Flag Manifold Sigma Models and Nilpotent Orbits

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To Andrei Alekseevich Slavnov on the occasion of his 80th birthday with respect and gratitude

Abstract—We study flag manifold sigma models that admit a zero-curvature representation. We show that these models can be naturally viewed as interacting (holomorphic and antiholomorphic) $\beta\gamma$-systems. In addition, using the theory of nilpotent orbits of complex Lie groups, we establish a relation of flag manifold sigma models to the principal chiral model.

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1. GAUGED LINEAR FORMULATION AND $\beta\gamma$-SYSTEMS

In the present paper we will consider sigma models whose target spaces are flag manifolds, i.e., homogeneous manifolds of the group $\text{SU}(N)$ of the form

$$\mathcal{F}(n_1,\ldots,n_m) := \text{SU}(N)/S(\text{U}(n_1) \times \cdots \times \text{U}(n_m)), \quad \sum_{i=1}^{m} n_i = N. \quad (1.1)$$

Sometimes for brevity we will also denote this quotient as $G/H$. This space admits a reductive metric, the so-called normal metric, which will be denoted by $\mathcal{G}$. It is defined as follows. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the standard decomposition of the Lie algebra ($\langle \mathfrak{m}, \mathfrak{h} \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the Killing metric). Then the normal metric on $G/H$ is induced from the restriction $\langle \cdot, \cdot \rangle|_{\mathfrak{m}}$ of the Killing metric. An interesting fact is that in this metric all geodesics are homogeneous; i.e., they are orbits of one-parameter subgroups of $G$ (see [1]). Quite generally, integrability of the geodesic flow on $G/H$ is a necessary condition for the integrability of the sigma model.

In order to define the models, we will also need a $G$-invariant (homogeneous) complex structure $\mathcal{I}$ on the target space. On the flag manifold (1.1) such complex structures are in one-to-one correspondence with the orderings of the set $\{n_1,\ldots,n_m\}$. Let us assume that the ordering is given by the index $i$ of $n_i$; otherwise we simply swap the indices. Then the space (1.1) allows an alternative definition as the quotient

$$\mathcal{F}(n_1,\ldots,n_m) \simeq \text{GL}(N,\mathbb{C})/P_{d_1,\ldots,d_m}, \quad (1.2)$$

where $P_{d_1,\ldots,d_m}$ is a parabolic subgroup of $\text{GL}(N,\mathbb{C})$ that stabilizes the flag of linear spaces $0 \subset L_1 \subset \cdots \subset L_m \simeq \mathbb{C}^N$, with $L_k \simeq \mathbb{C}^{d_k}$ and $d_k = \sum_{i=1}^{k} n_i$.

Schematically, the action has the following form [3] (here $X : \Sigma \to \mathcal{F}$):

$$\mathcal{S}[\mathcal{G},\mathcal{F}] := \int_{\Sigma} d^2z \left\| \partial X \right\|^2_{\mathcal{G}} + \int_{\Sigma} X^* \omega, \quad (1.3)$$

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where $\omega = \mathcal{G} \circ \mathcal{I}$ is the fundamental Hermitian form of the metric $\mathcal{G}$. It is closed if and only if $m = 2$, i.e., if the flag manifold is a Grassmannian [4]. In other cases the second term in the action (1.3) is not topological.

The $B$-field of the same form as in (1.3) was considered on other grounds in [23]. In addition, Lax pairs for models of type (1.3), in the special case of symmetric spaces, were apparently considered for the first time in [8] (in the $\mathbb{C}P^1$-case) and in [2] (mainly in the noncompact case). A detailed study of a similar case of paracompact target spaces can be found in the recent work [13].

In [4–6] we constructed the gauged linear sigma-model representation for the models of type (1.1)–(1.3). In the case when the target space is a Grassmannian, the metric $\mathcal{G}$ is Kähler and this representation is equivalent to the Kähler quotient $G(k, N) \cong \text{Hom}(\mathbb{C}^k, \mathbb{C}^N) // U(k)$. In the general case our construction leads to a quotient by a nonreductive group and to the “Killing” metric $\mathcal{G}$, which is not Kähler in general.

The construction is as follows. We introduce a field $U \in \text{Hom}(\mathbb{C}^M, \mathbb{C}^N)$, where $M = d_{m-1}$, satisfying the orthonormality condition $U^\dagger U = \mathbb{1}_M$, as well as the “gauge” field $A = A_z dz + A \bar{z} d\bar{z}$ of the following special form:

\[
A_z = \begin{pmatrix}
& & & & & & & 1 \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & 
\end{pmatrix}, \quad A = (A_z)^\dagger.
\] (1.4)

The Lagrangian reads

\[
\mathcal{L} = \text{Tr}((D_\bar{z} U)^2), \quad \text{where} \quad D_\bar{z} U = \partial_\bar{z} U + i U A_z.
\] (1.5)

This Lagrangian is equivalent to (1.3), which can be proved by eliminating the field $A$. Due to the orthonormality condition $U^\dagger U = \mathbb{1}_M$, the gauge group of the model is $U(n_1) \times \ldots \times U(n_m)$. A natural question is whether one can instead use a quotient by the complex group of upper/lower block triangular matrices. To answer this, we give up the orthonormality condition and assume that $U \in \text{Hom}(\mathbb{C}^M, \mathbb{C}^N)$ is an arbitrary complex matrix of rank $M$. We then write down the following Lagrangian:

\[
\mathcal{L} = \text{Tr} \left( (D_\bar{z} U)^\dagger D_\bar{z} U \frac{1}{U^\dagger U} \right).
\] (1.6)

It is easy to see that it is invariant with respect to complex gauge transformations $U \rightarrow U g$, where $g \in P_{d_1, \ldots, d_{m-1}}$. The Gram–Schmidt orthogonalization procedure brings the Lagrangian (1.6) to the form (1.5), but for a number of reasons the complex form is preferable. In order to get rid of the denominator in the Lagrangian, we introduce an auxiliary field $V \in \text{Hom}(\mathbb{C}^N, \mathbb{C}^M)$ and write down a new Lagrangian

\[
\mathcal{L} = \text{Tr}(VD_\bar{z} U) + \text{Tr}(VD_\bar{z} U)^\dagger - \text{Tr}(VV^\dagger U^\dagger U),
\] (1.7)

which turns into the original one upon elimination of the field $V$. Next we perform yet another quadratic transformation, in order to eliminate the quartic interaction. To this end we introduce the complex matrix field $\Phi_z \in \text{End}(\mathbb{C}^N)$ and its Hermitian conjugate $\Phi = (\Phi_z)^\dagger$. We write one more Lagrangian

\[
\mathcal{L} = \text{Tr}(VD_\bar{z} U) + \text{Tr}(VD_\bar{z} U)^\dagger + \text{Tr}(\Phi_\bar{z} \Phi_z),
\] (1.8)

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where $\mathcal{D}_z$ is the “elongated” covariant derivative

$$\mathcal{D}_zU = \partial_z U + iU A_z + i\Phi_z U.$$  \hspace{1cm} (1.9)

Let us clarify the geometric meaning of the Lagrangian (1.8). The first two terms correspond to a sum of the so-called $\beta\gamma$-systems on the flag manifold $\mathcal{F}$ in a background field $\Phi_z$ (see [22, 24]). By definition, such a system can be defined for an arbitrary complex manifold $\mathcal{M}$ ($\dim \mathcal{M} = m$) with the help of a complex fundamental $(1, 0)$-form $\theta = \sum_{i=1}^m p_i dq_i$ on $T^* \mathcal{M}$. Here $q_i$ are the complex coordinates on $\mathcal{M}$ and $p_i$ are the complex coordinates in the fiber of the holomorphic cotangent bundle. The action of the $\beta\gamma$-system is then just $S = \int_\Sigma d^2 z \sum_{i=1}^m p_i \partial_z q_i$. In the case of the flag manifold, this action can be most conveniently written using two matrices $U \in \text{Hom}(\mathbb{C}^M, \mathbb{C}^N)$ and $V \in \text{Hom}(\mathbb{C}^N, \mathbb{C}^M)$ and the gauge field $A_z$. Indeed, it will be shown in the next section that the fundamental $(1, 0)$-form can be written as $\theta = \text{Tr}(V dU)|_{\mu_C = 0}$, where $(U, V)$ satisfy the condition

$$\mu_C = VU|_{\mathfrak{t}^*} = 0, \quad \mathfrak{t} = \text{Lie}(P_{d_1, \ldots, d_{m-1}}),$$  \hspace{1cm} (1.10)

with $\mathfrak{t}$ the Lie algebra of the corresponding parabolic subgroup of $\text{GL}(M, \mathbb{C})$. It is also assumed that the space of matrices satisfying this condition is factorized with respect to the action of $P_{d_1, \ldots, d_{m-1}}$; i.e., one has a complex symplectic reduction. Condition (1.10) is precisely the condition of vanishing of the moment map $\mu_C = 0$ for the action of the parabolic group $P_{d_1, \ldots, d_{m-1}}$ on the space of matrices $(U, V)$ endowed with the symplectic form $\omega_0 = \text{Tr}(dU \wedge dV)$. As a result,

$$d\theta = \omega_{\text{red}}$$  \hspace{1cm} (1.11)

is the complex symplectic form arising after the reduction with respect to the parabolic group. In order to impose condition (1.10) at the level of the Lagrangian of the model, one needs the gauge field $A_z \in \text{Lie}(P_{d_1, \ldots, d_{m-1}})$. Indeed, differentiating the Lagrangian (1.8) with respect to $A_z$, one arrives at condition (1.10).

A large class of integrable sigma models in the $\beta\gamma$-formulation has been recently proposed in [11] (for the background material see also [9, 10]). In the terminology of that work our field $\Phi_z$ should be viewed as the component $A_z$ of the Chern–Simons gauge field along the “topological plane” (i.e., the worldsheet $\Sigma$). The quadratic form in the interaction term $\text{Tr}(\Phi_z \Phi_z)$ in (1.8) is in this context the inverse propagator of the field $A_z$, which in the present (rational) case is proportional to the identity matrix.

2. RELATION TO THE QUIVER FORMULATION

Before passing to further topics, let us clarify the relation of the complex symplectic form constructed using the symplectic quotient by a parabolic subgroup, as above, to the symplectic form that arises as a result of a reductive quotient defined by the so-called quiver. We recall that $T^* \mathcal{F}$ is a hyper-Kähler manifold that can be constructed by a hyper-Kähler quotient of a flat space (though we stress that the real symplectic form—the Kähler form—will not concern us here). This quotient is based on a linear quiver diagram of the following form:

\[
\begin{array}{cccccccccccc}
 & & & & & & & & & & & & & \\
V_1 & V_2 & \ldots & & & & & & & & & & & \\
L_1 & U_1 & L_2 & & & & & & & & & & & \\
& & & & & & & & & & & & & \\
\end{array}
\]

\hspace{1cm} (2.1)

where $\mathcal{F}$ is the complex symplectic form arising after the reduction with respect to the parabolic group. In order to impose condition (1.10) at the level of the Lagrangian of the model, one needs the gauge field $A_z \in \text{Lie}(P_{d_1, \ldots, d_{m-1}})$. Indeed, differentiating the Lagrangian (1.8) with respect to $A_z$, one arrives at condition (1.10).

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V_1 & V_2 & \ldots & & & & & & & & & & & \\
L_1 & U_1 & L_2 & & & & & & & & & & & \\
& & & & & & & & & & & & & \\
\end{array}
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In each node there is a vector space \( L_k \simeq \mathbb{C}^{d_k} \), and to each arrow from node \( i \) to node \( j \) there corresponds a field taking values in \( \text{Hom}(L_i, L_j) \). The full space of fields is therefore
\[
\mathcal{W}_0 := \bigoplus_{i=1}^{m-1} (\text{Hom}(L_i, L_{i+1}) \oplus \text{Hom}(L_{i+1}, L_i)).
\]

We can then consider the GIT-quotient \( \mathcal{W}_i := \mathcal{W}/G \) of the stable subset \( \mathcal{W} \subset \mathcal{W}_0 \) by the group \( G := \prod_{i=1}^{m-1} \text{GL}(L_i) \). In \( \mathcal{W}_i \) we define a submanifold given by the vanishing conditions for the moment maps \((U_0 = 0, V_0 = 0)\):
\[
\mathcal{F} := \{ \mu_i = U_{i-1}V_{i-1} - V_iU_i = 0, \ i = 1, \ldots, m-1 \} \subset \mathcal{W}_i.
\]

The (well-known) statement is that the resulting space is the flag manifold \((1.2)\), which is why we have denoted it by \( \mathcal{F} \). On \( \mathcal{W} \) there is a natural complex symplectic form (restriction of the symplectic form on \( \mathcal{W}_0 \))
\[
\Omega = \sum_{i=1}^{m-1} \text{Tr}(dU_i \wedge dV_i).
\]

The construction just described can be interpreted as the symplectic quotient by the complex group \( G \), and it endows \( \mathcal{F} \) with a symplectic form \( \Omega_{\mathcal{F}} \). We prove the following statement.

**Lemma.** \( \Omega_{\mathcal{F}} = \omega_{\text{red}} \), where \( \omega_{\text{red}} \) is the symplectic form \((1.11)\) that arises as a result of the reduction with respect to a parabolic subgroup of \( \text{GL}(M, \mathbb{C}) \).

**Proof.** First of all, consider the fields \( \{U_i\} \), where \( U_i \) is a matrix with \( d_i \) columns and \( d_{i+1} \) rows. By the definition of the stable set \( \mathcal{W} \subset \mathcal{W}_0 \), the rank of \( U_i \) is \( d_i \). Therefore, by the action of \( \text{GL}(d_{i+1}, \mathbb{C}) \) one can bring \( U_i \) to the form where the first \( d_{i+1} - d_i \) rows are zero and the last \( d_i \) rows represent the identity matrix. The stabilizer of this canonical form with respect to the joint (left–right) action of \( \text{GL}(d_{i+1}, \mathbb{C}) \times \text{GL}(d_i, \mathbb{C}) \) is the subgroup \( P_{d_i,d_{i+1}} \subset P_{d_i,d_{i+1}} \times \text{GL}(d_i, \mathbb{C}) \) embedded according to the rule \( g \rightarrow (g, \pi_i(g)) \), where \( \pi_i(g) \) is the projection on the block of size \( d_i \times d_i \). Iterating this procedure, i.e., bringing all matrices \( U_i, \ i = 1, \ldots, m \), to the canonical form, we arrive at the situation when one is left with a single nontrivial matrix \( U_{m-1} := U \), and the resulting symmetry group is precisely \( P_{d_1,\ldots,d_{m-1}} \). We also set \( V_{m-1} := V \). Now, let \( a \in \mathfrak{t} = \text{Lie}(P_{d_1,\ldots,d_{m-1}}) \).

By the definition of the stabilizer, \( aU_{m-2} = U_{m-2}\pi_{m-2}(a) \); therefore,
\[
\text{Tr}(aU_{m-2}V_{m-2}) = \text{Tr}(\pi_{m-2}(a)V_{m-2}U_{m-2}) = \text{Tr}(\pi_{m-2}(a)U_{m-3}V_{m-3}),
\]

where the second equality is based on the condition in \((2.3)\). Since \( \pi_{m-2}(a) \in \text{Stab}(U_{m-3}) \), we can iterate this procedure and finally obtain \( \text{Tr}(aU_{m-2}V_{m-2}) = 0 \). Due to the equation \( U_{m-2}V_{m-2} - VU = 0 \) we get \( VU|_{\mathfrak{t}^*} = 0 \), which coincides with \((1.10)\). Moreover, since the matrices \( U_i, \ i = 1, \ldots, m-2 \), are constant, the restriction of the symplectic form \( \Omega \) coincides with \( \text{Tr}(dU_{m-1} \wedge dV_{m-1}) = \text{Tr}(dU \wedge dV) \). \( \square \)

Let us clarify the role of the field \( \Phi_z \). Differentiating the Lagrangian \((1.8)\) with respect to \( \Phi_z \), we obtain
\[
\Phi_z = -iUV.
\]

This coincides with the expression for the \( z \)-component of the Noether current for the action of the group \( \text{GL}(N, \mathbb{C}) \) on the space of matrices \((U, V)\). For the \( \beta \gamma \)-system written above, \( \Phi_z \) is nothing but the moment map for the action of this group.
3. RELATION TO THE PRINCIPAL CHIRAL MODEL: NILPOTENT ORBITS

Recall that the equations of motion of the principal chiral model can be written as follows:

\[ j = -g^{-1}dg, \quad d*j = 0. \]  \hspace{1cm} (3.1)

Similarly one can write down the equations of motion for a sigma model with a symmetric target space. Let \( \sigma : G \to G \) be the Cartan homomorphism, \( \sigma^2 = 1 \). Then the equations of motion have the form

\[ j_s = -\hat{g}^{-1}d\hat{g}, \quad d*j_s = 0, \quad \text{where} \quad \hat{g} = \sigma(g)g^{-1}. \]  \hspace{1cm} (3.2)

Note that the map \( g \to \hat{g} \) in (3.2) is nothing but the Cartan embedding \( \tilde{\sigma} : G/H \hookrightarrow G \). Both \( j \) and \( j_s \) are the Noether currents of the sigma model, calculated using the standard action \( S = \int d^2z \| \partial X \|^2 \).

In other words, suppose \( X : \Sigma \to G \) is a harmonic map. If its image lies in the symmetric space \( G/H \), i.e., \( X(\Sigma) \subset G/H \subset \tilde{\sigma} G \), then the map \( X : \Sigma \to G/H \) is harmonic. The converse is also true: if \( X : \Sigma \to G/H \) is a harmonic map, then \( \tilde{\sigma} \circ X : \Sigma \to G \) is also harmonic. This can be alternatively understood by recalling that \( \tilde{\sigma} \) is a totally geodesic embedding. By definition, this means that the second fundamental form of \( \tilde{\sigma}(G/H) \subset G \) vanishes: \( (\nabla_X Y) \perp = 0 \) for any two vectors \( X, Y \in T(\tilde{\sigma}(G/H)) \). It is easy to check that if \( M \subset N \) is a totally geodesic submanifold and \( X : \Sigma \to M \) is a harmonic map, then \( \tilde{\sigma} \circ X : \Sigma \to N \) is also harmonic.

After this small digression let us return to formulas (1.10)–(2.5):

\[ \Phi_z = -iUV, \quad \mu_\zeta = VU|_{\mathfrak{g}} = 0, \]  \hspace{1cm} (3.3)

where \( \mathfrak{g} = \text{Lie}(P_{d_1,\ldots,d_{m-1}}) \).

3.1. Grassmannian. To begin with, we consider the case of a Grassmannian, i.e., \( m = 2 \). Then the vanishing of the moment map is expressed simply as \( VU = 0 \). Therefore, \( \Phi_z^2 = 0 \). From the expression for \( \Phi_z \) it also follows that \( \text{Im}(\Phi_z) \subset \text{Im}(U) \subset \text{Ker}(\Phi_z) \). As is well known,

\[ \{(U, \Phi_z) : \text{rk}(U) = M, \ \Phi_z^2 = 0, \ \text{Im}(\Phi_z) \subset \text{Im}(U) \subset \text{Ker}(\Phi_z)\} \simeq T^*G(M, N) \]  \hspace{1cm} (3.4)

is the cotangent bundle to a Grassmannian, and the forgetful map

\[ T^*G(M, N) \to \{\Phi_z : \Phi_z^2 = 0\} \]  \hspace{1cm} (3.5)

provides a resolution of singularities of the nilpotent orbit on the right-hand side (Springer resolution). The conditions on the left-hand side of (3.4) imply the factorization (3.3) for \( \Phi_z \), and the nonuniqueness in this factorization corresponds exactly to the gauge symmetry \( U \to Ug, V \to g^{-1}V \), where \( g \in \text{GL}(M, \mathbb{C}) \).

Let us now derive the equations of motion for the field \( \Phi_z \). First of all, the Lagrangian (1.8) implies the following equations of motion for the fields \( U \) and \( V \):

\[ \mathcal{D}_U U = 0, \quad \mathcal{D}_V V = 0. \]  \hspace{1cm} (3.6)

Therefore, \( \mathcal{D}_V \Phi_z = 0 \), i.e.,

\[ \partial_V \Phi_z + i[\Phi_z, \Phi_z] = 0. \]  \hspace{1cm} (3.7)

This equation is nothing but the equation of motion of the principal chiral field. Indeed, introduce a 1-form \( j = i(\Phi_z dz + \Phi_z d\zeta) \) with values in the Lie algebra \( \mathfrak{u}_N \). In this case (3.7) together with the Hermitian conjugate equation can be written in the form of two conditions

\[ d*j = 0, \quad dj - j \wedge j = 0, \]  \hspace{1cm} (3.8)
which are the equations of motion of the principal chiral field. This is consistent with the fact, proved in \cite{3}, that the Noether current of the model (1.3) is flat.

In other words, in the case of a Grassmannian sigma model, the field $\Phi_z$ satisfies the equations

$$ \partial_z \Phi_z + i [\Phi_z, \Phi_z] = 0, \quad \Phi_z^2 = 0. \quad (3.9) $$

It is fairly clear, though, that these equations do not completely characterize the Grassmannian sigma model. Indeed, for example, the solution $\Phi_z = 0$ of equations (3.9) corresponds in fact to a whole large class of sigma model solutions: $D_z U = 0$, $D_z V = 0$, $UV = 0$. Since, according to the assumption, $U^T U$ is a nondegenerate matrix, it follows from the last condition that $V = 0$, and as a result one is left with the equation $D_z U = 0$, i.e., a holomorphic map $U$ to $G(M, N)$. The converse is also true: if $D_z U = 0$, then $\Phi_z U = 0$, and it follows from $\Phi_z = UV$ that $\Phi_z \Phi_z = \Phi_z^2 \Phi_z = 0$, i.e., $\Phi_z = 0$. As a result, one has the correspondence

$$ \Phi_z = 0 \quad \leftrightarrow \quad \text{Holomorphic curves in } G(M, N). \quad (3.10) $$

Therefore, the equation for $U$ is essential in general. As for the field $V$, it can be completely excluded and replaced by the field $\Phi_z$. Indeed, from the conditions on the left-hand side of (3.4) one obtains the factorization $\Phi_z = UV$, and the equations $\mathcal{D}_z U = 0$ and $\mathcal{D}_z \Phi_z = 0$ imply $U \mathcal{D}_z V = 0$. The condition that $U$ is a matrix of rank $M$ leads to $\mathcal{D}_z V = 0$. In other words, an alternative formulation of the model is as follows: we consider the fields $(U, \Phi_z)$ satisfying the conditions

$$ \text{rk}(U) = M, \quad \Phi_z^2 = 0, \quad \text{Im}(\Phi_z) \subset \text{Im}(U) \subset \text{Ker}(\Phi_z). \quad (3.11) $$

In this case the equations of motion take the form

$$ \mathcal{D}_z U = 0, \quad \mathcal{D}_z \Phi_z = 0. \quad (3.12) $$

Next we describe a situation when the equation for $U$ is indeed redundant.

**Lemma.** The equation $\mathcal{D}_z U = 0$ follows from $\mathcal{D}_z \Phi_z = 0$ and from conditions (3.11) if and only if $\text{Ker}(\Phi_z) \simeq \text{Im}(U)$.

**Proof.** Multiplying $D_z U$ by $\Phi_z$ and using the equation $D_z \Phi_z = 0$ and the condition $\Phi_z U = 0$, we obtain $\Phi_z D_z U = D_z (\Phi_z U) = 0$. Therefore, the equation $\Phi_z D_z U = 0$ holds identically. It is equivalent to $\mathcal{D}_z U = 0$ if and only if $\text{Ker}(\Phi_z) \simeq \text{Im}(U)$. $\square$

The condition $\Phi_z^2 = 0$ means that the Jordan structure of $\Phi_z$ consists of $m$ cells of size $2 \times 2$ and $n$ cells of size $1 \times 1$. In this case $N = 2m + n$ and $\text{dim} \text{Ker}(\Phi_z) = m + n$. Since $\text{Im}(U) \subset \text{Ker}(\Phi_z)$ and $\text{rk}(U) = M$, we get the condition $M \leq m + n$. This easily leads to $m \leq N - M$ and $n \geq 2M - N$, and the inequalities are saturated precisely in the case $\text{Ker}(\Phi_z) \simeq \text{Im}(U)$ considered earlier. In this case the number of $2 \times 2$ cells is maximal and equal to $N - M$, and the number of cells of size $1 \times 1$ is $2M - N$. Note that this is only possible in the case $M \geq N/2$. A reduction of the number of $2 \times 2$ cells corresponds to the degeneration of the matrix $\Phi_z$.

The dynamical equation (3.9) imposes severe constraints on the way in which the Jordan structure of the matrix $\Phi_z$ can change as one varies the point $z, \bar{z}$ on the worldsheet. Indeed, it implies that $\Phi_z = kQ(z)k^{-1}$, where $Q(z)$ is a matrix that depends holomorphically on $z$. The Jordan structure of the matrix $Q(z)$ is the same as that of $\Phi_z$, and the vanishing of the Jordan blocks occurs holomorphically in $z$. In particular, the Jordan structure changes only at “special points,” isolated points on the worldsheet. As a result, “almost everywhere” the dimension of the kernel $\text{dim} \text{Ker}(\Phi_z) := \hat{M}$ is the same, and the map $\lambda: (z, \bar{z}) \rightarrow \text{Ker}(\Phi_z)$ is a map to the Grassmannian $G(M, N)$. A more careful analysis of the behavior of $\Phi_z$ at a special point would show that $\lambda$ can be extended to these points. We have come to the following conclusion: let $g(z, \bar{z})$ be a solution of
the principal chiral model, i.e., a harmonic map to the group $G$ satisfying the condition $\Phi_z^2 = 0$, where $\Phi_z := g^{-1}z\partial_z g$ is a component of the Noether current, and let the dimension of the kernel of $\Phi_z$ at a typical point of the worldsheet be $M$. Then one can construct a harmonic map to the Grassmannian $G(M, N)$ by the rule $(z, \overline{z}) \rightarrow \text{Ker}(\Phi_z)$.

3.2. The SU(2)-case. We consider the special case $N = 2$, when $G = \text{SU}(2)$. In this case $\text{Tr}(\Phi_z) = 0$, and the condition $\Phi_z^2 = 0$ is equivalent to $\text{Tr}(\Phi_z^2) = 0$. The latter condition is, in turn, the Virasoro constraint, i.e., the condition that the harmonic map is minimal. As a result, with a minimal surface in $G = \text{SU}(2)$ satisfying $\Phi_z \neq 0$, we can associate a harmonic map into $\mathbb{CP}^1$. Let us construct it explicitly. The Cartan embedding $\mathbb{CP}^1 \hookrightarrow \text{SU}(2)$ has the form $g = 1_2 - 2w \otimes \overline{w}$, where $w \in \mathbb{CP}^1 (\|w\| = 1)$. The Noether current is $\Phi_z = 2(w \otimes D_z w - D_z w \otimes \overline{w})$, where $D_z w = \partial_z w - (\overline{w} \circ \partial_z w) w$, and its square is $\Phi_z^2 = -4(D_z w \otimes D_z w - (D_z \overline{w} \circ D_z w) w \otimes \overline{w})$. Multiplying the condition $\Phi_z^2 = 0$ by $w$ from the right and taking into account that $D_z \overline{w} \circ D_z w = 0$, we get $D_z \overline{w} \circ D_z w = 0$; therefore, $D_z w \otimes D_z \overline{w} = 0$. We see that the map $w(z, \overline{z})$ is either holomorphic or antiholomorphic. If $D_z \overline{w} = 0$, then the null-vector $\chi$ of the matrix $\Phi_z(\Phi_z \chi = 0)$ satisfies the constraint $\overline{w} \circ \chi = 0$, i.e., $\chi$ is the antipodal point to $w$ on $\mathbb{CP}^1$. If $D_z w = 0$, then, as is easy to verify, $D_z w$ is a holomorphic map and the null-vector $\chi$ satisfies the condition $D_z \overline{w} \circ \chi = 0$. Therefore, in both cases we conclude that $\chi$ is an antiholomorphic map related to a holomorphic map $(w$ or $D_z w)$ by the antipodal involution.

3.3. The partial flag manifold. Let us extend the results of Subsection 3.1 to more general flag manifolds. We return first to equation (3.7) for $\Phi_z$, but this time we assume that the matrix $\Phi_z$ satisfies, at a typical point $(z, \overline{z}) \in \Sigma$, the condition

$$\Phi_z^m = 0 \quad \text{and} \quad \Phi_z^{m-1} \neq 0.$$  

(3.13)

The matrix $\Phi_z$ naturally defines a flag

$$f := \{0 \subset \text{Ker}(\Phi_z) \subset \text{Ker}(\Phi_z^2) \subset \ldots \subset \text{Ker}(\Phi_z^m) \simeq \mathbb{C}^N\}.$$  

(3.14)

**Proposition.** Given a matrix $\Phi_z$ satisfying (3.7) and (3.13), the map $(z, \overline{z}) \rightarrow f$ is a solution to the equations of motion of the flag manifold sigma model (1.8).

**Proof.** Consider a matrix $U$ of the form $U = (U_{m-1}|\ldots|U_1)$, where $U_i$ is a matrix whose columns are the linearly independent vectors from $\text{Ker}(\Phi_z^i)/\text{Ker}(\Phi_z^{i-1})$. Let us relate the dimensions of these spaces to the the number and dimensions of the Jordan cells of the matrix $\Phi_z$. To this end we bring $\Phi_z$ to the Jordan form

$$\Phi_z^{(0)} = \text{Diag}\{J_{s_1}, \ldots, J_{s_\ell}\}, \quad \sum_{j=1}^{\ell} s_j = N,$$  

(3.15)

where $J_s$ is a Jordan cell of size $s \times s$. We have chosen the ordering $s_1 \geq \ldots \geq s_\ell$, where, according to the assumption (3.13), $s_1 = m$. We denote by $\kappa_i$ the number of Jordan cells of size at least $i$ ($\kappa_1 = \ell$). The following two properties are obvious:

- $\kappa_{i+1} \leq \kappa_i$; i.e., $\kappa_1, \ldots, \kappa_m$ is a nonincreasing sequence;
- $\dim \text{Ker}(\Phi_z) = \kappa_1$, $\dim \text{Ker}(\Phi_z^2) = \kappa_1 + \kappa_2$, etc.; therefore, $\dim \text{Ker}(\Phi_z^i)/\text{Ker}(\Phi_z^{i-1}) = \kappa_i$.

Therefore, $U_i \in \text{Hom}(\mathbb{C}^\kappa_i, \mathbb{C}^M)$ and $U \in \text{Hom}(\mathbb{C}^M, \mathbb{C}^N)$, where $M = \sum_{i=1}^{m-1} \kappa_i$.

Since, by construction, $\text{Im}(U) \simeq \text{Ker}(\Phi_z^{m-1})$ and $\text{Im}(\Phi_z) \subset \text{Ker}(\Phi_z^{m-1})$, we have $\text{Im}(\Phi_z) \subset \text{Im}(U)$; i.e., there exists a matrix $V \in \text{Hom}(\mathbb{C}^N, \mathbb{C}^M)$ such that $\Phi_z = -iUV$. Let us now derive the equations

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1 Note that the condition $\Phi_z = 0$ implies $g(z, \overline{z}) = g_0$; i.e., in this case the map is trivial.
of motion for the matrices $U$ and $V$. Since $\Phi_z^k U_k = 0$ and $D_\tau \Phi_z = 0$, one has $\Phi_z^k U_k = 0$. The columns of the matrix $(U_k | \ldots | U_1)$ span the kernel of $\Phi_z^k$; hence $D_\tau U_k = -i \sum_{j \leq k} U_j A_{jk}$, where $A_{jk}$ are matrices of relevant sizes. Out of the matrices $A_{jk}$, $1 \leq j \leq k \leq m-1$, we form a single matrix $A_\tau$, which schematically looks as in (1.4). Then, clearly, the following equation holds:

$$D_\tau U = \partial_\tau U + iU A_\tau + i \Phi_z U = 0. \quad (3.16)$$

Since $\Phi_z = -iUV$, from the nondegeneracy of $U$ it follows that $D_\tau V = 0$. Next, we have $U_1, \ldots, U_k \subset \text{Ker}(\Phi_z^k)$, so in the matrix $\Phi_z^k U \sim U(VU)^k$ the last $\sum^k_{i=1} \kappa_i$ columns vanish; therefore, the matrix $(VU)^k$ is strictly lower triangular and has zeros on the first $k$ block diagonals (the main diagonal is counted as the first one). We denote by $\mathfrak{k}$ the parabolic subalgebra of $\mathfrak{gl}_M$ that stabilizes the subflag of (3.14) with the last element omitted. We have proved that $VU|_{\mathfrak{k}} = 0$. Therefore, a solution $\Phi_z(z, \tau)$ of system (3.7), (3.13) produces a solution $(U, V)$ to the equations of motion of the sigma model with target space given by the flag manifold

$$U(N)/U(\kappa_1) \times \ldots \times U(\kappa_m), \quad (3.17)$$

where $\kappa_j = \dim(\text{Ker}(\Phi_z^j)/\text{Ker}(\Phi_z^{j-1}))$ is a nonincreasing sequence. The complex structure on the flag is uniquely determined by the structure of the complex flag (3.14). $\square$

4. CONCLUSIONS

In the present paper we have obtained two principal results. First, we showed that the flag manifold sigma models introduced earlier by the author can be alternatively formulated as two coupled $\beta\gamma$-systems interacting via an auxiliary field $\Phi_z$. This proves the equivalence of these models and the flag manifold models described in [11] (in the terminology of that paper the field $\Phi_z$ should be interpreted as the holomorphic component of the gauge field $A_w$ along the “topological plane” that coincides with the worldsheet $\Sigma$ of the sigma model). We believe that the gauged linear formulation of the $\beta\gamma$-systems for flag manifold models introduced in the present paper will also be useful for the investigation of their trigonometric ($\eta$) deformations (there is a vast literature on the subject; see, for example, [14–16, 18–21]). The first steps in this direction were made in [7].

Second, we investigated the relation between the flag manifold sigma models and the principal chiral model. We showed that the solutions of the principal chiral model, which define a map into the nilpotent orbit of the corresponding complexified group Lie, correspond to solutions of the flag manifold sigma model (see [12] for a nice review of the theory of nilpotent orbits). It seems likely that the full analysis of this correspondence will require the theory of the Springer resolution (see, for example, [17]); this will be a subject of further investigation.

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REFERENCES

1. D. Alekseevsky and A. Arvanitoyeorgos, “Riemannian flag manifolds with homogeneous geodesics,” Trans. Am. Math. Soc. 359 (8), 3769–3789 (2007).

2. O. Brodbeck and M. Zagermann, “Dimensionally reduced gravity, Hermitian symmetric spaces and the Ashtekar variables,” Classical Quantum Gravity 17 (14), 2749–2763 (2000); arXiv: gr-qc/9911118.

3. D. Bykov, “Complex structures and zero-curvature equations for \( \sigma \)-models,” Phys. Lett. B 760, 341–344 (2016); arXiv:1605.01093 [hep-th].

4. D. V. Bykov, “A gauged linear formulation for flag-manifold \( \sigma \)-models,” Theor. Math. Phys. 193 (3), 1737–1753 (2017) [transl. from Teor. Mat. Fiz. 193 (3), 381–400 (2017)].

5. D. Bykov, “The \( 1/N \)-expansion for flag-manifold \( \sigma \)-models,” Theor. Math. Phys. 197 (3), 1691–1700 (2018) [transl. from Teor. Mat. Fiz. 197 (3), 345–355 (2018)].

6. D. Bykov, “Flag manifold \( \sigma \)-models: The \( 1/N \)-expansion and the anomaly two-form,” Nucl. Phys. B 941, 316–360 (2019); arXiv:1901.02861 [hep-th].

7. D. Bykov, “Quantum flag manifold \( \sigma \)-models and Hermitian Ricci flow,” arXiv:2006.14124 [hep-th].

8. A. G. Bytsko, “The zero-curvature representation for the nonlinear \( O(3) \) sigma-model,” J. Math. Sci. 85 (1), 1619–1628 (1997) [transl. from Zap. Nauchn. Semin. POMI 215, 100–114 (1994)]; arXiv:hep-th/9403101.

9. K. Costello, E. Witten, and M. Yamazaki, “Gauge theory and integrability. I,” ICCM Not. 6 (1), 46–119 (2018); arXiv:1709.09993 [hep-th].

10. K. Costello, E. Witten, and M. Yamazaki, “Gauge theory and integrability. II,” ICCM Not. 6 (1), 120–146 (2018); arXiv:1802.01579 [hep-th].

11. K. Costello and M. Yamazaki, “Gauge theory and integrability. III,” arXiv:1908.02289 [hep-th].

12. P. Crooks, “Complex adjoint orbits in Lie theory and geometry,” arXiv:1703.03390 [math.AG].

13. F. Delduc, T. Kameyama, S. Lacroix, M. Magro, and B. Vicedo, “Ultralocal Lax connection for para-complex \( \mathbb{Z}_2 \)-cosets,” arXiv:1909.00742 [hep-th].

14. F. Delduc, M. Magro, and B. Vicedo, “On classical \( q \)-deformations of integrable \( \sigma \)-models,” J. High Energy Phys. 2013 (11), 192 (2013); arXiv:1308.3581 [hep-th].

15. V. A. Fateev, “The sigma model (dual) representation for a two-parameter family of integrable quantum field theories,” Nucl. Phys. B 473 (3), 509–538 (1996).

16. V. A. Fateev, E. Onofri, and A. B. Zamolodchikov, “Integrable deformations of the \( O(3) \) sigma model. The sausage model,” Nucl. Phys. B 406 (3), 521–565 (1993).

17. V. Ginzburg, “Geometric methods in the representation theory of Hecke algebras and quantum groups,” arXiv:math/9802004 [math.AG].

18. B. Hoare, N. Levine, and A. A. Tseytlin, “Integrable 2d sigma models: Quantum corrections to geometry from RG flow,” Nucl. Phys. B 949, 114798 (2019); arXiv:1907.04737 [hep-th].

19. C. Klimčík, “On integrability of the Yang–Baxter \( \sigma \)-model,” J. Math. Phys. 50 (4), 043508 (2009); arXiv:0802.3518 [hep-th].

20. C. Klimčík, “Integrability of the bi-Yang–Baxter \( \sigma \)-model,” Lett. Math. Phys. 104 (9), 1095–1106 (2014); arXiv:1402.2105 [math-ph].

21. S. L. Lukyanov, “The integrable harmonic map problem versus Ricci flow,” Nucl. Phys. B 865 (2), 308–329 (2012); arXiv:1205.3201 [hep-th].

22. N. A. Nekrasov, “Lectures on curved beta–gamma systems, pure spinors, and anomalies,” arXiv:hep-th/0511008.

23. E. Witten, “Topological sigma models,” Commun. Math. Phys. 118 (3), 411–449 (1988).

24. E. Witten, “Two-dimensional models with (0, 2) supersymmetry: Perturbative aspects,” Adv. Theor. Math. Phys. 11 (1), 1–63 (2007); arXiv:hep-th/0504078.

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