A GLOBAL TORELLI THEOREM FOR CALABI-YAU MANIFOLDS

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ABSTRACT. We present a proof that the period map from the Teichmüller space of polarized and marked Calabi-Yau manifolds to the classifying space of polarized Hodge structures is an embedding. The proof is based on the constructions of holomorphic affine structure and global holomorphic affine flat coordinates on the Teichmüller space.

In this talk we study the global properties of the period map from the Teichmüller space of polarized and marked Calabi-Yau manifolds to the classifying space of polarized Hodge structures. Our method is based on a simple observation that there exists a natural holomorphic affine structure with global holomorphic affine flat coordinates on the Teichmüller space of Calabi-Yau manifolds. By using the very basic properties of simply connected holomorphic affine manifolds we can extend a natural holomorphic local affine embedding to a global holomorphic affine embedding of the Teichmüller space into a complex Euclidean space of the same dimension, which is explicitly given by the global holomorphic affine flat coordinates. From this the global Torelli theorem follows directly.

Although our method works for more general cases, for simplicity we will restrict our discussion to Calabi-Yau manifolds. More precisely a compact projective manifold $M$ of complex dimension $n$ with $n \geq 3$ is called Calabi-Yau in this paper, if it has a trivial canonical bundle and satisfies $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < n$. A polarized and marked Calabi-Yau manifold is a triple consisting of a Calabi-Yau manifold $M$, an ample line bundle $L$ over $M$ and a basis of integral homology group modulo torsion, $H_n(M, \mathbb{Z})/\text{Tor}.$

We will denote by $\mathcal{T}$ the Teichmüller space from the deformation of the complex structure on $M$. We will take $\mathcal{T}$ as the universal cover of the smooth moduli space $\mathcal{Z}_m$ constructed by Popp, Viehweg, and Szendroi, for example in Section 2 of [4]. The versal family $U \to \mathcal{T}$ of the polarized and marked Calabi-Yau manifolds is the pull-back of the versal family over $\mathcal{Z}_m$ constructed in [4]. Therefore $\mathcal{T}$ is a simply connected smooth complex manifold of complex dimension

$$\dim_{\mathbb{C}} \mathcal{T} = h^{n-1,1}(M) = N,$$

where $h^{n-1,1}(M) = \dim_{\mathbb{C}} H^{n-1,1}(M)$ with $H^{n-1,1}(M)$ the $(n-1,1)$ Dolbeault cohomology group of $M$.

Let $D$ be the classifying space of polarized Hodge structures of the weight $n$ primitive cohomology of $M$. The period map $\Phi : \mathcal{T} \to D$ assigns to each point in $\mathcal{T}$ the corresponding Hodge structure of the fiber. The main result of this paper is the proof of the following global Torelli theorem:

**Theorem 0.1.** The period map $\Phi : \mathcal{T} \to D$ is injective.

The main idea of our proof is the construction on the Teichmüller space $\mathcal{T}$ a holomorphic affine structure and the global holomorphic affine flat coordinates on $\mathcal{T}$. A
holomorphic affine structure on a complex manifold is a holomorphic coordinate cover with affine transition maps. By using the local Kuranishi deformation theory of Calabi-Yau manifolds, we introduce local holomorphic affine flat coordinate charts and define a holomorphic affine structure by verifying that the transition maps between any two holomorphic affine flat coordinate charts are holomorphic affine maps. The computation is based on the construction of the local canonical family of holomorphic \((n,0)\)-forms on the local Kuranishi family of Calabi-Yau manifolds. We call the coordinate cover of \(T\) given by this local holomorphic affine flat coordinate charts, the Kuranishi coordinate cover of the Teichmüller space.

It is known that the existence of a holomorphic affine structure is equivalent to the existence of a holomorphic torsion-free flat connection on the Teichmüller space. See for example page 215-219 in [2]. Then we use this holomorphic flat connection and the fact that \(T\) is simply connected to prove the existence of global holomorphic affine flat coordinates on \(T\), which gives the explicit embedding of \(T\) in \(\mathbb{C}^N\).

To be slightly more precise, let us take a point \(p \in T\), and denote by \(U_p\) the local holomorphic flat coordinate chart introduced in Section 3. We can identify the holomorphic tangent space \(T_{1,0}^p T\) to \(H^{n-1,1}(M_p) \cong \mathbb{C}^N\), where \(M_p\) is the fiber of the versal family \(U \to T\) at \(p\). In fact we use the Calabi-Yau metric to get a harmonic orthonormal basis of \(H^{n-1,1}(M_p)\), which gives us its natural identification to \(\mathbb{C}^N\), and we can also use this basis to construct the holomorphic affine flat coordinates on \(U_p\). This defines a natural local holomorphic affine map

\[
\rho_{U_p} : U_p \to T_{1,0}^p T \cong H^{n-1,1}(M_p) \cong \mathbb{C}^N.
\]

For \(q \in U_p\), under the above identifications \(\rho_{U_p}(q)\) is simply the \((n-1,1)\) component of the Hodge decomposition on \(M_p\) of the canonical holomorphic \((n,0)\) forms on \(M_q\). Through the above identifications, \(\rho_{U_p}\) is also the holomorphic affine flat coordinate map we introduced on \(U_p\). Since the Teichmüller space \(T\) is simply connected, a theorem in page 252-255 in [1] of Kobayashi and Nomizu tells us that this local holomorphic affine map can be extended to a global holomorphic affine map \(\rho_p : T \to T_{1,0}^p T \cong H^{n-1,1}(M_p)\). Then by using the global holomorphic affine flat coordinates, we prove that this holomorphic affine map is actually an embedding explicitly given by the global holomorphic affine flat coordinates, from which the global Torelli theorem, Theorem 0.1, follows immediately.

References

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A Global Torelli Theorem for Calabi-Yau Manifolds

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Main topics

I will present the proof of a global Torelli theorem for Calabi-Yau manifolds.

Plan of the lecture

- Introduction
- The period map
- Local geometry of Teichmüller space
- Holomorphic affine structure
- Global coordinates on Teichmüller space
- Holomorphic affine embedding of Teichmüller space in $\mathbb{C}^N$
- Proof of the global Torelli theorem.

The main idea is to patch the local properties of variation of Hodge structures and deformation of Calabi-Yau manifolds by using holomorphic affine structure.
Torelli problem has very long history, starting from 1914. For a compact Kähler manifold $M$, it asks whether the Hodge structure on its cohomology groups

$$H^r(M, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M)$$

can determine the complex structure of $M$.

- Andreotti proved global Torelli for Riemann surfaces, following Weil’s reformulation.
- Shafarevich and Piatetski-Shapiro, Looijenga, Burns-Rapoport proved global Torelli theorem for $K3$ surfaces.
- Voisin and Looijenga proved global Torelli theorem for cubic fourfolds.

There are many works on local, generic Torelli theorems.
Although our method works for compact projective manifolds with trivial canonical line bundle, we will focus on Calabi-Yau case.

- A compact projective manifold $M$ of complex dimension $n \geq 3$ is called Calabi-Yau, if it has a trivial canonical bundle and satisfies $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < n$.

- A pair $(M, L)$ consisting of a Calabi-Yau manifold $M$ of complex dimension $n$ and an ample line bundle $L$ over $M$ is called a polarized Calabi-Yau manifold.

- Let $\{\gamma_1, \cdots, \gamma_h\}$ be a basis of the integral homology group modulo torsion, $H_n(M, \mathbb{Z})/\text{Tor}$. The triple $(M, L, \{\gamma_1, \cdots, \gamma_h\})$ is called a polarized and marked Calabi-Yau manifold.
Let $M$ be a Calabi-Yau of complex dimension $n$. Still use $L$ to denote the first Chern class of $L$. It defines a map

$$L : H^n(M, \mathbb{Q}) \to H^{n+2}(M, \mathbb{Q})$$

given by $A \mapsto L \wedge A$ for any $A \in H^n(M, \mathbb{Q})$. Denote by $H^n_{pr}(M) = \ker(L)$ the primitive cohomology groups.

Let $H^{k,n-k}_{pr}(M) = H^{k,n-k}(M) \cap H^n_{pr}(M, \mathbb{C})$ and denote its dimension by $h^{k,n-k}$.

We have the induced Hodge decomposition

$$H^n_{pr}(M, \mathbb{C}) = H^n_{pr,0}(M) \oplus \cdots \oplus H^n_{pr,n}(M).$$
The Poincaré bilinear form $Q$ on $H^n_{pr}(M, \mathbb{Q})$ is defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_M u \wedge v$$

for any $d$-closed $n$-forms $u, v$ on $M$. $Q$ is non-degenerate and can be extended to $H^n_{pr}(M, \mathbb{C})$ bilinearly, and satisfies the Hodge-Riemann relations

$$Q \left( H^k_{pr}, n-k(M), H^l_{pr}, n-l(M) \right) = 0 \text{ unless } k + l = n,$$

and

$$\left( \sqrt{-1} \right)^{2k-n} Q(v, \bar{v}) > 0 \text{ for } v \in H^k_{pr}, n-k(M) \setminus \{0\}.$$
Hodge filtration

Let $f^k = \sum_{i=k}^{n} h^{i,n-i}$, and

$$F^k = F^k(M) = H_{pr}^{n,0}(M) \oplus \cdots \oplus H_{pr}^{k,n-k}(M)$$

from which we have the decreasing filtration

$$H_{pr}^n(M, \mathbb{C}) = F^0 \supset \cdots \supset F^n.$$

We know that

$$\dim_{\mathbb{C}} F^k = f^k,$$

(1)

$$H_{pr}^n(M, \mathbb{C}) = F^k \oplus \bar{F}^{n-k+1}$$

and

$$H_{pr}^{k,n-k}(M) = F^k \cap \bar{F}^{n-k}.$$
In term of the Hodge filtration \( F^n \subset \cdots \subset F^0 = H_{pr}^n(M, \mathbb{C}) \), the Hodge-Riemann relations can be written as

\[
Q \left( F^k, F^{n-k+1} \right) = 0
\]

and

\[
Q(Cv, \bar{v}) > 0 \text{ if } v \neq 0,
\]

where \( C \) is the Weil operator given by \( Cv = (\sqrt{-1})^{2k-n} v \) when \( v \in H_{pr}^{k,n-k}(M) \). The classifying space \( D \) for polarized Hodge structures is the space of all such Hodge filtrations

\[
D = \left\{ F^n \subset \cdots \subset F^0 = H_{pr}^n(X, \mathbb{C}) \mid (1), (2) \text{ and } (3) \text{ hold} \right\}.
\]
A basis of \((H_n(M, \mathbb{Z})/\text{Tor})/m(H_n(M, \mathbb{Z})/\text{Tor})\) is called a level \(m\) structure on the polarized Calabi-Yau manifold. We have the following theorem due to Popp, Viehweg and Szendroi.

**Theorem**

Let \(m \geq 3\) and \(M\) be polarized Calabi-Yau manifold with level \(m\) structure, then there exists a quasi-projective complex manifold \(Z_m\) with a versal family of Calabi-Yau manifolds,

\[ \mathcal{X}_{Z_m} \rightarrow Z_m, \]

containing \(M\) as a fiber, and polarized by an ample line bundle \(\mathcal{L}_{Z_m}\) on \(\mathcal{X}_{Z_m}\).
We define $\mathcal{T} = T_L(M)$ to be the universal cover of $Z_m$, and the family $U \to T_L(M)$ to be the pull-back family.

**Theorem**

The Teichmüller space $\mathcal{T}$ is a simply connected smooth complex manifold, and the family $U \to \mathcal{T}$ containing $M$ as a fiber, is local Kuranishi at each point of $\mathcal{T}$.

Local Kuranishi follows from unobstructedness of the deformation of Calabi-Yau manifolds. It means any local deformation of of $M_p$ for any $p \in \mathcal{T}$ can be induced from the family $U$. 
The period map from $\mathcal{T}$ to $D$ is defined by assigning each point $p \in \mathcal{T}$ the Hodge structure on $M_p$,

$$\Phi : \mathcal{T} \rightarrow D$$

with $\Phi(p) = \{F^n(M_p) \subset \cdots \subset F^0(M_p)\}$. Griffiths showed that $\Phi$ is a holomorphic map.

The main theorem of my talk:

**Theorem**

*The period map $\Phi$ constructed above is an embedding,*

$$\Phi : \mathcal{T} \hookrightarrow D.$$
For a point $p \in T$, we denote by $(M_p, L)$ the corresponding polarized and marked Calabi-Yau manifold as the fiber over $p$. Let $\omega_p \in L$ denote the Kähler form of the Calabi-Yau metric by Yau’s theorem.

Let $(U, z_1, \cdots, z_n)$ be the local coordinate chart, and $\Omega = f dz_1 \wedge \cdots \wedge dz_n$ be a smooth $(n, 0)$-form on $M$, $\phi = \sum_i \phi^i \frac{\partial}{\partial z_i} \in A^{0,1}(M, T^{1,0}M)$ be a Beltrami differential. We define

$$\phi \Omega = \sum_i (-1)^{i-1} f \phi^i \wedge dz_1 \wedge \cdots \wedge \hat{dz_i} \wedge \cdots \wedge dz_n.$$
Local deformation

The following lemma follows since $\Omega_p$ is parallel with respect to the Calabi-Yau metric. It also implies the local Torelli theorem for Calabi-Yau manifolds.

Lemma

Let $\Omega_p$ be a nowhere vanishing holomorphic $(n,0)$-form on $M_p$ such that

$$
\left(\frac{\sqrt{-1}}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} \Omega_p \wedge \bar{\Omega}_p = \omega_p^n.
$$

Then the map $\iota : A^{0,1}(M, T^{1,0}M) \rightarrow A^{n-1,1}(M)$ given by $\iota(\phi) = \phi \downarrow \Omega_p$ is an isometry with respect to the natural Hermitian inner product on both spaces induced by $\omega_p$. Furthermore, $\iota$ preserves the Hodge decomposition.

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Let $\phi_1, \cdots, \phi_N \in \mathbb{H}^{0,1} \left( M_p, T_{M_p}^{1,0} \right)$ be a basis of Harmonic Beltrami differentials. Then there is a unique power series

$$\phi(\tau) = \sum_{i=1}^{N} \tau_i \phi_i + \sum_{|I| \geq 2} \tau^I \phi_I$$

which converges for $|\tau|$ small. Here $\phi_I \in A^{0,1} \left( M_p, T_{M_p}^{1,0} \right)$, and

$$\bar{\partial}_{M_p} \phi(\tau) = \frac{1}{2} [\phi(\tau), \phi(\tau)],$$

$$\bar{\partial}_{M_p}^* \phi(\tau) = 0,$$

$$\phi_I \lrcorner \Omega_p = \partial_{M_p} \psi_I,$$

for each $|I| \geq 2$ where $\psi_I \in A^{n-2,1}(M_p)$ are smooth $(n - 2, 1)$-forms.
We fix on $M_p$ a nowhere vanishing holomorphic $(n, 0)$-form $\Omega_p$ and an orthonormal basis $\{\phi_i\}_{i=1}^N$ of $\mathbb{H}^1(M_p, T^{1,0}M_p)$. Let $\phi(\tau)$ be the Beltrami differentials defining a local deformation of $M_p$ which we denote by $M_\tau$.

Locally write $\Omega_p = dz_1 \wedge \cdots \wedge dz_n$. Introduce a smooth form on $M_\tau$,

$$\Omega^c_p(\tau) = (dz_1 + \phi(dz_1)) \wedge \cdots \wedge (dz_n + \phi(dz_n)).$$

Here $\phi(dz_i) = \phi^i \in A^{0,1}(M)$. Todorov proved that for $\tau$ small, $\Omega^c_p(\tau)$ is a well-defined nowhere vanishing holomorphic $(n, 0)$-form on $M_\tau$ depending on $\tau$ holomorphically.
Local holomorphic flat coordinates

The following expansion in cohomology follows easily, and it is crucial for our construction of the holomorphic affine structure on the Teichmüller space. We will use \( \{\tau_1, \cdots, \tau_N\} \) as the local holomorphic affine flat coordinates.

**Theorem**

We have the following expansion for \( |\tau| \) small,

\[
[\Omega^c_p(\tau)] = [\Omega_p] + \sum_{i=1}^{N} \tau_i [\phi_i \Omega_p] + A(\tau),
\]

where \( A(\tau) = O(|\tau|^2) \in \bigoplus_{k=2}^{n} H^{n-k,k}(M) \) denotes terms of order at least 2 in \( \tau \).

This expansion can also be derived from the local Torelli theorem for Calabi-Yau manifold and the Griffiths transversality.
Let $M$ be a differentiable manifold of real dimension $n$, if there is a coordinate cover $\{U_i, \phi_i; i \in I\}$ of $M$ satisfying that, $\phi_{ik} = \phi_i \circ \phi_k^{-1}$ is a real affine transformation on $\mathbb{R}^n$, whenever $U_i \cap U_k$ is not empty, then we say that $\{U_i, \phi_i; i \in I\}$ is a real affine coordinate cover and defines a real affine structure on $M$.

For the definition of holomorphic affine manifold we just replace the "real" by "holomorphic" and "$\mathbb{R}^n$" by "$\mathbb{C}^n$" in the definition of real affine manifold.
A linear connection $\nabla$ on a complex manifold is called a holomorphic linear connection if the following two more conditions are satisfied,

- \((\nabla_Y X)^{1,0} = \nabla_Y X^{1,0};\)
- If $U$ and $W$ are complex holomorphic vector fields defined on an open set $O$, then $\nabla_W U$ is also holomorphic.

A holomorphic linear connection is called a holomorphic flat connection if the following two additional conditions are satisfied,

- $\nabla_X Y - \nabla_Y X = [X, Y];$
- $\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = 0.$

This means that a holomorphic flat connection is a holomorphic linear connection which is torsion-free and has zero curvature.
The following theorem is due to Matsushima, Vitter. Its real analogue was due to Auslander in 50s.

**Theorem**

*Let $M$ be a complex manifold, then there is a one-to-one correspondence between the set of all holomorphic affine structures on $M$ and the set of all holomorphic flat torsion-free connections on $M$. The geodesics of a holomorphic flat connection are straight lines in each coordinate chart of the holomorphic affine coordinate cover.*
Let $f : M \to M'$ be a holomorphic affine map between holomorphic affine manifolds $M$ and $M'$. The following theorem is essentially due to Kobayashi-Nomizu.

**Theorem**

Let $M$ be a connected, simply connected holomorphic affine manifold and $M'$ be a complete holomorphic affine manifold. Then every holomorphic affine map $f_U$ of a connected open subset $U$ of $M$ into $M'$ can be uniquely extended to a holomorphic affine map $f$ of $M$ into $M'$.

Note that an affine map maps geodesics to geodesics, so straight lines to straight lines.
Let $e = \{e_0, \cdots, e_m\}$ be a basis of $F^0$ in the given Hodge filtration. We call $e$ a Hodge basis adapted to this Hodge filtration, if

$$\text{Span}_\mathbb{C}\{e_0, \cdots, e_{m_k}\} = F^{n-k},$$

where $m_k = f^{n-k} - 1$, for each $0 \leq k \leq n$.

A square matrix $T = [T_{\alpha,\beta}]$, with each $T_{\alpha,\beta}$ a submatrix, is called block upper triangular if $T_{\alpha,\beta}$ is zero matrix whenever $\alpha > \beta$. To clarify the notations, we will use $T_{ij}$ to denote the entries of the matrix $T$. We use such matrix to transform Hodge basis to Hodge basis.
The following is a linear algebra lemma.

**Lemma**

We fix a base point $p \in T$ and a Hodge basis $\{c_0(p), \cdots, c_m(p)\}$ of the Hodge filtration of $M_p$. Then there is an open neighborhood $U_p$ of $p$, such that for any $q \in U_p$, there exists a block upper triangular matrix $\sigma(q)$ such that the basis

\[
\begin{bmatrix}
c_0(q) \\
\vdots \\
c_m(q)
\end{bmatrix} = \sigma(q) 
\begin{bmatrix}
c_0(p) \\
\vdots \\
c_m(p)
\end{bmatrix}
\]

is a Hodge basis of the Hodge filtration of $M_q$.

It follows from Gauss eliminations or from Griffiths transversality.
For each \( p \in \mathcal{T} \), we denote by \( C_p \) the set consisting of all of the orthonormal bases for \( \mathbb{H}^{0,1}(M_p, T^{1,0}M_p) \). Then for each pair \( (p, \Psi) \), where \( p \in \mathcal{T} \) and \( \Psi \in C_p \), we call the following coordinate chart

\[
U_{p,\Psi} = (U_p, \{\tau_1, \cdots \tau_N\}) = (U_p, \tau)
\]

a holomorphic affine flat coordinate chart around \( p \). It gives us a coordinate cover of \( \mathcal{T} \), which we call Kuranishi coordinate cover of \( \mathcal{T} \).
Our first key observation is that the Kuranishi coordinate cover gives us a natural holomorphic affine coordinate cover of $\mathcal{T}$, therefore a holomorphic affine structure on $\mathcal{T}$.

**Theorem**

The Kuranishi coordinate cover on $\mathcal{T}$ is a holomorphic affine coordinate cover, thus defines a global holomorphic affine structure on $\mathcal{T}$.

The proof of this theorem uses the Calabi-Yau property in a crucial way. It is so simple that we can explain the detail of the argument.
The key to prove the above theorem is the following

**Lemma**

Let $p, q \in T$ be two points. If $q \in U_p$, then the transition map between $(U_p, \tau)$ and $(U_q, t)$ is a holomorphic affine map.

We take $\Phi_p = \{\phi_1, \cdots, \phi_N\}$ and $\Psi_q = \{\psi_1, \cdots, \psi_N\}$ to be the orthonormal bases of $\mathbb{H}^{0,1}(M_p, T_{1,0}^1 M_p)$ and $\mathbb{H}^{0,1}(M_q, T_{1,0}^1 M_q)$ respectively. Let $\Omega_p$ and $\Omega_q$ be respectively the holomorphic $(n, 0)$-forms on $M_p$ and $M_q$. Then we know that

$\{\phi_1 \perp \Omega_p, \cdots, \phi_N \perp \Omega_p\}$ and $\{\psi_1 \perp \Omega_q, \cdots, \psi_N \perp \Omega_q\}$ are respectively the orthonormal bases for $\mathbb{H}^{n-1,1}(M_p)$ and $\mathbb{H}^{n-1,1}(M_q)$. 
Write

\[ \eta_0 = [\Omega_p], \quad \eta_i = [\phi_i \Omega_p] \quad \text{for} \ 1 \leq i \leq N; \]
\[ \alpha_0 = [\Omega_q], \quad \alpha_i = [\psi_j \Omega_q] \quad \text{for} \ 1 \leq j \leq N. \]

We complete them to Hodge bases for \( M_p \) and \( M_q \) respectively,

\[ \eta = (\eta_0, \eta_1, \cdots, \eta_N, \cdots, \eta_m)^T \quad \text{for} \ M_p; \]
\[ \alpha = (\alpha_0, \alpha_1, \cdots, \alpha_N, \cdots, \alpha_m)^T \quad \text{for} \ M_q. \]

For any point \( r \in U_p \cap U_q \), let us compute the transition map between the holomorphic affine flat coordinates at \( r \),
\( (\tau_1(r), \cdots, \tau_N(r)) \) and \( (t_1(r), \cdots, t_N(r)) \).
Let $[\Omega_r] = [\Omega^c_p(\tau(r))] \in H^{n,0}(M_r)$, where $[\Omega^c_p(\tau)]$ is the canonical section of the holomorphic $(n,0)$-classes around $p$. Then we have the following identities:

$$[\Omega_r] = \eta_0 + \sum_{i=1}^{N} \tau_i(r)\eta_i + \sum_{k>N} f_k\eta_k;$$  \hspace{1cm} (9)$$

$$[\Omega_r] = \lambda \left( \alpha_0 + \sum_{i=1}^{N} t_i(r)\alpha_i + \sum_{k>N} g_k\alpha_k \right).$$  \hspace{1cm} (10)$$

Here each $f_k(r)$ is the coefficient of $\eta_k$ in the decomposition of $[\Omega_r]$ according to the Hodge decomposition on $M_p$, and each $g_k(r)$ is the coefficient of $\alpha_k$ in the decomposition of $\lambda^{-1}[\Omega_r]$ according to the Hodge decomposition on $M_q$, for $N + 1 \leq k \leq m$. 
Holomorphic affine structure on the Teichmüller space

We know that there is a Hodge basis corresponding to the Hodge structure of $M_q$,

$$C(q) = (c_0(q), \cdots, c_N(q), \cdots, c_m(q)),$$

such that $c_i(q) = \sum_j \sigma_{ij} \eta_j$, and the matrix $\sigma = [\sigma^{\alpha,\beta}]$ is block upper triangular and nonsingular. Since we have two bases of the same Hodge filtration on $M_q$, $C(q)$ and $\alpha$. They are related by a nonsingular block diagonal transition matrix of the form

$$A = \begin{bmatrix}
A^{0,0} & 0 & \cdots & 0 \\
0 & A^{1,1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A^{n,n}
\end{bmatrix}$$

where each $A^{\alpha,\alpha}$ is an invertible $h_{n-\alpha,\alpha} \times h_{n-\alpha,\alpha}$ matrix, for $0 \leq \alpha \leq n$. 
Thus we have

\[ \alpha = A \cdot C(q) \text{ and } C(q) = \sigma \cdot \eta, \]

from which we get the transition matrix between the basis \( \alpha \) and basis \( \eta \), \( \alpha = A\sigma\eta \).

It is clear that \( A\sigma \) is still a nonsingular block upper triangular matrix of the form

\[
\begin{bmatrix}
A^{0,0} \cdot \sigma^{0,0} & * & \ldots & * \\
0 & A^{1,1} \cdot \sigma^{1,1} & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A^{n,n} \cdot \sigma^{n,n}
\end{bmatrix}
\]
Let us denote \( A \sigma \) by \( T = [T^\alpha,\beta]_{0 \leq \alpha,\beta \leq n} \). Note that each \( T^\alpha,\beta \) is an \( h^{n-\alpha,\alpha} \times h^{n-\beta,\beta} \) matrix for \( 0 \leq \alpha, \beta \leq n \), and each \( T^{\alpha,\alpha} \) is invertible.

In particular note that the \( 1 \times 1 \) matrix \( T^{0,0} = [T_{00}] \) and the \( N \times N \) matrix \( T^{1,1} \) in \( T \) are nonsingular, which are used in the computation of holomorphic affine flat coordinate transformation in the following.

Then we project \( [\Omega_r] \) to \( F_p^{n-1} = H^{n,0}(M_p) \oplus H^{n-1,1}(M_p) \). Let \( P_p^{n-1} \) denote the projection from \( F_p^0 \) to \( F_p^{n-1} \).
From (9) and (10) we see that

\[
\eta_0 + \sum_{i=1}^{N} \tau_i \eta_i = \lambda P_p^{n-1} \left( \alpha_0 + \sum_{i=1}^{N} \alpha_i t_i + \sum_{k=N+1}^{m} g_k(t) \alpha_k \right)
\]

\[
= \lambda P_p^{n-1} \left( \sum_{j=0}^{m} T_{0j} \eta_j + \sum_{i=1}^{N} t_i \sum_{j=0}^{m} T_{ij} \eta_j + \sum_{k=N+1}^{m} g_k(t) \sum_{j=0}^{m} T_{kj} \eta_j \right)
\]

\[
= \lambda \left( \sum_{j=0}^{N} T_{0j} \eta_j + \sum_{i=1}^{N} t_i \sum_{j=0}^{N} T_{ij} \eta_j \right)
\]

\[
= \lambda T_{00} \eta_0 + \sum_{j=1}^{N} (\lambda T_{0j} + \sum_{i=1}^{N} \lambda T_{ij} t_i) \eta_j.
\]

Here for brevity we have dropped the notation \( r \) in the coordinates \( \tau_i(r) \) and \( t_i(r) \) in the above formulas.
By comparing the coefficients of the basis \( \{\eta_1, \cdots, \eta_N\} \) on both sides of the above identity, we get \( 1 = T_{00}\lambda \) and
\[
\tau_j = \lambda T_{0j} + \sum_{i=1}^{N} \lambda T_{ij}t_i.
\]
Therefore for \( 1 \leq j \leq N \), we have the identity,
\[
\tau_j = T_{00}^{-1} T_{0j} + \sum_{i=1}^{N} T_{00}^{-1} T_{ij}t_i.
\]
In this identity, one notes that the transition matrix \( T \), while depending on \( p \) and \( q \), is independent of \( r \). Thus we have proved that the coordinate transformation is a holomorphic affine transformation.
Let \((U_p, \tau)\) and \((U_q, t)\) be two holomorphic affine flat coordinate charts in the Kuranishi coordinate cover. We need to show that the transition map between them is a holomorphic affine map. If \(p = q\), it is easy to see the transition map between these two coordinate charts is a holomorphic affine map by a bases change matrix. If \(p \neq q\), then we use a smooth curve \(\gamma(s)\) to connect \(p = \gamma(0)\) and \(q = \gamma(1)\). Then we can easily choose

\[0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1,\]

such that \(\gamma(s_{l+1}) \in U_{\gamma(s_l)}\) and the transition maps \(\phi_{l, l+1}\) between the holomorphic affine flat coordinates in \(U_{\gamma(s_l)}\) and \(U_{\gamma(s_{l+1})}\) are holomorphic affine maps.
Then the transition map between the holomorphic affine flat coordinates in $U_p$ and $U_q$, which is the following compositions of holomorphic affine maps,

$$\phi_{pq} = \phi_{0,1} \circ \cdots \circ \phi_{k-1,k}$$

is also a holomorphic affine map, whenever $U_p \cap U_q$ is not empty. This completes the proof of the existence of holomorphic affine flat structure on $T$. 
Holomorphic affine embedding of $\mathcal{T}$ in $\mathbb{C}^N$

Take a base point $p \in \mathcal{T}$, and choose a holomorphic flat coordinate chart $(U_p, \{\tau_1, \cdots, \tau_N\})$ around $p$. We can define a local holomorphic affine embedding

$$\rho_{U_p} : U_p \to \mathbb{C}^N \cong H^{n-1,1}(M_p)$$

by letting $\rho_{U_p}(q) = (\tau_1(q), \cdots, \tau_N(q))$ for any $q \in U_p$. Here recall that we make the identification $H^{n-1,1}(M_p) \cong \mathbb{C}^N$ by using the orthonormal basis $\{[\phi_1 \wedge \Omega_p], \cdots, [\phi_N \wedge \Omega_p]\}$ of $H^{n-1,1}(M_p)$. We can extend $\rho_{U_p}$ to a holomorphic affine map by the theorem of Kobayashi-Nomizu,

$$\rho_p : \mathcal{T} \to \mathbb{C}^N,$$

such that when restricted to $U_p$, one has $\rho_p|_{U_p} = \rho_{U_p}$. 

Kefeng Liu
A Global Torelli Theorem for Calabi-Yau Manifolds
We have the following theorem which gives us the global holomorphic affine flat coordinates and the global holomorphic embedding of $\mathcal{T}$ in $\mathbb{C}^N$.

**Theorem**

The holomorphic affine map $\rho_p : \mathcal{T} \rightarrow \mathbb{C}^N$ is an embedding.

Our proof is to first construct global holomorphic affine flat coordinates on $\mathcal{T}$ centered at $p$, then we show that the $\rho_p$ is actually given by the coordinate map with $\rho_p(p) = 0$. So the standard holomorphic affine flat coordinates on $\mathbb{C}^N$ induce the global affine flat coordinates on $\mathcal{T}$. 
Because \((U_p, \{\tau_1, \cdots, \tau_N\})\) is a holomorphic affine flat coordinate chart, the holomorphic vector fields \(\{\partial/\partial \tau_1, \cdots, \partial/\partial \tau_N\}\) form a local parallel frame of \(T^{1,0}U_p\) with respect to the holomorphic flat connection on \(\mathcal{T}\). We can extend this local parallel frame, by parallel transports of the holomorphic flat connection, to a global holomorphic parallel frame \(\{V_1, \cdots, V_N\}\) of \(T^{1,0}\mathcal{T}\). We denote by \(\{\theta_1, \cdots, \theta_N\}\) the dual frame of \(\{V_1, \cdots, V_N\}\) defined by the condition

\[\theta_i(V_j) = \delta_{ij}, \quad \text{for} \quad 1 \leq i, j \leq N.\]
By using that the holomorphic flat connection is torsion-free and that $\mathcal{T}$ is simply connected, we see that these holomorphic one forms $\{\theta_1, \cdots, \theta_N\}$ are closed, therefore exact. This implies that there are holomorphic functions $\{t_1, \cdots, t_N\}$ uniquely defined on $\mathcal{T}$ with the following properties,

$$dt_i = \theta_i \quad \text{and} \quad t_i(p) = 0,$$

for each $1 \leq i \leq N$. By a straightforward application of the Frobenius theorem or by using the fact that $\{\theta_1, \cdots, \theta_N\}$ is a global frame of $T^{*1,0}\mathcal{T}$, we deduce that $\{t_1, \cdots, t_N\}$ give us global holomorphic flat coordinates on $\mathcal{T}$. 
Holomorphic affine embedding of $\mathcal{T}$ in $\mathbb{C}^N$

Note that $dt_i = \theta_i = d\tau_i$ on $U_p$. By the uniqueness of \{t_1, \cdots, t_N\}, we have

$$t_i(q) = \tau_i(q) \quad \text{for} \quad q \in U_p.$$ (12)

Recall that we have constructed the global holomorphic affine map $\rho_p : \mathcal{T} \to \mathbb{C}^N$ by extending $\rho_{U_p}$. We will show that $\rho_p$ is actually given by the global holomorphic affine flat coordinates, that is $\rho_p(q) = (t_1(q), \cdots, t_N(q))$ for any $q \in \mathcal{T}$. 
Holomorphic affine embedding of $\mathcal{T}$ in $\mathbb{C}^N$

First by definition we know that when restricted to $U_p$, we have

$$\rho_i(q) = \tau_i(q) \quad \text{for} \quad q \in U_p$$

where we write $\rho_p(q) = (\rho_1(q), \cdots, \rho_N(q)) \in \mathbb{C}^N$.

Note that for each $i$, the coordinate function $t_i$ and $\rho_i$ are both globally defined holomorphic functions on $\mathcal{T}$. This implies that

$$\rho_i(q) = t_i(q) \quad \text{for} \quad q \in \mathcal{T}.$$ 

Therefore the holomorphic affine map $\rho_p : \mathcal{T} \to \mathbb{C}^N$, actually given by the global holomorphic affine flat coordinates on $\mathcal{T}$, is an embedding.
Now we are ready to prove our main theorem.

**Theorem**

The period map $\Phi : T \to D$ is an embedding.

Recall that for arbitrary $p \in T$, the previous theorem gives us a global holomorphic affine flat coordinate system $\{t_1, \cdots, t_N\}$ in $T$ centered at $p$, and an embedding $\rho_p : T \to \mathbb{C}^N$ such that for any $q \in T$,

$$\rho_p(q) = (t_1(q), \cdots, t_N(q)).$$

In particular we have $\rho_p(q) = 0$ if and only if $q = p$. 
For any \( q \in T \) different form \( p \), we connect \( p \) and \( q \) by a smooth curve \( \gamma(s) \) with \( \gamma(0) = p \) and \( \gamma(1) = q \). We can choose

\[
0 = s_0 < s_1 < \cdots < s_k = 1
\]

such that, \( \gamma(s_{\zeta+1}) \in U_{\gamma(s_{\zeta})} \), where \( U_{\gamma(s_{\zeta})} \) is the holomorphic affine flat coordinate chart at \( \gamma(s_{\zeta}) \). We know that there exists a nonsingular block upper triangular matrices \( T_{\zeta} \), such that

\[
C(s_{\zeta+1}) = T_{\zeta} \cdot C(s_{\zeta}),
\]

where \( C(s_{\zeta}) = (c_0(s_{\zeta}), \cdots, c_m(s_{\zeta})) \) is a Hodge basis of the Calabi-Yau manifold \( M_{\gamma(s_{\zeta})} \) at the point \( \gamma(s_{\zeta}) \), for \( 0 \leq \zeta \leq k \).
The matrices $T_\zeta$ all have the same pattern, and we have
$C(q) = T \cdot C(p)$ where $C(p) = C(s_0)$, $C(q) = C(s_k)$ with

$$T = \begin{bmatrix} T^{\alpha,\beta} \end{bmatrix}_{0 \leq \alpha, \beta \leq n} = \prod_{\zeta=0}^{k-1} T_\zeta.$$

Here $T$ is a nonsingular block upper triangular matrix of the same pattern as $T_\zeta$, that is, $T^{\alpha,\beta}$ is an $h^{n-\alpha,\alpha} \times h^{n-\beta,\beta}$ matrix for each $0 \leq \alpha, \beta \leq n$, and $T^{\alpha,\alpha}$ is nonsingular for each $\alpha$. 
Comparing the first term of the basis we have

\[ c_0(q) = T_{00}c_0(p) + \sum_{j=1}^{N} T_{0j}c_j(p) + \sum_{l=N+1}^{m} T_{0l}c_l(p), \]  

(14)

where \( c_0(q) \) is a generator of \( H^{n,0}(M_q) \). Here recall that \( N \) is the dimension of \( H^{n-1,1}(M_p) \).
Let \( \{t'_1, \cdots, t'_N\} \) be the global holomorphic affine flat coordinate system centered at \( q \). Then we get the following holomorphic affine transformation relating the two global holomorphic affine flat coordinates,

\[
t_j = T_{00}^{-1} T_{0j} + \sum_{i=1}^{N} T_{00}^{-1} T_{ij} t'_i
\]  

(15)

for \( 1 \leq j \leq N \). This is the transition map of the two global holomorphic affine flat coordinate systems \( \{t_1, \cdots, t_N\} \) and \( \{t'_1, \cdots, t'_N\} \). The global holomorphic affine flat coordinates centered at \( p \) is given by \( t_j(q) = T_{00}^{-1} T_{0j} \), since \( t'_i(q) = 0 \) for each \( 1 \leq i \leq N \).
Suppose there exists a point \( q \in \mathcal{T} \) which is different from \( p \), such that \( H^{n,0}(M_q) = H^{n,0}(M_p) \). Then (14) shows that \( T_{0j} = 0 \), for \( 1 \leq j \leq N \). We get

\[
t_j(q) = T_{00}^{-1} T_{0j} = 0 \quad \text{for} \quad 1 \leq j \leq N.
\]

But by the definition of global holomorphic affine flat coordinate chart \( \{t_1, \cdots, t_N\} \), this means \( p = q \), which contradicts to that \( q \) is different from \( p \). This completes the proof of the theorem.
Thank You!