Supersymmetric extension of the nine-dimensional continuation of the Euler density with $\mathcal{N} = 2$.

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Abstract: A local supersymmetric extension with $\mathcal{N} = 2$ of the dimensional continuation of the Euler-Gauss-Bonnet density from eight to nine dimensions is constructed. The gravitational sector is invariant under local Poincaré translations, and the full field content is given by the vielbein, the spin connection, a complex gravitino, and an Abelian one-form. The local symmetry group is shown to be super Poincaré with $\mathcal{N} = 2$ and a $U(1)$ central extension, and the full supersymmetric Lagrangian can be written as a Chern-Simons form.
1. Introduction

The existence of new Lagrangians for supergravity in dimensions lower than eleven, which cannot be obtained by dimensional reduction from the standard Cremmer-Julia-Scherk theory [1], suggests that M-Theory may have new “cusps” from which these new theories would be obtained (see e.g. [2]). In this sense, exploring new dynamic and geometric structures with local supersymmetry in dimensions $d < 11$ deserves some attention. In this spirit, we construct a local $\mathcal{N} = 2$ supersymmetric extension of the Poincaré invariant gravity in nine dimensions, possessing a different structure than the supergravity theory obtained by dimensional reduction from standard eleven-dimensional supergravity (see e.g. [3]). The gravitational sector of standard $\mathcal{N} = 2$ supergravity in nine dimensions is described by the Einstein-Hilbert action, and its full field content is given by $(e^a_\mu, 2B_{\mu\nu}, A_{\mu\nu\rho}; 2\psi_\mu, 4\chi)$ where the scalar fields parametrize the coset $GL(2, R)/SO(2)$. In our case, the Lagrangian for the gravitational sector is given by the dimensional continuation of the Euler-Gauss-Bonnet density form eight to nine dimensions, which still gives second order field equations for the metric. The field content of this theory is the set $(e^a_\mu, \omega^{ab}_\mu, \psi_\mu, c_\mu)$, where $\psi_\mu$ is a complex gravitino, and $c_\mu$ is an Abelian one-form. Here $e^a_\mu$ is the vielbein, and the spin connection $\omega^{ab}_\mu$ is now regarded as an independent dynamical variable. The local supersymmetry algebra closes off-shell, and is given by the super Poincaré group with a $U(1)$ central extension. The field content of the theory, then can be regarded as the different components of a single connection for this group, and hence, the local symmetries can be read off from a gauge transformation. This fact, together with the existence of a fifth-rank invariant tensor for the supergroup, allows to write the Lagrangian as a Chern density. Note that for Chern-Simons theories the number of bosonic and fermionic degrees of freedom do not necessarily match, since there exists an alternative to the introduction of auxiliary fields (see e.g. [4]). Indeed, the matching may not occur when the dynamical fields are assumed to belong to a connection instead of a multiplet for the supergroup [5]. For a vanishing cosmological constant, supergravity theories sharing these last features have been constructed in three [6], [4] and higher odd dimensions [7, 8]. There exist also supergravity models with $\Lambda < 0$ in three [9], five [10] and higher odd dimensions [11], [2].

In the next section the gravitational sector of this theory is discussed, and its supersymmetric extension is given in Section III. Section IV is devoted to the discussion.

2. The gravitational sector

As it is well-known in three dimensions, in the absence of cosmological constant, it is possible to ensure the off-shell closure of the superalgebra without using auxiliary fields, by demanding invariance under local translations in tangent space. This is due to the fact that the three-dimensional Einstein-Hilbert action is invariant under the following transformations

$$\delta e^a = D\lambda^a = d\lambda^a + \omega^a_b \lambda^b; \quad \delta \omega^{ab} = 0,$$  \hspace{1cm} (2.1)
without imposing the vanishing torsion condition. In higher dimensions, these transformations are no longer a symmetry of the Einstein-Hilbert action. However, this symmetry is present in odd-dimensional spacetimes if one considers an action linear in the vielbein instead of the curvature. In particular, in nine dimensions the action reads

\[ I_G = \int \epsilon_{a_1 \cdots a_9} R^{a_1 a_2} \cdots R^{a_7 a_8} e^{a_9}. \tag{2.2} \]

Here, the Lagrangian is the dimensional continuation of the Euler-Gauss-Bonnet density from eight to nine dimensions. This action is singled out as the most general gravity theory, constructed out of the vielbein and the curvature, still leading to second order field equations for the metric, that possesses local invariance under the Poincaré group \([13]\). Note that the transformations (2.1) can be read off from a gauge transformation with parameter \( \lambda = \lambda^a P_a \), assuming that the vielbein and the spin connection belong to a connection of the Poincaré group, i.e., \( A = 1/2 \omega^{ab} J_{ab} + e^a P_a \).

Invariance under local translations (2.1) presents an advantage when one deals with the locally supersymmetric extension in nine dimensions, because the closure of the superalgebra can be attained without the need of auxiliary fields as it is discussed in the next section.

3. Local supersymmetric extension

The local supersymmetric extension of the gravitational action \( I_G \) in Eq. (2.2) is given by the sum of three pieces

\[ I_9 = I_G + I_\psi + I_c, \tag{3.1} \]

where the fermionic term \( I_\psi \) reads

\[ I_\psi = \int \frac{i}{3} R_{abc} \bar{\psi} \Gamma^{abc} D \psi + 12 \left( R_{ab} R^{ab} R_{cd} + 4 (R^3)_{cd} \right) \bar{\psi} \Gamma^{cd} D \psi + \text{h.c.} \]

and the bosonic term needed to close supersymmetry \( I_c \) is

\[ I_c = -12 \int \left[ (R^{ab} R^{ab})^2 - 4 R^d_{\ b} R^b_{\ c} R^d_{\ a} \right] c. \tag{3.2} \]

Here \( D \psi = d \psi + \frac{1}{4} \omega^{ab} \Gamma_{ab} \psi \) is the Lorentz covariant derivative, and as a shorthand we have defined \( (R^3)_b^a = R^a_b R^b_c R^c_d \), and \( R_{abc} = \epsilon_{abc a_1 \cdots a_9} R^{a_1 a_2} R^{a_3 a_4} R^{a_5 a_6} \).

The action (3.1) is invariant under the following local supersymmetry transformations

\[ \delta e^a = -i \left( \bar{\epsilon} \Gamma^a \psi - \bar{\psi} \Gamma^a \epsilon \right); \delta \psi = D \epsilon \]

\[ \delta c = (\bar{\epsilon} \psi - \bar{\psi} \epsilon); \delta \bar{\psi} = D \bar{\epsilon} \tag{3.3} \]

together with \( \delta \omega^{ab} = 0 \).

\[ ^1 \text{Here } e^a = e^a_\mu dx^\mu \text{ is the vielbein, and } R^{ab} = d \omega^{ab} + \omega^c_a \omega^{cb} \text{ stands for the curvature two-form. Wedge product between forms is assumed throughout.} \]
Note that, apart from diffeomorphisms, local Lorentz rotations, and local Poincaré transformations (2.1), this action possesses an extra local $U(1)$ symmetry whose only nonvanishing transformation reads

$$\delta c = du. \quad (3.4)$$

The full set of symmetries closes off-shell without requiring the introduction of auxiliary fields. Indeed, the commutator of two supersymmetry transformations acting on the fields is given by

$$[\delta_\epsilon, \delta_\eta] \left( \begin{array}{c} e^a \\ \omega^{ab} \\ c \\ \psi \end{array} \right) = \left( \begin{array}{c} D\lambda^a \\ 0 \\ du \\ 0 \end{array} \right),$$

where the parameter of the local translations is $\lambda^a = -i(\bar{\epsilon}\Gamma^a\eta - \bar{\eta}\Gamma^a\epsilon)$ while the parameter of the $U(1)$ symmetry is $u = \bar{\epsilon}\eta - \bar{\eta}\epsilon$. This means that the local symmetry group is spanned by the $N = 2$ super Poincaré algebra with a $U(1)$ central extension whose generators are given by the set $G_A = \{J_{ab}, P_a, Z, Q^\alpha, \bar{Q}_\alpha\}$. Here $J_{ab}$ and $P_a$ generate the Poincaré group, $Z$ is the $U(1)$ central charge, and $Q^\alpha, \bar{Q}_\alpha$ are complex fermionic generators whose anticommutator reads

$$\{Q^\alpha, \bar{Q}\beta\} = -i(\Gamma^a)^\alpha_\beta P_a + \delta^\alpha_\beta Z. \quad (3.5)$$

Let us point out that, assuming the dynamical fields to belong to a single connection for the supergroup, i.e.,

$$A = \frac{1}{2} \omega^{ab} J_{ab} + e^a P_a + cZ + \bar{\psi}Q - Q\psi, \quad (3.6)$$

all the local symmetries can be viewed as a gauge transformation $\delta_\lambda A = d\lambda + [A, \lambda]$, where $\lambda$ is a Lie algebra-valued parameter. Indeed, the supersymmetry transformations (3.3) are obtained from a gauge transformation with parameter $\lambda = \bar{\epsilon}Q - \bar{Q}\epsilon$. Analogously, the local Lorentz transformations, local translations (2.1), and the local $U(1)$ symmetry (3.4) are obtained for $\lambda = \frac{1}{2}\lambda^a J_{ab}$, $\lambda = \lambda^a P_a$ and $\lambda = uZ$, respectively.

It is worth mentioning that the Lagrangian (3.1) can be written as a Chern density, since it satisfies $dL_9 = \langle F^5 \rangle$, where $F = dA + A^2$ is the curvature two-form which in components reads

$$F = \frac{1}{2} R^{ab} J_{ab} + (T^a + i\bar{\psi}\Gamma^a\psi) P_a + (dc - \bar{\psi}\psi)Z + D\bar{\psi}Q - \bar{Q}D\psi. \quad (3.7)$$

Here the bracket $\langle \cdots \rangle$ stands for an invariant multilinear form for the super Poincaré group with $\mathcal{N} = 2$ with a $U(1)$ central extension, whose only nonvanishing components are given by

$$\langle J_{ab}, J_{cd}, J_{ef}, J_{gh}, P_i \rangle = \frac{16}{5}\epsilon_{abcdefghi},$$

$$\langle J_{ab}, J_{cd}, J_{ef}, J_{gh}, Z \rangle = -\frac{48}{5}\left[\delta^e_{abcd} - \delta^e_{ab} \delta^d_{cd}\right]. \quad (3.8)$$
\[
\langle Q^\alpha, J_{ab}, J_{cd}, J_{ef}, \tilde{Q}_\beta \rangle = -\frac{1}{5} (\Gamma_{abcdef})^\alpha_\beta - \frac{3}{10} \delta_{cde}^g (\Gamma_{gh})^\alpha_\beta + \frac{3}{10} \delta_{cde}^h (\Gamma_{gh})^\alpha_\beta ,
\]

where (anti)symmetrization under permutation of each pair of generators is understood when all the indices are lowered. As a consequence, the action constructed here can be seen as a gauge theory with fiber bundle structure. The field equations can be written in a manifestly covariant form as

\[
\langle F^4 G_A \rangle = 0 ,
\]

where \(G_A\) are the generators of the gauge group. As a direct result, the consistency of the fermionic equations reproduces the bosonic field equations associated to the generators appearing in the r.h.s. of the anticommutator in Eq.(3.5) without imposing additional constraints. In this case, these equations correspond to the variation with respect to the fields \(e^a\) and \(c\).

In the Appendix B, it is shown that, if one assumes the gauge group to be super-Poincaré admitting at most a \(U(1)\) central charge, theories featuring the properties considered here exist only in 3, 5 and 9 dimensions.

4. Discussion

Here, it has been shown that the coupling of a spin \(3/2\) particle to the graviton can be achieved consistently with local supersymmetry, when the gravitational sector is described by a Lagrangian that is linear in the vielbein rather than in the curvature, yielding second order field equations for the metric. This corresponds to a new \(\mathcal{N} = 2\) local supersymmetric extension of Poincaré invariant gravity in nine dimensions which can be formulated as a gauge theory with a fiber bundle structure as it occurs for Yang-Mills, and therefore it features some of the formal advantages of the three dimensional gravity \(\mathcal{N} = 2\).

The field equations of this theory admit a class of vacuum solutions of the form \(S^{8-d} \times X_{d+1}\) where \(X_{d+1}\) is a warped product of \(\mathbb{R}\) with a \(d\)-dimensional spacetime. For this class of geometries, a nontrivial propagator for the graviton exists only for \(d = 4\) and for a positive constant cosmological \(\Lambda\).

In nine dimensions, there exists another \(\mathcal{N} = 2\) theory having a similar structure. The corresponding gauge group is a supersymmetric extension of Poincaré containing a fifth-rank antisymmetric generator, so that the anticommutator of the fermionic generators reads

\[
\{ Q^\alpha, \tilde{Q}_\beta \} = -i (\Gamma^a)^\alpha_\beta P_a - i (\Gamma^{abcde})^\alpha_\beta Z_{abcde} . \quad (4.1)
\]

Apart from the vielbein, the spin connection and a complex gravitino, a bosonic one-form \(b_{\mu}^{abcde}\) which transforms as an antisymmetric fifth-rank tensor under local Lorentz rotations was considered. In the model described here, the role of this field is played by the Abelian

\footnote{This kind of fields has recently become relevant in the context of dual descriptions of linearized gravity.\cite{16,17,18}. In the last reference a generalization of the Poincaré lemma was also provided.}
one-form $c_\mu$. It would be desirable to explore whether a link between both theories can be established.

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5. Appendix A: Some useful formulas

Here $\Gamma^a$ stands for the Dirac matrices satisfying $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} I$, where $a, b = 1, 2, \cdots, d$, where $d$ is the spacetime dimension and $\eta^{ab} = \text{diag}(-, +, \cdots, +)$. For the Minkowskian signature, these matrices can be chosen such that $(\Gamma^a)^\dagger = \Gamma_0 \Gamma^a \Gamma_0$, and the Dirac conjugate is given by $\bar{\psi} = \psi^\dagger \Gamma_0$. The totally antisymmetric product of Gamma matrices is defined as

$$\Gamma^{a_1 \cdots a_p} = \frac{1}{p!} \sum_\sigma \text{sgn}(\sigma) \Gamma^{a_\sigma(1)} \cdots \Gamma^{a_\sigma(p)}. \quad (5.1)$$

For $d = 2n + 1$ dimensions it is always possible to find a representation of the Gamma matrices such that

$$\Gamma^{1 \cdots d} = (-i)^{n+1} I,$$

and hence, one obtains the following relation

$$\Gamma^{a_1 \cdots a_{d-p}} = (-1)^{\frac{p(p-1)}{2}} \frac{(-i)^{n+1}}{p!} \epsilon^{a_1 \cdots a_{d-p} \ b_1 \cdots b_p} \Gamma_{b_1 \cdots b_p}. \quad (5.2)$$

All fermionic terms in the actions considered here take the form

$$I^p_\psi = \int X_{a_1 \cdots a_p} \left[ \bar{\psi} \Gamma^{a_1 \cdots a_p} D\psi + \text{h.c.} \right],$$

where $X_{a_1 \cdots a_p}$ is a covariantly constant $(d-3)$-form satisfying $DX_{a_1 \cdots a_p} = 0$, which does not transform under supersymmetry, i.e., $\delta X_{a_1 \cdots a_p} = 0$. The variation of $I^p_\psi$ under the supersymmetry transformations (3.3) is given by

$$\delta I^p_\psi = -\frac{1}{4} \int X_{a_1 \cdots a_p} R_{ab} \left[ \bar{\epsilon} \left\{ \Gamma^{a_1 \cdots a_p}, \Gamma^{ab} \right\} \psi - \text{h.c.} \right] = -\frac{1}{2} \int X_{a_1 \cdots a_p} R_{ab} \left[ \bar{\epsilon} \Gamma^{a_1 \cdots a_p, ab} \psi - \text{h.c.} \right] \quad (5.3)$$

$$+ \frac{p(p-1)}{2} \int X_{a_1 \cdots a_{p-2}ab} R^{ab} \left[ \bar{\epsilon} \Gamma^{a_1 \cdots a_{p-2}} \psi - \text{h.c.} \right],$$

up to a boundary term. In our case, $X_{a_1 \cdots a_p}$ always involves the contraction of $(n-1)$ curvatures leaving $p$ free indices.
6. Appendix B: Three, five and nine dimensions

The only action for gravity, constructed out of the vielbein and the curvature, leading to second order field equations for the metric, and possessing local invariance under the Poincaré group exists only for \( d = 2n + 1 \) dimensions, and is given by

\[
I_G = \int \epsilon_{a_1 \ldots a_{2n+1}} R^{a_1 a_2} \ldots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}} = \int R_{abc} R^{ab} e^c ,
\]

where \( R_{abc} \) is a shorthand for \( R_{abc} = \epsilon_{abca_1 \ldots a_{2n} - 2} R^{a_1 a_2} \ldots R^{a_{2n-3} a_{2n-2}} . \)

We explore now whether this theory accepts a supersymmetric extension, for which the dynamical fields belong to a connection of the standard super-Poincaré group admitting at most a \( U(1) \) central charge. This means that the field content, apart from the vielbein, the spin connection and a complex gravitino, should eventually be supplemented by a one-form \( c_\mu \). In addition, the supersymmetry transformations remain the same as in Eq.(3.3), regardless the space-time dimension.

The variation of (6.1) under supersymmetry is

\[
\delta I_G = -i \int R_{abc} R^{ab} (\bar{\psi} \Gamma^c \psi - \text{h.c.}) ,
\]

which must be cancelled by the variation of a fermionic term, that is quadratic in the gravitini without involving the vielbein or the \( c \)-field. Lorentz covariance singles out this fermionic term to be

\[
\frac{i}{3} \int R_{abc} \left( \bar{\psi} \Gamma^{abc} D \psi + \text{h.c.} \right) .
\]

Its variation, in turn, produces an additional contribution of the form

\[
-\frac{i}{6} \int R_{abc} R_{de} \left( \bar{\psi} \Gamma^{abcde} \psi - \text{h.c.} \right) .
\]

In three dimensions, this term identically vanishes, and hence, the supergravity action does not require a central extension\(^4\). In five dimensions, since \( \Gamma^{abcde} \) is proportional to the Levi-Civita tensor, the extra bosonic term \( 2 \int R_{ab} R^{ab} \), must be necessarily added to cancel (6.4).

In this form, the minimal five dimensional supergravity is obtained for the \( U(1) \) centrally extended super-Poincaré group \( \mathbb{F} \).

In order to see whether the term (6.4) can be canceled in higher odd dimensions, it is useful to express it as a linear combination of

\[
\int (R^3)_{a_1 a_2} R_{a_3 a_4} \ldots R_{a_{4-d+5} a_{d-5}} (\bar{\psi} \Gamma^{a_1 \ldots a_{d-5}} \psi - \text{h.c.}) ,
\]

\(^3\)Lorentz-Chern-Simons forms, which are trivially invariant under supersymmetry, can also be considered. However, as it occurs in three dimensions \( \mathbb{F} \), this would yield third order field equations for the metric in the vanishing torsion sector.

\(^4\)Had we dealt with Majorana spinors, the same argument would have held, and the theory for \( \mathcal{N} = 1 \), discussed in Ref. \( \mathbb{F} \), is recovered.
and
\[ \int R^2 R_{a_1 a_2} R_{a_3 a_4} \cdots R_{a_{d-6} a_{d-5}} (\bar{\psi} \Gamma^{a_1 \cdots a_{d-5}} \psi - \text{h.c.)} \ . \quad (6.6) \]

In seven dimensions, even though the term (6.6) can be compensated by
\[ \int R^2 \left( \bar{\psi} D\psi + \text{h.c.)} \ , \quad (6.7) \]
the remaining one (6.5) can never be canceled. Indeed, this term reduces to
\[ \int (R^2)_{a b} \left( \bar{\psi} \Gamma^{a b} \psi - \text{h.c.)} \ , \quad (6.8) \]
and can not be eliminated by the variation of a bosonic piece. On the other hand, by virtue of formula (5.3), the only fermionic term, different from (6.3), whose variation is cubic in the curvature times the combination \( \bar{\psi} \Gamma^{a b} \psi \) is given by (6.7). However, its variation can never cancel the term in (6.8), and hence, under our assumptions, the seven dimensional case is ruled out.

Following the same procedure in higher dimensions, invariance under local supersymmetry would demand the introduction of a growing series of fermionic terms in the Lagrangian of the form
\[ I_p^\psi = \int X[\rho] \left[ \bar{\psi} \Gamma[\rho] D\psi + \text{h.c.} \right] , \quad (6.9) \]
where \( \Gamma[\rho] = \Gamma^{a_1 \cdots a_\rho} \), and \( X[\rho] \) is a \((d-3)\)-form constructed exclusively from curvatures \( R_{a b} \). Exhausting all the relevant combinations of the fermionic terms in (6.9) and also the bosonic ones containing the Abelian field \( c \), it is shown that supergravity theories with local invariance under super-Poincaré with a \( U(1) \) central charge do not exist for higher odd dimensions different from nine.

For dimensions \( d = 4k + 1 > 5 \), the variation (6.5) can only be canceled by
\[ \int \left( R^3 \right)_{a_1 a_2} R_{a_3 a_4} \cdots R_{a_{k-1} a_k} \left( \bar{\psi} \Gamma^{a_1 \cdots a_k} \psi + \text{h.c.)} \right) , \quad (6.10) \]
and following the Noether procedure, after a number of steps, a term of the form
\[ \int \left( R^3 \right)_{a b} \left( R^2 \right)_{c d} \left( \bar{\psi} \Gamma^{a b c d} \psi - \text{h.c.)} \right) , \quad (6.11) \]
necessarily appears in the variation\(^5\). For \( k > 2 \) this last expression can not be canceled by a bosonic term, while using formula (5.3), it might be compensated resorting to terms proportional to \( I_6^\psi \) and \( I_2^\psi \). However, none of the possibilities are satisfactory. In fact, in the first case, the explicit construction of the terms along \( \Gamma^{[6]} \) leads us back exactly to the expression that generates the term (6.11). For the second case, formula (5.3) implies that the variation of any of the possible terms contains at least one curvature that is not contracted

\(^5\)Here \( (R^m)^a_b = R^a_{c_1} R^c_{c_2} \cdots R^m_{c_{m-1}} \).
with another curvature, and therefore, the term (6.11) can never be canceled. Consequently, for \( k > 2 \), local supersymmetry is never attained.

The nine dimensional case \( (k = 2) \) is exceptional because the leftover term (6.11) involves a curvature that is not contracted with another one. Unlike the previous cases, this fact allows to cancel (6.11) by means of the additional term

\[
\int (R^3)_{ab} \left( \bar{\psi} \Gamma^{ab} D\psi + \text{h.c.} \right),
\]

that produces also a contribution which is compensated by a bosonic term containing the \( c \)-field given by

\[
\int (R^3)_{ab} R^{ab} c.
\]

Analogously, the remaining term in Eq.(6.6) is canceled by the variation of a fermionic term and a bosonic one depending on the Abelian field. In this case, the full supergravity action in Eq.(3.1) is recovered.

For the remaining dimensions, \( d = 4k - 1 > 7 \), in order to cancel the variation in Eq.(6.5), after a number of steps, it is inevitable to add the following term to the action

\[
\int (R^3)_{ab} (R^{2k-5})_{cd} \left( \bar{\psi} \Gamma^{abcd} D\psi + \text{h.c.} \right),
\]

whose variation contains the term

\[
\int \left( R^{2k-1} \right)_{ab} \left( \bar{\epsilon} \Gamma^{ab} \psi - \text{h.c.} \right).
\]

Here the argument to rule out these dimensions is similar to the one for \( d = 7 \). Indeed, this last term can not be canceled by a bosonic one, and by virtue of formula (5.3), the only fermionic term different from (6.12), whose variation leads to a \((2k - 1)\)-th power of the curvature times the combination \( \bar{\epsilon} \Gamma^{ab} \psi \) must be of the form given by \( I_0^0 \psi \) in Eq.(5.9). However, as it can be seen from formula (5.3), their variation always involves the combination \( R^{ab} \Gamma^{ab} \), and thus the term (6.13) can never be canceled.

In summary, in this Appendix we have shown that the supersymmetric extension of the action (6.4), that possesses local invariance under the super-Poincaré group admitting at most a \( U(1) \) central charge, exists only in 3, 5, and 9 dimensions.

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$$C^F_{BA_1}\Delta_{FA_2\cdots A_5} + \cdots + C^F_{BA_5}\Delta_{A_1\cdots A_4 F} = 0,$$

where $C^A_{BC}$ are the structure constants. In nine dimensions, the existence of this invariant tensor is guaranteed as long as the central charge $Z$ is present, and it can be written in terms of a multilinear form of the generators as $\Delta_{ABCDE} = \langle G_A, G_B, G_C, G_D, G_E \rangle$.

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