On the translation invariant operators in $\ell^p(\mathbb{Z}^d)$

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Abstract

In this paper we study boundedness of translation invariant operators in the discrete space $\ell^p(\mathbb{Z}^d)$. In this context a Mikhlin type multiplier theorem is given, yielding boundedness for certain known operators. We also give $\ell^p - \ell^q$ boundedness of a discrete wave equation.

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1 Introduction

It is well known that translation invariant operator from $L^p$ into $L^q$ may be represented by convolution with a tempered distribution, or equivalently by Fourier multiplier transformation. This was originally proved in the classical article of Hörmander [4]. Through many aspects of harmonic analysis, many studies have been devoted to the topic of the $L^p$-bounded of translation invariant operator. The most famous are the works of Calderón and Zygmund on the singular integral operators, with a large number of generalizations.

In this paper we consider translation invariant operator $T$ on $\mathbb{Z}^d$. The problem is essentially the multiplier problem,

$$F_{\mathbb{Z}^d}(T(f)) = mF_{\mathbb{Z}^d}(f)$$

where the function $m$ is defined on the Torus $\mathbb{R}^d/\mathbb{Z}^d$. In this setting a Hörmander’s type theorem for $\ell^p - \ell^q$ boundedness of $T$ and an $\ell^p$-theorem of Mikhlin- type are given. We apply our results to get $L^p$-estimate for the discrete wave equation.

We begin by introducing the following notations. Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the $d$-dimensional torus. Functions on $\mathbb{T}^d$ are functions $f$ on $\mathbb{R}^d$ that satisfy $f(x+n) = f(x)$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$. Such functions are called 1-periodic in every coordinate.
Haar measure on $\mathbb{T}^d$ is the restriction of $d$-dimensional Lebesgue measure to the set $[0,1)^d$. This measure is still denoted by $dx$ and given by

$$
\int_{\mathbb{T}^d} f(x)dx = \int_{[0,1)^d} f(x)dx.
$$

We denote by $L^p(\mathbb{T}^d)$, $1 \leq p \leq \infty$ the Lebesgue space $L^p([0,1)^d, dx)$. The inner product of the Hilbert space $L^2(\mathbb{T}^d)$ is given by

$$
\langle f, g \rangle = \int_{\mathbb{T}^d} f(\xi)\overline{g(\xi)}d\xi.
$$

The functions $\psi_n : \xi \rightarrow e^{2\pi i n \cdot \xi}$, indexed by $n \in \mathbb{Z}^d$, form a complete orthonormal system of $L^2(\mathbb{T}^d)$ where for $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{R}^d$

$$
x.y = x_1y_1 + \ldots + x_dy_d \quad \text{and} \quad |x| = (x.x)^{1/2}.
$$

By $\ell^p = \ell^p(\mathbb{Z}^d)$, $1 \leq p < \infty$, we denote the usual Banach space of $p$-summable complex-valued function $f = (f(n))_{n \in \mathbb{Z}^d}$ equipped with the norm

$$
\|f\|_{\ell^p} = \left( \sum_{n \in \mathbb{Z}^d} |f(n)|^p \right)^{\frac{1}{p}}
$$

and $\ell^\infty$ the space of bounded function on $\mathbb{Z}^d$ with $\|f\|_\infty = \sup_{n \in \mathbb{Z}^d} |f(n)|$. We note the following elementary embedding relations

$$
\ell^q \subset \ell^p; \quad \text{and} \quad \|f\|_{\ell^p} \leq \|f\|_{\ell^q}, \quad 1 \leq p \leq q \leq \infty. \quad (1.1)
$$

For $f \in \ell^2(\mathbb{Z}^d)$ its Fourier transform is given by

$$
\mathcal{F}_{\mathbb{Z}^d}(f)(\xi) = \sum_{n \in \mathbb{Z}^d} f(n)e^{2\pi i n \cdot \xi}, \quad \xi \in \mathbb{R}^d.
$$

The Fourier transform $\mathcal{F}_{\mathbb{Z}^d}$ is an isometry from $\ell^2(\mathbb{Z}^d)$ into $L^2(\mathbb{T}^d)$ and its inverse $\mathcal{F}_{\mathbb{Z}^d}^{-1}$ is given by

$$
\mathcal{F}_{\mathbb{Z}^d}^{-1}(u)(n) = \int_{\mathbb{T}^d} u(\xi)e^{-2\pi i n \cdot \xi}dx, \quad u \in L^2(\mathbb{T}^d).
$$

By Riesz-Thorin convexity theorem, the Fourier transform $\mathcal{F}_{\mathbb{Z}^d}$ and its inverse $\mathcal{F}_{\mathbb{Z}^d}^{-1}$ satisfy the Hausdorff-Young inequalities

$$
\|\mathcal{F}_{\mathbb{Z}^d}(f)\|_{L^{p'}} \leq \|f\|_{\ell^p}, \quad (1.2)
$$

and

$$
\|\mathcal{F}_{\mathbb{Z}^d}^{-1}(u)\|_{\ell^{p'}} \leq \|u\|_{L^{p}}, \quad (1.3)
$$

for $1 < p \leq 2$ and $1/p + 1/p' = 1$.

Convolution product of two functions $f$ and $g$ of $\ell^2(\mathbb{Z}^d)$ is defined by

$$
f \ast_{\mathbb{Z}^d} g(n) = g \ast_{\mathbb{Z}^d} f(n) = \sum_{k \in \mathbb{Z}^d} f(k)g(n-k); \quad n \in \mathbb{Z}^d.
$$

If $f, g \in \ell^1(\mathbb{Z}^d)$ then $f \ast_{\mathbb{Z}^d} g \in \ell^1(\mathbb{Z}^d)$ and

$$
\mathcal{F}_{\mathbb{Z}^d}(f \ast_{\mathbb{Z}^d} g) = \mathcal{F}_{\mathbb{Z}^d}(f)\mathcal{F}_{\mathbb{Z}^d}(g).
$$

Suppose $f \in \ell^p(\mathbb{Z}^d)$ and $g \in \ell^q(\mathbb{Z}^d)$ with $1 \leq p, q, r \leq \infty, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. Then

$$
\|f \ast_{\mathbb{Z}^d} g\|_{\ell^r} \leq \|f\|_{\ell^p}\|g\|_{\ell^q} \quad \text{(Young’s Inequality)}. \quad (1.4)
$$
2 Translation invariant operators

In this section we shall be concerned with the space of bounded operators $T$ from $\ell^p$ to $\ell^q$ for $p \leq q$, which commute with translations; that is, $\tau_n T = T \tau_n$ for all $n \in \mathbb{Z}^d$, where $\tau_n(f)(k) = f(n + k)$. It is not difficult to see that $T$ is a convolution operator. Indeed, let $K = T(1_{\{0\}})$, where $1_A$ is the characteristic function of a set $A$. Consider first, functions $f$ with compact support and write

$$f = 1_{\{0\}} \ast_{\mathbb{Z}^d} f = \sum_{n \in \mathbb{Z}^d} f(n) \tau_n(1_{\{0\}}).$$

Since $T$ is translation invariant operator then

$$T(f) = T(1_{\{0\}} \ast_{\mathbb{Z}^d} f) = \sum_{n \in \mathbb{Z}^d} f(n)(\tau_n(1_{\{0\}})) = \sum_{n \in \mathbb{Z}^d} f(n)\tau_n T(1_{\{0\}}) = K \ast_{\mathbb{Z}^d} f.$$

Now for function $f \in \ell^1(\mathbb{Z}^d)$ we let

$$f_j = \sum_{|n| \leq j} f(n)1_{\{n\}}, \quad j \geq 0$$

Clearly the sequence $(f_j)_j$ converges to $f$ in $\ell^r(\mathbb{Z}^d)$ for all $1 \leq r < \infty$ and

$$\|T(f_j) - K \ast_{\mathbb{Z}^d} f\|_r = \|K \ast_{\mathbb{Z}^d} (f_j - f)\|_r \leq \|K\|_r \|f_j - f\|_1$$

which implies that $(T(f_j))_j$ converges to $K \ast_{\mathbb{Z}^d} f$ in $\ell^q(\mathbb{Z}^d)$. But $T: \ell^p(\mathbb{Z}^d) \rightarrow \ell^q(\mathbb{Z}^d)$ is bounded and $(f_j)_j$ converges to $f$ in $\ell^p(\mathbb{Z}^d)$, thus by uniqueness of the limit we may have $T(f) = K \ast_{\mathbb{Z}^d} f$. Notice that the $\ell^p - \ell^q$ boundedness of $T$ implies that $K \in \ell^q(\mathbb{Z}^d)$. We state the following

**Theorem 2.1.** If $T$ is a bounded translation invariant operator from $\ell^p(\mathbb{Z}^d)$ to $\ell^q(\mathbb{Z}^d)$, $p \leq q$, then there exists a function $K \in \ell^q(\mathbb{Z}^d)$ such that

$$T(f) = K \ast_{\mathbb{Z}^d} f, \quad f \in \ell^1(\mathbb{Z}^d).$$

Translation invariant operator can also be described as Fourier multiplier transformation $T_m$ defined by

$$F_{\mathbb{Z}^d}(T_m(f)) = mF_{\mathbb{Z}^d}(f)$$

where $m$ is a bounded measurable function $m$ on $\mathbb{T}^d$. An important class of $T_m$ is given by $L^{r,\infty}(\mathbb{T}^d)$ for $r > 1$, that is the space of measurable functions $m$ such that for some constant $c > 0$,

$$\int_{\{\xi \in (0,1)^d, \ |m(\xi)| \geq s\}} dx \leq \frac{c}{s^r}, \quad s > 0. \quad (2.1)$$

In particular if $m$ satisfies the estimate

$$|m(\xi)| \leq c|\xi|^{-r}; \quad \xi \in (0,1)^d, \ \xi \neq 0, \quad (2.2)$$

for $0 < \alpha < d$ then $m \in L^{d/r,\infty}(\mathbb{T}^d)$. 


Theorem 2.2. If \( m \in L^{\alpha, \infty}(\mathbb{T}^d) \) with \( \alpha > 1 \) then \( T_m \) is a bounded operator from \( \ell^p(\mathbb{Z}^d) \) into \( \ell^q(\mathbb{Z}^d) \), provided that

\[
1 < p \leq 2 \leq q < \infty , \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{\alpha}.
\]

This result is originally proved in [1] for translation invariant operator on \( \mathbb{R}^d \) and in [?] for translation invariant operator on \( \mathbb{Z} \), for completeness we extended this result to \( \mathbb{Z}^d \). The proof of Theorem 2.2 follows closely the argument of [4].

Lemma 2.3. Let \( \varphi \geq 0 \) be a measurable function such that for some constant \( c > 0 \)

\[
\int_{\{\xi \in (0,1)^d, \varphi(\xi) \geq s\}} dx \leq \frac{c}{s}, \quad s > 0. \tag{2.3}
\]

Then for all \( 1 < p \leq 2 \) there exists a constant \( c_p > 0 \) such that

\[
\left( \int_{(0,1)^d} |\mathcal{F}_{\mathbb{Z}^d}(f)(\xi)|^p |\varphi(\xi)|^{2-p} d\xi \right)^\frac{1}{p} \leq c_p \|f\|_{\ell^p}; \quad f \in \ell^p(\mathbb{Z}^d). \tag{2.4}
\]

Proof. Put \( d\mu(\xi) = \varphi^2(\xi) \, d\xi \) and let \( T \) be the operator defined on \( \ell^1(\mathbb{Z}^d) \) by

\[
T(f) = \frac{\mathcal{F}_{\mathbb{Z}^d}(f)}{\varphi}.
\]

Noting that \( T(f) \) is well defined \( \mu \)-almost everywhere on \((0,1)^d\), since we have that \( \mu(\{\xi \in (0,1)^d, \varphi(\xi) = 0\}) = 0 \). In fact, for \( s > 0 \) we have

\[
\mu(\{\xi \in (0,1)^d, \varphi(\xi) \leq s\}) \begin{aligned}
= \int_{\{\xi \in (0,1)^d, \varphi(\xi) \leq s\}} \varphi(\xi)^2 \, d\xi \\
= 2 \int_{\{\xi \in (0,1)^d, \varphi(\xi) \leq s\}} \int_{0 \leq t \leq \varphi(\xi)} t \, dt \, d\xi \\
\leq 2 \int_0^s \int_{\{\xi \in (0,1)^d, \ t \leq \varphi(\xi)\}} t \, dt \, d\xi \\
\leq 2c \int_0^s dt = 2cs
\end{aligned}
\]

which implies that \( \mu(\{\xi \in (0,1)^d, \varphi(\xi) = 0\}) \leq 2cs \), for all \( s > 0 \).

Now for \( s > 0 \) and \( f \in \ell^1(\mathbb{Z}^d) \) we have

\[
\mu(\{\xi \in (0,1)^d, \ |T(f)(\xi)| \geq s\}) \leq \mu \left( \left\{ \xi \in (0,1)^d, \varphi(\xi) \leq \frac{\|f\|_{\ell^1}}{s} \right\} \right) \leq \frac{2c\|f\|_{\ell^1}}{s}.
\]

Hence \( T \) is of weak type \((1,1)\). In addition, from Plancherel Theorem

\[
\mu(\{\xi \in (0,1)^d, \ |T(f)(\xi)| \geq s\}) \leq \frac{\|f\|_{L^2}^2}{s^2},
\]

which mean that \( T \) is of weak type \((2,2)\). We thus obtain Lemma 2.3 by using Marcinkiewicz interpolation Theorem.
Lemma 2.4. If \( \varphi \) satisfies (2.3) and \( 1 < p < r < p' < \infty \), then we have
\[
\left( \int_{(0,1)^d} |\mathcal{F}_{\mathbb{Z}^d}(f)(\xi)(\varphi(\xi))^{(1/r-1/p')}|^r \, d\xi \right)^{1/r} \leq C_p \|f\|_{\ell^p} ; \quad f \in \ell^p(\mathbb{Z}^d).
\]

Proof. Put \( a = (p - p)/(p' - r) \) and \( a' \) its conjugate. We note the following
\[
\frac{p}{a} + \frac{p'}{a'} = r, \quad \left(1 - \frac{r}{p'} \right) a = 2 - p, \quad \left(r - \frac{p}{a} \right) a' = p'.
\]

Then using Holder’s inequality (2.5) and the Hausdorff-Young inequality (1.2)
\[
\left( \int_{(0,1)^d} |\mathcal{F}_{\mathbb{Z}^d}(f)(\xi)|^r |\varphi(\xi)|^{(1-r/p')} \, d\xi \right)^{1/r} \leq \left( \int_{(0,1)^d} |\mathcal{F}_{\mathbb{Z}^d}(f)(\xi)|^p |\varphi(\xi)|^{(2-p)} \, d\xi \right)^{1/ra} \left( \int_{(0,1)^d} |\mathcal{F}_{\mathbb{Z}^d}(f)(\xi)|^{p'} \, d\xi \right)^{1/ra'} \leq c_p \|f\|_{\ell^p}
\]
which is the desired statement.

Proof of Theorem 2.2. Assume first that \( p \leq q' \) and let \( \varphi = |m|^\alpha \). Clearly from (2.1) the function \( \varphi \) satisfies the condition (2.3). Hence using Lemma 2.4 with \( r = q' \) and the fact that \( 1/p - 1/q = 1/q' - 1/p' = 1/\alpha \), we obtain that
\[
\left( \int_{(0,1)^d} |m(\xi)\mathcal{F}_{\mathbb{Z}^d}(f)(\xi)|^{q'} \, d\xi \right)^{1/q'} \leq c \|f\|_{\ell^p}.
\]

Now the Hausdorff-Young inequality (1.3) implies
\[
\|T_m(f)\|_{\ell^q} \leq \|m\mathcal{F}(f)\|_{\ell^{q'}} \leq c_p \|f\|_{\ell^p}.
\]
When \( q' < p = (p')' \), we can apply the similar argument to the adjoint operator \( T_m^* = T_m^\ast \), since \( 1 < q' \leq 2 \leq p' < \infty \) and \( 1/q' - 1/p' = 1/\alpha \). Hence by duality it follows that
\[
\|T_m(f)\|_{\ell^q} \leq c_p \|f\|_{\ell^p}.
\]
This finishes the proof of Theorem 2.2

Corollary 2.5. If \( m \) satisfies (2.3) with \( 0 < r < d \) then \( T_m \) is bounded from \( \ell^p(\mathbb{Z}^d) \) into \( \ell^q(\mathbb{Z}^d) \), provided that
\[
1 < p \leq 2 \leq q < \infty, \quad \frac{1}{p} - \frac{1}{q} = \frac{r}{d}.
\]

Now observe that the inequality (2.1) can be restricted only to \( s \geq 1 \) which implies that \( L^{\alpha,\infty}(\mathbb{T}^d) \subset L^{\beta,\infty}(\mathbb{T}^d) \) for all \( 1 < \beta \leq \alpha \). Thus one can state

Corollary 2.6. If \( m \in L^{\alpha,\infty}(\mathbb{T}^d) \) with \( \alpha > 1 \) then the operator \( T_m \) is bounded from \( \ell^p(\mathbb{Z}^d) \) into \( \ell^q(\mathbb{Z}^d) \), provided that
\[
1 < p \leq 2 \leq q < \infty, \quad \frac{1}{p} - \frac{1}{q} \geq \frac{1}{\alpha}.
\]
We now study $L^p$-boundedness of the multiplier operator $T_m$. We begin by the following:

**Theorem 2.7.** If $m$ is a $C^{d+1}$-function on $\mathbb{T}^d$ then $T_m$ is a bounded operator from $\ell^p(\mathbb{Z}^d)$ into itself for all $1 \leq p \leq \infty$.

**Proof.** We note first that the kernel of $T_m$ is given by

$$K(n) = \int_{(0,1)^d} m(\xi)e^{-2\pi i \xi \cdot n} d\xi, \quad n \in \mathbb{Z}^d.$$  

Using integrations by parts we have

$$|n_1^{\gamma_1}...n_d^{\gamma_d}K(n)| = \left| \int_{(0,1)^d} \partial_{\xi_1}^{\gamma_1}...\partial_{\xi_d}^{\gamma_d} m(\xi) e^{-2\pi i \xi \cdot n} d\xi \right| \leq c$$

for all $\gamma_1, ..., \gamma_d \in \mathbb{N}$ with $\gamma_1 + \ldots + \gamma_d \leq d + 1$. It follows that

$$|K(n)| \leq \frac{c}{(1 + |n_1|)\ldots(1 + |n_j|)^2\ldots(1 + |n_d|)}$$

for all $j = 1, \ldots, d$. By varying $j$ from 1 to $d$ we deduce the following estimate

$$|K(n)| \leq \frac{c}{\left((1 + |n_1|)\ldots(1 + |n_j|)\ldots(1 + |n_d|)\right)^{1+1/d}}$$

which implies that the kernel $K$ is in $\ell^1(\mathbb{Z}^d)$. This yields the result. \hfill \Box

Our main result is the following Hörmander-Mihlin type multiplier theorem where we may consider $T_m$ as a Calderón-Zygmund operator.

**Theorem 2.8.** Let $m$ be a bounded function on the torus $\mathbb{T}^d$. We assume that $m$ is $C^{d+1}$-function on $\mathbb{R}^d \setminus \mathbb{Z}^d$ and satisfies the Mikhlin condition,

$$|\partial_\xi^\alpha m(\xi)| \leq c|\xi|^{-|\alpha|}, \quad \xi \in (-1/2, 1/2)^d \quad (2.6)$$

for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ with $|\alpha| = \alpha_1 + \ldots + \alpha_d \leq d + 1$. Then $T_m$ extended to a bounded operator from $\ell^p$ into itself for all $1 < p < \infty$.

**Proof.** Taking $\psi$ a $C^\infty$-function on $\mathbb{R}^d$ such that $\psi(\xi) = 1$ for $|\xi| \leq 1/16$ and $\psi(\xi) = 0$ for $|\xi| \geq 1/8$. In $(-1/2, 1/2)^d$ we split $m$ into

$$m = (1 - \psi)m + m\psi = m_1 + m_2.$$ 

Since $m_1 = m$ near the sides $|\xi_j| = 1/2$, then $m_1$ can be extended to $C^{d+1}$-function on $\mathbb{T}^d$ and by Theorem 2.7 the operator $T_{m_1}$ is bounded on $\ell^p$ for all $1 \leq p \leq \infty$. It is therefore enough to prove boundedness of $T_{m_2}$. Introduce the function $K$ by

$$K(x) = \int_{(-1/2,1/2)^d} m_2(\xi)e^{-2\pi i \xi \cdot x} d\xi, \quad x \in \mathbb{R}^d$$
and $K$ its restriction to $\mathbb{Z}^d$. One can write

$$T_{m_2}(f) = K *_{\mathbb{Z}^d} f, \quad f \in \ell^1.$$  

Our aim is to prove that $T_{m_2}$ is a Calderon-Zygmund operator. We consider here $\mathbb{Z}^d$ as a space of homogeneous type in the sense of Coifman and Weiss [?], equipped with the Euclidean metric $(r,s) \rightarrow |r - s|$ and the counting measure. Precisely, we will prove that the kernel $K$ satisfies the integral Hörmander condition: there exists constant $c > 0$ such that for all $s \in \mathbb{Z}^d$,

$$\sum_{r \in \mathbb{Z}^d, |r| \geq 2|s|} |K(r - s) - K(r)| \leq c. \quad (2.7)$$

To begin, let $\phi$ be a $C^\infty$- function on $\mathbb{R}$, such that $\mathrm{supp}(\phi) \subset \{ t \in \mathbb{R}; 1/2 \leq |t| \leq 2 \}$ and

$$\sum_{j \in \mathbb{Z}} \phi(2^{-j}t) = 1, \quad t \neq 0.$$

So we have

$$\sum_{j=3}^{\infty} \phi(2^{-j}|\xi|) = 1, \quad |\xi| \leq \frac{1}{8},$$

and we may write

$$m_2(\xi) = \sum_{j=3}^{\infty} m_2(\xi) \phi(2^{-j}|\xi|) = \sum_{j=3}^{\infty} m_j(\xi), \quad \xi \in (-1/2, 1/2)^d.$$  

Notice that

$$\sum_{j=3}^{\infty} |m_j(\xi)| \leq c \quad (2.8)$$

for some constant $c > 0$, since this sum contains at most three non-null terms. We now set

$$K_j(x) = \int_{(-1/2,1/2)^d} m_j(\xi)e^{-i2\pi\xi \cdot x}d\xi, \quad x \in \mathbb{R}^d.$$  

By Fubini’s Theorem and (2.8) the sum $\sum_j K_j$ converges and

$$\sum_{j=3}^{\infty} K_j(x) = \int_{(-1/2,1/2)^d} \sum_{j=3}^{\infty} m_j(\xi)e^{-i2\pi\xi \cdot x}d\xi = \int_{(-1/2,1/2)^d} m_2(\xi)e^{-i2\pi\xi \cdot x}d\xi = K(x),$$

Next we shall give estimates of the kernels $K$. Observe first that $m_2$ satisfies the condition (2.6) and from this estimate and the compactness of $\mathrm{supp}(\varphi)$ one can obtain the following

$$|\partial^\alpha m_j(\xi)| \leq c \, 2^{-j|\alpha|}, \quad \text{and} \quad |\partial^\alpha (\xi m_j(\xi))| \leq c \, 2^{-j(|\alpha|-1)};$$

for $i = 1, \ldots, d$ and for $|\alpha| \leq d + 1$. It follows that

$$|x|^s |K_j(x)| \leq c \sum_{|\alpha| = s} \|\partial^\alpha m_j\|_{L^1} \leq c \, 2^{j(d-s)} \quad (2.9)$$
and

\[ |x|^s \left| \frac{\partial K_j}{\partial x_i}(x) \right| \leq c \sum_{|\alpha|=s} \|\partial^\alpha (\xi_i m_j(\xi))\|_{L^1} \leq c 2^j(d+1-s). \]  \hspace{1cm} (2.10)

Using (2.9) with \( s = 0 \) and \( s = d + 1 \) we get

\[ \sum_{j=3}^{\infty} |K_j(x)| = \sum_{2^j \leq |x|^{-1}} |K_j(x)| + \sum_{2^j > |x|^{-1}} |K_j(x)| \leq c |x|^{-d}. \]

Similarly by (2.10) with \( s = 0 \) and \( s = d + 2 \)

\[ \sum_{j=3}^{\infty} \left| \frac{\partial K_j}{\partial x_i}(x) \right| = \sum_{2^j \leq |x|^{-1}} \left| \frac{\partial K_j}{\partial x_i}(x) \right| + \sum_{2^j > |x|^{-1}} \left| \frac{\partial K_j}{\partial x_i}(x) \right| \leq c |x|^{-d-1}. \]

Therefore we obtain the following estimates

\[ |K(x)| \leq c(1 + |x|)^{-d}, \quad \left| \frac{\partial K}{\partial x_i}(x) \right| \leq c(1 + |x|)^{-d-1}, \quad i = 1, \ldots, d. \]  \hspace{1cm} (2.11)

Note that \( K \) is a \( C^\infty \)-function and all of its derivatives are bounded.

Now we come to the proof of (2.7). Let \( r, s \in \mathbb{Z}^d \) with \( |r| \geq 2|s| \). By mean value Theorem we have

\[ |K(r - s) - K(r)| = |K(r - s) - K(r)| \leq |s| \int_0^1 \sum_{i=1}^d \left| \frac{\partial K}{\partial x_i}(r - ts) \right| dt. \]

Using (2.11) and the fact that

\[ |r - ts| \geq |r| - |s| \geq \frac{|r|}{2}, \]

we obtain the following

\[ |K(r - s) - K(r)| \leq \frac{c |s|}{(1 + |r|)^{d+1}}. \]  \hspace{1cm} (2.12)

Now use that

\[ \max_{1 \leq i \leq d} (|r_i|) \leq |r| \leq d^{1/2} \max_{1 \leq i \leq d} (|r_i|), \]

we have

\[ \sum_{|r| \geq 2|s|} \frac{1}{(1 + |r|)^{d+1}} \leq c \sum_{\max_{1 \leq i \leq d} \max_{1 \leq i \leq d} (|r_i|) \geq 2^{d-1/2}|s|} \frac{1}{(1 + \max_{1 \leq i \leq d} (|r_i|))^{d+1}}. \]

It is not hard to see that

\[ \sum_{r \in \mathbb{Z}^d, \max_{1 \leq i \leq d} (|r_i|) = k} \leq ck^{d-1}. \]
and from which
\[ \sum_{|r| \geq 2d^{-1/2}|s|} \frac{1}{(1 + |r|)^{d+1}} \leq c \sum_{k \geq 2d^{-1/2}|s|} \frac{1}{k^2} \leq \frac{c}{|s|}. \tag{2.13} \]

Combine (2.13) with (2.12), yield that
\[ \sum_{|r| \geq 2d^{-1/2}|s|} \left| K(r - s) - K(r) \right| \leq c \]
which proves (2.7). The proof of Theorem 2.8 follows.

In the next we shall be concerned with Mikhlin type multiplier on \( \mathbb{Z} \). Thus one can read Theorem 2.8 as follows

**Theorem 2.9.** If \( m \) is a bounded \( C^2 \)-function on \( (0, 1) \) such that
\[ (\xi(1 - \xi))^k |m^{(k)}(\xi)| \leq c; \quad 0 \leq k \leq 2 \tag{2.14} \]
then \( T_m \) is a bounded operator from \( \ell^p(\mathbb{Z}) \) into \( \ell^p(\mathbb{Z}) \) for \( 1 < p < \infty \).

Clearly condition (2.14) is exactly (2.6) when extending \( m \) to a periodic function.

Now we replace the interval \( (0, 1) \) by a bounded interval \( (a, b) \). For \( f \in \ell^2(\mathbb{Z}) \) we define its Fourier transform by
\[ \mathcal{F}_{\mathbb{Z}}^{a,b}(f)(\xi) = \sum_{n \in \mathbb{Z}} f(n) e^{i2\pi n \left( \frac{\xi - a}{b - a} \right)} = \mathcal{F}_{\mathbb{Z}}(f) \left( \frac{\xi - a}{b - a} \right) \]
and its inverse
\[ (\mathcal{F}_{\mathbb{Z}}^{a,b})^{-1}(f)(n) = \frac{1}{b - a} \int_a^b f(\xi) e^{i2\pi n \left( \frac{\xi - a}{b - a} \right)} d\xi, \quad n \in \mathbb{Z}. \]
For a bounded function \( m \) on \( (a, b) \) we define on \( \ell^2(\mathbb{Z}) \) the operator \( T_m^{a,b} \) by
\[ T_m^{a,b}(f)(n) = \frac{1}{b - a} \int_a^b m(\xi) \mathcal{F}_{\mathbb{Z}}^{a,b}(f)(\xi) e^{i2\pi n \left( \frac{\xi - a}{b - a} \right)} d\xi \]
\[ = \int_0^1 m(a + (b - a)\xi) \mathcal{F}_{\mathbb{Z}}^{a,b}(f)(a + (b - a)\xi) e^{i2\pi n d\xi} d\xi \]
\[ = \int_0^1 m(a + (b - a)\xi) \mathcal{F}_{\mathbb{Z}}(f)(\xi) e^{i2\pi n d\xi}, \]
for \( n \in \mathbb{Z} \). According to Theorem 2.9 we have the following

**Corollary 2.10.** If \( m \) is a bounded \( C^2 \)-function on a bounded interval \( (a, b) \), such that for some constant \( c > 0 \)
\[ (\xi - a)^k (b - \xi)^k |m^{(k)}(\xi)| \leq c, \quad 0 \leq k \leq 2 \]
then \( T_m^{a,b} \) is a bounded operator from \( \ell^p(\mathbb{Z}) \) into \( \ell^p(\mathbb{Z}) \) for \( 1 < p < \infty \).
As a typical example we have \( m = \chi_{(a,b)} \) the characteristic function of \((a,b)\). Theorem 2.9 can be generalized as follows.

**Theorem 2.11.** Let \((a_j)_{0 \leq j \leq s}\) be a subdivision of \([0, 1]\). If \(m\) is a bounded \(C^2\)-function on \((0, 1)\) such that, for some constant \(c > 0\),

\[
\left( \prod_{0 \leq j \leq s} |\xi - a_j|^k \right) |m^{(k)}(\xi)| \leq c; \quad 0 \leq k \leq 2.
\]

then \(T_m\) is a bounded operator from \(\ell^p(\mathbb{Z})\) into \(\ell^p(\mathbb{Z})\) for \(1 < p < \infty\).

**Proof.** Assume first that \(s = 2\) and let \(0 = a_0 < a_1 < a_2 = 1\). Let \(\varepsilon > 0\) and \(\varphi\) a be \(C^\infty\) function on \(\mathbb{R}\) such that \([a_1 - 2\varepsilon, a_1 + 2\varepsilon] \subseteq [0, 1]\) and \(\varphi(\xi) = 0\) for \(\xi \in [a_1 - \varepsilon, a_1 + \varepsilon]\) and \(\varphi(\xi) = 1\) for all \(\xi \notin [a_1 - 2\varepsilon, a_1 + 2\varepsilon]\). Put

\[
m(\xi) = m(\xi)\varphi(\xi) + m(\xi)(1 - \varphi(\xi)) = m_1(\xi) + m_2(\xi), \quad \xi \in (0, 1).
\]

and

\[
T_m = T_{m_1} + T_{m_2}
\]

Clearly the boundedness of \(T_{m_1}\) is a consequence of Theorem 2.9. However, if we consider \(\tilde{m}_2\) the 1- periodic function such that \(\tilde{m}_2(\xi) = m_2(\xi)\) for \(\xi \in [0, 1]\), then one can write

\[
T_{m_2}(f)(n) = \int_0^1 \tilde{m}_2(\xi)\mathcal{F}_Z(f)(\xi)e^{i2\pi n \xi}d\xi
\]

\[
= \int_0^{a_1 + 1} \tilde{m}_2(\xi)\mathcal{F}_Z(f)(\xi)e^{i2\pi n \xi}d\xi
\]

\[
= \int_0^1 \tilde{m}_2(\xi - a_1)\mathcal{F}_Z(f)(\xi + a_1)e^{i2\pi n(\xi - a_1)}d\xi
\]

\[
= e^{i2\pi a_1} \int_0^1 \tilde{m}_2(\xi + a_1)\mathcal{F}_Z(f)(\xi)e^{i2\pi n \xi}d\xi
\]

\[
= e^{i2\pi a_1} T_{m_2}^{a_1, a_1 + 1}(\tilde{f})(n)
\]

where \(\tilde{f}\) is the function given by \(\tilde{f}(n) = e^{i2\pi n a_1}f(n), n \in \mathbb{Z}\). Now observe that \(\tilde{m}_2\) satisfies the hypothesis of Corollary 2.10 on \((a_1, a_1 + 1)\), then the boundedness of \(T_{m_2}\) follows.

Now for \(s \geq 3\) we proceed as follows: choose \(\varepsilon > 0\) such that the intervals \([a_j - 2\varepsilon, a_j + 2\varepsilon]\) are disjoint for all \(1 \leq j \leq s - 1\) and \(C^\infty\) functions \(\varphi_j\) with \(\varphi_j(\xi) = 0\) for \(\xi \in [a_j - \varepsilon, a_j + \varepsilon]\) and \(\varphi_j(\xi) = 1\) for \(\xi \notin [a_j - 2\varepsilon, a_j + 2\varepsilon]\). Put

\[
\varphi = \frac{1}{s - 1} \sum_{j=1}^{s-1} \varphi_j
\]

and write

\[
m = m\varphi + m(1 - \varphi) = m\varphi + \sum_{j=1}^{s-1} \left( \frac{1 - \varphi_j}{s - 1} \right) m = m_0 + \sum_{j=1}^{s-1} m_j
\]
\[ T_m = T_{m_0} + \sum_{j=1}^{s-1} T_{m_j}. \]

Therefore from the above argument all the operators \( T_{m_j} \) are bounded on \( \ell^p(\mathbb{Z}) \), \( 1 < p < \infty \). \( \square \)

# 3 Applications

## 3.1 Discrete Riesz Transforms

For a complex-valued function \( f \) on \( \mathbb{Z}^d \) its discrete Laplacian is given by

\[
\Delta_d(f)(n) = \sum_{j=1}^{d} \partial_j \partial_j^* f(n) = \sum_{j=1}^{d} \partial_j^* \partial_j f(n), \quad n \in \mathbb{Z}^d
\]

where \( \partial_j f(n) = f(n + e_j) - f(n) \) and \( \partial_j^* f(n) = f(n) - f(n - e_j) \). We have

\[
\Delta_d(f)(n) = \sum_{j=1}^{d} \left( f(n + e_j) - 2df(n) + f(n - e_j) \right).
\]

The discrete Laplacian \( \Delta_d \) is a bounded self-adjoint operator on \( \ell^2(\mathbb{Z}^d) \) and one has

\[
\mathcal{F}_{\mathbb{Z}^d}(\Delta_d(f))(\xi) = -\sum_{j=1}^{d} |e^{i2\pi \xi_j} - 1|^2 \mathcal{F}_{\mathbb{Z}^d}(f)(\xi) = -4 \left( \sum_{j=1}^{d} \sin^2(\pi \xi_j) \right) \mathcal{F}_{\mathbb{Z}^d}(f)(\xi).
\]

The discrete Riesz transforms \( R_j \), \( j = 1, \ldots, d \), associated with \( \Delta_d \) are defined on \( \ell^2(\mathbb{Z}^d) \) as the multiplier operators

\[
\mathcal{F}_{\mathbb{Z}^d}(R_j(f))(\xi) = \frac{e^{-i\pi \xi_j} \sin(\pi \xi_j)}{2 \left( \sum_{k=1}^{d} \sin^2(\pi \xi_k) \right)^{1/2}} \mathcal{F}_{\mathbb{Z}^d}(f)(\xi),
\]

its can be interpret as \( R_j(f) = \partial_j \Delta_d^{-1/2} \). Let us set

\[
\psi_j(\xi) = \frac{e^{-i\pi \xi_j} \sin(\pi \xi_j)}{2 \left( \sum_{k=1}^{d} \sin^2(\pi \xi_k) \right)^{1/2}}
\]

and prove that \( \psi_j \) satisfies the Mikhlin condition (2.6). This can be seen by using the fact that \( \gamma \in \mathbb{N}^d \), \( \partial^\gamma \psi_j \) is a linear combination of the following functions

\[
e^{-i\pi \xi_j} \prod_{k=1}^{d} \sin^{\alpha_k}(\pi \xi_k) \prod_{k=1}^{d} \cos^{\beta_k}(\pi \xi_k) \left( \sum_{i=1}^{d} \sin^2(\pi \xi_k) \right)^{\frac{d}{2}} - \sum_{i=1}^{d} (\alpha_k + \beta_k)/2
\]
where $\sum_{k=1}^{d} \alpha_k \leq |\gamma| + 1$ and $\sum_{k=1}^{d} \beta_k \leq |\gamma|$. Hence using the fact that for $k = 1, \ldots, d$, $\xi_k \in (-1/2, 1/2)$ and $2|\xi_k| \leq |\sin \pi \xi| \leq \pi |\xi|$ we obtain

\[
\left| \prod_{k=1}^{d} \sin^{\alpha_k}(\pi \xi_k) \prod_{k=1}^{d} \cos^{\beta_k}(\pi \xi_k) \left( \sum_{k=1}^{d} \sin^2(\pi \xi_k) \right)^{-\frac{1}{2}} \right| \leq |\xi| \sum_{k=1}^{d} \beta_k \leq |\xi|^{-|\gamma|}.
\]

Therefore we can apply Theorem 2.8 to assert that $R_j$ is bounded on $\ell^p(\mathbb{Z}^d)$ for $1 < p < \infty$.

### 3.2 Imaginary powers of the discrete Laplace operator $\Delta_d$

Theorem 2.8 also applies to imaginary powers of the discrete Laplacian: $(-\Delta_d)^it$ for $t \in \mathbb{R}$, it is the multiplier operator with multiplier \(4 \sum_{j=1}^{d} \sin^2(\pi \xi_j)\)\(^it\).

### 3.3 Strichartz type estimates for discrete wave equation

We define the $d$-dimensional discrete wave equation by

\[
\Delta_d u(n, t) = \partial_t^2 u(n, t),
\]

\[
u(n, 0) = f(n), \quad \partial_t u(n, t) = g(n), \quad (n, t) \in \mathbb{Z}^d \times \mathbb{R}
\]

where $f$ and $g$ are a given suitable functions on $\mathbb{Z}^d$. Considered as a discrete counterpart of the continuous wave equation, many authors have been interested in studying this equation see, for example, [6, 5, 7] and the references therein. Putting

\[
\phi(\xi) = 2 \sqrt{\sum_{j=1}^{d} \sin^2(\pi \xi_j)}
\]

and applying the discrete Fourier transform, considering $t$ as a parameter, we deduce that the solution of (3.1) can be written (formally) in the form

\[
u(n, t) = \mathcal{F}_{\mathbb{Z}^d}^{-1} \left( \cos(t\phi(\cdot)) \mathcal{F}_{\mathbb{Z}^d}(f)(n) + \mathcal{F}_{\mathbb{Z}^d}^{-1} \left( \frac{\sin(t\phi(\cdot))}{\phi(\cdot)} \mathcal{F}_{\mathbb{Z}^d}(g)(n) \right) \right). \tag{3.2}
\]

We will prove the following version of the Strichartz estimates.

\[
\|u(\cdot, t)\|_{\ell^q} \leq c(t) \left( \|g\|_{\ell^p} + \sum_{j=1}^{d} \|\partial_j f\|_{\ell^p} \right), \tag{3.2}
\]

for all $1 < p \leq 2 \leq q < \infty$.

Let us observe first that $\xi \rightarrow \sin(t\phi(\xi))/\phi(\xi)$ is a $C^\infty$- function on $\mathbb{T}^d$ and then in view of Theorem 2.7 we have

\[
\left\| \mathcal{F}_{\mathbb{Z}^d}^{-1} \left( \frac{\sin(t\phi(\cdot))}{\phi(\cdot)} \mathcal{F}_{\mathbb{Z}^d}(g) \right) \right\|_{\ell^q} \leq c(t) \|g\|_{\ell^p}
\]
whenever $1 < p \leq q < \infty$. To prove (3.2) it suffices to show that

$$\left\| F_{Zd}^{-1} \left( \cos(t\phi(.))F_{Zd}(f) \right) \right\|_{\ell^q} \leq c(t) \sum_{j=1}^{d} \| \partial_j f \|_{\ell^p}.$$ 

We write

$$\cos(t\phi(\xi))F_{Zd}(f)(\xi) = \frac{\cos t\phi(\xi)}{\phi(\xi)} \sum_{j=1}^{d} F_{Zd}(R_j(\partial_j f))(\xi).$$

As

$$\left| \frac{\cos(\phi(\xi))}{\phi(\xi)} \right| \leq \frac{c}{|\xi|},$$

it follows from Theorem 2.2 that

$$\left\| F_{Zd}^{-1} \left( \cos(t\phi(.))F_{Zd}(f) \right) \right\|_{\ell^q} \leq c(t) \sum_{j=1}^{d} \| R_j(\partial_j f) \|_{\ell^p}$$

and by using $\ell^p$-boundedness of $R_j$,

$$\left\| F_{Zd}^{-1} \left( \cos(t\phi(.))F_{Zd}(f) \right) \right\|_{\ell^q} \leq c(t) \sum_{j=1}^{d} \| \partial_j f \|_{\ell^p}$$

which conclude the proof of (3.2).

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