Four-point functions in $N = 4$ supersymmetric Yang-Mills theory at two loops

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Abstract

Four-point functions of gauge-invariant operators in $D = 4, N = 4$ supersymmetric Yang-Mills theory are studied using $N = 2$ harmonic superspace perturbation theory. The results are expressed in terms of differential operators acting on a scalar two loop integral. The leading singular behaviour is obtained in the limit that two of the points approach one another. We find logarithmic singularities which do not cancel out in the sum of all diagrams. It is confirmed that Green’s functions of analytic operators are indeed analytic at this order in perturbation theory.

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1 Introduction

Supersymmetric Yang-Mills (SYM) theory with maximally extended \((N = 4)\) supersymmetry in four dimensions has long been known to have some very interesting properties. In particular, it is ultra-violet finite and hence superconformally invariant quantum mechanically \([1]\), it admits monopole solutions which fall into spin one multiplets \([2]\) and it exhibits Montonen-Olive duality \([3, 4]\). More recently, renewed interest in this theory as a superconformal field theory (SCFT) has been stimulated by the Maldacena conjecture via which it is related, in a certain limit, to IIB supergravity (SG) on \(AdS_5 \times S^5\) \([5]\). In this paper we shall study some \(N = 4\) SYM four-point correlation functions of gauge-invariant operators at two loops in perturbation theory. The motivation for this study is threefold: firstly, from a purely field-theoretic point of view, complete two-loop four-point calculations in four-dimensional gauge theories are not commonplace; secondly, it is of interest to see if, even qualitatively, there is any sign of agreement with supergravity calculations \([6, 7]\), and, thirdly, it provides a check in perturbation theory of the assumption of references \([8]\) that correlation functions of analytic operators are indeed analytic in the quantum theory.

The main results of the paper concern the evaluation of four-point functions of gauge-invariant bilinears constructed from \(N = 2\) hypermultiplets in the adjoint representation of the gauge group \(SU(N_c)\). (We recall that the \(N = 4\) SYM multiplet splits into the \(N = 2\) SYM multiplet and a hypermultiplet in the adjoint representation.) The calculation of two-loop four-point amplitudes is, by present standards, still a difficult task in any field theory, except for quantities which depend only on the ultra-violet divergent parts of diagrams, such as renormalisation group functions. Very few exact result have been obtained for the finite parts of such amplitudes, the notable exception being the case of gluon-gluon scattering in \(N = 4\) SYM theory for which the two-loop on-shell amplitude has been calculated in terms of the scalar double box integrals \([9]\).

Our calculation is carried out in the \(N = 2\) harmonic superspace formalism \([10, 11]\) and we arrive at answers expressed in terms of a second-order differential operator acting on the standard \(x\)-space scalar two-loop integral. From our results we are then able to deduce the asymptotic behaviour of the four-point functions as two of the points approach each other. In this limit the behaviour of the leading singularity can be found explicitly in terms of elementary functions. We find that it has the form \((x^2)^{-1}\ln x^2\) where \(x\) is the coordinate of the difference between the two coinciding points. This behaviour, although obtained in perturbation theory, is in line with the behaviour reported on in references \([6, 7]\) where some progress towards computing four-point functions in supergravity has been made. We also find purely logarithmic next-to-leading terms.

For the case of three-point functions it is possible to check the Maldacena conjecture by comparing SG results with perturbation theory for SCFT, because the higher loop corrections to the leading terms turn out to vanish for the three-supercurrent correlator and vanish at least to first non-trivial order for other correlators \([13, 14, 15]\). However, it is more difficult to compare SG results with perturbative SCFT for higher point functions. One technical difficulty that one encounters is that the easiest functions to compute in SCFT are the leading scalar terms of the four-point function of four supercurrent operators, whereas, on the SG side, it is easiest to compute amplitudes involving the axion and dilaton fields. In section 2 we shall show how one may compute the leading term of the \(N = 4\) four-point function from the three \(N = 2\) hyper-multiplet four-point functions that are considered in perturbation theory in section 4. On the grounds that there are no nilpotent superinvariants of four points \([5]\) this means that we can, in principle, determine the entire \(N = 4\) four-point correlation function from the three \(N = 2\) hy-
permultiplet correlation functions. Furthermore, we can also compute the four-point correlation functions of $N = 2$ operators constructed as bilinears in the $N = 2$ SYM field-strength tensor superfield. The leading terms of the non-vanishing amplitudes of this type are related to the hypermultiplet correlation functions due to their common origin in $N = 4$. In section 3 we then use a superinvariant argument to show how the $N = 2$ YM correlators can be constructed in their entirety from the leading terms. It is the highest components of these correlation functions in a $\theta$-expansion which correspond to the SG amplitudes currently being investigated.

In the $N = 4$ harmonic superspace approach to SYM one can construct a family of gauge-invariant operators which are described by single-component analytic superfields. These fields depend on only half the number of odd coordinates of standard $N = 4$ superspace and, in addition, depend analytically on the coordinates of an auxiliary bosonic space, the coset space $S(U(2) \times U(2))/SU(4)$, which is a compact complex manifold. Moreover, this family of operators is in one-to-one correspondence with the KK states of IIB supergravity compactified on $AdS_5 \times S^5$. In references [8] it has been argued that one might hope to get further constraints on correlation functions of these operators by using superconformal invariance and the assumption that analyticity is maintained in the quantum theory. However, this assumption is difficult to check directly in the $N = 4$ formalism because it is intrinsically on-shell. In a recent paper [15] analyticity was checked for certain three-point functions using $N = 2$ harmonic superspace (which is an off-shell formalism). The four-point functions computed in the current paper also preserve analyticity thereby lending further support to the $N = 4$ analyticity postulate.

The organisation of the paper is as follows: in the next section we shall show how the leading, scalar term of the $N = 4$ four-point correlation function can be determined from three $N = 2$ four-point functions; following this, we show that knowledge of these functions is also sufficient to determine the $N = 2$ four-point function with two $W^2$ operators and two $\bar{W}^2$ operators, $W$ being the chiral $N = 2$ Yang-Mills field-strength multiplet, and we also show that the leading scalar term of this correlation function can be used, in principle, to determine it completely. In section 4 we present the $N = 2$ harmonic superspace calculations of the two hypermultiplet correlation functions at two loops in some detail. We then discuss the asymptotic behaviour of the integrals that occur in these computations in order to find out the leading singularities that arise when two of the points approach each other. The paper ends with some further comments. The appendix collects some known results on massless one-loop and two-loop integrals which we have found useful.

2 $N = 4$ in terms of $N = 2$

In this section we show how one can compute the leading scalar term of the $N = 4$ four-point function of four supercurrents from the leading scalar terms of three $N = 2$ hypermultiplet four-point functions. We recall that in standard $N = 4$ superspace the $N = 4$ field strength superfield $W_{IJ} = -W_{JI}$, $I, J = 1 \ldots 4$ transforms under the real six-dimensional representation of the internal symmetry group $SU(4)$ (where the $I$ index labels the fundamental representation), i.e. it is self-dual, $\bar{W}_{IJ} = \frac{1}{2} \epsilon_{IJKLM} W_{KL}$. This superfield satisfies the (on-shell) constraint

$$D^I_\alpha W^{JK} = D^{[I}_\alpha W^{JK]}$$  \hspace{1cm} (2.1)

where $D^I_\alpha$ is the superspace covariant derivative. Strictly speaking this constraint holds only
for an Abelian gauge theory and in the non-Abelian case a connection needs to be included. However, the constraints satisfied by the gauge-invariant bilinears that we shall consider are in fact the same in the Abelian and non-Abelian cases.

In order to discuss the energy-momentum tensor it is convenient to use an $SO(6)$ formalism. The field-strength itself can be written as a vector of $SO(6)$, $W^A$, $A = 1, \ldots, 6$,

$$W^A = \frac{1}{2} (\sigma^A)_{JK} W^{JK}$$

(2.2)

The sigma-matrices have the following components

$$(\sigma^a)_{bc} = 0 \quad (\sigma^a)_{b4} = \delta^a_b \quad (2.3)$$

where the small Latin indices run from 1 to 3, and where the $SO(6)$ and $SU(4)$ indices split as $A = (a, \bar{a})$ (in a complex basis) and $I = (a, 4)$. An upper (lower) ($\bar{a}$) index is equivalent to a lower (upper) $a$ index and vice versa. The sigma-matrices are self-dual,

$$(\bar{\sigma}^A)^{IJ} = \frac{1}{2} \epsilon^{IJKL} (\sigma^A)_{KL}$$

(2.4)

In terms of the $SU(3)$ indices $a, \bar{a}$ one has the decompositions,

$$W^A \rightarrow (W^a, W^{\bar{a}} = \bar{W}_a)$$

(2.5)

and

$$W^{IJ} \rightarrow \left(W^{ab} = \epsilon^{abc} \bar{W}_c, \ W^{a4} = W^a\right)$$

(2.6)

The leading component of $W^a$ in an expansion in the fourth $\theta$ can be identified with the $N = 3$ SYM field-strength tensor. Decomposing once more to $N = 2$ one finds that $W^a$ splits into the $N = 2$ field-strength $W$ and the $N = 2$ hypermultiplet $\phi^i$,

$$W^a \rightarrow (W^i \equiv \phi^i, W^3 \equiv W)$$

(2.7)

From (2.4) it is easy to see that these superfields (evaluated with both the third and fourth $\theta$ variables set equal to zero) do indeed obey the required constraints, that is, $W$ is chiral and

$$D^{(i} \phi^{j)} = 0$$

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(2.8)
In addition, at the linearised level, the SYM field-strength $W$ also satisfies the equation of motion, $D_\alpha^i D^{\alpha j} W = 0$, and this also follows from (2.1). The $N = 4$ supercurrent is given by

$$T^{AB} = W^A W^B - \frac{1}{6} \delta^{AB} W^C W^C$$

(2.9)

and the four-point function we are going to consider is

$$G^{(N=4)} = \langle T^{A_1 B_1} T^{A_2 B_2} T^{A_3 B_3} T^{A_4 B_4} \rangle$$

(2.10)

where the numerical subscripts indicate the point concerned. This function can be expressed in terms of $SO(6)$ invariant tensors multiplied by scalar factors which are functions of the coordinates. The only $SO(6)$ invariant tensor that can arise is $\delta$, and there are two modes of hooking the indices up in $G^{(N=4)}$ each of which can occur in three combinations making six independent amplitudes in all. Thus we have

$$G^{(N=4)} = a_1 (\delta_{12})^2 (\delta_{34})^2 + a_2 (\delta_{13})^2 (\delta_{24})^2 + a_3 (\delta_{14})^2 (\delta_{23})^2 + b_1 \delta_{13} \delta_{14} \delta_{23} \delta_{24} + b_2 \delta_{12} \delta_{14} \delta_{32} \delta_{34} + b_3 \delta_{12} \delta_{13} \delta_{42} \delta_{43}$$

(2.11)

where, for example,

$$(\delta_{12})^2 (\delta_{34})^2 = \delta_{\{A_1 B_1\}}^{A_2 B_2} \delta_{\{A_3 B_3\}}^{A_4 B_4}$$

(2.12)

and

$$\delta_{13} \delta_{14} \delta_{23} \delta_{24} = \delta_{\{A_1 B_1\}}^{A_3 \{B_3 \}} \delta_{\{A_2 B_2\}}^{A_4 \{B_4 \}}$$

(2.13)

and where the brackets denote tracefree symmetrisation at each point.

In $N = 3$ the $N = 4$ supercurrent splits into two multiplets, the $N = 3$ supercurrent $T^{ab}$, and a second multiplet $T^{\bar{a} \bar{b}}$ which contains amongst other components the fourth supersymmetry current. The $N = 3$ supercurrent transforms under the real eight-dimensional representation of $SU(3)$ while the second multiplet transforms according to the complex six-dimensional representation. The $N = 4$ four-point function decomposes into several $N = 3$ four-point functions. Amongst them we find three which, when further decomposed under $N = 2$, will suffice to determine all the $a$ and $b$ functions in (2.12):

$$G_1^{(N=3)} = \langle T^{a_1 b_1} T^{a_2 b_2} T^{a_3 b_3} T^{a_4 b_4} \rangle$$

(2.14)

$$G_2^{(N=3)} = \langle T^{a_1 b_1} T^{\bar{a_2} \bar{b_2}} T^{a_3 b_3} T^{a_4 b_4} \rangle$$

(2.15)

$$G_3^{(N=3)} = \langle T^{a_1 b_1} T^{a_2 \bar{b_2}} T^{a_3 b_3} T^{a_4 \bar{b_4}} \rangle$$

(2.16)

In the $N = 2$ decomposition of $T^{ab}$ we find

\footnote{Here, and throughout the paper, bilinear expressions such as $WW$, $\phi\phi$ etc. are understood to include a trace over the Yang-Mills indices.}
\[ T^{ij} = \phi^i \phi^j \] (2.17)

while \( T^{33} = W^2 \). On the other hand, the \( N = 3 \) supercurrent contains another \( N = 2 \) hypermultiplet operator, also in a triplet of \( SU(2) \). This is obtained from \( T^{ab} \) by restricting the indices to run from 1 to 2 and then removing the trace over the \( i, j \) indices. In this way we can construct

\[ \hat{T}^{ij} = T^{ij} - \frac{1}{2} \delta_j^i T^{kk} = \phi^i \phi_j - \frac{1}{2} \delta_j^i \phi^k \phi_k \] (2.18)

The \( N = 2 \) harmonic superspace hypermultiplet \( q^+ \) is related to the \( N = 2 \) superfield \( \phi^i \) by

\[ q^+ = u_i^i \phi^i \] (2.19)
on-shell. Its conjugate \( \bar{q}^+ \) is given by

\[ \bar{q}^+ = u^{i+} \bar{\phi}_i \] (2.20)

where the details of \( N = 2 \) harmonic superspace are reviewed in section 4.

Restricting the indices on the three \( N = 3 \) four-point functions \( G^{(N=3)}_1 \) and \( G^{(N=3)}_2 \) to run from 1 to 2, removing the \( N = 2 \) traces at each point where necessary and multiplying the resulting functions by \( u_i^i u_j^j \) at each point we find three hypermultiplet four-point functions. The leading terms of these correlation functions are given in terms of the \( a \) and \( b \) functions introduced in equation (2.12) and it is a straightforward computation to show that, in the notation of section 4,

\[ G^{(N=2)}_1 \quad = \quad <q^+ \bar{q}^+ | q^+ \bar{q}^+ | q^+ \bar{q}^+ | q^+ \bar{q}^+ > 
\]

\[ = \quad (12)^2(34)^2 a_1 + (14)^2(23)^2 a_3 + (12)(23)(34)(41) b_2 \] (2.21)

while

\[ G^{(N=2)}_2 \quad = \quad <q^+ \bar{q}^+ | q^+ \bar{q}^+ | q^+ \bar{q}^+ | q^+ \bar{q}^+ > 
\]

\[ = \quad \frac{1}{4} \left( (12)^2(34)^2(-2a_1 - b_3) + (14)^2(23)^2 b_1 + (12)(23)(34)(41)(b_3 - b_1 - b_2) \right) \] (2.22)

and

\[ G^{(N=2)}_3 \quad = \quad <q^+ \bar{q}^+ | q^+ \bar{q}^+ | q^+ \bar{q}^+ | q^+ \bar{q}^+ > 
\]

\[ = \quad \frac{1}{8} \left( (12)^2(34)^2(2a_1 + 2a_2 + b_3) + (14)^2(23)^2(2a_2 + 2a_3 + b_1) + \right. \]
\[ + (12)(23)(34)(41)(b_2 - b_1 - b_3 - 4a_2) \] (2.23)

where, for example,
\[ (12) = u_1^{+i} u_2^{+j} \epsilon_{ij} \] 

(2.24)

In section 4 we shall see that each of these four-point functions can be written in terms of three functions of the coordinates, \( A_1, A_2, A_3 \), and hence all of the \( a \) and \( b \) functions that appear in the \( N = 4 \) four-point function can be determined by computing the above \( N = 2 \) correlators.

We now consider the \( N = 2 \) correlation functions involving the gauge-invariant operators \( W^2 \) and \( \bar{W}^2 \) constructed from the \( N = 2 \) SYM field-strength. The possible independent four-point functions of this type are

\[
\begin{align*}
G_{4}^{(N=2)} &= <W^2\bar{W}^2W^2\bar{W}^2> \\
G_{5}^{(N=2)} &= <W^2W^2W^2\bar{W}^2> \\
G_{6}^{(N=2)} &= <W^2W^2W^2W^2> 
\end{align*}
\]

(2.25)

The leading terms of \( G_{5}^{(N=2)} \) and \( G_{6}^{(N=2)} \) vanish as one can easily see by examining the leading terms of the \( N = 3 \) correlation functions from which they can be derived. In fact, it is to be expected that these correlation functions should vanish at all orders in \( \theta \). This can be argued from an \( N = 4 \) point of view from the absence of nilpotent \( N = 4 \) superinvariants, or directly in \( N = 2 \). For example, using just \( N = 2 \) superconformal invariance, it is possible to show that all correlation functions of gauge-invariant powers of \( W \) vanish.

We remark that the \( N = 2 \) four-point function \( G_{4}^{(N=2)} = <W^2\bar{W}^2W^2\bar{W}^2> \) is also obtainable from \( G_{2}^{(N=3)} \). In terms of the \( a \) and \( b \) functions it is given by

\[
< W^2\bar{W}^2W^2\bar{W}^2 > = a_1 + a_3 + b_2
\]

(2.26)

On-shell the \((\theta)^4\) component of \( W^2 \) is \( F_{\alpha\beta}F^{\alpha\beta} \) where \( F_{\alpha\beta} \) is the self-dual space-time Yang-Mills field-strength tensor, so that the top \((\theta)^{16}\) component of this correlation function is directly related to the dilaton and axion amplitudes in supergravity. Clearly any four-point function of the operators \( F_{\mu\nu}F^{\mu\nu} \) and \( \frac{1}{2} \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \) can be obtained from the above.

3 The \( N = 2 \) chiral-antichiral four-point function

In this section we show that the complete \( N = 2 \) four-point function \( G_{4}^{(N=2)} \) is determined by its leading term using only superconformal invariance. The coordinates of \( N = 2 \) superspace are \((x^{\alpha\dot{\alpha}}, \theta^i, \bar{\theta}^{\dot{i}})\) and an infinitesimal superconformal transformation in this space is given by a superconformal Killing vector field \( V = F^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} + \varphi_i^\alpha D_i^{\alpha} - \bar{\varphi}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}} \), where \( D_i^{\alpha} \) is the usual flat space supercovariant derivative. By definition, \( V \) satisfies the equation

\[
[D_i^{\alpha}, V] \equiv D_i^{\alpha}
\]

(3.27)

The chiral superfield \( W^2 \) transforms as

\[
\delta W^2 = VW^2 + \Delta W^2
\]

(3.28)
where $\Delta = \partial_{\dot{a}\dot{\alpha}} F^{\alpha\dot{\alpha}} - D^{\dot{a}}_\alpha \phi^{\alpha\dot{a}}$.

Since $G^{(N=2)}_4$ is chiral at points 1 and 3 and anti-chiral at points 2 and 4 it depends only on the chiral or anti-chiral coordinates at these points. These are given by

$$X^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + 2i \theta_i^\alpha \bar{\theta}^{\dot{i}\dot{\alpha}} \quad \text{chiral}$$

$$\bar{X}^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} - 2i \theta_i^\dot{\alpha} \bar{\theta}^{\dot{i}\alpha} \quad \text{anti - chiral}$$

At lowest order in Grassmann variables translational invariance in $x$ implies that $G^{(N=2)}_4$ depends only on the difference variables $x_r - x_s$, $r, s = 1 \ldots 4$, of which three are independent. Combining this with $Q$-supersymmetry one finds that $G^{(N=2)}_3$ depends only on the $Q$-supersymmetric extensions of these differences, which will be denoted $y_{rs}$, as well as the differences $\theta_{13} = \theta_1 - \theta_3$ and $\theta_{24} = \bar{\theta}_2 - \bar{\theta}_4$. The supersymmetric difference variables joining one chiral point ($r$) with one anti-chiral point ($s$) have the following form:

$$y_{rs} = X_r - X_s - 4i \theta_r \bar{\theta}_s$$

(3.30)

When the chirality of the two points is the same one has

$$y_{13} = X_1 - X_3 - 2i \theta_{13} (\bar{\theta}_2 + \bar{\theta}_4)$$

$$y_{24} = \bar{X}_2 - \bar{X}_4 + 2i (\theta_1 + \theta_3) \bar{\theta}_{24}$$

(3.31)

It is easy to find a free solution of the Ward Identity for $G^{(N=2)}_4$; it is given by

$$G^{(N=2)}_4 = \frac{1}{y_{14} y_{23}}$$

(3.32)

A general solution can be written in terms of this free solution in the form

$$G^{(N=2)}_4 = G^{(N=2)}_4 \times F$$

(3.33)

where $F$ is a function of superinvariants. At the lowest order, $F$ is a function of two conformal invariants which may be taken to be

$$S = \frac{x_{12} x_{24}}{x_{14} x_{23}^2}, \quad T = \frac{x_{13} x_{21}}{x_{14} x_{23}^2}$$

(3.34)

The strategy is now to show that these two conformal invariants can be extended to superconformal invariants and furthermore that there are no further superconformal invariants. Any new superinvariant would have to vanish at lowest order and would thus be nilpotent.

From the above discussion it is clear that $F$ can only depend on the $y_{rs}$ and the odd differences $\theta_{13}$ and $\bar{\theta}_{24}$. Furthermore, it has dilation weight and $R$-weight zero, the latter implying that it
depends on the odd coordinates in the combination $\theta_{13} \bar{\theta}_{24}$. This takes all of the symmetries into account except for $S$-supersymmetry and conformal symmetry. However, conformal transformations appear in the commutator of two $S$-supersymmetry transformations and so it is sufficient to check the latter. The $S$-supersymmetry transformations of the chiral-antichiral variables are rather simple:

$$\delta y^{\alpha \dot{\alpha}} = \theta^{\alpha}_{\alpha} \eta^{j}_{\beta} y^{\beta \dot{\alpha}}$$

(3.35)

The transformations of the (anti-)chiral-chiral variables are slightly more complicated and it is convenient to introduce new variables as follows:

$$\hat{y}_{13} = y_{12} + y_{23}$$

$$\hat{y}_{24} = -y_{12} + y_{14}$$

(3.36)

Under $S$-supersymmetry these transform as follows:

$$\delta \hat{y}^{\alpha \dot{\alpha}}_{13} = \theta^{\alpha}_{3} \eta^{j}_{\beta} y^{\beta \dot{\alpha}}_{13} + \theta^{\alpha}_{13} \eta^{i}_{\beta} y^{\beta \dot{\alpha}}_{12}$$

$$\delta \hat{y}^{\alpha \dot{\alpha}}_{24} = \theta^{\alpha}_{1} \eta^{j}_{\beta} y^{\beta \dot{\alpha}}_{24}$$

(3.37)

The transformations of the odd variables are

$$\delta \theta^{\alpha}_{i} = \theta_{j}^{\alpha} \eta^{j}_{\beta} \theta^{\beta}_{i}$$

$$\delta \bar{\theta}^{\alpha \dot{\alpha} i} = -i \eta^{i}_{\alpha} X^{\alpha \dot{\alpha}}$$

(3.38)

Using these transformations it is easy to extend $S$ to a superinvariant $\hat{S}$. It is

$$\hat{S} = \frac{y_{12}^{2} y_{24}^{2}}{y_{14}^{2} y_{23}^{2}}$$

(3.39)

However, the extension of the second invariant is slightly more complicated because it involves the (anti-)chiral-chiral differences. A straightforward computation shows that the required superinvariant is

$$\hat{T} = \frac{1}{y_{14} y_{23}} \left( \hat{y}_{13}^{2} \hat{y}_{24}^{2} - 16 i \theta_{13} \cdot \bar{\theta}_{24} \cdot (\hat{y}_{13} y_{12} \hat{y}_{24}) - 16 y_{12}^{2} (\theta_{13} \cdot \bar{\theta}_{24})^{2} \right)$$

(3.40)

Now consider the possibility of a nilpotent superinvariant. Under an $S$-supersymmetry transformation the leading term in the variation arises from the $\theta$-independent term in the variation of $\bar{\theta}$. On the assumption that the $x$-differences are invertible, it follows immediately that there can be no such invariants. Thus, the general form of the four-point function $G_{4}^{(N=2)}$ is

$$< W^{2} \bar{W}^{2} W^{2} \bar{W}^{2} > = \frac{1}{y_{14}^{2} y_{23}^{2}} F(\hat{S}, \hat{T})$$

(3.41)

and so is calculable from the leading term as claimed.
Computation of four-point $N = 2$ correlation functions

4.1 $N = 2$ harmonic superspace and Feynman rules

4.1.1

The $N = 4$ SYM multiplet reduces to an $N = 2$ SYM multiplet and a hypermultiplet. The latter are best described in $N = 2$ harmonic superspace \[10\]. In addition to the usual bosonic $(x^{\alpha\dot{\alpha}})$ and fermionic $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ coordinates, it involves $SU(2)$ harmonic ones:

$$SU(2) \ni u = (u^+_i, u^-_i) : u^-_i = u^+_i, \quad u^{+i}u^-_i = 1.$$  \hspace{1cm} (4.42)

They parametrise the coset space $SU(2)/U(1) \sim S^2$ in the following sense: the index $i$ transforms under the left $SU(2)$ and the index (“charge”) $\pm$ under the right $U(1)$; further, all functions of $u^\pm$ are homogeneous under the action of the right $U(1)$ group. The harmonic functions $f(u)$ are defined by their harmonic expansion (in general, infinite) on $S^2$.

The main advantage of harmonic superspace is that it allows us to define Grassmann-analytic (G-analytic) superfields satisfying the constraints

$$D^+_{\dot{\alpha}} \Phi(x, \theta, u) = \bar{D}^+_{\dot{\alpha}} \Phi(x, \theta, u) = 0 .$$ \hspace{1cm} (4.43)

Here

$$D^+_{\alpha, \dot{\alpha}} = u^+_i D^i_{\alpha, \dot{\alpha}}$$ \hspace{1cm} (4.44)

are covariant $U(1)$ harmonic projections of the spinor derivatives. The G-analyticity condition (4.43) is integrable, since by projecting the spinor derivative algebra

$$\{D^i_{\dot{\alpha}}, D^j_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}i}, \bar{D}_{\dot{\beta}j}\} = 0, \quad \{D^i_{\alpha}, \bar{D}^j_{\dot{\beta}}\} = -2i \delta^j_i \partial_{\alpha\dot{\beta}}$$ \hspace{1cm} (4.45)

one finds

$$\{D^+_\alpha, D^+_\beta\} = \{\bar{D}^+_\alpha, \bar{D}^+_\beta\} = \{D^+_\alpha, \bar{D}^+_\beta\} = 0 .$$ \hspace{1cm} (4.46)

Moreover, the G-analyticity condition (4.43) can be solved explicitly. To this end one introduces a new, analytic basis in harmonic superspace:

$$x^{\alpha\dot{\alpha}}_A = x^{\alpha\dot{\alpha}} - 2i \theta^\alpha (i \bar{\theta}^{\dot{\alpha}}) u^+_i u^-_i, \quad \theta^\pm_{\alpha, \dot{\alpha}} = u^+_i \theta^i_{\alpha, \dot{\alpha}} , \quad u^\pm_i .$$ \hspace{1cm} (4.47)

In this basis the constraints (4.43) just imply

$$\Phi^q = \Phi^q(x_A, \theta^+, \bar{\theta}^+, u^\pm),$$ \hspace{1cm} (4.48)

i.e. the solution is a Grassmann-analytic function of $\theta^+, \bar{\theta}^+$ (in the sense that it does not depend on the complex conjugates $\theta^-, \bar{\theta}^-$).
We emphasise that the superfield (4.48) is a non-trivial harmonic function carrying an external $U(1)$ charge $q = 0, \pm 1, \pm 2, \ldots$. Thus, $\Phi^q$ has an infinite harmonic expansion on the sphere $S^2$. Most of the terms in this expansion turn out to be auxiliary (in the case of the hypermultiplet) or pure gauge (in the SYM case) degrees of freedom. In order to obtain an ordinary superfield with a finite number of components one has to restrict the harmonic dependence. This is done with the help of the harmonic derivative (covariant derivative on $S^2$)

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} ,$$

(4.49)

or, in the analytic basis (4.47),

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} - 2i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} .$$

(4.50)

Thus, for instance, one can choose the $G$-analytic superfield $q^+(x_A, \theta^+, \bar{\theta}^+, u^\pm)$ carrying $U(1)$ charge $+1$ and impose the harmonic analyticity condition

$$D^{++} q^+ = 0 .$$

(4.51)

Due to the presence of a space-time derivative in the analytic form (4.50) of $D^{++}$, equation (4.51) not only “shortens” the harmonic superfield $q^+$, but also puts the remaining components on shell. In fact, (4.51) is the equation of motion for the $N = 2$ hypermultiplet. We remark that eq. (4.51) is compatible with the $G$-analyticity conditions (4.43) owing to the obvious commutation relations

$$[D^{++}, D_\alpha^{++}] = 0 .$$

(4.52)

Note also that in the old basis $(x, \theta_i, \bar{\theta}^i, u)$ the harmonic analyticity condition has the trivial solution given in (2.19); however, in this basis the $G$-analyticity condition (4.43) on $q^+$ becomes a non-trivial condition which in fact follows from the $N = 4$ SYM on-shell constraints (2.1).

4.1.2

A remarkable feature of harmonic superspace is that it allows us to have an off-shell version of the hypermultiplet. To this end it is sufficient to relax the on-shell condition (4.51) and subsequently obtain it as a variational equation from the action

$$S_{HM} = \int d^4x A du \theta^{+\alpha} \bar{q}^{+\alpha} D^{++} q^+ .$$

(4.53)

Here the integral is over the $G$-analytic superspace: $du$ means the standard invariant measure on $S^2$ and $d^4\theta^+ = (D^-)^4$. The special conjugation $\tilde{q}^+$ combines complex conjugation and the antipodal map on $S^2$ [10]. Its essential property is that it leaves the $G$-analytic superspace invariant (unlike simple complex conjugation). The reality of the action is established with the help of the conjugation rules $\tilde{\tilde{q}^+} = -q^+$ and $\tilde{D}^{++} = D^{++}$. Note that the $U(1)$ charge of the
Lagrangian in (4.53) is $+4$, which exactly matches that of the measure (otherwise the $SU(2)$ invariant harmonic integral would vanish).

The other ingredient of the $N = 4$ theory is the $N = 2$ SYM multiplet. It is described by a real $G$-analytic superfield $V^{++}(x_A, \theta^+, \bar{\theta}^+, u^+) = \bar{V}^{++}$ of $U(1)$ charge $+2$ subject to the following gauge transformation:

$$
\delta V^{++} = -D^{++} \Lambda + ig[\Lambda, V^{++}] 
$$

(4.54)

where $\Lambda(x_A, \theta^+, \bar{\theta}^+, u^+)$ is a $G$-analytic gauge parameter. It can be shown $[10]$ that what remains from the superfield $V^{++}$ in the Wess-Zumino gauge is just the (finite-component) $N = 2$ SYM multiplet. Under a gauge transformation (4.54), the matter (hypermultiplet) superfields transforms in the standard way

$$
\delta q^+ = i\Lambda q^+ .
$$

(4.55)

Thus $V^{++}$ has the interpretation of the gauge connection for the harmonic covariant derivative $D^{++} = D^{++} + igV^{++}$. All this suggests the standard minimal SYM-to-matter coupling which consists in covariantising the derivative in (4.53):

$$
S_{HM/SYM} = \int d^4x_A du d^4\theta^+ \left[ \bar{q}^+_{\dot{a}} (\delta_{ab} D^{++} + \frac{g}{2} f_{abc} V^{++}_c) q^+_b \right] .
$$

(4.56)

In order to reproduce the $N = 4$ theory we have chosen the matter $q^+ = q^+_a t_a$ in the adjoint representation of the gauge group $G$ with structure constant $f_{abc}$. 

We shall not explain here how to construct the gauge-invariant action for $V^{++} [10]$. We only present the form of the gauge-fixed kinetic term in the Fermi-Feynman gauge $[11]$:

$$
S_{SYM+GF} = -\frac{1}{2} \int d^4x_A du d^4\theta^+ V^{++}_{a\dot{a}} \square V^{++}_a .
$$

(4.57)

Of course, the full SYM theory includes gluon vertices of arbitrary order as well as Faddeev-Popov ghosts, but we shall not need them here (the details can be found in $[11]$).

4.1.3

The off-shell description of both theories above allows us to develop manifestly $N = 2$ supersymmetric Feynman rules. We begin with the gauge propagator. Since the corresponding kinetic operator in the FF gauge (4.57) is simply $\square$, the propagator is given by the Green’s function $1/4i\pi^2 x^2_{12}$:

$$
\square_1 \frac{1}{4i\pi^2 x^2_{12}} = \delta(x_1 - x_2)
$$

(4.58)

combined with the appropriate Grassmann and harmonic delta functions:

$^3$We use the definitions $[t_a, t_b] = if_{abc} t_c$, $\text{tr}(t_a t_b) = C(\text{Adj}) \delta_{ab}$, where the quadratic Casimir is normalised so that $C(\text{Adj}) = N_c$ for $SU(N_c)$.
\[ \langle V^+_{a}^{(1)}V^+_{b}^{(2)} \rangle = \]

\[ = \frac{i}{4\pi^2} \delta_{ab} (D_1^+)^4 \left( \frac{\delta_{12}}{x_{12}^2} \right) \delta^{(-2,2)}(u_1, u_2) = \frac{i}{4\pi^2} \delta_{ab} (D_2^+)^4 \left( \frac{\delta_{12}}{x_{12}^2} \right) \delta^{(2,-2)}(u_1, u_2), \]

where \( \delta_{12} \) is shorthand for the Grassmann delta function \( \delta^g(\theta_1 - \theta_2) \) and \( \delta^{(2,-2)}(u_1, u_2) \) is a harmonic delta function carrying \( U(1) \) charges +2 and −2 with respect to its first and second arguments. Note that the propagator is written down in the usual basis in superspace and not in the analytic one, so \( x_{1,2} \) appearing in (4.59) are the ordinary space-time variables and not the shifted \( x_A \) from (4.47). The G-analyticity of the propagator is assured by the projector

\[ (D^+)^4 = \frac{1}{16} D^{\alpha+} D_{\alpha}^+ D_{\bar{\alpha}}^{+\bar{\alpha}}. \]

The two forms given in (4.59) are equivalent owing to the presence of Grassmann and harmonic delta functions.\(^4\)

The matter propagator is somewhat more involved. The kinetic operator for the hypermultiplet is a harmonic derivative, so one should expect a harmonic distribution to appear in the propagator. Such distributions are simply given by the inverse powers of the \( SU(2) \) invariant combination \( u_1^+ u_2^+ \equiv (12) \) which vanishes when \( u_1 = u_2 \). One can prove the relations \(^{[1]}\):

\[ D_1^{++} \frac{1}{(12)^n} = \frac{1}{(n-1)!} (D_1^{--})^{n-1} \delta^{(n,n)}(u_1, u_2) \]

which are in a way the \( S^2 \) analogues of eq. (4.58). Here \( D^{--} = D^{++} \) is the other covariant derivative on \( S^2 \). So, the \( q^+ \) propagator is then given by

\[ \langle q_\alpha^+(1)q_b^+(2) \rangle = \]

\[ = \frac{i}{4\pi^2} \delta_{ab} (D_1^+)^4 (D_2^+)^4 \left( \frac{\delta_{12}}{x_{12}^2} \right) \equiv \Pi_{12} \delta_{ab}. \]

This time we need the presence of two G-analyticity projectors \((D_1^+)^4(D_2^+)^4\) because we do not have a harmonic delta function any more. In order to show that (4.62) is indeed the Green’s function for the operator \( D^{++} \), one uses (4.61) and the identity

\[ -\frac{1}{2} (D^+)^4 (D^{--})^2 \Phi = \Box \Phi \]

\(^{4}\)In certain cases the harmonic delta function needs to be regularised in order to avoid coincident harmonic singularities. In such cases one should use an equivalent form of the propagator (4.59) in which the analyticity with respect to both arguments is manifest.\(^{[12]}\)
on any G-analytic superfield $\Phi$.

Finally, the only vertex relevant to our two-loop calculation can be read off from the coupling term in (4.56):

\[ \frac{4}{2} f_{abc} \int d^4 x A_{1} du_1 d^4 \theta^+_1 \]

Figure 3

It involves an integral over the G-analytic superspace.

Note also the following useful relations. The full superspace Grassmann measure is related to the G-analytic one by

\[ d^8 \theta = d^4 \theta^+ (D^+)^4 = (D^-)^4 (D^+)^4 . \]

The Grassmann delta function $\delta^8(\theta_1 - \theta_2) \equiv \delta_{12}$ is defined as usual,

\[ \int d^8 \theta \delta^8(\theta) = (D^-)^4 (D^+)^4 \delta^8(\theta) \bigg|_{\theta=0} = 1 , \]

from which it is easy to derive

\[ (D^+_1)^4 (D^+_2)^4 \delta_{12} \bigg|_{\theta=0} = (12)^4 . \]

Using this relation as a starting point we find others, e.g.:

\[ (D^+_1)^2 (D^+_2)^4 (D^+_2)^4 \delta_{12} \bigg|_{\theta=0} = (D^+_3)^2 (D^+_1)^4 (D^+_2)^4 \delta_{12} \bigg|_{\theta=0} = 0 , \]

\[ D^+_3 \bar{D}^+_3 (D^+_1)^4 (D^+_2)^4 \delta_{12} \bigg|_{\theta=0} = -2(13)(23)(12)^3 \partial_{1\alpha\hat{\alpha}} , \]

\[ (D^+_3)^4 (D^+_1)^4 (D^+_2)^4 \delta_{12} \bigg|_{\theta=0} = -(13)^2 (23)^2 (12)^2 \Box_1 , \text{ etc.} \]

4.2 Four-point hypermultiplet correlators

In what follows we shall apply the above Feynman rules to compute four-point correlators of composite gauge invariant operators made out of two hypermultiplets. There are two types of such composite operators: $q^+ q^+$ (and its conjugate $\bar{q}^+ \bar{q}^+$) and $\bar{q}^+ q^+$. The structure of the hypermultiplet propagator (4.62) suggests that we need equal numbers of $q^+$’s and $\bar{q}^+$’s in order to form a closed four-point loop. Indeed, for instance, correlators of the type

\[ \langle q^+ q^+ | q^+ q^+ | q^+ q^+ | q^+ q^+ \rangle \]
must vanish in the free case as well as to all orders in perturbation theory. The reason is that
the only interaction the \( q^+ \)'s have is given by the vertex (4) and it is easy to see that there are
no possible graphs of this type. The same applies to any configuration with unequal numbers
of \( q^+ \)'s and \( \bar{q}^+ \)'s. So, the non-trivial ones are

\[
\langle \bar{q}^+ q^+ | q^+ q^+ | \bar{q}^+ q^+ | q^+ q^+ \rangle, \tag{4.69}
\]

\[
\langle \bar{q}^+ q^+ | q^+ q^+ | q^+ q^+ | \bar{q}^+ q^+ \rangle, \tag{4.70}
\]

\[
\langle q^+ q^+ | q^+ q^+ | \bar{q}^+ q^+ | q^+ q^+ \rangle. \tag{4.71}
\]

As explained in section 2, the correlators (4.69)-(4.71) are sufficient to determine the full cor-
relator of four \( N = 4 \) supercurrents. In fact, as we shall see later on, it is enough to compute
(4.69), the other two can then be obtained by permutations of the points and symmetrisation.
The relevant graph topologies for the computation of the correlator (4.69) are shown in Figure 4:

The graph (c) contains a vanishing two-point insertion (see [12]). The graphs (d) and (e) are
proportional to the trace of the structure constant \( f_{abb} \) and thus vanish unless the gauge group
contains a \( U(1) \) factor \([5]\). Thus, we only have to deal with the topologies (a) and (b). We shall
do the calculation in some detail for the case (a), the other one being very similar.

Here is a detailed drawing of the configurations having the topology of graph (a):

\[\text{Figure 4}\]

The hypermultiplet in the \( N = 4 \) multiplet has no electric charge, therefore a \( U(1) \) gauge factor corresponds
to a trivial free sector of the theory. Note, nonetheless, that if the graphs (d) and (e) are to be considered, they
contain divergent \( x \)-space integrals.
The expression corresponding to the first of them is (up to a factor containing the 't Hooft parameter $g^2 N_c$):

$$I_1 = -\Pi_{14}\Pi_{32}(D_1^+)^4 \int \frac{d^4 x_5 d^4 x_6 du_5 du_6 \theta_5^+ \theta_6^+ \Pi_{15}\Pi_{52}\Pi_{36}\Pi_{64}(D_6^+)^4 \left(\frac{\delta_{56}}{x_{56}^2}\right) \delta^{(2-2)}(u_5,u_6)}{(15)^3 x_{15}^2 x_{56}^3} \times \frac{\theta_{12}}{x_{12}^2} \frac{\theta_{31}}{x_{36}^2} \frac{\theta_{14}}{x_{64}^2}.$$  

(4.72)

The technique we shall use to evaluate this graph is similar to the usual $D$-algebra method employed in $N = 1$ supergraph calculations. First, since the propagators $\Pi$ are $G$-analytic, we can use the four spinor derivatives $(D_1^+)^4$ to restore the full Grassmann vertex $d^6 \theta_6$ (see (4.64)). Then we make use of the Grassmann and harmonic delta functions $\delta_{56}\delta^{(2-2)}(u_5,u_6)$ to do the integrals $\int du_6 d^6 \theta_6$. The next step is to pull the projector $(D_1^+)^4$ from the propagator $\Pi_{15}$ out of the integrals (it does not contain any integration variables and it only acts on the first propagator). After that the remaining projector $(D_5^+)^4$ from $\Pi_{15}$ can be used to restore the Grassmann vertex $d^6 \theta_5$ (everything else under the integral is $D_5^+$-analytic). In this way we can free the Grassmann delta function $\delta_{15}$ and then do the integral $\int d^6 \theta_5$. The resulting expression is (up to a numerical factor):

$$I_1 = -\Pi_{14}\Pi_{32}(D_1^+)^4 \int \frac{d^4 x_5 d^4 x_6 du_5 \theta_5^+ \theta_6^+ \Pi_{15}\Pi_{52}\Pi_{36}\Pi_{64}(D_6^+)^4 \left(\frac{\delta_{56}}{x_{56}^2}\right) \delta^{(2-2)}(u_5,u_6)}{(15)^3 x_{15}^2 x_{56}^3} \times \frac{\theta_{12}}{x_{12}^2} \frac{\theta_{31}}{x_{36}^2} \frac{\theta_{14}}{x_{64}^2}.$$  

Here all $D^+$s contain the same $\theta = \theta_1$ but different $u$’s, as indicated by their index. Next we distribute the four spinor derivatives $(D_1^+)^4$ over the three propagators and use the identities (4.67) (remember that we are only interested in the leading term of the correlator, therefore we set all $\theta$’s= 0). The result is:

$$I_1(\theta = 0) = -\frac{(14)(23)}{x_{23}^2 x_{14}^2} \int du_5 \times \left[ \frac{4i\pi^2}{x_{34}^2} (14)^2 (52)(53) g_3 + \frac{4i\pi^2}{x_{34}^2} (13)^2 (52)(54) g_4 + \frac{4i\pi^2}{x_{12}^2} (12)^2 (53)(54) g_1 \right] + (13)(14) \frac{(52)}{(51)} 2\partial_3 \cdot \partial_4 f + (12)(14) \frac{(53)}{(51)} 2\partial_2 \cdot \partial_4 f + (12)(13) \frac{(54)}{(51)} 2\partial_2 \cdot \partial_3 f.$$  

Here
\[ f(x_1, x_2, x_3, x_4) = \int \frac{d^4x_5 d^4x_6}{x_5^2 x_6^2 x_5^2 x_6^2 x_5^2 x_6^2} \]  
(4.73)

and, e.g.,
\[ g_1(x_2, x_3, x_4) = \frac{x_2^2}{4i\pi^2} \Box_2 f(x_1, x_2, x_3, x_4) = \int \frac{d^4x_5}{x_5^2 x_5^2 x_5^2 x_5^2} \]  
(4.74)

are two- and one-loop space-time integrals. The last step is to compute the harmonic integral. The way this is done is explained by the following example:

\[ \int du_5 \frac{d_5^+(5-2)}{(51)} = -\int du_5 (5-2) D_5^{++} \frac{1}{(51)} = -\int du_5 (5-2) \delta^{(1,1)}(5,1) = -(1-2), \]  
(4.75)

where \((1-2) \equiv u_1^- u_2^+\) and eq. (4.61) has been used. Other useful identities needed to simplify the result are, e.g.:

\[ (1-2)(13) = -(23) + (1-3)(12) \]  
(4.76)

(based on the cyclic property of the SU(2) traces and on the defining property \((11-) = 1\), see (4.43)) and

\[ (\partial_1 + \partial_2)^2 f = (\partial_3 + \partial_4)^2 f \Rightarrow 2\partial_1 \cdot \partial_2 f = 2\partial_3 \cdot \partial_4 f - \frac{4i\pi^2}{x_{12}^2} (g_1 + g_2) + \frac{4i\pi^2}{x_{34}^2} (g_3 + g_4) \]  
(4.77)

(based on the translational invariance of \(f\)). So, the end result for the first graph in Figure 2 is:

\[ I_1(\theta = 0) = \frac{(14)(23)}{x_{14} x_{23}} \left\{ (21-)(13)(14) \frac{4i\pi^2 g_2}{x_{12}^2} + (12-)(23)(24) \frac{4i\pi^2 g_1}{x_{12}^2} - (23-)(13)(34) \frac{4i\pi^2 g_4}{x_{34}^2} + (24-)(14)(34) \frac{4i\pi^2 g_3}{x_{34}^2} \right\} \]  
(4.78)

The second graph in Figure 5 is obtained by exchanging points 1 and 3:

\[ I_2 = I_1(3, 2, 1, 4). \]  
(4.79)

An important remark concerning these intermediate results is the fact that they do not satisfy the harmonic analyticity condition \((151)\), as one would expect from the property of the free on-shell hypermultiplets. Indeed, several terms in (4.78) contain negative-charged harmonics which are not annihilated by \(D^{++}\). As we shall see below, this important property of harmonic analyticity is only achieved after summing up all the relevant two-loop graphs. So, let us move on to the topology (b) in Figure 4. There are four such graphs shown in Figure 6:
The calculation is very similar to the one above, so we just give the end result:

\[ J_1(\theta = 0) = -\frac{(23)(34)}{x_{23}^2 x_{34}^2} \left[ (42^{-})(12)^2 \frac{4i\pi^2 g_3}{x_{12}^2} + (24^{-})(14)^2 \frac{4i\pi^2 g_3}{x_{14}^2} + (12)(14)2\partial_2 \cdot \partial_4 f(1, 2, 1, 4) \right]. \]

(4.80)

Notice the appearance of the two-loop integral \( f(4.73) \) with two points identified, \( x_1 = x_3 \). The other three graphs \( J_{2,3,4} \) in Figure 3 are obtained by cyclic permutation. Finally, putting everything together, using cyclic harmonic identities of the type \((4.76)\) as well as the identity (see the Appendix)

\[ \Box_1 f(1, 2, 1, 3) = \frac{x_{23}^2}{x_{12}^2 x_{13}^2} 4i\pi^2 g_4, \]

(4.81)

we arrive at the following final result:

\[ \langle \bar{q}^+ q^+ | q^+ q^+ | \bar{q}^+ q^+ | q^+ q^+ \rangle = I_1 + J_2 + J_1 + J_2 + J_3 + J_4 \]

\[ = \left[ (14)^2(23)^2 A_1 + (12)^2(34)^2 A_2 + (12)(23)(34)(41) A_3 \right]. \]

(4.82)

where

\[ A_1 = \frac{(\partial_1 + \partial_2)^2 f(1, 2, 3, 4)}{x_{14}^2 x_{23}^2}, \quad A_2 = \frac{(\partial_1 + \partial_4)^2 f(1, 4, 2, 3)}{x_{12}^2 x_{34}^2}, \]

\[ A_3 = 4i\pi^2 \left( \frac{x_{24}^2 - x_{14}^2 - x_{12}^2}{x_{14}^2 x_{23}^2 x_{34}^2} \right) g_3 + \left( \frac{x_{13}^2 - x_{12}^2 - x_{23}^2}{x_{14}^2 x_{23}^2 x_{34}^2} \right) g_4 + \left( \frac{x_{24}^2 - x_{23}^2 - x_{34}^2}{x_{14}^2 x_{23}^2 x_{34}^2} \right) g_1 + \left( \frac{x_{13}^2 - x_{14}^2 - x_{34}^2}{x_{14}^2 x_{23}^2 x_{34}^2} \right) g_2 \]

\[ + \frac{(\partial_2 + \partial_3)^2 f(1, 2, 3, 4)}{x_{14}^2 x_{23}^2} + \frac{(\partial_1 + \partial_2)^2 f(1, 4, 2, 3)}{x_{12}^2 x_{34}^2}. \]

(4.83)

(4.84)

As pointed out earlier, this result is manifestly harmonic analytic (there are only positive-charged harmonics in \((4.82)\)). It is also easy to see that the correlator is symmetric under the permutations \( 1 \leftrightarrow 3 \) or \( 2 \leftrightarrow 4 \) corresponding to exchanging the two \( \bar{q}^+ q^+ \) or \( q^+ q^+ \) vertices.

Finally, we turn to the two other correlators \((4.70)\) and \((4.71)\). The difference in the graph structure is the change of flow along some of the hypermultiplet lines. By examining the graphs
in Figures 5 and 6 it is easy to see that this amounts to an overall change of sign in the case (4.70) (due to an odd number of reversals of $q^+$ propagators and SYM-to-matter vertices) or to no change at all in the case (4.71) (an even number of reversals). Then one has to take into account the different symmetry of the new configurations which means that the three harmonic structures in (4.82) have to be symmetrised accordingly. This is done with the help of cyclic identities like

$$(14)(23)^{3+4} \rightarrow (13)(24) = (12)(34) + (14)(23). \quad (4.85)$$

5 Discussion of the results and asymptotic behaviour

An interesting technical feature of our calculation is that the space-time integrals resulting from the Grassmann and harmonic integrations can be written in terms of a second-order differential operator acting on the basic scalar two-loop integral. Ordinarily, this type of gauge theory calculation will produce a set of tensor integrals, which one would first need to reduce to scalar integrals algebraically, using algorithms such as in [17]. The reason for this unusual property can be traced back to an alternative form for the $q^+$ propagator which is obtained as follows.

Given two points in $x$ space, $x_1$ and $x_2$, one can define the supersymmetry-invariant difference

$$\hat{x}_{12} = x_{12} + \frac{2i}{(12)} \left[(1-2)\theta_1^+\bar{\theta}_1^+ - (12^-)\theta_2^+\bar{\theta}_2^+ + \theta_1^+\bar{\theta}_2^+ + \theta_2^+\bar{\theta}_1^+\right]. \quad (5.86)$$

Note the manifest G-analyticity of this expression with respect to both arguments. Now, with the help of (5.86) one can rewrite the propagator (4.62) in the equivalent form

$$\langle \tilde{q}^+(1)q^+(2) \rangle = \frac{(12)}{x_{12}^2}. \quad (5.87)$$

One sees that the whole $\theta$ dependence of the propagator is concentrated in the shift (5.86). Thus, doing the $\theta$ integrals in the above graph calculation effectively amounts to taking a couple of terms in the Taylor expansion of the scalar propagators. This explains the general structure of the resulting space-time integrals.

We now discuss the explicit space-time dependence of the correlation functions. The basic integral we encounter is the two-loop one (4.73). In principle, it could be obtained by Fourier transformation from the known result for the momentum space double box (see eq. (A.14) in the Appendix). Unfortunately, this appears to be a very difficult job. It is more useful to note that, rewritten as a momentum space diagram, the same integral is identical with the “diagonal box” diagram shown in Figure 7.
This diagram has not yet been calculated for the general off-shell case. However in the special case where either \( x_{23} = 0 \) or \( x_{41} = 0 \) it is known to be expressible in terms of the function \( \Phi^{(2)} \) defined in eq. (A.12) in the Appendix. For example, for \( x_{41} = 0 \) one has [22]

\[
f(x_1, x_2, x_3, x_1) = \frac{(i\pi^2)^2}{x_{23}^2} \Phi^{(2)} \left( \frac{x_{12}^2}{x_{23}^2}, \frac{x_{13}^2}{x_{23}^2} \right).
\]

(5.88)

In the same way, the one-loop integral \( g \) can, by eq. (A.2), be expressed in terms of another function \( \Phi^{(1)} \) defined in eq. (A.4) in the Appendix:

\[
g(x_1, x_2, x_3) \equiv \int \frac{dx_4}{x_{14}^2 x_{24}^2 x_{34}^2} = -\frac{i\pi^2}{x_{12}^2} \Phi^{(1)} \left( \frac{x_{23}^2}{x_{12}^2}, \frac{x_{31}^2}{x_{12}^2} \right).
\]

(5.89)

A further explicit function can be found in the case of some particular combination of derivatives on the two-loop integral \( f \). By exploiting the translation invariance of the \( x_5 \)-subintegral we can do the following manipulation on the integral, e.g.:

\[
(\partial_1 + \partial_2)^2 f(1, 2, 3, 4) = \int \frac{dx_6}{x_{36}^2 x_{46}^2} (\partial_1 + \partial_2)^2 \int \frac{dx_5}{x_{15}^2 x_{25}^2 x_{56}^2} \\
= \int \frac{dx_6}{x_{36}^2 x_{46}^2} \partial_6^2 \int \frac{dx_5}{x_{15}^2 x_{25}^2 x_{56}^2} \\
= 4i\pi^2 \int \frac{dx_6}{x_{36}^2 x_{46}^2} \int \frac{dx_5}{x_{15}^2 x_{25}^2} \delta(x_{56}) \\
= 4i\pi^2 \int \frac{dx_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}.
\]

(5.90)

This 4-point one-loop function is given by

\[
h(x_1, x_2, x_3, x_4) \equiv \int \frac{dx_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = -\frac{i\pi^2}{x_{12}^2 x_{24}^2} \Phi^{(1)} \left( \frac{x_{12}^2 x_{24}^2}{x_{13}^2 x_{24}^2}, \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2} \right).
\]

(5.91)

Unfortunately, no such trick exists in the case of the combination of derivatives, e.g., \((\partial_1 + \partial_3)^2 f(1, 2, 3, 4)\) and we do not know the corresponding explicit function. This technical problem
prevents us from demonstrating the manifest conformal invariance of the result. Indeed, if
the integrals appearing in the coefficients \( A_1 \) and \( A_2 \) (4.83) can be reduced to the form (4.91)
where the explicit dependence on the two conformal cross-ratios is visible, the same is not
obvious for the third coefficient \( A_3 \) (4.84). The property eq. (5.88) indicates that \( f \) itself has
not the form of a function of the conformal cross ratios times propagator factors. On the other
hand, without further information on \( f \) we cannot completely exclude the possibility that this
particular combination of derivatives of \( f \) also breaks down to conformally invariant one-loop
quantities, only in a less obvious way than it happens for the other ones.

The only qualitative information about the correlation functions we can obtain concerns their
asymptotic behaviour when two points approach each other. This is done in several steps.
Firstly, eq. (A.7) gives us information on the coincidence limits of \( g \), e.g., for \( x_3 \to x_1 \) one has

\[
g(x_1, x_2, x_3) \xrightarrow{x_3 \to 0} i\frac{\pi^2}{2} \ln \frac{x_2}{x_1} . \tag{5.92}
\]

Similarly, for the function \( h \) we find:

\[
h(x_1, x_2, x_3, x_4) \xrightarrow{x_4 \to 0} \frac{i\pi^2}{2} \ln \frac{x_2}{x_1} \ln \frac{x_3}{x_1} . \tag{5.93}
\]

This then allows us to determine the asymptotic behaviour of \( A_1 \) and \( A_2 \):

\[
A_1 \xrightarrow{x_4 \to 0} -4\pi^4 \ln \frac{x_2}{x_1} \frac{x_3}{x_4} ; \tag{5.94}
\]

\[
A_2 = 4i\pi^2 \frac{h(x_1, x_2, x_3, x_4)}{x_1^2 x_2 x_3} \xrightarrow{x_4 \to 0} -4\pi^4 \ln \frac{x_2}{x_1} \frac{x_3}{x_4} . \tag{5.95}
\]

The case of \( A_3 \) requires more work, since the derivatives of \( f \) appearing here cannot be used to
get rid of one integration. However, in the case of \((\partial_2 + \partial_3)^2 f(1, 2, 3, 4)\) the limit \( x_4 \to x_1 \) is
finite. We can therefore take this limit before differentiation. By a similar argument as above
one can show that

\[
(\partial_2 + \partial_3)^2 f(1, 2, 3, 1) = 4i\pi^2 \frac{x_2}{x_1^2 x_3} g_4 \tag{5.96}
\]

(this identity can also be derived using eq. (5.88) and differentiating under the integral in eq.
(A.11)). Thus we find

\[
\frac{(\partial_2 + \partial_3)^2 f(1, 2, 3, 4)}{x_1^2 x_2^2} \xrightarrow{x_4 \to 0} 4i\pi^2 \frac{g_4}{x_1^2 x_2 x_3} . \tag{5.97}
\]
This term is a pure pole term, without logarithmic corrections. The same procedure does not work for the last term in $A_3$, since here the limit is divergent. We evaluate this term by first symmetrising and then differentiating under the integral:

$$
(\partial_1 + \partial_2)^2 f(1, 4, 2, 3) = \frac{1}{2} \left[ (\partial_1 + \partial_2)^2 + (\partial_3 + \partial_4)^2 \right] f(1, 4, 2, 3)
= 2\pi^2 \left[ \frac{g_4}{x_{14}^2} + \frac{g_3}{x_{23}^2} + \frac{g_2}{x_{23}^2} + \frac{g_1}{x_{14}^2} \right]
+ 4 \left\{ \int \frac{dx_5 dx_6}{x_{45}^2 x_{56}^2 x_{36}^2 x_{15}^4 x_{26}^4} x_{15} x_{26} + \int \frac{dx_5 dx_6}{x_{26}^2 x_{15}^4 x_{36}^2 x_{15}^4} \right\}.
$$

The remaining integrals are still singular in the limit $x_4 \to x_1$ but do not contribute to the $\frac{1}{x_{14}^2}$ - pole. After combining the terms involving $g_4$ and $g_1$ with eq. (5.97) and the explicit $g_i$ - terms appearing in $A_3$ one finds that the leading $\frac{1}{x_{14}^2}$ - pole cancels out, leaving a subleading logarithmic singularity for $A_3$.

6 Conclusions

We have seen that both the $A_1$ and $A_2$ terms appearing in the four-point function calculations turn out to be reducible to one-loop quantities by various manipulations; they can be evaluated explicitly and are clearly conformally invariant. However, we have not succeeded in reducing the $A_3$ term to a one-loop form (although it is not ruled out that such a reduction may be possible) and it is consequently more difficult to verify conformal invariance for this term since the function $f(x_1, x_2, x_3, x_4)$ is not known explicitly.

Even though $A_3$ is not known explicitly as a function, we have seen that it is possible to evaluate its leading behaviour in the coincidence limit $x_{14} \sim 0$. The three different tensor structures in the correlation function have different singularities with the strongest being the one with the known coefficient $A_1$. This is given by $(x^2)^{-1} \ln x^2$. In section 2 we have shown how one may calculate the leading term (in a $\theta$ expansion) of the $N = 4$ supercurrent four-point function from $A_1$, $A_2$ and $A_3$, and in particular, that we can calculate the leading term of the $N = 2$ four-point function involving two $W^2$ operators and two $\bar{W}^2$ operators. The leading behaviour of the $\theta$-independent term of this four-point function in the coincidence limit is therefore determined by the leading behaviour of $A_1$. If this were to remain true for the higher-order terms in the $\theta$-expansion then, since $A_1$ is a known function of invariants, we can use the argument of section 3 to compute the asymptotic behaviour of four-point functions of $F^2$ and $\tilde{F}^2$ and this would put us in a better position to make a comparison with the SG computations when they are complete. This point is currently under investigation.

It is interesting to note that some of the qualitative features we have found here, such as one-loop box integrals and logarithmic asymptotic behaviour, have also been found in instanton calculations [13].

In the case of three-point functions it is believed that the corrections to the free term in perturbation theory cancel, at least at leading order in $1/N_c$. This is certainly not the case for four-point functions, and it is not clear precisely what relation the perturbative results reported...
on here should have to SG computations. It would be interesting to see what happens at three loops, and whether one gets similar asymptotic behaviour in the coincidence limit.

Finally, we note that the four-point functions computed here exhibit harmonic analyticity even though the underlying fields, the hypermultiplet and \( N = 2 \) SYM gauge field in harmonic superspace, are only Grassmann analytic. This is a more stringent check of the analyticity postulate of the \( N = 4 \) harmonic superspace formalism than the previous three-point check and obtaining a positive result on this point is encouraging.

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Note added: In a recent e-print [23], a special case of the amplitude considered here has been calculated using \( N = 1 \) superspace Feynman rules. Their result corresponds to our term \( A_2 \).

Note added in proof: We would like to emphasise that the three components of the amplitude \( A_1, A_2, A_3 \) are not trivially related to each other by the \( SU(4) \) invariance of the \( N = 4 \) theory (see Section 2), as has been erroneously assumed in the first reference in [18].

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A Calculation of massless $x$-space diagrams

The $x$-space Feynman graphs considered here are convergent for generic arguments, so that dimensional regularization is never needed. We can thus use the fact that in strictly four dimensions the (massless) $x$-space and momentum space propagators are identical,

$$\frac{1}{p^2 + i\epsilon} \leftrightarrow \frac{1}{x^2 - i\epsilon}$$

This fact allows us to formally rewrite our integrals as momentum space integrals, and to draw on results which are available for momentum space diagrams (the difference in the treatment of the propagator pole will cause some sign changes in transferring the formulas below to $x$-space). At the one-loop level, the massless 3-and 4-point functions have been known for a long time [19, 20]. At two loops, a number of exact results for 3-and 4-point diagrams were obtained by Davydychev and Ussyukina [21, 22].

Defining the one-loop 3-point Feynman integral with ingoing momenta $p_1, p_2, p_3$ (such that $p_1 + p_2 + p_3 = 0$) in 4 dimensions as
\[ C^{(1)}(p_1^2, p_2^2, p_3^2) = \int \frac{d^4q}{q^2(q + p_1)^2(q + p_1 + p_2)^2} \]  

(A.1)

one has

\[ C^{(1)}(p_1^2, p_2^2, p_3^2) = \frac{i\pi^2}{p_3^2} \Phi^{(1)}(x, y) \]  

(A.2)

Here the dimensionless variables \( x, y \) are defined by

\[ x \equiv \frac{p_1^2}{p_3^2}, \quad y \equiv \frac{p_2^2}{p_3^2} \]  

(A.3)

The function \( \Phi^{(1)}(x, y) \) can be represented explicitly in terms of dilogarithms [21],

\[ \Phi^{(1)}(x, y) = \frac{1}{\lambda} \left\{ 2 \left( \text{Li}_2(-\rho x) + \text{Li}_2(-\rho y) \right) + \ln \frac{y}{x} \ln \frac{1 + \rho y}{1 + \rho x} + \ln(\rho x) \ln(\rho y) + \frac{\pi^2}{3} \right\} \]  

(A.4)

where

\[ \lambda(x, y) \equiv \sqrt{(1 - x - y)^2 - 4xy}, \quad \rho(x, y) \equiv 2(1 - x - y + \lambda)^{-1} \]  

(A.5)

(here we assume \( \lambda^2 > 0 \); the case \( \lambda^2 < 0 \) requires an appropriate analytic continuation). Also the following parameter integral representation for \( \Phi^{(1)} \) is useful [21],

\[ \Phi^{(1)}(x, y) = -\int_0^1 \frac{d\xi}{\xi^2 + (1 - x - y)\xi + x} \left( \ln \frac{y}{x} + 2 \ln \xi \right) \]  

(A.6)

from which one can easily read off its asymptotic behaviour for \( p_2 \to 0 \),

\[ \Phi^{(1)}(x, y) \xrightarrow{p_2 \to 0} -\ln p_2^2 \]  

(A.7)

The 4-point function

\[ D^{(1)}(p_1^2, p_2^2, p_3^2, p_4^2, s, t) = \int \frac{d^4q}{q^2(q + p_1)^2(q + p_1 + p_2)^2(q + p_1 + p_2 + p_3)^2} \]  

(A.8)
\( s \equiv (p_1 + p_2)^2, \ t \equiv (p_2 + p_3)^2 \) can be expressed in terms of the same function as \[21\]

\[
D^{(1)}(p_1^2, p_2^2, p_3^2, p_4^2, s, t) = \frac{i\pi^2}{st} \Phi^{(1)}(X, Y)
\]

where now

\[
X \equiv \frac{p_2^2 p_3^2}{st}, \quad Y \equiv \frac{p_2^2 p_4^2}{st}
\]

Very similar results were obtained for the basic two-loop diagrams depicted in Figure 8 \[21, 22\].

\[
\Phi^{(2)}(x, y) = \frac{1}{\lambda} \left\{ 6 \left( \text{Li}_4(-\rho x) + \text{Li}_4(-\rho y) \right) + 3 \ln \frac{y}{x} \left( \text{Li}_3(-\rho x) - \text{Li}_3(-\rho y) \right) 
\right.
\]

\[
+ \frac{1}{2} \ln^2 \frac{y}{x} \left( \text{Li}_2(-\rho x) + \text{Li}_2(-\rho y) \right) + \frac{1}{4} \ln^2(\rho x) \ln^2(\rho y)
\]

\[
+ \frac{1}{2} \pi^2 \ln(\rho x) \ln(\rho y) + \frac{1}{12} \pi^2 \ln^2 \frac{y}{x} + \frac{7}{60} \pi^4 \right\}
\]

Here the polylogarithms are defined as

\[
\text{Li}_N(z) = \frac{(-1)^N}{(N-1)!} \int_0^1 d\xi \frac{\ln^{N-1}\xi}{\xi - z^{-1}}
\]
The momentum-space double box diagram Figure. 8b can be expressed in terms of the same function $\Phi^{(2)}$,

$$D^{(2)}(p_1^2, p_2^2, p_3^2, p_4^2, s, t) = t \left( \frac{i\pi^2}{st} \right)^2 \Phi^{(2)}(X, Y)$$

(A.14)