The dark side of fuzzball geometries

M. Bianchi\textsuperscript{a,b}, D. Consoli\textsuperscript{a,b}, A. Grillo\textsuperscript{a}, J.F. Morales\textsuperscript{b}

\textsuperscript{a}Dipartimento di Fisica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica, 00133 Roma, Italy
\textsuperscript{b}I.N.F.N. Sezione di Roma “Tor Vergata”, Via della Ricerca Scientifica, 00133 Roma, Italy

E-mail: massimo.bianchi@roma2.infn.it, dario.consoli@roma2.infn.it, alfredo.grillo89@gmail.com, morales@roma2.infn.it

Abstract: Black holes absorb any particle impinging with an impact parameter below a critical value. We show that 2- and 3-charge fuzzball geometries exhibit a similar trapping behaviour for a selected choice of the impact parameter of incoming massless particles. This suggests that the blackness property of black holes arises as a collective effect whereby each micro-state absorbs a specific channel. More interestingly, this universal property of fuzzball geometries suggests the possible existence of astrophysical objects behaving as gravitational filters obscuring only a band in the light spectrum of distant sources!

Keywords: Black holes, fuzzballs, D-branes, micro-states
1 Introduction

The interest in objects colloquially known as black holes (BH’s) has been revived not only by their role in the generation of the first gravitational wave signal detected by the LIGO-Virgo collaboration [1] but also by the possibility that primordial BH’s may account for a (small) fraction of the dark matter in the universe [2] and rotating BH’s and similar objects may accelerate cosmic rays thanks to Penrose mechanism [3, 4].

In String Theory it is natural to describe BH’s as ensembles of micro-states represented by smooth, horizonless geometries without closed time-like curves, the so called “fuzzballs” [5–11]. The counting of micro-states for extremal 3- and 4-charged black hole states in five and four dimensions has proven to be very successful [12–16], while the identification of the corresponding geometries in the supergravity regime has revealed to be much harder [17–37]. To go one step further one can probe fuzzball geometries with particles, waves and strings and test the proposal at the dynamical level [38]. Elaborating on our recent work on 2-charge systems [38], our present focus will be on the geodetic motion of massless particles...
on a class of 3-charge micro-state geometries introduced in [39]. This should capture the relevant physics not only for large impact parameters where the eikonal approximation of the scattering process is valid even quantitatively [40–51], but also for small impact parameters whereby the particles get trapped or absorbed, at least at a qualitative level. We leave the analysis of waves and strings or other classes of smooth geometries (such as JMaRT [52]) to the future.

The picture that emerges from our analysis is that the blackness property of black holes arises as a collective/statistical effect where each micro-state absorbs a specific channel. More interestingly, this universal property of fuzzball geometries suggests the possible existence of more exotic distributions of micro-state geometries looking effectively as gravitational filters obscuring only a band in the light spectrum of distant sources, or more bizarre black looking objects such as rings, spherical shells, etc. A more detailed analysis should take into account radiation damping, i.e. the energy lost in gravitational wave emission by an accelerated particle. Contrary to the case of an accelerated charged particle, we expect gravitational brems-strahlung to be anyway negligible for a vast range of kinematical parameters.

The paper is organised as follows. In section 2 we introduce the class of micro-state geometries we will consider, discuss the general behaviour of massless geodesics in these backgrounds and summarise our results. In particular we will introduce the notions of turning points and critical geodesics, characterising geodesics that either bounce back to infinity or get trapped spinning around the gravitational source, respectively. In Section 3-5 we analyse the behavior of massless geodesics in the case of 3-charge black holes, 2-charge and 3-charge fuzzballs respectively. The analysis of 2-charge fuzzballs is performed in full generality, while the analysis of the 3-charge case is restricted to geodesic motion along or perpendicular to the plane of the string profile characterising the fuzzball. The latter case lacks spherical symmetry and exhibits an intricate non-completely separable dynamics. A simple solution in this class is presented in some detail. Section 6 contains our conclusions and outlook.

2 Overview and summary of results

In this section we introduce the fuzzball geometries we will be interested in and summarise our results. We write down the general form of the metric, the Lagrangian governing the dynamics of massless neutral particles and the geodesic equations. We then identify the conjugate momenta and the Hamiltonian and describe how to take advantage of the isometries when present. We also discuss the classification of the geodesics when the system is integrable.
2.1 The 3-charge fuzzball metrics

We will consider 3-charge BPS micro-state geometries belonging to the general class constructed in\(^1 \) [39]. The ten-dimensional metric can be written in the form
\[
ds^2 = \frac{\sqrt{Z_1 Z_2}}{Z^2} ds_6^2 + \frac{Z_1}{Z_2} ds_{T_4}^2.
\]
where \( ds_{T_4}^2 \) is the metric on a \( T^4 \) torus (or a K3 surface, in fact) while the 6-dimensional metric \( ds_6^2 \) describes a 5-dimensional space-time times a compact circle of radius \( R_g \). This manifold can be parametrized with coordinates \( \{ t, \vec{X}, y \} \) or alternatively by introducing the null coordinates \( u = \frac{t+y}{\sqrt{2}} \) and \( v = \frac{t-y}{\sqrt{2}} \) and the oblate spheroidal coordinate system
\[
X_1 + iX_2 = \sqrt{\rho^2 + a^2} \sin \vartheta e^{i\varphi}, \quad X_3 + iX_4 = \rho \cos \vartheta e^{i\psi}.
\]
By doing so one obtains
\[
ds_6^2 = g_{mn} dx^m dx^n = -2 (dv + \beta_m dx^m) (du + \gamma_m dx^m) + Z^2 ds_4^2.
\]
where \( ds_4^2 \) is the flat metric of \( \mathbb{R}^4 \)
\[
ds_4^2 = (\rho^2 + a^2 \sin^2 \vartheta) \left( \frac{d\rho^2}{\rho^2 + a^2} + d\vartheta^2 \right) + (\rho^2 + a^2) \sin^2 \vartheta d\varphi^2 + \rho^2 \cos^2 \vartheta d\psi^2.
\]
The functions \( Z_1, Z_2, Z, \beta_m, \gamma_m \) depend on the coordinates \( \vec{x} \) of \( \mathbb{R}^4 \) and on \( v \), their explicit expression is as follows
\[
Z_1 = 1 + \frac{L_1^2}{\rho^2 + a^2 c_\vartheta^2} + \frac{\varepsilon_1 R^2 \Delta_n s_\vartheta^2 \cos 2\phi}{L_5^2 (\rho^2 + a^2 c_\vartheta^2)} \quad Z_2 = 1 + \frac{L_2^2}{\rho^2 + a^2 c_\vartheta^2}
\]
\[
Z_3^2 = \frac{2 \varepsilon_2^2 R^2 \Delta_n s_\vartheta^2 \cos^2 \phi}{(\rho^2 + a^2 c_\vartheta^2)^2} \quad Z^2 = Z_1 Z_2 - Z_3^2
\]
\[
\beta_\varphi = \frac{a^2 R s_\vartheta^2}{\rho^2 + a^2 c_\vartheta^2} \quad \beta_\psi = -\frac{a^2 R c_\vartheta^2}{\rho^2 + a^2 c_\vartheta^2}
\]
\[
\gamma_\varphi = \alpha \beta_\varphi - \frac{n \varepsilon_1 R}{2L_5^2} \Delta_n \cos 2\phi s_\vartheta^2 \quad \gamma_\psi = -\alpha \beta_\psi \quad \gamma_\vartheta = F_n
\]
\[
\gamma_\theta = -\frac{\varepsilon_1 R}{2L_5^2} \Delta_n \sin 2\phi s_\theta c_\vartheta \quad \gamma_\rho = -\frac{\varepsilon_1 R}{2L_5^2} \Delta_n \sin 2\phi s_\theta^2
\]
with \( s_\vartheta = \sin \vartheta, c_\vartheta = \cos \vartheta \) and
\[
\phi = \varphi + \frac{nv}{R} \quad R = \frac{R_g}{\sqrt{2}}
\]
\[
F_n = -\frac{\varepsilon_2^2}{2a^2} \left[ 1 - \left( \frac{\rho^2}{\rho^2 + a^2} \right)^n \right]
\]
\[
\Delta_n = \frac{a^2}{\rho^2 + a^2} \left( \frac{\rho^2}{\rho^2 + a^2} \right)^n
\]
\[
\alpha = 1 - F_n - \frac{n \varepsilon_1}{2L_5^2} \Delta_n \cos 2\phi s_\theta^2
\]
\[\text{\textsuperscript{1}}\text{In the notation of this reference, we focus on solutions with } k = 1, m = 0 \text{ and } n \text{ an arbitrary positive integer.}
\]
\[\text{\textsuperscript{2}}\text{For the class of solutions we are interested in the components } \beta_\varphi, \beta_\theta, \beta_\varrho, \beta_\varpi \text{ and } \gamma_u \text{ are identically zero.}\]
Regularity of the metric near $\rho = 0, \vartheta = \pi/2$ requires \[\text{[39]}\]

\[
a^2 = \frac{L_1^2 L_5^2}{2R^2} - \frac{\varepsilon_1^2}{2}, \quad \varepsilon_3 = \varepsilon_1 \left(1 + \frac{a^2 n}{L_5^2}\right). \tag{2.7}
\]

The conserved charges and the angular momenta $J$ and $\tilde{J}$ are given by

\[
Q_1 = L_1^2, \quad Q_5 = L_5^2, \quad Q_p = \frac{\varepsilon_3 n}{2}, \quad J = \tilde{J} = \frac{Ra^2}{\sqrt{2}} \neq 0. \tag{2.8}
\]

or equivalently

\[
J_\varphi = J + \tilde{J} = \sqrt{2}Ra^2, \quad J_\psi = J - \tilde{J} = 0 \quad (2.9)
\]

We will study the scattering of massless neutral particles in the following special cases of the family of BPS metrics introduced above:

- 3-charge non-rotating black holes: Recovered as the $a \to 0$ limit of the 3-charge metric.
- 2-charge fuzzball: Obtained by setting $\varepsilon_1 = n = 0$ in the 3-charge metric.
- 3-charge fuzzball: The general case restricted to the planes $\vartheta = 0$ and $\vartheta = \pi/2$.

### 2.2 The geodesics

We are interested in null geodesics in the 6-dimensional geometry that solve the Euler-Lagrange equations derived from the Lagrangian

\[
\mathcal{L} = \frac{1}{2}g_{mn} \dot{x}^m \dot{x}^n, \tag{2.10}
\]

with $g_{mn}$ the six-dimensional metric, and dots denoting derivatives with respect to an affine parameter $\tau$. Null geodesics are specified by solutions $x^m(\tau)$ of the Euler-Lagrange equations satisfying $\mathcal{L} = 0$. Equivalently one can introduce the Hamiltonian

\[
\mathcal{H} = P_m \dot{x}^m - \mathcal{L} = \frac{1}{2}g_{mn} P_m P_n \tag{2.11}
\]

expressed in terms of the conjugate momenta

\[
P_m = \frac{\partial \mathcal{L}}{\partial \dot{x}^m} = g_{mn} \dot{x}^n. \tag{2.12}
\]

It will prove useful to keep in mind that

\[
2P_u P_v = E^2 - P_y^2 \geq 0, \tag{2.13}
\]

where $E$ and $P_y$ are the momenta conjugate to $t$ and $y$, respectively. In the Hamiltonian formulation, geodesics are described by the velocities

\[
\dot{x}^m = \frac{\partial \mathcal{H}}{\partial P_m} \tag{2.14}
\]
with $P_m$ a solution of the system of equations$^3$

$$2\mathcal{H} = g^{mn} P_m P_n = 0 \quad (2.16)$$  

$$\dot{P}_m = - \frac{\partial \mathcal{H}}{\partial x^m} \quad (2.17)$$

The metric is independent of the variables $u$ and $\psi$, so the momenta $P_u$ and $P_\psi$ will always be conserved. The Hamiltonian can be written in the compact form

$$\mathcal{H} = - P_u \hat{P}_v + \frac{1}{2Z^2} \left[ \frac{(\rho^2 + a^2)\hat{P}_\rho^2}{\rho^2 + a^2 c_\rho^2} + \frac{\hat{P}_\varphi^2}{\rho^2 + a^2 c_\varphi^2} - \frac{\hat{P}_\psi^2}{\rho^2 c_\psi^2} \right] \quad (2.18)$$

in terms of the shifted momenta

$$\hat{P}_m = P_m - \beta_m P_v - (\gamma_v - \beta_m \gamma_u) P_u \quad (2.19)$$

The velocities become

$$\dot{\rho} = \frac{(\rho^2 + a^2)\hat{P}_\rho}{Z^2(\rho^2 + a^2 c_\rho^2)}, \quad \dot{\varphi} = \frac{\hat{P}_\varphi}{Z^2(\rho^2 + a^2 c_\varphi^2)}, \quad \dot{\psi} = \frac{\hat{P}_\psi}{Z^2 \rho^2 c_\psi^2} \quad (2.20)$$

with more involved formulae for $\dot{u}$ and $\dot{v}$. The Hamiltonian constraint $\mathcal{H} = 0$ can be solved by taking

$$\hat{P}_\rho = \pm\left(\frac{\rho^2 + a^2 c_\rho^2}{\rho^2 + a^2}\right)^{\frac{1}{2}} \left[ 2Z^2 P_u \hat{P}_v - \frac{\hat{P}_\varphi^2}{\rho^2 + a^2 c_\varphi^2} - \frac{\hat{P}_\psi^2}{(\rho^2 + a^2) c_\psi^2} \right]^{\frac{1}{2}} \quad (2.21)$$

with minus and plus signs for the branches along which the particle approaches or leaves the gravitational target, respectively. We notice that according to (2.20) $\hat{P}_\rho$ determines the radial velocity of the particle. Starting from infinity, $\rho(\tau)$ monotonously decreases until it reaches a point $\rho_*$ where $\hat{P}_\rho$ vanishes and flips sign. This is said to be an inversion (or turning) point. Since $\rho$ is a monotonous function along this branch it can be used in principle to parametrize the evolution time, expressing all remaining coordinates $x^m(\rho)$ as a function of $\rho$ instead of the affine parameter $\tau$. In practice, this is possible only when the system is integrable. Examples of integrable geodesics occur for BH’s with or without angular momenta, 2-charge circular fuzzballs and geodesics along the plane orthogonal to the string profile in the 3-charge system. The most difficult and interesting case (motion along the plane of the profile in the 3-charge case) eludes this simplistic analysis and will be addressed in section 5.3.

$^3$We notice that the equations of motion imply

$$\dot{\mathcal{H}} = g^{mn} P_m \left( \dot{P}_n + \frac{\partial \mathcal{H}}{\partial x^n} \right) = 0 \quad (2.15)$$

so, one of the equations of motion, let us say the one for $\rho$ can be replaced by $\mathcal{H} = 0$. 

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When the system is integrable and all the variables can be explicitly expressed in terms of $\rho$, the time (measured by an observer at infinity) required by a geodesic to reach the inversion (or turning) point $\rho_*$ starting from a point $\rho_0$ is given by

$$
\Delta t = \int_{\rho_0}^{\rho_*} d\rho \left( \frac{dt}{d\tau} \right) \frac{\rho^2 + a^2 c_2^2(\rho)}{\rho^2 + a^2} \frac{Z^2(\rho)}{P_\rho(\rho)}
$$

(2.23)

This integral may or may not diverge. Focusing for simplicity on geodesics with zero internal momenta ($P_y = 0$) and denoting by $K$ the total angular momentum of the incoming

\[\text{Figure 1: Geodesics in the black hole and fuzzball geometries for different values of the impact parameter } b.\]
particle the impact parameter is given by \( b = K/E \). We can distinguish three distinct scenari depending on the value of \( b \) (see figure 1):

- Scattering processes: They occur where either the geodesics encounter a turning point \( \rho_\ast > 0 \), i.e. a single zero of \( \tilde{P}_\rho^2(\rho) \) or when \( \tilde{P}_\rho(\rho) \) is positive everywhere and the time to reach \( \rho = 0 \) is finite. This includes all geodesics on black hole geometries with large enough impact parameter and generic geodesics in fuzzball geometries.

- Critical falling: They occur when geodesics encounter a critical point \( \rho_\ast \) defined as a double zero of \( \tilde{P}_\rho^2(\rho) \). In this case, the time to reach \( \rho_\ast \) is infinite and the particle asymptotically approaches \( \rho_\ast \) without ever reaching it. This class of geodesics exists for specific choices of the impact parameter, both for black holes and fuzzballs.

- Absorption processes: They occur for black hole geometries when geodesics find no turning point before the black hole horizon. In this case \( \tilde{P}_\rho(\rho) \) is positive everywhere and the time to reach the horizon is infinite.

3 Black hole geometry

In this section we consider massless geodesics in the 3-charge five-dimensional black hole geometry with and without angular momenta.

3.1 The non-rotating three charge black hole

The non-rotating 3-charge black hole metric is obtained by taking \( a = n = 0 \) in (2.1) and (2.3). The \( Z \)-functions and one-forms reduce to

\[
Z_1 = 1 + \frac{L_1^2}{\rho^2}, \quad Z_2 = 1 + \frac{L_5^2}{\rho^2}, \quad Z^2 = Z_1 Z_2
\]

\[
\gamma_m dx^m = \mathcal{F}_0 dv = -\frac{L_7^2}{\rho^2} dv, \quad \beta_m = 0
\]

(3.1)

For this choice the oblate radius \( \rho \) coincides with the spherical radius \( r \) everywhere and the solution is spherically symmetric. The solution corresponds to a non-rotating five-dimensional black hole with a horizon at \( \rho = 0 \) [53, 54]

The ‘dressed’ D1-brane charge \( Q_1 \), D5-brane charge \( Q_5 \) and Kaluza-Klein momentum \( Q_P \) are given by

\[
Q_1 = L_1^2, \quad Q_5 = L_5^2, \quad Q_P = L_p^2.
\]

(3.2)

The massless geodesic equation \( \mathcal{H} = 0 \) can be written in the separable form

\[
2\rho^2 Z^2 \mathcal{H} = \left[ -2\rho^2 Z^2 P_u (P_v - \mathcal{F}_0 P_u) + \rho^2 \tilde{P}_\rho^2 \right] + \left[ P_\vartheta^2 \vartheta^2 + \frac{P_\varphi^2}{\vartheta^2} + \frac{P_\psi^2}{\vartheta^2} \right] = 0
\]

(3.3)

where the two brackets account for \( \rho \) and \( \vartheta \) dependent terms, respectively. The former equation can be solved by imposing that the combinations inside the brackets be constant, i.e.

\[
K^2 = P_\vartheta^2 + \frac{P_\varphi^2}{\vartheta^2} + \frac{P_\psi^2}{\vartheta^2} = 2\rho^2 Z^2 P_u (P_v - \mathcal{F}_0 P_u) - \rho^2 \tilde{P}_\rho^2
\]

(3.4)
The right hand side equation can be solved for $P_{\rho}$

$$P_{\rho}^2 = - \frac{K^2}{\rho^2} + \frac{2P_u(\rho^2 + L_1^2)(\rho^2 + L_2^2)}{\rho^4} \left( P_v + \frac{L_2^2 P_u}{\rho^2} \right)$$

(3.5)

We notice that for

$$K^2 < 2P_u L_p^2 + 2P_v P_u (L_5^2 + L_1^2)$$

(3.6)

the function $P_{\rho}^2$ is positive everywhere, so the geodesics extend down to the horizon at \(\rho = 0\). The flight time down to the horizon diverges

$$\Delta t \approx - L_1 L_5 L_p \int_{\rho_0}^{\rho} \frac{d\rho}{\rho^5}$$

(3.7)

as expected for a black hole geometry.

### 3.2 The rotating supersymmetric black hole

The analysis of geodesics in more general black hole backgrounds, extremal or not, with or without charges and angular momenta, follows mutatis mutandis the same steps as before and the existence of a critical value for the total angular momentum of the incoming particles can be always displayed. In this section, we illustrate this universal feature by considering scattering from a three equal charge supersymmetric black hole with non-trivial angular momentum in five dimensions. The metric of this black hole reads [55]

$$ds_S^2 = - \left( 1 - \frac{\mu}{r^2} \right)^2 \left( dt - \frac{\mu \omega}{r^2 - \mu} \sin^2 \vartheta d\varphi - \frac{\mu \cos^2 \vartheta}{r^2 - \mu} d\psi \right)^2 + \left( 1 - \frac{\mu}{r^2} \right)^{-2} dr^2 + r^2 \left( d\vartheta^2 + \sin^2 \vartheta \sin^2 \varphi \right)$$

(3.8)

where $\mu$ is the mass parameter and $\omega$ accounts for the angular velocity. For concreteness, we focus on geodesics at constant $\vartheta$, let us say $\vartheta = 0^5$. Consistently, we set $\dot{\vartheta} = \dot{\phi} = 0$, i.e. $P_{\vartheta} = P_{\phi} = 0$. The corresponding Hamiltonian reduces to

$$\mathcal{H} = \frac{1}{2} g^{mn} P_m P_n = - \frac{1}{2} \left( 1 - \frac{\mu}{r^2} \right)^{-2} E^2 + \frac{1}{2} \left( 1 - \frac{\mu}{r^2} \right)^2 P^2 + \frac{1}{2r^2} \left( J - \frac{\mu \omega E}{r^2 - \mu} \right)^2$$

(3.9)

with

$$E = g_{\vartheta n} x^n = - \left( 1 - \frac{\mu}{r^2} \right)^2 \left( i - \frac{\mu \omega}{r^2 - \mu} \dot{\vartheta} \right)$$

$$J = g_{\psi n} x^n = \frac{\mu \omega}{r^2 - \mu} \left( i - \frac{\mu \omega}{r^2 - \mu} \dot{\psi} \right) + r^2 \dot{\psi}$$

(3.10)

$$P = g_{r n} x^n = \left( 1 - \frac{\mu}{r^2} \right)^{-2} \dot{r}.$$  

The momenta $E$ and $J$ are conserved while $P$ is determined by solving the null condition $\mathcal{H} = 0$ leading to (in the incoming branch)

$$P(r^2) = - \frac{r}{(r^2 - \mu)} \left[ E^2 r^6 - [J(r^2 - \mu) - \mu \omega E]^2 \right]^{\frac{1}{2}}$$

(3.11)

---

5The analysis for $\vartheta = \pi/2$ is identical exchanging $\varphi \leftrightarrow \psi$
We notice that, if $\omega^2 < \mu$, the polynomial inside the brackets is positive for large $r$ and negative for $r = \sqrt{\mu}$ and therefore it vanishes for some $r_* > \sqrt{\mu}$. For this choice, the particle either bounces back or gets trapped inside a critical trajectory before it reaches the horizon at $r = \sqrt{\mu}$. The trapping behaviour occurs if $J = J_c$ such that a point $r_*$ exists where $P(r_*) = P'(r_*) = 0$. Parametrising the angular momentum by means of the impact parameter $b = J/E$, the two equations are solved by taking

$$r_* = \left| \frac{2b_c}{3} \right|$$

with $b_c$ a solution of the cubic equation

$$4b_c^3 - 27\mu(b_c + \omega) = 0$$

The solutions are

$$b_c = -3\sqrt{\mu} \sin \left( \frac{1}{3} \arctan \frac{\omega}{\sqrt{\mu - \omega^2}} + \frac{2\pi}{3} C \right), \quad C = -1, 0, 1$$

It is easy to see that $C = 0$ leads to a zero $r_* < \sqrt{\mu}$ inside the horizon, so it should be discarded. The remaining two roots lead to critical geodesics of the black hole geometry.

4 Two-charge fuzzballs

In this section we consider massless geodesics along 2-charge fuzzball geometries obtained by setting $\varepsilon_1 = \varepsilon_4 = n = 0$ in the three-charge fuzzball solution.

4.1 The circular fuzzball solution

The general 2-charge geometry is specified by a profile function $\vec{F}(v)$ with values on $\mathbb{R}^4 \times T^4$. Here we choose a circular profile $\vec{F}(v)$ in $\mathbb{R}^4$

$$\vec{F}(v) = a \left( \cos \frac{2\pi v}{\lambda}, \sin \frac{2\pi v}{\lambda}, 0, 0 \right)$$
for which one has

\[
Z_1 = 1 + \frac{L_1}{\lambda} \int_0^\lambda \left| \frac{\dot{F}(v)}{X - \dot{F}(v)} \right|^2 dv = 1 + \frac{L_1^2}{\rho^2 + a^2 c_\phi^2}
\]

\[
Z_2 = 1 + \frac{L_5}{\lambda} \int_0^\lambda \frac{dv}{X - \dot{F}(v)} = 1 + \frac{L_5^2}{\rho^2 + a^2 c_\phi^2}
\]

(4.2)

and \(Z^2 = Z_1 Z_2\). Moreover the 1-forms \(\beta\) and \(\gamma\) are given by [56]

\[
\beta = \beta_m dx^m = \frac{a^2 R}{\rho^2 + a^2 c_\phi^2} \left( s_\phi^2 d\varphi - c_\phi^2 d\psi \right),
\]

\[
\gamma = \gamma_m dx^m = \frac{a^2 R}{\rho^2 + a^2 c_\phi^2} \left( s_\phi^2 d\varphi + c_\phi^2 d\psi \right)
\]

(4.3)

with \(R = R_y/\sqrt{2}\), \(R_y\) being the radius of \(S^1\) along the \(y\)-direction. The geometry has no horizon for

\[
a^2 = \frac{L_1^2 L_5^2}{2 R^2}
\]

(4.4)

4.2 The geodesic equations

The Hamiltonian depends only on \(\vartheta\) and \(\rho\), so the momenta \(P_u, P_v, P_\psi\) and \(P_\varphi\) are all conserved. The Hamiltonian can be separated [57–60] according to

\[
2Z^2 (\rho^2 + a^2 c_\phi^2) \mathcal{H} = \lambda_\rho(\rho, P_\rho) + \lambda_\vartheta(\vartheta, P_\vartheta)
\]

(4.5)

with

\[
\lambda_\vartheta(\vartheta, P_\vartheta) = P_\vartheta^2 + \frac{P_\psi^2}{\cos^2 \vartheta} + \frac{P_\rho^2}{\sin^2 \vartheta} + 2a^2 \sin^2 \vartheta P_u P_v
\]

(4.6)

\[
\lambda_\rho(\rho, P_\rho) = (\rho^2 + a^2) P_\rho^2 + \frac{a^2 P_\psi^2}{\rho^2} - \frac{a^2 P_\varphi^2}{\rho^2 + a^2} - 2(\rho^2 + a^2 + L_1^2 + L_5^2) P_u P_v
\]

(4.7)

and

\[
\tilde{P}_\psi = P_\psi + R (P_v - P_u), \quad \tilde{P}_\varphi = P_\varphi + R (P_v + P_u)
\]

(4.8)

Equation \(\mathcal{H} = 0\) can be solved by taking

\[
\lambda_\vartheta = -\lambda_\rho = K^2
\]

(4.9)

with \(K\) a constant, that can be interpreted as the total angular momentum. Equivalently one has

\[
P_\vartheta(\vartheta)^2 = K^2 - \frac{P_\psi^2}{c_\phi^2} - \frac{P_\varphi^2}{s_\phi^2} - 2P_u P_v a^2 s_\phi^2
\]

\[
P_\rho(\rho)^2 = \frac{a^2 \tilde{P}_\psi^2}{\rho^2 (\rho^2 + a^2)} + \frac{a^2 \tilde{P}_\varphi^2}{(\rho^2 + a^2)^2} + \frac{2(\rho^2 + L_1^2 + L_5^2 + a^2) P_u P_v - K^2}{\rho^2 + a^2}
\]

(4.10)
Expressing the velocities in terms of the momenta
\[ \dot{\theta} = \frac{P_\theta(\theta)}{Z^2(\rho^2 + a^2 c_\theta^2)} \, , \quad \dot{\rho} = \frac{\rho^2 + a^2}{\rho^2 + a^2 c_\theta^2} \frac{P_\rho(\rho)}{Z^2} \]
one finds the separable geodesic equation
\[ \frac{d\theta}{P_\theta(\theta)} = \frac{d\rho}{P_\rho(\rho)(\rho^2 + a^2)} \] (4.11)
that implicitly determines \( \theta(\rho) \) in terms of elliptic integrals. Finally, \( \varphi(\rho) \) and \( \psi(\rho) \) follow from
\[ d\psi = \frac{\rho^2 P_\psi + a^2 c_\theta^2 \tilde{P}_\psi}{P_\rho(\rho)\rho^2(\rho^2 + a^2)c_\theta^2} d\rho \, , \quad d\varphi = \frac{(\rho^2 + a^2)\tilde{P}_\varphi - a^2 s_\theta^2 \tilde{P}_\varphi}{P_\rho(\rho)\rho^2(\rho^2 + a^2)s_\theta^2} d\rho \] (4.12)
after integration over \( \rho \).

4.3 Critical geodesics
It is convenient to write
\[ P_\rho^2(\rho) = \frac{P_3(\rho^2)}{\rho^2(\rho^2 + a^2 c_\theta^2)^2} \] (4.13)
and set \( \rho^2 = x \) so that
\[ P_3(x) = A x^3 + B x^2 + C x + D \] (4.14)
with
\[ A = 2 P_u P_v \]
\[ B = 2P_u P_v (2a^2 + L_1^2 + L_5^2) - K^2 \]
\[ C = a^2 \left[ \tilde{P}_\rho^2 - \tilde{P}_\psi^2 - 2P_u P_v (a^2 + L_1^2 + L_5^2) - K^2 \right] \]
\[ D = -a^4 \tilde{P}_\psi^2 \] (4.15)
Since \( A > 0 \) and \( D < 0 \), the polynomial \( P_3(x) \) is positive for large \( x \) and negative for small \( x \). Therefore it has at least a zero \( x_* \) (the largest one) for positive \( x = \rho^2 \). This is in contrast with the behaviour observed for the black hole geometry, where \( P_\rho^2(\rho) \) was shown to be positive everywhere for small enough angular momenta \( K \). We conclude that massless probes in the fuzzball metric escape from the gravitational background, even for low values of the angular momentum \( K \). An exception occurs when the angular momentum is tuned such that \( x_* \) is a double zero of \( P_3(x) \), i.e.
\[ P_3(x_*) = P_3'(x_*) = 0 \] (4.16)
For this choice, the integral (2.23) diverges and the surface \( \rho_* = \sqrt{x_*} \) looks like a horizon for the massless geodesics. Indeed, for a critical value of \( K \) such that the two largest roots of \( P_3(x) \) collide, the particle winds around the target forever, asymptotically approaching the ‘circular’ orbit with radius \( \rho_* \). Such geodesics will be referred to as critical geodesics. In the remaining of this section we will display some explicit choices of kinematics exhibiting such trapping behaviour.
First, we notice that the conditions $A > 0$ and $D < 0$, together with the requirement that the largest root is double and positive, imply that all three roots are positive and

$$A, C > 0 \quad , \quad B, D < 0$$

(4.17)

Solving (4.16) for $x_*$ and $D$ one finds

$$x_* = \frac{1}{3A} \left(-B + \sqrt{B^2 - 3AC}\right)$$

$$D = \frac{2}{27A^2}(B^2 - 3AC)^{3/2} - \frac{B}{27A^2}(2B^2 - 9AC)$$

(4.18)

Solutions compatible with (4.17) exist if

$$4AC \geq B^2 \geq 3AC$$

(4.19)

The two extreme cases where the inequalities are saturated are easy to solve in analytic form:

- **Case I: $B^2 = 3AC$.** For this choice all three roots collide and $D = \frac{BC}{2A}$. From (4.15) one finds

$$\tilde{P}_\varphi^2 = \frac{[K^2 + 2(a^2 - L_1^2 - L_5^2)P_uP_v]^3}{108a^4P_u^2P_v^2}$$

$$\tilde{P}_\psi^2 = \frac{[K^2 - 2(2a^2 + L_1^2 + L_5^2)P_uP_v]^3}{108a^4P_u^2P_v^2}$$

(4.20)

and

$$\rho_*^2 = \frac{K^2}{6P_uP_v} - \frac{1}{3}(2a^2 + L_1^2 + L_5^2) > 0$$

(4.21)

We notice that a critical geodesic of this type exists for a large enough total angular momentum $K$.

- **Case II: $B^2 = 4AC$.** For this choice one finds $D = 0$,

$$\tilde{P}_\psi = 0$$

$$\tilde{P}_\varphi^2 = \frac{[K^2 - 2P_uP_v(L_1^2 + L_5^2)]^2}{8a^2P_uP_v}$$

(4.22)

and

$$\rho_*^2 = \frac{K^2}{4P_uP_v} - \frac{1}{3}(2a^2 + L_1^2 + L_5^2) > 0$$

(4.23)

### 4.4 An example of critical geodesics

To illustrate the trapping behaviour of fuzzballs, let us consider the critical geodesics along the plane $\vartheta = \pi/2$, for the choice

$$L_1 = L_5 = a \quad , \quad P_u = P_v \quad , \quad P_\psi = 0$$

(4.24)
Figure 3: Time delay between massless particles moving in a 2-charge fuzzball geometry and flat space-time as a function of the adimensionalised impact parameter $b/a$.

For this choice the velocity $\dot{y}$ of the particle along the compact circle can be set to zero along the full trajectory. The critical geodesics fall into case II above. Introducing the impact parameter

$$b = \frac{P_\varphi}{E} = \frac{P_\varphi}{\sqrt{2}P_u}$$

and using (4.13), (4.15), (4.10) one finds

$$\mathcal{P}_3(\rho) = 2P_u^2\rho^2 [\rho^4 + (3a^2-b^2)\rho^2 + (3a-2b)a^3]$$

with largest zero

$$\rho_*^2 = \frac{b^2 - 3a^2 + \sqrt{(b-a)^3(b+3a)}}{2}$$

The turning point exists for $b \leq -3a$ or $b \geq 3a/2$; when $b = 3a/2$ or $b = -3a$ a limit cycle exists at $\rho = 0$ and $\rho = \sqrt{3}a$ respectively. For values of $b$ in-between $P_u^2$ has no zeroes, the probe reaches $\rho = 0$ in a finite, possibly large, amount of time, surpasses it and gets scattered back at infinity. The time to reach $\rho_*$ is given by

$$\Delta t = \int_{\rho_0}^{\rho_*} d\rho \frac{P_u \rho + (3a-b)a^3}{\sqrt{2}P_u \rho^2 + a^2 \sqrt{\mathcal{P}_3(\rho^2)}}$$

In (Fig. 3) we display the difference between the total flight time in the fuzzball geometry and in flat space-time as a function of $b$ for a fixed large $\rho_0$. As expected, the closer a particle’s impact parameter approaches the critical one, the longer the time it will spend orbiting around the fuzzball. It is also clear that even though for $b < b_c$ the particle will eventually be scattered, it spends a considerable amount of time in the proximity of the fuzzball.

5 3-charge fuzzballs

In this section we consider scattering on 3-charge fuzzball geometries.
5.1 The Hamiltonian and momenta

Momenta and velocities in the 3-charge geometry are related by

\begin{align*}
    P_u &= -(\dot{u} + \beta_m x^m) \\
    \hat{P}_v &= -(\dot{v} + \gamma_m x^m) \\
    \hat{P}_\rho &= \frac{Z_2^2(\rho^2 + a^2 c_\rho^2)}{\rho^2 + a^2} \dot{\rho} \\
    \hat{P}_\varphi &= Z_2^2(\rho^2 + a^2) \frac{c_\varphi^2}{\rho^2} \dot{\varphi} \\
    \hat{P}_\vartheta &= Z_2^2(\rho^2 + a^2) \psi \\
    \hat{P}_\psi &= \frac{Z_2^2}{\rho^2 + a^2} (\rho^2 + a^2) \dot{\psi}
\end{align*}

(5.1)

The important difference with respect to the 2-charge case is that now \(\beta_m, \gamma_m\), and \(Z\), and therefore the Hamiltonian, explicitly depend on the combination \(\phi = \varphi + \frac{\omega_R}{\pi}\) and therefore \(P_v\) and \(P_\varphi\) are no longer conserved separately but only their combination \(P_\nu = P_v - \frac{\omega_R}{\pi} P_\varphi\) is. Indeed, the equations of motion become

\begin{align*}
    \dot{P}_u &= \dot{P}_v = \dot{P}_\psi = \mathcal{H} = 0 \\
    \dot{P}_\varphi &= -\frac{\partial \mathcal{H}}{\partial \varphi} = -\frac{R}{n} \frac{\partial \mathcal{H}}{\partial v} = \frac{R}{n} \dot{P}_v \\
    \dot{P}_\vartheta &= -\frac{\partial \mathcal{H}}{\partial \vartheta}
\end{align*}

(5.2)

We observe that the Hamiltonian \(\mathcal{H}\) is a rational function of \(\cos \vartheta^2\) and therefore

\[
\frac{\partial \mathcal{H}}{\partial \vartheta} \sim \cos \vartheta \sin \vartheta
\]

(5.3)

This implies that \(P_\vartheta\) is conserved for \(\vartheta = 0, \pi/2\). Moreover at \(\vartheta = 0, \pi/2\), \(\hat{P}_\vartheta = P_\vartheta\) and therefore constant \(P_\vartheta\) implies constant \(\dot{\vartheta}\). We conclude that geodesics starting at \(\vartheta = 0, \pi/2\) with zero initial \(\vartheta\) velocity, \(\dot{\vartheta} = 0\) keep \(\vartheta\) constant along the whole trajectory. In the following we restrict ourselves on geodesics along these two planes.

5.2 \(\vartheta = 0\) geodesics

Let us start by choosing \(n = 1\) and considering the geodesics in the plane \(\vartheta = 0\), orthogonal to the circular profile. The functions and forms defining the metric assume the following expression

\begin{align*}
    Z_4 &= 0 \\
    \beta &= -\frac{a^2 R}{\rho^2 + a^2} d\psi \\
    \gamma &= \frac{a^2 R}{\rho^2 + a^2} (1 - \mathcal{F}_1) d\psi + \mathcal{F}_1 dv \\
    \mathcal{F}_1 &= -\frac{\varepsilon_4^2}{2(\rho^2 + a^2)} \\
    Z^2 &= Z_1 Z_2 = \left(1 + \frac{L_1^2}{\rho^2 + a^2}\right) \left(1 + \frac{L_2^2}{\rho^2 + a^2}\right)
\end{align*}

(5.4)

Taking \(\hat{P}_\vartheta = P_\vartheta = 0\) and \(P_\varphi = \hat{P}_\varphi = 0\), the Hamiltonian becomes

\[
\mathcal{H} = -P_u \hat{P}_v + \frac{1}{2Z^2} \left(\frac{\hat{P}_\rho^2 + \hat{P}_\psi^2}{\rho^2}\right)
\]

(5.5)
with
\[
\hat{P}_v = P_v + \frac{\varepsilon^2}{2(\rho^2 + a^2)} P_u, \quad \hat{P}_\rho = P_\rho
\]
\[
\hat{P}_\psi = P_\psi - \frac{a^2 R}{\rho^2 + a^2} (P_u - P_v) .
\]

Recall that \( P_u, P_v, P_\psi \) are conserved quantities. Plugging this into (2.21) one finds
\[
P_\rho = \pm \left[ 2Z^2 P_u \hat{P}_v - \frac{\hat{P}_\psi^2}{\rho^2} \right]^{\frac{1}{2}} = \pm \frac{\mathcal{P}_4(\rho^2)^{\frac{1}{2}}}{\rho(\rho^2 + a^2)^{\frac{3}{2}}} \tag{5.6}
\]
with, setting \( \rho^2 = x \) as above,
\[
\mathcal{P}_4(x) = P_u x (x + a^2 + L_1^2)(x + a^2 + L_5^2) \left[ 2 P_u (x + a^2) + \varepsilon^2 P_u \right] - (x + a^2) \left[ P_\psi (x + a^2) - a^2 R (P_u - P_v) \right]^2 \tag{5.7}
\]
We notice that the polynomial \( \mathcal{P}_4(x) \) is positive for \( x \to \infty \) and negative for \( x \to 0 \). Therefore it has a zero somewhere on the positive \( x \) axis. Again we denote \( x_\ast \) the largest positive zero. If \( x_\ast \) is simple then it is a turning point and the particle gets deflected in the gravitational background. On the other hand for a critical choice of \( P_\psi \) for which \( x_\ast \) is a double zero the particle gets trapped in the gravitational background, asymptotically approaching \( \rho_\ast = \sqrt{x_\ast} \).

As an illustration of this critical behavior, let us consider a particle with no internal Kaluza-Klein momentum \( P_v = P_u \) and
\[
L_1^2 = L_5^2 = \varepsilon_4^2/2 = L^2 \geq 3a^2 . \tag{5.8}
\]
For this choice the polynomial \( \mathcal{P}_4(x) \) takes the simple form
\[
\mathcal{P}_4(x) = 2P_u^2 x (x + a^2 + L^2)^3 - (x + a^2)^3 P_\psi^2 \tag{5.9}
\]
Solving the critical conditions \( \mathcal{P}_4(x) = \mathcal{P}_4'(x) = 0 \) one finds a double zero at
\[
x_\ast = L^2 - a^2 + L \sqrt{L^2 - 3a^2} \tag{5.10}
\]
for the critical choice of angular momentum
\[
P_\psi = \sqrt{6} P_u L \left[ 1 + \frac{L^2}{9a^2} - \frac{L^2}{9a^2} \left( 1 - \frac{3a^2}{L^2} \right)^{3/2} \right] \tag{5.11}
\]
In other words, scattering massless particles off the fuzzball geometry, one finds that the components with \( P_\psi \) satisfying (5.11) are missing in the out-going spectrum, and the fuzzball geometry behaves effectively as a black object for the selected “channel”.

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5.3 $\vartheta = \pi/2$ geodesics

In this plane, the Hamiltonian, explicitly depends on the combination $\phi = \varphi + \frac{nv}{R}$, so it is convenient to introduce the canonically related variables $\phi, \nu$ (and their conjugate momenta)

$$\varphi = \phi - \frac{nv}{R}, \quad P_\varphi = P_\phi$$
$$v = \nu, \quad P_v = P_\nu + \frac{n}{R} P_\phi$$

(5.12)

In terms of these variables the equations of motion become

$$\dot{P}_u = \dot{P}_v = \dot{P}_\psi = 0$$
$$\dot{P}_\vartheta = -\frac{\partial H}{\partial \vartheta}$$
$$\dot{P}_\phi = -\frac{\partial H}{\partial \phi}$$

(5.13)

For motion in the plane of the string profile, the metric is given by (2.1) and (2.3) with

$$Z_1 = 1 + \frac{L_4^2}{\rho^2} + \frac{\varepsilon_1 R^2 \Delta_n \cos 2\phi}{L_5^2 \rho^2}$$
$$Z_2 = 1 + \frac{L_5^2}{\rho^2}$$
$$Z_3^2 = \frac{2 \varepsilon_1 R^2 \Delta_n \cos^2 \phi}{\rho^4}$$
$$Z_4 = Z_1 Z_2 - Z_3^2$$
$$\beta_\varphi = \frac{a^2 R}{\rho^2}$$
$$\beta_\psi = 0$$
$$\gamma_\rho = -\frac{\varepsilon_1 R}{2 \rho L_5^2} \Delta_n \sin 2\phi$$
$$\gamma_\psi = \gamma_\theta = 0$$
$$\gamma_\nu = \mathcal{F}_n$$

(5.14)

Taking $\hat{P}_\vartheta = P_\vartheta = 0, \hat{P}_\psi = P_\psi = 0$, the Hamiltonian reads

$$H = -P_u \hat{P}_u + (\rho^2 + a^2) \hat{P}_\rho^2 + \frac{\hat{P}_\varphi^2}{2Z^2(\rho^2 + a^2)}$$

(5.15)

where the hatted conjugate momenta have the form

$$\hat{P}_v = P_\nu + \frac{n}{R} P_\phi + \mathcal{F}_n P_u$$
$$\hat{P}_\rho = P_\rho + \frac{\varepsilon_1 R P_u \Delta_n \sin 2\phi}{2\rho L_5^2}$$
$$\hat{P}_\varphi = P_\phi - \frac{a^2}{\rho^2} (n P_\phi + R P_\nu + R P_u) + \frac{2a^2 R P_u}{\rho^2} \left[ \mathcal{F}_n + \frac{\varepsilon_1 \Delta_n (\rho^2 + a^2) \cos 2\phi}{4a^2 L_5^2} \right]$$

(5.16)

with $P_u$ and $P_\nu$ conserved quantities.
Let us focus on the truly dynamical variables $\rho$ and $\phi$. Their velocities are given by

$$
\dot{\rho} = \frac{\rho^2 + a^2}{Z^2 \rho^2} \hat{P}_\rho \\
\dot{\phi} = \frac{\hat{P}_\rho (\rho^2 - na^2)}{Z^2 \rho^2 (\rho^2 + a^2)} - \frac{n P_u}{R} 
$$

(5.18)

Choosing $\phi$ as independent variable, the equations of motion can be written in the form

$$
\frac{d\rho}{d\phi} = \frac{\hat{P}_\rho R (\rho^2 + a^2)^2}{\hat{P}_\rho R (\rho^2 - na^2) - P_u Z^2 \rho^2 (\rho^2 + a^2)} \\
\frac{dP_\phi}{d\phi} = -\frac{1}{\phi} \frac{\partial}{\partial \phi} \left[ \left( \frac{\rho^2 + a^2}{2Z^2 \rho^2} \hat{P}_\rho \right) + \frac{\hat{P}_\rho^2}{2Z^2 (\rho^2 + a^2)} \right] 
$$

(5.19)

and

$$
\hat{P}_\rho^2 = \frac{\rho^2}{(\rho^2 + a^2)^2} \left[ 2Z^2 P_u \hat{P}_\rho (\rho^2 + a^2) - \hat{P}_\rho^2 \right] 
$$

(5.20)

We are interested in solutions of the geodesic equations (5.19) characterised by trajectories trapped in the gravitational background. As before, we expect that for specific values of the incoming angular momentum $P_\phi$, there exists geodesics ending on trapping trajectories but now both the asymptotic trajectory and the angular momentum will in general vary with $\phi$. Due to the complexity of the problem, the exact solutions cannot be obtained analytically in full generality. In the remaining of this section we describe in detail a simple solution of the system as an illustration of the general case.

### 5.3.1 Asymptotic circular orbits

We consider first geodesics asymptotically reaching circular trajectories with constant angular velocity, i.e. $\dot{\rho} = 0$, $\dot{\phi} = \omega$. For concreteness we take

$$
L_1 = L_5 = L = a
$$

(5.21)

According to (5.18), a constant angular velocity can be found by taking

$$
\rho^2 = na^2 \quad \Rightarrow \quad \dot{\phi} = -\frac{n P_u}{R} 
$$

(5.22)

while $\dot{\rho} = 0$ requires

$$
\hat{P}_\rho = 0 
$$

(5.23)

or equivalently

$$
2Z^2 P_u \hat{P}_\rho (\rho^2 + a^2) - \hat{P}_\rho^2 = 0 
$$

(5.24)

---

The evolution of $\nu$, as well as of the other coordinates, follows from the one of $\rho$ and $\phi$. In particular

$$
\dot{\nu} = -P_u - \frac{a^2 R}{Z^2 \rho^2 (\rho^2 + a^2)} \hat{P}_\rho 
$$

(5.17)
We notice that at the critical point $\rho^2 = na^2$, $Z^2$ is constant and $\hat{P}_\varphi$ reduces to

$$\hat{P}_\varphi = \frac{R}{n} \left[ 2F_n P_u - P_u - P_\nu + \frac{\epsilon_1(n + 1)}{2a^2} P_\nu \Delta_n \cos 2\phi \right]$$  \hspace{1cm} (5.25)$$

Equation (5.24) can therefore be easily solved for $P_\phi$

$$P_\phi = \frac{R}{n} \left[ \frac{\hat{P}_\varphi^2}{2P_u Z^2 a^2(n + 1)} - P_\nu - F_n P_u \right]$$  \hspace{1cm} (5.26)$$

The two equations of motion (5.19) are satisfied for $\rho = \sqrt{na}$ and $P_\phi$ given by (5.26), quite remarkably this provides an exact solution for the non separable system. It would be interesting to find a solution interpolating between infinity and these closed trajectories.

6 Conclusions and outlook

Relying on a class of micro-state geometries for 3-charge systems in $D = 5$ constructed in [39], we have further tested the fuzzball proposal by studying massless geodesics in these backgrounds. In particular we have shown that 2- and 3-charge fuzzball geometries tend to trap massless neutral particles for a specific choice of their impact parameter. This is at variant with classical BH’s that trap all particles impinging with an impact parameter below a certain critical value of the order of the horizon radius. This suggests that the blackness property of black holes arises as a collective effect whereby each micro-state absorbs a specific channel.

The analysis has been performed in various steps. First we have reviewed the general form of the metric and written down the geodesic equations for massless neutral probes in both the Lagrangian and Hamiltonian forms. Then we focused on the cases of (singular) non-rotating BPS black-holes with 3-charge, on micro-states for 2-charge systems with a circular profile and finally on the 3-charge case.

We have (implicitly) integrated the geodesic equations for the 2-charge case for generic values of the initial value of $\vartheta_0$ and of the integration constant $K$ (playing the role of total angular momentum), thus generalising our previous results for $\vartheta = 0$ (plane orthogonal to the circular profile) and $\vartheta = \pi/2$ (plane of the circular profile).

In the 3-charge case we have fully analysed the geodesics for $\vartheta = 0$ (since they lead to separable equations of the same form as in the 2-charge case, previously analysed) and written down the equations for $\vartheta = \pi/2$, that lead to a non-separable system. A simple solution of this intricate system has been found.

We leave the analysis of waves and strings or to other classes of smooth geometries (such as JMaRT [52]) to the future.

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