DECOMPOSITION OF TOPOLOGICAL AZUMAYA ALGEBRAS WITH ORTHOGONAL INVOLUTION

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Abstract. Let \( A \) be a topological Azumaya algebra of degree \( mn \) with an orthogonal involution over a CW complex \( X \) of dimension less than or equal to \( \min\{m,n\} \). We give conditions for the positive integers \( m \) and \( n \) so that \( A \) can be decomposed as the tensor product of topological Azumaya algebras of degrees \( m \) and \( n \) with orthogonal involutions.

1. Introduction

The concept of a central simple algebra over a field was generalized by Azumaya [Azu51] and Auslander–Goldman [AG60] to the notion of an Azumaya algebra over a commutative ring, and Grothendieck [Gro66] defined this concept in the context of a locally-ringed topos, that is, in a category of sheaves equipped with a designated choice of sheaf of rings with enough points, \( \mathcal{O} \), having local rings as stalks. One recovers the case of Auslander–Goldman when the category is \((\text{Spec} \mathcal{O})_{\text{ét}} \) of étale sheaves over the spectrum of the ring \( R \). That is, an Azumaya algebra of degree \( n \) is a sheaf of \( \mathcal{O} \)-algebras that is locally isomorphic to \( M(n,\mathcal{O}) \). In the topological case, a topological Azumaya algebra of degree \( n \) is defined by taking the sheaf \( \mathcal{O} \) to be the sheaf of continuous functions with value \( \mathbb{C} \), that is, it is a bundle of associative and unital complex algebras over a topological space that is locally isomorphic to the matrix algebra \( M(n,\mathbb{C}) \), [Gro66, 1.1].

Example 1.1. The endomorphism bundle of a complex vector bundle of rank \( n \) is a topological Azumaya algebra of degree \( n \) over \( X \).

Example 1.2. Let \( G \) be a group acting on the left on \( M(n,\mathbb{C}) \) via algebra automorphisms. Let \( X \) be a topological space with a free right \( G \)-action. Suppose the projection \( p : X \to X/G \) is a principal \( G \)-bundle. Then the fiber bundle with fiber \( M(n,\mathbb{C}) \) associated to \( p, M(n,\mathbb{C}) \to X \times_{\text{ét}} M(n,\mathbb{C}) \to X/G \) is a topological Azumaya algebra of degree \( n \) over \( X/G \), where \( G \) acts on \( X \times M(n,\mathbb{C}) \) by \( (x,M) \cdot g := (x \cdot g,M) \) for all \( x \in X \) and \( M \in M(n,\mathbb{C}) \).

In the theory of central simple algebras over a field \( k \), a theorem of Wedderburn states that any central simple algebra \( A/k \) has the form \( M(n,D) \), where \( D/k \) is a division algebra that is unique up to isomorphism, [Sal99, Theorem 1.3]. This theorem reduces the classification problem for finite-dimensional central simple algebras over \( k \) to the classification problem for finite-dimensional central division algebras over \( k \). Two finite-dimensional central simple algebras \( A \cong M(m,D) \) and \( B \cong M(n,E) \) are Brauer equivalent if their division algebras \( D \) and \( E \) are isomorphic. The Brauer group of \( k \), \( \text{Br}(k) \), is the group of equivalence classes of finite-dimensional central simple algebras over \( k \) modulo Brauer equivalence, with the tensor product of algebras, [FD93, Proposition 4.3].

Given two topological Azumaya algebras \( A \) and \( A' \) over \( X \) of degrees \( m \) and \( n \), respectively, we can define the tensor product \( A \otimes A' \) by performing the tensor product of (Kronecker product of) matrices \( M(m,\mathbb{C}) \otimes M(n,\mathbb{C}) \) fiberwise. We say two topological Azumaya algebras \( A \) and \( A' \) are Brauer equivalent if there exist complex vector bundles \( V \) and \( V' \) such that \( A \otimes \text{End}(V) \cong A' \otimes \text{End}(V') \) as bundles of \( \mathbb{C} \)-algebras. The (topological) Brauer group of \( X \), \( \text{Br}(X) \), is the set of isomorphism classes of topological Azumaya algebras over \( X \) modulo Brauer equivalence with the tensor product operation. The identity of \( \text{Br}(X) \) is the class \( [\text{End}(V)] \) for all complex vector bundles \( V \), and \( [A]^{-1} = [A^{\text{op}}] \) for all classes \( [A] \in \text{Br}(X) \) since \( A \otimes A^{\text{op}} \cong \text{End}(A) \). We say a topological Azumaya \( A \) is Brauer trivial if its class \( [A] \in \text{Br}(X) \) is equal to the identity.

If \( D/k \) is a finite-dimensional central division algebra, then \( \dim_k(D) \) is a square, [FD93, Theorem 3.10]. The degree of \( D/k \) is defined by \( \sqrt{\dim_k(D)} \). The theory of central division algebras is equipped
with a structure theorem stating that every finite-dimensional central division algebra $D/k$ can be broken up into pieces corresponding to the prime factorization of its degree $\deg(D) = p_1^{i_1} \cdots p_r^{i_r}$, i.e., $D$ is isomorphic to $D_1 \otimes_k D_2 \otimes_k \cdots \otimes_k D_r$ where each $D_i$ is a division algebra of degree $p_i^{i_i}$, and the decomposition is unique up to isomorphism, [Sal99, Theorem 5.7] and [GS06, Proposition 4.5.16]. Saltman asked in [Sal99, page 35] whether the analogue to the prime decomposition theorem for central simple algebras over a field holds for Azumaya algebras over a commutative ring. Using algebraic topology and topological Azumaya algebras, Antieau–Williams showed that, in general, there is no prime decomposition for these algebras, [AW14, Corollary 1.3]. The author provided conditions for a positive integer $n$ and a topological space $X$ such that a topological Azumaya algebra of degree $n$ on $X$ has a non-unique tensor product decomposition, [AM22, Theorem 1.3, Remark 3.7].

An involution on a finite-dimensional central simple algebra $A/k$ is an anti-automorphism $\tau : A \to A$. In other words $\tau \circ \tau = \text{id}_A$, $\tau(a + b) = \tau(a) + \tau(b)$, and $\tau(ab) = \tau(b)\tau(a)$ for all $a, b \in A$. A central simple algebra with an involution is denoted by $(A, \tau)$. It can be checked that the center $k$ is preserved under $\tau$. The restriction of $\tau$ to $k$ is therefore an automorphism which is either the identity or of order 2. Involutions that leave the center invariant are called involutions of the first kind. Involutions whose restriction to $k$ is an automorphism of order 2 are called involutions of the second kind, [KMRT98]. A quaternion algebra over a field $k$ is a central simple algebra of degree 2. Any quaternion algebra has a symplectic involution called the quaternion conjugation, [KMRT98, page 26]. Let $(A_1, \tau_1), \ldots, (A_n, \tau_n)$ be central simple $k$-algebras with involution of the first kind. Then $\tau_1 \otimes \cdots \otimes \tau_n$ is an involution of the first kind on $A_1 \otimes_k \cdots \otimes_k A_n$, [KMRT98, Proposition 2.29(1)]. Knus–Parimala–Srinivas gave a necessary and sufficient decomposability condition for an involution on a tensor product of two quaternion algebras, [KPS91]. Merkurjev showed that every central simple algebra with involution is Brauer equivalent to a tensor product of quaternion algebras, [Mer81]. However, Amitsur–Rowen–Tignol provided examples of division algebras of degree 8 with involutions which do not decompose into tensor products of quaternion algebras, [ART79].

Knus–Parimala–Srinivas generalized the notion of a central simple algebra with an involution to Azumaya algebras over schemes, [KPS90]. Saltman presented a classification of involutions of Azumaya algebras over commutative rings into kinds, [Sal78, Section 3]. An involution on a topological Azumaya algebra is said to be an involution of the first kind or an involution of the second kind depending on whether or not there is a group action on the base space, respectively. All the involutions we discuss here are involutions of the first kind, which are classified as orthogonal and symplectic involutions.

Definition 1.3. Let $X$ be a connected topological space, and let $\mathcal{A}$ be a topological Azumaya algebra of degree $n$ over $X$. An involution on $\mathcal{A}$ is a morphism of fiber bundles $\tau : \mathcal{A} \to \mathcal{A}$ such that $\tau \circ \tau = \text{id}_\mathcal{A}$, and when restricted to fibers it is an anti-automorphism of complex algebras. In this case, $(\mathcal{A}, \tau)$ is called a topological Azumaya algebra with involution.

Definition 1.4. Let $X$ be a connected topological space, and let $(\mathcal{A}, \tau)$ be a topological Azumaya algebra with involution over $X$. The involution $\tau$ is said to be orthogonal (symplectic) if the restriction $\tau|_{\mathcal{A}^{-1}(x)} : \mathcal{A}^{-1}(x) \to \mathcal{A}^{-1}(x)$ is an orthogonal (a symplectic) involution of complex algebras for all $x \in X$, in the sense of the definition given in Subsection 2.1.

The goal of this paper is to provide conditions on a positive integer $n$ and a topological space $X$ such that a topological Azumaya algebra of degree $n$ over $X$ with an orthogonal involution has a tensor product decomposition of topological Azumaya algebras with orthogonal involution. The main result of this paper is the following theorem:

Theorem 1.5. Let $m$ and $n$ be relatively prime positive integers such that $m$ is even, and $n$ is odd. Let $X$ be a CW complex such that $\dim(X) \leq d$ where $d := \min\{m, n\}$. If $\mathcal{A}$ is a topological Azumaya algebra of degree $mn$ over $X$ with an orthogonal involution, then there exist topological Azumaya algebras $\mathcal{A}_m$ and $\mathcal{A}_n$ of degrees $m$ and $n$, respectively, such that $\mathcal{A}_m$ and $\mathcal{A}_n$ have orthogonal involutions, $\mathcal{A}_n$ is Brauer-trivial and $\mathcal{A} \cong \mathcal{A}_m \otimes \mathcal{A}_n$.

It is worth observing that Theorem 1.5 does not hold for CSAs with orthogonal involution. Specifically, for any $m, n > 1$, there exists a Brauer-trivial CSA with an orthogonal involution that does not decompose as the tensor product of degree-$m$ and degree-$n$ CSAs with orthogonal involution. Assuming the base field’s characteristic is not 2 for simplicity, if all Brauer-trivial CSAs with orthogonal involution of degree $mn$ could be decomposed as stated, then any $mn$-dimensional quadratic form would decompose as the tensor product of an $m$-dimensional and an $n$-dimensional quadratic form.
However, the essential dimension of a 'generic' \( mn \)-dimensional quadratic form is \( mn \), whereas the essential dimension of the tensor product of an \( m \)-dimensional and an \( n \)-dimensional quadratic form is at most \( m + n - 1 \). Hence, this is impossible whenever \( mn > m + n - 1 \), which is the case whenever \( m, n > 1 \).

From [Ste51, 8.2] there is a bijective correspondence

\[
\text{Isomorphism classes of degree-}n
\text{topological Azumaya algebras}
\text{over } X \text{ with an involution}
\text{ locally isomorphic to } \tau

\leftrightarrow
\text{Isomorphism classes of principal}
\text{Aut}(M(n, \mathbb{C}), \tau)-\text{bundles}
\text{over } X
\]

where \( \tau \) is an involution of the first kind on \( M(n, \mathbb{C}) \). Proposition 2.1 implies that topological Azumaya algebras over \( X \) with involution are classified by \([X, BPO(n, \mathbb{C})]\) in the orthogonal case, and \([X, BPSp(n, \mathbb{C})]\) in the symplectic case, where \( BPO(n, \mathbb{C}) \) and \( BPSp(n, \mathbb{C}) \) are the classifying spaces of the projective complex orthogonal group of degree \( n \) and the projective complex symplectic group of degree \( 2n \), respectively, see Subsection 1.2.

We prove in Theorem 1.5 that a map \( X \to BPO(mn, \mathbb{C}) \) can be lifted to \( BPO(m, \mathbb{C}) \times BSO(n, \mathbb{C}) \) when the dimension of \( X \) is less than \( d + 1 \). The proof of Theorem 1.5 relies in the description of the homomorphisms induced on homotopy groups by the \( r \)-fold direct sum of matrices \( \oplus^r : O(n, \mathbb{C}) \to O(rn, \mathbb{C}) \) in the range \( \{0, 1, \ldots, n - 1\} \). We call this set "the stable range for \( O(n, \mathbb{C}) \)."

This paper is organized as follows. The second section contains preliminaries. The third section presents preliminaries on the effect of direct sum and tensor product operations on homotopy groups of compact Lie groups related to the orthogonal groups. The fourth section is devoted to the proof of Theorem 1.5.

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1.2. Notation. Throughout this paper, we adopt the notation for the classical Lie groups as in [MT91]. We denote by \( M(n, \mathbb{C}) \) the set of \( n \times n \) complex matrices. We use \( GL(n, \mathbb{C}), O(n, \mathbb{C}), SO(n, \mathbb{C}), \) and \( Sp(n, \mathbb{C}) \) to denote the complex general linear group of degree \( n \), the complex orthogonal group of degree \( n \), the special complex orthogonal group of degree \( n \), and the complex symplectic group of degree \( 2n \), respectively. These groups are closed subgroups of \( GL(n, \mathbb{C}) \) in the orthogonal cases, and of \( GL(2n, \mathbb{C}) \) in the symplectic case. In matrix terms these groups are defined as

\[
\begin{align*}
GL(n, \mathbb{C}) &= \{ M \in M(n, \mathbb{C}) : M \text{ is invertible} \}, \\
O(n, \mathbb{C}) &= \{ M \in M(n, \mathbb{C}) : M^tM = MM^t = I_n \}, \\
SO(n, \mathbb{C}) &= \{ M \in O(n, \mathbb{C}) : \det(M) = 1 \}, \quad \text{and} \\
Sp(n, \mathbb{C}) &= \{ M \in M(2n, \mathbb{C}) : M^tJ_{2n}M = J_{2n} \},
\end{align*}
\]

where \( M^t \) denotes transposition, and

\[
J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.
\]

If \( G \) is a group, then \( Z(G) \) denotes its center. The projective complex orthogonal group of degree \( n \), the projective special complex orthogonal group of degree \( n \), and the projective complex symplectic group of degree \( 2n \) shall be denoted by \( PO(n, \mathbb{C}) := O(n, \mathbb{C})/Z(O(n, \mathbb{C})) \), \( PSO(n, \mathbb{C}) := SO(n, \mathbb{C})/Z(SO(n, \mathbb{C})) \), and \( PSp(n, \mathbb{C}) := Sp(n, \mathbb{C})/Z(Sp(n, \mathbb{C})) \), respectively.

All topological spaces will have the homotopy type of a CW complex. We fix basepoints for connected topological spaces, and for topological groups we take the identities as basepoints. We write \( \pi_i(X) \) in place of \( \pi_i(X, x_0) \).
2. Preliminaries

2.1. Involutions on matrix algebras. Let $A \in \text{GL}(n, \mathbb{C})$, we denote by $\text{Inn}_A : M(n, \mathbb{C}) \to M(n, \mathbb{C})$ the inner automorphism induced by $A$, $\text{Inn}_A(M) = AMA^{-1}$ for all $M \in M(n, \mathbb{C})$.

Observe that transposition, $\text{tr} : M(n, \mathbb{C}) \to M(n, \mathbb{C})$, is an involution of the first kind on $M(n, \mathbb{C})$. All automorphisms of $(M(n, \mathbb{C}), \text{tr})$ arise by conjugation by an invertible matrix: Let $\tau$ be an arbitrary involution on $M(n, \mathbb{C})$. Since $\text{tr} \circ \tau$ is an automorphism of $(M(n, \mathbb{C}), \text{tr})$, then $\text{tr} \circ \tau = \text{Inn}_A$ for some $A \in \text{GL}(n, \mathbb{C})$. Hence $\tau(M) = A^{-\text{tr}}M\text{tr}A^\text{tr}$. Since $\tau$ is an involution, then $M = (A^{-\text{tr}}A)M(A^{-1}A^\text{tr})$. Therefore $\text{Inn}_{A^{-1}A^\text{tr}} = \text{id}_{M(n, \mathbb{C})}$, this is $A^{-1}A^\text{tr} = \lambda I_n$ for some $\lambda \in \mathbb{C}$. This last equation implies $A^\text{tr} = \lambda A$, and thus $A = \lambda^2 A$. Hence $\lambda = 1$ or $\lambda = -1$, i.e. $A^\text{tr} = A$ or $A^\text{tr} = -A$.

Consider the subspaces
\[
\begin{align*}
(M(n, \mathbb{C}), \tau)^+ &= \left\{ M \in M(n, \mathbb{C}) : \tau(M) = M \right\} \quad \text{(symmetric elements), and} \\
(M(n, \mathbb{C}), \tau)^- &= \left\{ M \in M(n, \mathbb{C}) : \tau(M) = -M \right\} \quad \text{(skew symmetric elements)}.
\end{align*}
\]

Observe that
\[
(M(n, \mathbb{C}), \tau)^+ = \begin{cases} A(M(n, \mathbb{C}), \text{tr})^+ & \text{if } A^\text{tr} = A, \\ A(M(n, \mathbb{C}), \text{tr})^- & \text{if } A^\text{tr} = -A. \end{cases}
\]

where $\dim \mathbb{C}(M(n, \mathbb{C}), \text{tr})^+ = \frac{1}{2}n(n+1)$ and $\dim \mathbb{C}(M(n, \mathbb{C}), \text{tr})^- = \frac{1}{2}n(n-1)$.

In summary, we obtain the following proposition.

Proposition 2.1. Let $\tau$ be an involution of the first kind on $M(n, \mathbb{C})$.

1. The involution $\tau$ on $M(n, \mathbb{C})$ has the form $\tau = \text{Inn}_A \circ \text{tr}$ for some $A \in \text{GL}(n, \mathbb{C})$ such that $A^\text{tr} = \pm A$.

2. The subspace $(M(n, \mathbb{C}), \text{tr})^+$ has complex dimension $\frac{1}{2}n(n+1)$ if and only if $A^\text{tr} = A$.

3. The subspace $(M(n, \mathbb{C}), \text{tr})^+$ has complex dimension $\frac{1}{2}n(n-1)$ if and only if $A^\text{tr} = -A$.

If $\tau$ is an involution on $M(n, \mathbb{C})$. The type of $\tau$ is said to be orthogonal if $A^\text{tr} = A$, and symplectic if $A^\text{tr} = -A$. Up to isomorphism, the matrix algebra $M(n, \mathbb{C})$ can carry at most one involution (transposition) if $n$ is odd; and up to two involutions (orthogonal and symplectic) if $n$ is even, [KMRT98, Proposition I.2.20].

2.2. Non-degenerate bilinear forms. Let a pair $(V, B)$ denote a finite-dimensional complex vector space, $V$, with a non-degenerate bilinear form, $B$. If $\dim \mathbb{C}V$ is even, $V$ can be given both a skew-symmetric bilinear form and a symmetric bilinear form. If $\dim \mathbb{C}V$ is odd, $V$ can be given a symmetric bilinear form. These bilinear forms are unique up to isomorphism, given that $\mathbb{C}$ is algebraically closed, [Car17, Theorem 9.13].

The automorphism group of $(V, B)$ is
\[
\text{Aut}(V, B) = \left\{ T \in \text{GL}(n, \mathbb{C}) : B(Tv, Tw) = B(v, w) \text{ for all } v, w \in V \right\}.
\]

Let $(\mathbb{C}^n, B)$ and $(\mathbb{C}^{2n}, B')$ denote the spaces $\mathbb{C}^n$ and $\mathbb{C}^{2n}$ with the standard symmetric bilinear form and the standard skew-symmetric bilinear form, respectively. Then
\[
\text{Aut}(\mathbb{C}^n, B) \cong \text{O}(n, \mathbb{C}) \quad \text{and} \quad \text{Aut}(\mathbb{C}^{2n}, B') \cong \text{Sp}(n, \mathbb{C}).
\]

Let $(V, B)$ and $(V', B')$ be finite-dimensional complex vector spaces with non-degenerate bilinear forms. The tensor product $(V, B) \otimes (V', B')$ is a non-degenerate bilinear form over $\mathbb{C}$, where $(B \otimes B')(v \otimes v', w \otimes w') = B(v, w)B'(v', w')$ (multiplication in $\mathbb{C}$), [Gre67, Section 1.21, Section 1.27]. The tensor product is bifunctorial, so that one obtains induced maps
\[
\otimes : \text{Aut}(V, B) \times \text{Aut}(V', B') \longrightarrow \text{Aut}(V \otimes V', B \otimes B').
\]

The following tensor product operation is of interest in the subsequent sections. Let $(\mathbb{C}^m, B)$ and $(\mathbb{C}^n, B')$ be the spaces $\mathbb{C}^m$ and $\mathbb{C}^n$ with the standard symmetric bilinear forms. Then, the tensor product (1) induces a homomorphism
\[
\otimes : \text{O}(m, \mathbb{C}) \times \text{O}(n, \mathbb{C}) \longrightarrow \text{O}(mn, \mathbb{C}).
\]
Remark 2.2. The tensor product (1) also induces the following tensor products on Lie groups.

(1) Let \((\mathbb{C}^{2m}, B)\) be the space \(\mathbb{C}^{2m}\) with the standard skew-symmetric bilinear form, and \((\mathbb{C}^n, B')\) be the space \(\mathbb{C}^n\) with the standard symmetric bilinear form. Then, the tensor product (1) induces a homomorphism

\[ \otimes : \text{Sp}(m, \mathbb{C}) \times \text{O}(n, \mathbb{C}) \rightarrow \text{Sp}(mn, \mathbb{C}). \]

(2) Let \((\mathbb{C}^{2m}, B)\) and \((\mathbb{C}^{2n}, B')\) be the spaces \(\mathbb{C}^{2m}\) and \(\mathbb{C}^{2n}\) with the standard skew-symmetric bilinear forms. From the tensor product (1) there is a homomorphism

\[ \otimes : \text{Sp}(m, \mathbb{C}) \times \text{Sp}(n, \mathbb{C}) \rightarrow G, \]

where \(G \leq \text{GL}(4mn, \mathbb{C})\) denotes \(\text{Aut}(\mathbb{C}^{4mn}, B \otimes B')\). Observe that \(B \otimes B'\) is not the standard symmetric bilinear form on \(\mathbb{C}^{4mn}\). Let \(P \in \text{GL}(4mn, \mathbb{C})\) be matrix associated to changing the basis of \(\mathbb{C}^{4mn}\) to an orthonormal basis. Then we have the commutative square below

\[ \begin{array}{ccc}
\text{Inn}_P & \cong & \text{Inn}_P \\
\downarrow & & \downarrow \\
O(4mn, \mathbb{C}) & \rightarrow & \text{GL}(4mn, \mathbb{C}).
\end{array} \]

Thus, the composite \(\text{Inn}_P \circ \otimes : \text{Sp}(m, \mathbb{C}) \times \text{Sp}(n, \mathbb{C}) \rightarrow G \rightarrow O(4mn, \mathbb{C})\) gives a homomorphism

\[ \boxtimes : \text{Sp}(m, \mathbb{C}) \times \text{Sp}(n, \mathbb{C}) \rightarrow O(4mn, \mathbb{C}). \]

3. Stabilization of operations on the complex orthogonal group

In what follows, we shall drop \(\mathbb{C}\) from the notations \(O(n, \mathbb{C})\) and \(SO(n, \mathbb{C})\).

We compute the homotopy groups of \(\text{PO}(n)\) and \(\text{PSO}(n)\) in low degrees.

When \(n = 1\), then \(O(1) = S^0\), \(SO(1) = \{1\}\) and \(PO(1) = PSO(1) = \{1\}\).

When \(n = 2\), then \(SO(2) = U(1) = S^1\), \(PO(2) \cong O(2)\) and \(PSO(2) \cong S^1\). Let \(n \geq 3\) and \(i < n - 1\), the first homotopy groups of the orthogonal group are given by Bott periodicity,

\[ \pi_i(O(n)) \cong \begin{cases} 
0 & \text{if } i = 2, 4, 5, 6 \pmod{8}, \\
\mathbb{Z}/2 & \text{if } i = 0, 1 \pmod{8}, \\
\mathbb{Z} & \text{if } i = 3, 7 \pmod{8}.
\end{cases} \]

The special orthogonal group is the connected component of the identity of \(O(n)\), then \(\pi_0(SO(n))\) is trivial. Moreover, for \(n \geq 2\) there is a fiber sequence \(SO(n) \rightarrow O(n) \rightarrow \mathbb{Z}/2\) where \(\mathbb{Z}/2\) has the discrete topology, then we can use the long exact sequence associated to it to see that \(\pi_i(SO(n)) \cong \pi_i(O(n))\) for \(i \geq 1\).

To calculate the homotopy groups of the projective orthogonal group and the projective special orthogonal group in low degrees when \(n \geq 3\) consider the diagrams (2) and (3) below for \(k \geq 2\).

\[
\begin{array}{ccc}
\{\pm I_{2k}\} & \rightarrow & \{\pm I_{2k}\} \\
\downarrow & & \downarrow \\
\text{SO}(2k) & \rightarrow & \text{O}(2k) \\
\downarrow & & \downarrow \text{det} \\
\text{PSO}(2k) & \rightarrow & \text{PO}(2k) \\
\downarrow & & \downarrow \text{det} \\
& \rightarrow & \{\pm 1\}
\end{array}
\]
All columns as well as the two top rows of diagrams (2) and (3) are exact. The nine-lemma implies that the bottom rows in (2) and (3) are also exact. Therefore,

\[ \pi_i(\text{PO}(2k)) \cong \pi_i(\text{PSO}(2k)) \cong \pi_i(\text{SO}(2k)) \quad \text{for all } i \geq 2, \]

and

\[ \text{PO}(2k-1) \cong \text{PSO}(2k-1) \cong \text{SO}(2k-1) \]

The only calculation left is the one of the fundamental group of \( \text{PSO}(2k) \) for \( k \geq 2 \). Let \( n \geq 3 \). By definition, the \( n \)-th spinor group \( \text{Spin}(n) \) is the universal double covering group of \( \text{SO}(n) \) [MT91, Page 74]. This group is also the universal covering of \( \text{PSO}(n) \). Since we have an exact sequence \( \text{Ker}(p) \to \text{Spin}(n) \xrightarrow{p} \text{PSO}(n) \), then \( \pi_1(\text{PSO}(n)) = \text{Ker}(p) = Z(\text{Spin}(n)) \). From [MT91, Theorem II.4.4] the center of \( \text{Spin}(n) \) for \( n \geq 3 \) is given by

\[
Z(\text{Spin}(n)) = \begin{cases} 
\mathbb{Z}/2 & \text{if } n \text{ is odd}, \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } n \equiv 0 \pmod{4}, \\
\mathbb{Z}/4 & \text{if } n \equiv 2 \pmod{4}.
\end{cases}
\]

3.1. **First unstable homotopy group of** \( \text{O}(n, \mathbb{C}) \). The standard inclusion of the orthogonal group \( i : \text{O}(n) \hookrightarrow \text{O}(n + 1) \) is \((n - 1)\)-connected, hence it induces an isomorphism on homotopy groups in degrees less than \( n - 1 \) and an epimorphism in degree \( n - 1 \). This can be seen by observing the long exact sequence for the fibration \( \text{O}(n) \to \text{O}(n + 1) \to \text{O}(n + 1)/\text{O}(n) \cong S^n \).

The first unstable homotopy group of \( \text{O}(n) \) happens in degree \( n - 1 \). Consider the following segment of the long exact sequence

\[
\pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(\text{O}(n)) \xrightarrow{i_*} \pi_{n-1}(\text{O}(n + 1)) \to 0.
\]

By exactness of the sequence above, there is a short exact sequence

\[
0 \to \text{Ker} i_* \to \pi_{n-1}(\text{O}(n)) \xrightarrow{i_*} \pi_{n-1}(\text{O}) \to 0,
\]

where \( O := \text{colim}_{n \to \infty} \text{O}(n) \) and \( \pi_{n-1}(O) \cong \pi_{n-1}(\text{O}(n + 1)) \).

Let \( n = 3, 7 \). From [Whi78, Corollary IV.10.6, Theorem IV.10.8], \( \text{Ker} i_* \) is trivial. Thus \( \pi_{n-1}(\text{O}(n)) \cong \pi_{n-1}(\text{O}(n + 1)) \) is trivial.

Let \( n = 2 \), then \( \pi_1(\text{O}(2)) = \pi_1(S^1) = \mathbb{Z} \).

Let \( n \neq 1, 2, 3, 7 \). The short exact sequence (5) splits. Moreover, by [MT91, Corollary IV.6.14] we have

\[
\pi_{n-1}(\text{O}(n)) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z} & \text{if } n \equiv 0, 4 \pmod{8}, \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } n \equiv 1 \pmod{8}, \\
\mathbb{Z}/2 & \text{if } n \equiv 2 \pmod{8}, \\
\mathbb{Z} & \text{if } n \equiv 3, 5, 7 \pmod{8}, \\
\mathbb{Z} & \text{if } n \equiv 6 \pmod{8}.
\end{cases}
\]

Let \( G \in \{ \text{O}(n), \text{SO}(n), \text{PO}(n), \text{PSO}(n) \} \). Tables 1, 2, 3, and 4 summarize the previous results.

| \text{G} | \text{O}(n) | \text{SO}(n) | \text{PO}(2k) | \text{PSO}(2k) |
|---|---|---|---|---|
| \pi_0(G) | \mathbb{Z}/2 | * | \mathbb{Z}/2 | * |

**Table 1.** Connected components of compact Lie groups related to the complex orthogonal group
Table 2. Fundamental group of compact Lie groups related to the complex orthogonal group

| i > 1 and i (mod 8) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------------------|---|---|---|---|---|---|---|---|
| \( \pi_i(G) \)      | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) |

Table 3. Homotopy groups of compact Lie groups related to the complex orthogonal group for \( i = 2, \ldots, n - 2 \).

| n equals to | 2 | 3 | 4 | 5 | 6 | 7 |
|-------------|---|---|---|---|---|---|
| \( \pi_{n-1}(G) \) | \( \mathbb{Z} \) | \( \mathbb{Z} \oplus \mathbb{Z} \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z} \) | \( 0 \) |

| n > 7 and n (mod 8) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------------------|---|---|---|---|---|---|---|---|
| \( \pi_{n-1}(G) \)  | \( \mathbb{Z} \oplus \mathbb{Z} \) | \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \oplus \mathbb{Z} \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z} \oplus \mathbb{Z} \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) |

Table 4. First unstable homotopy group of compact Lie groups related to the complex orthogonal group

3.2. Stabilization. Let \( m, n \in \mathbb{N} \) and \( m \leq n \). Define the map \( s : \text{O}(m) \to \text{O}(m + n) \) by

\[
s(A) = \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}.
\]

Since the map \( s \) is equal to the composite of consecutive canonical inclusions, it follows that \( s \) is \( (m - 1) \)-connected.

Notation 3.1. Let \( \text{stab} \) denote the homomorphisms the map \( s \) induces on homotopy groups. From now on, we will identify \( \pi_i(\text{O}(m)) \) with \( \pi_i(\text{O}(m + n)) \) for all \( i < m - 1 \) through the isomorphism

\[
\text{stab} : \pi_i(\text{O}(m)) \xrightarrow{\cong} \pi_i(\text{O}(m + n)).
\]

The following lemma is a technical result that will be used to prove Lemma 3.3 and Lemma 3.9.

Lemma 3.2. Let \( G \) be a Lie group and let \( G_0 \) be the component of the identity. If \( r : G \to G \) is conjugation by \( P \in G_0 \), then there is a basepoint preserving homotopy \( H \) from \( r \) to \( \text{id}_G \) such that for all \( t \in [0, 1] \), \( H(-, t) \) is a homomorphism.

Proof. Since \( G_0 \) is path-connected, there exists a path \( \alpha \) from \( P \) to \( e_G \) in \( G \). Define \( H : G \times [0, 1] \to G \) by \( H(A, t) = \alpha(t) A \alpha(t)^{-1} \). Observe that \( H(-, t) : G \to G, A \mapsto H(A, t) \) is a homomorphism. Moreover, \( H \) is such that \( H(e_G, t) = e_G, H(A, 0) = \text{id}_G \), and \( H(A, 1) = A \).

Lemma 3.3. Let \( n, r \in \mathbb{N} \). For all \( j = 1, \ldots, r \) define \( s_j : \text{O}(n) \to \text{O}(rn) \) by

\[
s_j(A) = \text{diag}(I_n, \ldots, I_n, A, I_n, \ldots, I_n),
\]

where \( A \) is in the \( j \)-th position. Then the map \( s_j \) is pointed homotopic to \( s_{j+1} \) for \( j = 1, \ldots, r - 1 \).

Proof. The lemma is straightforward for \( n = 1 \). Suppose \( n \geq 2 \) or \( r \geq 3 \), and let \( P_j \) be the permutation matrix

\[
P_j = \begin{pmatrix} I_{(j-1)n} & 0 & I_n \\ 0 & I_n & 0 \\ I_{(r-j-1)n} \end{pmatrix}.
\]

We consider two cases according to the parity of \( n \). If \( n \) is even, then \( \det(P_j) = 1 \), and \( P_j \) is such that \( s_{j+1}(A) = P_j s_j(A) P_j \) for \( j = 1, \ldots, r - 1 \). In the case \( n \) is odd, \( \det(P_j) = -1 \), and the matrices

\[
W_d = \begin{pmatrix} I_{rn-2} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad W_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
are such that $\det(W_dP_j) = \det(W_uP_j) = 1$. Moreover,
\[ s_2(A) = (W_dP_j)s_1(A)(W_dP_j)^{-1}, \quad \text{and} \quad s_{j+1}(A) = (W_uP_j)s_j(A)(W_uP_j)^{-1} \]
for $j = 2, \ldots, r-1$. By Lemma 3.2 the result follows. \hfill \Box

**Notation 3.4.** We call the $s_j$ maps stabilization maps. As $s_1$ is equal to $s : O(n) \to O(n + (r-1)n)$, it follows that $s_j$ is $(n-1)$-connected for all $j = 1, \ldots, r$. From Lemma 3.3 the homomorphisms induced on homotopy groups by the stabilization maps are equal, hence stab also denotes $\pi_1(s_1) = \cdots = \pi_1(s_r)$. Thus we identify $\pi_i(O(n))$ with $\pi_i(O(rn))$ for $i < n-1$ through stab. The identification allows one to introduce a slight abuse of notation, namely to identify $x$ and stab$(x)$ for $x \in \pi_i(O(n))$ and $i < n-1$.

### 3.3 Operations on homotopy groups
We do not write proofs of some results in this section given that they are similar to the proofs of the results in [AM22].

**Proposition 3.5.** [AM22, Proposition 2.5] Let $i \in \mathbb{N}$. The homomorphism $\oplus_* : \pi_i(O(m)) \times \pi_i(O(n)) \to \pi_i(O(m+n))$ is equal to
\[ \oplus_*(x, y) = \text{stab}(x) + \text{stab}(y) \quad \text{for} \quad x \in \pi_i(O(m)) \quad \text{and} \quad y \in \pi_i(O(n)). \]

**Corollary 3.6.** [AM22, Corollary 2.6] If $m < n$ and $i < m-1$, then $\oplus_*(x, y) = x + y$ for $x \in \pi_i(O(m))$ and $y \in \pi_i(O(n))$.

**Proposition 3.7.** [AM22, Proposition 2.7] Let $i \in \mathbb{N}$. The homomorphism $\oplus^*_v : \pi_i(O(n)) \to \pi_i(O(rn))$ is equal to
\[ \oplus^*_v(x) = r \text{stab}(x) \quad \text{for} \quad x \in \pi_i(O(n)). \]

**Corollary 3.8.** [AM22, Corollary 2.8] If $i < n-1$, then $\oplus^*_v(x) = rx$ for $x \in \pi_i(O(n))$.

**Lemma 3.9.** Let $L, R : O(m) \to O(mn)$ be the homomorphisms $L(A) = A \otimes I_n$ and $R(A) = I_n \otimes A$. There is a basepoint preserving homotopy $H$ from $L$ to $R$ such that for all $t \in [0, 1]$, $H(-, t)$ is a homomorphism.

**Proof.** The lemma is clear for $m = 1$. Suppose $m \geq 2$, and let $A \in O(m)$.
\[
L(A) = \begin{pmatrix}
a_{11}I_n & \cdots & a_{1m}I_n \\
\vdots & \ddots & \vdots \\
a_{m1}I_n & \cdots & a_{mm}I_n
\end{pmatrix} \quad \text{and} \quad R(A) = \begin{pmatrix} A & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A
\end{pmatrix} = A^\oplus n.
\]

Let $P_{m,n}$ be the permutation matrix
\[
P_{m,n} = [e_1, e_{n+1}, e_{2n+1}, \ldots, e_{(m-1)n+1}, e_2, e_{n+2}, e_{2n+2}, \ldots, e_{(m-1)n+2},
\ldots,
\ldots,
\ldots,e_{n-1}, e_{2n-1}, e_{3n-1}, \ldots, e_{mn-1}, e_n, e_{2n}, e_{3n}, \ldots, e_{(m-1)n}, e_{mn}]
\]
where $e_i$ is the $i$-th standard basis vector of $\mathbb{C}^{mn}$ written as a column vector. Observe that $L(A) = P_{m,n}R(A)P_{m,n}^{-1}$. If $\det(P_{m,n}) = 1$, the result follows from Lemma 3.2. Suppose $\det(P_{m,n}) = -1$. Consider $W_d$ and $W_u$ from the proof of Lemma 3.3. Observe that $\det(P_{m,n}W_d) = \det(P_{m,n}W_u) = \det(W_d^{-1}W_u) = 1$, and $R(A) = s_1(A)s_2(A)\cdots s_n(A)$. Then
\[
L(A) = P_{m,n}R(A)P_{m,n}^d
= P_{m,n}s_1(A)s_2(A)\cdots s_n(A)P_{m,n}^{-1}
= P_{m,n}(W_d s_1(A)\cdots s_{n-1}(A)W_d^{-1})(W_u s_n(A)W_u^{-1})P_{m,n}^{-1}
= (P_{m,n}W_d) s_1(A)\cdots s_{n-1}(A)(P_{m,n}W_d)^{-1}(P_{m,n}W_u) s_n(A)(P_{m,n}W_u)^{-1}.
\]
Applying Lemma 3.2 twice yields the result. \hfill \Box

**Proposition 3.10.** [AM22, Proposition 2.10] Let $i \in \mathbb{N}$, the homomorphism $\otimes_* : \pi_i(O(m)) \times \pi_i(O(n)) \to \pi_i(O(mn))$ is given by
\[ \otimes_*(x, y) = nx + my \quad \text{for} \quad x \in \pi_i(O(m)) \quad \text{and} \quad y \in \pi_i(O(n)). \]

**Corollary 3.11.** [AM22, Corollary 2.11] If $m < n$ and $i < m-1$, then $\otimes_*(x, y) = nx + my$ for $x \in \pi_i(O(m))$ and $y \in \pi_i(O(n))$. 

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Remark 3.12. Under the hypothesis of Corollary 3.11, the homomorphism
\[ \otimes : \pi_{m-1}(O(m)) \times \pi_{m-1}(O(n)) \to \pi_{m-1}(O(mn)) \]
is given by \( \otimes(x, y) = n \text{stab}(x) + m \text{stab}(y) = n \text{stab}(x) + my. \) This can be seen by observing that \( \otimes \) is equal to the sum of the two paths around diagram (8).

\[
\begin{array}{c}
\pi_{m-1}(O(m)) \\
\downarrow \text{proj}_1 \\
\pi_{m-1}(O(mn)) \\
\times_m \\
\pi_{m-1}(O(mn)) \\
\downarrow \\
\pi_{m-1}(O(mn))
\end{array} \quad \begin{array}{c}
\pi_{m-1}(O(n)) \\
\downarrow \text{proj}_2 \\
\pi_{m-1}(O(mn)) \\
\times_m \\
\pi_{m-1}(O(mn)) \\
\downarrow \\
\pi_{m-1}(O(mn))
\end{array}
\]

\[ (8) \]

Proposition 3.13. [AM22, Proposition 2.12] Let \( i \in \mathbb{N} \). The homomorphism \( \otimes^r : \pi_i(O(n)) \to \pi_i(O(n^r)) \) is given by
\[ \otimes^r(x) = rn^{r-1} \text{stab}(x) \text{ for } x \in \pi_i(O(n)). \]

Corollary 3.14. [AM22, Corollary 2.13] If \( i < n-1 \), the map \( \otimes^r(x) = rn^{r-1}x \) for \( x \in \pi_i(O(n)) \).

3.4. Tensor product on the quotient. We want to describe the effect of the tensor product operation on the homotopy groups of the projective complex orthogonal group.

The methods used in [AM22] to establish decomposition of topological Azumaya algebras apply to those whose degrees are relatively prime. For this reason, we study the tensor product
\[ \otimes : PO(m) \times PO(n) \to PO(mn) \]
in two cases: when \( m \) and \( n \) are odd, and when \( m \) is even and \( n \) is odd.

3.4.1. Case \( m \) and \( n \) odd. Let \( m \) and \( n \) be positive integers such that \( m \) and \( n \) are odd. Since \( PO(m) \times PO(n) = SO(m) \times SO(n) \), the tensor product in (9) may be written as
\[ \otimes : SO(m) \times SO(n) \to SO(mn), \]
and by Proposition 3.10 we have

Proposition 3.15. Let \( i \in \mathbb{N} \). The homomorphism
\[ \otimes : \pi_i(SO(m)) \times \pi_i(SO(n)) \to \pi_i(SO(mn)) \]
is given by \( \otimes(x, y) = n \text{stab}(x) + m \text{stab}(y) \text{ for } x \in \pi_i(SO(m)) \text{ and } y \in \pi_i(SO(n)) \).

3.4.2. Case \( m \) even and \( n \) odd. Let \( m \) and \( n \) be positive integers such that \( m \) is even and \( n \) is odd. The tensor product operation \( \otimes : O(m) \times SO(n) \to O(mn) \) sends the center of \( O(m) \times SO(n) \) to the center of \( O(mn) \). As a consequence, the operation descends to the quotient
\[ \otimes : PO(m) \times SO(n) \to PO(mn). \]

Proposition 3.16. Let \( i > 1 \). The homomorphism
\[ \otimes : \pi_i(PO(m)) \times \pi_i(SO(n)) \to \pi_i(PO(mn)) \]
is given by \( \otimes(x, y) = n \text{stab}(x) + m \text{stab}(y) \text{ for } x \in \pi_i(PO(m)) \text{ and } y \in \pi_i(SO(n)). \)
Proof. There is a map of fibrations
\[ Z(O(m)) \times \{I_o\} \xrightarrow{\otimes} O(m) \times SO(n) \xrightarrow{} PO(m) \times SO(n) \]
(12)
\[ Z(O(mn)) \xrightarrow{} O(mn) \xrightarrow{} PO(mn). \]
From the homomorphism between the long exact sequences associated to the fibrations in diagram (12), we obtain a commutative square
\[ \pi_i(O(m)) \times \pi_i(SO(n)) \xrightarrow{\otimes} \pi_i(PO(m)) \times \pi_i(SO(n)) \]
\[ \pi_i(O(mn)) \xrightarrow{\otimes} \pi_i(PO(mn)), \]
for \( i > 1 \). From this diagram and Proposition 3.13 we have that for all \( i > 1 \), \( \otimes_\ast(x, y) = n\text{stab}(x) + m\text{stab}(y) \) for \( x \in \pi_i(PO(m)) \) and \( y \in \pi_i(SO(n)) \).

Proposition 3.17. The homomorphism
\[ \otimes_\ast : \pi_1(PO(m)) \times \pi_1(SO(n)) \xrightarrow{} \pi_1(PO(mn)) \]
is given by the following expressions.

1. If \( m \equiv 0 \mod 4 \) and \( n \) is odd, then \( \otimes_\ast(x, \beta) = x \) for \( x \in \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) and \( \beta \in \mathbb{Z}/2 \).
2. If \( m \equiv 2 \mod 4 \) and \( n \) is odd, then \( \otimes_\ast(\alpha, \beta) = n\alpha \) for \( \alpha \in \mathbb{Z}/4 \) and \( \beta \in \mathbb{Z}/2 \).

Proof. From the long exact sequence associated to diagram (12) there is a diagram of exact sequences
\[ \begin{array}{c}
\pi_1(PO(m)) \times \pi_1(SO(n)) \\
\pi_1(PO(m)) \times \pi_1(SO(n)) \\
\pi_0(PO(m)) \\
\pi_0(PO(m))
\end{array} \xrightarrow{\otimes_\ast} \begin{array}{c}
\pi_1(O(mn)) \\
\pi_1(O(mn)) \\
\pi_0(O(mn)) \\
\pi_0(O(mn))
\end{array} \]
(13)
Since \( Z(O(m)) \) is contained in the connected component of the identity of \( O(m) \), then the homomorphism \( \pi_0 Z(O(m)) \rightarrow \pi_0(O(m)) \) is trivial. Hence we have a diagram of short exact sequences
\[ \begin{array}{c}
\pi_1(O(m)) \times \pi_1(SO(n)) \\
\pi_1(O(mn)) \\
\pi_1(O(mn)) \\
\pi_1(O(mn))
\end{array} \xrightarrow{\otimes_\ast} \begin{array}{c}
\pi_1(PO(m)) \times \pi_1(SO(n)) \\
\pi_1(PO(mn)) \\
\pi_1(PO(mn)) \\
\pi_1(PO(mn))
\end{array} \xrightarrow{\otimes_\ast} \begin{array}{c}
\pi_0 Z(O(m)) \\
\pi_0 Z(O(mn)) \\
\pi_0 Z(O(mn)) \\
\pi_0 Z(O(mn))
\end{array} \]
(14)
By Corollary 3.11, and the parities of \( m \) and \( n \), the homomorphism \( \otimes_\ast : \pi_1(O(m)) \times \pi_1(SO(n)) \rightarrow \pi_1(O(mn)) \) is the projection onto the first coordinate. Diagram (14) becomes
\[ \begin{array}{c}
0 \\
\pi_1(O(mn)) \\
\pi_1(O(mn)) \\
\pi_1(O(mn))
\end{array} \xrightarrow{\otimes_\ast} \begin{array}{c}
\pi_1(PO(mn)) \\
\pi_1(PO(mn)) \\
\pi_1(PO(mn)) \\
\pi_1(PO(mn))
\end{array} \xrightarrow{\otimes_\ast} \begin{array}{c}
\pi_0 Z(O(mn)) \\
\pi_0 Z(O(mn)) \\
\pi_0 Z(O(mn)) \\
\pi_0 Z(O(mn))
\end{array} \]
(15)
Part 1. If \( m = 4k \), then diagram (15) takes the form

\[
\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}/2 \times \mathbb{Z}/2 & \to & (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 & \to & \mathbb{Z}/2 & \to & 0 \\
\text{proj}_1 & & \downarrow \kappa & & \downarrow \varphi & & \downarrow & & \\
0 & \to & \mathbb{Z}/2 & \to & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \to & \mathbb{Z}/2 & \to & 0.
\end{array}
\]

Let \( s : \mathbb{Z}/2 \to (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \) be a section of \( \varphi \) such that \( s(1) = (0 \oplus 1, 0) \). The composite \( \otimes \circ s \), \( \Delta \), and \( s \) make the short exact sequences in diagram (16) split compatibly. Thus

\[
(Z/2 \oplus Z/2) \times Z/2 \xrightarrow{\otimes} \text{Im } i \oplus \text{Im } s \\
\downarrow \otimes & \downarrow \text{proj}_1 \oplus \text{id} \\
Z/2 \oplus Z/2 \xrightarrow{\otimes} \text{Im } i \oplus \text{Im}(\otimes \circ s).
\]

Hence \( \otimes_s(\alpha \oplus \beta, \gamma) = \alpha \oplus \beta \).

Part 2. If \( m = 4k + 2 \), then diagram (15) takes the form

\[
\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}/2 \times \mathbb{Z}/2 & \to & \mathbb{Z}/4 \times \mathbb{Z}/2 & \to & \mathbb{Z}/2 & \to & 0 \\
\text{proj}_1 & & \downarrow \kappa & & \downarrow \varphi & & \downarrow & & \\
0 & \to & \mathbb{Z}/2 & \to & \mathbb{Z}/4 & \to & \mathbb{Z}/2 & \to & 0.
\end{array}
\]

By direct inspection we have \( \otimes_s(1, 0) = n \) and \( \otimes_s(0, 1) = 0 \). Therefore, \( \otimes_s(\alpha, \beta) = n\alpha \).

Proposition 3.18. Let \( n \) be a positive odd integer, then the homomorphism

\( \otimes_s : \pi_0(\text{PO}(m)) \times \pi_0(\text{SO}(n)) \to \pi_0(\text{PO}(mn)) \)

is a bijection.

4. PROOF OF THEOREM 1.5

Let \( m \) and \( n \) be positive integers. By applying the classifying-space functor to the homomorphism (9) we obtain a map

\( f_\otimes : \text{BPO}(m) \times \text{BPO}(n) \to \text{BPO}(mn). \)

Proposition 4.1. Let \( m \) and \( n \) be relatively prime positive integers such that \( m \) is even, and \( n \) is odd. There exists a homomorphism \( \tilde{T} : \text{PO}(m) \times \text{SO}(n) \to \text{SO}(N) \), for some a positive integer \( N \), such that the homomorphisms induced on homotopy groups

\( \tilde{T}_i : \pi_i(\text{PO}(m)) \times \pi_i(\text{SO}(n)) \to \pi_i(\text{SO}(N)) \)

are given by the following expressions. Let \( d \) denote \( \min\{m, n\} \).

(1) If \( 1 < i < d - 1 \),

\[
\tilde{T}_i(x, y) = 2umx + vy \quad \text{where} \quad x \in \pi_i(\text{PO}(m)), \ y \in \pi_i(\text{SO}(n)),
\]

and \( u, v \) are some positive integers, independent of \( i \), for which \( |vn - 2um^2| = 1 \).

(2) If \( i = 1 \), then

\[
\left\{ \begin{array}{ll}
\tilde{T}_1(\alpha \oplus \beta, \gamma) = z\beta + \gamma & \text{if } m \equiv 0 \pmod{4}, \\
\tilde{T}_1(\delta, \gamma) = z'\delta + \gamma & \text{if } m \equiv 2 \pmod{4}.
\end{array} \right.
\]

for \( z, z', \alpha, \beta, \gamma \in \mathbb{Z}/2 \) and \( \delta \in \mathbb{Z}/4 \).

Proof. We establish the result for \( m < n \); the case of \( n < m \) follows similarly. Since \( m \) and \( n \) are relatively prime, there exist positive integers \( u \) and \( v \) such that \( vn - 2um^2 = \pm 1 \). Let \( N \) denote
\(um^2 + vn\), and let \(T\) denote the composite
\[
\begin{align*}
O(m) \times SO(n) \\
\downarrow^{(\otimes^2, \text{id})} \\
SO(m^2) \times SO(n) \\
\downarrow^{(\otimes^v, \otimes^v)} \\
SO(um^2) \times SO(vn) \\
\downarrow^{\otimes} \\
SO(N).
\end{align*}
\]

Note that the elements \((\pm I_m, I_n)\) are sent to \((I_{m^2}, I_n^2)\) by \((\otimes^2, \text{id})\), hence to the identity by the composite \(T\) defined above. Hence \(T\) factors through \(PO(m) \times SO(n)\)
\[
\begin{align*}
O(m) \times SO(n) & \xrightarrow{T} \quad \text{PO}(m) \times SO(n) \\
\downarrow & \quad \downarrow^{\tilde{T}} \\
& \quad \text{SO}(N) \quad \text{SO}(N).
\end{align*}
\]

From Proposition 3.16, and Corollaries 3.6, 3.8 and 3.14 we have that \(T_i(x, y) = 2umx + vy\) for \(i < m - 1\).

The map of fibrations
\[
\begin{align*}
Z(O(m)) \times \{I_n\} & \xrightarrow{T} \quad O(m) \times SO(n) \xrightarrow{T} \quad \text{PO}(m) \times SO(n) \\
\downarrow & \quad \downarrow^{T} \\
\{I_N\} & \xrightarrow{T} \quad \text{SO}(N) \quad \text{SO}(N)
\end{align*}
\]
induces a commutative diagram
\[
\begin{align*}
\pi_i(O(m)) \times \pi_i(SO(n)) & \xrightarrow{T_i} \quad \pi_i(PO(m)) \times \pi_i(SO(n)) \\
\downarrow & \quad \downarrow^{T_i} \\
\pi_i(SO(N)) & \xrightarrow{T_i} \quad \pi_i(SO(N))
\end{align*}
\]
for \(i > 1\). Then \(\tilde{T}_1(x, y) = T_1(x, y) = 2umx + vy\) for \(1 < i < m - 1\).

For \(i = 1\), the map of fibrations induces the commutative diagram below
\[
\begin{align*}
\pi_1(O(m)) \times \pi_1(SO(n)) & \xrightarrow{T_1} \quad \pi_1(PO(m)) \times \pi_1(SO(n)) \\
\downarrow & \quad \downarrow^{\tilde{T}_1} \\
\pi_1(SO(N)) & \xrightarrow{T_1} \quad \pi_1(SO(N))
\end{align*}
\]
which takes the form
\[
\begin{align*}
\mathbb{Z}/2 \times \mathbb{Z}/2 & \xrightarrow{\tilde{T}_1} \quad Z(\text{Spin}(m)) \times \mathbb{Z}/2 \\
& \quad \downarrow^{\tilde{T}_1} \\
& \quad \mathbb{Z}/2.
\end{align*}
\]

Then observe that the equality \(vn - 2um^2 = \pm 1\) implies \(v\) is odd, then \(T_1(\alpha, \beta) = 2um\alpha + v\beta = \beta\), i.e. \(T_1\) is projection onto the second coordinate.

**Subcase i:** Suppose \(m = 4k\) for some \(k \in \mathbb{Z}\). Diagram (18) takes the form
\[
\begin{align*}
\mathbb{Z}/2 \times \mathbb{Z}/2 & \xrightarrow{T_1} \quad (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \\
& \quad \downarrow^{\tilde{T}_1} \\
& \quad \mathbb{Z}/2
\end{align*}
\]
where the horizontal homomorphism is the inclusion \( \alpha, \beta \mapsto (\alpha \oplus 0, \beta) \). Thus \( \tilde{T}_1(1 \oplus 0, 0) = 0 \) and \( \tilde{T}_1(0 \oplus 1, 0) = 1 \). Let \( z \) denote \( \tilde{T}_1(0 \oplus 1, 0) \). Hence \( \tilde{T}_1(\alpha \oplus \beta, \gamma) = z\beta + \gamma \).

**Subcase ii:** Suppose \( m = 4k + 2 \) for some \( k \in \mathbb{Z} \). Diagram (18) takes the form

\[
\begin{array}{ccc}
\mathbb{Z}/2 \times \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/4 \times \mathbb{Z}/2 \\
\downarrow_{\tilde{T}_1} & & \downarrow_{\tilde{T}_1} \\
\mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2
\end{array}
\]

where the horizontal homomorphism is the inclusion \( \alpha, \beta \mapsto (\alpha, \beta) \). Thus \( \tilde{T}_1(0, 1) = 1 \). Let \( z' \) denote \( \tilde{T}_1(1, 0) \). Hence \( \tilde{T}_1(\delta, \gamma) = z'\delta + \gamma \).

\[\square\]

4.1. **A \( d \)-connected map where \( d \) is the minimum of two positive integers of opposite parity.** Let \( J \) denote the map

\[J : \text{BPO}(m) \times \text{BPO}(n) \longrightarrow \text{BPO}(mn) \times \text{BSO}(N) \]

\[(x, y) \longmapsto (f_\otimes(x, y), \tilde{T}(x, y))\]

Let \( J_i \) denote the homomorphism induced on homotopy groups by \( J \).

\[J_i : \pi_i(\text{BPO}(m)) \times \pi_i(\text{BPO}(n)) \rightarrow \pi_i(\text{BPO}(mn)) \times \pi_i(\text{BSO}(N))\]

**Proposition 4.2.** Let \( m \) and \( n \) be relatively prime positive integers such that \( m \) is even, and \( n \) is odd. Let \( d \) denote \( \min\{m, n\} \). The homomorphism \( J_i \) is an isomorphism for all \( i < d \).

**Proof.** We establish the result for \( m < n \); the case of \( n < m \) follows similarly.

\[J_i : \pi_i(\text{BPO}(m)) \times \pi_i(\text{BSO}(N)) \rightarrow \pi_i(\text{BPO}(mn)) \times \pi_i(\text{BSO}(N))\]

Let \( i < m \). Given that the homotopy groups of the spaces involved are zero in degrees \( i \equiv 3, 5, 6, 7 \) (mod 8), it suffices to prove that \( J_i \) is an isomorphism for \( i = 1, 2 \), and for \( i \equiv 0, 1, 2, 4 \) (mod 8) with \( i > 2 \).

Let \( i = 1 \). By Proposition 3.18 the homomorphism \( J_1 : \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \) is the identity.

Let \( i = 2 \). The homomorphism (19) takes the form

\[J_2 : \mathbb{Z}(\text{Spin}(m)) \times \mathbb{Z}/2 \longrightarrow \mathbb{Z}(\text{Spin}(mn)) \times \mathbb{Z}/2.\]

**Case i:** Let \( m = 4k \) for some \( k \in \mathbb{Z} \). From Propositions 3.17 and 4.1, \( J_2 \) is given by

\[J_2 : (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \longrightarrow (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \]

\[(\alpha \oplus \beta, \gamma) \longmapsto (\alpha \oplus \beta, z\beta + \gamma),\]

i.e. \( J_2 \) is represented by the invertible matrix \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & 1 \end{pmatrix} \) where \( z \in \mathbb{Z}/2 \). Hence \( J_2 \) is an isomorphism.

**Case ii:** Let \( m = 4k + 2 \) for some \( k \in \mathbb{Z} \). From Proposition 3.17, \( f_\otimes(\alpha, \beta) = n\alpha \). Thus, by Proposition 4.1, \( J_2 \) is given by

\[J_2 : \mathbb{Z}/4 \times \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \times \mathbb{Z}/2 \]

\[(\alpha, \beta) \longmapsto (n\alpha, z'\alpha + \beta),\]

i.e. \( J_2 \) is represented by the invertible matrix \( \begin{pmatrix} n & 0 \\ z' & 1 \end{pmatrix} \) where \( z' \in \mathbb{Z}/2 \). Hence \( J_2 \) is an isomorphism.

Let \( 2 < i < m \). The homomorphism (19) takes the form \( J_i : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} \) if \( i \equiv 0, 4 \) (mod 8), and \( J_i : \mathbb{Z}/2 \times \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \) if \( i \equiv 1, 2 \) (mod 8). Note that both homomorphisms are represented by the matrix \( \begin{pmatrix} n & m \\ 2um & v \end{pmatrix} \), whose determinant equals \( vn - 2um^2 \). Thus, \( J_i \) is an isomorphism because, by Proposition 4.1, \( vn - 2um^2 = \pm 1 \).

\[\square\]
Proposition 4.3. Let $m$ and $n$ be relatively prime positive integers such that $m$ is even, and $n$ is odd. Let $d$ denote $\min\{m, n\}$. The induced homomorphism

\[
J_d : \pi_d(BPO(m)) \times \pi_d(BSO(n)) \longrightarrow \pi_d(BPO(mn)) \times \pi_d(BSO(N))
\]

is an epimorphism.

Proof. Suppose $m < n$. Propositions 3.7 and 4.1, and Corollaries 3.8 and 3.14 show that $T_{m-1} : \pi_{m-1}(O(m)) \times \pi_{m-1}(SO(n)) \rightarrow \pi_{m-1}(SO(N))$ is given by

\[
T_{m-1}(x, y) = \text{stab}(u \text{stab}(2m \text{stab}(x))) + \text{stab}(v \text{stab}(y)) = 2um \text{stab}(x) + vy,
\]

where $vn - 2um^2 = \pm1$. This can be seen by observing that $T_{m-1}$ is equal to the sum of the two paths around diagram (21).

Hence (22) is given by

\[
J_m : \pi_m(BPO(m)) \times \pi_m(BSO(n)) \longrightarrow \pi_m(BSO(mn)) \times \pi_m(BO(N))
\]

\[
(x, y) \longrightarrow (n \text{stab}_1(x) + my, 2um \text{stab}_2(x) + vy)
\]

where

\[
\text{stab}_1 : \pi_{m-1}(O(m)) \longrightarrow \pi_{m-1}(O(mn))
\]

and

\[
\text{stab}_2 : \pi_{m-1}(O(m)) \longrightarrow \pi_{m-1}(O(m^2))
\]

are epimorphisms.

Observe that for $m = 1$ or $m \equiv 3, 5, 6, 7 \mod 8$, the homomorphism $J_m$ has trivial target.

Let $m = 2$. Then $J_2 : \mathbb{Z} \times \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \times \mathbb{Z}/2$ is given by

\[
J_2(x, y) = (n \text{stab}_1(x) + 2y, y).
\]
Thus $J_2$ is an epimorphism. Let $i > 2$ and $m \equiv 0, 4 \pmod{8}$. Then $J_m : (\mathbb{Z} \oplus \mathbb{Z}) \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is given by

$$J_m(x \oplus y, z) = (n \text{stab}_1(x \oplus y) + mz, 2um \text{stab}_2(x \oplus y) + vz).$$

Note that $J_m$ factors as

$$(\mathbb{Z} \oplus \mathbb{Z}) \times \mathbb{Z} \xrightarrow{(\text{stab}, \text{id})} \mathbb{Z} \times \mathbb{Z} \xrightarrow{(n, 2um) \choose (m, v)} \mathbb{Z} \times \mathbb{Z}.$$ 

Hence $J_m$ is an epimorphism.

Let $i > 2$ and $m \equiv 2 \pmod{8}$. Then $J_m : (\mathbb{Z}/2 \oplus \mathbb{Z}) \times \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2$ is given by

$$J_m(x \oplus y, z) = (\text{stab}_1(x \oplus y), z).$$

Then $J_m$ is an epimorphism.

If we suppose $n < m$, then in the same manner it can be proved that

$$J_n : \pi_n(\text{B PO}(m)) \times \pi_n(\text{B SO}(n)) \to \pi_n(\text{B SO}(mn)) \times \pi_n(\text{BO}(N))$$

is an epimorphism, where

$$\text{stab}_1 : \pi_{n-1}(\text{SO}(n)) \to \pi_{n-1}(\text{SO}(mn))$$

and

$$\text{stab}_2 : \pi_{n-1}(\text{SO}(n)) \to \pi_{n-1}(\text{SO}(vn)).$$

\[
\square
\]

From Propositions 4.2, and 4.3 we obtain Corollary 4.4.

**Corollary 4.4.** Let $m$ and $n$ be relatively prime positive integers such that $m$ is even, and $n$ is odd. Let $d$ denote $\min\{m,n\}$. The map $J$ is $d$-connected.

**4.2. Factorization through** $f_\otimes : \text{B PO}(m, \mathbb{C}) \times \text{B SO}(n, \mathbb{C}) \to \text{B PO}(mn, \mathbb{C})$.

**Proof of Theorem 1.5.** Diagrammatically speaking, we want to find a map

$$\mathcal{A}_m \times \mathcal{A}_n : X \to \text{B PO}(m) \times \text{B SO}(n)$$

such that diagram (23) commutes up to homotopy

$$\begin{array}{ccc}
\mathcal{A}_m \times \mathcal{A}_n & \to & \text{B PO}(m) \times \text{B SO}(n) \\
X \downarrow & \cong & \downarrow f_\otimes \\
\mathcal{A} & \to & \text{B PO}(mn).
\end{array}$$

We establish the result for $m < n$; the case of $n < m$ follows similarly. Corollary 4.4 yields a map $J : \text{B PO}(m) \times \text{B SO}(n) \to \text{B PO}(mn) \times \text{B SO}(N)$ where $N$ is some positive integer so that $N \gg n > m$. Observe that $f_\otimes$ factors through $\text{B PO}(mn) \times \text{B SO}(N)$, so we can write $f_\otimes$ as the composite of $J$ and the projection $\text{proj}_1$ shown in diagram (24).

$$\begin{array}{ccc}
\text{B PO}(m) \times \text{B SO}(n) & \xrightarrow{J} & \text{B PO}(mn) \times \text{B SO}(N) \\
\text{B PO}(mn) \downarrow & \cong & \downarrow \text{proj}_1 \\
\text{B PO}(mn) & \xrightarrow{f_\otimes} & \text{B PO}(mn).
\end{array}$$

Since $J$ is $m$-connected and $\dim(X) \leq m$, then by Whitehead’s theorem

$$J_\# : [X, \text{B PO}(m) \times \text{B SO}(n)] \to [X, \text{B PO}(mn) \times \text{B SO}(N)]$$

is a surjection, [Spa81, Corollary 7.6.23].
Let $s$ denote a section of $\text{proj}_1$. The surjectivity of $J_s$ implies $s \circ A$ has a preimage $A_m \times A_n : X \to BPO(m) \times BSO(n)$ such that $J \circ (A_m \times A_n) \simeq s \circ A$.

Commutativity of diagram (23) follows from commutativity of diagram (24). Thus, the result follows. □

Remark 4.5. The map $f_\otimes : BPO(m) \times BSO(n) \to BPO(mn)$ does not necessarily have a section. We find examples for small values of $m$. Let $m$ and $n$ be positive integers such that $m \in \{2, 4, 6, 8, 10\}$, $n$ is odd and $n > 16$. Then $f_\otimes : BPO(m) \times BSO(n) \to BPO(mn)$ does not have a section.

Suppose $f_\otimes$ has a section, namely $\sigma$. By Proposition 3.16, the homomorphism induced on homotopy groups by $f_\otimes$ is given by $(x, y) \mapsto n \text{stab}(x) + m \text{stab}(y)$ for all $x \in \pi_i(BPO(m))$, $y \in \pi_i(BSO(n))$, and $i > 2$.

Let $C$ denote the set $\{(4, 2), (8, 4), (12, 6), (16, 8), (16, 10)\}$. From [MT63], [MSOl93], Table 6.VII, Appendix A] and Table 3, $\pi_i(BPO(m))$ is torsion and $\pi_i(BPO(n)) \cong \pi_i(BPO(mn)) \cong \mathbb{Z}$ for all $(i, m) \in C$. Since $\sigma$ is a section of $f_\otimes$, the image of $(f_\otimes)_* \circ \sigma_*$ is equal to $\pi_i(BPO(mn)) \cong \mathbb{Z}$ for $(i, m) \in C$. However, the image of $(f_\otimes)_*: \pi_i(BPO(m)) \times \pi_i(BSO(n)) \to \pi_i(BPO(mn))$ is $m\mathbb{Z}$ for all $(i, m) \in C$.

Remark 4.6. If $A$ is a topological Azumaya algebra of degree $mn$ with an orthogonal involution over a finite CW complex of dimension higher than $\min\{m, n\}$, then $A$ may not have a decomposition as the one in Theorem 1.5.

Let $m$ and $n$ be positive integers such that $m \in \{2, 4, 6, 8, 10\}$, $n$ is odd and $n > 16$. Let $\mathcal{S}$ be a topological Azumaya algebra of degree $mn$ with an orthogonal involution on $S^l$ such that $\mathcal{S}$ generates $\pi_i(BPO_{mn})$ for $(i, m) \in C$. An argument similar to the one used in Remark 4.5 can be used to prove that $\mathcal{S}$ cannot be decomposed as $A_m \otimes A_n$ for $A_m$ and $A_n$ topological Azumaya algebras of degrees $m$ and $n$, respectively, with orthogonal involutions.

5. Tensor Product Decomposition of Orthogonal Bundles

Proposition 5.1. Let $m$ and $n$ be relatively prime positive integers such that $m$ and $n$ are odd. Let $d$ denote $\min\{m, n\}$. There exists a homomorphism $T : \text{SO}(m) \times \text{SO}(n) \to \text{SO}(N)$, for some positive integer $N$, such that for all $i < d − 1$ the homomorphisms induced on homotopy groups

$$T_i : \pi_i(\text{SO}(m)) \times \pi_i(\text{SO}(n)) \longrightarrow \pi_i(\text{SO}(N))$$

are given by $T_i(x, y) = ux + vy$ where $x \in \pi_i(\text{SO}(m))$, $y \in \pi_i(\text{SO}(n))$, and $u, v$ are some positive integers, independent of $i$, for which $|vn − um| = 1$.

Proof. Assume without loss of generality $m < n$. Since $m$ and $n$ are relatively prime, there exist positive integers $u$ and $v$ such that $vn − um = ±1$. Let $N$ denote $um + vn$, and let $T$ denote the composite

$$\text{SO}(m) \times \text{SO}(n) \xrightarrow{(\oplus^n, \oplus^n)} \text{SO}(um) \times \text{SO}(vn) \xrightarrow{\oplus} \text{SO}(N).$$

Let $i < m − 1$. From Corollaries 3.6 and 3.8 we have that $T_i(x, y) = ux + vy$. □

Proposition 5.2. Let $m$ and $n$ be relatively prime positive integers such that $m$ and $n$ are odd. The map $J$ is $d$-connected where $d$ denotes $\min\{m, n\}$.

Proof. Without loss of generality, let $m < n$. Let $i < m$. By Propositions 3.15 and 5.1, and Corollary 3.11 the homomorphism

$$J_i : \pi_i(\text{BSO}(m)) \times \pi_i(\text{BSO}(n)) \to \pi_i(\text{BSO}(mn)) \times \pi_i(\text{BSO}(N))$$

is represented by the matrix $\begin{pmatrix} n & m \\ u & v \end{pmatrix}$.

Observe that this matrix is invertible for all $i < m$ because by Proposition 5.1 its determinant satisfies $nv − um = ±1$.

Let $i = m$. Propositions 3.7 and 5.1, and Corollary 3.8 show that the homomorphism $T_{m−1} : \pi_{m−1}(\text{SO}(m)) \times \pi_{m−1}(\text{SO}(n)) \to \pi_{m−1}(\text{SO}(N))$ is given by $T_{m−1}(x, y) = \text{stab}(u\text{stab}(x)) + \text{stab}(v\text{stab}(y)) = u\text{stab}(x) + vy$, where $vn − um = ±1$. This can be seen by observing that $T_{m−1}$ is equal to the sum of the two paths around diagram (25).
Theorem 5.3. Let $m$ and $n$ be relatively prime positive integers such that $m$ and $n$ are odd. Let $X$ be a CW complex such that $\dim(X) \leq d$ where $d \coloneqq \min\{m, n\}$. If $V$ is an orthogonal complex vector bundle rank $mn$ over $X$, then there exist orthogonal complex vector bundles $V_m$ and $V_n$ of ranks $m$ and $n$, respectively, such that $V \cong V_m \otimes V_n$.

Proof. We can use Propositions 5.1 and 5.2 to prove the result in a similar manner to the proof of Theorem 1.5.
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