Analytic structure of a family of hyperboloidal beams of potential interest for advanced LIGO

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This paper is concerned with a study of the analytic structure of a family of hyperboloidal beams introduced by Bondarescu and Thorne which generalizes the nearly-flat and nearly-concentric mesa beam configurations of interest for advanced LIGO. Capitalizing on certain results from the applied optics literature on flat-top beams, a physically-insightful and computationally-effective representation is derived in terms of rapidly-converging Gaussian-Laguerre expansions. A generalization (involving fractional Fourier transform operators of complex order) of some recently discovered duality relations between the nearly-flat and nearly-concentric mesa configurations is obtained. Possible implications for the advanced-LIGO optical cavity design are discussed.

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I. INTRODUCTION

Fabry-Perot optical cavities with nonspherical mirrors capable of supporting flat-top ("mesa") beams are being actively investigated [1] to be used in the baseline design of the advanced Laser Interferometer Gravitational-wave Observatory (LIGO) [2]. These configurations may actively be investigated [1] to be used in the baseline design of advanced LIGO. Capitalizing on certain results from the applied optics literature on flat-top beams, a physically-insightful and computationally-effective representation is derived in terms of rapidly-converging Gaussian-Laguerre expansions. A generalization (involving fractional Fourier transform operators of complex order) of some recently discovered duality relations between the nearly-flat and nearly-concentric mesa configurations is obtained. Possible implications for the advanced-LIGO optical cavity design are discussed.

Concerning flat-top beams, which have most likely not come to the attention of the gravitational-wave community. Specifically, the FM and CM beams belong to the class of flattened beams introduced in [10], and can therefore be represented in terms of the rapidly-converging Gaussian-Laguerre (GL) beam expansions derived therein. Based on this observation, we extend the approach in [10] to accommodate the more general family of BT hyperboloidal beams [9]. This leads to a generalization (we limit the analysis here to the dominant eigenmode) of the FM-CM duality relations in [7,8], which involves fractional Fourier transforms of complex order.

II. BT HYPERBOLOIDAL BEAMS

Referring to the problem geometry illustrated in Fig. 1, we consider a perfectly symmetric Fabry-Perot optical cavity composed of two nearly-spherical mirrors separated by a distance L along the z-axis of a Cartesian (x, y, z) (and associated cylindrical (r, θ, z)) coordinate system. The transverse coordinates at the waist (z = 0) and mirror (z = L/2) planes are denoted by $r_0 = r_0 \hat{x} + y_0 \hat{y} = r_0 \cos \theta_0 \hat{x} + r_0 \sin \theta_0 \hat{y}$ and $r = x\hat{x} + y\hat{y} = r \cos \theta \hat{x} + r \sin \theta \hat{y}$, respectively. Here and henceforth, $\hat{x}$, $\hat{y}$ and $\hat{z}$ denote the standard Cartesian unit vectors. Throughout the paper, an implicit time-harmonic $\exp(-i\omega t)$ dependence is assumed for all field quantities.

The BT construction [9], which generalizes the original idea in [3], is based on the superposition of minimum-superimposed Gaussian beams (GBs), which allow continuous spanning from the FM to the CM configurations, and including the standard GB case.

This paper elaborates on the analytic structure of the Bondarescu-Thorne (BT) hyperboloidal beams, building on a number of results from the applied optics literature concerning flat-top beams, which have most likely not come to the attention of the gravitational-wave community. Specifically, the FM and CM beams belong to the class of flattened beams introduced in [10], and can therefore be represented in terms of the rapidly-converging Gaussian-Laguerre (GL) beam expansions derived therein. Based on this observation, we extend the approach in [10] to accommodate the more general family of BT hyperboloidal beams [9]. This leads to a generalization (we limit the analysis here to the dominant eigenmode) of the FM-CM duality relations in [7,8], which involves fractional Fourier transforms of complex order.
For other values of \( \alpha \) the mirror location are roughly approximated by the “fiducial” spheroids \([9]\).

In (2), \( \Lambda \) is an \( \alpha \)-independent complex constant, and the GB spot size at the waist is chosen as \( w_0 = \sqrt{L/k_0} \) (\( k_0 = 2\pi/\lambda_0 \) denoting the free-space wavenumber, and \( \lambda_0 \) the free-space wavelength), so that the mirror plane is located exactly at the Rayleigh distance \([11]\), \( z_R \equiv k_0w_0^2/2 = L/2 \). For \( \alpha = \pi/2 \), the double integral in (2) can be computed in closed form, yielding a simple Gaussian \([9]\).

For other values of \( \alpha \), the radial integral in (2) can still be computed analytically, while the angular integral has to be evaluated numerically.

In order for the optical cavity to support a stable beam with a given profile as the fundamental eigenmode, its mirror profile has to match the beam wavefront; this can be achieved by applying a correction \([12]\)

\[
h_{\alpha}(r) = \frac{\arg[U_\alpha(r, S_\alpha)] - \arg[U_\alpha(0, S_\alpha)]}{k_0}
\]

\[
U_\alpha(r, z) = \Lambda \int_0^{R_0} dr_0 \int_0^{2\pi} d\theta_0 r_0 \exp \left[ i \frac{rr_0}{w_0} \sin \theta_0 \sin \alpha \right] - \left( \frac{r^2 + r_0^2}{2w_0^2} - 2\frac{r_0r}{r_0} \cos \theta_0 \right) \left( 1 - i \cos \alpha \right).
\]

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\]
The $A_m^{(\sigma)}$ are almost constant for $m \leq R_0^2/(2w_0^2)$, and fall off quite rapidly for $m > R_0^2/(2w_0^2)$ (see [10], Fig. 2). For the parametric range of interest for advanced LIGO ($w_0/R_0 = 0.25$), this results in a rapidly-converging ($m < 20$) expansion (5). A natural question then arises, as to whether the FM-CM expansion coefficient relation (10b) can be generalized to arbitrary values of the twist-angle $\alpha$, thereby allowing a GL representation for general BT hyperboloidal beams. In this connection, one notes that the mapping

$$A_m^{(\alpha)} = (-\cos \alpha)^m A_m^{(\pi)} \tag{11}$$

accounts correctly for the three notable cases $\alpha = 0$ (FM), $\alpha = \pi/2$ (GB) and $\alpha = \pi$ (CM). One is accordingly led to speculate whether it may hold for arbitrary values of the twist angle $\alpha$. This turns out to be indeed the case, as verified below by numerical comparison against the BT reference solution in (2). In all examples below, the relevant parameters (specified in the figure captions) were chosen as in [4,9]. For the truncation of the GL series involved, a simple criterion was utilized, requiring that the magnitude of the last retained $M$-th term is less than 0.1% of that of the leading term, $|A_m^{(\alpha)}| < 10^{-3}|A_0^{(\alpha)}|$. For the cases $\alpha = 0, \pi$ this yields $M = 18$. In view of the coefficient mapping in (11), the convergence becomes faster as $\alpha$ approaches the critical value of $\pi/2$ (pure GB, for which one obtains only one nonzero coefficient). As a reference solution, the BT integral representation in (2) was considered, with the radial integral computed analytically, the angular integration performed numerically, and the complex constant $\Lambda$ determined by enforcing the matching with the GL expansion at $r = 0$.

Some representative results for the field distribution are shown in Fig. 2. Specifically, Fig. 2(a) shows the GL-computed (via (5)) intensity distribution on the fiducial surface, for various values of the twist-angle $\alpha$, illustrating the gradual transition from the Gaussian ($\alpha = \pi/2$) to the mesa ($\alpha = 0, \pi$) profile. To quantify the agreement with the reference solution, Fig. 2(b) shows the relative error, which, consistently with the truncation criterion adopted, never exceeds 0.1% over the region of significant field intensity, and drops below numerical precision for the $\alpha = \pi/2$ (pure GB case). The results pertaining to the mirror profiles are shown Fig. 3. Specifically, Fig. 3(a) shows the GL-computed corrections $h_\sigma$ in (3), illustrating the gradual transition from the spherical ($\alpha = \pi/2$) to the Mexican-hat ($\alpha = 0, \pi$) mirror profile. Figure 3(b) shows the absolute error with respect to the reference solution, which never exceeds $10^{-4}\lambda_0$ over the significantly illuminated portion of the mirror. For the LIGO design ($\lambda_0 = 1064$ nm), this corresponds to errors $\sim 0.1$ nm, well within the typical fabrication tolerances.

The above results validate the analytic GL expansion in (5), which is obtained here for the first time, to the best of our knowledge, and sets the stage for a generalization, to arbitrary twist-angles, of the duality relations in [7,8]. In this framework, we consider a class of $\sigma$-parameterized generalized Hankel transform (HT) operators defined as

![FIG. 2.](image1)

![FIG. 3.](image2)
The generalized HT operator in (12) can be shown (see [14], p. 43 and [15] for more details) to admit as eigenfunctions the GL basis functions in (8),

\[
\mathcal{H}_{w_0}^{(\sigma)}[\psi_m(\sqrt{2}r/w_0)] = (-\sigma)^m \psi_m(\sqrt{2}r/w_0). \tag{13}
\]

Application of the generalized HT (12) to the GL expansion in (5) reveals, via (13), the functional relation between the field distributions at the waist plane pertaining to two BT hyperboloidal beams characterized by generic values, \(\alpha_1\) and \(\alpha_2\), of the twist-angle,

\[
U_{\alpha_2}(r, 0) \underset{\mathcal{H}_{w_0}^{(\sigma)}}{\longrightarrow} U_{\alpha_1}(r, 0), \quad \sigma = -\frac{\cos\alpha_2}{\cos\alpha_1}. \tag{14}
\]

The generalized HT in (14) extends, for the dominant eigenmode, the FM-CM duality relations in [7,8] to the most general case, and admits a suggestive analytic interpretation in terms of a fractional Fourier operator of complex order [16–18] (see also [15] for more details),

\[
\gamma = 1 + i \frac{\log(\sigma)}{\pi}, \tag{15}
\]

whose real part can either be 1 or 0, whereas the imaginary part is generally nonzero (except for the cases \(\alpha_1 = \alpha_2\) and \(\alpha_1 = \pi - \alpha_2\) for which the generalized HT operator in (12) reduces to the identity and ordinary HT operator, respectively).

IV. CONCLUDING REMARKS

In this paper, the analytic structure of a family of hyperboloidal beams of interest for advanced LIGO has been investigated, via the development of rapidly-converging GL beam expansions and a complex-Fourier-transform-based generalization of the duality relations in [7,8] for the dominant eigenmode. Extension to higher-order eigenmodes, aimed at the full generalization (for arbitrary twist-angles) of the duality relations in [7,8] is currently under investigation. It is hoped that the above results, which provide a physically-insightful and computationally-effective parameterization of the beam and mirror profiles, may be useful in addressing the optimization of the advanced-LIGO optical cavities in a broader perspective including a thorough parametric analysis of BT and other classes of flat-top beams (see, e.g., [19–21]), aimed at finding optimal design criteria in terms of thermal-noise and tilt-instability reduction.

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