POLICY OPTIMIZATION OVER GENERAL STATE AND ACTION SPACES *

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Abstract.
Reinforcement learning (RL) problems over general state and action spaces are notoriously challenging. In contrast to the tableau setting, one cannot enumerate all the states and then iteratively update the policies for each state. This prevents the application of many well-studied RL methods especially those with provable convergence guarantees. In this paper, we first present a substantial generalization of the recently developed policy mirror descent method to deal with general state and action spaces. We introduce new approaches to incorporate function approximation into this method, so that we do not need to use explicit policy parameterization at all. Moreover, we present a novel policy dual averaging method for which possibly simpler function approximation techniques can be applied. We establish linear convergence rate to global optimality or sublinear convergence to stationarity for these methods applied to solve different classes of RL problems under exact policy evaluation. We then define proper notions of approximation errors for policy evaluation and investigate their impact on the convergence of these methods applied to general-state RL problems with either finite-action or continuous-action spaces. To the best of our knowledge, the development of these algorithmic frameworks as well as their convergence analysis appear to be new in the literature. Preliminary numerical results demonstrate the robustness of the aforementioned methods and show they can be competitive with state-of-the-art RL algorithms.

Key words. Markov decision process, reinforcement learning, policy gradient, mirror descent, dual averaging.

AMS subject classifications. 90C40, 90C15, 90C26, 68Q25.

1. Introduction. Stochastic dynamic programming provides a general framework to model the interactions between an agent and its environment, and to improve the agent’s decisions through such interactions. More specifically, the status of the environment is described by either discrete or continuous state variables, while the agent’s behavior is described by actions. Upon the agent’s action, the system’s state gets updated, and the agent receives some reward or pays some cost. The goal of stochastic dynamic programming is to find the optimal policy which specifies the agent’s best action at a given state.

Markov decision process (MDP) is among the most widely used stochastic dynamic programming models. Consider the infinite-horizon discounted Markov decision process \( M = (S, A, P, c, \gamma) \), where \( S \) is the state space, \( A \) denotes the action space, \( P : S \times S \times A \to \mathbb{R} \) is transition model, \( c : S \times A \to \mathbb{R} \) is the cost function, and \( \gamma \in [0, 1) \) is the discount factor. A policy \( \pi : S \to A \) determines a particular action to be chosen at a given state. Since MDP can be used to model many important sequential decision making problems (e.g., inventory control, resource allocation, and organ donation and transplantation), it had been a classic topic in the area of operations research and stochastic control [6, 41]. While in the classic MDP literature, the transition model \( P \) is assumed to be given when searching for the optimal policy \( \pi^* \), we often have no access to exact information about \( P \) in the more recent reinforcement (RL) literature [49]. This can be viewed as the main difference between MDP and RL. During the past few years, RL has received widespread attention partly due to its success in computer games such as AlphaZero [47, 48].

In general, RL methods can be divided into model-based and model-free approaches. In model-based methods, one seeks to approximate the transition model \( P \) by using various statistical learning techniques and then to solve an approximate problem to the original MDP. In model-free methods, one attempts to search improved policies without formulating an approximate transition model first. The difference between model-based and model-free RL is similar to the one between sample average approximation (SAA) and stochastic approximation (SA) in the area of stochastic optimization. Model-based methods separate statistical analysis from the design of optimization algorithms, while model-free methods intend to carefully trade-off the effort needed for data collection and computation through a combined design and analysis. Model-free methods appear to be more applicable to online reinforcement learning where the policies need to be improved as streaming data is being collected. These types of methods are the main subject of this paper.

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In spite of the popularity of RL, the development of efficient RL methods with guaranteed convergence is still lacking in this area. Most existing theoretical studies focused only on problems with finite state and finite action spaces. In this setting, a rich set of model-free methods have been developed by incorporating different sampling schemes into the classic methods for Markov decision processes, including linear programming, value iteration (or Q-learning), and policy iteration. In addition, there has been considerable interest in the development of nonlinear optimization based methods that utilize the gradient information (i.e., Q function) for solving RL problems (e.g., [1, 2, 11, 14, 19, 20, 32, 46, 50, 52, 55]). While these methods are very popular in practice, it was only until recently that we know these stochastic gradient type methods can exhibit comparable and even superior performance than those based on classic dynamic programming and/or linear programming approaches. In particular, we show in [24] that a general class of policy mirror descent method can achieve linear convergence for solving both regularized and unregularized MDP problems. When applied to RL problems with unknown \( \mathcal{P} \), it can achieve the optimal \( \mathcal{O}(1/\epsilon^2) \) and \( \mathcal{O}(1/\epsilon) \) sample complexity for solving unregularized and regularized problems, respectively. It is worth noting the latter sample complexity has not been achieved even for model-based approaches. Further interesting developments of PMD type methods can be found in some more recent works (e.g., [10, 21, 25, 28, 31, 54, 56]).

RL problems with general state and action spaces are notoriously more challenging than the finite-state finite-action setting mentioned above. One significant difficulty associated with a general state space exists in that one cannot enumerate all the states and properly define the policy found for each state. In order to address this issue, the current practice is to apply function approximation, e.g., by using a neural network, for policy parameterization (see, e.g., [7,10]). Then one can possibly apply stochastic gradient type methods restricted to this policy class to search for the policy parameters. However, the incorporation of policy parameterization would destroy the possible convexity of action space, and make it difficult to satisfy the constraints imposed on policies. In many applications, to meet these constraints might be even more important than to find an optimal policy, since the latter goal is usually not achievable for problems with general state and action spaces due to their high nonconvexity. Moreover, with policy parameterization, it is often difficult to assess the quality of the solutions since the stationarity is defined w.r.t. the specific policy class rather than the original policy optimization problem. In addition, by using policy parameterization, we often end up with an optimization problem of very high dimension in the space of policy parameters, even if the dimension of the original action space is rather small. Finally, even with policy parameterization, another type of function approximation is also needed to evaluate a policy and compute the gradients, since one cannot save such gradient information in a tableau form. In general, there are very little theoretical guarantees (e.g., feasibility, optimality, stationarity) on the convergence of existing RL methods dealing with general state and action spaces.

This paper attempts to address some of the aforementioned issues by presenting novel algorithmic frameworks to solve RL problems over general state and action spaces. The main contributions of this paper mainly consist of the following two aspects. The first aspect of our contribution exists in a substantial generalization of the policy mirror descent (PMD) method. More specifically, we start with an idealized situation for policy optimization, where exact gradient information can be computed over general state and action spaces. We generalize the PMD method and show its linear convergence to the global optimality and sublinear convergence to a stationary point for solving different classes of RL problems depending on the curvature of the action-value function. We then discuss how function approximation can be incorporated into the PMD method. In contrast to existing methods, we will not incorporate policy parameterization into this algorithmic scheme at all. Instead, we apply function approximation only to the policy evaluation step applied to an augmented action-value function. The policies will be represented by a simple optimization problem, and will be computed, whenever needed, based on the parameters obtained during the policy evaluation step. For RL problems with general state and finite action spaces, this method converges to the global optimality up to function approximation errors. Moreover, its subproblems to compute the policy can have explicit solutions in this setting. Furthermore, for RL problems with general state and continuous action spaces, it converges to the global optimality up to function approximation errors if certain curvature condition is satisfied for the approximation functions. Otherwise, it will converge to a stationary point for the original policy optimization problem under certain conditions for the approximation error.

The second aspect of our contribution exists in the development of a novel class of policy dual averaging (PDA) methods for solving RL problems. The dual averaging method was originally developed for convex optimization [36] and later further studied for machine learning (e.g., [53]). Similar to the original dual
averaging method, the PDA method developed herein minimizes a weighted summation of the advantage functions plus a certain regularization term at each iteration. The PDA method is equivalent to PMD under the tabular setting with specific selection of distance generating functions, but these two methods are different in general. In comparison with PMD, the PDA method appears to be more amenable to function approximation as it only requires function approximation for the original action-value function, and the stepsize parameters will not impact function approximation. However, other than the tableau setting, the convergence of the PDA method has not been studied before in the literature. Through a novel analysis, we show that PDA also exhibits linear convergence to the global optimality and sublinear convergence to a stationary point (in a slightly different sense than PMD) depending on the curvature of the action-value function. We also study the impact of stochastic policy evaluation and function approximation on the convergence of the PDA method for solving RL problems with general state and finite action spaces, as well as general state and continuous action spaces. In general, PDA exhibits comparable convergence properties to PMD. In some cases it does have some advantages over PMD by either allowing more adaptive stepsize policy, e.g., when applied to general un-regularized RL problems with continuous action spaces, or requiring less restrictive assumptions on the function approximation error, e.g., for solving RL problems with continuous action spaces.

This paper is organized as follows. We first formally introduce the problems of interest in Section 2. Section 3 is devoted to the development of the PMD method for solving RL problems with general state and action spaces. The development of the PDA method is detailed in Section 4, whose structure is parallel to that of Section 3. Finally some concluding remarks are made in Section 5.

1.1. Notation and terminology. Let $\| \cdot \|$ be given norm in action space $A \subseteq \mathbb{R}^{n_A}$ (for some $n_A \geq 1$) and $\omega : A \rightarrow \mathbb{R}$ be a strongly convex function w.r.t. to given $\| \cdot \|$ s.t.

$$D(a_2, a_1) := \omega(a_1) - \omega(a_2) + \langle \omega'(a_2), a_1 - a_2 \rangle \geq \frac{1}{2} \|a_1 - a_2\|^2, \forall a_1, a_2 \in A. \tag{1.1}$$

For general-state and finite-action MDPs (see (2.1)-(2.2)), a common selection of $\omega$ for RL is the entropy function $\omega(a) = \sum_{i=1}^{n_A} a_i \log a_i$. With this selection, the Bregman distance becomes the KL divergence (see [24]) given by $D(a', a) = \text{KL}(a \mid\mid a') = \sum_{i=1}^{n_A} a_i \log \frac{a_i}{a'_i}$. For general-state and continuous-action MDPs, the selection of $\omega$ will be problem-dependent. A common section would be the Euclidean norm $\omega(a) = \sum_{i=1}^{n_A} a_i^2/2$, with the corresponding Bregman distance given by $D(a', a) = \frac{1}{2} \|a - a'\|^2$.

2. Problems of Interest. For a space $\Omega$ with a $\sigma$-algebra $\sigma_\Omega$, we define $\mathcal{M}(\Omega)$ as the set of all probability measures over $\sigma_\Omega$. In this paper, the term “measurable” refers to a Borel measurable set or function depending on the context.

2.1. Markov Decision Processes. An infinite-horizon discounted Markov decision process (MDP) is a five-tuple $(S, A, P, c, \gamma)$, where $S \subseteq \mathbb{R}^n$ is a measurable state space, $A \subseteq \mathbb{R}^{n_A}$ is a measurable action space, $P : S \times A \rightarrow \mathcal{M}(S)$ is a mapping with domain $S \times A$, $c : S \times A \rightarrow \mathbb{R}$ denote the cost function, and $\gamma \in [0, 1)$ is a discount factor. The transition kernel $P$ is a stochastic kernel [17, Definition C.1]. The measurable cost function $c(s, a)$ specifies the instantaneous cost associated with the selection of action $a$ in state $s$.

A policy $\pi : S \rightarrow A$ is a measurable mapping that determines a feasible action $\pi(s) \in A$ for each state $s \in S$. In this paper, we focus on the case when the action space $A \subseteq \mathbb{R}^{n_A}$ is a closed convex set and intend to find a deterministic policy that maps a given state $s$ to a specific selection of $a \in A$. It should be noted, however, that our development covers the finite set of actions $\{A_1, \ldots, A_{n_A}\}$ as an important special case. Indeed, when the set of possible actions is finite, a widely used approach in policy optimization is to seek a randomized policy that determines the probability of selecting a specific action. This is equivalent to specify a continuous action $a$ over the simplex set

$$A \equiv \Delta_{n_A} := \{ p \in \mathbb{R}^{n_A} : \sum_{i=1}^{n_A} p_i = 1, p_i \geq 0, i = 1, \ldots, n_A \}, \tag{2.1}$$

with each of its extreme point corresponding to an action $A_i$, $i = 1, \ldots, n_A$. Accordingly, the cost function and the transition kernel, respectively, are given by

$$c(s, a) = \sum_{i=1}^{n_A} a_i c_0(s, A_i) \quad \text{and} \quad P(s, a) = \sum_{i=1}^{n_A} a_i P_0(\cdot|s, A_i), \tag{2.2}$$

Here $c_0(s, A_i)$ and $P_0(\cdot|s, A_i)$ denote the cost and transition probability when the action $A_i$ is chosen. In this rest of the paper, we will refer to the aforementioned situation as general-state and finite-action MDP even
though the action space is given by the simplex as in (2.1). More generally, our development also covers an action space with mixed discrete and continuous actions by employing randomization over the discrete ones.

Let \( \Pi \) denote the set of all measurable stationary policies that are invariant of time epoch \( t \). For any \( \pi \in \Pi \), we measure its performance by the action-value function \( Q^\pi : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \) defined as

\[
Q^\pi(s, a) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t [c(s_t, a_t) + h^\pi(s_t)] \mid s_0 = s, a_0 = a, a_t = \pi(s_t), s_{t+1} \sim \mathcal{P}(\cdot|s_t, a_t)\right],
\]

where \( h^\pi(s) \) is a regularizer which is strongly convex w.r.t. \( a \) with modulus \( \mu_h \geq 0 \), i.e.,

\[
h^{{a_1}}(s) - h^{{a_2}}(s) - \langle (h^{{a_2}})'(s), a_1 - a_2 \rangle \geq \mu_h D(a_2, a_1).
\]

In particular, entropy regularization has been shown to incentivize safe exploration and to learn risk-sensitive policies \([3, 16, 38]\). Our regularization is more general, e.g., it can model penalty for constraints. Here we separate \( h^{{a_1}}(s_t) \) from \( c(s_t, a_t) \) in order to take the advantage of its strong convexity in the design and analysis of algorithms. Moreover, we define the state-value function \( V^\pi : \mathcal{S} \to \mathbb{R} \) associated with \( \pi \) as

\[
V^\pi(s) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t [c(s_t, a_t) + h^\pi(s_t)] \mid s_0 = s, a_t = \pi(s_t), s_{t+1} \sim \mathcal{P}(\cdot|s_t, a_t)\right].
\]

Throughout the paper, we assume that \( c(s, a) + h^\pi(s) \in [0, \bar{c}] \) so that \( V^\pi(s) \in [0, \bar{c} / (1 - \gamma)] \). It can be easily seen from the definitions of \( Q^\pi \) and \( V^\pi \) that

\[
V^\pi(s) = Q^\pi(s, \pi(s)),
\]

\[
Q^\pi(s, a) = c(s, a) + h^\pi(s) + \gamma \int \mathcal{P}(ds'|s, a)V^\pi(s').
\]

Without specific mention, we assume throughout the paper that the integral is taken over the state space \( \mathcal{S} \) and is well-defined. The latter is guaranteed since all corresponding integrands appearing in the paper are assumed to be finite.

The main objective in MDP is to find an optimal policy \( \pi^* : \mathcal{S} \to \mathcal{A} \) s.t.

\[
V^{\pi^*}(s) \leq V^\pi(s), \forall \pi \in \Pi, \forall s \in \mathcal{S}.
\]

Sufficient conditions that guarantee the existence of \( \pi^* \) have been intensively studied (e.g., \([6, 17, 41]\)). For example, one such sufficient condition is given by the existence of a measurable policy \( \pi \in \Pi \) such that

\[
c(s, \pi(s)) + h^\pi(s) + \gamma \int V(s')\mathcal{P}(ds'|s, \pi(s)) = \min_{a \in \mathcal{A}} \left[ c(s, a) + h^\pi(s) + \gamma \int V(s')\mathcal{P}(ds'|s, a) \right].
\]

The existence of such a \( \pi \) is guaranteed under the so-called the measurable selection condition \([17]\). Throughout the paper, we assume the measurable selection condition holds. Under these conditions, (2.6) can be formulated as a nonlinear optimization problem with a single objective function. More specifically, given an initial state distribution \( \rho \) over \( \mathcal{S} \), let \( f_\rho \) be defined as the functional

\[
f_\rho(\pi) := \int V^\pi(s)\rho(ds).
\]

Our goal is to solve the following policy optimization problem

\[
\min_{\pi(s) \in \mathcal{A}, \forall s \in \mathcal{S}} f_\rho(\pi).
\]

For a given policy \( \pi \), let \( \mathcal{P}^\pi \) be its associated transition kernel, i.e., \( \mathcal{P}^\pi(\cdot|s) = \mathcal{P}(\cdot|s, \pi(s)) \). A probability distribution \( \nu^\pi \) is said to be a stationary distribution of \( \mathcal{P}^\pi \) if for all \( B \in \sigma_\mathcal{S} \), \( \nu^\pi(B) = \int \mathcal{P}^\pi(B|s)\nu^\pi(ds) \). Existence and uniqueness of a stationary distribution \( \nu^\pi \) and convergence of the Markov chain to \( \nu^\pi \) is ensured by ergodicity conditions. A sufficient condition is that the Markov chain is Harris ergodic, i.e., it has a regeneration set, and is aperiodic and positively recurrent \([33]\). For discrete-state Markov chains, it is well-known that any aperiodic and positive recurrent Markov chain has a unique invariant stationary distribution. Throughout this paper, we assume that \( \nu^\pi \) is the unique stationary distribution associated with \( \mathcal{P}^\pi \).

In particular, we use \( \nu^* \equiv \nu^{\pi^*} \) to denote the stationary distribution induced by the optimal policy \( \pi^* \). To simplify our discussion, we choose \( \rho \) in (2.7) as the stationary distribution \( \nu^* \). While the distribution \( \rho \)}
can be arbitrarily chosen, it has been recently observed (e.g., [24]) the choice of \( \nu^* \) can simplify the analysis of RL algorithms. In this case, our problem of interest becomes
\[
(2.8) \quad f^* := \min_{\pi(s) \in \mathcal{A}, \nu \in \mathcal{S}} \{ f(\pi) := f_{\nu^*}(\pi) \}.
\]
It is clear that \( \pi^* \) must be an optimal solution of problem (2.8). Moreover, an optimal solution of (2.8) must be an optimal policy for our MDP if \( \nu^* \) is sufficiently random (i.e., supported over \( \mathcal{S} \)).

2.2. Performance Difference and Policy Gradient. Given a feasible policy \( \pi \in \Pi \), we define the discounted state visitation measure by
\[
(2.9) \quad \kappa^\pi_s(B) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \Pr^\pi(s_t \in B | s_0 = s)
\]
for any measurable set \( B \subset \mathcal{S} \). Here, \( \Pr^\pi(s_t \in \cdot | s_0 = s) \) denotes the distribution of the state at time \( s_t \) after we follow the policy \( \pi \) starting from state \( s_0 = s \). Since \( \Pr^\pi(s_t \in B | s_0 = s) \) is in \( [0, +\infty) \) and \( \gamma < 1 \), the sum in \( \kappa^\pi_s(B) \) is absolutely convergent and hence is well-defined. Note that \( \kappa^\pi_s(B) \) is a probability distribution over states and also satisfies a set of balance equations [7, Lemma 17]. Let \( \psi \) be the \( \pi \)-step transition kernel associated with \( \mathcal{P}^\pi \) defined recursively according to
\[
(\mathcal{P}^\pi)^t(B | s) = \int (\mathcal{P}^\pi)^{t-1} (ds') \mathcal{P}^\pi(B | s'), \ t \geq 2.
\]
Then we have \( \Pr^\pi(s_t \in B | s_0 = s) = (\mathcal{P}^\pi)^t(B | s) \).

We now state an important “performance difference” lemma which tells us the difference on the value functions for two different policies.

Lemma 1. Let \( \pi \) and \( \pi' \) be two feasible policies. Then we have
\[
V^\pi(s) - V^{\pi'}(s) = \frac{1}{1-\gamma} \int \psi^\pi(q, \pi'(q)) \kappa^\pi_s(dq), \forall s \in \mathcal{S},
\]
where
\[
\psi^\pi(q, a) := Q^\pi(q, a) - V^\pi(q) + h^a(q) - h^{\pi(q)}(q).
\]

Proof. By the definition of \( V^\pi \) in (2.3), we have
\[
V^\pi'(s) - V^\pi(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t c(s_t, a_t) + h^\pi'(s_t) \right] | s_0 = s, a_t = \pi'(s_t), s_{t+1} \sim \mathcal{P}(\cdot | s_t, a_t) - V^\pi(s)
\]
\[
= \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t [c(s_t, a_t) + h^\pi'(s_t) + \gamma V^\pi(s_{t+1}) - V^\pi(s_t)] \right]
\]
\[
| s_0 = s, a_t = \pi'(s_t), s_{t+1} \sim \mathcal{P}(\cdot | s_t, a_t)
\]
\[
+ \mathbb{E} [V^\pi(s_0) | s_0 = s, a_t = \pi'(s_t), s_{t+1} \sim \mathcal{P}(\cdot | s_t, a_t)] - V^\pi(s)
\]
\[
= \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t [c(s_t, a_t) + h^\pi(s_t) + \gamma V^\pi(s_{t+1}) - V^\pi(s_t) + h^{\pi'(s_t)}(s_t) - h^\pi(s_t)] \right]
\]
\[
| s_0 = s, a_t = \pi'(s_t), s_{t+1} \sim \mathcal{P}(\cdot | s_t, a_t),
\]
where the second identity follows from the cancelation of the terms by taking telescoping sum, and the last identity follows from \( \mathbb{E} [V^\pi(s_0) | s_0 = s, a_t = \pi'(s_t), s_{t+1} \sim \mathcal{P}(\cdot | s_t, a_t)] = V^\pi(s) \). Now using the above identity, (2.5) and (2.9), we have
\[
V^\pi'(s) - V^\pi(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t [Q^\pi(s_t, a_t) - V^\pi(s_t) + h^\pi'(s_t) - h^\pi(s_t)] \right]
\]
\[
| s_0 = s, a_t = \pi'(s_t), s_{t+1} \sim \mathcal{P}(\cdot | s_t, a_t)
\]
\[
= \frac{1}{1-\gamma} \int [Q^\pi(q, \pi'(q)) - V^\pi(q) + h^{\pi'(q)}(q) - h^{\pi(q)}(q)] \kappa^\pi_s(dq).
\]

We call \( \psi^\pi \) in (2.11) the advantage function. It is different than the advantage function used in the RL literature in that it deals with deterministic policies and involves the term \( h^a \).
Lemma 2. For any feasible policy \( \pi \), we have \( \mathbb{E}_{s \sim \nu^*} \left[ -\psi^\pi(s, \pi^*(s)) \right] = (1 - \gamma) \mathbb{E}_{s \sim \nu^*} \left[ V^\pi(s) - V^\pi^*(s) \right] \).

Proof. By Lemma 1 (with \( \pi' = \pi^* \)),

\[
(1 - \gamma) \mathbb{E}_{s \sim \nu^*} \left[ V^\pi(s) - V^\pi^*(s) \right] = \mathbb{E}_{q \sim \kappa^\pi_\nu} \left[ \psi^\pi(q, \pi^*(q)) \right].
\]

Taking expectation for w.r.t. \( \nu^* \) and noticing the stationarity of \( \pi^* \), we have

\[
(1 - \gamma) \mathbb{E}_{s \sim \nu^*} \left[ V^\pi(s) - V^\pi^*(s) \right] = \mathbb{E}_{s \sim \nu^*, q \sim \kappa^\pi_\nu} \left[ \psi^\pi(q, \pi^*(q)) \right] = \mathbb{E}_{s \sim \nu^*} \left[ \psi^\pi(s, \pi^*(s)) \right].
\]

When both the state and action space are finite, Lemma 1 can be used to establish the gradient of the objective function \( f \) in (2.7) w.r.t. policy \( \pi \) [24]. However, for general state-action spaces, the policy lies in an infinite-dimensional space, so establishing a gradient will be different than the finite setting. Let us do so now.

For simplicity, assume the action space \( \mathcal{A} \) is compact, so \( \pi' = \pi^* \) and \( \nabla \psi^\pi \) will be Fréchet differentiable w.r.t. \( \pi \). In order to guarantee that each step of PMD is well-defined, we need to assume through this section that \( \pi' \) is compact. Denote \( \delta = \pi' - \pi \) for any two (measurable) policies \( \pi', \pi \in \mathcal{P} \). Then \( \delta \) is measurable as well. Let \( \mathcal{H} \) be a Hilbert space w.r.t. the inner product \( \langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \), and suppose \( \pi, \pi', \delta \in \mathcal{H} \) (e.g., \( \mathcal{H} \) is the \( L^2 \) space when the action space is bounded). We wish to show the function \( V^\pi(s) \) is Fréchet differentiable w.r.t. \( \pi \) for any \( s \in \mathcal{S} \). By (2.10),

\[
V^\pi'(s) - V^\pi(s) = \int [Q^\pi(q, \pi'(q)) - Q^\pi(q, \pi(q)) + h^\pi(q)(q) - h^\pi(q)(q)] \kappa_q^\pi dq.
\]

Suppose \( Q^\pi(s, a) \) is differentiable w.r.t. the action, i.e., \( Q^\pi(q, \pi'(q)) - Q^\pi(q, \pi(q)) = \langle \nabla_a Q^\pi(q, \pi(q)), \delta(q) \rangle + o(||\delta(q)||) \) for any state \( q \in \mathcal{S} \) (see [37, Section 1.2.1] for the definition of the function \( o(\cdot) \)). Also assume that \( h^\pi \) is differentiable w.r.t. the action. These assumptions, along with compactness of \( \mathcal{A} \), ensure the gradients \( \nabla_a Q^\pi(q, \pi(q)) \) and \( \nabla_a h^\pi(q)(q) \) are bounded. Let \( \mathcal{K}_s \) be the space of discounted state visitation measures starting at state \( s \), as defined in (2.9). Consider the metric space \( (\mathcal{K}_s, ||\cdot||_1) \), and suppose \( \kappa_q^\pi \) is locally Lipschitz continuous, i.e., \( ||\kappa_q^\pi(\cdot) - \kappa_q^\pi(\cdot)||_1 \leq L_\pi ||\delta|| \). Note that sufficient conditions for differentiability are shown in Appendix A, and conditions for Lipschitz continuity have been studied in the literature (e.g., [7, 30, 39, 40]). In view of these discussions,

\[
\lim_{||\delta|| \to 0} \frac{1}{||\delta||} \left[ V^\pi'(s) - V^\pi(s) - \left\langle \int [\nabla_a Q^\pi(q, \pi(q)) + \nabla_a h^\pi(q)(q)] \kappa_q^\pi dq, \delta(q) \right\rangle \right] = 0.
\]

Consequently, \( V^\pi(s) \) is Fréchet differentiable w.r.t. the policy \( \pi \). We denote its Fréchet derivative as the function \( \nabla_a V^\pi(s) \in \mathcal{H} \), whose value at state \( q \in \mathcal{S} \) is

\[
(2.12) \quad [\nabla_a V^\pi(s)](q) := \int [\nabla_a Q^\pi(q, \pi(q)) + \nabla_a h^\pi(q)] \kappa_q^\pi(dq) \in \mathbb{R}^{n_a},
\]

and we can use it as the termination criterion to evaluate the efficiency of policy optimization algorithms.

Finally, by observing our assumptions imply \( Q^\pi(q, \cdot) \) and \( h(q, \cdot) \) have bounded gradients, one can show the objective function \( f \) from (2.7) is also Fréchet differentiable.

3. Policy Mirror Descent. The policy mirror descent method was initially designed for solving RL problems over finite state and finite action spaces. Our goal in this section is to generalize this algorithm for RL over general state and action spaces.

3.1. The Generic Algorithmic Scheme. For a given policy \( \pi_k \), the PMD method minimizes the advantage function \( \psi^\pi_k(s, a) \) plus the regularization term \( D(\pi_k(s), a) \) with a certain stepsize parameter \( \eta_k \geq 0 \). In order to guarantee that each step of PMD is well-defined, we need to assume through this section...
that the policy value function $Q^\pi(s, a)$ satisfies certain curvature conditions. In particular, we assume that for some finite $\mu_Q \geq 0$, the function
\begin{equation}
Q^\pi(s, a) + \mu_Q D(\pi(s), a)
\end{equation}
is convex w.r.t. $a$ for any $s \in S$ and any $\pi \in \Pi$. $Q^\pi(s, \cdot)$ is convex if $\mu_Q = 0$, and strongly convex if $\mu_Q < 0$. Otherwise, if $\mu_Q > 0$, $Q^\pi(s, \cdot)$ is not necessarily convex and function of this type is also referred to as being weakly convex. It is well-known that any differential function with Lipschitz continuous gradients is weakly convex [13, Lemma 4.2]. Later in subsection 3.4 when we incorporate function approximation, we only require the approximation of $Q^\pi$ (and not $Q^\pi$ itself) to be weakly convex. We introduce weak convexity now to gain intuition on how it affects convergence to a global or local solution. With that in mind, let us denote
\begin{equation}
\mu_d := \mu_h - \mu_Q.
\end{equation}
The advantage function $\psi^\pi(s, \cdot)$ defined in (2.11) is convex if $\mu_d \geq 0$, and weakly convex otherwise.

**Algorithm 1** The policy mirror descent method

**Input:** $\eta_k \geq 0$ and $\pi_0$. 
for $k = 0, 1, \ldots$, do

\begin{equation}
\pi_{k+1}(s) = \arg\min_{a \in A} \{\psi^\pi_k(s, a) + \frac{1}{\eta_k} D(\pi_k(s), a)\}
\end{equation}

\begin{equation}
= \arg\min_{a \in A} \{Q^\pi_k(s, a) + \frac{1}{\eta_k} D(\pi_k(s), a)\}, \forall s \in S.
\end{equation}
end for

With a properly chosen $\eta_k$, the objective function of the subproblem in (3.3) is strongly convex, and thus the new policy $\pi_{k+1}$ is well-defined. The following result is well-known for mirror descent methods.

**Lemma 3.** If $\eta_k$ in (3.3) satisfies
\begin{equation}
\mu_d + \frac{1}{\eta_k} \geq 0,
\end{equation}
then for any $a \in A$,
\begin{equation}
\psi^\pi_k(s, \pi_{k+1}(s)) + \frac{1}{\eta_k} D(\pi_k(s), \pi_{k+1}(s)) + (\mu_d + \frac{1}{\eta_k}) D(\pi_{k+1}(s), a) \leq \psi^\pi_k(s, a) + \frac{1}{\eta_k} D(\pi_k(s), a).
\end{equation}

**Proof.** Observe that the objective function of (3.3) is strongly convex w.r.t. $a \in A$ with modulus $\mu_d + 1/\eta_k$. The result then follows immediately from the optimality condition of (3.3) [23, Lemma 3.5].

We now show the progress made by each iteration of the PMD method.

**Proposition 1.** Suppose $\eta_k$ in the PMD method satisfies (3.5). Then for any $s \in S$,
\begin{equation}
V^{\pi_{k+1}}(s) - V^{\pi_k}(s) \leq \psi^\pi_k(s, \pi_{k+1}(s)) \leq -\left(\frac{1}{\eta_k} D(\pi_k(s), \pi_{k+1}(s)) + (\mu_d + \frac{1}{\eta_k}) D(\pi_{k+1}(s), \pi_k(s))\right).
\end{equation}

**Proof.** By (3.6) with $a = \pi_k(s)$, we have
\begin{equation}
\psi^\pi_k(s, \pi_{k+1}(s)) + \frac{1}{\eta_k} D(\pi_k(s), \pi_{k+1}(s)) + (\mu_d + \frac{1}{\eta_k}) D(\pi_{k+1}(s), \pi_k(s))
\end{equation}
\begin{equation}
\leq \psi^\pi_k(s, \pi_k(s)) + \frac{1}{\eta_k} D(\pi_k(s), \pi_k(s)) = 0,
\end{equation}
where the last identity follows from the fact that $\psi^\pi_k(s, \pi_k(s)) = 0$ due to (2.4) and (2.11). By Lemma 1, (3.8), and the fact that $\kappa_{\pi_{k+1}}^\pi(\{s\}) \geq 1 - \gamma$ due to (2.9), we have
\begin{equation}
V^{\pi_{k+1}}(s) - V^{\pi_k}(s) = \frac{1}{1-\gamma} \int \psi^\pi_k(q, \pi_{k+1}(q)) \kappa_{\pi_{k+1}}^\pi(dq)
\end{equation}
\begin{equation}
\leq \frac{1}{1-\gamma} \psi^\pi_k(s, \pi_{k+1}(s)) \kappa_{\pi_{k+1}}^\pi(\{s\}) \leq \psi^\pi_k(s, \pi_{k+1}(s)).
\end{equation}
The result then follows by combining (3.8) with (3.9).

In the following two results, we establish the convergence of the PMD method under the assumption $\mu_d \geq 0$ and $\mu_d < 0$, respectively.
\textbf{Theorem 1.} Suppose that $\mu_d \geq 0$ and $\eta_k$ in the PMD method satisfies (3.5). Then for any $k \geq 1$, \begin{align}
abla H \quad & (3.10) \quad \sum_{t=0}^{k-1} \{\eta[f(\pi_{t+1}) - f(\pi^*)] - \gamma \eta[f(\pi_t) - f(\pi^*)]\} + (\mu_d \eta_{k-1} + 1)D(\pi_k, \pi^*) \leq D(\pi_0, \pi^*), \\
abla H \quad & (3.11) \quad D(\pi, \pi') := E_{s \sim \nu^*}[D(\pi(s), \pi'(s))].
\end{align}
In particular, if $\eta_k = \gamma^{-k}$, then
\begin{align*}
\sum_{t=0}^{k-1} \{\eta[f(\pi_{t+1}) - f(\pi^*)] - \gamma \eta[f(\pi_t) - f(\pi^*)]\} + (\mu_d \gamma^{-k-1} + 1)D(\pi_k, \pi^*) \leq \gamma^k f(\pi_0) - f(\pi^*) + \gamma^{-k-1}D(\pi_0, \pi^*).
\end{align*}

\textbf{Proof.} Multiplying (3.6) by $\eta_k$ and then taking the telescopic sum (in view of $\mu_d \geq 0$), we obtain
\begin{align*}
\sum_{t=0}^{k-1} \eta f(\pi_{t+1}) - f(\pi_t)] + (\mu_d \eta_{k-1} + 1)D(\pi_k, \pi^*) \leq \sum_{t=0}^{k-1} \eta f(\pi_t) - f(\pi^*) + D(\pi_0, \pi^*),
\end{align*}
where
\begin{align*}
D(\pi_k, \pi^*) := E_{s \sim \nu^*}[D(\pi(s), \pi'(s))].
\end{align*}
Setting $a = \pi^*(s)$ in the above inequality, and using the first relation in (3.7), we then have
\begin{align*}
\sum_{t=0}^{k-1} \eta f(\pi_{t+1}) - f(\pi_t)] + (\mu_d \eta_{k-1} + 1)D(\pi_k, \pi^*) \leq \sum_{t=0}^{k-1} \eta f(\pi_t) - f(\pi^*) + D(\pi_0, \pi^*),
\end{align*}
which clearly implies (3.10) after rearranging the terms. \hfill \-box

In view of Theorem 1, the PMD method will exhibit a linear rate of convergence if the advantage function is convex w.r.t. $a \in A$. This result generalizes the previous linear rate of convergence for the PMD method in the tabular setting to more general problems. The following result further shows that this algorithm can possibly converge to a stationary point of problem (2.8) when $\mu_d < 0$. We refer to $\pi_k$ as a stationary point at state $s$ when the derivative $\nabla \pi V^\pi(s)$, whenever it exists (e.g. (2.12)), is small.

\textbf{Theorem 2.} Suppose that $\mu_d < 0$ and that $\eta_k = 1/(2|\mu_d|)$ for any $k \geq 0$. Then for any $s \in S$, there exist iteration indices $k(s)$ found by running $k$ iterations of the PMD method s.t.
\begin{align}
0 \leq -\psi^\pi(s, \pi_{k(s)+1}) \leq \frac{1}{k}[V^{\pi_0}(s) - V^\pi(s)],
\end{align}
\begin{align}
2D(\pi_{k(s)+1}) + D(\pi_{k(s)+1}) \leq \frac{1}{k|\mu_d|}[V^{\pi_0}(s) - V^\pi(s)].
\end{align}

\textbf{Proof.} Taking telescopic sum of (3.7), we obtain
\begin{align}
0 \leq -\psi^\pi(s, \pi_{k+1}) \leq \frac{1}{k}[V^{\pi_0}(s) - V^\pi(s)],
\end{align}
The iteration indices $k(s)$ in (3.12) are given by the ones with the smallest value of $-\psi^\pi(s, \pi_{k+1}, s)$, $t = 0, \ldots, k-1$. The relation in (3.13) follows directly from (3.7) and (3.12). \hfill \- \hfill \- \hfill \- \hfill \hfill \- \hfill \- \hfill \- \hfill \- \hfill \-

Observe that the relation in (3.13) tells us some information about the stationarity of the policy $\pi_j(s)$ for any $s \in S$ and some index $j(s)$. For simplicity, let us assume that $\omega(a) = \|a\|^2/2$, $A = \mathbb{R}^n A$, and that $Q^\pi(s, \cdot)$ and $h^s(s)$ have Lipschitz continuous gradient w.r.t. $a$. Then the optimality condition of problem (3.4) given by
\begin{align}
\nabla_a Q^\pi(s, \pi(s)+1) + \nabla_a h^\pi(s)+1(s) + 2|\mu_d||\pi(s)+1 - \pi(s)| = 0, \forall s \in S,
\end{align}
implies that
\begin{align}
||\nabla_a Q^\pi(s, \pi(s)) + \nabla_a h^\pi(s)(s) || \leq (L_Q + L_h + 2|\mu_d||\pi(s)+1 - \pi(s)|| = O(1/\sqrt{k}), \forall s \in S.
\end{align}
Hence, by (2.12), the norm of the derivative $\nabla_a V^\pi(s)$ is also decreasing as $O(1/\sqrt{k})$ with the selection of the policy $\pi = \pi_j(s)$, $s \in S$. It should be noted, however, that it might be difficult to implement this strategy in practice, since it requires us to specify a different index $j(s)$ for a different state $s \in S$. In practice, one can possibly select one random iterate for all different states. In addition, the strategy of choosing a different iteration index for a different state can possibly be implemented when function approximation is used in PMD as shown in the next subsection.
3.2. Function Approximation in PMD. In an ideal situation for the PMD method, we update a given policy $\pi_k$ by

$$
\pi_{k+1}(s) = \arg\min_{a \in A} \left\{ Q^\pi_k(s, a) - \frac{1}{\eta_k} \langle \nabla \omega(\pi_k(s)), a \rangle + h^\pi(s) + \frac{1}{\eta_k} \omega(a) \right\}, \forall s \in S
$$

for some properly chosen stepsize $\eta_k \geq 0$. However, for the general state space $S$, it is impossible to perform these updates since $S$ is possibly infinite. Current practice is to apply different function approximations for both estimating $Q^\pi_k(s, a)$ and computing a parameterized form of $\pi_k(s)$. While the idea of using a parameterized form of $\pi_k$ appears to be conceptually simple, it often results in a few significant limitations. First, it introduces additional approximation error due to the difficulty to identify the optimal policy class. Second, it will destroy the convexity of the action space $A$, and make it difficult to ensure the policy’s chosen action is feasible, i.e., lies in $A$, without further processing.

In order to deal with these issues for RL with general state spaces, we adopt the idea of using function approximation for the value functions. However, different from existing methods that require function approximation for both value function and policy parameterization, we will not use policy parameterization in the policy optimization step. Instead, as shown in Algorithm 2, we suggest to compute a function approximation for the value function $L^\pi_k(s, a)$ (rather than $Q^\pi_k(s, a)$) used to define the objective function in (3.15).

Algorithm 2 The policy mirror descent method with function approximation

**Input:** $\eta_k \geq 0$ and $\pi_0$.

**for** $k = 0, 1, \ldots$, **do**

Set

$$
L^\pi_k(s, a) = Q^\pi_k(s, a) + \frac{1}{\eta_k} \langle \nabla \omega(\pi_k(s)), a \rangle.
$$

Run a policy evaluation procedure to estimate $L^\pi_k(s, a)$ s.t.

$$
\tilde{L}(s, a; \theta_k) \approx L^\pi_k(s, a), \forall s \in S, \forall a \in A.
$$

Update the policy to

$$
\pi_{k+1}(s) = \arg\min_{a \in A} \{ \tilde{L}(s, a; \theta_k) + h^\pi(s) + \frac{1}{\eta_k} \omega(a) \}, \forall s \in S.
$$

**end for**

In (3.17), $\tilde{L}(s, a; \theta_k)$ represents a function approximation, parameterized by $\theta_k$, of the augmented action-value function $L^\pi_k(s, a)$. In the simplest form of function approximation, we are given a set of linearly independent basis functions $l_1(s, a), l_2(s, a), \ldots, l_d(s, a)$ (derived from feature construction [49] such as random features [43]) and the goal is to determine the coefficients $\theta_k \in \mathbb{R}^d$ so that $\tilde{L}(s, a; \theta_k) = \sum_{i=1}^d \theta_k^i l_i(s, a)$ approximates $L^\pi_k(s, a)$. Stochastic estimates of $L^\pi_k(s, a)$ can be constructed from samples along a trajectory in the environment [29], which are then used to fit $\tilde{L}(s, a; \theta_k)$; how to fit the function approximation will be further discussed in the upcoming two subsections. More powerful and sophisticated nonlinear function approximation methods, such as Reproducing Kernel Hilbert Space (RKHS) and neural networks, can also be used to estimate $L^\pi_k(s, a)$. We will study the impact of the error of approximating $L_k(s, a)$ on the convergence of the PMD method in the next few subsections.

The equation in (3.18) tells us how to compute the new policy $\pi_{k+1}$. In some cases, the solution of $\pi_{k+1}$ is given in an explicit form (see Section 3.3). In a more general setting, $\pi_{k+1}$ is represented as an optimal solution of problem (3.18). With a properly chosen $\eta_k$, the objective function of (3.18) is strongly convex, and hence the policy $\pi_{k+1}$ is uniquely defined. Moreover, any efficient numerical procedures (e.g., the Ellipsoid method and accelerated gradient method) can be used to solve (3.18). For the sake of simplicity, we assume throughout this paper that exact solutions can be computed for (3.18). However, our analysis can be extended to the case when only approximate solutions of (3.18) are available (see Section 6 of [24]) for an illustration).
Observe that, different from existing policy gradient methods, we do not actually compute $\pi_{k+1}(s)$ for any $s \in \mathcal{S}$. Since the function approximation $L(s; a; \theta_k)$ is given in a parametric form (through $\theta_k$’s), we can compute $\pi_{k+1}(s)$ using (3.18) whenever needed, e.g., inside the policy evaluation procedure. The idea of representing a policy by an optimization problem has been used in multi-stage stochastic optimization.

3.3. PMD for general state and finite action spaces. In this subsection, we focus on RL with a fine set of actions $\{A_1, \ldots, A_{n_A}\}$. As mentioned earlier, in this case we search for an optimal randomized policy in the probability simplex $\mathcal{A} = \Delta_{n_A}$ in (2.1), with the cost and transition probability defined in (2.2). For a given iterate $\pi_k(s) \in \mathcal{A}$, we use

\begin{align}
Q_{0}^{\pi_k}(s, A_i) := c_0(s, A_i) + h^\pi_k(s) + \gamma \int P_0(ds'|s, A_i)V^\pi_k(s'), i = 1, \ldots, n_A, \label{eq:Q_pik}
\end{align}

\begin{align}
L_{0}^{\pi_k}(s, A_i) := Q_{0}^{\pi_k}(s, A_i) + \frac{1}{\eta_k} \nabla_i \omega(\pi_k(s)), i = 1, \ldots, n_A, \label{eq:L_pik}
\end{align}

respectively, to denote the value function and augmented value function associated with each action $A_i$. We also denote $Q_{0}^{\pi_k}(s) = (Q_{0}^{\pi_k}(s, A_1), \ldots, Q_{0}^{\pi_k}(s, A_{n_A}))$ and $L_{0}^{\pi_k}(s) = (L_{0}^{\pi_k}(s, A_1), \ldots, L_{0}^{\pi_k}(s, A_{n_A}))$ for notational convenience. It then follows that $Q^\pi_k(s, a) = (Q_{0}^{\pi_k}(s, a), a)$ and

$$L^\pi_k(s, a) = Q_{0}^{\pi_k}(s, a) + \langle \nabla \omega(\pi_k(s)), a \rangle / \eta_k = L_{0}^{\pi_k}(s, a).$$

We suggest to approximate $L_{0}^{\pi_k}(s, A_i)$ by $\tilde{L}_0(s, A_i; \theta_k)$, $i = 1, \ldots, n_A$, where $\tilde{L}_0(s, A_i; \theta)$ denotes some function class parameterized by $\theta$. Then the subproblem in (3.4) reduces

\begin{align}
\pi_{k+1}(s) = \arg \min_{\theta \in \Theta_{\pi_k}} \mathbb{E}_{s \sim \rho} \left[ \min_{a \in A_{\pi_k}} \left( \langle \tilde{L}_0(s; \theta_k), a \rangle + h^\pi_k(s) + \frac{1}{\eta_k} \omega(a) \right) \right], \forall s \in \mathcal{S}, \label{eq:subproblem}
\end{align}

where $\tilde{L}_0(s; \theta_k) := (\tilde{L}_0(s, A_1; \theta_k), \ldots, \tilde{L}_0(s, A_{n_A}; \theta_k))$. The objective function of (3.21) is strongly convex with modulus $\mu_k + 1/\eta_k$. It is well-known that the above problem has an explicit solution, for example, when $\omega(a) = \sum_{i=1}^{n_A} a_i \log a_i$ and $h^\pi = \lambda \text{KL}(a || \pi_0(s))$ for some $\lambda \geq 0$.

In order to find the parameter $\theta_k$, we need to solve the following statistical learning/estimation problem

\begin{align}
\min_{\theta \in \Theta_{\pi_k}} \mathbb{E}_{s \sim \rho} \left[ \frac{1}{2} || \tilde{L}_0(s; \theta) - L_{0}^{\pi_k}(s) ||_2^2 \right] = \min_{\theta \in \Theta} \mathbb{E}_{s \sim \rho} \left[ \max_{y \geq 0, ||z|| \leq 1} \left( \max_{z} \left( \langle \tilde{L}_0(s; \theta) - L_{0}^{\pi_k}(s), z \rangle \right) - \frac{y^2}{\gamma} \right) \right], \label{eq:estimation}
\end{align}

for a given distribution $\rho$ over $\mathcal{S}$. We do not specify $\rho$ here, since it can take on multiple forms, such as the uniform distribution over states (i.e., off-policy sampling) or the steady state distribution of $\pi_k$ (i.e., on-policy sampling). Observe that there exists two types of uncertainty for the nested stochastic composite optimization (NSCO) problem in (3.22), including: a) the sampling of $s$ from $\mathcal{S}$ according to $\rho$ in the outer minimization problem, and b) the estimation of $L_{0}^{\pi_k}(s, A_i)$ through the sampling of the Markov Chain induced by $\pi_k$ for the inner maximization problem. We use $\xi_k$ to denote these random samples generated at iteration $k$ to solve problem (3.22), and $F_k := \{\xi_1, \ldots, \xi_k\}$ denote its associated filtration. Let $\theta^*_k$ be an optimal solution of (3.22) and $\theta_k$ be an approximate solution found by using some policy evaluation algorithms. Note that problem (3.22) is convex when linear function approximation is used, and we can apply either sample average approximation or stochastic gradient type methods to solve it. Further details are given in Appendix A.

It is worth noting that in the PMD method the stepsize $\eta_k$ appears in the definition of $L^\pi_k$, which may impact the magnitude of the stochastic first-order information of the policy evaluation problem (see (3.22)). As a result, the sample complexity of solving this statistical learning problem might increase as $\eta_k$ decreases. One can possibly alleviate this issue by using constant stepsize $\eta_k$, which might worsen the dependence on $1 - \gamma$ as noticed in [24]. Another possible solution is to use a larger sample size to approximate the term $\nabla \omega(\pi_k(s))$ than that for the action-value function $Q^\pi_k$, since the former sampling scheme does not involve the Markov chain and can be performed faster than collecting samples from the Markov chain associated with $\pi_k$. Yet we will propose another different approach that can possibly address this issue by developing the policy dual averaging method in the next section.

In this section we assume that the approximation error

\begin{align}
\delta_k(s) := \tilde{L}_0(s; \theta_k) - L_{0}^{\pi_k}(s) \label{eq:approximation_error}
\end{align}
can be decomposed into the following components:

\[
\delta_k(s) = \mathbb{E}_{\xi_k}[\tilde{L}_0(s;\theta_k)] - L_0^\pi(s) + \tilde{L}_0(s;\theta_k) - \mathbb{E}_{\xi_k}[\tilde{L}_0(s;\theta_k)]
\]

(3.24)

\[
= \tilde{L}_0(s;\theta_k^*) - L_0^\pi(s) + \mathbb{E}_{\xi_k}[\tilde{L}_0(s;\theta_k)] - \tilde{L}_0(s;\theta_k^*) + \tilde{L}_0(s;\theta_k) - \mathbb{E}_{\xi_k}[\tilde{L}_0(s;\theta_k)],
\]

where the expectation \(\mathbb{E}_{\xi_k}\) denote the expectation w.r.t. \(\xi_k\) conditional on \(F_{k-1}\). Note that \(\delta_k^{det}(s)\) and \(\delta_k^{sto}(s)\) denotes the deterministic and stochastic errors, respectively. Moreover, the deterministic error \(\delta_k^{det}(s)\) can be further decomposed into the error caused by function approximation \(\delta_k^{app}(s)\) and the bias term \(\delta_k^{bia}(s)\) when computing the estimation. One can possibly reduce \(\delta_k^{app}(s)\) by using more complex function class. In particular, we have \(\delta_k^{app}(s) = 0\) in the tabular case, and for some specific kernels in RKHS. While these two deterministic error terms are caused by different sources, their impact on the overall convergence of the PMD method is similar. We make the following assumptions about these error terms:

\[
\mathbb{E}_{\pi \sim \nu^*}[\|\delta_k^{det}(s)\|_\infty] \leq \zeta,
\]

(3.25)

\[
\mathbb{E}_{\xi_k}\left\{\mathbb{E}_{\pi \sim \nu^*}[\|\delta_k^{sto}(s)\|_\infty^2] \right\} \leq \sigma^2,
\]

(3.26)

for some \(\zeta \geq 0\) and \(\sigma \geq 0\). The term \(\zeta\) captures both the approximation error \(\delta_k^{app}(s)\) and bias \(\delta_k^{bia}(s)\), while the term \(\sigma\) bounds the statistical estimation error \(\delta_k^{sto}(s)\). Both the \(\delta_k^{bia}(s)\) and \(\delta_k^{sto}(s)\) terms become smaller with more samples, although \(\delta_k^{bia}(s)\) can usually be reduced much faster than \(\delta_k^{sto}(s)\) as observed previously in [24]. In Appendix A, we give sufficient conditions for these assumptions to hold.

The following result states an important recursion about each PMD iteration for solving RL problems with finite action space.

**Proposition 2.** For any \(s \in \mathcal{S}\), we have

\[
-\psi^+(s, a) + h^+(s, a) + \frac{1}{\eta_k}[\frac{1}{\eta_k} + M_k + \|\delta_k^{sto}(s)\|_\infty^2] + \|\delta_k^{det}(s)\|_\infty - \langle \delta_k^{sto}(s), \pi_k(s) - a \rangle, \forall a \in \mathcal{A}.
\]

(3.27)

**Proof.** By the optimality condition of (3.21), we have

\[
\langle L_0^\pi(s), \pi_{k+1}(s) - a \rangle = \sum_{a} Q^\pi_k(s, \pi_{k+1}(s)) - Q^\pi_k(s, a) + h^+(s, a) + \frac{1}{\eta_k}[\omega(\pi_{k+1}(s)) - \omega(a)] + (\mu_k + \frac{1}{\eta_k})D(\pi_{k+1}(s), a)
\]

(3.28)

Using the facts \(L_0^\pi(s, a) = H^+(s, a) = Q^\pi_k(s, a) + \frac{1}{\eta_k}\langle \nabla \omega(\pi_k), a \rangle\) in the above relation, we obtain

\[
Q^\pi_k(s, \pi_{k+1}(s)) - Q^\pi_k(s, a) + h^+(s, a) + \frac{1}{\eta_k}[\omega(\pi_{k+1}(s)) - \omega(a)] + (\mu_k + \frac{1}{\eta_k})D(\pi_{k+1}(s), a)
\]

\[
+ \frac{1}{\eta_k}\langle \nabla \omega(\pi_k(s)), a \rangle - a \rangle \leq 0.
\]

Noticing that

\[
\omega(\pi_{k+1}(s)) - \omega(a) + \langle \nabla \omega(\pi_k(s)), a \rangle - a = D(\pi_k(s), \pi_{k+1}(s)) - D(\pi_k(s), a),
\]

we have

\[
Q^\pi_k(s, \pi_{k+1}(s)) - Q^\pi_k(s, a) + h^+(s, a) + \frac{1}{\eta_k}[D(\pi_k(s), \pi_{k+1}(s)) - D(\pi_k(s), a)]
\]

\[
+ (\mu_k + \frac{1}{\eta_k})D(\pi_{k+1}(s), a) + \langle \delta_k(s), \pi_{k+1}(s) - a \rangle \leq 0,
\]

which, in view of (2.11), then implies that

\[
\psi^+(s, \pi_{k+1}(s)) - \psi^+(s, a) + h^+(s, a) + \frac{1}{\eta_k}[D(\pi_k(s), \pi_{k+1}(s)) - D(\pi_k(s), a)]
\]

(3.29)

\[
+ (\mu_k + \frac{1}{\eta_k})D(\pi_{k+1}(s), a) + \langle \delta_k(s), \pi_{k+1}(s) - a \rangle \leq 0.
\]
Noticing that
\[\psi^\pi(s, \pi_{k+1}(s)) = Q^\pi(s, \pi_{k+1}(s)) - V^\pi(s) = \langle Q_0^\pi(s, \pi_{k+1}(s) - \pi_k(s)), \]

we have
\[
\psi^\pi(s, \pi_{k+1}(s)) + h^{\pi_{k+1}}(s) - h^\pi(s) + \langle \delta_k(s), \pi_{k+1}(s) - a \rangle = \langle \delta_k^{det}(s), \pi_{k+1}(s) - \pi^*(s) \rangle + \langle \delta_k^{sto}(s), \pi_{k+1}(s) - \pi_k(s) \rangle
\]
\[+
\langle \delta_k^{sto}(s), \pi_k(s) - a \rangle.
\]

The result then follows by using the above relation in (3.29) and rearranging the terms.

Observe that in the proof of (3.27), we bound the term \( \psi^\pi(s, \pi_{k+1}(s)) \) and \( h^{\pi_{k+1}}(s) - h^\pi(s) \) in (3.30) and (3.31) simply by the Cauchy-Schwarz inequality and Lipschitz continuity of \( h^a \), respectively. These relations are sufficient to prove the sublinear rate of convergence for the PMD method. In order to prove the linear rate of convergence of the PMD method in terms of number of iterations for the stochastic setting, more sophisticated techniques to bound this term are needed as initially shown in [24].

**Theorem 3.** Assume that the stepsize \( \eta_k \) satisfies
\[\frac{\beta_k}{\eta_k} \leq \beta_{k-1}(\mu_h + \frac{1}{\eta_k}), k \geq 1,
\]
for some \( \beta_k \geq 0. \) Then
\[\sum_{i=0}^{k-1} \beta_i^{-1} \left\{ (1 - \gamma) \sum_{t=0}^{k-1} \beta_t E[f(\pi_t) - f^*] + \beta_{k-1}(\mu_h + \frac{1}{\eta_{k-1}})E[D(\pi_k, \pi^*)] \right\} \leq \sum_{i=0}^{k-1} \beta_i \eta_i \left[ \frac{\mu_h}{\gamma} + M_h \right]^2 + \zeta. \]

where \( D \) is defined in (3.11).

**Proof.** Fixing \( a = \pi^*(s) \), taking expectation w.r.t. \( a \sim \nu^* \) for (3.27), and using (3.25)-(3.26), we have
\[E_{\scriptscriptstyle \pi \sim \nu^*} \left[ -\psi^\pi(s, \pi^*(s)) + h^\pi(s) - h^*(s) + (\mu_h + \frac{1}{\eta_k})D(\pi_{k+1}(s), \pi^*(s)) \right] \leq \frac{1}{\eta_k} E_{\scriptscriptstyle \pi \sim \nu^*} D(\pi_k, \pi^*(s)) + \frac{\mu_h}{\gamma} \left[ \frac{\mu_h}{\gamma} + M_h \right]^2 + \zeta.
\]

Taking conditional expectation of (3.27) w.r.t. \( \xi_k \), it then follows from Lemma 2, (2.26), and \( E_{\scriptscriptstyle \xi_k} [\delta_k^{sto}(s)] = 0 \) that
\[\sum_{i=0}^{k-1} \beta_i \left\{ V^\pi(s) - V^*(s) + (\mu_h + \frac{1}{\eta_k})E_{\scriptscriptstyle \xi_k} D(\pi_{k+1}(s), \pi^*(s)) \right\} \leq \frac{1}{\eta_k} E_{\scriptscriptstyle \pi \sim \nu^*} D(\pi_0, \pi^*(s)) + \zeta \left( \frac{\mu_h}{\gamma} + M_h \right)^2 + \zeta.
\]

Taking full expectation w.r.t. \( F_k \) and the \( \beta_k \)-weighted sum of the above inequalities, it then follows from (3.33) and the definition of \( f \) in (2.8) that
\[\sum_{i=0}^{k-1} \beta_i \left[ \eta \left( \left[ \frac{\mu_h}{\gamma} + M_h \right]^2 + \zeta \right) \right],
\]
which implies the result follows immediately.

We specify the selection of \( \eta_k \) and \( \beta_k \) in the following result which follows immediately from Theorem 3 by straightforward calculations.

**Corollary 1.**

a) If \( \mu_h = 0 \), the number of iterations \( k \) is given a priori, and

\[
\eta_t = \sqrt{\frac{\mathcal{D}(\pi_t, \pi^\star)}{k(\epsilon/1-\gamma-\mu_h)^2+\sigma^2}} \quad \text{and} \quad \beta_t = 1, t = 0, \ldots, k-1,
\]

then

\[
(1-\gamma)\frac{1}{k} \sum_{t=0}^{k-1} |f(\pi_t) - f^*| \leq 2\sqrt{\frac{\mathcal{D}(\pi_t, \pi^\star)(\epsilon/1-\gamma+\mu_h)^2+\sigma^2}{k} + \zeta}.
\]

b) If \( \mu_h > 0 \) and \( \eta_t = \frac{1}{\mu_h(t+1)} \) and \( \beta_t = 1, t = 0, \ldots, k-1 \), then

\[
(1-\gamma)\frac{1}{k} \sum_{t=0}^{k-1} |f(\pi_t) - f^*| + \mu_h \mathbb{E}[\mathcal{D}(\pi_k, \pi^\star)] \leq \frac{1}{k} \left\{ \mu_h \mathcal{D}(\pi_1, \pi^\star) + \frac{3}{2} \left[ (\frac{\epsilon}{1-\gamma} + \mu_h)^2 + \sigma^2 \right] \log k \right\} + \zeta.
\]

c) If \( \mu_h > 0 \) and \( \eta_t = \frac{2}{\mu_h(t+1)} \) and \( \beta_t = t+2, t = 0, \ldots, k-1 \), then

\[
\frac{2(1-\gamma)}{k^2+3} \sum_{t=0}^{k-1} (t+2) \mathbb{E}[|f(\pi_t) - f^*) + \frac{3}{2} \mathbb{E}[\mathcal{D}(\pi_k, \pi^\star)] \leq \frac{2}{k^2+3} \mu_h \mathcal{D}(\pi_0, \pi^\star) + \frac{8}{\mu_h(k+3)} \left[ (\frac{\epsilon}{1-\gamma} + \mu_h)^2 + \sigma^2 \right] + \zeta.
\]

In view of the previous result, we can randomly select the output solution \( \pi_R \) from \( \{\pi_0, \ldots, \pi_{k-1}\} \) with \( \text{Prob}(R = t) = \sum_{t=0}^{k-1} \beta_t, t = 0, \ldots, k-1 \). Observe that both part a) and part b) also provides a so-called regret bound, i.e., \( \mathbb{E}g(h, \mu_h) \) for RL problems.

### 3.4. PMD for general state and continuous action spaces

In this subsection, we consider RL problems with continuous action spaces. Different from finite action spaces, the augmented-value function \( L_{\pi_k}(s, a) \) and its approximation \( \tilde{L}(s, a; \theta_k) \) are not necessarily linear w.r.t. to \( a \in A \) anymore. Instead, we assume that \( \tilde{L}(s, a; \theta_k) \) is given in the form of

\[
\tilde{L}(s, a; \theta_k) = \tilde{Q}(s, a; \theta_k) + \langle \tilde{\omega}(s; \theta_k), a \rangle,
\]

where \( \tilde{Q}(s, a; \theta_k) \) and \( \tilde{\omega}(s; \theta_k) \) approximate \( Q^\pi_k(s, a) \) and \( \nabla \omega(\pi_k(s))/\eta_k \), respectively. We assume that \( \tilde{Q}(s, a; \theta) \) is weakly convex, i.e.,

\[
\tilde{Q}(s, a; \theta) + \mu_{\tilde{Q}} \mathcal{D}(\pi_0(s), a)
\]

is convex w.r.t. \( a \in A \) for some \( \mu_{\tilde{Q}} \geq 0 \) and denote \( \tilde{\mu}_d := \mu_h - \mu_{\tilde{Q}} \). Moreover, we assume that \( \tilde{Q}(s, \cdot; \theta) \) and \( Q^\pi(s, \cdot) \) are Lipschitz continuous s.t.

\[
\begin{align*}
|\tilde{Q}(s, a_1; \theta) - \tilde{Q}(s, a_2; \theta)| &\leq M_{\tilde{Q}} \|a_1 - a_2\|, \forall a_1, a_2 \in A, \\
|Q^\pi(s, a_1) - Q^\pi(s, a_1)| &\leq M_{\tilde{Q}} \|a_1 - a_2\|, \forall a_1, a_2 \in A.
\end{align*}
\]

We make a couple remarks. First, note that the Lipschitz constant of \( L_{\pi_k}(s, a) \) can be very large, since \( \eta_k \) can possibly be small. That explains why we need to handle the two terms \( \tilde{Q}(s, a; \theta_k) \) and \( \langle \tilde{\omega}(s; \theta_k), a \rangle \) in the definition of \( \tilde{L} \) separately. Second, the aforementioned assumptions hold if \( Q^\pi(s, \cdot) \) is continuous and its approximation \( \tilde{Q}(s, \cdot; \theta_k) \) is continuous and Lipschitz smooth for any \( s \in S \) \cite[Lemma 4.2]{13}. Appendix A presents sufficient conditions for these assumptions to take place.

In order to find \( \theta_k \), we can possibly formulate the following statistical learning/estimation problem

\[
\min_{\theta \in \Theta} \left\{ \mathbb{E}_{s \sim P}[\|\tilde{Q}(s, \cdot; \theta) - Q^\pi(s, \cdot)\|^2 + \|\tilde{\omega}(s; \theta) - \nabla \omega(\pi_k(s))/\eta_k\|^2] \right\},
\]

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for a given distribution $\rho$ over $\mathcal{S}$, where $\| \cdot \|_*$ is the dual norm of a given arbitrary norm $\| \cdot \|$ of the action space. This problem is also convex if $\tilde{Q}(s; a) = \tilde{Q}(s, \theta_k) - Q_{\pi_k}(s, a)$, $\tilde{\omega}(s; \theta_k)$ and $\nabla \omega(s; \theta)$ are linear w.r.t. $\theta$. Let us denote

\begin{align}
(3.40) \quad & \delta^Q_k(s, a) := \tilde{Q}(s, a; \theta_k) - Q_{\pi_k}(s, a), \\
(3.41) \quad & \delta^\omega_k(s) = \tilde{\omega}(s; \theta_k) - \frac{1}{\eta_k} \nabla \omega(\pi_k(s)), \\
(3.42) \quad & \nabla \omega(s; \theta_k) = \left[ E[\nabla \omega(s; \theta_k)] - \frac{1}{\eta_k} \nabla \omega(\pi_k(s)) + \tilde{\omega}(s; \theta_k) - E[\tilde{\omega}(s; \theta_k)] \right].
\end{align}

We start with the convex case when $\tilde{\mu}_d \geq 0$. In this case, we need to make the following assumption about the approximation error $\delta^Q_k(s, a)$ and $\delta^\omega_k(s)$:

\begin{align}
(3.43) \quad & E_{\xi_k} \{ E_{s \sim \nu^*} [\| \delta_k(s, \pi_k(s)) \| + \| \delta_k(s, \pi^*(s)) \|] \} \leq \varsigma^Q, \\
(3.44) \quad & E_{\xi_k} \left\{ E_{s \sim \nu^*} [\| \delta^\omega_k(s) \|_*] \right\} \leq \varsigma^\omega, \\
(3.45) \quad & E_{\xi_k} \left\{ E_{s \sim \nu} [\| \delta^\omega_{k, \text{step}}(s) \|_*^2] \right\} \leq (\sigma^\omega)^2.
\end{align}

Similar to the finite action, Appendix A outlines sufficient conditions for these error bounds to hold.

**Theorem 4.** Suppose (3.43)-(3.45) hold. Also assume that the stepsize $\eta_k$ satisfies

\begin{equation}
(3.46) \quad \frac{\beta_k}{\eta_k} \leq \beta_{k-1}(\tilde{\mu}_d + \frac{1}{\eta_{k-1}}), k \geq 1,
\end{equation}

for some $\beta_k \geq 0$. Then

\begin{align}
& \left| \sum_{t=0}^{k-1} \beta_k \right|^{-1} \left\{ (1 - \gamma) \left| \sum_{t=0}^{k-1} \beta_k E[f(\pi_t) - f^*] + \beta_{k-1}(\tilde{\mu}_d + \frac{1}{\eta_{k-1}}) E[\tilde{D}(\pi_k, \pi^*)] \right\} \\
& \leq \left| \sum_{t=0}^{k-1} \beta_k \right|^{-1} \left\{ \frac{\beta_k}{\eta_k} E[D(\pi_0, \pi^*)] + \sum_{t=0}^{k-1} \beta_k \eta_k [(2MQ + M_Q + M_h)^2 + (\sigma^\omega)^2] \right\} + \varsigma^Q + \varsigma^\omega \tilde{D}_A,
\end{align}

where $\tilde{D}$ is defined in (3.11) and

\begin{equation}
(3.47) \quad \tilde{D}_A := \max_{a_1, a_2 \in A} D(a_1, a_2).
\end{equation}

**Proof.** By the optimality condition of (3.18), we have

\begin{align}
& \tilde{L}(s, \pi_{k+1}(s); \theta_k) - \tilde{L}(s, a; \theta_k) + h^{\pi_{k+1}(s)}(s) - h^a(s) + \frac{1}{\eta_k} [\omega(\pi_{k+1}(s)) - \omega(a)] + (\tilde{\mu}_d + \frac{1}{\eta_k}) D(\pi_{k+1}(s), a) \leq 0,
\end{align}

which implies that

\begin{align}
& L_k(s, \pi_{k+1}(s)) - L_k(s, a) + h^{\pi_{k+1}(s)}(s) - h^a(s) + \frac{1}{\eta_k} [\omega(\pi_{k+1}(s)) - \omega(a)] + (\tilde{\mu}_d + \frac{1}{\eta_k}) D(\pi_{k+1}(s), a) \\
& + \delta^Q_k(s, \pi_{k+1}(s)) - \delta^Q_k(s, a) + \langle \delta^\omega_k(s), \pi_{k+1}(s) - a \rangle \leq 0.
\end{align}

Similarly to (3.29), we can show that

\begin{equation}
(3.49) \quad \psi^{\pi_k}(s, \pi_{k+1}(s)) - \psi^{\pi_k}(s, a) + h^{\pi_{k+1}(s)}(s) - h^{\pi_k}(s) + \frac{1}{\eta_k} [D(\pi_k(s), \pi_{k+1}(s)) - D(\pi_k(s), a)] \\
+ (\tilde{\mu}_d + \frac{1}{\eta_k}) D(\pi_{k+1}(s), a) + \delta^Q_k(s, \pi_{k+1}(s)) - \delta^Q_k(s, a) + \langle \delta^\omega_k(s), \pi_{k+1}(s) - a \rangle \leq 0.
\end{equation}

Noting that

\begin{align}
(3.50) \quad & \psi^{\pi_k}(s, \pi_{k+1}(s)) = Q^{\pi_k}(s, \pi_{k+1}(s)) - Q^{\pi_k}(s, \pi_k(s)) \geq -M_Q \| \pi_{k+1}(s) - \pi_k(s) \|, \\
(3.51) \quad & h^{\pi_{k+1}(s)}(s) - h^{\pi_k}(s) \geq -M_h \| \pi_{k+1}(s) - \pi_k(s) \|, \\
& \delta^Q_k(s, \pi_{k+1}(s)) = \delta^Q_k(s, \pi_{k+1}(s)) - \delta^Q_k(s, \pi_k(s)) + \delta^Q_k(s, \pi_k(s)) \\
& \geq -(M_Q + M_Q) \| \pi_{k+1}(s) - \pi_k(s) \| + \delta^Q_k(s, \pi_k(s)), \\
(3.53) \quad & \langle \delta^\omega_k(s), \pi_{k+1}(s) - a \rangle = \langle \delta^\omega_k(s), \pi_{k+1}(s) - a \rangle + \langle \delta^\omega_k(s), \pi_k(s) - a \rangle \\
& + \langle \delta^\omega_{k, \text{step}}(s), \pi_{k+1}(s) - a \rangle \\
& \geq -\| \delta^\omega_{k, \text{step}}(s) \|_* \tilde{D}_A - \| \delta^\omega_{k, \text{step}}(s) \|_* \| \pi_{k+1}(s) - \pi_k(s) \| + \langle \delta^\omega_{k, \text{step}}(s), \pi_k(s) - a \rangle,
\end{align}

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we then conclude from (3.49) and Young’s inequality that
\[
- \psi^{n_k}(s, a) + (\tilde{\mu}_d + \frac{1}{\sqrt{k}}) D(\pi_{k+1}(s), a) \\
\leq \frac{1}{k_0} D(\pi_k(s), a) + \frac{m_k}{2} (2M_Q + M_\eta) + \|\delta_k^{\omega,sto}(s)\|_*^2 \\
- \delta_k^Q(s, \pi_k(s)) + \delta_k^{\omega, det}(s) \|_* \tilde{D}_A - \langle \delta_k^{\omega, sto}(s), \pi_k(s) - a \rangle.
\]

The rest of the proof is similar to that in Theorem 3.

The following result follows immediately from Theorem 3 by straightforward calculations, and it resembles Corollary 1 for RL with finite action spaces.

**Corollary 2.**

a) If \( \tilde{\mu}_d = 0 \), the number of iterations \( k \) is given a priori, and
\[
\eta_t = \sqrt{\frac{kD(\pi_1, \pi^*)}{k(2M_Q + M_\eta + M_h)^2 + (\sigma^*)^2}} \quad \text{and} \quad \beta_t = 1, t = 0, \ldots, k - 1,
\]
then
\[
(1 - \gamma) \frac{1}{k} \sum_{t=0}^{k-1} [f(\pi_t) - f^*] \leq 2 \sqrt{\frac{kD(\pi_0, \pi^*)[(2M_Q + M_\eta + M_h)^2 + (\sigma^*)^2]}{k}} + \tilde{\zeta} Q + \tilde{\zeta} \tilde{D}_A.
\]

b) If \( \tilde{\mu}_d > 0 \) and \( \eta_t = \frac{1}{\tilde{\mu}_d(t+1)} \) and \( \beta_t = 1, t = 0, \ldots, k - 1, \) then
\[
(1 - \gamma) \frac{1}{k} \sum_{t=0}^{k-1} [f(\pi_t) - f^*] + \tilde{\mu}_d E[D(\pi_k, \pi^*)] \\
\leq \frac{1}{k} \left\{ \tilde{\mu}_d D(\pi_1, \pi^*) + \frac{2}{\tilde{\mu}_d} [(2M_Q + M_\eta + M_h)^2 + (\sigma^*)^2] \log k \right\} + \zeta Q + \zeta \tilde{D}_A.
\]

c) If \( \tilde{\mu}_d > 0 \) and \( \eta_k = \frac{2}{\tilde{\mu}_d(k+3)} \) and \( \beta_t = t + 2, t = 0, \ldots, k - 1, \) then
\[
\frac{2(1 - \gamma)}{k(k+3)} \sum_{t=0}^{k-1} (t + 2) E[f(\pi_t) - f^*] + \frac{2}{k} \tilde{\mu}_d^2 E[D(\pi_k, \pi^*)] \\
\leq \frac{2}{k(k+3)} \tilde{\mu}_d D(\pi_0, \pi^*) + \frac{8}{\tilde{\mu}_d(k+3)} [(2M_Q + M_\eta + M_h)^2 + (\sigma^*)^2] + \zeta Q + \zeta \tilde{D}_A.
\]

We now consider the case when \( \tilde{\mu}_d < 0 \). In this case, our goal is to find an approximate stationary point and we need to assume estimation error terms \( \delta_k^{\omega, det}(s) \) and \( \delta_k^{\omega, sto}(s) \) have bounded support, i.e.,
\[
(3.55) \quad \|\delta_k^{\omega, det}(s)\|_* \leq \tilde{\zeta} \omega \quad \text{and} \quad \|\delta_k^{\omega, sto}(s)\|_* \leq \tilde{\sigma} \omega.
\]

On the other hand, we do not need to make any assumptions about the error term \( \delta_k^Q(s, a) \) as long as it is Lipschitz continuous w.r.t. \( a \).

**Theorem 5.** Let the number of iterations \( k \) is fixed a priori. Suppose that (3.55) holds. Also assume that \( \tilde{\mu}_d < 0 \) and that \( \eta = \frac{\eta d}{\sqrt{k}} \) for \( t = 0, \ldots, k - 1. \) Then for any \( s \in \mathcal{S} \), there exist iteration indices \( k(s) \) such that if we run the \( k \) iterations of the PMD method, s.t.
\[
(3.56) \quad - \psi^{n_k}(s, \pi_{k+1}(s)) \leq \frac{1}{k} [V^{n_0}(s) - V^{n^*}(s)] + \gamma \frac{1}{1-\gamma} \left[ \frac{(M_h + M_\eta + M_\omega)^2}{\sqrt{k}} + \tilde{\zeta} \omega \tilde{D}_A \right],
\]
\[
\frac{1}{2\eta} D(\pi_k(s), \pi_{k+1}(s)) + (\tilde{\mu}_d + \frac{1}{\sqrt{k}}) D(\pi_{k+1}(s), \pi_k(s)) \\
\leq \frac{1}{k} [V^{n_0}(s) - V^{n^*}(s)] + \frac{1}{1-\gamma} \left[ \frac{(M_h + M_\eta + M_\omega)^2}{\sqrt{k}} + \tilde{\zeta} \omega \tilde{D}_A \right],
\]

where \( \tilde{D}_A \) is defined in (3.47).

**Proof.** Setting \( a = \pi_k(s) \) in (3.49), and using the facts that \( \psi^{n_k}(s, \pi_k(s)) = 0 \) and \( D(\pi_k(s), \pi_k(s)) = 0 \), we obtain
\[
\psi^{n_k}(s, \pi_{k+1}(s)) + h^{n_{k+1}}(s) - h^{n_k}(s) + \frac{1}{\eta} D(\pi_k(s), \pi_{k+1}(s)) \\
+ (\tilde{\mu}_d + \frac{1}{\eta}) D(\pi_{k+1}(s), \pi_k(s)) + \delta_k^{\omega, det}(s, \pi_{k+1}(s)) - \delta_k^Q(s, \pi_{k+1}(s)) - \delta_k^Q(s, a) + \langle \delta_k^{\omega, sto}(s), \pi_{k+1}(s) - a \rangle \leq 0.
\]
Using the above inequality, (3.51)-(3.54), and Young’s inequality, we then have
\[
\psi_{\pi_k}(s, \pi_{k+1}(s)) + \frac{1}{2\mu_k}D(\pi_k(s), \pi_{k+1}(s)) + (\bar{\mu} + \frac{1}{\mu_k})D(\pi_k(s), \pi_k(s)) \\
\leq (M_h + M_Q + M_\bar{Q} + \|\delta_k^{\pi,e}(s)\|_1)\|\pi_{k+1}(s) - \pi_k(s)\| - \frac{1}{2\mu_k}D(\pi_k(s), \pi_{k+1}(s)) + \|\delta_k^{\pi,e}(s)\|_1\bar{D}_\mathcal{A} \\
\leq \eta_k(M_h + M_Q + M_\bar{Q} + \|\delta_k^{\pi,e}(s)\|_1)^2 + \|\delta_k^{\pi,e}(s)\|_1\bar{D}_\mathcal{A}.
\]
(3.58)

Similar to (4.10), we can show that
\[
V^{\pi_{k+1}}(s) - V^{\pi_k}(s) = \frac{1}{1-\gamma} \int \psi_{\pi_k}(s, \pi_{k+1}(q))n^{\pi_{k+1}}_s(dq) \\
= \frac{1}{1-\gamma} \int \psi_{\pi_k}(s, \pi_{k+1}(q)) - \left[\eta_k(M_h + M_Q + M_\bar{Q} + \bar{\omega})^2 + \bar{\omega} \bar{D}_\mathcal{A}\right]n^{\pi_{k+1}}_s(dq) \\
+ \frac{1}{1-\gamma} \left[\eta_k(M_h + M_Q + M_\bar{Q} + \bar{\omega})^2 + \bar{\omega} \bar{D}_\mathcal{A}\right] \\
\leq \psi_{\pi_k}(s, \pi_{k+1}(s)) + \frac{1}{1-\gamma} \left[\eta_k(M_h + M_Q + M_\bar{Q} + \bar{\omega})^2 + \bar{\omega} \bar{D}_\mathcal{A}\right].
\]
(3.59)

The result in (3.56) follows by taking the telescopic sum of (3.59), while the one in (3.57) follows directly from (3.56) and (3.58).

Observe that the assumptions in (3.55) are needed to provide a constant upper bound in (3.58). Otherwise, if we only make the weaker assumptions in (3.44) and (3.45), the relation in (3.59) does not necessarily hold since the distribution \(\kappa_s^{\pi_{k+1}}\) depends on the random sample \(\xi_k\) generated in the \(k\)-th iteration. In the next section, we will show that this issue can possibly be addressed by using the policy dual averaging method.

4. Policy Dual Averaging. Inspired by the dual averaging method for convex optimization [36,53], we present in this section a new class of policy gradient type method, referred to as policy dual averaging (PDA), for reinforcement learning. Similar to the previous section, we will first present the generic algorithmic scheme in subsection 4.1 and discuss the incorporation of function approximation into PDA in subsection 4.2. The convergence properties of PDA under stochastic policy evaluation will be established in subsection 4.3 and 4.4, respectively, for RL problems with finite and continuous action spaces.

4.1. The Generic Algorithmic Scheme. In this section, we assume that we have access to exact action-value function \(Q^\pi\) for a given policy \(\pi\). We present a basic policy dual averaging method for RL and establish its convergence properties.

Let \(\|\cdot\|\) be given norm in action space \(\mathcal{A}\) and \(\omega : \mathcal{A} \rightarrow \mathbb{R}\) be a strongly convex function w.r.t. to \(\|\cdot\|\) and \(D(\cdot, \cdot)\) be its associated Bregman distance defined in (1.1). At the \(k\)-iteration of the PDA method (see Algorithm 3), we minimize \(\Psi_k\) defined as the weighted sum of the advantage functions plus the regularization term \(\lambda_k D(\pi_0(s), a)\) (see (4.1)). For notational convenience, we set \(\Psi_{-1}(s, a) = \lambda_{-1} D(\pi_0(s), a)\) for some \(\lambda_{-1} \geq 0\). Since \(\psi_{\pi^*}(s, a)\) differs from \(Q^\pi(s, a) + h^\pi(s)\) by a constant independent of \(a\), the update in (4.1) is equivalent to (4.2), and we only need to evaluate the action-value function \(Q^\pi(s, a)\) at each iteration. Different from the PMD method in the previous section, the prox-center \(\pi_0(s)\) in the regularization term \(\lambda_k D(\pi_0(s), a)\) does not change over different iterations. This fact makes it more convenient to implement function approximation techniques in PDA. While it is easy to see that PDA is equivalent to PMD under certain specific settings (e.g. in the tabular setting with KL divergence as the Bregman distance), the algorithmic scheme of PDA is different from PMD in general, and its convergence for solving general RL problems has not been studied before in the literature.

In the remaining part of this subsection, our goal is to show that PDA achieves mostly comparable performance to the PMD method under exact policy evaluation. However, the analysis of PDA is significantly different from PMD as well as the dual averaging method for convex optimization.

Similar to the previous section, we need to assume that \(Q^\pi(s, \cdot)\) is \(\mu_Q\)-weakly convex and define \(\mu_d := \mu_h - \mu_Q\) (see (3.2)). We say that the advantage function \(\psi_{\pi^*}(s, \cdot)\) is convex if \(\mu_d \geq 0\), and weakly convex otherwise. Then, for some properly chosen \(\lambda_k \geq 0\), the subproblem in (4.1) is strongly convex and has a unique solution characterized by the following result.

Lemma 4. Suppose that \(\{\beta_t\}\) and \(\lambda_k\) are chosen such that
\[
\mu_k := \mu_d \sum_{t=0}^{k} \beta_t + \lambda_k \geq 0, \forall k \geq -1,
\]
(4.3)
Algorithm 3 The policy dual averaging (PDA) method

\textbf{Input:} $\beta_k \geq 0$, $\lambda_k \geq 0$ and initial policy $\pi_0(s) \in A$.

for $k = 0, 1, \ldots$, do

\begin{equation}
\pi_{k+1}(s) = \arg \min_{a \in A} \left\{ \Psi_k(s, a) := \sum_{t=0}^k \beta_t \psi \pi_t(s, a) + \lambda_k D(\pi_0(s), a) \right\}
\end{equation}

\begin{equation}
= \arg \min_{a \in A} \left\{ \sum_{t=0}^k \beta_t [Q^\pi(s, a) + h^a(s)] + \lambda_k D(\pi_0(s), a) \right\}, \forall s \in S.
\end{equation}

end for

where we denote $\mu_- = \lambda_-$. Then for any $k \geq -1$, we have

\begin{equation}
\Psi_k(s, \pi_{k+1}(s)) + \mu_k D(\pi_{k+1}(s), a) \leq \Psi_k(s, a), \forall s \in S, \forall a \in A.
\end{equation}

\textbf{Proof.} Note that the result holds obviously for $k = -1$. For any $k \geq 0$, observe that by (3.1), $\Psi_k(s, \cdot)$ is strongly convex w.r.t. modulus $\mu_k$. The result then follows immediately from the optimality condition of (4.1) [23, Lemma 3.5].

Proposition 3 establishes some descent property of the PDA method.

PROPOSITION 3. If $\beta_k > 0$, relation (4.3) holds, and

\begin{equation}
\lambda_k \geq \lambda_{k-1}, \forall k \geq 0,
\end{equation}

then

\begin{equation}
V^{\pi_{k+1}}(s) - V^{\pi_k}(s) \geq \frac{\gamma (\lambda_k - \lambda_{k-1})}{(1 - \gamma \beta_k)} \bar{D}_A \leq \psi^{\pi_k}(s, \pi_{k+1}(s))
\end{equation}

\begin{equation}
= -\frac{\mu_k}{\beta_k} D(\pi_{k+1}(s), \pi_k(s)) - \lambda_k D(\pi_k(s), \pi_{k+1}(s)) + \frac{\lambda_k - \lambda_{k-1}}{\beta_k} \bar{D}_A
\end{equation}

for any $k \geq 0$, where $\bar{D}_A$ is defined in (3.47)

\textbf{Proof.} First note that by the definition of $\Psi_k$ in (4.1) and Lemma 4 (applied to the $k - 1$ iteration), we have for any $k \geq 0,$

\begin{equation}
\beta_k \psi^{\pi_k}(s, a) = \Psi_k(s, a) - \Psi_{k-1}(s, a) - (\lambda_k - \lambda_{k-1}) D(\pi_0(s), a)
\end{equation}

\begin{equation}
\leq \Psi_k(s, a) - \Psi_{k-1}(s, \pi_k(s)) - \mu_{k-1} D(\pi_k(s), a) - (\lambda_k - \lambda_{k-1}) D(\pi_0(s), a).
\end{equation}

Setting $a = \pi_{k+1}(s)$ and using (4.5) in the above relation, we obtain

\begin{equation}
\beta_k \psi^{\pi_k}(s, \pi_{k+1}(s)) \leq \Psi_k(s, \pi_{k+1}(s)) - \Psi_{k-1}(s, \pi_k(s)) - \mu_{k-1} D(\pi_k(s), \pi_{k+1}(s)).
\end{equation}

By the above inequality, and Lemma 4 (with $a = \pi_k(s)$ in (4.4)), we have

\begin{equation}
\beta_k \psi^{\pi_k}(s, \pi_{k+1}(s)) \leq \Psi_k(s, \pi_k(s)) - \mu_k D(\pi_{k+1}(s), \pi_k(s)) - \Psi_{k-1}(s, \pi_k(s)) - \mu_{k-1} D(\pi_k(s), \pi_{k+1}(s))
\end{equation}

\begin{equation}
= -\frac{\mu_k}{\beta_k} D(\pi_{k+1}(s), \pi_k(s)) - \mu_{k-1} D(\pi_k(s), \pi_{k+1}(s)) + (\lambda_k - \lambda_{k-1}) D(\pi_0(s), \pi_k(s))
\end{equation}

\begin{equation}
\lambda_k \geq \lambda_{k-1}
\end{equation}

where the last two identities follows from the definition of $\Psi_k$ and the fact that $\psi^{\pi_k}(s, \pi_k(s)) = 0$. The previous conclusion and the assumption $\beta_k > 0$ then clearly implies that

\begin{equation}
\psi^{\pi_k}(s, \pi_{k+1}(s)) \leq \frac{1}{1 - \gamma} (\lambda_k - \lambda_{k-1}) \bar{D}_A \leq \frac{\mu_k}{\beta_k} D(\pi_{k+1}(s), \pi_k(s)) - \mu_{k-1} D(\pi_k(s), \pi_{k+1}(s)) \leq 0, \forall s \in S.
\end{equation}

By Lemma 1, (4.10), and the fact that $\kappa^{\pi_{k+1}}(\{s\}) \geq 1 - \gamma$ due to (2.9), we have

\begin{equation}
V^{\pi_{k+1}}(s) - V^{\pi_k}(s) = \frac{1}{1 - \gamma} \int [\psi^{\pi_k}(q, \pi_{k+1}(q)) - \frac{1}{\beta_k} (\lambda_k - \lambda_{k-1}) \bar{D}_A] \kappa^{\pi_{k+1}}(dq) + \frac{1}{(1 - \gamma) \beta_k} (\lambda_k - \lambda_{k-1}) \bar{D}_A
\end{equation}

\begin{equation}
\leq \frac{1}{1 - \gamma} [\psi^{\pi_k}(s, \pi_{k+1}(s)) - \frac{1}{\beta_k} (\lambda_k - \lambda_{k-1}) \bar{D}_A] \kappa^{\pi_{k+1}}(\{s\}) + \frac{1}{(1 - \gamma) \beta_k} (\lambda_k - \lambda_{k-1}) \bar{D}_A
\end{equation}

\begin{equation}
\leq \psi^{\pi_k}(s, \pi_{k+1}(s)) + \frac{\gamma}{(1 - \gamma) \beta_k} (\lambda_k - \lambda_{k-1}) \bar{D}_A.
\end{equation}
which together with (4.10) clearly imply the result.

Below we establish the linear convergence rate of PDA for the case when the advantage function defined in (2.11) is convex w.r.t. \( a \), i.e., \( \mu_d \geq 0 \) in (4.3).

**Theorem 6.** Suppose that \( \beta_k > 0 \) and that relations (4.3) and (4.5) hold for the PDA method. Then

\[
\sum_{t=0}^{k-1} \left\{ \beta_t \left[ f(\pi_{t+1}) - f(\pi^*) \right] - \gamma [f(\pi_t) - f(\pi^*)] \right\} - \frac{\gamma (\lambda_k - \lambda_{t+1})}{(1-\gamma) \beta_t} \bar{D}_A \right\} + \mu_{k-1} \mathcal{D}(\pi_k, \pi^*) \\
+ \sum_{t=0}^{k-1} [\mu_{t-1} \mathcal{D}(\pi_t, \pi_{t+1})] \leq \lambda_k \mathcal{D}(\pi_0, \pi^*),
\]

(4.12)

for any \( k \geq 1 \), where where \( \mathcal{D} \) and \( \bar{D}_A \) are defined in (3.11) and (3.47), respectively. In particular, if \( \mu_d \geq 0 \), \( \beta_k = \gamma^{-k} \), and \( \lambda_k = \lambda \) for some \( \lambda > 0 \) for any \( k \geq 1 \), then

\[
[f(\pi_k) - f(\pi^*)] + [(1 - \gamma^k) \mu_d / (1 - \gamma) + \lambda \gamma^{k-1}] \mathcal{D}(\pi_k, \pi^*) \\
\leq \gamma^{k-1} \lambda^2 \mathcal{D}(\pi_0, \pi^*) + \gamma^k [f(\pi_0) - f(\pi^*)], \quad \forall k \geq 1.
\]

If in addition, \( \lambda = 0 \) in the PDA algorithm, then

\[
[f(\pi_k) - f(\pi^*)] + (1 - \gamma^k) \mu_d / (1 - \gamma) \mathcal{D}(\pi_k, \pi^*) \leq \gamma^k [f(\pi_0) - f(\pi^*)], \quad \forall k \geq 1.
\]

**Proof.** We first show part a). By taking the telescopic sum of (4.8), we have

\[
\sum_{t=0}^{k-1} \beta_t \Psi_t(s, \pi_{t+1}(s)) \leq \Psi_{k-1}(s, \pi_k(s)) - \Psi_{-1}(s, \pi_0(s)) - \sum_{t=0}^{k-1} [\mu_{t-1} \mathcal{D}(\pi_t(s), \pi_{t+1}(s))] \\
= \Psi_{k-1}(s, \pi_k(s)) - \sum_{t=0}^{k-1} [\mu_{t-1} \mathcal{D}(\pi_t(s), \pi_{t+1}(s))],
\]

(4.14)

where the equality follows from the fact that \( \Psi_{-1}(s, \pi_0(s)) = \lambda^{-1} \mathcal{D}(\pi_0(s), \pi_0(s)) = 0 \). It then follows from the above conclusion and Lemma 4 that

\[
\sum_{t=0}^{k-1} \beta_t \Psi_t(s, \pi_{t+1}(s)) \\
\leq \Psi_{k-1}(s, a) - \mu_{k-1} \mathcal{D}(\pi_k(s), a) - \sum_{t=0}^{k-1} [\mu_{t-1} \mathcal{D}(\pi_t(s), \pi_{t+1}(s))] \\
= \sum_{t=0}^{k-1} \beta_t \Psi_t(s, a) + \lambda_{k-1} \mathcal{D}(\pi_0(s), a) - \mu_{k-1} \mathcal{D}(\pi_k(s), a) - \sum_{t=0}^{k-1} [\mu_{t-1} \mathcal{D}(\pi_t(s), \pi_{t+1}(s))],
\]

where the identity follows from the definition of \( \Psi_k \). Using (4.6) and the above inequality, we obtain

\[
\sum_{t=0}^{k-1} \beta_t [V^{\pi_{t+1}}(s) - V^{\pi^*}(s)] + \lambda_k \mathcal{D}(\pi_0(s), a) - \mu_{k-1} \mathcal{D}(\pi_k(s), a) \\
\leq \sum_{t=0}^{k-1} \beta_t [V^\pi(s) - V^{\pi^*}(s)] + \lambda_k \mathcal{D}(\pi_0(s), a) - \mu_{k-1} \mathcal{D}(\pi_k(s), a)
\]

(4.15)

for any \( s \in \mathcal{S} \) and \( a \in \mathcal{A} \). Setting \( a = \pi^*(s) \) in (4.15), taking expectation w.r.t. \( \nu^* \) on its both sides, and using Lemma 2, we then obtain

\[
\sum_{t=0}^{k-1} \left\{ \beta_t \int [V^{\pi_{t+1}}(s) - V^{\pi^*}(s)] \nu^*(ds) - \frac{\gamma (\lambda_k - \lambda_{t+1})}{(1-\gamma) \beta_t} \bar{D}_A \right\} + \sum_{t=0}^{k-1} \int [\mu_{t-1} \mathcal{D}(\pi_t(s), \pi_{t+1}(s))] \nu^*(ds) \\
\leq \sum_{t=0}^{k-1} \beta_t (1 - \gamma) \int [V^\pi(s) - V^{\pi^*}(s)] \nu^*(ds) + \lambda_k \int \mathcal{D}(\pi_0(s), \pi^*(s)) \nu^*(s) - \mu_{k-1} \int \mathcal{D}(\pi_k(s), \pi^*(s)) \nu^*(s).
\]

Using the definition of \( f \) and \( \mathcal{D} \), and rearranging the terms, we have

\[
\sum_{t=0}^{k-1} \left\{ \beta_t [f(\pi_{t+1}) - f(\pi^*)] - \frac{\gamma (\lambda_k - \lambda_{t+1})}{(1-\gamma) \beta_t} \bar{D}_A \right\} + \sum_{t=0}^{k-1} [\mu_{t-1} \mathcal{D}(\pi_t, \pi_{t+1})] \\
\leq \sum_{t=0}^{k-1} \beta_t (1 - \gamma) [f(\pi^*) - f(\pi_t)] + \lambda_k \mathcal{D}(\pi_0, \pi^*) - \mu_{k-1} \mathcal{D}(\pi_k, \pi^*),
\]

which implies (4.12) after rearranging the terms. Now using (4.12) and the selection of \( \beta_k \) and \( \lambda_k \), we have

\[
\gamma^{-(k-1)} [f(\pi_k) - f(\pi^*)] - \gamma [f(\pi_0) - f(\pi^*)] + \mu_{k-1} \mathcal{D}(\pi_k, \pi^*) \leq \lambda \mathcal{D}(\pi_0, \pi^*).
\]

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Also note that by (4.3), \( \mu_k = \mu_d \sum_{t=0}^{k-1} \gamma^{-t} + \lambda = \frac{\mu_d (1 - \gamma^k)}{\gamma t (1 - \gamma)} + \lambda \). Combining the above two relations, we obtain
\[
\gamma^{-(k-1)} [f(\pi_k) - f(\pi^*)] + [(1 - \gamma^k) \gamma^{-k} \mu_d / (1 - \gamma)] D(\pi_k, \pi^*) \\
\leq \lambda D(\pi_0, \pi^*) + \gamma [f(\pi_0) - f(\pi^*)],
\]
from which (4.13) follows immediately.

We add some remarks about Theorem 6. First, the PDA method achieves a linear rate of convergence with properly chosen \( \beta_k \). Since \( \beta_k \) in Theorem 6 is geometrically increasing, in the limiting case the PDA will be approaching the classic policy iteration method. Below we provide a simple and alternative convergence analysis of the policy iteration method as it might of some interest in its own right.

Corollary 3. Suppose \( \pi_{k+1}(s) = \arg \min_{a \in A} \psi_{\pi_k}(s, a) \) for any \( s \in S \). Then we have \( f(\pi_{k+1}) - f(\pi^*) \leq \gamma [f(\pi_k) - f(\pi^*)] \).

Proof. By the definition of \( \pi_{k+1} \), we have
\[
\psi_{\pi_k}(s, \pi_{k+1}(s)) - \psi_{\pi_k}(s, a) \leq 0, \forall a \in A.
\]
Setting \( a = \pi_k(s) \), we have \( \psi_{\pi_k}(s, \pi_{k+1}(s)) \leq \psi_{\pi_k}(s, \pi_k(s)) = 0 \). Therefore, similarly to (4.11), we have \( V_{\pi_{k+1}}(s) - V_{\pi_k}(s) \leq \psi_{\pi_k}(s, \pi_{k+1}(s)) \). Using the previous inequality in (4.16) (with \( a = \pi^*(s) \)), we have \( V_{\pi_{k+1}}(s) - V_{\pi_k}(s) \leq \psi_{\pi_k}(s, \pi^*(s)) \). Taking expectation w.r.t. \( s \sim \nu^* \), using Lemma 2 and the definition of \( f \), we conclude \( f(\pi_{k+1}) - f(\pi_k) \leq (1 - \gamma) [f(\pi^*) - f(\pi_k)] \), from which the result immediately follows.

Second, in view of Theorem 6, one can set \( \lambda = 0 \) and so it appears that adding the strongly convex term \( \lambda_k D(\pi_0(s), a) \) in the definition of \( \Psi_k \) is not needed. It should be noted, however, that this term will be critical in the following sense: a) for the convex settings (i.e., \( \mu_d \geq 0 \)), this term will help with suppressing the impact of the noisy estimation of \( Q_{\pi_k} \) as will be demonstrated later; and b) for the nonconvex setting (i.e., \( \mu_d < 0 \), adding \( \lambda_k D(\pi_0(s), a) \) will make the subproblem tractable and thus the policy easily computable. The following result discusses the convergence for PDA for RL problems with nonconvex advantage functions (i.e., \( \mu_d < 0 \)).

Theorem 7. Suppose that \( \mu_d < 0 \). Assume that the number of iterations \( k \) is fixed a priori, \( \lambda_t = \lambda = k(k + 1) / |\mu_d| \) and \( \beta_t = t + 1 \), \( t = 0, \ldots, k - 1 \). Then for any \( s \in S \), there exist iteration indices \( \bar{k}(s) \) s.t.
\[
\begin{align}
0 & \leq -\psi_{\pi_k(s)}(s, \pi_{\bar{k}(s)+1}(s)) \leq 2 V^{\pi_0(s)}(s) - V^\pi(s) \leq \frac{2 [V^{\pi_0(s)}(s) - V^\pi(s)]}{(1 - \gamma)(k + 1)}, \\
\frac{\mu_k}{\beta_k} D(\pi_{\bar{k}(s)+1}(s), \pi_{\bar{k}(s)}(s)) + \frac{\mu_k}{\beta_k} D(\pi_{\bar{k}(s)+1}(s), \pi_{\bar{k}(s)}(s)) & \leq 2 V^{\pi_0(s)} - V^\pi(s) \leq 2 V^{\pi_0(s)} - V^\pi(s) \leq \frac{2 [V^{\pi_0(s)}(s) - V^\pi(s)]}{(1 - \gamma)(k + 1)},
\end{align}
\]
where \( \mu_k = |\mu_d| k(k + 1) / 2 \).

Proof. Clearly, the relation (4.3) is satisfied. It follows from (4.6) that
\[
\beta_k \psi_{\pi_k(s)}(s, \pi_{\bar{k}(s)+1}(s)) \leq \psi_{\pi_k(s)}(s, \pi_{\bar{k}(s)+1}(s)) \leq 0.
\]
Taking telescopic sum, we have
\[
0 \leq \sum_{t=0}^{k-1} [-\beta_t \psi_{\pi_t(s)}(s, \pi_{t+1}(s))] \leq \frac{1}{1 - \beta_1} \sum_{t=0}^{k-1} \beta_t [V^{\pi_t}(s) - V^{\pi_{t+1}}(s)].
\]
The indices \( \bar{k}(s) \) are given by the one which minimizes \( -\beta_t \psi_{\pi_t(s)}(s, \pi_{t+1}(s)) \) for any \( s \in S \). Also notice that
\[
\sum_{t=0}^{k-1} \beta_t [V^{\pi_t}(s) - V^{\pi_{t+1}}(s)] = \beta_0 V^{\pi_0}(s) + \sum_{t=0}^{k-2} (\beta_{t+1} - \beta_t) V^{\pi_{t+1}}(s) - \beta_k V^{\pi_k}(s)
\]
\[
= \sum_{t=0}^{k-1} [V^{\pi_t}(s) - k V^{\pi_k}(s) \leq k [V^{\pi_0}(s) - V^{\pi_k}(s)],
\]
where the first inequality follows from the monotonicity of \( V^{\pi_k} \) due to (4.19) and the second inequality holds since \( V^{\pi_k}(s) \geq V^{\pi_0}(s) \). Combining the above two relations, we obtain the result in (4.17). The result in (4.18) follows immediately from (4.6) and (4.17).

Comparing Theorem 6 and Theorem 7 with Theorem 1 and Theorem 2, we can see that the PDA method exhibits similar convergence behavior to the PMD method. However, a possible limitation for the PDA method is that the relation in (4.18) does not necessarily imply the stationarity of the policy \( \pi_{\bar{k}(s)}(s) \) for the original problem in (2.8) and \( s \in S \), while the PMD method can possibly ensure stationarity (see the discussion after Theorem 2).
4.2. Function Approximation in PDA. In this subsection, we discuss how to incorporate function approximation into the PDA method described in the previous section to solve RL problems over general state spaces.

Similar to the PMD method, we also adopt the idea of using function approximation for the value functions in the PDA method. However, different from the PMD method that requires function approximation for an augmented value function $L^\pi(s, a)$, we will compute a function approximation $\hat{Q}(s, a; \theta_k)$, parameterized by $\theta_k$, directly for the action value function $Q^\pi(s, a)$ in (4.24), i.e.,

$$
(4.21) \quad \hat{Q}(s, a; \theta_k) \approx Q^\pi(s, a).
$$

As a result, the algorithmic parameters $\beta_k$ and $\lambda_k$ do not appear in the subjects that we need to approximate. On the other hand, in the PMD method the stepsize $\eta_k$ does enter into the definition of $L^\pi$, which may impact the magnitude of the stochastic first-order information of the policy evaluation problem (see the discussion after (3.22)). Moreover, the appearance of $\eta_k$ in $L^\pi$ also resulted stronger assumptions in (3.55) to guarantee the convergence of the PMD method for RL over continuous action spaces with $\mu_d < 0$.

**Algorithm 4** The policy dual averaging method with function approximation

**Input:** $\beta_k \geq 0$, $\lambda_k \geq 0$ and initial policy $\pi_0(s) \in \mathcal{A}$.

for $k = 0, 1, \ldots$, do

Define an approximate advantage function

$$
(4.22) \quad \hat{\psi}(s, a; \theta_k) := \hat{Q}(s, a; \theta_k) - V^\pi(s).
$$

Replace the update formulas in (4.1)-(4.2) by

$$
(4.23) \quad \pi_{k+1}(s) = \arg\min_{a \in \mathcal{A}} \left\{ \psi_k(s, a) := \sum_{t=0}^{k} \beta_t \hat{\psi}(s, a; \theta_t) + \lambda_k D(\pi_0(s), a) \right\}
$$

$$
(4.24) \quad = \arg\min_{a \in \mathcal{A}} \left\{ \sum_{t=0}^{k} \beta_t [\hat{Q}(s, a; \theta_t) + h^a(s)] + \lambda_k D(\pi_0(s), a) \right\}, \forall s \in \mathcal{S}.
$$

end for

Similar to PMD (c.f. the discussion in subsection 3.2), we can consider linearly independent basis functions or more powerful and sophisticated nonlinear function approximation methods, such as RKHS and neural networks, to estimate $Q^\pi(s, a)$ via $\hat{Q}(s, a; \theta_k)$. The equation in (4.24) then tells us how to compute the new policy $\pi_{k+1}$ from the parameters $\theta_{[k]} := \{\theta_0, \ldots, \theta_k\}$. Instead of saving every iteration of the policy $\pi_k(s)$ for all $s \in \mathcal{S}$, we only save these parameters used for function approximation. Similar to the PMD method, we suggest to compute $\pi_{k+1}(s)$ using (4.24) whenever needed, e.g., inside the policy evaluation procedure. In fact, given $\theta_{[k]}$, for any $s \in \mathcal{S}$, $\pi_{k+1}(s)$ can be computed by an explicit formula in some important cases. This happens, for example, when the approximation function $\hat{Q}(s, a; \theta)$ is linear w.r.t. to $a$ and $D$ is given by the KL divergence for solving RL problems with finite actions (see Section 4.3). In a more general setting, $\pi_{k+1}$ is represented as an optimal solution of problem (4.24). With a properly chosen $\lambda$, the objective function of (4.24) is strongly convex, and hence the policy $\pi_{k+1}$ is uniquely defined. Moreover, the dimension for the action space is usually relatively small, in comparison with the dimension of state space, for RL problems with continuous action space. Hence, any efficient numerical optimization procedures can be used to solve (4.24) quickly. For the sake of simplicity, we also assume throughout this section that exact solutions can be computed for (4.24).

Observe that when evaluating the term $\hat{Q}^k(s, a; \theta_{[k]}) := \sum_{t=0}^{k} \beta_t \hat{Q}(s, a; \theta_t)$ used in (4.24), one can possibly use the recursive formula

$$
(4.25) \quad \hat{Q}^k(s, a; \theta_{[k]}) = \hat{Q}^{k-1}(s, a; \theta_{[k-1]}) + \beta_k \hat{Q}(s, a; \theta_k)
$$

with $\hat{Q}^0(s, a; \theta_{[-1]}) = 0$. When $\hat{Q}(s, a; \theta)$ is a linear w.r.t. $\theta$, the above formula further simplifies to a recursive updating on $\sum_{t=0}^{k} \beta_t \theta_t$. In this way, we can also significantly save the memory required to store the parameters $\{\theta_1, \ldots, \theta_t\}$. For the more general case (e.g., $\hat{Q}(s, a; \theta)$ is given by a neural network), in order to
save both computation and memory, we can possibly modify the policy evaluation step in (4.21) so that we have a single set of parameters $\Theta_k$ satisfying
\begin{equation}
\hat{Q}(s, a; \Theta_k) \approx \tilde{Q}(s, a; \Theta_{k-1}) + \beta_k Q^\pi_k(s, a)
\end{equation}
at the $k$-th iteration with $\tilde{Q}(s, a; \Theta_{-1}) = 0$. In other words, we would apply function approximation directly for estimating $\sum_{t=0}^k \beta_t Q^\pi_t(s, a)$ in a recursive manner. However, this would require us to design a modified policy evaluation procedure and to consider the impact of the algorithmic parameters $\beta_k$ for function approximation. In this section we focus only on the policy evaluation step in (4.21) for the sake of simplicity.

### 4.3. PDA for General State and Finite Action Spaces.
In this subsection, we focus on RL with a finite set of actions $\{A_1, \ldots, A_n\}$. As mentioned earlier, in this case we search for an optimal randomized policy in the probability simplex $P = \Delta_{n,A}$ in (2.1), with the cost and transition probability defined in (2.2). For a given iterate $\pi \in \Pi$, we use
\begin{equation}
Q_0^\pi(s, A_i) := c_0(s, A_i) + h^\pi(s) + \gamma \int P_0(ds'|s, A_i)V^\pi_k(s') \quad i = 1, \ldots, nA,
\end{equation}
to denote the value function associated with each action $A_i$. For notational convenience, we also denote $Q_0^\pi(s) = (Q_0^\pi(s, A_1), \ldots, Q_0^\pi(s, A_n))$. It then follows that $Q^\pi(s, a) = \langle Q_0^\pi(s), a \rangle$. We suggest to approximate $Q^\pi(s, A_i)$ by $\hat{Q}_0(s, A_i; \theta_i)$, $i = 1, \ldots, nA$, where $\hat{Q}_0(s, A_i; \theta_i)$ denotes some function class parameterized by $\theta_i$. Let us denote $\hat{Q}_0(s; \theta_k) \equiv \langle \hat{Q}_0(s, A_1; \theta_k), \ldots, \hat{Q}_0(s, A_n; \theta_k) \rangle$ and $\hat{Q}(s, a; \theta_k) = \langle \hat{Q}(s; \theta_k), a \rangle$ also let $\tilde{Q}(s, a; \theta_k)$ be defined in (4.22). Then the subproblem in (4.23)-(4.24) reduces to
\begin{equation}
\pi_{k+1}(s) = \arg \min_{\alpha \in \Delta_{n,A}} \{\sum_{i=0}^k \beta_i \{[\hat{Q}_0(s; \theta_i), a] + h^\pi(s)\} + \lambda_k D(\pi_0(s), a)\}, \forall s \in S,
\end{equation}
It is well-known that the above problem has an explicit solution, for example, when $\omega(a) = \sum_{i=1}^{nA} a_i \log a_i$ and $h^\pi = \lambda K(a||\pi_0)$. In this section we focus only on the policy evaluation step in (4.21) for the sake of simplicity.
Theorem 8. Assume that $\beta_k$ and $\lambda_k$ satisfy

$$
\tilde{\mu}_k := \mu_h \sum_{l=0}^{k-1} \beta_l + \lambda_k \geq 0 \quad \text{and} \quad \lambda_{k+1} \geq \lambda_k, \forall k \geq -1,
$$

where we denote $\tilde{\mu}_1 = \lambda_{-1}$. Also let us denote $\tilde{\beta}_k := \sum_{l=1}^{k-1} \beta_l$. Then we have

$$
(1 - \gamma)\tilde{\beta}_k^{-1} \sum_{t=0}^{k-1} \beta_t E[f(\pi_t) - f(\pi^*)] + \tilde{\beta}_k^{-1} \tilde{\mu}_k \tilde{\beta}_k^{-1} E[D(\pi_k, \pi^*)]
$$

$$
\leq \lambda_{k-1} \tilde{\beta}_k^{-1} D(\pi_0, \pi^*) + \left[\frac{\bar{c}}{1 - \gamma} + M_h^2 + \sigma^2\right] \tilde{\beta}_k^{-1} \sum_{t=0}^{k-1} \left(\beta_t^2 / \tilde{\mu}_t + 1\right) + \zeta,
$$

where $D$ is defined in (3.11). In particular, if $\mu_h > 0$, $\beta_k = k + 1$ and $\lambda_k = \mu_h$, then

$$
\frac{2\lambda_k}{\gamma} \sum_{t=0}^{k-1} \left\{E[f(\pi_t) - f(\pi^*)]\right\} + \frac{4}{\mu_h(k+1)} \left[\frac{\bar{c}}{1 - \gamma} + M_h^2 + \sigma^2\right] + \zeta.
$$

Moreover, if $\mu_h = 0$, $\beta_k = k + 1$ and $\lambda_k = \lambda(k + 1)^{3/2}$ with $\lambda = \sqrt{2D(\pi_0, \pi^*)} [\frac{\bar{c}}{1 - \gamma} + M_h^2 + \sigma^2]^{-1/2}$, then

$$
\frac{2\lambda_k}{\gamma} \sum_{t=0}^{k-1} \left\{E[f(\pi_t) - f(\pi^*)]\right\} \leq 4 \left(\frac{\sqrt{\bar{c}}}{k+1} + \frac{2M_h}{k}\right) \sqrt{2D(\pi_0, \pi^*)} [\frac{\bar{c}}{1 - \gamma} + M_h^2 + \sigma^2] + \zeta.
$$

Proof. Similar to (4.14), we can show that

$$
\sum_{t=0}^{k-1} \beta_t \tilde{\psi}(s, \pi_{t+1}(s), \theta_t) \leq \sum_{t=0}^{k-1} \beta_t \tilde{\psi}(s, a, \theta_t) + \lambda_k - 1 D(\pi_0(s), a) - \tilde{\mu}_k - 1 D(\pi_k(s), a)
$$

Using the relation $\tilde{\psi}(s, a; \theta_k) = \psi^*(s, a) + \delta_k(s, a) = \psi^*(s, a) + \langle \delta_0(s, a) \rangle$ in (4.39), we have

$$
\sum_{t=0}^{k-1} \beta_t \psi^*(s, \pi_{t+1}(s)) \leq \sum_{t=0}^{k-1} \beta_t [\psi^*(s, a) + \langle \delta_0(s, a) \rangle] + \lambda_k - 1 D(\pi_0(s), a)
$$

Note that

$$
\psi^*(s, \pi_{t+1}(s)) = \langle Q^*_0(s), \pi_{t+1}(s) - \pi_t(s) \rangle + h^*(s) - h^*(s) \geq -\langle Q^*_0(s), \pi_{t+1}(s) - \pi_t(s) \rangle_1
$$

Also observe that

$$
\langle \delta_0(s), a - \pi_{t+1}(s) \rangle = \langle \delta_{0,t}(s) + \delta_{0,t}(s), a - \pi_{t+1}(s) \rangle
$$

Combining the above three relations, fixing $a = \pi^*(s)$ and rearranging the terms, we have

$$
(1 - \gamma) \sum_{t=0}^{k-1} \beta_t E[f(\pi_t) - f(\pi^*)] + \tilde{\mu}_k - 1 E[D(\pi_k, \pi^*)]
$$

$$
\leq \lambda_k - 1 D(\pi_0(s), \pi^*(s)) + \sum_{t=0}^{k-1} \beta_t [\langle \delta_{0,t}(s), \pi^*(s) - \pi_{t+1}(s) \rangle + \langle \delta_{0,t}(s), \pi^*(s) - \pi_t(s) \rangle]
$$

$$
+ \sum_{t=0}^{k-1} \beta_t \left(\frac{\bar{c}}{1 - \gamma} + M_h + \|\delta_{0,t}(s)\|_2\right) \pi_{t+1}(s) - \pi_t(s)\|_1 - \tilde{\mu}_t + 1 \pi_t(s) - \pi_{t+1}(s)\|_1
$$

$$
\leq \lambda_k - 1 D(\pi_0(s), \pi^*(s)) + \sum_{t=0}^{k-1} \beta_t \left[\|\delta_{0,t}(s)\|_2 + \|\delta_{0,t}(s)\|_2 \|\pi^*(s) - \pi_t(s)\|_1\right]
$$

$$
+ \sum_{t=0}^{k-1} \beta_t \left(\frac{\bar{c}}{1 - \gamma} + M_h + \|\delta_{0,t}(s)\|_2\right)\|\pi^*(s) - \pi_t(s)\|_1^2
$$

Taking expectation for $s$ w.r.t. $\nu^*$ and then taking expectation w.r.t. $F_k$, and using the assumptions in (4.33)-(4.34), and Lemma 2, we have

$$
(1 - \gamma) \sum_{t=0}^{k-1} \beta_t E[f(\pi_t) - f(\pi^*)] + \tilde{\mu}_k - 1 E[D(\pi_k, \pi^*)]
$$

$$
\leq \lambda_k - 1 D(\pi_0(s), \pi^*(s)) + \sum_{t=0}^{k-1} \beta_t \zeta + \sum_{t=0}^{k-1} \beta_t \left(\frac{\bar{c}}{1 - \gamma} + M_h^2 + \sigma^2\right)\|\pi^*(s) - \pi_t(s)\|_1^2,
$$
which clearly implies (4.36). The result in (4.37) follows from straightforward calculations. Similar to (4.51), we can show that

$$\frac{2(1-\gamma)}{k(k+1)} \sum_{t=0}^{k-1} (t\mathbb{E}[f(\pi_t) - f(\pi^*)]) \leq \frac{2\lambda D(\gamma_0(s),a)}{k+1} \sqrt{\frac{\sigma^2}{\lambda}} + \frac{4\sqrt{k+1}([\bar{t}\sqrt{\frac{\sigma^2}{\lambda}} + M_k])^2 + \sigma^2}{\lambda} + \zeta,$$

which implies (4.38) by the selection of $\lambda$.

Notice that in comparison with the PMD method, we do not need to fixed the number of iterations $k$ a priori when $\mu_h = 0$. As a result, the stepsize selection of the PDA method is more adaptive than PMD for this case. Note that an alternative way to bound $\psi^\pi(s,\pi_{t+1}(s))$ in (4.41) is to modify Proposition 3, similarly to how the linear convergence for the SPMD method was established in [24].

4.4. PDA for General State and Continuous Action Spaces. We now investigate the impact of the approximation error in (4.21) on the convergence of the PDA method for solving RL problems with continuous action spaces. Observe that when estimating $Q^\pi$ there exist two possible types of stochasticity including: a) the sampling of $(s,a)$ from $S \times A$ according to some given initial distribution $\rho_S \times \rho_A$; and b) the estimation of $Q^\pi(s,a)$ through the sampling of the Markov Chain induced by $\pi_k$. Similar to previous discussions, we use $\xi_k$ to denote these random samples generated at iteration $k$ and $F_k := \{\xi_1, \ldots, \xi_k\}$ denote its associated filtration. Let

$$\delta_k(s,a) := \hat{Q}(s,a;\theta_k) - Q^\pi(s,a).$$

We need to make the following assumptions on these error terms.

$$\mathbb{E}_{\xi_k} \{\mathbb{E}_{\sim \nu_k} [\|\delta_{k-1}(s,\pi_{k-1}(s)) + \delta_{k-1}(s,\pi^*(s))]\} \leq \zeta,$$

where $\mathbb{E}_{\xi_k}$ denotes the expectation w.r.t. $\xi_k$ conditional on $F_{k-1}$. In addition, we also need to assume that $\hat{Q}(s,a;\theta_k) + \tilde{\mu}_Q D(\gamma_0(s),a)$ is convex for some $\tilde{\mu}_Q \geq 0$ and denote $\hat{\mu}_d := \mu_h - \tilde{\mu}_Q$. Note that $\tilde{\mu}_Q$ can be viewed as an estimation of $\mu_h$.

Moreover, we assume that $\hat{Q}(s,\cdot;\theta)$ and $Q^\pi(s,a)$ are Lipschitz continuous s.t.

$$\|\hat{Q}(s,a_1;\theta) - \hat{Q}(s,a_2;\theta)\| \leq M_\hat{Q} ||a_1 - a_2||, \forall a_1, a_2 \in A,$$

$$\|Q^\pi(s,a_1) - Q^\pi(s,a_2)\| \leq M_Q ||a_1 - a_2||, \forall a_1, a_2 \in A.$$

We make a couple remarks. First, note that if $\hat{Q}$ is $M_\hat{Q}$-Lipschitz continuous, then $\tilde{\psi}$ is also $M_\hat{Q}$-Lipschitz continuous. Moreover, $\delta_k(s,a)$ is $(M_\hat{Q} + M_Q)$-Lipschitz continuous w.r.t. $a \in A$. Second, Appendix A presents sufficient conditions for the error bound (4.45) to be small, as well as sufficient conditions for (4.46) and (4.47) to hold.

**Theorem 9.** Suppose $\hat{\mu}_d \geq 0$ and that $\beta_k$ and $\lambda_k$ satisfy

$$\hat{\mu}_k := \hat{\mu}_d \sum_{t=0}^{k-1} \beta_t + \lambda_k \geq 0 \text{ and } \lambda_{k+1} \leq \lambda_k, \forall k \geq -1,$$

where we denote $\hat{\mu}_{-1} = \lambda_{-1}$. Also let us denote $\hat{\beta}_k := \sum_{t=0}^{k-1} \beta_t$. Then we have

$$\lambda_{k-1} \hat{\beta}_k^{-1} D(\gamma_0, \pi^*) + M_Q^2 \hat{\beta}_k^{-1} \sum_{t=0}^{k-1} (\beta_t^2 / \hat{\mu}_{t-1}) + \zeta,$$

where $D$ is defined in (3.11). In particular, if $\hat{\mu}_d > 0$, $\beta_k = k + 1$ and $\lambda_k = \hat{\mu}_d$, then

$$\frac{2(1-\gamma)}{k(k+1)} \sum_{t=0}^{k-1} (t \mathbb{E}[f(\pi_{t-1}) - f(\pi^*)]) + \hat{\mu}_d \mathbb{E}[D(\gamma_k, \pi^*)] \leq \frac{2\hat{\mu}_d D(\gamma_k, \pi^*)}{k(k+1)} + \frac{4M_Q^2}{\hat{\mu}_d (k+1)} + \zeta.$$

Moreover, if $\hat{\mu}_d = 0$, $\beta_k = k + 1$ and $\lambda_k = \lambda(k + 1)^{3/2}$ with $\lambda = M_Q / \sqrt{D(\gamma_0, \pi^*)}$, then

$$\frac{2(1-\gamma)}{k(k+1)} \sum_{t=0}^{k-1} (t \mathbb{E}[f(\pi_t) - f(\pi^*)]) \leq \frac{2\lambda D(\gamma_0, \pi^*)}{k(k+1)} + \frac{4M_Q^2}{\lambda k} + \zeta.$$
Proof. First observe that (4.39) still holds. Also note that

\[
\hat{\psi}(s, \pi_{t+1}(s), \theta_t) = \hat{\psi}(s, \pi_{t+1}(s), \theta_t) - \hat{\psi}(s, \pi_t(s), \theta_t) + \hat{\psi}(s, \pi_t(s), \theta_t)
\]

\[
\geq -\hat{M}_Q \|\pi_{t+1}(s) - \pi_t(s)\| + \hat{\psi}(s, \pi_t(s), \theta_t)
\]

\[
= -\hat{M}_Q \|\pi_{t+1}(s) - \pi_t(s)\| + \hat{\psi}^*(s, \pi_t(s)) + \delta_t(s, \pi_t(s))
\]

\[
= -\hat{M}_Q \|\pi_{t+1}(s) - \pi_t(s)\| + \delta_t(s, \pi_t(s)).
\]

Combining these relations and rearranging the terms, we obtain

\[
- \sum_{t=0}^{k-1} \beta_t \psi^*(s, a) + \hat{\mu}_{k-1} D(\pi_k(s), a) \leq \lambda_{k-1} D(\pi_0(s), a) + \sum_{t=0}^{k-1} \beta_t [\delta_t(s, a) - \delta_t(s, \pi_t(s))]
\]

\[
+ \sum_{t=0}^{k-1} \left[ \beta_t \hat{M}_Q \|\pi_{t+1}(s) - \pi_t(s)\| - \hat{\mu}_{t-1} D(\pi_t(s), \pi_{t+1}(s)) \right]
\]

\[
\leq \lambda_{k-1} D(\pi_0(s), a) + \sum_{t=0}^{k-1} \beta_t [\delta_t(s, a) - \delta_t(s, \pi_t(s))] + \hat{M}_Q^2 \sum_{t=0}^{k-1} (\beta_t^2 / \hat{\mu}_{t-1}).
\]

Setting \( a = \pi^*(s) \), taking expectation of \( s \) w.r.t. \( \nu^* \) and then w.r.t. \( F_k \), and using the assumption in (4.45) and Lemma 2, we then have

\[
(1 - \gamma) \sum_{t=0}^{k-1} \beta_t \mathbb{E}[f(\pi_t) - f(\pi^*)] + \hat{\mu}_{k-1} \mathbb{E}[D(\pi_k, \pi^*)] \leq \lambda_{k-1} D(\pi_0(s), a) + \sum_{t=0}^{k-1} \beta_t s + \hat{M}_Q^2 \sum_{t=0}^{k-1} (\beta_t^2 / \hat{\mu}_{t-1}),
\]

which clearly implies (4.49). The result in (4.50) follows from straightforward calculations, and (4.51) can be verified similarly in view of the observation that \( \sum_{t=0}^{k-1} (\beta_t^2 / \hat{\mu}_{t-1}) = \frac{1}{\gamma} \sum_{t=0}^{k-1} \sqrt{t + 1} \leq \frac{2}{\gamma} (k + 1)^{3/2} \). \( \blacksquare \)

We now consider the case when \( \hat{\mu}_d < 0 \). Unlike the counterpart Theorem 5 for PMD, here we only need to assume the Lipschitz continuity of \( \bar{Q}(s, a; \theta) \) and \( Q^*(s, a) \) w.r.t. \( \theta \) to establish a convergence guarantee.

Theorem 10. Suppose that \( \hat{\mu}_d < 0 \). Assume that the number of iterations \( k \) is fixed a priori, \( \lambda_t = \lambda = k(k + 1) |\hat{\mu}_d| \), and \( \beta_t = t + 1, t = 0, \ldots, k - 1 \). Then for any \( s \in \mathcal{S} \), there exist iteration indices \( k(s) \) s.t.

\[
(4.52) \quad \frac{(M_Q + \hat{M}_Q^2)^2}{|\hat{\mu}_d||k + 1|} \leq -\psi^*(s, \pi_{k(s)}(s)) \leq \frac{2 |V^*(s) - V^*(s+s)|}{k+1} + \frac{3(M_Q + \hat{M}_Q^2)^2}{(1-\gamma)|\hat{\mu}_d||k + 1|}.
\]

Moreover, we have

\[
\frac{\hat{\mu}_{k(s)}}{2\hat{\mu}_{k(s)}} D(\pi_{k(s)}(s), \pi_{k(s)}(s)) + \frac{\hat{\mu}_{k(s)-1}}{2\hat{\mu}_{k(s)}} D(\pi_{k(s)}(s), \pi_{k(s)+1}(s)) \leq \frac{2 |V^*(s) - V^*(s+s)|}{k+1} + \frac{3(M_Q + \hat{M}_Q^2)^2}{(1-\gamma)|\hat{\mu}_d||k + 1|} + \frac{2(M_Q + \hat{M}_Q^2)^2}{|\hat{\mu}_d||k + 1|},
\]

where \( \hat{\mu}_t = |t(t+1)|\hat{\mu}_d|/2 + k(k+1)|\hat{\mu}_d|, t = 0, \ldots, k - 1 \).

Proof. Similar to (4.10), we can show that

\[
\hat{\psi}(s, \pi_{k+1}(s), \theta_k) - \frac{1}{\beta_k} |\lambda_k + \lambda_k| \bar{D}_A \leq \hat{\psi}(s, \pi_k(s), \theta_k) - \frac{\hat{\mu}_k}{\beta_k} D(\pi_{k+1}(s), \pi_k(s)) - \frac{\hat{\mu}_{k-1}}{\beta_k} D(\pi_k(s), \pi_{k+1}(s)),
\]

which in view of the definition of \( \delta_k(s, a) \) in (4.44) and \( \psi^*(s, \pi_k(s)) \) = 0 then implies that

\[
\psi^*(s, \pi_{k+1}(s)) + \delta_k(s, \pi_{k+1}(s)) - \frac{1}{\beta_k} |\lambda_k + \lambda_k| \bar{D}_A \leq \delta_k(s, \pi_k(s)) - \frac{\hat{\mu}_k}{\beta_k} D(\pi_{k+1}(s), \pi_k(s)) - \frac{\hat{\mu}_{k-1}}{\beta_k} D(\pi_k(s), \pi_{k+1}(s))
\]

\[
\leq \delta_k(s, \pi_k(s)) - \frac{\hat{\mu}_k}{\beta_k} D(\pi_{k+1}(s), \pi_k(s)) - \frac{\hat{\mu}_{k-1}}{\beta_k} D(\pi_k(s), \pi_{k+1}(s))
\]

\[
- \frac{\hat{\mu}_k}{2\beta_k} D(\pi_{k+1}(s), \pi_k(s)) - \frac{\hat{\mu}_{k-1}}{2\beta_k} D(\pi_k(s), \pi_{k+1}(s)).
\]

Also note that

\[
\delta_k(s, \pi_k(s)) - \delta_k(s, \pi_{k+1}(s)) - \frac{1}{4\beta_k} (\hat{\mu}_k + \hat{\mu}_{k-1}) \|\pi_{k+1}(s) - \pi_k(s)\|^2 \leq (M_Q + \hat{M}_Q)^2 \|\pi_{k+1}(s) - \pi_k(s)\|^2 - \frac{1}{4\beta_k} (\hat{\mu}_k + \hat{\mu}_{k-1}) \|\pi_{k+1}(s) - \pi_k(s)\|^2 \leq \frac{\beta_k(M_Q + \hat{M}_Q)^2}{\beta_k^2 + \hat{\mu}_{k-1}}.
\]
Combining these two relations, we have
\[ \psi^\pi_k(s, \pi_{k+1}(s)) + \frac{\bar{\mu}_k}{2\beta_k} D(\pi_{k+1}(s), \pi_k(s)) + \frac{\bar{\mu}_{k-1}}{2\beta_k} D(\pi_k(s), \pi_{k+1}(s)) \]
\begin{equation}
\leq \frac{|\lambda_{k+1} - \lambda_k| D_A}{\beta_k} + \frac{\beta_k(M_Q + M_Q)^2}{(\bar{\mu}_k + \bar{\mu}_{k-1})^2}.
\end{equation}
(4.54)

Using the above relation, we can show similarly to (4.11) that
\[ V^\pi_{k+1}(s) - V^\pi_k(s) \leq \psi^\pi_k(s, \pi_{k+1}(s)) + \frac{2|\lambda_{k+1} - \lambda_k| D_A}{(1-\gamma)\beta_k} + \frac{\gamma \beta_k(M_Q + M_Q)^2}{(1-\gamma)(\bar{\mu}_k + \bar{\mu}_{k-1})^2}. \]
(4.57)

Using the fact that \( \lambda_k = \lambda_{k-1} \) in the previous two relations, we conclude that
\[ \beta_k[V^\pi_{k+1}(s) - V^\pi_k(s)] \leq \beta_k \psi^\pi_k(s, \pi_{k+1}(s)) \leq \frac{\beta_k^2(M_Q + M_Q)^2}{(1-\gamma)(\bar{\mu}_k + \bar{\mu}_{k-1})^2}. \]

Rearranging terms in the above inequality, we then have
\[ 0 \leq -\beta_k \left[ \psi^\pi_k(s, \pi_{k+1}(s)) - \frac{\beta_k(M_Q + M_Q)^2}{\bar{\mu}_k + \bar{\mu}_{k-1}} \right] \leq \beta_k[V^\pi_k(s) - V^\pi_{k+1}(s)] + \frac{\beta_k^2(M_Q + M_Q)^2}{(1-\gamma)(\bar{\mu}_k + \bar{\mu}_{k-1})^2}. \]
(4.56)

Moreover, summing up (4.54) and (4.55), we have
\[ V^\pi_k(s) \leq V^\pi_0(s) + \sum_{t=0}^{k-1} \beta_t(M_Q + M_Q)^2 \frac{1}{(1-\gamma)(\bar{\mu}_t + \bar{\mu}_{t+1})} \leq V^\pi_0(s) + \frac{(M_Q + M_Q)^2}{2(1-\gamma)\bar{\mu}_d}, \]
where the last inequality follows from \( \bar{\mu}_t = \bar{\mu}_d \sum_{j=0}^t \beta_j + \lambda_k \geq |\bar{\mu}_d|k(k+1)/2 \). It then follows from the selection of \( \beta_t = t+1 \) and the previous inequality that
\[ \sum_{t=0}^{k-1} \beta_t[V^\pi_t(s) - V^\pi_{t+1}(s)] = \sum_{t=0}^{k-1} V^\pi_t(s) - kV^\pi_k(s) \leq k[V^\pi_0(s) + \frac{(M_Q + M_Q)^2}{2(1-\gamma)\bar{\mu}_d} - V^\pi_k(s)] \leq k[V^\pi_0(s) + \frac{(M_Q + M_Q)^2}{2(1-\gamma)\bar{\mu}_d} - V^\pi_*(s)]. \]
(4.57)

Taking the telescopic sum of (4.56), and utilizing the above bound in (4.57), we then have
\[ -\sum_{t=0}^{k-1} \beta_k \left[ \psi^\pi_k(s, \pi_{k+1}(s)) - \frac{\beta_k(M_Q + M_Q)^2}{\mu_k + \mu_{k-1}} \right] \leq \sum_{t=0}^{k-1} \beta_t[V^\pi_t(s) - V^\pi_{t+1}(s)] + \sum_{t=0}^{k-1} \beta_t^2(M_Q + M_Q)^2 \frac{1}{2(1-\gamma)\bar{\mu}_d} \leq k[V^\pi_0(s) - V^\pi_*(s) + \frac{3(M_Q + M_Q)^2}{2(1-\gamma)\bar{\mu}_d}], \]
where the last inequality follows from the fact that \( \sum_{t=0}^{k-1} \beta_t^2 \leq \sum_{t=0}^{k-1} \frac{(t+1)^2}{\bar{\mu}_d(k+1)} \leq \frac{k}{\bar{\mu}_d}. \)

Noticing the first inequality in (4.56), it then follows from the above inequality by choosing \( \tilde{k}(s) \) as the iteration with the smallest value of \( -[\psi^\pi_*(s, \pi_{k+1}(s)) - \beta_k(M_Q + M_Q)^2] \) that
\[ -\frac{k(k+1)}{2}[\psi^\pi_*(s, \pi_{k}(s)+1(s)) - \frac{\beta_k(M_Q + M_Q)^2}{\mu_k(s) + \mu_{k-1}(s)}] \leq \tilde{k}[V^\pi_0(s) - V^\pi_*(s) + \frac{3(M_Q + M_Q)^2}{2(1-\gamma)\bar{\mu}_d}]. \]

The result in (4.52) then clearly follows from the above inequality and the fact that \( 0 \leq \frac{\beta_k(s)}{\mu_k(s) + \mu_{k-1}(s)} \leq \frac{1}{\bar{\mu}_d(k+1)}. \) The result in (4.53) follows immediately from (4.52) and (4.54).

We make a few remarks about the above results. First, Theorem 10 appears to stronger than the corresponding one in Theorem 5 for PMD in the sense that it requires much weaker assumptions. Second, it is unclear if the result above also implies convergence to a stationary point; see the discussion after Theorem 7. Third, while for the case \( \bar{\mu}_d \geq 0 \), the number of iterations \( k \) does not need to fixed a priori for the PDA method, Theorem 10 does require us to fix the number of iterations in order to reduce \( \lambda_{k+1} - \lambda_k \). It will be interesting to see if more adaptive selection of \( \lambda_k \) is possible for the PDA method when \( \bar{\mu}_d < 0 \).
5. Numerical experiments. We now compare the performance of policy dual averaging (PDA) against several algorithms from Stablebaselines3 [42], a state-of-the-art RL library. We implemented PDA over policy mirror descent (PMD) for a couple reasons. First, we only need to estimate the simpler state-action value function rather than the augmented form in (3.15). Second, when the Q-function is convex, the PDA stepsize does not need to specify the number of iterations a priori like in PMD.

Let us now provide some implementation details. The code is available at https://github.com/jucaleb4/RL-general-action-state.

We used a discount factor of $\gamma = 0.99$. For function approximation, we considered (linear) random Fourier features [43] and neural networks. More details about the former can be found in Appendix A. For the latter, we employed a fully connected neural network with 2 hidden layers, where each layer consists of an affine function of width 64 with tanh activation function, which is the same architecture used in Stablebaselines3 [42]. For policy evaluation, we estimated the value function by a simple Monte-Carlo approach while bootstrapping the remaining cost-to-go values. Then to update the linear function approximation, we used sklearn’s stochastic gradient descent with the default settings and 100 epochs. To update the neural network, we used the same update scheme as in Stablebaselines3. Finally, we tuned the PDA stepsizes via grid search. Throughout this section, we refer to “steps” as a timestep or sample from the environment.

5.1. Grid World with traps. We first consider a 2D Grid-World example over a 10x10 grid. An agent wants to reach a target by moving in one of the four cardinal directions each time step. The agent suffers a unit cost for each step and an addition cost of 5 when they land on a trap spot; see also [12]. Once the agent reaches the target, no further cost is incurred. The state spaces consists of the (x,y)-coordinates for both the agent and target, which are randomly chosen when the environment is initialized. Ten randomly placed traps (dependent only on the environment seed) are set. Thus, the state space is of size $|S| = 10^4$. To avoid state enumeration, we use linear function approximation where the strongly convex function $\omega$ from (1.1) is Shannon entropy. We also use a neural network where $\omega$ can either be Shannon entropy (so the Bregman divergence in (1.1) is the KL-divergence) or negative Tsallis entropy [29,51].

Figure 1 compares the various PDA algorithms against proximal policy optimization (PPO) [45] and deep Q-network (DQN) [35] as implemented in Stablebaselines3 [42]. We see the best performing PDA algorithm is with neural networks and Shannon entropy, labeled as \textit{pda-nn-kl}. It marginally outperforms PPO for the majority of training and then the two exhibit similar performance after step 150000. Similarly, PDA with neural networks and Tsallis entropy (labeled \textit{pda-nn-tsallis}) also improves but has subpar performance. In contrast, both DQN (labeled \textit{dqn}) and PDA with linear function approximation (labeled \textit{pda-lin}) cannot learn a good policy.

As seen from its performance’s variance, PPO has more variability in performance than PDA. To better understand this phenomena, we plot every seed’s performance for \textit{pda-nn-kl} and \textit{ppo} in Figure 2. Here, we can clearly see PDA has more consistent performance, where every seed obtains a relatively good policy. On the other hand, about half of PPO’s seeds seem to take much longer to find a good-performing policy. For instance, the fifth seed cannot converge to a good policy after 200000 steps.
5.2. Lunar Lander. Next, we consider the discrete action Lunar Lander problem, which is a classic rocket trajectory optimization problem. The goal is to land a rocket safely within a pre-defined landing spot, where the four actions are to not fire the engine, and to fire the left, right, main (bottom) engine. The rocket starts at the top center of the world with a random initial force applied to the center of mass. The agent receives various small costs for having the rocket be more stable, being closer to the landing site, and using the rocket less often; a large negative cost for safely landing the rocket; and a large positive cost for landing at the wrong site. More details can be found at https://gymnasium.farama.org. It is important to note that we are considering a discounted cost.

In Figure 3, we compare the various RL algorithms. This time, we see linear function approximation performs better in PDA than with neural networks. Furthermore, PDA’s cumulative discounted cost is not as high as either PPO or deep Q-Learning. In our experience, we found PPO performs better than PDA when the cost is sparse, meaning a large portion of the cost arrives during in a single time step later in the trajectory for accomplishing a particular task. This might be in part due to the fact PDA is designed to handle the discounted setting, while PPO may be implemented to perform well for such sparse cost settings.

5.3. Inverted Pendulum. Next, we consider inverted pendulum [4]. In this continuous state and action space environment, the goal is to balance a pole fixed to a cart subject to gravitational and inertial forces (whose physics are simulated by MuJoCo). The agent applies continuous forces (the action space is $\mathcal{A} = [-3, 3]$) to the cart to balance the pole for as long as possible, where a negative cost of -1 is received for every time step the pole remains on the cart. Clearly, the agent is incentivized to balance the pole for as long as possible. Note that this environment’s trajectory length is limited to 1000, so the minimum cumulative discounted cost is -100.

Before comparing the RL algorithms, let us discuss some implementation details in PDA. Since PPO adds Gaussian noise to their policy (i.e., use a randomized policy) to encourage action exploration, we similarly add independent zero-mean Gaussian noise whose covariance matrix is the identity matrix scaled by $1/\sqrt{t}$, where $t$ is current PDA iteration. This ensures the noise gradually decreases over time. In fact, we found that without such exploration, the policy evaluation is not accurate enough to consistently improve the policy. Also, the subproblem (4.24) is solved using a recently developed universal (i.e., parameter-free) smooth convex optimization solver [27], which has good empirical performance and does not require knowledge of unknown Lipschitz smoothness parameters. When tuning the stepsize, we considered both the stepsize when the Q-function is convex (see Theorem 9) and nonconvex (see Theorem 10). We found the former does better.

The results comparing PDA using neural networks and Stablebaselines3’s implementation of PPO and deep deterministic policy gradient (DDPG) – which is the analogue of DQN for continuous action space – are shown in Figure 4. The plot showcases PDA and DDPG’s superior performance to PPO early on. Later after around 20000 steps, PDA can still get marginally close to PPO, and meanwhile both PDA and PPO outperform DDPG.

5.4. Linear quadratic regulator for the longitudinal control of a wide-body aircraft. Our last experiment is another continuous state and action space problem called linear quadratic regulator (LQR). In our instantiation of LQR, we consider a state-feedback control problem of the longitudinal dynamics of a Boeing 747 aircraft. Due to space constraints, we omit details, but the full model description can be found in [18]. The main difference between inverted pendulum and LQR is the former has bounded state-action...
space while the latter has an unbounded space, e.g., $\mathbb{R}^{n_s+n_a}$. Thus, this environment explores how different algorithms choose actions when the state can diverge.

We again compare PDA with neural networks, PPO, and DDPG. Similar to PDA in Inverted Pendulum, we also use the randomized policy with decreasing noise. Unlike in Inverted Pendulum, we found PDA with the nonconvex stepsize from Theorem 10 works best. The algorithms are compared against one another in Figure 5. PDA exhibits the best long-term performance, while PPO seems to have good initial performance but it eventually diverges away. Meanwhile, DDPG is unable to address the unboundedness as seen by its unbounded cost.

6. Concluding Remarks. In this paper, we study new policy optimization methods for solving Markov decision process and reinforcement learning problems over general state and action spaces. We first present a generalization of the policy mirror descent method and suggest a novel function approximation scheme into it to deal with general state spaces. We then introduce a new class of policy dual averaging (PDA) methods for reinforcement learning that might be more amenable for function approximation in that it only requires the approximation of the action-value function. We establish the convergence properties of both methods under different situations, including the linear convergence to the global optimality under exact action-value function evaluation, the sublinear convergence to the global optimality up to function approximation error under certain curvature conditions, and the sublinear convergence to a stationary point up to function approximation if such curvature conditions are violated. Our numerical experiments also show PDA exhibits robust and consistent performance across a range of optimal control and planning problems, sometimes outperforming state-of-the-art RL algorithms.

Appendix A. Policy evaluation. In this section, we explore when policy evaluation can be accurate. For notational convenience, let $Z := S \times A$ be the state-action pair space. Throughout the appendix, let $\| \cdot \|_Z$ be the uniform norm and $\| \cdot \|_{L^2}$ be the induced norm for the space $L^2(Z, U(Z))$, which is the space of functions $f : Z \rightarrow \mathbb{R}$ that are square integrable w.r.t. $U(Z)$ (the uniform probability measure over $Z$).

Definition 1 (Continuous stochastic control MDP). Consider an MDP $M = (S, A, P, c, \gamma)$ with regularizer $h$, where both $c(\cdot, \cdot) : Z \rightarrow [0, 1]$ and $h(\cdot) : Z \rightarrow \mathbb{R}$ are continuous in $Z$. In addition, suppose the transition kernel is defined according to the transfer function $s' = f(s, a, \xi)$, where $\xi$ is an i.i.d. random variable (i.e., $P(s'|s, a) = P(\xi|s' = f(s, a, \xi))$) and $f(\cdot, \cdot, \xi)$ is continuous in $Z$ for any fixed $\xi$.

The continuous stochastic control MDP is a special class of the stochastic control environment [5], and it captures stochastic dynamical systems. Our goal is to show this class of MDPs has state-action value functions $Q^\pi$ that can be approximated arbitrarily well. To do so, we will utilize universal kernels, which are dense in the space of continuous functions over a compact set [34]. To start, we will show the state-action value $Q^\pi$ (defined in subsection 2.1) is continuous, which allows it to be approximated arbitrarily well.

Lemma 5. Given a continuous stochastic control MDP and a policy $\pi : S \rightarrow A$ that is continuous in $S$, then $Q^\pi$ is continuous in $Z$.

Proof. We sketch the proof. Define the $T$-truncated state-action value function, 

$$Q^\pi_T(s, a) := \mathbb{E}\left[\sum_{t=0}^{T-1} \gamma^t [c(s_t, a_t) + h^{\pi_t}(s_t)] \mid s_0 = s, a_0 = a, a_t = \pi(s_t), s_{t+1} \sim P(\cdot|s_t, a_t)\right],$$

By the properties of a continuous stochastic control MDP and continuity of $\pi$, a proof by induction shows that $Q^\pi_T$ is continuous on $Z$ for all $T \geq 1$. Since $\|Q^T\|_Z$ is bounded by $c/(1-\gamma)$ and $\gamma < 1$, then $Q^\pi_T$ uniformly converges to $Q^\pi$. By the uniform limit theorem, $Q^\pi$ is also continuous in $Z$.

The policy $\pi$ will be set to $\pi_k$ from PDA, which is the solution to an optimization subproblem of (4.24). We assume a solution to the subproblem exists. We will show $\pi_k$ is continuous, which in view of the previous lemma, verifies $Q^{\pi_k}$ is continuous as well. A similar result can be shown for PMD from (3.18). Before stating the result, we say the action set $A$ is polyhedral when $A := \{x \in \mathbb{R}^{n_a} : A x = b , x \geq 0\}$ for some full-rank matrix $A$ and vector $b$. A polyhedral set $A$ generalizes both the finite-action and certain general action space problems. For the following result, we skip its proof since it is a well-known result from sensitivity analysis; see [15, Theorem 5.1] and references therein.

Corollary 4. If $A$ is polyhedral, the function approximation $\tilde{Q}(\cdot, \cdot; \theta_k)$ and its derivative are continuously differentiable for all states $s$ and weights $\theta_k$, and $\pi_{k+1}(s)$ satisfies the strict complementary slackness condition for any state $s$, then $\pi_{k+1}$ is locally continuously differentiable.

The assumption on strict complementary slackness seems to be standard in the sensitivity analysis of optimization problems [8,15]. Moreover, it is satisfied in the finite-action case with $D(\cdot, \cdot)$ in (4.24) set to the
KL divergence, which ensures $\pi_{k+1}(s)$ is strictly positive.

**A.1. Generalization error bounds.** Denote $z \equiv [s, a] \in \mathcal{Z}$ as a state-action pair. Throughout the section, we make the following ergodicity assumption.

**Assumption 1.** The state-action space $\mathcal{Z}$ is compact. Additionally, the Markov chain $\{z_t\}_{t \geq 0}$ defined with any initial $z_0 \in \mathcal{Z}$, $a_t = \pi_k(s_t)$, and $s_{t+1} \sim \mathcal{P}(\cdot | s_t, a_t)$ for $t \geq 0$ is irreducible and aperiodic for all $k$.

The assumption guarantees a stationary distribution $\rho_k$ over states $\mathcal{S}$ exists for $\pi_k$, and that mixing the Markov chain will converge to $\rho_k$ [26, Theorem 4.9]. The assumption is true when $\Pr\{s_t = s, s_{t+1} = s'\} > 0$ for any $s, s' \in \mathcal{S}$ and $t$. For example, suppose $s_{t+1} = f(s_t, a_t) + \xi_t$ for a continuous transfer function $f : \mathcal{Z} \to \mathcal{S}$ s.t. $\|f(s_t, a_t)\| < \|s_t\|$ for all $(s_t, a_t) \in \mathcal{Z}$ (e.g., stable time-invariant linear state-space systems [9]), and $\xi_t$ is an i.i.d. standard normal.

We consider the (linear) function approximation

$$Q(s, a; \theta) := \sum_{i=1}^{N} \theta_i \cdot \left( \frac{1}{D} \sum_{j=1}^{D} \phi_j(z_i)^T \phi_j(z) \right),$$

where $\theta \in \mathbb{R}^N$ are the weights, $\{\phi_j : \mathcal{Z} \to \mathbb{R}^d\}_{j=1}^D$ are some $D$ feature maps, and $\{\zeta_i \in \mathcal{Z}\}_{i=1}^N$ are $N$ some (possibly random) data points. By choosing random Fourier features for $\phi$ [43], then $(1)$ $\hat{Q}(\cdot; \theta)$ is infinitely differentiable, $(2)$ with high probability the gradient $\nabla_a Q(s, a; \theta)$ is bounded and hence $\hat{Q}(s, \cdot; \theta)$ is Lipschitz smooth, and $(3)$ $\hat{Q}(\cdot; \theta)$ approximates a Gaussian kernel, which is a universal kernel, i.e., forms a space that is dense in space of continuous functions over a compact domain [34].

We will discuss how to select $\theta$. For policy $\pi_k$, let $\xi_k := \{(z_i, y_i)\}_{i=1}^N$ be i.i.d. samples, where $z_i \sim \rho_k \times U(A)$, $U(A)$ is the uniform distribution over actions, and $y_i$ is an i.i.d. unbiased estimates of $Q^{\pi_k}(s_i, a_i)$. This procedure can be implemented by (1) mixing the Markov chain until the (random) state $s_t \sim \rho_k$, $(2)$ a Monte Carlo estimation $y_i$ of $Q^\pi_k$ [29, 49] and $(3)$ resetting the MDP to a random initial state to obtain i.i.d samples. As an alternative to $(3)$, one can use conditional sampling if there is Markovian noise [22].

With this dataset, we retrieve $\theta_k$ by solving ridge regression on (A.1). See [44] for more details. We are now ready to bound the error of $\hat{Q}(\cdot; \theta_k)$. Recall for any probability measures $\rho$ and $\rho'$ over a measurable space where $\rho$ is absolutely continuous w.r.t. $\rho'$, the Radon-Nikodym derivative is $\frac{d\rho}{d\rho'}$.

**Proposition 4.** Let $\rho^*$ be a probability measure over $\mathcal{S}$. If Assumption 1 is true, $Q^{\pi_k}(\cdot)$ is continuous and bounded, and $N$ is a sufficiently large constant, then there exists $D = O(\sqrt{N \log(N)})$ and universal positive constant $c_1$ such that

$$E_{\xi_k, \xi \sim \rho^*} \left[ \max_{a} (\hat{Q}(s, a; \theta_k) - Q^\pi(s, a))^2 \right] \leq \frac{41c_1|A|M_{\infty}}{\sqrt{N}},$$

where $D$ and $N$ are from (A.1) and $M_{\infty} := \|\frac{d\rho^*}{d\rho_k}\|_{\infty} < +\infty$.

**Proof.** First, Assumption 1 ensures the stationary distribution $\rho_k$ has positive measure almost everywhere (a.e.); see Proposition 1.7 and 1.14 from [26]. So, $\rho^*$ is absolutely continuous w.r.t. $\rho_k$ and $\|\frac{d\rho^*}{d\rho_k}\|_{\infty} < +\infty$.

Let $\mathcal{H}$ be reproducing kernel Hilbert space (mapping $\mathcal{Z}$ to reals) defined as the completion of the linear span of a Gaussian (i.e., universal) kernel, let $\mathcal{H}_R := \{ f : f \in \mathcal{H}, \|f\|_{L^2} \leq R \}$ for a constant $R$, and let the parameterized optimization problem be $\phi(R) := \min_{f \in \mathcal{H}_R} \{ \mathcal{E}(f) := \int (f(s, a) - Q^{\pi_k}(s, a))^2 \rho_k(ds)da \}$, where the integral is taken over the state-action pair space $\mathcal{Z}$. For every finite $R$, a minimizer exists [44]. Now, we want to show for every $\epsilon > 0$, there exists a finite $R$ s.t. $\phi(R) \leq \epsilon$ (to avoid making existence assumptions like in [44]). Denote $U(A)$ as the uniform probability measure over $\mathcal{A}$, which has positive support a.e. by compactness of $\mathcal{A}$. By a change-in-measure argument,

$$\|\frac{d\rho}{d\mathcal{U}(A)}\|_{\infty}^2 \|f - Q^{\pi_k}\|_{L^2}^2 \leq \mathcal{E}(f) \leq \|\frac{d\mathcal{U}(A)}{d\rho_k}\|_{\infty}^2 \|f - Q^{\pi_k}\|_{L^2}^2 \leq |\mathcal{Z}| \|\frac{d\mathcal{U}(A)}{d\rho_k}\|_{\infty} \|f - Q^{\pi_k}\|_{L^2}$$

for every $f \in \mathcal{H}$ s.t. $\|f - Q^{\pi_k}\|_{L^2}$ is bounded. Since $\mathcal{H}$ is a universal kernel and $Q^{\pi_k}$ is bounded and continuous, then for every $\epsilon > 0$ there exists $f \in \mathcal{H}$ such that $\|f - Q^{\pi_k}\|_{L^2} \leq \epsilon / (|\mathcal{Z}| \|\frac{d\mathcal{U}(A)}{d\rho_k}\|_{\infty})$ [34]. Hence, $\mathcal{E}(f) \leq \epsilon$.

Using the above inequalities and the bound $p^2 \leq 2[(p-q)^2 + q^2]$ for reals $p, q$, we derive

$$\|\hat{f}\|_{L^2}^2 \leq 2\|\hat{f} - Q^{\pi_k}\|_{L^2} + \|Q^{\pi_k}\|_{L^2}^2 \leq 2(\|\frac{d\rho_k}{d\mathcal{U}(A)}\|_{\infty} \epsilon + \|Q^{\pi_k}\|_{L^2}^2) := R(\epsilon)^2 < +\infty,$$

\(^1\)Bounded uniform norm guarantees $f - Q^{\pi_k}$ is square integrable over the compact domain $\mathcal{Z}$. 

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where we used boundedness of $Q^{*\epsilon}$. Thus, we conclude $\hat{f} \in \mathcal{H}_{R(\epsilon)}$, implying $\phi(R(\epsilon)) \leq \epsilon$, which is what we wanted to show. With this fact in hand, we have
\[
\mathbb{E}_{\xi_k} \mathbb{E}_{s \sim \rho_k}\left[\max(Q(s, a; \theta_k) - Q^{*\epsilon}(s, a))^2\right]
\leq |A| \left\| \frac{d\nu}{d\rho_k} \right\|_{\infty} \mathbb{E}_{\xi_k} \mathbb{E}_{s \sim \rho_k, a \sim U(A)}\left[\left(\hat{Q}(s, a; \theta_k) - Q^{*\epsilon}(s, a)\right)^2\right]
\leq |A|M \left(40c_1/\sqrt{N} + \min_{f \in \mathcal{H}_{R(1/\sqrt{\epsilon})}} \mathcal{E}(f)\right)
\leq 41c_1 |A|M \frac{1}{\sqrt{N}},
\]
where the second line is by a change-in-measure and the third is by [44, Corollary 1].

We make some remarks. We have already shown when $Q^{*\epsilon}$ is continuous and bounded. More details about the minimum size of $N$ is in [44, Corollary 1]. Now, the proposition can bound the general error (4.45) with $\rho' = \nu^*$. Similarly, the proposition can bound both the bias (4.33) and stochastic error (4.34). For the latter, the choice of $D$ will determine the stochastic error; see [43, Theorem 1]. Thus, obtaining $N = O(\epsilon^{-2})$ samples at every iteration ensures the aforementioned error terms are at most $\epsilon$.

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