A note on the semitotal domination number of trees

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Abstract
In this paper, we show that if $T$ is a tree that is not a star, then $\gamma_{t2}(T) \leq 2\gamma(T) - 1$, and provide a constructive characterization of the trees achieving equality in the bound. In addition, we also study the semitotal domination multisubdivision number of trees, $msd_{\gamma_{t2}}(T)$. We show that for any tree $T$ of order at least three, $1 \leq msd_{\gamma_{t2}}(T) \leq 3$, and characterize the trees whose semitotal domination multisubdivision number is 3.

Keywords domination number, semitotal domination number, semitotal domination multisubdivision number, tree

1 Introduction

Let $G = (V, E)$ be a simple graph without isolated vertices, and let $v$ be a vertex in $G$. The open neighborhood of $v$ is $N(v) = \{u \in V \mid uv \in E\}$ and the degree of $v$ is $d(v) = |N(v)|$. For two vertices $u$ and $v$ in a connected graph $G$, the distance $d(u, v)$ between $u$ and $v$ is the length of a shortest $(u, v)$-path in $G$. The maximum distance among all pairs of vertices of $G$ is the diameter of a graph $G$ which is denoted by $diam(G)$. A leaf of $G$ is a vertex of degree 1, and a support vertex of $G$ is a vertex adjacent to a leaf.

A set $S$ of vertices of a graph $G$ is called a dominating set (respectively, total dominating set) of $G$ if every vertex in $V(G) \setminus S$ (respectively, $V(G)$) is adjacent to at least one vertex in $S$. The domination number (respectively, total domination number) of $G$, denoted by $\gamma(G)$ (respectively, $\gamma_t(G)$), is the minimum cardinality of a dominating set (respectively, total dominating set) of $G$.

The concept of semitotal domination in graphs was introduced by Goddard et al. A set $S$ of vertices in $G$ is a semitotal dominating set of $G$ if it is a dominating set

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of $G$ and every vertex in $S$ is within distance 2 of another vertex of $S$. The semitotal domination number, $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of $G$. We observe that $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$. A semitotal dominating set (respectively, dominating set, total dominating set) of $G$ of cardinality $\gamma_{t2}(G)$ (respectively, $\gamma(G)$, $\gamma_t(G)$) is called a $\gamma_{t2}(G)$-set (respectively, $\gamma(G)$-set, $\gamma_t(G)$-set).

The domination multisubdivision number of a graph $G$ was defined in [5] as the minimum positive integer $k$ such that there exists an edge which must be subdivided $k$ times to increase the domination number of $G$. Subsequently, the total domination multisubdivision number was introduced by Alaminos et al. [1]. One of the purposes of our paper is to initialize the study of the semitotal domination multisubdivision number. The semitotal domination multisubdivision number of a graph $G$ is the minimum positive integer $k$ such that there exists an edge which must be subdivided $k$ times to increase the semitotal domination number of $G$.

An area of research in domination of graphs that has received considerable attention is the study of classes of graphs with equal domination parameters, or the ratio between two domination parameters, some related results can be referred to [2 – 4, 8, 10 – 13]. Motivated by the above papers, we are ready to consider the ratio between domination number and semitotal domination number. In this paper, we show that if $T$ is a tree that is not a star, then $\gamma_{t2}(T) \leq 2\gamma(T) - 1$, and provide a constructive characterization of the trees achieving equality in the bound. In addition, we show that for any tree $T$ of order at least three, $1 \leq msd_{\gamma_{t2}}(T) \leq 3$, and characterize the trees whose semitotal domination multisubdivision number is 3.

## 2 Domination versus semitotal domination in trees

From the definitions of domination number and semitotal domination number, we have the following observations.

**Observation 2.1** Let $G$ be a connected graph that is not a star. Then,

(i) there is a $\gamma$-set that contains no leaf of $G$, and

(ii) there is a $\gamma_{t2}$-set that contains no leaf of $G$.

Our aim in this section is to present a tight upper bound for the semitotal domination number of a non-star tree in terms of its domination number, and provide a constructive characterization of the trees achieving equality in this bound. For our purposes, we define a labeling of a tree $T$ as a partition $S = (S_A, S_B, S_C, S_D, S_E)$ of $V(T)$ (This idea of labeling the vertices is introduced in [6]). We will refer to the pair $(T, S)$ as a labeled tree. The label or status of a vertex $v$, denoted $sta(v)$, is the letter $x \in \{A, B, C, D, E\}$ such that $v \in S_x$. 

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Let $\mathcal{T}$ be the family of labeled trees that: (i) contains $(P_6, S_0)$ where $S_0$ is the labeling that assigns to the two leaves of the path $P_6$ status $C$, to the two support vertices status $A$ and $B$ respectively, and to the two center vertices status $D$ and $E$ respectively (see Fig.1(a)); and (ii) is closed under the two operations $O_1$ and $O_2$ that are listed below, which extend the tree $T'$ to a tree $T$ by attaching a tree to the vertex $v \in V(T')$.

**Operation $O_1$:** Let $v$ be a vertex with $\text{sta}(v) = A$ or $B$. Add a vertex $u$ and the edge $uv$. Let $\text{sta}(u) = C$.

**Operation $O_2$:** Let $v$ be a vertex with $\text{sta}(v) = A$. Add a path $u_1u_2u_3u_4$ and the edge $u_1v$. Let $\text{sta}(u_1) = D$, $\text{sta}(u_2) = E$, $\text{sta}(u_3) = B$ and $\text{sta}(u_4) = C$.

The two operations $O_1$ and $O_2$ are illustrated in Fig.1(b) and (c).

Let $(T, S) \in \mathcal{T}$ be a labeled tree for some labeling $S$. Then there is a sequence of labeled trees $(P_6, S_0), (T_1, S_1), \ldots, (T_{k-1}, S_{k-1}), (T_k, S_k)$ such that $(T_k, S_k) = (T, S)$. The labeled tree $(T_i, S_i)$ can be obtained from $(T_{i-1}, S_{i-1})$ by one of the operations $O_1$ and $O_2$, where $i \in \{1, 2, \ldots, k\}$, $T_0 = P_6$. We remark that a sequence of labeled trees used to construct $(T, S)$ is not necessarily unique. The graph in Fig.2 is an example which belongs to $\mathcal{T}$. 
In what follows, we present a few preliminary results.

**Observation 2.2.** Let $T$ be a tree of order at least 6 and $S$ be a labeling of $T$ such that $(T, S) \in \mathcal{T}$. Then, $T$ has the following properties:

(a) A vertex is labeled $A$ or $B$ if and only if it is a support vertex.

(b) A vertex is labeled $C$ if and only if it is a leaf.

(c) $|S_A| = 1$, $|S_B| = |S_D| = |S_E|$.

(d) The set $S_A \cup S_B$ is the unique $\gamma$-set of $T$.

(e) The set $S_A \cup S_B \cup S_D$ is a $\gamma_{t2}$-set of $T$.

(f) If a vertex has status $A$ (respectively, $B$), then each of its non-leaf neighbors is labeled $D$ (respectively, $E$).

(g) If a vertex has status $D$ (respectively, $E$), then it has degree two and the two neighbors are labeled $A$ and $E$ (respectively, $B$ and $D$).

From Observation 2.2 (c), (d) and (e), the following corollary can be derived immediately.

**Corollary 2.3.** Let $T$ be a tree and $S$ be a labeling of $T$ such that $(T, S) \in \mathcal{T}$. Then, $\gamma_{t2}(T) = 2\gamma(T) - 1$.

**Theorem 2.4.** Let $T$ be a tree that is not a star, we have that $\gamma_{t2}(T) \leq 2\gamma(T) - 1$. Moreover, the trees $T$ satisfying $\gamma_{t2}(T) = 2\gamma(T) - 1$ are precisely those trees $T$ such that $(T, S) \in \mathcal{T}$ for some labeling $S$.

**Proof.** We proceed by induction on the order of $T$. If $|T| \leq 6$, it is easy to verify that $\gamma_{t2}(T) \leq 2\gamma(T) - 1$, and $T = P_6$ when the equality holds. So we let $|T| \geq 7$ and assume that for every non-star tree $T'$ of order less than $|T|$ we have $\gamma_{t2}(T') \leq 2\gamma(T') - 1$, with
equality if and only if \((T', S') \in \mathcal{F}\) for some labeling \(S'\). By Observation 2.1(i), there exists a \(\gamma\)-set of \(T\) which contains no leaf, say \(D\).

Claim 1. Each support vertex has exactly one leaf-neighbor.

Suppose that \(v\) is a support vertex which has at least two leaf-neighbors, say \(v_1, v_2\). Let \(T' = T - v_1\) and \(R\) be a \(\gamma_{t2}\)-set of \(T'\) containing no leaf. Then, \(R\) is also a semitotal dominating set of \(T\). Hence, \(\gamma_{t2}(T) \leq \gamma_{t2}(T')\). Combining the fact that \(\gamma(T') \leq \gamma(T)\), we have that \(\gamma_{t2}(T) \leq \gamma_{t2}(T') \leq 2\gamma(T') - 1 \leq 2\gamma(T) - 1\). If \(\gamma_{t2}(T) = 2\gamma(T) - 1\), then we have that \(\gamma_{t2}(T') = 2\gamma(T') - 1\). By the inductive hypothesis, \((T', S') \in \mathcal{F}\) for some labeling \(S'\).

It follows from Observation 2.2(a) that \(v\) has status \(A\) or \(B\) in \(S'\). Let \(S\) be obtained from \(S'\) by labeling the vertex \(v_1\) with label \(C\). Then, \((T, S)\) can be obtained from \((T', S')\) by operation \(\mathcal{E}_1\). Thus, \((T, S) \in \mathcal{F}\).

We suppose that \(\text{diam}(T) \geq 6\) (the result is trivial when \(\text{diam}(T) \leq 5\)) and \(P = v_1v_2v_3 \cdots v_t\) be a longest path in \(T\) such that \(d(v_3)\) as large as possible.

Claim 2. \(d(v_3) = 2\).

Assume that \(d(v_3) > 2\). Let \(T' = T - \{v_1, v_2\}\) and \(R'\) be a \(\gamma_{t2}\)-set of \(T'\) containing no leaf. Note that \(D \setminus \{v_2\}\) is a dominating set of \(T'\). On the other hand, \(R' \cup \{v_2\}\) be a semitotal dominating set of \(T\). Thus, \(\gamma_{t2}(T) \leq \gamma_{t2}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2\gamma(T) - 2\).

Claim 3. \(d(v_4) = 2\).

Assume that \(d(v_4) > 2\) and \(u_1\) is a neighbor of \(v_4\) outside \(P\). Let \(T' = T - \{v_1, v_2, v_3\}\) and \(R'\) be a \(\gamma_{t2}\)-set of \(T'\) containing no leaf. From Claim 1 and the choice of \(P\), we have that at least one of the two conditions as follows holds:

1. \(u_1\) is a leaf;
2. \(u_1\) is a support vertex of degree two;
3. \(u_1\) has degree two and is adjacent to a support vertex of degree two, say \(u_2\).

In the first case, \(v_4\) belongs to \(D\) and \(R'\). Hence, \(\gamma(T) - 1 \geq \gamma(T')\) and \(\gamma_{t2}(T') + 1 \geq \gamma_{t2}(T)\). Similar to the proof of Claim 2, we have that \(\gamma_{t2}(T) \leq 2\gamma(T) - 2\).

In the second case, \(u_1\) belongs to \(D\) and \(R'\). Then, \(\gamma(T) - 1 \geq \gamma(T')\) and \(\gamma_{t2}(T') + 2 \geq \gamma_{t2}(T)\). It means that \(\gamma_{t2}(T) \leq \gamma_{t2}(T') + 2 \leq 2\gamma(T') - 1 + 2 \leq 2\gamma(T) - 1\). Suppose next that \(\gamma_{t2}(T) = 2\gamma(T) - 1\). Then we have equality throughout the above inequality chain. In particular, \(\gamma_{t2}(T') = 2\gamma(T') - 1\). By induction, \((T', S') \in \mathcal{F}\) for some labeling \(S'\). Since \(u_1\) is a support vertex in \(T'\), it follows from Observation 2.2 (a) that \(u_1\) has status \(A\) or \(B\) in \(S'\).

If \(\text{sta}(u_1) = A\), from \(d(u_1) = 2\) and Claim 1, we have that \(T' = P_6\) and \(v_4\) has status \(D\) in \(S'\). Then, \(T\) is the tree obtained from a star of order four by subdividing two edges twice and the remaining edge once. But in this case, \(\gamma_{t2}(T) = 4\) and \(\gamma(T) = 3\), a contradiction.
If \( \text{sta}(u_1) = B \), then \( v_4 \) has status \( E \) in \( S' \). It is easy to check that \( \gamma_{t_2}(T) = 2\gamma(T) - 2 \), a contradiction.

In the third case, \( u_2 \) belongs to \( D \) and \( R' \), \( |\{u_1, v_4\} \cap R'| = 1 \). Without loss of generality, we let \( v_4 \in R' \) (If \( u_1 \in R' \), then we can replace \( u_1 \) in \( R' \) by \( v_4 \)). It follows that \( \gamma(T) - 1 \geq \gamma(T') \) and \( \gamma_{t_2}(T') + 1 \geq \gamma_{t_2}(T) \). It means that \( \gamma_{t_2}(T) \leq 2\gamma(T) - 2 \). \( \square \)

Now, we let \( T' = T - \{v_1, v_2, v_3, v_4\} \) and \( R' \) be a \( \gamma_{t_2} \)-set of \( T' \). Clearly, \( |\{v_3, v_4, v_5\} \cap D| = 1 \). Without loss of generality, \( v_5 \in D \) (If \( v_5 \) or \( v_4 \) belongs to \( D \), then we can replace it in \( D \) by \( v_5 \)). It implies that \( \gamma(T) - 1 \geq \gamma(T') \). On the other hand, \( R' \cup \{v_2, v_3\} \) be a semitotal dominating set of \( T \). Hence, \( \gamma_{t_2}(T) \leq \gamma_{t_2}(T') + 2 \leq 2\gamma(T') - 1 + 2 \leq 2\gamma(T) - 1 \). Suppose next that \( \gamma_{t_2}(T) = 2\gamma(T) - 1 \). Then we have equality throughout the above inequality chain. In particular, \( \gamma_{t_2}(T') = 2\gamma(T') - 1 \). By induction, \( (T', S') \in \mathcal{T} \) for some labeling \( S' \).

If \( v_5 \) has status \( A \) in \( S' \), let \( S \) be obtained from \( S' \) by labeling the vertices \( v_1, v_2, v_3, v_4 \) with label \( C, B, E, D \), respectively. Then, \( (T, S) \) can be obtained from \( (T', S') \) by operation \( \mathcal{O}_2 \). Thus, \( (T, S) \in \mathcal{T} \). If \( \text{sta}(v_5) \in \{C, D, E\} \) in \( S' \), or \( \text{sta}(v_5) = B \) in \( S' \) and \( |T'| > 6 \), we always have that \( \gamma_{t_2}(T) \leq 2\gamma(T) - 2 \), a contradiction. If \( \text{sta}(v_5) = B \) in \( S' \) and \( |T'| = 6 \), then \( T' = P_6 \). Moreover, \( v_6, v_7, v_8 \) have status \( E, D, A \) in \( S' \), respectively. Let \( S'' \) be obtained from \( S' \) by relabeling the vertices \( v_5, v_6, v_7, v_8 \) with label \( A, D, E, B \), respectively. And let \( S \) be obtained from \( S'' \) by labeling the vertices \( v_1, v_2, v_3, v_4 \) with label \( C, B, E, D \), respectively. Thus, we can also obtain that \( (T, S) \in \mathcal{T} \). \( \square \)

3 The semitotal domination multisubdivision number of trees

From the definition of semitotal domination multisubdivision number, we have the following observations.

**Observation 3.1** (1) Let \( T \) be a tree, \( u, v \) be two adjacent support vertices. If either \( u \) or \( v \) has degree two, then \( msd_{\gamma_{t_2}}(T) \leq 2 \).

(2) Let \( T \) be a tree, \( u, v \) be two support vertices at distance two apart. If either \( u \) or \( v \) has degree two, then \( msd_{\gamma_{t_2}}(T) \leq 2 \).

Next, we present the upper bound of \( msd_{\gamma_{t_2}}(T) \).

**Theorem 3.2** For any tree \( T \) of order at least 3, \( msd_{\gamma_{t_2}}(T) \leq 3 \).

**Proof.** The result is trivial when \( \text{diam}(T) \leq 3 \), so we assume that \( \text{diam}(T) \geq 4 \). Let \( P = v_1v_2v_3 \cdots v_t \) be a longest path of \( T \). Let \( T' \) be obtained from \( T \) by subdividing the edge \( v_2v_3 \) with vertices \( x, y \) and \( z \), and let \( D \) be a \( \gamma_{t_2} \)-set of \( T' \) which contains no leaf.
Then, \(v_2 \in D\). Moreover, \(|\{x, y\} \cap D| = 1\). Without loss of generality, let \(y \in D\) (If \(x\) belongs to \(D\), then we can replace it in \(D\) by \(y\)). Set \(D' = (D \setminus \{z\}) \cup \{v_3\}\) when \(z \in D\) and \(D' = D\) when \(z \notin D\). Note that \(D' \setminus \{y\}\) is a semitotal dominating set of \(T\). That is, \(\gamma_{t2}(T) \leq \gamma_{t2}(T') - 1\). The proof is completed.

\[\square\]

**Fig. 3**

Trees are classified as Class 1, Class 2 and Class 3 depending on whether their semitotal domination multisubdivision number is 1, 2 or 3, respectively. In the following, we are ready to provide a constructive characterization of trees in Class 3.

Let \(\mathcal{U}\) be the family of labeled trees that: (i) contains \((P_3, S'_0)\) where \(S'_0\) is the labeling that assigns to the two leaves of the path \(P_3\) status \(C\) and to the support vertex status \(A\) (see Fig. 3(a)); and (ii) is closed under the operations \(\mathcal{P}_1, \mathcal{P}_2\) and \(\mathcal{P}_3\) that are listed below, which extend the tree \(T'\) to a tree \(T\) by attaching a tree to the vertex \(v \in V(T')\).

**Operation \(\mathcal{P}_1\):** Let \(v\) be a vertex with \(\text{sta}(v) = A\). Add a vertex \(u\) and the edge \(uv\). Let \(\text{sta}(u) = C\).

**Operation \(\mathcal{P}_2\):** Let \(v\) be a vertex with \(\text{sta}(v) = B\). Add a path \(v_1v_2v_3v_4\) and the edge \(vv_1\). Let \(\text{sta}(v_1) = B, \text{sta}(v_2) = \text{sta}(v_4) = C,\) and \(\text{sta}(v_3) = A\).

**Operation \(\mathcal{P}_3\):** Let \(v\) be a vertex with \(\text{sta}(v) = C\). Add a path \(v_1v_2v_3v_4v_5\) and the edge \(vv_1\). Let \(\text{sta}(v_1) = \text{sta}(v_2) = B, \text{sta}(v_3) = \text{sta}(v_5) = C\) and \(\text{sta}(v_4) = A\).
The graph in Fig. 4 is an example which belongs to $U$.

In what follows, we present a few preliminary results.

**Observation 3.3** Let $T$ be a tree of order at least 3 and $S$ be a labeling of $T$ such that $(T, S) \in U$. Then, $T$ has the following properties:

(a) If a vertex $v$ is a support vertex, then $\text{sta}(v) = A$, and each of its neighbors has status $C$.

(b) If a vertex $v$ is a leaf, then $v$ has status $C$.

(c) If a vertex $v$ is labeled $C$, then it is a leaf or a vertex all of whose neighbors are labeled $B$ except for one, which is labeled $A$.

(d) If a vertex $v$ has status $B$, then all of the neighbors of $v$ are labeled $B$ except for one, which is labeled $C$.

(e) $S_A$ and $S_C$ are two independent sets of $T$.

Before giving the following lemma, we shall need an additional notation. We call $D$ an almost semitotal dominating set of a graph $G$ relative to a vertex $v$ if $D$ is a dominating set of $G$ and every vertex in $D$ is within distance 2 of another vertex of $D$, except for $v$.

**Lemma 3.4** If $T$ is a tree such that $(T, S) \in U$ for some labeling $S$, then for any vertex $x \in S_A$, there exists an almost semitotal dominating set of $T$ relative to $x$ with cardinality $\gamma_{t2}(T) - 1$.

**Proof.** We know that $(T, S) \in U$ for some labeling $S$, and as mentioned in section 2, a sequence of labeled trees used to construct $(T, S)$ is not necessarily unique. So we
select a sequence used to construct \((T, S)\): \((T_0, S_0), (T_1, S_1), \ldots, (T_{k-1}, S_{k-1}), (T_k, S_k)\),
where \((T_0, S_0) = (P_3, S'_0)\) and \((T_k, S_k) = (T, S)\), such that the vertex \(x \in V(T_i) \setminus V(T_{i-1})\)
\((i \in \{0, 1, 2, \ldots, k\})\) and the number \(i\) as small as possible (this condition is essential for
the following algorithm).

Now, we construct a set \(H\) as follows.

(1) Set \(P := \emptyset\) and \(H := \{t\}\), where \(t\) is the vertex of \(V(T_0)\) which has status \(A\) in \(S_0\).
Set \(j := 1\).

(II) We query whether \(j > i\) or not.
— If the answer to the query is ‘yes’,
then go to (IV).
— If the answer to the query is ‘no’,
then go to (III).

(III) We query which operation is used at the \(j\)-th step.
— If \((T_j, S_j)\) is obtained from \((T_{j-1}, S_{j-1})\) by operation \(\mathcal{P}_1\).
then set \(j := j + 1\). Go to (II).
— If \((T_j, S_j)\) is obtained from \((T_{j-1}, S_{j-1})\) by operation \(\mathcal{P}_2\).
then set \(H := H \cup P \cup \{y\}\), where \(y\) is the vertex of \(V(T_j) \setminus V(T_{j-1})\) which has status \(A\) in \(S_j\). Set \(P := \emptyset\) and put the vertex of \(V(T_j) \setminus V(T_{j-1})\) which at distance 2 from \(y\)
into \(P\). Set \(j := j + 1\). Go to (II).
— If \((T_j, S_j)\) is obtained from \((T_{j-1}, S_{j-1})\) by operation \(\mathcal{P}_3\).
then set \(H := H \cup \{y, z\}\), where \(y\) is the vertex of \(V(T_j) \setminus V(T_{j-1})\) which has status \(A\)
in \(S_j\) and \(z\) is the vertex of \(V(T_j) \setminus V(T_{j-1})\) which at distance 3 from \(y\). Set \(P := \emptyset\) and put the vertex of \(V(T_j) \setminus V(T_{j-1})\) which at distance 2 from \(y\)
into \(P\). Set \(j := j + 1\). Go to (II).

(IV) We query whether \(j > k\) or not.
— If the answer to the query is ‘yes’,
then we terminate.
— If the answer to the query is ‘no’,
then go to (V).

(V) We query whether \(|V(T_j) \setminus V(T_{j-1})| = 1\) or not.
— If the answer to the query is ‘yes’,
then set \(j := j + 1\). Go to (IV).
— If the answer to the query is ‘no’,
then set \(H := H \cup \{w, h\}\), where \(w\) is the vertex of \(V(T_j) \setminus V(T_{j-1})\) which has status \(A\) in \(S_j\), and \(h\) is the vertex of \(V(T_j) \setminus V(T_{j-1})\) which at distance 2 from \(w\). Set \(j := j + 1\). Go to (IV).

After the end of this procedure, the set \(H\) is a desire set. Moreover, it follows from
the method of constructing the set \(H\) that \(H\) contains all vertices of \(S_A\).
Lemma 3.5 If \( T \) is a tree such that \((T, S) \in \mathcal{U}\) for some labeling \( S \), then \( T \) is in Class 3.

Proof. Let \( T^* \) be obtained from \( T \) by subdividing any edge \( w \) of \( T \) twice. It is easy to see that \( \gamma_{t2}(T) \leq \gamma_{t2}(T^*) \). In order to show that \( T \) is in Class 3, we need to show that \( \gamma_{t2}(T) \geq \gamma_{t2}(T^*) \).

Since \((T, S) \in \mathcal{U}\) for some labeling \( S \), we can select a sequence of labeled trees used to construct \((T, S)\): \((T_0, S_0), (T_1, S_1), \ldots, (T_{k-1}, S_{k-1}), (T_k, S_k)\), where \((T_0, S_0) = (P_3, S'_0)\) and \((T_k, S_k) = (T, S)\) such that \( w \in E(T_i) \setminus E(T_{i-1}) \) and the number \( i \) as small as possible.

If \( i = 0 \), \( w \) is a pendant edge in \( T_0 \), say \( xx_1 \), where \( x \) is the support vertex in \( T_0 \). Let \( y_1, y_2 \) be the two new vertices resulting from subdividing the edge \( xx_1 \). By Lemma 3.4, there exists an almost semitotal dominating set of \( T \) relative to \( x \) with cardinality \( \gamma_{t2}(T) - 1 \), say \( X \). Then, the set \( X \cup \{y_2\} \) is a semitotal dominating set of \( T^* \). Hence, \( \gamma_{t2}(T^*) \leq \gamma_{t2}(T) \). So we consider the case of \( i \neq 0 \).

If \( T_i \) is obtained from \( T_{i-1} \) by adding a vertex \( x_1 \) and joining it to a vertex \( x_2 \) of \( T_{i-1} \), which has status \( A \) in \( S_{i-1} \), then \( w = x_1x_2 \). We construct an almost semitotal dominating set \( H \) of \( T \) relative to \( x_2 \) with cardinality \( \gamma_{t2}(T) - 1 \), the method of constructing the set \( H \) is as mentioned in the algorithm of Lemma 3.4. Let \( y_1, y_2 \) be the two new vertices resulting from subdividing the edge \( x_1x_2 \). Then, \( H \cup \{y_1\} \) is a semitotal dominating set of \( T^* \). That is, \( \gamma_{t2}(T^*) \leq \gamma_{t2}(T) \).

If \( T_i \) is obtained from \( T_{i-1} \) by adding a path \( x_1x_2x_3x_4x_5 \) and an edge \( x_1x \), where \( x \) has status \( C \) in \( S_{i-1} \). We construct an almost semitotal dominating set \( H \) of \( T \) relative to \( x_4 \) with cardinality \( \gamma_{t2}(T) - 1 \), the method of constructing the set \( H \) is as mentioned in the algorithm of Lemma 3.4. It follows from the construction method of \( H \) and the definition of almost semitotal dominating set that \( x_1 \in H \). Let \( y_1, y_2 \) be the two new vertices resulting from subdividing the edge \( w \). If \( w = xx_1 \), then \( (H \setminus \{x_1\}) \cup \{y_1, y_2\} \) is a semitotal dominating set of \( T^* \). If \( w = x_1x_2 \), then \( H \cup \{x_2\} \) is a semitotal dominating set of \( T^* \). If \( w \in \{x_2x_3, x_3x_4, x_4x_5\} \), the proof is similar to the argument as above. In either case, we have that \( \gamma_{t2}(T^*) \leq \gamma_{t2}(T) \).

If \( T_i \) is obtained from \( T_{i-1} \) by adding a path \( x_1x_2x_3x_4 \) and an edge \( x_1x \), where \( x \) has status \( B \) in \( S_{i-1} \), the proof is similar to the argument as above. \( \square \)

Lemma 3.6 If a tree \( T \) of order at least 3 is in Class 3, then \((T, S) \in \mathcal{U}\) for some labeling \( S \).

Proof. We proceed by induction on the order \( n \) of \( T \). If \( T \) is a star of order at least 3, then it is in Class 3, and \((T, S) \in \mathcal{U}\), where \( S \) is the labeling that assigns to the support vertex of \( T \) status \( A \) and to the leaves status \( C \). It is easy to verify that no tree whose diameter is at most 6 is in Class 3, except for the stars of order at least 3. So we consider the case that \( diam(T) \geq 7 \). Assume that for any tree \( T' \) in Class 3 with order less than \( |T| \), we always have that \((T', S') \in \mathcal{U}\) for some labeling \( S' \).
Claim 1. Each support vertex has exactly one leaf-neighbor.

If not, assume that there is a support vertex \( u \) which is adjacent to at least two leaves. Deleting one of its leaf-neighbors, say \( v \), and denote the resulting tree by \( T' \). Take an edge \( w \in E(T') \), let \( T^* \) (respectively, \( T'^* \)) be obtained from \( T \) (respectively, \( T' \)) by subdividing the edge \( w \) twice. Let \( D \) be a \( \gamma_t \)-set of \( T' \) containing no leaf. Clearly, \( D \) is a semitotal dominating set of \( T \). Then, we have that \( \gamma_t(T) \leq \gamma_t(T') \leq \gamma_t(T'^*) \leq \gamma_t(T^*) = \gamma_t(T) \). Thus we must have equality throughout this inequality chain, whence \( \gamma_t(T') = \gamma_t(T'^*) \). That is, \( T' \) is in Class 3. By the inductive hypothesis, \((T', S) \in \mathcal{W}\) for some labeling \( S \). Let \( S \) be obtained from the labeling \( S' \) by labeling the vertex \( u_1 \) with label \( C \). Then, \((T, S) \in \mathcal{W}\).

Let \( P = v_1 v_2 v_3 \cdots v_4 \) be a longest path in \( T \) such that
(i) \( d(v_3) \) as large as possible, and subject to this condition
(ii) \( d(v_4) \) as large as possible.

By Claim 1, \( d(v_2) = 2 \). It follows from Observation 3.1 that \( d(v_3) = 2 \).

Claim 2. \( d(v_4) = 2 \).

Assume that \( d(v_4) > 2 \). From Observation 3.1(2), \( v_4 \) is not a support vertex. Let \( u \) be a neighbor of \( v_4 \) outside \( P \). Then, either \( u \) is a support vertex of degree two, or \( d(u) = 2 \) and it is adjacent to a support vertex of degree two outside \( P \).

In either case, we subdivide the edge \( uv_4 \) twice, and denote the resulting tree by \( T^* \). Clearly, \( \gamma_t(T^*) - 1 \geq \gamma_t(T) \). Contradicting to the condition that \( T \) is in Class 3.

Claim 3. \( d(v_5) = 2 \).

Assume that \( d(v_5) > 2 \). Let \( u \) be a neighbor of \( v_5 \) outside \( P \). If \( u \) is a leaf or a support vertex, we subdivide the edge \( uv_5 \) twice, and denote the resulting tree by \( T^* \). Clearly, \( \gamma_t(T^*) - 1 \geq \gamma_t(T) \). Contradicting to the condition that \( T \) is in Class 3.

Since \( d(v_5) > 2 \), there exists the leaves outside \( P \), say \( a_1, a_2, \cdots, a_{l} \), such that for each \( i \in \{1, 2, \cdots, l\} \), \( V(P_i) \cap V(P) = \{v_5\} \), where \( P_i \) is the shortest path between \( a_i \) and \( v_5 \). Without loss of generality, assume that \( P_1 = v_5 u_s u_{s-1} \cdots u_1 \) be the longest path among all \( P_i \), where \( u_1 = a_1 \). Note that \( s = 3 \) or \( 4 \).

From Observation 3.1, Claim 1 and the choice of \( P \), we only need to consider the case that each \( u_i \) has degree two, where \( i = 2, 3, \cdots, s \).

Let \( T' = T - \{v_1, v_2, v_3, v_4\} \). Clearly, \( \gamma_t(T) \leq \gamma_t(T') + 2 \). Let \( T^* \) (respectively, \( T'^* \)) be obtained from \( T \) (respectively, \( T' \)) by subdividing an edge \( w \in E(T') \) twice. Next, we ready to show that \( \gamma_t(T^*) - 2 \geq \gamma_t(T'^*) \). If \( w \not\in \{v_5 u_s, u_s u_{s-1}, \cdots, u_2 u_1\} \), then we are done. So \( w \in \{v_5 u_s, u_s u_{s-1}, \cdots, u_2 u_1\} \), without loss of generality, assume that \( w = v_5 u_s \).
That is, \( T^* \) (respectively, \( T'^* \)) be obtained from \( T \) (respectively, \( T' \)) by subdividing the edge \( v_5 u_s \) with vertices \( x_1, x_2 \).

If \( s = 3 \), let \( H \) be obtained from \( T \) by subdividing the edge \( v_5 v_6 \) with vertices \( y_1, y_2 \). Let \( D \) be a \( \gamma_t(H) \)-set which contains no leaf. Then, \( v_2 \in D \). Note that \(|\{v_3, v_4\} \cap D| = 1 \).
Without loss of generality, let \( v_4 \in D \) (If \( v_3 \) belongs to \( D \), then we can replace it in \( D \) by \( v_4 \)). Similarly, we have that \( u_2, v_5 \in D \). Clearly, \(|\{y_1, y_2\} \cap D| \leq 1\). If \(|\{y_1, y_2\} \cap D| = 1 \) and \( v_6 \notin D \), we can simply replace \( x \) in \( D \) by \( v_6 \), where \( x \in \{y_1, y_2\} \cap D \). If \(|\{y_1, y_2\} \cap D| = 1 \) and \( v_6 \in D \), we can simply replace \( x \) in \( D \) by \( y \), where \( x \in \{y_1, y_2\} \cap D \) and \( y \in N_H(v_6) \setminus \{y_2\} \). (Note that if \(|\{y_1, y_2\} \cap D = \emptyset\), then \( v_6 \in D \). Let \( D' = (D \setminus \{v_5\}) \cup \{x_2\} \) and \( D'' = (D \setminus \{v_2, v_4, v_5\}) \cup \{x_2\} \). The set \( D' \) is a \( \gamma_{t2} \)-set of \( T^* \), and the set \( D'' \) is a semitotal dominating set of \( T^* \). That is, \( \gamma_{t2}(T^*) - 2 \geq \gamma_{t2}(T'') \).

If \( s = 4 \), by a similar argument as above, we can also obtain the same conclusion. That is, \( \gamma_{t2}(T^*) - 2 \geq \gamma_{t2}(T'') \).

In summary, we have that \( \gamma_{t2}(T) \leq \gamma_{t2}(T') + 2 \leq \gamma_{t2}(T'') + 2 \leq \gamma_{t2}(T^*) = \gamma_{t2}(T) \). Consequently, we must have equality throughout this inequality chain, whence \( \gamma_{t2}(T') = \gamma_{t2}(T'') \). It follows that \( T' \) is in Class 3. By induction, \((T', S') \in \mathcal{W}\) for some labeling \( S' \). And then, \( u_1, u_2 \) have status \( C \) and \( A \), respectively. Moreover, by Observation 3.1(a), (c) and (d), \( u_3, v_5 \) have status \( C, B \) respectively when \( s = 3 \), and \( u_3, u_4, v_5 \) have status \( C, B, B \) respectively when \( s = 4 \). In either case, let \( S \) be obtained from the labeling \( S' \) by labeling the vertex \( v_1, v_2, v_3, v_4 \) with label \( C, A, C, B \), respectively. Then, \((T, S) \) can be obtained from \((T', S') \) by operation \( \mathcal{P}_2 \). Thus, \((T, S) \in \mathcal{F} \).

Now we let \( T' = T - \{v_1, v_2, v_3, v_4, v_5\} \), let \( w \in E(T') \) and \( T^* \) (respectively, \( T'' \)) be obtained from \( T \) (respectively, \( T' \)) by subdividing the edge \( w \) twice. Clearly, we have that \( \gamma_{t2}(T') + 2 \geq \gamma_{t2}(T) \).

If \( d(v_6) > 2 \), then \( v_6 \) is not a support vertex. Otherwise, let \( H \) be obtained from \( T \) by subdividing the edge \( v_1v_2 \) twice. It is easy to verify that \( \gamma_{t2}(H) - 1 \geq \gamma_{t2}(T) \). Contradicting the assumption that \( T \) is in Class 3. Since \( d(v_6) > 2 \), there exists the leaves, say \( b_1, b_2, \ldots, b_l \), such that for each \( i \in \{1, 2, \ldots, l\}, V(P'_i) \cap V(P) = \{v_6\} \), where \( P'_i \) is the shortest path between \( b_i \) and \( v_6 \). Without loss of generality, assume that \( P'_1 = v_6u_0u_1 \cdots u_s \) be the longest path among all \( P'_i \), where \( u_s = b_1 \). Note that \( s \leq 4 \). We only need to consider the case that \( d(u_i) = 2 \) for \( i = 0, 1, \ldots, s - 1 \) (otherwise, the proof is similar to the previous arguments).

If \( s = 2 \) or \( 3 \), we can obtain a similar contradiction as above. So we only need to consider the case that \( d(v_6) = 2 \), or \( d(v_6) > 2 \) and \( s = 1, 4 \). In these cases, by similar arguments as in Claim 3, we have that \( \gamma_{t2}(T^*) - 2 \geq \gamma_{t2}(T'') \). Hence, \( \gamma_{t2}(T) \leq \gamma_{t2}(T') + 2 \leq \gamma_{t2}(T'') + 2 \leq \gamma_{t2}(T^*) = \gamma_{t2}(T) \). Thus we must have equality throughout this inequality chain, whence \( \gamma_{t2}(T') = \gamma_{t2}(T'') \). That is, \( T' \) is in Class 3. By the inductive hypothesis, \((T', S') \in \mathcal{W}\) for some labeling \( S' \).

If \( d(v_6) = 2 \), or \( d(v_6) > 2 \) and \( s = 1 \), then \( v_6 \) has status \( C \) in \( S' \). If \( d(v_6) > 2 \) and \( s = 4 \), the vertices \( u_4, u_3, u_2, u_4, u_0 \) have status \( C, A, C, B, B \) in \( S' \), respectively. Since \( d(u_0) = 2 \), by Observation 3.3(d), \( v_6 \) has status \( C \). In either case, let \( S \) be obtained from the labeling \( S' \) by labeling the vertices \( v_1, v_2, v_3, v_4, v_5 \) with label \( C, A, C, B, B \), respectively. Then, \((T, S) \) can be obtained from \((T', S') \) by operation \( \mathcal{P}_3 \). Thus, \((T, S) \in \mathcal{F}\). □
As an immediate consequence of Lemmas 3.5 and 3.6 we have the following conclusion.

**Theorem 3.7** A tree $T$ of order at least 3 is in Class 3 if and only if $(T, S) \in \mathcal{U}$ for some labeling $S$.

**References**

[1] D. A. Alaminos, M. Dettlaff, M. Lemańska, R. Zuazua, *Total domination multisubdivision number of a graph*, Discuss. Math. Graph T., **35** (2015) 315-327.

[2] C. Brause, M. A. Henning, M. Krzywkowski, *A characterization of trees with equal 2-domination and 2-independence numbers*, Discrete Math. Theor., **19** (2017) 1-14.

[3] M. Chellali, O. Favaron, T. W. Haynes, *Ratios of some domination parameters in trees*, Discrete Math., **308** (2008) 3879-3887.

[4] J. Cyman, J. Raczek, *Total outer-connected domination numbers of trees*, Discrete Appl. Math., **157** (2009) 3198-3202.

[5] M. Dettlaff, J. Raczek, J. Topp, *Domination subdivision and multisubdivision numbers of graphs*, Discuss. Math. Graph T., **39** (2019) 829-839.

[6] M. Dorfling, W. Goddard, M. A. Henning, C. M. Mynhardt, *Construction of trees and graphs with equal domination parameters*, Discrete Math., **306** (2006) 2647-2654.

[7] W. Goddard, M. A. Henning, C. A. McPillan, *Semitotal domination in graphs*, Util. Math., **94** (2014) 67-81.

[8] T. W. Haynes, M. A. Henning, P. J. Slater, *Trees with equal domination and paired-domination numbers*, Ars Combinatoria, **76** (2005) 169-175.

[9] M. A. Henning, A. J. Marcon, *On matching and semitotal domination in graphs*, Discrete Math., **324** (2014) 13-18.

[10] M. A. Henning, S. A. Marcon, *Domination versus disjunctive domination in trees*, Discrete Math., **184** (2015) 171-177.

[11] X. Hou, *A characterization of trees with equal domination and total domination numbers*, Ars Combinatoria, **97** (2010) 499-508.

[12] M. Krzywkowski, *On the ratio between 2-domination and total outer-independent domination numbers of trees*, Chinese Aann. Math. B, **34** (2013) 765-776.

[13] Z. Li, J. Xu, *A characterization of trees with equal independent domination and secure domination numbers*, Inf. Process. Lett., **119** (2017) 14-18.