Visualizing the turbulent energy cascade: Fourier and physical space

Ryan McKeown\(^1\), Alain Pumir\(^2\), Shmuel M. Rubinstein\(^3\), Michael P. Brenner\(^1\), and Rodolfo Ostilla-Mónico\(^4,5\)

\(^1\)School of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138, USA
\(^2\)Univ Lyon, ENS de Lyon, Laboratoire de Physique, F-69342 Lyon, France
\(^3\)Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem, Israel
\(^4\)Department of Mechanical Engineering, University of Houston, Houston, TX 77204, USA
\(^5\)Escuela Superior de Ingeniería, Universidad de Cádiz, Cádiz, Spain

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The transfer of kinetic energy from large to small scales is a hallmark of turbulent fluids. While it is generally well understood in a statistical sense, a mechanistic understanding of the exact processes is still lacking. In this manuscript, we filter the velocity field in bands of wavenumbers distributed logarithmically to investigate the transfer of energy in Fourier space, an approach which also provides a way of visualizing the energy cascade in real space. We use this method to study how energy is transferred to small scales in different flow configurations. In the case of a statistically steady homogeneous isotropic turbulent flow at moderate Reynolds numbers, we demonstrate that the transfer between bands of wavenumbers is well described by a phenomenological model originally proposed by Tennekes & Lumley\(^{[1972]}\). We then apply this method to investigate the formation of small scales during the interaction between two vortex tubes and show that the transfer of energy in these problems correlates with the dynamics of vorticity formation at ever decreasing scales. This enticing correlation between vortices and energy transfer, however, does not fully capture the transfer of energy in a structureless homogeneous isotropic turbulent flow, where strain also plays an important role.

Key words: Turbulent cascade, vortex dynamics, energy transfer

1. Introduction

The cornerstone of our understanding of turbulence rests on the notion that for three-dimensional incompressible flows at a very high Reynolds number, energy is transferred from the large scale at which the flow is stirred down to very small scales where fluid motion is damped by viscosity (Taylor\(^{[1935]}\); de Karman & Howarth\(^{[1938]}\); Kolmogorov\(^{[1941]}\)). This notion was originally applied to model the decay of turbulence, in particular that of homogeneous isotropic flows, and later extended to turbulent flows in general (Batchelor & Townsend\(^{[1947]}\); [1948a, b]; Moffatt\(^{[2002]}\)). Due to this, the canonical problem of decaying turbulence generated by a grid in a wind tunnel has proved an ideal testing ground for ideas and concepts around the behaviour of turbulence (Uehiro\(^{[1963]}\); van Atta & Chen\(^{[1968]}\); Warhaft & Lumley\(^{[1978]}\); Sreenivasan \textit{et al.}\(^{[1980]}\); Antonia \textit{et al.}\(^{[2003]}\); Sinhuber\(^{[2008]}\)).
However, important questions concerning the structure of turbulence and the means by which energy transfer occurs remain unanswered (Lumley 1992; Cardesa et al. 2015; Vassilicos 2015; Yang et al. 2018). The importance of understanding the nature of how energy cascades down to the dissipative scale cannot be overstated. In fact, the development of subgrid-scale models in the context of large-eddy simulations (LES), one of the most used numerical techniques in fluid dynamics, amounts to providing a parameterization of the energy transfer from the resolved to the unresolved scales of a flow in a numerical simulation (Smagorinsky 1963; Bardina et al. 1980; Metais & Lesieur 1996). Due to its significance, this problem has been extensively studied, even in the simplest case of homogeneous, isotropic turbulence, both experimentally (Liu et al. 1994; Tao et al. 2002; Katz & Sheng 2010) and numerically (Menon et al. 1994; Kerr et al. 1996; Meneveau & Katz 2000). One of the main research questions is quantifying the locality of the energy transfer in Fourier space. Detailed studies of the contribution of triads of wavenumbers in the energy transfer point to a subtle structure: some of the triads contribute to a nonlocal energy transfer (Domaradzki & Rogallo 1990), with possible consequences to the structure and dynamics of the smallest structures in the flow (Brasseur & Wei 1994; Yeung et al. 1995). The locality of the energy transfer in Fourier space can also be directly studied by representing the flux of energy between wavenumber bands (Favier et al. 2014; Verma et al. 2018; Alexakis et al. 2005), a method which we will use extensively here. We note that the analysis in terms of wavenumber bands can also be used for more complex flows, e.g. involving convection and rotation (Favier et al. 2014), and in the case of magnetohydrodynamic turbulence, this approach reveals subtle exchanges between kinetic and magnetic energy (Mininni et al. 2005).

The notion of energy transfer is also particularly useful when investigating the fundamental question of how an ordered, laminar flow evolves in time to become turbulent. We stress that visualization tools, from experiments as well as from direct numerical simulations (DNS) of the Navier-Stokes equations, have proven essential to identify the physical mechanisms governing the formation of small-scale vortices. It was recently realized that the conceptually simple configuration of two colliding vortex rings (Lim & Nickels 1992; McKeown et al. 2018; 2020) leads, at sufficiently high Reynolds numbers, to the development of a disordered flow with a $k^{-5/3}$ energy spectrum, consistent with Kolmogorov prediction for turbulence (Kolmogorov 1941). Remarkably, this spectrum results from a cascade of instabilities (Tao 2016; Brenner et al. 2016) during the interaction between the vortices, which is responsible for the formation of small-scale flow structures (McKeown et al. 2020). The instability that mediates the cascade, known as the elliptical instability (Kerswell 2002; Leweke et al. 2016), is induced by the deformation of the core of each filament due to the strain generated by the opposing vortex (McKeown et al. 2020). In another related flow, i.e. the reconnection between two vortex filaments at sufficiently high Reynolds number, Yao & Hussain (2020) observed the development of small scales of motion with a Kolmogorov $k^{-5/3}$ spectrum, and also suggested an “avalanche” of iterative reconnections as a possible mechanism for the cascade.

An understanding of the underlying energy transfer mechanisms in these conceptually simple flow configurations could possibly lead to insights on the cascading motion in turbulence (Ostilla-Monico et al. 2021; Yao & Hussain 2022). In this respect, it is appropriate to stress that the complex motion in turbulent flows has been conjectured to involve elementary coherent structures (Hussain 1986); vortex tubes are likely candidates to capture these important dynamical processes (Siggia 1981; Jimenez et al. 1993; Goto 2008; Buaria et al. 2020). This has provided an essential motivation for the study of the interactions of vortex tubes (Siggia & Pumir 1985; Brenner et al. 2016; Moffatt &
Kimura (2019) and suggests the use of a common approach to study the transfer of energy in both fully turbulent flows as well as various configurations of interacting vortex tubes. By comparing the energy transfer in these systems of varying complexity, we can thereby investigate whether iterative self-similar processes play a role. Indeed, from a theoretical point of view, it is often assumed that the transfer of energy occurs in a self-similar manner. This hypothesis is clearly made in the phenomenological approach of Tennekes & Lumley (1972), who derived from elementary fluid mechanical considerations an explicit scaling expression for the energy transferred between scales. The work of Yoneda et al. (2021) also postulates a hierarchical structure of vortices to describe vortex stretching, inspired by the numerical results of Goto et al. (2017) and Motoori & Goto (2019).

To investigate these issues, we proceed by decomposing the velocity field into logarithmically spaced shells of wavenumbers, \( I_P \), such that \( 2^{-1} k_f/\sqrt{2} \leq |k| \leq 2^{-1} k_f \times \sqrt{2} \), where \( k_f \) is a fixed wavenumber corresponding to the largest scale flow structure. We introduce the corresponding band-passed filtered velocity fields \( u_P \), from which we can define associated quantities such as the vorticity, rate of strain, and kinetic energy. The rate of energy transferred from the band of modes \( I_P \) to the band \( I_Q \) is \( T_{P,Q} = -u_Q \cdot (\nabla \times u_P) \). This equation allows us to identify the regions of the flow where \( T_{P,Q} \) is positive, indicating that energy is transferred from the band \( I_P \) to \( I_Q \) (Favier et al. 2014; Kunnen et al. 2016). We can then use this to correlate the regions where the flow transfers energy with the presence of regions of high vorticity or high strain in the flow, and elucidate the mechanisms for energy transfer across scales.

Our work is organized as follows. In the Methods section (Sec. 2), we present our approach to study energy transfer, and describe our numerical methods. The case of a statistically stationary homogeneous turbulent flow at moderately large Reynolds number (\( Re_\lambda = 210 \)) is considered in Section 3. In particular, we quantify the locality of the energy transfer in Fourier space, and demonstrate that the phenomenological approach of Tennekes & Lumley (1972) provides a compelling description of our data. We consider next in Sec. 4 the case of statistically unsteady flows, starting from simple vortex tube configurations: one with two tubes initially at a \( \pi/2 \) angle, and another one with two antiparallel tubes. We analyze the generation of small-scales in the flow in Fourier space, and show that intense energy transfer correlates with the appearance of intense vortex tubes. This correlation persists, up to the point where the flow develops a turbulent regime, with a \( k^{-5/3} \) spectrum. We demonstrate, however, that this simple correlation between the observed vortical structures and energy scales does not hold in more general cases of homogeneous and isotropic turbulence and contextualize our results with a recent analysis of vortex stretching. Finally, we present our conclusions in Section 5.

2. Methods

2.1. Analysis of energy transfer

To study the transfer from one band of modes to another, we consider shells of wavenumbers, \( I_P \), defined by:

\[
I_P = \{(k_x, k_y, k_z), \text{ such that } 2^{P-1} k_f/\sqrt{2} \leq |k| = \sqrt{k_x^2 + k_y^2 + k_z^2} \leq 2^{P-1} k_f \times \sqrt{2} \}
\]  

(2.1)

where \( k_f \) is a fixed wavenumber, chosen here to be \( k_f = 2.3 \). In elementary terms, the sets \( I_P \) correspond to spherical shells in wavenumber space, around the wavenumber \( K_P = 2^{P-1} k_f \), and of width \( \sim k_f \). With the value of \( k_f \) chosen here, the number of Fourier modes in the lowest shell, \( I_1 \), is \( \sim 200 \), which minimizes the effect of discretization.
Furthermore, we define the band-passed filter $u_P$ as:

$$u_P(x,t) = \sum_{(k_x,k_y,k_z) \in I_P} \hat{u}(k_x,k_y,k_z) \exp[i(k_xx + k_yy + k_zz)]$$  \hspace{1cm} (2.2)

In the same spirit, we define $\omega_P$ as the curl of $u_P$, the strain rate $S_P$ as the symmetric part of the derivative tensor of $u_P$, and the kinetic energy as $\epsilon_P(x,t) = \frac{1}{2}u_P(x,t)^2$.

To study the rate of energy transfer between scales, one can simply deduce the equation of evolution for $u_P(x,t)$ directly from the Navier-Stokes equations. Each of the velocity component, $u_P$, filtered in the band $I_P$ is advected by the full velocity field $u$. To analyze the energy transfer between scales, we derive the evolution equation for $e_P(x,t)$ from elementary manipulations of the Navier-Stokes equations:

$$\frac{\partial}{\partial t} e_P(x,t) = \sum_Q T_{P,Q} + D_P + F_P$$  \hspace{1cm} (2.3)

where the sum in the RHS of Eq. (2.3) extends over all shells. The quantity $T_{P,Q}$ is the rate of energy transfer between the modes in band $P$ and band $Q$:

$$T_{P,Q}(x,t) = -u_Q \cdot (u \cdot \nabla)u_P(x,t)$$  \hspace{1cm} (2.4)

while $D_P$ and $F_P$ represent the rates of dissipation and forcing acting on the modes in the band $I_P$. An elementary estimate of the dissipation is given by $D_P \approx -\nu K_P^2 \epsilon_P$.

It is customary to integrate $T_{P,Q}$ over space to define the total rate of exchange of energy between shells:

$$T_{P,Q}(t) = \int T_{P,Q}(x,t) dx$$  \hspace{1cm} (2.5)

A simple integration by parts of Eq. (2.5) shows that $T_{P,Q} = -T_{Q,P}$. This demonstrates that the rate of energy transfer between bands of wavenumbers conserves energy overall. For our purpose, it is convenient to represent the total rate of energy transfer $T_{P,Q}$, as introduced e.g. by Favier et al. (2014), in the $(P,Q)$ plane to represent the exchange of energy between different modes. In practice, this is done by color coding the regions of the plane $(P,Q)$, such that red corresponds to a positive transfer from mode $P$ to mode $Q$, and blue corresponds to a negative transfer from mode $P$ to mode $Q$ (c.f. Fig. 1). Furthermore, because in a turbulent flow, energy is ultimately transferred down to very large wavenumbers (very small scales) where it is dissipated by viscosity, we systematically normalize $T_{P,Q}$ by dividing by the dissipation rate in the flow, $\epsilon = \nu \langle \omega^2 \rangle$.

All of the previously defined quantities have been determined numerically in our DNS, and are presented in the analysis below. The novel aspect of our work consists in representing not only the integral quantities $T_{P,Q}$, but also in visualizing the field $T_{P,Q}(x,t)$ in physical space. This quantity allows us to monitor also where the energy transfer occurs in real space. We note, however, that whereas the energy transfer integrated over bands $P$ and $Q$, $T_{P,Q}$ is unambiguously defined as an integral by Eq. (2.5), $T_{P,Q}$ is defined only up to a divergence. An elementary calculation leads us to the equality: $T_{P,Q} + T_{Q,P} = -\nabla \cdot [u(u_P \cdot u_Q)]$. The divergence term integrates to 0, which guarantees that the global balance $T_{P,Q} + T_{Q,P} = 0$, even if locally $T_{P,Q} + T_{Q,P}$ are not equal to zero.

2.2. Details of the DNS

The DNS for the two antiparallel vortex tubes were carried out using the energy conserving finite-difference code described in Appendix C of McKeown et al. (2018). In essence, this is a centered, energy-conserving, second-order finite difference code which has a fractional time-stepping mechanism. The nonlinear terms are treated explicitly
using a third-order Runge-Kutta scheme, while the viscous term is treated implicitly using a second-order Crank-Nicholson method. The simulations shown here were obtained in a cubic periodic domain of side length $L$, with up to $540^3$ grid points for the circulation Reynolds number $Re_f = \Gamma/\nu = 6000$, where $\Gamma$ is the circulation of the vortices and $\nu$ the kinematic viscosity of the fluid. The other parameter that defines the system is the ratio $a/b$, where $a$ is the radius of the vortex cores, and $b$ is the initial distance between the vortices. For this configuration, $a/b = 0.4$, and $a/L = 0.06$.

We simulated statistically stationary turbulence with two types of forcing: a random isotropic forcing (Eswaran & Pope 1988), with the forcing parameters chosen following Chouippe & Uhlmann (2015), and we also used a forcing inspired by Goto et al. (2017) of the form:

$$f = \begin{bmatrix} -\sin(x) \cos(y) \\ \cos(x) \sin(y) \\ 0 \end{bmatrix}.$$  

This forcing favors the appearance in the box of 4 vortex tubes parallel to the $z$–axis, and centered in the $(x,y)$ plane at $\pi/(2(n+1,2m+1)$ with $n, m = 0$ or 1.

We also used a spectral code to perform DNS of the Navier-Stokes equations, both to simulate homogeneous isotropic turbulence (HIT), and also to investigate the configuration with two vortices oriented at an angle $\pi/2$ (Ostilla-Mónico et al. 2021). The code solves the Navier-Stokes equations in a triply periodic box $[-\pi,\pi]^3$, as described in (Pumir 1994). The DNS study of HIT was carried out using $512^3$ Fourier modes, at a Reynolds number $Re_\lambda = 210$. The quality of the spatial resolution can be judged by the product $k_{max}\eta$, where $k_{max}$ is the largest Fourier mode included in the simulation, and $\eta = (\nu^3/\epsilon)^{1/4}$ is the Kolmogorov length. The value of $k_{max}\eta \approx 1.5$ shows that the small-scales are satisfactorily resolved. We compared the results of the two different codes to study HIT, as shown in the following section, and verified that the results presented here are robust.

To simulate two vortex tubes originally at an angle $\pi/2$, we inserted two vortex tubes in the cubic domain, whose centerlines are given by $(x = y, z = d/2)$ and $(x = -y, z = -d/2)$, both with a circulation $\Gamma$, see (Ostilla-Mónico et al. 2021). The signs of the circulations are selected so that the two vortices close to $x = 0$ and $y = 0$ move in the direction $x > 0$. Numerically, we set $\Gamma = 1$ and $d = 0.9$. We run the simulation at several resolutions, with $256^3$ up to $512^3$ grid points. The core size of the vortices was chosen to be $\sigma = 0.4/\sqrt{2}$ slightly smaller than $d/2$.

For each flow configuration, the resulting flow parameters (i.e. the vorticity modulus or energy transfer rate) are examined in the 3D visualization program Dragonfly (Object Research Systems).

3. Energy transfer in statistically stationary turbulent flows

In this section, we consider the energy transfer in the conceptually simple case of statistically stationary flows. We first re-analyze the transfer of energy for hydrodynamic turbulence at a moderate Taylor Reynolds number ($Re_\lambda = 210$), as this allows us to provide a relevant point of comparison for the study of time-dependent flows investigated in the following Section (Sec. 4).

3.1. Numerical results

Fig. 1a shows the energy transfer in a simulation of an homogeneous isotropic flow at $Re_\lambda = 210$. The quantity $T_{P,Q}$, defined by Eq. (2.4) and (2.5), was calculated by
analysing the flow using 120 snapshots at different times, uniformly distributed over a total duration of approximately one large eddy turnover time. In this flow, the range of inertial scales extends in wavenumber space, from $k \sim 1$, corresponding to the large scale, up to $k \sim 1/(10\eta)$, which corresponds to a range $1 \lesssim k \lesssim 16$. With the value of $k_f = 2.3$ chosen to define the bands of wavenumbers, it means that the four first bands of wavenumbers correspond to the inertial range, and the subsequent ones correspond to dissipative scales. To double-check this, we confirmed that the dissipation $D_P$ in Eq. (2.3) is less that the transfer to small scales for $P \leq 4$.

In all cases, we find that the rate of energy transfer is positive from bands of wavenumbers $P$ to bands of wavenumbers $Q$, with $P < Q$. This corresponds to a transfer of energy towards small-scales, a characteristic property of turbulent flows in three-spatial dimensions. Interestingly, Fig. 1a shows that the transfer of energy is mostly concentrated around the diagonals, with a sharp decay away from the principal diagonal. The dominance of the $T_{PQ}$ terms close to the diagonal, i.e. from $P \rightarrow Q = P + 1$, is qualitatively consistent with the notion that energy transfer is local, as documented many times [Verma et al. 2018; Alexakis et al. 2005]. We nonetheless observe that the transfer extends to larger modes, $P \rightarrow Q = P + 2$, $P \rightarrow Q = P + 3$ etc. The mechanism through which energy is transferred from a band of wavenumber to a more remote band in Fourier space provides a way to quantify the locality of the transfer.

To study this question, Fig. 1b shows the amount of transfer from $P = 1$ (“+” symbols), or $P = 2$ (“×” symbols) to $Q = P + n$. Fig. 1b suggests an exponential decay of $T_{PQ}$ ($Q = P + n$) when $n$ increases. Since $n$ refers to bands centered around a wavenumber $\sim 2^n$ in Fourier space, the trend observed in Fig. 1b in effect suggests a power law behavior as a function of the wavenumber. This decay of the energy transfer $T_{PQ}$ from one wavenumber, $k_P$, to higher wavenumbers $k_Q$ is consistent with data obtained at lower Reynolds numbers (see e.g. Figs. 2 and 3 of Alexakis et al. 2005) and shows that energy can in fact flow to bands of significantly larger wavenumbers. We also note that the exponential decays seem to differ slightly between $P = 1$ and $P = 2$. We interpret this as an effect of the limited inertial range in the simulation.
3.2. The Tennekes and Lumley phenomenological model of energy transfer

We can contextualize these results with existing theories. Much work has been done to describe the energy transfer in Fourier space using the interaction between wavenumbers, originating from the nonlinear term in the Navier-Stokes equations, yet without reference to physical space. Interestingly, [Tennekes & Lumley (1972)](see Chapter 8, section 2) have proposed an approach of energy transfer from physical space considerations, which is very useful for analyzing our own data.

We begin by introducing the standard energy spectrum of the flow, $E(k)$. In the following, we will estimate the kinetic energy in a band of wavenumbers centered around $k$, i.e. between $k/\sqrt{2}$ and $k\times\sqrt{2}$, as $\sim E(k)\times k$. With this notation, the strain, $s$, generated by eddies in a band of wavenumbers centered at $k$ scales as $s(k)\sim (k^3 E(k))^{1/2}$.

[Tennekes & Lumley (1972)] argued that energy transfer from a band of wavenumbers centered around $k_P$ (corresponding to scales $\sim 2\pi/k_P$) to a band of wavenumbers centered around $k_Q$ (scales $\sim 2\pi/k_Q$) results from the deformation induced by the strain originating from the large scale, $s(k_P)$, on smaller scale eddies. This deformation, however, is opposed by the tendency of strain at smaller scales $s(k_Q)$ to restore local isotropy. To take into account these two competing effects, [Tennekes & Lumley (1972)] postulate the following elementary form for the rate of energy transferred to eddies of scales $\sim 2\pi/k_P$ from eddies of scale $\sim 2\pi/k_Q$:

$$T_{P,Q} \sim \frac{s(k_P)^2}{s(k_Q)} \times [k_Q E(k_Q)].$$  \hspace{1cm} (3.1)

In a homogeneous and isotropic turbulent flow as studied here, and restricting ourselves to wavenumbers in the inertial range, so that the energy spectrum behaves as $E(k)\sim \epsilon^{2/3}k^{-5/3}$, Eq. (3.1) reduces to the simple scaling form:

$$T_{P,Q} \sim \epsilon \left(\frac{k_P}{k_Q}\right)^{4/3}.$$  \hspace{1cm} (3.2)

Organizing the wavenumber space in bands by octave, as introduced by Eq. (2.1), the energy transfer between bands of wavenumbers $k_P$ and $k_Q$ is a simple power law:

$$T_{P,Q} \sim \epsilon 2^{-4(Q-P)/3}.$$  \hspace{1cm} (3.3)

Fig. 1b shows that Eq. (3.3) captures very well the dependence on $Q$ of $T_{1,Q}$. As suggested before, the slightly faster decay of $T_{2,Q}$ can plausibly be attributed to the limited range of bands of inertial scales in our flow. It would be interesting to carry out a similar analysis at flows at a higher Reynolds number, to see whether the predictions of [Tennekes & Lumley (1972)] (Eq. (3.3)) apply throughout the inertial range in very turbulent flows.

4. Time-dependent flows: interaction between vortex tubes.

In this Section, we analyze the energy transfer during the interaction of vortex tubes. Specifically, we focus on two configurations, which correspond to the two qualitatively different dynamics observed when two initially straight tubes at a given angle to each other interact [Ostilla-Mónico et al. (2021)]: We also compare with the results obtained for HIT flows in Sec. 3. The Reynolds number $Re_T$ is defined here as the dimensionless ratio between the circulation of the tubes at $t = 0$, $\Gamma$, and viscosity: $Re_T = \Gamma/\nu$. 
4.1. Vortex tubes originally at a 90° angle: Reconnection and energy transfer

We first consider the configuration of two tubes originally perpendicular to each other. In this configuration, the dynamics lead to the formation of very intense vortex sheets as the tubes initially stretch and flatten prior to contacting one another. These vortex sheets then burst, leading to a reconnection of the vortex tubes which leaves behind a cloud of small scale vortices (Ostilla-Mónico et al. 2021). This evolution shares many features with the more classical setup used to study reconnection that starts with two initially close antiparallel vortices with a perturbation in their position (Yao & Hussain 2022). The run analyzed here was described elsewhere by Ostilla-Mónico et al. (2021). The Reynolds number of this run, $Re = 4000$, is sufficient to see most of the phenomena observed in symmetric configurations studied at higher Reynolds numbers, e.g. by Yao & Hussain (2020). However, it is still too low to observe evidence of a genuinely turbulent flow.

Fig. 2a shows the energy transfer, $T_{PQ}$ at 3 different stages of the evolution: as the vortices approach each other ($t = 13$, left), shortly before they reconnect ($t = 18.5$, center), and after reconnection ($t = 28.5$, right), while the vorticity field at these times is shown in (b). The maximum kinetic energy dissipation, defined as $\epsilon = \nu \langle \omega^2 \rangle$, grows from $\sim 6 \times 10^{-5}$ to $\sim 9 \times 10^{-5}$ shortly after the time shown in the central column, in the time range $20 \sim t \sim 24$.

Even at the earliest time shown, some significant energy transfer takes place between modes $P = 1$ and $Q = 2$, and between $P = 2$ and $Q = 3$, with a much weaker energy transfer from higher modes ($P \geq 3$) to $Q > P$. When the strong interaction between the tubes starts, at $t = 18.5$, the transfer of energy between $P = 1$ and $Q = 2$ is very small, and actually slightly negative. On the other hand, most of the transfer originates from the mode $P = 2$, see also Fig. 2 of (Ostilla-Mónico et al. 2021).

As the energy transfer between wavenumber bands is mostly between $P = 2$ and $Q = 3$ and $P = 3$ and $Q = 4$, in Fig. 2-d we superimpose on the vorticity field the regions of intense energy transfer $T_{PQ}$ towards smaller scales, i.e. $T_{2,3} > 0$ for 2c or $T_{3,4} > 0$ for 2d, in blue, and the regions of intense energy transfer towards larger scales in green. At the earliest time shown, much of the energy transfer from $P = 2$ towards $Q = 3$ is observed in the regions where the tubes come together, visible as the blue region between the filaments on the left column of Fig. 2: On the other hand, a negative transfer is observed on the other sides of the tubes, with evidence of a positive energy transfer further away from the interaction regions. At this early time, we do not observe any significant transfer between modes $P = 3$ and $Q = 4$, as seen in Fig. 2d.

At $t = 18.5$, when the vortex tubes have paired, the band structure of regions of positive and negative energy transfer between $P = 2$ and $Q = 3$ persists. The positive transfer of energy between the tubes, however, is no longer as strong as it was at $t = 13$. The negative transfer, seen on the other side of the tubes gets amplified, along with the positive transfer further away from the interacting region. The interaction between the tubes gives rise to a strong transfer between $P = 3$ and $Q = 4$, as shown in the center of panel (d).

At later times, $t = 28.5$, after the strong interaction between the vortices which results in the reconnection and separation of the two tubes, Fig. 2 reveals that an active transfer takes place over a broader range of wavenumber bands, and not just around $P = 2$. At this time, the solution exhibits a characteristic bow-like structure behind the reconnected vortices, with active small-scale motion (Ostilla-Mónico et al. 2021). The right most column of Fig. 2: shows that regions of positive and negative energy transfer between $P = 2$ and $Q = 3$ are located along the two separating tubes, particularly in the regions.
Figure 2. Energy transfer for two vortex tubes originally at a 90° angle. The solution is shown at three times: $t = 13$ as the tubes are approaching (left column); $t = 18.5$ as the tubes begin to interact (central column), and $t = 28.5$, after reconnection (right column). (a) 2D energy transfer spectrum. The transfer is originally confined to low wavenumber bands ($Q, P \leq 3$), and extends to higher wavenumber bands as time goes on. (b) Front view of the evolution of the vorticity modulus, showing the tubes as they approach, interact, and separate after reconnection, leaving behind a bow structure with intense small-scale motion. (c) Regions of intense positive energy transfer to smaller scales, from $P = 2$ to $Q = 3$ (blue), and of intense negative energy transfer (to larger scales) from $P = 3$ to $Q = 2$ (green). For $t \lesssim 20$, intense transfer towards small scales happens between the two tubes, whereas some transfer towards larger scales on the outside parts of the tubes. At the latest stage, $t = 28.5$, energy transfer to small scales is occurring on the separating tubes. (d) Regions of intense positive energy transfer to smaller scales from $P = 3$ to $Q = 4$ (blue) and of intense negative energy transfer from $P = 4$ to $Q = 3$ (green). Transfer builds up as the vortices begin to interact ($t = 18.5$), and persist mostly on the bow regions at $t = 28.5$. (e) Unfiltered rate of strain magnitude (magenta), which is most intense during and after the reconnection.
where the two interacting tubes reconnect. Also, a significant amount of energy transfer is observed in the two braids behind the regions where the vortices are interacting. As suggested by Yao & Hussain (2020), this is the region where one observes, at much higher Reynolds numbers, a proliferation of very small scale motion, leading to a $k^{-5/3}$ energy spectrum. It would be interesting to study the energy transfer rate at higher Reynolds numbers in order to understand whether these dynamics come from a cascade of instabilities, similar to the one documented when two parallel tubes interact (McKeown et al. 2020).

4.2. Anti-parallel vortex tubes: Elliptical instability and disintegration

Initially anti-parallel vortex tubes lead to very different dynamics than those considered in the previous subsection. The tubes are subjected to an elliptical instability, leading to the formation of anti-parallel tubes, perpendicular to the original vortices (McKeown et al. 2018). The pattern reproduces itself in a cascade-like manner, eventually giving rise to turbulence (McKeown et al. 2020). In the following, we distinguish the two phases of the evolution and characterize the corresponding transfers of energy between bands of wavenumbers. We analyze the run at $Re_\Gamma = 6000$ from (McKeown et al. 2020). The time $t$ is expressed here in code units, and relates to the dimensionless time, $t^* \approx 45.5 \times t$.

4.2.1. Early stage: development of the instability

Fig. 3 illustrates the evolution of the two tubes during the phase where the dissipation rate strongly increases. Fig. 3a shows the energy transfer, $T_{PQ}$ in Fourier space, whereas Fig. 3b shows the development of the instabilities through volumetric visualizations of the vorticity modulus. At the earliest time shown ($t = 1.1$), the instability is just starting to develop, and little energy transfer is taking place. At the intermediate time, $t = 1.33$, the most pronounced transfer is between the band $P = 1$ and the band $Q = 3$. This corresponds to a wavelength equal to roughly $1/8$ the size of the computational box, which is essentially the wavelength emerging from the instability. The trend seen at $t = 1.33$ intensifies at a later time ($t = 1.50$). The energy transfer also involves other pairs of modes, which all correspond to energy transfer toward smaller scales.

As the most intense energy transfer over the time interval illustrated in Fig. 3a occurs between $P = 1$ and $Q = 3$, panel c shows in blue (green) the regions of intense positive (negative) energy transfer from $P = 1$ to $Q = 3$. As the flow develops, an alternating pattern of positive and negative transfer perpendicular to the interacting tubes can be seen. At the early time of the instability development, the transfer of energy tends to be negative between pairs of vortex tubes. This appears at first sight contradictory with the observations in Fig. 2 (panel (c) at $t = 13.0$ and panel (d) at $t = 18.5$). We recall that the quantities $T_{PQ}$ are scaled here by the instantaneous energy dissipation, which remains relatively weak up to $t = 1.33$. As a consequence, the exchange of energy is not particularly strong. The growth of energy in the band of wavenumbers $K = 3$ mostly comes from the instability. However, at later stages ($t = 1.50$), a positive energy transfer between the tubes (blue) begins to overwhelm the negative energy transfer (green). This observation can be interpreted as a sign that the two tubes begin to strongly interact, in a way which is reminiscent of that observed for the two tubes initially at $90^\circ$, see Fig. 2. It is interesting to note that the rate of strain filtered at the $K = 3$ band, visualized in magenta in Fig. 3c, very closely follows the pattern of positive and negative energy transfer seen in Fig. 3c. In fact, at the early times shown ($t = 1.1$ and $t = 1.33$), the
Figure 3. Energy transfer for two vortex tubes originally antiparallel, at times $t = 1.10$ (left column), when the first instability begins to develop; $t = 1.33$ (central column), as the transverse filaments grow, and $t = 1.50$ (right column), as the second stage of the iterative instability is proceeding. (a) 2D energy transfer spectrum. (b) Front view (top) and cross-sectional top view (bottom) through the dashed white line showing the evolution of the vorticity modulus, as the elliptical instability develops. (c) Regions of intense positive energy transfer to smaller scales, from $P = 1$ to $Q = 3$ (blue), and of intense negative energy transfer (to larger scales) from $P = 3$ to $Q = 1$ (green). The presence of a strong energy transfer correlates precisely with the formation of perpendicular secondary vortex tubes. (d) Filtered rate of strain magnitude at the $I_3$ band (magenta). Regions of intense strain correlate strongly between interacting secondary filaments.
development of intense strain regions precedes the formation of intense tubes, clearly visible at \( t = 1.50 \).

4.2.2. Later stage: the turbulent regime

When the kinetic energy dissipation rate peaks, the flow exhibits the classical \( E(k) \propto \epsilon^{2/3}k^{-5/3} \) spectrum (McKeown et al. 2020). It is therefore of fundamental interest to use this configuration as a test ground to study energy transfer, on par with other classical flows, such as statistically stationary or decaying turbulence. For future comparison with the configuration studied in Section 3, we note that estimating the Taylor Reynolds number of the flow studied in (McKeown et al. 2020; Ostilla-Monico et al. 2021) leads to a value of the order of \( Re_\lambda \approx 50 \), which indicates that the inertial range encompasses a much more restricted range of spatial scales compared to our HIT run at \( Re_\lambda = 210 \), a fact which we will return to later.

Fig. 4 shows the flow at time \( t = 1.83 \), which corresponds to the peak in the energy dissipation, and to the moment where the flow has developed a \( k^{-5/3} \) spectrum (McKeown et al. 2020), which indicates that it has become turbulent in a classical sense. Fig. 4a shows the normalized 2D energy transfer spectrum, \( T_{PQ}/\epsilon \), at this time, which reveals an ongoing transfer of energy towards small scales, mostly concentrated along the diagonal. Fig. 4b shows the complicated structure of the vorticity modulus at this time. The solution results from the iterative cascade of instabilities documented in (McKeown et al. 2020). Furthermore, Fig. 4 shows the energy transfer from \( P = 1 \) to \( Q = 2 \) (c), to \( Q = 3 \) (d), and to \( Q = 4 \) (e); from \( P = 2 \) to \( Q = 3 \) (f) and to \( Q = 4 \) (g) and from \( P = 3 \) to \( Q = 4 \) (h). Remarkably, at all scales represented, the intense contributions to the energy transfer are observed in the vicinity of the interacting vortex pairs which were formed as a result of the instabilities. This is suggestive of the important role played by these vortex pairs and by their iterative interactions in transferring energy across scales. Although it is clear that most intense regions of energy transfer, be it towards small scales or towards large scales (blue or green, respectively) occur very often between pairs of vortex tubes, it is not clear whether a given pair of vortex tubes will give rise to a transfer towards small or towards large scales; both cases can be observed, depending on the particular pair \( (P,Q) \) considered.

4.2.3. Comparing the energy transfer with HIT: Implications for turbulent flows

It is worth asking whether the clear correlation between vortex tubes and intense energy transfer, observed during the late-stage evolution of the flow with two antiparallel vortex tubes, can be generalized to a fully-developed turbulent flow. To consider this question, we turn to homogeneous, isotropic flow in a triply periodic box, as described in Sec. 2.2.

For the sake of comparison with Fig. 4, Fig. 5a shows the 2D energy transfer spectrum, \( T_{PQ} \) normalized by the dissipation rate, \( \epsilon \), already presented in Fig. 1a. This panel showcases a qualitative resemblance between the two turbulent flows, resulting from the interaction of two vortex tubes, and from forced HIT. Despite this resemblance, the structure of the regions of intense vorticity magnitude in the HIT flow, see Fig. 5b, strongly differ from those observed for the two interacting vortex tubes in Fig. 4b. The nature of the vorticity field at \( Re_\lambda = 210 \) is far more complicated and involves a much larger hierarchy of scales. This is expected from elementary estimates of the Taylor Reynolds number, the classical estimate for the ratio between the large scales of the flow, \( L \), and the Kolmogorov scale, \( \eta \), as \( L/\eta \propto Re_\lambda^3/2 \), which allows us to compare the range of inertial scales between different turbulent flows. \( Re_\lambda \) does not exceed 50 for antiparallel tube configuration, implying that the range of inertial scales in Fig. 4 is reduced by a
Figure 4. Energy transfer for the turbulent state reached with two initially antiparallel vortex tubes, when the energy dissipation rate is maximized, at $t = 1.83$. (a) 2D energy transfer spectrum. The transfer is mostly localized to neighboring bands. (b) Visualization of the vorticity modulus. (c-h) Localized regions of intense energy transfer for (c) $T_{1,2}$, (d) $T_{1,3}$, (e) $T_{1,4}$, (f) $T_{2,3}$, (g) $T_{2,4}$, (h) $T_{3,4}$. Regions of intense positive energy transfer from larger to smaller scales are blue and regions of intense negative energy transfer from smaller to larger scales are green.
Figure 5. Energy transfer for DNS of homogeneous isotropic turbulence where Re$_\lambda$ = 210, at $t = 6.18$. (a) 2D energy transfer spectrum. The transfer is mostly localized to neighboring bands. (b) Visualization of the vorticity modulus. (c-h) Localized regions of intense energy transfer for (c) $T_{1,2}$, (d) $T_{1,3}$, (e) $T_{1,4}$, (f) $T_{2,3}$, (g) $T_{2,4}$, (h) $T_{3,4}$. Regions of intense positive energy transfer from larger to smaller scales are blue and regions of intense negative energy transfer from smaller to larger scales are green.
factor $\sim 8$, compared to the HIT flow in Fig. 5. We also note that despite the statistical homogeneity of the flow, one can notice in Fig. 5b that the distribution of intense vorticity is inhomogeneous in the volume, with some regions of weak activity, and some regions of much larger activity. In fact, the regions of low intensity, correspond to regions of low kinetic energy in the fluid.

The regions of intense energy transfer rate, $T_{PQ}$, between nearby shells are shown in Fig. 5c-h, with the convention that blue regions correspond to a positive energy transfer towards small scales ($Q > P$), and green to a negative energy transfer. These are localized in regions of intense activity. Nonetheless, the regions of most intense transfer $T_{PQ}$ with $P = 1$ and $Q = 2$, $P = 1$ and $Q = 3$, and $P = 2$ and $Q = 3$, shown in Fig. 5c, d and f respectively, are of much larger spatial extent than the small vortex filaments (“worms”) present in the HIT flow. The energy transfer between bands of low wavenumbers therefore does not correlate particularly well with the smallest scale vortices. As one considers bands of higher wavenumbers, such as $P = 3$ and $Q = 4$, see Fig. 5h, the scales of the regions of intense transfer become smaller but remain much larger than the size of the smallest vortex tubes.

The qualitative discrepancy between the spatial extent of the regions of intense energy transfer and the size of the most intense vortices demonstrates that the simple scheme suggested by Fig. 4 cannot be mapped one-to-one onto the cases of turbulent flow at a much higher Reynolds number. This can be understood by noticing that one of the striking aspects of the transfer in Fig. 4 is the prevalence of large vortex-shaped structures, which appear as the result of an iteration of instabilities documented in Fig. 3 and correspond to the band of wavenumbers $P = 3$. We also observe in Fig. 3a a strong transfer of energy from $P = 1$ to $Q = 3$. In fact, $T_{1,3}/\epsilon$ is significantly higher for the antiparallel tubes configuration than what was found in HIT in Fig. 1a. This suggests that, despite its relatively complex structure, the flow shown in Fig. 4b is still reminiscent of the instability that gave rise to the very strong vortices perpendicular to the two original antiparallel tubes. This specific feature of the flow may not have any obvious equivalent in the HIT flow.

4.2.4. The role of large-scale vorticity

Whereas from our previous discussion, the most intense structures of the vorticity field do not appear to be particularly important to understand the energy transfer in HIT, one may expect that the filtered vorticity $\omega_K$, obtained by keeping only Fourier modes in bands $I_K$, with $P \leq K \leq Q$ (or equivalently, the rate of strain), may correlate better with $T_{PQ}/\epsilon$ and therefore be more relevant to the energy transfer mechanisms. An intermediate possibility between the transient problem of two interacting antiparallel vortex tubes, which shows strong signatures of the initial conditions, and a randomly forced flow that shows no large-scale vortices was discussed in Goto et al. (2017). This work employed a statistically stationary flow designed to maintain relatively simple and large-scale vortices which are continuously energized in order to drive the energy transfer to smaller scales. In particular, Goto et al. (2017) used a configuration forced by a simple mode, $k = (1, 1, 0)$ in Fourier space to demonstrate that the forcing had a strong influence in organizing the vorticity field filtered in bands of higher wavenumbers (with $K > 1$). Furthermore, in this statistically stationary flow, the structure of the vorticity field can still be characterized in a hierarchical manner (Goto et al. 2017; Yoneda et al. 2021). Namely, the stretching that amplifies the vorticity in a waveband $Q$ is induced by vorticity in bands $P$, smaller than yet close to $Q$. This self-similar picture is reminiscent in spirit to the one introduced by Tennekes & Lumley (1972), although it rests on a very different role for the strain in contributing to the energy transfer.
Figure 6. Energy transfer for DNS with the forcing used in Goto et al. [2017], at $t = 8.20$. (a) 2D energy transfer spectrum. The transfer is mostly localized to neighboring bands. (b) Visualization of the vorticity modulus. (c-h) Localized regions of intense energy transfer for (c) $T_{1,2}$, (d) $T_{1,3}$, (e) $T_{1,4}$, (f) $T_{2,3}$, (g) $T_{2,4}$, (h) $T_{3,4}$. Regions of intense positive energy transfer from larger to smaller scales are blue and regions of intense negative energy transfer from smaller to larger scales are green.
To examine the possibility that the configuration introduced by Goto et al. (2017) could show that large vortex tubes play a crucial role in the transfer of energy, we generated a turbulent flow using the same forcing as in Goto et al. (2017) (see Eq. 2.6 for more details). Fig. 6 summarizes the results of our analysis of energy transfer at a given time instant, in the statistically steady state of the flow. Fig. 6 shows the 2D energy transfer spectrum, $T_{PQ}/\epsilon$. Beyond a qualitative resemblance with comparable figures, we notice that the transfer from $P = 1$ to $Q > P$ does not decay with increasing $Q$ as fast as in the case of HIT, see Fig. 5a, but rather extends to values beyond $Q = 3$. This is reminiscent of the antiparallel tube interaction, see Fig. 4a. In these two configurations, the organized nature of the flow in the band $P = 1$ leads to an efficient transfer of energy, reaching out to scales (values of $Q$) beyond what would occur in the case of an HIT flow with a much less structured forcing at the largest scales.

The complexity of the vorticity field, shown in Fig. 6b, is intermediate between the one in Fig. 4b, (turbulence resulting from the interaction between two antiparallel tubes, $Re_\lambda \sim 50$) and in Fig. 5b (HIT at $Re_\lambda = 210$). This observation can be easily explained by the Reynolds number of the flow illustrated in Fig. 6, $Re_\lambda \approx 140$, which is intermediate between the two flows studied in Figs. 4 and 5.

As was the case in Fig. 4c-h and Fig. 5c-h, Fig. 6c-h shows that regions of intense energy transfer from $P$ to $Q > P$ (blue regions), and from $Q > P$ to $P$ (green regions) are somewhat spread throughout the flow, without any strong correlation with the most intense structures of the vorticity field. Our attempts to correlate e.g. the energy transfer rate $T_{12}$ with the regions where $\omega_1$ (obtained by keeping only wavenumbers in the band $I_1$ of vorticity) did not reveal as strong a correlation as the one observed by Goto et al. (2017) when analyzing vortex stretching. As was the case in the HIT configuration, see Fig. 5c-h, we did not observe any convincing correlation between regions where $T_{PQ}$ is intense, and regions of intense values of $\omega_K$, where $P \leq K \leq Q$. This leads us to the conclusion that the organized structure of vortex stretching found by Goto et al. (2017) does not extend to the energy transfer rate studied here.

To conclude, in the flow forced at $k = (1, 1, 0)$, as in the flow generated by two initially anti-parallel vortices, we observe the predominance of the transfer from the lowest band of wavenumbers ($P = 1$) to higher modes, in the sense that $T_{1,Q}/\epsilon$ decays significantly slower than what is observed in HIT. This is consistent with the results of Goto et al. (2017), demonstrating that the vorticity structure in the lowest band of wavenumbers plays a strong role in organizing the Fourier modes in wavenumber bands close to $P = 1$.

This feature, however, suggests a deviation from what is observed in HIT and from the phenomenological description of Tennekes & Lumley (1972), presented in Section 3.2. In this sense, the remarkably simple picture, observed in the interaction of antiparallel vortex tubes and in the configuration studied by Goto et al. (2017), appears to be the result of a strong influence of the forcing (or initial conditions) at a low wavenumber. The standard phenomenology (Tennekes & Lumley 1972) predicts that at much higher Reynolds numbers, the structure of the flow in the inertial range, at high wavenumbers, is less sensitive to the details of the forcing, and closer to what is observed in HIT, see Fig. 5.

5. Summary and conclusion

This work investigates the transfer of energy between bands of wavenumbers in several turbulent flows. To this end, we utilize tools which we developed to visualize the energy transfer both in Fourier and in real space. We first decompose the velocity fields by splitting the wavenumber space into shells of wavenumbers, $I_P$, centered around $k_f \times 2^P$—
The energy transfer rate $T_{PQ}$ between bands of wavenumbers $I_P$ and $I_Q$ is then written as an integral over a scalar quantity, $T_{PQ}(x,t)$. This decomposition allows us to visualize the transfer not only in Fourier space, by representing $T_{PQ}$ as a function of $P$ and $Q$, but also in real space, thus characterizing the regions where $T_{PQ}(x,t)$ is large and correlating this with the structure of the velocity field.

We first used this decomposition to consider the canonical configuration of a statistically stationary homogeneous isotropic turbulent flow (in the presence of an external forcing), at a moderately high Reynolds number, $Re_\lambda = 210$. In agreement with several earlier studies, we demonstrated that in Fourier space, the energy transfer is mostly local, i.e., $T_{PQ}$ decreases very fast with $|P - Q|$. Remarkably, our numerical results are consistent with the prediction of Tennekes & Lumley (1972), who proposed a phenomenological model to describe the transfer of energy as resulting from competing effects of the strain at small and large scales acting on vortices, and derived scaling laws for $T_{PQ}$ (Eq. 3.3).

We then considered the transfer of energy in time-dependent flows, taking as initial condition two laminar vortices, which rapidly develop intense small-scale motions. Using our method, we co-localize the intense vortex structures, which develop in the flow as a result of the interaction between the two original vortices, with the regions of intense energy transfer directed mostly towards small scales. This simple correlation between intense vortices and regions of intense energy transfer persists even when a turbulent regime develops a $k^{-5/3}$ velocity spectrum.

While our results may appear to be in qualitative agreement with the conclusions reached by Goto et al. (2017) and Motoori & Goto (2019) in their analyses of the generation of vortices of smaller and smaller sizes, it is important to stress that the energy transfer rate, $T_{PQ}$ (and the corresponding scalar function, $T_{PQ}$), fundamentally differs from the vorticity-based observables used by Goto et al. (2017) and Yoneda et al. (2021). As stressed recently (Carbone & Bragg 2020; Johnson 2021), vortex stretching does not coincide with energy transfer, but in fact, nonlinear interactions involving strain contribute more to energy transfer. Hence, what is revealed by looking at enstrophy in other studies could be unrelated to energy transfer. Indeed, when subjecting the flow used in Goto et al. (2017) to our analysis and restricting ourselves to comparable wavenumber bands, we could not find any particularly strong correlation between regions of intense energy transfer and of intense vorticity.

Our observations are further confirmed by analyzing the energy transfer in real space for HIT at $Re_\lambda = 210$, where vortex interactions occur over a wider range of scales and correlating spatially local events with energy transfer fails. While spatially localized cascade events, similar to those analyzed in McKeown et al. (2020), are not ruled out from these results, their importance may be reduced in a more general class of turbulent flows, which do not have carefully selected initial conditions. The evidence of a strong cascade that coincides with the production of transverse vortices in the time-dependent problem of interacting vortex tubes obtained from the analysis of vortex stretching does not extend simply to flows involving a wide range of scales, such as those found in homogeneous isotropic turbulence. The structure of such flows is far more complicated, and in particular, the full nonlinear interactions which are responsible for energy transfer do not reduce to the coupling between strain and vorticity underlying vortex stretching but also include other phenomena such as strain self-amplification.

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Visualizing the turbulent energy cascade: Fourier and physical space

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