A new construction of the moonshine vertex operator algebra over the real number field.

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Abstract

We give a new construction of the moonshine module vertex operator algebra $V^\#$ over the real field, which was originally constructed in [FLM2]. The advantage of our construction is that we can easily prove the facts that $V^\#$ has a positive definite invariant bilinear form and Aut($V^\#$) is the Monster simple group. In addition, we construct a lot of conformal vectors in $V^\#$ which give rise to 2A-involutions. We also construct an infinite series of holomorphic VOAs. Each of them has exactly one irreducible module and its full automorphism group is finite. At the end of the paper, we will calculate the character of a 3C element of the Monster simple group.

1 Introduction

All VOAs in this paper are defined over the real number field $\mathbb{R}$ and $\mathbb{C}V$ denotes the VOA $\mathbb{C} \otimes_{\mathbb{R}} V$ for a VOA $V$.

The most interesting example of vertex operator algebra (VOA) is the moonshine module VOA $V^\# = \sum_{i=0}^{\infty} V^\#_i$. Although it has many interesting properties, the original construction [FLM2] essentially depends on the actions of the centralizer $2^{1+24}Co.1$ of an involution called $2B$ of the Monster simple group and so it is hard to see the actions of the other elements explicitly. The Monster simple group has the other conjugacy class of involutions called 2A. We will construct the Moonshine VOA $V^\#$ from the point of view of elementary abelian 2-group generated by 2A-elements.

The simplest example of VOA is the rational Virasoro VOA $L(\frac{1}{2}, 0)$ of the minimal series with central charge $\frac{1}{2}$. It has only three irreducible modules $L(\frac{1}{2}, 0)$, $L(\frac{1}{2}, \frac{1}{2})$, and $L(\frac{1}{2}, \frac{1}{16})$, where the first entry is the central charge and the second entry denotes the
lowest weights. Its fusion rules (or fusion products) are known as, see [BPZ] or [DMZ]:

\[(1) \quad L(\frac{1}{2}, 0) \text{ is identity,} \]
\[(2) \quad L(\frac{1}{2}, \frac{1}{2}) \times L(\frac{1}{2}, \frac{1}{2}) = L(\frac{1}{2}, 0), \]
\[(3) \quad L(\frac{1}{2}, \frac{1}{2}) \times L(\frac{1}{2}, \frac{1}{16}) = L(\frac{1}{2}, 0), \]
\[(4) \quad L(\frac{1}{2}, \frac{1}{16}) \times L(\frac{1}{2}, \frac{1}{16}) = L(\frac{1}{2}, 0) + L(\frac{1}{2}, \frac{1}{2}). \]

\[\text{(1.1)} \]

For a VOA \(V\), we call \(e \in V_2\) a rational conformal vector if a sub VOA \(<e>\) generated by \(e\) is a rational VOA and \(e\) is the Virasoro element of \(<e>\). We are essentially interested in a rational conformal vector \(e\) with central charge \(\frac{1}{2}\). Under this assumption, \(<e>\) is isomorphic to \(L(\frac{1}{2}, 0)\) and we can view \(V\) as a \(<e>-\text{module}\). The fusion rules of \(L(\frac{1}{2}, 0)\) will then play an important role in our arguments. In particular, we will use an automorphism \(\tau_e\) of \(V\), which is defined by the author in [Mi1], for each rational conformal vector \(e\) with central charge \(\frac{1}{2}\), where \(\tau_e\) is given by

\[\tau_e : \begin{cases} 
1 & \text{on all } <e>-\text{submodules isomorphic to } L(\frac{1}{2}, 0) \text{ or } L(\frac{1}{2}, \frac{1}{2}) \\
-1 & \text{on all } <e>-\text{submodules isomorphic to } L(\frac{1}{2}, \frac{1}{16}) 
\end{cases} \]

Since we will treat only rational VOAs \(V\), the tensor product of two \(V\)-modules \(W^1\) and \(W^2\) is well-defined [Li] and it is equal to the fusion product \(W^1 \times W^2\). Therefore, we will also consider a fusion product as a module. From now on, \(\otimes_{i=1}^n W^i\) means a \(\otimes_{i=1}^n V^i\)-module for \(V^i\)-modules \(W^i\).

In this paper, we will consider a set of mutually orthogonal rational conformal vectors \(\{e^i : i = 1, ..., n\}\) with central charge \(\frac{1}{2}\) such that the sum \(\sum e^i\) is Virasoro element of \(V\). Here, ”orthogonal” means \(e^i e^j = 0\) for \(i \neq j\). We will call such a set of conformal vectors ”a coordinate set”. Thus, the sub VOA \(T = <e^1, ..., e^n>\) is isomorphic to \(L(\frac{1}{2}, 0)^\otimes \otimes\) and it is known that every irreducible \(T\)-module \(W\) is a tensor product \(\otimes_{i=1}^n L(\frac{1}{2}, h^i)\) of irreducible \(L(\frac{1}{2}, 0)\)-modules \(L(\frac{1}{2}, h^i)\), see [DMZ]. Define a binary word

\[\tilde{\tau}_T(W) = (a_1, ..., a_n) \quad (1.2)\]

by \(a_i = 1\) if \(h_i = \frac{1}{16}\) and \(a_i = 0\) if \(h_i = 0\) or \(\frac{1}{2}\). We call it a (binary) \(\tau\)-word since it is corresponding to the actions of automorphisms \(\tau_{e^i}\). (The author once called it a word of \(\frac{1}{16}\)-positions and denoted it by \(\tilde{h}(W)\) in [Mi3] and [Mi4].) We note that \(T\) is rational and the fusion product is given by

\[\left(\otimes_{i=1}^n W^i\right) \times \left(\otimes_{i=1}^n U^i\right) = \otimes_{i=1}^n (W^i \times U^i)\]

as Dong, Mason and Zhu proved in [DMZ].
We will construct the moonshine VOA $V^\sharp$ over the real number field $\mathbb{R}$ as a direct sum of irreducible $T$-modules. It is not difficult to construct the underlining space $V^\sharp$ as a direct sum of irreducible $T$-modules. Originally, it has shown by Dong, Mason and Zhu [DMZ] that the moonshine VOA $V^\sharp$ of rank 24 contains 48 mutually orthogonal conformal vectors $e^i$ with the central charge $\frac{1}{2}$ such that the sum is the Virasoro element of $V^\sharp$ and the author determined the multiplicities of all irreducible $T$-submodules of $V^\sharp$ for some $T$ in [Mi4].

The reason why we will treat a VOA over $\mathbb{R}$ is that a positive definite invariant bilinear form on a VOA is very useful to determine an automorphism group. For example, Frenkel, Lepowsky and Meurman constructed the moonshine VOA $V^\sharp$ over the real number field (in fact, they constructed it over the rational number field) and have shown that $V^\sharp$ has a positive definite invariant bilinear form in [FLM2]. One of our important tools in determining the full automorphism group is the uniqueness theorem for a VOA satisfying Hypotheses I mentioned later. This holds for VOAs over the complex number field without assuming the positive definite invariant bilinear form (see [Mi5]). However, it is not uniquely determined for VOAs over the real number field. In order to avoid this anomaly, we will treat only VOAs over $\mathbb{R}$ with positive definite invariant bilinear forms. For example, the code VOAs which the author defined in [Mi2] have a positive definite invariant bilinear form if we construct them over $\mathbb{R}$. This setting offers us exactly the same situation as in VOAs over the complex field.

One of our tools is a code VOA $M_D$ and its representation theory for an even linear binary code $D$ of length $n$ ([Mi2] and [Mi3]). We will briefly explain it in §3. The characterization of a code VOA $M_D$ is that it is a simple VOA $V$ containing $T = L(\frac{1}{2}, 0)^{\otimes n}$ and $\tilde{\tau}(V) = (0^n)$. Any irreducible $M_D$-module $W$ is a direct sum of irreducible $T$-modules $U^i$ since $T$ is rational. By the fusion rules of Ising models (1.1), $\tilde{\tau}(U^i)$ is uniquely determined and so we use the same notation $\tilde{\tau}(W)$ for it. If a simple VOA $V$ contains a coordinate set $\{e^1, ..., e^n\}$, then $P = \langle \tau_{e^i} : i = 1, ..., n \rangle$ is an elementary abelian automorphism group. Decompose $V$ into a direct sum

$$V = \bigoplus_{\chi \in \text{Irr}(P)} V^\chi$$

of eigenspaces of $P$, where $V^\chi = \{v \in V : gv = \chi(g)v \text{ for } g \in P\}$ and $V^1 = V^P$ is the set of $P$-invariants. It is known by [DM2] that $V^\chi$ is a nonzero irreducible $V^P$-module. It follows from the definition of $\tau_{e^i}$ that $V^P$ contains $T = \langle e^1, ..., e^n \rangle$ and is isomorphic to a code VOA. Moreover, if $\tilde{\tau}(V^\chi) = (a_i)$ then $\chi(\tau_{e^i}) = (-1)^{a_i}$. Therefore, the representation theory of code VOA plays an essential role in the study of such VOAs.

Another tool is "induced VOA". In [Mi3], we introduced a concept of the induced
\[ \mathbb{C}M_D - \text{module } \text{Ind}_D^E(\mathbb{C}W) \] for a subcode \( E \) containing a maximal self-orthogonal subcode of \( D_\beta = \{ \alpha \in D | \text{Supp}(\alpha) \subseteq \text{Supp}(\beta) \} \) and an \( \mathbb{C}M_E - \text{module } \mathbb{C}W \) satisfying \( \langle \tilde{\tau}(W), D \rangle = 0 \). This is a special case of the concept of induced modules defined in [DL]. We will also define an induced module for code VOAs over \( \mathbb{R} \) and apply it to a VOA here. Namely, \( \text{Ind}_D^E(W) \) becomes a VOA if \( W \) is a VOA under some conditions. An advantage of this construction is that it keeps a positive definite invariant bilinear form. As applications, we will construct several holomorphic VOAs from a VOA. For example, we will construct a lattice VOA \( V_\Lambda \) of the Leech lattice from \( V^2 \) by restricting and defining an induced VOA.

We should note that it is possible to construct \( V^2 \) over the rational number field by our way. However, it makes us add several conditions to get the uniqueness theory and we will avoid such complications.

We will prove that our VOA \( V^2 \) over \( \mathbb{R} \) is a holomorphic VOA of rank 24. We will also prove that the full automorphism group is the Monster simple group \( M \). These information are enough to show that our VOA \( V^2 \) is isomorphic to the moonshine module VOA constructed in \([FLM2]\).

We are now in position to mention the outline of this paper. In this paper, we will not only construct the moonshine VOA but also VOAs with similar structures. Our essential tool is the following theorem, which was proved for VOAs over \( \mathbb{C} \) by the author in \([Mi5]\). We will show that this theorem is also true for VOAs over \( \mathbb{R} \) with positive definite invariant bilinear forms.

**Hypotheses I**

1. \( D \) and \( S \) are both even linear codes of length \( 8k \) and \( S \subseteq D \cap D^{'\perp} \).
2. For any \( \alpha, \beta \in S \), \( \alpha \neq \beta \), there is a self-dual subcode \( E = E_\alpha \oplus E_{\alpha^c} \) of \( D \) and maximal self-orthogonal (doubly even) subcodes \( H_\beta \) and \( H_{\alpha^+\beta} \) of \( D_\beta \) and \( D_{\alpha^+\beta} \) containing \( E_\beta \) and \( E_{\alpha^+\beta} \), respectively, such that
   1. \( E_\alpha \) and \( E_{\alpha^c} \) are direct sums of \([8, 4, 4]\)-Hamming codes,
   2. \( H_\beta + E = H_{\alpha^+\beta} + E \),
   where \( \alpha^c \) denotes the complement \( (1^8k) - \alpha \) and \( S_\delta \) denotes a subcode \( \{ \gamma \in D : \text{Supp}(\gamma) \subseteq \text{Supp}(\delta) \} \) of a code \( S \) and
3. There is an \( S \)-graded \( M_D \)-module \( V = \bigoplus_{\alpha \in S} V^\alpha \) such that each \( V^\alpha \) is an \( M_D \)-submodule with \( \tilde{\tau}(V^\alpha) = \alpha \). In particular, \( V^{(0^8k)} \cong M_D \) as \( M_D \)-modules.
4. For \( \alpha, \beta \in S - \{ (0^8) \} \) and \( \alpha \neq \beta \),
   \[ V^{\alpha,\beta} = M_D \oplus V^\alpha \oplus V^\beta \oplus V^{\alpha^+\beta} \]
has a simple VOA structure containing $M_D$ as a sub VOA. 

(5) $V^{\alpha,\beta}$ has a positive definite invariant bilinear form.

**Theorem 3.3** Under the above assumptions (1)~(5) of Hypotheses I, we obtain the fusion product $V^{\alpha} \times V^{\beta} = V^{\alpha+\beta}$ for $\alpha, \beta \in D$ and

$$V = \bigoplus_{\alpha \in S} V^{\alpha}$$

has a structure of simple VOA with $M_D$ as a sub VOA and it has a positive definite invariant bilinear form. The structure of VOA on $V$ with a positive definite invariant bilinear form is uniquely determined up to $M_D$-isomorphisms.

We note that an important assertion of Theorem 3.3 is that the fusion product $V^{\alpha} \times V^{\beta}$ is irreducible, that is, if $I(\ast, z) \in I\left( V^{\alpha+\beta}, \begin{pmatrix} V^{\alpha} & V^{\beta} \end{pmatrix} \right)$ is a nonzero intertwining operator, then for any VOA structure $(V, \tilde{Y})$ there is a scalar $\lambda$ such that $\tilde{Y}(v, z)|_{V^{\beta}} = \lambda I(v, z)$. As we will show that the uniqueness of the VOA structure on $V$ comes from this property.

The assumptions (1) and (2) are conditions on the codes $D$ and $S$. So our construction is just to collect a set $\{V^{\alpha} : \alpha \in S\}$ of $M_D$-modules satisfying (4) and (5). In order to prove the condition (4), we will use the following theorems. These are also essentially based on the irreducibility of fusion products.

**Theorem 3.2** Assume that (1) and (2) of Hypotheses I hold for $(S, D)$. Choose $\alpha, \beta \in S$ so that dim $< \alpha, \beta > = 2$, where $< \alpha, \beta >$ is the code generated by $\alpha$ and $\beta$. Let $F$ be an even linear code containing $D$ and assume $\alpha, \beta \in F^\perp$. If $U = M_D \oplus W^{\alpha} \oplus W^{\beta} \oplus W^{\alpha+\beta}$ has a simple VOA structure satisfying $\tilde{\tau}(W^\gamma) = \gamma$ for $\gamma \in < \alpha, \beta >$, then

$$\text{Ind}_D^F(U) = M_F \oplus \text{Ind}_{MD}^F(W^{\alpha}) \oplus \text{Ind}_{MD}^F(W^{\beta}) \oplus \text{Ind}_{MD}^F(W^{\alpha+\beta})$$

has a simple VOA structure.

**Theorem 4.1** Under the assumptions in Theorem 3.2, if a VOA $U$ has a positive definite invariant bilinear form, then $\text{Ind}_D^F(U)$ has a VOA structure with a positive definite invariant bilinear form. Furthermore, such a VOA is uniquely determined up to $M_F$-isomorphisms.

In order to construct a VOA by using Theorem 3.3, it is sufficient to collect $M_D$-modules satisfying (4) and (5) for a small code $D$ as we showed in Theorem 4.1. We will
gather such modules from the lattice VOA $\tilde{V}_{E_8}$ with a positive definite invariant bilinear form constructed from a even unimodular lattice of type $E_8$. We will also prove that $\tilde{V}_{E_8}$ has a structure satisfying Hypotheses I. Namely, $\tilde{V}_{E_8}$ contains 16 mutually orthogonal conformal vectors $\{e^i\}$ such that

1. The order of $P = \langle \tau_e : i = 1, \ldots, 8 \rangle$ is 32,
2. $(\tilde{V}_{E_8})^P$ is isomorphic to a code VOA $M_{D_{E_8}}$, where $D_{E_8}$ is a Reed Muller code $RM(4, 2)$ and
3. $\tilde{V}_{E_8} = \bigoplus_{\alpha \in S_{E_8}} \tilde{V}_{E_8}^\alpha$, where $S_{E_8} = D^\perp \cong RM(4, 1)$, $\tilde{V}_{E_8}^{(0)} \cong M_{RM(4, 2)}$ and $\tilde{V}_{E_8}^\alpha$ are irreducible $M_{RM(4, 2)}$-modules. Note that $S_{E_8} = \langle (16), (0^6 1)^8, (\{0^4 1^4\})^2, (\{0^2 1^2\})^4, (\{01\})^8 \rangle$ (1.3) and the weight enumerator of $S_{E_8}$ is $x^{16} + 30x^8y^8 + y^{16}$. Moreover, the minimal weight of $D_{E_8}$ is 4 and the pair $(D_{E_8}, S_{E_8})$ satisfies the conditions (1) and (2) of Hypotheses I, see Lemma 5.1. Therefore, a VOA structure on the $M_{D_{E_8}}$-module $\tilde{V}_{E_8} = \bigoplus_{\alpha \in S_{E_8}} \tilde{V}_{E_8}^\alpha$ is uniquely determined by Theorem 3.3. We also have a fusion product $\tilde{V}_{E_8}^\alpha \times \tilde{V}_{E_8}^\beta = \tilde{V}_{E_8}^{\alpha + \beta}$ of $M_{D_{E_8}}$-modules for any $\alpha, \beta \in S_{E_8}$.

We will next explain how to construct the moonshine VOA. In order to define the moonshine VOA $V^\natural$, we will set

$$S^2 = \{ (\alpha, \alpha, \alpha), (\alpha, \alpha, \alpha^c), (\alpha, \alpha^c, \alpha), (\alpha^c, \alpha, \alpha) : \alpha \in S_{E_8} \}$$ (1.4)

where $\alpha^c$ is the complement of $\alpha$. Set $D^2 = (S^2)^\perp$ and call it a moonshine code. It is of dimension 41 and contains $D^2_{E_8} = D_{E_8} \oplus D_{E_8} \oplus D_{E_8}$. We note that $S^2$ and $D^2$ are even linear codes of length 48. Clearly, the pair $(D^2_{E_8}, S^2)$ satisfies the conditions (1) and (2) of Hypotheses I.

Our construction consists of the following three steps.

First, since $\tilde{V}_{E_8}^\alpha \times \tilde{V}_{E_8}^\beta = \tilde{V}_{E_8}^{\alpha + \beta}$ for $\alpha, \beta \in S_{E_8}$,

$$V^1 = \bigoplus_{(\alpha, \beta, \gamma) \in S^2} (\tilde{V}_{E_8}^\alpha \otimes \tilde{V}_{E_8}^\beta \otimes \tilde{V}_{E_8}^\gamma)$$ (1.5)

is a sub VOA of $\tilde{V}_{E_8} \otimes \tilde{V}_{E_8} \otimes \tilde{V}_{E_8}$. Clearly, $V^1$ has a positive definite invariant bilinear form. Our second step is to twist it. Namely, set $\xi_1 = (10^{15})$ and let $R = M_{D_{E_8} + \xi_1}$ be a coset module. To simplify the notation, denote $R \times \tilde{V}_{E_8}^\gamma$ by $R\tilde{V}_{E_8}^\gamma$. Set

$$Q = \langle (\xi_1\xi_1 0^{16}), (0^{16}\xi_1\xi_1) \rangle \subseteq \mathbb{Z}_2^{48}.$$ We induce $V^1$ to

$$V^2 = \text{Ind}_{D^2_{E_8}}^{D^2_{E_8} + Q}(V^1).$$

6
Although $V^2$ is not a VOA, we can find the following $M_{D_3}$-submodules in $V^2$:
\[
W^{(\alpha,\alpha,\alpha)} = \tilde{V}^\alpha_{E_8} \otimes \tilde{V}^\alpha_{E_8} \otimes \tilde{V}^\alpha_{E_8}
\]
\[
W^{(\alpha,\alpha,\alpha^c)} = (R\tilde{V}^\alpha_{E_8}) \otimes (R\tilde{V}^\alpha_{E_8}) \otimes \tilde{V}^{\alpha^c}_{E_8}
\]
\[
W^{(\alpha^c,\alpha,\alpha)} = (R\tilde{V}^\alpha_{E_8}) \otimes \tilde{V}^\alpha_{E_8} \otimes (R\tilde{V}^\alpha_{E_8})
\]
\[
W^{(\alpha^c,\alpha,\alpha^c)} = (\tilde{V}^\alpha_{E_8}) \otimes (R\tilde{V}^\alpha_{E_8}) \otimes (R\tilde{V}^\alpha_{E_8}).
\]
for $\alpha \in S_{E_8}$. At the end, we set
\[
(V^2)^\chi = \text{Ind}_{D_3}^{D_2}(W^\chi)
\]
for $\chi \in S^2$. We will show that these $M_{D_3}$-modules $(V^2)^\chi$ satisfy the condition (4) of Hypotheses I. Therefore, we obtain a VOA
\[
V^2 = \bigoplus_{\chi \in S^2} (V^2)^\chi
\]
which possesses a positive definite invariant bilinear form. Since we construct $V^2$ under the condition $S^2 = (D^2)^{-1}$, $V^2$ is the only irreducible $V^2$-module by Theorem 6.1. From the construction, we will see that $\dim(V^2)_0 = 1$ and $(V^2)_1 = 0$. It comes from the structure of $V^2$ and the multiplicity of irreducible $M_{D_3}$-submodules that $q^{-1}c_{V^2} = J(q) = q^{-1} + 196884q + \ldots$ is the J-function. We will also see that the full automorphism group of $V^2$ is the Monster simple group. Although it is not easy to determine the full automorphism groups of VOAs in general, our construction has certain advantages. For example, it is easy to prove that the full automorphism group of a VOA satisfying Hypotheses I is finite if $V_1 = 0$ (Theorem 9.2). Furthermore, if $S$ is a subcode of $\{(\alpha, \alpha) : \alpha \in \mathbb{Z}_{2}^{n/2}\}$ by rearranging the order of coordinates, then we will show that our VOA is a sub VOA of some lattice VOA with rank $n$ by the uniqueness of VOA structures. Also since our VOA $V$ contains a lot of rational conformal vectors $\{e^i : i \in I\}$ with central charge $\frac{1}{2}$, $V$ has a large automorphism group generated by $\{\tau_e^i : i \in I\}$, which is clearly a normal subgroup of $\text{Aut}(V)$. Using these properties, we will prove that the space $(V^2)^{<\delta>}$ of $\delta$-invariant is isomorphic to $V^\theta_{\Lambda}$ for a lattice VOA $V_{\Lambda}$ of the Leech lattice and an automorphism $\theta$ of $V_{\Lambda}$ induced from $-1$ on $\Lambda$ for $\delta = \tau_{e^1} \tau_{e^2}$. For a conformal vector $e \in (V^2)^{<\delta>} \cong V^\theta_{\Lambda}$, we can define automorphisms $\tau_e \in \text{Aut}(V^2)$ and $\tilde{\tau}_e \in \text{Aut}(V_\Lambda)$. By this correspondence, we can calculate $C_{\text{Aut}(V^2)}(\delta)$. Also, we can calculate $N_{\text{Aut}(V^2)}(\langle \tau_{e^1} \tau_{e^2}, \tau_{e^1} \tau_{e^3} \rangle)$ and $N_{\text{Aut}(V^2)}(\langle \tau_{e^1} \tau_{e^2}, \tau_{e^1} \tau_{e^3}, \tau_{e^1} \tau_{e^5} \rangle)$. By this information, we can conclude that $\text{Aut}(V^2)$ is the Monster simple group and $V^2$ coincides with the moonshine module VOA constructed in [FLM2]. Thus, this is a new construction of the moonshine VOA and the monster simple group.
Remark  It is possible to induce $V^1$ in (1.5) into a VOA

$$\tilde{V} = \text{Ind}_{D^3}^{D^3}(V^1)$$

directly. It follows from a direct calculation and the fusion rule (1.1) that $\tilde{V}_1$ is a commutative Lie algebra of dimension 24. Since $\tilde{V}$ is a holomorphic VOA by Theorem 6.1, $\tilde{V}$ is isomorphic to the lattice VOA $V_\Lambda$ of Leech lattice $\Lambda$ by [Mo], (see Section 9).

Another important theorem in this paper is that if $S = D^\perp$ then a simple VOA $V$ satisfying Hypotheses I has the exactly one irreducible $V$-module $V$, see Theorem 6.1. Since Dong, Griess and Höhn [DGH] have proved that a simple VOA satisfying Hypotheses I is rational, the VOAs $V = \bigoplus_{m=0}^{\infty} V_m$ satisfying $S = D^\perp$ are holomorphic and so $q^{-n/48} \sum (\dim V_m) q^m$ is a modular function of $SL_2(\mathbb{Z})$ with a linear character by [Z].

In §4, we construct a VOA $V_{E_8}$ with a positive definite invariant bilinear form. In §5, we investigate the structure of $V_{E_8}$. In §7, we construct the moonshine VOA $V^\natural$. In §8, we will construct a lot of rational conformal vectors of $V^\natural$ explicitly. In §9, we prove that $\text{Aut}(V^\natural)$ is the Monster simple group and $V^\natural$ is equal to the one constructed in [FLM2]. In §10, we will construct an infinite series of holomorphic VOAs with finite full automorphism groups. In §11, we will calculate the characters of some elements of the Monster simple group.

Acknowledgment
The author wishes to thank K. Harada, T. Kondo and H. Yamaki for their helpful advises. The author also would like to express the appreciation to J. Lepowsky for his useful comments.

2 Notation and preliminary results

We adopt all notation and results from [Mi3] and recall the construction of a lattice VOA.
2.1 Notation

\( \alpha^c \) The complement \((1^n) - \alpha\) of a binary word \( \alpha \).

\( D, D(m) \) Even binary linear codes, also see §4.

\( D_\beta = \{ \alpha \in D : \text{Supp}(\alpha) \subseteq \text{Supp}(\beta) \} \).

\( D^3 = \{ (\alpha, \beta, \gamma) : \alpha, \beta, \gamma \in D \} \).

\( D^3, S^3 \) The moonshine codes. See (1.4).

\( D_E, S_E \) See (1.3).

\( \{ e_i \mid i = 1, \ldots, n \} \) A set of mutually orthogonal rational conformal vectors with central charge \( \frac{1}{2} \).

\( e^\pm(x) = \frac{1}{16} ( - 1 )^1 \pm \frac{1}{4} \iota(x) + \iota(-x) \in V_L \) the conformal vectors defined by \( x \in L \) with \( \langle x, x \rangle = 4 \).

\( E_8, E_8(m) \) An even unimodular lattice of type \( E_8 \), also see (5.1).

\( \{ f^i : i \}, \{ d^i : i \} \) The other sets of mutually orthogonal eight conformal vectors in a Hamming code VOA \( M_{H_8} \), see \[ Mi5 \].

\( H_8 \) The \( [8, 4, 4] \)-Hamming code.

\( H(\frac{1}{2}, \alpha), H(\frac{1}{16}, \beta) \) The irreducible \( V_{H_8} \)-modules, see Def.13 in \[ Mi5 \].

\( \text{Ind}_{M_D}^{M_E} (U) \) The induced \( M_D \)-module from an \( M_E \)-module \( U \), see Sec.5.2 in Sec.6.2 in \[ Mi5 \].

\( \iota(x) \) A vector in a lattice VOA \( V_L = \bigoplus_{x \in L} M(1) \iota(x) \), see \[ FLM2 \].

\( L \) A lattice.

\( M_{\beta+D} \) A coset module \( \bigoplus_{(a^i)} \in \beta+D \left( (\otimes_{i=1}^n M_{a^i}) \otimes e(a^i) \right) \).

\( M_D \) A code VOA, see §3.

\( Q = < (10^{15}10^{15}0^{16}), (10^{15}0^{16}10^{15}) > \).

\( R \) \( M(10^7)_{+D} \).

\( RV^\alpha_{E_8} \) \( R \times V^\alpha_{E_8} \).

\( \tilde{\tau}(W) \) A \( \tau \)-word \((a_1, \ldots, a_n)\), see (1.2).

\( T = \otimes_{i=1}^n L(\frac{1}{2}, 0) \).

\( \times \) A fusion rule or a tensor product.

\( A(x, z) \sim B(x, z) \) \((x - z)^n (A(x, z) - B(x, z)) = 0 \) for an \( n \in \mathbb{N} \).

\( \theta \) An automorphism of \( V_L \) defined by \(-1\) on \( L \).

\( V_L \) A lattice VOA \( \bigoplus_{x \in L} M(1) \iota(x) \), see \[ FLM2 \] and §2.2.

\( \xi_i \) A word which is 1 in the \( i \)-th entry and 0 everywhere else, for example, \((0^i-10^{-n-i}), (0^i-10^{8-i}), (0^i-116^{-i})\).

\((1^m0^n) = (1 \cdots 10 \cdots 0) \).

\((\{abc\}^n*) = (abcabc \cdots abc*) \).
2.2 Lattice VOA

Let $L$ be a lattice with a bilinear form $\langle \cdot, \cdot \rangle$. Viewing $H = \mathbb{R} \otimes \mathbb{Z} L$ as a commutative Lie algebra with a bilinear form $<,>$, we define the affine Lie algebra

$\hat{H} = H[t, t^{-1}] + \mathbb{R}C$

$[C, \hat{H}] = 0, \quad [ht^n, h't^m] = \delta_{m+n,0} \langle h, h' \rangle C$

associated with $H$ and the symmetric tensor algebra $M(1) = S(\hat{H}^-)$ of $\hat{H}^-$, where $\hat{H}^- = H[t^{-1}]t^{-1}$. As in [FLM2], we shall define the Fock space $V_L = \bigoplus_{x \in L} M(1) \iota(x)$ with the vacuum $1 = \iota(0)$ and the vertex operators $Y(\ast, z)$ as follows: The vertex operator of $\iota(a)$ is given by

$Y(\iota(a), z) = \exp \left( \sum_{n \in \mathbb{Z}^+} \frac{a(-n)}{n} z^n \right) \exp \left( \sum_{n \in \mathbb{Z}^+} \frac{a(n)}{-n} z^{-n} \right) e^a z^a$

and that of $a(-1)\iota(0)$ is

$Y(a(-1)\iota(0), z) = a(z) = \sum a(n) z^{-n-1}$.

The vertex operators of other elements are defined by the normal product:

$Y(a(n)v, z) = a(z)_n Y(v, z) = \text{Res}_x \{ (x-z)^n a(x)Y(v, z) - (z-x)^n Y(v, z)a(x) \}.$

Here the operator of $a \otimes t^n$ on $M(1)\iota(b)$ are denoted by $a(n)$ and

$a(n)\iota(b) = 0$ for $n > 0$

$a(0)\iota(b) = < a, b > \iota(b)$

$e^a \iota(b) = c(a, b) \iota(a + b)$ for some cocycle $c(a, b)$

$z^a \iota(b) = \iota(b) z^{<a, b>}.$

We note that the above definition of vertex operator is very general and so we may think

$Y(v, z) \in \text{End}(V_{\mathbb{R} \otimes L})\{z\}$

for $v \in \mathbb{R} \otimes L$, where $V_{\mathbb{R} \otimes L} = \sum_{a \in \mathbb{R} \otimes L} M(1)\iota(a)$. Set $1 = \iota(0)$. It is worthy to note that if we set $Y(v, z) = \sum_{n \in \mathbb{R}} v_n z^{-n-1}$, then $v_{-1}\iota(0) = v$ for any $v \in \mathbb{R} \otimes L$.

2.3 $L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{1}{2}, \frac{1}{16})$

In this subsection, we assume $L = \mathbb{Z}x$ with $\langle x, x \rangle = 1$ and we don’t use a cocycle $c(a, b)$ since $\iota(mx)$ is generated by one element $\iota(x)$ and $\iota(x) \in (V_L)_{\frac{1}{2}}$. As mentioned in [DMZ],
we can find two mutually orthogonal conformal vectors
\[
e^+(2x) = \frac{1}{4}x(-1)^2 1 + \frac{1}{4}(\nu(2x) + \nu(-2x)) \quad \text{and} \quad e^-(2x) = \frac{1}{4}x(-1)^2 1 - \frac{1}{4}(\nu(2x) + \nu(-2x))
\]
with central charge $\frac{1}{2}$ such that $w = e^+(2x) + e^-(2x) = \frac{1}{2}x(-1)^2 1$ is the Virasoro element of $V_{2x}$. Let $\theta$ be the automorphism of $V_L$ induced from the automorphism $-1$ on $L$, which is given by $\theta(x(-n_1) \cdots x(-n_i)\nu(v)) = (-1)^ix(-n_1) \cdots x(-n_i)\nu(-v)$. We should note that $\theta$ is usually defined by $\theta(x(-n_1) \cdots x(-n_i)\nu(v)) = (-1)^{i+k}x(-n_1) \cdots x(-n_i)\nu(-v)$ for $\nu(v) \in (V_L)_k$, but we here have a half integer weight $k$. Take the fixed point space $(V_L)^\theta$ of $V_L$ by $\theta$. We note that each $e^\pm(2x)$ generates a simple vertex operator subalgebra $< e^\pm(2x) >$ isomorphic to $L(\frac{1}{2}, 0)$ since it is contained in $(V_{2x})^\theta$, which has a positive definite invariant bilinear form as we will see in the next subsection. As we mentioned in the introduction, $< e^+(2x) > \cong L(\frac{1}{2}, 0)$ has only three irreducible modules $L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2}), L(\frac{1}{2}, \frac{1}{10})$. By calculating the dimensions of weight spaces, there are no $L(\frac{1}{2}, \frac{1}{10})$ in $V_L$ since all elements $v \in V_L$ have integer or half integer weights. Since $\dim(V_L)_0 = 1$, $\dim(V_L)_1 = 1$, and $\dim(V_L)_{1/2} = 2$, we conclude that $V_L$ is isomorphic to the direct sum of the tensor products
\[
\left( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \right) \oplus \left( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, \frac{1}{2}) \right) \oplus \left( L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, 0) \right) \oplus \left( L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}) \right)
\]
as $< e^+(2x) > \otimes < e^-(2x) >$-modules by the actions of $e^\pm(2x)$ on $(V_L)^\theta$. Since $\theta$ fixes $e^\pm(2x)$ and $x(-1)(\nu(x) - \nu(-x))$, it keeps the above four irreducible $< e^+(2x) > \otimes < e^-(2x) >$-submodules invariant. Hence we obtain the decomposition:
\[
(V_L)^\theta \cong \left( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \right) \oplus \left( L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, 0) \right)
\]
as $< e^+(2x) > \otimes < e^-(2x) >$-modules, see (4.11). Take the subspace $M = \{ v \in (V_L)^\theta \mid e^{-}(2x)_1v = 0 \}$. Since $V_L$ is a SVOA, $M$ is a SVOA with the Virasoro element $e^+(2x)$ and we see
\[
M = M_0 \oplus M_1, \quad M_0 \cong L(\frac{1}{2}, 0) \quad \text{and} \quad M_1 \cong L(\frac{1}{2}, \frac{1}{2})
\]
as $< e^+(2x) >$-modules. We note that $q = \nu(x) + \nu(-x)$ is a lowest degree vector of $M_1$ and $q_0q = 2\nu(0)$.

It follows from the definition of vertex operators that $V_{2x+\frac{3}{4}x}$ and $V_{2x-\frac{3}{4}x}$ are irre-
ducible $V_{2\mathbb{Z}x}$-modules. Hence, we have the following correspondence:

| $x(-1)1$ | $\in L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})$ | $\theta$ |
|----------|---------------------------------|---------|
| $\iota(x) - \iota(-x)$ | $\in L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, \frac{1}{2})$ | $-1$ |
| $\iota(x) + \iota(-x)$ | $\in L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, 0)$ | $+1$ |
| $\iota(\pm x/2)$ | $\in L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{1}{2}, \frac{1}{16}) \oplus L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{1}{2}, \frac{1}{16})$ | $+1$ |

Fix the lowest weight vectors $\iota(\frac{1}{2}x)$ and $\iota(-\frac{1}{2}x)$ of $V_{2\mathbb{Z}x+x/2}$ and $V_{2\mathbb{Z}x-x/2}$, respectively. By restricting $v$ in $M_1 \cong L(\frac{1}{2}, \frac{1}{2})$ and taking the eigenspace $W$ of $e^{-}(2x)$ with an eigenvalue $\frac{1}{16}$, $Y(v, z)$ defines the following three intertwining operators:

$$I^{\frac{1}{2}, 0}(\ast, z) \in I \left( \begin{array}{cc} L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, 0) \\ L(\frac{1}{2}, 0) & L(\frac{1}{2}, \frac{1}{2}) \end{array} \right),$$

$$I^{\frac{1}{2}, \frac{1}{2}}(\ast, z) \in I \left( \begin{array}{cc} L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, \frac{1}{2}) \\ L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, 0) \end{array} \right),$$

$$I^{\frac{1}{2}, \frac{3}{2}}(\ast, z) \in I \left( \begin{array}{cc} L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, \frac{1}{2}) \\ L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, \frac{1}{2}) \end{array} \right).$$

Also, the restriction to $M_0 \cong L(\frac{1}{2}, 0)$ defines the following intertwining operators:

$$I^{0, 0}(\ast, z) \in I \left( \begin{array}{cc} L(\frac{1}{2}, 0) & L(\frac{1}{2}, 0) \\ L(\frac{1}{2}, 0) & L(\frac{1}{2}, 0) \end{array} \right),$$

$$I^{0, \frac{1}{2}}(\ast, z) \in I \left( \begin{array}{cc} L(\frac{1}{2}, 0) & L(\frac{1}{2}, \frac{1}{2}) \\ L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, 0) \end{array} \right),$$

$$I^{0, \frac{3}{2}}(\ast, z) \in I \left( \begin{array}{cc} L(\frac{1}{2}, 0) & L(\frac{1}{2}, \frac{1}{2}) \\ L(\frac{1}{2}, \frac{1}{2}) & L(\frac{1}{2}, \frac{1}{2}) \end{array} \right),$$

which are actually module vertex operators of $< e^{+}(2x) >$. We fix these intertwining operators throughout this paper.

We recall their properties from [Mi3].

**Proposition 2.1** (1) The powers of $z$ in $I^{0, \ast}(\ast, z)$, $I^{\frac{1}{2}, 0}(\ast, z)$ and $I^{\frac{1}{2}, \frac{1}{2}}(\ast, z)$ are all integers and those of $z$ in $I^{\frac{1}{2}, \frac{3}{2}}(\ast, z)$ are half-integers, that is, in $\frac{1}{2} + \mathbb{Z}$.

(2) $I^{\ast, \ast}(\ast, z)$ satisfies the $L(-1)$-derivative property.

(3) $I^{\frac{1}{2}, \frac{1}{16}}(\ast, z)$ satisfies the supercommutativity:

$$I^{0, \frac{1}{16}}(v, z_1)I^{0, \frac{1}{16}}(v', z_2) \sim I^{0, \frac{1}{16}}(v', z_2)I^{0, \frac{1}{16}}(v, z_1),$$

$$I^{0, \frac{1}{16}}(v, z_1)I^{\frac{1}{2}, \frac{1}{16}}(u, z_2) \sim I^{\frac{1}{2}, \frac{1}{16}}(u, z_2)I^{0, \frac{1}{16}}(v, z_1),$$

$$I^{\frac{1}{2}, \frac{1}{16}}(u, z_1)I^{\frac{1}{2}, \frac{1}{16}}(u', z_2) \sim -I^{\frac{1}{2}, \frac{1}{16}}(u', z_2)I^{\frac{1}{2}, \frac{1}{16}}(u, z_1),$$

for $v, v' \in M_0$ and $u, u' \in M_1$. 

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2.4 A lattice VOA with a positive definite invariant bilinear form

As we will see, we will gather the pieces from \( \tilde{V}_{E_8} \) to construct \( V^2 \). In order to construct \( V^2 \) with a positive definite invariant form, we will show that there is a VOA \( V_{E_8} \) over \( \mathbb{R} \) with a positive definite invariant bilinear form.

We should note that \( V_{E_8} \) in (1) is slightly different from an ordinary lattice VOA \( V_{E_8} \) constructed from a lattice of type \( E_8 \). If we construct a lattice VOA \( V_{E_8} \) over \( \mathbb{R} \) by the construction in \( \text{FLM2} \), then \( \iota(v)_{2k-1}\iota(v) \in S(\hat{H}^{-})\iota(2v) \cap (V_{E_8})_0 = \{0\} \) for any element \( 0 \neq v \in L \) and \( \langle v, v \rangle = 2k \) and so \( \langle \iota(v), \iota(v) \rangle = \langle 1, (-1)^k \iota(v)_{2k-1}\iota(v) \rangle = 0 \). Namely, \( V_{E_8} \) does not have a positive definite invariant bilinear form.

**Proposition 2.2** Let \( L \) be an even lattice. Then there is a VOA \( \tilde{V}_L \) which has a positive definite invariant bilinear form such that \( \mathbb{C} \otimes \tilde{V}_L \cong \mathcal{CV}_L \).

**[Proof]** A lattice VOA \( V_L = \bigoplus_{v \in L} S(\mathbb{R} \otimes \mathbb{Z} L^+)^{\iota(v)} \) constructed by the lattice construction in \( \text{FLM2} \) has an invariant bilinear form \( \langle \cdot, \cdot \rangle \). That is, it satisfies

\[
\langle Y(a, z)u, v \rangle = \langle u, Y(e^{zL(1)}(-z^{-2})L(0)a, z^{-1})v \rangle
\]

for \( a, u, v \in V_L \), see \( \text{FLM2} \). \( Y^+(a, z) = Y(e^{zL(1)}(-z^{-2})L(0)a, z^{-1}) = \sum a_i^1 z^{-n-1} \) is called the adjoint vertex operator. For \( v \in \mathbb{R} \otimes L \), identify it with \( v(-1)\iota(0) \in (V_L)_1 \). Since \( L(1)v(-1)\iota(0) = 0 \) and \( L(0)v(-1)\iota(0) = v(-1)\iota(0) \), \( Y^+(v, z) = -z^{-2}Y(v, z^{-1}) \) and so we have \( v^!(n) = -v^!(n) \). In the definition of \( V_L \) in \( \text{FLM2} \), they used a group extension satisfying \( \iota(w)\iota(u) = (-1)^{<u',u>}\iota(u)\iota(u') \) and \( \iota(v)\iota(-v) = \iota(0) \) for \( \iota(v) \in (V_L)_k \). Namely,

\[
\iota(v)_{2k-1}\iota(-v) = \iota(-v)_{2k-1}\iota(v) = \iota(0).
\]

By definition, \( Y^+(\iota(v), z) = (-z^{-2})^{\langle v, v \rangle/2}Y(\iota(v), z^{-1}) \). Hence, for \( \iota(v) \in V_k \), we have

\[
(\iota(v))^n = (-1)^k(\iota(v))_{2k-n-2}
\]

and

\[
\langle \iota(v) + \iota(-v), \iota(v) + \iota(-v) \rangle \iota(0) = (-1)^k(\iota(v) + \iota(-v))_{2k-1}(\iota(v) + \iota(-v)) = (-1)^k2\iota(0)
\]

and

\[
\langle \iota(v) - \iota(-v), \iota(v) - \iota(-v) \rangle = (-1)^{k+1}2\iota(0).
\]

Let \( \theta \) be an automorphism of \( V_L \) induced from \(-1\) on \( L \), which is given by

\[
\theta(v^1(-i_1) \cdots v^m(-i_m)\iota(x)) = (-1)^{k+m}v^1(-i_1) \cdots v^m(-i_m)\iota(-x).
\]
Hence, the space \( V^0 = (V_L)^{\theta} \) of \( \theta \)-invariants is spanned by the elements of the forms
\[
v^1(-n_1) \cdots v^{2m}(-n_{2m})(\iota(v) + (-1)^k \iota(-v)) \quad \text{and} \quad v^1(-n_1) \cdots v^{2m+1}(-n_{2m+1})(\iota(v) - (-1)^k \iota(-v))
\]
for \( \iota(v) \in V_k \) and so \( V^0 \) has a positive definite invariant form. Similarly, \( V^1 = (V_L)^{-} \) has a negative definite invariant bilinear form, where \( (V_L)^{-} = \{ v \in V_L : \theta(v) = -v \} \). Since \( V_L = V^0 \oplus V^1 \) has a \( \mathbb{Z}_2 \)-grade, it is possible to denote the vertex operator of \( v \in V^0 \) by \( \begin{pmatrix} Y^{11}(v, z) & 0 \\ 0 & Y^{22}(v, z) \end{pmatrix} \) and the vertex operator of \( u \in V^1 \) by \( \begin{pmatrix} 0 & Y^{21}(u, z) \\ Y^{12}(u, z) & 0 \end{pmatrix} \), where \( Y^{ij}(v, z) \in \text{Hom}(V^i, V^j)[[z, z^{-1}]] \). Define new vertex operators by
\[
\tilde{Y}(v, z) = \begin{pmatrix} Y^{11}(v, z) & 0 \\ 0 & Y^{22}(v, z) \end{pmatrix}
\]
for \( v \in V^0 \) and
\[
\tilde{Y}(u, z) = \begin{pmatrix} 0 & -Y^{21}(u, z) \\ Y^{12}(u, z) & 0 \end{pmatrix}
\]
for \( u \in V^1 \). Then \( (V, \tilde{Y}) \) is a VOA with a positive definite invariant bilinear form. This is the desired VOA.

Q.E.D.

In the remaining of this paper, \( \tilde{V}_{E_8} \) denotes the above VOA \((V_{E_8}, \tilde{Y})\) with a positive definite invariant bilinear form. Since we mainly treat a VOA with a positive definite invariant bilinear form, we sometimes denote \( V_L \) by \((\tilde{V}_L)^{\theta} \oplus \sqrt{-1} \tilde{V}_L^{-}\), where \( \tilde{V}_L^{-} = \{ v \in \tilde{V}_L : \theta(v) = -v \} \).

## 3 Code VOAs with positive definite invariant bilinear forms

In this section, we recall and prove several results from \([\text{Mi}2] \sim [\text{Mi}5]\). We will first construct a code VOA \( M_D \) with a positive definite invariant bilinear form for an even linear binary code \( D \) of length \( n \). Set \( M_0 = L(\frac{1}{2}, 0) \) and \( M_1 = L(\frac{1}{2}, \frac{1}{2}) \). It is known that \( F = M_0 \oplus M_1 \) has a super VOA structure \((F, Y^F)\), see (2.2). Although a SVOA structure on \( CF \) is uniquely determined, a SVOA structure on \( F \) is not unique. Since \( F \) has a \( \mathbb{Z}_2 \)-grade, we can express a vertex operator \( Y(v, z) \) by a \( 2 \times 2 \)-matrix:
\[
Y(v, z) = \begin{pmatrix} Y^{00}(v, z) & 0 \\ 0 & Y^{11}(v, z) \end{pmatrix} \quad \text{for} \quad v \in M_0,
\]
\[
Y(v, z) = \begin{pmatrix} Y^{01}(v, z) & 0 \\ 0 & Y^{10}(v, z) \end{pmatrix} \quad \text{for} \quad v \in M_1.
\]
If we define new vertex operators $Y'(v, z)$ by

$$
Y'(v, z) = \begin{pmatrix}
Y^{00}(v, z) & 0 \\
0 & Y^{11}(v, z) \\
0 & -Y^{10}(v, z)
\end{pmatrix}
$$

for $v \in M_0$,

$$
Y'(v, z) = \begin{pmatrix}
Y^{01}(v, z) & 0 \\
0 & Y^{10}(v, z)
\end{pmatrix}
$$

for $v \in M_1$,

then $(F, Y')$ is also a SVOA and it is not isomorphic to $(F, Y)$. So we choose one of them satisfying $q_0q \in \mathbb{R}^+1$, where $q$ is a highest weight vector of $M_1$ and $\mathbb{R}^+ = \{ r \in \mathbb{R} | r > 0 \}$.

An essential property is a super-commutativity:

$$
Y^F(v, z_1)Y^F(u, z_2) \sim (-1)^{|v||u|}Y^F(u, z_2)Y^F(v, z_1)
$$

(3.1)

for $|u|, |v| = 0, 1$ and $v \in M_{|v|}$ and $u \in M_{|u|}$. Here $A(z_1, z_2) \sim B(z_1, z_2)$ means $(z_1 - z_2)^N A(z_1, z_2) = (z_1 - z_2)^N B(z_1, z_2)$ for a sufficiently large integer $N$. For a binary word $\alpha = (a_1, ..., a_n) \in Z_2^n$, set $\hat{M}_\alpha = \oplus_{i=1}^n M_{a_i}$, which is a subspace of $F_{\otimes^n} = (M_0 \oplus M_1)_{\otimes^n} = \bigoplus_{\alpha \in Z_2^n} \hat{M}_\alpha$.

Define a vertex operator $Y_{\otimes^n}(u, z)$ of $u \in F_{\otimes^n}$ by

$$
Y_{\otimes^n}(\otimes_{i=1}^n u^i, z)(\otimes_{i=1}^n u^i) = \otimes_{i=1}^n (Y^F(v, z)u^i)
$$

(3.2)

for $u^i, v^i \in F$ and extend it to the whole space $F_{\otimes^n}$ linearly. It follows from (3.1) that for $v \in \hat{M}_\alpha$ and $u \in \hat{M}_\beta$, we have the super commutativity:

$$
Y_{\otimes^n}(v, z_1)Y_{\otimes^n}(u, z_2) \sim (-1)^{\alpha \beta}Y_{\otimes^n}(u, z_2)Y_{\otimes^n}(v, z_1).
$$

(3.3)

Viewing $D$ as an elementary abelian 2-group with an invariant form, we shall use a central extension $\hat{D} = \{ \pm e^\alpha : \alpha \in D \}$ of $D$ by $\pm 1$ in order to modify the supercommutativity (3.3). Let $\xi_i$ ($i = 1, ..., n$) denote a word $(0^{i-1}10^{n-i})$ and $e^\xi_i$ a formal element satisfying $e^\xi_ie^\xi_i = 1$ and $e^\xi_ie^\xi_j = -e^\xi_je^\xi_i$ for $i \neq j$. For a word $\alpha = \xi_{j_1} + \cdots + \xi_{j_k}$ with $j_1 < \cdots < j_k$, set

$$
e^\alpha = e^{\xi_{j_1}}e^{\xi_{j_2}} \cdots e^{\xi_{j_k}}.
$$

(3.4)

It is straightforward to check the following:

**Lemma 3.1** [Mi3] For $\alpha, \beta$,

$$
e^\alpha e^\beta = (-1)^{|\alpha|\beta + |\alpha||\beta|}e^\beta e^\alpha
$$

$$
e^\alpha e^\alpha = (-1)^{\frac{k(k-1)}{2}} \text{ for } |\alpha| = k.
$$

(3.5)
In order to combine (3.3) and (3.5), set

\[ M_\delta = \hat{M}_\delta \otimes e^\delta \]  

and

\[ M_D = \bigoplus_{\delta \in D} M_\delta. \]  

Define a new vertex operator \( Y(u, z) \) of \( u \in M_D \) by setting

\[ Y(v \otimes e^\beta, z) = Y \otimes n(v, z) \otimes e^\beta \]  

for \( v \otimes e^\beta \in M_\beta = \hat{M}_\beta \otimes e^\beta \) and extending it linearly. We then obtain the desired commutativity:

\[ Y(v, z_1)Y(w, z_2) \sim Y(w, z_2)Y(v, z_1) \]  

for \( v, w \in M_D \). It is not difficult to see that

\[ w = \sum_{i=1}^{n}(1^1 \otimes \ldots \otimes 1^{i-1} \otimes w^i \otimes 1^{i+1} \otimes \ldots \otimes 1^n) \otimes e^0 \]  

is Virasoro element of \( M_D \) and

\[ 1 = (1^1 \otimes \ldots \otimes 1^n) \otimes e^0 \]  

is the vacuum of \( M_D \), where \( w^i \) and \( 1^i \) are Virasoro element and the vacuum of \( M_i \), respectively. So we have proved the following theorem in [Mi2].

**Theorem 3.1** If \( D \) is an even binary linear code, then \( (M_D, Y, w, 1) \) is a simple VOA.

It follows from the construction that \( M_\beta \ominus D \) is an irreducible \( M_D \)-module and we will call it a coset module of \( M_D \). From the choice of our cocycle, we can easily prove the following lemma.

**Lemma 3.2** If \( g \in \text{Aut}(D) \), there is an automorphism \( \tilde{g} \) of a code VOA \( M_D \) such that \( \tilde{g}(e^i) = e^{g(i)} \) and \( \tilde{g}(M_\alpha) = M_{g(\alpha)} \).

**[Proof]** For \( g \in \text{Aut}(D) \), we define a permutation \( g_1 \) on \( \{ \hat{M}_\alpha : \alpha \in D \} \) by \( g_1(\otimes M_\alpha) = \otimes M_{g(\alpha)} \) and an automorphism \( g_2 \) of \( \hat{D} \) by \( g_2(e^{\xi_1} \ldots e^{\xi_n}) = e^{g(\xi_1)} \ldots e^{g(\xi_n)} \). Combining the both action on \( M_D = \sum_{\alpha \in D} \hat{M}_\alpha \otimes e^\alpha \), \( \tilde{g} = g_1 \otimes g_2 \) becomes an automorphism of \( M_D \).

Q.E.D.
We will next construct an invariant bilinear form on $M_D$. Let $(M, Y_M)$ be a module of $(V, Y)$. A bilinear form $\langle \cdot, \cdot \rangle$ on $M$ is said to be invariant if
\[ \langle Y_M(a, z)v, u \rangle = \langle u, Y_M(e^{zL(1)}(-z^{-2})L(0)v, z^{-1})v \rangle \quad \text{for } a \in V, u, v \in M, \] (3.12)
where $L(n) = w_{n+1}$. It was proved in [Li] that any invariant bilinear form on a VOA is automatically symmetric and there is a one-to-one correspondence between invariant bilinear forms and elements of $\text{Hom}(V_0/L(1)V_1, \mathbb{R})$. Since $\dim V_0 = 1$ and $L(1)V_1 = 0$ for a code VOA $V = M_D$, there is a unique invariant bilinear form $\langle \cdot, \cdot \rangle$ satisfying $\langle 1, 1 \rangle = 1$. Using (3.12), it is given by
\[ \langle u, v \rangle 1 = \langle u_{-1}1, v \rangle 1 = \text{Res}_zz^{-1}(Y(e^{zL_1}(-z^{-2})L_0u, z^{-1})v. \] (3.13)
Set $B = \langle L(1), L(0), L(-1) \rangle$. Since $B \cong \mathfrak{sl}_2(\mathbb{R})$ and $L(1)(M_D)_1 = 0$, $M_D$ is a direct sum of irreducible $B$-modules. Let $U$ be an irreducible $B$-submodule of $M_D$. Then there is an element $u \in (M_D)_k$ satisfying $L(1)u = 0$ such that $U$ is spanned by $\{L(-1)^s u : s = 0, 1, \ldots \}$. For any $v \in V_k$,
\[ \langle u, v \rangle 1 = \langle u_{-1}1, v \rangle 1 = \text{Res}_zz^{-1}(Y(((1)^kz^{-2k})u, z^{-1})z^{-1}v = (-1)^ku_{2k-1}v. \] (3.14)
Also we note
\[ \langle L(-1)^i u, L(-1)^j v \rangle = \langle L(-1)^{i-1} u, L(1)L(-1)^j v \rangle = (2kj + j^2 - j)\langle L(-1)^{i-1} u, L(-1)^{j-1} v \rangle \] (3.15)
and $2kj + j^2 - j > 0$. Thus, $\langle \cdot, \cdot \rangle$ is positive definite if and only if
\[ u_{2k-1} \in (-1)^k \mathbb{R}^+ 1 \] (3.16)
for $0 \neq u \in V_k$ satisfying $L(1)u = 0$.

We first prove the $\mathbb{R}$-version of Theorem 4.5 in [Mi3].

**Proposition 3.1** 2) Let $V = \bigoplus_{m=0}^{\infty} V_m$ be a simple VOA over $\mathbb{R}$ with $\dim V_0 = 1$. Assume that $V$ contains a set of mutually orthogonal conformal vectors $\{e^1, \ldots, e^n\}$ so that the sum of them is the Virasoro element of $V$ and $\{e^1, \ldots, e^n\}$ generates $T = L(\frac{1}{2}, 0)^{\otimes n}$. Assume further that $V$ has a positive definite invariant bilinear form and $\bar{\tau}(V) = (0^n)$. Then there is an even linear code $D$ such that $V$ is isomorphic to a code VOA $M_D$.

**Proof** Since $\bar{\tau}(V) = (0^n)$, $\tau_e^i = 1$ and so we can define automorphism $\sigma_{e^i}$ for $i = 1, \ldots, n$. Set $Q = \langle \sigma_{e^i} : i = 1, \ldots, n \rangle$. $Q$ is an elementary abelian 2-group and let
\[ V = \bigoplus_{x \in \text{Irr}(Q)} V^x. \]
be the decomposition of \( V \) into the direct sum of eigenspaces of \( Q \). Since \( \dim V_0 = 1 \) and \( V^\chi \) is an irreducible \( V^\chi \)-module by [DM2], we have \( V^\chi = T \) and \( V^\chi \cong \otimes L(\frac{1}{2}, h^i/2) \) as \( T \)-modules. Here \( h^i \in \{0, 1\} \) is given by \( \chi(\sigma^i) = (-1)^{h^i} \). Let \( q \) denote a highest weight vector of \( M \) such that \( q_0 q = 1 \in M_0 \). For a binary word \( \alpha = (a^i) \), \( \alpha^{(1)} \) denotes \( \otimes \alpha^{(1)} \in M_\alpha \), where \( q_0 = 1 \) and \( q^1 = q \). Identifying \( \chi \) and \( (h^i) \), \( V^\chi \cong M_\chi \otimes \tilde{\chi} \) as \( T \)-modules such that \( \langle q^\chi \otimes \tilde{\chi}, q^\chi \otimes \tilde{\chi} \rangle = 1 \).

Assume \( |\chi| = 2k \). By the choice of \( q^\chi \otimes \tilde{\chi} \) and \( q_{2k-1}^\chi = 1 \), we have

\[
1 = \langle q^\chi \otimes \tilde{\alpha}, q^\chi \otimes \tilde{\alpha} \rangle 1 \\
= \langle 1, (-1)^k (q^\chi \otimes \tilde{\alpha})_{2k-1} q^\chi \otimes \tilde{\alpha} \rangle 1 \\
= \langle 1, (-1)^k \tilde{\alpha} \tilde{\alpha} \rangle 1.
\]

Hence, \( \tilde{\alpha} \tilde{\alpha} = (-1)^k \tilde{a}^0 \), which uniquely determine a cocycle that coincides with (3.17). This completes the proof of Proposition 3.1.

Q.E.D.

As a corollary, we have

**Corollary 3.1** For an even linear code \( D \), \( M_D \) has a positive definite invariant bilinear form. In particular, if \( \alpha \) is even, then the coset module \( M_{D+\alpha} \) also has a positive definite invariant bilinear form.

**[Proof]** Recall that for a word \( \alpha \) with \( |\alpha| = 2k \), say \( \alpha = (1^{2k}\underline{0}^{n-2k}) \),

\[
e^\alpha \tilde{e}^\alpha = e^{\xi_1} \ldots e^{\xi_{2k}} e^{\xi_1} \ldots e^{\xi_{2k}} = (-1)^{k(2k-1)} = (-1)^k.
\]

(3.17)

Let \( S^n \) be the set of all even words of length \( n \). Since all code VOAs are subVOAs of the code VOA \( M_{S^n} \), it is sufficient to prove the assertion for the code \( S^n \). Also, since \( M_{S^n} \cong M_{S^n} \otimes (\mathbb{R}1)^{\otimes n} \subseteq M_{S^{2n}} \) as sub VOAs, we may assume that \( D \) is the set of all even words of length \( 2n \). Let \( \{x^1, \ldots, x^n\} \) be an orthonormal basis of an Euclidean space of dimension \( n \) and set

\[
L = \{ \sum a_i x^i : a_i \in \mathbb{Z}, \sum a_i \equiv 0 \pmod{2} \}.
\]

(3.18)

Let \( V_L \) be a lattice VOA constructed from \( L \), (see §2.2). Let \( \theta \) be an automorphism of \( V_L \) induced from \(-1\) on \( L \) and decompose \( V_L \) into \( (V_L)^\theta \oplus (V_L)^- \), where \( (V_L)^- = \{ v \in V_L | \theta(v) = -v \} \). \((V_L)^\theta \) contains \( 2n \) mutually orthogonal rational conformal vectors

\[
e^{(2x^i)^\pm} = \frac{1}{4} x^i (-1)^2 1 \pm \frac{1}{4} (\iota(2x^i) + \iota(-2x^i))
\]

(3.19)
with central charge $\frac{1}{2}$ by (2.1). Set $\tilde{V}_L = (V_L)^\theta \oplus \sqrt{-1}(V_L)^-$. Then $\tilde{V}_L$ is a VOA with a positive definite invariant bilinear form containing

$$T = \langle e(2x^i)\tilde{e} : i = 1, ..., n \rangle$$

by Proposition 2.2. Since $\langle v, 2x^j \rangle \in 2\mathbb{Z}$ for $v \in L$, (2.3) implies $\tilde{\tau}(\tilde{V}_L) = (0^{2n})$. By Proposition 3.1, a code VOA $M_{S2n}^2$ is isomorphic to $\tilde{V}_L$ which has a positive definite invariant bilinear form.

Q.E.D.

If $\alpha \in D$ is a codeword of weight 2, say $\alpha = (110^{n-2})$, then $(M_{\alpha})_1 \neq 0$. Set $E = \{(00), (11)\}$, then $M_E$ is isomorphic to $V_{Zx}$ with $\langle x, x \rangle = 1$ given in §2.3. We note $V_{Zx} \cong M_{Zx}^2$ and $\exp(\pi i x(0))$ keeps $V_{Zx}$ invariant. We also note that $x(-1)1 \in (V_{Zx})_1$ and $x(0) = (x(-1)1)_0$. It follows from a direct calculation that

$$\exp(\pi i x(0)) = (-1)^{(\beta, (11))}$$

for $\beta \in \mathbb{Z}_2^n$. As long as a VOA $V$ contains a vector $v$ of weight 1, we can define automorphism $\exp(v_0)$ of $CV$. Hence, we have the following lemma.

**Lemma 3.3** If a VOA contains a code $M_D$ and $D$ contains a codeword $\xi_i + \xi_j$ of weight 2, then $CV$ contains an automorphism $g$ such that

$$g = (-1)^{(\beta, \xi_i + \xi_j)}$$

on $M_\beta$.

In particular, it coincides with $\sigma_{e^i}\sigma_{e^j}$ on $M_D$.

**Conjecture 1** If a simple VOA $V$ contains a code VOA $M_D$ and $\beta \in D$, then there is an automorphism $g$ of $V$ such that $g = \prod_{i \in \text{Supp}(\beta)} \sigma_{e^i}$ on $M_D$.

An important property of our cocycle is that if a maximal self-orthogonal subcode $H$ of $D$ is doubly even, (for example, a Hamming code), then $\hat{H} = \{\pm e^\alpha : \alpha \in H\}$ is a maximal normal (elementary) abelian 2-subgroup of $\hat{D}$ and so every irreducible $\mathbb{R}\hat{D}$-module is induced from a linear $\mathbb{R}\hat{H}$-module. In the remainder of this section, we assume that for an $M_D$-module $W$, one of maximal self-orthogonal subcode of $D_{\tilde{\tau}(W)}$ is doubly even and we denote it by $H$.

We recall the structures of irreducible $CM_D$-modules from [Mi3]. Let $CW$ be an irreducible $M_D$-module with $\tilde{\tau}(CW) = \mu$. If $H$ is a maximal self-orthogonal subcode of
Let $\mu$ and $U'$ be an irreducible $CM_H$-submodule of $CW$, then the author showed in [Mi3] that $CW = \text{Ind}_H^D(U')$ and every irreducible $CM_H$-module is irreducible as a $CT$-module. We will show that these results also hold for an irreducible $M_D$-modules under the assumption that $H$ is doubly even.

**Theorem 3.2** Let $(X,Y^X)$ be an $M_D$-module with $\tau(X) = \mu$ and $\{X^i : i = 1,\ldots,m\}$ the set of all non-isomorphic irreducible $T$-submodules of $X$. Then there are representations $\phi^i : \hat{D}_\mu \to \text{End}(Q^i)$ with $\phi^i(-e^0) = -I$ for $i = 1,\ldots,m$ such that $X \cong \bigoplus_{i=1}^m (X^i \otimes Q^i)$ as $M_{D_H}$-modules. Moreover, if $X$ is irreducible, then all $\phi^i$ are irreducible. For $\alpha \in D_\mu$, the module vertex operator $Y^X(q^\alpha, z)$ of $q^\alpha = (\otimes q^{a_i}) \otimes e^\alpha \in M_\alpha$ on $X^j \otimes Q^j$ is given by

$$\otimes_{i=1}^n \mu_{a_i/2*}(q^{a_i}, z) \otimes \phi^j(e^\alpha)$$

for $\alpha = (a_1,\ldots,a_n)$. Here $q^0$ denotes the vacuum of $M_0$ and $q^1$ denotes the lowest degree vector $q$ in $M_1$. See §2.3 for $\otimes_{i=1}^n \mu_{a_i/2*}(q^{a_i}, z)$.

**Proof** Let $U$ be a homogeneous component of $X$ generated by all $T$-submodules isomorphic to $X^1$ and let $U = \bigoplus_{i=1}^k U^i$ be a decomposition of $U$ into a direct sum of irreducible $T$-submodules $U^i$. By (1.1), $U$ is an $M_{D_H}$-module. Let $Q^1$ be the lowest degree space of $U$. Since the dimension of the lowest degree space of $X^1$ is one, $\dim(U^i \cap Q^1) = 1$ and $U \cong Q^1 \otimes X^1$ as vector spaces. Let $u^i$ be a nonzero lowest degree vector of $U^i$, then $\{u^1,\ldots,u^k\}$ is a basis of $Q^1$. Let $\pi_j : U \to U^j = u^j \otimes X^1$ be a projection of $U$, that is,

$$\pi_j(u^i \otimes v) = \delta_{ij}u^i \otimes v \text{ for } v \in X^1.$$

By (1.1), $Y^X(q^\alpha, z)U \subset U[[z, z^{-1}]]$ for $q^\alpha \in M_\alpha$ and $\alpha \in D_{\mu}$. Since $\pi_j(Y^X(q^\alpha, z)|_{u^j \otimes X^1})$ is an intertwining operator of type $\left(\begin{array}{c}X^1 \\ M_\alpha \end{array}\right)$ for $\alpha \in D_{\mu}$, the vertex operator $Y^X(q^\alpha, z)|_U$ of $q^\alpha$ has an expression

$$Y^X(q^\alpha, z) = A(e^\alpha) \otimes ((\otimes I)(\hat{q}^\alpha, z)),$$

where $A(e^\alpha)$ is a $k \times k$-matrix acting on $Q^1 = \mathbb{R}u^1 \oplus \cdots \mathbb{R}u^k$ and $(\otimes I)(\hat{q}^\alpha, z)$ is the tensor product $\otimes I(\hat{q}^\alpha, z)$ of the fixed intertwining operators in §2.3 for $\hat{q}^\alpha = \otimes q^{a_i}$. We note that $Y^X$ is uniquely determined by $\{Y^X(q^\alpha, z) : \alpha \in D\}$. Since $Y^X(q^\alpha, z)$ satisfies the commutativity

$$Y^X(q^\alpha, z)Y^X(q^\beta, w) \sim Y^X(q^\beta, w)Y^X(q^\alpha, z)$$

and $(\otimes I)(\hat{q}^\alpha, z)$ satisfies the super-commutativity

$$(\otimes I)(\hat{q}^\alpha, z)(\otimes I)(\hat{q}^\beta, w) \sim (-1)^{(\alpha,\beta)}(\otimes I)(\hat{q}^\beta, w)(\otimes I)(\hat{q}^\alpha, z),$$

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we obtain the supercommutativity:

\[ A(e^\alpha)A(e^\beta) = (-1)^{\langle \alpha, \beta \rangle} A(e^\beta)A(e^\alpha). \]

Moreover, since \( Y^X(*, z) \) satisfies the associativity

\[ Y^X(q^\alpha_m q^\beta, z) = Y^X(q^\alpha, z)Y^X(q^\beta, z) \]

and \( (\otimes I)(\hat{q}^\alpha, z) \) satisfies the superassociativity by [MIM], we have

\[
A(e^\alpha e^\beta)(\otimes I)(\hat{q}^\alpha_m \hat{q}^\beta, z) = Y^X(q^\alpha_m q^\beta, z) = Y^X(q^\alpha, z)Y^X(q^\beta, z) = A(e^\alpha)(\otimes I)(\hat{q}^\alpha, z)A(e^\beta)(\otimes I)(\hat{q}^\beta, z)
\]

\[ = \text{Res}_x \{(x - z)^m A(e^\alpha)(\otimes I)(\hat{q}^\alpha, x)A(e^\beta)(\otimes I)(\hat{q}^\beta, x)\}
\]

\[ = A(e^\alpha)A(e^\beta) \text{Res}_x \{(x - z)^m (\otimes I)(\hat{q}^\alpha, x)(\otimes I)(\hat{q}^\beta, x)\}
\]

\[ = A(e^\alpha)A(e^\beta)[-(-z + x)^m (\otimes I)(\hat{q}^\beta, x)]
\]

\[ = A(e^\alpha)A(e^\beta)(\otimes I)(\hat{q}^\alpha_m \hat{q}^\beta, z).
\]

Hence we have the associativity:

\[ A(e^\alpha)A(e^\beta) = A(e^\alpha e^\beta) \]

and \( A(e^\alpha)A(e^\alpha) = (-1)^{|\alpha|/2} I \) for all \( \alpha, \beta \in D_\mu \), where \( I \) is the identity matrix. Hence \( A \) is a matrix representation of the central extension \( \hat{D}_\mu \) on \( Q^1 \). We next assume that \( X \) is irreducible. Let \( Q^0 \) be an irreducible \( \hat{D}_\mu \)-submodule and \( W \) the subspace spanned by \( \{ v_n w : v \in M_D, w \in Q^0 \otimes X^1, n \in \mathbb{Z} \} \). Proposition 4.1 in [DM2] implies \( X = W \). On the other hand, the tensor product \( M_{\beta + D_\mu} \times (Q^0 \otimes X^1) \) does not contains a submodule isomorphic to \( X^1 \) for \( \beta \not\in D_\mu \) by (1.1) and so \( U = W \cap U = Q^0 \otimes X^1 \). Hence, \( Q^1 \) is an irreducible \( \hat{D}_\mu \)-module on which \(-e^0\) acts as \(-1\).

Q.E.D.

As a corollary, we have the followings:

**Corollary 3.2** If \( D \) is a doubly even code and \( W \) is an irreducible \( M_D \)-module, then \( W \) is also irreducible as a \( T \)-module.
Note that if \( H \) is a maximal self-orthogonal subcode (doubly even) of \( D \), then every irreducible \( \mathbb{R}H \)-module \( W \) is induced from a \( \mathbb{R}H \)-module and \( W \) is a direct sum of distinct irreducible \( \mathbb{R}H \)-modules. Hence, we have the following corollary.

**Corollary 3.3** If \((W,Y^W)\) is an irreducible \( M_D \)-module with \( \tau(W) = (1^n) \), then there is an irreducible representation \( \phi : \rD \rightarrow \text{End}(Q) \) satisfying \( \phi(-e^0) = -I \) such that \( W \cong L(\frac{1}{2}, \frac{1}{16})^{\otimes n} \otimes Q \) as \( M_D \)-modules. Here the module vertex operator \( Y^X(q^\alpha, z) \) of \( q^\alpha = (\otimes q^a_i) \otimes e^\alpha \in M_\alpha \) on \( L(\frac{1}{2}, \frac{1}{16})^{\otimes n} \otimes Q \) is given by

\[
\otimes_{i=1}^n I^{a_i/2, \frac{1}{16}}(q^a_i, z) \otimes \phi(e^\alpha)
\]

for \( \alpha = (a_1, ..., a_n) \). In particular, \( Y^W \) is uniquely determined by an irreducible \( M_H \)-submodule.

Conversely, we will prove the following proposition:

**Proposition 3.2** Let \( \mu \) be a word such that \( \langle D, \mu \rangle = 0 \). Assume that \( H \) is a maximal self-orthogonal (doubly even) subcode of \( D_\mu \) and \( U \) is an irreducible \( M_H \)-module with \( \tau(U) = \mu \). Then there is an irreducible \( M_D \)-module \( W \) containing \( U \) as an \( M_H \)-submodule.

**[Proof]** We may assume \( \mu = (0^{n-m}1^m) \). By the above lemmas, there is a binary word \((a_1, ..., a_{n-m})\) such that \( U = (\otimes_{i=1}^{n-m} L(\frac{1}{2}, \frac{a_i}{2})) \otimes (L(\frac{1}{2}, \frac{1}{16})^{\otimes m}) \otimes \mathbb{R}_X \). Since \( D \subseteq \mu \subseteq \frac{1}{2} \) and \( D \) is even, \( D \subseteq S_{n-m} \oplus S_m \), where \( S_r \) denotes the set of all even words of length \( r \). If \( n = m \), then \( L(\frac{1}{2}, \frac{1}{16})^{\otimes m} \otimes \text{Ind}_H^D(\mathbb{R}_X) \) is the desired \( M_D \)-module. If \( m = 0 \), then a coset module \( M_{(a_i)+D} \) is the desired \( M_D \)-module. For general cases, let \( K \) be a maximal self-orthogonal subcode of \( S_m \) containing \( H \) and choose an irreducible \( \rK \)-module \( Q \) containing \( \mathbb{R}_X \). The tensor product \( M_{(a_i)+S_{n-m}} \otimes \text{Ind}_K^S(L(\frac{1}{2}, \frac{1}{16})^{\otimes m} \otimes Q) \) is an \( M_{S_{n-m}} \oplus S_m \)-module containing \( U \). By Theorem 3.2, there is an irreducible \( M_D \)-submodule containing \( U \), which is the desired \( M_D \)-module.

Q.E.D.

Our next aim is to prove that an \( M_D \)-module \( W \) satisfying the above condition is uniquely determined. We will call it an *induced module* and denote it by \( \text{Ind}_H^D(U) \). Applying Proposition 11.9 in [DL] into our case, we have the following lemma (see [Mi3]).

**Lemma 3.4** Let \( E \) be a subcode of \( D \). Let \( W^1, W^2, W^3 \) be irreducible \( M_D \)-module and \( U^1, U^2 \) irreducible \( M_E \)-submodules of \( W^1 \) and \( W^2 \), respectively, then there is an injection map:

\[
\phi : I_{M_D} \left( \begin{array}{c} W^3 \\ W^1 \\ W^2 \end{array} \right) \rightarrow I_{M_E} \left( \begin{array}{c} W^3 \\ U^1 \\ U^2 \end{array} \right).
\]

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Using the above lemma, we will prove the following fusion rule, which was proved in [Mi3] for a code VOA $\mathbb{C}_D$.

**Theorem 3.3** If $X$ is an irreducible $M_D$-module, then the fusion product

$$M_{\alpha+D} \times X$$

is an irreducible $M_D$-module for any $\alpha$.

Set $\mu = \tilde{\tau}(X)$. We will first prove the following lemmas, (Lemma 3.5~3.7).

**Lemma 3.5** Assume that $D$ is a doubly even code and $\text{Supp}(D) \subseteq \text{Supp}(\mu)$. Then $M_{\alpha+H} \times X$ is irreducible.

**Proof** By Corollary 3.2 and 3.3, $W \cong (\otimes L(\frac{1}{2}, h)) \otimes \mathbb{R}_\chi$ as $M_D$-modules, where $\chi$ is a linear representation of $\hat{D}$ on $\mathbb{R}_\chi$. Let $U$ be an irreducible $M_H$-module so that $0 \neq I_{M_H} \left( \begin{array}{c} U \\ M_{\alpha+H} \end{array} \right) W$. Clearly, $\tilde{\tau}(U) = \tilde{\tau}(W)$ and so $U \cong \otimes L(\frac{1}{2}, h^i + \frac{a_i}{2}) \otimes \mathbb{R}_\phi$ for some linear representation $\phi$ of $\hat{D}$. By Lemma 3.4, there is an injective map

$$\pi : I_{M_H} \left( \begin{array}{c} U \\ M_{\alpha+H} \end{array} \right) \rightarrow I_T \left( \begin{array}{c} U \\ M_{\alpha} \end{array} \right).$$

Since $M_{\alpha} = \otimes L(\frac{1}{2}, a_i/2)$, $M_{\alpha} \times W$ is irreducible as a $T$-module by (1.1) and so $M_{\alpha} \times W \cong U$ as $T$-modules. We fix a nonzero intertwining operator $J(*, z) \in I_T \left( \begin{array}{c} U \\ M_{\alpha} \end{array} \right)$. Then for any intertwining operator $I(*, z) \in I \left( \begin{array}{c} U \\ M_{\alpha+H} \end{array} \right)$ we may assume $I(v, z) = J(v, z)$ for $v \in M_{\alpha}$ by multiplying a scalar. Since $I$ satisfies the commutativity:

$$I^{\otimes n}(q^a, x)\phi(e^a)I(v, z) \sim I(v, z)I^{\otimes n}(q^a, x)\chi(e^a)$$

with the module vertex operators, $\phi$ is uniquely determined by $\chi$ and so $M_{\alpha+D} \times W$ is irreducible.

Q.E.D.

**Lemma 3.6** Assume that $\text{Supp}(D) \subseteq \text{Supp}(\mu)$. Then $M_{\alpha+D} \times W$ is irreducible.
3.2 and 3.3, \( W \cong \otimes L(\frac{1}{2}, h^i) \otimes Q_\chi \) as \( M_D \)-modules, where \( \chi \) is a representation of \( \hat{D} \) on \( Q_\chi \) such that \( \chi(-e^0) = -I \). Since \( \hat{H} \) is a maximal normal abelian subgroup of \( \hat{D} \), there is a \( \hat{H} \)-submodule \( Q^0 \) such that \( \text{Ind}_{\hat{H}}^D(Q^0) = Q_\chi \). Let \( U \) be an irreducible \( M_D \)-module such that \( 0 \neq \IM \left( \begin{array}{c} U \\ M_{\alpha+D} \end{array} \right) \). Clearly, \( \tau(U) = \mu \) and so \( U \cong \otimes L(\frac{1}{2}, h^i) \otimes Q_\phi \) for some \( \hat{D} \)-module \( Q_\phi \). By Lemma 3.6, \( M_{\alpha+H} \times (\otimes L(\frac{1}{2}, h^i) \otimes Q^0) \) is irreducible and so \( U \) contains an \( M_{H} \)-module \( M_{\alpha+H} \times (\otimes L(\frac{1}{2}, h^i) \otimes Q^0) \). Therefore, \( U \) is uniquely determined. Since \( Q_\phi \) is a direct sum of distinct irreducible \( \hat{H} \)-modules, \( \dim \left( \begin{array}{c} U \\ M_{\alpha+D} \end{array} \right) = 1 \) and so \( M_{\alpha+D} \times W \) is irreducible.

Q.E.D.

**Lemma 3.7** Let \( (W, Y^W) \) be an irreducible \( M_D \)-module with \( \tau(W) = \mu \) and let \( W = \oplus_{i=1}^s U_i \) be a decomposition of \( W \) into the direct sum of distinct homogeneous \( M_{D_{\alpha}} \)-submodules \( U_i \). Then \( U_i \) is irreducible and \( Y^W \) is uniquely determined by an \( M_{D_{\mu}} \)-module \( U_i \) for any \( i \).

**Proof** Let \( X \) be an irreducible \( M_{D_{\mu}} \)-submodule of \( U^1 \) and set \( X \cong \otimes L(\frac{1}{2}, h_i) \). By (1.1), \( U^0 \) is isomorphic to \( X \). By [DM2], \( \{ v_m u : u \in X, v \in M_{\alpha}, \alpha \in D \} \) spans \( W \). On the other hand, if \( \alpha = (a_i) \not\in D_{\beta} \), then irreducible \( T \)-submodule generated by \( v_m u \) is isomorphic to \( \otimes L(\frac{1}{2}, h_i + \frac{a_i}{2}) \) and so \( v_m u : u \in X, v \in M_{\alpha}, \alpha \in D \cap U^0 = X \), which proves \( U^0 = X \). Clearly, \( v_m u : u \in U^0, v \in M_{\alpha+D_{\mu}} \) is an irreducible \( M_{D_{\mu}} \)-module \( U^\alpha \) by the same argument. Lemma 3.8 implies that \( M_{\alpha+D_{\mu}} \times U^0 \) is irreducible. Since the restriction \( Y(v, z) : U^0 \to U^\alpha [[z, z^{-1}]] \) for \( v \in M_{\alpha+D_{\mu}} \) is a nonzero intertwining operator, we conclude \( M_{\alpha+D_{\mu}} \times U^\beta = U^{\alpha+\beta} \). Namely, if one of \( \{ U^i : i = 1, \ldots, r \} \) is given, then the other \( U^j \) are uniquely determined as \( M_{D_{\mu}} \)-modules. By Proposition 3.2, there is at least one \( M_D \)-module \( S \) such that \( S = \oplus_{\beta \in D/D_{\mu}} U^\beta \). Let \( Y^S \) be the module vertex operator of \( S \) and set \( I^{\alpha, \beta}(\ast, z) = Y^W(\ast, z) : U^\beta \to U^{\alpha+\beta} \) for \( v \in M_{\alpha+D_{\beta}} \) by restriction. Since \( \dim I \left( \begin{array}{c} U^{\alpha+\beta} \\ M_{D_{\mu}+\alpha} \end{array} \right) = 1 \), \( J^{\alpha, \beta}(v, z) = \lambda_{\beta, \beta+\alpha} I^{\alpha, \beta}(v, z) \) for \( v \in M_{\alpha+D_{\mu}} \). Let \( A(\alpha) \) be a matrix \( (\lambda_{\beta, \beta+\alpha}) \) whose \( (\beta, \beta+\alpha) \)-entry is \( \lambda_{\beta, \beta+\alpha} \) and 0 otherwise. Since \( I \) and \( J \) satisfy the mutually commutativity and the associativity, respectively, \( A : D/D_{\mu} \to M(n \times n, \mathbb{R}) \) is a regular representation and so we can reform \( A(\alpha) \) into a permutation matrix by changing the basis. Thus, \( J^{\alpha, \beta} = I^{\alpha, \beta} \) and so \( W \) is isomorphic to \( S \) as an \( M_D \)-module.
Proof of Theorem 3.3] Set $\alpha = (a_i)$ and let $\mu = \bar{\tau}(W)$. Let $H$ be a maximal self-orthogonal (doubly even) subcode of $D_\mu$. Let $W^1$ be an irreducible $M_{D_\mu}$-submodule of $W$ and let $U$ be an irreducible $M_D$-module such that $I \left( \begin{array}{cc} U \\ M_{\alpha+D} \ W \end{array} \right) \neq 0$. Clearly, $\bar{\tau}(U) = \mu$. By [DM2], there is an injective map:

$$\pi : I \left( \begin{array}{cc} U \\ M_{\alpha+D} \ W \end{array} \right) \rightarrow I_T \left( \begin{array}{cc} U \\ M_{\alpha+D_\mu} \ W^0 \end{array} \right).$$

By Lemma 3.7, $W^1 = M_{\alpha+D_\mu} \times W^0$ is irreducible and so $U$ contains $W^1$. Since $W^1$ determines $U$ uniquely and $U$ contains only one irreducible $M_{D_\mu}$-submodule isomorphic to $W^1$, $M_{\alpha+D_\mu} \times W = U$.

Q.E.D.

Combining the above arguments, we have the following theorem:

**Theorem 3.4** Let $W$ be an irreducible $M_D$-module with $\bar{\tau}(W) = \mu$. Let $E$ be an even linear code containing $D$ and assume $\langle E, \mu \rangle = 0$. Assume that there is a maximal self-orthogonal (doubly even) subcode $H$ of $D_\mu$ and $H$ is also a maximal self-orthogonal in $E_\mu$. Then there is a unique irreducible $M_E$-module $X$ containing $W$ as an $M_D$-submodule.

We will call $X$ as an induced $M_E$-module and denote it by $\text{Ind}_D^E(W)$.

We next quote the results about Hamming code VOA from [Mi2]. In this paper, a Hamming code means a $[8, 4, 4]$-Hamming code. Let $H$ be a Hamming code and $\{e^1, ..., e^8\}$ be a set of coordinate conformal vectors of a Hamming code VOA $M_H$. Let $W$ be an irreducible $M_{H_8}$-module. If $\bar{\tau}(W) = (0^8)$, then $W$ is isomorphic to a coset module $M_{H_8} + \alpha$. We denote it by $H(\frac{1}{2}, \alpha)$. If $\bar{\tau}(W) = (1^8)$, then there is a linear representation $\chi : H_8 \rightarrow \{\pm 1\}$ such that $W$ is isomorphic to $(L(\frac{1}{2}, \frac{1}{16} )^{\otimes 8}) \otimes \mathbb{R}_\chi$. If we fix a basis $\{\alpha^1, \alpha^2, \alpha^3, \alpha^4\}$ of $H_8$, then there is a word $\beta$ such that $\chi(\alpha^i) = (-1)^{\langle \beta, \alpha^i \rangle}$. We denote $W$ by $H(\frac{1}{16}, \beta)$. We should note that $H(\frac{1}{16}, \beta)$ depends on the choice of a basis of $H_8$. So, we will fix a basis $\{(1^8), (1^40^4), (1^20^21^20^2), ((10)^4)\}$ in this paper. Namely, we have the following result.

**Theorem 3.5** Let $W$ be an irreducible $M_{H_8}$-module. If $\bar{\tau}(W) = (0^8)$, then $W$ is isomorphic to one of

$$\{H(\frac{1}{2}, \alpha) : \alpha \in \mathbb{Z}_2^8\}.$$
If $\tilde{\tau}(W) = (1^8)$, then $W$ is isomorphic to one of

$$\{H(\frac{1}{16}, \alpha) : \alpha \in \mathbb{Z}_2^8\}.$$ \[H(\frac{1}{2}, \alpha) \cong H(\frac{1}{2}, \beta)\] if and only if $\alpha + \beta \in H_8$ and $H(\frac{1}{16}, \alpha) \cong H(\frac{1}{16}, \beta)$ if and only if $\alpha + \beta \in H_8$. $H(\frac{1}{2}, \alpha)$ is a coset module $M_\alpha$ and $H(\frac{1}{16}, \beta)$ is isomorphic to $L(\frac{1}{2}, \frac{1}{16})^{\otimes 8}$ as an $L(\frac{1}{2}, 0)^{\otimes 8}$-module.

In [Mi2], the author obtains the following fusion rules.

**Lemma 3.8** [Mi2]

- $H(\frac{1}{2}, \alpha) \times H(\frac{1}{2}, \beta) = H(\frac{1}{2}, \alpha + \beta)$
- $H(\frac{1}{16}, \alpha) \times H(\frac{1}{2}, \beta) = H(\frac{1}{16}, \alpha + \beta)$
- $H(\frac{1}{16}, \alpha) \times H(\frac{1}{16}, \beta) = H(\frac{1}{2}, \alpha + \beta)$

The proof is based on the nice properties of the Hamming code VOA $M_{H_8}$. To simplify the notation, we will choose another cocycle of $\hat{H}_8$ for a while. Set $\tilde{e}^\alpha = e^{\lambda_1\alpha_1} \cdots e^{\lambda_4\alpha_4}$ for $\alpha = \lambda_1\alpha_1 + \cdots + \lambda_4\alpha_4 \in H_8$, where $\{\alpha_1, ..., \alpha_4\}$ is a fixed basis of $H_8$. In $H_8$, there are 14 words of weight 4. For such a codeword (or a 4 points set) $\alpha$, set

$$\tilde{q}^\alpha = \frac{1}{4}(\otimes_{i=1}^8 q^{a_i}) \otimes \tilde{e}^\alpha.$$ 

It follows from a direct calculation that

$$s^\alpha = \frac{1}{8}(\tilde{e}^1 + ... + \tilde{e}^8) + \frac{1}{8} \sum_{\beta \in H_8, |\beta|=4} (-1)^{(\alpha, \beta)} \tilde{q}_\beta$$

is a conformal vector with central charge $\frac{1}{2}$ for a word $\alpha$ in [Mi2]. Clearly, $s^\alpha = s^\beta$ if and only if $\alpha + \beta \in H_8$. It is also straightforward to check that $\langle s^\alpha, s^\beta \rangle = 0$ if and only if $\alpha + \beta$ is even word. Therefore, there are two other sets of coordinate conformal vectors $\{d^1, ..., d^8\}$ and $\{f^1, ..., f^8\}$ in $M_{H_8}$. By the definition of the set of coordinate conformal vectors, $T_d = \langle d^1, ..., d^8 \rangle$ and $T_f = \langle f^1, ..., f^8 \rangle$ are coordinate sets of conformal vectors. Viewing an $M_{H_8}$-module as a $T_d$-module and a $T_f$-modules, we have the following correspondence: (see Proposition 2.2 and Lemma 2.7 in [Mi2]).

**Lemma 3.9** There are the other two sets of coordinate conformal vectors $\{d^1, ..., d^8\}$ and $\{f^1, ..., f^8\}$ in $M_{H_8}$ such that

- $H(\frac{1}{2}, (0^8))$ w.r.t. $\tilde{e}^i$ $\iff$ $H(\frac{1}{2}, (0^8))$ w.r.t. $d^i$ $\iff$ $H(\frac{1}{2}, (0^8))$ w.r.t. $f^i$
- $H(\frac{1}{2}, \xi_1)$ w.r.t. $\tilde{e}^i$ $\iff$ $H(\frac{1}{16}, (0^8))$ w.r.t. $d^i$ $\iff$ $H(\frac{1}{16}, \xi_1)$ w.r.t. $f^i$
- $H(\frac{1}{16}, (0^8))$ w.r.t. $\tilde{e}^i$ $\iff$ $H(\frac{1}{16}, \xi_1)$ w.r.t. $d^i$ $\iff$ $H(\frac{1}{16}, (0^8))$ w.r.t. $f^i$
- $H(\frac{1}{16}, \xi_1)$ w.r.t. $\tilde{e}^i$ $\iff$ $H(\frac{1}{2}, \xi_1)$ w.r.t. $d^i$ $\iff$ $H(\frac{1}{16}, (0^8))$ w.r.t. $f^i$,

where $\xi_1$ denotes $(10^7)$. 

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As a corollary, we have:

**Corollary 3.4** If \( W \) is an irreducible \( M_{h_8} \)-module, then \( W \times H(\frac{1}{2}, \alpha) \) and \( W \times H(\frac{1}{10}, \alpha) \) are irreducible for any \( \alpha \in \mathbb{Z}_2^8 \).

We next recall the following important theorem from [Mi5] and prove it as a corollary of the above results.

**Theorem 3.6** Let \( W^1 \) and \( W^2 \) be irreducible \( M_D \)-modules and assume that the pair \( (D, < \tau(W^1), \tau(W^2) >) \) satisfies (1) and (2) of Hypotheses I. Then \( W^1 \times W^2 \) is irreducible.

**[Proof]** Set \( \tau(W^1) = \alpha \) and \( \tau(W^2) = \beta \). If \( \alpha = 0 \) or \( \beta = 0 \), then the assertion follows from Theorem 3.3. We may assume \( \alpha = (1^80^8) \). Let \( U \) be an irreducible \( M_D \)-module so that \( 0 \neq I \left( \begin{array}{cc} U & \alpha \times \beta \end{array} \right) \). Clearly \( \tau(U) = \alpha + \beta \). By Hypotheses I, there is a self-dual subcode \( E = E_\alpha \oplus E_\alpha^e \) of \( D \) such that \( E \) is a direct sum of Hamming codes. Assume that \( E_\beta \) is a direct factor of \( E \). Then \( E = E_\beta \oplus E_{\beta^e} \). Let \( U^i \) be irreducible \( M_E \)-submodules of \( W^i \) for \( i = 1, 2 \). By Theorem 3.5, \( U^1 \cong (\bigotimes_{i=1}^r H(\frac{1}{10}, \alpha)) \otimes (\bigotimes_{j=1}^s H(\frac{1}{2}, \beta)) \) as \( M_E \)-modules and so \( U^1 \times U^2 \) is irreducible. Since \( U \) contains \( U^1 \times U^2 \), \( U \) is uniquely determined. Since \( U \) is a direct sum of distinct irreducible \( M_E \)-submodules, we have that \( W^1 \times W^2 = U \) is irreducible.

So we may assume that \( E_\beta \) is not a direct factor of \( E \). By Hypotheses I, there are maximal self-orthogonal subcodes \( H_\beta \) and \( H_{\alpha+\beta} \) of \( D_\beta \) and \( D_{\alpha+\beta} \) containing \( E_\beta \) and \( E_{\alpha+\beta} \), respectively, such that \( H_\beta + E = H_{\alpha+\beta} + E \). Since \( H_\beta + E \) satisfies Hypotheses I for < \( \alpha, \beta > \) and \( W^i \) and \( U \) are direct sums of distinct irreducible \( M_{H_\beta+E} \)-modules, we may assume \( D = H_\beta + E = H_{\alpha+\beta} + E \). We first assert the following:

**Claim:** \( W^2 \) and \( U \) are irreducible as \( M_E \)-modules.

Since the proofs are almost the same, we will prove the assertion for \( W^2 \). Set \( E = E_1 \oplus E_k \), where \( E_i \cong H_8 \). Assume first that \( E_\beta \) contains a direct factor of \( E \), say \( E_1 \). Namely, assume \( \beta = (1^8...). \) Then \( \alpha + \beta = (0^8...) \). Let \( \pi_\beta : (a^i) \rightarrow (a^i)_{i \in \text{Supp}(\beta)} \) be a projection. Since \( \pi(1^80^8k-s)(D) = \pi(H_{\alpha+\beta} + E) = E_1 \), \( D = E_1 \oplus D(0^81^{8k-8}) \) and so it is sufficient to prove the assertion for \( D(0^81^{8k-8}) \). By the induction and \( \langle \beta, \alpha \rangle = 0 \) for \( \alpha \in D \), we may assume \( \beta = (1^40^41^40^4...1^40^4) \). Since \( H_\beta \) contains \( E_\beta \) and \( D = H_\beta + E \), \( D_\beta = H_\beta \). Let \( X \) be an irreducible \( M_{H_\beta} \)-submodule of \( W^2 \), then \( W^2 = \text{Ind}_{H_\beta}^D(X) \) and \( X \) is irreducible as a \( T \)-module. In particular, \( X \) is irreducible as an \( M_{E_\beta} \)-module with
\[ \tilde{\tau}(X) = \mu. \] Hence \( \text{Ind}_{E_\beta}^E(X) \) is an irreducible \( M_E \)-submodule of \( W^2 \). On the other hand, since \( D/H^\beta \cong E/E_\beta \), \( \dim \text{Ind}_{E_\beta}^E(X) = \dim W^2 \), which proves the claim.

We now go back to the proof of Theorem 3.6. Set \( \gamma = \alpha + \beta \). Let \( X \) be an irreducible \( M_E \)-submodule of \( W^1 \). Since \( W^2 \) and \( U \) are both irreducible \( M_E \)-modules by the above claim,

\[
\dim I_{MD} \begin{pmatrix} U \\ W^1 \\ W^2 \end{pmatrix} \leq \dim I_{ME} \begin{pmatrix} U \\ X \\ W^2 \end{pmatrix} = 1
\]

and so \( U \cong X \times W^2 \) as \( M_E \)-modules. Fix a nonzero intertwining operator

\[
I^1(*, z) \in I_{ME} \begin{pmatrix} W^3 \\ X \\ W^2 \end{pmatrix}.
\]

For \( I(*, z) \in I_{MD} \begin{pmatrix} W^3 \\ W^1 \\ W^2 \end{pmatrix} \), there is a scalar \( \lambda \) such that \( I(v, z) = \lambda I^1(v, z) \) for \( v \in X \).

\[
Y^U(u, z)I(v, z) \sim I(v, z)Y^2(u, z)
\]

and so \( Y^U(u, z)I^1(v, z) = I^1(v, z)Y^2(u, z) \) for \( u \in M_D \) and \( v \in X \). Since \( \langle I^1(v, z)w : v \in X, w \in W^2 \rangle = U \), \( Y^U(u, z) \) is uniquely determined by \( Y^2(u, z) \) and so \( W^1 \times W^2 = W^3 \).

Q.E.D.

When we want to prove the condition (4) in Hypotheses I, the notion of induced VOA will be very useful as we mentioned in the introduction. Set \( S = \langle \alpha, \beta \rangle \). We assume that a pair \((D, S)\) satisfies the conditions (1) and (2) in Hypotheses I for a while. An important tool is Theorem 3.1. Namely, for any irreducible \( M_D \)-modules \( \gamma^\alpha \) and \( \gamma^\beta \) with \( \tilde{\tau}(\gamma^\alpha') = \alpha' \) and \( \tilde{\tau}(\gamma^\beta') = \beta' \), an \( M_D \)-module \( \gamma^\alpha \times \gamma^\beta \) is irreducible for \( \alpha', \beta' \in S \). Set \( \gamma^{\alpha+\beta} = \gamma^\alpha \times \gamma^\beta \). Then by the property of fusion rules, we have \( V^\gamma+\delta = V^\gamma \times V^\delta \) for \( \gamma, \delta \in S \). This implies that there is a unique nonzero intertwining operator of type

\[
\begin{pmatrix} V^\gamma & V^\delta \\ V^\gamma+\delta \end{pmatrix}
\]

up to scalar multiple for \( \delta, \gamma \in S \). So if we have an algebraic structure (like a VOA) on

\[
(M_D \oplus V^\alpha \oplus V^\beta \oplus V^{\alpha+\beta}, Y),
\]

then \( Y \) is uniquely determined up to an \( M_D \)-isomorphism.

The purpose of this section is to show the following theorem, which is a \( \mathbb{R} \)-version of Theorem 6.5 in [Mi3]:

**Theorem 3.7** Set \( S = \langle \alpha, \beta \rangle \) and assume the pair \((D, S)\) satisfies the conditions (1) and (2) of Hypotheses I. Let \( F \) be an even linear code containing \( D \) such that \( \langle F, S \rangle = 0 \). Assume that \( W = M_D \oplus W^\alpha \oplus W^\beta \oplus W^{\alpha+\beta} \) has a simple VOA structure. Then

\[
V = M_F \oplus \text{Ind}_{M_D}^{M_F}(W^\alpha) \oplus \text{Ind}_{M_D}^{M_F}(W^\beta) \oplus \text{Ind}_{M_D}^{M_F}(W^{\alpha+\beta})
\]

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also has a simple VOA structure up to an $M_F$-isomorphism.

In order to prove the above theorem, we will prove the following lemma:

**Lemma 3.10**\[\text{Ind}_D^F(W^\alpha) \times \text{Ind}_D^F(W^\beta) = \text{Ind}_D^F(W^\alpha+\beta).\]

**Proof**\[\text{Let } U \text{ be an irreducible } M_F\text{-module such that } I\begin{pmatrix} U \\ \text{Ind}_D^F(W^\alpha) \ & \text{Ind}_D^F(W^\beta) \end{pmatrix} \neq 0. \text{ Since } \text{Ind}_D^F(W^\alpha) \text{ and } \text{Ind}_D^F(W^\beta) \text{ are irreducible modules satisfying Hypotheses I, } \text{Ind}_D^F(W^\alpha) \times \text{Ind}_D^F(W^\beta) \text{ is irreducible. By Theorem 11.9 in [DL], we have an injective map}\]

\[\phi : I\begin{pmatrix} U \\ \text{Ind}(W^\alpha) \ & \text{Ind}(W^\beta) \end{pmatrix} \rightarrow I\begin{pmatrix} U \\ W^\alpha \ & \ W^\beta \end{pmatrix}\]

\[\text{and so } U \text{ contains } W^\alpha \times W^\beta = W^\alpha+\beta, \text{ which implies } U = \text{Ind}_D^F(W^\alpha+\beta).\]

Q.E.D.

**Proof of Theorem 3.7.**\[\text{Let } Y^W(v, z) \in \text{End}(W)[[z, z^{-1}]] \text{ be the given vertex operator of } v \in U \text{ and let}\]

\[J_{\alpha', \beta'}(v, z) \in I\begin{pmatrix} W^\alpha+\beta' \\ W^\alpha' \ & \ W^\beta' \end{pmatrix}\]

be the restriction of $Y(v, z)$ for $v \in W^\alpha'$ and $\alpha', \beta' \in S = \langle \alpha, \beta \rangle$. Since Theorem 11.9 in [DL] implies that $\phi : I\begin{pmatrix} \text{Ind}(W^\gamma) \\ \text{Ind}(W^\alpha') \ & \text{Ind}(W^\beta') \end{pmatrix} \rightarrow I\begin{pmatrix} W^\gamma \\ W^\alpha' \ & \ W^\beta' \end{pmatrix}$ is injective and the multiplicity of $W^\alpha' \times W^\beta'$ in $\text{Ind}(W^\alpha+\beta')$ is one, we can choose

\[I_{\alpha', \beta'}(*, z) \in I\begin{pmatrix} \text{Ind}(W^\alpha'+\beta') \\ \text{Ind}(W^\alpha') \ & \ \text{Ind}(W^\beta') \end{pmatrix}\]

such that $I_{\alpha', \beta'}(v, z)u = J_{\alpha', \beta'}(v, z)u$ for $v \in W^\alpha'$ and $u \in W^\beta'$. Define $Y(v, z) \in \text{End}(V)[[z, z^{-1}]]$ by $I(v, z)u = I_{\alpha', \beta'}(v, z)u$ for $v \in \text{Ind}_D^F(W^\alpha')$ and $u \in \text{Ind}_D^F(W^\beta')$. Note $Y(v, z)u = Y^U(v, z)u$ for $u, v \in U$. Moreover, the powers of $z$ in $Y(v, z)$ are all integers since $\langle \hat{\tau}(\text{Ind}(U)), F \rangle = 0$. For $u, v \in U$, $Y(u, z)$ and $Y(v, z)$ satisfy the commutativity on $U$. For $v \in V$, $Y(v, z)|_{\text{Ind}(W^\alpha)}$ is at least an intertwining operator and $Y(v, z)$ satisfies the commutativity with a vertex operator $Y(u, z)$ of $u \in M_F$. By this commutativity,

\[\{w \in \text{Ind}(U) : I(u', z)I(u, x)w \sim I(u, x)I(u', z)w\}\]

is a $M_F$-module for $u, u' \in U$. Since it contains $U$, it coincides with $V$. Namely, $\{Y(u, z) : u \in U \cap M_D\}$ satisfies the mutual commutativity on $V$. Clearly, $\{Y(v, z) : v \in M_D \cup U\}$
generates all intertwining operators by the normal products and so all $I(v, z)$ for $v \in V$ satisfy the mutually commutativity by Dong’s lemma. The other required conditions are also easy to check and so we have a VOA structure on $\text{Ind}(U)$.

Q.E.D.

Lemma 3.11 Let $V = \oplus_{\alpha \in S} V^\alpha$ be a VOA satisfying Hypotheses I and $W$ be an irreducible $V$-module. Then there is a word $\gamma$ and irreducible $M_D$-modules with $\tilde{\tau}(W^\beta) = \beta$ for $\beta \in S + \gamma$ such that $W = \oplus_{\beta \in S + \gamma} W^\beta$.

[Proof] Since $T$ is rational, $W$ is a direct sum of irreducible $T$-modules and so we have $W = \oplus_{\beta \in S'} W^\beta$ for some $S'$, where $W^\beta$ is the sum of all irreducible $T$-submodules $X$ with $\tilde{\tau}(X) = \beta$. By (1.1), $W^\beta$ is an $M_D$-module. By the similar arguments as in the proof of Theorem 3.2, $W^\beta$ is irreducible. Since $\tilde{\tau}(V^\alpha \times W^\beta) = \alpha + \beta$, $S' = S + \gamma$ for some $\gamma$.

Q.E.D.

Lemma 3.12 Let $V = \oplus_{\alpha \in S} V^\alpha$ be a VOA satisfying Hypotheses I and $W = \oplus_{\beta \in S + \gamma} W^\beta$ be an irreducible module. Assume that $< S + \mathbb{Z}_2 \gamma, D >$ satisfies Hypotheses I. Then $W$ is uniquely determined by a $W^\beta$ for some $\beta$.

[Proof] Since $V^\alpha \times W^\beta = W^{\alpha + \beta}$, $M_D$-module structure on $W$ is uniquely determined by $W^\beta$. By the similar arguments as in the proof of Theorem 3.4, we have the desired assertion.

Q.E.D.

Since the fusion rules (1.1) are all well-defined over $\mathbb{R}$ (even over $\mathbb{Q}$), we can rewrite Theorem 4.1 in [Mi3] into the following theorem.

Theorem 3.8 Under the assumptions (1)∼(4) of Hypotheses I, we obtain a fusion product $V^\alpha \times V^\beta = V^{\alpha + \beta}$ for $\alpha, \beta \in S$. Moreover, there is a simple VOA structure on

$$V = \bigoplus_{\alpha \in S} V^\alpha$$

such that it contains $M_D$ as a sub VOA $V^{(0^\alpha)}$ and has a positive definite invariant bilinear form. A simple VOA structure on $V$ with a positive definite invariant bilinear form is uniquely determined up to $M_D$-isomorphisms.
First, we fix module vertex operators $Y^V(v, z)$ for $v \in M_D$. Let $Y^{\alpha, \beta}$ be the vertex operator of the VOA $V^{\alpha, \beta} = M_D \oplus V^\alpha \oplus V^\beta \oplus V^{\alpha + \beta}$ such that $Y^{0, \beta}(v, z)u = Y^V(v, z)u$ for $v \in M_D$ and $u \in V^{\alpha, \beta}$. Since $V^\alpha \times V^\alpha = M_D$, there are two possible simple VOA structures on $M_D \oplus V^\alpha$. Moreover, since we assumed that $M_D \oplus V^\alpha$ has a positive definite invariant bilinear form, there is a unique VOA structure on $M_D \oplus V^\alpha$ up to $M_D$-isomorphisms. Namely, if we fix an orthonormal basis $\{u_i^\alpha : i \in I_\alpha\}$ of $V^\alpha$, then $Y^{\alpha, \beta}(u, z)v$ for $u, v \in V^\alpha$ does not depend on the choice of $\beta$. Define a nonzero intertwining operator

$$I^{\alpha, \beta}(*, z) \in I\left( \begin{array}{c} V^{\alpha + \beta} \\ V^\alpha \\ V^\beta \end{array} \right)$$

for $\alpha, \beta \in S$ by $I^{\alpha, \beta}(v, z)u = Y^{\alpha, \beta}(v, z)u$ for $v \in V^\alpha$, $u \in V^\beta$.

Our next step is to choose suitable scalars $\lambda^{\alpha, \beta}$ and define a new vertex operator $Y(v, z) \in \text{End}(V)[[z, z^{-1}]]$ by

$$Y(v, z)u = \lambda^{\alpha, \beta} I^{\alpha, \beta}(v, z)u$$

(3.1)

for $v \in V^\alpha$ and $u \in V^\beta$ such that $\{Y(v, z) : v \in V\}$ satisfies the mutual commutativity. Since intertwining operators satisfy the $L(-1)$-derivative property and the other conditions except the mutual commutativity, $(V, Y)$ becomes a simple VOA with a positive definite invariant bilinear form.

Set $\dim S = t$ and let $\{\alpha_1, \ldots, \alpha_t\}$ be a basis of $S$. Set $S_i = <\alpha_1, \ldots, \alpha_i>$ for $i = 0, 1, \ldots, t$ and $V^i = \oplus_{\alpha \in S_i} V^\alpha$. We will choose $\lambda^{\alpha, \beta}$ inductively. Since $V^\alpha$ are all $M_D$-modules, the module vertex operators $Y^V(v, z)$ of $v \in V^0 = M_D$ on $V$ satisfy the mutual commutativity if we choose $\lambda^{0, \alpha} = 1$. We next assume that there is an integer $r$ such that the vertex operators $\{Y(v, z) : v \in V^r\}$ satisfy the mutual commutativity by choosing $\lambda^{\alpha, \beta}$ for $\alpha \in S_r$. In particular, $V^r$ is a sub VOA and $V$ is a $V^r$-module by these vertex operators. It is clear that $V^{S^r + \delta} = \oplus_{\gamma \in S^r} V^{\delta + \gamma}$ are irreducible $V^r$-modules for any $\delta \in S$ by the fusion rules and $V$ decomposes into the direct sum of irreducible $V^r$-modules. By the fusion rule of $M_D$-modules $V^\beta$ and Lemma 3.12, we obtain a fusion rule:

$$V^{\delta + S^r} \times V^{\gamma + S^r} = V^{\delta + \gamma + S^r}$$

as $V^r$-modules.

Decompose $V^{r+1} = V^r \oplus V^{\alpha_r + 1 + S^r}$ as $V^r$-modules. To simplify the notation, set $\alpha = \alpha_r + 1$ and $I^{\alpha}(v, z) = I^{\alpha, \beta}(v, z)$ for a while. Let $\{\gamma^i \in S : i \in J\}$ be a set of representatives of cosets $S/S^{r+1}$. Since there is an injection

$$\pi : I\left( \begin{array}{c} V^{S^r + \alpha + \gamma^i} \\ V^{S^r + \alpha} \end{array} \right) \rightarrow I\left( \begin{array}{c} V^{S^r + \alpha + \gamma^i} \\ V^\alpha \end{array} \right)$$

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and \( \dim I \left( \begin{array}{cc} V^{S_r + \alpha + \gamma} \\
V^{\alpha} & V^{\gamma} \end{array} \right) = 1 \), we can choose a nonzero intertwining operator

\[
I^{\alpha + S_r, \gamma + S_r} (\ast, z) \in I \left( \begin{array}{cc} V^{S_r + \alpha + \gamma} \\
V^{\alpha} & V^{\gamma} \end{array} \right)
\]

such that \( I^{\alpha + S_r, \gamma + S_r} (v, z) = Y^{\alpha, \gamma} (v, z) u \) for \( v \in V^\alpha, u \in V^\gamma \). Restricting \( I^{\alpha + S_r, \gamma + S_r} (\ast, z) \) into \( V^{\alpha + \beta, \gamma + \delta} \) for \( \beta, \delta \in S_r \), we have a scalar \( \lambda_{\alpha + \beta, \gamma + \delta} \) such that \( I^{\alpha + S_r, \gamma + S_r} (v, z) = \lambda_{\alpha + \beta, \gamma + \delta} Y^{\alpha + \beta, \gamma + \delta} (v, z) u \) for \( v \in V^{\alpha + \beta} \) and \( u \in V^{\gamma + \delta} \). We will show that \( V^{r+1} \) is a sub VOA and \( V \) is a \( V^{r+1} \)-module by the above intertwining operators \( I^{\alpha + S_r, \gamma + S_r} (\ast, z) \).

Set

\[
Q = \{ w \in V | Y(u, z) Y(u', x) w \sim Y(u', x) Y(u, z) w \text{ for } u, u' \in V^\alpha \}.
\]

Since \( Y(\ast, z) \) is an intertwining operator of \( V^r \)-modules, \( Q \) is a \( V^r \)-module. On the other hand, by the choice of \( Y, Q \) contains \( V^{\gamma_i} \) for all \( i \). Hence, \( Q \) coincides with \( V \). In particular, all vertex operators in \( \{ Y(u, z) : u \in V^r \cup V^\alpha \} \) satisfy the mutual commutativity. Since \( V^{r+1} \) is generated by \( V^r \) and \( V^\alpha \), we have the desired result. This completes the construction of our VOA.

We next show that the VOA structures on \( V \) is unique. Assume that there are two VOA structures \((V, Y)\) and \((V, Y')\) on \( V \). We may assume that all \( V^{\alpha, \beta} \) are sub VOAs of \((V, Y)\). Since \( \dim I \left( \begin{array}{cc} V^{\alpha + \beta} \\
V^{\alpha} & V^{\beta} \end{array} \right) = 1 \), there are \( \lambda_{\alpha, \beta} \) such that \( Y'(v, z) u = \lambda_{\alpha, \beta} Y(v, z) u \) for \( v \in V^\alpha, u \in V^\beta \). We may assume \( \lambda_{0^\alpha, \beta} = 1 \) and so \( \lambda_{\beta, 0^\alpha} = 1 \) by the skew symmetry. We will show that by changing the sign of an orthonormal basis of \( V^{S_r + \alpha_{r+1}} \) if necessary we can get \( Y = Y' \). Define \( f = (-1)^\beta \in \text{End}(V) \) by \( (-1)^{\beta, \alpha} \) on \( V^\alpha \). It is clear that \( f \) is an automorphism. Assume \( Y|_{V^{S_r}} = Y'|_{V^{S_r}} \) and \( Y|_{V^{S_{r+1}}} \neq Y'|_{V^{S_{r+1}}} \). We assert \( \lambda_{\alpha} \in \{ \pm 1 \} \).

By changing the sign of an orthogonal basis of \( V^{\alpha + S_r} \), we may assume \( Y'|_{V^\alpha} = Y|_{V^{\alpha}} \).

By Lemma 3.12, \((V^{\alpha + S_r}, Y)\) is isomorphic to \((V^{\alpha + S_r}, Y')\) as a \( V^{S_r} \)-module. On the other hand, since \( V^{\alpha + S_r} \times V^{\alpha + S_r} = V^{S_r}, Y'|_{V^{\alpha + S_r}} = \lambda_{\alpha} Y|_{V^{\alpha + S_r}} \). Hence, \( Y = Y' \) on \( S^{r+1} \). Namely, \( \lambda_{\alpha} = 1 \). Let \( \beta \in S^r > 1 \) and \( (\beta, \alpha) = 1 \). Then by using an automorphism \((-1)^\beta\), we may assume \( Y = Y' \) on \( S^{r+1} \). By induction, we have \( Y = Y' \) on \( V \).

Q.E.D.

We will next show a relation between automorphisms of \( M_D \) and fusion product modules \( M_{\alpha + D} \times W \). For a word \( \alpha \), we can define an automorphism \( \sigma_{\alpha} \) of \( M_D \) by

\[
\sigma_{\alpha} : (-1)^{(\beta, \alpha)} \text{ on } M_{\beta},
\]

which coincides with \( \prod_{i \in \text{Supp}(\alpha)} \sigma_e^i \), where \( \sigma_e^i \) is an automorphism given in [MIL] of type 2.
Lemma 3.13 Suppose $\beta = \tilde{\tau}(W)$ and $D_\beta$ contains a maximal self orthogonal subcode $H$ which is doubly even and is orthogonal to $\alpha$, then $\sigma_\alpha W$ is isomorphic to $W$.

[Proof] Decompose $M_D$ into $M_D^+ \oplus M_D^-$, where $M_D^+ = \{v \in M_D : \sigma_\alpha(v) = \pm v\}$. Set $E = \{\beta \in D : \langle \beta, \alpha \rangle = 0\}$. Clearly, $M_D^+ = M_E$. Since $E$ contains a maximal self-orthogonal subcode $H$ of $D_\beta$ which is doubly even, there is an $M_E$-module $U$ such that $\text{Ind}_{M_E}(U) = W$ by Proposition 3.2. It follows from the definition of the induced modules that $\text{Ind}_{M_E}^M(U) \cong U \oplus (M_D^- \times U)$ as $M_E$-modules. The actions of $M_D^-$ switch $U$ and $M_D \times U$, that is, $u_n(U) \subseteq M_D \times U$ and $u_n(M_D^- \times U) \subseteq U$ for any $n \in \mathbb{Z}$ and $u \in M_D^-$. Moreover, $u_n\sigma_\alpha v = -u_n v$ for $u \in M_D^-$ and $v \in \text{Ind}_E^D(U)$. It is easy to check that $(1_U, -1_{M_D \times U})$ on $U \oplus M_D \times U$ is an isomorphism from $\sigma_\alpha(\text{Ind}_E^D(U))$ to $\text{Ind}_E^D(U)$.

Q.E.D.

For an irreducible $M_D$-module $W$, $\sigma_\alpha W$ is also an irreducible $M_D$-module. Clearly, $W$ and $\sigma_\alpha W$ are isomorphic as $T$-modules and $\sigma_\alpha = \sigma_\beta$ if and only if $\alpha + \beta \in D^\perp$. We next investigate an irreducible $M_D$-module $M_{D+\alpha} \times W$ for $\alpha$ satisfying $\text{Supp}(\alpha) \subseteq \text{Supp}(\tilde{\tau}(W))$. In this case, $M_{D+\alpha} \times W$ is isomorphic to $W$ as a $T$-module. The following lemma is important.

Lemma 3.14 Let $W$ be an irreducible $M_D$-module and assume $\text{Supp}(\alpha) \subseteq \text{Supp}(\tilde{\tau}(W))$. Then $M_{D+\alpha} \times W$ is isomorphic to $\sigma_\alpha W$ as an $M_D$-module.

[Proof] Set $U = M_{\alpha+D}$ and $\beta = \tilde{\tau}(W)$. Clearly, $\tilde{\tau}(M_{D+\alpha} \times W) = \tilde{\tau}(\sigma_\alpha W) = \beta$. By Theorem 3.3, $W' = U \times W$ is irreducible. Let $H_\beta$ be a maximal self-orthogonal (doubly even) subcode of $D_\beta$. Since an $M_D$-module $W$ with $\tilde{\tau}(W) = \beta$ is uniquely determined by an $M_{H_\beta}$-submodule, we may assume that $D$ is a self-orthogonal doubly even code and $\text{Supp}(D) \subseteq \text{Supp}(\beta)$. In particular, we may also assume that $W$ and $W'$ are both isomorphic to $L(1/2, 1/16)^{\otimes n}$ as $T$-modules. Since $1 \leq \dim I_{M_D}(W') \leq \dim I_T \left( \begin{array}{c} L(1/2, 1/16)^{\otimes n} \\ M_1 \end{array} \right) = 1$, an intertwining operator of type $I_T \left( \begin{array}{c} W' \\ M_1 \end{array} \right)$ is uniquely determined up to scalar multiples for $\gamma \in D + \alpha$. As shown in §2.3 or in [Mi3], we can choose a nonzero intertwining operator $I(\ast, z) \in I_T \left( \begin{array}{c} L(1/2, 1/16)^{\otimes n} \\ M_1 \end{array} \right)$ by

$I(q^\gamma, z) = I(\hat{q}^\gamma, z) = \otimes I^{q^{n/16}}(q^{n}, z),$

where $I^{q^{n/16}}(\ast, z)$ is a fixed intertwining operator of type $\left( \begin{array}{c} L(1/2, 1/16) \\ L(1/2, 1/16) \end{array} \right)$, see §2.3.
By Theorem 3.2, there are linear representations $\chi$ and $\phi$ of $\hat{D}$ such that $W \cong L(\frac{1}{2}, \frac{1}{16})^\otimes n \otimes Q_\chi$ and $W' \cong L(\frac{1}{2}, \frac{1}{16})^\otimes n \otimes Q_\phi$. By the associativity property of intertwining operators,

$$I(u_n q^\alpha, z) = \text{Res}_x \left \{(x-z)^n Y^W(q^\beta, x)I(q^\alpha, z) - (-z+x)^n I(q^\alpha, z)Y^W(q^\beta, x) \right \}$$

$$= \text{Res}_x \left \{(x-z)^n I^{\otimes n}(q^\beta, x)\phi(e^\beta)I(q^\alpha, z) - (-z+x)^n I(q^\alpha, z)I^{\otimes n}(q^\beta, x)\chi(e^\beta) \right \}$$

for $q^\beta \in M_\beta \subseteq M_D$ and $u \in M_\alpha$. In particular, for a sufficiently large $N$, we obtain

$$0 = \text{Res}_x \left \{(x-z)^N I^{\otimes n}(q^\beta, x)\phi(e^\beta)I(q^\alpha, z) - (-z+x)^N I(q^\alpha, z)I^{\otimes n}(q^\beta, x)\chi(e^\beta) \right \}.$$

On the other hand, as we showed in §2.3, $I(\ast, z)$ satisfies the super-commutativity:

$$(x-z)^N I^{\otimes n}(q^\beta, x)I^{\otimes n}(q^\alpha, z) - (-1)^{\langle \alpha, \beta \rangle} (-z+x)^N I^{\otimes n}(q^\alpha, z)I^{\otimes n}(q^\beta, x) = 0.$$

Therefore,

$$\text{Res}_x \left \{(x-z)^N \phi(e^\beta) - (-1)^{\langle \alpha, \beta \rangle} (-z+x)^N \chi(e^\beta) \right \} = 0$$

and so $\phi(e^\beta) = (-1)^{\langle \alpha, \beta \rangle} \chi(e^\beta)$ for $\beta \in D$. Hence, $W'$ is isomorphic to $\sigma_\alpha W$ as $M_D$-module.

Q.E.D.

Remark 1 The above lemma may look a little strange since we usually obtain relations $\sigma(W^1) \times \sigma(W^2) = \sigma(W^1 \times W^2)$ and $(M_{\alpha+D} \times W^1) \times (M_{\alpha+D} \times W^2) = (W^1 \times W^2)$ for an automorphism $\sigma$ and a coset module $M_{\alpha+D}$, respectively. However, if we have $\sigma(W^i) \cong M_{D+\alpha} \times W^i$ for $i = 1, 2$, then $W^1 \times W^2$ does not satisfy the condition of the above lemma by (1.1) and so $\sigma(W^1 \times W^2) = W^1 \otimes W^2$.

4 Positive definite invariant bilinear form

In order to construct $V^2$, we will use "induced VOAs". So, we will prove the following theorem.

Theorem 4.1 Assume that $W^\alpha$ is an irreducible $M_D$-module with $\tau(W^\alpha) = \alpha$ and $(D, < \alpha >)$ satisfies the conditions (1) and (2) of Hypotheses I. Let $F$ be an even linear code containing $D$ satisfying $(F, \alpha) = 0$. If a simple VOA $U = M_D \oplus W^\alpha$ has a positive definite invariant bilinear form, then $\text{Ind}_{M_D}^{M_F}(U) \cong M_F \oplus \text{Ind}_{M_D}^{M_F}(W^\alpha)$ also has a positive definite invariant bilinear form.
We note that if an irreducible $M_D$-module $W$ is not isomorphic to $M_D$, then the lowest degree of $W$ is greater than 0. Clearly, it is sufficient to prove the lemma for $F = \langle \alpha, (1^n) \rangle$. Since $< \alpha, (1^n) >$ is generated by words of weight 2, we may assume $F = D + \mathbb{Z} \beta$ such that $|\beta| = 2$ by induction. Say $\beta = (110^{n-2})$. Since $|\beta| = 2$ and $\langle \beta, \alpha \rangle = 0$, $\text{Supp}(\beta) \subseteq \text{Supp}(\alpha)$ or $\text{Supp}(\beta) \cap \text{Supp}(\alpha) = \emptyset$.

Since $D_\alpha$ contains a direct sum $E$ of Hamming codes such that $\text{Supp}(E) = \text{Supp}(\alpha)$, the orthogonal doubly even subcode of $D_\alpha$ is irreducible. Set

$$V = M_F \oplus \text{Ind}_D^F(W^\alpha).$$

By Theorem 3.7, $V$ has a VOA structure. Since $\text{Ind}_D^F(W^\alpha) \times \text{Ind}_D^F(W^\alpha) = M_F$ by Lemma 3.10, there are only two possible VOA structures on $V$. Namely, if one is $(M_F \oplus \text{Ind}_D^F(W^\alpha), Y)$, then the other is $(M_F \oplus \sqrt{-1}\text{Ind}_D^F(W^\alpha), Y)$. Since $W^\alpha \times W^\alpha = M_D$, we may assume $(M_F \oplus \text{Ind}_D^F(W^\alpha), Y)$ contains $U$ as a sub VOA. Let $E$ be a maximal self orthogonal doubly even subcode of $D_\alpha$. Then $W^\alpha$ is a direct sum $\oplus W^i$ of distinct irreducible $M_E$-modules $W^i$ and $V^i = M_E \oplus M_{E+\beta} \oplus W^i \oplus (M_{E+\beta} \times W^i)$ is a sub VOA of $V$ for each $i$. Since $(M_E \oplus W^i, Y)$ is a sub VOA of $M_D \oplus W^\alpha$, $(M_E \oplus W^i, Y)$ has a positive definite invariant bilinear form.

We will later show that a VOA structure $(V^i, Y)$ on $V^i$ has a positive definite invariant bilinear form. In particular, $W^i \oplus (M_{E+\beta} \times W^i)$ has an orthonormal basis with respect to $Y$. Then since $M_{E+\beta} \times W^\alpha$ coincides with $\oplus (M_{E+\beta} \times W^i)$, we have the desired result. Therefore, we may assume that $\text{Supp}(D) = \text{Supp}(\alpha)$ and $D$ is a direct sum $D = E^1 \oplus \cdots \oplus E^s$ of Hamming codes $E^i$ by Hypotheses I. In particular, $W^\alpha$ is irreducible as a $T$-module. Since a VOA structure $(V, Y)$ on $V$ containing $U$ is uniquely determined, it is sufficient to show that there exists a VOA structure on $(V, Y)$ with a positive definite invariant bilinear form. For if $(V, Y')$ is the other simple VOA structure on $V$, then $(W^\alpha, Y')$ has a negative definite invariant bilinear form and so $(V, Y')$ does not contain $U$. If $\text{Supp}(\beta) \cap \text{Supp}(\alpha) = \emptyset$, then $< D, \beta >$ is self-orthogonal. Let $D^0$ be the code of length $n - 2$ consisting of the codewords $\gamma$ such that $\langle 00\gamma \rangle \in D$. Then $M_D = L(1, 0) \oplus L(\frac{1}{2}, 0) \otimes M_{D^0}$ and $M_{D+\beta} = L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}) \otimes M_{D^0}$. By the above decompositions, we can write

$$W^\alpha \cong L(1, h^1) \otimes L(\frac{1}{2}, h^2) \otimes W'$$

and

$$M_{D+\beta} \times W^\alpha \cong L(\frac{1}{2}, h^1 + \frac{1}{2}) \otimes L(\frac{1}{2}, h^2 + \frac{1}{2}) \otimes W',$$

for some irreducible $M_{D^0}$-module $W'$ and $h^1, h^2 = 0, \frac{1}{2}$ and $h^1 + \frac{1}{2} = 0$ if $h^i = \frac{1}{2}$ and $h^i + \frac{1}{2} = \frac{1}{2}$ if $h^i = 0$. Since $L(\frac{1}{2}, 0)^{\otimes 2} \oplus L(\frac{1}{2}, \frac{1}{2})^{\otimes 2} \cong V_{22x} = (V_{22x})^0 \otimes \sqrt{-1}(V_{22x})^-$ for
\( \langle x, x \rangle = 1, \sqrt{-1}x(0) \) is an isomorphism from \( L(\frac{1}{4}, h^1) \otimes L(\frac{1}{4}, h^2) \) to \( L(\frac{1}{4}, h^1 + \frac{1}{2}) \otimes L(\frac{1}{4}, h^2 + \frac{1}{2}) \) and \( x(0)^2 \) acts diagonally on \( L(\frac{1}{4}, h^1) \otimes L(\frac{1}{4}, h^2) \) with a positive eigenvalues. Let \( \{v^i : i \in I\} \) be an orthogonal normal basis such that each \( v^i \) is in an eigenspaces of \( x(0)^2 \). Then \( \{\sqrt{-1}x(0)v^i : i \in I\} \) is a basis of \( L(\frac{1}{4}, h^1 + \frac{1}{2}) \otimes L(\frac{1}{4}, h^2 + \frac{1}{2}) \) and
\[
\langle \sqrt{-1}x(0)v^i, \sqrt{-1}x(0)v^j \rangle = \langle v^i, x(0)^2v^j \rangle = \delta_{ij}\langle v^i, x(0)^2v^j \rangle \geq 0.
\]
Hence, \( \text{Ind}_F^D(U) \) has a positive definite invariant bilinear form.

We next assume \( \text{Supp}(\beta) \subseteq \text{Supp}(\alpha) \). Since \( D \) is a direct sum of Hamming codes and the weight of \( \beta \) is 2, it is sufficient to treat the following two cases:

1. \( \text{Supp}(\beta) \subseteq \text{Supp}(E^1) \).
2. \( D = E_8 \oplus \cdots \oplus E_8 \) and \( \beta = (10^710^70^{n-16}) \).

Case (1). By Lemma 3.8, there is another set of coordinate conformal vectors \( \{d^i\} \) of \( M_D \) such that \( W \) is a coset module \( M_{D+\gamma} \) w.r.t. \( <d^i> \). Since \( \text{Supp}(\beta) \subseteq \text{Supp}(E^1) \) and \( \beta \) has an even weight, \( M_{\beta+D} \) is also a coset module \( M_{\beta+D} \). Namely, \( \text{Ind}_F^D(U) \) is a code VOA \( M_{<D,\beta,\gamma>} \) w.r.t. \( <d^i> \). Hence, it has a positive definite invariant bilinear form.

Case (2). By taking another set of coordinate conformal vectors, we may assume that \( \alpha = (1^{16}0^n-16) \) and \( \beta = (10^710^70^{n-16}) \). Since \( L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}) \) has a positive definite invariant bilinear form and the lowest weight is an integer, we may also assume that \( n = 16 \) and \( \alpha = (1^{16}) \). We will find such a VOA in \( V_{E_8} \) in the next section. This will complete the proof of Theorem.

Q.E.D.

5 \( E_8 \)-lattice VOA

As we mentioned in the introduction, we will gather the parts of \( V^2 \) from \( \tilde{V}_{E_8} \). Hence, the main aim of this section is to study the structure of \( V_{E_8} \) and \( \tilde{V}_{E_8} \). In particular, we will show that \( \tilde{V}_{E_8} \) satisfies the conditions (1)~(5) of Hypotheses I. Incidentally, we will see that the orbifold construction from VOA \( V_{E_8} \) coincides with the change of a set of coordinate conformal vectors of a Hamming code sub VOA of \( V_{E_8} \).

Let \( E_8 \) denote the root lattice of type \( E_8 \). It is known that \( E_8 \) is the unique positive definite unimodular even lattice of rank 8. We first define lattices \( E_8(m) : m = 1, 2, 3, 4, 5 \) and \( L(1) \). Let \( \{x^1, \ldots, x^8\} \) be an orthonormal basis and set \( E_8(1) =< \frac{1}{2}(\sum_{i=1}^{8} x^i), x^i \pm x^j : i, j = 1, \ldots, 8 > \) and \( L(1) =< x^i : i = 1, \ldots, 8 > \), where \( < u^i : i \in I > \) denotes a lattice
generated by \( \{u^i : i \in I\} \). It is easy to check that \( E_8(1) \) is isomorphic to \( E_8 \). We can define the other \( E_8 \)-lattices as follows:

\[
E_8(2) = \langle \frac{1}{2}(x^1 - x^2 - x^3 - x^4) + x^5, \frac{1}{2}(x^5 + x^6 + x^7 + x^8) + x^1,
\]
\[
x^i \pm x^j : i, j \in \{1, 2, 3, 4\}, \text{ or } i, j \in \{5, 6, 7, 8\} \rangle.
\]

\[
E_8(3) = \langle \frac{1}{2}(x^1 - x^2 - x^3 - x^4) + x^3, \frac{1}{2}(x^1 + x^2 - x^3 - x^4) - x^7,
\]
\[
\frac{1}{2}(x^1 + x^2 - x^3 - x^4) - x^7, \frac{1}{2}(x^1 + x^2 - x^3 - x^4) + x^1, x^1 + x^3 + x^5 + x^7, x^{2i-1} + x^{2i}, (i = 1, 2, 3, 4) \rangle
\]

\[
E_8(4) = \langle \frac{1}{2}(x^1 - x^3 - x^5 - x^7) + x^2, \frac{1}{2}(x^1 - x^2 + x^5 - x^6) - x^3,
\]
\[
\frac{1}{2}(x^1 - x^2 - x^3 - x^4) - x^7, \frac{1}{2}(x^1 + x^3 - x^6 + x^8) + x^5, 2x^1, ..., 2x^8 >
\]

Fix \( m = 1, 2, 3, 4 \) and set \( L = E_8(m) \). Let \( V_L \) be a lattice VOA constructed as in [FLM2] and \( \theta \) an automorphism of \( V_L \) induced from \( -1 \) on \( L \). Since \( E_8(m) \) contains \( \{2x^1, ..., 2x^8\} \), we obtain a set \( I = \{e^i : i = 1, ..., 16\} \) of 16 mutually orthogonal conformal vectors of \( V_L \), where

\[
e^{2i-j} = \frac{1}{4}x^i(-1)^21 - (-1)^j\frac{1}{4}(\iota(2x^i) + \iota(-2x^i))
\]

for \( i = 1, ..., 8, j = 1, 0 \) as given in [DMZ]. Since they are all in \( V_L^\theta \), we can also take this set as a set of coordinate conformal vectors of \( \tilde{V}_{E_8} \). Hence, the decompositions of \( V_L \) and \( V_{E_8} \) into the direct sum of irreducible \( T \)-submodules are the same, where \( T = \langle e^1, ..., e^{16} \rangle \), (see the proof of Proposition 2.2).

Let \( P(m) = \langle \tau_{e^i} : i = 1, ..., 16 \rangle \) and \( L(m) = E_8(m) \cap L(1) \). By (2.3), \( \tilde{V}_{L(m)} \) contains \( \langle e^1, ..., e^{16} \rangle \) and it is straightforward to check that \( (\tilde{V}_L)^{P(m)} \) coincides with \( \tilde{V}_{L(m)} \). Define a code \( D(m) \) of length 16 by

\[
M_{D(m)} \cong (V_{E_8})^{P(m)}.
\]

It is also not difficult to check that \( (\tilde{V}_L)^{P(m)} \) has a decomposition satisfying Hypotheses I with respect to \( (D(m), D(m)^\perp) \). However, this is not what we want because \( D(m) \) has a root and so \( (M_{D(m)})_1 \neq 0 \) for \( m = 1, 2, 3, 4 \). We are going to get a code \( D \) without roots. In order to find such a decomposition, we will change the set of coordinate conformal vectors. Incidentally, this process coincides with an orbifold construction as we will show.

Let’s explain the relation between the orbifold construction and changing the coordinate sets of conformal vectors. It is known that any orbifold construction from \( V_L \) is isomorphic to itself. Let’s explain the orbifold construction. Let \( \theta \) be an automorphism of \( V_L \) induced from \( -1 \) on \( L \). \( \theta \) fixes \( \iota(x^i) + \iota(-x^i) \) and acts as \( -1 \) on \( x^i(-1)1 \) and \( \iota(x^i) - \iota(-x^i) \). Hence, \( \theta \) acts on \( M_o \) as \( (-1)^{(\iota((\alpha = \{01\})^3))} \) and so the fixed point space \( M_{D(m)}^\theta \) is equal to the direct sum \( \bigoplus_{\alpha \in D(m,+)} M_o \), where \( D(m,+) = \{\alpha \in D(m) : \langle \alpha, \{\{01\}^8\} \rangle = 0\} \). Assume that the twisted part of the orbifold construction does not contain any coset modules. Suppose that \( V = \bigoplus_{\alpha \in S} V^\alpha \) is a VOA satisfying Hypotheses I such that \( \tilde{\tau}(V^\alpha) = \alpha \) and
\(V^{(0^{2n})} \cong M_D\), where \(D\) is a code of length \(2n\) containing \((0^{2i}110^{2n-2i-2})\) for all \(i = 1, \ldots, m\).

Set \(\beta = (\{01\}^n)\).

Then the orbifold construction is corresponding to the following three steps as we will see in the next example.

1. Take an half \(D_{(+)}\) of \(D\), where \(D(+) = \{\alpha \in D : \langle \alpha, \beta \rangle = 0\}\).
2. Take an \(M_{D(+)\text{-}}\)-module \(V^\beta\) with \(\hat{\tau}(V^\beta) = \beta\) and generate \(M_{D(+)\text{-}}\)-modules \(V^{\beta+\gamma}\) with \(\hat{\tau}(V^{\beta+\gamma}) = \beta + \gamma\) by \(V^{\beta+\gamma} = V^\beta \times V^\gamma\) for \(\gamma \in S\).
3. Construct a VOA structure on \(\tilde{V} = \oplus_{\alpha \in \langle S, \beta \rangle} V^\alpha\).

In the case of \(E_8(1)\), \(\hat{\tau}(V_{L(1)+v}) = (1^{16})\) for \(v = \frac{1}{2}(\sum_{i=1}^{16} x^i)\) and so \(S(1) = \langle (1^{16}) \rangle\) and \(D(1)\) is the set of all even words of length 16. \(D(1)\) contains a self dual subcode \(H = H^1_8 \oplus H^2_8\), where \(H^1_8\) are Hamming codes and \(\text{Supp}(H^1_8) = \{1, 2, \ldots, 8\}\) and \(\text{Supp}(H^2_8) = \{9, 10, 16\}\). Since \(\langle ((10)^{8}), \beta \rangle = 0\) for any \(\beta \in H\), we have \(M_H \subseteq V^8_L\). Therefore, the decompositions of \(V_L\) and \(\tilde{V}_L\) as \(M_{H\text{-}}\)-modules are exactly the same. Since \(D(1)\) consists of all even words, the center \(Z(D(1))\) is \(\langle \pm e^{(0^{16})}, \pm e^{(1^{16})} \rangle\) and so there are exactly 2 irreducible \(M_{D(1)\text{-}}\)-modules \(\text{Ind}^{M_{D(1)}}_{M_H}(H(\frac{1}{16}, (0^8)) \otimes H(\frac{1}{16}, (0^8)))\) and \(\text{Ind}^{M_{D(1)}}_{M_H}(H(\frac{1}{16}, (0^8)) \otimes H(\frac{1}{16}, (1^8)))\) by Theorem 3.2. The difference between them is judged by the action of \(q^{(16)} = (q^{(16)}) \otimes e^{(1^{16})}\). By (2.3) and the proof of Proposition 2.2, we have \(q^{(16)} = x^1(-1) \cdots x^8(-1) 1\) and \(x^i(-1)1 = \sqrt{-1}(q^{(2^i)} e^{(2^i-1)} e^{(2^{i+1})})\). Since the eigenvalue of \(q^{(16)}\) on \(\ell(\frac{1}{2} \sum x^i)\) is positive,

\[
V_{E_8} = M_{D(1)} \oplus \text{Ind}^{M_{D(1)}}_{M_H}(H(\frac{1}{16}, (0^8)) \otimes H(\frac{1}{16}, (0^8)))
\]

(5.4)

by the choice of \(E(1)\). We should note that the difference between the above two modules is given by the action of \(q^{(16)} = (\otimes_{i=1}^{16} q^i) \otimes e^{(1^{16})}\). By Lemma 3.8, \(M_{H_8}^\theta\) contains another set of coordinate conformal vectors \(\{f^1, \ldots, f^8\}\) such that \(H(\frac{1}{16}, (0^8))\) w.r.t. \(< e^i >\) is isomorphic to \(H(\frac{1}{2}, \xi_1)\) w.r.t. \(< f^i >\). We note that \(H(\frac{1}{2}, \alpha)\) is a coset module \(M_{H_8+\alpha}\). Take the set \(J = \{f^1, \ldots, f^8, e^9, \ldots, e^{16}\}\) as a new set of coordinate conformal vectors. Then for \(\beta \in D(1)\) satisfying \(\langle \beta, (1^80^8) \rangle = 1\), the \(\hat{\tau}(M_{H+\alpha})\) is also a coset module w.r.t. \(J\) and \(\hat{\tau}(H(\frac{1}{16}, (0^8)) \otimes H(\frac{1}{16}, (0^8)))\) w.r.t. \(J\) is \((0^{8}1^8)\).

Hence, the set \(\hat{\tau}(V_L)\) w.r.t. \(J\) is \(S^2 = \{(0^{16}), (1^{8}0^8), (0^81^8), (1^{16})\}\). Set \(P^2 = \langle \tau_{f_i}, \tau_{e_j} : i = 1, \ldots, 8, j = 9, \ldots, 16 \rangle\) and define a linear code \(D_2\) by \((V_L)^{P^2} \cong M_{D_2}\) w.r.t. \(J\), then \(D_2\) splits into a direct sum \(D_2^1 \oplus D_2^2\) such that \(D_2^1\) and \(D_2^2\) are the sets of all even words in \(\{1, 2, \ldots, 8\}\) and \(\{9, \ldots, 16\}\), respectively. We note that this process is corresponding to an
orthogonal transformation

\[
\begin{pmatrix}
1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{pmatrix}
\]

by (2.3). Therefore, this decomposition coincides with the decomposition given by \(E_8(2)\).

We note that \((1^{16}) \in D_2\) and \(M_{(1^{16})+E}\) w.r.t. \(e^i\) is still equal to \(M_{(1^{16})+E}\) w.r.t. \(J\).

We next consider the case of \(E_8(2)\) and \(S^2 = \langle (1^80^8), (0^81^8) \rangle\). We use the above decomposition again by renaming \(\{f^1, ..., f^8, e^9, ..., e^{16}\}\), \(J\) and \(D_2\) by \(\{e^1, ..., e^{16}\}\), \(I\) and \(D(2)\), respectively. Set

\[
I_1 = \{\alpha \in D(2) : \text{Supp}(\alpha) \subset \{1, ..., 4, 9, ..., 12\}\}
\]

\[
I_2 = \{\alpha \in D(2) : \text{Supp}(\alpha) \subset \{5, ..., 8, 13, ..., 16\}\}
\]

It is clear that \(I_i\) contains Hamming code \(H_i\) for \(i = 1, 2\). Take a new coordinate set \(\{f^1, ..., f^4, f^9, ..., f^{12}\}\) of \(H_1\) and define a new set

\[
J = \{f^1, ..., f^4, e^5, ..., e^8, f^9, ..., f^{12}, e^{13}, ..., e^{16}\}
\]

as a set of coordinate conformal vectors of \(V_L\). Then if an \(M_{H_i} \otimes M_{H_2}\)-module \(U\) has a \(\tau\)-word \((\alpha, \beta) \in \{1, ..., 4, 9, ..., 12\} \oplus \{5, ..., 8, 13, ..., 16\}\) w.r.t. \(I\), then the \(\tau\)-word w.r.t. \(J\) is either \((\alpha, \beta)\) or \((\alpha^e, \beta)\). Moreover, there is a submodule with a \(\tau\)-word \((1^40^41^40^4)\) w.r.t. \(J\). An example is \(M_{H_i \oplus H_2 + \alpha}\), where \(\alpha\) is a word with \(\langle \alpha, (1^40^41^40^4) \rangle = 1\). Therefore, we have

\[
D_3 = \langle D_1^1 \oplus D_1^2 \oplus D_3^3 \oplus D_3^4, \{1, 5, 9, 13\} \rangle
\]

where \(D_3^3\) is the set of all even words in \(\{4i - 3, 4i - 2, 4i - 1, 4i\}\) for \(i = 1, ..., 4\). We also obtain

\[
S^3 = \langle (1^{16}), (1^80^8), (1^40^41^40^4) \rangle
\]

This corresponds to the decomposition with respect to \(E_8(3)\) and \(D_3 = D(3)\). \(D(3)\) also contains two orthogonal Hamming codes \(H_1(3)\) and \(H_2(3)\) whose supports are

\[
\{1, 2, 5, 6, 9, 10, 13, 14\} \quad \text{and} \quad \{3, 4, 7, 8, 11, 12, 15, 16\}.
\]

Repeating the above arguments, we have

\[
S^4 = \langle (1^{16}), (1^80^8), (1^40^41^40^4), \{1^20^2\}^4 \rangle
\]

and \(D(4) = (S^4)_{\perp}^\perp\). \(D(4)\) still contains a direct sum of 2 Hamming codes whose supports are \((\{10\}^8)\) and \((\{01\}^8)\). Repeating the same arguments again, we obtain

\[
S^5 = \langle (1^{16}), (1^80^8), (1^40^41^40^4), (\{1100\}^4), (\{10\}^8) \rangle
\]
and $D(5) = (S^5)^\perp$.

**Remark 2** Since $D(5)$ does not contain a subcode of rank 8 consisting of the form $\{(\alpha, \alpha) : \alpha \in \mathbb{Z}_2^8\}$ for any splits of coordinates into 8 and 8, it is impossible to assign $x^i(-1)1$ to $L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})$ for all $i = 1, \ldots, 8$. Thus, we cannot construct $D(5)$ and $S^5$ from a lattice directly.

Let’s finish the proof of Theorem 4.1. Set $D = D(1)$ and $\beta = (1^80^8)$. Set $H = H_8 \oplus H_8$ as in (5.5). Viewing $V_{E_8}$ as an $M_H$-module, $V_{E_8}$ is a direct sum of distinct irreducible $M_H$-modules. Since $D$ is the set of all even words, $M_D$ contains $H(\frac{1}{2}, \xi_1) \otimes H(\frac{1}{2}, \xi_1)$ and so $V_{E_8}$ has a sub VOA isomorphic to

$$
(H(\frac{1}{2}, (0^8)) \otimes H(\frac{1}{2}, (0^8))) \oplus (H(\frac{1}{2}, \xi_1) \otimes H(\frac{1}{2}, \xi_1))
\oplus (H(\frac{1}{16}, (0^8)) \otimes H(\frac{1}{16}, (0^8))) \oplus (H(\frac{1}{16}, \xi_1) \otimes H(\frac{1}{16}, \xi_1)),
$$

where $\xi_1 = (10^7)$. This is the desired VOA in the proof of Theorem 4.1.

Set $D_{E_8} = D(5)$ and $S_{E_8} = S^5$. We will show that this pair $(D_{E_8}, S_{E_8})$ satisfies the conditions (1) and (2) of Hypotheses I. We note that $D_{E_8}$ is a Reed Muller code $RM(4, 3)$ and $S_{E_8}$ is a Reed Muller code $RM(4, 1)$.

**Lemma 5.1** The pair $(RM(4, 3), RM(4, 1))$ satisfies the condition (1) and (2) of Hypotheses I.

**[Proof]** Set $D = RM(4, 3)$ and $S = RM(4, 1)$. Condition (1) is clear. The weight enumerator of $RM(4, 1)$ is $x^{16} + 30x^8y^8 + y^{16}$. We note that for any $\beta \in RM(4, 1)$ with weight 8, $D_\beta$ and $D_{\beta^c}$ are $[8, 4, 4]$-Hamming codes. We always set $H_\gamma = E_\gamma = D_\gamma$ for $\gamma \in RM(4, 1)$ with weight 8. If $\gamma = (0^{16})$, then set $H_\gamma = E_\gamma = \{(0^{16})\}$. Let $\alpha, \beta \in RM(4, 1)$. We can always choose $H_\beta$, $H_{\alpha+\beta}$, $E_\alpha$ and $E_{\alpha^c}$ satisfying the conditions (2), (2.1) and (2.2) of Hypotheses I.

Assume $\alpha = (0^{16})$ or $(1^{16})$, then $D_\beta = D_{\beta+\alpha}$ or $D_\beta \oplus D_{\beta+\alpha} \subseteq D$. In particular, there is a direct sum $H$ of 2 Hamming codes containing some maximal self orthogonal subcodes $H_\beta$ and $H_{\alpha+\beta}$. Set $E_{(1^{16})} = H$. Clearly, since $E_\alpha + E_{\alpha^c} = H$ and $H_\beta, H_{\beta^c} \subseteq H$, they satisfy the condition (2.3) of Hypotheses I.

We next assume that the weight of $\alpha$ is 8. If $\beta = (0^{16}), (1^{16}), \alpha$ or $\alpha^c$, then set $H_{(1^{16})} = E_\alpha \oplus E_{\alpha^c}$. Then they satisfy (2.3).
The remaining case is that $\alpha, \beta, \alpha + \beta$ have weight 8. Say $\alpha = (1^80^8)$ and $\beta = (1^40^41^40^4)$. We use expressions
\[
Z_2^{16} = \{(\delta_1, \delta_2, \delta_3, \delta_4) : \delta \in \mathbb{Z}_2^4\}.
\]
Clearly, since $E_\gamma = H_\gamma = D_\gamma$ is a Hamming code for $\gamma \in S$ with $|\gamma| = 8$, we have
\[
E_\alpha = \{(\delta0^40^4), (\delta\delta0^40^4) : \delta \in \mathbb{Z}_2^4 \text{ even}\},
\]
\[
E_{\alpha c} = \{(0^40^4\delta0^4), (0^40^4\delta\delta0^4) : \delta \in \mathbb{Z}_2^4 \text{ even}\},
\]
\[
H_{\beta} = \{(\delta0^40^4), (\delta0^4\delta0^4) : \delta \in \mathbb{Z}_2^4 \text{ even}\},
\]
\[
H_{\alpha+\beta} = \{(0^4\delta0^4), (0^4\delta\delta0^4) : \delta \in \mathbb{Z}_2^4 \text{ even}\}.
\]
Since
\[
(0^4\delta0^4) - (\delta0^40^4) = (\delta0^40^4) \quad \text{and} \quad (0^4\delta\delta0^4) - (\delta0^4\delta0^4) + (\delta\delta0^40^4),
\]
we obtain $H_{\alpha+\beta} + E_\alpha = H_{\beta} + E_\alpha$ and so (2.3).

Q.E.D.

**Proposition 5.1** There are 16 mutually orthogonal conformal vectors $\{e^1, ..., e^{16}\}$ in $\tilde{V}_{Es}$ such that the decomposition
\[
V_{Es} = \bigoplus_{\chi \in S_{Es}} V_{Es}^\chi
\]
given by $\{e^1, ..., e^{16}\}$ satisfies Hypotheses I, where

1. the order of $P = \langle \tau_{e^i} : i = 1, ..., 8 \rangle$ is 32,
2. $D_{Es} \cong RM(4, 1), S_{Es} = D_{Es}^\perp$,
3. $(V_{Es})^P = V_{Es}^{(16)}$ is isomorphic to a code VOA $M_{Ds}$,
4. $\tilde{\tau}(V_{Es})^\chi = \chi$.

**[Proof]** We have already shown that there are 16 mutually orthogonal conformal vectors in $\tilde{V}_{Es}$ satisfying the conditions (1) $\sim$ (3). By Lemma 5.1, $(D_{Es}, S_{Es})$ satisfies the conditions (1) and (2) of Hypotheses I. Hence, they satisfy all conditions of Hypotheses I.

Q.E.D.

We next talk about the reverse of the above process. It is clear that we can reverse the process. However, there is another important step. Namely, let
\[
\tilde{V}_{Es} = \bigoplus_{\alpha \in S^n} V^\alpha
\]
be the decomposition such that $V^{(0\uparrow)} \cong M_{D^n}$. Let $\beta$ be an even word so that $< \beta >^\perp \cap S^n \cong S^{n-1}$. Set $\tilde{S}^{n-1} = < \beta > \cap S^n$ and $\tilde{D}^{n-1} = (\tilde{S}^{n-1})^\perp$. Then $V^+ = \bigoplus_{\alpha \in \tilde{S}^n} V^\alpha$ is a sub VOA and the induced VOA

$$\tilde{V}^{n-1} = \text{Ind}_{\tilde{D}^{n-1}} V^+$$

is also a VOA containing $M_{\tilde{D}^{n-1}}$.

At the end of this section, we will explain properties of the automorphisms of a lattice VOA $V_L$ for an even lattice $L$. Let $L_2$ denote the set of all elements of $L$ with squared length 4. As we showed, for any $a \in L_2$, we can define two conformal vectors

$$e^+(a) = \frac{1}{16} a(-1)^2 1 + \frac{1}{4} (\iota(a) + \iota(-a))$$
$$e^-(a) = \frac{1}{16} a(-1)^2 1 + \frac{1}{4} (\iota(a) + \iota(-a)).$$

Then we have :

**Lemma 5.2** $\tau_{e^+(a)} = \tau_{e^-(a)}$ on $V_L$. By setting $\tau_a = \tau_{e^+(a)}$, we obtain $[\tau_a, x(m)] = 0$ and

$$\tau_a : \iota(x) \mapsto (-1)^{(x,a)} \iota(x).$$

In particular, $< \tau_a : a \in L_2 >$ is an elementary abelian 2-subgroup of $\text{Aut}(V_L)$. If $\langle a, b \rangle$ is odd for $a, b \in L_2$, then $\tau_b(e^\pm(a)) = e^\mp(a)$.

**Proof** Since $\langle a, L \rangle \in \mathbb{Z}$ and $\langle a, a \rangle = 4$, $L \subseteq \frac{1}{4} \mathbb{Z}a \oplus \frac{1}{4} < a >^\perp$. In particular, we may view $V_L \subseteq V_{\frac{1}{4} \mathbb{Z}a} \oplus V_{\frac{1}{4} < a >^\perp}$. Recall (2.3)

$$\tau_{e_i^\pm} : \begin{cases} 1 \text{ on } a(-1)1, & \iota((\frac{1}{2} + \mathbb{Z})a), & \iota(\mathbb{Z}a) \\ -1 \text{ on } & \iota((\pm \frac{1}{4} + \mathbb{Z})a) \end{cases}$$

for $i = 1, 2$. Then, $[\tau_{e^\pm(a)}, x(m)] = 1$ and

$$\tau_{e^\pm(a)} : \iota(x) \mapsto (-1)^{(x,a)} \iota(x).$$

Therefore, we obtain the desired results.

Q.E.D.

**Theorem 5.1** For $g \in \text{Aut}(S_{E_8})$, there is an automorphism $\tilde{g}$ of $\tilde{V}_{E_8}$ such that $\tilde{g}(e^i) = e^{g(i)}$. 

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First we note that $S_{E_8}$ is isomorphic to the Reed Muller code $RM(4,1)$, which is defined as follows:

Let $F = \mathbb{Z}_2^4$ be a vector space over $\mathbb{Z}_2$ of dimension 4 and denote $(1000), (0100), (0010), (0001)$ by $v^1, v^2, v^3, v^4$, respectively. Define $\langle (a_i), (b_i) \rangle = \sum a_i b_i$. The coordinate set of Reed Muller code $RM(4,1)$ is the set of all 16 vectors of $F$ and the codewords of $RM(4,1)$ are given by hyperplanes. It is easy to see that $\text{Aut}(RM(4,1)) \cong GL(5,2)$ and it is generated by $g(i): v \in F \rightarrow v + v^i$ for $i \neq j$.

By reversing the above processes, we have a set of mutually orthogonal conformal vectors $\{\bar{e}^1, ..., \bar{e}^{16}\}$ such that $V_{E_8}$ has the following decomposition:

$$V_{E_8} = M_{D^1} \oplus \text{Ind}_{E}^{E}(H(\frac{1}{2},0)H(\frac{1}{2},0)) \text{ w.r.t. } <\bar{e}^i : i = 1, ..., 16>.$$ (5.11)

Here $E = H_8 \oplus H_8$.

Choose $g \in \text{Aut}(S_{E_8})$. It is easy to see that $g \in A_{16}$ and so $g(e^{(16)}) = e^{(16)}$. By Lemma 3.2, we may assume $g \in \text{Aut}(D_{E_8})$. For an $M_{D_{E_8}}$-module $W$, $g(W)$ denotes an $M_{D_{E_8}}$-module defined by $v_n g(u) = (v_n^{g}(u))$ for $v \in M_{D_{E_8}}$ and $u \in W$. Since

$$g(M_{D_{E_8}}) \oplus g(V_{E_8}^a) \oplus g(V_{E_8}^\beta) \oplus g(V_{E_8}^{a+\beta})$$

has a simple VOA structure with a positive definite invariant bilinear form, so does

$$g(V_{E_8}) = \oplus_{a \in S_{E_8}} g(V_{E_8}^a)$$

by Theorem 3.8. We note that $g(V_{E_8})$ contains $M_{D_{E_8}}$. Using the backward processes according to the sequence

$$S^5 = g(S^5) \supseteq g(S^4) \supseteq g(S^3) \supseteq g(S^2) \supseteq g(S^1),$$

we obtain a set $\{\bar{e}^1, ..., \bar{e}^{16}\}$ of mutually orthogonal conformal vectors such that $g(V_{E_8})$ has the decomposition

$$g(V_{E_8}) \cong M_{D^1} \oplus W \text{ w.r.t. } <\bar{e}^i : i = 1, ..., 16>,$$

where $D^1$ is the set of all even words of length 16 and $W$ is an irreducible $M_{D^1}$-module with $\bar{\tau}(W) = (1^{16})$. Since the signs of the actions of $M_{(1^{16})}$ on a module $U$ with $\bar{\tau}(U)$
are changing in each step, we can conclude that $W \cong \text{Ind}^{D_2}_{H} (H(\frac{1}{2}, (0^8)) H(\frac{1}{2}, (0^8)))$, which coincides with (5.11). Therefore, there is a VOA isomorphism

$$\phi : V_{Es} \to g(V_{Es})$$

such that $\phi(e^i) = \bar{e}^i$ for $i = 1, \ldots, 16$. By the process of changing the coordinate sets according to

$$S_1 \supseteq S_2 \supseteq \cdots \supseteq S_5$$

and

$$g(S_1) \subseteq g(S_2) \subseteq \cdots \subseteq g(S_5),$$

respectively, we have the desired automorphism of $V_{Es}$.

Q.E.D.

6 Holomorphic VOA

Let $V$ be a simple VOA containing a set of mutually orthogonal rational conformal vectors $\{e^i : i = 1, \ldots, n\}$ with central charge $\frac{1}{2}$ such that the sum of them is the Virasoro element. Set $P = \langle e^i : i = 1, \ldots, n \rangle$ and let $V = \bigoplus_{\chi \in \text{Irr}(P)} V_{\chi}$ be the decomposition of $V$ into the eigenspaces of $P$. From Proposition 3.1, the space $V^P$ of $P$-invariants is isomorphic to $M_D$ for some even linear code $D$ of length $n$. Assign a binary word $\alpha_\chi = (a_i)$ by $\chi(e^i) = (-1)^{a_i}$ to $\chi$, we can identify $\text{Irr}(P)$ and a linear code $S = \{\alpha_\chi : \chi \in \text{Irr}(P)\}$. As we showed in [Mi5], $S$ is orthogonal to $D$. We will treat the case $S = D^\perp$ in this section.

**Theorem 6.1** If $S = D^\perp$, then $V$ is the only irreducible $V$-module.

**[Proof]** Let $U$ be an irreducible $V$-module. Since $M_D$ is rational, $U$ is a direct sum of irreducible $M_D$-modules. Decompose $U$ into the direct sum $\oplus U^\beta$ of $M_D$-modules such that $\bar{\tau}(U^\beta) = \beta$. Since $U^\beta$ is a $M_D$-module, $\beta \in D^\perp = S$ and so $V^\beta \neq 0$. Since $U = \langle v_n u : v \in V^\alpha, n \in \mathbb{Z}, \alpha \in S \rangle$ for any $0 \neq u \in U^\beta$ by [DM2], $U^\beta = \langle v_n u : v \in M_D, n \in \mathbb{Z} \rangle$ and so $U^\beta$ is irreducible $M_D$-module. Since the restriction

$$I \begin{pmatrix} U & V \\ V & U \end{pmatrix} \to I \begin{pmatrix} U & V \\ V^\beta & U^\beta \end{pmatrix}$$

is injective, $U^{(0^\alpha)} \neq 0$. So we may assume $\beta = (0^\alpha)$. Hence $U^\beta$ is isomorphic to a coset module $M_{D+\alpha}$ for some word $\alpha \in \mathbb{Z}_2^n$. Using the skew symmetry, we can define a nonzero
intertwining operator \( I(v, z) \in I_{MD} \begin{pmatrix} U & V \\ U & V \end{pmatrix} \) with integer powers of \( z \). By restricting it to \( U^\beta \), we have a nonzero intertwining operator \( I^\gamma(v, z) \in I_{MD} \begin{pmatrix} U^\gamma & V^\gamma \\ M_{\alpha+D} & V^\gamma \end{pmatrix} \) for \( \gamma \in S \). Since its vertex operator has integer powers of \( z \), \( \alpha \) is orthogonal to \( S \) and so \( \alpha \in S(P)^\perp = D \). Hence \( U^{(0^n)} \) is isomorphic to \( M_D \). Let \( q \) be a highest weight vector of \( U^{(0^n)} \) corresponding to the Vacuum. Since \( L(-1)q = 0 \), \( I(q, z) \) is a scalar and so \( I(q, z) \in I \begin{pmatrix} U & V \\ U & V \end{pmatrix} \) gives an \( M_D \)-isomorphism of \( U \) to \( V \). This completes the proof of Theorem 6.1.

Q.E.D.

7 Construction of the moonshine VOA

In this section, we will construct a VOA \( V^2 \), which will be proved to be equal to the moonshine VOA constructed in [FLM2] in the next section. In the section 5, we found a set of 16 mutually orthogonal conformal vectors \( \{e^i : i = 1, ..., 16\} \) of \( V_E8 \) satisfying the following conditions:

1. \( D_{Es} = RM(4, 2) \)
2. \( P = < \tau_{e^i} : i = 1, ..., 16 > \) has the order \( 2^5 \).
3. \( V^P \cong M_{D_{Es}} \) and \( S_{Es} = D_{Es}^\perp \) is generated by \( \{(116), (0^81^8), (\{0^41^4\}^2), (\{0^21^2\}^4), (\{01\}^8)\} \).

To simplify the notation, we denote \( D_{Es} \) and \( S_{Es} \) by \( D \) and \( S \) in this section, respectively. We note that \( D \) and \( S \) are \( D(5) \) and \( S^5 \) in the section 4, respectively. For each codeword \( \alpha \in S \), \( V_{Es} \) contains an irreducible \( M_D \)-module \( V_{Es}^\alpha \) such that

\[
V_{Es} \cong \bigoplus_{\alpha \in S} V_{Es}^\alpha
\]

and \( V_{Es}^{(0^{16})} = M_D \). Since \( V_{Es} \) is a simple VOA, Theorem 3.6 implies

\[
V_{Es}^\alpha \times V_{Es}^\beta = V_{Es}^{\alpha+\beta}
\]

for \( \alpha, \beta \in S \).

We note that all codewords of \( S \) except \( (0^{16}) \) and \( (1^{16}) \) have weight 8. We define a new code \( S^3 \) of length 48 by

\[
S^3 = < (1^{16}0^{16}0^{16}), (0^{16}1^{16}0^{16}), (0^{16}0^{16}1^{16}), (\alpha, \alpha, \alpha) : \alpha \in S > .
\]
The weight enumerator of $S^\natural$ is $X^{48} + 3X^{32} + 120X^{24} + 3X^{16} + 1$ and there is another expression:

$$S^\natural = \{(\alpha, \alpha, \alpha), (\alpha, \alpha, \alpha^c), (\alpha, \alpha^c, \alpha), (\alpha^c, \alpha, \alpha) : \alpha \in S\}. \quad (7.4)$$

Set $D^\natural = (S^\natural)^\perp$ and call it “the moonshine code.” Let’s explain our choice of the codes $D^\natural$ and $S^\natural$. We may be able to construct the moonshine VOA from another pair $(D', S')$, but $(D^\natural, S^\natural)$ is very easy to handle when we calculate the characters of the elements of the Monster. Let’s continue the construction. $D^\natural$ contains $D^3_3 = \{(\alpha, \beta, \gamma) : \alpha, \beta, \gamma \in D, \alpha, \beta, \gamma \text{ is even}\}$. (7.5)

Hence $D^\natural$ is of dimension 41 and has no codewords of weight 2. We note that a pair $(D^3_3, S^\natural)$ satisfies the conditions (1) and (2) in Hypotheses I. Denote (1015) by $\xi_1$ and set

$$Q = \langle (\xi_1 0^{16}), (0^{16} \xi_1 0) \rangle. \quad (7.6)$$

To simplify the notation, let $R$ denote a coset module $M_{\xi_1 + D}$ and $RW$ denote the fusion product (tensor product) $R \times W$. As we explained in the introduction, our construction consists of the following steps.

At first, $V_{E_8} \otimes V_{E_8} \otimes V_{E_8}$ contains a set of 48 coordinate conformal vectors

$$\{e^i \otimes 1 \otimes 1, \ 1 \otimes e^j \otimes 1, \ 1 \otimes 1 \otimes e^k : i, j, k = 1, ..., 16\},$$

where 1 is the Vacuum of $V_{E_8}$. Decompose it into

$$V_{E_8} \otimes V_{E_8} \otimes V_{E_8} = \bigoplus_{\alpha, \beta, \gamma} (V_{E_8}^\alpha \otimes V_{E_8}^\beta \otimes V_{E_8}^\gamma), \quad (7.7)$$

By the fusion rules,

$$V^1 = \bigoplus_{(\alpha, \beta, \gamma) \in S^2} (V_{E_8}^\alpha \otimes V_{E_8}^\beta \otimes V_{E_8}^\gamma) \quad (7.8)$$

is a sub VOA. Let’s induce it to

$$V^2 = \text{Ind}_{D^3_3 + Q}^{D^3_3 + Q}(V^1). \quad (7.9)$$

We note that since $\langle Q, S^2 \rangle \neq 0$, a vertex operator of some element in $V^2$ does not have integer powers of $z$. In particular, $V^2$ is not a VOA. However, as $M_{D^3_3}$-modules, we have

$$\text{Ind}_{D^3_3 + Q}^{D^3_3 + Q}(V_{E_8}^\alpha \otimes V_{E_8}^\beta \otimes V_{E_8}^\gamma) = (V_{E_8}^\alpha \otimes V_{E_8}^\beta \otimes V_{E_8}^\gamma) \oplus (RV_{E_8}^\alpha \otimes RV_{E_8}^\beta \otimes V_{E_8}^\gamma) \oplus (V_{E_8}^\alpha \otimes RV_{E_8}^\beta \otimes RV_{E_8}^\gamma).$$
Therefore we may assume that $\chi \in \mathcal{S}$ and $\mu$ is definite. Then, it is sufficient to prove that $\langle \chi, \mu \rangle = 0$ for any $\mu$. To prove this, we will use Theorem 3.3. \[
abla \chi = \text{Ind}_{D^3}^D(W^\chi) \] for $\chi \in S^3$. Finally, set $V^\chi = \bigoplus_{\chi \in S^3}(V^\chi)\] and $V^\mu = \bigoplus_{\mu \in S^3}(V^\mu)$. \[V^\chi = \bigoplus_{\chi \in S^3}(V^\chi)\]

This is the desired Fock space.

Since $(D^3, S^3)$ satisfies the conditions (1) and (2) of Hypotheses I, the remaining thing we have to do is to prove that $V^\chi = M_{D^3} \oplus V^\chi \oplus V^\mu \oplus V^\chi + \mu$ has a simple VOA structure with a positive definite invariant bilinear form for any $\mu, \chi \in S^3$ with $\dim < \mu, \chi > = 2$. We note that since $M_{D^3} \oplus W^{(\alpha, \alpha, \alpha)}$ and $M_{D^3} \oplus W^{(\alpha, \alpha, \alpha^c)}$ are sub VOAs of $\text{Ind}_{D^3}^{D^3}(\langle \xi_1 \xi_0^16 \rangle)(M_{D^3} \oplus W^{(\alpha, \alpha, \alpha)})$, they have simple VOA structures with positive definite invariant bilinear forms. Take a sub VOA $W^{(\alpha, \alpha, \alpha^c)}$ of $\mathcal{V}$ and set $W^\chi = M_{D^3} \oplus (V^1)^\chi \oplus (V^1)^\mu \oplus (V^1)^{\chi + \mu}$ using (7.10). If $\langle \chi, \mu \rangle$ is orthogonal to $(\xi_1 \xi_0^16)$, then $\text{Ind}_{D^3}^{D^3+<\xi_1 \xi_0^16>}(V^1)^{\chi + \mu}$ is a VOA with the desired properties. Moreover it contains $W^\chi$ as a sub VOA. Similarly, if $\langle \chi, \mu \rangle$ is orthogonal to $(0^16 \xi_1)$ or $(\xi_0^16 \xi_1)$, then we have the desired properties. Therefore we may assume that $\chi = (\alpha, \alpha^c)$ and $\mu = (\beta, \beta^c, \beta)$. Set $\gamma = \alpha^c + \beta$ and assume that $\text{Supp}(\alpha) \cap \text{Supp}(\beta) \neq \emptyset$. Choose $t \in \text{Supp}(\alpha) \cap \text{Supp}(\beta)$. Then $t \in \text{Supp}(\gamma)$. Set $\xi_t = (0^110^{15}1-t)$ and $R^t = M_{D^3+\xi_t}$. Since $$(\xi_t \xi_0^16) + (\xi_1 \xi_0^16) \in D^3, \quad (\xi_t^16 \xi_t) + (\xi_1^16 \xi_1) \in D^3, \quad (0^16 \xi_t^1) + (0^16 \xi_1^1) \in D^3,$$
we have

\[ \text{Ind}_{D_3}^D(R^tV_{Es} \alpha \otimes R^tV_{Es} \alpha \otimes V_{Es} \alpha^\times) = \text{Ind}_{D_3}^D(RV_{Es} \alpha \otimes RV_{Es} \alpha \otimes V_{Es} \alpha^\times), \]
\[ \text{Ind}_{D_3}^D(R^tV_{Es} \beta \otimes V_{Es} \beta^\times \otimes R^tV_{Es} \beta) = \text{Ind}_{D_3}^D(RV_{Es} \beta \otimes V_{Es} \beta^\times \otimes RV_{Es} \beta), \]
\[ \text{Ind}_{D_3}^D(V_{Es} \gamma^\times \otimes R^tV_{Es} \gamma^\times \otimes R^tV_{Es} \gamma) = \text{Ind}_{D_3}^D(V_{Es} \gamma^\times \otimes RV_{Es} \gamma \otimes RV_{Es} \gamma). \]

Set

\[ \gamma_1 = (\xi_0 t_{016}), \quad \gamma_2 = (\xi_0 t_{016} \xi_0), \quad \gamma_3 = (0_{16} \xi_0 \xi_0). \]

Since \( \text{Supp}(\gamma_1) \subseteq \text{Supp}(\chi), \text{Supp}(\gamma_2) \subseteq \text{Supp}(\mu) \) and \( \text{Supp}(\gamma_3) \subseteq \text{Supp}(\chi + \mu) \), it follows from Lemma 3.13 that

\[ R^t(V_{Es})^\alpha \otimes (V_{Es})^{\alpha^\times} \cong \sigma_{\gamma_1}(V^{(1)}(\alpha, \alpha, \alpha^\times)), \]
\[ R^t(V_{Es})^\beta \otimes (V_{Es})^{\beta^\times} \otimes R^t(V_{Es})^\beta \cong \sigma_{\gamma_2}(V^{(1)}(\beta, \beta^\times, \beta)), \]
\[ (V_{Es})^{\gamma^\times} \otimes R^t(V_{Es})^\gamma \otimes (V_{Es})^\gamma \cong \sigma_{\gamma_3}(V^{(1)}(\gamma, \gamma, \gamma)). \]

Since \( M_{D_3} \oplus (V^{(1)}(\alpha, \alpha, \alpha^\times) \oplus (V^{(1)}(\beta, \beta^\times, \beta) \oplus (V^{(1)}(\gamma, \gamma, \gamma)) \) has a simple VOA structure with a positive definite invariant bilinear form, so does \( M_{D_3} \oplus \sigma_{\gamma_1}(V^{(1)}(\alpha, \alpha, \alpha^\times) \oplus \sigma_{\gamma_2}(V^{(1)}(\beta, \beta^\times, \beta) \oplus \sigma_{\gamma_3}(V^{(1)}(\gamma, \gamma, \gamma)). \)

Hence \( W^{\alpha, \beta} = M_{D_3} \oplus W^{(\alpha, \alpha, \alpha^\times) \oplus W^{(\beta, \beta^\times, \beta) \oplus W^{(\gamma, \gamma, \gamma) \) has the desired VOA structure and so does \( (V^{(1)} \chi, \mu). \)

Hence we assume \( \text{Supp}(\alpha) \cap \text{Supp}(\beta) = \emptyset. \) Then one of \( \{\alpha, \beta, \alpha + \beta^c\} \) is at least \( (0_{16}) \) since \( \alpha, \beta \in S. \) Set \( \gamma = \alpha + \beta^c. \) So we may assume \( \alpha = (0_{16}) \) and \( \gamma^c = \beta. \) It follows from the structure of \( D \) that there is a self dual subcode \( E \) of \( D^{(1)} \) which is a direct sum \( \bigoplus_{i=1}^6 E^i \) of \( 6 \{8, 4, 4\}-\text{Hamming codes} E^i \) such that \( E_\delta = \{\alpha \in E| \text{Supp}(\alpha) \subseteq \text{Supp}(\delta)\} \) is a direct factor of \( E \) for any \( \delta \in < \beta, \gamma >. \) In particular, there are \( M_E \)-modules \( U^\alpha, U^\beta, U^\gamma \) such that

\[ \text{Ind}_{E}^{D_3}(U^\alpha) = (V^{(1)}(\alpha, \alpha, \alpha^\times), \]
\[ \text{Ind}_{E}^{D_3}(U^\beta) = (V^{(1)}(\beta, \beta^\times, \beta)), \]
\[ \text{Ind}_{E}^{D_3}(U^\gamma) = (V^{(1)}(\gamma, \gamma, \gamma)). \]

In the following, we assume \( |\beta| = 8. \) We can prove the assertion for \( \beta = (0_{16}) \) or \( \beta = (1_{16}) \) by the similar arguments. We may assume \( \beta = (1_{16}0^8). \) As we showed in §5, we have a VOA \( U = V_{Es}^{(0_{16})} \oplus V_{Es}^{(1_{16}0^8)} \oplus V_{Es}^{(0^818)} \oplus V_{Es}^{(1_{16})} \) with a positive definite invariant bilinear form such that

\[ V_{Es}^{(0_{16})} = \text{Ind}_{F}^{D_3}(H(1_2, (0^8)) \otimes H(1_2, (0^8))) \]
\[ V_{Es}^{(1_{16}0^8)} = \text{Ind}_{F}^{D_3}(H(1_2, (0^8)) \otimes H(1_2, (0^8))) \]
\[ V_{Es}^{(0^818)} = \text{Ind}_{F}^{D_3}(H(1_2, (0^8)) \otimes H(1_2, (0^8))) \]
\[ V_{Es}^{(1_{16})} = \text{Ind}_{F}^{D_3}(H(1_2, (0^8)) \otimes H(1_2, (0^8))), \]

where \( F = D_{(1_{16}0^8)} \oplus D_{(0^818)} \) is a direct sum of two Hamming codes. In order to simplify the notation, we omit the notation "\( \otimes \)" between \( H(\ast, \ast) \) and \( H(\ast, \ast). \) In particular,
\[ \bar{U} = H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8)) \oplus H(\frac{1}{10}, \xi_1)H(\frac{1}{2}, \xi_1) \oplus H(\frac{1}{2}, \xi_1)H(\frac{1}{10}, \xi_1) \oplus H(\frac{1}{10}, (0^8))H(\frac{1}{10}, (0^8)) \]

has a VOA structure with a positive definite invariant bilinear form. Since \( \bar{W}^{(\alpha, \alpha')} \) is given by \( RV_{E_8}^{\alpha} \otimes RV_{E_8}^{\alpha} \otimes V_{E_8}^{\alpha'}, \)

\[ U^\alpha = H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, (0^8))H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, (0^8))H(\frac{1}{10}, (0^8))H(\frac{1}{10}, (0^8)). \]

We similarly obtain

\[ U^\beta = H(\frac{1}{10}, (0^8))H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, (0^8))H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8)), \]

\[ U^\gamma = H(\frac{1}{10}, \xi_1)H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8))H(\frac{1}{2}, \xi_1). \]

By changing the order of the components, \((123456) \rightarrow (243516)\), we have

\[
\begin{align*}
M_E &= H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8)), \\
U^\alpha &= H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8))H(\frac{1}{2}, (0^8)), \\
U^\beta &= H(\frac{1}{2}, (\xi_1))H(\frac{1}{10}, (\xi_1))H(\frac{1}{2}, (\xi_1))H(\frac{1}{2}, (\xi_1))H(\frac{1}{2}, (\xi_1))H(\frac{1}{2}, (\xi_1)), \\
U^\gamma &= H(\frac{1}{10}, (\xi_1))H(\frac{1}{2}, (\xi_1))H(\frac{1}{2}, (\xi_1))H(\frac{1}{2}, (\xi_1))H(\frac{1}{2}, (\xi_1))H(\frac{1}{2}, (\xi_1)).
\end{align*}
\]

By Lemma 3.8, there is another coordinate set of conformal vectors \( \{d^1, \ldots, d^8\} \) in \( M_{H_8} \) such that

\[
\begin{align*}
H(\frac{1}{2}, (\xi_1)) \text{ w.r.t. } < e^i > &\cong H(\frac{1}{10}, (0^8)) \text{ w.r.t. } < d^i > \\
H(\frac{1}{10}, (\xi_1)) \text{ w.r.t. } < e^i > &\cong H(\frac{1}{2}, (\xi_1)) \text{ w.r.t. } < d^i > \\
H(\frac{1}{10}, (0^8)) \text{ w.r.t. } < e^i > &\cong H(\frac{1}{10}, (\xi_1)) \text{ w.r.t. } < d^i > .
\end{align*}
\]

Changing the coordinate sets, we have

\[
\begin{align*}
\tilde{V}_{E_8}^{(0^8)} &= \text{Ind}_F^\mathbb{E}(H(\frac{1}{2}, (0^8)) \oplus H(\frac{1}{2}, (0^8))) \\
\tilde{V}_{E_8}^{(\xi^8)} &= \text{Ind}_F^\mathbb{E}(H(\frac{1}{2}, (\xi_1)) \otimes H(\frac{1}{10}, (0^8))) \\
\tilde{V}_{E_8}^{(0^8\xi^8)} &= \text{Ind}_F^\mathbb{E}(H(\frac{1}{16}, (0^8)) \otimes H(\frac{1}{2}, (\xi_1))) \\
\tilde{V}_{E_8}^{(116)} &= \text{Ind}_F^\mathbb{E}(H(\frac{1}{16}, (\xi_1)) \otimes H(\frac{1}{16}, (\xi_1))).
\end{align*}
\]

with respect to \( \{d^1, \ldots, d^8, d^9, \ldots, d^{10}\} \). Therefore, \( \bar{U} = M_E \oplus U^\alpha \oplus U^\beta \oplus U^\gamma \) is a subset of a VOA \( \tilde{V}_{E_8} \otimes \tilde{V}_{E_8} \otimes \tilde{V}_{E_8} \). It is also easy to check that \( \bar{U} \) is closed under the products. Hence, \( \bar{U} \) is a VOA with a positive definite invariant bilinear form and so does \( (V^*)^{\alpha, \beta} = \text{Ind}_E^D(\bar{U}) \).

This completes the construction of \( V^2 \).

Q.E.D.

**Corollary 7.1** \( V^2 \) has a positive definite invariant bilinear form.
Remark 3 Because of our construction, a VOA satisfying Hypotheses I is a direct sum of the tensor product of \( L(\frac{1}{2},0), L(\frac{1}{2}, \frac{1}{2}), L(\frac{1}{2}, \frac{1}{16}) \) and we know the multiplicities of irreducible \( L(\frac{1}{2},0)^{\otimes n} \)-modules by Theorem 3.2, (c.f. Corollary 5.2 in [Mi3]). Hence it is not difficult to calculate its character
\[
\chi_V(z) = e^{2\pi i z/(\text{rank}(V))} \left( \sum_{n=0}^{\infty} \dim V_n e^{2\pi i z n} \right).
\]

For example, let’s show \( (V^2)_1 = 0 \). We first have \( (M_{D^2})_1 = 0 \) since \( D^2 \) has no codewords of weight 2. Also, if \( (V^2)_1 \neq 0 \), then the weight of \( \chi \) is equal to 16 and so \( \chi \) is one of \((161016016), (01616016) \) or \((01601616) \). Say \( \chi = (161016016) \). Since \( (V^2)^{\chi} = \text{Ind}_{D^2_E}^{D} (V^{16}_E \otimes M_{D_E} \oplus M_{D_E} \oplus \xi_1) \) and \( D^2 \) does not contains any words of the form \((\alpha, \xi_1, \xi_1) \), the minimal weight of \( (V^2)^{\chi} \) is greater than 1. Therefore, we obtain \( V_1^2 = 0 \).

8 Conformal vectors

Since each rational conformal vector \( e \in V \) with central charge \( \frac{1}{2} \) offers an automorphism \( \tau_e \), it is very important to find such conformal vectors for studying the automorphism group \( \text{Aut}(V) \). Therefore, we will construct several conformal vectors of \( V^2 \) explicitly.

8.1 Case I

Set \( D_1 = \langle H_8 \oplus H_8, (\xi_1, \xi_1) \rangle \) and \( S = \langle (16) \rangle \), where \( \xi_1 = (10^7) \). Then the pair \((D_1, S)\) satisfies the conditions (1) and (2) of Hypotheses I. Set
\[
U = H(\frac{1}{2}, 0)H(\frac{1}{2}, 0) \oplus H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1) \oplus H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0) \oplus H(\frac{1}{16}, 0)H(\frac{1}{16}, \xi_1).
\]

\( U \) is a sub VOA of \( V_{E_8} \). It is easy to see that \( \dim(H(\frac{1}{2}, 0)H(\frac{1}{2}, 0))_1 = 0 \) and \( \dim(H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1))_1 = \dim(H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0))_1 = \dim(H(\frac{1}{16}, 0)H(\frac{1}{16}, \xi_1))_1 = 1 \). Hence \( U_1 \) is isomorphic to \( \text{sl}(2) \). Viewing \( (H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1))_1 \) as a Cartan subalgebra of \( \text{sl}(2) \), \( H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0) \oplus H(\frac{1}{16}, 0)H(\frac{1}{16}, \xi_1) \) contains two roots \( \iota(x) \) and \( \iota(-x) \). Take a sub lattice VOA of type \( A_1 \) generated by \( U_1 \), we may obtain the following elements:
\[
x(-1)1 \in (H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1))_1,
\iota(x) + \iota(-x) \in (H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0))_1, \text{ and }
\iota(x) - \iota(-x) \in (H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0))_1.
\]

Take another copy of them and set
\[
y(-1)1 \in (H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1))_1,
\iota(y) + \iota(-y) \in (H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0))_1, \text{ and }
\iota(y) - \iota(-y) \in (H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0))_1.
\]
Then we have
\[\iota(\pm x) \otimes \iota(\pm y) + \iota(\mp x) \otimes \iota(\mp y) \in H(\frac{1}{16}, 0)H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0)H(\frac{1}{16}, \xi_1) + H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0)H(\frac{1}{16}, \xi_1)H(\frac{1}{16}, 0),\]
\[x(-1)y(-1) \in H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1), \text{ and}\]
\[x(-1)^2 \mathbf{1}, \ y(-1)^2 \mathbf{1} \in H(\frac{1}{2}, 0)H(\frac{1}{2}, 0)H(\frac{1}{2}, 0)H(\frac{1}{2}, 0).\]
Since \(x \pm y, x \pm y = 2, e^+(x \pm y) = \frac{1}{16}(x \pm y)(-1)^2 \mathbf{1} + \frac{1}{4}(\iota(x \pm y) + \iota(-x \mp y))\) and \(e^-(x \pm y) = \frac{1}{16}(x \pm y)(-1)^2 \mathbf{1} - \frac{1}{4}(\iota(x \pm y) + \iota(-x \mp y))\) are rational conformal vectors with central charge \(\frac{1}{2}\). Hence, we obtain four rational conformal vectors \(e^\pm(x \pm y)\) in
\[H(\frac{1}{2}, 0)H(\frac{1}{2}, 0)H(\frac{1}{2}, 0)H(\frac{1}{2}, 0) \oplus H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1)H(\frac{1}{2}, \xi_1) \oplus H(\frac{1}{16}, 0)H(\frac{1}{16}, 0)H(\frac{1}{16}, 0)H(\frac{1}{16}, 0).\]

### 8.2 Case II

We first treat the first component \(V_{E_8} \otimes \mathbf{1} \otimes \mathbf{1}\) of \(V_{E_8} \otimes V_{E_8} \otimes V_{E_8}\). We denote \(D_{E_8}, S_{E_8}\) and \(V_{E_8}\) by \(D, S, V\) here, respectively. Let \(\alpha, \beta \in S\) so that \(|\alpha| = |\beta| = |\alpha + \beta|\).

By rearranging the coordinate sets, we may assume \(\alpha = (1^80^8), \beta = (1^40^41^40^4)\). As we showed, \(V\) contains a sub VOA
\[U = M_D \oplus V^\alpha \oplus V^\beta \oplus V^{\alpha+\beta}\]
for \(\alpha, \beta \in S\). Since \(D_\alpha, D_\beta\) and \(D_{\alpha+\beta}\) are all isomorphic to \(H_8\), the multiplicities of the irreducible \(L(\frac{1}{2}, 0)^\otimes 8\)-modules in \(V^\alpha \oplus V^\beta \oplus V^{\alpha+\beta}\) are all one by Theorem 3.2. Hence \(\text{dim}(V^\alpha)_1 = \text{dim}(V^\beta)_1 = \text{dim}(V^{\alpha+\beta})_1 = 8\). Since \(D\) does not contain any words of weight 2, \((M_D)_1 = 0\) and so \((V^\alpha)_1, (V^\beta)_1\) and \((V^{\alpha+\beta})_1\) are all commutative Lie algebras. Since \(U\) is a sub VOA of a lattice VOA \(V\) of rank 8 and so \(U\) is isomorphic to \(sl(2)^8\). Viewing \((V^{\alpha+\beta})_1\) as a Cartan subalgebra and embedding it into a lattice VOA \(V_{A_1^8}\) of root lattice \(A_1^8\), we denote the positive roots by \(\iota(x_1), ..., \iota(x_8)\) and the negative roots by \(\iota(-x_1), ..., \iota(-x_8)\). In addition, we may assume
\[x_i(-1) \in V_1^{\alpha+\beta}\]
\[\iota(x_i) + \iota(-x_i) \in V_1^{\alpha}\]
\[\iota(x_i) - \iota(-x_i) \in V_1^{\beta}\]
for \(i = 1, ..., 8\).

We next treat the second and third components of \(V_{E_8} \otimes V_{E_8} \otimes V_{E_8}\). By the similar arguments as in the construction of the moonshine VOA, \(M_{D^2} \oplus W^{(\alpha, \alpha)} \otimes W^{(\beta, \beta)} \oplus W^{(\alpha+\beta, \alpha+\beta)}\) has a simple VOA structure, where \(W^{(\alpha, \alpha)} = RV_{E_8}^{\alpha} \otimes RV_{E_8}^{\alpha}, W^{(\beta, \beta)} = RV_{E_8}^{\beta} \otimes RV_{E_8}^{\beta}\) and
$W^{(\alpha+\beta,\alpha+\beta)} = V_{E_8}^{\alpha+\beta} \otimes V_{E_8}^{\alpha+\beta}$. Set $F = \{ (\alpha', \beta') : \alpha' + \beta' \in D, \alpha', \beta' \text{ even} \}$. Then $M_F$ does not contain any roots and $D \oplus F \subseteq D^2$. Set $U_{\gamma, \gamma} = \text{Ind}_{M_{D_2}}^{M_F} (W_{\gamma, \gamma})$ for $\gamma \in \{ \alpha, \beta, \alpha + \beta \}$. By Theorem 3.7, we have a VOA $U = M_F \oplus U_{\alpha, \alpha} \otimes U_{\beta, \beta} \oplus U_{\alpha + \beta, \alpha + \beta}$. Since $|F_{(\alpha, \alpha)}| = |F_{(\beta, \beta)}| = |F_{((\alpha + \beta), (\alpha + \beta))}| = 2^{11}$, the multiplicities of irreducible $L(\frac{\lambda}{2}, 0)^{\otimes 16}$-submodules is 8. Hence, $\dim(U_{\gamma, \gamma}) = 8$ for $\gamma \in \{ \alpha, \beta, \alpha + \beta \}$. Set $U_{\alpha, \beta} = \text{Ind}_{M_{D_2}}^{M_F} (R_{E_8}^{(16)} \otimes R_{E_8}^{(16)})$, $U_{\alpha, \alpha} = \text{Ind}_{M_{D_2}}^{M_F} (V_{E_8}^{\alpha} \otimes V_{E_8}^{\alpha})$, $U_{\beta, \beta} = \text{Ind}_{M_{D_2}}^{M_F} (V_{E_8}^{\beta} \otimes V_{E_8}^{\beta})$ and $U_{\alpha + \beta, \alpha + \beta} = \text{Ind}_{M_{D_2}}^{M_F} (R_{E_8}^{\alpha+\beta} \otimes R_{E_8}^{\alpha+\beta})$. Then, $X = M_F \oplus U_{(16)}^{(16)} \oplus U_{\alpha, \alpha} \oplus U_{\beta, \beta} \oplus U_{\alpha + \beta, \alpha + \beta} \oplus U_{\beta, \beta} \oplus U_{\alpha + \beta, \alpha + \beta}$ has a VOA structure. Since $(M_F \oplus U_{(16)}^{(16)} )_1 = 0$, $(U_{\alpha + \beta, \alpha + \beta} \oplus U_{\alpha + \beta, \alpha + \beta})_1$ is of dimension 16. Since $X$ is a sub VOA of a lattice VOA $V$ of rank 16, $X_1$ is isomorphic to $sl(2)^{16}$ and $U_1$ is isomorphic to $sl(2)^8$. Viewing $(U_{\alpha + \beta, \alpha + \beta})_1$ as a Cartan subalgebra and embedding them in a lattice VOA $V_{A_1}$ of the root lattice $A_1$, we denote the positive roots by $u(y_1), \ldots, u(y_8)$ and the negative roots by $u(-y_1), \ldots, u(-y_8)$. We may also assume that

\[
y_i(-1) \in (U_{\alpha + \beta, \alpha + \beta})_1
\]

for $i = 1, \ldots, 8$.

Set

\[
W^\alpha = V_{E_8}^{\alpha} \otimes U_{\alpha, \alpha}, \quad W^\beta = V_{E_8}^{\beta} \otimes U_{\beta, \beta} \quad \text{and} \quad W^{\alpha + \beta} = V_{E_8}^{\alpha + \beta} \otimes U_{\alpha + \beta, \alpha + \beta}.
\]

Then

\[
V^1 = M_{D \oplus F} \oplus W^\alpha \oplus W^\beta \oplus W^{\alpha + \beta}
\]

is a sub VOA of $V^2$. We have

\[
\begin{align*}
x_i(-1)^2 & \in M_D, \\
y_i(-1)^2 & \in M_F, \\
x_i(-1)y_i(-1) & \in W^{\alpha + \beta}, \\
(u(x^i) + u(-x^i))(u(y^i) + u(-y^i)) & \in W^\alpha \quad \text{and} \\
(u(x^i) - u(-x^i))(u(y^i) - u(-y^i)) & \in W^\beta.
\end{align*}
\]

By the same arguments as in the case I, we have 32 mutually orthogonal conformal vectors

\[
\begin{align*}
d^{4i-3} & = \frac{1}{16}(x^i + y^i)(-1)^2 1 + \frac{1}{4}(u(x^i + y^i) + u(-x^i - y^i)) \\
d^{4i-2} & = \frac{1}{16}(x^i + y^i)(-1)^2 1 - \frac{1}{4}(u(x^i + y^i) + u(-x^i - y^i)) \\
d^{4i-1} & = \frac{1}{16}(x^i - y^i)(-1)^2 1 + \frac{1}{4}(u(x^i - y^i) + u(-x^i + y^i)) \\
d^{4i} & = \frac{1}{16}(x^i - y^i)(-1)^2 1 - \frac{1}{4}(u(x^i - y^i) + u(-x^i + y^i))
\end{align*}
\]

in $V^1$, where $u(x^i + y^i)$ denotes $u(x^i) \otimes u(y^i)$. 

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9 Automorphism group

In this section, we will prove that the full automorphism group of $V^\natural$ is the Monster simple group. We first quote the following two theorems about the finiteness of automorphism group from [Mi4].

Hypotheses II
(1) $V = \sum_{i=0}^{\infty} V_i$ is a VOA over $\mathbb{R}$.
(2) $\dim V_0 = 1$.
(3) $V_1 = 0$.
(4) $V$ has a positive definite invariant bilinear form $\langle \cdot, \cdot \rangle$.
(5) The Virasoro element is a sum of mutually orthogonal conformal vectors with central charge $\frac{1}{2}$.

Under the above Hypotheses II, we recall the following results from [Mi4].

**Theorem 9.1** Let $e, f$ be two distinct conformal vectors with central charge $\frac{1}{2}$. Then we have
\[ \langle e, f \rangle \leq \frac{1}{12} \quad \text{and} \quad \langle e - f, e - f \rangle \geq \frac{1}{3}. \]
In particular, there are only finitely many conformal vectors with central charge $\frac{1}{2}$.

**Proof** Using the product $ab = a_1 b$ and the inner product $\langle a, b \rangle 1 = a_3 b$ for $a, b \in V_2$, $V_2$ becomes a commutative algebra called Griess algebra. Let $V_2 = \mathbb{R} e \oplus \mathbb{R} e^\perp$ be the decomposition of $V_2$, where $\mathbb{R} e^\perp = \{ v \in V_2 | \langle v, e \rangle = 0 \}$. For $f$, there are $r \in \mathbb{R}$ and $w \in \mathbb{R} e^\perp$ such that
\[ f = re + w. \]
Since $\langle ew, e \rangle = \langle w, e^2 \rangle = \langle w, 2e \rangle = 0$, we have $ew \in \mathbb{R} e^\perp$ and so
\[ 2re + 2w = 2f = f^2 = \{ r^2 2e + w_e^2 \} + \{ w^2 - w_e^2 \} + 2rew, \]
where $w_e^2$ denotes the first entry of $w^2$ in the decomposition $\mathbb{R} e \oplus \mathbb{R} e^\perp$. Hence,
\[ r^2/2 + \langle e, w_e^2 \rangle = \langle e, 2r^2 e + w_e^2 \rangle = \langle e, f^2 \rangle = \langle e, 2f \rangle = \langle e, 2re \rangle = r/2 \]
and so $\langle e, w_e^2 \rangle = r(1 - r)/2$. On the other hand,
\[ \frac{1}{4} = \langle f, f \rangle = r^2 \frac{1}{4} + \langle w, w \rangle, \]

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and so $\langle w, w \rangle = \frac{1}{2}(1 - r^2)$. Since $e > L(\frac{1}{2}, 0)$ and every irreducible $L(\frac{1}{2}, 0)$-module is isomorphic to one of $L(\frac{1}{2}, 0), L(\frac{1}{2}, \frac{1}{2}), L(\frac{1}{2}, \frac{1}{10})$ and $w - e$ is a sum of rational conformal vectors with central charge $\frac{1}{2}$, the eigenvalues of $L(0) - e_1$ is nonnegative. Hence, the eigenvalues of $e_1$ on $V_2$ are $0, 1, 2, \frac{1}{2}, \frac{1}{2} \pm 1, \frac{1}{10}, \frac{1}{10} + 1$. If $e_1 v = (\frac{1}{2} + 1)v$ or $(\frac{1}{10} + 1)v$, then $e_2 v \neq 0$, which contradicts to $e_2 v \in V_1 = 0$. If $e_1 v = 2v$, then $e_3 v \neq 0$, which contradicts to $v \in \mathbb{R}e_1$. If $e_1 v = v$, then $v \in (L(\frac{1}{2}, 0))_1 = 0$. Hence, the eigenvalues of $e$ on $\mathbb{R}e_1$ are $0, \frac{1}{2}, \frac{1}{2}$, or $\frac{1}{10}$. Hence, we obtain

$$r/2 - r^2/2 = \langle e, w^2 e \rangle = \langle e, w^2 \rangle = \langle we, w \rangle \leq \frac{1}{2} \langle w, w \rangle = \frac{1}{8} (1 - r^2)$$

and so $3r^2 - 4r + 1 \geq 0$. This implies $r \geq 1$ or $r \leq \frac{1}{3}$. If $r \geq 1$, then it contradicts $\langle w, w \rangle > 0$. We hence have $r \leq \frac{1}{3}$ and so $\langle e, f \rangle \leq \frac{1}{12}$, which implies $\langle e - f, e - f \rangle \geq \frac{1}{3}$. Hence, there are only finitely many conformal vectors with central charge $\frac{1}{2}$ since $\{v \in V_2 | \langle v, v \rangle = 4\}$ is a compact space.

Q.E.D.

**Theorem 9.2** If $V$ satisfies Hypothesis II, then $\text{Aut}(V)$ is finite.

**[Proof]** Suppose false and let $G$ be an automorphism group of $V$ of infinite order. Since $G$ acts on the set $J$ of all conformal vectors with central charge $\frac{1}{2}$ and $J$ is a finite set by Theorem 9.1, we may assume that $G$ fixes all conformal vectors with central charge $\frac{1}{2}$. In particular, $G$ fixes all coordinate conformal vectors $e^i$ for $i = 1, ..., n$. Set $P = \langle \tau^a : i = 1, ..., n \rangle$. By the definition of $\tau^a$, $P$ is an elementary abelian 2-group.

Let $V = \bigoplus_{\chi \in \text{Irr}(P)} V^\chi$ be the decomposition of $V$ into the eigenspaces of $P$, where $\text{Irr}(P)$ is the set of all linear characters of $P$ and $V^\chi = \{v \in V : gv = \chi(g)v \ \forall g \in P\}$. As we mentioned in the introduction, $\tilde{\tau}(V^\chi) = (a_1, ..., a_n)$ is given by $(-1)^{a_i} = \chi(e^i)$. Since $G$ fixes all $e^i$ and $g^{-1} \tau^a g = \tau^a(e^i)$ for $g \in \text{Aut}(V)$ by the definition, $[G, P] = 1$ and so $G$ leaves all $V^\chi$ invariant. In particular, $G$ acts on $V^{1G}$. We think over the action of $G$ on $V^{1G} (= V^P)$ for a while. Set $T = \langle e^1, ..., e^n \rangle \cong L(\frac{1}{2}, 0)^{\otimes n}$. Since $\dim V_0 = 1$, $T$ is the only irreducible $T$-submodule of $V$ isomorphic to $L(\frac{1}{2}, 0)^{\otimes n}$. By the hypotheses, $V$ has a positive definite invariant bilinear form and so $V^P$ is simple. Hence, $V^P$ is isomorphic to a code VOA $M_D$ for some even linear code $D$. In particular, $V^P$ is a direct sum of finite distinct irreducible $T$-modules $M_\alpha$. Since $T$ is generated by $\{e^i : i = 1, ..., n\}$ and $G$ fixes all $e^i$, $G$ fixes all vectors of $T$ and so the action of $g \in G$ on $M_\alpha$ is a scalar $\lambda_\alpha$. Since $V$ has a positive definite invariant bilinear form, we have $0 \neq \langle v, v \rangle = \langle g(v), g(v) \rangle = \lambda_\alpha^2 \langle v, v \rangle$ and so $\lambda_\alpha = \pm 1$. Since $|D|$ is finite, a finite index subgroup of $G$ fixes all vectors of $V^P$.  

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So we may assume that $G$ fixes all vectors in $V^P$. Since $V^\chi$ is an irreducible $V^P$-module by [DM2], $g \in G$ acts on $V^\chi$ as a scalar $\lambda^\chi$. By the same arguments as above, we have a contradiction.

Q.E.D.

In §3, we proved that we can induce every automorphism of $D$ into an automorphism of $M_D$. We will here show that we can induce every automorphism of $S^2$ into an automorphism of $V^2$.

**Lemma 9.1** For any $g \in \text{Aut}(S^2)$, there is an automorphism $\tilde{g}$ of $V^2$ such that $\tilde{g}(e^i) = e^{g(i)}$.

**Proof** By Lemma 3.2, we may assume that $g$ is an automorphism of $M_{D^3}$. Let $g((V^2)^\chi)$ be an $M_{D^3}$-module defined by $v_m(g \cdot u) = g \cdot (g^{-1}(v)_m u)$ for $v \in M_{D^3}$ and $u \in (V^2)^\chi$. Clearly, $\tilde{\tau}(g((V^2)^\chi)) = g^{-1}(\chi)$ and

$$g(V^2) = \bigoplus_{\chi \in S^2} g((V^2)^\chi)$$

has a VOA structure containing $g(M_{D^3}) = M_{D^3}$ by Theorem 3.8. We will prove that there is an $M_{D^3}$-isomorphism

$$\pi_\chi : g((V^2)^\chi) \rightarrow (V^2)^g(\chi)$$

for $\chi \in S^2$. Then, by the uniqueness theorem (Theorem 3.3), there are scalars $\lambda_\chi$ such that

$$\phi : g(V^2) \rightarrow V^2$$

given by $\phi = \lambda_\chi \pi_\chi$ on $g((V^2)^\chi)$ is a VOA-isomorphism. Hence, $\tilde{g}(v) = \phi(g \cdot v)$ for $v \in V^2$ is one of the desired automorphisms of $V^2$.

Since $S^2 = \{(\alpha, \beta, \gamma) : \alpha, \beta, \gamma \in S_{E_8}, \beta, \gamma = \alpha \text{ or } \alpha^c\}$, $\text{Aut}(S^2) = S_3 \times \text{Aut}(S_{E_8})$, where $S_3$ is the symmetric group on three letters. As we showed in §5, $\text{Aut}(S_{E_8}) \cong GL(5, 2)_1 = \{g \in GL(5, 2) : g^t(10000) = t(10000)\}$. In particular, $g$ leaves $D^3 = D_{E_8} \oplus D_{E_8} \oplus D_{E_8}$ and $D^2$ invariant. Set $\chi = (\alpha, \beta, \gamma)$. We first assume that $g \in S_3$. Since $(V^2)^\chi = \text{Ind}_{D^3}^{M_{D^3}}(W(\alpha, \beta, \gamma))$ and $W(\alpha, \beta, \gamma)$ is given by (7.10), we have $g(W(\alpha, \beta, \gamma)) \cong W^{g(\alpha, \beta, \gamma)}$ as $M_{D^3}$-modules and so we have the desired isomorphism for $g \in S_3$. Now, assume $g = (h, h, h)$ with $h \in \text{Aut}(S_{E_8})$. Set $j = h(1)$. By Lemma 5.1, $h(V^a_{E_8}) \cong V_{E_8}^{h(a)}$ and so $g(W(\alpha, \alpha, \alpha)) \cong W^{h(\alpha), h(\alpha), h(\alpha)}$. Hence, we may assume $\chi = (\alpha, \alpha, \alpha^c)$. By the definition,

$$g(W(\alpha, \alpha, \alpha^c)) = h(RV_{E_8}^\alpha) \otimes h(RV_{E_8}^{\alpha^c}) \otimes h(V_{E_8}^{\alpha^c})$$

$$\cong (h(R))V_{E_8}^{h(\alpha)} \otimes (h(R))V_{E_8}^{h(\alpha^c)} \otimes V_{E_8}^{h(\alpha^c)}$$

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as $M_{D_{6a}} \otimes M_{D_{6a}} \otimes M_{D_{6a}}$-modules. Since $R = M_{D_{6a}} + \xi_j$, $h(R) = M_{D_{6a}} + \xi_j$, where $\xi_j = (0^j10^{16-j})$. Since $(\xi_1 + \xi_j, \xi_1 + \xi_j, 0^{16}) \in D^\delta$, $(R \times h(R)) \otimes (R \times h(R)) \otimes M_{D_{6a}}$ is a submodule $M_{D^\delta + (\xi_1 + \xi_1 + \xi_j, 0^{16})}$ of $M_{D^\delta}$ and so we have

$$g(V^x) = g(\text{Ind}_{D_{6a}}^{D_3} W^{(\alpha, \alpha, \alpha)}(0))$$

$$= \text{Ind}_{D_{6a}}^{D_3} (h(R))V^{h}_E \otimes (h(R))V^{h}_E \otimes (V^{h}_E)$$

$$\cong \text{Ind}_{D_{6a}}^{D_3} RV^{h}_E \otimes RV^{h}_E \otimes V^{h}_E$$

$$\cong (V^x)g(x).$$

Q.E.D.

Let $\Lambda$ be the Leech lattice and let $V_\Lambda$ be a lattice VOA constructed from $\Lambda$. The following result easily comes from the construction of $V_\Lambda$ in $[FLM2]$.

**Lemma 9.2** \(\text{Aut}(V_\Lambda) \cong (\mathbb{R}^\times)^{\otimes 24})\Co.0, \text{ where } \mathbb{R}^\times = \mathbb{R} - \{0\}\text{ is the multiplicative group of } \mathbb{R}.\)

**[Proof]** Since $(V_\Lambda)_1$ is a commutative Lie algebra $\mathbb{R} \Lambda$ of rank 24 and $\exp(\alpha(0)) = \sum_{i=0}^{\infty} \frac{1}{i!}(\alpha(0))^i$ is an automorphism acting $\iota(x)$ as $\exp(\langle \alpha, x \rangle)\iota(x)$ for $\alpha \in (V_\Lambda)_1$ and $x \in \Lambda$, we have an automorphism group $\mathbb{R}^\times^{\otimes 24}$, which is a normal subgroup of $\text{Aut}(V_\Lambda)$. On the other hand, Frenkel, Lepowsky and Meurman $[FLM2]$ induced $g \in \text{Aut}(\Lambda)$ into an automorphism of the group extension $\hat{\Lambda} = \{\pm \iota(x) : x \in \Lambda\}$ and also into an automorphism of $V_\Lambda$ using cocycles. Hence, $V_\Lambda$ has an automorphism group $(\mathbb{R}^\times^{\otimes 24})\Co.0$. We note that this is not split extension. Conversely, choose $g \in \text{Aut}(V_\Lambda) - (\mathbb{R}^\times^{\otimes 24})\Co.0$, then $g$ leaves $(V_\Lambda)_1$ invariant and so it leaves a sub VOA $< (V_\Lambda)_1 >$ of free bosons and so $g$ acts on the lattice of highest weights of $< (V_\Lambda)_1 >$ in $V_\Lambda$, which is isomorphic to the Leech lattice. Multiplying an element of $\Co.0$, we may assume that $g$ fixes all highest weights vectors $\iota(x) : x \in \Lambda$ up to scalar multiple and so $g$ acts on the lattice of highest weights of $< (V_\Lambda)_1 >$ in $V_\Lambda$, which is isomorphic to the Leech lattice. Hence, $g$ fixes all elements of $(V_\Lambda)_1$ and so $g \in (\mathbb{R}^\times^{\otimes 24})$.

Q.E.D.

**Theorem 9.3** $\text{Aut}(V^x)$ is the Monster simple group.

**[Proof]** As we proved, the full automorphism group of $V^x$ is finite. Set $\delta = \tau_{e^1}\tau_{e^2}$ and decompose $V^x$ into the direct sum

$$V^x = V^+ \oplus V^-$$

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of the eigenspaces of $\delta$, where $V^\pm = \{ v \in V^\natural : \delta(v) = \pm v \}$. By the definition of $\tau_{e_i}$,

$$V^+ = \sum_{\alpha \in S^\natural, \langle \alpha, (110^{46}) \rangle = 0} (V^\natural)^\alpha.$$ 

Set $S_\Lambda = <(110^{46}) >^\perp \cap S^\natural$ and $D_\Lambda = S^\perp_\Lambda$. Since

$$S^\perp = \{ (\alpha, \beta, \gamma) : \alpha, \beta, \gamma \in S_{E_8}, \beta, \gamma \in \{ \alpha, \alpha^c \} \}$$

and

$$S_{E_8} = <(1^{16}), (1^{8}0^{8}), (1^{4}0^{4})^2, (1^{2}0^{2})^4, (10)^8 >,$$

we have the expression:

$$S_\Lambda = \{ (a_1, ..., a_{24}) \in S^\natural : a_i = (00), (11) \}.$$ 

In particular, $\delta$ is equal to $\tau_{e_{2m-1}e_{2m}}$ for $m = 1, ..., 24$. It is straightforward to check that $V^+$ has the structure given in Hypotheses I for $S = S_\Lambda$ and $D = D^\natural$. Since $S^\perp_\Lambda$ is larger than $D$, we can construct an induced VOA $\tilde{V}_\Lambda = \text{Ind}_{D^\perp_\Lambda}^{D_\Lambda}(V^\perp)$. 

Since $(S_\Lambda)^\perp = D_\Lambda$, $\tilde{V}_\Lambda$ is a holomorphic VOA of rank 24 by Theorem 6.1. It follows from the direct calculation that the codewords of $D_\Lambda$ of weight 2 are

$$\{(110^{46}), (00110^{44}), ..., (0^{46}11)\}.$$ 

We assert that $(\text{Ind}_{D^\perp_\Lambda}^{D_\Lambda}(V^\natural)^\alpha)_{1} = 0$ for $\alpha \neq 0$. Suppose $(\text{Ind}_{D^\perp_\Lambda}^{D_\Lambda}(V^\natural)^\alpha)_{1} \neq 0$ for some $\alpha$. Then $|\alpha| = 16$ and so $\alpha$ is one of $(1^{16}0^{32}), (0^{16}1^{16}), (0^{32}1^{16})$, say $\alpha = (1^{16}0^{32})$. Since $(V^\natural)^\alpha$ is given by $\text{Ind}_{D^\perp_\Lambda}^{D_\Lambda}(V_{E_8}^{(1^{16})} \otimes M_{D_{E_8}+\xi_1} \otimes M_{D_{E_8}+\xi_1})$ and $D_\Lambda$ does not contain any word of form $(\ast \xi_1 \xi_1)$, $(\text{Ind}_{D^\perp_\Lambda}^{D_\Lambda}(V^\natural)^\alpha)_{1} = 0$. Consequently,

$$\mathcal{G} = (\tilde{V}_\Lambda)_{1} = (M_{D_\Lambda})_{1} = \oplus_{\alpha \in D_\Lambda, |\alpha| = 2} (M_{\alpha})_{1}$$

is a commutative Lie algebra of rank 24. and $< (\tilde{V}_\Lambda)_{1} >$ is a VOA of free bosons of rank 24. We note that $\mathcal{G}$ has a positive definite invariant bilinear form $\langle \cdot, \cdot \rangle$ given by $v_1 u = \langle v, u \rangle 1$ since $\tilde{V}_\Lambda$ has a positive definite invariant bilinear form. Hence, $\mathbb{C}\tilde{V}_\Lambda$ is isomorphic to a lattice VOA $\mathbb{C}V_\Lambda$ of the Leech lattice $\Lambda$ by \cite{Mo}. More precisely, we will show the following lemmas in order to continue the proof of the theorem.
Lemma 9.3 $\tilde{V}_\Lambda$ is isomorphic to the lattice VOA $\check{V}_\Lambda$ of Leech lattice given in Proposition 2.2. In particular, we can choose a set of mutually orthogonal vectors $\{x^1, \ldots, x^{24}\}$ in $\Lambda$ of squared length 4 such that

$$e^{2j-i} = \frac{1}{16} x^j (-1)^2 e^0 + (-1)^i (\iota(x^j) + \iota(-x^j))$$

for $j = 1, \ldots, 24$ and $j = 0, 1$. Moreover, $(b_1 b_2 b_3 \cdots b_{24}) \in S_\Lambda$ if and only if there is $(a_i) \in \mathbb{Z}_{24}^2$ such that $x = \frac{1}{2} \sum a_i x^i + \frac{1}{4} \sum b_i x^i \in \Lambda$.

[Proof] Set

$$W = \{ v \in \check{V}_\Lambda : x(n)v = 0 \text{ for all } x \in G \text{ and } n > 0 \}.$$ 

Then the actions of $\{x(0) : x \in G\}$ on $CW$ is diagonalizable since $G$ is commutative. Let $L$ be the set of highest weights of $G$ in $CW$. It is easy to see that $L$ is an even unimodular positive definite lattice without roots since $(\check{V}_\Lambda)_1 = 0$. Hence, $L$ is the Leech lattice and $\mathbb{C}V_\Lambda \cong \mathbb{C}L$.

On the other hand, from Theorem 4.1, $\check{V}_\Lambda$ has a positive definite invariant bilinear form and it also has a $\mathbb{Z}_2$-grading

$$\check{V}_\Lambda = (V^2)^{\langle \delta \rangle} \oplus V^-_\Lambda$$

by the definition of induced VOAs, where $V^-_\Lambda = M_{(11046)}^{+D} \times (V^2)^{\langle \delta \rangle}$.

Let $\theta$ be an automorphism defined by 1 on $(V^2)^{\langle \delta \rangle}$ and -1 on $\check{V}_\Lambda^-$. Since $\theta$ is acting on $(\check{V}_\Lambda)_1$ as -1 and so it is equal to the automorphism of $\mathbb{C}V_\Lambda$ induced from -1 on $\Lambda$. Set $V = (V^2)^{\langle \delta \rangle} \oplus \sqrt{-1} \check{V}_\Lambda^-$. It is also a subVOA of $\mathbb{C}V_\Lambda$. Let $\iota(x)$ denote a highest weight vector of $G$ in $\mathbb{C}V_\Lambda$ with a highest weight $x \in \Lambda$. Namely, $u(0) \iota(x) = \langle u, x \rangle \iota(x)$ for $u \in G$.

We note that $\theta(\iota(x)) = (-1)^k \iota(x)$ for $\langle x, x \rangle = 2k$. As a $G$-module, the space $W$ of highest weight vectors is a direct sum of irreducible $G$-modules $W^i$ whose dimension are less than or equal to 2. If $\dim W^i = 1$, then $CW^i = \mathbb{C}t(x)$ for some $x \in \Lambda$. If $\dim W^i = 2$, then $CW^i = \mathbb{C}t(x) + \mathbb{C}t(y)$. Since $W^i$ is irreducible, $\iota(x)$ and $\iota(y)$ are in the same homogenous space $\mathbb{C}(\check{V}_\Lambda)_k$ for some $k$. Since $\mathbb{C}G = \mathbb{C}A$, we have $\mathbb{Z}x = \mathbb{Z}y$ and so $y = -x$. So $W^i$ has a basis $\{a\iota(x) + b\iota(-x), c\iota(x) + d\iota(-x)\}$ for some $a, b, c, d \in \mathbb{C}$. We may assume that $a \in \mathbb{R}$. Since $\check{V}_\Lambda$ has a positive definite invariant bilinear form, we have assume that $\{\frac{1}{\sqrt{2}} (a\iota(x) + b\iota(-x)), \frac{1}{\sqrt{2}} (c\iota(x) + d\iota(-x))\}$ is an orthonormal basis. Therefore, $b = (-1)^ka^{-1}$, $d = (-1)^kc^{-1}$ and $ad + bc = (-1)^k(ac^{-1} + a^{-1}c) = 0$. Hence, $a^2 = -c^2 > 0$ and so we have $c = \sqrt{-1}a$ and $d = -\sqrt{-1}b$. Since $CW^i = \mathbb{C}t(x) + \mathbb{C}t(-x)$ and $W^i = CW^i \cap \check{V}_\Lambda$, $\theta$ keeps $W^i$ invariant. Hence, $\theta(a\iota(x) + (-1)^ka^{-1}\iota(-x)) = a^{-1}\iota(x) + (-1)^kb\iota(-x) \in W^i$. 

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W^i and so we have \( a = \pm 1 \). Hence, \( \iota(x) + (-1)^k \iota(-x), \sqrt{-1}(\iota(x) - (-1)^k \iota(-x)) \in W \) and \( \sqrt{-1}x(0) \in G \) for \( x \in \Lambda \). Consequently, \( \tilde V_\Lambda \) coincides with the lattice VOA \( V_\Lambda \) defined in Proposition 2.2. We recall the structure \( V_{zz} \cong L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}) \) and \( (L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}))_1 = \mathbb{R}\sqrt{-1}(x(-1))I \) for a VOA \( V_{zz} \) with \( \langle x, x \rangle = 4 \). Since \( (\tilde V_\Lambda)_1 = (M_{D_\Lambda})_1 = \bigoplus_{i=1}^{24} (M_{\xi_2, \cdots, \xi_{24}})_1 \), \( e^{2j} - e^{2j-1} \in L = \{v \in V_\Lambda|x(n)v = 0 \text{ for all } x \in (V_\Lambda)_1 \text{ and } n > 0\} \) and \( \mathbb{R}(e^{2j} - e^{2j-1}) + \sqrt{-1}\mathbb{R}x(0)(e^{2j} - e^{2j-1}) \) is irreducible \( G \)-submodule of \( L \). Hence, by the above arguments we have

\[
e^{2j-i} = \frac{1}{16} x^i (-1)^2 e^0 + (-1)^i \frac{1}{4} (\iota(x^i) + \iota(-x^i))\]

for some \( x^j \in \Lambda \). Since

\[
0 = (e^{2j-1} + e^{2j})_1 (e^{2k} - e^{2k-1}) = \frac{1}{64} (x^j, x^k)^2 (\iota(x^k) + \iota(-x^k))
\]

for \( k \neq j \), we have \( \langle x^j, x^k \rangle = 0 \). Namely, \( \{x^1, \ldots, x^{24}\} \) is a set of mutually orthogonal vectors of \( \Lambda \) with squared length 4. If \( y = \sum c_i x^i \in \Lambda \), then \( c_i \in \mathbb{Q} \). Assume that \( y = \frac{1}{2} \sum b_i x^i + \frac{i}{2} \sum a_i x^i \) is in \( \Lambda \) and set \( W = V_{<x^1, \ldots, x^{24}>+y} \) and \( T^j = \langle e^{2j-1}, e^{2j} \rangle \). As we showed in §2,

1. \( b_j = 1 \) if and only if irreducible \( T^j \)-submodule of \( W \) is isomorphic to \( L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}) \).
2. \( b_j = 0 \) and \( a_j = 1 \) if and only if irreducible \( T^j \)-submodule of \( W \) is isomorphic to \( L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, 0) \) or \( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, \frac{1}{2}) \).

3. \( b_i = 0 \) and \( a_j = 0 \) if and only if irreducible \( T^j \)-submodule of \( W \) is isomorphic to \( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \) or \( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, \frac{1}{2}) \).

In particular, we have \( (b_1 b_2 b_3 b_4 b_{24}) \in S_\Lambda \).

Conversely, if \( \gamma = (b_1 b_2 b_3 b_4 b_{24}) \in S_\Lambda \), then \( \langle (\tilde V_\Lambda)_1 \rangle \) acts on \( (\tilde V_\Lambda)^\gamma \) and so \( (\tilde V_\Lambda)^\gamma \cap L \neq 0 \). Hence, by the above arguments, there is an element \( x \in \Lambda \) such that \( \iota(x) \in \tilde V_\Lambda \) or \( \iota(x) + (-1)^{|x|/2} \iota(-x) \in (\tilde V_\Lambda)^\gamma \). We can also find \( (a_i) \in \mathbb{Z}^{24} \) such that \( x = \frac{1}{2} \sum a_i x^i + \frac{i}{4} \sum b_i x^i \).

Q.E.D.

**Lemma 9.4** For any \( y \in \Lambda \) with squared length 4, \( \tau_{\epsilon(y)}^+ = \tau_{\epsilon(y)}^- \in \text{Aut}(V_\Lambda) \) and \( \tau_{\epsilon(y)}^+ \in <\pm 1>^{24} \subseteq (\mathbb{R}^\Lambda)^{24} \).

**[Proof]** Since \( Co.0 \) acts on the set of all vectors in \( \Lambda \) with squared length 4 transitively, we may assume that \( y = x^1 \) and \( \epsilon(y)^+ = \epsilon^1 \) and \( \epsilon(y)^- = \epsilon^2 \), where \( \{x^1, \ldots, x^{24}\} \) is the set defined in the above lemma. By the arguments in the proof of the above lemma, it is clear that \( \tau_{\epsilon(y)}^+ = \tau_{\epsilon(y)}^- \). Since \( \tau_{\epsilon^1} \iota(x) = (-1)^{\langle x^1, x \rangle} \iota(x) \) and \( [\tau_{\epsilon^1}, x(-n)] = 0 \), we have \( \tau_{\epsilon^1} \in <\pm 1>^{24} \).
Let's go back to the proof of the theorem 9.3. Set $V_\Lambda = (V^2)^\delta \oplus \sqrt{-1}V^-$. By the proof of Proposition 2.2 and the above lemma, $V_\Lambda$ is isomorphic to a lattice VOA of the Leech lattice which is given by the ordinary construction. Let $\theta$ be an automorphism of $V_\Lambda$ defined by 1 on $(V^2)^\delta$ and $-1$ on $\sqrt{-1}V^-$. We identify $(V^2)_{<\theta, \tau_1, \tau_2, \ldots, \tau_n>}$ and $V^\theta_{\Lambda}$. Let $J$ be the set of all rational conformal vectors in $(V^2)_{<\theta>}$ with central charge $0$. Set $K^2 = \langle \tau_e : e \in J \rangle \subseteq Aut(V^2)$, $K = \langle \tau_e : e \in J \rangle \subseteq Aut((V^2)^{<\theta>})$ and $K_\Lambda = \langle \tau_e : e \in J \rangle \subseteq Aut(V_\Lambda)$. By Lemma 9.2, we have epimorphisms $\pi^i : K^2 \to K$ and $\pi_\Lambda : K_\Lambda \to K$. By [DM2], $Ker(\pi^i) = <\delta>$ and $Ker(\pi_\Lambda) = <\theta > \cap K_\Lambda$. So we have the following diagram:

\[
\begin{array}{ccc}
G = Aut(V^2) & \xrightarrow{\pi^i} & Aut((V^2)^{<\theta>}) \\
| & & | \\
C_G(\delta) & \xrightarrow{\pi^i} & C_H(\delta) \\
| & & | \\
K^2 & \xrightarrow{\pi^i} & K \\
\langle \delta \rangle & \xrightarrow{\pi^i} & \langle \delta \rangle \\
\langle \theta \rangle & \xrightarrow{\pi^i} & \langle \theta \rangle \\
H = Aut(V_\Lambda) & \xrightarrow{\pi_\Lambda} & H \\
\end{array}
\]

First, we will show that $K_\Lambda \not\subseteq 2^{24} <\theta>$. Let $g$ be a permutation on 48 letters $\{1, \ldots, 48\}$ such that $g$ fixes all $1 + \ell m$ and $3 + \ell m$ and switches $2 + 4 m$ and $4 + 4 m$ for $m = 0, \ldots, 11$. It is straightforward to check that $g$ is an automorphism of $S^3$. By Lemma 9.1, there is an automorphism $\tilde{g} \in Aut(V^2)$ such that $\tilde{g}(e^i) = e^{g(i)}$. Set $\delta' = \tau_1 \tau_4 (= \tilde{g}(\delta))$ and $L'_\Lambda = g(L_\Lambda)$ and then apply the above arguments. By the above lemma, there is a set of mutually orthogonal vectors $\{\tilde{x}^1, \ldots, \tilde{x}^{24}\}$ in $\Lambda$ of squared length 4 such that

\[
e^{2j-i} = \frac{1}{16}\tilde{x}^j(-1)^2i(0) + (-1)^j\frac{1}{4}(i(\tilde{x}^j) + i(-\tilde{x}^j)).
\]

It is easy to see that $\gamma = (0^81^80^81^80^8) \in S_\Lambda$. Since $((V^2)^\gamma)_2 \neq 0$, there is $y \in \Lambda$ of squared length 4 such that $\langle y, x^i \rangle \equiv 1 \pmod{2}$ if and only if $i \in Supp(\gamma)$. Then $e(y) = \frac{1}{16}y(-1)^2i(0) + \frac{1}{4}(i(y) + i(-y))$ is a rational conformal vector in $(V_\Lambda)^{<\theta, \tau_1, \tau_2, \ldots, \tau_n>}$. Q.E.D.
In particular, \( g(e(y)) \in (V^2)^{<\delta>} \). Since \( \langle y, x^3 \rangle \equiv 1 \pmod{2} \), we have \( \tau_{e(y)}(t(\pm x^5)) = -t(\pm x^5) \) and so \( \tau_{e(y)} \) switches \( e^9 \) and \( e^{10} \). On the other hand, \( \tilde{g} \) fixes \( e^9 \) and switches \( e^{10} \) and \( e^{12} \). Hence, \( \tau_2(e(y)) \) switches \( e^9 \) and \( e^{12} \) and so \( \tilde{g}\tilde{g}(e(y)) \) does not belong to \( 2^{24} \langle \theta \rangle \).

Since \( K_\Lambda \) is generated by all conformal vectors in \( (V_\Lambda)^{<\theta>} \), \( K_\Lambda \) is a normal subgroup of \( C_H(\langle \theta \rangle) \cong 2^{24} Co.0 \) and so we have \( K_\Lambda = C_H(\langle \theta \rangle) \). Hence, \( K \cong 2^{24} Co.1 \) and so we have \( K^\perp = 2^{1+24} Co.1 \). If \( O_2(K^\perp) \) is an Abelian 2-group, then \( 2^{1+24} = \langle \delta \rangle \oplus \mathbb{Z}_2^{24} \) as a \( Co.1 \)-module. Let \( y \) be a vector of \( \Lambda \) of squared length 4 and \( \langle y, x^{24} \rangle = 1 \). Then \( e^\pm(y) \in (V^2)^{<\delta>} \) and \( \tau_{e^+(y)} \) fixes \( \delta = \tau_{e^1}\tau_{e^2} = \tau_{e^4}\tau_{e^8} \) and switches \( e^{47} \) and \( e^{48} \).

By Lemma 9.4, \( \tau_{e^{47}}, \tau_{e^{48}} \in 2^{1+24} \). Since \( \delta = \tau_{e^4}\tau_{e^8} \), we may assume \( e^{47} \in \mathbb{Z}_2^{24} \) and \( e^{48} \notin \mathbb{Z}_2^{24} \), which contradicts that \( \tau_{e(y)} \) switches \( e^{47} \) and \( e^{48} \). Hence, \( 2^{1+24} \) is a non-abelian and so \( 2^{1+24} \) are isomorphic to a central extension of \( \Lambda/2\Lambda \) using the inner product, since \( Co.1 \) acts on faithfully. By Lemma 9.1, \( \text{Aut}(V^2) \) contains a subgroup whose restriction on \( \{e^1, ..., e^{48}\} \) is isomorphic to \( GL(5,2)_1 \times S_3 \), where \( S_3 \) permutes 3 components of \( V^{\otimes 3} \) and \( GL(5,2)_1 = \{ A \in GL(5,2) : Av = v \text{ for } v = (10000) \} \). Set \( \delta_1 = \tau_{e^1}\tau_{e^3} \text{ and } B^2 = \langle \delta, \delta_1 \rangle \). Denote \( \delta \) and \( \delta_1 \) by \( \delta_0 \) and \( \delta_2 \), respectively. Since a subgroup of \( GL(5,2)_1 \) acts on \( \{\delta_0, \delta_1, \delta_2\} \) transitively and \( e^3 \) is given by a vector of \( \Lambda \) of squared length 4, we have \( N_{\text{Aut}(V^2)}(B^2) \cong 2^{2+12+22}(S_3 \times M_{24}) \) from the structure of \( C_{\text{Aut}(V^2)}(\delta) \cong 2^{1+24} Co.1 \). Similarly, all nontrivial elements of \( B^3 = \langle \tau_{e^1}\tau_{e^2}, \tau_{e^1}\tau_{e^3}, \tau_{e^1}\tau_{e^5} \rangle \) are conjugate by the actions of \( GL(5,2)_1 \subseteq \text{Aut}(V^2) \) and so \( N_{\text{Aut}(V^2)}(B^3) \cong 2^{3+6+12+18}(3S_6 \times L_3(2)) \). By the same arguments, we can calculate the normalizer of \( B^4 = \langle \tau_{e^1}\tau_{e^2}, \tau_{e^1}\tau_{e^3}, \tau_{e^1}\tau_{e^5}, \tau_{e^3}\tau_{e^9} \rangle \).

We leave these calculation to the reader.

We will next prove that \( \text{Aut}(V^2) \) is a simple group. Let \( H \) be a nontrivial minimal normal subgroup of \( \text{Aut}(V^2) \). Then \( C_H(\delta_i) \) is a normal subgroup of \( C(\delta_i) \cong 2^{1+24} Co.1 \) for \( i = 0, 1, 2 \). Hence, \( C_H(\delta_i) \cong 2^{1+24} Co.1 \) or \( C_H(\delta_i) \cong 2^{1+24} \) or \( C_H(\delta_i) = < \delta_i > \). We note that \( \delta_i (i = 0, 1, 2) \) are conjugate to each other in \( \text{Aut}(V^2) \) and so \( C_H(\delta_i) \cong C_H(\delta_0) \) for \( i = 1, 2 \). In any cases, \( \delta_i \in H \) and so \( C_H(\delta_i) \neq < \delta_i > \) since \( \delta_j \in < C_H(\delta_i) : i = 1, 2, 3 >= H \). If \( P = C_H(\delta_1) \cong 2^{1+24} \) then \( P \) is a Sylow 2-subgroup of \( H \). Since \( |P : C_P(\delta_2)| = 2 \) and \( C_P(\delta_2) \) is not abelian, \( [C_P(\delta_2), C_P(\delta_2)] = < \delta_1 > \), which contradicts \( [C_H(\delta_2), C_H(\delta_2)] = < \delta_2 > \). Hence we have \( C_H(\delta_1) = 2^{1+24} Co.1 \). Since \( < \delta_1 > \) is a characteristic subgroup of a Sylow 2-subgroup of \( H \), we have \( H = \text{Aut}(V^2) \) and so \( \text{Aut}(V^2) \) is a simple group. By the characterization of the Monster simple group and the above facts, we know \( \text{Aut}(V^2) \) is the Monster simple group, see [1], [2], [3].

Q.E.D.

Since \( V^2 \) is a holomorphic VOA with rank 24 with \( (V^2)_1 = 0 \) and the Monster simple group acts on \( B = V_2^\natural \) faithfully, \( B \) is isomorphic to the Griess algebra constructed in
We have also proved that \((V^z)^\delta\) is isomorphic to \((V_\lambda)^\theta\). Hence, \(V^z\) is equal to the moonshine VOA constructed in [FLM2].

\section{Meromorphic VOAs}

In this section, we will construct an infinite series of holomorphic VOAs whose full automorphism groups are finite. We will adopt the notation from §7 and repeat the similar constructions as in §7.

For \(n = 1, 2, \cdots\), set
\[
S^\sharp(n) = < \{0^{16}\}1^{16}\{0^{16}\}2^{n-1} \}, \{\alpha\}^{2n+1} : \alpha \in S(P), i = 1, \cdots, 2n > .
\]
\(S^\sharp(n)\) is an even linear code of length \(16 + 32n\) and \((S^\sharp(n))^\perp\) contains a direct sum \(D^{2n+1}\) of \(2n + 1\) copies of \(D\) for each \(n\). Let \(\gamma\) be an element of \(S^\sharp(n)\), then there is \(\alpha \in S(P)\) such that
\[
\gamma = (\beta_1, ..., \beta_{2n+1}),
\]
where \(\beta_i \in \{\alpha, \alpha^c\}\). We may assume that the number of \(\beta_i\) satisfying \(\beta_i = \alpha\) is odd. Set
\[
W^\gamma = \bigotimes_{i=1}^{2n+1} \tilde{W}^{\beta_i},
\]
where
\[
\tilde{W}^{\beta_i} = V_{E_8}^\alpha \quad \text{if} \quad \beta_i = \alpha \quad \text{and} \quad \tilde{W}^{\beta_i} = RV_{E_8}^\alpha \quad \text{if} \quad \beta_i = \alpha^c.
\]
Set
\[
V^\beta(n) = \bigoplus_{\gamma \in S^\sharp(n)} W^\gamma
\]
and
\[
V^z(n) = \text{Ind}_{M_D^{2n+1}}^{M_{S^\sharp(n)^\perp}} (V^\beta(n)).
\]
Then we can show that \(V^z(n)\) has a VOA structure by exactly the same proof as in the construction of \(V^z\). It also satisfies \((V^z(n))_1 = 0\). Moreover, it is a holomorphic VOA by Theorem 6.1 and its full automorphism group is finite by Theorem 9.2.

\section{Characters}

In this section, we will calculate the characters of \(3C\) element and \(2B\) element of the Monster simple group. It follows from our construction that we can induce an automorphism of \(D^z\) into an automorphism of \(V^z\).
11.1 3C

Clearly, \( \hat{g} = (1,17,33)(2,18,34)...(16,32,48) \) is an automorphism of \( D^3 \). Let \( g \) be an automorphism of \( V^3 \) induced from \( \hat{g} \). By the definition, \( g \) acts on \( \{ e^i : i = 1, ..., 48 \} \) as \( (1, 17, 33)(2, 18, 34)...(16, 32, 48) \).

\( V^3 \) contains \( M_D^3 = M_D \otimes M_D \otimes M_D \), where \( D = D_{Es} \). We view \( V^3 \) as an \( M_D \otimes M_D \otimes M_D \)-module. Since \( g \) permutes \( \{ V^\chi : \chi \in S^3 \} \), we obtain

\[
\text{ch}_{V^3}(g, z) = \text{tr}_{g,z}(V^3) = \text{tr}_{g,z}(\bigoplus_{\chi \in S^3} V^\chi) = \text{tr}_{g,z}(\bigoplus_{\alpha \in D_{Es}} V^{(\alpha,\alpha,\alpha)}),
\]

where \( \text{tr}_{g,z}(V) = \sum (g|_{V_n}) e^{2\pi inz} \) for \( V = \bigoplus V_n \).

By the definition of \( V^{(\alpha,\alpha,\alpha)} \),

\[
V^{\alpha,\alpha,\alpha} = \text{Ind}_{M_D}^{M_{D^3}}(V_{Es} \otimes V_{Es} \otimes V_{Es}).
\]

It follows from the definition of induced modules,

\[
\text{Ind}_{M_D}^{M_{D^3}}(U) \cong \bigoplus_{\mu \in D^3/D^3} M_{D^3+\mu} \times U
\]
as \( M_{D^3} \)-modules. Since \( D^3 = \{ (\alpha, \beta, \gamma) : \alpha + \beta + \gamma \in D, \alpha, \beta, \delta \text{ even} \} \), we obtain that \( g(D^3 + \mu) = D^3 + \mu \) if and only if \( \mu \in D^3 \). Hence,

\[
\text{tr}_{g,z}(V^{(\alpha,\alpha,\alpha)}) = \text{tr}_{g,z}(V_{Es}^\alpha \otimes V_{Es}^\alpha \otimes V_{Es}^\alpha) = \text{tr}_{g,3z}(V_{Es}^\alpha).
\]

Therefore, we have

\[
\text{ch}_{V^3}(g, z) = \sum_{\alpha \in D_{Es}} \text{tr}_{g,3z}(V_{Es}^\alpha) = \text{tr}_{g,3z}(V_{Es}) = \text{ch}_{V_{Es}}(1, 3z).
\]

11.2 1 and 2B

Let \( \delta = \tau e^1 \tau e^2 \). We proved that \( (V^3)^{<\delta>} \) is isomorphic to \( (V_A)^{<\delta>} \). Hence,

\[
\text{ch} ((V^3)^{<\delta>}) = 1 + 98580q^2 + ....
\]

So we will calculate the character of \( (V^3)^- = \{ v \in V^3 : \delta(v) = -v \} \). It follows from the definition of \( \tau e^i \) that

\[
\text{ch} ((V^3)^-) = \sum_{\langle \chi, (11046) \rangle = 1} \text{ch} ((V^3)^\chi).
\]
Set $\chi = (\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2^{16}$. Assume $\langle \chi, (110^{46}) \rangle = 1$. Then the weight of $\alpha$ is 8 and so the weight of $\chi$ is 24. Hence, $\dim D^\chi = 7 + 7 + 4$ and so the multiplicity of every irreducible $T$-submodule of $(V^\chi)^2$ is $2^6$. Let $U$ be an irreducible $T$-submodule of $(V^\chi)^2$. It follows from the total degree that the number of $L(\frac{1}{2}, \frac{1}{2})$ in $U = \bigotimes_{i=1}^{48} L(\frac{1}{2}, h^i)$ is odd.

On the other hand, let $\gamma$ be an odd word with $\text{Supp}(\gamma) \cap \text{Supp}(\chi) = \emptyset$. By the action of $M_{D^\chi}$, there exists an irreducible $T$-submodule isomorphic to $\bigotimes L(\frac{1}{2}, h^i)$ with $h^i = \frac{1}{2}$ for $i \in \text{Supp}(\gamma)$, $h^i = \frac{1}{16}$ for $i \in \text{Supp}(\chi)$ and $h^i = 0$ for $i \notin \text{Supp}(\gamma + \gamma)$. Hence

$$\text{ch}((V^\chi)^2) = 2^6 \text{ch} \{ L(\frac{1}{2}, 0) \} \otimes 2^4 \{ L(\frac{1}{2}, 0) + L(\frac{1}{2}, \frac{1}{2}) \} \otimes 2^4 - \{ L(\frac{1}{2}, 0) - L(\frac{1}{2}, \frac{1}{2}) \} \otimes 2^4 \}$$

Since there are 64 codewords $\chi$ such that $\langle \chi, (110^{46}) \rangle = 1$, we have

$$\text{ch}((V^\chi)^{-}) = 2^{11} q^{3/2} \prod_{n \in \mathbb{N}} (1 + q^n) \frac{1}{2} (1 + q^n)^{24} - \prod_{n \in \mathbb{N} + \frac{1}{2}} (1 - q^n)^{24}$$

In particular, we obtain $(V^\chi)^1 = 0$ and $\dim (V^\chi)^2 = 196884$.

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