GAMOW VECTORS AND BOREL SUMMATION

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Abstract. We analyze the detailed time dependence of the wave function \( \psi(x, t) \) for one dimensional Hamiltonians \( H = -\partial_x^2 + V(x) \) where \( V \) (for example modeling barriers or wells) and \( \psi(x, 0) \) are compactly supported.

We show that the dispersive part of \( \psi(x, t) \), its asymptotic series in powers of \( t^{-1/2} \), is Borel summable. The remainder, the difference between \( \psi \) and the Borel sum, is a convergent expansion of the form \( \sum_{k=0}^{\infty} g_k \Gamma_k(x) e^{-\gamma_k t} \), where \( \Gamma_k \) are the Gamow vectors of \( H \), and \( \gamma_k \) are the associated resonances; generally, all \( g_k \) are nonzero. For large \( k \), \( \gamma_k \sim \text{const} \cdot k \log k + k^2 \pi^2 i/4 \). The effect of the Gamow vectors is visible when time is not very large, and the decomposition defines rigorously resonances and Gamow vectors in a nonperturbative regime, in a physically relevant way.

The decomposition allows for calculating \( \psi \) for moderate and large \( t \), to any prescribed exponential accuracy, using optimal truncation of power series plus finitely many Gamow vectors contributions.

The analytic structure of \( \psi \) is perhaps surprising: in general (even in simple examples such as square wells), \( \psi(x, t) \) turns out to be \( C^\infty \) in \( t \) but nowhere analytic on \( \mathbb{R}^+ \). In fact, \( \psi \) is \( t \)-analytic in a sector in the lower half plane and has the whole of \( \mathbb{R}^+ \) a natural boundary.

Extension to other types of potentials, for instance analytic at infinity, is briefly discussed, and in the process we study the singularity structure of the Green’s function in a neighborhood of zero, in energy space.

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1. INTRODUCTION

Resonances play a major role in the physics of metastable states and their decay. From a mathematical standpoint, there is a good number of definitions of resonances and resonant states. In most approaches, they are based on the properties of the scattering matrix, on Gelfand triples (rigged Hilbert spaces), or on the complex analytic singular structure of the Green’s function beyond the spectrum of the resolvent. The pole positions of the Green’s function, “resonances” are pseudo-eigenvalues, and their residues (Gamow vectors) are pseudo-eigenvectors of the Hamiltonian with “purely growing” conditions at infinity. There is a vast literature on the subject, see e.g. the concise overview [13] and the references therein. See also [9] for a surprising consequence of resonant states, and for a clear description of the physical relevance of Gamow vectors.

By and large, the different mathematical definitions provide equivalent objects. However, there are conceptual difficulties in all rigorous approaches, and these lie in connecting (a) the mathematical definition, (b) the natural properties of the underlying quantum Hamiltonian, and (c) the physical phenomenon. In fact, Howland’s Razor, a principle so dubbed by B. Simon, cf. [19], states that no satisfactory definition of resonance can depend on the structure of a single operator on an abstract Hilbert space. Slightly oversimplifying the argument, the reason is that the analytic structure of the Green’s function, or of quantities obtained through dilation-analyticity, needed in most approaches, are by no means unitarily invariant. Unitary invariance plays of course an important role in quantum mechanics since observables are represented by self-adjoint operators on Hilbert spaces, the isomorphisms of which are precisely the family of all unitary transformations.

A concise and very illuminating critical analysis of the various mathematical attempts at rigorous definitions is found in [19].

1.1. Resonances and asymptotic expansions. We note however that many relevant physical quantities are not and need not be defined in a unitarily-equivalent way. As already mentioned, resonances are used in measuring the time decay of the probability distribution in physical space. In any interpretation of quantum mechanics, $L^2(\mathbb{R}^3)$ plays a distinguished role, when $\mathbb{R}^3$ is a representation of the space where we, and macroscopic apparatuses, lie.

A definition based on time behavior is natural to the underlying physics and avoids Howland’s razor since it rests on (i) a particular representation of the Hamiltonian–as an operator on $L^2$ of our $\mathbb{R}^3$, (ii) on a second observable, say $1_A$, the characteristic function of the set $A \in \mathbb{R}^3$ and (iii) on a specific mathematical question–the time decay of $\langle \psi | 1_A | \psi \rangle$. This triad is not (at least not manifestly) a property of a single operator. Nonetheless, $L^2(\mathbb{R}^3)$ and dependence on time are canonical objects in analyzing scattering or decay problems.

At the present time however rigorous definitions based on time behavior only exist in a perturbative regime, [20], [14]; see also below.

In this paper, for compactly supported potentials in one dimension, we show that the difference between the wave function and the Borel sum of its asymptotic series in powers of $t^{-1/2}$ is a convergent expansion in Gamow vectors. The resonances thus defined turn out to be independent of the initial condition. Gamow vectors are not $L^2$ functions; neither is the Borel sum (see [15]) of the power series. The expansion is valid uniformly on compact sets instead.
The representation as a Borel summed series plus Gamow vectors expansion is valid not only for large \( t \), but, in fact, simply for \( t > 0 \), though the convergence rate of the whole expansion is rapid enough only if \( t \) is not too small.

After completing a manuscript we found that decompositions in energy space in terms of Gamow vectors and a continuous part have been proposed in the physics literature, see [8], to our knowledge without completely rigorous, mathematical, proofs or study of Borel summability, and with a different interpretation and suggested physical meaning; cf. Note 9 below. Without Borel summability, uniqueness of a decomposition in terms of a continuum integral and a sum of exponentials generally does not hold, see §5.

The \( t^{-1/2} \) power series expansion roughly corresponds to the decay of a free particle. Indeed, if time is very long and the point spectrum of \( H \) is empty, then, eventually, the overlap between the wave function and the support of the potential becomes negligible. The specifics of the potential are seen while the particle has a fair probability of it being near the potential. This is why it is natural to subtract out the power series, “free” decay. But, generally, the series has zero radius of convergence.

If parameters are such that a resonance (complex generalized eigenvalue, [14]) is at a small distance \( \epsilon \) to the spectrum of \( H \), the setting is called perturbative and there is a time scale, roughly given by \( e^{-\epsilon t} \gg t^{-3/2} \), during which in a finite spatial interval, the decay of the position probability follows an exponential law. This corresponds to a transient, metastable state. The Gamow vector corresponding to such a resonance describes the wave function on increasingly larger spatial regions, see [9], §9. Only metastable states with long enough survival time are captured however in this way. (Rigorously speaking, we are dealing with a double limit, in which time goes to infinity and an external parameter goes to zero in some correlated fashion.) Borel summation provides an exact representation for all \( t > 0 \), as well as practical ways to calculate the wave function for times of order one, see §5.1; the influence of resonances which are not necessarily close to the spectrum is measurable.

For showing Borel summability, perhaps the most delicate part is the analysis of the Green’s function in the fourth quadrant in the energy parameter, where infinitely many poles recede rapidly to infinity; sharp estimates are needed in order to control a needed Bromwich contour integral.

Extension to other potentials with sufficient analyticity and decay is discussed in §5.4.

2. Setting and main results

We consider the one-dimensional Schrödinger equation

\[
\frac{i\hbar}{\partial t} \psi(x, t) = -\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) + V(x) \psi(x, t)
\]

where:

1. The influence of the potential—however distant—is still present in the “initial state” at some very late time \( t_i \gg 1 \), from which the almost free particle decays; the state at \( t_i \) is responsible for the generic disappearance of the zero energy resonance.

2. If the potential is unbounded, such as a dipole \( V(x) = Ex \), then the power series may be identically zero, see [11], [21] and references therein. Another exception is \( V = 0 \), for which the asymptotic \( t^{-1/2} \) series converges on compact sets in \( x \).
(a) The nonzero potential $V$ is independent of time, compactly supported and $C^2$ on its support. ($V$ is allowed to be discontinuous at the endpoints provided that it is one-sided $C^2$ at the endpoints.)

(b) The initial condition $\psi_0(x)$ is compactly supported and $C^2$ on its support.

We normalize the equation to

\begin{equation}
\frac{i}{\partial t} \psi(x,t) = -\frac{\partial^2}{\partial x^2} \psi(x,t) + V(x)\psi(x,t) = (H\psi)(x,t)
\end{equation}

where supp($V$) $\subset [-1,1]$. Under the assumptions above, we have the following results.

**Proposition 1.** For large $t$, the wave function $\psi(x,t)$ is $O(t^{-1/2})$ (in the generic case of absence of zero energy resonance \cite{10}, $\psi(x,t) = O(t^{-3/2})$), and $\psi(x,t)$ has a Borel summable asymptotic series $\tilde{\psi}(x,t)$ in powers of $t^{-1/2}$.

We denote as usual by $\mathcal{LB}$ the Borel summation operator. Let $t^{-1/2}\varphi(x,t) = \mathcal{LB}\tilde{\psi}(x,t)$ where $\varphi(x,\cdot)$ is bounded. As seen below, $\psi(x,t) - t^{-1/2}\varphi(x,t)$ is nonzero, and is a convergent combination of Gamow vectors, the residues at the poles of the analytic continuation of the resolvent of $H$.

Let $\{E_k\}_{k=1,\ldots,N}$ be the eigenvalues of $H$ and $\{\psi_k\}_{k=1,\ldots,N}$ be the corresponding eigenfunctions. (We convene to set $N = 0$ if these two sets are empty.) Let also $\gamma_k, \text{Re}\gamma_k > 0$ be the generalized eigenvalues (resonances) corresponding to the Gamow vectors $\Gamma_k(x)$.

**Theorem 1.** (i) For all $t > 0$ we have

\begin{equation}
\psi(x,t) - t^{-1/2}\varphi(x,t) = \sum_{k=1}^{N} b_k \psi_k(x)e^{-E_k t} + \sum_{k=1}^{\infty} g_k \Gamma_k(x)e^{-\gamma_k t}
\end{equation}

The infinite sum in (2) is uniformly convergent on compact sets in $x$—rapidly so if $t$ is large. (The coefficients $b_k$ and $g_k$ depend on $\psi$ and typically $g_k \neq 0$ for all $k$.)

(ii) $\psi_k(x),\Gamma_k(x), \varphi(x,t)$ are twice differentiable in $x$.

(iii) We have

\begin{equation}
\gamma_k \sim \text{const} \cdot k \log k + k^2 \pi^2 i/4 \quad \text{as} \quad k \to +\infty
\end{equation}

(\text{Higher orders depend on $V$, see Proposition 3}) The $\gamma_k$ are independent of $\psi_0$, and the constant depends on the endpoint behavior of $V$.

The series in (2), though valid for all $t$, converges poorly if $t \to 0$: this is not the regime it is intended for.

Let

\begin{equation}
E(u,t) = \sqrt{\frac{x^2}{t} + e^{-u^2 t}} \left( u^2 \sqrt{\pi t} E_2(-u^2 t) + u E_1(-u^2 t) \right)
\end{equation}

where $E_n$ is the $n$-exponential integral and $\text{arg} \ u \in (-\pi,0) \cup (0,\pi)$. \cite{4} \cite{4}
Proposition 2 \((o(e^{-Mt})\) accuracy, for arbitrary \(M\)). For any \(M\) there exists \(m, m_1, m_2\) such that

\[
\psi(x,t) = \sum_{k=1}^{m} b_k \psi_k(x)e^{-E_k t} + \sum_{k=1}^{m_1} g_k \Gamma_k(x)e^{-\gamma_k t} - \sum_{k=1}^{m_2} r_k \mathcal{E}(\gamma_k t) + \psi_M(x,t)
\]

Here \(\{\gamma_k\}_{k\leq m}\) are the poles of the Green’s function (resonances) on the first and second Riemann sheet with \(|\gamma_k| \leq M\), \(\{\gamma_k\}_{k\leq m_1}\) are the subset of them on the first Riemann sheet, \(r_k\) are the corresponding residues, and \(\psi_M\) differs by \(o(e^{-Mt})\) from its \(\psi(x,t)\) with \(x > t\), optimally truncated (see \[5.7\]).

**Note 1.** (i) We see that, as exponential contributions, only the resonances on the first Riemann sheet appear but both sheets contribute to the dispersive part.

(ii) The expression \(\psi(x,t)\) is not analytic on the Riemann surface of the log: it has a jump on \(\mathbb{R}^+\), compensated by an opposite jump of \(\psi_M\). These jumps are mandated by least term summability requirements.

**Corollary 3.** Any number of resonances can be calculated from \(\psi(x,t)\), if \(\psi\) is known with correspondingly high accuracy. Conversely, \(\psi\) can be calculated in principle with arbitrary accuracy from the contribution of a finite number of bound states, resonances, exponential integrals and optimal truncation of power series.

**Note 2** \(\psi(x,t)\) for arbitrary \(M\). It follows from the proof that \(\ln |g_k \Gamma_k| = O(\sqrt{p_k})\). Thus, since \(\varphi\) is manifestly analytic for \(\text{Re}\ t > 0\), it follows immediately from [2] that \(\psi\) is \(C^\infty\) in \(t\). Now, since \(\text{Im}\ -\gamma_k \sim -k^2\), near \(\mathbb{R}^+\), \(\psi\) equals a function analytic in the right half plane (the Laplace transform) plus a lacunary Dirichlet series, convergent for \(\text{Im}\ t < 0\) (the “heat-like” direction). For generic \(x\), the coefficients of the Dirichlet series are bounded below by \(e^{-\text{const}(x)\sqrt{|p|}}\) (all functions involved are of exponential order 1/2; the lower bounds follow relatively easily from the proofs, but we omit the details). Then the Dirichlet series does not converge past \(\mathbb{R}^+\); general theorems on lacunary series, see e.g. [15] imply then that \(\mathbb{R}^+\) is a natural boundary. See also Proposition [8]. Using similar estimates it can be checked that the Taylor coefficients of \(\psi\) at a point \(t_0\) behave roughly like \(\left(\frac{\pi^4 k^2}{4e^2 t_0^2 \ln^2 k}\right)^k\), showing once more that there is no point of analyticity on \(\mathbb{R}^+\).

This is another way to see the contribution of the Gamow vectors to the properties of \(\psi\). (More details about this are part of a future paper.)

### 3. Proofs of Main Results

#### 3.1. Integral reformulation of the problem.

\(H\) satisfies the assumptions of Theorem X.71, [15] v.2 pp 290. Thus, for any \(t\), \(\psi(t,\cdot)\) is in the domain of \(-d^2/dx^2\).

This implies continuity in \(x\) of \(\psi(t,x)\) and of its \(t\)--Laplace transform. It also follows that the unitary propagator \(U(t)\) is strongly differentiable in \(t\). Existence of a strongly differentiable unitary propagator for \(\psi(x,t)\) implies existence of the Laplace transform

\[
\hat{\psi}(x,p) = \int_0^\infty e^{-pt} \psi(x,t)dt = \left(\int_0^\infty e^{-pt} U(t)dt\right) \psi_0(x)
\]

for \(\text{Re} p > 0\). Taking the Laplace transform of \(\psi(x,t)\) we obtain

\[
-ip\hat{\psi}(x,p) = \hat{\psi}_0(x) = -\frac{\partial^2}{\partial x^2} \hat{\psi}(x,p) + V(x)\hat{\psi}(x,p)
\]

(6)
where \( \psi_0(x) \) is the initial condition. Treating \( p \) as a parameter, we write \( \psi(x, p) = y(x; p) =: y(x) \), and obtain

\[
y''(x) - (V(x) - ip) y(x) = i\psi_0(x)
\]

where \( y(x) \in L^2(\mathbb{R}) \). The associated homogeneous equation is

\[
y''(x) = (V(x) - ip) y(x)
\]

If \( y_+(x), y_-(x) \) are two linearly independent solutions of (8) with the additional restrictions (and the usual branch of the log)

\[
y_+(x) = e^{-\sqrt{-ip}x} \text{ when } x > 1
\]

\[
y_-(x) = e^{\sqrt{-ip}x} \text{ when } x < -1
\]

then, for \( \text{Re } p > 0 \), the \( L^2 \) solution of (6) (or equivalently of (7)) is

\[
\hat{\psi}(x, p) = \frac{i}{W_p} \left( y_-(x) \int_{-\infty}^x y_+(s)\psi_0(s)ds - y_+(x) \int_{-\infty}^x y_-(s)\psi_0(s)ds \right)
\]

where the Wronskian \( W_p = y_+(x)y'_-(x) - y_-(x)y'_+(x) \) is easily seen to be independent of \( x \).

As we shall see, this solution is meromorphic in \( p \) except for a possible branch point at 0, and for fixed \( x \) it has sub-exponential bounds in the left half \( p \)-plane (when not close to poles). The function \( \psi \) is the inverse Laplace transform of \( \hat{\psi} \), and it can be written in the form \( \psi(x, t) = \frac{1}{2\pi i} \int_{\gamma_0+i\infty}^{\gamma_0-i\infty} \hat{\psi}(x, p)e^{pt}dp \). We show that the contour of integration can be pushed through the left half plane; collecting the contributions from poles and branch points, the decomposition follows.

**Note 3.** The domain of interest in \( p \) is a sector on the Riemann surface of the square root, centered on \( \mathbb{R}^+ \) and of opening slightly more than \( 2\pi \), which, in the variable \( \sqrt{-ip} \) translates into a sector of opening more that \( \pi \) centered at \( \sqrt{-i} \).

### 3.2. Analyticity of \( \hat{\psi} \) on the Riemann surface of \( \sqrt{p} \)

We start with the analyticity properties of \( \hat{\psi} \). The more delicate analysis of the asymptotic behavior of the analytic continuation of \( \hat{\psi} \) on the Riemann surface of the log at zero is done in [3.3] The existence of a square root branch point at zero is typical in this type of problems. For our analysis, in proving Borel summability, we need to show that \( \hat{\psi} \) is meromorphic in \( \sqrt{p} \).

**Proposition 4.** \( \hat{\psi}(x, p) \) is meromorphic in \( p \) on the Riemann surface of the square root at zero, \( \mathbb{C}_{1/2,0} \) and zero is a possible square root branch point.

**Proof.** This follows from the following simple argument. Note first that continuity of \( y \) and \( y' \) imply the following matching conditions:

\[
\begin{align*}
y_+(1) &= e^{-\sqrt{-ip}} \\
y'_+(1) &= -\sqrt{-ip}e^{-\sqrt{-ip}} \\
y_-(1) &= e^{\sqrt{-ip}} \\
y'_-(1) &= \sqrt{-ip}e^{\sqrt{-ip}}
\end{align*}
\]

Consider now the solutions \( f_1 \) and \( f_2 \) of (8) with initial conditions \( f_1(-1) = 1, f'_1(-1) = 0 \) and \( f_2(-1) = 0, f'_2(-1) = 1 \). By standard results on analytic parametric-dependence of solutions of differential equations (see, e.g. [12]), we see
that \( f_1 \) and \( f_2 \) are defined on \( \mathbb{R} \) and for fixed \( x \) they are entire in \( p \). We note that, by construction, the Wronskian \([f_1, f_2]\) is one. Then,
\[
y_+(x) = C_1 f_1(x) + C_2 f_2(x), \quad y_-(x) = C_3 f_1(x) + C_4 f_2(x)
\]
where
\[
C_1 = \sqrt{-ip} e^{-\sqrt{-ip}} (f_2(1) - f_2'(1)) \\
C_2 = -\sqrt{-ip} e^{-\sqrt{-ip}} (f_1(1) - f_1'(1)) \\
C_3 = -\sqrt{-ip} e^{-\sqrt{-ip}} \\
C_4 = \sqrt{-ip} e^{-\sqrt{-ip}}
\]
Furthermore,
\[\tag{11} W_p = -e^{-2\sqrt{-ip}} \left( ip(f_2(1) + f_2'(1)) - \sqrt{-ip}(f_1(1) + f_2'(1)) \right)\]
Thus \( y_+ \) and \( W_p \) are analytic in \( C_{1/2,0} \) with a possible branch point at zero. The same follows for \( \hat{\psi} \), by inspection, if we rewrite its expression as
\[\tag{12} \hat{\psi}(x,p) = \frac{i}{W_p} \left( y_-(x) \left( \int_1^x y_+(s)\psi_0(s)ds + \int_{-\infty}^1 e^{-\sqrt{-ip}s}\psi_0(s)ds \right) \\
- y_+(x) \left( \int_1^x y_-(s)\psi_0(s)ds + \int_{-\infty}^1 e^{-\sqrt{-ip}s}\psi_0(s)ds \right) \right) \]
\]
3.3. The poles for large \( p \) in the left half plane. To effectively calculate the asymptotic position of poles as \( p \to \infty \) in the left half plane, we need a more convenient choice for \( f_1, f_2 \). In the previous subsection they were chosen to be analytic in \( p \). Here we choose a new pair of \( f_1, f_2 \) for which the asymptotic behavior as \( p \to \infty \) is manifest.

**Note 4.** It is straightforward to check that if \( f_1(x) \) and \( f_2(x) \) are solutions of (5), and their Wronskian \( W_{f_1, f_2} = [f_1, f_2] \) is nonzero, then in the decomposition \( y_+(x) = C_1 f_1(x) + C_2 f_2(x), \ y_-(x) = C_3 f_1(x) + C_4 f_2(x) \) we have
\[
C_1 = \sqrt{-ip} e^{-\sqrt{-ip}} W_{f_1, f_2} (f_2(1) - f_2'(1)) \\
C_2 = -\sqrt{-ip} e^{-\sqrt{-ip}} W_{f_1, f_2} (f_1(1) - f_1'(1)) \\
C_3 = -\sqrt{-ip} e^{-\sqrt{-ip}} W_{f_1, f_2} (f_2(-1) + f_2'(1)) \\
C_4 = \sqrt{-ip} e^{-\sqrt{-ip}} W_{f_1, f_2} (f_1(-1) + f_1'(1))
\]
Furthermore, \( W_p = [y_+, y_-] = (C_1 C_4 - C_2 C_3) [f_1, f_2] \) is given by
\[\tag{13} W_p = \frac{e^{-2\sqrt{-ip}}}{W_{f_1, f_2}} \left( \sigma(f_1(1)f_2(-1) - f_1(-1)f_2(1)) - f_1'(1)f_2'(1) - f_1'(1)f_2(-1) + f_1(-1)f_2'(1) \right) \]
Proposition 5 (WKB solutions). In $S_+ = \{p : \text{Re}(\sqrt{-ip}) \geq 0\}$ there exist two linearly independent solutions of (3) of the form

$$f_1(x) = e^{-\sqrt{-ip}x} \left(1 - \frac{1}{2i\sqrt{p}} \int_0^x \sqrt{i}V(s)ds + \frac{1}{p}g_1(x)\right)$$  \hspace{1cm} (14)$$

$$f_2(x) = e^{\sqrt{-ip}x} \left(1 + \frac{1}{2i\sqrt{p}} \int_0^x \sqrt{i}V(s)ds + \frac{1}{p}g_2(x)\right)$$  \hspace{1cm} (15)$$

where $g_1(x), g'_1(x), g_2(x), g'_2(x)$ are bounded in $p$ as $p \to \infty$ in $S_+$. A similar statement holds $S_- = \{p : \text{Re}(\sqrt{-ip}) \leq 0\}$.

Note 5. This is in a sense standard WKB; however, since details about the regularity of the terms expansion are needed we provide a complete proof.

Proof. We will only prove the conclusion for $g_1$, since the proof for $g_2$ follows analogously. Substituting (14) into (3), we obtain the equation for $g_1$:

$$g''_1(x) - 2\sqrt{-ip} g'_1(x) - V(x)g_1(x) + \frac{1}{2i\sqrt{p}} \left(\int_0^x V(s)ds - V'(x)\right) = 0$$  \hspace{1cm} (16)$$

We rewrite this equation as an integral equation for $g'_1$:

$$g'_1(x) = e^{2\sqrt{-ip}x} \int_{x_0}^x e^{-2\sqrt{-ip}u} \left[ V(u) \int_0^u g'_1(s)du - \frac{\sqrt{i}}{2\sqrt{p}} \left(\int_0^u V(s)du - V'(s)\right)\right] du$$  \hspace{1cm} (17)$$

where $x_0 = 1$ if $-\pi/2 < \text{arg}p < 3\pi/2$, and $x_0 = -1$ if $-5\pi/2 < \text{arg}p < -\pi/2$. Note that $|e^{-2\sqrt{-ip}(s-x)}| \leq 1$ for all $s$ between $x_0$ and $x$. Using integration by parts we obtain

$$g'_1(x) = -\frac{1}{2\sqrt{-ip}} \left(V(x) \int_0^x g'_1(s)du - e^{-2\sqrt{-ip}(x_0 - x)}V(x_0) \int_0^{x_0} g'_1(s)du\right)$$

$$+ \frac{1}{2\sqrt{-ip}} \int_{x_0}^x e^{-2\sqrt{-ip}(s-x)} \left(V'(s) \int_0^s g'_1(s)du + V(s)g'_1(s)\right) ds$$

$$- \frac{1}{4i} \left(\int_0^x V(u)du - V'(x)\right) - e^{-2\sqrt{-ip}(x_0 - x)} \left(\int_0^{x_0} V(u)du - V'(x_0)\right)$$

$$+ \frac{1}{4i} \int_{x_0}^x e^{-2\sqrt{-ip}(s-x)} (V(s) - V''(s)) ds$$  \hspace{1cm} (18)$$

For large $p$, under the norm $\|f\| = \sup_{x \in [-1,1]} |f(x)|$ the above integral equation is easily seen to be contractive inside the ball

$$\|f\| \leq \sup_{-1 \leq x \leq 1} \left(\|V(x)\| + \|V'(x)\| + \|V''(x)\|\right)$$

Therefore $g'_1(x)$ and $g_1(x) = \int_0^x g'_1(s)du$ are both bounded in $p$ as $p \to \infty$. \qed

Remark 6. Higher order terms in the asymptotic expansion of $f_1, f_2$ can be similarly obtained, provided that $V$ is sufficiently smooth.
Recalling \((12)\), we see that for large \(p\), the poles of \(\psi\) can only come from the zeros of \(W_p\). Substituting \((14)\) and \((15)\) into \((13)\), we see that
\[
W_p = \frac{1}{p^2 h_3(p)} \left( e^{-4\sqrt{-ip}} h_1(p) + h_2(p) \right)
\]
where
\[
h_1(p) = \left( \sqrt{\pi} \frac{\sqrt{i}}{2} V(1) - g_1(1) \right) \left( \sqrt{\pi} \frac{\sqrt{i}}{2} V(-1) + g_2(1) \right)
\]
\[
h_2(p) = 4ip^3 + 2i\sqrt{i} (V(1) - V(-1)) p^{5/2} + O(p^2)
\]
and
\[
h_3(p) = -2\sqrt{-ip} + o(1)
\]

**Proposition 6.** In the generic case when \(h_1 \neq 0\), \(W_p\) has infinitely many zeros in the left half plane. Their asymptotic behavior is
\[
p = \begin{cases} 
-\frac{\pi^2}{3} k^2 - \pi k \log k + a_v k + o(k), & V(1)V(-1) \neq 0; \\
-\frac{\pi^2}{3} k^2 - \frac{5\pi}{4} k \log k + b_v k + o(k), & \text{exactly one of } V(\pm 1) \text{ is zero; } \\
-\frac{\pi^2}{3} k^2 - \frac{5\pi}{4} k \log k + c_v k + o(k), & V(1) = V(-1) = 0.
\end{cases}
\]
where \(k \in \mathbb{N}\) and \(k \to \infty\), and \(a_v, b_v, c_v\) are constants.

**Proof.** The equation \(W_p = 0\) reads
\[
e^{-4\sqrt{-ip}} = -\frac{h_2(p)}{h_1(p)}
\]
A simple analysis shows that this can only happen if \(p\) is near the negative imaginary line with \(p \sim -k^2 \pi^2 i/4\) where \(k \in \mathbb{N}\). We let \(p = -i(k\pi/2 + z)^2\) and rewrite \((24)\) in terms of \(z\):
\[
z = \frac{1}{4i} \log \left( \frac{h_2(-i(k\pi/2 + z)^2)}{h_1(-i(k\pi/2 + z)^2)} \right)
\]
Recalling \((20)\) and \((21)\), we easily see that the right hand side of the above equation is contractive for large \(k\).

One can find the asymptotic behavior of \(z\) by iteration. First assume \(V(1)V(-1) \neq 0\). It is easy to see that
\[
\frac{h_2(-i(k\pi/2)^2)}{h_1(-i(k\pi/2)^2)} = \frac{\pi^4 k^4}{4V(1)V(-1)} (1 + O(1/k))
\]
Therefore \(z \sim -i \log k\). Further iteration implies \(z = -i \log k + \tilde{a}_v + o(1)\).

Similarly, if exactly one of \(V(\pm 1)\) is zero, then \(z = -\frac{5}{4} i \log k + \tilde{b}_v + o(1)\). If \(V(1) = V(-1) = 0\) then \(z = -\frac{5}{4} i \log k + \tilde{c}_v + o(1)\); Eq. \((23)\) follows.

\(\square\)

The above analysis shows that all zeros of \(W_p\) for large \(p\) are in the left half plane. Thus we have

**Corollary 7.** There are only finitely many bound states (this, of course can be simple shown by standard spectral techniques).

We may now proceed to consider the order of these poles as well as their residues.
Proposition 8. The poles of $\hat{\psi}$ for large $p$ are simple, and the residues grow sub-exponentially. The residues of $1/W_p$ grow at most polynomially. (In fact, they grow exactly polynomially, since the asymptotic expansions in (19)–(22) are differentiable.)

Proof. Recalling (19), we notice that

$$W_p = W'_p(p_k)(p - p_k)(1 + o(1))$$

where it can be easily checked that $W'_p(p_k) \neq 0$. Then $1/W'_p(p_k)$ grows at most polynomially, and this together with the bounds on $y_\pm$ (see Lemma 9) show that the residues of $\hat{\psi}$ are bounded by $O(e^{R e\sqrt{-ip}|x|+2})$.

The polynomial growth of residues, along with the analyticity of $\hat{\psi}$, show convergence of the sum in (2) as well as its Borel-summability.

3.4. Asymptotics of $\hat{\psi}$. We will show that $\hat{\psi}$ has sufficient decay to allow for inverse Laplace transform as well as the desired bending of contour leading to Borel summation. First we rewrite (10) as

$$-iW_p\hat{\psi}(x, p) = y_-(x) \int_M y_+(s)\psi_0(s)ds - y_+(x) \int_M y_-(s)\psi_0(s)ds$$

assuming supp $\psi_0 \in [-M, M]$.

Lemma 9. $y_\pm = O\left(\sqrt{p}e^{2|\text{Re}\sqrt{-ip}|(|x|+2)}\right)$ for large $p \in \mathbb{C}$.

Proof. We will prove the lemma for $y_+$ using matching conditions. The proof for $y_-$ follows analogously.

The result is obviously true for $x > 1$, where $y_+(x) = e^{-\sqrt{ip}x}$. For $-1 \leq x \leq 1$, we have

$$y_+(x) = C_1f_1(x) + C_2f_2(x)$$

where

$$C_1 = \sqrt{-ip}e^{-\sqrt{-ip}}\frac{W_{f_1}}{W_{f_2}}(f_2(1) - f_2'(1))$$

$$C_2 = -\sqrt{-ip}e^{-\sqrt{-ip}}\frac{W_{f_1}}{W_{f_2}}(f_1(1) - f_1'(1))$$

and $W_{f_1} = f_1(x)f_2'(x) - f_2(x)f_1'(x)$.

It is easy to see, using Proposition 5, that $C_{1,2} = O(\sqrt{p})$. The bounds follow from (14) and (15).

For $x < -1$ we have

$$y_+(x) = C_5e^{\sqrt{-ip}x} + C_6e^{-\sqrt{-ip}x}$$

where

$$C_5 = \frac{1}{2\sqrt{p}}e^{\sqrt{-ip}(\sqrt{ip}y_+(-1) + \sqrt{ip}y'_+(-1))}$$

$$C_6 = \frac{1}{2\sqrt{p}}e^{-\sqrt{-ip}(\sqrt{ip}y_+(-1) - \sqrt{ip}y'_+(-1))}$$

The result is shown by estimating $y_+(1)$ and $y'_+(1)$ with Proposition 5.

Lemma 10. $p^{-1}W_p\hat{\psi}(x, p) = O\left(e^{2|\text{Re}\sqrt{-ip}|(|x|+M+2)}\right)$ for large $p \in \mathbb{C}$. 

Proof. A straightforward estimate from (26) and Lemma 9.

Lemma 11. (i) There exists a set of curves \( p_k(s) \) parameterized by \( s \in [0, 1] \) with \( p(0) \) on the negative imaginary axis, \( p(1) \) on the negative real axis, \( |p_k(s)| \geq k \), so that \( 1/W_p \) is bounded uniformly in \( k \) by a polynomial in \( p \) along these curves. Here \( k \in \mathbb{N} \) can be chosen to be arbitrarily large.

(ii) Moreover, \( 1/W_p \) is bounded by a polynomial in the region \( \{ p : \arg p \in [\pi, \alpha] \text{ and } |p| > P_\alpha \} \) where \( -\pi < \alpha < -\pi/2 \) and \( P_\alpha > 0 \) depends only on \( \alpha \).

Proof. We rewrite (19) as

\[
W_p = \frac{h_2(p)}{p^2 h_3(p)} \left( e^{-4\sqrt{-ip}h_1(p)}/h_2(p) + 1 \right)
\]

We only need to show that

\[
\left| e^{-4\sqrt{-ip}h_1(p)}/h_2(p) + 1 \right| \geq 1
\]
on a chosen set of curves.

Recalling the asymptotic expressions for \( h_{1,2} \), we have \( \frac{h_1(k^2\pi^2i/4)}{h_2(k^2\pi^2i/4)} \sim c_0 k^n \) where \( c_0, n \) are constants.

Let \( p_k(s) = -i(k\pi/2 - \frac{1}{4} \arg c_0)^2(1 - is)^2 \) and we have 

\[
-4\sqrt{-ip_k} = (2k\pi - i \arg c_0)(1 - is) = (2k\pi - \arg c_0)s + 2k\pi i - i \arg c_0.
\]

Thus for \( s \in [0, 1] \) we have

\[
e^{-4\sqrt{-ip}h_1(p)}/h_2(p) \sim e^{2k\pi s}|c_0|k^n
\]

for all \( k \in \mathbb{N} \), while for \( s \in [1/\sqrt{k}, 1] \) we have

\[
\left| e^{-4\sqrt{-ip}h_1(p)}/h_2(p) \right| \geq e^{2\sqrt{k}\pi}
\]

for all \( k > 0 \).

The second part of the lemma follows from the above inequality since \( k \) may be taken to any large real number. Note also that \( \text{Re}(-\sqrt{-ip}) = (k\pi/2 - \frac{1}{4} \arg c_0)s \).

It is easy to see that \( p_k(s) \) also satisfy the other conditions specified in the lemma.

Collecting the above results we obtain

Lemma 12. \( \hat{\psi}(x,p) = O \left( e^{2|\text{Re}(-\sqrt{-ip})(|x|+M/2)} \right) \) for large \( p \), in any given sector \( \arg p \in [-\pi, \alpha], |p| > P_\alpha \) where \( -\pi < \alpha < -\pi/2 \) as well as along curves \( p_k(s) \) as shown in the previous lemma.

3.5. The inverse Laplace transform. To obtain the transseries of \( \psi \) from our \( \hat{\psi} \), we take the inverse Laplace transform, \( \frac{1}{2\pi i} \int_{a_0-i\infty}^{a_0+i\infty} e^{pt}\hat{\psi}(p)dp \) and push the contour into the left half plane. We will justify this procedure in this section.

First we rewrite (7) as an integral equation

\[
y = T(Vy + i\psi_0)
\]
where

\begin{equation}
T(f)(x) := \frac{1}{2\sqrt{-ip}}e^{-\sqrt{-ip}x} \int_{-\infty}^{x} e^{-\sqrt{-ip}s} f(s)ds - \frac{1}{2\sqrt{-ip}} e^{-\sqrt{-ip}x} \int_{-\infty}^{x} e^{\sqrt{-ip}s} f(s)ds
\end{equation}

We further let \( y(x) = T(i\psi_0)(x) + p^{-3/2}h(x) \) and rewrite the integral equation as

\begin{equation}
h = p^{3/2}T(V \cdot T(i\psi_0)) + T(Vh)
\end{equation}

We start with a simple observation.

**Remark 7.** \( e^{-\sqrt{-ip}} \) is bounded in the region \( \Omega := \{ p \in \mathbb{C} : -\pi/2 \leq \text{arg} p \leq \pi \} \cup \{ p \in \mathbb{C} : -\text{Re} p > \text{const} \cdot (\text{Re} p)^2 \}. \) (In the following, we will choose const = 1/9)

Note that

\[
\text{Re}(-\sqrt{-ip}) = -\frac{1}{2\text{Im} p} \sqrt{\text{Re} p} + |p|(|\text{Im} p| + |p|)
\]

We denote \( \mu = \sup_{p \in \Omega} |e^{-\sqrt{-ip}}| \).

**Lemma 13.** Assume \( f \) and \( g \) are locally bounded functions and \( fg \) is compactly supported, with \( \text{supp}(fg) \in [-b,b] \) where \( b > 0 \). Let \( a \geq b \) be an arbitrary number, \( \Omega' = \Omega \cup \{ p \in \mathbb{C} : |p| > p_v > 1 \} \). We then have

\[
|T(fg)| \leq \frac{2h_b \mathbf{sup}_{x \in [-b,b]} |g(x)||f||}{\sqrt{p_v}}
\]

where \( ||f|| := \sup_{p \in \Omega', x \in [-a,a]} |f(x,p)| \).
Proof. By (27) we have

\begin{equation}
|T(fg)(x,p)| \leq \frac{1}{2\sqrt{p}} \int_{0}^{b} |e^{-\sqrt{-1}px}||g(u+x)||f(u+x)|du
+ \frac{1}{2\sqrt{p}} \int_{-b}^{0} |e^{\sqrt{-1}px}||g(u+x)||f(u+x)|du
\leq \frac{\mu b}{\sqrt{p}} \int_{-b}^{b} |g(s)||f(s)||ds \leq \frac{2b\mu b^{\frac{1}{2}} \sup_{x \in [-b,b]} |g(x)|}{\sqrt{p}} ||f||
\end{equation}

Lemma 14. For compactly supported and twice differentiable \( \psi_0 \), we have

\[ T(i\psi_0)(x) = \frac{1}{p} \psi_0(x) + \frac{1}{p^{3/2}} G_1(x,p) \]

where \( |G_1(x,p)| \leq 2M \sup |\psi_0''| \sup_{x \in [0,M+|x|]} |e^{-\sqrt{-1}px}|, \) assuming \( \text{supp } \psi_0 \in (-M,M) \).

Proof. This is shown by repeated integration by parts to (27). Note that

\begin{equation}
T(i\psi_0)(x) = \frac{1}{p} \psi_0(x) - \frac{1}{2p} e^{\sqrt{-1}px} \int_{-M}^{x} e^{\sqrt{-1}px} \psi_0'(s)ds
- \frac{1}{2p} e^{-\sqrt{-1}px} \int_{-x}^{M} e^{-\sqrt{-1}px} \psi_0'(s)ds
= \frac{1}{p} \psi_0(x) + \frac{1}{2(ip)^{3/2}} e^{\sqrt{-1}px} \int_{M}^{x} e^{\sqrt{-1}px} \psi_0''(s)ds
- \frac{1}{2(ip)^{3/2}} e^{-\sqrt{-1}px} \int_{-M}^{-x} e^{-\sqrt{-1}px} \psi_0''(s)ds
\end{equation}

With the above lemmas, we have

Proposition 15. Let \( \Omega_0 = \Omega \cup \{p \in \mathbb{C} : |p| > p_c \} \) where \( p_c = 9(\sup_{x \in [-1,1]} V(x) + \mu + 1)^2 \). Let \( x_1 > 0 \) be an arbitrary real number. The integral equation (28) is contractive in the space of functions analytic in \( p \in \Omega_0 \) equipped with the sup norm \( ||f|| = \sup_{p \in \Omega_0, x \in [-x_1,x_1]} |f(x,p)| \), within a ball of size

\[ 2\mu \sup_{x \in [-1,1]} |V(x)| + 2M \sup |\psi_0''| \sup_{x \in [0,M+|x|]} |e^{-\sqrt{-1}px}| \]

In particular, the solution \( h \) is bounded as \( x_1 \to \infty \) if \( \text{Re} p > 0 \). See Fig. 3.5.

Proof. The estimates of \( p^{3/2} T(V \cdot T(i\psi_0)) \) follow from lemma 14 and 13, with \( f = V, g = \psi_0 \) and \( g = G_1 \) separately.

The contractivity of \( T \) follows from lemma 13 with \( f = V, g = h \). Note that analyticity in \( p \) is preserved by \( T \) and convergence in the sup norm.

We therefore have the following results.

Proposition 16. (i) \( \hat{\psi} \) (as in section 3.7) has the following decomposition:

\[ \hat{\psi}(x,p) = \frac{1}{p} \psi_0(x) + \frac{1}{p^{3/2}} G_2(x,p) \]
where \( G_2(x, p) \) is bounded in \( p \in \Omega_0, x \in [-x_1, x_1] \) where \( x_1 > 0 \) is arbitrary.

(ii)

\[
\psi(x, t) = \psi_0(x) + \frac{1}{2\pi i} \int_{a_0-i\infty}^{a_0+i\infty} \frac{G_2(x, p)}{p^{3/2}} e^{pt} dp
\]

is the solution to (7). Here \( a_0 > 0 \) is a constant.

**Proof.** We only need to show that in \( \Omega_0 \) the solution \( \hat{\psi} \) is identical to the solution \( y \) obtained in this section, the decomposition of which has already been shown. Part (ii) then follows immediately from properties of the inverse Laplace transform.

To this end, note that the general solution to (7) can be written in the form of

\[
y_{gen}(x, p) = \hat{\psi}(x, p) + c_1(p)y_+(x, p) + c_2(p)y_-(x, p)
\]

where \( y_+ \) and \( y_- \) are the homogeneous solutions defined in section 3.1 (with a slight abuse of notation). This implies

\[
y(x, p) = \hat{\psi}(x, p) + c_1(p)y_+(x, p) + c_2(p)y_-(x, p)
\]

where \( y(x, p) \) is the solution obtained earlier in this section. Since in the region \( \{ \text{Re}(p) > p_v, x < 1 \} \), \( y_+ \) is unbounded and \( y_- \) is bounded, and in \( \{ \text{Re}(p) > p_v, x > 1 \} \) \( y_+ \) is bounded and \( y_- \) is unbounded, while both \( y \) and \( \hat{\psi} \) are bounded (the boundedness of \( \hat{\psi} \) follows easily from (26)), we must have \( c_1 = c_2 = 0 \) in \( \text{Re}(p) > p_v \). Thus \( \hat{\psi} \) and \( y \) coincide in \( \text{Re}(p) > p_v \) and also in \( \Omega_0 \) by uniqueness of analytic continuation.

\[\Box\]

**Lemma 17.** In the expression

\[
\frac{1}{2\pi i} \int_{a_0-i\infty}^{a_0+i\infty} \frac{h_2(x, p)}{p^{3/2}} e^{pt} dp
\]

we may deform the contour to one which goes from \(-\infty\) below the real axis, turns counterclockwise around the origin and goes towards \(-\infty\) above the real axis. In the process we collect all residues from all the poles in the left half plane.

**Proof.** The deformation of the upper half of the contour is justified by Proposition 16 since \( \Omega \) contains the second quadrant.

In the third quadrant, recall that \( p_k(s) = -(k\pi/2 - \frac{1}{4} \text{arg} c_0)^2(1 - is)^2 \) and \( \text{Re}(\sqrt{-ip_k}) = (k\pi/2 - \frac{1}{4} \text{arg} c_0)s \). Thus \( \text{Re}p_k(s) \sim \text{const}.k^2s \) and \( \text{Re}(\sqrt{-ip_k}) = O(\text{Re}p_k(s)/k) \) for all \( s > 0 \). Therefore we may choose part of the curve \( p_k(s) \) where \( s \in [1/k, 1] \) and join it with a curve in \( \{ p \in \mathbb{C} : -\text{Im}p > (\text{Re}p)^2/9 \} \cap \Omega_0 \), say a vertical line downward to infinity. These two curves, along with the one from below the real axis to the origin and lower half of the original contour, surround all poles in the third quadrant as \( k \to \infty \). Decay along the \( p_k(s) \) curve is ensured by the term \( e^{pt} \), since \( e^{p_k(s)t\pm\sqrt{-ip_k(s)}}M_0 = O(e^{-kt}) \) for arbitrarily large \( M_0 \). Note also that the length of \( p_k(s) \) is of order \( k^2 \).

To prove Theorem 1 we further write \( \psi_0(x) = \frac{1}{2\pi i} \int_C \frac{h_2(x, p)}{p^{3/2}} e^{pt} dp \), where the contour of integration is the horizontal part around the negative real axis described above. This contour can be deformed to "0 to \(-\infty\)" in an upper and a lower sheets of the Riemann surface, which yields a Borel-summable power series in \( t^{-1/2} \).
Figure 2. A sketch of the contour deformation used. The shaded area is the contractivity region.

3.6. Connection with Gamow Vectors. Classically, Gamow vectors are obtained as solutions to (8) with “purely outgoing boundary conditions” as \( x \to \pm \infty \). In our case, this means such a solution (after rescaling) equals \( y_+(x) \) for \( x > 1 \) and a nonzero constant times \( y_-(x) \) for \( x < 1 \), \( y_\pm \) being as in section (3.1). The existence of such a solution, therefore, is equivalent to the linear dependence of \( y_+ \) and \( y_- \) (cf. Lemma 4), which in turn is equivalent to the vanishing of the Wronskian: \( W_p = 0 \). Thus the \( \gamma_k \) found from the poles of \( \hat{\psi} \) are exactly the resonances corresponding to the Gamow vectors, a constant multiple of \( y_+ \). The latter are easily seen to be multiples of the residues of \( \hat{\psi} \) for example by simplifying (26):

\[
-i W_p \hat{\psi}(x, p) = c y_+(x) \int_{-M}^{x} y_+(s) \psi_0(s) ds - y_+(x) \int_{-M}^{x} c y_+(s) \psi_0(s) ds
\]

\[
= -e \left( \int_{-M}^{M} y_+(s) \psi_0(s) ds \right) y_+(x)
\]

3.7. Proof of Proposition 2

Proof. This follows straightforwardly from Lemmas 18 and 5.1 after extracting a suitable number of poles from \( \hat{\psi} \). All poles are simple, and the contribution of a pole of residue \( r_k \) and position \( p_k \) is

\[
J_k(t) = r_k \int_{0}^{\infty} \frac{e^{-pt} dp}{\sqrt{p - p_k}}
\]

The representation of \( J_k \) in terms of special functions is perhaps most conveniently shown by solving the first order differential equation it satisfies, and determining the free constant from the asymptotic behavior in \( p_k \).
4. Example: the square barrier

Here we take as a simple example the Schrödinger equation with a square bump potential \( V(x) = \chi_{[-1, 1]} \), \( \chi \) being the indicator function. (One of few cases where explicit solutions exist.)

\[
y_+(x) = \begin{cases} 
A_1 e^{-ipx} + A_2 e^{-\sqrt{1-ip}x}, & x \leq -1; \\
A_3 e^{\sqrt{1-ip}x} + A_4 e^{-\sqrt{1-1-ip}x}, & -1 < x < 1; \\
e^{-\sqrt{1-ip}x}, & x \geq 1.
\end{cases}
\]

\[
y_-(x) = \begin{cases} 
B_1 e^{ipx} + B_2 e^{-\sqrt{1-ip}x}, & x \leq -1; \\
B_3 e^{\sqrt{1-ip}x} + B_4 e^{-\sqrt{1-1-ip}x}, & -1 < x < 1; \\
& x \geq 1.
\end{cases}
\]

where the coefficients \( A_j, B_j \) are determined by matching solutions at the endpoints, \( \pm 1 \). For example,

\[
A_3 = \frac{\sqrt{i + p} - \sqrt{p}}{2\sqrt{i + p}} (e^{-\sqrt{1-ip}x} - e^{-\sqrt{1-ip}x})
\]

The other coefficients are similar (and obtained in a similar way) and we omit them.

It follows that the Wronskian \( W_p \) has an explicit expression

\[
W_p = \frac{\sqrt{-ie^{-2\sqrt{1-ip}+2\sqrt{1-ip}}} e^{-4\sqrt{1-ip}(i + 2p - 2\sqrt{p}\sqrt{i + p}) - i - 2p - 2\sqrt{p}\sqrt{i + p}}}{2\sqrt{i + p}}
\]

We may find the asymptotic positions of the resonances by iterating

\[
z_k = \frac{1}{4i} \log \left( \frac{i + 2p_k + 2\sqrt{p_k}\sqrt{i + p_k}}{i + 2p_k - 2\sqrt{p_k}\sqrt{i + p_k}} \right)
\]

where \( p_k = -i(k\pi/2 + z_k)^2 \).

We also calculate the residues of \( 1/W_p \) by differentiating \( W_p \):

\[
1/W_p \sim \frac{\sqrt{p_k}(i + p_k)(i + 2p_k - 2\sqrt{p_k}\sqrt{i + p_k})}{\sqrt{-ie^{-2\sqrt{1-ip}+2\sqrt{1-ip}}(1 + \sqrt{1-ip})}} \frac{1}{p - p_k}
\]

Here we calculate the positions and residues of a series of poles using the above formulas and compare them to the asymptotic behavior \(-\pi k \log(\pi k) - i\pi^2 k^2/4\), as in Proposition \( \ref{prop:asymptotics} \). Then we plot these poles together with a density graph.

The asymptotic pole location formula gives (increasingly) good accuracy starting with the 15th pole or so, where it predicts the position \(-181 - 555i\), whereas the exact value is about \(-180 - 532i\).

The first resonance, the one closest to the imaginary line (in \( p \)-plane), may have a visible effect on the wave function \( \psi \) even if this resonance does not correspond to a (long-lived) metastable state. We will demonstrate this phenomenon, as well as the computational effectiveness of the Borel summation approach, using (near-) optimal truncation, see \( \ref{prop:optimal} \) on the example of the square barrier potential, where we choose the initial condition to be \( \psi_0(x) = \chi_{[-\frac{1}{2}, \frac{1}{2}]} \) for simplicity.

In our example, the first pole of \( 1/W_p \) is located at \( p_0 = -1.70018 - 0.805871i \). This can be found by standard iterative arguments.
Figure 3. Density graph of \(1/W_p\). Dark dots indicate poles calculated from the asymptotic formula.

We will demonstrate the effect of this pole in the region \(x > 1\), where (cf. (10))

\[
\hat{\psi}(x, p) = -ie^{-\sqrt{-ipx}/W_p} \int_{-\frac{1}{2}}^{\frac{1}{2}} y_-(s; p) ds
\]

and

\[
\psi(x, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{pt-\sqrt{-ipx}}}{W_p} \int_{-\frac{1}{2}}^{\frac{1}{2}} y_-(s; p) dsdp = -\frac{1}{2\pi} \int_{-\infty}^{0} \frac{e^{pt-\sqrt{-ipx}}}{W_p} \int_{-\frac{1}{2}}^{\frac{1}{2}} y_-(s; p) dsdp
\]

\[
+ \frac{1}{2\pi i} \lim_{p \to p_0} (p - p_0)\hat{\psi}(x, p) e^{pt} (1 + o(1))
\]

for large \(t\).

We may calculate the power series by expanding \(\hat{\psi}(x, p)\) near \(p = 0\) and using Watson’s Lemma. For instance, for \(x = 8\) we obtain the series

\[
(34)
(0.735266 + 0.735266i) \frac{1}{t^{3/2}} - (12.3883 - 12.3883i) \frac{1}{t^{5/2}} - (98.5277 + 98.5277i) \frac{1}{t^{7/2}}
\]

\[
+ (471.935 - 471.935i) \frac{1}{t^{9/2}} + (1429.08 + 1429.08i) \frac{1}{t^{11/2}}
\]

\[- (2690.72 - 2690.72i) \frac{1}{t^{13/2}} - (4000.95 + 4000.95i) \frac{1}{t^{15/2}} + O(t^{-8})
\]

**Note 8.** Taking \(x = 7\), the contribution of the first resonance to the power series is visible, about 3%, for \(t = 7\), and evidently decreases exponentially thereafter. The overall precision increases rapidly with \(t\), if \(x\) is fixed or does not increase faster than \(t\), unlike most direct numerical calculations.
An expansion is Borel summable if, by definition, it is the asymptotic expansion of the Laplace transform \( L \) of a function which is real-analytic on \( \mathbb{R}^+ \), exponentially bounded, and which has a convergent series at zero, in (ramified) powers of the variable and possibly logs (Frobenius series). The Borel summation operator, \( LB \), is essentially \( L S L^{-1} \) where \( L^{-1} \) applied to a series is understood in the formal sense, as the term-by-term transform, and \( S \) is convergent summation. Since \( LB \) is conjugated with usual summation, which commutes with virtually all operations, the same is true for \( LB \). Uniqueness of the Borel sum stems from uniqueness of the sum of a convergent series. Borel summation is a canonical extension of usual summation. The decomposition of a function in a Borel summed part and a sum of exponentials, when possible, is also unique and canonical, see \( \S \) 5 and for a detailed analysis e.g. [5].

5.1. **Borel summation and least term truncation.** Borel summation allows for exponentially accurate calculations of the associated function by truncating the series near its least term. We first briefly explain the reason and refer to [7] for more details. The accuracy, even for \( t \) not so large, is illustrated in \( \S \) 4.

Here we consider a Borel summed series of the type that intervenes in our problem, namely let

\[
(35) \quad f(t) = \int_{0}^{\infty} e^{-pt} F(p) dp
\]

where \( F(p) = g(p^\beta); \text{Re}\beta > 0 \) and \( g \) is analytic at the origin and meromorphic in \( \mathbb{C} \). The asymptotic behavior of \( f \) is, by Watson’s lemma, of the form

\[
(36) \quad f(t) \sim \tilde{f}(t) = \sum_{k=0}^{\infty} c_k t^{-k\beta}
\]

We let \( S_n(t; \tilde{f}) \) be the truncate of the power series, up to the power \( t^n \):

\[
(37) \quad S_n(t; \tilde{f}) = \sum_{0 \leq k \text{Re}\beta \leq n} c_k t^{-k\beta}
\]

We want to estimate the error by calculating the function from its power series by optimal truncation, or **truncation to the least term**. For a series in which the coefficient of \( t^k \) grows roughly like \( b^k/k! \), means using the truncation \( S_n(t; \tilde{f}) \). For example using Stirling’s formula we see that for \( F(p) = 1/(p - p_0) \), the general term of \( \tilde{f} \) grows like \( k!/|p_0|^k \) and its least term is near \( n = t|p_0| \); this location of the least term is the same regardless of the nature of the singularity, for all algebraic-logarithmic type of singularities.

We show that the error in approximating \( f \) by \( S_n(t; \tilde{f}) \) in this way is of the same order of magnitude as this least term, which is exponentially small in \( t \). In similar contexts this is known quite generally for Borel summed series, see [7] and references therein. Least term truncation provides a practical way to calculate functions with very high accuracy even for \( t \) of moderate size. In spite of the generality considered in [7], our case is not covered (because of ramification at zero). Instead of describing the adaptation of that proof, for convenience of the reader, we provide a complete argument in our case.
Lemma 18. Assume \( \beta \in (0, 1) \) and \( H \) is analytic in disk \( \overline{D}_A \). Let \( H_M \) be the maximum of \( H \) on a disk of radius \( A \in (A_1, A_2) \). Then

\[
\int_0^{A_1^\alpha} H(p^\beta) \exp(-tp) dp - S_{(n+1)\beta}(t, \tilde{H}(p^\beta)) = O \left( \frac{H_M \Gamma(n\beta + \beta + 1)}{A_1^{n+1}t^{n\beta + \beta + 1}} \right)
\]

Proof. Indeed, by using Taylor series with Cauchy integral remainder we have

\[
\int_0^{A_1^\alpha} H(p^\beta) \exp(-tp) dp - S_n(H(p^\beta)) = \frac{1}{2\pi i} \int_0^{A_1^\alpha} \exp(-tp)p^{(n+1)\beta} \frac{H(s)ds}{s^{n+1}(s-p^\beta)} dp = J
\]

where \( C \) is a circle of radius \( A \). If \( H_M \) is the maximum of \( H \) on \( C \), we have

\[
|J| \leq \frac{H_M}{2\pi A_1^\alpha(A_1 - A)} \int_0^\infty e^{-\eta p^{(n+1)\beta}} dp
\]

and the result follows. \( \square \)

Lemma 19. Assume \( \beta \in (0, 1) \) and let \( F(p) = (p^\beta - p_0)^{-1} \) where \( p_0 \notin \mathbb{R}^+ \) and let \( f = \mathcal{L}F \). Then,

\[
|f(t) - S_{(n+1)\beta}(t, \tilde{f})| \leq \frac{\Gamma(n\beta + \beta + 1)}{p_0^{n+1}t^{n\beta + \beta + 1}|\text{Im}p_0|}
\]

Proof. Writing

\[
\frac{1}{p^\beta - p_0} = \sum_{j=0}^{n} \frac{p^{j\beta}}{p_0^{j+1}} + \frac{p^{(n+1)\beta}}{p_0^{n+1}(p^\beta - p_0)}
\]

this follows from straightforward integration and estimates. \( \square \)

5.2. A class of level one transseries. We only need an especially simple subclass of transseries, exponential power series of the type

\[
\tilde{f}(t) = \sum_{k=0}^\infty e^{-\gamma_k t} t^{\gamma_k} \tilde{f}_k(t)
\]

where \( \tilde{f}_k(t) \) are formal power (integer or noninteger) series in \( 1/t \), where, for disambiguation purposes, the real part of the leading power of \( 1/t \) in \( \tilde{f}_k(t) \) is chosen to be \( 1 \). Agreeing that \( \text{Im} \tilde{f}(t) \) is exactly zero and the \( \gamma_k \) are distinct, it is required that the exponentials \( e^{-\gamma_k t} \) are well ordered, in the sense that \( \text{Re}(\gamma_k) \geq \text{Re}(\gamma_{k'}) \) if \( k \geq k' \), and every \( \text{Re}(\gamma_k) \) has a predecessor, the smallest \( \text{Re}(\gamma_j) \) greater than it. In our context the sets \( \{ j : \text{Re}(\gamma_j) = \text{Re}(\gamma_k) \} \) turn out to be finite.

The transseries \( \tilde{f} \) is Ecalle-Borel summable if (a) \( \tilde{f}_k(t), k \in \mathbb{N} \) are simultaneously Ecalle-Borel summable (in fact, simply Borel summable, in our case), and (b) upon replacing each \( \tilde{f}_k(t) \) by its sum, the resulting function series is uniformly convergent. We give precise definitions in \( \S 5 \). Transseries and Ecalle-Borel summability were introduced by Ecalle in the 1980s and there has been substantial development since. For an elementary introduction see [5].

The transseries is (Ecalle-Borel) summable if for some \( T > 0 \) we have the following.
(i) $\tilde{f}_k(t)$ are simultaneously Borel summable, that is there exists a $T$ so that $\tilde{f}_k(t)$ are the asymptotic expansions for large $t$ of Laplace transforms,

$$f_k(t) = \int_0^\infty F_k(p)e^{-pt}dp =: \mathcal{LB}\tilde{f}_k(t)$$

where

(ii) $F_k$ are ramified-analytic at zero, and real analytic on $\mathbb{R}^+$ with the uniform bound $\|F_k(p)\| \leq C_ke^{\nu|p|}$.

(iii) For some $\nu \in \mathbb{R}$ we have $|F_k(t)| \leq C_ke^{\nu|p|}$.

(iv) The series

$$\sum_{k=0}^{\infty} |e^{-\gamma_k T}|C_k$$

converges for some $T > 0$ (and thus for all $t \geq T$). We recall that, by convention, $F_k(p) = c_k(1 + o(1))$ as $p \to 0$, where $c_k \neq 0$.

Therefore, the sum

$$f = \mathcal{LB}\sum_{k=0}^{\infty} t^{a_k}e^{-\gamma_k t}\tilde{f}_k(t) := \sum_{k=0}^{\infty} e^{-\gamma_k t}t^{a_k} \mathcal{LB}\tilde{f}_k(t) = \sum_{k=0}^{\infty} e^{-\gamma_k t}t^{a_k} f_k(t)$$

converges absolutely for $t > T$.

The operator $\mathcal{LB}$ is a proper extension of the Borel summation operator. In particular, it allows for non-accumulating singularities on the axis of summation, in which case analytic continuation is replaced by Ecalle’s universal averaging. Super-exponential growth of $F$ of a controlled type is allowed, using Ecalle’s acceleration operators.

With these extensions, Borel summation is an extended isomorphism between series, or more generally transseries, and a class of functions (analyzable functions), commuting essentially with all operations with which analytic continuation does. In this sense, Ecalle-Borel summable transseries substitute successfully for convergent expansions; in particular the Ecalle-Borel sum of a formal solution of a problem (within certain known classes of problems such as ODEs and PDEs) is an actual solution of the same problem. It is known that the fundamental decaying solution of a nonlinear differential equation at a generic singularity is given, uniquely, by Borel summable transseries [3].

5.3. Uniqueness of the transseries representation. In the same way as the asymptotic power series of a function, when a series exists, is unique one function can only have one transseries representation, if at all. We sketch a proof that a representation of the form (46) of a given $f$ is unique. We assume of course that the transseries are in canonical form, as explained above. By linearity, it suffices to show that if $f$ given in (46) is identically zero, then all $\tilde{f}_k$, and thus all $f_k$ are identically zero. We assume by contradiction that some $\tilde{f}_k$ are nonzero. Since the Re($\gamma_k$) are well ordered, cf. §5.2 we choose the largest Re($\gamma_k$) such that $\tilde{f}_k \neq 0$. There are only finitely many $\gamma_k$ with the same Re($\gamma_k$), cf. again §5.2. We can assume without loss of generality that these $\lambda$s have indices $0, \ldots, n$, and assume that we have ordered the terms in the transseries so that Re$\gamma_i \leq$ Re$\gamma_{i+1}$ for all $i$. 
We write

$$f = \sum_{k=0}^{n} e^{-\gamma_k t^{\alpha_k}} f_k(t) + \sum_{k=n+1}^{\infty} e^{-\gamma_k t^{\alpha_k}} f_k(t)$$

Note that for any $\epsilon > 0$ small enough we have

$$\left| \sum_{k=n+1}^{\infty} e^{-\gamma_k (T+\tau)^{\alpha_k}} f_k(t) \right| \leq \text{const} |e^{-\gamma_{n+1}\tau}| = o \left( |e^{-\gamma_0 (T+\tau)}| \right)$$

as $\tau \to \infty$, since $\text{Re}(\gamma_0) < \text{Re}(\gamma_{n+1})$. Dividing (47) by $e^{-\text{Re} \gamma_0 t}$ we get

$$\sum_{k=0}^{n} e^{-i\alpha_k t^{\alpha_k}} f_k(t) = o(1), \quad (t \to \infty)$$

where $\alpha_k = \text{Im} \gamma_k$. For each $k$ we choose $\beta_k$ to be the smallest power of $p$ (in absolute value) with nonzero coefficient, $c_k$ in the expansion of $F_k$. Of course, if all coefficients in the Puiseux series of $F_k$ vanish, then $F_k$ vanishes near zero, and thus everywhere by analyticity. We arrange that there is no $k$ such that $F_k \equiv 0$. Then, by Watson’s lemma, $F_k = c_k \Gamma(\beta_k + 1) t^{-\beta_k - 1} (1 + o(1))$ for large $t$. We choose the largest $\beta_j$, in the sense above, and divide by $\Gamma(\beta_j + 1) t^{-\text{Re} \beta_j - 1}$. We get, by Watson’s Lemma,

$$\sum_{\text{Re} \beta_j = \text{Re} \beta_k; k \leq n} c_k e^{-i\alpha_k t^{\alpha_k}} = o(1)$$

where $\theta_k = \text{Im} \beta_k$. We now prove a lemma in more generality than needed here, in view of future generalizations to time dependent potentials.

**Lemma 20.** Assume $\sum_{k=0}^{\infty} |c_k|^2 < \infty$ and that

$$f(t) = \sum_{k=0}^{\infty} c_k e^{-i\alpha_k t^{\alpha_k} - i\theta_k} = o(1)$$

where $\alpha_k, \theta_k \in \mathbb{R}$, as $t \to \infty$. Then $f(t) \equiv 0$.

**Proof.** We first look at the simpler case where all $\theta_k = 0$; as we shall see, the general case is similar. We see, by explicit integration and dominated convergence, that for large $t_0$ and $t \to \infty$ we get from (54) that

$$\int_{t_0}^{t} |f(s)|^2 ds = \sum_{k=0}^{\infty} |c_k|^2 t + O \left( \left( \sum_{k=0}^{\infty} |c_k|^2 \right)^2 \right) = o(t)$$

which is only possible if

$$\sum_{k=0}^{\infty} |c_k|^2 = 0$$

To generalize to the case $\theta_k \neq 0$, we simply note that (51) implies

$$f(e^s) = \sum_{k=0}^{\infty} c_k e^{-i\alpha_k e^s} e^{-i\theta_k s} = o(1)$$
as $s \to \infty$ and that, still as $s \to \infty$ we have (e.g. by integration by parts) that, for $	heta \neq 0$, we have

$$
\int_{s_0}^{s} e^{-i\alpha u} e^{-i\theta u} du = \frac{i}{\theta} e^{-i\alpha s} e^{-i\theta s} (1 + o(1))
$$

\[\square\]

**Borel summation and usual summation: the underlying isomorphism.** Furthermore, there is the following important point. When a Borel summable transseries of a function exists, functions and their transseries have the same properties. That is, there exists an extended isomorphism between transseriable functions and transseries similar in many ways to the one between germs of analytic functions, and their local convergent Taylor series regarded as formal algebraic objects. This latter isomorphism is so flawless that we do not distinguish notationally a convergent sum as a formal sum, from its sum as a function. Borel summation is a proper extension of usual summation, carrying further these isomorphism features.

The isomorphism, provided by Ecalle-Borel summability, which recovers the function from its transseries, justifies the usage of the term complete asymptotics. Borel summation is a canonical way to sum factorially divergent series, cf. also \[5\].

**Independence of method.** Finally, the nontrivial terms in the transseries of a function can be exhibited by many other exponential asymptotic techniques some of which having of substantial calculational value, such as hyperasymptotics, a set of methods improving and refining optimal truncation of series, cf. \[2\], \[6\], \[17\], and references therein.

In the language of generalized Borel summability, the wave function asymptotics is given in all amplitude regimes by an Ecalle-Borel summable transseries, valid for $t > 0$, and this transseries turns out to rest on a Gamow vector decomposition.

**Note 9.** Sometimes a given series can be Borel summed with respect to different powers, or more generally functions, of the variable. For instance,

$$
\int_0^{\infty} e^{-pt} e^{-i\nu^2/4} dp = \int_0^{\infty} e^{-pt^2} \frac{dp}{2\sqrt{\pi p} (i - p)}
$$

Since the integrals are equal, they have the same asymptotic series for large $t$; both integrals are Borel sums of the same asymptotic series, on the left interpreted as a series in $1/t$, while on the left it is thought of as a series in $1/t^2$.

Using the connection with Gevrey asymptotics, \[5\], it is easy to see that a series has a unique Borel sum, with respect to any variable in which it is Borel summable, even when allowing for ramified-analytic functions. Ramified-analytic functions are real analytic, and near $p = 0$ of the form $F(p_1, ..., p_m)$ where $p_j = p^{a_j} \log p^{b_j}$, $\text{Re}(a_i) > 0$, and $F$ is analytic at $0$. It is an easy exercise to show that $e^{-i\nu^s}$, $\text{Re} \alpha > 0$ cannot be represented as a Laplace transform of a ramified analytic function.

But beyond ramified analyticity uniqueness of the representation as a “continuum” (to use physics terminology) plus exponentials does not hold. We have, e.g.,

$$
e^{-t} = \pi^{-1/2} \int_0^{\infty} p^{-3/2} e^{-1/p} e^{-pt^2} dp$$

which is a continuum type integral.

The Borel sum gives consistent results, and remaining exponential terms are uniquely defined.
Ramified analyticity of $\hat{\psi}$ follows from the formulas in [16] and [8]. However, the techniques [8] are more involved and, along those lines, there appear to be significant gaps in estimates leading to a mathematical proof of Borel summability (which, in fact, is not the intention of those works). The purpose of the analysis in [8], [16] and related literature is different: the extension of a spectral-like theory and a "spectral calculus" beyond the continuous spectrum.

5.4. Analytic potentials. Since the wave function is the solution of a PDE which mixes space and time information, finding the detailed time behavior of $\psi$ is contingent on detailed information about $V(x)$ and $\psi_0(x)$. It seems likely that generalized (multi-) summability of the large time (trans-)series of $\psi$ should hold whenever $V$ has a multisummable transseries as well. In this paper though we consider potentials analytic at infinity and with sufficient decay. For simplicity, we write $-ip = \epsilon^2$. Equation (8) reads:

$$y''(x) = (V(x) + \epsilon^2) y(x)$$

From the form of the Green’s function, it is clear that the analytic properties of the Green’s function at $p = 0$ follow from those of the Jost functions (defined as in (9)): $y^+$ is the solution that behaves like $e^{-\sqrt{-px}}$ as $x \to \infty$, when $p \in \mathbb{R}^+$. We analyze potentials of the form $V(x) = a/x^m, m \in \mathbb{N}$, and we discuss how essentially the same arguments would work for any potential which is analytic at infinity and $O(x^{-m})$. The value $m = 2$ marks in a sense a threshold, making the transition between convergent and divergent expansions in energy at the bottom of the continuous spectrum. For $m = 1, 2$ the equation can be solved in terms of simple special functions; the slow decay in the case $m = 1$ implies that zero is an accumulation point of poles; no convergent Frobenius expansion is possible.

Proposition 21. For $m \geq 2$ the Jost functions have convergent Frobenius expansions in $\epsilon$ (series in ramified powers of $\epsilon$ and $\epsilon \log \epsilon$ and are bounded by $\text{const} \epsilon^{1/|x|}$ uniformly in a sector $\arg \epsilon \in (-3\pi/4, 3\pi/4)$.

For $V(x) = a/x^2$, (58) is solved by Bessel functions; the solution that decays like $e^{-\epsilon x}$ as $x \to \infty$ is given in terms of the Bessel function $K$ as

$$y^+ = 2cx/\pi K_{\alpha}(cx); \quad \alpha = \sqrt{a+1}/4$$

For small $\epsilon$ and fixed $x$, $y^+$ has the form

$$C_1(x)\epsilon^{1+\alpha}A_1(\epsilon) + C_2(x)\epsilon^{1-\alpha}A_2(\epsilon)$$

with $A_1, A_2$ analytic. For $m \geq 3$, there are no ramified powers of $\epsilon$ in the expansions, but all powers of $\epsilon \log \epsilon$ intervene.

Proposition 22. For fixed $x$ and $m \geq 3$, the function $s(x; \epsilon)$ is of the form $G(\epsilon, \epsilon \log \epsilon)$ where $G(u, v)$ is analytic for small $(u, v)$. The Jost functions are bounded by $\text{const} \epsilon^{1/|x|}$ uniformly in a sector $\arg \epsilon \in (-3\pi/4, 3\pi/4)$.

Proof. The question is the dependence of the Jost function in $\epsilon$, for small $\epsilon \in \mathbb{C}$. As a mathematical question, this is a connection problem: the definition of the Jost function is given in terms of the asymptotic behavior as $x \to \infty$ while the analyticity properties in $\epsilon$ are sought globally in $x$.

It is convenient to treat this problem by Borel summation once again, this time in $x$, to transform it into a pure analyticity question. We analyze the Jost function given, for $\epsilon > 0$, by $y(x) \sim e^{-\epsilon x}(1 + s(x; \epsilon))$ where $s(x; \epsilon)$ is an $o(1)$ power series.
in $1/x$, as $x \to \infty$. It is easy to see that such a solution (whose existence is known and also follows from the argument below) is unique.

For Borel summability, we have to extract $s$. We thus write $y(x; \epsilon) = e^{-\epsilon x}(1 + s(x; \epsilon))$ and obtain

$$s'' - 2\epsilon s' - V(x)s = V(x)$$

To simplify even further the presentation we take $m = 3$, but there is nothing special about this choice, and the extension to other values of $m$ is immediate.

We inverse Laplace transform (60) (the legitimacy of which is justified “backwards” by showing that the Laplace transform of the solution of the thus obtained equation solves (60), which has a unique small solution) and obtain

$$H(q) = \frac{1}{2}p^2 + \frac{1}{2}p^3 \frac{H(q)}{q(q + 2\epsilon)}$$

where $PF$ is the definite antiderivative of $F$ which is zero at zero. With the change of variable $\tau = \epsilon q$, $H(q) = F(\tau)$, we obtain

$$F(\tau) = \frac{\epsilon^2 \tau^2}{2} + \epsilon p^3 \frac{F(\tau)}{\tau(\tau + 2)}$$

We look for a solution which are $O(\epsilon^2 \tau^2)$ for small $\tau$. Consider the space $B$ of functions of the form $f(\tau) = \tau^2 G(\tau)$ where $G$ is analytic for, say, $|\tau| < 1$ with the norm $\|f\| = \sup_{|\tau| < 1} |G(\tau)|$. We see that this is a Banach space, and eq. (61) is contractive in $B$. It is also unique in the space of functions of the form $\tau^2 G(\tau)$ with $G$ defined in $L^1[0,1]$. The solution of (61) is unique, and analytic for small $\tau$. As a differential equation this reads

$$F''' = \frac{\epsilon F}{\tau(\tau + 2)}$$

The argument above, or Frobenius theory, shows that (63) also has a unique solution which is of the form $\frac{1}{2} e^{2\tau^2}(1 + o(1))$ for small $\tau$. The solution is obviously analytic for $\Re \tau > -2$, since there the only singularity of the equation is $\tau = 0$.

By standard ODE asymptotic results [22] we see that any solution of (63) is uniformly bounded in $\mathbb{C}$ by

$$C|\tau|^{2/3} e^{3|\tau|^{1/3}}$$

for some $C$. This ensures the necessary (sub)exponential bounds for taking the Laplace transform.

On the other hand, we look for solutions of (63) in the form

$$F = \epsilon^2 F_2 + \sum_{j \geq 3} \epsilon^j F_j(\tau)$$

The functions $F_j$ satisfy the recurrence

$$F'''_{j+1} = \frac{F_j}{\tau(\tau + 2)}, j \geq 3$$

With our initial condition, we get $F_2 = \tau^2/2$ and

$$F_{j+1} = p^3 \frac{F_j}{\tau(\tau + 2)}, j \geq 3$$
For now, we take $\tau$ in the right half plane, $\mathbb{H}$. It can be checked by induction that $F_j$ are analytic in $\mathbb{H}$ and at zero, and

$$|F_j| \leq 4^{j/3} \tau^j$$

It follows that the series (68) converges uniformly on any compact set in $\mathbb{H}$. Moreover, we see that the function series

$$H(q) = \frac{q^2}{2} + \sum_{j \geq 3} \epsilon^j F_j(q/\epsilon)$$

also converges uniformly in $q$ on any compact set in $\mathbb{H}$. The Laplace transform of $H$ reads

$$\int_0^\infty e^{-s\tau} H(q) dq = \frac{1}{x^3} + \sum_{j \geq 3} \epsilon^j \int_0^\infty e^{-s\tau} F_j(q/\epsilon) dq =: \frac{1}{x^3} + \sum f_j(x; \epsilon)$$

where, once more, the interchange of summation and integration is justified by the bound (68). We fix $x$, drop it from the notations, and note that in the last sum we have $|f_j| \leq \text{const}(j!)^{-2}$. We thus need to study the analyticity of $f_j$. We claim that $f_j(\epsilon) = G_j(\epsilon, \epsilon \log \epsilon)$ where $G_j(u, v)$ is analytic in small $(u, v)$. Dominated convergence ensures that the integral on the left side of (70) is of the same form.

We will use the following Lemma which applies at the other end of Watson’s Lemma setting.

**Lemma 23.** Assume $F$ is bounded on $0, M$ and analytic in a sector $\{z : |z| > R; \arg(z) \in (a, b)\}$ and $f$ is of the form $z^N G(z^{-1}, z^{-1} \log z)$ where $G(u, v)$ is analytic in the polydisk $\{(u, v) : |u| < M_1^{-1}, |v| < M_1^{-1}\}$, let $M_1 > M$ and consider

$$f(s) = \int_0^\infty e^{-sp} F(p) dp$$

Then

$$f(s) = s^{-N-1} H(s, s \log s)$$

where $H(u, v)$ is analytic for small $u, v$. (In a very similar way, the lemma could accommodate for fractional powers of $z^{-1}$.)

Note that the convergence of the series in $1/x, x^{-1} \log x$ entails that $F$ extends analytically on the Riemann surface of the log in a neighborhood of infinity.

**Proof.** The proof is elementary. Let $M > M_1$, and first note that $\int_0^M e^{-sp} F(s) ds$ is entire, and we only need to consider the integral from $M$ to infinity. Since for some $C > 0$ and all $(l, j) > (0, 0)$ we have

$$\int_M^\infty e^{-sp} p^{N-n-l} \log^l (p) dp \leq \text{const} M^{N-n-l} \log^l (N) \frac{1}{s}$$

the series

$$F(p) = p^N \sum_{k, l} c_{k, l} x^{-k} (x^{-1} \log x)^l$$

can be integrated term by term and uniform convergence easily entails that it is enough to show the property for a single term, of the form

$$Q(s) = \int_M^\infty e^{-sp} p^{N-n-l} \log^l (p) dp$$
A finite number of integrations by parts, multiplications by $s$ and differentiations in $s$ brings (73) to
\begin{equation}
\int_{M}^{\infty} e^{-sp} dp = e^{-Ms/s} = \frac{1}{s} + \text{entire}(s)
\end{equation}
Undoing the operations above on the last expression in (75) easily completes the proof.

For large enough $a$ we now write
\begin{equation}
f_j = c_{j+1} \int_{0}^{\infty} e^{-sx} F_j(s) ds = c_{j+1} \int_{0}^{M} e^{-sx} F_j(s) ds + c_{j+1} \int_{M}^{\infty} e^{-sx} F_j(s) ds
\end{equation}
where we choose $M$ large enough.

The first integral is manifestly entire in $\epsilon$. For the second term, we have the following.

Lemma 24. $F_j(s) = s^j W_j(s^{-1}, s^{-1} \log s)$ where $W_j(u, v)$ is analytic for small $u, v$.

Proof. Induction from (67): The right side operations on the right side consist in multiplication by $\tau^{-1}(\tau + 1)^{-1}$, and three definite antiderivatives (from zero). It is sufficient to show that each of these operations preserves the structure above, aside from the leading order behavior which follows from straightforward power counting. Multiplication by $\tau^{-1}(\tau + 1)^{-1}$ clearly preserves the structure mentioned.

\begin{equation}
\int_{0}^{\tau} = \int_{0}^{M} + \int_{M}^{\tau}
\end{equation}
where the first integral is a mere constant, and $M$ is chosen so that $W_j$ is analytic for $|u| < 1/M$ and $|v| < 1/M$. We then write $F_j = s^j \sum_{k \leq j+1, l} c_{jkl} u^k v^l + F_{j1}$ with $u = s^{-1}$, $v = s^{-1} \log s$ and where we see that $F_{j1} = O(s^{-2} \log s^k)$. The sum contains finitely many terms, and for it the structure follows by explicit integration.

For the second we write
\begin{equation}
\int_{M}^{\tau} = \int_{M}^{\infty} - \int_{\tau}^{\infty}
\end{equation}
where the first integral on the right is a constant. For the second one, the structure for large $t$ follows from term by term integration and straightforward estimates.

The rest of the proof follows from Lemma 23 noting that, for fixed $l$, there are only finitely many terms in the expansion at infinity of $f$ for which the total power of $\tau$ exceeds $-l$.

It is clear that all the arguments above go through if $x^{-3}$ is replaced by $x^{-m}, m > 3$, except for (64) where the exponent will be $p^{1/m}$ and the power of the prefactor changes. The bounds for the Jost functions follow immediately from the Laplace representation of $s(x)$ and contour deformation.

If the potential is analytic and $O(x^{-m})$ at infinity, then the function $H$ will, in general, have exponential order one, rather than fractional, and the bounds (68) are “worse”, the power of the factorial becoming one. This can be shown similarly, using a roughly similar recurrence. In the $O(x^{-3})$ case, one would get recurrence of the form
\begin{equation}
F_{j+1} = P^3 \frac{F_j}{\tau(\tau + 2)} + a_1 P^4 \frac{F_{j-1}}{\tau(\tau + 2)} + \cdots + a_{j-2} P^j \frac{F_2}{\tau(\tau + 2)}, j \geq 3
\end{equation}
where $a_j$ grow at most geometrically. The rest of the proof is roughly the same, but the details are more cumbersome.

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REFERENCES

[1] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables* New York : Wiley-Interscience (1984).
[2] M.V. Berry, C.J. Howls, *Proc. R. Soc. Lond. A* 430, pp. 653-668, (1990).
[3] O. Costin, *Duke Math. J.*, 93, 2, (1998).
[4] O. Costin, R.D. Costin and J. Lebowitz, *J. Stat. Phys.* 1–4 pp. 283-310 (2004).
[5] O. Costin, *Asymptotics and Borel Summability* CRC Press LLC, (2008).
[6] O. Costin, M.D. Kruskal, *Proc. R. Soc. Lond. A* 455, pp. 1931–1956 (1999).
[7] O. Costin and M.D. Kruskal, *Proc. R. Soc. Lond. A* 455, 1931-1956 (1999).
[8] G. García-Calderón and R. Peierls, Nuclear Physics A, 265 pp. 463-460 (1976).
[9] P. Garrido, S. Goldstein, J. Lukkarinen and R. Tumulka *Paradoxical Reflection in Quantum Mechanics* [arXiv:0808.0610]
[10] M. Goldberg, *Proceedings of the AMS*, 135, pp. 3173179, (2007).
[11] I. Herbst, in *Rigorous Atomic and Molecular Physics*, G. Velo and A.S. Wightman ed., Plenum Press, London (1981).
[12] E. Hille, *Ordinary Differential Equations in the Complex Domain*, Dover, New York (1997).
[13] R. de la Madrid and M. Gadella, , *Amer. J. Phys.* 70 no. 6, pp. 626–638, (2002).
[14] R. de la Madrid *European J. Phys.* 26 pp. 287–312, no. 2, (2005).
[15] S. Mandelbrojt *Séries lacunaires*, Actualités scientifiques et industrielles, Paris, 305 (1936).
[16] R. G. Newton, *Scattering theory of waves and particles*, McGraw Hill New York (1966).
[17] A.B. Olde Daalhuis, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 454 no. 1968, 1–29 (1998) (see also note at [http://www.maths.ed.ac.uk/~adri/public.html](http://www.maths.ed.ac.uk/~adri/public.html))
[18] M. Reed and B Simon, *Methods of modern mathematical physics* Academic Press, New York, (1972).
[19] B. Simon, *International Journal of Quantum Chemistry*, 14, 4, pp. 529–542 (1978).
[20] E. Skibsted, Comm. Math. Phys. 104 no. 4, pp 591–604 (1986).
[21] A. Rokhlenko, preprint.
[22] W Wasow, *Asymptotic expansions for ordinary differential equations*, Interscience Publishers (1968).

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