Classical and Quantum Mechanics of
Non-Abelian Chern-Simons Particles

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Abstract

We investigate the classical and quantum properties of a system of $SU(N)$ non-Abelian Chern-Simons (NACS) particles. After a brief introduction to the subject of NACS particles, we first discuss about the symplectic structure of various $SU(N)$ coadjoint orbits which are the reduced phase space of $SU(N)$ internal degrees of freedom or isospins. A complete Dirac’s constraint analysis is carried out on each orbit and the Dirac bracket relations among the isospin variables are calculated. Then, the spatial degrees of freedom and interaction with external gauge field are introduced by considering the total reduced phase space which is given by an associated bundle whose fiber is one of the coadjoint orbits. Finally, the theory is quantized by using the coherent state method and various quantum mechanical properties are discussed in this approach. In particular, a coherent state representation of the Knizhnik-Zamolodchikov equation is given and possible solutions in this representation are discussed.
I. INTRODUCTION

It is well known that the angular momentum is not quantized in two spatial dimensions because the rotation group $SO(2)$ is Abelian and this leads to one of the peculiar quantum mechanical properties of physical systems in two spatial dimensions: the existence of anyon and fractional spin and braid statistics [1,2]. They found many applications to various areas of physics [3] and in particular the anyon could be realized in the fractional quantum Hall effect and perhaps in high temperature superconductivity [4,5].

The notion of anyon can be generalized to non-Abelian anyon, that is anyon with internal degrees of freedom [6,10]. Especially in Ref. [6], the quantum mechanical model [11] of non-relativistic $SU(2)$ non-Abelian Chern-Simons (NACS) particles which carry non-Abelian charges and interact with each other through the non-Abelian Chern-Simons terms [12,13] was derived from a classical action principle and a detailed analysis of the model showed that they lead to the non-Abelian generalization [14] of fractional spin and braid statistics. Later, a model of $SU(N)$ NACS particles was constructed by considering the internal degrees of freedom defined on complex projective space $CP(N-1)$ [7]. This was generalized to an arbitrary group with invariant nonsingular metric [8] and an equivalent field-theoretic description of NACS particles was given [8,9]. Also, in Ref. [10], a Hamiltonian formalism of $SU(N)$ NACS particles on complex projective space was pursued by studying the symplectic structure of the reduced phase space of NACS particles which is given by an associated bundle [15]. The purpose of this paper is to extend the Hamiltonian analysis of the previous work [10] to other possible symplectic manifolds with $SU(N)$ symmetry and perform a rigorous coherent state quantization [16] of resulting classical NACS particles.

We start by giving a brief review of salient features of $SU(2)$ NACS particles theory to make this article self-complete. Let us first consider the anyon case. Anyons can be realized as particles carrying both charge and magnetic flux and a possible quantum mechanical model for them can be constructed [3] by considering a system of non-relativistic charged point particles coupled with the Abelian Chern-Simons gauge field [17]:
\[ L' = \sum_{\alpha} \frac{1}{2} m_{\alpha} \dot{q}_{\alpha}^2 + \int d^2 x A_\mu(x, t) j^\mu(x, t) + \frac{\theta}{2} \int d^2 x \varepsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho \] (1.1)

where

\[ j^\mu(x, t) \equiv (\rho(x, t), \mathbf{j}(x, t)) = \sum_\alpha e_{\alpha} x^\mu \delta^2(x - q_{\alpha}). \] (1.2)

e_{\alpha} (\alpha = 1, \ldots, N_p) is the charge of each particle. The Hamiltonian is given by

\[ H' = \sum_{\alpha} \frac{1}{2m_{\alpha}} \left( p_{\alpha}^i - e_{\alpha} A^i(q_{\alpha}) \right)^2 - \int d^2 x A_0(x, t) G(x, t) \] (1.3)

where \( G(x, t) \) is the Gauss law constraints

\[- \theta B(x, t) + \rho(x, t) = 0 \] (1.4)

with the magnetic field \( B(x, t) \). The Gauss constraints can be solved in the Coulomb gauge \( \nabla \cdot \mathbf{A} = 0 \) and we find

\[ A_i(x, t) = \frac{1}{2\pi \theta} \sum_\alpha \frac{e_{\alpha} \varepsilon_{ij}(x - q_{\alpha}) j}{|x - q_{\alpha}|^2}. \] (1.5)

This potential endows each particle with magnetic flux \( \Phi_{\alpha} = e_{\alpha}/\theta \) and provides statistical interactions between anyons. Alternatively, we can eliminate the interaction by a singular gauge transformation

\[ \psi'(q_1, \ldots, q_{N_p}) = \prod_{\alpha < \beta} \exp \left[ i e_{\alpha} \frac{\pi}{\theta} \Theta_{\alpha \beta} \right] \psi(q_1, \ldots, q_{N_p}) \] (1.6)

where \( \Theta_{\alpha \beta} \) is the relative polar angle between particles \( \alpha \) and \( \beta \). In this case, the Hamiltonian becomes free but the wave function \( \psi' \) is multi-valued and this is the description of anyon in the so-called anyon gauge [3].

In the non-Abelian case, we expect

\[ H_0 = \sum_{\alpha} \frac{1}{2m_{\alpha}} \left( p_{\alpha}^i - A^{ai}(q_{\alpha}) Q_{\alpha}^a \right)^2 \] (1.7)

where \( A^a_\mu \) is the non-Abelian gauge field and \( Q^a \) is the generator of a non-Abelian gauge group \( G \) with structure constants \( f_{abc} \)'s: \( [Q^a_{\alpha}, Q^b_{\beta}] = i f^{abc} Q^c_{\alpha} \delta_{\alpha \beta} \). The Gauss constraints would be
\[-\kappa B^a(x, t) + \sum_\alpha Q_\alpha^a \delta(x - q_\alpha) = 0 \quad (1.8)\]

for some constant $\kappa$. It turns out the above Hamiltonian and Gauss constraints can be derived from a classical action principle \[8\]. It can be constructed in terms of their spatial coordinates $q_\alpha$'s and the isospin functions $Q_\alpha^a$'s which transform under the adjoint representation of the internal symmetry group. Defining the isospin functions directly on the reduced phase space which is $S^2$ for the internal symmetry $SU(2)$,

\[
Q_1^a = J_\alpha \sin \theta_\alpha \cos \phi_\alpha, \quad Q_2^a = J_\alpha \sin \theta_\alpha \sin \phi_\alpha, \quad Q_3^a = J_\alpha \cos \theta_\alpha \quad (1.9)
\]

where $\theta_\alpha, \phi_\alpha$ are the coordinates of the internal $S^2$ and $J_\alpha$ is a constant, one may write the Lagrangian as \[8\]

\[
L = \sum_\alpha \left( \frac{1}{2} m_\alpha \dot{q}_\alpha^2 + J_\alpha \cos \theta_\alpha \dot{\phi}_\alpha \right) - \kappa \int d^2x \epsilon^{\mu
u\lambda} \text{tr} \left( A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right) - \kappa \int d^2x \sum_\alpha \left( A_1^a(t, x) \dot{q}_1^\alpha - A_0^a(t, x) \right) Q_\alpha^a \delta(x - q_\alpha). \quad (1.10)
\]

Here $\kappa = k/4\pi$, $k =$ integer, $A_\mu = A_1^a T^a$, $[T^a, T^b] = -\epsilon^{abc} T^c$ and tr($T^a T^b$) = $-1/2\delta_{ab}$. The equations of motion from the Lagrangian Eq. (1.10) contain Wong’s equations \[18\].

The above Gauss constraints can be solved explicitly in two gauge conditions. The first one is the axial gauge \[19,8\] in which, for example, we set $A_1^a = 0$. The remaining $A_2^a$ field becomes highly singular with strings attached to each source. The less singular solutions can be obtained by performing an analytic continuation of the gauge fields. Introducing complex spatial coordinates, $z = x + iy$, $\bar{z} = x - iy$, $z_\alpha = q_1^\alpha + i q_2^\alpha$, $\bar{z}_\alpha = q_1^\alpha - i q_2^\alpha$, $A_2^a = \frac{1}{2}(A_1^a - i A_2^a)$, $A_2^a = \frac{1}{2}(A_1^a + i A_2^a)$, analytic continuation means that $A_2^a$ and $A_2^\alpha$ are treated as independent variables which is consistent with the coherent state quantization scheme \[20\]. We choose $A_2^a = 0$ as a gauge fixing condition which was called holomorphic gauge in Ref. \[8\]. The solution of the Gauss constraints

\[
\Phi^a(z) = -\kappa \partial z A_2^a + \sum_\alpha Q_\alpha^a \delta(z - z_\alpha) = 0 \quad (1.11)
\]

in holomorphic gauge turns out to be \[8\].
\[ A_{z}^{\alpha}(z, \bar{z}) = \frac{i}{2\pi \kappa} \sum_{\alpha} Q_{\alpha}^2 \frac{1}{z - z_{\alpha}} + P(z) \]  

(1.12)

where \( P(z) \) is an arbitrary holomorphic polynomial in \( z \). The further choice of \( P(z) = 0 \), which is usually known as Knizhnik-Zamolodchikov (KZ) connection, results in a quantum mechanical model \([11,6]\) which provides a unified framework for fractional spin, braid statistics and KZ equation \([21]\).

Substituting the above solution into the \( N_{p} \) particle Hamiltonian Eq. (1.7), we obtain

\[ H = \sum_{\alpha} \frac{2}{m_{\alpha}} p_{\alpha}^{z} \left( p_{\alpha}^{z} - \frac{i}{2\pi \kappa} \sum_{\beta} Q_{\alpha}^2 Q_{\beta}^2 \frac{1}{z_{\alpha} - z_{\beta}} \right) \]  

(1.13)

Quantum mechanically, the dynamics of the NACS particles are governed by the operator version \( \hat{H} \) of the Hamiltonian Eq. (1.13) \([11,6–9]\)

\[ \hat{H} = - \sum_{\alpha} \frac{1}{m_{\alpha}} (\nabla_{z_{\alpha}} \nabla_{z_{\alpha}} + \nabla_{z_{\alpha}} \nabla_{\bar{z}_{\alpha}}) \]  

\[ \nabla_{z_{\alpha}} = \frac{\partial}{\partial z_{\alpha}} + \frac{1}{2\pi \kappa} \left( \sum_{\beta \neq \alpha} \hat{Q}_{\alpha}^a \hat{Q}_{\alpha}^b \frac{1}{z_{\alpha} - z_{\beta}} + \hat{Q}_{\alpha}^2 a_{z}(z_{\alpha}) \right) \]  

(1.14)

where \( a_{z}(z_{\alpha}) = \lim_{z \to \bar{z}_{\alpha}} 1/(z - z_{\alpha}) \) and the isospin operators \( \hat{Q}^a \)'s satisfy the \( SU(2) \) algebra, \([\hat{Q}_{\alpha}^a, \hat{Q}_{\beta}^b] = i\epsilon^{abc} \hat{Q}_{\alpha}^c \delta_{\alpha\beta} \) upon quantizing the classical Poisson bracket of isospin functions (1.9). The second term and the third term in \( \nabla_{z_{\alpha}} \) are responsible for the non-Abelian statistics and the exotic spins of the NACS particles respectively. This can be seen if the wave function \( \Psi_{h} \) for the NACS particles in the holomorphic gauge is expressed as follows:

\[ \Psi_{h}(z_{1}, \ldots, z_{N_{p}}) = U^{-1}(z_{1}, \ldots, z_{N_{p}}) U_{s}^{-1} \Psi_{u}(z_{1}, \ldots, z_{N_{p}}) \]  

\[ U_{s}^{-1} = \exp \left( -\frac{1}{2\pi \kappa} \sum_{\alpha} \lim_{z \to \bar{z}_{\alpha}} \int_{z}^{\bar{z}_{\alpha}} \hat{Q}_{\alpha}^2 \frac{1}{z - z_{\alpha}} dz \right) \]  

(1.15)

where \( U^{-1}(z_{1}, \ldots, z_{N_{p}}) \) satisfies the KZ equation \([21]\)

\[ \left( \frac{\partial}{\partial z_{\alpha}} + \frac{1}{2\pi \kappa} \sum_{\beta \neq \alpha} \hat{Q}_{\alpha}^a \hat{Q}_{\alpha}^b \frac{1}{z_{\alpha} - z_{\beta}} \right) U^{-1}(z_{1}, \ldots, z_{N_{p}}) = 0. \]  

(1.16)

The KZ equation has a formal solution which is expressed as a path ordered line integral in the \( N_{p} \)-dimensional complex space
\[ U^{-1}(z_1, \ldots, z_{N_p}) = P \exp \left[ -\frac{1}{2\pi \kappa} \int_{\Gamma} \sum_{\alpha} dz'_{\alpha} \sum_{\beta \neq \alpha} \hat{Q}^a_{\alpha} \hat{Q}^a_{\beta} \frac{1}{z'_{\alpha} - z'_{\beta}} \right] \]  

(1.17)

where \( \Gamma \) is a path in the \( N_p \)-dimensional complex space with one end point fixed and the other being \( z_f = (z_1, \ldots, z_{N_p}) \). Explicit evaluation \[ 3 \] of the above formal expression gives the monodromy matrices or the braid matrices. We see that \( \Psi_a \) obeys the the non-Abelian braid statistics due to \( U(z_1, \ldots, z_{N_p}) \) while the wave function \( \Psi_h \) obeys the ordinary statistics and the Hamiltonian for NACS particles becomes free in terms of \( \Psi_a(z_1, \ldots, z_{N_p}) \). We also observe these particles carry fractional spin \( 2j_\alpha(j_\alpha + 1)/k \), because

\[
\exp \left( -\frac{1}{2\pi \kappa} \lim_{z \to z_\alpha} \int_{z}^{z'} \frac{\hat{Q}^2_{\alpha}}{z - z'} dz \right)
\]

acquires a non-trivial phase

\[
\frac{-\hat{Q}^2_{\alpha}}{\kappa} i = -2\pi i \left( \frac{2j_\alpha(j_\alpha + 1)}{k} \right)
\]

under \( 2\pi \) rotation. In analogy with the Abelian Chern-Simons particle theory we may call \( \Psi_a \) the NACS particle wave function in the anyon gauge. Therefore we have two equivalent descriptions for the NACS particles as in the case of the Abelian Chern-Simons particles: in the holomorphic gauge and in the anyon gauge. \( U(z_1, \ldots, z_{N_p}) \) is the singular and non-unitary transformation function between the two gauges. It also defines an inner product in the holomorphic gauge

\[
\langle \Psi_1 | \Psi_2 \rangle = \int \prod_{\alpha} dz_{\alpha} d\bar{z}_{\alpha} \Psi_1(z) U^\dagger(z) U(z) \Psi_2(z).
\]  

(1.18)

The Hamiltonian (1.14) in the holomorphic gauge is hermitian with respect to this inner product. The detailed analysis of the above quantum mechanical model was performed in Ref. \[ 3 \].

In this paper, we generalize the above model to a system of \( SU(N) \) NACS particles and quantize it using the coherent state quantization method \[ 16 \]. To describe the \( SU(N) \) internal degrees of freedom, we first have to identify the phase space for them. The natural candidates are the coadjoint orbits of \( SU(N) \) group because they are symplectic manifolds \[ 22 \] with \( SU(N) \) symmetry and thus can be considered as the reduced phase space of generalized Hamiltonian dynamics \[ 23 \]. In fact, it can be shown that they are the reduced phase spaces of the cotangent bundle \( T^*SU(N) \) by the method of symplectic reduction \[ 23 \]. Having identified the coadjoint orbits as the reduced phase spaces by symplectic reductions, one
could proceed to formulate the whole theory by coupling with the spatial and external gauge degrees of freedom in the Lagrangian method with the first order formalism. In this approach, one writes down $SU(N)$ generalizations of the $SU(2)$ isospin functions (1.3) and the Lagrangian (1.10) on the $SU(N)$ coadjoint orbits. However, since not much is known about the coordinatization of them, we will not have explicit expressions of symplectic structure and isospin functions with a few exception, for example, like the case of $CP(N-1)$ [7]. This could pose a practical difficulty for this approach. One could still pursue the Lagrangian formulation by using the unreduced $SU(N)$ coordinates but with many constraints corresponding to each orbit put in the Lagrangian by Lagrange multipliers. This would make the theory look rather cumbersome. Instead we perform the Hamiltonian analysis in this paper.

The outline of the paper is as follows. We first consider various coadjoint orbits of $SU(N)$ group and study the symplectic structure of them. Total phase space will be obtained by coupling the internal degrees of freedom with the spatial degrees of freedom in the external gauge field. We will not include the phase space of the gauge field itself to make the presentation simple. Then, we quantize the system by the coherent state quantization method [16] and discuss about the various quantum mechanical properties in this approach. In Section 2, we start from symplectic reduction of the cotangent bundle of $SU(N)$ group to identify the coadjoint orbits as the reduced phase spaces of internal degrees of freedom and investigate the symplectic structure of the most general coadjoint orbit of $SU(N)$ group. The Dirac’s constraint analysis is carried out on each orbit. In Section 3, the symplectic structure on the total phase space which is an associated bundle is explicitly given. This section is drawn mostly from Ref. [10] and included in this paper for completeness. In Section 4, coherent state quantization method is applied to the NACS particles system defined on the reduced phase space of the associated bundle of Section 3 and quantum mechanical properties are discussed. In particular, we represent the KZ equation as a coherent state differential equation and discuss about the possible solutions of KZ equation in this method. Section 5 contains conclusion and discussion.
II. SYMPLECTIC REDUCTION AND $SU(N)$ ISOSPIN

We start from the configuration space for the internal degrees of freedom which can be taken as the group manifold $G$. To do analysis in a canonical approach, consider $T^*G \cong G \times G^*$, where $G^*$ is the dual of the Lie algebra $G$ of the group $G$ \cite{23}. A natural symplectic left group action on $T^*G$ can be defined as

$$G \times (G \times G^*) \longrightarrow G \times G^*$$

$$(g, (h, a)) \mapsto (gh, a). \quad (2.1)$$

Let us define the moment map $\rho : T^*G \rightarrow G^*$ via

$$<X, \rho(m)> = m \left( \frac{d}{dt} \bigg|_{t=0} \exp tX \circ g \right) \quad (2.2)$$

where $X \in \mathcal{G}$ and $m \in T^*_gG$ is the linear map of $\mathcal{G} \rightarrow \mathbb{R}$. The reduced phase space for isospin degrees of freedom can be obtained as the quotient space $\rho^{-1}(x)/G_x$ which is well defined for regular value of $x$. Here $G_x$ is the stabilizer group of the point $x \in G^*$. The above procedure is called a symplectic reduction. It can be shown that the reduced phase space is naturally identifiable with the coadjoint orbit $\mathcal{O}_x \equiv G \cdot x \subset G^*$ \cite{23}:

$$\rho^{-1}(x)/G_x \cong G/G_x \cong G \cdot x. \quad (2.3)$$

The same reduction can be achieved using the Dirac’s constraint analysis. According to Dirac \cite{24}, in general, there arise first class and second class constraints in the reduction of the phase space. In our case, momentum maps associated with $G_x$ are the first class constraints and the rest are second class. To see this, let us separate the momentum map $\rho_a(m) \equiv <T^a, \rho(m)>$ with $T^a$ being the generator of the Lie algebra $\mathcal{G}$, $[T^a, T^b] = -f^{ab}_c T^c$ with $\text{Tr}(T^a T^b) = -1/2\delta_{ab}$ into two groups: $T^\alpha$’s and $T^i$’s where $T^\alpha$’s belong to the Lie algebra of stabilizer subgroup $G_x$ and $T^i$’s are the rest. The indices $\alpha, \beta, \ldots, i, j, \ldots$ will be used repeatedly unless confusion arises. Then we have the following:

$$[T^\alpha, T^\beta] = -f^{\alpha\beta\gamma} T^\gamma, \quad [T^\alpha, T^i] = -f^{\alpha i j} T^j, \quad [T^i, T^j] = -f^{ijk} T^k - f^{ij\alpha} T^\alpha. \quad (2.4)$$
We also have the Lie algebra homomorphism on $T^*G$ \[23\]:

$$\{\rho_a, \rho_b\} = -f_{ab}^c \rho_c.$$ \tag{2.5}

Let us define $x_a = (x_\alpha, x_i)$. Then the constrained space $\rho^{-1}(x)$ is a subspace of $T^*G$ given by the level set $\rho_a = x_\alpha, \rho_i = x_i$. We can rewrite these equations in terms of the constraints $\Gamma_a \equiv \rho_a - x_\alpha \approx 0$. Since the group $G_x$ is the stabilizer group of the point $x \in G^*$, we have $\text{Ad}^*(T^a)(x_a) = 0$ and it gives $f_{\alpha \beta}^\gamma x_\gamma = f_{ai}^j x_j = 0$. So we have the following constraints algebra:

$$\{\Gamma_\alpha, \Gamma_\beta\} \approx -f_{\alpha \beta}^\gamma \Gamma_\gamma, \quad \{\Gamma_\alpha, \Gamma_i\} \approx -f_{ai}^j \Gamma_j,$$

$$\{\Gamma_i, \Gamma_j\} \approx -f_{ij}^k \Gamma_k - f_{ij}^a \Gamma_a - c_{ij}$$ \tag{2.6}

where we have $c_{ij} = f_{ij}^a x_\alpha + f_{ij}^k x_k \neq 0$. We see that $\Gamma_\alpha \approx 0$ are the first class constraints while $\Gamma_i \approx 0$ are second class constraints. Reduction to $\rho^{-1}(x)/G_x \cong G/G_x$ is achieved with a suitable gauge choice corresponding to the first class constraints $\Gamma_\alpha$’s.

Let us consider possible $SU(N)$ coadjoint orbits of type $O_{\{n_1, n_2, \ldots, n_l\}} \equiv SU(N)/SU(n_1) \times \cdots \times SU(n_l) \times U(1)^{l-1}$. Here we have $\sum_{i=1}^l n_i = N$ and the rank of the subgroup $H \equiv SU(n_1) \times \cdots \times SU(n_l) \times U(1)^{l-1}$ is equal to $N - 1$. It is well known that there is a natural symplectic structure on the coadjoint orbits of a Lie group \[22\]. They also have the complex structure inherited from the complex representation of $O_{\{n_1, n_2, \ldots, n_l\}} = SL(N, \mathbb{C})/P_{\{n_1, n_2, \ldots, n_l\}}$, where $SL(N, \mathbb{C})$ is the complexification of $SU(N)$ and $P_{\{n_1, n_2, \ldots, n_l\}}$ is a parabolic subgroup of $SL(N, \mathbb{C})$ which is the subgroup of block upper triangular matrices in the $(n_1 + n_2 + \cdots + n_l) \times (n_1 + n_2 + \cdots + n_l)$ block decomposition. Borel subgroup $B$ corresponds $P_{\{1,1,\ldots,1\}}$. Together with the symplectic structure, they become Kähler manifolds. Let us assume that the symplectic two form is given in the local complex coordinate $(\xi, \bar{\xi})$ by the Kähler form

$$\Omega = \sum_{i,j} \Omega_{ij} d\xi^i \wedge d\bar{\xi}^j$$ \tag{2.7}

where $\Omega_{ij}$ can be expressed in terms of Kähler potential $W$ by

$$\Omega_{ij} = i \partial_i \bar{\partial}_j W.$$ \tag{2.8}
Then the Poisson bracket can be defined via

\[ \{f, g\} = \sum_{i,k} \Omega^{ki} \left( \frac{\partial f}{\partial \xi^k} \frac{\partial g}{\partial \bar{\xi}^i} - \frac{\partial g}{\partial \xi^k} \frac{\partial f}{\partial \bar{\xi}^i} \right) \]  \hspace{1cm} (2.9)

where the inverse \( \Omega^{ki} \) satisfies \( \Omega^{ik} \Omega^{kj} = \delta^j_i \).

Isospin degrees of freedom on the coadjoint orbit \( O_{\{n_1, n_2, \ldots, n_l\}} \) can be defined as

\[ Q = \text{Ad}^*(x)g = gxg^{-1} \quad g \in SU(N) \]  \hspace{1cm} (2.10)

where \( x = i \text{diag}(x_1, x_2, \ldots, x_N) \). Here \( \sum_{i=1}^N x_i = 0 \) and we choose without loss of generality \( x_1 = x_2 = \cdots x_{n_1} > x_{n_1+1} = x_{n_1+2} = \cdots x_{n_2} = \cdots = x_{n_l-1} = x_{n_l} \). The restriction is \( n_i \leq N - 1 \). When \( n_1 = N - 1, n_2 = 1 \) or \( n_1 = 1, n_2 = N - 1 \), the orbit corresponds to the minimal orbit which is a complex projective space \( CP(N) \). When \( n_1 = n_2 = \cdots n_l = 1 \), it corresponds to the maximal orbit which is a flag manifold. It is convenient to express the element \( g \) of \( SU(N) \) by \( N \) column vectors \( (Z_1, Z_2, \ldots, Z_N) \), \( Z_p \in \mathbb{C}^N \) \( (p, q = 1, \cdots, N) \) such that

\[ \bar{Z}_p Z_q = \delta_{pq}, \quad \det(Z_1, Z_2, \ldots, Z_N) = 1. \]  \hspace{1cm} (2.11)

Let us consider the canonical one form \( \theta \)

\[ \theta = \text{Tr}(x g^{-1} dg) = i \sum_{p=1}^N x_p \bar{Z}_p dZ_p. \]  \hspace{1cm} (2.12)

Using the second equation of (2.11), we find

\[ \theta = i \sum_{p=1}^{N-1} J_p \bar{Z}_p dZ_p, \quad J_p = x_1 + \cdots + 2x_p + \cdots + x_{N-1} \geq 0 \]  \hspace{1cm} (2.13)

and the symplectic two-form \( \Omega' \):

\[ \Omega' = d\theta = -i \sum_{p=1}^{N-1} J_p dZ_p \wedge d\bar{Z}_p. \]  \hspace{1cm} (2.14)

Note the inequality \( J_{n_1} > J_{n_2} > \cdots > J_{n_l} \) and there still exist constraints \( \bar{Z}_p Z_q - \delta_{pq} \approx 0 \) \( (p, q = 1, \cdots, N - 1) \). Also \( Q \) can be expressed as

\[ Q = i \sum_{p=1}^{N-1} (J_p Z_p \bar{Z}_p - (J_p/N) I). \]  \hspace{1cm} (2.15)
We define the isospin functions $Q^a$'s by

$$Q^a = \text{Tr}(QT^a).$$  \hfill (2.16)

In the case of $SU(2)$ example, we have

$$Q^1 = -\frac{J}{2}(\xi^0\xi^1 + \xi^1\xi^0), Q^2 = -i\frac{J}{2}(\xi^0\xi^1 - \xi^1\xi^0), Q^3 = -\frac{J}{2}(\xi^0\xi^0 - \xi^1\xi^1)$$  \hfill (2.17)

with the constraint $|\xi^0|^2 + |\xi^1|^2 = 1$. This is the Hopf map of $S^3 \to S^2$.

To find the Poisson bracket relations among the isospin functions $Q^a$'s, we define the canonical Poisson bracket relations by \[26\to27\]

$$\{\bar{Z}_p^i, Z_q^j\} = (i/J_p)\delta_{pq}\delta^{ij} \quad (p, q = 1, \cdots N - 1; i, j = 1, \cdots, N)$$  \hfill (2.18)

with the constraints $\bar{Z}_p Z_q - \delta_{pq} \approx 0$. We suppose that none of the $J_p$'s are equal to zero for the time being. The case in which some of the $J_p$'s are equal to zero will be considered shortly after. Let us divide the constraints into

$$\Psi_p = \bar{Z}_p Z_p - 1 \approx 0, \quad \Phi_{pq} = \bar{Z}_p Z_q \approx 0 \; (p \neq q).$$  \hfill (2.19)

Using Eq. (2.18), we can check that the following constraint algebra holds:

$$\{\Psi_p, \Psi_q\} \approx 0$$

$$\{\Psi_p, \Phi_{qr}\} = (i/J_p)(\delta_{pq}\Phi_{qr} - \delta_{pr}\Phi_{pq})$$

$$\{\Phi_{pq}, \Phi_{rs}\} = (i/J_p)\delta_{ps}\bar{Z}_r Z_q - (i/J_q)\delta_{qr}\bar{Z}_p Z_s.$$  \hfill (2.20)

We see that each of $\Psi_p \approx 0$ is a first class constraint. Also from the third equation of (2.20), we deduce that each of $\Phi_{n_i} \equiv \Phi_{pq}(\sum_{j=1}^{l-1} n_j \leq p, q \leq \sum_{j=1}^l n_j; i = 1, 2, \cdots, l; n_0 \neq 0)$ is a first class constraint and the rest $\Phi_{pq}$'s are second class ones. So there are $(N - 1) + \sum_{i=1}^l n_i(n_i - 1)$ first class constraints and $(N - 1)(N - 2) - \sum_{i=1}^l n_i(n_i - 1)$ second class constraints. Note that the dimension of the reduced phase space is $2N(N - 1) - 2[(N - 1) + \sum_{i=1}^l n_i(n_i - 1)] - [(N - 1)(N - 2) - \sum_{i=1}^l n_i(n_i - 1)] = N^2 - \sum_{i=1}^l n_i^2$ which coincides with the dimension of the coadjoint orbit $O_{n_1, n_2, \cdots, n_l} = SU(N)/SU(n_1) \times \cdots \times SU(n_l) \times U(1)^{l-1}$. The first class
constraints generate the subgroup \( SU(n_1) \times \cdots \times SU(n_l) \times U(1)^{l-1} \). A Dirac’s constraint analysis on the maximal coadjoint orbit of dimensions \( N^2 - N \) with only first class constraints was performed before \([27]\). We can eliminate the second class constraints all together by using the Dirac bracket \([24]\):

\[
\{f, g\}^* = \{f, g\} - \sum_{pqrs} \{f, \Phi_{pq}\} D_{pq,rs}^{-1} \{\Phi_{rs}, g\}
\]

with

\[
D_{pq,rs}^{-1} \equiv \frac{i J_p J_q}{J_q - J_p} \delta_{pq,rs} \quad (p \neq q).
\]

Here \( \iota \) denotes that the first class constraints \( \Phi_n \)'s do not appear in the sum and \( \delta_{pq,rs} = \delta_{ps} \delta_{qr} \). Using the expression for isospin functions \((2.16)\), we find

\[
\{Q^a, Q^b\}^* = -f^{ab}_c Q^c.
\]

We note that the above equation implies that the relation \( \{Q^a, Q^b\} = -f^{ab}_c Q^c \) results if one calculates directly on the reduced phase space using the Eq. \((2.9)\).

When some of the \( J_p \)'s are zero, we assume that all the \( Z_p \)'s for which \( J_p = 0 \) can be eliminated by the Eq. \((2.19)\) in terms of \( Z_q \)'s for which \( J_q \neq 0 \) and these variables do not appear in the consequent analysis. This assumption is safe in view of the fact that the internal degrees of freedom defined in the Eq. \((2.13)\) does not contain the variables \( Z_p \)'s for which \( J_p = 0 \) and one does not have to consider the Dirac brackets which contain these variables. Note that the constraint algebra \((2.20)\) is also restricted to the constraints which do not contain these variables. A nice example of the above procedure is the case of \( CP(N-1) \). Consider \( x = i \text{diag}(N-1, -1, \cdots, -1) \) so that \( J_1 = N-1 \) and \( J_2 = \cdots = J_{N-1} = 0 \). This orbit is the \( CP(N-1) = SU(N)/SU(N-1) \times U(1) \). According to our prescription, the only remaining variables are \( Z_1 \in \mathbb{C}^N \) with the constraints \( \Psi_1 = Z_1 Z_1 - 1 \approx 0 \) which is obviously first class and generates a circle action. So the reduced phase space becomes \( S^{2N-1}/S^1 \) which is another representation of \( CP(N-1) \). The constraint equation \( \Psi_1 = Z_1 Z_1 - 1 \approx 0 \) with \( Z^T = (\Xi_0, \Xi_1, \cdots, \Xi_{N-1}) \) can be solved explicitly in the gauge choice \( \Xi_0 = \Xi_0(\neq 0) \) with the results
Here $\xi^i = \Xi^i/\Xi_0$ is the inhomogeneous coordinate for $CP(N - 1)$. The isospin function (2.16) is expressed as

$$Q^a(\xi, \bar{\xi}) = J_i \sum_{I,K=0}^{N-1} \Xi_i T_{IK}^a \Xi_K \quad (I = 0, i)$$  \hspace{1cm} (2.25)$$

where Eq. (2.24) is substituted into the final expression.

III. COUPLING WITH THE SPATIAL DEGREE OF FREEDOM

Consider two dimensional configuration space $M$ which we assume to be a plane. (We present one particle case and later extend to $N_p$ particles in a straightforward manner.) To account for the spatial degree of freedom and external gauge field, let us consider the principal $G = SU(N)$ bundle $P$ over $M$: $P \to M$. The universal phase space for isospin particles in external gauge field can be defined as the direct product $T^*P \times O\{n_1, n_2, \ldots, n_l\}$. The left action of $G$ on $P$ defined by $(g \cdot p) = p \cdot g^{-1}$ can be lifted to $T^*P$ and let us denote the momentum map for this action by $-\nu$. Also let us denote the momentum map for the $G$ action on $O\{n_1, n_2, \ldots, n_l\}$ by $\mu$. Applying the symplectic reduction procedure [23], we consider the constrained manifold $(-\nu + \mu)^{-1}(0)$ and dividing by $G$, we get the reduced phase space $(-\nu + \mu)^{-1}(0)/G$ [23]. When a connection on $P$ is chosen, the reduced phase space becomes diffeomorphic to the associated bundle $\mathcal{P} \equiv \bar{P} \times_G O\{n_1, n_2, \ldots, n_l\}$ where $\bar{P} \to M$ is the pull-back bundle of the bundle $P$ by the projection $\pi' : T^*M \to M$. $\mathcal{P}$ is Sternberg’s reduced phase space [13] and it can be shown that a given connection $\Theta$ on $P$ determines a unique symplectic structure on $\mathcal{P}$.

The essence of Sternberg’s reduced phase space is that there exist a unique form $\Omega_\Theta$ on $\mathcal{P}$ such that

$$d(\Theta, Q) + \pi^*\Omega = \bar{\pi}^*\Omega_\Theta$$ \hspace{1cm} (3.1)
where \( \pi \) is the projection map: \( \bar{P} \times O_{\{n_1, n_2, \ldots, n_l\}} \to O_{\{n_1, n_2, \ldots, n_l\}} \) and \( \bar{\pi} \) is the projection map \( \bar{\pi} : \bar{P} \times O_{\{n_1, n_2, \ldots, n_l\}} \to \bar{P} \). \( \Omega \) is the symplectic two form on the coadjoint orbit \( [2,7] \). It can be shown that the two form \( \Omega_\Theta \) is closed and nondegenerate in the case when \( \bar{P} \) is the pull-back of the bundle \( P \) as above and the connection is the pull-back of a connection defined on \( P \) \([13]\). Denoting \( \tilde{\omega} \) for the pull-back of \( \omega \) which is the canonical symplectic structure defined on \( T^*M \) to \( P \) via the projection onto \( T^*M \), we have a symplectic structure on \( P \) as

\[
\Omega_T = \tilde{\omega} + \Omega_\Theta. \tag{3.2}
\]

When \( M \) is a plane, \( T^*M \) is contractible and every associated bundle is trivial. So we have

\[
P = P \times_G O_{\{n_1, n_2, \ldots, n_l\}} = T^*M \times O_{\{n_1, n_2, \ldots, n_l\}}. \tag{3.3}
\]

In fact, the above holds for arbitrary Riemann surfaces \( M \) \([10]\). Hence we have \( \tilde{\omega} = \omega \) and

\[
\Omega_T = \omega + \sigma^*(d(\Theta, Q) + \pi^*\Omega)
= \omega + d(A^a Q^a) + \Omega \tag{3.4}
\]

where \( \sigma \) is the cross section : \( P \times_G O_{\{n_1, n_2, \ldots, n_l\}} \to P \times O_{\{n_1, n_2, \ldots, n_l\}} \) and we used \( \sigma^*\Theta = A \), the gauge field one form on \( M \). Notice that \( \omega + \Omega \) is not gauge invariant. We must have Sternberg’s two form \( d(\Theta, Q) \) to achieve the gauge invariance. Physically, this term describes the interaction between isospin charge and the external gauge field.

Now, we explicitly calculate the symplectic structure on \( P \) and prove the minimal substitutions for the non-Abelian case. We start from the two form on \( P \) given by

\[
\Omega_T = dp_i \wedge dq^i + d(A_i^a Q^a dq^i) + \Omega. \tag{3.5}
\]

To achieve the notational simplifications, we introduce \( \eta^I = (p_i, q^i) \) and \( x^M = (\xi^A, \bar{\xi}^B, \eta^I) \). \( \xi^A \)'s and \( \bar{\xi}^B \)'s are the internal coordinates. Then we can write \( \Omega_T = \frac{1}{2} \Omega_{MN} dx^M \wedge dx^N \). Using Eq. (3.3), one finds the following inverse matrix \( \Omega^{MN} \):

\[
\Omega^{MN} = \begin{pmatrix}
\Omega^{AB} & -F^{KJ} \Omega^{AC} A_K^a (\partial Q^a / \partial \xi^C) \\
F^{KI} \Omega^{BD} A_K^a (\partial Q^a / \partial \xi^D) & F^{IJ}
\end{pmatrix} \tag{3.6}
\]
where $F^{IJ}$ is the inverse matrix of $F_{IJ} \equiv \omega_{IJ} - f_{abc} A_i^a A_j^b Q^c$ and is given by

$$F^{IJ} = \begin{pmatrix} F^a_{ij} Q^a & -I \\ I & 0 \end{pmatrix}. \quad (3.7)$$

Here $F^a_{ij} \equiv \partial_j A_i^a - \partial_i A_j^a - f_{bc} A_i^b A_j^c$ is the Yang-Mills field strength.

The Poisson bracket on $\mathcal{P}$ is defined by the use of inverse matrix $\Omega^{MN}$ as before

$$\{F, H\} = \Omega^{MN} \frac{\partial F}{\partial x^M} \frac{\partial H}{\partial x^N}. \quad (3.8)$$

And we find the following Poisson brackets along with $\{Q^a, Q^b\} = -f_{ab}^c Q^c$,

$$\{Q^a, p_i\} = -f_{bc}^a A_i^b Q^c, \quad \{Q^a, q^i\} = 0 \quad (3.9)$$

$$\{p_i, p_j\} = F^a_{ij} Q^a, \quad \{p_i, q^j\} = -\delta^j_i, \quad \{q^i, q^j\} = 0.$$

The above relations are in accordance with the minimal substitution

$$p_i \rightarrow P_i = p_i - A_i^a Q^a. \quad (3.10)$$

In terms of canonical momentum $P_i$, we have, among others,

$$\{Q^a, P_i\} = 0 \quad \{P_i, P_j\} = 0. \quad \{P_i, q^j\} = -\delta^j_i. \quad (3.11)$$

Thus, one can work in $(p_i, q^i, Q^a)$ coordinates using the symplectic structure given by Eqs. (3.6) and (3.7) or with $(P_i, q^i, Q^a)$ using the canonical symplectic structure without mixing between $P_i$ and $Q^a$. The two procedures are equivalent [15].

Consider, for example, the free Hamiltonian $H = (1/2m)p^2$ with the symplectic structure given by Eqs. (3.6) and (3.7). The Hamiltonian equations of motion

$$\dot{x}^M = \Omega^{MN} \frac{\partial H}{\partial x^N} \quad (3.12)$$

reproduce the well known Wong’s equations [18]

$$m \ddot{q}_i = F^a_{ij} Q^a \dot{q}^j, \quad \dot{Q}^a = -f_{bc}^a A_i^b \dot{q}^i Q^c \quad (3.13)$$
which describe the dynamics of an isospin particle in an external gauge fields $A^a_i$. Minimal substitution implies that alternatively, we can work with

$$H = \frac{1}{2m} (P_i - A^a_i Q^a)^2$$  (3.14)

with canonical symplectic structure Eq. (3.11). Obviously, we get the same equations of motions. The above procedures can be generalized to a system of many particles in an obvious manner and can be applied to a system of NACS particles. We end up with the Hamiltonian Eq. (1.7) where the $Q^a$’s are now given by $SU(N)$ isospin functions (2.16) on each $O_{\{n_1, n_2, \ldots, n_l\}}$. Also, $SU(N)$ Gauss law constraint and its solution in complex spatial coordinates are in the same form as the equations (1.11) and (1.12) with $SU(N) Q^a$’s. Then, the quantum mechanical Hamiltonian of a system of NACS particles is obtained in the same expression as the equation (1.14) with the isospin operators satisfying $SU(N)$ algebra;

$$[\hat{Q}_a^a, \hat{Q}_b^b] = if^{ab}_c \hat{Q}_c^c \delta_{\alpha \beta}$$. We can infer that most of the quantum mechanical properties of $SU(2)$ NACS particles carry qualitatively over to $SU(N)$ case and in particular, a system of $SU(N) NACS$ particles also exhibit $SU(N)$ braid statistics described by $SU(N)$ KZ equation.

It is worth mentioning the origin of the Gauss law constraint (1.11) in our Hamiltonian formulation at this point. It can be shown [10] that it is the condition of the vanishing momentum map of gauge transformations in the total phase space in which the phase space of gauge connection is also included. In our Hamiltonian approach, we neglected the space of gauge connection for simplicity and reduced phase space of a system of $N_p$ NACS particles is given by an associated bundle $\prod_\alpha P_\alpha \equiv \prod_\alpha T^* M_\alpha \times O_{\{n_1, n_2, \ldots, n_l\}}^\alpha (\alpha = 1, \ldots, N_p)$ with the gauge connection given by the KZ connection, Eq. (1.12) with $P(z) = 0$. When the NACS particles are indistinguishable, configurations that differ by the interchange of two particles must be identified and the phase space is given by [24]

$$T^* \left[ \frac{\prod_\alpha M_\alpha - D}{S_{N_p}} \right] \times O_{\{n_1, n_2, \ldots, n_l\}}^\alpha$$.  (3.15)

In the above equation, $D$ is the set of points where $q_\alpha = q_\beta$ for some $\alpha, \beta$ and $S_{N_p}$ is the permutation group of $N_p$ objects.
IV. COHERENT STATE QUANTIZATION

In this section, we quantize the Hamiltonian (1.7) with $Q^a$’s given by the isospin functions (2.16) in the coherent state quantization method [16]. This method is used only for the internal degrees of freedom for convenience. The external gauge field $A_i^a$ will be arbitrary for the time being. Later, when we consider the NACS particles, it will be substituted by the KZ connection (1.12). Let us consider the propagator

$$K_{FI} = <q_F, \bar{\xi}_F | e^{-i\hat{H}(t_F-t_I)} | q_I, \xi_I>$$  (4.1)

where $|q_I, \xi_I> = \prod_{\alpha=1}^{N} |q_{\alpha I} > |\xi_{\alpha I}>$, and similarly for $<q_F, \bar{\xi}_F|\xi_I>$. $\xi_{\alpha I}$ is the internal complex coordinate on the coadjoint orbit of $\alpha$-th particle $O_{\{n_1,n_2,\cdots,n_l\}} = SU(N)/SU(n_1) \times \cdots \times SU(n_l) \times U(1)^{l-1}$. $\hat{H}$ is the quantum mechanical operator of the Hamiltonian (1.7). Finally, $|\xi> = \Lambda$ is the generalized coherent state [16] defined by

$$|\xi> = \exp(\xi \cdot E) |\Lambda>.$$  (4.2)

$\Lambda$ is the highest weight which can be expressed as

$$\Lambda = \sum_s \mu_s f_s.$$  (4.3)

Here, $\mu_s$ is a non-negative integer and $f_s$ is the highest weight of the fundamental representation. The summation over $s$ is done in such a way that that the maximum stability group of $|\Lambda>$ is $SU(n_1) \times \cdots \times SU(n_l) \times U(1)^{l-1}$ and the corresponding geometry of coherent state is $SU(N)/SU(n_1) \times \cdots \times SU(n_l) \times U(1)^{l-1}$. The existence of such a weight is guaranteed by the Borel-Weil-Bott theorem [30]. The normalization for the coherent state is chosen for convenience in the following manner

$$<\bar{\xi}|\xi> = e^{W(\bar{\xi}, \xi)}$$  (4.4)

where $W$ is the Kähler potential on the orbit $O_{\{n_1,n_2,\cdots,n_l\}}$. Note that with this normalization in Eq. (1.2), $|\xi>$ is a holomorphic function of $\xi$ and $<\bar{\xi}| = (|\xi>)^\dagger$ an antiholomorphic
state. Then, \( W(\bar{\xi}', \xi) \) is holomorphic in \( \xi \) and antiholomorphic in \( \xi' \). The resolution of unity for the coherent state is expressed as

\[
I = \int d\mu(\bar{\xi}, \xi)|\xi > < \bar{\xi}|e^{-W(\bar{\xi}, \xi)}. \tag{4.5}
\]

Also we have \( I_q = \int dq |q > < q|, I_p = \int dp |p > < p| \). We will not sometimes write bold faces for \( p, q \) and particle indices \( \alpha \) and \( \beta \) unless confusion arises.

We first perform the lattice evaluation of the propagator. Divide the time \( T \equiv t_F - t_I \) into \( \tilde{N} + 1 \) steps of equal length \( \epsilon \) so that \((\tilde{N} + 1)\epsilon = T, t_1 = t_I \) and \( t_{\tilde{N}+1} = t_F \). The boundary value is given by \( \bar{\xi}(t_{\tilde{N}+1}) = \bar{\xi}_F \) and \( \xi(t_1) = \xi_I \). Inserting the resolution of unity \( I \times I_q \) and writing \( \xi(t_n) \equiv \xi(n) \), we have

\[
K_{FI} = \int \cdots \int d\mu(\bar{\xi}(n), \xi(n)) dq(n) e^{-W(\bar{\xi}(n), \xi(n))} \prod_{n=1}^{\tilde{N}+1} < \bar{\xi}(n)|\xi(n-1) > \times
\]

\[
\left[ < q(n)|q(n-1) > -i\epsilon < q(n)\bar{\xi}(n)|\hat{H}|q(n-1)\xi(n-1) > < \xi(n)|\xi(n-1) > \right]. \tag{4.6}
\]

Using the kernel \( < \bar{\xi}(n)|\xi(n-1) > = \exp(W(\bar{\xi}(n), \xi(n-1))) \), \( \bar{\xi}(n) = \bar{\xi}(n-1) + d\xi(n-1) \) and treating the space part in the standard manner by inserting \( I_p \) repeatedly, we have in the continuum limit \( (\epsilon \rightarrow 0) \)

\[
K_{FI} = C \int d\mu(\bar{\xi}, \xi) dpdq e^{-\log W(\bar{\xi}_F, \xi_F)} e^{i \int_{t_I}^{t_F} L dt} \tag{4.7}
\]

where the Lagrangian is given by

\[
L = \sum_\alpha (p_\alpha \cdot \dot{q}_\alpha - i \frac{\partial W(\bar{\xi}_\alpha, \xi_\alpha)}{\partial \xi_\alpha} \cdot \dot{\bar{\xi}}_\alpha) - H. \tag{4.8}
\]

The Hamiltonian is given by

\[
H = \sum_\alpha < q_\alpha, \bar{\xi}_\alpha |\hat{H}|q_\alpha, \xi_\alpha > \tag{4.9}
\]

with \( \hat{H} \) given by the operator form of Eq. (1.7)

\[
\hat{H} = \sum_\alpha \frac{1}{2m_\alpha} (\hat{p}_\alpha^i - A_\alpha^n (\hat{q}_\alpha^n) \hat{\hat{q}}_\alpha^n)^2. \tag{4.10}
\]
It is to be noticed that in the above $\hat{H}$ the operator $\hat{Q}_a^a$ is the coherent state representation expressed in terms of the internal coordinates $\xi_\alpha$’s of the coadjoint orbit $O^{n_1,n_2,\ldots,n_l}$. We are interested in the differential operator representation satisfying $[\hat{Q}_a^a, \hat{Q}_b^b] = i f^{ab}_c \hat{Q}_c^c \delta_{\alpha\beta}$ and assume such a representation is possible.

Using the complex coordinates for spatial part and KZ connection given in Eq. (1.12) with $P(z) = 0$, we again recover the quantum mechanical model given by Eq. (1.14) where the isospin operators $\hat{Q}_a^a$’s satisfy the $SU(N)$ algebra: $[\hat{Q}_a^a, \hat{Q}_b^b] = i f^{ab}_c \hat{Q}_c^c \delta_{\alpha\beta}$. But as mentioned just before, $\hat{Q}_a^a$’s are now a differential operator and the wave function is now function of both spatial coordinates and internal coordinates: $\Psi \equiv \Psi(z_1, \ldots, z_{N_p}, \bar{\xi}_1, \ldots, \xi_{N_p}, \bar{\xi}_1, \ldots, \bar{\xi}_{N_p})$. The $\bar{z}$ dependence is dropped for convenience. It is a single component wave function. It is easy to show that the KZ equation can be written as follows:

$$\frac{\partial U^{-1}}{\partial z_\alpha} + \frac{1}{2 \pi \kappa} \sum_{\beta \neq \alpha} \frac{\hat{Q}^a(\xi_\alpha, \bar{\xi}_\alpha) \hat{Q}^a(\xi_\beta, \bar{\xi}_\beta)}{z_\alpha - z_\beta} U^{-1} = 0$$

(4.11)

where $\hat{Q}^a(\xi_\alpha, \bar{\xi}_\alpha)$ is the differential operator of $\hat{Q}_a^a$ in the coherent state representation. The holomorphicity of the state $< \xi | = \Lambda | \exp(\xi \cdot E^\dagger)$ enables one to choose the holomorphic polarization of $U^{-1}(z, \xi) \equiv < \xi | U^{-1}(z) >= \int d\mu(\bar{\xi}', \xi') e^{W(\bar{\xi}, \xi') - W(\bar{\xi}', \xi')} < \xi' | U^{-1}(z) >$ which is obviously holomorphic in $\xi$. Note that this choice is possible, because we have chosen the normalization given by the Eq. (4.4). From now on, we will be working on the holomorphic polarization: $U^{-1} \equiv U^{-1}(z_1, \ldots, z_{N_p}, \bar{\xi}_1, \ldots, \xi_{N_p})$. Also, the wave function is a holomorphic function: $\Psi \equiv \Psi(z_1, \ldots, z_{N_p}, \bar{\xi}_1, \ldots, \xi_{N_p})$. The inner product Eq. (1.18) is modified into

$$< \Psi_1 | \Psi_2 >= \int \prod \alpha dz_\alpha d\bar{z}_\alpha d\mu(\bar{\xi}_\alpha, \xi_\alpha) \Psi_1(z, \bar{\xi}) U^\dagger(z, \bar{\xi}) U(z, \xi) \Psi_2(z, \xi) e^{-W(\bar{\xi}, \xi)}.$$ 

(4.12)

The Hamiltonian Eq. (1.14) with $SU(N)$ $\hat{Q}_a^a$’s is hermitian with respect to this inner product assuming the hermiticity of $\hat{Q}_a^a(\xi_\alpha, \bar{\xi}_\alpha)$.

We discuss about $CP(N-1)$ case in details. The Kähler potential $W$ is given by

$$W(\bar{\xi}, \xi) = \sum \alpha J_\alpha \log(1 + \bar{\xi}_\alpha \xi_\alpha)$$

(4.13)
for some integer $J_{\alpha}$. The Lagrangian is expressed as

$$L = \sum_{\alpha} \left( p^{\bar{\alpha}}_\alpha \dot{z}_\alpha + p^{\bar{\alpha}}_\alpha \dot{\bar{z}}_\alpha - iJ_{\alpha} \frac{\xi_\alpha \cdot \dot{\bar{\xi}}_\alpha}{1 + |\xi_\alpha|^2} \right) - H$$  \hspace{1cm} (4.14)

with the Hamiltonian $H$ given by Eq. (1.13) with the isospin functions being given by the Eq. (2.25)

$$Q^a(\xi_\alpha, \bar{\xi}_\alpha) = i \sum_{\alpha} \sum_{I,K=0}^{N-1} J_{\alpha} \Xi_{aI} T_{IK} \Xi_{aK}$$  \hspace{1cm} (4.15)

where $\Xi_{\alpha0} = \frac{1}{\sqrt{1+|\xi_\alpha|^2}}$ and $\Xi_{\alphaI} = \frac{\xi_I}{\sqrt{1+|\xi_\alpha|^2}}$ are substituted into the final expression. The quantum mechanical Hamiltonian is again given by the Eq. (1.14) but with the quantum mechanical isospin operator being expressed by

$$\hat{Q}^a(\xi_\alpha) = i[T^a_{\alpha0} + T^a_{\alpha j} \xi_j - T^a_{0\alpha} \xi_j + T^a_{0j} \xi_j \xi_\alpha] \frac{\partial}{\partial \xi^a_\alpha} + iJ_{\alpha} T^a_{00} + iJ_{\alpha} T^a_{0j} \xi_j.$$  \hspace{1cm} (4.16)

The above differential operator satisfy $[\hat{Q}^a(\xi_\alpha),\hat{Q}^b(\xi_\beta)] = if^{abc}_\alpha \hat{Q}^c(\xi_\alpha)\delta_{\alpha\beta}.$

The explicit form of the differential KZ equation (1.11) in this representation can be given: for example, in $SU(2)$ case,

$$\frac{\partial U^{-1}}{\partial z_\alpha} + \frac{1}{2\pi\kappa} \sum_{\beta \neq \alpha} \frac{1}{2} \frac{(\hat{Q}^+(\xi_\alpha)\hat{Q}^-(\xi_\beta) + \hat{Q}^-(\xi_\alpha)\hat{Q}^+(\xi_\beta)) + \hat{Q}^3(\xi_\alpha)\hat{Q}^3(\xi_\beta)}{z_\alpha - z_\beta} U^{-1} = 0$$  \hspace{1cm} (4.17)

where we defined $\hat{Q}^\pm(\xi_\alpha) = \hat{Q}^1(\xi_\alpha) \pm i\hat{Q}^2(\xi_\alpha)$ and they are given by

$$\hat{Q}^+(\xi_\alpha) = \xi_\alpha^2 \frac{\partial}{\partial \xi_\alpha} - J_{\alpha} \xi_\alpha, \quad \hat{Q}^-(\xi_\alpha) = -\frac{\partial}{\partial \xi_\alpha}, \quad \hat{Q}^3(\xi_\alpha) = \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \frac{J_{\alpha}}{2}.$$  \hspace{1cm} (4.18)

We see that the above is the $SU(2)$ generalization of the Bargmann representation [16]. For example, the $(J + 1)$-dimensional irreducible representation of the operator is given by the holomorphic polynomial of order $J$: $\psi_J(\xi) = \sum_n a_n \xi^n$. The highest weight state is given by $\psi_{J+1}(\xi) = a_J \xi^J$ and they satisfy $\hat{Q}^+(\xi)\psi_J(\xi) = 0$, $\hat{Q}^-(\xi)\psi_J(\xi) = -J\psi_{J-1}(\xi)$, $\hat{Q}^3(\xi) = (J/2)\psi_J(\xi)$. We also have $\hat{Q}^a(\xi)\hat{Q}^b(\xi) = (J/2)(J/2 + 1)$. Note the correspondence with the matrix representation: the usual angular momentum $j = J/2$, $m = M - (J/2)$, and $\psi_J \rightarrow |j,m>$. 

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It is to be noted that antiholomorphic representation is given by the complex conjugation of the above equation, \( \hat{Q}^{a*}(\xi_\alpha) \)'s, and they satisfy \( [\hat{Q}^{a*}(\xi_\alpha), \hat{Q}^{a*}(\xi_\beta)] = -if^{ab}_c \hat{Q}^{c*}(\xi_\alpha)\delta_{\alpha\beta} \). Also, the operator (4.16) is Hermitian with respect to the following inner product:

\[
< \psi_1 | \psi_2 > = \int d\mu(\xi, \xi) \tilde{\psi}_1(\xi) \psi_2(\xi) e^{-W(\xi, \bar{\xi})}.
\]  

(4.19)

Finally, it can be easily checked that

\[
< \xi_\alpha | \hat{Q}^a_{\alpha} | \xi_\alpha > \equiv < \xi_\alpha | \hat{Q}^a_{\alpha} | \xi_\alpha > |_{\xi_\alpha \rightarrow \xi_\alpha} = (\hat{Q}^a(\xi'_\alpha) < \xi_\alpha | \xi_\alpha >)^* |_{\xi'_\alpha = \xi_\alpha} = Q^a(\xi_\alpha, \bar{\xi}_\alpha) < \xi_\alpha | \xi_\alpha >
\]  

(4.20)

using the reproducing kernel of \( CP(N-1) \)

\[
< \xi_\alpha | \xi_\alpha > = (1 + \xi_\alpha \cdot \xi_\alpha)^J_\alpha.
\]  

(4.21)

Now let us discuss the possible solution of the KZ equation in the \( CP(N-1) \) case with the differential operator given by (4.16). For our purpose, we rewrite the KZ equation (4.11) in the following integral differential equation

\[
U^{-1}(z_1, \ldots, z_{N_p}; \xi_1^i, \ldots, \xi_{N_p}^i) = U^{-1}_0(\xi_1^i, \ldots, \xi_{N_p}^i) = \frac{1}{2\pi \kappa} \int_{\Gamma} \sum_{a} dz'_a \sum_{\beta \neq \alpha} \frac{1}{z'_\alpha - z'_\beta} \times
\]

\[
\times \hat{Q}^a(\xi_\alpha) \hat{Q}^a(\xi_\beta) U^{-1}(\xi_1^1, \ldots, z_{N_p}^i; \xi_1^i, \ldots, \xi_{N_p}^i)
\]  

(4.22)

where \( \Gamma \) is a path in the \( N_p \)-dimensional complex space with one end point fixed and the other being \( z_f = (z_1, \ldots, z_{N_p}) \). \( U^{-1}_0(\xi_1^i, \ldots, \xi_{N_p}^i) \) is independent of spatial coordinates and necessary to take care of the boundary condition. It is well known that the exact solution can be achieved in the two body case. It can be easily inferred that the solution is given by

\[
U^{-1}(z_1, z_2; \xi_1^i, \xi_2^i) = \exp \left( -\frac{1}{2\pi \kappa} (\log(z_1 - z_2) + c_{12}) \hat{Q}^a(\xi_1) \hat{Q}^a(\xi_2) \right) U^{-1}_0(\xi_1^i, \xi_2^i)
\]  

(4.23)

where \( c_{12} \) is a constant which is inserted to adjust the boundary condition. An exact expression can be obtained for a couple of cases. Following the discussion of \( SU(2) \) Bargmann representation, let us consider the case in which we have the Clebsch-Gordan series

\[
U^{-1}_0(\xi_1^i, \xi_2^i) = \psi_{JM} = \sum_{M_1, M_2} C(J_1, J_2; M_1, M_2) \psi_{J_1 M_1} \psi_{J_2 M_2}
\]  

(4.24)
where \( C(\frac{\lambda}{2} m_1, \frac{\lambda}{2} m_2; \frac{\lambda}{2} m) \) is the Clebsch-Gordan coefficients. Then it can be easily shown that 
\[
\hat{Q}^a(\xi_1)\hat{Q}^a(\xi_2)U^{-1}_0(\xi^i_1, \xi^i_2) = (1/2)[\frac{\lambda}{2} (\frac{\lambda}{2} + 1) - \frac{\lambda}{2} (\frac{\lambda}{2} + 1) - \frac{\lambda}{2} (\frac{\lambda}{2} + 1)]U^{-1}_0(\xi^i_1, \xi^i_2)
\]
and the solution is given by the Eq. (4.23) with \( \hat{Q}^a(\xi_1)\hat{Q}^a(\xi_2) \) replaced by \( (1/2)[\frac{\lambda}{2} (\frac{\lambda}{2} + 1) - \frac{\lambda}{2} (\frac{\lambda}{2} + 1) - \frac{\lambda}{2} (\frac{\lambda}{2} + 1)] \). The extension to general \( CP(N-1) \) is immediate and will be reported elsewhere. Another case of interest would be the one in which
\[
U^{-1}_0(\xi^i_1, \xi^i_2) = <\xi_1|\bar{\zeta}_1 > <\xi_2|\bar{\zeta}_2 >
\]
for some \(|\bar{\zeta}_1 >| and \(|\bar{\zeta}_2 >| . The solution can be written as (with \( c_{12} = 0 \) for convenience)
\[
U^{-1}(z_1, z_2; \xi^i_1, \xi^i_2) = \sum_{n=0}^{\infty} (\frac{-\log(z_1 - z_2)}{2\pi\kappa})^n (\hat{Q}^a(\xi_1)\hat{Q}^a(\xi_2))^n (1 + \xi_1 \cdot \bar{\zeta}_1)^J_1 (1 + \xi_2 \cdot \bar{\zeta}_2)^J_2.
\]
We see that in the \( SU(2) \) case, for example, in the large \( \kappa \) limit neglecting terms of order \( (1/\kappa)^2 \), \( U^{-1}(z_1, z_2; \xi^i_1, \xi^i_2) \) is a polynomial of order \( J_1 \) and \( J_2 \) in \( \xi_1 \) and \( \xi_2 \) respectively. It would be interesting if one could find general solutions of Eq. (4.22).

V. CONCLUSION

In this paper, we investigated in detail the classical and quantum aspect of a system of \( SU(N) \) NACS particles. We discussed about the most general phase space of \( SU(N) \) internal degrees of freedom which can be identified as one of the coadjoint orbits of \( SU(N) \) group by the method of symplectic reduction. A detailed constraint analysis on each orbit by Dirac method was given. The quantum aspect of the theory was explored by using the method of the coherent state for the internal degrees of freedom. Coherent state corresponding to the geometry of each coadjoint orbit was introduced and an explicit path integral representation was derived. Especially, a coherent state representation of the KZ equation was given and possible solutions in this representation were discussed.

There remain several topics to be discussed further in this approach. First of all, it would be interesting if the constraint analysis and coherent state quantization approach of
this paper could be generalized to arbitrary groups including the non-compact ones and applied to give an explicit construction of the corresponding Darboux variables on each coadjoint orbit as was done on the maximal orbits of unitary and orthogonal group \[27\]. The results could give the functional integral quantization of spin \[31\] on the most general coadjoint orbits.

We performed coherent state quantization of NACS particles and obtained a path integral representation of NACS particles in Section 4. For example, by using the complex spatial coordinate and substituting the KZ connection (1.12) with \(P(z) = 0\) in the holomorphic gauge into Eq. (4.8), we obtain the desired propagator (4.7). Since the NACS particles are the non-Abelian generalizations of anyons and the propagator of a system of indistinguishable anyons is a representation of the braid group \[32\], our propagator should also provide a non-Abelian generalization of path integral representation of the braid group on the phase space Eq. (3.15). It could be called the coherent state representation of the braid group. The detailed analysis of a system of indistinguishable NACS particles will be reported elsewhere \[33\].

Note that in the usual expression of KZ equation (1.16), the braid operator or monodromy \(\exp(i\hat{Q}_a^\alpha \hat{Q}_\beta^\alpha / \kappa)\) is given as a matrix representation whereas in the coherent state approach this is an holomorphic differential operator. The relation between the two approach is connected by the simple exchange of the usual angular momentum basis and coherent state. This could have two applications. First, we recall that the difficulty in finding the solution of the KZ equation in matrix approach lies in the non-existence of the common eigenvectors of the braid operators in general \[19\]. This difficulty may be cured in our approach because it is replaced with finding the possible solutions of the KZ differential equation. Second, since the braid operator satisfies the Yang-Baxter equation and exhibits the non-Abelian braid statistics \[34\], our approach could give a new interpretation of the Yang-Baxter equation. The possibility of exact solutions and detailed property of non-Abelian braid statistics in this approach will be reported elsewhere.

Finally, it may be interesting if the generalized Bargmann representation can be explicitly
obtained for other coadjoint orbit as well and all the holomorphic irreducible representations are explicitly calculated. For example, for the maximal orbit, the Bruhat coordinatization \[35\] can be used in the construction of the coherent state and an explicit holomorphic representation of \(\hat{Q}^{a}(\xi)\) can be obtained by using the method of geometric quantization \[36\]. The details will be reported elsewhere \[33\].

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