Localization of Massless Spinning 
Particles and the Berry Phase

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Abstract

The components of the position operator, at a fixed time, for a massless and 
spinning particle with given helicity \( \lambda \) described in terms of bosonic degrees of 
freedom have an anomalous commutator proportional to \( \lambda \). The position operator 
generates translations in momentum space. We show that a ray-representation for 
these translations emerges due to the non-commuting components of the position 
operators and relate this to the Berry-phase for photons. The Tomita-Chiao exper-
iment then gives support for this relativistic and quantum mechanical description 
of photons in terms of non-commuting position operators.

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The concept of localization of relativistic elementary systems has a long and intriguing history in physics (see e.g. Refs [1, 2, 3, 4]). Observations of physical phenomena takes place in space and time. The notion of localizability of particles, elementary or not, then refers to the empirical fact that particles, at a given instance of time, appear to be localizable in the physical space.

In the realm of non-relativistic quantum mechanics the concept of localizability of particles is built into the theory at a very fundamental level and is expressed in terms of the canonical commutation relation between a position operator and the corresponding generator of translations, i.e. the canonical momentum of a particle. In relativistic theories the concept of localizability of physical systems is deeply connected to our notion of space-time, the arena of physical phenomena, as a 4-dimensional continuum. In the context of the general theory of relativity the localization of light rays in space-time is e.g. a fundamental ingredient. In fact, it has been argued [5] that the Riemannian metric is basically determined by basic properties of lightpropagation.

In a fundamental paper by Newton and Wigner [1] it was argued that in the context of relativistic quantum mechanics a notion of point-like localization of a single particle can be, uniquely, determined by kinematics. Wightman [2] extended this notion to localization to finite domains of space and it was, rigourously, shown that massive particles are always localizable if they are elementary, i.e. if they are described in terms of irreducible representations of the Poincaré group [6]. Massless elementary systems with non-zero helicity, like a gluon, graviton, neutrino or a photon, are not localizable in the sense of Wightman. The axioms used by Wightman can, of course, be weakened. It was actually shown by Jauch, Piron and Amrein [3] that in such a sense the photon is weakly localizable. The notion of weak localizability essentially corresponds to allowing for non-commuting observables in order to characterize the localization of massless and spinning particles in general.

Localization of relativistic particles, at a fixed time, as alluded to above, has been
shown to be incompatible with a natural notion of (Einstein-) causality \[7\]. If relativistic elementary system has an exponentially small tail outside a finite domain of localization at \( t = 0 \), then, according to the hypothesis of a weaker form of causality, this should remain true at later times, i.e. the tail should only be shifted further out to infinity. As was shown by Hegerfeldt \[8\], even this notion of causality is incompatible with the notion of a positive and bounded observable whose expectation value gives the probability to a find a particle inside a finite domain of space at a given instant of time.

In the present paper we will reconsider some of these questions related to the concept of localizability in terms of a quantum mechanical description of a massless particle with given helicity \( \lambda \) \[9, 10, 11\]. We will show how one can extend this description to include both positive and negative helicities. We will then be able to e.g. describe the motion of a linearly polarized photon in the framework of relativistic quantum mechanics.

2 Position Operators for Massless Particles

It is easy to show that the components of the position operators for a massless particle must be non-commuting if the helicity \( \lambda \neq 0 \). If \( J_k \) are the generators of rotations and \( p_k \) the diagonal momentum, \( k = 1, 2, 3 \), then \( J \cdot p = \pm \lambda \). Here \( J = (J_1, J_2, J_3) \) and \( p = (p_1, p_2, p_3) \) and \( (\hbar = 1) \)

\[
[J_k, p_l] = i\epsilon_{klm}p_m .
\]  

(2.1)

If a canonical position operator \( x \) exists with components \( x_k \) such that

\[
[x_k, x_l] = 0 ,
\]  

(2.2)

\[
[x_k, p_l] = i\delta_{kl} ,
\]  

(2.3)

\[
[J_k, x_l] = i\epsilon_{klm}x_m ,
\]  

(2.4)

then we can define generators of orbital angular momentum in the conventional way, i.e.

\[
L_k = \epsilon_{klm}x_lp_m .
\]  

(2.5)

\[\text{This argument has, as far as we know, first been suggested by N. Mukunda.}\]
"Spin"-generators are then defined by
\[ S_k = J_k - L_k \] (2.6)
They fulfill the algebra
\[ [S_k, S_l] = i \epsilon_{klm} S_m \] (2.7)
and they, furthermore, commute with \( \mathbf{x} \) and \( \mathbf{p} \). Then, however, the spectrum of \( \mathbf{S} \cdot \mathbf{p} \) is \( \lambda, \lambda - 1, ..., -\lambda \), which contradicts the requirement \( \mathbf{J} \cdot \mathbf{p} = \pm \lambda \) since, by construction, \( \mathbf{J} \cdot \mathbf{p} = \mathbf{S} \cdot \mathbf{p} \).

As has been discussed in detail in the literature, the non-zero commutator of two components of the position operator for a massless particle primarily emerge due to the non-trivial topology of the momentum space [9, 10, 11]. The irreducible representations of the Poincaré group for massless particles [6] can be constructed from a knowledge of the little group \( G \) of a light-like momentum four-vector \( p = (p^0, \mathbf{p}) \). This group is the Euclidean group \( E(2) \). Physically, we are interested in possible finite-dimensional representations of the covering of the little group. We therefore restrict ourselves to the compact subgroup, i.e. we represent the \( E(2) \)-translations trivially and consider \( G = SO(2) = U(1) \). Since the origin in the momentum space is excluded for massless particles one is therefore led to consider \( G \)-bundles over \( S^2 \) since the energy of the particle can be kept fixed. Such \( G \)-bundles are classified by mappings from the equator to \( G \), i.e. by the first homotopy group \( \Pi_1(U(1)) = \mathbb{Z} \), where it turns out that each integer corresponds to twice the helicity of the particle. A massless particle with helicity \( \lambda \) and sharp momentum is thus described in terms of a non-trivial line bundle characterized by \( \Pi_1(U(1)) = \{2\lambda\} \) [12].

This consideration can easily be extended to higher space-time dimensions. If \( D \) is the number of space-time dimensions, the corresponding \( G \)-bundles are classified by the homotopy groups \( \Pi_{D-3}(Spin(D - 2)) \). These homotopy groups are in general non-trivial. It is remarkable fact that the only trivial homotopy groups of this form in higher space-time dimensions correspond to \( D = 5 \) and \( D = 9 \) due to the existence of quaternions and the Cayley numbers (see e.g. Ref. [13]). In these space-time dimensions, and for \( D = 3 \), it
then turns that one can explicitly construct canonical *and* commuting position operators for massless particles \[11\]. The mathematical fact that the spheres \(S^1, S^3 \text{ and } S^7\) are parallelizable can then be expressed in terms of the existence of canonical *and* commuting position operators for massless spinning particles in \(D = 3, D = 5\) and \(D = 9\) space-time dimensions.

In terms of a canonical momentum \(p_i\) and coordinates \(x_j\) satisfying the canonical commutation relation Eq.\(2.3\) we can easily derive the commutator of two components of the position operator \(x\) by making use of a simple consistency argument as follows. If the massless particle has a given helicity \(\lambda\), then the generators of angular momentum is given by:

\[ J_k = \epsilon_{klm} x_l p_m + \lambda \frac{p_k}{|p|} \quad (2.8) \]

The canonical momentum then transforms as a vector under rotations, i.e.

\[ [J_k, p_l] = i \epsilon_{klm} p_m \quad (2.9) \]

without any condition on the commutator of two components of the position operator \(x\). The position operator will, however, not transform like a vector unless the following commutator is postulated

\[ i[x_k, x_l] = \lambda \epsilon_{klm} \frac{p_m}{|p|^3} \quad (2.10) \]

where we notice that commutator formally corresponds to a pointlike Dirac magnetic monopole \[14\] localized at the origin in momentum space with strength \(4\pi \lambda\). The energy \(p^0\) of the massless particle is, of course, given by \(\omega = |p|\). In terms of a singular \(U(1)\) connection \(A_l \equiv A_l(p)\) we can write

\[ x_k = i \partial_k - A_k \quad (2.11) \]

where \(\partial_k = \partial/\partial p_k\) and

\[ \partial_k A_l - \partial_l A_k = \lambda \epsilon_{klm} \frac{p_m}{|p|^3} \quad (2.12) \]

Out of the observables \(x_k\) and the energy \(\omega\) one can easily construct the generators (at time \(t = 0\)) of Lorentz boosts, i.e.

\[ K_m = (x_m \omega + \omega x_m) / 2 \quad (2.13) \]
and verify that $J_l$ and $K_m$ lead to a realization of the Lie algebra of the Lorentz group, i.e.

\[ [J_k, J_l] = i\epsilon_{klm}J_m \quad , \]
\[ [J_k, K_l] = i\epsilon_{klm}K_m \quad , \]
\[ [K_k, K_l] = -i\epsilon_{klm}J_m \quad . \]

The components of the Pauli-Plebanski operator $W_\mu$ are given by

\[ W_\mu = (W^0, \mathbf{W}) = (\mathbf{J} \cdot \mathbf{p}, J^0 + \mathbf{K} \times \mathbf{p}) = \lambda \rho^\mu \quad , \]

i.e. we also obtain an irreducible representation of the Poincaré group. The additional non-zero commutators are

\[ [K_k, \omega] = ip_k \quad , \]
\[ [K_k, p_l] = i\delta_{kl}\omega \quad . \]

At $t \equiv x^0(\tau) \neq 0$ the Lorentz boost generators $K_m$ as given by Eq.(2.13) are extended to

\[ K_m = (x_m\omega + \omega x_m)/2 - tp_m \quad . \]

In the Heisenberg picture, the quantum equation of motion of an observable $\mathcal{O}(t)$ is obtained by using

\[ \frac{d\mathcal{O}(t)}{dt} = \frac{\partial\mathcal{O}(t)}{\partial t} + i[H, \mathcal{O}(t)] \quad , \]

where the Hamiltonian $H$ is given by the $\omega$. One then finds that all generators of the Poincaré group are conserved as they should. The equation of motion for $\mathbf{x}(t)$ is

\[ \frac{d}{dt}\mathbf{x}(t) = \frac{\mathbf{P}}{\omega} \quad , \]

which is an expected equation of motion for a massless particle.

The non-commuting components $x_k$ of the position operator $\mathbf{x}$ transform as the components of a vector under spatial rotations. Under Lorentz boost we find in addition that

\[ i[K_k, x_l] = \frac{1}{2} \left( x_k \frac{p_l}{\omega} + \frac{p_l}{\omega} x_k \right) - t\delta_{kl} + \lambda \epsilon_{klm} \frac{p_m}{|\mathbf{P}|^2} \quad . \]
The first two terms in Eq. (2.23) correspond to the correct limit for \( \lambda = 0 \) since the proper-time condition \( x^0(\tau) \approx \tau \) is not Lorentz invariant \([13]\). The last term in Eq. (2.23) is due to the non-zero commutator Eq. (2.10). This anomalous term can be dealt with by introducing an appropriate two-cocycle for finite transformations consisting of translations generated by the position operator \( x \), rotations generated by \( J \) and Lorentz boost generated by \( K \). For pure translations this two-cocycle will be explicitly constructed in Section 3.

The algebra discussed above can be extended in a rather straightforward manner to incorporate both positive and negative helicities needed in order to describe e.g. linearly polarized light. As we now will see this extension corresponds to a replacement of the Dirac monopole in momentum space with a SU(2) Wu-Yang \([13]\) monopole. The procedure below follows a rather standard method of imbedding the singular \( U(1) \) connection \( A_\ell \) into a regular \( SU(2) \) connection. Let us specifically consider a massless, spin-one particle. The Hilbert space, \( \mathcal{H} \), of one-particle transverse wave-functions \( \phi_\alpha(p) \), \( \alpha = 1, 2, 3 \) is defined in terms of a scalar product

\[
(\phi, \psi) = \int d^3 p \phi^*_\alpha(p) \psi_\alpha(p),
\]

(2.24)

where \( \phi^*_\alpha(p) \) denotes the complex conjugated \( \phi_\alpha(p) \). In terms of a Wu-Yang connection \( A_k^\alpha \equiv A_k^\alpha(p) \), i.e.

\[
A_k^\alpha(p) = \epsilon_{\alpha kl} \frac{p_l}{|p|^2},
\]

(2.25)

Eq. (2.11) is extended to

\[
x_k = i \partial_k - A_k^3(p) S_3,
\]

(2.26)

where

\[
(S_3)_{kl} = -i \epsilon_{akl}
\]

(2.27)

are the spin-one generators. By means of a singular gauge-transformation the Wu-Yang connection can be transformed into the singular \( U(1) \)-connection \( A_\ell \) times the third component of the spin generators \( S_3 \). This position operator defined by Eq. (2.26) is compatible with the transversality condition on the one-particle wave-functions, i.e. \( x_k \phi_\alpha(p) \) is transverse. With suitable conditions on the one-particle wave-functions the position operator
\[ i[x_k, x_l] = F^a_{kl} S^a = \epsilon_{klm} \frac{p_m}{p^3} \hat{p} \cdot S, \]  
(2.28)

where

\[ F^a_{kl} = \partial_k A^a_l - \partial_l A^a_k - \epsilon_{abc} A^b_k A^c_l = \epsilon_{klm} \frac{p_m p_a}{|p|^4}, \]  
(2.29)

is the non-Abelian SU(2) field strength tensor and \( \hat{p} \) is a unit vector in the direction of the particle momentum \( p \). The generators of angular momentum are now defined as follows

\[ J_k = \epsilon_{klm} x_l p_m + \frac{p_k}{|p|} \hat{p} \cdot S. \]  
(2.30)

The helicity operator \( \Sigma \equiv \hat{p} \cdot S \) is covariantly constant, i.e.

\[ \partial_k \Sigma + i [A_k, \Sigma] = 0, \]  
(2.31)

where \( A_k \equiv A^a_k(p) S_a \). The position operator \( x \) therefore commutes with \( \hat{p} \cdot S \). One can therefore verify in a straightforward manner that the observables \( p_k, \omega, J_l \) and \( K_m = (x_m \omega + \omega x_m)/2 \) close to the Poincaré group. At \( t \neq 0 \) the Lorentz boost generators \( K_m \) are defined as in Eq.(2.20) and Eq.(2.23) is extended to

\[ i[K_k, x_l] = \frac{1}{2} \left( x_k \frac{p_l}{\omega} + \frac{p_l}{\omega} x_k \right) - t \delta_{kl} + i \omega [x_k, x_l]. \]  
(2.32)

For helicities \( \hat{p} \cdot S = \pm \lambda \) one extends the previous considerations by considering \( S \) in the spin \( |\lambda| \)-representation. Eqs.(2.28), (2.30) and (2.32) are then valid in general. A reducible representation for the generators of the Poincaré group for an arbitrary spin has therefore been constructed for a massless particle. We observe that the helicity operator \( \Sigma \) can be interpreted as a generalized “magnetic charge”, and since \( \Sigma \) is covariantly conserved one can use the general theory of topological quantum numbers [17] and derive the quantization condition

\[ \exp(i 4\pi \Sigma) = 1, \]  
(2.33)

i.e. the helicity is properly quantized.
3 Topological Spin

Coadjoint orbits on a group $G$ admit a symplectic two-form (see e.g. [18]) which can be used to construct topological Lagrangians, i.e. Lagrangians constructed by means of Wess-Zumino terms [13] (for a general account see e.g. [20]). Let us illustrate the basic ideas for a non-relativistic spin and $G = SU(2)$. Let $K$ be an element of the Lie algebra $\mathcal{G}$ of $G$ in the fundamental representation. Without loss of generality we can write $K = \lambda_\alpha \sigma_\alpha = \lambda \sigma_3$, where $\sigma_\alpha, \alpha = 1, 2, 3$ denotes the three Pauli spin matrices. Let $H$ be the little group of $K$. Then the coset space $G/H$ is isomorphic to $S^2$ and defines an adjoint orbit (for semi-simple Lie groups adjoint and coadjoint representations are equivalent due to the existence of the non-degenerate Cartan-Killig form). The action for the spin degrees of freedom is then expressed in terms of the group $G$ itself, i.e.

$$S_p = -i \int \langle K, g^{-1}(\tau) dg(\tau)/d\tau \rangle d\tau ,$$

where $\langle A, B \rangle$ denotes the trace-operation of two Lie-algebra elements $A$ and $B$ in $\mathcal{G}$ and where

$$g(\tau) = \exp(i \sigma_\alpha \xi_\alpha(\tau))$$

defines the (proper-)time dependent dynamical group element. We observe that $S_p$ has a gauge-invariance, i.e. the transformation

$$g(\tau) \rightarrow g(\tau) \exp(i \theta(\tau) \sigma_3)$$

only change the Lagrangian density $\langle K, g^{-1}(\tau) dg(\tau)/d\tau \rangle$ by a total time derivative. The gauge-invariant components of spin, $S_k(\tau)$, are defined in terms of $K$ by the relation

$$S(\tau) \equiv S_k(\tau) \sigma_k = \lambda g(\tau) \sigma_3 g^{-1}(\tau) ,$$

such that

$$S^2 \equiv S_k(\tau) S_k(\tau) = \lambda^2 .$$

By adding a non-relativistic particle kinetic term as well as a conventional magnetic moment interaction term to the action $S_p$, one can verify that the components $S_k(\tau)$ obey the correct classical equations of motion for spinprecession [9, 20].
Let $M = \{ \sigma, \tau | \sigma \in [0, 1] \}$ and $(\sigma, \tau) \to g(\sigma, \tau)$ parametrize $\tau$-dependent paths in $G$ such that $g(0, \tau) = g_0$ is an arbitrary reference element and $g(1, \tau) = g(\tau)$. The Wess-Zumino term in this case is given by

$$\omega_{WZ} = -id \langle K, g^{-1}(\sigma, \tau) dg(\sigma, \tau) \rangle = i \langle K, (g^{-1}(\sigma, \tau) dg(\sigma, \tau))^2 \rangle \quad ,$$

(3.6)

where $d$ denotes exterior differentiation and where now

$$g(\sigma, \tau) = \exp(i\sigma_\alpha \xi_\alpha(\sigma, \tau)) \quad .$$

(3.7)

Apart from boundary terms which do not contribute to the equations of motion, we then have that

$$S_P = S_{WZ} \equiv \int_M \omega_{WZ} = -i \int_{\partial M} \langle K, g^{-1}(\tau) dg(\tau) \rangle \quad ,$$

(3.8)

where the one-dimensional boundary $\partial M$ of $M$, parametrized by $\tau$, can play the role of (proper-) time. $\omega_{WZ}$ is now gauge-invariant under a larger $U(1)$ symmetry, i.e. Eq.(3.3) is now extended to

$$g(\sigma, \tau) \longrightarrow g(\sigma, \tau) \exp (i\theta(\sigma, \tau)\sigma_3) \quad .$$

(3.9)

$\omega_{WZ}$ is therefore a two-form defined on the coset space $G/H$. A canonical analysis then shows that there are no gauge-invariant dynamical degrees of freedom in the interior of $M$. The Wess-Zumino action Eq.(3.8) is the topological action for spin degrees of freedom.

As for the quantization of the theory described by the action Eq.(3.8), one may use methods from geometrical quantization and especially the Borel-Weil-Bott theory of representations of compact Lie groups [18, 20]. One then finds that $\lambda$ is half an integer, i.e. $|\lambda|$ corresponds to the spin. This quantization of $\lambda$ also naturally emerges by demanding that the action Eq.(3.8) is welldefined in quantum mechanics for periodic motion as recently was discussed by e.g. Klauder [21], i.e.

$$4\pi \lambda = \int_{S^2} \omega_{WZ} = 2\pi n \quad ,$$

(3.10)

where $n$ is an integer. The symplectic two-form $\omega_{WZ}$ must then belong to an integer class cohomology. This geometrical approach is in principal straightforward, but it requires
explicit coordinates on $G/H$. An alternative approach, as used in [1, 20], is a canonical Dirac analysis and quantization [13]. This procedure leads to the condition $\lambda^2 = s(s+1)$, where $s$ is half an integer. The fact that one can arrive at different answers for $\lambda$ illustrates a certain lack of uniqueness in the quantization procedure of the action Eq.(3.8). The quantum theories obtained describes, however, the same physical system namely one irreducible representation of the group $G$.

The action Eq.(3.8) was first proposed in [22]. The action can be derived quite naturally in terms of a coherent state path integral (for a review see e.g. Ref.[23]) using spin coherent states. It is interesting to notice that structure of the action Eq.(3.8) actually appears in such a language already in a paper by Klauder on continuous representation theory [24].

A classical action which after quantization leads to a description of a massless particle in terms of an irreducible representations of the Poincaré group can be constructed in a similar fashion [9]. Since the Poincaré group is non-compact the geometrical analysis referred to above for non-relativistic spin must be extended and one should consider coadjoint orbits instead of adjoint orbits (D=3 appears to be an exceptional case due to the existence of a non-degenerate bilinear form on the D=3 Poincaré group Lie algebra [25]. In this case there is a topological action for irreducible representations of the form Eq.(3.8) [26]). The action then takes the form

$$ S = \int d\tau \left( p_\mu(\tau) \dot{x}^\mu(\tau) + \frac{i}{2} \text{Tr}[K\Lambda^{-1}(\tau) \frac{d}{d\tau} \Lambda(\tau)] \right) . $$

(3.11)

Here $[\sigma_{\alpha\beta}]_{\mu\nu} = -i(\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu})$ are the Lorentz group generators in the spin one representation and $\eta_{\mu\nu} = (-1, 1, 1, 1)$ is the Minkowski metric. The Lorentz group Lie-algebra element $K$ is chosen to be $\lambda \sigma_{12}$. The $\tau$-dependence of the Lorentz group element $\Lambda_{\mu\nu}(\tau)$ is defined by

$$ \Lambda_{\mu\nu}(\tau) = \left[ \exp \left( i\sigma_{\alpha\beta} \xi^{\alpha\beta}(\tau) \right) \right]_{\mu\nu} . $$

(3.12)

The momentum variable $p_\mu(\tau)$ is defined by

$$ p_\mu(\tau) = \Lambda_{\mu\nu}(\tau) k^\nu , $$

(3.13)
where the constant reference momentum $k^\nu$ is given by

$$k^\nu = (\omega, 0, 0, |k|) ,$$  \hspace{1cm} (3.14)

where $\omega = |k|$. The momentum $p_\mu(\tau)$ is the light-like by construction. The action Eq.(3.11) leads to the equations of motion

$$\frac{d}{d\tau} p_\mu(\tau) = 0 ,$$  \hspace{1cm} (3.15)

and

$$\frac{d}{d\tau} \{ x_\mu(\tau)p_\nu(\tau) - x_\nu(\tau)p_\mu(\tau) + S_{\mu\nu}(\tau) \} = 0 .$$  \hspace{1cm} (3.16)

Here we have defined gauge-invariant spin degrees of freedom $S_{\mu\nu}(\tau)$ by

$$S_{\mu\nu}(\tau) = \frac{1}{2} \text{Tr}[\Lambda(\tau)K\Lambda^{-1}(\tau)\sigma_{\mu\nu}]$$  \hspace{1cm} (3.17)

in analogy with Eq.(3.4). These spin degrees of freedom satisfy the relations

$$p_\mu(\tau)S^{\mu\nu}(\tau) = 0 ,$$  \hspace{1cm} (3.18)

and

$$\frac{1}{2} S_{\mu\nu}(\tau)S^{\mu\nu}(\tau) = \lambda^2 .$$  \hspace{1cm} (3.19)

Inclusion of external electromagnetic and gravitational fields leads to the classical Bargman-Michel-Telegdi and Papapetrou equations of motion respectively \cite{9}. Since the equations derived are expressed in terms of bosonic variables these equations of motion admit a straightforward classical interpretation. (An alternative bosonic treatment of internal degrees of freedom can be found in Ref.[27].)

Canonical quantization of the system described by bosonic degrees of freedom and the action Eq.(3.11) leads to a realization of the Poincaré Lie algebra with generators $p_\mu$ and $J_{\mu\nu}$ where

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu} .$$  \hspace{1cm} (3.20)

The four vectors $x_\mu$ and $p_\nu$ commute with the spin generators $S_{\mu\nu}$ and are canonical, i.e.

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0 ,$$  \hspace{1cm} (3.21)

$$[x_\mu, p_\nu] = i\eta_{\mu\nu} .$$  \hspace{1cm} (3.22)
The spin generators $S_{\mu\nu}$ fulfil the conventional algebra

$$[S_{\mu\nu}, S_{\lambda\rho}] = i(\eta_{\mu\lambda}S_{\nu\rho} + \eta_{\nu\rho}S_{\mu\lambda} - \eta_{\mu\rho}S_{\nu\lambda} - \eta_{\nu\lambda}S_{\mu\rho}) \quad .$$

(3.23)

The mass-shell condition $p^2 = 0$ as well as the constraints Eq.(3.18) and Eq.(3.19) are all first-class constraints [15]. In the proper-time gauge $x^0(\tau) \approx \tau$ one obtains the system described in Section 2, i.e. we obtain an irreducible representation of the Poincaré group with helicity $\lambda$ [9]. For half-integer helicity, i.e. for fermions, one can verify in a straightforward manner that the wave-functions obtained change with a minus-sign under a $2\pi$ rotation [9, 11, 20] as they should.

### 4 The Berry Phase for Photons

We have constructed a set of $O(3)$-covariant position operators of massless particles corresponding to a reducible representation of the Poincaré group corresponding to a combination of positive and negative helicity. It is interesting to notice that the construction above leads to observable effects. Let us specifically consider photons and the motion of photons along an optical fibre. Berry has argued [28] that a spin in an adiabatically changing magnetic field leads to the appearance of an observable phase factor, called the Berry phase. It was suggested [29] that a similar geometric phase could appear for photons. We will now, within the framework of relativistic quantum mechanics, provide for a derivation of this geometrical phase in terms of the operator realization of the Poincaré discussed above. The Berry phase for photons can then be obtained as follows. We consider the motion of a photon with fixed energy moving in an optical fibre. We assume that as the photon moves in the fibre, the momentum vector traces out a closed loop in momentum space on the constant energy surface, i.e. on a two-sphere $S^2$. We therefore consider wave-functions $|\mathbf{p}\rangle$ which are diagonal in momentum. We also define the translation operators $U(\mathbf{a}) = \exp(i\mathbf{a} \cdot \mathbf{x})$. It is straightforward to show using Eq.(2.10) that

$$U(\mathbf{a})U(\mathbf{b}) |\mathbf{p}\rangle = \exp(i\gamma[\mathbf{a}, \mathbf{b}; \mathbf{p}]) |\mathbf{p} + \mathbf{a} + \mathbf{b}\rangle \quad ,$$

(4.1)
where the two-cocycle phase \( \gamma[a, b; p] \) is equal to the flux of the magnetic monopole in momentum space through the simplex spanned by the vectors \( a \) and \( b \) localized at the point \( p \), i.e.

\[
\gamma[a, b; p] = \lambda \int_0^1 \int_0^1 d\xi_1 d\xi_2 a_k b_l \epsilon_{ikm} B_m(p + \xi_1 a + \xi_2 b),
\]

where \( B_m(p) = p_m/|p|^3 \). The non-trivial phase appears because the second de Rham cohomology group of \( S^2 \) is non-trivial. The two-cocycle phase \( \gamma[a, b; p] \) is therefore not a coboundary and hence it cannot be removed by a redefinition of \( U(a) \). This result has a close analogy in the theory of magnetic monopoles \([30]\). The anomalous commutator Eq.(2.10) therefore leads to a ray-representation of the translations in momentum space.

A closed loop in momentum space, starting and ending at \( p \), can then be obtained by using a sequence of infinitesimal translations \( U(\delta a) |p\rangle = |p + \delta a\rangle \) such that \( \delta a \) is orthogonal to argument of the wave-function on which it acts (this defines the adiabatic transport of the system). The momentum vector \( p \) then traces out a closed curve on the constant energy surface \( S^2 \) in momentum space. The total phase of these translations then gives a phase \( \gamma \) which is the \( \lambda \) times the solid angle of the closed curve the momentum vector traces out on the constant energy surface. This phase does not depend on Planck's constant. This is precisely the Berry phase for the photon with a given helicity \( \lambda \). In the experiment by Tomita and Chiao \([31]\) one considers a linearly polarized photon. The same line of arguments above but making use Eq.(2.28) instead of Eq.(2.10) leads to the desired change of polarization as the photon moves along the optical fibre. An alternative derivation of the Berry phase for photons is based on observation that the covariantly conserved helicity operator \( \Sigma \) can be interpreted as a generalized "magnetic charge".

Let \( \Gamma \) denote a closed path in momentum space parametrized by \( \sigma \in [0, 1] \) such that \( p(\sigma = 0) = p(\sigma = 1) = p_0 \) is fixed. The parallel transport of a one-particle state \( \phi_\alpha(p) \) along the path \( \Gamma \) is then determined by a path-ordered exponential, i.e.

\[
\phi_\alpha(p_0) \longrightarrow \left[ P \exp \left( i \int_\Gamma A_k(p(\sigma)) \frac{dp_k(\sigma)}{d\sigma} d\sigma \right) \right]_{\alpha\beta} \phi_\beta(p_0),
\]

where \( A_k(p(\sigma)) \equiv A_k^a(p(\sigma)) S_a \). By making use of a non-Abelian version of Stokes theorem
one can then show that

\[ P \exp \left( i \int_{\Gamma} A_k(p(\sigma)) \frac{dp_k(\sigma)}{d\sigma} d\sigma \right) = \exp( i \Sigma \Omega[\Gamma] ) \]  \hspace{1cm} (4.4)

where \( \Omega[\Gamma] \) is the solid angle subtended by the path \( \Gamma \) on the two-sphere \( S^2 \). This result leads again to the desired change of linear polarization as the photon moves along the path described by \( \Gamma \). This derivation does not require that \( |p(\sigma)| \) is constant along the path.

In the experiments so far considered the photon flux is large. In order to strictly apply our results under such conditions one can consider a second quantized version of the theory we have presented following e.g. the discussion of Amrein [3]. By making use of coherent states of the electromagnetic field in a standard and straightforward manner (see e.g. Ref. [23]) one then realize that our considerations survive. This is so since the coherent states are parametrized in terms of the one-particle states. By construction the coherent states then inherits the transformation properties of the one-particle states discussed above.

\section{Final Remarks and Conclusions}

In the analysis of Wightman, corresponding to commuting position variables, the natural mathematical tool turned out to be systems of imprimitivity for the representations of the three-dimensional Euclidean group. In the case of non-commuting position operators we have also seen that notions from differential geometry are important. It is interesting to see that such a broad range of mathematical methods enters into the study of the notion of localizability of physical systems. We have in particular argued that Abelian as well as non-Abelian magnetic monopole field configurations reveal themselves in a description of localizability of massless spinning particles. Concerning the physical existence of magnetic monopoles Dirac remarked in 1981 [32] that “I am inclined now to believe that monopoles do not exist. So many years have gone by without any encouragement from the experimental side”. The “monopoles” we are considering appear, however, as
mathematical objects in the momentum space of the massless particles. Their existence, we have argued, is then only indirectly revealed to us by the properties of e.g. the photons moving along optical fibres.

Localized states of massless particles will necessarily develop non-exponential tails in space as a consequence of the Hegerfeldts theorem [8]. Various number operators representing the number of massless, spinning particles localized in a finite volume $V$ at time $t$ has been discussed in the literature. The non-commuting position observables we have discussed for photons correspond to the pointlike limit of the weak localizability of Jauch, Piron and Amrein [3]. This is so since our construction, as we have seen in Section 2, corresponds to an explicit enforcement of the transversality condition of the one-particle wave-functions.

In a finite volume, photon number operators appropriate for weak localization [3] do not agree with the photon number operator introduced by Mandel [33] for sufficiently small wavelengths as compared to the linear dimension of the localization volume. It would be interesting to see if there are measurable differences. A necessary ingredient in answering such a question would be the experimental realization of a localized one-photon state. It is interesting to notice that such states can be generated in the laboratory [34].

In concluding we find it appealing that the quantization of the system describing a photon, in general linearly polarized, is “geometry; after all ” [21].

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