ZIMMERMANN TYPE CANCELLATION IN THE FREE FAÀ DI BRUNO ALGEBRA

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Abstract. The $N$-variable Hopf algebra introduced by Brouder, Fabretti, and Krattenaler (BFK) in the context of non-commutative Lagrange inversion can be identified with the inverse of the incidence algebra of $N$-colored interval partitions. The (BFK) antipode and its reflection determine the (generally distinct) left and right inverses of power series with non-commuting coefficients and $N$ non-commuting variables. As in the case of the Faà di Bruno Hopf algebra, there is an analogue of the Zimmermann cancellation formula. The summands of the (BFK) antipode can indexed by the depth first ordering of vertices on contracted planar trees, and the same applies to the interval partition antipode. Both can also be indexed by the breadth first ordering of vertices in the non-order contractible planar trees in which precisely one non-degenerate vertex occurs on each level.

1. Introduction

Non-commutative power series play an important role in a number of areas, including combinatorics, free probability, and quantum field theory. A striking aspect of these applications is that one can effectively manipulate series in which neither the coefficients nor the variables commute. Such calculations are often simplified through the use of combinatorial indices such as trees and graphs. In turn, these somewhat ad hoc techniques can frequently be systematized by using Hopf algebras. This approach was pioneered by Rota and his colleagues in their studies of combinatorics [5]. More recently Kreimer [7], and Kreimer and Connes [2] have used Hopf-theoretic methods to rationalize various Feynman diagram methods used in perturbative quantum field theory.

An important example of this theory was described by Haiman and Schmitt [4], who showed that calculating the antipode for the reduced Faà di Bruno Hopf algebra is equivalent to finding an explicit Lagrange inversion formula for factorial power series with commuting coefficients. For this purpose they realized the Hopf algebra as the incidence algebra of the colored partitions of finite colored sets, and they proved a summation formula for the antipode in which the terms are indexed by colored trees. They showed that many of the terms in this sum cancel, and that it suffices to use the reduced trees, in which each non-leaf vertex is non-degenerate, i.e., has more than one offspring (see [4], Theorem 7). In their argument they passed from general trees to reduced trees by contracting the appropriate edges. As emphasized in [3] (see the discussion of Zimmermann’s formula in §12.2),
Haiman and Schmitt’s procedure may be regarded as an elementary illustration of the cancellations that play such an important role in perturbative quantum field theory.

In a recent paper, Brouder, Fabretti, and Krattenaler [1] described a Hopf algebra, called the left Lagrange algebra \( \mathcal{L} = \mathcal{L}^N \) below, which is related to Lagrange inversion for power series with non-commuting coefficients and \( N \) non-commuting variables. As they pointed out, the situation is more delicate, since the “composition” of such polynomials is generally not associative. Furthermore, they made the important observation that the antipode \( S_\mathcal{L} \) is not involutory, i.e., \( S_\mathcal{L}^2 \neq \text{id} \) (correcting the statement [10], Prop. 4.4). As a result, each left Lagrange algebra has a corresponding inverse Hopf algebra \( \mathcal{H} = \mathcal{H}^N \) associated with the antipode \( S_\mathcal{H} = S_\mathcal{L}^{-1} \) (see §§5-6). As we will see, \( \mathcal{H}^N \) is just the incidence Hopf algebra of the \( N \)-colored interval (i.e., ordered) partitions.

As one might expect, one has a formula for the antipode \( S_\mathcal{H} \) in which the terms are indexed by planar trees, but one must in addition keep track of the breadth first ordering of the vertices (see (6.5)). In contrast to the commutative situation considered by Haiman and Schmitt, one cannot contract the singular edges in these trees since that procedure can disrupt the ordering.

The primary result in this paper is that despite the new complications, a non-commutative analogue of Haiman and Schmitt’s reduced tree formula is valid provided one instead uses the depth first ordering on the vertices (Corollary 7.3). This is initially proved by induction. We also show that the reduced tree antipode formulae for the the three Hopf algebras \( \mathcal{H}, \mathcal{L} \) and the related right Lagrange Hopf algebra \( \mathcal{R} \) can be derived from each other.

In §§8-9 we show that the reduced tree formulae can be derived by suitable “ordered” cancellations. We begin by using order-preserving contractions and expansions to cancel out all but the “order reduced simple layered trees” \( \text{OST} \) in the breadth first formula (Theorem 5.2). In §9 we prove that \( \text{OST} \) is precisely the class of layered trees on which left-to-right breadth first and right-to-left depth first orderings coincide on the non-degenerate vertices. We then show that contraction of all of the singular edges provides a one-to-one correspondence between the trees in \( \text{OST} \) and the reduced planar trees \( \text{RT} \). This leads to a constructive proof of the reduced tree formula for \( S_\mathcal{H} \), which suggests that similar arguments might be used for studying antipodes in the non-commutative versions of the Connes-Kreimer algebras.

In §10 we prove that despite the fact that the substitution operation for non-commutative power series is not multiplicatively associative, one can still use the antipodes of the Lagrange Hopf algebra \( \mathcal{L} \) (which is associative) and its reflection \( \mathcal{R} \) to find the left and right substitutional inverses of power series in which neither the variables nor the constants commute. As we illustrate, these inverses are generally distinct.

In order to make the material more accessible to functional analysts and mathematical physicists, we have included a summary of the relevant constructions from algebraic combinatorics. Those familiar with this material might prefer to skip the initial sections.
2. ORDERED SETS AND THEIR COLORINGS

A partially ordered set \((P, \leq)\) is a set \(P\) together with a relation \(\leq\) such that \(x \leq y\) and \(y \leq x\) if and only if \(x = y\), and \(x \leq y \leq z\) implies \(x \leq z\). We say \(P\) is a linearly ordered set or simply an ordered set if \(x \leq y\) or \(y \leq x\) for all \(x, y \in X\). If a partially ordered set \(P\) has a minimum element, we denote it by \(0_P\), and similarly we write \(1_P\) for a maximum element. Given \(x, y \in P\) with \(x \leq y\), we let \([x, y]\) denote the segment \(\{z \in P : x \leq z \leq y\}\). We denote a finite ordered set \(S\) by \((x_1, \ldots, x_p)\) where \(x_1 < \cdots < x_p\). In particular if \(p \in \mathbb{N}\), we let \([p]\) = \((1, \ldots, p)\).

Given partially ordered sets \(P\) and \(Q\), we let \(P \times Q\) have the product partial ordering \((x_1, x_2) \leq (y_1, y_2)\) if \(x_1 \leq y_1\) and \(x_2 \leq y_2\).

We fix a finite (unordered) set of “colors” \(\Gamma\), which in most cases will be \(\langle N \rangle = \{1, 2, 3, \ldots, N\}\). A colored (or \(\Gamma\)-colored) ordered set \((S, c)\) is an ordered set \(S\) together with a function \(c = c_S : S \rightarrow \Gamma\) (we place no restrictions on \(c\)). We refer to \(c(x)\) as the color of a point \(x \in S\), and we say that \(c\) is a coloring of \(S\). If \(S = (x_1, \ldots, x_p)\) and we let \(v(i) = c(x_i)\), \(c\) is determined by the word \(v = v(1) \cdots v(p)\) in the free monoid \(\Gamma^*\) generated by \(\Gamma\). We will often identify \(c : S \rightarrow \Gamma\) with the word \(v\). The identity 1 of this monoid is the empty word \(\emptyset\). We write \(|v|\) for the length of an element \(v\) in \(\Gamma^*\). Two colored ordered sets \((S, v)\) and \((T, w)\) are isomorphic or have the same coloring if there is an order isomorphism of \(\theta : S \rightarrow T\) that preserves the coloring, i.e., \(c_T(\theta(s)) = c_S(s)\). Since these are linearly ordered sets, the mapping \(\theta\) will necessarily be unique.

Given a finite colored ordered set \(S\), any subset \(R \subseteq S\) is itself linearly ordered in the relative order, and we let \(R\) have the restricted coloring \(c|R\). Given disjoint \(\Gamma\)-colored sets \(S\) and \(T\), with colorings \(v\) and \(w\), we let \(S \sqcup T\) denote the union with the left to right ordering, and the coloring \(c_{S,\sqcup T} = vw\).

3. PLANAR TREES AND THEIR COLORINGS.

The planar forests (or simply “forests”) are defined recursively. For transparency we use terms associated with botanical and genealogical trees. To construct a plane forest \(F\) we first choose an ordered set \(L_0\) of vertices \((x_1, \ldots, x_r)\) called the roots or the zeroth level of \(F\). For each root \(x_i\) in \(L_0\) we then choose a possibly empty ordered set of vertices \((x_{i1}, \ldots, x_{ir_i})\) called the offspring or children of \(x_i\). The entire collection \(L_1\) of these offspring is called the first level, which we linearly order first by their parents and then among children of a given parent by the given order. Having chosen the \((n - 1)\)-st level, we choose an ordered set of vertices for each vertex in that set. These new vertices constitute the \(n\)-th level, and we order them in the same manner. We consider only finite forests. Owing to the conventions we have adopted, we regard the \((n - 1)\)-st level as being “higher” than the \(n\)-th level. A tree is a forest with only one root.

We identify a forest with a graph in the plane in the usual manner. The levels are placed in horizontal rows, parents are joined by edges to their offspring, and the left to right order reflects the recursively defined order on the parents, and the given order on the offspring in each family. A typical planar forest is illustrated below:
We use the notations $V(F)$ and $E(F)$ for the vertices and edges of a forest $F$. A vertex $x \in V(F)$ is a leaf if it has no offspring, it is unary if it has precisely one offspring, and it is non-degenerate if it has more than one offspring. The offspring of a vertex are said to be siblings. An edge is singular if it descends from a unary vertex.

A tree $T$ is said to be reduced if it has no unary vertices, or equivalently, all of its non-leaf vertices are non-degenerate. Given an arbitrary tree $T$, we let $\rho(T)$ be the corresponding reduced tree, in which each singular edge from a unary vertex $x$ is contracted to the vertex $x$.

A layer (respectively, branch) is a forest (respectively, tree) in which all vertices are roots or leaves. We define the $n$-th layer of a forest $F$ to be the forest obtained by considering only the vertices in the $(n - 1)$-st and $n$-th levels together with the edges joining them in $F$. We say that a forest $F$ is uniformly layered or simply layered if leaves occur only at the same lowest level. Equivalently, any vertex at a higher level must have at least one offspring.

We say that a layered tree is proper if each non-leaf level has at least one non-degenerate vertex (or equivalently, the $n$-th level is larger than the $(n - 1)$-st level for each $n$). Unless otherwise indicated, all layered forests in this paper are assumed to be proper.

A non-degenerate vertex is simple if it is the only non-degenerate vertex on that level. If that is the case, we say that the corresponding level is simple. A forest is simple if all of its non-degenerate vertices are simple.

The ascending breadth first ordering $\ll$ on the vertices of a layered forest $F$ is defined as follows. We write $x \ll y$ if $x$ is on a lower level than $y$, or $x$ is in the same level as $y$ and lies to the left of $y$ as illustrated in the forest below, in which $w_1 \ll w_2 \ll \ldots$. In the literature the term breadth first usually refers to the corresponding reverse or descending ordering $\gg$.

We will also use the left and right depth first total orderings of an arbitrary tree $T$ and their inverse orderings. Given distinct vertices $x$ and $y$ in $T$, we write $x \rightarrow y$ (respectively, $x \leftarrow y$) if either $y$ is an ancestor of $x$, or letting $z$ be the first common ancestor of $x$ and $y$, the branch headed towards $x$ lies to the left (respectively, right) of the branch headed towards $y$. We define the descending orderings $\rightarrow$ and $\leftarrow$ to be the inverses of $\leftarrow$ and $\rightarrow$, respectively. We have, for example, that
and are known as the (descending) right and left orderings. If two vertices \( v \) and \( w \) colored by \( u \), \( v \) or \( w \), respectively, with roots \( x \) (3.1)

→ linear orderings

\[ x \rightarrow \text{LT} \rightarrow \text{TR} \rightarrow \text{TL} \rightarrow \text{LR} \rightarrow \text{RL} \rightarrow \text{X} \]

\[-223 \text{ Each "reflected" vertex } x \rightarrow \text{reverse the order of the vertices at each level, keeping the same parental relation.} \]

\[-17 \text{ Given } u, v \in \langle N \rangle ^*, \text{ we let } F_u^v, R_u^v, S_u^v, \text{ and } L_u^v \text{ denote the equivalence classes of general, reduced, simple and (proper) layered forests, respectively, with roots colored by } v \text{ and leaves colored by } u. \text{ If } v = i \in \langle N \rangle, \text{ we let } T_u^i, R_u^i, S_u^i, \text{ and } L_u^i \text{ denote the corresponding classes of trees.} \]

\[-199 \text{ Given a colored tree } T, \text{ we define } T^* \text{ to be the “reflected” tree in which we reverse the order of the vertices at each level, keeping the same parental relation. Each “reflected” vertex } x^* \text{ is given the color of } x. \text{ It is evident that if } x \text{ and } y \text{ are}

\[ x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \ldots \text{ and } y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow \ldots \text{ in the following trees,} \]

\[ \begin{array}{c}
  x_1 \\
  x_2 \quad x_5 \\
  x_3 \quad x_4 \\
  x_6 \quad x_7 \\
  x_8 \quad x_9 \\
  x_{10} \\
\end{array} \quad \begin{array}{c}
  y_1 \\
  y_2 \quad y_5 \\
  y_3 \quad y_6 \\
  y_7 \quad y_8 \\
  y_{10} \\
\end{array} \]

\[ \text{whereas } x_{10}, x_9, x_8, \ldots \text{ and } y_1, y_2, y_3, \ldots \text{ in those same diagrams. The linear orderings } \ll \text{ and } \gg \text{ are known as the (descending) right and left depth first orderings. To numerically label the vertices according to } x, y \text{ one begins at the root, and then successively chooses right branches going to successive generations, backtracking to enumerate the next right-most uncounted vertex when necessary.} \]

\[ \text{It is important to remember that if } x \rightarrow y \text{ or } x^* \rightarrow y, \text{ one cannot conclude that the level of } x \text{ is lower than or equal to that of } y. \text{ Nevertheless the root is the greatest element in both the } \rightarrow \text{ and } \leftarrow \text{ orderings. If two vertices } x \text{ and } y \text{ are on the same level, then } x \ll y \text{ if and only if } x < y. \text{ Finally } x \rightarrow y \text{ and } x^* \rightarrow y \text{ will both hold if and only if } x \text{ is a descendant of } y. \text{ We will write } x \ll y \text{ if } x \rightarrow y \text{ but it is not true that } x^* \rightarrow y, \text{ i.e., } x \text{ is not a descendant of } y. \]

\[ A \text{ colored forest } (F, c) \text{ (or simply } F) \text{ consists of a forest } F \text{ together with a coloring of the vertices } c : F \rightarrow \Gamma \text{ such that if } x \text{ is a unary vertex with offspring } y, \text{ then } c(y) = c(x). \text{ The following is a 2-colored layered tree.} \]

\[ \begin{array}{c}
  1 \\
  2 \\
  1 \\
  1 \\
  2 \\
\end{array} \]

\[ \begin{array}{c}
  1 \\
  1 \\
  1 \\
  2 \\
\end{array} \]

\[ \text{We say that colored layered forests } F \text{ and } G \text{ are compatible if the set of roots of } F \text{ and the leaves of } G \text{ are colored and order isomorphic. If that is the case, we define the right join } F \triangleright G \text{ be the forest that results if one identifies these two sets (} G \text{ lies “over” } F \text{ in the resulting forest). It is apparent that every layered forest } F \text{ has a unique decomposition } F = L_1 \triangleright L_2 \triangleright \ldots \triangleright L_r, \text{ where } L_k \text{ is the } (r - k + 1) \text{ level of } F. \]

\[ \text{If the vertices of compatible layered forests } F \text{ and } G \text{ are given by } x_1 \ll \ldots \ll x_m \text{ and } y_1 \ll \ldots \ll y_n, \text{ then the vertices of } F \triangleright G \text{ are ordered by} \]

\[ x_1 \ll \ldots \ll x_m \ll y_1 \ll \ldots \ll y_n. \quad (3.1) \]

\[ \text{Given } u, v \in \langle N \rangle ^*, \text{ we let } F_u^v, R_u^v, S_u^v, \text{ and } L_u^v \text{ denote the equivalence classes of general, reduced, simple and (proper) layered forests, respectively, with roots colored by } v \text{ and leaves colored by } u. \text{ If } v = i \in \langle N \rangle, \text{ we let } T_u^i, R_u^i, S_u^i, \text{ and } L_u^i \text{ denote the corresponding classes of trees.} \]

\[ \text{Given a colored tree } T, \text{ we define } T^* \text{ to be the “reflected” tree in which we reverse the order of the vertices at each level, keeping the same parental relation. Each “reflected” vertex } x^* \text{ is given the color of } x. \text{ It is evident that if } x \text{ and } y \text{ are} \]
vertices in $T$, then $x \rightarrow y$ in $T$ if and only if $y^* \mathbf{1} x^*$ in $T^*$, and that $x \leftrightarrow y$ if and only if $y^* \mathbf{1} x^*$.

4. Colored ordered partitions and their segments

An ordered (or interval) partition $\sigma = (B_1, \ldots, B_q)$ of an ordered set $S$ is a collection of subsets for which $\bigcup B_k = S$ and $B_1 < \ldots < B_q$ in the given “left to right” ordering. We may use the layer (see §3)

\[(4.1)\]

or the parenthetical expression

\[(12)(345)(6)(789)\]

to denote the partition $\sigma = (B_1, \ldots, B_q)$ of $[10]$ with

$B_1 = (1, 2), B_2 = (3, 4, 5), B_3 = (6), B_4 = (7, 8, 9, 10)$

(we regard a block as an ordered set). We identify a partition with the corresponding equivalence relation on $S$ and the ordered quotient set $S/\sigma$ with $(B_1, \ldots, B_q)$. We say that a block is a singleton if it has only one element, and that its element is unary.

There is an alternative approach to partitions that is useful. We define an (abstract) partition $\sigma$ of an ordered set $S$ to be an increasing map $f_\sigma : S \rightarrow S_\sigma$ of $S$ onto an ordered set $T = S_\sigma$. We then have the ordered partition $(B_t)_{t \in T}$, where $B_t = f_\sigma^{-1}(t)$. Conversely given a partition $\sigma = (B_1, \ldots, B_q)$ in our initial sense, we have a corresponding increasing surjection $f = f_\sigma : S \rightarrow S/\sigma = (B_1, \ldots, B_q)$, where $f_\sigma(x) = B_j$ if $x \in B_j$. Moreover may be regarded as the mapping diagram of $f_\sigma$ in that example.

Given a second ordered partition $\pi = (C_1, \ldots, C_r)$ of $S$, we write $\sigma \triangleright \pi$ if every set $B_j$ is contained in some set $C_k$, i.e., $\sigma$ is a refinement of $\pi$. Equivalently, $f_\pi = g \circ f_\sigma$ for some increasing function $g : S_\sigma \rightarrow S_\pi$. We write $\sigma \triangleleft \pi$ if $\sigma \triangleright \pi$ and $\sigma \neq \pi$. Given partitions $\sigma$ of $S$ and $\pi$ of $T$,

$$\sigma \uplus \pi = \{ B \cap C : B \in \sigma, C \in \pi \}$$

is an ordered partition of $S \uplus T$ (we use the left to right ordering).

A colored partition $(\sigma, c_\sigma)$ of a colored ordered set $(S, c_S)$ is an ordered partition $\sigma = (B_1, \ldots, B_q)$ of $S$ together with a coloring $c_\sigma$ of $(B_1, \ldots, B_q)$, such that if $B_j = (x)$, then $c_\sigma(B_j) = c_S(x)$, i.e., each singleton has the same color as its unique element. Equivalently, we have a colored ordered set $S_\sigma$ and an order preserving surjection $f = f_\sigma : S \rightarrow S_\sigma$ with the property that if $x$ is unary, then $c(f(x)) = c(x)$. Given colored ordered sets $S$ and $T$, we say that a mapping $f : S \rightarrow T$ is proper, and write $f : S \rightarrow T$ if it satisfies these conditions, i.e., it is an order-preserving surjection with the singleton color condition. We say that $\sigma$ is a partition of $S$ with the coloring $w = c_\sigma$ and that $(S/\sigma, w)$ is a colored ordered set. We have that $(\sigma \uplus \pi, w)$ is a colored ordered partition of the colored ordered set $S \uplus T$.

We use the colored planar forest
or the parenthetical expression

$$(31)_{2}(233)_{1}(2)_{2}(1223)_{3}$$

to indicate the colored ordered partition $\sigma = ((12), (345), (6), (789\overline{10}))$, $w = c_{\sigma} = 2123$ of the colored set

$$([10], 3123321223).$$

In this example we have the colored order isomorphism

$$([10]/\sigma, w) \cong ([4], 2123).$$

If $S$ is a colored ordered set, we define $\mathcal{Y}(S)$ (respectively $\mathcal{Y}_{q}(S)$) to be the collection of all colored ordered partitions $\sigma$ of the colored set $S$ (respectively, with $q$ blocks). We partially order $\mathcal{Y}(S)$ by $(\sigma, v) \preceq (\pi, w)$ if (1) $\sigma \preceq \pi$, and (2) for any $B \in \sigma \cap \pi$, $c_{\sigma}(B) = c_{\pi}(B)$ (i.e., the common block $B$ has the same color in either partition). For simplicity we simply write $\sigma \preceq \pi$. Owing to the second condition, if $\sigma \preceq \pi$ and $\pi \preceq \sigma$, then $\sigma = \pi$ as colored sets. It is evident that $\sigma \preceq \pi$ if and only if $f_{\pi} = g \circ f_{\sigma}$, for a (necessarily unique) proper function $g : S_{\sigma} \rightarrow S_{\pi}$. If $S = (x_{1}, ..., x_{p})$ has the coloring $v = v(1) \cdots v(p) \in \Gamma^{*}$, then $\mathcal{Y}(S)$ has the minimum element $0_{v} = (((x_{1}) \cdots (x_{p})), v)$ and the maximal elements $1_{j} = ((x_{1} \cdots x_{p}), j)$ for each $j \in \Gamma$.

We turn next to segments of colored partitions $P = [\sigma, \tau]$, where $\sigma \preceq \tau \in \mathcal{Y}(S)$. $P$ has the relative partial ordering $\preceq$, and an element $\lambda \in P$ may be regarded as a colored ordered set, as a colored partition of the elements in $S/\sigma$, or as a colored partition of $S$. We say that segments $P \subseteq \mathcal{Y}(S)$ and $Q \subseteq \mathcal{Y}(T)$ are isomorphic and write $P \cong Q$, if there exists an order isomorphism $\theta : P \rightarrow Q$ such that for each $\lambda \in P$, $\lambda$ and $\theta(\lambda)$ are isomorphic colored ordered sets. In particular, for any partitions $\sigma \in \mathcal{Y}(S)$ and $\tau \in \mathcal{Y}(T)$, the segments $P = [\sigma, \sigma]$ and $Q = [\tau, \tau]$ are isomorphic if and only if $\sigma$ and $\tau$ are isomorphic colored ordered sets. Given a coloring $v$ of $[p]$ and $j \in [p]$, we let $Y_{v}^{j} = [0_{v}, 1_{j}]$. If $\sigma \preceq \pi$ and $\sigma' \preceq \pi'$, then we have a unique order isomorphism

$$(4.3) \quad \theta : [\sigma, \pi] \times [\sigma', \pi'] \simeq [\sigma \sqcup \sigma', \pi \sqcup \pi'],$$

where for each $(\lambda, \lambda')$, $\theta((\lambda, \lambda'))$ and $(\lambda, \lambda')$ are isomorphic colored ordered sets.

Lemma 4.1. Let us suppose that $S$ is a colored ordered set and that $(\sigma, v) \preceq (\pi, w)$ in $\mathcal{Y}(S)$. Then letting $\pi = (C_{1}, \ldots, C_{q})$, and $v_{k} = v|C_{k}$ we have a natural isomorphism

$$(4.4) \quad \theta : [\sigma, \pi] \cong Y_{v_{1}}^{w(1)} \times \cdots \times Y_{v_{q}}^{w(q)}$$

where for each $\lambda \in [\sigma, \pi]$, $\lambda$ and $\theta(\lambda) = (\lambda_{1}, \ldots, \lambda_{q})$ are isomorphic colored ordered sets.

Proof. Consider the mapping $g : S_{\sigma} \rightarrow S_{\pi}$ described above. Let us identify $S_{\pi}$ with $[q]$. The intermediate colored partitions correspond to factorizations $S_{\pi} \rightarrow T \rightarrow [q]$. To construct such a diagram it suffices to choose for each $j \in [q]$ a factorization $g^{-1}(j) \rightarrow T_{j} \rightarrow (j)$, i.e., an element $\lambda_{j}$ of $Y_{v_{j}}^{w}$, where $v_{j}$ is the coloring of the interval $g^{-1}(j)$. It is evident that we have a one-to-one order preserving correspondence $\lambda \leftrightarrow (\lambda_{1}, \ldots, \lambda_{q})$ with the desired coloring property.
If $S = ((x), j)$, then $π_j = ((x), j)$ is the only element in $\mathcal{Y}(S)$, and the only segment is isomorphic to $Y_j^2$. In order to obtain the incidence Hopf algebras, it is necessary to impose a more inclusive equivalence relation, which identifies all such intervals with a multiplicative identity. Given a colored partition $(σ, v)$ of $S$, we let $S_{ns}$ be the union of the non-singleton blocks in $σ$, $σ_{ns}$ be the collection of non-singleton blocks, and $v_{ns}$ be the restriction of $v$ to $σ_{ns}$. We say that order intervals $P = [σ, τ] ⊆ \mathcal{Y}(S)$ and $Q = [σ', τ'] ⊆ \mathcal{Y}(T)$ are similar, and write $P \sim Q$, if there is an order-preserving bijection $θ : P \to Q$ such that for each $λ ∈ P$, $λ_{ns}$ and $θ(λ)_{ns}$ are isomorphic colored sets. If the non-singleton sets $S_{ns}$ and $T_{ns}$ are empty, the latter is a vacuous restriction.

For any colored partition $σ$ and segment $P$,

$$[σ, σ] \times P \sim P.$$  

To prove this, consider the mapping

$$θ : [σ, σ] \times P \to P : σ \cup λ \mapsto λ.$$  

This is clearly a bijection and order preserving. Since $(σ \cup λ)_{ns} = (λ)_{ns}$, $θ$ satisfies the coloring condition. Similarly we have that $P \times [σ, σ] \sim P$ for any $σ$ and $P$. Finally it is easy to verify that if $P \sim P'$ and $Q \sim Q'$, then

$$(4.5) \quad P \times Q \sim P' \times Q'.$$  

We let $P = P^N$ denote the collection of all similarity classes $P_σ$ of segments $P = [σ, π] \subseteq \mathcal{Y}(S)$ for arbitrary finite $N$-colored sets $S$. We define an associative product on $P$ by

$$(4.6) \quad [σ, τ]_~[σ', τ']_~ = [σ \times σ', τ \times τ']_~.$$  

It follows from (4.5) that this is well-defined. The corresponding multiplicative identity is given by $1 = [Y^i_u]$, where $i \in \{N\}$ is arbitrary.

The intervals $Y^i_u$ with $|u| > 1$ are all non-similar. When confusion is unlikely, we will dispense with the similarity class notation $[ ]_~$. It is evident that $P$ is just the free monoid on the symbols $Y^i_u$ with $|u| > 1$.

Given a segment $P = [σ, τ]$ in $Y^i_v$ we define a chain $γ$ in $P$ to be a sequence $σ = σ_0 \prec σ_1 \prec \ldots \prec σ_r = τ$. Given such a chain, the blocks of $σ_{k−1}$ form a colored partition of the blocks in $σ_k$. Thus the interval $[σ_{k−1}, σ_k]$ determines a one-layer colored forest $L_k$ (see (4.1)). The colored layers $L_1, \ldots, L_r$ are consecutively compatible, and letting $v$ be the colors of $L_1$ and $w$ be the colors of the roots of $L_r$, we may associate the layered forest

$$F(γ) = L_1 \triangleright L_2 \triangleright \ldots \triangleright L_r \in L \mathbf{F^w}_v(N)$$  

with $γ$. Conversely for each $F \in L \mathbf{F^w}_v$, we may use the layers of $F$ to determine a unique chain $γ(F)$ in $[σ, τ]$ with $F(γ(F)) = F$.

We define the length of a chain $γ$ in a segment $P$ to be the number of intervals in $γ$, or equivalently the number of layers in $F(γ)$. The grading $ρ(P)$ of $P$ is the maximal length of a chain in $P$. If $P = Y^i_u$, any chain $γ|_{inP}$ can be extended to a chain $γ'$ such that for each $k, σ_{k−1}^i$ is obtained from $σ_k^i$ by splitting precisely one of the blocks of $σ_k$ into two blocks. The corresponding colored trees are simple, layered, and each non-degenerate vertex has two offspring. Examining the tree of such a chain it follows that the maximal chains in $Y^i_u$ all have $|u| − 1$ elements, and
thus \( \rho(Y^i_u) = |u| - 1 \). Given an arbitrary segment \( P = [\sigma, \tau] \cong Y^w_{\sigma_1} \cdots Y^w_{\tau_q} \), it is evident that
\[
\rho(P) = \rho(Y^w_{\sigma_1}) + \cdots + \rho(Y^w_{\tau_q}).
\]

5. HOPF ALGEBRAS

We briefly recall some elementary notions from the theory of Hopf algebras. More complete discussions can be found in \[6\], \[11\], \[3\], and \[9\].

Given a vector space \( V \), we let \( L(V) \) denote the algebra of all linear mappings \( T: V \to V \). The identity mapping \( I: V \to V \) is a multiplicative identity for \( L(V) \).

If \((A, 1)\) is a unital algebra, the tensor product algebra \( A \otimes A \) is given the associative multiplication
\[
(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1x_2 \otimes y_1y_2
\]
and the multiplicative unit \( 1 \otimes 1 \).

A bialgebra \((H, m, \eta, \Delta, \varepsilon)\) consists of a vector space \( A \) with an associative product \( m : H \otimes H \to H \), a homomorphism \( \eta : \mathbb{C} \to H : \alpha \to \alpha \), where \( 1 \) is a multiplicative unit for \( H \), a coassociative coproduct \( \Delta : H \to H \otimes H \), and a counit \( \varepsilon : H \to \mathbb{C} \) with the linking property that \( \Delta : H \to H \otimes H \) is a unital homomorphism. We employ Sweedler’s notation
\[
\Delta a = \sum_{(a)} a_{(1)} \otimes a_{(2)}.
\]

An antipode for a bialgebra \( H \) is a mapping \( S : H \to H \) such that for any \( a \in H \)
\[
\sum_{(a)} S(a_{(1)})a_{(2)} = \sum_{(a)} a_{(1)}S(a_{(2)}) = \varepsilon(a)1.
\]
or equivalently, \( m(S \otimes I)\Delta = m(I \otimes S)\Delta = \eta \circ \varepsilon \). We say that \( S \) is a left antipode if one just has the first and third terms are equal, and a right antipode, if one has the second equality. If \( H \) has an antipode, then any left (respectively right) antipode automatically coincides with \( S \), and in particular, \( S \) is unique. An antipode \( S \) is automatically a unital antihomomorphism, i.e., we have
\[
S(gh) = S(h)S(g)
\]
\[
S(1) = 1
\]
(see \[11\], Prop. 4.0.1). A Hopf algebra \((H, m, \eta, \Delta, \varepsilon, S)\) is a bialgebra \((H, m, \eta, \Delta, \varepsilon)\) together with an antipode \( S \).

Given \( \varphi, \psi \in L(H) \), we define the convolution \( \varphi * \psi \in L(H) \) by
\[
\varphi * \psi(a) = \sum_{(a)} \varphi(a_{(1)})\psi(a_{(2)})
\]
or equivalently, \( \varphi * \psi = m \circ (\varphi \otimes \psi) \circ \Delta \). This determines an associative product on \( L(H) \) with the multiplicative unit \( u = \eta \circ \varepsilon \). It is evident from the definition that \( S \) is an antipode if and only if \( S \ast I = I \ast S = u \), i.e., it is the convolution inverse of \( I \).

Given any vector space \( V \), we define the flip \( \tau : V \otimes V \to V \otimes V \) by \( \tau(v \otimes w) = w \otimes v \). Given a bialgebra \( H = (H, m, 1, \Delta, \varepsilon) \), we let \( m^{op} = m \circ \tau \) and \( \Delta^{cop} = \tau \circ \Delta \)
and We define the opposite and co-opposite bialgebras to be \( H^{op} = (H, m^{op}, 1, \Delta, \varepsilon) \) and \( H^{cop} = (H, m, 1, \Delta^{cop}, \varepsilon) \). It is shown in \[6\] Cor. 3.5.5 that these are indeed bialgebras, and the antipode \( S : S : H^{op} \to H^{cop} \) is a Hopf algebra homomorphism. The latter corresponds to the fact that the antipode \( S : H \to H \) is both an antihomomorphism and an anti-coendomorphism. If the antipode \( S : H \to H \) is a linear
isomorphism, then $S^{-1}$ is the antipode for both $H^{op}$ and $H^{cop}$, and $S : H^{op} \rightarrow H^{cop}$ is a Hopf algebraic isomorphism. We refer to either Hopf algebra as the inverse Hopf algebra of $H$.

The following will play an important role in what follows.

**Proposition 5.1.** If $H$ is a bialgebra for which both $H$ and $H^{cop}$ have antipodes $S$ and $S^{cop}$, respectively, then $S$ is invertible, and $S^{cop} = S^{-1}$.

**Proof.** See [8], Exercise 1.3.3 (the solution is provided).

A bialgebra $(H, m, \eta, \Delta, \varepsilon, S)$ is said to be filtered if one has an increasing sequence of subspaces $H^n$ ($n \geq 0$) with $\bigcup H^n = H$, for which

$$H^m H^n \subseteq H^{m+n},$$
$$\Delta H^n \subseteq \sum_{p+q=n} H^p \otimes H^q.$$ 

and it is said to be connected if in addition $H^0 = \mathbb{C}1$. We let $H_+$ denote the kernel of $\varepsilon$, and $H^+_n = H^n \cap H_+$.

A connected filtered bialgebra is automatically a Hopf algebra, i.e., it has an antipode. The following is proved in [3] (see also [10], p. 238).

**Theorem 5.2.** If $H$ is a connected filtered bialgebra, then it has an antipode given by the “geometric series”

$$Sa = (u - (u - I))^{-1}(a) = \sum_{k=0}^{\infty} (u - I)^{kn} a.$$ 

The sum is finite for each $a$ since if $a \in H^n$, then $(u - I)^{(n+1)}(a) = 0$.

**Corollary 5.3.** The antipode of a connected filtered Hopf algebra is invertible.

**Proof.** We have that $H^{cop}$ is a filtered bialgebra since

$$\Delta^{cop} H^n \subseteq \sum_{p+q=n} H^q \otimes H^p.$$ 

Thus from Theorem 5.2 $H^{cop}$ has an antipode and from Proposition 5.1 $S$ is invertible.

The proof of Theorem 5.2 in [3] is based upon the following lemma. We have included a proof since we will need the calculation in the discussion that follows.

**Lemma 5.4.** Suppose that $H$ is a connected Hopf algebra. Then for any $a \in H_+^n$,

$$\Delta a = a \otimes 1 + 1 \otimes a + y$$

where $y \in H_+^{n-1} \otimes H_+^{n-1}$.

**Proof.** Let us define $y$ by this relation. Since $\Delta a \in \sum H^k \otimes H^{n-k}$,

$$y = \Delta a - a \otimes 1 - 1 \otimes a \in \sum_{k=0}^{n} H^k \otimes H^{n-k},$$

and we have $y = \sum b_k \otimes c_k$, where $b_k \in H_k$ and $c_k \in H_{n-k}$. Applying the right coidentity relation to $a \in H_+$,

$$a = \varepsilon \otimes id(\Delta a) = \varepsilon(a)1 + a + \sum \varepsilon(b_k)c_k = a + \sum \varepsilon(b_k)c_k,$$
and thus $\sum \varepsilon(b_k)c_k = 0$. It follows that
\[
y = \sum (b_k - \varepsilon(b_k)1) \otimes c_k = \sum b'_k \otimes c_k
\]
where $b'_k = b_k - \varepsilon(b_k)1 \in H^+_k$. Similarly, from the right coidentity relation,
\[
y = \sum b'_k \otimes c'_k
\]
with $c'_k = c_k - \varepsilon(c_k)1 \in H^{n-k}_+$. Since $H^0_+ = \{0\}$, we obtain
\[
\Delta y \in \sum_{k=1}^{n-1} H^+_k \otimes H^{n-k}_+ \subseteq H^{n-1}_+ \otimes H^{n-1}_+.
\]
\[\square\]

There is a simple recursive characterization of the antipode in a connected filtered Hopf algebra. If $a \in H^+_n$, then from (5.2),
\[
\Delta a = a \otimes 1 + 1 \otimes a + \sum_{k=1}^{n-1} a_k \otimes b_{n-k}
\]
where $a_k \in H^k$, $b_{n-k} \in H^{n-k}$, and thus
\[
0 = \varepsilon(a)1 = S(a) + a + \sum_{k=1}^{n-1} S(a_k)b_{n-k}.
\]
$S$ is thus recursively determined by $S(1) = 1$ and if $a \in H_n$ with $\varepsilon(a) = 0$, then
\[
S(a) = -a - \sum_{k=1}^{n-1} S(a_k)b_{n-k},
\]
where $a_k \in H^k$ and $b_{n-k} \in H^{n-k}$.

Restricting our attention to the co-opposite algebra, the antipode $S^{-1}$ is characterized by the relations
\[
\sum_{(a)} S^{-1}(a(2))a(1) = \sum_{(a)} a(2)S^{-1}(a(1)) = \varepsilon(a)1.
\]
In particular assuming that $H$ is a connected filtered Hopf algebra, it is recursively determined by $S^{-1}(1) = 1$ and if $a \in H^+_n$ then
\[
S^{-1}(a) = -a - \sum_{k=1}^{n-1} b_kS^{-1}(a_{n-k}),
\]
where $a_k \in H^k$, $b_{n-k} \in H^{n-k}$.

A bialgebra $H$ is graded if there are subspaces $H_n$ of $H$ with $H_n \cap H_m = \{0\}$ and $\sum H_n = H$ such that
\[
H_mH_n \subseteq H_{m+n},
\]
\[
\Delta H_n \subseteq \sum_{p+q=n} H_p \otimes H_q.
\]
It is immediate that the subspaces $H^n = \sum_{i=0}^n H_i$ determine a filtration of $H$.

Finally let us suppose that $H$ is a Hopf algebra and that $\theta : H \to H$ is a linear isomorphism such that $\theta(1) = 1$ and $\varepsilon \circ \theta = \varepsilon$. It is evident that $H^\theta = (H, m^\theta, 1, \Delta^\theta, \varepsilon, S^\theta)$ is again a Hopf algebra, where $m^\theta = \theta \circ m \circ (\theta^{-1} \otimes \theta^{-1})$. 
\[ \Delta^\theta = (\theta \otimes \theta) \circ \Delta \circ \theta^{-1}, \text{ and } S = \theta \circ S \circ \theta^{-1}. \] We refer to \( H^\theta \) as the \( \theta \)-transformed Hopf algebra. In all of the cases considered below, \( \theta \) is involutory, i.e., \( \theta^2 = id \) and thus \( \theta^{-1} = \theta \).

6. THE INTERVAL AND LAGRANGE HOPF INCIDENCE ALGEBRAS

We begin by constructing the relevant incidence Hopf algebra (see [10] for more details). As in §4, we let \( \mathcal{P} = \mathcal{P}^N \) be the collection of all similarity classes of segments \([\sigma, \tau]\) where \( \sigma \preceq \tau \) are colored ordered partitions of \( N \)-colored sets. To be more explicit, we may assume that each \([\sigma, \tau]\) is a subset of some \( \mathcal{Y}(S_u) \), where \( S_u \) is the ordered set \([r] = (1, \ldots, r)\) with the \( N \)-coloring \( u \). We define the interval partition incidence bialgebra \( \mathcal{H} = \mathcal{H}^N \) to be the vector space with basis (labelled by) the elements of \( \mathcal{P} \). The monoid operation on \( \mathcal{P} \) determined by \( \mathcal{H} \) determines a multiplication on \( \mathcal{H} \). It is evident that \( \mathcal{H} \) is the free unital algebra on the segments \( Y_u^i \) \( (1 \leq i \leq N, |u| \geq 1) \). Following Joni and Rota [5], p. 98, we define a coproduct

\[ \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \]

by letting

\[ \Delta(P) = \sum_{0_P \leq \pi \leq 1_P} [0_P, \pi] \otimes [\pi, 1_P], \]

for \( P \in \mathcal{P} \), and then extending linearly to \( \mathcal{H} \). Given \( P_1 \) and \( P_2 \in \mathcal{P} \), the equality \( \Delta(P_1P_2) = \Delta(P_1) \Delta(P_2) \) is a simple consequence of (4.6), hence \( \Delta \) is a homomorphism. We also have a coidentity homomorphism \( \varepsilon : \mathcal{H} \to \mathbb{C} \) determined by \( \varepsilon(1) = 1 \) and \( \varepsilon(Y_u^1) = 0 \), for \( |u| \geq 2 \) and we see that \( \mathcal{H} \) is a bialgebra. As an algebra, \( \mathcal{H} \) is the free associative unital algebra generated by the segments \( Y_u^i \) \( (1 \leq i \leq N, |u| \geq 2) \) with unit 1. In order to simplify the notation below we let \( Y_u^i = \delta_{ij}1 \) if \( i, j \in \langle N \rangle \).

Given \((\pi, w) \in Y_u^i\), \( \pi = (C_1, \ldots, C_q) \) is an ordered partition of the colored set \(([p], u) \) where \( p = |u| \), and \( w = w(1) \cdots w(q) \) is a coloring of \( \pi \). We let \( u_i = u|C_i \).

From Lemma 1, \([0_u, \pi] \sim Y_v^{u(1)} \cdots Y_v^{u(q)} \) and \([\pi, 1_v] = [0_v, 1^v] \sim Y_v^i \) from which we conclude that

\[ \Delta(Y_u^i) = \sum_{q=1}^{p} \sum_{\pi \in \mathcal{Y}(p)} \sum_{v \in \langle N \rangle^q} Y_v^{u(1)} \cdots Y_v^{u(q)} \otimes Y_v^i \]

Owing to our convention that \( Y_u^1 = \delta_{ij}1 \), for \( i, j \in \langle N \rangle \), many of these terms are zero. For example if we let \( q = 1 \), then \( \mathcal{Y}_1(p) = \{((1, \ldots, p), j)\} \) and we have only the summand

\[ \sum_{v \in \langle N \rangle} Y_v^u \otimes Y_v^1 = \sum_{v} \delta_{vu}1 \otimes Y_v^i = 1 \otimes Y_v^i. \]

On the other hand if \( p = q \), then \( \mathcal{Y}_q(p) = \{(((1, \ldots, p), u(1)) \cdots u(p))\} \) and

\[ \sum_{v \in \langle N \rangle^p} Y_v^{u(1)} \cdots Y_v^{u(p)} \otimes Y_v^i = \sum_{v \in \langle N \rangle^p} \delta_{u(1)}^{v(1)} \cdots \delta_{u(p)}^{v(p)} 1 \otimes Y_v^i = 1 \otimes Y_v^i. \]

It follows that

\[ \Delta(Y_u^i) = Y_u^i \otimes 1 + 1 \otimes Y_u^i + \sum_{q=2}^{p-1} \sum_{\pi \in \mathcal{Y}(p)} \sum_{v \in \langle N \rangle^q} Y_v^{u(1)} \cdots Y_v^{u(q)} \otimes Y_v^i \]

\[ = 1 \otimes Y_u^i + \sum_{q=2}^{p-1} \sum_{\pi \in \mathcal{Y}(p)} \sum_{v \in \langle N \rangle^q} Y_v^{u(1)} \cdots Y_v^{u(q-1)} \otimes Y_v^{u(q)} \]

\[ = 1 \otimes Y_u^i + \sum_{q=2}^{p-1} \sum_{\pi \in \mathcal{Y}(p)} \sum_{v \in \langle N \rangle^q} Y_v^{u(1)} \cdots Y_v^{u(q)} \otimes Y_v^i. \]
Furthermore, since we have identified the factors $Y_{u_1}^{(1)} \cdots Y_{u_q}^{(q)} + \rho(Y_v) = (|u_1| - 1) + \cdots + |u_q| - 1) + (|v| - 1)
and thus
\[
\Delta(Y_u^i) \in \sum_{p+q=n} H_p \otimes H_q.
\]
For any $y \in H_p$, $y' \in H_p'$, $z \in H_q$, $z' \in H_q'$,
\[
(x \otimes y)(x' \otimes y') = xx' \otimes yy' \in H_{p+p'} \otimes H_{q+q'}.
\]
Since $\Delta$ is a multiplicative homomorphism, we conclude that if $P$ is an arbitrary interval, i.e., a product of terms of the form $Y_u^i$, and $\rho(P) = m$, then
\[
\Delta(P) \in \sum_{p'+q'=m} H_{p'} \otimes H_{q'}
\]
and thus $\mathfrak{H}$ is a graded and connected bialgebra. From the previous section, $\mathfrak{H}$ has an invertible antipode $S_{\mathfrak{H}}$, and in particular it is a Hopf algebra.

The following antipode formula of Schmitt may be regarded as a transcription of (5.1) (see [3], §11.1 and [10], Th. 4.1).

\[
(6.3) \quad S_{\mathfrak{H}}(Y_u^i) = \sum_{k \geq 0} \sum_{\sigma_0 = \sigma_1 \cdots \sigma_k = 1} (-1)^k \prod_{h=1}^{k} [\sigma_h^{-1}, \sigma_h].
\]

As we have seen in §4, there is a one-to-one correspondence between the chains $\gamma = (0_u = \sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_k = 1_i)$ in $Y_u^i$ and the colored layered trees $T(\gamma) = F_1 \triangleright \cdots \triangleright F_k \in \text{LT}_u^i(N)$, where the one layered forest $F_h$ corresponds to the interval $[\sigma_h^{-1}, \sigma_h]$. The roots $x_{1}^{h} \ll \cdots \ll x_{q_h}^{h}$ in $F_h$ determine the factors in the decomposition
\[
[\sigma_h^{-1}, \sigma_h] = Y(x_{1}^{h}) \cdots Y(x_{q_h}^{h}).
\]
Furthermore, since we have identified the factors $Y_j^i$ with the multiplicative identity in $\mathfrak{H}$, we may employ just the non-degenerate vertices. Relabelling the non-degenerate vertices in $T(\gamma)$ by $x_1 \ll \cdots \ll x_k$ and letting
\[
(6.4) \quad \Omega(T(\gamma)) = \prod_{h} Y(x) = Y(x_1) \cdots Y(x_k),
\]
we conclude that
\[
\prod_{h=1}^{k} [\sigma_h^{-1}, \sigma_h] = \Omega(T(\gamma)),
\]
and we may rewrite (6.3) in the form:
\[
(6.5) \quad S_{\mathfrak{H}}(Y_u^i) = \sum_{T \in \text{LT}_u^i} (-1)^{i(T)} \Omega(T).
\]
We call the inverse Hopf algebra
\[
\mathcal{L} = \mathcal{L}_N = (\mathfrak{H}, m, \varepsilon, \Delta, \eta, S_{\mathcal{L}} = S^{-1})
\]
the left Lagrange Hopf algebra. We may also regard $L$ as a transformed algebra of $H$, as we next show.

Given a word $u = u_1 \cdots u_n \in \langle N \rangle^*$, we let $u^*$ be its reflection $u_n \cdots u_1$. Since $H$ is freely generated by the $Y_u^i$, the inclusion mapping

$$s : Y_u^i \mapsto Y_{u^*}^i \in H^{op}$$

extends to an involutory algebra isomorphism $s : H \to H^{op}$. This may be regarded as an antismorphism $s : H \to H$ satisfying

$$s(Y_u^i \cdots Y_{u_n}^i) = Y_{u_n}^{i_n} \cdots Y_u^{i_1}.$$  \tag{6.6}

**Lemma 6.1.** We have that $m^{op} = s o m (s \otimes s)$, $\Delta = (s \otimes s) \circ \Delta \circ s$, and $S_L = s S_H s$.

**Proof.** The first result is immediate since $s$ is an antihomomorphism.

If $u = u(1) \cdots u(p)$ then $u^* = u(p) \cdots u(1)$ i.e., $u^*(k) = u(p + 1 - k)$ and

$$(s \otimes s) \circ \Delta \circ s(Y_u^i) = (s \otimes s) \circ \Delta(Y_{u^*}^i)$$

$$= (s \otimes s) \sum_v Y_{u^*}^{i_1} \cdots Y_{u^*}^{i_q} \otimes Y_{v}^i$$

$$= \sum_v Y_{u^*}^{i_q} \otimes \cdots \otimes Y_{u^*}^{i_1} \otimes Y_{v}^i.$$  

Given that $B_1 = (1, \ldots, j_{p(1)}, \ldots, B_q = (j_{p(q-1)} + 1, \ldots, j_{p(q)} = p)$ we have that

$$u^*|B_1 = u(p) u(p-1) \cdots u(p-j_{p(1)}+1),$$

$$u^*|B_2 = u(p-j_{p(1)}) \cdots u(p-j_{p(2)}+1)$$

$$\cdots$$

$$u^*|B_q = u(p-j_{p(q-1)}) \cdots u(1)$$

and thus

$$(u^*|B_q)^* = u(1) \cdots u(p-j_{p(q-1)}) = u(C_1)$$

$$\cdots$$

$$(u^*|B_1)^* = u(p-j_{p(1)}+1) \cdots u(p) = u(C_q).$$

where $C_k = B_{q-k+1}$. If we let $u(k) = p + 1 - k$, we conclude that

$$(s \otimes s) \circ \Delta \circ s(Y_u^i) = \sum_v Y_{u|C_1}^{i_1} \cdots Y_{u|C_q}^{i_q} \otimes Y_v^i = \Delta(Y_u^i).$$

Since $s$ is an antihomomorphism and $\Delta$ is a homomorphism, $(s \otimes s) \circ \Delta \circ s$ is a homomorphism and thus the relation holds for all elements of $H$.

Finally from (31),

$$\sum_{(u)} sS_{(u)} s((Y_u^i)_{(2)})(Y_u^i)_{(1)} = \sum_v s(S_{(v)} (Y_{(v_n \cdots v_1)}^i)) Y_{u_1}^{i_1} \cdots Y_{u_q}^{i_q}$$

$$= s( \sum_v Y_{u^*_v}^{i_q} \cdots Y_{u_1}^{i_1} S_{(v^*)} (Y_{v^*}^i))$$

$$= s( \sum_{(u)} Y_{u^*_v}^{i_q} \cdots Y_{u_1}^{i_1} S_{(v^*)} ((Y_{u^*_v}^i)_{(2)}))$$

$$= s(\varepsilon(Y_{u^*_v}^i)) = \delta_{u^*, v},$$

$\square$
Corollary 6.2. The left Lagrange algebra $\mathcal{L}$ is isomorphic to the $s$-transformed algebra $\mathcal{H}^s$.

Proof. As we remarked in §5, the mapping

$$ S_{\mathcal{H}} : (\mathcal{H}, m^{op}, 1, \Delta, S_{\mathcal{H}}^{-1}, \varepsilon) \to (\mathcal{H}, m, 1, \Delta^{op}, S_{\mathcal{H}}^{-1}, \varepsilon) = \mathcal{L} $$

is a Hopf algebraic isomorphism. \qed

The left Lagrange algebra can also be regarded as the incidence algebra of the reversed interval partitions. For this purpose we define a new ordering of colored segments of $\mathcal{H}$. To be more specific, let us notationally identify the segments $[\tau, \sigma]_\preceq$ with the segments $[\sigma, \tau]_\prec$. The underlying algebra of the Hopf algebra $\mathcal{H}_\preceq$ associated with this reordered system is then identified with the free algebra on the symbols $Y^i_u$ with $|u| \geq 2$. On the other hand the comultiplication is determined by

$$ \Delta_{\preceq}([\tau, \sigma]_\preceq) = \sum_{\tau \preceq \mu \preceq \sigma} [\tau, \mu]_\preceq \otimes [\mu, \sigma]_\preceq = \sum_{\sigma \preceq \mu \preceq \tau} [\mu, \tau] \otimes [\sigma, \mu] = \Delta^{op}([\sigma, \mu]). $$

We have a related involutory anti-isomorphism $t : \mathcal{H} \to \mathcal{H}$ determined by the identity mapping

$$ t : Y^i_u \mapsto Y^i_u \in \mathcal{L}^{op}, $$

or equivalently,

$$ t(Y^{i(1)}_{u_1} \cdots Y^{i(n)}_{u_n}) = Y^{i(n)}_{u_n} \cdots Y^{i(1)}_{u_1}. $$

We have that $t = \alpha s = s \alpha$, where $\alpha : \mathcal{H} \to \mathcal{H}$ is the involutory automorphism determined by $\alpha(Y^i_u) = Y^{\alpha(i)}_{\alpha(u)}$.

We define the right Lagrange algebra $\mathcal{R} = \mathcal{R}^N$ to be the $t$-transformed Hopf algebra, i.e.,

$$ \mathcal{R}^N = (\mathcal{H}^N, m^t, \varepsilon, \Delta^t, \eta, tSt) $$

We note that $m^t$ coincides with $m^{op}$ since $t$ is an antiisomorphism and

$$ \mathcal{R}^N = ((\mathcal{H}^N)^t)^\alpha = (\mathcal{L}^N)^\alpha. $$

Owing to the latter relation, we also refer to $\mathcal{R}^N$ as the reflection of $\mathcal{L}^N$.

7. The Reduced Tree Formulae for the Antipodes

Given a reduced colored tree $T$, each $x \in \mathbf{V}(T)$ determines a generator $Y(x) = Y^j_x$, where $j$ is the color of $x$, and $u = u(1) \cdots u(k)$ are the colors (in order) of its offspring. For any reduced tree $T$ we let $\mathbf{v}(T)$ denote the number of non-leaf vertices in $\mathbf{V}(T)$.

We define

$$ \Lambda_j(T) = \prod_{x \in \mathbf{V}(T)} Y(x) = Y(x_1) \cdots Y(x_r), $$

where $x_1, \ldots, x_r$ are the non-leaf vertices of $T$ (and thus $x_1$ is the root).

We may regard an ordered set of $n$ reduced trees $(T_1, \ldots, T_n)$ as a forest. Letting $T_x$ be a branch (see §3) with root $x$ and offspring indexed by the colored roots $x_1, \ldots, x_n$ of $T_1, \ldots, T_n$,

$$ c_x(T_1, \ldots, T_n) = (T_1, \ldots, T_n) \triangleright T_x $$
is the tree obtained by introducing a new colored root \( x \) with color \( i \), and edges joining each of the roots \( x_j \) (with color \( i_j \)) of \( T_j \) to \( x \). It is evident that with the exception of the unique one layer tree \( T_u^i \in \mathbb{R}_u^i \), every tree \( T \in \mathbb{R}_u^i \) has a unique representation of the form \( T = c_x(T_1, \ldots, T_n) \) with \( T_j \in \mathbb{R}_u^i \).

**Lemma 7.1.** Suppose that we are given an ordered \( n \)-tuple of reduced trees \((T_1, \ldots, T_n)\) \((n \geq 2)\), and that the root \( x_j \) of \( T_j \) has color \( i_j \). Then

\[
\Lambda_j(c_x(T_1, \ldots, T_n)) = Y_{i_1, \ldots, i_n} \Lambda_j(T_n) \ldots \Lambda_j(T_1)
\]

and \( v(c_x(T_1, \ldots, T_n)) = \sum v(T_j) + 1 \).

**Proof.** Let us suppose that the non-leaf vertices of \( T_k \) are given by \( x_{k,1} \leftarrow \ldots \leftarrow x_{k,p_k} \), and thus \( x_{k,1} \) is the root of \( T_k \). The new root \( x \) is not a leaf in the tree \( T = c_x(T_1, \ldots, T_n) \) and we have the ordering

\[
x \leftarrow x_{n,1} \leftarrow x_{n,2} \leftarrow \ldots \leftarrow x_{n,p_n} \leftarrow x_{n-1,1} \leftarrow x_{n-1,2} \leftarrow \ldots \leftarrow x_{1,p_1}.
\]

It follows that

\[
\Lambda_j(T) = Y(x)Y(x_{n,1})Y(x_{n,2}) \ldots Y(x_{n-1,1}) \ldots Y(x_{1,p_1})
\]

\[
= Y_{i_1, \ldots, i_n} \Lambda_j(T_n) \ldots \Lambda_j(T_1).
\]

The second relation is immediate. \( \square \)

**Theorem 7.2.** The left Lagrange antipode is given by

\[
S_L(Y_u^i) = \sum_{T \in \mathbb{R}_u^i} (-1)^v(T) \Lambda_j(T).
\]

**Proof.** Since \( S_L \) satisfies \( S_L(1) = 1 \), and it is an antihomomorphism, this relation indeed determines \( S_L \) on \( \mathcal{L} \). We use the recursive characterization (5.5) for \( S_L = S_{\mathcal{L}}^{-1} \). If \( u = jk \), \( \mathbb{R}_u^i \) contains only the branch \( T = T_j^k \) and \( \Lambda(T_j^k) = Y_{jk}^i \). We have from (5.5)

\[
S_{\mathcal{L}}^{-1}(Y_{jk}^i) = -Y_{jk}^i = (-1)^v(T) \Lambda_j(T),
\]

which coincides with the right side of (7.3).

Let us suppose that the formula is true for \( \rho(u) = p - 1 \). From (5.5), if \( u = u(1) \ldots u(p+1) \) (and thus \( \rho(u) \leq p \))

\[
S_{\mathcal{L}}^{-1}(Y_u^i) = -Y_u^i - \sum' (Y_u^i)_{(2)} S_{\mathcal{L}}^{-1}((Y_u^i)_{(1)})
\]
where the prime indicates that we are considering sums of terms \( b_k S_{\gamma k}^{-1}(a_k) \) with \( a(k), b(k) \in \mathcal{H}_+ \). It follows from [22] that

\[
S_{\gamma k}^{-1}(Y_u^i) = -Y_u^i - \sum_{q=2,\pi,w}^{p-1} Y_u^i S_{\gamma k}^{-1}(Y_{u_1}^w \cdots Y_{u_q}^w)
\]

\[
= -Y_u^i - \sum_{q=2,\pi,w}^{p-1} Y_u^i S_{\gamma k}^{-1}(Y_{u_1}^w) \cdots S_{\gamma k}^{-1}(Y_{u_1}^w)
\]

\[
= -Y_u^i - \sum_{q=2,\pi,w}^{p-1} \sum_{T_k \in RT_u^{w(k)}} (-1)^{\nu(T_k)} Y_u^i \Lambda_\gamma(T_q) \cdots \Lambda_\gamma(T_1)
\]

\[
= -Y_u^i + \sum_{T \in RT_u^i \setminus \{T_u^i\}} (-1)^{\nu(T)} \Lambda_\gamma(T)
\]

where the one layer tree \( T_u^i \in RT_u^i \) is not assembled from non-trivial reduced subtrees, and on the other hand, a reduced tree cannot have the form \( c_x(T_1) \), i.e., \( q > 1 \). Since \( \Lambda_\gamma(T_u^i) = Y_u^i \) and \( v(T_u^i) = 1 \),

\[
S_\gamma(Y_u^i) = S_{\gamma k}^{-1}(Y_u^i) = \sum_{T \in RT_u^i} (-1)^{\nu(T)} \Lambda_\gamma(T).
\]

\[\square\]

If \( y_1^i \; y_2^i \; \ldots \; y_r^i \) are the non-degenerate vertices in \( T \) (and thus \( y_r^i \) is the root), we define

\[\Lambda -1(T) = Y(y_1) \cdots Y(y_r)\]

Given \( T \in RT_u^i \), then \( T^* \in RT_u^i \), and this determines a one-to-one correspondence between these two families of colored ordered trees.

**Corollary 7.3.** The antipode of the Hopf interval algebra is determined by

\[
S_C(Y_u^i) = \sum_{T \in RT_u^i} (-1)^{\nu(T)} \Lambda -1(T)
\]

**Proof.** From Lemma 5, \( S_C = sS_\gamma s \), and since \( v(T^*) = v(T) \),

\[
S_C(Y_u^i) = s(S_\gamma(Y_u^i)) = s \sum_{T \in RT_u^i} (-1)^{\nu(T)} \Lambda_\gamma(T) = \sum_{T \in RT_u^i} (-1)^{\nu(T)} s \Lambda_\gamma(T^*).
\]

If \( y_1^i \; y_2^i \; \ldots \; y_r^i \) are the non-degenerate vertices in \( T \), then \( y_1^i \; \gamma^i \; \ldots \; \gamma^i \; y_r^i \) in \( T^* \), or changing notation, \( y_r^i \; \gamma^i \; \ldots \; \gamma^i \; y_1^i \). It follows that

\[
s \Lambda_\gamma(T^*) = sY(y_r^i) \cdots Y(y_1^i) = Y(y_1) \cdots Y(y_r) = \Lambda -1(T),
\]

and the desired formula follows. \[\square\]
If $z_1, z_2, \ldots, z_r$ are the non-degenerate vertices of $T$ we define $\Lambda_{L_i}(T) = Y(z_1) \cdots Y(z_r)$.

**Corollary 7.4.** The antipode of the right Lagrange algebra $R$ is given by

$$S_R(Y_i^u) = \sum_{T \in R \Gamma_{i}^{u}} (-1)^{v(T)} \Lambda_{L_i}(T).$$

**Proof.** From the definition of the reflected algebra $R$, $S_R = tS_Ht$, and thus

$$S_R(Y_i^u) = t(S_H(Y_i^u)) = \sum_{T \in R \Gamma_{i}^{u}} (-1)^{v(T)} \Lambda_1(T) = \sum_{T \in R \Gamma_{i}^{u}} (-1)^{v(T)} t(\Lambda_1(T))$$

If $z_1, z_2, \ldots, z_r$ are the non-degenerate vertices of $T$ then $z_r \cdots z_2 z_1$ implies that

$$t(\Lambda_1(T)) = t(Y(z_r) \cdots Y(z_1)) = Y(z_1) \cdots Y(z_r) = \Lambda_{L_i}(T)$$

and we are done. \( \Box \)

8. Cancellations in the breadth first antipodal formula

Given $i \in \langle N \rangle$ and $v \in \langle N \rangle^*$, we indexed the summands of $S_H(Y_i^u)$ by trees in $R \Gamma_{i}^{u}$, the layered trees with root and leaf colorings $i$ and $u$ (see (6.5)). We wish to show that it suffices to use “order reduced” simple trees.

It would be tempting to attempt to use (7.3) to obtain a formula with reduced trees by simply contracting the edges in layered trees, keeping track of the resulting “multiplicities”. In the commutative situation considered by Haiman and Schmitt, one has that $\Omega(\rho(T)) = \Omega(T)$ and thus in the formula for the antipode one may collect all the terms with the same reduced tree into a multiple of $\Omega(\rho(T))$. The coefficient is a sum of positive and negative 1’s, and using a combinatorial argument they showed that all of the non-reduced trees cancel.

In our situation, arbitrary contractions can disturb the $x \ll y$ ordering on the non-degenerate leaves, and thus one need not have that $\Omega(\rho(T)) = \Omega(T)$. This can be seen in the third tree of the diagram below, which was disordered by an “improper” contraction. One must therefore use only contractions which are $\ll$ order preserving. In our reduction we will also modify the contraction so that the tree remains layered.

Let us suppose that $T$ is a layered tree and that $x$ is a non-degenerate vertex in $T$. We say that $T$ is order contractible at $x$ if

a) its parent $x'$ is unary, i.e., $x$ has no siblings,

b) there does not exist a non-degenerate vertex to the right of $x$

c) there does not exist a non-degenerate vertex to the left of $x'$.

If $x$ is a vertex in the $k$-th row which satisfies these conditions, the order contraction $\kappa(T)$$ = \kappa_x(T)$ is the layered tree obtained in the following manner:

1) move $x$ to the position of its parent in the $(k - 1)$-st row,
2) attach each offspring $y$ of $x$ by a single line to a unary vertex $x'$ in the $k$-th row,
3) leave all other vertices and edges alone,
4) if there are no other non-degenerate vertices in the $k$-th row, delete it.

Conditions a)-c) guarantee that the $\ll$ ordering on the non-degenerate vertices is preserved. Thus the contraction on the non-degenerate vertex $x_2$ in the first tree below is allowed. On the other hand contracting on the vertex $x_1$ would transpose the $\ll$ ordering for the two non-degenerate vertices $x_1$ and $x_2$.

Since we will not consider general contractions in this section, we will simply use the terms contractible and contractions for the corresponding order preserving notions.

Given a tree $T \in \mathbf{LT}^+_j$ we define the canonical expansion $\Phi(T) \in \mathbf{LT}^+_j$ as follows. If $T$ is simple we let $\Phi(T) = T$. If $T$ is not simple, let $x_r$ be the first non-simple non-degenerate vertex in the $\ll$ ordering, and let us suppose that it is on the $k$-th level. We introduce a new level $L$ between the $k$-th and $(k+1)$-st levels in the following manner.

a) We move $x_r$ down to the level $L$ and we connect it to a new unary vertex $x'_r$ on the $k$-th level, and to the offspring of $x_r$ on the $(k+1)$-st level,

b) If $y$ is a vertex on the $(k+1)$-st that is not an offspring of $x_r$, we connect it by a single edge to a new unary vertex $y'$ on the level $L$, which we then connect to the parent of $y$ on the $k$-th level.

We define $\Phi(T)$ to be the new tree.

It should be noted that since there are no non-degenerate vertices to the left of $x_r$, this operation will not affect the $\ll$ ordering on the non-degenerate vertices. It also preserves the orderings $\rightarrow$ and $\leftarrow$ on the non-degenerate vertices, as is evident from the above diagram.

We have that $x_r$ is a contractible vertex in $\Phi(T)$ because all the other vertices on the new level are unary, and there are no non-degenerate vertices to the left of $x'_r$. If one contracts on this vertex, the new level will contain only unary vertices, and thus will itself be deleted (see the primed row in the right tree below). In this manner we see that if we contract $\Phi(T)$ at the vertex $x_r$, we recover $T$. This is
illustrated in the following diagram, in which $e = x_{r-1}$ and $e' = x'_{r-1}$.

Let us suppose that $x_1 \ll \ldots \ll x_p$ are the non-degenerate vertices of a tree $T$. Turning to the breadth first ordering, there is a unique sequence of indices $n_1 < \ldots < n_q$ with

\begin{align*}
\ldots x_{n_1} \approx x_{n_1+1} \approx x_{n_1+2} \approx \ldots \approx x_{n_2} \approx x_{n_2+1} \ldots
\end{align*}

where we let $x \approx y$ if $x$ is a descendant of $y$. We call a maximal sequence of the form $x_{n_h-1} \approx x_{n_h} \approx x_{n_h+2} \approx \ldots \approx x_{n_h}$ an irreducible string and we say that $x_{n_h}$ is its right end.

**Lemma 8.1.** Suppose that $T$ is an arbitrary tree in $\text{LT}^i_u$ with its non-degenerate vertices $x_1 \ll \ldots \ll x_p$ satisfying (\ref{eq:order}).

(i) Any contractible vertex in $T$ is a right end of an irreducible string.

(ii) If $T = E$ is simple then all of its right ends without siblings are contractible.

(iii) If one has $y \ll x$ in $T$ and both $y$ and $x$ are contractible, then after a contraction at $x$, $y$ will still be contractible.

**Proof.**

(i) Let us suppose that $x = x_j$ is a contractible non-degenerate vertex in $T$ on level $k$. Then its parent $x'_j$ is unary and there are no non-degenerate vertices to the left of $x'_j$. Since every level is assumed to have a non-degenerate vertex, $x_{j+1}$ must lie on the $(k - 1)$-st row of $E$ to the right of $x'_j$. It follows that $x_j \approx x_{j+1}$, and thus $x_j = x_{n_h}$ for some $h$.

(ii) Let us suppose that $x_{n_h}$ is a non-degenerate vertex on the $k$-th level. There are no non-degenerate vertices to the right of $x_{n_h}$ on its level since the level is simple. We have that $x_{n_h+1}$ must lie on the previous level. Since $x_{n_h} \approx x_{n_h+1}$, the parent $x'_{n_h}$ of $x_{n_h}$ lies to the left of $x_{n_h+1}$, and $x_{n_h}$ is (order) contractible.

(iii) If $x$ is simple, then the contraction at $x$ will simply raise the level of each vertex $y$ with $y \ll x$. If $x$ is not simple, then $x$ is the only non-degenerate vertex that is affected. Since $y$ is assumed contractible, there will not be any vertices to the left of it on its level. On the other hand if $y$ is on level $k + 1$, then by the same assumption, $x$ must lie to the right of the parent $y'$. This will still be the case when one contracts $x$ to a higher level. \hfill \Box

For each simple layered tree $E$ we let $\mathbf{T}_E$ be all the trees $T \in \text{LT}^i_u$ with $E = \Phi^n(T)$ for some $n$. It is evident that if $T$ has $n$ vertices and $k$ levels, then $E = \Phi^{n-k}(T)$ is a simple tree, and reversing the expansions as above, $T$ can be obtained by a particular sequence of contractions of $E$. More precisely, let $y_1 \ll \ldots \ll y_q$ be the right vertices of $T$ (or equivalently of $E$). We have that there is a subsequence $y_{m_1} \ll \ldots \ll y_{m_p}$ with

\begin{align*}
T = \kappa_{y_{m_1}} \ldots \kappa_{y_{m_p}}(E).
\end{align*}

Conversely given any such sequence, the subsequent right vertices remain contractible as one proceeds, and we get a corresponding tree $T$. The tree $T$ uniquely determines the sequence $y_{m_1} \ll \ldots \ll y_{m_p}$, since the latter are by definition the non-simple non-degenerate vertices of $T$ in their given $\ll$ order.
We say that a layered tree is **order reduced** if it does not have any non-trivial ordered contractions, and we let $\text{OST}^j_u$ be the set of all ordered reduced simple trees.

**Theorem 8.2.** The antipode in $\mathcal{H}$ is given by

$$S_{\mathcal{H}}(Y_j^v) = \sum_{E \in \text{OST}^j_u} (-1)^{\ell(E)}\Omega(E)$$

where $\ell(E)$ is the number of layers in $E$.

**Proof.** It is evident that

$$\text{LT}^j_u = \bigsqcup \{ T_E : E \in \text{ST}^j_u \}$$

and that for any $T \in T_E$ we have that $\Omega(T) = \Omega(E)$. Thus it suffices to show that if $E$ has contractions, then

$$\sum_{T \in T_E} (-1)^{\ell(T)}\Omega(T) = \Omega(E) \sum_{T \in T_E} (-1)^{\ell(T)} = 0.$$

From our earlier discussion, $T_E$ is in one-to-one correspondence with the sequences $y_{m_1} \ll \ldots \ll y_{m_p}$ drawn from the $q$ right vertices in $E$, or equivalently subsets drawn from $1, \ldots, q$. If the simple tree $E$ has $n$ non-degenerate vertices and thus $n$ levels, the tree $T(m_1, \ldots, m_p)$ has $n - p$ levels. There will be $\binom{q}{p}$ such sequence and thus

$$\sum_{T \in T_E} (-1)^{\ell(T)} = (-1)^{n-q} \sum_{p} \binom{q}{p} (-1)^{q-p} = (-1)^{n-q}(1 - 1)^q = 0,$$

and we have proved the desired result. \hfill \Box

9. **Breadth first and depth first duality**

There is a natural one-to-one correspondence between the reduced trees $\text{RT}^j_u$ and the order-reduced simple layered trees $\text{OST}^j_u$. Since the colorings as well as the set of leaves is unaffected by the operations, we will use the notations $T$, $\text{RT}$ and $\text{OST}$ for the proper trees, the reduced trees, and the order-reduced simple layered $N$-colored trees, respectively, with a given colored root and a given set of colored leaves. We let $\rho : \text{OST} \to \text{RT}$ be the contraction mapping described in §3.

We say that a vertex $x$ in a proper tree $T$ is **weakly contractible** if it has no siblings, i.e., its parent $x'$ is unary. We say that $T$ is **reduced** let $\rho : T \to \text{RT}$ be the contraction mapping. In this situation we say that any vertex without siblings (this applies to leaves as well) is said to be contractible. If $\rho(T') = T$, we may identify the non-leaf vertices of $T$ with the non-degenerate vertices in $T'$. Although we have seen that $\rho$ can disrupt the ordering $\ll$, it is evident that the depth first ordering $\prec$ is unaffected.

**Theorem 9.1.** Let $T$ be a layered tree.

(a) The breadth first ordering $\ll$ coincides with the depth first ordering $\prec$ on the non-degenerate vertices of $T$ if and only if $T \in \text{OST}$.

(b) The contraction mapping $\rho$ is a bijection of $\text{OST}$ onto $\text{RT}$.

(c) If $T'$ is the unique tree in $\text{OST}$ with $\rho(T') = T$, then $\Omega(T') = \Lambda^{-1}(T)$ and $\ell(T') = v(T)$. 

Proof. (a) Since these orderings are inverse to each other on the vertices in a given row, there will be only one non-degenerate vertex in that row, i.e., $T$ is simple. Thus it suffices to prove the equivalence for simple trees.

Let us suppose that the non-degenerate vertices of $T$ are given by $x_1 ≪ \ldots ≪ x_n$. Since $T$ is simple, these vertices will lie on the successively higher levels of $T$. A non-degenerate non-root vertex $x_j$ is non-order contractible if and only if either its parent $x_j'$ is non-degenerate, and thus coincides with $x_{j+1}$, or $x_{j+1}$ lies to the left of $x_j'$. These are precisely the conditions that $x_j \inter x_{j+1}$.

(b) We define $\Psi : RT \to OST$ by sequentially inserting singular edges. Let us suppose that $y_1 \inter \ldots \inter y_n$ are the non-degenerate vertices in $T$ (see the diagram below). Counting down from $n$, let us suppose that $i$ is the last index for which $y_{i-1} ≫ y_i$ (this would be $y_3 ≫ y_4$ in the diagram). Then we lower $y_{i-1}$ and the subtree from which it is a root by inserting the minimal number of unary vertices and singular edges so that the resulting vertex $y'_{i-1}$ satisfies $y_{i-1} ≪ y_i$. If the resulting tree is not proper, we eliminate the redundant rows having only unary vertices. Relabelling, we may assume that $y_{i-1} ≪ y_i$. By using singular edges to push down leaves to the bottom level, we obtain a layered tree. Furthermore, by eliminating “redundant” levels, we may assume that the layered tree is proper. If $y_{i-1} ≫ y_i$ is the next occurrence of the relation $≫$ in the sequence, we again “lower” the vertex $y_{i-1}$, and we proceed as before. This will not affect the fact that $y_{i-1} ≪ y_i$ and lowering does not change the $\inter$ relation. It should be pointed out that “lowering” will in general disrupt the ordering in general, but we are only concerned with successive terms in the given sequence. After “correcting” all of the reverse orderings, we obtain $T' = \Psi(T)$. It is evident that $T'$ satisfies the conditions in (a), and thus lies in $OST$.

The mapping $\Psi$ is illustrated in the following diagram, in which one initially has $y_1 \inter y_2 \inter y_3 \inter y_4 \inter y_5$ in both trees, $y_1 ≪ y_2 ≫ y_3 ≫ y_4 ≪ y_5$ in $T$, and $y_1 ≪ y_2 ≪ y_3 ≲ y_4 ≪ y_5$ in $T'$,

It is evident that $\rho(\Psi(T)) = T$, and thus $\rho$ is surjective.

Let us suppose that $T'$ is any layered tree with $\rho(T') = T \in RT$. As can be see from the above diagrams (this does not depend upon $T'$ being order reduced), $T'$ may be constructed by inserting $d(v)$ singular edges between each vertex $v$ and its parent $v'$ i.e., $T'$ is characterized by the function $d : OST \to \mathbb{N} \cup \{0\}$. For our purposes it is not necessary to characterize the functions $d$ that arise in this manner.

Let us suppose $T' \in OST$ and that $x_1 \inter \ldots \inter x_n$ are the non-degenerate (i.e., non-leaf) vertices of the reduced tree $T$. Then $x_1 \inter \ldots \inter x_n$ are the non-degenerate vertices of $T'$, and from (a), $x_1 ≪ \ldots ≪ x_n$. Since $T'$ is simple, each is on a different level, i.e., they reside on the successive levels upwards.

Let us suppose that $x = x_i$ is a nonleaf vertex in $T$ with parent $x' = x_j$ (recall that all the non-leaf vertices in $T$ are non-degenerate). Since $x_i$ is an offspring of $x_j$,
and $x_i \cdot \mathbf{1} x_j$ and thus $i < j$. It follows that in $T'$ there are $j - i$ levels between them. It hence $d(x) = j - i - 1$. On the other hand if $x$ is a leaf, then $d(x)$ is just the number of levels between $x$ and the bottom level (there are precisely $n$ levels in $T'$). Thus if $x$ is a leaf on the $k$-th level of $T$, then $d(x) = n - k$. Thus the function $d$ and the tree $T'$ are uniquely determined by the orderings $\ll$ and $\preceq$ on the non-degenerate vertices in $T$.

(c) Letting $x_1 \ll \ldots \ll x_n$ be the non-leaf vertices in $T$, we have that $x_1 \ll \ldots \ll x_n$ are the non-degenerate vertices in $T'$, and thus

$$\Omega(T') = Y(x_1) \ldots Y(x_n) = \Lambda'(T).$$

The number $\ell(T')$ of layers in $T'$ is equal to the number of non-degenerate vertices in $T'$, and thus the number $\nu(T)$ of non-degenerate vertices in $T$. \hfill \square

If one uses the above result, Corollary \text[6.2]{} is an immediate consequence of Theorem \text[6.5]{}. In this sense, the Haiman-Schmitt approach to the reduced formula for $C$ is obtained by taking a colored ordered interval partition $(((1, \ldots, p), j_1 \ldots j_q))$ of $((1, \ldots, p), w)$. We wish to compute the coefficients of substituted series. For this purpose we consider the coefficient $h_w$ of $z_w$. A typical summand of $h_w$ where $w = w(1) \ldots w(p)$ is obtained by taking a colored ordered interval partition $((C_1, \ldots, C_q), j_1 \ldots j_q)$ of $((1, \ldots, p), w)$. If $C_1 = (1, \ldots, p_1)$ then

$$g_{w(1) \ldots w(p_1)}^{j_1} z_{w(1)} \ldots z_{w(p_1)} = g_{w|C_1}^{j_1} z_w^{|C_1|},$$

where $z_w^{|C_1|}$ is the product of the $z_w$ in $w$ corresponding to the vertices in $C_1$.
and we have corresponding factors for $C_2, \ldots, C_q$. The relevant summand of $h_w$ is given by

$$f_{j_1 \ldots j_q} g_{w[C_1]}^{j_1} \cdots g_{w[C_q]}^{j_q}.$$ 

We conclude that

$$h_w = \sum_{q} \sum_{\pi=(C_k) \in \mathcal{Y}_q([p])} f_{j_1 \ldots j_q} g_{w[C_1]}^{j_1} \cdots g_{w[C_q]}^{j_q}.$$ 

More generally we may substitute $G$ into an $N$-tuple $F(z) = (F^1(z), \ldots, F^n(z))$, obtaining $H = F \circ G$, where

$$h_w' = \sum f_{j_1}^i g_{w[C_1]}^{j_1} \cdots g_{w[C_q]}^{j_q},$$

where we sum over all interval colored partitions

$$((C_1, \ldots, C_q), j_1 \cdots j_q) \ 1 \leq q \leq p$$

of the colored set $([p], u(1) \cdots u(p))$.

Given an algebra $A$, we let $\mathcal{S}_N^{dif}(A)$ denote the set of power series $F = F(z)$ with

$$F^j(z) = z_j + \sum_{|u| \geq 2} f_u^j z_u, \ \ (f_u^j \in A)$$

i.e., without constant terms and with $f^j_u = \delta^j_u$. Substitution of $G$ into $F$ provides us with a non-associative product $(F, G) \mapsto F \circ G$ on $\mathcal{S}_N^{dif}(A)$. From above,

$$(10.1) \quad (F \circ G)_u^j = z_i + \sum f_{wu}^{j} g_{w[C]}^{w(1)} \cdots g_{w[C]}^{w(q)}$$

where we sum over all interval colored partitions

$$((C_1, \ldots, C_q), w(1) \cdots w(q)) \ 1 \leq q \leq p$$

of the colored set $([p], u(1) \cdots u(p))$.

Each generator $Y_{u}^{i} \in \mathcal{L}^N (|u| > 1)$ may be used to select a corresponding coefficient $f_{u}^{i}$ in a power series $F(z)$. To be more precise, we define a linear mapping

$$\theta(Y_{u}^{i}) : \mathcal{S}_N^{dif}(A) \rightarrow A$$

by letting $\theta(Y_{u}^{i})(F) = f_{u}^{i}$. Since $\mathcal{L}^N$ is the free algebra on these generators, we extend this to the basis elements $Y_{u_1}^{i_1} \cdots Y_{u_q}^{i_q}$ by letting

$$\theta(Y_{u_1}^{i_1} \cdots Y_{u_q}^{i_q}) : \mathcal{S}_N^{dif}(A) \rightarrow A : F \mapsto f_{u_1}^{i_1} \cdots f_{u_q}^{i_q}$$

Extending linearly, we have a corresponding homomorphism

$$\theta : \mathcal{L}^N \rightarrow \text{Lin}(\mathcal{S}_N^{dif}(A), A)$$

and thus a bilinear mapping

$$\langle, \rangle : \mathcal{L}^N \times \mathcal{S}_N^{dif} \rightarrow A : (a, F) \mapsto \theta(a)(F).$$

From our definitions we have that

$$\langle ab, f \rangle = \langle a, f \rangle \langle b, f \rangle = m_A \langle a \otimes b, f \otimes f \rangle.$$
Theorem 10.1. Given $F \in S_N^{df}(A)$, the left substitutional inverse of $F$ is given by the power series $G(z)$, where $g^i = \langle S_{\mathcal{L}}(Y_{u}^i), f \rangle$ where $S_{\mathcal{L}} = sS_{2N}S$ is the antipode of the left Lagrange Hopf algebra $\mathcal{L}^N$. The right substitutional inverse of $F$ is given by $H(z)$, where $h^i = \langle S_{\mathcal{R}}(Y_{u}^i), f \rangle$, and $S_{\mathcal{R}} = tS_{2N}t$ is the antipode of the right Legendre Hopf algebra $\mathcal{R}^N$.

Proof. Let $m_A$ denote the multiplication in $A$. Defining $G$ as above, we have that

$$
\langle Y_u^i, G \circ F \rangle = m_A(\Delta_{A}(Y_u^i), G \otimes F)
$$

$$
= m_A(\sum(Y_i^u \otimes Y_{w(1)}^{w(s)}Y_{w(s)}^{u|C_s}, G \otimes F))
$$

$$
= \sum(\langle S_{-1}^{-1}(Y_u^i), F \rangle Y_{w(1)}^{w(1)} \cdots Y_{w(s)}^{u|C_s}, F)
$$

$$
= \sum(\langle S_{-1}^{-1}((Y_u^i(2)), Y_u^i(1)), F \rangle)
$$

$$
= \langle \varepsilon(Y_u^i)(1), F \rangle
$$

and thus $F$ is the left substitutional inverse of $G$.

On the other hand, if $H$ is defined as above, then using the fact that $tS$ is an algebraic homomorphism and that $t(Y_u^i) = Y_u^i$,

$$
\langle Y_u^i, F \circ H \rangle = m_A(\Delta_{A}(Y_u^i), F \otimes H)
$$

$$
= m_A(\sum(Y_i^u \otimes Y_{w(1)}^{w(s)}Y_{w(s)}^{u|C_s}, F \otimes H))
$$

$$
= m_A(\sum((Y_i^u \otimes (tS)(Y_{w(1)}^{w(1)} \cdots Y_{w(s)}^{u|C_s})), F \otimes F))
$$

$$
= \langle (tS)(Y_{w(1)}^{w(1)} \cdots Y_{w(s)}^{u|C_s}Y_u^i), F \rangle
$$

$$
= \langle (tS)(Y_u^i(1)), Y_u^i(2) \rangle
$$

$$
= \langle \varepsilon(Y_u^i)(1), F \rangle
$$

and $H$ is the right substitutional inverse of $F$. \qed

Corollary 10.2. If the number of variables $N$ is greater than 1, then the left and right substitutional inverses of a power series are generally distinct.

Proof. It suffices to show that

$$
s \circ S_{2N} \circ s(Y_{1234}^1) \neq t \circ S_{2N} \circ t(Y_{1234}^1)
$$

In the following calculation we have used boldface subscripts to indicate corresponding terms that equal. The bracketed terms cancel (these correspond to the unique order preserving contraction). The sums are over the set of colors, i.e., $k, \ell = 1, 2, 3, 4$ and they involve 13 varieties of layered trees.

$$
S(Y_{1234}^1) = -Y_{1234}^1 + \sum Y_{123}^k Y_{4k}^1 + \sum Y_{234}^k Y_{1k}^1 + \sum Y_{12}^k Y_{3k}^1
$$

$$
+ \sum Y_{23}^k Y_{1k}^1 + \sum Y_{34}^k Y_{1k}^1 + \sum Y_{12}^k Y_{34}^l Y_{4k}^1 - \sum Y_{34}^k Y_{12}^l Y_{4k}^1 - \sum Y_{12}^k Y_{34}^l Y_{1k}^1 - \sum Y_{34}^k Y_{12}^l Y_{1k}^1
$$

- \sum Y_{12}^k Y_{3k}^1 Y_{4l}^1 - \sum Y_{23}^k Y_{1k}^1 Y_{4l}^1 - \sum Y_{23}^k Y_{3k}^1 Y_{1l}^1 - \sum Y_{34}^k Y_{12}^l Y_{1l}^1 - \sum Y_{34}^k Y_{12}^l Y_{1l}^1
\[ sSs(Y_{1234}^{1}) = -Y_{1234}^{1} + p \sum_{k \neq 1} Y_{1k}^{1}V_{k34}^{1} + q \sum_{k \neq 1} Y_{1k}^{1}V_{134}^{1} + a \sum Y_{1k}^{1}Y_{k34}^{1} + b \sum Y_{1k}^{1}V_{k23}^{1} + c \sum Y_{1k}^{1}Y_{k34}^{1} + e \sum_{k \neq 1} Y_{1k}^{1}V_{134}^{1} - \sum_{k \neq 1} Y_{1k}^{1}V_{k23}^{1} \]

\[ t(Y_{1234}^{1}) = Y_{4321}^{1} \]

\[ St(Y_{1234}^{1}) = S(Y_{1234}^{1}) - Y_{1234}^{1} + \sum Y_{234}^{1}Y_{1k}^{1} + \sum Y_{324}^{1}Y_{1k}^{1} + \sum Y_{432}^{1}Y_{1k}^{1} + \sum Y_{432}^{1}Y_{1k}^{1} \]

\[ tSt(Y_{1234}^{1}) = -Y_{1234}^{1} + q \sum Y_{1k}^{1}V_{k34}^{1} + p \sum Y_{1k}^{1}V_{k23}^{1} + c \sum Y_{1k}^{1}V_{k34}^{1} + b \sum Y_{1k}^{1}V_{k23}^{1} + a \sum Y_{1k}^{1}Y_{k34}^{1} + e \sum Y_{1k}^{1}Y_{k34}^{1} - \sum Y_{1k}^{1}Y_{k34}^{1} \]

It follows that

\[ sSs(Y_{1234}^{1}) - tSt(Y_{1234}^{1}) = - \sum Y_{1k}^{1}V_{k23}^{1} - \sum Y_{1k}^{1}V_{k34}^{1} - \sum Y_{1k}^{1}Y_{k34}^{1} \]

In particular, one can check that if \( a \) and \( b \) and \( x \) and \( y \) do not commute, the substitional left and right inverses of the two-variable polynomial function

\[ u = x + ax^2 + by^2 \]

\[ v = y \]

do not agree in the fourth order terms.

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