The bizarre anti–de Sitter spacetime

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Abstract

Anti–de Sitter spacetime is important in general relativity and modern field theory. We review its geometrical features and properties of light signals and free particles moving in it. Applying only elementary tools of tensor calculus we derive ab initio all these properties and show that they are really weird. One finds superluminal velocities of light and particles, infinite particle energy necessary to escape at infinite distance and spacetime regions inaccessible by a free fall, though reachable by an accelerated spaceship. Radial timelike geodesics are identical to the circular ones and actually all timelike geodesics are identical to one circle in a fictitious five–dimensional space. Employing the latter space one is able to explain these bizarre features of anti–de Sitter spacetime; in this sense the spacetime is not self–contained. This is not a physical world.

Keywords: general relativity, exact solutions, geometry of anti–de Sitter space, timelike and null geodesics

1 Introduction

The anti–de Sitter spacetime is one of the simplest and most symmetric solutions of Einstein’s field equations including the cosmological constant. For this reason it is important for general relativity and it has its own mathematical relevance. After 1998 this spacetime has drawn attention of high energy physicists due to the conjectured anti–de Sitter space/conformal field theory (AdS/CFT) correspondence suggesting that fundamental particle interactions may be described in geometrical terms with the aid of this spacetime [1]. This idea has given rise to a great number of works on this spacetime.
which take into account only those geometrical features of it that are relevant in this quantum field theory aspect and seem to disregard all its other properties. We shall not discuss the correspondence, we wish only to emphasize that this spacetime, which has become one of the most fundamental spacetimes in physics, has rather bizarre geometrical properties and is weird also from the physical viewpoint. By the latter we mean motions of material (classical) bodies and propagation of light signals in this background.

In this spacetime almost everything is bizarre including its name. In the older literature, particularly mathematical, it was termed *de Sitter spacetime of the second kind* and the current name has been given to it to stress that its geometrical properties are opposite to those of de Sitter spacetime (which was studied earlier and more frequently as it better fits our intuition) though at first sight the two spaces should be similar. (To the best of our knowledge the term appeared for the first time in Ref. [2]). These bizarre properties were discovered by mathematicians rather long ago and exist in the literature which is now not easy to find. This is why this paper is written: its purpose is to collect and present in a possibly systematic way those features of the spacetime which are geometrically and physically important and can be expounded in almost elementary terms without resorting to sophisticated mathematics. In consequence its contents are hardly new, nonetheless we give rather few references. We find it easier to explicitly derive *ab initio* each result than to seek it in the dispersed literature; thus in most cases we cannot pretend to originality. Some very recently published and unpublished results are presented in sections 7, 8, 9 and in Appendix.

We first give the geometrical construction of the spacetime and show some of its global features. Then we focus our interest on motions: what an observer would see if he occurred to be there. We present all these effects in analytic form and our figures are simple diagrams illustrating these expressions. The reader interested in various images of the spacetime is referred to Ref. [3]. We assume that the reader is familiar with fundamentals of general relativity and tensor calculus.

2 Geometrical construction and various coordinate systems

The name of the spacetime will be abbreviated to *AdS space* and the term *space* will mean *spacetime* whenever there will be no risk of confusing it with the *physical space* of the spacetime. AdS space may be defined in any
number of spacetime dimensions equal to or larger than 2. Here we will be dealing only with the physical case of 4 dimensions. First one introduces an auxiliary unphysical 5–dimensional flat space $\mathbb{R}^{3,2}$ with Cartesian coordinates $(U, V, X, Y, Z)$ having two timelike dimensions $U$ and $V$ and three spatial ones $X, Y, Z$. Accordingly, the line element (the square of the spacetime interval) or the metric is

$$ds^2 = dU^2 + dV^2 - dX^2 - dY^2 - dZ^2.$$  \hspace{1cm} (1)

AdS is defined as a 4–dimensional hypersurface in $\mathbb{R}^{3,2}$ given by the equation

$$U^2 + V^2 - X^2 - Y^2 - Z^2 = a^2.$$  \hspace{1cm} (2)

The constant $a$ has dimension of length and determines, as we shall see, the curvature scale of AdS. The hypersurface is the locus of points equidistant (in this metric) to the origin of the Cartesian coordinate system and is legitimately termed pseudosphere. Yet if one takes the equation $U^2 + V^2 - X^2 = a^2$ in the euclidean 3–space $(U, V, X)$, the equation represents a one–sheeted hyperboloid and by this analogy the hypersurface of eq. (2) is also dubbed hyperboloid. One can parametrize points of the pseudosphere by means of four parameters which are so chosen that eq. (2) holds identically. Different parameterizations correspond to distinct coordinate systems on AdS. Here we present 5 different systems and each of them is most suitable for displaying a distinct geometrical feature.

Before doing it a comment on a distinction between reference frames and coordinate systems is in order. A reference frame is an ordered structure of material bodies, either point particles or extended bodies (rigid or not), covering the entire space of the spacetime, together with an infinite set of clocks densely located in the space and remaining at rest with respect to nearby bodies of the frame (in general the clocks and the bodies to which they are attached may move with respect to distant bodies of the frame—in the sense that the distance between them may vary in time\footnote{This definition is intuitive, a precise one is more complicated.}. The reference frame is a physical system which, at least in principle, can be built out of massive particles and which is the essential structure to make any physical measurements and to label spacetime points (events). The fundamental example is any inertial frame of reference in special relativity, being a dense infinite grid of rigid rods, equipped with clocks located at the intersection points of the grid; the whole system is free of accelerations and nonrotating. In a curved spacetime the collection of reference frames must be much wider and clearly there are no inertial frames. Yet a coordinate system is a purely mathematical way of labelling points in the spacetime (in the mathematical language...
it is a coordinate chart on a differential manifold, with all the charts forming the atlas). Each physical reference frame allows to introduce infinite number of coordinate systems. For instance, in an inertial frame, the standard Cartesian coordinates \((t, x, y, z)\), where \(t\) is the physical (i.e. proper) time measured by clocks in this frame, one can introduce coordinates \((t', r, \theta, \phi)\), where \((r, \theta, \phi)\) are curvilinear spatial coordinates defined as given functions of \(x, y, z\), e.g. the spherical ones and \(t' = f(t)\) with monotonously growing function \(f\). We emphasize that to assign coordinates to points in a physical spacetime one must apply a material reference frame and in this sense most of coordinate systems that are used are connected to some frame. However the freedom to mathematically construct coordinate systems is larger than it is allowed by reference frames. This means that there are coordinate systems which are not generated by a reference frame, e.g. null coordinates defined in terms of a „null frame”; these coordinates are useful in some calculations, but they are not measurable.

1. Parameters \((t, r, \theta, \phi)\). Points of AdS in \(\mathbb{R}^{3,2}\) are represented by

\[
\begin{align*}
U &= a \sin \frac{t}{a} \cosh \frac{r}{a}, \\
V &= a \cos \frac{t}{a} \cosh \frac{r}{a}, \\
X &= a \sinh \frac{r}{a} \sin \theta \cos \phi, \\
Y &= a \sinh \frac{r}{a} \sin \theta \sin \phi, \\
Z &= a \sinh \frac{r}{a} \cos \theta,
\end{align*}
\]  

(3)

here \(-\pi a < t < \pi a\), \(r \geq 0\) and \(0 \leq \theta \leq \pi\) and \(0 \leq \phi < 2\pi\) are ordinary angular coordinates on the 2–sphere \(S^2\). Inserting eq. (3) into eq. (2) one finds that it holds identically. Yet inserting eq. (3) into the line element (1) one finds that the square of the interval between two close points, \((t, r, \theta, \phi)\) and \((t + dt, r + dr, \theta + d\theta, \phi + d\phi)\) on the pseudosphere is

\[
ds^2 = \cosh^2 \frac{r}{a} dt^2 - dr^2 - a^2 \sinh^2 \frac{r}{a} (d\theta^2 + \sin^2 \theta d\phi^2).
\]  

(4)

By comparison with the line element in Minkowski space in spherical coordinates \((t, r, \theta, \phi)\) one identifies \(t\) as a time coordinate and \(r, \theta, \phi\) as spatial coordinates and the angles \(\theta\) and \(\phi\) determine the metric on the unit sphere \(S^2\) as \(dl^2 = d\theta^2 + \sin^2 \theta d\phi^2 \equiv d\Omega^2\). Then \(r\) is interpreted as a radial coordinate, but this term does not determine the coordinate uniquely. The radial coordinate in the euclidean 3–space \(E^3\) has two features: if points of a sphere have the radial coordinate \(r = r_0\), then i) the length of the equator is \(2\pi r_0\) (and the area of the sphere is \(4\pi r_0^2\)) and ii) the radius of the sphere, i.e. the distance of each of its points to the centre is \(r_0\). These two features cannot hold together in a curved space and one must choose between them. A space is spherically symmetric (only then the notion of the radial coordinate makes
sense) if there exist coordinates, frequently denoted \((t, r, \theta, \phi)\), such that \(\theta\) and \(\phi\) are the angular coordinates on the sphere and the full metric depends on the angles via only one term \(g_{22}(t, r) \, d\Omega^2\) (actually the correct mathematical definition is more sophisticated and we omit it); then \(r\) deserves the name „radial”. Any transformation \(r' = f(r)\) with \(df/dr \neq 0\) gives rise to another radial variable. In the metric eq. (4) the coordinate \(r\) is equal to the radius of the sphere, whereas the length of the equator is \(2\pi \sinh r/a\). The following two coordinate systems differ from that of eq. (4) only by the choice of the radial coordinate.

Notice that the time \(t\) has dimension of length or is measured in „light seconds”. We do not explicitly introduce the light velocity \(c\) here and throughout the paper each time coordinate \(\tau\) should be interpreted as \(c\tau\).

2. The transformation \(\rho \equiv a \sinh r/a\) yields \(\rho \geq 0\) and

\[
ds^2 = \frac{\rho^2 + a^2}{a^2} dt^2 - \frac{a^2}{\rho^2 + a^2} \, d\rho^2 - \rho^2 \, d\Omega^2. \tag{5}
\]

Here the sphere \(\rho = \rho_0\) has the radius equal to the length of the spatial curve \(dt = d\theta = d\phi = 0\), or

\[
\int_0^{\rho_0} dl = \int_0^{\rho_0} \sqrt{-ds^2} = \int_0^{\rho_0} \frac{a \, d\rho}{\sqrt{\rho^2 + a^2}} = a \ln \left( \frac{\rho_0}{a} + \frac{1}{a} \sqrt{\rho_0^2 + a^2} \right), \tag{6}
\]

whereas the length of the equator is \(2\pi \rho_0\). In these coordinates one sees that the flat Minkowski space arises in the limit \(a \to \infty\), then \(\rho\) becomes the ordinary radial coordinate; it is less easy to notice this limit in the coordinate \(r\) of eq. (4).

3. The „radial” angle \(\psi\) is defined by \(\sinh r/a = \tan \psi\), then \(0 \leq \psi < \pi/2\) and

\[
ds^2 = \frac{dt^2}{\cos^2 \psi} - \frac{a^2}{\cos^2 \psi} (d\psi^2 + \sin^2 \psi \, d\Omega^2), \tag{7}
\]

now both the radius of the sphere and its circumference do not have their familiar forms.

The three coordinate systems represent the same physical reference frame and have common important features. In the defining equation (2) all the five coordinates range from \(-\infty\) to \(+\infty\) and the transformation (3) preserves this range. This implies that the charts (coordinate systems) (4), (5) and (7) cover the entire manifold (spacetime) besides the coordinate singularities such as \(r = \rho = \psi = 0\). The hypersurfaces of simultaneity \(t =\text{const}\) form the
physical 3–spaces with the metric defined as \(dt^2 \equiv -ds^2\) for \(dt = 0\). From eq. (4),
\[
dt^2 = dr^2 + a^2 \sinh^2 \frac{r}{a} \, d\Omega^2. \tag{8}
\]
This is Lobatchevsky (hyperbolic) space \(H^3\) with coordinates \(r, \theta, \phi\). The curvature tensor of \(H^3\) is (Greek indices are spacetime ones, \(\alpha, \beta, \mu, \nu = 0, 1, 2, 3\) and Latin lower case indices are spatial, \(i, j, k = 1, 2, 3\))
\[
R^{(3)}_{ijkl} = \frac{R^{(3)}}{6} (g_{ik}g_{jl} - g_{il}g_{jk}). \tag{9}
\]
The curvature scalar \(R^{(3)} \equiv g^{ik}g^{jl}R^{(3)}_{ijkl}\) for eq. (8) is equal to \(R^{(3)} = -6/a^2\) and this property together with eq. (9) is expressed by saying that the hyperbolic space \(H^3\) is a \textit{space of constant negative curvature}. The space AdS has an analogous property: its four–dimensional Riemann tensor is given by a similar expression,
\[
R_{\alpha\beta\mu\nu} = \frac{R}{12} (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}), \tag{10}
\]
where its metric \(g_{\mu\nu}\) is taken either from eq. (4), (5) or (7) (or any other coordinate system) and the 4–dimensional curvature scalar \(R = g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu} = 12/a^2\). Notice that the metric signature is chosen here as (+ − − −) since it is more suitable for dealing with timelike worldlines of massive particles, whereas in classical field theory the opposite signature is commonly used. Altering the signature results in the change of sign of the scalar \(R\) and this is why AdS space is frequently characterized as a \textit{spacetime of constant negative curvature}.

The metric of eqs. (4), (5) and (7) is time independent, what means that AdS space is \textit{stationary}. Furthermore, this spacetime is \textit{static}, i. e. the time inversion \(t \to -t\) does not change the form of the metric. The gravitational field of a motionless star is static (for instance Schwarzschild field), yet a uniformly rotating star generates a stationary field: it is time independent, but the time inversion makes the star rotate in the opposite direction and its gravitational field is changed (e. g. Kerr spacetime).

Now we introduce two further coordinate systems describing two different reference frames.

4. The Poincaré coordinates \((t', x, y, z)\). Instead of eq. (3) one applies
\[
U = \frac{1}{2z} (a^2 + x^2 + y^2 + z^2 - t'^2), \quad V = a \frac{t'}{z}, \quad X = a \frac{x}{z},
\]
\[
Y = a \frac{y}{z}, \quad Z = \frac{1}{2z} (a^2 - x^2 - y^2 - z^2 + t'^2). \tag{11}
\]
then the metric is
\[ ds^2 = \frac{a^2}{z^2} (dt'^2 - dx^2 - dy^2 - dz^2). \] (12)

Here \( t', x \) and \( y \) are real and \( z > 0 \). From eq. (11) one gets \( U + Z = a^2/z > 0 \), what implies that these coordinates cover only one half of AdS manifold. The other half needs a similar chart with \( z < 0 \). The reference system given in eq. (12) is moving with respect to that given in eq. (4) and their coordinate times, \( t \) and \( t' \) are distinct. The expression in the round brackets in eq. (12) represents the metric of flat Minkowski space expressed in Cartesian coordinates of an inertial reference frame. (At this moment we disregard the derivation of eq. (12) and discuss only its final form.) Thus the metric of AdS is proportional to the metric of the flat spacetime, the proportionality factor is a scalar function of the coordinates. This is a geometrical property of AdS space, valid in all coordinate systems. The Poincaré coordinates are distinguished by making this property explicit; it is rather hard to recognize it in other coordinates. By ,,hard" we mean that if one uses only the three above mentioned coordinate systems (or any other ones) and is unaware that the spacetime is the pseudosphere in \( \mathbb{R}^{3,2} \) and that it may be parametrized by the Poincaré coordinates, then finding out the transformation to the metric (12) is really difficult. Yet showing this property is actually quite easy if one uses the Weyl tensor: this tensor is related to the Riemann curvature one and if the proportionality property holds for a spacetime, then this tensor (computed in any coordinate system) vanishes. In short, if the Weyl tensor is zero, then the metric is proportional to the flat one. In this article we shall not apply this tensor. If two spacetimes, \( M \) and \( \bar{M} \), have their metric tensors (expressed in the same coordinates) proportional, \( \bar{g}_{\mu\nu}(x^\alpha) = \Omega^2(x)g_{\mu\nu}(x^\alpha) \), where \( \Omega(x) > 0 \) is a scalar function, then the two spacetimes are conformally related. Let two conformally related metric tensors be introduced on the same spacetime (considered as a ,,bare" manifold of points), then distances between any pair of points expressed in terms of these metrics will be different, yet the angles between any two curves are the same in both the metrics and this explains why the property is called conformity. AdS is conformally flat.

The space \( t' = \text{const} \) in Poincaré coordinates is conformal to a half of euclidean space \( E^3 \).

5. Finally one takes the following parametrization:

\[
\begin{align*}
U &= a \cos \frac{\tau}{a}, & V &= a \sin \frac{\tau}{a} \cosh \chi, & X &= a \sin \frac{\tau}{a} \sinh \chi \sin \theta \cos \phi, \\
Y &= a \sin \frac{\tau}{a} \sinh \chi \sin \theta \sin \phi, & Z &= a \sin \frac{\tau}{a} \sinh \chi \cos \theta,
\end{align*}
\] (13)
where $0 < \tau < \pi a$ and the radial coordinate $\chi > 0$ is dimensionless. The metric is now time dependent,

$$ds^2 = d\tau^2 - a^2 \sin^2 \frac{\tau}{a} (d\chi^2 + \sinh^2 \chi d\Omega^2).$$

(14)

The coordinates cover only a part of the spacetime since $-a < U < a$ and $V > 0$. The static nature of AdS becomes now invisible and at first sight these coordinates seem to be a mere complication. We shall see below, however, that $(\tau, \chi, \theta, \phi)$ are *comoving coordinates* and reveal an important property of motion of free particles. By comparing eqs. (4), (8) and (14) one sees that the space $\tau = \text{const}$ is the Lobatchevsky space $H^3$.

One may introduce a number of other coordinates, but the spherical angles $\theta$ and $\phi$ are never altered.

### 3 Global properties of the spacetime

AdS space as the pseudosphere in the ambient $\mathbb{R}^{3,2}$ is unbounded in each direction. Yet from eq. (3) one sees that the times $U$ and $V$ are parametrized by a *periodic* time $t$ on the pseudosphere: the two quadruples, $q_1 = (t, r, \theta, \phi)$ and $q_2 = (t + 2\pi a, r, \theta, \phi)$ represent the same point of it. This means that in AdS space, defined as a manifold of points $(t, r, \theta, \phi)$, one must identify $q_1$ and $q_2$. More precisely, the range of time is $-\pi a \leq t < \pi a$ and points $(-\pi a, r, \theta, \phi)$ and $(\pi a, r, \theta, \phi)$ are identified. In other terms, the coordinate lines of time $t$, where $r, \theta, \phi = \text{const}$, are closed—they form circles $S^1$. On the other hand the hyperbolic space $H^3$ has topology (in the sense of geometrical topology) of euclidean $\mathbb{R}^3$, then the entire AdS has the product topology $S^1 \times \mathbb{R}^3$. Closed timelike curves are very unpleasant from the physical viewpoint. Though it may be argued that they do not break the causality and need not to give rise to various paradoxes (,,to kill one’s own grandfather”), it is desired to remove them if possible. This may be achieved due to the fact that the metric (4) (as well as eq. (5) and (7)) is time independent and the periodicity in time is invisible. One simply unwraps all time circles $S^1$ and extends them in the line of real numbers, now $-\infty < t < +\infty$. Geometrically this means making infinite number of turns around the pseudosphere in its time direction. To avoid the periodic identification of points in this direction one discards the pseudosphere model and introduces a new spacetime: one discards the whole derivation of eq. (4) based on employing the ambient space $\mathbb{R}^{3,2}$ and constructing the pseudosphere in it. Instead one defines the manifold as a set of points $(t, r, \theta, \phi)$ with $-\infty < t < \infty$ equipped with the metric in eq. (4). The coordinate lines of time have now topology $\mathbb{R}^1$ and
the entire spacetime has topology $\mathbb{R}^4$. This spacetime is called a *universal covering space* of anti–de Sitter space, in short CAdS. In what follows we shall be mainly dealing with CAdS space (unless otherwise is stated). It will be quite surprising to see that replacing AdS by CAdS space is a merely verbal operation and the latter inherits most of the features of the former.

In the search for symmetries of CAdS space one may resort to the pseudosphere since symmetries are local isometric mappings of the space onto itself preserving the form of the metric and do not depend on the topology. Like the ordinary sphere in euclidean space, the pseudosphere has as its symmetries the rotational symmetry of the ambient space, in this case this is $SO(3, 2)$ group, which is analogous to $SO(3, 1)$ Lorentz group of Minkowski space. This group has 10 parameters, the maximal number of symmetries in four dimensions; equally high symmetry is characteristic for Minkowski and de Sitter spacetimes. CAdS is *maximally symmetric*.

An important global property of a spacetime is its structure at infinity. This is termed *conformal structure* and has been developed in an extended subject presented in advanced textbooks [4, 5]. Here we need only one, the simplest and most intuitive notion. In Minkowski spacetime the boundary of the space $t = \text{const}$ for $r \to \infty$ (it is convenient here to use the spherical coordinates $(t, r, \theta, \phi)$) is the *sphere at infinity*. The collection of these spheres for all values of time forms a 3–dimensional hypersurface $\mathcal{J}$ being a boundary of the spacetime. To investigate the geometry of $\mathcal{J}$ one considers a special metric conformally related to the flat one. The resulting geometry of $\mathcal{J}$ is somewhat complicated, whereas the corresponding boundary of CAdS space, termed *spatial infinity* and also denoted by $\mathcal{J}$, is geometrically simpler. The coordinates $(t, \psi, \theta, \phi)$ of eq. (7) are most suitable for dealing with the infinity $\psi = \pi/2$. On CAdS space one introduces a new metric conformally related to that of eq. (7), $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with $\Omega = \cos \psi$. In this way one gets a new spacetime with the metric

$$d\bar{s}^2 = dt^2 - a^2 (d\psi^2 + \sin^2 \psi d\Omega^2).$$

(15)

The new spacetime is larger than CAdS space since points $\psi = \pi/2$ are now of its regular points, whereas the metric (7) is divergent there. Points $\psi = \pi/2$ of the new spacetime form the *conformal spatial infinity* $\mathcal{J}$ of CAdS space. This hypersurface has the metric (15) with $\psi = \pi/2$,

$$d\bar{s}^2 = dt^2 - a^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(16)

In a spacetime any hypersurface defined by an equation $f(x^\alpha) = 0$ belongs to one of three classes of hypersurfaces depending on the vector $n^\alpha$ orthogonal
to it: if \( n^\alpha \) is timelike, \( n^\alpha n_\alpha > 0 \) (according to the signature \((+−−−)\)), then

the hypersurface is spacelike, if \( n^\alpha \) is spacelike, \( n^\alpha n_\alpha < 0 \), the hypersurface is
timelike (and is a 3–dimensional spacetime on its own), finally, if \( n^\alpha \) is
null, \( n^\alpha n_\alpha = 0 \), it lies on the hypersurface to which it is orthogonal and the
latter is null. \( n^\alpha \) is the gradient of \( f \), \( n_\alpha = \partial f/\partial x^\alpha \). In general the type
of a hypersurface may change from point to point. In GR we try to avoid
this pathological behaviour and only consider hypersurfaces which are of the
same type everywhere. In the geometry of eq. (15) one has \( f = \psi - \pi/2 \),
\( n_\alpha = (0, 1, 0, 0) \) and \( n^\alpha n_\alpha = \tilde{g}^{\alpha\beta} n_\alpha n_\beta = \tilde{g}^{11} = -1/a^2 < 0 \). The conformal
infinity \( J \) of CAdS is a timelike hypersurface and, as is seen from eq. (16),
it has topology \( R^1 \times S^2 \), where \( S^2 \) is the boundary at infinity of the space
\( H^3 \). The conformal boundaries of Minkowski and CAdS spaces are different.
A null vector in the given metric remains null in all other metrics conformally
related to that, hence a null line remains null. CAdS space is conformally
flat (eq. (12)), therefore the light cones formed by light rays emitted from
any point of that spacetime are the same as those in Minkowski space. In
particular the straight lines at \( ±45 \) degrees in the spacetime diagram repre-
sent null rays (radially directed photon worldlines).

Since the infinity \( J \) is actually timelike, the effect is that far future cannot
be predicted in CAdS space. Suppose one is interested in finding a unique
solution to Maxwell equations. To this end one chooses a spacelike hyper-
surface \( S \), given by \( t = t_0 \) in some coordinate system, gives the initial data
on it (values of the electric and magnetic fields at points of \( S \)) and evolves
the data by means of Maxwell equations to the future. The value of the elec-
tromagnetic field cannot be predicted in this way in far future since external
electromagnetic signals, not included in the initial data on \( S \), will interfere.
As is well known, the field is uniquely determined by the data on \( S \) only
in the spacetime region which on a two–dimensional diagram (see Fig. 1)
is represented by a „triangle” whose base is \( S \) and the other two sides are
future directed null lines (photon paths) emitted from the boundary points
of \( S \). This region is termed the domain of dependence in future of \( S \), \( D^+(S) \),
or the future Cauchy development of \( S \). In Minkowski spacetime the hyper-
surface \( S \) may be extended to the entire physical space (in an inertial frame)
\( t = t_0 \), then the electromagnetic field (and other physical fields) is uniquely
determined for arbitrarily distant future (and past), i. e. for all times. This
is possible because the conformal infinity consists there of two null cones
and no external signal can enter the spacetime from outside (i. e. from \( J \))
without crossing the space at \( t = t_0 \). Also in many curved spacetimes there
exist spacelike hypersurfaces (being sets of simultaneous events with respect
to some coordinate time) which, if treated as initial data surfaces, allow to
predict to whole future and past.
Figure 1: Two–dimensional representation of CAdS space for $\theta = \pi/2$. Besides points with $\psi = 0$ where there is a coordinate singularity, each point $(t, \psi)$ represents a half circle in $\phi$. The boundary cylinder $J$ is depicted as one line for some $\phi$ and as the antipodal line at $\phi + \pi$. The initial data hypersurface is the whole space $H^3$ at $t = t_0$. The null boundaries $H^+$ of $D^+(H^3)$ are null future lines emanating from the boundary infinity $S^2$ of $H^3$. Any electromagnetic signal $k$ entering CAdS from $J$ for $t > t_0$ moves outside $D^+(H^3)$ and affects the solution along its path.

This is not the case of CAdS space. In Fig. 1 the diagram in coordinates $(t - \psi)$ is presented. $\theta = \pi/2$ and each point of the diagram to the left and right of the line $\psi = 0$ represents a half–circle of coordinate $\phi$. The line $\psi = 0$
consists of single points because the coordinate system is singular there and the spheres of \((\theta, \phi)\) shrink to a point. The boundary \(J\) is shown as two lines, one for some fixed value of \(\phi\) and the other as opposite to it, \(\phi + \pi\). One takes the whole space \(H^3\) for some \(t_0\) as the initial data surface, then any physical field is uniquely determined in the domain of dependence \(D^+(H^3)\) bounded in the future by two null hypersurfaces \(H^+\) made of null rays emanating from the sphere being the intersection of \(H^3\) with \(J\). The region \(D^+(H^3)\) cannot cover the whole CAdS because any light signal emitted from \(J\) at \(t > t_0\) will perturb the field. Physics in CAdS is unpredictable. This is particularly troublesome for quantizing fields propagating in this world [6].

4 Uniformly accelerating observers

The static coordinate system of eqs. (4), (5) and (7) may be given a physical interpretation by showing that observers at rest, \(r, \theta, \phi = \text{const}\), are actually uniformly accelerating ones [7]. The notion of uniform acceleration is taken directly from special relativity (SR). In SR consider a motion of a particle in a fixed inertial frame of reference denoted by LAB. In this frame the particle has 4–velocity \(u^\alpha = dx^\alpha/ds = (\gamma, \gamma v/c)\) and 4–acceleration

\[
    w^\alpha \equiv \frac{du^\alpha}{ds} = \frac{1}{c^2} \gamma^4 \left[ \frac{1}{c} \mathbf{v} \cdot \mathbf{a} \left( \frac{1}{c} \mathbf{v} \cdot \mathbf{a} \right) \frac{\mathbf{v}}{c} + \frac{1}{\gamma^2} \mathbf{a} \right],
\]

where the Lorentz factor is \(\gamma = (1 - v^2/c^2)^{-1/2}\) and \(\mathbf{a} = d\mathbf{v}/dt\) is the ordinary 3–acceleration measured in LAB. The identity \(u^\alpha u_\alpha = \eta_{\alpha\beta} u^\alpha u^\beta = 1\), where \(\eta_{\alpha\beta} = \text{diag}[+1, -1, -1, -1]\) is the Minkowski metric, implies \(\eta_{\alpha\beta} w^\alpha w^\beta = 0\) and \(w^\alpha\) is a spacelike vector with the squared length

\[
    \eta_{\alpha\beta} w^\alpha w^\beta = -\frac{\gamma^4}{c^4} \left[ \gamma^2 \left( \frac{1}{c} \mathbf{v} \cdot \mathbf{a} \right)^2 + \mathbf{a}^2 \right] < 0.
\]

Whereas LAB is an arbitrary frame, the particle has a distinguished inertial frame, the local proper frame in which it is momentarily at rest. In the proper frame the particle has \(\mathbf{v} = \mathbf{0}\) and the acceleration is denoted by \(\mathbf{a} = \mathbf{A}\); in consequence \(w^\alpha w_\alpha = -A^2/c^4\). The particle is uniformly accelerated if \(\mathbf{A} = \text{const}\) and in the case of a one–dimensional motion it amounts to \(w^\alpha w_\alpha = \text{const} < 0\). In any curved spacetime again \(u^\alpha = dx^\alpha/ds\) and the acceleration vector is the absolute derivative with respect to \(s\) of the velocity vector,

\[
    w^\alpha = \frac{D}{ds} \frac{dx^\alpha}{ds} = \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds},
\]

12
where $\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols for the metric $g_{\alpha\beta}(x^\mu)$. Again $g_{\alpha\beta}u^\alpha u^\beta = 1$ and $g_{\alpha\beta}u^\alpha w^\beta = 0$. Take the CAdS metric as in eq. (7) and a static observer with $\psi = \psi_0 > 0$ and $\theta, \phi =$ const. Then along its worldline $ds = dt/ \cos \psi_0$ and $u^\alpha = (dx^\alpha/dt)(dt/ds) = [\cos \psi_0, 0, 0, 0]$. In the static coordinates the observer remains at rest and is uniformly accelerated iff $w^\alpha w_\alpha = \text{const} < 0$. One needs not to compute the Christoffel symbols since the covariant components of the acceleration are given by

$$w_\alpha = D ds u_\alpha = \frac{d}{ds} (g_{\alpha\beta} u^\beta) - \frac{1}{2} g_{\mu\nu,\alpha} u^\mu u^\nu. \quad (20)$$

One gets $w_\alpha = -\delta^1_\alpha \tan \psi_0$ and $w^\alpha = +\frac{1}{a^2} \delta^1_\alpha \sin \psi_0 \cos \psi_0$, then $w^\alpha w_\alpha = -\frac{1}{a^2} \sin^2 \psi_0$ and identifying this expression with $-A^2$ (one returns to $c = 1$) one finds that each static observer is subject to a uniform acceleration equal to $\frac{1}{a} \sin \psi_0$. The acceleration monotonically grows with $\psi_0$ and reaches maximum at the spatial infinity. For $\psi_0 = 0$ the acceleration vanishes.

### 5 Geodesic lines

In a curved spacetime the geodesic lines play the same role as straight lines do in euclidean spaces. The straight line has two fundamental properties: i) the vector tangent to it at any point, when parallelly transported along it to any other point, remains tangent to it, and ii) it is the shortest line between any pair of its points. The second feature cannot be implemented without some changes in a spacetime. Already in Minkowski spacetime a straight timelike line is the longest one between its points. The timelike geodesic maximizes the spacetime interval between its points. Along the null geodesic, as along any other null curve, the interval between any pair of points, is zero. Only the spacelike geodesic is the shortest line joining two points. Yet the first property is transferred unaltered into any spacetime: the geodesic is such a line that for any parametric representation of the line, $x^\alpha = x^\alpha(v), \alpha = 0, 1, 2, 3$, the acceleration vector (i.e. the absolute derivative with respect to $v$ of the tangent vector) is proportional to the tangent vector,

$$\frac{D}{dv} \frac{dx^\alpha}{dv} \equiv \frac{d^2 x^\alpha}{dv^2} + \Gamma^\alpha_{\mu\nu} \left( \frac{dx^\mu}{dv} \frac{dx^\nu}{dv} \right) = h(v) \frac{dx^\alpha}{dv}, \quad (21)$$

where $h(v)$ is a scalar function depending on the choice of the parameter $v$. The proportionality feature is exactly as in mechanics: a body in a rectilinear motion may either move uniformly, if the temporal parameter $t$ is appropriately chosen, or move non-uniformly with respect to a different parameter $t'$, say $t = \ln t'$. Guided by this analogy one can show that there exists such a
parametrization of the geodesic that the acceleration vanishes, \( h(v) \equiv 0 \), then \( v \) is termed canonical parameter. For a timelike geodesic the canonical parameter coincides with the arc length (the proper time), \( v = s \); for spacelike geodesics the parameter denoted by \( l \) is defined as \( dl^2 = -ds^2 > 0 \) and for null ones the parameter \( \sigma \) has no simple geometrical or physical interpretation. In practice one replaces eq. (21) (for the canonical \( v \)) by the equivalent form which avoids computing \( \Gamma^\alpha_{\mu\nu} \) symbols and arises from eq. (20),

\[
\frac{d}{dv} \left( g_{\alpha\beta} \frac{dx^\beta}{dv} \right) - \frac{1}{2} g_{\mu\nu,\alpha} \frac{dx^\mu}{dv} \frac{dx^\nu}{dv} = 0.
\]

(22)

The behavior of the three types of geodesics exhibits the fundamental geometrical properties of the spacetime under consideration.

We begin studying geodesics in CAdS space with the spacelike ones. We use them to determine the distance from any given point to the spatial infinity \( J \). The distance is defined as the length of a spacelike geodesic joining the given point \( P_0 \) at \( t = t_0 \) to any simultaneous point at \( J \). We use the reference frame in which the metric is explicitly static, eqs. (4), (5) or (7), hence we expect that all points of the geodesic are simultaneous, \( t = t_0 \). Since CAdS is spherically symmetric, we expect that the geodesic is radial, i.e. the angles \( \theta \) and \( \phi \) are constant along it and only the radial coordinate is variable. One then need not at all to solve the geodesic equation, it suffices to compute the length of the radial line. Using e.g. eq. (4) one gets the distance from \( r_0 \) to \( r_1 \) equal \( l(r_0, r_1) = r_1 - r_0 \). The distance from any internal point to \( J \) is infinite, as it should be expected.

From the explicit form of the geodesic equation one infers that circular spacelike geodesics, \( r = \text{const} > 0 \) and \( \theta = \pi/2 \), do not exist.

### 6 Null geodesics

Interpreting any null geodesic as a worldline of a photon (being in the classical approximation a point particle) and the tangent vector as the wave vector, one writes \( x^\alpha = x^\alpha(\sigma) \) and \( dx^\alpha/d\sigma = k^\alpha \), then the geodesic equation reads

\[
\frac{d}{d\sigma} \left( g_{\alpha\beta} k^\beta \right) - \frac{1}{2} g_{\mu\nu,\alpha} k^\mu k^\nu = 0.
\]

(23)

The canonical parameter is determined up to a linear transformation (change of units), hence one may assume that \( \sigma \) is dimensionless. One knows from section 2 that CAdS space is conformally flat. In general, if two spacetimes
are conformally related, then they have the same null geodesics (in the sense of the same null lines). In fact, assume that $\bar{g}_{\mu \nu} = \Omega^2 g_{\mu \nu}$ and eq. (23) holds. Then making an appropriate transformation of the canonical parameter, $\bar{\sigma} = f(\sigma)$ with $f' > 0$, one shows by a direct calculation that the transformed wave vector $\bar{k}^\alpha = dx^\alpha/d\bar{\sigma}$ satisfies the same equation for the rescaled metric,

$$\frac{d}{d\bar{\sigma}}(\bar{g}_{\alpha \beta} \bar{k}^\beta) - \frac{1}{2} \bar{g}_{\mu \nu,\alpha} \bar{k}^\mu \bar{k}^\nu = 0.$$  

(24)

The function $f(\sigma)$ is determined by $\Omega$ via a differential equation. Applying the Poincaré coordinates, eq. (11) and (12), one sees that null geodesics of CAdS coincide with those of Minkowski space in coordinates $x^\alpha = (t', x, y, z)$; these are straight lines $x^\alpha = a^\alpha \bar{\sigma} + \text{const}$, where a constant vector $a^\alpha = (a^0, a)$ is null, $(a^0)^2 - a^2 = 0$, and $-\infty < \bar{\sigma} < \infty$. Instead of determining $\sigma = f^{-1}(\bar{\sigma})$ we directly solve the geodesic equation in the global coordinate system, eq. (4). We consider a radial geodesic $x^\alpha = (t(\sigma), r(\sigma), \pi/2, 0)$, then $k^\alpha = (dt/d\sigma, dr/d\sigma, 0, 0)$. Since $(\partial/\partial t) g_{\mu \nu} = 0$, eq. (23) for $\alpha = 0$ is immediately integrated,

$$\frac{d}{d\sigma} \left( k^0 \cosh^2 \frac{r}{a} \right) = 0 \Rightarrow \frac{dt}{d\sigma} \cosh^2 \frac{r}{a} = \text{const} \equiv Ea > 0,$$

(25)

where a dimensionless $E$ is proportional to the conserved energy of the photon. The equations for $\theta$ and $\phi$ hold identically and the second order equation for $r$ is replaced by the constraint

$g_{\alpha \beta} k^\alpha k^\beta = 0 = l^2 \cosh^2 r/a - \dot{r}^2$

working as an integral of motion and assuming that the geodesic emanates from $r = r_0 \geq 0$ for $\sigma = 0$ with $\dot{r} = dr/d\sigma > 0$, and employing eq. (25) one gets

$$E\sigma = \sinh \frac{r}{a} - \sinh \frac{r_0}{a}, \quad r = a \ln \left( E\sigma + \sinh \frac{r_0}{a} + \sqrt{(E\sigma + \sinh \frac{r_0}{a})^2 + 1} \right).$$

(26)

It is convenient to use also the angular radial variable of eq. (7), $\tan \psi = \sinh r/a$, then one finds $E\sigma = \tan \psi - A$, where $A \equiv \sinh r_0/a = \tan \psi_0$. Due to the conformal invariance of the null geodesic equation (23), the solution is independent of the cosmological constant $\Lambda = -3/a^2$. A radial photon emanating from any point reaches the spatial infinity $\mathcal{J}$ for $\sigma \to \infty$, as expected. This means that $\mathcal{J}$ consists of endpoints of future and past directed radial null geodesics and coincides with the set of endpoints of radial spacelike geodesics. Yet integrating eq. (25) one gets $t(\sigma)$ and the simplest expression arises if the variables $\psi$ and $\psi_0$ are used,

$$\frac{dt}{d\sigma} = Ea[(E\sigma + A)^2 + 1]^{-1} \quad \text{and} \quad t - t_0 = a \arctan(E\sigma + A) - a\psi_0,$$

(27)
or \( t(\sigma) - t_0 = a(\psi - \psi_0) \). The light cone in the variables \((t, a\psi)\) consists of straight lines inclined at 45°, as in Minkowski space. The coordinate time interval of the photon flight from \( \psi = \psi_0 \) to \( \mathcal{J} \) is finite and its maximum value is \( t - t_0 = \pi a/2 \) for \( \psi_0 = 0 \). Let the photon be emitted from point \( A, t = t_A \) and \( \psi = \psi_0 \), moves radially outwards, reaches the spatial infinity where it is reflected by a mirror and returns to \( \psi = \psi_0 \) at the event \( B \) at \( t = t_B \), Fig. 2. The time of the flight is finite, \( t_B - t_A = (\pi - 2\psi_0)a \), though the distance from \( \psi_0 \) to \( \mathcal{J} \) (measured along a spacelike radial geodesic) is infinite, \( l(\psi_0, \pi/2) = \infty \). Also the proper time \( s \) measured by a clock staying at \( \psi = \psi_0 \) between the emission and return of the photon is finite; from eq. (7) one has

\[
ds^2 = \frac{dt^2}{\cos^2 \psi_0} \implies s(A, B) = \frac{t_B - t_A}{\cos \psi_0} = \frac{\pi - 2\psi_0}{\cos \psi_0} a.
\] (28)

\( s(A, B) \) decreases from \( \pi a \) for \( \psi_0 = 0 \) to \( 2a \) for \( \psi_0 \to \pi/2 \).

What kind of a curve in the ambient space \( \mathbb{R}^{3,2} \) is the radial null geodesic of eqs. (26) and (27)? By rotations of the spheres one can always put \( \theta = \pi/2 \) and \( \phi = 0 \) along the geodesic, then applying eqs. (3) one finds the parametric description of the geodesic in the ambient space (we employ the relationships between functions \( \arctan, \arcsin \) and \( \arccos \)),

\[
U = Ea\sigma, \quad V = a, \quad X = Ea\sigma, \quad Y = Z = 0.
\] (29)

This is a straight line which is null, since the tangent five–vector 
\((dU/d\sigma, \ldots, dZ/d\sigma) = E a(1, 0, 1, 0, 0)\) is null in the metric of eq. (1). It is well known that in euclidean 3–space the one–sheeted hyperboloid \( x^2 + y^2 - z^2 = 1 \) contains a 1–parameter family of straight lines, which are geodesic curves on both the hyperboloid and in the space. Analogously, AdS space contains a 1–parameter family of null geodesics (the parameter is the energy \( E \)) being null straight lines of the ambient \( \mathbb{R}^{3,2} \).

In Schwarzschild spacetime generated by a static star or a static black hole there exists one (unstable) circular null geodesic: if a photon is emitted from a point on the equator of the sphere with the radial coordinate \( r_0 = 3GM/c^2 \) in a direction tangent to the equator, the gravitational field of the central body of mass \( M \) will capture it and the photon will revolve for ever around it on the circular orbit. In CAdS space one verifies, using the metric of eq. (4), that circular null geodesics do not exist for any finite value of the radial variable \( r \). In fact, the radial component of the geodesic equation, i.e. the \( \alpha = 1 \) component of equation (23), together with the integral of motion \( g_{0\beta}k^\alpha k^\beta = 0 \) show that the assumption \( r = \text{const} = r_0 \) is consistent.
Figure 2: The photon is emitted from A, $t = t_A$ and $\psi = \psi_0$, radially outwards, reaches the spatial infinity at C, where it is reflected by a mirror and returns to $\psi = \psi_0$ at B, $t = t_B$. The time of flight, $t_B - t_A$, and the proper time $s(A, B)$ are both finite and bounded from above, though the distance to $\mathcal{J}$ is $l(\psi_0, \pi/2) = \infty$.

only if $\sinh r_0 = \cosh r_0$, or $r_0 = \infty$. Formally, a circular null geodesic exists only at the spatial infinity.

Properties of null geodesics are to some extent related to the problem of stability of CAdS space. Spacetimes that approach CAdS one at infinity are called asymptotically CAdS spacetimes (a rigorous definition is quite sophisticated). It has been shown that CAdS space is a ground state for asymptotically CAdS spacetimes, in the same sense as Minkowski space is the ground
state for spacetimes which are asymptotically flat. In any field theory the
ground state solution must be stable against small perturbations, otherwise
the theory is unphysical. For Minkowski space it has been proven after long
and sophisticated investigations that the space is stable since sufficiently
small initial perturbations vanish in distant future due to radiating off their
energy to infinity. The spatial infinity $J$ of CAdS space actually is a timelike
hypersurface and any radiation may either enter the space through $J$ or es-
cape through it. It is therefore crucial for the question of stability to correctly
choose a boundary condition at infinity. Most researchers assume reflective
boundary conditions: there is no energy flux across the conformal boundary
$J$, in other terms the boundary acts like a mirror at which outgoing fields
(perturbations) bounce off and return to the interior of the spacetime. Under
this assumption P. Bizoń recently received a renowned result: CAdS space
is unstable against formation of a black hole for a large class of arbitrarily
small perturbations [8].

We have a critical remark to this outcome. The instability is due to the
presence of matter in the form of the linear massless scalar field and it is
physically relevant provided it is not a peculiarity specific to the scalar field.
The instability must also develop for dust matter and electromagnetic per-
turbations (this has not been checked yet due to computational difficulties).
Suppose that the instability is triggered by high frequency electromagnetic
waves of small amplitude, these may be viewed as photons. Consider a pho-
ton belonging to the perturbation. As is depicted in Fig. 2 the outgoing
photon is subject at point C to the reflective boundary conditions and is
forced to come back. Since CAdS space is maximally symmetric, the pho-
ton has conserved both its energy and linear momentum. For the incoming
(returning) photon the spatial momentum has the opposite sign to that of
the outgoing photon and this is possible only if the photon meets at C a
physical mirror and is bounced off it. In other terms the reflective boundary
conditions mean that CAdS space is equivalent to a box with material walls.
Gravitational instability of perturbations closed in a box is less surprising.

7 Timelike geodesics

Consider a cloud of free test particles, each of unit rest mass, whose own
gravitational field is negligible, which move in CAdS space. The notion of
„negligible” is intuitively clear, but in the framework of GR it is based on
a deeper reasoning. First, in GR a point particle does not exist: a point
particle with a mass, no matter how small, actually is a black hole with
the event horizon and diverging curvature near the singularity. Therefore
a „point particle” is an approximation and means an extended body of a
diameter $d$ and one assumes that all distances under consideration have scale
$L \gg d$. In this sense GR is similar to celestial mechanics where planets are
viewed as pointlike objects provided the error $L$ of determining their orbits
is much larger than their diameters. If $L \approx d$ one must take into account the
physical nature of the object. Second, one compares the gravitational field
(the curvature) of the particle of mass $m$, computed at the distance $L$ from
it, to the external gravitational field, in the present case being the CAdS
space curvature. If the external curvature is much larger than that of each
particle, their gravitation is negligible and the particles are viewed as „test”
one. (In consequence, in the flat spacetime, particles are free and test ones
only if their gravitational interactions are completely neglected.) Assuming
that this is the case, each particle moves on a timelike geodesic of the CAdS
metric. Let one choose a reference frame adapted to the cloud: the frame
is comoving with the particles, what means that every particle has constant
spatial coordinates, then its worldline coincides with one coordinate time line.
Though the particles are „motionless” in this frame, the distances between
them vary in time as the cloud expands or shrinks, hence in the frame the
metric is time dependent. In this comoving frame the CAdS metric has the
form given in eq. (14). In fact, let a particle of the cloud be at rest in the
coordinate system $(\tau, \chi, \theta, \phi)$. Then along its worldline there is $ds^2 = d\tau^2$ or
$\tau - \tau_0 = s$ and the tangent vector is $u^a = dx^a/ds = dx^a/d\tau = (1, 0, 0, 0)$. The
timelike geodesic equation, according to eqs. (21) and (22), is
\[
\frac{d}{ds} \left( g_{\alpha\beta} u^\beta \right) - \frac{1}{2} g_{\mu\nu,\alpha} u^\mu u^\nu = 0
\]
and for this worldline it holds identically since it reduces to
\[
\frac{d}{ds} g_{00} - \frac{1}{2} g_{00,\alpha} \equiv 0.
\]
The curves $\tau - \tau_0 = s$ and $\chi, \theta, \phi = \text{const}$ are timelike geodesics and as such
these are the worldlines of free particles (actually the coordinate time lines
in the comoving system are geodesic worldlines also in the case of a self–
gravitating cloud of particles, but then the metric differs from that of CAdS
space). One notices that these geodesics are orthogonal to the physical spaces
$H^3$ given by $\tau = \text{const}$; this is why this comoving frame is termed Gaussian
normal geodesic (GNG) system. Furthermore the time coordinate $\tau$ is the
physical time measured by good clocks travelling along these geodesics since
it is equal to intervals of proper time, $\Delta \tau = \Delta s$.

In a generic spherically symmetric spacetime one usually singles out the
simplest timelike geodesic curves, radial and circular. These are geometri-
cally distinguished by the symmetry centre and their distinction is frame
independent. (In de Sitter space the circular geodesics do exist, but they are revealed in the GNG coordinates, whereas the frequently used static coordinates, covering only a half of the manifold, deceptively suggest that circular geodesics are excluded) \[9\]. It is therefore rather astonishing that in CAdS space the difference between radial and circular geodesics is merely coordinate dependent and geometrically they form the same curve. Furthermore, each ,,generic” timelike geodesic may be transformed into a circular or a radial one. The proof of that using ,,internal” methods, that is the four–dimensional metric of the space, is complicated and we shall apply the external approach based on the use of the ambient flat space $R^{3,2}$.

To this end we resort to AdS space as a pseudosphere in $R^{3,2}$ since this piece of CAdS is sufficient (recall that CAdS is the infinite chain of AdS spaces opened in the time direction and glued together). One describes any timelike geodesic $G$ on AdS space as a curve in the embedding $R^{3,2}$. Using the coordinates $X^A = (U,V,X,Y,Z)$, $A = 1, \ldots, 5$, the curve $G$ is parametrized by its length, $X^A = X^A(s)$. Clearly $G$ is not a geodesic (a straight line) of the flat ambient space $R^{3,2}$. The geodesic equation follows from a variational principle and its derivation may be performed in the ambient space, the result reads (see e. g. \[9\])

$$\ddot{X}^A + \frac{1}{a^2}X^A = 0,$$

(31)

where $\dot{X}^A = dX^A/ds$. These are five decoupled equations and their general solution depends on ten arbitrary constants. The solution describes the geodesic $G$ if it satisfies two constraints, the definition of AdS space given in eq. (2) and the normalization of the velocity five–vector $\dot{X}^A$, $\dot{U}^2 + \dot{V}^2 - \dot{X}^2 - \dot{Y}^2 - \dot{Z}^2 = 1$. The geodesic is then

$$X^A(s) = q^A \sin\left(\frac{s}{a} + c\right) + p^A \cos\left(\frac{s}{a} + c\right),$$

(32)

where $c$ is an integration constant and two constant directional five–vectors $q^A$ and $p^A$ (constancy of the two vectors has a geometrical meaning because the ambient space is flat and the coordinates $X^A$ are Cartesian) are subject to three conditions,

$$q^A q_A = a^2, \quad p^A p_A = a^2 \quad \text{and} \quad q^A p_A = 0,$$

(33)

here $q^A p_A = \eta_{AB} q^A p_B$ and $\eta_{AB} = \text{diag}[1,1,-1,-1,-1]$ is the metric tensor in eq. (1). Altogether the arbitrary geodesic $G$, eq. (32), depends on eight initial values.

Now one puts $c = 0$ for simplicity and employs the full SO(3,2) symmetry of $R^{3,2}$. Let $P_0 \in R^{3,2}$ be an initial point $(s = 0)$ of $G$. Take any transformation
of SO(3,2) which makes the coordinates of $P_0$ equal to $X = Y = Z = U = 0$ and $V = a$, the transformation is non–unique. Then by the remaining transformations leaving invariant the straight line joining $P_0$ with the origin $X^A = 0$ one makes the tangent to $G$ at $P_0$ vector $\dot{X}^A(0)$ tangent to the $U$ line through $P_0$, i.e. $\dot{U}(0) = 1$ and $\dot{V}(0) = \dot{X}(0) = \dot{Y}(0) = \dot{Z}(0) = 0$. Then the representation of $G$ is reduced to

$$U(s) = a \sin \frac{s}{a}, \quad V(s) = a \cos \frac{s}{a}, \quad X = Y = Z = 0. \quad (34)$$

Each timelike geodesic on AdS space is represented in $\mathbb{R}^{3,2}$ by a circle of the same radius $a$ (determined by the curvature of the space) on an appropriately chosen euclidean two–plane $(U,V)$ [9]. The distinction between radial, circular and „general” geodesics has no geometrical meaning and in this space there is only one kind of timelike geodesics, analogously to Minkowski space possessing only one geodesic, a timelike straight line, which may be identified with the time axis of an inertial reference frame. (Recall that Minkowski space arises in the limit $a \to \infty$.) In other terms each timelike geodesic of AdS space is the circle lying on a euclidean two–plane going through the origin $X^A = 0$ of the ambient space. In general two timelike geodesics do not intersect and this means that their two–planes do not intersect either and the planes have only one common point, the origin.

One can also find an explicit transformation in $\mathbb{R}^{3,2}$ recasting a circular geodesic into a radial one, see Appendix. We emphasize that these properties of timelike geodesics are easy to investigate in the embedding flat five–space, whereas the internal four–dimensional approach is rather difficult.

8 Further properties of timelike geodesics

First we draw an important conclusion from the fact that each geodesic on AdS is the circle in $\mathbb{R}^{3,2}$. Accordingly, the parametric description in eq. (32) shows that each geodesic is periodic with the period $\Delta s = 2\pi a$ corresponding to one turn around the circle. We now analytically show that any two timelike geodesics emanating from an arbitrary point of AdS space first diverge and then reconverge at the distance $s = \pi a$, again diverge from that point and finally return to the initial point in the ambient space for $s = 2\pi a$. Let two arbitrary geodesics, $G_1$ and $G_2$, emanate from an arbitrary point $P_0$. One chooses the coordinates $X^A$ adapted to $P_0$ and $G_1$: the coordinates of $P_0$ are $X^A(P_0) = (a,0,0,0,0)$ and the directional vectors of $G_1$ are directed along the axes $X^1 = U$ and $X^2 = V$ respectively, $p_1^A = (a,0,0,0,0)$ and
\( q_1^A = (0, a, 0, 0, 0) \). Then \( G_1 \) is

\[
X_1^A(s) = q_1^A \sin \frac{s}{a} + p_1^A \cos \frac{s}{a}.
\]  

(35)

This implies that \( G_2 \) has a generic form of eq. (32) with the vectors \( q^A \) and \( p^A \) related by

\[
p^1 = \frac{1}{\cos c} (a - q^1 \sin c), \quad p^i = -q^i \tan c, \quad i = 2, 3, 4, 5.
\]

(36)

The three conditions in eq. (33) imply that \((q^1)^2\) is determined by \( q^i \) and \( \sin c = q^1/a \), thus arbitrary \( G_2 \) starting from \( P_0 \) is determined by four arbitrary parameters \( q^i \), corresponding to four independent components of the initial velocity \( \dot{X}^A(0) \). One sees from eqs. (32) and (35) that at the distance \( s = \pi a \) counted along both the geodesics one has \( X_1^A(\pi a) = -p_1^A \) for \( G_1 \) and from eq. (36) one has \( X^A(\pi a) = \left(-(a, 0, 0, 0, 0) = X_1^A(\pi a) \right) \) for \( G_2 \), or the two geodesics intersect at this point. This is a point conjugate to \( P_0 \) on \( G_1 \) and \( G_2 \) and antipodal to \( P_0 \) in \( \mathbb{R}^{3,2} \). At the distance \( s = 2\pi a \) both the geodesics return to \( P_0 \), \( X_1^A(2\pi a) = X^A(2\pi a) = (a, 0, 0, 0, 0) \), or make a closed loop on the pseudosphere in \( \mathbb{R}^{3,2} \).

Geometrically this effect is obvious. \( G_1 \) and \( G_2 \) are circles of the same radius lying on two–planes \( \pi_1 \) and \( \pi_2 \) respectively. Since \( P_0 \) is the common point of the circles, \( \pi_1 \) and \( \pi_2 \) intersect along the straight line connecting \( P_0 \) to the origin. Then the antipodal to \( P_0 \) point \( P_1 \) (i.e. having \( X^A(P_1) = -X^A(P_0) \)) lies on this line and \( G_1 \) and \( G_2 \) go through \( P_1 \) after delineating a half–circle from \( P_0 \).

In CAdS space the periodic time is replaced by the infinite line. For timelike geodesics this implies that each geodesic does not return to the initial point at the distance \( \Delta s = 2\pi a \), but goes to a new point, which is the same point in the three–space (using the static coordinates of eqs. (4), (5) and (7)) and is shifted forward in the time. The geodesics have in CAdS space infinite extension, yet their relationships cannot be altered in comparison to these in AdS space. Two geodesics having a common initial point must intersect for \( \Delta s = 2\pi a \) and the intersections will repeat infinitely many times, always after the same interval of the proper time. It turns out that it is hard to show this effect in full generality using exclusively the internal four–dimensional description due to computational difficulties. Geometrically it is clear that it is sufficient to show the effect for radial geodesics (the case including circular geodesics is discussed in the next section) and to this end the comoving coordinates \( (\tau, \chi, \theta, \phi) \) of eq. (14) are most appropriate. Consider the geodesics orthogonal to \( \tau = \text{const} \) hypersurfaces, these are the coordinate
time $\tau$ lines, $\chi, \theta, \phi = \text{const}$. The distance between two neighboring geodesics (simultaneous points) is
\[ dl^2 = a^2 \sin^2 \frac{\tau}{a} (d\chi^2 + \sinh^2 \chi d\Omega^2) \] (37)
and is largest for $\tau = \pi a/2$ and tends to zero both in the past for $\tau \to 0$ and in future for $\tau \to \pi a$. This means that all these hypersurface orthogonal geodesics emanate from the common point $\tau = 0$ and diverge until $\tau = \pi a/2$, then reconverge at $\tau = \pi a$. The comoving coordinates are valid in the region between two hypersurfaces, $\tau = 0$ and $\tau = \pi a$, which metrically shrink to one point. In CAdS space these coordinates hold independently in each region between $\tau = n\pi a$ and $\tau = (n+1)\pi a$ for any integer $n$; together these regions form an infinite chain, which, as we saw in Sect. 2, cover only a small part of the entire manifold. The fact that the geodesics actually intersect after $\Delta \tau = \pi a$ rather than $2\pi a$ corresponds to the geometrical effect in AdS space that the circles intersect twice. The points $\tau = n\pi a$ form an infinite sequence of points conjugate to $\tau = 0$ along these geodesics.

We emphasize that although CAdS space is static with timelike lines infinitely extending and it is a solution to Einstein field equations which may be constructed without the intermediating stage of the pseudosphere in the flat five–space, nevertheless this geodesic reconvergence is a residual effect of the time periodicity of the AdS space as the pseudosphere. Without invoking the pseudosphere in $\mathbb{R}^{3.2}$ this property of CAdS space is incomprehensible.

The fact that in CAdS space all timelike geodesics starting from a common point can only recede from each other to a finite distance and then must intersect infinite many times, has two important consequences. First, a timelike geodesic cannot reach the spatial infinity $\mathcal{J}$. In fact, the infinity is for $r$ and $\rho \to \infty$ and according to eq. (3) all the coordinates $X^A$ are infinite there (except for discrete values of $t, \theta$ and $\phi$ where some $X^A$ vanish). Yet it is seen from eq. (32) that the coordinates $X^A(s)$ of a timelike geodesic are always finite. Another, purely four–dimensional proof in the case of a radial geodesic is given in sect. 9.

Second, there are points inside the future light cone of any $P_0$ that cannot be reached from $P_0$ by any timelike geodesic. We are accustomed to in Minkowski space and expect the same effect in any curved spacetime (as it occurs in the Schwarzschild field) that if two points can be connected by a timelike curve, they can also be connected by a geodesic. This is not the case of CAdS space. To show it we again employ the five–dimensional description since one AdS space is sufficient to this aim. Let a bunch of geodesics emanate from arbitrary $P_0$. We have seen that at the distance $\Delta \tau = \pi a = \Delta s$
from $P_0$ all geodesics intersect at $P_1$. Take any spacelike 3–dimensional hypersurface $S$ through $P_1$. The future light cone from $P_0$ intersects $S$ along a closed surface $\Sigma$ (having topology of the two–sphere) being a boundary of a 3–dimensional set $D$ in $S$. The set $D$, lying in the interior of the light cone, belongs to the chronological future of $P_0$, i.e. any point of $D$ may be connected to $P_0$ by a timelike curve. However, no point of $D$ besides $P_1$, can be connected to $P_0$ by a geodesic. In other words, a large part (an open region) of the interior of the future (past) light cone of $P_0$ is inaccessible from $P_0$ along a timelike geodesic.

9 The twin paradox

Finally we discuss a version of the twin paradox known from special relativity (SR). In SR the ,,paradox” has a purely geometrical nature and consists in determining the longest timelike curve joining two given points P and Q (providing Q lies in the chronological future of P). There is no shortest curve since a timelike curve from P to Q may have arbitrarily small length. The solution in SR is simple: it is the straight line connecting P to Q. Physically this means that the twin which gets older at the reunion is the twin which always stays at rest in the inertial reference frame where this line is a coordinate time line. In a curved spacetime the problem is more sophisticated since there are actually two separate problems: a local and a global one. In CAdS space, due to its maximal symmetry, the two problems coincide. We consider three twins (,,siblings”): twin A stays at rest at a fixed point in space, twin B revolves on a circular geodesic orbit around a chosen origin of spherically symmetric coordinates and twin C moves upwards and downwards on a radial geodesic in these coordinates. Their worldlines emanate from a common initial point $P_0$ and we study where they will intersect in the future [10]. We apply the static coordinates ($t, \rho, \theta, \phi$) of eq. (5).

The nongeodesic twin A remains at $\rho = \rho_0 > 0$, $\theta = \pi/2$ and $\phi = 0$ and in a coordinate time interval $T$ its worldline has length

$$s_A(T) = \sqrt{\left(\frac{\rho_0}{a}\right)^2 + 1} T. \quad (38)$$

Any twin following a geodesic has conserved energy and denoting its energy per unit mass by $k$ (dimensionless) one finds [10]

$$\dot{i} \equiv \frac{dt}{ds} = \frac{a^2 k}{\rho^2 + a^2}. \quad (39)$$

As the initial point we choose $P_0(t_0 = 0, \rho = \rho_0 > 0, \theta = \pi/2, \phi = 0)$. For the circular geodesic of B with $\rho = \rho_0$ one has $t = s$, $\phi = s/a = t/a$.
and its energy is related to the radius by $\rho_0 = \sqrt{k_B - 1}a$. The period of one revolution is $T = 2\pi a$ and the length of $B$ for one revolution is $s_B = T = 2\pi a$, the already known result. After one revolution the twins A and B meet and $s_A(2\pi a) > s_B$, or there are timelike curves longer than the geodesic $B$.

The twin $C$ moving on a radial geodesic has the radial velocity $\dot{\rho} \equiv d\rho/ds$ given by

$$\dot{\rho}^2 = k_C^2 - \left(\frac{\rho^2}{a^2} + 1\right)$$

following from $g_{\alpha\beta}u^\alpha u^\beta = 1$. Let at $P_0$ twin $C$ be initially at rest, $\dot{\rho}(0) = 0$, then its energy is $k_C^2 = (\rho_0/a)^2 + 1$ and from the radial component of the geodesic equation (30) it follows that its acceleration is directed downwards, $\ddot{\rho}(0) = -(\rho_0/a)^2 < 0$ and the twin falls down. This shows that gravitation in CA$^d$S space is attractive. (This is not trivial since in de Sitter space gravitational forces are repulsive.) We therefore consider a more general motion: $C$ radially flies away with $\dot{\rho}(0) = u > 0$, reaches a maximum height $\rho = \rho_M$, falls down back to $\rho = \rho_0$ and then to $\rho = 0$ and farther (for $\phi = \pi$). The highest point of the trajectory is, from eq. (40), $\rho_M^2 = (k_C^2 - 1)a^2$, and $\rho_M > \rho_0$ implies $k_C^2 > (\rho_0/a)^2 + 1$. One sees that a radial geodesic cannot reach the spatial infinity $\mathcal{J}$ since $\rho_M \rightarrow \infty$ requires infinite energy $k_C \rightarrow \infty$.

Moving in the opposite direction ($\phi = \pi$), $C$ reaches the same highest point, $\rho = \rho_M$, and falls down back to $(\rho_0, \phi = 0)$; in this way it oscillates between the antipodal in the 3–space points $(\rho_M, \phi = 0)$ and $(\rho_M, \phi = \pi)$ infinite many times. The coordinates of the geodesic $C$ may be parametrically described by $t = t(\eta)$, $\rho = \rho(\eta)$ and $s = s(\eta)$, where $\eta$ is an angular parameter $[10]$, here we use a simpler description. To this end we again apply the five–dimensional picture. The points of the geodesic have coordinates $X^A$ given in eq. (3), where one puts $\rho = a \sinh r/a$ and $\theta = \pi/2$ and $\phi = 0$, then

$$U = \sqrt{\rho^2(s) + a^2 \sin \frac{t}{a}}, \quad V = \sqrt{\rho^2(s) + a^2 \cos \frac{t}{a}}, \quad X = \rho(s), \quad Y = Z = 0.$$ (41)

On the other hand $C$ is described by eq. (32) with $c = 0$. By comparing the two expressions one finds $X = \rho = q^1 \sin s/a + p^1 \cos s/a$. To determine $q^1$ and $p^1$ one inserts this expression into both eq. (40) and into the radial component of the geodesic equation (30) and checks that it is a solution to these equations. Applying the initial conditions one gets

$$\rho(s) = \sqrt{\rho_M^2 - \rho_0^2 \sin \frac{s}{a} + \rho_0 \cos \frac{s}{a}},$$ (42)

and the highest point $\rho_M$ is reached for $\cos s_M/a = \rho_0/\rho_M$. Since the domain of the radial coordinate is $\rho \geq 0$, the values $\rho(s) < 0$ are assigned to points
with $\rho = |\rho(s)|$ and $\phi = \pi$. Clearly, the proper time interval between the highest points, $(\rho_M, \phi = 0)$ and $(\rho_M, \pi)$ is $\Delta s = \pi a$ and the same interval is between the initial point $(\rho_0, 0)$ and its antipodal one $(\rho_0, \pi)$, independently of the energy $k_C$ [10]. Notice that the special solution $\rho(s) = 0$ and $k = 1$ actually represents the „canonical” description of any timelike geodesic given in eq. (34). This shows that the detailed behavior of any geodesic revealed by the general solution in eq. (42) is merely coordinate dependent.

One can also integrate eq. (39) applying eq. (42) but then one gets a generic formula for $t(s)$ being a complicated expression involving functions arc tan, which is of little use. Instead one considers a special case of $\rho_0 = 0$, then eq. (42) is reduced to

$$\rho(s) = \sqrt{k^2 - 1} a \sin \frac{s}{a}$$

(43) and if the length of this geodesic is divided into intervals according to $s = (\sigma + \frac{1}{2} n \pi) a$, where $0 \leq \sigma < \pi/2$ and $n = 0, 1, 2, \ldots$, the time coordinate is

$$t(s) = a \arctan(k \tan \sigma) + \frac{1}{2} n \pi a.$$  

(44)

The coordinate time interval between the highest point, $\sqrt{k^2 - 1} a$, and its antipodal one (always in the 3-space), is $\Delta t = \pi a$, and clearly the same holds for the generic geodesic C. This means that the geodesics B and C will intersect first at $(t = \pi a, \phi = 0)$ and then at $(t = 2\pi a, \phi = 0)$ and later infinite many times. Whereas the twins B and C meet each other after the constant intervals $\Delta s = \pi a = \Delta t$, which are independent of C’s initial velocity, twin C meets A after the time interval [10]

$$\Delta t_1 = \pi a - 2a \arctan \left( \frac{k_C \rho_0}{\sqrt{\rho_M^2 - \rho_0^2}} \right)$$

(45) and the corresponding length of the geodesic C is

$$s_C = 2a \arccos \left( \frac{\rho_0}{\rho_M} \right).$$

(46)

These two expressions are so complicated that it is not easy to analytically compare the lengths of A and C for the interval $\Delta t_1$ (being the time of C’s flight on the route $\rho_0 \rightarrow \rho_M \rightarrow \rho_0$). It has been numerically shown that always $s_C > s_A(\Delta t_1)$, as it should be, since on this segment of the geodesic C there are no points conjugate to $P_0$. In Fig. 3 we depict two radial geodesics emanating from a common point.

Finally we illustrate the results of the paper with a numerical example. Let the curvature scale of CAdS space be $a = 10^{16} m = 1$ light year, then the
Figure 3: Two radial timelike geodesics with energies \( k = 1.2 \) and \( k = 1.5 \), emanating from a common point, chosen for simplicity as \( t = 0 = \rho \). The length scale is \( a = 1 \). The geodesics must return to \( \rho = 0 \) and intersect at \( t = \pi \), then they go to \( \rho < 0 \) corresponding to the direction \( \phi + \pi \) and again intersect at \( t = 2\pi, \rho = 0 \). This evolution repeats infinite many times. All points \((t = \pi, \rho \neq 0)\) and \((t = 2\pi, \rho \neq 0)\) are inaccessible from the initial point along a timelike geodesic.

The cosmological constant is \( \Lambda = -3/a^2 = -3 \cdot 10^{-32} \text{m}^{-2} \). Let from the coordinate origin \( \rho_0 = 0 \) be emitted simultaneously a photon and a particle of mass \( m \), both radially and in the same direction, they move along a null and a timelike geodesic, respectively. At a spatial point \( P \) at \( \rho = \rho_M = 10^6 a \) they are reflected backwards by a mirror and return to \( \rho_0 = 0 \). Employing formulae given above one finds that along the photon path there is \( \rho(\sigma) = E a \sigma \) and the value \( \rho_M \) at the highest point shows that \( E \sigma = 10^6 \), then the distance
between the origin and P (measured along the radial spacelike geodesic at $t = \text{const}$) is $l(0, \rho_M) \cong a \ln(2\rho_M/a) \cong 14.5a = 14.5 \text{ l.y.}$ The coordinate time of the photon flight to P and back is equal to the proper time measured by a clock staying at rest at $\rho_0 = 0$ and is $\Delta t = \Delta s = 2a \arctan(E\sigma)$ and is very slightly below $\pi a$ or slightly below $\pi$ years. The massive particle is ultrarelativistic and closely follows the photon; from $\rho_M = \sqrt{k^2 - 1}a$ one gets that its energy is $10^6mc^2$ and its proper time interval when reaching $\rho_M$ is exactly $s_M = \pi a/2$, then its total travel lasts $2s_M = \pi$ years. Both the photon and the particle travel the distance $2 \cdot 14.5 = 29 \text{ l.y.}$ and to go it they need a period of time not exceeding $\pi$ years. This outcome deceptively suggests that the photon and the particle move at superluminal velocities since their average velocity is $29/\pi \cong 9.2c$. Clearly the local velocity of light is always $c$ and this superluminal one is merely a result of the weird geometry of CAdS space.

10 Conclusions

Anti–de Sitter space is one of the three simplest, maximally symmetric solutions to vacuum Einstein field equations. Its metric is static with the time coordinate extending from $-\infty$ to $+\infty$, nonetheless most of its geometric properties are periodic in the time, something which is incomprehensible from the intrinsic four–dimensional viewpoint. The light seems to move at superluminal velocities since the photon may travel over arbitrarily large distances (to spatial infinity and back) in a finite time interval. In static coordinates covering the whole spacetime one can single out in the set of all timelike geodesics the radial and circular curves, yet it turns out that this distinction is geometrically irrelevant and is merely coordinate dependent. No timelike geodesic can escape to the spatial infinity unless it has infinite energy. Also a timelike geodesic may travel large distances at a superluminal average velocity. All timelike geodesics emanating from a common initial event $(t_0, \mathbf{x})$ return to the same point $\mathbf{x}$ in the space after the time interval $\Delta t = 2\pi a$; this means that all simultaneous events ($t = t_0 + 2\pi a$), though belonging to the chronological future of the initial event, are inaccessible from the latter by a timelike geodesic. In other words, any point $(t_0 + 2\pi a, y)$ cannot be reached from $(t_0, x)$ by a free fall in any direction and with any initial velocity, if the points $x$ and $y$ are different. These bizarre features become understandable only if one divides the whole spacetime into an infinite chain of segments and each of them is identified with the anti–de Sitter space proper and the latter is modelled as a pseudosphere in an unphysical five–dimensional space. In this space each timelike geodesic od AdS space forms a
circle of the same radius, which accounts for their weird properties. This necessity is in conflict with general relativity stating that a physical spacetime is four-dimensional and all its properties are intrinsically grounded, without resorting to a fictitious higher dimensional embedding space. Finally, if the boundary conditions are suitably chosen, AdS space is unstable and cannot be a ground state for spacetimes with $\Lambda < 0$. The conclusion, therefore, is unambiguous: this spacetime is unphysical and cannot describe a physical world. It may only serve as a mathematical tool in field theory, e.g. in the recent AdS/CFT correspondence.

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11 Appendix

Here we derive an explicit transformation recasting a circular timelike geodesic on AdS space into a radial one. To this end one uses the static coordinates $(t, \rho, \theta, \phi)$ of eq. (5), then points $X^A$ of the pseudosphere in the ambient space are parametrized by these variables according to eq. (3) with $\rho = a \sinh r/a$.

For any timelike geodesic the angles $\theta$ and $\phi$ may be so chosen that the curve lies in the two-surface $\theta = \pi/2$, then its points are

$$U = \sqrt{\rho^2 + a^2} \sin \frac{t}{a}, \quad V = \sqrt{\rho^2 + a^2} \cos \frac{t}{a}, \quad X = \rho \cos \phi, \quad Y = \rho \sin \phi, \quad Z = 0.$$  \hfill (47)

As in sect. 9 a circular geodesic $G_c$ has $\rho = \rho_0 > 0$, $t = s$, $\phi = s/a = t/a$, its radius is determined by the energy, $\rho_0 = \sqrt{k_c - 1} a$ and its coordinates $X^A_c$ are

$$X^A_c(s) = q^A_c \sin \frac{s}{a} + p^A_c \cos \frac{s}{a}. \quad \hfill (48)$$

To determine the directional five-vectors one compares eq. (47) for $\rho = \rho_0$, $t = s$ and $\phi = s/a$ with eq. (48) and gets $p^A_c = (0, \sqrt{k_c} a, \sqrt{k_c - 1} a, 0, 0)$ and $q^A_c = (\sqrt{k_c} a, 0, 0, \sqrt{k_c - 1} a, 0)$.

Now assume that in a Cartesian coordinate system $X'^A$ (different from $X^A$ one) a radial geodesic $G_r$ is described by

$$X'^A_r(s) = q'^A_r \sin \frac{s}{a} + p'^A_r \cos \frac{s}{a}.$$  \hfill (49)

and $X'^A_r$ are parametrized by $x'^\alpha = (t', \rho', \theta', \phi')$ as in eq. (47). One sets $\phi'(s) = 0$ at points of $G_r$ and assuming that it emanates from $\rho'(0) = 0$ with
\( p'(0) > 0 \) its coordinates \( \rho'(s) \) and \( t'(s) \) are given by the right-hand sides of eqs. (43) and (44), the latter holds for \( n = 0, 1, 2, 3 \). To determine the directional vectors in this case it is sufficient to take \( n = 0 \) in eq. (44) and apply the identity \( \arctan x \equiv \arcsin[x(1 + x^2)^{-1/2}] \), then

\[
\sin \frac{t'}{a} = \frac{k \sin \frac{\rho}{a}}{(\cos^2 \frac{\rho}{a} + k^2 \sin^2 \frac{\rho}{a})^{1/2}},
\]

Next one inserts the relationships (47) into eq. (49) with due replacements of \( X^A \) by \( X'^A, x^\alpha \) by \( x'^\alpha \) and with \( \phi' = 0 \) and employs there eq. (43) for \( \rho' \) and eq. (50). Finally the normalizations of eq. (33) provide \( q^A_r = (ak_r, 0, \sqrt{k_r^2 - 1} a, 0, 0) \) and \( p^A_r = (0, a, 0, 0, 0) \). If \( G_r \) and \( G_c \) are two different (coordinate dependent) descriptions of the same curve in \( \mathbb{R}^{3,2} \), there exists a linear transformation of the pair \( (q^A_c, p^A_c) \) into \( (q^A_r, p^A_r) \). One then seeks for a transformation \( L \in SO(3,2) \), \( X'_A = L A B X^B \) such that \( L A B q^B_c = q^A_r \) and \( L A B p^B_c = p^A_r \). According to the fundamental theorem both the geodesics are geometrically represented by circles with the same radius, hence all other their characteristics, such as the conserved energy, are coordinate dependent and irrelevant. One can therefore put \( k_c = k_r = k \). A simple and long computation results in \( L \) depending on one arbitrary parameter and setting it equal zero one gets the simplest form of the matrix,

\[
(L^A_B) = \begin{pmatrix}
  k^{3/2} & (k - 1)\sqrt{k + 1} & -\sqrt{k(k^2 - 1)} & -k\sqrt{k - 1} & 0 \\
  0 & \sqrt{k(k^2 - 1)} & -\sqrt{k - 1} & 0 & 0 \\
  \sqrt{k - 1} & k\sqrt{k - 1} & -k^{3/2} & -(k - 1)\sqrt{k + 1} & 0 \\
  0 & 0 & 0 & -\sqrt{k} & 0 \\
  0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\( \det L = +1 \). It is clear that both \( G_c \) and \( G_r \) emanate from the same point for \( s = 0 \). In fact, the initial point \( P_0 \) of \( G_c \) has coordinates \( X^A(P_0) = X^A_c(0) = p^A_c \) and after the transformation its coordinates are \( X'^A = L A B p^B_c = p^A_r \) and these are the coordinates of the initial point of \( G_r, X'^A_r(0) = p^A_r \). We notice that the transformation in AdS space from \( (t, \rho, \phi) \) to \( (t', \rho', \phi') \) (for \( \theta = \theta' = \pi/2 \)) is very intricate and hence useless.

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