Scattering of dislocated wavefronts by vertical vorticity and the
Aharonov-Bohm effect II: Dispersive waves

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Abstract

Previous results on the scattering of surface waves by vertical vorticity on
shallow water are generalized to the case of dispersive water waves. Disper-
sion effects are treated perturbatively around the shallow water limit, to first
order in the ratio of depth to wavelength. The dislocation of the incident
wavefront, analogous to the Aharonov-Bohm effect, is still observed. At short
wavelengths the scattering is qualitatively similar to the nondispersive case.
At moderate wavelengths, however, there are two markedly different scatter-
ing regimes according to whether the capillary length is smaller or larger than
$\sqrt{3}$ times depth. The dislocation is characterized by a parameter that de-
pends both on phase and group velocity. The validity range of the calculation
is the same as in the shallow water case: wavelengths small compared to vor-
tex radius, and low Mach number. The implications of these limitations are
carefully considered.

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I. INTRODUCTION

In a preceding paper [1], hereafter referred to as I, we studied the scattering of surface waves by a stationary vertical vortex in the long wavelength approximation: surface tension was neglected and the fluid depth was supposed to be small compared to wavelength. This is also called the shallow water approximation. There were two motivations for the study of shallow water waves scattering. First, they are non dispersive waves, like acoustic waves in fluids, and it was plausible that a generalization of calculations for sound scattering by vorticity [3] was feasible. Secondly, it was a first attempt towards a quantitative confirmation of the heuristic approach of Berry et al. [2]. The aim of this paper is to go beyond this approximation.

In actual experimental situations [4] the shallow water limit is hard to obtain and, if a quantitative comparison with experiment is desired, it becomes necessary to take into account the finite depth and the surface tension. The main difference between surface waves in shallow water and in deeper water lies in the fact that in the latter case dispersion effects are important: there are two length scales, one associated with depth and the other with surface tension, that are responsible for wave velocity depending on wavelength. In this paper we seek to describe the scattering of surface waves by vorticity in terms of a single differential equation in which surface elevation is the only dependent variable. This is possible in a perturbative treatment away from the shallow water case, and we here present results that correspond to first order corrections.

As in I, we consider the scattering of surface waves by a stationary vortex, in the limit of small Mach number (the velocities of fluid particles are small by comparison with the phase velocity of the waves), $M \ll 1$, and large wavenumber $k$, i.e. $\beta \equiv ka \gg 1$ where $a$ is a typical length associated with the vortex flow. The product $M\beta$ is assumed to be of order 1. In Sec. [4] we derive, from the full hydrodynamic set of equations, an approximation valid to order $O(M)$ (or $O(\beta^{-1})$). First, equations are linearized for small surface perturbations around a steady vertical vortex and then higher order terms in $M$ and $\beta^{-1}$ are discarded.
We shall pay particular attention to the orders of magnitude of the different terms, and will justify the neglect of dissipative effects. The recovery of the shallow water results is subtle since it involves taking the singular limit of vanishing surface tension. There appears a partial differential equation (Eq. (2.38) below) that contains a squared Laplacian, and it is reduced to our previous result (I-14) of ref. [1] in the shallow water limit, i.e. when the layer’s depth is small and surface tension is negligible.

The solution of equation (2.38) is given in Sec. III. The results, given by (3.7), (3.27) and (3.28) and the calculations of the Appendix, seem much more complicated than the similar shallow water results (I-20), (I-24) and (I-25) of [1]. However, this complexity is essentially algebraic, and actually the physical results are rather similar, except when dispersive effects are closely balanced by advection to yield a spiral pattern for the scattered waves. The wavefront dislocation is characterized by a parameter $\alpha$ which is a generalization of the one in [1], and tends towards it smoothly in the shallow water limit. In the dispersive case, $\alpha$ depends on both the phase and group velocity of the waves. We give a perturbative justification of the heuristic argument of Berry et al. [2]. The behaviour of the scattered wave however depends strongly on the ratio of depth to capillary length. We also exhibit two different behaviors, depending on the relative values of the fluid depth and capillary length. At each important step in the calculations, we verify that the shallow water limit is recovered. However, the partial differential equations (2.38) and (I-14) differ in the order of differentiation, with surface tension appearing as a coefficient of the highest derivative term in (2.38); the limit of null surface tension is thus singular. Graphical illustrations of the solution are given in Sec. IV for various values of the dislocation parameter $\alpha$, and for fluid depth larger and smaller than capillary length. An Appendix has some computational details.
II. WATER WAVES IN INTERACTION WITH A VERTICAL VORTEX

Equations for an incompressible fluid of equilibrium depth $h$, free surface $h + \eta(x, y, t)$ with origin of vertical coordinates ($z = 0$) at the bottom, lying in a (uniform) gravitational field $g$ are

\[
\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{\rho} \nabla P - g\hat{z},
\]

\[
\nabla \cdot \mathbf{V} = 0,
\]

(2.1) \hspace{1cm} (2.2)

where $\mathbf{V}$ is the fluid velocity, $P$ the pressure and $\rho$ the (constant) density.

We neglect viscous dissipation. This is justified if the viscous attenuation time of the wave is greater than a typical time for the scattering problem. The attenuation times for gravity waves (GW) and capillary waves (CW) of wavelength $\lambda$ are, respectively, \[ T_{GW} = \frac{\rho g^2 \lambda^4}{2(2\pi)^4 \mu c_{\phi}^4}, \quad T_{CW} = \frac{\rho \lambda^2}{2(2\pi)^2 \mu}, \]

(2.3)

where $\mu$ is the dynamic viscosity of the fluid and $c_{\phi}$ the waves phase velocity. In the case of water, $\mu = 0.01 \text{g/cm}s$, $g = 981 \text{cm/s}^2$ and $\rho = 1 \text{g/cm}^3$. The dispersion relation for capillary-gravity waves in a viscous fluid is fairly involved, but the sum of the two times gives a good estimate. Numerical estimates for waves of several wavelengths are given in Table I. The period of the wave is much smaller than the dissipation time in all cases, and the travel time on the vortex scale, which is period $\times (a/\lambda)$, is also smaller than the attenuation time, at least as long as the vortex radius does not become very large. Thus, there is a range of values of $ka$ where it is reasonable to neglect dissipation.

Boundary conditions are that fluid elements at the free surface of the fluid remain there, that pressure has a discontinuity that is exactly compensated by surface tension, and that there is no vertical velocity at the bottom:

\[
z = h + \eta: \quad V_z = \partial_t \eta + \mathbf{V}_\perp \cdot \nabla \perp \eta
\]

(2.4)

\[
z = h + \eta: \quad P = -\tau \nabla_\perp^2 \eta
\]

(2.5)

\[
z = 0: \quad V_z = 0.
\]

(2.6)
with \( \tau \) the surface tension, \( \mathbf{V}_\perp \) the horizontal velocity and \( \nabla_\perp \) the horizontal gradient. We are interested in small perturbations around a steady, axially symmetric, vertical vortex:

\[
\mathbf{V} = \mathbf{U} + \mathbf{v} \quad \nu \ll U \quad (2.7)
\]

\[
P = P_0 + p_1 \quad p_1 \ll P_0 \quad (2.8)
\]

\[
\eta = \eta_0 + \eta_1 \quad \eta_1 \ll \eta_0 \quad (2.9)
\]

The vertical vortex is given by the (divergenceless) flow

\[
\mathbf{U} = U_0(r)\hat{\theta} \quad (2.10)
\]

in cylindrical coordinates \((r, \theta, z)\), with \((\hat{r}, \hat{\theta}, \hat{z})\) the unit vectors in the radial, tangential and vertical direction respectively.

We first study the zero order situation, \( \mathbf{v} = 0 \):

\[
\mathbf{U} \cdot \nabla \mathbf{U} = -\frac{1}{\rho} \nabla P_0 - g\hat{z}. \quad (2.11)
\]

The \( \hat{\theta} \) component of this equation is an identity. The \( \hat{z} \) component is

\[
0 = -\frac{1}{\rho} \partial_z P_0 - g \quad (2.12)
\]

so that

\[
P_0 = -\rho gz + p_0(x, y, t), \quad (2.13)
\]

and the \( \hat{r} \) component is

\[
\frac{U_0^2}{r} = \frac{1}{\rho} \partial_r p_0. \quad (2.14)
\]

Given a specific function \( U_0 \) this is integrated at once. Concerning boundary conditions, the third boundary condition \((2.6)\) is satisfied identically. The first boundary condition \((2.4)\) says that the surface deformation is independent of polar angle \( \theta \), and the second boundary condition \((2.5)\) gives the free surface \( \eta_0 \) in terms of the pressure:

\[
p_0 = \rho g \eta_0 - \tau \nabla_\perp^2 \eta_0. \quad (2.15)
\]
Writing \( \mathbf{v} = (\mathbf{u}, w) \) and neglecting terms quadratic in \( v \) we have the equations to order one:

\[
(\partial_t + \mathbf{U} \cdot \nabla_\perp) \mathbf{u} + \mathbf{u} \cdot \nabla_\perp \mathbf{U} = -\frac{1}{\rho} \nabla_\perp p_1 \tag{2.16}
\]

\[
(\partial_t + \mathbf{U} \cdot \nabla_\perp) w = -\frac{1}{\rho} \partial_z p_1 \tag{2.17}
\]

\[
\nabla_\perp \cdot \mathbf{u} + \partial_z w = 0. \tag{2.18}
\]

Similarly, the boundary conditions to order one are

\[
z = h + \eta: \quad w = (\partial_t + \mathbf{U} \cdot \nabla_\perp) \eta_1 + \mathbf{u} \cdot \nabla_\perp \eta_0 \tag{2.19}
\]

\[
z = h + \eta: \quad p_1 = \rho g \eta_1 - \tau \nabla_\perp^2 \eta_1 \tag{2.20}
\]

\[
z = 0: \quad w = 0. \tag{2.21}
\]

Using (2.17) we see that the third boundary condition reads

\[
z = 0: \quad \partial_z p_1 = 0. \tag{2.22}
\]

Taking the divergence of Equations (2.16) and (2.17), and using (2.18) gives

\[
\nabla_\perp^2 p_1 + \partial_{zz} p_1 = -2\rho (\nabla_a \mathbf{U}_b) (\nabla_b u_a). \tag{2.23}
\]

Up to now, these equations are exact for linear surface waves interacting with a static vortex. It is the fact that linear waves exist that provides us with another parameter, the phase velocity, with which to compare \( U \). We will now simplify the problem by using the following two approximations: First, the typical velocity of the vortical flow, \( U_0 \), is supposed to be much less than the phase velocity of the wave \( c_\phi \), defined in (2.40). Second, the wavelength \( \lambda \) is supposed to be much smaller than a typical length associated with the vortex, \( a \). In practice, \( a \) will be the core radius of the vortex, and we assume \( ka \gg 1 \) where \( k \equiv 2\pi/\lambda \) is the wave vector. We will denote formally the small quantities by \( \epsilon \). We thus assume \( U_0/c_\phi \equiv M = O(\epsilon) \), where \( M \) will be called the Mach number in analogy with
acoustics, and $ka = O(1/\epsilon)$, and we search for corrections of order $\epsilon$ to the wave equation without permanent vortical flow. To get the relative importance of the terms that appear in the differential equations, we will use the following estimates:

$$
\nabla \perp f_0 \sim \frac{f_0}{a}, \quad \partial_t f_1 \sim \nu f_1, \quad \nabla \perp f_1 \sim kf_1, \quad \partial_z f_1 \sim kf_1,
$$

(2.24)

where $f_0$ is any scalar quantity referring to the vortical flow, $f_1$ any scalar quantity referring to the surface waves and $\nu$ is the wave frequency. We have assumed, as suggested by Eqn. (2.23) that length scales for vertical and horizontal variations of surface waves are the same.

With those estimates, we get from (2.17) that

$$
\frac{k}{\rho} p_1 \sim \nu w \implies p_1 \sim \rho c_\phi w.
$$

(2.25)

Injecting this result in (2.23), the order of magnitude of the left-hand side is $k^2 p_1 = k^2 \rho c_\phi w$, whereas the right-hand side is of order $\rho(U_0/a)kw = k^2 \rho c_\phi w(U_0/c_\phi)(1/ka)$; it is thus negligible, being of order $O(\epsilon^2)$, and (2.23) is replaced by

$$
\nabla^2 \perp p_1 + \partial_{zz} p_1 = 0,
$$

(2.26)

which is the same Laplace equation as in the problem of water waves without the vortex; it has the big advantage of being autonomous and linear in the pressure so that separation of variables can be attempted.

An estimate of the surface elevation $\eta_0$ for the vortex flow, may be obtained from (2.15) and (2.14); it reads

$$
\eta_0 \sim \frac{U_0^2}{g} \left(1 + \frac{l_c^2}{a^2}\right)^{-1},
$$

(2.27)

where we have introduced the capillary length $l_c \equiv \sqrt{\tau/\rho g}$. For water, $\tau = 74$ dyn/cm, so that $l_c \approx 0.32$ cm and the effect of surface tension on the surface deformation of a vortex of size $a \approx 1$ cm, is quite small, of order one per cent. The surface wave elevation, from (2.20) and (2.25), reads

$$
\eta_1 \sim \frac{c_\phi w}{g} \left(1 + k^2 l_c^2\right)^{-1},
$$

(2.28)
with $k^2/l_c^2$ of order one. In the following, we take $\eta_1 \sim c_\phi w/g$, which is numerically inexact but adequate for order of magnitude considerations. In (2.19), the respective orders of magnitude of the different terms are

$$ \partial_t \eta_1 \sim \frac{k c_\phi^2}{g} w, $$

$$ U \cdot \nabla_\perp \eta_1 \sim \frac{k c_\phi^2}{g} \frac{U_0}{c_\phi} w = O(\epsilon)(\partial_t \eta_1), $$

$$ u \cdot \nabla_\perp \eta_0 \sim \frac{k c_\phi^2}{g} \frac{1}{ka} \frac{U_0^2}{c_\phi^2} w = O(\epsilon^3)(\partial_t \eta_1), $$

(2.29)

and the relevant approximation for (2.19), valid to $O(\epsilon)$, reads

$$ z = h: \quad w = (\partial_t + U \cdot \nabla_\perp)\eta_1. $$

(2.30)

In the same approximation, we also write

$$ z = h: \quad p_1 = \rho g \eta_1 - \tau \nabla_\perp^2 \eta_1 $$

(2.31)

In those equations, we neglect $\eta$ in comparison with $h$. If we write $p_1(h+\eta_0) = p_1(h) + \delta p_1$ and use (2.27) we get

$$ \frac{\delta p_1}{p_1} = \frac{\eta_0}{p_1} \left. \frac{dp_1}{dz} \right|_h \sim k \eta_0 \sim k \frac{U_0^2}{g} \frac{U_0^2}{c_\phi^2} = O(\epsilon^2), $$

(2.32)

so that $\delta p_1$ is indeed negligible. The same is true for the other boundary equation.

Let us use now these approximate equations to describe the propagation of surface waves in the vortical flow. We will consider almost shallow water waves, that is, the next order in the small parameter $kh$ of the calculations of ref. (I). In this limit, the pressure is given as a power series in the vertical coordinate $z$,

$$ p_1(r, \theta, z, t) = \sum_{n=0}^{\infty} z^n \Pi_n(r, \theta, t). $$

(2.33)

A proper development would consider the dimensionless variable $z/L$ with $L$ a typical length scale for horizontal variations. However, (2.33) is good enough for our purposes. Should we wish to have exact results in the deep water limit, $kh \to \infty$, this is the step that would break down. Inserting this development in (2.26), we obtain the recursion
\[ \Pi_{n+2} = \frac{-\nabla^2 \Pi_n}{(n+2)(n+1)}, \]  
which, together with the boundary condition \( (2.22) \), gives

\[ p_1(r, \theta, z, t) = \sum_{m=0}^{\infty} (-1)^m z^{2m} \frac{\nabla^2 \Pi}{(2m)!}, \]  
where we have dropped the index ‘0’ from \( \Pi_0 \) for convenience.

We introduce the notation \( D_t \equiv \partial_t + U \cdot \nabla \). Taking only the leading order terms in the small parameter \( kh \) in \( (2.33) \), applying the differential operator \( D_t \) to \( (2.30) \) and taking Eqn. \( (2.17) \) for \( z = h \), we get

\[ D_t^2 \eta_1 = \frac{h}{\rho} \nabla^2 \Pi - \frac{h^3}{6\rho} \nabla^4 \Pi. \]  
Applying \( \nabla^2 \) to \( (2.31) \) and using \( (2.35) \) at the same order, we get

\[ gh \nabla^2 \eta_1 - \frac{\tau h}{\rho} \nabla^4 \eta_1 = \frac{h}{\rho} \nabla^2 \Pi - \frac{h^3}{2\rho} \nabla^4 \Pi. \]  
The surface tension term is considered under the assumption that the capillary length is of the same order of magnitude as the depth of the fluid layer. In this case, those two equations are valid up to order \( O[(kh)^2] \). It is thus legitimate to replace the pressure by its value at order \( O(1) \), \( \Pi = \rho g \eta_1 \), in the term \( \propto \nabla^4 \Pi \), that has the highest derivative. Eliminating the pressure in the resulting equations, we get the final result: a dispersive wave equation for surface elevation \( \eta_1 \) that is analogous to Eqn. \( (1-14) \) of I in the shallow water case. It reads

\[ gh \nabla^2 \eta_1 + \left( \frac{1}{3} gh^3 - \frac{\tau h}{\rho} \right) \nabla^4 \eta_1 - D_t^2 \eta_1 = 0. \]  
This equation includes the leading order correction to the shallow water case. It is valid under the same assumptions (see ref. [I]) concerning wavelength and fluid velocity. It describes the scattering of surface waves over water whose depth is small but not negligible with respect to wavelength, when the wavelength is small compared to the vortex size, when the velocity of the vortex flow is much less than the phase velocity of the waves, and when the waves are of small amplitude.
Without the vortex, when \( U = 0 \) and \( \partial_t = D_t \), plane progressive waves of the form

\[
\eta_1 \propto e^{i(\nu t - k_\perp \cdot r_\perp)}
\]

exist provided frequency \( \nu \) and wave vector \( k_\perp \) are related through the dispersion relation

\[
\nu^2 = ghk^2 + \left( \frac{\tau h}{\rho} - \frac{1}{3}gh^3 \right) k^4.
\] (2.39)

The phase velocity \( c_\phi \) is given by

\[
c_\phi^2 = \frac{\nu^2}{k^2} = gh + \left( \frac{\tau h}{\rho} - \frac{1}{3}gh^3 \right) k^2,
\] (2.40)

and the group velocity \( c_g \) by

\[
c_g = \frac{1}{c_\phi} \left[ gh + 2 \left( \frac{\tau h}{\rho} - \frac{1}{3}gh^3 \right) k^2 \right].
\] (2.41)

Of course, the dispersion relation (2.39) is the approximation to order \( O[(kh)^3] \) of the well known dispersion relation for capillary-gravity waves,

\[
\nu^2 = \left( gk + \frac{\tau k^3}{\rho} \right) \tanh kh.
\] (2.42)

The wave dispersion is thus characterized by a dimensionless parameter \( \delta \) defined by

\[
k^2 \left( \frac{\tau}{\rho g} - \frac{h^2}{3} \right) = \frac{1}{\delta}.
\] (2.43)

It is positive for \( h < \sqrt{3}l_c \), and negative otherwise. We shall consider both cases. In order to be consistent with our approximations, namely, that the fourth order term in (2.38) be a small correction to the other two, the absolute value of \( \delta \) must be large, and the shallow water limit corresponds to \( |\delta| \rightarrow \infty \). For positive \( \delta \), the phase and group velocity read

\[
c_\phi^2 = gh \frac{1 + \delta}{\delta}, \quad c_g = gh \frac{2 + \delta}{c_\phi \delta}, \quad (\delta > 0),
\] (2.44)

whereas for negative values of \( \delta \) they read

\[
c_\phi^2 = gh \frac{|\delta| - 1}{|\delta|}, \quad c_g = gh \frac{|\delta| - 2}{c_\phi |\delta|}, \quad (\delta < 0).
\] (2.45)
The full dispersion relation (2.42) for water waves is either convex or concave at small depth, depending on the sign of $\delta$. The crossover point $h = \sqrt{3}l_c$, derived from the approximate relation (2.39), separates two regions of opposite convexity. Both may be experimentally accessible. The approximation of the hyperbolic tangent is better than 1% for $kh < 0.5$, and better than 5% for $kh < 0.8$. It is thus easy to stay in the small depth limit, $\tanh(kh) \approx kh - (kh)^3/3$, while keeping the wavelength small in comparison with the vortex radius. Using a fluid with high surface tension, like water, leads to a positive $\delta$, that is $h < \sqrt{3}l_c$, whereas the same experiment with a fluid of small surface tension will give a negative $\delta$.

III. SCATTERING OF DISLOCATED WAVES BY A VORTEX

We will now proceed in close analogy with the calculations of ref. [1]. We consider scattering of surface waves by a circular uniform vortex with vorticity $\omega$ and radius $a$ surrounded by an irrotational flow. Using polar coordinates $(r, \theta)$, the background flow is given by

$$U = \begin{cases} \frac{1}{2} \omega r \hat{\theta} & \text{if } r \leq a \\ \frac{\Gamma}{2\pi r} \hat{\theta} & \text{if } r > a \end{cases} \tag{3.1}$$

where $\Gamma = \pi \omega a^2$ is the circulation. Eqn. (2.38) will be solved separately for $r < a$ and $r > a$, and the results matched with a continuity condition for $\eta_1$ and its first three derivatives, since (2.38) is of order four.

We look for solutions that evolve harmonically (with a single global frequency $\nu$) in time, and Fourier decompose them in the polar angle $\theta$:

$$\eta_1 = \text{Re} \left[ \sum_n \tilde{\eta}_1 n e^{i(n\theta - \nu t)} \right], \tag{3.2}$$

where Re stands for the real part. Inserting this development, and the background flow (3.1) for $r \leq a$, that is, inside the vortex, in (2.38), a little calculation shows that the resulting equation factorizes exactly as
\[
\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} + (k_\pm)^2 \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} + (k_-)^2 \right] \bar{\eta}_n = 0, \quad (3.3)
\]

with
\[
(k_\pm)^2 \equiv \frac{1}{2} k^2 \delta \left( -1 \pm \sqrt{1 + \frac{4 (\nu - n\omega/2)^2}{ghk^2\delta}} \right), \quad (\delta > 0), \quad (3.4)
\]
or
\[
(k_\pm)^2 \equiv \frac{1}{2} k^2 |\delta| \left( 1 \pm \sqrt{1 - \frac{4 (\nu - n\omega/2)^2}{ghk^2|\delta|}} \right), \quad (\delta < 0), \quad (3.5)
\]

The two differential operators in (3.3) commute, and the four independent solutions of this fourth order equation are thus given by the two pairs of solutions of the two corresponding second order differential equations.

Taking the shallow water limit \( \delta \to \infty \), we get for positive \( \delta \)
\[
(k_+)^2 \to \left( \frac{\nu - n\omega}{2} \right)^2 \frac{gh}{}, \quad (3.6)
\]
whereas \( k_- \) diverges. Thus \( k_+ \) tends towards the value of \( k_n \) corresponding to the shallow water case (see Eqn. (19) of ref. [I]) as it should, since this case must be recovered as a limiting case. The other constant \( k_- \) comes from the fact that (3.3) is a fourth order differential equation, unlike Eqn. (19) of ref. [I]. Its limit for \( \delta \to \infty \) is singular, reflecting the fact that surface tension \( \tau \) multiplies the highest derivative term in the differential equation (2.38). The respective role of \( k_+ \) and \( k_- \) are exchanged for negative \( \delta \).

From (3.4), we see that when \( \delta \) is positive \( k_+ \) is real whereas \( k_- \) is imaginary for all \( n \), whereas for negative \( \delta \) the two wavevectors \( k_\pm \) are real for small \( n \) and complex for large \( n \). For positive \( \delta \), (3.3) has Bessel and Neumann functions as solutions, together with hyperbolic Bessel and hyperbolic Neumann functions. The Neumann and hyperbolic Neumann functions must be discarded because of regularity at the origin. For negative \( \delta \), we take Bessel and Neumann functions of a complex argument, and discard the Neumann functions to ensure regularity at the origin. Thus
\[
\eta_1 = \text{Re} \left[ \sum_n \left( a_n \frac{J_n(k_nr)}{J_n(k_n a)} + b_n \frac{X_n(k_n r)}{X_n(k_n a)} \right) e^{i(n\theta - \nu t)} \right], \quad (3.7)
\]
where the $a_n$ and $b_n$ are undetermined coefficients in both cases, and where we have introduced the notation

\[ k_+ \equiv k_n, \quad k_- \equiv i\kappa_n, \quad X_n \equiv I_n, \quad (\delta > 0), \quad (3.8) \]

\[ k_- \equiv k_n, \quad k_+ \equiv \kappa_n, \quad X_n \equiv J_n, \quad (\delta < 0). \quad (3.9) \]

Outside the vortex, for $r > a$, using $U = \Gamma / (2\pi r) \hat{\theta}$ and the decomposition (3.2), and dropping terms of order $M^2$, we get that (2.38) reads

\[ \left[ \left( \mathcal{L} - \frac{n^2}{r^2} \right)^2 - \delta k^2 \left( \mathcal{L} - \frac{n^2}{r^2} \right) - A + \frac{B}{r^2} \right] \tilde{\eta}_1n = 0, \quad (3.10) \]

where, using the dispersion relation (2.39), we define the two constants

\[ A \equiv (\delta + 1)k^4, \quad B \equiv \frac{\delta k^2}{gh} \frac{2\Gamma \nu n}{2\pi}, \quad (3.11) \]

and the linear differential operator

\[ \mathcal{L} \equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}. \quad (3.12) \]

This equation may be written in the factorized form

\[ O_+ O_- \tilde{\eta}_1n = 0, \quad (3.13) \]

with

\[ O_\pm \equiv \mathcal{L} - \frac{m_\pm^2}{r^2} + q_\pm^2, \quad (3.14) \]

provided the unknown coefficients $m_+$, $m_-$, $q_+$ and $q_-$ satisfy the following relations:

\[ \left( \mathcal{L} \right) : \quad \implies \quad q_+^2 + q_-^2 = -\delta k^2, \quad (3.15) \]

\[ (1) : \quad \implies \quad q_+^2 q_-^2 = -A, \quad (3.16) \]

\[ \left( \frac{1}{r^2} \right) : \quad \implies \quad m_+^2 q_-^2 + m_-^2 q_+^2 = -\delta k^2 n^2 - B, \quad (3.17) \]

\[ \left( \frac{\mathcal{L}}{r^2} \right) : \quad \implies \quad m_+^2 + m_-^2 = 2n^2, \quad (3.18) \]

\[ \left( \frac{1}{r^3} \frac{d}{dr} \right) : \quad \implies \quad m_-^2 = n^2, \quad (3.19) \]

\[ \left( \frac{1}{r^4} \right) : \quad \implies \quad m_+^2 m_-^2 - 4m_-^2 = n^4 - 4n^2. \quad (3.20) \]
Here we have indicated on the left the portion of the linear differential operator that leads to each condition. There are six equations for only four unknowns and obviously they cannot be simultaneously satisfied in general. The last two equations, (3.19) and (3.20), correspond to terms that are negligible at large distance from the vortex. If we compare them to $L^2$, they are smaller than $1/\beta^2$ because $r > a$. Accordingly, we are justified in ignoring these two equations in our approximation $\beta \gg 1$ and we solve (3.15-3.18), which gives

$$q_+^2 = k^2 > 0, \quad q_-^2 \equiv (iq)^2 = -k^2(1 + \delta) < 0, \quad (m_\pm)^2 = n^2 \pm 2n\alpha, \quad (\delta > 0), \quad (3.21)$$

$$q_+^2 \equiv q^2 = (|\delta| - 1)k^2 > 0, \quad q_-^2 = k^2 > 0, \quad (m_\pm)^2 = n^2 \mp 2n\alpha, \quad (\delta < 0), \quad (3.22)$$

$$\alpha \equiv \frac{\Gamma \nu}{2\pi} \frac{1}{gh + 2(\tau/\rho - gh^2/3)hk^2} = \frac{\Gamma \nu}{2\pi} \frac{1}{c_\phi c_g}, \quad (3.23)$$

where we used (2.41) to write the last equality. It is important to note that the index $n^2 + 2n\alpha$ is always associated to the incident wavevector $k$. We will comment further on this result after Eqn. (3.27). From now on, we define

$$m_+ \equiv \sqrt{n^2 + 2n\alpha}, \quad m_- \equiv \sqrt{n^2 - 2n\alpha}, \quad (3.24)$$

so that in the negative $\delta$ case $m_-$ (resp. $m_+$) is associated with $q_+$ (resp. $q_-$), as shown by (3.22).

The dimensionless parameter $\alpha$ is defined in close analogy with I. We can write $\alpha = M\beta(c_\phi/c_g)$, which may be of order 1, with $M \ll 1$ and $\beta \gg 1$. This parameter has the same physical interpretation as in the shallow water case (see below) : it gives the amount of dislocation for the wavefronts far from the vortex. This calculation provides an explicit confirmation, in a perturbation expansion near the shallow water case, of the intuitive result of Berry et al. [2].

The two differential operators $O_\pm$ in (3.13) do not commute. Using the usual notation $[\cdot, \cdot]$ for the commutator of two operators, we get

$$[O_+, O_-] = (m_+^2 - m_-^2) \left[ L, \frac{1}{r^2} \right] = 4 \left( m_+^2 - m_-^2 \right) \left( \frac{1}{r^4} - \frac{1}{r^3} \frac{d}{dr} \right), \quad (3.25)$$
which is small, of the same order as the neglected terms (3.19, 3.20), and will also be neglected. Thus, in the same approximation, for positive $\delta$ the solution of (3.13) is a linear combination of Bessel, Neumann, hyperbolic Bessel and hyperbolic Neumann functions, because $q_+$ is real and $q_-$ is imaginary. Since the hyperbolic Bessel function diverges at infinity, it must be discarded. For negative $\delta$, the solution is a linear combination of Bessel and Neumann functions of argument $kr$ and $qr$. Since the wave number $q$ is that of a scattered wave, we discard the Bessel function of $qr$, keeping only the outgoing wave from the vortex.

Following Berry et al. [2] as in [1], we write the surface elevation outside the vortex in the form

$$\eta_1 = \text{Re}(\eta_{AB} + \eta_R),$$

(3.26)

where

$$\eta_{AB} = \sum_n c_n \frac{J_{m_+}(kr)}{J_{m_+}(\beta)} e^{i(n\theta - \nu t)},$$

(3.27)

with $\beta \equiv ka$. It does not depend on the sign of $\delta$, which is physically obvious because the amount of dislocated wavefront is linked to the circulation of the vortex, not to the curvature of the dispersion relation. Thus $m_+ = \sqrt{n^2 + 2n\alpha}$ is always the index of the functions involving the wavevector $k$. The other term of (3.26) depends on the sign of $\delta$, which is also physically clear since they represent the wave scattered by the vortex. They read

$$\eta_R = \sum_n \left( d_n \frac{H_{m_+}^1(kr)}{H_{m_+}^1(ka)} + e_n \frac{Y_{m_-}(qr)}{Y_{m_-}(qa)} \right) e^{i(n\theta - \nu t)},$$

(3.28)

Depending on the sign of $\delta$, we have the following definitions:

$$Y_{m_-} \equiv K_{m_-}, \quad q = k\sqrt{1 + \delta}, \quad (\delta > 0)$$

(3.29)

$$Y_{m_-} \equiv H_{m_-}^1, \quad q = k\sqrt{|\delta| - 1}, \quad (\delta < 0)$$

(3.30)
The coefficients $a_n$, $b_n$, $c_n$, $d_n$ and $e_n$ are defined so that they denote the amplitude of the wave components at the vortex boundary $r = a$. In order to obtain these coefficients, and since the equation (2.38) is of order four, the continuity of $\tilde{\eta}_1$ and its first three derivatives at $r = a$ is required. This gives four relations:

$$a_n + b_n - d_n - e_n = c_n,$$

(3.31)

$$a_n k_n J''(k_n a) + b_n \kappa_n X''(\kappa_n a) - d_n k_n J''(\kappa_n a) - e_n q_m J''(qa) = c_n k_n J''(\beta),$$

(3.32)

$$a_n k_n^2 J''(k_n a) + b_n \kappa_n^2 X''(\kappa_n a) - d_n k_n^2 J''(\kappa_n a) - e_n q_m^2 J''(qa) = c_n k_n^2 J''(\beta),$$

(3.33)

$$a_n k_n^3 J''(k_n a) + b_n \kappa_n^3 X''(\kappa_n a) - d_n k_n^3 J''(\kappa_n a) - e_n q_m^3 J''(qa) = c_n k_n^3 J''(\beta),$$

(3.34)

where $X$ (resp. $Y$) is defined either by (3.8) or (3.9) [resp. (3.29) or (3.30)].

The fifth, and last, boundary condition comes from the asymptotic behaviour of $\eta$ at infinity. We require that the asymptotics of $\eta_{AB}$ coincides with the dislocated wave incident from the right plus outgoing waves only. Exactly in the same way as in I, this leads to

$$\frac{c_n}{J_{m+}(\beta)} = (-i)^{m+},$$

(3.35)

It is important, in order to use this result, that either the coefficient $q_+^2$, for positive $\delta$, or $q_-^2$, for negative $\delta$, in (3.13) be equal to $k^2$, and that they be associated in each case to $m_+$. Otherwise it would have been impossible to recover the dislocated wave, which is a crucial physical requirement for the solution because we need $q = k$ to be a possible result of the factorization. This fact fully justifies the factorization in (3.13).

The solution of system (3.31–3.34) is thus known in principle, but it is not too illuminating, and it will not be displayed here. Details of the calculations may be found in the Appendix. To illustrate the solution, we use Mathematica [9] in order to do the calculations that will be indicated in Sec. [V]. Thus it is sufficient to have the coefficients expressed
as ratio of $4 \times 4$ determinants, as in (A3). We insist on the fact that the solution may be inaccurate at a few wavelengths away from the vortex, because of the approximate character of the factorization (3.13).

Let us discuss the asymptotic behavior of the solution (3.27, 3.28) for $r \to \infty$. The case of $\eta_{AB}$ is completely similar to the shallow water case. Indeed, the index of the Bessel function, $m_+(n) = \sqrt{n^2 + (\text{Const.}) \times n}$, has exactly the same structure as $m(n)$ in [I]. An important consequence is that the parameter $\alpha = \beta M(c_\phi/c_\varphi)$ has the same physical significance as in the shallow water (or acoustics) case: it quantifies the dislocation of the wavefronts in the forward direction, at large distances from the vortex. Other results may also be transposed in a straightforward fashion, and the asymptotics of $\eta_{AB}(r, \theta)$ for large $r$ is still given by Eqn. (38) of [I], with the proviso that the function $G(\theta, -\pi/2)$ be replaced by

$$G_{DW}(\theta, t) \equiv \sum_{|n| < N} e^{i\theta} \left[ e^{im_+(t-\pi/2)} - e^{im_{\text{old}}(t-\pi/2)} \right],$$

where $m_+$ is given by (3.24) and

$$m_{\text{old}} = \left| n + \frac{\Gamma \nu}{2\pi c_\phi c_\varphi} \right|.$$

The asymptotics of $\eta_R$ depends on the sign of $\delta$. If $\delta$ is positive, the hyperbolic Bessel function does not contribute to the scattered far field, because $K_p(z) \sim e^{-z}/\sqrt{z}$ for large $z$. Thus

$$\eta_R \to \left( \frac{2}{\pi ikr} \right)^{1/2} e^{i(kr-\nu t)} \sum_n \frac{d_n}{H_{m_+}^1(\beta)} e^{i(n\theta - \pi m_+/2)}. $$

In the next section, we will compare the correction to the Aharonov-Bohm scattering amplitude in the case of shallow water waves, given by Eqn. (43) of [I], and the correction for dispersive water waves, which reads

$$\tilde{f}_{DW}(\theta) = G_{DW}(\theta, -\pi/2) + 2 \sum_n \frac{d_n}{H_{m_+}^1(\beta)} e^{in\theta} (-i)^{m_+}. $$

If $\delta$ is negative, we must take into account the two outgoing Hankel functions. We get

$$\eta_R \to \left( \frac{2}{\pi ikr} \right)^{1/2} e^{i(kr-\nu t)} \sum_n \left[ \frac{d_n e^{-i\pi m_+/2}}{H_{m_+}^1(\beta)} + \frac{e_n e^{-i\pi m_-/2}}{H_{m_-}^1(\beta \sqrt{|\delta|} - 1)} \right] e^{i\theta}, $$

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and the correction for dispersive water waves now reads

$$\tilde{f}_{DW}(\theta) = G_{DW}(\theta, -\pi/2) + 2 \sum_n \left[ \frac{d_n(-i)^{m+}}{H^1_{m+}(\beta)} + \frac{e_n(-i)^{m-}}{(|\delta| - 1)^{1/4}H^1_{m-}(\beta \sqrt{|\delta| - 1})} \right] e^{in\theta}, \quad (3.41)$$

**IV. NUMERICAL EXAMPLES**

The solution to the scattering problem of surface waves by a uniform vertical vortex depends on four dimensionless parameters. A first set includes the dimensionless vortex radius $\beta \gg 1$ and the dislocation parameter $\alpha = \beta M(c_\phi/c_\kappa)$ which quantifies the wavefront dislocation. They already appeared in [I], with the same definitions and physical interpretations. A third parameter is the dimensionless capillary length $\ell \equiv k \ell_c$, and the last one is the dimensionless depth $kh$. In order to simplify somewhat the discussion, we use the single dimensionless parameter $\delta$, defined in (2.43), in place of the depth and capillary length. As an example, we take $\delta = 8$, which may correspond, for example, to $h = l_c, \ k h = \sqrt{3}/4$, and $\delta = -8$, which may correspond to $h = 3l_c, \ k h = 3/4$. In both cases, the hyperbolic tangent in (2.42) is approximated to better than five percent by the two leading terms, the ones we are keeping, in its series expansion.

Scaling radial distance with the vortex radius, $r' \equiv r/a$, the analytical expression of the surface displacement is summarized as follows, depending on the sign of $\delta$. Inside the vortex ($0 < r' \leq 1$) we have

$$\eta_1 = \text{Re } \eta_c.$$ For positive values of $\delta$,

$$\eta_c = \sum_n \left( a_n \frac{J_n(\tilde{\phi}_n r')}{J_n(\tilde{\varphi}_n)} + b_n \frac{I_n(\varphi_n r')}{I_n(\varphi_n)} \right) e^{i(n\theta - \nu t)}, \quad (4.1)$$

where we have defined the following dimensionless wave numbers:

$$k_n a \equiv \tilde{\phi}_n = \beta \sqrt{\frac{\delta}{2}} \left( -1 + \sqrt{1 + \frac{1 + \delta}{\delta^2} \left( 1 - \frac{\alpha}{\beta^2} \frac{2 + \delta}{1 + \delta} \right)^2} \right)^{1/2}, \quad (4.2)$$

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\[ \kappa_n a \equiv \varphi_n = \beta \sqrt{\frac{|\delta|}{2} \left( 1 + \sqrt{1 + 4 \frac{1 + \delta}{\delta^2} \left( 1 - n \frac{\alpha}{\beta^2} \frac{2 + \delta}{1 + \delta} \right)^2} \right)^{1/2}}. \]  

(4.3)

For negative values of \( \delta \),

\[ \eta_c = \sum_n \left( a_n \frac{J_n(\phi_n r')}{J_n(\phi_n)} + b_n \frac{J_n(\tilde{\phi}_n r')}{J_n(\tilde{\phi}_n)} \right) e^{i(n\theta - \nu t)}. \]  

(4.4)

whith new dimensionless wave numbers:

\[ k_n a \equiv \phi_n = \sqrt{\frac{|\delta|}{2} \beta \left( 1 - \sqrt{1 - 4 \frac{|\delta| - 1}{\delta^2} \left( 1 - n \frac{\alpha}{\beta^2} \frac{|\delta| - 2}{|\delta| - 1} \right)^2} \right)^{1/2}}. \]  

(4.5)

\[ \kappa_n a \equiv \tilde{\phi}_n = \sqrt{\frac{|\delta|}{2} \beta \left( 1 + \sqrt{1 + 4 \frac{|\delta| - 1}{\delta^2} \left( 1 - n \frac{\alpha}{\beta^2} \frac{|\delta| - 2}{|\delta| - 1} \right)^2} \right)^{1/2}}. \]  

(4.6)

Outside the vortex \((r' > 1)\)

\[ \eta_1 = \text{Re}(\eta_{AB} + \eta_R), \]

where, whatever the sign of \( \delta \),

\[ \eta_{AB} = \sum_n (-i)^m J_0(\beta r') e^{i(n\theta - \nu t)}. \]  

(4.7)

For positive values of \( \delta \),

\[ \eta_R = \sum_n \left( d_n \frac{H_{m+}^1(\beta r')}{H_{m+}^1(\beta)} + e_n \frac{K_{m-} \left( \beta \sqrt{1 + \delta r'} \right)}{K_{m-} \left( \beta \sqrt{1 + \delta} \right)} \right) e^{i(n\theta - \nu t)}. \]  

(4.8)

where we have used the fact that

\[ qa = \beta \sqrt{1 + \delta}. \]  

(4.9)

For negative values of \( \delta \),

\[ \eta_R = \sum_n \left( d_n \frac{H_{m+}^1(\beta r')}{H_{m+}^1(\beta)} + e_n \frac{H_{m-}^1 \left( \beta \sqrt{|\delta| - 1} r' \right)}{H_{m-}^1 \left( \beta \sqrt{|\delta| - 1} \right)} \right) e^{i(n\theta - \nu t)}. \]  

(4.10)

where we have used the fact that

\[ qa = \beta \sqrt{|\delta| - 1}. \]  

(4.11)
Some details of the calculations of the coefficients $a_n, b_n, d_n$ and $c_n$ are given in the Appendix. As an illustration, absolute values of the coefficients $a_n$ and $b_n$ (resp. $d_n$ and $e_n$) are plotted in Fig. 1 (resp. in Fig. 2), for positive $\delta$, and in Figs. 3 and 4 for negative $\delta$.

Since convergence of the series expansions for $\eta_{AB}$ and $\eta_R$ is not uniform, the number of terms to keep in the numerical evaluation of the infinite series depends on the value of $r'$. In practice, the convergence of the series is comparable to the case of ref. [I], and we use roughly the same number of terms. We compute the patterns of the surface displacement in the region $|x'|, |y'| \leq 5[(x', y') = (r' \cos \theta, r' \sin \theta)]$ by the finite sum of (4.1, 4.4) and (4.8, 1.10) with $|n| \leq 50$ for $\beta = 10$ and $|n| \leq 30$ for $\beta = 5$, but we keep more terms, $|n| \leq 90$ in (4.7).

Let us first consider the case of positive $\delta$. Fig. 5 shows the resulting displacements for $\delta = +8, \beta = 5$ and $\alpha = 0.5, 1, 1.5, 2$, and Fig. 6 for the same values of $\alpha$ and $\delta$, but $\beta = 10$. The dislocation of the incident wavefronts by an amount equal to $\alpha$ is clearly visible. The outward travelling scattered wave is also visible. The interference patterns between scattered and incident wave is very similar to the corresponding pictures of [I], for the same values of $\beta$ and $\alpha$. This is confirmed by the comparison of the scattering cross section displayed in Fig. 9, as discussed below. Taking into account the dispersion greatly modifies the numerical value of $\alpha$, but other corrections are small in the case of positive $\delta$.

In the case of negative $\delta$, the interference pattern is strongly modified. This is illustrated in Fig. 7, where we plot the surface displacement for $\delta = -8, \beta = 5$ and $\alpha = 0.5, 1, 1.5, 2$. The spiral wave which is clearly seen for $\alpha = 2$ (Fig. 7-d) is very different from the corresponding figure of [I] (see Fig. 2-d of [I]). For larger values of $\beta$, shown in Fig. 8 for which $\beta = 10$, with same values for $\delta$ and $\alpha$ as before, the pictures are much more similar to the shallow water case.

This is confirmed by the plot of the absolute value of the correction to the Aharonov-Bohm scattering amplitude in both cases. The dashed line in Fig. 9 shows this correction in the shallow water (non dispersive) case, and the solid line shows the same correction in the dispersive case, for a positive $\delta = +8$. Both corrections are almost the same, in agreement
with the results of Fig. 5 & 6. The same is done for a negative $\delta = -8$ in Fig. 10. The graphs, this time, differ enormously. In the dispersive case, the scattering is much more isotropic, and very different in amplitude. An obvious, but somewhat formal, explanation of this difference is the supplementary function $H_{m_-}^1(\beta \sqrt{|\delta| - 1})$ in (3.41) compared to (3.39). Also, the index of this function is $m_-(n)$ which takes imaginary values for small positive $n$, rather than for small negative $n$ as $m_+(n)$. This implies that the partial amplitudes for $\exp(-i n \theta)$ and $\exp(i n \theta)$ are much more similar to each other than in the shallow water case, for which $m_-$ is absent, and also more similar than in the case of positive $\delta$, where the decrease of the corresponding function is exponential. The appearance of an algebraically ($\propto 1/\sqrt{r}$) decreasing amplitude associated to $m_-$ in the negative $\delta$ case restores the symmetry of the scattered wave.

A more physical explanation is as follows: Consider a plane wave incident from the right on a counterclockwise vortex, as in figures 5–8. Above the vortex, the wavefront velocity is increased by advection, whereas it is decreased below the vortex. Consequently the wave fronts should bend towards the bottom of the picture, as they do. The other effect of the vortical flow is to add a wavelength below the vortex, which means that the wavenumber $k$ decreases below the vortex. Since $kh$ is assumed to be small, for positive $\delta$ the phase velocity increases with $k$. The phase velocity is thus smaller below the vortex than above, which enhances the bending of the wavefronts, and reinforces the strong asymmetry in the interference pattern of Fig. 5 & 6, and in the scattering amplitude of Fig. 9. For negative $\delta$, the phase velocity decreases with $k$, and the effect of the dislocation is to make the phase velocity smaller for the part of the wavefront above the vortex. This effect balances the effect of advection, and we understand why the interference pattern (see Fig. 7 & 8) and the scattering amplitude (see Fig. 10) are much more symmetric than in the positive $\delta$ case, or than in the nondispersive ($\delta = 0$) case. It is also reasonable that this effect should be more important for $\beta = 5$ than for $\beta = 10$, because the relative variation in $k$ due to the dislocation is greater in the former case. The spiral waves are observed for negative $\delta$ because the interference pattern almost keeps rotational symmetry while smoothing the
wavefront dislocation in the forward direction.

As a last illustration, we compare the wave patterns predicted by the shallow water approximation, and by its first correction in powers of the fluid depth, in an experimentally accessible situation. We suppose the fluid to be pure water, of depth 1 mm.; the vortex radius is 1 cm., and the wavelength is 2 cm. Thus \( kh = \pi/10 \), and the approximation of the dispersion relation is excellent. The price to pay is the rather small value \( \beta = \pi \). We take the vortex circulation such that \( \alpha = 1 \) in the shallow water approximation, for which 
\[
c \equiv \sqrt{gh} = 9.9 \text{ cm/s}.
\]
Taking the dispersion into account, we get \( \delta = 1.4 \), \( c_\phi = 13.0 \text{ cm/s} \), \( c_g = 18.4 \text{ cm/s} \) and \( \alpha = 0.41 \). The difference in the respective numerical values of \( \alpha \) is the leading effect. The result is shown in Fig. 11. To obtain quantitative agreement with an experimental situation, it may be sufficient to keep the shallow water approximation, but with the actual value of \( \alpha \) obtained in the dispersive case (3.23). It should be interesting to use a fluid with a very small surface tension, in order to obtain a negative value of \( \delta \) while keeping a small value of \( kh \), for which the wave pattern should be extremely different from the shallow fluid layer approximation. All calculations were performed using Mathematica [4].

V. CONCLUDING REMARKS

We have computed the surface displacement due to a dispersive surface wave interacting with a vertical vortex when the vortex core performs solid body rotation, the wavelength is small compared to the vortex core radius and the particle velocities associated with the wave are small compared to the particle velocities associated with the vortex. When the parameter \( \alpha = \nu \Gamma/2\pi c_\phi c_g \) is of order one or bigger, the wavefronts become dislocated. This parameter depends, in the dispersive case, both on the phase and group velocity of the wave, and tends smoothly toward the result of [1] in the nondispersive limit. We thus give a proof, in a perturbative fashion, of the heuristic derivation of Berry et al. [2].

Formally, we proceed perturbatively around the shallow water limit, to obtain a fourth
order partial differential equation for the surface elevation associated with the surface wave. However, apart from some technical details, the solution is very similar to the nondispersive case. The scattered waves interact strongly with the dislocated wavefronts and produce interference patterns. A dimensionless parameter $\delta$ quantifies the relation between fluid layer depth $h$ and capillary length $l_c$. When it is positive ($h < \sqrt{3} l_c$) the wave pattern is similar to the shallow water case. When it is negative, for large values of the circulation, the wave pattern is very different.

We hope that the calculations in the dispersive case will help the comparison with experiments. Our calculations are valid when the approximation $\tanh kh \simeq kh - (kh)^3/3$ holds. This is a restrictive condition, but we believe that, once dispersion is correctly taken into account in the definition of $\alpha$, the wave pattern given by the nondispersive case should be roughly similar to the observations. Some discrepancies are expected when the fluid depth is greater than the capillary length ($h > \sqrt{3} l_c$), but in this case it is necessary to correctly approximate the hyperbolic tangent by the first two terms of the series, and in practice the effect should be observable for a fluid of small surface tension (thus small capillary length) only.

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APPENDIX A: COMPUTATIONAL DETAILS

In this Appendix, we explain how to calculate the coefficients in the series representations of Sec. [III]. We also discuss briefly the convergence of the series involved in our solution (3.7) and (3.28), since the sum in (3.27) obviously converges as in the shallow water case [I].

To get the solution of eqns. (3.31–3.34), we introduce the column vectors \( V_x \) of the coefficients of the unknown \( x_n \). They involve multiple derivatives of the Bessel functions, and may be simplified with the help of the following formulae [10]:

\[
\begin{align*}
Z'_p &= \frac{1}{2}(Z_{p-1} - Z_{p+1}), \\
I'_p &= \frac{1}{2}(I_{p-1} + I_{p+1}), \\
K'_p &= -\frac{1}{2}(K_{p-1} + K_{p+1}),
\end{align*}
\]

(A1)

where \( z \) is the argument of the function, \( p \) its index, and \( Z \) represents either \( J \) or \( H_1 \).

The column vectors thus read

\[
V_a \equiv \begin{pmatrix}
1 \\
n - \phi_n \frac{J_{n+1}(\phi_n)}{J_n(\phi_n)} \\
n(n-1) - \phi_n^2 + \phi_n \frac{J_{n+1}(\phi_n)}{J_n(\phi_n)} \\
(n-1)[n(n-2) - \phi_n^2] + \phi_n [\phi_n^2 - (n^2 + 2)] \frac{J_{n+1}(\phi_n)}{J_n(\phi_n)}
\end{pmatrix},
\]

(A2)

in the negative \( \delta \) case, and the same expression with \( \tilde{\phi}_n \) in the positive \( \delta \) case,

\[
V_b \equiv \begin{pmatrix}
1 \\
n + \varphi_n \frac{I_{n+1}(\varphi_n)}{I_n(\varphi_n)} \\
n(n-1) + \varphi_n^2 - \varphi_n \frac{I_{n+1}(\varphi_n)}{I_n(\varphi_n)} \\
(n-1)[n(n-2) + \varphi_n^2] + \varphi_n [\varphi_n^2 + n^2 + 2] \frac{I_{n+1}(\varphi_n)}{I_n(\varphi_n)}
\end{pmatrix},
\]

(A3)

in the positive \( \delta \) case, or
\[
V_h \equiv \begin{pmatrix}
1 \\
n - \tilde{\varphi}_n \frac{J_{n+1}(\tilde{\varphi}_n)}{J_n(\tilde{\varphi}_n)} \\
n(n-1) - \tilde{\varphi}_n^2 + \tilde{\varphi}_n \frac{J_{n+1}(\tilde{\varphi}_n)}{J_n(\tilde{\varphi}_n)} \\
(n-1)[n(n-2) - \tilde{\varphi}_n^2] + \tilde{\varphi}_n[\tilde{\varphi}_n^2 - (n^2 + 2)] \frac{J_{n+1}(\tilde{\varphi}_n)}{J_n(\tilde{\varphi}_n)}
\end{pmatrix},
\]

in the negative \(\delta\) case,

\[
V_d \equiv \begin{pmatrix}
-1 \\
-m_+ + \beta \frac{H^1_{m_++1}(\beta)}{H^1_{m_+}(\beta)} \\
-m_+(m_+ - 1) + \beta^2 - \beta \frac{H^1_{m_++1}(\beta)}{H^1_{m_+}(\beta)} \\
(m_+ - 1)[\beta^2 - m_+(m_+ - 2)] + \beta(m_+^2 + 2 - \beta^2) \frac{H^1_{m_++1}(\beta)}{H^1_{m_+}(\beta)}
\end{pmatrix},
\]

in the positive \(\delta\) case, or

\[
V_c \equiv \begin{pmatrix}
-1 \\
-m_+ + \beta \frac{\sqrt{1 + \delta} K_{m_++1}(\beta \sqrt{1 + \delta})}{K_{m_+}(\beta \sqrt{1 + \delta})} \\
-m_-(m_- - 1) - \beta^2 (1 + \delta) - \beta \sqrt{1 + \delta} \frac{K_{m_-+1}(\beta \sqrt{1 + \delta})}{K_{m_-}(\beta \sqrt{1 + \delta})} \\
(1 - m_-) \left[\beta^2 (1 + \delta) + m_-(m_--2)\right] + \beta \sqrt{1 + \delta} \left[m_-^2 + 2 + \beta^2 (1 + \delta)\right] \frac{K_{m_-+1}(\beta \sqrt{1 + \delta})}{K_{m_-}(\beta \sqrt{1 + \delta})}
\end{pmatrix},
\]

in the negative \(\delta\) case. The column vector \(V_c\) for the coefficients of \(c_n\) in the right hand member of (3.31) (3.34) reads
\[
\frac{V_c}{(-i)^{m_+}} \equiv \begin{pmatrix}
J_{m_+}(\beta) \\
-m_+J_{m_+}(\beta) - \beta J_{m_+1}(\beta) \\
[-m_+(m_+ - 1) - \beta^2]J_{m_+}(\beta) + \beta J_{m_+1}(\beta) \\
(m_+ - 1)[m_+(m_+ - 2) - \beta^2]J_{m_+}(\beta) + \\
+ \beta [\beta^2 - (m_+^2 + 2)] J_{m_+1}(\beta)
\end{pmatrix}.
\] (A8)

Now, formally, the coefficient \(a_n\) (say) is expressed as the ratio of two determinants,
\[
a_n = \frac{|V_c V_b V_d V_e|}{|V_a V_b V_d V_e|},
\] (A9)
and is a function of \(n, \alpha, \beta\) and \(\delta\).

Let us discuss briefly the asymptotic behavior of those coefficients. For large \(n\), \(m_+ \sim n\), and we can safely assume that \(Z_{n+1}/Z_n = O(1)\), in the case of a constant argument, \(\beta\), \(\beta \sqrt{1 + \delta}, \beta \sqrt{|\delta| - 1}\), or in the case of an argument of order \(n\), such as \(\phi_n\) or \(\varphi_n\). This is demonstrated for ordinary Bessel functions in [I], and may be deduced in the same fashion for hyperbolic Bessel functions, which have similar behaviors for large values of the index \([11]\). From the calculations of ref. [I], we easily get the dominant behavior for large \(n\) of the vectors \(V_x\), and thus of the coefficients \(x_n\). If \(x_n\) is not equal to \(c_n\),
\[
V_x \sim \begin{pmatrix}
1 \\
n \\
n^2 \\
n^3
\end{pmatrix},
\] (A10)
so that the determinant in the denominator of any expression such as (A9) behaves like
\[
|V_a V_b V_d V_e| \sim n^6.
\] (A11)

We also have that
\[
V_c \sim \begin{pmatrix}
1 \\
n \\
n^2 \\
n^3
\end{pmatrix} J_n(\beta) \sim \begin{pmatrix}
1 \\
n \\
n^2 \\
n^3
\end{pmatrix} \frac{1}{\sqrt{n}} \left(\frac{e}{n}\right)^n,
\] (A12)
so that the determinant in the numerator of (A9) behaves like

$$|V_c V_b V_d V_e| \sim n^{5/2} \left( \frac{c}{n} \right)^n,$$

and the coefficients $x_n$ decrease sufficiently quickly to ensure the convergence of the series in (3.7) and (3.28). The convergence is uniform for $r < a$, because the support of the functions is compact, but not for $r > a$. 

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[6] It is tempting to take $z = h$ in (2.19) and (2.20); this is obviously consistent for linear water waves, for which $\eta$ is a small quantity, but here the surface elevation takes into account the vortical flow as well. The justification of this approximation requires an estimate for $\eta_0$. This is done in the discussion that follows Eqns. (2.30) and (2.31).

[7] In this case, the restriction $u \ll U$ imposed in Section II will break down when $r$ is very small or very large. At those points, however, the condition $u \ll c$, implicit in the derivation of Eqn. (2.38), assures that nonlinear terms can still be neglected.

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FIGURES

FIG. 1. Absolute magnitude of coefficients $a_n$ (a) and $c_n$ (b) versus $n$, in a log-linear plot, for a positive $\delta = +8$, and for values of dislocation parameter $\alpha$ and dimensionless radius $\beta$ $(\alpha, \beta) = (0.5, 10)$, denoted by dots, $(\alpha, \beta) = (1.5, 10)$, denoted by empty circles ◦, $(\alpha, \beta) = (0.5, 5)$, denoted by filled circles • and $(\alpha, \beta) = (1.5, 5)$, denoted by empty squares □. Note the asymmetry with respect to $n \to -n$

FIG. 2. Same as figure 1, for the coefficients $d_n$ (a) and $e_n$ (b).

FIG. 3. Same as figure 1, for a negative $\delta = -8$.

FIG. 4. Same as figure 2, for a negative $\delta = -8$.

FIG. 5. Density plot of the surface elevation for the total wave patterns for $\delta = +8$ and $\beta = 5$, for several values of $\alpha$. Respectively $\alpha = 0.5 : (a), \alpha = 1 : (b), \alpha = 1.5 : (c), \alpha = 2 : (d)$. The greyscale is linear with surface amplitude (arbitrary units). The dark ring indicates the vortex location, and the vortex rotates counterclockwise. The wave is incident from the right. The box size is $10 \times 10$ in units of $a$, the vortex radius.

FIG. 6. Same as figure 5, for $\delta = +8$, $\beta = 10$, and several values of $\alpha$. Respectively $\alpha = 0.5 : (a), \alpha = 1 : (b), \alpha = 1.5 : (c), \alpha = 2 : (d)$.

FIG. 7. Same as figure 5, for $\delta = -8$, $\beta = 5$, and several values of $\alpha$. Respectively $\alpha = 0.5 : (a), \alpha = 1 : (b), \alpha = 1.5 : (c), \alpha = 2 : (d)$.

FIG. 8. Same as figure 5, for $\delta = -8$, $\beta = 10$, and several values of $\alpha$. Respectively $\alpha = 0.5 : (a), \alpha = 1 : (b), \alpha = 1.5 : (c), \alpha = 2 : (d)$. 
FIG. 9. Polar plot of the absolute value of the correction to the Aharonov-Bohm (i.e. point) scattering amplitude, in the case of non dispersive waves (dashed line) and in the case of dispersive waves (solid line), in the case of a positive $\delta = +8$, for $\beta = 5$ and $\alpha = 0.25$ : (a), $\alpha = 0.5$ : (b), $\alpha = 1.0$ : (c), $\alpha = 1.25$ : (d). The vortex location is marked by the large dot; the vortex rotates counterclockwise.

FIG. 10. Same as figure 9, but for a negative $\delta = -8$, for $\beta = 5$ and $\alpha = 0.25$ : (a), $\alpha = 0.5$ : (b), $\alpha = 1.0$ : (c), $\alpha = 1.25$ : (d). The vortex location is marked by the large dot; the vortex rotates counterclockwise. In this case the dispersive wave scatters very differently from the nondispersive wave.

FIG. 11. Density plot of the surface elevation for the total wave pattern calculated in the shallow water approximation, (a), and to first order in fluids depth (b). The greyscale is linear with surface amplitude (arbitrary units). The dark ring indicates the vortex location, and the vortex rotates counterclockwise. The incident wave comes from the right. The box size is $20 \times 20$ in units of $a$, the vortex radius. $\beta = \pi$ in both cases, but $\alpha = 1$ in the shallow water case (a), and $\alpha = 0.41$, $\delta = 1.4$ in the dispersive case (b).
TABLE I. Attenuation times for capillary-gravity (C-G) waves and shallow water (S-W) waves, compared to the period of the wave. The attenuation times are defined by \[2.3\]
C. Coste et al., Figure 1
C. Coste et al., Figure 2
C. Coste et al., Figure [4]
C. Coste et al., Figure 5
C. Coste et al., Figure 6
C. Coste et al., Figure 8
C. Coste et al., Figure 9
C. Coste et al., Figure 10
C. Coste et al., Figure 11