Research Article

Strong Total Monophonic Problems in Product Graphs, Networks, and Its Computational Complexity

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1. Introduction

Consider a simple connected graph $G(V, E)$ with order $m$ and size $n$. We refer [1] for the basic graph theoretical definitions. We denote $i \in \{1, 2, \ldots, n\}$ as $i \in [m]$. Some of the variants for covering the vertices problem are referred in [2–19]. Some results on strong geodetic problems can be referred in [15–19]. A set $S \subseteq V(G)$ is a geodetic set if it covers all the vertices of $V(G)|S$, and the cardinality of smallest geodetic set is called the geodetic number, denoted by $g(G)$. This concept is introduced by Harary et al. [3]. If $S \subseteq V$, then for each pair of vertices $x, y \in S, x \neq y$, let $\tilde{g}(x, y)$ be a selected fixed shortest $x \sim y$ path. Then, we set $\tilde{I}(S) = \{\tilde{g}(x, y) : x, y \in S\}$, and let $V(\tilde{I}(S)) = \bigcup_{P \in \tilde{I}(S)} V(P)$. If $V(\tilde{I}(S)) = V$, for some $I(S)$, then the set $S$ is called a strong geodetic set, and the strong geodetic problem is to find a minimum strong geodetic set $S$ of $G$ [5].

A $p – q$ monophonic path is a chordless $p – q$ path. The monophonic distance is the number of edges of the chordless path, denoted by $d_m(p, q)$. The monophonic eccentricity of a vertex $p \in V(G)$ is the maximum monophonic distance between $p$ and another vertex in $G$. The monophonic diameter, $d_m$, and monophonic radius, $r_m$, of $G$ are the maximum and minimum monophonic eccentricity among the vertices in $G$, respectively. A set of vertices $S \subseteq V$ is defined as monophonic set (MS) if every vertex of $G\mid S$ lies on a monophonic path joining some pair of vertices in $S$, and the cardinality of such smallest set is called monophonic number, denoted by $m(G)$. We refer [6–10] for other parameters of monophonic number. If every vertex of $G\mid S$ lies on a unique fixed monophonic path between the pair of vertices in $S \subseteq V(G)$, then $S$ is said to be a strong monophonic set (SMS, for short) which is defined in [11]. Its minimum cardinality is called strong monophonic number, denoted by $sm(G)$.
2. Motivation

Graph theory plays a vital role in modelling our day to day activities into graphs. We can model the traffic system in a region using graphs where the vertices are junctions of roads. The police officers can be stationed in such a way that (i) two police officers are stationed at junctions adjacent so that they can support each other if they face any issue, and (ii) two police officers other than the former ones are stationed in such a way that they guard the junctions along the unique path than the former pair of police officers. Thus, we can assign minimum number of police officers as well as reduce the risk of traffic issues. For the above mentioned problem, the concept of strong total monophonic problem is helpful. This concept can also be useful in power supply station, water supply station, etc. The next section defines the problem, and its computational complexity and results are discussed in the following sections.

3. Strong Total Monophonic Number of a Graph

A total monophonic set (TMS, for short), $S \subseteq V(G)$, is a monophonic set whose induced subgraph does not contain any isolated vertices. The cardinality of minimum total monophonic set is called total monophonic number (TM, for short), denoted by $m_t(G)$. The set $T \subseteq V(G)$ is said to be a strong total monophonic set (STMS, for short) if $T$ is a strong monophonic set, and the subgraph induced by $T$ does not contain isolated vertices. The cardinality of such smallest STMS is called strong total monophonic number, $sm_t(G)$.

In Figure 1, the TMS is $\{a, b, f, g\}$ and $m_t(S) = 4$. The STMS is $\{a, b, c, d, e\}$ and $sm_t(S) = 5$.

We recall some terms and definitions required. Simplex vertices of a graph $G$ are those vertices whose neighbourhood induces a clique and is denoted by Ext($G$). The vertex adjacent to the vertex of degree 1, that is, end ($x$).

Proposition 1. Every $\Omega(G)$ and $\Omega_1(G)$ of a connected graph $G$ belongs to every STMS of $G$. If the set of all $\Omega(G)$ and $\Omega_1(G)$ forms a STMS, then it is the unique minimum strong total monophonic set of $G$.

Corollary 1. For complete graph $K_m, m \geq 2, sm_t(K_m) = m$.

Theorem 1. The set of all $\Omega(T)$ and $\Omega_1(T)$ is the unique minimum STMS for any nontrivial tree $T$.

Proof. Since extreme and stem vertices of a nontrivial tree $T$ form a STMS of $T$, the result is obtained from Proposition 1.

Result 1. For path $P_n, n \geq 4, sm_t(P_n) = 4$.

Result 2. For star graph $K_{1,n-1}, sm_t(K_{1,n-1}) = n$.

Theorem 2. Let $H$ be a connected graph of order $m \geq 2$, then

(i) A strong monophonic set, SMS, contains a minimum of $2$ vertices or more; thus, $sm(H) \geq 2$. Each STMS is a strong monophonic set; therefore, $sm(H) \leq sm_t(H)$. Since $V(H)$ is a STMS of $H$, it is obvious that $sm(H) \leq m$. Hence, $2 \leq sm(H) \leq sm_t(H) \leq m$.

(ii) Let $T = \{u_1, u_2, \ldots, u_n\}; n \leq m$ be the strong monophonic set of $H$. Then, $T \cup \{y_1, y_2, \ldots, y_n\} \in \Omega(G)$ is the STMS. Therefore, $sm_t(H) = 2sm(H)$.

4. Complexity Results

We are reducing the problem of strong total monophonic for general graphs from the decision problem, whether there exists an induced path between two vertices such that it passes through a third vertex. Using the above concept, we solve NP-completeness of STMS.

Theorem 3 (See [13]). Let $x, y, z$ be three distinct vertices in a graph $G$. Deciding whether there is an induced path from $x$ to $y$ passing through $z$ is NP-complete.

Theorem 4. The STM problem for general graphs is NP-complete.

Proof. Given a graph $H$ with distinct vertices $p, q, r$ and construct the graph $H'$ as follows: let $\{p', q'\} \in V(H')$ such that these vertices are adjacent to all the vertices in $V(H)[r]$. A pendant vertex $a$ is joined to the vertex $p'$. Similarly, pendant vertices $\{a_1, a_2, \ldots, a_{n-1}\}$ are joined to the vertex $v$, refer Figure 2. Let $S = \{p', q', a_1, a_2, \ldots, a_{n-1}\}$ since the set of vertices of $T$ forms a set of Ext($H$) and stem vertices, which belong to any STMS of $H'$. Also, the monophonic paths between the vertices in $T$ do not cover the vertex $r$. It is
straightforward to see that every induced path in $H$ is also an induced path in $H'$. The monophonic paths between $x$ and $x'$ where $x', x \in V(H)$ will not cover the vertex $r$. Consider $A = T \cup \{u, v: u, v \in V(H)\}$. Clearly, $u, v$ are nonextreme vertices in $A$. It is straightforward that $A$ is a STMS in $H'$ if and only if the induced path between $p$ and $q$ contains $r$ in $H$.

5. Results

**Theorem 5.** Let $H$ be a graph of order $n$ with monophonic diameter $d_m$, then $\text{sm}_1(H) \geq \left[2(d_m - 2) + \sqrt{4(d_m - 1)^2 + 8n(d_m - 1)/2(d_m - 1)}\right]$.

**Proof.** Let $H$ be a connected graph with the strong total monophonic set $S$, and let $|S| = s$. No isolated vertices are contained in $G[S]$. If $xy \in E(G[S]), x, y \in S$, then the monophonic path between $x$ and $y$ does not cover any vertices of $V/S$. Note that $|E(G[S])| \geq s/2$. For a, $a, b \in S$, the $a - b$ fixed monophonic path covers atmost $d_m - 1$ vertices and $n - s$ vertices are covered by atmost $\left(\frac{s}{2} - s/2\right)$ monophonic paths. Thus, $n - s \leq \left(\frac{s}{2} - s/2\right)(d_m - 1)$. Therefore, we get $\text{sm}_1(H) \geq 2(d_m - 2) + \sqrt{4(d_m - 1)^2 + 8n(d_m - 1)/2(d_m - 1)}$.

The above bound is sharp for Figure 3.

**Theorem 6.** $\text{sm}_1(H) = 2$ if and only if $H \equiv K_2$.

**Proof.** If $H = K_2$, then $\text{sm}_1(H) = 2$. Conversely, assume $S = \{x, y\}$ be a minimum STMS of $H$. Suppose the length of $x - y$ monophonic path is greater than $2$, then it has an internal vertex. Let it be $u$, such that $u \in V(H)\setminus S$. Therefore, $G[S]$ has isolated vertices, which is a contradiction. Therefore, the only possibility is $H \equiv K_2$.

**Theorem 7.** $\text{sm}_1(H) = 3$ if and only if $H \equiv C_n$ or $P_3$.

**Proof.** Clearly, for path $P_3, \text{sm}_1(H) = 3$.

Let $H$ be cycle $C_n$: $v_1, v_2, \ldots, v_n$ of order $n$. By Theorem 6, $\text{sm}_1(H) \geq 3$. Let $S = \{v_1, v_{n-1}, v_n\}$. The set $S$ is a strong monophonic set of $C_n$. Hence, $\text{sm}_1(C_n) = 3$.

Conversely, let $\text{sm}_1(H) = 3$. Let us assume that $H$ is not a path $P_3$. Let $S = \{a, b, c\}$ be a STMS of $G$. Since no isolated vertices are contained in $G[S]$, one of the vertex, say, $c$ must be adjacent to both $a$ and $b$. Therefore, only $a - b$ fixed monophonic path which covers the vertices of $H\setminus S$ exists. Therefore, $H \equiv C_n$.

**Corollary 2.** If $H$ is not isomorphic to $K_2, P_3$, and $C_n$, then $\text{sm}_1(H) \geq 4$.

**Proof.** Proof follows from Theorems 6 and 7.

**Theorem 8.** Let $H$ be a connected graph of order $m$. If $m_1(H) = m - 1$, then $\text{sm}_1(H) = m - 1$.

**Proof.** Let $X$ be a total monophonic set of $G$. Let $a \in V(G)\setminus X$. Then, there exists $b, c \in X$ such that $a$ lies on $b - c$ monophonic path. Since it is the unique fixed path, $X$ is the strong total monophonic set. Therefore, $\text{sm}_1(H) = m - 1$.

Converse of the theorem is not true. For $H = K_n - e, n \geq 5, \text{sm}_1(H) = n - 1$ but $m_1(H) = 3$.

Sum of two graphs $S$ and $T$ is a graph with vertices $V(S) + V(T)$ and edges $E(S) + E(T) + \{u_i v_j: u_i \in V(S), v_j \in V(T)\}$.

**Theorem 9.** Let $S$ and $T$ be two connected graphs. Let $P \in (V(S) + T)$ and $P \cap V(S) \neq \emptyset, P \cap V(T) \neq \emptyset$. If $P \cap V(S)$ is a monophonic set of $S$ or $P \cap V(T)$ is a monophonic set of $T$, then $P$ is a total monophonic set of $S + T$.

**Proof.** Let $X = P \cap V(S)$ is a MS of $S$. If $X = V(S)$, then obviously $P$ is a TMS of $S + T$. Let $X \neq V(S)$. Let $u \in V(S) \setminus X$. Then, there exists $a, b \in X$ such that $c$ lies in the $a - b$ monophonic path of $S$. Obviously, every monophonic path in $S$ is a monophonic path of $S + T$. Since $d_{S+T}(a, b) = 2$, all the vertices in $T$ lie in the $r - s$ monophonic path, where $r, s \in X$. Therefore, $X$ is a MS of $S + T$. Since $U \cap V(T) \neq \emptyset, U$ is a TMS of $S + T$.

But Theorem 9 is not applicable to a strong total monophonic number of $G + H$. Consider $P_3 + P_4$, as in Figure 4, the total monophonic set $S$ contains $\{a, c, d\}$ where $\{a, c\}$ is the monophonic set of $P_3$ and $S \cap P_4 = \{d\}$ whereas the strong total monophonic set is $\{a, c, d, g\}$.

**Theorem 10.** Let $S$ and $T$ be two noncomplete graphs, then $m_1(S + T) \leq \min \{m(S) + 1, m(T) + 1, 4\}$.
Theorem 11. Let $S$ and $T$ be two graphs, then $sm_t(S + T) \leq sm(S) + sm(T)$.

Proof. Let $Z$ and $Z'$ be the SMS of $S$ and $T$, respectively. Clearly, $Z \cup Z'$ is a strong monophonic set of $S + T$. Also $G[Z \cup Z']$ contains no isolated vertices. Hence, $sm_t(G + H) \leq sm(G) + sm(H)$.

The bound is sharp for $P_n + C_m$.

The corona product of two graphs $S$ and $T$, $ST$, is the graph obtained by joining every vertex of $k^{th}$ copy of $T$ to $k^{th}$ vertex of $S$.

Theorem 12. Let $X$ and $X'$ be two connected graphs, then $sm_t(X \times X') \leq n(X)(1 + sm(X'))$.

Proof. Let $o(X) = n, o(X') = m$. $X \times X'$ contains $n$ copies of $X'$, that is, $X'_1, X'_2, \ldots, X'_n$. Let $S = \{h_1, h_2, \ldots, h_n\}$ be a SMS of $X'$. Then $S_i = \{h_1, h_2, \ldots, h_i\}$ is a SMS of $X'_i$, $i \in [n]$. It is straightforward to see $V(X) \cup \bigcup_{i=1}^{n} S_i$ is a STMS of $X \times X'$. Therefore, $sm_t(X \times X') \leq n(X)(1 + sm(X'))$.

The sharpness for the above bound is obtained for $P_n \circ P_m$, $n, m \geq 3$.

Cartesian product of two graphs $S$ and $T$, $S \times T$ is a graph with vertex set $V(S) \times V(T)$, and two vertices $(s, t)$ and $(s', t')$ are adjacent in $S \times T$ if and only if $s = s'$ and $t$ is adjacent to $t'$ or $t = t'$ and $s$ is adjacent to $s'$.

Theorem 13 (See [11]). Let $G$ and $H$ be connected graphs with $sm(G) = p$ and $sm(H) = q$, $2 \leq p, q$. Then, $sm(\square \bigotimes H) \leq pq - 1$ and the bound are sharp.

Remark 1. By Theorem 13, any three corner vertices of $P_n \circ P_m$ are a strong monophonic set.

Theorem 14. Let $S$ and $T$ be graphs with $sm(S) = m$ and $sm(T) = n, m, n \geq 2$. Then, $sm_r(S \sqcap T) \leq (mn - 1)$.

Proof. By Theorem 13, $sm_r(S \sqcap T) \leq mn - 1$. By Theorem 2, $sm_r(S \sqcap T) \leq 2(mn - 1)$.

Theorem 15. Let $S$ and $T$ be two connected graphs. Then, $sm_r(S \sqcap T) \leq \min\{sm(S)sm_r(T), sm_r(S)sm_r(T)\}$.

Proof. Let $V(S \sqcap T) = \{(x, y) : x_e \in V(S), y_j \in [IV(S)], y_i \in V(T), j \in [IV(T)]\}$. Let $sm_r(S) = l$ and $sm_r(T) = l'$. Let $X = \{x^1, x^2, \ldots, x^l\}$ be the SMS of $G$ and $Y = \{y^1, y^2, \ldots, y^{l'}\}$ be the STMS of $H$. Let $(x, y) \in V(S \circ T)(X \times Y)$. Then, there exists a fixed monophonic path $P: (l_i, l_j)$ where $l_i \in X$ such that $x$ lies on $P$ path. Similarly, $y$ lies on a fixed monophonic path $Q: (l'_i, l'_j)$ where $l'_i \in Y$. By Remark 1, $(x, y)$ will lie on some fixed monophonic path with adjacent edges $P$ and $Q$ and corner vertices $(l_i, l'_j), (l_i, l'_j), (l_i, l'_j)$. Since $Y$ is a strong total monophonic set, $G[X \times Y]$ contains no isolated vertices. Hence, $sm_r(S \sqcap T) \leq l'$. By symmetry, $sm_r(S \sqcap T) \leq sm_r(S)sm_r(T)$. Therefore, $sm_r(S \sqcap T) \leq \min\{sm(S)sm_r(T), sm_r(S)sm_r(T)\}$.

The bound is sharp for $P_n \circ P_m$.

Theorem 16. For two connected graphs $S$ and $T$, $sm_r(S \circ T) \leq \min\{sm(S)sm_r(T), sm_r(S)sm_r(T)\}$.

The proof is similar to Theorem 15. Sharpness of the bound is obtained for $P_n \circ P_m$.

Theorem 17. $sm_r(P_n \circ P_m) \leq 8$, where $n, m \geq 4$.

Proof. Let the vertices of $P_n \circ P_m$ be $\{(a_i, b_j) : 1 \leq i \leq m; 1 \leq j \leq n\}$. By Theorem 7, $sm_r(G) \geq 4$. Denote $S = \{(a_i, b_j), (a_i, b_j), (a_m, b_n), (a_m, b_n)\}$. All the internal and external vertices are covered by a unique fixed chordless path between the pair of vertices in $S$. Thus, the all vertices in $P_n \circ P_m$ lie on some unique fixed monophonic path. Since $G[S]$ contains isolated vertices, at most one vertex adjacent to each vertex in the set $S$ is also considered along with $S$ to form the strong total monophonic number. Thus, $sm_r(P_n \circ P_m) \leq 8$.

Theorem 18. For Petersen graph, $sm_r(P(5, 5)) = 4$.

Proof. By Corollary 2, $sm_r(G) \geq 4$. Consider the set $S \subseteq V(G)$ such that $S = \{x, y, z, w\}$ and $x$ is adjacent to $y$ and $z$ is adjacent to $w$. The vertices are covered by unique fixed $x - w, z - w$, and $z - y$ monophonic paths as given in Figure 5. Thus, $S$ is a STMS, and $sm_r(G) = 4$.

Theorem 19. The strong total monophonic number of the Sierpinski graph $S(n, k)$ of dimension, $n \geq 2$, is $6$.
SL generality, assume that covers all the vertices in a unique fixed simplicial vertices. Also, Ext $\in \mathcal{S}$.

Theorem 20. For $n$-dimensional silicate network $SL(n)$, $sm_t(SL(n)) = 6n^2 + 6n + (n(n - 1)/2)$.

Proof. Let $S$ be STMS. By Proposition 1, $Ext(SL(n)) \in STMS$. The number of simplicial vertices in $SL(n) = 6n^2 + 6n$ since $S$ should not contain isolated vertices. The minimum number of vertices required for $S$ to be not isolated is $(n(n - 1)/2)$. Therefore, $sm_t(G) = 6n^2 + 6n + (n(n - 1)/2)$.

Theorem 21. For wheel graph $W_n$ on $n \geq 5$ vertices, $sm_t(W_n) = 4$.

Proof. By Corollary 2, $sm_t(W_n) \geq 4$. Let $W_n = C_{n-1} + K_1$ be the wheel graph of order $n$, $n \geq 5$ which is constructed as follows. Let $u_i$ be center vertex which is adjacent to $n$ vertices of $C_{n-1}, n \geq 5$ and the diameter of wheel graph is 2, for all $n \geq 5$. The vertices that lie on the circle are denoted by $u_1, u_2, \ldots, u_{n-1}$. The $u_1 - u_{n-2}$ unique fixed monophonic path covers all vertices except $u_0, u_{n-1}$. Thus, the set \{u_0, u_1, u_2, \ldots, u_{n-1}\} forms the strong total monophonic set. Thus, $sm_t(W_n) = 4$.

Result 3. For a circulant graph $C$, $sm_t(C) = 4$.

Theorem 22. For grid $G_{nm}$, $sm_t(G_{nm}) = 4$.

Proof. By Corollary 2, $sm_t(G_{nm}) \geq 4$. Assume $\{(r_j, s_k); 1 \leq j \leq n, 1 \leq k \leq m\}$ be the set of vertices of $G_{nm}$. Let $S = \{(r_1, s_1), (r_2, s_2), (r_{n-1}, s_m), (r_n, s_m)\}$. Without loss of generality, assume that $n \leq m$.

Case 1: when $n \equiv 0(\text{mod}4)$

Except the last odd column, all the vertices in the odd columns are covered by the $(r_1, s_1) - (r_{n-1}, s_m)$ unique fixed monophonic path ($m_p$, for short). The $m_p$ is as follows: $(r_1, s_1), (r_2, s_2), (r_3, s_3), \ldots, (r_{n-3}, s_{m-2}), (r_{n-2}, s_{m-1}), (r_{n-1}, s_m)$.

All the odd columns are covered by the unique fixed $(r_1, s_1) - (r_{n-1}, s_m)m_p$. The last even column is covered by $(r_1, s_1) - (r_n, s_m)$ fixed $m_p$. Thus, all vertices are covered by the set $S$. Therefore, $sm_t(G_{nm}) = 4$.

Case 2: when $n \equiv 1(\text{mod}4)$

All the vertices in the odd columns lie on the $(r_1, s_1) - (r_{n-1}, s_m)$ unique fixed $m_p$. The $m_p$ is as follows: $(r_1, s_1), (r_2, s_2), (r_3, s_3), \ldots, (r_{n-3}, s_{m-2}), (r_{n-2}, s_{m-1}), (r_{n-1}, s_m)$. The unique fixed $(r_1, s_1) - (r_{n-1}, s_m)m_p$ covers all the vertices in the even columns.

Case 3: when $n \equiv 2(\text{mod}4)$

It is similar to $n \equiv 1(\text{mod}4)$.

Case 4: when $n \equiv 3(\text{mod}4)$

All the vertices in the odd columns except the last odd column are covered by the $(r_1, s_1) - (r_{n-1}, s_m)$ unique fixed $m_p$. The $m_p$ is as follows: $(r_1, s_1), (r_2, s_2), (r_3, s_3), \ldots, (r_{n-2}, s_{m-1}), (r_{n-1}, s_m)$.

The unique fixed $(r_1, s_1) - (r_{n-1}, s_m)m_p$ covers all the vertices in the even columns, and the $m_p$ is as follows: $(r_1, s_1), (r_2, s_2), (r_3, s_3), \ldots, (r_{n-1}, s_{m-1}), (r_{n-2}, s_m), (r_{n-1}, s_m)$. The last odd column lies on the unique fixed $(r_2, s_1) - (r_{n-1}, s_m)m_p$. The last even column is covered by $(r_1, s_1) - (r_n, s_m)$ fixed $m_p$. Thus, all vertices are covered by the set $S$. Therefore, $sm_t(G_{nm}) = 4$.

Theorem 23. For cylinder $P_m \square C_n, sm_t(P_m \square C_n) = 4$.

Proof. By Corollary 2, $sm_t(P_m \square C_n) \geq 4$. Assume $\{(r_j, s_k); 1 \leq j \leq n, 1 \leq k \leq m\}$ be the set of vertices of $P_m \square C_n$. Let $T = \{(r_1, s_1), (r_1, s_2), \ldots, (r_1, s_m)\}$.

Case 1: when $m \equiv 0(\text{mod}4)$

All the vertices in the odd column except the last row are covered by unique fixed $(r_1, s_1) - (r_{n-1}, s_m)m_p$ where the path is as follows: $(r_1, s_1), (r_1, s_2), \ldots, (r_1, s_{m-2}), (r_{n-1}, s_{m-1}), (r_{n-1}, s_m)$.

A unique fixed $(r_1, s_1) - (r_{n-1}, s_m)m_p$ covers all vertices that lie on the last row. The vertices $(r_1, s_1) - (r_{n-2}, s_m)$ lie on the $(r_n, s_1) - (r_{n-1}, s_m)$ fixed $m_p$. Thus, all the vertices are covered by the set $T$. Therefore, $sm_t(P_m \square C_n) = 4$. 

Figure 5: Petersen graph $G$ with $sm_t(G) = 4$. 

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Case 2: when $m \equiv 1 \pmod{4}$

All the vertices in the odd columns except the last row are covered by a unique fixed $(r_1, s_1) - (r_{n-1}, s_m)$ as follows: $(r_1, s_1) - (r_{n-1}, s_1), (r_{n-1}, s_2), (r_{n-1}, s_3) - (r_1, s_1), \ldots, (r_1, s_m) - (r_{n-1}, s_m)$.

All vertices in the even columns except the last row are covered by $(r_1, s_1) - (r_{n-1}, s_m)$ where the path is as follows: $(r_1, s_1), (r_1, s_3), (r_1, s_5), \ldots, (r_1, s_{m-1}) - (r_1, s_{m-1}), (r_1, s_m), (r_1, s_m)$. A fixed $(r_1, s_1) - (r_{n-1}, s_m)$ covers all vertices that lie on the last row. Since all the vertices are covered by the set $T$, $\text{sm}_t(P_n \cup C_m) = 4$.

Case 3: when $m \equiv 2 \pmod{4}$

The vertices in the odd columns except the last row are covered by the unique fixed $(r_1, s_1) - (r_{n-1}, s_m)$ as follows: $(r_1, s_1) - (r_{n-1}, s_1), (r_{n-1}, s_2), (r_{n-1}, s_3) - (r_1, s_1), \ldots, (r_1, s_m) - (r_{n-1}, s_m), (r_1, s_m)$.

All vertices in the even columns except the last row are covered by $(r_1, s_1) - (r_{n-1}, s_m)$ where the path is as follows: $(r_1, s_1), (r_1, s_3), (r_1, s_5), \ldots, (r_1, s_{m-1}) - (r_1, s_{m-1}), (r_1, s_m), (r_1, s_m)$. A fixed $(r_1, s_1) - (r_{n-1}, s_m)$ covers all vertices that lie on the last row. Thus, all the vertices in $P_n \cup C_m$ are covered by the set $T$.

Hence, $\text{sm}_t(P_n \cup C_m) = 4$.

Case 4: when $m \equiv 3 \pmod{4}$

The vertices in the odd columns with an exception for last column and last row are covered by $(r_1, s_1) - (r_{n-1}, s_m)$ unique fixed $m_p$. All the vertices in the even columns are covered by $(r_1, s_1) - (r_{n-1}, s_m)$ unique fixed $m_p$. All the vertices in the last row are covered by $(r_1, s_1) - (r_{n-1}, s_m)$ unique fixed $m_p$. The vertices $(r_1, s_m) - (r_{n-1}, s_m)$ are covered by unique fixed $(r_1, s_1) - (r_{n-1}, s_m)$. Thus, the vertices are covered by the set $T$; therefore, $\text{sm}_t(P_n \cup C_m) = 4$.

Theorem 24. For torus graph, the strong total monophonic number is 4.

The proof is similar to Theorem 23. Thus, we omit the proof.

Theorem 25. For two positive integers $l, m$ where $5 \leq l \leq m \leq n$ and $m \leq 2l - 1$, there exists a connected graph $S$ of order $n$ with $\text{sm}(S) = l$ and $\text{sm}_t(S) = m$.

Proof

Case 1: if $l < m \leq n$. Consider a path $P_{n-m+2} = v_1, v_2, \ldots, v_{n-m+1}$ of order $n - m + 3$. Join a vertex $u$ to the vertices $v_1$ and $v_2$. Join $m - l - 1$ copies of path $P_2: u_1, u_2; 1 \leq i \leq m - l - 1$ and $2m - l - 1$ vertices $w_1, w_2, \ldots, w_{2m-1}$ to the vertex $v_{n-3}$, refer to Figure 6.

Let $U_1 = \{u; 1 \leq i \leq m - l - 1\} \cup \{w; 1 \leq j \leq 2m - l - 1\}$ denote the set of simplicial vertices. Let $U_2 = \{u; 1 \leq i \leq m - l - 1\} \cup \{v_{n-m+1}\}$ be the set of stem vertices. Then, $U_1 \cup \{u, v_1\}$ is a strong monophonic set, and $U_1 \cup \{u, v_1\} \cup U_2$ is a strong total monophonic set of $S$. Therefore, $\text{sm}(S) = l, \text{sm}_t(S) = m$.

Case 2: if $l = m < n$. Consider a path $P_{n-l+1} = v_1, v_2, \ldots, v_{n-l+1}$. Let $S$ be the graph of order $n$ obtained by joining $w_i, 1 \leq i \leq l - 3$ to the vertices $v_1$ and $v_2$ and joining each vertex of $P_2: u_1, u_2$ to the vertex $v_{n-l+1}$, refer to Figure 7.

The set $\{v_1, u_1, u_2\} \cup \{w; 1 \leq i \leq l - 3\}$ is the strong monophonic and strong total monophonic set of $S$.

Case 3: if $l = m = n$. When $G = K_n$, then $\text{sm}(S) = \text{sm}_t(S) = n$.

Theorem 26. For three positive integers $\text{rad}(S), \text{diam}(S), l$ where $\text{rad}(S) \leq \text{diam}(S) \leq 2 \text{rad}(S) < n$ and $l \geq d + 1$, there...
exists a connected graph $G = K_nG$ with radius $r(G) = r$, diameter $d(G) = d$, and $smt(G) = l$.

**Proof.** If $r = 1, d = 1$, then $G = K_n$.

If $r = 1, d = 2$, then $G = K_{1,n-1}$.

Consider a cycle $C_{2r}: u_1, u_2, \ldots, u_{2r}$. The graph $G$ is constructed by joining path $P_{d-r}: v_1, v_2, \ldots, v_{d-r}$ to the vertex $u_{2r}$ and joining the isolated vertices $w_1, w_2, \ldots, w_{l-4}$ to the vertex $v_{d-r-1}$, refer to Figure 8. The set \[ \{u_2, u_3, v_{d-r-1}, v_{d-r}, w_1, w_2, \ldots, w_{l-4}\} \] forms the STMS. Thus, $smt(G) = l$. □

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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