Minimum 2-edge strongly biconnected spanning directed subgraph problem

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Abstract

Wu and Grumbach introduced the concept of strongly biconnected directed graphs. A directed graph $G = (V, E)$ is called strongly biconnected if the directed graph $G$ is strongly connected and the underlying undirected graph of $G$ is biconnected. A strongly biconnected directed graph $G = (V, E)$ is said to be 2-edge-strongly biconnected if it has at least three vertices and the directed subgraph $(V, E \setminus \{e\})$ is strongly biconnected for all $e \in E$. Let $G = (V, E)$ be a 2-edge-strongly biconnected directed graph. In this paper we study the problem of computing a minimum size subset $H \subseteq E$ such that the directed subgraph $(V, H)$ is 2-edge-strongly biconnected.

Keywords: Directed graphs, Connectivity, Approximation algorithms, Graph algorithms, strongly connected graphs

1. Introduction

Wu and Grumbach [26] introduced the concept of strongly biconnected directed graphs. A directed graph $G = (V, E)$ is called strongly biconnected if the directed graph $G$ is strongly connected and the underlying undirected graph of $G$ is biconnected. An edge $e$ in a strongly biconnected directed graph $G = (V, E)$ is b-bridge if the subgraph $(V, E \setminus \{e\})$ is not strongly biconnected. A strongly biconnected directed graph $G = (V, E)$ is called $k$-edge-strongly biconnected if for each $L \subseteq E$ with $|L| < k$, the subgraph $(V, E \setminus L)$ is strongly biconnected. A strongly biconnected directed graph $G = (V, E)$ is therefore 2-edge-strongly biconnected if and only if it has no b-bridges. Given a $k$-edge-strongly biconnected directed graph $G = (V, E)$, the minimum $k$-edge-strongly biconnected spanning subgraph problem (denoted by MKESBSS) is to compute a minimum size subset $E_{ke} \subseteq E$ such that the directed subgraph $(V, E_{ke})$ is $k$-edge-strongly biconnected. In this paper we study the MKESBSS problem for $k = 2$. Note that optimal solutions for minimum 2-edge-connected spanning subgraph problem are not necessarily
feasible solutions for the 2-edge strongly biconnected spanning subgraph problem, as shown in Figure 1.

Let $G = (V, E)$ be a $k$-edge-connected directed graph. The problem of calculating a minimum size $k$-edge-connected spanning subgraph of $G$ is NP-hard for $k \geq 1$ [6, 1]. Clearly, the MKESBSS problem is NP-hard for $k \geq 1$. Results of Edmonds [2] and Mader [21] imply that the number of edges in each minimal $k$-edge-connected directed graph is at most $2kn$ [1]. Cheriyan and Thurimella [1] gave a $(1 + 4/\sqrt{k})$-approximation algorithm for the minimum $k$-edge-connected spanning subgraph problem. Moreover, Cheriyan and Thurimella [1] provided a $(1 + 1/k)$-approximation algorithm for the minimum $k$-vertex-connected spanning subgraph problem. Georgiadis [10] improved the running time of this algorithm for the minimum 2-vertex-connected spanning subgraph (M2VCSS) problem and presented an efficient linear time approximation algorithm for the M2VCSS problem. Georgiadis et al. [11] presented efficient approximation algorithms based on the results of [8, 7, 12, 4] for the M2VCSS problem. Strongly connected components of a directed graph and blocks of an undirected graph can be calculated in $O(n + m)$ time using Tarjan’s algorithm [24, 25]. Wu and Grumbach [26] introduced the concept of strongly biconnected directed graph and strongly biconnected components. Italiano et al. [12, 13] provided linear time algorithms for computing all the strong articulation points and strong bridges of a directed graph. Georgiadis et al. [11] gave efficient approximation algorithms based on the results of [8, 7, 12, 4, 3] for the minimum 2-vertex connected spanning subgraph problem. Jaberi [17, 15, 14] studied twinless articulation points and some related problems. Moreover, He studied $b$-bridges and some related problems [18, 19]. Georgiadis and Kosinas [8] proved that twinless articulation points and twinless bridges can be found in $O(n + m)$ time. Jaberi studied minimum 2-vertex strongly biconnected spanning directed subgraph problem in [16] and minimum 2-vertex-twinless connected spanning subgraph problem in [20]. In this paper we study the minimum 2-edge strongly biconnected spanning subgraph problem (denoted by M2ESBSS).
Figure 1: (a) A 2-edge strongly biconnected directed graph. (b) This subgraph is not 2-edge strongly biconnected. But it is an optimal solution for the minimum 2-edge-connected spanning subgraph problem. (c) An optimal solution for the minimum 2-edge strongly biconnected spanning subgraph problem.
2. Approximation algorithm for the M2VS BSS problem

In this section we present an approximation algorithm (Algorithm 2.2) for the M2ESBSS Problem.

Lemma 2.1. Let $G = (V, E)$ be a strongly biconnected directed graph and let $U \subseteq E$ such that the directed subgraph $G_1 = (V, U)$ is strongly connected but the underlying graph of $G_1$ is not biconnected. Let $(v, w)$ be an edge in $E \setminus U$ such that $v, w$ are not in the same strongly biconnected component of $G_1$. Then the directed subgraph $(V, U \cup \{(v, w)\})$ has fewer strongly biconnected components than $G_1$.

Proof. There is a simple path $p$ from $w$ to $v$ in $g_1$ since $G_1$ is strongly connected. Moreover, the edge $(v, w)$ and the path $p$ form a simple cycle in the subgraph $(V, U \cup \{(u, w)\})$. Since the underlying graph of the subgraph $(V, U \cup \{(u, w)\})$ contains a biconnected component that contains $v$ and $w$, the vertices $v$ and $w$ are in the same strongly biconnected component of the subgraph $(V, U \cup \{(u, w)\})$. □

Lemma 2.3. The output of Algorithm 2.2 is 2-edge strongly biconnected.

Proof. It follows from Lemma 2.1. □

The following lemma shows that any 2-edge-strongly biconnected directed subgraph of a 2-edge-strongly biconnected directed graph has at least $2n$ edges.

Lemma 2.4. Let $G = (V, E)$ be a 2-edge-strongly biconnected directed graph. Let $U \subseteq E$ be an optimal solution for the M2ESBSS problem. Then the subgraph $(V, U)$ has at least $2n$ edges.

Proof. The subgraph $(V, U)$ has no strong bridges since it has no b-bridges. Therefore, the subgraph $(V, U)$ is 2-edge connected. □

Let $i$ be the number of b-bridges in $H$. The following lemma shows that Algorithm 2.2 has an approximation factor of $(5 + i)/2$.

Theorem 2.5. Let $i$ be the number of b-bridges in $H$. Then, $|E_2e| \leq i(n - 1) + 5n$.

Proof. Results of Edmonds [2] and Mader [21] imply that the number of edges in $H$ is at most $4n$ [1]. Moreover, by Lemma 2.4, any 2-edge-strongly biconnected directed subgraph of a 2-edge-strongly biconnected directed graph has at least $2n$ edges. Algorithm 2.2 adds at most $n - 1$ edge to $E_2e$ in while
Algorithm 2.2.  
**Input:** A 2-edge strongly biconnected directed graph \( G = (V, E) \)  
**Output:** a 2-edge strongly biconnected subgraph of \( G \)  
1. identify a subset \( U \subseteq E \) such that \( H = (V, U) \) is a minimal 2-edge-connected subgraph of \( G \).  
2. if the subgraph \( H = (V, U) \) is 2-edge strongly biconnected then  
3. output \( H = (V, U) \)  
4. else  
5. \( E_{2e} \leftarrow U \)  
6. \( G_{2e} \leftarrow (V, E_{2e}) \)  
7. while the underlying graph of \( G_{2e} \) is not biconnected do  
8. compute the strongly biconnected components of \( G_{2e} \)  
9. find an edge \((v, w) \in E \setminus E_{2e}\) such that \( v, w \) are not in  
10. the same strongly biconnected components of \( G_{2e} \).  
11. add \((v, w)\) to \( E_{2e} \).  
12. compute the b-bridges of \( G_{2e} = (V, E_{2e}) \).  
13. for each b-bridge \( t \in V \) do  
14. while the underlying graph of \( G_{2e} \setminus \{t\} \) is not biconnected do  
15. compute the strongly biconnected components of \( G_{2e} \setminus \{t\} \)  
16. find an edge \((u, w) \in E \setminus E_{2e}\) such that \( u, w \) are not in  
17. the same strongly biconnected components of \( G_{2e} \setminus \{t\} \).  
18. add \((u, w)\) to \( E_{2e} \).  
19. output \( G_{2e} \).
loop of lines 8–13 since the number of strong biconnected components of any
directed graph is at most \( n \). For every b-bridge in line 16, Algorithm 2.2 adds
at most \( n - 1 \) edge to \( E_{2e} \) in while loop. Consequently, \( |E_{2e}| \leq i(n - 1) + 5n \).

\[ \square \]

**Theorem 2.6.** Algorithm 2.2 runs in \( O(n^2m) \) time.

**Proof.** A minimal 2-edge-connected subgraph can be found in time \( O(n^2) \) \[1\].
The strongly biconnected components of a directed graph can be found in linear time \[2\]. Moreover, by Lemma 2.1, lines 16–20 take \( O(nm) \) time. \( \square \)

3. Open Problems

Results of Mader \[22, 23\] imply that the number of edges in each minimal
\( k \)-vertex-connected undirected graph is at most \( kn \) \[1\]. Results of Edmonds
\[2\] and Mader \[21\] imply that the number of edges in each minimal \( k \)-vertex-connected
directed graph is at most \( 2kn \) \[1\]. Moreover, Results of Edmonds
\[2\] and Mader \[21\] imply that the number of edges in each minimal \( k \)-edge-connected
directed graph is at most \( 2kn \) \[1\]. Jaberi proved that each minimal
2-vertex-strongly biconnected directed graph has at most \( 7n \) edges. The proof
in is based on results of Mader \[22, 23, 21\] and Edmonds \[2\].

We leave as open problem whether the number of edges in each minimal
2-edge strongly biconnected directed graph is at most \( 7n \) edges.

Cheriyan and Thurimella \[1\] gave a \( (1 + \lfloor 4/\sqrt{k} \rfloor) \)-approximation algo-
rithm for the minimum \( k \)-edge-connected spanning subgraph problem for di-
rected. For \( k = 2 \), the second step of this algorithm can be modified in order
to obtain a feasible solution for the M2ESBSS problem. An open problem is
whether this algorithm has a good approximation factor for the M2ESBSS
problem.

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