Linear response theory
and
damped modes of stellar clusters

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Linear response theory
Klimontovich equation

Describing one realisation in phase space \( w = (x, v) \)

Empirical DF

\[
F_d(w, t) = \sum_{i=1}^{N} m \delta_D(w - w_i(t))
\]

Empirical Hamiltonian

\[
H_d(w, t) = U_{\text{ext}}(w) + \int d\mathbf{w}' F_d(w', t) U(w, w')
\]

Continuity equation in phase space

\[
\frac{\partial F_d}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \left( F_d \dot{\mathbf{w}} \right) = 0
\]

Exact Klimontovich equation

\[
\frac{\partial F_d}{\partial t} + [F_d, H_d] = 0
\]

3D gravitational systems

\[
U_{\text{ext}} = \frac{|v|^2}{2}
\]

\[
U = -\frac{G}{|r - r'|}
\]

Damped modes of stellar clusters
Solving Klimontovich

Perturbative expansion

\[
\begin{align*}
F_d &= F_0 + \delta F \text{ with } \langle \delta F \rangle = 0, \\
H_d &= H_0 + \delta H \text{ with } \langle \delta H \rangle = 0.
\end{align*}
\]

Adiabatic approximation

\[
\begin{align*}
F_0 &= F_0(J, t), \\
H_0 &= H_0(J, t).
\end{align*}
\]

Quasi-linear evolution equations

\[
\frac{\partial \delta F}{\partial t} + [\delta F, H_0] + [F_0, \delta H] = 0
\]
\[
\frac{\partial F_0}{\partial t} = -\langle [\delta F, \delta H] \rangle
\]

Timescale separation

\[
\begin{align*}
T_{\delta F} &\simeq T_{\text{dyn}} \\
T_{F_0} &\simeq (\sqrt{N})^2 \times T_{\delta F}
\end{align*}
\]
Dynamics of fluctuations

Fast evolution of **perturbations** (Linearised Klimontovich equation)

\[
\frac{\partial \delta F}{\partial t} + [\delta F, H_0] + [F_0, \delta H] = 0
\]

- **[\delta F, H_0]**: Mean-field advection
- **[F_0, \delta H]**: Collective effects

**Self-consistent** amplification

\[
\delta H = \delta H \left[ \delta F \right]
\]

**Timescale separation**

\[
\begin{align*}
F_0(J) &= \text{cst} \\
H_0(J) &= \text{cst}
\end{align*}
\]

**Phases Mixing**
Solving for the fluctuations

Linear amplification

\[
\delta F_k(J, \omega) = - \frac{\delta F_k(J, 0)}{i(\omega - k \cdot \Omega(J))} - \frac{k \cdot \partial F_0/\partial J}{\omega - k \cdot \Omega(J)} \delta H_k(J, \omega)
\]

with the self-consistency

\[
\delta H(w, t) = \int \delta F(w', t) U(w, w') \quad U = -\frac{G}{|r - r'|}
\]

Generic form of a Fredholm equation

\[
[\delta H(J)]_{\text{dressed}} = [\delta H(J)]_{\text{bare}} + \int dJ' M(J, J') [\delta H(J')]_{\text{dressed}}
\]

Dressing of perturbations

\[
[\delta H(\omega)]_{\text{dressed}} \approx \frac{[\delta H(\omega)]_{\text{bare}}}{1 - M(\omega)} = \frac{[\delta H(\omega)]_{\text{bare}}}{|\varepsilon(\omega)|}
\]
Solving for the fluctuations

Linear amplification

\[ \delta \hat{F}_k(J, \omega) = - \frac{\delta F_k(J,0)}{i(\omega - k \cdot \Omega(J))} - \frac{k \cdot \partial F_0/\partial J}{\omega - k \cdot \Omega(J)} \delta \hat{H}_k(J, \omega) \]

with the self-consistency

\[ \delta H(w, t) = \int dw' \delta F(w', t) U(w, w') \]

Generic form of a Fredholm equation

\[ [\delta H(J)]_{\text{dressed}} = [\delta H(J)]_{\text{bare}} + \int dJ' M(J, J') [\delta H(J')]_{\text{dressed}} \]

Plasma dielectric function

\[ \varepsilon_k(\omega) = 1 + \frac{1}{k^2 \lambda_D^2} \int dv \frac{k \cdot \partial F / \partial v}{k \cdot v - \omega} \]
Damped modes of stellar clusters

Gravitational response matrix

\[ \epsilon_{pq}(\omega) = I - \sum_k \int \frac{k \cdot \partial F/\partial J}{k \cdot \Omega(J) - \omega} \psi_k^*(J) \psi_k(J) \]

Some properties

- Sum over resonances
- Scan over orbital space
- Resonant amplification
- Long-range interaction

Mode

\[ \det[\epsilon(\omega)] = 0 \]

Type of modes

\[
\begin{cases}
\text{Im}[\omega] > 0 & \text{Unstable} \\
\text{Im}[\omega] = 0 & \text{Neutral} \\
\text{Im}[\omega] < 0 & \text{Damped}
\end{cases}
\]
**Plasmas**

**Orbital coordinates**

\[ (\vec{x}, \vec{v}) \]

**Basis decomposition**

\[
U(\vec{x}, \vec{x}') \propto \int \frac{dk}{k^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} 
\]

**Dielectric function**

\[
1 + \frac{1}{k^2} \int dv \frac{\vec{k} \cdot \partial F / \partial \vec{v}}{\vec{k} \cdot \vec{v} - \omega} 
\]

**Resonance condition**

\[
\delta_D(\vec{k} \cdot (\vec{v} - \vec{v}')) 
\]

**Galaxies**

**Orbital coordinates**

\[ (\theta, \vec{J}) \]

**Basis decomposition**

\[
U(\vec{w}, \vec{w}') = -\sum_p \psi^{(p)}(\vec{w}) \psi^{(p)*}(\vec{w}') 
\]

**Dielectric function**

\[
\delta_{pq} - \sum_k \int dJ \frac{\vec{k} \cdot \partial F / \partial J}{\vec{k} \cdot \vec{\Omega}(J) - \omega} \psi^{(p)*}_k (J) \psi^{(q)}_k (J) 
\]

**Resonance condition**

\[
\delta_D(\vec{k} \cdot \vec{\Omega}(J) - \vec{k}' \cdot \vec{\Omega}(J')) 
\]
How to compute the dispersion function?
Damped modes of stellar clusters

**Basis method** \((\psi^{(p)}(w), \rho^{(p)}(w))\)

\[
\begin{align*}
\psi^{(p)}(w) &= \int dw' U(w, w') \rho^{(p)}(w'), \\
\int dw \psi^{(p)}(w) \rho^{(q)*}(w) &= -\delta_{pq}.
\end{align*}
\]

``Separable`` pairwise interaction

\[
U(w, w') = -\sum_p \psi^{(p)}(w) \psi^{(p)*}(w')
\]

**Plasmas**

\[
U(x, x') = \frac{1}{|x - x'|}
\]

**Galaxies**

\[
\Delta \Phi = 4\pi G\rho
\]

Poisson equation
Basis method \((\psi^{(p)}(\mathbf{w}), \rho^{(p)}(\mathbf{w}))\)

\[
\begin{align*}
\psi^{(p)}(\mathbf{w}) &= \int \text{d}\mathbf{w}' \, U(\mathbf{w}, \mathbf{w}') \rho^{(p)}(\mathbf{w}'), \\
\int \text{d}\mathbf{w} \, \psi^{(p)}(\mathbf{w}) \rho^{(q)*}(\mathbf{w}) &= -\delta_{pq}.
\end{align*}
\]

Newtonian interaction

\[
U(\mathbf{r}, \mathbf{r}') = -\frac{G}{|\mathbf{r} - \mathbf{r}'|} = -\int \frac{\text{d}\mathbf{k}}{k^2} \, \text{e}^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} = -\sum_{\ell, m} Y_{\ell m}(\hat{\mathbf{r}}) Y_{\ell m}(\hat{\mathbf{r}}') \frac{\text{Min}[r, r']^\ell}{\text{Max}[r, r']^{\ell+1}}
\]

Scale invariance
Translation invariance
Rotation invariance
Damped modes of stellar clusters

Biorthogonal basis

What matters is the mean potential

\[
\begin{align*}
\rho_{\ell=0, n=1}(r) &= \rho_0(r), \\
\rho_{\ell=1, n=1}(r) &= \frac{d\rho_0}{dr}, \\
\ldots
\end{align*}
\]

cf. Self-consistent field simulations

What matters are the perturbations

\[
\delta \rho(r, t) = \sum_p A_p(t) \rho^{(p)}(r)
\]

cf. Linear response in time domain

What matters is the pairwise interaction

\[
U(r, r') = -\sum_p \psi^{(p)}(r) \psi^{(p)*}(r')
\]

cf. Kinetic theory

How to choose the basis?
Self-consistent amplification

Linear response

\[
\left[ \delta H(J) \right]_{\text{dressed}} = \left[ \delta H(J) \right]_{\text{bare}} + \int \, dJ' \, M(J, J') \left[ \delta H(J') \right]_{\text{dressed}}
\]

Amplification kernel

In terms of **coupling coefficients**

\[
\psi_{kk}^d(J, J', \omega) = \psi_{kk}(J, J') \\
+ (2\pi)^d \sum_{k''} \int \, dJ'' \frac{k'' \cdot \partial F/\partial J''}{k'' \cdot \Omega(J'') - \omega} \psi_{kk''}(J, J'') \psi_{k''k}(J'', J', \omega)
\]

\[
\psi_{kk'}(J, J') \quad \text{Bare coefficient, Landau}
\]

\[
\psi_{kk}^d(J, J', \omega) \quad \text{Dressed coefficient, Balescu-Lenard}
\]

Can one compute the dressed coefficients without any basis?
Damped modes of stellar clusters

Gravitational response matrix

\[ \varepsilon_{pq}(\omega) = I - \sum_k \left( \frac{\mathbf{k} \cdot \partial F/\partial \mathbf{J}}{\mathbf{k} \cdot \Omega(\mathbf{J}) - \omega} \right) \Psi_k^{(p)*}(\mathbf{J}) \Psi_k^{(q)}(\mathbf{J}) \]

Some properties

- \( \sum_k \) Sum over resonances
- \( \int d\mathbf{J} \) Scan over orbital space
- \( \mathbf{k} \cdot \Omega(\mathbf{J}) - \omega \) Resonant amplification
- \( \Psi_k^{(p)*}(\mathbf{J}) \Psi_k^{(q)}(\mathbf{J}) \) Long-range interaction

Mode

\[ \det[\varepsilon(\omega)] = 0 \]

Type of modes

\[ \begin{cases} \text{Im}[\omega] > 0 & \text{Unstable} \\ \text{Im}[\omega] = 0 & \text{Neutral} \\ \text{Im}[\omega] < 0 & \text{Damped} \end{cases} \]
Damped modes of stellar clusters

**Landau’s prescription**

\[ \mathcal{L} = \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} = \begin{cases} \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} & \text{if } \text{Im}[\omega] > 0 \\ \mathcal{P} \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} + i\pi G(\omega) & \text{if } \text{Im}[\omega] = 0 \\ \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} + 2i\pi G(\omega) & \text{if } \text{Im}[\omega] < 0 \end{cases} \]

- **Unstable**, e.g., ROI
- **Neutral**, e.g., BL
- **Damped**, e.g., sloshing

**Some remarks**

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**Causality breaking**

+ Im[\omega] vs - Im[\omega]

**Aligned** resonant denominator

\[ \frac{1}{u - \omega} vs \frac{1}{f(u) - \omega} \]

**Analytic** integrand

\[ G(\omega) \text{ for } \omega \in \mathbb{C} \]

**Infinite** frequency support

\[ \int_{-\infty}^{+\infty} du \text{ vs } \int_{-1}^{1} du \]
Landau’s prescription

\[ \mathcal{L} = \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} = \begin{cases} \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} & \text{if } \Im[\omega] > 0 \\ \mathcal{P} \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} + i\pi G(\omega) & \text{if } \Im[\omega] = 0 \\ \int_{-\infty}^{+\infty} du \frac{G(u)}{u - \omega} + 2i\pi G(\omega) & \text{if } \Im[\omega] < 0 \end{cases} \]

\text{Unstable, e.g., ROI}

\text{Neutral, e.g., BL}

\text{Damped, e.g., sloshing}

\[ \varepsilon_k(\omega) = 1 + \frac{1}{k^2 \lambda_D^2} \int dv \frac{k \cdot \partial F/\partial v}{k \cdot v - \omega} \]

Plasmas

Resonant denominator is aligned \( u = k \cdot v \)

Integrand is typically \textbf{analytic} \( F(v) \propto e^{-v^2/2} \)

Frequency support is typically \textbf{infinite} \( \int_{-\infty}^{+\infty} dv \)

``Vanilla'' Maxwellian case \( Z[zeta_] := I \text{Sqrt}[\text{Pi}] \text{Exp}[-zeta^2](1 + I \text{Erfi}[zeta]) \)
Aligning the denominator

One can equivalently label orbits with their frequencies

\[ M(\omega) = \int dJ \frac{G(J)}{k \cdot \Omega(J) - \omega} \]

To resonant frequency

\[ u \propto k \cdot \Omega(J) \]

\[ M(\omega) = \int_{-1}^{1} du \frac{G(u)}{u - \omega} \]
Damped modes of stellar clusters

Analytic continuation

Initial expression

\[ M(\omega) = \int_{-1}^{1} du \frac{G(u)}{u - \omega} \]

On \([-1, 1]\) with unit weight: Legendre projection

\[ G(u) = \sum_{k} a_k P_k(u) \]

Polynomial, therefore analytic

Hence the separable writing

\[ M(\omega) = \sum_{k} a_k D_k(\omega) \]  
\[ D_k(\omega) = \int_{-1}^{1} du \frac{P_k(u)}{u - \omega} \]

\{\{F(J), \Omega(J), \psi^{(p)}(r)\}\}
Damped modes of stellar clusters

The resonant integral

**Finite frequency-domain**

\[
D_0(\omega) = \int_{-1}^{1} du \frac{1}{u - \omega}
\]

**Real part**

**Imag. part**
Only one difficult integral

We know the one integral

\[
D_0(\omega) = \int_{-1}^{1} du \frac{1}{u - \omega}
\]

``Pain de sucre``

\[
D_1(\omega) = \int_{-1}^{1} du \frac{u}{u - \omega} = \int_{-1}^{1} du \frac{u - (\omega - \omega)}{u - \omega} = 2 + \omega D_0(\omega)
\]

Legendre recurrence gives \( P_{k+2}(\omega) = \text{Linear}[P_k(\omega), P_{k+1}(\omega)] \)

Hence, we know all

\[
D_k(\omega) = \int_{-1}^{1} du \frac{P_k(u)}{u - \omega}
\]
Damped modes of stellar clusters

Ready to compute

Generic expression

$$M_{pq}(\omega) = \sum_k \sum_k a_k[p, q, k] D_k(\omega_k)$$

Projection to get \(\{a_k\}\)

$$\mathcal{O}[K \times N_{radial}^2 \times k_1^{max} \times \ell_{max} \times K_u \times K_v]$$

Evaluation to get \(M(\omega)\)

$$\mathcal{O}[N_{radial}^2 \times k_1^{max} \times \ell_{max} \times K_u]$$

- **Sampling of the orbit-average**: \(K\)
- **Number of basis elements**: \(N_{radial}\)
- **Number of radial resonances**: \(k_1^{max}\)
- **Considered harmonics**: \(\ell_{max}\)
- **Number of Legendre functions**: \(K_u\)
- **Number of sampling 2nd dim.**: \(K_v\)
Damped modes in globular clusters
Globular clusters

**Dense**, spherical stellar systems

Radii ~ a few parsecs

Contains N~10^5 stars

Very old ~10^{10} yr

Crossing time ~10^5 yr

Relaxation time ~10^{10} yr

Expected to be **linearly stable**

No maximum entropy, i.e. no Maxwellian
Damped modes of stellar clusters

Dispersion function

(Landau) damped modes in a (periodic) plasma

\[ \gamma_M = \text{Im}[\omega_M] \]

\[ \Omega_M = \text{Re}[\omega_M] \]
Dispersion function

Isotropic isochrone cluster

\[ \text{det}[\mathbf{e}_\ell(\omega)] \]

\( \ell = 1 \) damped mode

\[ \omega_M/\Omega_0 = 0.0143 - 0.00142 i \]

Slow mode

\[ \text{Re}[\omega_M]/\Omega_0 \ll 1 \]

Weakly damped

\[ \text{Im}[\omega_M]/\text{Re}[\omega_M] \ll 1 \]
Damped modes of stellar clusters

Dispersion function

Isotropic isochrone cluster

$$\text{det}[\varepsilon_{\ell}(\omega)]$$

$\Delta \omega \simeq \Omega_0/K_u$

$\ell = 1$ damped mode

$$\omega_M/\Omega_0 = 0.0143 - 0.00142 i$$

Slow mode

$$\text{Re}[\omega_M]/\Omega_0 \ll 1$$

Weakly damped

$$\text{Im}[\omega_M]/\text{Re}[\omega_M] \ll 1$$

How to reduce the spurious oscillations stemming from Legendre?
Damped modes of stellar clusters

**Amplification**

- Im $[\omega]$
- Re $[\omega]$

- Linearly stable system

- Damped mode

**Susceptibility**

$$\frac{1}{|\varepsilon(\Omega_M)|} \gg 1$$

**Thermalisation**

$$[\delta H(t)]_{\text{trans.}} \simeq e^{\gamma_M t}$$
Damped modes of stellar clusters

How strong is the amplification?

Amplification eigenvalue

\[ \lambda(\omega) = \text{EigMax} \left[ \frac{1}{\varepsilon(\omega)} \right] \]

Why is such a simple ansatz so effective?
Weakly damped modes and Landau’s trick

Root of the dispersion function

\[ \epsilon(\Omega + i \gamma) = 0 \]

The mode is weakly damped \( \gamma \ll \Omega \)

\[ \epsilon(\Omega) + i \gamma \frac{\partial}{\partial \Omega} \epsilon(\Omega) = 0 \]

Self-consistent constraints for the mode’s frequency

\[ \text{Re} \left[ \epsilon(\Omega) \right] = 0 \]

\[ \gamma = - \frac{\text{Im} \left[ \epsilon(\Omega) \right]}{\frac{\partial \epsilon(\Omega)}{\partial \Omega}} \]

Can one infer the modes without ever going in the lower half of the complex plane?
Radial shape of the mode

To estimate the mode’s shape from N-body simulations

Radial shell projection for $\rho_M(r)$

Multipole projection for $\psi_M(r)$

Heggie+ (2020)

Lau+ (2020)
Mode vs overall shift

Whole cluster shift by $\delta x$

$$\delta \rho(x, y, z) = \rho_0(x - \delta x, y, z) - \rho_0(x, y, z)$$

$$= \frac{d\rho_0}{dr} x \delta x$$

Shift’s perturbation

$$\delta \rho \propto \frac{d\rho_0}{dr} Y_{\ell m}(\hat{r})$$

Mode’s perturbation

$$\delta \rho \propto \rho_M(r) Y_{\ell m}(\hat{r})$$

Why is the mode so similar to the density gradient?
Damped modes of stellar clusters

Constraints on the radial shape

Conserving the **linear momentum**

\[
\delta \mathbf{P} = \int \mathrm{d} \mathbf{r} \mathbf{r} \delta \rho(\mathbf{r}, t) = 0 \quad \Rightarrow \quad \int \mathrm{d} r r^3 \rho_M(r) = 0
\]

Mode’s **node**

\[
\rho_M(R_{\text{node}}) = 0
\]

Mode’s **corotation radius**

\[
\Omega_2^{\text{circ}}(R_{\text{COR}}) = \text{Re}[\omega_M]
\]

Constraint on **rotation**

\[
R_{\text{node}} \leq R_{\text{COR}}
\]
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How to reduce spurious oscillations from the basis elements?

\[ \Delta r \simeq R_b / N_{\text{radial}} \]
Damped modes of stellar clusters

Dynamics of the perturbation

Typical perturbation

\[ \delta \rho(r, t) = A_M(t) \rho_M(r) Y_{\ell m}(\hat{r}) \]

Wobble of the density centre
Dynamics of the density centre

**Time evolution** of the density centre

**Rotation timescale**

\[ T = \frac{2\pi}{\text{Re}[\omega_M]} \approx 50 \text{ HU} \]

**Growth timescale**

\[ T = \frac{3}{\text{Im}[\omega_M]} \approx 250 \text{ HU} \]
Damped modes of stellar clusters

Dynamics of the density centre

Time evolution of the density centre

Why is the centre’s dynamics so messy?
Dynamics of the density centre

Stochastic dynamics

\[
\frac{d^2 x_c}{dt^2} - \gamma_M \frac{dx_c}{dt} + \Omega_M^2 x_c = \eta(t)
\]

How to understand these large-scale excursions?

Perpetual Poisson noise
Outermost bound particles
Escapers
Relaxation...
Dynamics of the density centre

Power spectrum

\[ \langle |\hat{x}_c(\omega)|^2 \rangle \propto \frac{1}{|\left(\omega - \omega_M\right)\left(\omega + \omega_M^*\right)|^2} \]

In **N-body simulations** *(with time-filtering)*

![Graph showing power spectrum over time](Image)
What about QL diffusion?
Damped modes of stellar clusters

Long-term dynamics

Decomposing the fluctuations

\[ \delta \Phi_{\text{tot}}(t) = \delta \Phi_{\text{BL}}(t) + \delta \Phi_{\text{M}}(t) \]

Total fluctuations Drives Drives
\( \Phi_{\text{BL}}(t) \) \( \Phi_{\text{M}}(t) \)

Two sources of evolution

\[ \frac{\partial F}{\partial t} = \left( \frac{\partial F}{\partial t} \right)_{\text{BL}} + \left( \frac{\partial F}{\partial t} \right)_{\text{QL}} \]

Requires a splitting of the perturbations

\[ \{ x_i(t), v_i(t) \} \mapsto \{ \delta \Phi_{\text{BL}}(t), \delta \Phi_{\text{M}}(t) \} \]

How to measure the waves’ amplitude in N-body runs?
Damped modes of stellar clusters

Mode’s energy

Typical perturbation

\[ \delta \rho(\mathbf{r}, t) = A_M(t) \rho_M(r) Y_{\ell m}(\hat{\mathbf{r}}) \]

Energy in the mode

\[ E_M(t) = |A_M(t)|^2 \]

Wave equation \( \text{Hamilton+}(2020) \)

\[ \frac{dE_M}{dt} = 2\gamma_M E_M + S_M \]

Energy budget

- \( 2\gamma_M E_M \) \text{ Landau damping } (Resonant interaction)
- \( S_M \) \text{ Spontaneous emission (Perpetual Poisson noise)}

Energy budget

- \( E_{\text{steady}} \)
- \( E_{\text{bare}} \)
- \( t \)

Thermalisation

\( \sim \frac{3}{\gamma_M} \)

Quasi-stationary statistics
Damped modes of stellar clusters

Mode’s energy

Prediction

N-body simulations

Mode’s energy

\[ E_M(t) \]

\[ E_{\text{steady}} \]

\[ E_{\text{bare}} \]

\[ \sim 3/\gamma_M \]

How to check the wave equation in N-body simulations?

No saturation

Diffusive-like dynamics

\[ E_M(t) \]

\[ t \]

0.0 0.2 0.4 0.6 0.8 1.0

0 200 400 600 800 1000

\[ t \text{[HU]} \]
Damped modes of stellar clusters

**Dominating mode**

**Wave equation**
\[
\frac{dE_M}{dt} = 2\gamma_M E_M + S_M
\]

**Steady energy**
\[
E_{\text{steady}} = -\frac{S_M}{2\gamma_M}
\]

**Spontaneous emission**
\[
S_M = \frac{1}{N} \sum_k \int dJ \, \delta_D[k \cdot \Omega(J) - \Omega_M] F(J)
\]

Which mode is dominating?
What happens after thermalisation?

(Weakly damped) Quasilinear Theory \( \text{Hamilton}+(2020) \)

\[
\frac{\partial F(J)}{\partial t} = - \frac{\partial}{\partial J} \cdot \left[ \sum_k \delta_D(k \cdot \Omega(J) - \Omega_M) \left\{ \frac{1}{N} F(J) - E_M(t) \cdot \frac{k \cdot \partial F}{\partial J} \right\} \right]
\]

\( \delta_D(k \cdot \Omega(J) - \Omega_M) \) 
Resonant particle-wave interaction

\( F(J) \) 
Friction 
Cost of spontaneous emission

\( E_M(t) \cdot \frac{k \cdot \partial F}{\partial J} \) 
Diffusion 
Resonant absorption

How to integrate over the QL resonance condition?
Damped modes of stellar clusters

\[ \frac{\partial F(J)}{\partial t} = \frac{1}{N} \frac{\partial}{\partial J} \cdot \left[ \sum_{k,k'} dJ' \frac{\delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J'))}{|\varepsilon(k \cdot \Omega(J))|^2} \times ... \right] \]

\[ \frac{\partial F(J)}{\partial t} = \frac{1}{N} \frac{\partial}{\partial J} \cdot \left[ \sum_k k \delta_D(k \cdot \Omega(J) - \Omega_M) \times ... \right] \]

Close to a (weakly) damped mode

\[ \text{Im}[\omega_M] \ll \text{Re}[\omega_M] \implies \frac{1}{|\varepsilon(\Omega_M)|} \gg 1 \]

Can QL ever dominate over BL?
Damped modes of stellar clusters

Long-term relaxation

Diffusion in orbital space $\frac{\partial F(J)}{\partial t}$

Why don’t we find any trace of QL in numerical simulations?
Damped modes of stellar clusters

Conclusion

Alternative approach to analytic continuation

\[ M(\omega) = \sum_k a_k D_k(\omega) \]
\[ M(\omega) = \frac{P(\omega)}{Q(\omega)} \]

Legendre series  
Rational functions

(Weakly) damped modes are unavoidable in globular clusters
Damped modes of stellar clusters

Future works

Impact of **anisotropy**

Impact of **potential**

Less puffy (e.g., Plummer)
Truncated (e.g., King)
Cuspy (e.g., Hernquist)
Degenerate (e.g., quasi-Keplerian)

**Thermalisation** timescale

\[ \frac{1}{\gamma_M} \quad \text{vs} \quad F_{vK}(\mathbf{J}, \omega)_{\text{Lau}(2020)} \]

Landau \quad \text{van Kampen}

**Others**

Other harmonics
*What is so special with \( \ell = 1 \)?*
Disc dynamics
*Swing amplification*
QL theory and escapers