Safe Adaptive Learning-based Control for Constrained Linear Quadratic Regulators with Regret Guarantees

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Abstract

We study the adaptive control of an unknown linear system with a quadratic cost function subject to safety constraints on both the states and actions. The challenges of this problem arise from the tension among safety, exploration, performance, and computation. To address these challenges, we propose a polynomial-time algorithm that guarantees feasibility and constraint satisfaction with high probability under proper conditions. Our algorithm is implemented on a single trajectory and does not require system restarts. Further, we analyze the regret of our learning algorithm compared to the optimal safe linear controller with known model information. The proposed algorithm can achieve a \( \tilde{O}(T^{2/3}) \) regret, where \( T \) is the number of stages and \( \tilde{O}(\cdot) \) absorbs some logarithmic terms of \( T \).

1 Introduction

Recent years have witnessed great interest in learning-based, and a lot of results have been developed for unconstrained systems (Fazel et al., 2018; Dean et al., 2018, 2019a; Mania et al. 2019; Simchowitz et al., 2018, 2020; Cohen et al., 2019). However, practical systems usually face constraints on the states and control inputs, especially in safety-critical applications (Campbell et al., 2010; Vasic and Billard, 2013). For example, drones are not supposed to visit certain locations to avoid collision and the thrusts of drones are usually bounded. Therefore, it is crucial to study safe learning-based control for constrained systems.

As a starting point, this paper considers a linear quadratic regulator (LQR) with linear constraints on the states and actions, i.e.,

\[
D_x x_t \leq d_x, \quad D_u u_t \leq d_u.
\]

(1)

We consider a linear system \( x_{t+1} = A_s x_t + B_s u_t + w_t \) with bounded system disturbances \( w_t \in \mathcal{W} = \{w : \|w\|_{\infty} \leq w_{\text{max}}\} \) and unknown model \((A_s, B_s)\). We aim to design an adaptive control algorithm to minimize the quadratic cost \( \mathbb{E}[x_t^\top Q x_t + u_t^\top R u_t] \) with safety guarantees during the learning process, i.e. satisfying all the constraints for any disturbances \( w_t \in \mathcal{W} \).

The constraints on LQR bring great difficulties even when the model is known. Unlike unconstrained LQR, which enjoys closed-form optimal policies (Lewis et al., 2012), there is no computationally efficient method to solve the optimal policy for constrained LQR (Rawlings and Mayne, 2009). Thus, most literature sacrifices optimality for computation efficiency by designing policies with certain structures, e.g. linear policies (Dean et al., 2019b; Li et al., 2020); piecewise-affine policies in robust model predictive control (RMPC) (Bemporad and Morari, 1999; Rawlings and Mayne, 2009), etc.

Therefore, when the model is unknown, a reasonable goal is to learn and achieve what can be obtained with perfect model information. In this paper, we adopt the optimal safe linear policy as our benchmark/target and leave the discussions on RMPC as future work.

1Efficient algorithms exist for some special cases, e.g. when \( w_t = 0 \), the optimal controller is piecewise-affine and can be computed as in Bemporad et al. (2002).
The current literature on learning the optimal safe linear policies adopts an offline/non-adaptive learning approach, which does not improve the policies until the learning terminates (Dean et al., 2019b). To improve the control performance during learning, adaptive/online learning-based control algorithms should be designed. However, though adaptive learning for unconstrained LQR can be designed by direct conversions from offline algorithms (see e.g., Simchowitz and Foster, 2020; Mania et al., 2019; Dean et al., 2018), it is much more challenging for the constrained case because direct conversions may cause infeasibility and/or constraint violation for single-trajectory adaptive learning as noted in Dean et al. (2019b).

Our contributions. In this paper, we propose a single-trajectory adaptive learning algorithm for constrained LQR with feasibility and constraint satisfaction guarantees. Our algorithm estimates the model with a least-square-estimator (LSE) and updates the policies with improved model estimations based on certainty-equivalence (CE) with robust constraint satisfaction against model uncertainties. To ensure safe policy updates, we propose a SafeTransit algorithm by extending the slow-variation trick in Li et al. (2020) for a known model to the case with model uncertainties and varying model estimations. Our algorithms can be implemented in polynomial time at each stage.

Further, we provide performance guarantees for our learning algorithm by discussing the regret compared with the optimal safe linear policy with perfect model information. We obtain a sublinear regret bound of order $\tilde{O}(T^{2/3})$. Interestingly, our regret bound also holds when compared against an RMPC algorithm (RMPC) proposed in Mayne et al. (2005). Discussions on more general regret benchmarks are left for the future.

Lastly, when developing our theoretical results, we provide a model estimation error bound for general and possible nonlinear policies. This is to handle the potential nonlinearity in our designed controllers when the model errors are non-negligible. Our model error bound extends the existing results for linear policies in Dean et al. (2019a,b) and can be useful by its own.

Related work. Constrained LQR with linear policies is studied in Dean et al. (2019b; Li et al., 2020; Dean et al., 2019b) consider an unknown model and propose an offline learning method with sample complexity guarantees. In contrast, Li et al. (2020) study online constrained LQR with a known model and adopt a slow-variation trick for safe policy updates. However, it remains open how to ensure safe adaptive control under model uncertainties.

Constrained LQR by model predictive control (MPC). MPC and its variants are popular methods for constrained control, e.g., RMPC designed for hard constraints (Mayne et al., 2005; Limon et al., 2010; Rawlings and Mayne, 2009), and stochastic MPC methods for soft constraints (Mesbah, 2016; Oldewurtel et al., 2008). With model uncertainties, robust adaptive MPC (RAMPC) algorithms are proposed to learn the model and updates the policies (Zhang and Shi, 2020; Bujarbaruah et al., 2019; Köhler et al., 2019; Lu et al., 2019). Most RAMPC algorithms guarantee recursive feasibility and constraint satisfaction but lack non-asymptotic performance guarantees compared with the known-model case. In contrast, there are some recent results on non-asymptotic regret bounds by sacrificing feasibility and/or constraint satisfaction, e.g., Wabersich and Zeilinger (2020) establish a regret bound for an adaptive MPC algorithm that requires restarting the system to some safe feasible state, Muthirayan et al. (2020) provides a regret bound for an adaptive algorithm without considering state constraints.

Learning-based unconstrained LQR enjoys rich literature, so we only review the most related papers below. Firstly, our algorithm is related with the CE-based adaptive control (Dean et al., 2018; Mania et al., 2019; Cohen et al., 2019; Simchowitz and Foster, 2020), and this approach is shown to be optimal for the unconstrained LQR (Mania et al., 2019; Simchowitz and Foster, 2020). Further, similar to Agarwal et al. (2019a,b); Plevrakis and Hazan (2020), we adopt the disturbance-action policies to approximate linear policies.

Safe reinforcement learning (RL). Safety in RL has different definitions (Mihatsch and Neuneier, 2002; Garcia and Fernández, 2015). This paper is related with RL with constraints, which enjoys a lot of research but has limited results on both safety and non-asymptotic optimality guarantees (Marvi and Kiumarsi, 2021; Leurent et al., 2020; Fisac et al., 2018; Garcia and Fernández, 2015; Cheng et al., 2019; Fulton and Platzer, 2018).
2 Problem formulation

We consider the following constrained LQR problem,

$$\min_{u_0, u_1, \ldots} \lim_{T \to +\infty} \frac{1}{T} \sum_{t=0}^{T-1} E[l(x_t, u_t)]$$

subject to

$$x_{t+1} = A x_t + B u_t + w_t, \quad \forall \, t \geq 0,$$

$$D_x x_t \leq d_x, \quad D_u u_t \leq d_u, \quad \forall \{w_k \in \mathbb{W} \}_{k \geq 0}.$$ 

where $$l(x, u) = x^\top Q x + u^\top R u, \quad Q \text{ and } R$$ are positive definite matrices, $$x_t \in \mathbb{R}^n$$ is the state with a given initial state $$x_0$$, $$u_t \in \mathbb{R}^m$$ is the action, and $$w_t$$ is the disturbance. The parameters $$D_x, d_x$$, and $$D_u, d_u$$ determine the constraint sets of the state and action respectively, where $$d_x \in \mathbb{R}^{k_x}, d_u \in \mathbb{R}^{k_u}$$. Further, the constraint sets on the state and action are assumed to be bounded with $$x_{\max} = \sup_{D_x x \leq d_x} ||x||_2$$ and $$u_{\max} = \sup_{D_u u \leq d_u} ||u||_2$$. Besides, denote $$\theta := (A, B)$$ and in for simplicity. The model parameters $$\theta_i$$ are unknown but other parameters are known.

An algorithm/controller is called ‘safe’ if its induced states and actions satisfy the constraints for all $$t$$ under any possible disturbances $$w_t \in \mathbb{W}$$, which is also called robust constraint satisfaction under disturbances $$w_t$$.

Notice that even with known model $$\theta$$, the optimal policy to problem (2) cannot be computed efficiently, but there are efficient methods to compute sub-optimal policies, e.g., optimal safe linear policies by quadratic programs (Dean et al., 2019b; Li et al., 2020) and piecewise affine policies by RMPC (Mayne et al., 2005; Rawlings and Mayne, 2009). In this paper, we set the optimal safe linear policy as our learning goal and leave RMPC for future. We aim to achieve our learning goal by designing safe adaptive learning-based control. Further, we consider single-trajectory learning, which is more challenging since the system cannot be restarted to ensure feasibility and constraint satisfaction.

For simplicity, we assume $$x_0 = 0$$. Our results can be generalized to $$x_0$$ in a small neighborhood around 0.

Regret and benchmark. Roughly speaking, we measure the performance of our adaptive learning algorithm by comparing with the optimal safe linear policy $$u_t = -K^* x_t$$ computed with perfect model information.

To formally define the performance metric, we first define a quantitative characterization of matrix stability as in e.g., (Agarwal et al., 2019a; Cohen et al., 2019).

Definition 1. For $$\kappa \geq 1, \gamma \in [0, 1]$$, a matrix $$A$$ is called $$(\kappa, \gamma)$$-stable if $$\|A^t\|_2 \leq \kappa (1-\gamma)^t, \forall \, t \geq 0$$.

Consider the following benchmark policy set:

$$\mathcal{K} = \{K : (A - B K) \text{ is } (\kappa, \gamma)\text{-stable}, \|K\|_2 \leq \kappa, D_x x^K + d_x, D_u u^K \leq d_u, \forall \{w_k \in \mathbb{W} \}_{k \geq 0} \},$$

where $$x^K_t, u^K_t$$ are generated by policy $$u_t = -K^* x_t$$.

For any safe learning algorithm/controller $$A$$, we measure its performance by ‘regret’ as defined below:

$$\text{Regret} = \sum_{t=0}^{T-1} l(x^A_t, u^A_t) - T \min_{K \in \mathcal{K}} J(K)$$

where $$x^A_t, u^A_t$$ are generated by the algorithm $$A$$ and $$J(K) = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=0}^{T-1} E[l(x^K_t, u^K_t)]$$.

Next, we provide and discuss the assumptions.

Assumptions. Firstly, though the model $$\theta$$ is not perfectly known, we assume there is some prior knowledge, which is captured by a bounded model uncertainty set $$\Theta_{ini}$$ that contains $$\theta$$. It is widely acknowledged that without such prior knowledge, hard constraint satisfaction is extremely difficult, if not impossible (Dean et al., 2019b). We also assume that $$\Theta_{ini}$$ is small enough so that there exists a universal linear controller $$u_t = -K_{stab, x_0}$$ to stabilize any system in $$\Theta_{ini}$$. This is a common assumption in constrained LQR with model uncertainties (Köhler et al., 2019; Lu et al., 2019) and $$K_{stab}$$ can be computed by, e.g., linear matrix inequalities (LMIs) (see Caverly and Forbes, 2019) as a review).

Assumption 1. There is a known model uncertainty set $$\Theta_{ini} = \{\theta : \|\theta - \hat{\theta}_{ini}\|_F \leq r_{ini}\}$$ for some $$0 < r_{ini} < +\infty$$.

Footnote 2 will provide more discussions on nonzero $$x_0$$ and generalization to a small $$x_0$$. Here, we discuss the implication of considering a small $$x_0$$. Remember that state 0 represents a desirable equilibrium point of the system. By considering $$x_0$$ close to 0, this paper focuses on how to safely optimize the performance around the equilibrium instead of safely driving a distant state back to the equilibrium. As an example of applications, this paper studies how to safely maintain a drone around a target in the air despite wind disturbances with minimum battery consumption, instead of flying the drone safely to the target from a distance. In practice, one can first apply existing algorithms such as (Mayne et al., 2005) to safely steer the system to around 0 and then apply our algorithm to achieve optimality and safety around 0.

Footnote 3 In some literature, e.g., (Agarwal et al., 2019a), this property is called $$(\sqrt{\kappa}, \gamma)$$-strong stability.

Footnote 4 The symmetry of $$\Theta_{ini}$$ is assumed for simplicity and not restrictive. We only need $$\Theta_{ini}$$ to be small and contain $$\theta$$. 

3
such that (i) $\theta_x \in \Theta_{ini}$, and (ii) there exist $\kappa \geq 1, \gamma \in [0, 1)$, and $K_{stab}$ such that for any $(A, B) \in \Theta_{ini}$, $A - BK_{stab}$ is $(\kappa, \gamma)$-stable.

Next, to further simplify the technical exposition, we impose a technical assumption $K_{stab} = 0$, which basically requires that $A$ is open-loop stable for any $A \in \Theta_{ini}$. This assumption can be removed by a standard pre-stabilizing procedure: consider inputs as $u_t = -K_{stab}x_t + v_t$ and focus on designing $v_t$ (see e.g., [Agarwal et al., 2019b]).

**Assumption 2** (Technical assumption). $K_{stab} = 0$.

Further, we need to assume a feasible linear policy exists for our constrained LQR [2], otherwise our regret benchmark is not well-defined. Here, we impose a slightly stronger assumption of strict feasibility to allow approximation errors in our control design. This assumption can also be verified by LMIs [Caverly and Forbes, 2019].

**Assumption 3.** There exists $K_F \in K$ and $\epsilon_{F,x} > 0, \epsilon_{F,u} > 0$ such that $D_x x_t^K_F \leq d_x - \epsilon_{F,x}I_{k_x}$ and $D_u w_t^K_F \leq d_u - \epsilon_{F,u}I_{k_u}$ for all $t \geq 0$ under all $w_t \in \mathbb{W}$.

Lastly, we impose assumptions on disturbance $w_t$. We define a certain anti-concentration property around 0 as in [Abeille and Lazaric, 2017], which essentially requires a random vector $X$ to have large enough probability at a distance from 0 on all directions.

**Definition 2** (Anti-concentration). A random vector $X \in \mathbb{R}^n$ satisfies $(s, p)$-anti-concentration for some $s > 0$, $p \in (0, 1)$ if $\mathbb{P}(X \geq s) \geq p$ for any $\|x\|_2 = 1$.

We assume anti-concentrated $w_t$ below to provide excitation for learning the model, which is crucial for our estimation error bound under general policies.

**Assumption 4.** $w_t \in \mathbb{W}$ is i.i.d., $\sigma^2_{sub}$-sub-Gaussian, zero mean, and $(s_w, p_w)$-anti-concentration.\(^5\)

# 3 Preliminaries

This section reviews the existing results in [Li et al., 2020] on how to approximate the optimal safe linear policy computationally efficiently and how to do safe policy updates when assuming a known model.

## 3.1 Approximation of Optimal Linear Polices with Disturbance-action Policies

When the model is known, a computationally efficient way to approximate the optimal linear policy for problem [2] is to rewrite the action as a linear combination of the history disturbances, i.e.,

$$u_t = \sum_{k=1}^{H} M[k]w_{t-k},$$

and solving the optimal policy matrices $M = \{M[1], \ldots, M[H]\}$ by a linearly constrained quadratic program. When the model is known, we can compute the disturbance by $w_{t} = x_{t+1} - A_x x_t - B_x u_t$. The policy [3] is called a disturbance-action policy (DAP) in [Agarwal et al., 2019b], [Li et al., 2020] and $H$ is called the policy’s memory length.\(^6\)

There are different ways to construct the linearly constrained quadratic program (see e.g., [Li et al., 2020] Dean et al., 2019b; Mesbah, 2016]). This paper adopts the method in [Li et al., 2020], which is briefly reviewed below.

Under DAP, the state can be approximated by $\tilde{x}_t$, which is an affine function of $M$ as below.

**Proposition 1** [Agarwal et al., 2019a]. Under a time-invariant policy $M$, we have $x_t = A^H x_{t-H} + \tilde{x}_t(M; \theta_x)$, where $\tilde{x}_t(M; \theta_x) = \sum_{k=1}^{2H} \Phi_k(M; \theta_x)w_{t-k}$ and $\Phi_k(M; \theta_x) = A_{x}^{-1} I_{k \leq H} + \sum_{k=1}^{H} A_{x}^{-1} B x M[k - i]_{i \leq k - i \leq H}$.

**Safe policy set.** [Li et al., 2020] reformulated the linear constraints on the states and actions to polytopic constraints on the policy parameters as follows.

$$g_i^x(M; \theta_x) \leq d_{x,i} - \epsilon_x, \forall 1 \leq i \leq k_x$$

\(^5\)Notice that by $W = \{w : \|w\| \leq \text{wmax}\}$, we have $\sigma^2_{sub} \leq \sqrt{n} \text{wmax}$.

\(^6\)[3] is also called finite-impulse response in [Dean et al., 2019b] and affine-disturbance policy in [Mesbah, 2016].
where \( g^u_j(M) \) and \( g^u_j(M) \) represents each line of the state and action constraints as follows:

\[
g^u_j(M; \theta_s) := \sup_{w_k \in W} \sum_{t=1}^{H} \left\| D_{x,t}^T \Phi^T_k(M; \theta_s) \right\|_1 w_{max}
\]

Notice that \( g^u_j(\cdot) \) is defined based on the approximate state \( \hat{x}_t \) in Proposition 1 so a constraint-tightening term \( \epsilon_x \geq \epsilon_H(H) = \| D_x \|_{\infty} \kappa_{x,\text{max}} (1 - \gamma)^H \) is introduced in \( (4) \) to account for the approximation error. \( g^u_j(\cdot) \) is defined on the actual action \( u_t \) so \( \epsilon_u = 0 \) here.

In summary, \( \text{Li et al. (2020)} \) construct a safe policy set \( \Omega(\theta_s, \epsilon_x, \epsilon_u) = \{ M \in \mathcal{M}_H : \epsilon_x \geq \epsilon_H(H), \epsilon_u \geq 0, \mathcal{M}_H = \{ M : \| M[k] \|_{\infty} \leq 2 \sqrt{\kappa_{x,\text{max}} (1 - \gamma)^k}, \forall 1 \leq k \leq H \} \) for technical simplicity without losing generality. Notice that set (6) is a polytope.

**Quadratic program (QP) for optimal safe DAP.** The optimal safe DAP \( M^* \) can be solved by QP below:

\[
\begin{align*}
\min_{M} f(M; \theta_s) \\
\text{s.t.} \quad \Omega(\theta_s, \epsilon_x, \epsilon_u) = \{ M \in \mathcal{M}_H : \epsilon_x = \epsilon_H(H), \epsilon_u = 0 \},
\end{align*}
\]

where \( f(M; \theta_s) = \mathbb{E}[l(\hat{x}_t(M, \theta_s), u)] \) is defined by the expected cost of the approximate state and the action, which is a quadratic convex function of \( M \). \( \text{Li et al. (2020)} \) show that the optimal safe DAP \( M^* \) approximates the optimal safe linear policy \( K^* \) in the sense of \( J(M^*) - J(K^*) \leq O(1/T) \) for large enough \( H \).

### 3.2 Safe Policy Updates with Known Models

Notice that the constraints in \( (4) \) can only guarantee safety when a feasible policy \( M \) is implemented in a time-invariant fashion. It is known that time-varying policies \( \{ M_t \}_{t \geq 0} \) may still violate the constraints even if each \( M_t \in \Omega(\theta_s, \epsilon_x, \epsilon_u) \) \( \text{Li et al. (2020)} \). Intuitively, this is because the approximate state \( \hat{x}_{t+1} \) is affected by not only \( M_t \) but also \( M_{t-1}, M_{t-2}, \ldots \), causing the safety constraints coupled across the history policies (see \( \text{Li et al. (2020)} \) for more details).

When the model is known, \( \text{Li et al. (2020)} \) tackle the coupled constraints by a *slow-variation* trick. Roughly, this trick indicates that safety of a slowly varying sequence of policies \( \{ M_t \} \) can be guaranteed if each \( M_t \) belongs to the set (6) with an additional constraint tightening term \( \epsilon_i \) to allow small policy variations.

**Lemma 1 (Slow variation trick with perfect model \( \text{Li et al. (2020)} \)).** Consider a slowly varying DAP sequence \( \{ M_t \}_{t \geq 0} \) with \( \| M_t - M_{t-1} \|_{\infty} \leq \Delta_M \), where \( \Delta_M \) is called the policy variation budget. \( \{ M_t \}_{t \geq 0} \) is safe to implement if \( M_t \in \Omega(\theta_s, \epsilon_x, \epsilon_u) \) for all \( t \geq 0 \), where

\[
\epsilon_x \geq \epsilon_H(H) + \epsilon_v(\Delta_M, H), \epsilon_u \geq 0,
\]

and \( \epsilon_v(\Delta_M, H) = \sqrt{mH \Delta_M \cdot \| D_x \|_{\infty} w_{\text{max}} \kappa_B}, \kappa_B = \max_{(A,B) \in \Theta} \| A \|_2 \).
4 Safe Adaptive Control Algorithm

This section introduces our safe adaptive control algorithm for constrained LQR. Our algorithm design adopts a standard model-based approach as in, e.g., Mania et al. (2019); Simchowitz and Foster (2020). That is, we improve model estimations with newly collected data and update the policies by near-optimal policies computed based on the current model estimation.

However, the constraints in our problem bring additional challenges on, e.g., constraint satisfaction, feasibility, tradeoff among exploration, exploitation, and safety. To address these challenges, we design three components in Algorithm 1 that are different or irrelevant in the unconstrained case (see e.g., Mania et al. (2019); Simchowitz and Foster (2020)), i.e., (i) the RobustCE and ApproxDAP subroutines (Lines 3, 6, 8, 10), (ii) safe policy updates by Algorithm 2 (Line 4 & Line 9), (iii) Phase 2: pure exploitation (Line 8-11).

In the following, we explain the components (i) and (ii) in detail then discuss the overall algorithm. The component (iii) is motivated by our regret analysis and will be explained after our regret bound in Theorem 4.

(i) Approximate DAP and robust CE. When the model \( \theta_\ast \) is unknown but an estimation \( \hat{\theta} \) is available, we consider the following approximate DA policy:

\[
\begin{align*}
    u_t = \sum_{k=1}^{H} M[k] \hat{w}_{t-k} + \eta_t,
\end{align*}
\]

where we approximate the disturbances by \( \hat{w}_t = \Pi_\mathcal{W} (x_{t+1} - \hat{A} x_t - \hat{B} u_t) \) and the projection on \( \mathcal{W} \) benefits constraint satisfaction but introduces nonlinearity to the policy (8) with respect to the history states. Besides, we add an excitation noise \( \eta_t \sim \eta D_\eta \) in (8) to encourage exploration, where \( \eta \) is an excitation level, the distribution \( D_\eta \) has zero mean, \( (s_\eta, p_\eta) \)-anti-concentration for some \( s_\eta, p_\eta \), and bounded support such that \( \| \eta \|_\infty \leq \bar{\eta} \). Examples of \( D_\eta \) include truncated Gaussian, uniform distribution, etc.

### Algorithm 1: Safe Adaptive Control

- **Input:** \( \Theta_{\text{ini}}, \mathcal{D}_\eta, T(e), H(e), \bar{\eta}(e), \Delta_M(e), T_D(e), \forall e. \)
- **Initialize:** \( \hat{\theta}(0) = \hat{\theta}_{\text{ini}}, r(0) = r_{\text{ini}}, \Theta(0) = \Theta_{\text{ini}}. \) Define \( w_t = \hat{w}_t = 0 \) for \( t < 0, t_1^{(0)} = 0. \)
- **for** Episode \( e = 0, 1, 2, \ldots \) **do**
  - // Phase 1: exploration & exploitation
  - 3 \( (M_t^{(e)}, \Omega_t^{(e)}) \leftarrow \text{RobustCE}(\Theta^{(e)}, H^{(e)}, \bar{\eta}^{(e)}, \Delta_M^{(e)}). \)
  - 4 Run Algorithm 2 to safely update the policy to \( M_t^{(e)} \) and output \( t_1^{(e)} \) if \( e > 0 \). The algorithm inputs are \( (M(\epsilon - 1), \Omega(\epsilon - 1), \Theta^{(e)}, 0, \Delta_M^{(e - 1)}), (M_t^{(e)}, \Omega_t^{(e)}, \Theta^{(e)}, \bar{\eta}^{(e)}, \Delta_M^{(e)}), T^{(e)}. \)
  - 5 for \( t = t_1^{(e)}, \ldots, t_1^{(e)} + T_D^{(e)} - 1 \) do
  - 6 Run ApproxDAP\( (M_t^{(e)}, \hat{\theta}(e), \bar{\eta}(e)). \)
  - // Model Updates
  - 7\( \Theta(e - 1) \leftarrow \text{ModelEst} \{(x_k, u_k)_{k=t_1^{(e)}+T_D^{(e)}}^t, \hat{\theta}(e)\}. \)
  - // Phase 2: pure exploitation \( (\eta_t = 0) \)
  - 8 \( (M(\epsilon), \Omega^{(e)}) \leftarrow \text{RobustCE}(\Theta^{(e + 1)}, H^{(e)}, 0, \Delta_M^{(e)}). \)
  - 9 Run Algorithm 2 to safely update the policy to \( M^{(e + 1)} \) and output \( t_2^{(e + 1)} \). Algorithm 2’s inputs are \( (M_t^{(e)}, \Omega_t^{(e)}, \Theta^{(e)}, \bar{\eta}^{(e)}, \Delta_M^{(e)}, (M^{(e)}, \Omega^{(e)}, \Theta^{(e + 1)}, 0, \Delta_M^{(e)}, T_D^{(e)}). \)
  - 10 for \( t = t_2^{(e + 1)}, \ldots, T^{(e + 1)} - 1 \) do
  - 11 Run ApproxDAP\( (M^{(e)}, \hat{\theta}^{(e + 1)}, 0). \)

### Subroutine RobustCE\( (\Theta, H, \bar{\eta}, \Delta_M) \):

- Compute the optimal policy \( M \) to (10).
- Construct the robustly safe policy set:
  \[ \Omega = \Omega(\Theta, \epsilon_x, \epsilon_u), \] where \( (\epsilon_x, \epsilon_u) \) are defined by (10).
- return policy \( M \) and robustly safe policy set \( \Omega \).
Subroutine **ApproxDAP** \((M, \hat{\theta}, \tilde{\eta})\):

Implement \(u_t = \sum_{k=1}^{\hat{t}} M[k] \hat{w}_{t-k} + \eta_t\), with \(\eta_t \overset{\text{ind}}{\sim} \tilde{\eta}D_\eta\). Observe \(x_{t+1}\) and record \(\hat{w}_t = \Pi_{\theta}(x_{t+1} - \hat{A}x_t - \hat{B}u_t)\).

Subroutine **ModelEst** \(\{\{x_k, u_k\}_{k=0}^{\hat{t}}, \eta\}\):

Estimate the model by LSE with projection on \(\Theta_{\text{ini}}\):

\[ \hat{\theta} = \arg \min_{\theta} \sum_{k=0}^{\hat{t}-1} \|x_{k+1} - Ax_k - Bu_k\|_2^2; \quad \hat{\theta} = \Pi_{\Theta_{\text{ini}}} (\hat{\theta}). \]

Set the model uncertainty set as \(\Theta = \mathcal{B}(\hat{\theta}, r) \cap \Theta_{\text{ini}}\), where \(r = \hat{O}\left(\frac{\sqrt{n} + \alpha}{\sqrt{\tilde{\eta}}}\right)\) by Corollary 1.

Return the model uncertainty set \(\Theta\).

To ensure safety of (8) without knowing the true model, we rely on robust constraint satisfaction. Specifically, given a model uncertainty set \(\Theta = \mathcal{B}(\hat{\theta}, r) \cap \Theta_{\text{ini}}\), where \(r\) is the confidence radius of the estimated model \(\hat{\theta}\), we construct a robustly safe policy set to include policies that are safe for any model in \(\Theta\) and any excitation noises \(\|\eta_t\|_\infty \leq \tilde{\eta}\). This robustly safe policy set can be determined by \(\Omega(\hat{\theta}, \epsilon_x, \epsilon_u)\) for any

\[ \epsilon_x \geq \epsilon_{\theta}(r) + \epsilon_{\eta,x}(\tilde{\eta}) + \epsilon_H(H), \quad \epsilon_u \geq \epsilon_{\eta,u}(\tilde{\eta}). \]

Compared with the safe policy set \(\Omega(\theta_s, \epsilon_x, \epsilon_u)\) in (6), the robustly safe policy set replaces the unknown model with the estimated model \(\hat{\theta}\) and includes additional constraint tightening terms \(\epsilon_{\theta}(r), \epsilon_{\eta,x}, \epsilon_{\eta,u}\) in \(\epsilon_x, \epsilon_u\)’s lower bounds to be more conservative in the face of model uncertainties and excitation noises to guarantee safety. Formulas of these constraint-tightening terms can be determined below based on perturbation analysis, whose proof is provided in the supplementary.

Lemma 2 (Definitions of \(\epsilon_{\theta}, \epsilon_{\eta,x}, \epsilon_{\eta,u}\)). Define \(\epsilon_{\theta}(r) = c_1 \sqrt{mr}, \epsilon_{\eta,x}(\tilde{\eta}) = c_2 \sqrt{\tilde{\eta}}, \epsilon_{\eta,u}(\tilde{\eta}) = c_3 \tilde{\eta}\), where \(c_1, c_2, c_3\) are polynomials of \(\|D_x\|_\infty, \|D_u\|_\infty, \kappa, \kappa_B, \gamma^{-1}, w_{\text{max}}, x_{\text{max}}, u_{\text{max}}\)\footnote{Formulas of \(c_1, \ldots, c_3\) are provided in Appendix A.2}. For any \(M \in \Omega(\hat{\theta}, \epsilon_x, \epsilon_u)\) where \((\epsilon_x, \epsilon_u)\) satisfies (9), policy (8) is safe for any model in \(\Theta\), any \(\|\eta\|_\infty \leq \tilde{\eta}\), and any \(w_t \in \mathcal{W}\).

Based on the robustly safe policy set, we can compute a near-optimal and robustly safe DAP by the QP below:

\[
\begin{align*}
\min_{M} & \; f(M; \hat{\theta}) \\
\text{s.t.} & \; M \in \Omega(\hat{\theta}, \epsilon_x, \epsilon_u), \; \epsilon_x = \epsilon_{\theta}(r) + \epsilon_{\eta,x}(\tilde{\eta}) + \epsilon_H(H) + \epsilon_c(\Delta_M, H).
\end{align*}
\]

Compared with the optimal DAP problem (7), problem (10) replaces the true model \(\theta\), by the estimated model \(\hat{\theta}\) (certainty equivalence) and ensures robust constraint satisfaction on \(\Theta\) and \(\|\eta\|_\infty \leq \tilde{\eta}\). Hence, we call (10) robust CE\footnote{Our robust CE does not consider min-max cost.}.

Notice that we include an additional constraint tightening term \(\epsilon_c\) in (10) to ensure safety of slowly varying policies according to Section 3.2, which is crucial for our safe policy update algorithm below.

(ii) SafeTransit Algorithm. Notice that at the start of each phase in Algorithm 1 we compute a new policy to implement in this phase. However, directly changing the old policy to the new one may cause constraint violation as discussed in Section 3.2.

To address this, we design Algorithm 2 to ensure safe policy updates at the start of each phase. The high-level idea of Algorithm 2 is based on the slow-variation trick reviewed in Section 3.2. That is, we construct a policy path connecting the old policy to the new policy such that this policy path is contained in some robustly safe policy set with an additional constraint tightening term to allow slow policy variation, then by slowly varying the policies along this path, we are able to safely transit to the new policy.

Next, we focus on constructing such a policy path. We illustrate our construction by Figure 1. We follow the notations in Algorithm 2, i.e. the old policy is \(M\) in an old robustly safe policy set \(\Omega\) and the new policy is \(M'\) in \(\Omega'\). Notice that the straight line from \(M\) to \(M'\) does not satisfy the requirements of the slow variation trick because some parts of the line are outside both robustly safe policy sets. To address this, Algorithm 2 introduces an intermediate policy \(M_{\text{mid}}\) in \(\Omega \cap \Omega'\), and slowly moves the policy from the old one \(M\) to the intermediate one \(M_{\text{mid}}\) (Step 1), then
slowly moves from $M_{\text{mid}}$ to the new policy $M'$ (Step 2). In this way, all the path is included in at least one of the robustly safe policy sets, which allows safe transition from the old policy to the new policy.

The choice of $M_{\text{mid}}$ is not unique. In practice, we recommend selecting $M_{\text{mid}}$ with a shorter path length for quicker policy transition. The existence of $M_{\text{mid}}$ can be guaranteed if the first RobustCE program in Algorithm 1 (Phase 1 of episode 0) is strictly feasible. This is usually called recursive feasibility and will be formally proved in Theorem 2.

![Figure 1: Illustration for Algorithm 2](image)

**Algorithm 2: SafeTransit**

Input: $(M, \Omega, \Theta, \eta, \Delta_M)$. $(M', \Omega', \Theta', \eta', \Delta_M)$. $t_0$.

1. Set $\eta_{\text{min}} = \min(\eta, \eta')$, $\theta_{\text{min}} = \hat{\theta}_{(r \leq r')} + \hat{\theta}_{(r > r')}$. 
2. Set an intermediate policy as $M_{\text{mid}} \in \Omega \cap \Omega'$.

   // Step 1: slowly move from $M$ to $M_{\text{mid}}$

3. Define $W_1 = \max\left(\left\|\frac{M - M_{\text{mid}}}{\Delta_M}\right\|_F, H'\right)$. 
4. for $t = t_0, \ldots, t_0 + W_1 - 1$ do 
5.  Set $M_t = M_{t-1} + \frac{1}{W_1}(M_{\text{mid}} - M)$. 
6.  Run ApproxDAP($M_t, \hat{\theta}_{\text{min}}, \eta_{\text{min}}$). 

   // Step 2: slowly move from $M_{\text{mid}}$ to $M'$

7. Define $W_2 = \left\|\frac{M' - M_{\text{mid}}}{\Delta_M}\right\|_F$. 
8. for $t = t_0 + W_1, \ldots, t_0 + W_1 + W_2 - 1$ do 
9.  Set $M_t = M_{t-1} + \frac{1}{W_2}(M' - M_{\text{mid}})$. 
10. Run ApproxDAP($M_t, \hat{\theta}', \eta'$).

Output: Termination stage $t_1 = t_0 + W_1 + W_2$.

**Remark 1.** If the robustly safe policy sets are monotone, e.g., if $\Omega \subseteq \Omega'$, then we can let $M_{\text{mid}} = M$ and the path constructed by Algorithm 2 reduces to the straight line from $M$ to $M'$. Hence, a non-trivial $M_{\text{mid}}$ is only relevant when the robustly safe policy sets are not monotone, which can be caused by the non-monotone model uncertainty sets generated by LSE (even though the error bound $r$ of LSE decreases with more data, the change in the point estimator $\hat{\theta}$ may cause $\Theta' \nsubseteq \Theta$). Though one can enforce decreasing uncertainty sets by taking joints over all the history uncertainty sets, this approach leads to an increasing number of constraints when determining the robustly safe policy sets in RobustCE, thus being computationally demanding especially for large episode index $e$.

**Discussions on the overall algorithm.** Firstly, though Algorithm 1 is implemented by episodes, it is still a single-trajectory learning since no system starts are needed when new episodes start. The episodic design reduces the computational burden since a new policy is only computed once in a while. Further, the new policies are computed by solving (10), which is QP and enjoys polynomial-time solvers. Besides, $M_{\text{mid}}$ in Algorithm 2 can also be computed efficiently since the set $\Omega \cap \Omega'$ is a polytope. As for the model updates, we can use all the history data in practice, though we only use part of history in ModelEst for simpler analysis. ModelEst also projects the estimated model onto $\Theta_{\text{mi}}$ to ensure bounded estimation. We also note that Algorithm 2 is not the unique way to ensure safe policy updates: other path design or MPC can also work. More discussions are provided in the remarks below.

**Remark 2 (Comparison with literature).** Some constrained control methods construct safe state sets and safe action sets for the current state, e.g., control barrier functions (Ames et al. 2016), reachability-set-based methods (Akametalu et al. 2014), regulation maps (Kellett and Teel 2004), etc. In contrast, this paper constructs safe policy sets in the

space of policy parameters $\mathbf{M}$. This is possible because our policy structure (linear on history disturbances) allows a transformation from linear constraints on the states and actions to polytopic constraints on the policy parameters.

**Remark 3** (More discussions on safe policy updates). As mentioned in Section 3.2, for time-varying policies, state constraints require coupling policy constraints that depend on not only the current policy but also the history ones, so the decoupled constraints in (10) no longer guarantee safety. To address this, we adopt the slow-variation trick in [Li et al. (2020)] and slowly update the policies. Another possible idea is to introduce additional constraints to conditioning on the history policies and directly compute a new policy that is safe to implement given the current history policies. However, these additional constraints may reduce the sizes of the safe policy sets and thus sacrifice optimality.

5 Theoretical Analysis

In this section, we provide theoretical guarantees of our algorithms including model estimation errors, feasibility, constraint satisfaction, and a regret bound.

5.1 Estimation Error Bound

The estimation error bounds for linear policies have been studied in the literature [Dean et al. (2018)]. However, due to the projection in our disturbance approximation in (8), the policies implemented by our algorithms can be sometimes nonlinear. To cope with this, we provide an error bound below for general policies.

**Theorem 1** (Estimation error bound). Consider actions $u_t = \pi_t(x_0, \{w_k, \eta_k\}_{k=0}^{t-1}) + \eta_t$, where $\|\eta_t\|_\infty \leq \bar{\eta}$ is generated as discussed after (8) and policies $\pi_t(\cdot)$ ensure bounded states and actions, i.e. $\|(x_t^T, u_t^T)\|_2 \leq b_z$ for all $t \geq 0$. Let $\hat{\theta}_T = \min_{A,B} \sum_{t=0}^{T-1} \|x_{t+1} - Ax_t - Bu_t\|_2^2$ denote the model estimation. For any $0 < \delta < 1/3$, for $T \geq O((log(1/\delta) + (m + n) log(b_z/\bar{\eta})))$, with probability (w.p.) $1 - 3\delta$, we have

$$\|\hat{\theta}_T - \theta_*\|_2 \leq O\left(\sqrt{n+m} \sqrt{\log(b_z/\bar{\eta} + 1/\delta)} \right).$$

**Theorem 1** holds for both linear and nonlinear policies as long as the induced states and actions are bounded, which can be guaranteed by the stability of the policies. Further, the error bound in Theorem 1 is $\hat{O}(\sqrt{n+m}/\bar{\eta})$, which coincides with the error bound for linear policies in terms of $T, \bar{\eta}, n, m$ in [Dean et al. (2018)].

Based on **Theorem 1** we obtain a formula for the estimation error bound $r^{(e)}$ in Line 7 of Algorithm 1.

**Corollary 1** (Formula of $r^{(e)}$). Suppose $H^{(0)} \geq \log(2\kappa)/\log(1 - \gamma^{-1})$, $T^{(e+1)} \geq t_k^{(e)} + T_D^{(e)}$ and $T_D^{(e)}$ satisfies the condition on $T$ in **Theorem 2**. For any $0 < p < 1$ and $e \geq 1$, with probability at least $1 - \frac{1}{2e^2}$, we have $\|\hat{\theta}^{(e)} - \theta_*\|_F \leq r^{(e)}$, where

$$r^{(e)} = O\left(\sqrt{n+m} \sqrt{\log(\sqrt{mn}/\bar{\eta}(e^{-1}) + e^2/p)} \right).$$

**Corollary 1** considers the $\| \cdot \|_F$ norm because Algorithm 1 projects matrix $\hat{\theta}^{(e)}$ onto $\Theta_{m}$ and the $\| \cdot \|_F$ norm is more convenient to analyze and implement when matrix projections are involved. Due to the change of norms, the error bound has an additional $\sqrt{n}$ factor.

5.2 Feasibility and Constraint Satisfaction

This section provides feasibility and constraint satisfaction guarantees of our adaptive control algorithm.

**Theorem 2** (Feasibility). Algorithms 1 and 2 output feasible policies for all $t$ under the following conditions.

(i) **(Strict initial feasibility)** There exists $\epsilon_0 > 0$ such that $\Omega(\hat{\theta}^{(0)}, \epsilon^{(0)}_x + \epsilon_0, \epsilon^{(0)}_u) \neq \emptyset$, where $\epsilon^{(0)}_x, \epsilon^{(0)}_u$ are defined by 11 with initial parameters $r^{(0)}, \bar{\eta}^{(0)}, H^{(0)}, \Delta_M^{(0)}$.

(ii) **(Monotone parameters)** $\bar{\eta}^{(e)}, H^{(e)}, T_D^{(e)}, \Delta_M^{(e)}$ are selected s.t. $(H^{(e)})^{-1}, \sqrt{H^{(e)}}\Delta_M^{(e)}$, $\bar{\eta}^{(e)}$, $r^{(e)}$ are all non-increasing with $e$, and $r^{(1)} \leq \frac{\epsilon_0}{c_1 \sqrt{mn}}$, where $r^{(e)}$ is defined in 11, $c_1$ is defined in Lemma 2.
Furthermore, under Assumption 2, condition (i) above is satisfied if
\[ \epsilon_x^{(0)} + \epsilon_o \leq \epsilon_{F,x} - \epsilon_{\theta(r_{ini})} - \epsilon_P, \quad \epsilon_u^{(0)} \leq \epsilon_{F,u} - \epsilon_P, \]
where \( \epsilon_P = c_4 \sqrt{m} (1 - \gamma)^{H^{(0)}} \) and \( c_4 \) is a polynomial of \( \| D_x \|_{\infty}, \| D_u \|_{\infty}, \kappa, \kappa_B, \gamma, \epsilon_{u_{max}}, x_{max}, u_{max} \).

Condition (i) requires the initial policy set \( \Omega_x^{(0)} \) to contain a policy that strictly satisfies the constraints on \( q_f(M; \hat{\theta}^{(0)}) \). Condition (ii) requires monotonic parameters in later phases, where the non-increasing estimation error \( r^{(c)} \) requires an increasing number of exploration stages \( T^{(c)}_D \). Conditions (i) and (ii) together establish the recursive feasibility: if our algorithm is (strictly) feasible at the initial stage, then the algorithm is feasible in the future under proper parameters.

The last statement in Theorem 3 provides conditions (12) for strict initial feasibility, which is based on the \( \epsilon_{F,x} \) strictly safe policy \( u_t = -K_{Fx_i} \) in Assumption 3. The term \( \epsilon_{F} \) captures the difference between the policy \( K_F \) and its approximate DAP (12) requires large enough \( H^{(0)}, T^{(0)}_D \) and small enough \( \tilde{\eta}^{(0)}, \Delta_M^{(0)} \). Further, (12) implicitly requires a small enough initial uncertainty radius: we at least need \( \epsilon_{\theta(r_{ini})} < \epsilon_{F,x} \). If the initial uncertainty set is too large but a safe policy is available, one can first explore the system with the safe policy to reduce the model uncertainty until the strict initial feasibility is obtained and then apply our algorithm.

**Theorem 3 (Constraint Satisfaction).** Under the conditions in Theorem 2 and Corollary 1, when \( T^{(e+1)} \geq t^{(e)}_2 \), we have \( u_t \in \mathbb{U} \) for all \( t \geq 0 \) w.p.1 and \( x_t \in \mathbb{X} \) for all \( t \geq 0 \) w.p.1. Further, w.p. \((1 - \rho)\),

\[
\text{Regret} \leq \tilde{O}(T^{2/3})
\]

**On parameters.** Theorem 4 provides choices of parameters that ensure feasibility, safety, and the regret bound. Here, we choose exponentially increasing episode lengths \( T^{(c)} \), and explore for \( (T^{(c)})^{2/3} \) stages at each episode with a small enough constant excitation level \( \tilde{\eta}^{(c)} \). We select large enough memory lengths \( H^{(c)} \geq O(\log(T^{(c)})) \) and small enough variation budgets \( \Delta_M^{(c)} \leq O((T^{(c+1)})^{-1/3}) \).

**On regret.** Though our regret bound \( \tilde{O}(T^{2/3}) \) is worse than the \( \tilde{O}(\sqrt{T}) \) regret bound for unconstrained LQR, it has the same order in terms of \( T \) with the robust learning of unconstrained LQR in [Dean et al., 2018]. This motivates future work on lower bounds of learning-based control with safety/robustness guarantees.

**Discussions on the pure exploitation phase.** Algorithm 1 includes a pure exploitation phase with no excitation noise at each episode, which is not present in the unconstrained algorithms [Dean et al., 2018, Mania et al., 2019, Simchowitz and Foster, 2020]. This phase is motivated by our regret analysis: the algorithm can still work without this phase but will generate a worse regret bound. Specifically, consider a robust CE policy (10) with respect to \( \tilde{\eta} \) and uncertainty radius \( r \), the regret of this policy per stage can be roughly bounded by \( O(\tilde{\eta} + r) \) in the supplementary (we omit \( \Delta_M \) here for simplicity). With no pure exploitation phases, the regret in episode \( e \) can be roughly bounded by

\[
\tilde{O}(T^{(e)}(\tilde{\eta}^{(e)} + r^{(e)})), \quad \text{where } r^{(e)} = \tilde{O}(\sqrt{T^{(e-1)}\tilde{\eta}^{(e-1)}}) \] by Corollary 1 (hiding \( n, m \)). Therefore, the total regret can be
bounded by \( \sum_{e} (\frac{1}{\sqrt{T(e-1)}} + \tilde{\eta}^v) T(e) \approx \sum_{e} (\frac{1}{\sqrt{T(e-1)}} + \tilde{\eta}^v) T(e) \), which is minimized at \( \frac{1}{\sqrt{T(e-1)}} + \tilde{\eta}^v = (T(e))^{-1/4} \), leading to a worse regret bound \( \tilde{O}(T^{3/4}) \).

Notice that \( \tilde{\eta}^v \) chosen in Theorem 4 is a constant, so our algorithm suffers slightly larger stage regret during the Phase 1 (exploration & exploitation) compared with the no-pure-exploitation algorithm described above with \( \eta^{v^*} = (T^{v^*})^{-1/4} \to 0 \). Nevertheless, by constantly refining the models and reducing \( \Delta_{v^*}^{v^*} \), the performance during the phase 1 still improve over episodes.

6 Discussions and Future Work

Comparing with linear dynamical policies. Though we only considered linear static policies as our regret benchmark, it is straightforward to include linear dynamical policies as the benchmark and achieve similar regret bounds under proper conditions (see e.g., [Plevrakis and Hazan (2020)].

Comparing with RMPC in [Mayne et al. (2005)]. Mayne et al. (2005) propose a tube-based RMPC algorithm for constrained LQR with perfect model information. Though this algorithm generates piecewise-affine policies, we are able to show that the infinite-horizon averaged cost of this algorithm can be characterized by \( J(K) \), where \( K \) is a pre-fixed safe linear policy for tube construction in the tube-based RMPC (for more details, see [Mayne et al. (2005)].

In other words, our current regret benchmark \( K^\ast \) performs not worse than RMPC in [Mayne et al. (2005)] in terms of the infinite-horizon averaged cost, so the regret of our Algorithm 1 remains unchanged by including this RMPC to the benchmark policy set. The major strength of RMPC in [Mayne et al. (2005)] compared with our algorithm is the initial feasibility: RMPC can safely control the system starting from more distant states on the linear system starting from 0, but RMPC can be safe to implement starting from nonzero states.

Future work. There are many interesting future directions, e.g., (i) improving practical performance by reducing the constraint tightenings, (ii) regret analysis compared with other RMPC algorithms, (iii) algorithm design and analysis for large initial states, (iv) fundamental regret lower bounds, (v) safe adaptive control with performance guarantees for nonlinear systems, etc.

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\(^{10}\)Here, \( K \) is required to be safe on the linear system starting from 0, but RMPC can be safe to implement starting from nonzero states.
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A Preparations: State Approximation and Constraint-tightening Terms

This section provides results that will be useful throughout the rest of the appendices. Specifically, the first subsection defines and discusses the constraint-tightening terms in Lemma 2 and (10) based on the upper bounds of the constraint decomposition terms.

A.1 State Approximation and Constraint Decompositions

We consider a more general form of approximate DAP below than that in (3), i.e., an approximate DAP with time-varying policy matrices $M_t$, time-varying excitation levels $\bar{\eta}_t$, and time-varying model estimations $\hat{\theta}_t$.

\[ u_t = \sum_{i=1}^{H_t} M_t[k]\bar{w}_{t-k} + \eta_t, \quad \bar{w}_t = \Pi_{\mathcal{W}}(x_{t+1} - \hat{\theta}_t z_t), \quad \|\eta_t\|_\infty \leq \bar{\eta}_t, \quad t \geq 0, \tag{13} \]

where $M_t \in \mathcal{M}_{H_t}$, and $\{H_t\}_{t \geq 0}$ is non-decreasing.

When implementing the time-varying approximate DAP (13) to the system $x_{t+1} = A_* x_t + B_* u_t + w_t$, we have the following state approximation lemma.

Lemma 3 (State approximation under time-varying approximate DAP). When implementing the time-varying approximate DAP (13) to the system $x_{t+1} = A_* x_t + B_* u_t + w_t$, we have the following state approximation result:

\[ x_t = A_*^{H_t} x_{t-H_t} + \sum_{k=2}^{2H_t} \sum_{i=1}^{H_t} A_*^{-1} B_* M_{i-1} [k-i]\bar{w}_{t-k} I(1 \leq k-i \leq H_{i-1}) + \sum_{i=1}^{H_t} A_*^{-1} w_{t-i} + \sum_{i=1}^{H_t} A_*^{-1} B_* \eta_{t-i} \]
The lemma above is a straightforward extension from Proposition reviewed in Section for the case with perfect model information, thus the proof is omitted.

To simplify the exposition, we introduce the following notations for time-varying DAP.

\[
\hat{\Phi}_k^r(M_{t-H_t:t}; \theta) = A^{k-1} I_{(k < H_t)} + \sum_{i=1}^{H_t} A^{i-1} B M_{t-i} |k - i| I_{(1 \leq k - i \leq H_t)}, \quad \forall 1 \leq k \leq 2H_t, \quad (14)
\]

\[
\tilde{g}_t^r(M_{t-H_t:t-1}; \theta) = \sup_{\tilde{w}_t \in \mathcal{W}} D_{x,t}^\top \sum_{k=1}^{2H_t} \hat{\Phi}_k^r(M_{t-H_t:t-1}; \theta) \tilde{w}_{t-k} = \sum_{k=1}^{2H_t} \| D_{x,t}^\top \hat{\Phi}_k^r(M_{t-H_t:t-1}; \theta) \|_W \tilde{w}_{t-k}, \quad (15)
\]

where $1 \leq i \leq k_x$ and we define $M_t = M_0$ for $t < 0$ for notational simplicity. Notice that when $M_t = M$ and $H_t = H$ (the time-invariant case), the definitions of $\hat{\Phi}_k^r(M_{t-H_t:t}; \theta)$ and $\tilde{g}_t^r(M_{t-H_t:t-1}; \theta)$ above reduce to the definitions of $\hat{\Phi}_k^r(M; \theta)$ and $g_t^r(M; \theta)$ respectively in Section.

Based on Lemma 3 and the notations defined above, we can obtain the following corollary on the decompositions of the state constraints $D_{x,t}$ and action constraints $D_{u,t}$. The decompositions are crucial when defining our constraint-tightening terms and developing the constraint satisfaction guarantees.

**Corollary 2 (Constraint decomposition).** When implementing the time-varying approximate DAP to the system $x_t = A_s x_{t-1} + B_s u_{t-1} + w_t$, for each $1 \leq i \leq k_x$ and $1 \leq j \leq k_u$, we have the following decompositions:

\[
D_{x,t}^\top x_t \leq \overbrace{g_t^r(M_t; \hat{\theta}_t^r)}^{\text{estimated state constraint function}} + \underbrace{D_{x,t}^\top \sum_{k=1}^{H_t} A^{k-1}_s M_{t-k} |k| \tilde{w}_{t-k}}_{\text{model estimation errors}} + \underbrace{\sum_{i=1}^{H_t} D_{x,t}^\top A^i_s g_t^r(M_{t-i}; \hat{\theta}_t^r)}_{\text{excitation errors on the state}} + \underbrace{\sum_{i=1}^{H_t} D_{x,t}^\top A^i_s \hat{\theta}_t - \hat{\theta}_t}_{\text{history truncation errors}} + \underbrace{(\tilde{g}_t^r(M_{t-H_t:t-1}; \theta) - g_t^r(M_t; \theta))}_{\text{policy variation errors}},
\]

\[
D_{u,t}^\top u_t \leq \overbrace{g_t^u(M_t)}^{\text{action constraint function}} + \underbrace{D_{u,t}^\top \sum_{i=1}^{H_t} \eta \tilde{M}_{t-k}}_{\text{excitation error on the action}}.
\]

where $\hat{\theta}_t^r$ is an estimated model used to approximate the state constraint function, and we allow $\hat{\theta}_t^r \neq \hat{\theta}_t$ for generality.

**Proof.** The proof is by the definitions of $g_t^r$, $g_t^u$, $\tilde{g}_t^r$, and Lemma 3. For the action constraints, by the definition above, we have

\[
D_{u,t}^\top u_t = \sum_{i=1}^{H_t} D_{u,t}^\top \hat{M}_{t-k} \tilde{w}_{t-k} + D_{u,t}^\top \eta_t \leq \sum_{i=1}^{H_t} \| D_{u,t}^\top \hat{M}_{t-k} \|_W \tilde{w}_{t-k} + D_{u,t}^\top \eta_t = g_t^u(M_t) + D_{u,t}^\top \eta_t
\]

where the inequality is because $\tilde{w}_{t-k} \in \mathcal{W}$ and the Hölder’s inequality. The state constraints can be similarly proved: notice that we apply the Hölder’s inequality on $\tilde{w}_{t-k}$ instead of $w_{t-k}$.

**A.2 The Constraint-tightening Terms**

This subsection provides the definitions of the factors $c_1, c_2, c_3$ in the constraint-tightening terms introduced in Lemma 2. Further, this subsection provides explanations on all the constraint-tightening terms in (10) by showing that each constraint-tightening term serves as an upper bound on an error term in the constraint decompositions in Corollary 2.

**Definition and explanation of $\epsilon_\theta(r)$.** The next lemma formally shows that the constraint-tightening term $\epsilon_\theta(r)$ is an upper bound on the model estimation errors in the state constraint decomposition in Corollary 1 where $r$ is the model estimation error bound.

**Lemma 4 (Definition of $\epsilon_\theta(r)$).** Consider implementing the time-varying approximate DAP to the system $x_{t+1} = A_s x_t + B_s u_t + w_t$. For a fixed $t$, suppose $\theta_t \in \Theta$, $\| \theta_t - \theta_s \|_F \leq r$, and $\| \theta_t - \theta_s \|_F \leq r$ for all $1 \leq k \leq H_t$.
Further, suppose \( x_{t-k} \in \mathbb{X}, u_{t-k} \in \mathbb{U} \) for all \( 1 \leq k \leq H_t \). Then, we have

\[
g^x_t(M_t; \theta_s) - g^x_t(M_t; \hat{\theta}^p_t) \leq \epsilon_\theta(r), \quad \sum_{k=1}^{H_t} D^T_{x,i} A^{k-1}_s (w_{t-k} - \hat{w}_{t-k}) \leq \epsilon_\hat{w}(r),
\]

\[
(g^\gamma_t(M_t; \theta_s) - g^\gamma_t(M_t; \hat{\theta}^p_t)) + \sum_{k=1}^{H_t} D^T_{x,i} A^{k-1}_s (w_{t-k} - \hat{w}_{t-k}) \leq \epsilon_\theta(r)
\]

where \( \epsilon_\hat{w}(r) = \|D_x\|_\infty \max \frac{\kappa}{\gamma} \cdot r = O(r), \) \( \epsilon_\theta(r) = 5\kappa^4 \kappa_B \|D_x\|_\infty \max w_{\max} \sqrt{\kappa m} r = O(\sqrt{\kappa m} r), \) and \( \epsilon_\theta(r) = \epsilon_\theta(r) + \epsilon_\hat{w}(r) = O(\sqrt{\kappa m} r). \) We can let \( c_1 = \|D_x\|_\infty \max \kappa \frac{\gamma}{\gamma} + 5\kappa^4 \kappa_B \|D_x\|_\infty \max w_{\max} \sqrt{\gamma} \).

The proof of Lemma 4 is based on the perturbation analysis and deferred to Appendix [G.1]. When proving Lemma 4 we also establish the following lemma.

**Lemma 5** (Disturbance approximation error). Consider \( \hat{w}_t = \Pi_{\mathbb{W}}(x_{t+1} - \hat{\theta} z_t) \) and \( x_{t+1} = \theta_s z_t + w_t \). Suppose \( \|z_t\|_2 \leq b_z \) and \( \|\theta_s - \hat{\theta}\|_F \leq r \), then

\[
\|w_t - \hat{w}_t\|_2 \leq b_z r.
\]

**Proof.** By non-expansiveness of projection, we have \( \|w_t - \hat{w}_t\|_2 \leq \|x_{t+1} - \theta_s z_t - (x_{t+1} - \hat{\theta} z_t)\|_2 = \|(\hat{\theta} - \theta_s) z_t\|_2 \leq b_z r. \)

**Definition and explanation of \( \epsilon_{\eta,x}(\bar{\eta}) \) and \( \epsilon_{\eta,u}(\bar{\eta}) \)** The next lemma formally shows that the terms \( \epsilon_{\eta,x}(\bar{\eta}) \) and \( \epsilon_{\eta,u}(\bar{\eta}) \) bounds the excitation errors on the state and action constraint decompositions in Corollary [3].

**Lemma 6** (Definition of \( \epsilon_{\eta}(\bar{\eta}) \)). Consider implementing the time-varying approximate DAP \([13]\) to the system \( x_{t+1} = A_s x_t + B_s u_t + w_t \). For a fixed \( t \), suppose \( \|\eta_t\|_\infty \leq \bar{\eta} \) for all \( 0 \leq k \leq H_t \). Then,

\[
\sum_{i=1}^{H_t} D^T_{x,i} A^{i-1}_s B_s \eta_{t-i} \leq \epsilon_{\eta,x}(\bar{\eta}), \quad \sum_{i=1}^{H_t} D^T_{u,j} \eta_{t} \leq \epsilon_{\eta,u}(\bar{\eta}),
\]

where \( \epsilon_{\eta,x} = \|D_x\|_\infty \kappa B / \gamma \sqrt{\kappa m \bar{\eta}} \) = \( O(\sqrt{\kappa m} \bar{\eta}) \), \( \epsilon_{\eta,u} = \|D_u\|_\infty \bar{\eta} = O(\bar{\eta}) \) and we define \( \epsilon_\eta = (\epsilon_{\eta,x}, \epsilon_{\eta,u}) \), \( c_2 = \|D_x\|_\infty \kappa B / \gamma \), \( c_3 = \|D_u\|_\infty \).

**Proof.** The proof is provided below.

\[
\|D_x \sum_{i=1}^{H_t} A^{i-1}_s B_s \eta_{t-i}\|_\infty \leq \|D_x\|_\infty \sum_{i=1}^{H_t} \|A^{i-1}_s B_s\|_\infty \|\eta_{t-i}\|_\infty \leq \|D_x\|_\infty \sqrt{\kappa m} \sum_{i=1}^{H_t} \|A^{i-1}_s B_s\|_2 \|\eta_{t-i}\|_\infty \\
\leq \|D_x\|_\infty \sqrt{\kappa m} \sum_{i=1}^{H_t} \kappa (1 - \gamma)^i - \kappa_B \|\eta_{t-i}\|_\infty \leq \|D_x\|_\infty \sqrt{\kappa m B / \gamma \bar{\eta}} \\
\|D_u \eta_t\|_\infty \leq \|D_u\|_\infty \|\eta_t\|_\infty \leq \|D_u\|_\infty \bar{\eta} \]

**Definition of \( \epsilon_H(H) \)** The term \( \epsilon_H(H) \) has been introduced in [Li et al. 2020] for the known-model case to bound the history truncation errors in the state constraint decomposition. Here, we slightly improve its dependence on the problem dimensions and include our proof below.

**Lemma 7** (Definition of \( \epsilon_H \)). For any \( x_{t-H_t} \in \mathbb{X}, \) we have

\[
D^T_{x,i} A^{H_t}_{t-H_t} \leq \epsilon_H(H_t) = \|D_x\|_\infty \kappa x_{\max} (1 - \gamma)^{H_t} = O((1 - \gamma)^{H_t}).
\]
Proof.
\[ \| D_x A_{x_t}^{H_t} x_t - H_t \|_\infty \leq \| D_x \| \| A_{x_t}^{H_t} x_t - H_t \|_\infty \leq \| D_x \| \| A_{x_t}^{H_t} \|_2 \| x_t - H_t \|_2 \leq \| D_x \| \kappa (1 - \gamma)^{H_t} x_{\text{max}}. \]

\[ \square \]

**Definition of \( \epsilon_{\text{p}}(\Delta_M, H) \)** The error term \( \epsilon_{\text{p}}(\Delta_M, H) \) has also been introduced in Li et al. (2020) for the known-model case to bound the policy variation error. Here, we also slightly improve its dependence on the problem dimensions and the memory length in the next lemma. The proof is based on the perturbation analysis and will be provided in Appendix G.2.

**Lemma 8 (Definition of \( \epsilon_{\text{p}}(\Delta_M, H) \)).** Under the conditions in Lemma 3, suppose \( \Delta_M \geq \max_{1 \leq k \leq H_t} \| M_t - M_{t-k} \| p \), then we have
\[ \left\lfloor \bar{g}_i^S(M_{t-k}; \theta_x) - g_i^S(M_t; \theta_x) \right\rfloor \leq \epsilon_{\text{p}}(\Delta_M, H_t) \]
where \( \epsilon_{\text{p}}(\Delta_M, H_t) = \| D_x \| w_{\text{max}} \kappa \kappa_\text{B} / \gamma^2 \sqrt{\min H_t \Delta_M} = O(\sqrt{\min H_t \Delta_M}). \]

**B Estimation Error Bounds**

This section provides a proof for Theorem 1 and a proof for Corollary 1. When proving Corollary 1, we also establishes a.s. upper bounds on the state and action trajectories of our algorithm.

**B.1 Proof of Theorem 1**

Our proof of Theorem 1 relies on a recently developed least square estimation error bound for general time series satisfying a block martingale small-ball (BMSB) condition Simchowitz et al. (2018). The general error bound and the definition of BMSB are included below for completeness. In the literature (Dean et al., 2018, 2019b), only linear policies are considered and shown to satisfy the BMSB condition. Our contribution is to show that even for general policies, BMSB still holds as long as the corresponding states and actions are bounded (which is usually the case if certain stability properties are satisfied). By general policies, we allow time-varying policies, nonlinear policies, policies that depend on all the history, etc., (i.e. we consider \( u_t = \pi_t(x_0, \{ w_k, \eta_k \}_{k=0}^{t-1}) + \eta_t \)). More rigorous discussions are provided below.

**Definition 3 (Block Martingale Small-Ball (BMSB) (Definition 2.1 Simchowitz et al. (2018))).** Let \( \{ X_t \}_{t \geq 1} \) be an \( \{ \mathcal{F}_t \}_{t \geq 1} \)-adapted random process taking values in \( \mathbb{R}^d \). We say that it satisfies the \( (k, \Gamma_{sb}, p) \)-block martingale small-ball (BMSB) condition for \( \Gamma_{sb} > 0 \) if, for any fixed \( \lambda \in \mathbb{R}^d \) such that \( \| \lambda \|_2 = 1 \) and for any \( j \geq 0 \), one has
\[ \frac{1}{k} \sum_{i=1}^{k} \mathbb{P}(\| \lambda^T X_{j+i} \| \geq \sqrt{\lambda^T \Gamma_{sb} \lambda} \| \mathcal{F}_j \) \geq p \text{ almost surely.} \]

**Theorem 5 (Theorem 2.4 in Simchowitz et al. (2018)).** Fix \( \epsilon \in (0, 1) \), \( \delta \in (0, 1/3) \), \( T \geq 1 \), and \( 0 < \Gamma_{sb} < \Gamma \). Consider a random process \( \{ X_t, Y_t \}_{t \geq 1} \in (\mathbb{R}^d \times \mathbb{R}^d)^T \) and a filtration \( \{ \mathcal{F}_t \}_{t \geq 1} \). Suppose the following conditions hold,

1. \( Y_t = \theta_x X_t + \eta_t \), where \( \eta_t \), \( \mathcal{F}_t \) is \( \sigma_{sb}^2 \)-sub-Gaussian and mean zero,
2. \( \{ X_t \}_{t \geq 1} \) is an \( \{ \mathcal{F}_t \}_{t \geq 1} \)-adapted random process satisfying the \( (k, \Gamma_{sb}, p) \)-block martingale small-ball (BMSB) condition,
3. \( \mathbb{P}(\sum_{t=1}^{T} X_t X_t^T \not\preceq T \Gamma) \leq \delta. \)

Define the (ordinary) least square estimator as \( \hat{\theta} = \arg \min_{\theta \in \mathbb{R}^d} \sum_{t=1}^{T} \| Y_t - \theta X_t \|_2^2 \). Then if
\[ T \geq \frac{10k}{p^2} \left( \log \left( \frac{1}{\delta} \right) + 2d \log(10/p) + \log \det(\Gamma_{sb}^{-1}) \right), \]
we have
\[ \|\hat{\theta} - \theta_*\|_2 \leq \frac{90\sigma_{\text{sub}}}{p} \sqrt{\frac{n + d \log(10/p) + \log \det(\Gamma_{\text{sub}}^{-1}) + \log(1/\delta)}{TV_{\text{min}}(\Gamma_{\text{sub}})}} \]
with probability at least \( 1 - 3\delta \).

Next, we present a proof for our Theorem 1 by verifying the conditions in Theorem 5 for general nonlinear policies.

**Proof of Theorem 1**
Condition 1 is straightforward: \( x_{t+1} = \theta_* z_t + w_t \), and \( w_t | \mathcal{F}_t = w_t \) which is mean 0 and \( \sigma_{\text{sub}}^2 \)-sub-Gaussian by Assumption 4. Condition 3 is also straightforward. Notice that \( \nu_{\text{max}}(z_t z_t^\top) \leq \text{trace}(z_t z_t^\top) = \|z_t\|_2^2 \leq b_2 \). Therefore, we can define \( \Gamma = b_2^2 I_\text{in} + m \), and then \( \mathbb{P}(\sum_{t=1}^T z_t z_t^\top \leq T \Gamma) = 0 \leq \delta \).

The tricky part is Condition 2. Next, we will show the BMSB condition holds for our system. Then, by Theorem 5 we complete the proof.

**Lemma 9 (Verification of BMSB condition).** Define filtration \( \mathcal{F}_t = \{w_0, \ldots, w_{t-1}, \eta_0, \ldots, \eta_t\} \). Under the conditions in Theorem 5, \( \{z_t\}_{t\geq 0} \) satisfies the \((1, s_z^2 I_{n+m}, p_z)\)-BMSB condition, where \( p_z = \min(p_w, p_\eta) \), \( s_z = \min(s_w/4, \sqrt{s_\eta}, \frac{s_w}{4b_n} \bar{\eta}) \).

**Proof of Lemma 2** Define filtration \( \mathcal{F}^m_{t+1} = \mathcal{F}(w_0, \ldots, w_{t-1}, \eta_0, \ldots, \eta_{t-1}) \). Notice that the policy in Theorem 1 can be written as \( u_t = \pi_t(\mathcal{F}^m_t) + \eta_t \). Note that \( z_t \in \mathcal{F}_t \) is by definition. Next,
\[ z_{t+1} | \mathcal{F}_t = \begin{bmatrix} \mathcal{F}_{t+1} \\ u_{t+1} \end{bmatrix} | \mathcal{F}_t = \begin{bmatrix} \theta_* z_t + w_t | \mathcal{F}_t \\ \pi_{t+1}(\mathcal{F}^m_{t+1}) + \eta_{t+1} | \mathcal{F}_t \end{bmatrix}, \]
where \( \mathcal{F}^m_{t+1} = \mathcal{F}(w_0, \ldots, w_t, \eta_0, \ldots, \eta_t) \).

Notice that conditioning on \( \mathcal{F}_t \), the variable \( \theta_* z_t \) is determined, but the variable \( \pi_{t+1}(\mathcal{F}^m_{t+1}) + \eta_{t+1} \) is still random due to the randomness of \( w_t \). For the rest of the proof, we will always condition on \( \mathcal{F}_t \), and omit the conditioning notation, i.e., \( \cdot | \mathcal{F}_t \), for notational simplicity.

Consider any \( \lambda = (\lambda_1^\top, \lambda_2^\top)^\top \in \mathbb{R}^{m+n} \), where \( \lambda_1 \in \mathbb{R}^{n}, \lambda_2 \in \mathbb{R}^{m}, \|\lambda\|^2_2 = \|\lambda_1\|^2_2 + \|\lambda_2\|^2_2 = 1 \). Define \( k_0 = \max(2/\sqrt{3}, 4b_n/s_w) \). We consider three cases: (i) when \( \|\lambda_2\|^2_2 \leq 1/k_0 \) and \( \lambda_1^\top \theta_* z_t \geq 0 \), (ii) when \( \|\lambda_2\|^2_2 \leq 1/k_0 \) and \( \lambda_1^\top \theta_* z_t < 0 \), (iii) when \( \|\lambda_2\|^2_2 > 1/k_0 \). We will show in all three cases,
\[ \mathbb{P}(|\lambda^\top z_{t+1}| \geq s_z) \geq p_z \]
Consequently, by Definition 2.1 in Simchowitz et al. (2018), we have \( \{z_t\} \) is \((1, s_z^2 I, p_z)\)-BMSB.

**Case 1: when \( \|\lambda_2\|^2_2 \leq 1/k_0 \) and \( \lambda_1^\top \theta_* z_t \geq 0 \)**
\[ \lambda_1^\top w_t \leq \lambda_1^\top (w_t + \theta_* z_t) \leq |\lambda_1^\top (w_t + \theta_* z_t)| \]
\[ = |\lambda_1^\top z_{t+1} - \lambda_2^\top u_{t+1}| \leq |\lambda_1^\top z_{t+1}| + |\lambda_2^\top u_{t+1}| \leq |\lambda_1^\top z_{t+1}| + \|\lambda_2\|^2_2 b_u \]
\[ \leq |\lambda_1^\top z_{t+1}| + b_u/k_0 \leq |\lambda_1^\top z_{t+1}| + s_w/4 \]
where the last inequality uses \( k_0 \geq 4b_u/s_w \).

Further, notice that \( k_0 \geq 2/\sqrt{3} \), so \( \|\lambda_2\|^2_2 \leq 1/k_0 \leq 3/4 \), thus, \( \|\lambda_1\|^2_2 \geq 1/4 \), which means \( \|\lambda_1\|^2_2 \geq 1/2 \). Therefore,
\[ \mathbb{P}(\lambda_1^\top w_t \geq s_w/2) = \mathbb{P}(\lambda_1^\top w_t \geq \frac{s_w}{2\|\lambda_1\|^2_2}) \geq \mathbb{P}(\|\lambda_1\|^2_2 \geq s_w) = p_w \]

Then,
\[ \mathbb{P}(|\lambda^\top z_{t+1}| \geq s_z) \geq \mathbb{P}(|\lambda_1^\top z_{t+1}| \geq s_w/4) = \mathbb{P}(|\lambda_1^\top z_{t+1}| + s_w/4 \geq s_w/2) \]
\[ \geq \mathbb{P}(\|\lambda_1\|^2_2 \geq s_w/2) \geq p_w \]
which completes case 1.
Case 2: when $\|\lambda_2\|_2 \leq 1/k_0$ and $\lambda_1^T \theta_s z_t < 0$.

$$
\lambda_1^T w_t \geq \lambda_1^T (w_t + \theta_s z_t) \geq -|\lambda_1^T (w_t + \theta_s z_t)|
$$

$$
= -|\lambda_1^T z_{t+1} - \lambda_2^T u_{t+1}| \geq -|\lambda_1^T z_{t+1}| - |\lambda_2^T u_{t+1}| \geq -|\lambda_1^T z_{t+1}| - \|\lambda_2\|_2 b_u
$$

$$
\geq -|\lambda_1^T z_{t+1}| - b_u/k_0 \geq -|\lambda_1^T z_{t+1}| - s_w/4
$$

where the last inequality uses $k_0 \geq 4b_u/s_w$.

Further, notice that $k_0 \geq 2/\sqrt{3}$, so $\|\lambda_1\|_2^2 \leq 1/k_0^2 \leq 3/4$, thus, $\|\lambda_1\|_2^2 \geq 1/4$, which means $\|\lambda_1\|_2 \geq 1/2$. Therefore,

$$
P(\lambda_1^T w_t \leq -s_w/2) = P(\lambda_1^T w_t \leq -s_w/2) \geq P(\lambda_1^T w_t \leq -s_w) = P(-\lambda_1^T w_t \leq s_w) = p_w
$$

by $s_w/(2\|\lambda_1\|_2) \leq s_w$, and thus $-s_w/(2\|\lambda_1\|_2) \geq -s_w$, and Assumption 4.

Consequently,

$$
P(|\lambda_1^T z_{t+1}| \geq s_w) \geq P(|\lambda_1^T z_{t+1}| \geq s_w/4) = P(-|\lambda_1^T z_{t+1}| - s_w/4 \leq -s_w/2)
$$

$$
\geq P(\lambda_1^T w_t \leq -s_w/2) \geq p_w
$$

which completes case 2.

Case 3: when $\|\lambda_2\|_2 > 1/k_0$. Define $v = \tilde{q} s_\eta/k_0 = \min(\sqrt{3} \tilde{q} s_\eta/2, s_w \tilde{q} s_\eta/(4b_u))$. Define

$$
\Omega_1^2 = \{w_t \in \mathbb{R}^n \mid \lambda_1^T (w_t + \theta_s z_t) + \lambda_2^T (\pi_{t+1}(F^m_{t+1})) \geq 0\}
$$

$$
\Omega_2^2 = \{w_t \in \mathbb{R}^n \mid \lambda_1^T (w_t + \theta_s z_t) + \lambda_2^T (\pi_{t+1}(F^m_{t+1})) < 0\}
$$

Notice that $P(w_t \in \Omega_1^2) + P(w_t \in \Omega_2^2) = 1$.

$$
P(|\lambda_1^T z_{t+1}| \geq s_w) \geq P(|\lambda_1^T z_{t+1}| \geq v) = P(\lambda_1^T z_{t+1} \geq v) + P(\lambda_1^T z_{t+1} \leq -v)
$$

$$
\geq P(\lambda_1^T z_{t+1} \geq v, w_t \in \Omega_1^2) + P(\lambda_1^T z_{t+1} \leq -v, w_t \in \Omega_2^2)
$$

$$
\geq P(\lambda_2^T \tilde{q} \eta_{t+1} \geq v, w_t \in \Omega_1^2) + P(\lambda_2^T \tilde{q} \eta_{t+1} \leq -v, w_t \in \Omega_2^2)
$$

$$
= P(\lambda_2^T \tilde{q} \eta_{t+1} \geq v)P(w_t \in \Omega_1^2) + P(\lambda_2^T \tilde{q} \eta_{t+1} \leq -v)P(w_t \in \Omega_2^2)
$$

$$
\geq p_\eta
$$

where the last inequality is because of the following arguments. Notice that

$$
P(\lambda_2^T \tilde{q} \eta_{t+1} \geq v) = P(\lambda_2^T \tilde{q} \eta_{t+1}/\|\lambda_2\|_2 \geq v/\|\lambda_2\|_2)
$$

$$
= P(\lambda_2^T \tilde{q} \eta_{t+1}/\|\lambda_2\|_2 \geq v/(\|\lambda_2\|_2 \tilde{q} \eta))
$$

$$
\geq P(\lambda_2^T \tilde{q} \eta_{t+1}/\|\lambda_2\|_2 \geq k_0 v/(\tilde{q} \eta))
$$

$$
= P(\lambda_2^T \tilde{q} \eta_{t+1}/\|\lambda_2\|_2 \geq s_\eta) \geq p_\eta
$$

Then,

$$
P(\lambda_2^T \tilde{q} \eta_{t+1} \leq -v) = P(-\lambda_2^T \tilde{q} \eta_{t+1} \geq v) \geq p_\eta
$$

This completes the proof of Case 3. □

Finally, we apply Theorem 5. Notice that $d = m + n$ in our problem, and $\log \det(\hat{\Gamma}_{sb}^{-1}) = 2(m + n) \log (b_z/s_z) = O((m + n) \log (b_z/s_z))$ as $\tilde{\eta} \rightarrow 0$, $\nu_{\min}(\Gamma_{sb}) = s_z^2 = O(1/\tilde{\eta}^2)$ as $\tilde{\eta} \rightarrow 0$, and $p = p_z$ here. Therefore, for $T$ large enough, we have:

$$
\|\hat{\theta}_T - \theta_s\|_2 \leq O\left(\sqrt{n+m} \frac{\sqrt{\log(b_z/\tilde{\eta} + 1/\delta)}}{\sqrt{T \tilde{\eta}}}ight).
$$

□
B.2 Proof of Corollary 1

Corollary 1 follows directly from Theorem 1. We only need to verify the boundedness of the states and actions. In the following, we will show that \( u_t \in \mathbb{U} \) for all \( t \) and \( \|x_t\|_2 \leq O(\sqrt{m\bar{n}}) \) for all \( t \). Notice that though we can further show a much smaller bound \( \|x_t\|_2 \leq x_{\text{max}} \) with probability \((1 - p)\) in Theorem 3, Theorem 1 requires an almost sure bound and thus we provide a larger bound \( \|x_t\|_2 \leq O(\sqrt{m\bar{n}}) \) here.

In the following, we show that \( u_t \in \mathbb{U} \) for all \( t \) and \( \|x_t\|_2 \leq O(\sqrt{m\bar{n}}) \) for all \( t \).

**Lemma 10** (Action constraint satisfaction). When applying Algorithm 1, \( u_t \in \mathbb{U} \) for all \( t \) and for any \( w_k \in \mathcal{W} \).

**Proof.** Notice that \( u_t = \sum_{k=1}^{H^{(e-1)}} M_t[k] \hat{w}_{t-k} + \eta_t \). Hence, for any \( 1 \leq j \leq k_u \), we have

\[
D_{u,j}^{T} u_t = D_{u,j}^{T} \sum_{k=1}^{H^{(e-1)}} M_t[k] \hat{w}_{t-k} + D_{u,j}^{T} \eta_t \leq \sum_{k=1}^{H^{(e-1)}} \|D_{u,j}^{T} M_t[k]\|_1 \omega_{\text{max}} + \|D_u\|_\infty \|\eta_t\|_\infty = g_j^{u}(M_t) + \|D_u\|_\infty \|\eta_t\|_\infty
\]

Our goal is to show that \( g_j^{u}(M_t) + \|D_u\|_\infty \|\eta_t\|_\infty \leq d_{u,j} \) for all \( j \) and for all \( t \geq 0 \). This is straightforward when \( \bar{t}_1^{(e)} \leq t \leq \bar{t}_1^{(e)} + T_D^{(e)} - 1 \) and \( \bar{t}_2^{(e)} \leq t \leq T^{(e+1)} - 1 \). For example, when \( \bar{t}_1^{(e)} \leq t \leq \bar{t}_1^{(e)} + T_D^{(e)} - 1 \), we have \( M_t = M_1^{(e)} \) and \( \|\eta_t\|_\infty \leq \bar{\eta}^{(e)} \), which leads to \( g_j^{u}(M_1^{(e)}) + \|D_u\|_\infty \|\eta_t\|_\infty = g_j^{u}(M_1^{(e)}) + c_3 \|\bar{\eta}^{(e)}\|_\infty \leq d_{u,j} \) by RobustCE and Lemma 6. Similar results can be shown for \( \bar{t}_2^{(e)} \leq t \leq T^{(e+1)} - 1 \).

Next, we focus on the safe policy transition stages. It suffices to show that \( u_t = \sum_{k=1}^{H^{(e-1)}} M_t[k] \hat{w}_{t-k} + \eta_t \) in all stages of Algorithm 2. In the following, we will adopt the notations in Algorithm 2. In Step 1 of Algorithm 2, we have \( \|\eta_t\|_\infty \leq \bar{\eta}_{\text{min}} \leq \bar{\eta} \) and \( M_t \in \Omega \) by the convexity of \( \Omega \). Therefore, we have \( g_j^{u}(M_t) + \|D_u\|_\infty \|\eta_t\|_\infty = g_j^{u}(M_1^{(e)}) + c_3 \|\bar{\eta}^{(e)}\|_\infty \leq d_{u,j} \), where we used the definition of \( \Omega \) in RobustCE. In Step 2, we have \( \|\eta_t\|_\infty \leq \bar{\eta} \) and \( M_t \in \Omega ' \) by the convexity of \( \Omega ' \). Therefore, we have \( g_j^{u}(M_1^{(e)}) + \|D_u\|_\infty \|\eta_t\|_\infty = g_j^{u}(M_1^{(e)}) + c_3 \|\bar{\eta}^{(e)}\|_\infty \leq d_{u,j} \), where we used the definition of \( \Omega ' \) by RobustCE with input \( \bar{\eta} \).

**Lemma 11** (Almost sure upper bound on \( x_t \)). Consider DAP policy \( u_t = \sum_{k=1}^{H^{(e-1)}} M_t[k] \hat{w}_{t-k} + \eta_t \), where \( M_t \in \mathcal{M}_{H_t}, \{H_t\}_{t \geq 0} \) is non-decreasing, and \( \|\eta_t\|_\infty \leq \bar{\eta}_{\text{max}} \). Suppose \( H_0 \geq \log(2\kappa)/\log((1 - \gamma)^{-1}) \) and \( \bar{\eta}_{\text{max}} \leq \omega_{\text{max}}/\kappa_B \). Let \( \{x_t, u_t\}_{t \geq 0} \) denote the trajectory generated by this policy on the system with parameter \( \theta_s \) and disturbance \( w_t \). Then, there exists \( b_x = 4\sqrt{m\bar{n}}\omega_{\text{max}}/\gamma + 4\sqrt{m\bar{n}}\omega_{\text{max}}/\kappa_B \gamma/\gamma^2 = O(\sqrt{m\bar{n}}) \) such that

\[
\|x_t\|_2 \leq b_x, \quad \forall t \geq 0, \quad \forall w_k, \hat{w}_k \in \mathcal{W}
\]

This lemma is a natural extension of Lemma 2 in [Li et al., 2020] and the proof is deferred to Appendix G.3.

**Proof of Corollary 4** By letting \( \delta^{(e)} = \frac{p}{w^{(e)}} \) for \( e \geq 1 \), we have that \( \|\hat{\theta}^{(e)} - \theta_s\|_F \leq O(\sqrt{m\bar{n}}) \sqrt{\log(m\bar{n})} \sqrt{\log(1/\epsilon^{(e-1)})} \) w.p. \( 1 - p/(2e^2) \). Notice that \( \|\hat{\theta}^{(e)} - \theta_s\|_F \leq \|\hat{\theta}^{(e)} - \theta_s\|_2 \), which completes the proof.

C Feasibility

This appendix provides a proof for Theorem 2. We will first establish the recursive feasibility and then prove the initial feasibility. For notational simplicity, we define \( \Omega_0 := \Omega(\bar{\theta}^{(0)}, \epsilon_{x_s}^{(0)} + \epsilon_0, \epsilon_0^{(0)}) \).

**Proof of recursive feasibility:** To show that Algorithm 1 and 2 are feasible at all stages, we need to show that \( \Omega_1^{(e)}, \Omega_1^{(e)} \cap \Omega^{(e)}, \Omega_1^{(e)} \cap \Omega^{(e)}, \Omega_1^{(e+1)} \cap \Omega^{(e)} \) are all non-empty for \( e \geq 0 \). Notice that it suffices to show that \( \Omega_0 \subseteq \Omega_1^{(e)} \) and \( \Omega_0 \subseteq \Omega^{(e)} \) for all \( e \geq 0 \).

Consider \( \Omega_1^{(e)} \) for \( e \geq 0 \). Notice that \( \Omega_0 \subseteq \Omega_1^{(0)} \) by definition, so we will focus on \( e \geq 1 \) below. We first consider the action constraints. For any \( M \in \Omega_0 \), we have

\[
g_j^{u}(M) \leq d_{u,j} - \epsilon_{n,u}(\bar{\eta}^{(0)}), \quad \forall 1 \leq j \leq k_u.
\]

Since \( \bar{\eta}^{(0)} \geq \bar{\eta}^{(e)} \) by condition (ii) of Theorem 2, we have \( \epsilon_{n,u}(\bar{\eta}^{(0)}) \geq \epsilon_{n,u}(\bar{\eta}^{(e)}) \), so \( M \) satisfies the action constraints in \( \Omega_1^{(e)} \).

\[
g_j^{u}(M) \leq d_{u,j} - \epsilon_{n,u}(\bar{\eta}^{(e)}), \quad \forall 1 \leq j \leq k_u.
\]
Next, we consider the state constraints. Notice that \( \hat{\theta}(e) \in \Theta_{ini} \) by ModelEst, so \( \| \hat{\theta}(e) - \hat{\theta}(0) \|_F \leq r(0) \) for \( e \geq 1 \).

By Lemma \ref{lem:initial} for any \( M \in \Omega_0 \), we have
\[
g_i^r(M; \hat{\theta}(e)) \leq g_i^r(M; \hat{\theta}(0)) + \epsilon_e(r(0))
\leq d_{x,i} - \epsilon_e(0) - \epsilon_0 + \epsilon_e(r(0))
= d_{x,i} - \epsilon_e(0) - \epsilon_{\epsilon_e}(\hat{\theta}(e)) - \epsilon_e(H(0)) - \epsilon_e(\Delta_M(0),H(0)) - \epsilon_0 + \epsilon_e(r(0))
= d_{x,i} - \epsilon_{\epsilon_e}(\hat{\theta}(e)) - \epsilon_e(H(0)) - \epsilon_e(\Delta_M(0),H(0)) - \epsilon_0
\]
Further, since \( r(e) \leq r^{(1)} \leq \epsilon_0/(c_1\sqrt{m}) \) by condition (ii) of Theorem \ref{thm:main}, we have \( \epsilon_e(r(e)) \leq \epsilon_e(r^{(1)}) \leq \epsilon_0 \), so
\[
g_i^r(M; \hat{\theta}(e)) \leq d_{x,i} - \epsilon_{\epsilon_e}(\hat{\theta}(e)) - \epsilon_e(H(0)) - \epsilon_e(\Delta_M(0),H(0)) - \epsilon_0
\]
by condition (ii) of Theorem \ref{thm:main}. So \( M \in \Omega^{(e)}_1 \) for \( e \geq 1 \). Similarly, we can show \( M \in \Omega^{(e)}_0 \) for \( e \geq 0 \). This completes the recursive feasibility.

**Proof of initial feasibility.** By Lemma 4 and Corollary 2 in \cite{Li2020}, we can construct \( M_F \) with length \( H(0) \) based on \( K_F \) in Assumption \ref{ass:feasibility}, such that
\[
g_i^r(M_F; \theta_\star) \leq d_{x,i} - \epsilon_{F,x} + \epsilon_F(H(0))
g_i^n(M_F) \leq d_{u,j} - \epsilon_{F,u} + \epsilon_F(H(0)).
\]
Therefore, by Lemma \ref{lem:initial} we have
\[
g_i^r(M_F; \hat{\theta}(0)) \leq d_{x,i} - \epsilon_{F,x} + \epsilon_F(H(0)) + \epsilon_F(r(0))
g_i^n(M_F) \leq d_{u,j} - \epsilon_{F,u} + \epsilon_F(H(0)).
\]
Therefore, \( M_F \in \Omega_0 \) if \( (12) \) in Theorem \ref{thm:main} holds.

\section{Constraint Satisfaction}

This section provides a proof for the constraint satisfaction guarantee in Theorem \ref{thm:general}. Notice that the control constraint satisfaction has already been established in Lemma \ref{lem:control}. Hence, we will focus on state constraint satisfaction in this appendix. Firstly, we present and prove a general state constraint satisfaction lemma for time-varying approximate DAPs, which is more general than Lemma \ref{lem:state}. Secondly, we prove the state constraint satisfaction of our algorithms by showing that our algorithms satisfy the conditions in the general state constraint satisfaction lemma.

\subsection{A General State Constraint Satisfaction Lemma}

This subsection provides a general state constraint satisfaction lemma for time-varying approximate DAPs, which includes Lemma \ref{lem:state} as a special case.

**Lemma 12 (General State Constraint Satisfaction Lemma).** Consider the time-varying approximate DAPs in \ref{eq:DAP}, where \( M_t \in M_{H_t} \) for non-decreasing \( \{H_t\}_{t \geq 0} \), \( \hat{\theta}_t \in \Theta_{ini} \). Define
\[
\epsilon_{H,t} = (1 - \gamma)H_t \cdot \| D_x \|_{\infty} \kappa x_{max} \n
\epsilon_{v,t} = \sqrt{mn}H_t \Delta_{M,t} \cdot \| D_z \|_{\infty} w_{max} \kappa B \gamma^2 \n
\Delta_{M,t} = \max_{1 \leq k \leq H_t} \frac{\| M_t - M_{t-k} \|_F}{k}.
\]

\[\epsilon_1 + \epsilon_3 = O(\sqrt{m}H(1 - \gamma)^H) \text{ in } \cite{Li2020}, \text{ but we improve the bound to } \epsilon_F = O(\sqrt{m}H(1 - \gamma)^H). \text{ Specifically, } \epsilon_1 = O(\sqrt{m}(1 - \gamma)^H) \text{ remains unchanged, but we can show } \epsilon_1(H) = O(\sqrt{mn}(1 - \gamma)^H). \text{ This is because the proof of Lemma 1 in } \cite{Li2020} \text{ shows that } \epsilon_1 = O(b_\epsilon(1 - \gamma)^H). \text{ where } \| x_\epsilon \|_2 \leq b_\epsilon \text{ a.s.}. \text{ In Lemma } \ref{lem:general} \text{ we show } b_\epsilon = O(\sqrt{mn}) \text{ in this paper, so we have } \epsilon_1(H) = O(\sqrt{mn}(1 - \gamma)^H).\]
\[ \epsilon_{\theta,t} = c_1 \max_{0 \leq k \leq H_t} \| \hat{\theta}_{t-k} - \theta_* \|_F \]
\[ \epsilon_{\eta,x,t} = c_2 \sqrt{m} \max_{1 \leq k \leq H_t} \eta_{t-k}, \]

where \( c_1, c_2 \) are defined in Lemma 4 and Lemma 6 and we let \( \mathbf{M}_t = \mathbf{M}_0, \eta_t = 0, H_t = H_0, \hat{\theta}_t = \hat{\theta}_0, w_t = \hat{w}_t = x_t = 0 \), for \( t \leq -1 \).

For any \( t \geq 0, i \), if \( x_s \in \mathbb{X}, u_s \in \mathbb{U} \) for all \( s \leq t - 1 \) and

\[ g_t^*(\mathbf{M}_t; \hat{\theta}_t) \leq d_{x,i} - \epsilon_{H,t} - \epsilon_{\eta,x,t} - \epsilon_{\theta,t} - \epsilon_{v,t}, \quad \forall 1 \leq i \leq k_x, \quad (16) \]

then \( x_i \in \mathbb{X} \).

Consequently, if \( (16) \) holds and \( u_t \in \mathbb{U} \) for all \( t \geq 0 \), then \( x_t \in \mathbb{X} \) for all \( t \geq 0 \).

**Proof.** Consider stage \( t \geq 0 \). By Lemma 4 for any \( 1 \leq i \leq k_x \), we have

\[ D_{x,i}^T x_t = D_{x,i}^T A_{x,t^-1}^H x_t - H_t \]
\[ = \sum_{k=1}^{2H_t} \sum_{i=1}^{H_t} D_{x,i}^T A_{x}^i B_i M_{t-i} \| (k-i) \hat{w}_{t-k} K_{(1 \leq k-i \leq H_t-i)} \| + \sum_{i=1}^{H_t} D_{x,i}^T A_{x}^i w_{t-i} + \sum_{i=1}^{H_t} D_{x,i}^T A_{x}^i B_i \eta_{t-i} \]
\[ \leq \| D_x \|_{\infty} \kappa (1 - \gamma)^{H_t} x_{\max} + g_t^*(\mathbf{M}_{t-H_t:t-1}; \theta_*) \| D_x \|_{\infty} \kappa / \gamma \max_{1 \leq k \leq H_t} \| \hat{\theta}_{t-k} - \theta_* \|_F^2 \max_{1 \leq k \leq H_t} \| \eta_{t-k} \|_F \]
\[ \leq \epsilon_{H,t} + g_t^*(\mathbf{M}_t; \theta_*) + \epsilon_{v,t} + \| D_x \|_{\infty} \kappa / \gamma \max_{1 \leq k \leq H_t} \| \hat{\theta}_{t-k} - \theta_* \|_F^2 \max_{1 \leq k \leq H_t} \| \eta_{t-k} \|_F \]
\[ \leq \epsilon_{H,t} + g_t^*(\mathbf{M}_t; \hat{\theta}_t) + \epsilon_\theta + \| D_x \|_{\infty} \kappa / \gamma \max_{1 \leq k \leq H_t} \| \hat{\theta}_{t-k} - \theta_* \|_F^2 \max_{1 \leq k \leq H_t} \| \eta_{t-k} \|_F \]
\[ \leq \epsilon_{H,t} + g_t^*(\mathbf{M}_t; \hat{\theta}_t) + \epsilon_{\theta,t} + \epsilon_{v,t} + \epsilon_{\eta,x,t} \]

where we used Lemma 5, Lemma 4, and \( x_s \in \mathbb{X}, u_s \in \mathbb{U} \) for all \( s \leq t - 1 \), and (16). The last inequality guarantees \( x_t \in \mathbb{X} \).

Therefore, the proof can be completed by induction.

**D.2 Proof of Theorem 3**

Define an event

\[ \mathcal{E}_{\text{safe}} = \{ \theta_* \in \bigcap_{c=0}^{N-1} \Theta^{(c)} \}. \quad (17) \]

Notice that

\[ \mathbb{P}(\mathcal{E}_{\text{safe}}) = 1 - \mathbb{P}(\mathcal{E}_{\text{safe}}^c) \geq 1 - \sum_{c=0}^{N} \mathbb{P}(\theta_* \not\in \Theta^{(c)}) \geq 1 - \sum_{c=1}^{N} p/(2e^2) \geq 1 - p \]
where we used Corollary 1 and \( \theta_s \in \Theta^{(0)} = \Theta_{\text{ini}} \). In the following, we will condition on the event \( \mathcal{E}_{\text{safe}} \) and show \( x_t \in X \) for all \( t \geq 0 \) under this event. By Lemma 12, we only need to show (16) for any \( t \).

We discuss three possible cases based on the value of \( t \). We introduce some notations for our case-by-case discussion: let \( W_1^{(e)}, W_2^{(e)} \) denote the \( W_1, W_2 \) defined in Algorithm 2 during the transition in Phase 1, and let \( \tilde{W}_1^{(e)}, \tilde{W}_2^{(e)} \) denote the \( W_1, W_2 \) defined in Algorithm 2 during the transition in Phase 2.

**Case 1:** when \( T^{(e)} \leq t \leq T^{(e)} + W_1^{(e)} - 1 \). In this case, \( M_t \in \Omega^{(e-1)} \), so

\[
g_t^x(M_t; \hat{g}^{(e)}) \leq d_{x,t} - \epsilon_H(H^{(e-1)}) - \epsilon_v(\Delta^{(e-1)}_M) - \epsilon_\theta(r^{(e)})
\]

Notice that \( H_t = H^{(e-1)} \), so \( \epsilon_{H,t} = \epsilon_H(H^{(e-1)}) \). Further, by our algorithm design, \( \Delta_{M,t} \leq \Delta^{(e-1)}_M \). Since \( \tilde{W}_1^{(e)} \geq H^{(e-1)} \) and \( \eta_k = 0 \) for \( t^{(e)} + 1 \leq k \leq t \), we have \( \epsilon_{\eta,x,t} = \max_{1 \leq k \leq H_t} \epsilon_2 \sqrt{\eta_1} - \epsilon_{H,t} \). Next, since \( r^{(e)} \leq r^{(e-1)} \) by Condition 2 of Theorem 2 for \( e \geq 1 \), we satisfy (16).

**Case 2:** when \( T^{(e)} + W_1^{(e)} \leq t \leq t^{(e)} + T_2^{(e)} + \tilde{W}_1^{(e)} - 1 \). We have \( M_t \in \Omega^{(e)} \). So

\[
g_t^x(M_t; \hat{g}^{(e)}) \leq d_{x,t} - \epsilon_H(H^{(e)}) - \epsilon_v(\Delta^{(e)}_M) - \epsilon_{\eta,x,t}(\hat{\eta}^{(e)}) - \epsilon_\theta(r^{(e)})
\]

Next, \( H_t = H^{(e)} \), since \( W_1^{(e)} \geq H^{(e)} \), we have \( \epsilon_{\eta,t} \leq \epsilon_w(r^{(e)}) \), and \( \epsilon_{\eta,t} = \epsilon_v(\Delta^{(e)}_M) \). Since we take minimum over potential \( \hat{\eta} \) in Step 1 of Algorithm 2 and \( W_1^{(e)} \geq H^{(e)} \), we have \( \epsilon_{\eta,x,t} \leq \epsilon_{\eta}(\hat{\eta}^{(e)}) \). So we satisfy (16).

**Case 3:** when \( t^{(e)} + T_2^{(e)} + \tilde{W}_1^{(e)} \leq t \leq T^{(e+1)} - 1 \). We have \( M_t \in \Omega^{(e)} \). So

\[
g_t^x(M_t; \hat{g}^{(e+1)}) \leq d_{x,t} - \epsilon_H(H^{(e)}) - \epsilon_v(\Delta^{(e)}_M) - \epsilon_\theta(r^{(e+1)})
\]

Next, \( H_t = H^{(e)} \), since \( W_1^{(e)} \geq H^{(e)} \), we have \( \epsilon_{\eta,t} \leq \epsilon_w(r^{(e+1)}) \) by \( r^{(e+1)} \leq r^{(e)} \), and \( \epsilon_{\eta,t} \leq \epsilon_v(\Delta^{(e)}_M) \). Since we take minimum over potential \( \hat{\eta} \) in Step 1 of Algorithm 2 and \( W_1^{(e)} \geq H^{(e)} \), we have \( \epsilon_{\eta,x,t} = 0 \). So we satisfy (16).

In conclusion, we satisfy (16) for all \( t \geq 0 \). By Lemma 12 and Lemma 10, we can show state constraint satisfaction under \( \mathcal{E}_{\text{safe}} \).

### E Regret Analysis

In this section, we provide a proof for Theorem 4. Specifically, we first prove the regret bound and then verify the conditions for feasibility and constraint satisfaction.

#### E.1 Proof of the Regret Bound

Our proof of the regret bound relies on decomposing the regret into several parts and bounding each part. First, we decompose the \( T \) stages into two parts and decompose the regret accordingly. For \( e \geq 0 \), define

\[
T_1^{(e)} = \{ T^{(e)} \leq t \leq t_{2}(e) + H^{(e)} - 1 \}, \quad T_2^{(e)} = \{ t_{2}(e) + H^{(e)} \leq t \leq T^{(e+1)} - 1 \}.
\]

Then, decompose the regret by the stage decomposition below:

\[
\text{Regret} = \sum_{t=0}^{T-1} (l(x_t, u_t) - J^*) = \sum_{e=0}^{N-1} \sum_{t \in T_1^{(e)}} (l(x_t, u_t) - J^*) + \sum_{e=0}^{N-1} \sum_{t \in T_2^{(e)}} (l(x_t, u_t) - J^*) \tag{18}
\]

The first term can be bounded straightforwardly by the fact that the single-stage regret is bounded and the total number of stages in \( T_1^{(e)} \) for all \( e \) can be bounded by \( O(T^{2/3}) \).

**Lemma 13** (Regret Bound of the First Term). When the event \( \mathcal{E}_{\text{safe}} \) defined in (17) happens, under the conditions in Theorem 4, we have

\[
\sum_{e=0}^{N-1} \sum_{t \in T_1^{(e)}} (l(x_t, u_t) - J^*) \leq O(T^{2/3})
\]
Proof. When \( E_{\text{safe}} \) is true, by Theorem 3 we have \( x_t \in X \) and \( u_t \in \mathcal{U} \), thus \( \|x_t\|_2 \leq x_{\max} \) and \( \|u\|_2 \leq u_{\max} \) and \( l(x_t, u_t) - J^* \leq \|Q\|_2^2 x_{\max}^2 + \|R\|_2 u_{\max}^2 = O(1) \).

Next, we bound the number of stages in \( T_1^{(e)} \). Under the conditions in Theorem 4, the number of stages in \( T_1^{(e)} \) is \( T_D^{(e)} + H^{(e)} \) plus the safe policy transition stages in Phase 1 and Phase 2. Since \( \mathcal{M}_{H^{(e)}} \) is a bounded set, the number of stages in SafeTaxi between any two policies in \( \mathcal{M}_{H^{(e)}} \) can be bounded by \( O(\max(1/\Delta_M^{(e)}, H^{(e)})) = \tilde{O}(\sqrt{\text{min}(T^{(e+1)})^{1/3}}) \), where we used \( H^{(e)} = O(\log(T^{(e+1)})) \) and \( \Delta_M^{(e)} = O(\frac{\log \tilde{\epsilon} F}{\sqrt{\text{min}(H^{(e)}(T^{(e+1)})^{1/3})}}) \). Further, by \( T^{(e+1)} = 2T^{(e)} \), \( T_D^{(e)} = (T^{(e+1)} - T^{(e)})^{2/3} \), we have \( T_D^{(e)} = O((T^{(e+1)})^{2/3}) \). Consequently, the total number of stages in \( T_1^{(e)} \) can be bounded by \( O((T^{(e+1)})^{2/3}) \) (notice that \( T^{(e+1)})^{1/3} \geq \sqrt{\text{min}(T^{(e+1)})} \) by our condition of \( T^{(1)} \) in Theorem 4.

Finally, with the help of the algebraic fact in Lemma 14, we are able to bound the total regret in all episodes by \( O((T^{(e+1)})^{2/3}) \).

Lemma 14 is a technical fact that will be used throughout our regret proof.

Lemma 14 (An algebraic fact). When \( T^{(e)} = 2^{e-1} T^{(1)} \), and \( T^{(N)} > T > T^{(N-1)} \), \( N \leq O(\log T) \). Further, for any \( \alpha > 0 \), we have

\[
\sum_{e=1}^{N} (T^{(e)})^\alpha = O(T^\alpha)
\]

Proof. By \( T \geq T^{(N-1)} \geq 2^{(N-2)} \), we have \( \log T \geq (N - 2) \log(2) \), so \( N \leq O(\log(T)) \). Further, \( \sum_{e=1}^{N} (T^{(e)})^\alpha = \sum_{e=1}^{N} (2^{e-1})^\alpha (T^{(1)})^\alpha \leq O((2^N)^\alpha (T^{(1)})^\alpha) \leq O(T^\alpha) \).

The second term in \((18)\) is more complicated to bound, so we further decompose it into four parts as follows.

\[
\sum_{e=0}^{N-1} \sum_{t \in T_2^{(e)}} (l(x_t, u_t) - J^*) = \sum_{e=0}^{N-1} \sum_{t \in T_2^{(e)}} (l(x_t, u_t) - l(\hat{x}_t, \hat{u}_t)) + \sum_{e=0}^{N-1} \sum_{t \in T_2^{(e)}} (l(\hat{x}_t, \hat{u}_t) - f(M^{(e)}; \theta_*)) + \sum_{e=0}^{N-1} \sum_{t \in T_2^{(e)}} (f(M^{(e)}; \theta_*) - f(M_{H^{(e)}}^{*}; \theta_*) - J^*)
\]

where we introduced auxiliary states \( \hat{x}_t \) and actions \( \hat{u}_t \) defined as

\[
\hat{x}_t = \sum_{k=1}^{2H} \Phi_k^{(e)}(M^{(e)}; \theta_*) w_{t-k}, \quad \hat{u}_t = \sum_{k=1}^{H} M^{(e)}[k] w_{t-k},
\]

which are basically the approximate states and the actions generated by the disturbance-action policy \( M^{(e)} \) computed in Phase 2 of Algorithm 1 when the actual disturbances \( w_{t-k} \) are known. We also introduce an auxiliary policy \( M_{H^{(e)}}^{*} \) in Part iii, which is defined as the optimal DAP policy in \((3)\) with a memory length \( H = H^{(e)} \) under a known model, i.e.

\[
M_{H^{(e)}}^{*} = \arg \min_{\mathcal{M} \in \Omega_{\epsilon}^{(e)}} f(M; \theta_*), \quad \Omega_{\epsilon}^{(e)} \text{ is defined by } (6) \text{ with } H = H^{(e)}.
\]

The rest of the proof is to bound Parts i-iv. Establishing the bound on Part i is the major part of the proof and the bound on Part ii is the dominating term in our regret bound, so we will present our bound on Part ii first. Then, we will establish bounds on Parts i, ii, and iv.

E.1.1 Bound on Part iii

Notice that \( M^{(e)} \) is the solution to the CCE program in \((10)\) and \( M_{H^{(e)}}^{*} \) is the solution to the optimal DAP program in \((3)\). Further, the CCE program \((10)\) can be viewed as a slightly perturbed version of the optimal DAP program \((3)\) due to model estimation errors and constraint-tightening terms. Therefore, we can bound Part iii by the perturbation analysis.

Specifically, we establish the following general perturbation bound. This bound is not only useful for our bound on Part iii but also helps the discussions after Theorem 4 on the reasons for including the pure exploitation phases.
Lemma 15 (Perturbation analysis for CCE). Consider a fixed memory length $H \geq \log(2\kappa)/\log((1 - \gamma)^{-1})$ and $\theta_1, \theta_2 \in \Theta_m$. Consider two CCE programs $M_1 = \arg \min_{M \in \Omega(\theta_1, \epsilon_x, \epsilon_u)} f(M; \theta_1)$ and $M_2 = \arg \min_{M \in \Omega(\theta_2, \epsilon_x, \epsilon_u)} f(M; \theta_2)$. Suppose there exists $\epsilon_g > 0$ such that $\Omega(\theta_1, \epsilon_x + \epsilon_g, \epsilon_u + \epsilon_g) \cap \Omega(\theta_2, \epsilon_x + \epsilon_g, \epsilon_u + \epsilon_g)$ is non-empty. Then, we have

$$f(M_1, \theta_2) - f(M_2, \theta_2) \leq O(mn \log(mn) + (\sqrt{mn} + \sqrt{k_x + k_u})n \sqrt{mH} \max(|\epsilon_x - \epsilon_x|, |\epsilon_u - \epsilon_u|))/\epsilon_g$$

where $\|\theta_1 - \theta_2\|_F \leq r$.

Proof. Notice that both the objective functions and the constraints are different in the two CCE programs above, so we introduce an auxiliary policy $M_3 = \arg \min_{M \in \Omega(\theta_1, \epsilon_x, \epsilon_u)} f(M; \theta_2)$ to discuss the perturbation bounds on cost differences and constraint differences separately. We will first bound $f(M_1, \theta_2) - f(M_3, \theta_2)$ and then bound $f(M_3, \theta_2) - f(M_2, \theta_2)$ below.

**Perturbation on the cost functions.**

$$f(M_1, \theta_2) - f(M_3, \theta_2) = f(M_1, \theta_2) - f(M_1, \theta_1) + f(M_1, \theta_1) - f(M_3, \theta_1) + f(M_3, \theta_1) - f(M_3, \theta_2) \leq f(M_1, \theta_2) - f(M_1, \theta_1) + f(M_3, \theta_1) - f(M_3, \theta_2) \leq O(mn \||\theta_1 - \theta_2\||_F)$$

where the first inequality is because $M_1$ and $M_3$ are in the same set and $M_1$ minimizes the cost $f(M, \theta_1)$ in this set, and the second inequality is because of the following perturbation lemma on the cost functions.

Lemma 16 (Perturbation bound on $f$ with respect to $\theta$). For any $H \geq \log(2\kappa)/\log((1 - \gamma)^{-1})$, $M \in M_H$, any $\theta, \hat{\theta} \in \Theta_m$, we have

$$|f(M; \theta) - f(M; \hat{\theta})| \leq O(mn)$$

where $\||\theta - \hat{\theta}\||_F \leq r$.

Proof. We let $\tilde{x}_i(\theta)$ and $\tilde{x}_i(\hat{\theta})$ denote the approximate states defined by Proposition 11 and we omit $M$ in this proof for notational simplicity. Notice that

$$\|\tilde{x}(\theta) - \tilde{x}(\hat{\theta})\|_2 = \|\sum_{k=1}^{2H} (\Phi^k_\theta - \Phi^k_{\hat{\theta}})w_{t-k}\|_2 \leq \sum_{k=1}^{2H} \|\Phi^k_\theta - \Phi^k_{\hat{\theta}}\|_2 w_{t-k} \leq \sum_{k=1}^{2H} \|\Phi^k_\theta - \Phi^k_{\hat{\theta}}\|_2 w_{t-k}$$

$$\leq \sum_{k=1}^{2H} \|\Phi^k_\theta - \Phi^k_{\hat{\theta}}\|_2 \sqrt{\max w_{\text{max}}} + \sum_{k=1}^{2H} \|\Phi^k_\theta - \Phi^k_{\hat{\theta}}\|_2 \sqrt{\max w_{\text{max}}} \|\theta - \hat{\theta}\|_F$$

$$\leq \sum_{k=1}^{2H} \|\Phi^k_\theta - \Phi^k_{\hat{\theta}}\|_2 \sqrt{\max w_{\text{max}}} O(k(1 - \gamma)^{k-1}||\theta - \hat{\theta}\|_F) + \sum_{k=1}^{2H} \|\Phi^k_\theta - \Phi^k_{\hat{\theta}}\|_2 \sqrt{\max w_{\text{max}}} O(k(1 - \gamma)^{k-1}||\theta - \hat{\theta}\|_F)$$

$$\leq O(\|\theta - \hat{\theta}\|_F) + \sum_{k=1}^{2H} \|\theta - \hat{\theta}\|_F \sqrt{\max w_{\text{max}}} O(k(1 - \gamma)^{k-1}||\theta - \hat{\theta}\|_F)$$

where we used Lemma 25 in the third and fourth inequality.

**Perturbation on the constraints.** Our bound on $f(M_1, \theta_2) - f(M_2, \theta_2)$ is established based on Proposition 2 and Lemma 9 in [11 et al. (2020)]. The major difference is that [11 et al. (2020)] only considers changes in the right-hand-side of the constraint inequalities but in our setting, the left-hand-side of the constraint inequalities also change. To handle this difference, we introduce two auxiliary sets $\Omega_1$ and $\Omega_2$ with the same left-hand-sides in the constraint inequalities.

25
such that $\Omega_1 = \Omega(\theta_1, \epsilon_{x_1}, \epsilon_{u_1})$ and $\Omega_2 = \Omega(\theta_2, \epsilon_{x_2}, \epsilon_{u_2})$, which is achieved by adding inactive inequality constraints. Specifically, define

$$\Omega_1 = \Omega(\theta_1, \epsilon_{x_1}, \epsilon_{u_1}) \cap \Omega(\theta_2, \epsilon_{x_2} - \epsilon(r), \epsilon_{u_1}), \quad \Omega_2 = \Omega(\theta_2, \epsilon_{x_2}, \epsilon_{u_2}) \cap \Omega(\theta_1, \epsilon_{x_2} - \epsilon(r), \epsilon_{u_2}).$$

Notice that the constraints in $\Omega(\theta_2, \epsilon_{x_2} - \epsilon(r), \epsilon_{u_1})$ and $\Omega(\theta_1, \epsilon_{x_2} - \epsilon(r), \epsilon_{u_2})$ are inactive due to Lemma 4. Further, notice that $\Omega_1, \Omega_2$ are both polytopes.

Another difference between our setting and that in Proposition 2 of [Li et al. 2020] is that $\Omega_1$ may not be a subset of $\Omega_2$ and vice versa, so we need to generalize Proposition 2 to this setting as follows.

**Lemma 17 (Extension from Proposition 2 [Li et al. 2020]).** Consider two polytopes: $\Omega_1 = \{x : Cx \leq h - \Delta_1\}, \Omega_2 = \{x : Cx \leq h - \Delta_2\}$, where $\Delta_1, \Delta_2$ are two vectors. Define $\Delta_0 = \min(\Delta_1, \Delta_2)$ elementwise. Define $\Delta_3 = \max(\Delta_1, \Delta_2)$ elementwise. Suppose the $l_2$-diameter of $\Omega_0$ is $d_{\Omega_0}$, $f(x)$ is $L$-Lipschitz continuous, and there exists $x_F \in \Omega_3$, then we have

$$\left| \min_{\Omega_1} f(x) - \min_{\Omega_2} f(x) \right| \leq \frac{2Ld_{\Omega_0} \|\Delta_1 - \Delta_2\|_{\infty}}{\min_{i : (\Delta_1)_i \neq (\Delta_2)_i} (h - \Delta_3 - Cx_F)}.$$

The proof is deferred to Appendix G.5.

Now, we are ready to bound $f(M_3, \theta_2) - f(M_2, \theta_2)$. By our definitions and discussions above, we have $f(M_3, \theta_2) = \min_{\mathcal{M} \in \Theta_{mi}} f(M; \theta_2)$ and $f(M_2, \theta_2) = \min_{\mathcal{M} \in \Theta_{mi}} f(M; \theta_2)$. Further, notice that $\Omega_1, \Omega_2$ are both polytopes and can be written in the form of $\{\hat{P} : \hat{C} \hat{P} \leq h - \Delta_1\}, \{\hat{P} : \hat{C} \hat{P} \leq h - \Delta_2\}$ by introducing auxiliary variables to represent the absolute values (see Lemma 10 in [Li et al. 2020] for more details). Therefore, we can apply Lemma 17 to obtain the bound on $f(M_3, \theta_2) - f(M_2, \theta_2)$ by bounding the corresponding constants $d_{\Omega_0}, L, \|\Delta_1 - \Delta_2\|_{\infty}$, and $\min_{i : (\Delta_1)_i \neq (\Delta_2)_i} (h - \Delta_3 - Cx_F)$.

As the proof of Lemma 9, we can show the $l_2$-diameter of $\Omega_0$ is $d_{\Omega_0} = O(\sqrt{mn + k_x + k_u})$ and $\|\Delta_1 - \Delta_2\|_{\infty} = \max(|\epsilon_{x_1} - \epsilon_{x_2}|, |\epsilon_{u_1} - \epsilon_{u_2}|)$. Further, the Lipschitz factor $L$ can be obtained from the gradient bound $L = G_f = O(\sqrt{mnH})$ as provided below, whose proof is provided in Appendix G.4.

**Lemma 18 (Gradient bound of $f(M; \theta)$).** For any $H \geq 1$, $M \in \mathcal{M}_H$, $\theta \in \Theta_{mi}$, we have $\|\nabla f(M; \theta)\|_F \leq G_f = O(\sqrt{mnH})$.

Next, since $\Omega(\theta_1, \epsilon_{x_1} + \epsilon_g, \epsilon_{u_1} + \epsilon_g) \cap \Omega(\theta_2, \epsilon_{x_2} + \epsilon_g, \epsilon_{u_2} + \epsilon_g)$ is non-empty, there exists $\hat{P}_F \in \Omega_3$ such that $\min_{i : (\Delta_1)_i \neq (\Delta_2)_i} (h - \Delta_3 - Cx_F)_i \geq \epsilon_0$. Lastly, by applying the constants above to Lemma 17, we can show $f(M_3, \theta_2) - f(M_2, \theta_2) \leq O((\sqrt{mn + k_x + k_u}) \sqrt{nmH}) \max(|\epsilon_{x_1} - \epsilon_{x_2}|, |\epsilon_{u_1} - \epsilon_{u_2}|)/\epsilon_g$.

The proof of Lemma 15 is completed by combining the bounds on $f(M_1, \theta_2) - f(M_3, \theta_2)$ and $f(M_3, \theta_2) - f(M_2, \theta_2)$.

Based on Lemma 15, we can show the following bound on Part iii when $\mathcal{E}_{safe}$ is true.

$$\text{Part } iii = \sum_{e=0}^{N-1} \sum_{t \in \mathcal{T}_{(e)}} (f(M^{(e)}; \theta_2) - f(M^{*(H(e))}; \theta_2)) \leq O((n^2m^2 + n^{2.5}m^{1.5}) \sqrt{mn + k_x + k_u}T^{2/3}) \quad (19)$$

The proof for (19) is provided below. For each $e \geq 0$, notice that $M^{(e)} = \arg \min_{M \in \Omega(\theta^{(e+1)}, \epsilon^{(e)}, 0)} f(M; \theta^{(e+1)})$, where we define $\epsilon^{(e)} = \epsilon_0 + \epsilon_{H}(H^{(e)}) + \epsilon_v(\Delta^{(e)}_{H}, H^{(e)})$; and $M^{*(H(e))} = \arg \min_{M \in \Omega(\theta, \epsilon_{H}(H^{(e)}), 0)} f(M; \theta_e)$. Therefore, we can apply Lemma 15. Notice that $\theta^{(e+1)}$, $\theta_e \in \Theta_{mi}$ is large enough. Further, we have $\epsilon_g = \min(\epsilon_{F_x}, \epsilon_{F_u})/A > 0$ such that $\Omega(\theta^{(e+1)}, \epsilon^{(e)}, \epsilon_g, \epsilon_g) \cap \Omega(\theta_e, \epsilon_{H}(H^{(e)}))$ is not empty due to the feasibility conditions in Theorem 2. Therefore, when $\mathcal{E}_{safe}$ is true, by Lemma 15 under our choices of parameters in Theorem 4, we have

$$f(M^{(e)}; \theta_2) - f(M^{*(H(e))}; \theta_2) \leq O(mn^{(e+1)} + (\sqrt{mn + k_x + k_u}) n \sqrt{mnH} \epsilon_0(\Delta^{(e)}_{H}, H^{(e)}))$$

$$\leq \hat{O}(n \sqrt{mn + k_x + k_u}) n \sqrt{mn} T^{(e+1) - 1/3} \leq \hat{O} \left( (n^2m^2 + n^{2.5}m^{1.5}) \sqrt{mn + k_u} T^{(e+1) - 1/3} \right)$$

Consequently, by Lemma 14, we prove the bound (19).
E.1.2 Bound on Part i

When $E_{\text{safe}}$ is true, we are able to show

$$\text{Part } i = \sum_{\epsilon} \sum_{t \in T_{\epsilon}^{(c)}} l(x_{\epsilon}, u_{\epsilon}) - l(\hat{x}_{\epsilon}, \hat{u}_{\epsilon}) \leq \tilde{O}(n\sqrt{m}\sqrt{m + nT^{2/3}}).$$  \hspace{1cm} (20)

The proof is provided below.

Firstly, under $E_{\text{safe}}$, we have $x_{\epsilon} \in X$ and $u_{\epsilon} \in U$, so $\|x_{\epsilon}\|_2 \leq O(1)$, $\|u_{\epsilon}\|_2 \leq O(1)$. Further, by the definitions of $\hat{x}_{\epsilon}, \hat{u}_{\epsilon}$ and the proof of Theorem 3, we can also verify that $\hat{x}_{\epsilon} \in X$ and $\hat{u}_{\epsilon} \in U$ based on the proof of Lemma 10 and Lemma 12. Therefore, we have $\|\hat{x}_{\epsilon}\|_2 \leq O(1)$, $\|\hat{u}_{\epsilon}\|_2 \leq O(1)$

Next, by Lemma 8 for $t \in T_{\epsilon}^{(c)}$, we can write $x_{\epsilon}, \hat{x}_{\epsilon}, u_{\epsilon}, \hat{u}_{\epsilon}$ as

$$x_{\epsilon} = A_{\epsilon}^{H_{\epsilon}} x_{\epsilon-H_{\epsilon}} + \sum_{k=2}^{2H_{\epsilon}} \sum_{i=1}^{H_{\epsilon}} A_{\epsilon}^{i-1} B_i M_{\epsilon-i}[k-i] \hat{u}_{\epsilon-k} I_{\{1 \leq k-i \leq H_{\epsilon-i}\}} + \sum_{i=1}^{H_{\epsilon}} A_{\epsilon}^{i-1} w_{\epsilon-i},$$

$$\hat{x}_{\epsilon} = A_{\epsilon}^{H_{\epsilon}} x_{\epsilon-H_{\epsilon}} + \sum_{k=2}^{2H_{\epsilon}} \sum_{i=1}^{H_{\epsilon}} A_{\epsilon}^{i-1} B_i M_{\epsilon-i}[k-i] w_{\epsilon-k} I_{\{1 \leq k-i \leq H_{\epsilon-i}\}} + \sum_{i=1}^{H_{\epsilon}} A_{\epsilon}^{i-1} w_{\epsilon-i},$$

$$u_{\epsilon} = \sum_{t=1}^{H_{\epsilon}} M_{\epsilon}[k] \hat{u}_{\epsilon-k}, \quad \hat{u}_{\epsilon} = \sum_{t=1}^{H_{\epsilon}} M_{\epsilon}[k] w_{\epsilon-k}.$$

Hence, we can bound $\|x_{\epsilon} - \hat{x}_{\epsilon}\|_2$ and $\|u_{\epsilon} - \hat{u}_{\epsilon}\|_2$ by Lemma 5 below:

$$\|x_{\epsilon} - \hat{x}_{\epsilon}\|_2 \leq O((1 - \gamma)^{H_{\epsilon}} + \sqrt{mn}r_{\epsilon}^{(\epsilon+1)}) = \tilde{O}(nm\sqrt{m + n}(T^{(\epsilon+1)})^{-1/3})$$

$$\|u_{\epsilon} - \hat{u}_{\epsilon}\|_2 \leq O(\sqrt{mn}r_{\epsilon}^{(\epsilon+1)}) = \tilde{O}(nm\sqrt{m + n}(T^{(\epsilon+1)})^{-1/3})$$

Consequently, by applying Lemma 14 and the quadratic structure of $l(x, u)$, we can bound the Part i by

$$\sum_{\epsilon} \sum_{t \in T_{\epsilon}^{(c)}} (l(x_{\epsilon}, u_{\epsilon}) - l(\hat{x}_{\epsilon}, \hat{u}_{\epsilon})) \leq \sum_{\epsilon} T^{(\epsilon+1)} \tilde{O}(nm\sqrt{m + n}(T^{(\epsilon+1)})^{-1/3}) \leq \tilde{O}(nm\sqrt{m + nT^{2/3}}).$$

E.1.3 Bound on Part ii

Lemma 19 (Bound on Part ii). With probability $1 - p$, Part ii $\leq \tilde{O}(mn\sqrt{T}).$

Notice that this part is not a dominating term in the regret bound. The proof relies on a martingale concentration analysis and is very technical, so we defer it to Appendix G.6.

E.1.4 Bound on Part iv

In the following, we will show that

$$\text{Part } iv = \sum_{\epsilon=0}^{N-1} \sum_{t \in T_{\epsilon}^{(c)}} (f(M_{H_{\epsilon}}; \theta_{\epsilon}) - J^{*}) = \tilde{O}(n\sqrt{m\sqrt{mn + k_c\sqrt{n}}})$$  \hspace{1cm} (21)

The proof is provided below.

Remember that $J^{*}$ is generated by the optimal safe linear policy $K^{*}$. By Lemma 4 and Corollary 2 in Li et al. (2020), for a memory length $H^{(c)}$, we can define $M_{H^{(c)}}(K^{*}) \in \Omega(\theta_{\epsilon}, -\epsilon_{\epsilon}^{(\epsilon)}), 0)$, where $\epsilon_{\epsilon}^{(\epsilon)} = \sqrt{mn}(1 - \gamma)^{H^{(\epsilon)}}$ corresponds to $\epsilon_1 + \epsilon_3$ in Li et al. (2020) with $H = H^{(\epsilon)}$. Further, by Lemma 6 in Li et al. (2020), we have

$$f(M_{H^{(\epsilon)}}(K^{*}); \theta_{\epsilon}) - J^{*} = \lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(M_{H^{(\epsilon)}}(K^{*}); \theta_{\epsilon}) - \mathbb{E}(l(x^{*_t}, u^{*_t})) \leq O(n^2 m(H^{(\epsilon)})^{2}(1 - \gamma)^{H^{(\epsilon)}})^{2/3}\quad (12)$$

As discussed in footnote 11, our $\epsilon_{\epsilon}^{(\epsilon)}$ has smaller dependence on $n, m, H$ compared with Li et al. (2020).
In addition, we have
\[
f(M^*_H(\cdot; \theta_*) - f(M_H(\cdot; \theta_*) \leq f(M_H(\cdot; \theta_*)) - \min_{M \in \Omega(\theta_*, \epsilon^i(\cdot, \cdot), 0)} f(M; \theta_*)
\leq \hat{O}(n\sqrt{m} \sqrt{mn + k_x + k_u(\epsilon_P + \epsilon_H(H^{(e)})))}
\leq \hat{O}(n\sqrt{m} \sqrt{mn + k_v \sqrt{mn(1 - \gamma)}^H(\epsilon^i)))
\]
By combining the bounds above and by choosing \( H^{(e)} \geq \log(T^{(e+1)})/\log((1 - \gamma)^{-1}) \), we have
\[
f(M^*_H(\cdot; \theta_*) - J^* = f(M_H(\cdot; \theta_*) - f(M_H(\cdot; \theta_*)) + (M_H(\cdot; \theta_*); \theta_*) - J^*
\leq \hat{O}(n\sqrt{m} \sqrt{mn + k_v \sqrt{mn/T^{(e+1)}}}),
\]
which directly leads to the bound (21) on Part iv.

Completing the proof of the regret bound.

By combining (20), (19), (21), and Lemma 19 we obtain our regret bound in Theorem 4. Notice that (20), (19), (21) all condition on \( \epsilon_{safe} \), and \( \epsilon_{safe} \) holds w.p. \( 1 - p \). But Lemma 19 conditions on a different event and that event also holds with probability \( 1 - p \). Putting them together, we have that our regret bound holds w.p. \( 1 - 2p \).

E.2 Condition Verification for Feasibility and Constraint Satisfaction

In this subsection, we briefly show that there exist parameters characterized by Theorem 4 that satisfy the conditions for feasibility and constraint satisfaction in Theorem 2 and Theorem 3 which include: \( T^{(e)}_D \) satisfying the condition in Theorem 1 (Corollary 1, condition), condition (12), condition (ii) of Theorem 2 and \( T^{(e+1)} \geq t_2^{(e)} \).

Firstly, in Corollary 1 we need \( T^{(e)}_D \geq O(\log(2e^2/p) + (m + n) \log(1/\bar{\eta})) \) for \( e \geq 0 \). By our choices, we have \( T^{(e)}_D = (T^{(\min(e, 1)})^{2/3} + T^{(e+1)} = 2T^{(e)} \), so \( T^{(e)}_D \) increases exponentially. Therefore, \( T^{(e)}_D \geq O(\log(2e^2/p) + (m + n) \log(1/\bar{\eta})) \) can be guaranteed if \( T^{(1)} \geq O(\log(1/p) + (m + n) \log(1/\bar{\eta})) \) with some sufficiently large constant factor, which requires \( T^{(1)} \geq O((m + n)^{3/2}) \).

Secondly, for condition (12), we set \( \epsilon_0 = \epsilon_{F,x}/4 \) and let \( \epsilon_P + \epsilon_H(H^{(0)}) \leq \epsilon_{F,x}/12, \epsilon_{n,x} \leq \epsilon_{F,x}/12, \epsilon_P(\Delta^{(0)}_m, H^{(0)}) \leq \epsilon_{F,x}/12, \) and \( \epsilon_P \leq \epsilon_{F,u}/4, \epsilon_{n,u} \leq \epsilon_{F,u}/4 \). These conditions require \( H^{(0)} \geq O(\log(\sqrt{mn} / \min(\epsilon_P))), \bar{\eta}^{(e)} = O(\min(\epsilon_P/\sqrt{mn}, \epsilon_P)), \) and \( \Delta^{(e)}_M = O(\epsilon_{F,x}/\sqrt{mn}H^{(e)}(T^{(e+1)})^{1/3}) \).

Thirdly, for the condition (ii) of Theorem 3 the monotonicity for \( H^{(e)}, \sqrt{H^{(e)}} \Delta^{(e)}_M \) are satisfied and \( \bar{\eta}^{(e)} \) is a constant, so its monotonicity condition is also satisfied. With exponentially increasing \( T^{(e)}_D \), the decreasing \( r^{(e)} \) is also satisfied. We only need to verify that \( r^{(1)} \leq r_{ini} \). This requires \( T^{(1)} \geq \hat{O}((\sqrt{mn + n}/\bar{\eta})^3) \).

Lastly, for \( T^{(e+1)} \geq t_2^{(e)} \), notice that Phase 1 only takes \( (T^{(e+1)} - T^{(e)})^{2/3} \) stages, and the safe transitions only takes \( \hat{O}((T^{(e+1)})^{1/3}) \) stages, so \( T^{(e+1)} \geq t_2^{(e)} \) for all \( e \) for large enough initial \( T^{(1)} \).

F More Discussions

In this appendix, we briefly introduce RMPC in [Mayne et al., 2005] and show that its infinite-horizon averaged cost can be captured by \( J(\mathbb{X}) \) for some safe linear policy \( \mathbb{X} \). Therefore, algorithms with small regret compared with optimal safe linear policies can also achieve comparable performance with RMPC in [Mayne et al., 2005] for long horizons, which further motivates our choice of regret benchmarks as safe linear policies. Further, we discuss the implementation of our algorithm for non-zero \( x_0 \).

F.1 A brief review of RMPC in [Mayne et al., 2005]

RMPC is a popular method to handle constrained system with disturbances and/or other system uncertainties. Since we will include RMPC in the benchmark policy class, we assume the model \( \theta_\star \) is available here, but RMPC can also handle model uncertainties. Many different versions of RMPC have been proposed in the literature, (see Rawlings and Mayne...
(2009) for a review). In this appendix, we will focus on a tube-based RMPC defined in [Mayne et al. (2005)]. The RMPC method in [Mayne et al. (2005)] enjoys desirable theoretical guarantees, such as robust exponential stability, recursive feasibility, constraint satisfaction, and is thus commonly adopted. RMPC usually considers $x_0 \neq 0$. When considering RMPC for regulation problems, one goal of RMPC is to quickly and safely steer the states to a neighborhood of origin (due to the system disturbances, one cannot steer the state to the origin exactly).

Next, we briefly introduce the tube-based RMPC scheme. In most tube-based RMPC schemes (not just [Mayne et al. (2005)]), it is required to know a linear static controller $u_t = -Kx_t$ such that this controller is strictly safe if the system starts from the origin. A disturbance-invariant set for any $x_0 \in X_f$, implementing RMPC also requires the knowledge of a terminal set $X_f$ such that for any $x_0 \in X_f$, implementing RMPC is feasible when $x_0 \neq 0$. When considering the infinite-horizon averaged cost of RMPC in [Mayne et al. (2005)] equals the infinite-horizon averaged cost of $\mathcal{K}$, one cannot steer the state to the origin exactly).

Theorem 6 suggests that (RMPC Mayne et al. (2005)) can quickly reduce the distance between $x_t$ and $\Xi$, i.e. it can drive a large initial state $x_0 \neq 0$ quickly to a neighborhood around $\Xi$, which is also a neighborhood around the origin. Based on the robust exponential stability, we can build a connection between the infinite horizon averaged cost of RMPC and that of the safe linear policy $\mathcal{K}$.

Theorem 7 (Connection between RMPC in [Mayne et al. (2005)] and linear control’s infinite-horizon costs). Consider (RMPC Mayne et al. (2005)) defined above with $\mathcal{K}$ satisfying the requirements in [Mayne et al. (2005)]. For any $x_0 \in X_N$, the infinite-horizon averaged cost of RMPC in [Mayne et al. (2005)] equals the infinite-horizon averaged cost of $\mathcal{K}$, i.e.

$$J(\text{RMPC in Mayne et al. (2005)}) = J(\mathcal{K}),$$
The proof is deferred to the end of this appendix.

Notice that $K$ is a pre-fixed safe linear policy, so by Theorem 7, we have $J(K^*) \leq J$(RMPC) in [Mayne et al. (2005)], where $K^*$ is our regret benchmark, i.e., the optimal safe linear policy. This suggests that RMPC in [Mayne et al. (2005)] achieves similar or worse performance than the optimal safe linear policy in the long run. Since our adaptive control algorithm enjoys a sublinear regret compared to the optimal safe linear policy, Theorem 7 suggests that our algorithm achieves the same regret bound even if we include RMPC in [Mayne et al. (2005)] to the benchmark policy set. Further, if $K \neq K^*$, our adaptive algorithm can even achieve better performance than RMPC in [Mayne et al. (2005)] at around the equilibrium point 0.

Nevertheless, one major strength of RMPC in [Mayne et al. (2005)] compared with our algorithm is that RMPC can guarantee safety for large nonzero $x_0$ and can drive a large state exponentially to a small neighborhood of 0. Therefore, an interesting and natural idea is to combine RMPC in [Mayne et al. (2005)] with our algorithm to achieve the strengths of both methods: quickly and safely drive a large initial state to a neighborhood around 0, and learning to optimize the performance around 0. We leave more studies on this combination as future work.

Remark 4. Since our proof relies on the robust exponential stability property of RMPC in [Mayne et al. (2005)], for other RMPC schemes without this property, we still cannot include them to our benchmark policy class and generate a sublinear regret. We leave the regret analysis compared with other RMPC schemes without robust exponential stability as future work. Further, we note that there are a few papers on the regret analysis with RMPC as the benchmark, e.g., [Wabersich and Zeilinger (2018), Muthirayan et al. (2020)]. However, Wabersich and Zeilinger (2018) allows constraint violation during the learning process and allows restarts when policies are updated, and Muthirayan et al. (2020) does not consider state constraints and the proposed algorithm involves an intractable oracle. In conclusion, the regret analysis with RMPC as the benchmark is largely under-explored and is an important direction for future research.

Proof of Theorem 7. To prove Theorem 7, we introduce some necessary results from the existing literature and some lemmas based on these existing results.

Firstly, we review the structure of constrained LQR’s solution proved in [Bemporad et al. (2002)]. Consider (CLQR) with p.d. quadratic costs and polytopic constraints below:

$$
\min_{u_{t+k|t}} \sum_{k=0}^{W-1} l(x_{t+k|t}, u_{t+k|t}) + x_{t+W|t}^T P x_{t+W|t}
$$

s.t. $x_{t+k+1|t} = A_k x_{t+k|t} + B_k u_{t+k|t}, \quad k \geq 0$

$$
D_k x_{t+k|t} \leq d_k, \quad \forall 0 \leq k \leq W - 1
$$

$$
D_k u_{t+k|t} \leq d_k, \quad \forall 0 \leq k \leq W - 1
$$

$$
D_{term} x_{t+W|t} \leq d_{term}
$$

$x_{t|t} = x$

Denote the optimal policy as $\pi_{CLQR}(x) = u_{tt}^*$, and denote the feasible region as $X_N$. Then, $X_N$ is convex, and $\pi_{CLQR}(x)$ is continuous and PWA on a finite number of closed convex polytopic regions. That is,

$$
\pi_{CLQR}(x) = K_i x + b_i, \quad G_i x \leq h_i, \quad i = 0, 1, \ldots, N_{clqr}.
$$

Further, the number of different gain matrices can bounded by a constant $N_{clqr}$-gain that only depends on the dimensionality of the problem.

Based on Proposition 2, we have that $\pi_{CLQR}(x)$ is Lipschitz continuous with Lipschitz factor $L_{CLQR} = \max_i \|K_i\|_2$ since $\pi_{CLQR}(x)$ is continuous and piecewise-affine with respect to $x$.

Next, we will use the exponential convergence results of RMPC in [Mayne et al. (2005)].

Proposition 3 (See the proof of Theorem 1 in [Mayne et al. (2005)]). There exists $c_1 > 0$ and $\rho \in (0, 1)$ such that for any $x_0 \in X_N$, and for any admissible disturbances $w_k$, we have

$$
\|x_{t|t}^*(x_t)\|_2 \leq c_1 \rho^{t} \|x_{0|0}^*(x_0)\|_2.
$$

13 Though RMPC in [Mayne et al. (2005)] requires a known model, there are standard approaches to extend RMPC to handle model uncertainties, e.g., [Kohler et al. (2019), Lu et al. (2019)].
Based on this, we can also show the exponential decay of $u^*_{t|t}(x_t)$.

**Lemma 20.** There exists $c_2 > 0$ and $ρ ∈ (0,1)$ such that for any $x_0 ∈ X_N$, and for any admissible disturbances $w_k$, $u^*_{t|t}(x^*_{t|t})$ is Lipschitz continuous with a finite factor denoted as $L_{\text{rmfpc}}$ on a convex feasible set. Further, we have $∥u^*_{t|t}(x_t)∥_2 ≤ c_2 ρ^t$, where $c_2 = L_{\text{rmfpc}} c_1 x_{\text{max}}$.

**Proof.** First of all, we point out that for the (RMPC Mayne et al. (2005)) optimization, when $x^*_{t|t}$ is fixed, then $u^*_{t|t}$ can be viewed as $u^*_{t|t} = π_{\text{CLQR}}(x^*_{t|t})$ for a (CLQR) problem with the same polytopic constraints and strongly convex quadratic cost functions with (RMPC Mayne et al. (2005)). Therefore, $u^*_{t|t}(x^*_{t|t})$ is Lipschitz continuous with a finite factor denoted as $L_{\text{rmfpc}}$ on a convex feasible set.

Further, notice that $u^*_{t|t}(0) = 0$. Therefore,

$$∥u^*_{t|t}(x^*_{t|t})∥_2 = ∥u^*_{t|t}(x^*_{t|t}) - u^*_{t|t}(0)∥_2 ≤ L_{\text{rmfpc}} ∥x^*_{t|t}∥ _2 ≤ L_{\text{rmfpc}} c_1 ρ^t ∥x_{0|0}(x_0)∥_2 ≤ c_2 ρ^t$$

where $c_2 = L_{\text{rmfpc}} c_1 x_{\text{max}}$.

Lastly, a technical lemma of a standard results. The proof is very straightforward.

**Lemma 21.** Consider $y^+ = A_b y + w$, where $y_0 = x_0 ∈ X$ and $p = −K_y$. Since $X$ is $(κ, γ)$ strongly convex, both $y$ and $p$ are bounded by

$$∥y∥_2 ≤ ∥y∥_2 κ^2 /γ + κ^2 x_{\text{max}} = y_{\text{max}}, ∥p∥_2 ≤ ∥w∥_2 κ^3 /γ + κ^2 x_{\text{max}} = p_{\text{max}}.$$

Now, we are ready for the proof of Theorem 7.

**Proof of Theorem 7** The closed-loop system of (RMPC Mayne et al. (2005)) is

$$x_{t+1} = A_s x_t + B_s π_{\text{RMPC}}(x_t) + w_t = A_s x_t − B_s K x_t + B_s(x^*_{t|t}(x_t) + u^*_{t|t}(x_t)) + w_t.$$

Consider a possibly unsafe system:

$$y_{t+1} = A_s y_t + B_s p_t + w_t, \quad p_t = −K_y y_t$$

with the same sequence of disturbances and $y_0 = x_0$.

The dynamics of the error $e_t = x_t − y_t$ is

$$e_{t+1} = A_b e_t + v_t$$

where $A_b = A_s − B_s K$, and $v_t = B_s(π_{\text{CLQR}}(x_t) + u^*_{t|t}(x_t))$. Notice that by Proposition 3 and Lemma 20 we have

$$∥v_t∥_2 ≤ ∥B_s∥_2 (κc_1 ρ^t x_{\text{max}} + c_2 ρ^t) = c_3 ρ^t,$$

where $c_3 = ∥B_s∥_2 (κc_1 x_{\text{max}} + c_2)$.

Therefore,

$$∥e_t∥_2 = ∥v_{t−1} + A_b v_{t−2} + A_b^{t−1} v_{0}∥_2 ≤ c_3 ρ^{t−1} + κ^2 (1 − γ) c_3 ρ^{t−2} + \ldots ≤ c_3 κ^2 t \max(ρ, 1 − γ)^{t−1} = c_4 t ρ^{t−1}$$

where $ρ_0 = \max(ρ, 1 − γ) ∈ (0,1)$ and $c_4 = c_3 κ^2$. Further,

$$∥u_t − p_t∥_2 = ∥−K_e v_t + v_t∥_2 ≤ κc_4 t ρ^{t−1} + c_3 ρ^t ≤ c_5 t ρ^{t−1},$$

where $c_5 = c_4 κ + c_3 /ρ$.

Therefore, the stage cost difference is

$$∥l(x_t, u_t) − l(y_t, p_t)∥ ≤ ∥Q∥_2 ∥e_t∥_2 (x_{\text{max}} + y_{\text{max}}) + ∥R∥_2 ∥u_t − p_t∥_2 ∥u_{\text{max}} + p_{\text{max}}∥_2$$

$$≤ ∥Q∥_2 (x_{\text{max}} + y_{\text{max}}) c_4 t ρ^{t−1} + ∥R∥_2 ∥u_{\text{max}} + p_{\text{max}}∥_2 c_5 t ρ^{t−1} = c_6 t ρ^{t−1}$$
where \(c_6 = \|Q\|_2 (x_{\text{max}} + y_{\text{max}}) c_4 + \|R\|_2 \|u_{\text{max}} + p_{\text{max}}\|_2 c_5\).

Therefore,

\[
\frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} l(x_t, u_t) - l(y_t, p_t) \leq \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} \|l(x_t, u_t) - l(y_t, p_t)\|_1 \leq \frac{1}{T} c_6 / (1 - \rho_0)^2
\]

By taking \(T \to +\infty\), we have \(\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} l(x_t, u_t) - l(y_t, p_t) = 0\). Since \(\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} l(y_t, p_t) = J(K)\), we have \(\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} l(x_t, u_t) = J(K)\).

\[
\square
\]

\[
\square
\]

\section{Additional Proofs}

\subsection{Proof of Lemma \[4\]}

The proof relies on the following two lemmas.

\begin{lemma}[Definition of \(\epsilon_{\hat{\omega}}\).] \label{lemma22} \(\epsilon_{\hat{\omega}}(r)\)]

Under the conditions in Lemma \[6\],

\[
\sum_{k=1}^{H_t} D_{x,i}^T A_s^{-1}(w_{t-k} - \hat{w}_{t-k}) \leq \epsilon_{\hat{\omega}}(r)
\]

Proof.

\[
\|D_x \sum_{k=1}^{H_t} A_s^{-1}(w_{t-k} - \hat{w}_{t-k})\|_\infty \leq \|D_x\|_\infty \sum_{k=1}^{H_t} \|A_s^{-1}(w_{t-k} - \hat{w}_{t-k})\|_\infty
\]

\[
\leq \|D_x\|_\infty \sum_{k=1}^{H_t} \|A_s^{-1}(w_{t-k} - \hat{w}_{t-k})\|_2
\]

\[
\leq \|D_x\|_\infty \sum_{k=1}^{H_t} \kappa (1 - \gamma) k^{-1} r z_{\text{max}}
\]

\[
\leq \|D_x\|_\infty \kappa / \gamma z_{\text{max}} r = \epsilon_{\hat{\omega}}(r)
\]

\end{lemma}

\begin{lemma}[Definition of \(\epsilon_{\hat{\theta}}\).] \label{lemma23} \(\epsilon_{\hat{\theta}}(r)\)]

For any \(\mathbf{M} \in \mathcal{M}\), any \(\hat{\theta}, \theta \in \Theta^{(0)}\) such that \(\|\hat{\theta} - \theta\|_F \leq r\), we have

\[
|g^x_r(\mathbf{M}; \hat{\theta}) - g^x_r(\mathbf{M}; \theta)| \leq \epsilon_{\hat{\theta}}(r)
\]

where \(\epsilon_{\hat{\theta}}(r) = c_{\hat{\theta}} r \sqrt{mn}\).

Proof. Firstly, we show that it suffices to prove an upper bound of a simpler quantity.

\[
|g^x_r(\mathbf{M}; \hat{\theta}) - g^x_r(\mathbf{M}; \theta)| = \left| \sum_{k=1}^{2H} \|D_{x,i}^T \Phi_k^x(\mathbf{M}; \hat{\theta})\|_1 - \|D_{x,i}^T \Phi_k^x(\mathbf{M}; \theta)\|_1 \right| \|w_{\text{max}}\|
\]

\[
\leq \sum_{k=1}^{2H} \|D_{x,i}^T \Phi_k^x(\mathbf{M}; \hat{\theta})\|_1 - \|D_{x,i}^T \Phi_k^x(\mathbf{M}; \theta)\|_1 \|w_{\text{max}}\|
\]

\[
\leq \sum_{k=1}^{2H} \|D_{x,i}^T \Phi_k^x(\mathbf{M}; \hat{\theta}) - D_{x,i}^T \Phi_k^x(\mathbf{M}; \theta)\|_1 \|w_{\text{max}}\|
\]

\[
\leq \sum_{k=1}^{2H} \|D_x\|_\infty \|\Phi_k^x(\mathbf{M}; \hat{\theta}) - \Phi_k^x(\mathbf{M}; \theta)\|_\infty \|w_{\text{max}}\|
\]

\[
\square
\]
thus, it suffices to bound \( \sum_{k=1}^{2H} \| \Phi_k^\varepsilon (\mathbf{M}; \dot{\theta}) - \Phi_k^\varepsilon (\mathbf{M}; \theta) \|_\infty \). To bound this, we need several small lemmas below.

**Lemma 24.** When \( \| \theta - \dot{\theta} \|_F \leq r \), we have \( \max(\| \hat{A} - A \|_2, \| \hat{B} - B \|_2) \leq \max(\| \hat{A} - A \|_F, \| \hat{B} - B \|_F) \leq r \)

This is quite straightforward so the proof is omitted.

**Lemma 25.** For any \( k \geq 0 \), any \( \dot{\theta}, \theta \in \Theta^{(0)} \) such that \( \| \dot{\theta} - \theta \|_F \leq r \), we have

\[
\begin{align*}
\| A^k - \hat{A}^k \|_2 &\leq k \kappa^2 (1 - \gamma)^{k-1} r \mathbf{1}_{(k \geq 1)} \\
\| A^k B - \hat{A}^k \hat{B} \|_2 &\leq k \kappa^2 \kappa_B (1 - \gamma)^{k-1} r \mathbf{1}_{(k \geq 1)} + \kappa (1 - \gamma)^k r
\end{align*}
\]

**Proof.** When \( k = 0, \| A^0 - \hat{A}^0 \|_2 = 0 \). When \( k \geq 1 \),

\[
\| \hat{A}^k - A^k \|_2 = \| \sum_{i=0}^{k-1} \hat{A}^{k-i-1} (A - A) A^i \|_2 \\
\leq \sum_{i=0}^{k-1} \| \hat{A}^{k-i-1} \|_2 \| A - A \| \| A^i \|_2 \\
\leq \sum_{i=0}^{k-1} \kappa (1 - \gamma)^{k-i-1} \epsilon \kappa (1 - \gamma)^i \\
= k \kappa^2 r (1 - \gamma)^{k-1}
\]

\[
\| \hat{A}^k \hat{B} - A^k B \|_2 \leq \| \hat{A}^k \hat{B} - A^k \hat{B} \|_2 + \| A^k \hat{B} - \hat{A}^k \hat{B} \|_2 \\
\leq k \kappa^2 \kappa_B r (1 - \gamma)^{k-1} \mathbf{1}_{(k \geq 1)} + \kappa (1 - \gamma)^k r
\]


Now, we can bound \( \sum_{k=1}^{2H} \| \Phi_k^\varepsilon (\mathbf{M}; \dot{\theta}) - \Phi_k^\varepsilon (\mathbf{M}; \theta) \|_\infty \). For any \( 1 \leq k \leq 2H \),

\[
\| \Phi_k^\varepsilon (\mathbf{M}; \dot{\theta}) - \Phi_k^\varepsilon (\mathbf{M}; \theta) \|_\infty \\
= \| \hat{A}^{k-1} \mathbf{1}_{(k \leq H)} + \sum_{i=1}^{H} \hat{A}^{i-1} \hat{B} M_{i-1} [k - i] \mathbf{1}_{(1 \leq k-i \leq H)} - A^{k-1} \mathbf{1}_{(k \leq H)} - \sum_{i=1}^{H} A^{i-1} B M_{i-1} [k - i] \mathbf{1}_{(1 \leq k-i \leq H)} \|_\infty \\
\leq \| \hat{A}^{k-1} - A^{k-1} \|_\infty \mathbf{1}_{(k \leq H)} + \sum_{i=1}^{H} \| (\hat{A}^{i-1} \hat{B} - A^{i-1} B) M_{i-1} [k - i] \|_\infty \mathbf{1}_{(1 \leq k-i \leq H)} \\
\leq \sqrt{n} \| \hat{A}^{k-1} - A^{k-1} \|_2 \mathbf{1}_{(k \leq H)} + \sqrt{m} \sum_{i=1}^{H} \| \hat{A}^{i-1} \hat{B} - A^{i-1} B \|_2 2\sqrt{\kappa^2 (1 - \gamma)^{k-i-1} \mathbf{1}_{(1 \leq k-i \leq H)}}
\]

There are two terms in the last right-hand-side of the inequality above. We sum each term over \( k \) below.

\[
\sum_{k=1}^{2H} \sqrt{n} \| \hat{A}^{k-1} - A^{k-1} \|_2 \mathbf{1}_{(k \leq H)} \leq \sum_{k=1}^{2H} \sqrt{n} (k - 1) \kappa^2 (1 - \gamma)^{k-2} r \mathbf{1}_{(2 \leq k \leq H)} \leq \sqrt{n} \kappa^2 r / \gamma^2
\]

\[
\sum_{k=1}^{2H} \sqrt{m} \sum_{i=1}^{H} \| \hat{A}^{i-1} \hat{B} - A^{i-1} B \|_2 2\sqrt{\kappa^2 (1 - \gamma)^{k-i-1} \mathbf{1}_{(1 \leq k-i \leq H)}} \\
\leq \sum_{k=1}^{2H} \sqrt{m} \sum_{i=1}^{H} (i - 1) \kappa^2 \kappa_B (1 - \gamma)^{i-2} r \mathbf{1}_{(i \geq 2)} 2\sqrt{\kappa^2 (1 - \gamma)^{k-i-1} \mathbf{1}_{(1 \leq k-i \leq H)}}
\]

33
\[
\sum_{k=1}^{2H} \sqrt{m} \sum_{i=1}^{H} \kappa (1 - \gamma)^{i-1} r 2\sqrt{m} \kappa^2 (1 - \gamma)^{k-i-1} \mathbf{1}_{(1 \leq k-i \leq H)} \\
= 2\sqrt{m} \kappa^4 B r \sum_{i=1}^{H} \sum_{j=1}^{H} (i - 1)(1 - \gamma)^{i-2}(1 - \gamma)^{j-1} + 2\sqrt{m} \kappa^3 r \sum_{i} \sum_{j} (1 - \gamma)^{i-1}(1 - \gamma)^{j-1} \\
= 2\sqrt{m} \kappa^4 B r / \gamma^3 + 2\sqrt{m} \kappa^3 r / \gamma^2
\]

\[\square\]

### G.2 Proof of Lemma 8

For notational simplicity, we omit the subscript \(t\) in \(H_t\) in this proof. Remember that \(g_t^\ast (M_{t-H:t-1}; \theta) = \sum_{n=1}^{H} \|D_{x,i}^T \Phi_t^\ast (M_{t-H:t-1}; \theta)\|_1\left| w_{\max}\right|\)

\[
|\tilde{g}_t^\ast (M_{t-H:t-1}; \theta) - g_t^\ast (M; \theta)| = \left| \sum_{k=1}^{2H} \|D_{x,i}^T \Phi_t^k (M_{t-H:t-1}; \theta^*)\|_1 - \|D_{x,i}^T \Phi_t^k (M_t; \theta)\|_1 \right| w_{\max}
\]

\[
\leq \sum_{k=1}^{2H} \|D_{x,i}^T \Phi_t^k (M_{t-H:t-1}; \theta^*)\|_1 - \|D_{x,i}^T \Phi_t^k (M_t; \theta)\|_1 \left| w_{\max}\right|
\]

\[
\leq \sum_{k=1}^{2H} \|D_{x,i}^T (\tilde{\Phi}_k^\ast (M_{t-H:t-1}; \theta) - \Phi_k^\ast (M_t; \theta))\|_1 w_{\max}
\]

\[
\leq \sum_{k=1}^{2H} \|D_x\|_{\infty} \|\tilde{\Phi}_k^\ast (M_{t-H:t-1}; \theta) - \Phi_k^\ast (M_t; \theta)\|_{\infty} w_{\max}
\]

\[
\leq \sum_{k=1}^{2H} \|D_x\|_{\infty} \sum_{i=1}^{H} \sum_{i=1}^{H} A_i A_i^{-1} B_i (M_{t-i} - M_t) \|_{\infty} w_{\max} \mathbf{1}_{(1 \leq k-i \leq H)}
\]

\[
\leq \sum_{k=1}^{2H} \|D_x\|_{\infty} \sum_{i=1}^{H} \sum_{i=1}^{H} A_i A_i^{-1} B_i \|M_{t-i} - M_t\|_{\infty} \mathbf{1}_{(1 \leq k-i \leq H)} \mathbf{1}_{(1 \leq k-i \leq H)}
\]

\[
\leq \|D_x\|_{\infty} \sqrt{m} w_{\max} \kappa B \sum_{k=1}^{H} \sum_{i=1}^{H} (1 - \gamma)^{i-1} \kappa B \|M_{t-i} - M_t\|_{\infty} \mathbf{1}_{(1 \leq k-i \leq H)}
\]

\[
= \|D_x\|_{\infty} \sqrt{m} w_{\max} \kappa B \sum_{i=1}^{H} \sum_{i=1}^{H} (1 - \gamma)^{i-1} \|M_{t-i} - M_t\|_{\infty} \mathbf{1}_{(1 \leq k-i \leq H)}
\]

\[
\leq \|D_x\|_{\infty} \sqrt{m} w_{\max} \kappa B \sqrt{n} H \sum_{i=1}^{H} (1 - \gamma)^{i-1} \|M_{t-i} - M_t\|_{F}
\]

\[
\leq \|D_x\|_{\infty} \sqrt{m} w_{\max} \kappa B / \gamma^2 \Delta M
\]

where the third last inequality is because \(M[j] \in \mathbb{R}^{m \times n}\)

\[
\sum_{j=1}^{H} \|M[j]\|_{\infty} \leq \sum_{j=1}^{H} \|M[j]\|_{2} \sqrt{n} \leq \sum_{j=1}^{H} \|M[j]\|_{F} \sqrt{n} \leq \|M\|_{F} \sqrt{n} \sqrt{H}
\]
G.3 Proof of Lemma 11

For notational simplicity, we define $y_t = \sum_{i=1}^{H_t} A_i^{t-1} w_{t-i} + \sum_{k=2}^{2H_t} \sum_{i=1}^{H_t} A_i^{t-1} B_s M_{t-i} [k-i] \hat{w}_{t-k} 1_{1 \leq k-i \leq H_t} + \sum_{i=1}^{H_t} A_i^{t-1} B_s \eta_{t-i}$. Since $A_s$ is $(\kappa, \gamma)$-stable, we have

$$\|y_t\|_2 \leq \sum_{i=1}^{H_t} \|A_i^{t-1}\|_2 \|w_{t-i}\|_2 + \sum_{k=2}^{2H_t} \sum_{i=1}^{H_t} \|A_i^{t-1} B_s M_{t-i} [k-i] \hat{w}_{t-k}\|_2 1_{1 \leq k-i \leq H_t} + \sum_{i=1}^{H_t} \|A_i^{t-1} B_s \eta_{t-i}\|_2$$

$$\leq \sum_{i=1}^{H_t} \kappa(1-\gamma)^{i-1} \sqrt{n} w_{\text{max}} + \sum_{k=2}^{2H_t} \sum_{i=1}^{H_t} \|A_i^{t-1} B_s\|_2 \|M_{t-i} [k-i] \hat{w}_{t-k}\|_2 1_{1 \leq k-i \leq H_t} + \sum_{i=1}^{H_t} \|A_i^{t-1} B_s\|_2 \|\eta_{t-i}\|_2$$

$$\leq \kappa \sqrt{n} w_{\text{max}} / \gamma + \sum_{k=2}^{2H_t} \sum_{i=1}^{H_t} \kappa(1-\gamma)^{i-1} \kappa B \sqrt{m} \|M_{t-i} [k-i] \hat{w}_{t-k}\|_\infty 1_{1 \leq k-i \leq H_t} + \sum_{i=1}^{H_t} \|A_i^{t-1} B_s\|_2 \|\eta_{t-i}\|_2$$

$$\leq \kappa \sqrt{n} w_{\text{max}} / \gamma + \kappa B / \gamma \sqrt{m} \max_j (1-\gamma)^{j+1} + \kappa^2 \kappa B 2 \sqrt{mn} w_{\text{max}} \sum_{i=1}^{H_t} \sum_{j=1}^{H_t} (1-\gamma)^{i+j}$$

$$\leq \sqrt{n} (\kappa w_{\text{max}} + \kappa B \eta_{\text{max}}) / \gamma + \kappa^2 \kappa B 2 \sqrt{mn} w_{\text{max}} / \gamma^2$$

$$\leq 2 \sqrt{n} \kappa w_{\text{max}} / \gamma + \kappa^2 \kappa B 2 \sqrt{mn} w_{\text{max}} / \gamma^2$$

Remember that $x_t = A^t H_t x_{t-H_t} + y_t$ and $\|x_t\|_2 = 0 \leq b_x$ for $t \leq 0$. We prove the bound on $x_t$ by induction. Suppose at $t \geq 0$, $\|x_{t-H_t}\|_2 \leq b_x$, then

$$\|x_t\|_2 \leq \|A^t H_t\|_2 \|x_{t-H_t}\|_2 + \|y_t\|_2 \leq \kappa(1-\gamma)^{H_t} b_x + 2 \sqrt{\kappa w_{\text{max}}} / \gamma + \kappa^2 \kappa B 2 \sqrt{mn} w_{\text{max}} / \gamma^2$$

$$\leq b_x / 2 + 2 \sqrt{\kappa w_{\text{max}}} / \gamma + \kappa^2 \kappa B 2 \sqrt{mn} w_{\text{max}} / \gamma^2$$

where the last inequality is by $\kappa(1-\gamma)^{H_t} \leq 1/2$ when $H_t \geq \log(2\kappa) / \log((1-\gamma)^{-1})$. This completes the proof.

G.4 Proof of Lemma 18

Proof. We omit $\theta$ in this proof for simplicity of notations.

For any $H \geq 1$, define $M_{\text{out}, H} = \{M \in \mathbb{R}^{mnH} : \|M[k]\|_\infty \leq 4\kappa^2 \sqrt{n} (1-\gamma)^{k-1}\}$. Notice that $M_H \subset \text{interior}(M_{\text{out}, H})$. Therefore, for any $M \in M_H$,

$$\|\nabla f(M; \theta)\|_F = \sup_{\Delta M \neq 0, M + \Delta M \in M_{\text{out}, H}} \frac{\langle \nabla f(M; \theta), \Delta M \rangle}{\|\Delta M\|_F} \leq \sup_{\Delta M \neq 0, M + \Delta M \in M_{\text{out}, H}} \frac{f(M + \Delta M) - f(M)}{\|\Delta M\|_F}$$

For $M, M' \in M_{\text{out}, H}$, we bound the following.

$$\|\tilde{x} - \tilde{x}'\|_2 \leq \sum_{k=1}^{2H} \| (\Phi_k^x(M) - \Phi_k^x(M')) w_{t-k} \|_2$$

$$\leq \sum_{k=1}^{2H} \| \sum_{i=1}^{H(k)} A^{i-1} B(M[k-i] - M'[k-i]) 1_{1 \leq k-i \leq H} w_{t-k} \|_2$$

$$\leq \sum_{j=1}^{H} O(\sqrt{n}) \|M[j] - M'[j]\|_F$$

$$\leq \sum_{j=1}^{H} O(\sqrt{n}) \|M[j] - M'[j]\|_F$$
\[ \leq O(\sqrt{m} \sqrt{H})\|M - M'\|_F \]
\[ \|\hat{u} - \hat{u}'\|_2 \sum_{k=1}^{H} \|M[k] - M'[k]\|_2 \sqrt{w_{\text{max}}} \leq O(\sqrt{m} \sqrt{H})\|M - M'\|_F \]

where the third inequality uses \( \theta \in \Theta_{\text{ini}} \).

Further, even though we make \( M_{\text{out},H} \) larger, but we don’t change the dimension, so by Lemma 24, \( \|\tilde{x}\|_2 \leq \sqrt{m} \). Further, even when we don’t have additional conditions on \( M \), we still have \( \|\hat{u}\|_2 \leq O(\sqrt{m} \sqrt{H}) \). Therefore, for \( M, M' \in M_{\text{out},H} \),
\[ |f(M) - f(M')| \leq O(\sqrt{m} \sqrt{H})\|M - M'\|_F \]

Therefore,
\[ \|\nabla f(M; \theta)\|_F \leq \sup_{\Delta M \neq 0, M + \Delta M \in M_{\text{out},H}} \frac{f(M + \Delta M) - f(M)}{\|\Delta M\|_F} \leq \sup_{\Delta M \neq 0, M + \Delta M \in M_{\text{out},H}} O(\sqrt{m} \sqrt{H})\|\Delta M\|_F \leq O(n \sqrt{m} \sqrt{H}) \]

\[ \square \]

**G.5 Proof of Lemma 17**

*Proof.* Notice that \( \Omega_1 \) and \( \Omega_3 \) satisfies the conditions in Proposition 2 in \cite{Li et al. 2020}. Therefore,
\[ |\min_{\Omega_1} f(x) - \min_{\Omega_3} f(x)| \leq \frac{Ld_{\Omega_1} \|\Delta_1 - \Delta_3\|_\infty}{\min_{\{i: (\Delta_1)_i > (\Delta_3)_i\}} (h - \Delta_1 - \mathcal{C}_F)} \]

Notice that
\[ (\Delta_3)_i = \begin{cases} (\Delta_1)_i, & \text{if } (\Delta_1)_i \geq (\Delta_2)_i \\ (\Delta_2)_i, & \text{if } (\Delta_1)_i < (\Delta_2)_i \end{cases} \]

therefore, \( \|\Delta_1 - \Delta_3\|_\infty \leq \|\Delta_1 - \Delta_2\|_\infty \). Further, \( \{i: (\Delta_3)_i > (\Delta_1)_i\} = \{i: (\Delta_2)_i > (\Delta_1)_i\} \subseteq \{i: (\Delta_1)_i \neq (\Delta_2)_i\} \). So \( \min_{\{i: (\Delta_3)_i > (\Delta_1)_i\}} (h - \Delta_1 - \mathcal{C}_F)_i \geq \min_{\{i: (\Delta_1)_i \neq (\Delta_2)_i\}} (h - \Delta_1 - \mathcal{C}_F)_i \). Therefore,
\[ |\min_{\Omega_1} f(x) - \min_{\Omega_3} f(x)| \leq \frac{Ld_{\Omega_1} \|\Delta_1 - \Delta_3\|_\infty}{\min_{\{i: (\Delta_1)_i > (\Delta_3)_i\}} (h - \Delta_1 - \mathcal{C}_F)_i} \leq \frac{Ld_{\Omega_1} \|\Delta_1 - \Delta_2\|_\infty}{\min_{\{i: (\Delta_1)_i \neq (\Delta_2)_i\}} (h - \Delta_1 - \mathcal{C}_F)_i} \]

Similarly,
\[ |\min_{\Omega_2} f(x) - \min_{\Omega_3} f(x)| \leq \frac{Ld_{\Omega_2} \|\Delta_2 - \Delta_3\|_\infty}{\min_{\{i: (\Delta_2)_i > (\Delta_3)_i\}} (h - \Delta_2 - \mathcal{C}_F)_i} \leq \frac{Ld_{\Omega_2} \|\Delta_1 - \Delta_2\|_\infty}{\min_{\{i: (\Delta_1)_i \neq (\Delta_2)_i\}} (h - \Delta_1 - \mathcal{C}_F)_i} \]

which completes the bound. \[ \square \]

**G.6 Proof of Lemma 19**

In this subsection, we provide a proof for our bound on Part ii by martingale concentration inequalities.

**Lemma 26.** \textbf{In our Algorithm} \[ M^{(e)} \in \mathcal{F}(w_0, \ldots, w_{t^{(e)}_1 + T^{(e)}_D - 1}, \eta_0, \ldots, \eta_{t^{(e)}_1 + T^{(e)}_D - 1}) = \mathcal{F}^{(e)}_{t^{(e)}_1 + T^{(e)}_D} \subseteq \mathcal{F}^{(e)}_{t^{(e)}_1} - H^{(e)}. \]

*Proof.* By definition, we have the following fact: \( M^{(e)} \in \mathcal{F}(\hat{\theta}^{(e+1)}) = \mathcal{F}(\{z_k, x_{k+1}\}_{k=t^{(e)}_1}^{t^{(e)}_1 + T^{(e)}_D - 1}) = \mathcal{F}(w_0, \ldots, w_{t^{(e)}_1 + T^{(e)}_D - 1}, \eta_0, \ldots, \eta_{t^{(e)}_1 + T^{(e)}_D - 1}) \). By \( \hat{W}^{(e)}_1 \geq H^{(e)} \), we have \( t^{(e)}_1 + T^{(e)}_D + H^{(e)} \leq t^{(e)}_2 \), and since \( \mathcal{F}^{(e)}_t \subseteq \mathcal{F}_t \), we have the last claim. \[ \square \]
Lemma 27. When \( t \in \mathcal{T}_2^{(e)} \), \( w_{t-2H^{(e)}} \perp \mathcal{F}_{t_2^{(e)}-H^{(e)}} \)

Proof. When \( t \in \mathcal{T}_2^{(e)} \), \( t \geq t_2^{(e)} + H^{(e)} \), so \( t - 2H^{(e)} \geq t_2^{(e)} - H^{(e)} \). Since \( \mathcal{F}_t \) contains up to \( w_{t-1} \), we have \( w_{t-2H^{(e)}} \perp \mathcal{F}_{t_2^{(e)}-H^{(e)}} \).

\[ \Box \]

Lemma 28. In our Algorithm \([\square]\) when \( t \in \mathcal{T}_2^{(e)} \), we have \( \mathbb{E}[l(\hat{x}_t, \hat{u}_t) \mid \mathcal{F}_{t_2^{(e)}-H^{(e)}}] = f(M^{(e)}; \theta_s) \).

Proof. By our lemmas above, \( M^{(e)} \in \mathcal{F}_{t_2^{(e)}-H^{(e)}} \), but \( w_{t-2H^{(e)}} \perp \mathcal{F}_{t_2^{(e)}-H^{(e)}} \). Then, by our definition of \( \hat{x}_t, \hat{u}_t \) and \( f(M; \theta_s) \), we have the result.

\[ \Box \]

Definition 5 (Martingale). \( \{X_t\}_{t \geq 0} \) is a martingale wrt \( \{\mathcal{F}_t\}_{t \geq 0} \) if (i) \( \mathbb{E}[X_t] < +\infty \), (ii) \( X_t \in \mathcal{F}_t \), (iii) \( \mathbb{E}(X_{t+1} \mid \mathcal{F}_t) = X_t \) for \( t \geq 0 \).

Proposition 4 (Azuma-Hoeffding Inequality). \( \{X_t\}_{t \geq 0} \) is a martingale with respect to \( \{\mathcal{F}_t\}_{t \geq 0} \). If (i) \( X_0 = 0 \), (ii) \( |X_t - X_{t-1}| \leq \sigma \) for any \( t \geq 1 \), then, for any \( \alpha > 0 \), any \( t \geq 0 \),

\[ \mathbb{P}(|X_t| \geq \alpha) \leq 2 \exp \left(-\frac{\alpha^2}{2t\sigma^2}\right) \]

Corollary. \( \{X_t\}_{t \geq 0} \) is a martingale wrt \( \{\mathcal{F}_t\}_{t \geq 0} \). If (i) \( X_0 = 0 \), (ii) \( |X_t - X_{t-1}| \leq \sigma \) for any \( t \geq 1 \), then, for any \( \alpha > 0 \), any \( t \geq 0 \),

\[ |X_t| \leq \sqrt{2t\sigma}\sqrt{\log(2/\delta)} \]

w.p. at least \( 1 - \delta \).

Proof. The proof is by letting \( \alpha = \sqrt{2t\sigma^2\log(2/\delta)} \) in Proposition 4.

Lemma 29. Define \( q_t = l(\hat{x}_t, \hat{u}_t) - f(M^{(e)}; \theta_s) \). Then, \( |q_t| \leq O(\text{m}) \) w.p.l.

Proof. We can show that \( \|\hat{x}_t\|_2 \leq O(\sqrt{\text{m}}) \) a.s. and \( \hat{u}_t \in \text{U} \) a.s. by the proofs of Lemmas 10 and 11. Therefore, we have \( |l(\hat{x}_t, \hat{u}_t)| = O(\text{m}) \). Since \( f(M^{(e)}; \theta_s) = \mathbb{E}[l(\hat{x}_t, \hat{u}_t) \mid \mathcal{F}_{t_2^{(e)}-H^{(e)}}] \), we have \( |f(M^{(e)}; \theta_s)| = O(\text{m}) \). This completes the proof.

Notations and definitions. Define, for \( 0 \leq \text{h} \leq 2H^{(e)} - 1 \), that

\[ \mathcal{T}_2^{(e)} = \{ t \in \mathcal{T}_2^{(e)} : t \equiv h \mod (2H^{(e)}) \} = \{ t^{(e)}_h + 2H^{(e)}, \ldots, t^{(e)}_h + 2H^{(e)}k^{(e)}_h \} \]

Lemma 30. \( t^{(e)}_h \geq t^{(e)}_2 - H^{(e)} \) and \( t^{(e)}_h \leq T^{(e+1)}/(2H^{(e)}) \)

Proof. Notice that \( t^{(e)}_h + 2H^{(e)} \geq t^{(e)}_2 + H^{(e)} \), so the first inequality holds. Besides, notice that \( 2H^{(e)}k^{(e)}_h \leq t^{(e)}_h + 2H^{(e)}k^{(e)}_h \leq T^{(e+1)} \), so the second inequality holds.

Define

\[ \hat{q}^{(e)}_{h,j} = q_{t^{(e)}_h + j(2H^{(e)})} \quad \forall 1 \leq j \leq k^{(e)}_h \]

\[ S^{(e)}_{h,j} = \sum_{s=1}^j \hat{q}^{(e)}_{h,s} \quad \forall 0 \leq j \leq k^{(e)}_h \]

\[ \mathcal{F}^{(e)}_{h,j} = \mathcal{F}_{t^{(e)}_h + j(2H^{(e)})} \quad \forall 0 \leq j \leq k^{(e)}_h \]

where we define \( \sum_{s=1}^0 a_s = 0 \). By Lemma 30, we have \( \mathcal{F}^{(e)}_{h,0} = \mathcal{F}_{t^{(e)}_h} \supseteq \mathcal{F}_{t^{(e)}_2-H^{(e)}} \).

Lemma 31. \( S^{(e)}_{h,j} \) is a martingale wrt \( \mathcal{F}_{h,j} \) for \( j \geq 0 \). Further, \( S^{(e)}_{h,0} = 0 \), \( |S^{(e)}_{h,j+1} - S^{(e)}_{h,j}| \leq O(\text{m}) \).

Proof. Since \( |q_t| \leq O(\text{m}) \), \( \mathbb{E}[|S^{(e)}_{h,j}|] \leq O(\text{m}\text{m}) < +\infty \). Notice that, for \( t \in \mathcal{T}_2^{(e)} \), \( w_{t-1}, \ldots, w_{t-2H^{(e)}} \in \mathcal{F}_t \), and \( M^{(e)} \in \mathcal{F}_t \), so \( q_t \in \mathcal{F}_t \), so \( S^{(e)}_{h,j} \in \mathcal{F}^{(e)}_{h,j} \). Next, \( \mathbb{E}[S^{(e)}_{h,j+1} \mid \mathcal{F}^{(e)}_{h,j}] = S^{(e)}_{h,j} + \mathbb{E}[q^{(e)}_{h,j+1} \mid \mathcal{F}^{(e)}_{h,j}] = S^{(e)}_{h,j} \). So this is done. The rest is by definition, and \( q_t \)’s bound.

\[ \Box \]
Lemma 32. Consider our choice of $H^{(e)}$ in Theorem 3. Let $\delta = \frac{p}{2 \sum_{e=0}^{\infty} H^{(e)}}$, w.p. $1 - \delta$, we have $|S_{h,k_h}^{(e)}| \leq \tilde{O}\left(\sqrt{k_h^{(e)} mn}\right)$.

Proof. By Lemma 31 we can apply Corollary 3 and obtain the bound, where we used $\log(2/\delta) = \tilde{O}(1)$.

Lemma 33. Consider our choice of $H^{(e)}$ in Theorem 3. For any $e$, w.p. $1 - 2H^{(e)}\delta$, where $\delta = \frac{p}{2 \sum_{e=0}^{\infty} H^{(e)}},$

$$|\sum_{h=0}^{2H^{(e)}-1} S_{h,k_h}^{(e)}| \leq \tilde{O}\left(\sqrt{T^{(e+1)} mn}\right)$$

Proof. Define event

$$E_h^{(e)} = \{|S_{h,k_h}^{(e)}| \leq \tilde{O}\left(\sqrt{k_h^{(e)} mn}\right)\}$$

When $\cap_h E_h^{(e)}$ holds,

$$|\sum_{e \in T_2^{(e)}} q_t| = |\sum_{h=0}^{2H^{(e)}-1} S_{h,k_h}^{(e)}|\tilde{O}(mn \sqrt{\sum_h k_h^{(e)} \sqrt{2H^{(e)}}}) \leq \tilde{O}(mnT^{(e+1)})$$

where we used Lemma 30 and Cauchy Schwartz.

Then, we have

$$P(\cap_h E_h^{(e)}) = 1 - P(\cup_h (E_h^{(e)})^c) \geq 1 - \sum_h P((E_h^{(e)})^c) \geq 1 - 2H^{(e)}\delta$$

Now, we can prove Lemma 19. By Lemma 33, w.p. $1 - p$, we have $|\sum_{h=0}^{2H^{(e)}-1} S_{h,k_h}^{(e)}| \leq \tilde{O}\left(\sqrt{T^{(e+1)} mn}\right)$ for all $e$. Then, by Lemma 14 we completed the proof.