Background-Quantum Split Symmetry and Phase-Space Path-Integrals

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Phase-space path-integrals are used in order to illustrate various aspects of a recently proposed interpretation of quantum mechanics as a gauge theory of metaplectic spinor fields.

1 Introduction

In the framework of Hilbert bundles quantum mechanics can be reformulated as a Yang-Mills theory over a symplectic manifold \( (M, \omega) \), with an infinite dimensional gauge group and a nondynamical connection [1]. The “matter fields” in this gauge theory (metaplectic spinors) are local generalizations of states and observables. They assume values in a family of local Hilbert spaces (and their tensor products) which are attached to the points of the phase-space \( \mathcal{M} \). In this approach the rules of canonical quantization are replaced by two new postulates with a very simple group theoretical and differential geometrical interpretation. The first one relates classical mechanics and semiclassical quantum mechanics while the second one, invariance under the background-quantum split symmetry, constructs the exact quantum theory by consistently sewing together an infinity of local semiclassical expansions. In the following we illustrate some features of this theory by means of hamiltonian path-integrals. (For simplicity \( \mathcal{M} \) is taken to be the symplectic plane in these notes.)

2 Semiclassical Quantum Mechanics from Representation Theory

In our approach the passage from classical mechanics to semiclassical quantum mechanics is based upon the following facts and constructions:

(1) There exists a notion of spin bundles and spinor fields appropriate for phase-spaces. Under local frame rotations these so-called metaplectic spinors

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We identify the operators $H$ of the fluctuations $x$ of the Schrödinger equation in (4). This implies that $\eta^{\nu}(\phi) \equiv \langle x|\psi\rangle_\phi$ “lives” in the local Hilbert space $V_\phi$, a copy of $V$ attached to the point $\phi$.

(3) Every hamiltonian vector field $h^{\alpha} = \omega^{ab} \partial_b H$ gives rise to a Lie-derivative $\ell_{H} \psi = h^{a} \partial_a \psi + \frac{i}{2} \partial_a \partial_b H \Sigma^{ab} \psi$ and similarly for arbitrary spinors/tensors $\chi^{\nu,\ldots,b}(\phi)$. Their Lie-transport along the hamiltonian flow is governed by the equation $-\partial_t \chi(\phi, t) = \ell_{H} \chi(\phi, t)$. In particular, we consider singular fields $\psi^{\nu}(\phi, t) = \eta^{\nu}(t) \delta(\phi - \Phi_{cl}(t))$ localized along solutions $\Phi_{cl}(t)$ of Hamilton’s equation $\partial_t \Phi_{cl}^{a} = h^{a}(\Phi_{cl})$. The resulting evolution equation for the “world line spinor” $\eta$ reads $i\partial_t \eta = \frac{i}{2} \partial_t \partial_b H(\Phi_{cl}(t)) \Sigma^{ab} \eta$. It has the same structure as the classical Jacobi equation $\partial_t \delta \phi^a = \partial_b h^{ab}(\Phi_{cl}) \delta \phi^b$ but with the vector representation of $Sp(2N)$ replaced by its spinor representation.

(4) Given a hamiltonian $H$ we consider the standard phase-space path-integral $\langle q_2, t_2|q_1, t_1\rangle_H = \int Dp(t) \int Dq(t) \exp \left[ i \int_{t_1}^{t_2} dt \{ p^i q^j - H(p, q) \} \right]$ over paths $\phi(t) \equiv (p^i(t), q^j(t))$ satisfying $q(t_{1,2}) = q_{1,2}$. We shift $\phi(t)$ by an arbitrary classical trajectory $\Phi_{cl}(t) \equiv (p_{cl}(t), q_{cl}(t))$, i.e., $\phi^a(t) = \Phi_{cl}^a(t) + \varphi^a(t)$ and $\partial_t \varphi^a(t) = 0$. Without any approximation or assumption about the terminal points of $\Phi_{cl}$, we obtain $\langle q_2, t_2|q_1, t_1\rangle_H = \exp \left[ i \left( \{ S_{cl} + p_{cl}^i(t_2) x_2^i - p_{cl}^i(t_1) x_1^i \} \right) \right] K_H$. Here $K_H \equiv \langle x_2, t_2|x_1, t_1\rangle_H$ is the amplitude related to the shifted integral $K_H = \int D\pi \int Dx \exp \left[ i \int_{t_1}^{t_2} dt \{ \pi^i \dot{x}^i - H(\varphi; \Phi_{cl}) \} \right]$ with boundary conditions $x(t_{1,2}) = x_{1,2} \equiv q_{1,2} - q_{cl}(t_{1,2})$. It contains the hamiltonian $H(\varphi; \Phi_{cl}) \equiv H(\Phi_{cl} + \varphi) - \varphi^a \partial_a H(\Phi_{cl}) - H(\Phi_{cl})$. The Schrödinger equation equivalent to the shifted path-integral reads $[i\partial_t - H(\varphi; \Phi_{cl})]|\langle x, t|x_1, t_1\rangle_H = 0$.

(5) We identify the operators $\hat{\varphi}^a \equiv (\hat{\pi}^i, \hat{\pi}^i)$ appearing in the canonical quantization of the fluctuations $\varphi^a(t)$ with those of the auxiliary system which were introduced in order to represent the Clifford algebra.

(6) In the semiclassical approximation one has $H(\hat{\varphi}; \Phi_{cl}) = \frac{i}{2} \partial_a \partial_b H(\Phi_{cl}) \hat{\varphi}^a \hat{\varphi}^b + O(\hat{\varphi}^3)$ which is in the Lie algebra of $Mp(2N)$. This implies that $\eta^{\nu}(t) \equiv \langle x, t|x_1, t_1\rangle_H$ is a world-line spinor: the equation of motion given in (3) coincides exactly with the semiclassical approximation of the Schrödinger equation in (4).
The upshot of the preceding arguments is that the notion of semiclassical wave functions follows from the classical concept of the Jacobi fields by simply passing over from the vector to the spinor representation of \( \text{Sp}(2N) \).

### 3 Exact Quantum Mechanics from Split Symmetry

Within this Hilbert bundle approach, the natural description of semiclassical quantum mechanics is in terms of singular spinor fields with support along \( \Phi_{cl}(t) \) only. The exact theory is formulated in terms of smooth fields \( \psi^a(\phi) = \langle x|\psi\rangle_\phi \) defined everywhere on \( \mathcal{M} \). Loosely speaking, a single field \( \psi^a(\phi) \) encapsulates the information contents of the world line spinors along the totality of all classical paths. Heuristically, this can be understood as follows. Let us assume we know the solutions \( \eta(t) \) of the semiclassical Schrödinger equation for a congruence of neighboring classical trajectories \( \Phi(t) \). We expect the wave function \( \eta_1 \) belonging to some trajectory \( \Phi_1 \) to provide a reasonable approximation to the complete theory within a tubular neighborhood of \( \Phi_1(t) \). Furthermore, let us suppose that there is a nearby solution \( \Phi_2(t) \) such that the tubular neighborhood within which its wave function \( \eta_2(t) \) is valid overlaps with the one of \( \Phi_1 \). As a consequence, there is a region in phase-space where both semiclassical expansions apply and where their predictions must agree. For instance, the expectation value of the “position” is given by \( \Phi_1(t) + \hat{\eta}_1(t) \hat{\varphi}^a \eta_1(t) \) according to the first, and by \( \Phi_2(t) + \hat{\eta}_2(t) \hat{\varphi}^a \eta_2(t) \) according to the second expansion. (Here \( \hat{\varphi}^a \eta \equiv \int dx \int dy \eta_x^a(\varphi)^x_y \eta^y \).) Invariance under the background-quantum split symmetry means that these two values coincide.

The time-independent spinor field \( \psi(\phi) = |\psi\rangle_\phi \) which describes this situation in the exact theory is constructed as follows. Since at different times \( t \) the world-line spinor \( \eta(t) \in \mathcal{V}_{\Phi(t)} \) lives in different local Hilbert spaces along the trajectory \( \Phi(t) \) we define the value of \( \psi(\phi) \) along \( \phi = \Phi(t) \) by setting \( |\psi\rangle_{\Phi(t)} = \eta(t) \) for all \( t \). Then, in order to find \( \psi(\phi) \) in the directions transverse to \( \Phi(t) \), we repeat this construction for all pairs \( (\Phi, \eta) \) which are connected by the requirement of the split symmetry. It can be shown that this procedure leads to a spinor field \( \psi^a(\phi) \) which satisfies an equation of the form \( (\partial_a + i\hat{\Gamma}_a)\psi = 0 \) where \( \hat{\Gamma}_a \) is a set of hermitian operators on \( \mathcal{V} \). In the gauge theory formulation this condition means that \( \psi \) is covariantly constant with respect to a universal connection (gauge field) \( \hat{\Gamma}_a \). This connection can be found by looking at how the shifted path-integral responds to the replacement \( \Phi_{cl} \to \Phi_{cl} + \delta\phi \) where \( \delta\phi \equiv (\delta p_{c1}, \delta q_{c1}) \) is a Jacobi field. The result reads \( \delta K_H = [\delta q_{c1}(t_2)\partial_{x_2} + \delta q_{c1}(t_1)\partial_{x_1} - ix_2\delta p_{c1}(t_2) + ix_1\delta p_{c1}(t_1)]K_H \). This equation describes an infinitesimal parallel transport of the spinor \( K_H \) with the connection \( \hat{\Gamma}_a \). Here \( K_H \) is considered a function of \( x_2, t_2 \) with \( x_1, t_1 \) fixed. Thus we read off that the universal connection \( \hat{\Gamma} \) is given by \( \hat{\Gamma}_a = \omega_{ab} \hat{\varphi}^b \) which happens to be independent of \( \phi \) for a flat phase-space.
We interpret \( \langle \hat{\varphi}^a \rangle = \phi^a + \phi \langle \hat{\varphi}^a | \psi \rangle \) as the exact quantum mechanical expectation value of the position in phase-space. The requirement of the split symmetry means that this value is the same for all pairs \(( \phi, | \psi \rangle )\). When we go from the point \( \phi \) to another point \( \phi + \delta \phi \) the state \(| \psi \rangle \phi \in \mathcal{V}_\phi \) is replaced by a new state \(| \psi \rangle_{\phi + \delta \phi} \in \mathcal{V}_{\phi + \delta \phi} \) in such a way that the change of the \( \varphi \)-expectation value precisely cancels the change of the classical contribution, \( \delta \phi^a \). This condition is fulfilled if the state vector changes by an amount \( \delta | \psi \rangle \phi = -i \delta \phi^a \hat{\Gamma}_a | \psi \rangle \phi \), i.e. if it is parallel-transported with the connection \( \hat{\Gamma} \).

This parallel transport is "almost integrable". Since the curvature of \( \hat{\Gamma}_a \) is \( \omega_{ab} \) times the unit operator on \( \mathcal{V} \), only the (irrelevant) phase of the transported spinor is path-dependent.

Classical observables \( f(\phi) \) are represented by covariantly constant operators \( \mathcal{O}_f(\phi) \) which generalize \( \phi^a + \hat{\varphi}^a \) above. For \( \mathcal{M} \) flat one has \( \mathcal{O}_f(\phi) = f(\phi^a) \).

At the level of the \( f \)'s the operator product of the \( \mathcal{O} \)'s corresponds to the star product of the Weyl symbol calculus.

The equivalence of the gauge theory approach with the standard formulation of quantum mechanics is established by noting that the parallel transport with \( \hat{\Gamma} \) can be used to identify all local Hilbert spaces \( \mathcal{V}_\phi \) with a single reference Hilbert space at \( \phi = \phi_0 \), say. This reference Hilbert space is the one used in the standard approach and \( \psi^\tau (\phi_0) \equiv \Psi(x) \), regarded as a function of \( x \), is the ordinary wave function.

Time evolution is described by a local unitary frame rotation: \( i \partial_t \psi(\phi) = \mathcal{O}_H(\phi) \psi(\phi) \). This fundamental dynamical law unifies two rather different types of time evolution: the Lie-transport along the hamiltonian flow and the standard Schrödinger equation.

For further details and the generalization of arbitrary curved phase-spaces we refer to the literature [1]. One of the central results is that the structures of quantum mechanics emerge from those of classical mechanics by (i) switching from the vector to the spinor representation of \( \text{Sp}(2N) \) and (ii) imposing the split symmetry. On curved phase-spaces the latter involves a quantum-deformed symplectic analog of the exponential map.

[1] M. Reuter, Int. J. Mod. Phys. A13 (1998) 3835 and hep-th/9804036