LIFTING CURVES SIMPLY

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Abstract. We provide linear lower bounds for $f_\rho(L)$, the smallest integer so that every curve on a fixed hyperbolic surface $(S, \rho)$ of length at most $L$ lifts to a simple curve on a cover of degree at most $f_\rho(L)$. This bound is independent of hyperbolic structure $\rho$, and improves on a recent bound of Gupta-Kapovich [GK]. When $(S, \rho)$ is without punctures, using [Pat] we conclude asymptotically linear growth of $f_\rho$. When $(S, \rho)$ has a puncture, we obtain exponential lower bounds for $f_\rho$.

1. Introduction

Let $S$ be a topological surface of finite type and negative Euler characteristic, and let $\rho$ be a complete hyperbolic metric on $S$. Let $\mathcal{C}(S)$ indicate the set of closed curves on $S$, i.e. the set of free homotopy classes of the image of immersions of $S^1$ into $S$. For $\gamma \in \mathcal{C}(S)$, let $\ell_\rho(\gamma)$ indicate the length of the $\rho$-geodesic representative of $\gamma$ on $S$, and let $\iota(\gamma, \gamma)$ indicate the geometric self-intersection number of $\gamma$. A closed curve $\gamma \in \mathcal{C}(S)$ is simple when its self-intersection $\iota(\gamma, \gamma)$ is equal to zero.

It is a corollary of a celebrated theorem of Scott [Sco] that each closed curve $\gamma \in \mathcal{C}(S)$ lifts to a simple closed curve in some finite-sheeted cover (i.e. $\gamma$ "lifts simply"). Recent work has focused on making Scott’s result effective [Pat]. As such, for $\gamma \in \mathcal{C}(S)$, let $\deg(\gamma)$ indicate the minimum degree of a cover to which $\gamma$ lifts simply.

We focus on two functions $f_\rho$ and $f_S$. Let the integer $f_S(n)$ be the minimum $d$ so that every curve $\gamma$ of self-intersection number $\iota(\gamma, \gamma)$ at most $n$ has degree $\deg(\gamma)$ at most $d$, and let the integer $f_\rho(L)$ be the minimum $d$ so that every curve $\gamma$ of $\rho$-length $\ell_\rho(\gamma)$ at most $L$ has degree $\deg(\gamma)$ at most $d$. Gupta-Kapovich have recently shown:

**Theorem 1.** [GK, Thm. C, Cor. 1.1] There are constants $C_1 = C_1(\rho)$ and $C_2 = C_2(S)$ so that

$$f_\rho(L) \geq C_1 \cdot (\log L)^{1/3} \quad \text{and} \quad f_S(n) \geq C_2 \cdot (\log n)^{1/3}.$$

Their work analyzed the ‘primitivity index’ of a ‘random’ word in the free group, exploiting the many free subgroups of $\pi_1 S$ (e.g. subgroups corresponding to incompressible three-holed spheres, or pairs of pants) to obtain the above result. We also exploit the existence of free subgroups of $\pi_1 S$, but instead of following in their delicate analysis of random walks in the free group.
group, we analyze explicit curves on $S$. The chosen curves are sufficiently uncomplicated to allow a straightforward analysis of the degree of any cover to which the curves lift simply. As a consequence, we provide the improved lower bounds:

**Theorem 2.** We have $f_S(n) \geq n + 1$. Moreover, let $B = B(S)$ be a Bers constant for $(S, \rho)$, and let $\epsilon > 0$. Then there is an $L_0 = L_0(\rho, \epsilon)$ so that, for any $L \geq L_0$,

$$f_\rho(L) \geq \frac{L}{B + \epsilon}.$$

Recall the theorem of Bers [Ber]: There is a constant $B = B(S)$ so that, for every hyperbolic metric $\rho$ on $S$, there is a maximal collection of disjoint simple curves on $S$ with each curve of $\rho$-length at most $B$. Such a constant $B$ is called a **Bers constant**, and such a collection of curves is called a **Bers pants decomposition** for $(S, \rho)$. It is interesting to note that the constant in the lower bound for $f_\rho$ in Theorem 2 is independent of the metric $\rho$.

**Remark.** The proof of Theorem 2 follows from an analysis of an explicit sequence of curves $\{\gamma_n\}$. These curves are also analyzed by Basmajian [Bas], where it is shown that they are in some sense the 'shortest' curves of a given intersection number: The infimum of the length function $\ell(\gamma_n)$ on the Teichmüller space of $S$ is asymptotically the minimum possible among curves with self-intersection $\iota(\gamma_n, \gamma_n)$ [Bas, Cor. 1.4].

Combined with work of Patel [Pat, Thm. 1.1] (see the comment of [GK, p. 1]), in many cases Theorem 2 implies a determination of the order of growth of $f_\rho$. We have:

**Corollary 3 (Linear growth of $f_\rho$).** Suppose $(S, \rho)$ is without punctures. There exist constants $C_1 = C_1(S), C_2 = C_2(\rho)$, and $L_0 = L_0(\rho)$ so that, for any $L \geq L_0$,

$$C_1 \cdot L \leq f_\rho(L) \leq C_2 \cdot L.$$

Recall that, when $S$ has boundary, we say that $(S, \rho)$ has a **puncture** if the closed curve homotopic to a boundary component has no geodesic representative. When $(S, \rho)$ does have a puncture, we are not aware of upper bounds for $f_\rho$. In fact, the hypothesis above is essential. We show:

**Theorem 4.** Suppose $(S, \rho)$ is a hyperbolic surface with a puncture. For any $\epsilon > 0$, there is $L_0 = L_0(\rho, \epsilon)$ so that, for any $L \geq L_0$,

$$f_\rho(L) \geq e^{L/\epsilon^2}.$$

This theorem indicates at least that Patel's upper bounds cannot hold in the punctured setting: If $(S, \rho)$ is a hyperbolic surface with a puncture, the minimal degree of a cover to which a given curve $\gamma$ lifts to a simple curve cannot be bounded linearly in the curve’s length $\ell_\rho(\gamma)$.

There are other avenues for further investigation. It would be natural to seek upper bounds for $f_S(n)$ (cf. [Riv, p. 15]), since no such bound follows
from [Pat]. One might also investigate whether the constant $C_2$ in the upper bound in Corollary 3 can be made independent of $\rho$, as with the lower bound. Finally, one could explore the set of curves of self-intersection number exactly $n$. For instance: Among the finitely many mapping class group orbits of curves $\gamma$ with self-intersection $n$, which maximize $\deg(\gamma)$?

**Outline of the paper.** In §2 we introduce a sequence of curves $\{\gamma_n\}$ on a pair of pants and analyze the degrees $\deg(\gamma_n)$, and in §3 we deduce Theorem 2 and Theorem 4 as straightforward consequences.

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### 2. Analysis of a Certain Curve Family

Let $P_0$ be a pair of pants. Identify $\pi_1(P_0, p)$ with a rank-2 free group $F$, with generators $a$ and $b$ as pictured in Figure 1. Let $\gamma_n$ indicate the closed curve given by the equivalence class of $a \cdot b^n$.

![Figure 1. The pair of pants $P_0$, with generators $a$ and $b$.](image)

The following lemma is neither new (see [Bas, Prop. 4.2]) nor surprising (see Figure 2), but we include a sketch of a proof for completeness:

**Lemma 5.** For $n \geq 0$, the curve $\gamma_n$ has $\iota(\gamma_n, \gamma_n) = n$.

**Proof sketch.** It is not hard to pick a representative of $\gamma_n$ that has self-intersection $n$, so that $\iota(\gamma_n, \gamma_n) \leq n$ (see Figure 2 for $n = 4$). On the other hand, it is also not hard to check that there are no immersed bigons for this chosen representative of $\gamma_n$: For every pair of intersection points, the concatenation of any pair of arcs of $\gamma_n$ that connect the two points forms an essential curve. The ‘bigon criterion’ of [FM, §1.2.4] can be altered straightforwardly to an ‘immersed bigon criterion’ in the setting of curves with self-intersections, and so the lack of immersed bigons guarantees that the chosen representative of $\gamma_n$ is in minimal position. □
We use Lemma 5 to estimate $\deg(\gamma_n)$, a calculation reminiscent of [GK, Lemma 3.10]. The following proposition is the main tool in our analysis.

**Proposition 6.** We have $\deg(\gamma_n) \geq n + 1$.

**Proof.** Towards contradiction, suppose there is a cover $P' \to P_0$ of degree $k \leq n$, so that $\gamma_n$ lifts to a simple curve $\gamma$. Draw $P_0$ as a directed ribbon graph with one vertex $p$ and the two edges labeled by $a$ and $b$, and $P'$ as a directed ribbon graph with vertices $p_1, \ldots, p_k$ and $2k$ directed edges, $k$ with $a$ labels and $k$ with $b$ labels. Choose an orientation for $\gamma$ so that $\gamma$ consists of a directed $a$ edge followed by $n$ directed $b$ edges. After relabeling, we may assume that the unique $a$ edge of $\gamma$ is followed by $p_1$.

Starting from $p_1$ and reading the vertices visited by $\gamma$ in order, the vertex that immediately follows the $n$ consecutive $b$ edges of $\gamma$ is $p_l$, where $l$ is equivalent to $n + 1$ modulo $k$. Finally, $\gamma$ follows an $a$ edge from $p_l$ to $p_1$. See Figure 3 for a schematic.

This implies that there is an incompressible embedded pair of pants $P''$ in $P'$ that contains $\gamma$ (see Figure 4 in the case that $k \nmid n$ – the other case is straightforwardly similar). After identifying $P''$ with $P_0$ appropriately, the closed curve $\gamma$ is given by the equivalence class of $a \cdot b^s$, where $s = \left\lfloor \frac{n}{k} \right\rfloor \geq 1$. By Lemma 5 this curve is not simple, a contradiction. $\square$

**Remark.** In fact, one can show that $\deg(\gamma_n) = n + 1$, but the precise computation of $\deg(\gamma_n)$ is irrelevant.

3. **Proofs of Theorem 2 and Theorem 4**

**Proof of Theorem 2.** First suppose that $P$ is any pair of pants on $S$, with any choice of identification of $P$ with $P_0$, so that we may view $\{\gamma_n\}$ as a sequence of closed curves on $S$. Suppose that $\pi : S' \to S$ is a cover of $S$ so that $\gamma_n$ lifts to a simple curve $\gamma'$. Let $P'$ be the component of $\pi^{-1}(P)$ containing $\gamma'$. We obtain a cover $\pi|_P : P' \to P$, so that the degree of $S' \to S$ is at least the degree of $P' \to P$. By Proposition 6, the degree of $P' \to P$
is at least $n + 1$. Thus $\deg(\gamma_n) \geq n + 1$, and the bound for $f_S(n)$ follows immediately from Lemma 5.

We turn to the bound for $f_\rho(L)$. Let $P$ be a pair of pants with geodesic boundary in a Bers pants decomposition for the hyperbolic metric $\rho$. Let $\alpha$ and $\beta$ be two cuffs of $P$, and let $\delta$ indicate the simple arc connecting $\alpha$ to $\beta$. Let the $\rho$-length of $\delta$ be given by $\ell_\rho(\delta) = D$. Identify $P$ with $P_0$ so that $\alpha$ is in the conjugacy class of $a$ and $\beta$ is in the conjugacy class of $b$, and consider the closed curves $\{\gamma_n\}$ in $P$. Evidently,

$$\ell_\rho(\gamma_n) \leq \ell_\rho(\alpha) + n \cdot \ell_\rho(\beta) + 2\ell_\rho(\delta) \leq B(1 + n) + 2D.$$  

Given $\epsilon > 0$, for large $n$ the $\rho$-lengths satisfy $\ell_\rho(\gamma_n) \leq n \cdot (B + \epsilon)$. Let

$$n = n(L) = \left\lfloor \frac{L}{B + \epsilon} \right\rfloor,$$  

Figure 3. A supposedly simple lift $\gamma$ of $\gamma_n$ to the cover $P' \to P_0$, where $p_1$ is the point above $p$ that follows the unique $a$ edge of $\gamma$. 

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so that $\ell_\rho(\gamma_n) \leq L$ for large enough $L$. Thus, for large enough $L$, we have

$$f_\rho(L) \geq \deg(\gamma_n) \geq n + 1 \geq \frac{L}{B + \epsilon}$$

as desired.

**Proof of Theorem 4.** Assume first that $(S, \rho)$ is not the three-punctured sphere. As before, we choose a Bers pants decomposition for $(S, \rho)$, letting $P$ be a pair of pants containing a puncture as a boundary component. Note that by assumption there is a pants curve of $P$ with hyperbolic holonomy. Identify $P$ with $P_0$ so that $b$ is homotopic to a curve that winds once around the puncture, and $a$ is homotopic to a pants curve with hyperbolic holonomy. Consider again the sequence of curves $\{\gamma_n\}$ on $S$.

We assume the upper half plane model for the hyperbolic plane $\mathbb{H}^2$. By conjugating the holonomy representation of $\rho$ appropriately, we may arrange for the holonomy around the puncture to be the transformation $z \mapsto z + 1$, 

\[\text{Figure 4. An incompressible pair of pants } P'' \subset P' \text{ contains } \gamma.\]
and so that there is a lift of $a$ to the hyperbolic plane $\mathbb{H}^2$ that is contained in a Euclidean circle centered at 0, say $|z| = s$.

There is a lift of a curve freely homotopic to $b^n$ that starts at $is$, travels vertically along the imaginary axis to $iy$, travels horizontally to $n + iy$, and vertically down to $n + is$. Let $\beta_y'$ indicate the projection of this curve to $P$, and note that by construction its starting and ending point are in common, and on the geodesic cuff $\alpha$. We may thus concatenate (a parametrization of) $\alpha$ with $\beta_y'$, and the curve so obtained is homotopic to $\gamma_n$.

An elementary computation shows that
\[
\ell_\rho(\beta_y') = 2 \log\left(\frac{y}{s}\right) + \frac{n}{y}.
\]
Taking $y = n$ we find
\[
\ell_\rho(\gamma_n) \leq \ell_\rho(\alpha) + \ell_\rho(\beta_n') \leq B - 2 \log s + 1 + 2 \log n.
\]
Given $\epsilon > 0$, for large $n$ the $\rho$-lengths satisfy $\ell_\rho(\gamma_n) \leq (2 + \epsilon) \log n$, and the result follows as in the proof of Theorem 2: Let
\[
n = n(L) = \left\lfloor e^{\frac{L}{2}} \right\rfloor,
\]
so that $\ell_\rho(\gamma_n) \leq L$ for large enough $L$. Thus, for large enough $L$, we have
\[
f_\rho(L) \geq \deg(\gamma_n) \geq n + 1 \geq e^{\frac{L}{2}},
\]
as desired.

If, on the other hand, $(S, \rho)$ is the unique hyperbolic structure on the three-punctured sphere, then we identify $P_0$ with $S$ arbitrarily. A straightforward calculation (see [Bas, eq. (29)]) shows that
\[
\ell_\rho(\gamma_n) = 2 \cosh^{-1}(1 + 2n).
\]
The latter is asymptotic to $2 \log n$ as $n$ goes to infinity, and so, given $\epsilon > 0$, for large enough $n$ we have $\ell_\rho(\gamma_n) \leq (2 + \epsilon) \log n$. The result follows. □

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