ON CERTAIN UNIVERSAL GRAPH $C^*$-ALGEBRAS

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Abstract. For special universal $C^*$-algebras associated to higher rank graphs or semigraphs we state the universal representations, prove a Cuntz-Krieger uniqueness theorem and state the $K$-theory.

1. Introduction

In this paper we continue our study of higher rank semigraph $C^*$-algebras from [3]. The class of higher rank semigraph $C^*$-algebras present a flexible generalisation of Tomforde’s ultragraph algebras [8] or $C^*$-algebras of labelled graphs [1] to higher rank structures.

In this work we compute the $K$-theory of weakly free higher rank semigraph algebras in Theorem 5.5. Roughly speaking, a weakly free semigraph algebra is something like the Toeplitz graph algebra [7] in the theory of graph algebras. Actually, Theorem 5.5 extends the $K$-theory computation of Toeplitz graph algebras in [4]. Also the proof is not much work: it is merely an application of a theorem in [4].

The second aim of this paper covering the main work is the introduction of special higher rank semigraph algebras which we call quell semigraph algebras and which are induced by higher rank graphs or semigraphs. A quell semigraph algebra is defined to be universally generated by a $k$-semigraph as a generator set, thereby satisfying only a minimum quantity of relations such that it still contains naturally the inducing $k$-semigraph. We find a concrete faithful representation of a semigraph algebra on a Hilbert space as a left regular representation of a semimultiplicative set in Proposition 4.8. A quell semigraph algebra is free, weakly free, cancelling, and satisfies a Cuntz–Krieger theorem, see Theorem 4.13.

Every semigraph $C^*$-algebra is generated by partial isometries with commuting source and range projections (see Lemma 2.13). There is some recent interest in universal $C^*$-algebras generated by partial isometries and their $K$-theory, see for instance Cho and Jorgensen [5] and Brenken and Niu [2]. Although there is no immediate motivation for a quell semigraph algebra, we suspect it is somehow the freest Cuntz–Krieger like algebra which naturally embedds a given $k$-graph, see also Lemma 4.14. Moreover we think every universal algebra generated by partial isometries for which one knows

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the $K$-theory is a benefit; even more so, as there is (despite [9]) an ongoing program initiated by G. Elliott to classify certain subclasses of nuclear $C^*$-algebras (as the subclass of purely infinite nuclear $C^*$-algebras) where the $K$-theory of a $C^*$-algebra plays a major role.

We give a brief overview of this paper. In Section 2, we recall the theory of semigraph algebras. In Section 3, we consider two conditions for a semigraph algebra which we call weakly free (Definition 3.3) and free (Definition 3.6). Freeness implies weakly freeness (Proposition 3.9), and free semigraph algebras are cancelling (Proposition 3.10), and so satisfy a Cuntz–Krieger uniqueness theorem. In Section 4, we introduce the quell semigraph algebras (Definition 4.3) and show that they are free and cancelling (Theorem 4.13). In Section 5, we explicitly compute the $K$-theory of weakly free semigraph $C^*$-algebras in Theorem 5.5. In Section 6, we use Theorem 5.5 for the computation of the $K$-theory of a quell semigraph $C^*$-algebra in Theorem 6.2.

2. Semigraph algebras

In this section we recall briefly the definition and some basic facts about semigraph algebras [3] for further reference.

Definition 2.1. A semimultiplicative set $T$ is a set equipped with a subset $T^{(2)} \subseteq T$ and a multiplication $T^{(2)} \to T : (s,t) \mapsto st$, which is associative, that is, for all $s,t,u \in T$, $(st)u$ is defined if and only if $s(tu)$ is defined, and both expressions are equal if they are defined.

Definition 2.2. Let $k$ be an index set (which may be regarded as a natural number if $k$ is finite). A $k$-semigraph $T$ is a semimultiplicative set $T$ equipped with a degree map $d : T \to \mathbb{N}_0^k$ satisfying the unique factorisation property which consists of the following two conditions:

1. For all $x,y \in T$ for which the product $xy$ is defined one has $d(xy) = d(x) + d(y)$.

2. For all $x \in T$ and all $n_1,n_2 \in \mathbb{N}_0^k$ with $d(x) = n_1 + n_2$ there exist unique $x_1,x_2 \in T$ with $x = x_1x_2$ satisfying $d(x_1) = n_1$ and $d(x_2) = n_2$.

We call $k$-semigraph also a higher rank semigraph or just a semigraph. We shall also write $|t|$ rather than $d(t)$ for elements $t$ in a $k$-semigraph. Let $T$ be a $k$-semigraph. We denote the set of all elements of $T$ with degree $n$ by $T^{(n)}$ ($n \in \mathbb{N}_0^k$). If $x \in T$ and $0 \leq n_1 \leq n_2 \leq d(x)$ then there are unique $x_1,x_2,x_3 \in T$ such that $x = x_1x_2x_3$, $d(x_1) = n_1,d(x_2) = n_2 - n_1$ and $d(x_3) = d(x) - n_2$. $x_2$ is denoted by $x(n_1,n_2)$.

Definition 2.3. Let $T$ be a $k$-semigraph. For $s,t \in T$ we write $s \leq t$ if $as = t$ for some $a \in T$.

Lemma 2.4. The last relation is an order relation on a semigraph.

Proof. Transitivity is clear. So assume $s \leq t$ and $t \leq s$. Then there are $\alpha, \beta \in T$ such that $t = \alpha s$ and $s = \beta t$. Then $s = \beta \alpha s$, and so $d(s) = d(s) + \alpha s$.
\[ d(\alpha) + d(\beta), \] which implies \( d(\alpha) = d(\beta) = 0. \] Hence one has \( \beta \alpha s = \beta \alpha \beta \alpha s. \) By the unique factorisation property \( s = \alpha s. \) So \( s = t. \)

**Definition 2.5.** A \( k \)-semigraph \( T \) is called finitely aligned if for all \( x, y \in T \) the minimal common extension of \( x \) and \( y \), which is the set

\[ T^{(\text{min})}(x, y) = \{ (\alpha, \beta) \in T \times T \mid x \alpha \text{ and } y \beta \text{ are defined, } x \alpha = y \beta, d(x \alpha) = d(x) \lor d(y) \}, \]

is finite.

It might help to recall the meaning of the relation \( (\alpha, \beta) \in T^{(\text{min})}(x, y) \) by visualising it by the tautologies \( \alpha \leq x \alpha \) and \( \beta \leq y \beta. \)

**Definition 2.6.** \( T \) is called a non-unital \( k \)-semigraph if there exists a \( k \)-semigraph \( T_1 \) which has a unit \( 1 \in T_1 \) such that \( T = T_1 \setminus \{ 1 \}. \)

We shall use the following notions when we speak about algebras. A \( * \)-algebra means an algebra over \( \mathbb{C} \) endowed with an involution. An element \( s \) in a \( * \)-algebra is called a partial isometry if \( ss^*s = s \), and a projection \( p \) is an element with \( p = p^2 = p^* \). We define \( P_a := a^*a \) and \( Q_a := aa^* \) for elements \( a \) of a \( * \)-algebra. If \( I \) is a subset of a \( * \)-algebra then \( \langle I \rangle \) denotes the self-adjoint two-sided ideal generated by \( I \) in this \( * \)-algebra.

Assume that we are given a set \( \mathcal{P} \) and a nonunital \( k \)-semigraph \( \mathcal{T} \) with \( \mathcal{P} \cap \mathcal{T} = \emptyset. \) Recall that in a non-unital \( k \)-semigraph one has \( d(t) > 0 \) for all its elements \( t. \) We denote by \( \mathcal{T}_1 = \mathcal{T} \cup \{ 1 \} \) the unital \( k \)-semigraph of Definition 2.6. Define \( \mathbb{F} \) to be the free nonunital \( * \)-algebra generated by \( \mathcal{P} \cup \mathcal{T}. \) We call a \( * \)-monomial in the letters of \( \mathcal{P} \cup \mathcal{T} \) a word.

**Definition 2.7.** The degree \( d(x) \) of a word \( x = x_1 \ldots x_n \) in \( \mathbb{F} \) \( (n \geq 1, \) \( x_i \in \mathcal{P} \cup \mathcal{T} \cup \mathcal{P}^\ast \cup \mathcal{T}^\ast \) \) is defined to be \( d(x) = d(x_1) + \ldots + d(x_n), \) where \( d(x_i) \) is to be the semigraph-degree \( d(x_i) \) when \( x_i \in \mathcal{T}, d(x_i) = 0 \) if \( x_i \in \mathcal{P}, \) and \( d(x_i^\ast) = -d(x_i) \) for any \( x_i \in \mathcal{T} \cup \mathcal{P}. \)

**Definition 2.8.** The fiber space of \( \mathbb{F} \) is the union of all fibers \( \text{span}(W_n), \) where \( W_n \) denotes the set of words with degree \( n \in \mathbb{Z}^k. \)

**Definition 2.9.** Let \( \sigma : \mathbb{T}^k \rightarrow \text{Aut}(\mathbb{F}) \) be the gauge action defined by \( \sigma_\lambda(p) = p \) and \( \sigma_\lambda(t) = \lambda^d(t)t \) for all \( p \in \mathcal{P}, t \in \mathcal{T} \) and \( \lambda \in \mathbb{T}^k. \)

**Lemma 2.10** (3, Lemma 4.7). \( X \) is the quotient of \( \mathbb{F} \) by a subset of the fiber space if and only if there is a gauge action on \( X \) (defined like in Definition 2.9).

**Definition 2.11** (Semigraph algebra). A \( k \)-semigraph algebra \( X \) is a \( * \)-algebra which is generated by disjoint subsets \( \mathcal{P} \) and \( \mathcal{T} \) of \( X, \) where

(i) \( \mathcal{P} \) is a set of commuting projections closed under taking multiplications,

(ii) \( \mathcal{T} \) is a set of nonzero partial isometries closed under nonzero products,
(iii) $\mathcal{T}$ is a non-unital finitely aligned $k$-semigraph,
(iv) for all $x \in \mathcal{T}$ and all $p \in \mathcal{P}$ there is a $q \in \mathcal{P}$ such that $px = xq$,
(v) for all $x, y \in \mathcal{T}$ there exist $q_{x,y,\alpha,\beta} \in \mathcal{P}$ such that

\begin{equation}
    x^{*}y = \sum_{(\alpha,\beta) \in \mathcal{T}_{1}^{(\text{min})}(x,y)} \alpha q_{x,y,\alpha,\beta}^{*}, \quad \text{and}
\end{equation}

(vi) $X$ is canonically isomorphic to the quotient of $\mathcal{F}$ by a subset of the fiber space (Definition 2.8).

It is understood in identity (1) that the unit 1 in $\mathcal{T}_{1} = \mathcal{T} \sqcup \{1\}$ is also a unit for all elements of $X$. The universal $C^{*}$-algebra $C^{*}(X)$ generated by $X$ is called the $k$-semigraph $C^{*}$-algebra associated to $X$. We often write also $\sum_{s,t,1} x_{\alpha} = y_{\beta}$ rather than $\sum_{(\alpha,\beta) \in \mathcal{T}_{1}^{(\text{min})}(x,y)}$ in (1).

**Definition 2.12.** We call an element of \{spt$^{*} \in X|s,t \in \mathcal{T}_{1}, p \in \mathcal{P}\}$ a **standard word** (of the semigraph algebra $X$). We call an element of \{sp $\in X|s \in \mathcal{T}_{1}, p \in \mathcal{P}\}$ a **half-standard word**.

**Lemma 2.13** ([3], Lemma 5.8). (a) The word set of $X$ is an inverse semigroup of partial isometries.

(b) For each word $w$ there exist half-standard words $a_{i}, b_{i}$ and $c_{i}$ such that $ww^{*} = \sum_{i=1}^{n} a_{i}a_{i}^{*}$ and $w = \sum_{j=1}^{m} b_{j}c_{j}^{*}$ with $d(w) = d(b_{s}c_{s}^{*})$ for all $1 \leq s \leq m$.

**Corollary 2.14.** (a) A semigraph algebra is linearly spanned by its standard words.

(b) The range projection of a word is a sum of range projections of half-standard words.

(c) The source projection of a half-standard word is in $\mathcal{P}$.

**Corollary 2.15** ([3], Corollary 6.4). The core is the union of a net of finite dimensional $C^{*}$-algebras, each one allowing a matrix representation where each projection on the diagonal is a finite sum of mutually orthogonal standard projections. A $C^{*}$-representation of $X$ is injective on the core if and only if it is non-vanishing on nonzero standard projections.

**Definition 2.16.** A semigraph algebra $X$ is called cancelling if for every standard word $w$ with nonzero degree and every nonzero standard projection $p$ there is a nonzero standard projection $q$ such that $q \leq p$ and $qwq = 0$.

**Theorem 2.17** (Cuntz–Krieger uniqueness theorem, [3], Theorem 7.3). The universal representation $X \rightarrow C^{*}(X)$ is injective on the core, and so non-vanishing on the nonzero standard projections, and up to isomorphism this is the only existing representation of $X$ in a $C^{*}$-algebra which is non-vanishing on nonzero standard projections and has dense image.
3. Free semigraph algebras

In this section we consider a condition on a semigraph algebra called freeness, and a weaker one called weakly freeness. The rough idea of freeness is that range projections of generators should not sum up to a unit. This is similar as $P_{s_1} + P_{s_2} < 1$ in the Toeplitz version of the Cuntz algebra $O_2$ with generating isometries $s_1$ and $s_2$. The reader may be warned that this section tends to be very technical in a combinatorial and tedious sense. In this section $X$ denotes a semigraph algebra.

**Definition 3.1.** Write $X_{\emptyset}$ for the $\ast$-subalgebra of $X$ generated by the elements $P$ and the non-unital finitely aligned semigraph $T_{\emptyset}$, where $T_{\emptyset} = \{ t \in T | d(t);_i = 0 \}$ (the $i$th coordinate vanishes). We set $T_{\emptyset} = T_{\emptyset} \cup \{ 1 \}$.

**Lemma 3.2.** $X_{\emptyset}$ is a semigraph algebra.

**Proof.** All points (i)-(v) of Definition 2.11 are almost obvious. Point (vi) can be seen by Lemma 2.10 and the fact that the restriction of the gauge action on $X$ to $X_{\emptyset}$ is a gauge action on $X_{\emptyset}$. □

We shall regard $X_{\emptyset}$ as a sub-semigraph algebra of $X$. When we work in $X$ and say “$w$ is a word in $X_{\emptyset}$” (and so on) then “word” refers to the semigraph algebra $X_{\emptyset}$; so $w$ does not mean a word in $X$ which accidentally happens to be in $X_{\emptyset}$.

**Definition 3.3.** A semigraph algebra $X$ is called weakly free if for all coordinates $i \in k$, all nonzero standard projections $p$ of $X_{\emptyset}$ and all finite subsets $B \subseteq T^{(e_i)}$ such that $p \leq \sum_{b \in B} P_b$ does not hold.

The next two lemmas (Lemma 3.4 and Lemma 3.5) will only be used in the proof of Lemma 5.3. The reader only interested in Theorem 5.5 could go directly to Section 5 after these two lemmas.

**Lemma 3.4.** If $i \in k, a \in T^{(e_i)}$ and $w$ is a word in $X_{\emptyset}$ then there exist $b_1, \ldots, b_n \in T^{(e_i)}$ such that $P_a w = P_a w (P_{b_1} + \ldots + P_{b_n})$.

**Proof.** We may write $w = spt^*$ for $s, t \in T_{\emptyset}$ and $p \in P$. Then (by Definition 2.11 (v))

$$ (aa^*) (spt^*) = a \sum_{aa=s\beta} \alpha q_{a, s, \alpha, \beta}^* pt^* $$

$$ = \sum_{aa=s\beta, t\beta \neq 0} \alpha q_{a, s, \alpha, \beta}^* pt^* $$

$$ = \sum_{aa=s\beta, t\beta \neq 0} \alpha q_{a, s, \alpha, \beta}^* P_{t\beta} \sum_{aa=s\beta, t\beta \neq 0} P_{t\beta} $$

for $P_{t\beta} \in P$ with $p\beta = \beta p\beta$ (by Definition 2.11 (iv)), and where $d(\beta) = d(a) = e_i$, and $t\beta = \beta t' \beta$ by the unique factorisation property with $d(t') = e_i$ (if $t\beta \neq 0$, which, recall, is equivalent to $t\beta$ being a defined product in $T$). □
Lemma 3.5. If $X$ is weakly free then for every coordinate $i \in k$, every element $x$ of the fiber space of $X^\lambda_i$, and every finite subset $B$ of $T^{(e_i)}$ we have that $(\sum_{b \in B} P_b)x = x$ implies $x = 0$.

Proof. Suppose that $x$ is a nonzero element of the 0-fiber $X^\lambda_0$ (i.e. the core) of $X^\lambda$, $p = \sum_{b \in B} P_b$, and $px = x$. The element $x$ is in the core of $X^\lambda$ and so in a finite dimensional $C^*$-algebra $A$ as described in Corollary 2.15. Say, $x = \sum_{ij} \lambda_{ij}e_{ij}$ for matrix units $e_{ij}$, where each diagonal unit $e_{ii}$ is a sum of mutually orthogonal standard projections of $X^\lambda$ (Corollary 2.15). Since $x \neq 0$ we may suppose that $0 \neq \lambda_{i_0j_0} = 1$ for some fixed pair $(i_0, j_0)$. By Corollary 2.15, there is a nonzero standard projection $q$ in $X^\lambda$ (so $q \in P(T_1^\lambda P)$) such that $q \leq e_{i_0i_0}$. Note that $p$ commutes with $e_{i_0i_0}$ since standard projections commute. Then

$$q \leq e_{i_0i_0} = e_{i_0i_0}pxe_{j_0i_0} = e_{i_0i_0}pxe_{j_0i_0} = pe_{i_0i_0}xe_{j_0i_0} = pe_{i_0i_0} \leq p.$$ 

However, $q \leq p$ contradicts the weakly free condition in $X$. Thus $x = 0$.

Now assume that $x$ is in another fiber $X^\lambda_n$ ($n \in \mathbb{Z}^k$) and $px = x$ (where $p = \sum_{b \in B} P_b$ again). We may write $x$ as $x = \sum_{k=1}^l a_kx_kb_k^*$, where $x_k$ is an element of the core $X^\lambda_0$, $a_k \in T_1$ with $d(a_k) = \min(n, 0), b_k \in T_1$ with $d(b_k^*) = \min(n, 0)$, and the pairs $(a_k, b_k)$ are mutually distinct for different $k$’s, for all $1 \leq k \leq l$. Fix $1 \leq k_0 \leq l$. Then, since $px = x$, and by several applications of Lemma 3.3, there exists a subset $B' \subseteq T^{(e_i)}$ such that for $p' = \sum_{b \in B'} P_b$ one has

$$r_{k_0} := a_{k_0}^*x_kb_{k_0} = Qa_{k_0}x_kb_{k_0} = a_{k_0}^*pxkb_{k_0} = p'a_{k_0}^*pxkb_{k_0} = p'r_{k_0}.$$ 

Thus $r_{k_0} = p'r_{k_0}$ and $r_{k_0} \in X^\lambda_0$, and so by what we have already proved, $r_{k_0} = 0$. Since $k_0$ was arbitrary,

$$0 = \sum_{k_0=1}^l a_{k_0}r_{k_0}b_{k_0}^* = \sum_{k_0=1}^l a_{k_0}x_kb_{k_0}b_{k_0}^* = x.$$ 

$\square$

Definition 3.6. A semigraph algebra $X$ is called free if

(i) $P_a$ is not in $P$ for every half-standard word $a \notin P$, and

(ii) if $p \in P$ and $a_1, \ldots, a_n$ are half-standard words with $P_{a_i} < p$ for all $1 \leq i \leq n$ then $\bigvee_{i=1}^n P_{a_i} < p$.

Lemma 3.7. Let $X$ be free. If $a, b_1, \ldots, b_n$ are half-standard words satisfying $P_a(1 - P_{b_i}) \neq 0$ for all $1 \leq i \leq n$ then $P_a \prod_{i=1}^n (1 - P_{b_i}) \neq 0$.

Proof. We may write $a = xs$ and $b_i = y_ip_i$ for certain $x, y_i \in T_1$ and $s, p_i \in P$. Assume that $X$ is Toeplitz and $P_a(1 - P_{b_i}) \neq 0$ for all $1 \leq i \leq n$. We have
(by Definition 2.11 (v))

\[ P_a(1 - P_b) = x_s x^*(1 - y_i p_i y_i^*) \]
\[ = x_s x^* - \sum_{x_\alpha = y_i} x_s o q_{x,y_i,\alpha,\beta} p_i y_i^* \]
\[ = x_s x^* \left( 1 - \sum_{x_\alpha = y_i} x_s o q_{x,y_i,\alpha,\beta} p_i \alpha^* x^* \right) \]
\[ = x_s x^* \prod_{(\alpha, \beta) \in T_1^{(\min)}(x, y_i)} \left( 1 - x_s o q_{x,y_i,\alpha,\beta} p_i \alpha^* x^* \right), \]

where \( p_i, \beta \) is chosen in \( P \) such that \( p_i \beta = \beta p_i, \beta \) (by Definition 2.11 (iv)), and

\[ \text{where in the last identity we successively used the formula } (1 - p)(1 - q) = 1 - p - q \text{ for orthogonal projections } p \text{ and } q. \]

Since the above is nonzero by assumption, we have

\[ x_s o q_{x,y_i,\alpha,\beta} p_i,\beta \alpha^* x^* < x_s x^* \]

for all \( 1 \leq i \leq n \) and all \( (\alpha, \beta) \in T_1^{(\min)}(x, y_i) \). Multiplying here from the left and right with \( x^* \) and \( x \), respectively, we get \( x^* x_s o q_{x,y_i,\alpha,\beta} p_i,\beta \alpha^* x^* < x^* x s \).

By freeness (Definition 3.6) we conclude

\[ \bigvee_{i=1}^n \bigvee_{\alpha=1}^n x^* x_s o q_{x,y_i,\alpha,\beta} p_i,\beta \alpha^* x^* < x^* x s. \]

Thus \( \bigvee_{i,\alpha,\beta} x_s o q_{x,y_i,\alpha,\beta} p_i,\beta \alpha^* x^* < x_s x^* \), whence

\[ P_a \prod_{i=1}^n (1 - P_b) = x_s x^* \prod_{i=1}^n \prod_{x_\alpha = y_i} \left( 1 - x_s o q_{x,y_i,\alpha,\beta} p_i \alpha^* x^* \right) \]
\[ = x_s x^* \left( 1 - \bigvee_{i,\alpha,\beta} x_s o q_{x,y_i,\alpha,\beta} p_i,\beta \alpha^* x^* \right) \neq 0, \]

where we used de Morgan’s law \( \bigwedge(1 - \gamma) = (1 - \bigvee \gamma). \) \( \square \)

Similarly as for elements in a semigraph we write \( x_1 \leq x_2 \) for half-standard words \( x_1 \) and \( x_2 \) if they allow a representation \( x = t_1 p_1 \) and \( x_2 = t_2 p_2 \) \((t_1, t_2 \in T \text{ and } p_1, p_2 \in P)\) with \( t_1 \leq t_2 \). Only in the next corollary \( H \) denotes the set of half-standard words.

**Corollary 3.8.** If \( X \) is free then these two sets are the set of nonzero standard projections:

\[ \left\{ P_a \prod_{i=1}^n (1 - P_b) \in X \mid n \in \mathbb{N}_0, a, b_i \in H, P_a (1 - P_b) \neq 0 \text{ for all } i \right\}, \]
\[ = \left\{ P_a \prod_{i=1}^n (1 - P_b) \in X \mid n \in \mathbb{N}_0, a, b_i \in H, P_{b_i} < P_a \text{ and } a \leq b_i \forall i \right\}. \]
Proof. That the first set is the set of nonzero standard projections follows from Lemma 3.7. For the second set just recall the identity (2) how we can write down a standard projection. The assertion can be directly read off from this expansion. □

**Proposition 3.9.** If $X$ is free then $X$ is weakly free.

**Proof.** Fix $i \in k$. To check weakly freeness, consider a finite subset $B$ of $T(e_i)$ and a nonzero standard projection $p = xq x^* \prod_{i=k}^n (1 - y_k y_k^*)$ in $X^\setminus i$, where $x \in T_i, q \in P$ and the $y_k$’s are half-standard words in $X^\setminus i$. If $b \in B$ then

$$qx^* x \geq qx^* bb^* x = \sum_{(e,f) \in T_1^{(\text{min})}(x,b)} q(eq_{x,h,e,f} f^*)(f q_{x,h,e,f} e^*)$$

(4)

$$= \sum_{(e,f) \in T_1^{(\text{min})}(x,b)} P_{eq_{x,h,e,f} f^*} f < qx^* x,$$

(5)

where we used $qe = eq_e$ from Definition 2.11(iv). The last inequality is here by freeness. Indeed, $P_{eq_{x,h,e,f} f^*} f \in P$ by Definition 3.6(i). On the other hand, $P_{eq_{x,h,e,f} f^*} f \leq qx^* x \in P$ by (4), so $P_{eq_{x,h,e,f} f^*} f < qx^* x$. Hence, inequality (5) is true by Definition 3.6(ii).

We conclude from (4) and (5) that $q x^* P_b x \neq x^* xq$. Hence, applying here the operation $x(\cdot)x^*$ we get $xq x^* P_b \neq xq x^*$. Thus, $xq x^*(1 - P_b) \neq 0$ for all $b \in B$. Hence

$$p \left(1 - \sum_{b \in B} P_b \right) = p \prod_{b \in B} (1 - P_b) = xq x^* \prod_{i=k}^n (1 - y_k y_k^*) \prod_{b \in B} (1 - P_b) \neq 0$$

by Lemma 3.7. Consequently $p \leq \sum_{b \in B} P_b$ does not hold. □

**Proposition 3.10.** If $X$ is free then $X$ is cancelling.

**Proof.** We are going to check the cancelling condition, Definition 2.16. Let $w = \alpha q \beta^*$ be a standard word with $d(w) \neq 0$ ($\alpha, \beta \in T_1, q \in P$). Let $P$ be a nonzero standard projection. We must find a nonzero standard projection $Q$ with $Q \leq P$ and $Q x Q = 0$.

We may write $P = px x^* s x^*$, where $p = \prod_{k=1}^n (1 - y_k y_k^*)$, $x \in T_1, s \in P$ and the $y_k$’s are half-standard words. If already $P x P = 0$ then the cancelling condition is verified. So assume that

$$0 \neq P x P = px(x^* \alpha) q \beta^* p x x^*.$$

Then there is a pair $(e, f) \in T_1^{(\text{min})}(x, \alpha)$ such that

$$v := px e(q_{x,\alpha,e,f} f^*) q \beta^* p x x^* \neq 0$$

by (1). Consequently $px e \neq 0$, and so

$$0 \neq P' := px e x^* = px e e^* x^* = px' s' x^*$$

(6)
is a standard projection, where \( x' := xe \) and \( s' \in \mathcal{P} \) satisfies \( se = es' \). (We intensively use the fact that the word set forms an inverse semigroup, Lemma 2.13) Note that \( P' \leq P \). If already \( P'wP' = 0 \), then the cancelling condition is verified.

So assume \( P'wP' \neq 0 \). Note that \( |x'| = |xe| \geq |\alpha| \) since \( (e,f) \in T_{1}^{(min)}(x,\alpha) \). Note that by \( P' = px's'x' \) has the same shape as \( P \), but with \( |x'| \geq |\alpha| \). As we are going to search \( Q \leq P' \leq P \), we may assume without loss of generality that we are given \( P \) with \( PwP \neq 0 \) and \( |x| \geq |\alpha| \).

Similar computations as above on the \( \beta \)-side show that we may also assume, by choosing a smaller projection than \( P \), that also \( |x| \geq |\beta| \).

So assume without loss of generality that \( PwP \neq 0 \) and \( |x| \geq |\alpha|, |\beta| \). We have

\[
0 \neq PwP = pxsx^*(\alpha q\beta^*)xsx*p.
\]

Hence, there are decompositions \( x = \alpha x_1 = \beta x_2 \ (x_1, x_2 \in T_1) \), and a \( q' \in \mathcal{P} \) chosen to satisfy \( Q_\alpha q Q_\beta x_2 = x_2 q' \), such that

\[
PwP = pxs(\alpha x_1)^*\alpha q\beta^* \beta x_2 sx^*p = pxs(x_1^* x_2)q' sx^*p
\]

where \( K := \sum_{(e,f)\in T_{1}^{(min)}(x_1, x_2)} eq_{x_1, x_2, e, f} f^* q' s x^* p \).

Successively using the formula \( x s (1 - ee^*) (1 - e'e^*) s x^* = x s (1 - ee^*) s x^* s (1 - e'e^*) s x^* \), we get

\[
(7) \quad Q := pxsQ sx^* = p \prod_{(e,f)\in T_{1}^{(min)}(x_1, x_2)} (1 - xsee^* x^*) sx^*.
\]

\[
(8) \quad QwQ = Q(PwP)Q = QpxsK sx^* pQ = QpxsK sx^* p pxsQ' sx^* = 0.
\]

So \( Q \) is the sought standard projection that cancels \( u \). It remains to show \( Q \neq 0 \).

Since \( d(\alpha q \beta^*) \neq 0 \), we may assume without loss of generality that \( |x_1| < |x_1| \lor |x_2| \), that is, \(|e| \neq 0 \) for every \( e \) in \( (7) \). By freeness, Definition 3.6 (i), \((x^* x) see^* = P_{x^* x} \notin \mathcal{P} \), and so \((x^* x)see^* \neq x^* xsx^* \). Hence \( xsee^* x^* \neq x^* sx^* \).

This shows \( 0 \neq (1 - xsee^* x^*) sxx^* \). Also \( 0 \neq (1 - y_i \gamma_i') xsx^* \) since \( 0 \neq P = pxsx^* \). Hence, by formula \( (8) \) and Lemma 3.7, \( Q \neq 0 \). \( \square \)
Lemma 3.11. Let $X$ be a semigraph algebra and $\pi : X \rightarrow C^*(X)$ the universal representation. Then $\pi(X)$ is a semigraph algebra (which is free if $X$ is free).

Proof. By universality of the universal representation $\pi : X \rightarrow C^*(X)$, the gauge map $X \overset{\sigma_\lambda}{\rightarrow} X \overset{\pi}{\rightarrow} C^*(X)$ ($\lambda \in \mathbb{T}^k$), Definition 2.9 induces a gauge map $\bar{\sigma}_\lambda : C^*(X) \rightarrow C^*(X)$. Hence, by Lemma 2.10, $\pi(X)$ is a quotient of the free algebra $\mathbb{F}$ by a subset of the fiber space. Thus $\pi(X)$ must be a quotient of $X$ by a subset of its fiber space, and is thus a semigraph algebra by [3 Lemma 8.1]. Now assume that $X$ is free. By the uniqueness theorem (Theorem 2.17) the cores of $X$ and $\pi(X)$ are isomorphic, and so the validity of Definition 3.6 (ii) carries over from $X$ to $\pi(X)$. If a half-standard word $\pi(a)$ in $\pi(X)$ ($a$ being a half-standard word in $X$) is not in $\pi(P)$ then $d(a) = d(\pi(a)) > 0$, and so $P_a \notin P$ since $X$ is free, and thus $P_{\pi(a)} = \pi(P_a) \notin \pi(P)$. This verifies Definition 3.6 (i) for $\pi(X)$. □

4. QUELL SEMIGRAPH $C^*$-ALGEBRAS

In this section we define quell semigraph algebras. Let us anticipate roughly what it is. A quell semigraph algebra could be most simply explained by considering a higher rank graph $T$ [6]. Then the quell semigraph $C^*$-algebra $Q(T)$ for $T$ is a $C^*$-algebra which is similar to the Toeplitz graph algebra $TC^*(T)$ [7] but without the relations $Q_t = s(t)$ ($t \in T$). So the Toeplitz graph algebra is a quotient of the quell semigraph algebra.

Suppose that $T$ is a finitely aligned $k$-semigraph. Define $T^{(0)}$ to be the set of elements of $T$ which have degree zero ($d(t) = 0$). Note that if $e \in T^{(0)}$ and $e^2$ is defined then $e$ is automatically idempotent. Indeed, by the factorization property we may choose a unique decomposition $e = ab$ for certain $a, b \in T^{(0)}$. Then $e^2 = abab$. By the unique factorization property $e = a = b$, and so $e^2 = ab = e$, which proves the claim. Moreover, if $e \in T^{(0)}$ is idempotent and $x \in T$ then either $ex$ is undefined or $ex = x$ (since $ex = e^2x$ and so $x = ex$ by the unique factorization property). In particular, $ef$ must be undefined for distinct $e, f \in T^{(0)}$.

In this section it is assumed that a semigraph $T$ has only idempotent elements in $T^{(0)}$. Let us summarise the consequences in a lemma.

Lemma 4.1. The elements of $T^{(0)}$ are idempotent elements and mutually incomposable. If $e \in T^{(0)}, t \in T$ and $et$ is defined then $et = t$. (Similarly, $te = t$ if $te$ is defined.)

Definition 4.2. For $x \in T$ we define $r(x) = x(0, 0)$ (range of $x$) and $s(x) = x(d(x), d(x))$ (source of $x$).

Definition 4.3 (Quell semigraph algebra for $T$). Suppose that $T$ is a finitely aligned $k$-semigraph. Then one associates the quell semigraph algebra $X$ to $T$. It is the universal $*$-algebra $X$ generated by the set $T$ subject to the following relations.
(i) \( T \) consists of partial isometries,
(ii) \( T^{(0)} \) consists of projections,
(iii) \( X \) respects the multiplication of \( T \) (that is, if \( xy = z \) holds in \( T \) for \( x, y, z \in T \) then this identity should also hold in \( X \)),
(iv) \( xy = 0 \) for all \( x, y \in T \) whose product \( xy \) is undefined,
(v) \( Q_x \) and \( Q_y \) commute for all \( x, y \in T \), and
(vi)
\[
x^*y = \sum_{(e,f) \in T^{(0)}(x,y)} eQ_\gamma f^*
\]
for all \( x, y \in T \).

“Quelle” is the German word for source. Since the source projections play an extraordinary role in the last definition (as compared to ordinary graph algebras), we decided for the word quell. One may wish however to say “source semigraph algebra” rather than “quell semigraph algebra”.

Note that \( T \) is faithfully embedded in the free algebra \( F \), but could perhaps degenerate in the quotient \( X \). Soon we will see below (Corollary 4.9) however that \( T \) is also faithfully embedded in \( X \). That is why we should not like to distinguish the \( k \)-semigraph \( T \) and its embedding in \( X \). For the remainder of this section we shall assume that \( T \) is a finitely aligned \( k \)-semigraph and \( X \) its associated free semigraph algebra.

**Definition 4.4.** We shall denote an element \((y, \alpha_1, \ldots, \alpha_n) \in T^{n+1} (n \geq 1)\) symbolically by \( y_{\mu \alpha_1, \ldots, \alpha_n} \), or for brevity, by \( y_{\mu \alpha} \). Define
\[
\Delta_\mu = \{ y_{\mu \alpha_1, \ldots, \alpha_n} \mid n \in \mathbb{N}, y, \alpha_i \in T \text{ and } \exists i \in \{1, \ldots, n\} \text{ with } y \leq \alpha_i \}.
\]

**Definition 4.5.** Set \( \Delta = T \uplus \Delta_\mu \).

**Lemma 4.6.** \( \Delta \) is a semimultiplicative set.

**Proof.** We endow \( \Delta \) with the multiplication from \( T \) for products within \( T \) (as far as defined), and define
\[
x(y_{\mu \alpha}) = (xy)_{\mu \alpha}
\]
if \( x \in T, y_{\mu \alpha} \in \Delta_\mu \) and \( (xy)_{\mu \alpha} \in \Delta_\mu \). Other products in \( \Delta \) are not allowed (for example, products within \( \Delta_\mu \) are undefined, or products where an element of \( \Delta_\mu \) appears as a left factor are invalid).

We are going to check that \( \Delta \) is a semimultiplicative set. To this end we have to check associativity in the sense of Definition 2.1. Suppose that \( a, b, c \in \Delta \). If \( a, b, c \in T \) then \((ab)c\) is defined if and only if \( a(bc) \) is defined as \( T \) is a semimultiplicative set. If \( a \) or \( b \) is in \( \Delta_\mu \) then both \((ab)c\) and \( a(bc) \) are undefined. Suppose \( a, b \in T \) and \( c \in \Delta_\mu \). Write \( c = y_{\mu \alpha} \). If \((ab)c = ((ab)y)_{\mu \alpha} \) is defined then \((ab)y \leq \alpha_i \) for some \( i \). Thus \( \alpha_i = zaby \) for some \( z \in T \) and so also \( a(bc) = a((by)_{\mu \alpha}) = (a(by))_{\mu \alpha} \) is defined and we have \((ab)c = a(bc)\). Similarly we see the reverse conclusion. \( \square \)
If a semimultiplicative set $G$ has left cancellation, that means, $st_1 = st_2$ implies $t_1 = t_2$ (for all $s, t_1, t_2 \in G$) then we can associate a left reduced $C^*$-algebra to $G$ as defined next. (We shall write $e_i$ or $\delta_i$ for the delta function $1_{\{i\}}$.)

**Definition 4.7.** For a semimultiplicative set $G$ with left cancellation define $\lambda : G \rightarrow B(\ell^2(G))$ by

$$\lambda_s \left( \sum_{t \in G} \alpha_t \delta_t \right) = \sum_{t \in G, \text{st is defined}} \alpha_t \delta_{st}$$

for all $s \in G$ and $\alpha_t \in \mathbb{C}$. The sub-$C^*$-algebra of $B(\ell^2(G))$ generated by $\lambda(G)$ is called the left reduced $C^*$-algebra of $G$ and denoted by $C^*_r(G)$.

**Proposition 4.8.** There is a representation $\varphi : X \rightarrow C^*_r(\Delta) : \varphi(t) = \lambda_t$.

**Proof.** Of course, $\varphi : F \rightarrow C^*_r(\Delta)$ would be a well defined representation of the free algebra $F$ generated by $T$. We need to show that this $\varphi$ respects the defining relations of Definition 4.3. Note that the $\lambda_t$'s are partial isometries with commuting range and source projections (these are canonical projections onto $\ell^2(Z)$ for subsets $Z \subseteq \Delta$). So, by the property of $\Delta$ to be a semimultiplicative set and identity (10) the points (i)-(v) of Definition 4.3 (for $\varphi(T) = \lambda_T$ rather than $T$) are easy to see. (For (ii) recall Lemma 4.1.)

Let us write down the adjoint operators $\varphi(t)^*$. We have

$$(11) \quad \varphi(t)^* \delta_{ts\mu_a} = \delta_{s\mu_a}, \quad \varphi(t)^* \delta_{ts} = \delta_s, \quad \varphi(t)^* \delta_a = 0 \ (\text{else})$$

To check Definition 4.3 (vi), consider $x, y \in T$. Suppose

$$(12) \quad \varphi(xx^*yy^*)\delta_a = \delta_a$$

for an $a \in \Delta$. Then $a$ is a product $a = y_a$ (since $\varphi(y)^*\delta_a \neq 0$) for some $y_a \in \Delta$, and similarly $a = xa_x$ for some $a_x \in \Delta$. Say that $a = ys_y\mu_a = x\mu_a$. Then $\nu := y$s_y has degree $d(\nu) \geq d(x) \lor d(y)$, and so there must exist a minimal common extension $(e, f) \in T^{(\text{min})}(x, y)$ such that

$$\nu(0, |x| \lor |y|) = xe = yf.$$ 

Consequently we have

$$(13) \quad \varphi \left( \sum_{(e, f) \in T^{(\text{min})}(x, y)} (xe)(yf)^* \right) \delta_a = \delta_a.$$ 

On the other hand, (12) follows quite immediately from (13). Since, for arbitrary $a \in \Delta$, the right hand sides of (12) and (13) either give $\delta_a$ or zero, we conclude from the shown equivalence that

$$(14) \quad \varphi(xx^*yy^*) = \varphi \left( \sum_{(e, f) \in T^{(\text{min})}(x, y)} (xe)(yf)^* \right).$$

The range projection of $\varphi(ef^*y^*)$ is

$$\varphi(ef^*y^*) = \varphi(e(x)^*(xe)e^*) = \varphi(ee^*xexe^*) = \varphi(xexe^*),$$

as defined next. (We shall write $e_i$ or $\delta_i$ for the delta function $1_{\{i\}}$.)
which is a smaller projection than \( \varphi(x^* x) \). Thus,

\[
\varphi(x^* x e f^* y^* y) = \varphi(e f^* y^* y) = \varphi(e f^* y^* y f f^*).
\]

Hence, multiplying in \([14]\) from the left and right with \( x^* \) and \( y \), respectively, we see that the identity \([9]\) holds in the image of \( \varphi \).

\( \square \)

**Corollary 4.9.** The canonical map \( \iota: T \longrightarrow X \) is an injective \( k \)-semigraph homomorphism and non-degenerate.

**Proof.** We compose \( \iota \) with \( \varphi \) of Proposition \(4.8\) to see this. Let \( t \in T \). Let \( e = s(t) \), so \( t = te \). Then \( \varphi(\iota(t))\delta_e = \delta_t \neq 0 \). So \( \iota \) is non-degenerate. If \( s \in T \) is distinct from \( t \) then we easily see with Lemma \(4.1\) that \( \varphi(\iota(s))\delta_e = \delta_t \neq \varphi(\iota(s))\delta_e \). So \( \iota \) is injective.

\( \square \)

**Lemma 4.10.** \( X \) is a semigraph algebra with generators

\[
(15) \quad \mathcal{P} = \{ Q_{t_1} \ldots Q_{t_n} \in X \mid n \in \mathbb{N}, t_i \in T \} \cup \{ 0 \},
\]

\[
(16) \quad \mathcal{T} = T \setminus T^{(0)}
\]

**Proof.** We need to show Definition \(2.11\). That \( \mathcal{P} \) is a commuting set of projections closed under multiplications (Definition \(2.11\) (i)) follows from Definition \(4.3\) (i) and (v). That \( \mathcal{T} \) is a set of partial isometries closed under nonzero products (Definition \(2.11\) (ii)) follows from Definition \(4.3\) (i), (iii) and (iv). We are going to check that \( \mathcal{T}_1 \) is a semigraph (Definition \(2.11\) (iii)). Let \( t \in \mathcal{T}_1 \), and \( t = t_1 t_2 \) be the unique decomposition in \( T \) subject to \( m = d(t_1) > 0 \) and \( n = d(t_2) > 0 \). This is the required decomposition in \( \mathcal{T}_1 \) also. If however \( m = 0 \), say, then take the factorisation \( t = 1t \).

To prove \((1)\) of Definition \(2.11\) (v), consider \( x, y \in \mathcal{T} \). Note that in \((9)\) \( xe = yf \) and \( Qyf \in \mathcal{P} \), so \((9)\) looks already similar like \((1)\). We only have to take care whether \( \mathcal{T}_1^{(\min)} \) and \( T^{(\min)} \) make here a difference. If \( d(x) > d(y) \), say, then we have \( \{(1, f)\} = \mathcal{T}_1^{(\min)}(x, y) \) and \( \{(e', f)\} = T^{(\min)}(x, y) \). Thus \( xe' = yf \) so that \( e' \in T^{(0)} \) is a right unit for \( xe' = yf \) by Lemma \(4.1\) that is, \( yfe' = yf \). Hence, by the unique factorisation property in \( T \), even \( fe' = e' \). Thus \( 1f^* yf f^* = e'f^* yf f^* \), so there is no difference. The cases \( d(x) < d(y) \) and \( d(x) = d(y) \) are treated similarly. If \( d(x) \) and \( d(y) \) are incomparable then there is obviously no difference. To check Definition \(2.11\) (iv), just note that

\[ Qxy = (x^* x)y = x^*(xy) = yQxy1 = yQxy \]

by \((9)\) if \( x, y \in T \) and \( xy \neq 0 \). The algebra \( X \) is generated by \( \mathcal{P} \) and \( \mathcal{T} \) since \( e = e^* e = Q_e \in \mathcal{P} \), for \( e \in T^{(0)} \) by Definition \(4.3\) (ii). There is a gauge action \( \sigma \) on \( X \) given by \( \sigma_\lambda(t) = \lambda d(t)t \) for \( t \in T \), \( \lambda \in \mathbb{T}^k \). Indeed, it exists on the free algebra \( \mathbb{F} \) generated by \( T \), and so also on \( X \), as the relations of Definition \(2.11\) are invariant under the \( \sigma_\lambda \)'s. Hence, Lemma \(2.11\) verifies Definition \(2.11\) (vi).

\( \square \)
Definition 4.11. Let $T$ be a semigraph and $X$ its free semigraph algebra. Then $Q(T) := C^*(X)$ is called the quell semigraph $C^*$-algebra associated to $T$.

Let us use the following abbreviation. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in T^n$ then $\mu_\alpha$ denotes $s(\alpha_1)\mu_\alpha$. The next lemma shows us that the representation $\varphi$ can distinguish the elements of $\mathcal{P}$. The restriction $a \mapsto \varphi(a)|_{\ell_2(T)}$ is not able to do this, and this is why we considered $\Delta$ at all and not just $T$, which would have been much simpler.

Lemma 4.12. Let $p_\alpha = Q_{\alpha_1} \cdots Q_{\alpha_n}$ be nonzero for $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $T^n$. Then $\varphi(p_\alpha)\delta_\mu_\alpha = \delta_\mu_\alpha$ and $\varphi(q)\delta_\mu_\alpha = 0$ for every $q \in \mathcal{P}$ with $q < p_\alpha$.

Proof. Since $Q_{\alpha_i} = \alpha_i^*\alpha_i = \alpha_i^*\alpha_i s(\alpha_i)$, and $p_\alpha$ is nonzero, $s(\alpha_i) = s(\alpha_i)$ for all $i$ by the orthonality of the idempotent elements of $T$ (Lemma 4.11 and Definition 4.3 (iv)). Consequently,

\[(17) \quad \varphi(Q_{\alpha_i})\mu_\alpha = \varphi(\alpha_i^*)\varphi(\alpha_i)(s(\alpha_i)\mu_\alpha) = \varphi(\alpha_i^*)(\alpha_i s(\alpha_i)\mu_\alpha) = \mu_\alpha.
\]

This proves $\varphi(p_\alpha)\delta_\mu_\alpha = \delta_\mu_\alpha$. We may write $q = Q_{y_1} \cdots Q_{y_\ell}$ for some $y_i \in T$ (see (15)). Note that either $\varphi(q)\mu_\alpha = \mu_\alpha$ or $\varphi(q)\mu_\alpha = 0$. Assume the first case. Then by a similar computation as in (17) we see that for every $i$ there is a $j_i$ such that $y_i \leq \alpha_{i j_i}$. Thus, for every $i Q_{y_i} \geq Q_{\alpha_{i j_i}} \geq p_\alpha$. Hence $q \geq p_\alpha$, which is a contradiction to the assumption $q < p_\alpha$. \hfill \Box

Theorem 4.13. The quell semigraph algebra $X$ associated to a $k$-semigraph $T$ is free, weakly free and cancelling. It thus satisfies the Cuntz–Krieger uniqueness theorem, that is, there is only one $C^*$-representation of $X$ (up to isomorphism) which is non-vanishing on nonzero standard projections. This universal $C^*$-representation, which is also injective on the core, is the representation $\varphi$ from Proposition 4.8. In particular, there is an isomorphism between the quell semigraph $C^*$-algebra and the left reduced $C^*$-algebra of the semimultiplicative set $\Delta$, i.e. $Q(T) \cong C^*_\ell(\Delta)$.

Proof. We are going to check that $X$ is free (Definition 3.8). Let $p \in \mathcal{P}$ and $y_1, \ldots, y_n$ be half-standard words. Assume that $P_{y_i} < p$ for all $1 \leq i \leq n$. By (15) and Lemma 4.12 there is an $\alpha \in T^n$ such that $p = p_\alpha$. Let $\varphi : X \to C^*(\Delta)$ be the representation of Proposition 4.8. By (11) we have $\varphi(y_i^*)\delta_\mu_\alpha = 0$ if $d(y_i) > 0$. On the other hand, if $d(y_i) = 0$ then $P_{y_i} \in \mathcal{P}$, and so $\varphi(y_i)\delta_\mu_\alpha = 0$ by Lemma 4.12 (as $y_i = P_{y_i} < p_\alpha$). Summarising these facts we get

$$\varphi\left(\bigvee_{i=1}^n P_{y_i}\right)\delta_\mu_\alpha = 0$$

and $\varphi(p_\alpha)\delta_\mu_\alpha = \delta_\mu_\alpha$ (Lemma 4.12). Consequently $\bigvee_{i=1}^n P_{y_i} < p_\alpha$. This proves Definition 3.8 (ii).

If $y$ is a half-standard word with $d(y) > 0$ then $\varphi(P_y)\delta_\mu_\beta = 0$ for any $\beta$ by (11). Consequently, $P_y$ cannot be in $\mathcal{P}$ by Lemma 4.12. This verifies Definition 3.8 (i).
We are going to check that $\varphi$ is faithful on standard projections. By Lemma 3.8 a nonzero standard projection $p = P_a \prod_{i=1}^n (1 - P_{b_i})$ with $P_{b_i} < P_a$ and $a \leq b_i$. Say that $a = tp_\alpha = tQ_t p_\alpha$ for $t \in T_\lambda$ and some $\alpha \in T^n$ according to Lemma 4.12. We may incorporate $Q_t$ in $p_\alpha$ and assume that $\alpha_1 = t$. We have

$$\varphi(P_\alpha) \delta_{t\mu} = \delta_{t\mu}$$

by (11) and Lemma 4.12. On the other hand, if $d(b_i) > d(a)$ then

$$\varphi(P_{b_i}) \delta_{t\mu} = 0$$

by (11). If $d(b_i) > d(a)$ is not true then $b_i = tp_\beta$ (for some $p_\beta \in \mathcal{P}$) since $a \leq b_i$, and then, as $P_{b_i} < P_a$, i.e. $tp_\beta t^* < tp_\alpha t^*$, one has

$$t^*tp_\beta t^*t < t^*tp_\alpha t^*t = p_\alpha$$

(the last identity by the fact that $\alpha_1 = t$). Hence, $\varphi(p_\beta Q_t) \delta_{t\mu} = \varphi(p_\beta) \delta_{t\mu} = 0$ by Lemma 4.12. So also in this case we have (19). Identities (18) and (19) show that $\varphi(p) \neq 0$. By Corollary 2.15 $\varphi$ is injective on the core.

Hence also the universal representation of $X$ must be injective on the core. Since we have also checked that $X$ is free, $X$ is weakly free and cancelling by Propositions 3.9 and 3.10. Thus, by the Cuntz–Krieger uniqueness theorem (Theorem 2.17), $\varphi$ is the universal representation, which implies $Q(T) \cong \varphi(X)$.

$C^*_f(\Delta)$ is generated by the operators $(\lambda_t)_{t \in T}$, since the operators $\lambda_t$ are zero for $t \in \Delta_\mu$ (the composition $ts$ is invalid in $\Delta$ for any element $t \in \Delta_\mu$). Consequently, the image of $\varphi$ is dense in $C^*_f(\Delta)$ and so $Q(T) \cong \varphi(X) = C^*((\lambda_t)_{t \in T}) = C^*_f(\Delta)$. \hfill $\square$

The next lemma is intended to serve as an example for a particular quell semigraph algebra. Let $\zeta_n$ be the graph induced by the skeleton consisting of one vertex $\nu$ and $n$ arrows $s_1, \ldots, s_n$ starting and ending in this single vertex $\nu$; $n$ may be any cardinal number.

**Lemma 4.14.** The quell semigraph $C^*$-algebra $Q(\zeta_n)$ is the universal unital $C^*$-algebra generated by the free inverse semigroup (of partial isometries) of $n$ generators $t_1, \ldots, t_n$ with the additional relations that the range projections of these generators are mutually orthogonal, i.e. $P_{t_i} P_{t_j} = 0$ for all $i \neq j$.

**Proof.** Set $A = C^*((t_1, t_1, \ldots, t_n))$. In $A$, $1$ is a unit and the words in the letters $t_i$ form an inverse semigroup (where the inverse element should be the adjoint element in $A$, that is, inverse semigroup elements happen to be partial isometries); moreover $P_{t_i} P_{t_j} = 0$ for $i \neq j$. $A$ is universal with respect to these relations. The quell semigraph $C^*$-algebra $Q(\zeta_n)$ is a semigraph $C^*$-algebra (Lemma 4.10). We propose a homomorphism

$$\alpha : A \rightarrow Q(\zeta_n) : \alpha(1) = \nu, \alpha(t_i) = s_i.$$
Since the generators $s_i$ form an inverse semigroup by Lemma 2.13, $P_{s_i}P_{s_j} = 0$ by Definition 4.3 (vi), and $v$ is a unit in $Q(\zeta_n)$ by Definition 4.3 (iii) and the fact that $vs_i = s_iv = s_i$, the map $\alpha$ is a well-defined homomorphism.

We propose an inverse homomorphism

$$\beta : Q(\zeta_n) \to A : \beta(v) = 1, \beta(s_{i_1} \cdots s_{i_m}) = t_{i_1} \cdots t_{i_m}.$$  

$A$ is generated by $\beta(\{1, s_1, \ldots, s_n\})$, so $\beta$ is surjective. We need to show that the relations of Definition 4.3 for $\beta(\zeta_n)$ rather than $\zeta_n$, hold in $A$. Definition 4.3 (iii) is satisfied in the image of $\beta$ as $\beta$ is multiplicative. Definitions 4.3 (i)-(ii) and (iv)-(v) are obviously also correct in the image of $\beta$. Definition 4.3 (vi) is

$$x^*y = \begin{cases} 
\nu Q_x \nu = Q_x & \text{if } x = y \\
0 & \text{if } \zeta_n^{(\min)}(x, y) = \emptyset \\
aQ_y \nu = aa^*x^*xa & \text{if } xa = y \text{ for some } a \in \zeta_n
\end{cases}$$

The first case is tautological, the third one reduces to a tautology in an inverse semigroup, so holds in the image of $\beta$. The second case we demonstrate for $x = s_1s_2$ and $y = s_1s_3s_4$, say. One has $\beta(x)^*\beta(y) = \beta(x)^*\beta(x)^*\beta(y)\beta(y)^*\beta(y) = 0$ since

$$\beta(x)^*\beta(y)\beta(y)^* = t_1t_2^*t_2(t_1^*t_1)t_3t_4t_4^*t_3^*t_1^* = t_1t_2^*t_2t_3t_4t_4^*t_3^*(t_1^*t_1)t_1^* = 0,$$

where we have used inverse semigroup rules (commutativity of projections) and the fact that $P_{s_2}P_{s_3} = 0$. Since all relations of Definition 4.3 evidently hold in the image of $\beta$, $\beta$ is a well-defined homomorphism. This proves the lemma as $\alpha$ and $\beta$ are inverses to each other.  

5. $K$-theory of Weakly Free Semigraph $C^*$-algebras

In this section we are going to compute the $K$-theory of a weakly free semigraph $C^*$-algebra by an application of [4, Theorem 2.2]. Since the setting of [4] is somewhat lengthy, we do not recall it here but directly apply it to semigraph algebras.

**Definition 5.1.** For a semigraph algebra $X$ we say the source projections cover the generators if for every $p \in P$ there is a $t \in T$ such that $p \leq Q_t$.

Let $X$ be a $k$-semigraph algebra. Assume that $k$ is finite, the universal representation $X \to C^*(X)$ is injective (we shall regard $X$ as a subset of $C^*(X)$), and that the source projections cover the generators (Definition 5.1). Define $A_i = \{ tp \mid t \in T(e_i), p \in P \}\{0\}$ for $1 \leq i \leq k$.

**Lemma 5.2.** $X$ is generated by $A := A_1 \cup \ldots \cup A_k$.

**Proof.** Let $t \in T$. Then $t = t_1s_{i_1} \cdots t_ns_{i_n} = (t_1Q_{i_1}) \cdots (t_nQ_{i_n})$ for certain $t_i \in T$ with $d(t_i) = e_{i_1}$. So $t$ is a product of elements of $A$ as $Q_s \in P$ for any $s \in T$. Let $p \in P$. Since the source projections cover the generators by assumption, there is a $t \in T$ such that $p = pQ_t = pt^*tp = (tp)^*(tp)$. This is a product.
of elements of $A$ again as we may write $tp$ as $t_1(t_2p)$ where $t_1, t_2 \in T$, $d(t_2) = e_i$, so $t_2p \in A_i$, and $t_1$ may be further expanded as above.

Note that by the unique factorisation property in $T$ every standard word $w$ may be written as

$$w = a_1 \ldots a_nb^*_m \ldots b_1^*$$

for suitable letters $a_i, b_j$ in $\bigcup_{i=1}^{k} T^{(e_i)}$ and some $p \in P$.

Define $\mathcal{X}$ to be $C^*(X)$. By Lemma 5.2 $\mathcal{X}$ is generated by a finitely partitioned alphabet $A$. We have a gauge action (as defined in Definition 2.9) with respect to this alphabet on $\mathcal{X}$ and consequently a degree map determined by $d(a_i) = e_i$ for $a_i \in A_i$. We define $S$ to be the set of half-standard words. Their range projections commute by Corollary 2.14. The core is locally matrical by Corollary 2.15. We resolve the core by finite dimensional $C^*$-algebras as they are described in Corollary 2.15, and in particular choose $D$ to be the set of all their diagonal entries. Since the diagonal elements of these matrices of the core are expressable as direct sums of standard projections (Corollary 2.15), and every standard projection can be written as a sum of range projections of half-standard words (Lemma 2.13), we have $D \subseteq P$, where we set

$$P := \text{span}_\mathbb{Z}\{ P_x \in \mathcal{X} \mid x \in S \},$$

$$Q := \text{Alg}^*\{ Q_x \in \mathcal{X} \mid x \in S \}.$$  

We define $W'$ to be the set of standard words. They linearly span $X$ by Lemma 2.14. We have all requirements for [4, Theorem 2.2], except the technical conditions (a) and (b) from [4].

**Lemma 5.3.** If $X$ is weakly free then the technical conditions (a) and (b) from [4] hold.

**Proof.** The proof is very similar (an almost word by word translation) of [4, Lemma 2.4]. So we ask the reader to prove it along the lines of [4, Lemma 2.4] by using representation (20) and Lemma 3.4, Lemma 3.5 and Lemma 2.14 where necessary. \qed

We have now all requirements for [4, Theorem 2.2] which states the following.

**Theorem 5.4** ([4], Theorem 2.2). The identical embedding $\theta : C^*(Q) \to \mathcal{X}$ induces an isomorphism $K_0(\theta)$, and $K_1(\mathcal{X}) = 0$.

By (15) we have $Q = \text{span}(P)$. Since the projections $P$ commute, we have $K_0(C^*(Q)) = \text{Ring}(P)$, where $\text{Ring}(P)$ denotes the subring of $C^*(X)$ generated by $P$, regarded then as an abelian group under addition. Since $Q$ is a subset of the core, which is locally matrical, $C^*(Q)$ is an AF-algebra and thus $K_1(C^*(Q)) = 0$. Theorem 5.4 states that the $K$-theory of $C^*(X)$ is the $K$-theory of $C^*(Q)$, which we have now. Theorem 5.4, Lemma 5.3 and the above discussion now yield the following theorem.
Theorem 5.5. Let $X$ be a weakly free $k$-semigraph algebra (with $k < \infty$) whose universal representation $X \rightarrow C^*(X)$ is injective. Suppose that the source projections cover the generators. Then the semigraph $C^*$-algebra has the following $K$-theory:

$$K_1(C^*(X)) = 0, \quad K_0(C^*(X)) \cong \text{Ring}(\mathcal{P})$$

via $[p] \mapsto p$ for $p \in \mathcal{P}$. ($\text{Ring}(\mathcal{P})$ denotes the subring of $X$ generated by $\mathcal{P}$, regarded then as an abelian group under addition.)

We are going to write $\text{Ring}(\mathcal{P})$ as a direct limit of subrings by using common refinements of projections in $\mathcal{P}$. For each finite subset $\mathcal{Q}$ of $\mathcal{P}$ we consider the subring $\text{Ring}(\mathcal{Q})$ generated by $\mathcal{Q}$. This ring is generated by the base elements

$$(p_{\mathcal{Q}, A}) = \prod_{p \in A} p \prod_{q \in \mathcal{Q} \setminus A} (1 - q),$$

where $A$ is a nonzero subset of $\mathcal{Q}$. Note that these base elements are mutually orthogonal for different $A$'s. The projection $q_{\mathcal{Q}, A}$ is nonzero if and only if

$$(\prod_{p \in A} p) (1 - q) \neq 0$$

for all $q \in \mathcal{Q} \setminus A$. This follows from Lemma 3.7. Let $\hat{\mathcal{Q}}$ denote the family of all subsets $A$ of $\mathcal{Q}$ for which $p_{\mathcal{Q}, A}$ is nonzero. Since the the $p_{\mathcal{Q}, A}$'s are mutually orthogonal, we have $\text{Ring}(\mathcal{Q}) = \mathbb{Z}^{|\hat{\mathcal{Q}}|}$. We write $\text{Ring}(\mathcal{P})$ as a direct limit

$$\text{Ring}(\mathcal{P}) = \lim_{\longrightarrow} \mathbb{Z}^{|\mathcal{Q}|} \quad \text{for } \mathcal{Q} \subset \mathcal{P}. $$

6. $K$-theory of quell semigraph $C^*$-algebras

We aim now to apply Theorem 5.5 to the quell semigraph $C^*$-algebra. To this end consider the quell semigraph algebra $X$ associated to a finitely aligned $k$-semigraph $T$. We may go over to its image $X'$ in $Q(T)$, and write again $X$ rather than $X'$ for simplicity. This semigraph algebra is weakly free by Theorem 4.12 and Lemma 3.11. By (15) it is clear that the source projections cover the generators. We can thus apply Theorem 5.5 if $k$ is finite. If $k$ is infinite then we write $Q(T)$ as the direct limit

$$Q(T) \cong \lim_{\longrightarrow} X^{(k_0)},$$

where $k_0$ runs over the finite subsets of $k$, and $X^{(k_0)}$ denotes the sub-semigraph algebra of $X$ which is generated by all elements of $T$ which have degree zero at any coordinate outside of $k_0$ (same proof as Lemma 3.2). Again, in $X^{(k_0)}$ the source projections cover the generators by (15). Since $X$ is weakly free, $X^{(k_0)}$ is also weakly free. So we can apply Theorem 5.5 to each $C^*(X^{(k_0)})$ and get

$$K_0(Q(T)) = \lim_{\longrightarrow} K_0(C^*(X^{(k_0)})) = \lim_{\longrightarrow} \text{Ring}(P^{(k_0)}) = \text{Ring}(P).$$
Lemma 6.1. If \( s, t \in T \) then in \( Q(T) \) we have

\[
\begin{align*}
(25) & \quad Q_t = Q_s \iff s = t, \\
(26) & \quad Q_t < Q_s \iff s < t, \\
(27) & \quad Q_s Q_t < Q_t \text{ if } s \leq t \text{ is wrong}
\end{align*}
\]

Proof. If \( Q_s = Q_t \) then by Lemma 4.12, \( \varphi(Q_s) \delta_{\mu_s} = \delta_{\mu_s} = \varphi(Q_t) \delta_{\mu_s} \) and thus \( t \leq s \). Similarly, \( s \leq t \). We conclude that \( s = t \) by Lemma 2.4.

If \( Q_t < Q_s \) then by Lemma 4.12, \( \varphi(Q_t) \delta_{\mu_t} = \delta_{\mu_t} = \varphi(Q_s) \delta_{\mu_t} \) and thus \( s \leq t \). Since \( s \neq t \), we have \( s < t \). For the reverse implication, if \( s < t \) then \( t = a s \) for some \( a \in T \). So \( Q_t \leq Q_s \). Since \( s \neq t \), we have \( Q_t < Q_s \).

Suppose \( Q_t Q_s = Q_t \). Then by Lemma 4.12, \( \varphi(Q_t Q_s) \delta_{\mu_t} = \varphi(Q_t) \delta_{\mu_t} = \delta_{\mu_t} \). Hence \( s \leq t \).

\[ \square \]

Theorem 6.2. Let \( T \) be a \( k \)-semigraph. Then the quell semigraph \( C^* \)-algebra \( Q(T) \) has the following \( K \)-theory. If \( T \) is finite then \( C^*(T) \) is a finite dimensional \( C^* \)-algebra and

\[
(28) \quad K_0(Q(T)) \cong \mathbb{Z}^{[P]}, \quad K_1(Q(T)) = 0.
\]

If \( T \) is countably infinite then

\[
(29) \quad K_0(Q(T)) \cong \bigoplus_{\mathbb{N}} \mathbb{Z}, \quad K_1(Q(T)) = 0.
\]

Proof. We have already obtained (24). We aim to analyse \( \text{Ring}(P) \) further. By (23) we need to estimate the size of the set \( \hat{Q} \) for a finite subset \( Q \) of \( P \).

Suppose \( T \) is finite. Then the set of standard words is a finite set. Since they span \( Q(T) \), \( Q(T) \) is finite dimensional. (28) follows from (23).

Now suppose that \( T \) is countably infinite. Suppose \( t \in T \) with \( d(t) = n \delta_l \) for an \( n \in \mathbb{N} \) and some \( l \in k \). Then set \( Q_t = \{Q_{t(i, j \delta_l)} | 1 \leq i \leq m \} \) for \( 1 \leq m \leq n \). Note that \( Q_t \subseteq Q_{\hat{Q}} \). Then \( p_{Q_t, Q_{\hat{Q}}} \) of (21) is nonzero. Indeed, for every \( m < j \leq n \) (22) is

\[
(30) \quad \left( \prod_{i=1}^{m} Q_{t(i, j \delta_l)} \right)(1 - Q_{t(i, j \delta_l)}) = Q_{t(i, j \delta_l)}(1 - Q_{t(i, j \delta_l)}),
\]

which is nonzero by (27). Thus every \( Q_t \) is an element of \( \hat{Q} \). Consequently, \( |\hat{Q}_t| \geq n \) (or even exactly \( n \) as one may check). So, if \( k \) is finite then we will find a sequence of \( t \)'s with \( d(t_l) \to \infty \) for some coordinate \( l \in k \) and so (23) shows (29).

So suppose finally the case that \( k = \mathbb{N} \) and there is an infinite family of \( t_i \)'s such that \( d(t_i) = \delta_i \) for all \( i \in \mathbb{N} \). Set \( Q_n = \{Q_{t_1}, \ldots, Q_{t_n} \} \) for \( n \in \mathbb{N} \). Set \( A_m = \{Q_{t_m} \} \) for \( 1 \leq m \leq n \). Then \( A_m \in \hat{Q} \) since \( p_{Q_m, A_m} \) is nonzero. Indeed, (22) is \( Q_{t_m}(1 - Q_{t_j}) \), which is nonzero by (27). So we have \( |\hat{Q}_n| \geq n \). Again, (23) shows (29).

\[ \square \]
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