WORD HYPERBOLIC EXTENSIONS OF SURFACE GROUPS

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Abstract. Let $S$ be a closed surface of genus $g \geq 2$. A finitely generated group $\Gamma_S$ is an extension of the fundamental group $\pi_1(S)$ of $S$ if $\pi_1(S)$ is a normal subgroup of $\Gamma_S$. We show that the group $\Gamma_S$ is hyperbolic if and only if the orbit map for the action of the quotient group $\Gamma = \Gamma_S/\pi_1(S)$ on the complex of curves is a quasi-isometric embedding.

1. Introduction

Let $\Gamma$ be a finitely generated group. A finite symmetric set $G$ of generators induces a word norm $\|\|$ on $\Gamma$ by defining $\|\phi\|$ to be the smallest length of a word in the generating set $G$ which represents $\phi$. For $\phi, \psi \in \Gamma$ let $d(\phi, \psi) = \|\phi^{-1}\psi\|$; then $d$ is a distance function on $\Gamma$ which is invariant under the left action of $\Gamma$ on itself. Any two such distance functions on $\Gamma$ are bilipschitz equivalent. The group $\Gamma$ is called word hyperbolic if equipped with the distance induced by one (and hence every) word norm, $\Gamma$ is a hyperbolic metric space.

In this note we are interested in word hyperbolic groups which are extensions of the fundamental group $\pi_1(S)$ of a closed orientable surface $S$ of genus $g \geq 2$. By definition, this means that such a group $\Gamma_S$ contains $\pi_1(S)$ as a normal subgroup. Our main goal is to give a geometric characterization of such groups via the action of the quotient group $\Gamma = \Gamma_S/\pi_1(S)$ on the complex of curves for the surface $S$.

Our approach builds on earlier work of Mosher [Mo96, Mo03] and Farb and Mosher [FM02]. First, recall that by a classical result of Dehn-Nielsen-Baer (see [I02]), the extended mapping class group $M^0_g$ of all isotopy classes of diffeomorphisms of $S$ is just the group of outer automorphisms of the fundamental group $\pi_1(S)$ of $S$. Since the center of $\pi_1(S)$ is trivial we can identify $\pi_1(S)$ with its group of inner automorphisms and therefore we obtain an exact sequence

$$1 \to \pi_1(S) \to \text{Aut}(\pi_1(S)) \xrightarrow{\Pi} M^0_g \to 1.$$ 

In particular, for every subgroup $\Gamma$ of $M^0_g$ the pre-image $\Pi^{-1}(\Gamma)$ of $\Gamma$ under the projection $\Pi$ is an extension of $\pi_1(S)$ with quotient group $\Gamma$. Vice versa, if $\Gamma_S$ is any group which contains $\pi_1(S)$ as a normal subgroup then the quotient group $\Gamma = \Gamma_S/\pi_1(S)$ acts as a group of outer automorphisms on $\pi_1(S)$ and therefore there is a natural homomorphism $\rho : \Gamma \to M^0_g$. 

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Now consider any finitely generated extension $\Gamma_S$ of $\pi_1(S)$ with quotient group $\Gamma = \Gamma_S/\pi_1(S)$. If $\Gamma_S$ is word hyperbolic, then the kernel $K$ of the natural homomorphism $\rho : \Gamma \to \mathcal{M}_g^0$ is finite. Namely, $\Gamma_S$ contains the direct product $\pi_1(S) \times K$ as a subgroup and no word hyperbolic group can contain the direct product of two infinite subgroups (see [FM02]). As a consequence, the extension of $\pi_1(S)$ defined by the subgroup $\rho(\Gamma)$ of $\mathcal{M}_g^0$ is the quotient of $\Gamma_S$ by a finite normal subgroup. Since passing to the quotient by a finite normal subgroup preserves hyperbolicity, we may assume without loss of generality that $\Gamma = \Gamma_S/\pi_1(S)$ is a subgroup of $\mathcal{M}_g^0$.

The vertices of the complex of curves $\mathcal{C}(S)$ for $S$ are nontrivial free homotopy classes of simple closed curves on $S$. The simplices in $\mathcal{C}(S)$ are spanned by collections of such curves which can be realized disjointly. In the sequel we restrict our attention to the one-skeleton of $\mathcal{C}(S)$ which we denote again by $\mathcal{C}(S)$ by abuse of notation. Since $g \geq 2$ by assumption, $\mathcal{C}(S)$ is a nontrivial graph which moreover is connected [Ha81]. However, this graph is locally infinite. Namely, for every simple closed curve $\alpha$ on $S$ the surface $S - \alpha$ which we obtain by cutting $S$ open along $\alpha$ contains at least one connected component of Euler characteristic at most $-2$, and such a component contains infinitely many pairwise distinct free homotopy classes of simple closed curves which viewed as curves in $S$ are disjoint from $\alpha$.

Providing each edge in $\mathcal{C}(S)$ with the standard euclidean metric of diameter 1 equips the complex of curves with the structure of a geodesic metric space. Since $\mathcal{C}(S)$ is not locally finite, this metric space $(\mathcal{C}(S), d)$ is not locally compact. Masur and Minsky [MM99] showed that nevertheless its geometry can be understood quite explicitly. Namely, $\mathcal{C}(S)$ is hyperbolic of infinite diameter (see also [B02, H05] for alternative shorter proofs). The extended mapping class group $\mathcal{M}_g^0$ of all isotopy classes of orientation preserving diffeomorphisms of $S$ acts naturally on $\mathcal{C}(S)$ as a group of simplicial isometries. In fact, Ivanov showed that if $g \neq 2$ then $\mathcal{M}_g^0$ is precisely the isometry group of $\mathcal{C}(S)$ (see [I02] for a sketch of a proof and for references).

A map $\Phi$ of a finitely generated group $\Gamma$ into a metric space $(Y,d)$ is called a quasi-isometric embedding if for some (and hence every) choice of a word norm $\| \|$ for $\Gamma$ there exists a number $L > 1$ such that

$$d(\Phi \psi, \Phi \eta)/L - L \leq \|\psi^{-1} \eta\| \leq L d(\Phi \psi, \Phi \eta) + L.$$ 

Note that a quasi-isometric embedding need not be injective. The following definition extends the well known notion of a convex cocompact group of isometries of a simply connected Riemannian manifold of bounded negative curvature to subgroups of the extended mapping class group, viewed as the isometry group of the complex of curves.

**Definition:** A finitely generated subgroup $\Gamma$ of $\mathcal{M}_g^0$ is called convex cocompact if for some $\alpha \in \mathcal{C}(S)$ the orbit map $\varphi \in \Gamma \to \varphi \alpha \in \mathcal{C}(S)$ is a quasi-isometric embedding.

For every subgroup $\Gamma$ of $\mathcal{M}_g^0$, the intersection of $\Gamma$ with the mapping class group $\mathcal{M}_g$ of all isotopy classes of orientation preserving diffeomorphisms is a subgroup of $\Gamma$ of index at most 2 and hence this group is quasi-isometric to $\Gamma$. Thus in the sequel we may restrict our attention to subgroups of $\mathcal{M}_g$. Since $\mathcal{C}(S)$ is a
hyperbolic geodesic metric space, every convex cocompact subgroup $\Gamma$ of $\mathcal{M}_g$ is word hyperbolic.

Farb and Mosher [FM02] introduce another notion of a convex cocompact subgroup $\Gamma$ of $\mathcal{M}_g$ via its action on the Teichmüller space $\mathcal{T}_g$ of all marked hyperbolic metrics on $S$. Namely, they define a group $\Gamma < \mathcal{M}_g$ to be convex cocompact if a $\Gamma$-orbit in $\mathcal{T}_g$ is quasi-convex with respect to the Teichmüller metric; this means that for every fixed $h \in \mathcal{T}_g$ and any two elements $\varphi, \psi \in \Gamma$ the unique Teichmüller geodesic connecting $\varphi h$ to $\psi h$ is contained in a uniformly bounded neighborhood of the orbit $\Gamma h$. Answering among other things a question raised by Farb and Mosher, we show.

**Theorem:** For a finitely generated subgroup $\Gamma$ of $\mathcal{M}_g$, the following are equivalent.

1. $\Gamma$ is convex cocompact.
2. Some $\Gamma$-orbit on $\mathcal{T}_g$ is quasi-convex.
3. The natural extension of $\pi_1(S)$ with quotient group $\Gamma$ is word hyperbolic.

The implication 3) $\Rightarrow$ 2) in our theorem is due to Farb and Mosher [FM02] and was the main motivation for this work. In the particular case that the subgroup $\Gamma$ of $\mathcal{M}_g$ is free, the reverse implication 2) $\Rightarrow$ 3) is also shown in [FM02]. The equivalence of 1) and 2) was independently and at the same time established by Kent and Leininger [KL05], with a different proof.

Examples of convex cocompact subgroups of $\mathcal{M}_g$ are Schottky groups which are defined to be free convex cocompact subgroups of $\mathcal{M}_g$. There is an abundance of such groups: Since every pseudo-Anosov element of $\mathcal{M}_g$ acts with north-south dynamics on the Gromov boundary of the complex of curves, the classical ping-pong lemma shows that for any two non-commuting pseudo-Anosov elements $\varphi, \psi \in \mathcal{M}_g$ there are numbers $k \geq 1, \ell \geq 1$ such that the subgroup of $\mathcal{M}_g$ generated by $\varphi^k, \psi^\ell$ is free and convex cocompact. However, to our knowledge there are no known examples of convex cocompact groups which are not virtually free. On the other hand, there are examples of surface-subgroups of $\mathcal{M}_g$ with interesting geometric properties [GDH99, LR05], but these groups contain elements which are not pseudo-Anosov. Since the orbit on $\mathcal{C}(S)$ of an infinite cyclic subgroup of $\mathcal{M}_g$ generated by an element which is not pseudo-Anosov is bounded, these groups are not convex cocompact.

The organization of this paper is as follows. In Section 2, we define a map $\Psi : \mathcal{T}_g \to \mathcal{C}(S)$ which is roughly equivariant with respect to the action of $\mathcal{M}_g$. We characterize quasi-geodesics in Teichmüller space which are mapped by $\Psi$ to quasi-geodesics in the complex of curves and deduce from this an immediate corollary the equivalence of 1) and 2) in our theorem. In Section 3 we give a geometric description of hyperbolic fibrations with fibre a tree and base a hyperbolic geodesic space. This is used in Section 4 to show the equivalence of 1) and 3) in our theorem.
2. Quasi-geodesics in Teichmüller space which project to quasi-geodesics in the complex of curves

In this section we consider an oriented surface $S$ of genus $g \geq 0$ with $m \geq 0$ punctures. We require that $S$ is non-exceptional, i.e. that $3g - 3 + m \geq 2$. The Teichmüller space $T_{g,m}$ of all marked isometry classes of complete hyperbolic metrics on $S$ of finite volume is homeomorphic to $\mathbb{R}^{6g-6+2m}$. The mapping class group $\mathcal{M}_{g,m}$ of all isotopy classes of orientation preserving diffeomorphisms of $S$ acts properly discontinuously on $T_{g,m}$ preserving the Teichmüller metric. The Teichmüller metric is a complete Finsler metric on $T_{g,m}$.

The one-skeleton $C(S)$ of the complex of curves for $S$ is defined to be the metric graph whose vertices are free homotopy classes of simple closed essential curves, i.e. curves which are not contractible or homotopic into a puncture, and where two such vertices are connected by an edge of length 1 if and only if they can be realized disjointly. Since $S$ is non-exceptional by assumption, the graph $C(S)$ is connected. Moreover, as a metric space it is hyperbolic in the sense of Gromov [MM99, B02, H05].

For every marked hyperbolic metric $h \in T_{g,m}$, every essential free homotopy class $\alpha$ on $S$ can be represented by a closed geodesic which is unique up to parametrization. This geodesic is simple if the free homotopy class admits a simple representative. The $h$-length $\ell_h(\alpha)$ of the class is defined to be the length of its geodesic representative; equivalently, $\ell_h(\alpha)$ equals the minimum of the $h$-lengths of all closed curves representing the class $\alpha$.

A pants decomposition for $S$ is a collection of $3g - 3 + m$ pairwise disjoint simple closed essential curves on $S$ which decompose $S$ into $2g - 2 + m$ pairs of pants, i.e. planar surfaces homeomorphic to a three-holed sphere. By a classical result of Bers (see [Bn92]), there is a number $\chi > 0$ only depending on the topological type of $S$ such that for every complete hyperbolic metric $h$ on $S$ of finite volume there is a pants decomposition for $S$ consisting of simple closed curves of $h$-length at most $\chi$. Define a map $\Psi : T_{g,m} \to C(S)$ by associating to a complete hyperbolic metric $h$ on $S$ of finite volume an essential simple closed curve $\Psi(h) \in C(S)$ whose $h$-length is at most $\chi$. By the collar theorem for hyperbolic surfaces (see [Bn92]), the number of intersection points between any two simple closed geodesics of length at most $\chi$ is bounded from above by a universal constant. On the other hand, the distance between two curves $\alpha, \beta \in C(S)$ is bounded from above by the minimal number of intersection points between any representatives of $\alpha, \beta$ plus one [MM99, B02]. Thus the diameter in $C(S)$ of the set of all simple closed curves of $h$-length at most $\chi$ is bounded from above by a universal constant $R > 0$ and the map $\Psi$ is roughly equivariant with respect to the action of $\mathcal{M}_{g,m}$. This means that for every $\varphi \in \mathcal{M}_{g,m}$ and every $h \in T_{g,m}$ we have $d(\varphi(\Psi h), \Psi(\varphi h)) \leq R$.

Let $J \subset \mathbb{R}$ be a closed connected subset, i.e. either $J$ is a closed interval or a closed ray or the whole line. For some $p > 1$, a map $\gamma : J \to C(S)$ is called a $p$-quasi-geodesic if for all $s, t \in J$ we have

$$d(\gamma(s), \gamma(t))/p - p \leq |s - t| \leq pd(\gamma(s), \gamma(t)) + p.$$
The map $\gamma : J \to C(S)$ is called an unparametrized $p$-quasi-geodesic if there is a closed connected set $I \subset \mathbb{R}$ and a homeomorphism $\zeta : I \to J$ such that $\gamma \circ \zeta : I \to C(S)$ is a $p$-quasi-geodesic. By a result of Masur and Minsky (Theorem 2.6 and Theorem 2.3 of [MM99], see also [H05] for a more explicit statement with proof), there is a number $p > 0$ such that the image under $\Psi$ of every Teichmüller geodesic (i.e., every geodesic in $T_{g,m}$ with respect to the Teichmüller metric) is an unparametrized $p$-quasi-geodesic. However, in general this curve is not a quasi-geodesic with its proper parametrization (see [MM99]).

For $\epsilon > 0$ let $T^\epsilon_{g,m}$ be the collection of all hyperbolic metrics $h \in T_{g,m}$ for which the length of the shortest closed $h$-geodesic is at least $\epsilon$. Informally we think of $T^\epsilon_{g,m}$ as the $\epsilon$-thick part of Teichmüller space. The mapping class group preserves the set $T^\epsilon_{g,m}$ and acts on it cocompactly. Moreover, every $M_{g,m}$-invariant subset of $T_{g,m}$ on which $M_{g,m}$ acts cocompactly is contained in $T^\epsilon_{g,m}$ for some $\epsilon > 0$.

Define for $\epsilon > 0$ a quasi-convex curve in $T^\epsilon_{g,m}$ to be a closed subset of $T_{g,m}$ whose Hausdorff distance to the image of a geodesic arc $\zeta : J \to T^\epsilon_{g,m}$ is at most $1/\epsilon$. Recall that the Hausdorff distance between two closed subsets $A, B$ of a metric space is the infimum of all numbers $r > 0$ such that $A$ is contained in the $r$-neighborhood of $B$ and $B$ is contained in the $r$-neighborhood of $A$. The main goal of this section is to show the following result of independent interest.

**Theorem 2.1:**

1. For every $\nu > 1$ there is a constant $\epsilon = \epsilon(\nu) > 0$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set of diameter at least $1/\epsilon$ and let $\gamma : J \to T^\epsilon_{g,m}$ be a $\nu$-quasi-geodesic. If $\Psi \circ \gamma$ is a $\nu$-quasi-geodesic in $C(S)$ then $\gamma(J)$ is a quasi-convex curve in $T^\epsilon_{g,m}$.

2. For every $\epsilon > 0$ there is a constant $\nu(\epsilon) > 1$ with the following property. Let $\gamma : J \to T_{g,m}$ be a $1/\epsilon$-quasi-geodesic in $T_{g,m}$ whose image $\gamma(J)$ is a quasi-convex curve in $T^\epsilon_{g,m}$; then $\Psi \circ \gamma$ is a $\nu(\epsilon)$-quasi-geodesic in $C(S)$.

We begin with establishing the second part of our theorem. For this we need the following simple no-retraction lemma for quasi-geodesics in the hyperbolic geodesic metric space $C(S)$.

**Lemma 2.2:** For $p > 1$ there is a constant $c = c(p) > 0$ with the following property. Let $\gamma : J \to C(S)$ be any unparametrized $p$-quasi-geodesic; if $t_1 < t_2 < t_3 \in J$ then $d(\gamma(t_1), \gamma(t_3)) \geq d(\gamma(t_1), \gamma(t_2)) + d(\gamma(t_2), \gamma(t_3)) - c$.

**Proof:** Let $p > 1$; by the definition of an unparametrized $p$-quasi-geodesic, it is enough to show the existence of a number $c > 0$ such that for every (parametrized) $p$-quasi-geodesic $\gamma : [0, n] \to C(S)$ and all $0 < t < n$ we have $d(\gamma(0), \gamma(t)) \geq d(\gamma(0), \gamma(n)) - c$.

Since $C(S)$ is hyperbolic, there is a constant $R > 0$ only depending on $p$ such that the Hausdorff distance between every $p$-quasi-geodesic and every geodesic connecting the same endpoints is at most $R$. Let $\gamma : [0, n] \to C(S)$ be any $p$-quasi-geodesic
and let $\zeta : [0, m] \rightarrow \mathcal{C}(S)$ be a geodesic connecting $\gamma(0)$ to $\gamma(n)$. Then for every $t \in (0, n)$ there is a point $s \in (0, m)$ such that $d(\gamma(t), \zeta(s)) \leq R$. Thus we have $d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(n)) \leq d(\zeta(0), \zeta(s)) + d(\zeta(s), \zeta(m)) + 2R = d(\gamma(0), \gamma(n)) + 2R$ which shows the lemma. \hfill \Box

A geodesic lamination for a hyperbolic metric $h \in T_{g,m}$ is a compact subset of $S$ foliated by simple $h$-geodesics [CEG87]. A measured geodesic lamination $\mu$ on $S$ is a geodesic lamination together with a nontrivial transverse invariant measure. An example of a measured geodesic lamination on $S$ is a simple closed curve with the transverse counting measure. The space $\mathcal{ML}$ of measured geodesic laminations on $S$ can be equipped with the weak*-topology, and with this topology, it is homeomorphic to $\mathbb{R}^{6g−6+2m}−\{0\}$. There is a natural continuous action of the multiplicative group $(0, \infty)$ of positive reals on $\mathcal{ML}$ by scaling, and the quotient of $\mathcal{ML}$ under this action is the space $\mathcal{PML}$ of projective measured laminations which is homeomorphic to the sphere $S^{6g−7+2m}$. It can naturally be identified with the projectivized tangent space of $T_{g,m}$ at $h$. The space $\mathcal{PML}$ also is the boundary of a compactification of $T_{g,m}$, called the Thurston boundary of Teichmüller space. This is used to show.

**Lemma 2.3:** For every $\epsilon > 0$ there is a number $\nu_0 = \nu_0(\epsilon) > 0$ with the following property. Let $\gamma : J \rightarrow T_{g,m}^e$ be a Teichmüller geodesic; then the curve $\Psi \circ \gamma : J \rightarrow \mathcal{C}(S)$ is a $\nu_0$-quasi-geodesic.

**Proof:** Let $p > 1$ be such that the image under $\Psi$ of every Teichmüller geodesic is an unparametrized $p$-quasi-geodesic in $\mathcal{C}(S)$; such a number exists by the results of Masur and Minsky [MM92, H05]. Let $\epsilon = \epsilon(p) > 0$ be as in Lemma 2.2.

We claim that for every $\epsilon > 0$ there is a constant $k_0 = k_0(\epsilon) > 0$ with the following property. Let $k \geq k_0$ and let $\gamma : [0, k] \rightarrow T_{g,m}^e$ be a geodesic arc of length at least $k_0$; then $d(\Psi(\gamma(0)), \Psi(\gamma(k))) \geq 2\epsilon$.

To see that this is the case, we argue by contradiction and we assume otherwise. With the number $c$ as above, by Lemma 2.2 there is then a number $\epsilon > 0$ and for every $k > 0$ there is a geodesic arc $\gamma_k : [0, k] \rightarrow T_{g,m}^e$ such that $d(\Psi(\gamma_k(0)), \Psi(\gamma_k(t))) \leq 3c$ for every $t \in [0, k]$. Let $R > 0$ be an upper bound for the diameter in $\mathcal{C}(S)$ of the set of all simple closed curves whose length with respect to some fixed metric $h \in T_{g,m}$ is at most $\chi$. Since the action of $\mathcal{M}_{g,m}$ on $T_{g,m}^e$ is isometric and cocompact, via replacing our constant $3c$ by $3c + 2R$ we may assume that the initial points $\gamma_k(0)$ ($k \geq 1$) of the geodesic arcs $\gamma_k$ are contained in a fixed compact subset of $T_{g,m}^e$. Thus by passing to a subsequence we may assume that the geodesics $\gamma_k$ converge locally uniformly as $k \rightarrow \infty$ to a geodesic $\gamma : [0, \infty) \rightarrow T_{g,m}^e$. By the definition of the map $\Psi$ and continuity of the length functions on Teichmüller space we then have $d(\Psi(\gamma(s)), \Psi(\gamma(0))) \leq 3c + 4R$ for all $s \geq 0$.

Let $\lambda \in \mathcal{PML}$ be the projective measured geodesic lamination which defines the direction of $\gamma$ at $\gamma(0)$, viewed as a point in the projectivized tangent space of $T_{g,m}$ at $\gamma(0)$. Since $\gamma$ is cobounded, i.e. it projects into a compact subset of moduli space $T_{g,m}/\mathcal{M}_{g,m}$, by a result of Masur [Ma82a] the lamination $\lambda$ fills up $S$; this means that every simple closed curve on $S$ intersects $\lambda$ transversally. Moreover, $\gamma(t)$
converges as $t \to \infty$ in the Thurston compactification of $T_{g,m}$ to $\lambda$ \cite{Mas82}. By the
definition of the Thurston compactification of $T_{g,m}$ (see \cite{FP91}), this implies that
the curves $\Psi(\gamma(t))$, viewed as projective measured laminations, converge as $t \to \infty$
in $\mathcal{PML}$ to $\lambda$. As a consequence, the curve $\Psi \circ \gamma$ is an unparametrized quasi-
geodesic in $C(S)$ of infinite diameter (see \cite{K99, H04}) which is a contradiction and
shows our claim.

Now let $n > 0$ and let $\gamma : [0, k_0 n] \to T_{g,m}^*$ be any Teichmüller geodesic. The
image under $\Psi$ of every geodesic in $T_{g,m}$ is an unparametrized $p$-quasi-geodesic; thus
by the choice of $c$, for all $0 \leq s \leq t$ we have $d(\Psi(\gamma(t)), \Psi(\gamma(0))) \geq d(\Psi(\gamma(t)), \Psi(\gamma(s))) + d(\Psi(\gamma(s)), \Psi(\gamma(0))) - c$. On the other hand, from our above consideration and the choice
of $k_0$ we conclude that for every $u < n$ we have $d(\Psi(\gamma(u k_0)), \Psi(\gamma((u + 1)k_0))) \geq 2c$ and therefore $d(\Psi(\gamma((u + 1)k_0)), \Psi(\gamma(s))) \geq d(\Psi(\gamma(u k_0)), \Psi(\gamma(s))) + c$ for all $s \leq u k_0$.
Inductively we deduce that $d(\Psi(\gamma(u k_0)), \Psi(\gamma(v k_0))) \geq c|u - v|$ for all integers $u, v \leq n$.
The map $\Psi : T_{g,m} \to C(S)$ is coarsely Lipschitz by which we mean that there is a
constant $a > 0$ such that $d(\Psi h, \Psi h') \leq ad(h, h') + a$ for all $h, h' \in T_{g,m}$ and where
d$(h, h')$ denotes the Teichmüller distance between $h$ and $h'$. Together with above, it
follows that $\Psi \gamma$ is a $\nu_0$-quasi-geodesic for a constant $\nu_0 > 0$ only depending on $\epsilon$
(more precisely, we have $c|s - t|/k_0 - k_0 a - a \leq d(\Psi(\gamma(s)), \Psi(\gamma(t))) \leq a|s - t| + a$ for
all $s, t \in [0, k_0 n]$). This shows the lemma.

The following corollary shows the second part of Theorem 2.1.

**Corollary 2.4:** For every $\epsilon > 0$ there is a number $\nu = \nu(\epsilon) > 1$ with
the following property. Let $\gamma : J \to T_{g,m}^*$ be a $1/\epsilon$-quasi-geodesic such that $\gamma(J)$ is a
quasi-convex curve in $T_{g,m}^*$; then $\Psi \circ \gamma : J \to C(S)$ is a $\nu$-quasi-geodesic.

**Proof:** Let $\epsilon > 0$ and let $\gamma : J \to T_{g,m}^*$ be a $1/\epsilon$-quasi-geodesic such that $\gamma(J)$ is a
quasi-convex curve in $T_{g,m}^*$. Then there is a Teichmüller geodesic $\zeta : I \to T_{g,m}^*$
and a map $\rho : J \to I$ such that $d(\zeta(\rho(t)), \gamma(t)) \leq 1/\epsilon$ for all $t$. Since by assumption
the map $\gamma$ is a $1/\epsilon$-quasi-geodesic in $T_{g,m}$ and since $\zeta$ realizes the distance between
any of its points, the map $\rho$ is necessarily a $b$-quasi-isometry for a constant $b > 1$ only depending on $\epsilon$. On the other hand, the map $\Psi$ is coarsely Lipschitz and
therefore the distances $d(\Psi(\gamma(t)), \Psi(\zeta(\rho(t))))$ are bounded from above by a number
only depending on $\epsilon$. This implies by Lemma 2.3 that $\Psi \circ \gamma$ is a $\nu$-quasi-geodesic
for a constant $\nu > 0$ only depending on $\epsilon$.

To show the first part of Theorem 2.1, we begin again with a simple observation.

**Lemma 2.5:** For every $\nu > 1$ there is a number $\epsilon_0 = \epsilon_0(\nu) > 0$ with the following
properties. Let $\gamma : [0, n] \to T_{g,m}$ be a $\nu$-quasi-geodesic whose projection $\Psi \gamma$ to $C(S)$
is a $\nu$-quasi-geodesic. If $n \geq 1/\epsilon_0$ then $\gamma([0, n]) \subset T_{g,m}^*$.

**Proof:** Let $n > 0$, $\nu > 1$ and let $\gamma : [0, n] \to T_{g,m}$ be a $\nu$-quasi-geodesic such that
$\Psi \circ \gamma$ is a $\nu$-quasi-geodesic in $C(S)$. Then we have $d(\Psi(\gamma(s)), \Psi(\gamma(t))) \geq |s - t|/\nu - \nu$
for all $s, t \in [0, n]$. Let $R > 0$ be an upper bound for the diameter in $C(S)$ of the
collection of all simple closed curves on $S$ whose length with respect to some metric
$h \in T_{g,m}$ is at most $\chi$ where as before, $\chi > 0$ is determined by Bers’ theorem. Let
be an interval for which there is a simple closed curve \( \alpha \in \mathcal{C}(S) \) so that 
\[ \ell_{\gamma(t)}(\alpha) \leq \chi \]
for all \( t \in [a, b] \); then we have \( d((\Psi(\gamma(a)), \alpha) \leq R, d(\Psi(\gamma(b)), \alpha) \leq R \)
and therefore \( |b - a| \leq 2\nu R + \nu^2 \).

Now by a result of Wolpert (see [LT93]), for all \( \alpha \in \mathcal{C}(S) \) and all \( h, h' \in \mathcal{T}_{g,m} \) the
distance between \( h \) and \( h' \) is at least \( |\log \ell_h(\alpha) - \log \ell_{h'}(\alpha)| \).
Thus if there is a point \( t \in [0, n] \) with \( \log(\ell_{\gamma(t)}(\alpha)) < \log(\chi) - 2\nu R - 2\nu^2 \) then the \( \gamma(s) \)-length of \( \alpha \) is smaller
than \( \chi \) for every \( s \in [0, n] \) with \( |s - t| \leq 2\nu R + 2\nu^2 \) and consequently by our above
consideration, \( \Psi \circ \gamma \) is not a \( \nu \)-quasi-geodesic provided that \( n \geq 4\nu R + 4\nu^2 \). \( \square \)

Every Teichmüller geodesic line \( \gamma : \mathbb{R} \to \mathcal{T}_{g,m} \) is defined by a quadratic differential
on \( S \). More precisely, for each \( t \in \mathbb{R} \) there is a holomorphic quadratic differential
\( q_t \) on the Riemann surface \( \gamma(t) \) defining a singular euclidean metric on \( S \) in the
conformal class of \( \gamma(t) \) and of area one. The differential \( q_t \) and the corresponding
piecewise euclidean metric are determined by the horizontal and the vertical foliation of \( q_t \). These foliations have a common finite set of singular points and are
equipped with a transverse invariant measure. For \( s \neq t \), the horizontal foliation for \( q_s \) coincides with the horizontal foliation for \( q_t \), but its transverse measure is
obtained from the transverse measure for \( q_t \) by scaling with the factor \( e^{s-t} \). Similarly,
the vertical foliation of \( q_s \) coincides with the vertical foliation of \( q_t \), but its transverse measure is obtained from the transverse measure for \( q_t \) by scaling with the factor \( e^{s-t} \). We use this description of Teichmüller geodesics together with the
arguments in Section 3.9 of [Mo03] to show the first part of Theorem 2.1.

**Lemma 2.6:** For every \( \nu > 1 \) there is a constant \( \epsilon = \epsilon(\nu) > 0 \) with the following
property. Let \( J \subset \mathbb{R} \) be a closed connected subset of diameter at least \( 1/\epsilon \) and let
\( \gamma : J \to \mathcal{T}_{g,m} \) be a \( \nu \)-quasi-geodesic such that \( \Psi \circ \gamma : J \to \mathcal{C}(S) \) is a \( \nu \)-quasi-geodesic
in \( \mathcal{C}(S) \); then \( \gamma(J) \) is a quasi-convex curve in \( \mathcal{T}_{g,m} \).

**Proof:** For \( \nu > 1 \) define a \( \nu \)-Lipschitz curve in \( \mathcal{T}_{g,m} \) to be a \( \nu \)-Lipschitz map
\( \gamma : J \to \mathcal{T}_{g,m} \) with respect to the standard metric on \( \mathbb{R} \) and the Teichmüller metric
on \( \mathcal{T}_{g,m} \). Since \( \mathcal{T}_{g,m} \) is a smooth manifold and the Teichmüller metric is a
complete Finsler metric, every \( \nu \)-quasi-geodesic \( \gamma : J \to \mathcal{T}_{g,m} \) can be replaced by a
piecewise geodesic \( \zeta : J \to \mathcal{T}_{g,m} \) which is a \( 2\nu \)-Lipschitz curve and which satisfies
\( d(\gamma(t), \zeta(t)) \leq 2\nu \) for all \( t \in J \). Thus it is enough to show the statement of the
lemma for \( \nu \)-Lipschitz curves \( \gamma : J \to \mathcal{T}_{g,m} \) which are \( \nu \)-quasi-geodesics and such
that \( \Psi \circ \gamma \) is a \( \nu \)-quasi-geodesic in \( \mathcal{C}(S) \). In the sequel we also assume that the
diameter \( |J| \) of the set \( J \) is bigger than \( 1/\epsilon_0 \) where \( \epsilon_0 = \epsilon_0(\nu) \) is as in Lemma 2.5;
then \( \gamma(J) \subset \mathcal{T}_{g,m} \).

Since \( \mathcal{C}(S) \) is hyperbolic and \( \Psi \circ \gamma \) is a \( \nu \)-quasi-geodesic by assumption, there
is a geodesic arc in \( \mathcal{C}(S) \) whose Hausdorff distance to \( \Psi \circ \gamma(J) \) is bounded from
above by a universal constant. As a consequence, if \( J \) is one-sided infinite, say if
\( (0, \infty) \subset J \), then the points \( \Psi(\gamma(t)) \) converge as \( t \to \infty \) to a point in the Gromov boundary \( \partial \mathcal{C}(S) \) of \( \mathcal{C}(S) \) (see [BH] for the definition of the Gromov boundary of a
hyperbolic geodesic metric space). The Gromov boundary of \( \mathcal{C}(S) \) can naturally
be identified with the space of minimal geodesic laminations on \( S \) which fill up \( S \),
equipped with a coarse Hausdorff topology (see [HO3]). Here a geodesic lamination
is minimal if each of its half-leaves is dense, and it fills up \( S \) if it intersects every essential simple closed curve on \( S \) transversely.

A simple closed curve \( \alpha \in \mathcal{C}(S) \) defines a projective measured lamination which we denote by \([\alpha]\). Similarly, for a measured lamination \( \lambda \in \mathcal{ML} \) we denote by \([\lambda]\) the projective class of \( \lambda \). Following Mosher [Mo03], we say that the projective measured lamination \([\alpha]\) defined by a simple closed curve \( \alpha \in \mathcal{C}(S) \) is realized at some \( t \in J \) if the length of \( \alpha \) with respect to the metric \( \gamma(t) \in T_{g,m} \) is at most \( \chi \). Note that the number of projective measured laminations which are realized at a given point \( t \in J \) is uniformly bounded and that \([\Psi(\gamma(t))]\) is realized at \( \gamma(t) \). Similarly, we say that the projectivization \([\lambda]\) of a measured geodesic lamination \( \lambda \) is realized at an infinite “endpoint” of \( J \) if the support of \( \lambda \) equals the corresponding endpoint of the quasi-geodesic \( \Psi(\gamma(J)) \) in the Gromov boundary \( \partial \mathcal{C}(S) \) of \( \mathcal{C}(S) \), viewed as a minimal geodesic lamination. The set of projective measured laminations which are realized at an infinite endpoint of \( J \) is a nonempty closed subset of \( \mathcal{PML} \) (see [K99], [H04]). We call a projective measured lamination which is realized at a (finite or infinite) endpoint of \( J \) an \textit{endpoint lamination}.

Now \( \Psi(\gamma) \) is a \( \nu \)-quasi-geodesic in \( \mathcal{C}(S) \) by assumption and the diameter in \( \mathcal{C}(S) \) of the set of all curves of length at most \( \chi \) with respect to some fixed hyperbolic metric \( h \in T_{g,m} \) is bounded from above by a universal constant. Since any two curves \( \alpha, \beta \in \mathcal{C}(S) \) with \( d(\alpha, \beta) \geq 3 \) jointly fill up \( S \), i.e. are such that every simple closed essential curve \( \zeta \in \mathcal{C}(S) \) intersects either \( \alpha \) or \( \beta \) transversely, by possibly increasing the lower bound for the diameter of \( J \) we may assume that any two projective measured laminations \([\alpha], [\beta]\) which are realized at the two distinct endpoints of \( J \) jointly fill up \( S \).

There is a 1-1-correspondence between measured geodesic laminations and equivalence classes of \textit{measured foliations} on \( S \) (see e.g. [Ke92] for a precise statement and references). Via this identification, any pair of distinct points \([\lambda] \neq [\mu] \in \mathcal{PML} \) which jointly fill up the surface \( S \) define a unique Teichmüller geodesic line. Thus for every \( \nu \)-quasi-geodesic \( \zeta : J \to \mathcal{C}(S) \) in \( \mathcal{C}(S) \) with \( |J| \geq 1/\epsilon_0 \), any pair of projective measured laminations \([\lambda], [\mu]\) realized at the two (possibly infinite) endpoints of \( \zeta \) defines a unique Teichmüller geodesic \( \eta([\lambda], [\mu]) \).

Choose a number \( R > 2\chi \) and a smooth function \( \sigma : [0, \infty) \to [0, 1] \) with \( \sigma(0, \chi] \equiv 1 \) and \( \sigma(R, \infty) \equiv 0 \). For each \( h \in T_{g,m} \), the number of simple closed curves \( \alpha \) on \( S \) with \( \ell_h(\alpha) \leq R \) is bounded from above by a universal constant not depending on \( h \), and the diameter of the subset of \( \mathcal{C}(S) \) containing these curves is uniformly bounded as well. Thus we obtain for every \( h \in T_{g,m} \) a finite Borel measure \( \mu_h \) on \( \mathcal{C}(S) \) by defining \( \mu_h = \sum_\beta \sigma(\ell_h(\beta))\delta_\beta \) where \( \delta_\beta \) denotes the Dirac mass at \( \beta \). The total mass of \( \mu_h \) is bounded from above and below by a universal positive constant, and the diameter of the support of \( \mu_h \) in \( \mathcal{C}(S) \) is uniformly bounded as well. Moreover, the measures \( \mu_h \) depend continuously on \( h \in T_{g,m} \) in the weak*-topology. This means that for every bounded function \( f : \mathcal{C}(S) \to \mathbb{R} \) the function \( h \to \int f d\mu_h \) is continuous.
We define now a new “distance” function $\rho$ on $T_{g,m}$ by

$$\rho(h,h') = \int_{C(S) \times C(S)} d(\cdot,\cdot) d\mu_h \times d\mu_{h'} / \mu_h(C(S)) \mu_{h'}(C(S)).$$

Clearly the function $\rho$ is positive and continuous on $T_{g,m} \times T_{g,m}$ and invariant under the action of $\mathcal{M}_{g,m}$. Moreover, it is immediate that there is a universal constant $a > 0$ such that $\rho(h,h')/a - a \leq d(\Psi(h), \Psi(h')) \leq a\rho(h,h') + a$. As a consequence, for every $\nu > 1$ there is a constant $p = p(\nu) > 1$ with the following property. If $\gamma : J \to T_{g,m}$ is such that $\Psi \gamma$ is a $\nu$-quasi-geodesic, then $\gamma$ is a $p$-quasi-geodesic with respect to the “distance” function $\rho$. By this we mean that

$$\rho(\gamma(s), \gamma(t))/p - p \leq |s - t| \leq p\rho(\gamma(s), \gamma(t)) + p$$

for all $s, t \in J$. Moreover, for every $p > 1$ there is a constant $\nu = \nu(p) > 1$ such that if $\gamma : J \to T_{g,m}$ is a Lipschitz curve which is a $p$-quasi-geodesic with respect to $\rho$, then $\Psi \circ \gamma$ is a $\nu$-quasi-geodesic in $C(S)$.

Let $h \in T_{g,m}$ and let $\mu \in \mathcal{ML}$ be a measured geodesic lamination. The product of the transverse measure for $\mu$ together with the length element of $h$ defines a measure on the support of $\mu$ whose total mass is called the $h$-length of $\mu$; we denote it by $\ell_h(\mu)$. Following Mosher \cite{Mo03}, for $p > 1$ define $\Gamma_p$ to be the set of all triples $(\gamma : J \to T_{g,m}, \lambda_+, \lambda_-)$ with the following properties.

1. $0 \in J$ and the diameter of $J$ is at least $1/\epsilon_0$ where $\epsilon_0 = \epsilon_0(\nu(p))$ is as in Lemma 2.5.
2. $\gamma : J \to T_{g,m}$ is a $p$-Lipschitz curve which is a $p$-quasi-geodesic with respect to the “distance” $\rho$.
3. $\lambda_+, \lambda_- \in \mathcal{ML}$ are laminations of $\gamma(0)$-length 1, and the projective measured lamination $[\lambda_+]$ is realized at the right end, the projective measured lamination $[\lambda_-]$ is realized at the left end of $\gamma$.

We equip $\Gamma_p$ with the product topology, using the weak* topology on $\mathcal{ML}$ for the second and the third component of our triple and the compact-open topology for the arc $\gamma$ in $T_{g,m}$. Note that this topology is metrizable.

We follow Mosher (Proposition 3.17 of \cite{Mo03}) and show that the action of $\mathcal{M}_{g,m}$ on $\Gamma_p$ is cocompact. Namely, recall from Lemma 2.5 that there is a constant $\epsilon_0 > 0$ such that for every $(\gamma, \lambda_+, \lambda_-) \in \Gamma_p$ the image of $\gamma$ is contained in $T_{g,m}^{\epsilon_0}$. Since $\mathcal{M}_{g,m}$ acts cocompactly on $T_{g,m}^{\epsilon_0}$ it is therefore enough to show that the subset of $\Gamma_p$ consisting of triples with the additional property that $\gamma(0)$ is contained in a fixed compact subset $A$ of $T_{g,m}^{\epsilon_0}$ is compact. Since our topology is metrizable, this follows if every sequence of points $(\gamma, \lambda_+, \lambda_-)$ with $\gamma(0) \in A$ has a convergent subsequence.

However, by the Arzela-Ascoli theorem, the set of $p$-Lipschitz maps into $T_{g,m}$, issuing from a point in $A$ is compact. Moreover, the function $\rho$ on $T_{g,m} \times T_{g,m}$ is continuous and invariant under the action of $\mathcal{M}_{g,m}$ and hence if $\gamma_i$ converges locally uniformly to $\gamma$ and if $\gamma_i$ is a $p$-quasi-geodesic with respect to $\rho$ for all $i$ then the same is true for $\gamma$. Since the function on $T_{g,m} \times \mathcal{ML}$ which assigns to a metric $h \in T_{g,m}$ and a measured lamination $\mu \in \mathcal{ML}$ the $h$-length of $\mu$ is continuous and since for every fixed $h \in T_{g,m}$ the set of measured laminations of $h$-length 1 is compact and naturally homeomorphic to $\mathcal{PML}$, the action of $\mathcal{M}_{g,m}$ on $\Gamma_p$ is indeed cocompact.
provided that the following holds: If \((\gamma_i : J_i \to T_{g,m}^\gamma)\) is a sequence of \(p\)-Lipschitz curves which converge locally uniformly to \(\gamma : J \to T_{g,m}^\gamma\), if the projective measured lamination \([\lambda]\) is realized at the right endpoint of \(J_i\) and if \([\lambda_i] \to [\lambda]\) in \(\mathcal{PMC}\) \((i \to \infty)\) then \([\lambda]\) is realized at the right endpoint of \(J\).

To see that this is indeed the case, assume first that \(J \cap [0, \infty) = [0, b]\) for some \(b \in (0, \infty)\). Then for sufficiently large \(i\) we have \(J_i \cap [0, \infty) = [0, b_i]\) with \(b_i \in (0, \infty)\) and \(b_i \to b\). Thus \(\gamma_i(b_i) \to \gamma(b)\) \((i \to \infty)\) and therefore for sufficiently large \(i\) there is only a finite number of curves \(\alpha \in \mathcal{C}(S)\) whose length with respect to one of the metrics \(\gamma_j(b_j), \gamma(b)\) \((j \geq i)\) is at most \(\chi\). By passing to a subsequence we may assume that there is a simple closed curve \(\alpha \in \mathcal{C}(S)\) with \([\lambda_j] = [\alpha]\) for all large \(j\). The \(\gamma_i(b_j)\)-length of \(\alpha\) is at most \(\chi\) for all sufficiently large \(j\) and hence the same is true for the \(\gamma(b)\)-length of \(\alpha\) by continuity of the length function. As a consequence, the limit \([\lambda] = [\alpha]\) of the sequence \(([\lambda_i])\) is realized at the endpoint \(\gamma(b)\) of \(\gamma\).

In the case that \([0, \infty) \subset J\) we argue as before. Namely, assume without loss of generality that \(b_i < \infty\) for all \(i\) and that \(b_i \to \infty\). Recall that each of the curves \(\Psi \gamma_i\) is a uniform quasi-geodesic in \(\mathcal{C}(S)\) and that the map \(\Psi\) is coarsely Lipschitz. Let \(\alpha_i \in \mathcal{C}(S)\) be the simple closed curve such that \([\alpha_i] = [\lambda_i]\). Then for each \(i\), the curve \(\alpha_i\) is contained in a ball about \(\Psi(\gamma_i(b_i))\) of radius \(R > 0\) independent of \(i\) and hence as \(i \to \infty\), the curves \(\alpha_i\) converge in \(\mathcal{C}(S) \cup \partial \mathcal{C}(S)\) to the endpoint \(\mu \in \partial \mathcal{C}(S)\) of \(\Psi \circ \gamma\) in the Gromov boundary of \(\mathcal{C}(S)\). As a consequence, the curves \(\alpha_i\) converge to \(\mu\) in the coarse Hausdorff topology \([103]\). This means that every accumulation point of \((\alpha_i)\) in the Hausdorff topology contains \(\mu\) as a sublamination. Since \(\mu\) is a minimal geodesic lamination which fills up \(S\), the complement of \(\mu\) in every lamination \(\zeta\) containing \(\mu\) as a sublamination consists of a finite number of isolated leaves and therefore every transverse measure supported in \(\zeta\) is in fact supported in \(\mu\). Thus after passing to a subsequence, the projective measured laminations \([\lambda_i]\) converge as \(i \to \infty\) to a projective measured lamination supported in \(\mu\). But \([\lambda_i] \to [\lambda]\) in \(\mathcal{PMC}\) by assumption and hence the lamination \([\lambda]\) is realized at the endpoint of \(\gamma\). This shows our above claim and implies that the action of \(M_{g,m}\) on \(\Gamma_p\) is indeed cocompact.

Now we follow Section 3.10 of \([Mo03]\). Namely, each point \((\gamma, \lambda_-, \lambda_+) \in \Gamma_p\) determines the geodesic \(\eta([\lambda_-], [\lambda_+])\) in \(T_{g,m}\). This geodesic defines a family \(q_t\) of quadratic differentials whose horizontal foliation corresponds to the lamination \(e^{-t} \lambda_+\) and whose vertical foliation corresponds to \(e^t \lambda_-\) (note that the area of these differentials is not necessarily normalized, see \([Mo03]\)).

For \((\gamma, \lambda_-, \lambda_+)\) define \(\sigma(\gamma, \lambda_-, \lambda_+)\) to be the point on the geodesic \(\eta([\lambda_-], [\lambda_+])\) which up to normalization corresponds to the quadratic differential defined by \(\lambda_-, \lambda_+\). The map taking \((\gamma, \lambda_-, \lambda_+)\) to \((\gamma(0), \sigma(\gamma, \lambda_-, \lambda_+)) \in T_{g,m} \times T_{g,m}\) is continuous and equivariant with respect to the natural action of \(M_{g,m}\) on \(\Gamma_p\) and on \(T_{g,m} \times T_{g,m}\). Since the action of \(M_{g,m}\) on \(\Gamma_p\) is cocompact, the same is true for the action of \(M_{g,m}\) on the image of our map (see \([Mo03]\)). Thus the distance between \(\gamma(0)\) and \(\sigma(\gamma, \lambda_-, \lambda_+)\) is bounded from above by a universal constant \(R > 0\).
Let again \((\gamma, \lambda_-, \lambda_+) \in \Gamma_p\). For each \(s \in J\) define

\[
a_-(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_-)}, \quad a_+(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_+)}
\]

where as before, \(\ell_{\gamma(s)}(\lambda_{\pm})\) is the \(\gamma(s)\)-length of \(\lambda_{\pm}\). These are continuous functions of \(t \in J\). Define for \(s \in \mathbb{R}\) the shift \(\gamma'(t) = \gamma(t+s)\); then the ordered triple \((\gamma'(0), a_-(s)\lambda_-, a_+(s)\lambda_+)\) lies in the \(M_{g,m}\)-cocompact set \(\Gamma_p\) and hence the distance between \(\gamma(s)\) and a suitably chosen point on the geodesic \(\eta([\lambda_-], [\lambda_+])\) is at most \(R\). As a consequence, the arc \(\gamma\) is contained in the \(R\)-neighborhood of the geodesic \(\eta([\lambda_-], [\lambda_+])\). Since the curve \(\gamma\) is a \(p\)-quasi-geodesic, this implies that the Hausdorff distance between \(\gamma(J)\) and a subarc of \(\eta([\lambda_-], [\lambda_+])\) connecting the same endpoints is uniformly bounded and shows the lemma. \(\square\)

Recall that for every finitely generated group \(\Gamma\), every finite symmetric set of generators induces a word norm on \(\Gamma\), and any two such word norms are equivalent. As in the introduction, we define a convex cocompact subgroup of the mapping class group \(M_{g,m}\) for \(S\) as follows.

**Definition 2.7:** Let \(\Gamma\) be a finitely generated subgroup of \(M_{g,m}\). The group \(\Gamma\) is called **convex cocompact** if some orbit map \(\varphi \in \Gamma \rightarrow \varphi \alpha \in \mathcal{C}(S)\) for the action of \(\Gamma\) on \(\mathcal{C}(S)\) is a quasi-isometric embedding.

The following observation follows immediately from the fact that the map \(\Psi\) is coarsely Lipschitz.

**Lemma 2.8:** For every convex cocompact group \(\Gamma < M_{g,m}\) and every \(h \in T_{g,m}\), the orbit map \(\varphi \in \Gamma \rightarrow \varphi h \in T_{g,m}\) is a quasi-isometric embedding.

**Proof:** Let \(\Gamma < M_{g,m}\) be a convex cocompact group with a finite symmetric set \(\mathcal{G}\) of generators and let \(h \in T_{g,m}\). Write \(\ell = \max\{d(h, \varphi h) \mid \varphi \in \mathcal{G}\}\); since \(\Gamma\) acts on \(T_{g,m}\) as a group of isometries we have \(d(\varphi h, \psi h) \leq \ell \|\varphi^{-1} \psi\|\) for all \(\varphi, \psi \in \Gamma\). On the other hand, the map \(\Psi\) is coarsely equivariant with respect to the action of \(M_{g,m}\) on \(T_{g,m}\) and \(\mathcal{C}(S)\) and coarsely Lipschitz; therefore there is a number \(\nu > 0\) such that \(d(\varphi h, \psi h) \geq d(\varphi \Psi h, \psi \Psi h)/\nu - \nu\) for all \(\varphi, \psi \in M_{g,m}\). Since \(\Gamma\) is convex cocompact by assumption, we moreover have \(d(\varphi \Psi h, \psi \Psi h) \geq \|\varphi^{-1} \psi\|/\nu' - \nu'\) for some \(\nu' > 0\) and all \(\varphi, \psi \in \Gamma\). This shows the lemma. \(\square\)

The \(\Gamma\)-orbit \(\Gamma h\) of a finite generated subgroup \(\Gamma\) of \(M_{g,m}\) is called **quasi-convex** if for any two \(\varphi, \psi \in \Gamma\), the Teichmüller geodesic connecting \(\varphi\) to \(\psi\) is contained in a uniformly bounded neighborhood of \(\Gamma h\). The following result shows the first part of our theorem from the introduction in the more general context of non-exceptional surfaces of finite type. It was independently and at the same time obtained by Kent and Leininger [KL05], with a different proof.

**Theorem 2.9:** A finitely generated subgroup \(\Gamma\) of \(M_{g,m}\) is convex cocompact if and only if some \(\Gamma\)-orbit on \(T_{g,m}\) is quasi-convex.
Proof: Let $\Gamma$ be a finitely generated convex cocompact subgroup of $\mathcal{M}_{g,m}$. Let $h \in T'_{g,m}$ for some $\epsilon > 0$. By Lemma 2.8, the orbit map $\varphi \in \Gamma \to \varphi h \in T'_{g,m}$ is a quasi-isometric embedding. In particular, for any two $\varphi, \psi \in \Gamma$ the orbit of a geodesic in $\Gamma$ connecting $\varphi h$ to $\psi h$ is a uniform quasi-geodesic in $T'_{g,m}$ which is contained in $T'_{g,m}$ and is mapped by $\Psi$ to a uniform quasi-geodesic in $\mathcal{C}(S)$. By Theorem 2.1, this curve is contained in a uniformly bounded neighborhood of the Teichmüller geodesic connecting $\varphi h$ to $\psi h$. In other words, the orbit $\Gamma h \subset T'_{g,m}$ is quasi-convex.

Vice versa, let $\Gamma < \mathcal{M}_{g,m}$ be a finitely generated group and assume that there is some $h \in T'_{g,m}$ such that the orbit $\Gamma h$ of $h$ for the action of $\Gamma$ on $T'_{g,m}$ is quasi-convex. This means that there is a number $D > 0$ such that each Teichmüller geodesic with both endpoints in $\Gamma h$ is contained in the $D$-neighborhood of $\Gamma h$. Using the map $\Psi : T'_{g,m} \to \mathcal{C}(S)$ from the beginning of this Section, write $\alpha = \Psi h \in \mathcal{C}(S)$ and assume to the contrary that the orbit map $\varphi \to \varphi \alpha$ for the action of $\Gamma$ on the complex of curves is not a quasi-isometry. Choose a finite symmetric set $\mathcal{G}$ of generators for $\Gamma$ which induces the word norm $\| \|$. Since $\Gamma$ acts on $\mathcal{C}(S)$ by isometries, the orbit map is coarsely Lipschitz with respect to the word norm $\| \|$ on $\Gamma$ and the metric on $\mathcal{C}(S)$. Thus our assumption implies that for every $L > 0$ there is a word $w = w_1 \ldots w_p \in \Gamma$ in the generators $w_i \in \mathcal{G}$ with $\|w\| = p$ and such that $d(w, \alpha) \leq p/L$. Choose a geodesic $\zeta : [0, m] \to T_{g,m}$ connecting $h = \zeta(0) \in T_{g,m}$ to $wh = \zeta(m)$. Since the orbit $\Gamma h \subset T_{g,m}$ is quasi-convex by assumption, the geodesic is contained in the $D$-neighborhood of $\Gamma h$, in particular it is contained in $T_{g,m}$ for a universal number $\epsilon > 0$. By Theorem 2.1, there is a number $\nu > 1$ not depending on $w$ such that the length $m$ of $\zeta$ is at most $\nu p/L - 1$.

On the other hand, the number of elements $\varphi \in \mathcal{M}_{g,m}$ with $d(\varphi h, h) \leq 2D + 1$ is bounded from above by a constant $\kappa > 0$ only depending on the topological type of the surface $S$ and on $\epsilon$ [BM92]. Therefore the word-norm of an element $\varphi \in \Gamma$ with $d(\varphi h, h) \leq 2D + 1$ is bounded from above by a constant $\ell > 0$. For an integer $k < m$ choose some $\varphi(k) \in \Gamma$ with $d(\varphi(k)h, \zeta(k)) \leq D$. Then we have $d(\varphi(k)^{-1} \varphi(k+1) h, h) \leq 2D + 1$ and hence the word norm of $\varphi(k)^{-1} \varphi(k+1)$ is at most $\ell$. As a consequence, the word norm $p$ of $w$ is at most $\ell(m + 1) \leq \ell \nu p/L$. For $L > \ell \nu$, this is a contradiction which shows that $\Gamma$ is indeed convex cocompact. □

Improving earlier results of Farb and Mosher [FM02], Kent and Leininger [KL05] obtained another characterization of convex cocompact subgroups of $\mathcal{M}_{g,m}$. For the formulation of their result, let $\partial T_{g,m}$ be the Thurston boundary of Teichmüller space; recall that $\partial T_{g,m}$ is naturally homeomorphic to the space $\mathcal{PML}$ of projective measured laminations on $S$. The limit set $\Lambda$ for the action of a group $\Gamma < \mathcal{M}_{g,m}$ on Teichmüller space $T_{g,m}$ is the closure in $\mathcal{PML}$ of the set of fixed points of the pseudo-Anosov elements of $\Gamma$; it is a closed subset of $\partial T_{g,m} \sim \mathcal{PML}$ which is invariant under the natural action of $\Gamma$ (see [MPS03] for the construction of limit sets for a subgroup of $\mathcal{M}_{g,m}$ and their basic properties). Its weak convex hull in $T_{g,m}$ is defined to be the closure of the union of all geodesics in $T_{g,m}$ with both endpoints in $\Lambda$; it is a closed $\Gamma$-invariant subset of $T_{g,m}$.
Theorem 2.10 [KL05]: A finitely generated subgroup \( \Gamma \) of \( \mathcal{M}_{g,m} \) is convex cocompact if and only if the group \( \Gamma \) acts cocompactly on the weak convex hull in \( \mathcal{T}_{g,m} \) of its limit set \( \Lambda \subset \mathcal{PML} \).

We refer to [KL05] for additional characterizations of convex cocompact subgroups of \( \mathcal{M}_{g,m} \) which correspond to the various characterizations of convex cocompact Kleinian groups.

3. Hyperbolic \( \mathbb{R} \)-bundles over proper hyperbolic spaces

In [FM02], Farb and Mosher introduce metric fibrations as a generalization of Riemannian submersions between complete Riemannian manifolds. The following definition is adapted to our needs from their paper.

Definition 3.1: Let \((X,d)\) be a proper geodesic metric space. A metric fibration over \(X\) with fibre a topological space \(F\) is a geodesic metric space \((Y = X \times F, d)\) with the following properties.

(1) For all \(x, x' \in X\) and every \(y \in F\) we have \(d((x, y), (x', y)) = d(x, x') = d(\{x\} \times F, \{x'\} \times F)\).

(2) For each \(x \in X\), the metric on \(Y\) induces a complete geodesic metric on \(\{x\} \times F\) which defines the given topology on \(\{x\} \times F \sim F\).

The metric fibration is called bounded if there is a number \(n > 0\) such that for all \(x, x' \in X\), the natural map \(\{x\} \times F \to \{x'\} \times F\) is bilipschitz with Lipschitz constant bounded from above by \(e^{nd(x,x')}\).

Recall that a geodesic metric space \((X, d)\) is called \(\delta\)-hyperbolic for some \(\delta \geq 0\) if the \(\delta\)-thin triangle condition holds for \(X\): For every geodesic triangle in \(X\) with sides \(a, b, c\), the side \(a\) is contained in the \(\delta\)-neighborhood of \(b \cup c\). In this section we consider a metric fibration \(Y = X \times T \to X\) over a \(\delta\)-hyperbolic geodesic metric space \((X, d)\) with fibre a simplicial tree \(T\) of bounded valence. Our goal is to give a necessary and sufficient condition for the space \(Y\) to be hyperbolic.

We begin with analyzing the case when \(T\) is a closed subset of the real line \(\mathbb{R}\). We use an idea of Bestvina and Feighn who introduced the following "rectangle flare" condition. Let \((X,d)\) be a geodesic metric space and let \(r : X \to (0, \infty)\) be a positive function. Given \(\kappa > 1, n \in \mathbb{Z}_+\), the \(\kappa, n\)-flaring property for \(r\) with threshold \(A \geq 0\) says that if \(J \subset \mathbb{R}\) is a closed connected subset, if \(t - n, t, t + n \in J\) and if \(\gamma : J \to X\) is a geodesic so that \(r(\gamma(t)) \geq A\) then \(\max\{r(\gamma(t-n)), r(\gamma(t+n))\} \geq \kappa r(t)\). We say that \(r\) satisfies the bounded \(\kappa, n\)-flaring property with threshold \(A\) if in addition the growth of \(r\) is uniformly exponentially bounded with exponent \(n\), i.e. if \(r(y) \leq e^{d(x,y)\alpha} r(x) + nd(x,y)\) for all \(x, y \in X\). Note that this implies in particular that the function \(r\) is continuous.

For a constant \(c > 0\) define a subset \(B\) of \(X\) to be \(c\)-quasi-convex if every geodesic in \(X\) connecting two points in \(B\) is contained in the \(c\)-neighborhood of \(B\). We have.
Lemma 3.2: Let $X$ be a proper geodesic metric space and let $r : X \to (0, \infty)$ be a function which satisfies the bounded $\kappa, n$-flaring property with threshold $A > 0$. Let $\mu = \inf_{x \in X} r(x)$. There is a constant $D = D(\kappa, n) > 0$ only depending on $\kappa, n$ with the following properties.

1. If $\mu \geq A$ then the function $r$ assumes a minimum on $X$. The diameter of the set $\{ x \in X \mid r(x) = \mu \}$ is bounded from above by $D$.
2. If $\mu < A$ then the set $\{ x \mid r(x) \leq A \}$ is $D$-quasi-convex.

Proof: Let $(X, d)$ be a proper geodesic metric space and let $r : X \to (0, \infty)$ be a function which satisfies the bounded $\kappa, n$-flaring property with threshold $A$. Assume that $\mu = \inf_{x \in X} r(x) \geq A$. We have to show that $r$ assumes a minimum on $X$. Namely, let $x, y$ be two points in $X$ whose distance $\gamma$ is at least $2n$ and such that $r(x) < \kappa \mu, r(y) < \kappa \mu$. Let $\gamma : [0, \chi] \to X$ be a geodesic connecting $\gamma(0) = x$ to $\gamma(\chi) = y$ and let $\ell \geq 2$ be such that $\chi \in [\ell n, (\ell + 1)n)$. By the flaring property for $r$ and the fact that $r(\gamma(n)) \geq A$ we have $r(\gamma(2n)) \geq \kappa r(\gamma(n)) \geq \kappa \mu$ and inductively we conclude that $r(\gamma(\ell n)) \geq \kappa^{\ell - 1} \mu$. On the other hand, the growth of $r$ is uniformly exponentially bounded and therefore $\kappa \mu > r(\gamma(\ell n)) \geq e^{-n^2 r(\gamma(\ell n)) - n^2} \geq e^{-n^2 \kappa^{\ell - 1} \mu - n^2}$. This implies that the distance between $x$ and $y$ is bounded from above by a constant $D > 0$ only depending on $\kappa, n$. Since $X$ is proper and $r$ is continuous, we conclude that the function $r$ assumes a minimum, and the diameter of the set of points at which such a minimum is achieved is at most $D$.

Now assume that $\mu < A$ and let $E = \{ z \mid r(z) \leq A \}$. We have to show that $E$ is $D'$-quasi-convex for a constant $D' > 0$ only depending on $\kappa, n$. For this let $x, y \in E$ and let $\gamma : [0, \chi] \to X$ be a geodesic arc connecting $x$ to $y$. Let $\ell \geq 0$ be such that the length $\chi$ of $\gamma$ is contained in the interval $[\ell n, (\ell + 1)n)$. If $\ell \leq 1$ then there is nothing to show, so assume otherwise. If $\gamma(n) \not\in E$ then we have $r(\gamma(n)) > A$ and it follows as above from the flaring property that $r(y) \geq e^{-n^2 \kappa^{\ell - 1} A - n^2}$. Hence the distance $\chi$ between $x$ and $y$ is bounded from above by a universal constant. Otherwise we have $\gamma(n) \in E$ and we can apply the same consideration to the points $\gamma(n), y$. Inductively we conclude that the set $E$ is $D'$-quasi-convex for a constant $D' > 0$ only depending on $\kappa, n$. \hfill \Box

In the sequel, we will use the following criterion for hyperbolicity of a geodesic metric space (Proposition 3.5 in [H05]).

Lemma 3.3: Let $(Y, d)$ be a geodesic metric space. Assume that there is a number $D > 0$ and for every pair of points $x, y \in Y$ there is a path $c(x, y) : [0, 1] \to Y$ connecting $c(x, y)(0) = x$ to $c(x, y)(1) = y$ with the following properties.

1. If $d(x, y) \leq 1$ then the diameter of the set $c(x, y)[0, 1]$ is at most $D$.
2. For $x, y \in X$ and $0 \leq s \leq t \leq 1$, the Hausdorff distance between $c(x, y)[s, t]$ and $c(c(x, y)(s), c(x, y)(t))[0, 1]$ is at most $D$.
3. For any triple $(x, y, z)$ of points in $X$, the arc $c(x, y)[0, 1]$ is contained in the $D$-neighborhood of $c(x, z)[0, 1] \cup c(z, y)[0, 1]$. 
Then the space \((Y,d)\) is \(\delta\)-hyperbolic for a constant \(\delta > 0\) only depending on \(D\). Moreover, for all \(x,y \in Y\) the Hausdorff distance between \(c(x,y)\) and a geodesic connecting \(x\) to \(y\) is at most \(\delta\).

Now consider a metric fibration whose fibre \(J\) either is the closed interval \([0,1]\) or the half-line \([0,\infty)\). By assumption, for every compact interval \([s,t] \subset J\) and every \(x \in X\) the arc \(\{x\} \times [s,t]\) is rectifiable. As a consequence, we can define a function on \(X\) by associating to \(x \in X\) the length of the arc \(\{x\} \times [s,t]\); we call such a function a \textit{vertical distance function}. The next lemma is the main technical result of this section.

**Lemma 3.4:** Let \((X,d)\) be a proper \(\delta\)-hyperbolic geodesic metric space and let \(Y = X \times J \to X\) be a bounded metric fibration. Assume that the vertical distance functions satisfy the \(\kappa, n\)-flaring property with flaring threshold \(A\) for some \(\kappa > 1, n > 0, A > 0\). Assume moreover that the infimum of every vertical distance function is not bigger than \(A\). Then \(Y\) is \(\delta_0\)-hyperbolic for a number \(\delta_0 > 0\) only depending on \(\delta, \kappa, n, A\).

**Proof:** Let \((X,d)\) be a proper \(\delta\)-hyperbolic geodesic metric space and let \(Y = X \times J \to X\) be a bounded metric fibration with fibre \(J = [0,1]\) or \(J = [0,\infty)\) such that the vertical distance functions satisfy the \(\kappa, n\)-flaring property with threshold \(A > 0\) for some \(\kappa > 1, n > 0, A > 0\). By the definition of a bounded metric fibration, the vertical distance functions satisfy in fact the \textit{bounded} \(\kappa, n\)-flaring property with threshold \(A\). Denote the distance on \(Y\) again by \(d\). For \(t \in J\) let \(\ell_t : X \to [0,\infty)\) be the function which associates to a point \(x \in X\) the length of the vertical path \(\{x\} \times [0,t]\). By assumption, the function \(\ell_t\) is continuous. Write \(\mu(t) = \inf_{x \in X} \ell_t(x)\); the function \(t \to \mu(t)\) is continuous and increasing. We assume that \(\mu\) is bounded from above by \(A\). Our goal is to construct for any two points \(x,y \in Y\) a curve \(c(x,y)\) connecting \(x\) to \(y\) so that the resulting curve system satisfies the properties 1-3 in Lemma 3.3. For this we proceed in four steps.

**Step 1:**

In a first step, we construct for every \(y = (x,t) \in Y\) and every \(s \leq t\) a curve \(\eta_s(x,t) : [0,1] \to Y\) connecting \((x,t)\) to \(X \times \{s\}\). For this let \(\overline{X}\) be the union of \(X\) with its Gromov boundary \(\partial X\). Since \(X\) is proper, the space \(\overline{X}\) is compact. For \(s \geq 0\) define a set \(C_s \subset \overline{X}\) as follows. If \(J = [0,1]\) then define \(C_s = \{x \in X \mid (\ell_1 - \ell_s)(x) \leq A\}\). By Lemma 3.2, the set \(C_s\) is \(D_1\)-quasi-convex for a universal constant \(D_1 > 0\). If \(J = [0,\infty)\) then for \(t \geq s\) write \(Q_{t,s} = \{\ell_t - \ell_s \leq A\}\). By Lemma 3.2 the sets \(Q_{t,s}\) are \(D_1\)-quasi-convex for our constant \(D_1 > 0\). If we denote by \(\overline{Q}_{t,s}\) the closure of \(Q_{t,s}\) in \(\overline{X}\) then the sets \(\overline{Q}_{t,s}\) are compact and non-empty and we have \(\overline{Q}_{t,s} \subset \overline{Q}_{u,s}\) for \(t \leq u\). Thus \(C_s = \cap_{t \geq s} \overline{Q}_{t,s} \neq \emptyset\), moreover \(C_s\) is \(D_1\)-quasi-convex. This means that either \(C_s\) consists of a single point \(\zeta \in \partial X\) or \(C_s \cap X\) is non-empty and \(D_1\)-quasi-convex, with closure \(C_s\) in \(\overline{X}\).

For \((x,t) \in Y\) and \(s \leq t\) define now a curve \(\eta_s(x,t) : [0,1] \to Y\) connecting \((x,t)\) to \(\eta_s(x,t)(0) \in X \times \{s\}\) as follows. First, if \(t = s\) then let \(\eta_s(x,t)(\tau) = (x,t)\) for all \(\tau \in [0,1]\). If \(y = (x,t)\) for some \(t > s\) then choose a
minimal geodesic $\gamma_{x,s} : [0, \sigma] \to X$ connecting $\gamma_{x,s}(0) = x$ to the set $C_x$; if $C_x$ is a single point $\zeta \in \partial X$ then $\sigma = \infty$ and we require that $\gamma_{x,s}$ converges to $\zeta$. We assume that the choice of $\gamma_{x,s}$ only depends on $x, s$ but not on $t$. Since $\ell_t < \ell_{t'}$, for $t < t'$, there is a smallest number $\nu_{x,t,s} \geq 0$ so that $(\ell_t - \ell_s)(\gamma_{x,s}(\nu_{x,t,s})) \leq A$. Let $\eta_s(x, t)$ be a reparametrization on the interval $[0, 1]$ of the horizontal arc $\gamma_{x,s}[0, \nu_{x,t,s}] \times \{t\}$ with a vertical arc of length at most $A$ connecting $(\gamma_{x,s}(\nu_{x,t,s}), t)$ to $(\gamma_{x,s}(v_{x,t,s}), s) \in X \times \{s\}$.

**Step 2:**

In a second step, we show that for every $R > 0$, $s \in J$ and all $y, z \in X \times ([s, \infty) \cap J)$ with $d(y, z) \leq R$ the Hausdorff distance between $\eta_s(y)$ and $\eta_s(z)$ is bounded from above by a number $\tau(R) > 0$ only depending on $R$ but not on $s, y, z$.

We first consider the case that the points $y, z$ are contained in $X \times \{t\}$ for some fixed $t \geq s$. Thus let $R > 0$, let $t \geq 0$, let $x, u \in X$ with $d(x, u) \leq R$, let $y = (x, t), z = (u, t) \in Y$ and let $s \in [0, t]$. By hyperbolicity of $X$, the Hausdorff distance between the two geodesics $\gamma_{x,s}, \gamma_{u,s}$ of minimal length connecting the points $x, u$ of distance at most $R$ to the $D_1$-quasi-convex subset $C_x$ of $X$ is bounded from above by a universal constant $\tau_1(R) > 0$ only depending on $R$ but not on $x, u$.

There are smallest numbers $\nu_{x,t,s} \geq 0, \nu_{u,t,s} \geq 0$ such that $(\ell_t - \ell_s)(\gamma_{x,s}(\nu_{x,t,s})) \leq A (v = x, u)$. By the definition of the curves $\eta_s(y)$ it is now enough to show that the distance between $\gamma_{x,s}(\nu_{x,t,s})$ and $\gamma_{u,s}(\nu_{u,t,s})$ is bounded from above by a constant which only depends on $R$. Note that for $s = t$ we have $\nu_{x,t,s} = 0 = \nu_{u,t,s}$ and hence there is nothing to show, so assume that $s < t$.

Since by assumption the growth of the vertical distance functions is uniformly exponentially bounded, there is a universal number $\beta > 0$ only depending on $\tau_1(R)$ such that for any two points $v, w \in X$ with $d(v, w) \leq \tau_1(R)$ we have $(\ell_t - \ell_s)(v) \leq \beta(\ell_t - \ell_s)(w)$. By the definition of $\nu_{x,t,s}$ and the flaring property, this means that there is a number $\xi > 0$ only depending on $R$ such that $(\ell_t - \ell_s)(w) > A$ whenever $w \in X$ is such that $d(v, \gamma_{x,s}(\sigma)) \leq \tau_1(R)$ for some $\sigma \in [0, \nu_{x,t,s} - \xi]$ (compare the argument in the proof of Lemma 3.2). As a consequence, if $a \geq 0$ is such that $d(\gamma_{x,s}(a), \gamma_{u,s}(\nu_{u,t,s})) \leq \tau_1(R)$ then $a \geq \nu_{x,t,s} - \xi$. Since $\gamma_{x,s}, \gamma_{u,s}$ are geodesics of Hausdorff distance at most $\tau_1(R)$ we conclude that $d(\gamma_{u,s}(0), \gamma_{u,s}(\nu_{u,t,s})) \geq d(\gamma_{x,s}(0), \gamma_{x,s}(a)) - R - \tau_1(R) = a - R - \tau_1(R) \geq \nu_{x,t,s} - \xi - R - \tau_1(R)$. Exchanging the role of $x$ and $u$ then shows that $|\nu_{x,t,s} - \nu_{u,t,s}| \leq \xi + R + \tau_1(R)$.

Now consider nearby points $y, z$ contained in the same fibre of our metric fibration. Thus let $x \in X$ and let $t \geq 0, b > 0$ be such that the length of the vertical arc $\{x\} \times [t, t + b]$ is at most $A$. Write $y = (x, t), z = (x, t + b)$ and let $s \leq t$; we claim that the Hausdorff distance between $\eta_s(y)$ and $\eta_s(z)$ is bounded from above by a universal constant.
Namely, let again $\gamma_{x,s}$ be the minimal geodesic connecting $x$ to $C_s$ as in the definition of the arc $\eta_s(x,t)$. There is a minimal number $\sigma_0 = \nu_{x,t,s} \geq 0$ such that $(\ell_{t+b} - \ell_s)(\gamma_{x,s}(\sigma_0)) \leq A$ and hence $\ell_{t+b}(\gamma_{x,s}(\sigma_0)) - \ell_s(\gamma_{x,s}(\sigma_0)) \leq A$ for $\sigma = 0, \sigma_0$.

By the bounded $\kappa, \nu$-flaring property for vertical distances, this implies that there is a universal number $A' \geq A$ such that $(\ell_{t+b} - \ell_s)(\gamma_{x,s}(\sigma)) \leq A'$ for all $\sigma \in [0, \sigma_0]$. Let $\sigma_0 = \nu_{x,t,s} \leq \sigma_0$ be the minimal number such that $(\ell_t - \ell_s)(\gamma_{x,s}(\sigma_0)) \leq A$. Then $A \leq (\ell_{t+b} - \ell_s)(\gamma_{x,s}(\sigma)) \leq 2A'$ for every $\sigma \in [\sigma_0, \sigma_0]$ and consequently an application of the flaring property as in the proof of Lemma 3.2 for the function $\ell_{t+b} - \ell_s$ yields that $\sigma_0 \leq \sigma_0 + \xi$ for a universal number $\xi > 0$. It follows that the Hausdorff distance between $\eta_s(y)$ and $\eta_s(z)$ is bounded from above by a universal constant.

By our assumption that the growth of the vertical distance functions is uniformly exponentially bounded, for every $R > 0$ there is a number $\nu(R) > 0$ such that for every $y = (x,t) \in Y$ the $R$-ball about $y$ is contained in the set $\{z = (u,s) \in Y \mid d(x,u) \leq \nu(R), \|\ell_t - \ell_s\| \leq \nu(R)\}$. Together we conclude that for every $R > 0$ we can find a number $\tau(R) > 0$ with the following property. Let $y = (x,t), y' = (x',t') \in Y$ with $d(y,y') \leq R$, then for every $s \leq \min\{t,t'\}$ the Hausdorff distance between $\eta_s(y)$ and $\eta_s(z)$ is bounded from above by $\tau(R)$.

Step 3:

Define a system $c(y,z)$ of arcs connecting an arbitrary pair of points $y, z \in Y$ as follows. If $y = (x,t), z = (u,s) \in Y$ with $0 \leq s \leq t$ then define $c(y,z)$ to be a reparametrization on $[0,1]$ of the composition of the arc $\eta_s(y)$ with a geodesic in $X \times \{s\} \sim X$ connecting $\eta_s(y)(1)$ to $z$. Define also $c(z,y)$ to be the inverse of $c(y,z)$.

In our third step we show that for every $R > 0$ and all $y, y' \in Y$ with $d(y,y') \leq R$, all $z \in Y$ the Hausdorff distance between $c(y,z), c(y',z)$ is bounded from above by a constant $\chi(R) > 0$ only depending on $R$. For this let $R > 0$ and let $y, y', z \in Y$ with $d(y,y') \leq R$. We distinguish 3 cases.

Case 1: $z = (u,s), y = (x,t), y' = (x',t')$ with $0 \leq s \leq t \leq t'$.

By the definition of the curves $c(y,z)$ and Step 2, the curves $c(y,z), c(y',z)$ are composed of the arcs $\eta_s(y), \eta_s(y')$ of Hausdorff distance at most $\tau(R)$ and geodesic arcs in $X \times \{s\} \sim X$ connecting the points $\eta_s(y)(1), \eta_s(y')(1)$ of distance at most $\tau(R)$ to $z$. By $\delta$-hyperbolicity of $X \times \{s\} \sim X$, the Hausdorff distance between $c(y,z)$ and $c(y',z)$ is bounded from above by a constant $\chi_1(R) > 0$ only depending on $R$.

Case 2: $z = (u,s), y = (x,t), y' = (x',t')$ with $0 \leq t \leq s \leq t'$.

Since the distance between $y$ and $y'$ is at most $R$ and $Y$ is a geodesic metric space, there is a point $y'' = (x'',s) \in X \times \{s\}$ whose distance to both $y, y'$ is at most $R$. By Case 1 above, the Hausdorff distance between $c(y',z)$ and $c(y'',z)$ is at most $\chi_1(R)$. Thus we may assume without loss of generality that $t' = s$; then $c(y',z) = c(z, y)$ is the lift to $X \times \{s\}$ of a geodesic in $X$ connecting $u$ to $x'$. Since $d(y,y') \leq R$ and $y' = (x',s), y = (x,t)$, we have $d(x,x') \leq R$ and hence by hyperbolicity of $X$, the Hausdorff distance between a geodesic connecting $u$ to $x$
and a geodesic connecting \( u \) to \( x' \) is bounded by a uniform constant \( \chi_2(R) > 0 \). Thus the Hausdorff distance between \( c(y', z) \) and \( c((x, s), z) \) is at most \( \chi_2(R) \) and we may assume without loss of generality that \( x = x' \).

By the flaring property for vertical distances, the point \( x \) is contained in a uniformly bounded neighborhood of the set \( E = \{ \ell_s - \ell_t \leq A \} \). Since \( E \) is \( D_1 \)-quasi-convex, by hyperbolicity a geodesic connecting \( u \) to \( x \) is contained in a uniform neighborhood of the composition of a minimal geodesic \( \xi \) connecting \( u \) to \( E \) and a geodesic arc connecting the endpoint of \( \xi \) to \( x \). By construction, the curve \( c(y, z) = c(z, y') \) is composed of the lift to \( X \times \{ s \} \) of a minimal geodesic \( \xi : [0, \tau] \rightarrow X \) connecting \( u \) to \( E \), a vertical arc of length at most \( A \) connecting \( (\xi(\tau), s) \) to \( (\xi(\tau), t) \) and the lift to \( X \times \{ t \} \) of a geodesic \( \xi \) in \( X \) connecting \( \xi(\tau) \) to \( x \) which is contained in a uniformly bounded neighborhood of \( E \). It follows that the Hausdorff distance between \( c(y, z) \) and \( c((y, z), \xi) \) is bounded. Therefore the Hausdorff distance between \( c(y, z), c(y', z) \) is bounded from above by a constant \( \chi_3(R) > 0 \) only depending on \( R \).

Case 3: \( z = (u, s), y = (x, t), y' = (x', t') \) with \( 0 \leq t \leq t' \leq s \).

We claim that \( \eta(\tau) \) contains a subarc whose Hausdorff distance to \( \eta(\tau) \) is uniformly bounded. Namely, the sets \( C_t, C_{t'} \subset X \) are \( D_1 \)-quasi-convex and \( C_t \subset C_{t'} \). By hyperbolicity of \( X \), if \( u \in X - C_{t'} \) then a minimal geodesic \( \xi \) in \( X \) connecting \( u \) to \( C_t \) is contained in a uniformly bounded neighborhood of the composition of a minimal geodesic \( \xi_1 : [0, a] \rightarrow X \) connecting \( u \) to \( C_{t'} \) and a minimal geodesic \( \xi_2 \) connecting \( \xi_1(a) = \xi_2(0) \) to \( C_t \). From this and the definition of our arcs \( \eta_t \), the claim is immediate.

Denote by \( z'' = \eta(\tau)(\sigma) \ (\sigma \in [0, 1]) \) the endpoint of this subarc and write \( z' = \eta(\tau)(1) \in X \times \{ t' \} \). The curve \( c(y', z) \) is composed of the arcs \( \eta(\tau)(z') \) and \( c(y', z') \), and the curve \( c(y, z) \) is composed of the arcs \( \eta(\tau)(z) \) and \( c(y, z') \). Thus up to a constant only depending on \( R \), the Hausdorff distance between \( c(y, z) \) and \( c(y', z) \) is bounded from above by the Hausdorff distance between \( c(y, z') \) and \( c(y', z') \). Now the distance between \( z' \) and \( z'' \) is uniformly bounded and hence by Case 1 above, the Hausdorff distance between \( c(y, z') \) and \( c(y, z'') \) is bounded by a constant only depending on \( R \). In other words, for our estimate we may replace \( z \) by \( z' \), i.e. we may assume without loss of generality that \( s = t' \). However, this case is contained in Case 2 above.

Together we established an upper bound \( \chi(R) > 0 \) for the Hausdorff distance between \( c(y, z) \) and \( c(y', z) \) whenever \( d(y, y') \leq R \).

Step 4:

In a final step, we show that our system of curves satisfies the properties 1)-3) in Lemma 3.3. Namely, for \( y = z \) the curve \( c(y, z) \) is constant, and hence if \( d(y, z) \leq 1 \) then the diameter of \( c(y, z) \) is at most \( \chi(R) \) where \( \chi(R) > 0 \) is as in Step 3. This means that property 1 is valid with \( D = \chi(1) > 0 \).
Similarly, let \( y, z \in Y \) and let \( 0 \leq s \leq t \leq 1 \). Then either the restriction of the curve \( c(y, z) \) to \( [s, t] \) is obtained from our above procedure, i.e. from the same construction used for the curve \( c(c(y, z)(s), c(y, z)(t)) \), or one of the points \( c(y, z)(s), c(y, z)(t) \) is contained in the vertical subarc of \( c(y, z) \). In the first case it is immediate from Step 3 above that the Hausdorff distance between \( c(y, z)[s, t] \) and \( c(c(y, z)(s), c(y, z)(t)) \) is uniformly bounded. In the second case, if say the point \( c(y, z)(s) \) is contained in the vertical subarc of \( c(y, z) \) then there is some \( s' \geq s \) such that \( c(y, z)[s', t] \) is obtained from the above procedure and that the Hausdorff distance between \( c(y, z)[s, t] \) and \( c(y, z)[s', t] \) is bounded from above by a universal constant. By Step 3, the Hausdorff distance between \( c(c(y, z)(s), c(y, z)(t)) \) and \( c(y, z)[s', t] \) is bounded from above by a universal constant as well. As a consequence, there is a number \( \nu > 0 \) such that property 2 is valid with \( D = \nu \).

We are left with showing that the \( \delta_0 \)-thin triangle condition for a universal number \( \delta_0 > 0 \) also holds. For this let \( y_1, y_2, y_3 \) be any 3 points in \( Y \). Assume that \( y_i = (x_i, s_i) \) with \( 0 \leq s_1 \leq s_2 \leq s_3 \). By construction and Step 3 above, the curves \( c(y_1, y_3), c(y_2, y_3) \) both contain a subarc whose Hausdorff distance to \( \eta_{s_2}(y_3) \) is uniformly bounded. By our above estimate of Hausdorff distances, this means that for the purpose of establishing the thin triangle condition we may replace \( y_3 \) by \( \eta_{s_2}(y_3)(1) \), i.e. we may assume that in fact \( s_2 = s_3 = s \). Then the arc \( c(y_2, y_3) \) is the lift to \( X \times \{s\} \) of a geodesic \( \gamma \) in \( X \) connecting \( x_2 \) to \( x_3 \).

Let \( E = \{ u \in X \mid (\ell_s - \ell_{s_1})(u) \leq A \} \). Recall that \( E \) is \( D_1 \)-quasi-convex. Let \( \zeta_i : [0, \sigma_i] \rightarrow X \) \( (i = 2, 3) \) be a minimal geodesic connecting \( x_2, x_3 \) to \( E \). Then \( \gamma \) is contained in a uniformly bounded neighborhood of the union of \( \zeta_2[0, \sigma_2] \cup \zeta_3[0, \sigma_3] \) with a geodesic arc connecting \( \zeta_2(\sigma_2) \) to \( \zeta_3(\sigma_3) \). Moreover, by our above considerations the curves \( c(y_2, y_1) \) and \( c(y_3, y_1) \) contain each a subarc \( \nu_2, \nu_3 \subset X \times \{s\} \) whose Hausdorff distance to the arcs \( \zeta_2 \times \{s\}, \zeta_3 \times \{s\} \) is uniformly bounded. As a consequence, for the purpose of the thin triangle condition we may as well assume that the points \( y_2, y_3 \) are contained in a uniformly bounded neighborhood of \( E \). However, in this case the thin triangle condition is immediate from the definition of the curves \( c(x, y) \) and hyperbolicity of \( X \). As a consequence, our system of curves \( c(x, y) \) satisfies the properties 1)-3) in Lemma 3.3 for a number \( D > 0 \) only depending on \( \delta, \kappa, n, A \) and hence the space \( Y \) is \( \delta' \)-hyperbolic for a constant \( \delta' > 0 \) only depending on \( \delta, \kappa, n, A \).

Let \( X \) be a proper \( \delta \)-hyperbolic geodesic metric space. Recall that a closed subset \( E \) of \( X \) is strictly convex if every geodesic connecting two points in \( E \) is contained in \( E \). The following lemma shows that under suitable assumptions, hyperbolicity is preserved under glueing along strictly convex subsets. For its formulation, for a number \( R > 0 \) we call two closed strictly convex subsets \( D, E \) of a \( X \)-\( R \)-separated if \( D, E \) are disjoint and if moreover the following holds. Let \( \gamma : [0, a] \rightarrow X \) be a minimal geodesic connecting \( D \) to \( E \); then \( \gamma[0, a] \) is contained in the \( R \)-neighborhood of every geodesic connecting \( D \) to \( E \). For example, two non-intersecting geodesics in the hyperbolic plane are \( R \)-separated for a constant \( R > 0 \) which tends to infinity as the distance between the geodesics tends to zero. The two boundary geodesics of a flat strip in \( \mathbb{R}^2 \) are not \( R \)-separated for any \( R > 0 \). Note also that by the explicit construction of the curves \( c(x, y) \) in the proof of Lemma 3.4 the following holds. If \( Y = X \times [0, 1] \rightarrow X \) is a bounded metric fibration over a hyperbolic geodesic metric
space such that the vertical distance functions satisfy the $\kappa, n$-flaring property with threshold $A$ for some $\kappa > 1$, $n > 0$, $A > 0$ and if the infimum of the vertical length of the fibres equals $A$ then $Y$ is hyperbolic and the subsets $X \times \{0\}, X \times \{1\}$ are strictly convex and $R$-separated for a number $R > 0$ only depending on $\kappa, n, A$.

**Lemma 3.5:** Let $\delta > 0, R > 0$, let $I \subset \mathbb{Z}$ be any subset and let $X$ be a geodesic metric space with the following properties.

a) $X = \bigcup_{i \in I} X_i$ where for each $i \in I$, $X_i$ is a proper $\delta$-hyperbolic geodesic metric space.

b) For each $i \in I$ the intersection $X_i \cap X_{i+1}$ is a strictly convex closed subset of both $X_i, X_{i+1}$, and $X_i \cap X_j = \emptyset$ for $|i - j| \geq 2$.

c) For each $i \in I$, the sets $X_i \cap X_{i-1}$ and $X_i \cap X_{i+1}$ are $R$-separated in $X_i$.

Then $X$ is $\delta'$-hyperbolic for a constant $\delta' > 0$ only depending on $\delta, R$.

**Proof:** Let $X, I, X_i$ be as in the lemma. We may assume without loss of generality that $I = \mathbb{Z}$. Write $E_i = X_i \cap X_{i+1}$; by our assumption, $E_i$ is a strictly convex subset of the proper $\delta$-hyperbolic spaces $X_i, X_{i+1}$; moreover, the subsets $E_{i-1}, E_i$ of $X_i$ are $R$-separated for a constant $R > 0$ not depending on $i$. Thus after possibly enlarging $R$ the following properties are satisfied.

i) Every point $x \in X_i$ can be connected to $E_i$ by a geodesic $\zeta^+_i : [0, 1] \to X_i$ of minimal length and to $E_{i-1}$ by a geodesic $\zeta^-_i : [0, 1] \to X_i$ of minimal length. If the distance between $x, y$ is at most 1 then the Hausdorff distance between $\zeta^+_i$ and $\zeta^-_i$ is at most $R$.

ii) Let $x, y \in X_i$ and let $\gamma$ be a geodesic connecting $x$ to $y$. Then $\gamma$ is contained in the $R$-neighborhood of the piecewise geodesic $\tilde{\gamma}^+$ which is composed of the arc $\zeta^+_i$, a geodesic in $E_i$ connecting $\zeta^+_i(1)$ to $\zeta^+_y(1)$ and the inverse of $\zeta^+_y$. The geodesic $\gamma$ is also contained in the $R$-neighborhood of a piecewise geodesic $\tilde{\gamma}^-$ which is constructed in the same way using the geodesic arcs $\zeta^-_i, \zeta^-_y$ and a geodesic in $E_{i-1}$.

iii) Let $\gamma_i : [0, 1] \to X_i$ be a minimal geodesic connecting $E_{i-1}$ to $E_i$; then for every $x \in E_{i-1}$ the Hausdorff distance between a minimal geodesic connecting $x$ to $E_i$ and the composition with $\gamma_i$ of a geodesic in $E_{i-1}$ connecting $x$ to $\gamma_i(0)$ is not bigger than $R$.

We use once more the criterion for hyperbolicity from Lemma 3.3. Namely, we define in three steps for any pair of points $x, y \in X$ a curve $c(x, y)$ connecting $x$ to $y$ as follows.

**Step 1:** If there is some $i \in \mathbb{Z}$ such that $x, y \in X_i$ then define $c(x, y)$ to be a geodesic in $X_i$ connecting $x$ to $y$.

**Step 2:** If there is some $i \in \mathbb{Z}$ such that $x \in X_i - E_i, y \in X_{i+1} - E_i$ then define $c(x, y)$ to be the piecewise geodesic which is composed from the geodesic $\zeta^+_i$ connecting $x$ to $E_i$, a geodesic arc in $E_i$ connecting $\zeta^+_i(1)$ to $\zeta^-_y(1)$ and the inverse of the geodesic $\zeta^-_y$.
Step 3: If \( x \in X_i - E_i \), \( y \in X_j - E_{j-1} \) for some \( j \geq i + 1 \) then define inductively \( c(x, y) \) to be the piecewise geodesic which consists of the geodesic segment \( \zeta_{s}^+ \), a geodesic in \( E_i \) connecting \( \zeta_{s}^+ (1) \) to \( \gamma_{i+1} (0) \) and the arc \( c(\gamma_{i+1} (0), y) \).

Assume that the curves \( c(x, y) \) are all parametrized on the unit interval \([0,1]\). We claim that there is a number \( D > 0 \) only depending on \( \delta, \kappa, n, A \) such that the curves \( c(x, y) \) satisfy the three conditions in Lemma 3.3.

The first property is immediate from the definition of the curves \( c(x, y) \). To show that the second condition is valid as well, let \( x, y \in X \), let \( 0 \leq s \leq t \leq 1 \) and let \( x' = c(x, y)(s), y' = c(x, y)(t) \) be points on the curve \( c(x, y) \). We have to show that the Hausdorff distance between \( c(x, y)[s, t] \) and \( c(x', y')[0,1] \) is bounded from above by a constant \( D_1 > 0 \) only depending on \( \delta, \kappa, n, A \). For this we distinguish three cases.

First, if \( x', y' \in X_i \) for some \( i \in \mathbb{Z} \), then either \( c(x, y)[s, t] \) is a geodesic in \( X_i \) connecting \( x' \) to \( y' \) or \( x' \in E_{i-1} \) or \( y' \in E_i \) and by properties ii) and iii) above for \( X_i \), the Hausdorff distance between the arc \( c(x, y)[s, t] \) and the geodesic \( c(x', y') \) connecting \( x' \) to \( y' \) is bounded from above by a universal constant \( \chi_1 > 0 \).

Next assume that \( x' \in X_i - E_i \) for some \( i \in \mathbb{Z} \) and that \( y' \in X_{i+1} - E_i \). Let \( \zeta_{s}^+ \) be a geodesic of minimal length connecting \( x' \) to \( E_i \). By the definition of the curve \( c(x, y) \) and hyperbolicity of \( X_i \), there is a number \( s' > s \) such that \( c(x, y)(s') \in E_i \) and that the Hausdorff distance between \( c(x, y)[s, s'] \) and the geodesic \( \zeta_{s}^+ \) is at most \( R \). Similarly, by property iii) above and the definition of the curves \( c(v, w) \) there is a number \( t' \leq t \) such that \( c(x, y)(t') \in E_i \) and that the Hausdorff distance between \( c(x, y)[t', t] \) and the geodesic \( \zeta_{s}^- \) of minimal length connecting \( y' \) to \( E_i \) is bounded from above by \( R \). On the other hand, \( c(x', y') \) is composed of the arcs \( \zeta_{s}^+, \zeta_{s}^- \) and a geodesic arc in \( E_i \) connecting \( \zeta_{s}^+ (1) \) to \( \zeta_{s}^- (1) \); moreover, \( c(x, y)[s', t'] \) is a geodesic in \( E_i \) connecting \( c(x, y)(s') \) to \( c(x, y)(t') \). Since \( E_i \) is \( \delta \)-hyperbolic for a number \( \delta > 0 \) not depending on \( i \), the Hausdorff distance between any two compact geodesic arcs in \( E_i \) is up to an additive constant bounded from above by the sum of the distances between the endpoints of the arcs. Therefore the Hausdorff distance between \( c(x, y)[s, t] \) and \( c(x', y') \) is at most \( \chi_2 \) for a constant \( \chi_2 \geq \chi_1 \) only depending on \( \delta \).

Finally, the case that \( x' \in X_i - E_i \) and \( y' \in X_j - E_{j-1} \) for some \( j \geq i + 1 \) follows immediately from the above consideration. Namely, in this case there are numbers \( s \leq s' < t' \leq t, 0 \leq \sigma < \tau \leq 1 \) such that the arcs \( c(x, y)[s', t'], c(x', y')[\sigma, \tau] \) coincide and that moreover the above consideration can be applied to the curves \( c(x', y')[0, \sigma], c(x, y)[\tau, 1] \) and \( c(x, y)[0, s'], c(x, y)[t', 1] \). Thus the second condition in the proof of Lemma 3.3 is satisfied for our system of curves with a number \( D_1 > 0 \) only depending on \( \delta, \kappa, n, A \) (note that we can choose \( D_1 = 2\chi_2 \) where \( \chi_2 > 0 \) is as above).

We are left with showing the thin triangle condition for our system of curves \( c(x, y) \), i.e. we have to find a number \( D_2 > 0 \) such that for every triple of points \( x, y, z \in Y \) the curve \( c(x, y) \) is contained in the \( D_2 \)-neighborhood of \( c(y, z) \cup c(z, x) \). Consider first the case that the points \( x, y, z \) are all contained in \( X_i \) for some \( i \in \mathbb{Z} \). Then the curves \( c(x, y), c(y, z), c(z, x) \) are geodesics in \( X_i \) connecting these three
points and hence the curve \( c(x, y) \) is contained in a uniformly bounded neighborhood of \( c(y, z) \cup c(z, x) \) by hyperbolicity of \( X_i \). Next assume that two of the points, say the points \( x, y \), are contained in \( X_i \) but that the third point \( z \) is contained in \( X_j - E_{j-1} \) for some \( j \geq i + 1 \). Then the intersections with \( \cup_{p \geq i+1} X_p - E_i \) of the curves \( c(x, z), c(y, z) \) coincide. Let \( t_x, t_y \in [0, 1] \) be such that \( c(x, z)(t_x, 1) = c(x, z)[0, 1] \cap (\cup_{p \geq i+1} X_p - E_i) \) and similarly for \( c(y, z) \); then \( v = c(x, z)(t_x) = c(y, z)(t_y) \). Together with property 2 for our curve system established above we conclude that it is enough to establish the \( D_2 \)-thin triangle condition for the curves \( c(x, v), c(y, v), c(x, y) \). However, since \( x, y, v \in X_i \) this condition holds by our above consideration. The same argument can also be applied in the case that for each \( i \), the set \( X \times [t_{i-1}, t_i] \) contains at most one of the points \( x, y, z \). From this we immediately deduce that the third condition for our curve system is valid as well for a universal constant \( D_2 > 0 \) only depending on \( \delta, R \). As a consequence of Lemma 3.3, the space \( X \) is \( \delta' \)-hyperbolic for a constant \( \delta' \) only depending on \( \delta, R \). □

Let \( T \) be a simplicial tree of bounded valence. Then for any two points in \( T \), there is a unique simple path connecting these points. For every metric fibration \( Y = X \times T \to X \) and every point \( \tau \in T \), the set \( X \times \{ \tau \} \subset Y \) is strictly convex. We use these facts together with the glueing lemma to extend Lemma 3.4 as follows.

**Corollary 3.6:** Let \( X \times T \to X \) be a bounded metric fibration with fibre a simplicial tree of bounded valence. Assume that \( X \) is \( \delta \)-hyperbolic for some \( \delta > 0 \) and that vertical distances satisfy the \( \kappa, n \)-flaring property with threshold \( A > 0 \) for some \( \kappa > 1, n > 0, A > 0 \). Then \( Y \) is \( \delta_1 \)-hyperbolic for a number \( \delta_1 > 0 \) only depending on \( \kappa, n, \delta, A \).

**Proof:** We begin with showing the corollary in the particular case that the tree \( T \) is just an arbitrary closed connected subset \( J \) of the real line \( \mathbb{R} \). Thus let \( X \) be a \( \delta \)-hyperbolic geodesic metric space, let \( J \subset \mathbb{R} \) be an arbitrary closed connected set and let \( Y = X \times J \) be a metric fibration with fibre \( J \). Assume that vertical distances satisfy the \( \kappa, n \)-flaring property with threshold \( A \) for some \( \kappa > 1, n > 0, A > 0 \) and assume without loss of generality that \( \kappa \geq 1 \).

Let \( 0 \in J \) and assume that \( 0 \) is an endpoint of \( J \) if \( J \neq \mathbb{R} \). Assume moreover that in this case the set \( J \) is contained in \([0, \infty)\). For \( t \in J \) let \( \ell^1_t : X \to [0, \infty) \) be the function which associates to a point \( x \in X \) the length of the vertical path \( \{x\} \times [0, t] \). By assumption, the function \( \ell^1_t \) is continuous. Write \( \mu^1(t) = \inf_{x \in X} \ell^1_t(x) \); the function \( t \to \mu^1(t) \) is continuous and monotonously increasing on \([0, \infty)\), monotonously decreasing on \((-\infty, 0] \). Let \( t_1 \in (0, \infty) \) be the smallest positive number with \( \mu^1(t_1) = A \); here we write \( t_1 = \infty \) if \( \mu^1(t) < A \) for all \( t > 0 \). If \( t_1 < \infty \) then define for \( t \geq t_1 \) a new function \( \ell^2_t : X \to [0, \infty) \) by assigning to \( x \in X \) the length of the arc \( \{x\} \times [t_1, t] \). Let \( \mu_2(t) = \inf_{x \in X} \ell^2_t(x) \) and let \( t_2 \in (t_1, \infty) \) be the smallest number such that \( \mu^2(t_2) = A \). Inductively we construct in this way an increasing sequence \( 0 < t_1 < t_2 < \ldots \) and functions \( \mu^1, \ell^1_t \). The sequence might be trivial, finite or infinite. If \( J = \mathbb{R} \) then define in the same way a sequence \( 0 > t_{-1} > t_{-2} > \ldots \) and functions \( \mu^1, \ell^1_t \) (\( i \leq -1 \)).
By Lemma 3.4, there is a constant $\delta_0 > 0$ such that for each $i \in \mathbb{Z}$, the convex subset $X \times [t_{i-1}, t_i]$ of $X \times J$ is $\delta_0$-hyperbolic. The sets $X \times \{t_{i-1}\}, X \times \{t_i\}$ are strictly convex in $Y$. Moreover, by the remark preceding Lemma 3.5 they are also $R$-separated for a constant $R > 0$ only depending on $\kappa, n, A$. Thus we can apply Lemma 3.5 and conclude that the metric fibration $X \times J$ is $\delta_1$-hyperbolic for a constant $\delta_1 > 0$ only depending on $\delta, \kappa, n, A$.

Now let $T$ be a simplicial tree of bounded valence. Let $X \times T \to X$ be a metric fibration over a proper $\delta$-hyperbolic geodesic metric space $X$. Assume that vertical distances satisfy the $\kappa, n$-flaring property with threshold $A > 0$ for some $\kappa > 1, n > 0, A > 0$. Our goal is to show that $Y$ is $\delta_2$-hyperbolic for a constant $\delta_2 > 0$ only depending on $\delta, \kappa, n, A$.

For this let $y_1, y_2, y_3 \in Y$ be a triple of points and let $c(y_i, y_j)$ ($i, j = 1, 2, 3$) be geodesics in $X \times T$ connecting $y_i$ to $y_j$. We have to show that $c(y_1, y_2)$ is contained in a uniformly bounded neighborhood of $c(y_2, y_3) \cup c(y_3, y_1)$. Write $y_i = (x_i, \tau_i)$ with $x_i \in X, \tau_i \in T$. For $i = 1, 2, 3$ let $J_i$ be the unique embedded segment in $T$ connecting $\tau_i$ to $\tau_{i+1}$ (indices are taken mod 3). Then the intersection $\cap_i J_i$ consists of a unique point $\tau$. By our above consideration, the subsets $X \times J_i \subset Y$ of $Y$ are strictly convex and moreover $\delta_1$-hyperbolic for a universal constant $\delta_1 > 0$; they contain $X \times \{\tau\}$ as a strictly convex subset. Let $\rho(y_i, y_{i+1})$ be a piecewise geodesic which up to orientation and parametrization is composed of a minimal geodesic $\alpha_i : [0, 1] \to X \times J_i$ connecting $y_i$ to $X \times \{\tau\}$, a minimal geodesic $\alpha_{i+1} : [0, 1] \to X \times J_{i+1}$ connecting $y_{i+1}$ to $X \times \{\tau\}$ and a geodesic in $X \times \{\tau\}$ connecting $\alpha_i(1)$ to $\alpha_{i+1}(1)$. The Hausdorff distance between the geodesic $c(y_i, y_{i+1}) \subset X \times J_i$ and the piecewise geodesic $\rho(y_i, y_{i+1})$ is bounded from above by a constant only depending on $\delta, \kappa, n, A$. Since $X \times \{\tau\}$ is $\delta$-hyperbolic, from this hyperbolicity of $Y$ is immediate. This shows the corollary. □

We summarize the results of this section as follows.

**Corollary 3.7:** Let $X$ be a proper hyperbolic geodesic metric space and let $Y = X \times T \to X$ be a bounded metric fibration with fibre a simplicial tree of bounded valence. Then $Y$ is hyperbolic if and only if vertical distances satisfy the $\kappa, n$-flaring property with threshold $A$ for some $\kappa > 1, n > 0, A > 0$.

**Proof:** Let $X$ be a proper hyperbolic geodesic metric space and let $Y = X \times T \to X$ be a bounded metric fibration with fibre a simplicial tree of bounded valence. Lemma 5.2 of [1] shows that if $Y$ is hyperbolic, then vertical distances satisfy the $\kappa, n$-flaring property with threshold $A$ for some $\kappa > 0, n > 0, A > 0$. By Corollary 3.6, this condition is also sufficient for hyperbolicity of $Y$. □

4. PROOF OF THE THEOREM

In this final section we consider a closed surface of genus $g \geq 2$. Our goal is to show that for a convex cocompact subgroup $\Gamma$ of the mapping class group $\mathcal{M}_g$ for $S$, the natural $\pi_1(S)$-extension $\Gamma_S$ of $\Gamma$ is word hyperbolic. For this choose a finite symmetric generating set $G$ for $\Gamma$ and denote by $\|\|$ the induced word norm on $\Gamma$.
and by \( CG \) the corresponding Cayley graph. Choose a point \( h \) in the Teichmüller space \( T_g \) for \( S \) which does not admit any nontrivial automorphisms (recall that the set of such points is open and dense in \( T_g \)) and define a map \( \Theta : CG \rightarrow T_g \) by mapping a vertex \( \varphi \in \Gamma \) to the point \( \varphi h \in T_g \) and by mapping an edge \( e \) of \( CG \) to the Teichmüller geodesic arc connecting the image of the endpoints of \( e \). By Lemma 2.8, the map \( \Theta \) is a quasi-isometric embedding; moreover, the set \( \Theta CG \) is invariant under the action of \( \Gamma \).

There is a natural smooth marked surface bundle \( S \rightarrow T_g \) whose fibre \( S_z \) at a point \( z \in T_g \) is just the surface \( S \) with the marking defined by \( z \). The hyperbolic structure \( z \in T_g \) defines a smooth Riemannian metric on the fibre \( S_z \), and these metrics fit together to a smooth Riemannian metric on the vertical bundle of the fibration (i.e. the tangent bundle of the fibres). In other words, the vertical foliation of \( S \) into the fibres of our fibration admits a natural smooth Riemannian metric.

The action of the mapping class group \( M_g \) on \( T_g \) lifts to a unique action on \( S \) which is determined by the requirement that for every \( \varphi \in M_g \) and every \( z \in T_g \), the restriction of the lift of \( \varphi \) to \( S_z \) is the unique isometry of \( S_z \) onto \( S_{\varphi z} \) in the isotopy class determined by \( \varphi \). In particular, the Riemannian metric on the vertical foliation is invariant under the action of \( M_g \). The restriction \( S_{\Gamma} \) of the bundle \( S \) to \( \Theta CG \) is invariant under the action of the subgroup \( \Gamma \) of \( M_g \).

We equip now the bundle \( S_{\Gamma} \) with the following geodesic metric. First, recall that for a given edge \( b \) in \( CG \) the arc \( \Theta b \) is a geodesic, and its endpoints are marked hyperbolic metrics on the surface \( S \) which are isometric with an isometry in the class determined by the element of \( G \) corresponding to \( b \). If we identify the edge \( b \) with the unit interval \([0,1]\) then for all \( s,t \in [0,1] \) there is a unique Teichmüller map of minimal quasi-conformal dilatation which maps the fibre \( S_{\Theta(s)} \) to \( S_{\Theta(t)} \), and these maps combine to a smooth fibre preserving horizontal flow on the restriction \( S_{\Theta b} \) of \( S \) to \( \Theta b \). Defining the tangent of each of these flow-lines to be orthogonal to the fibres and of the same length as its projection to \( \Theta b \) defines a smooth Riemannian metric on \( S_{\Theta b} \) so that the canonical projection \( S_{\Theta b} \rightarrow \Theta b \) is a Riemannian submersion. If two edges are incident on the same vertex, then the metrics on the fibres over this vertex coincide. Therefore, the metrics naturally induce a complete length metric \( d \) on \( S_{\Gamma} \) which for every edge \( b \) of \( CG \) restricts to the length metric of the above Riemannian structure on \( S_{\Theta b} \).

The universal cover \( \mathcal{H} \) of \( S \) is a smooth fibre bundle \( \Pi : \mathcal{H} \rightarrow T_g \) whose fibre \( \mathcal{H}_z \) at a point \( z \in T_g \) equipped with the lift of the Riemannian metric on \( S_z \) is isometric to the hyperbolic plane \( \mathbb{H}^2 \). The group \( \Gamma_S \) acts on \( \mathcal{H} \) as a group of bundle isomorphisms preserving the metric of the vertical foliation. The pre-image of every \( \Gamma \)-invariant subset of \( T_g \) is invariant under the action of \( \Gamma_S \). In particular, the set \( \mathcal{H}_{\Gamma} = \Pi^{-1}(\Theta CG) \) is \( \Gamma_S \)-invariant. The metric on \( S_{\Gamma} \) lifts to a geodesic metric \( d \) on \( \mathcal{H}_{\Gamma} \). The group \( \Gamma_S \) acts on the geodesic metric space \((\mathcal{H}_{\Gamma}, d)\) isometrically, properly and cocompactly and hence we have (see [FM02]).

**Lemma 4.1:** \( \mathcal{H}_{\Gamma} \) is quasi-isometric to \( \Gamma_S \).
By Lemma 4.1 it is therefore enough to show that the bundle $\mathcal{H}_\Gamma$ with its $\Gamma_S$-invariant geodesic metric is hyperbolic.

To show that this is indeed the case, we use the results from Section 3, applied to suitably defined line-subbundles of $\mathcal{H}_\Gamma$. For the construction of these bundles, fix a standard system $a_1, b_1, \ldots, a_g, b_g$ of generators for the fundamental group $\pi_1(S)$ of $S$. For every $z \in T_g$ there is a unique isomorphism $\rho(z)$ of $\pi_1(S)$ onto a discrete subgroup $\Upsilon(z)$ of $\text{PSL}(2, \mathbb{R})$ such that the surface $H^2/\Upsilon(z)$ is isometric to $S_z$ and that moreover the following holds (see [IT99]).

a) The conjugacy class of the representation $\rho(z)$ is determined by the marking of $S_z$.

b) In the upper half-plane model for $H^2$, the points $0, \infty$ are attracting and repelling fixed points for the action of $\rho(z)(b_g)$, and the point 1 is an attracting fixed point for the action of $\rho(z)(a_g)$.

The representation $\rho(z)$ depends smoothly on $z$, and the surface bundle $S$ over $T_g$ is the quotient of the trivial bundle $T_g \times H^2$ under the action of the group $\pi_1(S)$ defined by $\varphi(z, v) = (z, \rho(z)(\varphi)(v))$ ($\varphi \in \pi_1(S), (z, v) \in T_g \times H^2$).

For every $z \in T_g$ the fibre $\mathcal{H}_z$ of the bundle $\mathcal{H}$ admits a compactification by adding the ideal boundary $\partial \mathcal{H}_z$. Every pair of distinct points in $\partial \mathcal{H}_z$ defines uniquely a geodesic line in $\mathcal{H}_z$. Let again $h \in T_g$ be a point whose $\Gamma$-orbit is the vertex set of $\Theta G$. For every $z \in \Theta G$, the isomorphism $\rho(z) \circ \rho(h)^{-1}$ of $\Upsilon(h)$ onto $\Upsilon(z)$ induces a homeomorphism $\omega(z)$ of $\partial \mathcal{H}_h$ onto $\partial \mathcal{H}_z$. For every pair of distinct points $\xi \neq \eta \in \partial \mathcal{H}_h$ we define a line subbundle $\mathcal{L}_{\xi, \eta}^z$ of $\mathcal{H}_\Gamma$ by requiring that its fibre $\mathcal{L}_{\xi, \eta}^z$ at $z$ is the geodesic line in $\mathcal{H}_z$ whose endpoints in $\partial \mathcal{H}_z$ are the images of the points $\xi, \eta$ under the homeomorphism $\omega(z)$. We equip $\mathcal{L}_{\xi, \eta}^z$ with a complete length metric whose restriction to each fibre coincides with the restriction of the metric on $\mathcal{H}$ and which is such that the following holds. For each $z \in \Theta G$ let $\mathcal{B}_z \subset \mathcal{H}_z$ be the tubular neighborhood of radius one about the fibre $\mathcal{L}_{\xi, \eta}^z$ of $\mathcal{L}_{\xi, \eta}^z$ at $z$; then $\mathcal{B} = \bigcup_{z \in \Theta G} \mathcal{B}_z$ is a fibre bundle over $\Theta G \mathcal{L}$ which is an open subset of $\mathcal{H}_\Gamma$. The restriction to $\mathcal{B}$ of the length structure on $\mathcal{H}_\Gamma$ induces a length metric on $\mathcal{B}$; we require that the inclusion $\mathcal{L}_{\xi, \eta}^z \to \mathcal{B}$ is a $p$-quasi-isometry for some constant $p \geq 1$ independent of $\xi, \eta$. We have.

**Lemma 4.2:** There is a number $q > 0$ so that for every pair $\xi \neq \eta \in \partial \mathcal{H}_h$ the following is satisfied.

1. The inclusion $\mathcal{L}_{\xi, \eta}^z \to \mathcal{H}_\Gamma$ is a $q$-quasi-isometric embedding.
2. The bundle $\mathcal{L}_{\xi, \eta}^z$ is $q$-hyperbolic.

**Proof:** Let $\xi \neq \eta \in \partial \mathcal{H}_h$. We begin with showing that there is a number $q_0 > 0$ not depending on $\xi, \eta$ so that the inclusion $\iota : \mathcal{L}_{\xi, \eta}^z \to \mathcal{H}_\Gamma$ is a $q_0$-quasi-isometric embedding. For this we have to show that for any two points $v, w \in \mathcal{L}_{\xi, \eta}^z$ the distance in $\mathcal{L}_{\xi, \eta}^z$ between $v, w$ is not bigger than $q_0$ times the distance between $\iota(v), \iota(w)$. By definition, for the neighborhood $\mathcal{B}$ of $\iota(\mathcal{L}_{\xi, \eta}^z)$ in $\mathcal{H}_\Gamma$ which intersects each fibre $\mathcal{H}_z$ in the neighborhood of radius one about the geodesic $\iota \mathcal{L}_{\xi, \eta}^z$, we have
to find a curve connecting \( \iota(v) \) to \( \iota(w) \) in \( \mathcal{B} \) whose length is bounded from above by a constant multiple of the distance between \( \iota(v) \) and \( \iota(w) \) in \( \mathcal{H}_\Gamma \).

For this note first that by construction of the metric on \( \mathcal{H}_\Gamma \), there is a universal number \( a > 0 \) with the following property. If \( y \in \iota(\mathcal{L}^{\xi,\eta}) \) and if \( \zeta : [0, a] \to \mathcal{H}_\Gamma \) is a horizontal curve of length at most \( a \) issuing from \( y \), then \( \zeta[0, a] \subset \mathcal{B} \).

Let \( P : \mathcal{H}_\Gamma \to \mathcal{L}^{\xi,\eta} \) be the unique bundle map whose restriction to a fibre \( \mathcal{H}_\zeta \) is the shortest distance projection of \( \mathcal{H}_\zeta \) onto \( \mathcal{L}^{\xi,\eta}_\zeta \). Let \( \zeta : [0, am] \to \mathcal{H}_\Gamma \) be any geodesic of length \( am \geq 0 \) connecting the points \( \iota(v), \iota(w) \in \mathcal{L}^{\xi,\eta}_\zeta \). Since there is a number \( a > 0 \) such that the horizontal transport of the fibres of the bundle \( \mathcal{H} \to \Theta \mathcal{C} \mathcal{G} \) along a geodesic arc in \( \Theta \mathcal{C} \mathcal{G} \) of length at most \( \ell \) is a bilipschitz map with bilipschitz constant bounded from above by \( e^{an} \), there is a curve \( \zeta_0 : [0, 2m] \to \mathcal{H}_\Gamma \) connecting \( \iota(v) \) to \( \iota(w) \) with the property that for every \( i < m \) the restriction of \( \zeta_0 \) to the interval \([2i, 2i + 1]\) is horizontal and of length at most \( a \), and the restriction of \( \zeta_0 \) to \([2i + 1, 2i + 2]\) is vertical and of length at most \( e^{an} \). The length of \( \zeta_0 \) is bounded from above by \( am(e^{an} + 1) \). Define a curve \( \zeta_1 : [0, 2m] \to \mathcal{H}_\Gamma \) by requiring that for each \( i \leq m \), the restriction of \( \zeta_1 \) to \([2i, 2i + 1]\) is the horizontal arc issuing from \( P\zeta_0(2i) \) whose projection to \( \Theta \mathcal{C} \mathcal{G} \) coincides with the projection of \( \zeta_0[2i, 2i + 1] \), and the restriction of \( \zeta_1 \) to \([2i + 1, 2i + 2]\) is the vertical geodesic which connects the point \( \zeta_1(2i + 1) \) to \( P\zeta_0(2i + 2) \). Note that this curve is entirely contained in \( \mathcal{B} \). To establish our claim it is now enough to show that the distance between the points \( \zeta_1(2i + 1) \) and \( P\zeta_0(2i + 2) \) is bounded from above by a universal constant. Namely, if this is the case then the length of \( \zeta_1 \) is bounded from above by a universal multiple of the length of the geodesic \( \zeta \).

For this fix for the moment an arbitrary number \( L > 1 \). Let \( \Psi : \mathbf{H}^2 \to \mathbf{H}^2 \) be an \( L \)-quasi-isometry which induces the homeomorphism \( \psi \) of the ideal boundary \( \partial \mathbf{H}^2 \) of \( \mathbf{H}^2 \). Let \( \gamma : \mathbb{R} \to \mathbf{H}^2 \) be a geodesic line and let \( R : \mathbf{H}^2 \to \gamma \) be the shortest distance projection. Let \( y \in \mathbf{H}^2 \) be such that \( R(y) = \gamma(0) = x \) and let \( \zeta : [0, \infty) \to \mathbf{H}^2 \) be the geodesic ray issuing from \( \zeta(0) = x \) and passing through \( y \). Then the concatenation \( \sigma_\pm \) of the inverse \( \zeta^{-1} \) of \( \zeta \) with the geodesic ray \( \gamma[0, \infty) \) and the inverse of the geodesic ray \( \gamma(-\infty, 0] \) is a uniform quasi-geodesic in \( \mathbf{H}^2 \) containing both \( x \) and \( y \). Since the isometry group of \( \mathbf{H}^2 \) acts triply transitive on the ideal boundary, via composing \( \Psi \) with an isometry we may assume that \( \psi \) fixes the endpoints \( \gamma(\pm \infty), \zeta(\infty) \) of \( \gamma, \zeta \). Now \( \Psi \) is an \( L \)-quasi-isometry and hence the image \( L\gamma \) of \( \gamma \) is an \( L \)-quasi-geodesic with the same endpoints as \( \gamma \). Thus the Hausdorff distance between \( \gamma \) and \( \Psi \gamma \) is uniformly bounded and hence the distance between \( \Psi x \) and \( R\Psi x \) is bounded from above by a universal constant \( p_0 > 0 \). Similarly, \( \Psi \sigma_\pm \) are uniform quasi-geodesics contained in a uniformly bounded neighborhood of \( \sigma_\pm \). Since \( \Psi x \) is contained in \( \Psi \sigma_+ \cap \Psi \sigma_- \cap \Psi \gamma \), we conclude that the distance between \( x \) and \( \Psi x \) is uniformly bounded and that the distance between \( R\Psi y \) and \( \Psi x \) is uniformly bounded as well.

Now for every arc \( \nu : [0, 1] \to \Theta \mathcal{C} \mathcal{G} \) of length at most \( a \), the homeomorphism \( \zeta : \mathcal{H}_\nu(0) \to \mathcal{H}_\nu(1) \) obtained by horizontal transport of the fibres along \( \nu \) is an \( L \)-quasi-isometry for a universal constant \( L > 1 \) which induces the boundary homeomorphism \( \omega(\nu(1)) \circ \omega(\nu(0))^{-1} \). By the definition of the line bundle \( \mathcal{L}^{\xi,\eta} \) and the above consideration, we conclude that the distance in \( \mathcal{H}_{\nu(2i+1)} \) between the points \( \zeta_1(2i + 1) \) and the point \( P\zeta_0(2i + 1) \) is uniformly bounded. As a consequence, the
Next we claim that \( \mathcal{L}^{\xi,\eta} \) is uniformly quasi-isometric to a metric fibration over \( \Theta CG \) with fibre \( \mathbb{R} \). Namely, fix a component \( A \) of \( \partial H_h - \{ \xi, \eta \} \). For \( z \in \Theta CG \) and \( \nu \in A \) define \( \Pi(\nu, z) \in \mathcal{L}^{\xi,\eta} \) to be the shortest distance projection of \( \rho(z)(\nu) \) to \( \mathcal{L}^{\xi,\eta} \). The map \( \nu \rightarrow \Pi(\nu, z) \) is a homeomorphism of \( A \) onto \( \mathcal{L}^{\xi,\eta} \). It follows from our above consideration that for every \( \varphi \in \mathcal{G} \) and every \( \nu \in A \) the distance between \( \Pi(h, \nu) \in H_h \) and \( \Pi(\varphi h, \nu) \in H_{\varphi h} \) is uniformly bounded. But the map \( \Pi \) is equivariant with respect to the action of \( \Gamma \) on \( \Theta CG \) and on \( \mathcal{H}_T \) and hence for every \( \psi \in \Gamma \), the distance between \( \Pi(\varphi \psi h, \nu) \) and \( \Pi(\psi h, \nu) \) is bounded from above by the same constant. As a consequence, for every fixed \( \nu \in A \) the map \( z \rightarrow \Pi(\nu, z) \) is a \( q_1 \)-quasi-isometric embedding of \( \Theta C L \) into \( \mathcal{L}^{\xi,\eta} \) for a number \( q_1 > 0 \) not depending on \( \nu \) and on \( \xi, \eta \), and these quasi-isometric embeddings can be used to define on \( \mathcal{L}^{\xi,\eta} \) the structure of a bounded metric fibration which is quasi-isometric to \( \mathcal{L}^{\xi,\eta} \) equipped with the metric induced from the metric on \( \mathcal{H}_T \).

By Corollary 3.6, to show that \( \mathcal{L}^{\xi,\eta} \) is \( q \)-hyperbolic for a constant \( q > 0 \) not depending on \( \xi, \eta \) we only have to show that vertical distances for our metric fibration satisfy the \( \kappa, n \)-flaring property with threshold \( A \) for numbers \( \kappa > 1, n > 0, A > 0 \) not depending on \( \xi, \eta \).

For this let \( \zeta : \mathbb{R} \rightarrow \Theta CG \) be any geodesic line. By the discussion in Section 2, \( \zeta \) is contained in a uniformly bounded neighborhood of a Teichmüller geodesic. By the results of Mosher [Mo03], the restriction \( H_\zeta \) of the bundle \( H_T \) to \( \zeta \) is \( \delta_0 \)-hyperbolic for a constant \( \delta_0 \) not depending on \( \zeta \). The above consideration can be applied to the restrictions of the bundles \( \mathcal{L}^{\xi,\eta} \) and \( \mathcal{H}_T \) to the geodesic \( \zeta \) and shows that the inclusion \( \mathcal{L}^{\xi,\eta}|\zeta \rightarrow H_\zeta \) is a quasi-isometric embedding. Since \( H_\zeta \) is hyperbolic, the bundle \( \mathcal{L}^{\xi,\eta}|\zeta \) is \( \delta_1 \)-hyperbolic for a universal constant \( \delta_1 \). Now the geodesic \( \zeta \) was arbitrary and therefore Lemma 5.2 of [FM02] shows that vertical distances in \( \mathcal{L}^{\xi,\eta} \) satisfy the \( \kappa, n \)-flaring property with threshold \( A \) for constants \( \kappa > 1, n > 0, A > 0 \) not depending on \( \xi, \eta \). As a consequence of this and Corollary 3.6, the line bundle \( \mathcal{L}^{\xi,\eta} \rightarrow \Theta CG \) is \( q \)-hyperbolic for a universal constant \( q > 0 \).

Now we are ready to show.

**Lemma 4.3:** The bundle \( \mathcal{H}_T \) is hyperbolic.

**Proof:** Fix a triple of pairwise distinct points \( \xi_1, \xi_2, \xi_3 \in \partial H_h \). Then the line bundles \( \mathcal{L}^{\xi_i,\xi_i+1} \) bound a subbundle \( \mathcal{V} \) of \( \mathcal{H}_T \) whose fibre at a point \( z \) is isometric to an ideal triangle in \( H_z \). The arguments in the proof of Lemma 4.2 show that for a suitable choice of a length metric on \( \mathcal{V} \) which restricts to the metric on the fibres induced from the metrics on \( \mathcal{H}_T \), the inclusion \( \mathcal{V} \rightarrow \mathcal{H}_T \) is a quasi-isometric embedding. We claim that the bundle \( \mathcal{V} \) is \( \delta_0 \)-hyperbolic for a number \( \delta_0 \) not depending on \( \xi_i \). Namely, an ideal hyperbolic triangle \( T \) is uniformly quasi-isometric to the tripod which consists of the unique point in the interior of \( T \) of equal distance to each of the three sides and three geodesic rays issuing from this point which make a mutual angle \( 2\pi/3 \). The arguments in the proof of Lemma 4.2 show that the
bundle $\mathcal{V}$ is quasi-isometric to a metric fibration over $\mathcal{CG}$ whose fibre is precisely this tripod.

Now as before, for every geodesic $\zeta$ in $\mathcal{CG}$ the restriction of $H_\Gamma$ to $\zeta$ is hyperbolic and hence the same is true for the restriction $\mathcal{V}_\zeta$ of the bundle $\mathcal{V}$ since $\mathcal{V}_\zeta \subset H_\zeta$ is quasi-isometrically embedded. As a consequence of this and Lemma 5.2 of [FM02], vertical distances in the tripod bundle satisfy the $\kappa,n$-flaring property with threshold $A$ for universal numbers $\kappa > 1, n > 0, A > 0$. Corollary 3.6 then shows that the bundle $\mathcal{V}$ is $\delta_0$-hyperbolic for a universal number $\delta_0 > 0$.

We now use once more Lemma 3.3. Namely, for $x, y \in H_\Gamma$ construct a curve $c(x, y)$ connecting $x$ to $y$ as follows. Fix once and for all a point $\xi \in \partial H_\Gamma$. Then $\cup_{\nu \neq \xi} L^\xi_{\nu} = H_\zeta$ for every $z \in \mathcal{CG}$ and therefore there are unique not necessarily distinct points $\nu_x, \nu_y \in \partial H_\Gamma \setminus \{\xi\}$ such that $x \in L^\xi_{\nu_x}, y \in L^\xi_{\nu_y}$. The points $\xi, \nu_x, \nu_y$ define a bundle over $\mathcal{CG}$ whose fibre is a (possibly degenerate) ideal hyperbolic triangle; this bundle is quasi-isometrically embedded in $H_\Gamma$. We define $c(x, y)$ to be a geodesic in this bundle connecting $x$ to $y$.

We claim that the system of curves $c(x, y)$ satisfies the properties listed in Lemma 3.3. Namely, property 1) is immediate from the fact that the bundles $\mathcal{V} \subset H_\Gamma$ as above are uniformly quasi-isometrically embedded. To show property 3) above, let $x, y, z$ be a triple of points. Then $x, y, z$ are contained in a subbundle of $H_\Gamma$ whose fibre at the point $z$ is the ideal quadrangle in $H_z$ with vertices $\xi, \nu_x, \nu_y, \nu_z$. As before, this bundle is $\delta_1$-hyperbolic for a universal number $\delta_1 > 0$, and the subbundles whose fibres are the ideal triangles with vertices $\xi, \nu_x, \nu_y$ and $\xi, \nu_x, \nu_z$ and $\xi, \nu_y, \nu_z$ are uniformly quasi-isometrically embedded. By definition of our curve system, property 3) for our curve system now follows from hyperbolicity of our bundle of quadrangles, and property 2) is obtained in the same way.

As a consequence, we obtain the second part of our theorem.

**Corollary 4.4:** For a finitely generated subgroup $\Gamma < M_g$, the following are equivalent.

1. The natural $\pi_1(S)$-extension of $\Gamma$ is hyperbolic.
2. $\Gamma$ is convex cocompact.

**Proof:** Farb and Mosher [FM02] show the following. If $\Gamma < M_g$ is any finitely generated group such that the $\pi_1(S)$-extension of $\Gamma$ is hyperbolic, then there is a quasi-convex orbit for the action of $\Gamma$ on $T_g$. By Theorem 2.9, this is equivalent to saying that $\Gamma$ is convex cocompact.

On the other hand, by Lemma 4.1 and Lemma 4.3 the $\pi_1(S)$-extension of a convex cocompact subgroup $\Gamma$ of $M_g$ is word hyperbolic.

**Remark:** As mentioned earlier, there is no example known of a convex cocompact subgroup of $M_g$ which is not virtually free. On might ask whether indeed all convex cocompact subgroups $\Gamma$ of $M_g$ are virtually free. One possible approach to study this question is via the Gromov boundary $\partial \Gamma$ of $\Gamma$. Namely, by Theorem
2.9, the Gromov boundary of $\Gamma$ embeds into the Gromov boundary of the complex of curves $\mathcal{C}(S)$, and its image is contained in the subset of all minimal geodesic laminations which are uniquely ergodic, i.e. which support up to multiple a unique transverse invariant measure.

As a consequence, $\partial \Gamma$ embeds into the set $\mathcal{UE} \subset \mathcal{PML}$ of uniquely ergodic projective measured laminations. Therefore, if $\mathcal{UE}$ is totally disconnected, then the same is true for the Gromov boundary of $\Gamma$ and consequently $\Gamma$ is virtually free. However, to my knowledge, nothing is known about the structure of the set $\mathcal{UE}$.

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