THE DISTRIBUTION OF PATH LENGTHS ON DIRECTED WEIGHTED GRAPHS

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ABSTRACT. We consider directed weighted graphs and define various families of path counting functions. Our main results are explicit formulas for the main term of the asymptotic growth rate of these counting functions, under some irrationality assumptions on the lengths of all closed orbits on the graph. In addition we assign transition probabilities to such graphs and compute statistics of the corresponding random walks. Some examples and applications are reviewed.

1. Introduction and Main Results

Questions regarding the distribution of path lengths on directed weighted graphs are encountered in various fields of study of mathematics and physics. They arise naturally in dynamics and the study of closed orbits of suspensions of shifts of finite type, see among others [PP1], [PP2] and [Gu], and the more recent [KS] and [BPP].

Our motivations for counting paths on weighted graphs are diverse. The second author’s motivation originates in the study of a model of mathematical quasicrystals which we call multiscale substitution tilings, and in the study of equidistribution of what is known as Kakutani’s partitions, first described in [Ka]. The connection to problems concerning path counting on weighted graphs is introduced in subsection 5.2. The third author’s motivation is rooted in theoretical physics, specifically in the spectral properties of the Schrödinger operator for systems which are chaotic in the classical limit and for metric graphs [GS]. Of particular relevance are studies of the distribution of delay (transit) times through chaotic scatterers such as e.g., the scattering of ultra-short electromagnetic pulses by complex molecules, or traversing networks of transmission lines [An] and [SmU].

1.1. Counting paths in graphs. Let $G = (V, E, l)$ be a directed weighted metric graph with a set $V = \{1, \ldots, n\}$ of vertices and a set $E$ of edges. A positive weight is assigned to each edge $\alpha \in E$, and we think of this weight as the length of $\alpha$. For a path $\gamma$ connecting two vertices in $V$, the path metric $l$ is defined to be the sum of the weights of the edges in $\gamma$. When considering paths which do not necessarily terminate at a vertex of $G$, the path metric is defined by $l(\gamma) = a$ if the path $\gamma$ is isomorphic to $[0, a] \subset \mathbb{R}$ in the obvious way. Throughout this paper $G$ is assumed to be strongly connected multigraph, that is a graph which admits a path from every vertex $i \in V$ to every vertex $j \in V$, and loops and multiple edges are allowed.

We say that $G$ is a graph of incommensurable orbits, or incommensurable for short, if there exist at least two closed paths in $G$ of lengths $a, b$ such that $a \notin \mathbb{Q}b$. This irrationality condition on the lengths of the edges is equivalent to the set of lengths of all closed orbits in $G$ not being a uniformly discrete subset of $\mathbb{R}$.

Let $i, j \in V$ be a pair of vertices in $G$, and assume that there are $k \geq 0$ edges $\alpha_1, \ldots, \alpha_k$ from $i$ to $j$. This irrationality condition on the lengths of the edges is equivalent to the set of lengths of all closed orbits in $G$ not being a uniformly discrete subset of $\mathbb{R}$.

The matrix valued function $M : \mathbb{C} \to M_n(\mathbb{C})$, which we call the graph matrix function of $G$, is defined by

$$M_{ij}(s) = e^{-s l(\alpha_1)} + \cdots + e^{-s l(\alpha_k)}$$

and $M_{ij}(s) = 0$ if $i$ is not connected to $j$ by an edge. Note that the restriction of $M$ to $\mathbb{R}$ is real valued.

**Theorem 1.** Let $G$ be a strongly connected incommensurable graph. There exist a positive constant $\lambda$ and a matrix $Q \in M_n(\mathbb{R})$ with positive entries such that

(i) The number of paths from $i \in V$ to $j \in V$ of length at most $x$ grows as

$$\frac{1}{\lambda} Q_{ij} e^{\lambda x} + o\left(e^{\lambda x}\right), \quad x \to \infty.$$
(ii) Let $\alpha \in \mathcal{E}$ be an edge in $G$ which originates in vertex $j \in \mathcal{V}$. The number of paths of length exactly $x$ from vertex $i$ to a point on the edge $\alpha$ grows as
\[
\frac{1 - e^{-l(\alpha)\lambda}}{\lambda} Q_{ij} e^{\lambda x} + o\left(e^{\lambda x}\right), \quad x \to \infty.
\]
The constant $\lambda$ is the maximal real value for which the spectral radius of $M$ is equal to 1, and
\[
Q = Q(M(\lambda)) = \text{adj}(I - M(\lambda))
\]
where $M'$ is the entry-wise derivative of $M$, and $\text{adj} A$ is the adjugate or classical adjoint matrix of $A$, that is the transpose of its cofactor matrix.

1.2. Weighted random walks on graphs. Let $\alpha \in \mathcal{E}$ be an edge which originates at $i \in \mathcal{V}$. Denote by $p_{\alpha} > 0$ the probability that a walker who is passing through vertex $i$ chooses to continue his walk through edge $\alpha$, and assume that the sum of the probabilities over all edges originating at a given vertex is less than or equal to 1, for all vertices in $G$. Let $\alpha_1, \ldots, \alpha_k$ be the edges connecting vertex $i$ to vertex $j$. The graph probability matrix function $N : \mathbb{C} \to M_n(\mathbb{C})$ is defined by
\[
N_{ij}(s) = p_{\alpha_1} e^{-s l(\alpha_1)} + \cdots + p_{\alpha_k} e^{-s l(\alpha_k)}
\]
and $N_{ij}(s) = 0$ if $i$ is not connected to $j$ by an edge. Note that the restriction of $N$ to $\mathbb{R}$ is real valued. If the sum of the probabilities over all edges originating at a given vertex is strictly less than 1, there is a positive probability that the walker does not choose any of the edges and instead leaves the graph.

Theorem 2. Let $G$ be a strongly connected incommensurable graph, and consider a walker on $G$ advancing at constant speed 1. There exist a non-positive constant $\lambda$ and a matrix $Q \in M_n(\mathbb{R})$ with positive entries such that

(i) The probability that a walker leaving $i \in \mathcal{V}$ at time $t = 0$ is exactly at $j \in \mathcal{V}$ at time $t = T$ decays as
\[
Q_{ij} e^{\lambda T} + o\left(e^{\lambda T}\right), \quad T \to \infty
\]
for values of $T$ in the countable set of times for which this probability is non-zero.

(ii) Let $\alpha \in \mathcal{E}$ be an edge in $G$ which originates in vertex $j \in \mathcal{V}$. The probability that a walker who has left vertex $i \in \mathcal{V}$ at time $t = 0$ is on the edge $\alpha \in \mathcal{E}$ at time $t = T$, where $\alpha$ originates at $j$ and has probability $p_{\alpha}$, decays as
\[
p_{\alpha} \frac{1 - e^{-l(\alpha)\lambda}}{\lambda} Q_{ij} e^{\lambda T} + o\left(e^{\lambda T}\right), \quad T \to \infty
\]
whenever $\lambda < 0$. In the case $\lambda = 0$, the probability tends to
\[
p_{\alpha} l(\alpha) Q_{ij}, \quad T \to \infty.
\]
As in the previous theorem, the constant $\lambda$ is the maximal real value for which the spectral radius of $N$ is equal to 1, and
\[
Q = Q(N(\lambda)) = \frac{\text{adj}(I - N(\lambda))}{-\text{tr}(\text{adj}(I - N(\lambda)) \cdot N'(\lambda))}.
\]

As a direct corollary we have

Corollary 1. In the settings of the previous theorem, denote by $\mathcal{E}(j)$ the set of edges in $G$ with origin at vertex $j \in \mathcal{V}$. The probability that a walker who has left vertex $i \in \mathcal{V}$ at time zero is still on the graph $G$ at time $T$ decays as
\[
\sum_{j \in \mathcal{V}} \sum_{\alpha \in \mathcal{E}(j)} p_{\alpha} \frac{1 - e^{-l(\alpha)\lambda}}{\lambda} Q_{ij} e^{\lambda T} + o\left(e^{\lambda T}\right), \quad T \to \infty
\]
whenever $\lambda < 0$.

Remark 1. It will follow from Remark [4] that in the case $\lambda = 0$ this probability is 1.

The random walk defined above can be generalized by considering a random walk on the edges, and the transition probability $p_{\beta, \alpha}$ from and edge $\alpha$ to and edge $\beta$ vanishes unless $\beta$ originates at the vertex where $\alpha$ terminates. Let $d$ be the number of directed edges on the graph. The graph probability matrix function $W : \mathbb{C} \to M_d(\mathbb{C})$ is defined by
\[
W_{\beta, \alpha}(s) = p_{\beta, \alpha} e^{-s l(\alpha)}.
\]
The analogue of Theorem 2 and its Corollary 1 in the present case follow directly from the discussion above.

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2. Matrices, the Theory of Perron-Frobenius and Graphs

A real valued matrix $A \in M_n(\mathbb{R})$ is called positive if all entries of $A$ are positive and non-negative if all entries of $A$ are non-negative. $A$ is called primitive if there exists $k \in \mathbb{N}$ for which $A^k$ is positive and irreducible if for every pair of indices $i, j$ there exists $k \in \mathbb{N}$ for which $(A^k)_{ij} > 0$.

2.1. The Perron-Frobenius Theorem. The following results are due to Perron and Frobenius (full statements and proofs can be found in chapter XIII of [Ga]).

**Theorem.** Let $A \in M_n(\mathbb{R})$ be a non-negative and irreducible matrix.

1. There exists $\mu > 0$ which is a simple eigenvalue of $A$, and $|\mu_j| \leq \mu$ for any other eigenvalue $\mu_j$. We call $\mu$ the Perron-Frobenius eigenvalue.

2. There exist $v, u \in \mathbb{R}^n$ with positive entries such that $Av = \mu v$ and $u^T A = \mu u^T$. Moreover every right eigenvector with non-negative entries must be a positive multiple of $v$ (similarly for left eigenvectors and $u$).

**Theorem.** Let $A \in M_n(\mathbb{R})$ be a primitive matrix.

1. There exists $\mu > 0$ which is a simple eigenvalue of $A$, and $|\mu_j| < \mu$ for any other eigenvalue $\mu_j$. We call $\mu$ the Perron-Frobenius eigenvalue.

2. There exist $v, u \in \mathbb{R}^n$ with positive entries such that $Av = \mu v$ and $u^T A = \mu u^T$. Moreover every right eigenvector with non-negative entries must be a positive multiple of $v$ (similarly for left eigenvectors and $u$).

3. The following limit holds

$$\lim_{k \to \infty} \left( \frac{1}{\mu} A \right)^k = \frac{vu^T}{u^Tv}.$$  

The limit matrix $P = \frac{vu^T}{u^Tv}$ is called the Perron projection of $A$.

2.2. Perron’s projection. Given an irreducible matrix $A$, there are additional ways to represent its Perron projection $P$, as shown below

**Lemma 1.** Let $A$ be an irreducible matrix with Perron-Frobenius eigenvalue $\mu$ and a Perron projection $P$. Then

$$P = \frac{\text{adj}(\mu I - A)}{\text{tr}(\text{adj}(\mu I - A))}.$$  

**Proof.** Let $v$ and $u$ be eigenvectors as in the Perron-Frobenius theorem. The columns of $P$ are scalar multiples of $v$, and the rows of $P$ are scalar multiples of $u^T$, and so the column space of $P$ is spanned by $v$ and the row space by $u$. Denote by $V$ the subspace of $M_n(\mathbb{R})$ consisting of matrices with these row and column spaces, and notice that $\dim V = 1$. Since $\mu$ is an eigenvalue of $A$ we have

$$(\mu I - A) \cdot \text{adj}(\mu I - A) = \det(\mu I - A) I = 0,$$

and so every column of $\text{adj}(\mu I - A)$ is an eigenvector of $A$ corresponding to $\mu$. By the Perron-Frobenius theorem all columns of $\text{adj}(\mu I - A)$ must be scalar multiples of $v$ and so the column space of $\text{adj}(\mu I - A)$ is spanned by $v$. Similarly, using $\text{adj}(\mu I - A) \cdot (\mu I - A) = 0$ we deduce that the row space of $\text{adj}(\mu I - A)$ is spanned by $u$, and so $\text{adj}(\mu I - A) \in V$. Since $V$ is one dimensional $\text{adj}(\mu I - A) = \alpha P$ for some $\alpha \in \mathbb{R}$. Next, since $Pv = v$, and $Pw = 0$ for every $w \in (\text{span}\{u\})^\perp$, the Perron projection $P$ is similar to the matrix $\text{diag}(1, 0, \ldots, 0)$. Therefore $\text{tr}P = 1$, and so

$$\text{tr}(\text{adj}(\mu I - A)) = \text{tr}(\alpha P) = \alpha \text{tr}P = \alpha,$$

finishing the proof.
Corollary 2. Let $p_A$ be the characteristic polynomial of $A$, then

$$P = \frac{\text{adj} (\mu I - A)}{\frac{d}{dx} p_A (x) |_{x=\mu}}$$

Proof. Jacobi’s formula for the derivative of the determinant of a matrix is given by

$$\frac{d}{dx} \det B (x) = \text{tr} (\text{adj} (B (x)) B' (x))$$

and so using this formula, the corollary follows from the previous lemma for $B (x) = x I - A$. \qed

Remark 2. This result and others concerning the theory of Perron-Frobenius may be found in [Se]. Another proof for Corollary 2 can be derived by direct computation using the following identity

$$\text{adj} (\lambda I - A) = A^{n-1} + (\lambda + p_{n-1}) A^{n-2} + \cdots + (\lambda^{n-1} + p_{n-1}\lambda^{n-2} + \cdots + p_1) I$$

where $p_A (x) = x^n + p_{n-1}x^{n-1} + \cdots + p_0$ and $\lambda \in \mathbb{R}$ (see [Ga] or [FDC]). Let $\mu$ be the Perron-Frobenius eigenvalue and $v$ a corresponding eigenvector, we compute

$$\text{adj} (\mu I - A) v = \left[ \mu^{n-1} + (\mu + p_{n-1}) \mu^{n-2} + \cdots + (\mu^{n-1} + p_{n-1}\mu^{n-2} + \cdots + p_1) \right] v$$

Since $\text{adj} (\mu I - A) = \alpha P$ for some $\alpha \in \mathbb{R}$. Since

$$p_A' (\mu) v = \text{adj} (\mu I - A) v = \alpha P v = \alpha v,$$

it follows that $\alpha = \text{tr} (\text{adj} (\mu I - A)) = p_A' (\mu)$.

Corollary 3. Let $(\mu, \mu_2, \ldots, \mu_n)$ be the eigenvalues of $A$, perhaps with repetitions. Then

$$P = \prod (A - \mu I) / \prod (\mu - \mu_i).$$

Proof. Using the representation of $\text{adj} (\mu I - A)$ as a polynomial of degree $n - 1$, it follows from Vieta’s polynomial formulas (see for example [Vi]) that

$$\text{adj} (\mu I - A) = (A - \mu_2 I) \cdots (A - \mu_n I).$$

Since $p_A' (\mu) = (\mu - \mu_2) \cdots (\mu - \mu_n)$ this gives the desired result. \qed

2.3. Path counting in non-weighted graphs. Let $G$ be a non-weighted graph. The adjacency matrix of $G$ is the square matrix $A \in M_n (\mathbb{R})$ indexed by the vertices of $G$, where $A_{ij}$ is the number of edges from vertex $i$ to vertex $j$. Note that the adjacency matrix of a strongly connected graph is irreducible, but not necessarily primitive. For primitivity of the adjacency matrix we must also assume that $G$ is acyclic, which means that the greatest common divisor of the set of lengths of all closed paths is 1. The number of paths from vertex $i$ to vertex $j$ consisting of exactly $k$ edges is $(A^k)_{ij}$, and so if the graph is strongly connected and acyclic, using the Perron-Frobenius theorem this number can be approximated by

$$P_{ij} \mu^k.$$

From this we deduce that the number of paths from vertex $i$ to vertex $j$ which are of length at most $x$ can be approximated by

$$\frac{P_{ij}}{\mu - 1} \mu^{\lfloor x \rfloor} + 1$$

and the number of paths originating at vertex $i$ such that a walker moving along the path at speed 1 would at time $T$ be on the edge $\alpha$ which originates at vertex $j$ is

$$P_{ij} \mu^{\lceil T \rceil}.$$
2.4. Comparison between the non-weighted case and the weighted case. It is interesting to compare the matrices $P$ and $Q$. Due to Jacobi's formula we can write

$$Q = \frac{\text{adj} (I - M (\lambda))}{-\text{tr} (\text{adj} (I - M (\lambda)) \cdot M' (\lambda))} = \frac{\text{adj} (I - M (\lambda))}{\frac{d}{dx} (\det (I - M (s)))}_{|s=\lambda}$$

and

$$P = \frac{\text{adj} (I - \frac{1}{\mu} A)}{\text{tr} (\text{adj} (I - \frac{1}{\mu} A))} = \frac{\text{adj} (I - \frac{1}{\mu} A)}{\frac{d}{dx} \det (I - \frac{1}{x} \mu A)}_{|x=1}$$

and the resemblance is clear. Note that for convergence in the case of non-weighted graphs we assume that the graph is strongly connected and acyclic, otherwise the corresponding adjacency matrix need not be primitive. In the case of weighted graphs we replace the assumption that $G$ is acyclic by the assumption of incommensurability.

As an example we look at the following case: Assume all edges in $G$ are of equal length $a > 0$. So $M (s) = e^{-as} A$ where $A$ is the adjacency matrix of the underlying non-weighted graph. Obviously $G$ is not incommensurable and the assumptions of Theorem 3.1 do not hold, but still we can calculate $Q$.

Let $\mu$ be the Perron-Frobenius eigenvalue of $A$, then the matrix $M \left( \frac{\log a}{a} \right) = \frac{1}{\mu} A$ has Perron-Frobenius eigenvalue 1, and so $\lambda = \frac{\log \mu}{a}$. Since

$$-\text{tr} (\text{adj} (I - M (\lambda)) \cdot M' (\lambda)) = -\text{tr} \left( \text{adj} \left( I - \frac{1}{\mu} A \right) \cdot \frac{d}{dx} A \right) = \text{tr} \left( \text{adj} \left( I - \frac{1}{\mu} A \right) \cdot \frac{1}{\mu} A \right) = \text{tr} \left( \text{adj} \left( I - \frac{1}{\mu} A \right) \right)$$

we get

$$Q = \frac{\text{adj} (I - M (\lambda))}{-\text{tr} (\text{adj} (I - M (\lambda)) \cdot M' (\lambda))} = \frac{\text{adj} (I - \frac{1}{\mu} A)}{\text{tr} (\text{adj} (I - \frac{1}{\mu} A))} = \frac{1}{a} P$$

and so if we think of a non-weighted graph as a weighted graph with edges all of length $a = 1$, we get $P = Q$.

3. The Wiener-Ikehara Theorem and the Laplace Transform

3.1. The Wiener-Ikehara Theorem. The proofs of our main results follow from this Tauberian theorem due to Wiener and Ikehara (see chapter 8.3 in [MV]):

Theorem. Let $f(x)$ be a non-negative and monotone function on $[0, \infty)$. Suppose that the Laplace transform of $f(x)$, given by

$$F(s) := \mathcal{L} \{ f(x) \} (s) = \int_0^\infty f(x) e^{-sx} dx,$$

converges for all $s$ with $\text{Re}(s) > \lambda$, and that there exists $c \in \mathbb{R}$ for which the function

$$F(s) = \frac{c}{s - \lambda}$$

extends to a continuous function in the closed half-plane $\text{Re}(s) \geq \lambda$. Then

$$f(x) = ce^{\lambda x} + o (e^{\lambda x}) \quad , \quad x \to \infty.$$  

3.2. The Laplace Transform of the counting and probability functions. Denote the set of paths originating at vertex $i \in V$ and terminating at vertex $j \in V$ by $\Gamma(i,j)$, and by $p(\gamma)$ the product of probabilities of the edges which define the path $\gamma$. We calculate the Laplace transforms of the counting and probability functions which appear in the theorems:

Let $A_{i,j}(x)$ denote the number of paths originating at vertex $i$ and terminating at vertex $j$ of length at most $x$. Then

$$A_{i,j}(x) = \sum_{\gamma \in \Gamma(i,j)} \chi([l(\gamma), \infty)) (x) = \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma(i,j) \text{ with } k \text{ edges}} \chi([l(\gamma), \infty)) (x)$$
where $\chi_A$ is the characteristic function of the set $A \subset \mathbb{R}$. The Laplace transform is

$$\mathcal{L} \{ A_{i,j} (x) \} (s) = \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma(i,j)} \frac{1}{s} e^{-l(\gamma)} = \frac{1}{s} \left( \sum_{k=0}^{\infty} M^k (s) \right)_{i,j}.$$

Let $\alpha$ be an edge originating at vertex $j$. Denote by $B_{i,\alpha} (x)$ the number of paths of length exactly $x$ from vertex $i$ to a point on the edge $\alpha$. Then

$$B_{i,\alpha} (x) = \sum_{\gamma \in \Gamma(i,j)} \chi(I(\gamma),I(\gamma)+I(\alpha)) (x) = \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma(i,j)} \chi(I(\gamma),I(\gamma)+I(\alpha)) (x).$$

The Laplace transform is

$$\mathcal{L} \{ B_{i,\alpha} (x) \} (s) = \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma(i,j)} \frac{1 - e^{-l(\alpha)s}}{e^{-l(\gamma)}} = \frac{1 - e^{-l(\alpha)s}}{s} \left( \sum_{k=0}^{\infty} M^k (s) \right)_{i,j}.$$

Denote by $C_{i,j} (T)$ the probability that a walker leaving $i$ at time zero and moving along the the graph at speed 1, would at time $T$ be exactly at vertex $j$. Then

$$C_{i,j} (T) = \sum_{\gamma \in \Gamma(i,j)} p(\gamma) \delta (T - l(\gamma)) = \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma(i,j)} p(\gamma) \delta (T - l(\gamma))$$

where $\delta (x - a)$ is the delta function centered at $a \in \mathbb{R}$. The Laplace transform is

$$\mathcal{L} \{ C_{i,j} (T) \} (s) = \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma(i,j)} p(\gamma) e^{-l(\gamma)} = \left( \sum_{k=0}^{\infty} N^k (s) \right)_{i,j}.$$

Denote by $D_{i,\alpha} (T)$ the probability that a walker leaving $i$ at time zero and moving along the graph at speed 1, would at time $T$ be on the edge $\alpha$ which originates at vertex $j$. Then

$$D_{i,\alpha} (T) = \sum_{\gamma \in \Gamma(i,j)} p(\gamma) p_{i,j} \chi(I(\gamma),I(\gamma)+I(\alpha)) (T) = p_{ij} \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma(i,j)} p(\gamma) \chi(I(\gamma),I(\gamma)+I(\alpha)) (T).$$

The Laplace transform is

$$\mathcal{L} \{ D_{i,\alpha} (T) \} (s) = \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma(i,j)} p_{j\alpha} \frac{1 - e^{-l(\alpha)s}}{s} p(\gamma) e^{-l(\gamma)} = p_{j\alpha} \frac{1 - e^{-l(\alpha)s}}{s} \left( \sum_{k=0}^{\infty} N^k (s) \right)_{i,j}.$$

It will be shown that the sums $\sum_{k=0}^{\infty} M^k (s)$ and $\sum_{k=0}^{\infty} N^k (s)$ converge absolutely for suitable values of $s$, and so we can change the order of summation and integration as implied in the calculations above. From this we deduce explicit formulas for the Laplace transforms, namely

$$\mathcal{L} \{ A_{i,j} (x) \} (s) = \frac{1}{s} \frac{\text{adj} \left( I - M (s) \right)_{ij}}{\det (I - M (s))},$$

$$\mathcal{L} \{ B_{i,\alpha} (x) \} (s) = \frac{1 - e^{-l(\alpha)s}}{s} \frac{\text{adj} \left( I - M (s) \right)_{ij}}{\det (I - M (s))},$$

$$\mathcal{L} \{ C_{i,j} (T) \} (s) = \frac{\text{adj} \left( I - N (s) \right)_{ij}}{\det (I - N (s))},$$

$$\mathcal{L} \{ D_{i,\alpha} (T) \} (s) = p_{j\alpha} \frac{1 - e^{-l(\alpha)s}}{s} \frac{\text{adj} \left( I - N (s) \right)_{ij}}{\det (I - N (s))},$$

where $\frac{1 - e^{-l(\alpha)s}}{s}$ is an entire function with value $l(\alpha)$ at $s = 0$.

We will show that the constant $\lambda$ as described in the statement of the theorem exists and that these Laplace transforms satisfy the conditions of the Wiener-Ikehara theorem with a simple pole at $s = \lambda$. 
4. Proof of the Theorem

Although some of the following results can be found in the literature, we include the full details for the sake of clarity.

**Lemma 2.** The matrix elements of powers of $M(s)$ for $s = \sigma + it$ (respectively $N(s)$) are bounded in absolute value by the corresponding matrix elements of powers of $M(\sigma)$ (respectively $N(\sigma)$).

*Proof.* Indeed for every $k \in \mathbb{N}$

$$\left| (M^k(s))_{ij} \right| = \left| \sum_{i_1,...,i_{k-1}} M_{i_1,i_2}(...,M_{i_{k-1},j}(s) \right| \leq \sum_{i_1,...,i_{k-1}} |M_{i_1,i_2}|...|M_{i_{k-1},j}(s)| \leq \sum_{i_1,...,i_{k-1}} M_{i_1,i_2}(\sigma)...M_{i_{k-1},j}(\sigma) = (M^k(\sigma))_{ij}$$

and similarly for $N$, as required. \qed

**Remark 3.** This lemma is contained in a result due to Wielandt which can be found in [Ga].

For $\sigma \in \mathbb{R}$ the matrices $M(\sigma)$ and $N(\sigma)$ are real, non-negative and irreducible (because the graph $G$ is strongly connected), and so by Perron-Frobenius there exists a dominant real eigenvalue $\mu(\sigma)$ of multiplicity 1 corresponding to a positive eigenvector $v(\sigma)$.

**Lemma 3.** Let $M(\sigma)$ be as above. Then there exists $\lambda > 0$ such that $\mu(\lambda) = 1$ and for every $\sigma > \lambda$ the corresponding dominant eigenvalue satisfies $\mu(\sigma) < 1$.

*Proof.* For all $\sigma \in \mathbb{R}$ there exists $\mu(\sigma)$ which by Perron-Frobenius is a simple eigenvalue of $M(\sigma)$. Let $v(\sigma)$ and $u(\sigma)$ be right and left positive eigenvectors, then since $M(\sigma)$ is differentiable, then by Theorem 6.3.12 in [Hi] $\mu(\sigma)$ is differentiable and the following formula holds

$$\frac{d}{d\sigma} \mu(\sigma) = \frac{u^T(\sigma)M'(\sigma)v(\sigma)}{u^T(\sigma)v(\sigma)}.$$ 

Since the eigenvectors are positive, and the entry-wise derivative of $M$ is strictly negative, we deduce that

$$\frac{d}{d\sigma} \mu(\sigma) < 0$$

and in particular $\mu$ is monotone decreasing. Recall that $\mu(0)$ is the largest eigenvalue of the adjacency matrix $M(0)$ of the strongly connected graph $G$ and so $\mu(0) > 1$. Moreover, since all elements of $M(\sigma)$ tend to zero as $\sigma$ tends to infinity, so does the Perron-Frobenius eigenvalue. Therefore there exists a finite $\lambda > 0$ for which $\mu(\lambda) = 1$ and $\mu(\sigma) < 1$ for all $\sigma > \lambda$. \qed

**Lemma 4.** Let $N(\sigma)$ be as above. Then there exists $\lambda \leq 0$ such that $\mu(\lambda) = 1$ and for every $\sigma > \lambda$ the corresponding dominant eigenvalue satisfies $\mu(\sigma) < 1$.

*Proof.* The proof is similar to the discussion about $M(\sigma)$, only here we must show that $\mu(0) \leq 1$ to verify that the value of $\lambda$ for which $\mu(\lambda) = 1$ is negative. This follows from our assumption that the sum of the probabilities of edges originating at a given vertex is less or equal to 1, and so the sum of the entries of any row in $N(0)$ is bounded by 1, therefore $\mu(0)$ is bounded by 1 (see Wielandt’s proof of Perron-Frobenius theorem which appears in [Ga]). \qed

**Remark 4.** Clearly $\mu(0) = 1$ if and only if the sum of all probabilities for edges originating at vertex $i$ is 1, for all $i$. In other words $\lambda = 0$ if and only if $N(0)$ is a right stochastic matrix, that is all its rows sum up to 1.

The following Lemmas are stated for $M$, but analogous statements and their proofs apply for $N$. 

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**THE DISTRIBUTION OF PATH LENGTHS ON DIRECTED WEIGHTED GRAPHS** 7
Lemma 5. There exists a minimal $\lambda \in \mathbb{R}$ such that $\sum_{k=0}^{\infty} M^k (\sigma)$ converges for all $s = \sigma + i \tau$ with $\sigma > \lambda$. In this case,
\[ \sum_{k=0}^{\infty} M^k (s) = (I - M (s))^{-1} = \frac{\text{adj} (I - M (s))}{\text{det} (I - M (s))} \]
and so the Laplace transforms of the counting functions described above are analytic in the half plane $\sigma > \lambda$.

Proof. The Lemma follows because for $\sigma > \lambda$, as in Lemma 3, the geometric sum $\sum_{k=0}^{\infty} M^k (\sigma)$ converges, and by Lemma 2 so does $\sum_{k=0}^{\infty} M^k (s)$. □

Lemma 6. The matrix $\text{adj} (I - M (\lambda))$ has positive entries.

Proof. Recall that $\mu (\lambda) = 1$, and so there exist positive vectors $v, u$ such that
\[ \text{adj} (I - M (\lambda)) = \frac{v u^T}{u^T v}. \]
It follows that all the entries of $\text{adj} (I - M (\lambda))$ are non-zero and have the same sign as $\text{tr} (\text{adj} (I - M (\lambda)))$, which is $\frac{1}{\sigma} \text{Perr}_{M\lambda} (x) |_{x=1}$ by Jacobi’s formula. But 1 is a simple root of the characteristic polynomial and is the largest one, and therefore its derivative at $x = 1$ is positive. □

Lemma 7. The Laplace transforms of the graph counting functions have a simple pole at $\lambda$.

Proof. The point $s = \lambda$ is a singular point of the Laplace transforms, because by the previous lemma the numerator $(\text{adj} (I - M (\lambda)))_{ij}$ is non-zero while the denominator has a zero at $\lambda$. It is thus enough to show that the zero of $\text{det} (I - M (s))$ at $\lambda$ is a simple one. The characteristic polynomial of $M (s)$ is given by
\[ \mu (s) = \text{det} (s I - M (s)) = (s - \mu (s)) (s - \mu_2 (s)) \cdots (s - \mu_n (s)) \]
where $\mu (s)$ is the Perron-Frobenius eigenvalue of $M (s)$. Therefore $\mu (\lambda) = 1$, and $\mu_j (s) \neq 1$ for $j \geq 2$ in a small neighborhood of $\lambda$, and
\[ \text{det} (I - M (s)) = (1 - \mu (s)) \cdots (1 - \mu_n (s)). \]
It follows from equation (4.1) that the function $1 - \mu (s)$ has a simple zero at $\lambda$, and the same holds for the function $\text{det} (I - M (s))$. □

Lemma 8. For all $t \neq 0$,
\[ \text{det} (I - M (\lambda + i t)) \neq 0, \]
that is the Laplace transforms have no other poles on the line $\text{Re} (s) = \lambda$ than at $s = \lambda$ itself.

Proof. For simplicity we assume that for every pair of vertices $i, j$ in the graph $G$ there is at most one edge from vertex $i$ to vertex $j$. There is no loss of generality here because such graphs can be used to cover general multigraphs, and since the corresponding counting functions differ only by a multiplicative constant, so do their Laplace transforms. As a result, the position of the poles of the Laplace transforms is not changed.

Put $M_{ij} = (M (\lambda))_{ij}$, and recall that under our assumptions $M_{ij} = e^{-\lambda t (\alpha)}$ if $\alpha \in E$ connects the vertex $i$ to the vertex $j$, and $M_{ij} = 0$ if there is no such edge. Let $v = (v_1, \ldots, v_n)$ be a positive eigenvector of $M (\lambda)$ corresponding to the eigenvalue $\mu (\lambda) = 1$ and define $D$ to be the following invertible matrix
\[ D = \text{diag} (v_1, \ldots, v_n), \quad D^{-1} = \text{diag} \left( \frac{1}{v_1}, \ldots, \frac{1}{v_n} \right). \]
The matrix given by $S = D^{-1} MD$ is a non-negative and right stochastic matrix. Indeed, since $S_{ij} = M_{ij} v_j / v_i$, it is clear that $S_{ij} \geq 0$ and that the sum of the elements of the $i$th row is
\[ \sum_{j=1}^{n} S_{ij} = \sum_{j=1}^{n} M_{ij} v_j / v_i = \frac{1}{v_i} \sum_{j=1}^{n} M_{ij} v_j = \frac{v_j}{v_i} = 1. \]
Let $S (s)$ be the matrix with coefficients from $S$ raised to the power of $s$, that is
\[ (S (s))_{ij} = (S_{ij})^s = (M_{ij})^s \left( \frac{v_j}{v_i} \right)^s. \]
Notice that
\[ S(s) = (D^{-1})^s M(\lambda s) D^s, \]
that is \( M(\lambda s) \) and \( S(s) \) are similar, and in particular they have the same characteristic polynomial \( p_{S(s)}(x) = p_{M(\lambda s)}(x) \). Recalling the definition of the characteristic polynomial and plugging \( x = 1 \) we see that
\[ \det(I - M(\lambda s)) = \det(I - S(s)) \]
and so it is enough to show that \( \det(I - S(1 + it)) \neq 0 \) for all \( t \neq 0 \).

The following argument is due to Parry (see [Pa]). Assume \( \det(I - S(1 + it)) = 0 \) for some \( t \neq 0 \). So there exists a non-zero vector \( u = (u_1, ..., u_n) \) for which
\[ S(1 + it) u = u, \]
that is for all \( i \)
\[ \sum_{j=1}^{n} S_{ij}^{1+it} u_j = \sum_{j=1}^{n} S_{ij}^{it} u_j = u_i. \]
By the triangle inequality, for all \( i \)
\[ |u_i| \leq \sum_{j=1}^{n} |S_{ij}^{1+it} u_j| = \sum_{j=1}^{n} S_{ij} |S_{ij}^{it} u_j| = \sum_{j=1}^{n} S_{ij} |u_j|, \]
and together with the equality
\[ \sum_{j=1}^{n} S_{ij} = 1 \]
this implies that
\[ |u_i| = ... = |u_n|. \]
Assume \( |u_j| = r > 0 \) for all \( j \), and notice that this means \( S_{ij}^{it} u_j \) are points on a circle of radius \( r \). We have therefore that every \( u_i \), which is itself a point on the circle of radius \( r \), is a convex combination (that is, a linear combination with positive coefficients all adding up to 1) of points on that same circle. This is only possible of course if \( S_{ij}^{it} u_j = u_i \) for all \( j \) such that \( S_{ij} \neq 0 \).

Now, for any closed orbit on the graph, let \( \alpha_1 = (k_1, k_2), ..., \alpha_m = (k_m, k_1) \) denote the corresponding sequence of edges. We get
\[ (S_{k_1 k_2}^{it} u_{k_2}) (S_{k_2 k_3}^{it} u_{k_3}) ... (S_{k_{m-1} k_m}^{it} u_{k_m}) (S_{k_m k_1}^{it} u_{k_1}) = u_{k_1} u_{k_2} ... u_{k_{m-1}} u_{k_m} \]
and so
\[ S_{k_1 k_2}^{it} ... S_{k_m k_1}^{it} = (S_{k_1 k_2} ... S_{k_m k_1})^{it} = 1, \]
which gives
\[ (M_{k_1 k_2} ... M_{k_m k_1})^{it} = 1. \]
But
\[ M_{k_1 k_2} ... M_{k_m k_1} = e^{-t((\alpha_1) + ... + (\alpha_m))} \]
and so there exists some \( l \in \mathbb{Z} \) for which
\[ t = \frac{2\pi l}{\sum (\alpha_1) + ... + (\alpha_m)}. \]
This hold for every closed orbit on the graph, which is a contradiction to our irrationality assumptions on the lengths of the closed orbits on our incommensurable graph \( G \).

\[ \square \]

Lemma 9. The residue of the function \( \frac{\text{adj}(I - M(s))}{\det(I - M(s))} \) at \( s = \lambda \) is
\[ Q_{ij} = \frac{(\text{adj}(I - M(\lambda)))_{ij}}{-\text{tr}(\text{adj}(I - M(\lambda)) - M'(\lambda))} \]
Proof. We have seen that the function has a simple pole at \( s = \lambda \), and so the residue at \( s = \lambda \) is
\[
\frac{(\text{adj} (I - M (\lambda)))_{ij}}{\frac{d}{ds} (\det (I - M (s))) |_{s = \lambda}}.
\]
Finally, use Jacobi’s formula to obtain
\[
\frac{d}{ds} (\det (I - M (s))) = \text{tr} (\text{adj} (I - M (\lambda)) \cdot (I - M (s))' (\lambda))
\]
\[
= -\text{tr} (\text{adj} (I - M (\lambda)) \cdot M' (\lambda)).
\]
Combining the above we get the desired formula for the residue at hand. \( \square \)

5. Examples and Applications

5.1. Examples. Let \( G \) be the directed weighted graph which appears in Figure 1.

![Graph with two vertices \( V = \{1, 2\} \) and four edges \( E = \{\alpha, \beta, \gamma_1, \gamma_2\} \).](image)

The graph matrix function of \( G \) is given by
\[
M (s) = \begin{pmatrix}
  e^{-l(\alpha)s} & e^{-l(\beta)s} \\
  e^{-l(\gamma_1)s} + e^{-l(\gamma_2)s} & 0
\end{pmatrix}.
\]

Putting for example
\[
l (\alpha) = \log 2 \quad l (\gamma_1) = \log \frac{3}{2} \\
l (\beta) = \log 2 \quad l (\gamma_2) = \log 3
\]
we get \( \lambda = 1 \) and
\[
Q = \frac{6}{\log 432} \begin{pmatrix}
  1 & \frac{1}{2} \\
  1 & \frac{1}{2}
\end{pmatrix},
\]
and so by the second part of Theorem 1 the number of paths of length exactly \( x \) from vertex 1 to a point on the edge \( \gamma_2 \) grows as
\[
\frac{1 - e^{-L(\gamma_2)s}}{\lambda} Q_{12} e^{\lambda x} + o \left( e^{\lambda x} \right) = \frac{e^{x}}{\log \sqrt{432}} + o \left( e^{x} \right), \quad x \to \infty.
\]

Summation over regions of Pascal triangle. The well known triangular array of binomial coefficients contains many patterns of numbers and properties of combinatorial interest. It is a straightforward observation that summation of the binomial coefficients in the triangle \( OBA \) of sides \( OA = \frac{a}{b} \) and \( OB = \frac{b}{a} \) (see Figure 2) is equivalent to counting paths of length at most \( x \) in a graph with a single vertex and two loops of lengths \( a \) and \( b \).

This easily generalizes to Pascal pyramids of higher dimension and weighted graphs with a single vertex and several loops, and it would be interesting to understand the full correspondence between weighted graphs and regions in Pascal pyramids.
5.2. Multiscale substitution. A tile $T$ in $\mathbb{R}^d$ is a set homeomorphic to a closed $d$ dimensional ball. Consider a finite set of tiles $\mathcal{F} = \{T_1, \ldots, T_n\}$ in $\mathbb{R}^d$ which we call prototiles, and assume $\text{vol} T_i = 1$. A tile $T$ is said to be of type $i$ if it maps to $T_i$ by a similarity map. A multiscale substitution is a tiling of each prototile by rescaled copies of tiles in $\mathcal{F}$, or more formally, it is a set of dissection rules which dictates a tiling of each prototile by finitely many tiles of types appearing in $\mathcal{F}$.

A multiscale substitution can be modeled using a directed weighted graph $G$ with vertex set indexed by elements of $\mathcal{F}$ and edge set defined by the multiscale substitution rule: if the tiling of $T_i$ includes a tile of type $j$, that is a copy of $\alpha T_j$ with $0 < \alpha \leq 1$, then $G$ admits a directed edge of length $a = -\log \alpha$ connecting vertex $i$ to $j$. A multiscale substitution is called irreducible if $G$ is strongly connected, and incommensurable if $G$ is incommensurable. An example of an incommensurable multiscale substitution on a single prototile with scales $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$ is shown in Figure 3.

Observe that if $G$ is a graph associated with a $d$ dimensional multiscale substitution, then $\lambda = d$ and the Perron-Frobenius eigenvector of $M(d)$ corresponding to $\mu = 1$ can be chosen to be $v = (1, \ldots, 1)$. These properties enable us to address questions concerning the geometrical objects described below.

Kakutani splitting procedure. Consider the unit interval $I = [0,1]$ and some $\alpha \in (0,1)$. Kakutani introduced the following splitting procedure which generates a sequence of partitions of $I$ which is known as Kakutani’s sequence of partitions (see [Ka]). Begin with $\pi_0 = I$ the trivial partition of $I$, and define $\pi_1$ to be the partition of $I$ one gets after splitting $I$ into two intervals of lengths $\alpha$ and $1 - \alpha$. Assume that the partition $\pi_n$ is defined, then $\pi_{n+1}$ is the partition of $I$ one gets from $\pi_n$ after splitting the interval of maximal length in $\pi_n$ into two parts, proportional to $\alpha$ and $1 - \alpha$. 

Figure 2. A region in Pascal’s triangle and the associated graph.

Figure 3. A multiscale substitution rule and the associated graph.
For example, the first few Kakutani partitions of the unit interval with $\alpha = \frac{1}{3}$ are shown in Figure 4, together with the associated graph. The dashed lines represent intervals of maximal length in each partition.

$$\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4 \quad \pi_5$$

Figure 4. Kakutani’s partition with $\alpha = \frac{1}{3}$ and the associated graph.

We say that a sequence $\pi_n$ of partitions of $I$ is uniformly distributed if for any continuous function $f$ on $I$ we have

$$\lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f \left( t_i^{(n)} \right) = \int_I f(t) \, dt$$

where $k(n)$ is the number of intervals in the partition $\pi_n$, and $t_i^{(n)}$ is the right endpoint of the $i$th interval in the partition $\pi_n$. The following result is due to Kakutani.

**Theorem.** For any $\alpha \in (0,1)$, the corresponding Kakutani sequence of partitions of $I$ is uniformly distributed.

Kakutani’s splitting procedure is generalized in various ways, see for example [CV] and [Vo]. An additional generalization comes from multiscale substitution, where we begin with an initial tile $T_1$, and define a sequence of partitions of $T_1$ using the substitution rule applied at each stage to tiles of maximal volume. In fact, the example given above of Kakutani’s original procedure, can be considered as a multiscale substitution in $\mathbb{R}^1$ with $\mathcal{F} = \{ I \}$ the unit interval and $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$.

Using Theorem 1, it is shown in [SmY] that Kakutani splitting procedures which correspond to in-commensurable multiscale substitutions are uniformly distributed.

**Multiscale substitution tilings of Euclidean spaces.** A multiscale substitution rule in $\mathbb{R}^d$ can be used to generate a tiling of the entire space. We define a sequence of tilings of finite regions of $\mathbb{R}^d$ which depends on a continuous time parameter $t$ in the following way: At $t = 0$ apply the substitution rule on an initial tile $T_1$, and inflate the resulting patch of tiles at a constant speed. Any tile which reaches volume 1 is then substituted as dictated by the multiscale substitution rule, and so on. An appropriate compact topology defined on closed subsets of the space allows us to take limits of sequences of these partial tilings, and these limits define tilings of $\mathbb{R}^d$. The generalization of the pinwheel tiling which is presented in [Sa] can be regarded as a multiscale substitution tiling.

Although there is no uniqueness in the construction of tilings using multiscale substitutions, all tilings defined this way share various properties which can be analyzed using the multiscale substitution itself and the underlying weighted graph. For example, tilings which are associated with incommensurable multiscale substitutions are of infinite local complexity, unlike classical substitution tilings or tilings defined using cut-and-project constructions (for more on tilings and mathematical models of quasicrystals see [BG]). Our Theorem 1 may be used to study various statistics of these tilings, see [SS] for more details, and [So1] and [So2] for relevant results concerning classical substitution tilings.

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