Pure quantum integrability

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Abstract

The correspondence between the integrability of classical mechanical systems and
their quantum counterparts is not a 1-1, although some close correspondencies exist.
If a classical mechanical system is integrable with invariants that are polynomial in
momenta one can construct a corresponding commuting set of differential operators.
Here we discuss some 2- or 3-dimensional purely quantum integrable systems (the
1-dimensional counterpart is the Lame equation). That is, we have an integrable po-
tential whose amplitude is not free but rather proportional to $\hbar^2$, and in the classical
limit the potential vanishes. Furthermore it turns out that some of these systems
actually have $N + 1$ commuting differential operators, connected by a nontrivial al-
gebraic relation. Some of them have been discussed recently by A.P. Veselov et. al.
from the point of view of Baker-Akheizer functions.

1 Introduction

In classical mechanics the most common definition of integrability is that of Liouville
integrability: Suppose we have a system with $N$ degrees of freedom, that is, we have $N$
coordinates $q_i$ and $N$ conjugate momenta $p_j$, with Poisson brackets $[q_i, p_j] = \delta_{ij}$. This
system is said to be Liouville integrable [1], if there are $N$ functions $F_k(p, q)$ such that 1)
$[F_i, F_j] = 0, \forall i, j$, and 2) the $F_i$ are functionally independent and 3) sufficiently regular.
One of the $F$’s is the Hamiltonian that gives the dynamics. Weaker and stronger types of integrability have also been used (e.g. partial integrability, algebraic integrability and super-integrability).

When the idea of integrability is applied to quantum mechanics the properties of commutativity and functional independence are mutually exclusive: According to a theorem of von Neumann [2], if a pair of self-adjoint operators $A$ and $B$ commutes, then there is a third self-adjoint operator $C$ such that $A = f(C)$, $B = g(C)$. This problem is avoided if we concentrate on differential operators, which anyway are the most interesting from practical point of view. Thus, even if $A = i\partial_x$ and $B = i\partial_y$ commute, the operator $C$ mentioned above is not a differential operator. In this paper we consider only differential operators and property 3) above is then automatically satisfied, but we should relax the independence requirement 2): we should only ignore cases where one of the commuting operator can be given as a polynomial of the others.

In this article we study the existence of operators commuting with a standard type two-dimensional Hamiltonian (Schrödinger operator)

$$\mathcal{H} = -\frac{1}{2}\hbar^2 \left( \partial_x^2 + \partial_y^2 \right) + V(x, y).$$

This may be considered as arising from the classical system $H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y)$ by the usual correspondence $p_\alpha \rightarrow -i\hbar \partial_\alpha$. (The classical integrability of such systems has been reviewed in [3].) The transition from classical to quantum integrability seems to be always possible, sometimes one just has to add correction terms to the Hamiltonian and to the operator commuting with it, but these correction terms are always $O(\hbar^2)$ [4]. Recently the existence of a differential operator commuting with (1) has been discussed at length in [5] (along with multi-dimensional generalizations). In that work strong assumptions were made on the symmetries of the potential (invariance under the Weyl group), but it turns out that the purely quantum integrable (PQI) systems discussed here do not have such symmetry properties.

An important difference between classical and quantum integrability lies in the question of independence as mentioned above. Thus it will be meaningful and interesting to have $N + 1$ commuting operators, even if they must be algebraically related (sometimes this is called algebraic integrability). This can be already illustrated for 1-dimensional systems.
They are always integrable (the Hamiltonian commutes with itself) but some systems are
more integrable that others. For example the Lame operator

$$\mathcal{L} = -\frac{1}{2} \hbar^2 \frac{d^2}{dx^2} + \hbar^2 \mathcal{P}(x),$$

where $\mathcal{P}$ is the Weierstrass elliptic function, commutes with

$$\mathcal{I} = -i\hbar^3 \frac{d^3}{dx^3} + \frac{3}{2} i\hbar^3 \left\{ \frac{d}{dx}, \mathcal{P}(x) \right\},$$

where the bracket $\{., .\}$ stands for an anticommutator. The operators $\mathcal{L}$ and $\mathcal{I}$ are alge-
braically related,

$$\mathcal{I}^2 = 8 \mathcal{L}^3 - \frac{1}{2} \hbar^4 g_2 \mathcal{L} + \frac{1}{4} \hbar^6 g_3,$$

(where $g_i$ are the constants characterizing $\mathcal{P}$) and this relation can be used to express the
wave-function in terms of other elliptic functions. The potential $\hbar^2 \mathcal{P}(x)$ is the only one
that has a third order commuting operator, for a fifth order operator generalizations are possible.

The classification of commuting ordinary differential operators was studied already in
the 1920’s and more recently this problem has been studied from the point of view of
soliton theory. The corresponding two dimensional problem (i.e., the existence of three
commuting differential operators) was studied in . In the one-dimensional case the
algebraically integrable cases are (trivially) special cases of normal (classical) integrability,
and in it was conjectured that this holds also in higher dimensions, but it turns out not
to be the case.

In Section 2 we present results from a search for a third order differential operator,
commuting with $\mathcal{H}$ of . We find that there are indeed some PQI systems, whose potential
is proportional to $\hbar^2$ and therefore vanishes in the classical limit. In section 3 we show, that
the integrable systems found in Section 2 also have a fourth order commuting operator,
and that the three operators are polynomially related. In Section 4 we consider fourth
order differential operators commuting with the Hamiltonian with a potential that is
a generalization of the rational of four term Calogero-Moser potential. Again some PQI
potentials are found.
2 Third order commuting operator

We will consider only those third order differential operators $\mathcal{I}_3$ whose leading order part has constant coefficients. The analysis of the leading terms is the same as that of the corresponding classical system; this follows from the Weyl correspondence and the fact that the leading part of the Moyal bracket (which represents the commutator) agrees with the Poisson bracket \[4\]. Since the second order operator $\mathcal{H}$ has no first order derivatives, we may assume that $\mathcal{I}_3$ has no second order derivatives, i.e.,

$$\mathcal{I}_3 = (\imath \hbar)^3 (c_3 \partial_x^3 + c_2 \partial_x^2 \partial_y + c_1 \partial_x \partial_y^2 + c_0 \partial_y^3) + d_{11} \partial_x + d_{10} \partial_y + d_0.$$  \hspace{1cm} (5)

From the leading terms of the commutation condition $[\mathcal{I}_3, \mathcal{H}] = 0$ one immediately derives a linear PDE for the potential $V$

$$c_1 V_{xxx} + (3c_0 - 2c_2) V_{xxy} + (3c_3 - 2c_1) V_{xyy} + c_2 V_{yyy} = 0, \hspace{1cm} (6)$$

where the subscripts stand for partial derivatives. The generic solution of this equation is

$$V = g_1(\alpha_1 x + \beta_1 y) + g_2(\alpha_2 x + \beta_2 y) + g_3(\alpha_3 x + \beta_3 y), \hspace{1cm} (7)$$

where $(\alpha_i, \beta_i)$ are the three solutions of $c_1 \alpha^3 + (3c_0 - 2c_2) \alpha^2 \beta + (3c_3 - 2c_1) \alpha \beta^2 + c_2 \beta^3 = 0$. (The non-generic cases need separate study.)

The potential (7) is form invariant under coordinate rotations, which just imply changes in the parameters $\alpha_i, \beta_i$. This suggests writing all formulae in a rotationally covariant form. Furthermore, we may assume that the parameters $\alpha_i, \beta_i$ satisfy the condition

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \beta_1 + \beta_2 + \beta_3 = 0. \hspace{1cm} (8)$$

(To obtain this it is only necessary to scale the arguments of the functions $g$.) The submanifold (8) is invariant under parameter rotations. Instead of $x, y$ and the corresponding partial derivatives let us use the quantities

$$X_i = \alpha_i x + \beta_i y, \quad \bar{X}_i = \beta_i x - \alpha_i y, \quad L_i = -\imath \hbar (\beta_i \partial_x - \alpha_i \partial_y), \quad \bar{L}_i = -\imath \hbar (\alpha_i \partial_x + \beta_i \partial_y). \hspace{1cm} (9)$$
which satisfy

\[ [L_i, X_j] = -[\bar{L}_i, \bar{X}_j] = i\hbar D_{ij}, \quad [L_i, \bar{X}_j] = [\bar{L}_i, X_j] = -i\hbar \Delta_{ij}, \tag{10} \]

where

\[ D_{ij} := \alpha_i \beta_j - \alpha_j \beta_i, \quad \Delta_{ij} := \alpha_i \alpha_j + \beta_i \beta_j. \tag{11} \]

Due to (8) we have \( D \equiv D_{12} = D_{23} = D_{31} = -D_{21} = -D_{32} = -D_{13}. \) On the manifold (8) we have \( \Delta_{1i} + \Delta_{2i} + \Delta_{3i} = 0 \) so that we have the following relations between the diagonal and non-diagonal elements:

\[ \Delta_{ii} = - (\Delta_{ij} + \Delta_{ki}), \quad \Delta_{ij} = \frac{1}{2} (-\Delta_{ii} - \Delta_{jj} + \Delta_{kk}), \tag{12} \]

where \( \{i, j, k\} \) is a permutation of \( \{1, 2, 3\} \). Furthermore

\[ L_1 + L_2 + L_3 = 0, \quad L_1^2 \Delta_{23} + L_2^2 \Delta_{31} + L_3^2 \Delta_{12} = \hbar^2 D(\partial_x^2 + \partial_y^2). \tag{13} \]

By a direct computation we find the following expression for \( I_3 \)

\[ I_3 = L_1^3 \Delta_{23}^2 + L_2^3 \Delta_{31}^2 + L_3^3 \Delta_{12}^2 + \frac{3}{2} D^2 \sum_{j=1}^{3} \left( L_{ij} \left( \sum_{k=1}^{3} g_k \Delta_{jk} \right) - 2 g_j \Delta_{jj} \right), \tag{14} \]

but there is one remaining condition that can be written as

\[
\begin{align*}
(\Delta_{22} - \Delta_{33})(g_1'' h^2 \Delta_{11} - 12 g_1' g_1) \\
+ (\Delta_{33} - \Delta_{11})(g_2'' h^2 \Delta_{22} - 12 g_2' g_2) \\
+ (\Delta_{11} - \Delta_{22})(g_3'' h^2 \Delta_{33} - 12 g_3' g_3) \\
- 12 [(g_1 g_2 - g_1' g_2) \Delta_{12} + (g_2 g_3 - g_2' g_3) \Delta_{23} + (g_3 g_1 - g_3' g_1) \Delta_{31}] = 0.
\end{align*}
\]

The complete solution of (15) is not known, but several solutions can be found when we note that the Weierstrass elliptic function \( P \) satisfies the equations

\[ P''' = 12P P', \quad \begin{vmatrix} 1 & P(u) & P(u)' \hline 1 & P(v) & P(v)' \hline 1 & P(w) & P(w)' \end{vmatrix} = 0, \text{ where } u + v + w = 0. \tag{16} \]
Thus if we assume that \( g_i = a_i \mathcal{P} \) we can identify the last three terms of (15) as the above determinant, if
\[
a_1a_2\Delta_{12} = a_2a_3\Delta_{23} = a_3a_1\Delta_{31}.
\]
(17)

The first three terms of (15) imply three further equation:
\[
(\Delta_{22} - \Delta_{33})(h^2\Delta_{11} - a_1) = 0,
\]
\[
(\Delta_{33} - \Delta_{11})(h^2\Delta_{22} - a_2) = 0,
\]
\[
(\Delta_{11} - \Delta_{22})(h^2\Delta_{33} - a_3) = 0.
\]
(18)

From (17) we immediately find that
\[
a_i = \Delta_{jk}X, \ (i, j, k \text{ cyclic}),
\]
(19)

so that in any case the potential will be of the form
\[
V \propto \Delta_{23}\mathcal{P}(\alpha_1x + \beta_1y) + \Delta_{31}\mathcal{P}(\alpha_2x + \beta_2y) + \Delta_{12}\mathcal{P}(\alpha_3x + \beta_3y).
\]
(20)

The solutions of (19, 18) can be divided into three groups:

Case 1: All \( \Delta_{ii} \)'s equal.

From (12) we find that the off-diagonal \( \Delta \)'s and therefore \( a_i \)'s are all equal. The equality of \( \Delta_{ii} \)'s allows the parameterization \( \alpha_i = 2A \cos(\theta_i), \beta_i = 2A \sin(\theta_i) \), and from (8) it follows that \( \theta_2 = \theta_1 + \frac{2}{3}\pi, \theta_3 = \theta_1 - \frac{2}{3}\pi \). This is the well known solution with a nonzero classical limit. In addition to the overall rotation there is the freedom of the amplitudes \( A \) and \( a \).

If we choose \( \theta_1 = \pi/6 \) we get the familiar form
\[
V_{3,1} = a[\mathcal{P}(A(\sqrt{3}x + y)) + \mathcal{P}(A(-\sqrt{3}x + y)) + \mathcal{P}(-2Ay)].
\]
(21)

Case 2: Two \( \Delta_{ii} \)'s equal.

Let us say \( \Delta_{11} = \Delta_{22} \neq \Delta_{33} \). Then from (13) \( X = h^2\Delta_{11}/\Delta_{23} = -2h^2\Delta_{11}/\Delta_{33} \) and \( a_1 = a_2 = h^2\Delta_{11}, a_3 = h^2\Delta_{11}(2\Delta_{11} - \Delta_{33})/\Delta_{33} \). Note the overall \( h^2 \) in the amplitudes \( a_i \).

One possible parameterization (after fixing the rotational freedom) is
\[
V_{3,2} = h^2\frac{A^2 + B^2}{2A^2}[2A^2\mathcal{P}(Bx + Ay) + 2A^2\mathcal{P}(-Bx + Ay) + (B^2 - A^2)\mathcal{P}(-2Ay)].
\]
(22)

This is the solution found by Veselov et. al. [9, 10] for the algebraic special case \( \mathcal{P}(x) = x^{-2} \).
Case 3: All $\Delta_{ii}$’s different.

Then we must have $a_i = h^2 \Delta_{ii}$ by (18), and (19) implies $\Delta_{11} + \Delta_{22} + \Delta_{33} = 0$. Thus some of the parameters must be complex. One parameterizations is given by

$$V_{3,3} = h^2 A^2 \left[2(-1 + i\sqrt{3}\sin(\theta))P(i\sqrt{3}A \cos(\theta)x + A(1 + i\sqrt{3}\sin(\theta))y) \right.$$

$$+ 2(-1 - i\sqrt{3}\sin(\theta))P(-i\sqrt{3}A \cos(\theta)x + A(1 - i\sqrt{3}\sin(\theta))y)$$

$$+ 4P(-2Ay)] . \tag{23}$$

Note that even though some $\alpha_i, \beta_i$ must be complex, it is possible to get a real potential $V$, e.g., if $P$ itself is real. This solution seems to be new.

Note that Case 2 intersects with Case 1 at $B = \sqrt{3}A$ ($a = 4h^2A^2$), and with Case 3 at $B = i\sqrt{3}A$ ($\theta = 0$ in (23)). At these special points the system should have more symmetries.

For Cases 2 and 3 the amplitudes are proportional to $h^2$ so they are PQI. Although the amplitudes are fixed, there are two nontrivial degrees of freedom as in case 1, but they are now all in the direction parameters $\alpha_i, \beta_i$.

### 2.1 Connection with a three dimensional system

A Hamiltonian of the form (17) (with the term $-\Delta_{12}\Delta_{23}\Delta_{31}p_z^2/D^2$ added) can also be written as a three dimensional system

$$H = -\frac{\hbar^2}{2} \left(\Delta_{23} \partial_1^2 + \Delta_{31} \partial_2^2 + \Delta_{12} \partial_3^2 \right) + g_1(q_2 - q_3) + g_2(q_3 - q_1) + g_3(q_1 - q_2), \tag{24}$$

through the linear transformation

$$\begin{align*}
  x &= -(q_1\beta_1 + q_2\beta_2 + q_3\beta_3)/D, \\
  y &= (q_1\alpha_1 + q_2\alpha_2 + q_3\alpha_3)/D, \\
  z &= (q_1 \Delta_{13}\Delta_{12} + q_2 \Delta_{23}\Delta_{12} + q_3 \Delta_{13}\Delta_{23})/D^2,
\end{align*} \tag{25}$$

and correspondingly

$$\begin{align*}
  \partial_x &= (\partial_1 \beta_1 \Delta_{23} + \partial_2 \beta_2 \Delta_{13} + \partial_3 \beta_3 \Delta_{12})/D, \\
  \partial_y &= -(\partial_1 \alpha_1 \Delta_{23} + \partial_2 \alpha_2 \Delta_{13} + \partial_3 \alpha_3 \Delta_{12})/D, \\
  \partial_z &= \partial_1 + \partial_2 + \partial_3.
\end{align*} \tag{26}$$
If we denote \( \mu_i = \Delta_{jk}, \ (i, j, k \ \text{cyclic}) \) and use (20) the results can be combined as
\[
H = -\frac{\hbar^2}{2} \left( \mu_1 \partial^2_1 + \mu_2 \partial^2_2 + \mu_3 \partial^2_3 \right) + A \left[ \mu_1 P(q_2 - q_3) + \mu_2 P(q_3 - q_1) + \mu_3 P(q_1 - q_2) \right],
\]
and the three integrable cases are characterized as follows:

1. The classical case: \( \mu_1 = \mu_2 = \mu_3, A \) free.
2. Elliptic PQI case: \( \mu_1 = \mu_2, A = -\hbar^2(\mu_1 + \mu_3)/\mu_1 \)
3. Hyperbolic PQI case: \( \mu_1 + \mu_2 + \mu_3 = 0, A = \hbar^2. \)

Note that for the two PQI cases the same expression \( A = \hbar^2[1 - (\mu_1 + \mu_2 + \mu_3)/\mu_1] \) works for the overall amplitude.

In [9, 10] the \( n \)-dimensional elliptic case was associated to a non-Coxeter configuration called \( A_n(m) \), similar geometric interpretation for the hyperbolic case would be interesting.

3 Fourth order commuting operator

In this section we consider the operator
\[
\mathcal{I}_4 = (i\hbar)^4 \sum_{i=0}^{4} c_i \partial^4_{x_i} \partial^4_{y_i} + \ldots
\]
and now the condition of commutativity \( [\mathcal{I}_4, \mathcal{H}] = 0 \) leads to the necessary condition
\[
c_1 V_{xxxx} + 2(2c_0 - c_2)V_{xxyy} + 3(c_3 - c_1)V_{xxyy} + 2(c_2 - 2c_4)V_{xyyy} - c_3 V_{yyyy} = 0.
\]
Although this is a fourth order equation it does not allow four arbitrary directions \((\alpha_i, \beta_i)\), because of the way the \( c_i \)'s enter in the equation. We will return to this problem later and first consider potential with three terms.

3.1 Three term potential

By direct calculation with (7,8) one finds that the invariant can be written as
\[
\mathcal{I}_4 = \sum_{j,k=1}^{3} \left\{ L_j^2, -g_j \Delta^2_{j+1,j+2} + g_k (\Delta_{k,j+1} \Delta_{k,j+2} + 3 \Delta^2_{j+1,j+2}) \right\} + C(x, y).
\]
Here the indices for $\Delta$ are to be taken modulo 3. The integrability condition for $C$ can be written as

$$(\Delta_{23}\tilde{L}_1 + \Delta_{31}\tilde{L}_2 + \Delta_{12}\tilde{L}_3) \Omega = 0$$

where

$$\Omega = \frac{3\Delta_i,_{i+1}\Delta_{i,_{i+2}}}{\Delta_{ii}\Delta_{i+1_{,i+2}} + 2\Delta_{i,_{i+1}}\Delta_{i,_{i+2}}}$$

(The denominator of $K_i$ can also be written as $[\Delta_{23}\tilde{L}_1 + \Delta_{31}\tilde{L}_2 + \Delta_{12}\tilde{L}_3, \tilde{X}_i]$.) The expression (32) is the same as (15), except for the extra coefficients $K_i$, which however play no role if (32) is solved by $g_i = a_i P$ with (17, 18). The integrability condition for $C$ may have other solutions, but we can say at least that all solutions obtained in Section 2 also have a fourth order invariant. However, since $\mathcal{H}^2$ is also a fourth order invariant commuting with $\mathcal{H}$ we must discuss the independence of the newly found invariants.

### 3.2 Relations between the invariants

Since the system is two dimensional there can be at most two algebraically independent commuting quantities. Indeed on finds relations between the three operators found above, for example in Case 1 we have

$$\mathcal{I}_4 = -\Delta_{11} D^4 \mathcal{H}^2,$$

so that in this case the new fourth order invariant is useless.

For the other cases the algebraic relation is less trivial and therefore the extra invariant provides additional information. The computations are rather extensive and we have verified only the relation connecting the leading terms of $\mathcal{I}_j$, denoted below by $I_j$. The results are as follows:
For case 2 when $\Delta_{22} = \Delta_{11}$ we have the relation

$$I_6^3 \Delta_{33}^3 (16\Delta_{11}^4 + 16\Delta_{11}^3 \Delta_{33} - 20\Delta_{11}^2 \Delta_{33}^2 - 12\Delta_{11} \Delta_{33}^3 + 9\Delta_{33}^4)
+ 12I_2^4 I_4 \Delta_{33}^3 (16\Delta_{11}^4 + 8\Delta_{11}^3 \Delta_{33} - 16\Delta_{11}^2 \Delta_{33}^2 - 2\Delta_{11} \Delta_{33}^3 + 3\Delta_{33}^4)
+ 32I_2^2 I_4^2 \Delta_{33}^3 (12\Delta_{11}^4 - 26\Delta_{11}^3 \Delta_{33} + 15\Delta_{11}^2 \Delta_{33}^2 - \Delta_{33}^4)
+ 48I_2^2 I_4^2 \Delta_{11} \Delta_{33} (16\Delta_{11}^3 - 9\Delta_{11} \Delta_{33}^2 + 2\Delta_{33}^3)
+ 384I_2^2 I_4 \Delta_{11} \Delta_{33} (4\Delta_{11}^3 - 9\Delta_{11} \Delta_{33}^2 + 6\Delta_{11} \Delta_{33}^2 - \Delta_{33}^3)
+ 256I_4 \Delta_{11} (4\Delta_{11}^4 - 13\Delta_{11}^3 \Delta_{33} + 15\Delta_{11}^2 \Delta_{33}^2 - 7\Delta_{11} \Delta_{33}^3 + \Delta_{33}^4)
+ 64I_4^2 \Delta_{11}^2 (16\Delta_{11}^3 - 8\Delta_{11} \Delta_{33} + \Delta_{33}^2) = 0,$$

which is of degree 12 in the derivatives. If $\Delta_{11} = \Delta_{33}$ (intersection with Case 1) this expression simplifies to $9\Delta_{11}^4 (\Delta_{11} I_2^2 + 4I_4)^3 = 0$, in agreement with (33). It simplifies also if $\Delta_{33} = 4\Delta_{11}$ but this corresponds to the trivial limit $B = 0$.

In case 3 we get the following complicated identity among the leading terms:

$$i_4^6 (a^2 - a + 1)^4 a^2 (a - 1)^2
- 6 i_4^2 i_2^2 (a^2 - a + 1)^3 a (a - 1) (a^6 - 3a^5 + 5a^3 - 3a + 1)
+ 12 i_4^4 i_2^2 (a^2 - a + 1)^3 a (a - 1) (2a^6 - 6a^5 + 3a^4 + 4a^3 + 3a^2 - 6a + 2)
+ 3 i_4^4 i_2^2 (a^2 - a + 1)^2 (3a^{12} - 18a^{11} + 19a^{10} + 70a^9 - 140a^8 - 58a^7 + 251a^6
- 58a^5 - 140a^4 + 70a^3 + 19a^2 - 18a + 3)
- 2 i_4^3 i_3^2 (a^2 - a + 1)^3 a (a - 1) (8a^6 - 24a^5 + 21a^4 - 2a^3 + 21a^2 - 24a + 8)
- 4 i_4^3 i_3^2 i_2^2 (a^2 - a + 1)^2 (2a^{12} - 12a^{11} - 44a^{10} + 330a^9 - 425a^8 - 412a^7 + 1124a^6
- 412a^5 - 425a^4 + 330a^3 - 44a^2 - 12a + 2)
+ 2 i_4^3 i_2^2 (a^2 - a + 1) a (a - 1) (35a^{12} - 210a^{11} + 283a^{10} + 510a^9 - 1268a^8 - 298a^7
+ 1931a^6 - 298a^5 - 1268a^4 + 510a^3 + 283a^2 - 210a + 35)
- 18 i_4^2 i_3^2 i_2^2 (a^2 - a + 1)^2 a^2 (a - 1)^2 (a^2 - a - 5) (5a^2 - 11a + 5) (5a^2 + a - 1)
- 12 i_4^2 i_3^2 i_2^2 (a^2 - a + 1) a (a - 1) (13a^{12} - 78a^{11} + 54a^{10} + 445a^9 - 786a^8 - 384a^7
+ 1485a^6 - 384a^5 - 786a^4 + 445a^3 + 54a^2 - 78a + 13)
+ 3 i_4^2 i_2^2 (2a^{18} - 18a^{17} + 112a^{16} - 488a^{15} + 1012a^{14} + 28a^{13} - 3599a^{12} + 4330a^{11}
+ 2433a^{10} - 7622a^9 + 2433a^8 + 4330a^7 - 3599a^6 + 28a^5 + 1012a^4
+ 28a^4 + 1012a^3 - 3599a^2 + 4330a - 7622a + 2433)
\[-488a^3 + 112a^2 - 18a + 2\]
\[+ 108 i_4 i_3^6 i_2(a^2 - a + 1)^2 a^2(a - 1)^2(a^2 - 4a + 1)(a^2 + 2a - 2)(2a^2 - 2a - 1)\]
\[+ 6 i_4 i_3^4 i_2^2(a^2 - a + 1)a(a - 1)(8a^{12} - 48a^{11} - 120a^{10} + 1040a^9 - 1323a^8 - 1476a^7\]
\[+ 3846a^6 - 1476a^5 - 1323a^4 + 1040a^3 - 120a^2 - 48a + 8)\]
\[-12 i_4 i_3^2 i_2^2(a^3 - 3a + 1)(a^3 - 3a^2 + 1)(2a^{12} - 12a^{11} + 41a^{10} - 95a^9 + 86a^8 + 94a^7\]
\[-230a^6 + 94a^5 + 86a^4 - 95a^3 + 41a^2 - 12a + 2)\]
\[+ 6 i_4 i_2^{10}(a - 2)^2(a + 1)^2(2a - 1)^2(a^2 - a + 1)(a^3 - 2a^2 - a + 1)(a^3 - a^2 - 2a + 1)\]
\[+ 729 i_3^2 a^4(a - 1)^2(a^2 - a + 1)^2\]
\[+ 4 i_3^3 i_2^2(a^2 - a + 1)a(a - 1)(8a^{12} - 48a^{11} + 195a^{10} - 535a^9 + 342a^8 + 1314a^7\]
\[- 2544a^6 + 1314a^5 + 342a^4 - 535a^3 + 195a^2 - 48a + 8)\]
\[+ 2 i_3^4 i_2^6(8a^{18} - 72a^{17} + 231a^{16} - 216a^{15} - 777a^{14} + 3507a^{13} - 6201a^{12} + 3003a^{11}\]
\[+ 7470a^{10} - 13898a^9 + 7470a^8 + 3003a^7 - 6201a^6 + 3507a^5 - 777a^4\]
\[- 216a^3 + 231a^2 - 72a + 8)\]
\[-4 i_3^2 i_2(a^2 - a + 1)^2a(a - 1)(a^3 - 3a + 1)(a^3 - 3a^2 + 1)\]
\[(5a^6 - 15a^5 - 3a^4 + 31a^3 - 3a^2 - 15a + 5)\]
\[+ i_2^{12}(a^2 - a + 1)^4(a^3 - 2a^2 - a + 1)^2(a^3 - a^2 - 2a + 1)^2 = 0,\] \hspace{1cm} (35)

where we have used the parameterization \(\Delta_{11} = (1 - a)b, \Delta_{22} = ab, \Delta_{33} = -b, \) and \(i_j = b^{1-j}I_j.\)

The conditions of Cases 2 and 3 intersect when \(\Delta_{ii} = \Delta_{jj} = -2\Delta_{kk}.\) The various permutations of this correspond to \(a = 1/2, -1\) and \(a = 2\) and for these special values \((34)\) becomes a square of \((34),\) for \(a = -1\) and \(2\) we get the relation

\[i_2^6 + 12 i_2^3 i_3^2 - 3 i_2^2 i_4^2 + 36 i_2 i_3 i_4 - 36 i_3^2 + 2 i_4^3 = 0\] \hspace{1cm} (36)

and for \(a = 1/2\) (i.e., \(\mu_1 = \mu_2 = -\frac{1}{2} \mu_3\))

\[i_2^6 - 24 i_2^3 i_3^2 - 12 i_2^2 i_4^2 + 144 i_2 i_3 i_4 - 144 i_3^2 - 16 i_4^3 = 0\] \hspace{1cm} (37)

(The second form is obtained from the first by \(i_3 \to \sqrt{-2}i_3, i_4 \to -2i_4.\) In fact one can show that the expression \((35)\) can be written as \(U^2 + (a + 1)^2(a - 1/2)^2(a - 2)^2V = 0,\)
where $U$ and $V$ are some polynomials of $a$ and the $i_j$’s. Whether (35) factorizes also for some other special values is an open question.

### 3.3 Four term potential

Let us now return to equation (29). Since it is a fourth order linear equation is should have solutions of type $V = \sum_{i=1}^{4} g(\alpha_i x + \beta_i y)$. This is indeed the case, but since we really have only four rather than five $c_i$’s at our disposal (we can use $\mathcal{H}^2$ to eliminate $c_2$, say) there will be a relation between $\alpha_i$, $\beta_i$. In order to simplify subsequent computations we will use rotational invariance to fix $\alpha_1 = 0$, and scale so that $\beta_1 = 1$, and $\alpha_j = 1$, when $j \neq 1$. The geometric relation can then be written as

$$3\beta_2\beta_3\beta_4 + \beta_2 + \beta_3 + \beta_4 = 0. \quad (38)$$

In the following we will also restrict our attention only to $g(x) \propto 1/x^2$, i.e.,

$$V(x, y) = \frac{a_1}{y^2} + \frac{a_2}{(x + \beta_2 y)^2} + \frac{a_3}{(x + \beta_3 y)^2} + \frac{a_4}{(x + \beta_4 y)^2}. \quad (39)$$

By direct computation we obtain the leading terms of the fourth order operator commuting with the Hamiltonian (1,39), as

$$\mathcal{I}_4 = \hbar^4[\partial_x^4 + 4\partial_x \partial_y^3 \beta_2 \beta_3 \beta_4 - \partial_y^4(\beta_2 \beta_3 + \beta_2 \beta_4 + \beta_3 \beta_4)]$$

$$-4\hbar^2 \left\{ \partial_x^2, g_2 + g_3 + g_4 \right\}$$

$$- \{\partial_x, \partial_y, ((\beta_2 + \beta_3 + \beta_4)g_1 + \beta_2 g_2 + \beta_3 g_3 + \beta_4 g_4) \}$$

$$- \{\partial_y, ((\beta_2 \beta_3 + \beta_2 \beta_4 + \beta_3 \beta_4)g_1 + \beta_2 (\beta_3 + \beta_4)g_2 + \beta_3 (\beta_2 + \beta_4)g_3 + \beta_4 (\beta_2 + \beta_3)g_4) \}$$

$$+ D(x, y). \quad (40)$$

This far we can proceed just with (38). The integrability condition for $D$ introduces further conditions, and leads to the following classification:
Case 1

Assume that one $\beta$ vanishes, let us say $\beta_4 = 0$. Then $\beta_3 = -\beta_2$ from (38) and we get two integrable potentials: First there is the well known classical one with $\beta_2 = -\beta_3 = 1$

\[ V_{4,1} = \frac{a_1}{y^2} + \frac{a_2}{(x+y)^2} + \frac{a_2}{(x-y)^2} + \frac{a_1}{x^2}. \]  

We note that there are two degrees of freedom, all in the amplitudes.

Case 2

There is another possibility with $\beta_3 = -\beta_2$ arbitrary:

\[ V_{4,2} = C \left( \frac{1}{x^2} + \frac{\beta_4^2}{y^2} \right) + \hbar^2 \left( \frac{1 + \beta_2^2}{(x + \beta_2 y)^2} + \frac{1 + \beta_2^2}{(x - \beta_2 y)^2} - \frac{1}{8x^2} - \frac{1}{8y^2} \right). \]  

This potential has the classical limit $V = Ax^{-2} + By^{-2}$, with is separable and therefore has a quadratic second invariant (in both classical and quantum picture). The result above indicates that in quantum mechanics there exists a non-separable generalization for it. There are now two free parameters, $C$ and $\beta_2$, the solution presented in [9] corresponds to the choice $C = \frac{\hbar^2}{2} (m + \frac{1}{2})^2$, $C\beta_2^4 = \frac{\hbar^2}{2} (l + \frac{1}{2})^2$

Case 3

Next we assume that $\beta_4 \neq 0$, and since due to rotational invariance we could rotate any of the vector to the position $y$, we may in fact assume that all $\beta$'s are nonzero (in other words that no pair of the vectors is orthogonal). From the equations it then follows that $a_i = \hbar^2 (1 + \beta_i^2)$, which normalizes the vectors. There is one solution of this type, it is defined by (38) and

\[ \beta_2^2 + \beta_3^2 + \beta_4^2 - \beta_2 \beta_3 - \beta_3 \beta_4 - \beta_4 \beta_2 = 0, \]  

which have the parameterization

\[ \beta_2 = \frac{1-a}{\sqrt{a^3} - 1}, \quad \beta_3 = \frac{1-a\xi}{\sqrt{a^3} - 1}, \quad \beta_4 = \frac{1-a\xi^2}{\sqrt{a^3} - 1}, \]  

where $\xi$ is a cubic root of unity $\neq 1$. This is a new result. For real $a > 1$ the potential is also real because the $g_3$ and $g_4$ terms are complex conjugates. There is only one free parameter, but this is probably due to the special form $1/x^2$, for example if one makes the same restriction in (22) [22] one can scale out $A$. 

13
4 Conclusions

We have discussed the quantum integrability in two dimensions, and in particular the existence of commuting differential operators of order 3 and 4. For the third order operator we have considered the generic case with constant coefficients in the leading term, and found the classical three term Calogero-Moser system (in terms of Weierstrass functions) and two PQI (three term) systems, one of which was first reported in [9]. These systems also have a fourth order commuting operator, but only for the PQI cases is the algebraic relation between the three operators nontrivial.

It is also possible to have integrable four term potentials if the commuting operator is of fourth order. Our analysis was here restricted to $1/x^2$-type potential terms and we have identified two PQI examples. Whether these potentials are also algebraically integrable is an open question: in principle one should find for them another, still higher order invariant, but this seems to be a formidable task.

With these results we have only scratched the surface of pure quantum integrability. What is missing in particular is a geometric characterization of the phenomena and comprehensive extensions to higher orders and dimensions. The classification based on Lie algebra root systems [11], that works so well in the classical case, does not include these new PQI systems.

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