C¹ convexity-preserving piecewise variable degree rational interpolation spline

Lianyun PENG* and Yuanpeng ZHU*

* School of Mathematics, South China University of Technology
Guangzhou, 510640, China
E-mail: ypzhu@scut.edu.cn

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Abstract

An explicit expression of a C¹ piecewise variable degree rational interpolation spline is developed, which produces a convex (concave) interpolant to given convex (concave) data directly. The interpolant contains two local shape parameters which serves as local tension factors. Convergence analysis shows that the interpolant has $O(h^2)$ or $O(h^3)$ approximation order.

Keywords: Convexity-preserving, Variable degree rational interpolation spline, Convergence analysis, Approximation order

1. Introduction

Over the past years, a lot of scholars are devoted to the study of interpolation curves for a given set of data in computer aided geometric design, scientific data visualization and different areas. The resulting interpolation curves are expected to have some good properties, such as monotonicity-preserving, convexity-preserving, a certain degree of smoothness and so on. Thus they can be better used in practical applications. In this paper, we focus our attention on the construction of convexity-preserving interpolation curves on the addition that the set of data points are in convex position. Also the curves are with C¹ continuity.

Various convexity-preserving interpolation spline methods have been proposed, such as the piecewise rational interpolation splines developed in (Clements, 1990; Sarfraz, 2002; Abbas et al., 2014; Han, 2008; Merrien and Sablonnière, 2013; Zhu and Han, 2015a, 2015b; Zhu, 2018), the piecewise cubic interpolation splines constructed in (Costantini, 1984; Brodlie, 1991), the piecewise weighted polynomial interpolation splines given in (Kvasov, 2014a; Kvasov, 2014b), the piecewise exponential interpolation splines presented in (Heß and Schmidt, 1986) and the piecewise polynomial interpolation splines of nonuniform degree proposed in (Kaklis, 1990). A common point of these methods is that in order to preserve convexity property, it is requested sufficient data dependent constraints on the shape parameters. Therefore, as the convex data changed, the shape parameters should be recomputed to ensure the convexity of the resulting interpolants.

In (Delbourgo, 1989), Delbourgo constructed a kind of rational quadratic/linear interpolation splines, which produces directly C¹ convex (concave) interpolant to convex (concave) data without any data dependent constraints and has $O(h^3)$ approximation order. However, the resulting rational quadratic/linear interpolant is unique to the given convex data, thus users can not interactively adjust the shape of the obtained convexity-preserving interpolation curves without changing the given convex data. Recently, some improvements on this rational quadratic/linear interpolation spline have been proposed, such as the shape preserving quartic rational splines developed in (Han, 2008; Zhu and Han, 2015a, 2015b; Zhu, 2018), which include the rational quadratic/linear interpolation spline as a special or a limit case. These improved quartic rational splines have extra local shape parameters for interactively modifying the shape of spline curves. Nevertheless, these improved quartic rational splines can not produce directly convex (concave) interpolant to convex (concave) data. And for convexity preserving, it is requested sufficient data dependent constraints on the shape parameters involved in the improved quartic rational splines. In this paper, by introducing two local exponential shape parameters into the rational quadratic/linear interpolation spline given in (Delbourgo, 1989), we give an explicit representation of a piecewise
variable degree rational interpolation splines, which can produce $C^1$ convex (concave) interpolant to convex (concave) data. The new introduced local shape parameters have foreseeable shape control effect on generating convexity-preserving interpolation curves and can provide users more degrees of freedom to fulfill their different design intent.

The rest of this paper is organized as follows. Section 2 gives the construction of the variable degree rational interpolation splines. The convexity-preserving property and shape control analysis are shown in section 3, the corresponding convergence analysis is discussed in detail. Several numerical examples are given in section 4 to prove the worth of the new developed schemes. Conclusion is given in the section 5.

2. Variable degree rational interpolation spline

2.1. Construction of the spline

Let $(x_i, f_i)$ be given real data and $d_i$ chosen derivative values at knots $x_i$, $i = 1, 2, \ldots, n$, where $a = x_1 < x_2 < \ldots < x_n = b$ is a partition of interval $[a, b]$. For $x \in [x_i, x_{i+1}], i = 1, 2, \ldots, n - 1$, let $h_i = x_{i+1} - x_i$, $t = (x - x_i)/h_i$, $\Delta_i = (f_{i+1} - f_i)/h_i$, $A_i = \Delta_i - d_i$, and $B_i = d_{i+1} - \Delta_i$, then a new piecewise variable degree rational interpolation spline is constructed as follows

$$S(x) = (1 - t)f_i + tf_{i+1} - \frac{h_i(1 - t)AtB_i}{(1 - t)^\alpha B_i + \rho^\alpha A_i},$$

where $\alpha, \beta \in (0, 1]$ serve as two local shape parameters.

From Eq. 1, it is obvious that $S(x_i^+) = f_i, S(x_{i+1}^-) = f_{i+1}$. Moreover, for $x \in [x_i, x_{i+1}]$, by direct computation, we have

$$S'(x) = \Delta_i - \frac{A_iB_i}{(1 - t)^\alpha B_i + \rho^\alpha A_i} \left[ B_i(1 - t)\alpha (1 - 2t + \alpha_t) + A_i\beta (1 - 2t - \beta_i + \beta_t) \right],$$

it follows that $S'(x_i^+) = \Delta_i - A_i = d_i$ and $S'(x_{i+1}^-) = \Delta_i + B_i = d_{i+1}$, which implies that $S(x) \in C^1[a, b]$.

**Remark 2.1** It is observed that for all $\alpha_i = \beta_i = 1$, the resulting interpolation spline $S(x)$ is precisely the $C^1$ convexity-preserving rational quadratic/linear interpolant given in (Delbourgo, 1989).

For generating $S(x)$, the derivative values $d_i, i = 1, 2, \ldots, n$ should be provided in advance. In this paper, they are computed by the following arithmetic mean method

$$\begin{align*}
d_1 &= \Delta_1 - \frac{h_1}{h_1 + h_2} (\Delta_2 - \Delta_1), \\
d_i &= \frac{h_i}{h_{i-1} + h_i} (\Delta_i - \Delta_{i-1}) + \frac{h_{i-1}}{h_{i-1} + h_i} \Delta_i, \\
d_n &= \Delta_{n-1} + \frac{h_n}{h_{n-1} + h_n} (\Delta_n - \Delta_{n-2}),
\end{align*}$$

where $i = 2, 3, \ldots, n - 1$.

From these, we have

$$\begin{align*}
A_1 &= \frac{h_i}{h_1 + h_2} (\Delta_2 - \Delta_1), \\
A_i &= \frac{h_i}{h_{i-1} + h_i} (\Delta_i - \Delta_{i-1}), \\
B_{i-1} &= \frac{h_{i-1}}{h_{i-1} + h_{i}} (\Delta_{i-1} - \Delta_{i-2}), \\
B_{n-1} &= \frac{h_n}{h_{n-1} + h_n} (\Delta_{n-1} - \Delta_{n-2}),
\end{align*}$$

for $i = 2, 3, \ldots, n - 1$.

2.2. Convexity-preserving property and shape control analysis

For simplicity of presentation, we assume a strictly convex set of data as follows

$$\Delta_1 < \Delta_2 < \cdots < \Delta_{n-1}.$$

From Eq. 3, we get

$$\begin{align*}
A_i > 0, & \quad B_i > 0, & \quad i = 1, 2, \cdots, n - 1. 
\end{align*}$$

For $x \in [x_i, x_{i+1}], i = 1, 2, \ldots, n - 1$, the expression of $S''(x)$ is as follows

$$S''(x) = \frac{A\beta}{h_i [1 - (1 - \alpha_t)A_i^\beta (1 - \alpha_t) + (1 - \beta_i)A_i^2 \beta^\alpha - 2 (1 - t) \alpha^\alpha - 1] \{ 1 - \alpha_i \} B_i^2 (1 - t)^{2\alpha - 1} (2 [1 - t + \alpha_t] + (1 - \beta_i) A_i^2 \beta^\alpha - 1) [2 - \beta_i (1 - t)] + \beta_i (1 + \beta_i) A_i B_i (1 - t)^{\alpha + \beta^\alpha - 1} + 2 [(1 - \alpha_i) + (1 - \beta_i) + 2 \alpha \beta_i] A_i B_i (1 - t)^{\alpha + \beta^\alpha} + \alpha_i (1 + \alpha_i) A_i B_i (1 - t)^{\alpha + \beta^\alpha - 1} \}.$$

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Thus we can see that for any $\alpha_i, \beta_i \in (0, 1)$, $S''(x) > 0, \forall x \in [x_i, x_{i+1}]$, $i = 1, 2, \ldots, n - 1$, which implies that the interpolant $S(x)$ is convexity-preserving for any $\alpha_i, \beta_i \in (0, 1)$.

From the above analysis, we can see that the new constructed piecewise variable degree rational interpolation spline produces $C^1$ convex interpolant to convex data directly. And the resulting convexity-preserving interpolant also provides two local shape parameters for further interactively adjusting the shape of the convexity-preserving interpolation curves. From the expression of the interpolant $S(x)$ given in Eq. 1, we can see that the changes of a local control parameter $\alpha_i$ or $\beta_i$ will only affect a curve segment $S(x), x \in [x_i, x_{i+1}]$. Moreover, we give some shape control analysis concerning the effects of the two local shape parameters on generating convexity-preserving interpolation curves. For any $x \in (x_i, x_{i+1})$, it is obvious that

$$\lim_{\alpha_i \to 0, \beta_i \to 0} S(x) = (1 - t)f_i + tf_{i+1} - \frac{h_i(1 - t)A_iB_i}{B_i + A_i}.$$  

Thus, the decreases in the shape parameters $\alpha_i$ and $\beta_i$ simultaneously will make the convexity-preserving interpolant $S(x)$ locally tends to a quadratic interpolation polynomial in the subinterval $(x_i, x_{i+1})$.

**Remark 2.2** Similarly, for a strictly concave set of data

$$\Delta_1 > \Delta_2 > \cdots > \Delta_{n-1},$$

we have $A_i < 0, B_i < 0, i = 1, 2, \cdots, n - 1$, thus for $x \in [x_i, x_{i+1}], i = 1, 2, \ldots, n - 1$, from the expression of $R''(x)$ given in Eq.5, we can easily conclude that $S''(x) > 0, \forall x \in [x_i, x_{i+1}], i = 1, 2, \ldots, n - 1$, which implies that the interpolant $S(x)$ is concavity-preserving for any $\alpha_i, \beta_i \in (0, 1)$.

### 3. Convergence analysis

Let $f(x) \in C^3[x_i, x_n]$ be a given convex function with which $S(x)$ is compared and $f_i := f(x_i), i = 1, 2, \ldots, n$, $\delta = \min_{1 \leq i \leq n} f''(x_i) > 0$. We write $\|f^{(r)}\| = \max_{1 \leq i \leq n} |f^{(r)}(x_i)|$ for $r = 2, 3$ and $h = \max_{1 \leq i \leq n-1} |h_i|$. For $x \in [x_i, x_{i+1}], t = (x - x_i)/h_i, i = 1, 2, \ldots, n - 1$, let

$$H(x) = (1 - t)f_i + tf_{i+1} - h_i(1 - t)f_i[(1 - t)A_i + tB_i].$$

After some manipulations, we have

$$S(x) = H(x) + \frac{h_i(1 - t)}{(1 - t)^{\alpha_i}B_i + \rho^A_i} \left[(1 - t)^\alpha_i A_i^2 + (1 - t)^\rho_i t A_i + (1 - t)^\rho_i \rho_i B_i^2 \right].$$

With the notation $f'_i = f'(x_i), i = 1, 2, \cdots, n$, let $H^*(x)$ be the classical cubic Hermite interpolant to $f(x)$

$$H^*(x) = (1 - t)f_i + tf_{i+1} - h_i(1 - t)f_i [(1 - t)(A_i - f'_i) + t(f'_{i+1} - A_i)].$$

Since $A_i > 0, B_i > 0$, we have

$$|S(x) - f(x)| \leq |H^*(x) - f(x)| + |H(x) - H^*(x)|$$

$$+ \frac{h_i(1 - t)^\alpha_i A_i^2 + (1 - t)^\rho_i \rho_i B_i^2}{(1 - t)^{\alpha_i}B_i + \rho^A_i} \left[(1 - t)^\alpha_i A_i + (1 - t)^\rho_i \rho_i B_i^2 \right].$$

(6)

For $H^*(x)$, we have

$$|H^*(x) - f(x)| \leq \frac{h_i^3 \|f^3\|}{96},$$

$$|H(x) - H^*(x)| \leq \frac{h_i}{4} \max \{ |f'_i - d_i|, |f'_{i+1} - d_{i+1}| \}.$$  

And for any $\alpha_i, \beta_i \in (0, 1)$, it is easy to check that

$$\left\{ \begin{array}{l} (1 - t)f_i [(1 - t)^\alpha_i + (1 - t)^\rho_i \rho_i - 1] \leq (1 - t)f_i [(1 - t)^2 - 1] \leq \frac{1}{8}, \\ (1 - t)^\alpha_i A_i^2 \leq (1 - t)^2 t^2 \leq \frac{4}{27}, \\ (1 - t)^2 t^2 \rho_i \rho_i \leq (1 - t)^2 t^2 \leq \frac{4}{27}, \\ (1 - t)^2 t^2 B_i + \rho_i A_i \geq (1 - t)B_i + tA_i \geq \min \{ B_i, A_i \} \end{array} \right.$$  

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Moreover, from Eq. 2 and Eq. 3, by using a Peano kernel analysis, we have

\[
\begin{align*}
  f''_i - d_i &= \frac{1}{6} h_i (h_i + h_{i+1}) f^{(3)}(\xi_i), \\
  d_i - f'_i &= \frac{1}{6} h_i f^{(2)}(\xi_i), \\
  f''_i - d_n &= \frac{1}{6} h_{n-2} (h_{n-2} + h_{n-1}) f^{(3)}(\xi_n), \\
  A_1 &= \frac{1}{6} h_1 f^{(2)}(\xi_1), \\
  A_i &= \frac{1}{6} h_i f^{(2)}(\xi_i), \\
  B_{i-1} &= \frac{1}{6} h_{i-1} f^{(2)}(\xi_{i-1}), \\
  B_{n-1} &= \frac{1}{6} h_{n-1} f^{(2)}(\xi_{n-1}),
\end{align*}
\]

where \(\xi_i \in (x_i, x_{i+1}), \xi_n \in (x_{n-2}, x_n)\) and \(\xi_i \in (x_{i-1}, x_i), i = 2, 3, \cdots, n-1\).

Therefore, from Eq. 6, we get

\[
|S(x) - f(x)| \leq \frac{\|f''\|}{96} h^3 + \frac{\|f''\|}{12} h^3 + \frac{91}{864\delta} h^2,
\]

which implies that \(f(x) - S(x) = O(h^2)\).

Specially, for all \(\alpha_i = \beta_i = 1\), from Eq. 6, we have

\[
|S(x) - f(x)| \leq |H'(x) - f(x)| + |H(x) - H'(x)| + \frac{h_i(1 - t)^2 t^2 (B_i - A_i)^2}{(1 - t) B_i + tA_i}.
\]

From these together with

\[
\begin{align*}
  d_1 + d_2 - 2\Delta_1 &= 0, \\
  d_i + d_{i+1} - 2\Delta_i &= \frac{1}{6} h_i (h_i + h_{i+1}) f^{(3)}(\eta_i), \\
  d_{n-2} + d_{n-1} - 2\Delta_{n-1} &= 0,
\end{align*}
\]

where \(i = 2, 3, \cdots, n-2\) and \(B_i - A_i = d_i + d_{i+1} - 2\Delta_i\), we can easily deduce the following error bound

\[
|S(x) - f(x)| \leq \frac{\|f''\|}{96} h^3 + \frac{\|f''\|}{12} h^3 + \frac{64\delta}{64\delta} h^4,
\]

which implies that \(f(x) - S(x) = O(h^2)\) for all \(\alpha_i = \beta_i = 1\).

4. Numerical example

In this section, we present the performances of the new developed \(C^1\) convexity-preserving variable degree rational interpolation splines on three numerical examples.

**Example 4.1** We consider the concave data set \([(0, 5), (1, 7), (5, 9), (8, 9), (10, 1)]\) given in (Delbourgo, 1989). Fig. 1(A) shows the \(C^1\) concavity-preserving interpolation curves generated by setting all \(\alpha_i = \beta_i = 1\) (solid lines), and the dashed lines generated by changing \(\beta_2\) and \(\alpha_3\) from 1 to 0.1. Fig. 1(B) and Fig. 1(C) shows the derivative curves and curvature curves of Fig. 1(A), respectively.

![Fig. 1 Numerical results for Example 4.1.](image-url)
Example 4.2 We consider the convex data set \((-8, 4.5), (-7, 4), (2.2, 3.55), (7, 4), (10, 4.5), (12, 5)\) given in (Abbas et al., 2014). Figure 2 shows the \(C^1\) convexity-preserving interpolation curves generated by setting all \(\alpha_i = \beta_i = 1\) (solid lines), and the dashed lines generated by changing \(\beta_2\) from 1 to 0.3. Fig. 2(B) and Fig. 2(C) shows the derivative curves and curvature curves of Fig. 2(A), respectively.

Example 4.3 In this example, we consider a set of data to show the approximability of the interpolation method. The set of data is taken from the function \(f(x) = \sqrt{0.01 + 4x - x^2}, 0 \leq x \leq 4\). We set \(n = 11, h = 0.4, x_i = (i-1)h, i = 1, 2, \ldots, 11\). Fig. 3(A) shows the curve \(f(x)\) (dashed line) and the cubic interpolation spline curve \(S_3(x)\) (solid line) with \(S_3'(x_1) = f'(x_1)\) and \(S_3'(x_{11}) = f'(x_{11})\). Obviously, the cubic spline interpolation curve is not shape-preserving. Fig. 3(B) shows the curve \(f(x)\) (dashed line) and the rational variable degree interpolation spline curve (solid line) with all \(\alpha_i = \beta_i = 0.9\). From Fig. 3(B), we can see that the proposed variable degree rational interpolation spline retains the concavity property of the initial data.

From Figs. 1 and 2, we can see that for the unchanged data, the shape of the convexity-preserving interpolation spline curves can be modified by changing the two local shape parameters. We conclude from the figures that the new given interpolation method appears to produce satisfying curves.

5. Conclusions

The new developed \(C^1\) variable degree rational interpolation spline produces convex (concave) interpolant to given convex (concave) data directly, which includes the rational quadratic/linear interpolation splines given in (Delbourgo, 1989) as a special case. The interpolant has \(O(h^2)\) or \(O(h^3)\) convergence. Without changing the convex data, the shape of the resulting convexity-preserving interpolation spline curves can be adjusted conveniently by using the two local shape parameters. Future work will concentrate on applying the new construed interpolation spline to generate convexity-preserving interpolation surfaces.
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