Long-time large deviations for the multi-asset Wishart stochastic volatility model and option pricing

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In this paper, we prove a large deviations principle for the class of multidimensional affine stochastic volatility models considered in (Gourieroux, C. and Sufana, R., J. Bus. Econ. Stat., 28(3), 2010), where the volatility matrix is modelled by a Wishart process. This class extends the very popular Heston model to the multivariate setting, thus allowing to model the joint behaviour of a basket of stocks or several interest rates. We then use the large deviation principle to obtain an asymptotic approximation for the implied volatility of basket options and to develop an asymptotically optimal importance sampling algorithm, to reduce the number of simulations when using Monte-Carlo methods to price derivatives.

Key words: Large deviations, Wishart process, Importance sampling, Basket options, Implied volatility

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1. Introduction

The Heston stochastic volatility model (Heston, 1993) is one of the most popular models in quantitative finance for the evolution of a single asset price. The Wishart stochastic volatility model is its natural extension to a basket of assets, since it coincides with the Heston model in dimension 1 and preserves the affine structure. This model, proposed in (Gourieroux and Sufana, 2010), assumes that under the risk-neutral probability, the vector of \( n \) asset prices is modelled as an Itô process

\[
dS_t = \text{Diag}(S_t) \left( r1 dt + \frac{X_t^{1/2}}{2} d\tilde{Z}_t \right),
\]

where the \( n \times n \) volatility matrix \((\tilde{X}_t)\) follows the Wishart process with dynamics

\[
d\tilde{X}_t = \left( \alpha a^\top a + \tilde{b}\tilde{X}_t + \tilde{X}_t\tilde{b}^\top \right) dt + \frac{X_t^{1/2}}{2} d\tilde{W}_t a + a^\top d\tilde{W}_t^\top \tilde{X}_t^{1/2},
\]

where \( \tilde{Z} \) and \( \tilde{W} \) are independent standard \( n \)-dimensional and \( n \times n \)-dimensional Brownian motions, and \text{Diag}(S_t)\) is the diagonal matrix whose diagonal elements are given by the vector \( S_t \in \mathbb{R}^n \).

The matrix process (1.2) has been introduced by (Bru, 1991) to model the perturbation of experimental biological data. As shown by (Bru, 1991) and (Cuchiero et al., 2011) in a more general framework, for \( \alpha \geq n + 1 \) (resp. \( \alpha \geq n - 1 \)), the SDE (1.2) has a unique strong (resp. weak) solution. Furthermore, since \( \tilde{X}_t \) is

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positive semi-definite (Bru, 1991, Prop. 4), Wishart processes turn out to be very suitable processes to model covariance matrices. This, and the affine property of the Wishart process, led several authors to use them in stochastic volatility models for a single asset, such as (Da Fonseca et al., 2008) and (Benabid et al., 2008) and in the Wishart stochastic volatility model for multiple assets (1.1)–(1.2). Subsequently, this model has been extended by (Da Fonseca et al., 2007) to include a constant correlation between $W$ and $Z$ in a way to preserve the affine structure.

By using the affine property, the Laplace transform of the model (1.1)–(1.2) is computed as follows (Da Fonseca et al., 2007).

$$
\mathbb{E} \left( e^{\theta^\top \log(S_t)} \right) = \exp \left( \beta_\theta(t) + \text{Tr} \left[ \gamma_\theta(t) \tilde{X}_0 \right] + \delta_\theta^\top (t) \log(S_t) \right),
$$

where $\beta_\theta, \gamma_\theta$ and $\delta_\theta$ satisfy the matrix Riccati equations

$$\begin{align*}
\partial_t \beta_\theta(t) &= r \delta_\theta^\top(t) \mathbf{1} + \alpha \text{Tr} [\gamma_\theta(t)] \\
\partial_t \gamma_\theta(t) &= \tilde{b}^\top \gamma_\theta(t) + \gamma_\theta(t) \tilde{b} + 2 \gamma_\theta(t) a^\top a \gamma_\theta(t) - \frac{1}{2} \left( \text{Diag}(\delta_\theta(t)) - \delta_\theta(t) \delta_\theta^\top(t) \right) \\
\partial_t \delta_\theta(t) &= 0,
\end{align*}
$$

with initial conditions $\beta_\theta(0) = 0, \gamma_\theta(0) = 0$ and $\delta_\theta(0) = \theta$. Since the Riccati equations can be solved explicitly, the Laplace transform can be expressed explicitly in terms of matrix exponentials and inverses.

The goal of the present paper is to prove a large deviations principle the Wishart stochastic volatility model (1.1)–(1.2) in the large-time asymptotic regime. Since the Laplace transform of the log-price vector in the Wishart model is known explicitly, a natural path towards a large deviations principle is via GÃ¤rtner-Ellis theorem. However, despite the explicit form of the Laplace transform, it is not easy to calculate its long-time asymptotics and to check the assumptions of the theorem because of the multi-dimensional setting. In this paper we therefore focus on a (large enough) subclass of the model (1.1)–(1.2) which enables us to obtain a simpler formula for the limiting Laplace transform and then prove a large deviations principle.

Beyond its theoretical interest, knowing that a given model satisfies a large deviations principle, and knowing the explicit form of the rate function, enables one to develop a number of important applications. One can mention e.g., efficient importance sampling methods for Monte Carlo option pricing; asymptotic formulas for option prices and implied volatilities in various asymptotic regimes, approximate evaluation of risk measures, simulation of rare events and others. We refer the reader to (Pham, 2007) for a review of various applications of large deviations methods in finance. In this paper we develop applications to variance reduction of Monte Carlo methods and to the asymptotic computation of implied volatilities far from maturity.

Our variance reduction method follows previous works of (Guasoni and Robertson, 2008), (Robertson, 2010) and (Genin and Tankov, 2016) and uses Varadhan’s lemma of large deviations theory to approximate the optimal measure change in the importance sampling algorithm. Note that since the Laplace tranform is known explicitly, Fourier inversion methods can be used, as explained in (Da Fonseca et al.,
However, these methods are much less competitive than in dimension 1 since they require to approximate an integral on $\mathbb{R}^n$. When, for complexity reasons, Fourier methods are not an option, the use of a large number of Monte-Carlo simulations is necessary. (Ahdida and Alfonsi, 2013) present an exact simulation method for Wishart processes and a second order scheme for the Gourierouex and Sufana model (1.1)–(1.2). Thus, it is possible to sample efficiently such processes, and it is relevant to develop variance reduction techniques to reduce computational costs.

The approximation of implied volatility far from maturity extends earlier results on the Heston model and the one-dimensional affine stochastic volatility models (Forde and Jacquier, 2011; Jacquier et al., 2013) to the multidimensional setting of Wishart model. Once again, this approach is more relevant in the multidimensional setting, since in one-dimensional affine models the implied volatility may be quickly computed by Fourier inversion.

In this paper, we denote $M_n$ the set of real squared $n \times n$ matrices, $S_n \subset M_n$ the set of symmetric matrices and $S_n^+$, (resp. $S_n^{++}$), the sets of symmetric an non-negative (resp.) positive definite. For a Borel set $A$, we denote by $\overline{A}$ the closure of $A$ and by $\mathcal{A}$ the interior of $A$.

The paper is structured as follows. In Section 2, we describe the model, make certain assumptions on the parameters and give some properties of the model. In Section 3, we prove that the asset log-price vector satisfies large deviations principle when maturity goes to infinity. In Section 4, we calculate the asymptotic put basket implied volatility, following the approach of (Jacquier et al., 2013). In Section 5, we develop the variance reduction method using Varadhan’s lemma. Finally, in Section 6, we test numerically the results of Sections 4 and 5.

2. The Wishart stochastic volatility model

In this section we introduce the subclass of the Wishart stochastic volatility models, in which we are interested in the present paper, and compute the Laplace transform of the log stock price process.

Let $(S_t)_{t \geq 0}$ be a $n$-dimensional vector stochastic process with dynamics

$$dS_t = \text{Diag}(S_t) \begin{pmatrix} r 1 dt + a^T X_t^{1/2} dZ_t \end{pmatrix}, \quad S_t^0 > 0, \ i = 1, \ldots, n,$$

where $1 = (1, \ldots, 1)^T$, $\text{Diag}(S_t)_{ij} = 1_{(i=j)}S_{ti}$, $Z_t$ is $n$-dimensional standard Brownian motion and the stochastic volatility matrix $X$ is a Wishart process with dynamics

$$dX_t = (\alpha I_n + bX_t + X_t b) dt + X_t^{1/2} dW_t + (dW_t)^T X_t^{1/2}, \quad X_0 = x.$$ (2.2)

with $\alpha > n - 1$, $a \in M_n$ invertible, $-b, x \in S_n^{++}$ and $W$ is a $n \times n$ matrix standard Brownian motion independent of $Z$. Note again that $X_t \in S_n^{++}$ (Bru, 1991, Prop. 4). Let us also assume that $a$ is such that $a^T a \in S_n^{++}$.

Remark 2.1. The model $(S, X)$ defined in (2.1) and (2.2) is a (quite large) subclass of the one defined in (1.1) and (1.2). Indeed, defining $\tilde{X}_t := a^T X_t a$, we have
\[ a^\top X_t^{1/2} dZ_t = \hat{X}_t^{1/2} d\hat{Z}_t, \text{ where } \hat{Z}_t \text{ is another } n\text{-dimensional standard Brownian motion and} \]
\[ d\hat{X}_t = \left( \alpha a^\top + b \hat{X}_t + \hat{X}_t b^\top \right) dt + \hat{X}_t^{1/2} d\hat{W}_t + a^\top (d\hat{W}_t)^\top \hat{X}_t^{1/2}, \quad \hat{X}_0 = a^\top x, \]
where \( b = a^\top b (a^\top)^{-1} \) and \( \hat{W}_t \) is another \( n \times n\)-Brownian motion.

**Remark 2.2.** In dimension one, the model defined by eqs. (2.1) and (2.2) corresponds to the famous Heston model (Heston, 1993) and \( b \) being negative definite yields the mean reversion property of the stochastic volatility process.

Defining the log-price \( Y_t^k := \log(S_t^k), k = 1, \ldots, n, \) a simple application of Itô’s lemma gives
\[ dY_t = \left( r 1 - \frac{1}{2} \left( (a^\top X_t a)_{11}, \ldots, (a^\top X_t a)_{nn} \right) \right) dt + a^\top X_t^{1/2} dZ_t. \quad (2.3) \]
We are interested in the Laplace transform of \( Y_t \). In order to calculate it, we first cite the following proposition.

**Proposition 2.3.** (Alfonsi et al., 2016, Prop. 5.1.). Let \( \alpha \geq n - 1, x \in S_n^+, b \in S_n \) and \( X \) with dynamics (2.2). Let \( v, w \in S_n \) be such that
\[ \exists m \in S_n, \quad \frac{v}{2} - mb - bm - 2m^2 \in S_n^+ \quad \text{and} \quad \frac{w}{2} + m \in S_n^+. \]
If \( R_t := \int_0^t X_s ds \), then we have for \( t \geq 0 \)
\[ \mathbb{E} \left[ \exp \left( -\frac{1}{2} \text{Tr} [w X_t] - \frac{1}{2} \text{Tr} [v R_t] \right) \right] = \exp \left( \frac{-1}{2} \text{Tr} \left[ \left( V'_{v,w}(t) V_{v,w}^{-1}(t) + b \right) x \right] \right) \]
with
\[ V_{v,w}(t) = \left( \sum_{k=0}^{\infty} t^{2k+1} \frac{\tilde{v}^k}{(2k+1)!} \right) \tilde{w} + \sum_{k=0}^{\infty} t^{2k} \frac{\tilde{v}^k}{(2k)!}, \quad \tilde{v} = v + b^2 \quad \text{and} \quad \tilde{w} = w - b. \]
If besides, \( \tilde{v} \in S_n^{+,*} \), then
\[ V_{v,w}(t) = \tilde{v}^{-1/2} \sinh \left( \tilde{v}^{1/2} t \right) \tilde{w} + \cosh \left( \tilde{v}^{1/2} t \right) \]
and
\[ V'_{v,w}(t) = \cosh \left( \tilde{v}^{1/2} t \right) \tilde{w} + \sinh \left( \tilde{v}^{1/2} t \right) \tilde{v}^{1/2}. \]
The following proposition provides an explicit formula for the Laplace transform of the log stock price \( Y_t \) in the model (2.1)–(2.2).

**Proposition 2.4.** Let \( \phi : \mathbb{R}^n \rightarrow S_n \) be the function defined by
\[ \phi(\theta) := b^2 + a \left( \text{Diag}(\theta) - \theta \theta^\top \right) a^\top \in S_n \]
Let \( U \subset \mathbb{R}^n \), be the set defined by
\[ U := \{ \theta \in \mathbb{R}^n : \phi(\theta) \in S_n^+ \}. \]
Then, for all $\theta \in \mathcal{U}$, the Laplace transform of $Y_t$ is
\[
E\left(e^{\theta^T Y_t}\right) = \frac{e^{\theta^T Y_0 + r^T \mathbf{1} t - \frac{1}{2} \int_0^t \frac{1}{2} \theta^T \left( (a^T X_s a)_{11}, \ldots, (a^T X_s a)_{mm} \right)^T - \theta^T a^T X_s a \theta \, ds}}{\det[V(t)]^{n/2}} ,
\]
where
\[
V(t) = \cosh \left( t \phi^{1/2}(\theta) \right) - \phi^{-1/2}(\theta) \sinh \left( t \phi^{1/2}(\theta) \right) b .
\]

**Proof.** By conditioning on the trajectory of $X$, we have
\[
E\left(e^{\theta^T Y_t}\right) = E\left(E\left(e^{\theta^T Y_t} \mid (X_s)_{s \leq t}\right) \right) ,
\]
where
\[
E\left(e^{\theta^T Y_t} \mid (X_s)_{s \leq t}\right) = e^{\theta^T Y_0 + r^T \mathbf{1} t - \frac{1}{2} \int_0^t \frac{1}{2} \theta^T \left( (a^T X_s a)_{11}, \ldots, (a^T X_s a)_{mm} \right)^T - \theta^T a^T X_s a \theta \, ds}
\]
\[
= e^{\theta^T Y_0 + r^T \mathbf{1} t - \frac{1}{2} \int_0^t \frac{1}{2} \theta^T \left[ \text{Diag}(\theta) a^T X_s a \right] - \theta^T a^T X_s a \theta \, ds}
\]
\[
= e^{\theta^T Y_0 + r^T \mathbf{1} t - \frac{1}{2} \int_0^t a \left( \text{Diag}(\theta) - \theta \theta^T \right) a^T R_t} .
\]

Let $m = -b/2$. Then $m \in S_n^+$ and
\[
a \left( \text{Diag}(\theta) - \theta \theta^T \right) a^T - mb - bm - 2m^2 = \frac{\phi(\theta)}{2} \in S_n^+ .
\]
Therefore, by Proposition 2.3,
\[
E\left(e^{\theta^T Y_t}\right) = e^{\theta^T Y_0 + r^T \mathbf{1} t} E\left(e^{-\frac{1}{2} \int_0^t a \left( \text{Diag}(\theta) - \theta \theta^T \right) a^T R_t} \right)
\]
\[
= e^{\theta^T Y_0 + r^T \mathbf{1} t} \exp\left( -\frac{1}{2} \int_0^t \text{Tr}\left[ b b^T \right] \right) \exp\left( -\frac{1}{2} \int_0^t \left[ (V'(t) V^{-1}(t) + b) x \right] \right)
\]
\[
(2.5)
\]
where
\[
\left\{ \begin{array}{l}
V(t) = \cosh \left( t \phi^{1/2}(\theta) \right) - \phi^{-1/2}(\theta) \sinh \left( t \phi^{1/2}(\theta) \right) b ,
V'(t) = \sinh \left( t \phi^{1/2}(\theta) \right) \phi^{1/2}(\theta) - \cosh \left( t \phi^{1/2}(\theta) \right) b .
\end{array} \right.
\]
Since $\phi(\theta) \in S_n^+$, we can write $\phi(\theta) = PDP^T$, where $D$ is diagonal, $P$ is orthonormal and $\hat{b} = -P^T b P \in S_n^{+,*}$. 
\[
\left\{ \begin{array}{l}
V(t) = P \left( \cosh \left( t D^{1/2} \right) + \sinh \left( t D^{1/2} \right) D^{-1/2} \hat{b} \right) P^T ,
V'(t) = P \left( \sinh \left( t D^{1/2} \right) D^{1/2} + \cosh \left( t D^{1/2} \right) \hat{b} \right) P^T
= \phi^{1/2}(\theta) V(t) - \exp\left( -t \phi^{1/2}(\theta) \right) \left( b + \phi^{1/2}(\theta) \right) .
\end{array} \right.
\]
Replacing $V'$ by the latter expression finishes the proof.

**Remark 2.5.** Note that, when $\phi(\theta) \in S_n^+ \setminus S_n^{+,*}$, $\phi^{1/2}(\theta)$ is not invertible. The notation $\phi^{-1/2}(\theta) \sinh \left( t \phi^{1/2}(\theta) \right)$ is therefore abusive and is to be interpreted as the finite limit
\[
\lim_{S_n^+ \ni \phi \to \phi(\theta)} \phi^{-1/2} \sinh \left( t \phi^{1/2} \right) = \sum_{k=0}^{\infty} \frac{\phi(\theta)^k t^{2k+1}}{(2k+1)!} .
\]
Remark 2.6. The set $\mathcal{U}$ is bounded. Indeed, let $\theta = \lambda \bar{\theta}$, with $\lambda > 0$ and $\|\bar{\theta}\| = 1$. Then, letting $u = (a^\top)^{-1} \bar{\theta}$, we have
\[
u^\top \phi(\theta) u = \|b(a^\top)^{-1} \bar{\theta}\|^2 + \lambda \bar{\theta}^\top \text{Diag}(\bar{\theta}) \bar{\theta} - \lambda^2 \leq \|b(a^\top)^{-1} \bar{\theta}\|^2 + \lambda - \lambda^2
\]
It follows that $\mathcal{U}$ is contained, e.g., in the set $\|\theta\| \leq \lambda^*$ with
\[
\lambda^* = \max\{2, \|b(a^\top)^{-1} \bar{\theta}\| \sqrt{2}\}.
\]

3. Long-time large deviations for the Wishart volatility model

In this section, we prove that the Wishart stochastic volatility model satisfies a large deviation principle when time tends to infinity.

3.1. Reminder of large deviations theory. Let us recall some standard definitions and results of large deviations theory. For a wider overview of large deviations theory, we refer the reader to (Dembo and Zeitouni, 1998). We consider a family $(X_\epsilon)_{\epsilon > 0}$ of random variables on a measurable space $(\mathcal{X}, \mathcal{B})$, where $\mathcal{X}$ is a topological space.

Definition 3.1 (Rate function). A rate function $\Lambda^*$ is a lower semi-continuous mapping $\Lambda^*: \mathcal{X} \to [0, \infty]$. A good rate function is a rate function such that, for every $a \in [0, \infty]$, \{x : $\Lambda^*(x) \leq a$\} is compact.

Definition 3.2 (Large deviation principle). $(X_\epsilon)_{\epsilon > 0}$ satisfies a large deviation principle with rate function $\Lambda^*$ if, for every $A \in \mathcal{B}$, denoting $\mathring{A}$ and $\bar{A}$ the interior and the closure of $A$,
\[
- \inf_{x \in \mathring{A}} \Lambda^*(x) \leq \liminf_{\epsilon \to 0} \epsilon \log P(X_\epsilon \in A) \leq \limsup_{\epsilon \to 0} \epsilon \log P(X_\epsilon \in A) \leq - \inf_{x \in \bar{A}} \Lambda^*(x).
\]

Definition 3.3. Let $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function with domain $D := \{x \in \mathbb{R}^n : f(x) < \infty\}$. $f$ is called essentially smooth if $f$ is differentiable on $\mathring{D} \neq \emptyset$ and for every $x \in \mathring{D} \setminus \mathring{D}$, $\lim_{y \to x} ||\nabla f(y)|| = +\infty$.

The following theorem is the celebrated Gärtner-Ellis theorem of the large deviations theory. (Dembo and Zeitouni, 1998) give a version of this theorem for a family of random variables parameterized by an integer number (see paragraph 2.3 in their book), but the version for families parameterized by a real number is easily deduced from the abstract Gärtner-Ellis theorem given in paragraph 4.5.3.

Theorem 3.4 (Gärtner-Ellis). Let $(X_\epsilon)_{\epsilon > 0}$ be a family of random vectors in $\mathbb{R}^n$. Assume that for each $\lambda \in \mathbb{R}^n$,
\[
\Lambda(\lambda) := \lim_{\epsilon \to 0} \epsilon \log E\left[ e^{(\lambda, X_\epsilon)} \right] \tag{3.1}
\]
exists as an extended real number. Assume also that $0$ belongs to the interior of $D_\Lambda := \{\lambda \in \mathbb{R}^n : \Lambda(\lambda) < \infty\}$. Denoting
\[
\Lambda^*(x) = \sup_{\lambda \in \mathbb{R}^n} (\lambda, x) - \Lambda(\lambda),
\]
the Fenchel-Legendre transform of $\Lambda$, the following hold.
(a) For any closed set $F$,
\[
\limsup_{\epsilon \to 0} \epsilon \log P(X^\epsilon \in F) \leq - \inf_{x \in F} \Lambda^\ast (x).
\]
(b) For any open set $G$,
\[
\liminf_{\epsilon \to 0} \epsilon \log P(X^\epsilon \in G) \geq - \inf_{x \in G \cap F} \Lambda^\ast (x),
\]
where $F$ is the set of exposed points of $\Lambda^\ast$, whose exposing hyperplane belongs to the interior of $D_\Lambda$.
(c) If $\Lambda$ is an essentially smooth, lower semi-continuous function, then $(X^\epsilon)_{\epsilon > 0}$ satisfies a large deviations principle with good rate function $\Lambda^\ast$.

Remark 3.5. The function $\Lambda$ of (3.1) is a convex function. Indeed, let $\lambda, \mu \in \mathbb{R}^n$ and $u \in (0, 1)$. A direct application of Hölder’s inequality yields
\[
E \left[ e^{\langle u \lambda + (1-u) \mu, X^\epsilon \rangle / \epsilon} \right] = E \left[ e^{\langle \lambda X^\epsilon \rangle / \epsilon} \right] \leq \left( E \left[ e^{\langle \lambda X^\epsilon \rangle / \epsilon} \right] \right)^u \left( E \left[ e^{\langle \mu X^\epsilon \rangle / \epsilon} \right] \right)^{1-u}.
\]
Applying the logarithm then proves that $\lambda \mapsto \log E \left[ e^{\langle \lambda, X^\epsilon \rangle / \epsilon} \right]$ and therefore $\Lambda$ are convex.

Theorem 3.6 (Varadhan’s Lemma, extension of (Guasoni and Robertson, 2008)). Let $(\mathcal{X}, \mathcal{B})$ be a metric space with its Borel $\sigma$-field. Let $(X^\epsilon)_{\epsilon > 0}$ be a family of $\mathcal{X}$-valued random variables that satisfies a large deviations principle with rate function $\Lambda^\ast$. If $\varphi : \mathcal{X} \to \mathbb{R} \cup \{-\infty\}$ is a continuous function which satisfies
\[
\limsup_{\epsilon \to 0} \epsilon \log E \left[ e^{\langle \gamma \varphi(X^\epsilon) / \epsilon \rangle} \right] < \infty
\]
for some $\gamma > 1$, then, for any $A \in \mathcal{B}$,
\[
sup_{x \in A^c} \{\varphi(x) - \Lambda^\ast (x)\} \leq \liminf_{\epsilon \to 0} \epsilon \log \int_{A^c} \exp \left( \frac{\varphi(z)}{\epsilon} \right) d\mu^\epsilon(z) \leq \limsup_{\epsilon \to 0} \epsilon \log \int_A \exp \left( \frac{\varphi(z)}{\epsilon} \right) d\mu^\epsilon(z) = \sup_{x \in A} \{\varphi(x) - \Lambda^\ast (x)\},
\]
where $\mu^\epsilon$ denotes the law of $X^\epsilon$.

3.2. Long-time behaviour of the Laplace transform of the log-price. Let $T > 0$ and define the transformation $Y^\epsilon_T := \epsilon Y_{T/t}$, which corresponds to the long-time behaviour of $Y_T$. We are interested in the function
\[
\theta \mapsto \lim_{\epsilon \to 0} \epsilon \log E \left[ e^{-\gamma \varphi (Y^\epsilon_T) / \epsilon} \right].
\]
We first give the following lemma.

Lemma 3.7. Let $A, B \in \mathcal{M}_n$ such that $A + tB$ est invertible for all $t \geq t_0$. Then, $(A + tB)^{-1}tB$ is bounded for all sufficiently large $t$.

Proof. Since $A + t_0B$ is invertible, for all $t \geq t_0$,
\[
(A + tB)^{-1}tB = \left( I + (t - t_0)B \right)^{-1}(t - t_0)B \frac{t}{t - t_0}.
\]
where \( \tilde{B} = (A + tB)^{-1}B \). Now, the fact that \( A + tB \) est invertible for \( t \geq t_0 \) means that the eigenvalues \( \lambda_i \) of \( \tilde{B} \) satisfy \( \lambda_i > 0 \) or \( \Im \lambda_i \neq 0 \) for all \( i \). This implies \( \det [I + (t - t_0)\tilde{B}] \xrightarrow{t \to + \infty} c t^n \) for some \( c \neq 0 \), and since the adjugate matrix of \( I + (t - t_0)\tilde{B} \) has coefficients of order \( O(t^n) \), we get that \( \left\{ I + (t - t_0)\tilde{B} \right\}^{-1} \) is bounded for \( t \geq t_0 \). Therefore, \( \left\{ I + (t - t_0)\tilde{B} \right\}^{-1} (t - t_0)\tilde{B} = I - \left\{ I + (t - t_0)\tilde{B} \right\}^{-1} \) is bounded, and \((A + tB)^{-1}tB\) as well, whenever \( t \) is sufficiently large. \( \square \)

We now characterise the asymptotic behaviour of the Laplace transform of \( Y_t^\epsilon \).

**Proposition 3.8.** Define
\[
\Lambda(\theta) := \left\{
\begin{array}{ll}
T (r \theta^\top 1 - \frac{\alpha}{2} \text{Tr} [b + \phi^1/2(\theta)]) & \text{if } \theta \in \mathcal{U} \\
\infty & \text{if } \theta \notin \mathcal{U}
\end{array}
\right.
\] (3.2)
For every \( \theta \in \mathcal{U} \),
\[
\lim_{\epsilon \to 0} \epsilon \log \mathbb{E} \left[ e^{\epsilon^{-1} \theta^\top Y_T^\epsilon} \right] = \Lambda(\theta).
\]

**Proof.** Let \( \theta \in \mathcal{U} \). By Proposition 2.4,
\[
\epsilon \log \mathbb{E} \left[ e^{\epsilon^{-1} \theta^\top Y_T^\epsilon} \right] = \epsilon \log \mathbb{E} \left[ e^{\theta^\top Y_T/\epsilon} \right]
= \epsilon \left( \theta^\top Y_0 - \frac{1}{2} \text{Tr} \left[ \left( b + \phi^{1/2}(\theta) \right) x \right] \right)
+ \frac{1}{2} \epsilon \text{Tr} \left[ \exp \left( -T/\epsilon \phi^{1/2}(\theta) \right) \left( b + \phi^{1/2}(\theta) \right) V^{-1}(T/\epsilon) x \right]
+ T \epsilon \theta^\top 1 - \frac{T}{2} \text{Tr} [b] - \frac{\alpha}{2} \epsilon \log \det [V(T/\epsilon)].
\] (3.3)
Write \( \phi(\theta) = PDP^\top \), where \( D \) is diagonal, \( P \) is orthonormal and let \( \hat{b} = -P^\top b P \in \mathcal{S}_n^{++} \). Then
\[
V(t) = P \left( \cosh \left( t D^{1/2} \right) + \sinh \left( t D^{1/2} \right) D^{-1/2} \hat{b} \right) P^\top,
\]
Let \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) be \( n \times n \) square matrices with \( \mathcal{E}_{ij} = \mathbb{1}_{\{i = j, D_{ii} = 0\}} \) and \( \tilde{\mathcal{E}}_{ij} = D_{ii}^{-1/2} \mathbb{1}_{\{i = j, D_{ii} \neq 0\}} \). We then have
\[
\cosh \left( t D^{1/2} \right) = \frac{e^{tD^{1/2}}}{2} \left( I_n + e^{-2tD^{1/2}} \right) = \frac{e^{tD^{1/2}}}{2} \left( I_n + \mathcal{E} + \mathcal{O} \left( t^{-1} \right) \right)
\]
and
\[
\sinh \left( t D^{1/2} \right) D^{-1/2} \left( I_n - e^{-2tD^{1/2}} \right) = \frac{e^{tD^{1/2}}}{2} \left( \tilde{\mathcal{E}} + 2t \mathcal{E} + \mathcal{O} \left( t^{-1} \right) \right).
\]
Therefore,
\[
V(t) = \frac{1}{2} Pe^{tD^{1/2}} \left( (I_n + \mathcal{E}) + (2t \mathcal{E} + \tilde{\mathcal{E}}) \hat{b} + \mathcal{O} \left( t^{-1} \right) \right) P^\top
= -\frac{1}{2} P (I_n + \mathcal{E}) e^{tD^{1/2}} \left( \hat{b}^{-1} + (t \mathcal{E} + \tilde{\mathcal{E}}) + \mathcal{O} \left( t^{-1} \right) \right) P^\top \hat{b}
\] (3.4)
and
\[ V^{-1}(t) = -2\hat{b}^{-1}P \left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) + o(t^{-1}) \right)^{-1} e^{-tD^{1/2}} \left( I_n - \frac{1}{2}\mathcal{E} \right) P^\top \]
where the invertibility of \( \left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) + o(t^{-1}) \right) \) is guaranteed for every \( t \geq 0 \) by the existence of the Laplace transform. Since \( \hat{b}^{-1} \in \mathcal{S}^+_n \) and \( (t\mathcal{E} + \hat{\mathcal{E}}) \in \mathcal{S}^+_n \), \( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) \in \mathcal{S}^+_n \) and is therefore invertible. Hence
\[ \left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) + o(t^{-1}) \right)^{-1} = \left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) \right)^{-1} (I_n + o(t^{-1})) \]
and
\[ V^{-1}(t) = -2\hat{b}^{-1}P \left( I_n + o(t^{-1}) \right) \left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) \right)^{-1} e^{-tD^{1/2}} \left( I_n - \frac{1}{2}\mathcal{E} \right) P^\top. \]
But
\[
\left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) \right)^{-1} e^{-tD^{1/2}} = \left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) \right)^{-1} (\mathcal{E} + (I_n - \mathcal{E})) e^{-tD^{1/2}} = t^{-1} \left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) \right)^{-1} t\mathcal{E} + \left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) \right)^{-1} (I_n - \mathcal{E}) e^{-tD^{1/2}},
\]
where \( \left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) \right)^{-1} t\mathcal{E} \) is bounded by Lemma 3.7. Therefore,
\[
\left( \hat{b}^{-1} + (t\mathcal{E} + \hat{\mathcal{E}}) \right)^{-1} e^{-tD^{1/2}} \to 0
\]
and \( V^{-1}(t) \to 0 \) as \( t \to \infty \). Using (3.4), we find
\[
\epsilon \log \det [V(T/\epsilon)] = T \text{Tr} \left[ D^{1/2} \right] + \epsilon \log \det \left[ \frac{1}{2} (I_n + \mathcal{E}) \left( I_n + (\epsilon^{-1}T\mathcal{E} + \hat{\mathcal{E}})\hat{b} + o(\epsilon) \right) \right]
\]
\[
= T \text{Tr} \left[ \phi^{1/2}(\theta) \right] + \epsilon \log \det \left[ \epsilon^{-1}T\mathcal{E}\hat{b} + \frac{1}{2} (I_n + \mathcal{E}) \left( I_n + \mathcal{E}\hat{b} \right) + o(\epsilon) \right]
\]
\[
= T \text{Tr} \left[ \phi^{1/2}(\theta) \right] - n\epsilon \log(\epsilon) + \epsilon \log \det \left[ T\mathcal{E}\hat{b} + \frac{\epsilon}{2} (I_n + \mathcal{E} + \mathcal{E}\hat{b}) + o(\epsilon^2) \right].
\]
We have det \( T\mathcal{E}\hat{b} + \frac{\epsilon}{2} (I_n + \mathcal{E} + \mathcal{E}\hat{b}) + o(\epsilon^2) \) \( \sim_{\epsilon \to 0} \) det \( T\mathcal{E}\hat{b} + \frac{\epsilon}{2} (I_n + \mathcal{E} + \mathcal{E}\hat{b}) \), since the latter determinant is a non-zero polynomial of \( \epsilon \) (for \( \epsilon = 2T \) the determinant is clearly positive). Thus, by passing to the limit, \( \lim_{\epsilon \to 0} \epsilon \log \det [V(T/\epsilon)] = T \text{Tr} \left[ \phi^{1/2}(\theta) \right] \). Furthermore, since \( \phi \in \mathcal{S}^+_n \), \( \exp \left( -\frac{T}{\epsilon} \phi^{1/2}(\theta) \right) \) is bounded. Therefore,
\[
\text{Tr} \left[ \exp \left( -\frac{T}{\epsilon} \phi^{1/2}(\theta) \right) (b + \phi^{1/2}(\theta)) V^{-1}(T/\epsilon) x \right] \to_{\epsilon \to 0} 0.
\]
Finally, passing to the limit in (3.3) finishes the proof. \( \square \)

The next proposition proves the essential smoothness of \( \Lambda \).

**Proposition 3.9.** The function \( \theta \mapsto \Lambda(\theta) \) defined in (3.2) is essentially smooth.
Therefore, we get by the triangular inequality

\[ \Lambda(\theta) = T \left( r \theta^T 1 - \frac{\alpha}{2} \text{Tr} \left[ b + \phi^{1/2}(\theta) \right] \right). \]

Then for every \( j \in \{1, ..., n\}, \)

\[ \partial_{\theta_j} \Lambda(\theta) = T \left( r - \frac{\alpha}{2} \text{Tr} \left[ \partial_{\theta_j} \left[ \phi^{1/2}(\theta) \right] \right] \right), \]

where \( \partial_{\theta_j} \left[ \phi^{1/2}(\theta) \right] \) satisfies

\[ \partial_{\theta_j} \phi(\theta) = \partial_{\theta_j} \left[ \phi^{1/2}(\theta) \phi^{1/2}(\theta) \right] = \phi^{1/2}(\theta) \partial_{\theta_j} \left[ \phi^{1/2}(\theta) \right] + \partial_{\theta_j} \left[ \phi^{1/2}(\theta) \right] \phi^{1/2}(\theta). \]

Multiplying this equation by \( \phi^{-1/2}(\theta) \) and using the cyclic property of the trace, we get

\[ \text{Tr} \left[ \partial_{\theta_j} \left[ \phi^{1/2}(\theta) \right] \right] = \frac{1}{2} \text{Tr} \left[ \phi^{-1/2}(\theta) \partial_{\theta_j} \phi(\theta) \right]. \]

and therefore

\[ \partial_{\theta_j} \Lambda(\theta) = T \left( r - \frac{\alpha}{2} \text{Tr} \left[ \partial_{\theta_j} \left[ \phi^{1/2}(\theta) \right] \right] \right) = T \left( r - \frac{\alpha}{4} \text{Tr} \left[ \phi^{-1/2}(\theta) \partial_{\theta_j} \phi(\theta) \right] \right), \]

where

\[ \partial_{\theta_j} \phi(\theta) = a \left( e_j e_j^\top - \theta e_j^\top - e_j \theta^\top \right) a^\top. \]

We write \( \phi(\theta) = PD\theta^\top \) with \( D \in \mathcal{S}_{++}^n \) diagonal and denote \( w = a^\top P \), which is invertible since \( P \) is orthonormal and \( a^\top a \in \mathcal{S}_{++}^n \). Then

\[ \text{Tr} \left[ \phi^{-1/2}(\theta) \partial_{\theta_j} \phi(\theta) \right] = \text{Tr} \left[ D^{-1/2} P^\top \partial_{\theta_j} \phi(\theta) P \right] \]
\[ = \text{Tr} \left[ D^{-1/2} w^\top (e_j e_j^\top - \theta e_j^\top - e_j \theta^\top) w \right] \]
\[ = \text{Tr} \left[ D^{-1/2} w^\top (e_j e_j^\top - 2e_j \theta^\top) w \right] = \sum_{i=1}^n D^{-1/2}_{ii} (w_{ji}^2 - 2w_{ji} (\theta^\top w e_i)). \]

Now, we observe that

\[ D_{ii} = P_i^\top \phi(\theta) P_i = \|b P_i\|^2 + e_i^\top w^\top \left( \text{Diag}(\theta) - \theta^\top \right) w e_i \]
\[ = \|b P_i\|^2 + \sum_{j=1}^n \theta_j w_{ji}^2 - (\theta^\top w e_i)^2 \]
\[ = \|b P_i\|^2 + (\theta^\top w e_i)^2 + \sum_{j=1}^n \theta_j (w_{ji}^2 - 2w_{ji} (\theta^\top w e_i)). \]

Therefore, we get by the triangular inequality

\[ \sum_{j=1}^n |\theta_j| \left| \text{Tr} \left[ \phi^{-1/2}(\theta) \partial_{\theta_j} \phi(\theta) \right] \right| \geq \sum_{j=1}^n \theta_j \sum_{i=1}^n D^{-1/2}_{ii} (w_{ji}^2 - 2w_{ji} (\theta^\top w e_i)). \]
\[ \sum_{i=1}^{n} D_{ii}^{1/2} \left( ||b P_i||^2 + (\theta^\top w e_i)^2 \right) \]

Then, if \( \theta \to \bar{\theta} \) with \( \bar{\theta} \in \mathcal{U}\setminus \hat{\mathcal{U}} \), there exists \( i \) such that \( D_{ii} \to 0 \) and therefore \( \sum_{i=1}^{n} D_{ii}^{1/2} \left( ||b P_i||^2 + (\theta^\top w e_i)^2 \right) \to -\infty \) since \( ||b P_i||^2 + (\theta^\top w e_i)^2 \geq \lambda (-b)^2 \) with \( \lambda > 0 \), where \( \lambda \) is the smallest eigenvalue of \( -b \in S_n^+ \). Therefore, \( \sum_{i=1}^{n} D_{ii}^{1/2} \left( ||b P_i||^2 + (\theta^\top w e_i)^2 \right) \to -\infty \) since \( ||b P_i||^2 + (\theta^\top w e_i)^2 \geq \lambda (-b)^2 \) with \( \lambda > 0 \), where \( \lambda \) is the smallest eigenvalue of \( -b \in S_n^+ \).

Remark 3.10. Since, by Remark 3.5, \( \theta \mapsto \lim_{\epsilon \to 0} \epsilon \log \mathbb{E} \left[ e^{\epsilon^{-1}\theta^\top Y^\epsilon_{T}} \right] \) is a convex function, and, by Proposition 3.9, \( \Lambda \) admits infinite derivative on \( \mathcal{U}\setminus \hat{\mathcal{U}} \), then for every \( \theta \in \mathbb{R}^n \setminus \mathcal{U}, \lim_{\epsilon \to 0} \epsilon \log \mathbb{E} \left[ e^{\epsilon^{-1}\theta^\top Y^\epsilon_{T}} \right] = \Lambda(\theta) = \infty \). Therefore, Proposition 3.8 does not only hold for \( \theta \in \mathcal{U} \), but for every \( \theta \in \mathbb{R}^n \).

3.3. Long-time large deviation principle for the log-price process. We now state the large deviation principle for the family \((Y^\epsilon_{T})_{\epsilon>0}\), when \( \epsilon \to 0 \).

**Theorem 3.11.** The family \((Y^\epsilon_{T})_{\epsilon>0}\) satisfies a large deviation principle, when \( \epsilon \to 0 \) with good rate function

\[ \Lambda^*(y) = \sup_{\lambda \in \mathbb{R}^n} (\lambda, y) - \Lambda(\lambda) \]

**Proof.** First note that \( \phi(0) = b^2 \in S_n^{+,*} \). But since \( \theta \mapsto \phi(\theta) := b^2 + a \left( \text{Diag}(\theta) - \theta \theta^\top \right) a^\top \) is a continuous function, there exists a neighbourhood \( B(0, \delta) \) of 0 such that \( \phi(\theta) \in S_n^{+,*} \) for every \( \theta \in B(0, \delta) \), hence \( 0 \in \mathcal{U} \). Furthermore, Proposition 3.8 together with the argument in Remark 3.10 prove that

\[ \Lambda(\theta) = \lim_{\epsilon \to 0} \epsilon \log \mathbb{E} \left[ e^{\epsilon^{-1}\theta^\top Y^\epsilon_{T}} \right], \]

where \( \Lambda \) is defined in (3.2). Finally, Proposition 3.9 yields the essential smoothness of \( \Lambda \). Therefore, by the Gärtner-Ellis Theorem 3.4, \((Y^\epsilon_{T})_{\epsilon>0}\) satisfies a large deviation principle, when \( \epsilon \to 0 \) with good rate function \( \Lambda^* \). \( \square \)

4. Asymptotic implied volatility of basket options

In this section, to simplify the formulas and without loss of generality, we assume that \( Y^0_{j} = 0 \) for \( j = 1, \ldots, n \) and \( r = 0 \) so that \((e^{Y^\epsilon_{T}})_{\epsilon>0}\) is a martingale with initial value 1 (this follows from Proposition 2.4). We are interested in the limiting behavior far from maturity of basket option prices and the corresponding implied volatilities in the Wishart model. The basket call option price with log strike \( k \)
and time to maturity $T$ is defined by
\[ C(T, k) = E \left[ \left( \sum_{i=1}^{n} \omega_i S_i^T - e^k \right)^+ \right], \]
and the corresponding put option price is defined by
\[ P(T, k) = E \left[ \left( e^k - \sum_{i=1}^{n} \omega_i S_i^T \right)^+ \right], \]
where $\omega \in (\mathbb{R}_+)^n$ with $\sum_{i=1}^{n} \omega_i = 1$.

The implied volatility of basket options is defined by comparing their price to the corresponding option price in the Black-Scholes model $dS_t = \sigma dW_t$:
\[ C^{BS}(T, k, \sigma) = N(d_1) - e^k N(d_2), \quad d_{12} = k \pm \frac{1}{2} \sigma^2 T, \]
where $N$ is the standard normal distribution function. The implied volatility for log strike $k$ and time to maturity $T$ is then defined as the unique value $\sigma(T, k)$ such that
\[ C^{BS}(T, k, \sigma(T, k)) = C(T, k). \]
It can be equivalently defined using the put option price.

It is well known that in most models, for fixed log strike $k$, the implied volatility converges to a constant value independent from $k$ as $T \to \infty$ (Tehranchi, 2009). To obtain a non-trivial limiting smile, we therefore follow (Jacquier et al., 2013) and use a renormalized log strike $k(T) = yT$. We are interested in computing the limiting implied volatility
\[ \sigma_\infty(y) = \lim_{T \to \infty} \sigma(T, yT). \]

### 4.1. Asymptotic price for the Wishart model

Introduce the renormalized log-price process in the stochastic volatility Wishart model: $\hat{Y}^j_t = T^{-1} Y^j_t$, $j = 1, \ldots, n$. Note that to simplify notation, in this section we avoid using an extra parameter $\epsilon$ and simply consider the asymptotics when $T \to \infty$. For this reason, the asymptotic Laplace exponent $\Lambda(\theta)$ will be given by equation (3.2) with $T = 1$ and $r = 0$.

Denote the basket log price by $\mathcal{B}_T := \log \sum_{j=1}^{n} \omega_j e^{Y^j_T}$, and the corresponding renormalized price by $\mathcal{B}_{T} := T^{-1} \log \sum_{j=1}^{n} \omega_j e^{Y^j_T}$. We first show some LDP-like bounds for this quantity. In the following lemma and below, we will use the fact that $\Lambda(0) = \Lambda(e_j) = 0$, which implies in particular that $\Lambda^*(x) \geq 0$ and $\Lambda^*(x) - x_j \geq 0$ for all $x \in \mathbb{R}^d$. Thus, we let $x^* = \Lambda'(0)$ and $\tilde{x}^*_j = \Lambda_j'(e_j)$ for $j = 1, \ldots, n$ and introduce three constants: $\beta^* = \max_j x_j^*$, $\tilde{\beta}^* = \min_j \tilde{x}^*_j$ and $\hat{\beta}^* = \max_j \tilde{x}^*_j$. It is easy to see from (3.5) that $x_j^* = -\tilde{x}^*_j < 0$ since $\phi(0) = \phi(e_j) = b^2$ is positive definite and $a$ is invertible. We get $\beta^* < 0 < \hat{\beta}^* \leq \tilde{\beta}^*$.

**Lemma 4.1.** The following estimates hold for $\mathcal{B}_T$. 

(1) If $\beta < \beta^*$ then
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{P} \left( \tilde{B}_T \in (-\infty, \beta) \right) = - \inf_{x \in (-\infty, \beta]^n} \Lambda^*(x)
\]
\[
= \inf_{\lambda \in \mathbb{R}^n, \lambda \leq 0, i=1,\ldots,n} \{ \Lambda(\lambda) - \beta \langle \lambda, 1 \rangle \} < 0; \quad (4.1)
\]
Otherwise
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{P} \left( \tilde{B}_T \in (-\infty, \beta) \right) = 0.
\]

(2) If $\beta \geq \beta^*$ then
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{P} \left( \tilde{B}_T \in (\beta, \infty) \right) = - \inf_{x \in (\beta, \infty]^n} \Lambda^*(x) = \max_{i=1,\ldots,n} \inf_{\lambda \in \mathbb{R}, \lambda \leq 0, i \neq j} \{ -\lambda \beta + \Lambda(\lambda e_i) \},
\]
otherwise
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{P} \left( \tilde{B}_T \in (\beta, \infty) \right) = 0.
\]

In addition, if $\beta \geq \beta^*$ and $\beta \neq \tilde{x}^*_i$ for all $i$, then
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{P} \left( \tilde{B}_T \in (\beta, \infty) \right) < -\beta.
\]

(3) Let $j \in \{1, \ldots, n\}$. Then,
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{E} \left[ e^{Y_j^T} \mathbb{I}_{\tilde{B}_T \in (-\infty, \beta]} \right] = - \inf_{x \in (-\infty, \beta]^n} \Lambda^*(x) - x_j
\]
\[
= \beta + \inf_{\lambda \leq 1, \lambda \leq 0, i \neq j} \{ \Lambda(\lambda) - \beta \langle \lambda, 1 \rangle \}. \quad (4.3)
\]
In addition, if $\tilde{x}^*_j > \beta$ then
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{E} \left[ e^{Y_j^T} \mathbb{I}_{\tilde{B}_T \in (-\infty, \beta]} \right] < 0.
\]

(4) Let $j \in \{1, \ldots, n\}$ and assume $\beta > \tilde{x}^*_j$. Then,
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{E} \left[ e^{Y_j^T} \mathbb{I}_{\tilde{B}_T \in (\beta, \infty]} \right] = - \inf_{x \in (\beta, \infty]^n} \Lambda^*(x) - x_j
\]
\[
= \max_{i=1,\ldots,n} \inf_{\lambda \in \mathbb{R}, \lambda \leq 0, i \neq j} \{ -\lambda \beta + \Lambda(\lambda e_i + e_j) \} < 0. \quad (4.4)
\]

**Proof.** (1) Since $\omega_{\min} e^{\max} Y_j^T \leq \sum_{j=1}^n \omega_j e^{Y_j^T} \leq n \omega_{\max} e^{\max} Y_j^T$ with $(\omega_{\min}, \omega_{\max}) := (\min_{j=1,\ldots,n} \omega_j, \max_{j=1,\ldots,n} \omega_j)$, we have for every $T > 0$ and $\beta \in \mathbb{R}$,
\[
\left( \tilde{Y}_T \in (-\infty, \beta - T^{-1} \log(n \omega_{\max}))^n \right) \subset (\tilde{B}_T < \beta)
\]
\[
\subset \left( \tilde{Y}_T \in (-\infty, \beta - T^{-1} \log \omega_{\min})^n \right).
\]
Therefore, we get for every $\delta > 0$ and $T$ sufficiently large,
\[
\mathbb{P} \left( \tilde{Y}_T \in (-\infty, \beta - \delta)^n \right) \leq \mathbb{P}(\tilde{B}_T < \beta) \leq \mathbb{P} \left( \tilde{Y}_T \in (-\infty, \beta + \delta)^n \right).
\]
Passing to the lim sup and lim inf, we get:
\[
\lim_{T \to \infty} \inf \lim_{T \to \infty} \mathbb{P} \left( \tilde{Y}_T \in (-\infty, \beta - \delta)^n \right) \leq \lim_{T \to \infty} \inf \mathbb{P}(\tilde{B}_T < \beta)
\]
\[
\leq \lim_{T \to \infty} \sup \lim_{T \to \infty} \mathbb{P}(\tilde{B}_T < \beta) \leq \lim_{T \to \infty} \sup \mathbb{P} \left( \tilde{Y}_T \in (-\infty, \beta + \delta)^n \right).
Using the large deviations principle for $\tilde{Y}_T$ (Theorem 3.11) further yields:

$$- \inf_{x \in (-\infty, \beta - \delta)^n} \Lambda^*(x) \leq \liminf_{T \to \infty} T^{-1} \log \mathbb{P}(\tilde{B}_T < \beta)$$

$$\leq \limsup_{T \to \infty} T^{-1} \log \mathbb{P}(\tilde{B}_T < \beta) \leq - \inf_{x \in (-\infty, \beta + \delta)^n} \Lambda^*(x),$$

and making $\delta$ tend to zero, we see that

$$- \inf_{x \in (-\infty, \beta)^n} \Lambda^*(x) \leq \liminf_{T \to \infty} T^{-1} \log \mathbb{P}(\tilde{B}_T < \beta)$$

$$\leq \limsup_{T \to \infty} T^{-1} \log \mathbb{P}(\tilde{B}_T < \beta) \leq - \inf_{x \in (-\infty, \beta)^n} \Lambda^*(x).$$

The fact that the domain of $\Lambda$ is bounded (Remark 2.6) implies that $\Lambda^*$ is locally bounded from above and therefore continuous. The first equality of (4.1) then follows by continuity of $\Lambda^*$. The second equality then follows from the definition of $\Lambda^*$ and the minimax theorem (see, e.g., Corollary 37.3.2 in (Rockafellar, 1970)) which can be applied because the domain of $\Lambda$ is bounded (cf. Remark 2.6). Finally, the inequality follows from the fact that the function $f(\lambda) = \Lambda(\lambda) - \beta(\lambda, 1)$ satisfies $f(0) = 0$ and $f'(0) = x^* - 1$. Under the condition $\beta < \beta^*$ at least one component of the derivative is strictly positive, and hence the minimum of $f$ over the set $\{\lambda_i \leq 0, i = 1, \ldots, n\}$ is strictly negative.

(2) The first equality in (4.2) follows similarly to the previous item. If $\beta < \beta^*$ then $x^* \notin (-\infty, \beta)^n$ and the infimum equals 0. Otherwise by convexity of $\Lambda^*$ the infimum is attained on the boundary of this set. Therefore, we can write:

$$- \inf_{x \in (-\infty, \beta)^n} \Lambda^*(x) = \max_{i = 1, \ldots, n} \sup_{x \in \mathbb{R}^n, x_i = \beta} \{ -\Lambda^*(x) \}$$

$$= \max_{i = 1, \ldots, n} \sup_{x \in \mathbb{R}^n, x_i = \beta} \{ -\langle \lambda, x \rangle + \lambda(\lambda) \}$$

$$= \max_{i = 1, \ldots, n} \inf_{\lambda \in \mathbb{R}^n} \{ -\lambda_j + \Lambda(\lambda e_i) \},$$

since the inf and sup may once again be interchanged in virtue of the minimax theorem and then the supremum on $x \in \mathbb{R}^n$ such that $x_i = \beta$ is clearly $+\infty$ when there is $j \neq i$ such that $\lambda_j \neq 0$. Consider the function $f_i : \mathbb{R} \to \mathbb{R}$, $f_i(\lambda) = -\lambda \beta + \Lambda(\lambda e_i)$. Since $f_i(1) = -\beta$ and $f'_i(1) = -\beta + \tilde{x}_i^*$, it follows that

$$\beta + \max_{i = 1, \ldots, n} \inf_{\lambda \in \mathbb{R}} \{ -\lambda \beta + \Lambda(\lambda e_i) \} < 0,$$

when $\beta \neq \tilde{x}_i^*$ for all $i$.

(3) For the first identity in (4.3), remark that, similarly to the first part, for $T$ sufficiently large, all $\delta > 0$ and $\beta \in \mathbb{R}$ we have,

$$\mathbb{E}[e^{Y^T_T \mathbb{1}_{\{Y_T \in (-\infty, \beta - \delta)^n\}}}] \geq \mathbb{E}[e^{Y^T_T \mathbb{1}_{\{B_T \leq \beta\}}}] \geq \mathbb{E}[e^{Y^T_T \mathbb{1}_{\{Y_T \in (-\infty, \beta + \delta)^n\}}}].$$

We can apply Theorem 3.6 with the function $H : x \mapsto x_j$ since $\Lambda(e_j) = 0$ and $\Lambda(\gamma e_j) < \infty$ for $\gamma > 1$ small enough. When $\delta$ goes to zero, we get

$$\sup_{x \in (-\infty, \beta)^n} \{ x_j - \Lambda^*(x) \} \leq \liminf_{T \to \infty} T^{-1} \log \mathbb{E}[e^{Y^T_T \mathbb{1}_{\{B_T \leq \beta\}}}].$$
\[
\limsup_{T \to \infty} T^{-1} \log \mathbb{E}[e^{Y^j \mathbf{1}_{B^j_T \leq \beta}}] \leq \sup_{x \in (-\infty, \beta]^n} \{x_j - \Lambda^*(x)\}.
\]

By continuity of \(\Lambda^*\), the lower and the upper bounds are equal. Since \(\Lambda^*(x) = \sup_{\lambda \in \mathbb{R}^n} \{\lambda^T x - \Lambda(x)\}\), we get
\[
\sup_{x \in (-\infty, \beta]^n} \{x_j - \Lambda^*(x)\} = \sup_{x \in (-\infty, \beta]^n} \inf_{\lambda \in \mathbb{R}^n} \Lambda(\lambda + x) - \langle \lambda, x \rangle.
\]

The second identity in (4.3) then follows from the minimax theorem as above. Finally, to show the inequality, remark that
\[
\inf_{\lambda \in \mathbb{R}^n} \{\Lambda(\lambda) - \beta \langle \lambda, 1 \rangle\} \leq \inf_{\lambda \leq 1} f_j(\lambda)
\]
and \(f_j'(1) = \tilde{x}_j^* - \beta > 0\).

(4) The first identity in (4.4) follows as in item (3). We have \(\Lambda^*(x) - x_j \geq 0\) and \(\Lambda^*(\Lambda'(e_j)) = \Lambda'_j(e_j) = \tilde{x}_j^*\) since \(e_j\) is a critical point of \(\lambda \mapsto \langle \lambda, \Lambda'(e_j) \rangle - \Lambda(\lambda)\).

Since \(\beta > \tilde{x}_j^*\) and \(\Lambda'(e_j) \notin (-\infty, \beta]^n\), the supremum is attained as in item (2) on the boundary:
\[
\sup_{x \in \mathbb{R}^n} x_j - \Lambda^*(x) = \max_{i=1, \ldots, n} \sup_{x \in \mathbb{R}^n: x_j = \beta} x_j - \Lambda^*(x) = \max_{i=1, \ldots, n} \sup_{x \in \mathbb{R}^n: x_i = \beta} \inf_{\lambda \in \mathbb{R}} \Lambda(\lambda + e_j) - \langle \lambda, x \rangle.
\]

The second identity in (4.4) holds true in virtue of the minimax theorem as above, like in item (2). To prove the negativity, we consider the functions \(g_i(\lambda) = -\lambda \beta + \Lambda(\lambda e_i + e_j)\). We have that \(g_i(0) = 0\) and \(g_i'(0) = -\beta + \Lambda'_i(e_j)\). We have \(g_i'(0) = -\beta + \tilde{x}_j^* < 0\). If \(g_i'(0) \neq 0\) for all \(i\), the result is clear. Otherwise, we can find \(\tilde{\beta} \in (\tilde{x}_j^*, \beta)\) such that \(\tilde{\beta} \neq \Lambda'_i(e_j)\) for all \(i\), and since \(e^{Y^j} \mathbf{1}_{B_T \leq \beta} \leq e^{Y^j} \mathbf{1}_{B_T \leq \beta^*}\), we get the claim.

\[\square\]

The following theorem characterizes the asymptotic behavior of basket call prices in the Wishart model. There are different asymptotic regimes to consider, depending on the position of \(y\) with respect to the constants \(\beta^*, \tilde{\beta}^*\) and \(\tilde{\beta}^*\).

**Theorem 4.2.** Assume that \(y \neq \tilde{x}_i^*\) for all \(i\). Then, as \(T \to \infty\), the call option price in the Wishart model satisfies
\[
\lim_{T \to \infty} \mathbb{E}[(e^{B^j_T} - e^{y_T})_+] = \sum_{i=1}^n \omega_i \mathbb{1}_{x_i > y}.
\]

In addition, if \(y < \beta^*\) then
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{E}[(e^{y_T} - e^{B^j_T})_+] = \lim_{T \to \infty} T^{-1} \log \{e^{y_T} - 1 + \mathbb{E}[(e^{B^j_T} - e^{y_T})_+]\} = y - \inf_{z \in (-\infty, y]} \Lambda^*(z) < y; \tag{4.6}
\]
if \(y > \tilde{\beta}^*\), then
\[
\lim_{T \to \infty} T^{-1} \log \mathbb{E}[(e^{B^j_T} - e^{y_T})_+] = \max_{i,j=1, \ldots, n} \inf_{\lambda \in \mathbb{R}} \{-\lambda y + \Lambda(\lambda e_i + e_j)\} < 0; \tag{4.7}
\]
and if \(y \in (\beta^*, \tilde{\beta}^*)\), then
\[
\lim_{T \to \infty} T^{-1} \log (1 - \mathbb{E}[(e^{B^j_T} - e^{y_T})_+]) = y + \max_{i,j=1, \ldots, n} \inf_{\lambda \in \mathbb{R}} \{-\lambda y + \Lambda(\lambda e_i)\} < \min(0, y). \tag{4.8}
\]
Proof.

Proof of (4.5). We remark that
\[
\mathbb{E}\left[(e^{B_T} - e^{yT})_+\right] = \mathbb{E}\left[e^{B_T} 1_{\hat{B}_T > y}\right] - e^{yT} \mathbb{P}\left[\hat{B}_T > y\right]
\tag{4.9}
\]
and consider the two terms separately. If \( y < 0 \), the second term clearly converges to zero. Assume then that \( y \geq 0 \). Since \( \beta^* \leq 0 \), by Lemma 4.1 part 2,
\[
\lim_{T \to \infty} T^{-1} \log e^{yT} \mathbb{P}\left(\hat{B}_T > y\right) < 0
\]
This proves that the second term in (4.9) converges to zero. We now focus on the first term, which satisfies
\[
\mathbb{E}\left[e^{B_T} 1_{\hat{B}_T > y}\right] = \sum_{i=1}^{n} \omega_i \mathbb{E}\left[e^{Y^*_i} 1_{\hat{B}_T > y}\right].
\]
Fix some \( i \in \{1, \ldots, n\} \). Then, by Lemma 4.1 parts 3 and 4, if \( y > \tilde{x}^*_i \) then
\[
\lim_{T \to \infty} \mathbb{E}\left[e^{Y^*_i} 1_{\hat{B}_T > y}\right] = 0,
\]
and if \( y < \tilde{x}^*_i \) then
\[
\lim_{T \to \infty} \mathbb{E}\left[e^{Y^*_i} 1_{\hat{B}_T \leq y}\right] = 0.
\]
Combining these estimates for different \( i \), the proof of (4.5) is complete.

Proof of (4.6). The equality
\[
e^{yT}(1 - e^{-\delta T}) 1_{\{\hat{B}_T < y - \delta\}} \leq (e^{yT} - e^{B_T})_+ \leq e^{yT} 1_{\{\hat{B}_T < y\}}
\]
holds for every \( \delta > 0 \) and \( T > 0 \). Then by successively taking the expectation, the logarithm and multiplying by \( T^{-1} \), we find
\[
y + T^{-1} \log(1 - e^{-\delta T}) + T^{-1} \log \mathbb{P}\left(\hat{B}_T < y - \delta\right)
\leq T^{-1} \log \mathbb{E}\left[(e^{yT} - e^{B_T})_+\right] \leq y + T^{-1} \log \mathbb{P}\left(\hat{B}_T < y\right).
\]
Passing to the limit \( T \to \infty \) and using Lemma 4.1 part 1, the proof is complete.

Proof of (4.7). We use the inequality
\[
e^{B_T}(1 - e^{-\delta T}) 1_{\{y < \hat{B}_T - \delta\}} \leq (e^{B_T} - e^{yT})_+ \leq e^{B_T} 1_{\{y < \hat{B}_T\}}.
\]
Consider for instance the upper bound. Taking the expectation and the logarithm, we obtain
\[
\log \mathbb{E}[e^{yT} 1_{\{\hat{B}_T > y\}}] = \log \sum_{j=1}^{n} \omega_j \mathbb{E}\left[e^{Y^*_j} 1_{\{\hat{B}_T > y\}}\right]
\]
and thus
\[
T^{-1} \log \mathbb{E}[e^{B_T} 1_{\{\hat{B}_T > y\}}] \leq \max_{j=1,\ldots,n} T^{-1} \log \mathbb{E}\left[e^{Y^*_j} 1_{\{\hat{B}_T > y\}}\right],
\]
\[
T^{-1} \log \mathbb{E}[e^{B_T} 1_{\{\hat{B}_T > y + \delta\}}] \geq \max_{j=1,\ldots,n} T^{-1} \log \mathbb{E}\left[e^{Y^*_j} 1_{\{\hat{B}_T > y + \delta\}}\right] + \log(\omega_j)/T.
\]
The result then follows from Lemma 4.1, part 4.

Proof of (4.8). We use the following identity.
\[
1 - \mathbb{E}[(e^{B_T} - e^{yT})_+] = \mathbb{E}[e^{B_T} - (e^{B_T} - e^{yT})_+]
\]
\[
= e^{yT} \mathbb{P}[\hat{B}_T > y] + \mathbb{E}[e^{B_T} 1_{\hat{B}_T \leq y}],
\]
By Lemma 4.1, part 2,

\[ \lim_{T \to \infty} T^{-1} \log e^{yT} \mathbb{P}[\bar{B}_T > y] = y + \max_{i=1, \ldots, n} \inf_{\lambda \in \mathbb{R}} \{ -\lambda y + \Lambda(\lambda e_i) \} < 0. \]

Consider the function \( f_i : \mathbb{R} \to \mathbb{R} \), \( f_i(\lambda) = -\lambda y + \Lambda(\lambda e_i) \). Since \( f_i(0) = 0 \) and \( f'_i(0) = -y + x^*_i < 0 \), it follows that also

\[ y + \max_{i=1, \ldots, n} \inf_{\lambda \in \mathbb{R}} \{ -\lambda y + \Lambda(\lambda e_i) \} < y. \]

On the other hand, by Lemma 4.1, part 3,

\[ \lim_{T \to \infty} T^{-1} \log e^{yT} \mathbb{P}[\bar{B}_T \leq y] = y + \max_{j=1, \ldots, n} \inf_{\lambda \in \mathbb{R}} \{ \Lambda(\lambda) - y(\lambda, 1) \} \]

\[ \leq y + \max_{j=1, \ldots, n} \inf_{\lambda \in \mathbb{R}} f_j(\lambda). \]

Since, for \( y \in (\beta^*, \hat{\beta}^*) \), \( f'_j(0) < 0 \) and \( f'_j(1) > 0 \), the infimum is attained on the interval \((0, 1)\), and the contribution of this term is less than the one of the first term. The properties of the logarithm allow to conclude the proof. \( \square \)

4.2. **Implied volatility asymptotics.** In the Black-Scholes model with volatility \( \sigma \), we have (see, e.g. (Forde and Jacquier, 2011), Corollary 2.12)

\[ \lim_{T \to \infty} T^{-1} \log (C^{BS}(T, yT, \sigma) + e^{yT} - 1) = -\frac{1}{2} \left( \frac{\sigma}{2} - \frac{y}{\sigma} \right)^2, \quad y \leq -\frac{\sigma^2}{2} \]

\[ \lim_{T \to \infty} T^{-1} \log C^{BS}(T, yT, \sigma) = -\frac{1}{2} \left( \frac{\sigma}{2} - \frac{y}{\sigma} \right)^2, \quad y \geq \frac{\sigma^2}{2} \]

\[ \lim_{T \to \infty} T^{-1} \log (1 - C^{BS}(T, yT, \sigma)) = -\frac{1}{2} \left( \frac{\sigma}{2} - \frac{y}{\sigma} \right)^2, \quad -\frac{\sigma^2}{2} < y < \frac{\sigma^2}{2}. \]

Under the Wishart model, for the basket option, we can write:

\[ \lim_{T \to \infty} T^{-1} \log \mathbb{E} \left[ (e^{yT} - e^{yT})_+ \right] = -L(y), \quad y \leq \beta^* \]

\[ \lim_{T \to \infty} T^{-1} \log \mathbb{E} \left[ (e^{yT} - e^{yT})_+ \right] = -L(y), \quad y \geq \hat{\beta}^* \]

\[ \lim_{T \to \infty} T^{-1} \log \left( 1 - \mathbb{E} \left[ (e^{yT} - e^{yT})_+ \right] \right) = -L(y), \quad \beta^* < y < \hat{\beta}^*, \]

where

\[ L(y) = \inf_{\lambda \in \mathbb{R}^n: \lambda \leq 0, i=1, \ldots, n} \{ \Lambda(\lambda) - y(\lambda, 1) \}, \quad y \leq \beta^* \]

\[ L(y) = \max_{i,j=1, \ldots, n} \inf_{\lambda \in \mathbb{R}} \{ -\lambda y + \Lambda(\lambda e_i + e_j) \}, \quad y \geq \hat{\beta}^* \]

\[ L(y) = -y - \max_{i=1, \ldots, n} \inf_{\lambda \in \mathbb{R}} \{ -\lambda y + \Lambda(\lambda e_i) \}, \quad \beta^* < y < \hat{\beta}^*. \]

We deduce (see Jacquier et al., 2013) for details) that the limiting implied volatility of a basket option in the Wishart model is given by

\[ \sigma_\infty(y) = \sqrt{2} \left( \xi \sqrt{L(y)} + y + \eta \sqrt{L(y)} \right), \quad (4.11) \]
where \( \xi \) and \( \eta \) are constants with \( \xi^2 = \eta^2 = 1 \), which must be chosen to satisfy the conditions
\[
\begin{align*}
y &\leq -\frac{\sigma_{\infty}^2(y)}{2} \quad \text{if } y \leq \beta^* \\
y &\geq \frac{\sigma_{\infty}^2(y)}{2} \quad \text{if } y \geq \beta^* \\
-\frac{\sigma_{\infty}^2(y)}{2} &< y < \frac{\sigma_{\infty}^2(y)}{2} \quad \text{if } \beta^* < y < \beta^*.
\end{align*}
\]
First of all remark that by taking \( \lambda = 0 \) and \( \lambda = \epsilon_i \) it follows that \( L(y) \geq y \) and \( L(y) \geq 0 \), so that the expressions under the square root sign are positive. It is easy to see that for \( \beta^* \leq y \leq \beta^* \), these conditions imply \( \xi = -1 \) and \( \eta = 1 \) since \( b^* < 0 \) and \( -y \leq L(y) \), and for \( y \geq \beta^* \) one has \( \xi = 1 \) and \( \eta = -1 \). For \( \beta^* < y < \beta^* \), we still have \(|y| \leq \max(L(y), L(y) + y)\) and to satisfy the conditions in this case and \( \sigma_{\infty}(y) > 0 \), one must take \( \xi = \eta = 1 \).

The case when \( \beta^* < y < \beta^* \) requires a specific treatment. It is characterized by the following proposition.

**Proposition 4.3.** Let \( \beta^* < y < \beta^* \). Then, \( \sigma_{\infty}(y) = \sqrt{2y} \) and
\[
\sigma(T, yT) = \sqrt{2y + N^{-1}(C_{\infty}(y))T^{-1/2}} + \mathcal{O}(T^{-1/2})
\]
as \( T \to \infty \), where \( C_{\infty}(y) = \sum_{i=1}^{n} \omega_i \mathbf{1}_{x_i > y} \).

**Proof.** We follow the arguments of the proof of Theorem 3.3 in (Jacquier and Keller-Ressel, 2018) with some minor changes. The Black-Scholes call option price satisfies
\[
C^{BS}(T, yT, \sigma) = \mathcal{N}\left(\frac{-y + \frac{\sigma^2}{2} \sqrt{T}}{\sigma}\right) - e^{yT} \mathcal{N}\left(\frac{-y - \frac{\sigma^2}{2} \sqrt{T}}{\sigma}\right).
\]
We have by definition of the implied volatility and equation (4.5),
\[
C^{BS}(T, yT, \sigma(t, yT)) = C(T, yT) \to_{T \to \infty} C_{\infty}(y).
\]
Since \( y > \beta^* > 0 \), as \( T \to \infty \), we get necessarily \( \frac{y + \frac{\sigma(T,yT)^2}{\sigma(T,yT)}}{\sqrt{T}} \to +\infty \). Using the classical bound on the Mills ratio \( \mathcal{N}(-x) \leq x^{-1} \phi(x) \) for \( x > 0 \), where \( \phi \) is the standard Gaussian density, we have
\[
e^{yT} \mathcal{N}\left(\frac{-y - \frac{\sigma(T,yT)^2}{\sigma(T,yT)} \sqrt{T}}{\phi}\right) \leq \phi\left(\frac{y - \frac{\sigma(T,yT)^2}{\sigma(T,yT)} \sqrt{T}}{\phi}\right) \to 0
\]
as \( T \to \infty \). Therefore,
\[
\frac{-y + \frac{\sigma(T,yT)^2}{\sigma(T,yT)}}{\sqrt{T}} = N^{-1}(C_{\infty}(y))T^{-1/2} + \mathcal{O}(T^{-1/2}).
\]
(4.12)
Consider now the function \( f(z) = -\frac{z}{2} + \frac{z^2}{2} \). Its inverse which is positive in the neighborhood of zero is given by
\[
f^{-1}(x) = x + \sqrt{x^2 + 2y}
\]
Applying $f^{-1}$ to both sides of (4.12) and neglecting terms of order $o(T^{-1/2})$, the proof is complete. \hfill \Box

5. Variance reduction

Denote $P(S_T)$ the payoff of a European option on $(S^1_T, ..., S^n_T)$. The price of an option is generally calculated as the expectation $\mathbb{E}(P(S_T))$ under a certain risk-neutral measure $\mathbb{P}$. When the number of assets $n$ is low, this expectation may be evaluated by Fourier inversion, however, when the dimension is large, as in the case of index options, Monte Carlo is the method of choice. The standard Monte Carlo estimator of $\mathbb{E}(P(S_T))$ with $N$ samples is given by

$$\hat{P}_N = \frac{1}{N} \sum_{j=1}^{N} P(S_T^{(j)})$$

where $S_T^{(j)}$ are i.i.d. samples of $S_T$ under the measure $\mathbb{P}$. The variance of the standard Monte Carlo estimator is given by

$$\text{Var}[\hat{P}_N] = \frac{1}{N} \text{Var}[P(S_T)]$$

and is often too high for real-time applications. To decrease the computational time, various variance reduction methods have been proposed, the most popular being importance sampling.

The importance sampling method is based on the following identity, valid for any probability measure $\mathbb{Q}$, with respect to which $\mathbb{P}$ is absolutely continuous.

$$\mathbb{E}[P(S_T)] = \mathbb{E}^\mathbb{Q}\left[ \frac{d\mathbb{P}}{d\mathbb{Q}} P(S_T) \right].$$

This allows one to define the importance sampling estimator

$$\hat{P}_N^\mathbb{Q} := \frac{1}{N} \sum_{j=1}^{N} \left[ \frac{d\mathbb{P}}{d\mathbb{Q}} \right]^{(j)} P(S_T^{(j),\mathbb{Q}}),$$

where $S_T^{(j),\mathbb{Q}}$ are i.i.d. samples of $S_T$ under the measure $\mathbb{Q}$. For efficient variance reduction, one needs then to find a probability measure $\mathbb{Q}$ such that $S_T$ is easy to simulate under $\mathbb{Q}$ and the variance

$$\text{Var}^\mathbb{Q}\left[ P(S_T) \frac{d\mathbb{P}}{d\mathbb{Q}} \right] = \mathbb{E}^\mathbb{P}\left[ P(S_T)^2 \frac{d\mathbb{P}}{d\mathbb{Q}} \right] - \left( \mathbb{E}^\mathbb{P}[P(S_T)] \right)^2$$

is considerably smaller than the original variance $\text{Var}^\mathbb{P}[P(S)]$.

In this paper we consider the class of measure changes $\{\mathbb{P}_\theta : \theta \in \mathbb{R}^n\}$, where

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} = \frac{e^{\theta^\top Y_T}}{\mathbb{E}[e^{\theta^\top Y_T}]}.$$ 

To find the optimal variance reduction parameter $\theta^*$, we therefore need to minimize the variance of the estimator under $\mathbb{Q}$, or, equivalently, the expectation

$$\mathbb{E}^\mathbb{P}\left[ P(S_T)^2 \frac{d\mathbb{P}}{d\mathbb{P}_{\theta^*}} \right].$$
5.1. **Asymptotic variance reduction.** Denoting \( H(Y_T) := \log P(e^{Y_T}) \), the optimization problem writes

\[
\inf_{\theta \in \mathbb{R}^n} \mathbb{E} \left[ \exp \left( 2H(Y_T) - \theta^\top Y_T + G_\epsilon(\theta) \right) \right],
\]

(5.1)

where

\[
G_\epsilon(\theta) := \epsilon \log \mathbb{E} \left[ e^{\theta^\top Y_T^\epsilon} \right].
\]

Since we cannot compute the minimizer for this expression explicitly, we instead choose to minimize an asymptotic proxy for the variance, based on Varadhan’s lemma (Theorem 3.6). This proxy is introduced in the following proposition.

**Proposition 5.1.** Let \( H : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\} \) be a continuous function and \( \theta \in \mathbb{R}^n \) be such that there exists \( \gamma > 1 \) with

\[
\limsup_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E} \left[ \exp \left\{ \frac{\gamma 2H(Y_T^\epsilon) - \theta^\top Y_T^\epsilon}{\epsilon} \right\} \right] < \infty.
\]

(5.2)

Then

\[
\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E} \left[ \exp \left\{ \frac{2H(Y_T^\epsilon) - \theta^\top Y_T^\epsilon + G_\epsilon(\theta)}{\epsilon} \right\} \right] = \sup_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^\top y - \Lambda^*(y) \right\} + \Lambda(\theta).
\]

**Proof.** By Theorem 3.6,

\[
\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E} \left[ \exp \left\{ \frac{2H(Y_T^\epsilon) - \theta^\top Y_T^\epsilon}{\epsilon} \right\} \right] = \sup_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^\top y - \Lambda^*(y) \right\}.
\]

(5.3)

Furthermore, by Proposition 3.8,

\[
\epsilon \log \mathbb{E} \left[ \exp \left\{ \frac{G_\epsilon(\theta)}{\epsilon} \right\} \right] = G_\epsilon(\theta) \xrightarrow{\epsilon \rightarrow 0} \Lambda(\theta).
\]

(5.4)

Multiplying (5.3) and (5.4) finishes the proof. \( \square \)

**Remark 5.2.** In particular, if \( H \) is continuous and bounded from above and \( \theta \) is such that \( \phi(-\theta) \in S_n^{+,*} \), condition (5.2) is met.

**Definition 5.3.** A parameter \( \theta^* \in \mathbb{R}^n \) is **asymptotically optimal** if it achieves the infimum in the minimisation problem

\[
\inf_{\theta \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^\top y - \Lambda^*(y) \right\} + \Lambda(\theta).
\]

(5.5)

**Theorem 5.4.** Let \( H \) be a concave upper semi-continuous function. Then

\[
\inf_{\theta \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^\top y - \Lambda^*(y) \right\} + \Lambda(\theta) = 2 \inf_{\theta \in \mathbb{R}^n} \left\{ \tilde{H}(\theta) + \Lambda(\theta) \right\},
\]

where

\[ \tilde{H}(\theta) = \sup_{y \in \mathbb{R}^n} \left\{ H(y) - \theta^\top y \right\}. \]

Furthermore, if \( \theta^* \) minimizes the right-hand side, it also minimizes the left-hand side.
Proof. We follow the idea of the proof of (Genin and Tankov, 2016, Theorem 8), with some major simplifications due to the present finite-dimensional setting. By definition of \( \Lambda^* \),
\[
\inf_{\theta \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^T y - \Lambda^*(y) + \Lambda(\theta) \right\}
\]
\[
= \inf_{\theta \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^T y - \sup_{\lambda \in \mathbb{R}^n} \left\{ \lambda^T y - \Lambda(\lambda) \right\} + \Lambda(\theta) \right\}
\]
\[
= \inf_{\theta \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^n} \inf_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^T y - \lambda^T y + \Lambda(\lambda) + \Lambda(\theta) \right\}.
\]
The function
\[
(y, \lambda) \mapsto 2H(y) - \theta^T y - \lambda^T y + \Lambda(\lambda) + \Lambda(\theta)
\]
is concave-convex on \( \mathbb{R}^n \times U \) where \( U \) is bounded by Remark 2.6 and both \( \mathbb{R}^n \) and \( U \) are convex. Therefore, by the minimax Theorem for concave-convex functions (see, e.g., Corollary 37.3.2 in (Rockafellar, 1970)),
\[
\sup_{y \in \mathbb{R}^n} \inf_{\lambda \in \mathbb{R}^n} \left\{ 2H(y) - \theta^T y - \lambda^T y + \Lambda(\lambda) + \Lambda(\theta) \right\}
\]
\[
= \inf_{\lambda \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^T y - \lambda^T y + \Lambda(\lambda) + \Lambda(\theta) \right\}.
\]
This allows us to rewrite
\[
\inf_{\theta \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^T y - \Lambda^*(y) + \Lambda(\theta) \right\}
\]
\[
= \inf_{\theta \in \mathbb{R}^n} \inf_{\lambda \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} \left\{ 2H(y) - \theta^T y - \lambda^T y + \Lambda(\lambda) + \Lambda(\theta) \right\}
\]
\[
= 2 \inf_{\theta \in \mathbb{R}^n} \inf_{\lambda \in \mathbb{R}^n} \left\{ \hat{H} \left( \frac{\theta + \lambda}{2} \right) + \Lambda(\lambda) + \Lambda(\theta) \right\} = 2 \inf_{\theta \in \mathbb{R}^n} \left\{ \hat{H}(\theta) + \Lambda(\theta) \right\},
\]
where the last equality is justified by the fact that, by convexity,
\[
\frac{\Lambda(\lambda) + \Lambda(\theta)}{2} \geq \Lambda \left( \frac{\lambda + \theta}{2} \right)
\]
with equality if \( \lambda = \theta \).

To prove the last statement of the theorem, assume that the infimum in the right-hand side of (5.6) is attained by \( \theta^* \). Then, using the equality of the right-hand side and the left-hand side, and taking \( \lambda = \theta^* \) in the left-hand side, we see that the same value \( \theta^* \) also attains the infimum in left-hand side. \( \square \)

Remark 5.5. Similarly to (Genin and Tankov, 2016, Definition 6) and to the discussion in Section 4 of (Robertson, 2010), it can be shown that the asymptotically optimal \( \theta \) in Theorem 5.4 reaches the asymptotic lower bound of the variance on the log-scale over all equivalent measure changes.

Let \( Q \sim P \) be an equivalent measure change. Then by Jensen’s inequality
\[
\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}^Q \left( e^{\frac{2H(Y_\epsilon)}{\epsilon}} \left( \frac{dP}{dQ} \right)^2 \right) \geq 2 \lim_{\epsilon \to 0} \epsilon \log \mathbb{E}^Q \left( e^{\frac{H(Y_\epsilon)}{\epsilon}} \left( \frac{dP}{dQ} \right) \right)
\]
\[
= 2 \lim_{\epsilon \to 0} \epsilon \log \mathbb{E}\left( e^{\frac{H(Y_1^\epsilon)}{\epsilon}} \right).
\]

By Theorem 3.6, the right-hand side is equal to
\[
2 \sup_{y \in \mathbb{R}^n} \{ H(y) - \Lambda^*(y) \} = 2 \sup_{y \in \mathbb{R}^n} \inf_{\theta \in \mathbb{R}^n} \left\{ H(y) - \theta^\top y + \Lambda(\theta) \right\}
= 2 \inf_{\theta \in \mathbb{R}^n} \left\{ \sup_{y \in \mathbb{R}^n} \left\{ H(y) - \theta^\top y \right\} + \Lambda(\theta) \right\},
\]
where the second equality is obtained by the minimax theorem for concave-convex functions (Rockafellar, 1970), already used in the proof of Theorem 5.4. But by the same Theorem 5.4, this bound is reached when \( \theta \) is asymptotically optimal.

6. Numerical results

6.1. Long-time implied volatility. Let us now fix the parameters of the model to the values
\[
b = -\begin{pmatrix} 1.0 & 0.7 \\ 0.7 & 0.7 \end{pmatrix}, \quad a = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix}
\]
and \( \alpha = 1.5 \), with initial values \( S_0 = 1 \) and \( x = I_2 \) and consider the problem of pricing a basket put option with log-payoff
\[
H(Y_T) = \log \left( K - \frac{1}{2} e^{Y_1^T} + \frac{1}{2} e^{Y_2^T} \right).
\]

Figure 6.1 shows the implied volatility smile for such an option, for \( T = \frac{1}{3} \), computed by Monte Carlo over 100,000 trajectories, together with the 95% confidence interval. To sample the paths of the process, we use the exact simulation of the Wishart process described in (Ahdida and Alfonsi, 2013), Algorithm 3. Thus, we obtain the values of \( X_t \) on the regular time grid \( t_i = i \Delta t \), with \( i \in \mathbb{N} \) and \( \Delta t > 0 \). Then, for the stock, we use a trapezoidal rule since it gives a second-order weak convergence (see Section 4.3 in (Ahdida and Alfonsi, 2013) for details):
\[
Y_{t_i+1} = Y_{t_i} - \frac{1}{2} \text{diag} \left[ a^\top X_{t_i} + X_{t_i+1} a \right] \Delta t + \text{Chol} \left( a^\top X_{t_i} + X_{t_i+1} a \right) (Z_{t_i+1} - Z_{t_i}),
\]
where \( Z \) is a Brownian motion sampled independently from \( X \) and \( \text{Chol}(M) \) is the Cholesky decomposition of a positive definite matrix \( M \).

We next analyze the convergence of the renormalized implied volatility smile to the long-maturity limit described in section 4.2. Figure 6.2, shows the renormalized smiles for different maturities together with the limiting smile. These smiles were computed by Monte Carlo with 100,000 trajectories and a discretization time step \( \Delta t = 0.1 \). We see that the convergence indeed appears to take place but it is quite slow: even for 50-year maturity using the limit as the approximation for the smile would lead to \( 10 - 15\% \) errors.
6.2. Variance reduction. We now wish to test numerically the variance reduction method to price basket put options. In order to do so, we first identify the law of the Wishart process under the measure $\mathbb{P}_\theta$ and then calculate the asymptotically optimal measure change to finally test the method through Euler Monte-Carlo simulations.

6.2.1. Change of measure. In order to simulate from the model under $\mathbb{P}_\theta$, we need the following result.

**Proposition 6.1.** Let $\theta \in \mathbb{R}^n$ be such that $\mathbb{E}[e^{\theta^\top Y_T}] < \infty$ and consider the change of measure $d\mathbb{P}_\theta = \frac{e^{\theta^\top Y_T}}{\mathbb{E}[e^{\theta^\top Y_T}]} d\mathbb{P}$. Under $\mathbb{P}_\theta$, the process $(Y_t, X_t)$ has dynamics

$$dY_t = \left(r1 - \frac{1}{2} \left((a^\top X_t a)_{11}, \ldots, (a^\top X_t a)_{nn}\right)^\top + a^\top X_t a \theta\right) dt + a^\top X_t^{1/2} dZ_t^\theta$$
and
\[dX_t = (\alpha I_n + (b + 2 \gamma_\theta(T - t))X_t + X_t(b + 2 \gamma_\theta(T - t))) \, dt + X_t^{1/2} \, dW_t^\theta + (dW_t^\theta)^T X_t^{-1/2}, \quad X_0 = x,\]
where \(\gamma_\theta(t) = -\frac{1}{2}(V'(t, \theta) V^{-1}(t, \theta) + b), V(t, \theta) = V(t)\) is given in Proposition 2.4 and \((Z_t^\theta)_{t \geq 0}\) and \((W_t^\theta)_{t \geq 0}\) are \(\mathbb{R}^n\) and \(\mathbb{R}^{n \times n}\)-dimensional independent standard \(\mathbb{P}_\theta\)-Brownian motions.

**Proof.** By Equation 2.5, the Radon-Nikodym density satisfies
\[\zeta_t := \frac{d\mathbb{P}_\theta}{d\mathbb{P}} \Big|_{F_t} = \frac{E[e^{\theta^T Y_T} | F_t]}{E[e^{\theta^T Y_T}]} = \frac{e^{\theta^T \text{Tr}[\theta(t-\theta)Y_0 - r\theta^T \text{Tr} (\gamma_\theta(T) x)]} \det[V(T, \theta)]^{\alpha/2} \det[V(T - t, \theta)]^{-\alpha/2}}{\det[V(T, \theta)]^{\alpha/2} \det[V(T - t, \theta)]^{-\alpha/2}} e^\theta Y_t + \text{Tr}[\gamma_\theta(T - t) X_t].\]
By Itô formula, the martingale property of \(\zeta_t\), Equations (2.2) and (2.3), and the properties of the trace, the dynamics of \(\zeta_t\) is
\[d\zeta_t = \zeta_t \left( \theta^T a^T X_t^{1/2} dZ_t + \text{Tr} \left[ \gamma_\theta(T - t) X_t^{1/2} dW_t \right] + \text{Tr} \left[ \gamma_\theta(T - t) (dW_t)^T X_t^{1/2} \right] \right) = \zeta_t \left( \theta^T a^T X_t^{1/2} dZ_t + 2 \text{Tr} \left[ X_t^{1/2} \gamma_\theta(T - t) X_t^{1/2} \right] \right).\]
Therefore, by Girsanov’s theorem,
\[Z_t^\theta := Z_t - \int_0^t X_s^{1/2} a \theta \, ds\]
and
\[W_t^\theta := W_t - 2 \int_0^t X_s^{1/2} \gamma_\theta(T - s) \, ds\]
are \(n\)-dimensional and \(n \times n\)-dimensional standard \(\mathbb{P}_\theta\)-Brownian motions. Replacing \(dZ_t\) and \(dW_t\) in (2.2) and (2.3) by their \(\mathbb{P}_\theta\) versions finishes the proof. \(\square\)

We note that \(X\) is no longer a Wishart process under the probability \(\mathbb{P}_\theta\), since its dynamics has time-dependent coefficients. To sample paths on the time interval \([t_i, t_{i+1}]\), we use the exact scheme for the Wishart process with the coefficient \(b + 2 \gamma_\theta(T - (t_i + t_{i+1})/2)\) instead of \(b\). As explained in (Alfonsi, 2015) subsection 3.3.4 in the case of the CIR process with time-dependent coefficients, this leads to a second order scheme for the weak error. Then, we can approximate \(Y\) in the same way as under \(\mathbb{P}\):
\[Y_{t_{i+1}} = Y_{t_i} + \left[r 1 - \frac{1}{2} \text{diag} \left(a^T X_{t_i} + X_{t_{i+1}} a \right) + a^T X_{t_i} + X_{t_{i+1}} a \theta \right] \Delta t + \text{Chol} \left(a^T X_{t_i} + X_{t_{i+1}} a \right) (Z_{t_{i+1}} - Z_{t_i}),\]
where \(Z\) is a Brownian motion sampled independently from \(X\). This gives a second order scheme for \((X, Y)\).
6.2.2. **Optimal variance reduction parameter for the European basket put option.** In this section, we compute the asymptotically optimal measure to price basket put options with log-payoff \( H(Y_T) = \log(K - \omega^T e^{Y_T})_+ \), for some \( \omega \in (\mathbb{R}^*_+)^n \). It is shown in (Genin and Tankov, 2016, Section 4) that the function \( H \) is concave and that its convex conjugate is given by

\[
\hat{H}(\theta) = \begin{cases} 
+\infty & \text{if } \theta_k \geq 0 \text{ for some } k \\
- \left(1 - \sum_k \theta_k\right) \log \frac{1 - \sum_k \theta_k}{K} - \sum_k \theta_k \log(-\theta_k/\omega_k) & \text{otherwise}.
\end{cases}
\]

To compute the asymptotically optimal measure change parameter \( \theta^* \) using Theorem 5.4 we then minimize \( \hat{H}(\theta) + \Lambda(\theta) \) with a numerical convex optimization algorithm.

6.2.3. **Numerical simulations.** Let us now fix the parameters of the model to the values

\[
b = -\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.5 \end{pmatrix}, \quad a = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.12 \end{pmatrix}
\]

and \( \alpha = 4.5 \), with initial values \( S_0 = 1 \) and \( x = I_2 \) and consider the problem of pricing a basket put option with log-payoff

\[
H(Y_T) = \log \left( K - \frac{1}{2} e^{Y_{1T}} + \frac{1}{2} e^{Y_{2T}} \right)_+.
\]

For a wide variety of maturities \( T \) and strikes \( K \), listed in Table 1, we simulate 100,000 trajectories, using the discretization scheme described above, with step size \( \Delta = \frac{1}{40} \), under both measures \( P \) and \( P_{\theta^*} \) for the asymptotically optimal \( \theta^* \). The results are presented in Table 1.

| Maturity, years | Strike | Price    | Std. dev.  | Var. ratio | Time, seconds |
|-----------------|--------|----------|------------|------------|---------------|
| 0.50            | 0.7    | 2.18e-07 | 3.37e-08   | 119        | 202           |
| 0.50            | 0.8    | 3.29e-05 | 9.5e-07    | 22.5       | 167           |
| 0.50            | 0.9    | 1.776e-03| 1.38e-05   | 5.28       | 169           |
| 0.50            | 1.0    | 2.6201e-02| 6.85e-05   | 3.15       | 167           |
| 0.50            | 1.1    | 1.0306e-01| 9.86e-05   | 3.96       | 167           |
| 0.50            | 1.2    | 2.0027e-01| 8.29e-05   | 6.68       | 167           |
| 0.50            | 1.3    | 3.0005e-01| 6.41e-05   | 11.3       | 180           |
| 0.50            | 1.4    | 3.9999e-01| 5.32e-05   | 16.5       | 168           |
| 0.25            | 1.0    | 1.730e-02 | 5.17e-05   | 2.42       | 92            |
| 1.00            | 1.0    | 4.115e-02 | 9.51e-05   | 3.76       | 319           |
| 2.00            | 1.0    | 6.423e-02 | 1.39e-04   | 3.86       | 618           |
| 3.00            | 1.0    | 8.319e-02 | 1.78e-04   | 3.63       | 934           |
| 5.00            | 1.0    | 1.1579e-01| 2.46e-04   | 3.22       | 1522          |

Table 1. The variance ratio as function of the maturity and the strike for the basket put option on the Wishart stochastic volatility model.

The variance ratio is the ratio of the variance under the original measure \( P \) to that under the asymptotically optimal measure \( P_{\theta^*} \). As expected, the performance of the
importance sampling algorithm is best for options far from the money, when the exercise is a rare event, but even for at the money options the variance reduction factor is significant, of the order of 3–4. The computational overhead for using the variance reduction algorithm is small: it does not exceed 20% for a small number of trajectories and decreases with the number of trajectories because some precomputation steps are performed only once.

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