On one ansatz for $sl_2$–invariant R–matrices

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Abstract

The spectral decomposition of regular $sl_2$–invariant R–matrices $R(\lambda)$ is studied by means of the method of reduction of the Yang–Baxter equation onto subspaces of a given spin. Restrictions on the possible structure of several highest coefficients in the spectral decomposition are derived. The origin and structure of the exceptional solution in the case of spin $s = 3$ are explained. Analogous analysis is performed for constant R–matrices. In particular, it is shown that the permutation matrix $P$ is a “rigid” solution.

§1. Introduction

The Yang–Baxter equation plays a key role in the quantum inverse scattering method (see, e.g., the reviews [1, 2]). Its braid group form looks as follows

$$R_{12}(\lambda) R_{23}(\lambda + \mu) R_{12}(\mu) = R_{23}(\mu) R_{12}(\lambda + \mu) R_{23}(\lambda).$$

(1)

In this article we will consider the Yang–Baxter equation (1) on the space $V_s \otimes^3$, where $V_s$ is an irreducible finite–dimensional representation of the algebra $sl_2$. The dimension of the representation $V_s$ is $(2s+1)$, where $s$ is a positive integer or semi–integer number (referred to below as spin). Here and below we use the standard notations: the lower indices of $R(\lambda)$ indicate the tensor components of $V_s \otimes^3$ where $R(\lambda)$ acts nontrivially.

An operator–valued function $R(\lambda) : \mathbb{C} \rightarrow \text{End} V_s \otimes^2$ that satisfies (1) is called an R–matrix. We will consider $sl_2$–invariant R–matrices, i.e., those that have the spectral decomposition of the form

$$R(\lambda) = \sum_{j=0}^{2s} r_j(\lambda) P^j.$$

(2)

Here $P^j$ is the projector onto $V_j$ which is the subspace of spin $j$ in $V_s \otimes^2$, and $r_j(\lambda)$ is a scalar function. Additionally, we assume that R–matrices under consideration are regular, unitary, and normalized, that is, the following relations are satisfied

$$r_j(0) = 1, \quad r_j(\lambda)r_j(-\lambda) = 1, \quad r_{2s}(\lambda) = 1.$$

(3)

Let us remark that unitarity is a consequence of regularity and normalization [3].

Since regular R–matrices can be used to construct local integrals of motion for lattice models, in particular for spin chains, the problem of finding all solutions of the

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Yang–Baxter equation satisfying the properties is important for the quantum inverse scattering method. At present, there are known four series of inequivalent \( sl_2 \)-invariant regular solutions and one exceptional solution for \( s = 3 \) (see [4] and references therein). A computer–based check [4] led to a conjecture that this list of solutions is exhausting. However, the corresponding classification theorem has not been proven yet. In the present article, applying the approach developed in [3], we will make some progress in this direction. In particular, we will explain the origin and the structure of the exceptional solution for \( s = 3 \).

The paper is organized as follows. §2 contains analysis of one ansatz for an R–matrix. Although the results presented here are well known, we provide all necessary technical details because our aim is to develop similar technique in a more general case. In §3 we remind briefly the main details of the approach developed in [3] for analysis of \( sl_2 \)– and \( U_q(sl_2) \)–invariant R–matrices. Here we also prove one useful additional relation (Lemma 3). In §4.1 we will demonstrate that analysis of some number of highest coefficients in the spectral decomposition of an R–matrix can be done in a way closely resembling the analysis described in §2. In §§4.2–4.3 we will give details of this analysis. In particular, it turns out that the exceptional solution arises as a consequence of degeneration of a certain set of matrices. In §5 we perform analogous analysis for constant R–matrices. In particular, it is shown that the permutation \( \mathbb{P} \) is a “rigid” solution. The Conclusion summarizes the main results.

§2. Analysis of one ansatz for an R–matrix

Let \( \mathbb{E} \) denote the identity operator on \( V_s \otimes V_s \). For \( s \geq 1 \) let us consider R–matrices of the following form

\[
R(\lambda) = \frac{1}{1 + f(\lambda)} (\mathbb{E} + f(\lambda) \mathbb{P} + g(\lambda) P^0) .
\]

Here \( \mathbb{P} \) is the permutation operator on \( V_s \otimes V_s \). Recall that it can be expressed in terms of projectors:

\[
\mathbb{P} = \sum_{j=0}^{2s} (-1)^{2s-j} P^j .
\]

If the scalar functions \( f(\lambda) \) and \( g(\lambda) \) satisfy the condition \( f(0) = g(0) = 0 \), then [4] is an ansatz for a solution of the Yang–Baxter equation in the class [3]. It turns out that all R–matrices of this type can be described explicitly.

**Lemma 1** The following relations hold on \( V_s \otimes V_s \)

\[
P^0_l P^0_l = P^0_l , \quad \mathbb{P}_l \mathbb{P}_l = \mathbb{E} , \quad P^0_l \mathbb{P}_l = \mathbb{P}_l P^0_l = \xi P^0_l ,
\]

\[
P^0_l \mathbb{P}_l \mathbb{P}_l = \mathbb{P}_l \mathbb{P}_l \mathbb{P}_l ,
\]

\[
\mathbb{P}_l P^0_l \mathbb{P}_l = \mathbb{P}_l P^0_l \mathbb{P}_l , \quad \mathbb{P}_l P^0_l \mathbb{P}_l = \mathbb{P}_l P^0_l \mathbb{P}_l ,
\]

\[
P^0_l P^0_l \mathbb{P}_l = \eta P^0_l , \quad P^0_l P^0_l P^0_l = \eta^2 P^0_l ,
\]

\[
P^0_l P^0_l \mathbb{P}_l = \xi \eta P^0_l \mathbb{P}_l , \quad \mathbb{P}_l P^0_l P^0_l = \xi \eta \mathbb{P}_l P^0_l .
\]
where \( l = \{12\} \), \( l' = \{23\} \) or \( l = \{23\} \), \( l' = \{12\} \), and \( \xi \) and \( \eta \) are scalar constants:

\[
\xi = (-1)^{2s}, \quad \eta = \frac{1}{2s+1}.
\]  

(11)

**Proof.** The third relation in (11) follows from (10). Equalities (7) and (8) are obvious. Relations (9) follow from the well–known relation (see, e.g. [3])

\[
P^0_{12} p^j_{23} p^0_{12} = \frac{2j + 1}{(2s + 1)^2} P^0_{12}.
\]  

(12)

Relation (10) can be derived as follows:

\[
P^0_{12} P^0_{13} P^0_{12} = \xi P^0_{12} \xi P^0_{12} P^0_{13} = \xi \eta P^0_{12} P^0_{13}.
\]  

(13)

□

Substituting (4) in (1) and using the relations of Lemma 1, it is not difficult to check that the Yang–Baxter equation for the ansatz under consideration is equivalent to the following equation

\[
F_{\lambda,\mu} F + G_{\lambda,\mu} G + H_{\lambda,\mu} H + H_{\mu,\lambda} \tilde{H} = 0,
\]  

(14)

where

\[
F = P^0_{12} - P^0_{23}, \quad G = P^0_{12} - P^0_{23}, \\
H = P^0_{12} p^0_{23} - P^0_{12} p^0_{23}, \quad \tilde{H} = P^0_{23} p^0_{12} - P^0_{23} p^0_{12},
\]  

(15)

and

\[
F_{\lambda,\mu} = f(\lambda) + f(\mu) - f(\lambda + \mu),
\]  

(16)

\[
G_{\lambda,\mu} = g(\lambda) + g(\mu) - g(\lambda + \mu) + \xi g(\lambda)g(\mu) + \xi g(\lambda)f(\mu)
\]  

\[
+ g(\lambda)g(\mu) + \eta g(\lambda)g(\mu)f(\lambda + \mu) + \eta^2 g(\lambda)g(\mu)g(\lambda + \mu),
\]  

(17)

\[
H_{\lambda,\mu} = g(\lambda)f(\lambda + \mu) - f(\lambda)g(\lambda + \mu) + \xi g(\lambda)f(\mu)g(\lambda + \mu).
\]  

(18)

**Lemma 2** For \( s \geq 1 \) the matrices \( F, G, H \) and \( \tilde{H} \) in (15) are linearly independent.

The proof is given in Appendix B. As a consequence of Lemma 2 it follows that equations (14) are equivalent to the following system of functional equations

\[
F_{\lambda,\mu} = 0, \quad (19)
\]  

\[
G_{\lambda,\mu} = 0, \quad (20)
\]  

\[
H_{\lambda,\mu} = H_{\mu,\lambda} = 0. \quad (21)
\]  

Analysis of system (19)–(21) is fairly simple. There are three non–trivial cases:

1) \( f(\lambda) \neq 0, g(\lambda) = 0 \). In this case it is obvious from (16) that \( f(\lambda) \) is a linear function. Without loss of generality one can choose \( f(\lambda) = \lambda \).

2) \( f(\lambda) = 0, g(\lambda) \neq 0 \). In this case there remains one equation on \( g(\lambda) \),

\[
g(\lambda) + g(\mu) - g(\lambda + \mu) + \eta g(\lambda)g(\mu) + \eta^2 g(\lambda)g(\mu)g(\lambda + \mu) = 0,
\]  

(22)
which has (for $\eta \neq \frac{1}{2}$) the following solution:

$$g(\lambda) = b \frac{1 - e^{\gamma \lambda}}{e^{\gamma \lambda} - b^2}, \quad b + b^{-1} = \eta^{-1}. \quad (23)$$

Here $\gamma$ is an arbitrary finite constant, which can be chosen unity without loss of generality.

3) $f(\lambda) \neq 0$, $g(\lambda) \neq 0$. One can again choose $f(\lambda) = \lambda$. Introducing a new function $h(\lambda) = f(\lambda)/g(\lambda)$, we can rewrite equations (21) in the following form

$$h(\lambda + \mu) = h(\lambda) - \xi \eta f(\mu) = h(\mu) - \xi \eta f(\lambda). \quad (24)$$

Hence we infer that $h(\lambda)$ is a linear function. Therefore, the solution for $g(\lambda)$ looks as follows

$$g(\lambda) = \frac{\lambda}{\beta - \xi \eta \lambda}. \quad (25)$$

This function solves equation (20) provided that the following restrictions are imposed

$$\xi^2 = 1, \quad \beta = \eta - \xi/2. \quad (26)$$

R–matrices corresponding to the cases 1), 2) and 3) are known as the R–matrices of Yang, Baxter, and Zamolodchikovs, respectively. Analysis presented above shows clearly that there are no other solutions of the form (4). It is remarkable that the ansatz (4) covers three out of the four known series of $sl_2$–invariant regular R–matrices. It is therefore natural to study its generalization that could be analysed in a similar way.

§3. On reduced Yang–Baxter equation

Let us remind the main details of the approach developed in [3] to analysis of $U_q(sl_2)$–invariant R–matrices (we will take into account from the very beginning that $q = 1$ in our case). Let the symbol $[t]$ denote the entire part of a number $t$. The subspace $W_n^{(s)} \subset V_8^{\otimes 3}$ for $n = 0, 1, \ldots, [3s]$ is defined as a linear span of highest weight vectors of spin $(3s - n)$, i.e.

$$W_n^{(s)} = \{ \psi \in V_8^{\otimes 3} \mid S_{123}^+ \psi = 0, \quad S_{123}^- \psi = (3s - n)\psi \}. \quad (27)$$

For a given R–matrix of form (2), we construct a set of diagonal matrices $D^{(n)}(\lambda)$ as follows

$$D^{(n)}_{kk'}(\lambda) = \delta_{kk'} r_{2s-k}(\lambda), \quad \text{where} \quad \begin{cases} 0 \leq k \leq n & \text{for } 0 \leq n \leq 2s; \\ n-2s \leq k \leq 4s-n & \text{for } 2s \leq n \leq [3s]. \end{cases} \quad (28)$$

We also introduce $\hat{D}^{(n)}(\lambda) \equiv A^{(s,n)} D^{(n)}(\lambda) A^{(s,n)}$, where $A^{(s,n)}$ is a certain special matrix with properties described below. Then the condition that the Yang–Baxter equation (11) is fulfilled on the subspace $W_n^{(s)}$ can be written as the following matrix equation

$$D^{(n)}(\lambda) \hat{D}^{(n)}(\lambda + \mu) D^{(n)}(\mu) = \hat{D}^{(n)}(\mu) D^{(n)}(\lambda + \mu) \hat{D}^{(n)}(\lambda), \quad (29)$$

which we will call the reduced Yang–Baxter equation (of level $n$). The initial equation (11) is equivalent to the system of reduced equations (29) with $n = 0, 1, \ldots, [3s]$. 

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The matrix $A^{(s,n)}$ which plays an important role in the outlined approach has the following basic properties. Its entries are expressed (for $q = 1$) in terms of 6–$j$ symbols of the algebra $sl_2$ as follows (see also Appendix A)

$$A^{(s,n)}_{kk'} = (-1)^{2s-n} \sqrt{(4s-2k+1)(4s-2k'+1)} \left\{ \begin{array}{ccc} s & s & 2s-k \\ s & 3s-n & 2s-k' \end{array} \right\}, \quad (30)$$

where $k, k'$ take values as in $[23]$. The matrix $A^{(s,n)}$ is orthogonal, symmetric, and coincides with its own inverse ($t$ stands for matrix transposition)

$$A^{(s,n)} = (A^{(s,n)})^t = (A^{(s,n)})^{-1}. \quad (31)$$

For the purpose of the present work we will need one more property which we formulate as follows.

**Lemma 3** For all $n = 0, \ldots, |3s|$ the following matrix relation holds

$$A^{(s,n)} D_0^{(n)} A^{(s,n)} = (-1)^n D_0^{(n)} A^{(s,n)} D_0^{(n)}, \quad (32)$$

where the diagonal matrix $D_0^{(n)}$ has the form

$$(D_0^{(n)})_{kk'} = (-1)^k \delta_{kk'}, \quad (33)$$

and $k, k'$ take values as in $[23]$.

**Proof.** Let us write out matrix entries of (32) taking into account that $A^{(s,n)}$ is symmetric:

$$\sum_m (-1)^m A^{(s,n)}_{km} A^{(s,n)}_{k'm} = (-1)^n+k+k' A^{(s,n)}_{kk'}. \quad (34)$$

Now, taking into account formula (30), it is easy to see that relation (34) can be reduced to the Racah identity for 6–$j$ symbols (see, e.g. [5])

$$\sum_p (-1)^p (2p+1) \left\{ \begin{array}{ccc} r_1 & r_3 & l \\ r_2 & r_4 & p \end{array} \right\} \left\{ \begin{array}{ccc} r_1 & r_2 & l' \\ r_3 & r_4 & p \end{array} \right\} = (-1)^{l+l'} \left\{ \begin{array}{ccc} r_3 & r_1 & l \\ r_2 & r_4 & l' \end{array} \right\}, \quad (35)$$

where we have to set $r_1 = r_2 = r_3 = s, r_4 = 3s-n, l = 2s-k, l' = 2s-k', p = 2s-m. \Box$

It is obvious from (5) that $D_0^{(n)}$ and $D_0^{(n)} \equiv A^{(s,n)} D_0^{(n)} A^{(s,n)}$ correspond to restriction of operators $\mathbb{P}_{12}$ and $\mathbb{P}_{23}$ onto $W_n^{(s)}$. In particular, reduction of equation (17) on the subspace $W_n^{(s)}$ leads to the following relation

$$D_0^{(n)} A^{(s,n)} D_0^{(n)} A^{(s,n)} D_0^{(n)} A^{(s,n)} D_0^{(n)} A^{(s,n)} D_0^{(n)} A^{(s,n)} = (-1)^n A^{(s,n)} D_0^{(n)} A^{(s,n)} D_0^{(n)} A^{(s,n)} D_0^{(n)} A^{(s,n)}, \quad (36)$$

correctness of which follows immediately from the statement of Lemma 3. Another corollary of Lemma 3 is that $(-1)^n A^{(s,n)}$ corresponds to restriction of the operator $\mathbb{P}_{13} = \mathbb{P}_{12} \mathbb{P}_{23} \mathbb{P}_{12}$ on $W_n^{(s)}$. 

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§4. Partial analysis of a general ansatz

4.1 Derivation of equations

Observe that any sl$_2$-invariant R–matrix of spin $s \geq 1$ can be represented by the following ansatz

$$R(\lambda) = \frac{1}{1 + f(\lambda)} (\mathbb{E} + f(\lambda) \mathbb{P} + g(\lambda) P^{2s-m} + \sum_{j=0}^{2s-m'} \tilde{r}_j(\lambda) P^j),$$  \hspace{1cm} (37)

where $2 \leq m \leq 2s$ and $m < m'$ (if $m = 2s$ then the last sum in (37) is omitted). Below we will assume that $g(\lambda) \neq 0$, since otherwise (37) belongs to the known case 1) in §2. The regularity requirement imposes the condition

$$f(0) = g(0) = \tilde{r}_j(0) = 0.$$ \hspace{1cm} (38)

Let $\pi^{(m,n)}$ denote a matrix such that $(\pi^{(m,n)})_{kk'} = \delta_{km}\delta_{k'm}$, $k = 0, \ldots, n$. Then $\pi^{(m,n)}$ and $\hat{\pi}^{(m,n)} \equiv A^{(s,n)}\pi^{(m,n)}A^{(s,n)}$ correspond to restriction of the operators $P^{2s-m}$ and $P^{2s-m}$ on $W_n^{(s)}$. Notice that $\pi^{(m,n)}$ and $\hat{\pi}^{(m,n)}$ are projectors of rank 1.

For $n < m'$, the matrices $D^{(n)}(\lambda)$ and $\hat{D}^{(n)}(\lambda)$ corresponding to the R–matrix (37) look as follows

$$D^{(n)}(\lambda) = \frac{1}{1 + f(\lambda)} (\mathbb{E} + f(\lambda) D_0^{(n)} + \theta_{mn} g(\lambda) \pi^{(m,n)}),$$

$$\hat{D}^{(n)}(\lambda) = \frac{1}{1 + f(\lambda)} (\mathbb{E} + f(\lambda) \hat{D}_0^{(n)} + \theta_{mn} g(\lambda) \hat{\pi}^{(m,n)}),$$  \hspace{1cm} (39)

where $\theta_{mn} = 0$ for $n < m$ and $\theta_{mn} = 1$ for $m \leq n < m'$.

The following observation is a key place of the present work: analysis of the reduced Yang–Baxter equation (29) for the ansatz (39) is absolutely analogous (except for one special case) to analysis of equation (11) for the ansatz (10) given in §2. This observation is based on the following assertion:

**Lemma 4** Relations (6)–(10) of Lemma 1 remain true after the replacement

$$\mathbb{P}_l \rightarrow D_0^{(n)}, \quad \mathbb{P}_r \rightarrow \hat{D}_0^{(n)}, \quad P_0^{l} \rightarrow \pi^{(m,n)}, \quad P_0^{r} \rightarrow \hat{\pi}^{(m,n)},$$  \hspace{1cm} (40)

as well as after the replacement

$$\mathbb{P}_l \rightarrow \hat{D}_0^{(n)}, \quad \mathbb{P}_r \rightarrow D_0^{(n)}, \quad P_0^{l} \rightarrow \hat{\pi}^{(m,n)}, \quad P_0^{r} \rightarrow \pi^{(m,n)}.$$ \hspace{1cm} (41)

The corresponding scalar constants $\xi$ and $\eta$ become dependent on $m$ and $n$:

$$\xi_m = (-1)^m, \quad \eta_{m,n} = (-1)^n A^{(s,n)}_{mm}.$$  \hspace{1cm} (42)

**Proof.** The analogues of relations (6) follow from the definition of matrices $D_0^{(n)}$, $\hat{D}_0^{(n)}$, $\pi^{(m,n)}$, $\hat{\pi}^{(m,n)}$ and the property $(A^{(s,n)})^2 = \mathbb{E}$. The analogue of relation (7) is identity (38) which we have established above. The analogues of relations (8) can be reduced to the identity

$$\pi^{(m,n)} A^{(s,n)} D_0^{(n)} A^{(s,n)} D_0^{(n)} A^{(s,n)} D_0^{(n)} A^{(s,n)} D_0^{(n)} A^{(s,n)} \pi^{(m,n)},$$  \hspace{1cm} (43)
which is easily verified with the help of relation (32) and the analogues of relations (10).

The analogues of relations (9) and (10) are derived as follows:

\[
\begin{align*}
\pi^{(m,n)}&=\pi^{(m,n)}A^{(s,n)}D_0^{(n)}A^{(s,n)}D_0^{(n)}\pi^{(m,n)}\overset{32}{=}(-1)^n\pi^{(m,n)}D_0^{(n)}A^{(s,n)}D_0^{(n)}\pi^{(m,n)} \\
&=(-1)^n\pi^{(m,n)}A^{(s,n)}\pi^{(m,n)}=(-1)^nA^{(s,n)}\pi^{(m,n)}=\eta_{m,n}\pi^{(m,n)}, \\
\pi^{(m,n)}\pi^{(m,n)}&=\pi^{(m,n)}A^{(s,n)}\pi^{(m,n)}A^{(s,n)}\pi^{(m,n)}=(A^{(s,n)})^2\pi^{(m,n)}=\eta^2_{m,n}\pi^{(m,n)}, \\
\pi^{(m,n)}\hat{\pi}^{(m,n)}D_0^{(n)}&=\pi^{(m,n)}A^{(s,n)}\pi^{(m,n)}A^{(s,n)}D_0^{(n)}D_0^{(n)}A^{(s,n)} \\
&=A_{mm}\pi^{(m,n)}A^{(s,n)}D_0^{(n)}\pi^{(m,n)}A^{(s,n)}A^{(s,n)}D_0^{(n)}\pi^{(m,n)}A^{(s,n)} \\
&=\xi_{m,n}\pi^{(m,n)}A^{(s,n)}D_0^{(n)}A^{(s,n)}\pi^{(m,n)}D_0^{(n)}.
\end{align*}
\]

Let us emphasize that Lemma 3 plays a key role in this proof. \(\square\)

The derivation of equations (14) is based only on relations of Lemma 1. Therefore, Lemma 4 implies that for the R–matrix (37) the reduced Yang–Baxter equation at levels \(n<\pi\) for the ansatz (37) to have

\[
\pi^{(m,n)}\pi^{(m,n)}\pi^{(m,n)}D_0^{(n)}=\pi^{(m,n)}A^{(s,n)}\pi^{(m,n)}A^{(s,n)}\pi^{(m,n)}A^{(s,n)}D_0^{(n)}D_0^{(n)}A^{(s,n)}A^{(s,n)}D_0^{(n)}A^{(s,n)}\pi^{(m,n)}\pi^{(m,n)}D_0^{(n)}.
\]

4.2 Analysis of equations in the case \(f(\lambda)=0\)

Assuming that \(g(\lambda)\neq0\) in (37), let us consider first the case \(f(\lambda)=0\). In this case equation (14) at level \(n=m\) is equivalent to equation \(G^{(m,m)}G^{(m,m)}=0\), i.e., to equation (22) for \(g(\lambda)\), where \(\eta\) has the form

\[
\eta_{m,m}=(-1)^{m}A^{(s,m)}m_{m}\pi^{(m,m)}=\frac{2s)!}{(2s-m)!}\frac{(4s-2m+1)!}{(4s-m+1)!}.
\]

For \(2\leq m\leq2S\) and \(S\geq1\) we have \(|A^{(s,m)}m_{m}|<\frac{1}{2}\). Therefore, \(g(\lambda)\) is given by (22), where \(\eta=\eta_{m,m}\).

Further analysis of the case \(f(\lambda)=0\) naturally leads to a question: is it possible for the ansatz (37) to have \(m'>(m+1)\)? This is possible only if the already found function \(g(\lambda)\) solves equation (22) at level \(n=m+1\), that is, only if \(\eta^2\) takes the same value for levels \(n=m\) and \(n=m+1\). According to (12), the condition \(\eta^2_{m,m}=\eta^2_{m,m+1}\) is equivalent to the requirement

\[
|A^{(s,m)}m_{m}|=|A^{(s,m+1)}m_{m}|.
\]
However, it is easy to derive from formula (52) that

\[ A^{(s,m+1)}_{mm} = \frac{m^2 - m - 3ms + s}{2s} A^{(s,m)}_{mm}. \]  

Since \( m^2 - m - 3ms + 3s < 0 \) for \( 2 \leq m \leq 2s \), we infer that (53) cannot hold for these values of \( m \). Thus, we conclude that \( m' = m + 1 \).

### 4.3 Analysis of equations in the case \( f(\lambda) \neq 0 \)

Let us now turn to the case \( f(\lambda) \neq 0 \). Equations (14) at levels \( n = 1, \ldots, m - 1 \) are equivalent to equation \( F_{\lambda,\mu} F^{(m,n)} = 0 \), i.e., to equation (19) for \( f(\lambda) \). Therefore, without loss of generality we can choose \( f(\lambda) = \lambda \).

In order to analyze equation (14) for \( n \geq m \), it is important to notice that the analogue of Lemma 2 is in general not true. That is, matrices (47) can be linearly dependent. Observe that matrices \( F^{(m,n)} \) and \( G^{(m,n)} \) are symmetric and obviously linearly independent, whereas \( H^{(m,n)} \) and \( \tilde{H}^{(m,n)} \) are transposed to each other: \( \tilde{H}^{(m,n)} = (H^{(m,n)})^t \). It turns out that the following relations

\[ \tilde{H}^{(m,n)} = H^{(m,n)}, \]
\[ H^{(m,n)} + \tilde{H}^{(m,n)} = \beta G^{(m,n)}, \]

where \( \beta \) is a scalar constant, can hold only simultaneously. The case in which these relations do take place we will call an exceptional one.

**Lemma 5** For \( m \geq 2 \), each of relations (51) and (52) holds only for \( m = 3, n = 4 \). In this case relation (52) holds in the following form

\[ H^{(3,4)} + \tilde{H}^{(3,4)} = 2G^{(3,4)}. \]

The proof is given in Appendix B.

In a generic case we have \( H^{(m,n)} \neq \tilde{H}^{(m,n)} \). Therefore, the antisymmetric matrix part of equation (14) imposes the condition

\[ H^{(m,n)}_{\lambda,\mu} = H^{(m,n)}_{\mu,\lambda}, \]

and its symmetric part looks like following

\[ F_{\lambda,\mu} F^{(m,n)} + G^{(m,n)}_{\lambda,\mu} G^{(m,n)} + \frac{1}{2} (H^{(m,n)}_{\lambda,\mu} + H^{(m,n)}_{\mu,\lambda}) (H^{(m,n)} + \tilde{H}^{(m,n)}) = 0. \]

If \( F^{(m,n)}, G^{(m,n)} \) and \((H^{(m,n)} + \tilde{H}^{(m,n)})\) are linearly independent, then equations (54)–(55) lead to the system of functional equations (19)–(21). If, however, \( H^{(m,n)} + \tilde{H}^{(m,n)} = \beta G^{(m,n)} + \beta F^{(m,n)} \), where \( \beta \neq 0 \), then (55) is equivalent to the following system

\[ 2 F_{\lambda,\mu} + \beta (H^{(m,n)}_{\lambda,\mu} + H^{(m,n)}_{\mu,\lambda}) = 0, \]
\[ 2 G^{(m,n)}_{\lambda,\mu} + \beta (H^{(m,n)}_{\lambda,\mu} + H^{(m,n)}_{\mu,\lambda}) = 0. \]

Since our choice \( f(\lambda) = \lambda \) has already ensured equality \( F_{\lambda,\mu} = 0 \), we infer that equations (54), (56)–(57) lead again to system (19)–(21). Thus, we conclude that analysis of a generic case is absolutely analogous to analysis of the case 3) in §2.
Since, by Lemma 5, the level \( n = m \) corresponds to a generic case, the function \( g(\lambda) \) is determined by system (19)–(21) uniquely and has the following form

\[
g(\lambda) = \frac{\lambda}{\eta_{m,m} - \xi_m/2 - \xi_m \eta_{m,m} \lambda},
\]

where \( \xi_m \) and \( \eta_{m,m} \) are given by formulae (42).

Further analysis of the case \( f(\lambda) = \lambda \) leads to a question: is it possible for the ansatz (37) to have \( \lambda' > (m + 1) \)? If \( m \neq 3 \), then the level \( n = m + 1 \) corresponds to a generic case. It is easy to check that the function (58) can satisfy system (19)–(21) for \( n = m + 1 \) only if \( \eta_{m,m} = \eta_{m,m+1} \). This is impossible, as it was shown in §4.2. However, for \( m = 3 \) this level corresponds to the exceptional case. In this case equation (55) is equivalent to the equation

\[
G^{(3,4)}_{\lambda,\mu} + H^{(3,4)}_{\lambda,\mu} + H^{(3,4)}_{\mu,\lambda} = 0,
\]

whilst condition (54) is not imposed (because (14) has no antisymmetric part). Substituting (58) into (59), it is easy to verify that equation (59) is true if \( (\eta_{3,3} - \eta_{3,4})(2\eta_{3,4} - 1) = 0 \). Interestingly, this condition is satisfied for all \( s \geq \frac{3}{2} \), since, according to (48) and (50), we have

\[
\eta_{3,4} = A^{(s,4)}_{33} = 1/2.
\]

Thus, for \( m = 3 \) and for all \( s \geq \frac{3}{2} \) we have \( \lambda' = 4 \) or \( \lambda' = 5 \) in the ansatz (57). Actually, \( \lambda' > 5 \) is possible only for \( s = 3 \). Indeed, the level \( n = 5 \) corresponds to a generic case and hence a necessary condition in order to have \( \lambda' = 6 \) is the equality

\[
A^{(s,3)}_{33} = A^{(s,5)}_{33}.
\]

However, it is not difficult to derive from (82) the following relation

\[
A^{(s,5)}_{33} = \frac{10s^2 - 32s + 21}{s(4s - 7)} A^{(s,3)}_{33},
\]

which shows that (61) can hold only if \( (s - 3)(6s - 7) = 0 \). Finally, since \( A^{(3,6)}_{33} \neq A^{(3,3)}_{33} \), we conclude that \( \lambda' \leq 6 \) for \( s = 3 \).

§5. Analysis of constant R–matrices

We call \( R \in \text{End} V_s \otimes V_s \) a constant R–matrix if it solves the following Yang–Baxter equation

\[
R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}.
\]

We will consider \( sl_2 \)–invariant R–matrices, i.e., those that have the spectral decomposition,

\[
R = \sum_{j=0}^{2s} r_j P^j,
\]

where \( r_j \) are scalar constants. In addition, we will assume that

\[
r_{2s} = 1.
\]

The technique of analysing the spectral decomposition described in §§3–4 is applicable to the case of constant R–matrices as well. In particular, the following remark explains why condition (65) is natural.
Lemma 6 There exist no nontrivial $sl_2$–invariant constant $R$–matrices such that $r_{2s} = 0$.

Proof. Indeed, if such an $R$–matrix exists, then by a suitable normalization it can be brought to the form $R = \mathbb{P} + g P^{2s-m} + \sum_{j=0}^{2s-m-1} r_j P^j$, where $0 < m \leq 2s$. Then the corresponding reduced Yang–Baxter equation (29) at the level $r$ has the following ansatz

$$R = \frac{1}{1 + f} \left( \mathbb{P} + f \mathbb{P} + g P^{2s-m} + \sum_{j=0}^{2s-m'} \tilde{r}_j P^j \right),$$

(66)

where $2 \leq m \leq 2s$ and $m < m'$ (if $m = 2s$, then the last sum in (66) is omitted).

Applying the same arguments as in §3, we can use Lemma 4 to show that the reduced Yang–Baxter equation for the $R$–matrix (66) at levels $n < m'$ is equivalent to the same equation (14), where the matrices $F$, $G$, $H$, $\theta$ are given by formulae (47), and the scalar coefficients are obtained from (42)–(46) by converting the functions $f(\lambda)$ and $g(\lambda)$ into constants $f$ and $g$, i.e.:

$$F = f,$$

(67)

$$G^{(m,n)} = \theta_{m,n} \left( g + 2\xi_m fg + g^2 + \eta_{m,n} g^2 f + \eta_{m,n} g^2 \right),$$

(68)

$$H^{(m,n)} = \theta_{m,n} \xi_m \eta_{m,n} g^2 f.$$

(69)

Equation (14) at levels $n < m$ is equivalent to equation $FE^{(m,n)} = 0$, which can hold only for $f = 0$. As a result, $G^{(m,n)}$ acquires the following form

$$G^{(m,n)} = \theta_{m,n} \left( g + g^2 + \eta_{m,n} g^2 \right),$$

(70)

and then equation (14) at levels $m \leq n < m'$ yields the equation $G^{(m,n)} G^{(m,n)} = 0$, that is, the following quadratic equation on $g$

$$1 + g + \eta_{m,n} g^2 = 0.$$

(71)

Whence for $n = m$ we find $g = \frac{1}{2} (1 \pm \sqrt{1 - 4\eta_{m,m}^2})$, where $\eta_{m,m}$ is given by formula (48). Since, as it was shown in §4.2, we have $\eta_{m,m} \neq \eta_{m,m+1}^2$ for $2 \leq m \leq 2s$, the obtained value of $g$ cannot satisfy (71) for $n > m$. Thus, we conclude that $m' = m + 1$ in (66).

The ansatz (66) covers not all $sl_2$–invariant constant $R$–matrices of spin $s \geq 1$ satisfying (65). Namely, if such an $R$–matrix has $r_{2s-1} = -1$, then it can be represented by the following ansatz

$$R = \mathbb{P} + g P^{2s-m} + \sum_{j=0}^{2s-m'} \tilde{r}_j P^j,$$

(72)

where $2 \leq m \leq 2s$ and $m < m'$ (if $m = 2s$, then the last sum in (72) is omitted).
Using relations of Lemma 4, it is not difficult to check that the reduced Yang–Baxter equation for the R–matrix (72) at the level \( n = m \) is equivalent to the following equation
\[
g^2(1 + \eta_{m,m}g)G + \xi_m g^2(H + \tilde{H}) = 0,
\]
where \( G, H, \tilde{H} \) are given by formulae (47). Since the level \( n = m \) corresponds to a generic case (cf. §4.3), we infer that the only solution of (73) is \( g = 0 \). That is, the ansatz (72) is a solution for (63) only if \( g = r_j = 0 \). Thus, we have shown that the permutation \( P \) is a "rigid" solution, which does not admit a "deformation" of its spectral decomposition in the order \( 2s - 2 \) and lower orders.

Conclusion

The results of analysis carried out in §§4–5 can be formulated as the following restrictions on the structure of the spectral decomposition of R–matrices.

Proposition 1 Let \( R \) be an \( sl_2 \)–invariant solution of equation (63) on \( V^\otimes 3 \) for an integer or half–integer spin \( s \geq 1 \), satisfying condition (65). Then either \( r_{2s-1} = 1 \) or \( r_{2s-1} = -1 \).

I. In the first case
\[
R = E + g P^{2s-m} + \sum_{j=0}^{2s-m-1} \tilde{r}_j P^j,
\]
where \( g \) is a solution of equation (71) and \( 2 \leq m \leq 2s \). If \( m < 2s \), then \( \tilde{r}_{2s-m-1} \neq 0 \).

II. In the second case
\[
R = P.
\]

Proposition 2 Let \( R(\lambda) \) be an \( sl_2 \)–invariant solution of equation (71) on \( V^\otimes 3 \) for an integer or half–integer spin \( s \geq 1 \), satisfying conditions (5). Then either \( r_{2s-1}(\lambda) = 1 \) or \( r_{2s-1}(\lambda) = \frac{1 - \gamma \lambda}{1 + \gamma \lambda} \).

I. In the first case
\[
R(\lambda) = E + g(\lambda) P^{2s-m} + \sum_{j=0}^{2s-m-1} \tilde{r}_j(\lambda) P^j,
\]
where \( 2 \leq m \leq 2s \). If \( m < 2s \), then \( \tilde{r}_{2s-m-1}(\lambda) \neq 0 \). The function \( g(\lambda) \) has the form
\[
g(\lambda) = b \frac{1 - e^{\gamma \lambda}}{e^{\gamma \lambda} - b^2}, \quad b + b^{-1} = \frac{1}{\eta_{m,m}},
\]
where \( \eta_{m,m} \) is given by (48), and \( \gamma \) is some finite constant.

II. In the second case either
\[
R(\lambda) = \frac{1}{1 + \gamma \lambda} (E + \gamma \lambda P),
\]
or
\[
R(\lambda) = \frac{1}{1 + \gamma \lambda} (E + \gamma \lambda P + \frac{\lambda}{\eta_{m,m}(1 - (-1)^m \gamma \lambda) - (-1)^m \frac{1}{2}} P^{2s-m} + \sum_{j=0}^{2s-m'} \tilde{r}_j(\lambda) P^j),
\]
where $\eta_{m,m}$ is given by (48), $\gamma$ is some finite constant, $2 \leq m \leq 2s$ and $m < m'$. If $m < 2s$, then $\bar{r}_{2s-m'}(\lambda) \neq 0$, and moreover

$$m' = m + 1, \quad \text{if } m \neq 3,$$
$$m' \leq 5, \quad \text{if } m = 3, s \neq 3,$$
$$m' \leq 6, \quad \text{if } m = 3, s = 3.$$

The constant $\gamma$ can be set unity without loss of generality.

Let us make several brief remarks concerning Propositions 1 and 2.

According to Lemma 6, if all coefficients in the spectral decomposition of $R(\lambda)$ tend to certain limit values when $\lambda \to \infty$ in some direction in the complex plane, then these values are finite. It follows from Propositions 1 and 2 that the corresponding limit $R(\infty)$ has the form (74) only for solutions of the type (76). In other cases we have $R(\infty) = \mathbb{P}$.

What concerns Proposition 2, we should remark that for $s = 3$ a solution with $m' = 6$ really exists [4]:

$$R(\lambda) = P^6 + \frac{1 - \lambda}{1 + \lambda} P^5 + \frac{4 - \lambda}{4 + \lambda} P^4 + \frac{1 - \lambda}{1 + \lambda} P^3 + \frac{1 - \lambda}{1 + \lambda} P^2 + \frac{1 - \lambda}{1 + \lambda} 6 - \lambda P^0,$$  \hspace{1cm} (80)

It is easy to see that the coefficient of $P^3$ agrees with formula (79). Apart from this case, it is not known whether there exist $R$–matrices of the form (79) with $2 < m < 2s$.

For $m = 2$, the three highest order coefficients in (79),

$$R(\lambda) = P^{2s} + \frac{1 - \lambda}{1 + \lambda} P^{2s-1} + \frac{1 - \lambda}{1 + \lambda} \frac{1 - \frac{2s}{2s-1} \lambda}{1 + \frac{2s}{2s-1} \lambda} P^{2s-2} + \ldots$$  \hspace{1cm} (81)

coincide with the corresponding coefficients of the Kulish–Reshetikhin–Sklyanin R–matrix [6]. Let us mention that it follows from Proposition 2 and the results of [3] that only $R$–matrices of the form (76) and (81) can have $U_q(sl_2)$–invariant analogues.

### Appendix A. Matrix $A^{(s,n)}$

The expression (50) for entries of the matrix $A^{(s,n)}$ can be rewritten in a more explicit form:

$$A_{kk'}^{(s,n)} = F^s_k F^{s}_{k'} \sum_{l=6s-n-mn(k,k')} (-1)^l (l+1)! \left( (l+4s+k)! (l+4s+k')! \right) (l+4s+k)! (l+4s+k')! (l-6s+n+k)! (l-6s+n+k')! (6s-n-l)! (8s-n-k-k'-l)!^{-1},$$  \hspace{1cm} (82)

where $k, k'$ take values as in (28) and

$$F^s_k = (2s-k)! \left( \frac{(k)! (n-k)! (2s-n+k)! (4s-n-k)!}{(4s-k+1)! (6s-n-k+1)!} \right)^{\frac{1}{2}}.$$

The summation in (82) is taken over those $l$ for which the arguments of factorials are nonnegative and it is understood that 0! = 1.
Appendix B. Proof of Lemma 2 and Lemma 5

Proof of Lemma 2 Using the relations \((a, b = 1, 2, 3)\)
\[
\text{tr}_a E_a = 2s + 1, \quad \text{tr}_a P^j_{ab} = \frac{2j + 1}{2s + 1} E_b, \quad \text{tr}_a P_{ab} = E_b,
\]  
we take the trace over the third tensor component of \(F, G, H\) and \(\tilde{H}\), which yields:
\[
\text{tr}_3 F = \eta^{-1} P - E, \quad \text{tr}_3 G = \eta^{-1} P^0 - \eta E, \quad \text{tr}_3 H = \text{tr}_3 \tilde{H} = P^0 - \eta P. \tag{85}
\]
Since \(E, P, P^0\) are linearly independent for \(s \geq 1\), we conclude that \(F, G, H\) and \(\tilde{H}\) can be linearly dependent only if the following equality holds
\[
\eta^2 F - \eta G + \alpha H + \tilde{\alpha} \tilde{H} = 0, \quad \alpha + \tilde{\alpha} = 1. \tag{86}
\]
Multiply \((86)\) by \(P^0_{12}\) from the left, taking into account relations \((6)\), and take the trace over the first tensor component. Using again linear independence of \(E, P, P^0\), we infer that \((86)\) can hold only if \(\alpha = \xi \eta\). Multiplying \((86)\) by \(P^0_{12}\) from the right, we infer analogously that \(\tilde{\alpha} = \xi \eta\). Thus, \((86)\) can hold only if \(\xi \eta = 1/2\), which is impossible as seen from \((11)\). \Box

Proof of Lemma 5 Let us write out entries of the matrices \(H, \tilde{H}\) and \(G\) explicitly:
\[
H_{kk'} = (-1)^{m+k'} \delta_{km} A_{kk'}^{(s,n)} - (-1)^k A_{km}^{(s,n)} A_{mk'}^{(s,n)}, \tag{87}
\]
\[
\tilde{H}_{kk'} = (-1)^{m+k} \delta_{km} A_{kk'}^{(s,n)} - (-1)^{k'} A_{km}^{(s,n)} A_{mk'}^{(s,n)}, \tag{88}
\]
\[
G_{kk'} = \delta_{km} \delta_{k'm} - A_{km}^{(s,n)} A_{mk'}^{(s,n)}. \tag{89}
\]
Recall that \(m \geq 2\) and \(k, k' = 0, 1, \ldots, n\). Comparing \((87)-(89)\) for \(k = k' = 0\), we notice that \((52)\) can hold only for \(\beta = 2\). Further, considering \((87)-(89)\) for \(k' \neq m\), it is easy to see that each of relations \((51)\) and \((52)\) can hold only if
\[
A_{km}^{(s,n)} A_{mk'}^{(s,n)} = 0, \tag{90}
\]
for all values of \(k, k'\) such that \(k, k' \neq m\) and \((-1)^k + (-1)^{k'} \neq 2\). In particular, \((90)\) must hold for \(k = k' = 1\), which implies that \(A_{1m}^{(s,n)} = 0\). As can be seen from \((82)\), the latter equality is possible only if the following condition is satisfied (for \(m < n\))
\[
2m^2 - 2m + n^2 - n = 8ms - 6ns. \tag{91}
\]
Observe that \(A_{km}^{(s,n)} \neq 0\) for all \(k\). Therefore, \((90)\) implies that \(m \neq n\), and also that \(n\) is an even number. Furthermore, if \(m \neq n - 1\), then \((91)\) must hold for \(k = k' = n - 1\), which implies that \(A_{1n-1}^{(s,n)} = 0\). But, as can be inferred again from \((82)\), this is possible only if the following condition is fulfilled
\[
m^2 - m = 4ms - ns. \tag{92}
\]
It is easy to see that the conditions \((91)\) and \((92)\) are incompatible because they imply the equality \(n^2 - n + 4ns = 0\). Thus, the only remaining possibility is the case in which \(m = n - 1\). In this case condition \((91)\) is satisfied only for \(n = 4\). A direct check shows that relations \((51)\) and \((53)\) indeed hold for \(m = 3, n = 4\). \Box
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