On the MacPhersonian

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In 2003 Daniel K. Biss published, in the Annals of Mathematics [1], what he thought was a solution of a long standing problem culminating a discovery by Gelfand and MacPherson [3]. Six years later he was encouraged to publish an “erratum” of his prove [2], observed by Nikolai Mnev; up to now, the homotopy type of the MacPhersonian had remained a mystery...

The aim of this lecture is to convince the attendee of the fact that, using a completely different aproach to those used before, we can prove that the (acyclic) MacPhersonian has the homotopy type of the (affine) Grassmannian.

1 Preliminaries

An oriented matroid $M = (E, C)$ consists of a set $E$ and a symmetric relation on its disjoint pairs of subsets $C \subset \binom{E}{2}$, called the circuits, which encodes the so-called minimal Radon partitions of the matroid; if a pair of subsets, $A$ and $B$, form a circuit, then we denote this fact by $A \uparrow B$ (cf. [7]).

The basic example of an oriented matroid is a family of points in Euclidian $d$-space $E \subset \mathbb{R}^d$; its circuits are the minimal (affine) dependences, encoded as Radon partitions, i.e.,

$$A \uparrow B \iff \text{conv}(A) \cap \text{conv}(B) \neq \emptyset.$$ 

However, oriented matroids are a bit more general than that, and there are much more oriented matroids than configurations of points. Indeed, to decide if an oriented matroid is representable with points is an NP-hard problem (see [8]).

1.1 The Grassmannian $G^d_n$

Given $n$ points $p_1, \ldots, p_n$ in dimension $d$, we can build an $d \times n$ matrix $M = (p_i)$ consisting of these $n$ column vectors. This matrix is naturally identified with the linear map which sends the canonical base of $\mathbb{R}^n$ onto the points, and two such functions $M, M'$ are affinely equivalent if, and only if, their respective kernels $K_M, K_{M'}$ intersect the hyperplane

$$\mathbb{H}^\perp := \{ x \in \mathbb{R}^n : x \cdot (1, \ldots, 1) = 0 \}$$
in exactly the same plane (cf. [5]): $K_M \cap \Pi^\perp = K_{M'} \cap \Pi^\perp$.

If the points span the $d$-dimensional space, the kernel of such a function is of dimension $n - d$, and we can identify the configuration with the $n - d - 1$ subspace of the hyperplane $\Pi^\perp \approx \mathbb{R}^{n-1}$; thus, we can identify each configuration with a point of the Grassmannian $\mathcal{G}^d_n$ which consists of all $n - d - 1$ planes of $\mathbb{R}^{n-1}$, with the natural topology.

Furthermore, the intersection of each such a $n - d - 1$ plane $K_M \cap \Pi^\perp$ with the $n$-octahedron
\[ O_n := \text{conv}(\{ \pm e_i : i = 1, \ldots, n \}) , \]
the convex hull of the canonical base of $\mathbb{R}^n$ and its negatives, define the so-called Radon complex, a CW-complex which is an $(n - d - 2)$-sphere (see [5]), whose 1-skeleton is the so called (co)circuit graph of the corresponding oriented matroid; the cocircuit graphs of oriented matroids where characterised in [4, 6].

Observe that $O_n \cap \Pi^\perp$ corresponds to the space of all configurations of $n = d + 2$ points in dimension $d$; it is the double cover of $\mathcal{G}^d_{d+2}$, which is homeomorphic to the $d$-projective space.

\begin{figure}
\centering
\includegraphics[width=0.2\textwidth]{figure}
\caption{An element of $\mathcal{G}^1_3$.}
\end{figure}

\subsection{The MacPhersonian $\mathcal{M}^d_n$}

The family of all oriented matroids on $n$ elements in dimension $d$ can be partially ordered by “specialitation”; viz., we say that $M \leq M'$ if every minimal Radon partition of $M'$ is a Radon partition of $M$. Given such a partial order, we can define a topological space, in the usual way, and in the case of all (acyclic) oriented matroids of $n$ elements in dimension $d$ we get the so-called (acyclic) MacPhersonian $\mathcal{M}^d_n$.

For example, consider all oriented matroids arising from 4 points in the affine plane: there are essentially 3 labeled configurations in general position with the four points in the boundary of the convex hull, and 4 configurations where one of them is in the interior of the convex hull of the other three. These 7 uniform oriented matroids can be represented by 3 squares and 4 triangles that, when joining them together, form the hemicubooctahedron... this is exactly $O_4 \cap \Pi^\perp$ after identifying antipodes; that is to say that $\mathcal{M}^4_4 = \mathcal{G}^2_4$. 

2 Stretching spheres

The main idea to better understand the MacPhersonian, is to identify each oriented matroid with its set of circuits. They are encoded in the vertices of the circuit graph, which is well known to be the 1-skeleton of a PL-sphere; indeed, we start with such a “wiggly” sphere and applying to it a Ricci-type flow to stretch it ending up with a configuration of points. This way, we can assign to each oriented matroid a configuration of points, and in such a way that we are defining a homotopical retraction from the MacPhersonian to the Grassmannian.

2.1 Oriented Matroids as spheres inside $G_{n-2}^n$.

In [6] we characterised the circuit graphs $G(M)$ of oriented matroids as some graphs which can be embedded in what we called there “the $k$-dual of the $n$-cube”; that was to say that its vertices can be labeled with the faces of $\mathcal{O}_n \cap \mathbb{I}^k := \{ x \in \mathbb{R}^n : \sum |x_i| = 2 \} \cap \{ x \in \mathbb{R}^n : \sum x_i = 0 \}$.

We will identify such circuits (and the corresponding vertices in the circuit graph) with the baricentre of such faces. The vertices of $G_{n-2}^n$ are points of the form

$$e_{ij} = e_i - e_j,$$

where $i \neq j \in \{1, \ldots, n\}$ and $e_i$ is an element of the canonical base of $\mathbb{R}^n$; and each face of it can be determined in terms of the convex hull of the vertices that it contain. Therefore, each circuit $A \uparrow B$ of a given oriented matroid $M$ can (and will) be coordinatised in terms of differences of the canonical base vectors of $\mathbb{R}^n$:

$$c_{A \uparrow B} = \sum \lambda_a e_a - \sum \mu_b e_b,$$

where $\sum \lambda_a = -\sum \mu_b = 1$.

If we “fill up the faces” of the circuit graph, considering all its vectors, we end up with a PL-complex which is well known to be an sphere of dimension $n - d - 2$; that is, we can
identify each oriented matroid on \( n \) elements in dimension \( d \) with a, not necessarily “flat”, \((n - d - 2)\)-sphere inside \( G_{n}^{n-2} \). An oriented matroid \( M \) is stretchable if and only if there is a flat \((n - d - 2)\)-sphere inside \( G_{n}^{n-2} \) whose 1-skeleton is the circuit graph \( G(M) \) (cf., [5]).

2.2 The “fat” MacPhersonian \( S_{n}^{d} \)

We now consider a space which is much more “fatty” than the MacPhersonian and the Grassmannian, but contains both of them:

\[
S_{n}^{d} := \left\{ S^{n-d-2} \hookrightarrow G_{n}^{n-2} : sk^{1}(S^{n-d-2}) = G(M), M \in \mathcal{M}_{n}^{d} \right\}.
\]

That is, we take several copies of each sphere representing an oriented matroid, while considering all possible embeddings of such a sphere inside its natural ambiance space \( G_{n}^{n-2} \), with the extra freedom of having its vertices in any point representing the corresponding circuit (not only the barycentre of it).

Clearly, by definition, \( S_{n}^{d} \) and \( \mathcal{M}_{n}^{d} \) have the same homotopy type, and \( G_{n}^{d} \) is embedded in \( S_{n}^{d} \) as it is in \( \mathcal{M}_{n}^{d} \); we have the following diagram:

![Diagram of S_n^d and G_n^d](image)

Figure 3: Stretching spheres with Ricci-type flows

It remains to show that \( G_{n}^{d} \) is a homotopical retract of \( S_{n}^{d} \).

2.3 Combinatorial Curvature Flow on \( G_{n}^{n-2} \)

Let \( S \in S_{n}^{d} \) be an element of the “fat” MacPhersonian, a “wiggly” PL-sphere of dimension \( n - d - 2 \) whose vertices lie on the faces of \( G_{n}^{n-2} \) of dimension at most \( d \). Each vertex \( v \in S \) represents a circuit \( A \uparrow B \) of an oriented matroid \( M = (E,C) \), \(|A \cup B| \leq d + 2\), and the local curvature of \( S \) at \( v \) will be modeled as a function which depends only on the neighbourhood \( N(v) = \{v_{1}, \ldots, v_{m}\} \) in \( sk^{1}(S) = G(M) \), the 1-skeleton of the complex which is the circuit graph of the oriented matroid.

It is well known that the order of \( N(v) \) is even, and each vertex \( v_{i} \in N(v) \) can be related to its “opposite” vertex \( v'_{i} \) (cf., [4]); furthermore, the edges of \( G(M) \) can be partitioned in disjoint cycles, and whenever one of such a cycles contains \( v \), it also contains a pair of opposite vertices in its neighbourhood. The aim of this section is to exhibit a flow to “stretch” all such cycles to become flat cycles (i.e., contained each in a 2-flat).
For, let \( u_i \) and \( u'_i \) be normal vectors to the planes spanned by \( \{ v, v_i \} \) and \( \{ v'_i, v \} \), respectively, and denote by \( \theta_i = \angle u_i u'_i \) the angle between these vectors (i.e., \( \theta_i = 0 \) if and only if \( v_i, v, v'_i \) are coplanar vectors with the origin). We use as a local measure of curvature the function \( \varsigma(v) = \sum \eta_i \), where \( \eta_i = \sin \theta_i \).

Consider the following system of ODE in \( \mathbb{R} \): for each \( v = c_{A \cup B} \in S \),

\[
\dot{v}(t) = \sum \eta_i \Pi_{A \cup B}(v_i + v'_i - 2v) = \eta \vec{N},
\]

where \( \Pi_{A \cup B} \) denotes the projection onto the plane spanned by \( \{ e_{ij} : i, j \in A \cup B \} \). Due to the Cauchy-Lipschitz theorem, this system of Ricci-type equations has a solution, for some \( T > 0 \), in the interval \( t \in [0, T] \). Furthermore, it can be proved that the curvatures \( \varsigma(v) \) tend to zero exponentially fast as the time \( t \) goes on. The fixed points of this system are the elements of \( G_n^d \) and therefore it defines a strong retraction of \( S_n^d \) onto \( G_n^d \).

References

[1] Biss, Daniel K.; The homotopy type of the matroid Grassmannian. Ann. Math. (2) 158, No. 3, 929-952 (2003).

[2] Biss, Daniel K.; Erratum to “The homotopy type of the matroid Grassmannian”. Annals of Mathematics, 170 (2009), 493?493.

[3] Gelfand, I.M.; MacPherson, Robert D.; A combinatorial formula for the Pontrjagin classes. Bull. Am. Math. Soc., New Ser. 26, No.2, 304-309 (1992).

[4] Knauer, Kolja; Montellano-Ballesteros, Juan José; Strausz, Ricardo; A graph-theoretical axiomatization of oriented matroids. Eur. J. Comb. 35, 388-391 (2014).

[5] Montellano-Ballesteros, Juan José; Strausz, Ricardo; Counting polytopes via the Radon complex. J. Comb. Theory, Ser. A 106, No. 1, 109-121 (2004).

[6] Montellano-Ballesteros, Juan José; Strausz, Ricardo; A characterization of cocircuit graphs of uniform oriented matroids. J. Comb. Theory, Ser. B 96, No. 4, 445-454 (2006).

[7] Nešetřil, Jaroslav; Strausz, Ricardo; Universality of separoids. Arch. Math., Brno 42, No. 1, 85-101 (2006).

[8] Shor, Peter W.; Stretchability of pseudolines is NP-hard. Applied geometry and discrete mathematics, Festschr. 65th Birthday Victor Klee, DIMACS, Ser. Discret. Math. Theor. Comput. Sci. 4, 531-554 (1991).