Topological nodal line semimetals in holography

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Abstract

We build a holographic model of a strongly coupled topological nodal line semimetal and show that the nodal line semimetal phase could go through a quantum phase transition to a topologically trivial state. The dual fermion spectral function confirms that the Fermi surface is a closed nodal loop in the nodal line semimetal phase. We expect that our model provides a general framework for generating strongly coupled topological semimetals in holography, which indicates that at strong coupling topologically nontrivial semimetal states generally exist. In this general framework the topological structure in the bulk is induced by the IR interplay between the dual mass operator and the operator that deforms the topology of the Fermi surface.

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1 Introduction

During the last decade, our understanding towards topological states of matter has grown enormously in condensed matter physics. Topological states of matter are a large class of phases that cannot be classified by the Landau-Ginzburg paradigm which characterizes different states of matter by the symmetry that they break. There is no local order parameter for topological states of matter and they distinguish from other phases by the nontrivial topology in their quantum wave functions. Symmetry also plays an important role in some topological states of matter but not in the way of symmetry breaking, e.g. there exist symmetry protected topological states of matter whose nontrivial topology is protected by certain kinds of symmetry. Topological states of matter possess properties that are insensitive to small perturbations unless passing through a quantum phase transition which destroys the nontrivial topology.

Examples of topological states of matter include topological insulators, topological Weyl/nodal line semimetals, anomalous Hall states, and topological superconductors (see e.g. [1] for a review and references therein). Most properties of the topological band systems that we have known were built on the weakly coupled picture. An important challenge in the classification of topological states of matter is to consider the effect of interactions, especially strong interactions, on the topological structures of these systems. In experiments a lot of possible strongly interacting topological states of matter have been found, e.g. iridium oxide materials [2], transition metal oxide heterostructures [3] and the Kondo insulator SmB$_6$ [4]. With strong interactions, it is possible that a weakly coupled nontrivial topological structure would be destroyed by interactions and they can adiabatically deform to a topologically trivial state under interactions. On the other hand, it is also possible that with interactions, new topological states of matter arise which do not exist at weak coupling. This has been a rapidly growing research area in condensed matter during the recent years. However, it is still quite difficult to attack this problem due to the technical difficulty at strong coupling in the condensed matter theory, especially for gapless topological systems, and some attempts in this direction can be found in [5, 6].

To answer this question, a very powerful tool from string theory, the AdS/CFT correspondence, which maps a strongly coupled $d + 1$ dimensional field theory to a $d + 2$ dimensional weakly coupled classical gravity theory, would be extremely helpful. As a strong-weak duality, AdS/CFT has obtained lots of success in its applications to strongly coupled condensed matter systems [7, 8, 9]. Strongly coupled many body systems can be dually described by a weakly coupled classical gravity theory with one higher dimension. In holography, there is no band theory or notion of quasiparticles for the dual strongly coupled system and it is an ideal system to study strongly coupled
topological states of matter. However, it is still a relatively undeveloped research area to try to add topological states of matter into the holographic dictionary.\(^3\) Besides checking if certain types of weakly coupled topological states of matter still exist at strong coupling, we also hope to expand the holographic dictionary further to propose a general framework for generalizing holographic topological states of matter.

In [12], we have shown that at strong coupling there could still exist a topological Weyl semimetal state and the strongly coupled Weyl semimetal could pass through a topological quantum phase transition to a trivial semimetal phase. The non-local order parameter is the anomalous Hall conductivity which is only nonzero in the topologically nontrivial phase. This holographic model does not only confirm that topologically nontrivial Weyl semimetal states still exist at strong coupling, but also gave an important prediction for the transport property of the system which could be used to detect mixed axial-gravitational anomaly in laboratories [13]. We showed that there is a substantial amount of odd viscosity near the vicinity of the quantum critical regime of the holographic Weyl semimetal and this odd viscosity is induced by the mixed axial-gravitational anomaly. More study on this holographic model could be found in [14, 15, 16, 17]. However, in previous work the topological structure in the gravity bulk has not been discussed and it is not known whether the holographic Weyl semimetal is just an isolated example of a topological state of matter or if there exists a general paradigm to obtain general topologically nontrivial states in holography.

As a first step in the attempt to provide a general framework in holography for topological states of matter, in this paper we hope to elaborate along this line. We will start from building a holographic model of topological nodal line semimetal, which is another kind of gapless topological state of matter whose shape of Fermi surface forms a closed loop in momentum space. We will show the behavior of the corresponding fermion spectral function which confirms that the dual Fermi surface is a circle in the momentum space at zero density. We will reveal the corresponding topological structure on the gravity side. Based on the mathematical pattern of topology in the gravity solution space that both the topological nodal line and Weyl semimetals possess, we expect that this could be generalized to a general paradigm in holography to describe strongly coupled gapless topological states of matter. The result confirms that the Weyl semimetal is not an isolated example of a holographic topological state of matter and general topological semimetal states still exist at strong coupling. We expect this general framework could also be generalized to the gapped case and be utilized for the classification of strongly coupled topological states of matter.

In the following of this paper we will first give an example of a new entry in the

\(^3\) Some previous attempts could be found in e.g. [10] for topological insulator and [11] for quantum Hall states.
holographic dictionary of topological states of matter: the holographic topological nodal line semimetal, and point out the general bulk topological structure of topological semimetal states in Sec. 2. Then the behavior of the corresponding fermion spectral functions of the nodal line semimetal will be shown in Sec. 3. Sec. 4 will be devoted to discussions and open questions.

2 Holographic topological nodal line semimetals

A nodal line semimetal [18] has a nontrivial shape of Fermi surface where Fermi nodal points connect to form a circle due to the existence of certain symmetries, e.g. mirror reflection symmetry (see [19] for a review). A topologically nontrivial nodal line semimetal cannot be gapped by small perturbations unless passing through a topological phase transition to a topologically trivial state. To get some hint for building the dictionary of this model, we first take a look at a simple weakly coupled field theory model that represents a nodal line semimetal.

A field theoretic model

The Lagrangian is

\[ \mathcal{L} = i \bar{\psi} \left( \gamma^\mu \partial_\mu - m - \gamma^{\mu\nu} b_{\mu\nu} \right) \psi \]  

where \( \gamma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \) and \( b_{\mu\nu} = -b_{\nu\mu} \) is an antisymmetric two form field. We turn on a nonzero constant \( b_{xy} \) component of the two form field and the energy spectrum of the eigenstates of this system are therefore

\[ E_\pm = \pm \sqrt{k_z^2 + (2b_{xy} \pm \sqrt{m^2 + k_x^2 + k_y^2})^2}. \]  

For \( m^2 < 4b_{xy}^2 \), the system is a topological nodal line semimetal with Fermi points forming a connected circle with radius \( \sqrt{4b_{xy}^2 - m^2} \) in the momentum space. In this parameter regime we can see that the system cannot be gapped by small perturbations like an ordinary Dirac field, i.e. a small perturbation in \( m \) cannot gap this system. For \( m^2 > 4b_{xy}^2 \), the system becomes an insulator and \( m^2 = 4b_{xy}^2 \) is the quantum transition point of this topological phase transition.

We can see that this should be a topologically nontrivial nodal line semimetal. Other components of \( b_{\mu\nu} \) would play a similar role and produce a nodal line semimetal with slight difference in the spectrum, e.g. \( b_{tz} \) would produce an accidental nodal line semimetal.

\[ ^4 \text{To be consistent with the gravity calculations, we take the } (-, +,+ ,+) \text{ signature.} \]
Figure 1: The energy spectrum as a function of $k_x, k_y$ for $k_z = 0$. Left: there is a nodal line at the band crossing when $m^2 < 4b_{xy}^2$. Right: for $m^2 > 4b_{xy}^2$ the system is gapped.

The new two form field $b_{\mu\nu}$ term does not break the conservation of the electric current $J^\mu = \bar{\psi}\gamma^\mu\psi$ while does not conserve the axial current $J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. We have the following conservation equations

$$\partial_\mu J^\mu = 0, \quad (2.3)$$
$$\partial_\mu J_5^\mu = -2m\bar{\psi}\gamma^5\psi - 2b_{\mu\nu}\bar{\psi}\gamma^{\mu\nu}\gamma^5\psi, \quad (2.4)$$

where we have ignored the anomaly terms.

**The holographic dictionary**

With the hint from the weakly coupled theory, we utilize a massive two form field $B_{ab}$ which is dual to the operator $\bar{\psi}\gamma^{\mu\nu}\psi$. We also introduce an axially charged scalar field to break the axial symmetry and this scalar field represents the effect of the $m\bar{\psi}\psi$ term. A source for the operator $\bar{\psi}\gamma^{\mu\nu}\psi$ breaks the time reversal as well as the charge conjugation symmetry. We consider

$$S = \int d^5x\sqrt{-g}\left[\frac{1}{2\kappa^2} \left( R + \frac{12}{L^2} \right) - \frac{1}{4} F^2 - \frac{1}{4} F_5^2 + \frac{\alpha}{3} \epsilon^{\mu\nu\rho\sigma} A_\mu \left( 3 F_{\nu\rho} F_{\sigma\tau} + F_{\nu\rho} F_{\sigma\tau} \right) - (D_\mu \Phi)^* (D^\mu \Phi) - V_1(\Phi) - \frac{1}{3\eta} \left( D_{(a} B_{bc)} \right)^* (D^{(a} B^{bc)}) - V_2(B_{ab}) - \lambda |\Phi|^2 B^*_{ab} B^{ab} \right]$$

where $F_{ab} = \partial_a V_b - \partial_b V_a$ is the vector gauge field, $F_{5a} = \partial_a A_b - \partial_b A_a$ is the axial gauge field, $D_a = \nabla_a - iq_1 A_a$, $D_a = \nabla_a - i\theta_2 A_a$ and

$$D_{(a} B_{bc)} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab} - i\theta_2 A_a B_{bc} - i\theta_2 A_b B_{ca} - i\theta_2 A_c B_{ab}. \quad (2.5)$$

The field $B_{ab}$ is also axially charged as can be seen from (2.4) that the dual operator

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5 We set $2\kappa^2 = L = 1$. We use $a, b = t, x, y, z, r$ to denote bulk indices, $\mu, \nu = t, x, y, z$ for boundary indices.
is expected to break the axial symmetry.\textsuperscript{6} The potential terms are
\begin{align}
V_1 &= m_1^2 |\Phi|^2 + \frac{\lambda_1}{2} |\Phi|^4, \\
V_2 &= m_2^2 B_{ab}^* B^{ab},
\end{align}
where $m_1$ is the mass of the scalar field and $m_2$ is the mass of the two form field. The field $B_{ab}$ has a mass term because it does not correspond to a conserved operator and we will turn on the $B_{xy}$ component in the following. The $\lambda$ term denotes the effect of interplay between the dual mass term $m \bar{\psi} \psi$ and the dual operator $\bar{\psi} \gamma^{\mu\nu} \psi$, which we will see is very important to the existence of the topological structure and the topological phase transition. Note that this interaction term comes from the effect of the dual mass term in the nonconservation equation for $\bar{\psi} \gamma^{\mu\nu} \psi$ and this does not correspond to the interaction of the dual fermions, which is always there and is strong according to the holographic dictionary. Similar to the holographic Weyl semimetal \cite{12}, we introduce the $\lambda_1$ term because we cannot find nonsingular solutions without this term and $\lambda_1$ denotes the number of UV degrees of freedom that does not gap in the IR and $\lambda_1 = 0$ implies that the system could become fully gapped in the IR for solutions with nonzero scalar field at the horizon. This indicates that this simple holographic system cannot be completely gapped in the IR and could only be partially gapped at most. The equations of motion can be found in the appendix A.

In this system, the scalar and the two form fields both have sources at the boundary, therefore for the system to be normalizable at the boundary and without loss of generality we choose the conformal dimension for the source of the dual mass term and the $\bar{\psi} \gamma^{\mu\nu} \psi$ term in (2.1) to be 1. Thus the massive scalar and 2-form fields in the bulk have the values for the mass to be $m_1^2 = -3$ and $m_2^2 = 1$. Furthermore, for simplicity, we set $q_1 = q_2 = 1$, $\lambda = 1$, $\lambda_1 = 0.1$ and $\eta = 1$.

After a variation of the total action with respect to the gauge fields, we can obtain the dual consistent currents and they satisfy
\begin{align*}
\partial_\mu J^\mu_{\text{con}} &= 0, \\
\partial_\mu J_{\text{5con}}^\mu &= \lim_{r \to \infty} \sqrt{-g} \left( -\frac{\alpha}{3} \epsilon^{\alpha\beta\rho\sigma} (F_{\alpha\beta} F_{\rho\sigma} + F_{\alpha\beta} F_{\rho\sigma}) + iq_1 \left[ \Phi^* (D^r \Phi) - \Phi (D^r \Phi)^* \right] + \\
&\quad + \frac{iq_2}{\eta} \left( B_{\mu\nu}^* D[r B^{\mu\nu}] - (D[\nu B^{\mu\nu}]^* B_{\mu\nu}) \right) \right) + \text{c.t.}.
\end{align*}
Here we have not explicitly shown the counterterm for simplicity and the above conservation can be further simplified in the radial gauge. The point is that the last two

\textsuperscript{6} Other interesting application of (massless) two form field in holography can be found in \cite{20, 21} for holographic descriptions of 3+1D magnetohydrodynamics and in \cite{22} for magnetization transitions in AdS$_4$/CFT$_3$. 

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terms contribute only when the non-normalisable mode of the scalar field or two-from field is switched on and it is straightforward to see that the above identities are of the same structure of the weakly coupled theory (2.3, 2.4). Thus this holographic model is expected to go beyond the weakly coupled theory to a strongly coupled nodal line semimetal model.

**Three types of solutions at zero temperature**

To study the quantum phase transition of the system, we shall focus on solutions at zero temperature. The simplest way to solve the system is to consider the probe limit $\kappa/L^{3/2} \ll 1$. However, the probe limit is not well defined near the horizon and even with another set of scaling dimensions where the probe limit is well defined near the horizon it is not valid near the critical point. We leave the probe limit analysis in the appendix B and will refer to it when we discuss the topological structure below.

Here with backreactions we choose the ansatz of the background to be

$$
\begin{align*}
    ds^2 &= u(-dt^2 + dz^2) + \frac{dr^2}{u} + f(dx^2 + dy^2) \\
    \Phi &= \phi(r), \\
    B_{xy} &= B(r).
\end{align*}
$$

(2.8)

At the boundary, the fields $\phi(r)$ and $B(r)$ behave as

$$
\begin{align*}
    \phi &= \frac{M}{r} + \cdots, \\
    B &= br + \cdots,
\end{align*}
$$

(2.9)

where $M$ and $b$ correspond to the source for the two dual operators. The more explicit near boundary behavior of these fields can be found in the appendix A. To solve these equations, we need to identify the near horizon boundary condition. It turns out we have the following three different kinds of near horizon geometries. With proper irrelevant deformations, these solutions flow to the UV AdS$_5$ with different boundary values of $M/b$.

**Topological phase.** The near horizon solution for the topological phase is

$$
\begin{align*}
    u &= \frac{1}{8} (11 + 3\sqrt{13}) r^2 \left( 1 + \delta w r^{\alpha_1} \right), \\
    f &= \sqrt{\frac{2\sqrt{13}}{3}} - 2 b_0 r^\alpha \left( 1 + \delta f r^{\alpha_1} \right), \\
    \phi &= \phi_0 r^\beta, \\
    B &= b_0 r^\alpha \left( 1 + \delta b r^{\alpha_1} \right),
\end{align*}
$$

where $(\alpha, \beta, \alpha_1) = (11 - 3\sqrt{13}, \frac{2(\sqrt{1+9\lambda}+3\sqrt{13\lambda}-\sqrt{13})}{3\sqrt{13}}, 1.273)$, $(\delta f, \delta b) = (-2.616, -0.302)\delta u$ for the parameter values that we have fixed above. Note that $b_0$ can set to be 1 using
the transformation \((x, y) \to c(x, y)\). Moreover, without the deformation term, the near horizon has a Lifshitz symmetry

\[
(t, z, r^{-1}) \to c(t, z, r^{-1}), \quad (x, y) \to c^{\alpha/2}(x, y),
\]

which can be used to set \(\delta u = \pm 1\) and only \(-1\) can be used to flow the geometry to \(\text{AdS}_5\). Thus in the IR, we only have a unique free parameter \(\phi_0\).

We integrate this solution to the boundary and read the value of \(M/b\) at the boundary. For this type of near horizon boundary conditions we can only find solutions for \(M/b < 1.717\). When the value of \(\phi_0\) is zero, the boundary reaches \(M/b = 0\). When \(\phi_0\) grows, \(M/b\) becomes larger and larger and finally \(M/b\) reaches the critical value 1.717. To see that this is the nodal line semimetal phase, we will give evidence from fermion spectral function calculations in Sec. 3. Nodal line semimetals could also be in a topological trivial phase that can be easily gapped, thus to see that this is indeed a topologically nontrivial phase, we will explain its topological structure in the next subsection. As we have explained above, in this simple holographic setup, it is difficult to fully gap the system and here a topological nontrivial semimetal phase means that it cannot be partially gapped by a small perturbation.

**Critical point.** The near horizon solution for the critical point including an irrelevant deformations is

\[
\begin{align*}
u &= u_0 r^2 (1 + \delta u r^\beta), \\
f &= f_0 r^\alpha (1 + \delta f r^\beta), \\
\phi &= \phi_0 (1 + \delta \phi r^\beta), \\
B &= b_0 r^\alpha (1 + \delta b r^\beta),
\end{align*}
\]

with

\[
(u_0, f_0, \alpha, \phi_0) \simeq (3.076, 0.828b_0, 0.292, 0.894),
\]

and

\[
\beta = 1.272, \quad (\delta u, \delta f, \delta b) = (1.177, -2.771, -0.409)\delta \phi.
\]

Using the transformation \((x, y) \to c(x, y)\), we can set \(b_0 = 1\). Note that without the deformation, the near horizon has the same Lifshitz symmetry (2.10) as the case of topological phase. Utilizing this symmetry, \(\delta \phi\) could be chosen to be \(\pm 1\) and \(\delta \phi = -1\) will flow the geometry to \(\text{AdS}_5\) at the boundary. There is no free parameter in the IR and the geometry is unique. At the boundary we can read that the critical \(M/b \simeq 1.717\).
**Trivial phase.** The near horizon solution for the trivial phase is

\[
\begin{align*}
u &= (1 + \frac{3}{8\lambda_1}) r^2, \\
f &= r^2, \\
\phi &= \sqrt{\frac{3}{\lambda_1}} + \phi_1 r \sqrt{\frac{160\lambda_1^2 + 84\lambda_1 + 9}{3 + 8\lambda_1}}^{-2}, \\
B &= b_1 r 2^{\sqrt{2}} \sqrt{\frac{\lambda_1 + \lambda_1}{3 + 8\lambda_1}}. 
\end{align*}
\]

Note that this solution (without $\phi_1$) is also an exact solution so that we need irrelevant deformations to flow this to asymptotic $\text{AdS}_5$ solutions. For this type of near horizon boundary conditions we can only find solutions if $M/b > 1.717$. As we mentioned above $\lambda_1$ denotes the number of gapless degrees of freedom in the IR and when $\lambda_1$ is not zero the system is at most partially gapped in the IR [15]. For the critical and the topological trivial solutions here, the scalar field $\phi$ is nonzero at the horizon and the system is partially gapped.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{An illustration of the bulk solutions. Starting from different boundary values of $M/b$ in the UV, the geometry flows to different IR solutions, which correspond to different phases.}
\end{figure}

**Free energy**

Fig. 2 is an illustration for the phase structure in the solution space in the bulk. In Fig. 3, we show the bulk behavior of the scalar field $\phi$ and the two form field $B/f$ for different values of $M/b$ which correspond to different near horizon geometries. Close to the critical $M/b$, the near horizon solution flows to the critical solution quickly.
For each value of $M/b$ from the boundary, there is only one bulk solution, nevertheless we can compute the free energy of this system to see whether the phase transition is a continuous one. To compute the free energy, we need to be careful with the boundary counterterms. The total action is

$$S_{\text{ren}} = S + S_{\text{GH}} + S_{\text{c.t.}}$$

with the Gibbons-Hawking boundary term $S_{\text{GH}} = \int_{r=r_{\infty}} d^4x \sqrt{-\gamma} (2K)$ and the counterterms

$$S_{\text{c.t.}} = \int_{r=r_{\infty}} d^4x \sqrt{-\gamma} \left( -6 - |\Phi|^2 + |B_{\mu\nu}|^2 + \frac{1}{2} (\log r^2) \left[ \frac{1}{4} F^2 + \frac{1}{4} F^2 + |D_\mu \Phi|^2 + \left( \frac{1}{3} + \frac{\lambda}{2} \right) |\Phi|^4 + \frac{1}{3\eta} (D_\mu B_{\alpha\beta})^* (D^{\mu} B^{\alpha\beta}) - |B_{\mu\nu}|^4 + \lambda |\Phi|^2 |B_{\mu\nu}|^2 \right] \right)$$

where $\gamma_{\mu\nu}$ is the metric induced by $g_{ab}$ on the boundary via $\gamma_{\mu\nu} = g_{\mu\nu} - n_a n_b$ where $n_a$ is outward pointing unit normal vector of the boundary. $K = \gamma^{\alpha\beta} \nabla_\alpha n_\beta$ is the trace of the extrinsic curvature with respect to the metric at the boundary.

From the $zz$ component of the Einstein equation (A.1), we have $R_{zz} - \frac{1}{2} g_{zz} \mathcal{L} = 0$. Thus the bulk on-shell action is a total derivative

$$S = \int d^4x dr \sqrt{-g} \mathcal{L} = - \int d^4x \int_{0}^{r_{\infty}} dr \left[ f u^{1/2} u' \right]' .$$

Taking into account the boundary terms and performing a Wick rotation, the free energy density can be obtained as

$$\frac{\Omega}{V} = \frac{7}{9} b^4 - \frac{4 b b_2}{3} + \frac{7 M^4}{36} - 3 u_2 + \frac{5 + 8 \lambda}{9} b^2 M^2 - 2 M \phi_2 .$$
The stress tensor for the dual field theory can be calculated as

\[ T_{\mu\nu} = 2(K_{\mu\nu} - \gamma_{\mu\nu}K) + \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{ct.}}{\delta \gamma_{\mu\nu}}. \]  

(2.16)

The total energy density is

\[ \epsilon = \lim_{r \to \infty} \sqrt{-\gamma} \langle T_{00} \rangle = \frac{7}{9} b^4 - \frac{4}{3} b^2 + \frac{7}{9} M^4 - 3u_2 + \frac{5 + 8\lambda}{9} b^2 M^2 - 2M\phi_2, \text{ thus } \frac{\Omega}{\nu} = \epsilon. \]

The free energy can be found in Fig. 4. The system is smooth when it crosses the phase transition.

Figure 4: The free energy density as a function of \( M/b \). The red dot is the value of free energy density at the critical point. Approaching the transition point from both the topological phase and the trivial phase, the system is continuous and smooth.

**Topological structure in the solution space**

Whether the solution is a nodal line semimetal can be found from the shape of the dual Fermi surface in the next section. One important and immediate question is where does the corresponding topology lie in the gravity side? We expect that the topological invariant could be defined from the dual fermion spectral function in the same way as it is defined for interacting topological systems in condensed matter physics [23]. For the Weyl semimetal case, the anomalous Hall conductivity is itself a topologically nontrivial transport coefficient. However, besides that topological structure in the momentum space of fermions and in the transport properties, we have a seemingly more intrinsic topological structure for this system which can be seen from the bulk configurations for the fields \( B_{ab} \) and \( \Phi \).\(^7\)

\(^7\)This topological structure is clearer in the probe limit and the readers can refer to the appendix B for simplicity. The logic in the backreaction case is the same, only being more complicated due to modified metric fields.
First let us analyze the three types of solutions above with more details. From the equations of motion (A.8 - A.9) for $B$ and $\phi$, we can see that the interaction between the two fields are very important in the IR. Ignoring the interaction term (and also the $\phi^4$ term), $B$ and $\phi$ would be independent of each other and their near horizon behavior is determined by the IR conformal dimensions $\delta_{\pm}^{B,\phi}$ of $B$ and $\phi$ in the backreacted background (2.8) respectively. The term $r^{-\delta_{\pm}^{B,\phi}}$ is too divergent in the IR and will lead to a singular geometry, thus in the IR we only keep the $c_{B,\phi} r^{-\delta_{\pm}^{B,\phi}}$ terms as IR solutions. Taking back the interaction $\lambda$ term, we find that $c_B$ and $c_\phi$ cannot both be nonzero as the interaction term would diverge. This feature divides the leading order near horizon solutions into three types: $c_\phi = 0$ while $c_B \neq 0$, $c_B = 0$ while $c_\phi \neq 0$ and $c_\phi = c_B = 0$. In other words, the interaction term always changes the IR scaling dimension of at least one of the fields and it is not possible to keep both scaling dimensions unchanged with this term. Here note that this structure is still true with the presence of $\lambda_1 \phi^4$ term where $\phi$ is at most $O(1)$ at leading order and we can substitute $c_\phi r^{-\delta_\phi}$ with $c_\phi$ in this case.

The topologically nontrivial phase corresponds to the near horizon solution with $B \sim c_B r^{-\delta_B^B}$ and $\phi$ at irrelevant subleading order sourced by $B$. With the $\phi^4$ term, the topologically trivial phase corresponds to the near horizon solution with nonzero $\phi \sim c_\phi$ and $B$ at subleading order sourced by $\phi$. The critical solution is that both $B$ and $\phi$ are zero at leading order and sourcing each other at subleading order. Note that the critical solution seems to also have the form of $B \sim r^\alpha$ and $\phi \sim c_\phi$, however, the power $\alpha$ is larger than the IR scaling dimension $\delta_B^B$ on the geometry of the critical solution and $c_\phi$ is smaller than the value for the trivial phase indicating that fewer degrees of freedom become gapped in the critical phase compared to the trivial phase.

From this mathematical structure of the system we can see where the topology lies: at the horizon, when $B$ has a leading order nonzero term $B \sim c_B r^{-\delta_B^B}$, $\phi$ cannot have a nonzero leading order term simultaneously and only has a subleading order term sourced by leading order of $B$. This means that when we have a nodal line semimetal solution with this near horizon behavior, we cannot find a small perturbation in this background with $\phi \sim c_\phi r^{-\delta_\phi}$ or $c_\phi$ in the case with $\lambda_1 \phi^4$ term, which could gap or partially gap the semimetal. As can be seen from the fermion spectral function calculations below in the nodal line semimetal phase as the subleading order of $\phi$ grows bigger the radius of the nodal line circle becomes smaller, which is consistent with the fact that small perturbations would not gap the system while only making the Fermi nodal circle bigger or smaller. This means that the interaction between $B$ and $\phi$ at IR is the intrinsic reason that the nodal line solution is topological. The topological nodal line semimetal phase could only become partially gapped after a quantum phase transition passing through the critical point where the nodal line becomes a nodal
This holographic topological nodal line semimetal system shares the same mathematical structure at the horizon as the holographic Weyl semimetal model, though for different kinds of fields there are differences, e.g., for the axial vector field in the holographic Weyl semimetal, the probe limit is a good limit for the same scaling dimensions of operators dual to $A_z$ and $B_{xy}$ except near the critical point. Also for the Weyl semimetal phase the near horizon geometry is $AdS_5$ while the near horizon geometry of the nodal line semimetal phase is a Lifshitz geometry. For the Weyl semimetal case, it is clear to see that the Weyl semimetal phase exists because there is a time reversal breaking operator while the topological structure arises because the mass operator breaks the axial symmetry, i.e. the dual mass operator couples with the time reversal breaking operator.

The mechanism for the topological quantum phase transition here is different from the BF bound mechanism [25, 26] for many holographic quantum phase transitions, including holographic superconductors, metal-insulator phase transitions, etc.. Here the IR scaling dimensions of the fields do not change for each phase and different phases are due to different values of sources at the UV. There is no local order operator associated with this phase transition which is a property that was also found in other quantum phase transitions in holography [27].

3 Fermion spectral functions

For nodal line semimetals there is no smoking gun transport coefficient like anomalous Hall conductivity for Weyl semimetals. An immediate and straightforward test would be to see if there is indeed a closed nodal line from the dual fermion spectral functions. We will show that the fermion spectral function in this system confirms that in the nodal line semimetal phase, there is indeed a peak\(^8\) at $k_x^2 + k_y^2 = k_0^2$ where the value of $k_0$ decreases with $M/b$ increasing and finally becomes zero at the critical point. We expect that we could also see the nontrivial topology in the structure of the fermion spectral functions, which we hope to report in the future work.

The prescription for calculating the holographic fermion spectral function could be found in [28, 29, 30]. In five dimensions, one bulk Dirac spinor corresponds to one chirality of boundary Weyl spinor. We utilize two spinors with the same mass and both with standard quantization to calculate the dual fermion spectral function for

\(^8\)In numerics we always keep a small value of $\omega$ and this peak might become a pole when the small value of $\omega$ is removed.
both chiralities, since in both the holographic Weyl/nodal line semimetal model, the two chiralities interact with each other due to the axial symmetry breaking terms. As can be seen from formula (2.4) both the mass term and the $b_{\mu\nu}$ term break the axial symmetry and couple spinors of two chiralities together. The action of the two spinors $\Psi_1$ and $\Psi_2$ are

\[ S_{\text{fermion}} = S_1 + S_2 + S_{\text{int}}, \]

where

\[ S_1 = \int d^5x \sqrt{-g} \bar{\Psi}_1 (\Gamma^a D_a - m_f) \Psi_1, \]

\[ S_2 = \int d^5x \sqrt{-g} \bar{\Psi}_2 (\Gamma^a D_a - m_f) \Psi_2, \]

\[ S_{\text{int}} = -\int d^5x \sqrt{-g} (i\eta_1 \Phi \bar{\Psi}_1 \Psi_2 + i\eta_1 \Phi^* \bar{\Psi}_2 \Psi_1 + i\eta_2 B_{ab} \bar{\Psi}_1 \Gamma^{ab} \Psi_2 + i\eta_2 B_{ab}^* \bar{\Psi}_2 \Gamma^{ab} \Psi_1), \]

with

\[ D_a = \partial_a - \frac{i}{4} \omega_{mn,a} \Gamma^{mn} - ig_3 A_a - ig_4 A^5_a \Gamma^5 \]

where $\omega_{mn,a}$ is the bulk spin connection. Note that $a$ and $m$ are the bulk spacetime index and the tangent space index respectively. $\bar{\Psi} = \Psi^\dagger \Gamma^t$, $\Gamma^{ab} = e_m^a e_n^b \Gamma^{mn}$. $m_f$ is the mass of the bulk spinor which determines the scaling dimension of the dual Fermi operator and we choose $m_f = -1/4$ so that there could be poles in the dual fermion spectral functions [28]. The equations of motion are

\[ (\Gamma^a D_a - m_f) \Psi_1 - (\eta_1 \Phi - \eta_2 \Gamma^{ab} B_{ab}) \Psi_2 = 0, \]

\[ (\Gamma^a D_a - m_f) \Psi_2 - (\eta_1 \Phi - \eta_2 \Gamma^{ab} B_{ab}) \Psi_1 = 0, \]

where the five dimensional $\Gamma$ functions are $\Gamma^a = \gamma^a$, $\Gamma^5 = \gamma^5$. We expand the fermion field as $\Psi = (uf)^{-1/2} \psi e^{-i\omega t + ik_x x + ik_y y + ik_z z}$ and the corresponding Dirac equation can be written as

\[ \left( \Gamma^a \partial_a + \frac{1}{u} \left( -i \omega \Gamma^4 + i k_z \Gamma^z \right) + \frac{i}{\sqrt{uf}} \left( k_x \Gamma^x + k_y \Gamma^y \right) - \frac{m_f}{\sqrt{u}} \right) \psi_{1,2} - \left( \frac{\eta_1 \Phi}{\sqrt{u}} + \frac{\eta_2 B}{\sqrt{uf}} \Gamma^{xy} \right) \psi_{2,1} = 0. \]

We can solve this as a coupled system of 8 functions. At the horizon the ingoing boundary condition depends on the near horizon geometry of each phase. For the topologically trivial phase, the near horizon ingoing solution for nonzero $k$ while $\omega \to 0$ is real just as the pure AdS$_5$ case in [28]. Thus the imaginary part of the Green’s
function is automatically zero for nonzero $k$ and the Fermi momentum should stay at $k = 0$. For the topologically nontrivial and critical phases, the near horizon ingoing boundary condition is

$$\psi_l \simeq e^{i \sqrt{\omega^2 - k^2} \frac{r}{\omega}} \begin{pmatrix} z_1^l (1 + \cdots) \\ z_2^l (1 + \cdots) \\ i \sqrt{\frac{\omega - k^2}{\omega}} z_1^l (1 + \cdots) \\ i \sqrt{\frac{\omega - k^2}{\omega}} z_2^l (1 + \cdots) \end{pmatrix} \quad (3.4)$$

with $l = (1, 2)$ for $w > k_z$, where “...” denote subleading terms. This is because the terms proportional to $k_x$ and $k_y$ are not so important compared to terms with $\omega$ and $k_z$ as $g^{xx}$ is not as divergent as $g^{zz}$ and $g^{tt}$ at the horizon. This also means that for nonzero $k_z$ and $\omega \to 0$, the retarded Green’s function is real and the Fermi surface could only stay at $k_z = 0$, which is consistent with the fact that this corresponds to a nodal line semimetal in the $x$-$y$ plane.

Near the boundary $r \to \infty$,

$$\psi_l = \begin{pmatrix} a_1^l r^{m_f} + \cdots \\ a_2^l r^{m_f} + \cdots \\ a_3^l r^{-m_f} + \cdots \\ a_4^l r^{-m_f} + \cdots \end{pmatrix}, \quad (3.5)$$

with $l = (1, 2)$.

Because the two chiralities couple to each other, $\psi_{1,2}$ will also source expectation values of $\psi_{2,1}$ respectively. To calculate the retarded Green’s function, we need four different horizon boundary conditions and get four sets of source and expectation values. We denote the four boundary conditions as I, II, III, IV respectively and the source and expectation matrix would look like

$$M_s = \begin{pmatrix} a_{1,1}^{1,1}, a_{1,1}^{1,II}, a_{1,1}^{1,III}, a_{1,1}^{1,IV} \\ a_{2,1}^{1,1}, a_{2,1}^{1,II}, a_{2,1}^{1,III}, a_{2,1}^{1,IV} \\ a_{2,2}^{1,1}, a_{2,2}^{1,II}, a_{2,2}^{1,III}, a_{2,2}^{1,IV} \\ a_{2,2}^{1,1}, a_{2,2}^{1,II}, a_{2,2}^{1,III}, a_{2,2}^{1,IV} \end{pmatrix} \quad \text{and} \quad M_e = \begin{pmatrix} a_{3,1}^{1,1}, a_{3,1}^{1,II}, a_{3,1}^{1,III}, a_{3,1}^{1,IV} \\ a_{4,1}^{1,1}, a_{4,1}^{1,II}, a_{4,1}^{1,III}, a_{4,1}^{1,IV} \\ a_{4,2}^{1,1}, a_{4,2}^{1,II}, a_{4,2}^{1,III}, a_{4,2}^{1,IV} \\ a_{4,2}^{1,1}, a_{4,2}^{1,II}, a_{4,2}^{1,III}, a_{4,2}^{1,IV} \end{pmatrix}.$$  

The Green’s function is determined by $G = -M_e M_s^{-1} \Gamma^t$. After getting $G$ we find eigenvalues of $G$ and read the imaginary part of the four eigenvalues.

We calculate the retarded Green’s function by numerics and keep a very small $\omega = 10^{-3}$ for numerical convenience. Our results show that in all the three phases, when $k_z$ is not zero, the retarded Green’s function is real and does not have an imaginary part. In the topological trivial phase, the near horizon geometry also guarantees that

for all values of $k_x, k_y, k_z \neq 0$, the retarded Green’s function is real. The pole should stay at $k_x = k_y = k_z = 0$ as this is not a fully gapped phase but a semimetal phase.

We fix $k_z = 0$ in the following calculations and focus on the topologically nontrivial and critical solutions. Our numerics show that for the nodal line semimetal phase, there is a peak in the imaginary part of two eigenvalues of the Green’s function at\(^\text{10}\) $k_x^2 + k_y^2 = k_0^2$ and the imaginary part of the other two eigenvalues are extremely small for all $k_x, k_y$ indicating that these two correspond to gapped degrees of freedom. The position of $k_0$ decreases to zero while approaching the quantum phase transition point. As the radius of the circle decreases, the height of the Green’s function grows higher and sharper. For the critical solution, we find that two branches of eigenvalues have peak in the imaginary part at $k_x = k_y = 0$ and the other two are still extremely small for all $k_x, k_y$. This confirms that the topologically nontrivial phase indeed corresponds to a nodal line semimetal and the critical solution corresponds to a trivial semimetal.

\[ k_F = \sqrt{k_x^2 + k_y^2} \]

Figure 5: The Fermi momentum $k_F = \sqrt{k_x^2 + k_y^2}$ in the holographic nodal line semimetals for $k_z = 0$. When the system is in the critical point or the topological trivial phase, there is no Fermi surface at finite $k$ and the pole of fermionic retarded Green’s function is located at $k = \omega = 0$.

4 Discussion

We have constructed a holographic model of strongly coupled topological nodal line semimetals, which implies that at strong coupling, there still exist topologically non-

\(^{10}\text{In numerics the } k_0 \text{’s for the two branches have a very small difference in value and we believe that this is due to the small value of } w \text{ and the two } k_0 \text{’s should coincide when } \omega = 0.\)
trivial nodal line semimetals which cannot be gapped by small perturbations. We showed that the topological structure lies in the IR behavior of the solutions, where the interplay between the dual $m\bar{\psi}\psi$ and $\bar{\psi}\gamma_{\mu\nu}\psi$ operators in the IR plays an important role. The calculation of the dual fermion spectral functions confirmed that the topologically nontrivial phase is indeed a nodal line semimetal.

In condensed matter physics, Weyl/nodal line semimetals are topological states of matter that can be understood in terms of free fermions and weakly coupled band theory. Our work expands this framework to the strongly coupled regime and shows that topological states of matter also exist at strong coupling when there is no notion of free fermions, quasiparticles or band structures. Note that this does not mean that the topology of a weakly coupled Weyl/nodal line semimetal is not destroyed at strong coupling but only shows that at strong coupling we still have topological systems with the same topological structure and most probably this strongly coupled topological system cannot adiabatically deform to the weakly coupled analog.

This topological nodal line semimetal shares very similar near horizon mathematical structure as the holographic Weyl semimetal model [12] and we believe that this framework provides a general framework for topological semimetal states in holography. The key features of the topological structure that a holographic system which could support a topologically nontrivial gapless state shares in general are:

I. A holographic system with at least two fields interacting with each other, and one of them usually corresponds to a mass operator which we denote as $\phi$ with the other corresponding to an operator that is responsible for the deformation of the topology of the Fermi surface, which we denote as $A$ for illustration. For example, in the Weyl semimetal case, $A$ is just the axial gauge potential $A_z$ whose source is responsible for the separation of the Weyl points in momentum space while in the nodal line semimetal case, $A$ is the two form field whose source deforms the topology of the Fermi surface from a point to a circle. At the IR the existence of this interaction between the two operators guarantees the topological structure of the solution space in the bulk.

II. At the horizon there are usually three types of solutions for a two field system (and more solutions with more fields coupled to each other): the first two types are $A$ (or $\phi$) is nonzero at leading order with $r^{-d_{A,\phi}}$ while $\phi$ (or $A$) is at subleading order sourced by leading order $A$ (or $\phi$); the third type which usually corresponds to a critical solution has two fields both at subleading order sourcing each other and there is only one such solution without any tuning parameter. As horizon solutions determine many transport properties, the fact that $A$ and $\phi$ cannot both be at leading order with $r^{-d_{A,\phi}}$ at the horizon implies that the semimetal phase cannot be gapped by small perturbations.

We expect that a general holographic topological semimetal state shares the prop-
erties above. Introducing more kinds of fields in the gravity side would produce more possible topological semimetal states. As there is a seemingly more intrinsic topological structure for the system that does not require any knowledge of the dual fermion spectral functions, it is possible that this framework would shed light on a better understanding of the deeper organizational principles of strongly coupled topological gapless states of matter and predict new kinds of strongly coupled topological semimetal states.

Open questions and future directions

In the holographic system that we considered it is difficult to find fully gapped solutions and we do not have gapped topological states of matter. As a first step to build a general paradigm for topological states of matter in the holographic dictionary, an immediate open question is to generalise this mechanism to accommodate other types of topological states of matter, including gapped topological insulators, topological superconductors and other possible kinds of topological gapless states, and use this holographic framework as a possible method for classification of strongly interacting gapped as well as gapless topological states of matter. It would be interesting to see if there could exist any novel strongly coupled topological states of matter which even do not exist at weak coupling. Another question is if this mechanism could also be generalized to lower dimensions and what would happen after including translation symmetry breaking operators, especially could gapped topological states of matter be found? Other relevant questions include calculating the fermion spectral function for the holographic Weyl semimetal models, calculating the topological invariants from dual fermion spectral functions, looking for holographic accidental semimetal states, the quantum phase diagram for competing holographic topological phases, study of the edge states of these topological states (for the holographic Weyl semimetal case the edge states have been studied in [17]). We hope to answer some of these open questions in future work.

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A Equations of motion

The equations of motion for the action in Sec. 2 are

\[ R_{ab} - \frac{1}{2} g_{ab} (R + 12) - T_{ab} = 0, \]  
\[ \nabla_b F^{ba} + 2 \alpha \epsilon^{abdef} F_{bc} F_{de} = 0, \]  
\[ -iq_1 (\Phi^* D^n \Phi - (D^n \Phi)^* \Phi) - \frac{iq_2}{\eta} (B_{ca} D^{[b} B^{ca]} - (D^{[b} B^{ca]})^* B_{ca}) = 0, \]  
\[ D_a D^n \Phi - \partial \Phi \cdot V_1 - \lambda \Phi B_{ab} B^{ab} = 0, \]  
\[ \frac{1}{\eta} D^n D_{[ab} B_{bc]} - m_2^2 B_{ab} - \lambda \Phi^* \Phi B_{ab} = 0, \]

where

\[ T_{ab} = \frac{1}{2} [F_{ac} F^c_b - \frac{1}{4} g_{ab} F^2] + \frac{1}{2} [F_{ac} F^c_b - \frac{1}{4} g_{ab} F^2] + \frac{1}{2} ((D_a \Phi)^* D_b \Phi + (D_b \Phi)^* D_a \Phi) \]
\[ + (m_2^2 + \lambda |\Phi|^2)(B_{ac}^* B_{bc}^* + B_{bc}^* B_{ac}^*) + \frac{1}{2\eta} ((D_{[a} B_{ca]})(D_{[b} B^{ca]} + (D_{[b} B^{ca]})(D_{[a} B^{ca]})) \]
\[ - \frac{1}{6\eta} (D_{[m} B_{cd]})(D_{[m} B_{cd]} g_{ab} - \frac{1}{2} ((D_c \Phi)^* (D^c \Phi) + V_1 + V_2 + \lambda |\Phi|^2 B_{cd}^* B^{cd}) g_{ab}. \]
A.1 Zero temperature

With the ansatz in (2.8) the corresponding equations of motion are

\[
\frac{f''}{f} - \frac{u''}{u} + \frac{f'u'}{2fu} - \frac{u'^2}{2u^2} + \frac{4}{f^2} \left( \frac{B^2}{\eta} + \frac{\lambda B^2 \phi^2}{u} + \frac{m_2^2 B^2}{u} \right) = 0, \quad (A.6)
\]

\[
\frac{\phi'^2}{2} + \frac{6}{u} \frac{u}{u} \left( \frac{f'}{f} + \frac{u'}{4u} \right) + \frac{B^2}{\eta f^2} - \frac{\phi'^2}{2u} \left( m_1^2 + \frac{\lambda_1}{2} \phi^2 \right) - \frac{f'^2}{u f^2} - \frac{B^2}{u f^2} \left( m_2^2 + \lambda \phi^2 \right) = 0, \quad (A.7)
\]

\[
\phi'' + \left( \frac{3u' + f'}{2u} + \frac{f'}{f} \right) \phi' - \left( m_1^2 + \frac{\lambda_1}{2} \phi^2 + \frac{2\lambda B^2}{f^2} \phi \right) = 0, \quad (A.8)
\]

\[
\frac{B''}{\eta} + \frac{B'}{\eta} \left( \frac{3u' + f'}{2u} - \frac{f'}{f} \right) - \frac{B}{u} \left( m_2^2 + \lambda \phi^2 \right) = 0. \quad (A.9)
\]

Close to the boundary \( r \to \infty \), we have the following boundary behavior of the fields

\[
u = r^2 - 2b^2 - \frac{M^2}{3} + \left( \frac{4b^4 + 2\lambda b^2 M^2 + M^4}{9} + \frac{\lambda_1 M^4}{6} \right) \ln \frac{r}{r^2} + \frac{u_2}{r^2} + \cdots, \quad (A.10)
\]

\[
f = r^2 - \frac{M^2}{3} + \left( \frac{4b^4 + 2\lambda b^2 M^2 + M^4}{9} + \frac{\lambda_1 M^4}{6} \right) \ln \frac{r}{r^2} + \frac{f_2}{r^2} + \cdots, \quad (A.11)
\]

\[
\phi = \frac{M}{r} + \left( - \frac{M^3}{3} - \frac{\lambda_1 M^3}{2} - b^2 M \lambda \right) \ln \frac{r}{r^3} + \frac{\phi_2}{r^3} + \cdots, \quad (A.12)
\]

\[
B = br + \left( \frac{2b^3 - \lambda b M^2}{2} \right) \ln \frac{r}{r^2} + \frac{b_2}{r^2} + \cdots. \quad (A.13)
\]

with \( f_2 = \frac{1}{144} \left( 56 b^4 + 48 b b_2 + 14 M^4 - 72 u_2 + 4 b^2 M^2 (4 + 7 \lambda) + 9 M^4 \lambda_1 - 72 M \phi_2 \right) \). Note that we have set the non-physical free parameter related to the shift symmetry \( r \to r + c \) in the bulk to be zero in the above expansion. When we extract the boundary data, we should carefully deal with this shift constant.

Radially conserved quantity \( \partial_r J^r = 0 \) with \( J^r = \sqrt{uu'f - u^3/2} f' - \frac{4u^3/2}{f} BB' \). From the near boundary behavior, we have\(^{11} \quad J^r = 4f_2 - 4u_2 + 2\lambda M^2 b^2 \). The following scaling symmetry is useful for rescaling the boundary to be asymptotic to standard AdS\(_5\) and \( b = 1 \).

1. \((x,y) \to a(x,y), (f, B_{xy}) \to a^{-2}(f, B_{xy});
2. \((r \to ar, (x,y,z) \to a(t,x,y,z), (u,f, B_{xy}) \to a^{-2}(u,f, B_{xy}).

\(^{11}\)With the near horizon conditions, we have \( f_2 - u_2 + \frac{\lambda}{2} M^2 b^2 = 0 \) which can be used to check the numerical code.
A.2 Finite temperature

Ansatz for the background solutions at finite temperature are

\[ ds^2 = -udt^2 + \frac{dr^2}{u} + f(dx^2 + dy^2) + hdz^2 \]
\[ \Phi = \phi(r), \]  \hspace{1cm} (A.14)
\[ B_{xy} = B(r). \]

The equations of motion are

\[ f'' - \frac{u''}{u} + \frac{f'h' - h'u'}{2fu} + \frac{4}{f^2} \left( \frac{B^2}{\eta} + \frac{\lambda B^2 \phi^2}{u} + \frac{m_2 B^2}{u} \right) = 0, \]  \hspace{1cm} (A.15)
\[ \frac{\phi'^2}{2} + \frac{6}{u} - \frac{u'}{2u} \left( \frac{f'}{f} + \frac{h'}{2h} \right) - \frac{f'h'}{2fh} + \frac{B'^2}{\eta f^2} - \frac{\phi^2}{2u} \left( \frac{m_1}{2} + \frac{\lambda_1 \phi^2}{2} \right) \]
\[ - \frac{f'^2}{4f^2} - \frac{B^2}{uf^2} \left( \frac{m_2 + \lambda \phi^2}{2} \right) = 0, \]  \hspace{1cm} (A.16)
\[ \phi'' + \left( \frac{f'}{f} + \frac{h'}{2h} + \frac{u'}{u} \right) \phi' - \left( \frac{m_1^2}{f^2} + \frac{2 \lambda B^2 \phi^2}{f^2} + \frac{\lambda_1 \phi^2}{u} \right) \frac{\phi}{u} = 0, \]  \hspace{1cm} (A.17)
\[ \frac{B''}{\eta} + \frac{B'}{\eta} \left( \frac{u'}{u} + \frac{h'}{2h} - \frac{f'}{f} \right) - \frac{B}{u} \left( \frac{m_2^2 + \lambda \phi^2}{2} \right) = 0. \]  \hspace{1cm} (A.18)

B Probe limit

In the probe limit, we assume that the Newton coupling constant is very small so that the backreaction to the geometry could be ignored. We need to check if the probe limit is satisfied for a given solution of \( B \) and \( \phi \). The equations of motion for \( B \) and \( \phi \) in the \( AdS_5 \) background is

\[ B'' + \frac{B'}{r} - \frac{m_b^2 B}{r^2} - \frac{\eta B \phi^2}{2r^2} = 0, \]  \hspace{1cm} (B.1)
\[ \phi'' + \frac{5 \phi'}{r} - \frac{m_5^2 \phi}{r^2} - \frac{\eta B^2 \phi}{r^6} = 0. \]  \hspace{1cm} (B.2)

We have to make sure that the source term of \( B \) and \( \phi \) at both the boundary and the horizon are small enough not to cause too much backreaction. In the probe limit, the IR scaling dimension of \( B \) and \( \phi \) is the same as the UV scaling. Thus it seems that there
is only one scaling dimension that we can use, which is zero for both fields, otherwise, the fields either backreact too much at the horizon or too much at the boundary. In this way, the two fields contribute at $\kappa^2$ order compared to the background and when $\kappa \to 0$ the probe limit is a physically well defined limit for finite solutions.

Let us focus on $m_\phi = 0$ and $m_B = 2$. There are three types of near horizon solutions. The first solution is

$$
\phi \simeq \phi_0 + \phi_1(\phi_0, b_1) r^{\sqrt{16+2\lambda\phi_0^2}-4} + \cdots, \quad (B.3)
$$

$$
B_{xy} \simeq b_1 r^{4+\lambda\phi_0^2/2} + \cdots, \quad (B.4)
$$

where $b_1$ is a tuning parameter and the second solution is

$$
\phi \simeq \phi_1 r^{\sqrt{4+\lambda\phi_0^2}-2} + \cdots, \quad (B.5)
$$

$$
B_{xy} \simeq b_0 r^2 + b_1(b_0, \phi_1) r^{2(\sqrt{4+\lambda\phi_0^2}-1)} \quad (B.6)
$$

where $\phi_1$ is a tuning parameter. The critical solution is $B_{xy} = 1/\sqrt{2}r^2\phi$ and the critical point is $M/b = \sqrt{2}$. This is a special property of the probe limit at the critical point, which is also true for the holographic Weyl semimetal model at the probe limit. Flowing this geometry to the boundary we find that at the boundary, though the scaling dimension of $B$ and $\phi$ guarantees that free $B$ or $\phi$ does not have backreaction in the probe limit, the interaction at the UV makes both $B$ and $\phi$ more divergent than their scaling dimensions, which means that it cannot flow to asymptotic $AdS_5$ solutions.

To solve this problem and make the probe limit well defined, we introduce another scalar field $\lambda$ which mediates the interaction between $B$ and $\phi$ and choose a scaling dimension for $\lambda$ so that the interaction term is not important in the UV so that the probe limit is still valid. We change the interaction term of $\phi$ and $B$ to $\lambda^2 B^2 \phi^2$ and introduce the kinetic term for $\lambda$ in the action with a relative minus sign. The equations of motion now become

$$
B'' + \frac{B'}{r} - \frac{m_B^2 B}{r^2} - \frac{\eta \lambda^2 B \phi^2}{2r^2} = 0, \quad (B.7)
$$

$$
\phi'' + \frac{5\phi'}{r} - \frac{m_\phi^2 \phi}{r^2} - \frac{\eta \lambda^2 B^2 \phi}{r^6} = 0, \quad (B.8)
$$

$$
\lambda'' + \frac{5\lambda'}{r} - \frac{m_\lambda \lambda}{r^2} - \frac{V_0 \lambda^3}{r^2} + \frac{\eta \lambda B^2 \phi^2}{r^6} = 0, \quad (B.9)
$$

where $B$ denotes $B_{xy}$. We choose $m_\lambda^2 = -3$ so that the interaction at the UV is not important. For the interaction to be not important in the IR we introduce a $\lambda^4$ potential term for the $\lambda$ scalar.

The near horizon boundary conditions now have a new field $\lambda = \sqrt{3/V_0} + \cdots$ where the subleading terms in $\lambda$ depends on the phase of the solution. For the other
two fields, the near horizon boundary condition does not change except to substitute $\eta$ with $3\eta/v_0$. Flowing these solutions to the boundary and we could also get a valid probe limit system with three different types of solutions. This probe limit is well defined for most of the parameter regime. However, near the critical point the probe limit becomes subtle because the solutions become larger and larger near the boundary when getting more and more close to the critical point. Thus considering backreactions is a better and more physical choice, and here we do not elaborate more on this probe limit.