From Identity to Difference: A Quantitative Interpretation of the Identity Type

Paolo Pistone
Università di Bologna, Italy
paolo.pistone2@unibo.it

Abstract

We explore a quantitative interpretation of 2-dimensional intuitionistic type theory (ITT) in which the identity type is interpreted as a “type of differences”. We show that a fragment of ITT, that we call difference type theory (dTT), yields a general logical framework to talk about quantitative properties of programs like approximate equivalence and metric preservation. To demonstrate this fact, we show that dTT can be used to capture compositional reasoning in presence of errors, since any program can be associated with a “derivative” relating errors in input with errors in output. Moreover, after relating the semantics of dTT to the standard weak factorization systems semantics of ITT, we describe the interpretation of dTT in some quantitative models developed for approximate program transformations, incremental computing, program differentiation and differential privacy.

1 Introduction

From Program Equivalence to Program Differences

In program semantics, a classical problem is to know whether two programs behave in the same way in all possible contexts. Yet, in several fields of computer science, especially those involving numerical and probabilistic forms of computation (like e.g. machine learning), it is often more important to be able to describe to which extent two programs behave in a similar, although non equivalent, way. Hence, a crucial aspect is to be able to measure the change in the overall result that is induced by the replacement of a (say, computationally expensive) program by some (more efficient but only) approximately correct one.

These observations have motivated much research on denotational semantics involving metric and differential aspects, that is, in which one can measure differences between programs, as well as their capacity of amplifying errors. For instance, it has been observed that a fundamental property for a protocol to ensure differential privacy [30, 7, 15] is that the associated program is not too sensitive to errors; this has led to an elegant semantics [5] where types are interpreted as metric spaces and programs are interpreted by functions with bounded derivative (i.e. Lipschitz-continuous functions). More generally, the recent literature in theoretical computer science has seen the blossoming of many different notions of “derivative” for higher-order programming languages, each accounting for some differential aspect of computation: from connections with linearity (e.g. the differential λ-calculus [14, 9, 10]) to incremental computation [11, 1]), from higher-order automatic differentiation [21, 27] to higher-order approximate program transformations [36, 13, 23].

Do all these differential approaches share a common logic? Is there some common notion of “derivative” for higher-order programs? In this paper we argue that a proper fragment of standard intuitionistic type theory (ITT in the following), that we call difference type theory (dTT in short) provides a convenient framework to formalize the compositional reasoning about program derivatives found in some of these semantics.

The Identity Type

Our quantitative approach to intuitionistic type theory relies on a non-standard interpretation of the identity type. When Martin-Löf introduced ITT
The identity type $I_A(t, u)$ was one of its main novelties. Under the Curry-Howard correspondence, the elements of $I_A(t, u)$ were interpreted as proofs of the fact that $t$ and $u$ denote the same object of type $A$. The introduction rule for the identity type constructs an object $\text{refl}(t) \in I_A(t, t)$ for all $t \in A$, witnessing the equality $t = t \in A$; instead, the elimination rule provides a computational interpretation of Leibniz’s indiscernibles principle by justifying a form of transport of identity: from an equality proof $p \in I_A(t, u)$ and a proof $q \in C(t, t, \text{refl}(t))$, one can construct a proof $\Xi(t, u, p, q) \in C(t, u, p)$.

As is well-known, ITT comes in two flavors: in the extensional version one has rules for passing to and from $p \in I_A(t, u)$ and $t = u \in A$, i.e. saying that a proof of $I_A(t, u)$ exists precisely when $t = u \in A$ holds; with these rules, one can show that any element of $I_A(t, u)$, if any, is of the form $\text{refl}(t)$. In the intensional version these additional rules are not present (with the significant advantage that type-checking becomes decidable), and this leaves space for non-standard interpretations, as we will see.

In more recent times, a new wave of interest in intensional ITT has spread in connection with an interpretation relating it to homotopy type theory [20]: by exploiting the transport of identity principle, one can prove that the dependent type $\text{Id}_A(t, u)$ carries the structure of a groupoid (i.e. a category with invertible arrows), and that proofs $f : A \to B$ lift to functors $I_A(f) : I_A(x, y) \to I_B(f(x), f(y))$ between the respective groupoids. This is the basic ground for a suggestive and well-investigated interpretation, where $I_A(t, u)$ becomes the space of homotopies between $t$ and $u$.

While the original semantics of ITT based on locally cartesian closed category [20] validates the extensionality rules (which make the homotopy interpretation trivial), an elegant semantics for intensional ITT has been established since [16, 4, 35] based on the theory of weak factorization systems, providing the basis for the construction of various homotopy-theoretic models.

**The Type of Differences** Without extensionality, there can be different ways of proving $I_A(t, u)$, as we saw. What if these were not seen as ways of proving that $t$ and $u$ denote the same object, but rather as ways of measuring the difference between $t$ and $u$? The main point of this paper is to convince the reader that this idea is not only consistent, but yields some interesting new interpretation of (a fragment of) intensional ITT.

For example, in presence of a type $\text{Real}$ of real numbers with constants $r$ for all $r \in \mathbb{R}$, we might wish to interpret an element $a \in I_{\text{Real}}(r, s)$ as a difference between $r$ and $s$, that is, as a positive real greater or equal to $|r - s|$. The introduction rule produces the element $\text{refl}(r) = 0 \in I_{\text{Real}}(r, r)$, the self-distance of $r$; the interpretation of the elimination rule is more delicate, as this rule makes reference to arbitrary predicates; yet, let us consider a predicate of the form $C(x, y, p) = I_{\text{Real}}(f(x), g(y))$, where $x, y \in \text{Real}$, $p \in I_A(x, y)$ and $f, g \in \text{Real} \to \text{Real}$ are two smooth (i.e. infinitely differentiable) functions. Then given a difference $a \in I_{\text{Real}}(r, s)$ and a difference $b \in I_{\text{Real}}(f(r), g(r))$ we can obtain a difference $c \in I_{\text{Real}}(f(r), g(s))$ by reasoning as follows: first, since $g$ is smooth, by standard analytical reasoning (read: the mean value theorem) we can find some positive real $L_{r,s} \cdot a \geq L_{r,s} \cdot |r - s| \geq |g(r) - g(s)| \in I_{\text{Real}}(g(r), g(s))$; hence the operation $r, s, a \mapsto L_{r,s} \cdot a$ yields a way to transport differences between $r$ and $s$ into differences between $g(r)$ and $g(s)$. This difference can now be used to produce a difference $J(r, s, a, b) = b + L_{r,s} \cdot a \in I_{\text{Real}}(f(r), g(s))$, as required by the elimination rule applied to $C(x, y, p)$.

More generally, we will see that by interpreting higher-order programs as suitably “differentiable” maps, one can justify different kinds of “transport of difference” arguments, yielding quantitative interpretations of the elimination rule for the identity type.

**Plan of the Paper** In Section 2 we introduce difference type theory dTT, a fragment of ITT in which $I_A(t, u)$ is seen as a type of differences, and we provide a short overview of the kind of compositional differential reasoning formalizable in this system. In Section 3 we introduce a notion of model for dTT (that we call a dTT-category), and we prove that any instance of this notion yields a form of weak factorization system, thus relating our semantics to the usual semantics of the identity type.

In later sections we sketch some differential models of dTT. In Sec. 4 we describe the
interpretation of a sub-exponential version of dTT in the metric semantics used for differential privacy [5], providing a formal language to express metric preservation. In Sec. 5 we show that dTT yields a natural language for differential logical relations [13, 23, 28], an approach to approximate program transformations in which program differences are themselves higher-order entities. In Sec. 6 we provide an interpretation of dTT in models of higher-order incremental computing [11], with program differences interpreted as increments; finally, in Sec. 7 we show an interpretation of dTT in models of the differential λ-calculus [13, 4, 10].

2 Difference Type Theory

Let us start with a motivating example: suppose \( H : (\text{Nat} \rightarrow \text{Real}) \rightarrow \text{Real} \) is a program that takes a function \( f \) from integers to reals and computes a value \( \overline{H}(f(0), \ldots, f(N)) \) depending on the first \( N + 1 \) outputs of \( f \). For instance, \( H \) might compute some aggregated value from a time series \( f \) (e.g. \( H \) computes the average temperature in London from a series of measures taken from \( f \). Since measuring \( f \) every, say, minute might be too expensive, it might be worth considering an approximated computation, in which \( f \) is only measured every \( k \) minutes, (i.e. \( f \) is applied only to values \( 0, k, 2k, \ldots, \lceil N/k \rceil \)), and each computed value is fed to \( H \) \( k \) times (this technique is well-known under the name of loop perforation [31]).

What is the error we can expect for the replacement of \( H(f) \) by its approximation? We will show that a fragment of ITT, that we call difference type theory (in short dTT), can be used to reason about this kind of situations in a natural and compositional way.

**The Syntax of dTT** The fragment of ITT we consider in this paper includes two universes of types Type and dType (whose elements will be indicated, respectively, as \( A, B, C, \ldots \) and as \( A, B, C, \ldots \)), with formation rules illustrated in Fig. 1. In our basic language the types \( A \in \text{Type} \) are just usual simple types (yet in our examples we will often consider extensions or variants of this language). The types \( A \in \text{dType} \) can depend on terms of some simple type; we will often refer to them as predicates; intuitively, an element of some predicate \( A(t, u) \), depending on terms \( t, u \) of some simple type, will be interpreted as denoting differences, or errors, between the terms \( t, u \).

The introduction, elimination and computation rules of dTT are those of standard intuitionistic type theory, restricted to the types of dTT (we recall them in the Appendix). We use \( D_A(t, u) \) instead of \( I_A(t, u) \) for the usual identity type, since we are thinking of it as a type of differences. We illustrate in Fig. 2 the rules for the difference type \( D_A(t, u) \). Terms are constructed starting from a countable set of term variables \( x, y, z, \ldots \) and a
countable (disjoint) set of difference variables \( \epsilon, \delta, \ldots \). The rules in Fig. 2 must be read as dependent on some context, which, for dTT, are of the form \( (x \in \Phi_0 | \epsilon \in \Phi_1(x)) \), where \( x \in \Phi_0 = (x_1 \in A_1, \ldots, x_n \in A_n) \) is a sequence of (non-type dependent) declarations for \( A_i \in \text{Type} \), and \( \epsilon \in \Phi_2(x) = (\epsilon_1 \in C_1(x), \ldots, \epsilon_m \in C_m(x)) \) is a sequence of declarations for \( (x \in \Phi_0)C_i(x) \in \text{dType} \).

A term \( (z \in \Phi_1)t \in A \), where \( A \in \text{Type} \), is just an ordinary \( \lambda \)-term with pairing. We call such terms program terms and we use \( t, u, v \) for them. We will consider variants of this basic language with other type and term primitives (e.g. ground types like \( \text{Nat}, \text{Bool} \) or the probabilistic monad \([\Box]\), as well as a sub-exponential variant \( \text{STLC}^\ast \), with bounded linear types of the form \( !_k A \rightarrow B \) (see [15, 30]), described in the Appendix.

A term \( (x \in \Phi_0 | \epsilon \in \Phi_1)A \in \text{dType} \), for some \( (x \in \Phi_0)A \in \text{dType} \) belongs to the grammar

\[
a := \epsilon | \lambda x.a | at | \lambda xye.a | atta | \langle a, a \rangle | \pi_1(a) | \pi_2(a) | \hat{c}(t) | J(t, t, a, [x]a)
\]

We call such terms difference terms and we will use \( a, b, c \) for them.

Intuitively, a term of the form \( \hat{c}(t) \) (i.e. \( \text{refl}(t) \) in ITT) indicates the self-difference of \( t \). For example, when considering semantics based on metric spaces, \( \hat{c}(t) \) will represent the null error, i.e. 0. However, in other models of dTT, \( \hat{c}(t) \) needs not be zero (in fact, 0 is not even part of our basic syntax). In particular, in the models from Section 5 and 6, for a higher-order function \( f : A \rightarrow B \), \( \hat{c}(f) \) will provide a measure of the sensitivity of \( f \) (in fact, in such model \( \hat{c}(f) \) coincides with the derivative of \( f \), see below).

The terms of the form \( J(t, u, a, [x]b) \) are the main computational objects (and also the least intuitive) of dTT. The idea behind the quantitative interpretation of this constructor is that, given self-differences \( b(x) \in C(x, x) \), \( J \) “transports” an error \( a \) between \( t \) and \( u \), measured in \( D_A \), onto an error between \( t \) and \( u \) measured in \( C \). For example, as discussed in the introduction, \( C(x, y) \) might be the type of differences \( D_B(f(x), g(y)) \), and \( J \) will thus transport a difference \( a \) between \( t \) and \( u \) onto a difference between \( ft \) and \( gu \).

A fundamental application of \( J \) is the following: for any function \( f : A \rightarrow B \), the derivative of \( f \) is the following difference term

\[
D[f] := \lambda xye.J(x, y, y, \epsilon, [x] \hat{c}(fx)) \in (\Pi x, y \in A)(D_A(x, y) \rightarrow D_B(f(x), f(y)))
\]

\( D[f] \) tracks errors in input into errors in output of \( f \), and can thus be taken as a measure of the sensitivity of \( f \). In the models of dTT described in the following sections \( D[f] \) will be interpreted by different notions of program derivative, including the “true” derivative of \( f \), when the latter encodes a real-valued smooth function.

From the computation rules of \( J \) we deduce the following computation rules for derivatives:

\[
\begin{align*}
D[f](t, t, \hat{c}(t)) &= \hat{c}(ft) & (\beta D) \\
D[\lambda x.a](t, u, a) &= a & (\eta D)
\end{align*}
\]

(\( \beta D \)) says that the derivative of \( f \) computed on the self-distance of a point is just the self-distance of the image of the point. When \( D[f] \) is seen as the “true” derivative, the self-distances \( \hat{c}(v) \) are just the null error 0, and so \( (\beta D) \) says that the derivative computed in 0 is 0. \( (\eta D) \) says that the error produced in output by the identity function is just the error in input (this is in accordance with the analytical intuition too).

Given \( f : A \rightarrow B \) and \( g \in B \rightarrow C \) the composition of \( D[f] \) with \( D[g] \) yields a difference of type \( (\Pi x, y \in A)(D_A(x, y) \rightarrow D_C(gf(x), gf(y)) \). Several models of dTT will satisfy the chain rule axiom below, which identifies the latter with the derivative of \( g \circ f \):

\[
D[x.g(f(x))] = \lambda xye.D[g](fx)(fy)(D[f]xye)
\]

\( (\text{Dchain}) \)

Example 2.1. For all \( f \in \text{Nat} \rightarrow \text{Real} \), let \( f^* \in \text{Nat} \rightarrow \text{Real} \) be defined by \( f^*(2i) = f(i) \) and \( f^*(2i + 1) = f^*(2i) \). The loop perforation of index 2 of \( H(f) \) is precisely \( H(f^*) \). We sketch how to construct a difference \( D(f^*) \) and \( H(f^*) \) in dTT.

The step function \( \Delta f(x) = |f(x + 1) - f(x)| \) can be defined as \( \Delta f(x) = D[f](x + 1, 1) \) (where we take a difference between \( x, y \in \text{Nat} \) to be any positive real \( \geq |x - y| \)). For all \( x \in \text{Nat} \), the function \( d \) with \( d(2x) = 0 \) and \( d(2x + 1) = \Delta f(2x) \) yields then an element
$d \in (\Pi x \in \mathbb{N})D_{\text{Real}}(f(x), f^*(x))$. Using this and the derivative of $\hat{H}$ we can compute a distance $b \in D_{\text{Real}}(\hat{H}(f), \hat{H}(f^*))$ by $b = D(\hat{H}((f(i), d(i)))_{i=0,...,N})$.

More generally, we can construct a function $c \in (\Pi f, g \in \mathbb{N} \rightarrow \text{Real})(D_{\text{Nat} \rightarrow \text{Real}}(f, g) \rightarrow D_{\text{Real}}(\hat{H}(f), \hat{H}(g))$: from a distance $\epsilon \in D_{\text{Nat} \rightarrow \text{Real}}(f, g)$ we can define a function $c(\epsilon) \in (\Pi x \in \mathbb{N})D_{\text{Real}}(f(x), g(x))$ by letting $c(\epsilon) = J(f, g, \epsilon, [x]\hat{C}(f(x)))$, and we define $c(f, g, \epsilon)$ by replacing $d$ by $c(\epsilon)$ in $b$.

**Example 2.2 (distance function).** In presence of a type Bool for Booleans, with constants $0, 1 \in \text{Bool}$ and $\text{case}_C : \text{Bool} \rightarrow C \rightarrow C \rightarrow C$ (with $\text{case}_C(0, x, y) = x$ and $\text{case}_C(1, x, y) = y$), and with a constant $\epsilon \in D_{\text{Bool}}(0, 1)$, it is possible to define a distance function $d_{\text{A}} \in (\Pi x, y \in A)D_{\text{A}}(x, y)$ for all simple type $A$, by letting $d_{\text{A}} = \lambda xy.J(0, 1, \epsilon, [x]\hat{C}(\text{case}_A(w, x, y)))$. If we admit the equational rule $\text{case}_C(w, x, x) = x$, then using the Equation $\text{Jm}$ (see below) we can deduce that $d_{\text{A}xx}$ coincides with the self-difference $\hat{\epsilon}(x)$.

**Predicates in dTT.** An important property of dTT is that any predicate $(x \in \Phi_0)C(x) \in \text{dType}$ is obtained from a special family of binary predicates, defined below.

**Definition 2.1.** A predicate is said pure for $A$ if it is of the form $(x, y \in A)C(x, y)$ and one of the following holds:

- $C(x, y) = D_A(x, y)$;
- $A = B_1 \times B_2$ and $C(x, y) = B_1(\pi_1(x), \pi_1(y)) \times B_2(\pi_2(x), \pi_2(y))$, where $B_1$ is pure for $B_1$ and $B_2$ is pure for $B_2$;
- $A = B \rightarrow C$ and $C(x, y) = (\Pi z \in A)B(x, yz)$, where $B(z, z')$ is pure for $C$;
- $A = B \rightarrow B \rightarrow C$ and $C(x, y) = (\Pi x', y' \in B)(D_A(x', y') \rightarrow B(xx', yxy'))$, where $B(x', y')$ is pure for $C$.

**Lemma 2.1.** For any predicate $(z \in \Phi_0)C(z)$ there exists a pure predicate $(x, y \in A)C^0(x, y)$ and terms $(z \in \Phi_0)t, u \in A$ such that $C(z) = C^0(t, u)$.

For example, the predicate $(z \in A, w \in A)(\Pi x \in B)D_C(f(z, x), g(z, w, x))$ is obtained from the pure predicate $(y, y' \in B \rightarrow C)(\Pi x \in B)D_C(y, y'x)$ and the functions $(z, w \in A)\lambda xy.J(f(z, x), g(z, w, x)) \rightarrow B$.

**Remark 2.1.** In standard ITT, the identity type $I_A(x, x)$ yields a groupoid1, i.e. a category with invertible arrows. In dTT one can only prove that $D_A(x, x)$ has the structure of a deductive system (i.e. a non-associative category, see 2) in which for each arrow $a \in D_{\text{A}}(t, u)$ there is a “transpose” arrow $a^* \in D_{\text{A}}(u, t)$, with $\hat{c}(t) = \hat{\epsilon}(t)$.

**Remark 2.2.** In some formulation of intutionistic type theory (e.g. see 20) one finds a stronger version of the $\eta$-rule, which in the fragment dTT would read as follows:

\[
\begin{array}{c}
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\begin{array}{c}
a \in D_{\text{A}}(t, u) \\
J(t, u, a, [x]c(x, x, \hat{\epsilon}(x))) = c(t, u, a) \in C(t, u)
\end{array}
\end{array}
\end{array}
\]

However, in presence of 3 one can deduce that $a = \hat{\epsilon}(t) \in D_{\text{A}}(t, t)$ holds for all $t \in A$ and $a \in D_{\text{A}}(t, t)$, hence trivializing the interpretation of $D_{\text{A}}(x, x)$. Moreover, we can see that the rule also trivializes the interpretation of $D[f]$ as the “true” derivative, since it implies $D[f] = \lambda x.\hat{C}(fx)$ (i.e. $D[f] = \lambda x.0$ when $\hat{\epsilon}(u)$ is interpreted as the null error).

However, the following instance of 4 is valid in all models we consider:

\[
J(t, u, a, [x]b) = b \quad (x \notin \text{FV}(b))
\]

\[1\text{In fact, one obtains a groupoid by considering elements } p \in I_A(t, u) \text{ up to the equivalence induced by 3-dimensional homotopies in } I_{I_A(t, u)}(p, q).
\]

\[2\text{This is proved as follows: by letting } c(x, y, \epsilon) = \epsilon \text{ and } d(x, y, \epsilon) = \hat{\epsilon}(x), \text{ from } c(x, x, \hat{\epsilon}(x)) = d(x, x, \hat{\epsilon}(x)), \text{ we deduce } a = c(t, t, a) = c(t, t, a, [x]c(x, x, \hat{\epsilon}(x))) = c(t, t, a, [x]d(x, x, \hat{\epsilon}(x))) = d(t, t, a) = \hat{\epsilon}(t).
\]

\[3\text{It suffices to take } c(x, y, \epsilon) = \hat{C}(fx).
\]
Function Extensionality In ITT the function extensionality axiom essentially asserts that from a proof that $f$ and $g$ send identical points into identical points, one can construct a proof that $f$ is identical to $g$. In the following sections we will consider models which satisfy two variants of this axiom, namely

$$D_{A \rightarrow B}(f, g) = (\Pi x \in A)D_B(f(x), g(x)) \tag{FExt1}$$

$$D_{A \rightarrow B}(f, g) = (\Pi x, y \in A)(D_A(x, y) \rightarrow D_B(f(x), g(y))) \tag{FExt2}$$

Axioms (FExt1) (resp. (FExt2)) says that a difference between two functions $f, g \in A \rightarrow B$ is the same as a map from a point $x \in A$ into a difference between $f(x)$ and $g(x)$ in $B$ (resp. a map from a difference $\epsilon$ between two points of $A$ into differences between their respective images). Observe that, without these axioms, one can still construct programs

$$E_1 \in (\Pi f, g \in A \rightarrow B)(D_{A \rightarrow B}(f, g) \rightarrow (\Pi x \in A)D_B(f(x), g(x)))$$

$$E_2 \in (\Pi f, g \in A \rightarrow B)(D_{A \rightarrow B}(f, g) \rightarrow (\Pi x, y \in A)(D_A(x, y) \rightarrow D_B(f(x), g(y))))$$

given by $E_1 = \lambda f g \xi. \lambda x. \delta(\lambda x. \epsilon(f(x)))$ and $E_2 = \lambda f g \xi \eta. \lambda h. \epsilon(\lambda h. \delta(h))$.

Moreover, some of the models we consider will also satisfy the axiom below

$$D_{A \times B}(t, u) = D_A(\sigma_1(t), \pi_1(u)) \times D_B(\sigma_2(t), \pi_2(u)) \tag{CExt}$$

stating that a difference between pairs is a pair of differences. Even without (CExt), one can construct terms $C_1, C_2$ to and from the types above (yet they do not define an isomorphism).

In presence of one or more of the extensionality axioms, it makes sense to consider further computational rules for the operators $J$ and $D$ (for instance, the rule stating that for a higher-order function $f \in A \rightarrow B$, $\partial(f) = D[\delta(f)]$, that we discuss in the Appendix.

3 Models of dTT and Weak Factorization Systems

The by now standard semantics of the identity type is based on weak factorization systems (in short, WFS). A WFS is a category endowed with two classes of arrows $\mathcal{L}$ and $\mathcal{R}$, such that any arrow factorizes as the composition of a $\mathcal{L}$-arrow and a $\mathcal{R}$-arrow (the typical example is $\text{Set}$, with $\mathcal{L}$ being the class of surjective functions and $\mathcal{R}$ the class of injective functions).

The goal of this section is to introduce a workable notion of model for dTT, as formal basis for the concrete models illustrated in the next sections, and to relate it to the standard WFS semantics of ITT. We will first present a basic setting, that we call a dTT-category, which allows for the interpretation of dTT (and roughly follows [35]). We then introduce a slight variant of WFS, that we call $\mathcal{U}$-WFS, where $\mathcal{U}$ is some monoidal functor. This variant is adapted to the ontology of dTT, where one has two distinct families of terms, and only requires that the ($\mathcal{U}$-image of the) arrows from the first family factor through the arrows of the second family. We finally show that any dTT-category gives rise to a $\mathcal{U}$-WFS.

The fundamental example of a dTT-category will be the context category of dTT, that is, the category $\text{Ctx}$ with objects being contexts and arrows $(x \in \Phi_0 \mid \epsilon \in \Phi_1(x)) \rightarrow (y \in \Psi_0 \mid \delta \in \Psi_1(y))$ being sequences $(t \mid a)$ of $\langle \beta \eta \rangle$-equivalence classes of terms such that $(x \in \Phi_0)t_i \in A_i$ and $(x \in \Phi_0 \mid \epsilon \in \Phi_1(x))a_j \in C_j(t)$ holds for all $A_i \in \text{Type occurring in } \Psi_0$ and $C_j(x) \in \text{dType occurring in } \Psi_1$. We let $\text{Ctx}_0$ be the full subcategory of $\text{Ctx}$ made of contexts of the form $(x \in \Phi_0 \mid \epsilon \in \Phi_1(x))$, and $\iota : \text{Ctx}_0 \rightarrow \text{Ctx}$ indicate the inclusion functor.

When considering the sub-exponential simply typed $\lambda$-calculus $\text{STAC}_\Gamma$ as base language, we let $\text{Ctx}_\Gamma$ indicate the category of $\text{STAC}_\Gamma$-typed terms and $H : \text{Ctx}_\Gamma \rightarrow \text{Ctx}_0 \rightarrow \text{Ctx}$ indicate the associated embedding inside $\text{Ctx}$ (where $\text{Ctx}_\Gamma \rightarrow \text{Ctx}_0$ corresponds to the “forgetful” embedding of $\text{STAC}_\Gamma$ inside $\text{STAC}$ - for more details, see the Appendix).

Observe that the category $\text{Ctx}_\Gamma$ is cartesian closed, while $\text{Ctx}_0$ is symmetric monoidal closed and $\text{Ctx}$ is only cartesian. Hence, the basic data to interpret dTT will be given by a strict monoidal functor $\mathcal{U} : \mathcal{C}_0 \rightarrow \mathcal{C}$ between a symmetric monoidal closed category $\mathcal{C}_0$ (interpreting either $\text{STAC}$ or $\text{STAC}_\Gamma$) and a cartesian category $\mathcal{C}$ (interpreting the difference terms). We will use $\Gamma, \Delta$ for the monoidal product of $\mathcal{C}_0$.

While $\mathcal{C}_0$ only accounts for simple types, $\mathcal{C}$ needs to have enough structure to account for type dependency: for all object $\Gamma$ of $\mathcal{C}_0$, we consider a collection $\mathcal{P}(\Gamma)$ of predicates over $\Gamma$.
such that, for all \( P \in \mathcal{P}(\Gamma) \), there exists an object \( \Gamma \mid P \) of \( \mathcal{C} \) and an arrow \( \pi_{\Gamma} : \Gamma \mid P \to UT \) called the projection of \( P \). We also require that for any predicate \( P \in \mathcal{P}(\Gamma) \) and \( f : \Delta \to \Gamma \), the pullback \((U f)^{\sharp}(\Gamma \mid P)\) exists and is generated by some object \( f^{\sharp}P \in \mathcal{P}(\Delta) \):

\[
\Delta \mid f^{\sharp}P \xrightarrow{f^{\ast}} \Gamma \mid P \\
\downarrow \pi_{\Delta} \downarrow \downarrow \pi_{\Gamma} \\
U \Delta \xrightarrow{U f} UT
\]

Moreover, we require that the equalities \( \text{id}f^{\sharp}P = P \), \((g \circ f)^{\sharp}P = g^{\sharp}(f^{\sharp}P)\), \( \text{id}f^{\ast} = \text{id}f \) hold. In the case of \( \text{Ctx} \), \( \mathcal{P}(\Phi_0) \) is the set of predicates \( x \in \Phi_0) \mathcal{C}(x) \in \text{dType} \). Given a predicate \((x, y \in A) \mathcal{C}(x, y) \) and simply typed terms \((t, u) : (y \in \Psi_0) \to (x, y \in A) \), the pullback \((t, u)^{\ast} \mathcal{C}\) corresponds to the predicate \((y \in \Psi_0) \mathcal{C}(t(y), u(y)) \).

Given predicates \( P \in \mathcal{P}(\Gamma) \) and \( Q \in \mathcal{P}(\Delta) \), we indicate an arrow \( h \in \mathcal{C}(\Gamma \mid P, \Delta \mid Q) \) as \( (h_0 \mid h_1) \) if \( h = h_0^{\circ} h_1 \), for some \( h_0 \in \mathcal{C}_0(UT, U \Delta) \) and \( h_1 \in \mathcal{C}(\Gamma \mid P, \Gamma \mid h_0^{\ast}Q) \) occurring in a commuting diagram as below.

\[
\begin{array}{c}
\Gamma \mid P \\
\downarrow \pi_{\Gamma} \\
UT
\end{array}
\begin{array}{c}
\Gamma \mid h_0^{\ast}Q \\
\downarrow \pi_{\Gamma} \\
U \Delta
\end{array}
\begin{array}{c}
\Delta \mid Q \\
\downarrow \pi_{\Delta} \\
UT
\end{array}
\]

In \( \text{Ctx} \) this precisely says that an arrow \((t \mid a) : (x \in \Phi_0) \mid (e \in \Phi_1(x)) \to (y \in \Psi_0) \mid (\delta \in \Psi_1(y)) \) is composed of arrows \( t \in \Phi_0 \to \Psi_1 \) and \( a \in \Phi_1(x) \to \Psi_1(t(x)) \).

To handle the difference types we need to make some further requirements. First, we consider a sub-family \( \mathcal{P}^0(\Gamma) \subseteq \mathcal{P}(\Gamma, \Gamma) \) of binary predicates, that we call pure predicates, which generates the family \( \mathcal{P}(\Lambda) \), in the sense that for all object \( \Gamma \) and predicate \( P \in \mathcal{P}(\Gamma) \) there exists an object \( \Delta \), a pure predicate \( P^0 \in \mathcal{P}(\Delta) \subseteq \mathcal{P}(\Delta, \Delta) \) and \( f \in \mathcal{C}_0(\Gamma, \Delta) \) such that \( P = f^{\ast}P^0 \). In the case of \( \text{Ctx} \) this is precisely what is asserted by Lemma 2.7.

For any \( \Gamma \), we require a choice of a pure predicate \( \Gamma^0 \in \mathcal{P}^0(\Gamma) \). The introduction rule requires the existence of an arrow \( \tau_{\Gamma} : UT \to (\Gamma, \Gamma \mid \Gamma^0) \) such that \( \pi_{\Gamma} \circ \tau_{\Gamma} \) coincides with the diagonal \( \delta_{UT} : UT \to (U \Gamma, \Gamma \mid U \Gamma) = U \times UT \). In \( \text{Ctx} \) \( \mathcal{P}(\Phi_0) \) is \( (x, y \in \Phi_0) \mid e \in D_{\Phi_0}(x, y) \), where \( D_{\Phi_0}(x, y) \) is the list of all \( D_{A_i}(x_i, y_i) \), for \( A_i \) occurring in \( x \Phi_0 \) and \( r_{\Phi_0} \) is given by \( (x, x) \mid \hat{c}(x) \) (where \( \hat{c}(x) = (\hat{c}(x_1), \ldots, \hat{c}(x_k)) \)). Actually, in order to handle contexts properly, we must consider a slightly more complex condition (see the Appendix). To handle the elimination rule, for any binary predicate \( P = f^{\ast}P^0 \in \mathcal{P}(\Gamma, \Gamma) \) and commutative diagram

\[
\begin{array}{c}
UT \\
\downarrow \tau_{\Gamma} \\
\Gamma, \Gamma \mid \Gamma^0
\end{array}
\begin{array}{c}
\Gamma, \Gamma \mid P \\
\downarrow \pi_{\Gamma, \Gamma} \\
UT \times UT
\end{array}
\]

we require the existence of a diagonal filler \( j : (\Gamma, \Gamma \mid \Gamma^0) \to (\Gamma, \Gamma \mid P) \) making both triangles commute. In \( \text{Ctx} \), \( P = f^{\ast}P^0 \) is a predicate \((x, y \in \Phi_0) \mathcal{C}(x, y) \mathcal{C}(x, y) = \mathcal{C}(f_1(x, y), f_2(x, y)) \), \( \mathcal{C} \) is of the form \((x, x \mid c'(x)) \), where \( c'(x) \in \mathcal{C}(x, x) \), and a diagonal filler is provided by \( j = (x, y) \mid J(x, y, e, [x]c') \). The commutation of the upper triangle \( j \circ \tau_{\Gamma} = c \) coincides then with the \( \beta \)-rule. The validity of the \( \eta \)-rule corresponds to the fact that, when \( P = \Gamma^0 \), \( f = \pi_1, g = \pi_2 \) and \( c = \tau_{\Gamma} \), \( j \) coincides with the identity arrow \( \text{id}_{\Gamma, \Gamma} \). We will not require the \( \eta \)-condition in general. Again, to handle contexts and substitutions properly, we must consider a slightly more complex construction, together with a few coherence conditions for \( r_{\Gamma} \) and \( j \) (see [4.35]), but we discuss these more technical aspects in the Appendix.

Finally, we must require that \( \mathcal{C} \) has enough structure to interpret the dependent products present in the fragment dTT; we describe this structure in the Appendix.

We let a dTT-category be a strict monoidal functor \( U : \mathcal{C}_0 \to \mathcal{C} \) together with collections of predicates \( \mathcal{P}(\Lambda), \mathcal{P}^0(\Lambda) \) and of difference structures \((\mathcal{D}, r_{\mathcal{D}}, \ldots)\) satisfying the properties above. The following proposition assures that one can interpret dTT in any dTT-category.

**Proposition 3.1.** For any dTT-category \( U : \mathcal{C}_0 \to \mathcal{C} \), if \( \mathcal{C}_0 \) is cartesian closed, any map \( m \) from base type variables to \( \text{Ob}(\mathcal{C}_0) \) extends into functors \( \llbracket m \rrbracket : \text{Ctx}_0 \to \mathcal{C}_0 \) and \( \llbracket m \rrbracket : \text{Ctx} \to \mathcal{C} \), satisfying \( U \circ \llbracket m \rrbracket = (\llbracket m \rrbracket \circ \iota) \) and preserving all relevant structure. If \( \mathcal{C}_0 \) is symmetric monoidal closed, the same holds with \( \text{Ctx}_0 \) replaced by \( \text{Ctx} \) and \( \iota \) replaced by \( H \).
To conclude our general presentation of the semantics of dTT, we show how it relates to WFS. We recall that, given a category $C$ and two arrows $f \in C(A, B)$ and $g \in C(C, D)$, $f$ is said to have the leftlifting property with respect to $g$ (and $g$ is said to have the rightlifting property with respect to $f$), if for every commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow{f} & & \downarrow{g} \\
B & \xrightarrow{k} & D
\end{array}
\]

there exists a diagonal filler $j \in C(B, C)$ making both triangles commute. Given a set $\mathcal{F}$ of arrows in a category, we let $\mathcal{F}^h$ (resp. $\mathcal{F}^h$) indicate the set of arrows $g$ such that any arrow in $\mathcal{F}$ has the left (resp. right) lifting property with respect to $g$. The following notion generalizes usual WFS:

**Definition 3.1.** Let $U : C \to \mathbb{D}$ be a functor. An $U$-weak factorization system (in short, $U$-WFS) for $C$ inside $\mathbb{D}$ is a pair of classes of maps $(\mathcal{L}, \mathcal{R})$ of $\mathbb{D}$ such that (1) for every morphism $f$ of $C$, $Uf = pf \circ if$, with $if \in \mathcal{L}$ and $pf \in \mathcal{R}$, and (2) $\mathcal{L}^h = \mathcal{R}$ and $\mathcal{L} = \mathcal{R}^h$.

Observe that a weak factorization system in the usual sense is just a 1d-WFS.

When a functor $U : C_0 \to C$ yields a dTT-category, it is possible to construct a $U$-WFS $(\mathcal{L}_\mathcal{P}, \mathcal{R}_\mathcal{P})$ for $C_0$ inside $C$ by letting $\mathcal{L}_\mathcal{P} = \mathcal{P}^h$ and $\mathcal{R}_\mathcal{P} = \mathcal{P}^h$, where $\mathcal{P}^h$ is made of all arrows obtained by composing the arrows $p_\mathcal{P} : \Gamma \to \Delta$ in $C_0$ given by $pf \circ i_f$, where $pf = p_2 \circ p_\mathcal{P} : \Gamma, \Delta | \Delta_f, \Delta_j$ is a suitable pullback, and $i_f$ is the arrow obtained by the universality of pullback in the diagram below:

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\Delta_f} & \Delta \\
\downarrow{p_\mathcal{P}} & & \downarrow{\pi_{\Delta,j}} \\
\Gamma \times \Delta & \xrightarrow{p_\mathcal{P}, \pi_{\Delta,j}} & \Delta \times \Delta
\end{array}
\]

where $\Delta_f = (f, \Delta)^h(\Delta^D)$. To show that $r_f \in \mathcal{L}_\mathcal{P}$ we must rely on the difference structure in an essential way: the required diagonal filler is obtained by an arrow of the form $j$. We describe this construction (which follows the argument from [16]) in the Appendix.

**Theorem 3.2.** For any dTT-category $U : C_0 \to C$, with collections of predicates $\mathcal{P}(\_)$, $\mathcal{P}^h(\_)$, the pair $(\mathcal{L}_\mathcal{P}, \mathcal{R}_\mathcal{P})$ forms a $U$-WFS of $C_0$ inside $C$.

## 4 Metric Preservation

We start our parade of models of dTT by considering metric models focusing on program sensitivity. In several situations it is important to know that a program is not too sensitive to small changes in the input. A key example is differential privacy: if $f : db \to \text{Real}$ is a program producing some aggregated information from some database $db$ (e.g. $f$ outputs the percentage of LGBTIQ+ people among the students of a given university), we wish the result of $f$ not to depend too much on any single item of $db$, so that information about single individuals cannot be leaked from the outputs of $f$.

A standard way to capture sensitivity is through the Lipschitz-condition: a function $f$ between metric spaces $(X, a)$ and $(Y, b)$ is $r$-Lipschitz, for some positive real $r$, when it satisfies $b(f(x), f(y)) \leq r \cdot a(x, y)$ for all $x, y \in X$. It is well-known that from a $r$-Lipschitz function $f : db \to \text{Real}$ one can obtain, by adding Laplace-distributed noise, a randomized function $f^* : db \to \text{Real}$ which is $r$-differentially private.$^4$

The type system Fuzz [30] was designed to ensure that well-typed programs correspond to Lipschitz functions. Formally, it is a variant of bounded linear logic [15], i.e. an affine

$^4$Formally, this means that for all two inputs $x, x' \in db$ that differ by at most one parameter and for all $S \subseteq \mathbb{R}$, the probability $P[f^*(x) \in S]$ that $f^*(x) \in S$ is bounded by $e^{r} : P[f^*(x') \in S]$. 

8
simply typed $\lambda$-calculus with a bounded exponential $!, A$, where a program $f \in !, A \rightarrow B$ corresponds to a $r$-Lipschitz function. We consider here a basic fragment STAC$^2$ of Fuzz (described in the Appendix).

Fuzz admits a natural and simple semantics in the symmetric monoidal closed category $\text{Met}$ of metric spaces and non-expansive (i.e. 1-Lipschitz) maps (with monoidal product $(X, a) \otimes (Y, b) = (X \times Y, a \circ b)$). In particular, the bounded exponential $!, A$ is interpreted as the re-scaling $rX$ of a metric space $(X, a)$ (i.e. with $ra(x, y) = r \cdot a(x, y)$), so that a non-expansive map from $rX$ to $Y$ is the same as a $r$-Lipschitz function from $X$ to $Y$.

We construct a model of a variant of dTT where we take program terms to be ST

$$\text{ST}$$

factors as $\text{ST}[\Gamma]$, see the Appendix.

For any metric space $X$, the pure predicate $X$ is made of pullbacks of $X$, where $X, Y \in X \otimes X$, with $X, y \in X$ and $X, x \in X$, for some metric space $(Y, b)$ and $X, g \in X$, and for any metric $c : X \rightarrow (X \otimes X) \rightarrow X$, we can define a diagonal filler $j : (X \otimes X \otimes X) \rightarrow (X \otimes X \otimes X)$ by

$$j(\langle X, y, x \rangle, r) = \langle \langle X, y, r \rangle, c(x) \rangle$$

In fact, from from $\langle f, g \rangle \in X \otimes X \otimes X$ we deduce $b(f(x, y), f(x', y')) + b(g(x, y), g(x', y')) \leq a(x, x') + b(y, y')$. Hence, from $c(x) \in b(f(x, x), g(x, x))$ and $b(f(x, y), f(x, x)) + b(g(x, y), g(x, x)) \leq a(x, x) + a(x, y) \leq r$, we deduce $j(\langle X, y, x \rangle, r) = \langle \langle X, y, r \rangle, c(x) \rangle$.

Remark 4.1. When $h \in \text{Met}(rX, Y)$ interprets some $r$-Lipschitz program $t \in !, A \rightarrow B$, the predicate $DB(t, x, y)$ corresponds to the pullback $\langle h \circ \pi_1, h \circ \pi_2 \rangle : X \rightarrow X \otimes X$, and the derivative $D[h]$ is interpreted then by the map $x, y, h \sim \epsilon$, as desired.

Remark 4.2. From Theorem 3.2 it follows that given metric spaces $(X, a)$ and $(Y, b)$, any $f \in \text{Met}(rX, Y)$ factors as $X \rightarrow \bigprod_{x \in X} Y$, where $i(x) = \langle X, x \rangle, f(x), 0 \rangle$.

Example 4.1. dTT can be used to formalize meta-theoretical reasoning about Fuzz as discussed in [20, 21]. For instance, we might extend simple types with the probability monad $\Box A$, adding suitable primitives. Then, by interpreting the type $\Box A(t, u)$ with the metric

$$d(\delta_1, \delta_2) = \frac{1}{r} \sup_{x \in X} \left| \ln \left( \frac{\delta_1(x)}{\delta_2(x)} \right) \right|$$

(with $\delta_1, \delta_2$ distributions over $A$), for any randomized program $f \in !, A \rightarrow \Box B$, the statement $D[f] \in \Pi x, y \in X)(DB(t, x, y) \rightarrow DB(f, x, y))$ expresses that for all $x, y \in A, r \geq d(x, y)$ and $b \in B$, $[P[f(x) = b] - P[f(y) = b]] \leq r^2$, that is, that $f$ is a re-differentially private function.

Example 4.2. Suppose $H(f)$ computes the average of the simulations $f(0), \ldots, f(N)$, i.e. $H(f) = \sum_{i=0}^{N} f(i)$; note that $H(x) = \sum_{i=0}^{N} x_i : \text{Real}^{N+1} \rightarrow \text{Real}$ is $\frac{1}{N}$-Lipschitz, and thus $D[H](x, y) \leq \frac{1}{N} \cdot \sum_{i=0}^{N} y_i$. We deduce then that for all $r$-Lipschitz function $f \in !, \text{Nat} \rightarrow \text{Real}$, the perforation error $b \in \text{DReal}(H(f), H(f^*))$ computed in Example 2.1 corresponds to $\frac{1}{N+1} \cdot \sum_{i=0}^{N} r \cdot D[f](x, y) = \frac{1}{N+1} \cdot r$.

---

5Actually, to handle the higher-order structure, we must consider parameterized (pseudo-)metric spaces over $X$, see the Appendix.
5 Differential Logical Relations

When studying approximate program transformations like loop perforation, the Lipschitz condition is often too restrictive. In fact, even basic operations of the simply typed \(\lambda\)-calculus can make this property fail: while the binary function \(f(k, x) = k \cdot x : \text{Real}^2 \rightarrow \text{Real}\) is \(|k|\)-Lipschitz in \(x\) for all \(k \in \text{R}\), the unary function \(g(x) = x^2\) obtained by “contracting” the variables \(k\) and \(x\) already fails to be Lipschitz. In fact, the distance between \(g(x)\) and \(g(x + \epsilon)\) is bounded by \(2|x|\epsilon + \epsilon^2\), hence not proportional to \(\epsilon\). Indeed, this kind of issues is due to the fact that \(\text{Met}\) is not a cartesian closed category, that is, a model of full STAC, but only of its sub-exponential variant Fuzz.

The theory of differential logical relations \([13, 23, 28]\) (in short, DLR), has been developed to overcome this kind of problems when investigating approximate transformations in STAC. A DLR is a ternary relation \(\rho \subseteq X \times L \times X\) relating the elements of some set \(X\) with the values of some complete lattice \(L\) of “errors over \(X\)”; intuitively, \(\rho(x, \epsilon, y)\) is to be read as the fact that the error of replacing \(x\) by \(y\) is bounded by \(\epsilon\). As the name suggests, DLR generalize usual logical relations, which can be seen as DLR where \(L = \{0 < 1\}\). Yet, due to the arbitrary choice of \(L\), a distance between two programs needs not be a Boolean nor a positive real (as in metric semantics); typically, a distance between two functional programs is itself a function, tracking distances in input into distances in output.

Since \(L\) is a complete lattice, for all \(x, y \in X\), one can define a function \(\| \cdot \|_\rho : X \times X \rightarrow L\) where \(\|x, y\|_\rho = \inf \tilde{\rho}(x, y)\), with \(\tilde{\rho}(x, y) = \{\epsilon \in L \mid \exists \delta \leq \epsilon \text{ s.t. } \rho(x, \delta, y)\}\). For instance, if we consider the DLR \((\mathbb{R}, \mathbb{R}^\geq_0, \rho_{\text{Euc}})\), where \(\rho_{\text{Euc}}(r, v, s)\) holds if \(v \geq |r - s|\), the associated distance function is the Euclidean metric. However, \(\|x, y\|_\rho\) needs not be a metric in general: first of all, the self-distances \(\|x, x\|_\rho\) (that we note simply as \(\|x\|_\rho\)) need not be zero (i.e. the bottom element of \(L\)); moreover, \(\|x, x\|_\rho\) needs not satisfy the usual triangular law of metric spaces (for a detailed comparison between DLR and - generalized \([33]\) - metric spaces, see \([28]\)). Here we will restrict our attention to separated DLR, i.e. such that \(\|i\|_\rho = \tilde{\rho}(i,j)\) (or \(\|j\|_\rho = \tilde{\rho}(i,j)\)) implies \(i = j\).

A map of DLR \((X, L, \rho)\) and \((Y, M, \mu)\) is given by a function \(f : X \rightarrow Y\) (hence no Lipschitz or other continuity conditions are asked) together with an auxiliary map \(\varphi : X \times X \times L \rightarrow Y\) which, intuitively, tracks errors in input into errors in output; more formally, \(\varphi\) must satisfy, for all \(x, y \in X\) and \(\epsilon \in L\), that if \(\rho(x, \epsilon, y)\) holds, then both \(\mu(f(x), \varphi(x, y, \epsilon), f(y))\) and \(\mu(f(x), \varphi(x, y, \epsilon), f(y))\) also hold. (Separated) DLR and their maps form a cartesian closed category DLR (see the Appendix, and \([13, 28]\) for further details).

The presence of the auxiliary map \(\varphi\) is what ensures the “transport” of errors: if \(\rho(t, \epsilon, u)\) holds for some \(t, u\) of type \(A\) and the context \(C[(\_\_)] : A \rightarrow B\) admits an auxiliary map \(\varphi\), then \(\varphi(t, u, \epsilon)\) provides an error bound between \(C[t]\) and \(C[u]\). For example, to the function \(g(x) = x^2\) one can associate the auxiliary map \(\varphi_{\rho}(x, y, \epsilon) = 2|x|\epsilon + \epsilon^2\).

To model dTT in terms of DLR we will interpret \(D_A(t, u)\) as the set \(\tilde{\rho}(t, u)\) of differences between \(t\) and \(u\); hence, the self-differences \(\hat{\epsilon}(t)\) will correspond to \(\|t\|_\rho\) and \(\|f\|_\rho\) will provide each program \(f\) with the auxiliary map \(x, y, \epsilon \mapsto \sup(\|f(x), f(z)\|_\rho \mid z \in X \wedge \rho(x, \epsilon, z))\). Moreover, due to the higher-order structure of DLR (recalled in the Appendix) the DLR models satisfies the extensionality axioms \([\text{Ext} 1]\) and \([\text{Ext} 2]\), as well as the equational rule \(\hat{\epsilon}(f) = D[f]\), for \(f\) a higher-order function (see \([25]\), Lemma IV.1).

We provide a sketch of the dTT-category structure of the forgetful functor \(U : \text{DLR} \rightarrow \text{Set}\) (given by \(U(X, L, \rho) = X\) and \(U(f, \varphi) = f\)), described in detail in the Appendix. For any DLR \((X, L, \rho)\), a pure predicates \(P \in \mathcal{P}(X)\) is just a DLR \((X, L, \rho)\), with \((X \times X \mid P) = \prod_{x,y \in X} \tilde{\rho}(x, y)\) and projection \(\pi_{X \times X} : (X \times X \mid P) \rightarrow X \times X\). For any set \(Y\), \(\mathcal{P}(Y)\) is made of pullbacks \((f, g)^2Q\), where \(Q = (X, L, \rho) \in \mathcal{P}(X)\) and \(f, g : Y \rightarrow X\), with \((Y \mid (f, g)^2Q) = \prod_{y \in Y} \tilde{\rho}(f(y), g(y))\) and associated projection \(\pi_Y : (Y \mid (f, g)^2Q) \rightarrow Y\).

For any separated DLR \((X, L, \rho), X^D \in \mathcal{P}(X)\) is \((X, L, \rho)\) itself, with \(\pi_X(x) = \langle(x, x), \|x\|_\rho\rangle\); moreover, for any binary predicate \(P \in (f, g)^2P \in \mathcal{P}(X \times X)\) (with \((X \times X \mid P) = \prod_{x,y \in X} \tilde{\rho}(f(x, y), g(x, y))\)) and function \(c(x) = \langle(x, x), c(x)\rangle : X \rightarrow (X \times X \mid P)\), so that \(c(x) \in \tilde{\rho}(f(x, x), g(x, x))\), we can define a diagonal filler \(j : (X \times X \mid X^D) \rightarrow (X \times X \mid P)\) by...
\[(j((x,y),\epsilon)))_{2} = \sup\{\epsilon'((x,z), g(x,z)) \mid z \in X \land \rho(x,\epsilon, z) \in \hat{\mu}(f(x,y),g(x,y))\}\]

The \(\beta\)-rule \(j((x,y),\langle (x,z),\mu \rangle)) = \epsilon(x)\) follows from the fact that \(\rho\) is separated. The \(\eta\)-rule holds only if \(\rho\) is complete, i.e. for all \(x \in X\) and \(\epsilon \in L\), \(\sup\{\hat{\mu}(x,y) \mid \rho(x,\epsilon, y) \}=\epsilon\).

**Remark 5.1.** When \(h : X \to Y\) interprets a program \(t \to A \to B\), the predicate \(D_B(f(x,y))\) corresponds to the pullback of \(D_B(h(x))\) along \(f(x,y) = h(y)\); moreover, since \(\lambda x.\hat{c}(x)\) is the function \(\hat{c}(x) = [h(x)]_{\mu}\), \(D_B\) yields then the map \(x,y,\epsilon \mapsto \sup\{[h(x),h(z)]_{\mu} \mid z \in X \land \rho(x,\epsilon, z)\}\), as desired.

**Remark 5.2.** From Theorem \(\ref{tcat}\) it follows that given DLR \((X,L,\rho)\) and \((Y,M,\mu)\), any function \(f : X \to Y\) factors as \(X \xrightarrow{i} \prod_{x \in X,y \in Y} \hat{\mu}(f(x,y),y) \xrightarrow{\pi_{x,y}} Y\) where \(i(f) = \langle (x,f(x))\rangle, [f(x)]_{\mu}\).

**Example 5.1.** Suppose \(\hat{\mathbb{H}}(\overline{x}) = (\sum_i x_i)^2\), so that its derivative \(\hat{\mathbb{D}}(\hat{\mathbb{H}})\) is interpreted by the function \(\varphi(x, \overline{c}) = 2(\sum x)(\sum r) + (\sum r)^2\) (where \(\sum y = \sum_{i=0}^{N} y_i\)). Then for any function \(f \in \text{Nat} \to \text{Real}\) with auxiliary map \(\psi(n,\overline{d})\), the perforation error \(b \in D_{\text{Real}}(\hat{\mathbb{H}}(f),\hat{\mathbb{H}}(f^*)\) from Example \(\ref{ex:derivative}\) corresponds to \(\varphi([f(i),\psi(i,1)])_{i=0,...,N}\).

**Example 5.2.** By interpreting \(\text{Bool}\) as the set \(\{0,1\}\) and \(D_{\text{Bool}}\) as the DLR \((2,2,\rho_{2})\) corresponding to the discrete metric, the distance function \(d_\mathbb{A} \in (\Pi_{x,y}(x,y) A_{\mathbb{A}}(x,y))\) from Example \(\ref{ex:distance}\) yields for any DLR \((X,L,\rho)\) the function \(x,y \mapsto [\|x,y\|_{\rho}] : X \times X \to \mathbb{L}\).

## 6 Change Structures and Incremental Computation

In many cases in programming it happens that, after running a program \(f\) on some input \(i\), one needs to re-run \(f\) over some slightly changed input \(i'\); incremental computation is about finding ways to optimize this second computation without having to re-run \(f\) from scratch on the novel input. For example (we take this example from \([11]\)) suppose \(f\) is a set of natural numbers \(x\). Suppose \(f\) has been run on \(x = \{1,2,3,4,5\}\), and now needs to be re-run on \(x' = \{2,3,4,5\}\); then we can compute \(f(x')\) incrementally as follows: first, let the \(\text{change}\) between \(x\) and \(x'\) be the pair of bags \(\text{dx} = \{-1\}\) describing what has to be changed to turn \(x\) into \(x'\). Then the change \(\text{dy} = \{-1\}\) between \(f(x)\) and \(f(x')\) is the value \(\text{dy} = -1\) and \(f(x')\) can be computed simply by adding \(\text{dy}\) to \(f(x)\). In particular, the operation \(df(x,\text{dx})\) taking a bag and a change and returning the change \(\text{dy}\) is called (once more!) a derivative of \(f\) (we call it a change derivative for clarity), as it describes the change to get from \(f(x)\) to \(f(y)\) as a function of \(x\) and the change \(\text{dx}\).

In \([11]\) these ideas have been turned into a change semantics for the simply typed \(\lambda\)-calculus, which was later generalized and simplified through the theory of change actions \([4,3]\). These approaches have been applied to model different forms of discrete and automated differentiation \([11,21]\), and more recently related to models of the differential \(\lambda\)-calculus \([23]\). We focus here on change structures from \([11]\) since they have a natural higher-order structure.

A change structure (in short, CS) is a tuple \((X, \Delta_X, \oplus, \otimes)\) where \(X\) is a set, for all \(x \in X\), \(\Delta_X x\) is a set of changes over \(x\), and \(\otimes : X \times \Delta_X \to X\) and \(\oplus : X \times X \to \Delta_X\) are operations satisfying (1) \(x \otimes y \in \Delta_X y\) and (2) \(x \oplus (y \otimes x) = y\).

Any function \(f : X \to Y\) admits a change derivative \(df : X \times \Delta_X \to \Delta Y\) defined by \(df(x,\text{dx}) = f(x \oplus \text{dx}) \otimes f(x)\) and satisfying \(f(x \otimes \text{dx}) = f(x) \oplus df(x,\text{dx})\). In other words, \(df(x,\text{dx})\) describes the change needed to pass from \(f(x)\) to \(f(x \oplus \text{dx})\).

For all \(x \in X\), we let \(O_\text{dx} = x \oplus x \in D_X x\); notice that \(x \oplus O_\text{dx} = x\) and \(df(x, O_\text{dx}) = O_{f(x)}\). Moreover, whenever \(x \otimes \text{dx} = y\), we let \(O_{\text{dx}} := x \oplus y\). Finally, given \(dx \in \Delta_X x\) and \(dy \in \Delta_X (x \oplus dx)\), we let \(dx + dy := ((x \oplus dx) \oplus dy) \otimes x\).

The CS semantics of dTT will interpret \(D_A(\_,\_\_)\) as the type of changes over \(A\), and a judgement \(a \in D_A(t,u)\) as expressing the fact that \(a\) is a change from \(t\) to \(u\) (i.e. that \(t \otimes a = u\)). Self-differences \(\partial(t)\) will correspond to the null change \(O_t\), and \(D[f]\) will correspond to the change derivative \(df\). Moreover, due to the higher-order structure of change structures (recalled in the Appendix), this model satisfies the extensionality axiom \((\xi\text{Ext})\).

We provide a sketch of the dTT-category structure of the forgetful functor \(U : CS \to \text{Set}\), leaving as usual most details to the Appendix. For any CS \(X\), a pure predicate \(P \in \mathcal{P}^\mathbb{P}(X)\)
is a CS \( P = (X, \Delta_X, \oplus, \ominus) \), with \( (X \times X \mid P) = \bigsqcup_{x,y \in X} \Delta_X(x,y) \) (where \( \Delta_X(x,y) = \{dx \in \Delta_X(x \mid x \oplus dx = y)\} \)), and projection \( \pi_X : (X \times X \mid P) \to X \times X \). For any CS \( Y \), \( \mathcal{P}(Y) \) is made of pullbacks \( \langle f,g \rangle^2P \), where \( P \in \mathcal{P}(X) \) is a CS \( P = (X, \Delta_X, \oplus, \ominus) \), and \( (Y \mid \langle f,g \rangle^2P) = \bigsqcup_{y \in Y} \Delta_X(f(y),g(y)) \), with associated projection \( \pi_Y : (Y \mid \langle f,g \rangle^2P) \to Y \).

For any CS \( X \), \( X^0 \in \mathcal{P}(X) \) is X itself, with \( r_X(x) = \langle \langle x,x \rangle, 0 \rangle \); moreover, for any binary predicate \( P = \langle f,g \rangle^2P \in \mathcal{P}(X \times X) \) (with \( (X \times X \mid P) = \bigsqcup_{x,y \in X} \Delta_Y(f(x,y),g(x,y)) \)) and function \( c : X \to (X \times X \mid P) \), where \( c(x) = \langle \langle x,x \rangle, c'(x) \rangle \), with \( c'(x) \in \Delta_Y(f(x,x),g(x,x)) \), a diagonal filler \( j : (X \times X \mid X^0) \to (X \times X \mid P) \) is defined by

\[
(j(\langle \langle x,x \rangle, dx \rangle))_3 = df(\langle \langle x,x \rangle, (0_x, dx) \rangle) + c'(x) + dg(\langle \langle x,x \rangle, (0_x, dx) \rangle) \in \Delta_Y(f(x,x),g(x,x))
\]

One can check that \( (j(\langle \langle x,x \rangle, 0 \rangle))_3 = c(x) \); the validity of the \( \eta \)-rule requires the further assumption that for all \( x, y, y \ominus x \) is the unique change from \( x \) to \( y \).

**Remark 6.1.** When \( h : X \to Y \) interprets some program \( f \in A \to B \), the interpretation of \( D[f] \) corresponds to constructing a diagonal filler as above with \( f(x,y) = h(x) \), \( g(x,y) = h(y) \) and \( c' = 0_{h(x)} \); then one obtains the map \( x, y, dx \mapsto dh(x, dx) \) as desired.

**Remark 6.2.** From Theorem \( 3.2 \) it follows that given change structures on \( X, Y \), any function \( f : X \to Y \) factors as \( X \xrightarrow{\delta} \bigsqcup_{x \in X, y \in Y} \Delta_Y(f(x,y), y) \xrightarrow{\pi_{\Delta,1}} Y \) where \( i_f(x) = \langle \langle x, f(x) \rangle, 0_{f(x)} \rangle \).

**Example 6.1.** Consider the change structure on \( \text{Real} \) where \( \Delta_{\text{Real}}x = \mathbb{R} \), \( \oplus \) is addition and \( \ominus \) is subtraction. Let \( f, g : \text{Nat} \to \text{Real} \) be such that \( g \) “increments” on \( f \) through some function \( F : \text{Nat} \times \text{Nat} \to \text{Real} \) (i.e. \( g(x) = f(x) + F(x,0) \)). In dTT \( F \) yields then an element of \( D_{\text{Nat} \to \text{Real}}(f,g) \). By reasoning as in Example \( 2.2 \) we can construct the increment from \( \mathbb{R}(f) \) to \( \mathbb{R}(g) \); if \( \mathbb{R}(f) \) is as in Example \( 4.2 \) this corresponds then to \( 1/N \cdot \sum_{i=0}^{N-1} F(i,0) \).

## 7 Cartesian Differential Categories

We conclude our sketch of differential models of dTT with the axiomatization of program derivatives provided by cartesian differential categories (CDC). The introduction of the differential \( \lambda \)-calculus \cite{11}, an extension of the \( \lambda \)-calculus with a differential operator \( dt \), has motivated much research on abstract axiomatizations of differentiation that generalize the usual derivatives from calculus to higher-order programming languages \cite{10} \cite{9} \cite{8}. CDC can be seen as a common ground for all these approaches, as they provide basic algebraic rules for derivatives in a cartesian setting.

We recall that a cartesian category \( C \) is left-additive when the Hom-objects of \( C \) are monoids, with the monoidal operations \( 0, + \) commuting with the cartesian structure (e.g. \( f + (g \times h) = (f + g) \times (f + h) \)), and satisfies left-additivity, i.e. \( 0 \circ f = 0 \) and \( (g + h) \circ f = (g \circ f) + (h \circ f) \). A CDC is a cartesian left-additive category endowed with a derivative operator \( d \) such that for all \( f : X \to Y \), \( df : X \times X \to Y \), satisfying a few axioms (D1)-(D7) (recalled in the Appendix). Intuitively, \( df(x,y) \) describes the differential \( f'(x) \cdot y \) of \( f \) at \( x \), so it should be a linear function in \( y \). This is reflected by the axiom (D2) stating, informally, that \( df \) is additive in its second variable, i.e. \( df(x,0) = 0 \) and \( df(x,y+y') = df(x,y) + df(x,y') \). Among the other axioms for \( df \) we find analogs of axioms \( \text{Dchain} \) (the chain rule) as well as axioms expressing the commutation of \( df \) with the cartesian structure, plus some other axioms concerning second derivatives. When a CDC is cartesian closed, one usually adds also axiom \( \text{DExt} \) (called D-Curry in \cite{10}), and one speaks of a differential \( \lambda \)-category.

We now describe the interpretation of dTT in a differential \( \lambda \)-category. In fact, everything works in any CDC if we forget about the higher-order structure. We will interpret \( D_A(t, u) \) as a sort of “tangent space” of \( t \) (notice that we ignore \( i \)); the self-difference \( \partial(t) \) will correspond to the zero vector \( 0 \), and the derivative \( D[f] \), which sends “vectors tangent to \( x \)” into “vectors tangent to \( fx \)”, will correspond to \( df \). Moreover, the resulting model satisfies the extensionality axioms \( \text{CExt} \) and \( \text{FExt} \) (see the Appendix for details).

\(^{6}\)Since Euclidean spaces \( \mathbb{R}^n \) and smooth functions form a CDC \cite{9} this shows in particular that one can consistently interpret \( D[f] \) as the “true” derivative from analysis.
Let \( C \) be a differential \( \lambda \)-category. We sketch the dTT-structure on the identity functor \( \text{id} : C \rightarrow C \), leaving all details to the Appendix. The classes \( \mathcal{P}(X) \) contain all objects of the form \( X^n \) (where \( X^0 = 1, X^{n+1} = X^n \times X \)), with \( (X^2 | X^n) = X^2 \times X^n \) and projection \( \pi_X : X^2 \times X^n \rightarrow X^2 \). \( \mathcal{P}(Y) \) is made of pullbacks \( \langle f, g \rangle \circ X^n \), for arrows \( f, g : Y \rightarrow X \), with \( (Y | \langle f, g \rangle \circ X^n) = Y \times X^n \) and projection \( \pi_Y : (Y | \langle f, g \rangle \circ X^n) \rightarrow Y \).

For any object \( X \), the pure predicate \( X^0 \in \mathcal{P}(X) \) is just \( X \), and \( r_X : X \rightarrow X^3 \) is \( r_X = \langle \langle \text{id}_X, \text{id}_X \rangle, 0 \rangle \). Moreover, for any binary predicate \( P = \langle f, g \rangle \circ Z^n \in \mathcal{P}(X^2) \), where \( f, g : X^2 \rightarrow Z \), for any arrow \( e : X \rightarrow (X^2 | P) = X^2 \times Z^n \), where \( c = \langle \text{id}_X, \text{id}_X, c' \rangle \), with \( c' : X \rightarrow Z^n \), we can define a diagonal filler \( j : X^3 \rightarrow X^2 \times Z^n \) by letting

\[
\langle \langle \text{id}_X, \text{id}_X \rangle, c' \rangle \circ (\pi_1 \circ \pi_1) + \langle \text{id}_X, c \rangle,
\]

Observe that \( (j \circ r_X)_2 = c \) and moreover, if \( Z^n = X \), \( f = \text{id}_X \) and \( c = r_X \), then \( j = \text{id}_X \times X \), so both the \( \beta \)- and \( \eta \)-rules are satisfied.

**Remark 7.1.** When \( h : X \rightarrow Y \) interprets \( t \in A \rightarrow B \), \( D[h] \) corresponds to a diagonal filler with \( f = h \circ \pi_1, g = \pi_2 \) and \( c = \langle \text{id}_X, 0 \rangle \), yielding the map \( dh : X \times X \rightarrow Y \).

**Remark 7.2.** From Theorem 5.2 it follows that the classes \( \mathcal{L}, \mathcal{R} \) form a WFS in \( C \), where \( f : X \rightarrow Y \) factors as \( X \xrightarrow{\lambda_f} (X \times Y) \times Y \xrightarrow{\pi_2 \circ c} X \), with \( i_f = \langle \langle \text{id}_X, f \rangle, 0 \rangle \).

### 8 Conclusions

**Related Work** dTT is definitely not the first proof system proposed to formalize relational reasoning for higher-order programs (nor the first one based on dependent types, e.g. \cite{22}). Among the many existing systems we can mention the logics for parametricity and logical relations \cite{20,21}, the refinement type systems for cryptography \cite{6}, differential privacy \cite{7} and relational cost analysis \cite{12}. Relational Hoare Type Theory \cite{32} and Relational Higher-Order Logic \cite{12}. In particular, it is tempting to look at dTT as a proof-relevant variant of (some fragment of) RHOL, since the latter is based on judgements of the form \( \Gamma \vdash t : A \sim u : B \vdash \phi \), where \( \Gamma \vdash t : A, u : B \) are typings in STLC, and \( \Psi, \phi \) are logical formulas depending on the variables in \( \Gamma \) as well as \( t \) and \( u \). Indeed, the main difference between dTT and such systems is that program differences are represented as proof objects. This looks a rather natural choice at least for those semantics (like e.g. DLR and CS) where program differences can be seen as being themselves some kind of programs.

Neither we are the first to observe formal correspondences between various notions of program derivative. For instance, a formalization of change structures in the context of DLR is discussed in \cite{23}; connections between metric semantics and DLR are studied in \cite{17,28}, based on generalized metric spaces and quantaloid-enriched categories \cite{53}. In particular, the DLR model sketched in Section 5 can be used to provide a “quantaloid-interpretation” of \( D_{\lambda(-, -)} \) (to be compared with the groupoid structure of the identity type in full ITT).

Recently, cartesian difference categories \cite{2} have been proposed as a general framework for program derivatives (unifying cartesian differential categories with approaches related to both discrete derivatives and incremental computation). It seems that our model in Section 5 can be extended to such categories in a straightforward way.

**Future Work** The main goal of this paper was to provide evidence that ITT could serve the purpose of formalizing differential reasoning. Yet, examples were left necessarily sketchy and more substantial formalization work (as well as implementations) needs to be addressed.

As dTT is a fragment of ITT, syntactic results like strong normalization follow. Yet, the problem should be addressed whether such results are stable also when further equations for derivatives (as those described in Section 2) are added. Moreover, it is well-known that a suitable formulation of 2-dimensional ITT satisfies a canonicity condition \cite{25}; a closed normal term of type Bool is either \( 0, 1 \); it would be interesting to see whether this result can be scaled to the dTT.
Finally, it seems worth exploring extensions of dTT with further structure, for instance with dependent types at the base level (e.g. following work on dependent types for differential privacy [15]), as well as with primitives for probabilistic reasoning (as in [30][7][6]).

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A Type Systems: Details

The typing rules of dTT can be divided into the rules for typing program terms and the rules for typing difference terms.
Given the linear type $A \overset{\triangleleft}{\to} A$.

There exists a “forgetful” translation * from ST$\lambda C$ to ST$\lambda C'$ given on terms by

\[
x^* = x \\
(\lambda x.t)^* = \lambda x.t^* \\
(tu)^* = t^*u^*
\]

and on types by

\[
X^* = X \\
(l_rA)^* = A^* \\
(A \rightarrow B)^* = A^* \rightarrow B^* \\
(A \otimes B)^* = A^* \times B^*
\]

This translation can be used to define the functor $H$ from Section 3 from the context category $\text{Ctx}_0$ of ST$\lambda C'$ to $\text{Ctx}$.

**Difference Types** As discussed in Section 2, the rules for difference terms are the standard rules of ITT, restricted to the language of dTT. We illustrate in Fig. 6 the rules for the difference type (where, compared to the rules sketched in Section 2, we highlight the role of contexts), and all other rules in Fig. 7. Finally, we illustrate $\beta$- and $\eta$-rules in Fig. 7.

**Proof.** Proof of Lemma 2.1

- if $C(z) = D_C(t, u)$, then $(t, u) : (z \in \Phi_0) \rightarrow (y, y' \in C)$ we let $\Psi_0 = C$ and $\Phi_0(y, y') = D_C(y, y')$.
We list a few equational rules for the operators $B$.

**Equational Rules for Derivatives: Details**

- if $\mathcal{C}(z) = C_1(z) \times C_2(z)$, then by induction hypothesis there exist pure predicates $(x, y \in A_1)C_1^0, (x, y \in A_2)C_2^0$, and terms $(t_1, u_1) : (z \in \Phi_0) \rightarrow (x, y \in A_1)$ and $(t_2, u_2) : (z \in \Phi_0) \rightarrow (x, y \in A_2)$ such that $C_1^0 = C_1^0(t_1, u_1)$ and $C_2^0(t_2, u_2)$. We can let then $A = A_1 \times A_2$, $t = \langle t_1, t_2 \rangle$, $u = \langle u_1, u_2 \rangle$ and $C^0(w, w') = C_1^0(\pi_1(w), \pi'_1(w')) \times C_2^0(\pi_2(w), \pi'_2(w'))$.

- if $\mathcal{C}(z) = (\Pi w \in D)B(z, w)$ then by induction hypothesis there exists a pure predicate $(x, y \in A')B^0(x, y)$ and terms $(t', u') : (z \in \Phi_0, w \in D) \rightarrow (x, y \in A')$ such that $B(z, w) = B^0(t', u')$. We let then $A = D \rightarrow A'$, $C^0(x, y) = (\Pi w \in A)B^0(xw, yw)$ and $t = \lambda w.t', u = \lambda w.u'$.

- if $\mathcal{C}(z) = (\Pi w, w' \in C)(D_{C}(w, w') \rightarrow B(z, w, w'))$ then by induction hypothesis there exists a pure predicate $(x, y \in A')B^0(x, y)$ and terms $(t', u') : (z \in \Phi_0, w, w' \in A) \rightarrow (x, y \in A')$ such that $B(z, w, w') = B^0(t', u')$. We let then $A = C \rightarrow (C \rightarrow A')$, $C^0(x, y) = (\Pi w, w' \in C)(D_{A}(w, w') \rightarrow B^0(xww', yww'))$ and $t = \lambda ww'.t', u = \lambda ww'.u'$.

\[\square\]

## B Equational Rules for Derivatives: Details

We list a few equational rules for the operators $J$ and $D$, that make sense under the validity of some of the extensionality axioms.

- in presence of $(\text{LExt})$ one can consider the following rules:

\[\hat{c}(\langle t, u \rangle) = \langle \hat{c}(t), \hat{c}(u) \rangle\]  
$J(t, u, a, [x]b, c) = (J(t, u, a, [x], b), J(t, u, a, [x], c))$  
$J(t, u, a, [x]b) = \pi_1(J(t, u, a, [x]b))$  

These rules say that the difference structure commutes with the cartesian structure.

18
\[
\begin{align*}
(x \in \Phi_0, x \in A &\mid e \in \Phi_1(x)) \vdash a \in A(x, x) \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash \lambda x.a \in (\Pi x \in A)A(x, x) \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash a \in (\Pi x \in A)A(x, x) \\
(x \in \Phi_0 &\vdash t \in A) \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash at \in A(x, t) \\
(x \in \Phi_0, x, y \in A &\mid e \in \Phi_1(x), e \in D_A(x, y) \vdash a \in A(x, x, y) \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash \lambda xye.a \in (\Pi x, y \in A)(D_A(x, y) \rightarrow A(x, x, y)) \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash a \in (\Pi x, y \in A)(D_A(x, y) \rightarrow A(x, x, y)) \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash b \in D_A(t, u) \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash atub \in A(x, t, u) \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash a \in A(\Pi e \in \Phi_1(x)) \vdash b \in A \\
(x \in \Phi_0 &\vdash e \in \Phi_1(x)) \vdash \langle a, b \rangle \in A \times B \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash a \in A_1 \times A_2 \\
(x \in \Phi_0 \mid e \in \Phi_1(x)) &\vdash \pi_i(a) \in A_i
\end{align*}
\]

Figure 6: Typing rules of dTT.

\[
\begin{align*}
(\lambda x.a)t &\simeq_b a[t/x] & (\lambda x.a) &\simeq_\eta a & (x \notin \text{FV}(a)) \\
(\lambda xye.a)tub &\simeq_b a[t/x, u/y, b/\epsilon] & (\lambda xye.axye) &\simeq_\eta a & (x, y, \epsilon \notin \text{FV}(a)) \\
\pi_i(\langle a_1, a_2 \rangle) &\simeq_\beta a_i & \langle \pi_1(a), \pi_2(a) \rangle &\simeq_\eta a \\
J(t, t, \check{c}(t), [x]b) &\simeq_b b[t/x] & J(t, u, a, [x]\check{c}(x)) &\simeq_\eta a
\end{align*}
\]

Figure 7: $\beta$- and $\eta$-rules for difference terms.
in presence of (\text{FExt1}) one can consider the following rules:

\[ \partial(\lambda x.t) = \lambda x.\partial(t) \quad (\text{J1a}) \]
\[ J(t, u, a, [x]y.g.b(x, y)) = y.g.J(t, y, \langle u, y \rangle, \langle a, \partial(y) \rangle, [z]b(\pi_1(z), \pi_2(z))) \quad (\text{J1b}) \]

from which it follows that the derivative “in \( x \)” \( D[\lambda xy.f(x, y)] \) of some binary function \( f(x, y) \) is the same as the derivative “in \( z = \langle x, y \rangle \)” i.e. \( D[\lambda z.f(\pi_1(z), \pi_2(z))] \), where the error on \( y \) is its self-difference, i.e. \( \lambda x'\delta J(\lambda z.f(\pi_1(z), \pi_2(z)), \langle x, y \rangle, \langle x', y \rangle, \epsilon, \partial(y)) \)

(when \( \partial(y) \) is interpreted as the null error 0, this coincides with axiom \( D \)-curry from \([10]):

in presence of (\text{FExt2}) and (\text{CExt1}) one can consider the following rules:

\[ \partial(\lambda x.t) = D[\lambda x.t] \quad (\text{J2a}) \]
\[ J(t, u, a, [x]y.g.b) = \lambda yy'.\partial J(\langle t, y \rangle, \langle u, y' \rangle, \langle a, \delta \rangle, [z]b(\pi_1(z)/x, \pi_2(z)/y)) \quad (\text{J2b}) \]

This shows that the axioms (\text{FExt1}) and (\text{FExt2}) might give rise to derivative with a rather different operational semantics. In particular, Eq. (\text{J2a}) says that the self-difference of a function coincides with its derivative (this will be the case in the models from Sec. 5 and 6); moreover, (\text{J2a}) says that the derivative “in \( x \)” of a binary function \( f(x, y) \) actually derives also in the variable \( y \). In particular one can deduce that \( D[\lambda xy.t(x, y)]x'y'y'\delta \) is the same as \( D[\lambda z.t(\pi_1(z), \pi_2(z))]\langle x, y \rangle \langle x', y' \rangle \langle \epsilon, \delta \rangle \).

C dTT-Categories: Details

In this section we define in detail the notion of dTT-category that was sketched in Section 3. Throughout this section we suppose \( U : \mathbb{C}_0 \rightarrow \mathbb{C} \) to be a strict monoidal functor, where \( \mathbb{C}_0 \) is a symmetric monoidal category and \( \mathbb{C} \) is a cartesian category. Moreover, we suppose that for any object \( \Gamma \) of \( \mathbb{C}_0 \) the following data is given:

- a collection \( \mathcal{P}(\Gamma) \) of predicates over \( \Gamma \), and for each \( P \in \mathcal{P}(\Gamma) \) an object \( \Gamma \mid P \) and an arrow \( \pi_P : \Gamma \mid P \rightarrow UT \); we further require that:
  - for all \( P \in \mathcal{P}(\Gamma) \) and \( f : U\Delta \rightarrow UT \), the pullback \( f^P \) exists and is in \( \mathcal{P}(\Delta) \);
  - when the monoidal product of \( \mathbb{C}_0 \) is not cartesian, we furthermore require that:
    - for all \( P \in \mathcal{P}(\Gamma) \), the pullback \( \pi_U^P \) exists and is in \( \mathcal{P}(\Delta, \Gamma) \), where \( \pi_U^P : U\Delta \times UT \rightarrow UT \);
    - for all \( P \in \mathcal{P}(\Gamma, \Gamma) \), the pullback \( \delta_U^P \) exists and is in \( \mathcal{P}(\Gamma) \), where \( \delta_U^P : UT \rightarrow UT \times UT \);
  - we require all mentioned pullbacks to be associative and unital;
  - for all \( P, Q \in \mathcal{P}(\Gamma) \), a predicate \( P \times Q \in \mathcal{P}(\Gamma) \) exists such that \( \Gamma \mid P \times Q \) is the cartesian product of \( \Gamma \mid P \) and \( \Gamma \mid Q \) in the slice category \( \mathbb{C}_0^\Gamma \), i.e. in the category of predicates in \( \mathcal{P}(\Gamma) \) and \textit{vertical} arrows (see Section 3), i.e. those arrows \( h : \Gamma \mid P \rightarrow \Gamma \mid Q \) making the diagram below commute

\[
\begin{array}{ccc}
\Gamma \mid P & \xrightarrow{h} & \Gamma \mid Q \\
\pi_P \downarrow & & \downarrow \pi_Q \\
UT & & UT
\end{array}
\]

- a sub-collection \( \mathcal{P}^p(\Gamma) \subseteq \mathcal{P}(\Gamma, \Gamma) \) of \textit{pure predicates} generating \( \mathcal{P} \), (i.e. such that any \( P \in \mathcal{P}(\Gamma) \) is of the form \( f^P \), where \( P' \in \mathcal{P}^p(\Delta) \) and \( f \in \mathbb{C}_0(\Delta, \Gamma, \Gamma) \)) together with a chosen pure predicate \( \Gamma^P \in \mathcal{P}^p(\Gamma) \), and closed with respect to the following conditions:
  - for all pure predicates \( P, Q \in \mathcal{P}^p(\Gamma) \), \( P \times Q \in \mathcal{P}^p(\Gamma) \);
– for all objects $\Delta, \Gamma$ of $C_0$ and pure predicate $P \in \mathcal{P}(\Gamma)$, a pure predicate $\Pi_\Delta P \in \mathcal{P}(\mathcal{P}(\Pi_\Delta \Delta))$ such that for all $Q \in \mathcal{P}(\Sigma)$ there is a bijection

$$C(\Delta, \Sigma | \pi^2_\Sigma Q, \Gamma, \Gamma | P) \simeq C(\Sigma | Q, \Gamma^\Delta, \Gamma^\Delta | \Pi_\Delta P)$$

where $\pi_\Sigma$ is the projection $\Delta, \Sigma \to \Sigma$. Moreover, we require that dependent products commute with pullbacks, i.e. for all $f \in C_0(\Gamma', \Gamma)$, $f^2(\Pi_\Delta P) = \Pi_\Delta (f^2 P)$.

– for all objects $\Delta, \Gamma$ of $C_0$ and pure predicate $P \in \mathcal{P}(\Gamma, \Gamma)$ a pure predicate $P(\Delta^0) \in \mathcal{P}(\mathcal{P}(\Delta^0))$ such that for all $Q \in \mathcal{P}(\Sigma)$ there is a bijection

$$C(\Delta, \Delta, \Sigma | \pi^2_\Sigma Q \times \Delta^0, \Gamma, \Gamma | P) \simeq C(\Sigma | Q, \Gamma^\Delta, \Gamma^\Delta | P(\Delta^0))$$

where $\pi_\Sigma$ is the projection $\Delta, \Delta, \Sigma \to \Sigma$. Moreover, we require that for all $f \in C_0(\Gamma', \Gamma)$, $f^2(P(\Delta^0)) = (f^2 P)(\Delta^0)$.

We will make extensive use of the following constructions: for all objects $\Gamma, \Delta$ of $C_0$ there exists

- a functor $\pi^2_\Delta \Gamma : C^\Delta_\Gamma \to C^\Delta_\Gamma \Gamma$ induced by the projection $\pi_\Delta : \Delta, \Gamma \to \Delta, \Gamma$ which deletes the third component of $\Delta, \Gamma, \Gamma$, together with an arrow $\pi^2_\Delta \Gamma(P) \to \Delta, \Gamma | P$ making the pullback diagram below commute:

$$\begin{array}{ccc}
\Delta, \Gamma | \pi^2_\Delta \Gamma(P) & \xrightarrow{\pi^R} & \Delta, \Gamma | P \\
\downarrow^{\pi_\Delta r} & & \downarrow^{\pi_\Delta r} \\
U \Delta, \Gamma \cup \Gamma & \xrightarrow{\pi^R} & U \Delta, \Gamma
\end{array}$$

- a functor $\delta^2_{\Delta, \Gamma | Q} : C^\Delta_\Gamma \Gamma \to C^\Delta_\Gamma \Gamma$ induced by the arrow $\delta_{\Delta, \Gamma} : \Delta, \Gamma \to \Delta, \Gamma, \Gamma$ which duplicates the second component of $\Delta, \Gamma$, together with an arrow $\delta^2_{\Delta, \Gamma | Q}(P) \to \Delta, \Gamma | Q \times P$ making the pullback diagram below commute:

$$\begin{array}{ccc}
\Delta, \Gamma | \delta^2_{\Delta, \Gamma | Q}(P) & \xrightarrow{\delta^R} & \Delta, \Gamma, \Gamma | P \\
\downarrow^{\pi_\Delta r} & & \downarrow^{\pi_\Delta r} \\
U \Delta, \Gamma \cup \Gamma & \xrightarrow{\delta^R} & U \Delta, \Gamma \cup \Gamma
\end{array}$$

Moreover, one has $\delta^2_{\Delta, \Gamma | Q} \circ \pi^2_\Delta \Gamma = \text{id}_{C^\Delta_\Gamma \Gamma}$.

In $\text{Ctx}$ the operation $\pi^2_\Delta \Gamma$ turns a predicate $C(z, x)$ into a predicate $C'(z, x, y) = C(z, x)$ by adding a “dummy” variable $y$; the operation $\delta^2_{\Delta, \Gamma}$ turns a predicate $C(z, x, y)$ into a predicate $C'(z, x) = P(z, x, y)$. Notice that $(C')_! = C$.

The difference structure is provided by the following data:

- for all objects $\Gamma, \Delta$ of $C_0$ and for all predicate $Q \in \mathcal{P}(\Delta, \Gamma, \Gamma)$, an arrow $r_{\Delta, \Delta | Q} : \Delta, \Gamma | \delta^2_{\Delta, \Gamma | Q}(P) \to \Delta, \Gamma, \Gamma | Q \times \Delta^0$, such that the composition of $r_{\Delta, \Delta | Q}$ with the projection $\pi_1 : \Delta, \Gamma, \Gamma | Q \times \Delta^0 \to \Delta, \Gamma, \Gamma | Q$ coincides with $\delta^2_{\Delta, \Gamma} : \Delta, \Gamma | \delta^2_{\Delta, \Gamma | Q}(P) \to \Delta, \Gamma, \Gamma | Q$

- for all objects $\Delta, \Gamma$ of $C_0$, predicates $Q \in \mathcal{P}(\Delta, \Gamma, \Gamma)$ and $P = f^2 P^0 \in \mathcal{P}(\Delta, \Gamma, \Gamma)$, and for any arrow $c : \Delta, \Gamma | \delta^2_{\Delta, \Gamma | Q}(P) \to \Delta, \Gamma, \Gamma | Q \times P$ making the diagram below commute

$$\begin{array}{ccc}
\Delta, \Gamma | \delta^2_{\Delta, \Gamma | Q}(P) & \xrightarrow{c} & \Delta, \Gamma, \Gamma | Q \times P \\
\downarrow^{r_{\Delta, \Delta | Q}} & & \downarrow^{\pi_1} \\
\Delta, \Gamma, \Gamma | Q \times \Delta^0 & \xrightarrow{\pi_2} & \Delta, \Gamma, \Gamma | Q
\end{array}$$

(notice that this implies that $c$ is of the form $(\delta^2_{\Delta, \Gamma | Q}(P), c')$) a choice of a diagonal filler $j_{\Delta, \Gamma, Q, f, P, c}$ making both triangles commute.
We require the data above to satisfy a few coherence conditions, namely that for all $g \in \mathbb{C}_0(\Sigma, \Delta)$ and vertical morphism $h \in \mathbb{C}^\Delta_{\mathfrak{F},\Gamma}(R, Q)$, $\Gamma^D = (\{g, \Gamma, \Gamma\})^D$, and the pullback diagrams below commute:

$$\Sigma.\Gamma \mid \delta^g_\Sigma.\Gamma(g^2Q) \xrightarrow{\delta^g_\Delta.\Gamma(g)} \Delta.\Gamma \mid \delta^g_\Delta.\Gamma(Q)$$

$$\Sigma.\Gamma.\Gamma \mid g^2Q \times \Gamma^D \xrightarrow{(Ug, \Gamma)^+} \Delta.\Gamma.\Gamma \mid Q \times \Gamma^D$$

$$\Delta.\Gamma \mid \delta^g_\Delta.\Gamma(R) \xrightarrow{\delta^g_\Delta.\Gamma(h)} \Delta.\Gamma \mid \delta^g_\Delta.\Gamma(Q)$$

$$\Delta.\Gamma.\Gamma \mid R \times \Gamma^D \xrightarrow{h \times \Gamma^D} \Delta.\Gamma.\Gamma \mid Q \times \Gamma^D$$

$$\Sigma.\Gamma.\Gamma \mid g^2Q \times \Gamma^D \xrightarrow{j_{\Sigma.\Gamma.\Gamma, q, f, p, e}^+} \Sigma.\Gamma.\Gamma \mid g^2Q \times g^2P$$

$$\Delta.\Gamma.\Gamma \mid Q \times \Gamma^D \xrightarrow{j_{\Delta.\Gamma.\Gamma, q, f, p, e}^+} \Delta.\Gamma.\Gamma \mid Q \times P$$

$$\Delta.\Gamma.\Gamma \mid R \times \Gamma^D \xrightarrow{j_{\Delta.\Gamma.\Gamma, r, f, p, e}^+} \Delta.\Gamma.\Gamma \mid R \times P$$

$$\Delta.\Gamma.\Gamma \mid Q \times \Gamma^D \xrightarrow{j_{\Delta.\Gamma.\Gamma, q, f, p, e}^+} \Delta.\Gamma.\Gamma \mid Q \times P$$

where the arrow $c^* : \Sigma.\Gamma \mid \delta^g_\Sigma.\Gamma(g^2Q) \to \Sigma.\Gamma.\Gamma \mid g^2Q \times g^2P$ is given by the universality of the pullback along $g$:

$$\begin{array}{c}
\Sigma.\Gamma \mid \delta^g_\Sigma.\Gamma(g^2Q) \\
\downarrow \delta^g_\Delta.\Gamma \\
\Sigma.\Gamma.\Gamma \mid g^2Q \times g^2P(\{g, \Gamma, \Gamma\})^+ \xrightarrow{\pi_{\Delta.\Gamma.\Gamma}} \Delta.\Gamma.\Gamma \mid Q \times P \\
\pi_{\Sigma.\Gamma.\Gamma} \downarrow \pi_{\Delta.\Gamma.\Gamma} \\
\Sigma.\Gamma.\Gamma \mid g^2Q \\
\downarrow c^* \\
\Sigma.\Gamma \mid \delta^g_\Sigma.\Gamma(Q)
\end{array}$$

These (admittedly complicated) conditions essentially say that $\Gamma^D$, $r_{\Delta, \Gamma}$ and $j_{\Delta, \Gamma, Q, f, p, e}$ “do not depend on” $\Delta$ and $Q$, i.e. are invariant under substitutions of the variables in $\Delta$ and $Q$. In other words, they assure the soundness of the equations below for the substitution operation:

$$\bar{c}(t)[v/y] = \bar{c}(t[v/y])$$

$$\bar{c}(t)[c/e] = \bar{c}(t)$$

$$J(t, u, a, [x]b)[v/y] = J(t[v/y], u[v/y], a[v/y], [x]b[v/y])$$

$$J(t, u, a, [x]b)[c/e] = J(t, u, a[c/e], [x]b[c/e])$$

**Proof of Proposition** [3, 4] The definition of the functor $[\lambda]_m : \text{Ctx}_0 \to \mathbb{C}_0$ is standard. The functor $(\llbracket \_ \rrbracket)_m : \text{Ctx} \to \mathbb{C}$ is defined as follows:

- to any pure predicate $(x, y \in A)C(x, y)$ we associate $(\mathcal{C})_m \in \mathcal{P}([A]_m)$ by induction as follows:
  - if $C(x, y) = D_A(x, y)$, then $(\mathcal{A})_m = ([A]_m)^D$;
  - if $C(x, y) = C_1(\pi_1(x), \pi_1(y)) \times C_2(\pi_2(x), \pi_2(y))$, then $(\mathcal{C})_m = \pi^1_1([C]_m) \times \pi^1_2([C]_m)$;
  - if $C(x, y) = (\Pi z \in B)C'(xz, yz)$, then $(\mathcal{C})_m = \Pi_{[B]_m}(\mathcal{C'})_m$;

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- if \( C(x, y) = (\Pi_w, w' \in B)(D_B(w, w') \to C'(xww', yyw')) \), then \( \langle C \rangle_m = \langle \langle C' \rangle_m \rangle_m \).

• For any arrow \((t | a) : (x \in \Phi_0 | e \in \Phi_1(x)) \to (y \in \Psi_0 | \delta \in C(y))\) we define an arrow \( (\llbracket t \rrbracket_m | \langle a \rangle_m) : (\langle \Phi_0 \rangle_m | \langle \Phi_1 \rangle_m) \to (\langle \Psi_0 \rangle_m | \langle \langle C \rangle_m \rangle_m) \) in \( C \), where \( \langle a \rangle_m \) is a vertical morphism in \( C_{\langle \Phi_1 \rangle_m} \), by induction on \( a \). We here only mention the cases related to the difference type:

- if \( a = \hat{c}(t) \), where \( \langle xt \rangle A = \langle \hat{a} \rangle_0 \), \( A = A_1 \circ \llbracket t \rrbracket_m \), where \( x \in \Phi_0 = x' \in \Phi_0, x \in A \);

- if \( a = J(t, u, b, [x]c) \), where \( a \in D_A(t, u) \), \( (x, y) \in \Phi_0 \), \( C(x, y) = C_A(t(x, y)) \) and \( (x \in A) c \in C(x, x) \), then \( \langle a \rangle = J[t, u, b, [x]c] A_1 \circ \llbracket t \rrbracket_m \).

\[ \Box \]

**Proof of Theorem**

We must show that \( i_f \in \mathcal{E} \), so let \( P \in \mathfrak{P}(\Sigma) \) be a predicate. The problem of finding a diagonal filler for a commutative diagram of the form

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{d} & \Sigma | P \\
\downarrow^{i_f} & \downarrow^{\pi_C} & \\
\Gamma, \Delta | \Delta_f & \xrightarrow{h} & \Sigma
\end{array}
\]

can be reduced, by pulling back along \( h \), to that of finding a diagonal filler \( j \) for a diagram of the form

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\ell} & \Gamma, \Delta | Q \\
\downarrow^{i_f} & \downarrow^{\pi_C} & \\
\Gamma, \Delta | \Delta_f & \xrightarrow{\pi_{\Gamma, \Delta}} & \Gamma, \Delta
\end{array}
\]

(with \( Q = h^2 P \)). We will obtain \( j \) by putting together a few commutative diagrams:

1. \[
\begin{array}{ccc}
\Gamma & \xrightarrow{i_f, \ell} & \Gamma, \Delta | \Delta_f, Q \\
\downarrow^{i_f} & \downarrow^{\pi_C} & \\
\Gamma, \Delta | \Delta_f & \xrightarrow{\delta_{\Delta} \circ (\delta_{\Delta} \circ c) \circ \pi_{\Gamma, \Delta}} & \Gamma, \Delta, \Delta | \pi_{\Gamma, \Delta}^2(\Delta_f), \pi_{\Gamma, \Delta}^2(Q), \Delta^0
\end{array}
\]

2. \[
\begin{array}{ccc}
\Gamma & \xrightarrow{\ell} & \Gamma, \Delta | Q \\
\downarrow^{i_f, \ell} & \downarrow^{\pi_C \circ \pi_3} & \\
\Gamma, \Delta | \Delta_f, Q & \xrightarrow{\delta_{\Delta} \circ (\delta_{\Delta} \circ \delta_{\Delta})} & \Gamma, \Delta, \Delta | \pi_{\Gamma, \Delta}^2(\Delta_f), \pi_{\Gamma, \Delta}^2(Q), \pi_{\Gamma, \Delta}^2(Q)
\end{array}
\]

where \( \pi_{\Gamma, \Delta} \) is the projection \( \Gamma, \Delta, \Delta \to \Gamma, \Delta \) and we use the fact that \( \delta_{\Delta} \circ (\delta_{\Delta} \circ \delta_{\Delta}) = \Delta_f \) and \( \delta_{\Delta} \circ (\delta_{\Delta} \circ \delta_{\Delta}) = Q \), so that \( \delta_{\Delta} : \Gamma, \Delta | \Delta_f \to \Gamma, \Delta, \Delta | \pi_{\Gamma, \Delta}^2(\Delta_f) \) and \( (\delta_{\Delta})^+ : \Gamma, \Delta | Q \to \Gamma, \Delta, \Delta | \pi_{\Gamma, \Delta}^2(Q) \).

3. \[
\begin{array}{ccc}
\Gamma, \Delta, \Delta | \pi_{\Gamma, \Delta}^2(\Delta_f), \pi_{\Gamma, \Delta}^2(Q), \pi_{\Gamma, \Delta}^2(Q) & \xrightarrow{\pi_{\Gamma, \Delta} \circ \pi_2 \circ \pi_3} & \Gamma, \Delta | \Delta_f, Q \\
\downarrow^{\pi_{\Gamma, \Delta} \circ \pi_3} & & \downarrow^{\pi_{\Gamma, \Delta}} \\
\Gamma, \Delta, \Delta & \xrightarrow{\pi_{\Gamma, \Delta}} & \Gamma, \Delta
\end{array}
\]

By putting all this together we can obtain a diagonal filler from the diagram illustrated in Fig. [S] where \( e = \delta_{\Delta} \circ (\delta_{\Delta} \circ \delta_{\Delta}) \), and where the central diagonal filler exists by hypothesis.
If \( h \in \mathcal{P} \), then by definition of \( \mathcal{L}_\mathcal{P} \) it has the right-lifting property with respect to all \( f \in \mathcal{L}_\mathcal{P} \); we deduce then \( \mathcal{P}^* \subseteq \mathcal{R}_\mathcal{P} \). This proves that \( p_f \in \mathcal{R}_\mathcal{P} \).

Since \( \mathcal{L}_\mathcal{P}^* = \mathcal{R}_\mathcal{P} \) holds by definition, it remains to prove that \( \mathcal{L}_\mathcal{P} = \mathcal{R}_\mathcal{P} \). On the one hand, from \( \mathcal{P} \subseteq \mathcal{R}_\mathcal{P} \), we deduce \( \mathcal{P} \subseteq \mathcal{P}^* \subseteq \mathcal{R}_\mathcal{P} \). For the converse direction, by the Retract Argument (Lemma 1.1.9 in [19]), any \( g \in \mathcal{R}_\mathcal{P} \) is a retract of a projection: from \( g = p_g \circ i_g \), and the fact that \( g \) has the right lifting property with respect to \( i_g \), we deduce that there exists \( h \) (a diagonal filler of \( p_g \circ i_g = g \circ \text{id}_X \)) such that \( g \circ h = p_g \). This implies in particular that there is a diagram of the form

\[
\begin{array}{ccc}
X & \xrightarrow{g} & A \\
\downarrow & & \downarrow h \\
Y & \xrightarrow{p_g} & X
\end{array}
\]

Using this we can show that given a diagram of the form

\[
\begin{array}{ccc}
C & \xrightarrow{f} & X \\
\downarrow & & \downarrow g \\
D & \xrightarrow{k'} & Y
\end{array}
\]

where \( f \in \mathcal{L}_\mathcal{P} \), we can construct a diagram

\[
\begin{array}{ccc}
C & \xrightarrow{k} & X & \xrightarrow{i_g} & A \\
\downarrow & & \downarrow h & \downarrow p_g \\
D & \xrightarrow{k'} & Y & \xrightarrow{g} & Y
\end{array}
\]

where the diagonal filler \( j \) exists since \( f \) has the left-lifting property with respect to \( p_g \), and from which we obtain a diagonal filler \( j' = h \circ j \) for the original diagram. We have thus shown that any \( f \in \mathcal{L}_\mathcal{P} \) has the left-lifting property with respect to any \( g \in \mathcal{R}_\mathcal{P} \), and thus \( \mathcal{L}_\mathcal{P} \subseteq \mathcal{R}_\mathcal{P} \). \( \square \)
Figure 8: Construction of the diagonal filler.
D Metric Preservation: Details

We will need the following generalization of the notion of metric space: a parameterized (pseudo-)metric space \(\text{PMS} (X, K, a)\), where \(a : X \times X \to (\mathbb{R}_{\geq 0})^K\) satisfies, for all \(k \in K\):

\[
\begin{align*}
a(x, x)(k) &= 0 \\
a(x, y)(k) &= a(y, x)(k) \\
a(x, y)(k) &\leq a(x, z)(k) + a(z, y)(k)
\end{align*}
\]

Usual (pseudo-)metric spaces can be identified with PMS \((X, K, a)\) where \(K\) is a singleton.

We recall that Met is the category of pseudo-metric spaces and non-expansive maps. Met is symmetric monoidal closed; its monoidal product is \((X, a) \otimes (Y, b) = (X \times Y, a + b)\) and the right-adjoint to \(\otimes\) is \([[(X, a), (Y, b)] = (\text{Met}(X, Y), b_{\text{sup}}]\), where \(b_{\text{sup}}(f, g) = \sup\{b(f(x), g(x)) \mid x \in X\}\). It is a standard fact that Met has enough structure to interpret \(\text{STAC}^1\), with \(!, A\) corresponding to the rescaling of a metric space by \(r\).

We now describe the dTT-structure associated with the forgetful functor \(U : \text{Met} \to \text{Set}\).

For any set \(X\), the class of pure predicates \(\mathcal{P}^\mathbb{R}(X)\) is made of all PMS of the form \(P = (X, K, a)\), with \((X \otimes X \mid P) = \prod_{x, x' \in X} \prod_{k \in K} a(x, x')(k)\) and projection \(\pi_X : (X \otimes X \mid P) \to X \times X\). The class \(\mathcal{P}(X)\) contains all pullbacks \(\langle f, g \rangle P\), for all \(P = (X, K, a) \in \mathcal{P}^\mathbb{R}(X)\) and \(\langle f, g \rangle \in \text{Met}(Y, X \otimes X)\), with \((Y \mid \langle f, g \rangle P) = \prod_{y \in Y} \prod_{k \in K} a(f(y), g(y))(k)\) and projection \(\pi_Y : (Y \mid \langle f, g \rangle P) \to Y\).

Observe that for any projection \(\pi : X \times Y \to X\) and PMS \((X, K, a)\), \((X, K, \pi^2 a)\) is still a PMS, where \(\pi^2 a(u, v)(k) = a(\pi_1(u), \pi_1(v))(k)\) (in fact, \(\pi^2 a(u, u) = a(\pi_1(u), \pi_1(u)) = 0\), \(\pi^2 a(u, v) = a(\pi_1(u), \pi_1(v)) = a(\pi_1(v), \pi_1(\pi_1(v))) = \pi^2 a(v, u)\), and \(\pi^2 a(u, v) = a(\pi_1(u), \pi_1(v)) \leq a(\pi_1(v), \pi_1(v)) + a(\pi_1(u), \pi_1(v)) = \pi^2 a(u, v) + \pi^2 a(v, u)\)).

Moreover, for all PMS \((X \times X, K, a)\), also \((X, K, \delta_X a)\) is a metric space, where \(\delta_X a(u, x)(k) = \pi^2 a(\pi_1(u), x)(k)\) and \(\prod_{y \in Y} \prod_{b \in B} b(f'(x), g'(x))(h)\) is given by a function \(\varphi : (X \mid P) \times X \to \mathbb{R}_{\geq 0}\) such that \(\varphi(x, \phi, h) \in b(f'(x), g'(x))(h)\). Given PMS \(P = (Y, K, a)\) and \(Q = (Z, H, b)\), let \(P * Q = (Y \times Z, K + H, a + b)\), where \(a + b(\langle y, z \rangle, \langle y', z' \rangle)(\langle 0, k \rangle) = a(y, y')(k) + a + b(y', z'')(\langle 1, h \rangle) = b(z, z')(k)\). One can check that:

- for all predicates \(P = (f, g) P\) and \(Q = (f', g') Q' \in \mathcal{P}(X)\), where \(P' \in \mathcal{P}^\mathbb{R}(Y)\) and \(Q' \in \mathcal{P}^\mathbb{R}(Z)\) are the PMS \((Y, K, a)\) and \((Z, H, b)\), the product \(P \times Q \in \mathcal{P}(X)\) is \(\langle f, f' \rangle(\langle f', g' \rangle P \times Q)\), with \((X \mid P \times Q) = \prod_{x \in X} \prod_{k \in K} a(f(x), g(x)) + b(f'(x), g'(x))(u)\).
- Observe that if \(P, Q \in \mathcal{P}(X)\), then \(P \times Q = P * Q \in \mathcal{P}(X)\).
- for any pure predicate \(P = (X, K, a) \in \mathcal{P}(X)\) and set \(I\), the dependent product \(\Pi_I P \in \mathcal{P}(X^I)\) is the PMS \((X^I, I \times K, \Pi_I a)\) where \((\Pi_I a)(f, g)(\langle i, k \rangle) = a(f(i), g(i))(k)\);
- for all PMS \(P = (X, K, a)\) and \(Q = (Y, K, b) \in \mathcal{P}(Y)\), the dependent product \(Q^{(X)} \in \mathcal{P}(X \times X)\) is the PMS \((Y \times X, H \times (X \otimes X \mid P), p_{a,b})\), where \(p_{a,b}(f, g)(\langle h, \langle y, x \rangle, \phi \rangle) = b(f(x, y), g(x, y))(h)\).

The difference structure is as follows:

- for all metric spaces \((X, a)\), \(X^D \in \mathcal{P}(X)\) is \((X, a)\) itself, seen as a PMS;
- for all metric spaces \((X, a)\), and set \(Y\), predicate \(P = (f, g) P' \in \mathcal{P}(Y \otimes (X \otimes X))\), (with \((Y \otimes X \otimes X \mid P) = \prod_{y \in Y} \prod_{x, x' \in X} \prod_{k \in K} a(f(y, x, x'), g(y, x, x'))\), the pullback \((Y \otimes X \mid \delta_X P) = \prod_{y \in Y} \prod_{x, x' \in X} \prod_{k \in K} a(f(y, x, x'), g(y, x, x'))\); the morphism \(r_{Y, X \mid P} : (Y \otimes X \mid \delta_X P) \to (Y \otimes X \otimes X \mid P \times X^D)\) is given by \(r_{Y, X \mid P}(\langle y, x \rangle, \langle x, x \rangle, \phi)(k) = \langle y, x, x \rangle, \phi + 0\).
for all metric spaces \((X, a), (Y, b)\) predicate \(P = \langle m, n \rangle^2 P' \in \mathcal{P}(Y \times X \times X)\) (with \((Y \times X \times X \mid P) = \prod_{y \in Y, x \in X} \prod_{x, x' \in X} \mathcal{P}(m(y, x, x'), n(y, x, x'))(h)\), pure predicate \(Q = (Z, K, \epsilon) \in \mathcal{P}(Z)\), non-expansive functions \((f, g) \in \text{Met}(Y \times X \times X, Z \times Z)\) and for any function \(\epsilon : (Y' \times X) / \delta' \times P' \rightarrow (Y \times X \times X \mid P \times (f, g)^2 Q)\) satisfying \(\epsilon(y, x, \phi) = \langle (y, x), c'_{\phi}(y, x, \phi), c'(y, x, \phi) \rangle\), with \(c'(y, x, \phi) : H + \text{K} \rightarrow \mathbb{R}_{\geq 0}\) satisfying \(c'(y, x, \phi)(0, k) = \phi(k)\) and \(c'(y, x, \phi)((1, k)) = \epsilon(f(y, x, x), g(y, x, x))\), for all \(y \in Y', x, x' \in X, \phi \in \mathcal{P}(\mathbb{R}^2)\) with \(\phi(k) \geq b'(x, x')(k) \geq b'(x, x)(k)\); we have \(c'(y, x, \phi)((1, k)) \geq c(f(y, x, x), g(y, x, x))(h)\).

In fact, since \(\phi(k) \geq b'(x, x')(k) \geq b'(x, x)(k)\), we have \(c'(y, x, \phi)((1, k)) \geq c(f(y, x, x), g(y, x, x))(h)\); moreover, from \((f, g) \in \text{Met}(Y \times X \times X, Z \times Z)\) it follows \(c(f(y, x, x'), f(y, x, x)) + e(g(y, x, x), g(y, x, x')) \leq b(y, y) + a(x, x') + a(x, x') = a(x, x') \leq r\); then, using the triangular law, we deduce \(c'(y, x, \phi)((1, k)) + r \geq c'(y, x, x')(h)\).

Observe that \((\langle y, x, \phi(\langle f, g \rangle, r) \rangle, \epsilon(\langle f, g \rangle, r)))_2 \geq c'(y, x, \phi)\). Moreover, when \(Q = X^P\), \(f(y, x, x') = x\), \(g(y, x, x') = x'\) and \(c'(y, x, \phi)((1, h)) = \epsilon(0, h)\), we deduce that \(\text{Met}(Y \times X, Z \times Z)\) is cartesian closed, with the product and exponential of \(\text{Met}(Y \times X, Z \times Z)\). This notion of map is a slight variation with respect to \([13]\), where the auxiliary map \(\varphi\) goes from \(X \times X\) to \(M\).

Yet, this change does not affect the higher-order structure of DLR (see below), and the two families of maps are related by a retraction \(M^{X \times L} \xrightarrow{h \leftarrow k} M^{X \times X \times L}\) where \(h(\varphi)(x, y, \epsilon) = \varphi(x, \epsilon)\) and \(k(\psi)(x, y, \epsilon) = sup\{\psi(x, y, \epsilon) : y \in X \times \rho(x, \epsilon, y)\}\).

We now describe the dTTT-structure associated with the forgetful functor \(U : \mathcal{P}(X) \rightarrow \text{Set}\).

For any set \(X\), \(\mathcal{P}(X)\) is made of all DLR of the form \(P = (X, L, \rho)\), with \(X \times X \mid P = \prod_{x, x' \in X} \hat{\rho}(x, x')\) and projection \(\pi_X : (X \times X \mid P) \rightarrow X \times X\). \(\mathcal{P}(Y)\) is made of all pullbacks \((f, g)^2 P\), where \(P = (X, L, \rho) \in \mathcal{P}(X)\), and \(f, g : Y \rightarrow X\), with \((Y \mid (f, g)^2 P) = \prod_{y \in Y} \hat{\rho}(f(y), g(y))\) and projection \(\pi_Y : (Y \mid (f, g)^2 P) \rightarrow Y\).

An arrow in the slice category \(\mathcal{P}(X)\) between \(\prod_{x \in X} \hat{\rho}(f(x), g(x))\) and \(\prod_{x \in X} \hat{\rho}(f'(x), g'(x))\) (for given DLR \((X, L, \rho)\) and \((Y, M, \mu)\)) is given by a function \(\varphi : X \times L \rightarrow M\) such that \(\rho(f(x), e, g(x))\) implies \(\mu(f'(x), \varphi(x, e), g'(x))\). One can check that:

- for all predicates \(P = (f, g)^2 P' \in \mathcal{P}(X)\) and \(Q = (f', g')^2 Q' \in \mathcal{P}(X)\), their product is the predicate \(\langle (f, f'), (g, g') \rangle^2 (P' \times Q') \in \mathcal{P}(X)\). Observe that if \(P\) and \(Q\) are in \(\mathcal{P}(X)\), then \(P \times Q \in \mathcal{P}(X)\);
• for all pure predicate $P = (X, L, \rho) \in \mathcal{P}(X)$ and set $I$, the dependent product $\Pi_I P \in \mathcal{P}(X^I)$ is the DLR $(X^I, L^I, \rho^I)$, where $\rho^I(f, \varphi, g) \text{iff } \forall i \in I, \rho(f(i), \varphi(i), g(i))$;

• for all DLR $(X, L, \rho)$ and pure predicate $Q = (Y, M, \mu) \in \mathcal{P}(Y)$, the dependent product $P(X^\mu) \in \mathcal{P}(Y \times X \times X)$ is the exponential of $(X, L, \rho)$ and $Q$ in DLR.

The difference structure is as follows:

• for all DLR $(X, L, \rho)$, $X^D \in \mathcal{P}(X)$ is just $(X, L, \rho)$;

• for all DLR $(X, L, \rho)$, set $Y$ and predicate $P = \langle f, g \rangle \mathcal{P}' \in \mathcal{P}(Y \times X \times X)$, where $(Y \times X \times X \mid P) = \bigsqcup_{y \in Y, x', x'' \in X} \tilde{\mu}(f(y, x, x'), g(y, x, x'))$, the pullback $(Y \times X \mid \delta^1_X P)$ is $\bigsqcup_{y \in Y, x', x'' \in X} \tilde{\mu}(f(y, x, x'), g(y, x, x'))$, whereas $(Y \times X \times X \mid P \times \mathcal{P}' \mathcal{D})$ is $\bigsqcup_{y \in Y, x', x'' \in X} \tilde{\mu}(f(y, x, x'), g(y, x, x')) \times \tilde{\mu}(x', x'')$; we let then $r_{Y \times X \mid P}(\langle y, x, x' \rangle, \langle y, x, x'' \rangle) = \langle y, x, x' \rangle, \langle y, x, x'' \rangle$;

• for all DLR $(X, L, \rho)$, sets $Y, Z, W$, predicate $Q = \langle m, n \rangle \mathcal{P}' \in \mathcal{P}(Y \times X \times X)$, with $(Y \times X \times X \mid Q) = \bigsqcup_{y \in Y, x', x'' \in X} \tilde{\lambda}(m(y, x, x'), n(y, x, x')) \in \mathcal{P}(Y \times X \times X)$, pure predicate $P = (Z, M, \mu) \in \mathcal{P}(Z)$ and arrows $f, g : Y \times X \times X \rightarrow Z$, for all morphisms $c : (Y \times X \mid \delta^1_X Q) \rightarrow (Y \times X \times X \mid Q \times \langle f, g \rangle \mathcal{P})$ such that $c(\langle y, x, x', \langle s, c'(y, x, s) \rangle \rangle)$, with $c'(y, x, s) \in \tilde{\mu}(f(y, x, x'), g(y, x, x'))$, we can define a diagonal filler $j : (Y \times X \times X \mid Q \times \mathcal{P}' \mathcal{D}) \rightarrow (Y \times X \times X \mid Q \times \langle f, g \rangle \mathcal{P})$ by

$$(j(\langle y, x, x' \rangle, \langle y, x, x'' \rangle)) = \langle s, \sup\{c'(y, x, s) \mid \inf \tilde{\mu}(f(y, x, x'), g(y, x, x')) = \|f(y, x, x'), g(y, x, x')\| \leq c'(y, x, s)\} \rangle$$

We have that $(j(\circ_{Y \times X \mid Q})((y, x))) = \langle s, c'(y, x, s) \rangle$, since by separatedness $\rho(x, |x|, w)$ implies $w = x$, and $\inf \tilde{\mu}(f(y, x, x'), g(y, x, x')) = \|f(y, x, x'), g(y, x, x')\| \leq c'(y, x, s)$.

We leave to the reader to check the validity of all required coherence conditions.

**Extensionality** The DLR model of dTT satisfies the extensionality axioms (Ext1) and (Ext2). In fact, given simple types $A, B$, the interpretation of $DA \times DB$ is generated by the cartesian product in DLR of the interpretations of $DA$ and $DB$, which coincides with the product in $\mathcal{P}(A)$. Moreover, the interpretation of $DA \rightarrow DB$ in DLR, which coincides with the pullback along $\langle h, h \rangle$ of the dependent product of the interpretation of $DA$ and $DB$, where $h, h : Y \rightarrow Y \times X$ is given by $h(f)(x, y) = f(x)$, and this in turn coincides with the interpretation of the type $(\Pi x, y \in A)(DA(x, y) \rightarrow DB(f(x), g(y)))$.

One can also check that derivatives satisfy the equational rules $(j x a), (j x b), (j x c)$, $(j x 2 a)$ and $(j x 2 b)$.

**F Change Structures: Details**

Change structures (as defined in Section F) and functions form a category CS which is cartesian in particular, given CS $(X, \Delta X, \mathcal{P}, \mathcal{D})$ and $(Y, \Delta Y, \mathcal{P}', \mathcal{D}')$, their cartesian product is $(X \times Y, \Delta X \times \Delta Y, \mathcal{P} \times \mathcal{P}', \mathcal{D} \times \mathcal{D}')$, and their exponential is $(X, \Delta X, \mathcal{P})$, where $\mathcal{P} = \{ f \times \Delta X \rightarrow \Delta Y \mid \text{for all } x \in X \}$.

We now describe the dTT-structure associated with the forgetful functor $U : CS \rightarrow Set$.

For any set $X$, $\mathcal{P}(X)$ is made of CS with base set $X$, where for a CS $P = (X, \Delta X, \mathcal{P})$, $(X \times X \mid P) = \bigsqcup_{x, x' \in X} \Delta X(x, x')$, with projection $\pi_X : X \times X \rightarrow X \times X$, $\mathcal{P}(Y)$ is made of all pullbacks $\langle f, g \rangle \mathcal{P}'$, where $P \in \mathcal{P}(X)$ is a CS and $f, g : Y \rightarrow X$, with $(Y \times \langle f, g \rangle \mathcal{P}') = \bigsqcup_{x \in Y} \Delta X(f(x), g(x))$, with projection $\pi_Y : (Y \times \langle f, g \rangle \mathcal{P}') \rightarrow Y$.

An arrow in the slice category between $\bigsqcup_{x \in Y} \Delta X(f(x), g(x))$ and $\bigsqcup_{x \in Y} \Delta X(f'(x), g'(x))$ is given by a function $\varphi : Y \times \Delta X \rightarrow \Delta Y$, such that whenever $\varphi(y, dx) \in \Delta X(f(x), g(x))$, $(\varphi(y, dx))_2 \in \Delta X(f'(x), g'(x))$. One can check that:

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for all predicates $P = \langle f, g \rangle P \in \mathcal{P}(X)$ and $Q = \langle f', g' \rangle Q' \in \mathcal{P}(X)$, their product is $P \times Q = \langle \langle f, f' \rangle, \langle g, g' \rangle \rangle (P' \times Q')$ (where $P' \times Q'$ indicates the product of $P'$ and $Q'$).

The CS model of dTT satisfies the extensionality axiom $\text{Ext}_0$.

for all pure predicate $P \in \mathcal{P}(Y)$, where $P = (X, \Delta_X, \emptyset, \emptyset)$ and set $I$, the dependent product $\Pi I P \in \mathcal{P}(X^I)$ is the CS $(X^I, \Delta_X^I, \emptyset^I, \emptyset^I)$ where $(\Delta_X^I f) i$ is made by all functions $\varphi : I \rightarrow \Delta_X$ such that for all $i \in I$, $\varphi(i) \in \Delta_X f(x)$, $(f \otimes^I \varphi)(i) = f(i) \oplus \varphi(i)$ and $(f \otimes^I g)(i) = f(i) \otimes g(i)$.

for all CS $(X, \Delta_X, \emptyset, \emptyset)$ and pure predicate $Q \in \mathcal{P}(Y)$, where $Q = (Y, \Delta_Y, \emptyset', \emptyset')$, the dependent product $\Pi Y \in \mathcal{P}(X \times Y)$ is the CS $(X \times Y, \Delta_X \times \Delta_Y, \emptyset^*, \emptyset^*)$, where $(\Delta_X \times \Delta_Y)^* (f \times g)$ contains all functions $\varphi : X \times X \times \Delta_X \times \Delta_Y \rightarrow Y$ such that for all $x, x' \in X$ and $\,dx \in \Delta_X(x, x')$, $\varphi(x, x', dx) \in \Delta_Y f(x, x')$, and $(f \otimes^* \varphi)(x, x') = f(x, x') \oplus \varphi(x, x', x \otimes x')$, $(f \otimes^* g)(x, x', dx) = f(x, x') \otimes g(x, x')$.

The difference structure is as follows:

for all CS $(X, \Delta_X, \emptyset, \emptyset)$, $X^D \in \mathcal{P}(X)$ is the CS itself;

for all CS $(X, \Delta_X, \emptyset, \emptyset)$, set $Y$ and predicate $P = \langle f, g \rangle P' \in \mathcal{P}(Y \times X \times X)$, with $P' = (W, \Delta_W, \emptyset', \emptyset')$, the pullback $(Y \times X \times X, \delta_X^Y P)$ is $\Pi Y \in \mathcal{P}(Y')$,

while $(Y \times X \times X, \Delta_W)$ is $\Pi Y \in \mathcal{P}(X)$.

In fact, given simple types $A, B$, the interpretation of $D_{A \times B}$ is generated by the cartesian product in DLR of the interpretations of $D_A$ and $D_B$, which coincides with the product in $\mathcal{P}([A])$.

The CS model does not satisfy either $\text{Ext}_1$ or $\text{Ext}_2$; indeed, if the CS $X$ and $Y$ interpret two simple types $A$ and $B$, it seems that the exponential of $X$ and $Y$ in CS cannot be captured by a type of dTT.

G Cartesian Differential Categories: Details

A cartesian differential category (C CDC) is a left-additive cartesian category $\mathcal{C}$ such that for all arrow $f : X \rightarrow Y$ there exists an arrow $df : X \times X \rightarrow Y$ satisfying the axioms below:

$D1$. $df + g = df + dg$, $d0 = 0$;

$D2$. $df \circ (h + k, v) = df \circ (h, v) + df \circ (k, v)$, and $df \circ (0, v) = 0$;
D3. \( d(\text{id}) = \pi_1, d(\pi_1) = \pi_1 \circ \pi_1, d(\pi_2) = \pi_2 \circ \pi_1; \)

D4. \( d(\langle f, g \rangle) = \langle df, dg \rangle; \)

D5. \( d(g \circ f) = dg \circ (dg, g \circ \pi_2); \)

D6. \( d(df) \circ \langle g, 0 \rangle, \langle h, k \rangle = df \circ \langle g, k \rangle; \)

D7. \( d(df) \circ \langle \langle 0, h \rangle, \langle g, k \rangle \rangle = d(df) \circ \langle \langle 0, g \rangle, \langle h, k \rangle \rangle. \)

For an intuitive explanation of the axioms see [9].

A differential \( \lambda \)-category [10] is a CDC which is also a closed category, and where \( df \) further satisfies the axiom below:

D-curry. \( d(\lambda(f)) = \lambda(df \circ \langle \pi_1 \times 0, \pi_2 \times \text{Id} \rangle). \)

We now describe the dTT-structure associated with the identity functor \( \text{id}_C : C \rightarrow C \) on a differential \( \lambda \)-category \( C \).

We let \( X^n \) be a shorthand for \( X \times \cdots \times X \) \( n \) times. For any object \( X \), \( \mathcal{P}^n(X) \) contains all \( X^n \), with associated object \( X^{n+2} \) and projection \( \pi_X : X^{n+2} \rightarrow X \times X; \) \( \mathcal{P}(X) \) is made of all objects of the form \( X \times C \), with associated projection \( \pi_X : X \times C \rightarrow X \). For all \( f, g : Y \rightarrow X \), the pullback \( \langle f, g \rangle^*X \times C \) is just \( Y \times C \).

An arrow in the slice category \( C^{X}_{/Y} \) between \( X \times C \) and \( X \times D \) is an arrow \( h : X \times C \rightarrow D; \) one can check then that:

- The product of \( X \times C \) and \( X \times D \) in \( C^{X}_{/Y} \) is \( X \times (C \times D) \); moreover, for all pure predicate \( X^{n+2}, X^{m+2} \in \mathcal{P}^n(X) \), their product \( X^{n+m+2} \in \mathcal{P}^n(X) \);
- For all objects \( X, I \) and pure predicate \( X^{n+2} \in \mathcal{P}^n(X) \), the dependent product \( \Pi_I(X^{n+2}) \) is the pure predicate \( (X^I)^{n+2} \in \mathcal{P}^n(X^I) \); in fact, by the cartesian closure of \( C \) we have that \( \mathcal{C}(\pi_{W}, \pi_{W})^2(W^{n+2}), (X^I)^{n+2} \) \( \simeq \mathcal{C}(W^{n+2}, (X^I)^{n+2}) \), where \( \pi_W : W \times I \rightarrow W; \)
- For all objects \( X \) and pure predicate \( Y^{n+2} \in \mathcal{P}^n(Y) \), their dependent product is the pure predicate \( (Y \times X)^{n+2} \in \mathcal{P}^n(Y \times X) \); in fact, by the cartesian closure of \( C \) we have \( \mathcal{C}(\pi_{W}, \pi_{W})^2W^{n+2}, Y^{n+2} \) \( \simeq \mathcal{C}(W^{n+2}, (Y \times X)^{n+2}) \), where \( \pi_{W} : W \times I \times X \rightarrow W \).

The difference structure is as follows:

- For all objects \( X, X^0 \) is \( X^3 \in \mathcal{P}(X); \)
- For all objects \( X,Y, \) and predicate \( Q = V \times C \in \mathcal{P}(Y \times X \times X), \) where \( V = Y \times X \times X \), observe that \( \delta_X^Q = (Y \times X) \times C; \) we let then \( r_{Y,X\mid Q} : (Y \times X) \times C \rightarrow (Y \times X^2) \times (Q \times X) \) be \( \mu \times (\text{id}_Q \times 0), \) where \( \mu : Y \times X \rightarrow Y \times X^2 \) is \( \text{id}_Y \times \delta_X; \)
- For all objects \( X,Y,Z, \) predicate \( Q = (Y \times X) \times C \in \mathcal{P}(Y \times X), \) pure predicate \( P = Z^{n+2} \in \mathcal{P}(Z), \) arrows \( f, g : Y \times X \times X \rightarrow Z, \) the pullback \( \langle f, g \rangle^*P \) is \( Y \times X^2 \times Z^n; \) for all morphisms \( c : (Y \times X) \times Q \rightarrow (Y \times X^2) \times (Q \times Z^n) \) such that \( c = \langle \mu, (\text{id}_Q \circ \pi_2, c') \rangle \) for some \( c' : (Y \times X) \times Q \rightarrow Z^n \), we can define a diagonal filler \( j : (Y \times X^2) \times (Q \times X) \rightarrow (Y \times X^2) \times (Q \times Z^n) \) by

\[
\begin{align*}
j_2 &= \langle \text{id}_Q \circ (\pi_1 \circ \pi_2), c' \circ \nu \circ \pi_1 + \langle df \circ \langle \pi_1, \langle \langle 0, \pi_2 \circ \pi_2 \rangle, \langle 0, 0 \rangle \rangle \rangle \rangle \rangle \rangle \\
\end{align*}
\]

where \( \nu : Y \times X^2 \rightarrow Y \times X \) is \( \text{id}_Y \times \pi_1 \) and \( \langle u \rangle^n = \langle u, u, \ldots, u \rangle \) for \( n \) times.

We have that \( \langle j \circ r_{Y,X\mid Q} \rangle_2 = \langle \text{id}_Q \circ \pi_1, c' \rangle \) (using the fact that \( \nu \circ \mu = \text{id}_Y \times X \)).

Moreover, when \( Z = X, n = 1, f = \pi_2 \circ \pi_1, g = \pi_2 \circ \pi_2 \) and \( c' = 0 \) we have the \( j_2 = \text{id}_Q \times X \circ \pi_1 \). Hence both the \( \beta \)- and \( \eta \)-rules are satisfied.

**Extensionality** The CDC model of dTT satisfies the extensionality axioms [8] and [9]. This follows from the fact that, if simple types \( A, B \) are interpret as objects \( X, Y, \) \( D_{A \times B} \) is interpret by the pure predicate \( X \times Y \in \mathcal{P}(X \times Y), \) and \( D_{A \rightarrow B} \) is interpreted by the pure predicate \( Y \times X \in \mathcal{P}(Y \times X). \)

Moreover, derivatives satisfy the equational rules \( \langle J \times a \rangle, \langle J \times b \rangle, \langle J \times c \rangle, \) as well as \( \langle J \Lambda_0 \rangle \) and \( \langle J \Lambda_0 \rangle. \)