Einstein–scalar–Gauss–Bonnet black holes: Analytical approximation for the metric and applications to calculations of shadows

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Recently, numerical solutions to the field equations of Einstein-scalar-Gauss-Bonnet (EsGB) gravity that correspond to black-holes with non-trivial scalar hair have been reported. Here, we employ the method of the continued-fraction expansion in terms of a compact coordinate in order to obtain an analytical approximation for the aforementioned solutions. For a wide variety of coupling functions to the Gauss-Bonnet term we were able to obtain analytical expressions for the metric functions and the scalar field. In addition we estimated the accuracy of these approximations by calculating the black-hole shadows for such black holes. Excellent agreement between the numerical solutions and analytical approximations has been found.

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I. INTRODUCTION

Nowadays black holes are the most important objects for understanding the regime of strong gravity. Observations in the gravitational [1–3] and electromagnetic [4, 5] spectra support General Relativity, but, at the same time, leave ample room for alternative theories of gravity [6, 7]. One of the most interesting alternative approaches is related to adding higher curvature corrections to the Einstein action. This kind of extensions of the Einstein gravity is inspired by the low energy limit of string theory [8, 9] and, presumably, could describe quantum corrected black holes. The lowest order correction is given by the (second order in curvature) Gauss-Bonnet term, which is pure divergence in four dimensional spacetimes, but, when coupled to a scalar field, it leads to modifications of the Einstein equations.

All the known black hole solutions in the four-dimensional Einstein-scalar-Gauss-Bonnet gravity are obtained either numerically [9–11], or perturbatively [12, 13], what makes it either difficult or impossible to use a number of tools for analysis of behavior of such solutions. Analytical expressions for such numerical black-hole metrics, which are valid in the whole space outside the event horizon, would allow us to see the explicit dependence of the metric on physical parameters of the system and to work with the metric as, essentially, with an exact solution. The approach to finding analytical approximations of numerical solutions was based on the general parametrization for spacetimes of static spherically symmetric black holes [14] and extended in [15] to axial symmetry. For spherical symmetry the parametrization uses a continued-fraction expansion in terms of a compactified radial coordinate. This choice leads to superior convergence properties and allows one to approximate a black-hole metric with a much smaller set of coefficients. This approach was used to construct the analytical approximation of numerical black-hole solutions in the Einstein-Weyl [16], Einstein-dilaton-Gauss-Bonnet [17] and Einstein-scalar-Maxwell [18] theories. Further studies of observables in these parametrized spacetimes [19–24] showed that usually it is sufficient only two-three orders of the continued fraction expansion in order to achieve reasonable accuracy.

In [17] the analytical approximation was found for the particular choice of the scalar field coupling - the dilaton, exponential coupling, which was considered numerically in [9]. Recently this approach was extended in [10] to various types of the scalar field function and allowed therefore, to look whether there are some common features for all the considered couplings of the scalar field. A similar problem was attacked numerically for case of the Einstein-scalar-Maxwell theory [25] and the study of its analytical approximation [18] showed that the radius of the black-hole shadow is increased for any of the considered couplings of the scalar field. Scalarization, that is the phenomenon of spontaneous acquiring of a scalar hair by the black hole as a result of the non-minimal coupling of a scalar field to the system, has been actively studied in [26–30].

Here we will generalize the procedure for finding the
analytical approximation in the Einstein-Gauss-Bonnet theory to the cases of various coupling functions of the scalar field. Then, we will apply the obtained parametrized black-hole metrics to the calculation of the radii of shadows in order to estimate the relative error due to the truncation of the continued fraction expansion which we used. We will also present the analytical expressions for both the radius of the photon sphere and the black-hole shadow.

The paper is organized as follows. In Sec. II we present the basics of the Einstein-scalar-Gauss-Bonnet theory. Sec. III is devoted to the introduction of the continued fraction expansion, while in Sec. IV we apply this procedure to the numerical solution of the EsGB black holes. Finally in Sec. V we find black hole shadows for the above numerical and parametrized black-hole metrics. In the Conclusions we summarize the obtained results and discuss the open questions.

II. BLACK HOLES IN EINSTEIN–SCALAR–GAUSS–BONNET GRAVITY

The Lagrangian for EsGB gravity reads

$$\mathcal{L} = R + f(\varphi)R_{GB}^2 - \frac{1}{2} \nabla^\alpha \varphi \nabla^\alpha \varphi,$$

(1)

where the Gauss-Bonnet (GB) term is defined as

$$R_{GB}^2 \equiv (R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}),$$

(2)

and $f(\varphi)$ is an arbitrary smooth function of the scalar field $\varphi$ corresponding to GB coupling.

In four dimensions, if $f(\varphi)$ is a constant, then the GB term is topological in the sense that it does not contribute to the field equations. In the case of an exponential coupling function $f(\varphi) \propto e^\varphi$ black-hole solutions with scalar hair emerge for EsGB gravity and the first solutions were obtained numerically in [9]. More recently, the authors of [31] have reported that regular black-hole solutions with scalar hair appear as a generic feature of the theory (1).

Let us start by considering the following line element for a static and spherically symmetric spacetime: $ds^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$ (3)

We also assume that the scalar field shares the symmetries of the underlying spacetime and it thus depends solely on the radial coordinate $r$.

The Einstein equations that are derived from the theory (1) are the following:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{1}{4} g_{\mu\nu} \partial_\rho \varphi \partial^\rho \varphi + \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) \eta^{\lambda\alpha\beta} \hat{R}^\gamma_{\alpha\beta} \partial_\gamma \varphi f(\varphi),$$

(4)

where

$$\hat{R}^\gamma_{\alpha\beta} = \eta^{\rho\sigma\tau} R_{\sigma\tau\alpha\beta} = \frac{e^{\rho\sigma\tau} R_{\sigma\tau\alpha\beta}}{\sqrt{-g}}.$$  

(5)

Also, the scalar field equation of motion is

$$\nabla^2 \varphi + f'(\varphi)R_{GB}^2 = 0,$$

(6)

where it is understood that throughout this article a prime indicates differentiation with respect to the argument of the function.

Numerical solutions to the field equations of EsGB gravity corresponding to black-holes with scalar hair have been recently found in [10] for a wide range of GB couplings. Here, by employing the method of [14] we shall obtain analytical approximations of these numerical solutions.

III. THE CONTINUED–FRACTIONS APPROXIMATION

In this section we outline the method of the continued-fractions approximation (CFA) [14] and introduce the notations we will use in the rest of the article.

In the original coordinate system of (3), the radius of the event horizon of the black hole $r_0$ is determined by the vanishing of the norm of the time-like Killing vector associated with the invariance of the metric under time translations. This condition eventually translates to $g_{tt}(r_0) = 0$. Then, we may perform a radial coordinate transformation and introduce the compact coordinate

$$x \equiv 1 - \frac{r_0}{r},$$

(7)

that ranges from $x = 0$ at the location of the horizon up to $x = 1$ at spatial infinity.

In the CFA, we consider a new metric ansatz that is suitable for approximating any spherically-symmetric metric to high-accuracy with only a small number of parameters [16, 17]. The metric coefficients of (3) are written in terms of the new set of functions $A(x)$ and $B(x)$ defined via the following relations:

$$g_{tt}(r) = x A(x),$$

$$g_{rr}(r) g_{rr}(r) = B(x)^2,$$

(8)

with

$$A(x) \equiv 1 - \epsilon (1 - x) + (a_0 - \epsilon)(1 - x)^2 + \tilde{A}(x)(1 - x)^3$$

$$B(x) \equiv 1 + b_0(1 - x) + \tilde{B}(x)(1 - x)^2,$$

(9)

where the parameter $\epsilon$ is determined by the value of the asymptotic mass $M$ of the black hole and the location of its event horizon $r_0$ as

$$\epsilon \equiv - \left(1 - \frac{2M}{r_0}\right).$$

(10)
The parameter $\epsilon$ clearly measures the amount of the deviation of the EsGB black-hole geometry from the Schwarzschild black hole, for which $r_0 = 2 M$. The parameters $a_0$ and $b_0$ are defined in terms of $\epsilon$ and the so-called parametrized post-Newtonian (PPN) parameters $\beta$ and $\gamma$ as

$$a_0 \equiv (\beta - \gamma)(1 + \epsilon)^2 \frac{1}{2},$$

$$b_0 \equiv (\gamma - 1)(1 + \epsilon)^2 \frac{1}{2}.$$  

The functions $\tilde{A}(x)$ and $\tilde{B}(x)$ have the delicate role of describing the metric near the horizon ($x = 0$) and are defined in terms of continued-fractions expansions as follows:

$$\tilde{A}(x) = \frac{a_1}{1 + \frac{a_2 x}{1 + \frac{a_3 x}{\ldots}}}.$$

$$\tilde{B}(x) = \frac{b_1}{1 + \frac{b_2 x}{1 + \frac{b_3 x}{\ldots}}}.$$  

The values of the parameters $a_i$ and $b_i$ for $i \geq 1$ can be obtained numerically upon expanding both sides of Eqs.(8) near the horizon and comparing coefficients of the same order in the expansion.

At this point let us mention that at spatial infinity the metric functions and the scalar field can be approximated as [31]

$$g_{tt}(r) = 1 - \frac{2 M}{r} + \frac{M D^2}{12 r^4} + \frac{24 M D f' + M^2 D^2}{6 r^4} + \mathcal{O}(1/r^5),$$

$$g_{rr}(r) = 1 + \frac{2 M}{r} + \frac{16 M^2 - D^2}{4 r^4} + \frac{32 M^3 - 5 M D^2}{4 r^3} + \frac{768 M^4 - 208 M^2 D^2 - 384 M D f' + 3 D^4}{48 r^4} + \mathcal{O}(1/r^5),$$

$$\varphi(r) = \varphi_\infty + \frac{D}{r} + \frac{M D}{r^2} + \frac{32 M^2 D - D^3}{24 r^3} + \frac{12 M^3 D - 24 M^2 f' - M D^3}{6 r^4} + \mathcal{O}(1/r^5),$$

where $\varphi_\infty$ is the asymptotic value of the scalar field and $D$ is its charge. Notice that the exact form of $f(\varphi)$ plays no role in the asymptotic expansions up to the third order. The form of Eqs.(14)-(15) implies that $\beta = \gamma = 1$ and thus $a_0 = b_0 = 0$ for any GB coupling function.

In the same spirit, an analytical approximation for the scalar field can also be obtained by means of the CFA [17]. One defines a new function of the compact coordinate that is related to the scalar field and its asymptotic value at spatial infinity via the following relation:

$$F(x) = e^{\varphi(0) - \varphi_\infty},$$

where the left hand side is expanded as

$$F(x) = 1 + f_0 (1 - x) + \tilde{F}(x) (1 - x)^2.$$  

The coefficient $f_0 = D/r_0$ is determined by the value of the charge of the scalar field and

$$\tilde{F}(x) = \frac{\sum_{i=1}^{f_1}}{1 - f_{2x}}.$$  

Again by expanding (17) near the event horizon one can obtain numerically the values of the coefficients $f_i$ for $i \geq 1$.

### IV. ANALYTICAL APPROXIMATIONS FOR ESGB BLACK HOLES

By employing the method described in the previous section we have derived analytical approximations for numerical black-hole solutions emerging in EsGB gravity. More precisely, for all the numerical solutions obtained for the different coupling functions studied in [10] we give here the approximate analytic metric coefficients.

Near the location of the event horizon we may expand the metric functions and the scalar field as follows:

$$g_{tt}(r) = p_1 \left[ (r - r_0) + \sum_{n=2}^{\infty} p_n (r - r_0)^n \right],$$

$$g_{rr}(r) = \sum_{n=1}^{\infty} q_n (r - r_0)^n,$$

and

$$\varphi(r) = \sum_{n=0}^{\infty} \varphi_n (r - r_0)^n,$$

where $\varphi_n \equiv \varphi^{(n)}(r_0)$ is the $n$–th order derivative of the scalar field evaluated on the location of the horizon.

The value of the scalar field on the horizon $\varphi_0$ is a free parameter, subject to the requirement $\varphi_1 \in \mathbb{R}$ in order for a black-hole solution to exist. Upon specifying the form of the coupling function $f(\varphi)$, the first derivative of the scalar field $\varphi_1$ on the horizon is uniquely determined for each value of $\varphi_0$ through the constraint [10]

$$\varphi_1 = \frac{r_0}{4 f'(\varphi_0)} \left( 1 - \frac{96 f'(\varphi_0)^2}{r_0^2} - 1 \right).$$
Once the values of the parameters $p_1, \varphi_0$, and $\varphi_1$ have been specified, the rest of the parameters $p_i, q_i$, and $\varphi_i$ in the expansions can be determined recursively up to an arbitrary order. This is achieved by substituting Eqs. (20) and (21) into the field equations and then solving the corresponding equations order by order in the expansion.

It is convenient to introduce a new dimensionless parameter $p \in [0, 1]$ instead of $\varphi_0$ to parameterize the family of black-hole solutions for each GB coupling function as follows:

$$p \equiv \frac{96f'(\varphi_0)^2}{r_0^4}, \quad (23)$$

with the Schwarzschild limit corresponding to $p = 0$ while for $p \to 1$ the maximal-coupling regime is approached.

For each value of $p$ we numerically integrate the field equations to obtain the accurate numerical solutions for the metric functions and the scalar field\footnote{The interested reader can find more details about the numerical black-hole solutions emerging in EsGB gravity in [10].}. The parameter $p_1$ is then fine-tuned such that for $r \to \infty$ we have $g_{tt}(r) \to 1$ and $g_{rr}(r) \to 1$.

With these solutions at hand, the next step is to determine the values of the asymptotic parameters of the system. The asymptotic mass $M$ is computed by expanding the solution for $g_{tt}(r)$ at large values of the radial coordinate and isolating the numerical coefficient of the term $\sim 1/r$. Then, according to (14), $M$ simply corresponds to $-1/2 \times \text{value of coefficient}$. This also determines the value of the parameter $\epsilon$ via (10). Similarly, the asymptotic values for $\varphi_\infty$ and $D$ of the scalar-field expansion (16) are determined via the corresponding coefficients of the expansion of the numerical solution for $\varphi(r)$.

The numerical values for the parameters $(p_i, q_i, \varphi_i)$ are thus determined as described above for each value of $p_0$ and $p_1$ and in this way through Eqs.(8) and (17) one finally ends up with numerical values for the set $(a_i, b_i, f_i)$. The above steps are repeated for different values of $p$ that span the allowed range $[0, 1]$ and numerical data are collected for $(a_i, b_i, f_i)$. Then, one is able to perform a fitting of these data in order to obtain analytical expressions for the CFA parameters as functions of $p$. It is then straightforward to write down approximate analytical expressions for the metric functions and the scalar field to the desired order in the CFA via (9) and (18).

**A. The even–polynomial coupling function**

The first case we study is the even-polynomial coupling function

$$f(\varphi) = \frac{1}{4} \varphi^{2n}, \quad n \in \mathbb{Z}^+, \quad (24)$$

The form of the dimensionless parameter (23) for this family of black-hole solutions is

$$p = \frac{(24n^2) \varphi_0^{4n-2}}{r_0^4}, \quad (25)$$

and the allowed values for $\varphi_0$ are thus

$$|\varphi_0| \lesssim \left( \frac{24n^2}{r_0^4} \right)^{\frac{1}{4n-2}}. \quad (26)$$

In order to be able to perform the analysis we need to reduce the number of free parameters and so we choose as an indicative value $n = 1$ in (24).

The obtained analytical expressions for the parameters of the CFA (9),(13),(18) and (19) up to second order are given below

$$a_1 = \frac{63p}{417} - \frac{23p^2}{143} + p - \frac{221}{279}, \quad (27)$$

$$a_2 = -\frac{234p^2}{107} + \frac{152p}{228} + 1, \quad (28)$$

$$\epsilon = \frac{p}{14} - \frac{p^2}{201} + \frac{1}{57p^7}. \quad (29)$$

The profile of the $\epsilon$ parameter with respect to $p$ is depicted in Fig.1 for all the GB-couplings we have studied in this article.

The above parameters alone suffice for the determination of the analytical representation of $g_{tt}(r)$. We point out that a general feature of the approximate expressions for both the metric functions and the scalar field is that the relative error (RE) increases with $p$. For the GB-coupling $f(\varphi) = \varphi^2/4$ when $p = 0.8$ in Fig.2 we plot the

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**FIG. 1.** The asymptotic parameter $\epsilon$ (10) as a function of the dimensionless parameter $p$ (23) for different GB-coupling functions.
RE between the fourth-order analytical approximation for the $g_{tt}(r)$ metric function and its accurate numerical solution. The maximum error occurs around the photon sphere radius at $r \approx 1.5r_0$ and is less than $0.24\%$.

In turn, the analytical approximation of $g_{rr}(r)$ emerges via (8) and thus requires also the expressions for the parameters $b_i$ that are listed below

$$b_1 = \frac{91p}{306} + \frac{85p^2}{318} + \frac{248p^3}{131} + p - \frac{632}{707}, \quad (30)$$

$$b_2 = -\frac{173p^2}{422} + \frac{47p^3}{222} - \frac{106p^4}{203} + 1, \quad (31)$$

Finally for the scalar field the analytically approximated parameters for the CFA are found to be

$$\phi_\infty = \frac{-9p^2}{65} + \frac{-p}{67} + \frac{1}{139}, \quad (32)$$

$$f_0 = -\frac{27p^3}{406} + \frac{7p^4}{61} + \frac{3p^5}{139} + \frac{1}{6338} + \frac{1}{105}, \quad (33)$$

$$f_1 = -\frac{23p^5}{30} + \frac{3p^6}{70} + \frac{1}{1103} + \frac{1}{4p^2} + p + \frac{1}{107}, \quad (34)$$

$$f_2 = -\frac{23p^6}{85} - \frac{30p^7}{97} + \frac{5p^8}{271} + \frac{1}{8997}, \quad (35)$$

In Fig. 3 we plot the corresponding REs for the $g_{rr}(r)$ metric function and the scalar field $\phi(r)$ both at fourth order in the CFA. The expressions for the second-order analytical approximations for the metric functions and the scalar field can be found in Appendix A for all the GB couplings studied in this article².

### B. The odd–polynomial coupling function

The odd–polynomial coupling function is

$$f(\varphi) = \frac{1}{4}\varphi^{2n+1}, \quad n \in \mathbb{N}. \quad (36)$$

For this coupling the dimensionless parameter has the following form:

$$p = \frac{6(2n + 1)^2}{r_0^4}\varphi_0^{4n}, \quad (37)$$

and the allowed values of $\varphi_0$ are

$$|\varphi_0| \leq \left(\frac{r_0^4}{6(2n + 1)^2}\right)^{\frac{1}{4n}}. \quad (38)$$

For $n = 1$, the approximate analytic expressions for the parameters are given below

$$a_1 = \frac{11p}{115} + \frac{79p^2}{658} + \frac{79p^3}{405} + p - \frac{231}{328}, \quad (39)$$

$$a_2 = -\frac{179p^3}{182} + \frac{97p^4}{271} + \frac{97p^5}{360} - \frac{41p^6}{212} + \frac{41p^7}{295} + \frac{41p^8}{360}, \quad (40)$$

$$c = -\frac{p^3}{196} + \frac{7p^4}{214} + \frac{3p^5}{406}, \quad (41)$$

$$b_1 = \frac{33p}{185} + \frac{58p^2}{201} + \frac{58p^3}{214} + p - \frac{217}{414}, \quad (42)$$

$$b_2 = \frac{p^3}{196} - \frac{119p^4}{247} + \frac{186p^5}{227} - \frac{186p^6}{127}, \quad (43)$$

$$\phi_\infty = \frac{p^3}{65} + \frac{10p^4}{91} + \frac{5p^5}{77} + \frac{6p^6}{243} + p + \frac{1}{1057}, \quad (44)$$

$$f_0 = -\frac{p^3}{29} + \frac{11p^4}{84} + \frac{13p^5}{175} + \frac{1}{1001}, \quad (45)$$

² We only give the second-order expressions in the appendix for reasons of compactness but in the accompanying Mathematica® file one can obtain the analytical expressions up to fourth order.
\[ f_1 = \frac{13p^2}{121} + \frac{17p}{17} + \frac{1}{904}, \quad (46) \]
\[ f_2 = -\frac{63p^3}{93} + \frac{8p^2}{117} + p + \frac{4}{177}, \quad (47) \]

C. The inverse–polynomial coupling function

The inverse-polynomial coupling function is
\[ f(\varphi) = \frac{1}{4} \varphi^{-n}, \quad n \in \mathbb{Z}^+, \quad (48) \]
and the dimensionless parameter for this family of black-hole solutions turns out to be
\[ p = \frac{6n^2}{r_0^2} \left( \frac{1}{\varphi_0} \right)^{2(n+1)}. \quad (49) \]

The allowed range of values for \( \varphi_0 \) in this case is
\[ |\varphi_0| \geq \left( \frac{6n^2}{r_0^2} \right)^{\frac{1}{2(n+1)}}. \quad (50) \]

Once again, we fix \( n = 1 \) in order to perform the analysis.

The approximate analytic expressions for the parameters in this case are given below
\[ a_1 = \frac{13p^3}{121} - \frac{4p^2}{19} - \frac{16p}{281} + 1, \quad (51) \]
\[ a_2 = -\frac{58p^3}{875} + \frac{p^2}{540} - \frac{31p}{292} - \frac{13}{307}, \quad (52) \]

D. The logarithmic coupling function

Finally we turn to the logarithmic coupling function
\[ f(\varphi) = \frac{1}{4} \log \varphi, \quad (60) \]
the dimensionless parameter for this type of black-holes is

\[ p = \frac{6}{1 + 4\phi_0}, \]

and the allowed values of \( \varphi_0 \) are

\[ |\varphi_0| \geq \frac{\sqrt{6}}{2}. \]

The approximate analytic expressions for the parameters in this case are given below

\[ a_1 = \frac{11p^3}{270} - \frac{113p^2}{288} + \frac{18p}{79} + \frac{605p^2}{1314} + p - \frac{425}{787}, \]

\[ a_2 = -\frac{586p^3}{815} + \frac{367p^2}{255} + p - \frac{384}{611}, \]

\[ \epsilon = \frac{105p}{1018} - \frac{44p^2}{121} + \frac{1 - 44p^2}{657}, \]

\[ b_1 = \frac{85p}{327} - \frac{57p^2}{305} - \frac{29p^2}{192} + p - \frac{222}{267}, \]

\[ b_2 = -\frac{66p^2}{197} + p - \frac{262}{383}, \]

\[ \varphi_{\infty} = -\frac{277p^3}{458} + \frac{159p^2}{246} + p + \frac{10}{253}, \]

\[ f_0 = \frac{94p^3}{305} - \frac{17p^2}{171} + \frac{73p}{347} + \frac{1}{30}, \]

\[ f_1 = -\frac{83p^3}{283} + \frac{169p^2}{239} + \frac{28p}{285}, \]

\[ f_2 = \frac{81p^3}{283} + \frac{37p^2}{378} - \frac{450p}{571} - \frac{29}{802}, \]

At this point, one must mention an important phenomenon, the eikonal instability, which takes place when the Gauss-Bonnet term is turned on. Once the Gauss-Bonnet coupling constant is not small enough, the black hole solution suffers from a dynamical instability: if linearly perturbed, the perturbation grows unboundedly.

The linear instability breaks down and the regime of small perturbations, indicating that the black hole cannot exist in this range of parameters.

The instability brought by the Gauss-Bonnet term is of special kind: it develops at high multipole numbers, so that the summation over the multipole numbers cannot be valid anymore. This kind of instability was first observed for the higher dimensional Einstein-Gauss-Bonnet black holes [32] and later observed for a number of other cases, including black branes [33], asymptotically dS and AdS black holes [34–36], black holes and branes in theories with higher than the second order in curvature corrections [37–40]. In some cases, the instability occurs not only for the gravitational perturbations, but also for the test scalar field [41].

As the eikonal instability is a very wide phenomena which, it seems, does not depend on a particular form of the higher curvature correction, we believe that it must be present also for the Einstein-scalar-Gauss-Bonnet theory at least once the scalar coupling is not strong enough.

Therefore, the regime of near extremal \( p \), corresponding to the maximal coupling, most probably does not represent any realistic stable black hole. Exactly in this regime our continued fraction expansion converges slowly. On the contrary, in the regime where one can expect stable configuration the second order expansion is sufficient to constrain the relative error by fractions of one percent.

In other words, our analytical approximation must very accurate already at the second order, once one is limited by stable configurations.

V. BLACK–HOLE SHADOWS AND ACCURACY OF THE ANALYTICAL APPROXIMATION

In the previous sections we have obtained approximate analytical expressions for the metric functions and the scalar field up to fourth order in the CFA. In all cases, we have found excellent agreement between the numerical and analytical solutions by computing the RE. Still, the metric itself is not gauge invariant and comparison of various metric functions does not allow us to determine the accuracy of the analytical approximation. For the latter one needs to consider some gauge invariant, observable quantity.

Recently black-holes shadows have been intensively studied for various theories of gravity and astrophysical environment [42–62]. In this section we perform the computation of the shadows cast by the EsGB black holes numerically. For different orders in the continued-fractions approximation we compute the shadows and compare them with the numerical ones. This way we have a gauge invariant measure of the accuracy of our approximation.

The radius of the photon sphere \( r_{ph} \) of a black hole in the coordinate system of (3) is determined by means of the following function: (see, for example, [63, 64] and
references therein)
\[ h^2(r) \equiv \frac{r^2}{g_{tt}(r)}, \]  
(72)
as the solution to the equation
\[ \frac{d}{dr} h^2(r) = 0. \]  
(73)
Then, the radius of the black-hole shadow \( R_{sh} \) as seen by a distant static observer located at \( r_O \) will be
\[ R_{sh} = \frac{h(r_{ph})r_O}{h(r_O)} = \frac{r_{ph} \sqrt{g_{tt}(r_O)}}{\sqrt{g_{tt}(r_{ph})}} \approx \frac{r_{ph}}{\sqrt{g_{tt}(r_{ph})}}, \]  
(74)
where in the last equation we have assumed that the observer is located sufficiently far away from the black hole so that she/he is deep in the asymptotically-flat regime i.e. \( g_{tt}(r_O) \approx 1 \).

In the case of the Schwarzschild black hole it is known that \( r_{ph} = 1.5 r_0 \) and so according to (74) the shadow is \( R_{sh} \approx 2.59808 r_0 \). For the EsGB black holes, the deviations from these two limiting values are expected to increase with the parameter \( p \) as we move further and further away from the Schwarzschild limit \((p = 0)\). This is indeed the case as the plots for the numerical values of \( r_{ph} \) and \( R_{sh} \) reveal in Fig.4. We point out that although \( r_{ph} \) is a non-observable auxiliary quantity, which is not gauge-invariant, it is very useful in many applications beyond the computation of black-hole shadows. To this end, its profile with \( p \) as depicted in Fig.4 provides useful information. Also, in Appendix C, the interested reader can find approximate analytical expressions for these two quantities.

Having obtained the accurate solutions for the shadows numerically we can now compare how each order in the CFA stands against the numerical solutions. By terminating the series of the expansion of \( A(x) \) (13) each time at \( a_2, a_3 \) and \( a_4 \) we obtain the second, third and fourth order analytical approximation for the \( g_{tt}(r) \) metric function respectively.

The absolute RE of the analytical approximation of the black-hole shadow \( R_{sh}(p) \) from the numerical solution \( R_{sh} \), at each value of the dimensionless parameter \( p \), and for the three different orders in the CFA is given in Fig.5.
(left panel) for \( f(\varphi) = \varphi^2/4 \), in Fig.5 (right panel) for \( f(\varphi) = \varphi^3/4 \), in Fig.6 (left panel) for \( f(\varphi) = \varphi^{-1}/4 \) and in Fig.6 (right panel) for \( f(\varphi) = \log(\varphi)/4 \).

The analytic expressions for the metric functions in the second and fourth order in CFA deviate from the numerical ones by less than 1% for almost the entirety of the GB couplings that we have studied in this work. Only for the inverse polynomial case, the deviation is slightly larger but still smaller than 1.4%. It is noteworthy that these maximal values for the RE actually emerge in the large-\( p \) regime where the black-holes are presumably unstable. Thus for viable black-hole solutions the RE is quite small.

Notice that the third order approximation gives slightly worse accuracy for some values of \( p \) than those of the second and fourth order. We believe that this happens because the fitting procedure keeps some uncertainties: there are a number of ways to make better fitting of the parameters at the lower orders, but worse at the higher ones and vice versa.

### VI. CONCLUSIONS

In the context of Einstein-scalar-Gauss-Bonnet gravity, a plethora of black hole solutions with non-trivial scalar hair emerge for different coupling functions to the Gauss-Bonnet (GB) term [31]. This has been recently demonstrated in [10] where numerical solutions to the field equations have been obtained for four different GB-couplings (even-, odd-, inverse-polynomial and logarithmic). In this work, we employed the powerful method of the continued-fractions approximation [14] in order to obtain analytic expressions for the metric functions and the scalar field for the aforementioned GB-couplings.

For each coupling function we parametrized the family of black hole solutions that emerge in terms of a dimensionless compact parameter \( p \) that ranges from 0 (Schwarzschild limit) to 1. The analytical representation is based on the continued fraction expansion which converges quickly for all values of \( p \) except the regime of near extremal coupling, when \( p \) is close to unity. It is known that in this regime, Gauss-Bonnet black holes (as well as all the other known higher curvature corrected black holes and branes whose gravitational perturbations were investigated) are unstable and, therefore, cannot exist. Although the (in)stability for the above considered couplings of the scalar field have not been studied in the literature so far, we assume that at least in the regime of the weak scalar field, the instability should remain. It would be interesting to check this supposition on the instability of EsGB black holes in the future.

We performed the computation up to the forth order in the continued-fractions expansion and we have found that the deviation of the analytic expressions from the accurate numerical ones is at most of the order of \( \mathcal{O}(1) \% \) for stable black-hole configurations. This observation alone is not sufficient to guaranty the high accuracy of the approximation since the metric coefficients are not gauge-invariant quantities.

To this end, in order to make a concrete and gauge invariant statement about the accuracy of the approximation we turned to the black-hole shadows cast by the EsGB black holes. We computed the shadows for four GB-couplings numerically and compared them against the approximate results obtained via the analytical approximation to second, third and fourth order. We found that already in the second order, the largest relative error for the analytical approximations emerges in the maximal coupling limit \( p \to 1 \) and is less than 1%.

We noticed that all the considered coupling functions lead to an increase of the radius of the black hole shadow. In addition, we have obtained analytical expressions for the photon sphere which increases for all the couplings as well. The obtained here analytical representation for the black-hole metrics and scalar fields in the Einstein-scalar-Gauss-Bonnet theory allows one to explore various analytical, semi-analytical and numerical tools in order...
to study various effects in the background of these solutions, such as accretion of matter, quasinormal modes, scattering, Hawking radiation and others.

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Appendix A: Analytical expressions for the metric functions and the scalar field to second order in the CFA

In this Appendix we give the explicit expressions for the analytical approximations for the metric functions and the scalar field in second order in the CFA.

These functions are rational functions of \( r \), which we give in the following form:

\[
g_{tt}(r) \approx \frac{N^{(1)}_{\text{exc}}}{D^{(1)}_{\text{exc}}} \left( 1 - \frac{r_0}{r} \right), \quad (A1)
\]

\[
\sqrt{g_{tt}(r)g_{rr}(r)} \approx \frac{N^{(2)}_{\text{exc}}}{D^{(2)}_{\text{exc}}}, \quad (A2)
\]

\[
e^{\phi(r) - \phi_{\infty}} \approx \frac{N^{(3)}_{\text{exc}}}{D^{(3)}_{\text{exc}}}, \quad (A3)
\]

where the numerators, \( N^{(1)}, N^{(2)}_{\text{exc}}, N^{(3)}_{\text{exc}} \), and the denominators, \( D^{(1)}, D^{(2)}, D^{(3)} \), for each of the functions are given for each coupling separately.

1. Even–polynomial GB coupling: \( f(\phi) = \phi^2 / 4 \)

\[
g_{tt}(r) \approx \frac{N^{(1)}_{\text{exc}}}{D^{(1)}_{\text{exc}}} \left( 1 - \frac{r_0}{r} \right), \quad (A4)
\]

\[
\sqrt{g_{tt}(r)g_{rr}(r)} \approx \frac{N^{(2)}_{\text{exc}}}{D^{(2)}_{\text{exc}}}, \quad (A5)
\]

\[
e^{\phi(r) - \phi_{\infty}} \approx \frac{N^{(3)}_{\text{exc}}}{D^{(3)}_{\text{exc}}}, \quad (A6)
\]

where

\[
N^{(1)}_{\text{exc}} = p^6 \left( 0.0273395 r_0^3 - 0.027395 r_0^2 r_0 + 0.0114842 r_0 + 0.273615 r_0^3 + 10.1241 r_0^2 r_0 + 0.121014 r_0^4 + 1.15224 r_0^5 + p^3 \left( 29.8132 r_0^3 - 34.4461 r_0^2 - 0.419751 r_0^2 - 3.0735 r_0^3 \right) + p^2 \left( -7.36175 r_0^3 + 25.9669 r_0^2 + 0.540161 r_0^2 + 3.678783 r_0^3 + 27.879 r_0^2 r_0 - 0.230379 r_0^2 - 1.48583 r_0^3 + 39.8966 r_0^3 - 29.9903 r_0^2 r_0 \right) \right)
\]

\[
D^{(1)}_{\text{exc}} = p^5 r^2 (r - r_0) + p^4 r^4 (10.8289 r_0 - 10.4088 r_0) + p^3 r^3 (29.8132 r_0 - 34.5843 r_0) + p^2 r^2 (24.7365 r_0 - 7.36175 r_0) + p r^2 (28.8068 r_0 - 51.7172 r_0 + r^2 (39.8966 r_0 - 29.9903 r_0), \quad (A7)
\]

\[
N^{(2)}_{\text{exc}} = p^6 \left( 5.97995 r_0 - 5.97995 r_0^2 \right) + p^5 \left( 9.25097 r_0^2 - 9.03038 r_0 + 0.233666 r_0^2 \right) + p^4 \left( -13.5895 r_0^2 + 12.1594 r_0^2 - 0.516068 r_0^2 \right) + p \left( 22.3021 r_0^2 - 19.9922 r_0 + 0.283453 r_0^2 \right) - 13.2868 r_0 + 12.1842 r_0, \quad (A8)
\]

\[
D^{(2)}_{\text{exc}} = p^5 r (r - r_0) + p^4 r^3 (5.97995 r_0 - 5.97995 r_0) + p^3 r (9.25097 r_0 - 9.03038 r_0 + 0.233666 r_0^2) + p^2 r (12.1594 r_0 - 13.5895 r_0) + p \left( 22.3021 r_0^2 - 19.9922 r_0 + r (12.1842 r_0 - 13.2868 r_0), \quad (A9)
\]

\[
N^{(3)}_{\text{exc}} = p^6 \left( 1.329 r_0^2 - 0.10258 r_0 + 0.27448 r_0^2 \right) + p^5 \left( 3.11961 r_0^2 - 1.86997 r_0 + 0.95122 r_0^2 \right) + p^4 \left( -7.3476 r_0^2 + 7.67617 r_0^2 - 1.40974 r_0^2 \right) + p^3 \left( -21.022 r_0^2 + 12.5531 r_0^2 + 0.029329 r_0^2 \right) + p^2 \left( 11.5435 r_0^2 - 9.17859 r_0 + 0.0986619 r_0^2 \right) + p^1 \left( 18.7555 r_0^2 - 13.3789 r_0 + 0.15665 r_0^2 \right) + p^0 \left( 1.93962 r_0^2 - 1.3871 r_0 + 0.00373914 r_0^2 \right) + p \left( 0.050048 r_0^2 - 0.0361114 r_0 + 0.0000582519 r_0^2 \right) + 4.035 \times 10^{-6} r_0 + 1.495 \times 10^{-8} r_0 + 5.495 \times 10^{-8} r_0, \quad (A10)
\]
\(\mathcal{D}_{\text{eve}}^{(3)} = p^8 r (1.0 - 1.14693 r_0) + p^7 r (3.11961 r - 2.21898 r_0) + p^6 r (8.47066 r_0 - 7.34762 r) + p^5 r (14.279 r_0 - 21.0622 r) + p^4 r (11.5435 r - 11.0756 r_0) + p^3 r (18.7555 r - 13.9269 r_0) + p^2 r (1.93962 r - 1.41739 r_0) + p r (0.050048 r - 0.0362987 r_0) + 4.035 \times 10^{-6} r^2.\)  

(A12)

2. Odd-polynomial GB coupling: \(f(\varphi) = \varphi^3/4\)

\[
\frac{g_{tt}(r)}{r} \approx \frac{\mathcal{N}^{(1)}_{\text{odd}} / \mathcal{D}^{(1)}_{\text{odd}}}{1 - \frac{r_0}{r}},
\]

(A13)

\[
\sqrt{g_{tt}(r) g_{rr}(r)} \approx \frac{\mathcal{N}^{(2)}_{\text{odd}} / \mathcal{D}^{(2)}_{\text{odd}}}{1 - \frac{r_0}{r}},
\]

(A14)

\[
e^{-r(r)} \approx \frac{\mathcal{N}^{(3)}_{\text{odd}} / \mathcal{D}^{(3)}_{\text{odd}}}{1 - \frac{r_0}{r}},
\]

(A15)

where

\[
\mathcal{N}^{(1)}_{\text{odd}} = p^8 (0.00537634 r_0^2 - 0.00537634 r_0^4) + p^7 (-0.0518836 r_0^2 + 0.000105719 r_0^4 + 0.0529408 r_0^6) + p^6 (1.03^3 - 0.86139 r_0 - 0.010476 r_0^2 - 0.149018 r_0^4 + 0.029677 r_0^6 + 0.211035 r_0^8) + p^5 (3.89682 r_0^3 - 4.75407 r_0^2 - 0.0293161 r_0^4 - 0.143895 r_0^6 + 0.0567227 r_0^8 + 0.467441 r_0^{10} + 0.00542841 r_0^{12} + 0.0284754 r_0^{14} + 0.13561 r_0^3 + 1.05084 r_0^5 + 0.00352281 r_0^7 + 0.00513373 r_0^9 + 0.0000480655 r_0^{11} + 0.0000822131 r_0^{13}),
\]

(A16)

\[
\mathcal{D}^{(1)}_{\text{odd}} = p^8 r^2 (1.0 - r_0) + p^7 r^2 (4.15743 r_0 - 3.96079 r_0 + 4.71385 r_0) + p^5 r^2 (0.572272 r_0 + 0.426992 r_0) + p^2 r^2 (1.04068 r_0 - 1.35561 r_0 + p^2 (0.113474 r_0 - 0.17693 r_0 - 0.000822131 r_0^3),
\]

(A17)

\[
\mathcal{N}^{(2)}_{\text{odd}} = p^5 (r^2 - r_0) + p^4 (-5.85214 r_0 + 5.95711 r_0 + 0.108754 r_0 + 0.641291 r_0 - 7.08335 r_0 - 0.238311 r_0^2 + p^3 (2.69182 r_0 - 1.62168 r_0 + 0.127734 r_0 + p^2 (-4.37086 r_0 + 3.87431 r_0 - 0.0023899 r_0^2) - 0.0482146 r_0^2 + 0.0388754 r_0),
\]

(A18)

\[
\mathcal{D}^{(2)}_{\text{odd}} = p^5 r (1.0 - r_0) + p^4 r (5.95711 r_0 - 5.85214 r_0 + p^3 r (6.41291 r_0 - 7.08335 r_0) + p^2 r (2.69182 r_0 - 1.62168 r_0) + pr (3.87431 r_0 - 4.37086 r_0) + r (0.0388754 r_0 - 0.0482146 r_0),
\]

(A19)

\[
\mathcal{N}^{(3)}_{\text{odd}} = p^8 (0.0422665 r_0^2 - 0.0376495 r_0) + p^7 (1.0089 r_0 - 0.268148 r_0^2) + p^6 (3.89855 r_0^2 - 2.80431 r_0 - 0.150165 r_0^3 + p^5 (2.58704 r_0^2 - 0.806081 r_0 + 0.336338 r_0^2 + p^4 (-4.89664 r_0^2 + 3.59708 r_0 + 0.219547 r_0^2) + p^3 (-5.34556 r_0^2 + 3.27195 r_0 + 0.0555038 r_0^2 + p^2 (-0.719236 r_0^2 + 0.412732 r_0 + 0.00227847 r_0) + p^2 (-0.33562 r_0^2 + 0.178078 r_0 - 0.0000548838 r_0^2) - 0.000520093 r_0^2 - 0.000246922 r_0 - 4.958 \times 10^{-7} r_0^2,
\]

(A20)

\[
\mathcal{D}^{(3)}_{\text{odd}} = p^7 r (1.0 - 1.12263 r_0) + p^6 r (3.89855 r_0 - 3.52943 r_0 + p^5 r (2.58704 r_0 - 1.08552 r_0) + p^4 r (4.61171 r_0 - 4.89664 r_0 + p^3 r (3.73444 r_0 - 5.34556 r_0 + p^2 r (4.55550 r_0 - 0.719236 r_0 + pr (0.019046 r_0 - 0.033562 r_0 + r (0.000255419 r_0 - 0.000520093 r_0).\)

(A21)

3. Inverse-polynomial GB coupling \(f(\varphi) = \varphi^{-1/4}\)

\[
\frac{g_{tt}(r)}{r} \approx \frac{\mathcal{N}^{(1)}_{\text{inv}} / \mathcal{D}^{(1)}_{\text{inv}}}{1 - \frac{r_0}{r}},
\]

(A22)

\[
\sqrt{g_{tt}(r) g_{rr}(r)} \approx \frac{\mathcal{N}^{(2)}_{\text{inv}} / \mathcal{D}^{(2)}_{\text{inv}}}{1 - \frac{r_0}{r}},
\]

(A23)

\[
e^{-r(r)} \approx \frac{\mathcal{N}^{(3)}_{\text{inv}} / \mathcal{D}^{(3)}_{\text{inv}}}{1 - \frac{r_0}{r}},
\]

(A24)
where
\[ \mathcal{N}^{(1)}_{\text{inv}} = p^8(0.0630915r^2r_0 - 0.00630915r_0^3) + p^7(-0.270873r^2r_0 + 0.00158747rr_0^2 + 0.272461r_0^3) + p^6(r^3 + 1.68132r^2r_0 - 0.0676888r_0^2 - 1.87658r_0^3) + p^5(-30.771r^3 + 27.3744rr_0^2 + 0.56521r_0^3 - 3.28027r_0^4) + p^4(52.7707r^3 - 0.1195r_0^2 - 2.09652r_0^2) + p^3(-24.1726r^2 + 25.4044r_0^2 + 0.435715rr_0^2 + 0.619006r_0^3) + p(-0.865964r^3 - 3.69793r^2r_0 + 0.0327455rr_0^2 + 0.11195r_0^3) + 0.604044r^3 - 0.930292r^2r_0, \] (A25)
\[ \mathcal{D}^{(1)}_{\text{inv}} = p^5r^2(r - r_0) + (25.2874r_0 - 24.1726r) + p^4r^2(11.1237r_0 - 0.028509r) + p^3r^2(0.046444r^2 - 0.865964r_0 - 3.6373r_0) + (0.604044r_0 - 0.930292r_0), \] (A26)
\[ \mathcal{N}^{(2)}_{\text{inv}} = p^4(r^2 - 1.16356rr_0 - 0.14853r_0^2) + p^3(-4.69714r^2 + 6.11751r_0 + 0.733313r_0^2) + p^2(1.51773r^2 - 5.7017rr_0 - 1.01164r_0^2) + p(9.046444r^2 - 0.497894rr_0 + 0.487947r_0^2) - 7.0095 r^2 + 0.50739 r r_0, \] (A27)
\[ \mathcal{D}^{(2)}_{\text{inv}} = p^4(r - 1.16356r_0) + p^3r(6.11751r_0 - 4.69714r) + p^2r(1.51773r_0 - 5.7017r_0) + pr(9.046444r - 4.16948r_0) + (5.05739r_0 - 7.0095r), \] (A28)
\[ \mathcal{N}^{(3)}_{\text{inv}} = p^4(0.0565104rr_0 - 0.0806291r_0^2) + p^3(r^2 + 0.0248978rr_0 - 0.563046r_0^2) + p^2(-2.60975r^2 + 2.71404rr_0 + 0.472884r_0^2) + p^3(-0.28073r^2 - 1.28601rr_0 + 0.483577r_0^2) + p^4(2.90881r_0^2 - 1.41739r_0 - 0.221653r_0^2) + p^3(0.96914r^2 - 0.349375rr_0 - 0.0428696r_0^2) + p^2(0.0963716r^2 - 0.0319405rr_0 - 0.00210459r_0^2) + p(0.00313102r^2 - 0.00092518r_0r - 0.000521411r_0r + 0.0000138482r^2 - 0.0030495957rr_0 - 0.00000138482r^2 - 0.0000138482r - 7.881 - 10.857r_0), \] (A29)
\[ \mathcal{D}^{(3)}_{\text{inv}} = p^7r(r + 0.79726r_0) + p^6r(0.64892r_0 - 2.06975r + 0.575r_0 - 0.717579r_0) + p^5r(2.90881r - 0.698231r_0) + p^4r(0.69641r - 0.210491r_0) + p^3r(0.0963716r - 0.0234361r_0) + pr(0.00313102r - 0.00083285r_0) + r(0.0000138482r - 3.737 - 10.857r_0). \] (A30)

4. Logarithmic GB coupling: \(f(\varphi) = \log(\varphi)/4\)
\[ g_{tt}(r) \approx \mathcal{N}_{\log}^{(1)}/\mathcal{D}_{\log}^{(1)} \left( 1 - \frac{r_0}{r} \right), \] (A31)
\[ \sqrt{g_{tt}(r)g_{rr}(r)} \approx \mathcal{N}_{\log}^{(2)}/\mathcal{D}_{\log}^{(2)}, \] (A32)
\[ e^{\varphi(r) - \varphi} \approx \mathcal{N}_{\log}^{(3)}/\mathcal{D}_{\log}^{(3)}, \] (A33)

where
\[ \mathcal{N}_{\log}^{(1)} = p^7(0.0626084r_0^3 - 0.0626084r_0^2r_0) + p^6(r^3 - 0.621986r_0^2r_0 - 0.378014r_0^3) + p^5(-4.81894r^3 + 4.13961r^2r_0 - 0.0186288r_0^2 + 0.65135r_0^3) + p^4(7.27136r^3 - 6.76272r^2r_0 + 0.117437rr_0 - 0.047656r_0^2) + p^3(-1.24665r^3 + 0.285682r_0^2 - 0.245893r_0^3 - 0.910056r_0^4) + p^2(-6.33774r^3 + 8.6188r^2r_0 + 0.219636rr_0^2 + 0.845321r_0^3) + p(5.38018r^3 - 7.52226r^2r_0 - 0.0698525rr_0^2 - 0.230411r_0^3) - 1.24797r^3 + 1.92521r_0^2, \] (A34)
\[ \mathcal{D}_{\log}^{(1)} = p^6r^2(1.r - 1.r_0) + p^5r^2(4.81894r - 4.81894r_0) + p^4r^2(7.27136r^3 - 6.97382r_0) + p^3r^2(-1.24665r - 0.266442r_0) + p^2r^2(9.14686r_0 - 6.33774r) + pr^2(5.38018r - 7.65098r_0) + r^2(1.92521r_0 - 1.24797r_0), \] (A35)
\[ \mathcal{N}_{\log}^{(2)} = p^4(1.r - 0.593401r_0 + 0.394399r_0^3) + p^3(-7.90712r^2 + 4.88145rr_0 - 1.33585r_0^2) + p^2(20.3089r^2 - 13.2461r_0 + 1.50407r_0^2) + p(-21.2712r^2 + 14.5616rr_0 - 0.562773r_0^2 + 7.88315r^2 - 5.61692r_0), \] (A36)
\[ D_{\log}^{(2)} = p^4r(1. r - 0.593401r_0) + p^3r(4.88145r_0 - 7.9072r) + p^2r(20.3089r - 13.2461r_0) + pr(14.5616r_0 - 21.2712r) + r(7.88315 - 5.61692 r_0), \]  
\[ N_{\log}^{(3)} = p^8(0.308197rr_0 - 1.55526rr_0^2) + p^7(r^2 - 0.883943rr_0 + 6.6752rr_0^2) + p^6(4.20077r^2 - 1.14941rr_0 - 10.6532r_0^2) + p^5(-21.718r^2 + 6.07481r_0r - 7.16324rr_0^2) + p^4(25.0109r_0^2 - 6.1987rr_0r - 1.39585r_0^3) + p^3(-6.22803r_0^2 + 1.22793rrr_0 - 0.154846r_0^3) + p^2(-2.12542r_0^2 + 0.581306rr_0 - 0.0716687r_0^2) + p(-0.142358r_0^2 + 0.0402579rr_0 - 0.00455675r_0^3) - 0.00204968r_0^2 + 0.000583126rr_0 + 0.000015759r_0^2, \]  
\[ D_{\log}^{(3)} = p^7(1. r - 2.35737r_0) + p^6r(4.20077r + 4.80947r_0) + p^5r(2.31625r_0 - 21.718r) + p^4r(25.0109r_0 - 9.11216r_0) + p^3r(3.44067r_0 - 6.22803r) + p^2r(0.858209r_0 - 2.12542r) + pr(0.0474343r_0 - 0.142358r) + r(0.000634068r_0 - 0.00204968r_0). \]  

Appendix B: Analytical expressions for the higher-order CFA coefficients up to fourth order

1. Even–polynomial GB coupling: \( f(\varphi) = \varphi^2/4 \)

\[ a_3 = \frac{-13\varphi^2}{177} + p - \frac{37}{125}, \]  
\[ a_4 = \frac{-463\varphi^4}{1015} + p^3 - \frac{23\varphi^3}{318} + \frac{47\varphi}{56} - \frac{1}{180}, \]  
\[ b_3 = \frac{-32\varphi^2}{89} + p - \frac{79}{125}, \]  
\[ b_4 = \frac{p^2 - \frac{37\varphi}{90}}{10} + \frac{17}{250}, \]  
\[ f_3 = \frac{-18\varphi^2}{133} + p - \frac{59}{187}, \]  
\[ f_4 = \frac{-54\varphi^4}{115} + p^3 - \frac{50\varphi^3}{77} + \frac{10\varphi}{97} - \frac{1}{177}. \]

2. Odd–polynomial GB coupling: \( f(\varphi) = \varphi^3/4 \)

\[ a_3 = \frac{127\varphi^3}{188} - \frac{257\varphi^2}{270} + p - \frac{34}{213}, \]  
\[ b_3 = \frac{37\varphi^2}{270} + p - \frac{101}{207}, \]  
\[ b_4 = \frac{-206\varphi^4}{353} + p^3 - \frac{77\varphi^3}{325} + \frac{5}{35}. \]

3. Inverse–polynomial GB coupling: \( f(\varphi) = \varphi^{-1}/4 \)

\[ a_3 = \frac{-70\varphi^3}{131} + p^2 - \frac{167\varphi}{311} - \frac{35}{367}, \]  
\[ b_3 = \frac{p^2 - \frac{7\varphi}{200} - \frac{74}{197}}{99} + \frac{29}{296} + \frac{17}{264}, \]  
\[ a_4 = \frac{-99\varphi^3}{817} + \frac{70\varphi^2}{207} + p + \frac{37}{524}, \]  
\[ a_4 = \frac{-110\varphi^3}{197} + p^3 - \frac{13\varphi^2}{43} - \frac{2}{151}, \]  
\[ b_3 = \frac{63\varphi^2}{211} + p - \frac{101}{207} + \frac{16p}{35} + \frac{69}{245}, \]  
\[ b_4 = \frac{-206\varphi^4}{353} + p^3 - \frac{77\varphi^3}{325} + \frac{5}{35}. \]
accurate results. Instead, we employed only the actual data at each value of the dimensionless parameter $p$. Eventually, as any fitting procedure unavoidably introduces some error we have also included Fig. 7 to quantify the accuracy of the fitting of the numerical data at each value of the dimensionless parameter $p$.

1. Even–polynomial GB coupling: $f(\varphi) = \varphi^2/4$

\[ r_{ph} = -\frac{5p^2}{252} - \frac{306p}{148} + 1, \quad (C1) \]

\[ R_{sh} = -\frac{3p^2}{235} - \frac{221p}{148} + 1, \quad (C2) \]

2. Odd–polynomial GB coupling: $f(\varphi) = \varphi^3/4$

\[ r_{ph} = \frac{8p^3}{303} + p^2 - \frac{29p}{148} - \frac{249}{277}, \quad (C3) \]

\[ R_{sh} = -\frac{3p^3}{20p} - \frac{31p^2}{72} + \frac{183p}{575}, \quad (C4) \]

3. Inverse–polynomial GB coupling: $f(\varphi) = \varphi^{-1}/4$

\[ r_{ph} = \frac{3p^3}{115} - \frac{61p^2}{208} - \frac{146p}{577} + 1, \quad (C5) \]

\[ R_{sh} = -\frac{9p^3}{407} + \frac{61p^2}{171} + \frac{106p}{278} + 1, \quad (C6) \]

4. Logarithmic GB coupling: $f(\varphi) = \log(\varphi)/4$

\[ r_{ph} = -\frac{3p^3}{206} - \frac{131p^2}{412} - \frac{41p}{229} + 1, \quad (C7) \]

\[ R_{sh} = -\frac{2p^3}{259} - \frac{89p^2}{606} + p - \frac{523}{231}, \quad (C8) \]

Appendix C: Analytical expressions for the photon–sphere radii and the black–hole shadows

Here we present approximate analytical expressions for the radius of the photon sphere and the black-hole shadow for the four GB-couplings that we have considered in this work.

Notice that in order to obtain these analytical expressions no approximate expression for the metric functions has been involved. Instead, we employed only the accurate numerical solution for $g_{tt}(r)$ aiming to get the most accurate results.

For various values of $p$ we computed the corresponding values of $r_{ph}$ and $R_{sh}$ and in turn we performed a fitting of the collected data. Eventually, as any fitting procedure unavoidably introduces some error we have also included Fig. 7 to quantify the accuracy of the fitting of the numerical data at each value of the dimensionless parameter $p$. 

\[ f_3 = \frac{85p^3}{117} + p^2 - \frac{9p}{110} - \frac{1}{225}, \quad (B17) \]

\[ f_4 = \frac{p^3 - 55p^2 + 58p}{31p^3 + 83p^2 + \frac{31p}{104} + \frac{5}{98}}. \quad (B18) \]

4. Logarithmic GB coupling: $f(\varphi) = \log(\varphi)/4$

\[ a_3 = \frac{p^3 - 142p^2}{96} - \frac{39p}{72} - \frac{5}{33}, \quad (B19) \]

\[ a_4 = \frac{-637p^3}{906} + p^2 - \frac{132p}{335} + \frac{45}{49}, \quad (B20) \]

\[ b_3 = \frac{-109p^2}{8p^2} + \frac{p}{23} - \frac{63}{886}, \quad (B21) \]

\[ b_4 = \frac{93p^3}{119} + p^2 - \frac{243}{106} + \frac{7}{33}, \quad (B22) \]

\[ f_3 = \frac{203p^3}{147} + p^2 - \frac{5p}{63} - \frac{3}{101}, \quad (B23) \]

\[ f_4 = \frac{143p^2}{331} + p^2 - \frac{108p}{206} + \frac{7}{225}. \quad (B24) \]
FIG. 7. The absolute relative error between numerical values and the analytical expressions of Eqs.(C1)-(C8) that have been obtained by fitting the accurate numerical values for the photon-sphere radius (left) and the black-hole shadow (right).

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