TIME-DEPENDENT MULTI-GROUP MULTI-DIMENSIONAL RELATIVISTIC RADIATIVE TRANSFER CODE 
BASED ON SPHERICAL HARMONIC DISCRETE ORDINATE METHOD

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ABSTRACT

We develop a time-dependent, multi-group, multi-dimensional relativistic radiative transfer code, which is required to numerically investigate radiation from relativistic fluids that are involved in, e.g., gamma-ray bursts and active galactic nuclei. The code is based on the spherical harmonic discrete ordinate method (SHDOM) which evaluates a source function including anisotropic scattering in spherical harmonics and implicitly solves the static radiative transfer equation with ray tracing in discrete ordinates. We implement treatments of time dependence, multi-frequency bins, Lorentz transformation, and elastic Thomson and inelastic Compton scattering to the publicly available SHDOM code. Our code adopts a mixed-frame approach; the source function is evaluated in the comoving frame, whereas the radiative transfer equation is solved in the laboratory frame. This implementation is validated using various test problems and comparisons with the results from a relativistic Monte Carlo code. These validations confirm that the code correctly calculates the intensity and its evolution in the computational domain. The code enables us to obtain an Eddington tensor that relates the first and third moments of intensity (energy density and radiation pressure) and is frequently used as a closure relation in radiation hydrodynamics calculations.

Key words: radiative transfer – relativistic processes – shock waves

1. INTRODUCTION

Radiative transfer is an important aspect of physics used to describe how an astronomical object is observed. Also, radiation often affects the dynamical behavior of an astronomical object. Thus, radiative transfer has been studied in many astrophysical fields. However, the radiative transfer equation is intrinsically a six-dimensional Boltzmann equation that is computationally expensive. Therefore, the radiative transfer equation is frequently solved in a simplified and approximated form appropriate for an object of interest.

Methods for multidimensional radiative transfer and radiation hydrodynamics have been developed in various fields, e.g., cosmological structure formation (e.g., Iliev et al. 2006, 2009, for a review), star formation (e.g., Krumholz et al. 2007; Tomida et al. 2013), stellar and solar atmospheres (e.g., Asplund et al. 2000; Nordlund et al. 2009), and a terrestrial atmosphere (e.g., Clough et al. 2005; Collins et al. 2006). These methods are optimized for individual research fields and involve various sophisticated physics for individual phenomena, e.g., a chemical network for cosmological structure formation and star formation, fine-structure lines for stellar and solar atmospheres, and molecular lines and scattering by dust for a terrestrial atmosphere. On the other hand, they ignore terms of higher order than $O(v/c)$, i.e., they assume that the fluid velocity is much slower than the light velocity. This is a commonly used, appropriate assumption for simplifying the radiative transfer equation but is inapplicable to radiative transfer in a relativistic flow.

Emission from a relativistic flow recently attracted the attention of researchers following the advent of a gamma-ray burst (GRB). A GRB is a phenomenon emitting $\gamma$-ray photons from relativistic jets in a short period and is one of the brightest objects in the universe. GRBs have been detected in the distant universe with redshifts as high as $z \sim 8.2$ (Salvaterra et al. 2009; Tanvir et al. 2009; $z \sim 9.4$ for a photometric redshift, Cucchiara et al. 2011) and are believed to probe the high-$z$ universe as well as quasars and galaxies. Furthermore, interestingly, it has been demonstrated observationally that $\gamma$-ray emission has correlations in spectral peak energy versus isotropic radiation energy (Amati et al. 2002) and spectral peak energy versus peak luminosity (Yonetoku et al. 2004). A correlation between the X-ray luminosity and the break time has also been suggested (Dainotti et al. 2008). These correlations lead researchers to believe that GRBs can be standardizable candles detectable at higher redshifts than Type Ia supernovae (e.g., Amati et al. 2008).

The mechanism of GRB prompt emission and the origin of correlations have been intensively studied. Observationally, many satellites and telescopes report large variations of the prompt emission, for example, spectra with thermal (e.g., Ryde et al. 2010), non-thermal (e.g., Abdo et al. 2009b; Zhang & Pe’er 2009), and high-energy components (e.g., Abdo et al. 2009a; Fun et al. 2013), polarization (e.g., Yonetoku et al. 2011; Uehara et al. 2012), and duration (e.g., ultra-long GRBs, Stratta et al. 2013; Levan et al. 2014). A correlation between optical and $\gamma$-ray light curves is also exhibited (e.g., Vestrand et al. 2005; Woźniak et al. 2009; Gorbovskoy et al. 2012). Theoretically, the emission mechanism has been investigated by analytic studies (e.g., Mészáros 2006, and references therein) and numerical studies with various assumptions: superposing blackbody radiation from a scattering photosphere (Blinnikov et al. 1999; Mizuta et al. 2011; Nagakura et al. 2011; Lazzati et al. 2013), solving a radiative transfer

\footnote{The emission from a relativistic flow is also interesting for, e.g., active galactic nuclei (AGNs).}

\footnote{The production site of photons is much deeper than the scattering photosphere in a relativistic flow (e.g., Shibata et al. 2014).}
equation in a spherical steady flow (Beloborodov 2011), transferring photons in a steady flow with a relativistic Monte Carlo (RMC) method (Giannios 2006; Pe’er 2008; Beloborodov 2011; Ito et al. 2013; Lundman et al. 2013; S. Shibata & N. Tominaga 2015, in preparation), and calculating spherical relativistic radiation hydrodynamics (Tolstov 2005; Tolstov et al. 2013).

In spite of plenty of observations and theoretical studies over the course of many years, the mechanism of GRB prompt emission is still under debate (e.g., Zhang 2014). This is mainly because most studies are restricted to being qualitative and there are few quantitative studies taking into account the structure of relativistic jets. In order to investigate the GRB prompt emission quantitatively, a multi-dimensional relativistic radiation hydrodynamics calculation is essentially required. This is because a GRB is a relativistic and multi-dimensional phenomenon and the radiation, the energy of which dominates the matter energy, is closely coupled with the matter. Therefore, it is necessary to develop a radiative transfer code optimized for GRBs which fully includes terms higher than $O(v/c)$.

Much progress has been made recently toward a multi-dimensional radiation hydrodynamics calculation: for example, (1) special relativistic three-dimensional radiation magnetohydrodynamics (Takahashi & Ohsuga 2013; Takahashi et al. 2013), (2) general relativistic three-dimensional radiation hydrodynamics (Sadowski et al. 2013), (3) three-dimensional radiation magnetohydrodynamics (Davis et al. 2012; Jiang et al. 2012, 2014), (4) RMC transport coupled with hydrodynamics (Roth & Kasen 2015), and (5) three-dimensional special relativistic Boltzmann hydrodynamics (Nagakura et al. 2014). However, calculations (1) and (2) are based on the M1 closure method, which can treat an anisotropic radiation field but not intersecting radiation from various sources. This is not suitable for GRBs because the material surrounding the relativistic jet, e.g., a cocoon, is hot and emits thermal photons in various directions. Calculation (3) adopts the variable Eddington tensor (VET) method which can treat intersecting radiation from multiple sources and non-local radiation equilibrium, but ignores terms with orders higher than $O(v/c)$. Recently, Jiang et al. (2014) implemented a radiative transfer code involving time dependence and velocity dependent source terms, but the method is still only accurate up to $O(v/c)$. Calculation (4) implicitly couples an RMC transport method with hydrodynamics solvers, and calculation (5) is adopted for neutrino transport in a collapsing massive star. Although calculations (4) and (5) would be applicable for GRBs, calculation (4) can involve Monte Carlo noise even when using a reduction technique which they developed, and calculation (5) is time-consuming and it might be difficult to increase the number of mesh points because it involves an inversion of a huge matrix. There are also general relativistic radiative transfer (RRT) codes to describe emission from the areas surrounding black holes (e.g., Dexter & Agol 2009; Shcherbakov & Huang 2011). The codes first derive photon geodesic trajectories in curved spacetime and integrate an RRT equation along the geodesic paths. These would also be applicable for GRBs. However, they are a waste of time and computational resources because it is only necessary to calculate the trajectory in flat spacetime in GRBs.

In this paper, we develop an implicit time-dependent, multi-group, multi-dimensional special RRT code using the mixed-frame approach. The RRT code can calculate an Eddington tensor using the intensity, and thus could be the first step toward a time-dependent, special relativistic, multi-group, multi-dimensional radiation hydrodynamics code optimized for GRBs. The RRT code is based on the spherical harmonic discrete ordinate method (SHDOM) and takes into account ray tracing, time dependence, Lorentz transformation, and elastic Thomson and inelastic Compton scattering. We present numerical test problems with the RRT code.

We describe our method in Section 2 and present test problems in Section 3. Our results are summarized in Section 4.

2. METHOD

The RRT code is based on the SHDOM (Evans 1998; Pincus & Evans 2009)8 code, which is a publicly available static monochromatic radiation transfer code solving the following equation using the $\Lambda$ iteration (Picard iteration) method:

\[ n \cdot \nabla I_\nu(s, n) = -\chi_\nu(s, n)I_\nu(s, n) + \eta_\nu(s, n), \]

where $s$ is a total path along a ray, $n$ is the direction of travel of the photon, and $I_\nu$, $\chi_\nu$, and $\eta_\nu$ are an intensity coefficient, an extinction coefficient, and an emission coefficient at frequency $\nu$, respectively. Here, the extinction coefficient is the net absorption coefficient of $\alpha_\nu + \sigma_\nu$, where $\alpha_\nu$ and $\sigma_\nu$ are absorption and scattering coefficients, respectively. The SHDOM code was originally developed for radiative transfer in the terrestrial atmosphere, and thus correctly treats anisotropic source terms including emission and scattering with spherical harmonics. Equation (1) is integrated for each ray described with discrete ordinates $(\theta, \phi)$ using a short characteristic method (ray tracing). The SHDOM code transforms an intensity and a source function between spherical harmonics and discrete ordinates at every time step. The scheme of the SHDOM code, including the transformation between spherical harmonics and discrete ordinates, the ray tracing, etc., is comprehensively described and validated in Evans (1998) and Pincus & Evans (2009).

In order to apply the SHDOM code to a special relativistic phenomenon, especially a GRB, the following processes and physics must be included: (1) time dependence because a time step must be short enough to capture a relativistic fluid that moves at the speed of light, i.e., the light-crossing time is as long as the characteristic dynamical time; (2) Lorentz transformation between laboratory and comoving frames, which results in relativistic beaming and variation of the photon frequencies; and (3) anisotropic and inelastic Compton scattering because the photon energy is comparable to the electron rest-mass energy, and thus the assumption of Thomson scattering is not valid, and scattering dominates the opacity in the relativistic jets of GRBs (e.g., Beloborodov 2013).

We set a computational domain to be translationally symmetric along the $y$ axis, but the ray of radiation is solved in three dimensions and described with the polar coordinates $\theta$ and $\phi$. The coordinates are set to have a zenith direction of the $z$ axis and an azimuth angle measured from the $x$ axis to the $y$ axis. The radiative transfer equation is solved using the mixed-frame approach (e.g., Mihalas & Mihalas 1984; Hubeny & Burrows 2007); the photon ray is traced in the laboratory frame.

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8 http://net.colorado.edu/shdom.html
and the source term is evaluated in the comoving frame. In addition to the above implementation, we update the ray tracing scheme to a cubic Bezier interpolant method (de la Cruz Rodríguez & Piskunov 2013) and include an acceleration scheme of Λ iteration (Ng acceleration, Ng 1974), but we omit the adaptive treatment of mesh points and parallelization with the Message Passing Interface for simplicity.

2.1. Time Dependence

A time-dependent radiative transfer equation can be written as

\[
\frac{1}{c} \frac{\partial I_\nu(t, s, n)}{\partial t} + n \cdot \nabla I_\nu(t, s, n) = -\chi_\nu(t, s, n)I_\nu(t, s, n) + \eta_\nu(t, s, n),
\]

where \( c \) is the light speed. A finite difference approximation to the time derivative is written as

\[
\frac{\partial I_\nu(t, s, n)}{\partial t} \approx \frac{I_\nu(t + \Delta t, s, n) - I_\nu(t, s, n)}{\Delta t}.
\] \hspace{1cm} (3)

The time-dependent radiative transfer equation is deformed to the same shape as the static radiative transfer equation (Equation (1)) with modified absorption and emission coefficients as

\[
n \cdot \nabla I_\nu(t + \Delta t, s, n) = -\tilde{\chi}_\nu I_\nu(t + \Delta t, s, n) + \tilde{\eta}_\nu,
\] \hspace{1cm} (4)
The source function due to scattering is evaluated using spherical harmonics as in original SHDOM code. Since the prescription is outlined with many details in Evans (1993, 1998), we briefly describe the procedure. The source function \( S(\mu, \phi) \) is expanded in spherical harmonics space as

\[
S(\mu, \phi) = \sum_{lm} Y_{lm}(\mu, \phi) S_{lm},
\]

where \( Y_{lm}(\mu, \phi) \) are orthonormal real-valued spherical harmonics functions, whereas the phase function of the scattering \( \frac{d\sigma}{d\Omega} \) is expanded in a Legendre series in the scattering angle as

\[
\frac{d\sigma}{d\Omega} = \sum_{l=0}^{N_L} X_l P_l,
\]

where \( N_L \) and \( P_l \) are the maximum order of Legendre polynomials and Legendre polynomials, respectively. The source function is computed as

\[
S_{lm} = \frac{\omega X_l}{2l+1} I_{lm} + T_{lm},
\]

where \( \omega \) is the single scattering albedo, and \( I_{lm} \) and \( T_{lm} \) are the intensity and thermal emission expanded in spherical harmonics, respectively.

We adopt the phase functions of Thomson and Compton scattering. The phase function of Thomson scattering is

\[
\frac{d\sigma_T}{d\Omega} = \frac{r_0^2}{2} \left( 1 + \cos^2 \Theta \right),
\]

where \( r_0 \) is the classical electron radius and \( \Theta \) is the scattering angle (Rybicki & Lightman 1985). A Klein–Nishina scattering differential cross section is adopted for the Compton scattering. The equations are

\[
\nu_1 = \frac{\nu}{1 + \frac{\hbar \nu}{m c^2} (1 - \cos \Theta)} \quad (13)
\]

\[
\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2} \nu_1 \left( \frac{\nu}{\nu_1} + \frac{\nu_1}{\nu} - \sin^2 \Theta \right), \quad (14)
\]

\footnote{The subroutine is taken from http://cernlib.web.cern.ch/cernlib/version.html.}
where $\nu_f$ is the photon frequency after the scattering and $m_e$ is the rest mass of the electron (Rybicki & Lightman 1985).

A change to the photon frequency and dependence of the scattering kernel on the photon frequency are essential features of Compton scattering. Therefore, we implement a multi-group treatment for test calculations of the Compton scattering. The differential cross section $\frac{d\sigma}{d\Omega}^{(i,j)}$ of an incident photon in the $i$-th frequency bin with a range of $[\nu_f^{(i)}, \nu_f^{(i)} + \Delta\nu_f^{(i)}]$ to the $j$-th frequency bin with a range of $[\nu_f^{(j)}, \nu_f^{(j)} + \Delta\nu_f^{(j)}]$ is a frequency-dependent scattering kernel defined by Equation (14). $\frac{d\sigma}{d\Omega}^{(i,j)}$ is expanded in a Legendre series in the scattering angle with a frequency-dependent single scattering albedo. In the mixed-frame approach, the photon exchange between frequency bins takes place only in the comoving frame.

3. TEST PROBLEMS

Evans (1998) and Pincus & Evans (2009) have intensively tested the original SHDOM code and investigated its efficiency, accuracy, and scalability. Therefore, we focus on validation tests of time dependence, Lorentz transformation, and Thomson and Compton scattering in this paper.
3.1. Searchlight Beam Test

A searchlight beam test has been performed in various studies (e.g., Richling et al. 2001; Turner & Stone 2001; Takahashi & Ohsuga 2013; Takahashi et al. 2013). A narrow beam of light is introduced into the computational domain at a certain angle and the beam crosses the domain. This test examines whether a code can solve a time-dependent radiative transfer equation and how radiation disperses along the path.

We set a computational domain of $10^{10}$ cm square, for which the $x$ and $z$ axes are divided by 512 mesh points. The number of angular mesh points is $(N_\theta, N_\phi) = (4, 8)$. The origin is located at the bottom left corner of the domain. A beam of light is injected from the bottom boundary at $1.5 \times 10^9 \text{ cm} < x < 4.5 \times 10^9 \text{ cm}$ along a ray with an angle of $(\theta, \phi) = (0.17\pi, 0)$. Two cases of absorption coefficients are tested; the domain is uniformly filled with a medium with $\alpha = 0$ or $\alpha = 10^{-10} \text{ cm}^{-1}$.

Figures 1(a)–(d) show snapshots of the mean intensity $J$ in the domain at $t = 0.1 \text{ s}$ and $0.3 \text{ s}$. The beam properly crosses the domain with time at the injected angle. Figure 2 shows the profile section at $z = 5.0 \times 10^9 \text{ cm}$ and $t = 0.25 \text{ s}$ for the case of $\alpha = 0$. The radiation disperses slightly laterally because the RRT code adopts the short characteristic method.

Figures 3(a) and (b) show $J$ along $x = \tan(0.17\pi)z + 3.0 \times 10^9 \text{ cm}$. The figures demonstrate that a wave front proceeds with the light speed. However, the wave front is smeared. This is because the time dependence is taken into account with $\tilde{r}$ and $\tilde{t}$, which exponentially reduce and/or enhance the intensity as a function of path length $s$. The widths of the smooth profile at the wave front are identical for the cases of $\alpha = 0$ and $\alpha = 10^{-10} \text{ cm}^{-1}$. The mean intensity of the beam is constant for the case of $\alpha = 0$ and is reduced in accordance with $\exp(-\alpha s)$ for the case of $\alpha = 10^{-10} \text{ cm}^{-1}$. The test confirms that the time dependence and radiation attenuation are correctly solved by the RRT code, although the wave front is smeared.

3.2. Two Beam Shadow Test

A shadow test was proposed in Hayes & Norman (2003). This test examines a reproduction of a shadow behind an optically thick blob when plane-parallel radiation illuminates the blob. One must properly take into account at least up to the first moment of intensity in order to reproduce the shadow. Thus, for instance, a method with a diffusion approximation fails this test.

On the other hand, a two beam test has been performed, e.g., in Davis et al. (2012), Jiang et al. (2012), Sadowski et al. (2013), and Jiang et al. (2014). This test examines whether two independent beams proceeding at different angles pass through without any interactions when they intersect. To describe this phenomenon, it is necessary to properly account for at least up to the second moment of intensity. Thus, approximate schemes with a closure relation treating up to the first moment, e.g., the M1 closure method, fail this test.

We solve a two beam with shadow test combining the above tests, which were performed in Davis et al. (2012), Jiang et al. (2012), Sadowski et al. (2013), and Jiang et al. (2014). We set an optically thin ($\alpha = 0$) computational domain of $10^{10}$ cm square, where the $x$ and $z$ axes are divided by 512 mesh points. The number of angular mesh points is $(N_\theta, N_\phi) = (4, 8)$. The origin is located at the bottom left corner of the domain. An optically thick absorptive cylinder with a radius of $1.5 \times 10^9 \text{ cm}$ is located at $(x, z) = (5.0 \times 10^9 \text{ cm}, 3.3 \times 10^9 \text{ cm})$ perpendicular to the $xz$ plane. The optical depth of the cylinder is set to be $\tau = 100$ with the diameter, i.e., $\alpha = 3.3 \times 10^{-8} \text{ cm}^{-1}$. Plane-parallel radiation is injected from the bottom boundary at angles of $(\theta, \phi) = (0.17\pi, 0)$ and $(0.17\pi, \pi)$.

Figures 4(a) and (b) show snapshots of $J$ in the domain at $t = 0.2 \text{ s}$ and $1.0 \text{ s}$. The plane-parallel radiation proceeds properly with time and the wave speed is the speed of light. A shadow develops behind the cylinder along the directions of the two beams and the two beams cross around $(x, z) = (5.0 \times 10^9 \text{ cm}, 7.4 \times 10^9 \text{ cm})$ without any interaction. The test confirms that the calculation properly solves the radiation transfer equation for intensity.

3.3. Radiative Pulse Test

The RRT code is tested with the evolution of a radiative pulse initially having a Gaussian profile in an optically thin medium and a scattering-dominated optically thick medium. The tests were performed in Sadowski et al. (2013) and verify the treatment of time dependence and scattering.

3.3.1. Optically Thin Medium

In the optically thin limit, the isotropic radiative pulse spreads with the speed of light and the mean intensity decreases inversely proportionally to the radius in the translational symmetry.

We set an optically thin computational domain of $10^{10}$ cm square with $\alpha = 0$. The origin is at the center of the domain. Each axis is divided by 128 mesh points. The number of angular mesh points is $(N_\theta, N_\phi) = (64, 128)$. An initial radiative pulse is set according to the equation

$$I_0(x, n) = 100 \exp \left[ -\left( \frac{r}{w} \right)^2 \right],$$

(15)

where $r = |x|$ is the distance from the origin and $w = 9.0 \times 10^8 \text{ cm}$. The intensity of the pulse is initially isotropic at each mesh point.

The top panels of Figure 5 show snapshots of $J$ at $t = 0.1 \text{ s}$, $0.2 \text{ s}$, and $0.3 \text{ s}$. The expected size of the pulse is also shown by...
a green circle with a radius of $ct$. The bottom panels of Figure 5 show $J$ along the $x$ axis, the $z$ axis, and a line with $x = z$ at $t = 0.1, 0.2, \text{and } 0.3$ s. These profiles of $J$ are identical and the peak of $J$ decreases with the expansion of the pulse according to $\mu / Jr^1$, as expected. The result demonstrates that the pulse isotropically propagates with the speed of light and that the geometrical dilution of radiation is correctly followed with the RRT code.

3.3.2. Optically Thick Medium

In the scattering-dominated, optically thick medium, the radiative pulse diffuses out. A one-dimensional diffusion equation $\frac{\partial J}{\partial t} = \chi \frac{\partial^2 J}{\partial x^2}$ can be solved analytically and the solution is written as

$$J(t, x) = \frac{1}{2\sqrt{\pi \chi t}} \int_{-\infty}^{\infty} J_0(x') \exp \left\{ -\frac{(x - x')^2}{4\chi t} \right\} dx'$$  \hspace{1cm} (16)

where $\chi = \frac{1}{3 \sigma}$ and $J_0(x)$ is the initial profile of the radiative pulse.

We set the optically thick computational domain of $10^{10}$ cm square with $\tau = 300$ along a side, i.e., $\sigma = 3 \times 10^{-8}$ cm$^{-1}$. The origin is set at the center of the domain. Each axis is divided by 128 mesh points. The number of angular mesh points is ($N_\theta, N_\phi$) = (32, 64). Each mesh point is optically thick, $\tau = 2.3$, so that most of the photons are scattered at least once in the mesh point. Here, we adopt a spherical scattering kernel. A radiative pulse is initially set according to

$$I_0(x, n) = 100 \exp \left[ -\left( \frac{x}{w} \right)^2 \right],$$  \hspace{1cm} (17)

where $w = 9.0 \times 10^8$ cm. The intensity of the pulse is initially isotropic at each mesh point.

Figure 6 shows the time evolution of $J$ along the $x$ axis. The radiation gradually diffuses in the numerical simulation and the distribution of $J$ well reproduces that of the analytic solution at every time step until the radiation reaches the boundaries of the computational domain.

3.4. Relativistic Beaming Test

We validate the ability of the RRT code to treat relativistic beaming as a result of the Lorentz transformation. This is a characteristic feature of a special RRT calculation. Although the propagation of a beam of light along a curved trajectory is demonstrated in general RRT calculations (e.g., Sadowski et al. 2013), this is the first attempt to confirm relativistic beaming in special relativistic calculations.

We set a computational domain of $10^{10}$ cm square filled with an optically thin medium with $\alpha = 0$ and consider a cylindrical light source with a radius of $R = 5 \times 10^8$ cm at the center of a computational domain. The center of the computational domain is set to be the origin. In the cylinder, particles move with $v = (\nu_x, \nu_z)$ with respect to the laboratory frame and emit photons isotropically in the comoving frame of each particle, whereas they do not emit any photons and do not interact with

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Figure 7. Snapshots of the mean intensity normalized with the maximum mean intensity $J/J_{\text{max}}$ for the relativistic beaming test for cases (a) $(\nu_x, \nu_z) = (0, 0)$, (b) $(\nu_x, \nu_z) = (0.1c, 0)$, (c) $(\nu_x, \nu_z) = (0.5c, 0)$, (d) $(\nu_x, \nu_z) = (0.9c, 0)$, (e) $(\nu_x, \nu_z) = (0.99c, 0)$, and (f) $(\nu_x, \nu_z) = (0.995c, 0)$. Asymptotic analytic expressions of the beaming effect are also shown (green lines).
photons outside the cylinder. Each axis is divided by 128 mesh points and the number of angular mesh points is \( q_f = N^3 \), where \( q_f \) is the number of angular mesh points.

Figures 7(a)–(f) show snapshots of the normalized mean intensity \( \text{intensity} \) at \( t = 0.2 \) s for the cases with \( (v_x, v_z) = (0, 0), (0.1c, 0), (0.5c, 0), (0.9c, 0), (0.99c, 0), \) and \( (0.995c, 0) \), respectively, where \( J_{\text{max}} \) is the maximum mean intensity in the computational domain. The corresponding Lorentz factors are \( \Gamma = 1, 1.005, 1.15, 2.29, 7.09, \) and \( 10.0 \). It is clearly shown that the mean intensity is high along the direction of \( v \). An analytic expression of the beaming effect (asymptotically \( \theta \sim \frac{1}{\Gamma} \); Rybicki & Lightman 1985) is also shown by green lines circumscribing the light source. The biased distribution of \( J \) in the calculation is consistent with the analytic expression. The test shows that the beaming effect is correctly taken into account in the RRT code.

The speed of \( 0.995c \) is close to the maximum speed that can be correctly solved with \( N_0 = 32 \) because the half-angle of the concentration of the radiation for the case of \( v = 0.9952c \) is comparable to the interval between angular mesh points. Although the test is limited due to computational resources, a Lorentz factor as high as that encountered in GRB models can be resolved by the RRT code if a larger number of angular mesh points is adopted because the mapping subroutine for the Lorentz transformation is valid even for a high Lorentz factor.

Figure 8 shows the angular distribution of the intensity \( I \) in the laboratory frame for cases with \( (v_x, c - v_z) = (0, 5 \times 10^{-5}c) \) (\( \Gamma = 10 \)), \( (0, 5 \times 10^{-5}c) \) (\( \Gamma = 100 \)), \( (0, 6 \times 10^{-6}c) \) (\( \Gamma = 300 \)), and \( (0, 5 \times 10^{-7}c) \) (\( \Gamma = 1000 \)), which is transformed from an isotropic intensity \( I_0 \) in the comoving frame. The deviation from the analytical solution \( I(\mu) = (\Gamma(1 - \mu v/c)^{-5/2} I_0 \) is less than \( 10^{-6}I_0 \). Here, we adopt a large number of angular mesh points, \( (N_0, N_\phi) = (4096, 8192) \). We note that the mapping subroutine correctly transforms isotropic radiation for any Lorentz factor even with a small number of angular mesh points.

### 3.5. Comparison with Monte Carlo Method

The original SHDOM code and the Monte Carlo method have been carefully compared and their drawbacks and advantages have been presented in Evans (1998) and Pincus & Evans (2009). Therefore, we test the RRT code only for the treatments of the Thomson and Compton scattering and Lorentz transformation, and the implementation of multiple frequency groups for the Compton scattering tests, by comparing solutions of a shadow test of the RRT code with those of an RMC code (S. Shibata & N. Tominaga 2015, in preparation; see Appendix B for a test of the RMC code). In this subsection, we omit the time dependence and adopt the static version of the RRT code because the RMC code currently does not follow the time evolution of the radiation.

Here, we adopt an ideal test problem. A scattering-dominated, optically thick cylinder with \( \sigma = \sigma_0 \exp\left(-\frac{r^2}{w^2}\right) \)
where $s_0 = 2 \times 10^{-7}$ cm$^{-1}$ and $w = 2.5 \times 10^8$ cm, is placed at the center of an optically thin computational domain of $10^{10}$ cm square with $s = 2 \times 10^{-15}$ cm$^{-1}$. Scattering particles rest or flow with $v$ in the scattering cylinder. Plane-parallel radiation is injected from the left and bottom boundaries at a single angle of $(\theta, \phi) = (0, 0)$.

Figure 9. Snapshots of $z$-oriented fluxes normalized with the maximum flux $F_{z,\text{max}}$ in the medium with $(v_x, v_z) = (0, 0)$ ((a) and (b)) and $(v_x, v_z) = (0, 0.9c)$ ((c) and (d)). The panels show the results of the RRT code ((a) and (c)) and the RMC code ((b) and (d)). Here, the Thomson scattering kernel is adopted and the plane-parallel radiation is injected from the left and bottom boundaries at a single angle of $(\theta, \phi) = (0.24\pi, 0)$.

3.5.1. Thomson Scattering

In this subsection, we adopt a kernel of Thomson scattering. The computational domain is divided by $128 \times 128$ mesh points and the number of angular mesh points is $(N_\theta, N_\phi) = (160, 320)$.

Figures 9(a)–(d) show normalized $z$-oriented fluxes $F_z(x, z)/F_{z,\text{max}}$ obtained by the RRT and RMC calculations in cases with $(v_x, v_z) = (0, 0)$ and $(0, 0.9c)$, respectively, where $F_{z,\text{max}}$ is the maximum $z$-oriented flux in the computational domain. Figures 9(a) and (b) demonstrate that the $z$-oriented fluxes are small below the cylinder due to scattered photons propagating in the $-z$ direction, whereas the $z$-oriented fluxes are high at the top left side of the cylinder. Since the number density of scattered photons decreases with distance from the scattering cylinder as $1/r$, the $z$-oriented fluxes are prominently modified around the cylinder. Also, the shadow behind the cylinder consistently forms in both calculations. Figures 9(c) and (d) show that the scattered photons are concentrated in the $+z$ direction due to the relativistic beaming in both calculations. These demonstrate that the RRT code well solves the anisotropic scattering and the Lorentz transformation.

3.5.2. Compton Scattering

In this section, we adopt frequency-dependent kernels of Compton scattering and the multi-group treatment. The computational domain is divided into $64 \times 64$ mesh points. We test the RRT code for two cases with different velocity of scattering particles with $(v_x, v_z) = (0, 0)$ and $(0, 0.9c)$. The...
number of angular mesh points is \((N_{\theta}, N_{\varphi}) = (128, 256)\). The frequency range is equally divided by 10 bins in both cases but the maximum frequency is \(h\nu = 1.1m_e c^2\) for the case with \((\nu_e, \nu_c) = (0, 0)\) and \(4.0m_e c^2\) for the case with \((\nu_e, \nu_c) = (0, 0.9c)\). Monochromatic light with \(h\nu = 1.05m_e c^2\) is injected for the case with \((\nu_e, \nu_c) = (0, 0)\) and \(h\nu = 1.0m_e c^2\) for the case with \((\nu_e, \nu_c) = (0, 0.9c)\).

Figures 10(a)–(f) and 11(a)–(f) show frequency-dependent normalized z-oriented fluxes \(F_{\nu z}(x, z)/F_{\nu z, \text{max}}\) in the cases with \((\nu_e, \nu_c) = (0, 0)\) and \((0, 0.9c)\), respectively, where \(F_{\nu z, \text{max}}\) is the

**Figure 10.** Snapshots of frequency-dependent, z-oriented fluxes normalized with the maximum flux \(F_{\nu z}(x, z)/F_{\nu z, \text{max}}\) in the cases with \((\nu_e, \nu_c) = (0, 0)\). The panels represent fluxes in energy bins with \([0.88m_e c^2, 0.99m_e c^2]\) \((a)\) and \((b)\), \([0.66m_e c^2, 0.77m_e c^2]\) \((c)\) and \((d)\), and \([0.44m_e c^2, 0.55m_e c^2]\) \((e)\) and \((f)\). The panels show the results of the RRT code \((a)\), \((c)\), and \((e)\) and the RMC code \((b)\), \((d)\), and \((f)\). Here, frequency-dependent Compton scattering kernels are adopted.
maximum $z$-oriented flux in a frequency bin in the computational domain. Each panel shows the fluxes in different energy bins. There was no light in these energy bins before scattering. This is why no shadow appears behind the cylinder. The negative fluxes due to the scattered photons produce the black region below the cylinder in Figures 10(e) and (f). Although the shape of $F_{\nu, z}$ is smeared in the RRT code, especially in the case of $(\nu_x, \nu_z) = (0, 0.9c)$, because the frequency of photons is converted to the central frequency of each bin at every time step, it provides similar snapshots of the $z$-oriented flux in the

Figure 11. Same as Figure 10, but for cases with $(\nu_x, \nu_z) = (0, 0.9c)$. The panels represent fluxes in energy bins with $[2.0m_e c^2, 2.4m_e c^2]$ ((a) and (b)), $[1.2m_e c^2, 1.6m_e c^2]$ ((c) and (d)), and $[0.4m_e c^2, 0.8m_e c^2]$ ((e) and (f)).
RMC code. These demonstrate that the RRT code correctly solves for Compton scattering and Lorentz transformation, and treats the multi-group radiation transfer equation well.

4. SUMMARY

We develop a time-dependent multi-group multidimensional RRT code by implementing treatments of time dependence, multi-frequency bins, Lorentz transformation, and elastic Thomson and inelastic Compton scattering in the publicly available SHDOM code. The SHDOM code evaluates a source function in spherical harmonics and solves a static radiative transfer equation with ray tracing in discrete ordinates. The RRT code is validated using various tests and a comparison with the RMC calculations. The searchlight beam, two beam with shadow, radiative pulse, and relativistic beaming tests are successfully passed by the RRT code and confirm that the RRT code correctly handles the time dependence and the Lorentz transformation. The results of the RRT code are consistent with those of the RMC code and the comparisons verify the implementation of elastic Thomson and inelastic Compton scattering and multi-group treatment in the RRT code. The RMC code, in turn, is validated against the EGS tools (Appendix B).

The RRT code enables us to obtain the evolution of the intensity, and thus to self-consistently derive an Eddington tensor without approximations as in the flux limited diffusion or M1 closure methods. We emphasize that the radiation tends to be more anisotropic in the relativistic fluid because of the Lorentz transformation and Compton scattering of the γ-ray photons, and thus the angular distribution of the radiation should be properly taken into account. Combining the Eddington tensors with relativistic hydrodynamics calculations (e.g., Tominaga 2009), a relativistic radiation hydrodynamics will be realized with the variable Eddington tensor method. Furthermore, the RRT code implicitly solves the radiative transfer equation, and thus can follow radiative transfer in a non-relativistic fluid like a supernova without adopting unnecessarily short time steps. Such a method will be useful to clarify the connection between GRBs and supernovae.

It is currently difficult to increase the number of frequency bins and angular and spatial mesh points due to the available memory resources. These difficulties are solved if adaptive treatment of mesh points and parallelization with distributed memory are implemented because ray tracing in the laboratory frame and evaluation of the source function in the comoving frame are independent of each ray and each mesh point, respectively. However, we note that the low resolution of the frequency is an intrinsic drawback of the multi-group treatment compared to the Monte Carlo method, in which the frequency of each photon changes continuously. Thus, the Monte Carlo method is superior to the RRT code for spectral synthesis calculations of lines and fine spectral features. However, the method intrinsically involves noise and it is expensive to reduce the noise because the reduction is realized only proportional to the square root of the number of photon packets. Also, the large number of photon packets is necessary to follow the time dependence because the photon packets are emitted at each time step. Furthermore, the radiation contributes to the hydrodynamics with an integration of radiation over the entire frequency and the dynamical effects of radiation in GRBs are not dominated by narrow spectral lines. Thus, we propose a post-processing Monte Carlo calculation for spectral synthesis after a time-dependent, matter-coupled RRT calculation.

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APPENDIX A

TIME INTEGRATION

We treat the time dependence with the modified absorption and emission coefficients as shown in Section 2.1, and implicitly obtain the self-consistent, time-dependent intensity at \( t + \Delta t \). Although the time derivative is differentiated in the first order, we adopt the fourth-order Runge–Kutta scheme to progress the time step from \( t \) to \( t + \Delta t \) by dividing the time interval to four steps in order to increase the accuracy by adopting smaller time intervals:

\[
I_{\nu}(t + \Delta t, s, n) = I_{\nu}(t, s, n) + \Delta I_{\nu}^{(1)} + \Delta I_{\nu}^{(2)} + \Delta I_{\nu}^{(3)} + \Delta I_{\nu}^{(4)},
\]

where

\[
\Delta I_{\nu}^{(1)} = \mathcal{A}_{\nu}(\Delta t/6, I_{\nu}(t, s, n)) - I_{\nu}(t, s, n) \tag{19}
\]

\[
I_{\nu}^{(1)} = I_{\nu}(t, s, n) + 3\Delta I_{\nu}^{(1)} \tag{20}
\]

\[
\Delta I_{\nu}^{(2)} = \mathcal{A}_{\nu}(\Delta t/3, I_{\nu}^{(1)}) - I_{\nu}^{(1)} \tag{21}
\]

\[
I_{\nu}^{(2)} = I_{\nu}(t, s, n) + \frac{3}{2}\Delta I_{\nu}^{(2)} \tag{22}
\]

\[\text{Appendix B}\]

\[\text{References}\]

[11] Several techniques have been suggested to reduce noise (e.g., Steinacker et al. 2013; Roth & Kasen 2015).
\[ \Delta I_v^3 = \frac{\Delta I_v}{\sqrt{3}} \]  

(23)

\[ I_v^3 = I_v(t, s, n) + 3 \Delta I_v^3 \]  

(24)

\[ \Delta I_v^4 = \Delta I_v \left( \frac{\Delta t}{6}, I_v^3 \right) - I_v^3. \]  

(25)

Here, \( \Delta I_v(\Delta t, I_v) \) is the solution of Equation (4) with \( I_v \) and \( \Delta t \), i.e., \( \Delta I_v(\Delta t, I_v(t, s, n)) = I_v(t + \Delta t, s, n) \).

**APPENDIX B**

**COMPARISON WITH ELECTRON GAMMA SHOWER (EGS) SOFTWARE**

We compare the result of the RMC code with that of the National Research Council’s electron gamma shower (EGS) software tool\(^\text{12}\) to confirm its validity, especially for the treatment of scattering. EGS software is a publicly available code that enables sophisticated treatment of photon, electron, and positron transfer in a complicated medium and provides graphic tools to show numerical results. Here, we adopt the flu\text{u}rz\text{nc} package only with Compton scattering by electrons at rest in the EGS software.

We set a cylindrical computational domain with a radius of \( 2 \times 10^3 \) cm and a depth of \( 2 \times 10^3 \) cm. The cylinder is filled with H atoms with a density of \( 8.37 \times 10^{-5} \) g cm\(^{-3} \). The incident photons with 50 keV are vertically injected from the top boundary with \( r < 1 \times 10^4 \) cm, where \( r \) is the distance from the center of the top boundary.

Figure 12 shows a comparison with flu\text{u}nc as a function of the depth in rings with \( r < 1 \times 10^4 \) cm, \( 1 \times 10^4 \) cm < \( r < 5 \times 10^4 \) cm, \( 5 \times 10^4 \) cm < \( r < 1 \times 10^5 \) cm, and \( 1 \times 10^5 \) cm < \( r < 2 \times 10^5 \) cm, normalized by the maximum flu\text{u}nc. The flu\text{u}nc is defined as a summation of \( 1/\cos \theta \) for all of the photons per unit area at each boundary of the slabs, where \( \theta \) is the angle the photon makes normal to the plane. We adopt \( 6 \times 10^5 \) photons for both the RMC calculation and the calculation by the EGS tool. The RMC codes give results consistent with those obtained by the EGS software.

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\( ^{12} \) http://www.nrc-cnrc.gc.ca/eng/solutions/advisory/egsnrc_index.html

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