Herz-slice spaces and applications

Yuan Lu, Jiang Zhou, Songbai Wang

Abstract Let $\alpha \in \mathbb{R}$, $t \in (0, \infty)$, $p \in (0, \infty]$, $r \in (1, \infty)$ and $q \in [1, \infty]$. We introduce the homogeneous Herz-slice space $(\dot{K}E_{\alpha}^{p,q,r}_t)(\mathbb{R}^n)$, the non-homogeneous Herz-slice space $(K E_{\alpha}^{p,q,r}_t)(\mathbb{R}^n)$ and show some properties of them. As an application, the bounds for the Hardy–Littlewood maximal operator on these spaces is considered.

1 Introduction

The homogeneous Herz-slice space $(\dot{K}E_{\alpha}^{p,q,r}_t)(\mathbb{R}^n)$ and the non-homogeneous Herz-slice space $(K E_{\alpha}^{p,q,r}_t)(\mathbb{R}^n)$ studied in this paper are associated with classical Herz spaces and the slice space. We will first give a short history of the slice space.

In 1926, Wiener [19] first introduced amalgam spaces to formulate his generalized harmonic analysis. In general, for $p, q \in (0, \infty)$, the amalgam space $(L^p, \ell^q)(\mathbb{R})$ is defined by

$$(L^p, \ell^q)(\mathbb{R}) := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}) : \left( \sum_{n \in \mathbb{Z}} \| f1_{[n,n+1)} \|_{L^p(\mathbb{R})} \right)^{\frac{1}{q}} < \infty \right\}.$$ 

But the first systematic study of these spaces was undertaken by Holland [9] in 1975. In recognition of Wiener’s first use of amalgams, Feichtinger initially called these spaces "Wiener-type spaces", and then, adopted the name "Wiener amalgam spaces", that’s also the most general definition of the amalgam space so far which provided by Feichtinger in the early 1980’s in a series of papers [4–7].

In 2014, Auscher and Mourgoglou [1] introduced particular cases of Wiener amalgam spaces [6], the slice space $E^p_t(\mathbb{R}^n)$ (also $(E^p_t)(\mathbb{R}^n)$), to study the classification of weak solutions in the natural classes for the boundary value problems of a t-independent elliptic system in the upper half-space. Moreover, in 2017, Auscher and Prisuelos-Arribas [2] introduced a more general slice space $(E^q_t)(\mathbb{R}^n)$, that is, for $t \in (0, \infty)$, $r \in (1, \infty)$ and $q \in [1, \infty]$.
$q \in [1, \infty]$, the slice space $(E^q_\gamma, (\mathbb{R}^n)$ is defined by the set of all measurable functions $f$ such that

$$
\|f\|_{E^q_\gamma, (\mathbb{R}^n)} := \left\| \frac{1}{|B(\cdot, r)|} \int_{B(\cdot, r)} |f(y)|^q \, dy \right\|_{L^q(\mathbb{R}^n)}^{1/q} < \infty,
$$

with the usual modification when $q = \infty$, in fact, when $r = 2, q = p$, that is $E^p_\gamma(\mathbb{R}^n)$ in [1], furthermore, the authors also show the boundedness of some classical operators over these spaces. For more studies and developments about the slice space we may consult [12, 22] and the references therein.

In 1968, Herz [8] introduced Herz spaces in the study of absolutely convergent Fourier transforms. For $\alpha \in \mathbb{R}, p, q \in (0, \infty]$, the homogeneous Herz space $(K^\alpha_q)(\mathbb{R}^n)$ is defined by

$$
(K^\alpha_q)(\mathbb{R}^n) := \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \sum_{k=\infty}^{\infty} 2^{k\alpha p} \| f 1_{S_k} \|_{L^q}^{p} < \infty \right\},
$$

and the non-homogeneous Herz space $(K^\alpha_q)(\mathbb{R}^n)$ is defined by

$$
(K^\alpha_q)(\mathbb{R}^n) := \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n) : \sum_{k=\infty}^{\infty} 2^{k\alpha p} \| f 1_{S_k} \|_{L^q}^{p} < \infty \right\},
$$

but the first study of these spaces was undertaken by Beurling [3]. In the 1990’s, Lu and Yang [14] introduced the preceeding of the homogeneous Herz space and the non-homogeneous Herz space with general indices and established the block decomposition of the Herz space, from which they showed many properties of these spaces. In recent years, a series of papers have paid attention to the study of the Herz-type space, we refer to [11, 15, 16, 20, 21] and so on.

In this paper, the homogeneous Herz-slice space and the non-homogeneous Herz-slice space are introduced, we further show some properties over these spaces, such as the relationship between the homogeneous Herz-slice space and the non-homogeneous Herz-slice space, their dual spaces, a decomposition characterization of these spaces. Moreover, the bounds for the Hardy–Littlewood maximal operator over the homogeneous Herz-slice space and the non-homogeneous Herz-slice space is obtained.

This paper is organized as follows. The definition of Herz-slice spaces, weak Herz-slice spaces and their main remarks will be given in Section 2. In Section 3, we show main properties of Herz-slice spaces. In Section 4, we obtain the dual of Herz-slice spaces and a decomposition characterization of Herz-slice spaces. In the finial section, the boundedness of the Hardy–Littlewood maximal function is given on the homogeneous Herz-slice space $(K^\alpha_q)(\mathbb{R}^n)$ and the non-homogeneous Herz-slice space $(K^\alpha_q)(\mathbb{R}^n)$.

Finally, we make some conventions on notation. Let $B_k = B(0, 2^k) = \{ x \in \mathbb{R}^n : |x| \leq 2^k \}$ and $S_k := B_k \setminus B_{k-1}$ for any $k \in \mathbb{Z}$. Denote $1_k = 1_{S_k}$ for $k \in \mathbb{Z}$, and $1_{S_0} = 1_{B_0}$, where $1_k$ is the characteristic function of $S_k$. We write $A \lesssim B$ to mean that there exists a positive
constant $C$ such that $A \leq CB$. $A \sim B$ denotes that $A \leq B$ and $B \leq A$. Throughout this paper, the letter $C$ will be used for positive constants independent of relevant variables that may change from one occurrence to another.

2 Main definitions

To state the definition of the Herz-slice space, we recall some necessary definitions. For $p \in (0, \infty)$, the Lebesgue space $L^p(\mathbb{R}^n)$ is defined as the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} |f(x)|^p \, dx \right]^{\frac{1}{p}} < \infty.$$ 

The weak Lebesgue space $L^{p,\infty}(\mathbb{R}^n)$ is defined as the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} := \sup_{\alpha > 0} \alpha \left| \left\{ x \in \mathbb{R}^n : |f(x)| > \alpha \right\} \right|^{\frac{1}{p}} < \infty.$$ 

For $p = \infty$,

$$\|f\|_{L^{\infty}(\mathbb{R}^n)} := \text{ess sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$ 

**Definition 2.1.** Let $\alpha \in \mathbb{R}$, $t \in (0, \infty)$, $p \in (0, \infty]$, $r \in (1, \infty)$ and $q \in [1, \infty]$. The homogeneous Herz-slice space $(\dot{K}_E^{\alpha, p}, r)(\mathbb{R}^n)$ is defined as the set of $f \in L^r_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{(\dot{K}_E^{\alpha, p}, r)(\mathbb{R}^n)} := \left[ \sum_{k = -\infty}^{\infty} 2^{k \alpha p} \|f 1_{S_k}\|_{(E^p_q, r)(\mathbb{R}^n)}^p \right]^{\frac{1}{p}} < \infty,$$

with the usual modification when $q, p = \infty$.

The non-homogeneous Herz-slice space $(K_{E^{\alpha, p}}, r)(\mathbb{R}^n)$ is defined as the set of all measurable functions $f$ such that

$$\|f\|_{(K_{E^{\alpha, p}}, r)(\mathbb{R}^n)} := \left[ \sum_{k = 0}^{\infty} 2^{k \alpha p} \|f 1_{S_k}\|_{(E^p_q, r)(\mathbb{R}^n)}^p \right]^{\frac{1}{p}} < \infty,$$

with the usual modification when $q, p = \infty$.

**Remark 2.1.** Let $\alpha \in \mathbb{R}$, $t \in (0, \infty)$, $p \in (0, \infty]$, $r \in (1, \infty)$ and $q \in [1, \infty]$.

(1) For any $s \in (0, \infty)$, we see

$$\|f\|_{(K_{E^{\alpha, p}}, r)(\mathbb{R}^n)}^{\frac{1}{s}} = \|f\|_{(K_{E^{\alpha/s, p}}, r)(\mathbb{R}^n)}^{\frac{1}{s}};$$

(2.1)
(2) \((KE^{α,q}_q)_p(\mathbb{R}^n) = E^q_p(\mathbb{R}^n)_r\), and for \(α \in \mathbb{R}\), 
\((KE^{α,q}_q)_p(\mathbb{R}^n) = (L'_r, L^q_w)_r(\mathbb{R}^n)\) with \(w\) is a power weight \(|x|^α\) [18];

(3) For \(α \in \mathbb{R}\), if \(r = q\), \((KE^{α,q}_q)_p(\mathbb{R}^n) = (K^α_q)_p(\mathbb{R}^n)\), and \((KE^{α,q}_q)_p(\mathbb{R}^n) = (K^α_q)_p(\mathbb{R}^n)\):

(4) \((KE^{α,q}_q)_p(\mathbb{R}^n) = L^q_p(\mathbb{R}^n)\), and for \(α \in \mathbb{R}\), 
\((KE^{α,q}_q)_p(\mathbb{R}^n) = L^q_w(\mathbb{R}^n)\) with \(w = |x|^α\).

**Definition 2.2.** Let \(t \in (0, \infty)\), \(r \in (1, \infty)\) and \(q \in [1, \infty]\), the weak slice space \(W(E^q_r)_r(\mathbb{R}^n)\) is defined as the set of all measurable functions \(f\) such that

\[
\|f\|_{W(E^q_r)_r(\mathbb{R}^n)} := \sup_{λ > 0} \lambda \left\| 1_{\{x \in \mathbb{R}^n : |f(x)| > λ\}} \right\|_{E^q_r(\mathbb{R}^n)} < \infty.
\]

In fact, the weak slice space is the weak amalgam space \(W(L^r_w, L^q_w)_r(\mathbb{R}^n)\) with \(w = v = 1\) in [18].

**Definition 2.3.** Let \(α \in \mathbb{R}\), \(t \in (0, \infty)\), \(p \in (0, \infty)\), \(r \in (1, \infty)\) and \(q \in [1, \infty]\). The homogeneous weak Herz-slice space \((KE^{α,q}_q)_p(\mathbb{R}^n)\) is defined as the set of all measurable functions \(f\) such that

\[
\|f\|_{W(KE^{α,q}_q)_p(\mathbb{R}^n)} := \sup_{λ > 0} \left( 2^{kαp} \left\| 1_{\{x \in \mathbb{R}^n : |f(x)| > λ\}} \right\|_{E^q_r(\mathbb{R}^n)} \right)^{\frac{1}{p}} < \infty.
\]

The non-homogeneous weak Herz-slice space \((KE^{α,q}_q)_p(\mathbb{R}^n)\) is defined as the set of all measurable functions \(f\) such that

\[
\|f\|_{W(KE^{α,q}_q)_p(\mathbb{R}^n)} := \sup_{λ > 0} \left( 2^{kαp} \left\| 1_{\{x \in \mathbb{R}^n : |f(x)| > λ\}} \right\|_{E^q_r(\mathbb{R}^n)} \right)^{\frac{1}{p}} < \infty.
\]

It is obvious that \(W(KE^{0,q}_q)_p(\mathbb{R}^n) = W(KE^{0,q}_q)_p(\mathbb{R}^n) = W(E^p)_p(\mathbb{R}^n)\), and for \(r = q\), \(W(KE^{α,q}_q)_p(\mathbb{R}^n)\) and \(W(KE^{α,q}_q)_p(\mathbb{R}^n)\) are the homogeneous weak Herz space \((K^α_q)_p(\mathbb{R}^n)\) and the non-homogeneous weak Herz space \((K^α_q)_p(\mathbb{R}^n)\) in [17], respectively.

## 3 Properties of Herz-slice spaces

In this section, some main properties of the homogeneous Herz-slice space \((KE^{α,q}_q)_p(\mathbb{R}^n)\) and the non-homogeneous Herz-slice space \((KE^{α,q}_q)_p(\mathbb{R}^n)\) will be given.

**Proposition 3.1.** \((KE^{α,q}_q)_p(\mathbb{R}^n)\) and \((KE^{α,q}_q)_p(\mathbb{R}^n)\) are quasi-Banach function spaces, and if \(p, q, r \geq 1\), \((KE^{α,q}_q)_p(\mathbb{R}^n)\) and \((KE^{α,q}_q)_p(\mathbb{R}^n)\) are Banach spaces.

Since the proof is analogue to that of the classical Herz space, and the details are omitted.
Proposition 3.2. Let $\alpha \in \mathbb{R}$, $p \in (0, \infty]$, $t \in (0, \infty)$, $r \in (1, \infty)$ and $q \in [1, \infty]$. The following inclusions are valid:

1. if $p_1 \leq p_2$, then $(KE_{q,r}^{\alpha,p_1})(\mathbb{R}^n) \subset (KE_{q,r}^{\alpha,p_2})(\mathbb{R}^n)$ and $(KE_{q,r}^{\alpha,p_1})(\mathbb{R}^n) \subset (KE_{q,r}^{\alpha,p_2})(\mathbb{R}^n)$;
2. if $\alpha_2 \leq \alpha_1$, then $(KE_{q,r}^{\alpha_1,p})(\mathbb{R}^n) \subset (KE_{q,r}^{\alpha_2,p})(\mathbb{R}^n)$.

Proof. This proposition can be proved by fairly simple computation. In fact, (1) is a consequence of the inequality in [13]

$$\left(\sum_{k=1}^{\infty} |a_k|^r \right)^{1/r} \leq \sum_{k=1}^{\infty} |a_k|^r, \quad \text{if } 0 < r < 1. \quad (3.1)$$

(2) can be deduced from the Hölder inequality directly. \qed

Proposition 3.3. Let $0 < \alpha, t < \infty$, $p \in (0, \infty]$, $r \in (1, \infty)$ and $q \in [1, \infty]$. Then

$$(KE_{q,r}^{\alpha,p})(\mathbb{R}^n) = (KE_{q,r}^{\alpha,p})(\mathbb{R}^n) \cap (E_{p}^{q})(\mathbb{R}^n),$$

and for $f \in (KE_{q,r}^{\alpha,p})(\mathbb{R}^n) \cap (E_{p}^{q})(\mathbb{R}^n)$,

$$\|f\|_{(KE_{q,r}^{\alpha,p})(\mathbb{R}^n)} \sim \|f\|_{(KE_{q,r}^{\alpha,p})(\mathbb{R}^n)} + \|f\|_{(E_{p}^{q})(\mathbb{R}^n)}.$$

Proof. If $f \in (KE_{q,r}^{\alpha,p})(\mathbb{R}^n) \cap (E_{p}^{q})(\mathbb{R}^n)$, then

$$\|f\|_{(KE_{q,r}^{\alpha,p})(\mathbb{R}^n)}^p = \|f 1_{S_0}\|_{(E_{p}^{q})(\mathbb{R}^n)} + \sum_{k=1}^{\infty} 2^{k\alpha p} \|f 1_{S_k}\|_{(E_{p}^{q})(\mathbb{R}^n)} \leq \|f\|_{(KE_{q,r}^{\alpha,p})(\mathbb{R}^n)}^p + \|f\|_{(E_{p}^{q})(\mathbb{R}^n)}^p.$$

We claim that

$$\sum_{k=-\infty}^{0} 2^{k\alpha p} \|f 1_{S_k}\|_{(E_{p}^{q})(\mathbb{R}^n)}^p \leq C \|f 1_{S_0}\|_{(E_{p}^{q})(\mathbb{R}^n)}^p.$$

If $1 \leq q < p$, by (3.1), then

$$\sum_{k=-\infty}^{0} 2^{k\alpha p} \|f 1_{S_k}\|_{(E_{p}^{q})(\mathbb{R}^n)}^p \leq \left(\sum_{k=-\infty}^{0} 2^{k\alpha p/q}\right)^{p/q} \|f 1_{S_0}\|_{(E_{p}^{q})(\mathbb{R}^n)}^p \leq C \|f 1_{S_0}\|_{(E_{p}^{q})(\mathbb{R}^n)}^p.$$

Hölder’s inequality yields

$$\sum_{k=-\infty}^{0} 2^{k\alpha p} \|f 1_{S_k}\|_{(E_{p}^{q})(\mathbb{R}^n)}^p \leq \left(\sum_{k=-\infty}^{0} 2^{k(p/q)}\right)^{1/(p/q)} \left(\sum_{k=-\infty}^{0} \|f 1_{S_k}\|_{(E_{p}^{q})(\mathbb{R}^n)}^q\right)^{p/q} \leq C \|f 1_{S_0}\|_{(E_{p}^{q})(\mathbb{R}^n)}^p.$$

This proves our claim. From the fact that $\|f 1_{S_0}\|_{(E_{p}^{q})(\mathbb{R}^n)}^p \leq \|f\|_{(KE_{q,r}^{\alpha,p})(\mathbb{R}^n)}$, to obtain $f \in (KE_{q,r}^{\alpha,p})(\mathbb{R}^n)$ deduces $f \in (KE_{q,r}^{\alpha,p})(\mathbb{R}^n) \cap (E_{p}^{q})(\mathbb{R}^n)$, it suffices to prove that

$$\|f 1_{S_0}\|_{(E_{p}^{q})(\mathbb{R}^n)} \leq \|f\|_{(KE_{q,r}^{\alpha,p})(\mathbb{R}^n)}.$$
For $1 \leq q < p$, it follows from the Hölder inequality and (2.1) that
\[
\|f 1_{S_0}\|_{(E^p_q, \lambda)} = \sum_{k=1}^{\infty} \|f 1_{S_k}\|_{(E^p_q, \lambda)} \\
\leq \left( \sum_{k=1}^{\infty} 2^{-\kappa_a p/q} \right)^{1/p} \left( \sum_{k=1}^{\infty} 2^{\kappa_a p} \|f 1_{S_k}\|_{(E^p_q, \lambda)}^{p/q} \right)^{q/p} \\
\leq C\|f\|_{(KE^p_q, \lambda)}^{q/p}.
\]

For $0 < p \leq q$, using (3.1) one can see
\[
\|f 1_{S_0}\|_{(E^p_q, \lambda)}^q \leq \left( \sum_{k=1}^{\infty} \|f 1_{S_k}\|_{(E^p_q, \lambda)}^p \right)^{q/p} \leq \|f\|_{(KE^p_q, \lambda)}^{q/p}.
\]

This completes the proof of proposition 3.3. \hfill \Box

4 Block decompositions and dual spaces

This section is devoted to the decomposition characterizations and the dual spaces for Herz-slice spaces as given in Definition 2.1.

**Definition 4.1.** Let $0 < \alpha, t < \infty$, $1 < r < \infty$ and $1 \leq q < \infty$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q, r)$-block if

1. $\text{supp}(a) \subset B(0, R)$, for some $R > 0$;
2. $\|a\|_{(E^p_q, \lambda)} \leq CR^{-\alpha}$.

(ii) A function $b(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q, r)$-block of restrict type if

1. $\text{supp}(b) \subset B(0, R)$ for some $R \geq 1$;
2. $\|b\|_{(E^p_q, \lambda)} \leq CR^{-\alpha}$.

If $R = 2^k$ for some $k \in \mathbb{Z}$, then the corresponding central block is called a dyadic central block.

**Theorem 4.1.** Let $0 < \alpha, t, p < \infty$, $1 < r < \infty$ and $1 \leq q < \infty$. The following statements are equivalent:

1. $f \in (KE^p_q, \lambda)(\mathbb{R}^n)$.
2. $f$ can be represented by
\[
f(x) = \sum_{k \in \mathbb{Z}} \lambda_k b_k(x), \tag{4.1}
\]
where $\sum_{k \in \mathbb{Z}} |\lambda_k|^p < \infty$ and each $b_k$ is a dyadic central $(\alpha, q, r)$-block with support contained in $B_k$. 
Proof. We suppose initially that \( f \in (\mathcal{KE}_{q,p}^α,\mathcal{R}^n) \), write

\[
f(x) = \sum_k f(x)1_{S_k}(x) = \sum_k |B_k|^{\alpha/n} \left\| f1_{S_k}\right\|_{(E_j^p)_{\mathcal{R}^n}} = \sum_k \lambda_k b_k(x),
\]

where

\[
\lambda_k = |B_k|^{\alpha/n} \left\| f1_{S_k}\right\|_{(E_j^p)_{\mathcal{R}^n}} \quad \text{and} \quad b_k(x) = \frac{f(x)1_{S_k}(x)}{|B_k|^{\alpha/n} \left\| f1_{S_k}\right\|_{(E_j^p)_{\mathcal{R}^n}}}.
\]

 Obviously, supp \((b_k) \subset B_k\), and \(||b_k||_{(E_j^p)_{\mathcal{R}^n}} = |B_k|^{-\alpha/n}\). Thus, each \(b_k\) is a dyadic central \((\alpha, q, r)\)-block with the support \(B_k\) and

\[
\sum |\lambda_k|^p = \sum \left( |B_k|^{\alpha/n} \left\| f1_{S_k}\right\|_{(E_j^p)_{\mathcal{R}^n}} \right)^p = \sum |B_k|^{\alpha/n} \left\| f1_{S_k}\right\|^p_{(E_j^p)_{\mathcal{R}^n}} = \|f\|^p_{(\mathcal{KE}_{q,p}^α,\mathcal{R}^n)} < \infty.
\]

(2) \(\Rightarrow\) (1). Let \(f(x) = \sum k \lambda_k b_k(x)\) be a decomposition of \(f\) which satisfies the hypothesis (2) of Theorem 4.1. For each \(j \in \mathbb{Z}\), it is readily to see that

\[
\left\| f1_{S_j}\right\|_{(E_j^p)_{\mathcal{R}^n}} \leq \sum_{k \geq j} |\lambda_k| ||b_k||_{(E_j^p)_{\mathcal{R}^n}}. \tag{4.2}
\]

For \(0 < p \leq 1\), from (4.2), it follows that

\[
\|f\|^p_{(\mathcal{KE}_{q,p}^α,\mathcal{R}^n)} = \sum_{k \in \mathbb{Z}} 2^{kp} \left\| f1_{S_k}\right\|^p_{(E_j^p)_{\mathcal{R}^n}} \leq \sum_{k \in \mathbb{Z}} 2^{kp} \left( \sum_{j \geq k} |\lambda_k|^p ||b_j||^p_{(E_j^p)_{\mathcal{R}^n}} \right) \leq \sum_{k \in \mathbb{Z}} 2^{kp} \left( \sum_{j \geq k} |\lambda_k|^p \right) \leq C \sum_{k \in \mathbb{Z}} |\lambda_k|^p < \infty.
\]

For the case of \(1 < p < \infty\), again by (4.2) and the Hölder inequality,

\[
\left\| f1_{S_j}\right\|_{(E_j^p)_{\mathcal{R}^n}} \leq \left( \sum_{k \geq j} |\lambda_k|^p ||b_k||^\frac{p}{2}_{(E_j^p)_{\mathcal{R}^n}} \right)^\frac{1}{2} \left( \sum_{k \geq j} ||b_k||^\frac{p}{2}_{(E_j^p)_{\mathcal{R}^n}} \right)^\frac{1}{2} \leq C \sum_{k \geq j} |\lambda_k|^p 2^{-\frac{akp}{2}j} \left( \sum_{k \geq j} 2^{-\frac{akp}{2}} \right)^\frac{1}{p}.
\]

Therefore,

\[
\|f\|^p_{(\mathcal{KE}_{q,p}^α,\mathcal{R}^n)} \leq C \sum_{j \in \mathbb{Z}} 2^{ajp} \left( \sum_{k \geq j} |\lambda_k|^p 2^{-\frac{akp}{2}j} \right)^\frac{1}{p} \left( \sum_{k \geq j} 2^{-\frac{akp}{2}} \right)^\frac{1}{p} \leq C \sum_{j \in \mathbb{Z}} |\lambda_k|^p \sum_{j \leq k} 2^{\alpha(j-k)p/2} \leq C \sum_{k \in \mathbb{Z}} |\lambda_k|^p < \infty.
\]

This leads to that \(f \in (\mathcal{KE}_{q,p}^α,\mathcal{R}^n)\) and then completes the proof of Theorem 4.1. \(\square\)
**Remark 4.1.** From the proof of Theorem 4.1, it’s easy to see that

$$
\|f\|_{(KE_{\alpha}^0,p)_t(\mathbb{R}^n)} \sim \left( \sum_{k \in \mathbb{Z}} |\lambda_k|^p \right)^{\frac{1}{p}}.
$$

Similarly for the decompositional characterizations of the homogeneous Herz-slice space, it shows that the decompositional over the non-homogeneous Herz-slice spaces as follows.

**Theorem 4.2.** Let $0 < \alpha$, $t$, $p \in (0, \infty)$, $1 < r < \infty$ and $1 \leq q < \infty$. The following statements are equivalent:

1. $f \in (KE_{q,t}^{\alpha,p})(\mathbb{R}^n)$.
2. $f$ can be represented by

$$
 f(x) = \sum_{k=0}^{\infty} \lambda_k b_k(x),
$$

where $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$ and each $b_k$ is a dyadic central $(\alpha, q, r)$-block of restrict type with support contained in $B_k$. Moreover,

$$
\|f\|_{(KE_{\alpha}^0,p)_t(\mathbb{R}^n)} \sim \left( \sum_{k \in \mathbb{Z}} |\lambda_k|^p \right)^{\frac{1}{p}}.
$$

For stating the dual spaces of Herz-slice spaces, we recall the Hölder inequality over the slice space in the following.

**Lemma 4.1.** [10] Given $1 \leq q \leq \infty$ and $1 < r < \infty$,

$$
\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{(KE_{\alpha}^0,p)_t(\mathbb{R}^n)} \|g\|_{(KE_{q,r}^{\alpha,p})'_t(\mathbb{R}^n)},
$$

where $\frac{1}{q} + \frac{1}{q'} = \frac{1}{r} + \frac{1}{r'} = 1$.

**Theorem 4.3.** Let $\alpha \in \mathbb{R}$, $t$, $p \in (0, \infty)$, $r \in (1, \infty)$ and $q \in [1, \infty)$. The dual space of $(KE_{q,r}^{\alpha,p})_t(\mathbb{R}^n)$ is

$$
\left((KE_{q,r}^{\alpha,p})_t(\mathbb{R}^n)\right)^* = \begin{cases} 
(KE_{q,r}^{\alpha,p'})_t(\mathbb{R}^n), & 1 < p < \infty; \\
(KE_{q,r}^{\alpha,0})_t(\mathbb{R}^n), & 0 < p \leq 1.
\end{cases}
$$

**Proof.** We first prove it for the case $1 < p < \infty$. Let $T \in (KE_{q,r}^{\alpha,p'})_t(\mathbb{R}^n)$. For any $f \in (KE_{q,r}^{\alpha,p})_t(\mathbb{R}^n)$, we define

$$
(T, f) := \int_{\mathbb{R}^n} T(x)f(x)\,dx = \sum_{k=-\infty}^{\infty} \int_{S_k} T(x)f(x)\,dx.
$$
By Lemma 4.1, we have
\[
|\langle T, f \rangle| \leq \sum_{k=-\infty}^{\infty} 2^{-k\alpha} \|T\|\|f\|_{(E_{\alpha}^{\sigma},(\mathbb{R}^{\nu}))} \leq \|T\|_{((KE_{\alpha}^{\sigma,p},(\mathbb{R}^{\nu}))^{*})} \|f\|_{((KE_{\alpha}^{\sigma,p},(\mathbb{R}^{\nu}))^{*})}.
\]
This shows that \( T \in ((KE_{\alpha}^{\sigma,p},(\mathbb{R}^{\nu}))^{*})^{*}. \)

Let \( T \in ((KE_{\alpha}^{\sigma,p},(\mathbb{R}^{\nu}))^{*})^{*}. \) For any \( k \in \mathbb{Z} \) and \( f_k \in (E_{\alpha}^{\sigma},(\mathbb{R}^{\nu})) \), we define \( f_k := f_k 1_{S_k} \), then \( f_k \in (KE_{\alpha}^{\sigma,p},(\mathbb{R}^{\nu}))^{*} \) and \( \|f_k\|_{(KE_{\alpha}^{\sigma,p},(\mathbb{R}^{\nu}))^{*}} = 2^{k\alpha} \|f_k\|_{(E_{\alpha}^{\sigma},(\mathbb{R}^{\nu}))}. \) Let \( (T_k, f_k) := (T, f_k). \) It is easy to check that \( T_k \in (E_{\alpha}^{\sigma},(\mathbb{R}^{\nu}))^{*} \) and \( \|T_k\|_{(E_{\alpha}^{\sigma},(\mathbb{R}^{\nu}))} \leq 2^{k\alpha} \|T\|_{((KE_{\alpha}^{\sigma,p},(\mathbb{R}^{\nu}))^{*})}. \) Now, for any \( f \in (KE_{\alpha}^{\sigma,p},(\mathbb{R}^{\nu}))^{*} \), we have \( f 1_{S_k} \in (E_{\alpha}^{\sigma},(\mathbb{R}^{\nu}))^{*}. \) Then, for any given \( N, M \in \mathbb{N}, \)
\[
\sum_{k=-N}^{M} (T_k, f 1_{S_k}) = \left( T, \sum_{k=-N}^{M} f 1_{S_k} \right) \cdot
\]
For any \( k \in \mathbb{Z}, \) choose \( f_k^{*} \in (E_{\nu}^{\sigma},(\mathbb{R}^{\nu})) \) with \( \text{supp} (f_k) \subset S_k \) and \( \|f_k^{*}\|_{(E_{\nu}^{\sigma},(\mathbb{R}^{\nu}))} = 2^{-k\alpha} \) such that
\[
(T_k, f_k^{*}) \geq 2^{-k\alpha} \|T_k\|_{(E_{\nu}^{\sigma},(\mathbb{R}^{\nu}))} - \varepsilon_k,
\]
where \( \varepsilon_k > 0 \) is determined later. Let
\[
f_k := \left( 2^{-k\alpha} \|T_k\|_{(E_{\nu}^{\sigma},(\mathbb{R}^{\nu}))} ^{\nu-1} \right) f_k^{*}.
\]
For any given \( \varepsilon \in (0, \infty), \) select \( \varepsilon_k > 0 \) small enough such that
\[
(T_k, f_k) + 2^{-|k|} \varepsilon \geq 2^{-k\alpha} \|T_k\|_{(E_{\nu}^{\sigma},(\mathbb{R}^{\nu}))}^{\nu} = 2^{k\alpha} \|f_k\|_{(E_{\nu}^{\sigma},(\mathbb{R}^{\nu}))}^{\nu}.
\]
Then we obtain that
\[
\sum_{k=-N}^{M} 2^{-k\alpha} \|T_k 1_{S_k}\|_{(E_{\nu}^{\sigma},(\mathbb{R}^{\nu}))}^{\nu} \leq 4\varepsilon + \left( T, \sum_{k=-N}^{M} f_k 1_{S_k} \right) \cdot
\]
\[
\leq 4\varepsilon + \|T\|_{((KE_{\alpha}^{\sigma,p}(\mathbb{R}^{\nu}))^{*})} \|f_k\|_{(KE_{\alpha}^{\sigma,p}(\mathbb{R}^{\nu}))^{*}} \cdot
\]
\[
\leq 4\varepsilon + \|T\|_{((KE_{\alpha}^{\sigma,p}(\mathbb{R}^{\nu}))^{*})} \left( \sum_{k=-N}^{M} 2^{k\alpha} \|f_k 1_{S_k}\|_{(E_{\nu}^{\sigma},(\mathbb{R}^{\nu}))}^{\nu} \right) ^{\frac{1}{\nu}} \cdot
\]
\[
= 4\varepsilon + \|T\|_{((KE_{\alpha}^{\sigma,p}(\mathbb{R}^{\nu}))^{*})} \left( \sum_{k=-N}^{M} 2^{-k\alpha} \|T_k 1_{S_k}\|_{(E_{\nu}^{\sigma},(\mathbb{R}^{\nu}))}^{\nu} \right) ^{\frac{1}{\nu}} .
\]
Letting $\varepsilon \to 0$ and $N, M \to \infty$, we have
\[
\left\| \sum_{k=-\infty}^{\infty} 2^{-kq'} \| T_k 1_{S_k} \|_{(KE_{q,r})_t((\mathbb{R}^n))}^{q'} \right\|^{\frac{1}{q'}} \leq \| T \|_{((KE_{q,r})_t((\mathbb{R}^n))}^{\frac{1}{r'}}. \tag{4.4}
\]

Then we define
\[
\widetilde{T}(x) := \sum_{k=-\infty}^{\infty} T_k(x) 1_{S_k}(x).
\]

It follows from (4.4) that $\widetilde{T} \in (KE_{q,r})_t((\mathbb{R}^n))$. Moreover, for any $f \in (KE_{q,r})_t((\mathbb{R}^n))$, it can see that
\[
(\widetilde{T}, f) = \sum_{k=-\infty}^{\infty} T_k(x) 1_{S_k}(x) f(x) \, dx = \sum_{k=-\infty}^{\infty} \int_{S_k} T_k(x) f(x) \, dx = \sum_{k=-\infty}^{\infty} (T, f 1_{S_k}) = (T, f).
\]

This shows that $T$ is also in $(KE_{q,r})_t((\mathbb{R}^n))$. Hence we prove the result for the case $1 < p < \infty$. For the case $0 < p \leq 1$, the proof is similar and omitted. \hfill \Box

By the closed-graph theorem, we easily get the following corollary.

**Corollary 4.1.** Let $\alpha \in \mathbb{R}$, $t \in (0, \infty)$, $p, q \in [1, \infty)$, $r \in (1, \infty)$ and $1/p + 1/p' = 1/r + 1/r' = 1/q + 1/q' = 1$. Then $f \in (KE_{q,r})_t((\mathbb{R}^n))$ if and only if
\[
\int_{\mathbb{R}^n} f(x) g(x) \, dx < \infty,
\]
where $g \in (KE_{q,r})_t((\mathbb{R}^n))$, and
\[
\|f\|_{(KE_{q,r})_t((\mathbb{R}^n))} = \sup \left\{ \int_{\mathbb{R}^n} f(x) g(x) \, dx : \|g\|_{(KE_{q,r})_t((\mathbb{R}^n))} \leq 1 \right\}.
\]

A similar result for the non-homogeneous Herz-slice space is given as follows, and hence the details are omitted.

**Theorem 4.4.** Let $\alpha \in \mathbb{R}$, $t \in (0, \infty)$, $p \in (0, \infty)$, $r \in (1, \infty)$ and $q \in [1, \infty)$. The dual space of $(KE_{q,r})_t((\mathbb{R}^n))$ is
\[
\left( (KE_{q,r})_t((\mathbb{R}^n)) \right)^* = \begin{cases} 
(KE_{q,r'})_t((\mathbb{R}^n)), & 1 < p < \infty; \\
(KE_{q,r'})_t((\mathbb{R}^n)), & 0 < p \leq 1.
\end{cases}
\]

Furthermore, if $\alpha \in \mathbb{R}$, $t \in (0, \infty)$, $p, q \in [1, \infty)$, $r \in (1, \infty)$ and $1/p + 1/p' = 1/r + 1/r' = 1/q + 1/q' = 1$. Then $f \in (KE_{q,r})_t((\mathbb{R}^n))$ if and only if
\[
\left| \int_{\mathbb{R}^n} f(x) g(x) \, dx \right| < \infty,
\]
where $g \in (KE_{q,r'})_t((\mathbb{R}^n))$, and
\[
\|f\|_{(KE_{q,r})_t((\mathbb{R}^n))} = \sup \left\{ \left| \int_{\mathbb{R}^n} f(x) g(x) \, dx \right| : \|g\|_{(KE_{q,r'})_t((\mathbb{R}^n))} \leq 1 \right\}.
\]
5 Maximal functions on Herz-slice spaces

In this section, we obtain the bounds for the Hardy–Littlewood maximal operator over Herz-slice spaces. We begin with the definition of the Hardy–Littlewood maximal operator.

For a locally integrable function $f$, the Hardy–Littlewood maximal operator is defined by setting, for almost every $x \in \mathbb{R}^n$,

$$Mf(x) := \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy.$$ 

Theorem 5.1. Let $t$, $p \in (0, \infty)$, $q \in [1, \infty)$, $r \in (1, \infty)$ and $-n/q < \alpha < n(1 - 1/q)$. Then the Hardy–Littlewood maximal function is bounded on $(KE_{\alpha}^{q, p}, \mathbb{R}^n)$.

Lemma 5.1. [2, 18] Let $t \in (0, \infty)$, $r \in (1, \infty)$.

1. If $1 < q < \infty$, the Hardy–Littlewood maximal function $M$ is bounded on $(E_r^q, \mathbb{R}^n)$.

2. If $q = 1$, $M$ is bounded from $(E_r^q, \mathbb{R}^n)$ to $W(E_r^q, \mathbb{R}^n)$.

Lemma 5.2. Let $t \in (0, \infty)$, $r \in (1, \infty)$, for all $x \in \mathbb{R}^n$, $1 < r_0 < \infty$, a characteristic function on $B(x_0, r_0)$ satisfies

$$\|1_{B(x_0, r_0)}\|_{(E_r^q), \mathbb{R}^n} \leq Cr_0^{n/q}.$$ 

Proof. For any $x \in \mathbb{R}^n$, we have

$$\|1_{B(x_0, r_0)}\|_{(E_r^q), \mathbb{R}^n} = \left\{ \int_{\mathbb{R}^n} \left[ \|1_{B(x_0, r_0)}1_{B(x, r)}\|_{L^q} \right]^q \, dx \right\}^{1/q}$$

$$= \frac{1}{\|1_{B(0, r)}\|_{L^q(\mathbb{R}^n)}} \left[ \int_{\mathbb{R}^n} \|1_{B(0, r)}1_{B(x-x_0, r)}\|_{L^q} \, dx \right]^{1/q}$$

$$= \frac{1}{\|1_{B(0, r)}\|_{L^q(\mathbb{R}^n)}} \left[ \int_{\mathbb{R}^n} \|1_{B(0, r)}1_{B(x, r)}\|_{L^q} \, dx \right]^{1/q}$$

By this, without loss of generality, suppose that $B_1 := B(\vec{0}, 1)$ and $B_2 := B(\vec{0}, r_0)$ with $1 < r_0 < \infty$. By the geometric property, we know that there exist $M \in \mathbb{N}$ with $M \sim |B_2|^n$ and $\{x_1, \ldots, x_M\}$ such that $B(\vec{0}, r_0) \subset \bigcup_{j=1}^M B(x_j, 1)$, which implies that

$$\|1_{B(\vec{0}, r_0)}\|_{(E_r^q), \mathbb{R}^n} = \|1_{B_2}\|_{(E_r^q), \mathbb{R}^n}$$

$$\leq \sum_{j=1}^M \|1_{B(x_j, 1)}\|_{(E_r^q), \mathbb{R}^n}^q \leq \left( \sum_{j=1}^M \|1_{B(x_j, 1)}\|_{(E_r^q), \mathbb{R}^n}^q \right)^{1/q}$$

$$\sim |B_2|^q \|1_{B(\vec{0}, 1)}\|_{(E_r^q), \mathbb{R}^n} \sim r_0^{n/q}.$$
This completes the proof.

□

**Remark 5.1.** Let \( t \in (0, \infty) \), \( r, q \in (1, \infty) \), \( k \in \mathbb{Z} \), a characteristic function on \( S_k \) satisfies

\[
\|1_{S_k}\|_{L^q(\mathbb{R}^n)} \leq \|1_{B_k}\|_{L^q(\mathbb{R}^n)} \leq C 2^{kn/q}.
\]

**Proof of Theorem 5.1.** Write

\[
f(x) = \sum_{k \in \mathbb{Z}} f(x) 1_{S_k}(x) := \sum_{k \in \mathbb{Z}} f_k(x).
\]

For \( q \in (1, \infty) \), we get

\[
\|MF\|_{(KE_{q,R}^p)^c(\mathbb{R}^n)} = \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \|MF 1_{S_k}\|_{L^q(\mathbb{R}^n)}^p \right)^{\frac{1}{p}} = \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \sum_{l=-\infty}^\infty \|MF 1_{S_l}\|_{L^q(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=-k}^{k-2} \|MF 1_{S_l}\|_{L^q(\mathbb{R}^n)}^p \right) \right)^{\frac{1}{p}} + \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=k+1}^{\infty} \|MF 1_{S_l}\|_{L^q(\mathbb{R}^n)}^p \right) \right)^{\frac{1}{p}}
\]

\[
+ \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=-\infty}^{\infty} \|MF 1_{S_l}\|_{L^q(\mathbb{R}^n)}^p \right) \right)^{\frac{1}{p}} := I + II + III.
\]

For \( p \in (0, 1] \), applying the fact that for \( l \leq k - 2 \) and \( x \in S_k \), \(|MF 1_{S_l}(x)| \leq C 2^{-k\alpha} \|f\|_{L^1(\mathbb{R}^n)}\), and using Lemmas 4.1 and 5.2 one can deduce that

\[
I = \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=-\infty}^{k-2} \|f\|_{L^1(\mathbb{R}^n)} 2^{-kn} 1_{S_k} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=-\infty}^{k-2} \|f\|_{L^1(\mathbb{R}^n)} 2^{-kn} 1_{S_k} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=-\infty}^{k-2} \|f\|_{L^1(\mathbb{R}^n)} 2^{-kn} 1_{S_k} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=-\infty}^{k-2} \|f\|_{L^1(\mathbb{R}^n)} 2^{-kn} 1_{S_k} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=-\infty}^{k-2} \|f\|_{L^1(\mathbb{R}^n)} 2^{-kn} 1_{S_k} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=-\infty}^{k-2} \|f\|_{L^1(\mathbb{R}^n)} 2^{-kn} 1_{S_k} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha} \left( \sum_{l=-\infty}^{k-2} \|f\|_{L^1(\mathbb{R}^n)} 2^{-kn} 1_{S_k} \right) \right)^{\frac{1}{p}}
\]
and for $p \in (1, \infty)$,

\[
I = \left( \sum_{k \in \mathbb{Z}} 2^{kap} \left( \sum_{l=-\infty}^{k-2} \| f \|_p (E^l_{1,\mathbb{R}^n}) 2^{(k-l)p(1/q-1)n} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{kap} \left( \sum_{l=-\infty}^{k-2} \| f \|_p (E^l_{1,\mathbb{R}^n}) \|1_S \|_{(E^l_{1,\mathbb{R}^n})} 2^{-kn} 1_{S_k} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{kap} \left( \sum_{l=-\infty}^{k-2} \| f \|_p (E^l_{1,\mathbb{R}^n}) \|1_{B_k} \|_{(E^l_{1,\mathbb{R}^n})} 2^{-kn} 1_{B_k} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{kap} \left( \sum_{l=-\infty}^{k-2} \| f \|_p (E^l_{1,\mathbb{R}^n}) 2^{(k-l)p(1/q-1)n} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{kap} \left( \sum_{l=-\infty}^{k-2} \| f \|_p (E^l_{1,\mathbb{R}^n}) 2^{(k-l)(1/q-1)n} \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} 2^{kap} \left( \sum_{l=-\infty}^{k-2} \| f \|_p (E^l_{1,\mathbb{R}^n}) \left( \sum_{l=-\infty}^{k-2} 2^{(k-l)(1/q-1)n} \right)^{p/p'} \right)^{\frac{1}{p'}} \right)^{\frac{1}{p}}
\]

\[
\leq \| f \|_p (KE^0_{+1,\mathbb{R}^n}).
\]
Using Lemma 5.1, we obtain the following estimate for $II$:

$$II \lesssim \left( \sum_{k \in \mathbb{Z}} 2^{k \alpha p} \left\| \sum_{l=k+1}^{k+1} M f_l \right\|_{L^p(E_n^t)(\mathbb{R}^n)} \right)^{\frac{1}{p}} \lesssim \left( \sum_{k \in \mathbb{Z}} \sum_{l=k-1}^{k+1} 2^{(l-k) \alpha p} 2^{l \alpha p} \| f_l \|_{L^p(E_n^t)(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}$$

$$\lesssim \left( \sum_{l \in \mathbb{Z}} 2^{l \alpha p} \| f_l \|_{L^p(E_n^t)(\mathbb{R}^n)} \right)^{\frac{1}{p}} \lesssim \| f \|_{(KE_n^t)_{q, s}(\mathbb{R}^n)}.$$

We then prove $III$. Note that if $l \geq k + 2$, then

$$|M(f_l)(x)| \mathbf{1}_{S_k} \leq C 2^{-nl} \| f_l \|_{L^1(\mathbb{R}^n)}.$$

For $0 < p \leq 1$, (3.1) and Lemma 4.1 yield

$$III \lesssim \left( \sum_{k \in \mathbb{Z}} 2^{k \alpha p} \left\| \sum_{l=k+1}^{\infty} M f_l \mathbf{1}_{S_k} \right\|_{L^p(E_n^t)(\mathbb{R}^n)} \right)^{\frac{1}{p}} \lesssim \left( \sum_{k \in \mathbb{Z}} \sum_{l=k+2}^{\infty} 2^{-nl} \| f_l \|_{L^1(\mathbb{R}^n)} \mathbf{1}_{S_k} \left\| \mathbf{1}_{B_l} \right\|_{L^p(E_n^t)(\mathbb{R}^n)} \right)^{\frac{1}{p}}$$

$$\lesssim \left( \sum_{k \in \mathbb{Z}} \sum_{l=k+2}^{\infty} 2^{l \alpha p} \left\| f_l \right\|_{L^p(E_n^t)(\mathbb{R}^n)} \left\| \mathbf{1}_{B_l} \right\|_{L^p(E_n^t)(\mathbb{R}^n)} \right)^{\frac{1}{p}} \lesssim \left( \sum_{k \in \mathbb{Z}} \sum_{l=k+2}^{\infty} 2^{l \alpha p} \left\| f_l \right\|_{L^p(E_n^t)(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$
and by Lemma 4.1 and Hölder’s inequality for $1 < p < \infty$,

$$III \leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left( \sum_{l=k+2}^{\infty} \left\| M f_l 1_{S_l} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{p}} \right)^{\frac{1}{\beta}}$$

$$\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left( \sum_{l=k+2}^{\infty} \left\| f_l \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)} \left\| 1_{S_l} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)} 2^{-ln} 1_{S_k} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}$$

$$\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left( \sum_{l=k+2}^{\infty} \left\| f_l \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)} \left\| 1_{S_l} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)} 2^{-ln} 1_{S_k} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}$$

$$\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left( \sum_{l=k+2}^{\infty} \left\| f_l \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)} \left\| 1_{S_l} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)} 2^{-ln} 1_{S_k} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}$$

$$\leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left( \sum_{l=k+2}^{\infty} \left\| f_l \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)} \left\| 1_{S_l} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)} 2^{-ln} 1_{S_k} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}$$

For $q = 1$, for any $\lambda > 0$,

$$\lambda \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left\| 1_{\{x \in S_k : |M f_l f_l(\mathbb{R}^n)}^p \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{\beta}} \leq \lambda \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left\| 1_{\{x \in S_k : M l_{S_l} f_l f_l(x) > \lambda/3 \}} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{\beta}}$$

$$+ \lambda \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left\| 1_{\{x \in S_k : M l_{S_l} f_l f_l(x) > \lambda/3 \}} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{\beta}}$$

$$+ \lambda \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left\| 1_{\{x \in S_k : M l_{S_l} f_l f_l(x) > \lambda/3 \}} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{\beta}}$$

$$:= I + II + III.$$

Since $M$ is bounded from $(E_{r}^{\alpha})(\mathbb{R}^n)$ to $W(E_{r}^{\alpha})(\mathbb{R}^n)$ in Lemma 5.1, then

$$II \leq \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left( \sum_{l=k+2}^{\infty} \left\| f_l 1_{S_l} \right\|_{(E_{0}^{\alpha})(\mathbb{R}^n)}^p \right)^{\frac{1}{\beta}} \leq \left\| f \right\|_{(K E_{r}^{\alpha})(\mathbb{R}^n)}.$$
The \((KE^{\alpha,p}_{q,r})(\mathbb{R}^n))\) boundedness of the Hardy–Littlewood maximal function \(M\) is valid in the following, via a similar argument to that given in the proof of Theorem 5.1.

**Theorem 5.2.** Let \(t, p \in (0, \infty), q \in [1, \infty), r \in (1, \infty)\) and \(-n/q < \alpha < n(1 - 1/q)\). Then the Hardy–Littlewood maximal function is bounded on \((KE^{\alpha,p}_{q,r})(\mathbb{R}^n))\).

**Competing interests**
The authors declare that they have no competing interests.

**Funding**
The research was supported by the National Natural Science Foundation of China (Grant No. 12061069) and the Natural Science Foundation Project of Chongqing, China (Grant No. cstc2021jcyj-msxmX0705).

**Authors contributions**
All authors contributed equality and significantly in writing this paper. All authors read and approved the final manuscript.

**Acknowledgments**
All authors would like to express their thanks to the referees for valuable advice regarding previous version of this paper.

**Authors details**
Yuan Lu(Luyuan_y@163.com) and Jiang Zhou(zhoujiang@xju.edu.cn), College of Mathematics and System Science, Xinjiang University, Urumqi, 830046, P.R China.
Songbai Wang(haiyansongbai@163.com), College of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing 404130, P.R China.

**References**

[1] P. Auscher, M. Mourougoglou, Representation and uniqueness for boundary value elliptic problems via first order systems, Rev. Mat. Iberoam, 35: 241-315 (2014).

[2] P. Auscher, C. Prisuelos-Arribas, Tent space boundedness via extrapolation, Math. Z., 286: 1575-1604 (2017).

[3] A. Beurling, Construction and analysis of some convolution algebras, Ann I Fourier, 14:1-32 (1964).

[4] H.G. Feichtinger, A characterization of minimal homogeneous Banach spaces, Proc. Am. Math. Soc., 81: 55-61 (1981).
[5] H.G. Feichtinger, Banach spaces of distributions of Wiener’s type and interpolation, In: Functional Analysis and Approximation (Oberwolfach 1980), Internat. Ser. Numer. Math. 60, Birkhauser, Basel-Boston, 153-165 (1981).

[6] H.G. Feichtinger, Banach convolution algebras of Wiener type, In: Functions, Series, Operators, Proc. Int. Conf., Budapest 1980, Vol. I, Colloq. Math. Soc. Janos Bolyai, 35: 509-524 (1983).

[7] H.G. Feichtinger, An elementary approach to Wiener’s third Tauberian theorem for Euclidean n-spaces, Proc. of Conf. at Cortona, 1984, Symposia Math. Vol. 29, New York, Academic Press, 267-301 (1987).

[8] C.S. Herz, Lipschitz spaces and Bernstein’s theorem on absolutely convergent Fourier series, J. Math. Mech., 18:283-324 (1968).

[9] F. Holland, Harmonic analysis on amalgams of $L^p$ and $\ell^q$, J. Lond Math Soc, 10(2): 295-305 (1975).

[10] C. Heil, Wiener amalgam spaces in generalized harmonic analysis and wavelet theory, Ph.D. thesis, University of Maryland, College Park, MD (1990).

[11] E. Hernández, D.C. Yang, Interpolation of Herz spaces and applications, Math. Nachr., 205: 69-87 (1999).

[12] K.P. Ho, Operators on Orlicz-slice spaces and Orlicz-slice Hardy spaces, J. Math. Anal. Appl., 503, 125279 (2021).

[13] J.C. Kuang, Applied Inequalities (4rd ed), Shangdong Science Press, Jinan, (2010).

[14] S.Z. Lu, D.C. Yang, Decomposition of weighted Herz spaces and its applications. Science of China, Ser. A., 38: 147-158 (1995).

[15] S.Z. Lu, Herz type spaces, Adv Math, 33: 257-272 (2004).

[16] S.Z. Lu, L.F. Xu, Boundedness of rough singular integral operators on the homogeneous Morrey-Herz spaces, Hokkaido Math J., 34(2): 299-314 (2005).

[17] S.Z. Lu, D.C. Yang, G.E. Hu, Herz type Spaces and Their Applications, Science Press, Beijing, China, (2008).

[18] Y. Lu, S.B. Wang, J. Zhou, Boundedness of some operators on weighted amalgam spaces, arXiv:2110.01193 (2021).

[19] N. Wiener, On the representation of functions by trigonometrical integrals, Math. Z., 24(1): 575-16 (1926).
[20] H.B.Wang, Weak type estimates of commutators on Herz type spaces with variable exponent, Appl Anal, 2020: 1-20 (2020).

[21] M.Q.Wei, A characterization of via the commutator of Hardy-type operators on mixed Herz spaces, Appl Anal, 2021(3): 1-16 (2021).

[22] Y.Y.Zhang, D.C.Yang, W.Yuan, S.B.Wang, Real-Variable Characterizations of Orlicz-Slice Hardy Spaces, Anal. Appl., 17: 597-664 (2019).

Yuan Lu  
College of Mathematics and System Sciences  
Xinjiang University  
Urumqi 830046, China  
Email address: Luyuan_y@163.com

Jiang Zhou  
College of Mathematics and System Sciences  
Xinjiang University  
Urumqi 830046, China  
Email address: zhoujiang@xju.edu.cn

Songbai Wang  
Chongqing Three Gorges University  
Chongqing 404130, China  
Email address: haiyansongbai@163.com