Dark energy scenario consistent with GW170817 in theories beyond Horndeski gravity

Ryotaro Kase and Shinji Tsujikawa
Department of Physics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku, Tokyo 162-8601, Japan
(Dated: May 3, 2018)

The Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories up to quartic order are the general scheme of scalar-tensor theories allowing the possibility for realizing the tensor propagation speed $c_t$ equivalent to 1 on the isotropic cosmological background. We propose a dark energy model in which the late-time cosmic acceleration occurs by a simple k-essence Lagrangian analogous to the ghost condensate with cubic and quartic Galileons in the framework of GLPV theories. We show that a wide variety of the variation of the dark energy equation of state $w_{DE}$ including the entry to the region $w_{DE} < -1$ can be realized without violating conditions for the absence of ghosts and Laplacian instabilities. The approach to the tracker equation of state $w_{DE} = -2$ during the matter era, which is disfavored by observational data, can be avoided by the existence of a quadratic k-essence Lagrangian $X^2$. We study the evolution of nonrelativistic matter perturbations for the model $c_t^2 = 1$ and show that the two quantities $\mu$ and $\Sigma$, which are related to the Newtonian and weak lensing gravitational potentials respectively, are practically equivalent to each other, such that $\mu \simeq \Sigma > 1$. For the case in which the deviation of $w_{DE}$ from $-1$ is significant at a later cosmological epoch, the values of $\mu$ and $\Sigma$ tend to be larger at low redshifts. We also find that our dark energy model can be consistent with the bounds on the deviation parameter $\alpha_H$ from Horndeski theories arising from the modification of gravitational law inside massive objects.

I. INTRODUCTION

Two decades have passed after the first observational discovery of cosmic acceleration by the supernovae type Ia (SN Ia) [1, 2]. With the SN Ia data alone, the equation of state of dark energy $w_{DE}$ has not been strongly constrained. However, the joint data analysis combined with the observations of Cosmic Microwave Background (CMB) [3, 4] and Baryon Acoustic Oscillations (BAO) [3] placed tighter bounds on $w_{DE}$. The cosmological constant has been overall consistent with the data, but the deviation of $w_{DE}$ from $-1$ is also allowed [6].

From the theoretical side, there have been many attempts for constructing models of late-time cosmic acceleration in General Relativity (GR) or modified gravitational theories [7–12]. In GR, the representative dark energy models are quintessence [13–18] and k-essence [19–21], in which the potential energy and the kinetic energy of a scalar field $\phi$ drives the acceleration, respectively. Provided that the ghost instability is absent, the dark energy equation of state of quintessence and k-essence is in the range $w_{DE} > -1$. In current observations, there has been no statistically strong observational evidence that the models with $w_{DE} > -1$ are favored over the cosmological constant [22].

In modified gravitational theories, it is possible to realize $w_{DE}$ less than $-1$, while satisfying conditions for the absence of ghosts and Laplacian instabilities. The models with $w_{DE} < -1$ can reduce tensions of the Hubble constant $H_0$ between the CMB and the direct measurements of $H_0$ at low redshifts, so the best-fit model can be in the range $w_{DE} < -1$ depending on the data analysis [3, 23]. In this sense, it is worthwhile to construct theoretically consistent dark energy models in modified gravitational theories and confront them with observations.

In the presence of a scalar field $\phi$ coupled to gravity, Horndeski theories [24] are the most general scalar-tensor theories with second-order equations of motion [25–27]. There have been many attempts for constructing models of late-time cosmic acceleration in the framework of Horndeski theories, e.g., those in $f(R)$ gravity [28–31], Brans-Dicke theories [32–33], and Galileons [34–39]. At the background level, the dark energy equation of state in these models enters the region $w_{DE} < -1$, so they can be distinguished from the $\Lambda$CDM model. Different modified gravity models also lead to different cosmic growth histories, so the observations of large-scale structure and weak lensing allow one to distinguish between the models. For instance, the covariant Galileon is in strong tension with the data [40–47] due to a very different structure formation pattern compared to the $\Lambda$CDM model [18] and the large deviation of $w_{DE}$ from $-1$ for tracker solutions [38, 39].

One can perform a healthy extension of Horndeski theories in such a way that the number of propagating degrees of freedom is not increased [49–53]. In GLPV theories [50], for example, there are two additional Lagrangians beyond the domain of Horndeski theories. These beyond-Horndeski Lagrangians give rise to several distinguished features such as the mixing between the scalar and matter sound speeds [50–54], the breaking of the Vainshtein mechanism inside an astrophysical object [55], and the appearance of a solid angle deficit singularity at the center of a compact body [60, 61]. Even with these restrictions, it is possible to construct viable dark energy models in GLPV theories without theoretical inconsistencies [62].

The recent gravitational-wave (GW) event GW170817 [63] from a neutron star merger together with the gamma-ray
burst GRB 170817A \cite{light} significantly constrained the deviation of the propagation speed \( c_t \) of GWs. In the natural unit where the speed of light is 1, the constraint on \( c_t \) from the measurements of LIGO and Virgo is given by \cite{light}:

\[-3 \times 10^{-15} \leq c_t - 1 \leq 7 \times 10^{-16}.\]  

(1.1)

Since the GWs have propagated over the cosmological distance from the redshift \( z \approx 0.009 \) to us, the bound (1.1) can be applied to dark energy models in which \( c_t \) is modified from the GR value (\( c_t = 1 \)) for \( z \lesssim 10^{-2} \).

The modification from \( c_t = 1 \) occurs in quartic- and quintic-order Horndeski theories containing the dependence of the field kinetic energy \( X = \nabla^\mu \phi \nabla_\mu \phi \) in the couplings \( G_4 \) and \( G_5 \) \cite{light,light}. If we impose that \( c_t^2 = 1 \) is strictly equivalent to \( 1 \), the allowed Horndeski Lagrangians are up to the cubic interaction \( G_4(\phi, X) \nabla^\phi \nabla^\phi \nabla^\phi \nabla^\phi \) with the nonminimal coupling \( G_4(\phi) R \), where \( R \) is the Ricci scalar \cite{light,light} (see also Refs. \cite{light,light}). The \( f(R) \) gravity and Brans-Dicke theories belong to this class, but the quartic- and quintic-order covariant Galileons lead to the deviation of \( c_t^2 \) from 1.

In GLPV theories, the existence of an additional contribution to the quartic-order Horndeski Lagrangian gives rise to a self-tuning cosmological model in which \( c_t^2 \) is exactly equivalent to 1 \cite{light,light}. In such theories, it was recently shown that the speed of GWs also equals to 1 on the background of an exact Schwarzschild-de Sitter solution \cite{light}. This implies that the condition for imposing \( c_t^2 = 1 \) on the cosmological background can be sufficient to realize the same value in the vicinity of a strong gravitational source. In this case, the time delay does not occur when the GWs pass nearby massive objects. It is of interest to study whether the construction of viable dark energy models is possible in such self-tuning beyond-Horndeski theories.

In this paper, we propose a simple dark energy model with a self-accelerating de Sitter solution in the framework of GLPV theories. Since the quintic Lagrangian in GLPV theories leads to \( c_t^2 \) different from 1 \cite{light,light,light,light}, our analysis up to quartic-order Lagrangians is a most general scheme allowing the exact value \( c_t^2 = 1 \) in the domain of GLPV theories. We will not restrict the models from the beginning and derive the background and perturbation equations of motion in a general way by exploiting useful dimensionless parameters introduced in Ref. \cite{light}.

Our dark energy model is a simple extension of covariant Galileons up to quartic order with the \( a_2 X^2 \) term in the quadratic Lagrangian \( L_2 \). Since the linear Galileon term \( a_1 X \) is also present in \( L_2 \), the quadratic Lagrangian is similar to the ghost condensate model \cite{light}. Our model also contains the beyond-Horndeski coupling \( F_4 \), which is constant. The self-tuning cosmological model mentioned above corresponds to the specific choice of \( F_4 \). We are primarily interested in this self-tuning theory, but we leave the constant \( F_4 \) arbitrary to discuss also how the theories with \( c_t^2 \neq 1 \) can be constrained from the observational bound (1.1).

We would like to stress that the \( a_2 X^2 \) term in \( L_2 \) is fundamentally important to realize the background solution different from the tracker arising in covariant Galileons. The tracker solution of covariant Galileons has the equation of state \( w_{\text{DE}} = -2 \) during the matter era \cite{light,light}, which is ruled out from the joint data analysis of SN Ia, CMB, and BAO \cite{light}. In our model, the solutions can approach a self-accelerating de Sitter attractor (\( X = \text{constant} \)) before reaching the tracker due to the presence of the \( a_2 X^2 \) term. We will show that a variety of the dark energy equation of state including the entry to the region \( w_{\text{DE}} < -1 \) can be realized, depending on the moment at which the term \( a_2 X^2 \) dominates over cubic and quartic Galileon interactions. Even if the dominant contribution to the field energy density in the early Universe corresponds to the quartic Galileon interaction, the conditions for the absence of ghosts and Laplacian instabilities can be consistently satisfied throughout the cosmic expansion history.

We will also study the evolution of linear cosmological perturbations relevant to the observations of large-scale structure and compute the two quantities \( \mu = G_{\text{eff}} / G \) and \( \Sigma = G_{\text{light}} / G \) commonly used in the EFTCAMB code \cite{light,light}, where \( G_{\text{eff}} \) and \( G_{\text{light}} \) are gravitational couplings associated with the growth of matter perturbations and the light bending, respectively, with \( G \) being the Newton constant. For the self-tuning model (\( c_t^2 = 1 \)), we will show that the gravitational potentials \( \Psi \) and \( \Phi \) are almost equivalent to each other, in which case \( \mu \approx \Sigma > 1 \). If the dominance of the term \( a_2 X^2 \) occurs at a later cosmic epoch, \( \mu \) and \( \Sigma \) tend to deviate from 1. Hence it is possible to distinguish our model from the \( \Lambda \)CDM model from both the background and cosmic growth histories.

Nonlinear derivative interactions beyond Horndeski also have impacts on the gravitational law in regions of high density \cite{light}. We will study constraints on the deviation from Horndeski theories arising from the modification of gravitational potentials inside massive objects \cite{light,light,light,light} and show that our dark energy model with \( c_t^2 = 1 \) is well consistent with such bounds.

This paper is organized as follows. In Sec. [II] we revisit the no-ghost condition and the propagation speed of tensor perturbations in GLPV theories up to quartic-order. In Sec. [III] we present the background and scalar perturbation equations without specifying the model and obtain general expressions of \( \mu \) and \( \Sigma \) under the quasi-static approximation on sub-horizon scales. In Sec. [IV] we propose a concrete dark energy model in GLPV theories allowing for \( c_t^2 = 1 \). We study the background cosmological dynamics paying particular attention to the evolution of \( w_{\text{DE}} \) and investigate whether the conditions for the absence of ghosts and Laplacian instabilities are satisfied. In Sec. [V] we discuss the evolution of linear cosmological perturbations and numerically compute the quantities \( \mu \) and \( \Sigma \) to confront the model with the observations of large-scale structure and weak lensing. In Sec. [VI] we study constraints on our model.
arising from corrections to gravitational potentials inside massive objects induced by the beyond-Horndeski nonlinear derivative interaction. Sec. VII is devoted to conclusions.

II. GLPV THEORIES AND THE SPEED OF GRAVITATIONAL WAVES

The GLPV theories up to quartic order is given by the action 54

\[ S = \int d^4x \sqrt{-g} \sum_{i=2}^{4} L_i + S_M, \tag{2.1} \]

where \( g \) is a determinant of the metric \( g_{\mu\nu} \) and \( S_M \) is the matter action. We assume that the matter sector is minimally coupled to gravity. The Lagrangians \( L_{2,3,4} \) are given, respectively, by

\[
\begin{align*}
L_2 &= G_2(\phi, X), \\
L_3 &= G_3(\phi, X) \Box \phi, \\
L_4 &= G_4(\phi, X) R - 2G_4, X(\phi, X) \left[ (\Box \phi - \nabla^\mu \nabla_\mu \phi) \nabla_\mu \phi \right] \\
&\quad + F_4(\phi, X) \epsilon^{\mu\nu\rho\sigma} \nabla_\mu \phi \nabla_\nu \phi \nabla_\rho \phi \nabla_\sigma \phi, \tag{2.4}
\end{align*}
\]

where \( G_{2,3,4} \) and \( F_4 \) depend on the scalar field \( \phi \) and its kinetic energy \( X \equiv \nabla^\mu \phi \nabla_\mu \phi \) with the partial derivative \( G_{4, X} \equiv \partial G_4 / \partial X \), and \( R, G_{\mu\nu}, \epsilon_{\mu\nu\rho\sigma} \) are the Ricci scalar, the Einstein tensor, the totally antisymmetric Levi-Civita tensor satisfying the normalization \( \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} = +4! \), respectively. The last term of Eq. (2.4) arises in theories beyond Horndeski. The quintic-order Lagrangian \( L_5 \) in GLPV theories gives rise to the speed \( c_1 \) of GWs different from \( c_1 = 1 \) \cite{54, 55, 77, 78}, so the action (2.1) corresponds to the most general scalar-tensor theories allowing for \( c_1 = 1 \) in the framework of GLPV theories.

The action (2.1) can be also written in terms of scalar quantities arising from the 3+1 Arnowitt-Deser-Misner (ADM) decomposition of space-time \cite{93} with the foliation of constant-time hypersurfaces \( \Sigma_t \). The line element in the ADM formalism is given by \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \), where \( N \) is the lapse, \( N^i \) is the shift, and \( h_{ij} \) is the 3-dimensional spatial metric. The extrinsic curvature and intrinsic curvature are defined, respectively, as \( K_{\mu\nu} = h^k_{\mu} n_{k;\nu} \) and \( R_{\mu\nu} = (3) R_{\mu\nu} \), where \( n_{\mu} = (-N, 0, 0, 0) \) is a normal vector orthogonal to \( \Sigma_t \) and \( (3) R_{\mu\nu} \) is the 3-dimensional Ricci tensor on \( \Sigma_t \). There are several scalar quantities constructed from \( K_{\mu\nu} \) and \( R_{\mu\nu} \), as

\[
K \equiv g^{\mu\nu} K_{\mu\nu}, \quad S \equiv K_{\mu\nu} K^{\mu\nu}, \quad R \equiv g^{\mu\nu} R_{\mu\nu}. \tag{2.5}
\]

By choosing the so-called unitary gauge \( (\phi = \phi(t)) \) for a time-like scalar field, the action (2.1) is expressed in the form \( S = \int d^4x \sqrt{-g} L + S_m \), where \( 54, 17 \)

\[
L = A_2(N, t) + A_3(N, t) K + A_4(N, t)(K^2 - S) + B_4(N, t) R, \tag{2.6}
\]

with

\[
A_2 = G_2 - X E_3, \phi, \quad A_3 = 2 |X|^{3/2} \left( E_{3, X} + \frac{G_{4, \phi}}{X} \right), \quad A_4 = -G_4 + 2 X G_{4, X} - X^2 F_4, \quad B_4 = G_4. \tag{2.7}
\]

The auxiliary function \( E_3(\phi, X) \) obeys

\[
G_3 = E_3 + 2 X E_{3, X}. \tag{2.8}
\]

For the time-dependent scalar field \( \phi \), we have \( X = -N^{-2}(\dot{\phi} / dt)^2 < 0 \) and hence \( A_3 = 2(-X)^{3/2} E_{3, X} - 2 \sqrt{-X} G_{4, \phi} \). In the unitary gauge, the \( \phi, X \) dependence in the functions \( G_{2,3,4} \) and \( F_4 \) translates to the \( N, t \) dependence in the functions \( A_2, A_3, A_4 \) and \( B_4 \). Thus, the Lagrangian (2.6) depends on \( N, t \) and \( K, S, R \). The Horndeski theories satisfy the condition \( F_4 = 0 \), so that \( A_4 = -B_4 + 2 X B_{4, X} \), while \( F_4 \neq 0 \) in GLPV theories.

We study the background dynamics of dark energy driven by the field \( \phi \) as well as the evolution of linear cosmological perturbations for the perturbed line element:

\[
ds^2 = -(1 + 2 \delta N) dt^2 + 2 \delta \psi dtdx^i + a^2(t) [(1 + 2\zeta)\delta_{ij} + \gamma_{ij}] dx^i dx^j, \tag{2.9}
\]

1 The sign of \( \epsilon^{\mu\nu\rho\sigma} \) is opposite to that used in Ref. 92 in the context of beyond generalized Proca theories.
where $a(t)$ is the time-dependent scale factor, $\delta N, \psi, \zeta$ are scalar metric perturbations, and $\gamma_{ij}$ are tensor perturbations. The scalar perturbation $E$, which appears as the form $\partial_t \partial_i E \, dx^i \, dx^3$ in Eq. (2.9), is set to 0, so that the spatial component of a gauge-transformation vector $\xi^\mu$ is fixed. We also choose the unitary gauge in which the perturbation $\delta \phi$ of the scalar field $\phi$ vanishes, under which the temporal part of $\xi^\mu$ is fixed. In GLPV theories, there are no dynamical vector degrees of freedom, so we do not consider vector perturbations in our analysis.

The second-order tensor action derived by expanding (2.1) up to quadratic order in $\gamma_{ij}$ is

$$S_2^{\text{(T)}} = \int d^4 x \, a^3 \frac{Q_i}{4} \delta^{kl} \left( \gamma_{ij} \gamma_{kl} - \frac{c_i^2}{a^2} \partial \gamma_{ij} \partial \gamma_{kl} \right),$$

where a dot represents the derivative with respect to $t$, and

$$Q_i \equiv L_{,S} = -A_4, \quad c_i^2 \equiv \frac{L_{,X}}{L_{,S}} = -\frac{B_4}{A_4}.$$  \hfill (2.11) \hfill (2.12)

The condition for the absence of tensor ghosts corresponds to $Q_i > 0$, i.e., $A_4 < 0$. In GR we have $-A_4 = B_4 = 1/(16\pi G)$, so $c_i^2$ is equivalent to 1. For the theories in which $-A_4$ is different from $B_4$, there is the deviation of $c_i^2$ from 1. The GLPV theories satisfying $c_i^2 = 1$ have the following relation

$$F_4 = \frac{2G_{4,X}}{X},$$ \hfill (2.13)

for $X \neq 0$. For example, if we consider the case $G_4(X) = B_4(X) = 1/(16\pi G) + b_4X^2$ ($b_4$ is a constant), then the function $F_4 = 4b_4$ satisfies the condition (2.13). In Horndeski theories we have $F_4 = 0$, so $G_4$ depends on $\phi$ alone. The existence of the additional Lagrangian containing the $F_4$ term in GLPV theories allows the possibility for realizing $c_i^2 = 1$ even for the function $G_4$ containing the $X$ dependence.

### III. BACKGROUND AND SCALAR PERTURBATION EQUATIONS

In this section, we present the background and perturbation equations of motion in GLPV theories given by the action (2.7). For the matter sector, we take into account a perfect fluid whose background density and pressure are given, respectively, by $\rho$ and $P$. The scalar perturbations of the matter energy-momentum tensor $T^\mu_\nu$ are expressed in the form

$$\delta T^0_0 = -\delta \rho, \quad \delta T^0_i = \partial_i \delta q, \quad \delta T^j_j = \delta P \delta^j_j.$$ \hfill (3.1)

In the line element (2.9), there are also three scalar perturbations $\delta N, \psi, \zeta$ arising from the gravity sector.

We first revisit the general equations of motion without imposing the condition (2.13) and analytically estimate the quantities $\mu$ and $\Sigma$ under the condition that the deviation from Horndeski theories is not significant for the perturbations deep inside the Hubble radius. We then discuss how the constraint (1.1) of GWs puts restrictions on the values of $\mu$ and $\Sigma$.

#### A. Background equations

In the ADM language, the background equations of motion are expressed in the forms

$$\dot{L} + L_{,N} - 3HF = \rho,$$ \hfill (3.2)

$$\dot{\bar{L}} - \bar{F} - 3HF = -P,$$ \hfill (3.3)

where $H \equiv \dot{a}/a$ is the Hubble parameter, $F \equiv L_{,K} + 2HL_{,S}$, and a bar represents quantities of the background. The matter sector obeys the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0.$$ \hfill (3.4)

In GLPV theories given by the Lagrangian (2.6), Eqs. (3.2) and (3.3) reduce, respectively, to

$$A_2 - 6H^2A_4 + 2\dot{\phi}^2 \left( A_{2,X} + 3HA_{3,X} + 6H^2A_{4,X} \right) = \rho,$$ \hfill (3.5)

$$A_2 - 6H^2A_4 - \dot{A}_3 - 4\dot{H}A_4 - 4HA_4 = -P,$$ \hfill (3.6)
where we used $\tilde{N} = 1$, $\tilde{K} = 3H$, $\tilde{S} = 3H^2$, and $\tilde{R} = 0$. Since the function $B_4$ does not appear in Eqs. (3.8) and (3.9), the two theories with same $A_{2,3,4}$ but with different $B_4$ cannot be distinguished from each other [55]. In other words, Horndeski theories and GLPV theories with same functions $A_{2,3,4}$ lead to the same background cosmological dynamics.

### B. Scalar perturbation equations

The second-order action of GLPV theories expanded up to quadratic order in scalar perturbations was already computed in Refs. [54, 55, 74, 78]. To discuss the perturbation equations of motion for the Lagrangian [26] containing the $N, t, K, S, R$ dependence, it is convenient to define the following dimensionless quantities [73]

$$
\alpha_M = \frac{\dot{Q}_t}{HQ_t}, \quad \alpha_B = \frac{L_{KN} + 2HL_{SN}}{4HQ_t}, \quad \alpha_K = \frac{2L_N + L_{NN}}{2H^2Q_t}, \quad \alpha_H = \frac{L_R + L_{NR}}{L_S} - 1.
$$

(3.7)

The parameters $\alpha_M, \alpha_B, \alpha_K$ correspond to the running of the gravitational constant, the kinetic mixing between the scalar field and gravity, and the kinetic term for the scalar, respectively. The parameter $\alpha_H$ characterizes the deviation from Horndeski theories. In quartic-order GLPV theories given by the Lagrangian [26], it follows that

$$
\alpha_H = \frac{2XB_{4,X} - A_4 - B_4}{A_4} = \frac{X^2F_4}{A_4},
$$

(3.8)

which does not vanish for $F_4 \neq 0$.

In Fourier space with the comoving wavenumber $k$, the perturbation equations of motion corresponding to Hamiltonian and momentum constraints, which are derived by varying the second-order action with respect to $\delta N$ and $\partial^2 \psi$, are given by [41]

$$
2H^2Q_t (\alpha_K - 12\alpha_B - 6) \delta N + 4HQ_t (1 + \alpha_B) \left(3\dot{\zeta} + \frac{k^2}{a^2} \zeta \right) + 4Q_t (1 + \alpha_H) \frac{k^2}{a^2} \zeta = \delta \rho,
$$

(3.9)

$$
4HQ_t (1 + \alpha_B) \delta N - 4Q_t \dot{\zeta} = -\delta q,
$$

(3.10)

respectively. From the continuity equations $\delta T^\mu_{\alpha \mu} = 0$ and $\delta T^\mu_{i \mu} = 0$, we obtain

$$
\dot{\delta \rho} + 3H (\delta \rho + \delta P) = -(\rho + P) \left(3\dot{\zeta} + \frac{k^2}{a^2} \zeta \right) + \frac{k^2}{a^2} \delta q,
$$

(3.11)

$$
\dot{\delta q} + 3H \delta q = -(\rho + P) \delta N - \delta P.
$$

(3.12)

Varying the second-order action of scalar perturbations with respect to $\zeta$ and using Eq. (3.12), we obtain

$$
H (1 + \alpha_M) \psi + \dot{\psi} + (1 + \alpha_H) \delta N + c_s^2 \zeta = 0.
$$

(3.13)

The function $B_4$ appears in Eqs. (3.9) and (3.10) through the terms $L_R = B_4$ and $L_{NR} = B_{4,N}$. Then, the two theories with same $A_{2,3,4}$ and with different $B_4$ can be distinguished from each other at the level of perturbations. By specifying the functional forms of $A_{2,3,4}$ and $B_4$ with a given equation of state $w = P/\rho$ and a matter sound speed squared $c_s^2 = \delta P/\delta \rho$, we can solve Eqs. (3.9)-(3.10) and (3.11)-(3.13) together with the background Eqs. (3.5)- (3.6) to determine $\delta N, \psi, \zeta, \delta \rho, \delta q$.

### C. Conditions for the absence of ghosts and Laplacian instabilities

To study whether ghosts and Laplacian instabilities of scalar perturbations do not arise from the radiation era to today, we need to take into account both radiation and nonrelativistic matter in the action $S_M$. The perfect fluids of radiation and nonrelativistic matter can be modeled by the purely k-essence Lagrangians (pressures) $P_r(Y_r) = b_rY_r^2$ and $P_m(Y_m) = b_m(Y_m - Y_0)^2$, where $b_r, b_m, -Y_0$ are positive constants and $Y_r = \nabla^\mu \chi_r \nabla_\mu \chi_r$ and $Y_m = \nabla^\mu \chi_m \nabla_\mu \chi_m$ are the kinetic energies of two scalar fields $\chi_r$ and $\chi_m$ [55, 73, 77]. Since the background densities $\rho_i$ and the sound speed squares $c_i^2$ for $i = r, m$ are given, respectively, by $\rho_i = 2Y_iP_{i,Y_i} - P_i$ and $c_i^2 = P_{i,Y_i}/(P_{i,Y_i} - 2\chi_i^2 P_{i,Y_i})$, it follows that

$$
w_r \equiv \frac{P_r}{\rho_r} = \frac{1}{3}, \quad c_r^2 = \frac{1}{3} \quad \text{(for radiation)},
$$

(3.14)

$$
w_m \equiv \frac{P_m}{\rho_m} = \frac{Y_m - Y_0}{3Y_m + Y_0}, \quad c_m^2 = \frac{Y_m - Y_0}{3Y_m - Y_0} \quad \text{(for nonrelativistic matter)}.
$$

(3.15)
Provided that \(|(Y_m - Y_0)/Y_0| \ll 1\), both \(w_m\) and \(c_m^2\) are close to 0.

We also expand the matter action \(S_M\) up to second order in scalar perturbations by considering the field perturbations \(\delta \chi_r\) and \(\delta \chi_m\) of radiation and nonrelativistic matter, respectively. By using Eqs. (3.10) and (3.11) to eliminate \(\delta N\) and \(\psi\) from the second-order scalar action of Eq. (2.1), the resulting action can be expressed in terms of the three perturbations \(\delta \chi_r, \delta \chi_m, \zeta\) and their derivatives. The no-ghost conditions associated with the perturbations \(\delta \chi_r\) and \(\delta \chi_m\) are trivially satisfied for \(b_r > 0\) and \(b_m > 0\), respectively. The no-ghost condition for the dynamical scalar degree of freedom \(\zeta\) corresponds to (54, 55, 77, 78).

\[
Q_s = \frac{Q_t(\alpha_K + 6\alpha_B^2)}{(1 + \alpha_B)^2} > 0.
\] (3.16)

Taking the small-scale limit in the second-order scalar action, we can also derive the three propagation speed squares \(c_s^2, c_r^2, c_m^2\) associated with the perturbations \(\zeta, \delta \chi_r, \delta \chi_m\), respectively. In GLPV theories with \(\alpha_H \neq 0\), these propagation speeds are generally mixed with each other (50, 54, 55). In the limit that \(c_m^2 \to 0\), the matter sound speed squared \(c_m^2\) is decoupled from the others, such that \(c_m^2 = 0\) (94). The other two propagation speed squares are given, respectively, by (62, 94)

\[
c_s^2 = \frac{1}{2} \left[ c_r^2 + c_H^2 - \beta_H - \sqrt{\left(c_r^2 - c_H^2 + \beta_H\right)^2 + 2c_r^2\delta \chi_r^2} \right],
\] (3.17)
\[
c_r^2 = \frac{1}{2} \left[ c_s^2 + c_H^2 + \beta_H + \sqrt{\left(c_s^2 - c_H^2 + \beta_H\right)^2 + 2c_s^2\delta \chi_r^2} \right],
\] (3.18)

where

\[
c_H^2 = \frac{2}{Q_s} \left[ M + H M - c_t^2 Q_t - \frac{\rho_r + P_r + \rho_m + P_m}{4H^2(1 + \alpha_B)} \right], \quad M = \frac{Q_t(1 + \alpha_H)}{H(1 + \alpha_B)},
\]
\[
\quad \beta_H = \beta_r + \beta_m, \quad \beta_r = \frac{\alpha_H (\rho_r + P_r)}{Q_s H^2(1 + \alpha_B)^2}, \quad \beta_m = \frac{\alpha_H (\rho_m + P_m)}{Q_s H^2(1 + \alpha_B)^2}.
\] (3.19)

In Horndeski theories we have \(\alpha_H = \beta_H = 0\), so Eqs. (3.17) and (3.18) reduce to \(c_s^2 = c_H^2\) and \(c_r^2 = c_s^2 = 1/3\), respectively. In GLPV theories with \(|\alpha_H| \ll 1\), it follows that

\[
c_s^2 \simeq c_H^2 - \beta_H + \alpha_H \beta_r \frac{c_r^2}{2(c_r^2 - c_H^2 - \beta_H)},
\] (3.20)
\[
c_r^2 \simeq c_s^2 - \alpha_H \beta_r \frac{c_s^2}{2(c_s^2 - c_H^2 - \beta_H)}.
\] (3.21)

Even when \(|\alpha_H| \ll 1\), the term \(\beta_H\) is not necessarily much smaller than 1 (52). Hence the deviation from Horndeski theories affects \(c_s^2\) such that \(c_s^2 \simeq c_H^2 - \beta_H\), whereas \(c_r^2\) is close to \(c_s^2\).

### D. Sub-horizon perturbations

We discuss the evolution of linear scalar perturbations for the modes relevant to the growth of large-scale structures. We assume that the perfect fluid is described by nonrelativistic matter satisfying \(P = 0\) and \(\delta P = 0\), without taking into account the radiation. We introduce the gauge-invariant matter density contrast \(\delta_m\) and gravitational potentials \(\Psi, \Phi\), as

\[
\delta_m \equiv \delta - 3V_m, \quad \Psi \equiv \delta N + \dot{\psi}, \quad \Phi \equiv \zeta + H\psi,
\] (3.22)

where \(\delta \equiv \delta \rho/\rho\) and \(V_m \equiv H\delta q/\rho\). Taking the time derivative of Eq. (3.11) and using Eq. (3.12), it follows that

\[
\dot{\delta}_m + 2H\delta_m + \frac{k^2}{a^2}\Psi = -3 \left( \dot{B} + 2HB \right),
\] (3.23)

where \(B \equiv \zeta + V_m\). The matter density contrast \(\delta_m\) grows due to the gravitational instability associated with the potential \(\Psi\). This relation is quantified by the modified Poisson equation

\[
\frac{k^2}{a^2}\Psi = -4\pi\mu G \rho_m \delta_m, \quad \text{with} \quad \mu = \frac{G_{\text{eff}}}{G},
\] (3.24)
where \( G_{\text{eff}} \) is the effective gravitational coupling generally different from the Newton constant \( G \). Introducing the gravitational slip parameter

\[
\eta = -\frac{\Phi}{\Psi},
\]

(3.25)

the effective potential \( \psi_{\text{eff}} = \Phi - \Psi \) associated with the light bending in weak lensing observations obeys

\[
\frac{k^2}{a^2} \psi_{\text{eff}} = 8\pi G \Sigma \rho_m \delta_m, \quad \text{with} \quad \Sigma = \frac{1+\eta}{2} \mu.
\]

(3.26)

On using the gravitational potentials \( \Psi \) and \( \Phi \), we can express Eq. 3.18 as

\[
(1 + \alpha_H) \Psi + (1 + \alpha_M) \Phi + (c_s^2 - 1 - \alpha_M) \zeta = \alpha_H \dot{\psi}.
\]

(3.27)

Since we are interested in the evolution of perturbations for the modes deep inside the Hubble radius, we employ the approximation that the dominant contributions to perturbation equations are those containing \( k^2/a^2 \) and \( \dot{\rho} \). This is called the quasi-static approximation [98, 99], which is valid for the modes deep inside the sound horizon \((c_s^2 k^2/a^2 \gg H^2)\) in a strict sense. On using Eq. 3.10, Eqs. 3.9 and 3.11 reduce, respectively, to

\[
\delta \rho \simeq 4Q_1 \frac{k^2}{a^2} [(1 + \alpha_B) \Phi - (\alpha_B - \alpha_H) \zeta],
\]

(3.28)

\[
\dot{\delta} \rho + 3H \delta \rho - \frac{k^2}{a^2} \left[ \frac{\rho}{H} (\zeta - \Phi) + 4Q_1 \dot{\zeta} - 4HQ_1(1 + \alpha_B)(\Psi - \dot{\psi}) \right] \simeq 0.
\]

(3.29)

Substituting Eq. 3.28 and its time derivative into Eq. 3.20, we obtain

\[
(1 + \alpha_B) \Psi + \left[ (1 + \alpha_M + h) (1 + \alpha_B) + \frac{\dot{\alpha}_B}{H} + \frac{3}{2} \dot{\Omega}_m \right] \Phi
- \left[ \alpha_B (1 + \alpha_M + h) + h + \alpha_B \right] \frac{\dot{\zeta}}{H} - \frac{3}{2} \dot{\Omega}_m - \alpha_H (1 + \alpha_M) \frac{\dot{\alpha}_H}{H} \zeta \simeq \frac{\alpha_H}{H} \dot{\zeta}.
\]

(3.30)

where

\[
h \equiv \frac{\dot{H}}{H^2}, \quad \dot{\Omega}_m \equiv \frac{\rho_m}{6H^2 Q_t}.
\]

(3.31)

Since \( \beta_r = 0 \) in the absence of radiation, the sound speed squared 3.17 reduces to \( c_s^2 = c_s^2 - \beta_m \), which is expressed in the form

\[
c_s^2 = \frac{2Q_1(1 + \alpha_H)}{Q_s(1 + \alpha_B)} \left[ 1 + \alpha_M - h - \frac{\dot{\alpha}_B}{H(1 + \alpha_B)} + \frac{\dot{\alpha}_H}{H(1 + \alpha_M)} - c_s^2 \frac{1 + \alpha_B}{1 + \alpha_M} - \frac{3(1 + 2\alpha_H)}{2(1 + \alpha_B)(1 + \alpha_M)} \dot{\Omega}_m \right].
\]

(3.32)

In GLPV theories with \( \alpha_H \neq 0 \), there are two time derivatives on the r.h.s. of Eqs. 3.27, 3.28, and 3.30, so we cannot solve Eqs. 3.27, 3.28, and 3.30 for \( \Psi, \Phi, \zeta \). If the parameter \( |\alpha_H| \) is not much smaller than 1, the oscillating mode of perturbations cannot be neglected relative to the mode induced by matter perturbations \( \delta \rho \) in Eq. 3.28. Indeed, the quasi-static approximation breaks down for the models in which \( A_1 \) and \( B_1 \) are constants different from each other with \( |\alpha_H| = \mathcal{O}(1) \) [10]. For the models with \( |\alpha_H| \ll 1 \), the oscillating mode can be suppressed relative to the matter-induced mode [62].

In Horndeski theories or GLPV theories where \( |\alpha_H| \) is very much smaller than 1 throughout the cosmic expansion history, we can derive the closed-form expressions of \( \Psi, \Phi, \zeta \) by taking the limits \( \alpha_H \to 0 \) and \( \dot{\alpha}_H \to 0 \) in Eqs. 3.27, 3.28, and 3.30. In doing so, we employ the approximation that \( \delta_m \simeq \delta \) for sub-horizon perturbations and eliminate the \( \dot{\alpha}_B \) term in Eq. 3.30 by using Eq. 3.32. Then, the quantities \( \mu, \eta, \) and \( \Sigma \) reduce, respectively, to

\[
\mu = \frac{c_s^2}{16\pi G Q_t} \left[ 1 + \frac{2Q_1(c_s^2(1 + \alpha_B) - 1 - \alpha_M)^2}{Q_s c_s^2 c_t^2(1 + \alpha_B)^2} \right],
\]

(3.33)

\[
\eta = \frac{Q_s c_s^2(1 + \alpha_B)^2 + 2Q_1 A_B c_s^2(1 + \alpha_B) - 1 - \alpha_M}{Q_s c_s^2 c_t^2(1 + \alpha_B)^2 + 2Q_1(c_s^2(1 + \alpha_B) - 1 - \alpha_M)^2},
\]

(3.34)

\[
\Sigma = \frac{1 + c_s^2}{32\pi G Q_t} \left[ 1 + \frac{2Q_1(c_s^2(1 + \alpha_B) + \alpha_B - 1 - \alpha_M)(c_s^2(1 + \alpha_B) - 1 - \alpha_M)}{Q_s c_s^2 c_t^2(1 + \alpha_B)^2} \right],
\]

(3.35)
which are also valid in full Horndeski theories including the quintic Lagrangian $L_5$ \[ \text{[100, 101]} \]. In GR with the functions $-A_4 = B_4 = 1/(16\pi G)$ and $A_3 = 0$, we have $Q_t = 1/(16\pi G)$, $\alpha^2 = 1$, $\alpha_B = \alpha_M = 0$, and hence $\mu = \eta = \Sigma = 1$. In modified gravitational theories, the normalized effective gravitational coupling $\mu = G_{\text{eff}}/G$ is composed of the “tensor” part $c_t^2/(16\pi GQ_t)$ and the “scalar-matter interaction” given by the second term in the square bracket of Eq. (3.33). The scalar-matter interaction term is positive under the stability conditions $Q_s > 0$, $Q_t > 0$, $c_s^2 > 0$, $c_t^2 > 0$. This reflects the fact that the fifth force induced by the scalar degree of freedom is attractive.

For the theories in which $c_t^2 = 1$, Eqs. (3.33)-(3.35) further simplify to

\[
\begin{align*}
\mu & = \frac{1}{16\pi GQ_t} \left[1 + \frac{2Q_t(\alpha_B - \alpha_M)^2}{Q_s c_s^2(1 + \alpha_B)^2}\right], \\
\eta & = \frac{Q_s c_s^2(1 + \alpha_B)^2 + 2Q_t(\alpha_B - \alpha_M)}{Q_s c_s^2(1 + \alpha_B)^2 + 2Q_t(\alpha_B - \alpha_M)^2}, \\
\Sigma & = \frac{1}{16\pi GQ_t} \left[1 + \frac{4Q_t(2\alpha_B - \alpha_M)(\alpha_B - \alpha_M)}{Q_s c_s^2(1 + \alpha_B)^2}\right],
\end{align*}
\]

respectively. Under the stability conditions $Q_s > 0$, $Q_t > 0$, $c_s^2 > 0$, the necessary condition for realizing $G_{\text{eff}}$ smaller than $G$ corresponds to $1/(16\pi GQ_t) < 1$, i.e.,

\[ Q_t > \frac{1}{16\pi G}. \] (3.39)

As long as $\alpha_B \neq \alpha_M$, the scalar-matter interaction term in Eq. (3.36) is positive, so the condition (3.39) is not sufficient to realize $G_{\text{eff}} < G$. For the opposite inequality to Eq. (3.39), $G_{\text{eff}}$ is always larger than $G$. The quantity $\Sigma$ is greater than $1/(16\pi GQ_t)$ for $(2\alpha_B - \alpha_M)(\alpha_B - \alpha_M) > 0$.

For the theories with $c_t^2 \neq 1$, the necessary condition for $\mu < 1$ is modified to $Q_t > c_t^2/(16\pi G)$. In Ref. [101], the model with $c_t^2 < 1$ was proposed for realizing the small cosmic growth rate consistent with the observations of redshift-space distortions (RSDs) and CMB. This possibility is excluded after the GW170817 event, so we are left with only one parameter $Q_t$ for realizing $G_{\text{eff}} < G$.

Modifications of the quantities $\mu$ and $\Sigma$ compared to those in GR lead to the different evolution of $\delta_m$ and $\psi_{\text{eff}}$ through Eqs. (3.28) and (3.29). In Sec. V, we will study the dynamics of cosmological perturbations for a dark energy model with $\alpha_H \neq 0$ without resorting to the approximation used for the derivation of Eqs. (3.33)-(3.35).

IV. CONCRETE DARK ENERGY MODEL

In this section, we study the background cosmology for a concrete dark energy model given by the functions

\[ G_2 = a_1 X + a_2 X^2, \quad G_3 = 3a_3 X, \quad G_4 = \frac{1}{16\pi G} + b_4 X^2, \quad F_4 = 3b_4 - a_4, \] (4.1)

where $a_{1,2,3,4}$ and $b_4$ are constants. The model contains the Galileon interactions \[ \text{[35]} \] with the additional $a_2 X^2$ term. For $a_1 > 0$ and $a_2 > 0$, the function $G_2 = a_1 X + a_2 X^2$ corresponds to the ghost condensate scenario \[ \text{[80]} \] in which the cosmic acceleration is realized by the de Sitter fixed point satisfying $G_{2, X} = 0$, i.e., $X = -a_1/(2a_2)$. The existence of cubic and quartic Galileon interactions modifies the way how the solutions approach the de Sitter fixed point. Alternatively, we can consider a quintessence scenario in which the cosmic acceleration is driven by a scalar potential $V(\phi)$ \[ \text{[62]} \], i.e., $G_2 = -X/2 - V(\phi)$. It is also possible to generalize the Einstein-Hilbert term in $G_4$ to a nonminimal coupling of the form $F(\phi)/(16\pi G)$. In this paper, we are interested in the self-accelerating solution approaching a constant value $X$, so the explicit $\phi$ dependence is not included in the functions $G_{2,3,4}$ and $F_4$.

On using the correspondence \[ \text{[24]} \], the model given by the functions \[ \text{[101]} \] is equivalent to the Lagrangian \[ \text{[2.6]} \] with

\[ A_2 = a_1 X + a_2 X^2, \quad A_3 = 2a_3|X|^{3/2}, \quad A_4 = -\frac{1}{16\pi G} + a_4 X^2, \quad B_4 = \frac{1}{16\pi G} + b_4 X^2, \] (4.2)

and $E_3 = a_3 X$. The tensor propagation speed squared $c_t^2$ and the parameter $\alpha_H$ are given, respectively, by

\[ c_t^2 = \frac{1 + 16\pi G a_4 X^2}{1 - 16\pi G a_4 X^2}, \quad \alpha_H = \frac{16\pi G a_4 X^2}{1 - 16\pi G a_4 X^2(1 - 3s)}, \] (4.3)

where

\[ s = \frac{b_4}{a_4}. \] (4.4)
For the theories with \( s = -1 \), \( c_t^2 \) is equivalent to 1, but \( \alpha_H \) is different from 0. The covariant Galileon in Horndeski theories corresponds to \( s = 1/3 \), in which case \( c_t^2 \) deviates from 1.

### A. Dynamical system

To discuss the background cosmological dynamics, we take into account radiation (density \( \rho_r \) and pressure \( P_r = \rho_r/3 \)) and nonrelativistic matter (density \( \rho_m \) and pressure \( P_m = 0 \)). Substituting the functions (4.2) into Eqs. (3.5) and (3.6), the background equations of motion are

\[
\begin{align*}
3H^2 &= 8\pi G (\rho_{DE} + \rho_r + \rho_m) , \\
2\dot{H} + 3H^2 &= -8\pi G (P_{DE} + P_r) ,
\end{align*}
\]

where

\[
\begin{align*}
\rho_{DE} &= -a_1 \dot{\phi}^2 + 3a_2 \dot{\phi}^4 + 18a_3 H \ddot{\phi}^3 + 30a_4 H^2 \dot{\phi}^4 , \\
P_{DE} &= -a_1 \dot{\phi}^2 + a_2 \dot{\phi}^4 - 6a_3 \ddot{\phi}^3 \dot{\phi} - 2a_4 \dot{\phi}^3 \left[ 8H \dddot{\phi} + \dot{\phi}(2\dot{H} + 3H^2) \right] .
\end{align*}
\]

We define the following dimensionless quantities:

\[
\begin{align*}
x_1 &= -\frac{8\pi Ga_1 \dot{\phi}^2}{3H^2} , \\
x_2 &= \frac{8\pi Ga_2 \dot{\phi}^4}{H^2} , \\
x_3 &= \frac{48\pi Ga_3 H \ddot{\phi}^3}{H} , \\
x_4 &= \frac{80\pi Ga_4 \dot{\phi}^4}{H} , \\
\Omega_r &= \frac{8\pi G \rho_r}{3H^2} ,
\end{align*}
\]

which correspond to density parameters arising from the couplings \( a_1 X, a_2 X^2, 2a_3 |X|^{3/2}, a_4 X^2 \), and radiation, respectively. It is convenient to use \( x_1, x_2, x_3, x_4 \) rather than the coupling constants \( a_1, a_2, a_3, a_4 \) themselves, since we can easily make a comparison between different energy densities arising from different couplings. The standard minimally coupled scalar field without the potential \( (a_1 = -1/2) \) is recovered by taking the limits \( x_1 \to 4\pi G \dot{\phi}^2/(3H^2) \), \( x_2 \to 0 \), \( x_3 \to 0 \), \( x_4 \to 0 \).

From Eq. (4.5), it follows that

\[
\Omega_m \equiv \frac{8\pi G \rho_m}{3H^2} = 1 - \Omega_{DE} - \Omega_r , \\
\Omega_{DE} \equiv x_1 + x_2 + x_3 + x_4 .
\]

The quantities \( x_1, x_2, x_3, x_4 \) and \( \Omega_r \) obey the differential equations

\[
\begin{align*}
x'_1 &= 2x_1 (\epsilon_\phi - h) , \\
x'_2 &= 2x_2 (2\epsilon_\phi - h) , \\
x'_3 &= x_3 (3\epsilon_\phi - h) , \\
x'_4 &= 4x_4 \epsilon_\phi , \\
\Omega'_r &= -2\Omega_r (2 + h) ,
\end{align*}
\]

where a prime represents a derivative with respect to \( \mathcal{N} = \ln a \), and

\[
\epsilon_\phi \equiv \frac{\ddot{\phi}}{H \dot{\phi}} = -\frac{1}{q_s} \left[ 20(3x_1 + 2x_2) - 5x_3(3x_1 + x_2 + \Omega_r - 3) - x_4(36x_1 + 16x_2 + 3x_3 + 8\Omega_r) \right] ,
\]

\[
h \equiv \frac{\dot{H}}{H^2} = -\frac{1}{q_s} \left[ 10(3x_1 + x_2 + \Omega_r + 3)(x_1 + 2x_2) + 10x_3(6x_1 + 3x_2 + \Omega_r + 3) + 15x_3^2 \\
+ x_4(78x_1 + 32x_2 + 30x_3 + 12\Omega_r + 36) + 12x_4^2 \right] ,
\]

with

\[
q_s \equiv 20(x_1 + 2x_2 + x_3) + 4x_4(6 - x_1 - 2x_2 + 3x_3) + 5x_3^2 + 8x_4^2 .
\]

The dark energy equation of state is given by

\[
w_{DE} \equiv \frac{P_{DE}}{\rho_{DE}} = \frac{5(3x_1 + x_2 - \epsilon_\phi x_3) - (3 + 8\epsilon_\phi + 2h)x_4}{15(x_1 + x_2 + x_3 + x_4)} .
\]
B. de Sitter fixed point

For the dynamical system (4.11)-(4.15), there exists a de Sitter fixed point satisfying $\epsilon_\phi = 0$, $h = 0$, and $\Omega_r = 0$, i.e., both $\dot{\phi}$ and $H$ are constants. From Eqs. (4.16) and (4.17), we obtain the two relations among four constants $x_1, x_2, x_3, x_4$, as

$$x_4 = 1 - x_1 - x_2 - x_3, \quad x_3 = -\frac{1}{3} (18x_1 + 8x_2 + 12),$$ (4.20)

on the de Sitter solution. From Eq. (4.19), it follows that $\Omega_{DE} = 1$ and $\Omega_m = 0$. Substituting the two relations (4.20) with $\epsilon_\phi = 0$ and $h = 0$ into Eq. (4.19), we have $w_{DE} = -1$ as expected.

The stability of the de Sitter solution is known by considering the homogenous perturbations $\delta \epsilon_\phi, \delta h, \delta \Omega_r$ of the quantities $\epsilon_\phi, h, \Omega_r$, respectively. Since $\epsilon_\phi = h = 0$ on the de Sitter fixed point, the perturbation $\delta x_1$ of the variable $x_1$ obeys $\delta x_1^2 = 2x_1(\delta \epsilon_\phi - \delta h)$. On using the fact that the similar properties hold for $\delta x_2, \delta x_3, \delta x_4, \delta \Omega_r$ and taking the $N$ derivative of Eqs. (4.16) and (4.17), it follows that the perturbations $\dot{X} = (\delta \epsilon_\phi, \delta h, \delta \Omega_r)$ around the de Sitter fixed point obey

$$X' = AX,$$ (4.21)

where $A$ is the $3 \times 3$ matrix whose nonvanishing components are given by

$$A_{11} = -3, \quad A_{22} = -3, \quad A_{33} = -4,$$
$$A_{13} = \frac{18(x_1 + 2)}{3x_1(5x_2 - 6) + 6x_2 + 4x^2_2 - 36}, \quad A_{23} = -\frac{6(3x_1 + 4x_2 + 6)}{3x_1(5x_2 - 6) + 6x_2 + 4x^2_2 - 36}. \quad (4.22)$$

Since the eigenvalues of $A$ are $-3, -3, -4$, the de Sitter fixed point is always stable. This means that the solutions finally approach the de Sitter attractor, independent of the initial conditions of $x_1, x_2, x_3, x_4$.

C. Dark energy dynamics

We study the dynamics of dark energy from the radiation-dominated epoch to today. As in the ghost condensate model [80], we will focus on the case in which $x_1$ is negative, while $x_2, x_3, x_4$ are positive.

In the radiation and deep matter eras, let us consider the situation in which the quantities $x_3$ and $x_4$ dominate over $|x_1|$ and $x_2$. We also use the fact that $x_3$ and $x_4$ are much smaller than 1 in these epochs, by reflecting that $\Omega_{DE} \ll 1$. Then, the quantities (4.16) and (4.17) reduce, respectively, to

$$\epsilon_\phi \simeq \frac{5(\Omega_r - 3)x_3 + 8\Omega_r x_4}{4(5x_3 + 6x_4)}, \quad h \simeq -\frac{1}{2} (3 + \Omega_r). \quad (4.23)$$

If $x_4 \gg \{|x_1|, x_2, x_3\}$, then we have $\epsilon_\phi \simeq \Omega_r / 3$. In this case, $\epsilon_\phi \simeq 1/3$ and $h \simeq -2$ during the radiation domination ($\Omega_r \simeq 1$), so Eqs. (4.11)-(4.13) can be solved to give $|x_1| \propto a^{1/3}, x_2 \propto a^{16/3}, x_3 \propto a^3$, and $x_4 \propto a^{4/3}$. This behavior of $x_1, x_2, x_3, x_4$ can be confirmed in the numerical simulations of Figs. 1 and 2 at the redshift $z \gtrsim 3000$. From Eq. (4.19), the dark energy equation of state in the regime $x_4 \gg \{|x_1|, x_2, x_3\}$ is given by

$$w_{DE} \simeq -\frac{1}{9} \Omega_r. \quad (4.24)$$

In the numerical simulation of Figs. 1 and 2, $x_4$ dominates over $|x_1|, x_2, x_3$ during the radiation domination, so that $w_{DE} \simeq -1/9$. If the condition $x_3 \gg \{|x_1|, x_2, x_3\}$ is satisfied in the radiation and deep matter eras, the dark energy equation of state (4.19) yields

$$w_{DE} = -\frac{1}{9} \frac{1}{12} \Omega_r. \quad (4.25)$$

In Fig. 1 the dominance of $x_3$ over $x_4, |x_1|, x_2$ starts to occur right after the onset of the matter-dominated epoch, so $w_{DE}$ temporarily approaches the value $1/4$. In Fig. 2 $x_4$ dominates over $|x_1|, x_2, x_3$ by the redshift $z \approx 100$, so the evolution of $w_{DE}$ for $z \gtrsim 100$ is approximately given by Eq. (4.24). In the limit that $\Omega_r \rightarrow 0$, Eq. (4.24) reduces to 0, so the term $x_4$ arising from the quartic Galileon works as dark matter during the matter dominance. In Fig. 2 we can confirm that the variable $x_4$ stays nearly constant around 0.01 for $100 \lesssim z \lesssim 1000$. 

After $x_3$ dominates over $|x_1|, x_2, x_4$ during the matter era, the quantities given in Eq. (4.23) reduce to $\epsilon_\phi \simeq -3/4$ and $h \simeq -3/2$, so the solutions to Eqs. (4.11)-(4.14) are given, respectively, by $|x_1| \propto a^{3/2}, x_2 \propto a^0, x_3 \propto a^{-3/4}$, and $x_4 \propto a^{-3}$. Then, the quantities $|x_1|$ and $x_2$ eventually catch up with $x_3$ and $x_4$. After $|x_1|$ and $x_2$ grow to the order of 1, the energy density associated with the Lagrangian $G_2 = a_1 X + a_2 X^2$ becomes the main source for the late-time cosmic acceleration. There are also contributions to the dark energy density arising from the cubic and quartic Galileon terms. Since $x_4$ decreases more significantly than $x_3$ during the matter era, today’s value of $x_4$ is typically much smaller than $x_3$ and $|x_1|, x_2$, see Figs. 1 and 2. Demanding that the maximum value of $x_4$ reached during the matter era does not exceed the order of 0.1, today’s value of $x_4$ is constrained to be $x_4^{(0)} \lesssim 10^{-3}$.

In Fig. 1 the dark energy equation of state enters the region $w_{DE} < -1$ first, takes a minimum, and then finally approaches the de Sitter value $-1$. In Fig. 2 the solution reaches the de Sitter attractor without entering the region $w_{DE} < -1$. This difference comes from how much extent the cubic Galileon term $x_3$ contributes to the dark energy density at late times. If there is a long period in which $x_3$ dominates over $x_2$ and $|x_1|$ as in the case of Fig. 1, the solutions enter the region $w_{DE} < -1$. This reflects the fact that the tracker present for $x_2 \to 0$ has the equation of state smaller than $-1$ \cite{38, 39}. However, if this period is short as in the case of Fig. 2 the k-essence terms $x_2$ and $|x_1|$ start to dominate over $x_3$ before the entry to the region $w_{DE} < -1$.

The tracker solution, which exists for $x_2 \to 0$, satisfies the relation \cite{38, 39}

$$H \dot{\phi} = \text{constant} \quad i.e., \quad \epsilon_\phi = -h.$$ \hspace{1cm} (4.26)

On using Eqs. (4.16) and (4.17) and taking the limit $x_2 \to 0$, this condition translates to

$$10 x_1 + 5 x_3 + 4 x_4 = 0.$$ \hspace{1cm} (4.27)

Substituting the relations (4.26) and (4.27) into Eq. (4.19), it follows that

$$w_{DE} = -1 + \frac{2}{3} h,$$ \hspace{1cm} (4.28)

which is equivalent to $-2$ during the matter dominance $(h \simeq -3/2)$.

Now, we are considering the theories with $x_2 \neq 0$, so the approach to the tracker is prevented by the $a_2 X^2$ term. In Fig. 3 we plot the evolution of $w_{DE}$ for several different initial values of $x_2$ with same initial conditions of $x_1, x_3, x_4, \Omega_r$ as those used in Fig. 1. In the limit that $x_2 \to 0$, the solutions approach the tracker equation of state $w_{DE} = -2$.
FIG. 2. The same as Fig. 1 but with the different initial condition $x_4 = 2.2 \times 10^{-4}$ at $z = 1.3 \times 10^5$.

during the matter era. This tracker equation of state is in tension with the joint data analysis of SNIa, CMB, and BAO \[40\]. In the present case, however, the $a_2 X^2$ term leads to $w_{DE}$ larger than $-2$. For smaller $x_2$, the solutions enter the stage in which the terms $x_2$ and $|x_1|$ dominate over $x_3$ and $x_4$ later, so the minimum values of $w_{DE}$ tend to be smaller. This property can be confirmed in the numerical simulation of Fig. 3. In case (a) of Fig. 3 which corresponds to the initial conditions of Fig. 1 the dark energy equation of state takes a minimum $w_{DE} \simeq -1.2$ around the redshift $z \simeq 20$ and then approaches the asymptotic value $-1$. In case (d), the minimum value $w_{DE} \simeq -1.83$ is reached around $z \simeq 5$. It will be of interest to study how the joint data analysis of SNIa, CMB, and BAO place
bounds on today’s values of of $x_1, x_2, x_3, x_4$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4}
\caption{Evolution of the quantities $Q_s$, $c_s^2$, and $c_r^2$ versus $z + 1$ for the model $s = -1$ (i.e., $c_t^2 = 1$). The initial conditions of $x_1, x_2, x_3, x_4$ and $\Omega_r$ are the same as those used in Fig. 1.}
\end{figure}

D. Stability conditions

We study whether the conditions for the absence of ghosts and Laplacian instabilities are satisfied for the background cosmology discussed above. The quantities (2.11) and (2.12) reduce, respectively, to

$$Q_t = \frac{5 - x_4}{80\pi G}, \quad c_t^2 = \frac{5 + sx_4}{5 - x_4},$$

(4.29)

The ghost and Laplacian instabilities of tensor perturbations are absent under the conditions

$$x_4 < 5,$$

(4.30)

$$5 + sx_4 > 0.$$  

(4.31)

Since $x_4$ does not exceed the order of 1 for the realization of viable cosmology, the two conditions (4.30) and (4.31) are well satisfied for $|s| = O(1)$. The recent detection of gravitational waves from a binary neutron star merger placed the bound (1.1) for the redshift $z < 0.009$. On using Eq. (4.29), today’s value of $x_4$, denoted as $x_4^{(0)}$, is constrained to be

$$\left| (s + 1)x_4^{(0)} \right| \lesssim 10^{-14}.$$  

(4.32)

For the theories with $s = -1$, $c_t^2$ is equivalent to 1. In this case, the bound (4.32) is trivially satisfied irrespective of the value of $x_4^{(0)}$. However, for the theories in which $s$ is different from $-1$, $x_4^{(0)}$ is constrained to be smaller than the order of $10^{-14}$ for $s = O(1)$. This is the case for covariant Galileons in Horndeski theories ($s = 1/3$). Under such a tight bound of $x_4^{(0)}$, the contribution from $x_4$ to the total energy density at low redshifts is significantly suppressed relative to $|x_1|, x_2, x_3$. Hence, for the theories with $s \neq -1$, the quartic derivative interaction $a_4 X^2$ is practically irrelevant to the dynamics of dark energy after the matter-dominated epoch. We caution, however, that $x_4$ tends to dominate over $|x_1|, x_2, x_3$ as we go back to the past (see Figs. 1 and 2), so it can affect the stability conditions in the deep radiation era even with the value of $x_4^{(0)}$ of order $10^{-14}$. 

The quantity \(\frac{3(5 - x_4)q_s}{200\pi G(x_3' + 2x_4' - 2)^2}\) associated with the no-ghost condition of scalar perturbations yields

\[Q_s = \frac{3(5 - x_4)q_s}{200\pi G(x_3' + 2x_4' - 2)^2},\]  

(4.33)

where \(q_s\) is defined by Eq. (4.18). Under the condition \(q_s > 0\), the scalar ghost is absent for

\[q_s > 0,\]  

(4.34)

which does not depend on \(s\). In Fig. 1, we plot the evolution of the quantity \(Q_s\) for the background cosmology corresponding to Fig. 1. The quantity \(Q_s\) remains positive throughout the cosmic expansion history, by reflecting the fact that the contributions to \(Q_s\) from positive values of \(x_2, x_3, x_4\) dominate over \(|x_1|\).

For the scalar propagation speed squared \(c_s^2\), we first consider the regime in which the condition \(x_4 \gg |x_1|, x_2, x_3\) is satisfied in the early Universe. In this case, we obtain

\[c_s^2 \simeq \frac{1}{36} [5 + s + (3 - 5s)\Omega_r],\]  

(4.35)

so that \(c_s^2 \simeq (2 - s)/9\) during the radiation dominance and \(c_s^2 \simeq (5 + s)/36\) during the matter dominance, respectively. To avoid the Laplacian instability in these epochs, we require that the ratio \(s = b_4/a_4\) needs to be in the range

\[-5 \leq s \leq 2.\]  

(4.36)

The model with \(c_s^2 = 1\) corresponds to \(s = -1\), so it satisfies the condition \(4.36\). In Fig. 1, we plot the evolution of \(c_s^2\) for \(s = -1\) with the background initial conditions same as those in Fig. 1. We observe that \(c_s^2\) starts to evolve from the value close to 1/3 in the deep radiation era and decreases according to \(c_s^2 \simeq (1 + 2\Omega_r)/9\) by the moment at which \(x_3\) dominates over \(x_4\).

In the regime characterized by \(x_3 \gg |x_1|, x_2, x_3\), Eq. (3.17) reduces to

\[c_s^2 \simeq \frac{1}{12} (5 + \Omega_r),\]  

(4.37)

which is positive. In Fig. 1, we can confirm that, during the matter dominance, \(c_s^2\) temporally approaches the value 5/12 after \(x_3\) dominates over \(x_4\).

On the de Sitter attractor, there are two relations given by Eq. (4.20) with \(\Omega_r = 0\). In this case, the quantity \(\beta_H\) in Eq. (3.17) is equivalent to 0 and hence

\[c_s^2 = c_H^2 = \frac{(6 + 6x_1 + x_2)[18 + 9x_1 + s(3x_1 + x_2 + 3)(6 + 15x_1 + 4x_2)]}{3(3x_1 + x_2)[4x_2^2 + 6x_2 - 36 + 3x_1(5x_2 - 6)]},\]  

(4.38)

which is required to be positive to ensure the Laplacian stability on the de Sitter fixed point. In the numerical simulation of Fig. 1, the asymptotic future values of \(x_1\) and \(x_2\) are given, respectively, \(x_1 = -1.96556397\) and \(x_2 = 2.89669243\), so that \(c_s^2 = 2.84 \times 10^{-5} > 0\) from Eq. (4.38). In Fig. 1, we observe that \(c_s^2\) is positive from the radiation era to the de Sitter attractor. The other propagation speed squared \(c_T^2\) is always close to 1/3. Thus, the model discussed above suffers neither ghost nor Laplacian instabilities.

We also computed the quantities \(Q_1, Q_s, c_s^2\), and \(c_T^2\) for all the four cases shown in Fig. 3 and confirmed that, for \(s = -1\), they remain to be positive during the cosmic expansion history. Thus, in our model, it is possible to realize a wide variety of the dark energy equation of state including the entry to the region \(w_{DE} < -1\), while avoiding the appearance of ghosts and Laplacian instabilities. For larger initial conditions of \(x_4\), there is a tendency that the scalar propagation speed squared temporally enters the region \(c_T^2 < 0\) right after the end of the dominance of \(x_4\) over \(|x_1|, x_2, x_3\). To avoid this behavior, today’s values \(x_4\) are typically in the range

\[x_4^{(0)} \lesssim 10^{-4}.\]  

(4.39)

The numerical simulation of Fig. 2 corresponds to the marginal case in which the condition \(c_T^2 > 0\) is always satisfied. Since we have not searched for the whole parameter space of initial conditions, the condition \(4.39\) should be regarded only as a criterion for avoiding \(c_T^2 < 0\). To derive precise bounds on \(x_4^{(0)}\), we need to carry out the likelihood analysis by considering all possible initial conditions.
V. GROWTH OF LINEAR COSMOLOGICAL PERTURBATIONS

We study the evolution of linear perturbations relevant to the observations of RSDs and weak lensing for the dark energy model given by the functions \( f(R) \). Since we are interested in the growth of matter perturbations long after the end of the radiation era, we will neglect the radiation and consider nonrelativistic matter alone satisfying \( w_m = 0 \) and \( \epsilon^2_m = 0 \). The parameters \( \alpha_M, \alpha_B, \alpha_K, \) and \( \alpha_H \) defined in Eq. (5.7) reduce, respectively, to

\[
\begin{align*}
\alpha_M &= -\frac{4x_4c_\phi}{5 - x_4}, \\
\alpha_B &= \frac{5x_3 + 8x_4}{5 - x_4}, \\
\alpha_K &= \frac{6(5x_1 + 10x_2 + 5x_3 + 6x_4)}{5 - x_4}, \\
\alpha_H &= \frac{(1 - 3s)x_4}{5 - x_4}.
\end{align*}
\]  

(5.1)

From Eqs. (5.9)-(5.13), the perturbations \( \zeta, \chi \equiv H \psi, \delta, \) and \( V_m \) obey the first-order differential equations

\[
\begin{align*}
\zeta' &= \frac{5[3\Omega_m V_m + (2 - x_3 - 2x_4)\delta N]}{2(5 - x_4)}, \\
\chi' &= \frac{5(h - 1) + (1 - h + 4c_\phi)x_4}{5 - x_4} \chi - (1 + \alpha_H) \delta N - c_\delta^2 \zeta, \\
\delta' &= -\frac{15[3\Omega_m V_m + (2 - x_3 - 2x_4)\delta N]}{2(5 - x_4)} + K^2 (V_m - \chi), \\
V_m' &= hV_m - \delta N,
\end{align*}
\]  

(5.2)-(5.5)

where \( K \equiv k/(aH) \), and

\[
\delta N = \frac{30\Omega_m(5 - x_4)\delta - 4(1 + \alpha_H)(5 - x_4)K^2 \zeta - 5(2 - x_3 - 2x_4)[2(5 - x_4)K^2 \chi + 45\Omega_m V_m]}{60(5 - x_4)(x_1 + 2x_2) + 75x_3(4 + x_3) + 180(x_3 + 2)x_4 + 120x_4^2}.
\]  

(5.6)

Taking the \( \mathcal{N} \) derivative of Eq. (5.6) and using other perturbation equations of motion, we obtain

\[
V_m'' + \alpha_1 V_m' + \alpha_2 V_m = \beta_1 \chi + \beta_2 \zeta,
\]  

(5.7)

where \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are time-dependent functions. The general solution to Eq. (5.7) can be written as the sum of the homogenous solution to Eq. (5.7) and the special solution induced by the r.h.s. of Eq. (5.7). The deviation from Horndeski theories generally gives rise to a term proportional to \( \alpha_H \Omega_m K^2 \) in \( \alpha_2 \). In the present model, such a term (denoted as \( \tilde{\alpha}_2 \)) is given by

\[
\tilde{\alpha}_2 = -\frac{2}{q^s} (5 - x_4) \alpha_H \Omega_m K^2.
\]  

(5.8)

Under the conditions \( \epsilon^2_m, \alpha_2 \) and \( \epsilon^2_m, \beta_2 \), \( \tilde{\alpha}_2 \) is negative for \( \alpha_H > 0 \), whereas \( \tilde{\alpha}_2 > 0 \) for \( \alpha_H < 0 \). If \( x_4 \) dominates over \( |x_1|, x_2, x_3 \), then the quantity \( \tilde{\alpha}_2 \) approximately reduces to \( \tilde{\alpha}_2 \approx -(1 - 3s)\Omega_m K^2/12 \), so the coefficient in front of the \( K^2 \) term is smaller than the order of unity for \( |s| \lesssim \mathcal{O}(1) \). If either of \( |x_1|, x_2, x_3 \) dominates over \( x_4 \), the coefficient is much smaller than the order of 1. Thus, unlike the model of Ref. [9], the perturbation \( V_m \) is not plagued by the problem of a heavy oscillation arising from a large positive coefficient in front of \( \alpha_H \Omega_m K^2 \).

If \( \tilde{\alpha}_2 \) is negative, one may worry that the term \( \tilde{\alpha}_2 \) can lead to the instability in the small-scale limit \( (K \to 0) \). However, the coefficients \( \beta_1 \) and \( \beta_2 \) also contain contributions proportional to \( K^2 \), so the term \( \tilde{\alpha}_2 V_m \) can balance with the r.h.s. of Eq. (5.7). This is consistent with the fact that the matter sound speed squared \( c^2_m \) in the small-scale limit vanishes for \( c^2_m \to 0 \), which implies the absence of Laplacian instabilities for matter perturbations.

In the following, we choose the initial conditions satisfying \( \alpha_2 V_m = \beta_1 \chi + \beta_2 \zeta \), \( \zeta' = 0 \), and \( \chi' = 0 \) in the deep matter-dominated epoch. From Eq. (5.2), the condition \( \zeta' = 0 \) translates to \( \delta N \simeq -3\Omega_m V_m/2 \) for \( x_3, x_4 \ll 1 \), which corresponds to the same initial condition as that in GR. In this case, we obtain \( V''_m \simeq 0 \) from Eq. (5.7) in the deep matter era \( (\Omega_m \simeq 1 \) and \( h \simeq -3/2) \). Since the initial condition \( \alpha_2 V_m = \beta_1 \chi + \beta_2 \zeta \) is chosen, we have \( V''_m \simeq 0 \) from Eq. (5.7). Using also the initial condition \( \chi' \simeq 0 \) mimicking the behavior of GR, the initial values of \( V_m, \zeta, \) and \( \chi \) can be expressed in terms of \( \delta \). The initial value of \( \delta \) is chosen such that today’s amplitude of \( \delta \) is consistent with the bound constrained from CMB.

In Fig. 5, we plot the evolution of \( -\Psi, \Phi, \delta_m, V_m \) for \( s = -1 \) (i.e., \( c^2_m = 1 \)) corresponding to the background cosmology of Fig. 4. The initial conditions of perturbations are chosen as those explained above around the redshift \( z = 100 \), with today’s values \( \delta_m(z = 0) = 0.815 \) and \( K(z = 0) = 200 \). In the deep matter era, the velocity potential \( V_m \) stays nearly constant without heavy oscillations, so that \( V'_m \simeq 0 \) and \( V''_m \simeq 0 \) in Eq. (5.7). The variation of \( V_m \) starts to occur after the dark energy density contributes to the background density at low redshifts. Since the condition \( x_4 \ll \{|x_1|, x_2, x_3\} \) is satisfied for \( z \lesssim 100 \), \( \tilde{\alpha}_2 \) is much smaller than the order of \( K^2 \).
FIG. 5. Evolution of the two gravitational potentials $-\Psi$ and $\Phi$, the density contrast $\delta_m$, and the velocity potential $V_m$ versus $z + 1$ for the model $c_s^2 = 1$. The background cosmology corresponds to that in Fig. 1 without radiation. We start to integrate the perturbation equations of motion (5.2)-(5.5) with the initial conditions explained in the main text at the redshift $z = 100$. Today’s values of $\delta_m$ and $K$ are $\delta_m = 0.815$ and $K = 200$, respectively.

The numerical simulation of Fig. 5 shows that the two gauge-invariant gravitational potentials $-\Psi$ and $\Phi$ are almost identical to each other. This can be understood as follows. In the definition of $\alpha_M$ in Eq. (5.1), the quantity $\epsilon_\phi$ is at most of the order 1, so that $|\alpha_M| \ll x_4 \ll 1$. As we see in Figs. 1 and 2, $x_3$ is much larger than $x_4$ at low redshifts and hence the quantity $|\alpha_B|$ is of the order of $x_3$. Then, the ratio $|\alpha_M/\alpha_B|$ is in the range $\alpha_M/\alpha_B \ll 1$.

In the present model, the parameter $\alpha_B$ appearing on the r.h.s. of Eqs. (3.27) and (3.30) is much smaller than 1 at low redshifts, so the results (3.33)-(3.35) derived under the quasi-static approximation should not lose their validity. Since we are now considering the case $c_s^2 = 1$, the gravitational slip parameter is given by Eq. (3.37). Taking the limit $|\alpha_M| \ll |\alpha_B|$ in Eq. (3.37), we obtain

$$\eta \simeq 1,$$

and hence $-\Psi \simeq \Phi$. We note that taking the same limit in Eq. (3.34) leads to $\eta$ generally different from 1. This means that the condition $c_s^2 = 1$ plays an important role for realizing the value (5.10).

On using the property (5.10), it follows that the two quantities $\mu$ and $\Sigma$ are almost equivalent to each other. Indeed, taking the limit $|\alpha_M| \ll |\alpha_B|$ in Eqs. (3.36) and (3.38) gives

$$\mu \simeq \Sigma \simeq \frac{1}{16\pi GQ_t} \left[1 + \frac{2Q_t\alpha_B^2}{Q_s c_s^2 (1 + \alpha_B)^2}\right].$$

Since $1/(16\pi GQ_t) = 5/(5 - x_4) > 1$ for $x_4 > 0$, both $\mu$ and $\Sigma$ are larger than 1 under the conditions $Q_s > 0$, $Q_t > 0$, $c_s^2 > 0$. In Ref. [53], it was shown that the observational data generally favor the region $(\mu - 1)(\Sigma - 1) \geq 0$, so our model is consistent with this property. The parameter $\alpha_B$, which can be estimated as $\alpha_B \simeq -x_3/2$ for $x_3 \gg x_4$, gives rise to important modifications to $\mu$ and $\Sigma$ compared to those in GR.

In Fig. 6, we show the evolution of $\Sigma$ for the four different backgrounds corresponding to those in Fig. 3. We solved the full perturbation equations of motion from the redshift $z \simeq 100$ without employing the quasi-static approximation and found that the numerically derived values of $\mu$ and $\Sigma$ are very close to each other. Moreover, we confirmed that Eqs. (3.36)-(3.38) derived under the quasi-static approximation without the terms on the r.h.s. of Eqs. (3.27) and
FIG. 6. Evolution of the quantity $\Sigma$ versus $z + 1$ for the model $c_t^2 = 1$. Each curve corresponds to the background cosmologies shown as the cases (a), (b), (c), (d) in Fig. 3 without radiation. We start to integrate the perturbation equations of motion at the redshift $z = 100$ by using the initial conditions of $\zeta, \chi, V_m$ satisfying $\zeta' = 0$, $\chi' = 0$, and $V_m = (\beta_1 \chi + \beta_2 \zeta)/\alpha_2$. Today’s values of $\delta_m$ and $K$ are $\delta_m = 0.815$ and $K = 200$, respectively.

FIG. 7. Evolution of $f \sigma_8$ versus $z$ for the model $c_t^2 = 1$. Each curve corresponds to the cases (a), (b), (c), (d) explained in the caption of Fig. 6.

Equations (3.30) provide good estimates for the evolution of $\mu$ and $\Sigma$ by today. In Fig. 6 we observe that the deviation of $\Sigma$ from 1 is more significant for the cases in which the dark energy equation of state largely deviates from $-1$ at lower redshifts (see Fig. 3). This reflects the fact that today’s value of $x_3$ (denoted as $x_3^{(0)}$) tends to be larger for $w_{DE}$ entering the region $w_{DE} < -1$ at later epochs, e.g., $x_3^{(0)} = 5.6 \times 10^{-2}$ and $x_3^{(0)} = 0.82$ for the cases (a) and (d), respectively. For larger $x_3^{(0)}$, the quantities $\mu$ and $\Sigma$ increase according to Eq. (5.11).

In Fig. 7 we plot the evolution of $f \sigma_8$ corresponding to the cases (a)-(d) in Fig. 6 where $f = \dot{\delta}_m/(H\delta_m)$ and $\sigma_8$ is the
amplitude of over-density at the comoving $8\ h^{-1}\ Mpc$ scale ($h$ is today’s normalized Hubble parameter $H_0 = 100\ h\ km\ sec^{-1}\ Mpc^{-1}$). Today’s value of $\sigma_8$ is chosen to be $\sigma_8(z = 0) = 0.815$ for the consistency with the constraint from CMB $[6, 102]$. Since the normalized effective gravitational coupling $\mu = G_{\text{eff}}/G$ at low redshifts increases for larger $x_3^{(0)}$, it is possible to distinguish the several different cases plotted in Fig. 7 from the RSD measurements (see, e.g., Ref. $[103]$). We note that the cubic coupling $\alpha_3 = 3a_3X$ is crucially important to modify the values of $\mu$ and $\Sigma$ compared to those in GR. It remains to be seen how the observation data of RSDs and weak lensing place constraints on the values $x_3^{(0)}$ and $x_4^{(0)}$.

VI. CONSTRAINTS ON $\alpha_H$ FROM MASSIVE OBJECTS

Finally, we discuss constraints on our dark energy model arising from the change of gravitational law inside local massive objects. In such regimes, the nonlinear beyond-Horndeski derivative interaction leads to modifications to the gravitational potentials of GR $[56–59]$ in a way different from those derived for linear cosmological perturbations in Sec. V. Hence it is possible to place constraints on the parameter $\alpha_H$ from several astrophysical observations associated with nonrelativistic compact objects.

Let us first consider a scalar field $\phi$ with the radial dependence $r$ alone around a spherically symmetric and static body. For the model in which the constant difference between $-A_4$ and $B_4$ is present with the beyond-Horndeski interaction $F_4 \propto X^{-2} \bar{B}^4$, the scalar curvature exhibits a divergence of the form $R = -2\alpha_H/r^2$ at $r = 0 [60, 61]$. This is attributed to the appearance of a solid angle deficit singularity, which is related to the violation of the geometric structure of space-time. In our model (4.1), which corresponds to $F_4 = \text{constant}$, the parameter $\alpha_H$ is proportional to $X^2$. Since the regularity of $r$-dependent scalar field requires that $d\phi/dr = 0$ at $r = 0$, this means that the parameter $\alpha_H$, which is proportional to $(d\phi/dr)^2$, vanishes at the center of body. Hence our model is free from the problem of the solid angle deficit singularity (as in the model studied in Ref. $[62]$).

More importantly, in our model, the field kinetic energy $X$ drives the cosmic acceleration. In this case, $X$ contains the cosmological time derivative $(d\phi/dt)^2$ as well as the radial derivative $(d\phi/dr)^2$ around a local source. This situation is analogous to what happens in generalized Proca theories, in which a temporal vector component $\phi^\alpha$ is present besides a longitudinal scalar mode $(d\chi/dr)^2$ $[104]$. In such cases, it was shown that the solid angle deficit singularity is generally absent by the dominance of $\phi^\alpha$ over $(d\chi/dr)^2$ in the limit that $(d\chi/dr)^2 \to 0$ $[105]$. In GLPV theories where the time derivative of the scalar leads to cosmic acceleration, the problem of the solid angle deficit singularity does not arise either.

In GLPV theories, the corrections to gravitational potentials $\Psi$ and $\Phi$ inside massive objects induced by the beyond-Horndeski nonlinear derivative interaction were already derived in Refs. $[56–59, 90, 91]$. We consider the Newtonian gravitational potentials $\Psi(r)$ and $\Phi(r)$ with the line element

$$ds^2 = -[1 + 2\Psi(r)][dt^2 + [1 + 2\Phi(r)]d\chi d\chi + \delta_{ij}dx^idx^j], \tag{6.1}$$

where $r = \sqrt{\delta_{ij}x^ix^j}$ is the radial coordinate. The scalar field $\phi$ and the matter density $\rho_m$ acquire perturbations by the existence of a compact object, as $\phi = \delta\phi(t) + \chi(r)$ and $\rho_m = \rho_m(t) + \delta\rho_m(r)$. The time derivative $\dot{\phi}(t)$, which is the source for dark energy, works as a cosmological boundary term.

On small scales relevant to the physics of compact objects, the spatial derivatives of perturbations dominate over their time derivatives. We neglect such time derivatives and keep all the terms with second-order derivatives preserving the Galileon symmetry. Using the equation of motion for $\chi$, the field derivative $d\chi/dr$ in the region of over density can be expressed in terms of the mass function $M(r) = \int r^2 4\pi r^2 \delta\rho_m(r)dr$ and its $r$ derivative $dM(r)/dr$. Substituting this relation into the equations of $\Psi$ and $\Phi$, their $r$-derivatives can be written as

$$\frac{d\Psi(r)}{dr} = G_N \left[ \frac{M(r)}{r^2} + \gamma_1 \frac{d^2M(r)}{dr^2} \right], \tag{6.2}$$

$$\frac{d\Phi(r)}{dr} = -G_N \left[ \frac{M(r)}{r^2} + \gamma_2 \frac{1}{r} \frac{dM(r)}{dr} \right]. \tag{6.3}$$
where
\[ G_N = \frac{1}{16\pi(-A_4)(1 + \alpha_V)}, \quad (6.4) \]
\[ \gamma_1 = \frac{\alpha_H}{c_t^2(1 + \alpha_V) - 1 - \alpha_H}, \quad (6.5) \]
\[ \gamma_2 = \frac{\alpha_H(\alpha_V - \alpha_H)}{c_t^2(1 + \alpha_V) - 1 - \alpha_H}, \quad (6.6) \]
with
\[ \alpha_V = -\frac{A_4N}{A_4}. \quad (6.7) \]

In Refs. [55, 59], the authors studied the case of quartic GLPV theories without the cubic Lagrangian \( (x_3 = 0) \), so the quantity \( \alpha_B \) in Eq. (6.1) is equivalent to \( \alpha_V = -4x_4/(5 - x_4) \). Now, we are considering more general theories with \( x_3 \neq 0 \), in which case the terms \( \alpha_V \) appearing in Eqs. (6.5)-(6.6) cannot be replaced with \( \alpha_B \). Since \( -A_4 = (5 - x_4)/(80\pi G) \) in our model, the effective Newton constant is given by
\[ G_N = \frac{G}{1 - x_4}. \quad (6.8) \]

This is different from the effective gravitational coupling \( G_{\text{eff}} \) derived for linear matter density perturbations, see Eq. (5.11). The gravitational coupling \( G_{\text{eff}} \), which contains \( \alpha_B \), is affected by the cubic derivative coupling \( G_3 = 3\alpha_3X \), while this is not the case for \( G_N \). The latter property is attributed to the fact that, apart from the beyond-Horndeski interaction, the propagation of fifth forces is suppressed by the operation of the Vainshtein mechanism.

For the model with \( s = -1 \), we have
\[ c_t^2 = 1, \quad \alpha_H = -\alpha_V = \frac{4x_4}{5 - x_4}, \quad (6.9) \]
where we used Eq. (6.1). Hence the two parameters \( (6.3, 6.4) \) reduce, respectively, to
\[ -2\gamma_1 = \gamma_2 = \alpha_H = \frac{4x_4}{5 - x_4}, \quad (6.10) \]

In GR, the two gravitational potentials \( -\Psi \) and \( \Phi \) are equivalent to each other. This equivalence is broken in GLPV theories due to the existence of two terms containing \( \gamma_1 \) and \( \gamma_2 \) in Eqs. (6.5-6.6). The mass of galaxy clusters measured by weak lensing (sensitive to \( \psi_{\text{eff}} = \Phi - \Psi \)) and X-ray observations (sensitive to \( -\Psi \)) does not coincide for \( \alpha_H \neq 0 \), so it is possible to place bounds on the values of \( \gamma_1 \) and \( \gamma_2 \). From the CFHTLenS weak lensing data and the X-ray data from XMM-Newton in the redshift range \( 0.1 < z < 1.2 \), these parameters are constrained to be \( -0.78 \leq 4\gamma_1 \leq 0.82 \) and \( -1.41 \leq -4\gamma_2/5 \leq 1.00 \) [88]. These constraints translate to
\[ -0.57 \leq x_4 \leq 0.44, \quad \text{and} \quad -2.27 \leq x_4 \leq 1.53, \quad (6.11) \]
respectively.

There are also bounds on \( \gamma_1 \) arising from the modification to the mass of nonrelativistic stars. For the consistency of the white dwarf of lowest mass with the Chandrasekhar limit and the consistency of the minimum mass for hydrogen burning in stars with the red dwarf of lowest mass, the parameter \( \gamma_1 \) is constrained to be \( -0.51 < 4\gamma_1 < 1.6 \) [85, 86]. Then, we obtain the bound
\[ -1.25 \leq x_4 \leq 0.30, \quad (6.12) \]
at \( z = 0 \).

There is also a constraint on \( x_4 \) from the binary pulsars. The orbital period of binary stars is proportional to \( (-A_4G_Nc_t)^{-1} \), so for the model with \( c_t^2 = 1 \), it is simply related to the parameter \( \gamma_0 \equiv 1 + \alpha_V \). The observations of

\[ \text{References:} \quad [55, 59, 57-59, 85-86, 88]. \]

\[ \text{Note:} \quad \text{We updated the upper bound of} \ 4\gamma_1 \ \text{according to the arXiv version} \ 3 \ \text{of Ref.} \ [82]. \ \text{We thank Jeremy Sakstein for pointing it out.} \]

\[ \text{End of note.} \]
Hulse-Taylor binary pulsar PSR B1913+16 provided the bound \(-7.5 \times 10^{-3} \leq \gamma_0 - 1 \leq 2.5 \times 10^{-3}\), so this translates to
\[
-0.0031 \leq x_4 \leq 0.0094, \tag{6.13}
\]
at \(z \ll 1\).

The constraints \(4.11\), \(4.12\), and \(4.13\) need to be satisfied in the redshift range \(z \leq 1\). The tightest one comes from binary pulsars, so that all the astrophysical bounds discussed above can be satisfied for \(|x_4| \lesssim 10^{-3}\) at \(z \lesssim 1\). This is weaker than the criterion \(4.39\) derived for the realization of cosmological dynamics without theoretical inconsistencies.

For the models with \(s \neq -1\), the quantities \(\gamma_0, \gamma_1, \text{ and } \gamma_2\) are given, respectively, by
\[
\gamma_0 - 1 = -\frac{4x_4}{5 - x_4}, \quad \gamma_1 = -\frac{x_4(1 - 3s)^2}{20(1 - s) + 8x_4 s}, \quad \gamma_2 = \frac{x_4(5 - 3s)(1 - 3s)}{20(1 - s) + 8x_4 s}. \tag{6.14}
\]

Since we are now considering the case \(c_s^2 \neq 1\), today’s value of \(x_4\) is constrained to be smaller than the order of \(10^{-14}\) from the speed of gravitational waves. Then, the binary pulsar bound on \(\gamma_0\) is trivially satisfied. Provided that \(s\) is not very close to 1, both \(|\gamma_1|\) and \(|\gamma_2|\) are at most of the order of \(x_4\), so the other constraints on \(\gamma_1\) and \(\gamma_2\) are also satisfied.

\section{Conclusions}

In this paper, we proposed a dark energy model in the framework of GLPV theories consistent with the recent GW170817 bound on the tensor propagation speed. The existence of a beyond-Horndeski quartic coupling \(F_4\) allows the possibility for realizing \(c_s^2 = 1\) on the isotropic cosmological background under the condition \(2.15\). Our model contains the Galileon interactions up to quartic order as well as the \(a_2X^2\) term in the quadratic Lagrangian \(G_2\). In the 3+1 ADM language, the model is given by the Lagrangian \(2.6\) with the functions \(4.2\). For the model with \(s = b_4/a_4 = -1\), the tensor propagation speed squared in Eq. \(4.3\) is equivalent to 1.

In Sec. \(III\) we presented general background and scalar perturbation equations of motion in quartic-order GLPV theories without specifying the model. We also revisited conditions for the absence of ghosts and Laplacian instabilities and the behavior of perturbations for the modes deep inside the (sound) horizon. The deviation from Horndeski theories, weighted by the parameter \(\alpha_H\), generates additional time derivatives to the scalar perturbation equations even under the quasi-static approximation. In the case where \(|\alpha_H|\) is much smaller than 1, we derived general expressions of two quantities \(\mu\) and \(\Sigma\) by using the dimensionless variables given by Eq. \(3.7\). If \(c_s^2 = 1\), \(\mu\) and \(\Sigma\) reduce, respectively, to \(4.30\) and \(4.31\), so they are different from the value 1/(16\(\pi G Q_t\)) for the theories with \(\alpha_H \neq \alpha_M\). The necessary condition for realizing \(\mu = G_{\text{eff}}/G < 1\) corresponds to \(Q_t > 1/(16\pi G)\).

In Sec. \(IV\) we studied the background cosmological dynamics for the dark energy model given by the functions \(4.2\). There exists a de Sitter fixed point satisfying the conditions \(4.20\), which is always a stable attractor. In the early Universe, the cubic and quartic Galileon couplings with the density parameters \(x_3\) and \(x_4\) provide important contributions to dark energy relative to the k-essence density parameters \(x_1\) and \(x_2\). In the radiation and matter eras, the dark energy equation of state is given by \(w_{DE} \simeq -\Omega_r/9\) for \(x_4 \gg \{x_1, x_2, x_3\}\), whereas \(w_{DE} \simeq 1/4 - \Omega_r/12\) for \(x_3 \gg \{x_1, x_2, x_4\}\). Depending on the moment of the dominance of \(x_3\) over \(x_4\) and on the values of \(|x_1|\) and \(x_2\), the dark energy equation of state either enters the regime \(w_{DE} < -1\) or stays in the region \(w_{DE} > -1\) (see Figs. \(1\) and \(2\)). For smaller \(x_2\), the dominance of the \(a_1X^2\) term in \(G_2\) tends to occur at later epochs, so the larger deviation of \(w_{DE}\) from \(-1\) occurs at lower redshifts (see Fig. \(3\)).

In Sec. \(IV\) we also showed that the tensor ghost and Laplacian instability are absent under the conditions \(4.30\) and \(4.31\). From the observational bound of \(c_t\), today’s value of \(x_4\) is constrained to be \(|(s + 1)x_4(0)| \lesssim 10^{-14}\). To avoid the Laplacian instability of scalar perturbations in the radiation and matter eras, the parameter \(s\) needs to be in the range \(-5 \leq s \leq 2\), which includes the case \(s = -1\). For the background cosmologies plotted in Fig. \(3\) the model with \(s = -1\) is plagued by neither ghost nor Laplacian instabilities throughout the cosmic expansion history. With increasing \(x_4(0)\), there is a tendency that the scalar propagation speed squared temporally enters the region \(c_s^2 < 0\) after the end of dominance of \(x_4\) over \(|x_1|, x_2, x_3\). To avoid this behavior, today’s value of \(x_4\) is typically in the range \(x_4(0) \lesssim 10^{-4}\).

In Sec. \(V\) we studied the evolution of linear cosmological perturbations for our dark energy model with \(s = -1\) and showed that the velocity potential \(V_m\) does not suffer from the problem of a heavy oscillation induced by the beyond-Horndeski term \(\alpha_H\). Since \(|\alpha_H| \ll 1\) in the matter era, the analytic estimations \(4.30, 4.31\) provide good approximations for discussing the evolution of gravitational potentials and matter density perturbations. Since the
condition $|\alpha_M| \ll |\alpha_B|$ is satisfied in our model, the gravitational slip parameter $\eta$ is close to 1 and hence $\mu \simeq \Sigma > 1$ for $x_4 > 0$. The deviations of $\mu$ and $\Sigma$ from those in GR ($\mu = \Sigma = 1$) mostly come from the density parameter $x_4$ of cubic Galileons in $\alpha_B$. As we see in Figs. 6 and 7, the cosmic growth rate tends to be larger for the cases in which the deviation of $w_{DE}$ from $-1$ occurs significantly at lower redshifts.

In Sec. VI, we discussed constraints on $\alpha_B$ arising from the change of gravitational law inside massive objects. The beyond-Horndeski nonlinear derivative interaction gives rise to modifications to the radial derivatives of gravitational potentials $\Psi$ and $\Phi$ in the forms (6.2)-(6.3). For the model with $c_4^2 = 1$, the tightest bound on $x_4$ in the redshift range $0 \leq z \lesssim 1$ comes from the orbital period of Hulse-Taylor binary pulsar, such that $-0.0031 \leq x_4 \leq 0.0094$. This bound is consistent with the criterion $x_4^{(0)} \lesssim 10^{-4}$ derived for the realization of viable cosmological solutions without Laplacian instabilities.

We have thus shown that our proposed model in GLPV theories offers interesting possibilities for realizing a wide variety of the dark energy equation of state without theoretical pathologies, while satisfying the observational bound of $c_4$. Moreover, the model gives rise to the distinguished cosmic growth history characterized by $\mu \simeq \Sigma > 1$, which can be tested with future high-precision observational data of RSDs and weak lensing. It will be of interest to put precise observational bounds on today’s density parameters $x_3$ and $x_4$ to understand how much extent the Galileon and beyond-Horndeski interactions can contribute to the physics of late-time cosmic acceleration.

ACKNOWLEDGEMENTS

RK is supported by the Grant-in-Aid for Young Scientists B of the JSPS No.17K14297. ST is supported by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 16K05359 and MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas “Cosmic Acceleration” (No.15H05890).

[1] A. G. Riess et al., Astron. J. 116, 1009 (1998) [astro-ph/9805201].
[2] S. Perlmutter et al., Astrophys. J. 517, 565 (1999) [astro-ph/9812133].
[3] D. N. Spergel et al., Astrophys. J. Suppl. 148, 175 (2003) [astro-ph/0302209].
[4] P. A. R. Ade et al., Astron. Astrophys. 571, A16 (2014) [arXiv:1304.5076 [astro-ph.CO]].
[5] D. J. Eisenstein et al., Astrophys. J. 635, 560 (2005) [astro-ph/0501171].
[6] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 594, A14 (2016) [arXiv:1502.01590 [astro-ph.CO]].
[7] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006) [hep-th/0603057].
[8] A. Silvestri and M. Trodden, Rept. Prog. Phys. 72, 096901 (2009) [arXiv:0904.0021 [astro-ph.CO]].
[9] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010) [arXiv:0805.1726 [gr-qc]].
[10] A. De Felice and S. Tsujikawa, Living Rev. Rel. 13, 3 (2010) [arXiv:1002.4398 [gr-qc]].
[11] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. 513, 1 (2012) [arXiv:1106.2476 [astro-ph.CO]].
[12] A. Joyce, B. Jain, J. Khoury and M. Trodden, Phys. Rept. 568, 1 (2015) [arXiv:1407.0059 [astro-ph.CO]].
[13] Y. Fujii, Phys. Rev. D 26, 2580 (1982).
[14] L. H. Ford, Phys. Rev. D 35, 2339 (1987).
[15] C. Wetterich, Nucl. Phys. B 302, 668 (1988).
[16] T. Chiba, N. Sugiyama and T. Nakamura, Mon. Not. Roy. Astron. Soc. 289, L5 (1997) [astro-ph/9704199].
[17] P. G. Ferreira and M. Joyce, Phys. Rev. Lett. 79, 4740 (1997) [astro-ph/9707286].
[18] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998) [astro-ph/9708069].
[19] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B 458, 209 (1999) [hep-th/9904075].
[20] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D 62, 023511 (2000) [astro-ph/9912463].
[21] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2000) [astro-ph/0001134].
[22] T. Chiba, A. De Felice and S. Tsujikawa, Phys. Rev. D 87, no. 8, 083505 (2013) [arXiv:1210.3859 [astro-ph.CO]].
[23] A. De Felice, L. Heisenberg and S. Tsujikawa, Phys. Rev. D 95, 123540 (2017) [arXiv:1703.09573 [astro-ph.CO]].
[24] G. W. Horndeski, Int. J. Theor. Phys. 10, 363 (1974).
[25] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, Phys. Rev. D 84, 064039 (2011) [arXiv:1103.3260 [hep-th]].
[26] T. Kobayashi, M. Yamaguchi and J. i. Yokoyama, Prog. Theor. Phys. 126, 511 (2011) [arXiv:1105.5723 [hep-th]].
[27] C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, Phys. Rev. Lett. 108, 051101 (2012) [arXiv:1106.2000 [hep-th]].
[28] W. Hu and I. Sawicki, Phys. Rev. D 76, 064004 (2007) [arXiv:0705.1158 [astro-ph]].
[29] A. A. Starobinsky, JETP Lett. 86, 157 (2007) [arXiv:0706.2041 [astro-ph]].
[30] S. A. Appleby and R. A. Battye, Phys. Lett. B 654, 7 (2007) [arXiv:0705.3199 [astro-ph]].
[31] S. Tsujikawa, Phys. Rev. D 77, 023507 (2008) [arXiv:0709.1391 [astro-ph]].
[32] S. Tsujikawa, K. Uddin, S. Mizuno, R. Tavakol and J. Yokoyama, Phys. Rev. D 77, 103009 (2008) [arXiv:0803.1106 [astro-ph]].
[91] A. Dima and F. Vernizzi, arXiv:1712.04731 [gr-qc].
[92] L. Heisenberg, R. Kase and S. Tsujikawa, Phys. Lett. B 760, 617 (2016) arXiv:1605.05565 [hep-th].
[93] R. L. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. 116, 1322 (1959).
[94] A. De Felice, K. Koyama and S. Tsujikawa, JCAP 1505, 058 (2015) arXiv:1503.06539 [gr-qc].
[95] R. J. Scherrer, Phys. Rev. Lett. 93, 011301 (2004).
[96] D. Giannakis and W. Hu, Phys. Rev. D 72, 063502 (2005) astro-ph/0501423.
[97] F. Arroja and M. Sasaki, Phys. Rev. D 81, 107301 (2010) arXiv:1002.1376 [astro-ph.CO].
[98] B. Boisseau, G. Esposito-Farese, D. Polarski and A. A. Starobinsky, Phys. Rev. Lett. 85, 2236 (2000) gr-qc/0001066.
[99] A. De Felice, T. Kobayashi and S. Tsujikawa, Phys. Lett. B 706, 123 (2011) arXiv:1108.4242 [gr-qc].
[100] J. Gleyzes, D. Langlois and F. Vernizzi, Int. J. Mod. Phys. D 23, no. 13, 1443010 (2014) arXiv:1411.3712 [hep-th].
[101] S. Tsujikawa, Phys. Rev. D 92, 044029 (2015) arXiv:1505.02459 [astro-ph.CO].
[102] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 594, A13 (2016) arXiv:1502.01589 [astro-ph.CO].
[103] T. Okumura et al., Publ. Astron. Soc. Jap. 68, no. 3, 38, 24 (2016) arXiv:1511.08083 [astro-ph.CO].
[104] A. De Felice, L. Heisenberg, R. Kase, S. Tsujikawa, Y. l. Zhang and G. B. Zhao, Phys. Rev. D 93, 104016 (2016) arXiv:1602.00371 [gr-qc].
[105] L. Heisenberg, R. Kase and S. Tsujikawa, Phys. Rev. D 94, 123513 (2016) arXiv:1608.08390 [gr-qc].
[106] R. A. Hulse and J. H. Taylor, Astrophys. J. 195, L51-L53 (1975).