The myth about nonlinear differential equations
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September 13, 2001

Abstract

Taking the example of Koretweg–de Vries equation, it is shown that soliton solutions need not always be the consequence of the trade-off between the nonlinear terms and the dispersive term in the nonlinear differential equation. Even the ordinary one dimensional linear partial differential equation can produce a soliton.

Solitary waves and solitons are often described [1-8] as a consequence of the trade-off between nonlinear and dispersive terms in the nonlinear differential equations. This has given rise to the myth that solitary waves and solitons can be obtained as solutions of nonlinear differential equations only and not as solutions of linear differential equations. An associated misunderstanding is that only nonlinear differential equations are capable of describing nonlinear physical phenomena and nonlinear differential equations are more powerful than linear differential equations in describing physical phenomena. The observation that, in nature, linear phenomena are often only approximations to nonlinear phenomena probably gave birth to this belief. As a consequence, physicists are constructing more and more nonlinear differential equations to describe nonlinear physical phenomena. It is, of course true that nonlinear phenomena are more general, than linear phenomena. But linear differential equations are, in general neither approximations nor particular cases of nonlinear differential equations. Also, it is to be emphasised that solutions of both linear and nonlinear differential equations are functions which depend nonlinearly on the independent variable (the only exception being the straight line solution). We can construct linear as well as nonlinear differential equations from the same function. For example, from the function

\[ y(x, t) = x^2 + t^2 \quad (1) \]

we can construct the linear second order partial differential equation (PDE)
\[ \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \]  

and the first order, second degree nonlinear differential equation

\[ \left( \frac{\partial y}{\partial x} \right)^2 + \left( \frac{\partial y}{\partial t} \right)^2 = 4y \]  

Conversely,

\[ y(x,t) = x^2 + t^2 \]

is a solution of the linear differential equation (2) and the nonlinear differential equation (3). This simple example illustrates that the distinction between linear and nonlinear differential equations appears only in the differential equation level and not in the solution level. Looking at equations (1) - (3), it is obvious that nonlinearity is hidden in the second order linear differential equation (3) and the higher the order of the differential equation, the higher the nonlinearity it contains.

Any physical phenomena is described within a model. Within this model, the physical phenomenon, in general, can be described with the help of a linear differential equation or a nonlinear differential equation or a function. The information content in the linear differential equation with its initial and boundary conditions or in the equivalent nonlinear differential equation with its initial and boundary conditions or in the solution function will be the same. Therefore, the claim that a particular physical phenomenon can be described only by a nonlinear differential equation, and not by any linear differential equation is not tenable, provided a linear differential equation with the same solution as that of the nonlinear differential equation exists.

The same philosophy, as above, is at work, in the efforts of pure mathematicians to find linear operators equivalent to nonlinear operators. Incidentally, linearization is the oldest and most popular method of solving nonlinear differential equations. Linearization is essentially the setting up of a linear differential equation with its solutions the same as some of the solutions of the nonlinear differential equation.

For the purpose of this letter, a linear differential equation and a nonlinear differential equation having the same solutions are called equivalent differential equations. Throughout this letter, to save space and time, we use the Koretwe–de Vries (KdV) equation and its solutions in the form given by Drazin and Johnson [1].

The KdV equation

\[ u_t - 6uu_x + u_{xxx} = 0 \]  

has the soliton solution

\[ u(x,t) = -\frac{1}{2}csech^2 \left\{ \frac{1}{2}c^2(x - ct - x_0) \right\} \]
It can be easily verified that (5) is also a solution of the first order one dimensional wave equation

$$u_x = -\frac{1}{c} u_t \quad (6)$$

Therefore, the nonlinear differential equation (4) or the linear differential equation (6) or the solution (5) equally well represent the soliton. Hence, the argument that the solitons are produced only as a result of the trade-off between the nonlinear terms and the dispersive term of the nonlinear differential equation is not valid at least, for the soliton represented by the function (5). From the point of view of the operator formalism, we can say that the nonlinear KdV operator,

$$\frac{\partial}{\partial t} - 6u \frac{\partial^3}{\partial x^3} + \frac{\partial}{\partial x^3}$$

and the linear operator

$$\frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t}$$

are identical so far as the function (5) is concerned. Equation (5) is also a solution of the $n$th order wave equation

$$\frac{\partial^n u}{\partial x^n} = (-1)^n \frac{1}{c^n} \frac{\partial^n u}{\partial t^n} \quad (7)$$

for any value of $n$. Therefore, there are infinite number of linear operators

$$\frac{\partial}{\partial x} = -1 \frac{\partial}{c \partial t}, \quad \frac{\partial^2}{\partial x^2} = -1 \frac{\partial^2}{c^2 \partial t^2}, \quad \cdots$$

equivalent to the nonlinear KdV operator for the soliton solution (5).

The singular soliton solution [1],

$$u(x, t) = 2k^2 \text{cosech}^2 \left\{ k(x - 4k^2t) \right\} \quad (8)$$

of the KdV equation (4) is also a solution of the $n$th order linear differential equation (7) for any value of $n$.

Other nonlinear differential equations with soliton solutions have also equivalent differential equations. Since one example is enough to establish our point we do not cite further examples here.

As we have already mentioned, there may be soliton solutions for which a linear PDE may not exist. As an interesting example for the rational solution [1]

$$u(x, t) = \frac{6x(x^3 - 24t)}{(x^3 + 12t)^2} \quad (9)$$

of the KdV equation

$$u_x = \frac{1}{\kappa} u_t \quad \text{where } \kappa \text{ is a constant} \quad (10)$$
is not an equivalent linear equation.

It is interesting to point out in this connection that Davidov[9] has made the following observation about solitons from linear differential equations:

“Solitonic-type solutions give a second stable branch of solutions of the Schrodinger equation. In a certain sense, these solutions are isolated from the boundary conditions due to their localisation on a rather small region of the chain.

A soliton is described by a wave whose profile \( \phi(\xi) \) is unchanged under propagation. Such waves refer to the stationary ones, for which the following relation holds.

\[
\frac{\partial \phi}{\partial t} = -V \frac{\partial \phi}{\partial x}
\]

where \( V \) is the velocity of propagation.”

The significance of this finding is that solitons and many other complex entities associated with non-linear physical phenomena can be studied, predicted, and described in terms of linear differential equations. The additional advantage of using linear differential equations is that several techniques and tools have been developed in the past 300 years which can be effectively used to solve complex non-linear phenomena.

Acknowledgements

I have been benefitted by the criticism offered by G. Mohanachandran. I wish to thank E. Krishnan for the preparation of the manuscript and helpful discussions. The last touches to this letter were given during a short visit to TIFR, Mumbai.

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