LOW-TEMPERATURE PHASE TRANSITIONS IN THE QUADRATIC FAMILY

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ABSTRACT. We give the first example of a quadratic map having a phase transition after the first zero of the geometric pressure function. This implies that several dimension spectra and large deviation rate functions associated to this map are not (expected to be) real analytic, in contrast to the uniformly hyperbolic case. The quadratic map we study has a non-recurrent critical point, so it is non-uniformly hyperbolic in a strong sense.

1. Introduction

In their pioneer works, Sinai, Bowen and Ruelle [Sin72, Bow75, Rue76] initiated the thermodynamic formalism of smooth dynamical systems. They gave a complete description in the case of a uniformly hyperbolic diffeomorphism and a Hölder continuous potential. In the last decades there has been a substantial progress in extending the theory beyond this setting. A complete picture is emerging in real and complex dimension 1, see [BT09, IT11, MS00, MS03, PS08, PRL11] and references therein. See also [Sar11, UZ09, VV10] and references therein for (recent) results in higher dimensions.

In this paper we focus in the quadratic family; one of the simplest and yet challenging families of smooth one-dimensional maps. For a real parameter $c$ we consider 2 dynamical systems arising from the complex quadratic polynomial

$$f_c(z) := z^2 + c;$$

the action of $f_c$ on $\mathbb{R}$ and the action of $f_c$ on its complex Julia set. For each of these dynamical systems and for a varying real number $t$, we consider the pressure of the geometric potential $-t \log |Df_c|$. There are thus 2 pressure functions associated to $f_c$: one in the real setting and another one in the complex setting. In what follows we use “geometric pressure function” to refer to any of these functions; precise definitions and statements are given in §1.1.

Our main interest are “phase transitions” in the statistical mechanics sense: For a real number $t_*$ the map $f_c$ has a phase transition at $t = t_*$ if the geometric pressure function is not real analytic at $t = t_*$. In the real case, phase transitions might be caused by lack of transitivity, see for example [Dob09]. Since these phase transitions are well understood, we restrict our discussion to parameters for which the real map is transitive. For $c = -2$ the map $f_{-2}$ is a Chebyshev polynomial and it has a phase transition at $t = -1$.\footnote{For $c = -2$ the Julia set of $f_{-2}$ is the interval $[-2, 2]$ and both, the real and complex geometric pressure functions of $f_{-2}$ are given by $t \mapsto \max\{-t \log 4, (1-t) \log 2\}$.} The mechanism behind this phase transition, and of any phase transition in the complex setting that occurs at a negative value
of \( t \), was explained by Makarov and Smirnov, see [MS00, Theorem B].\(^1\) Combining the results of Makarov and Smirnov with recent results of Przytycki and the second named author, it follows that for every real parameter \( c \neq -2 \) the map \( f_c \) has at most 1 phase transition; moreover, if \( f_c \) has a phase transition, then it occurs at some \( t > 0 \). See [PRL11, §A.3] for the complex case and [PRL12] for the real case.

To describe the possible phase transitions for \( c \neq -2 \), it is useful to distinguish 3 complementary cases: \( f_c \) uniformly hyperbolic, \( f_c \) satisfying the Collet-Eckmann condition:

\[
\chi_{\text{crit}}(c) := \liminf_{n \to +\infty} \frac{1}{n} \log |Df^n_c(c)| > 0,
\]

and the remaining case, when \( f_c \) is not uniformly hyperbolic and does not satisfy the Collet-Eckmann condition. The Collet-Eckmann condition is one of the strongest and most studied non-uniform hyperbolicity conditions in dimension 1, see for example [AM05, BC85, GS11, NS98, PRLS03] and references therein. Benedicks and Carleson showed that the set of real parameters \( c \) such that \( f_c \) satisfies the Collet-Eckmann condition has positive Lebesgue measure, see [BC85]. Moreover, Avila and Moreira showed that the set of real parameters \( c \) such that \( f_c \) is not uniformly hyperbolic and does not satisfy the Collet-Eckmann condition has zero Lebesgue measure, see [AM05] and also [GS11].

When \( f_c \) is uniformly hyperbolic, the work of Sinai, Bowen and Ruelle can be adapted to show that the geometric pressure function is real analytic at every real number, see for example [PU10, §6.4]. That is, if \( f_c \) is uniformly hyperbolic, then it has no phase transitions.

If \( f_c \) is not uniformly hyperbolic and does not satisfy the Collet-Eckmann condition, then the geometric pressure function is non-negative and vanishes for large values of \( t \), see [NS98, Theorem A] or [RL12, Corollary 1.3] for the real case and [PRLS03, Main Theorem] for the complex case. Thus in this case \( f_c \) has a phase transition at the first zero of the geometric pressure function. Note that this phase transition is associated to the lack of (non-uniform) expansion of \( f_c \).

This paper is focused in the remaining case, when \( f_c \) satisfies the Collet-Eckmann condition. We show that, contrary to a widespread belief, such a map can have a phase transition at some \( t > 0 \). As a consequence, several dimension spectra and large deviation rate functions associated to such a \( f_c \) are not (expected to be) real analytic, see Remark 1.1. In the complex setting it also follows that the corresponding integral means spectrum is not real analytic either.

Our construction is very flexible. We give the simplest example here, of a “first-order” phase transition: The geometric pressure function is not differentiable at the phase transition. In the companion paper [CRL12] we modify our construction

\(^1\) Makarov and Smirnov showed this type of phase transition is caused by the existence of a gap in the Lyapunov spectrum; more precisely, they showed that if a complex rational map has a phase transition at some \( t < 0 \), then there is a finite set of periodic points \( F \) such that there is a definite distance separating the Lyapunov exponents of periodic points in \( F \) and the Lyapunov exponents of measures that do not charge \( F \). Makarov and Smirnov also showed that this type of phase transition is removable in the following sense: The function obtained by omitting the measures that charge \( F \) in the supremum defining the geometric pressure function is real analytic on \((-\infty, 0)\) and coincides with the geometric pressure function up to the phase transition.

\(^{\dagger}\) In the complex setting the Collet-Eckmann condition is usually formulated in such a way that a uniformly hyperbolic map satisfies the Collet-Eckmann condition by vacuity. Here we use the usual terminology in the real setting, for which a uniformly hyperbolic map does not satisfy the Collet-Eckmann condition.
to obtain an “infinite-order” phase transition; that is, a quadratic map having a phase transition at which the geometric pressure function is infinitely differentiable. To the best of our knowledge it is the first example of a smooth dynamical system having an infinite-order phase transition. Our construction is also robust: In every sufficiently small perturbation of the quadratic family there is a Collet-Eckmann parameter having a phase transition.

The quadratic maps studied here are largely inspired by the conformal Cantor sets with analogous properties studied by Makarov and Smirnov, see [MS03, §5]. There are however several important differences. Most notably, the conformal Cantor set studied by Makarov and Smirnov is defined through a map having 2 affine branches, something that cannot be replicated in a complex polynomial or rational map.

These examples show that lack of (non-uniform) expansion is not the only source of phase transitions. In fact, the quadratic maps studied here satisfy a property that is even stronger than the Collet-Eckmann condition: The critical point is non-recurrent. Thus, no slow recurrence condition, such as the one studied by Benedicks and Carleson [BC85] or by Yoccoz and by Pesin and Senti [PS08], is sufficient to avoid phase transitions.

1.1. Statements of results. We consider a set of real parameters close to \( c = -2 \), for which the critical point \( z = 0 \) is mapped to a certain uniformly expanding set under forward iteration by \( f_c \), see §3 for details. For such a parameter \( c \) we have \( f_c(c) > c \), the interval \( I_c := [c, f_c(c)] \) is invariant by \( f_c \), and \( f_c \) is topologically exact on this set. We consider both, the interval map \( f_c|_{I_c} \) and the complex quadratic polynomial \( f_c \) acting on its Julia set \( J_c \).

For a real parameter \( c \) denote by \( \mathcal{M}_c^\mathbb{R} \) the space of probability measures supported on \( I_c \) that are invariant by \( f_c \). For a measure \( \mu \) in \( \mathcal{M}_c^\mathbb{R} \) denote by \( h_\mu(f_c) \) the measure theoretic entropy of \( f_c \) with respect to \( \mu \) and for each \( t \) in \( \mathbb{R} \) put

\[
P_c^\mathbb{R}(t) := \sup \left\{ h_\mu(f_c) - t \int \log |Df_c| d\mu \mid \mu \in \mathcal{M}_c^\mathbb{R} \right\},
\]

which is finite. The function \( P_c^\mathbb{R} : \mathbb{R} \to \mathbb{R} \) so defined is called the geometric pressure function of \( f_c|_{I_c} \); it is convex and non-increasing. An invariant probability measure supported on \( I_c \) is an equilibrium state of \( f_c|_{I_c} \) for the potential \(-\log |Df_c|\), if the supremum above is attained at this measure.

Similarly, denote by \( \mathcal{M}_c^\mathbb{C} \) the space of probability measures supported on \( J_c \) that are invariant by \( f_c \) and for a measure \( \mu \) in \( \mathcal{M}_c^\mathbb{C} \) denote by \( h_\mu(f_c) \) the measure theoretic entropy of \( f_c \) with respect to \( \mu \). The geometric pressure function \( P_c^\mathbb{C} : \mathbb{R} \to \mathbb{R} \) of \( f_c|_{J_c} \) is defined by

\[
P_c^\mathbb{C}(t) := \sup \left\{ h_\mu(f_c) - t \int \log |Df_c| d\mu \mid \mu \in \mathcal{M}_c^\mathbb{C} \right\}.
\]

An invariant probability measure supported on \( J_c \) is an equilibrium state of \( f_c|_{J_c} \) for the potential \( -t \log |Df_c| \) if the supremum above is attained at this measure.

\[\text{In some sense, the phase transitions studied here, as those studied by Makarov and Smirnov, are caused by fine combinatorial properties of the critical orbit.}\]

\[\text{This is usually called the “Misiurewicz condition” and it is known to imply the Collet-Eckmann condition, see [Mis81] for the real setting and [Man93] for the complex one.}\]
Following the usual terminology in statistical mechanics, for a given \( t_* > 0 \) the map \( f_c|_{I_c} \) (resp. \( f_c|_{J_c} \)) has a phase transition at \( t_* \) if \( P_c^R \) (resp. \( P_c^C \)) is not real analytic at \( t = t_* \). As mentioned above, if \( c \neq -2 \) and if \( f_c \) is not uniformly hyperbolic and does not satisfy the Collet-Eckmann condition, then \( f_c|_{I_c} \) (resp. \( f_c|_{J_c} \)) has a phase transition at the first zero of \( P_c^R \) (resp. \( P_c^C \)) and it has no other phase transitions. In accordance with the usual interpretation of \( t > 0 \) as the inverse of the temperature in statistical mechanics, we say that such a phase transition is of high temperature. For a real parameter \( c \) and for \( t_* > 0 \) the map \( f_{c,|I_c} \) (resp. \( f_{c,|J_c} \)) has a low-temperature phase transition at \( t_* \), if it has a phase transition at \( t_* \) and \( P_c^R(t_*) < 0 \) (resp. \( P_c^C(t_*) < 0 \)). Note that if \( f_{c,|I_c} \) (resp. \( f_{c,|J_c} \)) has a low-temperature phase transition, then \( f_c \) satisfies the Collet-Eckmann condition.

**Main Theorem.** There is a real parameter \( c \) such that the critical point of \( f_c \) is non-recurrent and such that both, \( f_{c,|I_c} \) and \( f_{c,|J_c} \) have a low-temperature phase transition. Furthermore, the parameter \( c \) can be chosen so that each of the functions \( P_c^R \) and \( P_c^C \) is non-differentiable at the phase transition and so that each of the maps \( f_{c,|I_c} \) and \( f_{c,|J_c} \) has a unique equilibrium state at the phase transition.

For the parameter \( c \) we use to prove the Main Theorem, we show that the equilibrium state at the phase transition is ergodic and mixing and that its measure theoretic entropy is strictly positive, see Proposition A in §4. Combined with results of Young [You99] and Gouezel [Gou04, Théorème 2.3.1], our estimates imply that the decay of correlations of this measure is (at most) stretch exponential.

In the companion paper [CRL12], we use the results of this paper to show that there is a real parameter \( c \) and \( t_* > 0 \) such that both, \( f_{c,|I_c} \) and \( f_{c,|J_c} \) have a low-temperature phase transition at \( t = t_* \) and such that each of the functions \( P_c^R \) and \( P_c^C \) is infinitely differentiable at \( t = t_* \). In that case there is no equilibrium state at \( t = t_* \), see [IRRL11, Corollary 1.3].

**Remark 1.1.** For a parameter \( c \) in \( \mathbb{C} \) the dimension spectrum for Lyapunov exponents of the complex quadratic polynomial \( f_c(z) = z^2 + c \) is essentially the Legendre transform of \( P_c^R(t) \), see [GPR10, Theorem 1] for a precise statement and [Pes97] for the general theory. So, for a complex quadratic polynomial as in the Main Theorem the dimension spectrum for Lyapunov exponents is not real analytic. A similar behavior is expected for the dimension spectrum for Lyapunov exponents of an interval map as in the Main Theorem. The Legendre transform of \( P_c^R \) (or \( P_c^C \)) is also related to the dimension spectrum for pointwise dimension and the rate function in certain large deviation principles; see for example [MS00, §5] for the former and [KN92, Theorem 1.2 or 1.3] and [PRL11, Corollary B.4] for the latter. So for a map as in the Main Theorem we expect these functions not to be real analytic either. Finally, note that in the complex setting the integral means spectrum associated to \( f_c \) is an affine function of \( P_c^C \), see [BMS03, Lemma 2]. So

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*The following argument shows that for \( c \) as in the Main Theorem, the Legendre transform of \( P_c^C \) is not real analytic. Since \( P_c^C \) is not differentiable at the phase transition, there is an interval on which the Legendre transform of \( P_c^C \) is affine. So, if the Legendre transform was real analytic, then it would be affine on all of its domain of definition. This can only happen if \( P_c^C \) is affine up to the phase transition. But [MS00, Theorem C] or [PRL11, Theorem D] imply that this is not the case.*

**More precisely,** we expect the dimension spectrum of Lyapunov exponents not to be real analytic at the left end point of the interval \( A \) appearing in [IT11, Theorem A].
for a parameter $c$ as in the Main Theorem the integral means spectrum associated to $f_c$ is not real analytic.

1.2. Organization. After recalling some well-known facts in §2, we define in §3 the set of parameters from which we choose the parameter fulfilling the properties in the Main Theorem. For a parameter $c$ in this set, the critical value of $f_c$ is mapped to a uniformly expanding set after a certain number of iterates; this uniformly expanding set is denoted by $\Lambda_c$ and it is defined in §3.3.

In §4 we reduce the proof of the Main Theorem to a criterion stated solely in terms of the behavior of the derivatives along the orbit of the critical value; we state this result as Proposition A. To deduce the Main Theorem from Proposition A, in §4.2, we choose a parameter $c$ for which the orbit of the critical value of $f_c$ has a very particular behavior in $\Lambda_c$, described in Lemma 4.4. In this respect the quadratic map in the Main Theorem is similar to the conformal Cantor sets studied by Makarov and Smirnov in [MS03, §5]. In the proof of the Main Theorem we use a uniform distortion bound shown in §4.1. We also use an explicit calculation relegated to Appendix A.

The proof of Proposition A occupies §§5, 6 and 7. One of the main ingredients is the first return map to a neighborhood of the critical point denoted by $V_c$, see §6.1. In §5 we estimate the derivatives of the first landing map to $V_c$ (Proposition B). These estimates allow us to show a Bowen type formula relating the geometric pressure of the quadratic map $f_c$ to a certain 2 variables pressure function of the first return map to $V_c$ (Proposition C in §6.2). Using this result we estimate in §7 the geometric pressure function of the quadratic map in terms of the derivatives along the orbit of the critical value (Proposition D). Proposition A is easily deduced from Proposition D and well-known arguments.

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2. Preliminaries

We use $\mathbb{N}$ to denote the set of integers that are greater than or equal to 1 and put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For an annulus $A$ contained in $\mathbb{C}$ we use $\text{mod}(A)$ to denote the conformal modulus of $A$.

2.1. Koebe principle. We use the following version of Koebe distortion theorem that can be found, for example, in [McM94]. Given an open subset $G$ of $\mathbb{C}$ and a univalent map $f : G \to \mathbb{C}$, the distortion of $f$ on a subset $C$ of $G$ is by definition

$$ \sup_{x, y \in C} |Df(x)|/|Df(y)|. $$

Koebe Distortion Theorem. For each $A > 0$ there is a constant $\Delta > 1$ such that for each topological disk $\tilde{W}$ contained in $\mathbb{C}$ and each compact set $K$ contained in $\tilde{W}$ and such that $\tilde{W} \setminus K$ is an annulus of modulus at least $A$, the following property
holds: For each open topological disk \( U \) contained in \( \mathbb{C} \) and every biholomorphic map \( f : U \to \hat{\mathbb{W}} \), the distortion of \( f \) on \( f^{-1}(K) \) is bounded by \( \Delta \).

2.2. Quadratic polynomials, Green functions and Böttcher coordinates.

In this subsection and the next we recall some basic facts about the dynamics of complex quadratic polynomials, see for instance [CG93] or [Mil06] for references.

For \( c \) in \( \mathbb{C} \) denote by \( f_c \) the complex quadratic polynomial

\[
f_c(z) = z^2 + c,
\]

and by \( K_c \) the filled Julia set of \( f_c \); that is, the set of all points \( z \) in \( \mathbb{C} \) whose forward orbit under \( f_c \) is bounded in \( \mathbb{C} \). The set \( K_c \) is compact and its complement is the connected set consisting of all points whose orbit converges to infinity in the Riemann sphere. Furthermore, we have \( f_c^{-1}(K_c) = K_c \) and \( f_c(K_c) = K_c \). The boundary \( J_c \) of \( K_c \) is the Julia set of \( f_c \).

For a parameter \( c \) in \( \mathbb{C} \), the Green function of \( K_c \) is the function \( G_c : \mathbb{C} \to [0, +\infty) \) that is identically 0 on \( K_c \) and that for \( z \) outside \( K_c \) is given by the limit

\[
G_c(z) = \lim_{n \to +\infty} \frac{1}{2^n} \log |f_c^n(z)| > 0.
\]

The function \( G_c \) is continuous, subharmonic, satisfies \( G_c \circ f_c = 2G_c \) on \( \mathbb{C} \), and it is harmonic and strictly positive outside \( K_c \). On the other hand, the critical values are bounded from above by \( G_c(0) \) and the open set

\[
U_c := \{ z \in \mathbb{C} \mid G_c(z) > G_c(0) \}
\]

is homeomorphic to a punctured disk. Notice that \( G_c(c) = 2G_c(0) \), thus \( U_c \) contains \( c \). Moreover, the Green’s functions of \( f_c \) and \( f_{\overline{c}} \) are related by \( G_{\overline{c}}(z) = G_c(\overline{z}) \).

By Böttcher’s Theorem there is a unique conformal representation

\[
\varphi_c : U_c \to \{ z \in \mathbb{C} \mid |z| > \exp(G_c(0)) \},
\]

that conjugates \( f_c \) to \( z \mapsto z^2 \). It is called the Böttcher coordinate of \( f_c \) and satisfies \( G_c = \log|\varphi_c| \). Note that \( U_{\overline{c}} = \overline{U_c} \) and \( \varphi_{\overline{c}} = \overline{\varphi_c} \).

2.3. External rays and equipotentials. Let \( c \) be in \( \mathbb{C} \). For \( v > 0 \) the equipotential \( v \) of \( f_c \) is by definition \( G_c^{-1}(v) \). A Green’s line of \( G_c \) is a smooth curve on the complement of \( K_c \) in \( \mathbb{C} \) that is orthogonal to the equipotentials of \( G_c \) and that is maximal with this property. Given \( t \) in \( \mathbb{R}/\mathbb{Z} \), the external ray of angle \( t \) of \( f_c \), denoted by \( R_c(t) \), is the Green’s line of \( G_c \) containing

\[
\{ \varphi_c^{-1}(r \exp(2\pi i t)) \mid \exp(G_c(0)) < r < +\infty \}.
\]

By the identity \( G_c \circ f_c = 2G_c \), for each \( v > 0 \) and each \( t \) in \( \mathbb{R}/\mathbb{Z} \) the map \( f_c \) maps the equipotential \( v \) to the equipotential \( 2v \) and maps \( R_c(t) \) to \( R_c(2t) \). For \( t \) in \( \mathbb{R}/\mathbb{Z} \) the external ray \( R_c(t) \) lands at a point \( z \), if \( G_c : R_c(t) \to (0, +\infty) \) is a bijection and if \( G_c|_{R_c(t)}(v) \) converges to \( z \) as \( v \) converges to \( 0 \) in \( (0, +\infty) \). By the continuity of \( G_c \), every landing point is in \( J_c = \partial K_c \).

We use the following general lemma several times.

**Lemma 2.1.** Let \( c \) be a parameter in \( \mathbb{C} \), let \( t \) be in \( \mathbb{R}/\mathbb{Z} \) and suppose that the external ray \( R_c(t) \) lands at a point \( z_0 \) of \( K_c \) different from \( c \); so \( f_c^{-1}(z_0) \) consists of 2 distinct points. Then each point of \( f_c^{-1}(z_0) \) is the landing point of precisely 1 of the external rays \( R_c(t/2) \) or \( R_c((t + 1)/2) \).
Proof. Since $f_c^{-1}(z_0)$ consists of 2 distinct points, it is enough to show that each point $z$ of $f_c^{-1}(z_0)$ is the landing point of either $R_c(t/2)$ or $R_c((t+1)/2)$. Since $z_0$ is different from $c$, there is an open neighborhood $U$ of $z$ and an open neighborhood $U_0$ of $z_0$ such that $f_c$ maps $U$ biholomorphically to $U_0$. Reducing $U$ and $U_0$ if necessary, it follows that $f_c^{-1}(R_c(t))$ is contained in an external ray landing at $z$. It must be either $R_c(t/2)$ or $R_c((t+1)/2)$.

The Mandelbrot set $\mathcal{M}$ is the subset of $\mathbb{C}$ of those parameters $c$ for which $K_c$ is connected. The function

$$\Phi : \mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \text{cl}(\mathbb{D})$$

is a conformal representation, see [DH84, VIII, Théorème 1]. Since for each parameter $c \in \mathbb{C}$ we have $\varphi_\alpha = \varphi_\beta$, it follows that $\Phi(\alpha) = \Phi(\beta)$; that is, $\Phi$ is real. For $v > 0$ the equipotential $v$ of $\mathcal{M}$ is by definition

$$\mathcal{E}(v) := \Phi^{-1}(\{z \in \mathbb{C} ~|~ |z| = v\}).$$

On the other hand, for $t$ in $\mathbb{R}/\mathbb{Z}$ the set

$$\mathcal{R}(t) := \Phi^{-1}(\{r \exp(2\pi it) ~|~ r > 1\}).$$

is called the external ray of angle $t$ of $\mathcal{M}$. We say that $\mathcal{R}(t)$ lands at a point $z$ in $\mathbb{C}$ if $\Phi^{-1}(r \exp(2\pi it))$ converges to $z$ as $r \searrow 1$. When this happens $z$ belongs to $\partial \mathcal{M}$.

2.4. The wake $1/2$. In this subsection we recall a few facts that can be found for example in [DH84] or [Mil00].

Both external rays $\mathcal{R}(1/3)$ and $\mathcal{R}(2/3)$ of $\mathcal{M}$ land at the parameter $c = -3/4$ and these are the only external rays of $\mathcal{M}$ that land at this point, see for example [Mil00, Théorème 1.2]. In particular, the complement in $\mathbb{C}$ of the set

$$\mathcal{R}(1/3) \cup \mathcal{R}(2/3) \cup \{-3/4\}$$

has 2 connected components; denote by $\mathcal{W}$ the connected component containing the point $c = -2$ of $\mathcal{M}$.

For each parameter $c$ in $\mathcal{W}$ the map $f_c$ has 2 distinct fixed points; one of the them is the landing point of the external ray $R_c(0)$ and it is denoted by $\beta(c)$; the other one is denoted by $\alpha(c)$. The only external ray landing at $\beta(c)$ is $R_c(0)$. Lemma 2.1 implies that the only external ray landing at $-\beta(c)$ is $R_c(1/2)$.

For the following fact, see for example [Mil00, Théorème 1.2].

**Theorem 1.** Let $c$ be a parameter in $\mathcal{W}$. Then the only external rays of $f_c$ landing at $\alpha(c)$ are $R_c(1/3)$ and $R_c(2/3)$.

For $c$ in $\mathcal{W}$, the complement of $R_c(1/3) \cup R_c(2/3) \cup \{\alpha(c)\}$ in $\mathbb{C}$ has 2 connected components; on containing $-\beta(c)$ and $z = c$, and the other one containing $\beta(c)$ and $z = 0$. On the other hand, the point $\alpha(c)$ has 2 pre-images by $f_c$: itself and $\tilde{\alpha}(c) := -\alpha(c)$. Together with Lemma 2.1, the theorem above implies that $R_c(1/6)$ and $R_c(5/6)$ are the only external rays landing at $\tilde{\alpha}(c)$.

**Theorem 2** ([DH84], VIII, Théorème 2 and XIII, §1). Let $p$ and $q$ be integers without common factors, with $q$ even. Then the external ray $\mathcal{R}(p/q)$ of $\mathcal{M}$ lands and the landing point $c$ is such that the critical point of $f_c$ is eventually periodic but not periodic and such that the critical value $c$ of $f_c$ is the landing point of the external ray $R_c(p/q)$ of $f_c$. Conversely, if $c$ is a parameter in $\mathbb{C}$ such that the critical
point of $f_c$ is eventually periodic but not periodic, then there are integers $p$ and $q$
without common factors and with $q$ even, such that the critical value $c$ of $f_c$ is the
landing point of $R_c(p/q)$; moreover, every external ray of $f_c$ landing at $c$ is of this
form. In this case the parameter $c$ is the landing point of the external ray $R_c(p/q)$
of $\mathcal{M}$.

Note that for the parameter $c = -2$ we have $c = -\beta(c)$, so the theorem above
implies that $\mathcal{R}(1/2)$ is the only external ray of $\mathcal{M}$ that lands at $-2$.

2.5. Yoccoz puzzles and para-puzzle. In this subsection we recall the definitions of Yoccoz puzzle and para-puzzle. We follow [Roe00].

Definition 2.2 (Yoccoz puzzles). Fix $c$ in $W$ and consider the open region $X_c := \{z \in \mathbb{C} \mid G_c(z) < 1\}$. The Yoccoz puzzle of $f_c$ is given by the following sequence of graphs $(I_c,n)_{n=0}^{+\infty}$ defined for $n = 0$ by:

$$I_{c,0} := \partial X_c \cup (X_c \cap \cl(R_c(1/3)) \cap \cl(R_c(2/3)))$$

and for $n \geq 1$ by $I_{c,n} := f_{c^{-n}}(I_{c,0})$. The puzzle pieces of depth $n$ are the connected components of $f_{c^{-n}}(X_c) \setminus I_{c,n}$. The puzzle piece of depth $n$ containing a point $z$ is denoted by $P_{c,n}(z)$.

Note that for a real parameter $c$, every puzzle piece intersecting the real line is invariant under complex conjugation. Since puzzle pieces are simply-connected, it follows that the intersection of such a puzzle piece with $\mathbb{R}$ is an interval.

Definition 2.3 (Yoccoz para-puzzles). $^{11}$ Given an integer $n \geq 0$ put

$$J_n := \{t \in [1/3, 2/3] \mid 2^n t \pmod{1} \in \{1/3, 2/3\}\},$$

let $\mathcal{X}_n$ be the intersection of $\mathcal{W}$ with the open region in the parameter plane bounded by the equipotential $\mathcal{E}(2^{-n})$ of $\mathcal{M}$, and put

$$I_n := \partial \mathcal{X}_n \cup \left(\mathcal{X}_n \cap \bigcup_{t \in J_n} \cl(\mathcal{R}(t))\right).$$

Then the Yoccoz para-puzzle of $\mathcal{W}$ is the sequence of graphs $(I_n)_{n=0}^{+\infty}$. The para-puzzle pieces of depth $n$ are the connected components of $\mathcal{X}_n \setminus I_n$. The para-puzzle piece of depth $n$ containing a parameter $c$ is denoted by $P_{c,n}(c)$.

Observe that there is only 1 para-puzzle piece of depth 0 and only 1 para-puzzle piece of depth 1; they are bounded by the same external rays but different equipotentials. Both of them contain $c = -2$.

Definition 2.4 (Holomorphic motion). Let $\mathcal{C}$ be a complex manifold and fix $c_0$ in $\mathcal{C}$. Given a subset $Z$ of $\mathbb{C}$, a map

$$h : \mathcal{C} \times Z \to \mathcal{C} \times \mathbb{C}$$

of the form $(c, z) \mapsto (c, h^c(z))$ is a holomorphic motion based at $c_0$ if $h^{c_0}$ is the identity on $Z$, if for each $z$ in $Z$ its restriction to $\mathcal{C} \times \{z\}$ is holomorphic and if for each $c \in \mathcal{C}$ its restriction to $\{c\} \times Z$ is injective.

$^{11}$In contrast with [Roe00], we only consider para-puzzles contained in $\mathcal{W}$. 
See [Roe00] for a reference to the following lemma; the statement here is slightly different from the statement in [Roe00] since we extend the domain of definition of the holomorphic motions, but the proof is the same. For each integer \( n \geq 1 \) put
\[
V_n := \{ w \in \mathbb{C} \mid \log^+ |w| \geq 2^{-n} \}.
\]

**Lemma 2.5.** Let \( n \geq 0 \) be an integer and \( c_0 \) a parameter contained in a para-puzzle of depth \( n \). Then there exists a holomorphic motion
\[
h_n : \mathcal{P}_n(c_0) \times (I_{c_0,n+1} \cup \varphi_{c_0}^{-1}(V_{n+1})) \to \mathcal{P}_n(c_0) \times \mathbb{C}
\]
\[
h_n(c, z) := (c, h_n^c(z))
\]
such that for every \( c \) in \( \mathcal{P}_n(c_0) \) the function \( h_n^c \) is an extension of the restriction of \( \varphi_{c_0}^{-1} \circ \varphi_{c_0} \) to \( I_{c_0,n} \cup \varphi_{c_0}^{-1}(V_{n+1}) \) that satisfies
\[
I_{c_0,n+1} = h_n^c(I_{c_0,n+1}).
\]
Moreover, when \( n \geq 1 \) the map \( h_n \) coincides with \( h_{n-1} \) on \( \mathcal{P}_n(c_0) \times (I_{c_0,n} \cup \varphi_{c_0}^{-1}(V_n)) \) and for each \( c \) in \( \mathcal{P}_n(c_0) \) we have \( f_c \circ h_n^c = h_{n-1}^c \circ f_c \) on \( I_{c_0,n+1} \cup \varphi_{c_0}^{-1}(V_{n+1}) \).

3. Parameters

In this section we study the set of parameters from which we choose the parameter in the Main Theorem and at the same time introduce some notation used in the rest of the paper.

Given an integer \( n \geq 3 \), let \( K_n \) be the set of all those real parameters \( c \) such that the following properties hold:

1. We have \( c < 0 \) and for each \( j \) in \( \{1, \ldots, n-1\} \) we have \( f_c^j(c) > 0 \).
2. For every integer \( k \geq 0 \) we have
\[
f_c^{n+3k+1}(c) < 0 \quad \text{and} \quad f_c^{n+3k+2}(c) > 0.
\]

Note that for a parameter \( c \) in \( K_n \) the critical point of \( f_c \) cannot be asymptotic to a periodic point, see [MT88, §8]. This implies that all the periodic points of \( f_c \) in \( \mathbb{C} \) are hyperbolic repelling and therefore that \( K_n = J_c \), see [Mil06]. On the other hand, we have \( f_c(c) > c \) and the interval \( I_c = [c, f_c(c)] \) is invariant by \( f_c \).

This implies that \( I_c \) is contained in \( J_c \) and hence that for every real number \( t \) we have
\[
P_{c}^R(t) \leq P_{c}^L(t).
\]
Note also that \( f_c|_{I_c} \) is not renormalizable, so \( f_c \) is topologically exact on \( I_c \), see for example [dMvS93, Theorem III.4.1].

Since for \( c \) in \( K_n \) the critical point of \( f_c \) is not periodic, we can define the sequence \( \iota(c) \) in \( \{0,1\}^{\mathbb{N}} \) for each \( k \geq 0 \) by
\[
\iota(c)_k = \begin{cases} 0 & \text{if } f_c^{n+3k}(c) < 0; \\ 1 & \text{if } f_c^{n+3k}(c) > 0. \end{cases}
\]

The remainder of this section is devoted to prove the following proposition.

**Proposition 3.1.** For each integer \( n \geq 3 \) the set \( K_n \) is a compact subset of
\[
\mathcal{P}_n(-2) \cap (-2, -3/4)
\]
and for every sequence \( x \) in \( \{0,1\}^{\mathbb{N}} \) there is a unique parameter \( c \) in \( K_n \) such that \( \iota(c) = x \). Finally, for each \( \delta > 0 \) there is \( n_1 \geq 3 \) such that for each integer \( n \geq n_1 \) the set \( \overline{K_n} \) is contained in the interval \( (-2, -2 + \delta) \).

After defining some sequences of puzzle pieces that are important for the rest of this paper in §3.1, we study the para-puzzle pieces containing \( c = -2 \) in §3.2 and the maximal invariant set of \( f_3^c \) in \( P_{c,1}(0) \) in §3.3. The proof of Proposition 3.1 is in §3.4.
3.1. First landing domains to \( P_{c,1}(0) \). Fix a parameter \( c \) in \( \mathcal{P}_0(-2) \).

The following are consequences of the facts recalled in §2.4. There are precisely 2 puzzle pieces of depth 0: \( P_{c,0}(\beta(c)) \) and \( P_{c,0}(-\beta(c)) \). Each of them is bounded by the equipotential 1 and by the closures of the external rays landing at \( \alpha(c) \). Furthermore, the critical value \( c \) of \( f_c \) is contained in \( P_{c,0}(-\beta(c)) \) and the critical point in \( P_{c,0}(\beta(c)) \). It follows that the set \( f_c^{-1}(P_{c,0}(\beta(c))) \) is the disjoint union of \( P_{c,1}(-\beta(c)) \) and \( P_{c,1}(\beta(c)) \), so \( f_c \) maps each of the sets \( P_{c,1}(-\beta(c)) \) and \( P_{c,1}(\beta(c)) \) biholomorphically to \( P_{c,0}(\beta(c)) \). Moreover, there are precisely 3 puzzle pieces of depth 1:

\[
P_{c,1}(-\beta(c)), P_{c,1}(0) \text{ and } P_{c,1}(\beta(c));
\]

\( P_{c,1}(-\beta(c)) \) is bounded by the equipotential 1/2 and by the closures of the external rays that land at \( \alpha(c) \); \( P_{c,1}(\beta(c)) \) is bounded by the equipotential 1/2 and by the closures of the external rays that land at \( \alpha(c) \); and \( P_{c,1}(0) \) is bounded by the equipotential 1/2 and by the closures of the external rays that land at \( \alpha(c) \) and at \( \alpha(c) \). In particular, the closure of \( P_{c,1}(\beta(c)) \) is contained in \( P_{c,0}(\beta(c)) \).

Put

\[
\phi_c := f_c|_{P_{c,1}(-\beta(c))}^{-1} \text{ and } \bar{\phi}_c := f_c|_{P_{c,1}(\beta(c))}^{-1}.
\]

Since the closure of \( \bar{\phi}_c(P_{c,0}(\beta(c))) = P_{c,1}(\beta(c)) \) is contained in \( P_{c,0}(\beta(c)) \), all the iterates of \( \bar{\phi}_c \) are defined on \( P_{c,0}(\beta(c)) \) and take images in \( P_{c,1}(\beta(c)) \). Put \( \alpha_0(c) := \alpha(c), \bar{\alpha}_0(c) := \bar{\alpha}(c) \) and for each integer \( n \geq 1 \) put

\[
\bar{\alpha}_n(c) := \bar{\phi}_c^n(\bar{\alpha}_0(c)), \alpha_n(c) := \phi_c(\bar{\alpha}_{n-1}(c)),
\]

\[
\bar{V}_{c,n} := \bar{\phi}_c^n(P_{c,1}(0)) \text{ and } V_{c,n} := \phi_c(\bar{\phi}_c^n(P_{c,1}(0))).
\]

Note that \( f_c^n \) maps each of the sets \( V_{c,n} \) and \( \bar{V}_{c,n} \) biholomorphically to \( P_{c,1}(0) \); so both, \( V_{c,n} \) and \( \bar{V}_{c,n} \) are puzzle pieces of depth \( n + 1 \). On the other hand, \( f_c^n \) maps each of the sets \( P_{c,n}(-\beta(c)) \) and \( P_{c,n}(\beta(c)) \) biholomorphically to \( P_{c,0}(\beta(c)) \). Note moreover that if \( c \) is real, then \( f_c \), \( \phi_c \) and \( \bar{\phi}_c \) are all real, so \( \alpha(c), \bar{\alpha}(c), \alpha_n(c) \) and \( \bar{\alpha}_n(c) \) are also real and each of the sets \( V_{c,n}, \bar{V}_{c,n} \) are puzzle pieces of depth \( n + 1 \) and \( \bar{\alpha}_n(c) \) is invariant by complex conjugation and intersects \( \mathbb{R} \).

**Lemma 3.2.** Let \( c \) be a parameter in \( \mathcal{P}_0(-2) \). Then for every integer \( n \geq 0 \) the only external rays that land at \( \alpha_n(c) \) are \( R_c \left( \frac{1}{3} \frac{2^{n+1}}{2^{n+1}+1} \right) \) and \( R_c \left( \frac{2^{n+1}}{3} \frac{2^{n+1}}{2^{n+1}+1} \right) \) and the only external rays that land at \( \bar{\alpha}_n(c) \) are \( R_c \left( \frac{1}{2} \frac{2^{n+1}}{2^{n+1}+1} \right) \) and \( R_c \left( \frac{2^{n+1}}{3} \frac{2^{n+1}}{2^{n+1}+1} \right) \). Furthermore, for each integer \( n \geq 1 \) the following properties hold.

1. The only puzzle pieces of depth \( n + 1 \) contained in \( P_{c,n}(-\beta(c)) \) are \( P_{c,n+1}(-\beta(c)) \) and \( V_{c,n} \). Moreover, the closure of \( P_{c,n+1}(-\beta(c)) \) is contained in the open set \( P_{c,n}(-\beta(c)) \).
2. The puzzle piece \( P_{c,n}(-\beta(c)) \) is bounded by the external rays landing at \( \alpha_{n-1}(c) \) and the equipotential 1/2\(^{n+1} \); the puzzle piece \( P_{c,n}(\beta(c)) \) is bounded by the closure of the external rays landing at \( \bar{\alpha}_{n-1}(c) \) and the equipotential 1/2\(^{n+1} \).

**Proof.** For an integer \( n \geq 0 \) put \( \theta_n := \frac{1}{3} \frac{2^{n+1}}{2^{n+1}+1} \) and \( \theta'_n := 1 - \theta_n \).

The proof of the first assertion is by induction. When \( n = 0 \) the assertion is shown in §2.4. Given an integer \( n \geq 0 \) assume that the only external rays that land at \( \bar{\alpha}_n(c) \) are those of angles \( \theta_n \) and \( \theta'_n \). Since \( f_c^{-1}(\bar{\alpha}_n(c)) = \{ \alpha_{n+1}(c), \bar{\alpha}_{n+1}(c) \} \), by
Lemma 2.1 the only external rays landing at \( \alpha_{n+1}(c) \) or \( \tilde{\alpha}_{n+1}(c) \) are those of angles \( \theta_n \), \( \theta_n' \), \( \frac{\theta_n + 1}{2} \) and \( \frac{\theta_n' + 1}{2} \). Since

\[
\frac{\theta_n}{2} < \frac{1}{6} < \frac{\theta_n'}{2} < \frac{\theta_n + 1}{2} < \frac{5}{6} < \frac{\theta_n' + 1}{2}
\]

and since \( \tilde{\alpha}_{n+1}(c) \) is in \( P_{c,1}(\beta(c)) \), the external rays of angles \( \theta_n \) and \( \frac{\theta_n + 1}{2} \) land at \( \tilde{\alpha}_{n+1}(c) \). By Lemma 2.1 it follows that the external rays of angles \( \frac{\theta_n'}{2} \) and \( \frac{\theta_n' + 1}{2} \) land at \( \alpha_{n+1}(c) \). This completes the proof of the induction step and of the first assertion of the lemma.

To prove the rest of the assertions, assume \( n \geq 1 \). Since \( f^n \) maps \( P_{c,n}(\beta(c)) \) biholomorphically to \( P_{c,0}(\beta(c)) \), \( V_{c,n} \) biholomorphically to \( P_{c,1}(0) \), and \( P_{c,n+1}(\beta(c)) \) biholomorphically to \( P_{c,1}(\beta(c)) \), it follows that the only puzzle pieces of depth \( n+1 \) contained in \( P_{c,n}(\beta(c)) \) are \( V_{c,n} \) and \( P_{c,n+1}(\beta(c)) \). On the other hand, since the closure of \( P_{c,1}(\beta(c)) \) is contained in \( P_{c,0}(\beta(c)) \), it follows that the closure of \( P_{c,n+1}(\beta(c)) \) is contained in \( P_{c,n}(\beta(c)) \). We have thus proved part 1. When \( n = 1 \) part 2 follows from the considerations above. To prove part 2 when \( n \geq 2 \), recall that \( f^{n-1} \) maps each of the sets \( P_{c,n-1}(\beta(c)) \) and \( P_{c,n-1}(\beta'(c)) \) biholomorphically to \( P_{c,0}(\beta(c)) \). Since the closure of \( P_{c,1}(\beta(c)) \) is contained in \( P_{c,0}(\beta(c)) \) and since \( \tilde{\alpha}(c) \) is the only point in the boundary of \( P_{c,1}(\beta(c)) \) that is in the Julia set of \( f_c \), it follows that \( \alpha_{n-1}(c) \) is the only point in the boundary of \( P_{c,n}(\beta(c)) \) that is in the Julia set of \( f_c \) and that \( \tilde{\alpha}_{n-1}(c) \) is the only point in the boundary of \( P_{c,n}(\beta(c)) \) that is in the Julia set of \( f_c \). This implies part 2 and completes the proof of the lemma.

### 3.2. Para-puzzle pieces containing \( c = -2 \)

The purpose of this subsection is to prove the following lemma.

**Lemma 3.3.** For each integer \( n \geq 1 \) the following properties hold.

1. The para-puzzle piece \( P_n(-2) \) contains the closure of \( P_{n+1}(-2) \).
2. For each parameter \( c \in P_n(-2) \) the critical value \( c \) is in \( P_{c,n}(-\beta(c)) \).

The proof of this lemma is given after the following one.

For each integer \( n \geq 0 \) put

\[
t_n := \frac{3 \cdot 2^n - 1}{3 \cdot 2^{n+1}} \quad \text{and} \quad t'_n := \frac{3 \cdot 2^n + 1}{3 \cdot 2^{n+1}}
\]

**Lemma 3.4.** Fix an integer \( n \geq 1 \). Then the parameter \( c = -2 \) is contained in a para-puzzle piece of depth \( n \) and there is a unique point \( \tilde{\alpha}_{n-1} \) in the boundary of \( P_n(-2) \) that is contained in \( M \). Furthermore, \( \tilde{\alpha}_{n-1} \) is in \( \mathbb{R} \), the only external rays of \( M \) that land at \( \tilde{\alpha}_{n-1} \) are \( R(t_{n-1}) \) and \( R(t'_{n-1}) \), and \( P_n(-2) \) is invariant under complex conjugation. In particular, \( P_n(-2) \) is bounded by the equilateral polygon \( 1/2^n \) and the closures of the other external rays \( R(t_{n-1}) \) and \( R(t'_{n-1}) \) of \( M \).

**Proof.** Since \( R(1/2) \) is the only external ray of \( M \) that lands at \( c = -2 \) and since \( t = 1/2 \) is not in \( J_n \), it follows that \( c = -2 \) is contained in a para-puzzle of level \( n \). On the other hand, by Theorem 2 the external ray \( R(t_{n-1}) \) of \( M \) lands at a parameter in \( M \), denoted by \( \tilde{\alpha}_{n-1} \), and the external ray \( R_{\tilde{\alpha}_{n-1}}(t_{n-1}) \) of \( f_{\tilde{\alpha}_{n-1}} \) lands at the critical value \( \tilde{\alpha}_{n-1} \) of \( f_{\tilde{\alpha}_{n-1}} \). By Theorem 2, we have that \( \alpha_{n-1}(\tilde{\alpha}_{n-1}) = \tilde{\alpha}_{n-1} \) and that \( R_{\tilde{\alpha}_{n-1}}(t_{n-1}) \) and \( R_{\tilde{\alpha}_{n-1}}(t'_{n-1}) \) are the only external rays of \( f_{\tilde{\alpha}_{n-1}} \) landing at \( \tilde{\alpha}_{n-1} \). Using Theorem 2 again, we conclude that \( R(t_{n-1}) \) and \( R(t'_{n-1}) \) are the only external rays of \( M \) landing at \( \tilde{\alpha}_{n-1} \). Since \( \Phi \) is real, we have \( R(t'_n) = \overline{R(t_n)} \) and therefore \( \tilde{\alpha}_{n-1} \) is in \( \mathbb{R} \). On the other hand, since the interval \( (t_{n-1}, t'_{n-1}) \)
is disjoint from \( J_\alpha \) and contains 1/2, the closures of the external rays \( \mathcal{R}(t_{n-1}) \) and \( \mathcal{R}(t_{n-1}') \) of \( \mathcal{M} \) and the equipotential 1/2 of \( \mathcal{M} \) bound a para-puzzle piece of depth \( n \) that contains \( c = -2 \); that is, they bound \( P_n(2) \). It follows that \( \tilde{V}_{n-1} \) is the only point in the boundary of \( P_n(-2) \) in \( \mathcal{M} \). That \( P_n(-2) \) is invariant under complex conjugation follows from the fact that \( \tilde{V}_{n-1} \) and \( \Phi \) are real. □

**Proof of Lemma 3.3.** Part 1 follows from the descriptions of \( P_n(-2) \) and \( P_{n+1}(-2) \) in terms of external rays and equipotentials given by Lemma 3.4.

To prove part 2, we use the following direct consequence of the definitions of the puzzle and the para-puzzle and of Theorem 2: A parameter \( c \) in \( \mathcal{W} \) is in a para-puzzle piece of depth \( n \) if and only if the critical value \( f_c \) is in a puzzle piece of depth \( n \) of \( f_c \). Note that for \( c = -2 \) the critical value of \( f_{-2} \) is equal to \( -\beta(-2) \) and hence it is in \( P_{-2,n}(-\beta(-2)) \). Since \( P_{c,n}(-\beta(c)) \) depends continuously with \( c \) on \( P_{n-1}(-2) \) (Lemma 2.5) and since \( P_n(-2) \) is connected by definition, it follows that for each \( c \) in \( P_n(-2) \) the critical value \( c \) of \( f_c \) is in \( P_{c,n}(-\beta(c)) \). □

### 3.3. The uniformly expanding Cantor set

Let \( c \) be a parameter in \( \mathcal{P}_3(-2) \). In this subsection we study the maximal invariant set \( \Lambda_c \) of \( f_c^3 \) in \( P_{c,1}(0) \).

Note first that the critical value \( c \) of \( f_c \) is in \( P_{c,3}(-\beta(c)) \) (part 2 of Lemma 3.3) and hence in \( P_{c,2}(-\beta(c)) \). Since \( \alpha_1(c) \) is in the boundary of \( P_{c,2}(-\beta(c)) \) and since the only external rays that land at this point are \( R_c(5/12) \) and \( R_c(7/12) \) (Lemma 3.2), by Lemma 2.1 the external rays \( R_c(7/24) \) and \( R_c(17/24) \) land at the same point, denoted by \( \omega(c) \), and these are the only external rays that land at this point. Similarly, the external rays \( R_c(5/24) \) and \( R_c(19/24) \) land at the same point, denoted by \( \tilde{\omega}(c) \), and these are the only external rays that land at \( \tilde{\omega}(c) \). Note that if in addition \( c \) is real, then the points \( \alpha(c) \) and \( \alpha_1(c) \) are both real (§3.1); together with the fact that \( c \) is in \( P_{c,2}(-\beta(c)) \), this implies \( c < \alpha_1(c) < \alpha(c) \) (cf., part 2 of Lemma 3.2). It follows that the points \( \omega(c) \) and \( \tilde{\omega}(c) \) are both real and that the set \( P_{c,3}(0) = f_{c}^{-1}(P_{c,2}(-\beta(c))) \) satisfies

\[
P_{c,3}(0) \cap \mathbb{R} = (\omega(c), \tilde{\omega}(c)).
\]

**Lemma 3.5.** Let \( c \) be a parameter in \( \mathcal{P}_3(-2) \). Then there are precisely 2 connected components of \( f_c^{-3}(P_{c,1}(0)) \) contained in \( P_{c,1}(0) \): one containing \( \omega(c) \) in its closure, denoted by \( Y_{\omega} \), and another one containing \( \tilde{\omega}(c) \) in its closure, denoted by \( \tilde{Y}_{\tilde{\omega}} \); the map \( f_c^3 \) maps each of the sets \( Y_{\omega} \) and \( \tilde{Y}_{\tilde{\omega}} \) biholomorphically to \( P_{c,1}(0) \). Moreover, the closures of \( Y_{\omega} \) and of \( \tilde{Y}_{\tilde{\omega}} \) are disjoint and contained in \( P_{c,1}(0) \) and the set \( Y_{\omega} \cup \tilde{Y}_{\tilde{\omega}} \) is contained in \( P_{c,3}(0) \) and it is disjoint from \( P_{c,4}(0) \). Finally, if \( c \) is real, then each of the sets \( Y_{\omega} \) and \( \tilde{Y}_{\tilde{\omega}} \) is invariant by complex conjugation and intersects \( \mathbb{R} \).

**Proof.** We prove first

\[
(3.1) \quad f_c^{-3}(P_{c,1}(0)) \cap P_{c,1}(0) = f_c^{-1}(V_{c,2}).
\]

First notice that, since \( f_c^3 \) maps \( V_{c,2} \) biholomorphically to \( P_{c,1}(0) \), the set \( f_c^{-1}(V_{c,2}) \) is contained in \( f_c^{-3}(P_{c,1}(0)) \). On the other hand, the set \( f_c(P_{c,1}(0)) = P_{c,0}(-\beta(c)) \) contains \( V_{c,2} \), so \( f_c^{-1}(V_{c,2}) \) is contained in \( P_{c,1}(0) \). This proves that the set in the right hand side of (3.1) is contained in the set in the left hand side. To prove the reverse inclusion, let \( z \) be a point in \( P_{c,1}(0) \) such that \( f_c^3(z) \) is in \( P_{c,1}(0) \). Then \( z \) is in a puzzle piece of depth 4 and \( f_c(z) \) is in a puzzle piece of depth 3 contained in \( P_{c,1}(-\beta(c)) \). This implies \( f_c^2(z) \) is in \( P_{c,0}(\beta(c)) \). On the other hand, \( f_c^2(z) \) is in \( f_c^{-1}(P_{c,1}(0)) = V_{c,1} \cup \tilde{V}_{c,1} \) and \( V_{c,1} \) is contained in \( P_{c,1}(-\beta(c)) \subset P_{c,0}(-\beta(c)) \)
(part 1 of Lemma 3.2), so we conclude that \( f_c^2(z) \) is in \( \tilde{V}_{c,1} \) and hence that \( f_c(z) \)

is in \( f_c^{-1}(\tilde{V}_{c,1}) = V_{c,2} \cup \tilde{V}_{c,2} \). Since \( f_c(z) \) is in \( P_{c,1}(\beta(c)) \) and \( \tilde{V}_{c,2} \) is contained in \( P_{c,2}(\beta(c)) \subset P_{c,1}(\beta(c)) \) (part 1 of Lemma 3.2), we conclude that \( f_c(z) \) is in \( V_{c,2} \) and hence that \( z \) is in \( f_c^{-1}(V_{c,2}) \). This completes the proof of (3.1).

To prove the assertions of the lemma, note that by part 2 of Lemma 3.3 the critical value \( c \) of \( f_c \) is in \( P_{c,3}(\beta(c)) \), so it is not in the closure of \( V_{c,2} \). This implies that \( f_c^{-1}(V_{c,2}) \) has 2 connected components whose closures are disjoint. On the other hand, \( V_{c,2} \) contains \( \alpha_1(c) \) in its closure (cf., parts 1 and 2 of Lemma 3.2), so one of the connected components of \( f_c^{-1}(V_{c,2}) \) contains \( \omega(c) \) in its closure and the other one contains \( \tilde{\omega}(c) \) in its closure; denote them by \( Y_c \) and \( \tilde{Y}_c \), respectively. It follows that \( f_c^3 \) maps each of the sets \( Y_c \) and \( \tilde{Y}_c \) biholomorphically to \( P_{c,1}(0) \). From the fact that \( V_{c,2} \) is contained in \( P_{c,2}(\beta(c)) \) and that the closure of this last set is contained in \( P_{c,1}(\beta(c)) \) (part 1 of Lemma 3.2 with \( n = 1 \) and \( n = 2 \)), it follows that closures of \( Y_c \) and \( \tilde{Y}_c \) are both contained in \( P_{c,2}(0) = f_c^{-1}(P_{c,1}(\beta(c))) \). Note also that \( V_{c,2} \) is contained in \( P_{c,2}(\beta(c)) \) and it is disjoint from \( P_{c,3}(\beta(c)) \) (part 1 of Lemma 3.2), so \( Y_c \cup \tilde{Y}_c \) is contained in \( P_{c,3}(0) \) and it is disjoint from \( P_{c,4}(0) \). To prove the last statement of the lemma, suppose \( c \) is real. Then \( f_c \) and \( \alpha_1(c) \) are real and \( V_{c,2} \) is invariant by complex conjugation (§3.1). Since \( c \) is in \( P_{c,3}(\beta(c)) \) we also have \( c < \alpha_1(c) \). It follows that each of the sets \( Y_c \) and \( \tilde{Y}_c \) is invariant by complex conjugation and intersects \( \mathbb{R} \). This completes the proof of the lemma. □

For a parameter \( c \) in \( \mathcal{P}_3(-3) \) define

\[
\begin{align*}
g_c : Y_c \cup \tilde{Y}_c & \to P_{c,1}(0) \\
z & \mapsto g_c(z) := f_c^3(z).
\end{align*}
\]

Lemma 3.5 implies that \( g_c \) maps each of the sets \( Y_c \) and \( \tilde{Y}_c \) biholomorphically to \( P_{c,1}(0) \) and that

\[
\Lambda_c = \bigcap_{n \in \mathbb{N}} g_c^{-n}(\text{cl}(P_{c,1}(0))).
\]

In particular, \( \Lambda_c \) is contained in \( Y_c \cup \tilde{Y}_c \). So Lemma 3.5 implies that \( \Lambda_c \) is contained in \( P_{c,3}(0) \) and that it is disjoint from \( P_{c,4}(0) \). Moreover, Lemma 3.5 also implies that \( g_c \) is a Markov map, so \( \Lambda_c \) is a Cantor set and \( g_c \) is uniformly expanding on \( \Lambda_c \), see for instance [dFdM08]. In particular, \( g_c \) has a unique fixed point in \( Y_c \) and a unique fixed point in \( \tilde{Y}_c \). Finally, note that if \( c \) is real, then \( g_c \) is real and \( \Lambda_c \) is contained in \( \mathbb{R} \).

3.4. Proof of Proposition 3.1.

**Lemma 3.6.** There is a constant \( \Delta_1 > 1 \) such that for each parameter \( c \) in \( \mathcal{P}_2(-2) \) the following properties hold for each integer \( k \geq 2 \): We have

\[
\Delta_1^{-1}|Df_c(\beta(c))|^{-k} \leq \text{diam}(P_{c,k}(\beta(c))) \leq \Delta_1|Df_c(\beta(c))|^{-k}
\]

and for each point \( y \) in \( P_{c,k}(\beta(c)) \) or in \( P_{c,k}(\beta(c)) \) we have

\[
\Delta_1^{-1}|Df_c(\beta(c))|^{-k} \leq |Df_c^k(y)| \leq \Delta_1|Df_c(\beta(c))|^{-k}.
\]

**Proof.** Since \( P_{c,1}(\beta(c)) \) depends continuously with \( c \) on \( \mathcal{P}_0(-2) \) (cf., Lemma 2.5) and since \( \mathcal{P}_0(-2) \) contains the closure of \( \mathcal{P}_2(-2) \) (part 1 of Lemma 3.3), we have

\[
\Xi_1 := \sup_{c \in \mathcal{P}_2(-2)} \sup_{z \in P_{c,1}(\beta(c))} |Df_c(z)| < +\infty
\]
\[ \Xi_2 := \inf_{c \in \mathcal{P}_2(-2)} \inf_{z \in \mathcal{P}_{c,1}(\beta(c))} |Df_c(z)| > 0, \]
\[ \Xi_3 := \sup_{c \in \mathcal{P}_2(-2)} \text{diam}(\mathcal{P}_{c,1}(\beta(c))) < +\infty. \]

and
\[ \Xi_4 := \inf_{c \in \mathcal{P}_2(-2)} \text{diam}(\mathcal{P}_{c,1}(\beta(c))) > 0. \]

On the other hand, since for each \( c \) in \( \mathcal{P}_0(-2) \) the set \( \mathcal{P}_{c,0}(\beta(c)) \) contains the closure of \( \mathcal{P}_{c,1}(\beta(c)) \) (cf., §3.1), we have
\[ \Xi_5 := \inf_{c \in \mathcal{P}_2(-2)} \text{mod}(\mathcal{P}_{c,0}(\beta(c)) \setminus \text{cl}(\mathcal{P}_{c,1}(\beta(c)))) > 0. \]

Let \( \Delta > 1 \) be the constant given by Koebe Distortion Theorem with \( A = \Xi_5 \).

Let \( c \) be a parameter in \( \mathcal{P}_2(-2) \) and let \( k \geq 2 \) be an integer. Since \( f_{c}^{k-1} \) maps each of the sets \( \mathcal{P}_{c,k-1}(\beta(c)) \) and \( \mathcal{P}_{c,k-1}(\beta(c)) \) biholomorphically to \( \mathcal{P}_{c,0}(\beta(c)) \), the distortion of \( f_{c}^{k-1} \) on \( \mathcal{P}_{c,k}(\beta(c)) \) is bounded by \( \Delta \). So for each \( y \) in \( \mathcal{P}_{c,k}(\beta(c)) \) or in \( \mathcal{P}_{c,k}(\beta(c)) \) we have
\[ \Delta^{-1}|Df_c(\beta(c))|^{k-1} \leq |Df_{c}^{k-1}(y)| \leq \Delta|Df_c(\beta(c))|^{k-1}. \]

This implies the first assertion of the lemma with \( \Delta_1 = \Delta \max\{\Xi_1, \Xi_2^{-1}, \Xi_4^{-1}\} \) and second with \( \Delta_1 = \Delta \Xi_1 \Xi_2^{-1} \).

**Proof of Proposition 3.1.** By the monotonicity of the kneading invariant, the set \( \mathcal{K}_n \) is contained in \( (-2, \tilde{\alpha}_{n-1}) \), see [MT88, Theorem 13.1]. Combined with Lemma 3.4 this implies that \( \mathcal{K}_n \) is contained in \( \mathcal{P}_n(-2) \). Since \( \tilde{\alpha}_1 = -3/4 \), we also have \( \mathcal{K}_n \subset (-2, -3/4) \). To prove that \( \mathcal{K}_n \) is compact, just observe that from the definitions we have
\[ \mathcal{K}_n = \{c \in [-2, \tilde{\alpha}_{n-1}] \mid f_{c}^{n}(c) \in \Lambda \}, \]

For a given \( \tilde{x} \) in \( \{0, 1\}^{\mathbb{N}_0} \) the existence and uniqueness of \( c \) in \( \mathcal{K}_n \) such that \( \iota(c) = \tilde{x} \) is a direct consequence of general results of Milnor and Thurston and of Yoccoz, see for example [MT88, dMvS93] and [Hub93].

To prove the last statement of the proposition we show that \( \text{diam}(\mathcal{P}_n(-2)) \to 0 \) as \( n \to +\infty \). To do this, let \( \Delta_1 > 1 \) be given by Lemma 3.6, put
\[ \Xi := \inf_{c \in \mathcal{P}_2(-2)} |Df_{c}(\beta(c))| > 1, \]
and let \( \tau : \mathcal{P}_0(-2) \to \mathbb{C} \) be the holomorphic function defined by \( \tau(c) := c + \beta(c) \).

A direct computation shows that \( c = -2 \) is the only zero of \( \tau \) and that \( \tau(-2) \neq 0 \). Since the closure of \( \mathcal{P}_2(-2) \) is contained in \( \mathcal{P}_0(-2) \) (part 1 of Lemma 3.3), there is a constant \( C > 0 \) such that for every \( c \) in \( \mathcal{P}_2(-2) \) we have
\[ |c - (-2)| \leq C \frac{|\tau(c)|}{|\tau(-2)|}. \]

Let \( n \geq 2 \) be an integer and \( c \) a parameter in \( \mathcal{P}_n(-2) \). By part 2 of Lemma 3.3 we have \( c \in \mathcal{P}_{c,n}(-\beta(c)) \). So by Lemma 3.6 with \( k = n \) and the definition of \( \Xi \) we have,
\[ |\tau(c)| = |c - (-\beta(c))| \leq \Delta_1 |Df_{c}(\beta(c))|^{-n} \leq \Delta_1 \Xi^{-n}. \]

Combining this inequality with (3.2), we conclude that \( \text{diam}(\mathcal{P}_n(-2)) \to 0 \) as \( n \to +\infty \). This completes the proof of the proposition. \( \square \)
4. Reduced statement

The purpose of this section is to state a sufficient criterion for a quadratic map corresponding to a parameter in \( \bigcup_{n=3}^{\infty} \mathcal{K}_n \) to have a low-temperature phase transition (Proposition A). The rest of this section is devoted to prove the Main Theorem using this criterion. The proof of Proposition A occupies §§5, 6 and 7.

Recall that for a real parameter \( c \),
\[
\chi_{crit}(c) = \liminf_{m \to +\infty} \frac{1}{m} \log |Df_c^m(c)|.
\]

**Proposition A.** There is \( n_0 \geq 3 \) and a constant \( C_0 > 1 \) such that for every integer \( n \geq n_0 \) and every parameter \( c \) in \( \mathcal{K}_n \) the following property holds. Suppose that for every \( t > 0 \) sufficiently large the sum
\[
\sum_{k=0}^{+\infty} \exp((n+3k)t \chi_{crit}(c)/2)|Df_c^{n+3k}(c)|^{-t/2}
\]
is less than or equal to \( C_0^{-t} \) and that for some \( t_0 \geq 3 \) the sum above with \( t = t_0 \) is finite and greater than or equal to \( C_0^{t_0} \). Then there is \( t_* > t_0 \) such that \( f_c|_{\hat{U}_c} \) (resp. \( f_c|_{\hat{J}_c} \)) has a low-temperature phase transition at \( t = t_* \). If in addition the sum
\[
\sum_{k=0}^{+\infty} k \cdot \exp((n+3k)t_* \chi_{crit}(c)/2)|Df_c^{n+3k}(c)|^{-t_*/2}
\]
is finite, then \( P_c^{\hat{U}_c} \) (resp. \( P_c^{\hat{J}_c} \)) is not differentiable at \( t = t_* \) and there is a unique equilibrium state of \( f_c|_{\hat{U}_c} \) (resp. \( f_c|_{\hat{J}_c} \)) for the potential \(-t_* \log |Df_c|\). Furthermore, this measure is ergodic, mixing, and its measure theoretic entropy is strictly positive.

After making a uniform distortion bound in §4.1, we give the proof of the Main Theorem in §4.2.

4.1. Uniform distortion bound. In this subsection we prove a uniform distortion bound, stated as Lemma 4.3 below. We start with some preparatory lemmas. Recall that for a parameter \( c \) in \( \mathcal{P}_2(-2) \) the external rays \( R_c(7/24) \) and \( R_c(17/24) \) land at the point \( \omega(c) \) in \( P_{c,1}(0) \), see §3.3.

**Lemma 4.1.** For every parameter \( c \) in \( \mathcal{P}_2(-2) \) the following properties hold:

1. The open disk \( \hat{U}_c \) containing \( -\beta(c) \) that is bounded by the equipotential 2 and by
\[
R_c(7/24) \cup \{\omega(c)\} \cup R_c(17/24),
\]
contains the closure of \( P_{c,0}(-\beta(c)) \).

2. The open set \( \hat{W}_c := f_c^{-1}(\hat{U}_c) \) contains the closure of \( P_{c,1}(0) \) and it depends continuously with \( c \) on \( \mathcal{P}_3(-2) \).

**Proof.**

1. Since the puzzle piece \( P_{c,0}(-\beta(c)) \) is bounded by the equipotential 1 and by \( R_c(1/3) \cup \{\alpha(c)\} \cup R_c(2/3) \) (Theorem 1 and §3.1) and since \( 7/24 < 1/3 < 2/3 < 17/24 \), we deduce that \( \hat{U}_c \) contains the closure of \( P_{c,0}(-\beta(c)) \).

2. That \( \hat{W}_c \) contains the closure of \( P_{c,1}(0) = f_c^{-1}(P_{c,0}(-\beta(c))) \) is a direct consequence of part 1. To show that \( \hat{W}_c \) depends continuously with \( c \) on \( \mathcal{P}_3(-2) \), it
is enough to show that $\partial \hat{W}_c$ depends continuously with $c$ on $\mathcal{P}_3(-2)$. This last assertion follows directly from Lemma 2.5.

**Lemma 4.2.** Let $n \geq 3$ be an integer and let $c$ be a parameter in $\mathcal{K}_n$. Then for every integer $j \geq 1$ the point $f^j_c(0)$ is contained in

$$P_{c,1}(-\beta(c)) \cup \Lambda_c \cup P_{c,1}(\beta(c)).$$

Moreover, this set is disjoint from $\hat{U}_c \setminus P_{c,0}(-\beta(c))$ and from $P_{c,4}(0)$.

**Proof.** By part 2 of Lemma 3.3, the critical value $c$ of $f_c$ is in $P_{c,n}(-\beta(c))$. Thus for each $j$ in $\{1, \ldots, n-1\}$ the point $f^j_c(c)$ belongs to $P_{c,n-1}(\beta(c)) \subset P_{c,1}(\beta(c))$. Using the hypothesis that $c$ is in $\mathcal{K}_n$, we conclude that for every integer $k \geq 0$ the point $f^{n+3k}_c(c)$ belongs to $\Lambda_c$, that $f^{n+3k+1}_c(c)$ belongs to $V_{c,2} \subset P_{c,1}(-\beta(c))$ and that $f^{n+3k+2}_c(c)$ belongs to $\hat{V}_{c,1} \subset P_{c,1}(-\beta(c))$. This proves the first part of the lemma.

To prove the last assertion of the lemma, note that $P_{c,4}(0)$ is disjoint from $\Lambda_c$ (§3.3). On the other hand, $P_{c,4}(0)$ is contained in $P_{c,1}(0)$ and it is therefore disjoint from $P_{c,1}(-\beta(c)) \cup P_{c,1}(\beta(c))$. It remains to prove that (4.2) is disjoint from $\hat{U}_c \setminus P_{c,0}(-\beta(c))$. This last set is disjoint from $P_{c,1}(-\beta(c))$. To complete the proof, observe that the set (4.1) separates $\mathbb{C}$ into 2 connected components: one containing $-\beta(c)$, denoted by $H$, and another one containing $\beta(c)$, denoted by $\hat{H}$. Clearly $\hat{U}_c$ is contained in $H$. On the other hand, part 2 of Lemma 3.2 implies that $P_{c,1}(\beta(c))$ is contained in $\hat{H}$. Finally, note that $\Lambda_c$ is contained in $P_{c,3}(0)$ (§3.3) and that this last set is contained in $\hat{H}$, see the beginning of §3.3. This shows that $\Lambda_c$ and $P_{c,1}(\beta(c))$ are both disjoint from $\hat{U}_c$, and hence from $\hat{U}_c \setminus P_{c,0}(-\beta(c))$. This completes the proof of the lemma. □

**Lemma 4.3** (Uniform distortion bound). There is $\Delta_2 > 1$ such that for each integer $n \geq 4$ and each parameter $c$ in $\mathcal{K}_n$ the following properties hold: For each integer $m \geq 1$ and each connected component $W$ of $f^{-m}_c(P_{c,1}(0))$ on which $f^{-m}_c$ is univalent, $f^{-m}_c$ maps a neighborhood of $W$ biholomorphically to $\hat{W}_c$ and the distortion of this map on $W$ is bounded by $\Delta_2$.

**Proof.** Recall that for each parameter $c$ in $\mathcal{P}_3(-2)$ the set $\hat{W}_c$ contains the closure of $P_{c,1}(0)$ and that these sets depend continuously with $c$ on $\mathcal{P}_3(-2)$ (cf., part 2 of Lemma 4.1 and Lemma 2.5). As the closure of $\mathcal{P}_4(-2)$ is contained in $\mathcal{P}_3(-2)$ (part 1 of Lemma 3.3), we have

$$A := \inf_{c \in \mathcal{P}_4(-2)} \text{mod}(\hat{W}_c \setminus \text{cl}(P_{c,1}(0))) > 0.$$ 

Then the desired assertion follows from Lemma 4.2 and Koebe Distortion Theorem for this choice of the constant $A$. □

4.2. **Proof of Main Theorem assuming Proposition A.** The following elementary lemma describes the itinerary of the postcritical orbit, for the parameter $c$ for which we show there is a low-temperature phase transition.

**Lemma 4.4.** Let $N \geq 1$ and $\ell_0 \geq 1$ be given integers satisfying $2\ell_0 \geq N$. Define $(a_k)_{k=0}^{+\infty}$ as the sequence in $\{0,1\}^{\mathbb{N}_0}$ such that for $k$ in $\mathbb{N}_0$ we have $a_k = 0$ if and only if there is an integer $\ell \geq \ell_0$ such that

$$\ell^2 \leq k \leq \ell^2 + N - 1.$$
Moreover, let $N : \mathbb{N} \to \mathbb{N}_0$ be the function defined for $k \geq 1$ by

$$N(k) := \#\{j \in \{0, \ldots, k-1\} \mid a_j = 0\}$$

and let $B : \mathbb{N} \to \mathbb{N}$ be the function defined by $B(1) = 1$ and for $k \geq 2$ by

$$B(k) := 1 + \#\{j \in \{0, \ldots, k-2\} \mid a_j \neq a_{j+1}\}.$$

Then for every $k$ in $\{1, \ldots, \ell_0^2\}$ we have $N(k) = 0$ and $B(k) = 1$ and for every $k \geq \ell_0^2 + 1$ we have

$$B(k) \leq 2(\sqrt{k} - \ell_0) + 3 \quad \text{and} \quad N \cdot (\sqrt{k} - \ell_0) \leq N(k) \leq N \sqrt{k}.$$

**Proof.** The assertions for $k$ in $\{1, \ldots, \ell_0^2\}$ and the upper bound of $B(k)$ are straightforward consequences of the definitions. Let $k \geq \ell_0^2 + 1$ be a given integer. If there is an integer $\ell \geq \ell_0$ such that $\ell^2 \leq k - 1 \leq \ell^2 + N - 1$, then

$$N(k) = N \cdot (\ell - \ell_0) + k - \ell^2$$

and therefore

$$N \cdot (\sqrt{k} - \ell_0) + (\sqrt{k} - \ell)(\sqrt{k} + \ell - N) = N(k) \leq N \sqrt{k} + N \cdot (\ell + 1 - \sqrt{k} - \ell_0).$$

Using $N \leq 2\ell$ and $\ell + 1 - \sqrt{k} \leq 1$, we obtain the estimates for $N(k)$ in (4.3). Suppose there is an integer $\ell \geq \ell_0$ such that

$$\ell^2 + N \leq k - 1 \leq (\ell + 1)^2 - 1.$$

Then $N(k) = N \cdot (\ell - \ell_0 + 1)$ and we also get the bounds for $N(k)$ in (4.3). \qed

**Proof of the Main Theorem.** Let $n_0 \geq 3$ and $C_0 > 1$ be given by Proposition A and let $\Delta_1 > 1$ and $\Delta_2 > 1$ be given by Lemmas 3.6 and 4.3, respectively.

For a given parameter $c$ in $\mathcal{P}_3(-2)$ denote by $p(c)$ the unique fixed point of $g_c = f_c^2|_{Y_c \cup \tilde{Y}_c}$ in $Y_c$ and by $\tilde{p}(c)$ the unique fixed point of $g_c$ in $\tilde{Y}_c$, see §3.3. Each of the functions

$$p : \mathcal{P}_3(-2) \to \mathbb{C} \quad \text{and} \quad \tilde{p} : \mathcal{P}_3(-2) \to \mathbb{C}$$

so defined is holomorphic and real. By Lemma A.1 in Appendix A there is $\delta > 0$ such that for each parameter $c$ in the interval $(-2, -2 + \delta)$ we have

$$\eta_c := \frac{|Dg_c(p(c))|}{|Dg_c(\tilde{p}(c))|} > 1.$$

Since for $c = -2$ we have

$$|Dg_{-2}(\tilde{p}(-2))|^{1/3} = 2 \quad \text{and} \quad |Df_{-2}(\beta(-2))| = 4,$$

taking $\delta > 0$ smaller if necessary we assume that for each $c$ in $(-2, -2 + \delta)$ we have

(4.4) \quad \frac{2}{3} > |Dg_c(\tilde{p}(c))|^{1/3}/|f_c(\beta(c))| > 1/3.

By Proposition 3.1 there is $n_1 \geq 3$ such that for each integer $n \geq n_1$ the set $\mathcal{K}_n$ is contained in $(-2, -2 + \delta)$.

Fix a sufficiently large integer $n \geq \max\{n_0, n_1\}$ such that

(4.5) \quad \Delta_1^{1/2} \Delta_2^{3/2} (2/3)^{n/2} < C_0^{-1}/2.

Since $\mathcal{K}_n$ is compact, we have

$$\eta := \inf\{\eta_c \mid c \in \mathcal{K}_n\} > 1.$$
Let $N \geq 1$ be sufficiently large so that $\Delta_2^2 \eta^{-N} < 1$ and let $\ell_0 \geq 1$ be a sufficiently large integer so that

$$2\ell_0 \geq N \text{ and } \ell_0^2 > C_0^3(\Delta_1 \Delta_2 3^n)^{1/2}.$$ 

By Proposition 3.1 there is a unique parameter $c_0$ in $\mathcal{K}_n$ such that $\ell(c_0)$ is given by the sequence $(a_k)_{k=0}^{k-1}$ defined in Lemma 4.4 for these choices of $N$ and $\ell_0$. To prove the Main Theorem we just need to show that the hypotheses of Proposition A are satisfied for this choice of $n$ and for $c = c_0$ and $t_0 = 3$.

Note for each integer $k \geq 1$ the number $N(k)$ is equal to the number of 0’s in the sequence $(a_j)_{j=0}^{k-1}$ and that $B(k)$ is the number of blocks of 0’s or 1’s in this sequence. Let $k$ be an integer satisfying $k \geq \ell_0^2 + 1$. Applying Lemma 4.3 to each block of 0’s or 1’s in $(a_j)_{j=0}^{k-1}$, we obtain by (4.3) and by the definition of $\eta$,

$$\Delta_2^{2(\sqrt{k} - \ell_0) + 3} \left( \frac{|Dg_{c_0}(p(c_0))|}{|Dg_{c_0}(p(c_0))|} \right)^{N(\sqrt{k})} \geq \Delta_2^{-2(\sqrt{k} - \ell_0) - 3} \left( \frac{|Dg_{c_0}(p(c_0))|}{|Dg_{c_0}(p(c_0))|} \right)^{\Delta_2^{-3}(\Delta_2^2 \eta^{-N})^{(\sqrt{k} - \ell_0)/2}}.$$

This implies that,

$$\chi_{\text{crit}}(c_0) = \lim_{m \to +\infty} \frac{1}{m} \log |Df_{c_0}^m(c_0)| = \frac{1}{3} \lim_{k \to +\infty} \frac{1}{k} \log |Dg_{c_0}(f_{c_0}^k(c_0))| = \log |Dg_{c_0}(p(c_0))|^{1/3}$$

and, by Lemma 3.6 with $k = n$ and $y = c_0$ and by (4.4) and (4.5), that for each integer $k \geq \ell_0^2 + 1$ we have

$$\exp((n + 3k)\chi_{\text{crit}}(c_0)/2)|Df_{c_0}^{n+3k}(c_0)|^{-1/2} \leq (C_0^{-1}/2)(\Delta_2^2 \eta^{-N})^{(\sqrt{k} - \ell_0)/2}.$$

This implies that for every $t > 0$ the sum

$$\sum_{k=0}^{+\infty} k \cdot \exp(t(n + 3k)\chi_{\text{crit}}(c_0)/2)|Df_{c_0}^{n+3k}(c_0)|^{-t/2}$$

is finite and hence that the last hypothesis of Proposition A is automatically satisfied when the other ones are.

To prove that the rest of the hypotheses of Proposition A are satisfied, observe that by (4.7), by Lemma 4.3, by Lemma 3.6 with $k = n$ and $y = c_0$ and by (4.4) and (4.5), for each integer $k$ in $\{0, 1, \ldots, \ell_0^2\}$ we have

$$\Delta_1^{1/2} \Delta_2^{1/2} (1/3)^{n/2} \leq \exp((n + 3k)\chi_{\text{crit}}(c_0)/2)|Df_{c_0}^{n+3k}(c_0)|^{-1/2} \leq \Delta_1^{1/2} \Delta_2^{1/2} (2/3)^{n/2} < C_0^{-1}/2.$$ 

So, if for each $t > 0$ we put

$$S(t) := \sum_{k=\ell_0^2 + 1}^{+\infty} (\Delta_2^2 \eta^{-N})^t(\sqrt{k} - \ell_0)/2,$$
then by (4.8) we have

\begin{equation}
\sum_{k=0}^{+\infty} \exp(t(n + 3k)\chi_{\text{crit}}(c_0)/2)|Df_{c_0}^{n+3k}(c_0)|^{-t/2} < \frac{C_0 - t}{2^t}(t_0^2 + 1 + S(t)).
\end{equation}

By our choice of $N$ we have $\Delta_2^N \eta^{-N} < 1$ and hence $S(t) \to 0$ as $t \to +\infty$. Together with the fact that $C_0 > 1$, this proves that the sum above converges to $0$ as $t \to +\infty$. Finally, note that by (4.9) and our choice of $\ell_0$, the sum above with $t = 3$ is greater than $C_0^3$. This completes the proof that the hypotheses of Proposition A are satisfied with $c = c_0$ and $\ell_0 = 3$ and thus completes the proof of the Main Theorem. \qed

5. Expansion away from the critical point

For an integer $n \geq 4$ and a parameter $c$ in $\mathcal{P}_n(-2)$ put $V_c := P_{c,n+1}(0)$ and denote by $D'_c$ the set of all those points $z$ in $C \setminus V_c$ for which there is an integer $m \geq 1$ such that $f^m(z)$ is in $V_c$; for such $z$ denote by $m_c(z)$ the least integer $m$ with this property and call it the first landing time of $z$ to $V_c$. The first landing map to $V_c$ is the map $L_c : D'_c \to V_c$ defined by $L_c(z) = f_c^{m_c(z)}(z)$.

The purpose of this section is to prove the following proposition.

**Proposition B.** There is a constant $C_1 > 1$ such that for each $\varepsilon > 0$ there is $n_2 \geq 4$ such that the following property holds: For each integer $n \geq n_2$, each parameter $c$ in $\mathcal{K}_n$, and each $z$ in $L_c^{-1}(V_c)$ we have

$$|L_c'(z)| \geq C_1^{-1}2^{m_c(z)(1-\varepsilon)}.$$

To prove this proposition we first show that the restriction of $L_c$ to each connected component of its domain admits a univalent extension onto $P_{c,1}(0)$ (Lemma 5.1). The proof of the Proposition B is given in §5.2, after some derivative estimates stated as Lemmas 5.3 and 5.4.

5.1. Univalent pull-back property. Note that the domain $D'_c$ of $L_c$ is a disjoint union of puzzle pieces, so each connected component of $D'_c$ is a puzzle piece. Furthermore, for each connected component $W$ of $D'_c$ the first landing time to $V_c$ of all points in $W$ is the same; denote the common value by $m_c(W)$. So $L_c$ maps $W$ biholomorphically to $V_c$.

**Lemma 5.1** (Univalent pull-back property). For every integer $\tilde{n} \geq 0$ and every parameter $c$ in $\mathcal{P}_{\tilde{n}}(-2)$ the following property holds. Let $m \geq 1$ be an integer and $z$ a point in $f_{c,\tilde{n}}^{-m}(P_{c,1}(0))$ such that for each $j$ in $\{0, \ldots, m-1\}$ we have $f_{c,\tilde{n}}^j(z) \notin P_{c,\tilde{n}+1}(0)$. Then the puzzle piece $P$ of depth $m+1$ containing $z$ is such that for every $j$ in $\{0, \ldots, m-1\}$ the set $f_{c,\tilde{n}}^j(P)$ is disjoint from $P_{c,\tilde{n}+1}(0)$ and $f_{c,\tilde{n}}^m$ maps $P$ biholomorphically to $P_{c,1}(0)$. If in addition $c$ is real, then $P$ intersects the real line.

The proof of this lemma is below, after the following one.

**Lemma 5.2.** Let $c$ be a parameter in $\mathcal{P}_0(-2)$, let $\ell \geq 1$ be an integer, and let $z$ be a point in $f_{c,1}^{-\ell}(P_{c,1}(0))$ such that for each $j$ in $\{0, \ldots, \ell-1\}$ the point $f_{c,1}^j(z)$ is not in $P_{c,1}(0)$. Then $z$ is in $V_{c,\ell}$ or in $\tilde{V}_{c,\ell}$.

**Proof.** We proceed by induction in $\ell$. The case $\ell = 1$ follows from

$$f_{c,1}^{-1}(P_{c,1}(0)) = \phi_c(P_{c,1}(0)) \cup \tilde{\phi}_c(P_{c,1}(0)) = V_{c,1} \cup \tilde{V}_{c,1}.$$
Let \( \ell \geq 2 \) be an integer and suppose the desired assertion holds with \( \ell \) replaced by \( \ell - 1 \). If \( z \) is as in the lemma, then \( z \) is not in \( P_{c,1}(0) \) and \( G_{c,z}(z) \leq 1/2^\ell \leq 1/2 \). Therefore \( z \) is in either \( P_{c,1}(-\beta(c)) \) or \( P_{c,1}(\beta(c)) \); in both cases \( f_c(z) \) is in \( P_{c,0}(\beta(c)) \). Applying the induction hypothesis to \( f_c(z) \) we conclude that \( f_c(z) \) is in \( \tilde{V}_{c,\ell-1} \) and therefore that \( z \) is in

\[
f_c^{-1}(\tilde{V}_{c,\ell-1}) = V_{c,\ell} \cup \tilde{V}_{c,\ell}.
\]

This completes the proof of the induction step and of the lemma. \( \square \)

**Proof of Lemma 5.1.** We proceed by induction in \( m \). Since \( f_c^{-1}(P_{c,1}(0)) = V_{c,1} \cup \tilde{V}_{c,1} \), the desired assertions clearly hold for \( m = 1 \). Given an integer \( m \geq 2 \) suppose by induction that the desired assertions hold for every integer less than or equal to \( m - 1 \). Given \( z \) as in the statement of the lemma, let \( P \) be the puzzle piece of \( f_c \) of depth \( m + 1 \) containing \( z \), so that \( f_c^m(P) = P_{c,1}(0) \).

First, we prove that \( f_c^m \) maps \( P \) biholomorphically to \( P_{c,1}(0) \). Let \( \ell \geq 1 \) be the least integer such that \( f_c^\ell(P) \) is contained in \( P_{c,1}(0) \); we have \( \ell \leq m \). If \( P \) is not contained in \( P_{c,1}(0) \), then it is contained in \( V_{c,\ell} \) or \( \tilde{V}_{c,\ell} \) (Lemma 5.2); in both cases \( P \) is contained in a puzzle piece of depth \( \ell + 1 \) that is mapped biholomorphically to \( P_{c,1}(0) \) by \( f_c^\ell \). If \( P \) is contained in \( P_{c,1}(0) \), then \( \ell \geq 2 \). \( f_c(P) \) is contained in \( V_{c,\ell-1} \) (Lemma 5.2), and our hypotheses imply that \( \ell - 1 \leq \bar{n} \). So the puzzle piece of \( f_c \) of depth \( \ell + 1 \) containing \( P \) is mapped biholomorphically to \( V_{c,\ell-1} \) by \( f_c \); this proves that in all the cases \( f_c^\ell \) maps the puzzle piece of depth \( \ell + 1 \) containing \( P \) biholomorphically to \( P_{c,1}(0) \) and shows the inductive step in the case where \( \ell = m \).

If \( \ell \leq m - 1 \), then by the induction hypothesis applied to \( m - \ell \) instead of \( m \) and with \( f_c^\ell(z) \) instead of \( z \), we conclude that \( f_c^{m-\ell} \) maps \( f_c^\ell(P) \) biholomorphically to \( P_{c,1}(0) \). This completes the proof that \( f_c^m \) maps \( P \) biholomorphically to \( P_{c,1}(0) \).

Now we prove the other assertions of the lemma. For each \( j \in \{0, \ldots, m - 1\} \) we have \( f_c^{\ell}(z) \notin P_{c,\bar{n}+1}(0) \). Let \( P' \) be the puzzle piece of depth \( m \) containing \( z' := f_c(z) \). By our induction hypothesis we just need to prove that \( P \) is disjoint from \( P_{c,\bar{n}+1}(0) \), and if \( c \) is real, that \( P \) intersects \( \mathbb{R} \). Suppose \( z \) is not in \( P_{c,1}(0) \). Then by Lemma 5.2 there is an integer \( \ell \geq 1 \) such that \( z \) belongs to \( V_{c,\ell} \) or \( \tilde{V}_{c,\ell} \). Then \( m \geq \ell \) and \( P \) is contained in one of these sets; it follows that \( P \) is disjoint from \( P_{c,1}(0) \) and hence from \( P_{c,\bar{n}+1}(0) \). Suppose \( c \) is real. Then the maps \( \phi_c \) and \( \phi_{c'} \) are both real and by our induction hypothesis \( P' \) intersects \( \mathbb{R} \). Since \( P \) is equal to either \( \phi_c(P') \) or \( \phi_{c'}(P') \), it follows that \( P \) also intersects \( \mathbb{R} \). It remains to consider the case where \( z \) belongs to \( P_{c,1}(0) \). Since by hypothesis \( z \) is not in \( P_{c,\bar{n}+1}(0) \), the point \( z' \) is in \( P_{c,\bar{n}}(-\beta(c)) \). So there is an integer \( \ell \leq m - 1 \) in \( \{1, \ldots, \bar{n} - 1\} \) such that \( z' \) is in \( V_{c,\ell} \) (Lemma 5.2). It follows that \( P' \) is contained in \( V_{c,\ell} \) and that it is therefore disjoint from \( P_{c,\bar{n}}(-\beta(c)) \); this implies that \( P \) is disjoint from \( P_{c,\bar{n}+1}(0) \). If \( c \) is real, then by the induction hypothesis \( P' \) intersects \( \mathbb{R} \). Since \( P' \) is contained in \( V_{c,\ell} \) and \( \ell \leq \bar{n} - 1 \), it follows that \( P' \cap \mathbb{R} \) is contained in \( f_c(\mathbb{R}) \). This implies that \( P \) intersects \( \mathbb{R} \) and completes the proof of the induction step and of the lemma. \( \square \)

### 5.2. Derivatives estimates.

**Lemma 5.3.** There is a constant \( C_2 > 1 \) such that for every \( \varepsilon > 0 \) and every integer \( n_3 \geq 1 \) there is \( n_3 \geq 3 \) such that the following property holds for each integer \( n \geq n_3 \) and each parameter \( c \) in \( K_n \): For every integer \( m \geq 1 \) and every point \( z \) in \( f_c^{-m}(P_{c,1}(0)) \) such that for each \( j \in \{0, \ldots, m - 1\} \) we have \( f_c^j(z) \notin \mathbb{R} \).
Let $\Delta_2 > 1$ be the constant given by Lemma 4.3. Given a parameter $c$ in $[-2, 1/4]$ consider the smooth homeomorphism

$$h_c : [0, 1] \to [-\beta(c), \beta(c)]$$

$$\theta \mapsto h_c(\theta) := \beta(c) \cos(\pi \theta);$$

it depends smoothly on $c$ and when $c = -2$ we have

$$\inf_{x \in [-\beta(-2), \beta(-2)]} |Dh_{-2}(h_{-2}^{-1}(y))/|Dh_{-2}(h_{-2}^{-1}(x))| > 0.$$

So there is $\delta_0 \in (0, 9/4)$ such that

$$\kappa := \inf_{c \in [-2, -2+\delta_0]} \inf_{x \in [-\beta(c), \beta(c)]} \inf_{y \in [\alpha(c), \tilde{\alpha}(c)]} |Dh_c(h_c^{-1}(y))/|Dh_c(h_c^{-1}(x))| > 0$$

and

$$\tilde{\kappa} := \inf_{c \in [-2, -2+\delta_0]} \inf_{x \in [\alpha(c), \tilde{\alpha}(c)]} \inf_{y \in [\alpha(c), \tilde{\alpha}(c)]} |Dh_c(h_c^{-1}(y))/|Dh_c(h_c^{-1}(x))| < +\infty.$$

Let $\varepsilon > 0$ and let $m_1 \geq 1$ be given. Taking $m_1$ larger if necessary, we assume $m_1 \geq 4$. Since $P_{c,m_1}(0)$ depends continuously with $c$ on $P_{m_1-1}(-2)$ (cf., Lemma 2.5) and since this last set contains the closure of $P_{m_1-1}(-2)$ (part 1 of Lemma 3.3), there is $\tau > 0$ such that for each parameter $c$ in $P_{m_1-1}(-2) \cap [-2, 1/4]$ we have

$$[1/2 - \tau, 1/2 + \tau] \subset h_c^{-1}(P_{c,m_1}(0) \cap [-\beta(c), \beta(c)]).$$

For each parameter $c$ in $[-2, 1/4]$ let $T_c : [0, 1] \to [0, 1]$ be the map defined by $T_c = h_c^{-1} \circ f_c \circ h_c$. When $c = -2$ the map $T_{-2}$ is the tent map given by $T_{-2}(\theta) = 2\theta$ on $[0, 1/2]$ and $T_{-2}(\theta) = 2 - 2\theta$ on $[1/2, 1]$. When $c$ is not equal to $-2$, the map $T_c$ is smooth on $[0, 1]$. A direct computation shows that there is $\delta_1$ in $(0, \delta_0)$ such that for each parameter $c$ in $[-2, -2+\delta_1]$ and each $\theta$ in $[0, 1]$ satisfying $|\theta - 1/2| \geq \tau$, we have

$$2^{1-\varepsilon} \leq |DT_c(\theta)| \leq 2^{1+\varepsilon}.$$

Let $n_1$ be given by Proposition 3.1 with $\delta = \delta_1$.

Fix an integer $n \geq \max\{n_1, m_1\}$ and a parameter $c$ in $K_n$. By Proposition 3.1 we have $K_n \subset (-2, -2+\delta_1)$. Let $m \geq 1$ be an integer, $z$ a point in $f_{c,m}^m(P_{c,1}(0))$ such that for each $j$ in $\{0, \ldots, m-1\}$ we have $f_j(z) \not\in P_{c,m_1}(0)$ and let $P$ be the puzzle piece of $f_c$ of depth $m+1$ that contains $z$. By Lemma 5.1 with $m$ replaced by $m_1 - 1$ there is a real point $x$ in $P$ and for every $j$ in $\{0, \ldots, m-1\}$ the point $f_j(z)$ of $f_j(P)$ is not in $P_{c,m_1}(0)$; by our choice of $\tau$ it follows that $h_c^{-1}(f_j(z))$ is not in $[1/2 - \tau, 1/2 + \tau]$. On the other hand, $f_{c,m}^m$ maps $P$ biholomorphically to $P_{c,1}(0)$ and by Lemma 4.3 the distortion of $f_{c,m}^m$ on $P$ is bounded by $\Delta_2$. Since $x$ is in $[-\beta(c), \beta(c)]$ and

$$f_{c,m}^m(x) \in f_{c,m}^m(P) \cap \mathbb{R} = P_{c,1}(0) \cap \mathbb{R} = (\alpha(c), \tilde{\alpha}(c)),$$
by the considerations above we have by the definition of $\kappa$,
\[
|D f_c^m(z)| \geq \Delta_2^{-1}|D f_c^m(x)|
\]
\[
= \Delta_2^{-1}|D h_c(h^{-1}_c(f_c^m(x)))| \cdot |D T_c^m(h^{-1}_c(x))|/|D h_c(h_c^{-1}(x))| \geq \Delta_2^{-1}\kappa 2^{m(1-\epsilon)}.
\]

If in addition $z$ is in $P_{c,1}(0)$, then $x$ belongs to $(\alpha(c), \tilde{\alpha}(c))$ and by the considerations above we have by the definition of $\tilde{\kappa}$,
\[
|D f_c^m(z)| \leq \Delta_2|D f_c^m(x)|
\]
\[
= \Delta_2|D h_c(h^{-1}_c(f_c^m(x)))| \cdot |D T_c^m(h^{-1}_c(x))|/|D h_c(h_c^{-1}(x))| \leq \Delta_2 \tilde{\kappa}^{-1}2^{m(1+\epsilon)}.
\]

This proves the lemma with $n_2 = \max\{n_1, m_1\}$ and $C_2 = \Delta_2^{-1}\max\{\kappa^{-1}, \tilde{\kappa}\}$. $\square$

**Lemma 5.4.** There is $C_3 > 1$ such that for each integer $n \geq 4$ and each parameter $c$ in $K_n$ the following properties hold for each integer $q \geq 1$:

1. For each open set $W$ that is mapped biholomorphically to $P_{c,1}(0)$ by $f_c^q$ and each $x$ in $W$ we have
\[
|D f_c(x)| \geq C_3^{-1}|D f_c^{q-1}(f_c(x))|^{-1/2}.
\]

2. If $q \neq 1$, then for each point $x$ in $f_c^{q-1}(V_{c,q-1})$ we have
\[
|D f_c^q(x)| \geq C_3^{-1}|D f_c(\beta(c))|^{q/2}.
\]

**Proof.** Let $\Delta_1 > 1$ and $\Delta_2 > 1$ be the constants given by Lemmas 3.6 and 4.3, respectively.

Since the sets $P_{c,1}(\beta(c))$ and $P_{c,1}(0)$ are disjoint and depend continuously with $c$ on $P_0(-2)$ (cf., §2.5 and Lemma 2.5) and since $P_0(-2)$ contains the closure of $P_4(-2)$ (part 1 of Lemma 3.3), we have
\[
\Xi_1 := \inf_{c \in P_4(-2)} \text{diam}(P_{c,1}(0)) > 0 \text{ and } \Xi_2 := \sup_{c \in P_4(-2)} |D f_c(\beta(c))| < +\infty.
\]

On the other hand, for each $c$ in $P_3(-2)$ the closure of $P_{c,1}(0)$ is contained in $\hat{W}_c$ and $\hat{W}_c$ depends continuously with $c$ on $P_3(-2)$ (part 2 Lemma 4.1); so
\[
\Xi_3 := \inf_{c \in P_3(-2)} \text{mod}(\hat{W}_c \setminus \text{cl}(P_{c,1}(0))) > 0.
\]

Let $n \geq 4$ be an integer and $c$ a parameter in $K_n$.

1. Note that $f_c^q$ maps a neighborhood $\hat{W}$ of $W$ biholomorphically to $\hat{W}_c$ (Lemma 4.3). So if we put $\hat{W}':= f_c(\hat{W})$, then $c$ is not in $\hat{W}'$ and $f_c^{q-1}$ maps $\hat{W}'$ biholomorphically to $\hat{W}_c$; in particular we have
\[
\text{mod}(\hat{W}' \setminus \text{cl}(f_c(W))) = \text{mod}(\hat{W}_c \setminus \text{cl}(P_{c,1}(0))) \geq \Xi_3.
\]

Thus there is a constant $A_1 > 0$ independent of $n$, $c$ and $q$ such that for every $x$ in $W$, we have
\[
|f_c(x) - c| \geq \text{dist}(f_c(W), c) \geq \text{dist}(f_c(W), \partial \hat{W}') \geq A_1 \text{diam}(f_c(W))
\]
(cf., [LV73, Teichmüller's module theorem, §I.1.3]). Thus, if we put $A_2 := 2(A_1\Delta_2^{-1}\Xi_1)^{1/2}$, then by Lemma 4.3 with $m = q - 1$ and with $W$ replaced by $f_c(W)$ we have
\[
|D f_c(x)| \geq 2 A_1^{1/2} \text{diam}(f_c(W))^{1/2} \geq A_2|D f_c^{q-1}(f_c(x))|^{-1/2}.
\]
This proves part 1 with constant $C_3 = A_2^{-1}$.

2. Since $f_c(x)$ is in $V_{c,q-1}$ and this last set is contained in $P_{c,q-1}(-\beta(c))$ (part 1 of Lemma 3.2), by Lemma 3.6 with $y = f_c(x)$ and $k = q - 1$, we have

$$|Df_c^{\ell+1}(f_c(x))| \geq \Delta^{-1} |Df_c(\beta(c))|^q - 1.$$  

Our assumption $q - 1 \neq n$ implies that $f_c(0) = c$ is not in $V_{c,q-1}$, so $f_c^q$ maps the connected component $W$ of $f_c^{-1}(V_{c,q-1})$ containing $x$ biholomorphically to $P_{c,1}(0)$. So the desired assertion with $C_3$ replaced by $C_3(\Delta_1 \Xi_2)^{1/2}$ follows from (5.1) and from part 1. \hfill $\Box$

**Proof of Proposition B.** Let $C_2$ and $C_3$ be the constants given by Lemmas 5.3 and 5.4, respectively. Let $m_1 \geq 2$ be sufficiently large so that

$$2^{(m_1-1)\varepsilon/2} \geq C_2 C_3$$

and let $n_3$ be given by Lemma 5.3 for this choice of $m_1$. Notice that for $c = -2$ we have $Df_{-2}(\beta(-2)) = 4$. So, in view of Proposition 3.1, we can take $n_3$ larger if necessary and assume that for each parameter $c$ in $P_{n_3}(-2)$ we have

$$|Df_c(\beta(c))|^{1/2} \geq 2^{1-\varepsilon/2}$$

We prove the desired assertion with $n_2 = n_3$ and $C_1 = C_2$. To do this, let $n \geq n_3$ be an integer, $c$ a parameter in $K_n$, and let $z$ be a point in $L_c(V_c)$. If for every $j$ in $\{0, \ldots, m_c(z) - 1\}$ we have $f_c^j(z) \notin P_{c,m_c}(0)$, then the desired assertion follows from Lemma 5.3 with $m = m_c(z)$. So we assume that there is $\ell$ in $\{0, \ldots, m_c(z) - 1\}$ such that $f_c^{\ell}(z)$ belongs to $P_{c,m_c}(0)$. Let $k \geq 1$ be the number of all such integers, let $\ell_1 < \ell_2 < \cdots < \ell_k$ be the increasing sequence of all of these numbers, and put $\ell_{k+1} := m_c(z)$. Given $s$ in $\{1, \ldots, k\}$ let $\ell'_s$ be the least integer $\ell \geq \ell_s + 1$ such that $f_c^{\ell}(z)$ belongs to $P_{c,1}(0)$. Then $\ell'_s \leq \ell_{s+1}$, $\ell'_s - \ell_s \geq m_1 - 1$, and the point $f_c^{\ell'_s+1}(z)$ belongs to $V_{c,\ell'_s-\ell_s-1}$ (Lemma 5.2). By our choice of $z$, the point $f_c^{\ell'_s}(z)$ does not belong to $V_c = P_{c,m_c+1}(0)$ if $m_c(z) = 1$, and we have

$$\ell'_s - \ell_s \geq m_1 - 1, \quad |Df_c^{\ell'_s+1}(f_c^{\ell'_s}(z))| \geq C_3^{-1} |Df_c(\beta(c))|^{(\ell'_s - \ell_s)/2} \geq C_3^{-1} 2^{(\ell'_s - \ell_s)(1-\varepsilon/2)} \geq C_2 2^{(\ell'_s - \ell_s)(1-\varepsilon)}.$$  

When $\ell'_s = \ell_{s+1}$ we obtain

$$|Df_c^{\ell_{s+1} - \ell_s}(f_c^{\ell'_s}(z))| \geq 2^{(\ell_{s+1} - \ell_s)(1-\varepsilon)}.$$  

In the case where $\ell'_s \leq \ell_{s+1} - 1$, the point $f_c^{\ell'_s}(z)$ belongs to $P_{c,1}(0)$ but not to $P_{c,m_c+1}(0)$; so (5.2) together with Lemma 5.3 with $m = \ell_{s+1} - \ell'_s$ and with $z$ replaced by $f_c^{\ell'_s}(z)$ implies, by our choice of $n_3$, that

$$|Df_c^{\ell_{s+1} - \ell_s}(f_c^{\ell'_s}(z))| \geq |Df_c^{\ell_{s+1} - \ell'_s}(f_c^{\ell'_s}(z))| |Df_c(\beta(c))|^q \geq C_2 2^{(\ell_{s+1} - \ell'_s)(1-\varepsilon)}.$$  

So in all the cases we obtain (5.3) and therefore

$$|Df_c^{m_c(z) - \ell_1}(f_c^{\ell_1}(z))| = \prod_{s=1}^k |Df_c^{\ell_{s+1} - \ell_s}(f_c^{\ell'_s}(z))| \geq 2^{(m_c(z) - \ell_1)(1-\varepsilon)}.$$  

This proves the desired inequality in the case where $\ell_1 = 0$. If $\ell_1 \geq 1$, then by Lemma 5.3 with $m = \ell_1$ we have

$$|Df_c^{\ell_1}(z)| \geq C_2^{-1} 2^{\ell_1(1-\varepsilon)}.$$
Together with (5.4) this implies the desired inequality and completes the proof of the proposition.

6. Induced map

In this section, for a parameter $c$ in $\mathcal{P}_4(-2)$ we use the first return map $F_c$ of $f_c$ to $V_c$ to study $P^c_r$ and $P^c_c$. After some basic considerations in §6.1, we show that $P^c_r$ and $P^c_c$ are related to a 2 variables pressure function of $F_c$ through a Bowen type formula, see Proposition C in §6.2 and compare with [SU03] and [PRL11]. We do this by analyzing the convergence properties of a suitable Poincaré series (Lemma 6.5). In the proof of Proposition C we use a lower bound for $P^c_c$ (Proposition 6.2 in §6.3) that is used again in the next section.

6.1. Induced map. Let $n \geq 4$ be an integer and $c$ a parameter in $K_n$. Throughout the rest of this section put $\hat{V}_c := P_{c,4}(0)$. Since the critical value $c$ of $f_c$ is in $P_{c,n}(-\beta(c))$ (part 2 of Lemma 3.3), the closure of $V_c = P_{c,n+1}(0) = f_c^{-1}(P_{c,n}(-\beta(c)))$ is contained in $\hat{V}_c = f_c^{-1}(P_{c,3}(-\beta(c)))$ (cf., part 1 of Lemma 3.2).

Let $D_c$ be the set of all those points $z$ in $V_c$ for which there is an integer $m \geq 1$ such that $f^m_c(z)$ is in $V_c$. For $z$ in $D_c$ denote by $m_c(z)$ the least integer $m$ with this property and call it the first return time of $z$ to $V_c$. The first return map to $V_c$ is defined by

$$F_c : D_c \to V_c, \quad z \mapsto F_c(z) := f^m_c(z).$$

It is easy to see that $D_c$ is a disjoint union of puzzle pieces; so each connected component of $D_c$ is a puzzle piece. Note furthermore that in each of these puzzle pieces $W$, the return time function $m_c$ is constant; denote the common value of $m_c$ on $W$ by $m_c(W)$.

Lemma 6.1 (Uniform bounded distortion). There is a constant $\Delta_3 > 1$ such that for each integer $n \geq 5$ and each parameter $c$ in $K_n$ the following property holds: For every connected component $W$ of $D_c$ the map $F_c|_W$ is univalent and its distortion is bounded by $\Delta_3$. Furthermore, the inverse of $F_c|_W$ admits a univalent extension to $\hat{V}_c$ taking images in $V_c$. In particular, $F_c$ is uniformly expanding with respect to the hyperbolic metric on $\hat{V}_c$.

Proof. Recall that for each parameter $c$ in $\mathcal{P}_4(-2)$ the critical value $c$ of $f_c$ is in $P_{c,4}(-\beta(c))$ (part 2 of Lemma 3.3), so set $P_{c,4}(0) = f_c^{-1}(P_{c,3}(-\beta(c)))$ contains the closure of $P_{c,5}(0) = f_c^{-1}(P_{c,4}(-\beta(c)))$ (cf., part 1 of Lemma 3.2) and that these sets depend continuously with $c$ on $\mathcal{P}_4(-2)$ (cf,. Lemma 2.5). Since $\mathcal{P}_4(-2)$ contains the closure of $\mathcal{P}_5(-2)$ (part 1 of Lemma 3.3) we have

$$A := \inf_{c \in \mathcal{P}_4(-2)} \text{mod}(P_{c,4}(0) \setminus \text{cl}(P_{c,5}(0))) > 0.$$

Let $\Delta_3$ be the constant $\Delta$ given by Koebe Distortion Theorem for this value of $A$.

Since $\hat{V}_c$ is disjoint from the forward orbit of 0 (Lemma 4.2), for each connected component $W$ of $D_c$ the map $f^m_c(W)$ maps a neighborhood $\hat{W}$ of $W$ biholomorphically to $\hat{V}_c$. By Koebe Distortion Theorem the distortion of $f^m_c(W)$ on $W$ is bounded by $\Delta_3$. Note that $\hat{W}$ is a puzzle piece intersecting the puzzle piece $V_c$. Thus, these puzzle pieces are either equal or one is strictly contained in
the other. Since \( \hat{W} \) does not contain 0, it follows that \( \hat{W} \) is strictly contained in \( V_c \). Thus \( \left( f_{c_{V_c}}(W) \right)^{-1} \) is an extension of \( F_{c_{V_c}}^{-1} \) to \( \hat{V}_c \) taking images in \( V_c \). \( \square \)

6.2. **Pressure function of the induced map.** Let \( n \geq 4 \) be an integer and let \( c \) be a parameter in \( K_n \). In this subsection we state a Bowen type formula relating \( P^R_c \) (resp. \( P^C_c \)) to a certain 2 variables pressure of \( F_c \) (Proposition C) that is shown in §6.4.

Denote by \( \mathcal{D}_c \) the collection of connected components of \( D_c \) and by \( \mathcal{D}_c^R \) the sub-collection of \( \mathcal{D}_c \) of those sets intersecting \( \mathbb{R} \). For each \( W \) in \( \mathcal{D}_c \) denote by \( \phi_W : \hat{V}_c \rightarrow V_c \) the extension of \( F_{c_{|W}}^{-1} \) given by Lemma 6.1. Given an integer \( \ell \geq 1 \) denote by \( E_{c, \ell} \) (resp. \( E_{c, \ell}^R \)) the set of all words of length \( \ell \) in the alphabet \( \mathcal{D}_c \) (resp. \( \mathcal{D}_c^R \)). So, for each integer \( \ell \geq 1 \) and each word \( W_1 \cdots W_\ell \) in \( E_{c, \ell} \) the composition

\[
\phi_{W_1 \cdots W_\ell} = \phi_{W_1} \circ \cdots \circ \phi_{W_\ell}
\]

is defined on \( \hat{V}_c \). Put

\[
m_c(W_1 \cdots W_\ell) = m_c(W_1) + \cdots + m_c(W_\ell).
\]

For \( t, p \) in \( \mathbb{R} \) and an integer \( \ell \geq 1 \) put

\[
Z_{c, \ell}^R(t, p) := \sum_{W \in \mathcal{D}_c(\ell)} \exp(-m_c(W)p) \left( \sup\{|D\phi_W(z)| \mid z \in V_c\} \right)^t
\]

and

\[
Z_{c, \ell}^C(t, p) := \sum_{W \in E_{c, \ell}} \exp(-m_c(W)p) \left( \sup\{|D\phi_W(z)| \mid z \in V_c\} \right)^t.
\]

For a fixed \( t \) and \( p \) in \( \mathbb{R} \) the sequence

\[
\left( \frac{1}{\ell} \log Z_{c, \ell}^R(t, p) \right)_{\ell=1}^{+\infty} \quad \text{(resp.} \quad \left( \frac{1}{\ell} \log Z_{c, \ell}^C(t, p) \right)_{\ell=1}^{+\infty} \text{)}
\]

converges to the pressure function of \( F_c_{|D_c \cap \mathbb{R}} \) (resp. \( F_c \)) for the potential \(-t \log |D F_c| - p m_c;\) denote it by \( \mathcal{P}^R_c(t, p) \) (resp. \( \mathcal{P}^C_c(t, p) \)). On the set where it is finite, the function \( \mathcal{P}^R_c \) (resp. \( \mathcal{P}^C_c \)) so defined is strictly decreasing in each of its variables.

**Proposition C.** There is \( n_4 \geq 4 \) such that for every integer \( n \geq n_4 \) and every parameter \( c \) in \( K_n \) we have for each \( \ell \geq 3 \),

\[
P^R_c(t) = \inf \left\{ p \mid \mathcal{P}^R_c(t, p) \leq 0 \right\} \quad \text{(resp.} \quad P^C_c(t) = \inf \left\{ p \mid \mathcal{P}^C_c(t, p) \leq 0 \right\} \).
\]

The proof of this proposition is given in §6.4, after we give a lower bound on the pressure function in the next subsection.

6.3. **Critical line.** The purpose of this subsection is to prove the following proposition.

**Proposition 6.2.** For every integer \( n \geq 5 \) and every parameter \( c \) in \( K_n \) we have

\[
\chi_{inf}(c) := \inf \left\{ \int \log |D f_c| d\mu \mid \mu \in \mathcal{M}_c^R \right\} \leq \chi_{crit}(c)/2.
\]

In particular, for each \( t > 0 \) we have

\[
P^C_c(t) \geq P^R_c(t) \geq -t \chi_{crit}(c)/2.
\]

The proof of this proposition is given after the following lemma.
Lemma 6.3. There is a constant $C_4 > 0$ such that for each integer $n \geq 5$ and each parameter $c$ in $\mathcal{K}_n$, the following property holds: For every integer $k \geq 0$ there is a connected component $W$ of $D_c$ contained in $P_{c,n+3k+1}(0)$, that intersects $\mathbb{R}$ and such that $m_c(W) = n + 3k + 3$ and

$$\sup_{z \in W} |DF_c(z)| \leq C_4 |Df_c^{n+3k}(c)|^{1/2}. $$

Proof. Let $\Delta_2 > 1$ and $\Delta_3 > 1$ be the constants given by Lemmas 4.3 and 6.1, respectively. Since the set $P_{c,1}(0)$ depends continuously with $c$ on $\mathcal{P}_0(-2)$ (cf., Lemma 2.5) and since this last set contains the closure of $\mathcal{P}_4(-2)$ (part 1 of Lemma 3.3), we have

$$\Xi_0 := \sup_{c \in \mathcal{P}_4(-2)} \text{diam}(P_{c,1}(0)) < +\infty$$

and

$$\Xi_1 := \sup_{c \in \mathcal{P}_4(-2)} \sup_{z \in P_{c,1}(0)} |Df_c^2(z)| < +\infty.$$  

Fix an integer $n \geq 5$, a parameter $c$ in $\mathcal{K}_n$, and an integer $k \geq 0$. Then the parameter $c$ is real, so $\alpha(c)$ and $\bar{\alpha}(c)$ are both real ($\S$3.1) and the set $\Lambda_c$ is contained in the interval $P_{c,3}(0) \cap \mathbb{R} = (\omega(c), \bar{\omega}(c))$, see $\S$3.3. On the other hand, the point $\alpha_1(c)$ is real ($\S$3.1) and $c$ is in $P_{c,n}(-\beta(c))$ (part 2 of Lemma 3.3), so $c < \alpha_1(c)$. Note moreover that $V_{c,1}$ is invariant by complex conjugation and that $V_{c,1} \cap \mathbb{R} = (\alpha_1(c), \alpha(c))$ ($\S$3.1). It follows that $f_c^{-1}(V_{c,1})$ is the disjoint union of 2 puzzle pieces $X_c$ and $\tilde{X}_c$, such that

$$X_c \cap \mathbb{R} = (\alpha(c), \omega(c)) \text{ and } \tilde{X}_c \cap \mathbb{R} = (\bar{\omega}(c), \bar{\alpha}(c)).$$

Moreover, each of the puzzle pieces $X_c$ and $\tilde{X}_c$ is a connected component of $D_c'$ and $m_c(X_c) = m_c(\tilde{X}_c) = 2$.

Since $f_c^{n} \circ f_c^{3k+1}$ maps $P_{c,n+3k+2}(0)$ properly onto $P_{c,1}(0)$, it follows that $f_c^{n+3k+1}$ maps the end points of the interval $P_{c,n+3k+2}(0) \cap \mathbb{R}$ into $\partial P_{c,1}(0) \cap \mathbb{R} = \{\alpha(c), \bar{\alpha}(c)\}$. Since $f_c^{n+3k+1}(0)$ is in $\Lambda_c \subset Y_c \cup \tilde{Y}_c$, it follows that the interval $f_c^{n+3k+1}(P_{c,n+3k+2}(0) \cap \mathbb{R})$ contains either $X_c \cap \mathbb{R}$ or $\tilde{X}_c \cap \mathbb{R}$. This proves that there is a connected component $W$ of $D_c$ contained in $P_{n+3k+2}(0)$, that intersects $\mathbb{R}$ and such that $m_c(W) = n + 3k + 3$. Let $z_W$ be the unique point in $W$ such that $f_c^{n+3k+1}(z_W) = 0$. Then $f_c^{n+3k+1}(z_W)$ belongs to $P_{c,1}(0)$, so by definition of $\Xi_0$ we have

$$(6.1) \quad |f_c^{n+3k+1}(z_W) - f_c^{n+3k}(c)| \leq \text{diam}(P_{c,1}(0)) \leq \Xi_0.$$  

Since $f_c^n$ maps $V_{c,n} = P_{c,n+1}(c)$ biholomorphically to $P_{c,1}(0)$ and $f_c^n(c) \in \Lambda_c$, it follows that $f_c^{3k}$ maps $P_{c,n+3k+1}(c)$ biholomorphically to $P_{c,1}(0)$; so the distortion of $f_c^{n+3k}$ on $P_{c,n+3k+1}(c)$ is bounded by $\Delta_2$ (Lemma 4.3) and for each point $y$ in $P_{c,n+3k+1}(c)$ we have

$$(6.2) \quad \Delta_2 |Df_c^{n+3k}(c)| \leq |Df_c^{n+3k}(y)| \leq \Delta_2 |Df_c^{n+3k}(c)|.$$  

Together with (6.1) this implies that,

$$|f_c(z_W) - c| \leq \Delta_2 \Xi_0 |Df_c^{n+3k}(c)|^{-1}$$

and therefore that,

$$|Df_c(z_W)| \leq 2\Delta_2^{1/2} \Xi_0^{1/2} |Df_c^{n+3k}(c)|^{-1/2}. $$
Proof of Proposition 6.2. Let $C_4$ be given by Lemma 6.3 and for each integer $k \geq 0$ let $W_k$ be the element $W$ of $\mathcal{D}_c$ given by the same lemma. Since $W_k$ intersects $\mathbb{R}$ and $F_c|W_k = f_c^{n+3k+3}|W_k$ maps $W_k$ biholomorphically to $V_c$, we have $f_c^{n+3k+3}(W_k \cap \mathbb{R}) = V_c \cap \mathbb{R}$. On the other hand, since $W_k \subset V_c$, there is a periodic point $p_k$ of $f_c$ of period $n+3k+3$ in the closure of $W_k \cap \mathbb{R}$. Denoting by $\mu_k$ the invariant probability measure supported on the orbit of $p_k$, we have by Lemma 6.3 that for each $t > 0$

$$\chi_{\inf}(c) \leq \int \log |Df_c|d\mu_k = \frac{1}{n+3k+3} \log |Df_c^{n+3k+3}(p_k)| \leq \frac{1}{n+3k+3} (\log C_4 + \log |Df_c^{n+3k}(c)|^{1/2}).$$

We obtain the desired inequality by letting $k \to +\infty$. □

6.4. Proof of Proposition C.

Lemma 6.4. There are $n_5 \geq 4$ and $C_5 > 1$ such that for every integer $n \geq n_5$ and every parameter $c$ in $K_n$ the following property holds: For each $t \geq 3$, $p \geq -t(\chi_{\text{crit}}(c) + \frac{1}{3} \log 2)/2$, and $y$ in $V_c$ we have

$$L_{t,p}(y) := 1 + \sum_{z \in L_c^{-1}(y)} \exp(-m_c(z)p) |DL_c(z)|^{-t} \leq C_5^t.$$

Proof. Put $\varepsilon_0 := \frac{1}{15}$, let $C_1$ and $n_2$ be given by Proposition B with $\varepsilon = \varepsilon_0$, and let $n_3$ be given by Lemma 5.3 with $\varepsilon = \varepsilon_0$ and $m_1 = 4$. We prove the lemma with $n_5 := \max\{n_2, n_3\}$ and $C_5 := C_1 (1 - 2^{-1/10})^{-1/3}$.

Let $n$, $c$, $t$, $p$ and $y$ be as in the statement of the lemma. By Lemma 5.3 with $z = f_c^n(c)$ we have

$$\chi_{\text{crit}}(c) = \liminf_{m \to +\infty} \frac{1}{m} \log |Df_c^m(c)| \leq (1 + \varepsilon_0) \log 2.$$

On the other hand, for each integer $m \geq 1$ the set $\{z \in L_c^{-1}(y) \mid m_c(z) = m\}$ is contained in $f_c^{-m}(y)$ and therefore it contains at most $2^m$ points. So by Proposition B and the definition of $C_5$ we have

$$L_{t,p}(y) \leq C_5^t \sum_{m=0}^{+\infty} 2^m \left(1 - \frac{11}{30}\right) \leq C_5^t.$$

□
Lemma 6.5. Given an integer \( n \geq 5 \) and a parameter \( c \) in \( \mathcal{K}_n \) the following property holds for every \( t > 0 \) and every real number \( p \): If \( \mathcal{H}_c(t,p) > 0 \) (resp. \( \mathcal{H}_c(t,p) > 0 \)), then the series

\[
\sum_{j=1}^{+\infty} \exp(-jp) \sum_{y \in f^L_c(0)} |Df^j_c(y)|^{-t}
\]

(6.3)

diverges. On the other hand, there is \( n_6 \geq 5 \) such that if in addition \( n \geq n_6 \), then for every \( t \geq 3 \) and \( p \geq P^R_c(t) - t \frac{1}{10} \log 2 \) (resp. \( p \geq P^C_c(t) - t \frac{1}{10} \log 2 \)) satisfying \( \mathcal{H}_c(t,p) < 0 \) (resp. \( \mathcal{H}_c(t,p) < 0 \)), the series above converges.

Proof. We prove the assertions concerning \( f_c \). The arguments apply without change to \( f_c \). Let \( \Delta_3 > 1 \) be the constant given by Lemma 6.1.

Suppose first \( \mathcal{H}_c(t,p) > 0 \). Since for each integer \( \ell \geq 1 \) every point of \( F^{-\ell}(0) \) is a pre-image of 0 by an iterate of \( f_c \), by Lemma 6.1 the series (6.3) is bounded from below by,

\[
\sum_{\ell=1}^{+\infty} \sum_{y \in F^{-\ell}(0)} \exp(-(n_c(F^{\ell-1}_c(y)) + \cdots + m_c(y)p)|DF^\ell_c(y)|^{-t})
\]

\[
\geq \Delta_3^{-t} \sum_{\ell=1}^{+\infty} Z^C_{c,\ell}(t,p) = +\infty.
\]

To prove the last part of the lemma, let \( n_5 \) and \( C_5 > 1 \) be given by Lemma 6.4. We prove the desired assertion with \( n_6 = n_5 \). Suppose in addition we have \( n \geq n_5 \) and let \( t \geq 3 \) and \( p \geq P^C_c(t) - t \frac{1}{10} \log 2 \) be such that \( \mathcal{H}_c(t,p) < 0 \). By Proposition 6.2 we have \( p \geq p_c + \frac{1}{10} \log 2 \), so \( t \) and \( p \) satisfy the hypotheses of Lemma 6.4. Given an integer \( m \geq 1 \) and a point \( z \) in \( f^{-m}_c(0) \) denote by \( \ell(z) \) the number of those \( j \) in \( \{0, \ldots, m - 1\} \) such that \( f^j_c(z) \) is in \( V_c \). In the case where \( z \) is not in \( V_c \), this point is in the domain of \( L_c \) and we have \( \ell(z) = 0 \) if and only \( L_c(z) = 0 \). Moreover, if \( z \) is not in \( V_c \) and \( \ell(z) \geq 1 \), then \( L_c(z) \) is in the domain of \( F^\ell_c(z) \) and \( F^\ell_c(z)(L_c(z)) = 0 \). So, if \( z \) is not in \( V_c \) we have in all the cases

\[
|DF^m_c(z)| = |DF^\ell_c(L_c(z))| \cdot |DL_c(z)|.
\]

Then Lemma 6.4 implies that the series (6.3) is bounded from above by

\[
L_{t,p}(0) + \sum_{\ell=1}^{+\infty} \sum_{y \in F^{-\ell}(0)} L_{t,p}(y) \exp(-(m_c(F^{\ell-1}_c(y)) + \cdots + m_c(y)p)|DF^\ell_c(y)|^{-t})
\]

\[
\leq C_5 \left( 1 + \sum_{\ell=0}^{+\infty} Z^C_{c,\ell}(t,p) \right) < +\infty.
\]
Proof of Proposition C. We prove the assertion for $f_c|_{I_c}$; the arguments apply without change to $f_c|_{I_c}$. Let $\Delta_3 > 1$ be given by Lemma 6.1 and $n_6$ by Lemma 6.5. Let $n \geq n_6$ be an integer and let $c$ by a parameter in $K_n$. We use that fact that for each $t > 0$ we have

$$P_c(t) = \limsup_{m \to +\infty} \frac{1}{m} \log \sum_{y \in f_c^{-m}(0)} |Df_c^m(y)|^{-t},$$

see for example [Urb03] or [PRLS04].

Fix $t \geq 3$. We use the fact that the function $p \mapsto \mathcal{P}_c(t, p)$ is strictly decreasing where it is finite, see §6.2. In particular, for each $p$ satisfying $p < p_0 := \inf\{p : \mathcal{P}_c(t, p) \leq 0\}$ we have $\mathcal{P}_c(t, p) > 0$. Lemma 6.5 implies that for such $p$ the series (6.3) diverges and by (6.4) we have $P_c(t) \geq p > p_0$. To prove the reverse inequality, suppose by contradiction $p_0 < P_c(t)$ and let $p$ be in the interval $(p_0, P_c(t))$ satisfying $p \geq P_c(t) - t \frac{1}{2} \log 2$. Then $\mathcal{P}_c(t, p) < 0$ and by Lemma 6.5 the series (6.3) converges. Then (6.4) implies $P_c(t) \leq p$ and we obtain a contradiction that completes the proof of the proposition.

7. Estimating the Geometric Pressure Function

The purpose of this section is to prove the following proposition. The proof of Proposition A, at the end of this section, is based on this proposition, together with Propositions C and 6.2.

Recall that for a real parameter $c$,

$$\chi_{\text{crit}}(c) = \liminf_{m \to +\infty} \frac{1}{m} \log |Df_c^m(c)|.$$

Proposition D. There are $n_7 \geq 4$ and $C_6 > 1$ such that for every integer $n \geq n_7$ and every parameter $c$ in $K_n$ the following properties hold for each $t \geq 3$:

1. For $p$ in $[-\chi_{\text{crit}}(c)/2, 0)$ satisfying

$$\sum_{k=0}^{+\infty} \exp(-(n + 3k)p)|Df_c^{n+3k}(c)|^{-t/2} \geq C_6^t,$$

we have $\mathcal{P}_c(t, p) > 0$ and $P_c(t) \geq p$. If in addition the sum above is finite, then $\mathcal{P}_c(t, p)$ is finite and $P_c(t) > p$.

2. For $p \geq -\chi_{\text{crit}}(c)/2$ satisfying

$$\sum_{k=0}^{+\infty} \exp(-(n + 3k)p)|Df_c^{n+3k}(c)|^{-t/2} \leq C_6^{-t},$$

we have $\mathcal{P}_c(t, p) < 0$ and $P_c(t) \leq p$.

3. For $p \geq -\chi_{\text{crit}}(c)/2$ satisfying

$$\sum_{k=0}^{+\infty} k \cdot \exp(-(n + 3k)p)|Df_c^{n+3k}(c)|^{-t/2} < +\infty,$$

we have

$$\sum_{W \in \mathcal{B}_c} m_c(W) \cdot \exp(-m_c(W)p) \sup_{z \in W} |DF_c(z)|^{-t} \leq +\infty.$$
The proof of Proposition D is given after Lemma 7.1, below, which is used in the proof. The proof of Proposition A is given after the proof of Proposition D.

Let $n \geq 4$ be an integer and $c$ a parameter in $K_n$. Since the critical point $z = 0$ does not belong to $D_c$ (cf. Lemma 4.2), for each integer $\ell \geq 1$, each connected component of $D_c$ intersecting $P_{c,\ell}(0)$ is contained in $P_{c,\ell}(0)$. We define the level of a connected component $W$ of $D_c$ as the largest integer $k \geq 0$ such that $W$ is contained in $P_{c,n+3k+1}(0)$. Given an integer $k \geq 0$ denote by $D_{c,k}$ the collection of all connected components of $D_c$ of level $k$; we have $D_c = \bigcup_{k=0}^{\infty} D_{c,k}$.

**Lemma 7.1.** There is $C_7 > 0$ such that for each integer $n \geq 5$, each parameter $c$ in $K_n$, each integer $k \geq 0$, and each pair of real numbers $t > 0$ and $p$, we have

\[
\sum_{W \in D_{c,k}} \exp(-m_c(W)p) \sup_{z \in W} |DF_c(z)|^{-t} \leq 2C_7^t \exp(-(n+3k+1)p)|DF_c^{n+3k}(c)|^{-t/2} \cdot \left(1 + \sum_{w \in L_c^{-1}(0) \text{ in } P_{c,1}(0)} \exp(-m_c(w)p)|DL_c(w)|^{-t}\right).
\]

**Proof.** Let $\Delta_2$, $C_3$, and $\Delta_3$ be the constants given by Lemmas 4.3, 5.4 and 6.1, respectively.

Fix an integer $n \geq 4$, a parameter $c$ in $K_n$, and an integer $k \geq 0$. Note that there are precisely 2 elements $W'$ of $D_{c,k}$ such that $m_c(W') = n + 3k + 1$; denote them by $W_0$ and $W_0'$. Indeed, these sets are the connected components of the pre-image under $f_c$ of the set $(f_c^{n+3k}|_{P_{c,n+3k}(c)})^{-1}(V_c)$. For a connected component $W$ of $D_c$ of level $k$ denote by $z_W$ the unique point in $W$ such that $F_c(z_W) = 0$. If $W$ is different from $W_0$ and $W_0'$, then $z' := f_c^{n+3k+1}(z_W)$ is different from 0 and it is in the domain of definition $D_c'$ of the first landing map $L_c$ to $V_c$. So, denoting by $W'$ the connected component of $D_c'$ containing $z'$, there is a unique point $w_W$ in $W'$ such that $L_c(w_W) = 0$ and we have

\[m_c(W) = n + 3k + 1 + m_c(w_W).\]

Since $f_c^n$ maps $V_{c,n} = P_{c,n+1}(c)$ biholomorphically to $P_{c,1}(0)$ and $f_c^n(c) \in \Lambda_c$, it follows that $f_c^{n+3k}$ maps $P_{c,n+3k+1}(c)$ biholomorphically to $P_{c,1}(0)$; so the distortion of $f_c^{n+3k}$ on $P_{c,n+3k+1}(c)$ is bounded by $\Delta_2$ (Lemmas 4.3) and for each point $y$ in $P_{c,n+3k+1}(c)$ we have

\[|DF_c^{n+3k}(y)| \geq \Delta_2^{-1}|DF_c^{n+3k}(c)|.\]

On the other hand, by part 1 of Lemma 5.4 with $x = z_W$ and $q = n + 3k + 1$, we have

\[|DF_c^{n+3k+1}(z_W)| \geq C_3^{-1}|DF_c^{n+3k}(f_c(z_W))|^{1/2} \geq C_3^{-1}\Delta_2^{-1/2}|DF_c^{n+3k}(c)|^{1/2}.\]

Together with the inequality,

\[|DF_c^{m_c(w_W)}(f_c^{n+3k+1}(z_W))| \geq \Delta_2^{-1}|DL_c(w_W)|\]

given by Lemmas 4.3 and 5.1, this implies that if we put $C_0 := C_3^{-1}\Delta_2^{-3/2}$, then

\[|DF_c(z_W)| = |DF_c^{m_c(w_W)}(z_W)| \geq C_0|DF_c^{n+3k}(c)|^{1/2}|DL_c(w_W)|.\]
Since the distortion of $F_c|_W$ is bounded by $\Delta_3$ (Lemma 6.1) for each $p, t > 0$ we have,

$$(7.1) \quad \exp(-m_c(W)p) \sup_{z \in W} |DF_c(z)|^{-t}$$

$$\leq \Delta_3^t c_0^{-t} \exp(-(n + 3k + 1)p)|DF_{c,n+3k}(c)|^{-t/2} \cdot \exp(-m_c(w_N)p)|DL_c(w_N)|^{-t}.$$

To prove the desired inequality, observe that for each point $w$ of $L_{c, \tt 1}(0)$ in $P_{c,1}(0)$ there are precisely 2 connected components $W$ of $D_c$ in $\Omega_{c,k}$ such that $w_W = w$; in fact for each such $W$ the set $f_c(W)$ is uniquely determined as the pre-image by the univalent map $f_c^{n+3k}|_{P_{c,n+3k}(c)}$ of the connected component of $D_c$ containing $w_W$. Thus the first inequality of the lemma follows from (7.1) with constant $C_7 = \Delta_3 c_0^{-1}$.

**Proof of Proposition D.** Let $C_2$ and $n_3$ be given by Lemma 5.3 with $m_1 = 4$ and $\varepsilon = \frac{1}{2}$, let $n_4$ be given by Proposition C, let $n_5$ and $C_5 > 0$ be given by Lemma 6.4 and let $C_4$ and $C_5$ be given by Lemmas 6.3 and 7.1, respectively. We prove the proposition with $n_7 := \max\{n_3, n_4, n_5\}$. To do this, fix an integer $n \geq n_7$, a parameter $c$ in $K_n$ and $t \geq 3$.

To prove part 1, for each integer $k \geq 0$ let $W_k$ be the component of $W$ of $D_c$ given Lemma 6.3. Then for each $p < 0$ we have,

$$\sum_{W \in D_k} \exp(-m_c(W)p) \inf_{z \in W} |DF_c(z)|^{-t} \geq \sum_{k=0}^{+\infty} \exp(-m_c(W_k)p) \inf_{z \in W_k} |DF_c(z)|^{-t} \geq C_4^{-t} \sum_{k=0}^{+\infty} \exp(-(n + 3k + 3)p)|DF_{c,n+3k}(c)|^{-t/2} \geq C_4^{-t} \sum_{k=0}^{+\infty} \exp(-(n + 3k)p)|DF_{c,n+3k}(c)|^{-t/2}.$$

This shows that, if the sum

$$(7.2) \quad \sum_{k=0}^{+\infty} \exp(-(n + 3k)p)|DF_{c,n+3k}(c)|^{-t/2}$$

is greater than or equal to $C_4^t$, then $\mathcal{P}_{c,1}(t, p) > 0$ and by Proposition C we have $P_c^R(t) \geq p$. This proves the first part of part 1 with $C_6 = C_4$. To complete the proof of part 1, suppose (7.2) is finite and greater than or equal to $(2C_4)^t$. Then there is $p' > p$ such that (7.2) with $p$ replaced by $p'$ is greater than or equal to $C_4^t$. As shown above, this implies $P_c^R(t) \geq p' > p$. On the other hand, by Lemmas 6.4 and 7.1 the sum

$$\sum_{W \in D_{c,k}} \exp(-m_c(W)p) \sup_{z \in W} |DF_c(z)|^{-t}$$

is finite, so $\mathcal{P}_{c,1}(t, p)$ is also finite. This completes the proof of part 1 with $C_6 = 2C_4$.

To prove part 2, let $p \geq -t\chi_{\text{crit}}(c)/2$ be given. By Lemma 5.3 with $z = f_c^m(c)$ we have

$$(7.3) \quad \chi_{\text{crit}}(c) = \liminf_{m \to +\infty} \frac{1}{m} \log |DF_c^m(c)| \leq \frac{\delta}{\delta} \log 2.$$
Thus \( p \geq -t_3^2 \log 2 \) and therefore \( 2 \exp(-p) < 2^t \). So, if we put \( A_1 = 2C_7C_5 \), then by Lemmas 6.4 and 7.1, we have for each integer \( k \geq 0 \),

(7.4) \[
\sum_{W \in D_{c,k}} \exp(-m_c(W)p) \sup_{z \in W} |DF_c(z)|^{-t} < A_1^t \exp(-(n + 3k)p) |DF_c^{n+3k}(c)|^{-t/2}.
\]

This proves that, if (7.2) is less than or equal to \( A_1^{-t} \), then \( \mathcal{P}_c(t, p) < 0 \) and by Proposition C we have \( P_c^C(t) \leq p \). This proves part 2 of the proposition with \( C_6 = A_1 \).

To prove part 3, let \( p \geq -\chi_{\text{crit}}(c)/2 \) be given and observe that by Lemma 6.4 and by Lemma 7.1 with \( k = 0 \), the sum

\[
\sum_{W \in D_{c,0}} \exp(-m_c(W)p) \sup_{z \in W} |DF_c(z)|^{-t}
\]

is finite. Let \( A_2 > 0 \) be a constant such that for every pair of integers \( k \geq 1 \) and \( m \geq 3k + 1 \), we have

\[
m \leq A_2k2^{t(m-3k)/20}.
\]

By (7.3) we have

\[
p' := p - t \frac{1}{20} \log 2 \geq -t \frac{13}{20} \log 2 \quad \text{and} \quad 2 \exp(-p') < 2^t.
\]

Therefore for each integer \( k \geq 1 \), Lemmas 6.4 and 7.1 imply that we have (7.4) with \( p \) replaced by \( p' \), so

\[
\sum_{W \in D_{c,k}} m_c(W) \cdot \exp(-m_c(W)p) \sup_{z \in W} |DF_c(z)|^{-t}
\]

\[
\leq \sum_{W \in D_{c,k}} A_2k2^{t(m_c(W)-3k)/20} \exp(-m_c(W)p) \sup_{z \in W} |DF_c(z)|^{-t}
\]

\[
= A_2k2^{-3kt/20} \sum_{W \in D_{c,k}} \exp(-m_c(W)p') \sup_{z \in W} |DF_c(z)|^{-t}
\]

\[
\leq \left(A_2A_1^t2^{nt/20}\right) k \cdot \exp(-(n + 3k)p) |DF_c^{n+3k}(c)|^{-t/2}.
\]

Summing over \( k \geq 0 \) we obtain the desired assertion.

\[\square\]

**Proof of Proposition A.** We give the proof for \( f_c |_{\mathcal{L}_z} \); the proof for \( f_c |_{\mathcal{L}_f} \) is analogous. Let \( n_\gamma \) be given by Proposition C and let \( n_7 \) and \( C_6 \) be given by Proposition D. Put

\[
n_0 := \max\{5, n_4, n_\gamma\} \quad \text{and} \quad C_0 := C_6
\]

and let \( n \geq n_0 \) be an integer and \( c \) a parameter in \( \mathcal{K}_n \) for which the hypotheses of the proposition are satisfied.

The first hypothesis of the proposition together with part 2 of Proposition D with \( p = -\chi_{\text{crit}}(c)/2 \) imply that for every sufficiently large \( t > 0 \) we have

\[
P_c^C(t) \leq -t \chi_{\text{crit}}(c)/2.
\]

From Proposition 6.2 we deduce that for such \( t \) we have equality. The second hypothesis of the proposition together with part 1 of Proposition D with \( t = t_0 \) and \( p = -t_0 \chi_{\text{crit}}(c)/2 \), imply that \( f_c \) has a phase transition at some \( t_\ast > t_0 \) satisfying

\[
P_c^C(t_\ast) = -t_\ast \chi_{\text{crit}}(c)/2 < 0.
\]
This proves the first part of the proposition.

To prove the second part of the proposition, we first prove that there exists an equilibrium state of \( f_c \) for the potential \(-t_* \log |Df_c|\). Our additional hypothesis together with part 3 of Proposition D with \( t = t_* \) and \( p = -t_* \chi_{\text{crit}}(c)/2 \), imply that

\[
\sum_{W \in D_c} m_c(W) \exp(m_c(W)t_* \chi_{\text{crit}}(c)/2) \sup_{z \in W} |DF_c(z)|^{-t_*} < +\infty.
\]

The rest of the argument is now standard; we refer to [PRL11, §4] for precisions, see also Remark 7.2 below. Since \( \mathcal{P}_c^\infty(t_*, -t_* \chi_{\text{crit}}(c)/2) = 0 \), there is a \((t_*, -t_* \chi_{\text{crit}}(c)/2)\)-conformal measure \( \mu \) for \( f_c \) that assigns positive measure to the maximal invariant set of \( F_c \), see [PRL11, Theorem A in §4 and Proposition 4.3]. Standard considerations imply that there is an invariant probability measure \( \rho \) for \( F_c \) that is absolutely continuous with respect to \( \mu \). Thus (7.5) together with the bounded distortion property of \( F_c \) (Lemma 6.1) imply that the sum \( \sum_{W \in D_c} m_c(W) \rho(W) \) is finite. Therefore the measure

\[
\tilde{\rho} := \sum_{W \in D_c} \sum_{j=0}^{m_c(W)-1} (f_c^j)_*(\rho|W)
\]

is finite. This measure is invariant by \( f_c \) and it is absolutely continuous with respect to \( \mu \). To prove that the probability measure proportional to \( \tilde{\rho} \) is an equilibrium state of \( f_c \) for the potential \(-t \log |Df_c|\), first remark that \( \rho \) is an equilibrium state of \( F_c \) for the potential \(-t \log |Df_c| + (t_*) \chi_{\text{crit}}(c)/2 \) and that the measure theoretic entropy of this measure is strictly positive, see for example [MU03]. Then the generalized Abramov formula [Zwe05, Proposition 5.1] implies that the measure theoretic entropy of \( \tilde{\rho} \) is strictly positive and that the probability measure proportional to \( \tilde{\rho} \) is an equilibrium state of \( f_c \) for the potential \(-t \log |Df_c|\). That this measure is exact, and hence ergodic and mixing, is shown for example in [You99]. Finally, the uniqueness of the equilibrium state is given by [Dob08, Theorem 8] and by Ruelle’s inequality.

The non-differentiability of \( P_c^\infty \) at \( t = t_* \) follows from the existence of an equilibrium state of \( f_c \) for the potential \(-t \log |Df_c|\), see [IRRL11, Corollary 1.3].

\[\square\]

**Remark 7.2.** For completeness we show that for a parameter \( c \) as in the proof of Proposition A we have \( \chi^R_{\text{inf}}(c) = \chi_{\text{crit}}(c)/2 \) and

\[
\chi^C_{\text{inf}}(c) := \inf \left\{ \int \log |Df_c|d\mu \mid \mu \in \mathcal{M}_c^c \right\} = \chi_{\text{crit}}(c)/2,
\]

although this is not needed in the proof. For each invariant probability measure \( \mu \) of \( f_c \) supported on \( J_c \) and every sufficiently large \( t > 0 \) we have

\[
\int \log |Df_c|d\mu \geq (-P_c^\infty(t) + h_\mu(f_c))/t = \chi_{\text{crit}}(c)/2 + h_\mu(f_c)/t \geq \chi_{\text{crit}}(c)/2.
\]

This proves \( \chi^C_{\text{inf}}(c) \geq \chi_{\text{crit}}(c)/2 \). Together with Proposition 6.2 and with the inequality \( \chi^R_{\text{inf}}(c) \geq \chi^C_{\text{inf}}(c) \), this implies \( \chi^R_{\text{inf}}(c) = \chi^C_{\text{inf}}(c) = \chi_{\text{crit}}(c)/2 \).

**Appendix A. Multipliers of periodic orbits of period 3**

This appendix is devoted to prove the following lemma, used in §4.2. The functions \( p \) and \( \overline{p} \) appearing in the following lemma are defined in §4.2.
Lemma A.1. We have,
\[
\frac{\partial}{\partial c} |Df^3_c(p(c))|_{c=-2} > \frac{\partial}{\partial c} |Df^3_c(\tilde{p}(c))|_{c=-2}.
\]

Proof. Notice that for \(c = -2\)
\[
\mathcal{O} := \{2\cos(2\pi/7), 2\cos(4\pi/7), 2\cos(6\pi/7)\}
\]
and
\[
\tilde{\mathcal{O}} := \{2\cos(2\pi/9), 2\cos(4\pi/9), 2\cos(8\pi/9)\}
\]
are the only periodic orbits of minimal period 3 of \(f_{-2}\). Since,
\[
P_{-2,1}(0) \cap \mathbb{R} = [\alpha(-2), \tilde{\alpha}(-2)] = [-1, 1],
\]
it follows that \(x = 2\cos(4\pi/7)\) and \(x = 2\cos(4\pi/9)\) are the only periodic points of period 3 of \(f_{-2}\) in \(P_{-2,1}(0)\). On the other hand, the inequalities
\[
2\cos(4\pi/7) < 0 < 2\cos(4\pi/9)
\]
imply that \(p(-2) = 2\cos(4\pi/7)\) and \(\tilde{p}(-2) = 2\cos(4\pi/9)\) and that
\[
Df^3_{-2}(p(-2)) > Df^3_{-2}(\tilde{p}(-2)).
\]
Since both functions \(p\) and \(\tilde{p}\) are real, the desired assertion is equivalent to,
\[
(A.1) \quad \frac{\partial}{\partial c} |Df^3_c(p(c))|_{c=-2} > - \frac{\partial}{\partial c} |Df^3_c(\tilde{p}(c))|_{c=-2}.
\]
Let \(\pi_0\) be either one of the functions \(p, f_c \circ p, f_c^2 \circ p, \tilde{p}, f_c \circ \tilde{p}\), or \(f_c^2 \circ \tilde{p}\) and put
\[
\pi_1(c) = f_c \circ \pi_0(c) \text{ and } \pi_2(c) = f_c \circ \pi_1(c).
\]
Then \(f_c(\pi_2(c)) = \pi_0(c)\) and a direct computation shows that
\[
\pi'_0 = \frac{-1 + 2\pi_2 + 4\pi_1 \pi_2}{8\pi_0 \pi_1 \pi_2 - 1}.
\]
Therefore, for each \(c\) in \(\mathcal{P}_3(-2)\) we have,
\[
(\pi'_0, \pi_1, \pi_2)(c)
\]
\[
= \frac{-\pi_1(c)\pi_2(c) + 2\pi_0(c)\pi_1(c) + 4\pi_0(c)\pi_2(c) - 2c\pi_1(c) - 4c\pi_0(c) - 4c\pi_2(c) + 4c^2}{8\pi_0(c)\pi_1(c)\pi_2(c) - 1}.
\]
Using the formula above and the formula above with \(\pi_0\) replaced by \(\pi_1\) and then by \(\pi_2\), we obtain
\[
\frac{\partial}{\partial c} Df^3_c(\pi_0(c)) = 8(\pi_0 \pi_1 \pi_2)'(c)
\]
\[
= -8 \frac{7(\pi_0(c)\pi_1(c) + \pi_1(c)\pi_2(c) + \pi_2(c)\pi_0(c)) - 10c(\pi_0(c) + \pi_1(c) + \pi_2(c)) + 12c^2}{8\pi_0(c)\pi_1(c)\pi_2(c) - 1}.
\]
Thus, if for each \(j\) in \(\{1, 2, 3\}\) we denote by \(\sigma_j\) (resp. \(\tilde{\sigma}_j\)) the elementary symmetric function of degree \(j\) in the elements of \(\mathcal{O}\) (resp. \(\tilde{\mathcal{O}}\)), then by the above equation with \(\pi_0 = p\) (resp. \(\pi_0 = \tilde{p}\)) and \(c = -2\) we obtain,
\[
\left(\text{resp. } \frac{\partial}{\partial c} Df^3_c(\tilde{p}(c))\right)_{c=-2} = -\frac{8\tilde{\sigma}_2 + 20\tilde{\sigma}_1 + 48}{8\tilde{\sigma}_3 - 1}.
\]
To calculate these numbers, for a given an integer $n \geq 2$ let $T_n$ be the $n$-th Chebyshev polynomial, so that for every real number $\theta$, we have

$$T_n(\cos(\theta)) = \cos(n\theta).$$

Notice that the zeros of the polynomial $T_4(x/2) - T_3(x/2)$ different from $x = 2$ are precisely the elements of $\mathcal{O}$. We thus have the identity

$$\frac{2T_4(x/2) - 2T_3(x/2)}{x - 2} = x^3 + x^2 - 2x - 1 = x^3 - \sigma_1 x^2 - \sigma_2 x - \sigma_3.$$

So $\sigma_1 = -1$, $\sigma_2 = -2$, $\sigma_3 = 1$ and by the above

$$\frac{\partial}{\partial c} Df^3_c(p(c)) \bigg|_{c=-2} = -16.$$

On the other hand, the zeros of the polynomial $T_5(x/2) - T_4(x/2)$ different from $x = 2$ and $x = -1$ are precisely the elements of $\tilde{\mathcal{O}}$. Therefore we have the identity

$$\frac{2T_5(x/2) - 2T_4(x/2)}{(x - 2)(x + 1)} = x^3 - 3x + 1 = x^3 - \tilde{\sigma}_1 x^2 + \tilde{\sigma}_2 x - \tilde{\sigma}_3.$$

So $\tilde{\sigma}_1 = 0$, $\tilde{\sigma}_2 = -3$, $\tilde{\sigma}_3 = -1$ and

$$\frac{\partial}{\partial c} Df^3_c(\tilde{p}(c)) \bigg|_{c=-2} = 24.$$

This proves (A.1) and completes the proof of the lemma.

△

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