MINIMIZING INTERSECTION POINTS OF CURVES UNDER VIRTUAL HOMOTOPY

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Abstract. An elementary stabilization of a collection of curves \( \mathcal{L} \) on an oriented surface \( M \) is a surgery that results in attaching a handle to \( M \) along two discs away from \( \mathcal{L} \). The inverse operation is called an elementary destabilization. A virtual homotopy is a composition of elementary stabilizations, destabilizations, and homotopies.

Recently, Cahn and the first author [5] showed that, for a pair of curves \( (L_1, L_2) \) on a surface \( M \), the minimal number of intersection points of two curves homotopic to \( (L_1, L_2) \) equals the number of terms in the Anderson-Mattes-Reshetikhin (AMR) Poisson bracket \([L_1, L_2]\). We show that a similar statement is true for a generalization of the AMR bracket to the case of pairs of curves considered up to virtual homotopy.

Cahn [3] showed that the number of terms of \( \mu(K) \), her modification of the Turaev cobracket of a curve \( K \) on a surface \( M \), determines the minimal number of self intersection points of a curve homotopic to \( K \). We show that a similar formula holds for a generalization of the Cahn cobracket to the case of curves considered up to virtual homotopy.

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1. Virtual Homotopy and Irreducible Curves

Let \( M \) be a closed oriented surface, possibly non-connected, and \( \mathcal{L} \) a collection of closed curves on \( M \). An elementary stabilization of \( \mathcal{L} \) is a surgery on \( M \) induced by cutting out from \( M \) two discs away from \( \mathcal{L} \), and attaching a handle to \( M \) along the resulting boundary components. The inverse operation is called an elementary destabilization. More precisely, let \( A \) be a simple connected closed curve on \( M \) in the complement to \( \mathcal{L} \). An elementary destabilization of \( \mathcal{L} \) along \( A \) consists of cutting \( M \) open along \( A \) and then capping the resulting boundary circles. A virtual homotopy [6] is a composition of elementary stabilizations, destabilizations, and homotopies. The virtual homotopy class of a collection \( \mathcal{L} \) is called a flat virtual link and is denoted \([\mathcal{L}]_v\).

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An elementary destabilization of a collection \( \mathcal{L} \) is *trivial* if it chops off a sphere containing no components of \( \mathcal{L} \). We say that \( \mathcal{L} \) is *irreducible* if it admits only trivial destabilizations.

Analogously, one defines a *virtual homotopy class of a collection of (ordered) wedges of circles* \( \sqcup(S^1 \vee S^1) \to M \) on a surface, as well as the notion of an irreducible collection of (ordered) wedges of circles.

Motivated by a result of Kuperberg \([9]\), Ilyutko, Manturov, and Nikonov \([8]\) proved a uniqueness result for irreducible representatives of a flat virtual link. Their theorem is stated for *flat virtual knots* (i.e., one component links) but the proof works without change for multi-component links. Theorem 1.1 can also be established by a similar argument to that used by the first and third author \([7]\) to prove the uniqueness result for virtual Legendrian links.

**Theorem 1.1** (c.f. Theorem 1.2 in \([8]\)). *Every flat virtual link contains a unique (up to homotopy and an orientation preserving automorphism of \( M \)) irreducible representative. The irreducible representative can be obtained from any representative by a composition of destabilizations and homotopies.*

An argument similar to the one in the proof of Theorem 1.1 establishes Theorem 1.2.

**Theorem 1.2.** *Every virtual homotopy class of a collection of (ordered) wedges of circles on a surface contains a unique (up to homotopy and an orientation preserving automorphism of \( M \)) irreducible representative. The irreducible representative can be obtained from any representative by a composition of destabilizations and homotopies.*

An important consequence of Theorem 1.1 is that if two irreducible collections of curves \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) on surfaces \( M_1 \) and \( M_2 \) are virtually homotopic, then there is a homeomorphism \( \varphi: M_1 \to M_2 \) such that \( \varphi \circ \mathcal{L}_1 \) is homotopic to \( \mathcal{L}_2 \). A similar consequence holds true for a collection of (ordered) wedges of circles.

### 2. The Andersen–Mattes–Reshetikhin Bracket

A wedge on a surface \( M \) is a continuous map \( S^1 \vee S^1 \to M \). Let \( W(M) \) denote the set of homotopy classes of wedges on a surface \( M \), and \( FW(M) \) the free abelian group generated by \( W(M) \).

Let \( \mathcal{L} \) be a closed curve that consists of a pair of oriented components \( L_1 \) and \( L_2 \) on a surface \( M \), intersecting transversely. Then, for any intersection point \( p \), we define a wedge \( \mathcal{L}_p \) by taking the wedge sum of \( L_1 \) and \( L_2 \) at \( p \). Define \( \text{sign}(p) \) to be 1 if the orientation of the surface agrees with the orientation given by the tangent vectors of \( L_1 \) and \( L_2 \) at \( p \), and \(-1\) otherwise. The *Andersen–Mattes–Reshetikhin (AMR) bracket* is an element of \( FW(M) \) given by the formal expression

\[
[L_1, L_2] = \sum_p \text{sign}(p)[\mathcal{L}_p],
\]
where \( p \) ranges over the intersection points of \( L_1 \) and \( L_2 \).

Let \( W_V(M) \) be the set of virtual homotopy classes of wedges realized on a surface \( M \), and \( FW_V(M) \) the free abelian group on \( W_V(M) \). We generalize the AMR bracket to an operation \( [L_1, L_2]_V \) acting on a virtual homotopy class \( [\mathcal{L}]_V \) (with components \( L_1 \) and \( L_2 \)) in a similar way, except that we replace \( [L_p] \) by \( [L_p]_V \). In particular, \( [L_1, L_2]_V \) is an element of \( FW_V(M) \). Note that the generalized AMR operation is defined on a virtual homotopy class of a pair of curves rather than on a pair of virtual homotopy classes.

Let \( x \) be an element of the free abelian group \( F(S) \). Then there is a reduced expression for \( x \),

\[
x = \sum_{i=1}^{n} c_i s_i,
\]

where \( c_i \in \mathbb{Z}, s_i \in S \), and \( s_i \neq s_j \) for \( i \neq j \), and we define the terms of \( x \) to be

\[
terms(x) = \sum_{i=1}^{n} |c_i|.
\]

**Lemma 2.1.** Suppose a representative of the virtual homotopy class of a pair of curves \((L_1, L_2)\) is irreducible. Then \( \text{terms}[L_1, L_2] = \text{terms}[L_1, L_2]_V \).

**Proof.** By Theorem 1.1 we may assume that the pair \( \mathcal{L} = (L_1, L_2) \) on \( M \) is irreducible. Let \( p \) be a point in the intersection \( L_1 \cap L_2 \), and let \( \mathcal{L}_p \) be the wedge of \( L_1 \) and \( L_2 \) at \( p \). Since every homotopy of the wedge \( \mathcal{L}_p \) defines a homotopy of the irreducible curve \( \mathcal{L} \), we deduce that \( \mathcal{L}_p \) is irreducible. Consequently the collection of wedges \( \mathcal{L} = \sqcup \mathcal{L}_q \) on \( M \), where \( q \) ranges over all intersection points in \( L_1 \cup L_2 \), is irreducible.

By the uniqueness of the irreducible representative of a collection of wedges of circles (see Theorem 1.2), any other irreducible representative of \( \mathcal{L} \) is obtained from \( \mathcal{L} \) by the composition of a homotopy of the collection of wedges and an orientation preserving homeomorphism of \( M \) (i.e., no stabilizations or destabilizations are necessary). Under an orientation preserving homeomorphism of \( M \), even though the value \( [L_1, L_2] \) may change, the value of \( \text{terms}[L_1, L_2] \) does not change. Thus we conclude that \( \text{terms}[L_1, L_2] = \text{terms}[L_1, L_2]_V \). \( \square \)

**Remark 2.2.** When the representative \((L_1, L_2)\) of a virtual flat link is not irreducible, the value \( \text{terms}[L_1, L_2] \) may differ from the value \( \text{terms}[L_1, L_2]_V \). For example, consider the two curves on the genus three surface shown in Figure 1. They intersect in two points and one can show that \( \text{terms}[L_1, L_2] = 2 \). However, if one destabilizes the surface by deleting the central handle, then the two intersection points can be killed by a homotopy. Hence \( \text{terms}[L_1, L_2]_V = 0 \).

We are now in position to prove the following theorem.
Theorem 2.3. Assume that $[(L_1, L_2)]_V \neq [(L_2, L_1)]_V$, i.e., one can not exchange the curves $L_1$ and $L_2$ by homotopies and stabilizations/destabilizations of the pair. Then the minimal number of points in the intersection $L_1 \cap L_2$ equals $\text{terms}[L_1, L_2]_V$.

Theorem 2.3 generalizes the result of Cahn and the first author [5] saying that, given two non-homotopic curves $(L_1, L_2)$ on a surface $M$, $\text{terms}[L_1, L_2]$ equals the minimal number of intersection points of any pair of curves homotopic to the pair $(L_1, L_2)$.

Proof. By Theorem 1.1, we may assume that the representative $(L_1, L_2)$ is irreducible. Let $m$ denote the number of intersection points in $L_1 \cap L_2$. By Lemma 2.1 the value $\text{terms}[L_1, L_2]$ coincides with the value $\text{terms}[L_1, L_2]_V$. On the other hand, we know that $\text{terms}[L_1, L_2] = m$.

Let $m_V$ be the minimal number of intersection points between $L_1$ and $L_2$ in the virtual homotopy class of $(L_1, L_2)$. We have $m_V \leq m = \text{terms}[L_1, L_2]_V$. On the other hand, trivially, we have $m_V \geq \text{terms}[L_1, L_2]_V$. Thus $m_V = \text{terms}[L_1, L_2]_V$.

Remark 2.4. Assume that two curves on a surface are not homotopic to powers of a third curve. For a surface of genus greater than zero, the number of intersection points of two such curves is minimal when the curves are closed geodesics with respect to a metric of constant non-positive sectional curvature (see, for example, [11]). In the case of two curves which are homotopic to powers of a third curve, the two closed geodesics will be powers of the same closed geodesic and one has to parallel shift one of the curves slightly to realize the minimal number of intersection points. Combining this with Theorem 2.3, we get that the minimal number of intersection points of a pair of curves in a given virtual homotopy class is obtained when the underlying surface is irreducible and the two curves are closed geodesics.

3. The Cahn Cobracket

A curve on a surface is primitive if it is not homotopic to a (nontrivial) power of another curve. Cahn [3] generalized the Turaev cobracket to a cobracket $\mu$ and proved that the minimal number of self intersection points of a curve $K$ equals...
\[ \frac{1}{2} \text{ terms } \mu([K]) + (n - 1), \] where \( K \) is homotopic to \((K')^n\) with \( K' \) primitive and \( n > 0 \).

In [3], Cahn further generalized \( \mu \) to the case of flat virtual knots and conjectured that the minimal number of self intersection points of \([K]_V\) would be given by an analogous formula. We define an alternate generalization \( \mu_V \) of \( \mu \) to flat virtual knots and prove that the formula for the minimal number of self intersections extends to \( \mu_V \).

Let \( K \) be an oriented curve on a surface \( M \). For a self intersection \( e \) of \( K \) and \( \varepsilon = \pm \), let \( K^\varepsilon_e \) be the ordered wedge of circles with labeling \( \varepsilon \) as depicted in Figure 2. We denote the virtual homotopy class of \( K^\varepsilon_e \) by \([K^\varepsilon_e]_V\).

A self intersection \( e \) of \( K \) is semi-trivial if, after smoothing \( K \) at \( e \), one of the resulting components intersects neither itself nor the other component. By convention, say \([K^\varepsilon_e]_V = 0 \) if \( e \) is semi-trivial.

Let \( W'_V(M) \) be the set of virtual homotopy classes of ordered wedges realized on \( M \), and \( FW'_V(M) \) the free abelian group generated by \( W'_V(M) \). For \( K \) a representative of a flat virtual knot \([K]_V\) on a surface \( M \), we define \( \mu_V([K]_V) \) to be an element of \( FW'_V(M) \) given by the formal expression

\[ \mu_V([K]_V) = \sum_e [K^+_e]_V - [K^-_e]_V, \]

where \( e \) ranges over the self intersections of \( K \).

Similar to the proof of Theorem 2.3, we establish the following result.

**Theorem 3.1.** Let \([K]_V\) be a flat virtual knot and \( K \) an irreducible representative that is homotopic to \((K')^n\), where \( K' \) is primitive and \( n > 0 \). Then the minimal number of self intersection points of \([K]_V\) equals \( \frac{1}{2} \) terms \( \mu_V([K]_V) + (n - 1) \).

**Remark 3.2.** Similar to Remark 2.4, the minimal number of self intersection points for flat virtual knots \([K]_V\) with primitive irreducible representatives is obtained when the representative \( K \) is irreducible and a closed geodesic.

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