Absence of supercurrent sign reversal in a topological junction with a quantum dot

Schulenborg, J.; Flensberg, K.

Published in:
Physical Review B

DOI:
10.1103/PhysRevB.101.014512

Publication date:
2020

Document version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Schulenborg, J., & Flensberg, K. (2020). Absence of supercurrent sign reversal in a topological junction with a quantum dot. Physical Review B, 101(1), [014512]. https://doi.org/10.1103/PhysRevB.101.014512
Absence of supercurrent sign reversal in a topological junction with a quantum dot

J. Schulenborg and K. Flensberg

Center for Quantum Devices, Niels Bohr Institute, University of Copenhagen, 2100 Copenhagen, Denmark

(Received 15 October 2019; revised manuscript received 6 January 2020; published 28 January 2020)

Experimental techniques to verify Majoranas are of current interest. A prominent test is the effect of Majoranas on the Josephson current between two wires linked via a normal junction. Here, we study the case of a quantum dot connecting the two superconductors and the sign of the supercurrent in the trivial and topological regimes under grand-canonical equilibrium conditions, explicitly allowing for parity changes due to, e.g., quasiparticle poisoning. We find that the well-known supercurrent reversal for odd occupancy of the quantum dot (π junction) in the trivial case does not occur in the presence of Majoranas in the wires. However, we also find this to be a mere consequence of Majoranas being zero energy states. Therefore, the lack of supercurrent sign reversal can also be caused by trivial bound states and is thus not a discriminating signature of Majoranas.

DOI: 10.1103/PhysRevB.101.014512

I. INTRODUCTION

Majorana bound states in condensed-matter systems have attracted enormous interest in the last decade [1–5], owing mostly to their fundamental, quantum statistical properties, and their potential application in fault-tolerant, topological quantum computing [1,6,7]. A promising candidate among several potential physical systems [6,8–19] to host such states are semiconducting, quasi one-dimensional nanowires with proximity-induced s-wave superconductivity, Rashba spin-orbit coupling, as well as a specifically tuned parallel magnetic field [9,10,20,21]. However, despite considerable experimental evidence for the presence of Majoranas in such wires [18,21–27], unambiguously distinguishing Majoranas from regular Andreev subgap states [22,28–36] remains challenging.

In principle, a desired way to unambiguously trace and exploit Majoranas in solid-state systems is to measure and manipulate observables directly affected by the most distinctive properties of Majoranas—their statistics and their insusceptibility to local decoherence [1,6,18]. Two key long-term goals of this line of research are the possibility of braiding [6,37–43] and the successful implementation of fault-tolerant, Majorana-based qubits [7,43–46]. However, one major challenge is that building and operating such devices needs many components and fine tuning. It is thus important to find more easily applicable test criteria that, while not always fully conclusive, still provide sufficient confidence to further advance in conceiving and building the final device.

First experiments aiming to find zero-energy Majorana states in nanowires focused on measuring the corresponding zero-bias conductance peak [21,47,48]. Since the nanowire-based devices of interest are typically integrated into electrical circuits anyhow, such measurements have since become a relatively straightforward consistency check for the existence of Majoranas. Yet, due to the many possible causes of zero-bias peaks including trivial zero-energy Andreev bound states, they are far from conclusive. Alternatively, one can measure the $4\pi$-phase periodicity in the Josephson current between two topological, effectively $p$-wave superconducting wires separated by a semiconducting interface [20,23,24,36,49–52]. However, just as for zero-bias peaks, the $4\pi$ periodicity is only a necessary, but not sufficient condition for Majoranas [36]. Furthermore, experimental issues arise from the critical dependence on the conservation of parity, and thus on the absence of quasiparticle poisoning during the phase sweep. While difficult under typical equilibrium conditions, measurement of the response to a phase changing on timescales faster than the typical poisoning time [23,24,53] requires that the $4\pi$ periodicity is not caused by Landau-Zener...
transitions between topologically trivial subgap states. Finally, most recent advances into entropy measurements of nanoscale systems [54] promise to allow [55] to distinguish regular zero-energy bound states from Majoranas by accessing their fractional entropy [55–58], \( \Delta S = (k_B/2) \ln(2) \).

From a practical point of view, finding evidence of Majoranas in devices consisting of several nanowires that couple via quantum dots (QDs) or Coulomb islands is of particular interest [25,32,52,59–70]. Namely, such systems form the basic building blocks in many braiding and quantum-computation related proposals [40,43–46,71,72]. Measurable signatures of Majoranas hybridizing with the dot/junction states in these setups are both in the subgap spectra [67,70,73–76] and the equilibrium supercurrent [73,77–81]. Our main focus here is on whether and how the Majoranas interfere with the formation of so-called \( \pi \) junctions [82–84], that is, the supercurrent sign reversal concomitant with parity flips in a QD that connects two superconducting wires as in Fig. 1. Specifically, it was suggested that a notable absence of such sign changes could be a signature of Majoranas in the wires [77,79,81], both in the absence and presence of quasiparticle poisoning, including the case of conserved time-reversal symmetry [77].

In this paper, we more closely examine the case of a B-field-induced, time-reversal symmetry-breaking topological phase in grand-canonical equilibrium [81], explicitly accounting for quasiparticle poisoning. Performing numerical analyses of the Josephson current through the system displayed in Fig. 1, we consider local Coulomb interaction and a tunable potential \( \epsilon \) in the dot, theoretically allowing us to change the dot occupation one by one for arbitrary superconducting phase differences \( \delta \phi = \phi_L - \phi_R \) and parallel magnetic fields \( B \). We find that steplike sign reversals of the supercurrent (formation of \( \pi \) junctions) indeed disappear in the presence of Majoranas. However, as we elucidate with the help of effective low-energy models, this can be solely due to zero-energy wire states hybridizing with the dot, and thereby preventing total parity changes in the set of subgap eigenstates mediating most of the supercurrent. Topologically trivial subgap states in the wires as discussed in, e.g., Refs. [33–35] could hence have the same effect.

II. MODEL AND METHOD

The system of interest is modeled in Fig. 1. Two approximately one-dimensional nanowires \( L \) and \( R \) (length \( l_w \)) with Rashba spin-orbit coupling and proximity-induced \( s \)-wave superconductivity are connected by a confined junction of length \( l \) that forms a QD. The setup is assumed to be embedded in Rashba spin-orbit coupling and proximity-induced \( s \)-wave superconducting lead coupled to the central junction, see Fig. 1. The system of interest is modeled in Fig. 1. Two approximatel
calculate the averages \( \langle \ldots \rangle_{eq} \) numerically using the equation-of-motion method for Matsubara Green’s functions, see Appendix.

The transmission coefficient of the normal lead is obtained by approximating the latter as a 1D chain with a single site weakly and energy-independently coupled to only one site in the center of the normal junction, as suggested by the peaked shape of the lead in Fig. 1. The relevant \( 4 \times 4 \)-subblock \( S \) of the scattering matrix is determined by the Mahaux-Weidenmüller formula [89], see also comment at the end of Appendix.

\[
S = 1 - 2 \pi i W[U - H + i \pi W W^\dagger]^{-1} W. \tag{3}
\]

The couplings \( \delta_{x,y} = \text{Kronecker delta} \)

\[
(W)_{\text{mp,q,s'}} = -\eta \sqrt{\rho}/\pi \delta_{m,q} \delta_{s,s'} (t_{m} \delta_{\sigma\sigma'} - \sigma J_{0} \delta_{\sigma\sigma'}) \tag{4}
\]

are determined by the direct and spin-flip tunneling amplitudes \( t_{m}, J_{0} \) at which the particles \( (\eta = +)/\text{holes} \) \( (\eta = -) \) of spin \( \sigma = \uparrow/\downarrow \) at site \( m \) in the wire-junction-wire system couple to any of the four states associated with particles \( (\eta' = +)/\text{holes} \) \( (\eta' = -) \) of spin \( \sigma' \) at the first site of the normal lead (Fig. 1). The lead density of states \( \rho \) is assumed to be energy independent (wide band limit). The transmission coefficient is finally obtained as

\[
T(E) = 2 - \text{Tr}[S_{e}^\dagger S_{e} + \text{Tr}[S_{h}^\dagger S_{h}]]. \tag{5}
\]

where the \( 2 \times 2 \) matrices \( S_{e} \) and \( S_{h} \) contain all elements of \( S \) representing electrons from the normal lead respecting as electrons \( (ee) \) or holes \( (he) \) back into the lead.

To understand the behavior of \( I \) and \( T \), we relate them to the fermion parity \( p_{GS} = (-1)^{N} \) in the ground state (GS) of \( H \), calculated using the Pfaffian as in Ref. [20]. Moreover, we compare the system in the topological regime to a simplified Hamiltonian \( H_{t} \) in which the (quasi)continuum of the wires for energies \( |E| > \Delta \) is projected out. As such, \( H_{t} \) includes one Majorana for each wire, \( \gamma_{L} \) and \( \gamma_{R} \), and a single fermionic orbital \( f_{s} \), \( f_{p} \) with majority/minority spin \( \sigma = \pm \), corresponding to the direction antiparallel \( (+) \) and parallel \( (-) \) to the wire. For simplicity, we assume the Majoranas to be polarized antiparallel to the external field in this simple model. The dot is subject to strong onsite Coulomb interaction \( U/\Delta \gg 1 \) and couples via spin-conserving and spin-flip tunneling to both Majoranas. Introducing \( n_{s} = d_{s}^\dagger d_{s} \) and the nonlocally fused fermion \( f_{M} = (\gamma_{L} - i\gamma_{R})/\sqrt{2} \) with \( n_{M} = f_{M}^\dagger f_{M} \), the three-body Hamiltonian reads

\[
H_{t} = \sum_{s=\pm} (\epsilon - \sigma B \sin(\theta)) n_{s} + Un_{s}n_{-.}
+ \sin\left(\frac{\delta\phi}{4}\right)[f_{M}^\dagger(t_{M}f_{M} + J_{0}f_{p}) + \text{H.c.}]
+ \cos\left(\frac{\delta\phi}{4}\right)[if_{M}^\dagger(t_{M}f_{M} - J_{0}f_{p}) + \text{H.c.}]. \tag{6}
\]

The low-energy projection of the equilibrium supercurrent Eq. (2) is given by

\[
I_{C} = \text{Re}\left[\exp\left(\frac{i}{4}\delta\phi \right)(t_{M}f_{M}^\dagger - J_{0}f_{p})(f_{M} + f_{M})_{eq}\right]. \tag{7}
\]

To qualitatively compare this to the current \( I \) in the simulation [Eq. (2)], \( \delta\phi \) and \( \eta_{i} \) entering the couplings \( t_{M} = 1/(2m^{*}d_{M}^{2}) \), \( J_{0} = \eta_{i}/(2d_{M}) \) are set such that the peaks in \( I_{C} \) as a function of \( \delta\phi \) and \( \eta_{i} \) for fixed Zeeman energy \( B \) and \( U \) deviate \( \lesssim 1\% \) in height from those of \( I \). Furthermore, for a qualitative assessment of the mean-field treatment of the interaction in the simulations, we evaluate the grand-canonical ensemble averages \( \langle \ldots \rangle_{eq} \sim e^{-H_{t}/T} \) with the two-particle dot-on-site interaction \( \sim U > 0 \) fully accounted for, exploiting that \( H_{t} \) decouples into two \( 4 \times 4 \) blocks of opposite fermion parity. Note, however, that \( H_{t} \) does not capture the internal spin-orbit coupling affecting hopping within the normal region in the simulation. This leads to an underestimated magnitude of minority-spin tunneling and to a smaller level splitting compared to the simulation when choosing \( SB = U/2 \) and \( B_{L} = B \). We compensate this by enhancing \( J_{0} \) with a factor \( \alpha_{i} > \alpha \) and by setting \( \delta B < U/2 \).

### III. RESULTS

#### A. Supercurrent and parity flips

Figure 2(a) compares the equilibrium supercurrent \( I(\delta\phi, \epsilon) \) obtained from the simulation in the topological regime \( B > B_{C} \) to the one in the trivial regime \( B < B_{C} \). Most noticeable in the latter are sharp steps between positive and negative currents as a function of \( \epsilon \) for constant \( \delta\phi \). These well-understood \( \pi \)-phase shifts of the supercurrent [82–84] directly coincide with flips of the fermion parity in the junction, reflecting also in the total ground-state parity \( p_{GS} \). Given a splitting \( \sim 2(B + \delta B) \) due to both magnetic field and interaction, the resonances of the two levels with the chemical potential define a \( \epsilon \)-interval in which this parity is odd, and the current sign is inverted. By striking contrast, in the topological regime \( B > B_{C} \) there is no sign change as a function of \( \epsilon \), and instead we find two peaks and a more abrupt, nonsinusoidal sign change at phase \( \delta\phi = \pi \). The ground-state parity \( p_{GS} \) still flips at two different levels \( \epsilon \), but not anymore close to the current peak positions.

The question when and why flips of which parities correspond to a rapid sign change in the supercurrent can be answered with the general current expression

\[
I = 2\left[\frac{\partial H}{\partial \delta\phi}\right]_{eq} \rightarrow -\sum_{\epsilon_{i} > 0} \tanh\left(\frac{\epsilon_{i}}{2T}\right) \frac{\partial \epsilon_{i}}{\partial \delta\phi}. \tag{8}
\]

that holds within the mean-field treatment of the Coulomb interaction in the dot adopted here. The \( \epsilon_{i} \) are the quasiparticle energies with respect to the chemical potential \( \mu \), i.e., the eigenvalues of the Bogoliubov-de Gennes Hamiltonian \( H \). These always come in pairs \( \epsilon_{i,\pm} \) with opposite signs, \( \epsilon_{i,+} = -\epsilon_{i,-} \), representing particles and holes with opposite phase derivatives, \( \partial \epsilon_{i}/\partial \delta\phi = -\partial \epsilon_{i,-}/\partial \delta\phi \). Given a grand-canonical ensemble at low temperatures, the parity of a single state \( s \) flips if its energies \( \epsilon_{i,\pm} \) as a function of the system
parameters cross zero and change sign. Since Eq. (8) sums only over non-negative energies, the contribution of the state $s$ then swaps between the one from $\epsilon_{s,+}$ and from $\epsilon_{s,-}$, thereby leading to a sign change due to the opposite sign of $\partial\epsilon_s/\partial\delta \phi$. This, in any case, results in a (temperature broadened) step in the supercurrent. If the particular state $s$ gives the dominant contribution to $I$, meaning it has a relatively large phase derivative, this steplike transition even causes an overall sign change of $I$.

The main point is now that the energies with the largest phase derivative belong to the subgap states localized close to the dot-wire interfaces. In Fig. 2(c), we plot the $\epsilon$ dependence of these subgap-state energies as probed by the transmission $T(E)$ obtained from Eq. (3). In the trivial regime $B = 0$, there are two Yu-Shiba-Rusinov subgap states [90–92] forming in a dot with effective charging energy larger than the gap, $2\delta B = 6\Delta \gg \Delta$. These states cross zero at the two levels $\epsilon$ at which Fig. 2(a) indicates parity flips with concomitant supercurrent sign changes. In the topological regime $B > B_C$, the visible energies in Fig. 2(c) do not cross zero anymore, and instead form a diamond shape. This shape has previously been highlighted in the context of how Majorana fermions and their nonlocality influence the quasiparticle spectra [67,70]. Here, Figs. 2(b) and 2(c) show that these states provide the dominant current contribution and do not flip parity as a function of $\epsilon$, thereby also leading to an absent sign change in the supercurrent.

The abrupt current sign flip at $\delta \phi = \pi$ has previously been addressed in, e.g., Ref. [73]. It stems from the fact that the Majoranas in the wire decouple from the dot at phase difference $\delta \phi = \pi$, typically causing a zero-energy crossing and thus a parity flip. As we see in the next section, Sec. III B, this parity flip is localized to the Majoranas close to the dot. Moreover, as the constant ground-state parity $p_{GS}$ around $\delta \phi = \pi$ suggests, it is compensated by another parity flip of the Majoranas at the outer ends of the wires, which for any finite wire length have a strongly suppressed, yet not exactly vanishing phase dependence.

As further illustrated in the next section, Sec. III B, the absence of steplike $\epsilon$ dependencies (leading to current sign changes) in the topological phase can be explained by the effect of hybridization and level repulsion. In the trivial regime, the wire states are sufficiently far away from $E = 0$ on the scale of the hybridization with the dot. The dot levels increasing linearly with $\epsilon$ around $E = 0$ can therefore cross 0 without being level-repelled, and this crossing leads to a parity flip and a current sign change in the grand-canonical equilibrium. In the topological regime, two of the altogether four Majorana modes are, however, close to the dot, and therefore hybridize enough with the dot to repel the dot levels from zero energy. The total ground-state parity $p_{GS}$ still changes, but only due to the parity change in the weakly phase-dependent states formed by the Majoranas at the outer ends of the wires, which, as stated above, are practically irrelevant for the supercurrent by Eq. (8). This also explains why, unlike for the trivial regime, the level positions at which the $p_{GS}$ flips in the lower-right panel of Fig. 2(a) are located do not coincide with the current peaks in the topological regime.
Our main finding is thus that Majorana modes hybridizing with the QD formed by the normal junction prevents parity flips in and around the dot through level repulsion, thereby avoiding abrupt supercurrent sign changes as a function of the dot level. We, however, stress and show in the following section, Sec. III B, that this is only related to the existence of robust zero energy states at the dot-wire boundary, an absence of a sign change or any steplike transition with or without overall sign change does not imply the existence of Majoranas.

B. Low-energy approximation

We now turn to low-energy approximations to get a better physical picture of the current-carrying subgap states in both the trivial and topological regime. We start with the latter, topological case, which is captured by the Hamiltonian Eq. (6); important expectation values are plotted in Fig. 3. Parameters are set to $d_l = 102.5$ nm, $\alpha_l = 1.6\alpha$, $U = 8.6\Delta$, $\epsilon_0 = -4\Delta$, $m^* = 0.026m_e$, $\alpha = 16$ meV nm, $\Delta = 0.2$ meV, $\mu = 0.4$ meV, $T = 0.1$ K $\approx 0.043\Delta$.

![Fig. 3.](image1.png)

FIG. 3. Equilibrium supercurrent $I_e$, equilibrium parity $\langle p_+ \rangle_{eq}$, and equilibrium occupations $\langle n_+ \rangle_{eq}, \langle n_M \rangle_{eq}$ of the simplified model Eq. (6) as a function of phase difference $\delta \phi$ and dot level $\epsilon$. Parameters are set to $d_l = 102.5$ nm, $\alpha_l = 1.6\alpha$, $U = 8.6\Delta$, $\epsilon_0 = -4\Delta$, $m^* = 0.026m_e$, $\alpha = 16$ meV nm, $\Delta = 0.2$ meV, $\mu = 0.4$ meV, $T = 0.1$ K $\approx 0.043\Delta$.

The logic is thus that while the presence of an abrupt sign change as a function $\epsilon$ indicates the nonexistence of robust zero energy states at the dot-wire boundary, an absence of a sign change or any steplike transition with or without overall sign change does not imply the existence of Majoranas.

In Fig. 4, we plot both the supercurrent $I$ and the spectrum of the Bogoliubov-de Gennes Hamiltonian $H$ as function of $\epsilon$ and $\delta \phi$ for $B = 0$ and $B = B_0$ with $2t - \mu \approx -0.6\Delta \neq 0$. Without magnetic field, we find — similarly to the case
from left to right in our system. The symbols $V_j = V(x_j)$, $B_j = B(x_j)$, $\Delta_j = \Delta(x_j)$, $\phi_j = \phi(x_j)$ denote, respectively, the corresponding potential, Zeeman field, superconducting gap and superconducting phase at these points, as given in Fig. I and in Sec. II. The sets of points $x_j$ belonging to the left wire, dot, and right wire are determined by the total number of sites $M$ weighted by the corresponding length ratios.

For more efficient numerical computations, we switch to the Majorana basis by introducing

$$
\gamma_{j\sigma} = \frac{\delta_{j+} - i\delta_{j-}}{\sqrt{2}} (e^{-i\pi/2} c_{j\sigma} + \eta e^{i\pi/2} c_{j\sigma}^\dagger). \tag{A2}
$$

The operators Eq. (A2) are Hermitian, $\gamma_{j\sigma}^\dagger = \gamma_{j\sigma}$ and fulfill the anticommutation relations $\{\gamma_{j\sigma}, \gamma_{j'\sigma'}\} = \delta_{jj'}\delta_{\sigma\sigma'}$. The back transform to regular creation and annihilation operators reads

$$
c_{j\sigma} = \frac{1}{\sqrt{2}} (\gamma_{j+\sigma} + i\gamma_{j-\sigma}) e^{i\pi/2}. \tag{A3}
$$

We define the imaginary-time, Matsubara Green’s function for the Majoranas Eq. (A3) as

$$
G_{j\sigma, j'\sigma'}(\tau, \tau') = -\text{Tr}[\tau \gamma_{j\sigma}(\tau)\gamma_{j'\sigma'}(\tau')]\rho_{\text{eq}}. \tag{A4}
$$

where $\rho_{\text{eq}} = e^{-\hat{H}/\beta}/\text{Tr}[e^{-\hat{H}/\beta}]$ is the equilibrium state, $\tau$ is the time-ordering operator that shifts operators at larger $\tau$ further to the left, and $\gamma_{j\sigma}(\tau) = e^{i\hat{H}\tau} \gamma_{j\sigma} e^{-i\hat{H}\tau}$.

The Majorana equilibrium average is thus

$$
\langle \gamma_{j\sigma}, \gamma_{j'\sigma'} \rangle_{\text{eq}} = -G_{j\sigma, j'\sigma'}^{0}(\tau, \tau) := -\lim_{\tau \to \beta} G_{j\sigma, j'\sigma'}(\tau, 0). \tag{A5}
$$

The equilibrium averages entering the current Eq. (2) then follow from Eq. (A3):

$$
\langle c_{j\sigma} c_{j'\sigma'} \rangle_{\text{eq}} = -\frac{1}{2} \left[ G_{j+\sigma, j'+\sigma'}^{0} + G_{j-\sigma, j'-\sigma'}^{0} + t G_{j+\sigma, j'-\sigma'}^{0} - t G_{j-\sigma, j'+\sigma'}^{0} \right] e^{i\pi/2}. \tag{A6}
$$

Since the Hamiltonian Eq. (A1) is quadratic in the fields, the full matrix of Green’s functions $G(\tau, 0)$ and hence $G^{0}$ (i.e., $G_{j\sigma, j'\sigma'}(\tau, 0) = (G^{0})_{j\sigma, j'\sigma'}(\tau, 0)$) can be obtained from the equation of motion

$$
\frac{d}{d\tau} G_{j\sigma, j'\sigma'}(\tau, 0) = -\delta(\tau) \delta_{jj'} \delta_{\sigma\sigma'}
$$

$$
- \frac{d}{d\tau} \left[ \sum_{j=1}^{M} \sum_{\sigma=\pm, \tilde{\sigma}=\pm} \mathcal{H}_{j\sigma, j\tilde{\sigma}} \cdot G_{j\tilde{\sigma}, j'\sigma'}(\tau, 0), \tag{A7}
\right]
$$

with the matrix $\mathcal{H}$ identified from the commutator of the fields with the Hamiltonian, which has the general form

$$
[\gamma_{j\sigma}, \hat{H}] = \sum_{j=1}^{M} \sum_{\sigma=\pm, \tilde{\sigma}=\pm} \mathcal{H}_{j\sigma, j\tilde{\sigma}} \cdot \gamma_{j\tilde{\sigma}}. \tag{A8}
$$
Equation (A7) is solved by first solving the corresponding algebraic equation in the Matsubara frequency domain \((d/d\tau \rightarrow -i\omega_n)\) and then by back transforming to the matrix of Green’s functions \(G^0\) at \(\tau \rightarrow 0_+\):

\[
G^0 = T \sum_{n=-\infty}^{\infty} \frac{1}{i\omega_n - \mathcal{H}}, \quad \omega_n = T\pi(2n + 1), \quad n \in \mathbb{Z}.
\]

(A9)

Numerical simplifications now arise from the Hermiticity of the Majoranas Eq. (A2), leading to \(\mathcal{H} = i\tilde{\mathcal{H}}\) with \(\text{Im} \tilde{\mathcal{H}} = 0\) and \(\tilde{\mathcal{H}}^T = -\mathcal{H}\) (\(\bullet^T\) = transposition). Since inversion and transposition commute, and since every positive frequency in the sum in Eqs. (A9) has a corresponding negative frequency with equal absolute value, we find

\[
G^0 = -iT \tilde{G}^0, \quad \tilde{G}^0 = \sum_{n=0}^{\infty} \left[ \frac{1}{\omega_n - \tilde{\mathcal{H}}} - \left( \frac{1}{\omega_n - \mathcal{H}} \right)^T \right].
\]

(A10)

The matrix inversion in Eqs. (A10) is performed only once for positive frequencies, and only on a purely real matrix, which is numerically faster than the complex matrix inversion in Eqs. (A9) for both positive and negative frequencies. Moreover, since \(\tilde{\mathcal{H}}\) only couples nearest-neighbor sites, \((\omega_n - \tilde{\mathcal{H}})\) has a simple block-band structure,

\[
\mathcal{R} = \omega_n - \tilde{\mathcal{H}} = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & \cdots & \cdots & \cdots \\
\cdots & 0 & \mathcal{R}_{c,j}^T & \mathcal{R}_{d,j} & -\mathcal{R}_{c,j} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \mathcal{R}_{c,j+1}^T & \mathcal{R}_{d,j+1} & -\mathcal{R}_{c,j+1} & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & \mathcal{R}_{c,j}^T & \mathcal{R}_{d,j+1} & -\mathcal{R}_{c,j+1} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix},
\]

(A11)

with the \(4 \times 4\) subblocks,

\[
\mathcal{R}_{d,j} = \begin{pmatrix}
\omega_n & \mu - 2\tau - V_j & 0 & \Delta_j - B_j \\
2\tau - \mu + V_j & \omega_n & \Delta_j + B_j & 0 \\
0 & -\Delta_j - B_j & \omega_n & \mu - 2\tau - V_j \\
B_j - \Delta_j & 0 & 2\tau - \mu + V_j & \omega_n
\end{pmatrix}
\]

(A12)

and

\[
\mathcal{R}_{c,j} = \begin{pmatrix}
t_{j-} & -t_{j+} & -J_{j-} & J_{j+} \\
t_{j+} & t_{j-} & -J_{j+} & -J_{j-} \\
J_{j-} & -J_{j+} & t_{j-} & -t_{j+} \\
J_{j+} & J_{j-} & t_{j+} & t_{j-}
\end{pmatrix}
\]

(A13)

in the Majorana basis (+↑, −↑, +↓, −↓), where

\[
t_{j\pm} = t \cos \left( \frac{\phi_j - \phi_{j+1}}{2} + \frac{\pm \pi - \pi}{4} \right), \quad J_{j\pm} = J \cos \left( \frac{\phi_j - \phi_{j+1}}{2} + \frac{\pm \pi - \pi}{4} \right).
\]

(A14)

Due to this block-band structure Eq. (A11), the \(4 \times 4\) subblocks of the full inverses in \(\tilde{G}_0 = \sum_n (\omega_n - \tilde{\mathcal{H}})^{-1}\) required to evaluate the averages in the current formula Eq. (2), using Eqs. (A6) and (A10) can be efficiently determined by iteratively applying the block-inversion scheme,

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.
\]

(A15)

with matrices \(A, B, C, D\). Finally, we also use the Majorana basis Eq. (A2) and the block inversion Eq. (A15) for

\[
\frac{1}{E - \mathcal{H} + i\pi WW^\dagger} = i \frac{\tilde{\mathcal{H}} - \mathcal{H}}{iE + \mathcal{H} - \pi WW^\dagger}
\]

(A16)

in Eq. (3). Due to the broadening \(i\pi WW^\dagger\) from the normal lead, the matrix to invert is not real valued, but the result for \(-E\) is simply obtained from Eq. (A16) for \(+E\) by complex conjugation, as \(\text{Im}(WW^\dagger) = 0\) both in the particle-hole [Eq. (4)] and Majorana basis.
[61] Y. Cao, P. Wang, G. Xiong, M. Gong, and X.-Q. Li, Phys. Rev. B 86, 115311 (2012).

[62] M. Lee, J. S. Lim, and R. López, Phys. Rev. B 87, 241402(R) (2013).

[63] E. Vernek, P. H. Penteado, A. C. Seridonio, and J. C. Egues, Phys. Rev. B 89, 165314 (2014).

[64] D. A. Ruiz-Tijerina, E. Vernek, L. G. G. V. Dias da Silva, and J. C. Egues, Phys. Rev. B 91, 115435 (2015).

[65] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. M. Marcus, Nature 531, 206 (2016).

[66] S. Hoffmann, D. Chevallier, D. Loss, and J. Klinovaja, Phys. Rev. B 96, 045440 (2017).

[67] E. Prada, R. Aguado, and P. San-Jose, Phys. Rev. B 96, 085418 (2017).

[68] D. J. Clarke, Phys. Rev. B 96, 201109(R) (2017).

[69] D. Chevallier, P. Szumniak, S. Hoffmann, D. Loss, and J. Klinovaja, Phys. Rev. B 97, 045404 (2018).

[70] M.-T. Deng, S. Vaitiekėnas, E. Prada, P. San-Jose, J. Nygård, P. Krogstrup, R. Aguado, and C. M. Marcus, Phys. Rev. B 98, 085125 (2018).

[71] B. M. Terhal, F. Hassler, and D. P. DiVincenzo, Phys. Rev. Lett. 108, 260504 (2012).

[72] J. D. Sau and S. Das Sarma, Nat. Commun. 3, 964 (2012).

[73] J. Cayao, A. M. Black-Schaffer, E. Prada, and R. Aguado, Beilstein J. Nanotechnol. 9, 1339 (2018).

[74] E. C. T. O’Farrell, A. C. C. Drachmann, M. Hell, A. Fornieri, A. M. Whiticar, E. B. Hansen, S. Gronin, G. C. Gardner, C. Thomas, M. J. Manfra, K. Flensberg, C. M. Marcus, and F. Nichele, Phys. Rev. Lett. 121, 256803 (2018).

[75] L. S. Ricco, M. de Souza, M. S. Figueira, I. A. Shelykh, and A. C. Seridonio, Phys. Rev. B 99, 155159 (2019).

[76] K. Yavilberg, E. Ginossar, and E. Grosfeld, Phys. Rev. B 100, 241408 (2019).

[77] A. Camjayi, L. Arrachea, A. Aligia, and F. von Oppen, Phys. Rev. Lett. 119, 046801 (2017).

[78] J. D. Sau and S. Das Sarma, Nat. Commun. 3, 964 (2012).

[79] J. Cayao, A. M. Black-Schaffer, E. Prada, and R. Aguado, Beilstein J. Nanotechnol. 9, 1339 (2018).

[80] C. Mahaux and H. A. Weidenmüller, Phys. Rev. 170, 847 (1968).

[81] L. Yu, Acta Phys. Sin. 114, 75 (1965).

[82] H. Shiba, Prog. Theor. Phys. 40, 435 (1968).

[83] A. I. Rusinov, ZhETF Pis. Red. 9, 146 (1969) [JETP Lett. 9, 85 (1969)].