Dynamical Glass Phase and Ergodization Times in Josephson Junction Chains

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Models of Josephson junction chains turn integrable in the limit of large energy densities or small Josephson energies. Close to these limits the Josephson coupling between the superconducting grains induces a short range nonintegrable network. We compute distributions of finite time averages of grain charges and extract the ergodization time $T_E$ which controls their convergence to ergodic distributions. We relate $T_E$ to the statistics of fluctuation times of the charges, which are dominated by fat tails. $T_E$ is growing anomalously fast upon approaching the integrable limit, as compared to the Lyapunov time $T_\Lambda$ - the inverse of the largest Lyapunov exponent - reaching astonishing ratios $T_E/T_\Lambda \lesssim 10^8$. The microscopic reason for the observed dynamical glass phase is routed in a growing number of grains evolving over long times in a regular almost integrable fashion due to the low probability of resonant interactions with the nearest neighbors. We conjecture that the observed dynamical glass phase is a generic property of Josephson junction networks irrespective of their space dimensionality.

Ergodicity is a core concept of statistical physics of many body systems. It demands infinite time averages of observables during a microcanonical evolution to match with their proper phase space averages [1]. Any laboratory or computational experiment is however constrained to finite averaging times. Are these sufficient or not? How much time is needed for a trajectory to visit the majority of the available microcanonical states, and for the finite-time average of an observable to be reasonably close to its statistical average? Can we define an ergodization time scale $T_E$ on which these properties manifest? What is the ergodization time depending on? Doubts on the applicability of the ergodic hypothesis itself were discussed for such simple cases as a mole of Ne at room temperature (see [2] and references therein). Glassy dynamics have been reported in a large variety of Hamiltonian systems [3–6]. Further, spin-glasses [7] and stochastic Levy processes [8–13] reveal that the ergodization time (and even ergodicity itself) may be affected by heavy-tailed distributions of lifetimes of typical excitations. The aim of this work is to address the above issues using a simple and paradigmatic dynamical many-body system testbed.

Josephson junction networks are devices that are known for their wide applicability over various fields such as superconductivity, cold atoms, optics and metamaterials, among others [14–16] (for a recent survey on experimental results, see [17]). Synchronization has been studied in Ref. [18, 19], discrete breathers were observed and studied in Ref. [20–23], qubit dynamics was analyzed in Ref. [24, 25] and thermal conductivity was computed in Ref. [26, 27]. In particular, a recent study conducted by Pino et al. [28] showed the existence of a non-ergodic/bad metal region in the high-temperature regime of a quantum chain of Josephson junctions, that exists as a prelude to a many-body localization phase [29]. Notably, in [28] it has been conjectured that the bad metal regime persists as a non-ergodic phase in the classical limit of the model - the large energy density regime of a chain of coupled rotors, close to an integrable limit. A similar prediction of a nonergodic phase (called weak coupling phase) in the same model was obtained in [30]. Further in [31], a faster decay of thermal conductivity in the high temperature regime is observed. On the other side, a novel dynamical glass phase has been recently identified in the proximity of such a limit if the nonintegrable perturbation spans a short range network between corresponding actions [32]. This dynamical glass phase is still ergodic, though being characterized by rapidly increasing ergodization time scales. The limit of weak Josephson coupling or high temperature is precisely corresponding to that short range network case. Is the Josephson junction chain then ergodic or not?

In this letter we demonstrate the existence of a dynamical glass phase in a Josephson junction chain of coupled rotors. We evaluate the convergence of distributions of finite time averages of the superconducting grain charges, and extract an ergodization time scale $T_E$. We show that this time scale is related to the properties of the statistics of charge fluctuation times. Such fluctuation event statistics was introduced in Refs. [33, 34]. We compute the Lyapunov time $T_\Lambda$ - the inverse of the largest Lyapunov exponent $\Lambda$ [35, 36]. The Lyapunov time is a lower bound for the ergodization time: $T_\Lambda \leq T_E$. In the reported dynamical glass phase, the dynamics stays ergodic and $T_E$ is finite. However, $T_E$ is growing anomalously fast upon approaching the integrable limit, as compared to $T_\Lambda$ reaching astonishing ratios $T_E/T_\Lambda \geq 10^8$. We show that $T_E$ is controlled by fat tails of charge fluctuation time distributions. We compute the spatio-temporal evolution of nonlinear resonances between interacting grains [30, 37]. The microscopic reason for the observed dynamical glass phase is routed in a growing number of grains evolving over long times in a regular almost integrable fashion due to the low probability of resonant interactions with nearest neighbors. The dynamical glass phase is expected to be a generic property of a large class of dynamical systems, where ergodization time scales depend sensitively
on control parameters. At the same time, the concept of ergodicity is preserved, and statistical physics continues to work - it is all just a matter of time scales.

We consider the Hamiltonian

$$H(q, p) = \sum_{n=1}^{N} \left[ \frac{p_n^2}{2} + E_J(1 - \cos(q_{n+1} - q_n)) \right], \quad (1)$$

describing the dynamics of a chain of $N$ superconducting islands with weak nearest neighbor Josephson coupling in its classical limit. We note that this model is equivalent to a XY chain or similarly to a coupled rotor chain, where the grain charging energies turn into the above kinetic energy terms [38, 37]. We apply periodic boundary conditions $p_1 = p_{N+1}$ and $q_1 = q_{N+1}$ for the conjugate angles $q_n$ and momenta $p_n$. $E_J$ controls the strength of Josephson coupling. The corresponding equations of motion of Eq. (1) are

$$\dot{q}_n = p_n, \quad \dot{p}_n = E_J \left[ \sin(q_{n+1} - q_n) + \sin(q_{n-1} - q_n) \right]. \quad (2)$$

This system has two conserved quantities: the total energy $H$ and the total angular momentum $L = \sum_{n=1}^{N} p_n$. We will choose $L = 0$ without loss of generality. Exact expressions for average full $h$ and kinetic $k$ energy densities as functions of temperature are obtained using a Gibbs distribution [38] and yield

$$h = k + E_J \left( 1 - \frac{I_1(E_J/2k)}{I_0(E_J/2k)} \right), \quad (3)$$

with $I_{0,1}$ being the modified Bessel functions of the first kind. We investigate the equilibrium dynamics of the above system in proximity to two integrable limits: $h \to \infty$, or $E_J \to 0$. At these limits, the system reduces to a set of uncoupled superconducting grains $H_0 = \sum_{n=1}^{N} \frac{p_n^2}{2}$ [37]. In proximity to these limits the Josephson terms induce a nonintegrable perturbation through a short-range interaction network of actions $\{p_n\}_n$ [34]. We consider the kinetic energies $k_n = \frac{p_n^2}{2}$ as a set of time-dependent observables. Due to the discrete translational invariance of $H$ all $k_n$ variables are statistically equivalent, fluctuating around their equilibrium value $k$. We will integrate the equations of motion using symplectic integrators [38]. Unless otherwise stated, we use the system size $N = 2^{10}$.

To quantitatively assess the ergodization time $T_E$, we compute finite time averages $\bar{k}_{n,T} = \frac{1}{T} \int_{0}^{T} k_n(t) dt$ for a set of $R$ different trajectories at given $h, E_J$. The corresponding distribution $\rho(k; T)$ is characterized by its 1st moment $\mu_k(T)$ and the standard deviation $\sigma_k(T)$. Assuming ergodicity, $\mu_k(T \to \infty) = k$ and $\sigma_k(T \to \infty) = 0$, since the distribution $\rho(k; T \to \infty) = \delta(k - k)$. In the inset of Fig. 1 we show the distributions $\rho(k; T)$ for $h = 1.2$ at two different averaging times $T = 10^5, 10^8$. As expected the distribution $\rho(k, T)$ converges to a delta function, centered around $k$. We then use the fluctuation index $q(T) = \frac{\sigma^2(T)}{\mu^2(T)}$ as a quantitative dimensionless measure of the above convergence properties.

In Fig. 1 we show $q(T)$ for different values of $h$ with $E_J = 1$. We find $q(T < T_E) = q(0)$ and $q(T \gg T_E) \sim T_E/T$ where $T_E$ is our definition of the ergodization time scale. We rescale and fit the different curves $q(T)$ and extract $T_E$ [38]. The result is plotted in Fig. 3(a) with green squares. $T_E$ quickly grows by orders of magnitude upon increasing the energy density $h$ in a rather moderate window of values, with a loose power law fit $T_E \sim h^6$. When fixing $h = 1$ and varying $E_J$, we make similar observations [38], with $T_E \sim E_J^{-6.5}$ as shown in Fig. 3(b). With our results, we validate ergodic dynamics in the considered system. Previous reports [28, 30] were not addressing the quickly growing time scale $T_E$ upon approaching the integrable limit.

Let us study the fluctuation statistics of the observables $k_n(t)$. Each of them has to fluctuate around their common average $k$. This allows to segment the trajectory of the whole system phase space into consecutive excursions [33, 34]. Note that for each site $n$ the segmenting is different, and we account for all of them. We measure the consecutive piercing times $t_n^i$ at which $k_n(t) = k$. We then compute the excursion times $\tau_n^\pm(i) = t_n^{i+1} - t_n^i$ for a trajectory of excursion events during which $k(t) > k(\tau^+)$ and $k(t) < k(\tau^-)$ respectively. Fig. 2 shows the distributions for $E_J = 1$ and various energy densities $h$. As $h$ increases both distributions increase their tail weights, with $P_+$ dominating over $P_-$. Further, the distributions develop intermediate tail structures close to a $1/\tau^2$. (inset in Fig. 2).
We can now compute the following two time scales: the average excursion time, $\mu_\tau$ and the standard deviation $\sigma_\tau$ of the distribution $P_\tau$, which are shown in Fig. 3(a) (orange diamonds and blue triangles) as functions of $h$. We observe that $\sigma_\tau$ equals with $\mu_\tau$ at $h \approx 1$ and quickly overgrows $\mu_\tau$ for $h > 1$, signaling the proximity to an integrable limit, where the dynamics is dominated by fluctuations rather than the means. Indeed, if the distributions $P_\tau(\tau)$ asymptotically reach a $1/\tau^2$ dependence in the integrable limit, then not only the time scales $\mu_\tau$ and $\sigma_\tau$ have to diverge, but their ratio $\sigma_\tau/\mu_\tau$ will diverge as well.

The above time scales are related to the ergodization time scale $T_E$ as

$$T_E \sim \tau_q \equiv \frac{\sigma_\tau^2}{\mu_\tau}. \quad (4)$$

This relation can be obtained e.g. after approximating the dependence $k_n(t)$ by a telegraph random process with excursion time distributions $P_\pm$ [32]. We plot $A \tau_q$ versus $h$ in Fig. 3(a) (black circles) with a fitting parameter $A = 130$. The curve is strikingly close to the dependence $T_E(h)$ for $h > 1$. We thus independently reconfirm that the considered system dynamics is ergodic, yet with quickly growing time scales of ergodization. When fixing $h = 1$ and varying $E_J$, we make similar observations as shown in Fig. 3(b).

With ergodicity being restored, the question remains: what is the microscopic origin of the enormously fast growing ergodization time? Since the considered system is nonintegrable, its dynamics must be chaotic. Therefore there is a Lyapunov time scale $T_\Lambda = 1/\Lambda$ dictated by the largest Lyapunov exponent $\Lambda$. $T_\Lambda$ can be expected to serve as a lower bound for the ergodization time scale. We compute $\Lambda$ using standard techniques [38] and also compare it with theoretical predictions [35, 36]. We plot the Lyapunov time $T_\Lambda$ versus $h$ in Fig. 3(a) (black circles) versus $E_J$ in Fig. 3(b) (magenta stars). The surprising finding is that $T_\Lambda \lesssim T_E$, e.g. for $E_J = 1$ and $h = 10$ we find $T_\Lambda \sim 1$ and $T_E \sim 10^8$. The inset of Fig. 3(b) demonstrates the above findings where $T_E$, $A \tau_q$, $\mu_\tau$ and $\sigma_\tau$ are plotted versus $T_\Lambda$ and in units of $T_\Lambda$. These are typical features of the dynamical glass phase (DGP), introduced in [32], which starts at $h \approx E_J$.

In order to advance, we analyze the spatiotemporal dynamics of $k_n(t)$ in Fig. 4(a). We plot black points during

![Figure 2](image2.png)

**Figure 2.** (Color online) $P_\tau(\tau)$ (solid lines) and $P_\tau(\tau)$ (dashed lines) for various energy densities $h = 1.2$ (magenta), $h = 2.4$ (blue) and $h = 5.4$ (red). Here $E_J = 1$. The black line corresponds to a power law decay $\tau^{-2}$ and guides the eye. Inset: $d(\log_{10} P_\tau)/d(\log_{10} \tau)$. The black solid horizontal line guides the eye at value -2.

![Figure 3](image3.png)

**Figure 3.** (Color online) (a) Time scales $T_E$ (green squares), $A \tau_q$ (black circles), Lyapunov time $T_\Lambda$ (magenta stars), $\mu_\tau$ (orange diamonds) and $\sigma_\tau$ (blue triangles) vs the energy density $h$ for $E_J = 1$ and $A = 130$. (b) Same as in (a) but vs $E_J$ at the fixed energy density $h = 1$ and with $A = 235$. Inset: Rescaled times ($A \tau_q, T_E, \mu_\tau, m_\tau$ and $\sigma_\tau$) in units of the Lyapunov time $T_\Lambda$. 

The large ergodization time $T_E$ could be related to a small density of chaotic spots, and/or to a weak interaction between the spots. The density of chaotic spots was calculated in [30] as $D = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y^2} dy$ where $y = \frac{10^5}{\pi}$ for $E_J = 1$ and $\beta$ is the inverse temperature [38]. Note that $1/\beta \approx 2h$ for $h \gg 1$. It follows that $D \sim 1/\sqrt{h}$ for $h \gg 1$. This decay is way too slow to explain the rapid increase of the ergodization time $T_E$ upon increasing $h$ in Fig. 3(a). There $h$ increases by one order of magnitude, $D$ decreases by a factor of 3, but $T_E$ increases by six orders of magnitude. Therefore the ergodization time in the dynamical glass phase must be controlled by a very weak interaction between chaotic spots, which have to penetrate silent non-chaotic regions formed by breather like events.

To conclude, the classical dynamics of a Josephson junction chain at large temperatures (i.e. energy densities) or likewise at weak Josephson coupling is characterized by a dynamical glass phase in its proximity to corresponding integrable limits. This phase is induced by the short range of the nonintegrable perturbation network spanned between the actions which turn integrals of motion at the very integrable limit. The dynamics of the system remains ergodic, albeit with rapidly increasing ergodization time $T_E$. We relate $T_E$ to time scales extracted from the fluctuations of the actions. We also show, that the Lyapunov time, which is marking the onset of chaos in the system, is orders of magnitude shorter than $T_E$. The reason for the rapidly growing ergodization time is rooted in the slowing down of interactions between chaotic spots. By virtue of the short range network we expect our results to hold as well in higher space dimensions. A quantitative theory for the dependence of the ergodization time on the control parameters in the proximity to the discussed integrable limits is a challenging future task, and as intriguing as the question about the fate of the dynamical glass phase in the related quantum many body problem.

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**SUPPLEMENTAL MATERIAL**

I. STATISTICAL ANALYSIS

The energy density $h$ is calculated with the microcanonical partition function

$$Z = \int_{-\infty}^{\infty} \prod_{n} dp_n dq_n e^{-\beta H}$$

as

$$h = -\frac{1}{N} \frac{\partial \ln(Z)}{\partial \beta} = \frac{1}{2\beta} + E_f (1 - \frac{I_1(\beta E_f)}{I_0(\beta E_f)})$$

with average potential energy density

$$u = E_f (1 - \frac{I_1(\beta E_f)}{I_0(\beta E_f)})$$

and average kinetic energy density

$$k = \frac{1}{2\beta}.$$  \hspace{1cm} (8)

In terms of $k$ we rewrite Eq. 6 as

$$h = k + E_f \left(1 - \frac{I_1(E_f/2k)}{I_0(E_f/2k)}\right).$$  \hspace{1cm} (9)

II. INTEGRATION

We split Eq. 1 in the main text as

$$A = \sum_{n=1}^{N} \frac{p_n^2}{2}, \quad B = E_f \sum_{n=1}^{N} (1 - \cos(q_{n+1} - q_n)).$$ \hspace{1cm} (10)

As discussed in [40], this separation leads to a symplectic integration scheme called SBAB$_2$, where

$$e^{\Delta t H} = e^{\Delta t (A+B)} \approx e^{\Delta t L_B} e^{-\Delta t L_A} e^{\Delta t L_B}$$

where $d_1 = \frac{1}{6}$, $d_2 = \frac{2}{3}$, $c_2 = \frac{1}{2}$. The operators $e^{\Delta t L_A}$ and $e^{\Delta t L_B}$ which propagate the set of initial conditions $(q_n, p_n)$ from Eq. (10) at the time $t$ to the final values $(q_n, p_n)$ at the time $t + \Delta t$ are

$$e^{\Delta t L_A} : \begin{cases} q_n' = q_n + p_n \Delta t \\ p_n = p_n \end{cases} \quad e^{\Delta t L_B} : \begin{cases} q_n' = q_n \\ p_n = p_n + E_f [\sin(q_{n+1} - q_n) + \sin(q_{n-1} - q_n)] \Delta t \end{cases}$$

We then introduce a corrector $C = \{(A, B), B \}$. Following [40], this term applies

$$SBAB_2 C = e^{-\frac{\Delta t}{2} L_C} SBAB_2 e^{-\frac{\Delta t}{2} L_C}$$

for $g = 1/72$. The corrector term is

$$C = -\sum_{n=1}^{N} \frac{\partial \{A, B\}}{\partial p_n} \frac{\partial B}{\partial q_n} = \sum_{n=1}^{N} \left(\frac{\partial B}{\partial q_n}\right)^2$$

$$= E_f^2 \sum_{n=1}^{N} \left[\sin(q_{n+1} - q_n) + \sin(q_{n-1} - q_n)\right]^2.$$

The corrector operator $C$ yields to the following resolvent operator

$$e^{tL_C} : \begin{cases} q_n' = q_n \\ p_n = p_n + E_f^2 \left\{2 \left[\sin(q_{n+1} - q_n) + \sin(q_{n-1} - q_n)\right] \cdot \left[\cos(q_{n+1} - q_n) + \cos(q_{n-1} - q_n)\right] - 2 \left[\sin(q_{n+2} - q_{n+1}) + \sin(q_{n} - q_{n+1})\right] \cdot \cos(q_{n} - q_{n+1}) - 2 \left[\sin(q_{n} - q_{n-1}) + \sin(q_{n-2} - q_{n-1})\right] \cdot \cos(q_{n-2} - q_{n-1})\right\} \Delta t \end{cases}$$

the equations of motion are

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = -\frac{\partial V(q)}{\partial q}$$ \hspace{1cm} (16)

The corresponding variational Hamiltonian and equations of motion are

$$H_V(\delta q, \delta p) = \sum_{n=1}^{N} \left[\frac{1}{2} \delta p_n^2 + \frac{1}{2} \sum_{m=1}^{N} D_{nm}^2 (\delta q_n) \delta q_m \delta q_m \right],$$ \hspace{1cm} (17)

III. CALCULATION OF MAXIMAL LCE: TANGENT MAP METHOD AND VARIATIONAL EQUATIONS

If the autonomous Hamiltonian has the form [40]

$$H(q, p) = \sum_{n=1}^{N} \left[\frac{1}{2} p_n^2 + V(q)\right],$$  \hspace{1cm} (15)
and 
\[ \begin{bmatrix} \delta \dot{q} \\ \delta \dot{p} \end{bmatrix} = \begin{bmatrix} \delta \tilde{p} \\ -D^2_C(\tilde{q})\delta \tilde{q} \end{bmatrix}, \] 
(18)

respectively. Here,
\[ D^2_C(q^*(t))_{nm} = \frac{\partial^2 V(q)}{\partial q_n \partial q_m}|_{q^*(t)}. \]
(19)

For the Hamiltonian Eq. (1) the variational equations of motion are

following resolvent operator
\[ e^{\Delta t L_C} : \begin{cases} \delta \dot{q}' = \delta \tilde{q} \\ \delta \dot{p}' = \delta \tilde{p} - D^2_C(\tilde{q})\delta \tilde{q} \Delta t \end{cases} \]
(22)

Here \( D^2_C(\tilde{q}) = \frac{\partial^2 C}{\partial q_n \partial q_m} \) is the Hessian.

Following [40], the corrector operator \( C \) yields the fol-

From Eq. 22, we get
\[ e^{\Delta t L_C} : \begin{cases} \delta q'_n = \delta q_n \\ \delta p'_n = \delta p_n - E^2_f \left\{ 2 \cos(q_{n-2} - q_{n-1}) \cos(q_n - q_{n-1}) \right\} \delta q_{n-2} \\ &\quad + \left[ -2 \cos(q_{n-1} - 2q_n + q_{n+1}) - 4 \cos(2(q_n - q_{n-1})) - 2 \cos(q_{n-2} - 2q_{n-1} + q_n) \right] \delta q_{n-1} \\ &\quad + \left[ 4 \cos(2(q_{n+1} - q_n)) + 4 \cos(q_{n-1} - 2q_n + q_{n+1}) + 4 \cos(2(q_{n-1} - q_n)) - 2 \sin(q_{n-1} - q_n) \sin(q_n - q_{n-1}) \right] \delta q_n \\ &\quad + \left[ -4 \cos(2(q_{n+1} - q_n)) - 2 \cos(q_{n-1} - 2q_n + q_{n+1}) - 2 \cos(q_{n+2} - 2q_{n+1} + q_n) \right] \delta q_{n+1} \\ &\quad + \left[ 2 \cos(q_{n+2} - q_{n+1}) \cos(q_n - q_{n+1}) \right] \delta q_{n+2} \end{cases} \Delta t \]
(23)

IV. NUMERICAL SIMULATION

We simulate Eqs. (2) with periodic boundary conditions \( p_1 = p_{N+1} \) and \( q_1 = q_{N+1} \) and time step \( \Delta t = 0.1 \). In the simulation, the relative energy error \( \Delta E = |E(t) - E(0)| \) is kept lower than \( 10^{-4} \). The initial conditions follow by fixing the positions to zero \( q_n = 0 \) and by choosing the moments \( p_n \) according Maxwell’s distribution. The total angular momentum \( L = \sum_{n=1}^{N} p_n \) is set zero by a proper shift of all momenta \( p_n - L/N \). Finally, we rescale \( |p_n| \rightarrow \alpha |p_n| \) to precisely hit the desired energy (density).

![Figure 5](Figure 5. (Color online) Fluctuation index \( q \) for fixed the energy density \( h = 1 \) with \( R = 12 \). From top to bottom: \( E_f = 0.1 \) (green), \( E_f = 0.5 \) (red), \( E_f = 1.0 \) (blue), \( E_f = 2.0 \) (magenta) and \( E_f = 3.0 \) (cyan).)
Figure 6. (Color online) a) \(q(T/T_E)\) for fixed \(E_J = 1\) with energy densities 0.1 (black) 1.2 (red), 2.4 (green), 3.8 (blue), 5.4 (magenta), and 8.5 (cyan) (corresponding to Fig. 1 in the main body); b) \(q(T/T_E)\) for fixed energy density \(h = 1\) with \(E_J = 0.5\), (red), \(E_J = 1.0\), (blue), \(E_J = 2.0\), (magenta) and \(E_J = 3.0\), (cyan) (corresponding to Fig. 5).

**V. FINITE TIME AVERAGE FOR \(h = 1\)**

Fig. 5 shows the index \(q(T)\) for fixed energy \(h = 1\) with varying coupling strengths, \(E_J\). It is similar to Fig.1 from the main text for fixed \(E_J\) and varying \(h\).

**VI. EVALUATION OF THE ERGODIZATION TIME**

We rescale and fit the fluctuation index \(q(T)\) shown in Figs. 1 (main body) and 5 (in the supplement). We choose a parameter set with a clearly observed asymptotic \(q(T) \approx T_E/T\) dependence, and fit this dependence to obtain \(T_E\). We then rescale the variable \(T \to xT\) for all other lines to obtain the best overlap with the initially chosen line as shown in Fig. 6. The scaling parameters \(x\) are then used to compute the corresponding ergodization times.

**VII. LYAPUNOV EXPONENT COMPUTATION**

Figure 7. (Color online) Lyapunov time for fixed \(E_J = 1\). The black circles represent our numerical results, and the red squares represent the analytical results from [35]. The black line guides the eye.

We compute the largest Lyapunov exponent \(\Lambda\) by numerically solving the variational equations

\[
\dot{w}(t) = \left[J_{2N} \cdot D_{\tilde{H}}(x(t))\right] \cdot w(t)
\]

for a small amplitude deviation \(w(t) = (\gamma q(t), \gamma p(t))\) coordinates. The largest Lyapunov exponent \(\Lambda\) follows

\[
\Lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{||w(t)||}{||w(0)||}.
\]

In Fig. 7 we show the Lyapunov time \(T_\Lambda = 1/\Lambda\) versus energy densities \(h\) for given \(E_J = 1\). It matches well with the analytical results obtained in [35].