REMARKS ON A SYSTEM OF QUASI-LINEAR WAVE EQUATIONS IN 3D SATISFYING THE WEAK NULL CONDITION

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Abstract. We give an alternative proof of the global existence result originally due to Hidano and Yokoyama for the Cauchy problem for a system of quasi-linear wave equations in three space dimensions satisfying the weak null condition. The feature of the new proof lies in that it never uses the Lorentz boost operator in the energy integral argument. The proof presented here has an advantage over the former one in that the assumption of compactness of the support of data can be eliminated and the amount of regularity of data can be lowered in a straightforward manner. A recent result of Zha for the scalar unknowns is also refined.

1. Introduction. We consider the Cauchy problem for a system of quasi-linear wave equations in three space dimensions satisfying the weak null condition given by Lindblad and Rodnianski [14]

\[
\begin{align*}
\Box u_1 &+ G_{11,\alpha\beta}^{11}(\partial_\alpha u_1)(\partial_\beta u_1) + G_{12,\alpha\beta}^{21}(\partial_\alpha u_2)(\partial_\beta u_1) + H_{11,\alpha\beta}^{11}(\partial_\alpha u_1)(\partial_\beta u_1) + H_{12,\alpha\beta}^{21}(\partial_\alpha u_2)(\partial_\beta u_2) = 0, \\
\Box u_2 &+ G_{21,\alpha\beta}^{12}(\partial_\alpha u_1)(\partial_\beta u_1) + G_{22,\alpha\beta}^{22}(\partial_\alpha u_2)(\partial_\beta u_2) + H_{21,\alpha\beta}^{12}(\partial_\alpha u_1)(\partial_\beta u_1) + H_{22,\alpha\beta}^{22}(\partial_\alpha u_2)(\partial_\beta u_2) = 0,
\end{align*}
\]

(1)

with data given at \( t = 0 \). Here, \( \Box := \partial_\tau^2 - \Delta \), \( \partial_\tau := \partial/\partial t \), \( \partial_i := \partial/\partial x_i \), \( i = 1, 2, 3 \). We always use the summation convention: when the same index is above and below, summation over this index is assumed from 0 to 3. Since our concern is in classical solutions, we may assume the symmetry condition without loss of generality: there hold \( G_{11,\alpha\beta}^{11} = G_{11,\beta\alpha}^{11} \), \( G_{21,\alpha\beta}^{21} = G_{21,\beta\alpha}^{21} \), and \( G_{22,\alpha\beta}^{12} = G_{22,\beta\alpha}^{12} \). Using the energy inequality obtained by the method of ghost weight due to Alinhac (see [1, 2]), Yokoyama and the first author have proved in [7]:

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Theorem 1.1. Suppose
\[ G^{11,\alpha\beta\gamma}_{1}X_{\alpha}X_{\beta}X_{\gamma} = G^{21,\alpha\beta\gamma}_{1}X_{\alpha}X_{\beta}X_{\gamma} = G^{22,\alpha\beta\gamma}_{2}X_{\alpha}X_{\beta}X_{\gamma} = 0, \tag{2} \]
\[ H^{1,\alpha\beta}_{1}X_{\alpha}X_{\beta} = H^{12,\alpha\beta}_{1}X_{\alpha}X_{\beta} = H^{22,\alpha\beta}_{2}X_{\alpha}X_{\beta} = 0 \tag{3} \]
for any \( X = (X_0, \ldots, X_3) \in \mathbb{R}^4 \) with \( X_0^2 = X_1^2 + X_2^2 + X_3^2 \). Let \( 0 < \eta < 1/6, \ 0 < \delta < 1/6 \) so that \( \eta + 2\delta < 1/2 \). Then, there exist constants \( C > 0, \ 0 < \varepsilon \) depending only on the coefficients of the system (1), \( \delta \), and \( \eta \) such that if compactly supported smooth data satisfy \( W_4(u_1(0)) + W_4(u_2(0)) < \varepsilon \), then the Cauchy problem for (1) admits a unique global smooth solution \((u_1(t, x), u_2(t, x))\) satisfying for all \( t > 0, \ T > 0 \)
\[ W_4(u_1(t)) + (1 + t)^{-\delta}W_4(u_2(t)) \]
\[ + \sum_{i=1}^{3} \sum_{|a| \leq 3} \left( \| (t - r)^{-\eta} T_i \Gamma^a u_1 \|_{L^2((0,\infty) \times \mathbb{R}^3)} \right) \]
\[ + \left. (1 + T)^{-\delta} \| (t - r)^{-\eta} T_i \Gamma^a u_2 \|_{L^2((0,T) \times \mathbb{R}^3)} \right) \]
\[ \leq C( W_4(u_1(0)) + W_4(u_2(0)) ). \]
Here \( T_i = \partial_i + (x_i/|x|) \partial_1 \).

Remark 1. In Section 3 of [7], thanks to compactness of the support of initial data together with the finite speed of propagation, the proof of Theorem 1.1 was able to employ the standard local existence theorem in solving locally (in time) the Cauchy problem with data given at \( t = 0 \) and in continuing the local solutions to a larger strip, though some partial differential operators with “weight” (see just below) were naturally used. We should remark that the constant \( \varepsilon \) in the above theorem is independent of the “radius” of the support of given data \((u_i(0), \partial_t u_i(0)) = (f_i, g_i) \) \((i = 1, 2)\), that is, \( R_* := \inf \{ r > 0 : \text{supp} \{ f_1, g_1, f_2, g_2 \} \subset \{ x \in \mathbb{R}^3 : |x| < r \} \} \).

Here we explain the notation used in the statement of Theorem 1.1. We set
\[ E_1(u(t)) := \frac{1}{2} \int_{\mathbb{R}^3} \left( (\partial_t u(t, x))^2 + |\nabla u(t, x)|^2 \right) dx, \tag{5} \]
\[ W_\kappa(u(t)) := \sum_{|a| \leq \kappa - 1} E_1^{1/2}(\Gamma^a u(t)), \quad \kappa = 2, 3, \ldots \tag{6} \]
By \( \Gamma \), we mean the set of the operators \( \partial_a \) \((\alpha = 0, \ldots, 3), \ \Omega_{ij} := x_i \partial_j - x_j \partial_i \) \((1 \leq i < j \leq 3), \ L_k := x_4 \partial_k + t \partial_k \) \((k = 1, 2, 3), \ S := t \partial_4 + x \cdot \nabla \). Also, for a multi-index \( a \), \( \Gamma^a \) stands for any product of the \(|a|\) these operators. We remark that \( \partial^k_a u_i(0, x) \) for \( i = 1, 2 \) and \( k = 2, 3, 4 \) can be calculated with the help of the equation (1), and thus the quantity \( W_4(u_1(0)) + W_4(u_2(0)) \) appearing in (1) is determined by the given initial data.

We note that the proof of Theorem 1.1 fully exploits the Lorentz invariance in the sense that it uses the operators \( \Omega_{ij} \) and \( L_k \), in addition to \( \partial_a \) and \( S \). When it comes to the Cauchy problem for a nonrelativistic system satisfying the weak null condition (see, e.g., (2.8) of [10]) or the initial-boundary value problems in a domain exterior to an obstacle (see, e.g., [19, 11, 16]), the use of \( L_k \) should be avoided. The purpose of this paper is to revisit the Cauchy problem for (1) and prove global existence without relying upon \( L_k \). Moreover, we also aim at eliminating compactness of the
support of data and lowering the amount of regularity of data. To state the main theorem precisely, we set the notation. As in [5], we define

\[ N_1(u(t)) := \sqrt{E_1(u(t))}, \quad N_2(u(t)) := \left( \sum_{|\alpha|+|\beta|+d \leq 1} E_1(\partial^\alpha_x \partial^\beta_t u(t)) \right)^{1/2}, \]

\[ N_4(u(t)) := \left( \sum_{|\alpha|+|\beta|+d \leq 3} E_1(\partial^\alpha_x \partial^\beta_t \Delta^d u(t)) \right)^{1/2}, \]

where, for \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \), \( \partial^a := \partial_1^{a_1} \partial_2^{a_2} \partial_3^{a_3} \), \( \Omega^b := \Omega_{12}^{b_1} \Omega_{13}^{b_2} \Omega_{23}^{b_3} \).

We also define for a pair of time-independent functions \((v(x), w(x))\)

\[ D(v, w) := \left( \int_{\mathbb{R}^3} |\nabla \partial^a_{\Omega^b} \Delta^d v(x)|^2 dx + \int_{\mathbb{R}^3} |\partial^a_{\Omega^b} \Delta^d w(x)|^2 dx \right)^{1/2}. \]

Here, we have set \( \Lambda := x \cdot \nabla \), which can be regarded as a time-independent analogue of \( S \). Since \( \partial_t \partial_u = \partial_t u + \Lambda \partial_u u \) at \( t = 0 \), there obviously exists a numerical constant \( C_D > 0 \) such that \( N_4(u(0)) \leq C_D D(u(0), \partial_t u(0)) \) for smooth functions \( u(t, x) \).

**Theorem 1.2.** Suppose \((2), (3)\) for any \( X = (X_0, \ldots, X_3) \in \mathbb{R}^4 \) with \( X_0^2 = X_1^2 + X_2^2 + X_3^2 \). Then, there exists \( \varepsilon \in (0, 1) \) such that if \( f_1, f_2 \in L^6(\mathbb{R}^3) \) and \( D(f_1, g_1) + D(f_2, g_2) < \varepsilon \), then the Cauchy problem for \((1)\) with data \((u_i, \partial_t u_i) = (f_i, g_i) \) \((i = 1, 2)\) given at \( t = 0 \) admits a unique global solution \( u(t, x) = (u_1(t, x), u_2(t, x)) \) satisfying

\[
\text{ess sup}_{t > 0} N(u(t)) + G_T(u) + L_T(u) \leq C \sum_{i=1}^{2} D(f_i, g_i)
\]

for all \( T > 0 \), with a constant \( C > 0 \) independent of \( T \).

For the definition of \( N(u), G_T(u), \) and \( L_T(u) \), see (56), (87), and (88), respectively. (We remark that the constant \( \delta \) appearing in (56), (87)–(88) is smaller than in Theorem 1.1.) Compared with \( W_4(u_1(0)) \) (see Theorem 1.1 above), the seminorm \( D(v, w) \) has a couple of advantages. Firstly, by the standard way we can easily find a sequence \( \{v_j, w_j\} \in C^\infty_c(\mathbb{R}^3) \times C^\infty_c(\mathbb{R}^3) \) such that \( D(v - v_j, w - w_j) \to 0 \) as \( j \to \infty \) when \( v \in L^6(\mathbb{R}^3) \) and \( \nabla \partial^a_{\Omega^b} \Delta^d v, \partial^a_{\Omega^b} \Delta^d w \in L^2(\mathbb{R}^3) \) for any \( |a| + |b| + d \leq 3 \) with \( d \leq 1 \). (We remark that the corresponding procedure becomes rather complicated when we employ \( W_4 \) (see (6)), as in [7], to measure the size of data.) We are naturally led to proving Theorem 1.2 first for compactly supported smooth data (because the proof of global existence becomes easier for such initial data), and then we use this helpful property to complete its proof by passing to the limit of a sequence of compactly supported (for any fixed time) smooth solutions. See Section 8. Secondly, thanks to the limitation of the number of \( \Lambda \) to 1 in the definition of \( D(v, w) \), we easily see that the size condition in Theorem 1.2 is satisfied whenever the initial data is radially symmetric about \( x = 0 \) and its norm with the low weight \( \langle x \rangle := \sqrt{1 + |x|^2} \)

\[
\sum_{i=1,2} \left( \sum_{1 \leq |\alpha| \leq 4} \| \langle x \rangle \partial^\alpha_x f_i \|_{L^2} + \sum_{|\alpha| \leq 3} \| \langle x \rangle \partial^\alpha_x g_i \|_{L^2} \right)
\]
is small enough. Note that, thanks to its low weight, we can allow such an oscillating
and slowly decaying data as \( q_1(x) = \langle x \rangle^{-d} \sin(\langle x \rangle) \) with \( d > 5/2 \). Naturally, it results
from the limitation of the number of \( S \) to 1 in the definition of \( N(u), \mathcal{G}_T(u) \), and \( L_T(u) \).

Note that by setting \( H_{11}^{11,\alpha \beta} = 0 \) for all \( \alpha, \beta \) and choosing the trivial data
\( u_2(0, x) = \partial_1 u_2(0, x) = 0 \) and thus considering the trivial solution \( u_2(t, x) \equiv 0 \), we
can go back to the wave equation for the scalar unknowns

\[
\Box u + G^{\alpha \beta \gamma}(\partial_\gamma u)(\partial_{\alpha \beta}^2 u) + H^{\alpha \beta}(\partial_\alpha u)(\partial_\beta u) = 0, \ t > 0, \ x \in \mathbb{R}^3
\]

and thus obtain:

**Theorem 1.3.** Suppose the symmetry condition \( G^{\alpha \beta \gamma} = G^{\beta \alpha \gamma} \). Also, suppose the
null condition: there holds

\[
G^{\alpha \beta \gamma} X_\alpha X_\beta X_\gamma = H^{\alpha \beta} X_\alpha X_\beta = 0
\]

for any \( X = (X_0, \ldots, X_3) \in \mathbb{R}^4 \) with \( X_0^2 = X_1^2 + X_2^2 + X_3^2 \). Let \( \delta, \eta \) and \( \mu \)
be sufficiently small positive constants. Then, there exists \( \varepsilon \in (0, 1) \) such that if
\( f \in L^2(\mathbb{R}^3) \) and \( D(f, g) \leq \varepsilon \), then the Cauchy problem (11) with initial data \((f, g)\)
given at \( t = 0 \) admits a unique global solution \( u(t, x) \) satisfying

\[
\text{ess sup}_{t > 0} N_4(u(t)) + \left( \int_0^\infty \sum_{i=1}^3 \sum_{\|d\| + \|d\| \leq 3 \atop d \leq 1} \| (t - r)^{-1/2} - \eta T_i \tilde{Z}^a S^d u(t) \|_{L^2(\mathbb{R}^3)}^2 dt \right)^{1/2}
\]

\[+
\left( \int_0^\infty \sum_{i=1}^3 \sum_{\|d\| + \|d\| \leq 2 \atop d \leq 1} \| (t - r)^{-1/2} - \eta T_i \partial_\alpha \tilde{Z}^a S^d u(t) \|_{L^2(\mathbb{R}^3)}^2 dt \right)^{1/2}
\]

\[+
\sup_{t > 0} t^{-\delta} \left( \int_0^t \sum_{\|d\| + \|d\| \leq 3 \atop d \leq 1} \| r^{-(3/2) + \mu} \tilde{Z}^a S^d u(\tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau \right)^{1/2}
\]

\[+
\| r^{-(1/2) + \mu} \partial_\alpha \tilde{Z}^a S^d u(\tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau \right)^{1/2}
\]

\[
\leq CD(f, g).
\]

See the beginning of the next section for the definition of \( \tilde{Z} \). This improves
Theorem 1.1 of the second author [25] which says global existence of solutions to
(11) in the absence of the semi-linear term \( H^{\alpha \beta}(\partial_\alpha u)(\partial_\beta u) \) for small data with
higher regularity than is assumed in Theorem 1.3. Since we no longer assume
compactness of the support of initial data, Theorem 1.3 is also an improvement of
global existence results of [2] and [7] for (11) (see [2, p. 94] and [7, Theorem 1.5]).

The operators \( L_k \) together with the other elements of \( \Gamma \) played an essential role
in the proof of Theorem 1.1. Namely, the use of all the elements of \( \Gamma \) was crucial
for the purpose of getting time decay estimates for local solutions with the help
of the inequality of Klainerman [12] and its \( H^1-L^3 \) version due to Ginibre and Velo [4].
Since we avoid the use of the operators \( L_k \), some good substitutes for
these inequalities are necessary. In fact, there already exist two major ways of
obtaining time decay estimates without relying upon \( L_k \). One is to use point-wise
decay estimates for homogeneous and inhomogeneous wave equations (see, e.g.,
[24]). The other is to use the Klainerman-Sideris inequality [13] in combination
with some Sobolev-type inequalities with weights such as \( \langle t - r \rangle, \langle r \rangle \langle t - r \rangle^{1/2} \) (see, e.g., [21]). As in [25], we proceed along the latter approach to compensate for the absence of \( L_k \) in the list of the available differential operators and intend to combine the ghost weight method of Alinhac with the Klainerman-Sideris method. Actually, such an attempt of combining these two methods has been already made in [25].

With the help of some observations in [5] and [7], we adjust the machinery thereby such an attempt of combining these two methods has been already made in [25].

The next lemma states that the null form is preserved under the differentiation.

Lemma 2.1. Suppose that \( \{G^{\alpha \beta \gamma}\} \) and \( \{H^{\alpha \beta}\} \) satisfy the null condition (see (2), (3), and (12) above). For any \( Z_i \ (i = 1, \ldots, 7) \), the equality

\[
Z_i G^{\alpha \beta \gamma} (\partial_{\alpha} v)(\partial_{\beta,\gamma}^2 w) = G^{\alpha \beta \gamma} (\partial_i Z_i v)(\partial_{\beta,\gamma}^2 w) + G^{\alpha \beta \gamma} (\partial_i v)(\partial_{\alpha,\gamma}^2 Z_i w) + \tilde{G}^{\alpha \beta \gamma}_i (\partial_i v)(\partial_{\alpha,\gamma}^2 w)
\]

holds with the new coefficients \( \{\tilde{G}^{\alpha \beta \gamma}_i\} \) also satisfying the null condition. Also, the equality

\[
Z_i H^{\alpha \beta} (\partial_{\alpha} v)(\partial_{\beta} w) = H^{\alpha \beta} (\partial_i Z_i v)(\partial_{\beta} w) + H^{\alpha \beta} (\partial_i v)(\partial_{\beta} Z_i w) + \tilde{H}^{\alpha \beta}_i (\partial_i v)(\partial_{\beta} w)
\]

holds with the new coefficients \( \{\tilde{H}^{\alpha \beta}_i\} \) also satisfying the null condition.

This paper is organized as follows. In the next section, we prove some basic inequalities. In Section 3, we consider the bound for the weighted \( L^2 \) norm of the second or higher-order derivatives of local solutions. Sections 4–5 and 6–7 are devoted to the energy estimate and the space-time \( L^2 \) estimate for local solutions, respectively. In Section 8, we complete the proof of Theorem 1.2 by the continuity argument.

2. Preliminaries. As mentioned in Section 1, we use \( \partial_1, \partial_2, \partial_3, \Omega_{12}, \Omega_{23}, \Omega_{13} \) and \( S \), and we denote these by \( Z_1, Z_2, \ldots, Z_7 \) in this order. The set \( \{Z_1, Z_2, \ldots, Z_7\} \) is denoted by \( Z \). Note that \( \partial_i \notin Z \). For a multi-index \( a = (a_1, \ldots, a_7) \), we set \( Z_a := Z_{a_1} \cdots Z_{a_7} \). We also set \( \bar{Z} := \{Z_1, Z_2, \ldots, Z_6\} = Z \setminus \{S\} \), and \( \bar{Z}_a := Z_{a_1} \cdots Z_{a_6} \) for \( a = (a_1, \ldots, a_6) \).

We need the commutation relations. Let \([·, ·]\) be the commutator: \([A, B] := AB - BA\). It is easy to verify that

\[
[Z_i, \Box] = 0 \quad \text{for} \quad i = 1, \ldots, 6, \quad [S, \Box] = -2\Box,
\]

\[
[Z_j, Z_k] = \sum_{i=1}^{\mu} C_{i}^{j,k} Z_i, \quad j, k = 1, \ldots, 7,
\]

\[
[Z_j, \partial_k] = \sum_{i=1}^{n} C_{i}^{j,k} \partial_i, \quad j = 1, \ldots, 7, \quad k = 1, 2, 3,
\]

\[
[Z_j, \partial_1] = 0, \quad j = 1, \ldots, 6, \quad [S, \partial_1] = -\partial_1.
\]

Here \( C_{i}^{j,k} \) denotes a constant depending on \( i, j, \) and \( k \).

The next lemma states that the null form is preserved under the differentiation.

Lemma 2.1. Suppose that \( \{G^{\alpha \beta \gamma}\} \) and \( \{H^{\alpha \beta}\} \) satisfy the null condition (see (2), (3), and (12) above). For any \( Z_i \ (i = 1, \ldots, 7) \), the equality

\[
Z_i G^{\alpha \beta \gamma} (\partial_{\alpha} v)(\partial_{\beta,\gamma}^2 w) = G^{\alpha \beta \gamma} (\partial_i Z_i v)(\partial_{\beta,\gamma}^2 w) + G^{\alpha \beta \gamma} (\partial_i v)(\partial_{\alpha,\gamma}^2 Z_i w) + \tilde{G}^{\alpha \beta \gamma}_i (\partial_i v)(\partial_{\alpha,\gamma}^2 w)
\]

holds with the new coefficients \( \{\tilde{G}^{\alpha \beta \gamma}_i\} \) also satisfying the null condition. Also, the equality

\[
Z_i H^{\alpha \beta} (\partial_{\alpha} v)(\partial_{\beta} w) = H^{\alpha \beta} (\partial_i Z_i v)(\partial_{\beta} w) + H^{\alpha \beta} (\partial_i v)(\partial_{\beta} Z_i w) + \tilde{H}^{\alpha \beta}_i (\partial_i v)(\partial_{\beta} w)
\]

holds with the new coefficients \( \{\tilde{H}^{\alpha \beta}_i\} \) also satisfying the null condition.
For the proof, see, e.g., [2, p. 91]. It is possible to show the following lemma essentially in the same way as in [2, pp. 90–91]. Together with it, we will later exploit the fact that for local solutions \( u \), the special derivatives \( T_iu \) have better space-time \( L^2 \) integrability and improved time decay property of their \( L^\infty(\mathbb{R}^3) \) norms.

**Lemma 2.2.** Set \( \omega_0 = -1, \omega_k = x_k/|x|, k = 1, 2, 3 \). Suppose that \( \{G^{\alpha\beta\gamma}\}, \{H^{\alpha\beta}\} \) satisfy the null condition. Then, we have for smooth functions \( w_i(t, x) \) (i = 1, 2, 3)

\[
|G^{\alpha\beta\gamma}(\partial_{\alpha}w_1)(\partial_{\alpha}^2w_2)| \leq C(|Tw_1||\partial^2w_2| + |\partial w_1||T\partial w_2|),
\]

(20)

\[
|G^{\alpha\beta\gamma}(\partial_{\alpha}^2w_1)(\partial_{\beta}w_2)| \leq C(|T\partial w_1||\partial w_2| + |\partial^2w_1||T\partial w_2|),
\]

(21)

\[
|G^{\alpha\beta\gamma}(\partial_{\alpha}w_1)(\partial_{\beta}w_2)(\partial_{\gamma}w_3)|, \ |G^{\alpha\beta\gamma}(\partial_{\gamma}w_1)(\partial_{\beta}w_2)(-\omega_\alpha)(\partial_{\gamma}w_3)| \leq C(|Tw_1||\partial w_2||\partial w_3| + |\partial w_1||T\partial w_2||\partial w_3| + |\partial w_1||\partial w_2||T\partial w_3|),
\]

(22)

\[
|H^{\alpha\beta}(\partial_{\alpha}v)(\partial_{\beta}w)| \leq C(|Tv||\partial w| + |\partial v||T\partial w|).
\]

(23)

Here, and in the following, we use the notation \( \partial v := (\partial_0v, \ldots, \partial_3v), \)

\[
|Tv| := \left( \sum_{k=1}^3 |T_kv|^2 \right)^{1/2}, \quad |T\partial v| := \left( \sum_{k=1}^3 \sum_{\gamma=0}^3 |T_k\partial_{\gamma}v|^2 \right)^{1/2}
\]

(24)

**Lemma 2.3** (Lemma 2.2 of [25]). The inequality

\[
|Tv(t, x)| \leq C(t)^{-1}|(\partial_xv(t, x)| + |\partial_t v(t, x)| + \sum_{|\beta|=1} |\Omega^\beta v(t, x)|
\]

\[
+ |Sv(t, x)| + \langle t-r \rangle |\partial_x v(t, x)|
\]

(25)

holds for smooth functions \( v(t, x) \).

The following lemma is concerned with Sobolev-type or trace-type inequalities. We use these inequalities in combination with the Klainerman-Sideris inequality (see (38) below). The auxiliary norms

\[
M_2(v(t)) = \sum_{\alpha\beta\gamma, i,j,k=1} ||(t-x)||^2_{L^2(\mathbb{R}^3)}
\]

(26)

\[
M_4(v(t)) = \sum_{|\alpha| \leq 2} M_2(\tilde{Z}^\alpha v(t)),
\]

(27)

which appear in the following discussion, play an intermediate role. We remark that \( S \) and \( \partial_r^2 \) are absent in the right-hand side above. We also use the notation \( \partial_r := (x/|x|) \cdot \nabla, \)

\[
\|w\|_{L^\infty L^2(\mathbb{R}^3)} := \sup_{r > 0} \|w(r)\|_{L^\infty(\mathbb{R}^2)}, \quad \|w\|_{L^2 L^\infty(\mathbb{R}^3)} := \left( \int_0^\infty \|w(r)\|^2_{L^2(\mathbb{R}^2)} r^2 dr \right)^{1/2}.
\]

(28)

(29)

**Lemma 2.4.** Suppose that \( v \) decays sufficiently fast as \( |x| \to \infty \). The following inequalities hold for \( \alpha = 0, 1, 2, 3 \)

\[
\langle t-r \rangle |\partial_\alpha v(t)| \leq C \left( N_1(v(t)) + M_2(v(t)) \right),
\]

(30)

\[
\langle t-r \rangle |\partial_\alpha v(t, x)| \leq C \left( \sum_{|\alpha| \leq 1} N_1(\partial^2_\alpha v(t)) + \sum_{|\alpha| \leq 1} M_2(\partial^2_\alpha v(t)) \right).
\]

(31)
Moreover, we have
\[ \| r \partial_x v(t) \|_{L^\infty L^2_x(\mathbb{R}^3)} \leq C \sum_{|a| \leq 1} N_1(\bar{Z}^a v(t)), \]  
(32)

\[ \langle r \rangle | \partial_x v(t, x) | \leq C \sum_{|a| \leq 2} N_1(\bar{Z}^a v(t)). \]  
(33)

These inequalities have been already employed in the literature. For the proof of (30), see (2.10) of [5]. For the proof of (31), see (37) of [25], (2.13) of [5]. See (3.19) of [20] for the proof of (32). Finally, combining (3.14b) of [20] with the Sobolev embedding \( H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \), we obtain (33).

We also need the following inequality.

**Lemma 2.5.** Suppose that \( v \) decays sufficiently fast as \( |x| \to \infty \). For any \( \theta \) with \( 0 \leq \theta \leq 1/2 \), there exists a constant \( C > 0 \) such that the inequality
\[ r^{(1/2) + \theta}(t - r)^{-\theta} \| \partial_x v(t, r) \|_{L^4(S^2)} \leq C \left( \sum_{|a| \leq 1} N_1(\Omega^a v(t)) + M_2(v(t)) \right) \]  
(34)
holds.

Following the proof of (3.19) in [20], we are able to obtain this inequality for \( \theta = 1/2 \). The next lemma with \( v = (t - r) \partial_\alpha w \) immediately yields (34) for \( \theta = 0 \). We follow the idea in Section 2 of [15] and obtain (34) for \( \theta \in (0, 1/2) \) by interpolation.

In our proof, the trace-type inequality also plays an important role. (For the proof, see, e.g., (3.16) of [20].)

**Lemma 2.6.** There exists a positive constant \( C \) such that if \( v = v(x) \) decays sufficiently fast as \( |x| \to \infty \), then the inequality
\[ r^{1/2} \| v(r \cdot) \|_{L^4(S^2)} \leq C \| \nabla v \|_{L^2(\mathbb{R}^3)} \]  
(35)
holds.

We also need the space-time \( L^2 \) estimates for the variable-coefficient operator \( P \) defined as
\[ P := \partial_t^2 - \Delta + h^{\alpha\beta}(t, x) \partial_{\alpha\beta}. \]  
(36)

Let \( h^{\alpha\beta} \in C^\infty((0, T) \times \mathbb{R}^3) \) \( (\alpha, \beta = 0, 1, 2, 3) \), and suppose the symmetry condition \( h^{\alpha\beta} = h^{\beta\alpha} \) and the size condition \( \sum |h^{\alpha\beta}(t, x)| \leq 1/2 \). We have the following:

**Lemma 2.7** (Theorem 2.1 of [6]). For \( 0 < \mu < 1/2 \), there exists a positive constant \( C \) such that the inequality
\[ \begin{align*}
&(1 + T)^{-\mu} \left( \| r^{-(3/2) + \mu} u \|_{L^2((0, T) \times \mathbb{R}^3)}^2 + \| r^{-(1/2) + \mu} \partial_t u \|_{L^2((0, T) \times \mathbb{R}^3)}^2 \right) \\
&\quad \leq C \| u(0, \cdot) \|_{L^2(\mathbb{R}^3)}^2 + C \int_0^T \int_{\mathbb{R}^3} \left( |\partial u| |Pu| + \frac{|u||Pu|}{r^{1-2\mu}(r^2)^{3\mu}} + |\partial h||\partial u| \right)^2 + \frac{|\partial h||u\partial u|}{r^{1-2\mu}(r^2)^{3\mu}} + \frac{|h||u\partial u|}{r^{1-2\mu}(r^2)^{3\mu}} + \frac{|h||u\partial u|}{r^{1-2\mu}(r^2)^{3\mu}} \right) dx dt
\end{align*} \]  
(37)
holds for smooth and compactly supported (for any fixed time) functions \( u(t, x) \).

See also [17] for an earlier and related estimate. The estimate (37) was proved by the geometric multiplier method of Rodnianski (see Appendix of [23]). At first sight, the above estimate may appear useless for the proof of global existence, because of the presence of the factor \( (1 + T)^{-2\mu} \). Combined with Lemma 2.5 and the useful
idea of dyadic decomposition of the time interval (see (124) below), the estimate (37) actually works effectively for the proof of global existence with no use of $L_j$ and with limitation of the occurrence of $S$ to 1 in the definition of $N_4(u(t))$.

The following was proved by Klainerman and Sideris, and will be used in the proof of Proposition 1 below. By setting $t = 0$ in (38), we get the simple inequality

$$M_2(v(0)) \leq C_{KS} N_2(v(0)) \quad \text{which, together with Proposition 1, will be used in the proof of Proposition 6 below.}$$

**Lemma 2.8 (Klainerman-Sideris inequality [13]).** There exists a constant $C_{KS} > 0$ such that the inequality

$$M_2(v(t)) \leq C_{KS} \{ N_2(v(t)) + t \| \Box v(t) \|_{L^2(\mathbb{R}^3)} \}$$

holds for smooth functions $v = v(t, x)$ decaying sufficiently fast as $|x| \to \infty$.

3. **Bound for $M_4(u(t))$.** Since the second order quasi-linear hyperbolic system (1) can be written in the form of the first order quasi-linear symmetric hyperbolic system (see, e.g., (5.9) of Racke [18]), the standard local existence theorem (see, e.g., Theorem 5.8 of [18]) applies to the Cauchy problem for (1). To begin with, we assume that the initial data are smooth, compactly supported, and small so that

$$\sum_{i=1}^{2} N_4(u_i(0), \partial_t u_i(0)) \leq C_D \sum_{i=1}^{2} D(f_i, g_i)$$

$$\leq \varepsilon_0 := \min \left\{ \frac{\min \{ \varepsilon_1^*, \varepsilon_2^* \}}{2AC_2(1 + 2A)} + \varepsilon_3^*, \frac{1 - \frac{4C_0}{\sigma^2}}{2AC_3(1 + 2A)} \right\}$$

may hold. See the inequality following (8) for the constant $C_D$. See (40), (48), and (59) for the constants $\varepsilon_1^*$, $\varepsilon_2^*$, and $\varepsilon_3^*$, respectively. See (169) for $A$, and see (165) for $C_0$ and $C_1$. See (172) and the inequality following it for $C_2$ and $C_3$. Note that $\varepsilon_0$ is independent of $R_\varepsilon$ (see Remark 1).

We know that a unique, smooth solution to (1) exists at least for a short time interval, and it is compactly supported at any fixed time by the finite speed of propagation.

Before entering into the energy estimate in the next section, we must refer to an elementary result concerning point-wise estimates for $u_1$ and $u_2$. It compensates for the absence of $\partial^2_t v(t, x)$ ($i = 2, 3, 4$) in the definition of the norms $N_4(v(t))$, $M_4(v(t))$, $G(v(t))$, and $L(v(t))$ (see (7), (27), (84)–(85)).

**Lemma 3.1.** There exists a constant $\varepsilon_1^* > 0$ depending on the coefficients of (1) with the following property: whenever smooth solutions $u = (u_1, u_2)$ to (1) satisfy

$$\max \{ |\partial_\alpha Z^b u_k(t, x)| : |b| \leq 1, 0 \leq \alpha \leq 3, k = 1, 2 \} \leq \varepsilon_1^*,$$

the following point-wise inequalities (i)–(iv) hold for $i = 1, 2$.

(i) The inequalities

$$|\partial^2_t u_i(t, x)| \leq C |\partial_\alpha u_i(t, x)| + C \sum_{k=1}^{2} |\partial u_k(t, x)|,$$

$$|\partial^3_t u_i(t, x)| \leq C \sum_{|a|=1}^{2} |\partial^2_\alpha u_i(t, x)| + C \sum_{k=1}^{2} |\partial u_k(t, x)|$$

(41)
hold.

(ii) There hold

\[ |\partial_t^2 \bar{Z}^a u_i(t,x)| \leq C \sum_{|b| \leq |a|} \left( |\partial_x \bar{Z}^b u_i(t,x)| + \sum_{k=1}^{2} |\partial \bar{Z}^b u_k(t,x)| \right), \quad |a| = 1, 2, \quad (43) \]

\[ |\partial_t^2 \bar{Z}^a u_i(t,x)| \leq C \sum_{|b| = 2} \sum_{|c| \leq 1} |\partial \bar{Z}^b u_i(t,x)| C \sum_{|b| \leq 1} \sum_{k=1}^{2} |\partial \bar{Z}^b u_k(t,x)|, \quad |a| = 1. \quad (44) \]

(iii) The inequality

\[ |\partial_t^2 S u_i(t,x)| \leq C \sum_{d \leq 1} \left( |\partial_x S^d u_i(t,x)| + \sum_{k=1}^{2} |\partial S^d u_k(t,x)| \right), \quad i = 1, 2. \quad (45) \]

holds.

(iv) The inequality

\[ |T_j \partial_t^2 u_i(t,x)| \leq C |T_j \partial_x \bar{Z}^a u_i(t,x)| + C \sum_{k=1}^{2} |T_j \partial u_k(t,x)| \quad (46) \]

holds for \( j = 1, 2, 3 \). Also, for \( |a| = 1 \)

\[ |T_j \partial_t^2 \bar{Z}^a u_i(t,x)| \leq C |T_j \partial_x \bar{Z}^a u_i(t,x)| + C \sum_{k=1}^{2} \sum_{|b| \leq 1} |T_j \partial \bar{Z}^b u_k(t,x)| + C \left( \sum_{k=1}^{2} |T_j \partial u_k(t,x)| \right) \left( \sum_{|b| = 1} \sum_{|c| \leq 1} |\partial \bar{Z}^b u_k(t,x)| \right). \quad (47) \]

We must not assume smallness of \( |\partial S u_2(t,x)| \) (see, e.g., (54)–(55) below, where we allow \( \|\partial S u_2(t)\|_{L^\infty(\mathbb{R}^2)} \) to grow with \( t \), and therefore we treat point-wise estimates for \( \partial_t^2 \bar{Z}^a u_i(t,x) \) \( (|a| = 1) \), \( \partial_t^2 S u_i(t,x) \), and \( T_j \partial_t^2 S u_i(t,x) \), separately.

**Lemma 3.2.** There exists a constant \( \varepsilon_2 > 0 \) depending on the coefficients of (1) with the following property: whenever smooth solutions \( u = (u_1, u_2) \) to (1) satisfy

\[ \max \{ |\partial_x \bar{Z}^b u_k(t,x)| : |b| \leq 1, 0 \leq \alpha \leq 3, k = 1, 2 \} \leq \varepsilon_2, \quad (48) \]

then the inequality

\[ |\partial_t^2 \bar{Z}^a S u_i(t,x)| \leq C \sum_{|b|, d \leq 1} \left( |\partial_x \bar{Z}^b S^d u_i(t,x)| + \sum_{k=1}^{2} |\partial \bar{Z}^b S^d u_k(t,x)| \right) \]

\[ + C \left( \sum_{k=1}^{2} |\partial S u_k(t,x)| \right) \left( \sum_{|b| = 1} \sum_{|c| \leq 1} |\partial \bar{Z}^b u_i(t,x)| \right) \quad (49) \]

holds for \( |a| = 1 \) and \( i = 1, 2 \). Also, we have

\[ |\partial_t^3 S u_i(t,x)| \leq C \sum_{|a| = 2} |\partial_x \partial_x^2 S u_i(t,x)| + C \sum_{k=1}^{2} \sum_{|b| + d \leq 2} \sum_{d \leq 1} |\partial \partial_x \partial_x S^d u_k(t,x)| \]

\[ + C \left( \sum_{k=1}^{2} |\partial S u_k(t,x)| \right) \left( \sum_{|a| = 1} |\partial_x \partial_x^2 u_i(t,x)| + \sum_{k=1}^{2} |\partial u_k(t,x)| \right), \quad (50) \]
\[ |T_j \partial^2 S u_i(t, x)| \leq C \sum_{d \leq 1} |T_j \partial_x \partial^d S u_i(t, x)| + C \sum_{k=1}^2 |T_j \partial S^d u_k(t, x)| \]
\[ + C \left( \sum_{k=1}^2 |\partial S u_k(t, x)| \right) \left( |T_j \partial_x u_i(t, x)| + \sum_{k=1}^2 |T_j \partial u_k(t, x)| \right) \]  
(51)
\[ + C \left( \sum_{k=1}^2 |T_j \partial u_k(t, x)| \right) \left( |\partial_x S u_i(t, x)| + \sum_{k=1}^2 |\partial S u_k(t, x)| \right). \]

For the proof of Lemmas 3.1 and 3.2, we have only to repeat essentially the same argument as in the proof of Lemma 3.2 of [5]. We thus omit the proof.

Using the above point-wise inequalities, let us next consider the bound for \( M_4(u_1(t)) \) and \( M_4(u_2(t)) \). Taking (14) into account, we have for \(|a| + d \leq 3\)
\[ \Box \bar{Z}^a S^d u_1 + G^{11, \alpha \beta \gamma}_1 (\partial_{\alpha} u_1)(\partial_{\beta} \bar{Z}^a S^d u_1) + G^{21, \alpha \beta \gamma}_1 (\partial_{\alpha} u_2)(\partial_{\beta} \bar{Z}^a S^d u_1) \]
\[ + \sum' \tilde{G}^{\alpha \beta \gamma}_1 (\partial_{\alpha} \bar{Z}^a S^d u_1)(\partial_{\beta} \bar{Z}^{a''} S^{d''} u_1) \]
\[ + \sum' \tilde{G}^{\alpha \beta \gamma}_1 (\partial_{\alpha} \bar{Z}^a S^d u_2)(\partial_{\beta} \bar{Z}^{a''} S^{d''} u_1) \]
\[ + \sum'' \tilde{H}^{\alpha \beta}_1 (\partial_{\alpha} \bar{Z}^a S^d u_1)(\partial_{\beta} \bar{Z}^{a''} S^{d''} u_1) \]
\[ + \sum'' \tilde{H}^{\alpha \beta}_1 (\partial_{\alpha} \bar{Z}^a S^d u_2)(\partial_{\beta} \bar{Z}^{a''} S^{d''} u_2) = 0. \]  
(52)

Here \( \sum' \) stands for the summation over \( \alpha', a'', d' \) and \( d'' \) satisfying \(|\alpha'| + |a''| + d' + d'' \leq |a| + d, \ |a''| + d'' < |a| + d \), and \( d' + d'' \leq d \). Similarly, \( \sum'' \) stands for the summation over \(|\alpha'| + |a''| + d' + d'' \leq |a| + d \) and \( d' + d'' \leq d \). Just for simplicity of notation, we have omitted dependence of the coefficients \( G^{\alpha \beta \gamma}_1 = \tilde{G}^{11, \alpha \beta \gamma}_1, \ldots, \tilde{H}^{\alpha \beta}_1 = \tilde{H}^{22, \alpha \beta}_1 \) on \( \alpha', a'', d' \) and \( d'' \). Similarly, we have for \(|a| + d \leq 3\)
\[ \Box \bar{Z}^a S^d u_2 + G^{12, \alpha \beta \gamma}_2 (\partial_{\alpha} u_1)(\partial_{\beta} \bar{Z}^a S^d u_2) + G^{22, \alpha \beta \gamma}_2 (\partial_{\alpha} u_2)(\partial_{\beta} \bar{Z}^a S^d u_2) \]
\[ + \sum' \tilde{G}^{\alpha \beta \gamma}_2 (\partial_{\alpha} \bar{Z}^a S^d u_1)(\partial_{\beta} \bar{Z}^{a''} S^{d''} u_2) \]
\[ + \sum' \tilde{G}^{\alpha \beta \gamma}_2 (\partial_{\alpha} \bar{Z}^a S^d u_2)(\partial_{\beta} \bar{Z}^{a''} S^{d''} u_2) \]
\[ + \sum'' \tilde{H}^{\alpha \beta}_2 (\partial_{\alpha} \bar{Z}^a S^d u_1)(\partial_{\beta} \bar{Z}^{a''} S^{d''} u_1) \]
\[ + \sum'' \tilde{H}^{\alpha \beta}_2 (\partial_{\alpha} \bar{Z}^a S^d u_2)(\partial_{\beta} \bar{Z}^{a''} S^{d''} u_2) = 0. \]  
(53)

Here, \( \tilde{G}^{\alpha \beta \gamma}_1 = \tilde{G}^{12, \alpha \beta \gamma}_2, \ldots, \tilde{H}^{\alpha \beta}_1 = \tilde{H}^{22, \alpha \beta}_2 \). In what follows, by \( \delta, \eta, \) and \( \mu \), we mean sufficiently small positive constants such that \( \delta < 1/9, \eta < 5/18, \) and \( \mu < 1/4 \). We use the following quantities for local solutions \( u = (u_1, u_2) \):
\[ \langle \langle u(t) \rangle \rangle := \langle \langle u_1(t) \rangle \rangle + \sum_{|b| \leq 1} \| \|x\| \partial \bar{Z}^b u_1(t)\| \|_{L^\infty(\mathbb{R}^3)} + \sum_{|b| \leq 2} \| \|x\| \partial \bar{Z}^b u_1(t)\| \|_{L^\infty L^4(\mathbb{R}^3)} \]
\[ + (1 + t)^{-\delta} \langle \langle u_2(t) \rangle \rangle, \]  
(54)
where for a scalar function \( w(t, x) \)
\[ \langle \langle w(t) \rangle \rangle := (1 + t) \sum_{|b| \leq 1} \| \partial \bar{Z}^b w(t)\| \|_{L^\infty(\mathbb{R}^3)} + \sum_{|b| \leq 1} \| \|x\| (t-r)^{1/2} \partial \bar{Z}^b w(t)\| \|_{L^\infty(\mathbb{R}^3)} \]
Proof. We prove Proposition 1. Corollary 1 is an immediate consequence of it, because 
\( C \langle \langle u(t) \rangle \rangle \mathcal{M}(u(t)) \) can be absorbed into the left-hand side of (58) for small 
\( \langle \langle u(t) \rangle \rangle \). We use (52), (53) with \( |a| \leq 2, d = 0 \). Obviously, it suffices to explain how 
to bound \( M_2(\tilde{Z}^a u_1(t)) \) for \( |a| = 2, i = 1, 2 \).

We first bound \( M_2(\tilde{Z}^a u_1(t)) \). In view of the Klainerman-Sideris inequality (38),
our task is to bound the \( L^2(\mathbb{R}^3) \) norm of the 2nd, 3rd, \ldots, and 8th terms on the 
left-hand side of (52) for \( |a| = 2, d = 0 \). In fact, it is enough to bound the 5th and 
8th terms for \( |a'| + |a''| = 2 \) because the others can be handled similarly. For any 
\[ \sum_{|b| \leq 1} \left\| x^{1/2} \partial Z^b w(t) \right\|_{L^\infty(\mathbb{R}^3)} + \sum_{|b| \leq 2} \left\| x^{1/2} \partial \tilde{Z}^b w(t) \right\|_{L^\infty(\mathbb{R}^3)} \]
\[ + \sum_{|b| \leq 3} \left\| x^{1/2} S \tilde{Z}^b w(t) \right\|_{L^\infty(\mathbb{R}^3)} + \sum_{|b| \leq 2} \left\| x^{1/2} S \tilde{Z}^b w(t) \right\|_{L^\infty(\mathbb{R}^3)} \]
\[ + \sum_{|b| \leq 1} \left\| x^{1/2} (t-r) \partial \tilde{Z}^b w(t) \right\|_{L^\infty(\mathbb{R}^3)} \]
\[ + \sum_{|b| \leq 2} \left\| x^{1/2} (t-r) \partial \tilde{Z}^b w(t) \right\|_{L^\infty(\mathbb{R}^3)} \]
\[ + \sum_{i=1}^2 \sum_{|b| \leq 1} \left\| x^{(1/2)+\theta_i} (t-r)^{1-\theta_i} \partial \tilde{Z}^b w(t) \right\|_{L^\infty(\mathbb{R}^3)} \]
\[ + \sum_{i=1}^2 \sum_{|b| \leq 2} \left\| x^{(1/2)+\theta_i} (t-r)^{1-\theta_i} \partial \tilde{Z}^b w(t) \right\|_{L^\infty(\mathbb{R}^3)} \]
\[ + \| \partial S w(t) \|_{L^\infty(\mathbb{R}^3)} + \| r \partial S w(t) \|_{L^\infty(\mathbb{R}^3)} + \sum_{|b| \leq 1} \| r \partial \tilde{Z}^b S w(t) \|_{L^\infty(\mathbb{R}^3)} \]
where \( \theta_1 := (1/2) - 2\mu, \theta_2 := (1/2) - \eta \),
\[ \mathcal{N}(u(t)) := N_4(u_1(t)) + (t)^{-\delta} N_4(u_2(t)), \]
\[ \mathcal{M}(u(t)) := M_4(u_1(t)) + (t)^{-\delta} M_4(u_2(t)). \]
fixed \( t \in (0, T) \), we bound their \( L^2 \) norm over the set \( \{ x \in \mathbb{R}^3 : |x| \leq (t + 1)/2 \} \) and its complement, separately. Let \( \chi_1(x) \) be the characteristic function of this set, and we set \( \chi_2(x) := 1 - \chi_1(x) \). Recall that we now have \( |a'| + |a''| \leq 2 \) at the 5th term on the left-hand side of (52). For \( |a'| \leq 1 \), we get by (43)

\[
\| \chi_1(\partial \bar{Z}^a u_2(t))(\partial^2 \bar{Z}^{a''} u_1(t)) \|_{L^2(\mathbb{R}^3)} \\
\leq C \sum_{|b| \leq |a''|} \langle t \rangle^{-2+\delta} \langle t \rangle^{-1-\delta} \| \partial \bar{Z}^{a''} u_2(t) \|_{L^\infty} \| (t-r) \partial_x \bar{Z}^b u_1(t) \|_{L^2} \\
+ C \sum_{|k| \leq |a''|} \langle t \rangle^{-(3/2)+2\delta} \langle t \rangle^{-\delta} \| \partial \bar{Z}^{a''} u_2(t) \|_{L^\infty} \\
\times \langle t \rangle^{-\delta} \| x \|^{-1} (t-r) \partial \bar{Z}^b u_k(t) \|_{L^2} \\
\leq C \langle t \rangle^{-(3/2)+2\delta} \langle \| u(t) \| \rangle (\mathcal{M}(u(t)) + \mathcal{N}(u(t))),
\]

where we have used the Hardy inequality at the last inequality. For \( |a'| = 2 \) (therefore \( |a''| = 0 \)), we get by (41) and the Hardy inequality

\[
\| \chi_1(\partial \bar{Z}^a u_2(t))(\partial^2 u_1(t)) \|_{L^2(\mathbb{R}^3)} \\
\leq C \langle t \rangle^{-(3/2)+\delta} \langle \| x \| \rangle^{-1} (t-r) \partial \bar{Z}^a u_2(t) \|_{L^2} \| \| x \|^{-1} (t-r) \partial \bar{Z}^a u_1(t) \|_{L^\infty} \\
+ C \langle t \rangle^{-(3/2)+2\delta} \langle \| x \| \rangle^{-1} (t-r) \partial \bar{Z}^a u_2(t) \|_{L^2} \\
\times \langle \| x \| \rangle^{-1} (t-r) \partial \bar{Z}^a u_1(t) \|_{L^\infty} \langle \| x \| \rangle^{-\delta} \| x \|^{-1} (t-r) \partial \bar{Z}^a u_2(t) \|_{L^\infty} \\
\leq C \langle t \rangle^{-(3/2)+2\delta} \langle \| u(t) \| \rangle (\mathcal{M}(u(t)) + \mathcal{N}(u(t))).
\]

For the 8th term on the left-hand side of (52), we get, assuming \( |a'| \leq |a''| \) (therefore, \( |a'|, |a''| \leq 1 \) without loss of generality

\[
\| \chi_1(\partial \bar{Z}^a u_2(t))(\partial^2 \bar{Z}^{a''} u_1(t)) \|_{L^2} \\
\leq C \langle \| x \| \rangle^{-(3/2)+2\delta} \langle \| x \| \rangle^{-\delta} \| x \|^{-1} (t-r) \partial \bar{Z}^a u_2(t) \|_{L^\infty} \langle \| x \| \rangle^{-1} (t-r) \partial \bar{Z}^a u_1(t) \|_{L^2} \\
\times \langle \| x \| \rangle^{-\delta} \| x \|^{-1} (t-r) \partial \bar{Z}^a u_2(t) \|_{L^2} \\
\leq C \langle \| x \| \rangle^{-(3/2)+2\delta} \langle \| u(t) \| \rangle (\langle \| x \| \rangle^{-\delta} \| x \|^{-1} (t-r) \partial \bar{Z}^a u_1(t) \|_{L^\infty} + \langle \| x \| \rangle^{-\delta} \| x \|^{-1} (t-r) \partial \bar{Z}^a u_2(t) \|_{L^\infty})
\]

in the same way as above.

Next, let us consider the estimate over the set \( \{ x \in \mathbb{R}^3 : |x| > (t + 1)/2 \} \) for any fixed \( t \in (0, T) \). Recall that \( \chi_2(x) = 1 - \chi_1(x) \). Since the coefficients \( \tilde{G}^{\alpha\beta\gamma} \) satisfy the null condition thanks to Lemma 2.1, we can first use Lemma 2.2 to get

\[
\| \chi_2(\tilde{G}^{a\beta\gamma}(\partial_x \bar{Z}^a u_2(t))(\partial^2 \bar{Z}^{a''} u_1(t))) \|_{L^2(\mathbb{R}^3)} \\
\leq C \langle \| \chi_2(T\bar{Z}^{a''} u_2(t))(\partial^2 \bar{Z}^{a''} u_1(t)) \|_{L^2} + \| \chi_2(\partial \bar{Z}^a u_2(t))(T \partial \bar{Z}^{a''} u_1(t)) \|_{L^2} \rangle
\]

and then we use Lemma 2.3 to get

\[
\| \chi_2(T\bar{Z}^{a''} u_2(t))(\partial^2 \bar{Z}^{a''} u_1(t)) \|_{L^2} \leq C \langle \langle t \rangle^{-1} \| \chi_2(\partial \bar{Z}^a u_2(t))(\partial^2 \bar{Z}^{a''} u_1(t)) \|_{L^2} \\
+ \sum_{|b|=1} |\chi_2(\bar{G}^{a\beta} \bar{Z}^a u_2(t))(\partial^2 \bar{Z}^{a''} u_1(t)) \|_{L^2} \\
+ \| \chi_2(S \bar{Z}^{a''} u_2(t))(\partial^2 \bar{Z}^{a''} u_1(t)) \|_{L^2} \\
+ \| \chi_2(t-r)(\partial_x \bar{Z}^a u_2(t))(\partial^2 \bar{Z}^{a''} u_1(t)) \|_{L^2} \rangle
\]

(65)
If we are assuming the second last inequality, because the other terms can be estimated in a similar way.

If \(|a'| + |a''| \leq 2\), \(|a''| \leq 1\). If \(|a'| \leq 1\), then we get

\[
\| \chi_2(S \partial_z a u_2(t)) (\partial^2 \tilde{Z}^{a''} u_1(t)) \|_{L^2} \leq C(t)^{-1/2} \| x \|_{L^\infty} \| \partial \tilde{Z}^b u_k(t) \|_{L^2} + \sum_{k=1,2} \| \partial \tilde{Z}^b u_k(t) \|_{L^2}
\]

or

\[
\| \chi_2(t - r) (\partial Z a u_2(t)) (\partial z \tilde{Z}^{a''} u_1(t)) \|_{L^2} \leq C(t)^{-1/2} \| x \|_{L^\infty} \| \partial Z a u_1(t) \|_{L^2} + \sum_{k=1,2} \| \partial Z a u_k(t) \|_{L^2}
\]

It suffices to show how to treat the 3rd and 4th terms on the right-hand side of the second last inequality, because the other terms can be estimated in a similar way.

Recall that we are assuming \(|a'| + |a''| \leq 2\), \(|a''| \leq 1\). If \(|a'| \leq 1\), then we get

\[
\| \chi_2(S \partial_z a u_2(t)) (\partial^2 \tilde{Z}^{a''} u_1(t)) \|_{L^2} \leq C(t)^{-1/2} \| x \|_{L^\infty} \| \partial \tilde{Z}^b u_k(t) \|_{L^2} + \sum_{k=1,2} \| \partial \tilde{Z}^b u_k(t) \|_{L^2}
\]

or

\[
\| \chi_2(t - r) (\partial Z a u_2(t)) (\partial z \tilde{Z}^{a''} u_1(t)) \|_{L^2} \leq C(t)^{-1/2} \| x \|_{L^\infty} \| \partial Z a u_1(t) \|_{L^2} + \sum_{k=1,2} \| \partial Z a u_k(t) \|_{L^2}
\]

We thus conclude that

\[
\| \chi_2 \tilde{H}^{a \beta \gamma} (\partial Z a u_2(t)) (\partial_{\partial z} \tilde{Z}^{a''} u_1(t)) \|_{L^2(R^3)} \leq C(t)^{-3/2} \| \chi_2 \| N(u(t)).
\]

Similarly, the coefficients \(\tilde{H}^{a \beta \gamma}\) satisfy the null condition and thus we can use (23) to get the inequality

\[
\| \chi_2 \tilde{H}^{a \beta \gamma} (\partial Z a u_2(t)) (\partial_{\partial z} \tilde{Z}^{a''} u_1(t)) \|_{L^2(R^3)} \leq C(t)^{-3/2} \| \chi_2 \| N(u(t)).
\]

for \(|a'| + |a''| \leq 2\) in the same way as above.

Let us turn our attention to the bound for \(M_2(\tilde{Z} a u_2(t))\), \(|a| \leq 2\). Naturally, we may focus on the 2nd, 4th, 6th, and 7th terms on the left-hand side of (53) whose
coefficients do not necessarily satisfy the null condition. We will show how to treat the 4th and 6th terms, because the 2nd and 7th terms can be handled similarly. For the 3rd, 5th, and 8th terms whose coefficients satisfy the null condition, we have only to proceed as we did in the treatment of \(M_2(\bar{Z}^n u_1(t))\), thus we may omit the details.

Let us resume with the estimate of the 4th and 6th terms. Recall that Lemma 2.2 has played no role in (61)–(63) and it has played an essential role in (64)–(72). Since we can no longer use Lemma 2.2, our task is to consider their bound over the set \(\{x \in \mathbb{R}^3 : |x| > (t + 1)/2\}\). If \(|a'| \leq 1\), then we get by (43)

\[
\|\chi_2(\partial Z^a u_1(t)) (\partial^2 Z^{a''} u_2(t))\|_{L^2} \\
\leq C(t)^{-1} \|x\| \|\partial Z^a u_1(t)\|_{L^\infty} \\
\times \sum_{|b| \leq |a''|} (\|\partial \bar{Z}^b u_2(t)\|_{L^2} + \sum_{k=1,2} \|\partial \bar{Z}^b u_k(t)\|_{L^2}) \\
\leq C(t)^{-1+\delta} \|u(t)\|_N(u(t)).
\]

(73)

If \(|a'| = 2\), then we get by (41)

\[
\|\chi_2(\partial Z^a u_1(t)) (\partial^2 u_2(t))\|_{L^2} \\
\leq C(t)^{-1} \|x\| \|\partial Z^a u_1(t)\|_{L^\infty} (\|\partial \bar{Z} u_2(t)\|_{L^2 L^\infty} + \sum_{k=1,2} \|\partial u_k(t)\|_{L^2 L^\infty}) \\
\leq C(t)^{-1+\delta} \|u(t)\|_N(u(t)).
\]

(74)

Similarly, we obtain

\[
\|\chi_2(\partial Z^a u_1(t)) (\partial^2 Z^{a''} u_2(t))\|_{L^2} \leq C(t)^{-1+\delta} \|u(t)\|_N(u(t)).
\]

(75)

Summing up, we have finished the proof of Proposition 1.

4. Energy estimate for \(u_1\). From now on, we focus on the energy estimate of the highest order \(|a| + d = 3\); the energy estimate of the lower order is easier. Following the argument in page 93 of [2], we obtain for the function \(g = g(t-r)\) chosen below (see (91))

\[
\frac{1}{2} \partial_t \left\{ e^\theta \left( (\partial_\gamma \bar{Z}^a S^d u_1)^2 + |\nabla \bar{Z}^a S^d u_1|^2 \right) \\
- G_1^{1,\alpha \beta \gamma}(\partial_\alpha u_1)(\partial_\beta \bar{Z}^a S^d u_1)(\partial_\gamma \bar{Z}^a S^d u_1) \\
+ 2G_1^{1,\alpha \beta \gamma}(\partial_\alpha u_1)(\partial_\beta \bar{Z}^a S^d u_1)(\partial_\gamma \bar{Z}^a S^d u_1) \right\} \\
+ \nabla \cdot \{ \cdots \} + e^\theta q + e^\theta (J_1,1 + J_{1,2} + \cdots + J_{1,5}) = 0,
\]

where, and later on as well, summation over the repeated index \(i\) is assumed from 1 to 2. Here, \(q = q_1 - (1/2)g'(t-r)q_2\),

\[
q_1 = \frac{1}{2} G_1^{1,\alpha \beta \gamma}(\partial_\gamma^2 u_1)(\partial_\beta \bar{Z}^a S^d u_1)(\partial_\alpha \bar{Z}^a S^d u_1) \\
- G_1^{1,\alpha \beta \gamma}(\partial_\gamma u_1)(\partial_\beta \bar{Z}^a S^d u_1)(\partial_\alpha \bar{Z}^a S^d u_1) \\
+ 2G_1^{1,\alpha \beta \gamma}(\partial_\gamma u_1)(\partial_\beta \bar{Z}^a S^d u_1)(\partial_\alpha \bar{Z}^a S^d u_1) \\
+ \sum_{j=1}^3 (T_j \bar{Z}^a S^d u_1)^2 - G_1^{1,\alpha \beta \gamma}(\partial_\alpha u_1)(\partial_\beta \bar{Z}^a S^d u_1)(\partial_\gamma \bar{Z}^a S^d u_1) \\
+ 2G_1^{1,\alpha \beta \gamma}(\partial_\alpha u_1)(\partial_\beta \bar{Z}^a S^d u_1)(-\omega_\alpha)(\partial_\gamma \bar{Z}^a S^d u_1)
\]

(77)

(78)
where, as explained in Lemma 2.2 above, $\omega_0 = -1$, $\omega_k = x_k/|x|$, $k = 1, 2, 3$. Also, (see (52) for $\sum$, $\sum''$)

\[
J_{1,1} = \sum' \tilde{G}^{\alpha\beta\gamma}(\partial_{\gamma}Z^a S^d u_1)(\partial_{\alpha}\tilde{Z}^a S^d u_1),
\]

\[
J_{1,2} = \sum' \tilde{G}^{\alpha\beta\gamma}(\partial_{\gamma}Z^a S^d u_2)(\partial_{\alpha}\tilde{Z}^a S^d u_1),
\]

\[
J_{1,3} = \sum'' \tilde{H}^{\alpha\beta}(\partial_{\alpha}Z^a S^d u_1)(\partial_{\beta}\tilde{Z}^a S^d u_1),
\]

\[
J_{1,4} = \sum'' \tilde{H}^{\alpha\beta}(\partial_{\alpha}Z^a S^d u_1)(\partial_{\beta}\tilde{Z}^a S^d u_2),
\]

\[
J_{1,5} = \sum'' \tilde{H}^{\alpha\beta}(\partial_{\alpha}Z^a S^d u_2)(\partial_{\beta}\tilde{Z}^a S^d u_1).
\]

In the following, we use the following $G(v(t))$ and $L(v(t))$ (recall $\eta < 5/18$, $\mu < 1/4$), which are related to the ghost energy and localized energy, respectively:

\[
G(v(t)) := \left\{ \sum_{j=1}^{3} \left( \sum_{|\alpha|+d \leq 3} \frac{\|\langle t-r \rangle^{-(1/2)-\eta}T_{\gamma} Z^a S^d v(t)\|_{L^2(\mathbb{R}^3)}^2}{d!} \right)^{1/2} + \sum_{d \leq 1} \frac{\|\langle t-r \rangle^{-(1/2)-\eta}T_{\gamma} \partial_t Z^a S^d v(t)\|_{L^2(\mathbb{R}^3)}^2}{d!} \right\},
\]

\[
L(v(t)) := \left\{ \sum_{d \leq 1} \left( \|\langle t-r \rangle^{-(1/2)+\mu} \partial Z^a S^d v(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\langle t-r \rangle^{-(1/2)+\mu} \partial \tilde{Z}^a S^d v(t)\|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \right\}.
\]

We remark that the norm $\|\langle t-r \rangle^{-(1/2)-\eta}T_{\gamma} \partial_t Z^a S^d v(t)\|_{L^2(\mathbb{R}^3)}$ ($|\alpha| + d \leq 2$, $d \leq 1$), which requires a separate and careful treatment, naturally comes up later. See, e.g., (113) below. Just for simplicity, we denote for local solutions $u = (u_1, u_2)$, $N(u_k(t)) := N_4(u_k(t))$ and $M(u_k(t)) := M_4(u_k(t))$. Also, we use the notation (recall $\delta < 1/3$)

\[
N_T(u) := \sup_{0 < t < T} N(u(t)),
\]

\[
G_T(u) := \left( \int_0^T G(u_1(t))^2 dt \right)^{1/2} + \sup_{0 < t < T} \langle t \rangle^{-\delta} \left( \int_0^t G(u_2(\tau))^2 d\tau \right)^{1/2},
\]

\[
L_T(u) := \sup_{0 < t < T} \langle t \rangle^{-\mu-\delta} \left( \int_0^t L(u_1(\tau))^2 d\tau \right)^{1/2} + \sup_{0 < t < T} \langle t \rangle^{-\mu-(3\delta/2)} \left( \int_0^t L(u_2(\tau))^2 d\tau \right)^{1/2}.
\]

The purpose of this section is to show the following:

**Proposition 2.** The following inequality holds for smooth local solutions to (1) $u = (u_1, u_2)$, as long as they satisfy (59) for some time interval $(0, T)$:

\[
\sup_{0 < t < T} N(u_1(t))^2 + \int_0^T G(u_1(t))^2 dt \leq CN(u_1(0))^2 + C \sup_{0 < t < T} \langle t \rangle \langle N_T(u)^2 + G_T(u)^2 + L_T(u)^2 \rangle + CN_T(u)^3 + C G_T(u) N_T(u)^2.
\]
The rest of this section is devoted to the proof of this proposition. We postpone the proof of the required estimate for \( \| (t-r)^{-\gamma} T_j \partial_t Z^\alpha S' u_1 \|_{L^1((0,T) \times \mathbb{R}^3)} \), \(|a| + d \leq 2, d \leq 1\) until the end of this section because it should be treated separately.

For any fixed time \( t \), estimates are carried out over the set \( \{ x \in \mathbb{R}^3 \ | \ |x| < (t + 1)/2 \} \) and its complement, separately. We thus use the functions \( \chi_1(x) \) and \( \chi_2(x) \) again.  

**Estimate over \( \{ x \in \mathbb{R}^3 \ | \ |x| < (t + 1)/2 \} \).** Recall that \( q = q_1 - (1/2) g'(t-r) q_2 \) for \( q_1 \) and \( q_2 \) defined in (77), (78).

**Estimate of \( \chi_1 q \).** Recall \( \theta_1 := (1/2) - 2\mu \) (see (55)). Using (41), we estimate \( \chi_1 q_1 \) as follows:

\[
\begin{align*}
&\| \chi_1 (\partial^2 u_i(t)) (\partial Z^a S^d u_1(t)) (\partial Z^a S^d u_1(t)) \|_{L^1(\mathbb{R}^3)} \\
\leq & C (t)^{-1+\theta_1} (t-r)^{-\theta_1} \| \partial Z^a S^d u_1(t) \|_{L^\infty} \| r^{-1/4} (t-r)^{-\theta_1} \partial Z^a S^d u_1(t) \|_{L^2}^2 \\
\leq & C (t)^{-1/2 - 2\mu} \left( \| r(t) \|_{L^\infty} + \sum_{k=1,2} \| r(t) \|_{L^\infty} \| \| (t-r)^{-\theta_1} \partial Z^a S^d u_1(t) \|_{L^\infty} \right) \\
& \times \| r(t) \|_{L^\infty} \| \| (t-r)^{-\theta_1} \partial Z^a S^d u_1(t) \|_{L^\infty} \|_{L^2} \\
\leq & C (t)^{-1/2 - 2\mu + \delta} \| u(t) \|_{L^2} \left( \| u(t) \|_{L^2} \right)^2, \quad i = 1, 2.
\end{align*}
\]

(90)

For the estimate of \( g'(t-r) q_2 \), we choose \( g = 1/2 \) (\( \rho \in \mathbb{R} \)) so that

\[
g'(\rho) = -\langle \rho \rangle^{-1 - 2\eta}.
\]

(91)

We then obtain

\[
\begin{align*}
&\| \chi_1 g'(t-r) (\partial u_i(t)) (\partial Z^a S^d u_1(t)) \|_{L^1(\mathbb{R}^3)} \\
\leq & C (t)^{-1 - 2\eta} \| \partial u_i(t) \|_{L^\infty} \| \partial Z^a S^d u_1(t) \|_{L^2}^2 \leq C (t)^{-2 - 2\eta + \delta} \| u(t) \|_{L^2}^2 \| u(t) \|_{L^2}^2.
\end{align*}
\]

(92)

We have finished the estimate of \( \chi_1 q \).

**Estimate of \( \chi_1 J_1, k \).** Next, let us consider the estimate of \( \chi_1 J_{1,k} \) \((k = 1, \ldots, 5)\). It suffices to deal with \( \chi_1 J_{1,2} \) and \( \chi_1 J_{1,5} \), because the others can be handled similarly.

For the estimate of \( \chi_1 J_{1,2} \), we must proceed carefully, paying attention on the number of occurrence of \( S \). Obviously, we may focus on the case \( d' + d'' = 1 \); in other cases, the estimate becomes much easier. Recall that we are considering the highest-order energy, i.e., \(|a| + d = 3\). We thus see \(|a| \leq 2 \) when \( d' + d'' = 1 \).

Case 1. \( d' = 1, d'' = 0 \).

Case 1-1. \(|a'| = 0, |a''| \leq 2\). We use (43) and the Sobolev embedding on \( S^2 \), to get

\[
\begin{align*}
&\| \chi_1 (\partial S u_2(t)) (\partial Z^a S' u_1(t)) (\partial Z^a S' u_1(t)) \|_{L^1(\mathbb{R}^3)} \\
\leq & C \sum_{|b| \leq |a''|} \left( \| \chi_1 (\partial S u_2(t)) (\partial Z^a S' u_1(t)) (\partial Z^a S' u_1(t)) \|_{L^1(\mathbb{R}^3)} \right) \\
& \quad + \sum_{k=1}^2 \| \chi_1 (\partial S u_2(t)) (\partial Z^a S' u_1(t)) (\partial Z^a S' u_1(t)) \|_{L^1(\mathbb{R}^3)} \\
\leq & C \sum_{|b| \leq |a''|} \left( \| (t-r)^{-1} \partial S u_2(t) \|_{L^\infty} \| (t-r) \partial Z^a S' u_1(t) \|_{L^2} \| (t-r)^{-1} \partial Z^a S' u_1(t) \|_{L^2} \right).
\end{align*}
\]
+ \sum_{k=1}^{2} \langle t \rangle^{-1+\theta_1} \| r^{-1/2} + \mu \partial Su_2(t) \|_{L^2_t L^\infty_x} \| r^{(1/2)+\theta_1} \langle t \rangle^{-1-\theta_1} \partial Z^b u_k(t) \|_{L^\infty_t L^2_x} \times \| r^{-(1/2)+\mu \partial_x Z^a u_1(t)} \|_{L^2_x} \right) \\
\leq C(\langle t \rangle^{-1+2\delta} \| u(t) \| N(\langle t \rangle \| L(u_1(t)) + C(\langle t \rangle^{-1/2-(2\mu+\delta)} \| u(t) \| L(u_1(t)) L(u_2(t)), \tag{93}

where we have used (60) at the last inequality.

Case 1-2. \(|a'|, |a''| \leq 1\). Using \(|\langle r \rangle \partial Z^a u_2(t)\|_{L^\infty_t L^4_x} \| (t-r)^{1-\theta_1} \partial Z^b u_1(t) \|_{L^2_t L^\infty_x} \) (\(|b| \leq |a''|\)), we get by suitably modifying the argument in Case 1-1 above

\| \chi_1(\partial Z^a u_2(t)(\partial^2 Z^{a''} u_1(t))(\partial \tilde{Z}^a S u_1(t))\|_{L^1(\mathbb{R}^3)} \\
\leq C(\langle t \rangle^{-1+2\delta} \| u(t) \| N(\langle t \rangle \| L(u_1(t)) + C(\langle t \rangle^{-1/2-(2\mu+\delta)} \| u(t) \| L(u_1(t)) L(u_2(t)). \tag{94}

Case 1-3. \(|a'| \leq 2, |a''| = 0\). We use \(|\chi_1 \langle r \rangle^{1/2+\theta_1} \langle t-r \rangle r^{-1-\theta_1} \partial^2 u_1(t) \|_{L^\infty(\mathbb{R}^3)} \) and obtain

\| \chi_1(\partial Z^a u_2(t)(\partial^2 u_1(t))(\partial \tilde{Z}^a S u_1(t))\|_{L^1(\mathbb{R}^3)} \\
\leq C(\langle t \rangle^{-1/2-(2\mu+\delta)} \| u(t) \| L(u_1(t)) L(u_2(t)). \tag{95}

Case 2. \(d' = 0, d'' = 1\). Recall that we are discussing the case \(|a| + d = 3\). Since we always have \(|a''| + d'' < |a| + d\), we know \(|a''| \leq 1\) under the condition \(d'' = 1\).

Case 2-1. \(|a'|, |a''| \leq 1\). We employ (49) to deal with \(\partial_2^2 \tilde{Z} S u_1(t, x)\). We get

\| \chi_1(\partial \tilde{Z} u_2(t))(\partial^2 \tilde{Z}^{a''} u_1(t))(\partial \tilde{Z}^a S u_1(t))\|_{L^1(\mathbb{R}^3)} \\
\leq C(\langle t \rangle^{-1/2-(2\mu+\delta)} \| u(t) \| L(u_1(t)) \times \| r^{-(1/2)+\mu \partial \tilde{Z}^{a''} S u_1(t)} \|_{L^2_x} \| r^{-(1/2)+(2\mu+\delta)} \| \partial \tilde{Z}^a S u_1(t) \|_{L^2_x} \left( \sum_{k=1}^{2} (\| r^{-(1/2)+\mu \partial \tilde{Z}^b S u_1(t)} \|_{L^2_x} + \| r^{-(1/2)+\mu \partial \tilde{Z}^b S u_1(t)} \|_{L^2_x}) \right) \\
\leq C(\langle t \rangle^{-1/2-(2\mu+\delta)} \| u(t) \| (L(u_1(t)) + L(u_2(t))) L(u_1(t)). \tag{96}

At the last inequality, we have used \(\| \partial S u_2(t) \|_{L^\infty} \leq \langle t \rangle^\delta \| u(t) \| \). We have also used \(\| u(t) \|^2 \leq \| u(t) \| \) because we are assuming smallness of \(\| u(t) \| \) (see (59)).

Case 2-2. \(|a'| \leq 2, |a''| = 0\). Using \(|\chi_1 \langle r \rangle^{1/2+\theta_1} \langle t-r \rangle \partial Z^a u_2(t) \|_{L^\infty_t L^4_x} \) (\(|a'| + |a''| \leq 2\).
We use $$\|\chi_1 r^{(1/2)+\theta_1} (t-r)^{-1-\theta_1} \partial \bar{Z}^a \psi u_2(t)\|_{L^\infty(\mathbb{R}^2)}$$ for $$|a'| \leq 1$$ and $$\|\chi_1 r^{(1/2)+\theta_1} (t-r)^{-1-\theta_1} \partial \bar{Z}^a \psi u_2(t)\|_{L^p L_t^q(\mathbb{R}^3)}$$ for $$|a'| = 2$$ to get

$$\|\chi_1 (\partial \bar{Z}^a u_2(t)) (\partial \bar{Z}^a S \psi u_1(t)) (\partial \bar{Z}^a S \psi u_1(t))\|_{L^1(\mathbb{R}^3)}$$

$$\leq C(t)^{-1-2\eta-\theta_1} \|u(t)\| L(u_1(t)) L(u_2(t)).$$ (98)

We have finished the estimate for $$\chi_1 J_{1,k}, k = 1, \ldots, 5$$. 

**Estimate over \{x \in \mathbb{R}^3 \mid |x| > (t + 1/2)\}.**

**Estimate of \(\chi_2 q\).** By virtue of the null condition, we can use (22) together with (25), (41) and obtain similarly to (64)–(65)

$$\|\chi_2 G_1^{3,1,\alpha\beta\gamma} (\partial \bar{Z}^a S \psi u_1(t)) (\partial \bar{Z}^a S \psi u_1(t))\|_{L^1(\mathbb{R}^3)}$$

$$\leq C(t)^{-1-2\eta} \sum_{k=1}^3 \|\chi_2 \partial_\bar{Z}^a S \psi u_k(t)\|_{L^\infty} + \sum_{d \leq 1} \|\chi_2 T \partial S \psi u_d(t)\|_{L^\infty}$$

$$+ \sum_{d \leq 1} \|\chi_2 (t-r) \partial \bar{Z}^a S \psi u_d(t)\|_{L^\infty}$$

$$\leq C(t)^{-1-2\eta} \|u(t)\| N(u_1(t)) + C(t)^{-1-2\eta} \|u(t)\| T \bar{Z}^a S \psi u_1(t)$$

$$\leq C(t)^{-1-2\eta} \|u(t)\| N(u_1(t)).$$ (99)

Here we have employed the norm $$\|x|^{(1/2)+\theta_2} (t-r)^{-1-\theta_2} \partial \bar{Z}^a \psi u(t)\|_{L^\infty}, \theta_2 = (1/2) - \eta.$$ Naturally, by using (21) instead of (22), we have a similar estimate for the second term on the right-hand side of (77).

As for the treatment of $$-(1/2) q' (t-r) q_2$$, which is

$$\frac{1}{2} (t-r)^{-1-2\eta} \sum_{j=1}^3 (T \bar{Z}^a S \psi u_1(t))^2$$

$$- (t-r)^{-1-2\eta} G_1^{3,1,\alpha\beta\gamma} (\partial \bar{Z}^a S \psi u_1(t)) (\partial \bar{Z}^a S \psi u_1(t))$$

$$+ 2(t-r)^{-1-2\eta} G_1^{3,1,\alpha\beta\gamma} (\partial \bar{Z}^a S \psi u_1(t)) (\partial \bar{Z}^a S \psi u_1(t))$$

(see (78)) we proceed as above, in order to treat the second and third terms on the right-hand side above. Namely, we first employ (22). We then use (25), the simple inequality $$|t-r|^{-1-2\eta} (t-r) \leq 1$$ to get

$$\leq C(t)^{-1-2\eta} \|T \bar{Z}^a S \psi u_1(t)\|$$

$$\leq C(t)^{-1-2\eta} \|u(t)\| N(u_1(t))$$

$$\leq C(t)^{-1-2\eta} \|u(t)\| N(u_1(t)).$$ (100)

Using this inequality, we easily obtain

$$\|\chi_2 (t-r)^{-1-2\eta} G_1^{3,1,\alpha\beta\gamma} (\partial \bar{Z}^a S \psi u_1(t)) (\partial \bar{Z}^a S \psi u_1(t))\|_{L^1(\mathbb{R}^3)}$$

$$\leq C(t)^{-1-2\eta} \|u(t)\| N(u_1(t))$$

$$+ C(t)^{-1-2\eta} \|u(t)\| T \bar{Z}^a S \psi u_1(t)$$

Essentially the same estimate as above remains true for the third term on the right-hand side of (78).
Estimate of $\chi_2J_{1,k}$. For the estimate of $\chi_2J_{1,k}$ ($k = 1, \ldots, 5$) we must proceed carefully. As we did for $\chi_1J_{1,2}$ and $\chi_2J_{1,5}$, we first obtain for $\chi_2J_{1,2}$

$$
\|\chi_2\tilde{G}^{\alpha\beta}(\partial_x \tilde{Z}^{\alpha'} S^{\alpha''} u_2(t)) (\partial_\tau \tilde{Z}^{\alpha''} S^{\alpha'''} u_1(t)) (\partial_\xi \tilde{Z}^{\alpha} S^{\alpha'} u_1(t))\|_{L^1(\mathbb{R}^3)} 
\leq C\left(\|\chi_2(T \tilde{Z}^{\alpha'} S^{\alpha''} u_2(t)) (\partial_\tau \tilde{Z}^{\alpha''} S^{\alpha'''} u_1(t))\|_{L^2} + \|\chi_2(T \partial_x \tilde{Z}^{\alpha''} S^{\alpha'''} u_1(t))\|_{L^2}\right)\|\partial_\xi \tilde{Z}^{\alpha} S^{\alpha'} u_1(t)\|_{L^2} + C(K_1 + K_2)\|\partial_\xi \tilde{Z}^{\alpha} S^{\alpha'} u_1(t)\|_{L^2}.
$$

Recall that we are considering the highest-order energy $|a| + d = 3$, which means that we are discussing the case $|a| + |a''| + d' + d'' \leq 3, |a''| + d'' \leq 2$, and $d' + d'' \leq d$. Again, we have only to deal with the case $d' + d'' = d = 1$; in other cases, the argument becomes much easier.

Case 1, $d' = 1$ and $d'' = 0$.

Case 1-1. $|a'| = 0$ and $|a''| \leq 2$. Recall $\theta_2 = (1/2) - \eta$. We get by (43)

$$
K_1 = \|\chi_2(TSu_2(t))(\partial^2 \tilde{Z}^{\alpha''} u_1(t))\|_{L^2} 
\leq C(t)^{-1}\|\{r(t - r)^{-1}TSu_2(t)\|_{L^\infty}(t - r)\partial_\xi \tilde{Z}^{\alpha''} u_1(t)\|_{L^2} + C(t)^{-1+\eta}\|\{r(t - r)^{-1/2-\eta}TSu_2(t)\|_{L^2}\|L^2_{L^2}\|L^2_{L^2}).
$$

Here, to handle $\|r(t - r)^{-1}TSu_2(t)\|_{L^\infty}$, we have used the following (see, e.g., (27), (28) in [25])

$$
[\Omega_{ij}, T_k] = \delta_{ij}T_k - \delta_{jk}T_i, \quad \partial_\xi T_i = \sum_{k=1}^3 \frac{x_k}{r} T_i \partial_k
$$

and the Sobolev-type inequality (see, e.g., (3.19) and (3.14b) in [20])

$$
r\|v(r)\|_{L^2} \leq C\|\partial_x v\|_{L^2(\mathbb{R}^3)}^{1/2} \left(\sum_{|b| \leq 1} \|\Omega^b v\|_{L^2(\mathbb{R}^3)}^{1/2}\right)
$$

(105) together with the Sobolev embedding on $S^2$. We also get by (25), (43), and (60)

$$
K_2 = \|\chi_2(T\partial_x \tilde{Z}^{\alpha''} u_1(t))\|_{L^2} 
\leq C(t)^{-1}\|\{r\partial_x Su_2(t)\|_{L^\infty}(T\partial_x \tilde{Z}^{\alpha''} u_1(t))\|_{L^2} \leq C(t)^{-2+2\delta}\|\langle u(t)\rangle\|_{N(u(t))}.
$$

Case 1-2. $|a'| \leq 1$ and $|a''| \leq 1$. Employing $\|r(t - r)^{-1}TSu_2(t)\|_{L^\infty L^2}$ and $\|\{t - r\}\partial_\xi \tilde{Z}^{\alpha''} u_1(t)\|_{L^2 L^2}$, we can similarly modify the argument in Case 1-1 above to get

$$
K_1 = \|\chi_2(T\tilde{Z}^{\alpha'} Su_2(t))(\partial^2 \tilde{Z}^{\alpha''} u_1(t))\|_{L^2} 
\leq C(t)^{-1}G(u_2(t))M(u_1(t)) + \langle t \rangle^{1-\eta+\delta}G(u_2(t))\|\langle u(t)\rangle\|_{N(u(t))}.
$$

Similarly, we obtain

$$
K_2 \leq C(t)^{-1}\|\{r\\partial_x \tilde{Z}^{\alpha'} Su_2(t)\|_{L^\infty L^2} (T\partial_x \tilde{Z}^{\alpha''} u_1(t))\|_{L^2 L^2} \leq C(t)^{-2+2\delta}\|\langle u(t)\rangle\|_{N(u(t))}.
$$
Case 1-3. $|a'| \leq 2$ and $|a''| = 0$. We easily get by using (41)

$$K_1 \leq C(t)^{-1+\eta}(t-r)^{-(1/2)-\eta T}Z^aS_u(t)\|_{L^2}\|r^{1-\eta}(t-r)^{(1/2)+\eta}\partial^2 u_1(t)\|_{L^\infty}$$

$$\leq C(t)^{-1+\eta+\delta}|G(u_2(t))\|_{u(t)}.$$  \hspace{1cm} (110)

We also get by (25)

$$K_2 \leq \|\partial^2 \bar{Z}^a S_u(t)\|_{L^2}\|\chi_2 T\partial u_1(t)\|_{L^\infty} \leq C(t)^{-(3/2)+2\delta} N(u(t))\|_{u(t)}.$$  \hspace{1cm} (111)

Case 2. $d' = 0$ and $d'' = 1$. We note $|a''| \leq 1$ in this case.

Case 2-1. $|a'|, |a''| \leq 1$. Using (25), (45), (49), we get

$$K_1 = \|\chi_2(T \bar{Z}^a u_2(t))(\partial^2 \bar{Z}^a S_u(t))\|_{L^2}$$

$$\leq C(t)^{-1/2}\|r^{1/2}T \bar{Z}^a u_2(t)\|_{L^\infty}\|\chi_2 \partial^2 \bar{Z}^a S_u(t)\|_{L^2}$$

$$\leq C(t)^{-1+\eta+\delta}|G(u_2(t))G(u_1(t)).$$  \hspace{1cm} (112)

where we have used (49) along with $\|\chi_2 \partial S_u(t)\|_{L^\infty} \leq C(t)^{-1}\|\|r\| \partial S_u(t)\|_{L^\infty} \leq C(t)^{-1+\delta}|G(u_2(t))G(u_2(t))$. We also get

$$K_2 = \|\chi_2(\partial \bar{Z}^a u_2(t))(T \partial \bar{Z}^a S_u(t))\|_{L^2}$$

$$\leq C(t)^{-1+\eta}\|r^{1-\eta}(t-r)^{(1/2)+\eta}\partial \bar{Z}^a u_2(t)\|_{L^\infty}$$

$$\times \|\|t-r\|^{-(1/2)-\eta}T \partial S_u(t)\|_{L^2}L^2$$

$$\leq C(t)^{-1+\eta+\delta}|G(u_2(t))G(u_1(t)).$$  \hspace{1cm} (113)

We note that this is one of the places where we encounter a little troublesome norm $\|u(t)|_{L^2}(|a| + d \leq 2).

Case 2-2. $|a'| \leq 2$ and $|a''| = 0$. We naturally modify the argument in Case 2-1 and obtain

$$K_1 \leq C(t)^{-1/2}\|r^{1/2}T \bar{Z}^a u_2(t)\|_{L^\infty}\|\chi_2 \partial^2 S_u(t)\|_{L^2L^2}$$

$$\leq C(t)^{-1+\eta+\delta}|G(u_2(t))G(u_1(t)).$$  \hspace{1cm} (114)

and

$$K_2 \leq C(t)^{-1+\eta}\|r^{1-\eta}(t-r)^{(1/2)+\eta}\partial \bar{Z}^a u_2(t)\|_{L^\infty}\|\|t-r\|^{-(1/2)+\eta}T \partial S_u(t)\|_{L^2L^2}$$

$$\leq C(t)^{-1+\eta+\delta}|G(u_2(t))G(u_1(t)).$$  \hspace{1cm} (115)

We have obtained the estimate of $\chi_2 J_{1,2}$. As for the semi-linear terms $\chi_2 J_{1,5}$, we first note that due to (23), the inequality

$$\|\chi_2 H^{\alpha\beta}(\partial \bar{Z}^a S^d u_2(t))(\partial_\beta \bar{Z}^a S^d u_2(t))\|_{L^2(\mathbb{R}^2)}$$

$$\leq C\|\chi_2(T \bar{Z}^d S^d u_2(t))(\partial \bar{Z}^a S^d u_2(t))\|_{L^2}$$

$$+ C\|\chi_2(\partial \bar{Z}^a S^d u_2(t))(T \bar{Z}^a S^d u_2(t))\|_{L^2}$$

holds. Due to symmetry, we may suppose $d' = 1, d'' = 0$. When $|a'| = 0$ and $|a''| \leq 2$ or $|a'| \leq 1$ and $|a''| \leq 1$, we get as in (115)

$$\|\chi_2(T \bar{Z}^a S_u(t))(\partial \bar{Z}^a u_2(t))\|_{L^2} \leq C(t)^{-1+\eta}\|\|t-r\|^{-(1/2)+\eta}T \bar{Z}^a S_u(t)\|_{L^2 L^2}$$

$$\leq C(t)^{-1+\eta+\delta}|G(u_2(t))G(u_2(t)).$$  \hspace{1cm} (116)
Also, by (35), (25), (60) and the commutation relation
\[ [\partial_j, T_k] = \frac{1}{r} \left( \delta_{jk} - \frac{x_j x_k}{r^2} \right) \partial_t, \] (118)
we get
\[
\| \chi_2(\partial \tilde{Z}^\alpha Su_2(t))(T \tilde{Z}^\beta u_2(t)) \|_{L^2(\mathbb{R}^3)} \\
\leq C(t)^{-1/2}\| \partial \tilde{Z}^\alpha Su_2(t) \|_{L^2(\mathbb{R}^3)} + \| r^{1/2} \tilde{\chi}_2 T \tilde{Z}^\alpha u_2(t) \|_{L^\infty} \\
\leq C(t)^{-1/2}\| N(u(t)) \|_{L^\infty} + C(t)^{-1/2}N(u(t))^2. 
\] (119)

Here, we have used \( \tilde{\chi}_2 \) which is defined as \( \tilde{\chi}_2(x) := \chi((2/(t+1))x) \) for a smooth, radially symmetric function \( \chi(x) \) such that \( \chi(x) = 0 \) for \( |x| \leq 1/2, \chi(x) = 1 \) for \( |x| \geq 1. \)

When \( |a'| \leq 2 \) and \( |a''| = 0 \), we obtain
\[
\| \chi_2(T \tilde{Z}^\alpha Su_2(t))(\partial u_2(t)) \|_{L^2(\mathbb{R}^3)} \leq C(t)^{-1+\eta}G(u_2(t)) \| r^{1-\eta}(t-r)^{(1/2)+\eta} \partial u_2(t) \|_{L^\infty} \\
\leq C(t)^{-1+\eta+\delta}G(u_2(t))\| u(t) \|. 
\] (120)

We also get in the same way as in (119)
\[
\| \chi_2(\partial \tilde{Z}^\alpha Su_2(t))(T u_2(t)) \|_{L^2(\mathbb{R}^3)} \\
\leq C(t)^{-1/2}N(u_2(t)) + C(t)^{-1+\eta+\delta}N(u(t))^2. 
\] (121)

We are ready to complete the proof of Proposition 2. First, we note that owing to (91), the function \( g(t-r) \) is bounded, which means that the function \( e^{g(t-r)} \) appearing in (76) satisfies \( c \leq e^{g(t-r)} \leq C \) for some positive constants \( c \) and \( C \).

Second, we must mention how to deal with rather troublesome terms
\[
\int_0^t \langle \tau \rangle^{-(1/2)-2\mu+2\delta} L(u_1(\tau)) L(u_2(\tau)) d\tau, 
\] (122)
\[
\int_0^t \langle \tau \rangle^{-1+\eta+\delta} G(u_2(\tau)) d\tau, 
\] (123)
which naturally come from the integration of such terms as in (96) and (108) with respect to the time variable. As in [22, p. 363], the idea of dyadic decomposition of the interval \((0, t)\) plays a useful role. Without loss of generality, we suppose \( T > 1 \).

For any \( t \in (1, T) \), we see
\[
\int_1^t \langle \tau \rangle^{-(1/2)-2\mu+2\delta} L(u_1(\tau)) L(u_2(\tau)) d\tau = \sum_{j=0}^N \int_{2^j}^{2^{j+1}} \cdots \int_2^\infty \langle \sigma \rangle^{-\mu-\delta} \left( \int_{2^j}^{2^{j+1}} L(u_1(\tau))^2 d\tau \right)^{1/2} \\
\leq \sum_{j=0}^N (2^j)^{-(1/2)+(9\delta/2)} \left\{ (2^j)^{-\mu-\delta} \left( \int_{2^j}^{2^{j+1}} L(u_1(\tau))^2 d\tau \right)^{1/2} \right\} \\
\times \left\{ (2^j)^{-\mu-(3\delta/2)} \left( \int_{2^0}^{2^j} L(u_2(\tau))^2 d\tau \right)^{1/2} \right\} \\
\leq C \left( \sum_{j=0}^\infty (2^j)^{-(1/2)+(9\delta/2)} \right) \sup_{1<\sigma<T} \langle \sigma \rangle^{-\mu-\delta} \left( \int_1^\sigma L(u_1(\tau))^2 d\tau \right)^{1/2} \\
\times \sup_{1<\sigma<T} \langle \sigma \rangle^{-\mu-(3\delta/2)} \left( \int_1^\sigma L(u_2(\tau))^2 d\tau \right)^{1/2}. 
\] (124)
Here, and later on as well, we abuse the notation to mean $t$ by $2^{N+1}$. Because $\delta$ is a sufficiently small positive number, we are able to obtain the desired estimate. Also,

$$
\int_1^t \langle \tau \rangle^{-1+\eta+\delta} G(u_2(\tau)) d\tau \\
\leq C \sum_{j=0}^N \left( \int_2^{2^{j+1}} \tau^{-2+2(\eta+\delta)} d\tau \right)^{1/2} \left( \int_2^{2^{j+1}} G(u_2(\tau))^2 d\tau \right)^{1/2} \\
\leq C \sum_{j=0}^N (2^j)^{-(1/2)+\eta+2\delta} (2^j)^{-\delta} \left( \int_2^{2^{j+1}} G(u_2(\tau))^2 d\tau \right)^{1/2} \\
\leq C \sup_{1<\sigma<T} (\sigma)^{-\delta} \left( \int_1^\sigma G(u_2(\tau))^2 d\tau \right)^{1/2},
$$

(125)

because $\delta$ and $\eta$ are sufficiently small positive numbers. The estimate of the integration from 0 to 1 is much easier, thus we omit it. Integrating (76) over $(0, t) \times \mathbb{R}^3$, we are now able to obtain (89), except the required estimate for $\|T^i_\eta \partial^a \bar{Z}^a u_k\|_{L^2((0, T) \times \mathbb{R}^3)}$. After employing (50) together with $\|\chi_2 \partial^a \bar{Z}^a u_k\|_{L^\infty} \leq C \langle u(t) \rangle$, we can proceed in exactly the same way as we have done above. The proof of Proposition 2 has been finished.

5. Energy estimate of $u_2$. This section is devoted to the energy estimate of $u_2$. We will show:

**Proposition 3.** The following inequality holds for smooth local solutions to (1) $u = (u_1, u_2)$, as long as they satisfy (59) for some time interval $(0, T)$:

$$
\sup_{0 < t < T} \langle t \rangle^{-2\delta} N(u_2(t))^2 + \sup_{0 < t < T} \langle t \rangle^{-2\delta} \int_0^t G(u_2(\tau))^2 d\tau \\
\leq CN(u_2(0))^2 + C \sup_{0 < t < T} \langle u(t) \rangle \langle N_T(u)^2 + G_T(u)^2 + E_T(u)^2 \rangle \\
+ C N_T(u) T + C G_T(u) T N_T(u)^2.
$$

(126)

The rest of this section is devoted to the proof of this proposition. As in the previous section, we have only to deal with the highest-order energy. In the same way as in (76), we get

$$
\frac{1}{2} \partial_t \{ e^\varphi (\partial^a \bar{Z}^a u_2)^2 \} + |\nabla \bar{Z}^a u_2|^2 \\
= G_2^{\alpha \beta \gamma} (\partial_\alpha \bar{Z}^a u_2)(\partial_\beta \bar{Z}^a u_2)(\partial_\gamma \bar{Z}^a u_2)
$$
Here, \( g = g(t - r) \) is the same as in (91), \( \tilde{q} = q_3 - (1/2)q'(t - r)q_4 \),

\[
q_3 = \frac{1}{2} G_2^{\alpha \beta \gamma} (\partial_\gamma u_i)(\partial_\beta \tilde{Z}^a S^d u_2)(\partial_\alpha \tilde{Z}^a S^d u_2) - G_2^{\alpha \beta \gamma} (\partial_\alpha u_i)(\partial_\beta \tilde{Z}^a S^d u_2)(\partial_\gamma \tilde{Z}^a S^d u_2),
\]

(128)

\[
q_4 = \sum_{j=1}^{3}(T_j \tilde{Z}^a S^d u_2)^2 - G_2^{\alpha \beta \gamma} (\partial_\gamma u_i)(\partial_\beta \tilde{Z}^a S^d u_2)(\partial_\alpha \tilde{Z}^a S^d u_2) + 2G_2^{\alpha \beta \gamma} (\partial_\gamma u_i)(\partial_\beta \tilde{Z}^a S^d u_2)(-\omega_\alpha)(\partial_\gamma \tilde{Z}^a S^d u_2),
\]

(129)

and

\[
J_{2.1} = \sum \tilde{G}^{\alpha \beta \gamma} (\partial_\gamma \tilde{Z}^a S^d u_2)(\partial_\alpha \tilde{Z}^a S^d u_2)(\partial_\beta \tilde{Z}^a S^d u_2),
\]

(130)

\[
J_{2.2} = \sum \tilde{G}^{\alpha \beta \gamma} (\partial_\beta \tilde{Z}^a S^d u_2)(\partial_\alpha \tilde{Z}^a S^d u_2)(\partial_\gamma \tilde{Z}^a S^d u_2),
\]

(131)

\[
J_{2.3} = \sum \tilde{G}^{\alpha \beta \gamma} (\partial_\gamma \tilde{Z}^a S^d u_2)(\partial_\alpha \tilde{Z}^a S^d u_2)(\partial_\beta \tilde{Z}^a S^d u_2),
\]

(132)

\[
J_{2.4} = \sum \tilde{G}^{\alpha \beta \gamma} (\partial_\beta \tilde{Z}^a S^d u_2)(\partial_\alpha \tilde{Z}^a S^d u_2)(\partial_\gamma \tilde{Z}^a S^d u_2),
\]

(133)

\[
J_{2.5} = \sum \tilde{G}^{\alpha \beta \gamma} (\partial_\alpha \tilde{Z}^a S^d u_2)(\partial_\beta \tilde{Z}^a S^d u_2)(\partial_\gamma \tilde{Z}^a S^d u_2).
\]

(134)

Recall that we have dealt with \( \chi_1 \tilde{q} \) and \( \chi_1 J_{1.1}, \ldots, \chi_1 J_{1.5} \) without relying upon the null condition. Therefore, it is possible to handle \( \chi_1 \tilde{q} \) and \( \chi_1 J_{2.1}, \ldots, \chi_1 J_{2.5} \) as before. We may thus focus on the estimate of \( \chi_2 \tilde{q} \) and \( \chi_2 J_{2.1}, \ldots, \chi_2 J_{2.5} \). For the estimate of \( \chi_2 \tilde{q} \), it suffices to show how to handle the terms with the coefficients \( \{G_2^{\alpha \beta \gamma}\} \), because the coefficients \( \{G_2^{\alpha \beta \gamma}\} \) satisfy the null condition and thus we are able to treat all the terms with the coefficients \( \{G_2^{\alpha \beta \gamma}\} \) in the same way as before.

Using the first equation in (1) to represent \( \partial_\gamma^2 u_1 \) as \( \Delta u_1 + \) (higher-order terms) and then use (41) to represent \( \partial_\gamma^2 u_1 \) appearing in these higher-order terms, we obtain

\[
\|\chi_2 G_2^{\alpha \beta \gamma} (\partial_\gamma u_1(t))(\partial_\beta \tilde{Z}^a S^d u_2(t))(\partial_\alpha \tilde{Z}^a S^d u_2(t))\|_{L^1(\mathbb{R}^3)}
\]

\[
\leq C(t)^{-1}\|\chi_2 |x||\partial_\gamma^2 u_1(t))(\partial\tilde{Z}^a S^d u_2(t))(\partial\tilde{Z}^a S^d u_2(t))\|_{L^1(\mathbb{R}^3)}
\]

\[
+ C(t)^{-1}\|\chi_2 |x|(\partial_\gamma \tilde{Z}^a S^d u_2(t))(\partial\tilde{Z}^a S^d u_2(t))(\partial\tilde{Z}^a S^d u_2(t))\|_{L^1(\mathbb{R}^3)}
\]

(135)

\[
\leq C(t)^{-1+2\delta} \|u\| N(u(t))^2.
\]

(If we employed (41) directly, it would meet with the troublesome factor \( t^{-1+3\delta} \) on the right-hand side above. This is the reason why we have used the first equation in (1) to represent \( \partial_\gamma^2 u_1 \) as \( \Delta u_1 + \) (higher-order terms).) Here, we have used the assumption that \( \|u\| \) is small, so that we have \( \|u\|^2 \leq \|u\| \). In the same way, we get

\[
\|\chi_2 G_2^{\alpha \beta \gamma} (\partial_\alpha u_1(t))(\partial_\beta \tilde{Z}^a S^d u_2(t))(\partial_\gamma \tilde{Z}^a S^d u_2(t))\|_{L^1(\mathbb{R}^3)}
\]

\[
\leq C(t)^{-1+2\delta} \|u\| N(u(t))^2.
\]

(136)
It is easy to show
\[\| \chi G(\partial_t \tilde{Z}^a S^d u_1(t))(\partial_{\alpha \beta} \tilde{Z}^a S^d u_2(t))\|_{L^1(R^3)},\]
\[\| \chi G(\partial_t \tilde{Z}^a S^d u_1(t))(\partial_{\alpha \beta} \tilde{Z}^a S^d u_2(t))\|_{L^1(R^3)} \leq C\langle t \rangle^{-1+2\delta} \langle u \rangle N(u(t))^2.\]  

We have finished the estimate of \(\chi G\).

We next deal with \(\chi_2 J_{2,1}, \ldots, \chi_2 J_{2,5}\). We may focus on \(\chi_2 J_{2,1}, \chi_2 J_{2,3},\) and \(\chi_2 J_{2,4}\) because the coefficients \(\{G_{\alpha \beta}\}\) and \(\{H_{\alpha \beta}\}\) satisfy the null condition and it is therefore possible to handle \(\chi_2 J_{2,2}\) and \(\chi_2 J_{2,5}\) in the same way as before. Let us first deal with \(\chi_2 J_{2,1}\). When \(|a''| + d'' = 2\) (and thus \(|a'| + d' \leq 1\), we get by (43), (49)
\[\| \chi_2(\partial_t \tilde{Z}^a S^d u_1(t))(\partial_{\alpha \beta} \tilde{Z}^a S^d u_2(t))\|_{L^1(R^3)} \leq C\langle t \rangle^{-1} \| \chi_2 |x| \partial_t \tilde{Z}^a S^d u_1(t)\|_{L^\infty(R^3)} \times \| \chi_2 \partial_{\alpha \beta} \tilde{Z}^a S^d u_2(t)\|_{L^2(R^3)} \| \partial_t \tilde{Z}^a S^d u_2(t)\|_{L^2(R^3)} \leq C\langle t \rangle^{-1+2\delta} \langle u(t) \rangle N(u(t))^2.\]  

Note that, to handle \(\chi_2 \partial_t \tilde{Z}^a S^d u_2(t)\|_{L^2(R^3)}\) (\(|a''| = 1\), we have again used (49) along with \(\| \chi_2 \partial_t \tilde{Z}^a S^d u_2(t)\|_{L^\infty} \leq C\langle u(t) \rangle\) (see (112)) and smallness of \(\langle u(t) \rangle\).

On the other hand, when \(|a''| + d'' \leq 1\) (and thus \(|a'| + d' \leq 2\), we get
\[\| \chi_2(\partial_t \tilde{Z}^a S^d u_1(t))(\partial_{\alpha \beta} \tilde{Z}^a S^d u_2(t))\|_{L^1(R^3)} \leq C\langle t \rangle^{-1} \| \chi_2 |x| \partial_t \tilde{Z}^a S^d u_1(t)\|_{L^\infty L^5(R^3)} \times \| \partial_{\alpha \beta} \tilde{Z}^a S^d u_2(t)\|_{L^2 L^5(R^3)} \| \partial_t \tilde{Z}^a S^d u_2(t)\|_{L^2(R^3)} \leq C\langle t \rangle^{-1+2\delta} \langle u(t) \rangle N(u(t))^2.\]  

It is easy to obtain a similar estimate for \(\chi_2 J_{2,3}\) and \(\chi_2 J_{2,4}\).

Using the basic fact that the integration of \((1 + \tau)^{-1+2\delta}\) from 0 to \(t\) is \(O(t^{2\delta})\) for large \(t\), we can now complete the proof of Proposition 3, except the required estimate for \(\langle t - r \rangle^{-(1/2) - \eta} \tilde{T}_j \partial_t \tilde{Z}^a S^d u_2\|_{L^2((0,T) \times R^3)}, \). How to handle the similar norm for \(u_1\) has been dwelt on at the end of the last section, and we have only to follow the same approach as there. The proof of Proposition 3 has been finished.

6. Space-time \(L^2\) estimates of \(u_1\). In this section, we will prove the following:

Proposition 4. The following inequality holds for smooth local solutions to (1) \(u = (u_1, u_2),\) as long as they satisfy (59) for some time interval \((0, T)\):
\[\left( \sup_{0 < \tau < T} \langle t \rangle^{-1+b} \left( \int_0^t L(u_1(\tau))^2 d\tau \right)^{1/2} \right)^2 \leq C N(u_1(0))^2 + C \left( \sup_{0 < \tau < T} N(u_1(\tau)) \right) N_T(u)^2 + C \left( \sup_{0 < \tau < T} \langle u(\tau) \rangle \right) \left( \sup_{0 < \tau < T} N(u_1(\tau)) \right)^2 + C \left( \sup_{0 < \tau < T} \langle u(\tau) \rangle \right) \left( \sup_{0 < \tau < T} \langle t \rangle^{-1+b} \left( \int_0^t L(u_1(\tau))^2 d\tau \right)^{1/2} \right)^2.\]  

(140)
Let us first consider the case \( |a| = 2 \) and \( d = 1 \), when considering the estimate of \( ||r^{-(3/2) + \mu} \bar{Z}^a S^k u_1||_{L^2((0,t) \times \mathbb{R}^3)} \) and \( ||r^{-(1/2) + \mu} \partial \bar{Z}^a S^k u_1||_{L^2((0,t) \times \mathbb{R}^3)} \). By Lemma 2.7, we see that for \( |a| = 2 \)

\[
(1 + t)^{-2\mu} \left( ||r^{-(3/2) + \mu} \bar{Z}^a S^k u_1||_{L^2((0,t) \times \mathbb{R}^3)}^2 \right. \\
+ ||r^{-(1/2) + \mu} \partial \bar{Z}^a S^k u_1||_{L^2((0,t) \times \mathbb{R}^3)}^2 \\
\le C \left( ||\partial \bar{Z}^a S u_1(0)||_{L^2(\mathbb{R}^3)}^2 \\
+ C \sum_{k=1}^2 \sum_{j=1}^3 \int_{0}^{t} \int_{\mathbb{R}^3} |\partial \bar{Z}^a S u_1||\partial \bar{Z}^a S^\ell u_j| ||\partial \bar{Z}^a S^\ell u_j| |dx d\tau |\right.
\]

Estimate of the set \( K \). Estimate of \( K_{1,l} \) for \( l = 1, \ldots, 6 \). By \( \chi_1 \), we mean the characteristic function of the set \( \{ x \in \mathbb{R}^3 : |x| < (\tau + 1)/2 \} \) for any fixed \( \tau \in (0,t) \).

We again separate \( \mathbb{R}^3 \) into the two pieces \( \{ x \in \mathbb{R}^3 : |x| < (\tau + 1)/2 \} \) and its complement for the estimate of \( K_{1,l} \) for \( l = 1, \ldots, 6 \).

Estimate of \( \chi_1 K_{1,l} \) for \( l = 1, \ldots, 6 \). Let us first consider the case \( d' = 0 \) (and hence \( d'' \le 1 \)). If \( |a'| = 2 \) (and hence \( |a''| = 0 \)), then we get by (30), (45)

\[
||\chi_1 K_{1,1}||_{L^1(\mathbb{R}^3)} \le C(\tau)^{-1} \sum_{k=1}^2 \sum_{j=1}^3 N(u_1(\tau)) ||(\tau - r) \partial \bar{Z}^a S^\ell u_k(r)||_{L^3} ||\partial \bar{Z}^a S^\ell u_1(r)||_{L^3}
\]

\[
\le C(\tau)^{-1 + 2d} N(u_1(\tau)) N(u(\tau))^2.
\]

If \( |a'| \le 1 \) (and hence \( |a''| \le 1 \)), then we get by (31) and (49)

\[
||\chi_1 K_{1,1}||_{L^1(\mathbb{R}^3)}
\]
We can obtain

\[ \chi \] for (59) and (49) together with \( \| \partial Su_k(\tau) \|_{L^\infty} \leq \langle \tau \rangle^\delta \| \langle u(\tau) \rangle \| \leq \epsilon_3^* \langle \tau \rangle^\delta. \) (See (59) for \( \epsilon_3^* \).

Let us next consider the case \( d'' = 0 \) (and hence \( d' \leq 1 \)). By considering the case \( |d''| = 2 \) (and hence \( |a'| = 0 \)) and \( |a''| \leq 1 \) (and hence \( |a'| \leq 1 \)) separately, we are able to obtain

\[ \| \chi_1 K_{1,1} \|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+2\delta} N(u_1(\tau)) N(u(\tau))^2 \] (144)

in the same way as above.

Estimate of \( \chi_1 K_{1,2} \) We can obtain

\[ \| \chi_1 K_{1,2} \|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+2\delta} N(u_1(\tau)) N(u(\tau))^2 \] (145)

in a similar way.

Estimate of \( \chi_1 K_{1,3} \) Using the Hardy inequality and proceeding as above, we can obtain

\[ \| \chi_1 K_{1,3} \|_{L^1(\mathbb{R}^3)} \leq \sum_{k=1}^2 \sum_{|a'|+d' \leq 1} \| r^{-1} \tilde{Z}^a S u_k(t) \|_{L^2} \| \chi_1 (\partial \tilde{Z}^a S^d u_k) (\partial^2 \tilde{Z}^{a''} S^{d''} u_1) \|_{L^2} \] (146)

Estimate of \( \chi_1 K_{1,4} \) In the same way as in (146), we obtain

\[ \| \chi_1 K_{1,4} \|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+2\delta} N(u_1(\tau)) N(u(\tau))^2. \] (147)

Estimate of \( \chi_1 K_{1,5} \) Using (31) and (41), we get

\[ \| \chi_1 K_{1,5} \|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+\delta} N(u(\tau)) N(u_1(\tau))^2. \] (148)

Estimate of \( \chi_1 K_{1,6} \) Using the Hardy inequality, we can obtain

\[ \| \chi_1 K_{1,6} \|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+\delta} N(u(\tau)) N(u_1(\tau))^2. \] (149)

**Estimate of \( \chi_2 K_{1,l} \) for \( l = 1, \ldots, 6 \).** We next consider \( \chi_2 K_{1,l} \) for \( l = 1, \ldots, 6 \).

Estimate of \( \chi_2 K_{1,1} \) If \( |a''|+d'' = 2 \) (and hence \( |a'|+d' \leq 1 \)), then we get by (33) and (49)

\[ \| \chi_2 K_{1,1} \|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1} \sum_{k=1}^2 \sum_{|a'|+d' \leq 1} \| \chi_2(\tau) \|_{L^\infty} \| \chi_2(\tau) \|_{L^\infty} \| \chi_2(\tau) \|_{L^\infty} \| \chi_2(\tau) \|_{L^\infty} \] (150)

Here we have used (49) along with \( \| \chi_2 \|_{L^1(\mathbb{R}^3)} \leq C(\| u(\tau) \|) \leq C \epsilon_3^* \). If \( |a''|+d'' = 1 \) (and hence \( |a'|+d' \leq 2 \)), then we get by (32)

\[ \| \chi_2 K_{1,1} \|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1} \sum_{k=1}^2 \sum_{|a''|+d'' \leq 1} \| \chi_2(\tau) \|_{L^\infty} \| \chi_2(\tau) \|_{L^\infty} \| \chi_2(\tau) \|_{L^\infty} \] (151)

\[ \leq C(\tau)^{-1+2\delta} N(u_1(\tau)) N(u(\tau))^2. \]
On the other hand, if $|a''| + d'' = 0$ (and hence $|a'| + d' \leq 3$), then we get by using (33) and (41)
\[
\|\chi_2 K_{1,1}\|_{L^1(\mathbb{R}^3)} \\
\leq C(\tau)^{-1} \sum_{k=1}^{2} \sum_{|a'| + d' \leq 3} N(u_1(\tau))\|\partial Z^a S^d u_k(\tau)\|_{L^2} \|\chi_2(\tau)\|_{L^\infty} \|\partial^2 u_1(\tau)\|_{L^2} \\
\leq C(\tau)^{-1+2\delta} N(u_1(\tau))N(u(\tau))^2.
\] (152)

Estimate of $\chi_2 K_{1,2}$ Using (33), we can easily get
\[
\|\chi_2 K_{1,2}\|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+2\delta} N(u_1(\tau))N(u(\tau))^2.
\] (153)

Estimate of $\chi_2 K_{1,3}$ Arguing as in (150)–(152) and using the Hardy inequality, we get
\[
\|\chi_2 K_{1,3}\|_{L^1(\mathbb{R}^3)} \\
\leq C \sum_{k=1}^{2} \sum_{|a'| + d' \leq 3} \|r^{-1} Z^a S u_1(\tau)\|_{L^2} \|\chi_2(\tau)\|_{L^2} \|\partial Z^a S^d u_k(\tau)\|_{L^2} \|\partial^2 Z^a S^d u_1(\tau)\|_{L^2} \\
\leq C(\tau)^{-1+2\delta} N(u_1(\tau))N(u(\tau))^2.
\] (154)

Estimate of $\chi_2 K_{1,4}$ Using the Hardy inequality and proceeding as in (153), we can obtain
\[
\|\chi_2 K_{1,4}\|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+2\delta} N(u_1(\tau))N(u(\tau))^2.
\] (155)

Estimate of $\chi_2 K_{1,5}$ Using (33) and (41), we easily get
\[
\|\chi_2 K_{1,5}\|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+\delta} \|u(\tau)\|_{L^1(\mathbb{R}^3)} N(u_1(\tau))^2.
\] (156)

Estimate of $\chi_2 K_{1,6}$ By using the Hardy inequality, we can obtain
\[
\|\chi_2 K_{1,6}\|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+\delta} \|u(\tau)\|_{L^1(\mathbb{R}^3)} N(u_1(\tau))^2
\] (157)
in the same way as in (156).

**Estimate of $K_{1,7}$ and $K_{1,8}$.** It is easy to get by (31) and (33)
\[
\|K_{1,7}\|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+\delta} \|u(\tau)\|_{L^1(\mathbb{R}^3)} L(u_1(\tau))^2.
\] (158)

We also get
\[
\|K_{1,8}\|_{L^1(\mathbb{R}^3)} \leq C(\tau)^{-1+\delta} \|u(\tau)\|_{L^1(\mathbb{R}^3)} \|\partial Z^a S u_1(\tau)\|_{L^2(\mathbb{R}^3)} L(u_1(\tau)) \\
\leq C(\tau)^{-1+\delta} \|u(\tau)\|_{L^1(\mathbb{R}^3)} L(u_1(\tau))^2.
\] (159)

Now we are ready to complete the proof of Proposition 4. In view of (141)–(159), we have only to explain how to handle the integral over $(0, t)$ of $\|K_{1,7}(\tau)\|_{L^1(\mathbb{R}^3)}$ and $\|K_{1,8}(\tau)\|_{L^1(\mathbb{R}^3)}$. Without loss of generality, we may suppose $1 < t < T$. It follows from (158) that
\[
\int_0^t \|K_{1,7}(\tau)\|_{L^1(\mathbb{R}^3)} d\tau \leq C \sup_{0 < \tau < T} \|u(\tau)\| \int_0^t L(u_1(\tau))^2 d\tau,
\] (160)
and
\[
\int_1^t \|K_{1,7}(\tau)\|_{L^1(\mathbb{R}^3)} d\tau \\
\leq C \sup_{0 < \tau < T} \|u(\tau)\| \sum_{j=0}^{\infty} (2^j)^{-1+\delta} \int_{2^j}^{2^{j+1}} L(u_1(\tau))^2 d\tau \\
\leq C \sup_{0 < \tau < T} \|u(\tau)\| \left( \sum_{j=0}^{\infty} (2^j)^{-1+\delta+2(\mu+\delta)} \right) \\
\times \left( \sup_{1 < \sigma < T} \langle \sigma \rangle^{-\mu-\delta} \left( \int_1^\sigma L(u_1(\tau))^2 d\tau \right)^{1/2} \right)^2.
\]
(161)

Similarly, we get by (159)
\[
\int_0^t \|K_{1,8}(\tau)\|_{L^1(\mathbb{R}^3)} d\tau \leq C \sup_{0 < \tau < T} \|u(\tau)\| \int_0^1 L(u_1(\tau))^2 d\tau
\]
and
\[
\int_1^t \|K_{1,8}(\tau)\|_{L^1(\mathbb{R}^3)} d\tau \leq C \sup_{1 < \tau < T} \|u(\tau)\| \left( \sum_{j=0}^{\infty} (2^j)^{-1+\delta+2(\mu+\delta)} \right) \\
\times \left( \sup_{1 < \sigma < T} \langle \sigma \rangle^{-\mu-\delta} \left( \int_1^\sigma L(u_1(\tau))^2 d\tau \right)^{1/2} \right)^2.
\]
(163)

Since \(\delta\) and \(\mu\) are sufficiently small positive numbers, we see that the series in (161) and (163) converges. Therefore we have finished the proof of (140).

7. Space-time \(L^2\) estimates of \(u_2\). In this section, we consider the space-time \(L^2\) estimates of \(u_2\). We can prove:

**Proposition 5.** The following inequality holds for smooth local solutions to (1) \(u = (u_1, u_2)\), as long as they satisfy (59) for some time interval \((0, T)\):

\[
\left( \sup_{0 < t < T} \langle t \rangle^{-\mu-(3\delta/2)} \left( \int_0^t L(u_2(\tau))^2 d\tau \right)^{1/2} \right)^2 \\
\leq CN(u_2(0))^2 + C \left( \sup_{0 < t < T} \langle t \rangle^{-\delta} N(u_2(t)) \right)^2 \\
+ C \left( \sup_{0 < t < T} \|u(t)\| \right) \left( \sup_{0 < t < T} \langle t \rangle^{-\delta} N(u_2(t)) \right)^2 \\
+ C \left( \sup_{0 < t < T} \|u(t)\| \right) \left( \sup_{0 < t < T} \langle t \rangle^{-\mu-(3\delta/2)} \left( \int_0^t L(u_2(\tau))^2 d\tau \right)^{1/2} \right)^2.
\]
(164)

We have only to repeat essentially the same argument as in Section 6. We thus omit the proof.

8. Proof of Theorem 1.3. So far, we have proved that local solutions to (1) defined for \((t, x) \in [0, T) \times \mathbb{R}^3\) with compactly supported smooth data satisfy

\[
\mathcal{N}_T(u)^2 + \mathcal{G}_T(u)^2 + \mathcal{L}_T(u)^2 \leq C_0 (N_4(u_1(0))^2 + N_4(u_2(0))^2) \\
+ C_1 \left( \sup_{0 < t < T} \|u(t)\| \right) \left( \mathcal{N}_T(u)^2 + \mathcal{G}_T(u)^2 + \mathcal{L}_T(u)^2 \right) + \mathcal{N}_T(u)^3 + \mathcal{G}_T(u) \mathcal{N}_T(u)^2
\]
(165)
for suitable constants $C_0, C_1 > 0$, provided that

$$\sup_{0 < t < T} \|u(t)\| \leq \varepsilon^*_0.$$  \hfill (166)

See (59) for $\varepsilon^*_0$. In order to get the key a priori estimate (see (175) below), we must show that $\|u(t)\|$ is small (at least for a short time interval), whenever $N_4(u_1(0)) + N_4(u_2(0))$ is small enough. (See (167) below.)

Since initial data belong to $C_0^{\infty}(\mathbb{R}^3) \times C_0^{\infty}(\mathbb{R}^3)$ and the uniqueness theorem of $C^2$-solutions and its corollary in [9, p. 53] apply to the system (1), smooth local solutions satisfy

$$u_1(t, x) = u_2(t, x) = 0, \quad 0 \leq t < T, \quad |x| \geq R + t,$$  \hfill (167)

where $R > 0$ is a constant such that $u_i(0, x) = \partial_t u_i(0, x) = 0$ ($i = 1, 2$) for $|x| \geq R$. (Remark: All the constants $C$ appearing below will be independent of $R$.) Moreover, thanks to (167), we can easily verify

$$\mathcal{N}(u(t)) \in C([0, T]).$$  \hfill (168)

(Actually, the last property can be seen as a direct consequence of the fact $N_i(u_i(t))^2 \in C_\infty([0, T]), \ i = 1, 2$.) Due to (39) and (168), we know

$$\mathcal{N}(u(t)) \leq 2A\varepsilon_0 \quad (A := \max\{\sqrt{C_{0}}, C_{KS}, 1\})$$  \hfill (169)

(see (167), (38) for the constants $C_0, C_{KS}$) at least for a short time interval, which means

$$\{T > 0 : \text{For given data } (f_i, g_i) \in C_0^{\infty}(\mathbb{R}^3) \times C_0^{\infty}(\mathbb{R}^3) \ (i = 1, 2) \text{ satisfying (39), there exists a unique smooth solution } (u_1, u_2) \text{ to (1) defined for all } (t, x) \in [0, T] \times \mathbb{R}^3 \text{ satisfying } \mathcal{N}(u(t)) \leq 2A\varepsilon_0 \text{ for any } t \in [0, T]\} \neq \emptyset.$$

We define $T_*$ as the supremum of this non-empty set. In order to establish the key estimates (174) and (175), we must first prove:

**Proposition 6.** Suppose (39) for compactly supported smooth data. Then the local solution to (1) satisfies

$$\mathcal{M}(u(t)) \leq 2AV(u(t)), \quad 0 < t < T_*.$$  \hfill (170)

For the constant $A$, see (169) above.

**Proof.** When the initial data is identically zero and hence the corresponding solution identically vanishes, we obviously get (170). We may therefore suppose without loss of generality that the smooth initial data is not identically zero. We thus have $\mathcal{N}(u(0)) > 0$. Moreover, we actually know $\mathcal{N}(u(t)) > 0$ for all $t \in (0, T_*)$. Indeed, suppose $\mathcal{N}(u(T_0)) = 0$ for some $T_0 \in (0, T_*)$. Since $\partial u(T_0, x)$ is identically zero and $u(T_0, x)$ has compact support, $u(T_0, x)$ and $\partial_t u(T_0, x)$ are also identically zero. Define $w(t, x) := u(T_0 - t, x)$. We then see that $w$ satisfies a system of quasi-linear wave equations to which the above-mentioned uniqueness theorem of $C^2$-solutions [9] applies. Since $w(0, x)$ and $\partial_t w(0, x)$ are identically zero, we know by this uniqueness theorem that $w$ is a trivial solution, which in particular means $w(T_0, x)$ and $\partial_t w(T_0, x)$ are identically zero. This contradicts the fact that the initial data $(u(0, x), \partial_t u(0, x))$ is non-trivial.
Note that $\mathcal{N}(u(t)) > 0$ for $t \in [0, T_\ast)$ in the following discussion. Since Lemma 2.8 yields $\mathcal{M}(u(0)) \leq AN(u(0))$, that is, $\mathcal{M}(u(0))/\mathcal{N}(u(0)) \leq A$, and $\mathcal{M}(u(t))$, $\mathcal{N}(u(t))$, and $\mathcal{M}(u(t))/\mathcal{N}(u(t))$ are continuous on the interval $[0, T_\ast)$, we see

$$\mathcal{M}(u(t))/\mathcal{N}(u(t)) \leq 2A,$$

that is,

$$\mathcal{M}(u(t)) \leq 2AN(u(t)) \quad (171)$$

at least for a short time interval $\subset [0, T_\ast)$. It remains to show that in fact, the last inequality holds for all $t \in [0, T_\ast)$. Let

$$\hat{T} := \sup\{ T \in (0, T_\ast) : \mathcal{M}(u(t)) \leq 2AN(u(t)) \text{ for all } t \in [0, T) \}. $$

By definition we know $\hat{T} \leq T_\ast$. To show $\hat{T} = T_\ast$, we proceed as follows. Since we have $\mathcal{N}(u(t)) \leq 2A\varepsilon_0 (0 < t < T_\ast)$, we obtain by Lemmas 2.4–2.6

$$\langle \langle u(t) \rangle \rangle \leq C_2(\mathcal{N}(u(t)) + \mathcal{M}(u(t)))$$

$$\leq C_2(1 + 2A)\mathcal{N}(u(t)) \leq 2AC_2(1 + 2A)\varepsilon_0, \quad 0 < t < \hat{T}$$

for a constant $C_2 > 0$. Because of $2AC_2(1 + 2A)\varepsilon_0 \leq \min\{\varepsilon_1, \varepsilon_2\}$ (see (39)), we can use Proposition 1 with $T = \hat{T}$ to get

$$\mathcal{M}(u(t)) \leq A\mathcal{N}(u(t)) + 2AC_2\mathcal{C}_3(1 + 2A)\varepsilon_0\{\mathcal{M}(u(t)) + \mathcal{N}(u(t))\}, \quad 0 < t < \hat{T}$$

for a constant $C_3 > 0$, which yields owing to the definition of $\varepsilon_0$ (see (39))

$$\mathcal{M}(u(t)) \leq \frac{A + 2AC_2\mathcal{C}_3(1 + 2A)\varepsilon_0}{1 - 2AC_2\mathcal{C}_3(1 + 2A)\varepsilon_0}\mathcal{N}(u(t)) \leq \frac{3}{2}A\mathcal{N}(u(t)), \quad 0 < t < \hat{T}.$$  \hspace{1em} (173)

Since $\mathcal{M}(u(t))/\mathcal{N}(u(t)) \in C([0, T_\ast))$, we finally arrive at the conclusion $\hat{T} = T_\ast$. (If we assume $\hat{T} < T_\ast$, the estimate (173) contradicts the definition of $\hat{T}$.) We have finished the proof of Proposition 6. \hfill \Box

Now we are in a position to complete the proof of the key a priori estimate (175) below. As in (172), we get by Proposition 6 and the definition of $\varepsilon_0$

$$\langle \langle u(t) \rangle \rangle \leq 2AC_2(1 + 2A)\varepsilon_0 \leq \varepsilon_3^\ast, \quad 0 < t < T_\ast.$$ \hspace{1em} (174)

We can use (165) with $T = T_\ast$ owing to (174). Using the inequalities $\mathcal{N}(u(t)) \leq 2A\varepsilon_0$, $\langle \langle u(t) \rangle \rangle \leq 2AC_2(1 + 2A)\varepsilon_0 \{0 < t < T_\ast\}$, we get from (165)

$$\mathcal{N}_T(u)^2 + \mathcal{G}_T(u)^2 + \mathcal{L}_T(u)^2 \leq \frac{C_0\{N_4(u_1(0))^2 + N_4(u_2(0))^2\}}{1 - 2AC_1\mathcal{C}_2(1 + 2A)\varepsilon_0} - 3AC_1\varepsilon_0,$$

which yields owing to the definition of $\varepsilon_0$

$$\mathcal{N}(u(t)) \leq \frac{3}{2}A\mathcal{N}(u(0)) \leq \frac{3}{2}A\varepsilon_0, \quad 0 < t < T_\ast.$$  \hspace{1em} (175)

Now we are in a position to show $T_\ast = \infty$. Assume $T_\ast < \infty$. By solving (1) with data $(u_1(T_\ast - \delta, x), \partial_x u_1(T_\ast - \delta, x)) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$ (see (167)) given at $t = T_\ast - \delta$ ($\delta > 0$ is sufficiently small), we can extend this local solution smoothly to a larger strip, say, $\{(t, x) : 0 < t < \hat{T}, x \in \mathbb{R}^3\}$, where $\hat{T} > T_\ast$. Such a smooth local solution defined for $(t, x) \in (0, \hat{T}) \times \mathbb{R}^3$ satisfies $\mathcal{N}(u(t)) \in C([0, \hat{T}))$. Moreover, because of $\mathcal{N}(u(T_\ast)) \leq 3A\varepsilon_0/2$ by (175), we see that there exists $T' \in (T_\ast, \hat{T}]$ such that $\mathcal{N}(u(t)) \leq 2A\varepsilon_0, 0 < t < T'$, which contradicts the definition of $T_\ast$. Hence we have $T_\ast = \infty$.

To complete the proof of Theorem 1.2, we must relax the regularity of data and eliminate compactness of the support of data. Naturally, we employ the standard
mollifier and cut-off idea (see, e.g., [3, p. 12] and [8, p. 122]). Then, we easily see that, for any \((f_i, g_i)\) \((i = 1, 2)\) satisfying \(f_1, f_2 \in L^6(\mathbb{R}^3)\) and

\[
C_D \sum_{i=1,2} D(f_i, g_i) \leq \frac{\varepsilon_0}{2}
\]

(see (39) for the constant \(C_D\)), there exists a sequence \((f_{i,n}, g_{i,n}) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)\) \((n = 1, 2, \ldots)\) such that

\[
C_D \sum_{i=1,2} D(f_{i,n}, g_{i,n}) \leq \varepsilon_0
\]

(176)

for sufficiently large \(n\), and

\[
\sum_{i=1,2} D(f_{i,n} - f_i, g_{i,n} - g_i) \to 0 \quad (n \to \infty).
\]

(177)

(We must keep in mind that this procedure becomes rather complicated when we employ \(W_4\) (see (6)), as in [7], to measure the size of data.) Thanks to (176), we know that the Cauchy problem (1) with data \((u_0(t), \partial_t u_0(t)) = (f_{i,n}, g_{i,n})\) \((i = 1, 2)\) admits a unique solution, which is denoted by \(u_n(t, x) = (u_{1,n}(t, x), u_{2,n}(t, x))\) for every large \(n\). Also, we have

\[
N_T(u_n) + G_T(u_n) + L_T(u_n) \leq C \sum_{i=1,2} D(f_{i,n}, g_{i,n}) \leq C\varepsilon_0
\]

(178)

for all \(T > 0\), with a constant \(C > 0\) independent of \(n\) and \(T\). Furthermore, owing to (178) and \(M(u_n(t)) \leq CN(u_n(t))\) for \(0 < t < \infty\) (see (170)), we obtain by the same argument as (in fact, essentially simpler argument than) in Sections 4–7, with a few obvious modifications

\[
\sup_{t > 0} N_1(u_{1,m}(t) - u_{1,n}(t)) + \sup_{t > 0} t^{-\delta} N_1(u_{2,m}(t) - u_{2,n}(t))
\]

\[
+ \left(\int_0^\infty \sum_{i=1}^3 \| (t-r)^{-(1/2) - \eta} T_i(u_{1,m}(t) - u_{1,n}(t)) \|^2_{L^2(\mathbb{R}^3)} dt \right)^{1/2}
\]

\[
+ \sup_{t > 0} t^{-\delta} \left(\int_0^t \sum_{i=1}^3 \| (\tau-r)^{-(1/2) - \eta} T_i(u_{2,m}(\tau) - u_{2,n}(\tau)) \|^2_{L^2(\mathbb{R}^3)} d\tau \right)^{1/2}
\]

\[
+ \sup_{t > 0} t^{-\mu - \delta} \left(\int_0^t \| r^{-(3/2) + \mu} (u_{1,m}(\tau) - u_{1,n}(\tau)) \|^2_{L^2(\mathbb{R}^3)} d\tau \right)^{1/2}
\]

\[
+ \sup_{t > 0} t^{-\mu - \delta} \left(\int_0^t \| r^{-(1/2) + \mu} \partial (u_{1,m}(\tau) - u_{1,n}(\tau)) \|^2_{L^2(\mathbb{R}^3)} d\tau \right)^{1/2}
\]

\[
+ \sup_{t > 0} t^{-\mu - (3\delta/2)} \left(\int_0^t \| r^{-(3/2) + \mu} (u_{2,m}(\tau) - u_{2,n}(\tau)) \|^2_{L^2(\mathbb{R}^3)} d\tau \right)^{1/2}
\]

\[
+ \sup_{t > 0} t^{-\mu - (3\delta/2)} \left(\int_0^t \| r^{-(1/2) + \mu} \partial (u_{2,m}(\tau) - u_{2,n}(\tau)) \|^2_{L^2(\mathbb{R}^3)} d\tau \right)^{1/2}
\]

\[
\leq C \sum_{i=1,2} \| \nabla (f_{i,m} - f_{i,n}) \|_{L^2(\mathbb{R}^3)} + \| g_{i,m} - g_{i,n} \|_{L^2(\mathbb{R}^3)}
\]

(179)

for sufficiently large \(m, n\), with a constant \(C\) independent of \(m, n\). (When showing (179), we are supposed to choose \(\varepsilon_0\) smaller than before, if necessary.) We thus see by the standard argument that \(u_n = (u_{1,n}, u_{2,n})\) has the limit that is the solution
to (1) with the data \((f_i, g_i)\) \((i = 1, 2)\) given at \(t = 0\). The proof of Theorem 1.2 has been completed.

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