On a weighted linear matroid intersection algorithm
by deg-det computation

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Abstract
In this paper, we address the weighted linear matroid intersection problem from the computation of the degree of the determinants of a symbolic matrix. We show that a generic algorithm computing the degree of noncommutative determinants, proposed by the second author, becomes an $O(mn^3 \log n)$ time algorithm for the weighted linear matroid intersection problem, where two matroids are given by column vectors $n \times m$ matrices $A, B$. We reveal that our algorithm is viewed as a “nonstandard” implementation of Frank’s weight splitting algorithm for linear matroids. This gives a linear algebraic reasoning to Frank’s algorithm. Although our algorithm is slower than existing algorithms in the worst case estimate, it has a notable feature: Contrary to existing algorithms, our algorithm works on different matroids represented by another “sparse” matrices $A^0, B^0$, which skips unnecessary Gaussian eliminations for constructing residual graphs.

Keywords: combinatorial optimization, polynomial time algorithm, weighted matroid intersection, the degree of determinant, weight splitting

1 Introduction
Several basic combinatorial optimization problems have linear algebraic formulations. It is classically known [1] that the maximum cardinality of a matching in a bipartite graph $G = (U, V; E)$ with color classes $U = [n], V = [n']$ is equal to the rank of the matrix $A = \sum_{e \in E} A_e x_e$, where $x_e (e \in E)$ are variables and $A_e$ is an $n \times n'$ matrix with $(A_e)_{ij} := 1$ if $e = ij$ and zero otherwise. Such a rank interpretation is known for the linear matroid intersection, nonbipartite matching, and linear matroid matching problems; see [1].

The degree of the determinant of a polynomial (or rational) matrix is a weighted counterpart of rank, and can formulate weighted versions of combinatorial optimization problems. The maximum weight perfect matching problem in a bipartite graph
$G = ([n], [n]; E)$ with integer weights $c_e$ $(e \in E)$ corresponds to computing the degree of the determinant of the (rational) matrix $A(t) := \sum_{e \in E} A_e x_e^{c_e}$. Again, the weighted linear matroid intersection, nonbipartite matching, and linear matroid matching problems have such formulations.

Inspired by the recent advance of a noncommutative approach to symbolic rank computation, the second author introduced the problem of computing the degree of the Dieudonné determinant $\text{Det}(A(t))$ of a matrix $A(t) = \sum_i A_i(t)x_i$, where $x_i$ are pairwise noncommutative variables and $A_i(t)$ is a rational matrix with commuting variable $t$. He established a general min-max formula for $\text{deg}_t \text{Det}(A(t))$, presented a conceptually simple and generic algorithm, referred here to $\text{Deg-Det}$, for computing $\text{deg}_t \text{Det}(A(t))$, and showed that $\text{deg}_t \text{det}(A(t)) = \text{deg}_t \text{Det}(A(t))$ holds if $A(t)$ corresponds to an instance of the weighted linear matroid intersection problem. In particular, $\text{Deg-Det}$ gives rise to a pseudo-polynomial time algorithm for the weighted linear matroid intersection problem. In the first version of the paper, the second author asked (i) whether $\text{Deg-Det}$ can be a (strongly) polynomial time algorithm for the weighted linear matroid intersection, and (ii) how $\text{Deg-Det}$ is related to the existing algorithms for this problem. He pointed out some connection of $\text{Deg-Det}$ to the primal-dual algorithm by Lawler but the precise relation was not clear.

The main contribution of this paper is to answer the questions (i) and (ii):

- We show that $\text{Deg-Det}$ becomes an $O(nm^3 \log n)$ time algorithm for the weighted linear matroid intersection problem, where the two matroids are represented and given by two $n \times m$ matrices $A, B$. This answers affirmatively the first question.

- For the second question, we reveal the relation between our algorithm and the weight splitting algorithm by Frank. This gives a linear algebraic reasoning to Frank’s algorithm. We show that the behavior of our algorithm is precisely the same as that of a slightly modified version of Frank’s algorithm. Our algorithm is different from the standard implementation of Frank’s algorithm for linear matroids. This relationship was unexpected and nontrivial for us, since the two algorithms look quite different.

Although our algorithm is slower than the standard $O(nm^3)$-time implementation of Frank’s algorithm in the worst case estimate, it has a notable feature. Frank’s algorithm works on a subgraph $\tilde{G}_X$ of the residual graph $G_X$, where $G_X$ is determined by Gaussian elimination for $A, B$ and $\tilde{G}_X$ is determined by a splitting of the weight. On the other hand, our algorithm does not compute the residual graph $G_X$, and computes a non-redundant subgraph $G^0_X$ of $\tilde{G}_X$, which is the residual graph of different matroids represented by another “sparse” matrices $A^0, B^0$. Consequently, our algorithm applies fewer elimination operations than the standard one, which will be a practical advantage.

Related work. The essence of $\text{Deg-Det}$ comes from the combinatorial relaxation algorithm by Murota, which is an algorithm computing the degree of the (ordinary) determinant of a polynomial/rational matrix; see Section 7.1.

Several algorithms have been proposed for the general weighted matroid intersection problem under the independence oracle model; see e.g., Section 41.3 and the references therein. For linear matroids given by two $n \times m$ matrices, the current fastest algorithms (as far as we know) are an $O(nm^{\omega})$-time implementation of Frank’s algorithm.
using fast matrix multiplication and an $O\left(\frac{nm}{\omega} \cdot \log \frac{n}{\omega-1} \cdot n \log mC\right)$-time algorithm by Gabow and Xu [4], where $C$ is the maximum absolute value of weights $c_i$. Huang, Kakimura, and Kamiyama [7] gave an $O\left(nm \log n + Cmn^{\omega-1}\right)$-time algorithm, which is faster for the case of small $C$.

**Organization.** The rest of this paper is organized as follows. In Section 2 we introduce algorithm Deg-Det, and describe basics of the unweighted (linear) matroid intersection problem from a linear algebraic viewpoint; our algorithm treats the unweighted problem as a subproblem. In Section 3 we first formulate the weighted linear matroid intersection problem as the degree of the determinant of a rational matrix $A$, and show that Deg-Det computes $\text{deg} \det A$ correctly. Then we present our algorithm by specializing Deg-Det, analyze its time complexity, and reveal its relationship to Frank’s algorithm.

## 2 Preliminaries

### 2.1 Notation

Let $\mathbb{Q}$ and $\mathbb{Z}$ denote the sets of rationals and integers, respectively. Let $0 \in \mathbb{Q}^n$ denote the zero vector. For $I \subseteq [n] := \{1, 2, ..., n\}$, let $1_I \in \mathbb{Q}^n$ denote the characteristic vector of $I$, that is, $(1_I)_k := 1$ if $k \in I$ and 0 otherwise. Here, $1_{[n]}$ is simply denoted by $1$.

For a polynomial $p = \sum_{i=0}^{k} a_it^i \in \mathbb{Q}[t]$ with $a_k \neq 0$, the degree $\text{deg} \ p$ with respect to $t$ is defined as $k$. The degree $\text{deg} \ p/q$ of a rational function $p/q \in \mathbb{Q}(t)$ with polynomials $p, q \in \mathbb{Q}[t]$ is defined as $\text{deg} \ p - \text{deg} \ q$.

A rational function $p/q$ is called *proper* if $\text{deg} \ p/q \leq 0$. A rational matrix $Q \in \mathbb{Q}(t)^{n \times m}$ is called proper if each entry of $Q$ is proper. For a proper rational matrix $Q \in \mathbb{Q}(t)^{n \times m}$, there is a unique matrix over $\mathbb{Q}$, denoted by $Q^0$, such that $Q = Q^0 + t^{-1}Q'$, where $Q'$ is a proper matrix.

For an integer vector $\alpha \in \mathbb{Z}^n$, let $(t^\alpha)$ denote the $n \times n$ diagonal matrix having diagonals $t^{\alpha_1}, t^{\alpha_2}, \ldots, t^{\alpha_n}$ in order, that is,

$$(t^\alpha) = \begin{pmatrix} t^{\alpha_1} & & \\ & t^{\alpha_2} & \\ & & \ddots \\ & & & t^{\alpha_n} \end{pmatrix}.$$

For a matrix $A \in \mathbb{Q}^{n \times m}$ and $J \subseteq [m]$, let $A[J]$ denote the submatrix of $A$ consisting of the $j$-th columns for $j \in J$. Additionally, for $I \subseteq [n]$, let $A[I, J]$ denote the submatrix of $A$ consisting of the $(i, j)$-entries for $i \in I, j \in J$.

### 2.2 Algorithm Deg-Det

Given $n \times n$ rational matrices $M_1, M_2, \ldots, M_m \in \mathbb{Q}(t)^{n \times n}$, consider the following matrix $\begin{align*} M := M_1x_1 + M_2x_2 + \cdots + M_mx_m & \in \mathbb{Q}(t, x_1, x_2, \ldots, x_m), \end{align*}$
where \(x_1, x_2, \ldots, x_m\) are variables and \(M\) is regarded as a multivariate rational matrix with (pairwise commutative) variables \(t, x_1, x_2, \ldots, x_m\). We address the computation of the degree of the determinant of \(M\) with respect to \(t\).

Consider the following optimization problem:

\[
\text{(P)} \quad \text{Max. } \deg \det P + \deg \det Q \\
\text{s.t. } PMQ : \text{ proper,} \\
P, Q \in \mathbb{Q}(t)^{n \times n} : \text{ nonsingular.}
\]

This problem gives an upper bound of \(\deg \det M\). Indeed, if \(PMQ\) is proper, then \(\deg \det PMQ \leq 0\), and \(\deg \det M \leq -\deg \det P - \deg \det Q\). In fact, it is shown \([6]\) that the optimal value of (P) is interpreted as the negative of the degree of the Dieudonné determinant of \(M\) for the case where \(x_1, x_2, \ldots, x_n\) are pairwise noncommutative variables.

The following algorithm for (P) is due to \([6]\), which is viewed as a simplification of the combinatorial relaxation algorithm by Murota \([9]\); see also \([10, \text{Section 7.1}]\).

**Algorithm: Deg-Det**

**Input:** \(M = M_1x_1 + M_2x_2 + \cdots + M_mx_m\), where \(M_i \in \mathbb{Q}(t)^{n \times n}\) for \(i \in [m]\).

**Output:** An upper bound of \(\deg \det M\) (the negative of an optimal value of (P)).

1: Let \(P := t^{-d}I\) and \(Q := I\), where \(d\) is the maximum degree of entries in \(M\). Let \(D^* := nd\).

2: Solve the following problem:

\[
\text{(P)} \quad \text{Max. } r + s \\
\text{s.t. } K(\text{PMQ})^0L \text{ has an } r \times s \text{ zero submatrix,} \\
K, L \in \mathbb{Q}^{n \times n} : \text{ nonsingular,}
\]

and obtain optimal matrices \(K, L\); recall the notation \((\cdot)^0\) in Section 2.1.

3: If the optimal value \(r + s\) is at most \(n\), then stop and output \(D^*\).

4: Let \(I\) and \(J\) be the sets of row and column indices, respectively, of the \(r \times s\) zero submatrix of \(K(\text{PMQ})^0L\). Find the maximum integer \(\kappa(\geq 1)\) such that \((t^{\kappa 1})KPML(t^{-\kappa 1[n \setminus j]})\) is proper.

Let \(P \leftarrow (t^{\kappa 1})KP\) and \(Q \leftarrow QL(t^{-\kappa 1[n \setminus j]}).\) If \(\kappa\) is unbounded, then output \(-\infty\). Go to step 1 otherwise.

Observe that in each iteration \((P, Q)\) is a feasible solution of (P), and \(D^*\) equals \(-\deg \det P - \deg \det Q\). Thus, (P) actually gives an upper bound of \(\deg \det M\). We are interested in the case where the algorithm outputs \(\deg \det M\) correctly.

**Lemma 2.1** \([6]\). In step 2 of Deg-Det, the following holds:

1. If \(r + s > n\), then \((\text{PMQ})^0\) is singular (over \(\mathbb{Q}(x_1, x_2, \ldots, x_m)\)).
2. If \((\text{PMQ})^0\) is nonsingular, then \(r + s \leq n\) and \(D^* = \deg \det M\).
Proof. (1). It is obvious that any $n \times n$ matrix is singular if it has an $r \times s$ zero submatrix with $r + s > n$.

(2). $PMQ$ is written as $(PMQ)^0 + t^{-1}N$ for a proper $N$. If $(PMQ)^0$ is nonsingular, then $\deg\det PMQ = \deg\det(PMQ)^0 = 0$, and hence $\deg\det M = -\deg\det P - \deg\det Q = D^*$.

\[ \square \]

2.3 Algebraic formulation for linear matroid intersection

Let $A = (a_1 \ a_2 \ \cdots \ a_m)$ be an $n \times m$ matrix over $\mathbb{Q}$. Let $M(A) = ([m], \mathcal{I}(A))$ denote the linear matroid represented by $A$. Specifically, the ground set of $M(A)$ is the set $[m]$ of the column indices, and the family $\mathcal{I}(A)$ of independent sets of $M(A)$ consists of subsets $X \subseteq [m]$ such that the corresponding column vectors $a_i$ ($i \in X$) are linearly independent. Let $\rho_A : 2^{[m]} \to \mathbb{Z}$ denote the rank function of $M(A)$, that is, $\rho_A(X) := \max\{|Y| \mid Y \in \mathcal{I}(A),Y \subseteq X\}$. A minimal (linearly) dependent subset is called a circuit. See, e.g., [14, Chapter39] for basics on matroids.

Suppose that we are given another $n \times m$ matrix $B = (b_1 \ b_2 \ \cdots \ b_m) \in \mathbb{Q}^{n \times m}$. Let $M(B) = ([m], \mathcal{I}(B))$ be the corresponding linear matroid. A common independent set of $M(A)$ and $M(B)$ is a subset $X \subseteq [m]$ such that $X$ is independent for both $M(A)$ and $M(B)$. The linear matroid intersection problem is to find a common independent set of the maximum cardinality. To formulate this problem linear algebraically, define an $n \times n$ matrix $M = M(A,B)$ by

$$M := \sum_{i=1}^{m} a_i b_i^\top x_i,$$

where $x_1, x_2, \ldots, x_m$ are variables. The following is the matroid intersection theorem and its linear algebraic sharpening.

**Theorem 2.2** ([2]; see also [11]). The following quantities are equal:

1. The maximum cardinality of a common independent set of $M(A)$ and $M(B)$.

2. The minimum of $\rho_A(J) + \rho_B([m] \setminus J)$ over $J \subseteq E$.

(1') rank $M$.

(2') $2n$ minus the maximum of $r + s$ such that $KML$ has an $r \times s$ zero submatrix for some nonsingular matrices $K, L \in \mathbb{Q}^{n\times n}$.

**Sketch of Proof.** (1) = (2) is nothing but the matroid intersection theorem.

(1) = (1'). A $k \times k$ submatrix $M'$ of $M$ is represented by $M' = A'DB'^\top$, where $A', B'$ are the corresponding $k \times m$ submatrices of $A, B$, and $D$ is the diagonal matrix with diagonals $x_1, x_2, \ldots, x_m$ (in order). From Binet-Cauchy formula, we see that $\det M' \neq 0$ if and only if there is a $k$-element subset $X \subseteq [m]$ such that $\det A'[X] \det B'[X] \neq 0$. Thus, rank $M \geq k$ if and only if there is a common independent set of size $k$.

(2) $\geq$ (2'). Take a basis $u_1, u_2, \ldots, u_r$ of the orthogonal complement of the vector space spanned by $\{a_i \mid i \in J\}$, and extend it to a basis $u_1, u_2, \ldots, u_n$ of $\mathbb{Q}^n$, where $r = n - \rho_A(J)$. Similarly, take a basis $v_1, v_2, \ldots, v_s$ of the orthogonal complement of the vector space spanned by $\{b_i \mid i \in [m] \setminus J\}$, where
Then $u_k^T a_i b_i^T v_\ell = 0$ for all $k \in [s]$, $\ell \in [t]$, and $i \in [m]$. This means that $KML$ has an $r \times s$ zero submatrix for $K = (u_1 u_2 \cdots u_n)^\top$ and $L = (v_1 v_2 \cdots v_n)$.

$(2') \geq (1')$. If $KML$ has an $r \times s$ zero submatrix, then $\text{rank } M = \text{rank } KML \leq n - r + n - s$.

Let us briefly explain Edmonds’ algorithm to obtain a common independent set of the maximum cardinality. For any common independent set $X$, the auxiliary (di)graph $G_X = G_X(A,B)$ is defined as follows. The set $V(G_X)$ of nodes of $G_X$ is equal to the ground set $[m]$ of the matroids, and the set $E(G_X)$ of arcs is given by: $(i,j) \in E(G_X)$ if and only if one of the following holds:

- $i \in X$, $j \not\in X$, and $i,j$ belong to a circuit of $M(A)$.
- $i \not\in X$, $j \in X$, and $i,j$ belong to a circuit of $M(B)$.

Let $S_X = S_X(A)$ denote the subset of nodes $i \in E \setminus X$ such that $X \cup \{i\}$ is independent in $M(A)$, and $T_X = T_X(B)$ denote the subset of nodes $i \in E \setminus X$ such that $X \cup \{i\}$ is independent in $M(B)$. See Figure 1 for $G_X$, $S_X$, and $T_X$.

**Lemma 2.3** ([2]). Let $X$ be a common independent set, and let $R$ be the set of nodes reachable from $S_X$ in $G_X$.

1. Suppose that $R \cap T_X \neq \emptyset$. For a shortest path $P$ from $S_X$ to $T_X$, the set $X \triangle V(P)$ is a common independent set with $|X \triangle V(P)| = |X| + 1$.

2. Suppose that $R \cap T_X = \emptyset$. Then $X$ is a maximum common independent set and $R$ attains the $\min_{J \subseteq [m]} \rho_A(J) + \rho_B([m] \setminus J)$.

Here $\triangle$ denotes the symmetric difference. According to this lemma, Edmonds’ algorithm is as follows:
• Find a shortest path $P$ in $G_X$ from $S_X$ to $T_X$ (by BFS).

• If it exists, then replace $X$ by $X \triangle V(P)$, and repeat. Otherwise, $X$ is a common independent set of the maximum cardinality.

In our case, the auxiliary graph $G_X$ and optimal matrices $K, L$ in (2') are naturally obtained by applying elementary row operation to matrices $A, B$ as follows. Since $X$ is a common independent set, both $A[X]$ and $B[X]$ have column full rank $|X|$. Therefore, by multiplying nonsingular matrices $K$ and $L$ to $A$ and $B$ from left, respectively, we can make $A$ and $B$ diagonal in the position $X$, that is, for some injective map $\sigma : X \rightarrow [n]$, it holds $(KA)_{ki} = (LB)_{ki} = 1$ if $k = \sigma(i)$ and zero otherwise. Such matrices are said to be $X$-diagonal. Notice that these operations do not change the matroids $M(A)$ and $M(B)$. See Figure 2 where the columns and rows are permuted appropriately.

Then the auxiliary graph $G_X$ is constructed from the nonzero patterns of $KA$ and
As follows. For $i \in X$, arc $(i, j)$ (resp. $(j, i)$) exists if and only if $(KA)_{\sigma(i)j} \neq 0$ (resp. $(LB)_{\sigma(i)j} \neq 0$). Additionally, $S_X$ (resp. $T_X$) consists of nodes $i$ with $(KA)_{ki} \neq 0$ (resp. $(LB)_{ki} \neq 0$) for some $k \in [n] \setminus \sigma(X)$.

Moreover, in the case where $R \cap T_X = \emptyset$, the matrices $K, L^\top$ attain the maximum in (2'). Indeed, define $I^*, J^*$, $I$ and $J$ by

$$I^* := [m] \setminus \sigma(X),$$

$$J^* := [m] \setminus \sigma(X),$$

$$I := \sigma(R \cap X) \cup I^*,$$

$$J := \sigma(R \setminus X) \cup J^*.$$  

Then the submatrix $(KML^\top)[I, J]$ is an $(n - |R \setminus X|) \times (n - |R \cap X|)$ zero submatrix as in Figure 3.

## 3 Algorithm

In this section, we consider the weighted linear matroid intersection problem. In Section 3.1 we formulate the problem as the computation of the degree of the determinant of a rational matrix associated with given two linear matroids and weight. In Section 3.2 we specialize Deg-Det to present our algorithm for the weighted linear matroid intersection weight splitting problem. Its time complexity is analyzed in Section 3.3, and its relation to Frank’s algorithm is discussed in Section 3.4.

### 3.1 Algebraic formulation of weighted linear matroid intersection

Let $A, B$ be $n \times m$ matrices over $Q$ as in Section 2.3 and let $M(A)$ and $M(B)$ be the associated linear matroids on $[m]$. We assume that both $A$ and $B$ have no zero columns. In addition to $A, B$, we are further given integer weights $c_i \in \mathbb{Z}$ for $i \in [m]$. The goal of the weighted linear matroid intersection problem is to maximize the weight $c(X) := \sum_{i \in X} c_i$ over all common independent sets $X$.

Here we consider a restricted situation when the maximum is taken over all common independent sets of cardinality $n$. In this case, the maximum weight is interpreted as the degree of the determinant of the following rational matrix. Let $M$ be an $n \times n$ rational matrix defined by

$$M := \sum_{i=1}^m a_i b_i^\top x_i c^i.$$

**Lemma 3.1.** deg det $M$ is equal to the maximum of the weight $c(X)$ over all common independent sets $X$ of cardinality $n$.

**Proof.** As in the proof of Theorem 2.2 by Binet-Cauchy formula applied to $M$, we obtain det $M = \sum_{X \subseteq [m] : |X| = n} \det A[X] \det B[X] \det c(X) \prod_{i \in X} x_i$, and

$$\deg \det M = \max\{c(X) \mid X \subseteq [m] : \det A[X] \det [X] \neq 0\}.$$  

□
Lemma 3.2 ([1]). For the setting \( M_i := a_ib_i^T t^i \) \((i \in [m])\), the algorithm \textbf{Deg-Det} outputs \( \text{deg det} \ M \).

Proof. Consider step 2 of \textbf{Deg-Det}. Here \((PM_iQ)^0\) is also written as \( a_i^0b_i^T \) for some \( a_i^0, b_i^0 \in \mathbb{Q}^n \); see the next subsection. In particular, \((PMQ)^0 = \sum_{i=1}^m a_i^0b_i^T x_i \). Therefore, by Theorem 2.2, \((PMQ)^0\) is nonsingular if and only if \( r+s \leq n \). Thus, if the algorithm terminates, then \((PMQ)^0\) is nonsingular and \( D^* = \text{deg det} \ M \) by Lemma 2.1 \( \Box \)

3.2 Algorithm description

Here we present our algorithm by specializing \textbf{Deg-Det}. The basic idea is to apply Edmonds’ algorithm to solve the problem \((P^0)\) for \((PMQ)^0 = \sum_{i=1}^m (PM_iQ)^0 x_i \). We first consider the case where \( P \) and \( Q \) are diagonal matrices represented as \( P = (t^\alpha) \) and \( Q = (t^\beta) \) for some \( \alpha, \beta \in \mathbb{Z}^n \). In this case, \((PMQ)^0\) is explicitly written as follows. Observe that the properness of \( PMQ \) is equivalent to

\[
\alpha_k + \beta_\ell + c_i \leq 0 \quad (i \in [m], k, \ell \in [n] : (a_i)_k(b_i)_\ell \neq 0).
\]

(3.1)

For \( i \in [m] \), define \( a_i^0, b_i^0 \in \mathbb{Q}^n \) by

\[
(a_i)_k^0 := \begin{cases} (a_i)_k \text{ if } \exists \ell \in [n], (a_i)_k(b_i)_\ell \neq 0, \\ 0 \text{ otherwise,} \end{cases} \quad \text{(3.2)}
\]

\[
(b_i)_\ell^0 := \begin{cases} (b_i)_\ell \text{ if } \exists k \in [n], (a_i)_k(b_i)_\ell \neq 0, \\ 0 \text{ otherwise.} \end{cases} \quad \text{(3.3)}
\]

Then \((PM_iQ)^0 = a_i^0b_i^T\). Namely we have

\[
(PMQ)^0 = \sum_{i=1}^m a_i^0b_i^T x_i.
\]

Therefore the step 1 of \textbf{Deg-Det} can be executed by solving the unweighted linear matroid intersection problem for two matroids \( M(A^0) \) and \( M(B^0) \), where the matrices \( A^0, B^0 \) are defined by

\[
A^0 := (a_1^0 a_2^0 \cdots a_m^0), \quad B^0 := (b_1^0 b_2^0 \cdots b_m^0).
\]

Suppose that we are given a common independent set \( X \) of \( M(A^0) \) and \( M(B^0) \). According to Edmonds’ algorithm (given after Lemma 2.3), construct the residual graph \( G^0_X := G_X(A^0, B^0) \) with node sets \( S^0_X := S_X(A^0) \) and \( T^0_X := T_X(B^0) \). Then we can increase \( X \) or obtain \( K, L \) that are optimal to the problem \((P^0)\). A key observation here is that \( K \) and \( L \) commute \((t^\alpha) \) and \((t^\beta) \), respectively:

\[
K(t^\alpha) = (t^\alpha) K, \quad L(t^\beta) = (t^\beta) L.
\]

(3.4)

Indeed, by the definition \((3.2), (3.3)\), if \((a_i^0)_k \) and \((a_i^0)_k' \) are nonzero, then \( \alpha_k = \alpha_k' \) must hold. Therefore, each elementary row operation for \( A^0 \) is done between rows \( k, k' \) with \( \alpha_k = \alpha_k' \). Consequently, the commutation \((3.4)\) hold.

Hence, by updating \( A \leftarrow KA \) and \( B \leftarrow LB \), we can keep \( P, Q \) the form \((t^\alpha), (t^\beta)\) in the next iteration. Now the algorithm is as follows.
Algorithm: Deg-Det-WMI

Input: $n \times m$ matrices $A = (a_1 \ a_2 \ \cdots \ a_m)$, $B = (b_1 \ b_2 \ \cdots \ b_m)$, and weights $c_i \in \mathbb{Z}$ ($i = 1, 2, \ldots, m$).

Output: deg det $M$ for $M := \sum_{i=1}^{m} a_i b_i^\top x_i t^{c_i}$.

0: $X = \emptyset$, $\alpha := -\max_i c_i 1$ and $\beta := 0$.

1: If $|X| = n$, then output $-\sum_{i=1}^{n} (\alpha_i + \beta_i)$ and stop. Otherwise, according to (3.2), (3.3) decompose $A,B$ as $A = A^0 + A'$, $B = B^0 + B'$. Apply elementary row operations to $A,B$ so that $A^0, B^0$ are $X$-diagonal forms.

2: From $A^0, B^0$, construct the residual graph $G_X^0$ and node sets $S_X^0, T_X^0$. Let $R^0$ be the set of nodes reachable from $S_X^0$ in $G_X^0$.

2-1. If $R^0 \cap T_X^0 \neq \emptyset$: Taking a shortest path $P$ from $S_X^0$ to $T_X^0$, let $X \leftarrow X \Delta V(P)$, and go to step 1.

2-2. If $R^0 \cap T_X^0 = \emptyset$: Then $R^0$ determines the zero submatrix $((t^\alpha)M(t^\beta))^0[I,J]$ of maximum size $|I| + |J| \geq (n)$ by (2.3) and (2.4); see also Figures 2 and 3.

Letting $\alpha \leftarrow \alpha + \kappa 1_I$, $\beta \leftarrow \beta - \kappa 1_{[n] \setminus J}$, increase $\kappa$ from 0 until a nonzero entry appears in the zero submatrix. If $\kappa = \infty$ or $-\sum_{i=1}^{n} (\alpha_i + \beta_i) \leq n \min_i c_i$, then output $-\infty$ and stop. Otherwise go to step 1.

It is clear that $X$ is always a common independent set of $M(A^0)$ and $M(B^0)$ and that the algorithm correctly outputs deg det $M$.

Moreover, $X$ is a common independent set of $M(A)$ and $M(B)$ having the maximum weight among all common independent sets of cardinality $|X|$. We show this fact by using the idea of weight splitting [3].

Lemma 3.3. In step 1, define weight splitting $c_i = c_i^1 + c_i^2$ for each $i \in [m]$ by

\begin{align*}
  c_i^1 & := c_i - c_i^2, \quad (3.5) \\
  c_i^2 & := -\max\{\beta \ell \mid \ell \in [n] : (b_i)\ell \neq 0\}. \quad (3.6)
\end{align*}

Then $X$ is a common independent set of $M(A)$ and $M(B)$ such that $c^1(X) = \max\{c^1(Y) \mid Y \in \mathcal{I}(A), |Y| = |X|\}$ and $c^2(X) = \max\{c^2(Y) \mid Y \in \mathcal{I}(B), |Y| = |X|\}$. Thus $X$ maximizes the weight $c(X)$ over all common independent sets of size $|X|$.

Proof. We first verify that $X$ is a common independent set of $M(A)$ and $M(B)$. We may assume $X = \{1, 2, \ldots, h\}$. Since $X$ is commonly independent of $M(A^0)$ and $M(B^0)$, we can assume that $A^0[[h],X] = B^0[[h],X] = I$ in the $X$-diagonal forms. Then $I^* = J^* = \{h+1, \ldots, n\}$. We can further assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_h$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_h$. Necessarily $A[X]$ and $B[X]$ are lower-triangular matrices with nonzero diagonals. Hence $X$ is commonly independent in $M(A)$ and $M(B)$.

Next we make some observations to prove the statement. Observe from the definition (3.5) (3.6) and the properness (3.1) that

\begin{align*}
  c_i^1 & \leq -\alpha_k \quad (\forall k : (a_i)_k \neq 0), \quad (3.7) \\
  c_i^2 & \leq -\beta_\ell \quad (\forall \ell : (b_i)\ell \neq 0), \quad (3.8)
\end{align*}
and
\[c_1 i = -\alpha_k, \quad c_2 j = -\beta_{\ell} \quad (\exists k, \ell : (a_i^0)\kappa (b_j^0)\ell \neq 0).\] (3.9)

We also observe
\[
\max_{k \in [n]} \alpha_k = \alpha_{k'} (k' \in I^*), \quad \max_{\ell \in [n]} \beta_{\ell} = \beta_{\ell'} (\ell' \in J^*).
\] (3.10)

This follows from the way of update \(\alpha \leftarrow \alpha + 1, \beta \leftarrow \beta - 1\) with the initialization \(\alpha = -\max, c_1, \beta = 0\) of the algorithm, and the fact that both \(I^* \subseteq I\) and \(J^* \subseteq J\) monotonically decrease.

Finally we prove that \(X\) maximizes both weights \(c_1\) and \(c_2\). It suffices to show
\[
c_1^1 (X) \geq c_1^1 (X \cup \{i \} \setminus \{j \}) \quad (i \not\in X, j \in X : X \cup \{i \} \setminus \{j \} \in \mathcal{I}(A)), \quad (3.11)
\]
\[
c_2^2 (X) \geq c_2^2 (X \cup \{i \} \setminus \{j \}) \quad (i \not\in X, j \in X : X \cup \{i \} \setminus \{j \} \in \mathcal{I}(B)). \quad (3.12)
\]

Indeed, this is the well-known optimality criterion of the maximum(minimum) weight base problem on an oriented graph. Take \(i, j\) with \(X \cup \{i \} \setminus \{j \} \in \mathcal{I}(A)\). If there is a nonzero element \(a_{k^\ast i} \neq 0\) for \(k^\ast \in I^*\), then by \((3.7)\) and \((3.10)\) it holds \(c_1^1 \leq -\alpha_{k^\ast} \leq -\alpha_j = c_2^1\), and \((3.11)\) holds. Suppose not. Let \(k \in [n]\) be the smallest index such that \(a_{ij} \neq 0\). Then \(c_1^1 \leq -\alpha_k\). Now \(A[[h], X]\) is lower triangular. Additionally, by \((3.10)\), \(A[I^*, X] = A^0[I^*, X]\) is a zero matrix. Therefore, it must hold \(j \geq k\) for \(i, j\) to belong to a circuit in \(X \cup \{i\}\). Hence, \(c_1^1 = -\alpha_j \geq -\alpha_k \geq c_2^1\), and where the first equality follows from \((3.9)\) and \((a_j^i)\) \(j = 1\). Thus \((3.11)\) holds. \((3.12)\) is similarly shown.

\[\Box\]

### 3.3 Analysis

We analyze the time complexity of \textit{Deg-Det-WMI}. It is obvious that if \(R^0 \cap T^0_X \neq \emptyset\) (step 2-1) occurs, then the rank of \((t^\alpha) M(t^\beta))^0\) increases. Therefore the algorithm goes step 2-1 at most \(n\) times. The main analysis concerns step 2-2, particularly, how nonzero entries appear, how they affect \(A^0\), \(B^0\), and \(G^0_X\), and how many times these scenarios occur until \(R^0 \cap T^0_X \neq \emptyset\).

As \(\kappa\) increases, the submatrix \((t^\alpha) M(t^\beta))^0[[n] \setminus I, [n] \setminus J]\) becomes a zero block, since the degree of each element decreases. Accordingly, \(A^0[[n] \setminus I, R^0 \setminus X]\) and \(B^0[[n] \setminus J, E \setminus R^0]\) become zero blocks; see Figure 4. Then, in \(G^0_X\), all arcs entering \(R^0\) disappear. In particular, \(S^0_X, T^0_X\), and \(R^0\) do not change.

Next we analyze the moment when a non-zero element appears in \((t^\alpha) M(t^\beta))^0[I, J]$. Then, in the next step 1, it holds

\[(a_i^0)\kappa (b_j^0)\ell \neq 0\]

for some \(i \in [m], k \in I, \ell \in J\). In this case, a new nonzero element appears in the \(i\)-th column of \(A^0\) or \(B^0\).

#### (a-1) If \(i \not\in R^0\) and \(i \in X\): In the next step 1, Gaussian elimination for \(A^0\) makes the new nonzero element \((a_i^0)\kappa = (a_i^0)\kappa \neq 0\). Since \(A^0[[n] \setminus I, R^0] = 0\), this does not affect \(A^0[R]\). Therefore \(R^0\) is still reachable from \(S^0_X\). There may appear nonzero elements in \(A^0[[n] \setminus I, E \setminus R^0]\), which will make \(R^0\) or \(S^0_X\) larger in the next step 2.

#### (a-2) If \(i \not\in R^0\) and \(i \not\in X\): By \((a_i^0)\kappa \neq 0\) if \(k \in I^*\), then \(i\) is included to \(S^0_X\). Otherwise there appears an arc in \(G^0_X\) from \(X \cap R^0\) to \(i\). By \(\ell \in J\), if \(\ell \in J^*\), then \(i\) belongs to \(T^0_X\). Otherwise there is an arc from \(i\) to \(X \setminus R^0\). Then, \(R^0 \cap T^0_X\) becomes nonempty for the former case, \(|X \cap R^0|\) increases for the latter case.
(b-1) If $i \in R^0$ and $i \in X$: Similar to the analysis of (a-1) above, Gaussian elimination for $B^0$ makes $(b_i)_k = (b_i)_k$ zero, and $R^0$ and $T_X^0$ increase or do not change.

(b-2) If $i \in R^0$ and $i \not\in X$: By $(b_i)_\ell \neq 0$, if $\ell \in J^*$, then $i$ is included to $T_X^0$. Otherwise there appears an arc from $i$ to $X \setminus R^0$. If $k \in I^*$, then $R^0 \cap T_X^0$ becomes nonempty. Otherwise $|X \cap R^0|$ increases.

Therefore, if the case (a-2) or (b-2) occurs, then $T_X^0 \cap R^0 \neq \emptyset$ or $|X \cap R^0|$ increases. After $O(n)$ occurrences of the cases (a-2) and (b-2), $T_X^0 \cap R^0$ becomes nonempty and $|X|$ increases. When $X$ is updated, Gaussian elimination constructs the $X$-diagonal forms of $A^0, B^0$ in $O(mn^2)$ time.

We analyze the occurrences of (a-1) and (b-1). When $(a_i)_k$ for $i \in X \setminus R^0, k \in I$ becomes nonzero, it is eliminated by the row operation, and $(a_i)_k = (a_i)_k$ never becomes nonzero. Therefore, (a-1) and (b-1) occur at most $O(n|X|)$ time until $X$ is updated, where the row operation is executed in $O(m)$ time per each occurrence. The total time for the elimination is $O(nm|X|)$. The augmentation $\kappa$ and the identification of the next nonzero elements are computed in $O(nm)$ time by searching nonzero elements in $A, B$, which is needed when one of (a-1), (a-2), (b-1), and (b-2) occurs. Thus, by the naive implementation, Deg-Det-WMI runs in $O(mn^3)$ time.

We improve this complexity to $O(mn^3 \log n)$ as follows. Observe first that $\kappa$ is given by

$$\kappa = -\max\{c_i + \alpha_k + \beta_\ell | i \in [m], k \in I, \ell \in J : (a_i)_k(b_i)_\ell \neq 0\}.$$ 

The main idea is to sort indices $(i, k, \ell) \in [n] \times I \times J$ according to $c_i + \alpha_k + \beta_\ell$ and keep in a binary heap potential indices that attain $\kappa$. Notice that even if $(a_i)_k(b_i)_\ell$ is zero in a moment, it will become nonzero by row operations in (a-1) and (b-1) and can appear
in \(((t^n)M(t^3))^0[I, J]\) later. On the other hand, any index \((i, k, \ell)\) with \(c_i + \alpha_k + \beta_\ell > 0\) keeps \((a_i)_k(b_i)_\ell = 0\) and is irrelevant until \(X\) is updated.

Suppose now that \(X, A^0, B^0,\) and \(G^0_X\) were updated in step 1. By BFS for \(G^0_X\), we determine the reachable set \(R^0\) and the index sets \(I, J\). By sorting \(c_i + \beta_\ell \quad (i \in [m], \ell \in J)\) in \(O(mn \log n)\) time, we construct an array \(p\) such that the \(e\)-th entry \(p[e]\) has all indices \((i, k, \ell)\) with \(e\)-th largest \(c_i + \beta_\ell\) as a linked list. For each \(k \in I\), let \(p_k\) denote the copy of the array \(p\), where \(p_k[e]\) also has the value \(v_{k,e} := c_i + \alpha_k + \beta_\ell\) for indices \((i, \ell)\) in \(p_k[e]\). By the head index of \(p_k\) (relative to \(\alpha, \beta, I, J\)), we mean the minimum index \(e_k\) such that \(p_k[e_k]\) has the value \(v_{k,e_k}\) less than 0 and an index \((i, \ell)\) with \(\ell \in J\), where \(J\) will change later. Notice that if \(p_k[e]\) has the value \(v_{k,e} \geq 0\), then \((a_i)_k(b_i)_\ell = 0\) for all indices \((i, \ell)\) in \(p_k[e]\). Construct a binary (max) heap consisting of the pointers to the head indices \(e_k\) for all \(k \in I\), where the key is the value \(v_{k,e_k}\) of \(p_k[e_k]\). In the construction of the heap, if the key \(v_{k,e_k}\) of a node is equal to the key \(v_{k',e_k'}\) of its parent node, then the two nodes are combined as a single node and the corresponding pointers are also combined as a single list. Then, by referring to the root of the heap, we know all indices \((i, k, \ell)\in [m] \times I \times J\) having the maximum negative value. Increase \(\kappa\) to the negative of this value. If the root has no index \((i, k, \ell)\) with \((a_i)_k(b_i)_\ell \neq 0\), then delete the root from the heap, update the head index of each \(p_k\) indicated by the (deleted) root, and add the pointers of new head indices to the heap. Suppose that the root has an index \((i, k, \ell)\) with \((a_i)_k(b_i)_\ell \neq 0\); then \(\kappa = -c_i - \alpha_k - \beta_\ell\). If \(i \in X\) then execute the row operation to make \((a_i)_k(b_i)_\ell\) zero. After such row operations, the root has no index \((i, k, \ell)\) with \(i \in X\) and \((a_i)_k(b_i)_\ell \neq 0\). Suppose that there is \((i, k, \ell)\) with \(i \notin X\) and \((a_i)_k(b_i)_\ell \neq 0\). Then \(G^0_X, R^0, I,\) and \(J\) are updated. In particular, \(I\) increases and \(J\) decreases. For each newly added \(k \in I\), construct array \(p_k\) (from \(p\)), identify the head index of \(p_k\), and add the pointer to the heap. In this way, until \(X\) increases, each index \((i, k, \ell)\) is referred to at most twice, and the heap is updated in \(O(\log n)\) time par the reference. In total, \(O(mn^2 \log n)\) time is required. Thus we have:

**Theorem 3.4.** Algorithm Deg-Det-WMI runs in \(O(mn^3 \log n)\) time.

### 3.4 Relation to Frank’s algorithm

In this subsection, we reveal the relation between our algorithm Deg-Det and Frank’s weight splitting algorithm [3]. We show that the common independent sets \(X\) obtained by Deg-Det are the same as the ones obtained by a slightly modified version of Frank’s algorithm. This means in a sense that Deg-Det is a nonstandard specialization of Frank’s algorithm to linear matroids.

Let us briefly explain Frank’s algorithm; our presentation basically follows [12 section 13.7]. His algorithm keeps a weight splitting \(c_i = c^1_i + c^2_i\) for each \(i \in E\) and a common independent set \(X\) such that \(X\) is maximum for both \(c^1_i\) and \(c^2_i\) over all common independent set of size \(|X|\).

0: \(c^1_i := c_i, c^2_i := 0\) for \(i \in E\) and \(X := \emptyset\).

1: By applying elementary row operations to \(A, B,\) construct the residual graph \(G_X\), and node sets \(S_X, T_X\) as in Section 2.2.

2: From the weight splitting \(c = c^1 + c^2\), construct subgraph \(\bar{G}_X\) of \(G_X\) and node subsets \(\bar{S}_X \subseteq S_X, \bar{T}_X \subseteq T_X\) by: \(\bar{G}_X\) consists of arcs \(ij\) with \(i \in X \neq j\) and \(c^1_i = c^2_j\) or
\[\begin{align*}
i \notin X \in j \text{ and } c_i^2 &= c_j^2, \\
\bar{S}_X &:= \{i \in S_X \mid \forall j \in S_X, c_i^1 \geq c_j^1\}, \\
\bar{T}_X &:= \{i \in T_X \mid \forall j \in T_X, c_i^2 \geq c_j^2\}.
\end{align*}\] (3.13) (3.14)

3: Let \( R \) be the set of nodes reachable from \( \bar{S}_X \) in \( \bar{G}_X \).

4-1: If \( \bar{R} \cap \bar{T}_X \neq \emptyset \), for a shortest path \( P \) from \( S_X \) to \( T_X \), replace \( X \) by \( X \Delta V(P) \); go to step 1.

4-2: If \( \bar{R} \cap \bar{T}_X = \emptyset \), then let \( c_i^1 := c_i^1 - \epsilon, \ c_i^2 := c_i^2 + \epsilon \) for \( i \in \bar{R} \), and increase \( \epsilon \) from 0 until \( \bar{R} \) increases. Go to step 2.

We consider a modified update of weight splitting. Let \( R' \) be the subset of nodes \( i \in E \setminus (X \cup \bar{R}) \) such that it holds \( j \in X \cap R \) for all arcs \( ij \) leaving \( i \). Then the step 4-2 can be replaced by the following:

4-2': If \( \bar{R} \cap \bar{T}_X = \emptyset \), then let \( c_i^1 := c_i^1 - \epsilon, \ c_i^2 := c_i^2 + \epsilon \) for \( i \in \bar{R} \cup \bar{R}' \), and increase \( \epsilon \) from 0 until \( \bar{R} \) increases or \( \bar{R}' \) changes. Repeat until \( \bar{R} \) increases and go to step 2.

One can easily check that \( X \) keeps the optimality (3.11), (3.12) in the modified update. Hence, the modified algorithm using 4-2' is also correct.

We prove that \( G_0^0, S_0^0, T_0^0 \) in our algorithm and \( \bar{G}_X, \bar{S}_X, \bar{T}_X \) in modified Frank’s algorithm are the same up to an obvious redundancy. Here an arc in \( \bar{G}_X \) is said to be redundant if it leaves a node \( i \) that has no arc entering \( i \).

**Proposition 3.5.** Suppose that \( X, \alpha \) and \( \beta \) are obtained in an iteration of \( \text{Deg-Det} \). Define weight splitting \( c_i = c_i^1 + c_i^2 \) by (3.5), (3.6) and \( \bar{G}_X, \bar{S}_X \) and \( \bar{T}_X \) by (3.13), (3.14). Then we have the following:

1. \( G_0^0_X \) is equal to the subgraph of \( \bar{G}_X \) obtained by removing redundant arcs.
2. \( S_0^0_X \) is equal to \( \bar{S}_X \).
3. \( T_0^0_X \) is equal to the subset of \( \bar{T}_X \) obtained by removing isolate nodes.
4. \( R_0^0 \) is equal to \( \bar{R} \).
5. The total sum of increases \( \kappa \) until \( R_0^0 \) changes is equal to that of increases \( \epsilon \) until \( \bar{R} \) changes in the modified Frank’s algorithm.

**Proof.** Recall (the proof of) Lemma 3.3 that \( X \) is a common independent set of \( M(A) \) and \( M(B) \). Suppose that \( X = \{1, 2, \ldots, h\} \) and \( A^0[[h], X] = B^0[[h], X] = I \) with \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_h \) and \( \beta_1 \geq \beta_2 \geq \cdots \geq \beta_h \). Observe first that \( S_0^0_X \subseteq S_0^X \). Indeed, from (3.10), \( A[I^*, X] \) is a zero matrix. Therefore, if \( a_0^0 \) has a nonzero vector in a row in \( I^* \), i.e., \( i \in S_0^X \), then \( a_i \) is independent from \( a_{i'} (i' \in X) \), i.e., \( i \in S_X \). Consider the weight splitting of nodes in \( S_0^X \). For \( i \in S_0^X \), \( c_i^1 = -\alpha_k (k \in I^*) \), and \( -\alpha_k \geq c_j^2 \) for \( j \in S_X \) by (3.10). Thus \( S_0^X \subseteq S_X \). Also, for \( i' \in S_X \), it holds \( c_i^1 < -\alpha_k \) for \( k \in I^* \) since \( a_0^0 \) is a zero vector. Then \( c_i^1 < -\alpha_k = c_j^1 \) for \( i \in S_0^X \). Thus we have (2).

Showing (3) is similar. As above, we see that \( T_0^0_X \subseteq T_X \) and for \( i \in T_0^0_X \), \( c_i^2 = -\beta_{i'} \) (\( \ell \in J^* \)). Then \( T_0^0_X \subseteq \bar{T}_X \). Let \( i \in T_X \setminus T_0^0_X \). Then \( b_i^0 \) is a zero vector, and so is \( a_0^0 \).
Suppose that arc $ji$ for $j \in X$ exists in $G_X$. Recall that $A[[h], X]$ and $B[[h], X]$ are lower triangular. Then $j$ is at least the minimum index $k$ with $(a_k)_k \neq 0$. Then for $\ell \in J^*$, $c^1_j = -\alpha_j \geq -\alpha_k > c_i + \beta_\ell = c_i - c^2_i = c^1_i$, where the strictly inequality follows from the fact that $a^0_k$ and $b^0_i$ are zero vectors. Then $ji$ does not exist in $\bar{G}_X$. Similarly, arc $ij$ does not exist in $\bar{G}_X$. Thus we have (3).

Next we compare $G^0_X$ and $\bar{G}_X$ to prove (1) and (4). Consider a node $i \in E \setminus X$ such that $a^0_i$ and $b^0_i$ are nonzero. Suppose that arc $ki$ exists in $G^0_X$, i.e., $(a^0_i)_k \neq 0$ for $k \in [h] = X$. Then $c^1_i = -\alpha_k = c^1_k$. We show that $ki$ exists also in $\bar{G}_X$. Since $\alpha_j \geq \alpha_k$ $(j \neq k)$ implies $(a_k)_j = 0$, Gaussian elimination making $A$ X-diagonal does not affect $(a_i)_k$. Thus the arcs $ki$ exists in $G_X$ and in $\bar{G}_X$. Similarly, if $il$ exists in $G^0_X$, then $il$ exists in $\bar{G}_X$. Therefore for any node $i \in E \setminus X$ with nonzero $a^0_i, b^0_i$, the arcs incident to $i$ are the same in $G^0_X$ and $\bar{G}_X$. Consider $i \in E \setminus X$ such that $a^0_i$ and $b^0_i$ are zero vectors. In $G^0_X$, there are no arcs incident to $i$. For $k \in X, \ell \in [n]$ with $(a_k)_\ell (b_\ell)_\ell \neq 0$, it holds $c^1_k = -\alpha_k > \beta_\ell + c_i \geq -c^2_i + c_i = c^1_i$. This means that arcs $ki$ entering $i$ do not exist in $G_X$, implying (1). From (1), (2), and (3), we have (4).

Finally we prove (5). The step 2-2 in Deg-Det-WMI changes $\alpha, \beta$ as $\alpha \leftarrow \alpha + \kappa 1_j, \beta \leftarrow \beta - \kappa 1_{[n] \setminus J}$. We analyze the corresponding change of the weight splitting $c = c^1 + c^2$ defined by (3.5), (3.6). Consider $i \in [n]$ such that $a^0_i$ and $b^0_i$ are nonzero vectors. Suppose that $i \in R^0 = R$. Then $a^0_i$ and $b^0_i$ have nonzero entries in a row in $I$ and in $[n] \setminus J$, respectively; see Figure 4. Therefore $c^1_i = -\alpha_k$ for $k \in I$ and $c^2_i = -\beta_\ell$ for $\ell \in [n] \setminus J$, and $c^1_i, c^2_i$ are changed as $c^1_i \leftarrow c^1_i - \kappa, c^2_i \leftarrow c^2_i + \kappa$. Suppose that $i \notin R^0$. Then $a^0_i$ and $b^0_i$ have nonzero entries in a row in $[n] \setminus I$ and in $J$, respectively. In particular, $c^1_i = -\alpha_k$ for $k \in [n] \setminus I$ and $c^2_i = -\beta_\ell$ for $\ell \in J$. Then the weight splitting does not change. Thus, for any node $i$ with nonzero $a^0_i, b^0_i$, the update corresponds to the step 4-2 or 4-2'.

Consider a node $i$ with $a^0_i = b^0_i = 0$. Let $\Lambda$ be the set of indices $k$ that attain $\max_{k \in [n] \setminus [a_k] \neq 0} \alpha_k$, and let $\Pi$ be the set of indices $\ell$ that attain $\max_{\ell \in [n] \setminus (b_\ell) \neq 0} \beta_\ell = -c^2_\ell$.

Case 1: $\Pi \cap J \neq \emptyset$. Then $c^2_\ell$ does not change and so does $c^1_\ell$. If $A \cap I \neq \emptyset$, then $\kappa$ can increase until $c^1_\ell$ becomes $-\alpha_k$ for some $k \in \Lambda$.

Case 2: $\Pi \cap J = \emptyset$ ($\iff \Pi \subseteq [n] \setminus J$). Then $c^2_\ell$ changes as $c^2_\ell \leftarrow c^2_\ell + \kappa$, and hence $c^1_\ell$ changes as $c^1_\ell \leftarrow c^1_\ell - \kappa$. Here $\kappa$ can increase until $\Pi \cap J \neq \emptyset$; then the situation goes to (Case 1).

Notice that the node $i$ in the case 2 is precisely a node in $\bar{R}'$. Therefore the changes of the weight splitting are the same in Deg-Det-WMI and in the modified Frank’s algorithm (using step 4-2'). The steps are iterated for the same zero submatrix until $R^0$ changes. Therefore, the total sum of $\kappa$ is the same as that of $\epsilon$ in the modified Frank’s algorithm.

By this property, the obtained sequences of common independent sets $X$ can be the same in Deg-Det-WMI and the modified Frank’s algorithm. Therefore Deg-Det-WMI can also be viewed as yet another implementation of Frank’s algorithm for linear matroids. A notable feature of Deg-Det-WMI is to skip unnecessary eliminations in constructing the residual graphs. To see this fact, consider the partition $\{\sigma_1, \sigma_2, \ldots, \sigma_{n'}\}$ of $[n]$ such that $k, k' \in [n]$ belong to the same part if and only if $\alpha_k = \alpha_{k'}$. Then the elimination matrix $K$ is a block diagonal matrix with block diagonals of size $|\sigma_i| \times |\sigma_i|$. This means that the Gaussian elimination for $A$ after $X$ changes is done in $O(m \sum_i |\sigma_i|^2)$.

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time. Therefore, if values $\alpha_k, \beta_{\ell}$ are scattered, then $K, L$ are very sparse, and the update of $G_X^0$ after $X$ changes is very fast. On the other hand, necessary eliminations skipped at this moment will be done in the occurrences of (a-1) and (b-1). Hence, Deg-Det-WMI reduces eliminations compared with the usual implementation of Frank’s algorithm to linear matroids. More thorough analysis is left to a future work.

We close this paper by giving an example in which the elimination results are actually different in the two algorithms.

**Example 3.6.** Consider matrices

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and weight $c = (3 2 3 1 1)$. The both algorithms for this input can reach at $\alpha = (-2 -2 -2 -2), \beta = (-1 0 0 0)$ and $X = \{1, 2\}$ without elimination. Consider Deg-Det-WMI from this moment. The matrices $A^0$ and $B^0$ are given by

$$A^0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, B^0 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$  

The Gaussian elimination makes $(a_2^0)_2$ zero. Then $G_X^0$ consists of one arc 31, and $S_X^0 = \{3\}$ and $T_X^0 = \emptyset$. The reachable set $R^0$ is determined as $R^0 = \{1, 3\}$, and $I, J$ are given by $I = \{1, 2, 3\}, J = \{2, 3, 4\}, I^* = \{1, 3\}$, and $J^* = \{2, 3\}$. Then $\alpha, \beta$ are changed as $\alpha = (-1 -1 -1 -2), \beta = (-2 0 0 0)$ without occurrences of (a-1) and (b-1). Nonzero elements appear in $A^0[I^*, \{4, 5\}]$ and $B^0[J^*, \{4, 5\}]$, which implies $S_X^0 \cap T_X^0 = \{4, 5\}$. So $X$ is increased.

Therefore Deg-Det-WMI succeeds the augmentation without eliminating $(b_2)_1$, whereas Frank’s algorithm eliminates this element in constructing $G_X$.

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**References**

[1] J. Edmonds: Systems of distinct representatives and linear algebra, *Journal of Research of the National Bureau of Standards* **71B** (1967), 241–245.

[2] J. Edmonds: Matroid intersection. *Annals of Discrete Mathematics* **4** (1979), 39–49.

[3] A. Frank: A weighted matroid intersection algorithm. *Journal of Algorithms* **2** (1981), 328–336.

[4] H. N. Gabow and Y. Xu: Efficient theoretic and practical algorithms for linear matroid intersection problems. *Journal of Computer and System Sciences* **53** (1996), 129–147.
[5] A. Garg, L. Gurvits, R. Oliveira, and A. Wigderson: Operator scaling: theory and applications, *Foundations of Computational Mathematics*, (2019).

[6] H. Hirai: Computing the degree of determinants via discrete convex optimization on Euclidean buildings. *SIAM Journal on Applied Geometry and Algebra*, to appear.

[7] C.-C. Huang, N. Kakimura, and N. Kamiyama: Exact and approximation algorithms for weighted matroid intersection. *Mathematical Programming, Series A* **177** (2019), 85–112.

[8] G. Ivanyos, Y. Qiao, and K. V. Subrahmanyam: Constructive noncommutative rank computation in deterministic polynomial time over fields of arbitrary characteristics, *Computational Complexity* **27** (2018), 561–593.

[9] K. Murota: Computing the degree of determinants via combinatorial relaxation, *SIAM Journal on Computing* **24** (1995), 765–796.

[10] K. Murota: *Matrices and Matroids for Systems Analysis*, Springer-Verlag, Berlin, 2000.

[11] L. Lovász: Singular spaces of matrices and their application in combinatorics, *Boletim da Sociedade Brasileira de Matemática* **20** (1989), 87–99.

[12] B. Korte and J. Vygen: *Combinatorial Optimization: Theory and Algorithms, Sixth Edition*, Springer-Verlag, Berlin, 2018.

[13] E. L. Lawler: Matroid intersection algorithms, *Mathematical Programming* **9** (1975), 31–56.

[14] A. Schrijver: *Combinatorial Optimization—Polyhedra and Efficiency*, Springer-Verlag, Berlin, 2003.