Three-arc graphs: characterization and domination

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Abstract

An arc of a graph is an oriented edge and a 3-arc is a 4-tuple \((v,u,x,y)\) of vertices such that both \((v,u,x)\) and \((u,x,y)\) are paths of length two. The 3-arc graph of a graph \(G\) is defined to have vertices the arcs of \(G\) such that two arcs \(uv,xy\) are adjacent if and only if \((v,u,x,y)\) is a 3-arc of \(G\). In this paper we give a characterization of 3-arc graphs and obtain sharp upper bounds on the domination number of the 3-arc graph of a graph \(G\) in terms that of \(G\).

Key words: 3-arc graph; domination number; graph operation

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1 Introduction

The 3-arc construction \([10]\) is a relatively new graph operation that has been used in the classification or characterization of several families of arc-transitive graphs \([5, 7, 10, 12, 21, 22]\). (A graph is arc-transitive if its automorphism group acts transitively on the set of oriented edges.) As noted in \([8]\), although this operation was first introduced in the context of graph symmetry, it is also of interest for general (not necessarily arc-transitive) graphs, and many problems on this new operation remain unexplored. In this paper we give partial solutions to two problems posed in \([8]\) regarding 3-arc graphs.

An arc of a graph \(G\) is an ordered pair of adjacent vertices. For adjacent vertices \(u, v\) of \(G\), we use \(uv\) to denote the arc from \(u\) to \(v\), \(vu\) (\(\neq uv\)) the arc from \(v\) to \(u\), and \(\{u, v\}\) the edge between \(u\) and \(v\). A 3-arc of \(G\) is a 4-tuple \((v,u,x,y)\) of vertices, possibly with \(v = y\), such that both \((v,u,x)\) and \((u,x,y)\) are paths of \(G\). Let \(\Delta\) be a set of 3-arcs of \(G\). Suppose that \(\Delta\) is self-paired in the sense that \((y,x,u,v)\in\Delta\) whenever \((v,u,x,y)\in\Delta\). Then the 3-arc graph of \(G\) relative to \(\Delta\), denoted by \(X(G,\Delta)\), is defined \([10]\) to be the (undirected) graph whose vertex set is the set of arcs of \(G\) such that two vertices corresponding to arcs \(uv\) and \(xy\) are adjacent if and only if \((v,u,x,y)\in\Delta\). In the context of graph symmetry, \(\Delta\) is usually a self-paired orbit on the set of 3-arcs under the action of an automorphism group of \(G\). In the case where \(\Delta\) is the set of all 3-arcs of \(G\), we call \(X(G,\Delta)\) the 3-arc graph \([9]\) of \(G\) and denote it by \(X(G)\).

The first study of 3-arc graphs of general graphs was conducted by Knor and Zhou in \([9]\). Among other things they proved that if \(G\) has vertex-connectivity \(\kappa(G) \geq 3\) then its 3-arc graph has vertex-connectivity \(\kappa(X(G)) \geq (\kappa(G) - 1)^2\), and if \(G\) is connected of minimum
degree \( \delta(G) \geq 3 \) then the diameter \( \text{diam}(X(G)) \) of \( X(G) \) is equal to \( \text{diam}(G) + 1 \) or \( \text{diam}(G) + 2 \). In [2], Ballbuena, García-Vázquez and Montejano improved the bound on the vertex-connectivity by proving \( \kappa(X(G)) \geq \min\{\kappa(G)(\delta(G) - 1), (\delta(G) - 1)^2\} \) for any connected graph \( G \) with \( \delta(G) \geq 3 \). They also proved [2] that for such a graph the edge-connectivity of \( X(G) \) satisfies \( \lambda(X(G)) \geq (\delta(G) - 1)^2 \), and they further gave a lower bound on the restricted edge-connectivity of \( X(G) \) in the case when \( G \) is 2-connected. In [8], Knor, Xu and Zhou studied the independence, domination and chromatic numbers of 3-arc graphs.

In a recent paper [20] we obtained a necessary and sufficient condition [20, Theorem 1] for \( X(G) \) to be Hamiltonian. In particular, we proved [20, Theorem 2] that a 3-arc graph is Hamiltonian if and only if it is connected, and that if \( G \) is connected with \( \delta(G) \geq 3 \) then all its iterative 3-arc graphs \( X^i(G) \) are Hamiltonian, \( i \geq 1 \). (The iterative 3-arc graphs are recursively defined by \( X^1(G) := X(G) \) and \( X^{i+1}(G) := X(X^i(G)) \) for \( i \geq 1 \).) As a consequence we obtained [20, Corollary 2] that if a vertex-transitive graph is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three, then it is Hamiltonian. This provides new support to the well-known Lovász-Thomassen conjecture [17] which asserts that all connected vertex-transitive graphs, with finitely many exceptions, are Hamiltonian. We also proved (as a consequence of a more general result) [20, Theorem 4] that if a graph \( G \) with at least four vertices is Hamilton-connected, then so are its iterative 3-arc graphs \( X^i(G) \), \( i \geq 1 \).

The 3-arc construction was generalized to directed graphs in [8]. Given a directed graph \( D \), the 3-arc graph \( [8] \) of \( D \), denoted by \( X(D) \), is defined to be the undirected graph whose vertex set is the set of arcs of \( D \) such that two vertices corresponding to arcs \( uv, xy \) of \( D \) are adjacent if and only if \( v \neq x, y \neq u \) and \( u, x \) are adjacent in \( D \). Recently, we proved with Wood [19] that the well-known Hadwiger’s graph colouring conjecture [18] is true for the 3-arc graph of any directed graph with no loops.

In spite of the results above, compared with the well-known line graph operation [6] and the 2-path graph operation [11], [11], our knowledge of 3-arc graphs is quite limited and many problems on them are yet to be explored. For instance, the following problems were posed in [8]:

**Problem 1** Characterize 3-arc graphs of connected graphs.

**Problem 2** Give a sharp upper bound on \( \gamma(X(G)) \) in terms of \( \gamma(G) \) for any connected graph \( G \) with \( \delta(G) \geq 2 \), where \( \gamma \) denotes the domination number.

In this paper we give partial solutions to these problems. We first show that there is no forbidden subgraph characterization of 3-arc graphs (Proposition [1]), and then we provide a descriptive characterization of 3-arc graphs (Theorem [2]). We give a sharp upper bound for \( \gamma(X(G)) \) in terms of \( \gamma(G) \) (Theorem [5]) for any graph \( G \) with \( \delta(G) \geq 2 \), and more upper bounds for \( \gamma(X(G)) \) in terms of \( \gamma(G) \) and the maximum degree \( \Delta(G) \) when \( 2 \leq \delta(G) \leq 4 \) (Theorem [6]). Finally, we prove that if \( G \) is claw-free with \( \delta(G) \geq 2 \), then \( \gamma(X(G)) \leq 4\gamma(G) \) and moreover this bound is sharp (Theorem [7]).

All graphs in the paper are finite and undirected with no loops or multiple edges. The order of a graph is the number of vertices in the graph. As usual, the minimum and maximum degrees of a graph \( G = (V(G), E(G)) \) are denoted by \( \delta(G) \) and \( \Delta(G) \), respectively. The degree
of a vertex \( v \in V(G) \) in \( G \) is denoted by \( \text{deg}(v) \). The neighbourhood of \( v \) in \( G \), denoted by \( N(v) \), is the set of vertices of \( G \) adjacent to \( v \), and the closed neighbourhood of \( v \) is defined as \( N[v] := N(v) \cup \{v\} \). We say that \( v \) dominates every vertex in \( N(v) \), or every vertex in \( N[v] \) is dominated by \( v \). For a subset \( S \) of \( V(G) \), denote \( N(S) := \cup_{v \in S} N(v) \) and \( N[S] := N(S) \cup S \).

We may add subscript \( G \) to these notations (e.g. \( \text{deg}_G(v) \)) to indicate the underlying graph when there is a risk of confusion. If \( N[S] = V(G) \), then \( S \) is called a dominating set of \( G \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \); a dominating set of \( G \) with cardinality \( \gamma(G) \) is called a \( \gamma(G) \)-set of \( G \).

The reader is referred to [18] for undefined notation and terminology.

2 A characterization of 3-arc graphs

It is well known that line graphs can be characterized by a finite set of forbidden induced subgraphs [6]. In contrast, a similar characterization does not exist for 3-arcs graphs, as we show in the following result.

**Proposition 1** There is no characterization of 3-arc graphs by a finite set of forbidden induced subgraphs. More specifically, any graph is isomorphic to an induced subgraph of some 3-arc graph.

**Proof** Let \( H \) be any graph. Define \( H^* \) to be the graph obtained from \( H \) by adding a new vertex \( x \) and an edge joining \( x \) and each vertex of \( H \). It is not hard see that \( u, v \in V(H) \) are adjacent in \( H \) if and only if the arcs \( ux, vx \) of \( H^* \) are adjacent in \( X(H^*) \). Thus the subgraph of \( X(H^*) \) induced by \( A := \{vx : v \in V(H)\} \subseteq V(X(H^*)) \) is isomorphic to \( H \) via the bijection \( v \leftrightarrow vx \) between \( V(H) \) and \( A \). Since \( H \) is arbitrary, this means that any graph is isomorphic to an induced subgraph of some 3-arc graph, and so the result follows. \( \square \)

Next we give a descriptive characterization of 3-arc graphs. To avoid triviality we assume that the graph under consideration has at least one edge.

**Theorem 2** A graph \( G \) having at least one edge is isomorphic to the 3-arc graph of some graph if and only if \( V(G) \) admits a partition \( V := V_1 \cup V_2 \) and \( E(G) \) admits a partition \( E \) such that the following hold:

(a) each element of \( V_1 \) contains exactly one vertex of \( G \), and each element of \( V_2 \) is an independent set of \( G \) with at least two vertices;

(b) each \( E_i \in E \) induces a complete bipartite subgraph \( B_i \) of \( G \) with each part of the bipartition a subset of some \( V \in V_2 \) with size \( |V| - 1 \);

(c) if \( v \in V \in V_2 \), then \( v \) belongs to at most \( |V| - 1 \) complete bipartite graphs described in (b);

(d) if two distinct complete bipartite graphs \( B_i \) and \( B_j \) in (b) have parts contained in the same \( V \in V_2 \), then \( B_i \) and \( B_j \) have exactly \( |V| - 2 \) common vertices, and all of them are in \( V \);
(e) \(2|E| = \sum_{x \in V_2} |V| - |V_1|\).

**Proof** For a graph \(H\) and a vertex \(v\) of \(H\), denote by \(A_H(v)\) the set of arcs of \(H\) with tail \(v\).

**Necessity:** Suppose that \(G\) is isomorphic to \(X(H)\) for some graph \(H\). We identify \(G\) with \(X(H)\). Let \(\mathcal{V}_1 := \{A_H(v) : \deg_H(v) = 1\}\) and \(\mathcal{V}_2 := \{A_H(v) : \deg_H(v) \geq 2\}\). Then each element of \(\mathcal{V}_1 \cup \mathcal{V}_2\) is an independent set of \(G\), and each edge of \(G\) occurs only between distinct elements \(A_H(u), A_H(v)\) of \(\mathcal{V}_2\) with \(u, v\) adjacent in \(H\). For each pair of adjacent vertices \(u, v\) of \(H\) with \(\deg_H(u) \geq 2\) and \(\deg_H(v) \geq 2\), the set \(E_{\{u,v\}}\) of edges of \(G\) between \(A_H(u)\) and \(A_H(v)\) induces a complete bipartite subgraph of \(G\). Denote by \(\mathcal{E}\) the family of such \(E_{\{u,v\}}\). It is straightforward to verify that \(\mathcal{V} := \mathcal{V}_1 \cup \mathcal{V}_2\) is a partition of \(V(G)\) and \(\mathcal{E}\) is a partition of \(E(G)\) such that (a)-(e) are satisfied.

**Sufficiency:** Suppose that \(V(G)\) admits a partition \(\mathcal{V} := \mathcal{V}_1 \cup \mathcal{V}_2\) and \(E(G)\) admits a partition \(\mathcal{E}\) satisfying (a)-(e). We construct a graph \(H\) such that \(X(H)\) is isomorphic to \(G\).

We construct for each \(V_x \in \mathcal{V}\) a vertex \(x\) of \(H\). We say that \(x\) represents \(V_x\), and that \(x\) is of type \(V_1\) or \(V_2\) according to whether \(V_x\) belongs to \(\mathcal{V}_1\) or \(\mathcal{V}_2\). For each bipartite graph \(B_i\) as in (b), there are distinct elements \(V_x, y \in V_2\) and vertices \(v_{i,x} \in V_x, v_{i,y} \in V_y\) such that \(\{x, y\} = \{v_{i,x}\}, \{v_{i,y}\}\) is the bipartition of \(B_i\). The pair \(\{x, y\}\) is determined uniquely by \(B_i\), and vice versa (so we may write \(i = i(x, y)\)), and we add the edge \(\{x, y\}\) to \(H\). For each \(V_x \in \mathcal{V}_2\), denote by \(b_x\) the number of complete bipartite graphs \(B_i\) as in (b) that contain at least one vertex of \(V_x\). By (b), one part of the bipartition of each of such graphs \(B_i\) must be a subset of \(V_x\) with size \(|V_x| - 1\). On the other hand, by (c) each \(v \in V_x\) belongs to at most \(|V_x| - 1\) such complete bipartite graphs \(B_i\). By counting the number of ordered pairs \((v, B_i)\) with \(v \in V_x \cap V(B_i)\), we obtain \(b_x(|V_x| - 1) \geq |V_x|(|V_x| - 1),\) yielding \(b_x \leq |V_x|\). We then add \(|V_x| - b_x\) edges to \(H\) joining \(x\) to \(|V_x| - b_x\) vertices of type \(V_1\), in such a way that no vertex of type \(V_1\) is repeatedly used. Thus each vertex \(x\) of \(H\) of type \(V_2\) has degree \(|V_x|\) in \(H\). Note that the sum \(\sum_{V_x \in \mathcal{V}_2} |V_x|\) counts each edge between two vertices of type \(V_2\) twice, and each edge with one end-vertex of type \(V_1\) once. The total number of vertices of type \(V_1\) required is \(\sum_{V_x \in \mathcal{V}_2} |V_x| - 2|E|\), which agrees with \(|\mathcal{V}_1|\) by (e). This completes the construction of \(H\).

We now prove that \(X(H)\) is isomorphic to \(G\). By the construction above, each vertex \(x\) of \(H\) has degree \(|V_x|\) in \(H\), and the set \(A_H(x)\) of arcs of \(H\) outgoing from \(x\) is an independent set of \(X(H)\) with size \(|V_x|\). Obviously, such independent sets \(A_H(x)\) of \(X(H)\) are in one-to-one correspondence with the elements \(V_x\) of \(\mathcal{V}\). Note that \(\{A_H(x) : x \in V(H)\}\) is a partition of the vertex set \(A(H)\) of \(X(H)\) which corresponds to the partition \(\mathcal{V} = \{V_x : x \in V(H)\}\) of the vertex set of \(G\).

Let \(\{x, y\}\) be an edge of \(H\) with \(\deg_H(x) \geq 2\) and \(\deg_H(y) \geq 2\). Then \(A_H(x) \cup A_H(y)\) induces a complete bipartite subgraph \(B(x, y)\) of \(X(H)\) with bipartition \(\{A_H(x) - \{xy\}, A_H(y) - \{yx\}\}\). On the other hand, by the definition of \(H\), the edges of \(G\) between \(V_x - \{v_{i,x}\}\) and \(V_y - \{v_{i,y}\}\) induce a complete bipartite subgraph of \(G\) that is equal to \(B_i\) as in (b) with \(i = i(x, y)\). It can be verified that \(\{x, y\} \mapsto B_{i(x, y)}\) defines a bijection from the set of edges \(\{x, y\}\) of \(H\) with \(\deg_H(x) \geq 2\) and \(\deg_H(y) \geq 2\) to the set of complete bipartite graphs as in (b).

For a fixed \(V_x \in \mathcal{V}_2\), the corresponding vertex \(x\) has degree at least 2 in \(H\). For each neighbour \(y\) of \(x\) in \(H\) with \(V_y \in \mathcal{V}_2\), the complete bipartite graph \(B_i\) with \(i = i(x, y)\) has
bipartition \( \{V_x \setminus \{v_i,x\}, V_y \setminus \{v_i,y\}\} \). By the construction of \( H \), one can verify that the mapping \( xy \mapsto v_i, x \) (where \( i = i(x, y) \)) is a bijection from \( \{xy : y \in N_H(x), \deg_H(y) \geq 2\} \) (which is a subset of \( A_H(x) \)) to \( V_x \). Let \( L := \{y \in N_H(x) : \deg_H(y) = 1\} \) be the set of leaf-neighbours of \( x \), and \( W_x \) be the set of vertices \( w \in V_x \) such that there exists no \( B_i \) as in (b) with \( V(B_i) \cap V_x = V_x \setminus \{w\} \). Then \( |L| = |W_x| \) and so we can choose a bijection (in an arbitrary manner) between \( L \) and \( W_x \). Finally, we choose an arbitrary bijection between the set of arcs of \( H \) starting from leaves and \( V_1 \). Then we have defined a bijection between the vertices of \( X(H) \) and the vertices of \( G \). From the way this bijection is defined it is straightforward to verify that it is an isomorphism between \( X(H) \) and \( G \).

\[ \square \]

### 3 Domination number of 3-arc graphs

Given a graph \( G \), denote by \( G \circ K_1 \) the graph obtained from \( G \) by adding for each \( x \in V(G) \) a new vertex \( x' \) and a new edge joining \( x \) and \( x' \). Define \( \mathcal{A} \) to be the family of graphs depicted in Figure 1.

![Figure 1: Graphs in family \( \mathcal{A} \).](image)

**Lemma 3** Let \( G \) be a connected graph of order \( n \).

(a) If \( \delta(G) \geq 1 \), then \( \gamma(G) \leq \frac{n}{2} \) (\[14\] p.206), and \( \gamma(G) = \frac{n}{2} \) if and only if \( G \cong C_4 \) or \( H \circ K_1 \) for some graph \( H \) (\[4\, 15\]).

(b) if \( \delta(G) \geq 2 \) and \( G \notin \mathcal{A} \), then \( \gamma(G) \leq \frac{2n}{5} \) (\[13\]).

(c) if \( \delta(G) \geq 3 \), then \( \gamma(G) \leq \frac{3n}{8} \) (\[16\]).

Given a graph \( G \), define

\[ V_i(G) := \{x \in V(G) : \deg(x) \geq i\}, \quad i \geq 0. \tag{1} \]

For a fixed subset \( U \) of \( V(G) \), a subset \( D \subseteq V(G) \) is called a \( (G : U) \)-dominating set if \( U \subseteq N[D] \). The \( (G : U) \)-domination number, denoted by \( \gamma(G : U) \), is the minimum cardinality of a \( (G : U) \)-dominating set. Note that a \( (G : U) \)-dominating set needs not be a subset of \( U \), and a \( (G : V(G)) \)-dominating set is precisely an ordinary dominating set of \( G \).

**Lemma 4** Let \( G \) be a graph with order \( n \). Then

\[ \]
(a) \( \gamma(G : V_1(G)) \leq \frac{n}{2}, \) and equality holds if and only if each component of \( G \) is isomorphic to \( C_4 \) or \( H \circ K_1 \) for some connected graph \( H \) (which relies on the component);

(b) \( \gamma(G : V_2(G)) \leq \frac{2n}{5} \) if each component of \( G \) is not isomorphic to a graph in the family \( \mathcal{A} \);

(c) \( \gamma(G : V_3(G)) \leq \frac{3(n+2)}{8} \).

**Proof** Let \( G' \) be the graph obtained from \( G \) by deleting all isolated vertices. Then \( |V(G')| \leq n \) and \( \gamma(G : V_i(G)) \leq \gamma(G') \) for each \( i \geq 1 \).

(a) By Lemma 3(a), \( \gamma(G : V_1(G)) \leq \gamma(G') \leq |V(G')|/2 \leq n/2. \) Thus \( \gamma(G : V_1(G)) = n/2 \) if and only if all equalities throughout this inequality chain hold. The first equality in the chain holds if and only if \( G \) contains no isolated vertex, and the second equality holds if and only if (see Lemma 3(a)) each component of \( G' \) is isomorphic to \( C_4 \) or \( H \circ K_1 \) for some connected graph \( H \). Hence the result in (a) follows.

(b) Denote \( G_2 := G[V_2(G)] \) and \( W := N(V_2(G)) - V_2(G) \). Then \( W \) contains all vertices of \( G \) outside \( V_2(G) \) that have exactly one neighbour in \( V_2(G) \). Thus \( G[W] \) consists of isolated vertices, say, \( x_1, x_2, \ldots, x_l \), where \( l \geq 0 \). Denote by \( x'_i \) the unique neighbour of \( x_i \in W \) in \( V_2(G) \).

Note that it may happen that \( x'_i = x'_j \) for distinct \( x_i, x_j \in W \).

Let \( G_2 := G[V_2(G) \cup W] \). Then each vertex in \( V_2(G) \) has degree at least 2 in \( G_2 \). Define a new graph \( J \) as follows. If \( l = 0 \), set \( J := G_2 \); if \( l \geq 2 \), let \( J \) be the graph obtained from \( G_2 \) by adding all possible edges to \( G[W] \) until it becomes a path of length \( l - 1 \); if \( l = 1 \), let \( J \) be the graph obtained from \( G_2 \) by joining \( x_1 \) to a neighbour of \( x'_1 \) other than \( x_1 \) in \( G_2 \) (such a neighbour exists because \( \deg_{G_2}(x'_1) \geq 2 \)).

It is easy to see that \( J \) has minimum degree 2. Let \( R_1, R_2, \ldots, R_s \) be the components of \( J \). Let \( r_j \) be the order of \( R_j \), and let \( T_j := V_2(R_j) \cap V_2(G) \), for \( 1 \leq j \leq s \). Then \( \gamma(R_j : T_j) \leq \gamma(R_j : V_2(R_j)) = \gamma(R_j) \). We claim that \( \gamma(R_j : T_j) \leq 2r_j/5 \). In fact, if \( R_j \notin \mathcal{A} \), then by Lemma 3(b) we have \( \gamma(R_j : T_j) \leq \gamma(R_j) \leq 2r_j/5 \). Suppose that \( R_j \in \mathcal{A} \). Since no component of \( G \) is isomorphic to a graph in \( \mathcal{A} \), by examining each graph in \( \mathcal{A} \), we see that at least one edge joining two degree-two vertices in \( R_j \) is not in \( G \) (such an edge was introduced in the construction of \( J \)). Thus at least one vertex of \( R_j \) is not in \( T_j \). It is readily seen that \( \gamma(R_j : T_j) \leq 1 < 8/5 = 2r_j/5 \) if \( R_j \cong C_4 \in \mathcal{A} \), and \( \gamma(R_j : T_j) \leq 2 < 14/5 = 2r_j/5 \) otherwise. Since the union of all \( \gamma(R_j : T_j) \)-sets is a \( (G_2 : V_2(G)) \)-dominating set, it follows that \( \gamma(G : V_2(G)) \leq 2|V(G_2)|/5 \leq 2n/5 \).

(c) Denote \( G_3 := G[V_3(G)] \) and \( W := N(V_3(G)) - V_3(G) \). Since each vertex of \( W \) has at least one neighbour in \( V_3(G) \), if a vertex of \( W \) has degree two in \( G[W] \), then it must be in \( V_3(G) \) and so cannot be in \( W \), a contradiction. Therefore, \( \Delta(G[W]) = 1 \) and \( G[W] \) consists of isolated vertices and independent edges. Denote \( W = \{x_1, x_2, \ldots, x_l\} \), where \( l \geq 0 \), and for each \( i \) choose \( x'_i \) to be any (but fixed) neighbour of \( x_i \) in \( V_3(G) \).

Let \( G_3 := G[V_3(G) \cup W] \). Define a new graph \( J \) as follows: If \( l = 0 \), set \( J := G_3 \); if \( l \geq 3 \), let \( J \) be the graph obtained from \( G_3 \) by adding all possible edges to \( G[W] \) until it becomes a cycle of length \( l \); if \( l = 1 \), let \( J \) be the graph obtained from \( G_3 \) by adding two new vertices \( u_1, u_2 \) together with edges \( \{u_1, u_2\}, \{x_1, u_1\}, \{x_1, u_2\}, \{u_1, x'_1\} \) and \( \{u_2, x'_1\} \); if \( l = 2 \), let \( J \) be the graph obtained from \( G_3 \) by adding two new vertices \( u_1, u_2 \) together with all possible edges among \( u_1, u_2, x_1 \) and \( x_2 \).
Note that \( \delta(J) = 3 \). By Lemma 3(c), we have \( \gamma(J) \leq 3|V(J)|/8 \). Let \( D \) be a \( \gamma(J) \)-set. If \( D \cap \{u_1, u_2\} = \emptyset \), then \( D \) is a dominating set of \( G_3 \) and so \( \gamma(G_3) \leq \gamma(J) \). If \( D \cap \{u_1, u_2\} \neq \emptyset \), then by the minimality of \( D \) we have \( D \cap \{x_1, x_2\} = \emptyset \) and \( D \) contains exactly one of \( u_1 \) and \( u_2 \). Thus \((D - \{u_1, u_2\}) \cup \{x_1\}\) is a dominating set of \( G_3 \), and again we have \( \gamma(G_3) \leq \gamma(J) \). Since any dominating set of \( G_3 \) is also a \((G_3 : V_3(G))\)-dominating set, we obtain that \( \gamma(G : V_3(G)) \leq \gamma(G_3) \leq 3|V(J)|/8 \leq 3(|V(G_3)| + 2)/8 \leq 3(n + 2)/8 \). \( \Box \)

**Theorem 5** Let \( G \) be a graph with minimum degree \( \delta := \delta(G) \geq 2 \). Then

\[
\gamma(X(G)) \leq 3\gamma(G) + \min_{S \in Q} \{\gamma(G_S : V_{\delta-1}(G_S))\} - 1, \tag{2}
\]

where \( G_S = G - S \) and \( Q \) is the set of \( \gamma(G) \)-sets of \( G \). Moreover, this bound is attainable.

**Proof** Let \( S \) be a \( \gamma(G) \)-set of \( G \) and denote \( H := G_S \) for simplicity. As in (1), let \( V_j(H) \) denote the set of vertices of \( V(H) \) that have degree at least \( j \) in \( H \). With each \( x \in S \) we associate a set

\[
A(x) := \{xx_1, xx_2, xx_2x_3\} \tag{3}
\]

of three arcs of \( G \), where \( x_1, x_2 \) are distinct neighbours of \( x \), and \( x_3 \) is a neighbour of \( x_2 \) other than \( x \). Since \( \delta \geq 2 \), such a set \( A(x) \) exists for every \( x \in S \). (In general, many sets \( A(x) \) may be obtained this way. We choose one of them arbitrarily and fix it.) Define

\[
A(S) := \cup_{x \in S} A(x) \tag{4}
\]

so that \( |A(S)| = 3|S| = 3\gamma(G) \). Set

\[
W := \{w \in V(G) - S : |N(w) \cap S| = 1\} \tag{5}
\]

\[
U := V(G) - (S \cup W). \tag{6}
\]

Then \( S, W \) and \( U \) form a partition of \( V(G) \). Since each \( w \in W \) has exactly one neighbour in \( S \), it has at least \( \delta - 1 \) neighbours in \( H \). Therefore,

\[
W \subseteq V_{\delta-1}(H). \tag{7}
\]

Thus every \((H : V_{\delta-1}(H))\)-dominating set is also an \((H : W)\)-dominating set, and therefore

\[
\gamma(H : W) \leq \gamma(H : V_{\delta-1}(H)). \tag{8}
\]

Let \( D \) be a minimum \((H : W)\)-dominating set in \( H \). With each vertex \( y \in D \) we choose an arc \( yy' \) such that \( y' \in N(y) \cap S \). Such a vertex \( y' \) exists for every \( y \in D \) because \( y \) is dominated by a vertex in \( S \). Define

\[
A(D) := \{yy' : y \in D\}. \tag{9}
\]

Then \( |A(D)| = |D| = \gamma(H : W) \), and so \( |A(D)| \leq \gamma(H : V_{\delta-1}(H)) \) by (7).

**Claim 1**: \( A(S) \cup A(D) \) is a dominating set of \( X(G) \). Hence

\[
\gamma(X(G)) \leq |A(S) \cup A(D)| \leq |A(S)| + |A(D)| \leq 3\gamma(G) + \gamma(H : V_{\delta-1}(H)). \tag{9}
\]
Proof of Claim 1: It suffices to show that each arc with tail in $S$, $W$ or $U$ is dominated in $X(G)$ by at least one element of $A(S) \cup A(D)$. Let $ab$ be an arc of $G$. When $a \in S$, if $b = a_1$ or $a_2$ (where $a_1$ and $a_2$ are the vertices in the definition of $A(a)$), then clearly $ab \in A(S)$; if $b$ is a neighbour of a other than $a_1$ and $a_2$, then $ab$ is dominated by $a_2a'_2 \in A(S)$. If $a \in U$, then $a$ has at least two neighbours in $S$ and so at least one of them, say, $z \in S$ is different from $b$. Thus $ab$ is dominated by $zv \in A(S)$, where $v \in \{z_1, z_2\} - \{a\}$ with $z_1, z_2$ the neighbours of $z$ used in the definition of $A(z)$. Suppose that $a \in W$ and let $z$ be the unique neighbour of $a$ in $S$. If $b \notin S$, then $ab$ is dominated by $zv \in A(S)$, where $v \in \{z_1, z_2\} - \{a\}$. So we assume $b \in S$, so that $b = z$. If $a \in D$, then $ab \in A(D)$ by the definition of $A(D)$. If $a \notin D$, then $a$ is dominated by some vertex $y \in D$ as $D$ dominates all vertices of $W$ in $H$ including $a$. Hence $ab$ is dominated by $yy' \in A(D)$ in $X(G)$. This completes the proof of Claim 1.

In what follows we will show that the upper bound in (9) can be decreased by one. In fact, in the case when $A(S) \cap A(D) \neq \emptyset$, we have $|A(S) \cup A(D)| \leq |A(S)| + |A(D)| - 1$ and similar to (8) we obtain $\gamma(X(G)) \leq 3\gamma(G) + \gamma(H : V_{\delta - 1}(H)) - 1$. We now prove that the same bound holds when $A(S) \cap A(D) = \emptyset$. This is achieved by proving:

Claim 2: If $A(S) \cap A(D) = \emptyset$, then we can modify $A(S) \cup A(D)$ to obtain a new dominating set $A_1(S) \cup A_1(D)$ of $X(G)$ with size at most $|A(S) \cup A(D)| - 1$.

Proof of Claim 2: We first deal with the case where $|S| = 1$ (that is, $\gamma(G) = 1$). In this case, $D \neq \emptyset$ and $W = V(G) - S$ (and so $U = \emptyset$). Denote $S = \{x\}$ and let $y \in D$. Since $\delta \geq 2$, we may choose a neighbour $z \neq x$ of $y$. Let $A_1(S) := \{xz, zy, xy\}$. Note that $A(D)$ contains an arc $yy'$ with $y' \in N(y) \cap S$. If $yy' \neq xy$, then let $A_1(D)$ be obtained from $A(D)$ by replacing this particular arc $yy'$ by $yx$ but retaining all other $uv \in A(D)$ in (8); otherwise, let $A_1(D) := A(D)$. Since $yx$ appears in both $A_1(S)$ and $A_1(D)$, we have $|A_1(S) \cup A_1(D)| = |A(S) \cup A(D)| - 1$. We claim that each arc of $G$ is dominated by $A_1(S) \cup A_1(D)$. In fact, let $ab$ be an arbitrary arc of $G$. Since $W = V(G) - S$, either $a \in S$ or $a \in W$. If $a = x \in S$, then $ab$ is dominated by $zy$ or $ab = xz$. Suppose that $a \in W$. If $b = x \in S$, then $ab$ is dominated by $zy$ or $ab = xz$. If $b \notin S$, then $ab$ is dominated by either $xz$ or $yx$. Therefore, $A_1(S) \cup A_1(D)$ is a dominating set of $X(G)$.

In the rest proof of Claim 2, we assume that $|S| \geq 2$.

Case 1: $S$ is not an independent set of $G$. That is, there is an edge joining two vertices, say $x$ and $y$, of $S$. Let $x' \neq y$ be a neighbour of $x$ and $y' \neq x$ be a neighbour of $y$. Let $A_1(S)$ be obtained from $A(S)$ by replacing $A(x)$ by $\{xx', xy, yy'\}$, $A(y)$ by $\{yy', yx, xx'\}$, but leaving other $A(u)$ for $u \in S - \{x, y\}$ unchanged, in (8). Since $\{xx', xy, yy'\}$ and $\{yy', yx, xx'\}$ have two common arcs, we have $|A_1(S)| = |A(S)| - 2$. Set $A_1(D) := A(D)$. Then $|A_1(S) \cup A_1(D)| = |A(S) \cup A(D)| - 2$ and one can show that $A_1(S) \cup A_1(D)$ is a dominating set of $X(G)$.

Case 2: $S$ is an independent set of $G$ and $U \neq \emptyset$. Let $z \in U$ and let $x, y \in S$ be distinct neighbours of $z$. If $\deg(z) \geq 3$, let $z'$ be a neighbour of $z$ other than $x$ and $y$. Since $\delta \geq 2$, we may choose a neighbour $x' \neq z$ of $x$ and a neighbour $y' \neq z$ of $y$. Let $A_1(S)$ be obtained from $A(S)$ by replacing $A(x)$ by $\{xx', xz, zz'\}$, $A(y)$ by $\{yy', yz, zz'\}$, but leaving other $A(u)$ with $u \in S - \{x, y\}$ unchanged, in (8). Since $\{xx', xz, zz'\}$ and $\{yy', yz, zz'\}$ have one arc in common, we have $|A_1(S)| = |A(S)| - 1$. If $\deg(z) = 2$, then let $A_1(S)$ be obtained from $A(S)$
by replacing $A(x)$ by $\{xz,zy\}$, $A(y)$ by $\{yz,zx\}$, but leaving other $A(u)$ with $u \in S - \{x,y\}$ unchanged, in \([4]\). Then $|A_1(S)| = |A(S)| - 2$. Set $A_1(D) := A(D)$ regardless of the degree of $z$. Then $|A_1(S) \cup A_1(D)| \leq |A(S) \cup A(D)| - 1$ and $A_1(S) \cup A_1(D)$ is a dominating set of $X(G)$.

**Case 3:** $S$ is an independent set of $G$ and $U = \emptyset$. Then $W = V(G) - S \neq \emptyset$, $|A(D)| \geq 1$ and $|W| \geq 1$. Choose a vertex $z \in W - D$. Since $S$ is a dominating set of $G$, we may choose a neighbour $x \in S$ of $z$. Similarly, we may choose a neighbour $v \in D$ of $z$ in $H$ and a neighbour $u \in S$ of $v$ in $G$. It may happen that $u = x$, but this will not affect our subsequent proof.

Let $A_1(S)$ be obtained from $A(S)$ by replacing $A(x)$ by $\{xz,zy\}$, but leaving other $A(y)$ with $y \in S - \{x\}$ unchanged, in \([4]\). Since $v \in D$, $A(D)$ contains an arc $vv'$ with $v' \in N(v) \cap S$. If $vv' \neq vu$, then let $A_1(D)$ be obtained from $A(D)$ by replacing this particular arc $vv'$ by $vu$ but retaining all other $yy' \in A(D)$ in \([4]\); otherwise, let $A_1(D) := A(D)$. Since $vu$ appears in both $A_1(D)$ and $A_1(S)$, we have $|A_1(S) \cup A_1(D)| = |A(S) \cup A(D)| - 1$. We now show that $A_1(S) \cup A_1(D)$ is a dominating set of $X(G)$. In fact, by the definition of $A_1(D)$ and $A_1(S)$ one can see that every arc with tail not in $N[x]$ that is dominated by $A(S) \cup A(D)$ is now dominated by $A_1(S) \cup A_1(D)$. Let $ab$ be an arc with $a \in N[x]$. If $a = x$, then $ab = xz \in A_1(S)$ or $ab$ is dominated by $zv \in A_1(S)$. Suppose that $a \in N(x)$. If $a = z$, then $ab = zv$ or $ab$ is dominated by $vu \in A_1(S)$. Suppose that $a \in N(x) - \{z\}$. If $b \neq x$, then $ab$ is dominated by $xz \in A_1(S)$. If $b = x$, then either $ab \in A_1(D)$ when $a \in D$, or $ab$ is dominated by $yy' \in A_1(D)$, where $y$ is a vertex in $D$ that dominates $a$ in $H$. Therefore, $A_1(S) \cup A_1(D)$ is a dominating set of $X(G)$.

So far we have completed the proof of Claim 2. By this claim (and the discussion before it) and \([3]\), we obtain $\gamma(X(G)) \leq |A_1(S) \cup A_1(D)| \leq |A(S) \cup A(D)| - 1 \leq 3\gamma(G) + \gamma(H : V_{\delta - 1}(H)) - 1$. Since this holds for any $\gamma(G)$-set $S$ of $G$ and since $H = G_S$, we obtain \([2]\) immediately.

It is easy to see that the bound in \([2]\) is achieved by the 3-cycle $C_3$ and in general by any friendship graph (that is, a graph obtained from a number of copies of $C_3$ by identifying one vertex from each copy to form a single vertex).

It was proved in \([8]\) that, for any connected graph $G$ of order $n \geq 4$ and minimum degree at least 2, we have $\gamma(X(G)) \leq n$. Combining this with $n/(1 + \Delta(G)) \leq \gamma(G)$ \([3]\), we then have

$$\gamma(X(G)) \leq (1 + \Delta(G))\gamma(G).$$

(10)

The following theorem improves this bound for any graph $G$ with $\Delta(G) \geq \delta(G) = 2, 3$ or 4. In Theorem \([7]\) we will give a further improved bound for any claw-free graph with minimum degree at least two.

**Theorem 6** Let $G$ be a graph.

(a) If $\delta(G) = 2$, then $\gamma(X(G)) \leq \left(\frac{\Delta(G)}{2} + 3\right)\gamma(G) - 1$;

(b) if $\delta(G) = 3$, then $\gamma(X(G)) \leq \left(\frac{2\Delta(G)}{5} + 3\right)\gamma(G) - 1$;

(c) if $\delta(G) = 4$, then $\gamma(X(G)) \leq \left(\frac{3(\Delta(G) + 2)}{8} + 3\right)\gamma(G) - 1$. 

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Moreover, the bounds in (a) and (b) are attainable.

**Proof** Denote \( \delta := \delta(G) \). As in the proof of Theorem 5, let \( S \) be a \( \gamma(G) \)-set of \( G \) and denote \( H := G - S \). Let \( H' \) be obtained from \( H \) by deleting all isolated vertices. Then \( \gamma(H : V_1(H)) \leq \gamma(H') \).

(a) If \( \delta = 2 \), then by Theorems 4 and Lemma 3(a) and the fact that \( |V(G)| \leq (\Delta(G) + 1)\gamma(G) \), we have

\[
\gamma(X(G)) \leq 3\gamma(G) + \gamma(H : V_1(H)) - 1 \\
\leq 3\gamma(G) + \frac{n - \gamma(G)}{2} - 1 \\
\leq 3\gamma(G) + \frac{(\Delta(G) + 1)\gamma(G) - \gamma(G)}{2} - 1 \\
= \left( \frac{\Delta(G)}{2} + 3 \right) \gamma(G) - 1.
\]

(b) Assume \( \delta = 3 \). Let \( A(S) \) and \( W \) be defined by (4) and (5), respectively, with \( A(x) \) as given in (3) for each \( x \in S \). By (6), we have \( W \subseteq V_2(H) \).

Let \( R_1, R_2, \ldots, R_s \) be the set of components of \( H' \), and let \( r_j \) be the order of \( R_j \). We are going to prove that we can choose an appropriate subset \( D_j \) of \( V(R_j) \) for each \( j \) such that \( |D_j| \leq 2r_j/5 \) and \( A(S) \cup A(\bigcup_{j=1}^s D_j) \) is a dominating set of \( X(G) \).

In fact, if \( R_j \) is not isomorphic to any graph in the family \( \mathcal{A} \) (see Figure 1), then choose any minimum dominating set of \( R_j : V_2(R_j) \) to be a minimum \( (R_j : V_2(R_j)) \)-dominating set. By Lemma 3(b), we have \( |D_j| = \gamma(R_j : V_2(R_j)) \leq 2r_j/5 \).

Suppose that a component \( R_j \) is isomorphic to some graph in \( \mathcal{A} \). If \( R_j \) contains a vertex \( z \) which is not in \( W \), then we choose \( D_j \) to be a minimum dominating set of \( R_j - \{z\} \). Since in this case \( D_j \) is also an \((R_j : W \cap V(R_j))\)-dominating set, we have \( \gamma(R_j : W \cap V(R_j)) \leq \gamma(R_j - \{z\}) = |D_j| \leq 2r_j/5 \) by noting that \( r_j = 4 \) or 7.

Now assume that all vertices of \( R_j \) are in \( W \). Let \( z \) be an arbitrary vertex of \( R_j \) and \( D_j \) a minimum dominating set of \( R_j - \{z\} \). Let \( A(D_j) \) be the set of arcs \( xx' \) of \( G \) such that \( x \in D_j \) and \( x' \) is the unique neighbour of \( x \) in \( S \). It is not hard to verify that \( |D_j| < 2r_j/5 \). Let \( u \) be a neighbour of \( z \) in \( R_j \). Since \( \delta(R_j) = 2 \) (see Figure 1), we can choose a neighbour \( y \) of \( u \) in \( G \) other than \( u' \) and \( z \) (so that \( y \) is in \( R_j \)), where \( u' \) is the unique neighbour of \( u \) in \( S \). In forming \( A(S) \) by (4), we choose \( A(u) \) to be \( \{u'u, uy, u'v\} \), where \( v \) is a neighbour of \( u' \) other than \( u \). Similar to what we did in the proof of Theorem 5 when necessary we may modify \( A(S) \) to obtain a new set \( A(S) \) (also denoted by \( A(S) \)) such that every arc of \( G \) with tail \( z \) is dominated in \( X(G) \) by either \( uy \) or an arc in \( A(z') \), where \( z' \) is the unique neighbour of \( z \) in \( S \). It can be verified that we can always choose an appropriate pair of vertices \( z, u \) such that this happens. In this way we ensure that all arcs emanating from vertices of \( R_j \) are dominated by \( A(S) \cup A(D_j) \) in \( X(G) \).

With \( D_j \) as above we now set \( D := \bigcup_{j=1}^s D_j \). As in (8), choose a set of arcs of \( G \) by setting \( A(D_j) = \{xx' : x \in D_j\} \), where \( x' \) is a neighbour of \( x \) in \( S \). (As seen in the previous paragraph, such a set \( A(D_j) \) is unique when \( R_j \) is isomorphic to some graph in \( \mathcal{A} \) and \( V(R_j) \subseteq W \).) Set \( A(D) = \bigcup_{j=1}^s A(D_j) \). Since \( |D_j| \leq 2r_j/5 \) for each \( j \), we have \( |A(D)| = |D| = \Sigma_{j=1}^s |D_j| \leq \Sigma_{j=1}^s 2r_j/5 \).
2|V(H')|/5 ≤ 2|V(G) − S|/5. Similar to the proof of Theorem 5, once can verify that \( A(S) \cup A(D) \) is a dominating set of \( X(G) \). Thus

\[
\gamma(X(G)) \leq |A(S)| + |A(D)| - 1 \\
\leq 3\gamma(G) + \frac{2|V(G) - S|}{5} - 1 \\
\leq 3\gamma(G) + \frac{2((\Delta(G) + 1)\gamma(G) - \gamma(G))}{5} - 1 \\
= \left( \frac{2\Delta(G)}{5} + 3 \right) \gamma(G) - 1.
\]

(c) If \( \delta = 4 \), then by Theorem 5 and Lemma 4(c),

\[
\gamma(X(G)) \leq 3\gamma(G) + \gamma(H : V_3(H)) - 1 \\
\leq 3\gamma(G) + \frac{3(n - \gamma(G) + 2)}{8} - 1 \\
\leq 3\gamma(G) + \frac{3((\Delta(G) + 1)\gamma(G) - \gamma(G) + 2)}{8} - 1 \\
= \left( \frac{3(\Delta(G) + 2)}{8} + 3 \right) \gamma(G) - 1.
\]

The bounds in (a) and (b) are attained by the complete graphs \( K_3 \) and \( K_4 \) respectively. This completes the proof. \( \square \)

A graph is **claw-free** if it does not contain the complete bipartite graph \( K_{1,3} \) as an induced subgraph. The following result significantly improves the upper bounds in Theorem 3 for claw-free graphs, and it does not require the minimum degree to be 2, 3 or 4.

**Theorem 7** Let \( G \) be a claw-free graph with \( \delta(G) \geq 2 \). Then

\[
\gamma(X(G)) \leq 4\gamma(G)
\]  

and this bound is attainable.

**Proof** Let \( S \) be a \( \gamma(G) \)-set of \( G \). We prove (11) by constructing a dominating set of \( X(G) \) with size at most \( 4\gamma(G) \).

Since \( G \) is claw-free, for each \( x \in V(G) \), the induced subgraph \( G[N(x)] \) has independence number, and hence domination number, at most 2. Thus, for each \( x \in S \), we can choose a dominating set of \( G[N(x)] \) with size 2, say, \( D_x := \{x_1, x_2\} \), where \( x_1, x_2 \in N(x) \). We then associate \( x \) with a set \( A(x) \) of four arcs of \( G \) in the following way. If \( \deg(x) = 2 \), then let \( A(x) = \{xx_1, xx_2, x_1x, x_2x\} \). If \( \deg(x) \geq 3 \), then let \( x_3 \in N(x) - D_x \). Since \( D_x \) is a dominating set of \( G[N(x)] \), \( x_3 \) is adjacent to at least one of \( x_1 \) and \( x_2 \). We may assume without loss of generality that \( x_3 \) is adjacent to \( x_1 \), and then we set \( A(x) = \{xx_1, x_1x_3, x_3x, x_2x\} \). Define

\[
A(S) := \cup_{x \in S} A(x).
\]

We now show that \( A(S) \) is a dominating set of \( X(G) \). To this end it suffices to show that any arc \( ab \) of \( G \) outside \( A(S) \) is dominated in \( X(G) \) by at least one arc in \( A(S) \). In fact, if \( a \in S \), say, \( a = x \in S \), then either \( ab = xx_1 \in A(x) \), or \( b \neq x_1 \) and \( ab \) is dominated by \( x_1x_3 \in A(x) \).
Suppose then that $a \notin S$. Since $S$ is a dominating set of $G$, $a$ has at least one neighbour in $S$, say, $x$. Consider the case $b = x$ first. In this case, if $a \in \{x_2, x_3\}$ then $ab = x_3x$ is dominated by $x_3x$; and if $a \notin \{x_1, x_2, x_3\}$ then one of $x_1x_3$ and $x_2x$ dominates $ab$ since $\{x_1, x_2\}$ is a dominating set of $G[N(x)]$. In the case where $b \neq x$, if $a \neq x_1$ then $ab$ is dominated by $xx_1$, and if $a = x_1$ then either $ab = x_1x_3 \in A(S)$ or $ab$ is dominated by $x_3x$.

So far we have proved that every arc of $G$ outside $A(S)$ is dominated by $A(S)$. Therefore, $\gamma(X(G)) \leq |A(S)| \leq 4|S| = 4\gamma(G)$ and (11) is established.

To show that (11) is sharp, let $G^*$ be obtained from two vertex-disjoint complete graphs $K_s$ and $K_t$ of orders $s, t \geq 2$ respectively by identifying a vertex of $K_s$ with a vertex of $K_t$. Since $\gamma(X(H)) \geq 3$ for any graph $H$ with $\delta(H) \geq 2$ [8, Theorem 7], we have $\gamma(X(G^*)) \geq 3$. However, no three arcs of $G^*$ dominate all other arcs of $G^*$. Therefore, $\gamma(X(G^*)) = 4 = 4\gamma(G^*)$. 

Since line graphs are claw-free, the bound (11) holds in particular when $G$ is the line graph of some graph of minimum degree at least two.

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