Singular and regular vortices on top of a background pulled to the center

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Abstract

A recent analysis has revealed singular but physically relevant localized two-dimensional vortex states with density $\sim r^{-4/3}$ at $r \to 0$ and a convergent total norm, which are maintained by the interplay of the potential of the attraction to the center, $\sim -r^{-2}$, and a self-repulsive quartic nonlinearity, produced by the Lee-Huang-Yang correction to the mean-field dynamics of Bose–Einstein condensates. In optics, a similar setting, with the density singularity $\sim r^{-1}$, is realized with the help of quintic self-defocusing. Here we present physically relevant antidark singular-vortex states in these systems, existing on top of a flat background. Numerical solutions for them are very accurately approximated by the Thomas–Fermi wave function. Their stability exactly obeys an analytical criterion derived from the consideration of small perturbations. The singular-vortex states exist as well in the case when the effective potential is weakly repulsive. It is demonstrated that the singular vortices can be excited by the input in the form of the ordinary nonsingular vortices, hence the singular modes can be created in the experiment. We also consider regular (dark) vortices maintained by the flat background, under the action of the repulsive central potential $\sim +r^{-2}$. The dark modes with vorticities $l = 0$ and 1 are completely stable. In the case when the central potential is attractive, but the effective one, which includes the centrifugal term, is repulsive, and, in addition, a weak trapping potential $\sim r^2$ is applied, dark vortices with $l = 1$ feature an intricate pattern of alternating stability and instability regions. Under the action of the instability, states with $l = 1$ travel along tangled trajectories, which stay in a finite area defined by the trap. The analysis is also reported for dark vortices with $l = 2$, which feature a complex structure of alternating intervals of stability and instability against splitting. Lastly, simple but novel flat vortices are found at the border between the anidark and dark ones.

Keywords: dark vortex, quantum collapse, Thomas–Fermi approximation, stability, singular vortex

(Some figures may appear in colour only in the online journal)

1. Introduction

In typical cases, vortex states exist, in self-defocusing optical media [1–6] and Bose–Einstein condensates (BECs, i.e. in optics of coherent matter waves) [7–9], as two-dimensional (2D) dark solitons, supported by a modulationally stable background. In terms of polar coordinates $(r, \theta)$, the wave function of a vortex with integer winding number (topological charge) $l \geq 0$ has the standard asymptotic form at $r \to 0$: $\psi \sim \text{const} \cdot r^l e^{i \theta}$. Unlike this classical situation, physically relevant singular vortices, as well as zero-vorticity states, were predicted in 2D models which combine the attractive potential...
with $U_0 > 0$, and a self-repulsion term in the underlying Gross–Pitaevskii (GP) [10] or nonlinear Schrödinger (NLS) equation, which must be stronger than cubic, i.e.

$$U(r) = -U_0/\left(2r^2\right),$$

(2)

with $U_0 > 0$, and a self-repulsion term in the underlying Gross–Pitaevskii (GP) [10] or nonlinear Schrödinger (NLS) equation, which must be stronger than cubic, i.e.

$$N_{\text{3D}} = 4\pi \int_0^\infty |\psi(r)|^2 r^2 dr,$$

(5)

converges at $r \to 0$ under the condition of $\alpha > 2$, i.e., as mentioned above, the self-repulsion must be stronger than cubic. In work [11], the existence of such normalizable singular vortex modes was demonstrated for $\alpha = 4$ in equation (3), which corresponds to the quintic self-defocusing, well known in diverse forms in optics [18–22], or the effect of three-body collisions in BEC [23]. Then, a similar result was reported in work [12] for the GP equation including the beyond-mean-field Lee-Huang-Yang (LHY) correction, produced by an effect of quantum fluctuations in binary BEC. The latter term amounts to the quartic nonlinearity, with $\alpha = 3$ in equation (3) [24–27]. This nonlinearity corresponds to experiments producing stable self-trapped quantum droplets [28–32]. In addition to the systematic numerical investigation, exact analytical results which determine a stability boundary for the singular vortices, and approximate results for the shape of the vortices, were also reported in [12].

It is relevant to mention that the three-dimensional (3D) GP-like equation with the same attractive potential (2) and the self-repulsive term (3) has fundamental (zero-vorticity) isotropic solutions with the singular asymptotic form [11]

$$\psi_{\text{3D}} \approx \left[\frac{1}{2} \left(U_0 + \frac{4}{\alpha} \left(\frac{2}{\alpha} - 1\right)\right)\right]^{1/\alpha} r^{-2/\alpha} e^{-i\mu t+i\theta},$$

(6)

(7)

(cf equation (4). The convergence of the respective 3D norm, at $r \to 0$ is secured by condition $\alpha > 4/3$. The nonlinearity with $\alpha = 4/3$ corresponds to the effective self-repulsion in the quantum Fermi gas, according to the density-functional approximation [33–36], but it still produces a weak logarithmic divergence of the 3D norm. Obviously, the physically relevant cubic, quartic, and quintic nonlinearities all satisfy the convergence condition, while there is no physically relevant example of the nonlinearity with $4/3 < \alpha < 2$.

A noteworthy feature of the numerical and analytical results presented in [12] for the 2D setting is the fact that the singular vortices exist not only in the case of $U_0 > 0$ (see equation (5)), when they are maintained by the pull to the center, but also in the interval of

$$-4/9 < U_1 < 0,$$

(8)

(9)

(in terms of the notation adopted below in equation (10)), where the effective potential is slightly repulsive. In particular, for $l \geq 1$ the underlying potential (2), corresponding to interval (9), is still attractive, while it is repulsive, indeed, for the zero-vorticity modes, with $l = 0$. This counter-intuitive finding can be explained by the fact that the NLS equation with the self-repulsive nonlinearity may give rise to bright singular-soliton solutions [37, 38]. It is also relevant to mention, in this connection, that, in the 3D case with the cubic self-repulsion ($\alpha = 2$), the physically acceptable singular state (7) exists precisely at $U_0 > 0$, while the quartic or quintic terms ($\alpha = 3$ or 4 in equation (3)) provide for the convergence of the norm if the attraction strength exceeds a finite threshold, viz., $U_0 > 2/9$ or 1/4, respectively.

Note that the concept of vortices with singular cores is also known in theoretical studies of the 2D dynamics in spinor (three-component) BEC [39, 40]. However, the meaning of such modes is different in that context, as their densities remain finite, while the singularity implies splitting of vorticity axes of the three components.

In works [11] and [12], the singular zero-vorticity and vortex states in the 2D system combining potential (2) and the quintic or LHY nonlinearity were found, respectively, as ones localized at $r \to \infty$, i.e. with a negative chemical potential, $\mu < 0$. A natural possibility, which is the subject of the present work, is to construct physically relevant 2D singular states, especially vortical ones, featuring the asymptotic form (4), on top of a modulationally stable background with a finite density at $r \to \infty$, which corresponds to $\mu > 0$. In that sense, these states may be called antidark ones [41–43]. Parallel to the consideration of them, we also produce usual (regular) vortices of the dark type, i.e. solutions subject to the asymptotic form (1)
at \( r \to 0 \), and quite simple but novel solutions in the form of flat vortices, which exist at the border between the antidark and dark vortex modes.

The subsequent presentation is organized as follows. The model is formulated in section 2, where we also present some analytical results—in particular, those obtained by means of the Thomas–Fermi (TF) approximation. Numerical results for the existence and stability of singular and regular (antidark and dark, respectively) states are reported in section 3. An essential novel numerical result concerns the excitation of the singular vortex from an input, which may only be a regular (nonsingular) optical vortex in free space. This issue, which is crucially important for predicting the possibility to create the singular vortex in the experiment, was not addressed in previous works. Direct simulations reported in section 3 clearly demonstrate that stable singular vortices are readily excited by nonsingular inputs carrying the vorticity. The paper is concluded by section 4.

2. Basic equations and analytical approximations

2.1. The nonlinear-Schrödinger/GP equation

The underlying 2D NLS/GP equation for the mean-field wave function, \( \psi \), including the above-mentioned ingredients, i.e. potential (2) and the nonlinear term (3), was introduced in works [11] and [12]. Here we write the scaled equation in the polar coordinates:

\[
\frac{\partial \psi}{\partial t} = -\frac{1}{2} \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) - \frac{U_0}{2r^2} \psi + |\psi|^6 \psi + \frac{k}{2} r^2 \psi. \tag{10}
\]

We focus on the most essential case when equation (10) does not include a cubic term. In particular, the optical nonlinearity of the colloidal suspension of metallic nanoparticles can be accurately adjusted so as to eliminate the cubic part, keeping only the quintic one, with \( \alpha = 4 \) in equation (10). In BEC, the cubic mean-field intra-component self-repulsion and inter-component attraction in the binary condensate may be brought in full balance, the quartic term (\( \alpha = 3 \)) being the single nonlinear one in the system (the LHY liquid [44]).

As concerns the BEC realization, the limit of extremely tight confinement in the transverse direction, which provides the reduction of the underlying 3D GP equation to the effective 2D form, corresponds to the case when the confinement size, \( a_L \), is much smaller than the BEC healing length, \( \xi \). In this limit, the nonlinearity in the LHY-corrected 2D GP equation is different from that given by equation (3). Instead, it is written as \( \ln \left( 1 + \frac{|\psi|^2}{\xi^2} \right) \cdot |\psi|^4 \psi \) [25–27]. On the other hand, experiments are usually conducted under the opposite condition, \( \xi \ll a_L \). In this case, one can still use the quartic nonlinearity in equation (10), assuming that a characteristic lateral size of patterns produced by this equation is much larger than \( a_L \), which is a realistic condition [12].

Equation (10) includes the trapping harmonic-oscillator (HO) potential with strength \( k \to 0 \). Aiming to consider modes supported by the flat background, the trap may be dropped. In terms of the experimental realization, which always includes the HO trap [10] in BEC, or a cladding which confines the waveguide in optics, that may be approximated by the last term in equation (10) [45], setting \( k = 0 \) means that the characteristic size of the mode’s core, which is either the singular peak of the anti-dark vortex, or the ‘hole’ of the regular dark one, is localized in an area of a size \( \ll k^{-1/2} \). Nevertheless, the HO potential plays an essential role in the analysis of stability of dark vortices [46], which is also shown below.

To address, as said above, both singular (antidark) and regular (dark) modes, we here consider positive and negative values of \( U_0 \) in equation (10). In optics the sign of \( U_0 \) is determined by the sign of the detuning between the carrier electromagnetic wave and resonant dopants. In terms of the above-mentioned realization of the setup in BEC, based on the magnetic polarizability of atoms, \( U_0 < 0 \) corresponds to diamagnetic susceptibility [47, 48].

Stationary solutions with chemical potential \( \mu > 0 \) and integer vorticity \( l \geq 0 \) (a.k.a. the photonic angular momentum, in terms of the corresponding optics models [6]) are looked for as

\[
\psi(r, l) = u(r)e^{-il\theta}, \tag{11}
\]

where real radial function \( u(r) > 0 \) satisfies equation

\[
\mu u = -\frac{1}{2} \left( \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \frac{U_0}{4} r^2 u \right) + u^{\alpha+1} + \frac{k}{2} r^2 u. \tag{12}
\]

Obviously, for \( \mu > 0 \) equation (12) with \( k = 0 \) admits the presence of the modulationally stable flat background,

\[
u^2(r \to \infty) = \mu^{2/\alpha}. \tag{13}\]

The respective asymptotic form of the solution to equation (12) at \( r \to \infty \) is

\[
u = \mu^{1/\alpha} + \frac{U_l}{2\alpha \mu^{1-1/\alpha}} r^{-2} + O \left( r^{-4} \right). \tag{14}\]

For \( U_l + 4/\alpha > 0 \), the asymptotic form of the singular solution to equation (12) at \( r \to 0 \), which extends the above expression (4), is

\[
u = \left[ \frac{1}{2} \left( U_l + \frac{4}{\alpha^2} \right) \right]^{1/\alpha} r^{-2/\alpha} + \frac{2^{1-1/\alpha}}{3} \mu^{1/\alpha} \frac{\left( U_l + 4/\alpha^2 \right)^{1/\alpha}}{\alpha U_l + 4(3/\alpha - 1)} r^{-2/\alpha} + O \left( r^{10/3} \right). \tag{15}\]

In the special case of the quintic nonlinearity (\( \alpha = 4 \)) and \( U_l = 1/4 \), the correction term in equation (15) diverges, being replaced by one \( \sim \sqrt{\mu} \). Similarly, the correction term diverges in the case of the quartic nonlinearity (\( \alpha = 3 \)) and \( U_l = 0 \), being replaced by a term \( \sim \sqrt{\mu} r^{1/3} \). It is relevant to mention that the linearized version of equation (10) produces no counterpart of
this asymptotic solution, hence it does not bifurcate from any solution of a linear equation.

Note that, in the special case of \( U_l = 0 \), i.e., \( U_0 = \hat{P} \) (see equation (5)), equation (12) with \( \mu > 0 \) and \( k = 0 \) admits a simple but novel solution, which features a flat density profile but, nevertheless, carries the vorticity. This solution can be written in terms of the polar coordinates, as well as in the Cartesian coordinates, \((x, y)\):

\[
\psi_{\text{flat}} = \mu^{1/\alpha} e^{-i(\alpha t + \mu \theta)} = \mu^{1/\alpha} e^{-i\alpha t} \left( \frac{x + iy}{x - iy} \right)^{1/2}.
\]

The flat-vortex state is an intermediate one between the singular (antidark) and regular (dark) ones.

It is relevant to mention that, for \( U_l < 0 \), the linearized version of equation (12) admits an exact ground-state (GS) solution,

\[
\psi_{\text{GS}}(r) = u_0 \sqrt{-U_l} \exp \left( -\frac{1}{2} \sqrt{k} r^2 \right),
\]

\[
\mu_{\text{GS}} = \sqrt{k} (1 + \sqrt{-U_l}),
\]

with arbitrary constant \( u_0 \). An attempt to extend wave function (17) to \( U_l > 0 \) produces a formal wave function with asymptotic form \( \sim \cos (\sqrt{U_l} \ln r) \) at \( r \to 0 \), which actually does not make sense. The non-existence of the meaningful wave function produced by the linearized equation (12) for \( U_l > 0 \) implies the onset of the quantum collapse, alias ‘fall onto the center’ [17], in the framework of the 2D linear Schrödinger equation with the attractive potential (2). This fact stresses the crucial role of the nonlinear term in equation (12), which makes it possible to create the wave function with the meaningful asymptotic form (15) for \( U_l > 0 \).

As suggested in work [12], the consideration of singular solutions is facilitated by the substitution of

\[
u(r) \equiv r^{-2/\alpha} \chi(r),
\]

which transforms equation (12) into

\[
\mu \chi = -\frac{1}{2} \frac{d^2 \chi}{dr^2} - \left( \frac{4}{\alpha} - 1 \right) \frac{1}{r} \frac{d \chi}{dr} + \frac{(U_0 + 4/\alpha^2)}{r^2} \chi + \frac{\chi^{\alpha+1}}{r^2}.
\]

In equation (20) \( k = 0 \) is set, as the effect of the trapping potential on the singularity structure is negligible. Substitution (19) separates the singular factor, which is integrable (i.e. the respective norm converges), and the singularity-free function \( \chi(r) \), which takes a finite value at \( r = 0 \):

\[
\chi(r = 0) = \left[ \frac{1}{2} \left( U_1 + \frac{4}{\alpha^2} \right) \right]^{1/\alpha},
\]

cf equation (15).

In what follows below, we present detailed results chiefly for the case of \( \alpha = 3 \), which makes the asymptotic form (15) more singular, hence more interesting. The results for \( \alpha = 4 \), i.e. the quintic nonlinearity, which is relevant for the optical model, are presented below too, in a more compact form.

2.2. The stability problem

2.2.1. The Bogoliubov–de Gennes equations. The stability of stationary states can be addressed by taking perturbed solutions as

\[
\psi(r, \theta, t) = e^{-i(\mu t + \mu \theta)} r^{-2/\alpha} \times \left[ \chi(r) + v_1(r) \exp (\Lambda t + im\theta) \right.
\]

\[
\left. + v_2^* r \exp (\Lambda^* t - im\theta) \right]
\]

where \( \chi(r) \) is a solution of equation (20) (cf equation (19)), \( m \) is an integer angular index of small perturbations represented by the eigenmode with components \( v_{1,2}(r) \) (the asterisk stands for the complex conjugate), and \( \Lambda \) is the respective eigenvalue, which may be complex. Instability takes place if there is at least one pair of eigenvalues with \( \text{Re} \Lambda \neq 0 \).

The substitution of expression (22) in equation (10) and linearization with respect to \( v_{1,2} \) leads to the Bogoliubov–de Gennes (BdG) equations [10]. First, in the model with the quartic nonlinearity (\( \alpha = 3 \)) they are

\[
(i \Lambda + \mu)v_1 = -\frac{1}{2} \left[ \frac{d^2}{dr^2} + \frac{1}{3r} \frac{d}{dr} + \frac{U_0 + 4/9 - (l + m)^2}{r^2} \right] v_1
\]

\[
+ \frac{k}{2} r^2 v_1 + \frac{1}{2r} \chi^2(r) (5v_1 + 3v_2),
\]

\[
(i \Lambda - \mu)v_2 = -\frac{1}{2} \left[ \frac{d^2}{dr^2} + \frac{1}{3r} \frac{d}{dr} + \frac{U_0 + 4/9 - (l - m)^2}{r^2} \right] v_2
\]

\[
+ \frac{k}{2} r^2 v_2 + \frac{1}{2r} \chi^2(r) (5v_2 + 3v_1).
\]

Further, for the underlying quintic nonlinearity (\( \alpha = 4 \)) the BdG equations are

\[
(i \Lambda + \mu)v_1 = -\frac{1}{2} \left[ \frac{d^2}{dr^2} + \frac{U_0 + 1/4 - (l + m)^2}{r^2} \right] v_1
\]

\[
+ \frac{k}{2} r^2 v_1 + \frac{1}{2r} \chi^2(r) (3v_1 + 2v_2),
\]

\[
(i \Lambda + \mu)v_2 = -\frac{1}{2} \left[ \frac{d^2}{dr^2} + \frac{U_0 + 1/4 - (l - m)^2}{r^2} \right] v_2
\]

\[
+ \frac{k}{2} r^2 v_2 + \frac{1}{2r} \chi^2(r) (3v_2 + 2v_1).
\]

For the special flat-density solution given by equation (16) (i.e. with \( k = 0 \) and \( U_0 = \hat{P} \)), the perturbed solution is introduced as (cf equation (22))

\[
\psi(r, \theta, t) = e^{-i(\mu t + \mu \theta)} \times \left[ \mu^{1/\alpha} + v_1(r) \exp (\Lambda t + im\theta) \right.
\]

\[
\left. + v_2^* (r) \exp (\Lambda^* t - im\theta) \right]
\]
and the BdG equations take a simpler form,
\[ i \lambda v_1 = -\frac{1}{2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2 + 2lm}{r^2} \right) v_1 + \beta \mu (v_1 + v_2), \]
\[ -i \lambda v_2 = -\frac{1}{2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2 - 2lm}{r^2} \right) v_2 + \beta \mu (v_2 + v_1), \]
where \( \beta (\alpha = 3) \equiv 3/2 \) and \( \beta (\alpha = 4) \equiv 2 \).

2.2.2. Analytical results for the stability. While BdG equations \((23), (24)\) and \((26)\) should be solved numerically, partial results for the stability can be obtained in an analytical form, in spite of the apparent complexity of the equations. In particular, for the singular vortex states (with \( l \geq 1 \)) an asymptotic analysis of equation \((23)\) at \( r \to 0 \) was performed in work \([12]\), looking for solutions as
\[ v_{1,2}(r) \approx v_{1,2}^{(0)} r^\gamma, \] \((27)\)
with \( \gamma \) determined by equations
\[ \gamma^2 - \frac{4}{3} \gamma - 3 \chi^3(r = 0) - m^2 = \pm \sqrt{4l^2 m^2 + 9 \chi^6(r = 0)}, \] \((28)\)
where \( \chi^3(r = 0) \) is taken as per equation \((21)\). Relevant solutions of equation \((28)\) are ones with \( \Re(\gamma) \geq 0 \) (otherwise, eigenmode \((27)\) is inappropriate). The emergence of an eigenmode which may bring instability is signaled by a solution of equation \((28)\) crossing \( \gamma = 0 \). The critical role is played by the modes with \( m = \pm 1 \), which account for drift instability of the vortex (the onset of spontaneous motion of the vortex’ pivot along a spiral trajectory, drifting away from \( r = 0 \), see figures \(2\)(c), \(8\)(c) and (d) presented below; in certain cases, usual (regular) dark vortices may be subject to a similar instability \([49]\)). Thus, a straightforward consideration of equation \((28)\) with \( m^2 = 1 \) readily demonstrates that the singular vortex may be stable, as a solution to equation \((10)\) with \( \alpha = 3 \), in the region of
\[ U_0 \geq (U_0)_{\min} = (7/9) (3l^2 - 1), \] \((29)\)
where the drift-perturbation mode does not exist. Because the asymptotic consideration at \( r \to 0 \), which leads to equation \((29)\), is not altered in the presence of the background far from \( r = 0 \), condition \((29)\) is equally relevant for the singular vortices in the present case. Numerical results, presented in the next section, confirm that this condition accurately determines the stability of the vortices with \( l \geq 1 \).

A similar analysis can be performed using the BdG equations \((24)\) for singular vortices produced by equation \((10)\) with \( \alpha = 4 \) (the quintic nonlinearity). In this case, the substitution of ansatz \((27)\) for the asymptotic eigenmodes leads to the following equation, instead of equation \((28)\):
\[ \gamma^2 - \gamma - 4 \chi^4(r = 0) - m^2 = \pm 2 \sqrt{2 \chi^2 + 4 \chi^4(r = 0)}, \] \((30)\)
where \( \chi (r = 0) \) should again be taken from equation \((21)\). The drifting instability, corresponding to \( m^2 = 1 \), is absent in the region of
\[ U_0 \geq 2l^2 - 1/2, \] \((31)\)
the boundary of which is also defined as the zero-crossing of \( \gamma \). It is worthy to note the similarity of the stability conditions \((29)\) and \((31)\).

For the special flat-vortex state \((16)\) numerical solution of the BdG equations \((26)\) around this state is a challenging issue, as the result is not stable enough against variation of technical details, such as the mesh size of the underlying numerical scheme. On the other hand, there is a simple argument in favor of stability of the flat vortex. Indeed, it is easy to find an asymptotic form of eigenmodes produced by equation \((26)\) at \( r \to 0 \), choosing \( m > 0 \), for the sake of the definiteness:
\[ v_2 \approx \text{const} \cdot r^{\sqrt{m^2 - 2m}} \] 
\[ v_1 \approx \frac{\beta \mu}{2} \cdot \text{const} \cdot r^{\sqrt{m^2 - 2m} + 2} - \text{const} \cdot r^{\sqrt{m^2 - 2m} + 2} \] \((32)\)

The eigenmodes given by equation \((32)\) are relevant ones for \( m \geq 2l \), otherwise they produce singular expressions. The latter condition excludes \( m = 1 \), hence the above-mentioned drift instability in ruled out. Further, usual dark vortices with \( l \geq 2 \) tend to be unstable against spontaneous splitting into a set of \( l \) unitary eddies, as demonstrated in detail in various settings \([46, 50-54]\) (see also figure \(12\) below). Because a growing perturbation eigenmode with azimuthal index \( m \geq 2 \) splits the multiple vortex into a set of \( m \) fragments, the perturbation azimuthal index which might be responsible for the splitting instability is \( m = l \); hence the condition \( m \geq 2l \) excludes this instability as well.

2.3. The TF approximation
The TF approximation may be applied either to equation \((20)\) by dropping derivatives in it, or similarly to equation \((12)\). In the former case, the result is
\[ U_{\text{TF}}(r) \equiv r^{-2/3} \chi_{\text{TF}}(r) \approx \left( \mu + \frac{1}{2} \left( U_l + \frac{4}{\alpha^2} \right) r^{-2} \right)^{1/\alpha}. \] \((33)\)
For \( U_l + 4/\alpha^2 > 0 \), this approximation is valid at all values of \( r \), assuming that \( k = 0 \) is set in equation \((12)\). In particular, it yields the correct singular (but physically acceptable) form of the solution at \( r \to 0 \), which is fully tantamount to equation \((4)\). If the TF approximation is applied directly to equation \((12)\) with \( k = 0 \), the result is different:
\[ U_{\text{TF}}(r) \approx \left( \mu + \frac{U_l}{2} r^{-2} \right)^{1/\alpha}. \] \((34)\)
While TF approximation \((33)\) is more accurate at \( r \to 0 \), the expansion of the alternative approximation \((34)\) with \( k = 0 \) at \( r \to \infty \) yields a result which is fully tantamount to the correct asymptotic form \((14)\).

A nontrivial issue which is suggested by the combination of both versions of the TF approximation is a possibility of
the existence of the antidark singular modes in interval (9), where equation (33) predicts a profile which is a monotonously decaying function of \( r \), while both equations (34) and (14) demonstrate that, at \( r \to \infty \), \( u(r) \) approaches the background value \( \mu^{1/3} \) from below. These conflicting predictions suggest that, in interval (9), there may exist a mode with a non-monotonous radial profile, i.e. with a local minimum of \( u(r) \) at some finite \( r \). Numerical results displayed below confirm this conjecture (see figure 3(c)).

For regular dark vortices existing at \( U_1 < 0 \), with \( u(r) \) vanishing at \( r \to 0 \), substitution (19) is irrelevant, therefore the appropriate TF approximation is the one applied to equation (12). It predicts a structure which, as usual, includes artifacts in the form of an inner hole at small \( r \) and zero tail at large \( r \) (see [9]):

\[
\tilde{u}_{TF}(r) = \begin{cases} 
0, & \text{at } r^2 > r^2_\alpha \equiv k^{-1} \left( \mu + \sqrt{\mu^2 - k|U_1|} \right), \\
\left[ \mu - (|U_1|/2)r^2 - (k/2)r^3 \right]^{1/\alpha}, & \text{at } r^2 < r^2 < r^2_\alpha, \\
0, & \text{at } r^2 < r^2_\alpha \equiv k^{-1} \left( \mu - \sqrt{\mu^2 - k|U_1|} \right). 
\end{cases} 
\]  

This TF solution exists above the threshold,

\[
\mu > \mu_{thr} \equiv \sqrt{k|U_1|},
\]  

which is the TF limit of the GS eigenvalue (18) produced by the linearized version of equation (12). Equation (35) predicts a maximum of the TF field at

\[
(r_{max})_{TF} = \left( \frac{|U_1|}{k} \right)^{1/4}. 
\]  

Note that, in the TF approximation, expressions (36) and (37) do not depend on \( \alpha \).

In the case of \( k = 0 \), the above-mentioned zero-tail artifact disappears. In this case, \( \mu_{thr} \), as given by equation (36), vanishes too, and the TF approximation simplifies to

\[
\tilde{u}_{TF}(r) = \begin{cases} 
\left[ \mu - (|U_1|/2)r^2 \right]^{1/\alpha}, & \text{at } r^2 > r^2_\alpha (k = 0) \equiv |U_1|/(2\mu), \\
0, & \text{at } r^2 < |U_1|/(2\mu). 
\end{cases} 
\]  

Note that the TF solution (38) agrees with the correct asymptotic form (14) at \( r \to \infty \).

True dark-vortex solutions include no hole at small \( r \). Instead, the correct asymptotic form at \( r \to 0 \) is

\[
u(r) \approx \text{const} \cdot r^{\sqrt{-U_1}}, \]  

as per equation (17). In the case of \( U_0 = 0 \), i.e. \( U_1 = -l^2 \) (see equation (5)), this asymptotic form carries over into the usual one for the dark vortices, given by equation (1). Note also that the regular dark vortices exist in the presence of the attractive potential \( (U_0 > 0) \), provided that it is not too strong, viz., \( U_0 < l^2 \), making \( U_1 \) negative.

3. Numerical results

Stationary solutions of equation (10) were obtained by dint of the Newton’s iteration method, and their stability was identified through numerical solution of BdG equations (23). Then, the predicted (in)stability was verified in simulations of equation (10) for perturbed evolution of the solutions in question. The simulations were carried out in the Cartesian coordinates, using the split-step fast-Fourier-transform method. The numerical results are presented below chiefly for the quartic nonlinearity \( (\alpha = 3) \), because, as mentioned above, the solutions are more singular in this case, make it more interesting. For the quintic nonlinearity \( (\alpha = 4) \), numerical results, which are not shown here in detail, are quite similar.

3.1. Singular and intermediate states

At all values of the parameters, i.e. \( \mu > 0, U_0 > 0 \) (interval (9) is considered separately below), and \( l \geq 0 \), the numerically found profiles of the antidark singular states are virtually identical to their TF counterparts given by equation (33). The same is true for the quintic nonlinearity, with \( \alpha = 4 \) in equation (2) (not shown here in detail). A typical example is displayed in figure 1 for \( U_1 = 1 \) (see equation (5)) and \( \mu = 0.5 \). While panel 1(a), which shows the global shape of the singular solution, does not make it possible to discern details of the singular peak, this is shown by the lower lines in panel 1(b), by means of...
Figure 2. (a), (b) The same as in figures 1(b) and (c), but for $U_0 = 1.533$. This state is unstable, in agreement with equation (29). Panel (c) shows the trajectory of spontaneous motion of the vortex’ pivot, produced by simulations of equation (10) for total evolution time $t = 300$.

Figure 3. Panels (a) and (b): the same as in figures 1(a) and (b), but for $l = 0$, $\mu = 1$, and $U_0 = -0.2$ (this value belongs to interval (9)). Panel (c) shows a shallow local minimum in the numerically found radial profile of the solution. This state is stable, as are all solutions with $l = 0$ belonging to interval (9).

of function $\chi(r)$, from which the singular factor is eliminated with the help of substitution (19).

Further, up to the numerical accuracy, the results obtained for the stability of the singular states exactly agree with the analytical stability condition given by equation (29). This conclusion is illustrated by figures 1(b), (c) and 2, where the singular vortices with $l = 1$, $\mu = 1$ and spectra of their stability eigenvalues are displayed, respectively, for $U_0 = 1.578$ and $U_0 = 1.533$, while the respective stability-boundary value is $(U_0)_{\text{min}} = 14/9 \approx 1.556$. It is seen that, accordingly, the former solution is stable, while the latter one is not. Additionally, figure 2(c) displays the trajectory of spontaneous motion of the pivot caused by the drift instability of the stationary vortex shown in figure 2(a). The drift character of the instability complies with the fact that the unstable eigenvalues, shown in panel 2(b), are produced by small perturbations with $m = 1$ in equation (22).

Numerical results confirm as well the existence of the ‘counter-intuitive’ singular antidark modes in interval (9). An example, and its comparison to the TF approximation, as given by equation (33), are displayed in figure 3. In particular, the zoom of a segment of the modal profile shown in panel (c) clearly confirms the existence of a shallow local minimum of $u(r)$. As mentioned above, this feature is made necessary by the impossibility to directly match the asymptotic form of the singularity, given by equation (15), and the asymptotic expansion (14) at $r \to \infty$. The singular states with $l = 0$ are stable in interval (9), while their counterparts with $l \geqslant 1$ are not, in accordance with equation (29).

The above singular states are produced by setting $k = 0$ in equations (10) and (2), as the HO trapping potential does not play a significant role for the singular states. On the other hand,
the HO potential produces an essential effect if applied to the flat-vortex mode (16), since it distorts the flat profile, as shown in figure 4. Note that the width of the profile is close to that of the GS wave function of the linearized equation, given by equation (17), while its amplitude is accurately predicted by the TF approximation, which yields $u_{TF}(x = 0) = \mu^{1/3} \approx 1.59$ in this case.

### 3.2. Excitation of singular vortices by regular vortex inputs

An issue which is crucially important for the possibility of the creation of singular (antidark) vortices in the experiment, using the nonlinear light propagation and BEC alike, is a scenario for the excitation of such ‘extraordinary’ modes, as incident optical beams, which are employed for the creation of the usual photonic [1–6, 55, 56] and matter-wave [7–9, 57, 58] vortices, may carry only the ordinary (nonsingular) vorticity.

Thus, it is relevant to simulate equation (10) with $U_0 > 0$ and the input taken as a usual vortex beam, i.e. a stationary solution of the linearized version of equation (10) with $U_0 = 0$:

$$\psi_0(r, \theta) = a_0 r^l e^{il\theta} \exp\left(-\frac{1}{2} \sqrt{\kappa} r^2\right),$$

(40)

where $a_0$ is an arbitrary constant. To the best of our knowledge, such simulations were not reported in previous works. Note that input (40) cannot generate an ordinary (dark nonsingular) vortex state, as it does not exist in the case of $U_0 > 0$, according to equation (39).

Direct simulations of equation (10) readily confirm that, at all values $U_0 > 0$, and for both nonlinearities considered here, which correspond to $\alpha = 3$ and 4, initial condition (40) gives rise to perturbed vortex states, close to the expected singular ones, which keep the initial vorticity, $l$. A typical example is presented in figure 5, which displays amplitude profiles of the input and output vortex modes (panels (a) and (c), respectively) and the overall image of the spatiotemporal evolution from the regular initial condition to a quasi-singular output, shown by means of the radial cross section in panel (b).

In figure 5(c), it is observed that the emerging central singular vortex (in the simulation, its amplitude remains finite due to the effect of the Cartesian numerical mesh) is surrounded by an additional set of radial waves. The central core and radial undulations are separated by a zero-amplitude ring, whose radius may be predicted by the TF approximation, applied to equation (12) with $\mu = 0$. Indeed, such an approximation predicts $u(r) = 0$ at $r_0 = (U_1/k)^{1/4}$. For parameters of figure 5, this formula yields $r_0 \approx 2.34$, which is close to the zero-amplitude radius observed in figure 5(c).

The fact that the simulations displayed in figure 5 were performed in the Cartesian coordinates explains some deviation from the axial symmetry observed in the figure (in particular,
Figure 7. (a) The numerically found shape of the dark mode, obtained as a solution of equation (12) with $k = 0$ (no HO trapping potential), $\alpha = 3$, $U_0 = -1$, $l = 0$, and $\mu = 1$. Its TF counterpart is displayed too, as produced by equation (38). (b) The spectrum of stability eigenvalues for small perturbations around this mode, $\Lambda \equiv \Lambda_r + i\Lambda_i$, as produced by the numerical solution of the respective BdG equations. The spectrum includes no unstable eigenvalues with $\Lambda_r \neq 0$. (c) The simulated evolution of the same mode, with 5% initial random perturbations added to it, confirms its stability.

Figure 8. Panels (a) and (b) show the same as (a) and (b) in figure 7, but for an unstable dark vortex, produced by the numerical solution of equations (12) and (23) with $U_0 = 0.03$, $k = 0.02$, for $l = 1$ and $\mu = 1$. The TF profile in (a) is produced by equations (35) and (37). The unstable pair of eigenvalues in (b) corresponds to the azimuthal perturbation index $m = 1$, cf equation (22). Trajectories of the motion of the central point (pivot) of an unstable dark vortex from (a), as produced by simulations of equation (10) with total evolution time $t = 1000$ and 4000 are displayed in (c) and (d), respectively.

the pivot of the emerging singular vortex is somewhat shifted off the central point and performs circular motion around it.

The simulations used an absorber installed at edges of the spatial domain. As a result, about 2% of the initial norm was lost in the course of the simulations, by $t = 50$.

3.3. Regular dark vortices

We address the regular dark modes, first, in the case of the repulsive central potential, i.e. $U_0 < 0$, with $l = 0$ and 1, supported by the flat background (14) in the absence of the HO
Figure 9. The real instability growth rate, $\Lambda$, for the dark-vortex mode with $l = 1$, $\mu = 1$ and $m = 1$, cf equation (22), produced by the numerical solution of the BdG equations, vs. strength $U_0$ of the pulling-to-the-center potential (2). The strength of the HO trapping potential is fixed as $k = 0.02$. Panels (a) and (b) show two regions of complex instability, which are, approximately, $0.01 < U_0 < 0.17$ and $0.43 < U_0 < 1$.

Figure 10. (a) The numerically found profile of a stable dark vortex with $l = 2$ and $\mu = 2$, obtained at $U_0 = 0$ (hence, $U_i = -4$), $k = 0.02$, and $\alpha = 3$ in equation (10). The TF counterpart of this profile, produced by equations (35) and (37), is shown too. (b) Stable evolution of this double vortex with initially added random perturbations.

Figure 11. The same as in figures 8(a) and (b), but for the unstable double dark vortex ($l = 2$), with $\mu = 3.2$. The parameters are the same as in figure 10: $U_0 = 0$, $k = 0.02$, $\alpha = 3$. The unstable pair of eigenvalues corresponds to the azimuthal perturbation index $m = 2$, cf equation (22). There are no unstable eigenvalues with $m = 1$. 
Figure 12. A set of snapshots illustrating the evolution of the unstable dark double vortex from figure 11. It features a periodic chain of splittings into a rotating pair of unitary vortices (as observed at $t = 250$ and 600), alternating with recombinations back into the double vortex (as seen at $t = 0$ and 400).

Figure 13. The same as in figure 11, but for the unstable double dark vortex ($l = 2$), with $\mu = 2$, $U_0 = 1$, $k = 0.02$, and $\alpha = 3$. (b) The spectrum of stability eigenvalues, the unstable real pair corresponding to the azimuthal perturbation index $m = 1$, cf equation (22). There are no unstable eigenvalues for $m = 2$.

Figure 14. A set of snapshots illustrating the evolution of the unstable dark double vortex from figure 13. It splits in two separate unitary vortices, which are expelled from the center one by one, performing circular motion in a confined region. Both the calculation of the stability eigenvalues through the numerical solution of equation (23) and direct simulations demonstrate that all the dark modes with $l = 0$ and 1 are stable as solutions of equation (10) with $U_0 < 0$ and $k = 0$. An example is presented in figure 7, which demonstrates the shape, spectrum of eigenvalues, and perturbed evolution of a typical solution of such a type.

As mentioned above, the dark vortex with $l \geq 1$ may exist in the case of the attractive central potential, $\mu$. Different values of $U_0$ and $l$, which amount to the same value of combination (5), give rise to identical shapes of the modes (while their stability depends on $U_l$ and $l$ separately, as shown below). In figure 6 we plot a set of profiles of the dark modes produced by the numerical solution of equation (12) for a fixed value of the chemical potential, $\mu = 0.5$, and three different values of $U_l$, viz., $-1$, $-2$, and $-3$. In the same figure, the numerically found profiles are compared to the TF approximation, as given by equation (38).
which corresponds to $0 < -U_0 < \bar{\rho}$, as per equation (5). Even for the simplest case of $l = 1$, results for the stability are complicated in this case. The numerical analysis performed for $l = 1$ reveals an alternation of stability and instability regions in the interval of $0 < U_0 < 1$. The analysis was performed with the inclusion of a weak but finite trapping HO potential in equations (10) and (12) and the corresponding BdG equations, as, in the absence of the trap, the necessity to maintain the vertical boundary conditions at $r \to \infty$ makes it difficult to solve the stability problem, see work [46]. As an example, figure 8 presents the shape and spectrum of the stability eigenvalues for the dark vortex, as obtained at a small positive value of the attraction strength, $U_0 = 0.03$, with $\mu = 1$ and a weak trapping potential, corresponding to $k = 0.02$ in equation (10).

The instability represented by the pair of real eigenvalues in figure 8(b) corresponds to azimuthal index $m = 1$ of small perturbations, cf equation (22), which, as said above, implies drift instability. In accordance with the expectation, in direct simulations the pivot of the unstable dark vortex spontaneously starts motion along a spiral-like unwinding trajectory, which however, remains trapped in the area of $r < r_{\text{trap}} = k^{-1/4} \approx 2.7$, where $r_{\text{trap}}$ is the trapping radius imposed by the HO potential, see equation (17). Motion of the pivot along the confined trajectory of an apparently irregular shape was going on as long as the simulations were running.

In interval (41) corresponding to $l = 1$, i.e. $0 < U_0 < 1$, the calculation of stability eigenvalues produced by the numerical solution of the BdG equations produces a complex structure (feasibly, a fractal one) of alternating stability and instability windows, as shown in figure 9. This structure is confined to two sub-intervals, viz., $0.01 < U_0 < 0.17$ and $0.43 < U_0 < 1$.

The complex structure observed in figure 9 may be interpreted as a result of interplay and possible resonance between two basic oscillatory modes admitted by the GP equation (10). This possibility is suggested by the affinity of the dynamical setting under the consideration to a similar 1D system, in which the counterpart of the dark vortex state with $l = 1$ is the dark soliton [59]. In the 1D case, one basic mode represents collective dipole oscillations of the condensate as a whole (‘sloshing’) in the HO trap [60, 61], while the other mode amounts to oscillatory motion of the dark soliton under the action of the same trapping potential [62, 63]. The possibility of producing complex dynamics by the interplay of these 1D modes was demonstrated in various setups [64–66]. In the present 2D situation, perusal of the simulations demonstrates that the quasi-spiral motion of the vortex’ pivot, in the case of the instability of the stationary vortex, is indeed coupled to sloshing-rotating motion of the trapped condensate as a whole, therefore the resonance between the circular motion of the vortex displaced from the center and the collective oscillatory-rotational motion of the condensate is a feasible cause of the complex structure displayed in figure 9. Detailed analysis of this conjecture should be a subject of a separate work.

Another essential point of the analysis is instability of double dark vortices, with $l = 2$, against splitting into unitary ones, which occurs in many models supporting dark vortex modes [46, 50–54]. In this case too, the HO trapping potential should be kept in equation (10), to secure the robustness of the numerical scheme (cf a similar situation in the case of the 2D GP equation with the cubic term [46]). A weak trap with $k = 0.02$ is sufficient to support stable double vortices at particular values of $U_0$ and $\mu$, provided that $U_l$ is negative, see equation (5). An example of the profile of a stable dark double vortex and its stable perturbed propagation is presented in figure 10.

The increase of the chemical potential from $\mu = 2$, corresponding to the stable double vortex in figure 10, to $\mu = 3.2$ makes the dark double vortex unstable against perturbations with azimuthal index $m = 2$, cf equation (22). The vortex’ profile and spectrum of the respective (in)stability eigenvalues are displayed in figure 11.

Because the instability of the double vortex in figure 11 is driven by the perturbation eigenmode with azimuthal index $m = 2$, it splits the initial vortex ring into a pair of unitary
vortices. As shown in figure 12, the splitting is followed by recombination back into the original vortex, thus initiating a periodic chain of splittings and recombinations. Simultaneously, the vortex pair in the split state rotates persistently.

Unlike the motion of the unstable vortex with \( l = 1 \), which is shown above in figures 8(c) and (d), the periodic fission-fusion evolution of the double vortex, presented in figure 12, is not coupled to conspicuous sloshing motion of the condensate as a whole. This circumstance may be explained by the fact that monopole modes of stirring perturbations, produced by the motion of two unitary vortices (separated by a relatively small distance in figure 12) in opposite directions, cancel each other through destructive interference. The remaining dipole stirring mode produces a much weaker sloshing effect.

At other parameter values, another instability eigenmode of a double vortex is possible. An example of such a vortex is shown in figure 13. In this case, the instability corresponds to azimuthal index \( m = 1 \) (rather than \( m = 2 \)). Accordingly, direct simulations displayed in figure 14 show that the double vortex splits in a pair of unitary vortices. One of them is expelled from the central position first, which is later followed by the expulsion of the second one. Both secondary vortices perform persistent orbital motion in a confined region. Unlike what is observed in figure 12, the splitting is irreversible in figure 14 i.e. the unitary vortices never recombine back into a single double vortex.

The alternation of stability and instability regions for the double vortices, following the variation of \( U_0 \), forms a complex structure, which is plotted in figure 15, separately for eigenmodes with the azimuthal index \( m = 2 \) in (a), and \( m = 1 \) in (b). Note that there is no instability at \( U_0 < -1/2 \), and the dark double vortex does not exist at \( U_0 \geq 4 \). The structure observed in the figure may be interpreted as a result of the interplay and possible resonance between two oscillatory modes. One represents internal oscillations in the pair of separated unitary vortices, while the other mode corresponds to rotation of the vortex pair.

4. Conclusion

The main objective of this work is to extend the family of singular but physically relevant singular vortices in the 2D models of optical and matter waves, which combine attractive potential (2) and the quartic (\( \alpha = 3 \)) or quintic (\( \alpha = 4 \)) self-repulsion in equation (10), for the antidark vortex states built on top of a finite background. It is demonstrated that the TF approximation provides a very accurate fit for the numerically found singular states. Their stability exactly follows the recently proposed analytical condition, given by equation (29) for \( \alpha = 3 \), and by the newly derived equation (31) for \( \alpha = 4 \). A nontrivial finding is the existence of the singular states in the case when the effective potential strength, given by equation (5), is negative, corresponding to weak repulsion, instead of the pull to the center. In this case, the shape of the mode features a shallow local minimum, as shown in figure 3(c). An essential novel result is the scenario for the excitation of perturbed singular vortices by the ordinary (nonsingular) vortex input, which is shown in figure 5. Parallel to that, the analysis is performed for the usual regular (dark) vortices, which are also supported by a finite background, in the case when the effective central potential is repulsive. The dark states with vorticities \( l = 0 \) and \( l = 1 \) are completely stable for \( U_0 < 0 \) in equation (12). In the interval of \( 0 < U_0 < 1 \), where the effective potential’s strength, including the centrifugal term, \( U_0-1 \), corresponds to the repulsion, the accurate stability analysis for the dark vortex with \( l = 1 \) is possible in the presence of the weak confining HO potential. In this case, the vortex features an intricate pattern of alternating stability and instability windows, as shown in figure 9. Unstable vortices spontaneously move along complex trajectories, which do not leave the central area confined by the HO trap (an example is displayed in figures 8(c) and (d)). The presence of the weak trap is also necessary for the accurate analysis of the stability of dark double vortices. In this case too, stability and instability regions compose a complex pattern. Unstable double vortices periodically split into rotating pairs of unitary vortices and recombine back. Finally, simple but new solutions for flat vortices are obtained at the border between the antidark singular vortices and dark regular ones.

A challenging direction for extension of the present work may be the analysis of antidark vortex states in the 3D model with the same central potential.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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