$, intersection of two nested digraphs, obtain, using the two pseudocodes above, $\text{max}G(v)$.

We denote the vertices of the intersection by $v^m_i$, where $m$ is a level and $i$ is the position of the vertex in the level $m$ and $v = v^0_1$.

**First Pseudocode:** From $m := 0$, step 1, to $k-2$, do:
For all edges $u^m_i \leftarrow v^m_j$, consider the vertices that form an ascending sequence of length 2 with $u^m_i \leftarrow v^m_j$, say, the set $\{w^m_1, \ldots, w^m_{24}\}$. For $s$ ranging from 1 to $l$, consider as provisional labels,

$$\text{lbl}_{i,j,s}(w^m_s \rightarrow v^m_j) = \text{label}(w^m_s \rightarrow v^m_j) \cap \text{label}(w^m_j \rightarrow v^m_i)$$

Define,

$$\text{label}'(u^m_j \rightarrow v^m_i) = \cup_s \text{lbl}_{i,j,s}(w^m_s \rightarrow v^m_j) \cup \{v^m_j\}$$

Once we defined all labels, $\text{label}'(u^m_j \rightarrow v^m_i)$, define

$$\text{label}(w^m_s \rightarrow v^m_j) = \cup_s \text{lbl}_{i,j,s}(w^m_s \rightarrow v^m_j)$$

and, finally, $\text{label}(u^m_j \rightarrow v^m_i) \leftarrow \text{label}'(u^m_j \rightarrow v^m_i)$.

The final result of applying the first pseudocode is the union of linear digraphs ordered by $\Sigma$.

**Second Pseudocode:** We search backwards, for $m := k$ STEP $-1$ UNTIL 2.
For all $v^m_i$, let $L_{mi}$ be the set of all edges

$$\{v^m_{j-1} \rightarrow u^m_i \mid u^m_i \in \text{label}(u^m_i \rightarrow v^m_{j-1})\}$$

so that $u^m_i \in L_{mi}$.

For all $v^m_{j-1} \in L_{mi}$, define $G_{u^m_i,v^m_{j-1}}$ as the union of all linear sequences

$$w^k_{r_k} \rightarrow w^k_{r_{k-1}} \rightarrow \cdots \rightarrow u^m_i \rightarrow v^m_{j-1} \rightarrow \cdots \rightarrow v$$

where $\{u^m_i, v^m_{j-1}\}$ is contained in all labels of degree lower than $m - 1$.

For all $e$ in $Gr_{u^m_i,v^m_{j-1}}$, define,

$$\text{label}(e) = (\text{label}(e) \setminus \{u^m_p, v^m_{q-1} \mid p \neq i \text{ AND } q \neq j\})$$

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Define \(G_{m,m-1}(v)\) as \(\cup_{1 \leq i \leq m} (\cup_{v_{i}^{m-1}} E_{m} G_{r_{i}^{m-1} v_{j}^{m-1}})\)

Go to let \(G \leftarrow G_{m,m-1}(v)\) and perform step \(m - 1\). Repeat until level 2.

In the second procedure, we rule out all edges whose labels do not comply with the rules of nested digraphs.

We show that we obtain a nested digraph that is the maximal nested digraph contained in the intersection of two nested digraphs that share the same root.

Proposition 3.25 \((\max\!G(v))\). Using Pseudocodes [3.24], given two nested digraphs, \(G(v)'\) and \(G(v)''\), obtain \(\max\!G(v)\).

Proof. Define,

\[\text{lbl}_{i,j,s}(w_{s}^{m+2} \rightarrow v_{j}^{m+1}) = \text{label}(w_{s}^{m+2} \rightarrow v_{j}^{m+1}) \cap \text{label}(u_{i}^{m+1} \rightarrow v_{i}^{m})\]

\[\text{label}(w_{s}^{m+2} \rightarrow v_{j}^{m+1}) = \cup_{i} \text{lbl}(w_{s}^{m+2} \rightarrow v_{j}^{m+1})\]

\[\text{label}(u_{i}^{m+1} \rightarrow v_{i}^{m}) = \cup_{l} \text{lbl}(w_{l}^{m+2} \rightarrow v_{i}^{m+1}) \cup \{v_{j}^{m+1}\}\]

Deduce from the previous sentences that \(\text{label}(u_{j}^{m+1} \rightarrow v_{i}^{m})\) is the union, ranging over all \(s\), of \(\text{lbl}(w_{s}^{m+2} \rightarrow v_{j}^{m+1})\) and \(\text{label}(w_{s}^{m+2} \rightarrow v_{j}^{m+1})\) is the union of all temporally labels \(\text{lbl}(w_{l}^{m+2} \rightarrow v_{j}^{m+1})\) and, thus, we obtain, using the first algorithm, the union of linear digraphs whose labels are ordered by \(\subseteq\).

Using the second pseudocode, each \(G_{u_{i}^{m} v_{j}^{m-1}}\) is built after a selection of all linear chains so that \(\{u_{i}^{m}, v_{j}^{m-1}\}\) is contained in the label of all edges of dept lower than \(m - 1\). \(G_{u_{i}^{m} v_{j}^{m-1}}\) is modified by ruling out all labels

\[\{w_{r}^{s}|(r = m \text{ IMPLIES } s \neq i) \text{ AND } (r = m - 1 \text{ IMPLIES } s \neq j)\}\]

In this way, we keep only the linear arrays of depth \(k\) so that it its edges of length less than \(m\) contain \(\{u_{i}^{m}, v_{j}^{m-1}\}\) in their label, and, we rule out sequences that are not maximal and, thus, fulfill nested digraphs requirements.

Define \(G(v)\) as the union of all \(G_{u_{i}^{m} v_{j}^{m-1}}\). Conclude that, as we work backward, from \(k = m\) step \(-1\) that we obtain a nested digraph.

The search for \(\max\!G(v)\) is polynomially bounded, as we show in the analysis of computational boundaries in Proposition 5.8.

After writing the Linearized Digraph, we filter all compatible antichains.

We write a nested compatible digraph associated with each vertex \(v\).

These digraphs associated with root \(v_{j}^{m}\) are recursively created in iteration with each vertex \(v_{i}^{n}\), for 1 to \(m - 1\), for \(1 \leq i \leq k_{m}\). A non-empty
final result generates an output there are compatible antichains and, otherwise, there are no compatible antichains. Write a nested digraph associated with each edge of this digraph from the Linearized Digraph. Edges will play the role of vertices and the arrows connecting the vertices are labeled. We will obtain labeled digraphs whose labels are the pointers of compatible antichains.

The search for \( \max G(v) \) is polynomially bounded, as we show in the analysis of computational boundaries in Proposition 5.8.

**Procedure 3.26 (Linear Digraphs).** Given a linearized digraph endowed with \( p \) branches, \( Br = \{ B_1, \ldots, B_k \} \), for all \( 1 < i \leq l \), we will write a set of nested digraphs for each vertex \( e_j^t \), \( 1 < t \leq k \) and \( 1 \leq j \leq l_t \). The \( k \)th step generates either:

1. **Empty digraphs and an output, there are no compatible antichains** or
2. **Non empty digraphs and an output, there are compatible antichains**.

**Step 1:** For all \( e_j^1 \) in the first branch, \( Br_1 \), for all \( e_j^2, 2 \leq j \leq l \), if \( e_j^1 \) and \( e_j^2 \) are compatible, write the digraph,

\[
G(e_j^1, e_j^2) = (\{ e_j^1, e_j^2 \}, \{ e_j^2 \to e_j^1 \}, (\text{label}^1_j)(e_j^2 \to e_j^1) = \{ e_j^1 \})
\]

Define \( G(e_j^r) \) as the union of all \( G(e_j^r, e_j^{1i}) \), \( 1 \leq i \leq k_1 \).

All vertices in the columns \( i \), \( 2 \leq i \leq k \) are form an edge with the compatible vertices in the first column.

**Step r:** Suppose we performed all steps from 1 to \( r - 1 \) and obtained nested digraphs.

Let \( G(v_j^r) = \{ V_{v_j^r}, E_{v_j^r}, \text{label}_{v_j^r} \} \) be nested rooted digraph of depth \( r - 1 \), associated to the vertices \( v_j^s \), \( r \leq s \leq k \), \( 1 \leq j \leq l_r \).

For all \( G(v_j^s) \), \( 1 \leq i \leq l_r \), for all \( v_j^s, s > r, 1 \leq j \leq k_s \), if \( v_j^i \) and \( v_j^s \) are compatible, replace the (root) vertex \( v_j^r \) in \( G(v_j^r) \) by \( v_j^s \). Obtain the digraph \( G(v_j^r/v_j^s) \).

1. Let \( G(v_j^r, v_j^s)^m \) be the intersection of \( G(v_j^r/v_j^s) \) and \( G(v_j^s) \);

2. Let \( \max G(v_j^r, v_j^s) = (V_j^{\max})^r_i, (E_j^{\max})^r_i, (\text{label}^{\max})^r_i \) be the maximal nested digraph contained in \( G(v_j^r, v_j^s)^m \);

3. Define \( G(v_j^r, v_j^s)^r = (V_j^{\max})^r_i, (E_j^{\max})^r_i, (\text{label}^{\max})^r_i \) as

\[
\begin{align*}
(V_j^{\max})^r_i &= (V_j^{\max})^r_i \cup \{ v_i^r \} \\
(E_j^{\max})^r_i &= (E_j^{\max})^r_i \cup \{ v \leftarrow v_i^r | v \text{ belongs to level } r - 1 \} \\
(\text{label}^{\max})^r_i(e) &= \text{label}' v_i^s(e) \cup \{ v_i^r \} \\
(\text{label}^{\max})^r_i(v_i^r \to v) &= \{ v_i^r \}
\end{align*}
\]
Define \( G(v^*_j) \) as the union of all \( G(v^*_j, v^*_i)' \).

Call the set of nested antichains to the set of digraphs obtained at each vertex of level \( p \).

**Theorem 3.27.** For all vertex \( v \), the digraph \( G(v) \) is a compatible nested digraph.

**Proof.** In the first step, the sequences have the form \( e \rightarrow v^1 \), if both edges are compatible. The label associated to the sequences \( e \rightarrow v^1 \) is \( v^1 \). The union of such digraphs is a compatible nested digraph.

At the step \( r \), we replace the vertex \( G(v^*_r) \) by \( v^*_j \), intersect \( G(v^*_r) \) and \( G(v^*_j) \). By construction, \( \text{max}(G(v^*_r) \cap G(v^*_j)) \) is the biggest compatible nested digraph contained in \( G(v^*_r) \cap G(v^*_j) \).

The digraph \( G(v^*_j, v^*_i)' \) was obtained by adding the vertex \( v^*_i \), compatible with all vertices in \( G(v^*_r) \cap G(v^*_j) \).

Lastly, the union all \( G(v^*_j, v^*_r)' \) is compatible because it is the union of compatible digraphs. 

Finally, highlight that we reached our claim. We show that the nested digraphs, if non-empty, contain all compatible antichains.

**Definition 3.28.** Call a maximal nested path over a nested digraph to any maximal path
\[
v^k \rightarrow v^{k-1} \rightarrow \cdots \rightarrow v^2 \rightarrow v
\]
so that for all \( 1 \leq i \leq k-2 \), label \((v^i, v^{i+1}) \subseteq \text{label}(v^{i+1}, v^{i+2})\).

Identify the set of all maximal nested paths over a nested digraph with the set of maximal compatible antichains. Conclude,

**Theorem 3.29.** A pivoted \( 3 \text{-sat} \) \( \Psi \) is unsatisfiable if and only if the \( k \text{th} \) iteration of nested antichains is empty.

## 4 A Pivoted Strong Version of a \( 3 \text{-sat} \) Formula

Given a \( 3 \text{-sat} \) formula \( \Psi \), we show that there are at least one formula \( \Psi_T \) so that if \( \Psi \) is unsatisfiable, so is \( \Psi_T \).

Arbitrarily choose pairs of literals \( S^0 = \{p_{01}, \neg p_{01}, \ldots, p_{0k}, \neg p_{0k}\} \) in \( \Psi \) and factorize \( \Psi \),

\[
(p_{01} \lor S_{p_{01}}) \land (\neg p_{01} \lor S_{\neg p_{01}}) \land \cdots \land (p_{0k} \lor S_{p_{0k}}) \land (\neg p_{0k} \lor S_{\neg p_{0k}}) \land S^1_3
\]
so that none of the literals of $S^0$ appears in any formula in the set of formulas 
\{S_{p01}, S_{\neg p01}, \ldots, S_{pok}, S_{\neg pok}, S^1_3\}.

Successively, do the partitions:

\[ S^1_3 = (p_{11} \lor S_{p_{11}}) \land (\neg p_{11} \lor S_{\neg p_{11}}) \land \cdots \land (p_{1k_1} \lor S_{p_{1k_1}}) \land (\neg p_{1k_1} \lor S_{\neg p_{1k_1}}) \]

\[ S^2_3 = (p_{21} \lor S_{p_{21}}) \land (\neg p_{21} \lor S_{\neg p_{21}}) \land \cdots \land (p_{2k_2} \lor S_{p_{2k_2}}) \land (\neg p_{2k_2} \lor S_{\neg p_{2k_2}}) \]

\[ \vdots \]

\[ S^h_3 = (p_{h1} \lor S_{p_{h1}}) \land (\neg p_{h1} \lor S_{\neg p_{h1}}) \land \cdots \land (p_{hk_h} \lor S_{p_{hk_h}}) \land (\neg p_{hk_h} \lor S_{\neg p_{hk_h}}) \]

where no literal in the set of conjugated pairs \{p_{r1}, \neg p_{r1}, \ldots, p_{rk_r}, \neg p_{rk_r}\} belongs to the set of literals of $S^a_{3, r}$ for any $s > r$ or the set of literals of any $\cup\{S_{p_r}, S_{\neg p_r} | 1 \leq r \leq k_r\}$. Moreover $S_T$ is a 2-sat formula.

Finally, obtain the below partition:

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$p_{01}$ & $S_{p_{01}}$ & $\neg p_{01}$ \\
\hline
$p_{02}$ & $S_{p_{02}}$ & $\neg p_{02}$ \\
\hline
\vdots & \vdots & \vdots \\
\hline
$p_{0k}$ & $S_{p_{0k}}$ & $\neg p_{0k}$ \\
\hline
$p_{11}$ & $S_{p_{11}}$ & $\neg p_{11}$ \\
\hline
$p_{21}$ & $S_{p_{21}}$ & $\neg p_{21}$ \\
\hline
\vdots & \vdots & \vdots \\
\hline
$p_{1k_1}$ & $S_{p_{1k_1}}$ & $\neg p_{1k_1}$ \\
\hline
\vdots & \vdots & \vdots \\
\hline
$p_{h1}$ & $S_{p_{h1}}$ & $\neg p_{h1}$ \\
\hline
$p_{h2}$ & $S_{p_{h2}}$ & $\neg p_{h2}$ \\
\hline
\vdots & \vdots & \vdots \\
\hline
$p_{hk_h}$ & $S_{p_{hk_h}}$ & $\neg p_{hk_h}$ \\
\hline
\hline
$S_T$ & & \\
\hline
\end{tabular}
\end{center}

We show that there is a modified pivoted 3-sat, $\Psi_T$ so that $\Psi_T$ is unsatisfiable if and only if $\Psi$ is unsatisfiable.

Let \[ \{r_{21}, \neg r_{21}, \ldots, r_{1k_1}, \neg r_{1k_1}, \ldots, r_{h1}, \neg r_{h1}, \ldots, r_{hk_h}, \neg r_{hk_h}\} \]
be a set of literals disjoint from the set $\text{Letter}(\Psi)$. Let $\Psi_T$ be,

$$(p_{11} \vee S_{p_{11}}) \land (\neg p_{11} \vee S_{\neg p_{11}}) \land \cdots \land (p_{1k} \vee S_{p_{1k}}) \land (\neg p_{1k} \vee S_{\neg p_{1k}}) \land$$

$$(r_{21} \vee (S_{p_{21}} \land \neg p_{21})) \land (\neg r_{21} \vee (S_{\neg p_{21}} \land \neg p_{21})) \land \cdots \land$$

$$(r_{2k} \vee (S_{p_{2k}} \land \neg p_{2k})) \land (\neg r_{2k} \vee (S_{\neg p_{2k}} \land \neg p_{2k})) \land \cdots \land$$

$$(r_{h1} \vee (S_{p_{h1}} \land \neg p_{h1})) \land (\neg r_{h1} \vee (S_{\neg p_{h1}} \land \neg p_{h1})) \land (t \lor S_T) \land (\neg t \lor S_T)$$

Write $\Psi$ as its factorized version,

$$((p_{11} \land \neg p_{11}) \land \cdots \land (p_{1k} \land \neg p_{1k}) \land \neg p_{1k} \land S_{\neg p_{1k}}) \land (p_{21} \land \neg p_{21}) \land \cdots \land (p_{2k} \land \neg p_{2k}) \land \neg p_{2k} \land S_{\neg p_{2k}}) \land$$

$$\cdots$$

$$(p_{h1} \land \neg p_{h1}) \land \cdots \land (p_{hk} \land \neg p_{hk}) \land \neg p_{hk} \land S_{\neg p_{hk}}) \land S_T$$

We show that $\Psi_T$ is unsatisfiable if and only if $\Psi$ is unsatisfiable.

Let $\Sigma$ be the set of all mappings from \{11, \ldots, 1k_1, h1, \ldots, hk_h\} onto \{True, False\}. Note that $\Sigma$ has $2^{k_1+h+2k_h}$ elements.

Let $\sigma \in \Sigma$. For all conjugated pair $\{p_{uv}, \neg p_{uv}\}$, $1 \leq u \leq h$ and $1 \leq v \leq k_v$, let $\epsilon_{uv} = p_{uv}$ if $\sigma(p_{uv}) = True$ and, otherwise, $\epsilon_{uv} = \neg p_{uv}$. Let $v_{uv}$ be $r_{uv}$ if $\epsilon_{uv} = p_{uv}$ and, otherwise $v_{uv} = \neg r_{uv}$. Given a Boolean formula $F$, $F/\sigma$ denotes the substitution of all literals in $F$ by its value over $\sigma$.

Using the valuation $\sigma$, we have, respectively, for $\Psi$ and $\Psi_T$,

$$S_{\neg e_{11}}/\sigma \land \cdots \land S_{\neg e_{1k_1}}/\sigma \land S_{\neg e_{21}}/\sigma \land \cdots \land S_{\neg e_{2k_2}}/\sigma \land \cdots \land$$

$$S_{\neg e_{h1}}/\sigma \land \cdots \land S_{\neg e_{hk}}/\sigma \land S_T$$

and

$$S_{\neg e_{11}}/\sigma \land \cdots \land S_{\neg e_{1k_1}}/\sigma \land \neg v_{21} \land (v_{21} \land \neg S_{\neg e_{21}}/\sigma) \land \cdots \land$$

$$\neg v_{2k_2} \land (v_{2k_2} \land \neg S_{\neg e_{2k_2}}/\sigma) \land \neg v_{h1} \land (v_{h1} \land \neg S_{\neg e_{h1}}/\sigma) \land \cdots \land$$

$$\neg v_{hk} \land (v_{hk} \land \neg S_{\neg e_{hk}}/\sigma) \land S_T$$

Whether at least one $\neg v_{lm}$ is false then, $\Psi_T$ is false. Otherwise, all $\neg v_{lm}$ is true and $\Psi_T$ is

$$S_{\neg e_{11}}/\sigma \land \cdots \land S_{\neg e_{1k_1}}/\sigma \land S_{\neg e_{21}}/\sigma \land \cdots \land S_{\neg e_{2k_2}}/\sigma \land \cdots \land$$

$$S_{\neg e_{h1}}/\sigma \land \cdots \land S_{\neg e_{hk}}/\sigma \land S_T$$

and, under this conditions of valuation, $\Psi$ is unsatisfiable if and only if $\Psi_T$ is unsatisfiable.
5 Bounds on Computation

Open this section with the study of the bounds in Space, $\text{SPACE}(f(n))$ and in time, $\mathcal{O}(f(n))$. Both bounds ensure that we do not trade space by time or vice-versa in our considerations.

Given a pivoted 3-sat formula, we follow the polynomially bounded steps,

1. Write the cylindrical digraph;
2. Write the closed digraphs;
3. Write the linearized digraph;
4. Write the maximal nested digraphs. A special care has to be taken on writing the maximal nested digraph contained in the intersection.

We prove less straightforward bounds in time and space.

Lemma 5.1. The cylindrical digraph, $\text{Clndr} = (V, E, \text{label})$ is written in polynomial time and space.

Proof. The size of $\text{Clndr}$-graph is given by

1. The size of $V$, the set of vertices is the size of literals, $L$;
2. The size of $E$ is bounded by the square of the number of literals, $L$, $|E| \leq |L|^2$. Indeed, the arrows are bounded by the number of arrows in a polyhedron with $|L|$ sides.

There is an algorithm polynomially bounded in time to write a closed digraph. We describe the, polynomial in time, search for nonempty intervals $[a, \neg a]$. Recall that no loops are allowed. Indeed, if we write a loop, that means that if an incompatible combination $\text{Inc} = i_{j1}, \ldots, i_{jr}$, any compatible combination that contains $\text{Inc}$ is likewise incompatible.

Divide the search algorithm into two procedures,

1. Write $[a, q]$, the subdigraph of $\mathcal{CSD}$ of all sequences connected to $a$ (no specified end, just source, $a$);
2. If $[a, q] \neq \emptyset$, from $[a, q]$, write $[a, \neg a]$, the subdigraph of paths between $a$ and $\neg a$ contained in $[a, q]$. 
Pseudocode 5.2. Part I, write an Interval $[a, q]$

Input cylindrical digraph $Cyl = (V, E, label)$.

$[a, q] = (\{a\}, \emptyset, \emptyset) = (V_{[a, q]}, E_{[a, q]}, label_{[a, q]})$ # Initial State

$E_{aux} = E \setminus E_{[a, q]}$

$V_{aux} = V \setminus V_{[a, q]}$ # No loops

while $E_{aux} \neq \emptyset$

$E_{[a, q]} \leftarrow E_{[a, q]} \cup \{c \Rightarrow b \in V_{aux} \text{ AND } c \Rightarrow b \in E\}$

$V_{[a, q]} \leftarrow V_{[a, q]} \cup \{b \exists c \in V_{[a, q]}(c \Rightarrow b \in E_{aux})\}$

$label_{[a, q]} \leftarrow label_{[a, q]} \cup \{label(c \Rightarrow b) \exists b \in V_{[a, q]}(c \Rightarrow b \in E_{aux})\}$

$E_{aux} = E \setminus E_{[a, q]}$

$V_{aux} = V \setminus V_{[a, q]}$

end while

Once we obtain $[a, q] = (V_{[a, q]}, E_{[a, q]}, label_{[a, q]})$, if $-a \in V_{[a, q]}$, we perform the above algorithm backwards, that is we start with $[q, -a] = (\{-a\}, \emptyset, \emptyset) = (V_{[q, -a]}, E_{[q, -a]}, label_{[q, -a]})$ and perform Pseudocode 5.2 taking care to search over arrows $b \Rightarrow -a$ and obtain $[a, -a]$.

Observation 5.3. As Pseudocode 5.2 at each iteration, updates the set $E_{aux}$, we avoid sequences of the form,

$r \Rightarrow q_1 \Rightarrow \cdots \Rightarrow r$

that is, we do not have loops.

Let us scrutinize the size of the closed digraph.

Lemma 5.4. The set of closed digraphs is the union of at most $|V|^2$ digraphs with $|V|^2$ vertices and $|V|^3$ edges.

Proof. The set of necessarily true pairs has, at most, the size of $|V|$, the number of literals. Each interval $[-p, p]$ has its size bounded by the size of the cylindrical digraph, that is, at most $|V|$ vertices and $|V|^2$ edges, that is, a total of $|V|^2$ vertices and $|V|^3$ edges.

Obtain the union of, at most, $|V|$ digraphs with $|V|$ vertices and $|V|^2$ edges, SPACE$(|V|^3)$. □

Lemma 5.5. The search for the set of closed digraphs is polynomially in time.

Proof. We perform at most $|V|$ searches, one to each interval $[p, -p]$ for $p$ necessarily true.
We search over the number of vertices and obtain,

\[
\begin{align*}
|V| - 1 \\
|V| - 1 - r_1 \\
\vdots \\
|V| - 1 - r_1 - \cdots - r_q
\end{align*}
\]

where \(1, r_1, \ldots, r_q\) are the vertices we removed from the vertices that form the set of edges \(E_{aux}\).

The maximum under the conditioning \(1 + r_1 + \cdots + r_q = V\), leads to a maximum \(r_1 = \cdots = r_q = V/q\) and the search goes to a top of \(|V|^2\) searches over \(|V|\) vertices and the search is bonded by \(|V|^3\) in time.

Continue reviewing the effort to Lift, Multiply Branches and the Addition of Label. After that, we must analyze the whole process of writing the nested digraphs associated with each edge.

**Proposition 5.6.** Given a closed digraph, the operations of Lifting, Adding Labels and Multiplying Branches are linearly bounded.

**Proof.** Lifting together with Multiplication of branches multiply the number of any branching \(a\) accordingly to the maximum of branches in \(Up(a)\) or \(Down(a)\).

Adding Label adds several new labels to, at most the number of edges in the closed digraph its addition is bounded by the number of branching and Multiplication of branches is bounded by the maximum number of branches, that is, the maximum number of edges, \(|V|^3\).

Our next question is: What the size of the Linearized Digraph is?

The set of closed digraphs is the union of at most \(|V|\) digraphs. Each closed digraph has the size bounded by \(|V|^2\) vertices and \(|V|^3\) edges, by Lemma 5.4. As we deal with linear digraphs, the number of edges is bounded by the number of vertices in the linearized digraph.

We consider a maximum multiplication of branches. The maximum of multiplication operations we perform is bounded by \(|V|^2\) vertices. Suppose, in the very pessimistic valuation we have \(|V|^2\) columns with \(|V|^2\) edges.

The edges are bounded by \(|V|^2\) because we have linear digraphs and the vertices do not form branching.

**Lemma 5.7.** Given a Linearized digraph endowed with \(|V|^2\) columns with \(|V|^2\) edges. Then, the process of building the nested digraphs has the following bounds,
\textbf{Proof. First Step:} In the first step, each vertex in the first column interacts with each vertex on the remainder columns, that is, we build at most $|V| \times (|V| \times (|V| - 1))$ nested digraphs. Each rooted digraph, has a maximum of $|V|$ edges because all vertices are connected to a single root. Henceforth, we use a maximum of $|V|^2$ edges to keep optimization simpler.

\textbf{$r^{th}$ Step:} We have at most $|V|$ interactions over $|V| \times (|V| - r - 1)$ vertices that form the root associated to the vertex of each column. That is, we generate $|V| \times (|V| - r - 1)$ rooted digraphs. Each digraph has depth $r$ and each column has a maximum of $|V|$ vertices. We have a total bounded by a maximum of $r \times |V|$ vertices and $r \times |V|^2$ edges.

\textbf{$|V|^2$th Step:} In the $|V| - 1$ step, obtain a maximum of $|V|$ digraphs with depth $|V|$. We have a top of $|V|$ vertices in each column, that is, $|V|^2$ vertices and $|V|^3$ edges.

\textbf{Proposition 5.8.} Given two nested rooted digraphs, whose root is $v$, endowed with $R \times T$ vertices, located in $T$ columns, $R$ vertices at each column. $R^2 \times T$ edges, then the operation of writing the maximal nested digraph contained in their intersection is polynomially bounded.

\textbf{Proof.} The first algorithm shows the union of linear digraphs whose labels are ordered by $\subseteq$.

From $m$ ranging from levels 1 and 2 until $T - 2$ and $T - 1$, search over a maximum of $S \times T$ edges. At each edge, we perform searches over $S$ edges whose vertices lie in levels $m + 1, m + 2$. Obtain at most $S^2$ interactions over levels $m + 1, m + 2$. At the end of the process, obtain a total of $S^2 \times T$ searches.

In the second pseudocode, for edges in the levels $m = T$ and $T - 1$, step -1 until levels $m = 2$ and 1, write, for any edge $u^m_i \rightarrow v^{m-1}_j$, the subdigraph $G_{u^m_i, v^{m-1}_j}$ that contains all maximal sequences that start at level $k$, pass through $u^m_i \rightarrow v^{m-1}_j$ and end in $v$. Rule out from the set of labels any edge in the level $m$, except $u^m_i$ and in the level $m - 1$, except $v^{m-1}_j$.

Each edge visits, in a maximum of $|V|^2$ edges. We have, in the worst case, are bounded by the number of closed digraphs, and the use of dimension $|V|^3$. so, we search $|V|^5$ in time. \qed
6 Examples

Example 6.1. Consider the CSD,

We do not focus on the compatibility among vertices. It is not relevant for this analysis.

The task of writing the chains is clearly expensive, as we show below

Solve using our method. Multiply roots,

Add labels, obtain,
Lifting,

Add labels, obtain,

Lifting,

Lifting,
The only possible compatible combinations, according to compatibility of labels we added, not considering compatibility among the original label, is,

\[
\begin{align*}
p_5 & \Rightarrow p_7, p_6 \Rightarrow p_7, p_5 \Rightarrow p_8, p_6 \Rightarrow p_8 \\
p_5 & \Rightarrow p_7, p_4 \Rightarrow p_6, p_3 \Rightarrow p_6, p_5 \Rightarrow p_8 \\
p_3 & \Rightarrow p_5, p_6 \Rightarrow p_7, p_6 \Rightarrow p_8, p_4 \Rightarrow p_5 \\
p_3 & \Rightarrow p_5, p_4 \Rightarrow p_6, p_3 \Rightarrow p_6, p_4 \Rightarrow p_5 \\
p_3 & \Rightarrow p_5, p_1 \Rightarrow p_4, p_2 \Rightarrow p_3, p_4 \Rightarrow p_5 \\
p_1 & \Rightarrow p_3, p_4 \Rightarrow p_5, p_3 \Rightarrow p_6, p_2 \Rightarrow p_4 \\
p_1 & \Rightarrow p_3, p_1 \Rightarrow p_4, p_2 \Rightarrow p_3, p_2 \Rightarrow p_4
\end{align*}
\]

Example 6.2. Consider the two pivoted formulas,

\[
\begin{align*}
\Psi_1 &= \ a_1 \lor ((p_1 \lor q_1) \land (p_1 \lor \neg q_1)) \land \neg a_1 \lor ((p_1 \lor q_1) \land (p_1 \lor \neg q_1)) \land \\
&\quad \ a_2 \lor ((\neg p_1 \lor q_2) \land (\neg p_1 \lor \neg q_2)) \land \neg a_2 \lor ((p_1 \lor q_2) \land (p_1 \lor \neg q_2)) \\
\Psi_2 &= \ a_1 \lor ((p_1 \lor q_1) \land (\neg p_1 \lor q_2)) \land \neg a_1 \lor ((p_1 \lor q_1) \land (\neg p_1 \lor q_2)) \land \\
&\quad \ a_2 \lor ((p_1 \lor \neg q_1) \land (\neg p_1 \lor \neg q_2)) \land \neg a_2 \lor ((p_1 \lor \neg q_1) \land (\neg p_1 \lor q_2))
\end{align*}
\]

The formulas \( \Psi_1 \) and \( \Psi_2 \) are, respectively, unsatisfiable and satisfiable. Write a fragment of the closed digraph,

```
A --p1-- B --> q1
C --> p1
D --> q1

E <-> F
G --> -p1
H --> -q2
```

In the case of \( \Psi_1 \), the labels in \( A, B, C \) and \( D \) are \( a_1, \neg a_1 \) and \( E, F, G \) and \( H \), are \( a_2, \neg a_2 \).

In the case of \( \Psi_2 \), the labels in are shown below, \( A = a_1, \neg a_1 \), \( B = a_2, \neg a_2 \), \( C = a_2, \neg a_2 \), \( D = a_1, \neg a_1 \), \( E = a_1, \neg a_1, a_2, \neg a_2 \), \( F = a_2 \), \( G = a_2 \), \( H = a_1, \neg a_1, \neg a_2 \).
After lifting and adding labels, we obtain,

\[
\begin{array}{cccc}
\neg p_1 & \neg p_1 \\
\text{\text{\text{\text{A}}} r} & \text{\text{\text{B}}} r \\
\text{\text{q}_1} & \text{\text{\neg q}_1} \\
\text{\text{\text{\text{C}}} r} & \text{\text{\text{D}}} r \\
\text{\text{\text{\text{p}_1}}} & \text{\text{\text{p}_1}} \\
\text{\text{\text{\text{E}}} r} & \text{\text{\text{F}}} r \\
\text{\text{\text{q}_2}} & \text{\text{\neg q}_2} \\
\text{\text{\text{G}}} r & \text{\text{\text{H}}} r \\
\text{\text{\neg p}_1} & \text{\text{\neg p}_1}
\end{array}
\]

Case \(\Psi_2\), there are compatible combinations and, in the case of \(\Psi_1\), there is no compatible combination. We can infer \(\Psi_1\) is not satisfiable and, in the case of \(\Psi_2\), the conclusion is based in writing all combinations.

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