Invariance of quantum correlations under local channel for a bipartite quantum state

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Abstract – We show that the quantum discord in a bipartite quantum state is invariant under the action of a local quantum channel if and only if the channel is invertible. We also show that the geometric measure of discord is preserved under the action of a local channel if the channel is invertible and if the metric used to define the geometric discord is monotonic. In particular, both versions of quantum discord are invariant under a local unitary channel.

Introduction. – The characterization and quantification of quantum correlations in multipartite quantum systems is a fundamental problem facing quantum information theory [1–5]. Two principal approaches regarding quantum correlations consist of the entanglement separability paradigm and the quantum vs. classical paradigm [2–22]. In the last decade the later approach generated intense research activity, as it was found that quantum correlations involved in separable states play a crucial role in some applications [23–25]. In the quantum vs. classical paradigm, the quantum correlations form the part of total correlations which arise over and above the classical correlations implied by a multipartite quantum state. Such quantum correlations are characterized by quantum discord (QD) [3,4], measurement-induced non-locality (MiN) [5], quantum deficit [26], etc. These quantum correlations can be a resource for a number of quantum information applications [6,7,27–29]. In this context it becomes important to understand the dynamics of these quantum correlations under local noise (local quantum operations).

The basic problem we are concerned with is the relation of the local quantum operation (a local CPTP map or local quantum channel) on a part of a bipartite system in a given state with the resulting change in quantum correlations as the state changes under such local operations. In particular, we are concerned with the possibility of changing quantum discord under such local quantum operations. For the two-qubit case it is shown that the qubit channel that preserves commutativity is either unital, i.e., mapping a maximally mixed state to a maximally mixed state, or a completely decohering channel that nullifies QD in that state [30]. For \( m \times 3 \) system, it is shown that a channel acting on a second subsystem cannot create QD for zero QD states if and only if it is either a completely decohering channel or an isotropic channel [31]. It was further proved that a channel \( \Lambda \) transforms a zero QD state to a zero QD state if and only if \( \Lambda \) preserves commutativity [31,32]. The exact forms of unital channel for a qubit system and that for the completely decohering channel for any system are also obtained [33].

In this paper we obtain a necessary and sufficient condition on the local quantum operation (channel) operating on any bipartite system to preserve quantum correlations, namely the quantum discord. This result is completely general and applies to any bipartite state. The results obtained tackle an interesting problem, that of characterizing the class of operations that do not change the measures of non-classical correlations such as quantum discord [34,35].

Main results. – Our idea is to express the mutual information \( I(\rho) \) and the classical correlations \( C_{A,B}(\rho) \) (defined below) in a bipartite state \( \rho \) in terms of the relative entropy involving two states \( \rho \) and \( \sigma \) given by

\[
S(\rho||\sigma) = -S(\rho) - \text{tr}(\rho \log \sigma),
\]
where \( S(\rho) = -\text{tr}(\rho \log \rho) \) is the von-Neumann entropy of the state \( \rho \). We then seek condition on the local quantum operation so as to preserve \( I(\rho) \) and \( C_{A,B}(\rho) \) expressed in terms of the relative entropy. We need the following.

**Theorem 1** (Petz [36–38]): For states \( \rho \) and \( \sigma \) and a local quantum operation \( T \)

\[
S(\rho|\sigma) = S(T\rho|T\sigma)
\]

if and only if there exists a local quantum operation \( \tilde{T} \) such that

\[
\tilde{T}\rho = \rho; \quad \tilde{T}\sigma = \sigma.
\]

We now state and prove the main results.

**Lemma 1**: Let \( \rho_{AB} \) be a bipartite quantum state, \( \Lambda_a: \rho_A \mapsto \Lambda_a \rho_A; \Lambda_b: \rho_B \mapsto \Lambda_b \rho_B \) be local quantum channels and \( \rho_{A,B} = \text{tr}_{B,A}(\rho_{A,B}) \), making \( \Lambda_a \) and \( \Lambda_b \) to be local channels acting on parts \( A \) and \( B \) of the system, respectively. Then

\[
I((\Lambda_a \otimes \Lambda_b)\rho_{AB}) = I(\rho_{AB})
\]

if and only if there exist quantum operations \( \Lambda_{a,b}: \rho_{A,B} \mapsto \Lambda_{a,b}^{\star} \rho_{A,B} \) such that

\[
(\Lambda_a^* \otimes \Lambda_b^*)(\Lambda_a \otimes \Lambda_b)\rho_{AB} = \rho_{AB},
\]

and

\[
(\Lambda_a^* \otimes \Lambda_b^*)(\Lambda_a \otimes \Lambda_b)\rho_A \otimes \rho_B = \rho_A \otimes \rho_B.
\]

Here \( I(\rho) \) is the quantum mutual information in the state \( \rho \).

**Proof**: We use the definition of the quantum mutual information via relative entropy, namely,

\[
I(\rho_{AB}) = S(\rho_{AB}) - S(\rho_{A}) - S(\rho_{B}).
\]

Then the result is immediate from Theorem 1. We just have to replace \( \rho \) by \( \rho_{AB} \), \( \sigma \) by \( \rho_A \otimes \rho_B \) and the quantum operation \( T \) by \( \Lambda_a \otimes \Lambda_b \).

Next we obtain the conditions on the map \( \Lambda_a \otimes \Lambda_b \) which leave the classical correlations in a bipartite quantum state \( \rho_{AB} \) unchanged. Classical correlations in a bipartite quantum state \( \rho_{AB} \), with a POVM measurement defined by the set of positive operators \( \{A_i^a\} \) \( \{B_j^b\} \) carried out on part \( A(B) \) can be expressed as [4]

\[
C_{A,B}(\rho_{AB}) = \max_{\{A_i^a\},\{B_j^b\}} \left( S(\rho_{B,A}) - \sum_i p_i S(\rho_{B,A}^i) \right)
= \max_{\{A_i^a\},\{B_j^b\}} \sum_i p_i S(\rho_{B,A}^i),
\]

where \( \rho_{B,A}^i = \text{tr}_{B,A}(\rho_{B,A}^i) \) is the reduced density operator of the density operator \( \rho_{B,A}^i \) prepared by the POVM measurements \( \{A_i^a\}, \{B_j^b\} \) after the \( i \)-th outcome.

**Lemma 2**: Let \( \rho_{AB} \) be a bipartite quantum state and \( \Lambda_{a,b}: \rho_{A,B} \mapsto \Lambda_{a,b} \rho_{A,B} \) be the local quantum operations where \( \rho_{A,B} = \text{tr}_{B,A}(\rho_{A,B}) \). Then

\[
C_{A,B}(\Lambda_a \otimes \Lambda_b)\rho_{AB}) = C_{A,B}(\rho_{AB})
\]

if and only if there exist local quantum operations \( \Lambda_{a,b}^\star: \rho_{A,B} \mapsto \Lambda_{a,b}^{\star} \rho_{A,B} \) such that

\[
(\Lambda_a^* \otimes \Lambda_b^*)(\Lambda_a \otimes \Lambda_b)\rho_{AB} = \rho_{AB},
\]

and

\[
(\Lambda_a^* \otimes \Lambda_b^*)(\Lambda_a \otimes \Lambda_b)\rho_A \otimes \rho_B = \rho_A \otimes \rho_B.
\]

Here \( C_{A,B}(\rho) \) denote the classical correlations in the state \( \rho \) as defined in eq. (1).

**Proof**: Note that \( \rho_{AB} \) and hence \( \rho_A \) and \( \rho_B \) are arbitrary density operators so that the domain and the range of \( \Lambda \) and \( \Lambda^\star \) cover the corresponding set of all density operators. In particular, for the states \( \rho_{B,A}^i \) occurring in eq. (1) \( \Lambda_{a,b}^\star(\rho_{B,A}^i) = \rho_{B,A}^i \) and similarly for \( \rho_{B,A} \). The result follows because by Theorem 1 each term in the sum defining \( C_{A,B}(\rho_{AB}^i) \) (eq. (1)) is invariant under the action of the local channels \( \Lambda_a \) and \( \Lambda_b \) as defined in the statement of the lemma. Hence the maxima over the corresponding POVMs \( \{A_i^a\}, \{B_j^b\} \) are also invariant.

**Theorem 2**: Let \( \rho_{AB} \) be a bipartite quantum state and \( \Lambda_{a,b}: \rho_{A,B} \mapsto \Lambda_{a,b} \rho_{A,B} \) be the local quantum operations where \( \rho_{A,B} = \text{tr}_{B,A}(\rho_{A,B}) \). Then

\[
D_{A,B}(\Lambda_a \otimes \Lambda_b)\rho_{AB}) = D_{A,B}(\rho_{AB})
\]

if and only if there exist local quantum operations \( \Lambda_{a,b}^\star: \rho_{A,B} \mapsto \Lambda_{a,b}^{\star} \rho_{A,B} \) such that

\[
(\Lambda_a^* \otimes \Lambda_b^*)(\Lambda_a \otimes \Lambda_b)\rho_{AB} = \rho_{AB},
\]

and

\[
(\Lambda_a^* \otimes \Lambda_b^*)(\Lambda_a \otimes \Lambda_b)\rho_A \otimes \rho_B = \rho_A \otimes \rho_B.
\]

Here \( D_{A,B}(\rho_{AB}) = I(\rho_{AB}) - C_{A,B}(\rho_{AB}) \) denote the quantum discord in the state \( \rho_{AB} \). We also assume that the domain and the range of all the local maps cover the corresponding set of all density operators.

**Proof**: The if part follows immediately from Lemma 1 and Lemma 2. For the only if part, we have to cater to the possibility that a non-invertible local channel can preserve discord by lowering the classical correlation (a local operation cannot increase the classical correlation see [4]) and mutual information by equal amount, or, in other words, by lowering the classical correlation alone, without changing the quantum correlation. Thus, we must show that there is non-invertible local channel which preserves quantum discord by changing the classical correlation alone for all states (although a particular non-invertible local map may preserve discord of a particular state by reducing \( I(\rho_{AB}) \) and \( C_{A,B}(\rho_{AB}) \) by the same amount). To do this, we partition all the non-invertible local channels into two classes, namely the commutativity-preserving channels [31] and those which are not commutativity-preserving. The non-commutativity-preserving channels are shown to create discord in zero discord (classical quantum) states of a bipartite system [31]. Hence these local channels do not preserve discord in all bipartite
states. The commutativity-preserving channels are shown to have one of the two forms, namely completely decohering channel and the isotropic channel [31,33]. A completely decohering channel nullifies the quantum discord in any bipartite state. So it cannot preserve quantum discord in all discordant states. An isotropic local channel which is of the form [31] $\Lambda^{iso}(\rho) = p\Gamma(\rho) + (1-p)\frac{1}{2}$, where $\Gamma$ is any linear channel that preserves the eigenvalues of $\rho$ and $d$ the dimension of Hilbert space, decreases quantum correlation. To see that, we act by $I \otimes \Lambda^{iso}(\rho)$ on the maximally quantum correlated pure state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ which gives a mixed state which has less quantum correlation. So, the isotropic local channel cannot preserve quantum correlation for all states. Then, the commutativity-preserving channel does not preserve quantum correlation for all states (although it preserves discord for a zero discord state). This covers all the non-invertible local channels and completes the proof.

As an example, fig. 1 displays the discord in the two-qubit pure state,

$$\cos(\theta)|01\rangle + \sin(\theta)|10\rangle$$

and that in the state obtained by the local action of the Hadamard gate (denoted $\Phi$ in the figure) on a qubit. We see that the discord in both cases coincides for all values of $\theta$.

It is interesting to investigate the conditions on local channels so as to preserve other measures of quantum correlations, especially geometric discord which we define generically as

$$D(\rho) = \min_{\{\chi\}} d(\rho, \chi),$$

where $d(\rho, \sigma)$ is a suitable measure of distance (metric) between states while $\{\chi\}$ is the set of zero discord states [39]. The following lemma is easily proved.

**Lemma 3:** Let $d(\rho, \sigma)$ be a metric on the space of bipartite states, which is monotonic. Let $\Lambda_a$ and $\Lambda_b$ be the local channels as defined in Lemma 1. If $\Lambda_a$ and $\Lambda_b$ are invertible, then the action of $\Lambda_a \otimes \Lambda_b$ preserves the distance between states.

**Proof:** Let $\Lambda_a^*$ and $\Lambda_b^*$ be the inverses of $\Lambda_a$ and $\Lambda_b$, respectively. Using the monotonicity of $d(\rho, \sigma)$ we have

$$d(\rho, \sigma) \leq d((\Lambda_a \otimes \Lambda_b)\rho, (\Lambda_a \otimes \Lambda_b)\sigma) \leq d((\Lambda_a^* \otimes \Lambda_b^*)(\Lambda_a \otimes \Lambda_b)\rho, (\Lambda_a^* \otimes \Lambda_b^*)(\Lambda_a \otimes \Lambda_b)\sigma) = d(\rho, \sigma).$$

or

$$d((\Lambda_a \otimes \Lambda_b)\rho, (\Lambda_a \otimes \Lambda_b)\sigma) = d(\rho, \sigma).$$

**Lemma 4:** The set $\{\chi\}$ of zero discord states is invariant under the action of an invertible local channel.

**Proof:** A local invertible channel preserves the orthogonality between the pure states on which it acts, because by Lemma 3 it preserves the distance between orthogonal pure states which is equal for all orthogonal pure states (orthogonal projectors) [40]. This means that the action of a local invertible channel preserves the form of a zero discord state [39],

$$\chi_k = \sum_{l=1}^{d_k} p_l |l\rangle \otimes \rho, \quad k = 1, 2,$$

where $k$ runs over the parts of the bipartite system, $|l\rangle, l = 1, \ldots, d_k$ is an arbitrary orthonormal basis in $H_k$ and $\rho$ acts on the Hilbert space of the remaining part.

Using the definition of geometric discord and by Lemma 3 and Lemma 4 we immediately see that the action of the local invertible channels $\Lambda_a$ and $\Lambda_b$ preserves geometric discord provided the metric used is monotonic. Unfortunately, the square norm distance defined using the Hilbert-Schmidt norm is not monotonic. However, the trace distance $d(\rho, \sigma) = \frac{1}{2}tr(\rho - \sigma)$, where $|| \cdot ||$ is the Hilbert-Schmidt norm, is monotonic. Similarly, the metric defined via fidelity $F(\rho, \sigma) = \sqrt{\rho^{1/2}\sigma\rho^{1/2}}$ is also monotonic [1]. Thus, the geometric discord defined above using the trace distance or fidelity is preserved under the action of an invertible local channel.

The reverse implication, namely that a distance-preserving local channel must be invertible, is proved in [41], for the case of trace distance, under certain conditions. If such a reverse implication were available in general, it could be used to show that a geometric-discord-preserving local channel must be invertible, completing both ways the implication.

Thus, we see that, the geometric discord of a bipartite quantum state is preserved under a local channel if the channel is invertible, provided the metric used in the definition of discord is monotonic.

The most useful invariant local quantum channels are the local unitary operators $\rho \mapsto U\rho U^\dagger$ and the antiunitary operator $\Theta = UK$, with $U$ unitary and $K$ the conjugation operator $\rho \mapsto UK\rho KU^\dagger$ [42]. However, the

Fig. 1: Discord in a two-qubit pure state coincides with that in a state obtained by locally operating by a Hadamard gate on a single qubit (see text).
anti-unitary operator, which corresponds to the time reversal, is unphysical if applied only to a part of a quantum system. Therefore, the anti-unitary local operators must act on both parts of the system. Thus, we have shown, in particular, that the local unitary channel preserves quantum discord and the geometric measure of discord, provided the metric involved is monotonic. Finally, it will be interesting to search for the characteristics of local channels which increase discord in (possibly some class of) bipartite quantum states with non-zero discord.

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