A NOTE ON A NON-LINEAR KREIN-RUTMAN THEOREM

RAJESH MAHADEVAN

Abstract. In this note we will present an extension of the Krein-Rutman theorem for an abstract non-linear, compact, positively 1-homogeneous operators on a Banach space having the properties of being increasing with respect to a convex cone $K$ and such that there is a non-zero $u \in K$ for which $MTu \geq u$ for some positive constant $M$. This will provide a uniform framework for recovering the Krein-Rutman-like theorems proved for many non-linear differential operators of elliptic type, like the $p$-Laplacian cf. Anane [1], Hardy-Sobolev operator cf. Sreenadh [13], Pucci’s operator cf. Felmer et. al. [6]. Our proof follows the same lines as in the linear case cf. Rabinowitz [12] and is based on a bifurcation theorem.

1. Introduction

Let $X$ be a real Banach space. Let $K$ be a closed convex cone in $X$ with vertex at 0, that is a set having the properties:

(i) $0 \in K$, (ii) $x \in K, t \in \mathbb{R}^+ \implies tx \in K$, and (iii) $x, y \in K \implies x + y \in K$.

We further assume that

(A) \quad $K \cap -K = \{0\}$.

The cone $K$ induces an ordering $\preceq$ on $X$ defined as follows. Given any $x, y \in X$ we say that

\begin{equation}
  x \preceq y \iff y - x \in K.
\end{equation}

The ordering in (1) is said to be strict if $x \preceq y$ and $x \neq y$ and this will be denoted by $x \prec y$. A mapping $T : X \to X$ is said to be increasing if $x \preceq y \implies T(x) \preceq T(y)$ and it is said to be strictly increasing if $x \prec y$ implies $Tx \prec Ty$. The mapping is said to be compact if it takes bounded subsets of $X$ into relatively compact subsets of $X$. We say that the mapping is positively 1-homogeneous if it satisfies the relation $T(tx) = tT(x)$ for all $x \in X$ and $t \in \mathbb{R}^+$. We say that a real number $\lambda$ is an eigenvalue of the operator if there exists a non-zero $x \in X$ such that $\lambda Tx = x$.

Let us end this section by stating a simple and obvious fact concerning closed convex sets which will be used later on.

Lemma 1. Given any $0 \prec x$ and $y \notin K$ there exists a unique $\delta$ with $0 \leq \delta(y)$ such that

\begin{align*}
  x + \lambda y & \in K \text{ if } 0 \leq \lambda \leq \delta(y) \\
  x + \lambda y & \notin K \text{ if } \lambda > \delta(y).
\end{align*}
Furthermore, we shall have $\delta > 0$ if $x$ belongs to the interior of $K$ denoted by $\hat{K}$.

2. **Main Theorem**

Let $X$ be a real Banach space and let $K$ be a closed convex cone satisfying the assumption (A) given in the previous section.

**Theorem 2.** Let $T : X \to X$ be an increasing, positively 1-homogeneous compact continuous operator (non-linear) on $X$ for which there exists a non-zero $u \in K$ and $M > 0$ such that

$$(H) \quad MTu \succ u.$$ 

Then, $T$ has a non-zero eigenvector $x_0$ in $K$. Furthermore, if $K$ has non-empty interior and if $T$ maps $K \setminus \{0\}$ into $\hat{K}$ and is strictly increasing, then $x_0$ is the unique positive eigenvalue in $K$ up to a multiplicative constant. And, finally if $\mu_0$ be the corresponding eigenvalue, then it can be characterized as the eigenvalue having the smallest absolute value and furthermore, it is simple.

We prove this theorem in exactly the same way as it is done in the notes of Rabinowitz [12] for linear operators and relies on the following result cf. Rabinowitz [12], Corollaire 1.

**Proposition 3.** Let $X$ and $K$ be as in the statement of the previous theorem. Let us be given a mapping $F : \mathbb{R} \times K \to K$ which is compact, continuous and such that $F(0,x) = 0$ for all $x \in K$. Then, the equation $F(\lambda,x) = x$ has a non-trivial connected unbounded component of solutions $C^+$ in $\mathbb{R}^+ \times K$ containing the point $(0,0)$.

**Proof of Theorem**

Step 1: Let $u \in K$ ($u \neq 0$) be as in the hypothesis (H) of the theorem. Let $\varepsilon > 0$ be a parameter. We define a parametrized family of operators $F_\varepsilon : \mathbb{R}^+ \times K \to K$ as follows

$$(1) \quad F_\varepsilon(\lambda,x) := \lambda T(x + \varepsilon u).$$ 

Then, since $T$ is compact and continuous, each of these operators $F_\varepsilon$ is clearly compact and continuous on $\mathbb{R}^+ \times K$. Also they map $\mathbb{R}^+ \times K$ into $K$ since $K + \varepsilon u \subset K$ and $T$ maps $K$ into itself, which follows from the fact that $T$ is increasing and $T(0) = 0$. Let, by the above proposition, $C^+_\varepsilon \subset \mathbb{R}^+ \times K$ be a connected unbounded branch of solutions to the equation

$$(2) \quad F_\varepsilon(\lambda,x)(:= \lambda T(x + \varepsilon u)) = x.$$ 

**Claim:** Then $C^+_\varepsilon \subset [0,M] \times K$ for all $\varepsilon > 0$.

We shall prove the claim by proving the following estimate: whenever, $(\lambda,x) \in C^+_\varepsilon$ we have

$$(3) \quad x \succ \left( \frac{\lambda}{M} \right)^n \varepsilon u \quad \forall n \in \mathbb{N}.$$ 

Indeed, if the estimate (3) is true then we cannot have $\lambda > M$. Otherwise, letting $n \to +\infty$ we will have $u \in -K$ and hence, $u \in K \cap (-K)$ contradicting the hypothesis (A). So, we are left to prove the estimate (3). We shall do it by induction. Let $(\lambda,x)$ belong to $C^+_\varepsilon$. Then, $x = \lambda T(x + \varepsilon u)$ and we obtain, from the properties of $T$ and the inequalities $x + \varepsilon u \succ \varepsilon u, x + \varepsilon u \succ x$, respectively that

$$(4) \quad x \gg \lambda \varepsilon Tu,$$

$$(5) \quad x \gg \lambda Tx.$$
Now, using (H), it follows from (6) that
\[ x \succ \lambda \frac{\varepsilon}{M} u. \]
and we obtain our claim for \( n = 1 \). Let us now assume that (6) is true for \( n = m \).
Operating \( T \) on (6), we obtain
\[ T x \succ \left( \frac{\lambda}{M} \right)^m \varepsilon T u \succ \left( \frac{\lambda^m}{M^{m+1}} \right) \varepsilon u \]
So, using (6), we obtain
\[ x \succ \left( \frac{\lambda}{M} \right)^{m+1} \varepsilon u. \]
This completes the induction step and proves (8).

Step 2: As we have shown that \( C_\varepsilon^+ \subseteq [0, M] \times K \) for every \( \varepsilon > 0 \) and besides, the branch is connected and infinite starting from \((0,0)\), there must necessarily exist \( x_\varepsilon \) with \( \|x_\varepsilon\| = 1 \) and \( \lambda_\varepsilon \in [0, M] \) such that \((\lambda_\varepsilon, x_\varepsilon) \in C_\varepsilon^+ \). That is we have,
\[ x_\varepsilon = \lambda_\varepsilon T(x_\varepsilon + \varepsilon u), \quad \|x_\varepsilon\| = 1. \]
Now, if we consider a sequence \( \varepsilon \) which tends to 0, then the sequence \( x_\varepsilon + \varepsilon u \) is bounded in \( X \). As the operator \( T \) is compact, we may assume that \( T(x_\varepsilon + \varepsilon u) \) converges for a subsequence and we suppose also that \( \lambda_\varepsilon \) converges to some \( \lambda_0 \) for an induced subsequence. Without loss of generality, indexing this subsequence again by the same \( \varepsilon \), we obtain that \( x_\varepsilon \) converges to some \( x_0 \in K \) with \( \|x_0\| = 1 \). Further, since \( T \) is a continuous operator, letting \( \varepsilon \to 0 \) in (7) we obtain
\[ x_0 = \lambda_0 T x_0, \quad \|x_0\| = 1, x_0 \in K. \]
Thus, we have shown the existence of an eigenvector in \( K \setminus \{0\} \). Further, it follows from (8) that \( \lambda_0 \neq 0 \).

From now on we shall assume that \( K \) is non-empty and that \( T \) maps \( K \setminus \{0\} \) into \( K \) and is strictly increasing.

Step 3: We shall now prove that \( x_0 \) obtained above is the unique eigenvector in \( K \) up to a multiplicative constant. Since, \( x_0 \) is non-zero and we have assumed that \( T \) maps \( K \setminus \{0\} \) into \( K \), we obtain first of all that \( x_0 \in K \). By the same arguments, if \( y \in K \) is any other eigenvector having eigenvalue \( \lambda \) then, \( \lambda \neq 0 \) and \( y \in K \). First, using the fact that \( x \in K \) and \(-y \notin K \), we apply Lemma 1 and obtain \( \delta(-y) > 0 \) such that \( x_0 - \delta(-y) y \in K \). We claim that \( x_0 = \delta(-y) y \). Otherwise, since \( T \) is strictly increasing, we will have
\[ Tx_0 \succ T(\delta(-y) y). \]
From this we deduce that \((\lambda_0)^{-1} x_0 \succ (\lambda)^{-1} \delta(-y)y \) which, again by Lemma 1, yields
\[ \lambda_0 < \lambda. \]
On the other hand, similarly, starting from the fact that \( y - \delta(-x_0) x_0 \in K \), we obtain
\( (\lambda)^{-1} y \succ (\lambda_0)^{-1} \delta(-x_0) \) yielding
\[ \lambda \leq \lambda_0. \]
Since (8) and (9) contradict each other, it follows that our claim \( x_0 = \delta(-y) y \) must be true. This shows that any other eigenvector in \( K \) is a multiple of \( x_0 \).

Step 4: Let us now show the simplicity of \( \lambda_0 \). From the arguments of the previous paragraph if \( y \) is any other eigenvector corresponding to \( \lambda_0 \) and if \( y \in K \) then \( y \) is a
multiple of $x_0$. So, we deal now with the case that $y \notin K$. We claim that $x_0 = \delta(y) y$. If not, arguing as in the previous paragraph, we will obtain $\lambda_0 < \lambda_0$ which is absurd. Thus, we have proved that $\lambda_0$ is simple.

Step 5: Let us now show that $\lambda_0$ has the smallest absolute value among all eigenvalues of the operator $T$. Let $\lambda \in \mathbb{R}$, different from $\lambda_0$, be any eigenvalue of $T$ with corresponding eigenvector $y$. Since, $\lambda \neq \lambda_0$, this would mean that neither $y$ nor $-y$ belongs to $K$. So, we have, $x_0 \pm \delta(\pm y) y \in K$, which gives

\[
x_0 \pm \frac{\lambda_0}{\lambda} \delta(\pm y) y \in K.
\]

Let us first consider the case $\lambda > 0$. From (11), using Lemma 1, we get $\lambda_0 \leq \lambda$ showing that $\lambda_0$ has a smaller absolute value. In the case when $\lambda < 0$, from (11), we can obtain the inequalities

\[
\frac{\lambda_0}{-\lambda} \delta(+y) \leq \delta(-y), \quad \text{and} \quad \frac{\lambda_0}{-\lambda} \delta(-y) \leq \delta(+y).
\]

From the above, we get

\[
\frac{\lambda_0}{-\lambda} \leq \frac{\delta(-y)}{\delta(+y)} \leq \frac{-\lambda}{\lambda_0}.
\]

This, gives $|\lambda_0| \leq |\lambda|$. Thus, we have shown that $\lambda_0$ has the smallest absolute value among all eigenvalues of $T$. □

Remark 4. If the operator $T$ maps $K \setminus \{0\}$ into $K$ then it is called strongly positive. For such operators, the hypothesis (H) holds. This can be shown easily by contradiction. However, it is not necessary that $T$ be strongly positive for the hypothesis (H) to hold. In the next section, we shall see some examples of operators which fail to be strongly positive in the first place but we shall obtain the existence of positive eigenvectors as they satisfy the hypothesis (H). As far as the uniqueness and simplicity is concerned, we still need the operator to be strongly positive and strictly increasing. For a linear operator, strong positivity implies, automatically, that it is strictly increasing. □

3. Applications

It has been shown in several papers [1, 6, 13] that many non-linear elliptic operators such as the $p$-Laplacian operator, Hardy-Sobolev operator, Pucci’s maximal operators etc. verify the Krein-Rutman theorem. The methods that have been used to prove a Krein-Rutman theorem for the $p$-Laplace operator or the Hardy-Sobolev operator have relied very much on the variational structure of the operators and on the use of special identities such as the Picone’s identity [2], thus marking a complete difference with the methods used for proving the same result for the Pucci’s maximal operators which are completely non-linear and have no variational structure. It is in this context that our main theorem comes to illustrate the main features that a non-linear operator should possess in order to have a unique(upto a multiplicative constant) positive eigenvector.

The existence part is guaranteed as long as the operator is positively 1-homogeneous, is monotone with respect to a convex cone which does not necessarily have non-empty interior, and satisfies the condition (H) of the theorem. When it comes to checking whether an operator satisfies the hypotheses of the theorem, it is quite clear that homogeneity is a straightforward condition to verify while, the monotonicity depends on the operator space and the cone chosen. Often, the natural choice is the cone of non-negative functions in the function space and verifying this condition is, in general, a task of seeing
whether the weak comparison principle holds. This is, usually, not a difficult problem. Verifying the condition (H) on the other hand may involve a little more work and for this, although not necessary, it is helpful if the operator satisfies a strong maximum principle.

The uniqueness part requires the operator to be strictly increasing which is easily satisfied when the operator is increasing and injective. It also requires that the operator be strongly positive. This is a tougher condition to verify and to begin with, the cone definition has to be changed. A cone which has non-empty interior in the space of continuous functions on a domain $\Omega$ vanishing on the boundary is, for example,

$$(1) \quad K^* = \left\{ w \in C^1(\Omega) \mid w \geq 0 \text{ in } \Omega, \frac{\partial w}{\partial n} \leq 0 \text{ on } \partial \Omega \right\}.$$ 

To be able to work this cone the operator should have enough regularity. Admitting this, the operator shall be strongly positive for this cone if, for example, it satisfies a Hopf maximum principle which readily follows from a strong maximum principle if the boundary of the domain satisfies an interior sphere condition. Thus, to sum up, our conclusion is that the full Krein-Rutman theorem must be available for many non-linear elliptic operators defined on fairly regular domains satisfying the weak comparison principle, having good regularity properties and a strong maximum principle. We shall illustrate this in the case of the $p$-Laplace operator, Hardy-Sobolev operator and the Pucci operators.

Let $\Omega$ be a bounded domain having a smooth, connected boundary in $\mathbb{R}^n$. Let us look at the Dirichlet eigenvalue problem for the $p$-Laplace operator,

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

(2)

Or, now, let 0 belong to the domain and let us look at the Dirichlet eigenvalue problem for more general operators called the Hardy-Sobolev operators. For this recall the Hardy-Sobolev inequality \[8, 7\]

$$(3) \quad \int_{\Omega} |\nabla u|^p dx \geq \left( \frac{n-p}{p} \right)^p \int_{\Omega} w(x)|u|^p dx \quad \text{for all } u \in W^{1,p}_0(\Omega).$$

where $w$ is the weight function

$$w = \begin{cases} \frac{1}{|x| \log \left( \frac{R}{|x|} \right)} & \text{when } p = n \\ \frac{1}{|x|^p} & \text{when } 1 < p < n, \end{cases}$$

and $\left( \frac{n-p}{p} \right)^p$ is the best constant for the inequality. Let $\mu$ be a number strictly smaller than the best constant. Then one defines the Hardy-Sobolev operator $L_\mu$ by setting

$$L_\mu := -\Delta_p u - \mu w(x)|u|^{p-2}u.$$ 

In particular, we recover the $p$-Laplacian operator if we take $\mu = 0$. We may consider the Dirichlet eigenvalue problem

$$L_\mu u = \lambda V(x) |u|^{p-2}u \quad \text{in } \Omega, \quad u \in W^{1,p}_0(\Omega), \quad u \neq 0.$$ 

(4)

for a positive singular weight function $V$ whose singularity at 0 is not worse than that of $w$ above, in the sense that, $\lim_{x \to 0} w(x)V(x) = 0$, as in \[11, 13\]. The full Krein-Rutman for this problem has been proved, using variational techniques, by Sreenadh \[13\]. We now
show that the same can be recovered from our theorem by introducing a suitable operator framework. For this let us consider the weighted reflexive Banach space $L^p(\Omega, V)$ and define a non-linear operator $T$ on it by setting $Tf := (L_\mu)^{-1}(f) = (\int |f|^p dx)^{1/p}$. It can be shown that this is a well defined, compact, continuous operator by using arguments similar to those used in [11]. In fact, it has been shown in [11] that if $f \in W_0^{1,p}(\Omega)$, then $V(x)|f|^p - f \in W^{-1,\alpha}(\Omega)$ for the above class of singular weights $V$. In order that $(L_\mu)^{-1}$ be well defined we need to examine the existence and uniqueness of solution for the Dirichlet problem

\begin{equation}
L_\mu u = g \quad \text{in } \Omega, u \in W_0^{1,p}(\Omega)
\end{equation}

given any $g \in W^{-1,\alpha}(\Omega)$. A solution to this problem can be obtained by minimizing the energy functional

$$J(v) = \frac{1}{p} \int_\Omega |\nabla v|^p dx - \frac{1}{q} \int_\Omega |v|^q dx - \int_\Omega g v dx .$$

By the Hardy-Sobolev inequality \[3\] and the fact that $\mu$ is smaller than the best constant in \[3\], it follows that the functional is coercive. However, the problem is that it is not weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ as, except when $p = 2$, the functional is not convex for any $\mu > 0$. Thus to show that the minimum is attained one has to exhibit a minimizing sequence which converges strongly in $W_0^{1,p}(\Omega)$. For this, one chooses, using the Ekeland variational principle, a minimizing sequence $v_n$ for which $J'(v_n) \to 0$ and it can be checked, similarly as in [11], that such a sequence $v_n$ converges strongly in $W_0^{1,p}(\Omega)$. Also, it can be argued that what is important for the theorem is that $L_\mu^{-1}$ be well defined on the positive cone and the desired uniqueness of positive solutions follows from the weak comparison principle (WCP) proved for positive solutions of such operators in [5]. The same WCP also shows that the operator $T$ is monotone increasing on the positive cone. Now, it is quite easy to check that $T$ is positively 1-homogeneous and its compactness follows from the fact the image of this operator is contained in $W_0^{1,p}(\Omega)$ which is compactly imbedded in $L^p(\Omega, V)$ (cf. [11]). Therefore, it remains to verify the condition (H) to be able to obtain the existence of a positive eigenvector. For this, we use the $C^{1,\alpha}$ regularity of solutions to the Dirichlet problem [3] proved by Tolksdorf [15] and the strong maximum principle for such an operator [14, 16]. In fact, if $f$ is any non-negative smooth function with compact support then, by the above two results, $Tf$ will be a strictly positive function on $\Omega$ and we obtain (H) for a suitable constant $M$. If now we would like to obtain also the uniqueness (upto a multiplicative constant) of positive eigenvector, we need to change the setting to the space of continuous functions vanishing on the boundary. Using the $C^{1,\alpha}$ regularity and the Arzela-Ascoli theorem, we obtain the compactness of the operator $T$. The operator $T$ is strictly increasing because it is increasing and injective. Finally, $T$ sends $K^* \setminus \{0\}$ into the interior of $K^*$ because of the strong maximum principle. So, the full Krein-Rutman theorem follows.

Now we apply the results of our theorem to the case of fully non-linear elliptic operators where no variational methods can be used. Consider the Dirichlet eigenvalue problem

\begin{equation}
F(D^2u) = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\end{equation}

where $F$ is a fully non-linear elliptic operator like in the monograph of Caffarelli and Cabré [4] and which we assume, further, is positively 1-homogeneous. For example, let us consider the Pucci’s maximal operators $\mathcal{M}_\pm$. The Pucci’s maximal operator $\mathcal{M}_+^{1,\alpha}$
$(\mathcal{M}_{\lambda,A})$ is convex (respectively, concave) and hence, viscosity solutions of

$$F(D^2 u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial \Omega$$

(7)

are in fact $C^{2,\alpha}$ regular (cf. Section 6 [4]). So, we may look at the eigenvalue problem as corresponding to the eigenvalue problem $\lambda T u = u$ for the solution operator of the Dirichlet problem $\mathcal{D}$ in $C^2(\Omega)$. The compactness of $T$ follows from the $C^{2,\alpha}$ regularity. Also, it is known that the weak comparison principle holds for these operators (cf. Proposition 2.2. [6]). For all these reasons, we may now apply our theorem and conclude the existence of a unique (upto a multiplicative constant) positive eigenvector for these operators.

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