THE CHERN-RICCI FLOW ON SMOOTH MINIMAL MODELS OF GENERAL TYPE

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Abstract. We show that on a smooth Hermitian minimal model of general type the Chern-Ricci flow converges to a closed positive current on $M$. Moreover, the flow converges smoothly to a Kähler-Einstein metric on compact sets away from the null locus of $K_M$. This generalizes work of Tsuji and Tian-Zhang to Hermitian manifolds, providing further evidence that the Chern-Ricci flow is a natural generalization of the Kähler-Ricci flow.

1. Introduction

Let $(M, g_0)$ be a complex manifold of dimension $n$ with a Hermitian metric $g_0$. We define a real $(1,1)$ form $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}} dz^i \wedge d\bar{z}^j$ on $M$. The normalized Chern-Ricci flow is

(1.1) \[
\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) - \omega, \quad \omega|_{t=0} = \omega_0
\]

where $\text{Ric}(\omega) := -\sqrt{-1}\partial\bar{\partial}\log \det g$ is the Chern-Ricci form of $\omega$. When the initial metric $\omega_0$ is Kähler ($d\omega_0 = 0$), then (1.1) is the normalized Kähler-Ricci flow. Another flow of Hermitian metrics, the pluriclosed flow, has been considered by Streets-Tian [36, 37, 38] (see also Liu-Yang [25]). The unnormalized Chern-Ricci flow was introduced in [14]. The overall hope is that the Chern-Ricci flow will be useful in the classification of complex surfaces much like the Ricci flow in real dimension three [17, 18, 19, 26, 27, 28].

Recently, the Chern-Ricci flow has been shown to have many properties in common with the Kähler-Ricci flow, especially in the case of complex surfaces. When the first Bott-Chern class is zero, the flow was shown to exist for all time and converge smoothly to a Chern-Ricci flat metric [14] using estimates for the elliptic Monge-Ampère equation [5, 16, 41]. This generalized the Kähler case considered by Cao [4], whose proof made use of the estimates of Yau [48]. The work of Tosatti-Weinkove [42, 43] contains several explicit examples of the Chern-Ricci flow and many results generalizing those of the Kähler-Ricci flow. In particular, that the flow exists on some maximal time interval that depends on the Bott-Chern class of the initial metric. If the first Chern class of the manifold is negative, then the

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flow starting with any Hermitian metric converges smoothly to a Kähler-Einstein metric. On complex surfaces with an initial Gauduchon metric, the flow exists either for all time or until the volume or a curve of negative self-intersection tends to zero. Starting with an elliptic bundle over a Riemann surface of genus greater than one, the Chern-Ricci flow converges exponentially fast to a Kähler-Einstein metric on the base [44]. Local Calabi and curvature estimates are also known for the flow [31]. Analogous results for the Kähler-Ricci flow can be found in [4, 9, 10, 15, 30, 32, 33, 34, 35, 40].

If the first Bott-Chern class of the canonical bundle $K_M$ is nef, we say that $M$ is a minimal model. When $M$ is a minimal model, the normalized Chern-Ricci flow has a smooth solution for all time [44]. Additionally, if $K_M$ is a big line bundle, we say that $M$ is of general type. The null locus of $K_M$, $\text{Null}(K_M)$, is the union over all positive dimensional irreducible analytic subvarieties $V \subset M$ of dimension $k$ where

$$\int_V (c_1(K_M))^k = 0.$$ 

We assume that $M$ is a Hermitian smooth minimal model of general type and prove the following theorem:

**Theorem 1.1.** Let $(M, \omega_0)$ be a smooth Hermitian minimal model of general type of dimension $n$ with Hermitian metric $\omega_0$. Then the normalized Chern-Ricci flow (1.1) has a smooth solution for all time and there exists a closed positive current $\omega_{KE}$ on $M$ such that $\omega(t)$ converges $\omega_{KE}$ as currents as $t \to \infty$.

Moreover, letting $E = \text{Null}(K_M)$, $\omega(t)$ converges in $C^{\infty}_{\text{loc}}(M \setminus E)$ to a Kähler-Einstein metric $\omega_{KE}$ away from $E$ satisfying

$$\text{Ric}(\omega_{KE}) = -\omega_{KE}.$$ 

The null locus of $K_M$ is the smallest possible choice for $E$.

As an immediate corollary, we see that every smooth Hermitian minimal model of general type has a closed positive current which is a Kähler-Einstein metric away from the null locus of $K_M$. Additionally, $\omega_{KE}$ is unique in a sense that will be defined at the end of the introduction. The statement that the null locus of $K_M$ is the smallest choice for $E$ follows from the recent work of Collins-Tosatti [6].

In dimension $n = 2$, $M$ is projective. This is not true in general for $n > 2$. If $M$ is Kähler and we start the flow with a Kähler metric this is the result of Tsuji [35] and Tian-Zhang [40]. The difference in the above theorem is that $M$ need not be Kähler. If $M$ is Kähler and the initial metric is not Kähler, the main theorem implies that the Chern-Ricci flow still tends to the same limit as in the work of Tsuji and Tian-Zhang. This suggests that the Chern-Ricci flow is a natural object of study.
We now provide a brief outline of the proof. As in the Kähler case, we reduce to a complex parabolic Monge-Ampère equation
\[
\frac{\partial}{\partial t} \phi = \log \left( \frac{\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \phi}{\Omega} \right) - \phi, \quad \phi|_{t=0} = 0, \quad \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \phi > 0,
\]
where \( \hat{\omega}_t \) is a reference metric and \( \Omega \) is a volume form. Following the Kähler case we have uniform upper bounds for \( \phi, \dot{\phi} \) and \( \omega^n \). Applying a trick from Collins-Tosatti [6], we find a closed positive current
\[
T = \hat{\omega}_\infty + \sqrt{-1} \partial \bar{\partial} \psi \in -c_{BC}^1(M)
\]
with \( T \geq c_0 \omega \) as currents on \( M \). Here \( \psi \) is an upper-semi continuous function in \( L^1(M) \) with \( \sup_M \psi = 0 \) and is smooth away from \( E = \text{Null}(K_M) \). We find uniform bounds for \( \phi, \dot{\phi} \) and \( \omega^n \) in terms of \( \psi \). Letting \( \tilde{\phi} = \phi - \psi \) and using the Phong-Sturm term [29]
\[
\frac{1}{\tilde{\phi} + C_0}
\]
we define the quantity
\[
Q = \log \text{tr}_{\omega_0} \omega - A \tilde{\phi} + \frac{1}{\tilde{\phi} + C_0}
\]
as in [42]. Using the maximum principle we obtain the estimate
\[
\text{tr}_{\omega_0} \omega \leq C' e^{C \psi}.
\]
Applying the higher order estimates from [14] and the bounds for \( \dot{\phi} \) on compact subsets of \( M \setminus E \), we prove smooth convergence. We also have the following uniqueness result which follows immediately:

**Theorem 1.2.** \( \omega_{KE} \) is the unique closed, positive current on \( M \) smooth on \( M \setminus E \) satisfying
\begin{enumerate}
  \item \( \omega_{KE} = -\text{Ric}(\omega_{KE}) \) on \( M \setminus E \) and
  \item \( \frac{1}{C_\varepsilon} e^{\varepsilon \psi} \Omega \leq \omega_{KE} \leq C \Omega \) for all \( \varepsilon \in (0, 1] \) on \( M \setminus E \).
\end{enumerate}
This is independent of choice of \( \Omega \).

2. Preliminaries

In this section we will review some of the notation used in the proof of the main theorem. For a more detailed discussion, we refer the reader to [42]. Every Hermitian metric \( g \) has an associated \( (1,1) \) form
\[
\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.
\]
The metric also has a Chern connection \( \nabla \) with Christoffel symbols
\[
\Gamma_{ij}^k = g^{k\ell} \partial_i g_{\ell j}.
\]
The torsion of the metric is the tensor
\[
T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.
\]
If $g$ is a Kähler metric, then the torsion of $g$ is zero. The Chern curvature of $g$ is

$$R_{k\ell}^p = -\partial_l \tau_{ki}^p,$$

and it obeys the usual commutation identities for curvature. For example,

$$[\nabla_k, \nabla_l^p] X^i = R_{k\ell j}^i X^j.$$

The Chern-Ricci curvature of $g$ is

$$R_{k\ell} = g^{ji} R_{k\ell j} = -\partial_k \partial_l \log \det g$$

with associated Chern-Ricci form

$$\text{Ric}(\omega) = \sqrt{-1} R_{k\ell} dz^k \wedge d\bar{z}^\ell.$$

3. Estimates

First, we need to choose an appropriate reference metric. Since $-c_1^{BC}(M)$ is nef and $K_M$ is big, $M$ is Moishezon, and we can apply [20] to find a non-negative $(1, 1)$ form $\hat{\omega}_\infty$ such that

$$[\hat{\omega}_\infty] = -c_1^{BC}(M).$$

Additionally, there exists a smooth volume form $\Omega$ such that

$$\sqrt{-1} \partial \bar{\partial} \log \Omega = \hat{\omega}_\infty, \quad \int_M \Omega = \int_M \omega^n_0.$$

Define a family of reference metrics

$$\hat{\omega}_t = e^{-t} \omega_0 + (1 - e^{-t}) \hat{\omega}_\infty.$$

If $\varphi$ solves

$$\frac{\partial}{\partial t} \varphi = \log \left( \frac{\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi^n}{\Omega} \right) - \varphi, \quad \varphi|_{t=0} = 0, \quad \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi > 0$$

then $\omega = \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi$ solves the normalized Chern-Ricci flow (1.1).

We require some standard estimates for $\varphi$ that follow as in the Kähler case [45, 40]. For a recent exposition of this result, see [35].

Lemma 3.1. There exists a uniform $C$ such that on $M \times [0, \infty)$,

(i) $\varphi \leq C$

(ii) $\dot{\varphi} \leq C te^{-t}$ when $t \geq t_1$ for some $t_1 > 0$. In particular, $\dot{\varphi} \leq C$

(iii) $\omega^n \leq C \Omega$.

We need a version of Tsuji’s trick [45] that will apply in this non-Kähler setting. The new trick comes from the work of Collins–Tosatti [6] and a theorem of Demailly [7] and Demailly-Păun [8]:

Since $K_M$ is big there exists a Kähler current

$$T = \hat{\omega}_\infty + \sqrt{-1} \partial \bar{\partial} \psi \geq c_0 \omega_0$$
for some \( c_0 > 0 \) as currents on \( M \) where \( \psi \) is an upper-semi continuous function in \( L^1(M) \). Moreover, \( \psi \) can be chosen to be smooth away from a closed analytic subvariety \( E = \{ \psi = -\infty \} \).

By adding a constant, we can assume that \( \sup_M \psi = 0 \). Since \( M \) is Moishezon, it is in Fujiki’s class \( C \) (\( M \) is bimeromorphic to a compact Kähler manifold) [11]. Using this fact, the main theorem of Collins-Tosatti implies that we can take

\[
E = \text{Null}(K_M)
\]

and that this is the smallest possible choice for \( E \) [6].

From the definition of \( T \) and \( \psi \) we have the following useful facts.

**Lemma 3.2.** There exists a uniform \( C > 0 \) such that

(i) \( \sqrt{-1} \partial \bar{\partial} \psi \geq -C \omega_0 \) as currents on \( M \) and

(ii) \( \hat{\omega}_\infty + \varepsilon \sqrt{-1} \partial \bar{\partial} \psi \geq \varepsilon c_0 \omega_0 \) as currents on \( M \) for all \( \varepsilon \in (0, 1] \).

We can find lower bounds for \( \varphi, \tilde{\varphi} \) and \( \omega^n \) in terms of \( \psi \) and \( \varepsilon \).

**Lemma 3.3.** There exists a uniform constant \( C_\varepsilon \) depending on \( \varepsilon \) such that on \( M \times [0, \infty) \),

(i) \( \varphi \geq \varepsilon \psi - C_\varepsilon \)

(ii) \( \tilde{\varphi} \geq \varepsilon \psi - C_\varepsilon \)

(iii) \( \omega^n \geq \frac{1}{C_\varepsilon} e^{\varepsilon \psi} \Omega \).

**Proof.** Define

\[
Q = \dot{\varphi} + \varphi - \varepsilon \psi = \log \frac{\omega^n}{e^{\varepsilon \psi} \Omega}.
\]

If we can find a uniform lower bound for \( Q \) we immediately prove (iii). (i) and (ii) then follow from Lemma 3.1. Computing the evolution equation for \( Q \),

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \varphi = \dot{\varphi} - n + \text{tr} \omega \hat{\omega}_t
\]
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \tilde{\varphi} = \text{tr} \omega \dot{\hat{\omega}}_\infty - \hat{\omega}_t - \dot{\tilde{\varphi}}.
\]

Adding these,

\[
\left( \frac{\partial}{\partial t} - \Delta \right) Q = \text{tr} \omega \dot{\omega}_\infty - n + \text{tr} \omega \varepsilon \sqrt{-1} \partial \bar{\partial} \psi
\]
\[
= \text{tr} \omega \left( \hat{\omega}_\infty + \varepsilon \sqrt{-1} \partial \bar{\partial} \psi \right) - n
\]
\[
\geq \varepsilon c_0 \text{tr} \omega \omega_0 - n.
\]

Since \( Q \to \infty \) as \( x \to E \), \( Q \) achieves a spatial minimum for each fixed time \( t_0 \). If \( Q \) attains a minimum at the point \( (x_0, t_0) \) in \( M \setminus E \) with \( t_0 > 0 \), at that point

\[
\text{tr} \omega(x_0, t_0) \omega_0(x_0, t_0) \leq \frac{n}{\varepsilon c_0}.
\]
Applying the geometric-arithmetic mean inequality,
\[
\left( \frac{\omega_0^n(x_0, t_0)}{\omega^n(x_0, t_0)} \right)^{1/n} \leq \frac{\text{tr}_{\omega(x_0, t_0)} \omega_0(x_0, t_0)}{n} \leq \frac{1}{\varepsilon c_0}.
\]
This gives a uniform lower bound for $Q$ since
\[
Q(x_0, t_0) = \log \frac{\omega_0^n(x_0, t_0)}{e^{\varepsilon \psi(x_0, t_0) \Omega(x_0, t_0)}} \geq \log \frac{\varepsilon c_0 \omega_0^n(x_0, t_0)}{e^{\varepsilon \psi(x_0, t_0) \Omega(x_0, t_0)}} \geq -C_\varepsilon.
\]

We define a family of positive $(1, 1)$-currents which will be useful in bounding $\text{tr}_{\omega_0} \omega$. Let
\[
S_t = \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \psi = e^{-t} \omega_0 + (1 - e^{-t}) \hat{\omega}_\infty + \sqrt{-1} \partial \bar{\partial} \psi.
\]

**Lemma 3.4.** There exists $T_0 > 0$ such that for all $t \geq T_0$
\[
S_t \geq \frac{c_0}{2} \omega_0
\]
as currents on $M$.

**Proof.** Choose $T_0$ sufficiently large so that for $t \geq T_0$
\[
e^{-t} (\omega_0 - \hat{\omega}_\infty) \geq -\frac{c_0}{2} \omega_0.
\]
Then
\[
S_t = \hat{\omega}_\infty + \sqrt{-1} \partial \bar{\partial} \psi + e^{-t} (\omega_0 - \hat{\omega}_\infty) \geq \frac{c_0}{2} \omega_0
\]
as currents on $M$. \hfill \qed

Now we can bound $\text{tr}_{\omega_0} \omega$ using a trick from Phong-Sturm [29].

**Lemma 3.5.** There exists uniform $C$ and $C'$ such that on $M \times [0, \infty)$,
\[
\text{tr}_{\omega_0} \omega \leq \frac{C'}{e^{C \psi}}.
\]
Moreover, there exists uniform $C''$ such that on $M \times [0, \infty)$,
\[
\frac{e^{C \psi}}{C''} \omega_0 \leq \omega \leq \frac{C''}{e^{C \psi}} \omega_0.
\]
Proof. We begin by calculating the evolution equation for $\log \text{tr}_{\omega} \omega$ following a method similar to [42].

$$
\left( \frac{\partial}{\partial t} - \Delta \right) \log \text{tr}_{\omega} \omega = \frac{1}{\text{tr}_{\omega} \omega} \left( - g^{ik}(g_0)^{jk}(\nabla_0)_{kij}(\nabla_0)_{1ji} \omega + g^{i\bar{j}}(g_0)^{jk}(\nabla_0)_{kij}(\nabla_0)_{1ji} \omega \right) + \frac{1}{\text{tr}_{\omega} \omega} g^{ik}(\nabla_0)_{k} (\text{tr}_{\omega} \omega) (\nabla_0)_{\omega}\omega \\
- 2 \text{Re} \left( g^{i\bar{j}}(g_0)^{jk}(T_0)^{pi}_{ki}(g_0)^{jp}_{q}(T_0)^{p}_{ik}(T_0)^{q}_{jl} \right) + \left[ g^{i\bar{j}}(g_0)^{jk}(T_0)^{pi}_{ki}(g_0)^{jp}_{q} \omega \right] \\
- \left[ g^{i\bar{j}}(g_0)^{jk} e^{-t} \left( (\nabla_0)_{i} \left( (T_0)^{p}_{jl}(g_0)^{jp}_{q} \right) + (\nabla_0)_{\omega} \left( (T_0)^{p}_{ik}(g_0)^{jp}_{q} \omega \right) \right) \right) \\
- g^{i\bar{j}}(g_0)^{jk} e^{-t} (T_0)^{q}_{jl} (T_0)^{p}_{ik} (g_0)^{jp}_{q} \omega - \text{tr}_{\omega} \omega \right).$$

(3.7)

There are two differences between this equation and the one in [42]. The third term in square brackets has a factor of $e^{-t}$ since

$$
\bar{\partial} \omega = \bar{\partial} \omega_t = e^{-t} \bar{\partial} \omega_0.
$$

Also, because we are considering the normalized Chern-Ricci flow we have the final $-\text{tr}_{\omega} \omega$ term.

Let $(I)$ denote the first term in square brackets in (3.7), $(II)$ the second, and $(III)$ the third, all including the $\frac{1}{\text{tr}_{\omega} \omega}$ out front. Using the estimates from [42] Proposition 3.1,

$$
(I) \leq \frac{2e^{-t}}{\text{tr}_{\omega} \omega^2} \text{Re} \left( g^{ik}(T_0)^{pi}_{kp} \partial_t \text{tr}_{\omega} \omega \right)
$$

(3.9)

and

$$
(II) \leq C \text{tr}_{\omega} \omega_0,
$$

(3.10)

and

$$
(III) \leq C e^{-t} \text{tr}_{\omega} \omega_0 \leq C \text{tr}_{\omega} \omega_0.
$$

(3.11)

Combining (3.9), (3.10) and (3.11) with (3.7),

$$
\left( \frac{\partial}{\partial t} - \Delta \right) \log \text{tr}_{\omega} \omega \leq \frac{2}{\text{tr}_{\omega} \omega^2} \text{Re} \left( g^{ik}(T_0)^{pi}_{kp} \partial_t \text{tr}_{\omega} \omega \right) + C \text{tr}_{\omega} \omega_0.
$$

(3.12)

Let

$$
\tilde{\varphi} = \varphi - \psi.
$$
Using the trick from Phong-Sturm [29], we consider the quantity

\begin{equation}
Q = \log \text{tr}_{\omega_0} \omega - A\tilde{\varphi} + \frac{1}{\tilde{\varphi} + C_0}.
\end{equation}

Here \(A\) is a large constant to be determined later and \(C_0\) is large enough so that \(\tilde{\varphi} + C_0 \geq 1\) which exists by Lemma 3.3. This choice is made so that

\[0 < \frac{1}{\tilde{\varphi} + C_0} \leq 1.\]

Fix a time \(T' > T_0\) where \(T_0\) is as in Lemma 3.3. Since \(Q \to -\infty\) as \(x \to E\), \(Q\) achieves a maximum at some point \((x_0, t_0) \in (M \setminus E) \times [0, T']\). If \(0 \leq t_0 \leq T_0\), then \(Q\) clearly has a uniform upper bound on \(M \times [0, T']\). It remains to show that \(Q\) is uniformly bounded above if \(t_0 > T_0\).

We compute the parts of the evolution equation for \(Q\) separately.

\begin{equation}
\left(\frac{\partial}{\partial t} - \Delta\right) \tilde{\varphi} = \tilde{\varphi} - \text{tr}_\omega \left(\sqrt{-1} \partial \tilde{\varphi} \partial \bar{\varphi} - \sqrt{-1} \partial \bar{\varphi} \partial \tilde{\varphi}\right)
= \tilde{\varphi} - \text{tr}_\omega \left(\omega - \omega_t - \sqrt{-1} \partial \bar{\varphi}\right)
= \tilde{\varphi} - n + \text{tr}_\omega S_t.
\end{equation}

Using the previous calculation that showed \(\Delta \tilde{\varphi} = n - \text{tr}_\omega S_t\),

\begin{equation}
\left(\frac{\partial}{\partial t} - \Delta\right) Q = \left(\frac{\partial}{\partial t} - \Delta\right) \log \text{tr}_{\omega_0} \omega - \left(A + \frac{1}{(\tilde{\varphi} + C_0)^2}\right) \tilde{\varphi} + \left(A + \frac{1}{(\tilde{\varphi} + C_0)^2}\right) (n - \text{tr}_\omega S_t) - \frac{2|\partial \tilde{\varphi}|_g^2}{(\tilde{\varphi} + C_0)^3}.
\end{equation}

Combining (3.13), (3.14) and (3.15),

\begin{equation}
0 = \frac{\partial}{\partial t} Q = \frac{1}{\text{tr}_{\omega_0} \omega} \partial_t \text{tr}_{\omega_0} \omega - A \partial_t \tilde{\varphi} - \frac{\partial \tilde{\varphi}}{(\tilde{\varphi} + C_0)^2}.
\end{equation}

At the maximum of \(Q\), \((x_0, t_0)\),

\begin{equation}
\frac{2}{(\text{tr}_{\omega_0} \omega)^2} \text{Re} \left( g^k (T_0)^p_{kp} \partial_t \text{tr}_{\omega_0} \omega \right)
= \left| \frac{2}{(\text{tr}_{\omega_0} \omega)^2} \text{Re} \left( g^k (T_0)^p_{kp} \left(A + \frac{1}{(\tilde{\varphi} + C_0)^2}\right) \partial_t \tilde{\varphi} \right) \right|
\leq \frac{|\partial \tilde{\varphi}|_g^2}{(\tilde{\varphi} + C_0)^3} + CA^2 (\tilde{\varphi} + C_0)^3 \frac{\text{tr}_{\omega_0} \omega_0}{(\text{tr}_{\omega_0} \omega)^2}.
\end{equation}
Now we break this in to two cases. If \((\text{tr}_\omega \omega) \leq A^2(\tilde{\phi} + C_0)^3\) at \((x_0, t_0)\), then
\[
Q \leq \log A + \frac{3}{2} \log(\tilde{\phi} + C_0) - A\tilde{\phi} + \frac{1}{\tilde{\phi} + C_0} \leq C
\]
where \(C\) is some constant depending on \(A\) since \(\tilde{\phi}\) is bounded below and the function \(x \mapsto \frac{3}{2} \log(x + C_0) - Ax\) is bounded above.

If instead \((\text{tr}_\omega \omega) \geq A^2(\tilde{\phi} + C_0)^3\) at \((x_0, t_0)\), substituting (3.18) into (3.16),
\[
\left(\frac{\partial}{\partial t} - \Delta\right) Q \leq |\partial \tilde{\phi}|^2 \frac{\partial \Omega}{\Omega} + C\text{tr}_\omega \omega_0 - \left(A + \frac{1}{(\tilde{\phi} + C_0)^2}\right) \phi
\]
\[
\leq C\text{tr}_\omega \omega_0 + (A + 1) |\tilde{\phi}| + (A + 1) (n - \text{tr}_\omega S_t).
\]

Using Lemma 3.3, we can choose \(A\) large enough so that \((A + 1) S_t \geq (C + 1) \omega_0\). At \((x_0, t_0)\),
\[
0 \leq \left(\frac{\partial}{\partial t} - \Delta\right) Q \leq -\text{tr}_\omega \omega_0 + C \left|\log \frac{\Omega}{\omega^n}\right| + C.
\]

At the maximum of \(Q\),
\[
\text{tr}_\omega \omega_0 \leq \frac{1}{(n - 1)!} (\text{tr}_\omega \omega_0)^{n-1} \omega^n_0 \leq C \frac{\omega^n}{\Omega} \left|\log \frac{\Omega}{\omega^n}\right|^{n-1} + C \leq C
\]
and since \(\omega^n \leq C\Omega\) and the function \(x \mapsto x |\log x|^{n-1}\) is bounded for small \(x > 0\) we obtain a uniform upper bound for \(Q\). In either of the cases, \(Q\) is uniformly bounded above, so using Lemma 3.1,
\[
\log \text{tr}_\omega \omega_0 \leq C + C\tilde{\phi} \leq C - C\psi.
\]
Exponentiating gives (3.5) and (3.6) follows. \(\square\)

Using these lower order estimates with the higher order estimates in [14] on compact subsets of \(M \setminus E\) we obtain uniform \(C_{10c}(M \setminus E)\) estimates for \(\varphi\).

4. CONVERGENCE AND UNIQUENESS

We now complete the proof of Theorem 1.1 by showing that \(\omega\) converges to a Kähler-Einstein metric on \(M \setminus E\) in \(C_{10c}(M \setminus E)\).

Proof. The quantity
\[
Q = \varphi + Cte^{-t}
\]
where \(C\) is the constant in Lemma 3.1 is uniformly bounded below on compact subsets of \(M \setminus E\) by Lemma 3.3. By Lemma 3.1 (ii),
\[
\partial_t Q = \tilde{\phi} - Cte^{-t} \leq 0
\]
so \( \varphi \) converges pointwise to a function \( \varphi_\infty \) at \( t \to \infty \) on \( M \setminus E \). Using the estimates from the previous section, we have convergence in \( C^\infty_{loc}(M \setminus E) \).

Since \( \varphi \) converges as \( t \to \infty \), \( \dot{\varphi} \to 0 \) similarly in \( C^\infty_{loc}(M \setminus E) \).

The above convergence for \( \varphi \) and \( \dot{\varphi} \) implies that

\[
\omega \to \omega_\infty := \hat{\omega}_\infty + \sqrt{-1} \partial \bar{\partial} \varphi_\infty
\]

and \( \frac{\partial}{\partial t} \omega \to 0 \) as \( t \to \infty \) in \( C^\infty_{loc}(M \setminus E) \). Taking \( t \to \infty \) in the normalized Chern-Ricci flow \( (1.1) \),

\[
\text{Ric}(\omega_\infty) = -\omega_\infty
\]
on \( M \setminus E \). Since \( \text{Ric}(\omega_\infty) \) is closed, \( \omega_\infty \) is a Kähler-Einstein metric on \( M \setminus E \).

Moreover, applying weak compactness of currents we can extend \( \omega_\infty \) to a closed, positive current on \( X \) and then \( \omega \to \omega_\infty \) as currents on \( X \). \( \square \)

We now show that \( \omega_\infty \) is unique in the sense of Theorem 1.2.

**Proof.** The proof of this result is similar to the Kähler case [45, 40] (see also [35]), but we provide the proof for the sake of completeness. Let \( \omega_\infty \) and \( \tilde{\omega}_\infty \) be two closed positive currents satisfying (i) and (ii). Define

\[
\theta = \log \frac{\omega_\infty^n}{\hat{\Omega}}, \quad \tilde{\theta} = \log \frac{\tilde{\omega}_\infty^n}{\tilde{\Omega}}.
\]

Taking \( \sqrt{-1} \partial \bar{\partial} \),

\[
\sqrt{-1} \partial \bar{\partial} \theta = -\text{Ric}(\omega_\infty) - \dot{\omega}_\infty
\]

and so

\[
\omega_\infty = \dot{\omega}_\infty + \sqrt{-1} \partial \bar{\partial} \theta.
\]

Similarly

\[
\tilde{\omega}_\infty = \tilde{\dot{\omega}}_\infty + \sqrt{-1} \partial \bar{\partial} \tilde{\theta}.
\]

Define the quantity

\[
Q = \theta - (1 - \delta)\tilde{\theta} - \delta \varepsilon \psi
\]

for some \( 0 < \delta < 1 \). \( Q \) is bounded below and \( Q \to \infty \) as \( x \to E \) so \( Q \) attains a minimum at a point \( x_0 \) in \( M \setminus E \). At \( x_0 \),

\[
\theta - \tilde{\theta} = \log \frac{\omega_\infty^n}{\tilde{\omega}_\infty^n}
\]

\[
= \log \left( \dot{\omega}_\infty + (1 - \delta)\sqrt{-1} \partial \bar{\partial} \theta + \delta \varepsilon \sqrt{-1} \partial \bar{\partial} \psi + \sqrt{-1} \partial \bar{\partial} Q \right)^n
\]

\[
= \log \left( (1 - \delta)\tilde{\dot{\omega}}_\infty + \delta \varepsilon \sqrt{-1} \partial \bar{\partial} \tilde{\psi} + \sqrt{-1} \partial \bar{\partial} Q \right)^n
\]

\[
\geq \log \frac{(1 - \delta)n\tilde{\omega}_\infty^n}{\tilde{\omega}_\infty^n} = n \log(1 - \delta).
\]

By Lemma 3.3, \( \delta \tilde{\theta}(x_0) - \delta \varepsilon \psi(x_0) \geq \delta C \varepsilon \). Then

\[
Q(x_0) = \theta(x_0) - \tilde{\theta}(x_0) + \delta \tilde{\theta}(x_0) - \delta \varepsilon \psi(x_0) \geq n \log(1 - \delta) - \delta C \varepsilon.
\]
Choosing $\delta$ sufficiently small so that $n \log(1 - \delta) > -\varepsilon/2$ and $\delta C_\varepsilon < \varepsilon/2$,
\[
Q(x_0) \geq -\varepsilon.
\]
Since $Q$ achieves its minimum at $x_0$,
\[
\theta \geq (1 - \delta)\tilde{\theta} + \delta \varepsilon \psi - \varepsilon.
\]
Taking $\delta \to 0$ and $\varepsilon \to 0$,
\[
\theta \geq \tilde{\theta}.
\]
Similarly, $\theta \leq \tilde{\theta}$.

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