Abstract: The co-prime array is a sub-Nyquist acquisition scheme for the estimation of second order statistics. It cannot generate all the difference values in the co-prime range and hence, one of the sub-array is extended to enable the estimation of the second order statistics at each difference value in the co-prime range. Recently, the difference set for the prototype co-prime array was studied and low latency temporal spectrum estimation was demonstrated. In this paper, we develop the fundamentals of the difference set for the extended co-prime array, also known as the conventional co-prime array, and provide expressions for the degrees of freedom and the range of continuous difference values. The closed-form expression is developed for the number of sample pairs that contribute to estimate the second order statistics (weight function). The closed-form expression of the bias window for correlogram spectral estimation is also derived. We show that the choice of $M \approx \frac{N}{d}$ generates a bias function with a large relative amplitude between the main lobe and side-lobe peaks, where $M$ and $N$ are co-prime pairs. Simulation results demonstrate low latency spectrum estimation using the correlogram method.

1 Introduction

Co-prime sensing scheme provides a framework for sub-Nyquist acquisition and estimation of the second order statistics of a signal [11]. It has several applications which include direction of arrival estimation [2–5], power spectrum estimation [6, 7], system identification [8], beamforming [9], cross-correlation estimation, time-delay, range, velocity, and acceleration estimation [12].

A detailed analysis of the difference set of the prototype co-prime array is provided in [13]. The spectral estimation using the correlogram method was demonstrated with low latency (around 10 snapshots) using the combined difference set. The closed-form expressions for the weight function and bias window were also developed.

However, the co-prime array cannot generate a difference set containing all the elements in the co-prime range $[-MN + 1, MN - 1]$ and hence, an extended co-prime array was proposed in [14]. It has one of the sub-arrays extended by an additional co-prime period. The extended co-prime array generates all the values in the co-prime range but this range does not represent the largest continuous range.

The weight function of the extended co-prime array is given in Table IV of [15] for $M = 4$ and $N = 5$, but does not provide a general expression for arbitrary values of $M$ and $N$. This paper develops the fundamentals of the difference set for the extended (also known as conventional) co-prime array and demonstrates low latency estimation. The closed-form expressions for the weight function and the bias window for the correlogram method are also provided.

2 Extended Co-Prime Concept

An extended co-prime array is shown in Fig. 2 where $M$ and $N$ are co-prime. The Nyquist distance between the antennas is denoted by $d$. It has two sub-arrays and the antenna locations are given by:

- \[ P1 = \{Mnd, 0 \leq n \leq N - 1\} \]
- \[ P2 = \{Nmd, 0 \leq m \leq 2M - 1\} \]

Note that the antenna at the zeroth location is common to both the sub-arrays. In this paper, we assume that the second array with inter-element spacing of $Nd$ is extended. However, the first array could have been extended instead; without affecting the concept that governs this scheme. In the temporal or spatial domain, one prototype co-prime period is $[0, MN - 1]$ while the extended co-prime period is $[0, 2MN - 1]$. It may be noted that the co-prime and extended co-prime range corresponding to one period for autocorrelation (or other second order statistics) is $[-(MN - 1), MN - 1]$ and $[-(2MN - 1), 2MN - 1]$ respectively. Fig. 2 describes this concept from a sampling perspective. Here $d$ represents the Nyquist sampling period and the structure implies that one of the sampler is turned off every alternate co-prime period, while the other sampler uniformly samples the incoming signal. Every extended co-prime period is referred to as a snapshot. The second order statistics (e.g. autocorrelation, power spectrum, etc.) is estimated for each snapshot. Averaging across several snapshots improves the estimate. In this paper, we demonstrate the results based on the sampling framework. Some of the contributions of this paper are:

1. We analyze the difference sets (self and cross differences) for the extended co-prime scheme.
2. Describe the number of unique difference values (degrees of freedom) and continuous range.
3. Develop the closed-form expression for the weight function for the entire difference set.
4. Develop the closed-form expression for the bias window that distorts the correlogram spectral estimate.

Fig. 1: Extended co-prime structure
This is also true for the mirrored sets $\mathcal{L}^+_C$ and $\mathcal{L}^-_C$. The sets $\mathcal{L}^+_S$ and $\mathcal{L}^-_S$ have $2(M + N - 1)$ unique differences, hence the union set $\mathcal{L}^+_S \cup \mathcal{L}^-_S$ has $2(2M + N - 1) - 1$ unique differences. The difference value of zero is common to each set and needs to be considered only once in the union set. This justifies the negation of one in the equation for unique differences. The set $\mathcal{L}^+_C$ has $2MN$ unique differences. This is also true for the mirrored set $\mathcal{L}^-_C$. Refer Fig. 4 for the discussion on the cross-difference set. As in the case of the prototype co-prime array, the self differences of the extended array are a subset of the cross differences. However, the number of unique differences in the union set $\mathcal{L}^+_C$ is not trivial. Hence, to provide a better understanding we define two new sets $\mathcal{L}^+_A$ and $\mathcal{L}^+_B$ with $\mathcal{L}^+_A$ and $\mathcal{L}^+_B$ representing their mirrored counterparts:

$$\mathcal{L}^+_A = \{l_c | l_c = Mn - Nm, n \in [0, N - 1], m \in [0, M - 1]\}$$

$$\mathcal{L}^+_B = \{l_c | l_c = Mm - Nm, n \in [0, N - 1], m \in [M, 2M - 1]\}$$

It is obvious that $\mathcal{L}^+_A$ is same as the cross-difference set of the prototype co-prime array. Hence, the properties of the cross difference set of a prototype co-prime array hold true even for the set $\mathcal{L}^+_C$.

For the extended co-prime array, $\mathcal{L}^+_A = \mathcal{L}^+_A \cup \mathcal{L}^+_B$ and $\mathcal{L}^+_C = \mathcal{L}^+_A \cup \mathcal{L}^+_B$. Since the set $\mathcal{L}^+_C$ is well understood, the focus here would be to understand the set $\mathcal{L}^+_B$. Some of the properties of the set $\mathcal{L}^+_B$ are given below:

**Property 1.**

1. $\{l_c < 0, \forall l_c \in \mathcal{L}^+_A\}$ and $\{l_c > 0, \forall l_c \in \mathcal{L}^+_B\}$
2. $\mathcal{L}^+_S = \{0\} \subseteq \mathcal{L}^+_B$ and $\mathcal{L}^+_S = \{0\} \subseteq \mathcal{L}^+_B$
3. $\{l_c \in \mathcal{L}^+_S, \forall m \in [M, 2M - 1]\} \subseteq \mathcal{L}^+_B$

According to Property I-1, the sets $\mathcal{L}^+_A$ and $\mathcal{L}^+_B$ do not have any common values. But they do have overlapping values with the set $\mathcal{L}^+_A$ and $\mathcal{L}^+_A$ respectively, which is captured in Property I-2. The Property I-2 implies that $(N - 1)$ values in $\mathcal{L}^+_B$ overlap with $(N - 1)$ values in $\mathcal{L}^+_A$ and the same holds true for the sets $\mathcal{L}^+_A$ and $\mathcal{L}^+_A$. The self differences of the array with inter-element spacing of $M$ and $N$ are partly present in $\mathcal{L}^+_A$ and $\mathcal{L}^+_A$ $(m \in [0, M - 1])$ and partly in $\mathcal{L}^+_A$ and $\mathcal{L}^+_A$ $(m \in [M, 2M - 1])$. This is also evident from Fig. 4. It implies that $2(MN + (N - 1))$ unique values are present in $\mathcal{L}^+_A \cup \mathcal{L}^+_B$ which is not available in set $\mathcal{L}^+_A$. Therefore, the set $\mathcal{L}^+_A$ has $M + N + N - 2 + 2(3MN) = 3MN + M + N$ unique values. A summary of the unique differences in each difference set for the extended array is given in Table 1. The unique differences for the prototype co-prime array is also provided for comparison. The prototype co-prime array had holes (missing difference values) in the co-prime range, while the extended co-prime array generates all the difference values in the co-prime range $[M + 1, MN - 1]$. However, it does not represent the maximum continuous range of the extended co-prime array. We present Proposition I which provides the expressions for the range of integers in the cross difference sets of an extended co-prime array and their continuity (i.e. the range without holes). The proposition holds for one extended co-prime period $[0, 2MN - 1]$ with $0 \leq n \leq N - 1$ and $0 \leq m \leq 2M - 1$. 

5. Demonstrate low latency estimation using the correlogram spectral estimation method.

### 3 Difference set Analysis

The difference sets defined for the prototype co-prime array [13], is also applicable to the extended co-prime array. However, the extension of one of the sub-arrays needs to be taken into account. In this section, we briefly describe the difference sets before analyzing the number of unique differences or degrees of freedom (dof) and the range of continuous difference values.

The outputs obtained from the two sub-arrays with inter-element spacings $M$ and $N$ is represented by $x(Mn)$ and $x(Nm)$ respectively. Their self difference set is given by, $\mathcal{L}^+_S = \{l_s | l_s = Mn\}$ and $\mathcal{L}^-_N = \{l_N | l_N = Nm\}$. The cross difference set is given by, $\mathcal{L}^+_C = \{l_c | l_c = Mm - Nm\}$. Note that $0 \leq m \leq 2M - 1$ and $0 \leq n \leq N - 1$. The sets containing the mirrored locations of the elements in the above defined sets are denoted by $\mathcal{L}^+_S\mathcal{N}$ and $\mathcal{L}^-_N$ respectively. In addition, we define two union sets as:

$$\mathcal{L}_S = \mathcal{L}^+_S \cup \mathcal{L}^-_S$$

$$\mathcal{L}_C = \mathcal{L}^+_C \cup \mathcal{L}^-_C$$

where $\mathcal{L}^+_S = \mathcal{L}^+_S \cup \mathcal{L}^+_S\mathcal{N}$

and $\mathcal{L}^-_N = \mathcal{L}^-_N \cup \mathcal{L}^-_N\mathcal{N}$.

The self difference sets are shown in Fig. 3. It is obvious that $\mathcal{L}^+_S\mathcal{N}$ and $\mathcal{L}^-_N\mathcal{N}$ have $N$ and $2M$ unique differences respectively.
Table 1: Summary of unique differences (or dot) per difference set

| Set                | \(L_{SM} \) and \(L_{SM}^c\) | \(L_{SN}^c\) and \(L_{SN} \) | \(L_S^c\) and \(L_S\) | \(L_C^c\) and \(L_C\) | \(L_C\) |
|--------------------|-------------------------------|-------------------------------|------------------------|------------------------|---------|
| \# Unique diffs. (Extended) | \(N\) | \(2M\) | \(2M + N - 1\) | \(2(2M + N - 1) - 1\) | \(2MN\) | \(3MN + M - N\) |
| \# Unique diffs. (Prototype) | \(N\) | \(M\) | \(M + N - 1\) | \(2(2M + N - 1) - 1\) | \(MN\) | \(MN + M + N - 2\) |

Fig. 5: Weight function: \(M > N\).

Proposition 1.

1. \(L_C^+\) has \(2MN\) distinct integers in the range \(-N(2M - 1) \leq l_c \leq M(N - 1)\).
2. \(L_C^-\) has \(2MN\) distinct integers in the range \(-M(N - 1) \leq l_c \leq N(2M - 1)\).
3. \(L_C^+\) has consecutive integers in the range \(-MN + M - 1 \leq l_c \leq MN\).
4. \(L_C^-\) has consecutive integers in the range \(-N - 1 \leq l_c \leq M - MN\).
5. \(L_C\) has consecutive integers in the range \(-MN + M - 1 \leq l_c \leq MN\) which implies that this set has its first hole at \[l_c = (MN + M)\).

Proposition I-(1) and I-(2) can be easily proved by substituting the values of \(n\) and \(m\) in \((Mn - Nm)\) and \((Nm - Mn)\) respectively, to generate the maximum and minimum values in the set.

\[\begin{align*}
    -(N - 1) &\leq Nm \leq M(N - 1) + (MN + M - 1) \\
    -1 &< m < 2M \\
    0 &\leq m \leq 2M - 1
\end{align*}\]

which satisfies the required range. Proposition I-(4) can be proved along similar lines as Proposition I-(3).

Proof of Proposition I-(5): Let \(l_c = \pm (Mn - Nm)\) be an element in the set \(L_C\) satisfying \(-MN + M - 1 \leq l_c \leq MN + M - 1\), where \(m \in [0, 2M - 1]\) and \(n \in [0, N - 1]\). First we need to prove that \(\pm (Mn + M)\) is indeed a hole and then show that the range \(-MN + M - 1 \leq l_c \leq MN + M - 1\) is continuous. To prove that \(\pm (Mn + M)\) is a hole in the set \(L_C\), it is sufficient to prove that it is a hole in \(L_C^+\). This holds true because \(L_C^+\) is a flipped version of \(L_C\). Let \(l_c = Mn - Nm\) be an element in set \(L_C^+\), and let us assume that \(\pm (Mn + M)\) is not a hole and exists in set \(L_C^+\). This implies that: \(Mn - Nm = \pm (MN + M)\) or \(M(n + 1) = N(m \pm M)\), hence: \(M = \frac{(m + M)}{(n + 1)}\). Since \(n - 1 < N\) and \(m - M < M\), the ratios \(\frac{(m + M)}{(n + 1)}\) and \(\frac{(m - M)}{(n + 1)}\) cannot be satisfied. The proof of continuity in the range \(-MN + M - 1 \leq l_c \leq MN + M - 1\) follows directly from Proposition I-(3) and I-(4).

4 Weight Function

Proposition III in [13] gives the expression for the weight function or the number of sample pairs that contribute to estimate the autocorrelation at each value in the difference set. This parameter can affects the convergence, accuracy, latency and bias of the estimate. In this section we present the number of sample pairs that contribute
to estimate the autocorrelation for the extended co-prime array and is given as Proposition 2.

**Proposition 2.** Let $z(l)$ denote the number of elements contributing to the estimation at difference value $l$.

1. For $l \in \mathcal{L}_SM^+ \cup \mathcal{L}_SM^- - \{0\}$:
   \[ z(l) = (N - i) + 1, \{1 \leq i \leq N - 1, l = \pm Mi \} \]

2. For $l \in \mathcal{L}_SN^+ \cup \mathcal{L}_SN^- - \{0\}$:
   \[ z(l) = (2M - i), \{1 \leq i \leq 2M - 1, l = \pm Ni \} \]

3. For $l \in \{l = 0\}$: $z(l) = 2M + N - 1$

4. For $l \in \mathcal{L}_C - \mathcal{L}_S$:
   \[ z(l) = 2, \{l \in \{\mathcal{L}_S^+ \cup \mathcal{L}_S^-\} - \mathcal{L}_S \} \]
   \[ z(l) = 1, \{l \in \{\mathcal{L}_B^+ \cup \mathcal{L}_B^-\} - \mathcal{L}_S \} \]

It is easy to conclude that the number of sample pairs that map to each difference value in the self difference set of signal $x(Mn)$ is given by $N - i$ as shown in Fig. 3(b). In addition, the cross difference set also has pairs of samples that contribute to the estimate at these self differences except at ‘0’, i.e. $\mathcal{L}_SM^+ - \{0\}$ and $\mathcal{L}_SM^- - \{0\}$ in Fig. 5. Thus justifying Proposition 3(1). Proposition 3(2) can be easily inferred from Fig. 3(b). The number of contributors at difference value ‘0’, Proposition 3(3), is the sum of the contributors of the self difference set of the two arrays at zero minus one pair which is common to both the self difference sets. $\mathcal{L}_A^+$ and $\mathcal{L}_A^-$ represent the cross difference set of the prototype co-prime array and Proposition III-(4) in [13] shows that $z(l) = 2$ for $\{l \in \{\mathcal{L}_S^+ \cup \mathcal{L}_S^-\} - \mathcal{L}_S \}$. On the other hand, we can establish that $z(l) = 1$ for $\{l \in \{\mathcal{L}_A^+ \cup \mathcal{L}_A^-\} - \mathcal{L}_S \}$ from Fig. 4 and Property 1. In Fig. 3 and 4, we provide some examples of the weight function of the extended co-prime array for $M > N$ and $N > M$ respectively. The closed-form expression for the weight function $z_e(l)$ is given by:

\[
z_e(l) = \sum_{n=-(N-1)}^{N-1} (N - |n| + 1)\delta(l - Mn) \]

\[ + \sum_{m=-(2M-1)}^{2M-1} (2M - |m|)\delta(l -Nm) \]

\[ + \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} 2\delta(l - (Mn -Nm)) - \delta(l) \]

\[ + \sum_{n=1}^{N-1} \sum_{m=M+1}^{2M-1} \delta(|l| - (Mn -Nm)) - \delta(l) \]

The expressions for $z_e$ is directly obtained from Proposition 3(2). ‘A’ relates to Proposition 3(1) and $l = 0$ generates a value $N + 1$. ‘B’ relates to Proposition 3(2) and $l = 0$ generates a value $2M$. ‘C’ relates to the set $\mathcal{L}_A^+$ in Proposition 3(3). This term gives a value equal to –1 when $l = 0$. ‘D’ represents single contributors in the set $\mathcal{L}_B^+$ as in Proposition 3(4). The use of $\delta(|l| - (Mn -Nm))$ is due to the fact that $(Mn -Nm)$ generates values in $\mathcal{L}_B^+ - \mathcal{L}_S$ which are all negative. Note that when $l = 0$, the expression gives Proposition 3(3).
Fig. 7: Bias window for the extended co-prime arrays: \((M, N)\) with values \((14, 13), (14, 5), (7, 13)\).

Fig. 8: Bias window for the extended co-prime arrays: \((M, N)\) with values \((13, 14), (13, 7), (5, 14)\).

Table 2 Relative distance \((R)\) between main-lobe and side-lobe peak:
A comparison between \(M > N\) and \(N > M\).

| \((M,N)\)     | \((14,13)\) | \((14,5)\) | \((7,13)\) | \((13,14)\) | \((13,7)\) | \((5,14)\) | \((13,7)\) |
|---------------|-------------|------------|------------|-------------|------------|------------|------------|
| \(R\)         | 0.537       | 0.287      | 0.734      | 0.580       | 0.641      | 0.387      |

5 Numerical simulations

Simulations are performed on a temporal signal model described in [7, 13]. The first step is to estimate the autocorrelation using the combined set. Next take the Fourier transform of the autocorrelation estimate to obtain the correlogram spectral estimate. Repeat this for \(L\) snapshots and compute the average correlogram. It may be noted that instead of calculating the correlogram using the autocorrelation, the Fourier transform of the acquired signal with zeros inserted at the holes (missing difference values) can be calculated, i.e., Periodogram (Refer Section IV in [13]). We demonstrate the results for the case with number of snapshots, \(L = 10\). Fig. 9 and Fig. 10 demonstrates single peak estimation for different values of \((M, N)\). We observe
\[ W_b(e^{j\omega}) = \frac{1}{s_b} \left\{ \left| \frac{\sin(\omega M N)}{\sin(\omega M/2)} \right|^2 + \frac{\sin(\omega (2N-1)/2)}{\sin(\omega M/2)} + \left| \frac{\sin(\omega M N)}{\sin(\omega M/2)} \right|^2 + 2(1 + \cos(\omega MN)) \frac{\sin(\omega (N-1)/2)\sin(\omega (M-1)/2)}{\sin(\omega M/2)\sin(\omega M/2)} - 2 \right\} \]

(2)

Fig. 9: Simulation results for spectral estimation (1 peak) with number of snapshots \( L = 10 \): \((M, N) = (14, 13), (14, 5), (13, 7)\).

Fig. 10: Simulation results for spectral estimation (1 peak) with number of snapshots \( L = 10 \): \((M, N) = (7, 13), (3, 8), (3, 7)\).

Fig. 11: Simulation results for spectral estimation (3 peaks) with number of snapshots \( L = 10 \): \((M, N) = (14, 13), (14, 5), (13, 7)\).

Fig. 12: Simulation results for spectral estimation (3 peaks) with number of snapshots \( L = 10 \): \((M, N) = (7, 13), (3, 8), (3, 7)\).

several spurious peaks and note that \( M \approx N \) seems to be a good choice. Furthermore, \((M, N) = (3, 7)\) has the lowest spurious peak for the examples considered.

Next, consider Fig. 11 and 12 for the estimation of three spectral peaks. Most of the examples considered here, have spurious peaks. \((M, N) = (3, 7)\) seems to be a good choice. Based on the analysis in this paper, we have the following thoughts:
1. On the basis of the bias window expression and the examples considered, we find that $M \approx \frac{N}{2}$ seems to be a good choice to reduce the side-lobes for the extended co-prime arrays.

2. The simulation results indicate that all values of $M \approx \frac{N}{2}$ will not work. For example, we find lower valued integer, i.e. $(M, N) = (3, 7)$ to be good.

3. Since the correlogram spectral estimate is the convolution of the bias window with the true spectrum (in a statistical sense), we conclude that the accuracy will also depend on the signal model. For example, $(M, N) = (3, 8)$ works for single peak estimation in Fig. 10 but fails for three peak estimation in Fig. 12.

In the next section, concluding remarks and possible questions for future research are considered.

6 Conclusion

This paper describes the difference set for the extended co-prime arrays. Low latency spectral estimation is demonstrated using the correlogram method. Smaller values of co-prime parameters $(M, N)$ with $M \approx \frac{N}{2}$ is found to be a good choice for the examples considered. We develop the closed-form expressions for the weight function and correlogram bias window using the combined difference set. The bias window expression describes the distortion in the estimate. This throws up several challenges which needs further investigation. For example, can the bias distortion be reduced? What could be the ways to achieve this? In addition, other spectral estimation methods can be investigated. Several applications can be explored for low latency estimation along similar lines.

7 References

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