A comment on ‘The Cauchy problem of $f(R)$ gravity’

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Abstract
A critical comment on (N Lanahan-Tremblay and V Faraoni 2007 Class. Quantum Grav. 24 5667) is given discussing the well-formulation of the Chauchy problem for $f(R)$-gravity in metric-affine theories.

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In [1], the authors question the viability of $f(R)$-theories of gravity in the Palatini metric-affine formulation, since the Cauchy problem, following their approach, cannot be well-formulated. From their viewpoint, this shortcoming is present also in vacuo. Such a conclusion can be debated and, on the basis of the results which we are going to present in this note, the well-formulation of the initial value problem can be demonstrated, and no objections can be made on the viability of metric-affine $f(R)$-gravity. On the other hand, the well-posedness of the Cauchy problem has to be investigated.

First, we recall that, in the metric-affine formulation of $f(R)$-gravity, the dynamical fields are given by the couple of functions $(g, \Gamma_1)$, where $g$ is the metric and $\Gamma_1$ is the linear connection. In vacuo, the field equations are obtained by varying the following action with respect to the metric and the connection:

$$A(g, \Gamma) = \int \sqrt{|g|} f(R) \, ds,$$

where $f(R)$ is a real function, $R(g, \Gamma) = g^{ij} R_{ij}$ (with $R_{ij} := R^b_{\ ijb}$) is the scalar curvature associated with the dynamical connection $\Gamma$.

More precisely, in the approach with torsion, one can ask for a metric connection $\Gamma$ with torsion different from zero while, in the Palatini approach, $\Gamma$ is non-metric but torsion is null [3].

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In vacuo, the field equations for \( f(R) \)-gravity with torsion are [3]

\[
f'(R)R_{ij} - \frac{1}{2} f(R)g_{ij} = 0, \tag{2}
\]

\[
T_{ij}^h = -\frac{1}{2} f' \frac{\partial f'}{\partial x^p} \left( \delta^p_i \delta^h_j - \delta^p_j \delta^h_i \right), \tag{3}
\]

while the field equations for \( f(R) \)-gravity in the manner of Palatini are [8–12]

\[
f'(R)R_{ij} - \frac{1}{2} f(R)g_{ij} = 0, \tag{4}
\]

\[
\nabla_k (f'(R)g_{ij}) = 0. \tag{5}
\]

In both cases, considering the trace of Einstein-like field equations (2) and (4), one gets

\[
f'(R)R - 2 f(R) = 0. \tag{6}
\]

It is easy to conclude that the scalar curvature \( R \) is a constant coinciding with the solution of equation (6). In this case, equations (3) and (5) imply that both dynamical connections coincide with the Levi-Civita connection associated with the metric \( g_{ij} \) which is the solution of the field equations.

In other words, both theories reduce to the Einstein theory plus cosmological constant. It is well known that general relativity in vacuo shows a well-formulated and well-posed Cauchy problem, both in the Lagrangian and Hamiltonian (3 + 1 ADM) formulation [5–7]. This last fact is inconsistent with the arguments in [1], where the authors state that Palatini \( f(R) \) gravity ‘has an ill-formulated Cauchy problem in vacuo and, therefore, can hardly be regarded as a viable theory’ (see section 4 of [1]).

It is worth noting that metric-affine \( f(R) \)-theories of gravity (in both the Palatini and with torsion approaches) reduce to the general relativity with the cosmological constant also for the coupling with an electromagnetic field (or, more in general, with Yang–Mills fields). Therefore, also in this case, the Cauchy problem is well-formulated and well-posed [5]. In general, the same conclusion holds anytime the trace of the energy–momentum tensor, associated with the matter sources, is constant since it is straightforward to show that, in such a circumstance, the theory reduces to the general relativity with a cosmological constant [3].

Moreover, in the Lagrangian formulation (covariant approach and second-order time evolution equations), it can be shown that the Cauchy problem is well-formulated also in the case of coupling with a perfect fluid or with a Klein–Gordon scalar field [13]. This result is achieved by applying the approach proposed in [4] to the Einstein-like field equations resulting from metric-affine \( f(R) \)-theories, and developing a second-order analysis (in the time derivatives). In [13], and in this note, for well-formulation of the Cauchy problem, we mean the possibility of solving the field equations starting from the assigned initial data on a Cauchy surface and using suitable constraints for evolution equations. The adopted notion is consistent with that assumed in [1, 2]. The only difference is that, in [1, 2], a first-order analysis has been developed.

In conclusion, the statements about the non-viability of metric-affine \( f(R) \)-gravity, given in [1], seem excessively severe, at least if it is based only on the supposed ill-formulation of the initial value problem.

In order to prove whether metric-affine \( f(R) \)-theories of gravity are viable or not, the well-posedness of the Cauchy problem has to be discussed. Namely, the continuous and causal dependence of the solutions on the initial data has to be proven (at least in some physically important cases such as the coupling with perfect-fluid matter or the coupling with a scalar field). In our knowledge, the latter is still an open question as well as an interesting future task.
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