LETTER

Mixed $\ell_p/\ell_1$ Norm Minimization Approach to Intra-Frame Super-Resolution

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SUMMARY This paper deals with the problem of reconstructing a high-resolution digital image from a single low-resolution digital image and proposes a new intra-frame super-resolution algorithm based on the mixed $\ell_p/\ell_1$ norm minimization. Introducing some assumptions, this paper formulates the super-resolution problem as a mixed $\ell_0/\ell_1$ norm minimization and relaxes the $\ell_0$ norm term to the $\ell_p$ norm to avoid ill-posedness. A heuristic iterative algorithm is proposed based on the iterative reweighted least squares (IRLS). Numerical examples show that the proposed algorithm achieves super-resolution efficiently.

key words: super-resolution, sparse optimization

1. Introduction

Digital image enlargement has a lot of applications such as satellite image processing [1] and medical image processing [2]. We can easily achieve the digital image enlargement by using the bilinear method and the bicubic method, which are implemented in several commercial image processing software products. However, these methods blur high-resolution digital image and cannot restore high-frequency components.

Super-resolution technology has attracted attention for its ability to restore the high-frequency components in a high-resolution digital image. Super-resolution can be classified into two types, multi-frame super-resolution and intra-frame super-resolution. The multi-frame super-resolution is the method to derive the high-resolution digital image from multiple low-resolution digital images, and the intra-frame super-resolution is the method to derive the high-resolution digital image from a single low-resolution digital image. This letter deals with the problem of the intra-frame super-resolution.

The total variation (TV) norm is often used for the intra-frame super-resolution [3], [4]. The TV norm based method reconstructs a high-resolution digital image by the total variation regularization. While this approach can restore a high-resolution digital image with high-frequency components, the resulting high-resolution image has jaggy and ringing artifacts, and high-frequency components are not restored adequately. In order to recover more precise high-resolution image, this letter focuses on the $\ell_0$ norm minimization. Candès et al. applied the $\ell_0$ norm minimization and proposes the reweighted $\ell_1$ minimization to recover images [5]. In [6], the mixed $\ell_0/\ell_1$ norm minimization has been applied to the image colorization problem. Motivated by these works, this letter formulates the super-resolution problem as the mixed $\ell_0/\ell_1$ norm minimization and relaxes it to the mixed $\ell_p/\ell_1$ norm minimization. A heuristic iterative algorithm is proposed using the iteratively reweighted least squares (IRLS) [7]. We demonstrate numerical examples to show that the proposed algorithm has a good performance.

2. Main Results

Let us consider the problem of reconstructing $M \times N$ high-resolution digital image $x \in \mathbb{R}^{MN}$ from single $K \times L$ low-resolution digital image $b \in \mathbb{R}^{KL}$. First we deal with the following problem,

$$\text{Minimize } \|Dx\|_0 \text{ subject to } Ax = b. \tag{1}$$

where $\| \cdot \|_0$ denotes the $\ell_0$ norm of a vector, that is, the number of elements in a vector, and $A \in \mathbb{R}^{KL \times MN}$ denotes a subsampling and 2D weighted moving average matrix. The matrices $D$ are defined by

$$Dx = \begin{bmatrix} D_h(x) \\ D_v(x) \end{bmatrix},$$

where $D_h$ and $D_v$ denote the operators to calculate a horizontal differential image and a vertical differential image, respectively. If the $\ell_0$ norm is replaced by the $\ell_1$ norm, this problem is equal to the total variation (TV) norm approach. The problem (1) restores the high resolution image such that the intensity value of each pixel is necessarily equal to one of its neighbor pixels, and therefore a super-resolution algorithm based on (1) is almost the same as the nearest neighbor method. While (1) recovers the high-frequency components, it does not recover smooth images similar to the nearest neighbor method. To obtain a smooth high resolution image, we assume that the pixel is in smooth area if the difference of intensity values between its neighbor pixels is smaller than given constant $v_1 > 0$ or if the second order difference of intensity values between its neighbor pixels is smaller than given constant $v_2 > 0$. Then this letter proposes

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the following mixed $\ell_0/\ell_1$ problem,

$$\text{Minimize} \quad \|\mathcal{F}_v(Dx)\|_0 + \mu_1\|\mathcal{F}_v(Dx)\|_1 + \mu_2\|\mathcal{F}_v(\bar{D}x)\|_1,$$

subject to \hspace{1cm} $Ax = b$ \hspace{1cm} (2)

where $\| \cdot \|_1$ denotes the $\ell_1$ norm of a vector, $\bar{D}$ is a diagonal matrix defined by

$$\bar{D}x = \begin{bmatrix} D_h(D_h(x)) \\ D_v(D_v(x)) \end{bmatrix}.$$ 

$\mathcal{F}_v$ is an operator to let the elements which are not less than $v$ to be 0s, and $\mathcal{F}_v$ is an operator to let the elements which are less than $v$ to be 0s. The second term of the objective function in (2) minimizes the differences of intensity values to make the image smooth, and the third term minimizes the second order differences of intensity values to provide the smooth gradation. The aim of the problem (2) is to obtain a smooth image with high-frequency components. However, this problem is ill-posed because the element of $x$ can take any value such that $Ax = b$ if the differences between its all neighbor pixels are not less than $v$. To avoid this ill-posedness, this letter relaxes the $\ell_0$ norm in (2) by the $\ell_p$ norm with $0 < p < 1$, and we obtain the following problem,

$$\text{Minimize} \quad \|\mathcal{F}_v(Dx)\|_p + \mu_1\|\mathcal{F}_v(Dx)\|_1 + \mu_2\|\mathcal{F}_v(\bar{D}x)\|_1,$$

subject to \hspace{1cm} $Ax = b$ \hspace{1cm} (3)

where $\| \cdot \|_p$ denotes the $\ell_p$ norm of a vector. If we let $p$ be close to 1, the problem is nearly equal to the $\ell_1$ norm minimization, and the high-frequency components are never restored. Letting $p$ be close to 0, (3) recovers a image with both smooth area and high-frequency components.

To solve (3), this paper provides a heuristic iterative algorithm based on the iterative reweighted least squares (IRLS) algorithm [5]. Let us consider the following weighted least squares problem,

$$\text{Minimize} \quad \|W_1^{(k)}j_1^{(k)}Dx\|_2^2 + \mu_1\|W_2^{(k)}j_2^{(k)}Dx\|_2^2 + \mu_2\|W_3^{(k)}j_3^{(k)}Dx\|_2^2,$$

subject to \hspace{1cm} $Ax = b$ \hspace{1cm} (4)

where $j_1^{(k)}$, $j_2^{(k)}$, $j_3^{(k)}$, $W_1^{(k)}$, $W_2^{(k)}$ and $W_3^{(k)}$ are diagonal weight matrices defined by

$$j_1^{(k)}_{ii} = \begin{cases} 1 & \text{if } (Dx^{(k)})_{i} \geq v_1 \\ 0 & \text{if } (Dx^{(k)})_{i} < v_1 \end{cases},$$

$$j_2^{(k)}_{ii} = \begin{cases} 0 & \text{if } (Dx^{(k)})_{i} \geq v_1 \\ 1 & \text{if } (Dx^{(k)})_{i} < v_1 \end{cases},$$

$$j_3^{(k)}_{ii} = \begin{cases} 0 & \text{if } (Dx^{(k)})_{i} \geq v_2 \\ 1 & \text{if } (Dx^{(k)})_{i} < v_2 \end{cases},$$

$$W_1^{(k)}_{ii} = (\|Dx^{(k-1)}\|_2^2 + \epsilon)^{p/2-1},$$

$$W_2^{(k)}_{ii} = (\|Dx^{(k-1)}\|_2^2 + \epsilon)^{-1/2},$$

$$W_3^{(k)}_{ii} = (\|Dx^{(k-1)}\|_2^2 + \epsilon)^{-1/2},$$

To solve (3), this paper provides a heuristic iterative algorithm based on the iterative reweighted least squares (IRLS) algorithm [5]. Let us consider the following weighted least squares problem,

$$\text{Minimize} \quad \|W_1^{(k)}j_1^{(k)}Dx\|_2^2 + \mu_1\|W_2^{(k)}j_2^{(k)}Dx\|_2^2$$

subject to \hspace{1cm} $Ax = b$ \hspace{1cm} (4)

where $j_1^{(k)}$, $j_2^{(k)}$, $W_1^{(k)}$, $W_2^{(k)}$ and $W_3^{(k)}$ are diagonal weight matrices defined by

$$j_1^{(k)}_{ii} = \begin{cases} 1 & \text{if } (Dx^{(k)})_{i} \geq v_1 \\ 0 & \text{if } (Dx^{(k)})_{i} < v_1 \end{cases},$$

$$j_2^{(k)}_{ii} = \begin{cases} 0 & \text{if } (Dx^{(k)})_{i} \geq v_1 \\ 1 & \text{if } (Dx^{(k)})_{i} < v_1 \end{cases},$$

$$j_3^{(k)}_{ii} = \begin{cases} 0 & \text{if } (Dx^{(k)})_{i} \geq v_2 \\ 1 & \text{if } (Dx^{(k)})_{i} < v_2 \end{cases},$$

$$W_1^{(k)}_{ii} = (\|Dx^{(k-1)}\|_2^2 + \epsilon)^{p/2-1},$$

$$W_2^{(k)}_{ii} = (\|Dx^{(k-1)}\|_2^2 + \epsilon)^{-1/2},$$

$$W_3^{(k)}_{ii} = (\|Dx^{(k-1)}\|_2^2 + \epsilon)^{-1/2},$$

In order to show the efficiency of the proposed algorithm, we compare Algorithm 1 with the bicubic method and the total variation regularization [8] by applying the algorithms to real images as shown in Fig. 1. In all experiments the algorithms enlarge low-resolution images generated by applying a simple moving average (SMA) filter of size $5 \times 5$ to the original images and subsampling them to reduce the resolution to one quarter of their original sizes. Algorithm

**Algorithm 1 Proposed algorithm**

Require: $b$, $\mu_1$, $\mu_2$, $v_1$, $v_2$, $\epsilon$, $k_{\max}$ and $x^{(0)}$

$k \leftarrow 0$

repeat

Update $j^{(k)}_i$, $j^{(k)}_j$ and $j^{(k)}_k$ using (5) - (7).

Update $W_1^{(k)}$, $W_2^{(k)}$ and $W_3^{(k)}$ using (8) - (10).

Obtain $x^{(k+1)}$ by solving (4).

$k \leftarrow k + 1$

until $k \leq k_{\max}$

Ensure: $x^{(k)}$

Fig. 1 Original images (600 × 600): (a) Roof, (b) Cat, (c) Stairs, (d) Castle, (e) Fruits, (f) Tree, (g) Tower, (h) Character and (i) Notice.

$x^{(k)}$ denotes the optimal solution of (4) at the previous iteration, $(\cdot)_{ii}$ denotes the $i$th diagonal element of a matrix, and $(\cdot)_i$ denotes the $i$th element of a vector. Then this letter proposes Algorithm 1. Since (4) is a quadratic programming with linear constraints, its optimal solution can be obtained analytically. Though this algorithm is not guaranteed to obtain an exact solution of (3), empirical results indicate that it recovers images with low-frequency and high-frequency components well.

3. Numerical Experiments

In order to show the efficiency of the proposed algorithm, we compare Algorithm 1 with the bicubic method and the total variation regularization [8] by applying the algorithms to real images as shown in Fig. 1. In all experiments the algorithms enlarge low-resolution images generated by applying a simple moving average (SMA) filter of size $5 \times 5$ to the original images and subsampling them to reduce the resolution to one quarter of their original sizes. Algorithm
use $p = 0.1, \mu_1 = 0.1, \mu_2 = 0.9, \nu_1 = 0.5, \nu_2 = 0.5, \\
\epsilon = 10^{-2}$ and $k_{\text{max}} = 3$. We let $A$ in (4) to be a subsampling 
and SMA filter of size $5 \times 5$ and the initial vector $x^{(0)}$ to be 
a vector of all ones. 

Figure 2 and 3 show the resulting images, and Table 1 
and 2 show the results of the peak signal-to-noise ratio
Let’s consider Table 1 and Table 2 for the comparison of PSNR [dB] and SSIM, respectively.

### Table 1: Comparison of PSNR [dB]

| Image  | bicubic | total variation | proposed algorithm |
|--------|---------|-----------------|--------------------|
| Roof   | 24.742  | 24.892          | 25.480             |
| Cat    | 25.630  | 25.713          | 26.012             |
| Stairs | 26.023  | 26.190          | 26.682             |
| Castle | 23.383  | 23.344          | 23.775             |
| Fruits | 24.342  | 24.323          | 24.915             |
| Tree   | 32.398  | 32.600          | 33.693             |
| Tower  | 29.661  | 30.059          | 30.911             |
| Character | 30.113 | 30.662        | 32.438             |
| Notice | 25.191  | 25.477          | 27.543             |

### Table 2: Comparison of SSIM

| Image  | bicubic | total variation | proposed algorithm |
|--------|---------|-----------------|--------------------|
| Roof   | 0.8463  | 0.8703          | 0.8820             |
| Cat    | 0.8753  | 0.8905          | 0.8973             |
| Stairs | 0.8522  | 0.8739          | 0.8792             |
| Castle | 0.8786  | 0.8946          | 0.9012             |
| Fruits | 0.8721  | 0.8860          | 0.9065             |
| Tree   | 0.9766  | 0.9795          | 0.9829             |
| Tower  | 0.9592  | 0.9658          | 0.9686             |
| Character | 0.9335 | 0.9439        | 0.9523             |
| Notice | 0.9285  | 0.9429          | 0.9596             |

The $\ell_p$ norm ($0 < p < 1$) minimization and the $\ell_1$ norm minimization force the image to be sharpened and to be smoothed, respectively, and therefore we obtain the high-resolution image which is partially smoothed and contains high-frequency components. Experimental results show that the proposed algorithm restores precise high-resolution images efficiently.

### 4. Conclusions

This paper has proposed a new intra-frame super-resolution algorithm based on the mixed $\ell_p/\ell_1$ norm minimization.

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