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Study of the structures of four-quark states in terms of the Born-Oppenheimer approximation

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Abstract: In this work, we use the Born-Oppenheimer approximation, where the potential between atoms can be approximated as a function of distance between the two nuclei, to study the four-quark bound states. By this approximation, Heitler and London calculated the spectrum of the hydrogen molecule, which includes two protons (heavy) and two electrons (light). Generally, the observed exotic mesons Z_b(10610), Z_b(10650), Z_c(3900) and Z_c(4020) (Z_c(4025)) may be molecular states made of two physical mesons and/or diquark-anti-diquark structures. Analogous to the Heitler-London method for calculating the mass of the hydrogen molecule, we investigate whether there exist energy minima for these two structures. Contrary to the hydrogen molecule case where only the spin-triplet possesses an energy minimum, there exist minima for both of these states. This implies that both molecule and tetraquark states can be stable objects. Since they have the same quantum numbers, however, the two states may mix to result in the physical states. A consequence would be that partner exotic states co-existing with Z_b(10610), Z_b(10650), Z_c(3900) and Z_c(4020) (Z_c(4025)) are predicted and should be experimentally observed.

Key words: exotic states, Born-Oppenheimer approximation, molecule, tetraquark

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1 Introduction

The naive quark model suggests that a meson is made of a quark and an anti-quark, whereas a baryon consists of three quarks. The constituents in hadrons are bound together by the QCD interaction to constitute a color singlet. However, neither the quark model nor the QCD theory ever forbids the existence of multi-quark states as long as they are color-singlets. The fact that after several years of hard work all experimental trials to observe pentaquarks have failed, has greatly encouraged high energy physicists and experimentalists, even though the idea of pentaquarks is really stimulating. One may ask if nature indeed only favors the most economic structures called exotic states.

The inner structure of the multi-quark states is more complicated than the regular mesons, in that the exotic states can be molecular states or tetraquarks or their mixtures. The molecular state is constructed by two color singlet mesons. A strong point in support of such a structure is that the mass of the newly discovered meson Z_b(10610) is close to the sum of the masses of B and \bar{B}, while the mass of Z_c(3900) is also close to a sum of D and \bar{D} masses. The study of the decay modes of such mesons, however, seems to support the tetraquark structure [5, 6]. To clarify the structures of those exotic states, one may need to investigate their overall characteristics based on fundamental dynamics, instead of simply considering the closeness of their masses to the sum of the constituents.
One observation may draw our attention. The resonances $Z_0(10610)$, $Z_0(10650)$, $Z_0(3900)$, $Z_0(4020)$ and $Z_0(4025)$ have been experimentally observed and confirmed as exotic four-quark states. Many authors [7–11] have assumed them to be molecular states of $B$, $B^*$, $D$ and $D^*$ (and the corresponding anti-mesons) which are well-measured experimentally. A common point is that the masses of the observed exotic mesons are larger than the sum of the supposed constituent mesons. Specifically, $10608.4\pm2.0$ MeV [1] (the mass of $Z_0(10610)$) is larger than the sum of the masses of $B$ and $B^*$ (10604.45 MeV); $10653.2\pm1.5$ MeV [1] (the mass of $Z_0(10650)$) is larger than the sum of the masses of $B^*$ and $B^*$ (10650.4 MeV); $3899\pm3.6\pm4.9$ MeV [2] (the mass of $Z_0(3900)$) is larger than the sum of $D$ and $D^*$ (3876.6 MeV); $4022.9\pm0.8\pm2.7$ MeV [3] (the mass of $Z_0(4020)$) is larger than the sum of $D^*$ and $D^*$ (4013.96 MeV). Generally, unless there exists a linearly increasing potential (such as the confinement potential for quarks) or a barrier, the binding energy of two constituent mesons which is caused by exchanging color-singlet hadrons must be negative. Thus, the mass of a composite meson should be smaller than the sum of the two (or more) constituent masses. Moreover, as estimated by some authors [12, 13], the masses of those exotic states are also larger than the sums of the masses of the diquark and anti-diquark concerned. In our calculation, even though the sum of the two diquark masses is larger than the mass of the corresponding exotic meson, the negative binding energy still makes the resultant total energy smaller than the exotic meson. This may imply that neither molecular nor tetraquark states alone correspond to the observed exotic mesons. Our study indicates that only their mixture provides a reasonable picture for the four-quark states. Thus, both molecular and tetraquark states should exist, even though they may not be the physical states which we observe in experiments.

By the Born-Oppenheimer [14] approximation, the potential between atoms can be approximated as a function of the distance between the two nuclei; by this scheme, Heitler and London [15] calculated the spectrum of the hydrogen molecule. In that case, the two protons are supposed to be at rest and the two electrons are moving. Since the two electrons are identical fermions, the wavefunction of the two-electron system must be totally anti-symmetric. It was found that there is only one energy minimum corresponding to the triplet. Namely, in the hydrogen molecule the two electrons must be in the spin-triplet.

Compared with the hydrogen molecule, $Z_0$ (or $Z_0$) is made of four quarks: $Q$, $\bar{Q}$, $u(\bar{u})$, $d(\bar{d})$, where $Q$ stands for $b$ or $c$ quark. Since $Q$, $\bar{Q}$ are much heavier than the light flavors, we can approximate them to be at rest. Thus it is natural that we separate the four quarks into two groups. One possibility is that each group is in a color singlet, which corresponds to a molecular state, whereas another possibility is that one group containing $Qu$ is in a color-anti-triplet (or a sextet) and the other group containing $Q\bar{d}$ is in a color-triplet (or an anti-sextet), i.e. the dipole-anti-dipole structure. Since $u$ and $d$ are not identical particles, the wavefunction does not need to be anti-symmetrized. By the Born-Oppenheimer approximation, the potential between two groups can be a function of distance between $Q$ and $\bar{Q}$ and interactions between the two groups are taken as a perturbation. Since the interactions between quarks are complicated, calculation of the energy spectrum of the exotic states is much more difficult than for the hydrogen molecule. It is noted that Braaten et al. [16] also consider the Born-Oppenheimer potential to deal with the four-quark states.

First we need to determine the wavefunctions of the color singlet of $Q\bar{d}(Qu)$ and the color-anti-triplet (or sextet) dipole $Q\bar{u}$ ($Q\bar{d}$-triplet or anti-sextet $Qd$). Here we use the Cornell potential [17, 18] as the interaction between the quarks and since the light flavors are relativistic, following the literature, we employ the Schrödinger-like equation with relativistic kinematics. The effective interaction between the quarks (quark-anti-quark) which belong to different groups is complicated, because not only is there the short-distance QCD interaction, but the long-distance interaction, which can be treated by exchanging color-singlet light mesons such as $\pi$, $\sigma$ and $\rho$ (for the molecule case) or the color-flux tube (for the tetraquark case), plays an important role. Here we do not involve the strange flavor. Analogous to the hydrogen molecule, we calculate the spectrum of the ground state of the four-quark system (the molecule and tetraquark separately). Our strategy is similar to the Heitler-London approximation, namely we take the products of the two meson wavefunctions (for the molecule) and diquark-anti-diquark wavefunctions (for the tetraquark) as two independent trial functions and calculate the interaction between the two groups to obtain the spectra as functions of the distance between $b$ and $\bar{b}$ ($c$ and $\bar{c}$). Our goal is to see whether the molecular state or tetraquark state can possess energy minima with respect to the distance between $Q$ and $\bar{Q}$, by which one can judge if the molecule or tetraquark is physically allowed. If there exist minima for both cases, we can conclude that both structures are probable and the real physical state could be a mixture of the two structures. (In fact, our computations confirm that there are minima for both.)

This work is organized as follows. After this long introduction, we formulate the expressions for the energy spectra. We first present relevant effective potentials for the meson and diquark composed of a heavy quark and
a light flavor, and then derive the Born-Oppenheimer potentials for both molecule and tetraquark. In Section 2, we discuss the explicit color and spin structures of the molecular and tetraquark states and solve the Schrödinger-like equation to obtain the spatial wavefunctions of color-singlet meson and color-anti-triplet diquark. In Section 3, along with all input parameters, we present our numerical results, which show that for both molecule and tetraquark there exist minima with respect to the distance between Q and Q. The last section is devoted to the discussion and conclusions.

2 Derivation of the relevant formulae

In this section, we derive the theoretical formulae for calculating the mass spectra and wavefunctions of both molecular and tetraquark states. We first solve the Schrödinger-like equations to obtain the mass spectra and wavefunctions of the mesons B, B∗, D and D∗ and diquark (anti-diquark), which will be the trial functions for later calculations. Note that since the diquark is not a physical state, we determine its mass spectrum and wavefunction via theoretical computations. We go on using the Born-Oppenheimer approximation to evaluate the mass spectra of molecular and tetraquark states as functions of the distance between the two heavy constituents Q and Q.

2.1 Derivation of potentials

Here we first obtain the effective potentials between the relevant constituents inside a color-singlet, i.e. mesons and color-triplet (anti-triplet), i.e. anti-diquark (diquark). Then we go on to derive the potential between constituents from the different groups. For the two distinct configurations (the molecular state and tetraquark (diquark-antidiquark) state) (see Fig. 1), the effective interactions are different.

Fig. 1. Configurations of the four-quark system (left: molecular state, right: tetraquark state).

2.1.1 Meson and diquark (antidiquark) states

In this subsection, let us first discuss the interactions among the constituents inside a meson (qQ(qQ)) or an (anti) diquark (qQ(qQ)). The general Hamiltonian can be written as

\[ H = \sqrt{p_i^2 + m_i^2} + \sqrt{P_j^2 + m_j^2} + V(r), \]

\[ i = q(q); j = Q(Q). \]

where the \( p_i \) and \( P_j \) are the 3-momenta of the light flavor q(q) and heavy flavor Q(Q) respectively. The interaction potential is

\[ V(r_{ij}) = V_{\text{me}}(r_{ij}) + V_{\text{con}}(r_{ij}), \]

and \( r_{ij} \) is the distance between the quarks (quark-antiquark). The one-gluon exchange (oge) term \( V_{\text{con}}(r_{ij}) \), which plays the main role at short distances, is [19]

\[ V_{\text{con}}(r_{ij}) = -4 \left( \lambda_i^2 \lambda_j^2 \right) b(r_{ij}) + c, \]

and the confinement part \( V_{\text{con}}(r_{ij}) \) takes the linear form [17]

\[ V_{\text{con}}(r_{ij}) = 1 \left( \lambda_i^2 \lambda_j^2 \right) (b r_{ij} + c) \]

where \( \lambda_i \) and \( \lambda_j \) are, respectively, the color SU(3) and spin operators acting on quark i, and \( m_i \) is the quark mass. \( b \) is the string tension, and \( c \) is a global zero-point energy. \( \alpha_s \) is the QCD running coupling constant, which depends on the re-normalization scale \( \mu^2 \) [20]

\[ \alpha_s(\mu^2) = \frac{\alpha_0}{\ln \left( \frac{\mu^2 + \mu_0^2}{\Lambda^2} \right)}, \]

where \( \mu = m_i m_j / (m_i + m_j) \) is the reduced mass of the qQ system and \( \Lambda, \alpha_0, \mu_0 \) are fitted parameters. The framework can be generalized to the case for a diquark (anti-diquark) which involves two quarks (two anti-quarks).

The \( \delta \)-function in Eq. (3) is replaced by a Gaussian smearing function [21] with a fitted parameter \( h \)

\[ \delta(r_{ij}) \rightarrow \frac{1}{\pi h^2} e^{-h^2 r_{ij}^2}. \]

2.1.2 Molecular states

Now we specify the interaction between the two mesons for the molecular structures (Fig. 1, left). Since the constituent mesons are in a color singlet, the quarks (antiquarks) in one meson do not interact with the quarks in another meson via exchanging a single gluon, thus the interaction between B∗B∗ (or D∗D∗) only comes from meson-exchange between the light flavors qQ.

The constituent quark model has been thoroughly studied by many authors, for example, Vijnade et al. [20, 22], and its successful applications to phenomenology are noted, thus we here employ it to derive the effective interaction between mesons. The interactions \( V_{\text{me}}(r_{ij}) \) induced by meson-exchange(m) between q and q include the contributions of pseudoscalar (p) and scalar (s),

\[ V_{\text{me}}(r_{ij}) = \sum_{a=1}^{3} \sum_{b=4}^{7} V_{a}(r_{ij}) F_{a}^{v} F_{b}^{v} + \sum_{a=4}^{7} V_{a}(r_{ij}) F_{a}^{v}, \]

\[ + V_{a}(r_{ij}) [\cos \theta_{\mu} (F_{a}^{v} F_{a}^{v}) - \sin \theta_{\mu}] + V_{a}(r_{ij}), \]
and the explicit forms of the interactions are

\[ V_\chi(r_{ij}) = \frac{g_{sb}^2}{4\pi} \frac{m_\chi^2}{12m_{m_j}} \frac{A_{\chi}^2}{m_\chi^2} \chi(\sigma, \sigma_j) \left[ Y(m_\chi r_{ij}) - \frac{A_{\chi}^2}{m_\chi^2} Y(A_{\chi} r_{ij}) \right] \]  

\[ V_\sigma(r_{ij}) = \frac{g_{sb}^2}{4\pi} \frac{A_{\sigma}^2}{m_\sigma^2} \sigma(\sigma, \sigma_j) \left[ Y(m_\sigma r_{ij}) - \frac{A_{\sigma}^2}{m_\sigma^2} Y(A_{\sigma} r_{ij}) \right] \]  

with \( \chi=\pi, \eta \) and \( \mathcal{F}^a(a=1, 2, \ldots, 8) \) being the SU(3) flavor matrices. \( Y(x) = e^{-x} / x \) is the Yukawa function, \( g_{sb} \) is the chiral coupling constant, \( \theta_\sigma \) is the mixing angle for the physical \( \eta \) and \( \eta' \), and the \( \tilde{A} \) are the chiral symmetry breaking scales. Once the potential between \( q\bar{q} \) is determined, the corresponding potential for \( q\bar{q} \) can also be obtained from a G-parity transformation \[23\]. It is noted that the employed framework is the SU(3) chiral quark model where the heavy quark (c or b) does not couple to the SU(3) mesons.

Furthermore, in the effective potential there also exists a part \( V_{\text{ann}}(r_{ij}) \) induced by quark-antiquark pair annihilation into light mesons which mediate interactions in the s-channel. To the lowest order the quark-antiquark pair resides in an S-wave state and the contribution of the \( \sigma \)-meson \( (J^{PC}=0^{-+}) \) can be neglected \[23\]. So here we only keep the contributions of \( \pi \) and \( \rho \) to the potential \[24, 25\]

\[ V_{\text{ann}, \pi}(r_{ij}) = \frac{g_{sb}^2}{4m_\pi^2 - m_\pi^2} \left[ \frac{1}{3} \left( \frac{1}{2} \hat{\lambda}_i \cdot \hat{\lambda}_j \right) \right] \times \left( \frac{1}{2} \hat{\sigma}_i \cdot \hat{\sigma}_j \right) \]  

\[ V_{\text{ann}, \rho}(r_{ij}) = \frac{g_{sb}^2}{4m_\rho^2 - m_\pi^2} \left[ \frac{1}{3} \left( \frac{1}{2} \hat{\lambda}_i \cdot \hat{\lambda}_j \right) \right] \times \left( \frac{3}{2} \hat{\sigma}_i \cdot \hat{\sigma}_j \right) \times \left( \frac{1}{2} \hat{\tau}_i \cdot \hat{\tau}_j \right) \]  

\( \tau \) is the isospin operator, and the \( \delta \)-function is also rewritten in the same form as Eq. (6).

Summing all the individual parts, the interaction between the two mesons of the molecule is

\[ H_{\text{int}}^{(\text{mol})} = V_\pi(r_{ij}) + V_\text{ann}(r_{ij}). \]  

2.1.3 Tetraquark states

For the case of the tetraquark, we are dealing with the interaction between the two groups \( qQ \) and \( qQ \). The key point is to derive an effective potential. The total Hamiltonian is written as

\[ H_{\text{int}}^{(\text{tetra})} = \sum_{i=x,y,z} \left[ V_{\text{ee}}(r_{ij}) + V_{\text{con}}(r_{ij}) \right]. \]  

The interaction among the constituents in the diquark and anti-diquark is not simply determined by perturbative QCD, because the short-distance and long-distance contributions exist simultaneously. Following Brodsky et al. \[26\], the flux tube model may properly describe the interaction for the tetraquark case. Meanwhile in this case the contribution of meson exchange can be safely ignored compared with that of gluon exchange \[27\]. The general form of Hamiltonian in the flux-tube model can also be decomposed into the Coulomb-type part, which is responsible for short distance interaction, and the confinement part, for long-distance interaction. As Brodsky et al. \[26\] suggested, in a “substantial separation”, diquark and anti-diquark are connected by the flux-tube. It is noted that in our pictures according to the Heitler-London approximation, we need to consider all the interactions among the constituents of different groups, thus we account for the interactions as shown on the right-hand side of Fig. 1. Obviously, summing over all the contributions a resultant Born-Oppenheimer potential would be obtained, which is also an effective flux tube between the diquark and anti-diquark and is the picture from Ref. \[26\]. Moreover, as is well known, when the tension on the string goes beyond a certain bound, the string will break into two strings and at the new ends a quark-anti-quark pair is created \[28, 29\]. One can use a step function to describe the breaking effect as

\[ \theta(r_{ij}+c)\theta(r-r_0), \]  

where \( r_0 \) is a parameter corresponding to the strength limit of the string. A typical scale for non-perturbative QCD is \( \Lambda_{\text{QCD}} \), therefore it is natural to consider \( r_0 = 1/\Lambda_{\text{QCD}} \). Just as for smearing the delta function, we need also to smear the step function. In fact

\[ \theta(r-r_0) = \lim_{\varepsilon \to 0} \frac{1}{e^{(r_{ij}-r_0)/\varepsilon} + 1}, \]  

so smearing the step function implies that we keep \( \varepsilon \) as a non-zero free parameter to be determined.

Here the interaction between \( q \) from the diquark and \( \bar{q} \) from the antidiquark at a relatively large distance is described by a modified form as

\[ V_{\text{con}}(r_{ij}) = \frac{1}{4} \left( \hat{\lambda}_i \cdot \hat{\lambda}_j \right) (br_{ij}+c) \frac{1}{e^{(r_{ij}-r_0)/\varepsilon} + 1}, \]  

where \( \varepsilon \) is a parameter in fm, and we set \( \Lambda_{\text{QCD}} = 280 \text{MeV} \) in this work.

2.2 Wave functions of four-quark states

Combining all the degrees of freedom of the constituent quarks, the total wave function is a direct product of the radial, spin, color, and isospin (flavor) parts:

\[ |\Psi_{\text{tetra}}\rangle = |C_a\rangle \otimes |I_a\rangle \otimes |\phi_a\rangle \otimes |S_a\rangle \rangle^M, \]  

\( a = (\text{mol}), (\text{tetra}) \).
For molecular state and tetraquark state separately, unlike the hydrogen molecules, the quarks (antiquarks) involved are not identical, so the Pauli principle does not impose any restrictions on the compositions.

2.2.1 Radial wave function

In the essence of the Born-Oppenheimer approximation, we can choose the product of the two clusters’ wavefunctions as the basis shown in Fig. 1

\[ \phi_{(\text{mol})} = \phi_{uQ} \otimes \phi_{\bar{d}Q}, \quad \phi_{(\text{tetra})} = \phi_{uQ} \otimes \phi_{\bar{d}Q}. \quad (17) \]

The radial wave function for each cluster is obtained by solving the Schrödinger-like equation

\[ \left[ \sqrt{P^2 + m^2} + \sqrt{P^2 + m^2} + V(r) \right] \phi_\kappa = E \phi_\kappa, \]

\[ \kappa = uQ.dQ, uQ.dQ, \quad (18) \]

where the potential \( V(r) \) takes the Cornell type potential (see Eq. (3) and Eq. (4)), \( m_u \) and \( m_d \) are the masses of light (u, d) and heavy (c, b) quarks. It applies to both meson and diquark cases with different color factors.

We solve the Schrödinger-like equation numerically using the program offered by the authors of Ref. [30, 31] to deduce the radial wavefunction \( u(r) \), defined as \( \phi_\kappa(r) = \frac{u_\kappa(r)}{r} Y_{lm}(\hat{r}) \), with \( l = 0 \). In Fig. 2 the wavefunctions of B* and D* are shown. The eigenvalues are given in Table 1 where the constituent quark masses are input parameters.

2.2.2 Color factors in the wave function

We now turn to discuss the color part of the four-quark states. The color singlet state of a four-quark system is constructed as follows:

\[ |3_uq \otimes 3_Q\rangle, \quad |6_uq \otimes 6_Q\rangle, \quad (19) \]

\[ |1_uq \otimes 1_Q\rangle, \quad |8_uq \otimes 8_Q\rangle, \quad (20) \]

\[ |1_uq \otimes 1_Q\rangle, \quad |8_uq \otimes 8_Q\rangle, \quad (21) \]

which stand as three orthonormal basis-vectors. The expression in Eq. (19) is the so-called tetraquark state with a diquark-anti-diquark structure; we only consider the state \( |3_uq \otimes 3_Q\rangle \) (denoted as \( C_{(\text{tetra})} \)) here [26]. Eq. (20) and Eq. (21) are for the molecular states with a meson-meson structure; in particular, the state \( |1_uq \otimes 1_Q\rangle \) (denoted as \( C_{(\text{mol})} \)) corresponds to the B*+B* (or D*+D*), which is the concern of this work.

The three basis vectors are related to each other through rearrangements [33]

\[ |1_uq \otimes 1_Q\rangle = \sqrt{\frac{2}{3}} |3_uq \otimes 3_Q\rangle + \sqrt{\frac{1}{3}} |6_uq \otimes 6_Q\rangle, \quad (22) \]

\[ |8_uq \otimes 8_Q\rangle = -\sqrt{\frac{2}{3}} |3_uq \otimes 3_Q\rangle + \sqrt{\frac{1}{3}} |6_uq \otimes 6_Q\rangle, \quad (23) \]

and

\[ |1_uq \otimes 1_Q\rangle = -\sqrt{\frac{1}{3}} |3_uq \otimes 3_Q\rangle + \sqrt{\frac{2}{3}} |6_uq \otimes 6_Q\rangle, \quad (24) \]

\[ |8_uq \otimes 8_Q\rangle = \sqrt{\frac{2}{3}} |3_uq \otimes 3_Q\rangle + \sqrt{\frac{1}{3}} |6_uq \otimes 6_Q\rangle. \quad (25) \]

The color matrix elements which we need in Sec. 3 are summarized in Table 2.

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**Fig. 2.** (color online) The reduced wavefunction \( u(r) \) in coordinate space. (a) The solid curve is for B and the dashed for B*. (b) The solid curve is for D and the dashed for D*.

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**Table 1.** Masses of heavy mesons and diquark (with spin-0 and spin-1) calculated by solving the Schrödinger-like equation, with experimental data [32] and results from QCD sum rules presented for comparison.

|                  | B                | B*               | D                | D*               |
|------------------|------------------|------------------|------------------|------------------|
| **Exp./MeV**     | 5279.26±0.17     | 5325.2±0.4       | 1864.84±0.7      | 2010.26±0.07     |
| **this work/GeV**| 5.279            | 5.325            | 1.863            | 2.010            |
| **diquarks**     | (bq)s=0          | (bq)s=1          | (cq)s=0          | (cq)s=1          |
| **this work/GeV**| 5.344            | 5.355            | 1.963            | 2.00             |
| **QCD sum rules [13]/GeV** | 5.08±0.04 | 5.08±0.04 | 1.86±0.05 | 1.87±0.10 |

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the next section confirm that neither molecular state nor
tetraquark but their mixtures correspond to the observed
clusters are taken as a perturbation. In this work, al-
tween the two clusters, we do assume it. We will de-
rive an explicit form for the effective interaction in a fu-
ture work. Using the wave function described above, we
can calculate the binding energy with the Heitler-London
method. The binding energy is

\[ W_{a} = \langle \psi_{a} | H_{\text{int}}^{a} | \psi_{a} \rangle, \]  

(26)

where \( H_{\text{int}}^{a} \) (see Sec.2.1 for details) is a perturbative term
for both the molecular structure and the tetraquark.

In this work, we take the meson-quark coupling con-
stants \( g_{a} \) and cut-off parameters \( \Lambda_{a} \) from Ref. [20].
The masses of the light mesons are taken from the Particle
Data Group (PDG) values [32], and the other para-
eters, such as \( b, c, h, \alpha_{0} \) etc., have been determined by
fitting the heavy meson spectra (see Table 1). They are
presented in Table 4.

3.1 Molecular structure

In this subsection, we discuss the case of molecular
structure. In terms of the obtained wave functions and
eigen-energies of the two constituent mesons, we estimate
the expectation values shown in Eq. (26). The color,
spin and flavor parts of Eq. (26) are shown in Table 2
and Table 3, and integration of the radial part is carried
out numerically. The binding energy of molecular states
\( W_{\text{mol}}^{a} \) versus the distance between \( Q \) and \( \bar{Q} \) is drawn
in Fig. 3. The plots indicate that there exist minima \( E_{\text{mol}}^{a} \) for all the states concerned. As we expect, in the
Born-Oppenheimer approximation, a molecular state
of the four-quark system possesses a minimum which cor-
responds to a stable structure. Then, the masses of the molecular states are \( M_{\text{mol}} = m_{1} + m_{2} + E_{\text{mol}}^{a} \), where \( m_{1} \)
and \( m_{2} \) are the masses of the constituent mesons, and
are presented in Table 5.

Here we define \( R \) as the distance between \( Q \) and \( \bar{Q} \).
The minima are located at around \( R \sim 1 \) fm, and the \( B^{(*)} \),
\( D^{(*)} \) structures can be considered as loosely
bound states with binding energies of \(-3--5 \) MeV.

3.2 Tetraquark structure

Now let us turn to discuss the tetraquark case. With
the same procedure as for the molecular states, we ob-
tain the dependence of the binding energies of tetraquark
\( W_{\text{tetra}} \) on the distance between \( Q \) and \( \bar{Q} \) with various
values of the parameter \( \varepsilon \). The results are shown in Fig. 4
and Fig. 5.

Interestingly, we find that there indeed exists a mini-
mum \( E_{\text{tetra}} \) with respect to the distance between \( Q \)

| Table 2. Color matrix elements [34]. |
|--------------------------------------|
| \( \langle \alpha_{a} | \lambda_{(1)} \rangle \) | \( \langle \lambda_{(2)} | \alpha_{a} \rangle \) |
| \( \langle \lambda_{(3)} | \lambda_{(4)} \rangle \) | \( \langle \alpha_{a} | \lambda_{(5)} \rangle \) | \( \langle \lambda_{(6)} | \alpha_{a} \rangle \) |
| \( \langle \lambda_{(7)} | \lambda_{(8)} \rangle \) | \( \langle \alpha_{a} | \lambda_{(9)} \rangle \) | \( \langle \lambda_{(10)} | \alpha_{a} \rangle \) |

| state | flavor configuration | spin wave function |
|-------|---------------------|-------------------|
| molecular | \( \frac{1}{\sqrt{2}}(B^{+}B^{*}-B^{*+}B) \) | \( \frac{1}{\sqrt{2}}(0_{bb} \otimes 1_{ul} + 1_{bb} \otimes 0_{ul}) \) [35] |
| tetraquark | \( (bu)\langle bd \rangle_{[10650]} \) | \( \frac{1}{\sqrt{2}}(0_{bb} \otimes 1_{\bar{b}d} - 1_{bb} \otimes 0_{\bar{b}d}) \) [36] |
| molecular | \( B^{*+}B^{*} \) | \( 1_{bb} \otimes 1_{\bar{b}d} \) [36] |
| tetraquark | \( (cu)\langle cd \rangle_{[3900]} \) | \( \frac{1}{\sqrt{2}}(0_{cc} \otimes 1_{ul} - 1_{cc} \otimes 0_{ul}) \) [38] |
| molecular | \( D^{+}D^{*} \) [39] | \( 1_{cc} \otimes 1_{\bar{c}d} \) |
| tetraquark | \( (cu)\langle cd \rangle_{[4020]} \) | \( 1_{cc} \otimes 1_{\bar{c}d} \) |
Fig. 3. (color online) The obtained binding energies for $B\bar{B}^*$, $B^*\bar{B}$, $D\bar{D}^*$ and $D^*\bar{D}^*$ molecular structures.

Table 4. Parameters of the model and masses of related mesons.

| $m_{\pi}(d)$ | $m_0$ | $m_c$ | $\mu_c$ | $\mu_N$ | $\mu_M$ | $\mu_N$ |
|--------------|-------|-------|---------|---------|---------|---------|
| 0.313 GeV | 4.80 GeV | 1.40 GeV | 490 MeV | 139.57 MeV | 547.862 MeV |

| $\mu_0$ | $g_{\pi}^{\pi}/4\pi$ | $\Lambda_R$ | $\Lambda_N$ | $\Lambda_\sigma$ | $\Lambda_\sigma$ |
|---------|----------------|-------------|-------------|----------------|-------------|
| 775.26 MeV | 0.54 | 4.2 fm$^{-1}$ | 5.2 fm$^{-1}$ | 0.113 fm$^{-1}$ |

| $\mu_0$ | $\alpha_0$ | $\alpha_0$ | $b$ | $c$ |
|---------|-------------|-------------|-----|-----|
| 36.976 MeV | 2.118 | 0.79 GeV | 0.148 GeV$^2$ | $-0.319$ GeV |

Table 5. Binding energy minima ($E_{\text{mol}}$ (MeV)), distance $R$ (fm) between Q$\bar{Q}$ and the calculated masses $M_{\text{mol}}$ (MeV) of $B\bar{B}^*$, $B^*\bar{B}$, $D\bar{D}^*$ and $D^*\bar{D}^*$ molecular structures.

Table 6. Binding energy minima ($E_{\text{tetra}}$ (MeV)), distance $R$ (fm) between Q$\bar{Q}$ and the calculated masses $M_{\text{tetra}}$ (MeV) of (bu)(bd)$_{[10610]}$, (bu)(bd)$_{[10650]}$, (cu)(cd)$_{[3900]}$ and (cu)(cd)$_{[4020]}$ tetraquark structures, with respect to the free parameter $\varepsilon$ (fm$^2$).
Variation of the obtained binding energy for (bu)(b¯d)[10610], (bu)(b¯d)[10650] tetraquark structures, for values of ε from 0.02 to 0.12 fm. Obtained binding energies for (cu)(c¯d)[3900] and (cu)(c¯d)[4020] tetraquark structures, for values of ε from 0.02 to 0.12 fm.

and ¯Q, and the stable point corresponds to the distance at R≈0.79–1.5 fm, which is comparable with that for molecular states, but is generally shorter. It seems reasonable. The masses of the tetraquark (defined as \(M_{(tetra)} = m_{D1} + m_{D2} + E_{(tetra)}\)), where \(m_{D1}\) and \(m_{D2}\) are the masses of the diquark and anti-diquark, are presented in Table 6.

### 4 Discussion and conclusion

As discussed in the introduction, many authors suggested that the newly observed four-quark states \(Z_{b}(10610)\), \(Z_{b}(10650)\), \(Z_{c}(3900)\), \(Z_{c}(4020)\) etc. are hadronic molecules, the reason being that their masses are close to the sums of some mesons B, B*, D, D*. However, for all of them the sum of the masses of the constituent mesons is smaller than the mass of the concerned exotic meson. By the potential model, the binding energy should be negative, and the calculated values of the binding energies shown in Table 5 confirm this allegation. Therefore, assuming them to be molecular states brings up an inconsistency. To solve this puzzle, there must be corresponding tetraquark states which mix with the molecular states to result in the observed physical hadrons.

The possible energy matrix is written as

\[
H = \begin{pmatrix} M_{(mol)} & \Delta_Q \\ \Delta_Q & M_{(tetra)} \end{pmatrix},
\]

(27)

where \(M_{(mol)}\) and \(M_{(tetra)}\) are the masses of a pure molecular state and a tetraquark calculated in the theoretical framework, and the off-diagonal element \(\Delta_Q\) whose subscript \(Q\) means that it may be flavor-dependent (b or c), and is a mixing parameter. Solving the secular equation:

\[
\begin{vmatrix} M_{(mol)} - \lambda & \Delta_Q \\ \Delta_Q & M_{(tetra)} - \lambda \end{vmatrix} = 0
\]

(28)

we obtain two eigenvalues

\[
\lambda_\pm = \frac{M_{(mol)} + M_{(tetra)} \pm \sqrt{(M_{(mol)} - M_{(tetra)})^2 + 4\Delta_Q^2}}{2},
\]

(29)

and \(\lambda_+\) are the masses of physical states i.e. mixtures of molecular states and tetraquarks.

It is noted that \(\lambda_+ > \max(M_{(mol)}, M_{(tetra)})\) and \(\lambda_- < \min(M_{(mol)}, M_{(tetra)})\). In our framework, the masses of both molecular states and tetraquark states are below those of the observed exotic mesons, so we expect that the \(\lambda_+\)’s correspond to the physical exotic states which are the experimentally observed \(Z_{b}(10610), Z_{b}(10650),\)
been discussed in the literature [40], and a more specific side in different groups (Fig. 6). Such a mechanism has induced by exchanging quarks and anti-quarks which re-mixing between molecular structure and tetraquark is exist, and a mixture would naturally be expected. The molecule and tetraquark structures, so both of them can the anarchy state.

Our numerical results indicate that for both molecule and tetraquark states, the functions of the binding energies possess minima. For the case of molecular states, the minimum occurs at $R \sim 1$ fm (for $Z_b$ and $Z_c$, see Table 5), whereas, for the tetraquark, $R=0.79$–1.5 fm depending on the parameter $\varepsilon$ where $R$ is the distance between $Q$ and $\bar{Q}$. The situations for $Z_b$ and $Z_c$ are slightly different, but the tendency is roughly the same. It is also noted that the resultant $R$ is flavor dependent, but no matter whether $c$ or $b$, it falls within a reasonable range i.e. roughly $1/A_{QCD}$.

The following are a few observations on the results. First, from Fig. 4 and Fig. 5, one notices that the local minima is a metastable one and for $R < 0.6$ fm, the binding energy drops drastically. This may imply that there could be an anarchy state for a four-quark system. This is only a qualitative inference, though, and in that case the computed value for the binding energy would not be reliable because here the adopted picture is only valid for the diquark-anti-diquark structure rather than the anarchy state.

The main conclusion is that there are minima for both molecule and tetraquark structures, so both of them can exist, and a mixture would naturally be expected. The mixing between molecular structure and tetraquark is induced by exchanging quarks and anti-quarks which reside in different groups (Fig. 6). Such a mechanism has been discussed in the literature [40], and a more specific study can be found in Ref. [41, 42]. Because it is a non-perturbative QCD effect, however, in this work we do not directly calculate $\Delta_Q$ from an underlying principle or a concrete model. Instead, we fix it phenomenologically; for example, using the values given in Table 5, Table 6 and $\lambda_\alpha$ of Eq. (29), we obtain $\Delta_Q=2$–35 MeV.

With the provided model, we predict the positions of the partners of $Z_b(10510)$, $Z_b(10650)$, $Z_c(3900)$ and $Z_c(4020)$, which weakly depend on the value of $\varepsilon$. Therefore, the key point to validate or negate our model is to look for the counter-partners of the observed exotic mesons. However, since the masses of the expected mesons are below the production thresholds of $B^{(*)}$-$\bar{B}^{(*)}$ or $D^{(*)}$-$\bar{D}^{(*)}$ (which can be realized in $Z_b$ decays but not in $Z_c$'s), one should look for them in the decay modes such as KK$\pi$ etc.

Table 7. The mixing parameter $\Delta_Q$ and the masses $M'$ of the predicted counterparts of $Z_b(10610)$, $Z_b(10650)$, $Z_c(3900)$ and $Z_c(4020)$, with respect to the free parameter $\varepsilon$.

| $\varepsilon$/fm | $Z_b(10610)$ | $Z_b(10650)$ | $Z_c(3900)$ | $Z_c(4020)$ |
|------------------|--------------|--------------|-------------|-------------|
| $\Delta_{Q}$/MeV | $M'/MeV$     | $\Delta_{Q}$/MeV | $M'/MeV$   | $\Delta_{Q}$/MeV | $M'/MeV$ |
| 0.02             | 8.0          | 10592.2      | 18.10       | 10603.9     | 35.63    | 3827.01 | 29.04 | 3888.58 |
| 0.03             | 5.81         | 10595.4      | 17.36       | 10607.2     | 34.17    | 3830.31 | 28.67 | 3891.86 |
| 0.04             | 1.92         | 10598.6      | 16.58       | 10610.6     | 32.65    | 3833.60 | 28.29 | 3895.15 |
| 0.05             | 15.75        | 10613.9      | 31.03       | 3836.94     | 27.90    | 3898.48 |
| 0.06             | 14.87        | 10617.3      | 29.29       | 3840.34     | 27.49    | 3901.84 | 27.08 | 3905.24 |
| 0.07             | 13.95        | 10620.7      | 27.43       | 3843.76     | 26.67    | 3908.63 |
| 0.08             | 12.96        | 10624.1      | 25.45       | 3847.14     | 26.25    | 3911.99 |
| 0.09             | 11.89        | 10627.4      | 23.30       | 3850.53     | 25.82    | 3915.34 |
| 0.10             | 10.76        | 10630.7      | 20.94       | 3853.91     | 25.40    | 3918.63 |
| 0.11             | 9.54         | 10633.8      | 18.36       | 3857.18     | 24.98    | 3921.84 |
| 0.12             | 8.20         | 10636.8      | 15.44       | 3860.37     | 24.57    | 3924.93 |
| 0.13             | 6.70         | 10639.6      | 11.98       | 3863.44     | 24.17    | 3927.87 |
| 0.14             | 4.89         | 10642.3      | 7.29        | 3866.37     | 23.42    | 3933.26 |
| 0.16             |              |              |             |             | 23.07    | 3935.69 |
For a quantitatively reliable conclusion, more information (theoretical and especially experimental) is needed. Indeed, more accurate data are being accumulated, and we hope that further measurements will be carried out at BES, SuperBelle and LHCb, as well as the other proposed colliders.

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