Dispersive estimates of solutions to the Schrödinger equation in dimensions \( n \geq 4 \)

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**Abstract**

We prove dispersive estimates for solutions to the Schrödinger equation with a real-valued potential \( V \in L^\infty(\mathbb{R}^n) \), \( n \geq 4 \), satisfying \( V(x) = O(\langle x \rangle^{-(n+2)/2-\epsilon}) \), \( \epsilon > 0 \).

1 Introduction and statement of results

Let \( V \in L^\infty(\mathbb{R}^n) \), \( n \geq 4 \), be a real-valued function satisfying

\[
|V(x)| \leq C \langle x \rangle^{-\delta}, \quad \forall x \in \mathbb{R}^n,
\]

with constants \( C > 0 \) and \( \delta > (n+2)/2 \), where \( \langle x \rangle = (1 + |x|^2)^{1/2} \). Denote by \( G_0 \) and \( G \) the self-adjoint realizations of the operators \( -\Delta \) and \( -\Delta + V(x) \) on \( L^2(\mathbb{R}^n) \). It is well known that the absolutely continuous spectrums of the operators \( G_0 \) and \( G \) coincide with the interval \([0, +\infty)\). Moreover, by Kato’s theorem the operator \( G \) has no strictly positive eigenvalues, which in turn implies that \( G \) has no strictly positive resonances neither. Throughout this paper, given \( 1 \leq p \leq +\infty \), \( L^p \) will denote the space \( L^p(\mathbb{R}^n) \). Also, given an \( a > 0 \) denote by \( \chi_a \in \mathcal{C}_\infty(\mathbb{R}) \) a real-valued function supported in the interval \([2a, +\infty)\), \( \chi_a = 1 \) on \([2a, +\infty)\). Our main result is the following

**Theorem 1.1** Assume (1.1) fulfilled. Then for every \( a > 0 \), \( 0 < \epsilon \ll 1 \), there exist constants \( C, C_\epsilon > 0 \) so that the following estimates hold

\[
\left\| e^{itG} G^{-(n-3)/4} \chi_a(G) \right\|_{L^1 \to L^\infty} \leq C|t|^{-n/2}, \quad \forall t \neq 0,
\]

(1.2)

\[
\left\| e^{itG} \chi_a(G)x^{-n/2-\epsilon} \right\|_{L^2 \to L^\infty} \leq C_\epsilon|t|^{-n/2}, \quad \forall t \neq 0.
\]

(1.3)

Moreover, for every \( 0 \leq q \leq (n-3)/2 \), \( 2 \leq p \leq \frac{2(n-1-2q)}{n-3-2q} \), we have

\[
\left\| e^{itG} G^{-aq/2} \chi_a(G) \right\|_{L^p' \to L^p} \leq C|t|^{-\alpha n/2}, \quad \forall t \neq 0,
\]

(1.4)

where \( 1/p + 1/p' = 1 \), \( \alpha = 1 - 2/p \).

**Remark 1.** The desired result would be to prove the estimate

\[
\left\| e^{itG} \chi_a(G) \right\|_{L^1 \to L^\infty} \leq C|t|^{-n/2}, \quad \forall t \neq 0.
\]

(1.5)
It is shown by Goldberg and Visan [3], however, that when \( n \geq 4 \) there exists a compactly supported potential \( V \in C^k(\mathbb{R}^n), \forall k < (n-3)/2 \), for which (1.5) fails to hold. In other words, in order that (1.5) holds true one needs to have a control of \( (n-3)/2 \) derivatives of \( V \). It seems that for potentials satisfying (1.1) only, our estimate (1.2) with a loss of \( (n-3)/2 \) derivatives is quite optimal.

**Remark 2.** It is natural to expect that if the zero is neither an eigenvalue nor a resonance of \( G \), the statements of Theorem 1.1 hold true with \( \chi_a \) replaced by the characteristic function, \( \chi \), of the interval \([0, +\infty)\) (the absolutely continuous spectrum of \( G \)), \( G^{-(n-3)/4} \) and \( G^{-\alpha q/2} \) replaced by \( \langle G \rangle^{-(n-3)/4} \) and \( \langle G \rangle^{-\alpha q/2} \), respectively. To prove this, it suffices to show that in this case the estimate (1.5) holds true with \( \chi_a \) replaced by \((1 - \chi_a)\chi\) for \( a > 0 \) small enough. The proof of such an estimate, however, requires different techniques than those developed in the present paper.

**Remark 3.** We believe that the estimate (1.4) still holds at the end point \( p = \frac{2(n-1-2q)}{n-3-2q} \) for \( 0 \leq q < (n-3)/2 \) and that our approach leads to such an estimate. In fact, it is not hard to see from the proof of (1.4) in Section 4 that the problem is reduced to estimating the \( L^2 \to L^2 \) norm of operators with explicitly given kernels.

The estimate (1.5) is proved in the case \( n = 2 \) by Schlag [7] for potentials satisfying (1.1) with \( \delta > 2 \). When \( n = 3 \), the estimate (1.5) is proved by Goldberg and Schlag [2] for potentials satisfying (1.1) with \( \delta > 3 \). Recently, (1.5) has been proved in this case by Vodev [8] and Yajima [11] for potentials satisfying (1.1) with \( \delta > 5/2 \), while Goldberg [1] has proved (1.5) for a very large class of potentials including those satisfying (1.1) with \( \delta > 2 \). The proofs in all these papers (except for [8]) are based on the very nice properties of the outgoing and incoming free resolvents when \( n = 2 \) and \( n = 3 \). When \( n \geq 4 \), however, these properties are no longer valid, and consequently one needs different methods to prove estimates like (1.5) (or like those in Theorem 1.1). The first result in this case is due to Journé, Sofer and Sogge [5], where they proved an analogue of (1.5) for potentials satisfying (1.1) with \( \delta > n + 4 \) as well as the regularity property \( \hat{V} \in L^1 \). This was later improved by Yajima [10] using the properties of the wave operators. Note also that the estimate (1.3) was proved by Jensen and Nakamura [4] for potentials satisfying (1.1) with \( \delta > n \) as well as an extra technical assumption.

To prove Theorem 1.1 we use the method of [8] together with some ideas from [9] where similar dispersive estimates have been proved for the wave group \( e^{it\sqrt{G}} \) for potentials satisfying (1.1) with \( \delta > (n + 1)/2 \). Roughly speaking, the method consists of reducing the dispersive estimates to uniform estimates for the Shrörödinger group (resp. the wave group) on weighted \( L^2 \) spaces, which in turn are proved by using some more or less known properties of the perturbed resolvent on weighted \( L^2 \) spaces (see Section 3). Note finally that in view of Goldberg’s result [1], one should expect that the statements of Theorem 1.1 hold true for the larger class of potentials satisfying (1.1) with \( \delta > (n + 1)/2 \). The proof of such estimates, however, would require a different approach than this one presented here.

## 2 Preliminary estimates

The following properties of the free Schrödinger group will play a key role in the proof of our dispersive estimates.

**Proposition 2.1** Let \( \psi \in C_0^\infty((0, +\infty)) \). For every \( 0 \leq s \leq (n-1)/2 \), \( 0 < \epsilon \ll 1 \), \( 0 < h \leq 1 \),
$t \neq 0$, we have
\[
\left\| e^{itG_0} \psi(h^2 G_0) \langle x \rangle^{-1/2-s-\epsilon} \right\|_{L^2 \to L^\infty} \leq C h^{-(n-1)/2+s} |t|^{-s-1/2}, \tag{2.1}
\]
with a constant $C > 0$ independent of $t$ and $h$. For every $s \geq 0$, $0 < h \leq 1$, $t \in \mathbb{R}$, we have
\[
\left\| \langle x \rangle^{-s} e^{itG_0} \psi(h^2 G_0) \langle x \rangle^{-s} \right\|_{L^2 \to L^2} \leq C (t/h)^{-s}, \tag{2.2}
\]
with a constant $C > 0$ independent of $t$ and $h$.

**Proof.** We are going to take advantage of the formula
\[
e^{itG_0} \psi(h^2 G_0) = (\pi i)^{-1} \int_0^\infty e^{it\lambda^2} \psi(h^2 \lambda^2) \left( R^+_0(\lambda) - R^-_0(\lambda) \right) \lambda d\lambda, \tag{2.3}
\]
where $R^+_0(\lambda)$ are the outgoing and incoming free resolvents with kernels given in terms of the Hankel functions by
\[
[R^+_0(\lambda)](x,y) = \pm \frac{i}{4} \left( \frac{\lambda}{2\pi|x-y|} \right)^\nu \mathcal{H}^\nu_0(\lambda|x-y|),
\]
where $\nu = (n-2)/2$. Hence, the kernel of the operator $e^{itG_0} \psi(h^2 G_0)$ is of the form $K_h(|x-y|, t)$, where
\[
K_h(\sigma, t) = \frac{\sigma^{-2\nu}}{(2\pi)\nu+1} \int_0^\infty e^{it\lambda^2} \psi(h^2 \lambda^2) \mathcal{J}_\nu(\sigma\lambda) \lambda d\lambda = h^{-n} K_1(\sigma h^{-1}, t h^{-2}), \tag{2.4}
\]
where $\mathcal{J}_\nu(z) = z^\nu J_\nu(z)$, $J_\nu(z) = (H^+_\nu(z) + H^-_\nu(z))/2$ is the Bessel function of order $\nu$. It is easy to see that (2.1) follows from the bound
\[
|K_h(\sigma, t)| \leq Ch^{s-(n-1)/2} |t|^{-s-1/2} \sigma^{-(n-1)/2+s}, \quad \forall t \neq 0, \sigma > 0, 0 < h \leq 1. \tag{2.5}
\]
In view of (2.4), it suffices to show (2.5) with $h = 1$. Let $m \geq 0$ be an integer. Integrating $m$ times in (2.4) we obtain
\[
(it)^m K_1(\sigma, t) = \frac{\sigma^{-2\nu}}{2(2\pi)\nu+1} \int_0^\infty e^{it\mu} \frac{d^m}{d\mu^m} (\psi(\mu) \mathcal{J}_\nu(\sigma\sqrt{\mu})) d\mu,
\]
\[
= \sum_{k=0}^m \sigma^{k-2\nu} \int_0^\infty e^{it\mu} \psi_{m-k}(\mu) \mathcal{J}^{(k)}_\nu(\sigma \sqrt{\mu}) d\mu,
\]
where $\psi_{m-k} \in C^\infty_0((0, +\infty))$, $\mathcal{J}^{(k)}_\nu(z) := (k!)^k \mathcal{J}_\nu(z)/dz^k$. Making the change $\mu = \lambda^2$ we can write the above identity in the form
\[
(it)^m K_1(\sigma, t) = \sum_{k=0}^m \sigma^{k-2\nu} \int_0^\infty e^{it\lambda^2} \varphi_{m-k}(\lambda) \mathcal{J}^{(k)}_\nu(\sigma \lambda) d\lambda, \tag{2.6}
\]
where $\varphi_k(\lambda) = 2\lambda \psi_k(\lambda^2)$. To estimate the integral in the RHS of (2.6), we will use that, given any $b > 0$ and a function $\varphi \in C^\infty_0([-b, b])$, we have the bound (see the proof of Lemma 2.4 of 3):
\[
\left| \int_{-\infty}^\infty e^{it\lambda^2 - ir\lambda} \varphi(\lambda) d\lambda \right| \leq C |t|^{-1/2} \| \varphi \|_{L^1}.
\]
\[ \leq C_{b}|t|^{-1/2} \sup_{0 \leq j \leq 1} \sup_{\lambda \in \mathbb{R}} \left| \partial_{\lambda}^{j} \varphi(\lambda) \right|, \quad \forall t \neq 0, \tau \in \mathbb{R}, \quad (2.7) \]

with a constant \( C_{b} > 0 \) independent of \( t, \tau \) and \( \varphi \). Consider first the case \( 0 < \sigma \leq 1 \). It is well known that near \( z = 0 \) the function \( J_{\nu}(z) \) is equal to \( z^{2\nu} \) times an analytic function. Therefore, we have, for \( 0 < z \leq z_{0} \),

\[ |\partial^{k}_{z} J_{\nu}(z)| \leq C z^{2\nu - k}, \quad (2.8) \]

for all integers \( 0 \leq k \leq n - 2 \), while for integers \( k \geq n - 1 \),

\[ |\partial^{k}_{z} J_{\nu}(z)| \leq C, \quad (2.9) \]

with constants \( C, C_{k} > 0 \) depending on \( z_{0} \). By (2.6)-(2.9), we obtain

\[ |K_{1}(\sigma, t)| \leq C_{m}|t|^{-m-1/2}, \quad \forall t \neq 0, 0 < \sigma \leq 1, \quad (2.10) \]

for all integers \( m \geq 0 \) with a constant \( C_{m} > 0 \) independent of \( t \) and \( \sigma \). Clearly, (2.10) holds with \( m = s \) for all real \( 0 \leq s \leq (n - 1)/2 \), which in turn implies (2.5) with \( h = 1 \) in this case.

Let now \( \sigma \geq 1 \). It is well known that \( J_{\nu}(z) = e^{izb^{+}_{\nu}(z)} + e^{-izb^{-}_{\nu}(z)} \) with functions \( b^{\pm}_{\nu}(z) \) satisfying

\[ |\partial^{j}_{z} b^{\pm}_{\nu}(z)| \leq C_{j} z^{(n-3)/2-j}, \quad \forall z \geq z_{0}, \quad (2.11) \]

for every integer \( j \geq 0 \) and every \( z_{0} > 0 \), with a constant \( C_{j} > 0 \) depending on \( j \) and \( z_{0} \) but independent of \( z \). We can write \( K_{1} = K_{1}^{+} + K_{1}^{-} \), where \( K_{1}^{\pm} \) is defined by replacing in (2.4) the function \( J_{\nu}(z) \) by \( e^{\pm izb^{\pm}_{\nu}(z)} \). We have

\[ J_{\nu}^{(k)}(z) = \sum_{\pm} e^{\pm izb^{\pm}_{\nu,k}(z)} \quad (2.12) \]

with functions \( b^{\pm}_{\nu,k} \) satisfying (2.11). Thus, by (2.6), (2.7), (2.11) and (2.12) we get

\[ |K_{1}^{\pm}(t, \sigma)| \leq C_{m}|t|^{-m-1/2} \sigma^{-(n-1)/2+m}, \quad \forall t \neq 0, \sigma \geq 1, \quad (2.13) \]

for all integers \( m \geq 0 \) with a constant \( C_{m} > 0 \) independent of \( t \) and \( \sigma \). Obviously, (2.13) holds true with \( m = s \) for all real \( 0 \leq s \leq (n - 1)/2 \), which in turn implies (2.5) with \( h = 1 \) in this case.

Given a set \( \mathcal{M} \subset \mathbb{R}^{n} \) denote by \( \eta(\mathcal{M}) \) the characteristic function of \( \mathcal{M} \). We have

\[ \left\| \langle x \rangle^{-s} e^{itG_{0}} \psi(h^{2}G_{0}) \langle x \rangle^{-s} \right\|_{L^{2} \rightarrow L^{2}} \leq \left\| \eta(|x| \leq \gamma|t|/2h) e^{itG_{0}} \psi(h^{2}G_{0}) \eta(|x| \leq \gamma|t|/2h) \right\|_{L^{2} \rightarrow L^{2}} + C \langle t/h \rangle^{-s}, \quad (2.14) \]

where \( \gamma > 0 \) is a constant to be fixed below. In view of Schur’s lemma the norm in the RHS of (2.14) is upper bounded by

\[ \sup_{|x| \leq \gamma|t|/2h} \int_{|y| \leq \gamma|t|/2h} |K_{h}(|x - y|, t)|dy \leq \int_{|\xi| \leq \gamma|t|/h} |K_{h}(|\xi|, t)|d\xi \leq C \int_{0}^{\gamma|t|/h} \sigma^{n-1} |K_{h}(\sigma, t)| d\sigma. \quad (2.15) \]
Thus, to prove (2.2) it suffices to show that the integral in the RHS of (2.15) is upper bounded by $O_m(t/h)^{-m}$ for all integers $m \geq 0$. In view of (2.4) this would follow from the bound
\[
\int_0^{\gamma|t|} \sigma^{n-1} |K_1(\sigma, t)| \, d\sigma \leq C_m\langle t \rangle^{-m}, \tag{2.16}
\]
provided $\gamma$ is properly chosen. Write the function $K_1^\pm$ in the form
\[
K_1^\pm(\sigma, t) = \frac{\sigma^{-2\nu} e^{-i\sigma^2/4t}}{(2\pi)^{\nu+1}} \int_0^{\infty} e^{it(\lambda \pm \sigma/2t)^2} \varphi(\lambda) b_{\nu}^\pm(\sigma \lambda) \, d\lambda, \tag{2.17}
\]
where $\varphi(\lambda) = \lambda \psi(\lambda^2)$. Clearly, we can fix now $\gamma > 0$ (depending on $\text{supp} \varphi$) so that $|\lambda \pm \sigma/2t| \geq \text{Const} > 0$ on $\text{supp} \varphi$ for $\sigma \leq \gamma|t|$. Therefore, integrating by parts $m$ times in (2.17) and using (2.11), one can easily obtain the bound
\[
\left| K_1^\pm(\sigma, t) \right| \leq C_m|t|^{-m}, \quad 1 \leq \sigma \leq \gamma|t|, \tag{2.18}
\]
for every integer $m \geq 0$. On the other hand, in the same way as in the proof of (2.10), one concludes that $K_1$ satisfies (2.18) for $0 < \sigma \leq 1$ as well. This clearly implies (2.16), and hence (2.2).

We will also need the following lemma proved in [9] (see also [8]).

**Lemma 2.2** Assume (1.1) fulfilled. Then, for every $\psi \in C_0^\infty((0, +\infty))$, $0 \leq s \leq \delta$, $1 \leq p \leq \infty$, $0 < h \leq 1$, we have
\[
\left\| \langle x \rangle^{-s} \psi(h^2 G_0) \langle x \rangle^s \right\|_{L^2 \to L^2} \leq C, \tag{2.19}
\]
\[
\left\| \langle x \rangle^{-s} \psi(h^2 G) \langle x \rangle^s \right\|_{L^2 \to L^2} \leq C, \tag{2.20}
\]
\[
\left\| \left( \psi(h^2 G_0) - \psi(h^2 G) \right) \langle x \rangle^s \right\|_{L^2 \to L^2} \leq C h^2, \tag{2.21}
\]
\[
\left\| \psi(h^2 G_0) \right\|_{L^p \to L^p} \leq C, \tag{2.22}
\]
\[
\left\| \psi(h^2 G) \right\|_{L^p \to L^p} \leq C, \tag{2.23}
\]
\[
\left\| \psi(h^2 G) - \psi(h^2 G_0) \right\|_{L^p \to L^p} \leq C h^2, \tag{2.24}
\]
\[
\left\| \psi(h^2 G_0) \right\|_{L^2 \to L^p} \leq C h^{-n\left|\frac{1}{2} - \frac{1}{p}\right|}, \tag{2.25}
\]
\[
\left\| \psi(h^2 G) \right\|_{L^2 \to L^p} \leq C h^{-n\left|\frac{1}{2} - \frac{1}{p}\right|}, \tag{2.26}
\]
\[
\left\| \left( \psi(h^2 G) - \psi(h^2 G_0) \right) \langle x \rangle^s \right\|_{L^2 \to L^p} \leq C h^{-n\left|\frac{1}{2} - \frac{1}{p}\right|}, \tag{2.27}
\]
with a constant $C > 0$ independent of $h$. 

5
3 $L^2 \to L^2$ estimates for the Schrödinger group

Given a parameter $0 < h \leq 1$, and a real-valued function $\psi \in C_0^\infty((0, +\infty))$, denote

$$
\Psi(t; h) = e^{itG}\psi(h^2G) - e^{itG_0}\psi(h^2G_0).
$$

We will first prove the following

**Theorem 3.1** Assume (1.1) fulfilled. Then, we have

$$
\|\Psi(t; h)\|_{L^2 \to L^2} \leq Ch, \quad \forall t, \ 0 < h \leq 1,
$$

with a constant $C > 0$ independent of $t$ and $h$.

**Proof.** We will derive (3.1) from the following

**Proposition 3.2** For every $s > 1/2$, $0 < h \leq 1$, we have

$$
\int_{-\infty}^{\infty} \|\langle x \rangle^{-s} e^{itG}\psi(h^2G)f\|^2_{L^2} \ dt \leq Ch\|f\|^2_{L^2}, \quad \forall f \in L^2,
$$

with a constant $C > 0$ independent of $f$ and $h$.

Let $\psi_1 \in C_0^\infty((0, +\infty))$ be a real-valued function such that $\psi_1 \psi \equiv \psi$. By Duhamel’s formula we obtain the identity

$$
\Psi(t; h) = \sum_{j=1}^{2} \Psi_j(t; h),
$$

where

$$
\Psi_1(t; h) = \left(\psi_1(h^2G) - \psi_1(h^2G_0)\right)\Psi(t; h)
$$

$$
+ \left(\psi_1(h^2G) - \psi_1(h^2G_0)\right)e^{itG_0}\psi(h^2G_0) - \psi_1(h^2G_0)e^{itG_0}\left(\psi(h^2G) - \psi(h^2G_0)\right),
$$

$$
\Psi_2(t; h) = i\int_0^t e^{i(t-\tau)G_0}\psi_1(h^2G_0)Ve^{i\tau G}\psi(h^2G)d\tau.
$$

In view of (2.21) we have

$$
\|\Psi_1(t; h)\|_{L^2 \to L^2} \leq Ch^2.
$$

For all nontrivial $f, g \in L^2$, we have with $0 < s - 1/2 \ll 1$, $\forall \gamma > 0$,

$$
\left|\langle \Psi_2(t; h)f, g \rangle\right| \leq \int_{-\infty}^{\infty} \left|\langle \langle x \rangle^sVe^{i\tau G}\psi(h^2G)f, \langle x \rangle^{-s}e^{i(t-\tau)G_0}\psi_1(h^2G_0)g \rangle\right| d\tau
$$

$$
\leq C\gamma \int_{-\infty}^{\infty} \|\langle x \rangle^{-s}e^{i\tau G}\psi(h^2G)f\|^2_{L^2} d\tau + C\gamma^{-1} \int_{-\infty}^{\infty} \|\langle x \rangle^{-s}e^{i\tau G_0}\psi_1(h^2G_0)g\|^2_{L^2} d\tau
$$

$$
\leq Ch\gamma\|f\|^2_{L^2} + Ch\gamma^{-1}\|g\|^2_{L^2} \leq O(h)\|f\|_{L^2}\|g\|_{L^2},
$$

if we choose $\gamma = \|g\|_{L^2}/\|f\|_{L^2}$, which implies (3.1).
Proof of Proposition 3.2. Denote by $\mathcal{H}$ the Hilbert space $L^2(\mathbb{R}; L^2)$. Clearly, (3.2) is equivalent to the fact that the operator $A_h : L^2 \to \mathcal{H}$ defined by

$$(A_h f)(x, t) = \langle x \rangle^{-s} e^{itG} \psi(h^2 G) f$$

is bounded with norm $O(h^{1/2})$. Observe that the adjoint $A_h^* : \mathcal{H} \to L^2$ is defined by

$$A_h^* f = \int_{-\infty}^{\infty} e^{-i\tau G} \psi(h^2 G) \langle x \rangle^{-s} f(\tau, x) d\tau,$$

so we have, for every $f, g \in \mathcal{H}$,

$$\langle A_h A_h^* f, g \rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} \langle \rho(t, \cdot), g(t, \cdot) \rangle_{L^2} dt,$$

where

$$\rho(t, x) = \int_{-\infty}^{\infty} \langle x \rangle^{-s} e^{i(t-\tau)G} \psi(h^2 G) \langle x \rangle^{-s} f(\tau, \cdot) d\tau.$$

Hence, for the Fourier transform, $\hat{\rho}(\lambda, x)$, of $\rho(t, x)$ with respect to the variable $t$ we have

$$\hat{\rho}(\lambda, x) = Q(\lambda) \hat{f}(\lambda, x),$$

where $Q(\lambda)$ is the Fourier transform of the operator

$$\langle x \rangle^{-s} e^{itG} \psi^2(h^2 G) \langle x \rangle^{-s}.$$

On the other hand, the formula

$$e^{itG} \psi^2(h^2 G) = \frac{1}{\pi i} \int_{-\infty}^{\infty} e^{it \lambda^2} \psi(h^2 \lambda^2) (R^+(\lambda) - R^-(\lambda)) \lambda d\lambda,$$

where

$$R^\pm(\lambda) = \lim_{\varepsilon \to 0^+} (G - \lambda^2 \pm i\varepsilon)^{-1} : \langle x \rangle^{-s} L^2 \to \langle x \rangle^s L^2, \quad s > 1/2,$$

shows that

$$Q(\lambda) = (2\pi i)^{-1} \psi^2(h^2 \lambda) \langle x \rangle^{-s} (R^+(\sqrt{\lambda}) - R^-(\sqrt{\lambda})) \langle x \rangle^{-s}. $$

Note that the limit exists in view of the limiting absorption principle. Moreover, we have the estimate (e.g. see Lemma 3.3 of [9])

$$\|\langle x \rangle^{-s} R^\pm(\lambda) \langle x \rangle^{-s}\|_{L^2 \to L^2} \leq C\lambda^{-1}, \quad \lambda \geq \lambda_0,$$

for every $s > 1/2$, $\lambda_0 > 0$, with a constant $C > 0$ independent of $\lambda$. By (3.9) and (3.10) we conclude

$$\|Q(\lambda)\|_{L^2 \to L^2} \leq Ch,$$

with a constant $C > 0$ independent of $\lambda$ and $h$. By (3.7) and (3.11),

$$\|\hat{\rho}(\lambda, \cdot)\|_{L^2} \leq Ch\|\hat{f}(\lambda, \cdot)\|_{L^2},$$

which together with (3.6) leads to

$$|\langle A_h A_h^* f, g \rangle_{\mathcal{H}}| = \left| \int_{-\infty}^{\infty} \langle \hat{\rho}(\lambda, \cdot), \hat{g}(\lambda, \cdot) \rangle_{L^2} d\lambda \right|$$
if we take $\gamma_A$ (3.13) that the operator $A > 0$ independent of $h$, $f$ and $g$. It follows from (3.13) that the operator $A_h A_h^* : \mathcal{H} \rightarrow \mathcal{H}$ is bounded with norm $O(h)$, and hence the operator $A_h : L^2 \rightarrow \mathcal{H}$ is bounded with norm $O(h^{1/2})$.

In what follows in this section we will prove the following

**Theorem 3.3** Assume (1.1) fulfilled. Then, for every real-valued function $\psi \in C^\infty_0(\mathbb{R})$ and every $0 \leq s \leq n/2$, $0 < \epsilon \ll 1$, we have

$$\left\| \langle x \rangle^{-s-\epsilon} e^{itG} \psi(h^2G) \langle x \rangle^{-s-\epsilon} \right\|_{L^2 \rightarrow L^2} \leq C\langle t/h \rangle^{-s}, \quad \forall t, 0 < h \leq 1,$$

with a constant $C > 0$ independent of $t$ and $h$.

**Proof.** We will derive (3.14) from the following estimates

**Proposition 3.4** Assume (1.1) fulfilled. Then, for every real-valued function $\psi \in C^\infty_0(\mathbb{R})$ and every $0 \leq s \leq n/2$, $0 < \epsilon \ll 1$, $0 < h \leq 1$, we have

$$\left\| \langle x \rangle^{-1/2-s-\epsilon} e^{itG} \psi(h^2G) \langle x \rangle^{-1/2-s-\epsilon} \right\|_{L^2 \rightarrow L^2} \leq C\langle t \rangle^{-s}, \quad \forall t,$$

$$\int_{-\infty}^{\infty} \langle t/h \rangle^{2s} \left\| \langle x \rangle^{-1/2-s-\epsilon} e^{itG} \psi(h^2G) \langle x \rangle^{-1/2-s-\epsilon} f \right\|_{L^2}^2 dt \leq C h \|f\|_{L^2}, \quad \forall f \in L^2,$$

with a constant $C > 0$ independent of $t$, $h$ and $f$.

By a standard interpolation argument (e.g. see the proof of Theorem 1.2 of [3] or the proof of Theorem 3.4 of [9]) one can easily conclude that it suffices to prove (3.14) with $s = n/2$, only. On the other hand, in view of (2.2), it suffices to prove (3.14) for the difference $\Psi(t; h)$. To do so, we will make use of (3.3). Using (1.1), (2.2) and (2.21), we obtain,

$$\left\| \langle x \rangle^{-s-\epsilon} \Psi_1(t; h) \langle x \rangle^{-s_1-\epsilon} f \right\|_{L^2} \leq O(h^2) \left\| \langle x \rangle^{-(n+2)/2-\epsilon} \Psi(t; h) \langle x \rangle^{-s_1-\epsilon} f \right\|_{L^2} + O(h^2) \langle t/h \rangle^{-s} \|f\|_{L^2}, \quad \forall f \in L^2,$$

for all $s_1 \geq s \geq 0$. Using (1.1), (2.2) and (3.16), we obtain, $\forall f, g \in L^2$, with $s = n/2$,

$$\langle t/h \rangle^s \left\| \langle x \rangle^{-s-\epsilon} \Psi_2(t; h) \langle x \rangle^{-s_1-1/2-\epsilon} f, g \right\| \leq \langle t/h \rangle^s \int_0^t \left\| \langle x \rangle^{-s-\epsilon} e^{i(t-\tau)G_0} \Psi_1(h^2G_0) V e^{i\tau G} \psi(h^2G) \langle x \rangle^{-s_1-1/2-\epsilon} f, g \right\| d\tau \leq C \int_0^{t/2} \langle t - \tau/h \rangle^s \left\| \langle x \rangle^{-n/2-\epsilon} e^{-i(t-\tau)G_0} \Psi_1(h^2G_0) \langle x \rangle^{-s-\epsilon} g \right\|_{L^2}.$$
if we choose $\gamma < h$

Hence, there exists a constant $0 < h$ in the RHS of (3.19), thus obtaining the estimate (for $0 < h \leq 1$)

$$
\| \langle x \rangle^{-s-\epsilon} e^{itG} \psi(h^2 G) \langle x \rangle^{-s-1/2-\epsilon} f \|_{L^2} \leq C h \gamma \| g \|_{L^2} + C h \gamma^{-1} \| f \|_{L^2} = 2 C h \| f \|_{L^2} \| g \|_{L^2},
$$

(3.18)

if we choose $\gamma = \| f \|_{L^2}/\| g \|_{L^2}$. By (3.17) and (3.18), we have (with $s = n/2$)

$$
\| \langle x \rangle^{-s-\epsilon} \Psi(t; h) \langle x \rangle^{-s-1/2-\epsilon} f \|_{L^2} \leq O(h^2) \| \langle x \rangle^{-(n+2)/2-\epsilon} \Psi(t; h) \langle x \rangle^{-s-1/2-\epsilon} f \|_{L^2} + O(h \| t/h \|^{-s}) \| f \|_{L^2}.
$$

(3.19)

Hence, there exists a constant $0 < h_0 < 1$ so that if $0 < h \leq h_0$, we can absorb the first term in the RHS of (3.19), thus obtaining the estimate (for $0 < h \leq h_0$)

$$
\| \langle x \rangle^{-s-\epsilon} \Psi(t; h) \langle x \rangle^{-s-1/2-\epsilon} f \|_{L^2} \leq C h \langle t/h \rangle^{-s}.
$$

(3.20)

Let now $h_0 \leq h \leq 1$. Without loss of generality we may suppose $h = 1$. By (2.2) and (3.15), the norm in the first term in the RHS of (3.19) is upper bounded by $C \langle t \rangle^{-s} \| f \|_{L^2}$, which again implies (3.20). By (2.2) and (3.20), we conclude

$$
\| \langle x \rangle^{-s-\epsilon} e^{itG} \psi(h^2 G) \langle x \rangle^{-1/2-s-\epsilon} \|_{L^2} \leq C \langle t/h \rangle^{-s},
$$

(3.21)

with $s = n/2$, and hence with all $0 \leq s \leq n/2$. To show that this implies (3.14) with $s = n/2$, we will proceed in the same way as in Section 3 of [9]. Let $r = |x|$ denote the radial variable and set $D_r = \langle r \rangle^{-1} r h \partial_r$. It is easy to see that (3.21) implies

$$
\| \langle x \rangle^{-s-\epsilon} D_r e^{itG} \psi(h^2 G) \langle x \rangle^{-1/2-s-\epsilon} \|_{L^2} \leq C \langle t/h \rangle^{-s},
$$

(3.22)

for all $0 \leq s \leq n/2$. Furthermore, using Duhamel's formula together with the identity

$$
-2\Delta + [r \partial_r, \Delta] = 0,
$$

we obtain

$$
\psi_1(h^2 G) [r \partial_r, e^{itG}] \psi(h^2 G) = \int_0^t \psi_1(h^2 G) e^{i(t-\tau)G} [r \partial_r, G] e^{itG} \psi(h^2 G) d\tau
$$
where the functions $\psi$ and $\psi_1$ are as above. Set $\tilde{\psi}_1(\sigma) = \sigma^{-1}\psi_1(\sigma)$. From the above identity we get (with $s = n/2$)
\[
2(t/h)(x)^{-s-\epsilon} e^{itG} \psi(h^2G)(x)^{-s-\epsilon} \\
= \left(\langle x \rangle^{-s-\epsilon} \tilde{\psi}_1(h^2G)(x)^{s+\epsilon}\right)\langle x \rangle^{-s+1-\epsilon} \mathcal{D}_r e^{itG} \psi(h^2G)(x)^{-s-\epsilon} \\
+ \langle x \rangle^{-s-\epsilon} \tilde{\psi}_1(h^2G) e^{itG} (D_r^s + O(h)) \langle x \rangle^{-s+1-\epsilon} \left(\langle x \rangle^{s+\epsilon} \psi(h^2G)(x)^{-s-\epsilon}\right) \\
+ h \int_0^\infty \langle x \rangle^{-s-\epsilon} \tilde{\psi}_1(h^2G) e^{i(t-r)G} V e^{irG} \psi(h^2G)(x)^{-s-\epsilon} d\tau \\
+ \int_0^\infty \langle x \rangle^{-s-\epsilon} \tilde{\psi}_1(h^2G) e^{i(t-r)G} (D_r^s + O(h)) \langle r \rangle V e^{irG} \psi(h^2G)(x)^{-s-\epsilon} d\tau \\
+ \int_0^\infty \langle x \rangle^{-s-\epsilon} \tilde{\psi}_1(h^2G) e^{i(t-r)G} \mathcal{D}_r e^{irG} \psi(h^2G)(x)^{-s-\epsilon} d\tau. \tag{3.23}
\]
By (2.20), (3.21) and (3.22), we have that the L$^2$ norm of each of the first two terms in the RHS of (3.23) is upper bounded by $O(\langle t/h \rangle^{-s+1})$. Similarly, in view of (1.1), we also have that the $L^2$ norm of each integral in the RHS of (3.23) is upper bounded by $O(\langle t/h \rangle^{-s+1/2})$. Therefore, (3.14) with $s = n/2$ follows from (3.23) together with the estimates (3.21) and (3.22).

**Proof of Proposition 3.4.** We will derive (3.15) from the following lemma which can be proved in precisely the same way as Lemma 3.6 of [9].

**Lemma 3.5** Assume (1.1) fulfilled and let $0 \leq s \leq n/2$. Let also $m \geq 0$ denote the biggest integer $\leq s$ and set $\mu = s - m$. Then, the operator-valued function
\[
\mathcal{R}^\pm_s(\lambda) = \lambda \langle x \rangle^{-1/2-s-\epsilon} R^\pm_s(\lambda) \langle x \rangle^{-1/2-s-\epsilon} : L^2 \to L^2
\]
is $C^m$ in $\lambda$ for $\lambda > 0$, $\partial^m_\lambda \mathcal{R}^\pm_s$ is Hölder of order $\mu$, and satisfies the estimates
\[
\|\partial^j_\lambda \mathcal{R}^\pm_s(\lambda)\|_{L^2 \to L^2} \leq C, \quad \lambda \geq \lambda_0, \; 0 \leq j \leq m, \tag{3.24}
\]
\[
\|\partial^m_\lambda \mathcal{R}^\pm_s(\lambda_2) - \partial^m_\lambda \mathcal{R}^\pm_s(\lambda_1)\|_{L^2 \to L^2} \leq C|\lambda_2 - \lambda_1|^\mu, \quad \lambda_2 > \lambda_1 \geq \lambda_0, \tag{3.25}
\]
for every $\lambda_0 > 0$, with a constant $C > 0$ independent of $\lambda, \lambda_1$ and $\lambda_2$.

We are going to take advantage of the formula (3.8) with $h = 1$ and $\psi^2$ replaced by $\psi$. Set
\[
T(\lambda) = T^+(\lambda) - T^-(\lambda), \quad T^\pm(\lambda) = (2\pi i)^{-1}\lambda^{-1/2} \mathcal{R}^\pm_s(\sqrt{\lambda}), \tag{3.26}
\]
and choose a real-valued function $\phi \in C^\infty_0([1/3, 1/2]), \phi \geq 0$, such that $\int \phi(\sigma) d\sigma = 1$. Then the function
\[
T^\pm_{\theta}(\lambda) = \theta^{-1} \int T^\pm(\lambda + \sigma) \phi(\sigma/\theta) d\sigma, \quad 0 < \theta \leq 1,
\]
is $C^\infty$ in $\lambda$ with values in $L(L^2)$ and, in view of (3.24) and (3.25), satisfies the estimates
\[
\|\partial_\lambda^jT^\pm_\theta(\lambda)\|_{L^2 \to L^2} \leq C\lambda^{-(j+1)/2}, \quad 0 \leq j \leq m, \tag{3.27}
\]
\[
\|\partial_\lambda^jT^\pm_\theta(\lambda) - \partial_\lambda^jT^\pm_\theta(\lambda)\|_{L^2 \to L^2} \leq \theta^{-1} \int \|\partial_\lambda^jT^\pm_\theta(\lambda + \sigma) - \partial_\lambda^jT^\pm_\theta(\lambda)\|_{L^2 \to L^2}\phi(\sigma/\theta)d\sigma \leq C\lambda^{-(-m+1)/2}\theta^m, \tag{3.28}
\]
\[
\|\partial_\lambda^mT^\pm_\theta(\lambda)\|_{L^2 \to L^2} \leq \theta^{-1} \int \|\partial_\lambda^mT^\pm_\theta(\lambda + \sigma) - \partial_\lambda^mT^\pm_\theta(\lambda)\|_{L^2 \to L^2}\phi(\sigma/\theta)d\sigma \leq C\lambda^{-(-m+1)/2}\theta \tag{3.29}
\]
\[
\|\partial_\lambda^{m+1}T^\pm_\theta(\lambda)\|_{L^2 \to L^2} \leq \theta^{-2} \int \|\partial_\lambda^{m}T^\pm_\theta(\lambda + \sigma) - \partial_\lambda^{m}T^\pm_\theta(\lambda)\|_{L^2 \to L^2}\phi(\sigma/\theta)d\sigma \leq C\lambda^{-(-m+2)/2}\theta^2 \tag{3.30}
\]
Integrating by parts $m$ times and using (3.28) and (3.29), we get
\[
\left\|\int_0^\infty e^{it\lambda}\psi(\lambda)\left(T^\pm_\theta(\lambda) - T^\pm(\lambda)\right) d\lambda\right\|_{L^2 \to L^2} = \left\|t^{-m}\int_0^\infty e^{it\lambda}\frac{d^m}{d\lambda^m}\left(\psi(\lambda)\left(T^\pm_\theta(\lambda) - T^\pm(\lambda)\right)\right) d\lambda\right\|_{L^2 \to L^2} \leq C\theta^m|t|^{-m}. \tag{3.31}
\]
Similarly, integrating by parts $m + 1$ times and using (3.27) and (3.30), we get
\[
\left\|\int_0^\infty e^{it\lambda}\psi(\lambda)T^\pm_\theta(\lambda)d\lambda\right\|_{L^2 \to L^2} = \left\|t^{-(m+1)}\int_0^\infty e^{it\lambda}\frac{d^{m+1}}{d\lambda^{m+1}}\left(\psi(\lambda)T^\pm_\theta(\lambda)\right) d\lambda\right\|_{L^2 \to L^2} \leq C\theta^{-1+\mu}|t|^{-m-1}. \tag{3.32}
\]
By (3.26), (3.31) and (3.32),
\[
\left\|\int_0^\infty e^{it\lambda}\psi(\lambda)T(\lambda)d\lambda\right\|_{L^2 \to L^2} \leq C\theta^\mu|t|^{-m}(1 + |t|^{-1}\theta^{-1}) \leq C|t|^{-m-\mu}, \tag{3.33}
\]
if we take $\theta = |t|^{-1}$, which clearly implies (3.15).

In what follows in this section we will derive (3.16) from Lemma 3.5. Let $0 \leq s \leq n/2$ and let $m \geq 0$ be the bigest integer $\leq s$. Remark that the function $\partial_\lambda^mR^\pm_\theta$ satisfies (3.25) with $\mu = s - m + \epsilon/2$. Consequently, the estimates (3.28) and (3.30) are valid with $\mu = s - m + \epsilon/2$. Let $\phi_+ \in C^\infty(\mathbb{R})$, $\phi_+(t) = 0$ for $t \leq 1$, $\phi_+(t) = 1$ for $t \geq 2$. Given any function $f \in L^2$, set
\[
u(t; h) = \langle x \rangle^{-1/2-s-\epsilon}\phi_+(t/h)\theta^\epsilon\varphi(h^2G)\langle x \rangle^{-1/2-s-\epsilon}f.
\]
We have
\[
(\partial_t - iG)\langle x \rangle^{1/2+s+\epsilon}\nu(t; h)
\]
\[
h^{-1}\phi_+(t/h)\theta^\epsilon\varphi(h^2G)\langle x \rangle^{-1/2-s-\epsilon}f = h^{-1}\langle x \rangle^{-1/2-s-\epsilon}\nu(t; h). \tag{3.34}
\]
Clearly, the support of the function \(v(t; h)\) with respect to the variable \(t\) is contained in the interval \([h, 2h]\), and by (2.20) we have
\[
\|v(t; h)\|_{L^2} \leq C\|f\|_{L^2}, \quad 1 \leq t/h \leq 2, \tag{3.35}
\]
with a constant \(C > 0\) independent of \(t, h\) and \(f\). Using Duhamel's formula we deduce from (3.34),
\[
u(t; h) = h^{-1} \int_0^t \langle x \rangle^{-1/2-s-\epsilon} \psi_1(h^2G)e^{it(t-x)G} \langle x \rangle^{-1/2-s-\epsilon} v(\tau; h) \, d\tau,
\tag{3.36}
\]
where the function \(\psi_1\) is as above. It follows from (3.36) that the Fourier transforms of the functions \(u(t; h)\) and \(v(t; h)\) satisfy the identity
\[
h^{-1} Q^+(\lambda) \hat{v}(\lambda; h), \tag{3.37}
\]
where \(Q^+(\lambda)\) is the Fourier transform of the operator
\[
\langle x \rangle^{-1/2-s-\epsilon} \psi_1(h^2G) \eta_+ \langle t \rangle e^{itG} \langle x \rangle^{-1/2-s-\epsilon},
\]
\(\eta_+\) being the characteristic function of the interval \([0, +\infty)\). It is easy to see that
\[
Q^+(\lambda) = (2\pi i)^{-1} \langle x \rangle^{-1/2-s-\epsilon} \psi_1(h^2G) R^+(\sqrt{\lambda} \langle x \rangle^{-1/2-s-\epsilon} = B(h) T^+(\lambda),
\tag{3.38}
\]
where the operator
\[
B(h) = \langle x \rangle^{-1/2-s-\epsilon} \psi_1(h^2G) \langle x \rangle^{1/2+s+\epsilon} : L^2 \to L^2
\]
is bounded uniformly in \(h\) in view of (2.20). Fix a constant \(0 < \gamma < 1\) such that \(\text{supp} \psi_1 \subset (\gamma, \gamma^{-1})\). Then, for \(\lambda h^2 \in \mathbb{R} \setminus (\gamma, \gamma^{-1})\), we have
\[
\left\| \frac{d^k Q^+(\lambda)}{d\lambda^k} \right\|_{L^2 \to L^2} \leq C \left\| \psi_1(h^2G)(G - \lambda)^{-k-1} \right\|_{L^2 \to L^2}
\leq C \sup_{\sigma \in \mathbb{R}} \left| \psi_1(h^2\sigma)(\sigma - \lambda)^{-k-1} \right| \leq C h^{2k+2}, \tag{3.39}
\]
for every integer \(k \geq 0\) with a constant \(C_k > 0\) independent of \(\lambda\) and \(h\). Set \(Q^+_\theta(\lambda) = B(h) T^+_\theta(\lambda)\) and define the function \(u_\theta(t; h)\) via the formula
\[
\hat{u}_\theta(\lambda; h) = Q^+_\theta(\lambda) \hat{v}(\lambda; h).
\]
Using (3.27)-(3.30) when \(\lambda h^2 \in (\gamma, \gamma^{-1})\) and (3.39) when \(\lambda h^2 \in \mathbb{R} \setminus (\gamma, \gamma^{-1})\), we obtain
\[
\left\| \partial^j_{x} Q^+_\theta(\lambda) \right\|_{L^2 \to L^2} \leq C h^{j+1}, \quad 0 \leq j \leq m, \tag{3.40}
\]
\[
\left\| \partial^m_{x} Q^+_\theta(\lambda) - \partial^m_{x} Q^+(\lambda) \right\|_{L^2 \to L^2} \leq \theta^{-1} \int \left\| \partial^m_{x} Q^+(\lambda + \sigma) - \partial^m_{x} Q^+(\lambda) \right\|_{L^2 \to L^2} \phi(\sigma/\theta) \, d\sigma
\leq C h^{m+1} \theta^{-1} \int \sigma^m \phi(\sigma/\theta) \, d\sigma \leq C h^{m+1} \theta^m, \tag{3.41}
\]
\[
\left\| \partial^j_{x} \partial^m_{x} Q^+_\theta(\lambda) - \partial^j_{x} Q^+(\lambda) \right\|_{L^2 \to L^2} \leq C h^{j+1} \theta, \quad 0 \leq j \leq m - 1, \tag{3.42}
\]

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In the same way, we obtain
\[ \int_{-\infty}^{\infty} |t|^{2m} \| v(t; h) \|_{L^2}^2 dt = \int_{-\infty}^{\infty} \| \partial_{\lambda}^{m-k} \tilde{v}(\lambda; h) \|_{L^2}^2 d\lambda \]
\[ \leq C h^{2m+2} \theta^{-2+2\mu} \sum_{k=0}^{m+1} \int_{-\infty}^{\infty} |t|^{2m+2-2k} \| v(t; h) \|_{L^2}^2 dt \]
\[ \leq C h^{2m+3} \theta^{-2+2\mu} \| f \|_{L^2}^2, \]
with a constant \( C > 0 \) independent of \( h, \theta \) and \( f \). By (3.44) we get, \( \forall A \geq 1, \)
\[ \int_{A}^{2A} \| u_\theta(t; h) - u(t; h) \|_{L^2}^2 dt \leq C h A^{-2m} \theta^{-2+2\mu} \| f \|_{L^2}^2. \]
Summing up (3.49) leads to
\[
\int_h^\infty (t/h)^{2s} \|u(t; h)\|^2_{L^2} dt \leq C h \|f\|^2_{L^2}, \tag{3.50}
\]
which clearly implies (3.16).

## 4 Proof of Theorem 1.1

We will first prove the following

**Proposition 4.1** For every \(0 < \epsilon \ll 1, \frac{1}{2} - \epsilon / 2 \leq s \leq (n - 1)/2, 0 < h \leq 1, t \neq 0\), we have
\[
\left\| \Psi(t; h) \langle x \rangle^{-s-1/2-\epsilon} \right\|_{L^2 \to L^\infty} \leq C h^{-(n-3)/2+s-\epsilon} |t|^{-s-1/2}, \tag{4.1}
\]
with a constant \(C > 0\) independent of \(t\) and \(h\).

**Proof.** By (2.1) and (2.24), we have
\[
\left\| \Psi_1(t; h) \langle x \rangle^{-s-1/2-\epsilon} f \right\|_{L^\infty} \leq O(h^2) \left\| \Psi(t; h) \langle x \rangle^{-s-1/2-\epsilon} f \right\|_{L^\infty} + C h^{-(n-3)/2+s-\epsilon} |t|^{-s-1/2} \|f\|_{L^2}, \quad \forall f \in L^2. \tag{4.2}
\]
Using (2.1) and (3.14), we obtain
\[
\left\| \Psi_2(t; h) \langle x \rangle^{-s-1/2-\epsilon} \right\|_{L^2 \to L^\infty} \leq C \int_0^{t/2} \left\| e^{i(t-\tau)G_0} \psi_1(h^2 G_0) \langle x \rangle^{-s-1/2-\epsilon} \right\|_{L^2 \to L^\infty} \left\| \langle x \rangle^{-1-\epsilon} e^{i\tau G} \psi(h^2 G) \langle x \rangle^{-1/2} \right\|_{L^2 \to L^2} d\tau
+ C \int_0^{t/2} \left\| e^{i\tau G_0} \psi(h^2 G_0) \langle x \rangle^{-1-\epsilon} \right\|_{L^2 \to L^\infty} \left\| \langle x \rangle^{-1/2-s-\epsilon} e^{i(t-\tau)G} \psi(h^2 G) \langle x \rangle^{-1/2-s} \right\|_{L^2 \to L^2} d\tau
\leq C h^{s-(n-1)/2} |t|^{-s-1/2} \int_0^{t/2} \langle \tau/h \rangle^{-1-\epsilon} d\tau + C h^{-(n-2)/2-s} \langle t/h \rangle^{-s-1/2} \int_0^{t/2} |\tau|^{-1+\epsilon/2} (\tau)^{-\epsilon} d\tau
\leq C h^{s-(n-3)/2-\epsilon} |t|^{-s-1/2}. \tag{4.3}
\]
By (4.2) and (4.3),
\[
\left\| \Psi(t; h) \langle x \rangle^{-s-1/2-\epsilon} f \right\|_{L^\infty} \leq O(h^2) \left\| \Psi(t; h) \langle x \rangle^{-s-1/2-\epsilon} f \right\|_{L^\infty} + C h^{-(n-3)/2+s-\epsilon} |t|^{-s-1/2} \|f\|_{L^2}, \quad \forall f \in L^2. \tag{4.4}
\]
Hence, there exists a constant \(0 < h_0 < 1\) so that for \(0 < h \leq h_0\) we can absorb the first term in the RHS of (4.4), thus obtaining (4.1) in this case. Let now \(h_0 \leq h \leq 1\). Without loss of generality we may suppose \(h = 1\). Then the only term we need to estimate is
\[
\left\| (\psi_1(G) - \psi_1(G_0)) e^{\mu G} \psi(G) \langle x \rangle^{-s-1/2-\epsilon} f \right\|_{L^\infty}. \tag{4.5}
\]
Proposition 4.2 In view of (2.27), this is reduced to estimating
\[ \left\| \langle x \rangle^{-(n+2)/2-\epsilon} e^{itG} \psi(G) \langle x \rangle^{-s-1/2-\epsilon} f \right\|_{L^2}, \]
which, in view of Theorem 3.3, is upper bounded by \( O(|t|^{-s-1/2}) \|f\|_{L^2} \).

Write \( \Psi_2 = \Psi_3 + \Psi_4 \), where
\[
\Psi_3(t; h) = i \int_0^t e^{i(t-\tau)G_0} \psi_1(h^2G_0)Ve^{i\tau G_0} \psi(h^2G_0) d\tau,
\]
\[
\Psi_4(t; h) = i \int_0^t e^{i(t-\tau)G_0} \psi_1(h^2G_0) V\Psi(\tau; h) d\tau.
\]

Proposition 4.2 For every \( 0 < \epsilon \ll 1 \), \( 0 < h \leq 1 \), \( t \neq 0 \), we have
\[
\|\Psi_1(t; h)\|_{L^1 \rightarrow L^\infty} \leq Ch^{2-n/2} |t|^{-n/2},
\]
(4.5)
\[
\|\Psi_4(t; h)\|_{L^1 \rightarrow L^\infty} \leq Ch^{2-n/2-\epsilon} |t|^{-n/2},
\]
(4.6)
with a constant \( C > 0 \) independent of \( t \) and \( h \).

Proof. By (2.24), (2.27) and (4.1), we have
\[
\|\Psi_1(t; h)\|_{L^1 \rightarrow L^\infty} \leq Ch^{2-n/2} \left\| \langle x \rangle^{-(n+2)/2-\epsilon} \Psi(t; h) \right\|_{L^1 \rightarrow L^2}
\]
\[ + Ch^{2} |t|^{-n/2} \leq Ch^{3-n/2-\epsilon} |t|^{-n/2}, \]
which implies (4.5). By (2.1) and (4.1), we have
\[
\|\Psi_4(t; h)\|_{L^1 \rightarrow L^\infty}
\]
\[ \leq C \int_0^{t/2} \| e^{i(t-\tau)G_0} \psi_1(h^2G_0) \langle x \rangle^{-n/2-\epsilon} \|_{L^2 \rightarrow L^\infty} \left\| \langle x \rangle^{-1-\epsilon} e^{i\tau G} \psi(h^2G) \right\|_{L^1 \rightarrow L^2} d\tau \]
\[ + C \int_0^{t/2} \| e^{i\tau G_0} \psi_1(h^2G_0) \langle x \rangle^{-1-\epsilon} \|_{L^2 \rightarrow L^\infty} \left\| \langle x \rangle^{-2-\epsilon} e^{i(t-\tau)G} \psi(h^2G) \right\|_{L^1 \rightarrow L^2} d\tau \]
\[ \leq 2Ch^{-(n-4)/2-\epsilon} |t|^{-n/2} \int_0^{t/2} |\tau|^{-1+\epsilon/2} (\tau)^{-\epsilon} d\tau \leq C_\epsilon h^{-(n-4)/2-\epsilon} |t|^{-n/2}, \]
\( \forall 0 < \epsilon \ll 1 \), with a constant \( C_\epsilon > 0 \) independent of \( t \) and \( h \).

We will now derive Theorem 1.1 from the estimates (4.1), (4.5), (4.6) and the following

Proposition 4.3 For every \( 0 < h \leq 1 \), \( t \neq 0 \), we have
\[
\|\Psi_3(t; h)\|_{L^1 \rightarrow L^\infty} \leq Ch^{-(n-3)/2} |t|^{-n/2},
\]
(4.7)
with a constant \( C > 0 \) independent of \( t \) and \( h \). Moreover, the operator \( \Psi_3 \) is of the form
\[
\Psi_3(t; h) = E(t; h) + F(t; h),
\]
(4.8)
where the operator $E$ has a kernel of the form

$$\int_{\mathbb{R}^n} \int w(\lambda, t, |x - \xi|, |y - \xi|) \lambda^{(n-3)/4} \psi(t^2 \lambda) V(\xi) d\lambda d\xi,$$

(4.9)

with a function $w$ independent of $h$ and satisfying the bound, $\forall t \neq 0, \sigma_1, \sigma_2 > 0,$

$$\left| \int w(\lambda, t, \sigma_1, \sigma_2) \chi_\alpha(\lambda) d\lambda \right| \leq C |t|^{-n/2} \left( \sigma_1^{-(n-1)/2} + \sigma_1^{-n+1} + \sigma_2^{-(n-1)/2} + \sigma_2^{-n+1} \right).$$

(4.10)

The operator $F$ satisfies

$$
\|F(t; h)\|_{L^1 \to L^\infty} \leq Ch^{2-n/2}|t|^{-n/2}, \quad \forall t \neq 0, 0 < h \leq 1,
$$

(4.11)

with a constant $C > 0$ independent of $t$ and $h.$

Writting the function $\chi_\alpha$ as

$$\chi_\alpha(\sigma) = \int_0^1 \psi(\sigma \theta) \frac{d\theta}{\theta},$$

where $\psi(\sigma) = \sigma \chi'_\alpha(\sigma) \in C_0^\infty((0, +\infty))$, we obtain by (4.1) with $s = (n - 1)/2, 0 < \epsilon \ll 1,$

$$\left\| e^{itG} \chi_\alpha(G) \langle x \rangle^{-n/2 - \epsilon} - e^{itG_0} \chi_\alpha(G_0) \langle x \rangle^{-n/2 - \epsilon} \right\|_{L^2 \to L^\infty}$$

$$\leq \int_0^1 \left\| \Psi(t; \sqrt{\theta}) \langle x \rangle^{-n/2 - \epsilon} \right\|_{L^2 \to L^\infty} \frac{d\theta}{\theta} \leq C |t|^{-n/2} \int_0^1 \theta^{-1/2 - \epsilon/2} d\theta \leq C |t|^{-n/2},$$

(4.12)

which implies (1.3).

Take $\psi(\sigma) = \sigma^{1-(n-3)/4} \chi'_\alpha(\sigma)$ and denote by $\mathcal{E}(t)$ the operator with kernel defined by replacing in (4.9) the function $\lambda^{(n-3)/4} \psi(t^2 \lambda)$ by $\chi_\alpha(\lambda).$ By (1.1) and (4.10), we have

$$\|\mathcal{E}(t)\|_{L^1 \to L^\infty} \leq C |t|^{-n/2}, \quad \forall t \neq 0.$$  

(4.13)

By (4.5), (4.6) and (4.11)

$$\left\| G^{-(n-3)/4} e^{itG} \chi_\alpha(G) - G_0^{-(n-3)/4} e^{itG_0} \chi_\alpha(G_0) - \mathcal{E}(t) \right\|_{L^1 \to L^\infty}$$

$$\leq \int_0^1 \left\| \Psi(t; \sqrt{\theta}) - E(t; \sqrt{\theta}) \right\|_{L^1 \to L^\infty} \theta^{-1+(n-3)/4} d\theta$$

$$\leq C |t|^{-n/2} \int_0^1 \theta^{-3/4 - \epsilon/2} d\theta \leq C |t|^{-n/2},$$

(4.14)

which together with (4.13) and the fact that the operators $G_0^{-(n-3)/4} \chi_\alpha(G_0)$ and $\chi_\alpha(G_0)$ are bounded on $L^p, 1 \leq p \leq \infty,$ imply (1.2).

To prove (1.4), observe that by (4.5), (4.6) and (4.7) we have

$$\|\Psi(t; h)\|_{L^1 \to L^\infty} \leq Ch^{-(n-3)/2}|t|^{-n/2}, \quad \forall t \neq 0, 0 < h \leq 1.$$  

(4.15)

By interpolation between (3.1) and (4.15), we get

$$\|\Psi(t; h)\|_{L^p \to L^p} \leq Ch^{1-\alpha(n-1)/2}|t|^{-\alpha n/2}, \quad \forall t \neq 0, 0 < h \leq 1,$$

(4.16)
for every $2 \leq p \leq +\infty$, where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$. As above, taking $\psi(\sigma) = \sigma^{1-\alpha q/2}\chi'_a(\sigma)$ and using (4.16), we get

$$\left\|G^{-\alpha q/2}e^{itG}\chi_a(G) - G_0^{-\alpha q/2}e^{itG_0}\chi_a(G_0)\right\|_{L^p \rightarrow L^p} \leq \int_0^1 \left\|\Psi(t; \sqrt{\theta})\right\|_{L^p \rightarrow L^p} \theta^{-1+\alpha q/2} d\theta$$

$$\leq C|t|^{-\alpha n/2} \int_0^1 \theta^{-1/2-\alpha(n-1)/4+\alpha q/2} d\theta \leq C|t|^{-\alpha n/2}$$

if $1/2 + \alpha(n - 1)/4 - \alpha q/2 < 1$, that is, for $2 \leq p < 2(n - 1 - 2q)/(n - 3 - 2q)$. This clearly proves (1.4).

5 Proof of Proposition 4.3

The kernel of the operator $\Psi_3$ is of the form

$$\int_{\mathbb{R}^n} U_h(|x - \xi|, |y - \xi|; t)V(\xi)d\xi,$$

where

$$U_h(\sigma_1, \sigma_2; t) = i \int_0^t \tilde{K}_h(\sigma_1, t - \tau)K_h(\sigma_2, \tau)d\tau = h^{-2n+2}U_1(\sigma_1 h^{-1}, \sigma_2 h^{-1}; th^{-2}),$$

where $K_h$ and $\tilde{K}_h$ are defined by (2.4) (in the case of $\tilde{K}_h$ the function $\psi$ is replaced by $\psi_1$). It is easy to see that to prove the proposition it suffices to show that the function $U_1$ can be decomposed as $U_1 = W_1 + L_1$ with a function $W_1$ of the form

$$W_1(\sigma_1, \sigma_2; t) = \int w(\lambda, t, \sigma_1, \sigma_2)\lambda^{(n-3)/4}\psi(\lambda)d\lambda,$$

with a function $w$ independent of $\psi$, satisfying (4.10) and

$$w(h^2\lambda, th^{-2}, \sigma_1 h^{-1}, \sigma_2 h^{-1}) = h^{(3n-5)/2}w(\lambda, t, \sigma_1, \sigma_2),$$

while the function $L_1$ satisfies

$$|L_1(\sigma_1, \sigma_2; t)| \leq C|t|^{-n/2} \left(\frac{\langle \sigma_1 \rangle^{(n-2)/2}}{\sigma_1^{n-1}} + \frac{\langle \sigma_2 \rangle^{(n-2)/2}}{\sigma_2^{n-1}}\right), \forall t \neq 0, \sigma_1, \sigma_2 > 0.$$

To do so, observe that the function $U_1$ is of the form $U_1 = U_1^{(1)} + U_1^{(2)}$, where

$$U_1^{(1)}(\sigma_1, \sigma_2; t) = \frac{\langle \sigma_1 \sigma_2 \rangle^{-2\nu}}{(2\pi)^n} \int \int e^{it\lambda_1^2} \psi_1(\lambda_1^2)\psi(\lambda_2^2)\mathcal{J}_\nu(\sigma_1 \lambda_1)\mathcal{J}_\nu(\sigma_2 \lambda_2)\frac{\lambda_1 \lambda_2}{\lambda_1^2 - \lambda_2^2} d\lambda_1 d\lambda_2,$$

$$U_1^{(2)}(\sigma_1, \sigma_2; t) = \frac{\langle \sigma_1 \sigma_2 \rangle^{-2\nu}}{(2\pi)^n} \int \int e^{it\lambda_1^2} \psi_1(\lambda_1^2)\psi(\lambda_2^2)\mathcal{J}_\nu(\sigma_1 \lambda_1)\mathcal{J}_\nu(\sigma_2 \lambda_2)\frac{\lambda_1 \lambda_2}{\lambda_1^2 - \lambda_2^2} d\lambda_1 d\lambda_2.$$

Recall that $\mathcal{J}_\nu(z) = e^{izb^+_\nu(z)} + e^{-izb^-_\nu(z)}$ with functions $b^\pm_\nu$ satisfying (2.11). Set $\mathcal{N}_\nu(z) = e^{izb^+_\nu(z)} - e^{-izb^-_\nu(z)}$, and

$$a^\pm(\lambda_1, \lambda_2; \sigma_2) = (\lambda_1 - \lambda_2)^{-1} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \psi(\lambda_2^2)b^+_\nu(\sigma_2 \lambda_2) - \frac{1}{2} \psi(\lambda_2^2)b^+_\nu(\sigma_2 \lambda_1)\right).$$
We have
\[
\int_{-\infty}^{\infty} \psi(\lambda_2^2) J_\nu(\sigma_2 \lambda_2) \frac{\lambda_2 d\lambda_2}{\lambda_1^2 - \lambda_2^2} = \pm \int_{-\infty}^{\infty} e^{\pm i \sigma_2 \lambda_2} a^\pm(\lambda_1, \lambda_2; \sigma_2) d\lambda_2 \\
+ \sum_{\pm} \frac{1}{2} \psi(\lambda_1^2) b^\pm(\sigma_2 \lambda_1) \int_{-\infty}^{\infty} e^{\pm i \sigma_2 \lambda_2} \frac{d\lambda_2}{\lambda_1 - \lambda_2}
\]
\[
= \sum_{\pm} \int_{-\infty}^{\infty} e^{\pm i \sigma_2 \lambda_2} a^\pm(\lambda_1, \lambda_2; \sigma_2) d\lambda_2 + \text{Const} \sum_{\pm} \pm \psi(\lambda_1^2) b^\pm(\sigma_2 \lambda_1) e^{\pm i \sigma_2 \lambda_1}
\]
\[
=: \sum_{\pm} A^\pm(\lambda_1, \sigma_2) + \text{Const} \psi(\lambda_1^2) N_\nu(\sigma_2 \lambda_1),
\]
(5.6)
where we have used that, for any \( \sigma > 0 \),
\[
\int_{-\infty}^{\infty} e^{\pm i \sigma \mu} \frac{d\mu}{\mu} = \pm 2i \int_{0}^{\infty} \frac{\sin(\sigma \mu)}{\mu} d\mu = \pm 2i \int_{0}^{\infty} \frac{\sin \mu}{\mu} d\mu = \pm 2i \text{Const}.
\]
In view of (5.6) we can write the function \( U_1^{(1)} \) in the form
\[
U_1^{(1)}(\sigma_1, \sigma_2; t) = \text{Const} (\sigma_1 \sigma_2)^{-2\nu} \int e^{it \lambda_1^2} \psi(\lambda_1^2) J_\nu(\sigma_1 \lambda_1) N_\nu(\sigma_2 \lambda_1) \lambda_1 d\lambda_1
\]
\[
+ \frac{(\sigma_1 \sigma_2)^{-2\nu}}{(2\pi)^n} \int e^{it \lambda_1^2} \psi(\lambda_1^2) J_\nu(\sigma_1 \lambda_1) A(\lambda_1, \sigma_2) \lambda_1 d\lambda_1
\]
\[
=: W_1^{(1)}(\sigma_1, \sigma_2; t) + L_1^{(1)}(\sigma_1, \sigma_2; t),
\]
(5.7)
where \( A = A^+ + A^- \). We will now show that the function \( L_1^{(1)} \) satisfies (5.5). Observe first that the functions \( a^\pm \) satisfy (for \( \lambda_1^2 \in \text{supp } \psi_1 \))
\[
\left| \frac{\partial^{\alpha_1} \partial^{\alpha_2} a^\pm(\lambda_1, \lambda_2; \sigma)}{\sigma^2} \right| \leq C(\sigma)^{(n-3)/2} \langle \lambda_2 \rangle^{-1-\alpha_2}, \quad \forall \lambda_2 \in \mathbb{R}, \sigma > 0,
\]
(5.8)
for all multi-indices \( (\alpha_1, \alpha_2) \). Indeed, it is easy to see that for \( \sigma \geq 1 \) the bound (5.8) follows from (2.11), while for \( 0 < \sigma \leq 1 \) one needs to use the fact that near \( z = 0 \) the functions \( b^\pm_\nu \) are of the form
\[
b^\pm_\nu(z) = b^\pm_{\nu,1}(z) + z^{n-2} \log z b^\pm_{\nu,2}(z),
\]
(5.9)
where the functions \( b^\pm_{\nu,j} \) are analytic at \( z = 0 \), \( b^\pm_{\nu,2} \equiv 0 \) if \( n \) is odd. Therefore, we have (for \( \lambda^2 \in \text{supp } \psi_1 \))
\[
\left| \frac{d^k}{d\lambda^k} b^\pm(\sigma \lambda) \right| \leq C_k, \quad 0 < \sigma \leq 1,
\]
(5.10)
for every integer \( k \) with a constant \( C_k > 0 \) independent of \( \sigma \). Clearly, (5.8) for \( 0 < \sigma \leq 1 \) follows from (5.10). Furthermore, an integration by parts together with (5.8) lead to the following bounds for the functions \( A^\pm \) (for \( \lambda^2 \in \text{supp } \psi_1 \))
\[
\left| \frac{\partial^{\alpha}}{\partial \lambda^k} A^\pm(\lambda, \sigma) \right| \leq C_{\alpha,k}(\sigma)^{(n-3)/2} \sigma^{-k}, \quad \forall \sigma > 0,
\]
(5.11)
for every integers \( \alpha \geq 0, k \geq 1 \). Hence,
\[
\left| \frac{\partial^{\alpha}}{\partial \lambda^k} A^\pm(\lambda, \sigma) \right| \leq C_{\alpha} \sigma^{-1}, \quad \forall \sigma > 0.
\]
(5.12)
On the other hand, by (2.8) and (2.11), we have
\[ \left| \frac{d^k}{d\lambda^k} \mathcal{J}_\nu(z) \right| \leq C z^{n-2-k} (z)^{k-(n-1)/2}, \quad \forall z > 0, \quad (5.13) \]
for every integer \( 0 \leq k \leq n - 2 \), while for \( k \geq n - 1 \) we have
\[ \left| \frac{d^k}{d\lambda^k} \mathcal{J}_\nu(z) \right| \leq C_k (z)^{(n-3)/2}, \quad \forall z > 0. \quad (5.14) \]
Thus, by (5.12)-(5.14) we obtain (for \( \lambda^2 \in \text{supp} \psi \))
\[ \left| \frac{d^m}{d\lambda^m} (\mathcal{J}_\nu(\lambda_1 A(\lambda, \sigma_2)) \right| \leq C m \sum_{k=0}^m \left| \frac{d^k}{d\lambda^k} \mathcal{J}_\nu(\lambda_1 A(\lambda, \sigma_2)) \right| \leq C \sigma_1^{-n} \sigma_2^{-n} \langle \sigma_1 \rangle^{m-(n-1)/2}, \quad (5.15) \]
for every integer \( m \geq 0 \). In the same way as in the proof of (2.5), using (2.7) together with (5.15), we deduce
\[ \left| L_1^{(1)}(\sigma_1, \sigma_2; t) \right| \leq C m |t|^{-m-1/2} \sigma_2^{-n+1} \langle \sigma_1 \rangle^{m-(n-1)/2}, \quad (5.16) \]
for every integer \( m \geq 0 \), and hence for all real \( m \geq 0 \). Taking \( m = (n - 1)/2 \) in (5.16) we obtain the desired bound. Let \( \phi \in C_0^\infty([-1, 1]), \phi = 1 \) on \([-1/2, 1/2]\). We further decompose the function \( W_1^{(1)} \) as follows
\[ W_1^{(1)}(\sigma_1, \sigma_2; t) = \text{Const} \langle \sigma_1 \rangle^{-n+2} \int e^{it\psi(\lambda^2)} \mathcal{J}_\nu(\lambda_1 A(\lambda, \sigma_2)) \phi(\sigma_2 \lambda) \lambda d\lambda \]
\[ + \text{Const} \langle \sigma_1 \rangle^{-n+2} \int e^{it\psi(\lambda^2)} \mathcal{J}_\nu(\lambda_1 A(\lambda, \sigma_2)) \phi(\sigma_2 \lambda) \lambda d\lambda \]
\[ + \text{Const} \langle \sigma_1 \rangle^{-n+2} \int e^{it\psi(\lambda^2)} \mathcal{J}_\nu(\lambda_1 A(\lambda, \sigma_2)) \phi(\sigma_2 \lambda) \lambda d\lambda \]
\[ =: M_1^{(1)}(\sigma_1, \sigma_2; t) + N_1^{(1)}(\sigma_1, \sigma_2; t) + \tilde{W}_1^{(1)}(\sigma_1, \sigma_2; t). \quad (5.17) \]
By (5.10) we have (for \( \lambda^2 \in \text{supp} \psi \))
\[ \left| \frac{d^k}{d\lambda^k} (\mathcal{N}_\nu(\phi)(\sigma_\lambda)) \right| \leq C_k, \quad \forall \sigma > 0, \quad (5.18) \]
for every integer \( k \) with a constant \( C_k > 0 \) independent of \( \sigma \). By (5.13), (5.14) and (5.18) we get (for \( \lambda^2 \in \text{supp} \psi \))
\[ \left| \frac{d^m}{d\lambda^m} (\mathcal{J}_\nu(\lambda_1 A(\lambda_2 \phi)(\sigma_2 \lambda)) \right| \leq C \sigma_1^{-n} \sigma_2^{-n} \langle \sigma_1 \rangle^{m-(n-1)/2}, \quad (5.19) \]
for every integer \( m \geq 0 \). In the same way as above we deduce from (5.19)
\[ \left| M_1^{(1)}(\sigma_1, \sigma_2; t) \right| \leq C_m |t|^{-m-1/2} \sigma_2^{-n+2} \langle \sigma_1 \rangle^{m-(n-1)/2}, \quad (5.20) \]
for all real \( m \geq 0 \). Taking \( m = (n - 1)/2 \) we conclude that \( M_1^{(1)} \) satisfies (5.5). Furthermore, in view of (5.13) and (5.14), we have (for \( \lambda^2 \in \text{supp} \psi \))
\[ \left| \frac{d^k}{d\lambda^k} (\mathcal{J}_\nu(\phi)(\sigma_\lambda)) \right| \leq C_k \sigma_1^{-n}, \quad \forall \sigma > 0, \quad (5.21) \]
for every integer $k$ with a constant $C_k > 0$ independent of $\sigma$, while (2.11) leads to the bound
\[
\left| \frac{d^k}{d\lambda^k}((1-\phi)N_\nu)(\sigma\lambda) \right| \leq C_k \langle \sigma \rangle^{k-(n-1)/2}, \quad \forall \sigma > 0.
\] (5.22)

By (5.21) and (5.22), for $\lambda^2 \in \text{supp } \psi$
\[
\left| \frac{d^m}{d\lambda^m} ((\phi J_\nu)(\sigma_1\lambda)((1-\phi)N_\nu)(\sigma_2\lambda)) \right| \leq C \sigma^{-n-2} \langle \sigma_2 \rangle^{m-(n-1)/2},
\] for every integer $m \geq 0$. By (5.23) we get
\[
\left| N_1^{(1)}(\sigma_1, \sigma_2; t) \right| \leq C_m |t|^{-m-1/2} \sigma_2^{m-2} \langle \sigma_2 \rangle^{m-(n-1)/2},
\] for all real $m \geq 0$. Taking $m = (n-1)/2$ we conclude that $N_1^{(1)}$ satisfies (5.5), too.

We will now decompose $\tilde{W}_1^{(1)}$ using that the functions $b_\nu^\pm$ admit the expansion
\[
b_\nu^\pm(z) = \pm c_\nu z^{(n-3)/2} + O \left( z^{(n-5)/2} \right), \quad z \to +\infty,
\]
where $c_\nu$ is some constant. More precisely, we have
\[
\left| \partial_z^k \left( b_\nu^\pm(z) \mp c_\nu z^{(n-3)/2} \right) \right| \leq C_k z^{(n-5)/2-k}, \quad z \geq z_0,
\] for every integer $k \geq 0$ and every $z_0 > 0$, with a constant $C_k > 0$ independent of $z$ but depending on $k$ and $z_0$. Thus we can write
\[
(\sigma_1\sigma_2)^{-n+2} \lambda((1-\phi)J_\nu)(\sigma_1\lambda)((1-\phi)N_\nu)(\sigma_2\lambda)
= c_\nu^2(\sigma_1\sigma_2)^{-(n-1)/2} \lambda^{-2} \left( e^{i\sigma_1\lambda} - e^{-i\sigma_1\lambda} \right) \left( e^{i\sigma_2\lambda} + e^{-i\sigma_2\lambda} \right)
+ c_\nu^2(\sigma_1\sigma_2)^{-(n-1)/2} \lambda^{-2} \left( -\phi(\sigma_1\lambda) - \phi(\sigma_2\lambda) + \phi(\sigma_1\lambda)\phi(\sigma_2\lambda) \right) \left( e^{i\sigma_1\lambda} - e^{-i\sigma_1\lambda} \right) \left( e^{i\sigma_2\lambda} + e^{-i\sigma_2\lambda} \right)
+ (\sigma_1\sigma_2)^{-(n-2)} \lambda^{-2} \phi(\sigma_1\lambda)(1-\phi)(\sigma_2\lambda)
\times \left( e^{i\sigma_2\lambda} \left( b_\nu^+(\sigma_2\lambda) - c_\nu(\sigma_2\lambda)(n-3)/2 \right) + e^{-i\sigma_2\lambda} \left( b_\nu^-(\sigma_2\lambda) + c_\nu(\sigma_2\lambda)(n-3)/2 \right) \right)
+ c_\nu(\sigma_1\sigma_2)^{-(n-1)/2} \lambda^{-2} \left( 1-\phi(\sigma_1\lambda) \right) \left( 1-\phi(\sigma_2\lambda) \right) \left( e^{i\sigma_2\lambda} + e^{-i\sigma_2\lambda} \right)
\times \left( e^{i\sigma_1\lambda} \left( b_\nu^+(\sigma_1\lambda) - c_\nu(\sigma_1\lambda)(n-3)/2 \right) - e^{-i\sigma_1\lambda} \left( b_\nu^-(\sigma_1\lambda) + c_\nu(\sigma_1\lambda)(n-3)/2 \right) \right)
\overset{=}{=} X(\lambda; \sigma_1, \sigma_2) + Y(\lambda; \sigma_1, \sigma_2) + Z(\lambda; \sigma_1, \sigma_2),
\] (5.26)
where $X$ denotes the first term in the RHS, $Y$ denotes the second one, while $Z = Z_1 + Z_2$ denotes the remainder ($Z_2$ being the last term). In view of (5.25) we have (for $\lambda^2 \in \text{supp } \psi$)
\[
\left| \frac{d^m}{d\lambda^m} Z_1(\lambda; \sigma_1, \sigma_2) \right| \leq C \sum_{k=0}^{m} \langle \sigma_1 \rangle^{k-(n-1)/2} \langle \sigma_2 \rangle^{m-k-(n+1)/2}
\leq C \langle \sigma_1 \rangle^{-(n-1)/2} \langle \sigma_2 \rangle^{-(n+1)/2} \left( \langle \sigma_1 \rangle + \langle \sigma_2 \rangle \right)^m,
\] (5.27)
Therefore, the integral in the LHS of (5.30) satisfies (5.5). The terms corresponding to where the function \( w \) for every integer \( m \geq 0 \). Hence,

\[
\left| \int e^{it\lambda^2} \psi(\lambda^2) Z_1(\lambda; \sigma_1, \sigma_2) d\lambda \right| \leq C |t|^{-m-1/2} \langle \sigma_1 \rangle^{-(n-1)/2} \langle \sigma_2 \rangle^{-(n+1)/2} (\langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^m, \tag{5.28}
\]

for all real \( m \geq 0 \). Take \( m = (n - 1)/2 \) and observe that

\[
\langle \sigma_1 \rangle^{-(n-1)/2} \langle \sigma_2 \rangle^{-(n+1)/2} (\langle \sigma_1 \rangle + \langle \sigma_2 \rangle)^{(n-1)/2} \leq C \langle \sigma_1 \rangle^{-(n+1)/2} + C \langle \sigma_2 \rangle^{-(n+1)/2}.
\]

Therefore, the integral in the LHS of (5.28) satisfies (5.5). The function \( Z_2 \) is treated in precisely the same way. Furthermore, we decompose the function \( Y \) as \( Y = Y_1 + Y_2 + Y_3 \), with \( Y_1 \) corresponding to the term \( \phi(\sigma_1 \lambda) \), \( Y_2 \) corresponding to the term \( \phi(\sigma_2 \lambda) \), and \( Y_3 \) being the remainder. We have (for \( \lambda^2 \in \text{supp} \psi \))

\[
\left| \frac{d^m}{d\lambda^m} Y_1(\lambda; \sigma_1, \sigma_2) \right| \leq C \sigma_1^{-(n-1)/2} \sigma_2^{-(n-1)/2} \langle \sigma_2 \rangle^m, \tag{5.29}
\]

for every integer \( m \geq 0 \), and hence,

\[
\left| \int e^{it\lambda^2} \psi(\lambda^2) Y_1(\lambda; \sigma_1, \sigma_2) d\lambda \right| \leq C |t|^{-m-1/2} \sigma_1^{-(n-1)/2} \sigma_2^{-(n-1)/2} \langle \sigma_2 \rangle^m, \tag{5.30}
\]

for all real \( m \geq 0 \). Since \( \sigma_1 \) is bounded as long as \( \sigma_1 \lambda \in \text{supp} \phi \) and \( \lambda^2 \in \text{supp} \psi \), we can bound the RHS of (5.30) (for \( m = (n - 1)/2 \) by

\[
C |t|^{-n/2} \left( \sigma_1^{-(n-1)/2} \sigma_2^{-(n-1)/2} + \sigma_2^{-(n-1)/2} \right) \leq C |t|^{-n/2} \left( \sigma_1^{-(n+1)} + \sigma_2^{-(n+1)} \right).
\]

Therefore, the integral in the LHS of (5.30) satisfies (5.5). The terms corresponding to \( Y_2 \) and \( Y_3 \) can be treated in precisely the same way.

Furthermore, we write

\[
\int e^{it\lambda^2} \psi(\lambda^2) X(\lambda; \sigma_1, \sigma_2) d\lambda = \int w^{(1)}(\mu, t, \sigma_1, \sigma_2) \mu^{(n-3)/4} \psi(\mu) d\mu, \tag{5.31}
\]

where the function \( w^{(1)} \) is of the form

\[
w^{(1)}(\lambda^2, t, \sigma_1, \sigma_2) = \sum_{\pm} \sum_{\pm} \text{Const}(\sigma_1\sigma_2)^{-(n-1)/2} e^{it\lambda^2 + i\lambda(\pm \sigma_1 \pm \sigma_2)} \lambda^{(n-3)/2}. \tag{5.32}
\]

Clearly, the function \( w^{(1)} \) satisfies (5.4). To prove that \( w^{(1)} \) satisfies (4.10), it suffices to show that

\[
\left| \int_0^\infty e^{it\lambda^2 + i\sigma \lambda} \lambda^{(n-1)/2} \chi_\alpha(\lambda^2) d\lambda \right| \leq C |t|^{-n/2} \langle \sigma \rangle^{(n-1)/2}, \quad \forall t \neq 0, \sigma \in \mathbb{R}, \tag{5.33}
\]

with a constant \( C > 0 \) independent of \( t \) and \( \sigma \). Consider first the case of \( n \) odd, and set \( m = (n - 1)/2 \). Integrating by parts \( m \) times, we get

\[
2(it)^m \int_0^\infty e^{it\lambda^2 + i\sigma \lambda} \lambda^{(n-1)/2} \chi_\alpha(\lambda^2) d\lambda = (it)^m \int_0^\infty e^{it\mu + i\sigma \sqrt{\mu}} \mu^{(n-3)/4} \chi_\alpha(\mu) d\mu
\]
\[
\sum_{j=0}^{m-1} \sigma^j \int_0^\infty e^{it\lambda^2+ix\lambda} \varphi_j(\lambda) d\lambda + 2 \int_0^\infty e^{it\lambda^2+i\sigma \lambda} g_m(\lambda^2, \sigma) \chi_a(\lambda^2) \lambda d\lambda,
\]  
(5.34)

where \(\varphi_j \in C_0^\infty((0, +\infty))\), and

\[
g_m(\mu, \sigma) = e^{-i\sigma \sqrt{\mu}} \frac{d^{m}}{d\mu^m} \left( e^{i\sigma \sqrt{\mu} (n-3)/4} \right).
\]

By (2.7), each integral in the sum in the RHS of (5.34) is bounded by \(O(|t|^{-1/2})\). To bound the remainder, observe that \(g_m\) is of the form

\[
g_m(\mu, \sigma) = \mu^{-1/2} \sum_{j=0}^{m} \gamma_j \sigma^j \mu^{-j/2},
\]

where \(\gamma_j\) are independent of \(\mu\) and \(\sigma\). Therefore, we have (for \(\lambda^2 \in \text{supp} \chi_a\))

\[
\left| \frac{d^j}{d\lambda^j} \left( \lambda g_m(\lambda^2, \sigma) \right) \right| \leq C(\sigma)^m \lambda^{-2j}, \quad j = 0, 1,
\]

(5.35)

and hence

\[
\left| \frac{d}{d\lambda} \left( \lambda g_m(\lambda^2, \sigma) \chi_a(\lambda^2) \right) \right| \leq C(\sigma)^m \lambda^{-2}.
\]

(5.36)

We now write the last integral in the RHS of (5.34) as

\[
\int_0^\infty k(\lambda, \sigma, t) \frac{d}{d\lambda} \left( \lambda g_m(\lambda^2, \sigma) \chi_a(\lambda^2) \right) d\lambda,
\]

(5.37)

where

\[
k(\lambda, \sigma, t) = \int_0^\lambda e^{itx^2+i\sigma x} dx
\]

\[
= e^{-i\sigma^2/4t} \int_0^\lambda e^{i(x+\sigma/2)t^2} dx = |t|^{-1/2} e^{-i\sigma^2/4t} \int_{-\sigma/2|t|}^{\lambda/2|t|} e^{iy^2} dy,
\]

where \(\varepsilon = \text{sign} \ t\). Using the well known bound

\[
\left| \int_0^a e^{iy^2} dy \right| \leq C, \quad \forall a \in \mathbb{R},
\]

with a constant \(C > 0\) independent of \(a\), we get

\[
|k(\lambda, \sigma, t)| \leq C|t|^{-1/2},
\]

(5.38)

with a constant \(C > 0\) independent of \(\lambda\), \(\sigma\) and \(t\). By (5.36) and (5.38), the integral in (5.37) is bounded by \(C(\sigma)^m |t|^{-1/2}\), which clearly implies (5.33) in this case.

Let now \(n\) be even and set \(m = (n - 2)/2\). Then (5.34) still holds and each integral in the sum in the RHS is bounded by \(C_k(\sigma)^k |t|^{-k-1/2}\) for all real \(k \geq 0\), and in particular for \(k = 1/2\). Therefore, it suffices to show that the last integral in the RHS of (5.34) is bounded in this case by \(C(\sigma)^{m+1/2} |t|^{-1}\). The function \(g_m\) in this case is of the form

\[
g_m(\mu, \sigma) = \mu^{-1/4} \sum_{j=0}^{m} \gamma_j \sigma^j \mu^{-j/2}.
\]
Thus, it suffices to show that

$$\left| \int_0^\infty e^{it\lambda^2+i\sigma \lambda} \lambda^{1/2-j} \chi_a(\lambda^2) d\lambda \right| \leq C|t|^{-1}(\sigma)^{1/2+j}, \quad 0 \leq j \leq m. \quad (5.39)$$

When $j \geq 1$, (5.39) follows easily by integrating once by parts. To prove (5.39) for $j = 0$, we proceed as follows (if $\sigma/t < 0$

$$\int_0^\infty e^{it\lambda^2+i\sigma \lambda} \lambda^{1/2} \chi_a(\lambda^2) d\lambda = e^{-\sigma^2/4t} \int_0^\infty e^{it(\lambda^2+\sigma/2t)^2} \lambda^{1/2} \chi_a(\lambda^2) d\lambda$$

$$\quad \quad \quad \quad \quad + e^{-\sigma^2/4t} \int_0^\infty e^{it(\lambda^2+\sigma/2t)^2} \lambda^{1/2} \chi_a(\lambda^2) d\lambda$$

$$\quad \quad \quad \quad \quad = (-\sigma/2t)^{1/2} e^{-\sigma^2/4t} \int_0^\infty e^{it(\lambda^2+\sigma/2t)^2} \lambda^{1/2} \chi_a(\lambda^2) d\lambda$$

$$\quad \quad \quad \quad \quad + (-\sigma/2t)^{1/2} e^{-\sigma^2/4t} \int_0^\infty e^{it(\lambda^2+\sigma/2t)^2} \lambda^{1/2} \chi_a(\lambda^2) d\lambda$$

$$\quad \quad \quad \quad \quad -(it)^{-1} e^{-\sigma^2/4t} \int_0^\infty e^{it(\lambda^2+\sigma/2t)^2} \frac{d}{d\lambda} \left( \frac{\lambda^{1/2} \chi_a(\lambda^2)}{\lambda^{1/2} + (-\sigma/2t)^{1/2}} \right) d\lambda. \quad (5.40)$$

We bound the integral in the first term in the RHS of (5.40) by $O(|t|^{-1/2})$ in the same way as the integral (5.37) above, so the first term itself is bounded by $C(\sigma)^{1/2}|t|^{-1}$. The second term is bounded by $O(|t|^{-1})$ because of the bound

$$\left| \frac{d}{d\lambda} \left( \frac{\chi_a(\lambda^2)}{\lambda^{1/2} + (-\sigma/2t)^{1/2}} \right) \right| \leq C(\lambda)^{-3/2},$$

with a constant $C > 0$ independent of $\lambda$, $\sigma$ and $t$. When $\sigma/t \geq 0$, we write

$$\int_0^\infty e^{it\lambda^2+i\sigma \lambda} \lambda^{1/2} \chi_a(\lambda^2) d\lambda$$

$$\quad \quad \quad \quad \quad = -(it)^{-1} e^{-\sigma^2/4t} \int_0^\infty e^{it(\lambda^2+\sigma/2t)^2} \frac{d}{d\lambda} \left( \frac{\lambda^{1/2} \chi_a(\lambda^2)}{\lambda + \sigma/2t} \right) d\lambda, \quad (5.41)$$

so the integral in the LHS of (5.41) is bounded by $O(|t|^{-1})$ because of the bound

$$\left| \frac{d}{d\lambda} \left( \frac{\lambda^{1/2} \chi_a(\lambda^2)}{\lambda + \sigma/2t} \right) \right| \leq C(\lambda)^{-3/2},$$

with a constant $C > 0$ independent of $\lambda$, $\sigma$ and $t$. This completes the proof of (5.33). Since the function $U_1^{(2)}$ can be treated in precisely the same way as $U_1^{(1)}$, the proof of the proposition is completed.
References

[1] M. Goldberg, Dispersion bounds for the three-dimensional Schrödinger equation with almost critical potentials, GAFA, to appear.

[2] M. Goldberg and W. Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, Commun. Math. Phys. 251 (2004), 157-178.

[3] M. Goldberg and M. Visan, A counterexample to dispersive estimates for Schrödinger operators in higher dimensions, preprint.

[4] A. Jensen and S. Nakamura, $L^p$-mapping properties of functions of Schrödinger operators and their applications to scattering theory, J. Math. Soc. Japan 47 (1995), 253-273.

[5] J.-L. Journé, A. Soffer and C. Sogge, Decay estimates for Schrödinger operators, Commun. Pure Appl. Math. 44 (1991), 573-604.

[6] I. Rodnianski and W. Schlag, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials, Invent. Math. 155 (2004), 451-513.

[7] W. Schlag, Dispersive estimates for Schrödinger operators in two dimensions, Commun. Math. Phys. 257 (2005), 87-117.

[8] G. Vodev, Dispersive estimates of solutions to the Schrödinger equation, Ann. Henri Poincaré 6 (2005), 1179-1196.

[9] G. Vodev, Dispersive estimates of solutions to the wave equation with a potential in dimensions $n \geq 4$, Commun. Partial Diff. Equations, to appear.

[10] K. Yajima, The $W^{k,p}$-continuity of wave operators for Schrödinger operators, J. Math. Soc. Japan 47 (1995), 551-581.

[11] K. Yajima, Dispersive estimates for Schrödinger equations with threshold resonance and eigenvalue, Commun. Math. Phys. 259 (2005), 475-509.

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