Cyclic and Abelian CLT groups

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Abstract

A group $G$ of order $n$ is said to be an ACLT (CCLT) group, if for every divisor $d$ of $n$, where $d < n$, $G$ has an abelian (cyclic) subgroup of order $d$. A natural number $n$ is said to be an ACLT (CCLT) number if every group of order $n$ is an ACLT (CCLT) group. In this paper we find all ACLT and CCLT numbers and study various properties of ACLT (CCLT) groups.

Key Words: CLT groups, CLT numbers, Cyclic numbers, Abelian numbers

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1 Introduction and preliminaries

One of the important theorems of group theory is Lagrange’s theorem which states that order of a subgroup of a finite group divides the order of that group. However the converse of Lagrange’s theorem (CLT) is not true in general and over the years, different researchers have investigated this issue from a variety of angles. A group is said to be a CLT group if it satisfies the converse of Lagrange’s theorem. For further details about CLT groups refer [5]. A natural number $n$ is said to be a cyclic (abelian, CLT) number if every group of order $n$ is a cyclic (abelian, CLT) group. Berger in 1978 and Nganou in 2017 studied CLT numbers and gave a complete characterization of CLT numbers (see [3], [13]). In fact, Berger showed that every squarefree number and a number which is divisible by only one prime is a CLT number.

Throughout this paper all groups are finite. In this paper we generalize the notions of CLT groups and CLT numbers to the notions of ACLT (CCLT) groups and ACLT (CCLT) numbers respectively. A finite group of order $n$ is said to be an Abelian (Cyclic) CLT group, if for each divisor $d$ of $n$, where $d < n$, there exists an abelian (cyclic) subgroup of order $d$. We call such abelian CLT and cyclic CLT groups as ACLT and CCLT groups respectively. A natural number $n$ is said to be an ACLT (CCLT) number if every group of order $n$ is an ACLT (CCLT) group.
This paper is organized as follows. In Section 2, we find all CCLT numbers and CCLT groups. We also investigate important properties of CCLT groups. In Section 3 we characterize ACLT numbers, study various properties of ACLT groups and discuss some of their concrete examples. In section 4, we summarize the relationship between cyclic, abelian, CLT, CCLT and ACLT numbers using Venn diagram.

However in this work, all groups which arise out of semidirect products are non-abelian. Furthermore, depending on the context, we advise the reader to choose suitable semidirect product. We denote cyclic group of order $n$ by $C_n$. For the examples of groups of given order and for their properties refer [10].

2 CCLT Groups

In this section we find all CCLT numbers. We characterize all CCLT groups, discuss their properties and produce some related examples. We begin with the following observations about CCLT numbers.

It is obvious that every cyclic group is a CCLT group, hence every cyclic number is a CCLT number. The smallest number which is not a cyclic number but CCLT number is 4 and the smallest CCLT number which is not an abelian number is 6. If $p$ is a prime number, then the group $C_p \times C_p \times C_p$ is not a CCLT group as it has no cyclic subgroup of order $p^2$. Hence $p^3$ is not a CCLT number. But it is interesting to see that every other group of order $p^3$ is a CCLT group.

Theorem 1. A natural number $n$ is a CCLT number if and only if either $n$ is a cyclic number or $n = pq$, where $p$ and $q$ are prime numbers, need not be distinct.

Proof. If $n$ is a cyclic number or $n = p^2$ or $n = pq$, where $p$ and $q$ are distinct prime numbers, then it is easy to prove that $n$ is a CCLT number. For the converse, we prove that if $n$ is not a cyclic number or $n$ is not of the form $p^2$ or $pq$, then we give an example of a non-CCLT group of that order.

Let $n = p^k$, where $k \geq 3$. Then the group $G = C_p \times C_p \times C_{p^{k-2}}$, does not contain a cyclic subgroup of order $p^{k-1}$. Suppose $n$ is a square free number of the form $n = p_1 \times p_2 \times \cdots \times p_k$, where $k \geq 3$. Since $n$ is not a cyclic number, there exists $p_i, p_j \in \{p_1, p_2, \ldots, p_k\}$ such that $p_i | p_j - 1$. If we consider $G = H \times C_{n^{p_i}}$, where $H$ is a non-abelian group of order $p_i p_j$, then it is easy to see that $G$ does not contain an abelian subgroup of order $p_i p_j$. Finally if $n$ is of the form $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $a_1 \cdot a_2 \cdots a_k \geq 2$, then $G = \left(C_{p_j} \times C_{p_j^{a_j-1}}\right) \times C_{p_j^{a_j-1}}$, where $a_j \geq 2$ does not contain a cyclic subgroup of order $p_j^{a_j}$.

Now we have $n \in \mathbb{N}$ is a CCLT number if and only if either $\gcd(n, \varphi(n)) = 1$ or $n = pq$, where $p, q$ are primes need not be distinct. For every natural number $n$, there is a cyclic group of order $n$, hence a CCLT group of order $n$. The following are few nontrivial examples of CCLT groups of order $n$, where $n$ need not be a CCLT number.
Example 2. 1. Let $G$ be a dihedral group $D_n$ of order $2n$ or a Dicyclic group $Dic_n$ of order $4n$. Then $G$ is a CCLT group if and only if either $n$ is a prime number or $n = 2^k$, where $k \in \mathbb{N}$.

Proof. Since $m = 2p$, where $p$ is prime is a CCLT number, every group of order $2p$ is a CCLT group. If $n = 2^k$, then $D_n$ is a group of order $2^{k+1}$ and it contains a cyclic subgroup of order $2^k$ hence $D_n$ is a CCLT group.

If $n = 2p$ by definition, $Dic_n$ contains an element of order $4$ and an element of order $2p$. Hence it is CCLT group. If $n = 2^k$, then $Dic_n$ contains cyclic subgroup of order $2^{k+1}$.

Conversely suppose that if $n$ is not of the above forms, then $n$ is divisible by an odd prime say $p$ and $n > p$. Then it is easy to see that $D_n$ (or $Dic_n$) has no cyclic subgroup of order $2^{m+1}p^k$ (or $2^{m+2}p^k$), where $2^m | n$ but $2^{m+1} \nmid n$, $k \in \{0, 1\}$. □

2. The group $G = C_{p^{n-1}} \times C_p$ is a CCLT group, where $p$ is a prime number and $n \in \mathbb{N}$.

It is well known that cyclic groups $\subset$ abelian groups $\subset$ nilpotent groups $\subset$ CLT groups $\subset$ solvable groups. By definition CCLT and ACLT groups are CLT groups. We already know that abelian group need not be a CCLT group and $S_3$ is a CCLT group but not a nilpotent group. Also recall that $S_4$ is a CLT group, but its subgroup $A_4$ is not a CLT group, whereas every subgroup of a supersolvable group is supersolvable. A natural question: is every subgroup of a CCLT group is CCLT? To answer this question, we first prove that every CCLT group is supersolvable, in fact we prove more. Consequently, every subgroup of CCLT group is CLT.

Recall that a group $G$ is a metacyclic if $G$ contains a cyclic normal subgroup $H$ such that $G/H$ is cyclic. Every cyclic group is metacyclic. Let $G$ be a CCLT group of order $n$, $p$ is the smallest prime dividing $n$ and $H$ be a cyclic subgroup of order $p^{n-1}$. Then it is clear that $H$ is normal in $G$ and $G/H$ is cyclic. Hence the following result.

Proposition 3. Every CCLT group is metacyclic hence supersolvable.

An interesting observation is that most of the CCLT groups have a property that all its proper subgroups are cyclic. It is not true in general as groups of prime power order provides a counterexample. In particular, if $n \geq 3$ then the group $C_p \times C_p^{n-1}$ is a CCLT group of order $p^n$ and its subgroups $C_p^{n-1}$ and $C_p \times C_p^{n-2}$ of order $p^{n-1}$ are not isomorphic. In the literature non-cyclic groups all of whose proper subgroups are cyclic are known as minimal non-cyclic groups. These groups are completely characterized by Theorem 2.1 of [18].

Theorem 4. Let $G$ be a CCLT group of order $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, where $k \geq 2$. Then every Sylow $p$-subgroup of $G$, where $p \in \{p_1, p_2, \ldots, p_k\}$ is cyclic.

Proof. Let $G$ be a CCLT group of order $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, where $k \geq 2$ and $H_i$ be a cyclic Sylow $p_i$-subgroup of $G$. Then from Sylow’s theorem, every Sylow $p_i$-subgroup of $G$ is conjugate to $H_i$. Hence all Sylow $p$-subgroups of $G$ are cyclic. □
In the literature, the groups satisfying the property of hypothesis of Theorem 4 are called \(Z\)-groups and it is known that \(Z\)-groups are metacyclic. For examples of \(Z\)-groups (see [10]). Dihedral group \(D_{15}\) is an example of \(Z\)-group but not a CCLT group. Hence if we exclude groups of prime power order, then we have

cyclic groups \(\subset\) CCLT groups \(\subset\) \(Z\)-groups \(\subset\) metacyclic groups \(\subset\) supersolvable groups.

**Theorem 5.** Any two subgroups of same order of a \(Z\)-group are isomorphic.

*Proof.* Let \(G\) be a \(Z\)-group of order \(mn\). Then from Theorem 11 of Chapter V of [22] we have

\[
G = \langle a, b | a^m = b^n = e, bab^{-1} = a^r, \gcd((r - 1)n, m) = 1, r^n \equiv 1 \pmod{m} \rangle. \tag{1}
\]

Let \(H\) and \(K\) be subgroups of \(G\) of same order \(d\). If \(d|m\) or \(d|n\), then we are done as both \(H\) and \(K\) are cyclic groups of order \(d\). Suppose \(d = ts\), where \(t, s > 1\), \(t|m\) and \(s|n\). Then

\[
H = \langle a^{k_1}, b^{k_2} \rangle, K = \langle a^{v_1}, b^{v_2} \rangle, \text{ where } o(a^{k_1}) = o(a^{v_1}) = t, o(b^{k_2}) = o(b^{v_2}) = s.
\]

We define a map \(\psi : H \to K\) as \(\psi(a^{k_1}) = a^{v_1}\) and \(\psi(b^{k_2}) = b^{v_2}\). Then by using relations in Equation (1) we can see that \(\psi\) is an isomorphism. \(\square\)

**Corollary 6.** A CCLT group which is neither cyclic nor a \(p\)-group is minimal non-cyclic.

**Theorem 7.** Every subgroup of a CCLT group is CCLT.

*Proof.* Let \(G\) be a CCLT group. Let \(G\) be a \(p\)-group of order \(p^n\). It is clear that the result is true for \(n \leq 3\). Let \(G\) be a CCLT group of order \(p^n\), where \(n \geq 4\). We prove the result by the method of contradiction. Suppose \(k\) is the largest such that there exists a non-CCLT subgroup \(H\) of order \(p^k\). By Sylow's theorem \(H\) is a normal subgroup of a CCLT subgroup \(N\) of \(G\) of order \(p^{k+1}\). Let \(K\) be a cyclic subgroup of \(N\) of order \(p^k\). Then \(HK \subseteq N\) and

\[
|HK| = \frac{|H||K|}{|H \cap K|} = p^{2k-\alpha} \leq p^{k+1},
\]

where \(\alpha = |H \cap K|\). Also we can prove that \(\alpha = k - 1\). Thus we get a contradiction as \(H\) contains a cyclic subgroup of order \(p^{k-1}\).

\(G\) is not a \(p\)-group. In this case by Corollary 6 we can say that every proper subgroup of CCLT group which is not a \(p\)-group is cyclic. Hence the result follows. \(\square\)

**Remark 8.** Before proceeding by Proposition 2.5 of [11] and Theorem 2 we can see that order of every CCLT group which is neither cyclic nor a \(p\)-group is of the form \(p^r q\), where \(p, q\) are primes and \(p|q - 1\). Further there exists a unique non-cyclic CCLT group of that order. With these observations, we can make the following conclusions. Let
\(G_{CCLT}(n)\) denotes the number of non-isomorphic CCLT groups of order \(n\). If \(n\) is not a prime power, then we have

\[
G_{CCLT}(n) = \begin{cases} 
  2 & \text{if } n = p^r q \text{ and } p|(q-1) \\
  1 & \text{otherwise.} 
\end{cases}
\]

We know that from Theorem 1, \(p\) and \(p^2\) are CCLT number so every group of order \(p\) and \(p^2\) are CCLT. We also observe that if \(p\) is a prime number and \(k \in \mathbb{N}\) then

\[
G_{CCLT}(p^k) \geq \begin{cases} 
  6 & \text{if } p = 2 \text{ and } k > 3 \\
  3 & \text{if } p \neq 2 \text{ and } k > 2.
\end{cases}
\]

**Theorem 9.** Let \(G\) be a CCLT group of order \(n\) and \(S(G)\) denotes the number of subgroups of \(G\). Then

\[
S(G) = \begin{cases} 
  \tau(n) & \text{if } G \text{ is cyclic} \\
  2 + (p+1)(k-1) & \text{if } G \text{ is abelian and } n = p^k \\
  2r + q + 1 & \text{if } n = p^r q, p \neq q \text{ and } p|(q-1).
\end{cases}
\]

*Proof.* If \(G\) is cyclic then it is a well known fact that \(S(G) = \tau(n)\) (see [8]). However, when \(G\) is non-cyclic abelian group, then one can easily deduce that \(G\) is a \(p\)-group as well as it is isomorphic to \(C_p \times C_p^{k-1}\), therefore by Theorem A of [1] it is immediate that \(S(G) = 2 + (p+1)(k-1)\).

On the other hand if \(G\) is non-abelian, then Corollary 6 and Remark 8 yields that \(o(G) = p^r q\) and all proper subgroups of \(G\) are cyclic. According to a result given in [19] it can be concluded that \(G\) has a unique cyclic subgroup \(H\) of order \(p^{r-1}q\). Since any subgroup of \(G\) of order \(p^{r-1}q\) is maximal, therefore all other proper subgroups of \(G\) are contained in \(H\) except the subgroups of order \(p^r\). Now it is easy to see that number of subgroups of \(H\) is \(\tau(p^{r-1}q) = 2r\). Also, number of elements of order \(p^r\) in \(G\) is \(p^r q - p^{r-1}q\), which implies that number of subgroups of order \(p^r\) in \(G\) is \(q\). Hence, \(S(G) = 2r + q + 1\). \(\square\)

**Corollary 10.** Let \(G\) be a CCLT group and \(C(G)\) denotes the set of all cyclic subgroups of \(G\). Then

\[
|C(G)| = \begin{cases} 
  \tau(n) & \text{if } G \text{ is cyclic} \\
  (k-1)p + 2 & \text{if } G \text{ is abelian and } n = p^k \\
  2r + q & \text{if } n = p^r q, p \neq q \text{ and } p|(q-1).
\end{cases}
\]

*Proof.* The proof is immediate by using Theorem 1 of [20] and Corollary 6. \(\square\)

The following result answers, when the direct product of CCLT groups is a CCLT group.
Theorem 11. Let $H$ and $K$ be groups of order $m$ and $n$ respectively. If $H \times K$ is a CCLT group, then both $H$ and $K$ are cyclic groups. Conversely if $H$ and $K$ are cyclic groups, then $H \times K$ is a CCLT group if one of the following conditions hold:

1. If $\gcd(m, n) = 1$.

2. If $\gcd(m, n) \neq 1$, then $H \times K \cong C_p \times C_p^k$, where $p$ is a prime number and $k \in \mathbb{N}$.

Proof. Let $H \times K$ be a CCLT group and $H$ be not a cyclic group. If $\gcd(m, n) = 1$ then it is easy to see that there is no cyclic subgroup of order $m$. If $\gcd(m, n) \neq 1$ and $p | \gcd(m, n)$, where $p$ is a prime number. Then $m = p^r x$ and $n = p^s y$, where $\gcd(p, x) = \gcd(p, y) = 1$. We can show that $H \times K$ has no cyclic subgroup of order $p^{\max\{r, s\}+1}$. Conversely, suppose that $H$ and $K$ are cyclic groups. If $\gcd(m, n) = 1$, then $H \times K$ is cyclic and hence CCLT. Let $p | \gcd(m, n)$, where $p$ is a prime number. Then $m = p^r x$ and $n = p^s y$, where $\gcd(p, x) = \gcd(p, y) = 1$. Now we claim that $xy = 1$ and either $r = 1$ or $s = 1$. Suppose $xy \neq 1$ or both $r > 1$ and $s > 1$, then $H \times K$ has no cyclic subgroup of order $p^{\max\{r, s\}+1}$. This contradicts $H \times K$ is a CCLT group. □

Now from Theorem 11 and Part 2 of Theorem 11, we can say that every abelian CCLT group is cyclic except $C_{p^{n-1}} \times C_p$.

Theorem 12. Quotient group of a CCLT group is CCLT.

Proof. Let $G$ be a CCLT group and $N$ be a normal subgroup of $G$.

$G$ is a $p$-group. Let $G$ be a group of order $p^n$, where $p$ is a prime number. It is sufficient to consider the case that $o(N) = p^m$, where $m \leq n - 2$. Since $G$ is a CCLT group, there exists an element $g \in G$ of order $p^{n-1}$. Let $o(gN) = p^k$, where $k \leq n - m$. Then $o(g^p) \leq p^m$. Consider,

$$o(g^p) = \frac{o(g)}{(o(g), p^k)} = \frac{p^{n-1}}{(p^{n-1}, p^k)} = \frac{p^{n-1}}{p^k} = p^{n-k-1} \leq p^m,$$

As a consequence $k \geq n - m - 1$. Which shows that $k = n - m - 1$ or $k = n - m$. In both the cases $G/N$ is CCLT.

$G$ is not a $p$-group. Then by Corollary 6 all proper subgroups of $G$ are cyclic. If $N$ is a normal subgroup of $G$ then by Correspondence theorem every proper subgroup of $G/N$ is cyclic. Hence $G/N$ is CCLT. □

3 ACLT Groups

In this section we characterize all ACLT numbers. Later we find out important properties of ACLT groups and discuss some related examples. We begin with the necessary and sufficient condition for a natural number $n$ to be an ACLT number.
Theorem 13. A natural number \( n \) is an ACLT number if and only if one of the following conditions hold

1. \( n \) is an abelian number.
2. \( n = pq \), where \( p \) and \( q \) are distinct primes.
3. \( n = p^m \), where \( p \) is prime \( m \in \{0, 1, 2, 3, 4\} \).
4. \( n = p^2q \), where \( p \) and \( q \) are distinct primes and
   \[ (a) \quad p = 2, q \in \{4k + 3; k \in \mathbb{N}\}. \]
   \[ (b) \quad p|q - 1, \quad p^2 \nmid q - 1, \quad p \nmid q + 1 \text{ and } q \nmid p^2 - 1. \]

Proof. First we show that if \( n \) is one of the given form, then \( n \) is an ACLT number. If \( n \) is an abelian number or \( n = pq \) or \( n = p^3 \), then it is easy to see that \( n \) is an ACLT number. If \( n = p^4 \), then in order to show that \( n \) is an ACLT number it is sufficient to prove that it contains an abelian subgroup of order \( p^3 \). But it is known that a group of order \( p^m \) has an abelian subgroup of order \( p^k \), where \( k(k + 1) \geq 2m \) (see [6] or [4]). Thus \( n = p^4 \) is an ACLT number. If \( n = p^2q \), then every group of order \( n \) contains subgroups of order 1, \( p, q, p^2 \) and they are abelian. Hence in order to show that \( n \) is an ACLT number it is sufficient to show that it contains an abelian subgroup of order \( pq \).

If \( n = 4q \), where \( q \in \{4k + 3; k \in \mathbb{N}\} \), then \( G(n) = 4 \), where \( G(k) \) denotes the number of groups of order \( k \). Further, these four distinct groups up to isomorphism are \( C_{4q}, C_2 \times C_2q, Dic_q \) or \( D_{2q} \). From Example 2 and Example 14 we know that the groups \( Dic_q \) and \( D_{2q} \) are ACLT respectively. If \( n = p^2q \), where \( p|q - 1, \quad p^2 \nmid q - 1, \quad p \nmid q + 1 \text{ and } q \nmid p^2 - 1 \), then \( G(n) = 4 \) and if \( G \) is a group of order \( n \), then \( G \) is isomorphic to \( C_{2^2q} \) or \( C_p \times C_{pq} \) or \( C_q \times C_{p2} \) or \( C_q \times (C_p \times C_p) \). It is easy to verify that all these groups contain an abelian subgroup of order \( pq \). Hence they are ACLT groups.

To prove the converse part, we show that whenever \( n \) is not in any one of the forms given in the hypotheses, then there exist a non-ACLT group of order \( n \).

Suppose \( n \) is a squarefree number of the form \( p_1 \times p_2 \times p_3 \times \cdots \times p_m \), where \( m \geq 3 \). Since \( n \) is not an abelian number there exists \( p_i, p_j \in \{p_1, p_2, \ldots, p_m\} \) such that \( p_i \nmid p_j - 1 \). In this case if we consider the group \( G = (C_{p_i} \times C_{p_j}) \times C_{\frac{n}{p_i p_j}} \), then \( G \) does not contain an abelian subgroup of order \( p_i p_j \).

If \( n = p^m \), where \( p \) is prime and \( m \geq 5 \),

\[
G = \begin{cases} 
(C_2)^3 \times C_4 \times C_{2^{m-5}} & \text{if } p = 2 \\
(C_p^3) \times C_{p^3} \times C_{p^{m-5}} & \text{if } p \neq 2 
\end{cases}
\]

Then \( G \) has no abelian subgroup of order \( p^{m-1} \).

Suppose \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \), where \( k \geq 2 \) and \( a_j \geq 3 \) for some \( j \). Then if we consider the group \( G = H \times C_{\frac{n}{p_j}} \), where \( H \) is a non-abelian group of order \( p_j^{a_j} \), then \( G \) does not contain an abelian subgroup of order \( p_j^{a_j} \). Now onwards we suppose that \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \),
where \( k \geq 2 \) and \( a_j \leq 2 \). Let us assume that \( k \geq 3 \). Since \( n \) is not an abelian number, there exists \( p_i \) and \( p_j \) such that \( p_i | p_j^{a_j} - 1 \) and hence there exists a non-abelian group \( H \) of order \( p_i p_j^{a_j} \). If we consider \( G = H \times C_{p_i p_j} \), then \( G \) does not contain an abelian subgroup of order \( p_i p_j^{a_j} \). Finally the problem boils down to the cases \( n = p^2 q \) or \( n = p^2 q^2 \), it is known that some of these numbers are non-CLT numbers (see [2, 17, 15, 16]). Hence we confine to CLT numbers of this form.

Suppose \( n = 4q \), where \( q \in \{4k + 1 | k \in \mathbb{N}\} \). Consider the group \( G = C_q \times C_4 \) of order \( n \) which has no abelian subgroup of order \( 2q \). The groups \( G = C_{p^2} \times C_q \) or \( G = (C_p \times C_p) \times C_q \), where \( p > 2 \) have no abelian subgroups of order \( pq \). If \( n = p^2 q^2 \), where \( p | q - 1, p^2 \nmid q - 1, p \nmid q + 1, q \nmid p^2 - 1 \), then it is easy to see that the group \( G = (C_q \times C_q) \times C_{p^2} \) has no abelian subgroup of order \( p^2 q \). For all other CLT groups of order \( p^2 q^2 \) there exists a non-ACLT group \( H \) of order \( p^2 q \). Therefore in this case the group \( G = H \times C_q \) has no abelian subgroup of order \( pq^2 \). Hence the result.

The following are few nontrivial examples of ACLT groups of order \( n \), where \( n \) need not be an ACLT number.

**Example 14.**

1. A finite group \( G \) is said to be minimal non-abelian if \( G \) is non-abelian but every proper subgroup of \( G \) is abelian. These groups are completely classified in [12] and [14]. Every minimal non-abelian group is an ACLT group.

2. Dihedral group \( D_n \) of order \( 2n \) is an ACLT group if and only if \( n \) is one of the following
   
   (a) \( n \) is a prime number.
   
   (b) \( n = 2^k, k \in \mathbb{N} \).
   
   (c) \( n = 2p \), where \( p \) is a prime number.

   **Proof.** From Part [1] of Example 2 we already have the equivalence of (a), (b) and CCLT of \( D_n \). If \( n = 2p \), then \( D_n \) has a cyclic subgroup of order \( 2p \) hence it is an ACLT group. Note that \( D_n \) does not have a cyclic subgroup of order 4 hence it is not a CCLT group. Conversely suppose that \( n \) has two odd prime divisors \( p \) and \( q \) or \( n = 2^k p \), where \( k > 1 \). Then any subgroup of order \( 2^{m+1} p^\delta \) is not abelian, where \( 2^m | n, 2^{m+1} \nmid n \) and \( \delta \in \{0, 1\} \).

3. The group \( G = C_3^3 \times C_2^2 \) is an ACLT group.

4. The semi-dihedral group \( S_{2n} \) and generalised quaternion group \( Q_{2n} \) of order \( 2^n \) are ACLT groups.

A group whose commutator subgroup is an abelian group is called **metabelian group**. It is known that a group \( G \) is metabelian if and only if there is an abelian normal subgroup \( A \) of \( G \) such that the quotient group \( G/A \) is abelian. With this result in hand we have a necessary condition for a group to be an ACLT group. Let \( G \) be an ACLT
group of order \( n \). Suppose that \( p \) is the smallest prime dividing \( n \) and \( H \) is an abelian subgroup of \( G \) of order \( \frac{n}{p} \). Then it is clear that \( H \) is normal in \( G \) and \( G/H \) is abelian. Hence the following result.

**Proposition 15.** Every ACLT group is a metabelian group.

We know that direct product of abelian groups is abelian. The group \( D_p \times C_p \) is suggesting that \( H \times K \) can be an ACLT group even if one of \( H \) and \( K \) is not an abelian group. For example, if \( H \) is an ACLT, non-abelian group of order \( m \) and \( K \) is an abelian group of order \( n \) with \( \gcd(m, n) = 1 \), then \( H \times K \) does not contain an abelian subgroup of order \( m \). The group \( D_p \times C_p \) is not ACLT as it does not contain abelian subgroup of order \( 2p^2 \) even though \( D_p \) is ACLT, \( C_p \) is abelian and \( (2p, pq) > 1 \). We summarize these observations and state in the following result.

**Theorem 16.** Let \( H \) and \( K \) be groups with \( m \) and \( n \) elements respectively. If \( H \times K \) is an ACLT group, then both \( H \) and \( K \) are ACLT and one of them is abelian. Conversely suppose \( H \) is an ACLT group and \( K \) is an abelian group. Then \( H \times K \) is an ACLT group if one of the following conditions hold.

1. If \( H \) is an abelian group.
2. If \( H \) is non-abelian then if \( p|n \), where \( p \) is a prime number, then \( p|m \).

**Proof.** If \( H \) is non-ACLTL group then there exists \( d \in \mathbb{N} \) such that \( d|m \) and \( H \) has no abelian subgroup of order \( d \). Then we can show that \( H \times K \) has no abelian subgroup of order \( d^{k+1} \), where \( d \) divides \( n \) but \( d^{k+1} \) does not divide \( n \). Let \( H \) and \( K \) be non-abelian groups. Then it is obvious that \( H \times K \) has no abelian subgroup of order \( \frac{mn}{p} \), where \( p \) is the smallest prime dividing \( m \). Therefore \( H \times K \) is ACLT then both \( H \) and \( K \) are ACLT and one of them is abelian. Conversely, if \( H \) and \( K \) both are abelian then \( H \times K \) is abelian hence ACLT.

Let \( \pi(n) \) denotes the set of all primes dividing \( n \). Our objective is to prove that if \( G \) is a non-abelian ACLT group, then \( |\pi(|G|)| \leq 2 \). By using this result we can show that every ACLT group is supersolvable and every subgroup of an ACLT group is CLT.

**Theorem 17.** If \( G \) is a non-abelian, ACLT group then order of \( G \) has at most two prime divisors.

Before proving the above theorem we see few more properties of ACLT groups. Similar to Theorem 4 we have every Sylow \( p \)- subgroup of an ACLT group is abelian. In the literature, groups of this nature are called \( A \)-groups. Thus if we exclude \( p \)-groups, every ACLT group is an \( A \)-group. For examples of various \( A \)-groups refer [10]. We need following result on \( A \)-groups which is a collection of several results or observations of \( A \)-groups given by D.R. Taunt [19].

**Theorem 18** (D.R. Taunt [19]). 1. Any subgroup or homomorphic image of an \( A \)-group is \( A \)-group.
2. Every solvable $A$-group contains unique maximal abelian normal subgroup.

3. Let $Z[G]$ and $G'$ denote the center and commutator subgroups of $G$ respectively. Then $Z[G] \cap G' = \{e\}$, whenever $G$ is an $A$-group.

A subgroup $H$ of a group $G$ is Carter, if $H$ is both self-normalizing and nilpotent. Carter \[7\] proved that every finite solvable group has a Carter subgroup and all its Carter subgroups are conjugate.

**Lemma 19.** Let $G$ be a finite, non-abelian, CLT and $A$-group of order $n$, where $|\pi(n)| \geq 3$. Then for some $s \in \pi(n)$ all the subgroups of index $s$ in $G$ are non-abelian.

**Proof.** Let $S$ be the set of all non-isomorphic prime index abelian subgroups of $G$. Since every element in $S$ is a maximal subgroup of $G$, hence they are either normal or self-normalizing. Suppose the hypothesis is not true. Then for every $s \in \pi(n)$ there exists an abelian subgroup of index $s$ in $G$. Therefore $|S| \geq 3$. Further, from Part 2 of Theorem 18, $S$ contains exactly one normal subgroup. Also if any two elements of $S$ are self-normalizing, then they are Carter subgroups, thus they are conjugate to each other. Which shows that $|S| \leq 2$. Hence we got a contradiction. \[\square\]

Now the proof of Theorem 17 is immediate. Our next goal is to show that every ACLT group is supersolvable. We prove the following lemma to prove the same.

**Lemma 20.** Let $G$ be an ACLT group and $q$ be the largest prime divisor of $o(G)$, then

1. Sylow $q$-subgroup is normal in $G$.

2. If $K$ is the Sylow $q$-subgroup of $G$, then $G' \leq K$.

3. Further, if $G$ is a non-abelian group, then $C_G(K)$, the centralizer of $K$ in $G$, is the index $p$ abelian subgroup of $G$.

**Proof.** Proof of Part 1. If $G$ is abelian or $p = q$, then we are done. Let $G$ be a non-abelian group and $o(G) = p^a q^b$, where $p < q$. Since $G$ is an ACLT group, it has an abelian subgroup $H$ of index $p$. If $K$ is a subgroup of $H$ of order $q^b$, then $K$ is also a Sylow $q$-subgroup of $G$. Further, $H \leq N_G(K)$ follows from the fact that $H$ is abelian. Which shows that either $N_G(K) = H$ or $N_G(K) = G$. If $N_G(K) = H$, then number of Sylow-$q$ subgroups of $G$ is $p$ which is not possible from third Sylow theorem. Hence $K$ is normal in $G$.

Proof of Part 2. We observe from Part 1 that $G$ is isomorphic to $K \times L$, where $L$ is a Sylow $p$-subgroup of $G$. We can check from the definition of commutator subgroup that $G' \leq K$. Hence the result follows.

Proof of Part 3. Since $H$ is abelian so we can see that $H \leq C_G(K)$. Consequently, either $C_G(K) = H$ or $C_G(K) = G$. If $C_G(K) = G$, then $K \leq Z(G)$. But from Part 2 we have $G' \leq K$. Hence we got contradiction from part 3 of Theorem 18 and the fact that $G$ is non-abelian. \[\square\]
Theorem 21. Every ACLT group is supersolvable.

Proof. If $G$ is abelian, then we are done. Let $G$ be a non-abelian ACLT group of order $p^aq^b$, where $p$ and $q$ are prime numbers. If $p = q$, then $G$ is supersolvable. If $p < q$ then by Part 1 of Lemma 20 $G$ has a normal Sylow $q$-subgroup say $K$. If $L$ is a Sylow $p$-subgroup of $G$, then $KL = G$. Which shows that $K$ and $KL$ are normal in $G$. Therefore $G$ has a Sylow-tower $\{e\} \leq K \leq KL$. Finally from Part 3 of Lemma 20 we have $N_G(K)/C_G(K)$ is a cyclic group of order $p$, thus $N_G(K)/C_G(K)$ is strictly $p$-closed. Also if $L$ be any Sylow $p$-subgroup of $G$ and $N$ be an abelian subgroup of index $q$ in $G$. Then we can show that $N_G(L) = C_G(L) = N$. Therefore $N_G(L)/C_G(L)$ is also strictly $p$-closed. Hence the result follows from Theorem 1.12 of [21].

Theorem 22. Every subgroup of an ACLT group is ACLT.

We need few lemmas before proving Theorem 22.

Lemma 23. Every subgroup of an ACLT group of order $p^n$, where $p$ is a prime number and $n \in \mathbb{N}$ is ACLT.

Proof. The proof of this is same as the proof of Lemma 7.

Lemma 24. Let $G$ be a finite group of order $n = p^aq^b$, where $p$ and $q$ are prime numbers. If $G$ has abelian subgroups of order $n_1 = p^{a-1}q^b$ and $n_2 = p^aq^{b-1}$ then $G$ is ACLT.

Proof. Let $H$ and $K$ be abelian subgroups of $G$ order $n_1$ and $n_2$ respectively. We can easily show that if $d|n$ and $d < n$ then either $d|n_1$ or $d|n_2$. Thus $G$ has an abelian subgroup of order $d$. Hence $G$ is ACLT.

Lemma 25. Let $G$ be an ACLT group of order $p^aq^b$, where $p$ and $q$ are prime numbers and $p < q$. Then all maximal subgroups of $G$ are ACLT.

Proof. From Theorem 21 $G$ is supersolvable, hence it follows from Theorem 1.7 of [21] that all maximal subgroups of $G$ are of prime index. Thus order of every maximal subgroup of $G$ is either $p^{a-1}q^b$ or $p^aq^{b-1}$. Let $M$ and $N$ be any two non-abelian subgroups of $G$ of order $p^{a-1}q^b$ and $p^aq^{b-1}$. Since $G$ is ACLT, it has abelian subgroups of order $p^{a-1}q^b$ and $p^aq^{b-1}$ say $H$ and $K$ respectively. Also $HM \subseteq G$ and $KM \subseteq G$ and

$$|HM| = \frac{o(H) \times o(M)}{o(H \cap M)} = \frac{p^{2a-2}q^{2b}}{o(H \cap M)} \leq p^aq^b.$$ 

Consequently, $o(H \cap M) = p^{a-2}q^b$, similarly one can see that $o(K \cap M) = p^{a-1}q^{b-1}$. With the similar approach one can show that $o(H \cap N) = p^{a-1}q^{b-1}$ and $o(K \cap N) = p^aq^{b-2}$. Therefore by Lemma 24 $M$ and $N$ are ACLT.

Now we will prove Theorem 22 by induction on the row index of the hasse diagram of $G$, starting from the top most row. The result is trivially true for the first row as $G$ is ACLT. Let us assume that the result is true for the $k^{th}$ row that is all the subgroups of $G$ in the $k^{th}$ row of the hasse diagram are ACLT. Let $H$ be a subgroup of $G$ in $(k+1)^{th}$ row. Then $H$ is a maximal subgroup of some subgroup of $G$ in the $k^{th}$ row say $N$. By induction hypothesis $N$ is ACLT. Then by Lemma 25 $H$ is ACLT. Hence all subgroups of $G$ are ACLT.
4 Conclusion

This section is devoted to summarize the relationship among cyclic, abelian CLT, ACLT and CCLT numbers using diagrammatic visualization.

1. Figure 1 summarize, how ACLT and CCLT numbers interact with other known numbers.

![Diagram](image)

Figure 1:

2. Given a natural number $n$ we define a graph $X_n = (V_n, E_n)$, where the vertex set $V_n = \mathcal{G}_n$, the set of all non-isomorphic groups of order $n$ and there is an edge between $G_i, G_j \in \mathcal{G}_n$ if $G_i \times G_j$ is an ACLT group. It is easy to check that $X_n$ is a complete graph if and only if $n$ is an abelian number or a CCLT number. For example,

$$\mathcal{G}_{28} = \{C_{28}, C_2 \times C_{14}, D_{14}, Dic_7\}.$$

Note $n = 28$ is an ACLT number, but one can see that corresponding graph is not a complete graph. It is also easy to see that the graph $X_n$ is a connected graph if and only if $n$ is an ACLT number. It is interesting to find various properties of these graphs.
Figure 2: Corresponding Graph of $G_{28}$.

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