MINIMAL SURFACES THAT ATTAIN EQUALITY  
IN THE CHERN-OSSERMAN INEQUALITY

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Abstract. In the previous paper, Takahasi and the authors generalized  
the theory of minimal surfaces in Euclidean $n$-space to that of surfaces with  
holomorphic Gauss map in certain class of non-compact symmetric spaces. It  
also includes the theory of constant mean curvature one surfaces in hyperbolic  
3-space. Moreover, a Chern-Osserman type inequality for such surfaces was  
shown. Though its equality condition is not solved yet, the authors have  
oticed that the equality condition of the original Chern-Osserman inequality  
itself is not found in any literature except for the case $n = 3$, in spite of its  
importance. In this paper, a simple geometric condition for minimal surfaces  
that attains equality in the Chern-Osserman inequality is given. The authors  
hope it will be a useful reference for readers.

The total curvature $TC(M)$ of any complete minimal surface $M$ in $\mathbb{R}^n$ has a  
value in $2\pi\mathbb{Z}$ and satisfies the following inequality called the Chern-Osserman  
inequality [CO]:  

\[ TC(M) \leq 2\pi(\chi_M - m), \]

where $\chi_M$ denotes the Euler number of $M$ and $m$ is the number of ends of $M$.  

Then it is natural to ask which surfaces attain the equality of the inequality (1). In the case of $n = 3$, Jorge and Meeks [JM] gave a geometric proof of (1)  
and proved that the equality holds if and only if all of the ends are embedded.  
However, for general $n > 3$, the authors do not know any references on it. The  
purpose of this paper is to give the following geometric condition for attaining  
equality in (1) for general $n$.

Main Theorem. A complete minimal surface in $\mathbb{R}^n$ attains equality in the  
Chern-Osserman inequality if and only if each end is asymptotic to a catenoid-  
type end or a planar end in some 3-dimensional subspace $\mathbb{R}^3$ in $\mathbb{R}^n$. In partic-  
ular, all ends are embedded.

For $n = 3$, according to Jorge-Meeks [JM] and Schoen [S], one can easily  
observe that embedded ends are all asymptotic to catenoids or planes (see App-  
endix). So our theorem generalizes the result in Jorge-Meeks. For general  
n ($> 3$), we remark that the embeddedness of ends is not a sufficient con-  
dition for the equality of (1). For example, an embedded holomorphic curve  
f: $\mathbb{C}\setminus\{0\} \to \mathbb{C}^2$ defined by $f(z) = (z, 1/z^2)$ (considered as a complete minimal  
surface in $\mathbb{R}^4$) has total curvature $-6\pi$. So it does not satisfy equality in (1).
Preliminaries

We shall review the properties of minimal surfaces in $\mathbb{R}^n$ (cf. [L]). Let $f = (f_1, \ldots, f_n): M \to \mathbb{R}^n$ be a conformal minimal immersion of a Riemann surface $M$, where $n \geq 3$ is an integer. Then $\partial f$ is a $\mathbb{C}^n$-valued holomorphic 1-form on $M$. We define the Gauss map $\nu: M \to \mathbb{CP}^{n-1}$ of $f$ as

$$\nu := [\partial f] = \left[ \frac{\partial f_1}{\partial z}, \frac{\partial f_2}{\partial z}, \cdots, \frac{\partial f_n}{\partial z} \right],$$

where $z$ is a complex coordinate of $M$. Since $f$ is conformal, we have

$$\langle \partial f, \partial f \rangle = \sum_{j=1}^{n} \left( \frac{\partial f_j}{\partial z} \right)^2 dz^2 = 0.$$ 

Thus, the Gauss map $\nu$ is valued in the complex quadric $Q^{n-2} \subset \mathbb{CP}^{n-1}$.

We assume that $f$ is complete and of finite total curvature. Under this assumption, the following properties are well-known:

- $M$ is biholomorphic to a compact Riemann surface $\overline{M}$ punctured at finitely many points $\{p_1, \ldots, p_m\}$. Each point $p_j$ is called an end.
- The Gauss map $\nu$ can be extended holomorphically on $\overline{M}$, and the total curvature is given by $-2\pi d$ where $d$ is the homology degree of $\nu(\overline{M})$ in $\mathbb{CP}^{n-1}$.
- For each end $p_j$, there exists a local complex coordinate $z$ on $\overline{M}$ centered at $p_j$ such that the first fundamental form $ds^2$ is written as

$$ds^2 = |z|^{2\mu_j} d\bar{z} d\bar{z} \quad (\mu_j \leq -2).$$

We call $\mu_j$ the order of the metric $ds^2$ at the end $p_j$ and denote by $\text{ord}_{p_j} ds^2 = \mu_j$. Since $ds^2 = 2\langle \partial f, \overline{\partial f} \rangle$, $\mu_j$ coincides with the order of $\partial f$ at the end $p_j$.

Definition 1. An end $p_j$ of $f: M = \overline{M} \setminus \{p_1, \ldots, p_m\} \to \mathbb{R}^n$ is said to be asymptotic to a catenoid-type (resp. planar) end if there exists a piece of the catenoid (resp. the plane)

$$f_0: \{|z - p_j| < \varepsilon\} \to \mathbb{R}^3 \subset \mathbb{R}^n$$

which is complete at $p_j$ such that $|f(z) - f_0(z)| = O(|z - p_j|)$, that is,

$$\frac{|f(z) - f_0(z)|}{|z - p_j|}$$

is bounded on $\{|z - p_j| < \varepsilon\}$ for sufficiently small $\varepsilon > 0$.

Proof of the Main Theorem

The Chern-Osserman inequality follows from the fact $\text{ord}_{p_j} ds^2 \leq -2$ at each end $p_j$. Moreover, equality holds if and only if $\text{ord}_{p_j} ds^2 = -2$ (see [L] pp. 135–136), for example. Thus the Main Theorem immediately follows from the following Lemma.
Lemma 2. Let \( f : \Delta^* \to \mathbb{R}^n \) be a conformal minimal immersion of a punctured disc \( \Delta^* = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \) into \( \mathbb{R}^n \) which is complete at the origin 0. Then \( \operatorname{ord}_0 ds^2 = -2 \) holds if and only if the end 0 is asymptotic to a catenoid-type end or a planar end in \( \mathbb{R}^3 (\subset \mathbb{R}^n) \). In particular, it is an embedded end.

Proof. Suppose that \( \operatorname{ord}_0 ds^2 = -2 \). It implies that the Laurent expansion of \( \partial f \) is given by

\[
\partial f = \left( \frac{1}{z^2} a_{-2} + \frac{1}{z} a_{-1} + \cdots \right) dz, \quad a_{-2} \in \mathbb{C}^n \setminus \{0\}, \ a_{-1} \in \mathbb{R}^n
\]

because the residue of \( \partial f \) must be real. Moreover, it follows from (4) that

\[
\langle a_{-2}, a_{-2} \rangle = 0, \quad \text{and} \quad \langle a_{-2}, a_{-1} \rangle = 0.
\]

Therefore we have

\[
|\operatorname{Re} a_{-2}| = |\operatorname{Im} a_{-2}|, \quad \langle \operatorname{Re} a_{-2}, \operatorname{Im} a_{-2} \rangle = 0, \quad \langle \operatorname{Re} a_{-2}, a_{-1} \rangle = 0, \quad \langle \operatorname{Im} a_{-2}, a_{-1} \rangle = 0.
\]

Hence we can choose an orthonormal basis \( e_1, \ldots, e_n \) of \( \mathbb{R}^n \) so that

\[
\operatorname{Re} a_{-2} = a e_1, \quad \operatorname{Im} a_{-2} = a e_2, \quad a_{-1} = b e_3
\]

for some real constants \( a(\neq 0), b \). With respect to this basis, we have

\[
\partial f = \left( \frac{a}{z^2} (e_1 + ie_2) + \frac{b}{z} e_3 + \cdots \right) dz, \quad a, b \in \mathbb{R}, (a \neq 0)
\]

Then using the polar coordinate \( z = re^{i\theta} \), we have

\[
f(z) = 2 \int_{z_0}^z \partial f = -\frac{2a \cos \theta}{r} e_1 - \frac{2a \sin \theta}{r} e_2 + 2b \log re_3 + O(r),
\]

where \( z_0 \) is a base point. Here, we have dropped the constant terms in \( f(z) \) by a suitable parallel translation. By Definition 1, the formula (4) implies that the surface \( f(\Delta^*) \) is asymptotic to the catenoid (resp. the plane) for the sufficiently small \( r \) if \( b \neq 0 \) (resp. if \( b = 0 \)).

Conversely, suppose that \( \operatorname{ord}_0 ds^2 \neq -2 \). It implies that \( \operatorname{ord}_0 ds^2 = -k \) \((k \geq 3)\) and

\[
\partial f = \left( \frac{1}{z^k} a_{-k} + \cdots + \frac{1}{z} a_{-1} + \cdots \right) dz, \quad a_{-k} \neq 0 \in \mathbb{C}^n, \ a_{-1} \in \mathbb{R}^n.
\]

It is obvious that the end is asymptotic to neither a catenoid-type end nor a planar end.

From now on, we shall prove that an end is embedded if it is asymptotic to a catenoid-type end or a planar end. Assume that the end is not embedded. Then there exist two sequences \( \{ z_j \}, \{ z'_j \} \) convergent to 0 such that \( f(z_j) = f(z'_j) \) for all \( j \). Then by (4), there exists a positive constant \( C \) such that

\[
\frac{\cos \theta_j}{r_j} - \frac{\cos \theta'_j}{r'_j} \leq C |r_j - r'_j|, \quad \frac{\sin \theta_j}{r_j} - \frac{\sin \theta'_j}{r'_j} \leq C |r_j - r'_j|,
\]
where \( z_j = r_j e^{i\theta_j} \) and \( z'_j = r'_j e^{i\theta'_j} \) \((j = 1, 2, \ldots)\). With these estimates, we have

\[
\left( \frac{1}{r_j} - \frac{1}{r'_j} \right)^2 \leq \frac{1}{r_j^2} + \frac{1}{r'_j^2} - \frac{2}{r_j r'_j} \cos(\theta_j - \theta'_j) = \left| \frac{\cos \theta_j - \cos \theta'_j}{r_j} \right|^2 + \left| \frac{\sin \theta_j - \sin \theta'_j}{r'_j} \right|^2 \leq 2C^2 |r_j - r'_j|^2,
\]

and then,

\[
\frac{1}{(r_j r'_j)^2} \leq 2C^2
\]

holds. However the left hand side of (6) diverges to \(+\infty\) as \( j \to \infty \). This is a contradiction. \( \Box \)

Besides the Chern-Osserman inequality (1), the following inequalities for fully immersed complete minimal surfaces are known. (We say that the immersion \( f \) is full if the image \( f(M) \) is not contained in any hyperplanes of \( \mathbb{R}^n \).

Gackstatter [G] proved that

\[
TC(M) \leq (2\chi_M + m - 1 - n)\pi.
\]

On the other hand, Ejiri [E] proved the inequality

\[
TC(M) \leq (\chi_M + m - 2n + 2l)\pi
\]

if its Gauss image \( \nu(M) \) is contained in an \((n - 1 - l)\)-dimensional subspace of \( \mathbb{C}P^{m-1} \).

Here, we shall give a new example of complete minimal surfaces which satisfies the equality both in the Chern-Osserman equality (1) and in the Ejiri inequality (7).

**Example** (Generalized Jorge-Meeks’ surface). For \( j = 0, 1, \ldots, m - 1 \), we put

\[
g_j(z) = \frac{z^j(1 - z^{2m-2j})}{(z^{m+1} - 1)^2}, \quad h_j(z) = \frac{iz^j(1 + z^{2m-2j})}{(z^{m+1} - 1)^2},
\]

and define a complete conformal minimal immersion by

\[
f_m := \text{Re} \int_{z_0}^{z} \left( g_0, h_0, g_1, h_1, \ldots, g_{m-1}, h_{m-1}, \frac{2\sqrt{m+1}}{(z^{m+1} - 1)^2} \right) \, dz.
\]

Then by similar computations as in [JM], the integrand of (8) has real residue at each pole, and then, \( f_m \) gives a conformal minimal immersion

\[
f_m : M = (\mathbb{C} \cup \{\infty\}) \setminus \{z; z^{m+1} = 1\} \longrightarrow \mathbb{R}^{2m+1}.
\]

Obviously, the genus of \( M \) is zero, the number of ends is \( m + 1 \), and \( f_m : M \to \mathbb{R}^{2m+1} \) is full.

Since the degree of the Gauss map of \( f_m \) is \( 2m \), the total curvature \( TC(M) \) is equal to \(-4m\pi\). Therefore it attains the equality in the Chern-Osserman inequality.
On the other hand, it is easy to see that \( f_m \) has non-degenerate Gauss map, that is, \( l = 0 \) in (7). Then the right hand side of (7) is \(-4m\pi\). Hence the equality in (7) holds.

**APPENDIX: EMBEDDED ENDS IN \( \mathbb{R}^3 \)**

For the case \( n = 3 \), embeddedness of the end 0 in Lemma 3 implies \( \text{ord}_0 \, ds^2 = -2 \), and consequently the end is asymptotic to a catenoid-type end or a planer end ([JM, Theorem 4] or [S, Proposition 1]). Here we shall give a simple proof of this fact, which is a mixture of Jorge-Meeks' and Schoen's. The authors hope that it will be helpful to readers. The crucial point of the Jorge-Meeks' proof is to show that the intersection of the end and the sphere of radius \( r \) centered at the origin converges to a finite covering of a great sphere as \( r \to \infty \). According to Schoen [S], we prove it via the Weierstrass representation directly.

Consider the Laurent expansion as (5) for \( k \geq 2 \). Without loss of generality, we may set \( a_{-k} = (a, ia, 0) \) \((a \in \mathbb{R}\setminus\{0\})\) because of (2). Integrating this, we have

\[
 f(re^{i\theta}) = \frac{1}{r^{k-1}} \left[ 2a \left( \cos(k-1)\theta, \sin(k-1)\theta, 0 \right) + o(1) \right],
\]

where \( o(1) \) means a term tending to 0 as \( r \to 0 \). Let \( S_R^2 \) be the sphere in \( \mathbb{R}^3 \) with radius \( R \) centered at the origin and consider the intersection of the surface and \( S_R^2 \):

\[
 E_R := \frac{1}{R} \left( S_R^2 \cap f(\Delta^*) \right) \subset S_1^2,
\]

which is normalized as a subset of the unit sphere.

Here, \( f \in S_R^2 \) if and only if

\[
 R^2 = f_1^2 + f_2^2 + f_3^2 = \frac{1}{r^{2k-2}} (4a^2 + o(1))
\]

holds. Then \( r \to 0 \) as \( R \to \infty \) when \( f(re^{i\theta}) \in S_R^2 \) because \( k \geq 2 \). In particular, \( \lim_{r \to \infty} R^2 r^{2k-2} = 4a^2 \) holds. Then under the condition \( f(z) \in S_R^2 \),

\[
 \lim_{r \to \infty} \frac{1}{R} f(re^{i\theta}) = (\cos(k-1)\theta, \sin(k-1)\theta, 0)
\]

holds. This implies that, for sufficiently large \( R \), \( E_R \) is a closed curve in a neighborhood of the equator of \( S_1^2 \) with rotation index \(|k-1|\), which is embedded if and only if \( k = 2 \).

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