BREGMAN NONEXPANSIVE TYPE ACTIONS OF SEMITOPOLOGICAL SEMIGROUPS

BUI NGOC MUOI AND NGAI-CHING WONG

Dedicated to Professor Anthony To-Ming Lau on the occasion of his retirement

Abstract. Let \( S \) be a semitopological semigroup, and let \( C \) be a nonempty closed convex subset of a reflexive Banach space. Under some amenability conditions on \( S \), we provide existence results of fixed points for several Bregman nonexpansive type actions \( S \times C \to C, (s,x) \mapsto T_sx \), of \( S \) on \( C \). The mappings \( T_s \) we discuss include those being Bregman generalized hybrid, Bregman nonspreading, and Bregman left asymptotically nonexpansive.

1. Introduction

Let \( S \) be a semitopological semigroup, i.e., \( S \) is a semigroup with a (Hausdorff) topology such that for each fixed \( t \in S \), the mappings \( s \to ts \) and \( s \to st \) are continuous. An action of \( S \) on a nonempty set \( C \) is a mapping of \( S \times C \) into \( C \), denoted by \((s,x) \mapsto T_sx\), such that \( T_{st}x = T_s(T_tx) \) for all \( s,t \in S \) and \( x \in C \). A point \( x_0 \in C \) is called a common fixed point for \( S \) if \( T_sx_0 = x_0 \) for all \( s \in S \).

Let \( X \) be a left translation invariant normed vector subspace of \( \ell^\infty(S) \), i.e., \( l_sf \in X \) for all \( s \in S, f \in X \), where the left translation \( l_sf \) is given by \( l_sf(t) = f(st), \forall t \in S \). Assume also that \( X \) contains all the constant functions. For example, the Banach algebra \( \text{CB}(S) \) of continuous and bounded functions on \( S \) is a left translation invariant subspace of \( \ell^\infty(S) \) containing constants. A bounded linear functional \( \mu \) on \( X \) is called a mean if \( \|\mu\| = \mu(1) = 1 \). A mean \( \mu \) is left invariant, or a LIM in short, if \( \mu(f) = \mu(l_sf) \) for all \( s \in S \) and \( f \in X \). A mean \( \mu \) is called multiplicative if \( X \) is a subalgebra of \( \ell^\infty(S) \) and \( \mu(fg) = \mu(f)\mu(g), \forall f, g \in X \). We call \( S \) left amenable (resp. extremely left amenable) if \( \text{CB}(S) \) has a left invariant mean (resp. multiplicative left invariant mean).

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Corresponding author: Ngai-Ching Wong, E-mail: wong@math.nsysu.edu.tw.
It is a classical result of Mitchell [16] that $S$ is extremely left amenable if and only if every continuous action of $S$ on a compact set has a common fixed point. When $S$ has only a non-multiplicative left invariant mean instead, however, we need some nonexpansiveness of the action to guarantee a fixed point.

Lau and Takahashi [12] considered left asymptotically nonexpansive actions, while Lau and Zhang [15] considered generalized hybrid actions, of a left amenable semi-group $S$ on a nonempty set $C$ in Hilbert space. They showed that such an action always has a common fixed point whenever there exists a bounded orbit $O_c := \{T_s c : s \in S\}$ of some $c$ in $C$. On the other hand, we studied norm-nonexpansive type actions on Banach spaces in [18], and metric- and seminorm-nonexpansive type actions on Fréchet and general locally convex spaces in [19].

In this paper, we consider nonexpansive type actions with respect to Bregman distances on reflexive Banach spaces with bounded orbits. We note that Bregman distances (also called Bregman divergence), though not being symmetric or satisfying the triangle inequality in general, are recently popular and useful in the quantum information theory (see, e.g., [4]).

Fix a strictly convex and Gâteaux differentiable function $g : U \to \mathbb{R}$ defined on an open set $U$ in a Banach space $E$ (see section 2 for definitions and notations). The Bregman distance (see, e.g., [2]) $D_g$ on $U$ is defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y) \rangle, \quad \forall x, y \in U.$$ 

Let $S$ be a semitopological semigroup. Let $C$ be a nonempty subset of $U$. An action $(s, x) \mapsto T_s x$ of $S$ on $C$ is called

- **Bregman nonexpansive** if

$$D_g(T_s x, T_s y) \leq D_g(x, y), \quad \forall x, y \in C, \forall s \in S;$$

- **Bregman left asymptotically nonexpansive** if for given $\varepsilon > 0$ and $y \in C$, there exists an $s_\varepsilon$ in $S$ depending on $\varepsilon$ and also on $y$, such that

$$D_g(T_{s_\varepsilon} x, T_{s_\varepsilon} y) \leq D_g(x, y) + \varepsilon, \quad \forall s \in S, \forall x \in C; \quad (1.1)$$

- **Bregman nonspreading** [20] if

$$D_g(T_s x, T_s y) + D_g(T_s y, T_s x) \leq D_g(T_s x, y) + D_g(T_s y, x), \quad \forall x, y \in C, \forall s \in S; \quad (1.2)$$

- **Bregman generalized hybrid** [10] if there are real numbers $\alpha, \beta$ such that

$$\alpha D_g(T_s x, T_s y) + (1 - \alpha) D_g(x, T_s y) \leq \beta D_g(T_s x, y) + (1 - \beta) D_g(x, y), \quad \forall x, y \in C, \forall s \in S. \quad (1.3)$$
It is plain that nonexpansive, left asymptotically nonexpansive \cite{12}, nonsprea-
ding \cite{8,9,23} and generalized hybrid \cite{22} maps defined on Hilbert spaces are exactly those Bregman nonexpansive, Bregman left asymptotically nonexpansive, Bregman nonsprea-
ding and Bregman generalized hybrid maps with respect to the Bregman distance \(D_g\) with \(g(x) = \|x\|^2\). We also note that all Bregman nonexpansive mappings are Bregman left asymptotically nonexpansive and Bregman generalized hybrid, while Bregman nonsprea-
ding might be neither Bregman nonexpansive nor continuous; see, e.g., \cite{20, example 1}.

We will show that any action \((s,x) \mapsto T_s x\) of a left amenable semitopological semigroup \(S\) on a nonempty closed and convex subset \(C\) of a reflexive Banach space \(E\) has a common fixed point, provided that all \(T_s\) carry any one of the above Bregman nonexpansiveness and there is a bounded orbit \(O_c\) of some point \(c \in C\).

Here is a sketch of our reasoning. Let \(X\) be a left translation invariant subspace of \(\ell^\infty(S)\) containing constants and all coordinate functions \(s \mapsto \langle T_s c, x^* \rangle\) with \(x^* \in E^*\). Let \(\mu\) be a left invariant mean on \(X\). Then for the bounded linear functional \(x^* \mapsto \mu_s \langle T_s c, x^* \rangle\), there exists \(z_\mu \in E = E^{**}\) such that \(\langle z_\mu, x^* \rangle = \mu_s \langle T_s c, x^* \rangle\). We pretend for a moment that \(\mu\) is a probability measure on \(S\). Then we could write

\[
z_\mu = \int_S T_s c \, d\mu(s).
\]

Note that the point \(z_\mu \in C\) if \(C\) is closed and convex. For each \(t \in S\), we would have

\[
T_t z_\mu = T_t \int_S T_s c \, d\mu(s) = \int_S T_{ts} c \, d\mu(s) = \int_S T_s c \, d\mu(s) = z_\mu,
\]

since \(\mu\) is left invariant. In other words, \(z_\mu\) is a candidate of the common fixed point for \(S\). In the following, we will verify this claim.

In section 2, we describe briefly the Bregman distances on Banach spaces and their properties. With respect to these distances, in section 3, we study various Bregman type nonexpansive actions of a left amenable semitopological semigroup on a nonempty closed convex subset of a reflexive Banach space. We show that such an action has a common fixed point, if there is a bounded orbit and the left translation invariant function space \(X\) generated by the action on the orbit has a LIM. The uniqueness of the fixed point is also discussed. In section 4, we study the problem when the underlying function space \(X\) has a LIM by embedding \(X\) into some classical function spaces on \(S\). Finally, in section 5 we discuss some possible resolves for the case when we do not have any LIM in stock.
2. Bregman distances

Let $U$ be a nonempty open set in a Banach space $E$. A function $g : U \to \mathbb{R}$ is said to be Gâteaux differentiable at $y$ if there is a bounded linear functional $\nabla g(y)$ in $E^*$, called the gradient of $g$ at $y$, such that

$$
\langle x, \nabla g(y) \rangle = \lim_{t \to 0} \frac{g(y + tx) - g(y)}{t}, \quad \forall x \in E.
$$

We call $g$ Fréchet differentiable at $y$ if for each given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$
|g(x) - g(y) - \langle x - y, \nabla g(y) \rangle| \leq \varepsilon \|x - y\| \quad \text{whenever} \quad \|x - y\| \leq \delta.
$$

We call $g$ strictly convex if $g(\alpha x + (1 - \alpha)y) < \alpha g(x) + (1 - \alpha)g(y)$ for all distinct $x, y \in U$ and $\alpha \in (0, 1)$.

Let $g : U \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function on a nonempty open set $U$ in a Banach space $E$. The Bregman distance $D_g$ on $U$ is defined by

$$
D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y) \rangle, \quad \forall x, y \in U. \tag{2.1}
$$

It is known that if a strictly convex function $g$ is Gâteaux differentiable then its gradient $\nabla g$ is norm-to-weak* continuous (see, e.g., [3, Proposition 1.1.10]) and $D_g(x, y) \geq 0$, $\forall x, y \in U$; the equality holds exactly when $x = y$. If $g$ is Fréchet differentiable then $\nabla g$ is norm-to-norm continuous (see, e.g., [7, page 508]).

Some Bregman distances between positive definite matrices used in quantum information theory associated with the Bregman function $g(A) = \text{trace}(f(A))$ follows:

- **classical divergence**: $D_g(A, B) = \text{trace}(A^2) + \text{trace}(B^2) - 2\text{trace}(BA)$ while $f(x) = x^2$;
- **Umegaki relative entropy**: $D_g(A, B) = \text{trace} [A(\log A - \log B)]$ while $f(x) = x \log x$;
- **Quantum divergence**: $D_g(A, B) = \|\sqrt{A} - \sqrt{B}\|_2^2$ while $f(x) = (\sqrt{x} - 1)^2$, where $\| \cdot \|_2$ is the Hilbert-Schmidt norm of matrices.

In the following demonstrations, $E = \mathbb{R}$, $U = (0, +\infty)$ and thus $g = f$. 
Figure 1: The Bregman classical divergence is exactly the square Hilbert space distance.

Figures 2, 3: Umegaki relative entropy is not symmetric, $D_g(x, y) > D_g(y, x)$.

Although the Bregman distances $D_g(x, y)$ have deficiencies such as not necessarily being symmetric, not necessarily satisfying the triangle inequality and not necessarily translation invariant, they do carry the *Bregman-Opial property*. That is, for any weakly convergent sequence $x_n \to x$ in $U$, we have

$$\limsup_{n \to \infty} D_g(x_n, x) < \limsup_{n \to \infty} D_g(x_n, y), \quad \forall y \in U \setminus \{x\}.$$

For the function $g(x) = \|x\|^2$ on a Hilbert space, this reduces to the famous Opial property. However, unlike the Bregman-Opial property, the Opial property fails to hold in the general Banach space setting. See, e.g., [6].
We call a function \( g : E \to \mathbb{R} \) strongly coercive if \( \lim_{\|x_n\| \to \infty} g(x_n) = +\infty \), call \( g \) locally bounded if \( g \) is bounded on bounded sets, and call \( g \) a Bregman function \(^3\) if it satisfies the following conditions.

1. \( g \) is continuous, strictly convex and Gâteaux differentiable.
2. The set \( \{ y \in E : D_g(x, y) \leq r \} \) is bounded for all \( x \in E \) and all \( r > 0 \).

Lemma 2.1 (see \([3, \text{page 70}]\)). Let \( C \) be a nonempty, closed and convex subset of \( E \). Let \( g \) be a strongly coercive Bregman function on \( E \), then for each \( x \in E \), there exists a unique point \( \hat{x} \in C \) such that

\[
D_g(\hat{x}, x) = \min_{y \in C} D_g(y, x).
\]

In this case, \( D_g(y, \hat{x}) \leq D_g(y, x) + D_g(\hat{x}, x) \leq D_g(y, x) \) for all \( y \in C \).

We call \( \hat{x} \) the Bregman projection of \( x \) on \( C \) and denote it by \( P^g_C(x) \).

3. Fixed point properties for Bregman nonexpansive type actions

In this section, we always consider an action \( S \times C \to C \), denoted by \((s, x) \mapsto T_s x\), of a semitopological semigroup \( S \) on a nonempty closed convex subset \( C \) of a reflexive Banach space \( E \). Moreover, we always assume \( g : E \to \mathbb{R} \) is a strongly coercive and locally bounded Bregman function.

Assume that there exists a point \( c \in C \) such that the orbit \( O_c = \{ T_s c : s \in S \} \) is bounded. Let \( X \) be the intersection of all left translation invariant subspaces of \( \ell^\infty(S) \), which contains all the constant functions and all the coordinate functions

\[
s \to \langle T_s c, x^* \rangle \quad \text{and} \quad s \to D_g(T_s c, x), \quad \forall x \in E, \; x^* \in E^*
\]

(3.1)
generated by the orbit \( O_c \). We call \( X \) the Bregman subspace associated to the action on the bounded orbit \( O_c \). Note that the local boundedness of \( g \) ensures that the functions in (3.1) are bounded on \( S \). For any mean \( \mu \) on \( X \), we consider the bounded linear functional of the dual space \( E^* \) of \( E \) defined by \( x^* \mapsto \mu_s(T_s c, x^*) \).

We shall write \( \mu_s(f(s)) \) instead of \( \mu(f) \) for the value of \( \mu \) at the function \( s \mapsto f(s) \) in \( s \). Since \( E \) is reflexive, there exists a unique \( z_\mu \in E \) such that

\[
\mu_s(T_s c, x^*) = \langle z_\mu, x^* \rangle, \quad \forall x^* \in E^*.
\]

(3.2)

We call \( z_\mu \) the \( \mu \)-barycenter of the bounded orbit \( O_c \).

We are going to show that \( z_\mu \) is a common fixed point of the action, provided that some Bregman type nonexpansiveness is assumed. As an intermediate step,
we will show that \( z_\mu \) is a Bregman attractive point, namely it belongs to the set of Bregman attractive points defined by

\[
A^B_C(S) = \{ x \in E : D_g(x, T_s y) \leq D_g(x, y), \forall y \in C, \forall s \in S \}. 
\] (3.3)

Clearly, \( z_\mu \) is a common fixed point of the action exactly when \( z_\mu \in A^B_C(S) \cap C \).

The following lemma can be deduced from [24, page 209], we give a different proof since its arguments will be vital for some later parts.

**Lemma 3.1.** If \( C \) is closed and convex then \( z_\mu \in C \).

**Proof.** From (2.1), the following Bregman three-point identity holds for any \( x, y, z \in E \),

\[
D_g(x, z) = D_g(x, y) + D_g(y, z) + \langle x - y, \nabla g(y) - \nabla g(z) \rangle. 
\] (3.4)

Therefore, for each \( x \in E \) we have

\[
D_g(T_s c, x) - D_g(T_s c, z_\mu) = D_g(z_\mu, x) + \langle T_s c - z_\mu, \nabla g(z_\mu) - \nabla g(x) \rangle. 
\]

Taking \( \mu_s \) on both sides,

\[
\mu_s D_g(T_s c, x) - \mu_s D_g(T_s c, z_\mu) = D_g(z_\mu, x) + \mu_s \langle T_s c - z_\mu, \nabla g(z_\mu) - \nabla g(x) \rangle 
\]

\[
= D_g(z_\mu, x) + \langle z_\mu - z_\mu, \nabla g(z_\mu) - \nabla g(x) \rangle 
\]

\[
= D_g(z_\mu, x) \geq 0. 
\] (3.5)

Since \( D_g(z_\mu, x) = 0 \) exactly when \( x = z_\mu \), we have

\[
M_\mu := \{ x \in E : \mu_s D_g(T_s c, x) = \inf_{y \in E} \mu_s D_g(T_s c, y) \} = \{ z_\mu \}. 
\] (3.6)

Let \( P_C^g(z_\mu) \) be the Bregman projection of \( z_\mu \) on \( C \). By Lemma 2.1, since \( T_s c \in C \) we have \( D_g(T_s c, P_C^g(z_\mu)) \leq D_g(T_s c, z_\mu) \). Therefore, \( \mu_s D_g(T_s c, P_C^g(z_\mu)) \leq \mu_s D_g(T_s c, z_\mu) \).

Since \( z_\mu \) is the unique element of \( M_\mu \), we have \( P_C^g(z_\mu) = z_\mu \). Hence, \( z_\mu \in C \). □

**Theorem 3.2.** Let \( C \) be a closed convex subset of a reflexive Banach space \( E \). Let \( (s, x) \mapsto T_s x \) be a Bregman generalized hybrid action of a semitopological semigroup \( S \) on \( C \) with a bounded orbit \( O_c \). If the Bregman subspace \( X \) associated to \( O_c \) has a left invariant mean \( \mu \), then the \( \mu \)--barycenter \( z_\mu \) of \( O_c \) is a common fixed point of \( S \).

**Proof.** If suffices to show that \( z_\mu \) is a Bregman attractive point of the action; see (3.3). Indeed, for each \( t \in S \), from the three-point identity (3.4), we have

\[
D_g(z_\mu, T_t x) - D_g(z_\mu, x) = D_g(x, T_t x) + \langle z_\mu - x, \nabla g(x) - \nabla g(T_t x) \rangle. 
\]
From the definition of $z_\mu$ we have
\[
D_g(x, T_ix) + \langle z_\mu - x, \nabla g(x) - \nabla g(T_ix) \rangle \\
= D_g(x, T_ix) + \mu_s \langle T_sc - x, \nabla g(x) - \nabla g(T_ix) \rangle \\
= D_g(x, T_ix) + \mu_s \langle T_ts c - x, \nabla g(x) - \nabla g(T_ix) \rangle, \text{ since } \mu \text{ is a LIM,} \\
= D_g(x, T_ix) + \mu_s [D_g(T_ts c, T_ix) - D_g(T_ts c, x) - D_g(x, T_ix)] \\
= \mu_s D_g(T_ts c, T_ix) - \mu_s D_g(T_ts c, x), \text{ since } \mu \text{ is a LIM.}
\] (3.7)

Because the action is Bregman generalized hybrid, there exists $\alpha, \beta \in \mathbb{R}$ satisfying (1.3). Thus, by following an idea in [15, Lemma 5.1],
\[
\mu_s D_g(T_ts c, T_ix) = \mu_s [\alpha D_g(T_ts c, T_ix) + (1 - \alpha) D_g(T_ts c, T_ix)] \\
= \mu_s [\alpha D_g(T_t T_sc, T_ix) + (1 - \alpha) D_g(T_sc, T_ix)], \text{ since } \mu \text{ is a LIM,} \\
\leq \mu_s [\beta D_g(T_ts c, x) + (1 - \beta) D_g(T_sc, x)] \\
= \mu_s [\beta D_g(T_sc, x) + (1 - \beta) D_g(T_sc, x)], \text{ since } \mu \text{ is a LIM,} \\
= \mu_s D_g(T_sc, x).
\]
Therefore,
\[
D_g(z_\mu, T_ix) \leq D_g(z_\mu, x), \quad \forall t \in S, \ x \in C.
\]
In other words, $z_\mu$ is a Bregman attractive point.

By Lemma 3.1, $z_\mu \in C$, and thus $D_g(z_\mu, T_t z_\mu) \leq D_g(z_\mu, z_\mu) = 0$. Consequently, $T_t z_\mu = z_\mu$ for all $t \in S$. In other words, $z_\mu$ is a common fixed point of $S$. \( \square \)

In the following theorems, we need to assume, in addition, that the Bregman subspace $X$ contains all the functions $s \to D_g(x, T_sc), \forall x \in C$.

**Theorem 3.3.** Let $C$ be a closed convex subset of a reflexive Banach space $E$. Let $(s, x) \mapsto T_sc$ be a Bregman nonsparing action of a semitopological semigroup $S$ with a bounded orbit $O_c$. If the Bregman subspace $X$ associated to $O_c$ has a left invariant mean $\mu$, then the $\mu-$barycenter $z_\mu$ of $O_c$ is a common fixed point of $S$.

**Proof.** The proof goes exactly as in that of Theorem 3.2, except for (3.7). In other words, it suffices to verify that
\[
\mu_s D_g(T_ts c, T_ix) \leq \mu_s D_g(T_sc, x), \quad \forall t \in S, \ x \in C
\] (3.8)

Suppose it is not the case, and let $t_0 \in S$ and $x_0 \in C$ such that $\mu_s D_g(T_t_0 s c, T_t_0 x_0) > \mu_s D_g(T_sc, x_0)$. Since $T_t_0$ is Bregman nonsparing, by (1.2) we have
\[
D_g(T_t_0 s c, T_t_0 x_0) + D_g(T_t_0 x_0, T_t_0 s c) \leq D_g(T_t_0 s c, x_0) + D_g(T_t_0 x_0, T_sc).
\]
By taking $\mu_s$ on both sides, we have
\[
\mu_s D_g(T_{t_0}s, T_{t_0}x_0) + \mu_s D_g(T_{t_0}x_0, T_{t_0}s) \leq \mu_s D_g(T_{t_0}s, x_0) + \mu_s D_g(T_{t_0}x_0, T_sx)
\]
\[
= \mu_s D_g(T_s, x_0) + \mu_s D_g(T_{t_0}x_0, T_sx)
\]
\[
< \mu_s D_g(T_{t_0}s, T_{t_0}x_0) + \mu_s D_g(T_{t_0}x_0, T_sx).
\]
Thus $\mu_s D_g(T_{t_0}x_0, T_{t_0}s) < \mu_s D_g(T_{t_0}x_0, T_sx)$. However, this conflicts with the fact that $\mu$ is a LIM. This contradiction finishes the verification. \qed

Recall that a vector subspace $X$ of $\ell^\infty(S)$ is called right translation invariant if $rsf \in X$ for all $f \in X$, where $rsf(t) = f(ts), \forall t \in S$, is the right translation of $f$. A mean $\mu$ on a right translation invariant subspace is called right invariant if $\mu(rsf) = \mu(f)$ for all $s \in S, f \in X$. If $X$ is translation invariant, i.e., both left and right translation invariant, we call a mean $\mu$ on $X$ an invariant mean if it is both left and right invariant. On the other hand, we call $\mu$ faithful if $f = 0$ whenever $f \in X, f \geq 0$ and $\mu(f) = 0$.

**Theorem 3.4.** Let $C$ be a closed convex subset of a reflexive Banach space $E$. Let $(s, x) \mapsto T_s x$ be a Bregman left asymptotically nonexpansive action of a semitopological semigroup $S$ with a bounded orbit $O_c$. If the Bregman subspace $X$ associated to $O_c$ is translation invariant and has a faithful invariant mean $\mu$, then the $\mu-$barycenter $z_\mu$ of $O_c$ is a common fixed point of $S$.

**Proof.** By Lemma 3.1, $z_\mu \in C$. Since the action is Bregman left asymptotically nonexpansive, by (1.1) for each given $\varepsilon > 0$ there exists $s_\varepsilon \in S$ such that
\[
D_g(T_{s_\varepsilon}T_t c, T_{s_\varepsilon}z_\mu) \leq D_g(T_t c, z_\mu) + \varepsilon, \quad \forall s, t \in S.
\]
Then
\[
\mu_t D_g(T_t c, T_{s_\varepsilon}z_\mu) = \mu_t D_g(T_{s_\varepsilon}T_t c, T_{s_\varepsilon}z_\mu)
\]
\[
\leq \mu_t D_g(T_t c, z_\mu) + \varepsilon, \quad \forall s \in S.
\]
On the other hand, from (3.5) we have $D_g(z_\mu, x) = \mu_t D_g(T_t c, x) - \mu_t D_g(T_t c, z_\mu)$ for all $x \in E$. Hence, putting $x = T_{s_\varepsilon}z_\mu$ we get
\[
D_g(z_\mu, T_{s_\varepsilon}z_\mu) = \mu_t D_g(T_t c, T_{s_\varepsilon}z_\mu) - \mu_tD_g(T_t c, z_\mu)
\]
\[
\leq \mu_tD_g(T_t c, z_\mu) + \varepsilon - \mu_tD_g(T_t c, z_\mu) = \varepsilon.
\] (3.9)
Since $\mu$ is right invariant, $\mu_s D_g(z_\mu, T_s z_\mu) = \mu_s D_g(z_\mu, T_{s_\varepsilon}z_\mu) \leq \varepsilon$ for all $\varepsilon > 0$. Therefore $\mu_s D_g(z_\mu, T_s z_\mu) = 0$. Since $\mu$ is faithful and $D_g(z_\mu, T_s z_\mu) \geq 0$ for all $s \in S$, we have $D_g(z_\mu, T_s z_\mu) = 0$. This implies $z_\mu = T_s z_\mu$ for all $s \in S$. Then $z_\mu$ is a common fixed point for $S$. \qed

We are now going to discuss the uniqueness of the fixed point. In fact, as shown in Propositions 3.6 and 3.7 below, the common fixed point $z_\mu$, which is
the $\mu-$barycenter of a bounded orbit $O_c$, is the same for any choice of an invariant mean $\mu$.

**Lemma 3.5.** Let $C$ be a closed convex subset of a reflexive Banach space $E$. Let $(s, x) \mapsto T_s x$ be an action of a semitopological semigroup $S$ with a bounded orbit $O_c$. Let the Bregman subspace $X$ associated to $O_c$ be translation invariant and have an invariant mean $\mu$, and let $y$ be a common fixed point. Then

(i) $\sup_t \inf_s D_g(T_{ts} c, y) \leq \mu_t D_g(T_{ts} c, y) \leq \inf_s \sup_t D_g(T_{ts} c, y)$ if the action is Bregman generalized hybrid or Bregman nonspreading;

(ii) $\sup_t \inf_s D_g(T_{ts} c, y) = \mu_t D_g(T_{ts} c, y) = \inf_s \sup_t D_g(T_{ts} c, y)$ if the action is Bregman left asymptotically nonexpansive.

**Proof.** We follow an idea in [12] which works for the Hilbert space. For each $f \in X$ and $s \in S$, by the right invariance of $\mu$, we have

$$\mu_t D_g(T_{ts} c, y) = \mu_t D_g(T_{ts} c, y) \leq \sup_{t \in S} D_g(T_{ts} c, y).$$

Hence,

$$\mu_t D_g(T_{ts} c, y) \leq \inf_s \sup_t D_g(T_{ts} c, y).$$

(i) As shown in (3.7) and (3.8), for each $t, u \in S$ we have

$$\mu_s D_g(T_s c, y) \geq \mu_s D_g(T_{tu} T_s c, T_{tu} y) \geq \inf_s D_g(T_{ts} c, y).$$

Thus $\mu_s D_g(T_s c, y) \geq \sup_t \inf_s D_g(T_{ts} c, y)$.

(ii) Since $y$ is a common fixed point and the action is Bregman left asymptotically nonexpansive, for any $\epsilon > 0$ there is $u \in S$ such that

$$D_g(T_{ts} c, y) = D_g(T_{tu} T_{ts} c, T_{tu} y) \leq D_g(T_s c, y) + \epsilon, \quad \forall s, t \in S.$$

Consequently,

$$\inf_s \sup_{s'} D_g(T_{ts} c, y) \leq \sup_{s'} D_g(T_{ts} c, y) \leq \sup_t D_g(T_{ts} c, y) + \epsilon \leq D_g(T_s c, y) + \epsilon, \quad \forall s \in S.$$

Because $\epsilon > 0$ is arbitrary, $\inf_s \sup_{s'} D_g(T_{ts} c, y) \leq \mu_s D_g(T_s c, y)$. Hence $\mu_t D_g(T_t c, y) = \inf_s \sup_{s'} D_g(T_{ts} c, y)$. 


On the other hand, by the Bregman left asymptotically nonexpansiveness again, 
\[
\sup_t \inf_s D_g(T_t s c, y) \geq \inf_s D_g(T_{ts} c, y) \geq \inf_s \mu_t D_g(T_{ts} T_{us} c, T_{tu} y) - \epsilon \\
\geq \inf_s \mu_t D_g(T_{ts} s c, y) - \epsilon \\
= \inf_s \mu_t D_g(T t c, y) - \epsilon, \quad \text{since } \mu \text{ is right invariant,} \\
= \mu_t D_g(T t c, y) - \epsilon.
\]
Since \( \epsilon > 0 \) is arbitrary, we have
\[
\sup_t \inf_s D_g(T_{ts} c, y) \leq \inf_t \sup_s D_g(T_{ts} c, y) = \mu_t D_g(T t c, y) \leq \sup_t \inf_s D_g(T_{ts} c, y).
\]

\[\square\]

**Proposition 3.6.** Let \((s, x) \mapsto T_s x\) be an action of a semitopological semigroup \(S\) on a nonempty closed convex set \(C\) in a reflexive Banach space \(E\). Let the Bregman subspace \(X\) associated to a bounded orbit \(O_c\) be translation invariant. Assume that the barycenters \(z_\mu, z_\psi\) are common fixed points of \(S\) for invariant means \(\mu, \psi\) on \(X\), respectively. Assume that the action is either

(i) Bregman left asymptotically nonexpansive, or

(ii) Bregman generalized hybrid or Bregman nonspreading, such that
\[
\sup_t \inf_s D_g(T_{ts} c, z) = \inf_t \sup_s D_g(T_{ts} c, z) \quad \text{for } z = z_\mu \text{ or } z = z_\psi.
\]

Then, \(\mu_t D_g(T t c, z_\mu) = \psi_t D_g(T t c, z_\psi)\) and \(z_\mu = z_\psi\).

**Proof.** From Lemma 3.5 and the assumptions, in each case, we have
\[
\mu_t D_g(T t c, z_\mu) = \inf_t \sup_s D_g(T_{ts} c, z_\mu) \quad \text{and} \quad \psi_t D_g(T t c, z_\psi) = \inf_t \sup_s D_g(T_{ts} c, z_\psi).
\]

Following an idea in [12, Theorem 4.8], we obtain from (3.6) that
\[
\mu_t D_g(T t c, z_\mu) \leq \mu_t D_g(T t c, z_\psi) = \psi_t D_g(T_{ts} c, z_\psi) \leq \inf_t \sup_s D_g(T_{ts} c, z_\psi) \\
= \psi_t D_g(T t c, z_\psi) \leq \psi_t D_g(T t c, z_\mu) \leq \inf_t \sup_s D_g(T_{ts} c, z_\mu) \\
= \mu_t D_g(T t c, z_\mu).
\]

Hence, \(\mu_t D_g(T t c, z_\mu) = \psi_t D_g(T t c, z_\psi)\) and \(\mu_t D_g(T t c, z_\mu) = \mu_t D_g(T t c, z_\psi)\). Since \(M_\mu\) contains the unique point \(z_\mu\), we have \(z_\mu = z_\psi\). \[\square\]

The following proposition extends a result of Lau and Zhang [15, Theorem 4.11] in which they considered norm nonexpansive actions on Hilbert spaces. It implies that the barycenter \(z_\mu\) of a bounded orbit \(O_c\) does not depend on the choice of the invariant mean \(\mu\), provided that \(S\) is right reversible together with other assumptions. Recall that a semitopological semigroup \(S\) is called right reversible if
the intersection $\overline{a} \cap \overline{b}$ of two closed left ideals is nonempty for every $a, b \in S$. In this case, we can define a direct order on $S$ by letting $a \leq b$ if $b = a$ or $b \in \overline{a}$. It is easy to see that if $a \leq b$ then $a \leq tb$ for all $t \in S$. In particular, $ts \to \infty$ whenever $s \to \infty$ for any fixed $t$ in $S$.

Recall that if the Bregman function $g$ is Fréchet differentiable then $\nabla g$ is norm-to-norm continuous.

**Proposition 3.7.** Let $C$ be a nonempty closed convex subset of a reflexive Banach space $E$. Assume that $\nabla g$ is norm-to-norm continuous and the Bregman distance $D_g$ is symmetric. Let $(s, x) \mapsto T_s x$ be a Bregman generalized hybrid or Bregman nonspooling action of a right reversible semitopological semigroup $S$ on $C$ with a bounded orbit $O_c$. Assume that the Bregman subspace $X$ associated to $O_c$ is translation invariant and has an invariant mean $\mu$. Then $A^g_C(S)$ is nonempty, closed and convex. Moreover, $\lim_{t \to \infty} P_{A^g_C(S)}(T_t c) = z_{\mu}$ in norm.

**Proof.** It is shown in the proof of Theorems 3.2 and 3.3 that $z_{\mu} \in A^g_C(S)$. Hence $A^g_C(S)$ is nonempty. Furthermore, as shown in [15, Lemmas 4.9 and 4.10], see also [5, Lemmas 3.2 and 4.4], $A^g_C(S)$ is closed convex, and $P_{A^g_C(S)}(T_t c)$ converges strongly to some $u \in A^g_C(S)$. Since $D_g$ is symmetric, i.e., $D_g(x, y) = D_g(y, x), \forall x, y \in E$, by Lemma 2.1, for each $t, s \in S$ we have

$$D_g(T_t c, P_{A^g_C(S)}(T_t c)) + D_g(P_{A^g_C(S)}(T_t c), y) \leq D_g(T_t c, y).$$

By the three-point identity (3.4), we have

$$\langle T_t c - P_{A^g_C(S)}(T_t c), \nabla g(P_{A^g_C(S)}(T_t c)) - \nabla g(y) \rangle \geq 0, \quad \forall y \in A^g_C(S).$$

This implies

$$\langle T_t c - P_{A^g_C(S)}(T_t c), \nabla g(y) - \nabla g(u) \rangle$$

$$\quad \leq \langle T_t c - P_{A^g_C(S)}(T_t c), \nabla g(P_{A^g_C(S)}(T_t c)) - \nabla g(u) \rangle. \quad (3.10)$$

Since $O_c$ is bounded, it follows from condition (2) in the definition of a Bregman function and Lemma 2.1 that $\{P_{A^g_C(S)}(T_t c) : s \in S\}$ is also bounded. Hence, there exists $M > 0$ such that $\|T_t c\| + \|P_{A^g_C(S)}(T_t c)\| \leq M$ for all $t, s \in S$. It follows from (3.10) that

$$\langle T_t c - P_{A^g_C(S)}(T_t c), \nabla g(y) - \nabla g(u) \rangle \leq M \|\nabla g(P_{A^g_C(S)}(T_t c)) - \nabla g(u)\|.$$

Since $\mu$ is a mean on $X$, we have

$$\mu_t \langle T_t c - P_{A^g_C(S)}(T_t c), \nabla g(y) - \nabla g(u) \rangle \leq \sup_{t \in S} \langle T_t c - P_{A^g_C(S)}(T_t c), \nabla g(y) - \nabla g(u) \rangle$$

$$\quad \leq M \sup_{t \in S} \|\nabla g(P_{A^g_C(S)}(T_t c)) - \nabla g(u)\|. $$
Since $\mu$ is right invariant, this implies
\[
\mu_t \left( T_t c - P_{A^g_c(S)}^g(T_{ts}c), \nabla g(y) - \nabla g(u) \right) \leq M \sup_{t \in S} \| \nabla g(P_{A^g_c(S)}^g(T_{ts}c)) - \nabla g(u) \|.
\]
Because $P_{A^g_c(S)}^g(T_{ts}c)$ converges strongly (in the index $s$) to $u$ and because $\nabla g$ is norm-to-norm continuous, $\| \nabla g(P_{A^g_c(S)}^g(T_{ts}c)) - \nabla g(u) \| \to 0$ uniformly on $t \in S$ when $s \to \infty$. Then, by taking the limit in $s$ we have
\[
\mu_t \langle T_t c - u, \nabla g(y) - \nabla g(u) \rangle \leq 0.
\]
In other words,
\[
\langle z_\mu - u, \nabla g(y) - \nabla g(u) \rangle \leq 0, \quad \forall y \in A^g_c(S).
\]
Let $y = z_\mu$, we have
\[
0 \geq \langle z_\mu - u, \nabla g(z_\mu) - \nabla g(u) \rangle = D_g(z_\mu, u) + D_g(u, z_\mu).
\]
Hence $u = z_\mu$. \hfill \qed

4. Bregman subspaces associated to the action

In this section, we study under what conditions on an action $(s, x) \mapsto T_s x$, the coefficient functions in (3.1) belong to some classical function spaces defined on $S$. Since there are many well established amenability conditions concerning several classical function spaces on semitopological semigroups, see, e.g., [12,13,15], we can guarantee the existence of a LIM for the Bregman subspace $X$ and thus utilize the results in section 3.

Let $AP(S)$ (resp. $WAP(S)$) be the set of all almost periodic (resp. weakly almost periodic) functions on $S$, i.e., all those functions $f \in CB(S)$ such that its right orbit $\{r_tf : t \in S\}$ is precompact in the norm (resp. weak) topology of $CB(S)$. Let $RUC(S)$ be the set of all right uniformly continuous on $S$, i.e., all those functions $f \in CB(S)$ such that the map $s \mapsto r_s f$ from $S$ into $CB(S)$ is continuous when $CB(S)$ is equipped with the uniform norm topology. In general, $AP(S) \subset WAP(S) \subset CB(S)$ and $AP(S) \subset RUC(S) \subset CB(S)$. When $S$ is compact, we have $AP(S) = RUC(S) \subset WAP(S) \subset CB(S)$. The reader can see [1,11,13,17] for more discussion about these function spaces and their amenability.

Let $c \in C$. We call its orbit $O_c = \{T_sc : s \in S\}$ a continuous orbit if the map $s \to T_sc$ is continuous in norm topology, a nonexpansive orbit if $\|T_s T_tc - T_s T_t' c\| \leq \|T_tc - T_t' c\|$ for all $s, t, t' \in S$, and a precompact orbit if $O_c$ is norm precompact. Note that $O_c$ will be precompact in the weak topology of the reflexive Banach space $E$ whenever it is bounded.
Proposition 4.1. Let \((s, x) \mapsto T_s x\) be an action of a semitopological semigroup \(S\) on a nonempty subset \(C\) of a reflexive Banach space \(E\). Assume that there is a point \(c \in C\) with a bounded orbit \(O_c\). Let \(X\) be the Bregman subspace of \(\ell^\infty(S)\) associated to \(O_c\) for the Bregman distance \(D_g\). Then

(i) \(X \subset \text{CB}(S)\) if \(c\) has a continuous orbit;
(ii) \(X \subset \text{RUC}(S)\) if \(c\) has a continuous nonexpansive orbit and the Bregman function \(g\) is Lipschitz continuous on \(O_c\);
(iii) \(X \subset \text{AP}(S)\) if \(c\) has a nonexpansive precompact orbit and \(g\) is Lipschitz continuous on \(O_c\);
(iv) \(X \subset \text{WAP}(S)\) if each \(T_s\) is weak-to-weak continuous and \(g\) is weakly continuous on \(O_c\).

Proof. We show that, for each \(x \in E\) and each \(x^* \in E^*\), the coefficient functions

\[ s \mapsto f(s) = \langle T_s c, x^* \rangle \quad \text{and} \quad s \mapsto h(s) = D_g(T_s c, x) \]

are in the corresponding function spaces. The proof for \(s \mapsto D_g(x, T_s c)\) is similar.

(i) Trivial.

(ii) Let \(t_\lambda \to t\). For each \(s \in S\), we have

\[
|r_{t_\lambda} f(s) - r_t f(s)| = |\langle T_{st_\lambda} c - T_{st} c, x^* \rangle| \\
\leq \|x^*\| \|T_{t_\lambda} c - T_t c\| \\
\leq \|x^*\| \|T_{t_\lambda} c - T_t c\|.
\]

Hence \(\|r_{t_\lambda} f - r_t f\| \leq \|x^*\| \|T_{t_\lambda} c - T_t c\| \to 0\) and thus \(f \in \text{RUC}(S)\).

Let \(K > 0\) be the Lipschitz constant of \(g\). Then \(\|\nabla g(y)\| \leq K\) for all \(y \in E\). Observe

\[
|r_{t_\lambda} h(s) - r_t h(s)| = |D_g(T_{st_\lambda} c, x) - D_g(T_{st} c, x)| \\
= |g(T_{st_\lambda} c) - g(T_{st} c) - \langle T_{t_\lambda} c - T_t c, \nabla g(x) \rangle| \\
\leq 2K \|T_{t_\lambda} c - T_t c\| \\
\leq 2K \|T_{t_\lambda} c - T_t c\|.
\]

Hence \(\|r_{t_\lambda} h - r_t h\| \leq 2K \|T_{t_\lambda} c - T_t c\| \to 0\), and thus \(h \in \text{RUC}(S)\).

(iii) It follows from [11, Lemma 3.1] that \(f \in \text{AP}(S)\).

We show that the map \(L : O_c \to \text{CB}(S)\) given by \(L(z) = \varphi_z\) is norm-to-norm continuous where \(\varphi_z(s) = D_g(T_s z, x)\). Indeed, let \(\{z_n\}\) be a sequence converging to \(z\) in \(O_c\) in norm. For each \(s \in S\), since \(g\) is \(K\)-Lipschitz we have

\[
|D_g(T_s z_n, x) - D_g(T_s z, x)| = |g(T_s z_n) - g(T_s z) - \langle T_s z_n - T_s z, \nabla g(x) \rangle| \\
\leq 2K \|T_s z_n - T_s z\| \leq 2K \|z_n - z\|.
\]
Hence $L$ is norm-to-norm continuous. Since $O_c$ is norm precompact, so is $L(O_c)$. On the other hand, $r_th(s) = h(st) = D_y(T_s T_t c, x)$ for each $t \in S$. Hence $\{r_th : t \in S\}$ is contained in the precompact subset $L(O_c)$ of $CB(S)$. In particular, $h \in AP(S)$.

(iv) As in (iii) we define the map $L : O_c \to CB(S)$ by

$$z \mapsto D_y(T_s z, x) = g(T_s z) - g(x) - \langle T_s z - x, \nabla(x) \rangle.$$ 

Since both $g$ and $T_s$ are weak-weak continuous, so is $L$. Consequently, $L(O_c)$ is weakly precompact. Because $\{r_th : t \in S\} \subset L(O_c)$, it is also weakly precompact in $CB(S)$. In particular, $h \in WAP(S)$. In similar manner, we see that $f \in WAP(S)$. □

5. Further Discussion

In section 3, we have seen that a common fixed point exists if $X$ has a LIM. But, sometimes we only have an approximate left invariant mean instead, i.e., there is a net $\{\mu_\lambda\}$ of means on $X$ such that $\mu_\lambda(f - l_s f) \to 0$ for every $s \in S$ and every $f \in X$. In this situation, we ask the following questions.

Q1: Do we have a common fixed point?
Q2: When a fixed point exist, how do we locate it?

Proposition 5.1. Let $C$ be a nonempty closed convex subset of a reflexive Banach space $E$. Let $(s,x) \mapsto T_s x$ be an action of a semitopological semigroup $S$ on $C$ with a bounded continuous orbit $O_c$. Let $\{\mu_\lambda\}$ be an approximate left invariant mean on the Bregman subspace $X$ associated to $O_c$. Then

(i) any weak* cluster point $\mu$ of $\{\mu_\lambda\}$ is a left invariant mean and $z_{\mu_\lambda} \to z_\mu$ weakly;

(ii) if $\mu_\lambda \to \mu$ in norm then $z_{\mu_\lambda} \to z_\mu$ in norm.

Proof. (i) We note that there always exists a weak* cluster point of $\{\mu_\lambda\}$ since the closed unit ball of $X^*$ is weak* compact. It follows similarly as [21, page 883] that $\mu$ is a left invariant mean on $X$. By the definition of $z_\mu$ in (3.2), $z_{\mu_\lambda} \to z_\mu$ weakly.

(ii) If $\mu_\lambda$ converges to $\mu$ in norm then

$$\|z_{\mu_\lambda} - z_\mu\| = \sup_{\|x^*\| \leq 1} |\langle z_{\mu_\lambda}, x^* \rangle - \langle z_\mu, x^* \rangle|$$

$$= \sup_{\|x^*\| \leq 1} |\mu_\lambda (T_t c, x^*) - \mu (T_t c, x^*)|$$

$$\leq M \|\mu_\lambda - \mu\|,$$

where $M = \sup_{t \in S} \|T_t c\|$. This implies $z_{\mu_\lambda}$ converges to $z_\mu$ in norm. □
We end this paper with an open problem about a possible variance of our results. Let $X$ be a subspace of $\ell^\infty(S)$ containing all constant functions. A real valued function $\mu$ on $X$ is called a submean, see e.g. [12,14], if

1. $\mu(f + g) \leq \mu(f) + \mu(g)$ for all $f, g \in X$;
2. $\mu(\alpha f) = \alpha \mu(f)$ for all $f \in X$ and $\alpha \geq 0$;
3. $f, g \in X$ and $f \leq g$ imply $\mu(f) \leq \mu(g)$;
4. $\mu(c) = c$ for every constant function $c$.

By [12, proposition 3.6], if $S$ is a left reversible semitopological semigroup, then $\text{CB}(S)$ always has a left invariant submean, while, in general, it might not have any left invariant mean.

**Question 5.2.** Do we have similar results as in the Theorems 3.2, 3.3 and 3.4 when $X$ has a (left) invariant submean?

Here is the difficulty. Since we do not have the linearity of a submean $\mu$, the functional on $E^*$ defined by $x^* \mapsto \mu_s \langle T_s c, x^* \rangle$ in (3.2) is not linear. Therefore, we are unable to define the $\mu-$barycenter $z_\mu$ by the formula $\mu_s \langle T_s c, x^* \rangle = \langle z_\mu, x^* \rangle$ for all $x^* \in E^*$.

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W. Takahashi, N.-C. Wong and J.-C. Yao, Attractive point and mean convergence theorems for semigroups of mappings without continuity in Banach spaces, *J. Fixed Point Theory Appl.*, 16 (2014), 203–227.
(Bui Ngoc Muoi) Department of Mathematics, Hanoi Pedagogical University 2, Vinh Phuc, Vietnam, and Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.

E-mail address: muoisp2@gmail.com

(Ngai-Ching Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.

E-mail address: wong@math.nsysu.edu.tw