The Descent Set Polynomial Revisited

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Abstract

We continue to explore cyclotomic factors in the descent set polynomial $Q_n(t)$, which was introduced by Chebikin, Ehrenborg, Pylyavskyy and Readdy. We obtain large classes of factors of the form $\Phi_{2s}$ or $\Phi_{4s}$ where $s$ is an odd integer, with many of these being of the form $\Phi_{2p}$ where $p$ is a prime. We also show that if $\Phi_{2}$ is a factor of $Q_{2n}(t)$ then it is a double factor. Finally, we give conditions for an odd prime power $q = p^r$ for which $\Phi_{2p}$ is a double factor of $Q_{2q}(t)$ and of $Q_{q+1}(t)$.

1 Introduction

For a permutation $\pi$ in the symmetric group $S_n$, define the descent set of $\pi$ to be the subset of $[n-1] = \{1, 2, \ldots, n-1\}$ given by $\text{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$. The descent set statistics $\beta_n(S)$ are defined for subsets $S$ of $[n-1]$ by

$$\beta_n(S) = |\{\pi \in S_n : \text{Des}(\pi) = S\}|.$$

Chebikin, Ehrenborg, Pylyavskyy and Readdy defined the $n$th descent set polynomial to be

$$Q_n(t) = \sum_{S \subseteq [n-1]} t^{\beta_n(S)}.$$

They observed that this polynomial has many factors that are cyclotomic polynomials. The most common of these cyclotomic polynomials is $\Phi_2 = t + 1$. It is direct that having $\Phi_2$ as a factor implies that the number of subsets of $[n-1]$ having an even descent set statistic is the same as the number of subsets having an odd descent set statistic. Consider the proportion of odd entries among the descent set statistics in the symmetric group $S_n$, that is,

$$\rho(n) = \frac{|\{S \subseteq [n-1] : \beta_n(S) \equiv 1 \mod 2\}|}{2^{n-1}}.$$

Chebikin et al. showed that this proportion depends on the number of 1’s in the binary expansion of $n$. We quote their paper with the following table. Only the values $2^k - 1$ are included in the table since $\rho(2^k - 1)$ is the same as $\rho(n)$ if $n$ has $k$ 1’s in its binary expansion. Hence when $n$ has two or

| $n$  | 1  | 3  | 7  | 15  | 31  |
|-----|----|----|----|-----|-----|
| $\rho(n)$ | 1/2 | 1/2 | 1/2 | 29/2^6 | 3991/2^{13} |

Table 1: The proportion $\rho(n)$.

three 1’s in its binary expansion we obtain $\Phi_2$ as a factor in the descent set polynomial $Q_n(t)$. Note that the proportion is not known for six or more 1’s in the binary expansion.

Chebikin et al. gave more results for cyclotomic factors in the descent set polynomial:
that differ by only one element, stated as

\[ S - S \]

set statistic is straightforward to compute modulo the prime

general linear function; see Proposition 7.1. The underlying reason for these results is that the descent
two sets, that is, \( S - S \)

Let \[ (i) \text{ When } n = 2^j \geq 4 \text{ then } \Phi_4 \text{ divides } Q_n(t). \]

(ii) When \( q = p^r \) is an odd prime power with two or three 1’s in its binary expansion and \( q \neq 3 \) or 7,
then \( \Phi_{2p} \) divides \( Q_q(t) \).

(iii) When \( q = p^r \) is an odd prime power with two or three 1’s in its binary expansion, then \( \Phi_{2p} \)
divides \( Q_{2q}(t) \).

They also found cases when there were double factors in the descent set polynomial:

(iv) If the binary expansion of \( n \) has two 1’s in its binary expansion and \( n > 3 \), then \( \Phi_2 \) is a double
factor of \( Q_n(t) \).

(v) If \( n = 2^j \geq 4 \) then \( \Phi_4 \) is a double factor of \( Q_n(t) \).

(vi) When \( q = p^r \) is an odd prime power and \( q \) has two 1’s in its binary expansion, then \( \Phi_{2p} \) is a
double factor of \( Q_n(t) \).

We continue their work in explaining cyclotomic factors in these polynomials. In Section 2 we
review some preliminary notions and tools that will help in developing our results. We introduce
a simplicial complex in Section 3 that determines the parity of the descent statistics. Namely, the
reduced Euler characteristic of an induced subcomplex gives the descent statistics modulo 2. In
Section 4 we prove for \( s \) an odd integer when \( \Phi_{4s} \) is a factor of \( Q_n(t) \) with \( n \) being a power of 2. In
Section 5 we show for \( s \) an odd integer when \( \Phi_{2s} \) is a factor of \( Q_n(t) \) when \( n \) has two non-zero digits
in its binary expansion. We prove a multitude of cases in this section when we set \( s \) to be a prime
number \( p \). Similarly, when \( n \) has three digits in its binary expansion, we develop cases when \( \Phi_{2s} \), and
likewise \( \Phi_{2p} \), is a factor of \( Q_n(t) \) in Section 6.

We also continue the work on double factors in the descent set polynomial \( Q_n(t) \) in Sections 7
through 9. In fact, the two results (iv) and (vi) both need the condition that the number of 1’s in
the binary expansion of \( n \) is exactly two. Furthermore, the result (vi) applies only (so far) to the
group of five Fermat primes and the prime power \( 3^2 \), whereas our results apply when there are two or three
1’s in the binary expansion. First in Theorem 7.2 we show that if \( \Phi_2 \) is a factor of \( Q_{2n}(t) \) then it is
double factor. Next in Theorems 8.1 and 9.1 we find the double factor \( \Phi_{2p} \) in \( Q_{2q}(t) \) and \( Q_{q+1}(t) \)
where \( q = p^r \) is an odd prime power. The corresponding proofs in 3 depend on substituting values
for the variables in the index of the Boolean algebra, whereas our proofs rely on evaluating a more
general linear function; see Proposition 7.1. The underlying reason for these results is that the descent
set statistic is straightforward to compute modulo the prime \( p \); see Lemma 8.2 and equation 9.1.

A summary of cyclotomic factors of \( Q_n(t) \) that Chebikin et al. found, as well as which ones were
explained by their and our results, can be found in Table 6. We end with open questions in the
concluding remarks.

2 Preliminaries

Let \([i, j] \) denote the interval \( \{i, i + 1, \ldots, j \} \). Furthermore, let \( \Delta \) denote the symmetric difference of
two sets, that is, \( S \Delta T = S \cup T - S \cap T \). Finally, let \( S - k \) denote the shifting of the set by \( k \), that is,
\( S - k = \{s - k : s \in S\} \).

MacMahon’s Multiplication Theorem [9, Article 159] relates the descent set statistics of two sets
that differ by only one element, stated as

\[
\beta_n(S) + \beta_n(S \Delta \{k\}) = \binom{n}{k} \cdot \beta_k(S \cap [k - 1]) \cdot \beta_{n-k}(S \cap [k + 1, n - 1] - k).
\]
This result is usually written with the assumption \( k \not\in S \) and the left hand-side as \( \beta_n(S) + \beta_n(S \cup \{k\}) \), whereas we find it more convenient to work with the symmetric difference.

One way to compute the descent set statistics is via the flag \( f \)-vector of the Boolean algebra. For \( S = \{s_1 < s_2 < \cdots < s_k\} \subseteq \left\{ n - 1 \right\} \), let \( \cos(S) = \bar{c} = (c_1, c_2, \ldots, c_{k+1}) \) be the associated composition of \( n \) where \( c_i = s_i - s_{i-1} \), where we let \( s_0 = 0 \) and \( s_{k+1} = n \). Then the flag \( f \)-vector of the Boolean algebra \( B_n \) is given by the multinomial coefficient

\[
|S| - \frac{n}{\bar{c}} = \binom{n}{c_1, c_2, \ldots, c_{k+1}},
\]

and the descent set statistics is given by the inclusion-exclusion

\[
\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S - T|} \cdot f_T.
\] (2.1)

An efficient encoding of all the flag \( f \)-vector entries of the Boolean algebra is by the quasi-symmetric function. For a composition \( \bar{c} = (c_1, c_2, \ldots, c_k) \) let \( M_{\bar{c}} \) denote the monomial quasi-symmetric function defined by

\[
M_{\bar{c}} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1}^{c_1} \cdot x_{i_2}^{c_2} \cdots x_{i_k}^{c_k}.
\]

The algebra of quasi-symmetric functions is the linear span of the monomial quasi-symmetric functions. Multiplication of monomial quasi-symmetric functions is described in Lemma 3.3 in [4]. Now the quasi-symmetric function of the Boolean algebra is given in [4] by

\[
F(B_n) = (x_1 + x_2 + \cdots)^n = M^n_{(1)} = \sum_{\bar{c}} \binom{n}{\bar{c}} \cdot M_{\bar{c}}.
\]

The purpose of quasi-symmetric functions is that they allow efficient computations of the flag \( f \)-vector modulo a prime \( p \), using the classical relation \((x + y)^p \equiv x^p + y^p \mod p\). Finally, using the inclusion-exclusion equation (2.1), we obtain information about the descent set statistics. Below is a lemma, adapted from Lemma 3.2 in [3], to compute the quasi-symmetric function of the Boolean algebra \( F(B_n) = M^n_{(1)} \) modulo a prime.

\textbf{Lemma 2.1.} For \( p \) prime and \( n = d_1p^{j_1} + d_2p^{j_2} + \cdots + d_kp^{j_k} \) with \( j_1 > \cdots > j_k \geq 0 \), the quasi-symmetric function of the Boolean algebra \( B_n \) modulo \( p \) is given by \( F(B_n) \equiv \prod_{i=1}^k M^{d_i}_{(p^{j_i})} \mod p \).

\textit{Proof.} The congruence \((x + y)^p \equiv x^p + y^p \mod p\) extends to monomial quasi-symmetric functions as \( M^{n}_{(1)} \equiv M^{p^m}_{(p^m)} \mod p \). Hence the quasi-symmetric function of Boolean algebra \( B_n \) is evaluated as follows:

\[
F(B_n) = M^{d_1p^{j_1} + d_2p^{j_2} + \cdots + d_kp^{j_k}}_{(1)} = \left( M^{p^{j_1}}_{(1)} \right)^{d_1} \cdot \left( M^{p^{j_2}}_{(1)} \right)^{d_2} \cdots \left( M^{p^{j_k}}_{(1)} \right)^{d_k} \\
\equiv M^{d_1}_{(p^{j_1})} \cdot M^{d_2}_{(p^{j_2})} \cdots M^{d_k}_{(p^{j_k})} \mod p.
\]

Chebikin et al. defined essential elements in the case of base 2, and we extend this notion to base \( p \) for any prime \( p \).

\textbf{Definition 2.2.} Let \( p \) be a prime and \( 1 \leq k \leq n - 1 \). We say \( k \) is essential for \( n \) in base \( p \) if we expand both \( n \) and \( k \) in base \( p \), that is, \( n = \sum_{i \geq 0} n_i \cdot p^i \) and \( k = \sum_{i \geq 0} k_i \cdot p^i \) where \( 0 \leq k_i, n_i < p \), and the inequality \( k_i \leq n_i \) holds for all indices \( i \). Otherwise we say \( k \) is non-essential for \( n \) in base \( p \).
A different way to state that $k$ is essential for $n$ in base $p$ is that when adding $k$ and $n - k$ in base $p$ there are no carries. Directly from this interpretation we have the following natural symmetry:

**Lemma 2.3.** The element $k$ is essential for $n$ in base $p$ if and only if $n - k$ is essential for $n$ in base $p$.

Another alternative interpretation is as follows:

**Lemma 2.4.** The element $k$ is essential for $n$ in base $p$ if and only if $\binom{n}{k} \not\equiv 0 \mod p$.

*Proof.* By Lucas’ theorem, see [8, Chapter XXIII, Section 228], we have that
\[
\binom{n}{k} \equiv \prod_{i \geq 0} \binom{n_i}{k_i} \mod p.
\]
Observe that for $0 \leq k_i, n_i \leq p - 1$ we have that $\binom{n_i}{k_i} \not\equiv 0 \mod p$ if and only if $k_i \leq n_i$. \hfill \Box

Note that for an element $k$ which is non-essential in base $p$, the previous lemma implies that $p$ divides $\binom{n}{k}$. This allows the following lemma to apply for this number $k$ when we set the integer $m$ to be the prime $p$.

**Lemma 2.5.** Let $m$ and $k$ be positive integers such that $1 \leq k \leq n - 1$ and $m$ divides $\binom{n}{k}$. For a subset $S$ of $[n - 1]$ the following holds
\[
\beta_n(S) \equiv -\beta_n(S \triangle \{k\}) \mod m.
\]

*Proof.* By MacMahon’s multiplication theorem we have that
\[
\beta_n(S) + \beta_n(S \triangle \{k\}) = \binom{n}{k} \cdot \beta_k(S \cap [k - 1]) \cdot \beta_{n-k}(S \cap [k + 1, n - 1] - k),
\]
and the result follows by the assumption that $\binom{n}{k} \equiv 0 \mod m$. \hfill \Box

For $0 \leq j \leq m - 1$ define $a_{m,j}$ to be the number of subsets $S \subseteq [n - 1]$ such that $\beta_n(S) \equiv j \mod m$. Note that we suppress the dependency on $n$. Furthermore, if $m$ is clear from the context, we simply write $a_j$.

**Lemma 2.6.** Let $m$ be a positive integer and $1 \leq k \leq n - 1$. If $m$ divides $\binom{n}{k}$ then the equality $a_{m,j} = a_{m,-j}$ holds for all $j$.

*Proof.* By Lemma 2.5 we have that $\beta_n(S) \equiv -\beta_n(S \triangle \{k\}) \mod m$. Hence the map sending $S$ to the symmetric difference $S \triangle \{k\}$ yields a bijection between the sets counted by $a_{m,j}$ and $a_{m,-j}$. \hfill \Box

The following are consequences of Theorem 2.1 in [3], which gives information about the proportion of even or odd descent statistics $\beta_n(S)$ depending on the number of 1’s in the binary expansion of $n$. We apply their result to achieve equalities involving $a_{i,j}$.

**Theorem 2.7** (Chebikin et al.). (a) If $n$ has only one 1 in its binary expansion, i.e. $n = 2^a$, then $\beta_n(S) \equiv 1 \mod 2$ for all subsets $S \subseteq [n - 1]$. 
If \( n \) has either two or three 1’s in its binary expansion, then there is an identical number of even descent statistics as there is of odd descent statistics.

In terms of the proportion introduced in the introduction, we have \( \rho(2^a) = 1, \rho(2^b + 2^c) = 1/2 \) and \( \rho(2^c + 2^b + 2^a) = 1/2 \) for non-negative integers \( c > b > a \). As a direct corollary we have

**Corollary 2.8.** Let \( s \) be an odd positive integer.

(a) If \( n \) has only one 1 in its binary expansion, then for \( j \) even \( a_{2s,j} = 0 \) holds.

(b) If \( n \) has either two or three 1’s in its binary expansion, then

\[
\sum_{j=0}^{2s-2} a_{2s,j} = \sum_{j=0}^{2s-1} a_{2s,j}.
\]

We end with a well-known fact from algebra.

**Fact 2.9.** If \( f(t) \) is a polynomial in \( \mathbb{Q}[t] \) with \( e^{2\pi i/j} \) as a root of multiplicity \( r \) then the \( j \)th cyclotomic polynomial \( \Phi_j(t) \) is a factor of order \( r \) of \( f(t) \).

This follows since the cyclotomic polynomial is the minimal polynomial of \( e^{2\pi i/j} \) over the rational field \( \mathbb{Q} \).

3 The simplicial complex \( \Delta_n \)

We now introduce a simplicial complex, which will encode the descent set statistics modulo 2, via the reduced Euler characteristic. Let \( \Delta_n \) be a simplicial complex on the vertex set \([n - 1]\). Let \( F \) be a face of \( \Delta_n \) if when adding the entries of the associated composition \( \text{co}(F) = (c_1, c_2, \ldots, c_{k+1}) \), that is, the sum \( c_1 + c_2 + \cdots + c_{k+1} = n \) has no carries in base 2.

Notice that \( \{i\} \) is a vertex of \( \Delta_n \) if and only if \( i \) is an essential element of \( n \) in base 2. In fact, the simplicial complex \( \Delta_n \) is completely described by the number of 1’s in the binary expansion of \( n \). For \( n \) with \( k \) 1’s in its binary expansion, the complex \( \Delta_n \) is the barycentric subdivision of the boundary of a \((k - 1)\)-dimensional simplex. A different way to describe it is that \( \Delta_n \) is the boundary of the dual of the \((k - 1)\)-dimensional permutahedron.

**Theorem 3.1.** The quasi-symmetric function of \( B_n \) modulo 2, is given by

\[
F(B_n) \equiv \sum_{F \in \Delta_n} M_{\text{co}(F)} \mod 2.
\]

**Proof.** Write \( n \) as a sum of 2-powers, that is, \( n = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_k} \) where \( j_1 > j_2 > \cdots > j_k \). By Lemma 2.1 we have the identity

\[
F(B_n) \equiv M_{2^{j_1}} \cdot M_{2^{j_2}} \cdots M_{2^{j_k}} \mod 2.
\]

Now when multiplying out these \( k \) monomial quasi-symmetric functions we obtain a sum over monomial quasi-symmetric functions, where the indexing composition has parts consisting of sums of the 2-powers \( 2^{j_1}, 2^{j_2}, \ldots, 2^{j_k} \). Furthermore, each 2-power can only appear in exactly one part and only once in that part. Also note no composition can be created in two different ways; in the language of the article [6], the partition \( \{2^{j_1}, 2^{j_2}, \ldots, 2^{j_k}\} \) is a knapsack partition. Finally, translating the compositions of \( n \) into subsets of \([n - 1]\) proves the result. \( \square \)
In other words, the flag $f$-vector entry $f_S(B_n)$ is odd if and only if $S$ is a face of the complex $\Delta_n$. Let $\Delta_n|S$ denote the simplicial complex $\Delta_n$ restricted to vertex set $S$, that is, 

\[ \Delta_n|S = \{ F \subseteq S : F \in \Delta_n \}. \]

**Theorem 3.2.** The descent set statistic $\beta_n(S)$ modulo 2 is given by the reduced Euler characteristic of the induced subcomplex $\Delta_n|S$, that is, 

\[ \beta_n(S) \equiv \tilde{\chi}(\Delta_n|S) \mod 2. \]

**Proof.** By a direct computation

\[
\beta_n(S) \equiv \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T(B_n)
\]

\[
\equiv \sum_{T \subseteq S} (-1)^{|T|-1} \cdot f_T(B_n)
\]

\[
\equiv \sum_{T \subseteq S, T \in \Delta_n} (-1)^{|T|-1}
\]

\[
\equiv \tilde{\chi}(\Delta_n|S) \mod 2. \]

4 One binary digit

In this section we explore cyclotomic factors in the descent set polynomial $Q_n(t)$ where $n$ is a power of 2, that is, $n$ has one 1 in its binary expansion. First we have a result showing conditions on the values of $a_{m,j}$ when we have a cyclotomic factor in the general $n$th descent set polynomial. Note that we abbreviate $a_{m,j}$ as $a_j$.

**Lemma 4.1.** Let $m$ be an even positive integer. The cyclotomic polynomial $\Phi_m$ is a factor of the descent set polynomial $Q_n(t)$ if the following equations hold:

1. $a_j = a_{-j}$, \hspace{1cm} (4.1)
2. $a_j = a_{m/2-j}$, \hspace{1cm} (4.2)

for all integers $j$.

**Proof.** Consider the primitive $m$th root of unity $\omega = e^{i\pi/m}$. In order for $\Phi_m$ to be a factor of $Q_n(t)$, we must have $Q_n(\omega) = 0$. Since $\omega^m = 1$, we need to show

\[ Q_n(\omega) = \sum_{S \subseteq [n-1]} \omega^{\beta_n(S)} = a_0 + a_1 \cdot \omega + a_2 \cdot \omega^2 + \cdots + a_{m-1} \cdot \omega^{m-1} \]

is zero. By reflection in the real and the imaginary axis in the complex plane we have $\omega^{-j} + \omega^j + \omega^{m/2-j} + \omega^{m/2+j} = 0$, from which the result follows.

Assume that $s$ is an odd positive integer. We consider which values of $s$ such that the 4th cyclotomic polynomial, $\Phi_{4s}$, divides the descent set polynomial $Q_n(t)$ when $n$ is a power of 2.
Table 2: Examples of cyclotomic factors of $Q_n(t)$ of the form $\Phi_{4s}$ where $n = 2^a$.

| $n$  | $s$  | $k$  | Chebikin et al. statement | Our statement |
|------|------|------|--------------------------|--------------|
| 4    | 1    | 1    | Thm. 3.5                 | Thm. 4.2     |
| 8    | 1    | 1    | Thm. 3.5                 | Thm. 4.2     |
| 8    | 7    | 2    | Thm. 4.2                 |              |
| 16   | 1    | 1    | Thm. 3.5                 | Thm. 4.2     |
| 16   | 5, 11, 13, 55, 65, 143, 715 | 7    | Thm. 4.2                 |              |
| 16   | 3, 15 | 5    | Thm. 4.2                 |              |
| 16   | 39   | 2    | Thm. 4.2                 |              |
| 32   | all the divisors of 17678835 | 15   | Rem. 4.3                 |              |

**Theorem 4.2.** Let $n = 2^a$ where $a \geq 2$. Assume that $s$ is an odd integer such that $s$ divides the central binomial coefficient $\binom{n}{n/2}$ and $s$ divides $\binom{n}{k}$ for some $k \neq n/2$. Then the cyclotomic polynomial $\Phi_{4s}(t)$ divides the descent set polynomial $Q_n(t)$.

**Proof.** Observe that there is one carry in the addition $n/2 + n/2 = n$ in base 2. Hence by Kummer’s theorem, see [7, Pages 115–116], 2 is the largest 2-power dividing $\binom{n}{n/2}$. In other words, $\binom{n}{n/2} \equiv 2 \mod 4$. Combining this with the fact that $s$ divides this central binomial coefficient, we have $\binom{n}{n/2} \equiv 2s \mod 4s$. Thus, MacMahon’s multiplication theorem gives that

$$\beta_n(S) + \beta_n(S \triangle \{n/2\}) = \binom{n}{n/2} \cdot \beta_{n/2}(S \cap [1, n/2 - 1]) \cdot \beta_{n/2}(S \cap [n/2 + 1, n - 1] - n/2).$$

Since $\beta_{n/2}$ only takes odd values as shown in Theorem 2.7(a), we obtain that

$$\beta_n(S) + \beta_n(S \triangle \{n/2\}) \equiv 2s \mod 4s.$$

Thus, the statement $\beta_n(S) \equiv j \mod 4s$ is equivalent to $\beta_n(S \triangle \{n/2\}) \equiv 2s - j \mod 4s$. That is, the map $S \mapsto S \triangle \{n/2\}$ yields a bijection that proves $a_j = a_{2s-j}$ for all $j$.

Next since the addition $k + (n - k) = n$ in base 2 has at least two carries, we obtain that $2^2 = 4$ divides the binomial coefficient $\binom{n}{k}$. Hence, $4s$ divides $\binom{n}{k}$ and by Lemma 2.6 the equality $a_j = a_{-j}$ holds for all $j$. We now have that both equations (4.1) and (4.2) from Lemma 4.1 are upheld; thus, the cyclotomic polynomial $\Phi_{4s}$ divides $Q_n(t)$.

**Remark 4.3.** The case $n = 32 = 2^5$ and $k = 15$ is particularly nice. We have that $\binom{32}{16} = 2 \cdot 3^2 \cdot 5 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$ and $\binom{32}{15} = 16/17 \cdot \binom{32}{16}$. Hence, for any divisor $s$ of $3^2 \cdot 5 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$ and there are 96 such divisors, we obtain the cyclotomic factor $\Phi_{4s}$ of $Q_{32}(t)$. Furthermore, we do not obtain any more cyclotomic factors by changing $k$, that is, all the the odd divisors of $\binom{32}{k}$ for $k \leq 14$ are divisors of $\binom{32}{15}$.

See Table 2 for examples of cyclotomic factors of $Q_{2^a}(t)$ that are explained by Theorem 4.2 along the $k$ value in which $s$ divides $\binom{2^a}{k}$.
5 Two binary digits

Now we state the result that lets us deduce cases when the cyclotomic polynomial $\Phi_{2s}$, where $s$ is an odd positive integer, is a factor of the descent set polynomial $Q_n(t)$ when $n$ has two 1’s in its binary expansion.

**Theorem 5.1.** Let $n = 2^b + 2^a$, where $b > a$ and $s$ is an odd positive integer. Assume that $s$ divides $\binom{n}{2a}$. Furthermore, assume there is an integer $k$ which is non-essential in base 2 (that is, $k \neq 2^a, 2^b$) and such that $s$ divides $\binom{n}{k}$. Then the cyclotomic polynomial $\Phi_{2s}$ is a factor of $Q_n(t)$.

**Proof.** Since there are no carries in the addition $2^b + 2^a = n$ in base 2, by Kummer’s theorem we know that $\binom{n}{2a}$ is odd. Combining this fact with the congruence modulo $s$, we obtain $\binom{n}{2a} \equiv s \mod 2s$. Therefore, by MacMahon’s multiplication theorem, we have that

$$\beta_n(S) + \beta_n(S \triangle \{2^a\}) = \binom{n}{2a} \cdot \beta_{2s}(S \cap \{2^a - 1\}) \cdot \beta_{2s}(S \cap \{2^a + 1, n - 1\} - 2^a) \equiv s \mod 2s,$$

(5.1)

since both $\beta_{2s}$ and $\beta_{2s}$ are odd. Hence, we use the bijective map $S \mapsto S \triangle \{2^a\}$ to conclude that $a_j = a_{s-j}$ for all $j$.

Since the addition $k + (n-k)$ has at least one carry in base 2 the binomial coefficient $\binom{n}{k}$ is even. Hence $\binom{n}{k}$ is divisible by 2s. By Lemma 2.6 the inequality $a_j = a_{-j}$ holds for all $j$. Combining these two equalities using Lemma 4.1 the result follows. \hfill \Box

We begin by two remarkable examples.

**Remark 5.2.** Consider the case $n = 18 = 2^4 + 2^1$ and $k = 4$. Note that $\binom{18}{2} = 3^2 \cdot 17 = 153$. Furthermore note that $\binom{18}{4} = 2^2 \cdot 5 \cdot \binom{18}{2}$. Hence for any divisor $s$ of 153 we obtain that the cyclotomic polynomial $\Phi_{2s}$ divides the descent set polynomial $Q_{18}(t)$. This argument explains all the cyclotomic factors found in the descent set polynomial $Q_{18}(t)$; see Table 6.

**Remark 5.3.** Consider the case $n = 20 = 2^4 + 2^2$ and $k = 6$. Now we have $\binom{20}{4} = 3 \cdot 5 \cdot 17 \cdot 19 = 4845$ and $\binom{20}{6} = 2^3 \cdot \binom{20}{4}$. Hence for any divisor $s$ of 4845 the cyclotomic polynomial $\Phi_{2s}$ is a factor in the descent set polynomial $Q_{20}(t)$, explaining all the 16 known cyclotomic factors; see the longest row in Table 6.

We now continue to study the case when the integer $s$ is an odd prime $p$. Recall from Lemma 2.3 that $k$ being a non-essential element in base $p$ implies that $p$ divides $\binom{n}{k}$. Hence, to satisfy the assumptions in Theorem 5.1 for this case, we need to show that $2^a$ and $k$ are non-essential in base $p$ and that $k$ is non-essential in base 2.

Note however that for two relative prime integers $p$ and $q$, a carry in the addition $k + (n-k) = n$ in base $p \cdot q$ does not imply a carry for this addition in both base $p$ and $q$. An example the addition $12 + 3 = 15$. In base 15 there is a carry, where as in base 3 there is no carry.

The following lemma is useful in determining when $2^a$ is non-essential for $n$ in base $p$, where $p$ is prime, in order to apply Theorem 5.1. Although rarely cited during the subsequent arguments since we often need the actual value of $i + j \mod p$ instead of only the fact that it is at least $p$, it provides reasoning for finding particular values of $n$.

**Lemma 5.4.** For $n = 2^a + 2^b$, if $2^a \equiv i \mod p$ and $2^b \equiv j \mod p$ where $1 \leq i, j \leq p-1$ and $i+j \geq p$, then $2^a$ is non-essential for $n$ in base $p$. 

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**Proof.** Since \( i, j \leq p - 1 \) and \( i + j \geq p \), we have \( i > i + j \mod p \). Therefore, the last digit of the base \( p \) expansion of \( 2^a \) is larger than the last digit of the base \( p \) expansion of \( n \), causing \( 2^a \) to be non-essential in base \( p \). \( \square \)

The following theorems provide conditions for the prime \( p \), the multiplicative order \( g \) of \( 2 \) in \( \mathbb{Z}_p^* \), and the exponents \( a \) and \( b \) that allow Theorem \text{[5.1]} to be applied to show that \( \Phi_{2p} \) is a factor of \( Q_{n(t)} \).

**Theorem 5.5.** Assume that \( 2 \) has order \( g \) in the multiplicative group \( \mathbb{Z}_p^* \) where \( g \) is even. Let \( n = 2^b + 2^a \) where we assume \( b > a \) and \( n \geq 9 \). If we have \( \{a, b\} \equiv \{0, g/2\} \mod g \), then \( 2^a \) is non-essential in base \( p \). Furthermore, the element \( 7 \) is non-essential for both base 2 and base \( p \). Hence \( \Phi_{2p} \) is a factor of \( Q_{n(t)} \).

**Proof.** Since \( 2^{g/2} \not\equiv 1 \mod p \) and \((2^{g/2} - 1)\cdot(2^{g/2} + 1) = 2^g - 1 \equiv 0 \mod p \) we know that \( 2^{g/2} \equiv -1 \mod p \) using that \( p \) is a prime. Hence the last digits of \( 2^a \) and \( 2^b \) in their base \( p \) expansions are 1 and \( p - 1 \), in some order. Thus, we have \( n = 2^b + 2^a \equiv 1 + (p - 1) \equiv 0 \mod p \), that is, the last digit in the base \( p \) expansion of \( n \) is 0. Hence \( 2^a \) is non-essential in base \( p \).

Notice that \( 7 \) has three non-zero digits in its binary expansion compared to only 2 such digits for \( n \), making \( 7 \) non-essential for \( n \) in base 2. Since the order of \( 2 \) in \( \mathbb{Z}_p^* \) is 3, which is odd, we have \( p \neq 7 \). Finally, the last digit of the base \( p \) expansion of \( 7 \) is non-zero for all odd primes \( p \neq 7 \). Hence \( 7 \) is also non-essential for \( n \) in base \( p \), completing the result. \( \square \)

**Remark 5.6.** The assumption in Theorem \text{[5.5]} of \( n \geq 9 \) was needed in order for \( 7 \) to always be a non-essential element, but note that the theorem can still be applied when \( n = 6 \) if \( p = 3 \). The element \( 5 \) is instead chosen as the non-essential element in base 2 and in base \( p \).

**Theorem 5.7.** Assume that \( 2 \) has order \( g \) in the multiplicative group \( \mathbb{Z}_p^* \) where \( g \) is even. Let \( n = 2^b + 2^a \) where we assume \( b > a \) and \( n > 2p - 1 \). If we have \( a \equiv b \equiv g/2 \mod g \), then \( 2^a \) is non-essential in base \( p \). Furthermore, the element \( 2p - 1 \) is non-essential for both base 2 and base \( p \). Hence \( \Phi_{2p} \) is a factor of \( Q_{n(t)} \).

**Proof.** Similar to part of the previous proof, we have in this case that \( 2^a \equiv 2^b \equiv 2^{g/2} \equiv p - 1 \mod p \). Therefore, \( n = 2^a + 2^b \equiv (p - 1) + (p - 1) \equiv p - 2 \mod p \). Thus, the last digit of the base \( p \) expansion of \( n \) is \( p - 2 \) while the last digit of the expansion of \( 2^a \) is \( p - 1 \), making \( 2^a \) be non-essential in base \( p \).

Since \( 2p - 1 \) is odd, the last digit in its base 2 expansion is 1, but the last digit of the base 2 expansion of \( n \) is 0 because \( a, b \neq 0 \). Hence \( 2p - 1 \) is non-essential in base 2. Additionally, \( 2p - 1 \equiv p - 1 \equiv p - 2 \mod p \), thus it is non-essential in base \( p \) as well. \( \square \)

**Remark 5.8.** The equivalence conditions on the exponents within Theorems \text{[5.5] and 5.7]} are not the only such conditions that makes the theorem hold true when \( g \) is even. These are many such conditions, especially if \( 2 \) is a generator of \( \mathbb{Z}_p^* \) since the powers of \( 2 \) contain every possible non-zero value as the last digit, and all that is needed is for the argument in the proof of \( 2^a \) being non-essential in base \( p \) is for the sum of these digits to be at least \( p \), as shown in Lemma \text{[5.4]}. In this case of \( 2 \) being a generator of \( \mathbb{Z}_p^* \) for \( p = 2r + 1 \), there are exactly \( r \cdot (r + 1) \) of pairs of possible exponents modulo \( g \) that will work. One still needs to find element \( k \) that is non-essential in base 2 and in base \( p \). Finding this \( k \) value is easy if given a particular pair of \( n \) and \( p \) values, but this step causes a further generalization of the proof to be difficult.

Table 4 includes all of the equivalence conditions modulo the order \( g \) for four odd primes that lead to \( 2^a \) being non-essential for \( n \) in base \( p \). For examples of finding the non-essential \( k \) value, see Table 4.
| $p$ | $g$ | $\{a, b\} \mod g$ |
|-----|-----|-----------------|
| 3   | 2   | $\{0, 1\}, \{1, 1\}$ |
| 5   | 4   | $\{0, 5\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 5\}, \{2, 6\}, \{2, 7\}, \{3, 4\}, \{3, 3\}, \{3, 5\}, \{3, 6\}, \{3, 7\}, \{3, 8\}, \{3, 9\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{4, 9\}, \{5, 5\}, \{5, 6\}, \{5, 7\}, \{5, 8\}, \{5, 9\}, \{6, 6\}, \{6, 7\}, \{6, 8\}, \{6, 9\}, \{7, 7\}, \{7, 9\}, \{9, 9\}$ |
| 11  | 10  | $\{0, 4\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{3, 7\}, \{4, 4\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{5, 5\}, \{5, 6\}, \{5, 7\}, \{6, 6\}, \{6, 7\}$ |
| 17  | 8   | $\{0, 2\}, \{1, 2\}, \{1, 3\}, \{2, 2\}, \{2, 3\}, \{3, 3\}$ |

Table 3: Examples of equivalency conditions for small prime numbers.

**Remark 5.9.** If 2 has multiplicative order $g$ in $\mathbb{Z}_p^*$, then its order $G$ in $\mathbb{Z}_{p^l}^*$ is a divisor of $p^{l-1} \cdot g$. The order $g$ gives the length of the repeating sequence of the last digit of the base $p$ expansions of the powers $2^a$, and likewise, the order $G$ gives the length of the repeating sequence of the last $l$ digits of those powers of 2. Similar reasoning to Lemma 5.4 applies when adding together any pair of digits together, not just the last digit. Thus, there are equivalencies modulo $G$ that cause $2^a$ to be non-essential in base $p$ because of a carry in one of the last $l$ digits. As an example, when $p = 3$ the order of 2 in $\mathbb{Z}_3^*$ is 6, hence the last two digits of $2^a$ cycle through the six values 01, 02, 11, 22, 21 and 12 as $a$ increases. Therefore, when $\{a, b\} \equiv \{2, 4\} \mod 6$, the last two digits of $n$ in base 3 are $11 + 21 = 02$, so the second digit from the right is larger for $2^a$ than for $n$, making it non-essential in base $p$.

**Theorem 5.10.** Let $n = 2^b + 2^a$ where we assume $b > a$ and $n \geq 5$, and also assume that $p > 3$. If we have $a, b \equiv g - 1 \mod g$ where $g$ is the multiplicative order of 2, then $2^a$ is non-essential in base $p$. Furthermore, the element 3 is non-essential in both base 2 and base $p$. Hence $\Phi_{2^p}$ is a factor of $Q_n(t)$.

**Proof.** If the multiplicative order of 2 is $g$, then $g$ is the smallest integer so that $2^g \equiv 1 \mod p$. Thus, $2^a \equiv 2^b \equiv 2^{g-1} > 1 \mod p$, and $n = 2^a + 2^b \equiv 2^{g-1} + 2^{g-1} \equiv 2^g \equiv 1 \mod p$. Hence $2^a$ is non-essential in base $p$ because the last digit in its base $p$ expansion is larger than that of $n$.

The element 3 is non-essential for $n$ in base 2 since our assumption of $n \geq 5$ implies that $b \geq 2$. Because of our assumption that $p > 3$, the element 3 is also non-essential in base $p$ since the last digit of the base $p$ expansion for $n$ is $1 < 3$, concluding the result.

Note that we omitted $p = 3$ from the previous result because this was already proven for $p = 3$ in Theorem 5.7 due to the order of 2 being $g = 2$, making $g/2 = g - 1$.

**Remark 5.11.** Assuming $p > 3$, if $p$ is a Mersenne prime, that is, $p$ has the form $2^q - 1$ implying that $q$ is also a prime number, the equivalence condition on the exponents in Theorem 5.10 is the only such condition modulo $g$ for which $\Phi_{2^p}$ is a factor of $Q_n(t)$. The first examples of Mersenne primes after 3 are $p = 7, 31$ and 127.

Table 4 summarizes particular values of $n$ and $s$ for which $\Phi_{2^s}$ is a factor of $Q_n(t)$ with $n$ having two binary digits. An element $k$ that is non-essential in base 2 and base $p$ and the statement explaining why it is a factor are also included. For the cases in which $s$ is a prime $p$, the set of exponents modulo the multiplicative order $g$ of 2 is also listed. The top portion includes factors that were known by Chebikin et al., although many were left unexplained in their work. Cases that were proven by Chebikin et al. are included within the statement column. The bottom portion displays just a few examples of factors that are explained by our results that were previously unknown.
| $n$  | $s$  | $\{a, b\} \mod g$ | $k$ | Chebikin et al. statement | Our statement |
|------|------|--------------------|-----|---------------------------|--------------|
| 6    | 3    | $\{0, 1\}$        | 5   | Thm. 5.6                  | Rem. 5.6     |
| 6    | 5    | $\{1, 2\}$        | 3   | Rem. 5.8                  |              |
| 9    | 3    | $\{0, 1\}$        | 7   | Thm. 5.5                  | Thm. 5.5     |
| 9    | 9    |                    | 2   | Thm. 5.1                  |              |
| 10   | 3    | $\{1, 1\}$        | 5   | Thm. 5.7                  |              |
| 10   | 5    | $\{1, 3\}$        | 1   | Thm. 5.6                  | Rem. 5.8     |
| 10   | 9    |                    | 5   | Thm. 5.1                  |              |
| 10   | 15   |                    | 3   | Thm. 5.1                  |              |
| 12   | 3    | $\{0, 1\}$        | 7   | Thm. 5.5                  |              |
| 12   | 5    | $\{2, 3\}$        | 3   | Rem. 5.8                  |              |
| 12   | 11   | $\{2, 3\}$        | 2   | Rem. 5.8                  |              |
| 12   | 55   |                    | 3   | Thm. 5.1                  |              |
| 12   | 9, 33, 99 |                | 5   | Thm. 5.1                  |              |
| 17   | 17   | $\{0, 4\}$        | 7   | Thm. 5.5                  | Thm. 5.5     |
| 18   | 17   | $\{1, 4\}$        | 3   | Rem. 5.8                  |              |
| 18   | 9, 51, 153 |                | 4   | Rem. 5.2                  |              |
| 20   | 3    | $\{2, 4\} \mod 6$ | 3   | Rem. 5.9                  |              |
| 20   | 5    | $\{0, 2\}$        | 7   | Thm. 5.5                  |              |
| 20   | 17   | $\{2, 4\}$        | 5   | Rem. 5.8                  |              |
| 20   | 15, 19, 51, 57, 85, 95, 255, 285, 323, 969, 1615, 4845 | | 6 | Rem. 5.3 |
| 72   | 3    | $\{0, 1\}$        | 7   | Thm. 5.5                  |              |
| 528  | 31   | $\{4, 4\}$        | 3   | Thm. 5.10                 |              |
| 1088 | 5    | $\{2, 2\}$        | 9   | Thm. 5.7                  |              |

Table 4: Examples of cyclotomic factors of $Q_n(t)$ of the form $\Phi_{2s}$, where the binary expansion of $n$ has two 1’s.
6 Three binary digits

We now continue to explore cyclotomic factors Φ₂ₙ, where s is an odd positive integer, in the descent set polynomial Qₙ(t) where n has three 1s in its binary expansion.

Theorem 6.1. Let \( n = 2^c + 2^b + 2^a \) where \( c > b > a \) and \( s \) is an odd positive integer. Assume that \( s \) divides the three binomial coefficients \( \binom{n}{2^c} \), \( \binom{n}{2^b} \) and \( \binom{n}{2^a} \). Assume furthermore that there is an element \( k \) which is non-essential in base 2, that is, \( k \not\in \{2^a, 2^b, 2^a + 2^b, 2^c + 2^a, 2^c + 2^b\} \), such that \( s \) divides \( \binom{n}{k} \). Then the cyclotomic polynomial \( \Phi_{2s} \) is a factor of the descent set polynomial \( Qₙ(t) \).

Proof. Since there is an element \( k \) which is non-essential in base 2, we know that 2 divides \( \binom{n}{k} \). Thus \( 2s \) divides \( \binom{n}{k} \), and Lemma 2.5 gives that \( a_j = a_{s-j} \) for all \( j \). Next our major goal is to show that \( a_j = a_{s-j} \). We do that by constructing an involution \( \phi \) on all subsets of \( [n-1] \) such that \( \beta_n(S) + \beta_n(\phi(S)) \equiv s \mod 2s \). Hence for every contribution to \( a_j \) there is a corresponding contribution to \( a_{s-j} \). The form of the involution \( \phi \) will be \( \phi(S) = S \setminus X \) where the subset \( X \) depends on how \( S \) intersects the four element set \( \{2^a, 2^b, 2^c + 2^a, 2^c + 2^b\} \).

Since the elements \( 2^c \) and \( 2^b + 2^a \) are both essential in base 2, we apply MacMahon’s theorem to get

\[
\beta_n(S) + \beta_n(S \setminus \{2^b + 2^a\}) = \binom{n}{2^b + 2^a} \cdot \beta_{2^c}(S \setminus \{1, 2^b + 2^a - 1\}) \]

\[
\equiv \begin{cases} 
0 & \text{if } |S \setminus \{2^b, 2^a\}| = 1, \\
1 & \text{if } |S \setminus \{2^b, 2^a\}| = 0 \text{ or } 2 \mod 2,
\end{cases}
\]

\[
\beta_n(S) + \beta_n(S \setminus \{2^c\}) = \binom{n}{2^c} \cdot \beta_{2^c}(S \setminus \{2^c - 1\}) \cdot \beta_{2^c+2^a}(S \setminus \{2^c + 1, n - 1\} - 2^c) \]

\[
\equiv \begin{cases} 
0 & \text{if } |S \setminus \{2^c, 2^a, 2^c + 2^b\}| = 1, \\
1 & \text{if } |S \setminus \{2^c, 2^a, 2^c + 2^b\}| = 0 \text{ or } 2 \mod 2,
\end{cases}
\]

since the two binomial coefficients \( \binom{2^b + 2^a}{2^b + 2^a} = \binom{n}{2^b + 2^a} \) are both odd and the descent set statistics involving \( \beta_{2^c} \) are also odd by Theorem 2.7 (a). Therefore, the sums of these descent set statistics are determined by the values for \( \beta_{2^b + 2^a} \), which we examine by considering the complex \( \Delta_{2^b + 2^a} \) and using Theorem 3.2. This complex consists of only of two isolated vertices at \( 2^b \) and \( 2^a \). Thus, the induced subcomplex \( \Delta_n|S \setminus \{1, 2^b + 2^a - 1\} \) is a single vertex if \( |S \setminus \{2^b, 2^a\}| = 1 \) with a reduced Euler characteristic of 0. Otherwise, it is two isolated vertices or the empty complex, both of which have a reduced Euler characteristic of 1 mod 2. The reasoning behind the second sum is identical once the set \( S \) is shifted down by \( 2^c \).

Since \( s \) divides \( \binom{n}{2^c} = \binom{2^b + 2^a}{2^b + 2^a} \), we have by Lemma 2.5 that \( \beta_n(S) + \beta_n(S \setminus \{2^b + 2^a\}) \equiv \beta_n(S) + \beta_n(S \setminus \{2^c\}) \equiv 0 \mod s \). Combining this with the modulo 2 sums, we have the following results modulo 2s

\[
\beta_n(S) + \beta_n(S \setminus \{2^b + 2^a\}) \equiv \begin{cases} 
0 & \text{if } |S \setminus \{2^a, 2^b\}| = 1, \\
s & \text{if } |S \setminus \{2^a, 2^b\}| = 0 \text{ or } 2 \mod 2s,
\end{cases}
\]

\[
\beta_n(S) + \beta_n(S \setminus \{2^c\}) \equiv \begin{cases} 
0 & \text{if } |S \setminus \{2^c + 2^a, 2^c + 2^b\}| = 1, \\
s & \text{if } |S \setminus \{2^c + 2^a, 2^c + 2^b\}| = 0 \text{ or } 2 \mod 2s.
\end{cases}
\]
We now begin to construct the involution $\phi$. Assume that $|S \cap \{2^a, 2^b\}| = 0$ or 2. Then by equation (6.1), $\beta_n(S) + \beta_n(S \triangle \{2^b + 2^a\}) \equiv s \mod 2s$. Hence in this case let the involution be given by $\phi(S) = S \triangle \{2^b + 2^a\}$.

The symmetric case is as follows. Assume that we have $|S \cap \{2^a, 2^b\}| = 1$ and $|S \cap \{2^a + 2^a, 2^a + 2^b\}| = 0$ or 2. By equation (6.2), $\beta_n(S) + \beta_n(S \triangle \{2^c\}) \equiv s \mod 2s$ and let the involution be given by $\phi(S) = S \triangle \{2^c\}$.

By equations (6.1) and (6.2) we have that

$$\beta_n(S) \equiv \beta_n(S \triangle \{2^b + 2^a, 2^c\}) \equiv -\beta_n(S \triangle \{2^b + 2^a\}) \equiv -\beta_n(S \triangle \{2^c\}) \mod 2s.$$

Especially, these four descent set statistics all have the same parity. In order to determine this parity, we need to consider the complex $\Delta_n$, displayed in Figure 1 and then apply Theorem 3.2.

We now have four subcases to consider.

- First consider sets $S$ such that $S \cap \{2^a, 2^b, 2^c + 2^a, 2^c + 2^b\} \neq \emptyset$. Note that the four induced subcomplexes $\Delta_{\{2^a, 2^b, 2^c + 2^a\}}, \Delta_{\{2^a, 2^c + 2^a\}}, \Delta_{\{2^b, 2^c + 2^a\}}, \Delta_{\{2^b, 2^c + 2^b\}}$ are all contractible and hence have reduced Euler characteristic 0. Hence in this case $\beta_n(S), \beta_n(S \triangle \{2^b + 2^a\})$, $\beta_n(S \triangle \{2^b\})$ and $\beta_n(S \triangle \{2^b + 2^a\})$ are all even.

- Second, when $S \cap \{2^a, 2^b, 2^c + 2^a, 2^c + 2^b\} = \emptyset$, by considering the reverse sets of the previous case, the four sets $S, S \triangle \{2^a + 2^b\}, S \triangle \{2^b\}$ and $S \triangle \{2^b + 2^a\}$ have even descent set statistics because their corresponding induced induced subcomplexes are contractible.

- Third, consider sets $S$ such that $S \cap \{2^a, 2^b, 2^c + 2^a + 2^b, 2^c + 2^b\} \neq \emptyset$. Now the four induced subcomplexes $\Delta_{\{2^a, 2^b, 2^c + 2^a + 2^b\}}, \Delta_{\{2^a, 2^c + 2^a + 2^b\}}, \Delta_{\{2^b, 2^c + 2^a + 2^b\}}, \Delta_{\{2^b, 2^c + 2^a\}}$ are all homotopy equivalent to two points and hence have reduced Euler characteristic 1. Hence in this case the descent set statistics of the four sets $S, S \triangle \{2^b + 2^a\}, S \triangle \{2^a\}$ and $S \triangle \{2^b + 2^a\}$ are all odd.

- The fourth and last case is when $S \cap \{2^a, 2^b, 2^c + 2^a + 2^b, 2^b + 2^c\} = \emptyset$. Again, the four induced subcomplexes $\Delta_{\{2^a, 2^b, 2^c + 2^a + 2^b\}}, \Delta_{\{2^a, 2^c + 2^a + 2^b\}}, \Delta_{\{2^b, 2^c + 2^a + 2^b\}}, \Delta_{\{2^b, 2^c\}}$ are all homotopy equivalent to two points and hence have reduced Euler characteristic 1. Therefore, the descent set statistics of the four sets $S, S \cup \{2^b + 2^a\}, S \cup \{2^2\}$ and $S \cup \{2^b + 2^a, 2^c\}$ are all odd.

From these four subcases above we know that $\beta_n(S) \equiv 1 + \beta_n(S \triangle \{2^a, 2^b\}) \mod 2$. Next, since $2^a$ and $2^b$ both satisfy $\binom{n}{2^a} \equiv \binom{n}{2^b} \equiv 0 \mod s$, we have that $\beta_n(S) \equiv -\beta_n(S \triangle \{2^a\}) \equiv \beta_n(S \triangle \{2^a, 2^b\}) \mod s$. Combining these two statements and using that $2s$ divides $\binom{n}{k}$ we conclude that

$$\beta_n(S) \equiv s + \beta_n(S \triangle \{2^a, 2^b\}) \equiv s - \beta_n(S \triangle \{2^a, 2^b\}, k) \mod 2s.$$
Thus, the third and final case of the definition of \( \phi \) is \( \phi(S) = S \triangle \{2^a, 2^b, k\} \). This proves that the equality \( a_j = a_{s-j} \). With the proper equalities holding, Lemma \( 4.1 \) proves the theorem. \( \blacksquare \)

One might ask if it is possible for \( \Phi_{2s} \) to be a factor of \( Q_n(t) \) if the binary expansion of \( n \) has more than three binary digits. Although the equations within Lemma \( 4.1 \) are only sufficient conditions and not necessary conditions for this cyclotomic polynomial to be a factor, it is easy to see why this lemma cannot be used unless there are equal numbers of even and odd descent set statistics, as this is implied by the combination of equations \( (1.1) \) and \( (1.2) \). Chebikin et al. showed that there are not equal numbers when the binary expansion of \( n \) has 4 or 5 digits, although it is not known if there is a \( k > 3 \) for which this condition is true when \( n \) has \( k \) binary digits.

Similar to Remark \( 5.2 \) and \( 5.3 \) we have the next remark about \( 21 \) and \( 22 \).

**Remark 6.2.** Consider \( n = 21 = 2^4 + 2^2 + 1 \) and \( k = 2 \). Observe that \( \gcd \left( \left( \frac{21}{16} \right), \left( \frac{21}{4} \right), \left( \frac{21}{1} \right) \right) = 21 \). Furthermore, observe that \( \left( \frac{21}{2} \right) \) is a multiple of 21. Hence we obtain for each divisor \( s \) of 21 that the cyclotomic polynomial \( \Phi_{2s} \) divides \( Q_{21}(t) \). Similarly, for \( n = 22 = 2^4 + 2^2 + 2^1 \) and \( k = 3 \), we have that \( \gcd \left( \left( \frac{22}{16} \right), \left( \frac{22}{4} \right), \left( \frac{22}{2} \right) \right) = 77 \) divides \( \left( \frac{22}{3} \right) \). Hence for each divisor \( s \) of 77 we conclude that \( \Phi_{2s} \) divides \( Q_{22}(t) \).

We continue to consider the case when the integer \( s \) is an odd prime \( p \). The following theorems give conditions for \( p \) and the exponents \( a, b \) and \( c \) that provide the assumptions made for applying Theorem \( 6.1 \).

**Theorem 6.3.** Let \( n = 2^c + 2^b + 2^a \) where we assume \( c > b > a, n \geq 11 \), and that the order \( g \) of 2 in the multiplicative group \( \mathbb{Z}_p^* \) is even. If we have \( \{a, b, c\} \equiv \{1, g/2, g/2\} \mod g \), then \( 2^c, 2^b \) and \( 2^a \) are non-essential in base \( p \). Furthermore, the element 7 is non-essential for both base 2 and base \( p \). Hence \( \Phi_{2p} \) is a factor of \( Q_n(t) \).

**Proof.** Using the same congruences as in the proof of Theorem \( 5.5 \), we have

\[
n = 2^c + 2^b + 2^a \equiv 2^g/2 + 2^g/2 + 2^1 \equiv (p-1) + (p-1) + 2 \equiv 0 \mod p,
\]

hence the last digit in the base \( p \) expansion of \( n \) is 0. This makes \( 2^c, 2^b \) and \( 2^a \) be non-essential in base \( p \) since the last digit for these powers of two are each greater than 0.

The assumption that \( n \geq 11 \) implies that \( c \geq 3 \), hence the number 7 is non-essential for \( n \) in base 2. Additionally, the last digit of the base \( p \) expansion of 7 is non-zero except when \( p = 7 \), but this case is not included for this theorem since the order of 2 in \( \mathbb{Z}_7^* \) is odd. Therefore, 7 is non-essential in base \( p \) as well, which concludes the proof of the theorem. \( \blacksquare \)

**Theorem 6.4.** Let \( n = 2^c + 2^b + 2^a \) where \( c > b > a \), and assume that \( p \) is an odd prime greater than or equal to 5. If \( \{a, b, c\} \equiv \{g-2, g-2, g-1\} \mod g \) where \( g \) is the multiplicative order of 2 in \( \mathbb{Z}_p^* \), then \( 2^c, 2^b \) and \( 2^a \) are non-essential in base \( p \). Furthermore, the element 3 is non-essential in both base 2 and base \( p \). Hence \( \Phi_{2p} \) is a factor of \( Q_n(t) \).

**Proof.** We have

\[
n = 2^c + 2^b + 2^a \equiv 2^{g-2} + 2^{g-2} + 2^{g-1} \equiv 2^g \equiv 1 \mod p,
\]

hence the last digit of the base \( p \) expansion of \( n \) is 1. Since we assume \( p \geq 5 \), we must have \( g \geq 3 \), hence \( 2^{g-1} > 2^{g-2} > 1 \mod p \). Thus, \( 2^c, 2^b \) and \( 2^a \) are non-essential for \( n \) in base \( p \) since the last digit of their base \( p \) expansions larger than 1.
Also since we assume \( p \geq 5 \), the last digit of the base \( p \) expansion of 3 is greater than 1 as well, making it non-essential in base \( p \). The element 3 is also non-essential in base 2 since the fact that \( g \geq 3 \) implies that \( 2^a \neq 1 \).

**Remark 6.5.** With \( n \) having two binary digits, there were many equivalencies modulo \( m \) on the exponents \( a \) and \( b \) beyond what could be shown in results that held for all \( p \) or for all \( p \) with \( m \) being even. Likewise, many such equivalencies exist in the three binary digit case that cause each of \( 2^e \), \( 2^b \) and \( 2^a \) to be non-essential in base \( p \). Examples of these equivalencies include the following:

- \( \{a, b, c\} \equiv \{0, 0, 0\} \mod 2 \) when \( p = 3 \) since the final digit of \( n \) in base 3 is \( 1 + 1 + 1 \equiv 0 \mod 3 \)
- \( \{a, b, c\} \equiv \{1, 2, 4\} \mod 10 \) when \( p = 11 \) because the last digit of \( n \) is \( 2 + 4 + 16 \equiv 0 \mod 11 \)
- \( \{a, b, c\} \equiv \{1, 2, 3\} \mod 12 \) when \( p = 13 \) since the last digit of \( n \) in base 13 is \( 2 + 4 + 8 \equiv 1 \mod 13 \).

Of course, to show that \( Q_n(t) \) has \( \Phi_{2^p} \) as a factor, one still needs to find an element \( k \) that is non-essential in base 2 and \( p \), which is shown for these examples for a particular \( n \) value in Table 5.

**Remark 6.6.** As with Remark 5.9, we can also find equivalencies modulo \( G \) for the exponents \( a \), \( b \) and \( c \) when \( n \) has three binary digits. As an example, when \( p = 3 \) there are equivalencies such as \( \{a, b, c\} \equiv \{3, 4, 5\} \mod 6 \) that cause \( 2^c \), \( 2^b \) and \( 2^a \) to be non-essential in base \( p \). This one exists because the last two digits of \( n \) are \( 22 + 21 + 12 \equiv 02 \), whereas the second to last digit of \( 2^c \), \( 2^b \) and \( 2^a \) is each greater than 0.

**Theorem 6.7.** Let \( n = 2^c + 2^b + 2^a \) where \( c > b > a \) and \( n > 7 \), and assume that \( p = 2^e + 2^d + 1 \) where \( e > d \). If \( \{a, b, c\} \equiv \{0, d, e\} \mod g \) where \( g \) is the multiplicative order of \( 2 \) in \( \mathbb{Z}_p^* \), then \( 2^c \), \( 2^b \) and \( 2^a \) are non-essential in base \( p \). Furthermore, at least one of the elements \( 7 \) or \( 13 \) is non-essential in both base \( 2 \) and base \( p \). Hence \( \Phi_{2^p} \) is a factor of \( Q_n(t) \).

**Proof.** We have

\[
n = 2^c + 2^b + 2^a \equiv 2^e + 2^d + 1 \equiv p \equiv 0 \mod p,
\]

making the last digit in the base \( p \) expansion of \( n \) be 0. This causes \( 2^c \), \( 2^b \) and \( 2^a \) to be non-essential in base \( p \) since their last digits are \( 1, 2^d \) or \( 2^e \), all of which are greater than 0 mod \( p \).

First consider when \( p \neq 7 \). In this case, the element 7 is non-essential in base \( p \) because the last digit in its base \( p \) expansion is greater than 0, which is the last digit for \( n \). Since we assume \( n > 7 \) with three binary digits, the element 7 is also non-essential in base 2 since 7 also three digits in its binary expansion, completing the result in this case.

If we instead assume \( p = 7 \), then the element 13 is non-essential in base 7 since its base 7 expansion has a 6 as its final digit. The assumption of \( n > 7 \), the fact that \( d = 1 \) and \( e = 2 \), and that \( \{a, b, c\} \equiv \{0, d, e\} \mod 3 \) where the 3 is the order of 2 in \( \mathbb{Z}_7^* \), result in the smallest such value for \( n \) being 14. Since \( n \) and 13 each have three binary digits with \( n > 13 \), we have that 13 is also non-essential in base 2, concluding the proof of the theorem.

The following proposition explains the occurrence of another cyclotomic factor of the form \( \Phi_{2^p} \) which is an outlier compared to other such factors. When \( n = 11 \) and \( p = 3 \), observe from the base 2 and base 3 expansions of 11 = \( 2^3 + 2 + 1 = 3^2 + 2 - 1 \) that both 2 and 1 are essential in base \( p \). Therefore, Theorem 5.1 is not applicable. However, \( \Phi_6 \) is still a factor of the descent set polynomial for \( n = 11 \), as shown in Proposition 6.9 but we first need the following lemma to obtain certain descent set statistics modulo 3.

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Lemma 6.8. Let $R$ be a subset of the interval $[3,8]$. Then we have the following four evaluations of descent set statistics:

\[
\beta_{11}(R \cup \{1,9\}) \equiv \beta_{11}(R \cup \{2,10\}) \equiv (-1)^{|R|} \mod 3,
\]

\[
\beta_{11}(R \cup \{1,10\}) \equiv \beta_{11}(R \cup \{2,9\}) \equiv (-1)^{|R|} \mod 3.
\]

Especially, all these values are non-zero modulo 3.

Proof. We consider the quasi-symmetric function of $B_{11}$ modulo 3. Using Lemma 2.1 we have

\[
F(B_{11}) \equiv M(9) \cdot M^2(1) \\
\equiv M(9) \cdot (M(2) + 2M(1,1)) \\
\equiv M(11) + M(9,2) + M(2,9) + 2M(9,1,1) + 2M(10,1) \\
\quad + 2M(1,9,1) + 2M(1,10) + 2M(1,1,9) \mod 3,
\]

where the second and third step is expanding a product of monomial quasi-symmetric functions in terms of monomial quasi-symmetric functions; see [4, Lemma 3.3]. Reading of the coefficients of the quasi-symmetric functions, we have the following values for the flag $f$-vector:

\[
|S| \equiv \begin{cases} 1 & \text{if } S = \emptyset, \{9\}, \text{ or } \{2\}, \\ 2 & \text{if } S = \{9,10\}, \{10\}, \{1,10\}, \{1\}, \text{ or } \{1,2\}, \mod 3. \\ 0 & \text{otherwise.} \end{cases}
\]

Observe that only eight entries are non-zero modulo 3. Using inclusion-exclusion, the descent set statistic is given by

\[
\beta_{11}(R \cup \{1,9\}) \equiv \sum_{T \subseteq R \cup \{1,9\}} (-1)^{|R \cup \{1,9\} - T|} \cdot f_T \\
\equiv (-1)^{|R \cup \{1,9\}|} \cdot f_\emptyset + (-1)^{|R \cup \{9\}|} \cdot f_{\{1\}} + (-1)^{|R \cup \{1\}|} \cdot f_{\{9\}} \\
\equiv (-1)^{|R|} \mod 3.
\]

The three descent set statistics $\beta_{11}(R \cup \{1,10\})$, $\beta_{11}(R \cup \{2,9\})$ and $\beta_{11}(R \cup \{2,10\})$ can be computed similarly. □

Proposition 6.9. The cyclotomic polynomial $\Phi_6$ is a factor of the descent set polynomial $Q_{11}(t)$.

Proof. Observe from the base 2 and base 3 expansions of 11 that 3 is non-essential for 11 in base 2 and in base 3. Therefore, Lemma 2.6 implies that $a_j = a_{-j}$ for all $j$, or $a_1 = a_5$ and $a_2 = a_4$. We next focus on showing $a_0 = a_3$ before proving $a_j = a_{3-j}$ for all other $j$.

Similarly to equations (6.1) and (6.2), since 8 and 3 are essential for 11 in base 2 but non-essential in base 3, we have

\[
\beta_{11}(S) + \beta_{11}(S \Delta \{3\}) \equiv \begin{cases} 0 & \text{if } |S \cap \{1,2\}| = 1, \\ 3 & \text{if } |S \cap \{1,2\}| = 0 \text{ or } 2 \mod 6, \end{cases}
\]

\[
\beta_{11}(S) + \beta_{11}(S \Delta \{8\}) \equiv \begin{cases} 0 & \text{if } |S \cap \{9,10\}| = 1, \\ 3 & \text{if } |S \cap \{9,10\}| = 0 \text{ or } 2 \mod 6. \end{cases}
\]
Table 5: Examples of cyclotomic factors of $Q_n(t)$ of the form $\Phi_{2s}$ where the binary expansion of $n$ has three 1’s.

Assume $S \subseteq [10]$ in which $\beta_{11}(S) \equiv 0 \mod 3$. As in Theorem 6.1 if $|S \cap \{1, 2\}| = 0$ or 2, or if $|S \cap \{9, 10\}| = 0$ or 2, the descent set statistics $\beta_{11}(S)$, $\beta_{11}(S \triangle \{3\})$, $\beta_{11}(S \triangle \{8\})$, and $\beta_{11}(S \triangle \{3, 8\})$ contribute evenly between $a_0$ and $a_3$.

On the other hand, if $|S \cap \{1, 2\}| = 1$ and $|S \cap \{9, 10\}| = 1$, then $S$ is one of the four sets in Lemma 6.8. Therefore, the descent set statistic of the set $S$ is non-zero modulo 3, and does not contribute to either $a_0$ or $a_3$. In conclusion, the only possible sets that do contribute to $a_0$ and $a_3$ do so evenly, so $a_0 = a_3$.

It remains to show $a_1 = a_2$ and $a_4 = a_5$. Since 11 has three digits in its binary expansion, Corollary 2.8 gives that $a_0 + a_2 + a_4 = a_1 + a_3 + a_5$. Combining this equality with $a_0 = a_3$, $a_1 = a_5$ and $a_2 = a_4$, it follows that $a_1 = a_2$ and $a_4 = a_5$. Thus, Lemma 4.1 implies that $\Phi_6$ is a factor of $Q_{11}(t)$.

This result is particular to $n = 11$. Attempts to generalize to values of $n$ of the form $2^r + 2 + 1 = p^r + 2$ have so far failed. For these $n$ one can similarly show that $a_0 = a_p$. Unfortunately, this does not imply $a_j = a_{p-j}$, which is in fact not true for all $j$.

Table 5 summarizes particular values of $n$ and $s$ for which $\Phi_{2s}$ is a factor of $Q_n(t)$ with $n$ having three binary digits, as was done in Table 4 for $n$ with two binary digits.

7 The double factor $\Phi_2$ in the descent set polynomial

Our next results are about the occurrence of the double factors in the descent set polynomial $Q_n(t)$. Here we sharpen techniques of Chebikin et al. to explain more double factors.
We begin by recalling the \textbf{ab}- and the \textbf{cd}-index of the Boolean algebra. Let \( \mathbb{Z}\langle a, b \rangle \) denote the polynomial ring in the non-commutative variables \( a \) and \( b \). For \( S \) a subset of \([n-1]\) define the \textbf{ab}-monomial \( u_S = u_1 u_2 \cdots u_{n-1} \) where \( u_i = a \) if \( i \notin S \) and \( u_i = b \) if \( i \in S \). The polynomial \( \Psi(B_n) \) given by
\[
\Psi(B_n) = \sum_{S \subseteq [n-1]} \beta_n(S) \cdot u_S,
\]
is known as the \textbf{ab}-index of the Boolean algebra. The result we need is that \( \Psi(B_n) \) can be written in terms of \( c = a + b \) and \( d = ab + ba \), which is originally due to Bayer and Klapper \cite{BayerKlapper}. For ways to compute \( \Psi(B_n) \) see \cite{Glasby} Proposition 8.2 and \cite{Glasby2}. For more details see also Theorem 1.6.3 or Section 3.17 in \cite{Glasby3}.

Define a linear function \( L \) from \( \mathbb{Z}\langle a, b \rangle \) to \( \mathbb{Z} \) by
\[
L(u_S) = (-1)^{\beta_n(S)},
\]
where \( S \) is a subset of \([n-1]\) and \( u_S \) is the associated \textbf{ab}-monomial of degree \( n - 1 \). Note that we abuse notation such that for an \textbf{ab}-monomial \( u \) of degree \( n - 1 \), we write \( \beta_n(u) \) instead of \( \beta_n(S) \), where \( u = u_S \).

\textbf{Proposition 7.1.} Let \( w \) be a \textbf{cd}-monomial of degree \( 2n - 1 \) having \( j \) \textbf{d}'s. Then the following evaluation holds
\[
L(w) = 2^{2n-j-1} \cdot (1 - 2 \cdot \rho(n)).
\]

\textit{Proof.} Let \( u = u_1 u_2 \cdots u_{2n-1} \) be an \textbf{ab}-monomial in the expansion of \( w \). Let \( v \) be the \textbf{ab}-monomial formed by taking the letters in even positions from \( u \), that is, \( v = u_2 u_4 \cdots u_{2n-2} \). By Theorem 3.2 we have that
\[
\beta_{2n}(u) \equiv \bar{\chi}(\Delta_{2n}|S) \equiv \bar{\chi}(\Delta_n|T) \equiv \beta_n(v) \mod 2,
\]
since the two complexes \( \Delta_{2n}|S \) and \( \Delta_n|T \) are identical where \( u = u_S \) and \( v = u_T \). Furthermore, observe that every \textbf{ab}-monomial of degree \( n - 1 \) appears this way.

Given an \textbf{ab}-monomial \( v \) of degree \( n - 1 \), how many corresponding monomials \( u \) can we find within the expansion of the \textbf{cd}-monomial \( w \)? There are \( n \) odd positions in \( u \) to fill in. If an odd position is covered by a \textbf{d} in \( w \), there is a unique way to fill it in. Note that there are \( n - j \) odd positions in \( u \) associated with \textbf{c}'s in \( w \). Hence there are \( 2^{n-j} \) ways to fill in \( v \) to get an \textbf{ab}-monomial \( u \) in the expansion of \( w \). Now
\[
L(w) = \sum_u (-1)^{\beta_{2n}(u)}
= 2^{n-j} \cdot \sum_v (-1)^{\beta_n(v)}
= 2^{n-j} \cdot Q(n)(-1)
= 2^{n-j-1} \cdot (1 - 2 \cdot \rho(n)),
\]
where the first sum is over all \textbf{ab}-monomials \( u \) occurring in the expansion of \( w \) and the second sum is over all \textbf{ab}-monomials \( v \) of degree \( n - 1 \).

\textbf{Theorem 7.2.} If \( \Phi_2 \) is a factor of \( Q_{2n}(t) \) then \( \Phi_2 \) is a double factor of \( Q_{2n}(t) \).
Proof. Observe that

\[ Q'_{2n}(t) = \sum_S \beta_{2n}(S) \cdot t^{\beta_{2n}(S) - 1}. \]

Hence evaluating \( Q'_{2n}(t) \) at \( t = -1 \), we obtain

\[ Q'_{2n}(-1) = - \sum_S \beta_{2n}(S) \cdot (-1)^{\beta_{2n}(S)} \]
\[ = - \mathcal{L} \left( \sum_S \beta_{2n}(S) \cdot u_S \right) \]
\[ = - \mathcal{L}(\Psi(B_{2n})). \]

Now if \( \Phi_2 \) is a factor of \( Q_{2n}(t) \), we have \( \rho(n) = 1/2 \). Since \( \Psi(B_{2n}) \) can be expressed in terms of the two variables \( c \) and \( d \), we conclude that \( \mathcal{L}(\Psi(B_{2n})) = 0 \). Hence \(-1\) is a double root of \( Q_{2n}(t) \), yielding the conclusion.

Now extending Theorem 7.3 in [3] we have the next result.

**Corollary 7.3.** If the binary expansion of \( n \) has three 1’s then \( \Phi_2^2 \) divides \( Q_{2n}(t) \).

8 The double factor \( \Phi_{2p} \) in \( Q_{2q}(t) \)

Throughout this section, assume \( q \) is an odd prime power, that is, \( q = p^r \) where \( p \) is prime and \( r \) is a positive integer.

Observe that by Theorem 6.1, part (iv) in [3] the cyclotomic polynomial \( \Phi_{2p} \) is a factor of the descent set polynomial \( Q_{2q}(t) \). Hence we concentrate on extending Theorem 7.5 from [3] to show that \( \Phi_{2p} \) is a double factor in this section.

**Theorem 8.1.** If \( \rho(q) = 1/2 \), then the cyclotomic polynomial \( \Phi_{2p} \) is a double factor of the descent set polynomial \( Q_{2q}(t) \).

In order to prove this theorem we introduce two new linear functions \( C \) and \( S \) from \( ab \)-polynomials of degree \( 2q - 1 \) to the real field \( \mathbb{R} \) by

\[ C(u_S) = \cos(\pi/p \cdot \beta_{2q}(S)), \]
\[ S(u_S) = \sin(\pi/p \cdot \beta_{2q}(S)). \]

(8.1)
(8.2)

Our goal is to show that \( C(w) = S(w) = 0 \) for any \( cd \)-monomial \( w \) of degree \( 2q - 1 \). We do this by a series of lemmas. First from Corollary 5.3 in [3], we have the following result.

**Lemma 8.2.** The descent set statistic \( \beta_{2q} \) modulo \( p \) is given by

\[ \beta_{2q}(S) \equiv (-1)^{|S\setminus\{q\}|} \mod p. \]

This is straightforward to show using that \( (x_1 + x_2 + \cdots)^{2q} \equiv (x_1^q + x_2^q + \cdots)^2 \equiv M_{2q} + 2M_{(q,q)} \mod p \).

**Lemma 8.3.** For any \( ab \)-monomial \( u \) of degree \( 2q - 1 \), we have \( C(u) = - \cos(\pi/p) \cdot (-1)^{\beta_{2q}(u)} \).
Proof. According to Lemma 8.2 there are only four possible values for \( \beta_{2q}(u) \mod 2p \). When \( \beta_{2q}(u) \) is odd, then the only two values for \( \beta_{2q}(u) \) modulo \( 2p \) are \( \pm 1 \), in which case \( \mathcal{C}(u) = \cos(\pi/p) \). Similarly, when \( \beta_{2q}(u) \) is even, it can only take the values \( p \pm 1 \), and hence \( \mathcal{C}(u) = -\cos(\pi/p) \). \[ \square \]

Lemma 8.4. If \( \rho(q) = 1/2 \), then for a \( cd \)-monomial \( w \), we have \( \mathcal{C}(w) = 0 \).

Proof. Assume that the \( cd \)-monomial \( w \) has \( j \) \( d \)'s. Now by the previous lemma we have

\[
\mathcal{C}(w) = \sum_u \mathcal{C}(u) = -\cos(\pi/p) \cdot \sum_u (-1)^{\beta_{2q}(u)} = -\cos(\pi/p) \cdot \mathcal{L}(w) = -\cos(\pi/p) \cdot 2^{q-1} \cdot (1 - 2 \cdot \rho(q)).
\]

Since \( \rho(q) = 1/2 \), we obtain the conclusion \( \mathcal{C}(w) = 0 \). \[ \square \]

Lemma 8.5. Let \( u \) and \( v \) be two \( ab \)-monomials such that \( \deg(u) + \deg(v) = 2q - 2 \), both \( \deg(u) \) and \( \deg(v) \) are even, and both \( \deg(u) \) and \( \deg(v) \) differ from \( q - 1 \). Then the functional \( S \) applied to \( u \cdot c \cdot v \) is zero, that is, \( S(u \cdot c \cdot v) = 0 \).

Proof. Since \( \deg(u) + 1 \) is non-essential for \( 2 \cdot q \) both in base 2 and in base \( p \), we have by Lemma 2.5 that

\[
\beta_{2q}(u \cdot a \cdot v) = -\beta_{2q}(u \cdot b \cdot v) \mod 2p.
\]

Since \( \sin \) is an odd function, this identity directly implies \( S(u \cdot a \cdot v) = -S(u \cdot b \cdot v) \). \[ \square \]

Lemma 8.6. Let \( w \) be a \( cd \)-monomial of degree \( 2q - 1 \) different from the monomial \( d^{(q-1)/2}cd^{(q-1)/2} \). Then \( S(w) \) vanishes.

Proof. The monomial \( w \) has \( q \) odd positions and \( q - 1 \) even positions. Since a \( d \) covers both an odd position and an even position, there will always be a \( c \) in an odd position. Unless \( w \) is the monomial \( d^{(q-1)/2}cd^{(q-1)/2} \) we can find a \( c \) in an odd position different from \( q \). By the previous lemma we know \( S(u \cdot c \cdot v) = 0 \) for all \( ab \)-monomials \( u \) and \( v \) and hence by linearity we conclude \( S(w) = 0 \). \[ \square \]

Lemma 8.7. If \( \rho(q) = 1/2 \) then \( S(d^{(q-1)/2}cd^{(q-1)/2}) = 0 \).

Proof. If \( u \) is an \( ab \)-monomial occurring in the expansion of \( w = d^{(q-1)/2}cd^{(q-1)/2} \) then it has \( q - 1 \) or \( q \) \( b \)'s. In fact, it has \( q - 1 \) \( b \)'s in the positions different from the position \( q \) since this is the position of the \( c \) in \( w \).

Lemma 8.2 implies that \( \beta_{2q}(u) \equiv (−1)^{q−1} \equiv 1 \mod p \), also using that \( q \) is odd. Hence modulo \( 2p \) we have that \( \beta_{2q}(u) \equiv 1 \) or \( p + 1 \mod 2p \). That is, the value of \( \beta_{2q}(u) \) modulo \( 2p \) only depends on the value modulo 2. Hence we have the sum

\[
S(w) = \sum_u S(u) = \sum_u \sin(\pi/p \cdot \beta_{2q}(u)) = \sum_u -\sin(\pi/p \cdot (−1)^{\beta_{2q}(u)}) = -\sin(\pi/p \cdot \mathcal{L}(w)) = -\sin(\pi/p \cdot 2^q \cdot (1 - 2 \cdot \rho(q))).
\]

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Since \( \rho(q) = 1/2 \), we obtain \( S(w) = 0 \).

**Proof of Theorem 8.1.** Observe that
\[
e^{\pi i/p} \cdot Q_{2q}'(e^{\pi i/p}) = \sum_S \beta_{2q}(S) \cdot e^{\beta_{2q}(S) \cdot \pi i/p}
\]
\[
= \sum_S \beta_{2q}(S) \cdot (\mathcal{C}(u_S) + i \cdot S(u_S))
\]
\[
= (\mathcal{C} + i \cdot S) \left( \sum_S \beta_{2q}(S) \cdot u_S \right)
\]
\[
= (\mathcal{C} + i \cdot S)(\Phi_{2q}),
\]
which vanishes. Hence \( e^{\pi i/p} \) is a root of \( Q_{2q}' \), so \( e^{\pi i/p} \) is a double root of \( Q_{2q} \).

### 9 The (double) factor \( \Phi_{2q} \) in \( Q_{q+1}(t) \)

Let \( q = p^r \) be an odd prime power, that is, \( p \) is an odd prime and \( r \) a positive integer. Now we study the case of the cyclotomic factor \( \Phi_{2q} \) in \( Q_{q+1}(t) \).

**Theorem 9.1.** If \( \rho(q) = 1/2 \) then the cyclotomic polynomial \( \Phi_{2q} \) divides the descent set polynomial \( Q_{q+1}(t) \). Furthermore, if \( q \equiv 3 \mod 4 \), then \( \Phi_{2q} \) is a double factor in \( Q_{q+1}(t) \).

We start by explicitly expressing the flag \( f \)-vector of the Boolean algebra \( B_{q+1} \) modulo \( p \):
\[
F(B_{q+1}) \equiv (M_{(1)})^q \cdot M_{(1)} \equiv M_q \cdot M_{(1)} \equiv M_{q+1} + M_{(q,1)} + M_{(1,q)} \mod p.
\]
Hence the flag \( f \)-vector \( f(S) \equiv 1 \mod p \) if \( S \) is equal to \( \emptyset, \{1\} \) or \( \{q\} \), and zero otherwise. By inclusion-exclusion we obtain that the descent set statistic modulo \( p \) is given by
\[
\beta_{q+1}(S) \equiv \begin{cases} (-1)^S & \text{if } |S \cap \{1,q\}| = 0, \\ 0 & \text{if } |S \cap \{1,q\}| = 1, \mod p, \\ -(-1)^S & \text{if } |S \cap \{1,q\}| = 2, \end{cases} \tag{9.1}
\]
In terms of \( \text{ab} \)-monomials, this result can be stated as \( \beta_{q+1}(a \cdot v \cdot b) \equiv \beta_{q+1}(b \cdot v \cdot a) \equiv 0 \mod p \) and \( \beta_{q+1}(a \cdot v \cdot a) \equiv -\beta_{q+1}(b \cdot v \cdot b) \equiv (-1)^j \mod p \) where \( v \) is an \( \text{ab} \)-monomial of degree \( q-2 \) having \( j \) \( b \)'s.

Similarly to the previous section, we use two linear functions from \( \text{ab} \)-polynomials of degree \( q \) to the reals \( \mathbb{R} \), defined by
\[
\mathcal{C}(u_S) = \cos(\pi/p \cdot \beta_{q+1}(S)), \tag{9.2}
\]
\[
S(u_S) = \sin(\pi/p \cdot \beta_{q+1}(S)). \tag{9.3}
\]
Note that they differ from definitions (8.1) and (8.2) by replacing the descent set statistic \( \beta_{2q} \) by \( \beta_{q+1} \).

**Lemma 9.2.** Let \( w \) be a \( \text{cd} \)-monomial of degree \( q \) beginning or ending with the letter \( c \). If \( \rho(q+1) = 1/2 \) then \( \mathcal{C}(w) = 0 \).
Proof. It is enough to consider the case when \( w \) begins with a \( c \). Let \( u = u_1 u_2 \cdots u_q \) be an ab-monomial in the expansion of \( w \). If \( u_1 \) differs from \( u_q \), we have by (9.1) that \( \beta_{q+1}(u) \equiv 0 \mod p \). Hence \( C(u) = \cos(\pi/p \cdot \beta_{q+1}(u)) = (-1)^{\beta_{q+1}(u)} \). In the case in which the first and last letter of \( u \) are the same, we have that \( \beta_{q+1}(u) \equiv \pm 1 \mod p \) by (9.1). Hence \( \beta_{q+1}(u) \) takes one of the four values \( \pm 1, p \pm 1 \) modulo 2p and so \( C(u) = \cos(\pi/p \cdot \beta_{q+1}(u)) \) takes one of the two values \( \pm \cos(\pi/p) \). Note that if \( \beta_{q+1}(u) \) is even, then \( \beta_{q+1}(u) \) is \( p \pm 1 \) modulo 2p and hence \( C(u) = -\cos(\pi/p) \). Similarly, if \( \beta_{q+1}(u) \) is odd we have \( C(u) = \cos(\pi/p) \). To summarize these two cases when \( u_1 = u_q \), we have that \( C(u) = -\cos(\pi/p) \cdot (-1)^{\beta_{q+1}(u)} \).

Then we have the sum

\[
C(w) = \sum_{u : u_1 \neq u_q} (-1)^{\beta_{q+1}(u)} - \cos(\pi/p) \cdot \sum_{u : u_1 = u_q} (-1)^{\beta_{q+1}(u)},
\]

where both sums are over all ab-monomials \( u \) in the expansion of \( w \). Let \( \overline{u} \) denote the involution defined by \( \overline{a} = b \) and \( \overline{b} = a \). In each of the sums, also include the term \( \overline{u_1} u_2 \cdots u_q \). Since 1 is non-essential for \( q + 1 \) in base 2, we have \( \beta_{q+1}(\overline{u_1} u_2 \cdots u_q) \equiv \beta_{q+1}(u) \mod 2 \). Hence both sums will double to give us

\[
C(w) = \frac{1}{2} \cdot \sum_{u} (-1)^{\beta_{q+1}(u)} - \cos(\pi/p) \cdot \frac{1}{2} \cdot \sum_{u} (-1)^{\beta_{q+1}(u)}
= \frac{1}{2} \cdot (1 - \cos(\pi/p)) \cdot C(w),
\]

where both sums are over all \( u \) occurring in the expansion of \( w \). This works since \( w \) begins with the letter \( c \). By the assumption \( \rho(q + 1) = 1/2 \), this expression will vanish by Proposition 7.1. \[\square\]

Lemma 9.3. Let \( w \) be a cd-monomial of degree \( q \) beginning or ending with the letter \( d \). If \( \rho(q + 1) = 1/2 \) and \( q \equiv 3 \mod 4 \) then \( C(w) = 0 \).

Proof. Assume that \( w \) begins with a \( d \). The proof is the same as the proof of the previous lemma, except that \( q \equiv 3 \mod 4 \) implies that 2 is non-essential for \( q + 1 \) in base 2. In the end of the proof when we extend the two sums ranging over \( u = u_1 u_2 u_3 \cdots u_q \), also include the terms \( \overline{u_1} u_2 \cdots u_q \). Then the both sums become \( C(w) \) and the result follows. \[\square\]

Lemma 9.4. Let \( u \) and \( v \) be two ab-monomials such that \( \deg(u) + \deg(v) = q - 1 \), both \( \deg(u) \) and \( \deg(v) \) are even, and both \( \deg(u) \) and \( \deg(v) \) differ from zero. Then the functional \( S \) applied to \( u \cdot c \cdot v \) is zero, that is, \( S(u \cdot c \cdot v) = 0 \).

Proof. Since \( \deg(u) + 1 \) is non-essential for \( q + 1 \) both in base 2 and in base \( p \), we have by Lemma 2.5 that

\[
\beta_{q+1}(u \cdot a \cdot v) \equiv -\beta_{q+1}(u \cdot b \cdot v) \mod 2p.
\]

Since sin is an odd function, this identity directly implies \( S(u \cdot a \cdot v) = -S(u \cdot b \cdot v) \). \[\square\]

Lemma 9.5. Let \( w \) be a cd-monomial of degree \( q \) different from the monomials \( cd^{(q-1)/2}, d^{(q-1)/2}c \), and \( cd^{i} cd^{j} c \) where \( i + j = (q - 3)/2 \). Then \( S(w) \) vanishes.

Proof. If the monomial \( w \) has a \( c \) in an odd position \( i \), where \( 2 \leq i \leq q - 1 \), then \( S(w) \) vanishes by the previous lemma.

The monomial \( w \) has \( (q + 1)/2 \) odd positions and \( (q - 1)/2 \) even positions. Since a \( d \) covers both an odd position and an even position, there will always be a \( c \) in an odd position. However, this position
could be position 1 or position \( q \). In that situation, if there is only one \( c \) in \( w \), then it is either the monomial \( cd^{(q-1)/2} \) or \( d^{(q-1)/2}c \). If there are three \( c \)'s in \( w \) then two of \( c \)'s must be the first and last positions. That is \( w \) is of the form \( cd^i cd^j c \). Note that the middle \( c \) is in an even position and the previous lemma does not help.

**Lemma 9.6.** Let \( w \) be a \( cd \)-monomial of degree \( q \) beginning and ending with the letter \( c \). Then \( S(w) \) vanishes. In particular, \( S(cd^i cd^j c) = 0 \) for \( i + j = (q - 3)/2 \).

**Proof.** Let \( u \) be an \( ab \)-monomial occurring in the expansion of \( w \). Observe that if \( u \) has the form \( a \cdot v \cdot b \) or \( b \cdot v \cdot a \) then \( \beta_{q+1}(u) \equiv 0 \) mod \( p \) by (9.1). This implies that \( S(u) = \sin\left(\pi/p \cdot \beta_{q+1}(u)\right) = 0 \). Hence we have only to consider \( ab \)-monomials in the expansion of \( w \) that begin and end with the same letter. Again by (9.1) observe that \( \beta_{q+1}(ab \cdot v \cdot a) \equiv -\beta_{q+1}(b \cdot v \cdot b) \) mod \( p \). Since positions 1 and \( q \) are non-essential for \( q + 1 \) in base 2, we have \( \beta_{q+1}(ab \cdot v \cdot a) \equiv \beta_{q+1}(b \cdot v \cdot b) \) mod 2. Combining these two congruences to one statement modulo \( 2p \) we have \( \beta_{q+1}(ab \cdot v \cdot a) \equiv -\beta_{q+1}(b \cdot v \cdot b) \) mod \( 2p \). This implies that \( S(ab \cdot v \cdot a) = -S(b \cdot v \cdot b) \) and the statement of the lemma.

**Lemma 9.7.** If \( \rho(q + 1) = 1/2 \) and \( q \equiv 3 \) mod 4 then \( S(d^{(q-1)/2}c) = S(cd^{(q-1)/2}) = 0 \).

**Proof.** The congruence relation on \( q \) implies that 4 divides \( q + 1 \). Hence the element 2 is a non-essential element for \( q + 1 \) in base 2. We will use this fact, together with the facts that 1 and \( q \) are non-essential elements.

By symmetry it is enough to prove the lemma for \( w = d^{(q-1)/2}c \). Let \( u \) be an \( ab \)-monomial occurring in the expansion of \( w \). If \( u \) begins and ends with different letters, similar to the previous lemma, we have that \( S(u) = 0 \). Hence we have that \( u \) has the form \( ab \cdot v \cdot a \) or \( ba \cdot v \cdot b \). Next we have that \( \beta_{q+1}(ab \cdot v \cdot a) \equiv -\beta_{q+1}(ba \cdot v \cdot b) \equiv \beta_{q+1}(ba \cdot v \cdot b) \) mod \( p \) by (9.1). Furthermore, since the three elements 1, 2 and \( q \) are non-essential for \( q + 1 \) in base 2, we have that \( \beta_{q+1}(ab \cdot v \cdot a) \equiv \beta_{q+1}(ba \cdot v \cdot b) \) mod 2. That is, modulo \( 2p \) we have \( \beta_{q+1}(ab \cdot v \cdot a) \equiv \beta_{q+1}(ba \cdot v \cdot b) \) mod \( 2p \).

Hence these two cases \( ab \cdot v \cdot a \) and \( ba \cdot v \cdot b \) are the same, that is,

\[
S(w) = 2 \cdot \sum_{ab \cdot v \cdot a} \sin\left(\frac{\pi}{p} \cdot \beta_{q+1}(ab \cdot v \cdot a)\right).
\]

The monomial \( u = ab \cdot v \cdot a \) has \( (q - 1)/2 \) \( b \)‘s, so \( \beta_{q+1}(u) \equiv (-1)^{(q-1)/2} \) mod \( p \) by Lemma 8.2. By considering the four values \( \pm 1, p \pm 1 \) of \( \beta_{q+1}(u) \) modulo \( 2p \) we have that

\[
\sin\left(\frac{\pi}{p} \cdot \beta_{q+1}(u)\right) = -(-1)^{(q-1)/2} \cdot \sin\left(\frac{\pi}{p}\right) \cdot (-1)^{\beta_{q+1}(u)}.
\]

Hence \( S(w) \) is given by

\[
S(w) = -2 \cdot (-1)^{(q-1)/2} \cdot \sin\left(\frac{\pi}{p}\right) \cdot \sum_{ab \cdot v \cdot a} (-1)^{\beta_{q+1}(u)}.
\]

Again since the elements 1, 2 and \( q \) are non-essential for \( q + 1 \) in base 2, we can switch the letters in these places without changing the descent set statistic \( \beta_{q+1} \) modulo 2. Hence we have

\[
S(w) = -\frac{1}{2} \cdot (-1)^{(q-1)/2} \cdot \sin\left(\frac{\pi}{p}\right) \cdot \sum_{u} (-1)^{\beta_{q+1}(u)},
\]

where the sum is over all \( ab \)-monomials \( u \) in the expansion of \( w \). Now by the assumption that \( \rho(q + 1) = 1/2 \) and Proposition 7.1 this last sum is zero.
Proof of Theorem 9.1. Observe that

\[ Q_{q+1}(e^{\pi i/p}) = \sum_u e^{\pi i/p \cdot \beta_{q+1}(u)} = \sum_u (\cos(\pi/p \cdot \beta_{q+1}(u)) + i \cdot \sin(\pi/p \cdot \beta_{q+1}(u))) = (C + i \cdot S)(e^q), \]

since the first two sums is over all ab-monomials of degree q, that is, all the ab-monomials in the expansion of \( e^q \). Finally, the last expression vanishes by Lemmas 9.2 and 9.5.

With the added assumption \( q \equiv 3 \mod 4 \), Lemmas 9.2 and 9.3 imply that \( C \) applied to any cd-polynomial of degree q vanishes. Similarly, with the assumption Lemmas 9.5 through 9.7 imply that \( S \) applied to any cd-polynomial of degree q vanishes. Now we have that

\[ e^{\pi i/p} \cdot Q'_{q+1}(e^{\pi i/p}) = \sum_u \beta_{q+1}(u) \cdot e^{\pi i/p \cdot \beta_{q+1}(u)} = (C + i \cdot S)(\Psi(B_{q+1})) = 0, \]

since \( \Psi(B_{q+1}) \) can be written in terms of the variables c and d. Thus \( e^{\pi i/p} \) is a double root of \( Q_{q+1}(t) \).

10 Concluding remarks

By considering Table 6 one sees that there are two unexplained cyclotomic factors in this table. They are \( \Phi_4 \) and \( \Phi_{28} \), both dividing \( Q_{14}(t) \). Here it is straightforward to see \( a_{4,1} = a_{4,3} \), that is, \( Q_{14}(i) \) is a real number. But it remains to find an argument demonstrating that \( a_0 = a_2 \). Since 4 is a square, the Chinese Remainder Theorem cannot come to our rescue. Note that these factors seems to isolated \( n = 14 \) and does not occur among other n with three 1’s in their binary expansion.

Further consideration of Table 6 shows that all cyclotomic factors that appear in table with multiplicity have now been explained. Are there other square factors appearing beyond \( n = 23 \) that have not yet been explained?

Do Theorems 8.1 and 9.1 apply to infinitely many prime powers? As mentioned in the introduction, there are only 6 prime powers with with two 1’s in their binary expansion. However, there seems to be an infinite number of primes with three 1’s in their binary expansion; see the sequence A081091 in The On-Line Encyclopedia of Integer Sequences. However, this seems to be a hard number theory problem.

Chebikin et al. calculated the proportion for the number of odd entries in the descent set statistics \( \beta_n \) for \( n = 1, 3, 7, 15, 31 \); see Table 1 that is, for any integer with at most five 1’s in its binary expansion. Could the topological view of Theorem 3.2 help for calculating the next case \( n = 63 \)? From this topological viewpoint, is there a classification of simplicial complexes \( \Delta \) such that exactly half of the induced subcomplexes \( \Delta_S \) have an odd Euler characteristic?

In Chebikin et al. they also consider the signed descent set polynomial, that is,

\[ Q_n^\pm(t) = \sum_{S \subseteq [n]} t^{\beta_n^\pm(S)}, \]

where \( \beta_n^\pm(S) \) is the number of signed permutations in \( S_n^\pm \) with descent set \( S \). Can any of our techniques be extended to explain cyclotomic factors in this polynomial? There are plenty of such factors; see Table 3 in [3].
| $n$ | degree | cyclotomic factors of $Q_n(t)$ |
|-----|--------|-------------------------------|
| 3   | 2      | $\Phi_2$                      |
| 4   | 5      | $\Phi_2$                      |
| 5   | 16     | $\Phi_2 \cdot \Phi_{10}$      |
| 6   | 61     | $\Phi_2 \cdot \Phi_6 \cdot \Phi_{10}$ |
| 7   | 272    | $\Phi_2$                      |
| 8   | 1385   | $\Phi_2 \cdot \Phi_{28}$      |
| 9   | 7936   | $\Phi_2 \cdot \Phi_6 \cdot \Phi_{18}$ |
| 10  | 50521  | $\Phi_2 \cdot \Phi_6 \cdot \Phi_{10} \cdot \Phi_{18} \cdot \Phi_{30}$ |
| 11  | 353792 | $\Phi_2 \cdot \Phi_6$         |
| 12  | 2702765| $\Phi_2 \cdot \Phi_6 \cdot \Phi_{10} \cdot \Phi_{18} \cdot \Phi_{22} \cdot \Phi_{66} \cdot \Phi_{110} \cdot \Phi_{198}$ |
| 13  | 22368256| $\Phi_2 \cdot \Phi_{26}$    |
| 14  | $1.993 \cdot 10^8$ | $\Phi_2 \cdot \Phi_{14} \cdot \Phi_{18} \cdot \Phi_{22} \cdot \Phi_{66} \cdot \Phi_{110} \cdot \Phi_{198}$ |
| 15  | $1.904 \cdot 10^9$ | $-$                                                   |
| 16  | $1.939 \cdot 10^{10}$ | $\Phi_2 \cdot \Phi_9 \cdot \Phi_{12} \cdot \Phi_{20} \cdot \Phi_{44} \cdot \Phi_{52} \cdot \Phi_{60} \cdot \Phi_{156} \cdot \Phi_{220} \cdot \Phi_{260} \cdot \Phi_{572} \cdot \Phi_{2860}$ |
| 17  | $2.099 \cdot 10^{11}$ | $\Phi_2 \cdot \Phi_{34}$     |
| 18  | $2.405 \cdot 10^{12}$ | $\Phi_2 \cdot \Phi_9 \cdot \Phi_{18} \cdot \Phi_{34} \cdot \Phi_{102} \cdot \Phi_{306}$ |
| 19  | $2.909 \cdot 10^{13}$ | $\Phi_2 \cdot \Phi_{38}$     |
| 20  | $3.704 \cdot 10^{14}$ | $\Phi_2 \cdot \Phi_6 \cdot \Phi_{10} \cdot \Phi_{30} \cdot \Phi_{34} \cdot \Phi_{38} \cdot \Phi_{102} \cdot \Phi_{114} \cdot \Phi_{170}$ |
|     |        |                               | $\Phi_{190} \cdot \Phi_{510} \cdot \Phi_{570} \cdot \Phi_{646} \cdot \Phi_{1938} \cdot \Phi_{3230} \cdot \Phi_{9690}$ |
| 21  | $4.951 \cdot 10^{15}$ | $\Phi_2 \cdot \Phi_6 \cdot \Phi_{14} \cdot \Phi_{42}$ |
| 22  | $6.935 \cdot 10^{16}$ | $\Phi_2 \cdot \Phi_9 \cdot \Phi_{14} \cdot \Phi_{22} \cdot \Phi_{154}$ |
| 23  | $1.015 \cdot 10^{18}$ | $-$                                                   |

Table 6: Cyclotomic factors of $Q_n(t)$. This table is from Chebikin et al. [3], but the explained factors have been updated. These factors occur in **boldface**. Furthermore the factor $\Phi_{2860}$ in $Q_{16}(t)$ has been included, which was missing in the original table. Note that the two factors $\Phi_4$ and $\Phi_{28}$ in $Q_{14}(t)$ are still unexplained.

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