Constraining Quark-Hadron Duality at Large-$N_c$

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ABSTRACT

Quark-meson duality for two-point functions of vector and axial-vector QCD currents is investigated in the large-$N_c$ approximation. We find that the joint constraints of duality and chiral symmetry imply degeneracy of excited vector and axial-vector mesons in the large-$N_c$ limit. We compare model-independent constraints with expectations based on the Veneziano-Lovelace-Shapiro string model. Several models of duality are constructed, and phenomenological implications are discussed.
1. Introduction

In the large-$N_c$ limit, QCD correlators of quark bilinears can be expressed as sums of zero-width meson tree graphs [1]. These sums must be infinite in order to be consistent with perturbative QCD logarithms at large momentum transfer. The detailed matching of hadronic and partonic degrees of freedom, known as quark-hadron duality [2,3,4], has been explicitly verified in QCD in 1+1 dimensions in the large-$N_c$ limit [5]. It has long been thought that the exchange of an infinite number of vector mesons is in some sense dual to the perturbative QCD continuum [6]. Early work uncovered various intriguing similarities between the simplest models of quark-meson duality and hadronic string models. Given the widespread belief that large-$N_c$ QCD is in some sense equivalent to a string theory, these similarities have received recent attention [7]. Following Ref. 7, in this paper we investigate duality in the large-$N_c$ limit in the simplest correlators that have an operator product expansion (OPE); i.e. two-point functions of vector and axial-vector currents. We point out that there are non-trivial chiral symmetry constraints which must be satisfied in addition to those constraints implied by duality. We discuss the interesting dilemma raised by simultaneous satisfaction of all constraints. These constraints suggest that there is an infinite tower of degenerate vector and axial-vector mesons in the large-$N_c$ limit. The phenomenological implications of this conjecture are considered in a simple model. As an example of a system with an infinite spectrum of mesons we consider how chiral symmetry is satisfied in the Lovelace-Shapiro-Veneziano (LSV) string model [8,9] and we investigate the implications of that model for duality.

2. Duality Constraints

In large-$N_c$ QCD, mesons have the most general quantum numbers of the quark bilinear $\bar{q}\Gamma q$ where $\Gamma$ is some arbitrary spin structure [1]. Hence all mesons have zero or unit isospin and transform as $(2,2)$, $(3,1)$ and/or $(1,3)$ with respect to $SU(2) \times SU(2)$ (to be precise, large-$N_c$ QCD has a $U(2) \times U(2)$ chiral symmetry). We will assume that the order parameter of chiral symmetry breaking in QCD with two massless flavors transforms as $(2,2)$. Assuming confinement, it then follows that chiral symmetry is spontaneously broken [10]. The conserved vector and axial-vector currents, $V^a_\mu$ and $A^a_\mu$ form a six-dimensional multiplet; hence they transform as $(3,1) \oplus (1,3)$. Consider the time-ordered product of vector currents

$$
\Pi^{\mu\nu}_{VV}(q) \delta_{ab} = 2i \int d^4xe^{iqx} \langle 0 | T[V^\mu_a(x)V^\nu_b(0)] | 0 \rangle.
$$

(1)

Here $\Pi_{VV}$ transforms as $(1,1) \oplus (3,3) \oplus \ldots$ with respect to $SU(2) \times SU(2)$. Lorentz invariance and current conservation allow the decomposition
\[
\Pi_{V,V}^{\mu\nu}(q) = (q'^i q^i - g^{\mu\nu} q^2) \Pi_V(Q^2),
\]
where \(Q^2 = -q^2\). Identical considerations for the AA correlator lead to \(\Pi_A(Q^2)\). One can write a dispersive representation of the function \(\Pi_{V,A}(Q^2)\) and saturate with an infinite number of zero-width meson states. This dispersion relation requires one subtraction, however we will assume an unsubtracted dispersion relation and track the divergent part. We find, in the large-\(N_c\) limit,

\[
\Pi_V(Q^2) = 2 \sum_{n=0}^{\infty} \frac{F_{V}^2(n)}{Q^2 + M_V^2(n)},
\]

\[
\Pi_A(Q^2) = 2 \frac{F_{A}^2}{Q^2} + 2 \sum_{n=0}^{\infty} \frac{F_{A}^2(n)}{Q^2 + M_A^2(n)},
\]

where \(F_{V,A}(n)\) and \(M_{V,A}(n)\) are the vector and axial-vector decay constants and masses, respectively. Because the functions \(\Pi_{V,A}(Q^2)\) transform as \((1,1)\), they have perturbative components which are easily computed in QCD perturbation theory. The Euler-Maclaurin summation formula implies the duality matching condition

\[
2 \int_0^\infty dn \frac{F_{V,A}^2(n)}{Q^2 + M_{V,A}^2(n)} + \mathcal{O}(Q^{-2}) \xrightarrow{Q^2 \to \infty} - \frac{N_c}{12\pi^2} \log Q^2 + \ldots + \sum_{m=1}^{\infty} \frac{\langle \mathcal{O} \rangle_{d=2m}^{V,A}}{Q^{2m}}
\]

where the dots correspond to the logarithmic divergence which appears on both sides of the equation and the \(\langle \mathcal{O} \rangle\)'s are Wilson coefficients of mass-dimension \(d\). The coefficient of the logarithm is computed in perturbative QCD [11,7]. The duality matching condition implies

\[
F_{V,A}^2(n)/M_{V,A}^2(n) \xrightarrow{n \to \infty} n^{-1}.
\]

In addition to this asymptotic constraint, there are constraints on the \(n\) dependences of the couplings and masses: (i) the existence of an OPE implies that the sums over \(n\) in Eq. (3) must generate functions which, aside from perturbative logarithms, are analytic in \(1/Q^2\); (ii) the coefficients of the OPE must have factorial behavior in \(n\); (iii) chiral symmetry must be preserved. We will address the issue of chiral symmetry in detail in the next section.

\[1\] Ref. 12 points out that for \(F_{V,A}^2(n) = F_{V,A}^2\), the sums over \(n\) in Eq. (3) are Euler \(\psi\)-functions which satisfy (i) and (ii). The occurrence of gamma functions is reminiscent of hadronic string models.
3. Chiral Constraints

3.1 Matching to Perturbation Theory
In the $Q^2 \to \infty$ limit, duality dictates that the infinite sums over vector and axial-vector meson states match to a perturbative expansion in $\alpha_s$. This expansion is defined in the asymptotically-free phase where chiral symmetry is unbroken. Therefore, in the matching region, each vector meson in the infinite sum must be paired with a *degenerate* axial-vector chiral partner; pair-by-pair they fill out *irreducible* $(1,3) \oplus (3,1)$ representations of the chiral group. This leads to the asymptotic constraints

\[
\frac{F_V^2(n)}{F_A^2(n)} \xrightarrow{n \to \infty} 1, \quad (6a)
\]

\[
\frac{M_V^2(n)}{M_A^2(n)} \xrightarrow{n \to \infty} 1. \quad (6b)
\]

We will see that these constraints are naturally incorporated in more general statements of chiral symmetry which will be derived below. Notice that if $M_{V,A}(n)$ is linear in $n$, Eq. (6b) implies a “universal” slope parameter.

3.2 Matching to the OPE
The procedures of expanding in $1/Q^2$ and summing over $n$ in $\Pi_{V,A}(Q^2)$ do not commute. This is due to the presence of logarithms which reorder the $1/Q^2$ expansion. Matching to the OPE must be achieved by summing over $n$ and only then expanding in $1/Q^2$. However, this non-commutativity is not true of the correlator

\[
\Pi_{LR}(Q^2) \equiv \frac{1}{2} \left( \Pi_V(Q^2) - \Pi_A(Q^2) \right) \xrightarrow{Q^2 \to \infty} \sum_{m=1}^{\infty} \frac{\langle 0 \rangle_{d=2m}^{(3,3)}}{Q^{2m}}. \quad (7)
\]

The subscript labeling the Wilson coefficients indicates that this correlator transforms as $(3,3)$ and therefore contains no perturbative logarithm. Hence performing the sum over $n$ does not rearrange the $1/Q^2$ expansion, and one can expand in $1/Q^2$ before performing the infinite sum over $n$. We will see in the next section that this commutativity is protected by chiral symmetry. Since the first two OPE coefficients in Eq. (7) vanish in QCD in the chiral limit, one reads directly from Eq. (3) the spectral-function sum rules in the large-$N_c$ limit [13,14]

\[
\sum_{n=0}^{\infty} F_V^2(n) - \sum_{n=0}^{\infty} F_A^2(n) = F_π^2, \quad (8a)
\]

\[
\sum_{n=0}^{\infty} F_V^2(n) M_V^2(n) - \sum_{n=0}^{\infty} F_A^2(n) M_A^2(n) = 0. \quad (8b)
\]
These sum rules must be satisfied by any model of large-$N_c$ QCD consistent with chiral symmetry. The asymptotic constraints of Eq. (6) are enforced by these sum rules.

### 3.3 Constraints from the Chiral Algebra

It will prove useful to give a derivation of Eq. (8) which is independent of the OPE [15,16] as it will allow contact with hadronic string models. For this purpose, it is convenient to work in the infinite momentum frame. This is a natural choice given our interest in large (Euclidean) momenta. Of course, the results that we derive are true in all frames. A useful property of the infinite-momentum frame is that the axial charges annihilate the vacuum, $Q_5^a|0\rangle = 0$. If we boost the vector mesons along the 3-axis to $p_3 = (p_0, 0, 0, p_3)$, we can write, in the $p_3 \to \infty$ limit,

\begin{align}
\langle 0| A_\mu^a | \pi^b \rangle &= \delta^{ab} F_\pi p_\mu, \\
\langle 0| A_\mu^a | A^{b(0)} \rangle &= \delta^{ab} F_A M_a \epsilon^{(0)}_\mu = \delta^{ab} F_A p_\mu + O(p_3^{-1}), \\
\langle 0| V_\mu^a | V^{b(0)} \rangle &= \delta^{ab} F_V M_{V\mu}^{(0)} = \delta^{ab} F_V p_\mu + O(p_3^{-1}),
\end{align}

where the superscripts in parentheses label the helicity, $\lambda$. It will prove worthwhile to consider matrix elements of the axial charges as well. The matrix element for a transition from a meson $\beta$ to a meson $\alpha$ and a pion in the infinite-momentum frame is given by

\[
\mathcal{M} (\beta (\lambda') \to \alpha (\lambda) + \pi_a) = (F_\pi)^{-1} \left( M_\alpha^2 - M_\beta^2 \right)^{(\lambda')} \langle \beta | Q^a_5 | \alpha \rangle^{(\lambda)} \delta_{\lambda \lambda'},
\]

where the Kronecker delta ensures helicity conservation and $Q^a_5$ is a conserved axial charge. We define

\begin{align}
\langle \pi_b | Q^a_5 | S \rangle &= -i \delta_{ab} G_{S\pi} / F_\pi, \\
\langle \pi_b | Q^a_5 | V_c \rangle &= -i \epsilon_{abc} G_{V\pi} / F_\pi, \\
A_b Q^a_5 | V_c \rangle &= -i \epsilon_{abc} G_{VA} / F_\pi.
\end{align}

Here $S$, $V$ and $A$ represent meson states with $I^G (J^{PC})$ given by $0^+$(even++), $1^+$(odd--) and $1^-$ (odd++), respectively. We suppress the helicity labels on the states as we are interested only in zero-helicity transitions.

Consider the following matrix elements of the chiral algebra

\begin{align}
\langle 0| [Q^a_5, V^b_\mu] | A^c \rangle &= i \epsilon^{abc} \langle 0| A^c_\mu | A^c \rangle, \\
\langle 0| [Q^a_5, A^b_\mu] | V^c \rangle &= i \epsilon^{abc} \langle 0| V^c_\mu | V^c \rangle, \\
\langle 0| [Q^a_5, V^b_\mu] | \pi^c \rangle &= i \epsilon^{abc} \langle 0| \pi^c_\mu | \pi^c \rangle, \\
\langle \pi_e | [Q^a_5, Q^b_\delta] | \pi_d \rangle &= i \epsilon_{abc} \langle \pi_e | T_c | \pi_d \rangle.
\end{align}
By inserting a complete set of states in the commutators and using Eq. (9), Eq. (11) and 
\( \langle \pi_a | T_b | \pi_c \rangle = i \epsilon_{abc} \), it is easy to derive a cornucopia of sum rules [16]. Consider, as an example, Eq. (12a); there is a sum rule for each axial-vector state, labeled by \( n' \). Using 
\( Q_5^5 | 0 \rangle = 0 \) and inserting a complete set of states yields

\[
- \sum_{n=0}^{\infty} \langle 0 | V^b \mu | V^f ; n \rangle \delta_{J,1} \langle V^f ; n | Q_5^a | A^c ; n' \rangle = i \epsilon_{abc} \langle 0 | A^c \mu | A^e ; n' \rangle, 
\]

where the Kronecker delta constrains the sum to spin-one \( V \) states. Using Eq. (9) and Eq. (11), it is easy to derive

\[
\sum_{n=0}^{\infty} F_V (n) G_{VA}^{j=1} (n, n') = F_\pi F_A (n'), 
\]

where the superscript indicates that the sum is over spin-one \( V \) states. The sum rules from Eq. (12) which are of relevance to this paper are

\[
\sum_{n=0}^{\infty} F_V^2 (n) - \sum_{n=0}^{\infty} F_A^2 (n) = F_\pi^2, 
\]

\[
\sum_{n=0}^{\infty} F_V (n) G_{VA}^{j=1} (n) = F_\pi^2, 
\]

\[
\sum_{n=0}^{\infty} G_{S\pi}^2 (n) + \sum_{n=0}^{\infty} G_{V\pi}^2 (n) = F_\pi^2. 
\]

The first sum rule is the first spectral-function sum rule. We now see that, in the large-\( N_c \) limit, this sum rule is true independent of the OPE; it is a simple consequence of chiral symmetry, which is encoded in the commutators of Eq. (12). The second and third sum rules constrain the pion vector form factor and \( \pi - \pi \) scattering, respectively [16].

There are additional sum rules which involve the meson masses, and which can be derived without the OPE [16]. These sum rules require the assumption that the order parameter of chiral symmetry breaking transforms purely as \( (2, 2) \). Those of relevance here are

\[
\sum_{n=0}^{\infty} F_V^2 (n) M_V^2 (n) - \sum_{n=0}^{\infty} F_A^2 (n) M_A^2 (n) = 0, 
\]

\[
\sum_{n=0}^{\infty} G_{S\pi}^2 (n) M_S^2 (n) - \sum_{n=0}^{\infty} G_{V\pi}^2 (n) M_V^2 (n) = 0. 
\]
The first sum rule is the second spectral-function sum rule. The second sum rule constrains $\pi - \pi$ scattering [16].

4. **Trouble with Mass Splittings**

In this section, we consider how duality and chiral symmetry constrain the meson decay constants and masses when $M_{V,A}^2(n)$ is a linear function of $n$. The duality constraint, Eq. (5), allows the general parametrization

\[
F_V^2(n) = F_V^2 + \tilde{\chi}_V(n) + \chi_V(n), \quad (17a)
\]
\[
F_A^2(n) = F_A^2 + \tilde{\chi}_A(n) + \chi_A(n) \quad (17b)
\]

where $\tilde{\chi}_{V,A}(n)$ and $\chi_{V,A}(n)$ are functions which vanish as $n \to \infty$. The general decomposition is such that $\sum_{n=0}^{\infty} \tilde{\chi}_{V,A}(n)$ is divergent (with no finite part) while $\sum_{n=0}^{\infty} \chi_{V,A}(n)$ is convergent. The first spectral-function sum rule, Eq. (8a), implies

\[
F_V = F_A \equiv F, \quad (18a)
\]
\[
\tilde{\chi}_V(n) = \tilde{\chi}_A(n) \equiv \tilde{\chi}(n), \quad (18b)
\]
\[
\chi_V(n) - \chi_A(n) \equiv \chi(n), \quad (18c)
\]
\[
\sum_{n=0}^{\infty} \chi(n) = F_\pi^2, \quad (18d)
\]

while the second spectral-function sum rule, Eq. (8b), requires

\[
M_{V,A}^2(n) = M_{V,A}^2 + \Lambda^2 n, \quad (19a)
\]
\[
\sum_{n=0}^{N} n \chi(n) = \frac{(M_A^2 - M_V^2)}{\Lambda^2} \left( F^2 (N + 1) + \sum_{n=0}^{N} \tilde{\chi}(n) + \sum_{n=0}^{\infty} \chi_V(n) \right) - F_\pi^2 \frac{M_A^2}{\Lambda^2} \quad (19b)
\]

where $M_{V,A}^2$ and $\Lambda^2$ are free parameters, and we have imposed a cutoff, $N$, on the number of vector and axial-vector mesons; Eq. (19b) should be satisfied in the limit $N \to \infty$. Note that Eq. (18a) and Eq. (19a) ensure compliance with Eq. (6a) and Eq. (6b), respectively.

This parametrization illustrates the difficulty in satisfying the chiral constraints and the duality constraints simultaneously. There would appear to be no solution, $\chi(n)$, which satisfies Eq. (18) and Eq. (19) with $M_V \neq M_A$. For instance, by naive power counting, Eq. (18d) requires that $\chi(n)$ vanish faster than $n^{-1}$ for large $n$. But with this asymptotic behavior, the sum in Eq. (19b) cannot generate the linear divergence necessary to balance
the equation. Therefore, given the assumption that \(M_{V,A}^2(n)\) is linear in \(n\), we find no solution to the duality and chiral constraints in the large-\(N_c\) limit with \(M_V \neq M_A\).

If the vector and axial-vector mesons are degenerate, \(M_V = M_A \equiv M\), and Eq. (19b) becomes

\[
\sum_{n=0}^{\infty} n \chi(n) = -F_\pi^2 \frac{M^2}{\Lambda^2}. \tag{20}
\]

By naive power counting, Eq. (20) and Eq. (18d) can be satisfied simultaneously if \(\chi(n)\) vanishes faster than \(n^{-2}\). We will return to the degenerate case below.

Group-theoretically the situation is as follows. As pointed out above, if the vector and axial-vector mesons are degenerate, pair-by-pair they fill out irreducible \((1,3) \oplus (3,1)\) representations of the chiral group, which is rather trivial. In the absence of degeneracy, the vector and axial-vector mesons generally fill out infinite-dimensional reducible sums of \((1,3), (3,1)\) and \((2,2)\) representations.

5. The Lovelace-Shapiro-Veneziano String Model

Ideally, one would like to find a smooth ansatz for \(F_{V,A}^2(n)\) which generates both chiral physics and perturbative physics. For vector and axial-vector squared-masses linear in \(n\) and degenerate, this involves finding the function \(\chi(n)\), which satisfies Eq. (18d) and Eq. (20). Hadronic string models are an interesting place to look for clues. Generally these models are interesting for large-\(N_c\) QCD because there are an infinite number of mesons exchanged\(^2\). Consider the following representation of the \(\pi - \pi\) scattering amplitude

\[
A(s,t,u) = -\frac{1}{2} \lambda \{ \Phi(\alpha_s, \alpha_t) + \Phi(\alpha_s, \alpha_u) - \Phi(\alpha_t, \alpha_u) \} \tag{21}
\]

where

\[
\Phi(a,b) \equiv \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-a-b)} = (1-a-b) B(1-a,1-b) \tag{22}
\]

and the linear Regge trajectory is

\[
\alpha_s = \alpha_0 + \alpha s. \tag{23}
\]

The parameter \(\lambda\), the intercept \(\alpha_0\) and the slope \(\alpha\) determine scattering. Chiral symmetry requires that the amplitude have an Adler zero at the point \(s = t = u = 0\). This determines \(\alpha_0 = 1/2\). Scattering is then consistent with the low-energy theorems of chiral symmetry if one takes \(\pi \lambda \alpha = F_\pi^{-2}\). Normalizing the Regge slope to the lightest exchanged state gives \((2\alpha)^{-1} = M_\rho^2\). Using

\(^2\) Reviews of this model are given in Ref. 17 and Ref. 18.
\[ \text{Im } \Phi(\alpha_s, \alpha_t) = -\pi \sum_{n=1}^{\infty} \frac{\Gamma(\alpha_t + n)}{\Gamma(n) \Gamma(\alpha_t)} \delta(\alpha_s - n) \]  

(24)

It is straightforward to extract the generalized couplings and masses as a function of \( n \). We find

\[ G_{V\pi}^2(n) = G_{S\pi}^2(n) = \frac{1}{2} \chi_{LSV}(n) \quad n = 0, 1, \ldots \]  

(25)

where

\[ \chi_{LSV}(n) \equiv \frac{F_{\pi}^2}{\pi} \frac{\Gamma(\frac{1}{2} + n)}{\Gamma(1 + n) (n + \frac{1}{2})}. \]  

(26)

and

\[ M_V^2(n) = M_S^2(n) = M_\rho^2(1 + 2n) \quad n = 0, 1, \ldots \]  

(27)

The sum rule, Eq. (15c), is then

\[ \sum_{n=0}^{\infty} G_{S\pi}^2(n) + \sum_{n=0}^{\infty} G_{V\pi}^2(n) = \sum_{n=0}^{\infty} \chi_{LSV}(n) = F_{\pi}^2 \]  

(28)

which is indeed satisfied. The states that participate in the string amplitude are therefore in an infinite-dimensional representation of the chiral group. This representation includes states of all spins. Notice that the mass sum rule, Eq. (16b), is trivially satisfied by Eq. (25) and Eq. (27), a consequence of the fact that the amplitude with \( I = 2 \) in the t-channel vanishes, by construction, in the LSV model [17,18].

6. **Stringy Implications for Duality**

6.1 **The LSV Model**

The LSV model is notable in that it satisfies the chiral constraints with an infinite number of mesons and is therefore consistent with large-\( N_c \) QCD. Given the symmetric appearance of the chiral sum rules in Eq. (15) one might consider \( \chi_{LSV}(n) \) as an ansatz for duality in Eq. (17) when the vector and axial-vector mesons are degenerate\(^3\). However, for \( n \) large, \( \chi_{LSV}(n) \to n^{-3/2} \), and therefore the sum in Eq. (20) does not converge. There is a further related problem with this ansatz. The sum over \( n \) is easy to do in the correlators of Eq. (3). For large \( Q \) the resulting functions contain fractional powers of \( 1/Q^2 \) and therefore do not

\[^3\] A generalization of the LSV model to pion scattering on an arbitrary hadronic target suggests \( M_\lambda^2(n) - M_V^2(n) = (2\alpha)^{-1} \) [19].
match to the OPE. This is no surprise since $\chi_{LSV}(n)$ generates Regge asymptotic behavior in $\pi - \pi$ scattering.

The chiral sum rule, Eq. (15b), relates $F_V(n)$ to $\pi - \pi$ scattering and thus links duality and the LSV model. Consider the ansatz

$$F_V(n) G_{V\pi}^{j=1}(n) = \chi_{LSV}(n) \quad n = 0, 1, \ldots$$

which satisfies Eq. (15b). Using the duality matching condition, Eq. (5), this implies $(G_{V\pi}^{j=1}(n))^2 \to n^{-3}$ for $n$ large. We can immediately put this to the test in the LSV model; partial-wave projection yields

$$(G_{V\pi}^{j=1}(n))^2 = \frac{3F_V^2}{\pi} (n + \frac{1}{2})^{-4} \int_{-n}^{\frac{1}{2}} \Gamma(x + n + 1) \Gamma(n + 1) \Gamma(x) (2x - \frac{1}{2} + n).$$

(30)

We have not succeeded in evaluating this integral to a simple expression. Asymptotically, one finds [20]

$$(G_{V\pi}^{j=1}(n))^2 \to n^{-5/2} (\log n)^{-1} \quad n \to \infty$$

(31)

which is not (quite) consistent with Eq. (29).

6.2 A Generalization of the LSV Model

The success of the LSV model in incorporating the chiral symmetry constraints suggests that it might be profitable to search for simple generalizations of $\chi_{LSV}(n)$ that are consistent with duality as well. Consider, for instance [21],

$$\chi(n, N_M, \alpha_0) \equiv F_V^2 \frac{\Gamma(N_M + \alpha_0) (-1)^n}{\Gamma(\alpha_0) \Gamma(N_M - n) \Gamma(1 + n) (n + \alpha_0)} \quad N_M > 0.$$ 

(32)

For integral values of $N_M$, $\chi(n, N_M, \alpha_0)$ vanishes for $n > N_M$, while for non-integral values $\chi(n, N_M, \alpha_0)$ is non-vanishing for all $n$. Using this function one can define a one-parameter coupling which interpolates between a finite and an infinite number of mesons. Note that $\chi(n, 1/2, 1/2) = \chi_{LSV}(n)$. We now have the asymptotic behavior

$$\chi(n, N_M, \alpha_0) \to n^{-(N_M+1)} \quad n \to \infty.$$ 

(33)

Therefore $\chi(n, N_M, \alpha_0)$ with $N_M > 1$ serves as an ansatz for duality when the vector and axial-vector mesons are degenerate. In effect, we find

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4 Ref. 21 considers $F_V(n) G_{V\pi}^{j=1}(n) = \chi(n, N_M, 1/2)$ as an ansatz for the pion vector form factor. For integer values, $N_M$ counts the number of vector mesons which contribute to the form factor. Evidently, a fit to data gives $N_M \sim 1.3$ which implies an infinite number of vector mesons.
\[
\sum_{n=0}^{\infty} \chi(n, N_M, \alpha_0) = F_\pi^2 \tag{34a}
\]
\[
\sum_{n=0}^{\infty} n \chi(n, N_M, \alpha_0) = -\alpha_0 F_\pi^2 \quad N_M > 1 \tag{34b}
\]

which is in agreement with Eq. (18d) and Eq. (20) when \( \alpha_0 = M^2/\Lambda^2 \). With \( \alpha_0 = 1/2 \) one finds the spectrum, Eq. (27), of the LSV model. This is not really surprising since the sum rules of Eq. (34) are a statement of chiral symmetry, and the Regge intercept in Eq. (27) was fixed using chiral symmetry.

7. Models of Duality

7.1 A String-Inspired Model

In this section we build a model of duality which is consistent with chiral symmetry and which has no discontinuity in \( n \). The vector and axial-vector mesons are degenerate so it has little to do with the real world. In our model we choose \( \tilde{\chi}(n) = 0 \) in Eq. (17)\(^5\). An ansatz consistent with duality and chiral symmetry is

\[
M_{V,A}^2(n) = M^2 + \Lambda^2 n, \quad \tag{35a}
\]
\[
F_{V}^2(n) = \frac{1}{\Lambda^2} \psi \left( \frac{M^2 + Q^2}{\Lambda^2} \right) + \ldots
\]
\[
F_{A}^2(n) = \frac{1}{\Lambda^2} \psi \left( \frac{M^2 + Q^2}{\Lambda^2} \right) + \ldots
\]

where \( \eta \) is a free parameter and \( N_M > 1 \). Inserting this ansatz into Eq. (3) and doing the sums over \( n \) yields

\[
\Pi_{V,A}(Q^2) = -\frac{2F_\pi^2}{\Lambda^2} \psi \left( \frac{M^2 + Q^2}{\Lambda^2} \right) + \ldots
\]
\[
+ \frac{2\eta F_\pi^2}{Q^2} \left[ 1 - \epsilon_{V,A} \frac{\Gamma \left( \frac{M^2}{\Lambda^2} + N_M \right) \Gamma \left( \frac{M^2 + Q^2}{\Lambda^2} \right)}{\Gamma \left( \frac{M^2}{\Lambda^2} \right) \Gamma \left( N_M + \frac{M^2 + Q^2}{\Lambda^2} \right)} \right], \tag{36a}
\]
\[
\Pi_{LR}(Q^2) = -\frac{F_\pi^2}{Q^2} \frac{\Gamma \left( \frac{M^2}{\Lambda^2} + N_M \right) \Gamma \left( \frac{M^2 + Q^2}{\Lambda^2} \right)}{\Gamma \left( \frac{M^2}{\Lambda^2} \right) \Gamma \left( N_M + \frac{M^2 + Q^2}{\Lambda^2} \right)} \tag{36b}
\]

\(^5\) If, for instance, \( \tilde{\chi}(n) \to n^{-1} \) for \( n \) large, its effect on duality is to generate logarithmic corrections to the OPE coefficients.
where the dots represent a logarithmic divergence; \( \epsilon_v = 1 \) and \( \epsilon_A = (\eta - 1)/\eta \). At large \( Q^2 \) we then have

\[
\Pi_{V,A}(Q^2) = -\frac{2F^2}{\Lambda^2} \log Q^2 + \ldots + \left[ 2\eta F^2_{\pi} - \frac{2F^2}{\Lambda^2} \left( M^2 - \frac{1}{2} \Lambda^2 \right) \right] \frac{1}{Q^2} \\
+ \frac{F^2}{\Lambda^2} \left( M^4 - M^2 \Lambda^2 + \frac{1}{6} \Lambda^4 \right) \frac{1}{Q^4} - \frac{2F^2}{3\Lambda^2} \left( M^2 - \frac{1}{2} \Lambda^2 \right) (M^2 - \Lambda^2) M^2 \frac{1}{Q^6} \\
- 2\eta \epsilon_{V,A} F^2_{\pi} \frac{\Gamma \left( \frac{M^2}{\Lambda^2} + N_M \right)}{\Gamma \left( \frac{M^2}{\Lambda^2} \right)} \Lambda^{2N_M} \frac{1}{Q^{2N_M+2}} + \mathcal{O} \left( Q^{-2N_M-4}, Q^{-8} \right) , \tag{37a}
\]

\[
\Pi_{LR}(Q^2) = F^2_{\pi} \frac{\Gamma \left( \frac{M^2}{\Lambda^2} + N_M \right)}{\Gamma \left( \frac{M^2}{\Lambda^2} \right)} \Lambda^{2N_M} \frac{1}{Q^{2N_M+2}} + \mathcal{O} \left( Q^{-2N_M-4} \right) . \tag{37b}
\]

Here we see that \( N_M \) must be an integer in order to match to the OPE. Hence \( N_M \) counts the number of vector and axial-vector mesons which contribute to the \( \Pi_{LR}(Q^2) \) correlator. In principle, one would expect \( N_M \) to be infinite. Taking \( N_M \) (arbitrarily) large and matching to the OPE gives

\[
F^2 = \frac{N_c}{24\pi^2} \Lambda^2 , \tag{38a}
\]

\[
\langle O \rangle_{V,A}^{d=2} = 0 = 2\eta F^2_{\pi} - \frac{N_c}{12\pi^2} \left( M^2 - \frac{1}{2} \Lambda^2 \right) , \tag{38b}
\]

\[
\langle O \rangle_{V,A}^{d=4} = \frac{\alpha_s}{12\pi} \langle G_{\mu\nu} G^{\mu\nu} \rangle = \frac{N_c}{24\pi^2} \left( M^4 - M^2 \Lambda^2 + \frac{1}{6} \Lambda^4 \right) , \tag{38c}
\]

\[
\langle O \rangle_{V}^{d=6} = -\frac{28}{9} \pi \alpha_s \langle \bar{q}q \rangle^2 = -\frac{N_c}{36\pi^2} \left( M^2 - \frac{1}{2} \Lambda^2 \right) (M^2 - \Lambda^2) M^2 , \tag{38d}
\]

\[
\langle O \rangle_{A}^{d=6} = -\frac{44}{9} \pi \alpha_s \langle \bar{q}q \rangle^2 = -\frac{N_c}{36\pi^2} \left( M^2 - \frac{1}{2} \Lambda^2 \right) (M^2 - \Lambda^2) M^2 , \tag{38e}
\]

\[
\vdots
\]

\[
\langle O \rangle_{V}^{d=2N_M+2} = -2\eta F^2_{\pi} \frac{\Gamma \left( \frac{M^2}{\Lambda^2} + N_M \right)}{\Gamma \left( \frac{M^2}{\Lambda^2} \right)} \Lambda^{2N_M} + \ldots , \tag{38f}
\]

\[
\langle O \rangle_{A}^{d=2N_M+2} = -2 (\eta - 1) F^2_{\pi} \frac{\Gamma \left( \frac{M^2}{\Lambda^2} + N_M \right)}{\Gamma \left( \frac{M^2}{\Lambda^2} \right)} \Lambda^{2N_M} + \ldots . \tag{38g}
\]

Since there is no local QCD operator with \( d = 2 \), we can fix \( \eta \) using Eq. (38b). For large \( N_M \), there is no solution with \( \langle \bar{q}q \rangle \neq 0 \). This is is not inconsistent with degenerate vector
and axial-vector mesons. While this model is clearly unrealistic, it provides an existence proof of a smooth chirally-invariant ansatz for duality with an infinite number of mesons.

7.2 A Minimal Realistic Model

One way to satisfy all constraints is to make an artificial separation between the low-energy physics relevant to chiral symmetry and the high-energy physics relevant to duality [22,7]. This requires introducing a discontinuity in $n$. A simple ansatz [22] is

\[
F_{V,A}^2(n) = \begin{cases} 
F_{\rho,a_1}^2 & n = 0 \\
F_{V,A}^2 & n > 0
\end{cases}, \quad (39a)
\]

\[
M_{V,A}^2(n) = \begin{cases} 
M_{\rho,a_1}^2 & n = 0 \\
M_{V,A}^2 + \Lambda_{V,A}^2(n - 1) & n > 0.
\end{cases} \quad (39b)
\]

Here we have extracted the lowest-lying vector and axial-vector mesons, $\rho$ and $a_1$, respectively. This is the minimal non-trivial model consistent with chiral symmetry. The duality and chiral constraints then imply

\[
F_V^2 = F_A^2 = \frac{N_c}{24\pi^2} \Lambda^2, \quad (40a)
\]

\[
M_V = M_A \equiv M, \quad (40b)
\]

\[
\Lambda_V = \Lambda_A \equiv \Lambda \quad (40c)
\]

where we have matched to the coefficient of the perturbative logarithm, and

\[
F_\rho^2 - F_{a_1}^2 = F_\pi^2, \quad (41a)
\]

\[
F_\rho^2 M_\rho^2 - F_{a_1}^2 M_{a_1}^2 = 0. \quad (41b)
\]

Notice that the vector and axial-vector mesons in the infinite tower are degenerate. With respect to the $\Pi_{LR}(Q^2)$ correlator, this simple ansatz has been investigated in many places [13,23,24]. Here $\pi$, $\rho$ and $a_1$, together with an isoscalar $S$, fill out a reducible (10-dimensional) $(1,3) \oplus (3,1) \oplus (2,2)$ representation, while all other vector and axial-vector mesons are in irreducible $(1,3) \oplus (3,1)$ representations. It is interesting that the chiral symmetry constraints effectively decouple the hadronic parameters $M$ and $\Lambda$ from low-energy chiral physics.\(^6\) Inserting the ansatz, Eq. (39), in Eq. (3), doing the sums over $n$ and matching to the OPE gives

---

\(^6\) The authors of Ref. 7 consider an ansatz given by Eq. (39) with $F_{a_1}^2 = 0$, match to the OPE and experience no such decoupling. However, they do not impose the sum rules of Eq. (8); according to Eq. (40) and Eq. (41), consistency of their ansatz with chiral symmetry requires $F_V^2 = F_A^2$, $M_V^2(n) = M_A^2(n)$ and $M_{\rho}^2 = 0$. In this case, $\pi$ and $\rho$ are in an irreducible $(1,3) \oplus (3,1)$ representation.
These values differ from the experimental values by amounts consistent with 
we then find Λ = 1189 MeV, which predicts M\_F\_fitting \( \approx 1078 \) MeV, compared with the experimental values φ\_corrections. We also predict φ\_parametrized by a single mixing angle, by large uncertainties in the values of the condensates. The relations of Eq. (41) can be condensates with this (or other) simple parametrizations of duality. This is hampered for the first few Wilson coefficients. One can develop a phenomenology for the QCD condensates with this (or other) simple parametrizations of duality. This is hampered by large uncertainties in the values of the condensates. The relations of Eq. (41) can be parametrized by a single mixing angle, φ, via \( F_\pi = F_\rho \sin \phi, F_{a_1} = F_\pi \cot \phi \) and \( M_\rho = M_{a_1} \cos \phi \). The known vector excited states are \( \rho'(1450), \rho''(1700) \) and \( \rho'''(2150) \) [25]. Fitting to \( \rho'(1450) \) we have \( M = 1450 \) MeV. Using Eq. (42) with \( F_x, M_\rho \) and M as input we then find Λ = 1189 MeV, which predicts \( M_{\rho''} = 1875 \) MeV and \( M_{\rho'''} = 2220 \) MeV. These values differ from the experimental values by amounts consistent with O(1/N\_c) corrections. We also predict φ = 44.4°, compared to the value φ = 37.4° resulting from fitting \( F_\rho \) directly to \( \rho^0 \to e^+e^- \) [25]. One then predicts \( F_{a_1} = 95 \) MeV and \( M_{a_1} = 1078 \) MeV, compared with the experimental values \( F_{a_1} = 122 \pm 23 \) MeV and \( M_{a_1} = 1230 \pm 40 \) MeV. The predicted condensates are \( \alpha_s \langle G_{\mu\nu} G_{\mu\nu} \rangle = 0.06 \) GeV\(^4\) and \( \pi \alpha_s \langle \bar{q}q \rangle^2 = 1.5 \times 10^{-3} \) GeV\(^6\), respectively. These values are somewhat large; recent determinations give \( \alpha_s \langle G_{\mu\nu} G_{\mu\nu} \rangle = 0.048 \pm 0.03 \) GeV\(^4\) [26] and \( \pi \alpha_s \langle \bar{q}q \rangle^2 = 9 \pm 2 \times 10^{-4} \) GeV\(^6\) [27].

This model predicts excited axial-vector states with masses \( M_{a_1'} = 1450 \) MeV, \( M_{a_{1''}} = 1875 \) MeV and \( M_{a_{1'''}} = 2220 \) MeV. The particle data group lists one excited axial-vector state, \( a_1'(1640) \) [25]. The splitting between this state and \( \rho'(1450) \) is consistent with an O(1/N\_c) correction. It will be very interesting to have new data on the spectrum of excited vector and axial-vector mesons. It is expected that the masses and widths of the low-lying excited vectors and axial-vectors will be determined in the Hall D program at Jefferson Laboratory in the near future [28].

8. Conclusion

Two-point functions of conserved vector and axial-vector QCD currents offer an interesting system to investigate quark-hadron duality. In the large-N\_c limit, the duality matching conditions are tractable and, in contrast with QCD in 1+1-dimensions, there are chiral symmetry constraints, which take a particularly simple form. Finding a smooth ansatz for duality, consistent with all constraints, is equivalent to finding the infinite dimensional
matrix which mixes the irreducible chiral representations filled out by the vector and axial-vector mesons. We find no smooth solution consistent with the duality and chiral symmetry constraints when the vector and axial-vector squared-masses are linear in \( n \) and non-degenerate. To avoid this degeneracy it would appear necessary to go beyond the Regge-type linear-spacing ansatz for the squared masses. In the large-\( N_c \) limit, the basic constraints of duality and chiral symmetry require vector-axial-vector degeneracy in the meson spectrum at sufficiently high excitation energy\(^7\). The characteristic energy at which degeneracy should set in is unknown. A simple realistic model, which predicts a tower of degenerate vector and axial-vector mesons, is roughly consistent with existing data.

Although hadronic string models provide important insight into how correlators determined by sums of infinite numbers of simple poles can be consistent with chiral symmetry, they do not provide an easy analog which satisfies the constraints of duality as well. Fundamentally this is because string models exhibit Regge asymptotic behavior for four-point functions, which is governed by fractional powers of the momentum transfer variable \( Q^2 \), while duality for two-point functions involves the OPE, which does not see fractional powers of \( Q^2 \). Hadronic string models do suggest simple generalizations which give smooth solutions to the joint duality and chiral constraints in the degenerate limit. However, the relation, if any, between these models and large-\( N_c \) QCD remains unclear.

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