On $q$-component models on the Cayley tree: the general case

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Abstract. In this paper we generalize results of Rozikov (2005 Lett. Math. Phys. 71 27) for $q$-component models on a Cayley tree of order $k \geq 2$. We generalize them in two directions: (1) from $k = 2$ to any $k \geq 2$; (2) from concrete examples (Potts and SOS models) of $q$-component models to any $q$-component models (with nearest neighbour interactions). We give a set of periodic ground states for the model. Using the contour argument which was developed in Rozikov (2005 Lett. Math. Phys. 71 27) we show the existence of $q$ different Gibbs measures for $q$-component models on the Cayley tree of order $k \geq 2$.

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1. Introduction

This paper is the continuation of our investigations (see [12, 13]) on developing a contour method on the Cayley tree. We investigate \( q \)-component spin models on the Cayley tree. One of the key problems related to such spin models is the description of the set of Gibbs measures. The method used for the description of Gibbs measures on the Cayley tree is the method of Markov random field theory and recurrent equations of this theory (see e.g. [2, 8, 11]). If one considers a spin model with competing interactions, then description of Gibbs measures by the method becomes a difficult problem. The problem of description of Gibbs measures has a good connection with the problem of the description of the set of ground states, because the phase diagram of Gibbs measures is close to the phase diagram of the ground states for sufficiently small temperatures (see [4]–[7], [9, 10, 14] for details). A theory of phase transitions at low temperatures in general classical lattice (on \( \mathbb{Z}^d \)) systems was developed by Pirogov and Sinai. This theory is now globally known as the Pirogov–Sinai theory or contour arguments [10], [14]–[16].

In this paper we investigate \( q \)-component models on the Cayley tree. We generalize the results of paper [12]. The organization of the paper is as follows. In section 2 following [12] we recall all necessary definitions. In section 3 we generalize properties of contours of [12] from \( k = 2 \) to any \( k \geq 2 \). In section 4 we describe a set of ground states for the model. Section 5 is devoted to proving the existence of \( q \) different Gibbs measures for any \( q \)-component models with nearest neighbouring interactions on the Cayley tree of order \( k \geq 2 \). Note that in [12] this result was proved for Potts and SOS models on the Cayley tree of order 2.

2. Definitions

2.1. The Cayley tree

The Cayley tree \( \Gamma^k \) (see [1]) of order \( k \geq 1 \) is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly \( k + 1 \) edges issue. Let \( \Gamma^k = (V, L, i) \), where \( V \) is the
set of vertices of $\Gamma^k$, $L$ is the set of edges of $\Gamma^k$ and $i$ is the incidence function associating each edge $l \in L$ with its end-points $x, y \in V$. If $i(l) = \{x, y\}$, then $x$ and $y$ are called nearest neighbouring vertices and we write $l = \langle x, y \rangle$. The distance $d(x, y), x, y \in V$ on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d \exists x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \text{ such that } \langle x_0, x_1 \rangle, \ldots, \langle x_{d-1}, x_d \rangle\}.$$  

For fixed $x^0 \in V$ we set $W_n = \{x \in V \mid d(x, x^0) = n\}$,

$$V_n = \{x \in V \mid d(x, x^0) \leq n\}, \quad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}. \quad (1)$$

It is known that there exists a one-to-one correspondence between the set $V$ of vertices of the Cayley tree of order $k \geq 1$ and the group $G_k$ of the free products of $k + 1$ cyclic groups of second order with generators $a_1, a_2, \ldots, a_{k+1}$.

2.2. Configuration space and the model

We consider models where the spin takes values in the set $\Phi = \{v_1, v_2, \ldots, v_q\}, q \geq 2$. A configuration $\sigma$ on $V$ is then defined as a function $x \in V \rightarrow \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$.

Assume that the group of spatial shifts acts on $\Omega$. We define a $F_k$-periodic configuration as a configuration $\sigma(x)$ which is invariant under a subgroup of shifts $F_k \subset G_k$ of finite index. For a given periodic configuration the index of the subgroup is called the period of the configuration. A configuration that is invariant with respect to all shifts is called translational invariant.

The Hamiltonian of the $q$-component model has the form

$$H(\sigma) = \sum_{\langle x, y \rangle \in L} \lambda(\sigma(x), \sigma(y)) + \sum_{x \in V} h(\sigma(x)) \quad (2)$$

where $\lambda(v_i, v_j) = \lambda_{ij}$, $i, j = 1, \ldots, q$, is given by a symmetric matrix of order $q \times q$, $h(v_j) \equiv h_j \in R$, $j = 1, \ldots, q$, and $\sigma \in \Omega$.

2.3. Contours on the Cayley tree

Let $\Lambda \subset V$ be a finite set, $\Lambda^\prime = V \setminus \Lambda$ and $\omega_\Lambda = \{\omega(x), x \in \Lambda\}$, $\sigma_\Lambda = \{\sigma(x), x \in \Lambda\}$ a given configuration. The energy of the configuration $\sigma_\Lambda$ has the form

$$H_\Lambda(\sigma_\Lambda|\omega_\Lambda^\prime) = \sum_{\langle x, y \rangle \in \Lambda; x, y \in \Lambda} \lambda(\sigma(x), \sigma(y)) + \sum_{\langle x, y \rangle \in \Lambda; x \in \Lambda, y \in \Lambda^\prime} \lambda(\sigma(x), \omega(y)) + \sum_{x \in \Lambda} h(\sigma(x)). \quad (3)$$

Let $\omega_\Lambda^{(i)} \equiv v_i, i = 1, \ldots, q$, be a constant configuration outside $\Lambda$. For each $i$ we extend the configuration $\sigma_\Lambda$ inside $\Lambda$ to the entire tree by the $i$th constant configuration and denote this configuration by $\sigma_\Lambda^{(i)}$ and $\Omega_\Lambda^{(i)} = \{\sigma_\Lambda^{(i)}\}$. Now we describe a boundary of the configuration $\sigma_\Lambda^{(i)}$.

Consider $V_n$ and for a given configuration $\sigma_\Lambda^{(i)} \in \Omega_\Lambda^{(i)}$ define $V_n^{(i)} = V_n^{(i)}(\sigma_\Lambda^{(i)}) = \{t \in V_n; \sigma_\Lambda^{(i)}(t) = v_j\}, j = 1, \ldots, q$. Let $G_n = (V_n^{(i)}, L_n^{(i)})$ be the graph such that

$$L_n^{(j)} = \{l = \langle x, y \rangle \in L \mid x, y \in V_n^{(j)}\}, \quad j = 1, \ldots, q.$$
It is clear that for a fixed \( n \) the graph \( G^{n,j} \) contains a finite number (\( = m \)) of maximal connected subgraphs \( G^{n,j}_r \), i.e.

\[
G^{n,j} = \{G^{n,j}_1, \ldots, G^{n,j}_m\}, \quad G^{n,j}_r = (V^{(j)}_{n,r}, L^{(j)}_{n,r}), \quad r = 1, \ldots, m.
\]

Here \( V^{(j)}_{n,r} \) is the set of vertices and \( L^{(j)}_{n,r} \) the set of edges of \( G^{n,j}_r \).

For a set \( A \) denote by \( |A| \) the number of elements in \( A \).

Two edges \( l_1, l_2 \in L \) (\( l_1 \neq l_2 \)) are called nearest neighbouring edges if \( |i(l_1) \cap i(l_2)| = 1 \) and we write \( \langle l_1, l_2 \rangle_1 \).

For any connected component \( K \subset \Gamma^k \) denote by \( E(K) \) the set of edges of \( K \) and

\[
b(K) = \{l \in L \setminus E(K): \exists l_1 \in E(K) \text{ such that } \langle l, l_1 \rangle_1\}.
\]

**Definition 1.** An edge \( l = \langle x, y \rangle \in L_{n+1} \) is called a boundary edge of the configuration \( \sigma^{(i)}_{V_n} \) if \( \sigma^{(i)}_{V_n}(x) \neq \sigma^{(i)}_{V_n}(y) \). The set of boundary edges of the configuration is called the boundary \( \partial(\sigma^{(i)}_{V_n}) \equiv \Gamma \) of this configuration.

The boundary \( \Gamma \) consists of \( q(q - 1)/2 \) parts

\[
\partial(\sigma^{(i)}_{V_n}) \equiv \Gamma, \quad \epsilon \in \{ij: i < j; i, j = 1, \ldots, q\} \equiv Q_q,
\]

where for instance \( \Gamma_{12} \) is the set of edges \( l = \langle x, y \rangle \) with \( \sigma(x) = v_1 \) and \( \sigma(y) = v_2 \).

The (finite) sets \( b(G^{n,j}_r), j = 1, \ldots, q, r = 1, \ldots, m \) (together with an indication for each edge of this set of which part \( \Gamma_{\epsilon}, \epsilon \in Q_q, \) of the boundary contains this edge) are called subcontours of the boundary \( \Gamma \).

The set \( V^{(j)}_{n,r}, j = 1, \ldots, q, r = 1, \ldots, m, \) is called the interior \( \text{Int} b(G^{n,j}_r) \) of \( b(G^{n,j}_r) \).

The set of edges from a subcontour \( \gamma \) is denoted by \( \text{supp} \gamma \). The configuration \( \sigma^{(i)}_{V_n} \) takes the same value \( v_j, j = 1, \ldots, q, \) at all points of the connected component \( G^{n,j}_r \).

This value \( v = v(G^{n,j}_r) \) is called the mark of the subcontour and denoted by \( v(\gamma) \), where \( \gamma = b(G^{n,j}_r) \).

The collection of subcontours \( \tau = \tau(\sigma^{(i)}_{V_n}) = \{\gamma_r\} \) generated by the boundary \( \Gamma = \Gamma(\sigma^{(i)}_{V_n}) \) of the configuration \( \sigma^{(i)}_{V_n} \) has the following properties:

(a) every subcontour \( \gamma \in \tau \) lies inside the set \( V_{n+1} \);

(b) for every two subcontours \( \gamma_1, \gamma_2 \in \tau \) their supports \( \text{supp} \gamma_1 \) and \( \text{supp} \gamma_2 \) satisfy

\[
|\text{supp} \gamma_1 \cap \text{supp} \gamma_2| \in \{0, 1\};
\]

the subcontours \( \gamma_1, \gamma_2 \in \tau \) are called adjacent if \( |\text{supp} \gamma_1 \cap \text{supp} \gamma_2| = 1 \);

(c) for any two adjacent subcontours \( \gamma_1, \gamma_2 \in \tau \) we have \( v(\gamma_1) \neq v(\gamma_2) \).

A set of subcontours \( A \subset \tau \) is called connected if for any two subcontours \( \gamma_1, \gamma_2 \in A \) there is a sequence of subcontours \( \gamma_1 = \gamma_1, \gamma_2, \ldots, \gamma_l = \gamma_2 \) in the set \( A \) such that for each \( i = 1, \ldots, l - 1 \) the subcontours \( \gamma_i \) and \( \gamma_{i+1} \) are adjacent.

**Definition 2.** Any maximal connected set (component) of subcontours is called a contour of boundary \( \Gamma \).

Let \( \Upsilon = \{\gamma_r, r = 1, 2, \ldots\} \) (where \( \gamma_r \) is a subcontour) be a contour of \( \Gamma \) and define

\[
\text{Int} \Upsilon = \bigcup_j \text{Int} \gamma_j; \quad \text{supp} \Upsilon = \bigcup_j \text{supp} \gamma_j; \quad |\Upsilon| = |\text{supp} \Upsilon|.
\]
3. Properties of the contours

For $A \subset V$ define $\partial(A) = \{x \in V \setminus A : \exists y \in A, \text{such that} \langle x, y \rangle \}$. Let $G$ be a graph; denote the vertex and edge sets of the graph $G$ by $V(G)$ and $E(G)$, respectively.

**Lemma 3.** Let $K$ be a connected subgraph of the Cayley tree $\Gamma^k$ such that $|V(K)| = n$; then $|\partial(V(K))| = (k - 1)n + 2$.

**Proof.** We shall use the induction on $n$. For $n = 1$ and $2$ the assertion is trivial. Assume that the lemma is true for $n = m$, i.e. from $|K| = m$ it follows that $|\partial(K)| = (k - 1)m + 2$. We shall prove the assertion for $n = m + 1$, i.e. for $K = K \cup \{x\}$. Since $K$ is a connected graph we have $x \in \partial(K)$ and there is a unique $y \in S_1(x) = \{u \in V : d(x, u) = 1\}$ such that $y \in K$. Thus $\partial(K) = (\partial(K) \setminus \{x\}) \cup (S_1(x) \setminus \{y\})$. Consequently,

$$|\partial(K)| = |\partial(K)| - 1 + k = (k - 1)(m + 1) + 2.$$ 

□

**Lemma 4 ([3]).** Let $G$ be a countable graph of maximal degree $k + 1$ (i.e. each $x \in V(G)$ has at most $k + 1$ neighbours) and let $\tilde{N}_{n,G}(x)$ be the number of connected subgraphs $G' \subset G$ with $x \in V(G')$ and $|E(G')| = n$. Then

$$\tilde{N}_{n,G}(x) \leq (e \cdot k)^n.$$ 

For $x \in V$ we will write $x \in \Upsilon$ if there is $l \in \Upsilon$ such that $x \in i(l)$. Define $N_r(x) = \{|\Upsilon : x \in \Upsilon, |\Upsilon| = r\}$.

**Lemma 5.** For any $k \geq 2$ we have

$$N_r(x) \leq \theta \cdot \alpha^r,$$ 

where $\alpha = (2ke)^{k/(k - 1)}$, $\theta = 1/2 \sqrt[k]{\alpha - 1}$.

**Proof.** Denote by $K_\Upsilon$ the minimal connected subgraph of $\Gamma^k$, which contains a contour $\Upsilon$. It is easy to see that if $\Upsilon = \{\gamma_1, \ldots, \gamma_m\}$, $m \geq 1$, then

$$E(K_\Upsilon) = \sup \Upsilon \cup (\cup_{i=1}^m \langle x, y : x, y \in \text{Int} \gamma_i \rangle).$$ 

(5)

Using the fact that if $K$ is a connected subgraph of $\Gamma^k$ then the number of edges of $K$ is equal to $|K| - 1$, equality $\sum_{i=1}^m |\gamma_i| = |\Upsilon| + m - 1$ and lemma 3, we get

$$|E(K_\Upsilon)| = |\Upsilon| + \sum_{i=1}^m (|\text{Int} \gamma_i| - 1) = |\Upsilon| + \sum_{i=1}^m \left( \frac{|\gamma_i| - 2}{k - 1} - 1 \right) = \frac{k}{k - 1} |\Upsilon| - \frac{km + 1}{k - 1}. \quad (6)$$

Since $\Upsilon \subseteq K_\Upsilon$, we get $|\Upsilon| \leq |E(K_\Upsilon)| = (k/(k - 1))|\Upsilon| - (km + 1)/(k - 1)$. Consequently, $1 \leq m \leq (|\Upsilon| - 1)/k$. Combinatorial calculations show that

$$N_r(x) \leq \sum_{m=1}^{\lfloor (r - 1)/k \rfloor} \binom{|K_\Upsilon| - 1}{r} \tilde{N}_{|K_\Upsilon| - 1, r^k}(x), \quad (7)$$

where $[a]$ is the integer part of $a$. Using the inequality $\binom{n}{r} \leq 2^{n-1}, r \leq n$, and lemma 4 from (7) we get (4). □

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4. Ground states

For \( l = \langle x, y \rangle \in L \) and the configuration \( \sigma \in \Omega \) define \( \sigma_l = \{ \sigma(x), \sigma(y) \} \). Define the energy of the configuration \( \sigma_l = \{ v_i, v_j \} \) by
\[
U(\sigma_l) \equiv U_{ij} \equiv \lambda_{ij} + \frac{1}{k+1} (h_i + h_j).
\]
(8)

Then our Hamiltonian can be written as
\[
H(\sigma) = \sum_{l \in L} U(\sigma_l).
\]

For a pair of configurations \( \sigma \) and \( \varphi \) that coincide almost everywhere, i.e. everywhere except for at a finite number of positions, we consider a relative Hamiltonian \( H(\sigma, \varphi) \), the difference between the energies of the configurations \( \sigma, \varphi \), of the form
\[
H(\sigma, \varphi) = H(\sigma) - H(\varphi) = \sum_{l \in L} (U(\sigma_l) - U(\varphi_l)).
\]

**Definition 6.** A periodic configuration \( \varphi \) is called the ground state (for the relative Hamiltonian \( H \)) if \( H(\varphi, \sigma) \leq 0 \) for any configuration \( \sigma \) that coincides with \( \varphi \) almost everywhere.

**Lemma 7.** For any normal subgroup \( F_k \) with index \( r \), \( r \leq q \), of \( G_k \) there exist at least \( \frac{q!}{(q-r)!} \) \( F_k \)-periodic configurations.

**Proof.** Since \( F_k \) is the subgroup of index \( r \) in \( G_k \), the quotient group has the form \( G_k/F_k = \{ F_{k,0}, \ldots, F_{k,r-1} \} \) with the coset \( F_{k,0} = F_k \). An \( F_k \)-periodic configuration \( \sigma_{F_k} \) can be defined as \( \sigma_{F_k}(x) = v_i \) if \( x \in F_{k,i}, i = 0, \ldots, r-1 \). We have at least \( \binom{q}{r} \cdot r! = q!/(q-r)! \) possibilities for defining such a configuration combining the values \( v_1, \ldots, v_q \). This completes the proof. \( \Box \)

**Remark.** If \( r > q \) then one can construct an \( F_k \)-periodic configuration. But in this case one has to set values of the configuration the same on some cosets.

The following very simple lemma gives periodic ground states.

**Lemma 8.** An \( F_k \)-periodic configuration \( \sigma \) is a ground state if \( U(\sigma_l) = U_{\min} \) for any \( l = \langle x, y \rangle \in L \), where \( U_{\min} = \min \{ U_\epsilon : \epsilon \in Q_q \} \).

5. Non-uniqueness of the Gibbs measure

In this section we assume
\[
U_{ii} = U_{\min} < U_\epsilon, \quad i = 1, \ldots, q; \quad \epsilon \in Q_q,
\]
(9)
and thus the ground states of the model will all be constant configurations \( \sigma^{(m)} = \{ \sigma^{(m)}(x) = v_m, x \in V \} \), \( m = 1, \ldots, q \). Now we shall prove that every such ground state generates a Gibbs measure.
The energy $H_\Lambda(\sigma|\varphi)$ of the configuration $\sigma$ in the presence of boundary configuration $\varphi = \{\varphi(x), x \in V \setminus \Lambda\}$ is expressed by the formula

$$H_\Lambda(\sigma|\varphi) = \sum_{l=(x,y): x, y \in \Lambda} U(\sigma_l) + \sum_{l=(x,y): x \in \Lambda, y \in V \setminus \Lambda} U(\sigma_l).$$ (10)

The following lemma gives a contour representation of the Hamiltonian:

**Lemma 9.** The energy $H_n(\sigma_n) \equiv H_{V_n}(\sigma_n|\varphi_{V_n} = v_i)$ has the form

$$H_n(\sigma_n) = \sum_{\epsilon \in Q_q} (U_\epsilon - U_{ii})|\Gamma_\epsilon| + (|V_{n+1}| - 1)U_{ii},$$ (11)

where $|\Gamma_\epsilon|$ is defined in section 2.3.

**Proof.** We have

$$H_n(\sigma_n) = \sum_{l \in L_{n+1}} U(\sigma_{n,l}) = \sum_{\epsilon \in Q_q} U_\epsilon|\Gamma_\epsilon| + (|V_{n+1}| - 1 - |\Gamma|)U_{ii}. $$ (12)

Now using the equality $|\Gamma| = \sum_{\epsilon \in Q_q} |\Gamma_\epsilon|$ from (12) we get (11). $\square$

The Gibbs measure on the space $\Omega_\Lambda = \{v_1, \ldots, v_q\}^\Lambda$ with boundary condition $\varphi$ is defined as

$$\mu_{\Lambda, \beta}(\sigma/\varphi) \equiv \mu^{\varphi}_{\Lambda, \beta}(\sigma) = Z^{-1}(\Lambda, \beta, \varphi) \exp(-\beta H_\Lambda(\sigma|\varphi)),$$ (13)

where $Z(\Lambda, \beta, \varphi)$ is the normalizing factor (statistical sum).

Define $U = \{U_\epsilon : \epsilon \in Q_q\}$, $U^{\text{min}} = \min_{\epsilon \in Q_q} U_\epsilon$ and

$$\lambda_0 = \min \{U \setminus \{U_\epsilon : U_\epsilon = U^{\text{min}}\}\} - U^{\text{min}}.$$ (14)

**Lemma 10.** Assume that (9) is satisfied. Let $\gamma$ be a fixed contour and

$$p_i(\gamma) = \frac{\sum_{\sigma_n: \gamma \subset \Gamma} \exp\{-\beta H_n(\sigma_n)\}}{\sum_{\sigma_n} \exp\{-\beta H_n(\sigma_n)\}}.$$ 

Then

$$p_i(\gamma) \leq \exp\{-\beta \lambda_0 |\gamma|\},$$ (15)

where $\lambda_0$ is defined by formula (14) and $\beta = 1/T; T > 0$ is the temperature.

**Proof.** Put $\Omega_\gamma = \{\sigma_n: \gamma \subset \Gamma\}$, $\Omega_\gamma^0 = \{\sigma_n: \gamma \cap \Gamma = \emptyset\}$ and define a map $\chi_\gamma: \Omega_\gamma \to \Omega_\gamma^0$ by

$$\chi_\gamma(\sigma_n)(x) = \begin{cases} v_i & \text{if } x \in \text{Int } \gamma \\ \sigma_n(x) & \text{if } x \notin \text{Int } \gamma. \end{cases}$$

For a given $\gamma$ the map $\chi_\gamma$ is a one-to-one map. For any $\sigma_n \in \Omega_{V_n}$ we have

$$|\Gamma_\epsilon(\sigma_n)| = |\Gamma_\epsilon(\chi_\gamma(\sigma_n))| + |\gamma_\epsilon|, \quad \epsilon \in Q_q,$$ (16)

where $\gamma_\epsilon = \gamma \cap \Gamma_\epsilon$. 

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Using lemma 9 we have

\[
p_i(\gamma) = \frac{\sum_{\sigma_n \in \Omega_n} \exp\{-\beta \sum_{\epsilon \in Q} (U_\epsilon - U_{ii})|\Gamma_\epsilon(\sigma_n)|\}}{\sum_{\sigma_n} \exp\{-\beta \sum_{\epsilon \in Q} (U_\epsilon - U_{ii})|\Gamma_\epsilon(\sigma_n)|\}} \leq \frac{\sum_{\sigma_n \in \Omega_n} \exp\{-\beta \sum_{\epsilon \in Q} (U_\epsilon - U_{ii})|\Gamma_\epsilon(\chi(\sigma_n))|\}}{\sum_{\sigma_n \in \Omega_n} \exp\{-\beta \sum_{\epsilon \in Q} (U_\epsilon - U_{ii})|\Gamma_\epsilon(\sigma_n)|\}}.
\]  

(17)

By the assumption (9) we have \(U_\epsilon - U_{ii} \geq \lambda_0\) for any \(\epsilon \in Q, i = 1, \ldots, q\). Thus using this fact and (16) from (17) we get (15).

Using lemmas 5 and 10, by very similar argument of [12] one can prove:

**Theorem 11.** If (9) is satisfied then for all sufficiently large \(\beta\) there are at least \(q\) Gibbs measures for the model (2) on the Cayley tree of order \(k \geq 2\).

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**References**

[1] Baxter R J, 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[2] Bleher P M and Ganikhodjaev N N, *On pure phases of the Ising model on the Bethe lattice*, 1990 *Theor. Probab. Appl.* 35 216
[3] Borgs C, *Statistical physics expansion methods in combinatorics and computer science*, 2004 [http://research.microsoft.com/∼borgs/CBMS.pdf](http://research.microsoft.com/∼borgs/CBMS.pdf)
[4] Fernández R, *Contour ensembles and the description of Gibbsian probability distributions at low temperature*, 1998 [www.univ-rouen.fr/LMRS/persopage/Fernandez](http://www.univ-rouen.fr/LMRS/persopage/Fernandez)
[5] Holstynski W and Slawny J, *Feierls condition and the number of ground states*, 1978 *Commun. Math. Phys.* 61 177
[6] Kashapov I A, *Structure of ground states in three-dimensional Ising model with tree-step interaction*, 1977 *Theor. Math. Phys.* 33 912
[7] Minlos R A, 2000 *Introduction to Mathematical Statistical Physics* (University Lecture Series vol 19) ISSN 1047-3998
[8] Mukhamedov F M and Rozikov U A, *On Gibbs measures of models with competing ternary and binary interactions and corresponding von Neumann algebras. I, II*, 2004 *J. Stat. Phys.* 114 825
[9] Mukhamedov F M and Rozikov U A, 2005 *J. Stat. Phys.* 119 427
[10] Peierls R, *On Ising model of ferro magnetism*, 1936 *Proc. Camb. Phil. Soc.* 32 477
[11] Pirogov S A and Sinai Ya G, *Phase diagrams of classical lattice systems, I*, 1975 *Theor. Math. Phys.* 25 1185
[12] Rozikov U A and Suhov Yu M, *A hard-core model on a Cayley tree: an example of a loss network*, 2004 *Queueing Syst.* 46 197
[13] Rozikov U A, *On q-component models on Cayley tree: contour method*, 2005 *Lett. Math. Phys.* 71 27
[14] Sinai Ya G, 1982 *Theory of Phase Transitions: Rigorous Results* (Oxford: Pergamon)
[15] Zahradnik M, *An alternate version of Pirogov–Sinai theory*, 1984 *Commun. Math. Phys.* 93 559
[16] Zahradnik M, *A short course on the Pirogov–Sinai theory*, 1998 *Rend. Math. Ser. VII* 18 411

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