CHANGE THE COEFFICIENTS OF CONDITIONAL ENTROPIES IN EXTENSIVITY

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ABSTRACT. The Boltzmann–Gibbs entropy is a functional on the space of probability measures. When a state space is countable, one characterization of the Boltzmann–Gibbs entropy is given by the Shannon–Khinchin axioms, which consist of continuity, maximality, expandability and extensivity. Among these four properties, the extensivity is generalized in various ways. The extensivity of a functional is interpreted as the property that, for any random variables \((X, Y)\) taking finitely many values in \(\mathbb{N}\), the difference between the value of the functional at the joint law of \((X, Y)\) and that at the law of \(X\) coincides with the linear combinations of the values at the conditional laws of \(Y\) given \(X = n\) with coefficients given by the probabilities of each event \(X = n\). A generalization of the extensivity obtained by replacing the coefficients with a power of the probabilities of the events \(X = n\) provides a characterization of the Tsallis entropy.

In this paper, we first prove the impossibility to replace the coefficients with a non-power function of the probabilities of the events \(X = n\). Then we estimate the difference between the value at the joint law of \((X, Y)\) and that at the law of \(X\) for a general functional.

1. INTRODUCTION AND RESULTS

The notion of entropy is a fundamental ingredient in statistics, information theory and so on. The origin of entropy lies in thermodynamics, and the negative of the Boltzmann–Gibbs entropy is considered as an internal energy. For \(n \in \mathbb{N}\), set

\[
\mathcal{P}_n := \left\{ p = (p_j)_{j=1}^n \in \mathbb{R}^n \left| p_j \geq 0 \text{ for } 1 \leq j \leq n, \quad \sum_{j=1}^n p_j = 1 \right. \right\}, \quad \mathcal{P} := \bigcup_{n \in \mathbb{N}} \mathcal{P}_n.
\]

Then the Boltzmann–Gibbs entropy \(S_1 : \mathcal{P} \to \mathbb{R}\) is defined by

\[
S_1(p) = -\sum_{j=1}^n p_j \log p_j \quad \text{for } p \in \mathcal{P}_n,
\]

where by convention \(0 \log 0 := 0\). The following characterization of the Boltzmann–Gibbs entropy is called the Shannon–Khinchin axioms.

**Theorem 1.1.** ([3, Theorem 1]) The Boltzmann–Gibbs entropy is a unique functional \(S\) on \(\mathcal{P}\) that satisfies the following four properties:

(I) (continuity) \(S\) is continuous on \(\mathcal{P}_n\) for \(n \in \mathbb{N}\).

(II) (maximality) Define \(p^{(n)} \in \mathcal{P}_n\) by \(p_j^{(n)} = 1/n\). Then \(S(p) \leq S(p^{(n)})\) for \(p \in \mathcal{P}_n\).

(III) (expandability) For \(p = (p_j)_{j=1}^n \in \mathcal{P}_n\), \(S((p_1, \ldots, p_n, 0)) = S(p)\).

(IV) (extensivity) For \(P = (p_j^i)_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathcal{P}_{mn}\), if

\[
p_j := \sum_{i=1}^m p_j^i > 0 \quad \text{for } 1 \leq j \leq n,
\]

Then

\[
S(P) = \sum_{i=1}^m S((p_j^i)_{j=1}^n).
\]
then

\[ (1.2) \quad S(P) = S((p_1, \cdots, p_n)) + \sum_{j=1}^{n} p_j S\left(\left(\frac{p^1_j}{p_j}, \cdots, \frac{p^n_j}{p_j}\right)\right). \]

Here the uniqueness is up to a multiplicative positive constant.

If a functional \( S \) on \( \mathcal{P} \) has density and \(-S\) can be regarded as an internal energy, then the density should be concave and vanishes at 0 (for example, see [4]). In this case, \( S \) satisfies the properties (I)–(III) but may not the property (IV) (see Proposition 2.1). The property (IV) reduces to additivity for independent two systems. More generally, the property (IV) is interpreted as follows. For any random variables \( X_k \) on a probability space \((\Omega, \mathcal{P})\) taking values in \{1, \cdots, k\}, the difference between the value of the functional at the joint law of \((X_n, X_m)\) and that at the law of \(X_n\) coincides with the linear combinations of the values at the conditional laws of \(X_m\) given \(X_n = j\) with coefficients given by the probabilities of each event \(X_n = j\), where we consider

\[ p_j^i = \mathbb{P}(X_m = i, X_n = j), \quad p_j = \mathbb{P}(X_n = j), \quad \frac{p_j^i}{p_j} = \mathbb{P}(X_m = i \mid X_n = j). \]

The property (IV) is generalized in various ways. For example, see [1, 2, 8, 9] and references therein. In particular, for \(q > 0\) with \(q \neq 1\), if we replace the coefficients in (1.2) with the \(q\)th power of the probabilities of the events \(X_n = j\), that is, (1.2) is modified as

\[ (1.3) \quad S(P) = S((p_1, \cdots, p_n)) + \sum_{j=1}^{n} p_j^q S\left(\left(\frac{p^1_j}{p_j}, \cdots, \frac{p^n_j}{p_j}\right)\right), \]

then this property is satisfied by the Tsallis entropy \(S_q\) defined by

\[ S_q(p) := \frac{1}{q-1} \left(1 - \sum_{j=1}^{n} p_j^q\right) \quad \text{for} \quad p \in \mathcal{P}_n. \]

Note that

\[ S_q(p) \xrightarrow{q \to 1} S_1(p) \quad \text{for} \quad p \in \mathcal{P}. \]

Suyari [7] proved that the Tsallis entropy \(S_q\) is a unique functional that satisfies (1.3) in addition to the properties (I)—(III) up to a multiplicative positive constant.

In this paper, we first show the impossibility to replace the coefficients in (1.2) with a non-power function of the probabilities of the events \(X_n = j\) for all functionals on \(\mathcal{P}\) with \(C^2\)-density whose second derivative is negative.

**Theorem 1.2.** For \(s \in C([0,1]) \cap C^2((0,1])\) with \(s'' < 0\) on \((0,1]\), define a functional \(S : \mathcal{P} \to \mathbb{R}\) by

\[ S(p) := \sum_{j=1}^{n} s(p_j) \quad \text{for} \quad p \in \mathcal{P}_n. \]

If there exists \(f : (0,1] \to \mathbb{R}\) such that

\[ (1.4) \quad S(P) = S((p_1, \cdots, p_n)) + \sum_{j=1}^{n} f(p_j) S\left(\left(\frac{p^1_j}{p_j}, \cdots, \frac{p^n_j}{p_j}\right)\right) \]

holds for \(P \in \mathcal{P}_{mn}\) satisfying (1.1), then there exists \(q > 0\) such that \(f(r) = r^q\) on \((0,1]\) and \(S\) coincides with \(S_q\) up to an additive constant and a multiplicative positive constant.
We next estimate the difference between the value at the joint law of \((X_m, X_n)\) and that at the law of \(X_n\) for a general functional on \(\mathcal{P}\) which can be regarded as the negative of an internal energy. Let \(S\) be a functional on \(\mathcal{P}\) with density \(s\) such that \(s\) is concave on \([0, 1]\) and \(s(0) = 0\). We notice that
\[
\tilde{s}(r) := s(r) - s(1)r
\]
is also concave on \([0, 1]\) and \(\tilde{s}(0) = 0\). Moreover, \(\tilde{s}(1) = 0\). Since the functional \(\tilde{S}\) on \(\mathcal{P}\) with density \(\tilde{s}\) satisfies
\[
\tilde{S}(P) - \tilde{S}((p_1, \ldots, p_n)) = S(P) - S((p_1, \ldots, p_n))
\]
for \(P \in \mathcal{P}_{mn}\) satisfying (1.1), to estimate the difference \(S(P) - S((p_1, \ldots, p_n))\), we can assume that the density \(s\) of \(S\) vanishes at 1 without loss of generality. In addition to the concavity \(s\) on \([0, 1]\), we assume that \(s \in C^2([0, 1])\) with \(s'' < 0\) on \((0, 1]\) and the finiteness of \(\sup \{s''(rt)/s''(t) \mid t \in (0, 1]\} \) for \(r \in (0, 1]\). Then \(\inf \{s''(rt)/s''(t) \mid t \in (0, 1]\}\) is obviously nonnegative and finite for \(r \in (0, 1]\).

**Theorem 1.3.** For \(s \in C([0, 1]) \cap C^2((0, 1])\) with \(s(0) = s(1) = 0\) and \(s'' < 0\) on \((0, 1]\), define a functional \(S: \mathcal{P} \to \mathbb{R}\) by
\[
S(p) := \sum_{j=1}^{n} s(p_j) \quad \text{for } p \in \mathcal{P}_n.
\]
Assume the finiteness of \(\sup \{s''(rt)/s''(t) \mid t \in (0, 1]\} \) for \(r \in (0, 1]\) and define \(\underline{f, \overline{f}}: (0, 1] \to \mathbb{R}\) by
\[
\underline{f}(r) := r^2 \cdot \inf_{t \in (0, 1]} \frac{s''(rt)}{s''(t)} \quad \text{and} \quad \overline{f}(r) := r^2 \cdot \sup_{t \in (0, 1]} \frac{s''(rt)}{s''(t)},
\]
respectively. Then
\[
\sum_{j=1}^{n} \underline{f}(p_j)S \left( \left( \frac{p_j^1}{p_j}, \ldots, \frac{p_j^n}{p_j} \right) \right) + s'(1) \sum_{j=1}^{n} (\overline{f}(p_j) - \underline{f}(p_j)) \\
\leq S(P) - S((p_1, \ldots, p_n)) \\
\leq \sum_{j=1}^{n} \overline{f}(p_j)S \left( \left( \frac{p_j^1}{p_j}, \ldots, \frac{p_j^n}{p_j} \right) \right) - s'(1) \sum_{j=1}^{n} (\overline{f}(p_j) - \underline{f}(p_j))
\]
for \(P \in \mathcal{P}_{mn}\) satisfying (1.1).

**Remark 1.4.** (1) If \(S = S_q\) with \(q > 0\), then \(\underline{f}(r) = \overline{f}(r) = r^q\) on \((0, 1]\) and all inequalities in Theorem 1.3 become equality.

(2) For \(s \in C([0, 1]) \cap C^2((0, 1])\) with \(s'' < 0\) on \((0, 1]\), \(\sup \{s''(rt)/s''(t) \mid t \in (0, 1]\} = \infty\) may happen for some \(r \in (0, 1]\). Indeed, if we define
\[
s(r) := -\int_{0}^{r} \int_{0}^{t} u (|\cos(1/u)| + u) \, du \, dt \quad \text{for } r \in [0, 1],
\]
then \(s \in C([0, 1]) \cap C^2((0, 1])\) with
\[
s''(r) = -r (|\cos(1/r)| + r) < 0 \quad \text{for } r \in (0, 1].
\]
For \(t_k := \{(k + \frac{1}{2})\pi\}^{-1}\) with \(k \in \mathbb{N}\), we observe that
\[
\frac{s''(t_k/2)}{s''(t_k)} = \frac{\frac{1}{2}}{|\cos((k + \frac{1}{2})\pi)| + \frac{1}{(2k + 1)\pi}} = \frac{1}{2} \left\{ \left( k + \frac{1}{2} \right) \pi + \frac{1}{2} \right\} \xrightarrow{k \to \infty} \infty.
\]
implying \( \sup \{ s''(t/2)/s''(t) \mid t \in (0, 1) \} = \infty \).

(3) Naudts [5] discussed a generalization of the Boltzmann–Gibbs entropy \( S_1 \) via a continuous, nondecreasing function \( \phi : (0, \infty) \to (0, \infty) \). This generalized entropy is called the \( \phi \)-deformed entropy and denoted by \( S_\phi \). For a functional \( S \) on \( \mathcal{P} \) with density \( s \), if \( s \in C([0, 1]) \cap C^2((0, 1]) \), \( s'' \) is nondecreasing and \( s''(1) < 0 \), then \( S \) coincides with \( S_{-1/s''} \) on \( \mathcal{P} \) up to an additive constant and a multiplicative positive constant.

(4) Ohta and the author [6] classified \( \phi \)-deformed entropies in terms of a quantity
\[
\theta_\phi := \sup \left\{ \frac{r}{\phi(r)} \cdot \limsup_{\varepsilon \downarrow 0} \frac{\phi(r + \varepsilon) - \phi(r)}{\varepsilon} \mid r > 0 \right\} \in [0, \infty],
\]
and analyzed the Wasserstein gradient flow of \(-S_\phi \), where \( \theta_\phi < 2 \) is assumed. Note that the Wasserstein gradient flow of \(-S_\phi \) is the evolution equation associated to the internal energy \(-S_\phi \). When \( \phi = -1/s'' \), the assumption \( \theta_\phi < 2 \) guarantees the finiteness of \( \sup \{ s''(rt)/s''(t) \mid t \in (0, 1) \} \) for \( r \in (0, 1) \).

(5) If \( s \) is concave on \([0, 1]\) with \( s(0) = 0 \), then
\[
s(\lambda r) \geq (1 - \lambda)s(0) + \lambda s(r) = \lambda s(r) \quad \text{for } r \in (0, 1), \quad \lambda \in [0, 1].
\]
This observation implies
\[
S(P) = \sum_{j=1}^{n} \sum_{i=1}^{m} s(p_j^i) = \sum_{j=1}^{n} \sum_{i=1}^{m} s\left(\frac{p_j^i}{p_j^i}\right) \geq \sum_{j=1}^{n} \sum_{i=1}^{m} p_j^i s(p_j^i) = \sum_{j=1}^{n} s(p_j) = S((p_1, \ldots, p_n))
\]
for \( P \in \mathcal{P}_{mn} \) satisfying (1.1). Thus the first inequality in Theorem 1.3 is trivial unless
\[
(1.5) \quad \sum_{j=1}^{n} f(p_j) S\left(\left(\frac{p_1^j}{p_j^i}, \ldots, \frac{p_m^j}{p_j^i}\right)\right) + s'(1) \sum_{j=1}^{n} \left( f(p_j) - f(p_j^i) \right) \geq 0.
\]
However, the inequality (1.5) may fail even for a \( \phi \)-deformed entropy \( S_\phi \) with \( \theta_\phi < 2 \). Indeed, if we define
\[
s(r) := -\int_{0}^{r} \log \sin\left(\frac{\pi}{4} t\right) dt + Cr \quad \text{for } r \in [0, 1], \quad \text{where } C := \int_{0}^{1} \log \sin\left(\frac{\pi}{4} t\right) dt,
\]
then \( s \in C([0, 1]) \cap C^2((0, 1]) \) with \( s(0) = s(1) = 0 \) and
\[
s''(r) := -\frac{\pi}{4} \cot\left(\frac{\pi}{4} r\right) < 0 \quad \text{for } r \in (0, 1].
\]
This leads to \( s'(1) < 0 \). Define \( \phi : (0, \infty) \to (0, \infty) \) by
\[
\phi(r) := \begin{cases} 
\frac{4}{\pi} \tan\left(\frac{\pi}{4} r\right) & r \in (0, 1], \\
\frac{2}{r} + \frac{4}{\pi} & r > 1,
\end{cases}
\]
which is continuous, nondecreasing on \((0, \infty)\) with \( \theta_\phi = \pi/2 < 2 \) and \( s'' = -1/\phi \) on \([0, 1]\).

We find that
\[
2 = \inf_{t \in (0, 1]} \frac{s''(t/2)}{s''(t)} \leq \frac{s''(u/2)}{s''(u)} = \frac{\cos(\pi u/4) + 1}{\cos(\pi u/4)} \leq \sup_{t \in (0, 1]} \frac{s''(t/2)}{s''(t)} = 1 + \sqrt{2} \quad \text{for } u \in (0, 1].
\]
Then, for \( P \in \mathcal{P}_{22} \) defined by
\[
(1.6) \quad p_1^1 = \frac{1}{2}, \quad p_1^2 = 0, \quad p_2^1 = \frac{1}{2} x, \quad p_2^2 = \frac{1}{2} (1 - x), \quad \text{with } x \in (0, 1),
\]
it turns out that
\[ \sum_{j=1}^{2} f(p_j) S\left(\left(\frac{p^1_j}{p_j}, \frac{p^2_j}{p_j}\right)\right) + s'(1) \sum_{j=1}^{2} \left(\mathcal{F}(p_j) - f(p_j)\right) \]
\[ = \frac{2}{4} (s(x) + s(1 - x)) + 2s'(1) \cdot \frac{1}{4} \left(1 + \sqrt{2} - 2\right) x \rightarrow s'(1) \left(\sqrt{2} - 1\right) < 0. \]
Thus the inequality (1.5) fails for \( P \) given by (1.6) if \( x \in (0, 1) \) is small enough.

The rest of this paper is organized as follows. We first confirm that the negative of an internal energy should satisfy the properties (I)—(III) (Proposition 2.1). Then we prove Theorem 1.2 and Theorem 1.3.

2. Proofs

For a function \( s : [0, \infty) \to \mathbb{R} \), define a functional \( S \) on \( \mathcal{P} \) by
\[ S(p) := \sum_{j=1}^{n} s(p_j) \quad \text{for} \quad p \in \mathcal{P}_n. \]
This functional can be extended to a functional on the space of probability measures on a continuous state space. If we regard \( -S \) as an internal energy, then it is natural to assume \( s(0) = 0 \) since the energy of no matter should be zero. To be physical, the pressure function of \( -S \) should be nonnegative and nondecreasing, in which case \( s \) is concave (for example, see [4]). Under these conditions, \( S \) satisfies the properties (I)—(III). Note that if we restrict our attention to a countable state space, then it suffices that \( s \) is defined on \([0, 1]\), not on \([0, \infty)\).

**Proposition 2.1.** For a concave function \( s : [0, 1] \to \mathbb{R} \) with \( s(0) = 0 \), the functional \( S \) on \( \mathcal{P} \) with density \( s \) satisfies the three properties (I), (II) and (III).

**Proof.** The concavity of \( s \) on \([0, 1]\) leads to the continuity of \( s \) on \([0, 1]\). Consequently, \( S \) is continuous on \( \mathcal{P}_n \) and (I) follows.

We next confirm (II). For \( p = (p_j)_{j=1}^{n} \in \mathcal{P}_n \), it follows from the concavity of \( s \) that
\[ S(p) = \sum_{j=1}^{n} s(p_j) = n \sum_{j=1}^{n} \frac{1}{n} s(p_j) \leq ns\left(\sum_{j=1}^{n} \frac{1}{n} p_j\right) = ns(1/n) = \sum_{j=1}^{n} s(1/n) = S(p^n). \]
As for (III), we deduce from \( s(0) = 0 \) that
\[ S((p_1, \cdots, p_n, 0)) = S(p) \quad \text{for} \quad p = (p_j)_{j=1}^{n} \in \mathcal{P}_n. \]
This completes the proof of the proposition. \( \square \)

By Aleksandrov’s theorem, a concave function defined on an interval is twice differentiable almost everywhere and its second derivative is nonpositive. In Theorems 1.2 and 1.3 the density of a functional on \( \mathcal{P} \) is assumed to satisfy a slightly stronger condition than concavity, that is, the twice continuous differentiability and the negativity of the second derivative.

**Proof of Theorem 1.2.** Let \( S \) be a functional on \( \mathcal{P} \) with density \( s \in C([0, 1]) \cap C^2((0, 1]) \) such that \( s'' < 0 \) on \((0, 1]\). Assume that there exists \( f : (0, 1] \to \mathbb{R} \) satisfying (1.4). We find that \( f(1) = 1 \). Fix \( r \in (0, 1) \) and \( \xi \in (0, 1] \). For \( x \in (0, \xi) \), we define \( P \in \mathcal{P}_3 \) by
\[ p^1_i = rx, \quad p^2_i = r(\xi - x), \quad p^3_i = r(1 - \xi), \quad p^4_i = \frac{1 - r}{3} \quad \text{for} \quad 1 \leq i \leq 3. \]
We substitute this matrix $P$ into (1.4) and differentiate it twice with respect to $x$ to have
\begin{equation}
(2.1) \quad r^2 \{s''(rx) + s''(r(\xi - x)) \} = f(r) \{s''(x) + s''(\xi - x)\}.
\end{equation}

The choice of $x = \xi/2$ implies
\begin{equation}
(2.2) \quad \frac{f(r)}{r^2} = \frac{s''(rx)}{s''(x)} \quad \text{for} \quad x \in (0, 1/2], \quad r \in (0, 1).
\end{equation}

Hence $f$ is continuous on $(0, 1]$. In (2.1), if we take $x = t\xi$ with $\xi \in (0, 1/2]$ and $t \in [1/2, 1)$, then $t\xi \in (0, 1/2), (1-t)/t \in (0, 1]$ and
\[
\frac{f(r)}{r^2} = \frac{s''(rt\xi) + s''(r(1-t)\xi)}{s''(t\xi) + s''((1-t)\xi)} = \left( \frac{s''(rt\xi)}{s''(t\xi)} + \frac{s''(r(1-t)\xi)}{s''((1-t)\xi)} \right) \left( \frac{s''(t\xi)}{s''(t\xi)} + \frac{s''(r(1-t)\xi)}{s''((1-t)\xi)} \right) = \left( \frac{f(r)}{r^2} + \frac{f(r(1-t))}{(r(1-t))^2} \right) \left( 1 + \frac{f(1-t)}{(1-t)^2} \right)
\]
for $r \in (0, 1)$, which means that
\[
f(r)f(1-t) = f\left( r\frac{1-t}{t} \right) \quad \text{for} \quad r \in (0, 1], \quad t \in [1/2, 1).
\]

By the continuity of $f$ on $(0, 1]$, there exists $q \in \mathbb{R}$ such that $f(r) = r^q$ on $(0, 1]$. We deduce from (2.2) that
\[
s''(rx) = s''(x)r^{q-2} \quad \text{for} \quad x \in (0, 1/2], \quad r \in (0, 1].
\]

Substituting it into (2.1) and choosing $x \in (0, 1/2], \xi = 1$, yield that
\[
s''(r(1-x)) = s''(1-x)r^{q-2} \quad \text{for} \quad x \in (0, 1/2], \quad r \in (0, 1].
\]

Hence $s''(r) = s''(1)r^{q-2}$ holds on $(0, 1]$. Note that
\[
s'(1)(1-r) - s(1) + s(r) = \int_r^1 \left( s'(1) - s'(t) \right) dt = \int_r^1 \int_t^1 s''(u) du dt \quad \text{for} \quad r \in (0, 1].
\]

If $q = 1$, then
\[
s'(1)(1-r) - s(1) + s(r) = s''(1)(1 + r \log r - r),
\]
that is,
\[
s(r) = k_1 r \log r + a_1 r + b_1 \quad \text{for} \quad r \in [0, 1],
\]
where $k_1 := s''(1) < 0$, $a_1 := -s''(1) + s'(1)$, $b_1 := s''(1) - s'(1) + s(1)$.

Similarly, for $q = 0$, it turns out that
\[
s'(1)(1-r) - s(1) + s(r) = -s''(1)(1 - r + \log r) \quad \text{for} \quad r \in (0, 1].
\]

When $r \downarrow 0$, the right-hand side diverges while the left-hand side converges. Thus $q = 0$ is inappropriate. Finally, if $q \neq 0, 1$, then
\[
s'(1)(1-r) - s(1) + s(r) = s''(1) \left( \frac{1 - r - 1 - r^q}{q} \right) \quad \text{for} \quad r \in (0, 1].
\]

Since the limit of the left-hand side exists as $r \downarrow 0$, we find $q > 0$ and
\[
s(r) = k_q \frac{r^q - r}{q - 1} + a_q r + b_q \quad \text{for} \quad r \in (0, 1],
\]
where $k_q := \frac{s''(1)}{q} < 0$, $a_q := -\frac{s''(1)}{q} + s'(1)$, $b_q := \frac{s''(1)}{q} - s'(1) + s(1)$. 
Thus there exists \( q > 0 \) such that \( f(r) = r^q \) on \((0, 1]\) and
\[
S(p) = \sum_{j=1}^{n} s(p_j) = -k_qS_q(p) + a_q\sum_{j=1}^{n} p_j + b_q = -k_qS_q(p) + s(1) \quad \text{for } p \in \mathcal{P}_n.
\]
This proves the theorem. \( \square \)

**Proof of Theorem 1.3.** Let \( s \in C([0, 1]) \cap C^2((0, 1]) \) such that \( s(0) = s(1) = 0 \) and \( s'' < 0 \) on \((0, 1]\). Fix \( P \in \mathcal{P}_{mn} \) satisfying \((1.1)\) and \( 1 \leq j \leq n \). Given \( u \in (0, 1) \), we deduce from the definition of \( \overline{f}, \underline{f} \) together with the negativity of \( s'' \) that
\[
\underline{f}(p_j)s''(u) \geq p_j^2s''(p_ju) \geq \overline{f}(p_j)s''(u).
\]
For \( a, r \in (0, 1] \), since
\[
\int_0^1 \int_t^1 a^2s''(au)dudt = as'(a)r - s(ar),
\]
integrating the above inequalities on \( u \in [t, 1] \) and then \( t \in [0, r] \) gives
\[
(2.3) \quad \underline{f}(p_j)(s'(1)r - s(r)) \geq p_js'(p_j)r - s(p_jr) \geq \overline{f}(p_j)(s'(1)r - s(r)).
\]
Choosing \( r = p_j^2/p_j \) and summing over \( 1 \leq i \leq m \), we find that
\[
\underline{f}(p_j)s'(1) - \underline{f}(p_j)\sum_{i=1}^{m} s(p_j^i/p_j) \geq p_js'(p_j) - \sum_{i=1}^{m} s(p_j^i) \geq \overline{f}(p_j)s'(1) - \overline{f}(p_j)\sum_{i=1}^{m} s(p_j^i/p_j).
\]
The sum of these inequalities over \( 1 \leq j \leq n \) gives
\[
\sum_{j=1}^{n} \underline{f}(p_j)s'(1) - \underline{f}(p_j)S\left(P \right) \geq \sum_{j=1}^{n} p_js'(p_j) - S(P)
\]
\[
\geq \sum_{j=1}^{n} \overline{f}(p_j)s'(1) - \overline{f}(p_j)S\left(P \right),
\]
implying
\[
\sum_{j=1}^{n} \underline{f}(p_j)S\left(\frac{p_j^1}{p_j}, \ldots, \frac{p_j^m}{p_j}\right) + \sum_{j=1}^{n}
\left( -\underline{f}(p_j)s'(1) + p_j s'(p_j) - s(p_j) \right)
\leq S(P) - S(p_1, \ldots, p_j)
\]
\[
\leq \sum_{j=1}^{n} \overline{f}(p_j)S\left(\frac{p_j^1}{p_j}, \ldots, \frac{p_j^m}{p_j}\right) + \sum_{j=1}^{n}
\left( -\overline{f}(p_j)s'(1) + p_j s'(p_j) - s(p_j) \right).
\]
In \((2.3)\), if we choose \( r = 1 \), then
\[
\underline{f}(p_j)s'(1) \geq p_j s'(p_j) - s(p_j) \geq \overline{f}(p_j)s'(1).
\]
Substituting these into the above inequalities completes the proof of the theorem. \( \square \)

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