The Perron method associated with finely \( p \)-harmonic functions on finely open sets

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Abstract. Given a bounded finely open set \( V \) and a function \( f \) on the fine boundary of \( V \), we introduce four types of upper Perron solutions to the nonlinear Dirichlet problem for \( p \)-energy minimizers, \( 1 < p < \infty \), with \( f \) as boundary data. These solutions are given as pointwise infima of suitable families of fine \( p \)-superminimizers in \( V \). We show (under natural assumptions) that the four upper Perron solutions are equal quasieverywhere and that they are fine \( p \)-minimizers of the \( p \)-energy integral. We moreover show that the upper and lower Perron solutions coincide quasieverywhere for Sobolev and for uniformly continuous boundary data, i.e. that such boundary data are resolutive. For the uniformly continuous boundary data, the Perron solutions are also shown to be finely continuous and thus finely \( p \)-harmonic. We prove our results in a complete metric space \( X \) equipped with a doubling measure supporting a \( p \)-Poincaré inequality, but they are new also in unweighted \( \mathbb{R}^n \).

Keywords and phrases: Complete metric space, Dirichlet problem, doubling measure, fine \( p \)-minimizer, finely continuous, finely open set, finely \( p \)-harmonic function, nonlinear fine potential theory, Perron method, Poincaré inequality, resolutive.

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1. Introduction

The aim in this paper is to study the Perron method related to the nonlinear Dirichlet problem for finely \( p \)-harmonic functions on finely open sets. Recall that in unweighted \( \mathbb{R}^n \), \( p \)-harmonic functions on open sets minimize the \( p \)-energy \( \int |\nabla u|^p \, dx \) and are solutions of the \( p \)-Laplace equation

\[
\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u) = 0.
\]  

(1.1)

The \( p \)-superharmonic functions on open sets associated with this equation are known to be finely continuous, which has (especially in the classical linear case \( p = 2 \)) led to the development of fine potential theory. For \( p \neq 2 \), the equation (1.1) is nonlinear.
Given a bounded finely open set $V$ (with complement of positive capacity) and a function $f$ on the fine boundary $\partial_p V$, we seek a finely $p$-harmonic function in $V$ with $f$ as boundary data. For this purpose, we introduce four types of upper Perron solutions $\overline{P}f$, $\overline{Q}f$, $\overline{R}f$ and $\overline{S}f$, given as pointwise infima of suitable families of fine superminimizers in $V$. Out of these, $\overline{Q}f$ is the most important for our approach as it, surprisingly, combines several useful properties of the solutions. On the other hand, $\overline{P}f$ most closely mimics the traditional Perron solutions. The Perron solutions $\overline{S}f$ and $\overline{R}f$ provide useful connections between the other two solutions.

We perform our study in a complete metric space $X$ equipped with a doubling measure supporting a $p$-Poincaré inequality, where $1 < p < \infty$. However, as far as we know, the Perron method associated with $p$-harmonic functions has not been studied earlier on finely open sets beyond the linear case $p = 2$, not even on unweighted $\mathbb{R}^n$. The fine (super)minimizers considered in this paper coincide on unweighted $\mathbb{R}^n$ with the fine $p$-(super)solutions of $\Delta_p u = 0$ introduced in Kilpeläinen–Mály [24] in 1992.

The following is a special case of our main results and follows directly from Theorems 6.4 and 7.1. By "q.e." we mean "quasieverywhere", i.e. up to a set of zero capacity.

**Theorem 1.1.** Let $f : \partial_p V \to \mathbb{R}$ be bounded. Then
\[ \overline{P}f = \overline{Q}f = \overline{R}f = \overline{S}f \quad \text{q.e. in } V \]
and the four upper Perron solutions are fine minimizers in $V$.

The minimizing property of the Perron solutions is vital but nontrivial. Both parts of Theorem 1.1 hold also for large classes of unbounded functions, see Theorems 6.4 and 7.1 for precise statements.

On open sets, Perron solutions are defined using $p$-superharmonic functions, which essentially are lsc-regularized superminimizers of the $p$-energy integral. Here we have defined the upper Perron solutions $\overline{P}f$ and $\overline{S}f$ using finely lsc-regularized fine superminimizers. In contrast, the upper Perron solutions $\overline{Q}f$ and $\overline{R}f$ are defined using finely usc-regularized fine superminimizers. Note that (unlike in the linear case $p = 2$ or on open sets) it is not known whether fine (super)minimizers have finely continuous representatives, not even in unweighted $\mathbb{R}^n$.

It is perhaps a bit surprising that we have obtained the best results using usc-regularizations, but we have not been able to construct direct proofs showing e.g. that $\overline{P}f$ is a fine minimizer or that $\overline{P}f \leq \overline{Q}f$ q.e. On the other hand, $\overline{Q}f$ and $\overline{R}f$ are finely upper semicontinuous and can be written q.e. as decreasing limits of fine superminimizers (Lemma 5.2). This is then used as a key tool to obtain several properties of the Perron solutions, as well as comparisons between them, including Theorem 1.1. Our proofs are therefore rather different from the corresponding proofs for open sets in the literature, where the compactness of the boundary or convergence on countable dense subsets is used. Since countable sets often have zero capacity and are not seen by the fine topology, they cannot be used in our situation.

In order to use the Perron method, it is also important to study resolutivity, i.e. when the upper and lower Perron solutions agree (at least q.e.). The following result is a special case of Proposition 8.1 and Theorems 6.4 and 8.4. In what follows, $\overline{V} = V \cup \partial_p V$ is the fine closure of $V$ and $N^{1,p}(V)$ is the Newtonian–Sobolev space, defined by means of upper gradients, see Section 2.

**Theorem 1.2.** Let $f : \overline{V} \to [-\infty, \infty]$.

(a) If $f \in N^{1,p}(V)$ and
\[ f(x) = \text{fine lim}_{y \to x} f(y) \quad \text{for q.e. } x \in \partial_p V \]
The Perron method associated with finely $p$-harmonic functions on finely open sets

(which in particular holds if $f \in N^{1,p}(X)$), then

$$Pf = \overline{P}f = Qf = \overline{Q}f = Rf = \overline{R}f = Sf = \overline{S}f \quad \text{q.e. in } V. \quad (1.2)$$

(b) If $f$ is uniformly continuous on $\partial pV$ with respect to the metric topology, then

$$\overline{Q}f = \overline{S}f \quad \text{everywhere in } V,$$  

and this solution is finely continuous in $V$.

If we ignore the “q.e.” in (1.2), Theorem 1.2 corresponds to the two main resolutivity results known in the nonlinear case on open sets: for functions in $N^{1,p}$ and for continuous functions. There are also a few other resolutivity results on open sets, mainly for semicontinuous functions. (See e.g. [3, Chapter 10], or the references below.) When turning to finely open sets there are new issues involved and many open questions, see e.g. Section 9 in our paper [9].

Another question is invariance of the Perron solutions under perturbations of the boundary data on sets of zero capacity. On open sets, such invariance results for nonlinear problems were first obtained by Björn–Björn–Shanmugalingam [12] for functions in $N^{1,p}$ and for continuous functions. In our situation, such invariance for the $Q$- and $S$-Perron solutions follows immediately from their definitions. Moreover, these two upper Perron solutions can equivalently be defined using (finely usc/lsc-regularized) fine minimizers (see Remark 7.4).

Perron solutions of the Dirichlet problem were introduced by Perron [29] and Remak [30] (independently) in 1923 for harmonic functions on open subsets of the plane. The Perron method has later been used also for various linear and nonlinear elliptic and parabolic equations. Fuglede [18, p. 173] extended the Perron method to finely open sets in the linear axiomatic setting. An alternative approach, under weaker axioms, was given by Lukeš–Malý–Zajíček [28, Chapter 13]. They however needed to consider boundary values on the “bitopological” boundary $\partial V \setminus V$. Our approach is thus more in line with Fuglede’s.

However, the linear theory considered in [18] and [28] differs much from the nonlinear one since it is based on potentials and measures rather than on Sobolev spaces. At the same time for nonlinear problems, such as (1.1), Sobolev spaces are essential already for defining $p$-harmonic functions on $\mathbb{R}^n$, as seen in Heinonen–Kilpeläinen–Martio [21]. The metric space approach is particularly suitable for nonlinear fine potential theory since Newtonian (Sobolev) spaces can be defined directly on an arbitrary measurable subset $E$ by considering $E$ as a metric space in its own right. On the other hand, it follows from Björn–Björn [5, Theorem 7.3] and Björn–Björn–Latvala [9] that the study of the Dirichlet problem for $p$-harmonic functions is not natural beyond finely open or quasiopen sets. By Theorem 3.2, finely open sets and superlevel sets of global Newtonian functions are quasiopen. The use of fine limits in our definition of Perron solutions makes it natural to restrict attention to finely open sets.

As far as we know, Aronsson [1, Section 4] was the first who used the Perron method in connection with a nonlinear equation, namely for the $\infty$-Laplacian (also called Aronsson’s equation). In connection with $p$-harmonic functions (for $1 < p < \infty$) it was first used by Granlund–Lindqvist–Martio [19]. See also Kilpeläinen [23], Heinonen–Kilpeläinen–Martio [21], Björn–Björn–Shanmugalingam [12], [14], Björn–Björn–Sjödin [15] and Hansevi [20]. All of these studies were on open sets. The nonlinear Dirichlet problem for fine solutions/minimizers on quasiopen sets was studied in the Sobolev sense by Kilpeläinen–Malý [24] on unweighted $\mathbb{R}^n$, and on metric spaces by the authors in [9].

The outline of the paper is as follows. In Section 2 we present the main background definitions related to Newtonian functions and capacity in metric spaces, while in Section 3 we introduce the fine topology and finely semicontinuous regularizations. Following our papers [7] and [9], we introduce the fine local Newtonian
space $N_{\text{fine-loc}}^{1,p}$, fine superminimizers and the fine obstacle problem in Section 4. The main new contribution here is Proposition 4.8, which is a convergence result for solutions of obstacle problems that is important later on.

In Section 5 we introduce our Perron solutions and prove two auxiliary results for the upper Perron solution $\overline{Q}f$, whereas Section 6 contains comparisons between different Perron solutions. In Section 7 we give sufficient conditions for Perron solutions to be fine minimizers. Finally in Section 8 we prove Theorem 1.2 and in Section 9 we study the behaviour of our Perron solutions on open sets.

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2. Notation and preliminaries

In this section, we introduce the necessary metric space concepts used in this paper. For brevity, we refer to our papers [6] and [8] for more extensive introductions and references to the literature. See also the monographs Björn–Björn [3] and Heinonen–Koskela–Shanmugalingam–Tyson [22], where the theory of upper gradients and Newtonian (Sobolev) spaces on metric spaces is thoroughly developed with proofs.

Let $X$ be a metric space equipped with a metric $d$ and a positive complete Borel measure $\mu$ such that $\mu(B) < \infty$ for all balls $B \subset X$. We also assume that $1 < p < \infty$.

A curve is a continuous mapping from an interval, and a rectifiable curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable, and they can therefore be parameterized by their arc length $ds$. A property holds for $p$-almost every curve if the curve family $\Gamma$ for which it fails has zero $p$-modulus, i.e. there is $\rho \in L^p(X)$ such that $\int_{\gamma} \rho \, ds = \infty$ for every $\gamma \in \Gamma$.

A measurable function $g : X \to [0, \infty]$ is a $p$-weak upper gradient of $u : X \to \mathbb{R} := [-\infty, \infty]$ if for $p$-almost all curves $\gamma : [0, t_{\gamma}] \to X$,

$$|u(\gamma(0)) - u(\gamma(t_{\gamma}))| \leq \int_{\gamma} g \, ds,$$

where the left-hand side is $\infty$ whenever at least one of the terms therein is infinite. If $u$ has a $p$-weak upper gradient in $L^p_{\text{loc}}(X)$, then it has a minimal $p$-weak upper gradient $g_u \in L^p_{\text{loc}}(X)$ in the sense that $g_u \leq g$ a.e. for every $p$-weak upper gradient $g \in L^p_{\text{loc}}(X)$ of $u$. For such measurable $u$, we let

$$\|u\|_{N^{1,p}(X)} = \left( \int_X |u|^p \, d\mu + \int_X g^p_u \, d\mu \right)^{1/p}.$$

The Newtonian space on $X$ is

$$N^{1,p}(X) = \{ u : \|u\|_{N^{1,p}(X)} < \infty \}.$$

For a measurable set $E \subset X$, the Newtonian space $N^{1,p}(E)$ is defined by considering $(E, d_E, \mu|_E)$ as a metric space in its own right.

The space $N^{1,p}(X)/\sim$, where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice. In this paper it will be convenient to assume that functions in
$N^{1,p}(X)$ are defined everywhere (with values in $\overline{\mathbb{R}}$), not just up to an equivalence class in the corresponding function space. For an arbitrary set $A \subset X$, we let

$$N^{1,p}_0(A) = \{ u|_A : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus A \}.$$ 

Functions from $N^{1,p}_0(A)$ can be extended by zero in $X \setminus A$ and we will regard them in that sense when needed.

The Sobolev capacity of an arbitrary set $A \subset X$ is

$$C_p(A) = \inf_u \| u \|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on $A$. A property holds quasieverywhere (q.e.) if the set of points for which it fails has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions, namely $\| u \|_{N^{1,p}(X)} = 0$ if and only if $u = 0$ q.e. Moreover, if $u, v \in N^{1,p}(X)$ and $v = u$ a.e., then $v = u$ q.e.

The measure $\mu$ is doubling if there is $C > 0$ such that for all balls $B = B(x_0, r) := \{ x \in X : d(x, x_0) < r \}$ in $X$, we have $0 < \mu(2B) \leq C\mu(B) < \infty$, where $\lambda B = B(x_0, \lambda r)$. In this paper, all balls are open.

The space $X$ supports a $p$-Poincaré inequality if there are $C > 0$ and $\lambda > 1$ such that for all balls $B \subset X$, all integrable functions $u$ on $X$ and all $p$-weak upper gradients $g$ of $u$, we have $0 < \mu(B) < \infty$ and

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C \text{diam}(B) \left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p \, d\mu \right)^{1/p}, \quad (2.1)$$

where $u_B := \int_B u \, d\mu / \mu(B)$.

In $\mathbb{R}^n$ equipped with a doubling measure $d\mu = w \, dx$, the $p$-Poincaré inequality (2.1) is equivalent to the $p$-admissibility of the weight $w$ in the sense of Heinonen–Kilpeläinen–Martio [21], see Corollary 20.9 in [21] and Proposition A.17 in [3]. Moreover, in this case $g_u = |\nabla u|$ a.e. if $u \in N^{1,p}(\mathbb{R}^n, \mu)$, and the capacities in this paper coincide with the corresponding capacities in [21] (see [3, Theorem 6.7] and [4, Theorem 5.1]).

3. Fine topology

Throughout the rest of the paper, we assume that $X$ is complete and supports a $p$-Poincaré inequality with $1 < p < \infty$ and that $\mu$ is doubling. We also assume that $V$ is a nonempty finely open set, and from Section 4 onwards that in addition $V$ is bounded and $C_p(X \setminus V) > 0$.

To avoid pathological situations we also assume that $X$ contains at least two points (and thus must be uncountable due to the Poincaré inequality). In this section we recall the basic facts about the fine topology associated with Newtonian functions. The variational capacity of $A$ with respect to $B$ is defined by

$$\text{cap}_p(A, B) := \inf_u \int_X g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}_0(B)$ such that $u \geq 1$ on $A$.

**Definition 3.1.** A set $E \subset X$ is thin at $x \in X$ if

$$\int_0^1 \left( \frac{\text{cap}_p(E \cap B(x, r), B(x, 2r))}{\text{cap}_p(B(x, r), B(x, 2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty.$$

A set $V \subset X$ is finely open if $X \setminus V$ is thin at each point $x \in V$. 
In the definition of thinness, we use the convention that the integrand is 1 whenever \( \text{cap}_p(B(x,r), B(x,2r)) = 0 \). It is easy to see that the finely open sets give rise to a topology, which is called the fine topology. Every open set is finely open, but the converse is not true in general. A function \( u : V \to \mathbb{R} \), defined on a finely open set \( V \), is \emph{finely continuous} if it is continuous when \( V \) is equipped with the fine topology and \( \mathbb{R} \) with the usual topology. Pointwise fine (semi)continuity is defined analogously. The fine interior, fine boundary and fine closure of \( E \) are denoted \( \text{fine-int}_p E, \partial_p E \) and \( E_p \), respectively. See Björn–Björn [3, Section 11.6] and Björn–Björn–Latvala [6] for further discussion on thinness and the fine topology in metric spaces. Note that the fine topology is usually not metrizable.

A set \( U \subset X \) is \emph{quasiopen} if for every \( \varepsilon > 0 \) there is an open set \( G \subset X \) such that \( \text{cap}_p(G) < \varepsilon \) and \( G \cup U \) is open. Quasiopen sets are measurable by Lemma 9.3 in Björn–Björn [5]. Various characterizations of quasiopen sets can be found in Björn–Björn–Malý [11]. Therein it is also shown that the \( C_p \) capacities with respect to \( X \) and a quasiopen \( U \) have the same zero sets.

The following result explains the close connection between finely open and quasiopen sets. In particular, finely open sets are quasiopen and we therefore have at our disposal all earlier results obtained for quasiopen sets.

**Theorem 3.2.** (Theorem 3.4 in [9]) The following conditions are equivalent for any set \( U \subset X \):

(a) \( U \) is quasiopen;
(b) \( U = V \cup E \) for some finely open \( V \) and a set \( E \) with \( C_p(E) = 0 \);
(c) \( C_p(U \setminus \text{fine-int}_p U) = 0 \);
(d) \( U = \{ x : u(x) > 0 \} \) for some \( u \in N^{1,p}(X) \).

We will use the following principle several times as a substitute for compactness.

**Theorem 3.3.** (Quasi-Lindelöf principle, Theorem 3.4 in [7]) For each family \( \mathcal{V} \) of finely open sets there is a countable subfamily \( \mathcal{V}' \) such that \( \text{cap}_p \left( \bigcup_{V \in \mathcal{V}} V \setminus \bigcup_{V' \in \mathcal{V}'} V' \right) = 0 \).

We will also need the following characterization of \( N^{1,p}_0(V) \), which is a special case of Theorem 7.2 in [9].

**Proposition 3.4.** Let \( u \in N^{1,p}(V) \). Then \( u \in N^{1,p}_0(V) \) if and only if

\[
\text{fine lim}_{V \ni y \to x} u(y) = 0 \quad \text{for q.e. } x \in \partial_p V.
\]

In this paper, the notions of fine lim, fine lim sup and fine lim inf are defined using punctured fine neighbourhoods. Since

\[
\text{cap}_p(B(x,r) \setminus \{ x \}, B(x,2r)) = \text{cap}_p(B(x,r), B(x,2r)),
\]

there are no isolated points in the fine topology, i.e. no singleton sets are finely open. Moreover, if \( W \) is finely open and \( C_p(E) = 0 \), then \( W \setminus E \) is also finely open. Hence, if \( x \in \overline{V}^p \), then

\[
\text{fine lim inf}_{V \ni y \to x} u(y) = \sup_{\text{finely open } W \ni x} \inf_{(V \cap W) \setminus \{ x \}} u = \sup_{\text{finely open } W \ni x} \text{ess inf}_{(V \cap W) \setminus \{ x \}} C_p u,
\]

where

\[
C_p \text{ ess inf}_E u := \sup \{ k : u \geq k \text{ q.e. in } E \}.
\]
We will extensively use fine lsc- and usc-regularizations, which we define as follows. Let \( u : V \to \mathbb{R} \) and set for \( x \in V \),
\[
\begin{align*}
\underline{u}_2(x) & \equiv \liminf_{y \to x} u(y) = \sup_{\text{finely open } W \ni x} \mathcal{C}_{p^*} \text{ess inf } u, \\
\underline{u}_4(x) & \equiv \begin{cases} 
\underline{u}_2(x), & \text{if } \mathcal{C}_p(\{x\}) = 0, \\
\min\{\underline{u}_2(x), u(x)\}, & \text{if } \mathcal{C}_p(\{x\}) > 0,
\end{cases} \\
\overline{u}(x) & \equiv \limsup_{y \to x} u(y) = \sup_{\text{finely open } W \ni x} \mathcal{C}_{p^*} \text{ess inf } u.
\end{align*}
\]

Thus, \( \underline{u}_4 \) is defined using punctured neighbourhoods, while \( \overline{u} \) is using nonpunctured neighbourhoods. We also say that \( u \) is finely lsc-regularized if \( u = \underline{u}_4 \). Similarly, we define
\[
\underline{u}(x) = \limsup_{y \to x} u(y) \quad \text{and} \quad \overline{u}(x) = \begin{cases} 
\underline{u}(x), & \text{if } \mathcal{C}_p(\{x\}) = 0, \\
\max\{\underline{u}(x), u(x)\}, & \text{if } \mathcal{C}_p(\{x\}) > 0,
\end{cases}
\]

and say that \( u \) is finely usc-regularized if \( u = \overline{u} \).

It follows directly from the definition that for every \( a \in \mathbb{R} \) the set \( \{x \in V : u_*(x) > a\} \) is finely open and hence \( u_* \) is finely lower semicontinuous in \( V \). Moreover, \( u \) is finely lower semicontinuous in \( V \) if and only if \( u \leq u_* \) everywhere in \( V \). The same is true for \( \underline{u}_4 \) and similar results hold for \( u^* \) and \( \overline{u}^* \).

**Remark 3.5.** Assume that \( u \) and \( v \) are functions on \( V \) such that \( u \leq v \) q.e. in \( V \). Then
\[
\underline{u}_4 \leq \overline{u}_4, \quad \underline{u}_2 \leq \overline{u}_2, \quad \overline{u} \leq \overline{v} \quad \text{and} \quad \underline{u} \leq \overline{v} \quad \text{everywhere in } V.
\]

In fact, if \( \mathcal{C}_p(\{x\}) = 0 \), then
\[
\underline{u}_4(x) = \underline{u}_2(x) = \overline{u}_2(x) = \overline{v}_2(x) = \overline{v}_4(x) = u_*(x)
\]

since the fine lower limits do not see sets of zero capacity. If \( \mathcal{C}_p(\{x\}) > 0 \), then \( u(x) \leq \overline{v}(x) \) by assumption, and \( \underline{u}_4(x) \leq \overline{u}_2(x) \) as above. The upper semicontinuous case is similar.

The following lemma is the reason for introducing \( u_* \) with the more complicated definition than \( \underline{u}_4 \), since \( \underline{u}_4 = \underline{u}_2 \) is false in general, see Example 3.7 below. Note however that most of the functions we regularize are finely continuous q.e., and thus \( \underline{u}_4 = u_* \). In particular this holds for fine superminimizers and functions in Newtonian spaces. It is only when we regularize the Perron solutions that the notion of \( u_* \) is really needed.

**Lemma 3.6.** The function \( u_* \) is finely lsc-regularized in \( V \), i.e. \( u_{**} = u_* \). Similarly \( u^{**} = u^* \).

**Proof.** It suffices to show that \( u_* \geq u_{**} \) in \( V \), since the converse inequality follows from the fine lower semicontinuity of \( u_* \). Let \( x \in V \) and
\[
a < u_{**}(x) \quad \text{(3.1)}
\]

be arbitrary. By definition, there exists a finely open set \( W \ni x \) such that \( u_* > a \) q.e. in \( W \). Hence for q.e. \( y \in W \), there exists a finely open set \( W_y \ni y \) such that \( u > a \) q.e. in \( W_y \).

The sets \( W_y \) cover \( W \) up to a set of zero capacity. Therefore, by the quasi-Lindelöf property (Theorem 3.3), there are \( \{y_j\}_{j=1}^\infty \) such that the sets \( W_{y_j} \) cover \( W \) up to a set of zero capacity. Since \( \{y_j\}_{j=1}^\infty \) is countable, we conclude that \( u > a \) q.e. in \( W \). From this it follows that \( u_*(x) \geq a \) and taking supremum over all \( a \) admissible in (3.1) gives \( u_* \geq u_{**} \). The proof for \( u^* \) is similar. \( \square \)
**Example 3.7.** We shall now see that the statement corresponding to Lemma 3.6 is false in general for $u_2$. Consider the real line $\mathbb{R}$. Since $p > n = 1$, all points have positive capacity and the fine topology is the same as the Euclidean topology. Let $u(2^{-n}) = 0$, $n = 1, 2, \ldots$, and $u = 1$ otherwise. Then $u(0) = 0$ and $u_2 = 1$ otherwise, while $u_{\mathbb{R}} \equiv 1$, and thus $u_{\mathbb{R}} \neq u_2$.

Similarly, if we let $v(2^{-n} + 2^{-n-m}) = 0$, $n, m = 1, 2, \ldots$, and $v = 1$ otherwise, then $v_2 = u$ and thus $u_{\mathbb{R}} = u_2 \neq 1$, while $v_{\mathbb{R}} = u_{\mathbb{R}} \equiv 1$. One can create similar examples such that any fixed number of $\sharp$-lsc-regularizations is not $\sharp$-lsc-regularized.

**Remark 3.8.** A direct consequence of the definitions is that $u_\ast \leq u^*$ (and $u_2 \leq u^*$), a fact that we will use extensively when comparing the different Perron solutions later in the paper.

If $U$ is quasiopen, then points in $U \setminus \text{fine-int } U^{\partial}$ are isolated in the fine topology, and it is thus not natural to define fine limits and fine regularizations at such points. As a simple example, let $U \subset \mathbb{R}^n$ (unweighted) be the union of an open set $G$ and a point $x \notin \overline{G}$, where $1 < p \leq n$.

This is another reason for why we have restricted our attention to finely open sets in this paper.

**Proposition 3.9.** Assume that $C_p(\lambda B \cap \partial_p V) = 0$ for some ball $B$ with $B \cap V \neq \emptyset$, where $\lambda$ is the dilation constant in the $p$-Poincaré inequality. Then $C_p(B \setminus V) = 0$.

In particular, $C_p(\partial_p V) > 0$ if and only if $C_p(X \setminus V) > 0$.

We will not use Proposition 3.9 directly, but $C_p(\partial_p V) > 0$ is important for the definitions of the $S$- and $Q$-Perron solutions (see Definition 5.1) to make sense and the comparison principle (Theorem 6.2) to hold.

**Proof.** Since $C_p(\lambda B \cap \partial_p V) = 0$, both $\lambda B \setminus V$ and $\lambda B \cap V$ are quasiopen by Theorem 3.2. It then follows from Shanmugalingam [31, Remark 3.5] that for $p$-almost all curves $\gamma : [0, l_\gamma] \to \lambda B$, both $\gamma^{-1}(V)$ and $[0, l_\gamma] \setminus \gamma^{-1}(V)$ are relatively open subsets of $[0, l_\gamma]$. Since $[0, l_\gamma]$ is connected, $\gamma^{-1}(V)$ either equals $[0, l_\gamma]$ or $\emptyset$. Hence for $p$-almost all curves $\gamma \subset \lambda B$, either $\gamma \subset V$ or $\gamma \cap V = \emptyset$.

This implies that the characteristic function $\chi_V$ has $0$ as a $p$-weak upper gradient in $\lambda B$ and so $\chi_V \in N_1^p(\lambda B)$. The $p$-Poincaré inequality then implies that $\chi_V$ is a.e. constant in $B$ and thus also q.e. constant, see Section 2. Since $B \cap V$ is finely open and nonempty, we have $C_p(B \cap V) > 0$ and hence $C_p(B \setminus V) = 0$, which proves the first statement.

One implication in the second statement then follows by the countable subaditivity of the capacity upon letting $r \to \infty$ in $B = B(x, r)$, while the converse implication is trivial. \qed

4. **Fine (super)minimizers and the obstacle problem**

Recall the standing assumptions from the beginning of Section 3. In particular, $V$ is a bounded nonempty finely open set with $C_p(X \setminus V) > 0$.

To define fine (super)minimizers, we first need an appropriate fine local Sobolev space. Here $p$-strict subsets will play a key role, as a substitute for relatively compact subsets. The results in this section also hold for quasiopen sets, even though we only formulate them for finely open sets. Recall that $W \Subset V$ if $W$ is a compact subset of $V$.

**Definition 4.1.** A set $W \Subset V$ is a $p$-strict subset of $V$ if there is a function $\eta \in N_0^{1,p}(V)$ such that $\eta = 1$ on $W$. We will write $W \Subset V$. 

A function $u$ belongs to $N^{1,p}_{\text{fine-loc}}(V)$ if $u \in N^{1,p}(W)$ for all finely open $W \Subset V$.

By Lemma 3.3 in our paper [7], $V$ has a base of fine neighbourhoods $W \Subset V$. Functions in $N^{1,p}_{\text{fine-loc}}(V)$ are finite q.e., finely continuous q.e. and quasicontinuous, by Theorem 4.4 in [7]. Throughout the paper, we consider minimal $p$-weak upper gradients in $V$. For a function $u \in N^{1,p}_{\text{fine-loc}}(V)$ we say that $g_{u,V}$ is a minimal $p$-weak upper gradient of $u$ in $V$ if

$$g_{u,V} = g_{u,W} \text{ a.e. in } W$$

for every finely open $W \Subset V$.

where $g_{u,W}$ is the minimal $p$-weak upper gradient of $u \in N^{1,p}(W)$ with respect to $W$, defined in Section 2. See [7, Lemma 5.2 and Theorem 5.3] for the existence, a.e.-uniqueness and minimality of $g_{u,V}$ when $u \in N^{1,p}_{\text{fine-loc}}(V)$. If $u \in N^{1,p}(V)$, then this definition agrees with the definition of $g_u$ with respect to $V$ in Section 2. If moreover $u \in N^{1,p}(X)$, then the minimal $p$-weak upper gradients $g_{u,V}$ and $g_u$ with respect to $V$ and $X$, respectively, coincide a.e. in $V$, see [5, Corollary 3.7] or [7, Lemma 4.3].

For this reason we drop $V$ from the notation and simply write $g_u$ from now on.

If follows from the definition of $N^{1,p}_{\text{fine-loc}}(V)$ and the properties of $N^{1,p}$ (see Section 2) that functions in $N^{1,p}_{\text{fine-loc}}(V)$ are defined everywhere in $V$, and that if $u \in N^{1,p}_{\text{fine-loc}}(V)$ and $v = u$ q.e., then also $v \in N^{1,p}_{\text{fine-loc}}(V)$. Moreover, by the quasi-Lindelöf principle (Theorem 3.3) and the existence of a base of fine neighbourhoods in $V$, Corollary 2.21 in [3] extends to functions in $N^{1,p}_{\text{fine-loc}}(V)$. That is, if $u, v \in N^{1,p}_{\text{fine-loc}}(V)$ then

$$g_u = g_v \text{ a.e. on } \{x \in V : u(x) = v(x)\}.$$

In [9, Definition 5.1] we introduced the following definition.

**Definition 4.2.** A function $u \in N^{1,p}_{\text{fine-loc}}(V)$ is a fine minimizer (resp. fine superminimizer) in $V$ if

$$\int_W g_u^p \, d\mu \leq \int_W g_{u+\varphi}^p \, d\mu$$

for every finely open $W \Subset V$ and for every (resp. every nonnegative) $\varphi \in N^{1,p}_0(W)$. Moreover, $u$ is a fine subminimizer if $-u$ is a fine superminimizer.

A finely $p$-harmonic function is a finely continuous fine minimizer.

On unweighted $\mathbb{R}^n$, it follows from Proposition 5.3 in [9] that these fine (super)minimizers are exactly the fine $p$-(super)solutions of $\Delta_p u = 0$ introduced in Kilpeläinen–Malý [24] in 1992.

Note that, unlike for minimizers and $p$-harmonic functions on open sets, it is not known, even in unweighted $\mathbb{R}^n$, whether every fine minimizer can be modified on a set of zero capacity so that it becomes finely continuous and thus finely $p$-harmonic, see the discussion in [9, Section 9]. Of course, being a function from $N^{1,p}_{\text{fine-loc}}(V)$, every fine (super/sub)minimizer is finely continuous at q.e. point.

It is not difficult to see that a function is a fine minimizer if and only if it is both a fine subminimizer and a fine superminimizer, see Lemma 5.4 in [9]. Corollary 5.8 in [9] shows that the minimum of two fine superminimizers is also a fine superminimizer. We will use these facts without further ado. We refer to our papers [7], [9] and [10] for further discussion on fine (super)minimizers and the Newtonian space $N^{1,p}_{\text{fine-loc}}(V)$.

In order to prove Theorem 7.1, we will need the following special case of the results in [10, Section 7].
Theorem 4.3. Let \( \{h_j\}_{j=1}^\infty \) be a decreasing sequence of fine (super)minimizers in \( V \). Let \( h(x) = \lim_{j \to \infty} h_j(x) \). If either
\[
h \in N^{1,p}_{\text{fine-loc}}(V) \quad \text{or} \quad \operatorname{ess sup}_{x \in V} |h(x) - u(x)| < \infty,
\]
for some \( u \in N^{1,p}(V) \), then \( h \) is a fine (super)minimizer in \( V \).

Proof. When \( h \in N^{1,p}_{\text{fine-loc}}(V) \), this follows from \([10, \text{Proposition 7.1 and Corollary 7.3.}]\). In the second case, there is \( M \) such that \( |h - u| \leq M \) a.e. Hence, the conclusion follows from \([10, \text{Theorem 7.4(d) and Corollary 7.5(d)}]\) with \( f = u \), \( f_0 = u - M \) and \( f_1 = u + M \).

The obstacle problem is a fundamental tool when studying fine minimizers. See [5] and [9] for earlier studies of the obstacle problem on nonopen sets in metric spaces.

Definition 4.4. Let \( f \in N^{1,p}(V) \) and \( \psi : V \to \mathbb{R} \). Then we define
\[
\mathcal{K}_{\psi,f}(V) = \{ v \in N^{1,p}(V) : v - f \in N^{1,p}_0(V) \text{ and } v \geq \psi \text{ q.e. in } V \}.
\]
A function \( u \in \mathcal{K}_{\psi,f}(V) \) is a solution of the \( \mathcal{K}_{\psi,f}(V) \)-obstacle problem if
\[
\int_V g_u^p \, d\mu \leq \int_V g_v^p \, d\mu \quad \text{for all } v \in \mathcal{K}_{\psi,f}(V).
\]

The Dirichlet problem is a special case of the obstacle problem, with the trivial obstacle \( \psi \equiv -\infty \). Note that the boundary data \( f \) are only required to belong to \( N^{1,p}(V) \), i.e. \( f \) need not be defined on \( \partial V \) or the fine boundary \( \partial_p V \).

Theorem 4.5. ([5, \text{Theorem 4.2}] and [9, \text{Theorem 6.2}]) Let \( f \in N^{1,p}(V) \) and \( \psi : V \to \mathbb{R} \), and assume that \( \mathcal{K}_{\psi,f}(V) \neq \emptyset \). Then there exists a solution \( u \) of the \( \mathcal{K}_{\psi,f}(V) \)-obstacle problem, which is unique q.e. The solution \( u \) is a fine superminimizer in \( V \), and if \( \psi \equiv -\infty \) (i.e. for the Dirichlet problem) it is a fine minimizer.

We will also use the following converse of Theorem 4.5, which follows from Lemma 6.3 in Björn–Björn–Latvala [10].

Lemma 4.6. If \( u \in N^{1,p}(V) \) is a fine superminimizer in \( V \), then it is a solution of the \( \mathcal{K}_{u,u}(V) \)-obstacle problem for any finely open \( W \subset V \).

The following comparison principle was deduced in Björn–Björn [5, Corollary 4.3] for more general sets. We will only need it for finely open sets.

Lemma 4.7. (Comparison principle) Let \( f, f' \in N^{1,p}(V) \) and \( \psi, \psi' : V \to \mathbb{R} \) be such that \( f - f' \in N^{1,p}_0(V) \) and \( \psi \leq \psi' \) q.e. in \( V \). If \( u \) and \( u' \) are solutions of the \( \mathcal{K}_{\psi,f}(V) \)- and \( \mathcal{K}_{\psi',f'}(V) \)-obstacle problems, respectively, then \( u \leq u' \) q.e. in \( V \).

We end this section with the following convergence result for solutions of obstacle problems. It will be a vital tool when proving Lemma 6.3.

Proposition 4.8. Let \( \{f_j\}_{j=1}^\infty \) be a q.e. decreasing sequence in \( N^{1,p}(V) \) such that \( ||f_j - f_0||_{L^p(V)} \to 0 \), as \( j \to \infty \). Let \( u_j \) be a solution of the \( \mathcal{K}_{f_j,f_j}(V) \)-obstacle problem, \( j = 0, 1, \ldots \). Then \( \{u_j\}_{j=1}^\infty \) decreases q.e. to \( u_0 \).

This result holds, with the same proof, if \( V \) is just assumed to be a bounded measurable set with \( C_p(X \setminus V) > 0 \).
Proof. As \( f_j \geq f_{j+1} \geq f_0 \) q.e. in \( V \), when \( j \geq 1 \), it follows from the comparison principle (Lemma 4.7) that \( u_j \geq u_{j+1} \geq u_0 \) q.e. in \( V \). Hence \( \{ u_j \}_{j=1}^{\infty} \) is a q.e.-decreasing sequence and converges q.e. to a function \( v \) on \( V \). Moreover, \( f_j \to f_0 \) q.e., by Corollary 1.72 in [3].

Let \( w_j = u_j - f_j \) and \( w = v - f_0 \), all extended by zero outside of \( V \). We have \( w_j \in N_{0}^{1,p}(V) \subset N^{1,p}(X) \), \( j = 0, 1, \ldots \). Using the Poincaré inequality for \( N_{0}^{1,p} \) [3, Corollary 5.54] and the fact that \( u_j \) is a solution of the \( K_{f_j,f_j}(V) \)-obstacle problem, it follows that there is \( C > 0 \) such that

\[
\|w_j\|_{N^{1,p}(X)} = \left( \|w_j\|_{L^p(V)}^p + \|g_{w_j}\|_{L^p(V)}^p \right)^{1/p} \leq C\|g_{w_j}\|_{L^p(V)} \leq C(\|g_{u_j}\|_{L^p(V)} + \|g_{f_j}\|_{L^p(V)}) \leq 2C\|f_j\|_{N^{1,p}(V)}
\]

and hence

\[
\|w_j\|_{N^{1,p}(V)} \leq \|w_j\|_{N^{1,p}(V)} + \|f_j\|_{N^{1,p}(V)} \leq (1 + 2C)\|f_j\|_{N^{1,p}(V)}
\]

is a bounded sequence in \( N^{1,p}(V) \). Since \( u_j \to v \) and \( w_j \to w \) q.e. in \( X \), as \( j \to \infty \), it follows from Corollary 6.3 in [3] that \( v \in N^{1,p}(X) \), \( w \in N^{1,p}(X) \) and

\[
\|g_v\|_{L^p(V)} \leq \liminf_{j \to \infty} \|g_{u_j}\|_{L^p(V)}. \quad (4.1)
\]

From \( w = 0 \) q.e. in \( X \setminus V \) we see that \( v - f_0 \in N_{0}^{1,p}(V) \). As \( u_j \geq f_j \geq f_0 \) q.e. in \( V \), we have \( v \geq f_0 \) q.e. in \( V \), and hence \( v \in K_{f_0,f_0}(V) \).

Next set \( \varphi_j = f_j + u_0 - f_0 \in K_{f_j,f_j}(V) \). It follows that

\[
\int_V g_{\varphi_j}^p \, d\mu \leq \int_V g_{\varphi_j}^p \, d\mu. \quad (4.2)
\]

Moreover, \( \varphi_j - u_0 \to f_j - f_0 \to 0 \) in \( N^{1,p}(V) \) and in particular \( g_{\varphi_j} \to g_{u_0} \) in \( L^p(V) \), as \( j \to \infty \). Thus, using (4.1) and (4.2),

\[
\int_V g_{\varphi_j}^p \, d\mu \leq \liminf_{j \to \infty} \int_V g_{\varphi_j}^p \, d\mu \leq \liminf_{j \to \infty} \int_V g_{\varphi_j}^p \, d\mu = \int_V g_{\varphi_j}^p \, d\mu.
\]

Therefore, \( v \) is also a solution of the \( K_{f_0,f_0}(V) \)-obstacle problem, and hence \( v = u_0 \) q.e., by the uniqueness part of Theorem 4.5. \( \square \)

5. Perron solutions

We are now ready to define and study Perron solutions on finely open sets. We give four different definitions. As we shall see they give almost the same solutions, but this is far from immediate. Recall that \( \partial_{\text{p}}V \) denotes the fine boundary of \( V \), and that the fine regularizations \( u_* \) and \( u^* \) were defined in Section 3.

**Definition 5.1.** Given a function \( f : \partial_{\text{p}}V \to \mathbb{R} \), let \( \mathcal{U}_f \) be the set of all fine superminimizers \( u \in N^{1,p}(V) \) such that

\[
\lim\inf_{y \to x \in \partial_{\text{p}}V} u(y) \geq f(x) \quad \text{for all } x \in \partial_{\text{p}}V.
\]

Define the upper \( P- \) and \( R- \)Perron solutions of \( f \) by

\[
\overline{P}f(x) = \inf_{u \in \mathcal{U}_f} u_*(x) \quad \text{and} \quad \overline{R}f(x) = \inf_{u \in \mathcal{U}_f} u^*(x), \quad x \in V.
\]

Similarly, let \( \tilde{\mathcal{U}}_f \) be the set of all fine superminimizers \( u \in N^{1,p}(V) \) such that

\[
\lim\inf_{y \to x \in \partial_{\text{p}}V} u(y) \geq f(x) \quad \text{for q.e. } x \in \partial_{\text{p}}V,
\]

\[
\text{fine lim inf } u(y) \geq f(x) \quad \text{for q.e. } x \in \partial_{\text{p}}V, \quad (5.1)
\]
and define the upper $S$- and $Q$-Perron solutions as
\[
\overline{S}f(x) = \inf_{u \in \overline{U}_f} u_*(x) \quad \text{and} \quad \overline{Q}f(x) = \inf_{u \in \overline{U}_f} u^*(x), \quad x \in V.
\]

The lower Perron solutions are defined analogously using suprema of fine sub-minimizers in $N^{1,p}(V)$, or by letting
\[
Pf := -\overline{P}(-f), \quad Rf := -\overline{P}(-f), \quad \underline{S}f := -\overline{S}(-f) \quad \text{and} \quad \underline{Q}f := -\overline{Q}(-f).
\]

As usual, $\inf \emptyset = \infty$. We shall call the fine regularizations $u_*$ and $u^*$, appearing in the above infima, admissible in the definitions of the corresponding Perron solutions.

Note that since functions in $N^{1,p}(V)$ are finite q.e., the upper Perron solutions are either $\leq \infty$ q.e. or $\equiv \infty$. Clearly,
\[
\underline{S}f \leq Pf \leq Rf \quad \text{and} \quad \underline{S}f \leq \overline{Q}f \leq Rf \quad \text{everywhere in } V. \tag{5.2}
\]

We shall later see that under a very mild assumption, all these solutions are equal q.e. and finely continuous q.e. We postpone to Section 7 the proof that Perron solutions are indeed fine minimizers, under some growth conditions on the boundary data.

See Remark 6.7 for why we do not need to assume that functions in $U_f$ and $\tilde{U}_f$ are bounded from below. As we only have the comparison principle (Theorem 6.2 below) for functions in $N^{1,p}(V)$, we instead require that functions in the upper classes $U_f$ and $\tilde{U}_f$ belong to $N^{1,p}(V)$. This will be crucial when obtaining Theorem 6.1 comparing upper and lower $Q$-Perron solutions.

Since each function $u^* \in U_f$ is finely upper semicontinuous, so are $\overline{Q}f$ and $Rf$. Hence
\[
(\overline{Q}f)_* \leq (\overline{Q}f)^* \leq \overline{Q}f \quad \text{and} \quad (\overline{R}f)_* \leq (\overline{R}f)^* \leq \overline{R}f \quad \text{everywhere in } V. \tag{5.3}
\]

Similarly, $Qf$ and $Rf$ are finely lower semicontinuous and
\[
(Qf)^* \geq (Qf)_* \geq Qf \quad \text{and} \quad (Rf)^* \geq (Rf)_* \geq Rf \quad \text{everywhere in } V. \tag{5.4}
\]

In particular, if $\overline{Q}f = Qf$, then it is finely continuous. We shall see in Proposition 5.3 and Lemma 6.6 that equality in (5.3) and (5.4) holds q.e. under a very mild assumption.

The use of finely upper semicontinuous functions in the definitions of $\overline{Q}f$ and $Rf$ is a key tool for obtaining results for the Perron solutions, see Remark 7.5. Another practical reason for introducing $\overline{Q}f$ is that the inequality in (5.1) is only required q.e., which proves to be a useful tool. This also immediately implies that $Q$- and $S$-Perron solutions are invariant under perturbations on sets of zero capacity, i.e. that $\overline{Q}f = \overline{Q}f'$ and $\overline{S}f = \overline{S}f'$ whenever $f = f'$ q.e. on $\partial pV$.

At the same time, the $P$-Perron solutions most closely mimic the traditional definition of Perron solutions using lsc-regularized $p$-superharmonic functions dominating the boundary data $f$ everywhere on the boundary. This is the reason why we have reserved the letter $P$ for the $P$-Perron solutions. The letter $Q$ in the $Q$-Perron solutions comes from the q.e.-condition in their definition. It is also closely related to the $Q$-Perron solutions introduced in Björn–Björn–Shanmugalingam [12]. The $S$-Perron solutions also share this property, but for us the $Q$-Perron solutions are more important because of Lemma 5.2 and Theorem 6.1 below. Instead, $S$ comes from it being the smallest upper Perron solution. Finally, $R$ was chosen since the upper and the lower $R$-solutions are most relaxed (far away) with respect to each other, and as a tribute to Remak [30]. In addition, all four Perron solutions are closely
related to the Sobolev–Perron solutions introduced in Björn–Björn–Sjödin [15], see Remark 9.3.

It is the following countability lemma that makes it possible to obtain some fundamental results for \( \overline{Qf} \).

**Lemma 5.2.** Assume that \( f : \partial_p V \to \overline{\mathbb{R}} \) is such that \( \tilde{U}_f \neq \emptyset \). Then there is a decreasing sequence \( \{ u_j \}_{j=1}^\infty \) of functions in \( \tilde{U}_f \) such that
\[
\overline{Qf}(x) = \lim_{j \to \infty} u_j(x) \quad \text{for q.e. } x \in V.
\]

The same statement holds also for \( \overline{Rf} \) and \( u_j \in U_f \).

**Proof.** Since each function \( u^* \in \tilde{U}_f \), admissible for \( \overline{Qf} \), is finely upper semicontinuous, so is \( \overline{Qf} \). Moreover, for any \( q \in Q \),
\[
\{ x \in V : \overline{Qf}(x) < q \} = \bigcup_{u \in \tilde{U}_f} \{ x \in V : u^*(x) < q \} \tag{5.5}
\]
is a union of finely open sets. By the quasi-Lindelöf principle (Theorem 3.3), we can find a countable collection \( \{ u_{j,q} \}_{j=1}^\infty \) of functions in \( \tilde{U}_f \) and a set \( E_q \) such that \( C_p(E_q) = 0 \) and
\[
\{ x \in V : \overline{Qf}(x) < q \} = E_q \cup \bigcup_{j=1}^\infty \{ x \in V : u_{j,q}^*(x) < q \}.
\]

Letting \( E = \bigcup_{q \in Q} E_q \), we see that
\[
\overline{Qf}(x) = \inf_{j \in \mathbb{N}} u_{j,q}^*(x) \quad \text{for all } x \in V \setminus E.
\]

Next, we reorder the countable collection
\[
\{ u_{j,q} : q \in Q \text{ and } j = 1, 2, ... \}
\]
into a sequence \( \{ v_j \}_{j=1}^\infty \). Then \( u_j := \min\{v_1, ..., v_j\} \in \tilde{U}_f \) for each \( j \) and the sequence \( \{ u_j \}_{j=1}^\infty \) is decreasing. Since \( u_j^* \leq \min\{v_1^*, ..., v_j^*\} \to \overline{Qf} \) in \( V \setminus E \), it follows that \( u_j \to \overline{Qf} \) q.e. in \( V \). The proof for \( \overline{Rf} \) is the same. \( \square \)

We can now show that the \( Q \)-Perron solutions are finely continuous q.e. under a rather mild assumption.

**Proposition 5.3.** Let \( f : \partial_p V \to \overline{\mathbb{R}} \) and assume that \( (\overline{Qf})^* > -\infty \) q.e. Then
\[
(\overline{Qf})^* = (\overline{Qf})^* = \overline{Qf} \quad \text{q.e.}, \tag{5.6}
\]
and thus \( \overline{Qf} \) is finely continuous q.e. in \( V \). In particular, this holds if \( Qf \neq -\infty \).

It follows from Theorem 1.4 (b) in Björn–Björn–Latvala [8] that \( \overline{Qf} \) is also quasicontinuous on \( V \), under the assumptions above.

**Proof.** We may assume that \( \tilde{U}_f \neq \emptyset \), since otherwise \( \overline{Qf} \equiv \infty \) and there is nothing to prove. Let \( u_j \in \tilde{U}_f \) be the decreasing sequence of functions provided by Lemma 5.2. Since \( u_j \in N^{1,p}(V) \) is finite q.e. in \( V \), it follows that for q.e. \( x \in V \), there is an integer \( m \) such that
\[
x \in V_m := \{ y : (\overline{Qf})^*(y) + m > u_1^*(y) \}.
\]
Note that each \( V_m \) is finely open (because \((Qf)_* - u_1^*\) is finely lower semicontinuous) and that, by (5.3),
\[
\begin{align*}
    u_1 - m < (Qf)_* & \leq Qf \leq u_j \leq u_1 \quad \text{q.e. in } V_m.
\end{align*}
\]
Since each \( u_j \) is a fine superminimizer in \( V_m \), Theorem 4.3 implies that so is \( Qf \).
In particular, \( Qf \in N_1^{1,p}(V_m) \) and it is thus finely continuous q.e. in \( V_m \). This proves (5.6) in \( V_m \) and, by letting \( m \to \infty \), also in \( V \).

Finally, if \( v \in N^{1,p}(V) \) is admissible in the definition of \( Qf \not\equiv -\infty \), then (5.4) and the comparison in Theorem 6.1 below imply that
\[
\begin{align*}
    (Qf)_* & \geq (Qf)_* \geq Qf \geq v > -\infty \quad \text{q.e.}
\end{align*}
\]
(Note that Proposition 5.3 is not used when proving Theorem 6.1 below.)

An intriguing question is if \((Qf)_* = Qf\) everywhere. In Proposition 8.1 and Theorem 8.4 below we show this in some special cases. For Perron solutions on open sets, this is true for arbitrary \( f \), by Proposition 9.1 below. On the other hand, in the theory of balayage, which has many similarities with the Perron method, it is known that one needs to regularize even on open sets. We have found it helpful to use regularizations.

6. Comparing Perron solutions

Our primary aim in this section is to deduce the following fundamental inequalities between upper and lower Perron solutions.

**Theorem 6.1.** Let \( f : \partial_p V \to \overline{\mathbb{R}} \). Then
\[
Qf \leq Sf \leq \overline{Qf} \quad \text{everywhere in } V.
\]

A direct consequence of (5.2), Theorem 6.1 and the duality between upper and lower Perron solutions is that
\[
\begin{align*}
    Pf \leq Qf \leq \overline{Qf} \leq Qf \leq \overline{Qf} \leq \overline{Pf} \leq \overline{Qf} \quad \text{and } Qf \leq \overline{Pf} \leq \overline{Qf} \quad \text{(6.1)}
\end{align*}
\]
everywhere in \( V \).

We have not been able to prove that \( Pf \leq \overline{Pf} \) and \( Sf \leq \overline{Sf} \), not even for bounded \( f \). A reason for this is that the inequality in our comparison principle below holds only quasieverywhere. This problem appears already for fine \( p \)-supersolutions in the Euclidean case, see Theorem 5.17 in Latvala [27]. However when \( p = n \) on unweighted \( \mathbb{R}^n \), the inequality in the comparison principle was deduced everywhere in Corollary 7.13 in [27].

The key to deducing Theorem 6.1 is the following comparison principle. The assumption \( C_p(X \setminus V) > 0 \) from the beginning of Section 4 is essential, since otherwise (6.2) and (6.3) are void and the theorem fails, see also Proposition 3.9.

**Theorem 6.2.** (Comparison principle) Let \( u \in N^{1,p}(V) \) be a fine superminimizer and \( v \in N^{1,p}(V) \) be a fine subminimizer. Assume that for q.e. \( x \in \partial_p V \), either
\[
\begin{align*}
    \text{fine lim inf}_{V \ni y \to x} u(y) & \geq \text{fine lim sup}_{V \ni y \to x} v(y) \quad \text{(6.2)}
\end{align*}
\]
or
\[
\text{fine lim inf}_{V \ni y \to x} (u(y) - v(y)) \geq 0. \quad \text{(6.3)}
\]
Then \( u \geq v \) q.e. in \( V \), while \( u^* \geq v^* \geq v_* \) and \( u^* \geq u_* \geq v_* \) everywhere in \( V \).
Note that (6.2) does not imply (6.3) when the limits in (6.2) are infinite. The following lemma will be crucial for proving Theorem 6.2. It will also be used in the proof of Lemma 6.6 and in Remark 6.7.

**Lemma 6.3.** Let \( u \in N^{1,p}(V) \) be a fine superminimizer and let \( E \subset \partial_p V \) with \( C_p(E) = 0 \). Then there is a decreasing sequence \( \{u_j\}_{j=1}^\infty \) of finely lsc-regularized fine superminimizers in \( N^{1,p}(V) \) which are bounded from below and such that

\[
\lim_{j \to \infty} u_j = u \text{ q.e. in } V \quad \text{and} \quad \lim_{V \ni y \to x} u_j(y) = \infty \text{ for } x \in E \text{ and } j = 1, 2, \ldots.
\]

If moreover \( u \in \tilde{U}_f \) for some \( f : \partial_p V \to \mathbb{R} \), then in addition we can choose \( u_j \in U_f \).

**Proof.** By Corollary 1.3 in Björn–Björn–Shanmugalingam [13] (or [3, Theorem 5.31]), \( C_p \) is an outer capacity. Thus we can find a decreasing sequence of open sets \( \{G_k\}_{k=1}^\infty \) such that \( E \subset G_k \) and \( C_p(G_k) < 2^{-k}p, k = 1, 2, \ldots \). By Lemma 10.17 in [3] there is a decreasing sequence of nonnegative functions \( \{\psi_j\}_{j=1}^\infty \) such that \( \|\psi_j\|_{N^{1,p}(X)} < 2^{-j} \) and \( \psi_j \geq k - j \) in \( G_k \) whenever \( k > j \).

Let \( v_j = \psi_j + \max\{u, j\} \) and let \( u_j \) be the finely lsc-regularized solution of the \( \mathcal{K}_{u, v_j}(V) \)-obstacle problem. Then \( u_j \) is a fine superminimizer, by Theorem 4.5. Moreover, \( u_j \geq -j \) and it is thus bounded from below. If \( k > j \), then

\[
v_j \geq \psi_j - j \geq k - 2j \quad \text{in } G_k \cap V,
\]

and hence \( u_j \geq k - 2j \) in \( G_k \cap V \). Thus for each \( j \),

\[
\liminf_{V \ni y \to x} u_j(y) = \infty \quad \text{for } x \in E.
\]

(6.4)

Since \( v_j \searrow u \) in \( N^{1,p}(V) \), and \( u \) is a solution of the \( \mathcal{K}_{u, u}(V) \)-obstacle problem (by Lemma 4.6), it follows from Proposition 4.8 that \( \{u_j\}_{j=1}^\infty \) decreases q.e. to \( u \).

Finally, if \( u \in \tilde{U}_f \) then we let \( \tilde{E} = E \cup \{x : \liminf_{V \ni y \to x} u(y) < f(x)\} \) and construct \( u_j \) as above with \( \tilde{E} \) instead of \( E \). (Note that \( C_p(\tilde{E}) = 0 \), as \( u \in \tilde{U}_f \).) Thus

\[
\liminf_{V \ni y \to x} u_j(y) \geq \liminf_{V \ni y \to x} u(y) \geq f(x) \quad \text{for } x \in \partial_p V \setminus \tilde{E},
\]

which together with (6.4) (with \( E \) replaced by \( \tilde{E} \)) shows that \( u_j \in U_f \).

**Proof of Theorem 6.2.** Assume to start with that \( u \) is bounded from below and \( v \) from above. Then (6.2) \( \Rightarrow \) (6.3) and Proposition 3.4 shows that \( \max\{v - u, 0\} \in N^{1,p}_0(V) \).

Now let \( \tilde{v} \) be a solution of the Dirichlet \( \mathcal{K}_{-\infty, v}(V) \)-obstacle problem. Note that \( -\tilde{v} \) is a solution of the Dirichlet \( \mathcal{K}_{-\infty, -v}(V) \)-obstacle problem. By Lemma 4.6, \( u \) is a solution of the \( \mathcal{K}_{u, u}(V) \)-obstacle problem, and \( -v \) is a solution of the \( \mathcal{K}_{-v, -v}(V) \)-obstacle problem. By the comparison principle (Lemma 4.7), we therefore see that \( -\tilde{v} \leq -v \) and \( \tilde{v} \leq u \) q.e. in \( V \) and thus

\[
v \leq \tilde{v} \leq u \quad \text{q.e. in } V.
\]

Next, for general \( u \) and \( v \), Lemma 6.3 provides us with a decreasing sequence \( \{u_j\}_{j=1}^\infty \) of fine superminimizers bounded from below such that \( \lim_{j \to \infty} u_j(x) = u(x) \) q.e. in \( V \). In particular, \( u_j \geq u \) q.e. in \( V \). Similarly, there is an increasing sequence \( \{v_j\}_{j=1}^\infty \) of fine subminimizers bounded from above such that \( \lim_{j \to \infty} v_j(x) = v(x) \) q.e. in \( V \) and \( v_j \leq v \) q.e. in \( V \).

Applying the already proved bounded case to \( u_j \) and \( v_j \), we get \( u_j \geq v_j \) q.e. in \( V \). Letting \( j \to \infty \) shows that \( u \geq v \) q.e. in \( V \) also in this case. The statement about fine regularizations now follows immediately. \( \square \)
We are now ready to prove Theorem 6.1.

**Proof of Theorem 6.1.** Let \( u^* \) and \( v^* \) be admissible in the definitions of \( Qf \) and \( Sf \), respectively. Then

\[
\text{fine lim inf } u^*(y) \geq f(x) \geq \text{fine lim sup } v^*(y) \quad \text{for q.e. } x \in \partial_p V.
\]

It thus follows from the comparison principle (Theorem 6.2) that \( u^* \geq v^* \) everywhere in \( V \). Taking infimum resp. supremum over all admissible \( u^* \) and \( v^* \) yields that \( Qf \geq Sf \) in \( V \). By duality, the other inequality follows from (5.2). \( \square \)

We conclude this section by comparing the four definitions of Perron solutions and prove that they coincide q.e.

**Theorem 6.4.** Let \( f : \partial_p V \to \mathbb{R} \). Then

\[
\begin{align*}
Qf & \leq (Qf)_* \leq (Qf)_* \leq Sf \quad \text{everywhere in } V \quad (6.5) \\
(\overline{Q}f)_* = (\overline{f})_* \leq (\overline{Q}f)_* = (\overline{f})_* \quad \text{everywhere in } V. \quad (6.6)
\end{align*}
\]

If moreover \((\overline{Q}f)_* > -\infty \) q.e., then also

\[
Sf = \overline{f} = \overline{Q}f = \overline{R}f \quad \text{q.e. in } V.
\]

and these four upper Perron solutions are finely continuous q.e.

Note that by (6.6) the condition \((\overline{Q}f)_* > -\infty \) q.e. in the last part of Theorem 6.4, as well as in Proposition 5.3, can equivalently be expressed using any of the lsc-regularized upper Perron solutions. In order to prove Theorem 6.4 we will need the following two lemmas.

**Lemma 6.5.** Let \( f : \partial_p V \to \mathbb{R} \). Then

\[
(\overline{Q}f)_* \leq Sf \quad \text{and} \quad (\overline{R}f)_* \leq Pf \quad \text{everywhere in } V.
\]

**Proof.** Let \( u \in \tilde{U}_f \). Then \( u^* \) is admissible in the definition of \( \overline{Q}f \) and hence \((\overline{Q}f)_* \leq (u^*)_* = u_* \) in \( V \), by Remark 3.5 together with the fact that \( u \) is finely continuous q.e. in \( V \). Since \( u_* \) is admissible in the definition of \( Sf \), taking infimum over all \( u \in \tilde{U}_f \) shows that \((\overline{Q}f)_* \leq Sf \).

The inequality \((\overline{R}f)_* \leq Pf \) is shown similarly by considering \( u \in U_f \). \( \square \)

**Lemma 6.6.** Let \( f : \partial_p V \to \mathbb{R} \). Then \( \overline{Q}f = \overline{R}f \) q.e. in \( V \).

**Proof.** We may assume that \( \tilde{U}_f \neq \emptyset \). Lemma 5.2 provides us with a decreasing sequence \( \{u_j\}_{j=1}^\infty \) of functions in \( \tilde{U}_f \) such that

\[
\overline{Q}f(x) = \lim_{j \to \infty} u_j(x) \quad \text{for q.e. } x \in V.
\]

For each \( j = 1, 2, \ldots \), let \( u_{j,k} \in U_f \) be the decreasing sequence of functions provided by Lemma 6.3 so that

\[
u_j(x) = \lim_{k \to \infty} u_{j,k}(x) \quad \text{for q.e. } x \in V.
\]

Then

\[
v_k := \min_{j \leq k} u_{j,k} \to \overline{Q}f \quad \text{q.e. in } V, \quad \text{as } k \to \infty.
\]

Since \( v_k^* \) are admissible in the definition of \( \overline{R}f \) and \( v_k^* = v_k \) q.e., letting \( k \to \infty \) shows that \( \overline{Q}f \geq \overline{R}f \) q.e. The reverse inequality \( \overline{Q}f \leq \overline{R}f \) holds everywhere and is immediate from the definition. \( \square \)
The Perron method associated with finely $p$-harmonic functions on finely open sets

**Proof of Theorem 6.4.** The inequalities (6.5) follow directly from (5.4), Theorem 6.1 and Lemma 6.5. Next, Lemmas 6.5 and 6.6 together with (5.2) yield

$$ (Qf)_* \leq \underline{S} f \leq \overline{P} f = \overline{R} f \quad \text{q.e. in } V. \quad (6.7) $$

Applying the fine lsc-regularization to this chain of inequalities, together with Lemma 3.6, gives the equalities (6.6). The last statement follows from (6.7) and Proposition 5.3.

**Remark 6.7.** Traditionally, Perron solutions (on open sets) are defined using functions in the upper class that are bounded from below. This “lower boundedness” assumption was introduced by Brelot [17], who was the first to study Perron solutions for unbounded functions; see [17, p. 146] both for the definition and a remark explaining why it is essential. The first papers on Perron solutions, by Perron [29] and Remak [30], only dealt with continuous (and thus bounded) boundary data $f$, for which this assumption is redundant.

As we have seen, we do not need this assumption. The reason for this is that our upper and lower classes only contain Sobolev functions, for which the comparison principle (Theorem 6.2) holds without any boundedness assumption. On the other hand, even with this assumption, essentially all of the theory developed here would be true, including the results in this section. Let us denote such upper Perron solutions by $\hat{S} f$, $\hat{P} f$, $\hat{Q} f$ and $\hat{R} f$. By Lemmas 5.2 and 6.3 it follows (essentially as in the proof of Lemma 6.6) that $\hat{R} f \leq \hat{Q} f \text{ q.e.}$ Using also the trivial inequalities $Q f \leq \hat{Q} f \leq \hat{R} f$ gives

$$ \hat{Q} f = \hat{R} f = \overline{Q} f = \overline{R} f \quad \text{q.e.} \quad (6.8) $$

The $\hat{\cdot}$-version of Theorem 6.4 then yields that

$$ \hat{S} f = \hat{P} f = \hat{Q} f = \hat{R} f = \overline{Q} f \quad \text{q.e.,} \quad (6.9) $$

provided that $(\hat{Q} f)_* > -\infty \text{ q.e.}$ (or equivalently $(\overline{Q} f)_* > -\infty \text{ q.e.}$). See also Remark 9.2 below.

The only real difference after adding the “lower boundedness” assumption would be in Proposition 8.1 below, where $h^*$ with inf $h^* = -\infty$ is not admissible in the definition of $\hat{Q} f$ and thus we would not be able to conclude that $\hat{Q} f = h^*$ and $\hat{S} f = h^*$ everywhere for such $h$. It would still follow that $\hat{Q} f = \overline{Q} f = h^* \hat{S} f = \overline{S} f = h^*$ q.e., by (6.8) and (6.9).

### 7. Perron solutions are fine minimizers

Our next aim is to show that the Perron solutions are indeed fine minimizers, under rather general assumptions on the boundary data.

**Theorem 7.1.** Let $f : \partial_p V \to \overline{\mathbb{R}}$. Assume that one of the following conditions holds:

(a) $Q f \in N_{\text{fine-loc}}^{1,p}(V)$;
(b) there is $M > 0$ and $u \in N^{1,p}(V)$ such that $|Q f - u| \leq M$ a.e. in $V$;
(c) there is $M > 0$ and $u \in N^{1,p}(V)$ such that

$$ \text{fine lim sup}_{V \ni y \to x} u(y) - M \leq f(x) \leq \text{fine lim inf}_{V \ni y \to x} u(y) + M \quad \text{for q.e. } x \in \partial_p V. $$

Then $Q f$ is a fine minimizer. In particular, this holds if $f$ is q.e. bounded.

As a direct consequence we obtain the following characterization.

**Corollary 7.2.** Let $f : \partial_p V \to \overline{\mathbb{R}}$. Then $Q f$ is a fine minimizer if and only if $Q f \in N_{\text{fine-loc}}^{1,p}(V)$.
A weaker necessary condition for $\overline{Q} f$ to be a fine minimizer is that $\overline{Q} f$ is finite q.e. We do not know if this condition is sufficient.

To prove Theorem 7.1, we will need the following lemma.

**Lemma 7.3.** For $f : \partial_p V \to \overline{\mathbb{R}}$ and $u \in \overline{U}_f$, let $h$ be a solution of the Dirichlet $\mathcal{K}_{-\infty,u}(V)$-obstacle problem. Then $h \in \overline{U}_f$, $h \leq u$ q.e. in $V$ and $h$ is a fine minimizer.

**Proof.** By Theorem 4.5, $h$ is a fine minimizer. Since $u$ is a solution of the $\mathcal{K}_{u,u}(V)$-obstacle problem, by Lemma 4.6, the comparison principle (Lemma 4.7) implies that $h \leq u$ q.e. in $V$. By definition, $h - u \in N^1_0(V)$, and thus by Proposition 3.4, also $h \in \overline{U}_f$.

**Proof of Theorem 7.1.** By Lemma 5.2, there is a decreasing sequence $\{u_j\}_{j=1}^\infty$ of functions in $\overline{U}_f$ such that

$$\overline{Q} f(x) = \lim_{j \to \infty} u_j(x) \quad \text{for q.e. } x \in V.$$ 

If (a) or (b) in the statement of the theorem holds, then replace each $u_j$ by a fine minimizer $h_j \in \overline{U}_f$ provided by Lemma 7.3, so that $h_j \to \overline{Q} f$ q.e. in $V$ as $j \to \infty$. It follows from the comparison principle (Lemma 4.7) that (after redefinitions on sets of capacity zero) $\{h_j\}_{j=1}^\infty$ is also a decreasing sequence. Theorem 4.3 then shows that $\overline{Q} f$ is a fine minimizer.

Assume next that (c) holds and let $h \in N^{1,p}(V)$ be a solution of the Dirichlet $\mathcal{K}_{-\infty,u}(V)$-obstacle problem. Since $h - u \in N^1_0(V)$, Proposition 3.4 shows that

$$f(x) \leq \liminf_{y \to x} f(y) + M = \liminf_{y \to x} h(y) + M \quad \text{for q.e. } x \in \partial_p V,$$

thus $v_j := \min\{u_j, h + M\} \in \overline{U}_f$.

Let $h_j \in \overline{U}_f$ be a fine minimizer provided for $v_j$ by Lemma 7.3. As in (7.1), we have using (c) that

$$\limsup_{y \to x} h(y) - M \leq f(x) \leq \liminf_{y \to x} h_j(y) \quad \text{for q.e. } x \in \partial_p V.$$

The comparison principle (Theorem 6.2) then implies that

$$h - M \leq h_j \leq v_j \leq h + M \quad \text{q.e. in } V.$$ 

Since $h_j \to \overline{Q} f$ q.e. in $V$ as $j \to \infty$, Theorem 4.3 concludes the proof.

**Remark 7.4.** Another consequence of Lemma 7.3 and Proposition 3.4 is that

$$\overline{Q} f = \inf\{h^* \in \overline{U}_f : h \text{ is a fine minimizer}\} = \inf_u H^* u \quad \text{everywhere in } V,$$

where the second infimum is taken over all $u \in N^{1,p}(V)$ such that (5.1) holds, and $H^* u$ is the usc-regularized solution of the Dirichlet $\mathcal{K}_{-\infty,u}(V)$-obstacle problem. Similar identities hold for $\overline{S} f$. On the other hand, it is far from clear whether $\overline{P} f$ and $\overline{R} f$ can be expressed in such terms.

**Remark 7.5.** Let us compare the proof of Theorem 7.1 with the more traditional proofs for Perron solutions on open sets, as in Heinonen–Kilpeläinen–Martio [21, Theorem 9.2] and Björn–Björn–Shanmugalingam [12, Theorem 4.1] (or [3, Theorem 10.10]). A major idea in those proofs is to use convergence on a countable dense
subset of the open set $V$. (In [21] this appears in the proof of Choquet’s topological lemma [21, Lemma 8.3].) In many cases, countable subsets of $X$ have zero capacity and are therefore not seen by the fine topology. Hence there is no possibility to obtain general results using a countable finely dense subset of a finely open $V$. Instead, we use the countable dense subset $Q$ on the target side via Lemma 5.2 and obtain convergence q.e. in $V$. For this, the use of finely usc-regularized functions is essential as it makes it possible to deduce the crucial identity (5.5) in Lemma 5.2 with finely open sets in the right-hand side.

Another ingredient in the traditional proofs is the Poisson modification on an exhaustion by compactly contained open subsets of $V$, together with Harnack’s convergence principle. For fine minimizers, we have only weaker convergence theorems which require the additional assumptions in Theorem 7.1. At the same time, since we define the $Q$-Perron solutions using functions from $N^{1,p}(V)$ and require the boundary inequality in (5.1) only for q.e. $x \in \partial_p V$, the Poisson modification in the proof of Theorem 7.1 can be taken with respect to all of $V$ through Lemma 7.3.

8. Resolutivity of Sobolev and continuous boundary values

Next we study resolutivity, i.e. when the upper and lower Perron solutions agree, at least q.e. As we shall see, the $Q$-Perron solutions seem to behave slightly better than the other solutions. In combination with Theorem 6.4, they provide resolutivity and invariance results for the other Perron solutions as well.

Proposition 8.1. Let $f : V^p \to \mathbb{R}$. Assume that $f \in N^{1,p}(V)$ and

$$f(x) = \lim_{V \ni y \to x} f(y) \quad \text{for q.e. } x \in \partial_p V. \quad (8.1)$$

In particular, this holds if $f \in N^{1,p}(X)$.

Let $h$ be a solution of the Dirichlet $K_{-\infty,f}(V)$-obstacle problem. Then

$$\mathcal{Q}f = h^* = \mathcal{S}f \quad \text{and} \quad Qf = h_+ = \mathcal{S}f \quad \text{everywhere in } V \quad (8.2)$$

and $Qf = \mathcal{Q}f$ q.e. in $V$. Moreover,

$$\left(\mathcal{Q}f\right)^* = \left(\mathcal{Q}f\right)^* = \mathcal{Q}f \quad \text{and} \quad \left(\mathcal{Q}f\right)_* = \left(\mathcal{Q}f\right)_* = \mathcal{Q}f \quad \text{everywhere in } V. \quad (8.3)$$

Proof. Proposition 3.4, Theorem 4.5, and the assumptions on $f$ imply that $h^*$ is admissible in the definition of both $\mathcal{Q}f$ and $\mathcal{S}f$. Hence by Theorem 6.1,

$$\mathcal{Q}f \leq h^* \leq \mathcal{S}f \leq \mathcal{Q}f \quad \text{everywhere in } V,$$

which proves the first statement in (8.2). The second statement in (8.2) is shown similarly, or by applying the first one to $-f$. The equality $\mathcal{Q}f = \mathcal{Q}f$ q.e. then follows immediately from the fact that $h^* = h_+$ q.e. Applying the fine regularizations, together with Lemma 3.6, now gives

$$\left(\mathcal{Q}f\right)^* = \left(\mathcal{Q}f\right)^* = h^{**} = h^* = \mathcal{Q}f$$

as well as the other identities in (8.3).

Finally, if $f \in N^{1,p}(X)$, then $f$ is finely continuous q.e. in $X$ and thus (8.1) holds.

The following example shows that the fine limit in (8.1) need not exist for a large part of $\partial_p V$ when $f \in N^{1,p}(V)$. 


Example 8.2. Let (using complex notation)

\[ V = \{ z = re^{i\theta} : 0 < r < 1 \text{ and } 0 < \theta < 2\pi \} \]

be the slit disc in \( \mathbb{R}^2 \). Then \( f(re^{i\theta}) := r\theta \in N^{1,p}(V) \) for all \( p > 1 \), but the fine limits

\[ \lim_{V \ni y \to z} f(z) \]

do not exist for any \( z \) in the slit, but for the tip 0. This also shows that condition (c) in Theorem 7.1 would be less general if fine \( \lim \sup \) and fine \( \lim \inf \) were replaced by fine lim.

We shall now see that under additional assumptions on the boundary data, the \( Q \)-Perron solutions are finely continuous.

Proposition 8.3. Let

\[ V_0 = \{ z \in V : C_p(B(z, \rho) \cap \partial_p V) > 0 \text{ for all } \rho > 0 \}. \]

Also let \( f : \overline{V}^p \to \mathbb{R} \) be such that \( f \in N^{1,p}(V) \), (8.1) holds and

\[ C_p \text{-ess lim}_{\partial_p V \ni y \to x} f(x) \quad \text{exists for all } z \in V_0. \] (8.4)

In particular, this holds if \( f = f_0 \) q.e. in \( V \) for some \( f_0 \in \text{Lip}(\overline{V}^p) \).

Then \( h_* = h^* \) is finely continuous and

\[ \mathcal{S}f = \mathcal{Q}f = h_* = h^* = \overline{\mathcal{Q}}f = \overline{\mathcal{S}}f \quad \text{everywhere in } V. \] (8.5)

Here \( C_p \text{-ess lim} \) is taken with respect to the metric topology from \( X \) and up to sets of zero capacity, as in [9, Section 7]. Note that the limit in (8.4) does not make sense for \( z \in V \setminus V_0 \). The simplest example showing that \( V_0 \) can be nonempty is perhaps letting \( V = \Omega \cup \{ 0 \} \), where \( \Omega \) is the complement of the Lebesgue spine in \( \mathbb{R}^3 \) as in Example 13.4 in [3].

Proof. The assumption (8.1) implies that the extensions

\[ f_*(x) = \lim_{V \ni y \to x} f(y) \quad \text{and} \quad f^*(x) = \lim_{V \ni y \to x} \sup f(y) \quad \text{for } x \in \partial_p V \]

satisfy \( f_* = f = f^* \) q.e. on \( \partial_p V \). Hence, as (8.4) holds, all \( z \in V_0 \) satisfy condition (b) in [9, Theorem 7.5]. On the other hand, for \( z \in V \setminus V_0 \), condition (c) in [9, Theorem 7.5] holds. Theorem 7.5 in [9] then implies that \( h_* = h^* \) is finely continuous in \( V \). The remaining conclusions in (8.5) then immediately follow from (8.2).

Our next aim is to prove the following resolutivity and invariance result. Here \( C_{\text{unif}}(\partial_p V) \) is the space of uniformly continuous functions on \( \partial_p V \), with respect to the metric topology.

Theorem 8.4. Let \( f \in C_{\text{unif}}(\partial_p V) \) and \( k : \partial_p V \to \mathbb{R} \) be a function which vanishes q.e. Then

\[ \mathcal{S}f = \mathcal{Q}f = \overline{\mathcal{Q}}f = \overline{\mathcal{S}}f = \mathcal{S}(f + k) = \mathcal{Q}(f + k) = \overline{\mathcal{Q}}(f + k) \]

everywhere in \( V \), and this function is finely \( p \)-harmonic, i.e. a finely continuous fine minimizer.
A natural question is if some resolutivity can be shown for finely continuous functions on the boundary. However, even if \( V \subset \mathbb{R}^n \) is open, there are no such results available in the nonlinear theory. Another question is if it would be enough to require \( f \in C(\partial_p V) \). Due to the possible noncompactness of the fine boundary with respect to the metric topology, this is not equivalent to requiring that \( f \in C_{\text{unif}}(\partial_p V) \). Resolutivity for functions in \( C(\partial_p V) \) is unknown even for open \( V \), cf. the discussion in Björn [2].

**Proof of Theorem 8.4.** Since \( X \) is proper and \( V \) is bounded, \( \partial P V \) is totally bounded. Using that \( f \) is uniformly continuous, we can therefore for each \( j = 1, 2, \ldots \), find \( \delta_j > 0 \) and finitely many balls \( B_{i,j} := B(x_{i,j}, \delta_j) \) covering \( \partial_p V \), such that

\[
\text{osc}_{2B_{i,j} \cap \partial_p V} f < 1/j.
\]

Let

\[
\psi_{i,j}(x) = \max\{0, \min\{1, 2 - d(x, x_{i,j})/\delta_j\}\} \quad \text{and} \quad \eta_{i,j} = \frac{\psi_{i,j}}{\max\{1, \sum_i \psi_{i,j}\}}
\]

be a Lipschitz partition of unity on \( \bigcup_i B_{i,j} \) subordinate to \( 2B_{i,j} \). Then

\[
f_j := \sum_i f(x_{i,j})\eta_{i,j} \in \text{Lip}_c(X)
\]

and

\[
|f(x) - f_j(x)| \leq \sum_i |f(x) - f(x_{i,j})|\eta_{i,j}(x) < 1/j \quad \text{for} \ x \in \partial_p V.
\]

Hence, by Proposition 8.3, applied to \( f_j + 1/j \),

\[
\bar{Q}f = \bar{Q}(f_j + 1/j) = Q(f_j + 1/j) \leq 2/j + Qf \quad \text{in} \ V.
\]

Letting \( j \to \infty \) shows that \( \bar{Q}f = Qf \) in \( V \). As also the converse inequality holds, by Theorem 6.1, we see that \( \bar{Q}f = Qf \). The equalities \( Q(f + k) = Qf \) and \( \bar{Q}f = \bar{Q}(f + k) \) follow directly from the definition and the fact that \( k = 0 \) q.e. Finally, Theorem 6.1 and (6.1) include also \( \bar{S}f, \bar{F}f, \bar{S}(f + k) \) and \( \bar{S}(f + k) \) in the equalities.

**9. Perron solutions on open sets**

In addition to the standing assumptions from the beginning of Section 3, we assume in this section that \( V \) is a bounded open set with \( C_p(X \setminus V) > 0 \).

In this final section we will show that our four upper Perron solutions coincide if \( V \) is open. Note that for open sets \( V \), the definitions of the \( P \)-, \( Q \)- and \( S \)-Perron solutions in [3], [12] and Björn–Björn–Sjödin [15] are different from the ones considered here. One major difference is that our Perron solutions use fine limits on the fine boundary \( \partial_p V \) instead of ordinary limits on the full metric boundary \( \partial V \), see Remark 9.3 below.

**Proposition 9.1.** Assume that \( V \) is open and let \( f : \nabla^p \to \mathbb{R} \). Then

\[
\bar{S}f = \bar{P}f = \bar{Q}f = \bar{F}f \quad \text{everywhere in} \ V.
\]

Moreover, this upper Perron solution is in each component of \( V \) either identically \( \pm \infty \) or \( p \)-harmonic, i.e. a continuous minimizer.
Proof. We may assume that \( \tilde{\mathcal{U}}_f \neq \emptyset \), as otherwise \( \overline{\mathcal{S}} f = \overline{\mathcal{P}} f = \overline{\mathcal{Q}} f = \overline{\mathcal{R}} f = \infty \) everywhere in \( V \). Let \( u \in \tilde{\mathcal{U}}_f \). As \( V \) is open, \( u \) is a standard superminimizer (as e.g. in [3]), by Corollary 5.6 in [9]. It thus follows from Theorem 5.1 and Proposition 7.4 in Kinnunen–Martio [25] (or Theorem 8.22 and Proposition 9.4 in [3]) that \( u \) has a \( p \)-superharmonic representative, which is finely continuous by Björn [16, Theorem 4.4] or Korte [26, Theorem 4.3] (or [3, Theorem 11.38]), i.e. \( u^+ = u_* \). As this holds for all \( u \in \tilde{\mathcal{U}}_f \), and in particular for all \( u \in \mathcal{U}_f \), we get immediately from the definitions that

\[
\overline{\mathcal{S}} f = \overline{\mathcal{Q}} f \quad \text{and} \quad \overline{\mathcal{P}} f = \overline{\mathcal{R}} f \quad \text{everywhere in} \ V.
\]

Similarly, Lemma 7.3 implies that \( \overline{\mathcal{Q}} f \) is a pointwise infimum of continuous minimizers, and is thus upper semicontinuous in \( V \).

One can now proceed essentially verbatim as in Björn–Björn–Shanmugalingam [12, Theorem 4.1] (or [3, Theorem 10.10]) to show that the Perron solutions are \( p \)-harmonic. Alternatively one can argue as follows:

For \( \overline{\mathcal{R}} f \), Lemma 5.2 provides us with a decreasing sequence of \( p \)-superharmonic functions \( u_j \in \mathcal{U}_f \) such that \( u_j \to \overline{\mathcal{R}} f \) q.e. in \( V \). For each \( j \), let \( \tilde{u}_j \in \mathcal{U}_f \) be the Poisson modification of \( u_j \) in an open set \( \Omega \Subset V \), as in [3, Theorem 9.44]. Then \( \tilde{u}_j \leq u_j \) and \( \{ \tilde{u}_j \}_j \) is also a decreasing sequence. Let \( \tilde{u} = \lim_{j \to \infty} \tilde{u}_j \) in \( V \). It follows that \( \tilde{u} \geq \overline{\mathcal{R}} f \), \( \overline{\mathcal{Q}} f \) everywhere in \( V \). By Lemma 6.6, \( \tilde{u} = \overline{\mathcal{R}} f = \overline{\mathcal{Q}} f \) in \( \Omega \setminus E \) for some set \( E \) with \( C_p(E) = 0 \). Note that \( \Omega \setminus E \) is dense in \( V \).

Since each \( \tilde{u}_j \) is \( p \)-harmonic in \( \Omega \), [3, Corollary 9.38] implies that in every component of \( \Omega \), either \( \tilde{u} \) is \( p \)-harmonic or \( \tilde{u} \equiv -\infty \). We thus have in each such component, using the upper semicontinuity of \( \overline{\mathcal{Q}} f \) and the continuity of \( \tilde{u} \), that

\[
\tilde{u}(x) \geq \overline{\mathcal{R}} f(x) \geq \overline{\mathcal{Q}} f(x) = \limsup_{\Omega \setminus E \ni z \to x} \overline{\mathcal{Q}} f(z) = \limsup_{\Omega \setminus E \ni z \to x} \tilde{u}(z) = \tilde{u}(x), \quad x \in V.
\]

Thus \( \overline{\mathcal{R}} f = \overline{\mathcal{Q}} f = \tilde{u} \) is \( p \)-harmonic, or identically \( -\infty \), in each component of \( \Omega \) and as \( \Omega \Subset V \) was arbitrary, also in each component of \( V \). \( \square \)

Remark 9.2. Consider the upper Perron solutions \( \tilde{\mathcal{S}} f, \tilde{\mathcal{P}} f, \tilde{\mathcal{Q}} f \) and \( \tilde{\mathcal{R}} f \) introduced in Remark 6.7. The proof of Proposition 9.1 applies equally well to them, showing that \( \tilde{\mathcal{S}} f = \tilde{\mathcal{P}} f = \tilde{\mathcal{Q}} f = \tilde{\mathcal{R}} f \) everywhere in \( V \), and that this function is \( \overline{\mathcal{R}} \)-continuous. As shown in Remark 6.7, \( \tilde{\mathcal{Q}} f = \overline{\mathcal{Q}} f \) q.e., but since both functions are \( \overline{\mathcal{R}} \)-continuous they coincide everywhere. Hence, for open \( V \),

\[
\tilde{\mathcal{S}} f = \tilde{\mathcal{P}} f = \tilde{\mathcal{Q}} f = \tilde{\mathcal{R}} f = \overline{\mathcal{S}} f = \overline{\mathcal{P}} f = \overline{\mathcal{Q}} f = \overline{\mathcal{R}} f \quad \text{everywhere in} \ V.
\]

Remark 9.3. Upper Perron solutions on bounded open sets in metric spaces, defined using lower bounded \( p \)-superharmonic functions satisfying

\[
\liminf_{V \ni y \to x} u(y) \geq f(x)
\]

on the whole boundary \( \partial V \) rather than \( \partial_p V \), were first studied in Björn–Björn–Shanmugalingam [12]. Such solutions are traditionally denoted \( \overline{\mathcal{P}} f \), but to avoid confusion with our definitions, we use \( \tilde{\mathcal{P}} f \) for them. Similarly, in Björn–Björn–Sjödin [15], upper Sobolev–Perron solutions (denoted \( \tilde{\mathcal{S}} f \) therein) were defined by requiring in addition that the admissible \( p \)-superharmonic functions \( u \) have finite \( p \)-energy, which on bounded sets and for upper bounded boundary data is equivalent to \( u \in N^{1,p}(\partial_D V) \). We denote these solutions by \( \tilde{\mathcal{S}} f \). Examples 6.5 and 6.6 in [15] show that \( \tilde{\mathcal{P}} f \) and \( \tilde{\mathcal{S}} f \) are in general different and that there are \( \overline{\mathcal{P}} \)-resolutive boundary data which are not \( \tilde{\mathcal{S}} \)-resolutive. Corollary 7.2 in [15] shows that \( \tilde{\mathcal{S}} f \) can be equivalently defined by requiring (9.1) only for q.e. \( x \in \partial V \).
It follows quite easily from the definitions that for upper bounded $f$,
\[ \tilde{S}f \geq Rf. \tag{9.2} \]
Indeed, if $u$ is admissible for $\tilde{S}f$ (and $f \not\equiv -\infty$), then $\tilde{u} := \min\{u, \max_{\partial V} f\} \in N^{1,p}(V)$ is $p$-superharmonic and thus finely continuous in $V$ (see the proof of Proposition 9.1). Since also $\partial_p V \subset \partial V$ and fine lim inf $V \ni y \rightarrow x \tilde{u}(y) \geq \lim inf_{V \ni y \rightarrow x} \tilde{u}(y)$,

$\tilde{u}$ is admissible for $Rf$ according to Definition 5.1.

In combination with Theorem 6.1 and Proposition 9.1, inequality (9.2) together with its analogue for lower Perron solutions implies that if a bounded function $f$ is Sobolev-resolutive in the sense of [15], then $f$ is resolutive for any of our four Perron solutions, while our definition might give more resolutive functions.

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