Testing cosmic censorship conjecture near extremal black holes with cosmological constants

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Abstract

It has been shown previously that an extremal Reissner-Nordström or an extremal Kerr black hole cannot be overcharged or overspun by a test particle if radiative and self-force effects are neglected. In this paper, we consider extremal charged and rotating black holes with cosmological constants. By studying the motion of test particles, we find the following results: An extremal Reissner-Nordström anti-de Sitter (RN-AdS) black hole can be overcharged by a test particle but an extremal Reissner-Nordström de Sitter (RN-dS) black hole cannot be overcharged. We also show that both extremal Kerr-de-Sitter (Kerr-dS) and Kerr-anti-de-Sitter (Kerr-AdS) black holes can be overspun by a test particle, implying a possible breakdown of the cosmic censorship conjecture. For the Kerr-AdS case, the overspinning requires that the energy of the particle be negative, a reminiscent of the Penrose process. In contrast to the extremal RN and Kerr black holes, in which cases the cosmic censorship is upheld, our results suggest some subtle relations between the cosmological constants and the cosmic censorship. We also discuss the effect of radiation reaction for the Kerr-dS case and find that the magnitude of energy loss due to gravitational radiation may not be enough to prevent the violation of the cosmic censorship.

1 Introduction

It is generally believed that singularities appear only after the formation of black holes during gravitational collapses, i.e., singularities exist inside black hole horizons and cannot be seen by distant observers. This is the well-known "cosmic censorship" conjecture first proposed by Penrose [1]. One way of testing the cosmic censorship is to throw a test particle into an existing black hole. If the

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particle can pass through the horizon and change the parameters of the black hole such that the horizon disappears, then the cosmic censorship conjecture might break down. It has been shown by Wald\[2\] that a test particle cannot destroy the horizon of an extremal Kerr-Newman black hole. This issue has been revisited in recent years \[3\]-[18]. By noticing that linear approximation had been used in Wald’s analysis, we took into account higher-order terms and find possible violation of the cosmic censorship for extremal Kerr-Newman black holes \[19\].

The acceleration of our universe indicates the existence of dark energy, while the cosmological constant $\Lambda$ turns out to be a good candidate. The recent development of the AdS/CFT correspondence has given many new insights into the nature of black holes. Thus, the study of asymptotically de Sitter and anti-de Sitter black holes have become more important and realistic than ever. Previously \[3\], \[5\], \[19\], it has been shown that the horizon of an extremal RN or Kerr black hole cannot be destroyed by any test particles even higher order terms are considered. It is then interesting to know whether the presence of $\Lambda$ could make a difference. For charged black holes with cosmological constants, we study the extremal RN-dS and RN-AdS black holes and the conditions that the horizons can be destroyed by a test particle. We find that a particle could destroy the horizon of an extremal RN-AdS black hole, but not an extremal RN-dS black hole. In case of rotating black holes, Our analysis shows that both extremal Kerr-dS and Kerr-AdS black holes can be overspun by test particles. The interesting feature of overspinning an Kerr-AdS black hole is that the particle must possess negative energy, just like the Penrose process.

We further find that for a possible violation of the cosmic censorship, the allowed range for the particle’s energy $E$ is of order $Q\Lambda q^2$ or $L^2/M^3$, $ML\Lambda$. Since $\Lambda M^2 \ll 1$ and $M \sim Q$ in the extremal case, we see that the energy of the particle must be finely tuned. We study the orbit near the horizon of the Kerr-dS black hole and estimate the energy loss due to the radiative effect. Due to the coincidence of the location of the light ring and the extremal horizon, the radiative effect is negligible. Such analysis has also been made by other authors for nearly extremal Kerr and RN black holes \[20\]-[22], where the radiative effect is important at least for some orbits.

2 Degenerate horizons of RN-dS and RN-AdS black holes

A spherically symmetric black hole carrying mass $M$ and charge $Q$ and a nonvanishing cosmological constant $\Lambda$ is described by the RN-(anti-)dS metric \[23\]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2,$$ \hspace{1cm} (1)
where
\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2. \]  
\hspace{1cm} (2)

We first study the RN-dS metric ($\Lambda > 0$). If $\Lambda$ vanishes, the metric reduces to the RN solution. In the RN case, when $M > Q$, there are two horizons determined by $f(r_i) = 0$, $i = 1, 2$, which are called the Cauchy horizon and event horizon. For $M = Q$, the two horizons coincide with each other and $f'(r_i) = 0$. When a positive $\Lambda$ is turned on, the two roots of $f(r) = 0$ have negligible changes due to the fact $\Lambda M^2 \ll 1$. But with the increase of $r$, the last term in Eq. (2) finally dominates and a third horizon $r_3$ appears. This horizon is called the cosmological horizon.

![Figure 1: Function $f(r)$ of RN-dS black holes. (a): An ordinary RN-dS black hole with three horizons. (b): An extremal black hole containing a degenerate horizon.](image)

Fig. 1 (a) shows the function $f(r)$ of a RN-dS black hole with all the three horizons. If $r_1 = r_2$, the two horizons coincide and we have $f'(r_1) = f(r_1) = 0$. This means that the black hole possesses a degenerate horizon which corresponds to a zero temperature according to black hole thermodynamics. In this case, $f(r)$ is depicted in Fig. 1 (b). There is also a possibility that the event horizon coincides with the cosmological horizon, i.e., $r_2 = r_3$. We shall not discuss this case because an extremal RN-dS black hole usually refers to the degenerate horizon depicted in Fig. 1 (b) and the cosmological horizon is not relevant to the discussion of the cosmic censorship.

Now we investigate under what conditions an extremal black hole can exist. Denote $r_h \equiv r_1 = r_2$, which satisfies
\[ f(r_h) = 0 \]  
\hspace{1cm} (3)

and then leads to the relation
\[ M = \frac{3Q^2 + 3r_h^2 - \Lambda r_h^4}{6r_h}. \]  
\hspace{1cm} (4)

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An extremal black hole requires that \( f'(r_h) = 0 \). It is not difficult to verify that this condition is equivalent to
\[
\frac{\partial M}{\partial r_h} = 0. \tag{5}
\]
Thus, the degenerate event horizon is located at
\[
r_h = \sqrt{\frac{1 - \sqrt{1 - 4Q^2\Lambda}}{2\Lambda}}. \tag{6}
\]
Here and through this paper, we assume \(|\Lambda Q^2| \ll 1\). This means that \( \Lambda \) can be treated as a perturbation to the RN solution around the region \( r \sim r_h \). Using Eqs. (4) and (6), one can write \( M \) as a function of \( Q \):
\[
M = M(Q) = 1 + 4Q^2\Lambda - \sqrt{1 - 4Q^2\Lambda} \cdot \frac{3\sqrt{2\Lambda}}{\sqrt{1 - \sqrt{1 - 4Q^2\Lambda}}} \tag{7}
\]
If the black hole parameters satisfy this equation, we have an extremal black hole as shown in Fig. 1 (b). If we decrease the mass \( M \) by a small amount \( \Delta M \) while keeping \( Q \) fixed, Eq. (2) clearly shows that \( f(r) \) will increase everywhere and the degenerate root will disappear, implying the disappearance of the extremal horizon. Therefore, Eq. (7) gives the minimum mass for the existence of a horizon.

For a RN-AdS black hole \( \Lambda < 0 \), all the above arguments and results are valid except that there is no cosmological horizon.

### 3 Overcharging RN-(anti-)dS black holes

Assuming a test particle is moving towards a RN-(anti-)de Sitter black hole in the radial direction with four-velocity
\[
u^a = \dot{t} \left( \frac{\partial}{\partial t} \right)^a + \dot{r} \left( \frac{\partial}{\partial r} \right)^a. \tag{8}
\]
The four-potential of the vacuum electromagnetic field is
\[
A_a = A_t dt_a + A_\phi d\phi_a. \tag{9}
\]
The energy of the particle can be defined through the killing vector field \((\partial/\partial t)^a\)
\[
E = -t^a (mu_a + qA_a) = -mg_{tt}\dot{t} - qA_t = mif(r) + \frac{qQ}{r}, \tag{10}
\]

\(^{1}\)This solution is valid for either \( \Lambda > 0 \) or \( \Lambda < 0 \). There is another solution \( r_h = \sqrt{1 + \sqrt{1 - 4Q^2\Lambda}} \) for \( \Lambda > 0 \), which corresponds to the cosmological horizon as we discussed above.
where \( \dot{t} > 0 \) because \( u^a \) is future directed. Since \( f(r) > 0 \) outside the black hole horizon \( r = r_h \), we obtain from Eq. (10)

\[
E > \frac{qQ}{r_h} \equiv E_{\text{min}}. \tag{11}
\]

This is the condition that a particle can be captured by the black hole. To overcharge the black hole, we need another constraint on the parameters of the particle and black hole.

As argued at the end of Section 2, \( M(Q) \) defined by Eq. (7) is the critical mass for a given \( Q \). If the black hole mass is smaller than \( M(Q) \), there is no event horizon. In attempt to destroy the horizon of an existing extremal black hole, a particle with energy \( E \) and charge \( q \) must satisfy

\[
M(Q) + E < M(Q + q). \tag{12}
\]

So the upper limit for \( E \) is

\[
E < M(Q + q) - M(Q) \equiv E_{\text{max}}. \tag{13}
\]

To destroy the horizon, the particle must satisfy both inequalities (12) and (13), which means

\[
E_{\text{max}} - E_{\text{min}} > 0. \tag{14}
\]

Using Eqs. (6) and (7), we can express the left-hand side of Eq. (14) as a function of \((Q, q, \Lambda)\). Since \( q \) and \( \Lambda \) are small values, by Taylor expansion, we have the leading term

\[
E_{\text{max}} - E_{\text{min}} = -\frac{|Q|}{2} \Lambda q^2 + O(\Lambda^2 q^2) + O(\Lambda q^3) + \ldots. \tag{15}
\]

This shows that Eq. (14) holds only for \( \Lambda < 0 \). There is no violation of the cosmic censorship for extremal RN-dS black holes.

For confirmation, we take \( Q = 1, \Lambda = -10^{-4} \) and \( q = 1.3 \times 10^{-3} \). The mass of the extremal RN-AdS black hole is determined by Eq. (7). Substituting these data into Eq. (12), we find

\[
E_{\text{min}} \approx 1.30 \times 10^{-3}. \tag{16}
\]

Using Eq. (63) to compute \( E_{\text{max}} \), we have

\[
\Delta E = E_{\text{max}} - E_{\text{min}} = 8.45 \times 10^{-11}. \tag{17}
\]

Therefore, as long as \( E \) is chosen within this range, we can always have solutions satisfying both Eq. (62) and Eq. (63), meaning that the extremal RN-AdS black holes could be destroyed. Note that \( \Delta E \) is of order \( QAq^2 \). For a particle to attain an energy within this range, a fine tuning is required.
4 Overspinning Kerr-dS and Kerr-AdS black holes

Now we turn attention to rotating black holes with cosmological constants, i.e., Kerr-dS and Kerr-AdS solutions. Such a spacetime is described by the metric \[ds^2 = -\frac{\Delta_r}{\chi^2 \rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\Delta_\theta \sin^2 \theta}{\chi^2 \rho^2} (adt - (r^2 + a^2) d\phi)^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2\] (18)
with
\[
\chi = 1 + \frac{a^2 \Lambda}{3} \\
\rho^2 = r^2 + a^2 \cos^2 \theta \\
\Delta_r = (1 - \frac{\Lambda}{3} r^2)(r^2 + a^2) - 2Mr \\
\Delta_\theta = 1 + \frac{a^2 \Lambda}{3} \cos^2 \theta
\] (19)
where \(M\) is the mass of the black hole and \(a = J/M\) is the specific angular momentum.

A horizon is located at
\[\Delta_r(r_h) = 0. \] (20)
By the same argument as given in [19], we find the condition for the particle to be captured by the black hole is
\[E \geq -\frac{g_{t\phi}}{g_{\phi\phi}} \bigg|_{r=r_h}. \] (21)
By direct substitution, one finds
\[E \geq \frac{aL}{r_h^2 + a^2} \equiv E_{\text{min}}. \] (22)

In order to find the condition to overspin the black hole, we first use Eq. (20) to express the mass in terms of the horizon radius
\[M = \frac{a^2}{2r_h} + \frac{r_h}{2} - \frac{1}{6} a^2 Ar_h - \frac{\Lambda}{6} r_h^3. \] (23)

An extremal black hole satisfies \(f'(r_h) = 0\), which is equivalent to
\[\frac{\partial M}{\partial r_h} = \frac{1}{2} - \frac{a^2}{2r_h^2} - \frac{\Lambda}{6} a^2 - \frac{\Lambda}{2} r_h^2 = 0. \] (24)

By solving Eq. (24), one has the solution of the horizon radius
\[r_h = \sqrt{\frac{3 - \Lambda a^2 - \sqrt{\Lambda^2 a^4 - 42\Lambda a^2 + 9}}{6\Lambda}}. \] (25)
Eq. (23) together with Eq. (25) defines a function

\[ M = f_M(a). \] (26)

A black hole is extremal if and only if this relation is satisfied. For a fixed \( a \), Eq. (26) also gives the minimum mass for the existence of a horizon. Consider an extremal black hole with \( M \) and \( a \). A particle with \( E \) and \( L \) is dropped into the black hole. The final state of the black hole is described by

\[ M' = M + E \] (27)
\[ a' = \frac{Ma + L}{M + E}. \] (28)

Thus, the condition for a possible violation of the cosmic censorship is given by

\[ M' < f_M(a'), \] (29)

i.e.,

\[ M + E < f_M \left( \frac{Ma + L}{M + E} \right). \] (30)

To proceed, we need to simplify Eq. (26). By expanding \( M \) to the first order of \( \Lambda \), we find

\[ f_M(a) \approx |a| - \frac{1}{3} |a|^3 \Lambda, \] (31)

Without loss of generality, we shall assume \( a > 0 \) in the following calculation (Obviously, the sign of \( a \) should not change the nature of our analysis). Then Eq. (30) is approximated as

\[ M + E < \left( \frac{Ma + L}{M + E} \right)^3. \] (32)

Define

\[ W = (M + E)^2, \] (33)

which satisfies

\[ W^2 - (Ma + L)W + \frac{1}{3} (Ma + L)^3 < 0. \] (34)

The solution of this inequality is

\[ z_1 < W < z_2, \] (35)

where

\[ z_2 = \frac{1}{6} \left( 3L + 3aM + \sqrt{(-3L - 3aM)^2 - 12(L^3 \Lambda + 3aL^2M \Lambda + 3a^2LM^2 \Lambda + a^3M^3 \Lambda} \right). \] (36)
and $z_1 < 0$.

Similarly, Eq. (22) can be rewritten as

$$W \equiv (E + M)^2 \geq \left(\frac{aL}{r_h^2 + a^2} + M\right)^2 \equiv z_3.$$  (37)

Then the necessary and sufficient condition for the particle to destroy the horizon is

$$z_2 - z_3 > 0.$$  (38)

Using Eq. (31) to replace $M$ in Eqs. (36) and (37), Eq. (25) to replace $r_h$, and then Taylor expanding $z_2 - z_3$ around $\Lambda = L = 0$, we finally obtain

$$z_2 - z_3 = -\frac{L^2}{4a^2} + \frac{a^2}{3}L\Lambda + O(L\Lambda^2) + ...$$  (39)

This corresponds to

$$\Delta E \equiv E_{\text{max}} - E_{\text{min}} \sim L^2/M^3 \sim M\Lambda,$$  (40)

which means that the allowed energy window for the particle is very narrow.

Using Eq. (39), the solution of Eq. (38) is found to be

$$0 < L < \frac{4}{3}\Lambda a^4,$$  (41)

or

$$0 > L > \frac{4}{3}\Lambda a^4.$$  (42)

Clearly, Eq. (41) requires $\Lambda > 0$ and Eq. (42) requires $\Lambda < 0$. In the following, we shall discuss the two cases respectively.

1. **Kerr-de Sitter solution ($\Lambda > 0$)**

For $\Lambda > 0$, we may choose a small but positive $L$ to satisfy the inequality (41), leading to a possible violation of the cosmic censorship. One might suspect that the terms we dropped above could cause some serious error and then invalidate our conclusion. For clarification, we construct a numerical solution as follows.

We choose

$$a = 1, \quad \Lambda = 10^{-4}, \quad L = 10^{-4}$$  (43)

so that the inequality (41) holds. Eq. (26) gives the mass of the existing extremal Kerr-dS black hole as

$$M = 0.999967.$$  (44)

We use Eqs. (22) and (25) to find the lower limit of $E$

$$E_{\text{min}} = 4.99967 \times 10^{-5}.$$  (45)
Choose the energy of the particle to be
\[ E = E_{\text{min}} + 10^{-10} \] (46)
and we find
\[ f_M \left( \frac{Ma + L}{M + E} \right) - (M + E) = 6.336 \times 10^{-10} > 0. \] (47)

Therefore, we have found a solution such that Eqs. (30) and (22) both hold. Note that our numerical calculation only involves original formulas without any approximation.

2. Kerr-anti-de Sitter solution ($\Lambda < 0$)–Penrose process

Eq. (42) shows that both $L$ and $\Lambda$ are negative. As a consequence, the energy $E$ of the particle must be negative too. The reason is as follows.

In the absence of $\Lambda$, i.e., the Kerr solution, the overspinning condition simply reduces to
\[ M + E < \frac{Ma + L}{M + E}. \] (48)

With the extremal condition $M = a$, we have
\[ E < \frac{L}{2M}. \] (49)

Obviously, a negative angular momentum $L$ implies a negative energy $E$ in the Kerr case. On the other hand, one see immediately that Eq. (22) reduces to
\[ E \geq \frac{L}{2M}, \] (50)
because $M = a = r_h$ in the extreme Kerr solution. Therefore, no overspinning actually occurs in the absence of $\Lambda$.

Now consider the case $\Lambda < 0$ and $L < 0$. With the help of Eq. (31), one can expand Eq. (30) around $\Lambda = 0$ and $L = 0$:

\[ s = f_M \left( \frac{Ma + L}{M + E} \right) - (M + E) \]
\[ = -a - E + \frac{a^2}{a + E} + \frac{L}{a + E} + \left( \frac{a^3}{3} - \frac{a^6}{3(a + E)^3} + \frac{a^5}{3(a + E)^2} - \frac{a^4}{3(a + E)} \right) \Lambda + O(\Lambda L). \]

Since $E \ll a$, $s$ can be expanded as
\[ s = \frac{L}{a} + \left( -2 - \frac{L}{a^2} + \frac{2a^2 \Lambda}{3} \right) E. \] (51)

Noting that $|L| \ll a$ and $\Lambda a^2 \ll 1$, we see that $s > 0$ implies $E < 0$. Therefore, to overspin an extremal Kerr-AdS black hole, the energy of the particle must be negative. This is a reminiscent of the Penrose process [25]. In the Kerr-AdS
spacetime, an ergo sphere exists outside the black hole horizon. Thus, a particle with $E < 0$ can be made within the ergo sphere. In the Kerr spacetime, it is well-known that energy can be extracted from the black hole via the Penrose process. However, in the Kerr-AdS spacetime, our derivation suggests that a particle with finely tuned negative energy can enter the horizon and then make it disappear. This is an important difference caused by the negative cosmological constant in the Penrose process. This result is consistent with that obtained by Cardoso, Dias and Yoshida [26] who showed that small Kerr-AdS black holes are unstable against scalar perturbation, via a superradiant mechanism.

To verify the conclusion numerically, we choose

$$a = 1, \, \Lambda = -10^{-4}, \, L = 10^{-4}.$$  \hspace{1cm} (52)

The black hole mass determined by Eq. (26) is given by

$$M = 1.000033.$$  \hspace{1cm} (53)

Similarly to the case $\Lambda > 0$, we find that the lower limit of the particle’s energy is

$$E_{\text{min}} = -5.00033 \times 10^{-4}.$$  \hspace{1cm} (54)

Choose the energy of the particle to be slightly bigger than the lower limit

$$E = E_{\text{min}} + 10^{-10}.$$  \hspace{1cm} (55)

We then find

$$f_M \left( \frac{Ma + L}{M + E} \right) - (M + E) = 6.330 \times 10^{-10} > 0.$$  \hspace{1cm} (56)

Therefore, we have found a particle with negative energy which satisfies both Eqs. (50) and (22).

5 Discussion of radiative effects in the Kerr-dS case

In the above analysis, we treated the spacetime as a fixed background and neglected radiation reaction from the particle. We have mentioned that the allowed energy window is quite small for the Kerr-dS case. So if the gravitational radiation of the particle is of the same order, the overspinning might be prevented. Recently, Barausse, Cardoso and Khanna [20, 21] showed that the effect of radiation reaction and self-force could help prevent the formation of naked singularities. A key point in their analysis is that when possible violation of the cosmic censorship occurs, the particle always follows an almost circular orbit such that the particle may radiate significant energy before it reaches the horizon. Following the same spirit, we analyze the orbits near the horizon and
find that the particle will make tremendously large number of circles near the horizon and cause energy loss that is sufficient to break down the argument in the previous section.

We consider the the Kerr-dS case. Substituting Eq. (26) into Eq. (22) and expand at $\Lambda = 0$, we find

$$E_{\text{min}} = \frac{L}{2a} - \frac{1}{3}aL\Lambda,$$

(57)

Expanding Eq. (36) around $\Lambda = 0$ gives

$$E_{\text{max}} = \frac{L}{2a} - \frac{L^2}{8a^3} - \frac{aL}{6}\Lambda.$$

(58)

Then

$$E_{\text{max}} - E_{\text{min}} > 0$$

(59)

yields

$$0 < L < \frac{4}{3}a^4\Lambda,$$

(60)

which is the same as Eq. (41). We can take

$$L = a^4\Lambda$$

(61)

such that

$$E_{\text{min}} = \frac{L}{2a} - \frac{1}{3}a^5\Lambda^2,$$

(62)

$$E_{\text{max}} = \frac{L}{2a} - \frac{7}{24}a^5\Lambda^2.$$  

(63)

Then the allowed energy is written in the form

$$E = \frac{L}{2a} - \beta a^5\Lambda^2$$

(64)

with

$$\frac{7}{24} < \beta < \frac{1}{3}.$$  

(65)

The standard calculation shows that the equations of motion for a particle with mass $m$ on the $\theta = \frac{\pi}{2}$ in the Kerr-dS spacetime are given by

$$\dot{\varphi}^2 = \frac{E^2g_{\varphi\varphi} + 2Eg_{t\varphi}L - g_{t\varphi}^2m^2 + g_{tt}(L^2 + g_{\varphi\varphi}m^2)}{g_{rr}(g_{t\varphi}^2 - g_{\varphi\varphi}g_{tt})m^2},$$

$$\dot{\phi} = -\frac{Eg_{t\phi} + g_{tt}L}{(g_{t\phi}^2 - g_{\varphi\varphi}g_{tt})m}.$$  

(66)
Assuming $E \gg m$ [20, 21], we have
\[ \dot{r}^2 = \frac{E^2 g_{\phi\phi} + 2E g_{\phi t} L + g_{tt} L^2}{g_{rr}(g_{\phi\phi} - g_{\phi t} g_{tt}) m^2} \equiv V(r). \] (67)

Near the horizon $r = r_h$, $V(r)$ may be written as
\[ V(r) \approx V(r_h) + V'(r_h)(r - r_h). \] (68)

Substituting the metric (18), where $M$ satisfies Eq. (31), and using Eqs. (61) and (64), then expanding $V(r)$ around $\Lambda = 0$, we find
\[ V(r_h) \approx \frac{4a^{10} \Lambda^4}{9m^2} (1 - 3\beta)^2, \] (69)

and
\[ V'(r_h) = \frac{4a^7 (1 - 3\beta) \Lambda^3}{3m^2} > 0. \] (70)

The linear approximation (68) is valid within the range
\[ \Delta r = r - r_h \sim \frac{V'(r_h)}{V''(r_h)} \sim (1 - 3\beta) \Lambda a^3. \] (71)

On the other hand, it follows from Eq. (66) that
\[ \dot{\phi} \sim \frac{a^3 \Lambda}{m(r - r_h)} \sim \frac{1}{m}, \] (72)

at $r_h$. Therefore,
\[ \frac{d\phi}{dr} \sim \frac{1}{m \sqrt{V(r_h) + V'(r_h)(r - r_h)}}. \] (73)

Finally, the number of cycles made by the particle near the horizon is given by
\[ N = \int_{r_h}^{r_h + \Delta r} \frac{1}{m \sqrt{V(r_h) + V'(r_h)(r - r_h)}} \, dr \sim \frac{1}{\Lambda a^2}. \] (74)

For a black hole with 100 solar mass, one can show $\Lambda a^2 \sim 10^{-42}$. So $N$ is an extremely large number.

According to [20], the gravitational-wave loss of energy can be estimated by
\[ E_{rad} \sim N(r_0 - r_h) E^2 / a^2, \] (75)

where $r_0$ is the radius of the circular phonon orbit and we have divided the formula in [20] by $a^2$ to make the dimension consistent with our convention. However, as shown in Appendix A, $r_0 = r_h$ for the extremal Kerr-dS black hole, which makes $E_{rad}$ vanish. Unlike the results for nearly extremal Kerr black holes [20, 21], which suggest that the radiative effects can be neglected for some trajectories giving rise to naked singularities, we show that the radiative effects are negligible for all trajectories that lead to possible violation of the cosmic censorship.
6 Conclusions

We have extended the test of the cosmic censorship to black holes with cosmological constants. Previous works showed that extremal RN and Kerr black holes cannot serve as counterexamples of the cosmic censorship even if higher order terms are taken into account. In the presence of $\Lambda$, we derived the condition that a test particle can be absorbed by an extremal black hole and the condition that the black hole horizon might disappear after the absorption. For charged black holes, we show that an extremal RN-AdS black hole can be overcharged, implying possible violation of the cosmic censorship. In contrast, an extremal RN-dS black hole cannot be overcharged. For rotating black holes, both extremal Kerr-dS and Kerr-AdS black holes can be overspun. In particular, to overspin the Kerr-AdS black hole, the energy of the particle must be negative, i.e., the Penrose process is involved.

Similarly to the discussion in the absence of $\Lambda$, we have found that the allowed parameter range for the test particle is very narrow. By studying the orbits near the horizon of an extremal Kerr-dS black hole and using the light ring argument, we found that the radiative effect is not enough to prevent the violation of the cosmic censorship. As shown in [20]-[21], the self-force effect is important for nearly extremal black holes. Although we have not discussed the self-force effect in this paper, our results suggest some interesting regime in which such effect should be studied.

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A Circular photon orbit

We calculate the circular photon orbit (light ring). Consider the orbit of a photon on the $\theta = \pi/2$ plane in the Kerr-dS spacetime. The tangent vector $k^a$ to an orbit parameterized by $\lambda$ is written as

$$k^a = i \left( \frac{\partial}{\partial t} \right)^a + r \left( \frac{\partial}{\partial r} \right)^a + \phi \left( \frac{\partial}{\partial \phi} \right)^a.$$  \hspace{1cm} (76)

The conservation of energy and angular momentum leads to

$$E = -g_{ab} k^a \left( \frac{\partial}{\partial t} \right)^b,$$ \hspace{1cm} (77)

$$L = g_{ab} k^a \left( \frac{\partial}{\partial \phi} \right)^b.$$ \hspace{1cm} (78)
Together with the constraint

$$g_{ab}k^a k^b = 0$$  \hspace{1cm} (79)

and the metric Eq. (18), we find the radial motion for the photon is given by

$$\dot{r}^2 = \frac{1}{27r^3} V(r),$$  \hspace{1cm} (80)

where

$$V(r) = 9E^2 (6M + (\Lambda + 3)r (r^2 + 1)) - 6E(\Lambda + 3)L (6M + \Lambda r (r^2 + 1)) + (\Lambda + 3)^2 L^2 (6M + r (\Lambda + \Lambda r^2 - 3)).$$  \hspace{1cm} (81)

Without loss of generality, we have taken $a = 1$ in Eq. (81). The light ring refers to the circular orbit at $r = r_0$ satisfying

$$V'(r_0) = V(r_0) = 0.$$  \hspace{1cm} (82)

First, $V'(r_0) = 0$ yields

$$r_0 = \frac{\sqrt{-27E^2 + 27L^2} - 9E^2\Lambda + 18EL\Lambda + 9L^2\Lambda + 6E\Lambda^2 - 3L^2\Lambda^2 - L^2\Lambda^3}{3\sqrt{27E^2 + 9E^2\Lambda - 18EL\Lambda + 9L^2\Lambda - 6E\Lambda^2 + 6L^2\Lambda^2 + L^2\Lambda^3}}.$$  \hspace{1cm} (83)

Our purpose is to calculate $r_0$. So without loss of generality, we choose $E = 1$. By substituting Eq. (26) into Eq. (81), we can solve $V(r_0) = 0$ and Eq. (83) for $L$ and $r_0$. It is difficult to find the analytical solutions. However, we shall show that $r_0$ is coincident with the horizon radius $r_h$. For this purpose, we assume

$$r_0 = r_h,$$  \hspace{1cm} (84)

where $r_h$ is given by Eq. (25). Thus, $L$ can be solved as

$$L = \frac{\sqrt{32\Lambda^4 - 768\Lambda^3 - 96\sqrt{\Lambda^2 - 42\Lambda + 9}\Lambda^2 + 9\Lambda^3 - 6\Lambda^2}}{2(2\Lambda^3 + 6\Lambda^2)} + \frac{14\Lambda^2 + 2\sqrt{\Lambda^2 - 42\Lambda + 9} \Lambda}{2(2\Lambda^3 + 6\Lambda^2)}.$$  \hspace{1cm} (85)

Substituting $L$ into Eq. (81), we obtain

$$V(r_h) \propto \sqrt{2\Lambda^2 + \sqrt{2}\Lambda \sqrt{\Lambda^2 - 42\Lambda + 9} - \left(\sqrt{\Lambda^2 - 42\Lambda + 9} - 3\right) \cdot \left(3\sqrt{2} - \sqrt{\Lambda^2 + \left(\sqrt{\Lambda^2 - 42\Lambda + 9} - 24\right) \Lambda - 3\sqrt{\Lambda^2 - 42\Lambda + 9} + 9}\right)} + \left(\sqrt{\Lambda^2 + \left(\sqrt{\Lambda^2 - 42\Lambda + 9} - 24\right) \Lambda - 3\sqrt{\Lambda^2 - 42\Lambda + 9} + 9 - 24\sqrt{2}\right) \Lambda.$$  \hspace{1cm} (86)

By some algebraic manipulation, we find

$$V(r_h) = 0.$$  \hspace{1cm} (87)

for all $\Lambda$. Therefore, $r_0$ is equal to $r_h$. 14
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