Hodograph solutions of the dispersionless coupled KdV hierarchies, critical points and the Euler–Poisson–Darboux equation

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Abstract
It is shown that the hodograph solutions of the dispersionless coupled KdV (dcKdV) hierarchies describe critical and degenerate critical points of a scalar function which obeys the Euler–Poisson–Darboux equation. Singular sectors of each dcKdV hierarchy are found to be described by solutions of higher genus dcKdV hierarchies. Concrete solutions exhibiting shock-type singularities are presented.

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1. Introduction
In the present paper we study hierarchies of hydrodynamical systems describing quasiclassical deformations of hyperelliptic curves [1, 2]:

\[ p^2 = u(\lambda), \quad u(\lambda) := \lambda^m - \sum_{i=0}^{m-1} \lambda^i u_i, \quad m \geq 1. \]  

(1)

These hierarchies are of interest for several reasons. First, there are hierarchies of important hydrodynamical-type systems among them. For \( m = 1 \) one has the Burgers–Hopf hierarchy [3, 4] associated with the dispersionless KdV equation \( u_t = \frac{1}{2} uu_x \). For \( m = 2 \) one has the hierarchy of higher equations for the one-layer Benney system (classical long wave equation)

\[
\begin{align*}
    u_t + uu_x + v_x &= 0 \\
    v_t + (uv)_x &= 0.
\end{align*}
\]

(2)

The system (2) and the corresponding hierarchy are quasiclassical limits of the nonlinear Schrödinger (NLS) equation and the NLS hierarchy [5]. For \( m \geq 3 \), these hierarchies turn to describe the singular sectors of the above \( m = 1, 2 \) hierarchies [1].
Second, all these hierarchies are the dispersionless limits of integrable coupled KdV (cKdV) hierarchies [6–8] associated with Schrödinger spectral problems

$$\partial_{xx} \psi = v(\lambda, x) \psi,$$

with potentials which are polynomials in the spectral parameter $\lambda$

$$v(\lambda, x) := \lambda^m - m \sum_{i=0}^{m-1} \lambda^i v_i(x) \quad m \geq 1.$$

The cKdV hierarchies have been studied in [6–8], they have bi-Hamiltonian structures and, as a consequence of this property, the dispersionless expansions of their solutions possess interesting features such as the quasi-triviality property [9, 10]. Moreover, the cKdV hierarchies also arise in the study of the singular sectors of the KdV and AKNS hierarchies [11, 12]. Henceforth, we will refer to the hierarchies of hydrodynamical systems associated with the curves (1) for a fixed $m$ as the $m$th dispersionless coupled KdV (dcKdV$^m$) hierarchies.

The Hamiltonian structures of the dcKdV$^m$ hierarchies have been studied in [13]. At last, it should be noted that the dcKdV$^m$ hierarchies are closely connected with the higher genus Whitham hierarchies introduced in [14].

In our analysis of the hodograph equations for the dcKdV$^m$ hierarchies, we use Riemann invariants $\beta_i$ (roots of the polynomial $u(\lambda)$ in (1)) which provide a specially convenient system of coordinates. We show that the dcKdV$^m$ hodograph equations have the following form:

$$\frac{\partial W_m(t, \beta)}{\partial \beta_i} = 0, \quad i = 1, \ldots, m,$$

where $t = (t_1, t_2, \ldots)$ are times of the hierarchy and

$$W_m(t, \beta) := \oint \frac{d\lambda}{2i\pi} \frac{\sum_{n \geq 0} t_n \lambda^n}{\prod_{i=1}^m (1 - \beta_i/\lambda)}.$$

Here $\gamma$ denotes a large positively oriented circle $|\lambda| = r$. Thus, the hodograph solutions of the dcKdV$^m$ hierarchies describe critical points of the functions $W_m(t, \beta)$. These functions turn to be very special as they satisfy a well-known system of equations in differential geometry: the Euler–Poisson–Darboux (EPD) equations [15]

$$2(\beta_i - \beta_j) \frac{\partial^2 W_m}{\partial \beta_i \partial \beta_j} = \frac{\partial W_m}{\partial \beta_i} - \frac{\partial W_m}{\partial \beta_j}, \quad i, j = 1, \ldots, m.$$

The system (6) appeared in the theory of the Whitham equations arising in the small dispersion limit of the KdV equations [17–19], and in the theory of hydrodynamic chains [20, 21]. Equations of EPD type with $m = 2$ were used in [22] to generate solutions of some dcKdV equations.

The main part of our analysis is devoted to the study of the singular sectors $\mathcal{M}_m^{\text{sing}}$ of the spaces of hodograph solutions for the dcKdV$^m$ hierarchies. They are given by the points $(t, \beta)$ such that

$$\text{rank} \left( \frac{\partial^2 W_m(t, \beta)}{\partial \beta_i \partial \beta_j} \right) < m.$$

The use of equations (4)–(6) simplifies drastically our analysis and allows us to study the structure of singular sectors $\mathcal{M}_m^{\text{sing}}$ in detail. In particular, a nested sequence of subvarieties,

$$\mathcal{M}_m^{\text{sing}} \supset \mathcal{M}_{m,1}^{\text{sing}} \supset \mathcal{M}_{m,2}^{\text{sing}} \supset \cdots \supset \mathcal{M}_{m,q}^{\text{sing}} \supset \cdots,$$

is characterized, which represents subsets of the singular sector of the dcKdV$^m$ hierarchy with increasing singular degree $q$, such that $\mathcal{M}_{m,q}^{\text{sing}}$ is determined by a class of hodograph solutions.
of the dcKdV $m+2$ hierarchy. The varieties $\mathcal{M}_m^{\text{sing}}$ provide us with special classes of degenerate critical points of the function $W_m$ within the general theory of critical points developed by Arnold and others about 40 years ago [24, 25].

The paper is organized as follows. The dcKdV $m$ hierarchies are described in section 2. Equations (4)–(6) are derived in section 3. Section 4 deals with the analysis of the singular sectors of the dcKdV $m$ hierarchies in terms of their associated hodograph equations. The relation between singular points of the dcKdV $m$ hodograph equations and solutions of higher dcKdV $m+2$ hodograph equations is stated in section 4. Some concrete examples involving shock singularities of the Burgers–Hopf equation and the one-layer Benney system are presented in section 5.

2. The dcKdV $m$ hierarchies

Given a positive integer $m \geq 1$, we consider the set $M_m$ of algebraic curves (1). For $m = 2g + 1$ (odd case) and $m = 2g + 2$ (even case), these curves are, generically, hyperelliptic Riemann surfaces of genus $g$. We will denote by $q = (q_1, \ldots, q_m)$ any of the two sets of parameters $u := (u_0, \ldots, u_{m-1})$ or $\beta := (\beta_1, \ldots, \beta_m)$ which determine the curves (1):

$$u(\lambda) = \lambda^m - \sum_{i=0}^{m-1} \lambda^i u_i = \prod_{i=1}^{m} (\lambda - \beta_i).$$

(9)

Obviously, for any fixed $\beta$ all the permutations $\sigma(\beta) := (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(m)})$ represent the same element of $M_m$. Note also that

$$u_i = (-1)^{m-i} s_{m-i}(\beta),$$

(10)

where $s_k$ are the elementary symmetric polynomials:

$$s_k = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \beta_{i_1} \cdots \beta_{i_k}.$$

We next introduce the dcKdV $m$ hierarchy as particular systems of commuting flows, $q(t)$, $t := (x := t_0, t_1, t_2, \ldots)$, on $M_m$. In order to define these flows, we use the set $\mathcal{L}$ of formal power series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where

$$z := \lambda^{1/2} \quad \text{for} \quad m = 2g + 1; \quad z := \lambda \quad \text{for} \quad m = 2g + 2.$$

For any given $m \geq 1$, a distinguished element of $\mathcal{L}$ is provided by the branch of $p = \sqrt{u(\lambda)}$ such that as $z \to \infty$ it has an expansion of the form

$$p(z, q) = z^{2g+1} \left(1 + \sum_{n \geq 1} \frac{b_n(q)}{z^{2n}}\right), \quad m = 2g + 1,$$

$$p(z, q) = z^{g+1} \left(1 + \sum_{n \geq 1} \frac{b_n(q)}{z^n}\right), \quad m = 2g + 2.$$

(11)

We define the following splittings: $\mathcal{L} = \mathcal{L}_{(+q)} \oplus \mathcal{L}_{(-q)}$,

$$f_{(+q)}(z) := \left(\frac{f(z)}{p(z, q)}\right)_+ p(z, q), \quad f_{(-q)}(z) := \left(\frac{f(z)}{p(z, q)}\right)_- p(z, q).$$

(12)
where \( f_{\oplus} \) and \( f_{\ominus} \) stand for the standard projections on positive and strictly negative powers of \( z \), respectively:

\[
f_{\oplus}(z) := \sum_{n=0}^{\infty} c_n z^n, \quad f_{\ominus}(z) := \sum_{n=-\infty}^{-1} c_n z^n.
\]

The dKdV\(_m\) flows \( q(t) \) are characterized by the following condition: there exists a family of functions \( S(z, t, q(t)) \) in \( \mathcal{L} \) satisfying

\[
\partial_t S(z, t, q(t)) = \Omega_n(z, q(t)), \quad n \geq 0,
\]

where

\[
\Omega_n(z, q) := (\lambda(z)^{n+m/2}(\tau, q)) = \begin{cases} 
\left( z^{2n+2g+1}(\tau, q) \right), & m = 2g + 1 \\
\left( z^{n+1}(\tau, q) \right), & m = 2g + 2, \quad n \geq 0.
\end{cases}
\]

We note that

\[
\Omega_n(z, q) = (\lambda^n R(\lambda(z), q))_{\oplus} p,
\]

where \( R(\lambda, q) := \frac{\sqrt{\lambda^m}}{u(\lambda)} = \sum_{n \geq 0} R_n(q) \lambda^n, \lambda \to \infty. \]

The coefficients \( R_n(q) \) are polynomials in the coordinates \( q \), for example

\[
R_0 = 1, \quad R_1 = \frac{1}{2} u_m, \quad R_2 = \frac{1}{2} u_{m-2} + \frac{3}{8} u_{m-1}^3, \ldots.
\]

Functions \( S \) which satisfy (13) will be referred to as \textit{action functions} of the dKdV\(_m\) hierarchy. This kind of generating functions \( S \) has been already used in the theory of dispersionless integrable systems (see e.g. [14]). It can be proved [1] that (13) is a compatible system of equations for \( S \). In fact its general solution will be determined in the next section. We note that for \( n = 0 \), equation (13) reads

\[
\partial_t S(z, t, q(t)) = p(z, q(t)),
\]

so that (13) is equivalent to the system

\[
\partial_t \partial_n S(z, t, q(t)) = \partial_z \Omega_n(z, q(t)), \quad n \geq 0.
\]

We will henceforth refer to the dKdV\(_m\) hierarchy for \( m = 2g + 1 \) and \( m = 2g + 2 \) as the Burgers–Hopf (BH\(_g\)) and the dispersionless Jaulent–Miodek (dJM\(_g\)) hierarchies, respectively. Observe that both hierarchies, BH\(_g\) and dJM\(_g\), determine deformations of hyperelliptic Riemann surfaces of genus \( g \). In our work we always consider an arbitrary but finite number of these flows.

Since \( u = u(\lambda(z), q) = p(z, q)^2 \), the operator \( J = J(\lambda, u) \) defined by

\[
J := 2p \partial_z \cdot p = 2u \partial_z + u_z,
\]

\[
J = \sum_{i=0}^{m} \lambda^i J_i, \quad J_m = 20, \quad J_i = -(2u_i \partial_z + u_{i,z}), \quad u_m := -1,
\]

satisfies \( JR = 0 \). Then, from (18) it follows that

\[
\partial_u u = J(\lambda^n R(\lambda(z), u))_{\oplus} = -J(\lambda^n R(\lambda(z), u))_{\ominus},
\]

which constitutes the dKdV\(_m\) hierarchy in terms of the coordinates \( u \),

\[
\partial_u u_i = \sum_{l-k+1, k \geq 1} J_l R_{n+k}(u), \quad i = 0, \ldots, m - 1.
\]
From (18) it also follows that
\[ \partial_n \log p(z, q) = \partial_n [(\lambda(z)^\beta R(\lambda(z), q)) \otimes p], \]
and then, identifying the residues of both sides at \( \lambda = \beta_i \), we obtain
\[ \partial_n \beta_i = \omega_n,i(\beta) \partial_i \beta_i, \quad i = 1, \ldots, m, \tag{21} \]
where
\[ \omega_n,i(\beta) := (\lambda^n R(\lambda, \beta)) \otimes |_{\lambda = \beta_i}. \tag{22} \]
The systems (21) are the equations of the dcKdV_\beta hierarchy in terms of the coordinates \( \beta_i \). Observe that we have two dcKdV_\beta hierarchies, BH_1 and dJM_0, which determine deformations of hyperelliptic Riemann surfaces of genus \( g \). It can be shown [2, 13] that the dcKdV_\beta flows are bi-Hamiltonian systems.

We next present some examples of interesting flows in the dcKdV_\beta hierarchies. The dcKdV_1 hierarchy is associated with the curve
\[ p^2 - u(\lambda) = 0, \quad u(\lambda) = \lambda - v, \quad v := u_0 = \beta_1. \]
The corresponding flows are given by
\[ \partial_n v = c_n v^n u_x, \quad c_n := \frac{(2n + 1)!!}{2^n n!}, \quad n \geq 1, \]
and constitute the Burgers–Hopf hierarchy BH_0. In particular, the \( t_1 \)-flow is the Burgers–Hopf equation
\[ \partial_t v = \frac{1}{2} v u_x, \]
which is in turn the dispersionless limit of the KdV equation.

The dcKdV_2 (dJM_0) hierarchy is associated with the curve
\[ p^2 - u(\lambda) = 0, \quad u(\lambda) = \lambda^2 - \lambda u_1 - u_0 = (\lambda - \beta_1)(\lambda - \beta_2), \]
\[ u_1 = \beta_1 + \beta_2, \quad u_0 = -\beta_1 \beta_2. \]
The \( t_1 \)-flow of this hierarchy is given by the dispersionless Jaulent–Miodek system
\[ \begin{align*}
\partial_t u_0 &= u_0 u_{1x} + \frac{1}{2} u_1 u_{0x}, \\
\partial_t u_1 &= u_{0x} + \frac{3}{2} u_1 u_{1x},
\end{align*} \tag{23} \]
which under the changes of dependent variables,
\[ u = -u_1, \quad v = u_0 + \frac{u_2}{4}, \]
becomes the one-layer Benney system (2). In terms of the Riemann invariants \( \beta_1 \) and \( \beta_2 \) \((u = -\beta_1 + \beta_2), v = (\beta_1 - \beta_2)^2/4\), the system (2) takes the well-known form
\[ \begin{align*}
\partial_t \beta_1 &= \frac{1}{2} (3 \beta_1 + \beta_2) \beta_{1x}, \\
\partial_t \beta_2 &= \frac{1}{2} (3 \beta_2 + \beta_1) \beta_{2x}.
\end{align*} \tag{24} \]
For \( v > 0 \) the one-layer Benney system is hyperbolic while for \( v < 0 \) it is elliptic.

Finally, we consider the BH_1 hierarchy. Its associated curve is given by
\[ p^2 - u(\lambda) = 0, \quad u(\lambda) = \lambda^3 - \lambda^2 u_2 - \lambda u_1 - u_0 = (\lambda - \beta_1)(\lambda - \beta_2)(\lambda - \beta_3), \]
\[ u_1 = \beta_1 + \beta_2 + \beta_3, \quad u_2 = -\beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3), \quad u_3 = \beta_1 \beta_2 \beta_3. \]
The first flow takes the forms
\[ \begin{align*}
\partial_t u_0 &= \frac{1}{2} u_2 u_{0x} - u_0 u_{2x}, \\
\partial_t u_1 &= u_{0x} + \frac{1}{2} u_2 u_{1x} + u_1 u_{2x}, \quad \iff \partial_t \beta_1 = \frac{1}{2} (3 \beta_1 + \beta_2 + \beta_3) \beta_{1x}, \\
\partial_t u_2 &= u_{1x} + \frac{3}{2} u_2 u_{2x}, \quad \iff \partial_t \beta_2 = \frac{1}{2} (3 \beta_2 + \beta_1 + \beta_3) \beta_{2x}. \tag{25}
\end{align*} \]
3. Hodograph equations for dcKdV$_m$ hierarchies and the Euler–Poisson–Darboux equation

Let us introduce the function

$$W_m(t, q) := \int_\gamma \frac{d\lambda}{2i\pi} U(\lambda, t) R(\lambda, q) = \sum_{n \geq 0} t_n R_{n+1}(q),$$  \hspace{1cm} (26)$$

where $\gamma$ denotes a large positively oriented circle $|\lambda| = r$, $U(\lambda, t) := \sum_{n \geq 0} t_n \lambda^n$ and $R(\lambda, q)$ is the function defined in (16).

**Theorem 1.** If the functions $q(t) = (q_1(t), \ldots, q_m(t))$ satisfy the system of hodograph equations

$$\frac{\partial W_m(t, q)}{\partial q_i} = 0, \quad i = 1, \ldots, m,$$  \hspace{1cm} (27)$$

then $q(t)$ is a solution of the dcKdV$_m$ hierarchy.

**Proof.** We are going to prove that the function

$$S(z, t, q(t)) = \sum_{n \geq 0} t_n \Omega_n(z, q(t)) = (U(\lambda(z), t) R(\lambda(z), q(t))) \oplus p(z, q(t))$$  \hspace{1cm} (28)$$

is an action function for the dcKdV$_m$ hierarchy. By differentiating (28) with respect to $t_n$, we have that

$$\frac{\partial_n S}{\partial \beta_i} = \Omega_n + (U \partial_n R) \oplus p + (UR) \partial_n p.$$  \hspace{1cm} (29)$$

We now use the coordinates $\beta = (\beta_1, \ldots, \beta_m)$ so that we may take advantage of the identities

$$\partial_\beta p = -\frac{1}{2} \frac{p}{\lambda - \beta_i}, \quad \partial_\beta R = \frac{1}{2} \frac{R}{\lambda - \beta_i}.$$  \hspace{1cm} (30)$$

Thus, we deduce that

$$(U \partial_n R) \oplus p + (UR) \partial_n p = \frac{1}{2} \sum_{i=1}^m \left[ \frac{UR}{\lambda - \beta_i} \oplus \frac{UR}{\lambda - \beta_i} \right] p \partial_n \beta_i.$$  \hspace{1cm} (31)$$

On the other hand,

$$\frac{\partial W_m(t, \beta)}{\partial \beta_i} = \frac{1}{2} \int_\gamma \frac{d\lambda}{2i\pi} \frac{U(\lambda, t) R(\lambda, \beta)}{\lambda - \beta_i} = \frac{1}{2} \int_\gamma \frac{d\lambda}{2i\pi} \frac{(U(\lambda, t) R(\lambda, \beta)) \oplus}{\lambda - \beta_i}.$$  \hspace{1cm} (32)$$

Hence, the hodograph equations (27) can be written as

$$(U(\lambda, t) R(\lambda, \beta(t)) \oplus)_{|\lambda = \beta_i} = 0, \quad i = 1, \ldots, m.$$  \hspace{1cm} (33)$$

Thus, we have that $(U(\lambda, t) R(\lambda, \beta(t)) \oplus)$ is a polynomial in $\lambda$ which vanishes at $\lambda = \beta_i(t)$ for all $i$. As a consequence

$$(UR) \oplus_{\lambda - \beta_i} = \left( \frac{UR}{\lambda - \beta_i} \right)_\oplus = \left( \frac{UR}{\lambda - \beta_i} \right)_\oplus.$$  \hspace{1cm} (34)$$

Then, from (29) and (31) we deduce that $\partial_n S = \Omega_n$ and therefore the statement follows. $\square$

Using (26) we obtain that the hodograph equations (27) can be expressed as

$$\sum_{n \geq 0} t_n \frac{\partial R_{n+1}(q)}{\partial q_i} = 0, \quad i = 1, \ldots, m.$$  \hspace{1cm} (34)$$
Furthermore, from (21), (22) and (33) the hodograph equations (27) can also be written as [1]

\[ \sum_{n \geq 0} t_n \omega_{n,i}(\beta) = 0, \quad i = 1, \ldots, m, \]  

which represent the hodograph transform for the dcKdV\textsubscript{m} hierarchy of flows in the hydrodynamic form.

Note also that we may shift the time parameters \( t_n \to t_n - c_\alpha \) in (34) to get solutions depending on an arbitrary number of constants.

It is easy to see that the generating function

\[ R(\lambda, \beta) := \sqrt{\frac{\lambda^m}{u(\lambda)}} = \sqrt{\frac{\lambda^m}{\prod_{i=1}^{m}(\lambda - \beta_i)}} \]

is a symmetric solution of the EPD equation:

\[ 2(\beta_i - \beta_j) \frac{\partial^2 R}{\partial \beta_i \partial \beta_j} = \frac{\partial R}{\partial \beta_i} - \frac{\partial R}{\partial \beta_j}. \]  

Consequently, the same property is satisfied by \( W(t, \beta) \) for all \( t \). Thus, we have proved

**Theorem 2.** The solutions \((t, \beta)\) of the hodograph equations

\[ \frac{\partial W_m(t, \beta)}{\partial \beta_i} = 0, \quad i = 1, \ldots, m, \]  

are the critical points of the solution

\[ W_m(t, \beta) := \oint_{C} \frac{d\lambda}{2i\pi} \frac{U(\lambda, t)}{\sqrt{\prod_{i=1}^{m}(1 - \beta_i/\lambda)}} \]

of the EPD equation

\[ 2(\beta_i - \beta_j) \frac{\partial^2 W_m}{\partial \beta_i \partial \beta_j} = \frac{\partial W_m}{\partial \beta_i} - \frac{\partial W_m}{\partial \beta_j}. \]  

Let us denote by \( M_m \) the variety of points \((t, \beta) \in \mathbb{C}^{\infty} \times \mathbb{C}^m\) which satisfy the hodograph equations (37). From (32) it is clear that for any permutation \( \sigma \) of \([1, \ldots, m]\), the functions

\[ F_i(t, \beta) := \frac{\partial W_m(t, \beta)}{\partial \beta_i}, \]  

satisfy

\[ F_i(t, \sigma(\beta)) = F_{\sigma(i)}(t, \beta). \]  

Then, it is clear that \( M_m \) is invariant under the action of the group of permutations

\((t, \beta) \in M_m \implies (t, \sigma(\beta)) \in M_m.\)

If \((t, \beta)\) is a solution of (37) such that \( \beta_i \neq \beta_j \) for all \( i \neq j \), then it will be called an *unreduced* solution of (37). In this case the EPD equation (38) implies that

\[ \frac{\partial^2 W_m(t, \beta)}{\partial \beta_i \partial \beta_j} = 0, \quad \forall i \neq j. \]  

Given \( 2 \leq r \leq m \), a solution \((t, \beta)\) of (37) such that exactly \( r \) of its components are equal will be called an *r-reduced solution* of (37).

The formulation (27) of the hodograph equations for the dcKdV\textsubscript{m} hierarchies allows us to apply the theory of critical points of functions to analyze the solutions of these hierarchies, while (38) indicates that the functions \( W_m \) are of a very special class.
The EPD equation (38) arose in the study of cyclids [15], where solutions $W$ of the above form have been found too. Much later it appeared in the theory of Whitham equations describing the small dispersion limit of the KdV equation [16, 17, 19].

We note that hodograph equations of a form close to (27) have been presented in [20] and [23]. Furthermore, linear equations of the EPD type and their connection with hydrodynamic chains have been studied in [21] too.

Finally, we emphasize that the functions $W_m$ depend on the parameters $t_1, t_2, \ldots$ (times of the hierarchy). Since ‘degenerate critical points appear naturally in cases when the functions depend on parameters’ [24, 25], one should expect the existence of families of degenerate critical points for the functions $W_m$. Their connection with the singular sectors in the spaces of solutions for dcKdV$_m$ will be considered in the next section.

To illustrate the statements given above, we next present some simple examples. For the dcKdV$_2$ hierarchy, we have

$$W_2(t, \beta) = \frac{x}{2} (\beta_1 + \beta_2) + \frac{1}{8} t_1 \left(3 \beta_1^2 + 2 \beta_1 \beta_2 + 3 \beta_2^2\right) + \frac{1}{16} t_1 \left(5 \beta_1^2 + 3 \beta_1 \beta_2 + 3 \beta_1 \beta_2 + 5 \beta_2^2\right) + \frac{1}{128} t_1 \left(35 \beta_1^4 + 20 \beta_1^2 \beta_2 + 18 \beta_1 \beta_2^2 + 20 \beta_1 \beta_2 + 35 \beta_2^4\right) + \cdots. $$

The hodograph equations with $t_n = 0$, for $n \geq 4$, take the form

$$\begin{cases} 8x + 4t_1 (3 \beta_1 + \beta_2) + 3t_2 (5 \beta_1^2 + 2 \beta_1 \beta_2 + \beta_2^2) + \frac{1}{8} t_1 \left(140 \beta_1^4 + 60 \beta_1^2 \beta_2 + 18 \beta_1 \beta_2^2 + 20 \beta_2^4\right) = 0, \\
8x + 4t_1 (\beta_1 + 3 \beta_2) + 3t_2 (5 \beta_1^2 + 2 \beta_1 \beta_2 + 5 \beta_2^2) + \frac{1}{8} t_1 \left(140 \beta_1^4 + 60 \beta_2^2 \beta_1 + 18 \beta_2^4 \beta_1 + 20 \beta_1^4\right) = 0. \end{cases} $$

(42)

For the dcKdV$_3$ hierarchy, we have

$$W_3(t, \beta) = \frac{x}{2} (\beta_1 + \beta_2 + \beta_3) + \frac{1}{8} t_1 \left(3 \beta_1^2 + 3 \beta_2^2 + 3 \beta_3^2 + 2 \beta_1 \beta_2 + 2 \beta_i \beta_j + 2 \beta_2 \beta_3\right) + \frac{1}{16} t_1 \left(5 \beta_1^2 + 5 \beta_1 \beta_2 + 5 \beta_2^2 + 3 \beta_1 \beta_3 + 3 \beta_2 \beta_3 + 3 \beta_3^2 + 3 \beta_1 \beta_2 + 3 \beta_1 \beta_3 + 3 \beta_2 \beta_3 + 3 \beta_3^2\right) + \frac{1}{128} t_1 \left(35 \beta_1^4 + 35 \beta_1^2 \beta_2 + 35 \beta_1 \beta_2^2 + 35 \beta_2^4 + 35 \beta_1 \beta_2^2 + 35 \beta_1 \beta_3^2 + 35 \beta_2 \beta_3^2 + 35 \beta_3^4\right) + \cdots. $$

The hodograph equations with $t_n = 0$ for $n \geq 3$ are

$$\begin{cases} 8x + 4t_1 (3 \beta_1 + \beta_2 + \beta_3) + t_2 \left(15 \beta_1^2 + 3 \beta_2^2 + 3 \beta_3^2 + 6 \beta_1 \beta_2 + 6 \beta_1 \beta_3 + 2 \beta_2 \beta_3\right) = 0, \\
8x + 4t_1 (\beta_1 + 3 \beta_2 + \beta_3) + t_2 \left(15 \beta_1^2 + 3 \beta_2^2 + 3 \beta_3^2 + 6 \beta_1 \beta_2 + 2 \beta_1 \beta_3 + 6 \beta_2 \beta_3\right) = 0, \\
8x + 4t_1 (\beta_1 + \beta_2 + 3 \beta_3) + t_2 \left(15 \beta_1^2 + 3 \beta_2^2 + 15 \beta_3^2 + 2 \beta_1 \beta_2 + 6 \beta_1 \beta_3 + 6 \beta_2 \beta_3\right) = 0. \end{cases} $$

(43)

4. Singular sectors of dcKdV$_m$ hierarchies

We say that $(t, \beta) \in \mathcal{M}_m$ is a regular point if it is a nondegenerate critical point of the function $W_m$. That it is to say, if it satisfies [24, 25]

$$\det \left( \frac{\partial^2 W_m(t, \beta)}{\partial \beta_i \partial \beta_j} \right) \neq 0. $$

(44)

The set of regular points of $\mathcal{M}_m$ will be denoted by $\mathcal{M}_m^{\text{reg}}$ and the points of its complementary set $\mathcal{M}_m^{\text{sing}} := \mathcal{M}_m - \mathcal{M}_m^{\text{reg}}$, where the second differential of $W_m$ is a degenerate quadratic form, will be called singular points. We will also refer to $\mathcal{M}_m^{\text{reg}}$ and $\mathcal{M}_m^{\text{sing}}$ as the regular and singular sectors of the dcKdV$_m$ hierarchy, respectively. So $\mathcal{M}_m^{\text{reg}}$ describes families of degenerate critical points of the function $W_m$. Near a regular point the variety $\mathcal{M}_m^{\text{reg}}$ can be uniquely described as $(t, \beta(t))$ where $\beta(t)$ is a solution of the dcKdV$_m$ hierarchy.
Theorem 3. Let

From (41) it follows at once that

and

Furthermore, there are also unreduced singular points \((x, t_1, t_2, \beta_1, \beta_2)\) determined by the constraint

360x^2_t = -45t_3t_1^2 + 180t_1t_3t_2 + \sqrt{15}(8t_1t_3 - 3t_2^2)\sqrt{t_2^2 - t_3(t_3 - 8t_1t_3)},

and

\[
\beta_1 = -\frac{3t_2t_3 + \sqrt{15}\sqrt{t_2^2 - t_3(t_3 - 8t_1t_3)}}{12t_3}, \quad \beta_2 = -\frac{5t_2t_3 + \sqrt{15}\sqrt{t_2^2 - t_3(t_3 - 8t_1t_3)}}{20t_3}
\]

From (41) it follows at once that

**Theorem 3.** Let \((t, \beta)\) be an unreduced solution of the hodograph equations (37), then \((t, \beta)\) is a singular point if and only if at least one of the derivatives

\[
\frac{\partial^2 W_m(t, \beta)}{\partial \beta_i^2}, \quad i = 1, \ldots, m,
\]

vanishes.

Note that since the function \(W_m\) satisfies the EPD equation (38), its partial derivatives at unreduced points \((t, \beta)\)

\[
\frac{\partial^q W_m(t, \beta)}{\partial ^{\beta_1}_{q_1}\ldots\partial ^{\beta_m}_{q_m}}, \quad q := q_1 + \cdots + q_m,
\]

can always be expressed as a linear combination of diagonal derivatives \(\partial^k_{\beta_i} W_m\) with \(k_i \leq q_i\). Thus, for each vector \(q = (q_1, \ldots, q_m) \in \mathbb{N}^m\) with at least one \(q_i \geq 1\), it is natural to introduce an associated subvariety \(\mathcal{M}^\text{sing}_{m,q}\) of \(\mathcal{M}^\text{sing}_m\) defined as the set of unreduced solutions \((t, \beta)\) of the hodograph equations (37) such that

\[
\frac{\partial^k W_m(t, \beta)}{\partial ^{\beta_i}_{k_i}} = 0, \quad \forall k_i \leq q_i, + 1.
\]

(45)

In particular, for \(q = (0, \ldots, 0, q)\) with \(q \geq 1\) we denote by \(\mathcal{M}^\text{sing}_{m,q}\) the subvariety associated with \(q = (0, \ldots, 0, q)\). That is to say, \(\mathcal{M}^\text{sing}_{m,q}\) is the set of solutions \((t, \beta)\) of the hodograph equations (37) such that

\[
\frac{\partial^2 W_m(t, \beta)}{\partial ^{\beta_1}_{q_1}\partial ^{\beta_m}_{q_m}} = \frac{\partial^3 W_m(t, \beta)}{\partial ^{\beta_1}_{q_1}\partial ^{\beta_m}_{q_m}} = \cdots = \frac{\partial^q W_m(t, \beta)}{\partial ^{\beta_1}_{q_1}\ldots\partial ^{\beta_m}_{q_m}} = 0.
\]

(46)

These subvarieties define a nested sequence,

\[
\mathcal{M}^\text{sing}_m \supset \mathcal{M}^\text{sing}_{m,1} \supset \mathcal{M}^\text{sing}_{m,2} \supset \cdots \mathcal{M}^\text{sing}_{m,q} \supset \cdots,
\]

(47)
and represent sets of points whose singular degree increases with \( q \). Moreover, due to the covariance of the functions \( F_i = \partial_{\beta_i} W_m \) under permutations there is no need of introducing alternative sequences of form (46) based on systems of equations corresponding to the remaining coordinates \( \beta_j \) for \( j \neq m \).

The next result states that the varieties \( \mathcal{M}_{m,q}^{\text{sing}} \) of the dcKdV\(_m\) hierarchy are closely related to the \((2q+1)\)-reduced solutions of the dcKdV\(_{m+2q}\) hierarchy.

Note that given \( 2 \leq r \leq m \), the hodograph equations for \( r \)-reduced solutions \( \beta_{m-r+1} = \beta_{m-r+2} = \cdots = \beta_m \), of the dcKdV\(_m\) hierarchy reduce to the system

\[
F_i(t, \beta) = 0, \quad i = 1, \ldots, m - r + 1,
\]

of \( m - r + 1 \) equations for the \( m - r + 1 \) unknowns \( \beta_1, \ldots, \beta_{m-r+1} \). Now we prove

**Theorem 4.** If \((t, \beta) \in \mathcal{M}_{m,q}^{\text{sing}} \) where \( t = (t_0, t_1, \ldots) \) and \( \beta = (\beta_1, \ldots, \beta_m) \), then if we define

\[
t^{(m+2q)} := (t_2, t_{q+1}, \ldots), \quad \beta^{(m+2q)} := (\beta_1, \ldots, \beta_m, \beta_m, \ldots, \beta_m),
\]

it follows that \((t^{(m+2q)}, \beta^{(m+2q)})\) is a \((2q+1)\)-reduced solution of the hodograph equations for the dcKdV\(_{m+2q}\) hierarchy.

**Proof.** To proof this statement we will use the superscripts \((m)\) and \((m+2q)\) to distinguish objects corresponding to different hierarchies. By assumption we have that \((t^{(m)}, \beta^{(m)}) \in \mathcal{M}_{m,q}^{\text{sing}} \) where \( t = (t_0, t_1, \ldots, t_m) \) and \( \beta = (\beta_1, \ldots, \beta_m) \). Thus, taking (30) into account, we have that (46) can be rewritten as

\[
\begin{align*}
F^{(m)}_{i}(t^{(m)}, \beta^{(m)}) &:= \oint_{\gamma} \frac{d\lambda}{2i\pi} \frac{U^{(m)}(\lambda, t^{(m)}) R^{(m)}(\lambda, \beta^{(m)})}{\lambda - \beta^{(m)}_i} = 0, \quad i = 1, \ldots, m, \\
F^{(m)}_{m,j}(t^{(m)}, \beta^{(m)}) &:= \oint_{\gamma} \frac{d\lambda}{2i\pi} \frac{U^{(m)}(\lambda, t^{(m)}) R^{(m)}(\lambda, \beta^{(m)})}{(\lambda - \beta^{(m)}_m)^j} = 0, \quad j = 2, \ldots, q + 1.
\end{align*}
\]

Now a \((2q+1)\)-reduced solution of the hodograph equations for the dcKdV\(_{m+2q}\) is characterized by

\[
F^{(m+2q)}_{i}(t^{(m+2q)}, \beta^{(m+2q)}) := \oint_{\gamma} \frac{d\lambda}{2i\pi} \frac{U^{(m+2q)}(\lambda, t^{(m+2q)}) R^{(m+2q)}(\lambda, \beta^{(m+2q)})}{\lambda - \beta^{(m+2q)}_i} = 0,
\]

where \( i = 1, \ldots, m \). But it is clear that

\[
R^{(m+2q)}(\lambda, \beta^{(m+2q)}) = \frac{\lambda^q}{(\lambda - \beta^{(m+2q)}_m)^q} R^{(m)}(\lambda, \beta^{(m)}).
\]

Hence, if we set

\[
t^{\prime (m+2q)} := t^{\prime (m)}, \quad i \geq 0,
\]

we have

\[
U^{(m)}(\lambda, t^{(m)}) = x^{(m)} + \lambda t_1^{(m)} + \cdots + \lambda^{q-1} t_{q-1}^{(m)} + \lambda^q t^{(m+2q)}(\lambda, t^{(m+2q)}).
\]

Then, it follows that

\[
F^{(m+2q)}_{i}(t^{(m+2q)}, \beta^{(m+2q)}) = \oint_{\gamma} \frac{d\lambda}{2i\pi} \frac{U^{(m)}(\lambda, t^{(m)}) R^{(m)}(\lambda, \beta^{(m)})}{(\lambda - \beta^{(m)}_i)(\lambda - \beta^{(m)}_m)^q}, \quad i = 1, \ldots, m.
\]
Furthermore, for any given \( i = 1, \ldots, m \), we have

\[
F_{i}^{(m)}(t^{(m)}, \beta^{(m)}) = \oint d\lambda \frac{U^{(m)}(\lambda, t^{(m)})R^{(m)}(\lambda, \beta^{(m)})}{\lambda - \beta_{i}^{(m)}}
\]

\[
= \oint d\lambda \frac{(\lambda - \beta_{i}^{(m)}) U^{(m)}(\lambda, t^{(m)})R^{(m)}(\lambda, \beta^{(m)})}{(\lambda - \beta_{i}^{(m)})^{q} \lambda - \beta_{i}^{(m)}}
\]

\[
= \sum_{k=0}^{q} c_{i,k}(\beta^{(m)}) \frac{I_{i,k}(t^{(m)}, \beta^{(m)})}{1}
\]

and

\[
F_{m,j}^{(m)}(t^{(m)}, \beta^{(m)}) = \oint d\lambda \frac{U^{(m)}(\lambda, t^{(m)})R^{(m)}(\lambda, \beta^{(m)})}{\lambda - \beta_{m}^{(m)}}
\]

\[
= \oint d\lambda \frac{(\lambda - \beta_{m}^{(m)}) U^{(m)}(\lambda, t^{(m)})R^{(m)}(\lambda, \beta^{(m)})}{(\lambda - \beta_{m}^{(m)})^{q} \lambda - \beta_{m}^{(m)}}
\]

\[
= \sum_{k=0}^{q-j+1} c_{j,k}(\beta^{(m)}) \frac{I_{i,k}(t^{(m)}, \beta^{(m)})}{1}
\]

where the functions \( c_{i,k}(\beta^{(m)}) \) are the coefficients of the polynomials

\[
\begin{cases}
(\lambda - \beta_{m}^{(m)})^q = \sum_{k=0}^{q} c_{1,k}(\beta^{(m)}) \lambda^k, \\
(\lambda - \beta_{i}^{(m)}) (\lambda - \beta_{m}^{(m)})^{q-j} = \sum_{k=0}^{q-j} c_{j,k}(\beta^{(m)}) \lambda^k, & j = 2, \ldots, q + 1,
\end{cases}
\]

(53)

and

\[
I_{i,k}(t^{(m)}, \beta^{(m)}):= \oint d\lambda \frac{\lambda^k U^{(m)}(\lambda, t^{(m)})R^{(m)}(\lambda, \beta^{(m)})}{(\lambda - \beta_{i}^{(m)}) (\lambda - \beta_{m}^{(m)})^q}
\]

(54)

Now, for any given \( i = 1, \ldots, m \) the system (46) implies

\[
\begin{cases}
F_{i}^{(m)}(t^{(m)}, \beta^{(m)}) = 0, \\
F_{m,j}^{(m)}(t^{(m)}, \beta^{(m)}) = 0, & j = 2, \ldots, q + 1,
\end{cases}
\]

and, as a consequence, we deduce the following system of \( q \) homogeneous linear equations for the \( q \) functions \( I_{i,k}(t^{(m)}, \beta^{(m)}) \)

\[
\sum_{k=0}^{q-j+1} c_{j,k}(\beta^{(m)}) I_{i,k}(t^{(m)}, \beta^{(m)}) = 0, & j = 1, \ldots, q + 1.
\]

(52)

Because of the linear independence of the polynomials (53), these equations are linearly independent and, therefore, all the functions \( I_{i,k}(t^{(m)}, \beta^{(m)}) \) vanish. Finally, from (52) we conclude that \( I_{i,0}(t^{(m)}, \beta^{(m)}) = 0 \) is equivalent to \( F_{i}^{(m+2q)}(t^{(m+2q)}, \beta^{(m+2q)}) = 0 \) and the statement follows.

\[\square\]

5. Examples

5.1. dcKdV \(_1\) hierarchy

The hodograph equations for the dcKdV \(_1\) hierarchy with \( t_n = 0 \) for all \( n \geq 3 \) reduce to

\[
8x + 12t_1 \beta_1 + 15t_2 \beta_1^2 + 12t_3 \beta_1 = 0.
\]

(55)
The singular variety $\mathcal{M}^{\text{sing}}_{1,1}$ for (55) is determined by adding to (55) the equation
\[ 2t_1 + 5t_2 \beta_1 = 0, \] (56)
so that for $t_2 \neq 0$ we have $\beta_1 = -\frac{2}{5} t_2$. Substituting this result in (55), we find a constraint for the flow parameters
\[ x = \frac{3}{10} t_2^2, \]
which is the shock region for the solution of (55) given by
\[ \beta_1 = \frac{2}{15 t_2} \left(-3 t_1 + \sqrt{3(3 t_1^2 - 10 t_2 x)}\right). \] (57)

There are two sectors $\mathcal{M}^{\text{sing}}_{1,1,k}$ ($k = 1, 2$) in $\mathcal{M}^{\text{sing}}_{1,1}$
\[ \mathcal{M}^{\text{sing}}_{1,1,1} : \quad x = t_1 = t_2 = 0, \quad \beta_1 \text{ arbitrary}; \]
\[ \mathcal{M}^{\text{sing}}_{1,1,2} : \quad (x, t_1, t_2, \beta_1) \quad \text{such that} \quad t_2 \neq 0, \quad x = \frac{3}{5} \frac{t_1^2}{t_2}, \quad \beta_1 = -\frac{2}{5} \frac{t_1}{t_2}. \] (58)

To see the relationship with the dcKdV3 hierarchy, we note that
\[ x^{(3)} = t_1, \quad t_1^{(3)} = t_2, \]
and
\[ \beta^{(3)} = (\beta_1, \beta_1, \beta_1) = -\frac{2}{5} \frac{x^{(3)}}{t_1^{(3)}} (1, 1, 1), \]
which is a 3-reduced solution of the first flow (25) of the dcKdV3 hierarchy.

The dcKdV3 hodograph equation with $t_n = 0$ for all $n \geq 6$ is
\[ 693 t_3 \beta_1^5 + 630 t_3 \beta_1^4 + 560 t_3 \beta_1^3 + 480 t_3 \beta_1^2 + 384 t_3 \beta_1 + 256 x = 0. \]

Let us first consider the singular variety $\mathcal{M}^{\text{sing}}_{1,1}$ with $t_n = 0$ for all $n \geq 4$. It is determined by the equations
\[ 560 t_3 \beta_1^3 + 480 t_3 \beta_1^2 + 384 t_3 \beta_1 + 256 x = 0, \]
\[ 1680 t_3 \beta_1^2 + 960 t_3 \beta_1 + 384 t_3 = 0. \]

Thus, an open subset of $\mathcal{M}^{\text{sing}}_{1,1}$ can be parametrized by the equations
\[ x = \frac{-25 t_2^3 + 105 t_1 t_3 + \sqrt{5} \sqrt{125 t_2^6 - 1050 t_1 t_2^2 + 2940 t_1^2 t_2^2 + 2744 t_1^3 t_3^3}}{245 t_3^3}, \]
\[ \beta_1 = -\frac{2 \left(-25 t_2^3 + 70 t_1 t_2 t_3 + \sqrt{5} \sqrt{(5 t_2^2 - 14 t_1 t_2)^3}\right)}{35 t_3 (14 t_1 t_3 - 5 t_2^2)}. \]

It determines the following 3-reduced solution of the two first flows of the dcKdV3 hierarchy $(x^{(3)} = t_1, t_1^{(3)} = t_2, t_2^{(3)} = t_3)$
\[ \beta_1^{(3)} = \beta_2^{(3)} = \beta_3^{(3)} = -\frac{2 \left(-25 (t_1^{(3)})^3 + 70 x (t_1^{(3)}) t_2^{(3)} + \sqrt{5} \sqrt{(5 (t_1^{(3)})^2 - 14 x (t_1^{(3)}) t_2^{(3)})^3}\right)}{35 (14 x (t_1^{(3)}) t_2^{(3)} - 5 (t_1^{(3)})^2)}. \]

Next, for the sector $\mathcal{M}^{\text{sing}}_{1,2}$ if we set $t_n = 0$ for all $n \geq 5$, we obtain the equations
\[ 630 t_4 \beta_1^4 + 560 t_5 \beta_1^3 + 480 t_5 \beta_1^2 + 384 t_5 \beta_1 + 256 x = 0, \]
\[ 2520 t_4 \beta_1^3 + 1680 t_5 \beta_1^2 + 960 t_5 \beta_1 + 384 t_5 = 0, \]
\[ 7560 t_6 \beta_1^2 + 3360 t_5 \beta_1 + 960 t_5 = 0. \]
From these equations, we obtain
\[ t_1 = \frac{5(-49t_1^3 + 189t_2t_3t_4 + \sqrt{7}\sqrt{343t_1^6 - 2646t_2t_3t_4^2 + 6804t_2^2t_4^3} - 5832t_2^2t_4^2)}{1701t_4^2} , \]
\[ x = \frac{5(-98t_2^3 + 378t_2t_3t_4 + 2\sqrt{7}\sqrt{(7t_2^2 - 18t_2t_4)^3} t_1 - 243t_2^2t_4^2)}{10206t_4^2} , \]
\[ \beta_1 = \frac{-2(-49t_2^3 + 126t_2t_3t_4 + \sqrt{7}\sqrt{(7t_2^2 - 18t_2t_4)^3})}{63t_4(18t_2t_4 - 7t_2^2)} . \]

Then, the associated 5-reduced solution of the two first flows of the dcKdV5 hierarchy \((x, t_1^{(5)} = t_2, t_1^{(5)} = t_3, t_2^{(5)} = t_4)\) is given by
\[ \beta_i = \frac{-2(-49(t_1^{(5)})^3 + 126x(t_1^{(5)})t_2^{(5)}t_3^{(5)} + \sqrt{7}(7(t_1^{(5)})^2 - 18x(t_1^{(5)})t_2^{(5)}))}{63t_2^{(5)}(18x(t_1^{(5)})t_2^{(5)} - 7(t_1^{(5)})^2)} , \quad i = 1, \ldots, 5. \]

5.2. dcKdV2 hierarchy

Let us consider the hodograph equations for the dcKdV2 hierarchy with \(t_n = 0\) for all \(n \geq 3\).

From (42) we have that they take the form
\[
\begin{align*}
8x + 4t_1(3\beta_1 + \beta_2) + 3t_2(5\beta_1^2 + 2\beta_1\beta_2 + \beta_2^2) &= 0, \\
8x + 4t_1(\beta_1 + 3\beta_2) + 3t_2(\beta_1^2 + 2\beta_1\beta_2 + 5\beta_2^2) &= 0.
\end{align*}
\]
(59)
The singular variety \(M_2^{\text{sing}}\) is determined by (59) together with the additional equation
\[(2t_1 + 3t_2(\beta_1 + \beta_2))^2 + 9(2t_1 + t_2(5\beta_1 + \beta_2))(2t_1 + t_2(\beta_1 + 5\beta_2)) = 0. \]
(60)
The elements of \(M_2^{\text{sing}}\) are
\[ x = t_1 = t_2 = 0, \quad (\beta_0, \beta_1) \text{ arbitrary}; \]
\[ (x, t_1, t_2, \beta_1, \beta_2) \text{ such that } t_2 \neq 0, \quad x = \frac{t_1^2}{3t_2} \quad \text{and} \quad \beta_1 = \beta_2 = -\frac{t_1}{3t_2}. \]
(61)
The subvarieties \(M_2^{\text{sing}}_{k,q}\) are all equal and given by
\[ x = t_1 = t_2 = 0, \quad (\beta_0, \beta_1) \text{ arbitrary with } \beta_0 \neq \beta_1. \]

Note that the constraint \(x = \frac{t_1^2}{3t_2}\) determines the shock region for the following solution of (59):
\[ \beta_1 = \frac{-t_1 + \sqrt{2\sqrt{t_1^2 - 3t_2x}}}{3t_2}, \quad \beta_2 = \frac{-t_1 - \sqrt{2\sqrt{t_1^2 - 3t_2x}}}{3t_2}. \]
(62)

Let us now consider the system of hodograph equations (42) for the dcKdV2 hierarchy with \(t_n = 0\) for all \(n \geq 4\). It is possible to characterize the whole singular sector \(M_2^{\text{sing}}\) [26]. However, by lack of space we just present here the subvariety \(M_2^{\text{sing}}_{1,1}\). It is characterized by the system (42) and the additional constraint
\[ 8t_1 + 4t_2(\beta_1 + 5\beta_2) + t_3(3\beta_1^2 + 10\beta_1\beta_2 + 35\beta_2^2) = 0. \]
There are two sectors \( \mathcal{M}_{2,1}^{\text{sing}} \) \( (k = 1, 2) \) in \( \mathcal{M}_{2,1}^{\text{sing}} \):

\[
\mathcal{M}_{2,1}^{\text{sing}}: \quad x = \frac{-45t_3^3 + 180t_1t_2^2t_2 + \sqrt{15}(8t_1t_2 - 3t_2^2)\sqrt{t_2^2(3t_2^2 - 8t_1t_2)}}{360r_3^3},
\]

\[
\beta_1 = \frac{-3t_2t_3 + \sqrt{15}t_2^2(3t_2^2 - 8t_1t_2)}{12r_3^2}, \quad \beta_2 = \frac{-5t_2t_3 + \sqrt{15}t_2^2(3t_2^2 - 8t_1t_2)}{20r_3^2};
\]

\[
\mathcal{M}_{2,1}^{\text{sing}}: \quad x = \frac{-45t_3^3 + 180t_1t_2^2t_2 - \sqrt{15}(8t_1t_2 - 3t_2^2)\sqrt{t_2^2(3t_2^2 - 8t_1t_2)}}{360r_3^3},
\]

\[
\beta_1 = \frac{-3t_2t_3 + \sqrt{15}t_2^2(3t_2^2 - 8t_1t_2)}{12r_3^2}, \quad \beta_2 = \frac{-5t_2t_3 + \sqrt{15}t_2^2(3t_2^2 - 8t_1t_2)}{20r_3^2}.
\]

It is also possible to determine the solutions of the hodograph equations (42) whose shock regions are given by the constraints \( x = x(t_1, t_2, t_3) \) arising in \( \mathcal{M}_{2,1}^{\text{sing}} \) \( (k = 1, 2) \) [26].

To check the relationship with the dKdV\(_d\) hierarchy, we set

\[
x^{(4)} = t_1, \quad t^{(4)}_1 = t_2, \quad t^{(4)}_2 = t_3, \quad \beta^{(4)} = (\beta_1, \beta_2, \beta_2, \beta_2),
\]

and it is immediate to see that \( \beta^{(4)}(t^{(4)}) \) verifies the equations of the first flow of the dKdV\(_d\) hierarchy

\[
\frac{\partial \beta_i}{\partial t_1^{(4)}} = \left( \beta_i + \frac{1}{2} \sum_{k=1}^{4} \beta_k \right) \frac{\partial \beta_i}{\partial x^{(4)}}, \quad i = 1, \ldots, 4.
\]

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