Partial Traces and the Geometry of Entanglement; Sufficient Conditions for the Separability of Gaussian States

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January 11, 2022

Abstract

The notion of partial trace of a density operator is essential for the understanding of the entanglement and separability properties of quantum states. In this paper we investigate these notions putting an emphasis on the geometrical properties of the covariance ellipsoids of the reduced states. We thereafter focus on Gaussian states and we give new and easily numerically implementable sufficient conditions for the separability of all Gaussian states. Unlike the positive partial transposition criterion, none of these conditions is however necessary.

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Introduction

Mixed quantum states play a pivotal role in quantum mechanics and its applications (for instance teleportation, cryptography, quantum computation and optics, to name a few). Mixed states are identified for all practical purposes with their density operators (or matrices), which are positive semidefinite self-adjoint operators with trace one on a Hilbert space $H$. One of the most important problems in density operator theory, which is still largely open at the time being, is the characterization of the separability of density operators or, which amounts to the same, of the entanglement properties of mixed quantum states. In the case $H = L^2(\mathbb{R}^n)$ (which we assume from now on) necessary conditions for separability can be found in the literature; one of the oldest is the Peres–Horodecki criterion on the partial transpose of a density operator; more recently Werner and Wolf have proposed a geometric condition involving the covariance matrix of the state. This condition is also sufficient for separability for all density operators with Wigner distribution

$$\rho(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2} z^T \Sigma^{-1} z}$$  (1)
the covariance matrix $\Sigma$ being subjected to the quantum condition

$$\Sigma + \frac{ih}{2} J \geq 0 \quad (2)$$

(see §1.1 for a discussion of this condition). It requires that the covariance ellipsoid

$$\Omega = \{z : \frac{1}{2} \Sigma^{-1} z^2 \leq 1\}$$

has symplectic capacity at least $\pi \hbar$; this property, which is a topological formulation of the uncertainty principle, means that there exists a symplectic automorphisms of $\mathbb{R}^{2n}$ sending the phase space ball with radius $\sqrt{\hbar}$ inside $\Omega$.

We will discuss the partial traces $\hat{\rho}_A$ and $\hat{\rho}_B$ of a density operator $\hat{\rho}$ with respect to a splitting $\mathbb{R}^{2n} = \mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B}$ of phase space, and show that the covariance ellipsoids of $\hat{\rho}_A$ and $\hat{\rho}_B$ are the orthogonal projections of $\Omega$ onto the reduced phase spaces $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$. We will see that if in particular $\hat{\rho}$ is a Gaussian then these reduced states are themselves Gaussian states with Wigner distributions

$$\rho_A(z_A) = \frac{1}{(2\pi)^{n_A} \sqrt{\det \Sigma_{AA}}} e^{-\frac{1}{2} \Sigma_{AA}^{-1} z_A^2}$$

$$\rho_B(z_B) = \frac{1}{(2\pi)^{n_B} \sqrt{\det \Sigma_{BB}}} e^{-\frac{1}{2} \Sigma_{BB}^{-1} z_B^2}$$

where the reduced covariance matrices $\Sigma_{AA}$ and $\Sigma_{BB}$ are calculated from the total covariance matrix $\Sigma$ using the theory of Schur complements (see §1.2), and the corresponding covariance ellipsoids

$$\Omega_A = \{z_A \in \mathbb{R}^{2n_A} : \frac{1}{2} \Sigma_{AA}^{-1} z_A^2 \leq 1\}$$

$$\Omega_B = \{z_B \in \mathbb{R}^{2n_B} : \frac{1}{2} \Sigma_{BB}^{-1} z_B^2 \leq 1\}$$

are the orthogonal projections (or “shadows”) of the covariance ellipsoid $\Omega$ on the reduced phase spaces $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$, respectively.

The main new results are stated and proved in Sections 3 and 4. In these sections we discuss the separability of Gaussian states. In Section 3, we prove a necessary and sufficient condition for the separability of Gaussian states, which amounts to a refinement of the Werner-Wolf condition. In Section 4, we prove various sufficient conditions for the separability of Gaussian states, and show that, while sufficient, they are not necessary conditions.

**Notation 1** The standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$ is $\sigma = \sum_{j=1}^{n} dp_j \wedge dx_j$; in matrix notation $\sigma(z, z') = J z \cdot z' = (z')^T J z$ where $J = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{pmatrix}$
and \cdot denotes the Euclidean scalar product. We denote by $Sp(n)$ the symplectic group of $(\mathbb{R}^{2n}, \sigma)$. Given a tempered distribution $a \in S'(\mathbb{R}^{2n})$ we denote by $Op_W(a)$ the Weyl operator with symbol $a$. The metaplectic group $Mp(n)$ is a faithful unitary representation of the double cover of $Sp(n)$; elements of $Mp(n)$ are denoted by $\hat{S}$ and their projections on $Sp(n)$ by $S$. Given $S \in Sp(n)$, and $R > 0$, the symplectic ball $S(B^{2n}(R))$ is the ellipsoid:

$$S(B^{2n}(R)) = \{Sz : |z| \leq R \}.$$ 

1 Partial Traces and Reduced States

1.1 Density operators: basics

Let $\hat{\rho} \in L_1(L^2(\mathbb{R}^n))$ be a positive semidefinite operator with trace $\text{Tr}(\hat{\rho}) = 1$ on $L^2(\mathbb{R}^n)$. In particular $\hat{\rho}$ is self-adjoint and compact. Such operators represent the mixed states of quantum mechanics and we will freely identify them with these states. It follows from the spectral theorem that there exists a sequence $(\lambda_j)_{j \in I}$ (I a discrete index set) of nonnegative real numbers with $\sum_{j \in I} \lambda_j = 1$ and an orthonormal basis $(\psi_j)_{j \in I}$ of $L^2(\mathbb{R}^n)$ such that $\hat{\rho} = \sum_{j \in I} \lambda_j \hat{\Pi}_j$ where $\hat{\Pi}_j$ is the orthogonal projection on the ray $\mathbb{C}\psi_j$. The number

$$\mu(\hat{\rho}) = \sum_{j \in I} \lambda_j^2 = \text{Tr}(\hat{\rho}^2) \quad (3)$$

is called the purity of $\hat{\rho}$ and we have $\mu(\hat{\rho}) = 1$ if and only if one of the coefficients $\lambda_j$ is equal to one, in which case $\hat{\rho} = \hat{\Pi}_j$ is called a pure state. Density operators are Weyl operators in their own right; in fact $\hat{\rho} = (2\pi\hbar)^n Op_W(\rho)$ where

$$\rho = \sum_{j \in I} \lambda_j W\psi_j \quad (4)$$

the $W\psi_j \in L^2(\mathbb{R}^{2n})$ being the Wigner transforms of the functions $\psi_j$; it follows from Moyal’s identity [15] that the $W\psi_j$ form an orthonormal subset of $L^2(\mathbb{R}^{2n})$. The operator $\hat{\rho}$ is the bounded operator on $L^2(\mathbb{R}^n)$ with square-integrable distributional kernel

$$K(x,y) = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(p(x-y))\rho(\frac{1}{2}(x+y),p)} dp. \quad (5)$$

It is current practice in the physically oriented literature to write

$$\text{Tr}(\hat{\rho}) = \int_{\mathbb{R}^n} K(x,x) dx \quad (6)$$
which leads, setting \( x = y \) in (5), to
\[
\text{Tr}(\hat{\rho}) = \int_{\mathbb{R}^{2n}} \rho(z)dz = 1 .
\] (7)

One has however to view these formulas with a more than critical eye; they are generally false unless some additional conditions are imposed on \( \rho(z) \) (see [13, 16] and the references therein). Formula (7) however holds true if one makes the extra assumption that \( \rho \in L^1(\mathbb{R}^{2n}) \) (see [8]). We will use in this paper the following stronger result due to Shubin ([30], §27). Setting
\[
\langle z \rangle = (1 + |z|^2)^{1/2}
\]
for \( z \in \mathbb{R}^{2n} \) we have:

**Proposition 2 (Shubin)** Let \( \hat{\rho} \) be a bounded operator with Weyl symbol \((2\pi\hbar)^n \rho\) If \( \rho \in C^\infty(\mathbb{R}^{2n}) \) and all its z-derivatives \( \partial^\alpha z \rho \) satisfy estimates
\[
|\partial^\alpha z \rho(z)| \leq C_\alpha \langle z \rangle^{m-|\alpha|}
\] (8)
with \( m < -2n \) and \( C_\alpha > 0 \), then the operator \( \hat{\rho} \) is of trace class and we have
\[
\text{Tr}(\hat{\rho}) = \int_{\mathbb{R}^{2n}} \rho(z)dz .
\] (9)

The interest of this result comes from the fact that one does not have to assume from the beginning that \( \hat{\rho} \) is of trace class, let alone a density operator. Notice that the trace formula (9) automatically follows since the condition (8) implies that \( \rho \in L^1(\mathbb{R}^{2n}) \).

We will denote by \( \Gamma^m(\mathbb{R}^{2n}) \) the Shubin class of all functions \( \rho \in C^\infty(\mathbb{R}^{2n}) \) satisfying the estimates (8) for all \( \alpha \in \mathbb{N}^n \).

Let \( \hat{\rho} = (2\pi\hbar)^n \text{Op}_W(\rho) \) be a density operator. We assume that
\[
\int_{\mathbb{R}^{2n}} \langle z \rangle^2 |\rho(z)|dz < \infty ;
\] (10)
this ensures us of the existence of first and second order momenta. This condition holds if for instance \( \rho \) belongs to some Shubin symbol class \( \Gamma^m(\mathbb{R}^{2n}) \) with \( m < -2n - 2 \). Let \( \alpha, \beta = 1, ..., 2n \) and \( \tilde{z}_\alpha = x_\alpha \) for \( 1 \leq \alpha \leq n \) and \( \tilde{z}_\alpha = p_\alpha \) for \( n + 1 \leq \alpha \leq 2n \). The average value of \( \hat{\rho} \) is defined by \( \bar{z} = (\tilde{z}_1, ... , \tilde{z}_{2n}) \) where
\[
\bar{z}_\alpha = \int_{\mathbb{R}^{2n}} z_\alpha \rho(z)dz
\] (11)
and the covariances are given by the integrals

\[ \Sigma(z_\alpha, z_\beta) = \int_{\mathbb{R}^{2n}} (z_\alpha - \bar{z}_\alpha)(z_\beta - \bar{z}_\beta) \rho(z) dz . \]  

(12)

The covariance matrix of \( \hat{\rho} \) is, by definition, the \( 2n \times 2n \) matrix

\[ \Sigma = (\Sigma(z_\alpha, z_\beta))_{1 \leq \alpha, \beta \leq 2n} \]  

(13)

or, in more compact form,

\[ \Sigma = \int_{\mathbb{R}^{2n}} (z - \bar{z})(z - \bar{z})^T \rho(z) dz \]

where \( z \) is viewed as a column vector \((x_\rho)\). The condition \( \hat{\rho} \geq 0 \) requires that

\[ \Sigma + \frac{i\hbar}{2} J \geq 0 \]  

(14)

where \( \geq 0 \) means “is positive semidefinite” (note that all the eigenvalues of \( \Sigma + \frac{i\hbar}{2} J \) are real since it is a self-adjoint matrix). This condition implies, in particular, that \( \Sigma > 0 \); it is actually an equivalent form of the Robertson–Schrödinger inequalities \[12, 17\]. It is a symplectically invariant formulation of the uncertainty principle of quantum mechanics: introducing the covariance ellipsoid

\[ \Omega = \{ z \in \mathbb{R}^{2n} : \frac{1}{2} \Sigma^{-1} z^2 \leq 1 \} \]  

(15)

condition (14) can be rewritten as

\[ c(\Omega) \geq \pi \hbar \]  

(16)

where \( c(\Omega) \) is the symplectic capacity of the ellipsoid \( \Omega \) \[11, 12, 13, 17\]. Equivalently:

There exists \( S \in \text{Sp}(n) \) such that \( SB^{2n}(\sqrt{\hbar}) \subset \Omega \).

(17)

The symplectic balls \( SB^{2n}(\sqrt{\hbar}) \) are minimum uncertainty ellipsoids; it is convenient to use the following terminology \[14, 17\] as it simplifies many statements:

A quantum blob in \( \mathbb{R}^{2n} \) is a symplectic ball

\[ S(B^{2n}(R)) \text{ with radius } R = \sqrt{\hbar} . \]  

(18)

These properties all follow from the following observation:
Proposition 3  Let \(\lambda_1, \ldots, \lambda_n, \sigma\) be the symplectic eigenvalues of \(\Sigma\), that is, \(\lambda_j, \sigma > 0\) and \(\pm i \lambda_{j, \sigma}\) is an eigenvalue of \(J \Sigma\) for all \(j = 1, \ldots, n\). The condition \(\Sigma + \frac{i}{2} J \geq 0\) is equivalent to the conditions \(\lambda_j, \sigma \geq \frac{1}{2} \hbar\) for all \(j = 1, \ldots, n\).

Proof. See [11, 12]. It is based on the use of Williamson’s symplectic diagonalization theorem: \(M\) being positive definite there exists \(S \in \mathrm{Sp}(n)\) such that

\[
M = S^T DS, \quad D = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n, \sigma) \quad (19)
\]

(see for instance [9, 10, 11]). Notice that the eigenvalues of \(J \Sigma\) are those of the antisymmetric matrix \(\Sigma^{1/2} J \Sigma^{1/2}\) and are hence indeed of the type \(\pm i \lambda\) with \(\lambda > 0\). ■

1.2 Reduced density operators

Let \(n_A, n_B\) be two integers such that \(n = n_A + n_B\). We identify the direct sum \(\mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B}\) with \(\mathbb{R}^{2n}\) and the symplectic form \(\sigma\) with \(\sigma_A \oplus \sigma_B\) where \(\sigma_A\) (resp. \(\sigma_B\)) is the standard symplectic form on \(\mathbb{R}^{2n_A}\) (resp. \(\mathbb{R}^{2n_B}\)).

Let \(\rho\) be a density operator on \(L^2(\mathbb{R}^n)\) with Wigner distribution \(\rho\). Assuming that \(\rho\) satisfies the Shubin estimates \(\rho\) for some \(m < -2n\), we define the reduced density operator \(\tilde{\rho}_A\) by the formula

\[
\tilde{\rho}_A = (2\pi \hbar)^{n_A} \text{Op}_W(\rho_A) \quad (20)
\]

where we have set

\[
\rho_A(z) = \int_{\mathbb{R}^{2n_B}} \rho(z_A, z_B) dz_B . \quad (21)
\]

This terminology has of course to be justified (it is not a priori clear why \(\tilde{\rho}_A\) should be a density operator). Let us recall the following result from quantum harmonic analysis which reduces to a classical theorem of Bochner [4] on functions of positive type when \(\hbar = 0\):

Proposition 4 (KLM conditions)  Let \(a \in L^1(\mathbb{R}^{2n})\) and assume that \(\hat{A} = \text{Op}_W(a)\) is of trace class. We have \(\hat{A} \geq 0\) if and only if the symplectic Fourier transform \(a_\phi = F_\phi a\) defined by

\[
a_\phi(z) = \int_{\mathbb{R}^{2n}} e^{i \sigma(z, z')} a(z') dz'
\]
is continuous\footnote{With the assumption \( a \in L^1(\mathbb{R}^{2n}) \), this is automatically valid, via the Riemann-Lebesgue lemma.} and of \( h \)-positive type, that is if for every integer \( N \) the \( N \times N \) matrix \( \Lambda_{(N)} \) with entries

\[
\Lambda_{jk} = e^{-\frac{ih}{2} \sigma(z_j, z_k)} a_\sigma (z_j - z_k) \tag{23}
\]
is positive semidefinite for all choices of \((z_1, z_2, \ldots, z_N) \in (\mathbb{R}^{2n})^N \).

The proof of this result goes back to the seminal work of Kastler \cite{Kastler} and Loupias and Miracle-Sole \cite{LoupiasMiracleSole}. While these authors use the theory of \( C^* \)-algebras and hard functional analysis, one of us has recently given in \cite{Author} a conceptually simpler proof using the properties of the Heisenberg–Weyl displacement operators \( \hat{T}(z) = e^{-i\sigma(z, z)/\hbar} \) \cite{HeisenbergWeyl}.

**Proposition 5** Let \( \rho \in \Gamma^m(\mathbb{R}^{2n}) \) for some \( m < -2n \). The operator \( \hat{\rho}_A = (2\pi \hbar)^n \text{Op}_W(\rho_A) \) is a density operator on \( L^2(\mathbb{R}^{nA}) \) and we have \( \rho_A \in \Gamma^{m_A}(\mathbb{R}^{2nA}) \) for every \( m_A < -2n_A \).

**Proof.** The integral (21) is convergent in view of the trivial inequality \((1 + |z|^2)^m \leq (1 + |z_B|^2)^m\). Choosing \( m_A < -2n_A \) and \( m_B < -2n_B \) such that \( m = m_A + m_B \) we have \( \langle z \rangle^{m - |\alpha|} \leq \langle z_A \rangle^{m_A - |\alpha|} \langle z_B \rangle^{m_B} \) as follows from the inequality

\[
(1 + |z|^2)^{m - |\alpha|} \leq (1 + |z_A|^2)^{m_A - |\alpha|} (1 + |z_B|^2)^{m_B} .
\]

Using the Shubin estimates (8) we thus have

\[
\partial_{z_A}^\alpha \rho_A(z_A) = \int_{\mathbb{R}^{2n_B}} \partial_{z_A}^\alpha \rho(z_A, z_B) dz_B 
\leq C_{\alpha} \langle z \rangle^{m - |\alpha|} \int_{\mathbb{R}^{2n_B}} \langle z_B \rangle^{m_B} dz_B
\]

and hence \( \rho_A \in \Gamma^{m_A}(\mathbb{R}^{2nA}) \) since the integral over \( \mathbb{R}^{n_B} \) is convergent in view of the inequality \( m_B < -2n_B \). It follows from Proposition 2 that \( \hat{\rho}_A \) is a trace class operator whose trace is

\[
\text{Tr}_A(\hat{\rho}_A) = \int_{\mathbb{R}^{2n_A}} \rho_A(z_A) dz_A = 1 . \tag{24}
\]

There remains to show that \( \hat{\rho}_A > 0 \) (and hence \( \hat{\rho}_A^* = \hat{\rho}_A \)). In view of the KLM conditions (Proposition 4) it is sufficient to prove that the Fourier
transform \((\rho_A)_\emptyset\) is continuous and satisfies \(\Lambda^A_{(N)} \geq 0\) for every integer \(N > 0\) where \(\Lambda^A_{(N)} = (\Lambda^A_{jk})_{j,k}\) with
\[
\Lambda^A_{jk} = e^{-\frac{i}{\hbar} \sigma_A(z_{A,j} - z_{A,k})} (\rho_A)_{\emptyset} (z_{A,j} - z_{A,k})
\]
the vectors \(z_{A,j}\) and \(z_{A,k}\) of \(\mathbb{R}^{2n_A}\) being arbitrary. The continuity of \((\rho_A)_\emptyset\) being obvious (Riemann–Lebesgue Lemma) all we have to do is to show that \(\Lambda^A_{(N)} \geq 0\). We first observe that by Fubini’s theorem \((\rho_A)_\emptyset (z_A) = \rho_\emptyset (z_A \oplus 0)\) and hence
\[
\Lambda^A_{jk} = e^{-\frac{i}{\hbar} \sigma_A(z_{A,j} \oplus 0, z_{A,k} \oplus 0)} (\rho_A)_{\emptyset} (z_{A,j} \oplus 0) - (z_{A,k} \oplus 0)) ;
\]
the matrix \(\Lambda^A_{(N)}\) is thus the matrix \(\Lambda_{(N)}\) corresponding to the particular choices \(z_j = z_{A,j} \oplus 0\) and \(z_k = z_{A,k} \oplus 0\). Since \(\rho_\emptyset\) satisfies the KLM conditions we must have \(\Lambda^A_{(N)} \geq 0\), hence \((\rho_A)_\emptyset\) also satisfies them.

From now on we will write the covariance matrix \(\Sigma\) in the \(AB\)-ordering as
\[
\Sigma = \left( \begin{array}{cc} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{array} \right) \quad \text{with} \quad \Sigma_{BA} = \Sigma_{AB}^T
\]
the blocks \(\Sigma_{AA}, \Sigma_{AB}, \Sigma_{BA}, \Sigma_{BB}\) having dimensions \(2n_A \times 2n_A, 2n_A \times 2n_B, 2n_B \times 2n_A, 2n_B \times 2n_B\), respectively. In this notation the quantum condition (14) reads
\[
\Sigma + \frac{i \hbar}{2} J_{AB} \geq 0 \quad \text{with} \quad J_{AB} = \left( \begin{array}{cc} J_A & 0 \\ 0 & J_B \end{array} \right)
\]
(26)

The covariance matrices \(\Sigma_A\) and \(\Sigma_B\) of the reduced density operators are, respectively, the blocks \(\Sigma_{AA}\) and \(\Sigma_{BB}\) of \(\Sigma\) as immediately follows from the definitions (11) and (12) using the formulas
\[
\rho_A(z_A) = \int_{\mathbb{R}^{2n_B}} \rho(z_A, z_B) dz_B \quad \rho_B(z_B) = \int_{\mathbb{R}^{2n_A}} \rho(z_A, z_B) dz_A .
\]
These matrices satisfy the quantum conditions
\[
\Sigma_{AA} + \frac{i \hbar}{2} J_A \geq 0 \quad \text{and} \quad \Sigma_{BB} + \frac{i \hbar}{2} J_B \geq 0
\]
and the covariance ellipsoids of \(\tilde{\rho}_A\) and \(\tilde{\rho}_B\) are
\[
\Omega_A = \{ z_A : \frac{1}{2} \Sigma_{AA}^{-1} z_A^2 \leq 1 \} \quad \text{and} \quad \Omega_B = \{ z_B : \frac{1}{2} \Sigma_{BB}^{-1} z_B^2 \leq 1 \}
\]
we will see below that they are just the orthogonal projections on \(\mathbb{R}^{2n_A}\) and \(\mathbb{R}^{2n_A}\) of the covariance ellipsoid \(\Omega\). That the quantum conditions (27) hold
follows from the fact that $\hat{\rho}_A$ and $\hat{\rho}_B$ are bona fide density operators, but this can also be seen directly by noting that (26) can be written

$$
\begin{pmatrix}
\Sigma_{AA} + \frac{i}{\hbar} J_A & \Sigma_{AB} \\
\Sigma_{BA} & \Sigma_{BB} + \frac{i}{\hbar} J_B
\end{pmatrix} \geq 0 .
$$

The symmetric matrix

$$
\Sigma/\Sigma_{BB} = \Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{BA} .
$$

is called the Schur complement [19, 36] of the block $\Sigma_{BB}$ of $\Sigma$. Using the obvious factorization

$$
\Sigma = \begin{pmatrix}
I_A & \Sigma_{AB} \\
0 & I_B
\end{pmatrix}
\begin{pmatrix}
\Sigma/\Sigma_{BB} & 0 \\
0 & \Sigma_{BB}
\end{pmatrix}
\begin{pmatrix}
I_A & 0 \\
\Sigma_{BA} & I_B
\end{pmatrix}
(30)
$$

we readily get various formulas for the inverse of $\Sigma$; the one we will use here is

$$
\Sigma^{-1} = \begin{pmatrix}
(\Sigma/\Sigma_{BB})^{-1} & -(\Sigma/\Sigma_{BB})^{-1} \Sigma_{AB} \Sigma_{BB}^{-1} \\
-\Sigma_{BB}^{-1} \Sigma_{BA} (\Sigma/\Sigma_{BB})^{-1} & (\Sigma/\Sigma_{AA})^{-1}
\end{pmatrix}
(31)
$$

(see [33] for a review of various formulas for block-matrix inversion). Also note that it immediately follows from (30) that

$$
\det \Sigma = \det(\Sigma/\Sigma_{BB}) \det \Sigma_{BB} .
$$

1.3 The shadows of the covariance ellipse

In practice, we have to deal more often with the inverse of the covariance matrix than with the covariance matrix itself (this occurred already above in the definition of the covariance ellipsoid (15)). It is therefore useful to have an explicit formula for that inverse.

In particular, to study the orthogonal projections (“shadows”) of the covariance ellipsoid $\Omega$ on the reduced phase spaces $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$ it will be convenient to set $M = \frac{1}{2} \Sigma^{-1}$. We will write, using the $AB$-ordering $z = (z_A, z_B),

$$
M = \begin{pmatrix}
M_{AA} & M_{AB} \\
M_{BA} & M_{BB}
\end{pmatrix}
(33)
$$

where $M_{AA}, M_{AB}, M_{BA}, M_{BB}$ are, respectively, $2n_A \times 2n_A, 2n_A \times 2n_B, 2n_B \times 2n_A, 2n_B \times 2n_B$ matrices. In this notation the covariance ellipsoid of $\hat{\rho}$ is the set

$$
\Omega = \{ z \in \mathbb{R}^{2n} : Mz^2 \leq \hbar \}
$$

(34)
and the quantum condition $\Sigma + \frac{i}{\hbar} J_{AB} \geq 0$ becomes

$$M^{-1} + iJ_{AB} \geq 0$$

which is equivalent, in view of Proposition 3, to the statement:

The symplectic eigenvalues of $M$ are $\leq 1$. (35)

Notice that since $M$ is positive definite and symmetric (because $\Sigma$ is) the blocks $M_{AA}$ and $M_{BB}$ are also symmetric and positive definite and we have $M_{BA} = M_{AB}^T$.

The following general Lemma will be very useful in our geometric considerations about separability:

**Lemma 6** Let $\Pi_A$ (resp. $\Pi_B$) be the orthogonal projection $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n_A}$ (resp. $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n_B}$) and $\Omega_R$ the phase space ellipsoid $\{z \in \mathbb{R}^{2n} : Mz^2 \leq R^2\}$, for some $R > 0$. We have

$$\Pi_A \Omega_R = \{z_A \in \mathbb{R}^{2n_A} : (M/M_{BB}) z_A^2 \leq R^2\}$$

(36)

$$\Pi_B \Omega_R = \{z_B \in \mathbb{R}^{2n_B} : (M/M_{AA}) z_B^2 \leq R^2\}$$

(37)

where

$$M/M_{BB} = M_{AA} - M_{AB}M_{BB}^{-1}M_{BA}$$

(38)

$$M/M_{AA} = M_{BB} - M_{BA}M_{AA}^{-1}M_{AB}$$

(39)

are the Schur complements.

**Proof.** Let us set $Q(z) = Mz^2 - R^2$; the boundary $\partial \Omega_R$ of the hypersurface $Q(z) = 0$ is defined by

$$M_{AA} z_A^2 + 2M_{BA} z_A \cdot z_B + M_{BB} z_B^2 = R^2 .$$

(40)

A point $z_A$ belongs to $\partial \Pi_A \Omega_R$ if and only if the normal vector to $\partial \Omega_R$ at the point $z = (z_A, z_B) \in \Omega_R$ is parallel to $\mathbb{R}^{2n_A}$, hence the constraint $\partial_z Q(z) = 2Mz \in \mathbb{R}^{2n_A} \oplus 0$. This is equivalent to the condition $M_{BA} z_A + M_{BB} z_B = 0$, that is to $z_B = -M_{BB}^{-1} M_{BA} z_A$. Inserting $z_B$ in (40) shows that the boundary $\partial \Pi_A \Omega_R$ is the set $\Sigma_A = (M/M_{BB}) z_A^2 = R^2$ which yields (36). Formula (37) is proven in the same way.

It follows from Lemma 6 that the orthogonal projections on $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$ of the covariance ellipsoid $\Omega$ of $\hat{\rho}$ are just the covariance ellipsoids of the reduced operators $\hat{\rho}_A$ and $\hat{\rho}_B$:
Proposition 7  The covariance ellipsoids $\Omega_A$ and $\Omega_B$ of the reduced quantum states $\hat{\rho}_A$ and $\hat{\rho}_B$ are the orthogonal projections on $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$ of the covariance ellipsoid $\Omega$ of $\hat{\rho}$:

$$\Omega_A = \Pi_A \Omega = \{z_A \in \mathbb{R}^{2n_A} : (M/M_{BB}) z_A^2 \leq \hbar\}.$$  \hfill (41)

$$\Omega_B = \Pi_B \Omega = \{z_B \in \mathbb{R}^{2n_B} : (M/M_{AA}) z_B^2 \leq \hbar\}.$$  \hfill (42)

Proof. Let $M = \frac{\hbar}{2} \Sigma^{-1}$. Writing $M$ in block matrix form (33), its inverse has the form

$$M^{-1} = \begin{pmatrix} (M/M_{BB})^{-1} & * \\ * & (M/M_{AA})^{-1} \end{pmatrix}$$  \hfill (43)

(cf. formula (31)) and hence

$$(M/M_{BB})^{-1} = \frac{\hbar}{2} \Sigma_{AA}$$  and $$(M/M_{AA})^{-1} = \frac{\hbar}{2} \Sigma_{BB}.$$  

Formulas (41) and (42) follow using Lemma 6 with $R = \sqrt{\hbar}$. \qed

2 The $AB$-separability of a density operator

In this section we study two necessary conditions for bipartite separability of density operator on $L^2(\mathbb{R}^n)$. The first (Proposition 5) is the so-called “PPT criterion”, of which we give a rigorous proof, and the second (Proposition 11) is a non-trivial refinement of a result due to Werner and Wolf [35].

2.1 The Peres–Horodecki condition

We say that the operator $\hat{\rho}$ is “$AB$ separable” if there exist sequences of density operators $\hat{\rho}_j^A \in L_1(L^2(\mathbb{R}^{n_A}))$ and $\hat{\rho}_j^B \in L_1(L^2(\mathbb{R}^{n_B}))$ and real numbers $\alpha_j \geq 0$, $\sum_j \alpha_j = 1$ such that

$$\hat{\rho} = \sum_{j \in I} \alpha_j \hat{\rho}_j^A \otimes \hat{\rho}_j^B$$  \hfill (44)

where the convergence is for the norm of $L_1(L^2(\mathbb{R}^n))$.

Let us introduce some new notation. We denote by $I_A$ the identity $(x_A, p_A) \mapsto (x_A, p_A)$ and by $\overline{T}_B$ the involution $(x_B, p_B) \mapsto (x_B, -p_B)$. We set $\overline{T}_{AB} = I_A \oplus \overline{T}_B$ and, as before, $J_{AB} = J_A \oplus J_B$ where $J_A$ (resp. $J_B$) is the standard symplectic matrix in $\mathbb{R}^{2n_A}$ (resp. $\mathbb{R}^{2n_B}$).

Given a general density operator $\hat{\rho} = (2\pi \hbar)^n \text{Op}_W(\rho)$ there exists a necessary condition for $AB$-separability; it is known in the physical literature
as the PPT criterion (PPT stands for “positive partial transpose”) and was first precisely stated in [20, 21, 28]. Also see the paper [31] of Simon where it is shown that the PPT criterion is sufficient for separability of Gaussian states when \( n_A = n_B = 2 \) (also see Duan et al. [7]). Below we give a short and rigorous proof of this condition based on the (trivial) equality
\[
W\psi(I_B z_B) = W\overline{\psi}(z_B)
\] (45)
valid for all \( \psi \in L^2(\mathbb{R}^{n_B}) \).

**Proposition 8** Let \( \hat{\rho} = (2\pi\hbar)^n \text{Op}_W(\rho) \) be a density operator on \( \mathbb{R}^{2n} = \mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B} \). Suppose that the AB-separability condition
\[
\hat{\rho} = \sum_{j \in I} \lambda_j \hat{\rho}_j^A \otimes \hat{\rho}_j^B
\] (46)
holds. Then the operator
\[
\hat{\rho}^T_B = (2\pi\hbar)^n \text{Op}_W(\rho \circ I_{AB})
\]
is also a density operator on \( \mathbb{R}^{2n} = \mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B} \).

**Proof.** Suppose that (46) holds; then \( \rho = \sum_j \lambda_j \rho_j^A \otimes \rho_j^B \) and
\[
\rho_j^A = \sum_\ell \alpha_{j,\ell} W_A \psi_{j,\ell}^A, \quad \rho_j^B = \sum_m \beta_{j,m} W_B \psi_{j,m}^B
\]
with \( (\psi_{A,\ell}^A, \psi_{B,\ell}^B) \in L^2(\mathbb{R}^{n_A}) \times L^2(\mathbb{R}^{n_B}) \) and \( \alpha_{j,\ell}, \beta_{j,m} \geq 0 \); that is
\[
\rho = \sum_{j,\ell,m} \gamma_{j,\ell,m} W_A \psi_{j,\ell}^A \otimes W_B \psi_{j,m}^B
\]
where \( \gamma_{j,\ell,m} = \lambda_j \alpha_{j,\ell} \beta_{j,m} \geq 0 \). We have
\[
\rho(\overline{I_{AB} z}) = \sum_{j \in I} \lambda_j \rho_j^A(z_A) \rho_j^B(\overline{I_B z_B});
\]
using (45) we thus have
\[
\rho \circ \overline{I_{AB}} = \sum_{j,\ell,m} \gamma_{j,\ell,m} W(\psi_{j,\ell}^A \otimes \overline{\psi}_{j,m}^B)
\]
hence \( \text{Op}_W(\rho \circ \overline{I_{AB}}) \) is also a positive semidefinite trace class operator; that \( \text{Tr}(\hat{\rho}^T_B) = \text{Tr}(\hat{\rho}) = 1 \) is obvious. \( \blacksquare \)
Notice that we have \( \hat{\rho}^T_B = \sum_j \alpha_j \hat{\rho}_j^A \otimes (\hat{\rho}_j^B)^T \) where
\[
(\hat{\rho}_j^B)^T = (2\pi \hbar)^{n_B} \text{Op}_W(\rho_j \circ T_B)
\]
is the transpose of \( \hat{\rho}_j^B \), hence the denomination “partial positive transpose” for the operator \( \hat{\rho}^T_B \) used in the literature.

Proposition 8 has the following consequence. We set
\[
J_{AB} = J_A \oplus (-J_B) = T_{AB} J_{AB} T_{AB}
\]
(that is, \( J_{AB} \) is the standard symplectic matrix of the symplectic vector space \((\mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B}, \sigma_A \oplus (-\sigma_B))\)).

**Corollary 9** Let \( \hat{\rho} = (2\pi \hbar)^{n} \text{Op}_W(\rho) \) be a separable density operator. Then, in addition to (26), we have
\[
\Sigma + \frac{i\hbar}{2} J_{AB} \geq 0 ;
\]
or equivalently
\[
\Sigma + \frac{i\hbar}{2} J_{AB} \geq 0
\]
where \( \Sigma = T_{AB} \Sigma T_{AB} \) that is
\[
\Sigma = \begin{pmatrix}
\Sigma_{AA} & \Sigma_{AB} T_B \\
T_B \Sigma_{BA} & T_B \Sigma_{BB} T_B
\end{pmatrix}.
\]

**Proof.** Replacing \( \rho \) with \( \rho \circ T_B \) the matrix \( \Sigma_{AA} \) in (25) remains unchanged while \( \Sigma_{BB}, \Sigma_{AB}, \) and \( \Sigma_{BA} \) become \( T_B \Sigma_{BB} T_B, \Sigma_{AB} T_B, \) and \( T_B \Sigma_{BA} \) respectively. The covariance matrix (25) thus becomes \( \Sigma = T_{AB} \Sigma T_{AB} \). In view of Proposition 8 the operator \((2\pi \hbar)^{n} \text{Op}_W(\rho \circ T_B)\) is also positive semidefinite hence we must have \( \Sigma + \frac{i\hbar}{2} J_{AB} \geq 0 \) which is equivalent to \( \Sigma + \frac{i\hbar}{2} T_{AB} J_{AB} T_{AB} \geq 0 \). Since \( T_{AB} J_{AB} T_{AB} = J_{AB} \) this is equivalent to (47).

The ellipsoid
\[
\overline{\Omega} = \{ z \in \mathbb{R}^{2n} : \frac{1}{2} \Sigma^{-1} z^2 \leq 1 \}
\]
for the covariance matrix of the partial transpose \( \hat{\rho}^T_B \) can be expressed in terms of the matrix \( \overline{M} = \frac{1}{2} \Sigma^{-1} \) by
\[
\overline{\Omega} = \{ z \in \mathbb{R}^{2n} : \overline{M} z^2 \leq \hbar \}
\]
where \( \overline{M} = T_{AB} M T_{AB} \).
2.2 Werner and Wolf’s condition

Using techniques previously developed by Werner [34], Werner and Wolf [35] prove the following crucial necessary condition for separability (a different proof can be found in Serafini [29, p.178):

**Proposition 10 (Werner and Wolf)** Suppose that the density operator \( \hat{\rho} \) with covariance matrix \( \Sigma \) is separable. There exist two partial covariance matrices \( \Sigma_A \) and \( \Sigma_B \) of dimensions \( 2n_A \times 2n_A \) and \( 2n_B \times 2n_B \) satisfying the quantum conditions

\[
\Sigma_A + \frac{i\hbar}{2} J_A \geq 0 \quad \text{and} \quad \Sigma_B + \frac{i\hbar}{2} J_B \geq 0
\]

and such that

\[
\Sigma \geq \Sigma_A \oplus \Sigma_B.
\]

We are going to show that Werner and Wolf’s result can be considerably refined using the properties of the symplectic group. We first remark that the quantum condition \( \Sigma + \frac{i\hbar}{2} J \geq 0 \) on a covariance matrix is equivalent to the following property: there exists \( S \in \text{Sp}(n) \) such that \( \Sigma \geq \frac{\hbar}{2} (S^T S)^{-1} \) (see [12, 13]); this property is easily deduced from [17]. It is equivalent to saying that the covariance ellipsoid \( \Omega \) contains a quantum blob [14].

**Proposition 11** The Werner–Wolf condition \([53]\) is equivalent to the existence of two positive definite symplectic matrices

\[
P_A = (S_A^T S_A)^{-1}, \quad P_B = (S_B^T S_B)^{-1},
\]

with \( S_A \in \text{Sp}(n_A) \) and \( S_B \in \text{Sp}(n_B) \), such that

\[
\Sigma \geq \frac{\hbar}{2} (P_A \oplus P_B).
\]

Equivalently, the covariance ellipsoid \( \Omega \) contains a quantum blob of the form \((S_A \oplus S_B)(B^{2n}(\sqrt{\hbar}))\).

**Proof.** The sufficiency of the condition is clear since \( \Sigma_A = \frac{\hbar}{2} P_A \) and \( \Sigma_A = \frac{\hbar}{2} P_A \) satisfy the conditions \([52]\). Assume conversely that \( \Sigma \geq \Sigma_A \oplus \Sigma_B \) as in Proposition \([10]\). In view of Williamson’s diagonalization theorem \([9, 11]\) there exist \( S_A \in \text{Sp}(n_A) \) and \( S_B \in \text{Sp}(n_B) \) such that \( S_A \Sigma_A S_A^T = D_A \) and \( S_B \Sigma_B S_B^T = D_B \) where

\[
D_A = \begin{pmatrix} \Lambda_A & 0 \\ 0 & \Lambda_A \end{pmatrix}, \quad D_B = \begin{pmatrix} \Lambda_B & 0 \\ 0 & \Lambda_B \end{pmatrix}
\]

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and \( \Lambda_A, \Lambda_B \) being the diagonal matrices consisting of the symplectic eigenvalues \( \lambda_1^{\sigma_A}, ..., \lambda_n^{\sigma_A} \) of \( \Sigma_A \) and \( \lambda_1^{\sigma_B}, ..., \lambda_n^{\sigma_B} \) of \( \Sigma_B \) (i.e. the \( \pm i \lambda_i^{\sigma_A} \) are the eigenvalues of \( \Sigma_1^{1/2} J_A \Sigma_1^{1/2} \), see e.g. [11, 32]). Since

\[
S_A J_A S_A^T = J_A \quad \text{and} \quad S_B J_B S_B^T = J_B
\]

the conditions \( \Sigma_A + i\hbar J_A \geq 0 \) and \( \Sigma_B + i\hbar J_B \geq 0 \) are equivalent to \( D_A \geq \frac{\hbar}{2} I_A \) and \( D_B \geq \frac{\hbar}{2} I_B \) the characteristic equation of \( D_A + i\hbar J_A \) is

\[
\det \left( (\Lambda_A - \lambda I_A)^2 - \frac{1}{4} \hbar^2 I_A \right) = 0.
\]

Writing \( \Lambda_A = \text{diag}(\lambda_1^{\sigma_A}, ..., \lambda_n^{\sigma_A}) \) this equation is equivalent to the set of equations

\[
(\lambda_j^{\sigma_A} - \lambda)^2 - \frac{1}{4} \hbar^2 = 0, \quad 1 \leq j \leq n_A,
\]

whose solutions are the real numbers \( \lambda_j = \lambda_j^{\sigma_A} \pm \frac{\hbar}{2} \). Since \( \lambda_j \geq 0 \) we must thus have \( \lambda_j^{\sigma_A} \geq \frac{\hbar}{2} \) and hence \( D_A \geq \frac{\hbar}{2} I_A \). Similarly, \( D_B \geq \frac{\hbar}{2} I_B \) so we must have the inequalities

\[
\Sigma_A = S_A^{-1} D_A (S_A^T)^{-1} \geq \frac{\hbar}{2} (S_A^T S_A)^{-1}
\]

\[
\Sigma_B = S_B^{-1} D_B (S_B^T)^{-1} \geq \frac{\hbar}{2} (S_B^T S_B)^{-1}.
\]

Setting \( P_A = (S_A^T S_A)^{-1} \) and \( P_B = (S_B^T S_B)^{-1} \) the inequality (55) follows.

### 2.3 A property of the reduced covariance ellipsoids

The previous propositions have a very simple geometrical meaning, to which we will come back in Section 4. The conditions (52) mean that \( \Sigma_A \) and \( \Sigma_B \) are quantum covariances matrices, hence the sum

\[
\Sigma_A \oplus \Sigma_B \equiv \begin{pmatrix} \Sigma_A & 0 \\ 0 & \Sigma_B \end{pmatrix}
\]

is a quantum covariance matrix in its own right. It follows from (53) that the corresponding covariance ellipsoid, which we denote

\[
\Omega_{A \oplus B} = \left\{ z_A \oplus z_B : \frac{1}{2} \Sigma_A^{-1} z_A^2 + \frac{1}{2} \Sigma_B^{-1} z_B^2 \leq 1 \right\},
\]

is included in \( \Omega \).

Moreover, in view of (the proof of) Proposition 11 the ellipsoid \( \Omega_{A \oplus B} \) always contains a quantum blob of the form

\[
\Omega_{AB} = S_A \oplus S_B(2^n(\sqrt{\hbar})) = \left\{ z_A \oplus z_B : |S_A^{-1} z_A|^2 + |S_B^{-1} z_B|^2 \leq \hbar \right\}.
\]

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Hence, if the density operator $\hat{\rho}$ with covariance ellipsoid $\Omega$ is separable then there exist quantum covariance ellipsoids of the form (56) and (57) such that the following inclusions hold

$$\Omega \supset \Omega_{A\oplus B} \supset \Omega_{AB}.$$  \hspace{1cm} (58)

This result has an interesting consequence for the covariance ellipsoids

$$\Omega_A = \{ z_A : \frac{1}{2}\Sigma_A z_A^2 \leq 1 \} \quad \text{and} \quad \Omega_B = \{ z_B : \frac{1}{2}\Sigma_B z_B^2 \leq 1 \}.$$  \hspace{1cm}

of the reduced density operators $\hat{\rho}_A$ and $\hat{\rho}_B$. We first show that:

**Proposition 12** The orthogonal projections $\Pi_A\Omega_{AB}$ and $\Pi_B\Omega_{AB}$ of $\Omega_{AB}$ onto $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$ satisfy

$$\Pi_A\Omega_{AB} = S_A(B^{2n_A} (\sqrt{\hbar})) \quad \text{and} \quad \Pi_B\Omega_{AB} = S_B(B^{2n_B} (\sqrt{\hbar})).$$  \hspace{1cm} (59)

**Proof.** This result is easily proved directly from the definition of $\Omega_{AB}$. Alternatively we can use a recent result [6] which generalizes Gromov’s symplectic non-squeezing theorem [18] in the linear case, and which refines a previous result of Abbondandolo and his collaborators [1, 2]. It states that for every $S \in \text{Sp}(n)$ there exists $S_A \in \text{Sp}(n_A)$ such that

$$\Pi_A S(B^{2n} (\sqrt{\hbar})) \supset S_A(B^{2n_A} (\sqrt{\hbar}))$$  \hspace{1cm} (60)

with equality if and only if $S = S_A \oplus S_B$. The result (59) follows using the definition (57) of $\Omega_{AB}$. The same argument applies to $\Pi_B\Omega_{AB}$. \hspace{1cm} \blacksquare

Notice that since symplectic automorphisms are volume-preserving the result above implies that

$$\text{Vol}_{2n_A} \Pi_A\Omega_{AB} = \text{Vol}_{2n_A} B^{2n_A} (\sqrt{\hbar}) = \frac{(\pi\hbar)^{n_A}}{n_A!},$$

$$\text{Vol}_{2n_B} \Pi_B\Omega_{AB} = \text{Vol}_{2n_B} B^{2n_B} (\sqrt{\hbar}) = \frac{(\pi\hbar)^{n_B}}{n_B!}. $$

Likewise, the orthogonal projections of $\Omega_{A\oplus B}$ on $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$ are just the intersections of $\Omega_{A\oplus B}$ with the hyperplanes $z_B = 0$ and $z_A = 0$, respectively.

Finally, from (58) we easily conclude that the covariant ellipsoids $\Omega_A$ and $\Omega_B$ impose the following constraints on the symplectic matrices $S_A$ and $S_B$ of Proposition 11.
Corollary 13 Assume that the density operator $\hat{\rho}$ with covariant ellipsoid $\Omega$ is separable. Then the symplectic matrices $S_A$ and $S_B$ of Proposition 11 satisfy:

$$S_A B^{2n_A}(\sqrt{\hbar}) \subset \Omega_A \quad , \quad S_B B^{2n_B}(\sqrt{\hbar}) \subset \Omega_B$$

(61)

Proof. From (58) we have $\Omega_{AB} \subset \Omega$ and so:

$$\Pi_A \Omega_{AB} \subset \Pi_A \Omega = \Omega_A \quad \text{and} \quad \Pi_B \Omega_{AB} \subset \Pi_B \Omega = \Omega_B .$$

The result then follows from (59).

$\blacksquare$

3 Gaussian Quantum States

3.1 Generalities, a sufficient condition for separability

A simple, but very interesting case, occurs when $\rho$ is a Gaussian Wigner distribution

$$\rho(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2} \Sigma^{-1} (z-\bar{z})^2}$$

centered at $\bar{z} \in \mathbb{R}^{2n}$, where $\Sigma$ is a positive definite real symmetric $2n \times 2n$ matrix (the “covariance matrix”). The normalization factor preceding the exponential guarantees that $\text{Tr}(\hat{\rho}) = 1$. We will only consider the case $\bar{z} = 0$; the more general case is easily reduced to the former by a phase space translation. Hence we assume that

$$\rho(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2} \Sigma^{-1} z^2}$$

(62)

and, setting as usual $M = \frac{\hbar}{2} \Sigma^{-1}$, we can rewrite (62) as

$$\rho(z) = (\pi \hbar)^{-n} (\det M)^{1/2} e^{-\frac{1}{2} \hbar M z^2} .$$

(63)

Since $\rho$ is real, the Weyl operator $\hat{\rho} = (2\pi \hbar)^n \text{Op}_W(\rho)$ is self-adjoint. To ensure that $\hat{\rho}$ is positive semidefinite it is necessary and sufficient [25, 26, 27] that the covariance matrix satisfies the quantum condition (26), which we assume from now on. Notice that the general result (60) that was used in Proposition 12 also provides an alternative proof of the fact that the partial trace operators $\hat{\rho}_A$ and $\hat{\rho}_B$ are density operators. In fact, to prove this we had to use for the general case the KLM conditions (Proposition 4) in Section 1.2 to prove the positivity properties $\hat{\rho}_A \geq 0$ and $\hat{\rho}_B \geq 0$. In the
Gaussian case we can instead consider the quantum condition (14) which is equivalent to (17). From (60) it then follows that
\[
\Sigma_A + \frac{i\hbar}{2} J_A \geq 0, \Sigma_B + \frac{i\hbar}{2} J_B \geq 0
\]
(64)
hence \(\hat{\rho}_A\) and \(\hat{\rho}_B\) are (Gaussian) density operators.

The purity of \(\hat{\rho}\) is then given by
\[
\mu(\hat{\rho}) = (\frac{\hbar}{2})^n (\det \Sigma)^{-1/2} = \sqrt{\det M}
\]
(65)
(see e.g. [11], §9.3, p.301). That the terminology “covariance matrix” applied to \(\Sigma\) is justified in the quantum case as it is in classical statistical mechanics, follows from formulas (11) and (12). It is also clear that we have \(\rho \in \Gamma^m(\mathbb{R}^{2n})\) for every \(m < -2n\) hence \(\rho_A \in \Gamma^{m_A}(\mathbb{R}^{2n_A})\) for every \(m_A < -2n_A\) (see Proposition 5).

It turns out that Werner and Wolf’s conditions in Proposition 10 are sufficient for a Gaussian state to be separable:

**Proposition 14** Assume that there exist two partial covariance matrices \(\Sigma_A\) and \(\Sigma_B\) satisfying the quantum conditions (64) and such that
\[
\Sigma \geq \Sigma_A \oplus \Sigma_B
\]
(66)
Then the Gaussian state (62) is separable.

**Proof.** See [35] (Proposition 1).

### 3.2 Pure Gaussians

Let \(X\) and \(Y\) be real symmetric \(n \times n\) matrices, with \(X > 0\). To these matrices we associate the Gaussian function \(\phi_{X,Y}\) on \(\mathbb{R}^n\) defined by
\[
\phi_{X,Y}(x) = (\pi\hbar)^{-n/4} (\det X)^{1/4} e^{-\frac{1}{2\hbar}(X+iY)x^2}
\]
(67)
where we are writing \((X + iY)x^2\) for \((X + iY)x \cdot x\). This function is \(L^2\)-normalized: \(\|\phi_{X,Y}\|_{L^2(\mathbb{R}^n)} = 1\) and its Wigner transform is given by the well-known formula [3] [11] [15]
\[
W\phi_{X,Y}(z) = (\pi\hbar)^{-n} e^{-\frac{1}{\hbar} Gz^2}
\]
(68)
where \(G\) is the positive-definite symmetric matrix
\[
G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix}
\]
(69)
In fact $G = S^T S$ where

$$S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix} \in \text{Sp}(n)$$

hence $G$ is a positive definite symplectic matrix. Setting $\Sigma^{-1} = \frac{\hbar}{2} G$ we can rewrite (68) as

$$W_{\phi_{X,Y}}(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2} \Sigma^{-1} z^2}.$$  

Hence, to $\rho_{X,Y} = W_{\phi_{X,Y}}$ corresponds a Gaussian density operator $\hat{\rho}_{X,Y}$ (the quantum condition (14) becomes here $S^T S + iJ \geq 0$; since $(S^T)^{-1} JS^{-1} = J$ this is equivalent to $I + iJ \geq 0$ which is trivially satisfied).

**Lemma 15** A Gaussian state $\hat{\rho}$ is pure if and only if there exists $(X,Y)$ such that $\rho = W_{\phi_{X,Y}}$.

**Proof.** The sufficiency is clear, so all we have to do is to show that it is necessary. The purity formula (65) for Gaussians shows that $\mu(\hat{\rho}) = 1$ if and only if $\det \Sigma = (\hbar/2)^n$. Let $\lambda_1^\sigma, ..., \lambda_n^\sigma$ be the symplectic eigenvalues of $\Sigma$ (i.e. the numbers $\lambda_j^\sigma > 0$ such that the $\pm i\lambda_j^\sigma$ are the eigenvalues of $JM$); in view of Williamson’s symplectic diagonalization theorem there exists $S \in \text{Sp}(n)$ such that $\Sigma = (S^T)^{-1} DS^{-1}$ where $D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$ with $\Lambda = \text{diag}(\lambda_1^\sigma, ..., \lambda_n^\sigma)$. The quantum condition (14) is equivalent to $\lambda_j^\sigma \geq \hbar/2$ for all $j$ hence

$$\det \Sigma = (\lambda_1^\sigma)^2 \cdots (\lambda_n^\sigma)^2 = 1$$

if and only all the $\lambda_j^\sigma$ are equal to $\hbar/2$, hence $\Sigma = \frac{\hbar}{2} (S^T)^{-1} S^{-1}$. □

**Remark 16** The action of the metaplectic group $\text{Mp}(n)$ on the set of all Gaussians $\phi_{X,Y}$ is transitive [13, 15]. The Lemma above can thus be rephrased by saying that every pure Gaussian state is obtained from the standard Gaussian $\phi_0(x) = (\pi \hbar)^{-n/4} e^{-|x|^2/2\hbar}$ by some $\hat{S} \in \text{Mp}(n)$.

### 3.3 Separability of Gaussian states

Before we state and prove our main results, let us make the following simple observation:
Lemma 17 If the covariance ellipsoid
\[ \Omega = \{ z \in \mathbb{R}^{2n} : \frac{1}{2} \Sigma^{-1} z^2 \leq 1 \} \]
of a Gaussian state \( \hat{\rho} \) contains the ball \( B^{2n}(\sqrt{\hbar}) \), then \( \hat{\rho} \) is separable for all partitions \( (A, B) \).

Proof. Setting \( M = \frac{1}{2} \Sigma^{-1} \), the inclusion \( B^{2n}(\sqrt{\hbar}) \subset \Omega \) is equivalent to \( M \leq I \). Hence, the Werner–Wolf condition condition (53) is satisfied with \( \Sigma_A \oplus \Sigma_B = \frac{1}{2} I_{2n \times 2n} \). ■

More generally, there always exists \( S \in \text{Sp}(n) \) such that \( SB^{2n}(\sqrt{\hbar}) \subset \Omega \) (see condition (17)), but this does not ensure separability unless \( S = S_A \oplus S_B \) with \( S_A \in \text{Sp}(n_A) \) and \( S_B \in \text{Sp}(n_B) \). In this case we will have \( M \leq S_A \oplus S_B \) hence (53) is satisfied.

Next, we are going to show that for Gaussian states the necessary condition for separability in Proposition 11 is also sufficient.

Proposition 18 The Gaussian density operator \( \hat{\rho} \) is separable if and only if there exist positive definite symplectic matrices \( P_A \in \text{Sp}(n_A) \) and \( P_B \in \text{Sp}(n_B) \) such that
\[ \Sigma \geq \frac{\hbar}{2} (P_A \oplus P_B) . \] (71)

Proof. In view of Proposition 11, the condition (71) is equivalent to the Werner-Wolf condition (53). Since for Gaussians the Werner-Wolf condition is necessary and sufficient, this is also the case for the condition (71). ■

Suppose we have equality in (71). Then \( \hat{\rho} \) is a tensor product \( \hat{S}_A^{-1} \phi_{0,A} \otimes \hat{S}_B^{-1} \phi_{0,B} \) where
\[ \phi_{0,A}(x_A) = (\pi \hbar)^{-n_A/4} e^{-|x_A|^2/2\hbar} \]
\[ \phi_{0,B}(x_B) = (\pi \hbar)^{-n_B/4} e^{-|x_B|^2/2\hbar} \]
are the standard Gaussians on \( \mathbb{R}^{n_A} \) and \( \mathbb{R}^{n_B} \), and \( \hat{S}_A \in \text{Mp}(n_A) \) (resp. \( \hat{S}_B \in \text{Mp}(n_B) \)) is anyone of the two metaplectic operators covering \( S_A \) (resp. \( S_B \)). In fact, the Wigner distribution \( \rho \) becomes in this case
\[ \rho(z) = (\pi \hbar)^{-n} e^{-\frac{1}{\hbar}(S_A^T z_A z_A + S_B^T z_B z_B)} \]
\[ = W_A \phi_{0,A}(S_A z_A) W_B \phi_{0,B}(S_B z_B) \]
where $W_A\phi_{0,A}$ is the Wigner transform of $\phi_{0,A}$ and $W_B\phi_{0,B}$ that of $\phi_{0,B}$. It follows from the symplectic covariance property \[\text{[15]}\] of the Wigner transform that

$W_A\phi_{0,A} \circ S_A = W_A(\hat{S}_A^{-1}\phi_{0,A})$, $W_B\phi_{0,B} \circ S_B = W_A(\hat{S}_B^{-1}\phi_{0,B})$

hence $\rho$ is the Wigner transform of $\hat{S}_A^{-1}\phi_{0,A}$ $\otimes$ $\hat{S}_B^{-1}\phi_{0,B}$. The converse of this property is trivial. Notice that $\hat{S}_A^{-1}\phi_{0,A}$ and $\hat{S}_B^{-1}\phi_{0,B}$ are easily calculated \[\text{[11, 15]}\]: they are explicitly given by

$\hat{S}_A^{-1}\phi_{0,A}(x_A) = (\pi\hbar)^{-n_A/4}(\text{det } X_A)^{-1/4}e^{-\frac{1}{4\hbar}(X_A + iY_A)x_A \cdot x_A}$

$\hat{S}_B^{-1}\phi_{0,B}(x_B) = (\pi\hbar)^{-n_B/4}(\text{det } X_B)^{-1/4}e^{-\frac{1}{4\hbar}(X_B + iY_B)x_B \cdot x_B}$

where the real symmetric matrices $X_A > 0$, $X_B > 0$ and $Y_A, Y_B$ are obtained by solving the identities

$S_A^T S_A = \begin{pmatrix} X_A + Y_A X_A^{-1} Y_A & Y_A X_A^{-1} \\ X_A^{-1} Y_A & X_A^{-1} \end{pmatrix}$

$S_B^T S_B = \begin{pmatrix} X_B + Y_B X_B^{-1} Y_B & Y_B X_B^{-1} \\ X_B^{-1} Y_B & X_B^{-1} \end{pmatrix}$.

More generally the Gaussian state $\hat{\rho}$ is separable if and only if its Wigner distribution dominates a tensor product of two Gaussian states, up to a factor being the purity of $\hat{\rho}$:

**Theorem 19** The Gaussian state $\hat{\rho}$ is separable if and only if there exist pairs $(X_A, Y_A)$ and $(X_B, Y_B)$ such that

$\rho \geq \mu(\hat{\rho})(W_A\phi_{X_A,Y_A} \otimes W_B\phi_{X_B,Y_B})$ (72)

where

$\mu(\hat{\rho}) = \left(\frac{\hbar}{2}\right)^n (\text{det } \Sigma)^{-1/2}$

is the purity \[\text{[65]}\] of $\hat{\rho}$.

**Proof.** In view of the transitivity of the action of the metaplectic group on Gaussians, this is equivalent to proving that there exist $\hat{S}_A \in \text{Mp}(n_A)$ and $\hat{S}_B \in \text{Mp}(n_B)$ such that

$\rho \geq \mu(\hat{\rho})\left(W_A(\hat{S}_A^{-1}\phi_{0,A}) \otimes W_B(\hat{S}_B^{-1}\phi_{0,B})\right)$.

(73)
In view of Proposition [11], $\hat{\rho}$ is separable if and only if condition (71)

$$\Sigma \geq \frac{\hbar}{2} \left[ (S_A^T S_A)^{-1} \oplus (S_B^T S_B)^{-1} \right]$$

holds for some $S_A \in \text{Sp}(n_A)$ and $S_B \in \text{Sp}(n_B)$. Suppose this is the case; by definition (62) of $\rho$ we then have

$$\rho(z) \geq \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{\hbar} S_A^T S_A z_A z_A} e^{-\frac{1}{\hbar} S_B^T S_B z_B z_B}.$$  

We have [11, 15]

$$W_A \phi_{0,A}(S_A z_A) = (\pi \hbar)^{-n_A} e^{-\frac{1}{\hbar} |S_A z_A|^2}$$

$$W_B \phi_{0,B}(S_B z_B) = (\pi \hbar)^{-n_B} e^{-\frac{1}{\hbar} |S_B z_B|^2}$$

and hence

$$\rho(z) \geq \left( \frac{\hbar}{2} \right)^n (\det \Sigma)^{-1/2} W_A \phi_{0,A}(S_A z_A) W_B \phi_{0,B}(S_B z_B).$$  

(74)

Let now $\hat{S}_A \in \text{Mp}(n_A)$ (resp. $\hat{S}_B \in \text{Mp}(n_B)$) cover $S_A$ (resp. $S_B$); we have, using the symplectic covariance of the Wigner transform [9, 13, 15]

$$W_A \phi_{0,A}(S_A z_A) = W_A (\hat{S}_A^{-1} \phi)(z_A)$$

$$W_B \phi_{0,B}(S_B z_B) = W_B (\hat{S}_B^{-1} \phi)(z_B)$$

which shows that (73) must hold if the state $\hat{\rho}$ is separable. Suppose conversely that this inequality holds. Then we must have

$$e^{-\frac{1}{2} \Sigma^{-1} z \cdot z} \geq e^{-\frac{1}{\hbar} S_A^T S_A z_A z_A} e^{-\frac{1}{\hbar} S_B^T S_B z_B z_B}$$

which is equivalent to condition (71) in Proposition [18].

**Corollary 20** If the Gaussian state $\hat{\rho}$ is separable there exist Gaussians $\phi_{X_A,Y_A}$ and $\phi_{X_B,Y_B}$ such that

$$\rho_A \geq \mu(\hat{\rho}) W_A \phi_{X_A,Y_A}, \quad \rho_B \geq \mu(\hat{\rho}) W_B \phi_{X_B,Y_B}.$$  

(75)

**Proof.** It immediately follows from the inequality (72) integrating $\rho$ with respect to $z_B$ and $z_A$. ■

Let us describe in detail the reduced states of a Gaussian state:
Proposition 21 The reduced density operator $\hat{\rho}_A$ is a Gaussian state with Wigner distribution

$$\rho_A(z_A) = (\pi \hbar)^{-n_A}(\det M/M_{BB})^{1/2}e^{-\frac{1}{\hbar}(M/M_{BB})z_A^2}; \quad (76)$$

and its covariance ellipsoid

$$\Omega_A = \{z_A : (M/M_{BB})z_A^2 \leq \hbar\} \quad (77)$$
is the orthogonal projection $\Pi_A\Omega$ on $\mathbb{R}^{2n_A}$ of the covariance ellipsoid $\Omega$ of $\hat{\rho}$.

Proof. The result is in a sense rather obvious since the calculation of $\rho_A$ involves the integration of the Gaussian $\rho$ with respect to a partial set of variables, and thus yields a Gaussian. That this Gaussian is given by (76) then follows from the projection formula (41). Let us however give a direct analytical proof. Writing $z = z_A \oplus z_B$ we have

$$Mz^2 = M_{AA}z_A^2 + 2M_{BA}z_A \cdot z_B + M_{BB}z_B^2$$

so that

$$\int_{\mathbb{R}^{2n_B}} e^{-\frac{1}{\hbar}Mz^2} dz_B = e^{-\frac{1}{\hbar}M_{AA}z_A^2} \int_{\mathbb{R}^{2n_B}} e^{-\frac{1}{\hbar}(M_{BB}z_B^2 + 2M_{BA}z_A \cdot z_B)} dz_B.$$ 

Setting $z_B = u_B - M_{BB}^{-1}M_{BA}z_A$ we have

$$M_{BB}z_B^2 + 2M_{BA}z_A \cdot z_B = M_{BB}u_B^2 - M_{AB}M_{BB}^{-1}M_{BA}z_A^2$$

and hence, integrating with respect to the variables $z_B$,

$$\int_{\mathbb{R}^{2n_B}} e^{-\frac{1}{\hbar}Mz^2} dz_B = e^{-\frac{1}{\hbar}(M_{AA} - M_{AB}M_{BB}^{-1}M_{BA})z_A^2} \int_{\mathbb{R}^{2n_B}} e^{-\frac{1}{\hbar}M_{BB}u_B^2} du_B.$$ 

Using the classical formula (Folland [9], App. A)

$$\int_{\mathbb{R}^{2n_B}} e^{-\frac{1}{\hbar}M_{BB}u_B^2} du_B = (\pi \hbar)^{n_B}(\det M_{BB})^{-1/2}$$

we thus have

$$\int_{\mathbb{R}^{2n_B}} e^{-\frac{1}{\hbar}Mz^2} dz_B = (\pi \hbar)^{n_B}(\det M_{BB})^{-1/2}e^{-\frac{1}{\hbar}(M/M_{BB})z_A^2}$$

where $M/M_{BB}$ is the Schur complement [29] of $M_{BB}$ of $M$; the identity (76) now follows from formula (32). The covariance ellipsoid of the reduced state $\hat{\rho}_A$ is given by (77), and in view of Lemma 6 it is indeed the orthogonal projection $\Pi_A\Omega$ of $\Omega$ on $\mathbb{R}^{2n_A}$. \hfill \blacksquare
Corollary 22 The purity of the reduced density operator $\hat{\rho}_A$ is

$$\mu(\hat{\rho}_A) = (\det M/M_{BB})^{1/2}$$

(78)

and $\hat{\rho}_A$ is a pure state if and only if $M/M_{BB} \in \text{Sp}(n_A)$, in which case we have $\mu(\hat{\rho}) = \det M_{BB}$.

Proof. The purity of $\hat{\rho}_A$ is $\mu(\hat{\rho}_A) = \sqrt{\det M/M_{BB}}$; hence $\mu(\hat{\rho}_A) = 1$ if and only if $\det M/M_{BB} = 1$; by the same token as used in Lemma 15 we must then have $M/M_{BB} \in \text{Sp}(n_A)$. The equality $\mu(\hat{\rho}) = \det M_{BB}$ follows from the identity (32).

4 Sufficient Conditions for Separability of Gaussian states

In this section, we will derive a number of sufficient, albeit not necessary, conditions for the separability of Gaussian states.

We will write as usual

$$M = \frac{\hbar}{2} \Sigma^{-1} = \begin{pmatrix} M_{AA} & M_{AB} \\ M_{BA} & M_{BB} \end{pmatrix},$$

(79)

and it is presupposed that $M = M^T > 0$, and hence $M_{AA} > 0$, $M_{BB} > 0$ and $M_{BA} = M_{AB}^T$. It follows from Proposition 3 that:

$$\Sigma + \frac{i \hbar}{2} J_{AB} \geq 0 \iff \text{The symplectic eigenvalues } \lambda_{\sigma,j}(M) \text{ of } M \text{ are all } \leq 1.$$ (80)

We shall also assume, without loss of generality, that $n_B \geq n_A$. Let

$$\mu_1^{AB} \geq \mu_2^{AB} \geq \cdots \geq \mu_{2n_A}^{AB} \geq 0$$

(81)

be the singular values of $M_{AB}$, that is the positive square roots of the eigenvalues of the $2n_A \times 2n_A$ matrix $M_{AB}M_{AB}^T = M_{AB}M_{BA}$. Notice that, apart from the multiplicities of zero, the matrices $M_{AB}M_{BA}$ and $M_{BA}M_{AB}$ have the same eigenvalues, and so $M_{AB}$ and $M_{BA}$ have the same singular values.

We shall write, as customary, $|M_{AB}| = (M_{AB}M_{BA})^{1/2}$ and $|M_{BA}| = (M_{BA}M_{AB})^{1/2}$. In particular, we have:

$$\|M_{AB}\|_{op} = \sup_{z_B \neq 0} \frac{|M_{AB}z_B|}{|z_B|} = \mu_1^{AB} = \sup_{z_A \neq 0} \frac{|M_{BA}z_A|}{|z_A|} = \|M_{BA}\|_{op}.$$ (82)
By the singular value decomposition, there exist unitary matrices \( U \in \mathbb{C}^{2n_A \times 2n_A} \) and \( V \in \mathbb{C}^{2n_B \times 2n_B} \), such that
\[
M_{AB} = UD_{AB}V^*,
\]
where \( D_{AB} \in \mathbb{C}^{2n_A \times 2n_B} \) is the diagonal matrix of singular values, that is \((D_{AB})_{jj} = \mu_{AB}^j\), for \( j = 1, \cdots, 2n_A \), and \((D_{AB})_{jk} = 0\), for all \( j = 1, \cdots, 2n_A \) and \( k = 1, \cdots, 2n_B \), such that \( j \neq k \).

Given a set of positive numbers \( \epsilon = (\epsilon_1, \cdots, \epsilon_{2n_A}) \in \mathbb{R}^{2n_A}_+ \), we define the \( 2n_A \times 2n_A \) matrix \( |M^\epsilon_{AB}| \) and the \( 2n_B \times 2n_B \) matrix \( |M^{1/2}_{BA}| \) by:
\[
U^*|M^\epsilon_{AB}|U = \text{diag} \left( \epsilon_1 \mu_{1,AB}, \cdots, \epsilon_{2n_A} \mu_{2n_A,AB} \right),
\]
\[
V^*|M^{1/2}_{BA}|V = \text{diag} \left( \frac{\mu_{1,AB}}{\epsilon_1}, \cdots, \frac{\mu_{2n_A,AB}}{\epsilon_{2n_A}}, 0, \cdots, 0 \right).
\]
In particular, if we write \( 1 = (1, \cdots, 1) \) for \( \epsilon_j = 1 \), for all \( j = 1, \cdots, 2n_A \), then we have:
\[
|M^1_{AB}| = |M_{AB}| \text{ and } |M^{1/2}_{BA}| = |M_{BA}|.
\]

We will now derive a hierarchy of sufficient conditions for separability, which culminate in Theorem 25. The advantage of developing this hierarchy, instead of going directly to Theorem 25, is that in this manner we increase the computational complexity gradually.

### 4.1 The first separability criterion

Let us state the first criterion for separability of Gaussian states.

**Theorem 23** Let \( \tilde{M}_{AA} = M_{AA} + \|M_{AB}\|_{op} I_{n_A} \) and \( \tilde{M}_{BB} = M_{BB} + \|M_{BA}\|_{op} I_{n_B} \).

If
\[
\lambda_{\sigma,j} \left( \tilde{M}_{AA} \right) \leq 1 \text{ and } \lambda_{\sigma,k} \left( \tilde{M}_{BB} \right) \leq 1,
\]
for all \( j = 1, \cdots, n_A \) and all \( k = 1, \cdots, n_B \), then the Gaussian state \( \hat{\rho} \) with covariance ellipsoid
\[
\Omega = \{ z \in \mathbb{R}^{2n} : Mz^2 \leq \hbar \}
\]
is separable.

**Proof.** We have, by the Cauchy-Schwarz and the geometric-arithmetic mean inequalities,
\[
z_A \cdot M_{AB} z_B \leq |z_A \cdot M_{AB} z_B| \leq |z_A| \cdot |M_{AB} z_B| \leq \|M_{AB}\|_{op} |z_A| \cdot |z_B| \leq \frac{\|M_{AB}\|_{op}}{2} \left( |z_A|^2 + |z_B|^2 \right).
\]
It follows that
\[
Mz^2 = M_{AA}z_A^2 + 2z_A \cdot M_{AB}z_B + M_{BB}z_B^2
\]
\[
\leq M_{AA}z_A^2 + \|M_{AB}\|_{op}|z_A|^2 + \|M_{AB}\|_{op}|z_B|^2 + M_{BB}z_B^2 = (M_{AA} + \|M_{AB}\|_{op}I_A)z_A^2 + (M_{BB} + \|M_{BA}\|_{op}I_B)z_B^2,
\]
and thus:
\[
M \leq \tilde{M}_{AA} \oplus \tilde{M}_{BB}.
\] (89)
If conditions (86) hold, then $\tilde{M}_{AA}^{-1} + iJ_A \geq 0$ and $\tilde{M}_{BB}^{-1} + iJ_B \geq 0$. By the Werner-Wolf condition, the state $\hat{\rho}$ is separable. $
BBox$

4.2 Geometric interpretation

Here is a straightforward geometric interpretation of Theorem 23. It says that if the ellipsoid
\[
\tilde{\Omega} = \{ z \in \mathbb{R}^{2n} : \tilde{M}_{AA}z_A^2 + \tilde{M}_{BB}z_B^2 \leq \hbar \}
\]
is “large enough” to contain a “quantum blob” of the type $\Omega_{AB} = (S_A \oplus S_B)B^{2n}(\sqrt{\hbar})$, then the Gaussian state $\hat{\rho}$ with covariance ellipsoid $\tilde{\Omega}$ will be separable. Hence, we have the inclusions:
\[
\Omega_{AB} \subset \tilde{\Omega} \subset \Omega
\]
and it follows from the projection results discussed in the sections 1.3 and 2.3 that the following inclusions also hold
\[
S_A(B^{2n_A}(\sqrt{\hbar})) \subset \tilde{\Omega}_A \subset \Omega_A \text{ and } S_B(B^{2n_B}(\sqrt{\hbar})) \subset \tilde{\Omega}_B \subset \Omega_B.
\] (91)
where $\Omega_A$ and $\Omega_B$ are the covariance ellipsoids of the reduced density operators $\hat{\rho}_A$ and $\hat{\rho}_B$ (cf. (42),(41)) and
\[
\tilde{\Omega}_A = \{ z \in \mathbb{R}^{2n} : \tilde{M}_{AA}z_A^2 \leq \hbar \}
\] (92)
and likewise for $\tilde{\Omega}_B$.

4.3 The second separability criterion

We will now derive a second criterion and then use it to show that the previous criterion is not necessary for separability of a Gaussian state.

27
Theorem 24 Define $M_{AA}^a = M_{AA} + |M_{AB}|$ and $M_{BB}^a = M_{BB} + |M_{BA}|$. If their symplectic eigenvalues satisfy

$$\lambda_{\sigma_A,j} \left( M_{AA}^a \right) \leq 1 \quad \text{and} \quad \lambda_{\sigma_B,k} \left( M_{BB}^a \right) \quad \text{for all} \quad j = 1, \ldots, n_A \quad \text{and all} \quad k = 1, \ldots, n_B,$$

then the Gaussian state $\hat{\rho}$ with covariance ellipsoid (87) is separable.

Proof. With the previous notation, let $u_A = U^* z_A$ and $v_B = V^* z_B$. Then:

$$z_A \cdot M_{AB} z_B \leq |z_A \cdot M_{AB} z_B| = |u_A \cdot D_{AB} v_B| = \left| \sum_{j=1}^{2n_A} \mu_{A,j} u_{A,j} v_{B,j} \right| =$$

$$\leq \sum_{j=1}^{2n_A} \mu_{A,j} |u_{A,j}| |v_{B,j}| \leq \sum_{j=1}^{2n_A} \mu_{A,j} \left( \frac{|u_{A,j}|^2}{2} + \frac{|v_{B,j}|^2}{2} \right) =$$

$$= \frac{1}{2} \sum_{j=1}^{2n_A} u_{A,j} \mu_{A,j} u_{A,j} + \frac{1}{2} \sum_{j=1}^{2n_A} v_{B,j} \mu_{A,j} v_{B,j} = \frac{1}{2} |M_{AB}| z_A^2 + \frac{1}{2} |M_{BA}| z_B^2,$$

where we used (84) and (85).

Consequently:

$$M z^2 = M_{AA} z_A^2 + 2z_A \cdot M_{AB} z_B + M_{BB} z_B^2 \leq M_{AA} z_A^2 + |M_{AB}| z_A^2 + |M_{BA}| z_B^2 + M_{BB} z_B^2 = \left( M_{AA}^a \oplus M_{BB}^a \right) z^2,$$

and the rest follows as before. $\blacksquare$

4.4 An example of non-necessity

Let us now show that the separability criterion stated in Theorem 23 is sufficient but not necessary. We consider the particular case $n_A = n_B = 1$.

Let $M$ be the $4 \times 4$ matrix given by:

$$M = \left( \begin{array}{cccc}
\frac{1}{2} & 0 & \frac{2}{3} & 0 \\
0 & 1 & 0 & \frac{1}{2} \\
\frac{2}{3} & 0 & \frac{1}{3} & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4}
\end{array} \right).$$

With the previous notation, we have:

$$M_{AA} = \left( \begin{array}{cc}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{array} \right), \quad M_{BB} = \left( \begin{array}{cc}
\frac{1}{3} & 0 \\
0 & \frac{1}{4}
\end{array} \right),$$

$$M_{AB} = M_{BA} = |M_{AB}| = |M_{BA}| = \left( \begin{array}{cc}
\frac{2}{3} & 0 \\
0 & \frac{1}{4}
\end{array} \right).$$
Since $\mu_{1}^{AB} = \|M_{AB}\|_{op} = \|M_{BA}\|_{op} = \frac{2}{3}$, we have:

\[
\tilde{M}_{AA} = \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & \frac{7}{6} \end{pmatrix}, \quad \tilde{M}_{BB} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{11}{12} \end{pmatrix}.
\]

(98)

It follows that $\lambda_{\sigma}(\tilde{M}_{AA}) = \frac{7}{6} > 1$, while $\lambda_{\sigma}(\tilde{M}_{BB}) = \sqrt{\frac{11}{12}} < 1$. We conclude that $M$ does not satisfy the criterion of Theorem 23. Nevertheless, $M$ is associated with a separable state. This can be shown using the criterion of Theorem 24. Indeed, we have:

\[
M_{AA}^\sharp = \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}, \quad M_{BB}^\sharp = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.
\]

(99)

and hence

\[
\lambda_{\sigma}(M_{AA}^\sharp) = \sqrt{\frac{7}{8}} < 1 \quad \text{and} \quad \lambda_{\sigma}(M_{BB}^\sharp) = \frac{1}{\sqrt{2}} < 1.
\]

According to Theorem 24, the associated Gaussian state is separable.

4.5 The third separability criterion

In the previous criteria, we always used the geometric-arithmetic mean inequality $|ab| \leq (|a|^2 + |b|^2)/2$ to prove our results. This inequality places an upper bound on the product $|ab|$ with $|a|^2$ and $|b|^2$ on equal footing. However, it is perfectly conceivable that in some directions $M_{AA}$ is "too large" for us to have $M_{AA} + |M_{AB}|$ dominated by a positive symplectic matrix $P_A$ and that this may be compensated by the fact that $M_{BB}$ is "smaller". In this case, it may be more suitable to use the scaled geometric-arithmetic mean inequality:

\[
|ab| \leq \frac{|a|^2}{2\epsilon} + \frac{\epsilon |b|^2}{2},
\]

(100)

which holds for any $\epsilon > 0$. We will derive, using this inequality, another sufficient criterion for separability, which will permit us to prove that the criterion stated in Theorem 24 is again sufficient but not necessary for separability. With the same notation as previously, we have:

**Theorem 25** Let $\tilde{M}_{AA}^\epsilon$ be a $2n_A \times 2n_A$ matrix and $\tilde{M}_{BB}^\epsilon$ a $2n_B \times 2n_B$ defined by:

\[
\tilde{M}_{AA}^\epsilon = M_{AA} + |M_{AB}| \quad \text{and} \quad \tilde{M}_{BB}^\epsilon = M_{BB} + \left| M_{BA}^\epsilon \right|.
\]

(101)
If their symplectic eigenvalues satisfy
\[ \lambda_{\sigma_A,j}(\tilde{M}_{AA}) \leq 1 \text{ and } \lambda_{\sigma_B,k}(\tilde{M}_{BB}^{-1}) \]
for all \( j = 1, \cdots, n_A \) and all \( k = 1, \cdots, n_B \), then the Gaussian state \( \hat{\rho} \) with covariance ellipsoid \( \tilde{M}_{AA} \) is separable.

**Proof.** We proceed as in the previous proofs and apply this time the inequality (100) for the set of positive numbers \( \epsilon = (\epsilon_1, \cdots, \epsilon_{2n_A}) \in \mathbb{R}^{2n_A}_+ \).

\[
z_A \cdot M_{AB} z_B \leq \sum_{j=1}^{2n_A} \mu_j |u_{A,j}| |v_{B,j}| \\
\leq \sum_{j=1}^{2n_A} \mu_j \left( \frac{|u_{A,j}|^2}{2 \epsilon_j} + \frac{|v_{B,j}|^2}{2 \epsilon_j} \right) = |M_{AB}^\epsilon| z_A^2 + |M_{BA}^1| z_B^2. \tag{103}
\]

It follows that:

\[
M z^2 = M_{AA} z_A^2 + 2 z_A \cdot M_{AB} z_B + M_{BB} z_B^2 \\
\leq (M_{AA} + |M_{AB}^\epsilon|) z_A^2 + (M_{BB} + |M_{BA}^1|) z_B^2, \tag{104}
\]

which means that:

\[
M \leq \tilde{M}_{AA}^\epsilon \oplus \tilde{M}_{BB}^1. \tag{105}
\]

The rest follows as previously. ■

4.6 Another example of non-necessity

We will now show, with a particular example when \( n_A = n_B = 1 \), that the criterion stated in Theorem 24 is not necessary for separability.

Let \( M \) be given by:

\[
M = \begin{pmatrix}
\frac{2}{5} & 0 & \frac{1}{2} & 0 \\
0 & \frac{2}{3} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{8} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{8}
\end{pmatrix} \tag{106}
\]

With the previous notation, we have:

\[
M_{AA} = \begin{pmatrix}
\frac{2}{5} & 0 \\
0 & \frac{2}{3}
\end{pmatrix}, \quad M_{BB} = \begin{pmatrix}
\frac{1}{8} & 0 \\
0 & \frac{1}{8}
\end{pmatrix} \\
M_{AB} = M_{BA} = |M_{AB}| = |M_{BA}| = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}. \tag{107}
\]
We thus have:

\[ M_{AA}^\sharp = \left( \begin{array}{cc} 7/6 & 0 \\ 0 & 7/6 \end{array} \right), \quad M_{BB}^\sharp = \left( \begin{array}{cc} 5/8 & 0 \\ 0 & 5/8 \end{array} \right), \]  

which entails that \( \lambda_{\sigma_A}(M_{AA}^\sharp) = 7/6 > 1 \), while \( \lambda_{\sigma_B}(M_{BB}^\sharp) = 5/8 < 1 \). We conclude that the condition in the criterion of Theorem 24 is not respected. However, the Gaussian state associated with \( M \) is a separable state. Indeed, we have for \( \varepsilon = \left( \begin{array}{cc} 13/27, 13/27 \end{array} \right) \) and \( \frac{1}{\varepsilon} = \left( \begin{array}{cc} 21/13, 21/13 \end{array} \right) \):

\[ \tilde{M}_{AA}^\varepsilon = \left( \begin{array}{cc} 41/42 & 0 \\ 0 & 41/42 \end{array} \right), \quad \tilde{M}_{BB}^{1/\varepsilon} = \left( \begin{array}{cc} 95/104 & 0 \\ 0 & 95/104 \end{array} \right). \]  

We thus have:

\[ \lambda_{\sigma_A}(\tilde{M}_{AA}^\varepsilon) = 41/42 < 1, \quad \lambda_{\sigma_B}(\tilde{M}_{BB}^{1/\varepsilon}) = 95/104 < 1. \]  

From Theorem 25 we conclude that the associated Gaussian state is separable.

### 4.7 The fourth separability criterion: a particular case

In this section, we derive another sufficient criterion, which applies only to the particular case where all the blocks, \( M_{AA}, M_{AB} \) and \( M_{BB} \) are either diagonal or can be brought to a diagonal form by a symplectic transformation \( S_A \oplus S_B \). We illustrate this example with several pictures which highlight the geometric nature of the problem.

We will thus assume that there exist \( S_A \in \text{Sp}(n_A) \) and \( S_B \in \text{Sp}(n_B) \), such that, for \( S = S_A \oplus S_B \):

\[ SMST = M_D = \left( \begin{array}{cc} A & D \\ DT & B \end{array} \right), \]  

where we have the following diagonal blocks:

\[ A = \text{diag} \left( \Lambda_A, \Lambda_A \right), \quad B = \text{diag} \left( \Lambda_B, \Lambda_B \right), \]  

with

\[ \Lambda_A = \text{diag} \left( \lambda_{\sigma_A,1}(M_{AA}), ..., \lambda_{\sigma_A,n_A}(M_{AA}) \right) \]  

and

\[ \Lambda_B = \text{diag} \left( \lambda_{\sigma_B,1}(M_{BB}), ..., \lambda_{\sigma_B,n_B}(M_{BB}) \right). \]
and $D$ is a $2n_A \times 2n_B$ matrix of the form:

$$D = \begin{pmatrix} E & 0_{n_A \times n_B} \\ 0_{n_A \times n_B} & F \end{pmatrix} \tag{114}$$

where $E$ and $F$ are diagonal $n_A \times n_B$ matrices with entries:

$$E_{j,k} = d_j \delta_{j,k}, \quad F_{j,k} = d_j + n_A \delta_{j,k}, \quad j = 1, \cdots, n_A, \quad k = 1, \cdots, n_B. \tag{115}$$

In the sequel, we will need to consider the following $2 \times 2$ matrices for a set of numbers $a_j, b_j$:

$$Q_j(a_j, b_j) = \begin{pmatrix} a_j - \lambda_{\sigma_A,j} (M_{AA}) & -d_j \\ -d_j & b_j - \lambda_{\sigma_B,j} (M_{BB}) \end{pmatrix}, j = 1, \cdots, n_A \tag{116}$$

and

$$P_j(a_j, b_j) = \begin{pmatrix} \frac{1}{a_j} - \lambda_{\sigma_A,j} (M_{AA}) & -d_j + n_A \\ -d_j + n_A & \frac{1}{b_j} - \lambda_{\sigma_B,j} (M_{BB}) \end{pmatrix}, j = 1, \cdots, n_A \tag{117}$$

**Theorem 26** Suppose that there exist a set of numbers $a_1, \cdots, a_{n_A} > 0$ and $b_1, \cdots, b_{n_B} > 0$, such that:

$$\lambda_{\sigma_A,j} (M_{AA}) \leq a_j \leq \frac{1}{\lambda_{\sigma_A,j} (M_{AA})}, \quad j = 1, \cdots, n_A \tag{118}$$

$$\lambda_{\sigma_B,k} (M_{BB}) \leq b_k \leq \frac{1}{\lambda_{\sigma_B,k} (M_{BB})}, \quad k = 1, \cdots, n_B$$

and

$$\det Q_j(a_j, b_j) \geq 0, \quad \det P_j(a_j, b_j) \geq 0, \quad j = 1, \cdots, n_A. \tag{119}$$

Then the Gaussian state with covariance ellipsoid $[87]$ is separable.

**Proof.** First of all, notice that if the state is separable, then there exist $S'_A \in Sp(n_A)$ and $S'_B \in Sp(n_B)$, such that:

$$M \leq \left( (S'_A)^T S'_A \right) \oplus \left( (S'_B)^T S'_B \right) \leq \left( (S'_A S'_A)^T (S'_A S'_A)^T \right) \oplus \left( (S'_B S'_B)^T (S'_B S'_B)^T \right) \tag{120}$$

Thus, $\hat{\rho}$ is separable if and only if the Gaussian state with covariance ellipsoid given by the matrix $M_D = S S^T$ is separable. We may therefore assume that $M$ is of the form $[111]-[115]$. 
Next, consider the positive symplectic matrix $P_A \oplus P_B$, with

$$P_A = \text{diag} \left( a_1, \ldots, a_{n_A}, \frac{1}{a_1}, \ldots, \frac{1}{a_{n_A}} \right),$$

$$P_B = \text{diag} \left( b_1, \ldots, b_{n_B}, \frac{1}{b_1}, \ldots, \frac{1}{b_{n_B}} \right).$$

(121)

If $M \leq P_A \oplus P_B$, then $\hat{\rho}$ is a separable state. Writing $z = (z_A, z_B)$ and $z_A = (x_A, p_A)$, $z_B = (x_B, p_B)$, this is equivalent to:

$$\sum_{j=1}^{n_A} \lambda_{\sigma_{A,j}} (M_{AA}) \left( x_{A,j}^2 + p_{A,j}^2 \right) + 2 \sum_{j=1}^{n_A} d_j z_{A,j} z_{B,j} +$$

$$+ \sum_{j=1}^{n_B} \lambda_{\sigma_{B,j}} (M_{BB}) \left( x_{B,j}^2 + p_{B,j}^2 \right) \leq \sum_{j=1}^{n_A} \left( a_j x_{A,j}^2 + \frac{p_{A,j}^2}{a_j} \right) + \sum_{j=1}^{n_B} \left( b_j x_{B,j}^2 + \frac{p_{B,j}^2}{b_j} \right)$$

(122)

These equations can be decoupled for each $j$ and we obtain the set of inequalities:

$$\lambda_{\sigma_{A,j}} (M_{AA}) x_{A,j}^2 + 2d_j x_{A,j} x_{B,j} + \lambda_{\sigma_{B,j}} (M_{BB}) x_{B,j}^2$$

$$\leq a_j x_{A,j}^2 + b_j x_{B,j}^2, j = 1, \ldots, n_A ,$$

(123)

$$\lambda_{\sigma_{A,j}} (M_{AA}) p_{A,j}^2 + 2d_n a_j p_{A,j} p_{B,j} + \lambda_{\sigma_{B,j}} (M_{BB}) p_{B,j}^2$$

$$\leq \frac{p_{A,j}^2}{a_j} + \frac{p_{B,j}^2}{b_j}, j = 1, \ldots, n_A ,$$

(124)

and

$$\lambda_{\sigma_{B,j}} (M_{BB}) (x_{B,j}^2 + p_{B,j}^2) \leq b_j x_{B,j}^2 + \frac{p_{B,j}^2}{b_j}, j = n_A + 1, \ldots, n_B .$$

(125)

Inequalities (123)-(125) are equivalent to (118) and (119). ■

If $n_A = n_B$, then we can discard inequalities (125). If $n_B > n_A$, then we just have to find $b_{n_A+1}, \ldots, b_{n_B} > 0$, such that $\lambda_{\sigma_{B,j}} (M_{BB}) \leq b_j \leq \frac{1}{\lambda_{\sigma_{B,j}} (M_{BB})}$, $j = n_A + 1, \ldots, n_B$. The nontrivial part corresponds to determining the remaining constants, $a_j, b_j$ for $j = 1, \ldots, n_A$. In the general case, these are easily obtained numerically.

Each solution $a_j, b_k, j = 1, ..., n_A$ and $k = 1, ..., n_B$ determines an ellipsoid

$$\Omega_{AB} = \{ z \in \mathbb{R}^{2n} : P_A z_A^2 + P_B z_B^2 \leq \hbar \} \subset \Omega_D$$

(126)

33
where $P_A, P_B$ are given by (121), and $\Omega_D$ is the covariant ellipsoid of the matrix $M_D$. The projection of $\Omega_{AB}$ onto the plane $x_A, x_B$ determines an ellipse (of size $1/\sqrt{\alpha_j}, 1/\sqrt{\beta_j}$, if we assume $\hbar = 1$) and the projection onto the plane $p_A, p_B$ determines another ellipse, "conjugate" to the first one, and of size $\sqrt{\alpha_j}, \sqrt{\beta_j})$. These two ellipses are enclosed in the projections of $\Omega_D$ onto these two planes. We also conclude from (118) that (cf. (111,121)):

$$P_A \geq A \geq M_D / B = A - DB^{-1}D^T$$

and so $\Pi_A \Omega_{AB} \subset \Pi_A \Omega_D$. An equivalent result is valid for the projection $\Pi_B$. These geometrical relations are illustrated by the example at the end of this section.

A set of conditions equivalent to those of Theorem 26 is the following. We use the abbreviated notation $\lambda_j^A = \lambda_{\sigma, A, j}(M_{AA}), \lambda_j^B = \lambda_{\sigma, B, j}(M_{BB})$.

**Lemma 27** The following set of conditions are equivalent.

1. The matrices $Q_j(a, b)$ and $P_j(a, b)$ are positive semi-definite for some $a, b > 0$.

2. There exists $a_0 \in \left[\lambda_j^A, \frac{1}{\lambda_j^A}\right]$, such that $f(a_0) \geq 0$, where $f(x) = \alpha x^2 + \beta x + \gamma$, with:

$$\alpha = \lambda_j^A \lambda_j^B - \lambda_j^B d_{n_A + j} - \lambda_j^A$$

$$\beta = 1 + \left(\lambda_j^A\right)^2 - \left(\lambda_j^B\right)^2 + \left(\lambda_j^A \lambda_j^B - d_j^2\right) \cdot \left(\frac{d_j^2}{n_A + j} - \lambda_j^A\right) \quad (127)$$

$$\gamma = \left(\lambda_j^A \lambda_j^B - d_j^2\right) \lambda_j^B.$$

**Proof.** For simplicity, we write $\lambda_j^A = \lambda^A, \lambda_j^B = \lambda^B, d_j = d$ and $d_{n_A + j} = D$.

Conditions 1 are equivalent to

$$\lambda^A \leq a \leq \frac{1}{\lambda^A}, \quad \lambda^B \leq b \leq \frac{1}{\lambda^B}, \quad (128)$$

and

$$\left(a - \lambda^A\right) \cdot \left(b - \lambda^B\right) \geq d^2, \quad \left(\frac{1}{a} - \lambda^A\right) \cdot \left(\frac{1}{b} - \lambda^B\right) \geq D^2. \quad (129)$$
From the first inequality in (129), we obtain:

\[ \lambda^a + \frac{d^2}{a - \lambda^a} \leq b. \]  

(130)

Similarly, from the second inequality, we obtain:

\[ b \leq \frac{1}{\lambda^B + \frac{D^2}{a - \lambda^B}}. \]  

(131)

If \( \lambda^a \leq a \leq \frac{1}{\lambda^a} \), then we conclude from (130) and (131) that we have automatically \( \lambda^B \leq b \leq \frac{1}{\lambda^B} \). It follows that conditions 1 are equivalent to \( \lambda^a \leq a \leq \frac{1}{\lambda^a} \) and

\[ \lambda^a + \frac{d^2}{a - \lambda^a} \leq \frac{1}{\lambda^B + \frac{D^2}{a - \lambda^B}} \iff f(a) \geq 0, \]  

(132)

which concludes the proof. ■

As an example let us consider the case \( n_A = n_B = 1 \) with the matrix

\[
M = \begin{pmatrix}
\frac{1}{2} & 0 & \frac{2}{5} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{4} \\
\frac{2}{5} & 0 & \frac{17}{18} & 0 \\
0 & \frac{1}{4} & 0 & \frac{3}{16}
\end{pmatrix}.
\]  

(133)

The associated matrix \( M_D \) is:

\[
M_D = \begin{pmatrix}
\frac{1}{2} & 0 & \left( \frac{2}{5} \right)^{1/4} & 0 \\
0 & \frac{1}{2} & 0 & \frac{17}{54}^{1/4} \\
\left( \frac{2}{5} \right)^{1/4} & 0 & \frac{17/6}{4} & 0 \\
0 & \frac{17/54}^{1/4} & 0 & \frac{17/6}{4}
\end{pmatrix}
\]  

(134)

where we used the fact that \( \lambda_{\sigma_A,1}(M_{AA}) = 1/2 \) and \( \lambda_{\sigma_B,1}(M_{BB}) = \frac{\sqrt{17/6}}{4} \).

The matrices \( Q_1, P_1 \) for this case are:

\[
Q_1(a, b) = \begin{pmatrix}
a - 1/2 & \left( \frac{2}{5} \right)^{1/4} \\
\left( \frac{2}{5} \right)^{1/4} & b - \frac{\sqrt{17/6}}{4}
\end{pmatrix},
\]

\[
P_1(a, b) = \begin{pmatrix}
1/a - 1/2 & \frac{17/54}^{1/4} \\
\frac{17/54}^{1/4} & 1/b - \frac{\sqrt{17/6}}{4}
\end{pmatrix}
\]
We are looking for solutions of

\[ \det Q_1(a, b) \geq 0, \quad \det P_1(a, b) \geq 0 \]  

in the range

\[ \frac{1}{2} \leq a \leq 2 \quad \text{and} \quad \frac{\sqrt{17/6}}{4} \leq b \leq \frac{4}{\sqrt{17/6}}. \]

These solutions can be obtained numerically. They are given by the points between the two curves in Figure 1.

![Figure 1: The solutions of eq. (135) are given by the points between the two curves. Each point corresponds to an ellipsoid \( \Omega_{AB} \) (126) that is contained in the covariant ellipsoid \( \Omega_D \) associated to (134).](image)

Recall that \( \Omega_{AB} \) is given by (126) and that \( \Omega_D \) is the covariant ellipsoid associated to \( M_D \). In Figures 2.1 to 2.4 we consider the case \( a = 1.6, \ b = 0.6, \ h = 1 \) and plot the projections of \( \Omega_D \) and \( \Omega_{AB} \) onto the planes \( x_A,1p_A,1, \ x_B,1p_B,1, \ x_A,1x_B,1 \) and \( p_A,1p_B,1 \).

![Figures 2.1 and 2.2: Projections of \( \Omega_D \) and \( \Omega_{AB} \) onto the \( x_A,1p_A,1 \) plane (left) and onto the \( x_B,1p_B,1 \) plane (right) for the case \( a = 1.6, \ b = 0.6 \) and \( h = 1 \).](image)
Figures 2.3 and 2.4: Projections of $\Omega_D$ and $\Omega_{AB}$ onto the $x_{A,1}x_{B,1}$ plane and onto the $p_{A,1}p_{B,1}$ plane for the case $a = 1.6$, $b = 0.6$ and $\hbar = 1$.

In Figures 3.1 to 3.4 the plots represent the same projections of $\Omega_D$ and $\Omega_{AB}$ for another solution of (135): $a = 0.7$ and $b = 1.8$.

Figures 3.1 to 3.4: Projections of $\Omega_D$ and $\Omega_{AB}$ onto the planes $x_{A,1}x_{B,1}$, $x_{A,1}p_{A,1}$, $x_{A,1}x_{B,1}$ and $p_{A,1}p_{B,1}$ for the case $a = 0.7$, $b = 1.8$ and $\hbar = 1$.

Finally, Figure 4 displays the possible values of $a$ and $b$ of the enclosed ellipsoids $\Omega_{AB}$ for the example of Section 4.6.
Figure 4: Numerical solutions of eq. (135) for the case (106). Each point between the two curves corresponds to an ellipsoid $\Omega_{AB}$ that is enclosed in the covariant ellipsoid $\Omega = \Omega_D$ associated to (106).

5 Discussion

Since we have Theorem 25 $\Rightarrow$ Theorem 24 $\Rightarrow$ Theorem 23, but the converse is not valid, we conclude that only the criterion stated in Theorem 25 is a candidate for a necessary and sufficient condition for separability of Gaussian states. Bearing this fact in mind, one could be tempted to forget about Theorems 23 and 24 altogether and keep only Theorem 25 as a criterion. We nevertheless feel that this hierarchy of criteria may be useful, because the computational complexity increases from one criterion to the next. In particular, it may not be easy to determine the optimal choice of numbers $\varepsilon_1, \cdots, \varepsilon_n > 0$, to satisfy the condition of Theorem 25. So, if one is able to prove separability using, say, Theorem 23, then there is no need to apply the more complicated Theorems 24 and 25.

This situation is however in no way discouraging since it is always easy to check whether a Gaussian is a good candidate to be a separable state by using the very simple PPT criterion, which reduces to some trivial manipulations of the covariance matrix.

Acknowledgement 28 Part of this research was done while Maurice de Gosson held the Giovanni-Prodi-Lehrstuhl at the University of Würzburg. The work of N.C. Dias and J.N. Prata was supported by the Portuguese Science Foundation (FCT) grant PTDC/MAT-CAL/4334/2014.
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