Representations of the Riemann zeta function: A probabilistic approach

Jiamei Liu  Yuxia Huang  Chuancun Yin
School of Statistics, Qufu Normal University
Shandong 273165, China

February 1, 2019

Abstract In this paper, we give a short elementary proof of the well known Euler’s recurrence formula for the Riemann zeta function at positive even integers and integral representations of the Riemann zeta function at positive integers and at fractional points by means of probabilistic approach. The proof is based on the moment generating function and the characteristic function of logistic and half-logistic distributions in probability theory.

Keywords: Bernoulli numbers; (half-)logistic distribution; Integral representation; probabilistic approach; Riemann zeta function

1 Introduction

The well known Riemann zeta function $\zeta$ is defined by

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}, & \text{if } \Re(s) > 1, \\ \frac{1}{1-2!} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^s}, & \text{if } \Re(s) > 0, s \neq 1, \end{cases}$$

which can be continued meromorphically to the whole complex $s$-plane, except for a simple pole at $s = 1$ with its residue 1, see Srivastava (2003) and Choi et al. (2004) for details.
One of the most celebrated formulas, discovered by Euler in 1734, is the following formula for positive even integers

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1}}{(2n)!} \pi^{2n} B_{2n}, \quad n \in \mathbb{N}_0, \quad (1.1)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $B_n$ is the $n$th Bernoulli numbers. Here $\mathbb{N}$ is the set of natural numbers. Since then, many new proofs have been obtained, see, for example, Titchmarsh and Heath-Brown (1986), Amo et al. (2011), Arakawa et al. (2014) and Ribeiro (2018). On the contrary, however, no analogous closed forms representation of $\zeta(s)$ at odd integers or fractional points are known (cf. Srivastava and Choi (2012), P. 167). Even up to now, for positive odd integer arguments the Riemann zeta function can only be expressed by series and integral. One possible integral expression is established by Cvijović and Klinowski (2002) as follows

$$\zeta(2n + 1) = (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2(2n + 1)!} \int_0^1 B_{2n+1}(u) \cot(\pi u)\,du, \quad n \in \mathbb{N}, \quad (1.2)$$

where $B_n(x)$ are Bernoulli polynomials defined by the generating function (cf. Lu (2011))

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi.$$

The Bernoulli numbers $B_n = B_n(0)$ are well-tabulated (see, for example, Srivastava(2003)):

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_{2n+1} = 0 \quad (n = 1, 2, \ldots), \quad \cdots,$$

from which one can finds that

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \cdots.$$

The zeta function $\zeta(s)$ has also the following integral representation (cf. Srivastava and Choi (2012, p.172))

$$\zeta(s) = \frac{(1 - 2^{1-s})^{-1}}{\Gamma(s+1)} \int_0^\infty \frac{t^s e^t}{(e^t + 1)^2} \,dt, \quad R(s) > 0. \quad (1.3)$$

Note that there is an extra 2 in (51) of Srivastava and Choi (2012, p.172).

The aim of this note is to present a simple proof of the recurrence formula (1.1) for $\zeta(2n)$ and the integral representations for $\zeta(n)$ and $\zeta\left(n - \frac{1}{2}\right)$ by making use of probabilistic method.
2 The main results and their proofs

In this section we present a new elementary proof to the following well known results.

**Proposition 2.1.** For Riemann's zeta function $\zeta$, we have

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} \pi^{2n} B_{2n}}{(2n)!}, \quad n \in \mathbb{N}_0,$$

(2.1)

$$\zeta \left( n - \frac{1}{2} \right) = \frac{2^n \int_1^\infty \frac{(\ln y)^{n-\frac{1}{2}}}{(1+y)^2} dy}{\sqrt{\pi}(2n-1)!!(1 - 2^{1-2n})}, \quad n \in \mathbb{N},$$

(2.2)

and

$$\zeta(n) = \frac{(1 - 2^{1-n})^{-1}}{n!} \int_0^\infty \frac{x^n e^{-x}}{(1 + e^{-x})^2} dx, \quad n \in \mathbb{N}, n > 1,$$

(2.3)

where $B_n$ is the $n$th Bernoulli numbers.

To prove the proposition, we need the following three lemmas.

**Lemma 2.1.** We assume that random variable $X$ has the standard logistic distribution with the probability density function (pdf)

$$f(x) = \frac{\exp(-x)}{(1 + \exp(-x))^2}, \quad -\infty < x < \infty.$$  

(2.4)

Then the moment generating function (mgf) of $X$ is given by

$$E[\exp(tX)] = \frac{\pi t}{\sin(\pi t)}, \quad |t| < 1.$$  

(2.5)

**Proof** By the definition of mgf, for any $|t| < 1$, we get

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{e^{-x}}{(1 + e^{-x})^2} dx$$

$$= \int_0^1 y^{-t}(1 - y)^t dy = B(1 - t, 1 + t)$$

$$= \Gamma(1 - t)\Gamma(1 + t)$$

$$= \frac{\pi t}{\sin(\pi t)},$$
where $B(\cdot, \cdot)$ is the Beta function and $\Gamma(\cdot)$ is the $\Gamma$ function, we have used the fact that (see Gradshteyn and Ryzhik (1980) P.896))

$$\Gamma(1 - t)\Gamma(t) = \frac{\pi}{\sin \pi t}$$

in the last equality.

**Lemma 2.2.** We assume that random variable $X$ has the standard 1-dimensional elliptically symmetric logistic distribution with pdf

$$f(x) = c\frac{\exp(-x^2)}{(1 + \exp(-x^2))^2}, -\infty < x < \infty, \quad (2.6)$$

where

$$c = \left( \int_0^\infty t^{-\frac{3}{2}} \frac{e^{-t}}{(1 + e^{-t})^2} dt \right)^{-1}.$$ 

Then the characteristic function of $X$ is given by

$$E[\exp(itX)] = 1 + \sum_{n=1}^{\infty} (-1)^n c\sqrt{\pi} \frac{t^{2n}}{2^{2n+1} n!} \left( 1 - 2 \frac{2n-1}{2} \right) \zeta \left( n - \frac{1}{2} \right), \quad (2.7)$$

where $\zeta$ is the Riemann zeta function.

**Proof** Let $h(x) = (1 + \exp(-x^2))^2$, with the expansion for $x \neq 0$,

$$(1 + \exp(-x^2))^{-2} = \sum_{k=1}^{\infty} (-1)^{k-1} k \exp(-(k-1)x^2),$$

we rewrite (2.6) as

$$f(x) = c \sum_{k=1}^{\infty} (-1)^{k-1} k \exp(-kx^2).$$

Noting that $f(-x) = f(x), -\infty < x < \infty$, so that all the odd-order moments of $f$ are
zero. Hence, we only need to determine the even-order moments. We get

\[ E(X^{2m}) = 2 \int_0^\infty x^{2m} f(x) \, dx \]

\[ = 2c \int_0^\infty x^{2m} \sum_{k=1}^\infty (-1)^{k-1} k \exp(-kx^2) \, dx \]

\[ = c \sum_{k=1}^\infty (-1)^{k-1} \frac{\sqrt{\pi} (2m)!}{2^{2m} m!} k^{-2m-1} \]

\[ = -c \frac{\sqrt{\pi} (2m)!}{2^{2m} m!} \frac{1}{2^{m-1}} + c \sum_{k=1}^\infty \frac{\sqrt{\pi} (2m)!}{2^{2m} m!} (2k-1)^{-2m-1} \]

\[ = c \frac{\sqrt{\pi} (2m)!}{2^{2m} m!} \frac{1}{2^{m-1}} - c \sum_{k=1}^\infty \frac{\sqrt{\pi} (2m)!}{2^{2m} m!} (2k)^{-2m-1} \]

\[ = \frac{\sqrt{\pi} c (2m)!}{2^{2m} m!} (1 - 2^{-2m-3}) \zeta(m - \frac{1}{2}), \]

where we have used the fact that

\[ \int_0^\infty \exp(-bx^2)x^{2k} \, dx = \frac{\sqrt{\pi} \frac{1}{2} \frac{3}{2} \frac{1}{2} \ldots \frac{2k-1}{2}}{b^{\frac{2k+1}{2}}}. \]

For any \( t \in (-\infty, \infty) \), we get the characteristic function of \( X \) by performing the following calculations

\[ E[\exp(itX)] = \int_{-\infty}^\infty \exp(itx) f(x) \, dx \]

\[ = E[\exp(itX) + \exp(-itX)]/2 \]

\[ = E \left[ 1 + \sum_{n=1}^\infty (-1)^n \frac{t^{2n} X^{2n}}{(2n)!} \right] \]

\[ = 1 + \sum_{n=1}^\infty (-1)^n \frac{t^{2n} E(X^{2n})}{(2n)!} \]

\[ = 1 + \sum_{n=1}^\infty (-1)^n \frac{c \sqrt{\pi} (2n)!}{2^{2n} n! (2n)!} \left( 1 - 2^{-2n-3} \right) \zeta \left( n - \frac{1}{2} \right) \]

\[ = 1 + \sum_{n=1}^\infty (-1)^n \frac{c \sqrt{\pi} t^{2n}}{2^{2n} n!} \left( 1 - 2^{-2n-3} \right) \zeta \left( n - \frac{1}{2} \right). \]

This ends the proof of Lemma 2.2.

**Lemma 2.3.** We assume that random variable \( X \) has the standard half-logistic distribution with the pdf

\[ f(x) = \frac{2 \exp(-x)}{(1 + \exp(-x))^2}, \quad x > 0. \tag{2.8} \]
Then the mgf of $X$ is given by

$$E[\exp(tX)] = 1 + 2 \sum_{n=1}^{\infty} (1 - 2^{1-n})\zeta(n)t^n, \ |t| < 1,$$

(2.9)

where $\zeta$ is the Riemann zeta function.

**Proof** The mean of $X$ is given by

$$E(X) = 2 \int_{0}^{\infty} \frac{x e^{-x}}{(1 + \exp(-x))^2} dx = 2 \ln 2.$$

Using the expansion

$$f(x) = 2 \sum_{k=1}^{\infty} (-1)^{k-1} ke^{-kx}$$

$$= 2 \sum_{k=1}^{\infty} (2k-1)e^{-(2k-1)x} - 2 \sum_{k=1}^{\infty} 2ke^{-2kx}, \ x > 0,$$

we get, for any positive integer $n > 1$,

$$E(X^n) = 2 \int_{0}^{\infty} x^n f(x) dx$$

$$= 2 \sum_{k=1}^{\infty} (2k-1) \int_{0}^{\infty} x^n e^{-(2k-1)x} dx - 2 \sum_{k=1}^{\infty} 2k \int_{0}^{\infty} x^n e^{-2kx} dx$$

$$= 2n! \sum_{k=1}^{\infty} \frac{1}{(2k-1)^n} - 2n! \sum_{k=1}^{\infty} \frac{1}{(2k)^n}$$

$$= 2n!(1 - 2^{1-n})\zeta(n).$$

Then we have

$$E(e^{tX}) = 1 + \sum_{k=1}^{\infty} \frac{E(X^k)}{k!} t^k$$

$$= 1 + 2t \ln 2 + \sum_{k=2}^{\infty} \frac{2k!\zeta(k)(1-2^{1-k})}{k!} t^k$$

$$= 1 + 2t \ln 2 + 2 \sum_{k=2}^{\infty} (1 - 2^{1-k})\zeta(k)t^k, \ |t| < 1,$$

where we have used the fact

$$\lim_{s \to 1} (s-1)\zeta(s) = 1.$$

This completes the proof of Lemma 2.3.
Proof of Proposition 2.1

The mgf of the standard logistic distribution can be written as

\[ E(e^{tX}) = 1 + \sum_{n=1}^{\infty} \frac{[2^{2n-1} - 1] \zeta(2n)}{2^{2(n-1)}} t^{2n}, \quad (2.10) \]

see, for example, Ghosh, Choi and Li (2010). Comparing (2.4) and (2.7) yields

\[ g(t) = 1 + \sum_{n=1}^{\infty} \frac{[2^{2n-1} - 1] \zeta(2n)}{2^{2(n-1)}} t^{2n}, \]

and

\[ h(t) = \frac{\pi t}{\sin(\pi t)}. \]

Using the series expansion

\[ \frac{\pi t}{\sin(\pi t)} = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{2^{2k} - 2}{(2k)!} B_{2k}(\pi t)^{2k}, \]

where \( B_{2k} \) is the \( 2k \)th Bernoulli numbers, we have

\[ \sum_{n=1}^{\infty} \frac{[2^{2n-1} - 1] \zeta(2n)}{2^{2(n-1)}} t^{2n} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} - 2}{(2k)!} B_{2k}(\pi t)^{2k}, \quad |t| < 1, \]

from which we deduce that

\[ \zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} \pi^{2n} B_{2n}. \]

This completes the proof of (2.1).

Now we prove (2.2). Denoting by

\[ H(t) = \int_{-\infty}^{\infty} \exp(itx) f(x) dx, \]

and

\[ G(t) = 1 + \sum_{k=1}^{\infty} (-1)^{k} c \sqrt{\pi} \frac{t^{2k}}{2^{2k+1} k!} \left( 1 - 2 \frac{2k-1}{2} \right) \zeta \left( k - \frac{1}{2} \right), \]

where \( f \) is defined by (2.4). Taking \( 2n \)th and \( (2n + 1) \)th derivatives of the two functions with respect to \( t \), we get

\[ H^{(2n)}(t) = (-1)^{n} 2c \int_{0}^{\infty} x^{2n} \cos tx \frac{\exp(-x^2)}{(1 + \exp(-x^2))^2} dx, \]
\[ H^{(2n+1)}(t) = (-1)^{n+1} 2c \int_{0}^{\infty} x^{2n-1} \sin tx \frac{\exp(-x^2)}{(1+\exp(-x^2))^2} dx, \]

and

\[ G^{(2n-1)}(t) = \sum_{k=n}^{\infty} (-1)^k \frac{\sqrt{\pi} \prod_{l=0}^{2n-2} (2k-l)}{2^{2k} k!} (1 - 2^{2k-3}) \left( k - \frac{1}{2} \right) \zeta \left( k - \frac{1}{2} \right) t^{2k-2n+1}, \]

\[ G^{(2n)}(t) = \sum_{k=n}^{\infty} (-1)^k \frac{\sqrt{\pi} \prod_{l=0}^{2n-1} (2k-l)}{2^{2k} k!} (1 - 2^{2k-3}) \left( k - \frac{1}{2} \right) \zeta \left( k - \frac{1}{2} \right) t^{2k-2n}. \]

Note that \( H(t) = G(t) \) for any real \( t \), and thus \( H^{(n)}(t) = G^{(n)}(t) \) for any real \( t \) and any positive integers \( n \). In particular, \( H^{(n)}(0) = G^{(n)}(0) \). However, \( H^{(2n+1)}(0) = G^{(2n+1)}(0) = 0 \), and from \( H^{(2n)}(0) = G^{(2n)}(0) \) we have

\[ \zeta \left( \frac{2n - 1}{2} \right) = \frac{2^{2n+1} n! \int_{0}^{\infty} x^{2n} \frac{\exp(-x^2)}{(1+\exp(-x^2))^2} dx}{\sqrt{\pi}(2n)!(1 - 2^{2n-3})} = \frac{2^n \int_{0}^{\infty} x^{2n-1} \frac{\exp(-x)}{(1+\exp(-x))^2} dx}{\sqrt{\pi}(2n-1)!(1 - 2^{2n-3})} = \frac{2^n \int_{1}^{\infty} \frac{(\ln y)^n}{y^2} dy}{\sqrt{\pi}(2n-1)!(1 - 2^{2n-3})}, \quad n \in \mathbb{N}, \]

which concludes the proof of (2.2).

Finally we prove (2.3). Using (2.9) one has

\[ 2 \int_{0}^{\infty} e^{tx} \frac{e^{-x}}{(1 + e^{-x})^2} dx = 1 + 2t \ln 2 + 2 \sum_{k=2}^{\infty} (1 - 2^{1-k}) \zeta(k) t^k, \quad |t| < 1. \quad (2.11) \]

Taking \( n \)th derivative of both sides of (2.11) with respect to \( t \) and then setting \( t = 0 \) yields the desired result.

**Acknowledgements.** The research was supported by the National Natural Science Foundation of China (No. 11571198).

**References**

[1] Arakawa, T., Ibukiyama, T., Kaneko, M. (2014). Bernoulli numbers and zeta functions, with an appendix by Don Zagier. Springer Monographs in Mathematics, Springer, Tokyo.
[2] Choi, J., Cho, Y. J., Srivastava, H. M. (2004). Series involving the zeta function and multiple Gamma functions. Applied Mathematics and Computation 159, 509-537.

[3] Cvijović, D., Klinowski, J. (2002). Integral representations of the Riemann zeta function for odd-integer arguments. Journal of Computational and Applied Mathematics 142, 435-439.

[4] De Amo, E., Díaz Carrillo, M., Fernández-Sánchez, J. (2011). Another proof of Euler’s formula for $\zeta(2k)$. Proceedings of the American Mathematical Society 139, 1441-1444.

[5] Ghosh, M., Choi, K. P., Li, J. (2010). A commentary on the logistic distribution. In book: The Legacy of Alladi Ramakrishnan in the Mathematical Sciences, Alladi, K., Klauder, R., Rao, R. (eds.), Springer Science+Business Media.

[6] Gradshteyn, I. S., Ryzhik, I. M. (1980). Tables, Integrals, Series, and Products. Academic, New York.

[7] Lu, D. Q. (2011). Some properties of Bernoulli polynomials and their generalizations. Applied Mathematics Letters 24, 746-751.

[8] Ribeiro, P. (2018). Another proof of the famous formula for the zeta function at positive even integers. The American Mathematical Monthly 125(9), 839-841.

[9] Srivastava, H. M. (2003). Certain classes of series associated with the zeta and related functions. Applied Mathematics and Computation 141, 13-49.

[10] Srivastava, H. M., Choi, J. (2012). Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier Inc., New York.

[11] Titchmarsh, E. C., Heath-Brown, D. R. (1986). The Theory of the Riemann Zeta-function, 2nd ed., Oxford University Press, New York.