LOCALISATION OF THE DONALDSON’S INVARIANTS ALONG SEIBERG-WITTEN CLASSES.

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ABSTRACT. This article is a first step in establishing a link between the Donaldson polynomials and Seiberg-Witten invariants of a smooth 4-manifold.

§0. INTRODUCTION.

As the result of analysis of the structure of the Donaldson’s invariant of a smooth simply connected 4 manifold $X$ Kronheimer and Mrowka [K-M1] (see also [F-S]) proved the existence of $k_X$ classes

$$K_1, \ldots, K_{k_X} \in H_2(X, \mathbb{Z})$$

and $k_X$ polynomials

$$f_1, \ldots, f_{k_X} \in \mathbb{Q}[z]$$

such that the Donaldson’s invariant of $X$ is defined uniquely by these pairs.

These classes are called the basic classes of $X$ and the set $\{K_i\}$ for $i = 1, \ldots, k_X$, in $H^2(X, \mathbb{Z})$ is a diffeomorphism invariant. So the diffeomorphism group $Diff X$ admits a representation to the symmetric group on $k_X$ letters. A little bit later Seiberg and Witten introduced another set of classes $\{SW\}$ and polynomials and Witten predicted the shape of the Donaldson’s invariant in terms of these data. S-W classes are labelling the moduli spaces $\{\mathcal{M}_{SW}\}$ of the solutions of the Seiberg-Witten system of equations (see §1) and the dimensions of these moduli spaces are the degrees of the polynomials. Using other type of system of equations we will prove that the classes of Kronheimer and Mrowka and Seiberg-Witten coincide. This paper is the first step of this proof and contains those facts from gauge theory which we need for the proof.

At various stages of this project useful were conversations with Ed Witten, S.Donaldson, S.Bauer. We are grateful to S.Donaldson who invited us to the conference at the Newton Institute in December 1994 where we described these results and to I. Hambleton for his kind invitation to MPI, Bonn. First author aknowledges with gratitude financial support of AMS (fSU grant) and VW-stiftung during the work over this project as well as the kind invitation of S.Bauer and the hospitality of Bielefeld University.
§1. Configuration spaces and its cohomology.

Let $c \in H^2(X)$ be a $\text{Spin}^c$ structure on $X$, i.e. a pair of complex hermitian rank 2 bundles $W_\pm^c, c_1(W_\pm^c) = c$ (we shall omit subscript $c$ if there is no confusion). Then the tangent bundle

$$TX = \text{Hom}_h(W^-, W^+)$$

is the bundle of homomorphisms preserving the Hermitian structures.

Fix a pair of connections $\nabla^\pm$ on $W^\pm$ such that their tensor product is Levi Civita connection on $TX$ (which is the same as fixing the determinant $\nabla_{det} = det\nabla^\pm$).

This defines a Dirac operator:

$$D_g, \nabla_{det} : \Gamma(W^+) \rightarrow \Gamma(W^-)$$

as a composition of the connection $\nabla^+ : \Gamma(W^+) \rightarrow \Gamma(W^+ \otimes T^*X)$ and the convolution along (1.1). We shall omit the superscripts if this will not make a confusion.

**U(1) - monopoles.**

We shall remind that a configuration space $C_{detW^+}$ for $U(1)$-Seiberg-Witten moduli space is

$$A_{detW^+} \times \Gamma(W^+)/G_{detW^+} = N \times \Gamma(W^+)/S^1$$

where $N = \ker(d^* : \Omega^1 \rightarrow \Omega^0)$ is a slice of an action of $G_{detW^+}$. Seiberg-Witten moduli spaces $M_{SW}(0, c)$ is defined as a space of solutions of the system

$$D_{\nabla_{det}} = 0$$

$$F^+_{\nabla_{det}} = -(\phi \otimes \bar{\phi})_0,$$

modulo action of the gauge group, where $(\nabla_{det}, \phi) \in A_{detW^+} \times \Gamma(W^+)$. Over the subspace $C_{detW^+}^* = N \times \mathbb{P}(\Gamma(W^+))$ of non-reducible points of $C_E$ there is a complex linear bundle $\mathcal{O}_{E}(-1) = \mathcal{O}_{\mathbb{P}(\Gamma(W^+))}(1)$ with the first Chern class $t = c_1(\mathcal{O}_{E}(-1))$ generating cohomology ring $H^*(C^*, \mathbb{Q})$ in the case $X$ is simply connected.

The system of equations (1.4) can be gauge invariant perturbed as

$$D_{\nabla_{det}} = 0$$

$$(\omega + F^+_{\nabla_{det}})^+ = -(\phi \otimes \bar{\phi})_0,$$

where $\omega$ is any 2-form.

Seiberg-Witten moduli spaces $M_{SW}(\omega, c)$ is defined as a space of solutions of the perturbed system (1.4$\omega$) modulo action of the gauge group. For the generic form $\omega$ the moduli space is cut out transversally and therefore smooth.

We shall also consider another perturbation - perturbation of the metric $g$. The similar ”generic position” statement for this space of parametra is provided by the following

**Lemma 1.1.** Fix some 2-form $\omega$ and consider the map

$$Met_g \times \Omega^1 \times A_{detW^+} \times \Gamma(W^+) \rightarrow \Gamma(W^-) \times \Omega^2$$

(1.5)
given by sending
\[(g, \xi, \nabla_{det}, \phi) \rightarrow (D^{g,\nabla_{det}}\phi, (\omega + F_{\nabla_{det}})^+ + (\phi \otimes \overline{\phi})_0). \quad (1.6)\]

This map is a submersion.

Proof. We shall remark first that since principal symbol of Dirac operator, as well as of Levi-Civita connection, doesn’t depend on metric, dependence is given by the formula:
\[D^{g+r,\nabla_{det}} = D^{g,\nabla_{det}} + \delta_r \quad (1.7)\]
where \(\delta_r \in \Omega^1\). Thus
\[\delta f(r, \alpha, \delta \nabla_{det}, \delta r) = (\delta_1 f, \delta_2 f) \quad (1.8)\]
where
\[\delta_1 f = (\alpha + \delta \nabla_{det} + \delta_r) \otimes \phi + D^{g,\nabla_{det}}(\delta \phi), \]
\[\delta_2 f = r^*(F_{\nabla_{det}}^+ + \omega) + d_{\nabla_{det}}^+(\delta \nabla_{det}) + (\delta \phi \otimes \overline{\phi})_0 + (\phi \otimes \overline{\delta \phi})_0. \]

It is the standard observation that the restriction
\[\delta_1 f|_{r=\delta \nabla_{det}=0} = (\alpha + \delta_r) \otimes \phi + D^{g,\nabla_{det}}(\delta \phi)\]
is an epimorphism. Therefore it is enough to prove that the restriction
\[\delta_2 f|_{\alpha=\delta \phi=0}\]
is onto. This follows from the fact that for any non zero 2-form \((F_{\nabla_{det}}^+ + \omega)\) and for any harmonic self dual 2-form \(\chi\) one can find an infinitesimal deformation of the metric \(r\) such that the pairing
\[\langle r^*(F_{\nabla_{det}}^+ + \omega), \chi \rangle \neq 0. \]

Indeed, by Cor. 4.2.23 of [D-K] such \(\chi\) vanishes on an open set if only \(\chi = 0\) everywhere.

There is another gauge invariant perturbation of the equation. Let
\[T_{1/2} \subset \Gamma(\text{End}(\Lambda^0 \oplus \Lambda^2_+)) \quad (1.9)\]
be a subspace of endomorphisms \(t\) of the bundle \(\Lambda^0 \oplus \Lambda^2_+\) subject to the condition \(|t| \leq \frac{1}{2}\). Any such \(t\) defines equation as follows
\[D^{\nabla_{det}} \phi = 0 \quad (1.4_{\omega, t})\]

At points where \(|\phi|\) has its maximal value using Weitzenböck formula one has
\[0 \leq \Delta |\phi| \leq \frac{-s}{2} |\phi|^2 + \langle ((t - 1)(\phi \otimes \overline{\phi}))_0(\phi), \phi \rangle \leq \langle \phi^2 \rangle (-s - |\phi|^2 + |t||\phi|^2) \quad (1.10)\]
and as a consequence
\[ |\phi|^2 \leq \frac{-s}{(1 - |t|)} \leq -2s \]
provided our condition $|t| \leq \frac{1}{2}$ is true (compare [K-M2]).

Thus following well known properties of $\mathcal{M}_{SW}(\omega, c)$ for generic metric $g$ (see for example [K-M2], [W]) are true also for the moduli $\mathcal{M}_{SW}(\omega, t, c)$ of solutions of 1.4.1.

1) there exists only finite set of classes $c \in H^2(X)$ such that $\mathcal{M}_{SW}(\omega, t, c) \neq \emptyset$;
2) every moduli space $\mathcal{M}_{SW}(\omega, t, c)$ is compact;
3) if $X$ admits a metric with positive scalar curvature then $\mathcal{M}_{SW}(0, t, c) = \emptyset$ (in a relevant chamber in the case $b^+_2(X) = 1$).

So we can see that the Seiberg-Witten system of equations (1.4) and (1.4.1) is very strong and the space of solutions is very rigid.

The strategy to get a non trivial solution is to consider a degenerated solution $(\nabla_{\text{det}}, 0) \in \mathcal{M}_{SW}(\omega, c)$ for some $\omega$ and to use the description of Kuranishi map for this singular point.

**Non abelian monopoles.**

Now take a $PU(r)$-bundle $\xi$ on $X$ and fix its lift to a $U(r)$-bundle $E$ that is fix $c_1(E) \in H^2(X)$. As a configuration space consider the set of triples

\[ (a_0, \nabla_{\text{det}}, \phi) \in \mathcal{A}(P(E)) \times \mathcal{A}(\text{det}(E \otimes W^+)) \times \Gamma(E \otimes W^+) \]  

(1.11)

where $a_0$ is a traceless part of $U(r)$-connection, $\nabla_{\text{det}}$ is an abelian connection on the line bundle $\text{det}(E \otimes W^+)$ and $\phi$ is a section of the vector bundle $E \otimes W^+$. The first two components of a triple (1.11) define a coupled Dirac operator

\[ D_{a_0}^{\nabla_{\text{det}}} : \Gamma(E \otimes W^+) \rightarrow \Gamma(E \otimes W^-) \]  

(1.12)

by the Leibnitz rule for the action of the connection along $E$ and the ordinary Dirac operator (1.2) along $W^+$. Now we can consider the non abelian analogy of the system (1.4) (see [V-W]) :

\[ D_{a_0}^{\nabla_{\text{det}}} (\phi) = 0 \]

(1.13)

and the symmetry group of this system is given as the central extension

\[ 1 \rightarrow G_{\text{det}(E \otimes W^+)} \rightarrow G \rightarrow G_{\xi} \rightarrow 1. \]  

(1.14)

The moduli spaces $\mathcal{M}_{SW}(0, 2c_1 + rc, p_1)$ of Seiberg-Witten monopoles are defined as a space of solutions of this system modulo action of the gauge group (1.14), where $p_1$ is the Pontriagin number of $E$.

In the same vein we can define the $\omega, t$ perturbed system considering

\[ (F_{a_0} + \left( \frac{1}{2} F_{\nabla_{\text{det}}} + \omega \right) \otimes id_E)^+ = - (\phi \otimes \overline{\phi})_0, \]  

(1.13.\omega)

as the second equation of the system (1.13) and get the moduli space

\[ \mathcal{M}_{SW}(\omega, t, 2c_1 + rc, p_1) \]
In particular

\[ M_{\text{asd}} \subset M_{\text{SW}} \left(-\frac{1}{2}F_{\nabla_{\text{det}}} t, 2c_1 + rc, p_1 \right) \] (1.15)

as the subspace of degenerated solutions of type \((a_0, \nabla_{\text{det}}, 0)\).

Non abelian Seiberg-Witten equations are very strong too (in spite of fact that we lost a compactness). In particular it is easy to see that

1) in the symplectic (or Kähler) case we have nontrivial solution of (1.13) only if

\[ (2c_1 + rc) \cdot [\omega] \leq 0 \] (1.16)

where \([\omega]\) is the class of the symplectic (Kähler) form;

2) if \(p_1\) is negative enough then generic degenerated solution \((a_0, \nabla_{\text{det}}, 0)\) can be deformed to non trivial solution of the system (1.13) for any \(\omega\).

3) if \(X\) admits a metric with positive scalar curvature then \(M_{\text{SW}}(0, 2c_1 + rc, p_1) = \emptyset\) always.

To get more plastic moduli space we construct the system of equations which haven’t any analogy in the abelian case.

For simplicity we will consider the case of r\(k\)2 bundle only and use the following trick: if we fix some connection \(b_0\) on \(\text{det}W^+\) then the first pair of components (1.11) \((a_0, \nabla_{\text{det}})\) is given uniquely by a \(U(2)\)-connection \(a\) on \(E\). So instead the space of triples (1.11) we can consider the space of pairs

\[ (a, \phi) \in A(E) \times \Gamma(E \otimes W^+) \] (1.17)

as a configuration space and \(G_E\) as a gauge group.

Let \(\omega \in \Omega^2(X)\) be a two form in the cohomology class of \(-\pi ic_1(E)\). Denote by \(A_\omega \subset A_E\) a subspace of the space of all \(U(2)\) connections \(a\) on the bundle \(E\) subject to the condition

\[ trF_a = 2\omega, \] (1.18)

\(tr : \text{End}E \to \mathbb{C}\) is a trace. Traceless part of the curvature \(F_a - \frac{1}{2}tr(F_a)\) will be denoted as \((F_a)_0 = F_{a_0}\).

A gauge group \(G_E\) of \(E\) acts on both spaces \(A_E\) and \(A_\omega\) with orbit spaces, resp., \(B_E\) and \(B_\xi\).

Let us consider a natural action of this group on the space \(A_\omega \times \Gamma(W^+ \otimes E)\) with the orbit space denoted as \(C_E\). So \(C_E\) is the fibration

\[ C_E \to B_\xi \] (1.19)

with a fibre \(\Gamma(W^+ \otimes E)\).

Let \((a, \phi) \in A_\omega \times \Gamma(W^+ \otimes E)\). Denote by \(N_{(a, \phi)}\) a slice of the action at \((a, \phi)\):

\[ N_{(a, \phi)} = \text{Ker}(d_a^* \oplus m_\phi^*) \] (1.20)

where

\[ d_a \oplus m_\phi : \Omega^0(\text{ad}(E)) \to \Omega^1(\text{ad}(E)) \times \Gamma(W^+ \otimes E), \] (1.21)

is a tangent map of the action, \(m_\phi\) is a multiplication of a vector spinor \(\phi\) by the endomorphism.
Stabilisers of the gauge action.

The stabiliser $St = St_{(a, \phi)}$ of a point $(a, \phi)$ is at most $U(2)$ - the largest possible stabiliser of the connection itself. There are three possibilities for $St_a \subset U(2)$ (provided by the fact that $St_a$ is a centraliser of some subgroup of $U(2)$) except the trivial subgroup: $S^1, S^1 \times S^1, U(2)$. The last one corresponds to the pair consisting of trivial connection and zero spinor. Its neighbourhood is modelled as $N_{(0,0)}/U(2)$.

Centre $S^1 \subset U(2)$ of the structure group is the stabiliser of a point of the type $(a, 0)$, where $a$ is non-reducible. Orbits of such a points form the subspace of the zero-section of the fibration (1.19) $B_\xi \subset \mathcal{C}_E$ with the neighbourhood $U = \tilde{U}/S^1$ where $\tilde{U}$ is modelled on the total space of the bundle (1.19):

$$
\tilde{U} = \Gamma(W^+ \otimes E) \times_{G_E/S^1} \mathcal{A}_a^* \rightarrow B_\xi^*,
$$

(1.22)

$S^1$ acts with a weight one representation on fibres. The determinant of the universal bundle $detE$ can be described as $S^1$-equivariant bundle over $X \times \tilde{U}$ associated to a tautological representation of $S^1$ in $detE$.

If the stabiliser $S^1$ of $(a, \phi)$ is a different subgroup of $U(2)$, namely the one which maps onto a maximal torus of $PU(2)$, it means that connection $a = \lambda_1 \oplus \lambda_2$ is reducible due to some reduction of the bundle $E = L_1 \oplus L_2$, and with spinor of the form $(\phi_1 \oplus \phi_2), \phi_1 = 0, \phi_2 \neq 0$ with respect to the same reduction of the bundle. Subspace of all orbits of such a singularities is a subspace of

$$
\mathcal{A}_{L_1} \times \mathcal{A}_{L_2} \times \Gamma(W^+ \otimes L_2)/(\mathcal{G}_{L_1} \times \mathcal{G}_{L_2})
$$

(1.23)

consisting of points subject to the condition $F_{\lambda_1} + F_{\lambda_2} = 2\omega$ (which, given $\lambda_2$, determines $\lambda_1$ up to a gauge equivalence) i.e. isomorphic to $\mathcal{C}_{L_2}$. The neighbourhood $U$ of this locus is modelled as follows:

$$
U = \mathcal{N} \times (\Omega^1(L_1 \otimes L_2^{-1}) \times \Gamma(W^+ \otimes L_1) \times (\Gamma(W^+ \otimes L_2) - \{0\}))/(S^1 \times S^1).
$$

(1.24)

This is a total space of the real cone fibration

$$
(\Omega^1(L_1 \otimes L_2^{-1}) \otimes \mathcal{O}_{\mathcal{C}_E}^*_{det(W^+ \otimes L_2)} (-1) \oplus \Gamma(W^+ \otimes L_1) \otimes \mathcal{O}_{\mathcal{C}_E}^*_{det(W^+ \otimes L_2)})/S^1
$$

(1.25)

with weight one action of $S^1$ on fibres of the vector bundle, over $\mathcal{C}_{det(W^+ \otimes L_2)}^*$. The restriction of the universal bundle to $U$ is given as a $S^1 \times S^1$-equivariant fibration over $S^1 \times S^1$-space

$$
X \times \mathcal{N} \times (\Omega^1(L_1 \otimes L_2^{-1}) \times \Gamma(W^+ \otimes L_1) \times (\Gamma(W^+ \otimes L_2) - \{0\}))
$$

(1.26)

given by natural action of $S^1 \times S^1$ on $L_1 \oplus L_2$. This is equivalent to $S^1$-equivariant bundle $L_1 \oplus L_2 \otimes \mathcal{O}_{\mathcal{C}_E}^*_{det(W^+ \otimes L_2)}(1)$ over the total space of the bundle

$$
(\Omega^1(L_1 \otimes L_2^{-1}) \otimes \mathcal{O}_{\mathcal{C}_E}^*_{det(W^+ \otimes L_2)} (-1) \oplus \Gamma(W^+ \otimes L_1) \otimes \mathcal{O}_{\mathcal{C}_E}^*_{det(W^+ \otimes L_2)}).
$$

(1.27)
Cohomology ring of $C^*_E$.

Let $(A_\omega \times \Gamma(W^+ \otimes E))^*$ denotes the subspace of all points in $A_\omega \times \Gamma(W^+ \otimes E)$ with trivial stabiliser. Denote by $C^*_E$ corresponding orbit space. There is a universal bundle $\mathbb{E}$ over the space $X \times C^*_E$

$$\mathbb{E} = E \times G_E \cdot (A_\omega \times \Gamma(W^+ \otimes E))^*.$$  

(1.28)

The cohomology ring $H^*(C^*_E, \mathbb{Q})$ is generated by 2-dimensional classes

$$\mu(\Sigma) = \frac{1}{4} p_1(\mathbb{E})/\lbrack \Sigma \rbrack,$$

$$t = c_1(\mathbb{E})/\lbrack pt \rbrack$$  

(1.29)

and 4-dimensional class $\nu = \mu(pt) = \frac{1}{4} p_1(\mathbb{E})/\lbrack pt \rbrack$ as it follows from the fibration

$$\mathbb{R}_+ \times \mathbb{C}P^\infty \rightarrow (A_\omega \times \Gamma(W^+ \otimes E))^*/G_E \rightarrow A_\omega^*/G_\xi.$$  

(1.30)

Space

$$\mathcal{P}_E = (A_\omega \times S(\Gamma(W^+ \otimes E))^*/G_E),$$  

(1.31)

where $S()$ is the unit sphere bundle, is a deformational retract of $C^*_E$ and of the blow-up $\hat{C}_E$ (cf. §3 below). Therefore an inclusion $\mathcal{P}_E \rightarrow \hat{C}_E$ together with the projection $\hat{C}_E \rightarrow \mathcal{P}_E$ induces isomorphism in cohomologies. Thus the restriction of the $\mu, \nu$ classes to the locus of reducibles of the first type (as well as on it’s link in modified definition) coincides with those for $\mathcal{P}_E$. Restriction of the $t$ class is the generator of the fibre $\mathbb{C}P^\infty$ of the trivial fibration (1.30).

For the locus of reductions of the second type description (1.26-1.27) of the restriction of the universal bundle to the neighbourhood of this locus gives restriction of our generators. Link of this locus is a projective bundle

$$\mathbb{P}_{L_1,L_2} = \mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \otimes \mathcal{O}_{\mathbb{C}L_2^*}(-1) \oplus \Gamma(W^+ \otimes L_1) \otimes \mathcal{O})$$  

(1.32)

with the restriction of the universal bundle given as

$$L_1 \otimes \mathcal{O}_{\mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \otimes \mathcal{O}_{\mathbb{C}L_2^*}(-1) \oplus \Gamma(W^+ \otimes L_1) \otimes \mathcal{O})}(1) \oplus L_2 \otimes \mathcal{O}_{\mathbb{C}L_2^*}(1).$$  

(1.33)

A small computation shows

$$\mu(\Sigma) = \frac{1}{2}(l_1 - l_2, \Sigma)(t + c_1(\mathcal{O}_{\mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \otimes \mathcal{O}_{\mathbb{C}L_2^*}(-1) \oplus \Gamma(W^+ \otimes L_1) \otimes \mathcal{O})}(1)))$$  

(1.34)

$$\nu = (t + c_1(\mathcal{O}_{\mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \otimes \mathcal{O}_{\mathbb{C}L_2^*}(-1) \oplus \Gamma(W^+ \otimes L_1) \otimes \mathcal{O})}(1)))^2.$$  

(1.35)
§2. The equation.

Let \( a \in \mathcal{A}_E, \phi \in \Gamma(E \otimes W^+) \) and \( \nabla^\pm \in \mathcal{A}_{W^\pm} \). On the tensor product of bundles \( \text{Hom}(W^-, W^+) \otimes adE \) connections \( a \) and \( \nabla^\pm \) define a connection which is the tensor product of the Levi-Civita connection and the traceless part \( a_0 \) of \( a \). Dirac operator (1.12) coupled with a connection \( a \) is defined as a map:

\[
\mathcal{D}_{a_0}^{\text{det}(a \otimes \nabla^\pm)} : \Gamma(W^+ \otimes E) \to \Gamma(W^- \otimes E).
\]

(2.1)

Here the dependence on traceless part \( a_0 \) and \( U(1) \)-connection

\[
\nabla_{\text{det}} = \text{det}(a \otimes \nabla)
\]

on \( \text{det}(E \otimes W^+) \) will be exploited in different way and this is reflected in notations.

Fix any 2-form \( \omega \) in the cohomology class of \( -4\pi i f = -4\pi i(c_1 + c) \) and \( t \in T_\frac{1}{4} \).

Then our system of equations is:

\[
\begin{align*}
F_{\nabla_{\text{det}}} &= \omega, \\
\mathcal{D}_{a_0}^{\nabla_{\text{det}}} \phi &= 0, \\
F_{a_0}^+ &= ((t - 1)(\phi \otimes \overline{\phi}))_{00} \\
F_{a_0}^- &= 0
\end{align*}
\]

(2.2)

where \(((1 - t)(\phi \otimes \overline{\phi}))_{00}\) is a “double” traceless component of

\(((1 - t)(\phi \otimes \overline{\phi})) \in \text{sl}(E) \otimes \Lambda^2_+ = \text{su}(E) \otimes \text{su}(W^+) \subset u(E) \otimes u(W^+).\)

So the symmetry group of this system is given as the central extension (1.14) which we can identify as \( \mathcal{G}_E \) fixing any connection \( b_0 \) on \( \text{det}W^\pm \). Under this identification the coupled Dirac operator (2.1) depends on \( a \) and the metric \( g \) on \( X \).

So

\[
\mathcal{D}_{a_0}^{\nabla_{\text{det}}} = \mathcal{D}_a^{g, c + c_1(E)}.
\]

(2.3)

Definition 2.1. We shall denote moduli space of orbits of solutions of (2.2), modulo action of the gauge group as \( \mathcal{M}_{B}^{g, \omega, t}(c + c_1(E), p_1(E)) \).

The infinitesimal properties of this system such as the deformation complex, normal cones of singularities and so on are related closely to the properties of the following system of equations

\[
\begin{align*}
F_{\nabla_{\text{det}}} &= \omega, \\
\mathcal{D}_{a_0}^{\nabla_{\text{det}}} \phi &= 0, \\
F_{a_0}^+ &= 0
\end{align*}
\]

(2.20)

which was investigated in the series of papers [P-T], [T1-T5] and [P].

The moduli space of orbits of solutions of (2.20), modulo action of the gauge group was denoted as \( \mathcal{M}\mathcal{P}(X, p_1(E), c + c_1(E)) \) (the moduli space of jumping pairs, see Def. of §1 of [P] or (1.1.26) of [P-T]). Sending a pair \((a, \phi)\) to \( a_0 \) we get the map

\[
\pi : \mathcal{M}\mathcal{P}(X, p_1(E), c + c_1(E)) \to \mathcal{M}_{\text{asd}}
\]

(2.4)

to the moduli space of \( SO(3) \)-instantons and the image of this map

\[
\mathcal{M}_{1}^{g, \omega}(c + c_1(E), p_1(E)) = \pi(\mathcal{M}\mathcal{P}(X, p_1(E), c + c_1(E))) \subset \mathcal{M}_{\text{asd}}
\]

(2.5)

called the moduli space of jumping instantons.

On the set of solutions of the system (2.2) the gauge group action is not free. Possible stabiliser groups are \( S^1, S^1 \times S^1, U(2) \). The last one occurs as the stabilisers group of the solutions with trivial connection (on the trivial bundle) and zero spinor only.

\( S^1 \) may be a stabiliser of two types of solutions.
**Singularities of the first type.** Solutions with $\phi = 0$. This implies $(F_a^+)_0 = F_{a_0}^+ = 0$ i.e. (uniquely) corresponding $PU(2)$ connection $(a)_0$ is antiselfdual. Stabiliser is the centre of $U(2)$. We shall call these solutions reducible of the first type. The subspace of reducible solutions of the first type is diffeomorphic to the moduli of $asd$ connection for the adjoint bundle $\xi$.

Formally the set of singularities of this type is all $M_{asd}$ but let

$$M_B^{g,\omega,t}(c + c_1(E), p_1(E))_0 \subset M_B^{g,\omega}(c + c_1(E), p_1(E))$$

(2.6)

be the subset of solutions with $\phi \neq 0$. Then in $M_{asd}$ the subset

$$Sing_1 = lim M_B^{g,\omega,t}(c + c_1(E), p_1(E))_0$$

(2.7)

is defined correctly as the subset of limits of solutions with non trivial twisted spinor field $\phi$.

**Proposition 2.1.** For generic metric $g$ and generic form $\omega$

$$Sing_1 = M_1^{g,\omega}(c + c_1(E), p_1(E))$$

(2.8)

This Proposition is following immediately from the Transversality Theorem (see the next section) and the description of the Kuranishi family at a pair $(a, 0)$.

**Remark.** There exists the elementary proof of this statement using the iterative procedure providing by the third equation of (2.2).

**Singularities of the second type.** Let $Sing_2$ be the set of solutions with reducible connections, which we lift to $U(2)$-connection $a$, then the Levi-Civita $PU(2)$-connection on $W^+$ is lifted to $U(2)$-connection $\nabla^+$ and

$$F_{\nabla^+} + F_{det} = \frac{1}{2} \omega.$$ 

(2.9)

Moreover, $a = \lambda_1 \oplus \lambda_2$, due to some reduction of the bundle $E = L_1 \oplus L_2$, and with spinor of the form $\phi = 0 \oplus \phi_2$ with respect to the same reduction of the bundle. For a reducible connection the harmonic spinor always splits in this way and we require vanishing of the first component. Stabiliser here is of the form $S^1 \times 1 \subset S^1 \times S^1 \subset U(2)$. Let $l_1 = c_1(L_1)$, $a = l_1 - l_2$ and $f = c + c_1(E)$. Spinor $\phi_2 \in \Gamma(W^+_c \otimes L_2) = \Gamma(W^+_{c+2l_2})$ is a harmonic one:

$$D^{(2\lambda_2 \otimes det \nabla^* = \nabla_{det})} \phi_2 = 0.$$ 

Now

$$F_a = \left( \begin{array}{cc} F_{\lambda_1} & 0 \\ 0 & F_{\lambda_2} \end{array} \right) = \left( \begin{array}{ccc} \omega - F_{det} & F_{\lambda_2} & 0 \\ 0 & 0 & F_{\lambda_2} \end{array} \right),$$

$$(F_a)_0 = \frac{1}{2} \left( \begin{array}{cc} F_{\lambda_1} - F_{\lambda_2} & 0 \\ 0 & F_{\lambda_2} - F_{\lambda_1} \end{array} \right)$$

and

$$-(\phi \otimes \bar{\phi}) = \left( \begin{array}{cc} 0 & 0 \\ 0 & \phi \otimes \bar{\phi} \end{array} \right).$$
\[-(\phi \otimes \bar{\phi})_0 = \frac{1}{2} \begin{pmatrix} (\phi_2 \otimes \overline{\phi_2}) & 0 \\ 0 & -(\phi_2 \otimes \overline{\phi_2}) \end{pmatrix} \.\]

Therefore second our equation \((F_a^+)_0 = ((t-1)(\phi \otimes \bar{\phi}))_{00}\) provides
\[F_{2\lambda_2 + \text{det} \nabla^+}^+ = ((t-1)(\phi_2 \otimes \bar{\phi}_2))_{00} - \omega^+. \quad (2.10)\]
So a reducible solution of the second type produces a solution to \(U(1)\) - Seiberg Witten equation:
\[(2\lambda_2 + \text{det} \nabla^+, \phi_2) \in \mathcal{M}_{SW}(\omega^+, t, f - \alpha). \quad (2.11)\]
It is easy to see that this is one to one correspondence and we proved

**Proposition 2.2.**
\[\text{Sing}_2 = \bigcup_{\beta | (f - \beta)^2 = p_1} \mathcal{M}_{SW}(\omega^+, t, \beta). \quad (2.12)\]

**Remark.** The very important and very interesting case is when \(f = 0\). Here we can choose \(\omega = 0\) and we have the standard Seiberg - Witten equation (1.4). In this case the diffeomorphisms group \(\text{Diff}_X\) acts on the moduli space \(\mathcal{M}_{B,g,0}^{g,0}(0, p_1(E))\) and all its singularities.

And finally \(S^1 \times S^1\) occurs as a stabiliser of a solution consisting of an \(\text{asd}\) reducible connections and vanishing spinor.

**Linearization.** Linearization of the the system (2.2) is elliptic and, therefore, Fredholm:
\[\mathcal{D}_a \delta \phi + \delta a \star \phi = 0 \]
\[d_a^+ (\delta a) = ((t-1)(\delta \phi \otimes \bar{\phi} + \phi \otimes \delta \bar{\phi}))_{00} \quad (2.13)\]
where \(\delta a \in \Omega^1(u(E)), \delta \phi \in \Gamma(E \otimes W^+)\) are components of tangent vector at the point \((a, \phi)\).

This linear map together with the linearization of the action of gauge group gives following deformation complex, which is homotopic to a direct sum of a deformation complex for moduli space of \(\text{asd}\)-connections and Dirac operator:
\[\Omega^0(u(E)) \xrightarrow{\delta_1} \Omega^1(pu(E)) \times \Gamma(W^+ \otimes E) \xrightarrow{\delta_2} \Omega^2_+(pu(E)) \times \Gamma(W^- \otimes E), \quad (2.14)\]
with first map given by \(\delta_1 = d_a \oplus m_\phi\) and second by a matrix
\[\delta_2(\delta a, \delta \phi) = (d_a^+ (\delta a) + ((t-1)(\phi \otimes \delta \bar{\phi} + \delta \phi \otimes \bar{\phi})_{00}, \mathcal{D}_a(\delta \phi) + (\delta a) \cdot \phi) \quad (2.15)\]
Index of the complex is given by following formula:
\[\text{ind} = -\frac{3}{2}p_1 - 3(1 + b^2_2) + \frac{1}{2}(f^2 - \sigma_X) - 1 \quad (2.16)\]
where \(p_1 = p_1(E), f = c + c_1(E), \sigma_X\) - signature of \(X\).

Standard technique of Kuranishi models gives a model of a neighbourhood of the point \((a, \phi)\) as space of \(\text{St}_{(a, \phi)}\) orbits in a neighbourhood of zero in the preimage of
zero of certain real-analytic map \( H^1_{(a, \phi)} \xrightarrow{\Psi} H^2_{(a, \phi)} \), where \( H^i_{(a, \phi)} \) are homologies of the deformation complex.

For a reduction of the first type one has all cross-terms in deformation complex vanishing and it turns to be a direct sum of the deformation complex for the moduli of \( asd \)-connections and Dirac operator with stabiliser \( S^1 \) acting only on the spaces of sections of twisted spinors \( \Gamma(W^\pm \otimes E) \) with weight one.

Let \( (a, \phi) = (\lambda_1 \oplus \lambda_2, 0 \oplus \phi_2) \) be a reduction of the second type. Its deformation complex is a sum of the deformation complexes of \( U(1) \)-monopole equations and following complex:

\[
\Omega^0(L_1 \otimes L_2^{-1}) \xrightarrow{\delta_1} \Omega^1(L_1 \otimes L_2^{-1}) \times \Gamma(W^+ \otimes L_1) \xrightarrow{\delta_2} \Omega^2_+(L_1 \otimes L_2^{-1}) \times \Gamma(W^- \otimes L_1),
\]

with first map given by \( \delta_1 = d_\lambda \oplus m_{\phi_2} \) and second by a matrix

\[
\delta_2(\alpha, \delta \phi_1) = (d^+_\lambda(\alpha) + ((t - 1)(\phi_2 \otimes \delta \phi_1 + \delta \phi_1 \otimes \phi_2)_0, D_{\lambda_1}(\delta \phi_1) + (\alpha) \cdot \phi_2).
\]

\[\text{§3. Transversality.}\]

Let us return to the begining of the previous section. If we fix the connections \( \nabla^\pm \) on the spinor bundles \( W^\pm \) compatible with the Levi - Civita connection then the Dirac operator (2.1) will depend on \( U(2) \)-connection \( a \) and as the configuration space we can consider the space \( \mathcal{A}_\omega \times \Gamma(W^+ \otimes E) \) with condition (1.18) instead the first equation of (2.2). We will do it in this section.

From Kuranishi description it follows that the point \( (a, \phi) \) is smooth if the obstruction space vanishes: \( H^2_{(a, \phi)} = 0 \). Our moduli space depends also on different continuous parameters i.e. metric, connection on \( detW^+ \) etc. It is the task of this section to prove that for generic parameters one has \( H^2_{(a, \phi)} = 0 \) for all solutions \( (a, \phi) \).

First we shall discuss transversality at reducible points. In order to get it we shall change configuration space of our system. Following illustrates reasons in the case of reducible solutions of the first type.

Reductions of the first type, i.e. \( (a, 0) \) with \( asd \)-connection \( a \) occurs for a generic metric only if \( v.dim(\mathcal{M}_{asd}) = -2p_1(E) - 3(1 + b_2^+ \geq 0 \) (cf [FU], Thm 3.1). It also follows that for generic metric \( asd \)-component of the deformation complex has vanishing second cohomology \( H^2_a = 0 \). Therefore the only possibility for \( H^2_{(a, \phi)} \neq 0 \) is when \( \text{coker} D_a \neq 0 \). Generally it is the case since the Chern class of the minus index bundle \( c_{indD_a}(-1) \) does not vanish. This is a topological obstruction for vanishing of the obstruction space \( H^2_{(a, 0)} \) for all \( asd \) connections \( a \).

The Kuranishi description is given by a map

\[
\Psi : H^1_a \otimes ker D_a \rightarrow \text{coker} D_a
\]

with \( \Psi(\alpha, \sigma) = \alpha \ast \sigma + O((\alpha, \sigma)^3) \) has vanishing differential.

Thus we shall change equation, i.e. configuration space to get rid of this topological obstruction for \( H^2_{(a, 0)} \) to vanish everywhere. Take instead of the space \( C_E \mathcal{A}_\omega \times \Gamma(W^+ \otimes E) \) its blow up in the locus \( \mathcal{A}_{\omega, red} \times \{0\} \) of pairs consisting of the reducible connection and zero spinor:

\[
\tilde{\mathcal{A}}_\omega = \mathcal{A}_\omega \times \tilde{\Gamma}(W^+ \otimes E)
\]
Gauge group acts in a natural way on the blow up. Lift the equation to the blow up in a natural way. For example in the neighbourhood of reducibles of the first type for
\[(a, \phi, \psi) \in A_\omega \times \Gamma(W^+ \otimes E) \times \mathbb{P}(\Gamma(W^+ \otimes E))\]
take the restriction of solutions of the system
\[D_a \psi = 0\]
\[(F_a^+)_0 = - (\phi \otimes \overline{\phi})_0\]
to \(Y \subset \Gamma(W^+ \otimes E) \times \mathbb{P}(\Gamma(W^+ \otimes E))\) where \(Y\) is a blow up of \(\Gamma(W^+ \otimes E)\) in zero or equivalently the total space of the linear bundle \(\mathcal{O}_{\mathbb{P}(\Gamma(W^+ \otimes E))}(-1)\).

**Definition 3.1.** Denote by \(\hat{M}_{g, \omega}^{g, \omega}(c + c_1(E), p_1(E))\) moduli of solutions in the space (3.2) modulo the action of the gauge group.

There is an obvious projection
\[\pi : \hat{M}_{g, \omega}^{g, \omega}(c + c_1(E), p_1(E)) \rightarrow M_{g, \omega}^{g, \omega}(c + c_1(E), p_1(E))\] (3.3) (compare 2.4).

Now the space of reductions of the first type \(\pi^{-1}(M_{\text{red}})\) can be identified with the moduli space of pairs \(M\mathcal{P}(X, p_1(E), c + c_1(E))\) (2.4). In [P-T] it is proved that this moduli space is smooth in the sense that second homology of corresponding deformation complex vanishes.

Consider reducible solutions of the second type \(\pi^{-1}(M_{SW}(\beta))\). Let
\[(a = \lambda_1 \oplus \lambda_2, \phi = 0 \oplus \phi_2) \in M_{SW}(\beta)\]
and
\[(a, \phi_2, \langle \theta, \delta \phi_1 \rangle) \in \pi^{-1}(M_{SW}(\beta)),\] (3.4)
where
\[\langle \theta, \delta \phi_1 \rangle \in \mathbb{P}(\ker \delta_1^*) \subset \mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^+ \otimes L_1)).\]

The curvature of \(\lambda = \lambda_1 - \lambda_2\) has form \(\sigma = \sigma \otimes u\) for two-form \(\sigma\) and \(u \in \ker d_a\).

Deformations of the solution of the modified system is described by a sum of the deformation complex of Seiberg-Witten moduli space at the point \((\lambda_2, \phi_2)\) and following picture for the deformations in the normal direction (compare 2.17-2.18): take the bundle
\[
\begin{align*}
\Omega^2_+(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^- \otimes L_1) \otimes \mathcal{O}_{\mathbb{P}(1)} \\
\downarrow
\mathbb{P}(\ker \delta_1^*)
\end{align*}
\]
(3.5)
and take the section of this defined by the map \(\delta_2\), with \(\delta_2\) given by the formulas
\[
\delta_1 = d_\lambda \oplus m_{\phi_2}
\]
\[
\delta_2 (\theta, \delta \phi_1) = (\frac{\delta_2}{\delta_1} (\theta, \delta \phi_1)) + \frac{\delta_1}{\delta_1} (\delta_2 \phi_1) = (\theta, \phi_1).
\]
Now directions where deformation is not obstructed are given as zero set of this section denoted by \( s_{\delta_2} \).

Dependence on metric given through that of \( \delta_2 \) and dependence on 1-form \( \eta \) given by replacing \( \omega \) by \( \omega + d\eta \) presents \( \delta_2 \) as a section of

\[
\begin{align*}
&\left( \Omega^2_+(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^- \otimes L_1) \right) \otimes \mathcal{O}_P(1) \\
&\downarrow \\
&\mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^+ \otimes L_1)/\delta_1(\Omega^0(L_1 \otimes L_2^{-1}))) \times \text{Metr} \times \Omega^1
\end{align*}
\]

with linearization at the point (3.4) given by the differential

\[
\begin{align*}
&\left( \Omega^1(L_1 \otimes L_2^{-1}) \times \Gamma(W^+ \otimes L_1) \right) \times T\text{Metr} \times \Omega^1 \to \\
&\to \Omega^0(L_1 \otimes L_2^{-1}) \times \Omega^2_+(L_1 \otimes L_2^{-1}) \times \Gamma(W^- \otimes L_1),
\end{align*}
\]

(3.6)

\((\theta, \delta\phi_1, r, \eta) \mapsto (\delta_1^*(\theta, \delta\phi_1), (d_\lambda^* + r^*d_\lambda)\theta + \overline{\phi_2} \otimes \delta\phi_1, D_\lambda, \delta\phi_1 + \eta \ast \delta\phi_1 + \theta \ast \phi_2)\)

Assume that

\[
c_{dim\mathcal{M}_{SW}}(\text{Ind}(d_\lambda \oplus D_\lambda)) \neq 0.
\]

(3.7)

Assumption provides us with the nonzero point \((\theta, \delta\phi_1) \in \ker\delta_2\) for some solution \(a = \lambda_1 \oplus \lambda_2, \phi = 0 \oplus \phi_2\) or in other words with a point in \(\mathcal{M}_{SW}^\theta(g, \omega)(c + c_1(E), p_1(E))\) lying over the corresponding Seiberg - Witten moduli space.

**Lemma 3.1.** With the assumptions as above the differential (3.6) is epimorphic.

**Proof.** We shall remark first that if \(s_{\delta_2}(a, \phi_2, (\theta, \delta\phi_1)) = 0\) for \((a, \phi_2, (\theta, \delta\phi_1)) \in \mathbb{P}(\ker\delta_1^*)\) then

\((\theta, \delta\phi_1) \neq 0 \Rightarrow \theta \neq d_\lambda\xi \& \delta\phi_1 \neq 0.
\)

Indeed, assume \(\theta = d_\lambda\xi\). Vanishing of the section implies in particular

\[
d_\lambda^*\theta + m_{\phi_2}^*(\delta\phi_1) = 0, \ d_\lambda^*\theta + (\delta\phi_1 \otimes \phi_2)_0 = 0.
\]

This is equivalent to

\[
\begin{align*}
&d_\lambda^*d_\lambda\xi + (\delta\phi_1 \otimes \phi_2)_{tr} = 0 \\
&\sigma \otimes [u, \xi] + (\delta\phi_1 \otimes \phi_2)_0 = 0.
\end{align*}
\]

(3.8) (3.9)

Taking scalar product of (3.8) with \(\xi\) one has

\[
(\sigma \otimes [u, \xi], \xi) + ((\delta\phi_1 \otimes \phi_2)_0, \xi) = 0 + ((\delta\phi_1 \otimes \phi_2)_0, \xi) = 0
\]

and small calculation shows that

\[(\delta\phi_1 \otimes \phi_2)_0, \xi) = 0 \Rightarrow ((\delta\phi_1 \otimes \phi_2)_{tr}, \xi) = 0\]

Now taking scalar product of (3.9) with \(\xi\) one has

\[
(d_\lambda^*d_\lambda\xi, \xi) + ((\delta\phi_1 \otimes \phi_2), \xi) = (d_\lambda^*d_\lambda\xi, \xi) = 0 \Rightarrow |d_\lambda\xi| = 0 \Rightarrow \xi = 0.
\]
Substituting this to (3.8) and (3.9) we get \((\delta\phi_1 \otimes \phi_2) = 0\) which, provided \(\phi_2\) being a solution to an elliptic equations with Laplace - type symbol vanishes totally if vanishes on the open subset, means that \(\delta\phi_1 = 0\). This is a contradiction to the assumption \((\theta, \delta\phi_1) \neq 0\).

In a similar way let us assume \(\delta\phi_1 = 0\). From the vanishing of the section it follows in particular

\[ D_{\lambda_1}(\delta\phi_1) + (\theta) \cdot \phi_2 = 0 \]

which means \((\theta) \cdot \phi_2 = 0\) and therefore \(\theta = 0\) for the same reason as above.

Take the nonzero point \((\theta, \delta\phi_1) \in \ker d_{\lambda_1} \oplus D_{\lambda_1}\). Then, as we showed, \(\theta \neq d_{\lambda_1} \xi\) and by Thm 4.19 [FU] restriction of the differential of our map to infinitesimal variations of metrics \(g = g + r\) covers \(\Omega^2_+(L_1 \otimes L_2^{-1})\) : \(r \rightarrow r^*d_{\lambda}^{-1}\) is onto \(\text{coker} d_{\lambda}^{-1}\). As well \(\delta\phi_1 \neq 0\) and infinitesimal variation of 1-forms covers \(\Gamma(W^- \otimes L_1)\): \(\eta \rightarrow \eta \ast \delta\phi_1\) is onto \(\text{coker} D_{\lambda_1}\). Together with Lemma 1.1 this proves the statement.

Therefore one can assume that for generic metric and 1-form second homology group vanishes and reducibles of the second type are in general position. Thus for a generic parametra one has zero locus of the section \(s_{\delta_2}\) being smooth finite-dimensional submanifold of the mentioned projective fibration with the fibre \(\mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^+ \otimes L_1)/\delta_1(\Omega^0(L_1 \otimes L_2^{-1}))\) over the point of \(M_{SW}(\beta)\). This submanifold will be referred to as a link of \(M_{SW}(\beta)\) in \(\overline{M}_B\).

Following theorem shows that when some extra continuous parametra are added this is true also for non reducible solutions. As above dependence on the choice of metric and form \(\omega\) gives gauge invariant smooth map:

\[ T \times \text{Metr} \times \Omega^1 \times A_\omega \times \Gamma(W^+ \otimes E) \xrightarrow{w \oplus v} \Omega^2_+ (pu(E)) \times \Gamma(W^- \otimes E) \quad (3.10) \]

defined as

\[ (t, g, \omega, a, \phi) \rightarrow (D_a \phi, F_a^+ (t - 1)((-\phi \otimes \overline{\phi}))_0) \quad (3.11) \]

If we prove that it is a submersion it will follow from Sard-Smale theorem that for generic \(g\) and \(\omega\) the moduli space is smooth in the sense that the second homology of corresponding deformation complex vanishes.

**Theorem 3.1.** The map \(w\) is a submersion.

**Proof.** Let \(Dw, Dv\) denote a differentials of maps \(w, v\) resp. It follows from [P-T, Prop.1.3.5] that if \(D_{(a, \phi)} v|a=g=\delta t=0\) isn’t onto then the connection \(a\) is reducible \(a = \lambda_1 \oplus \lambda_2\) and the spinor has form \(\phi = 0 \oplus \phi_2\), that is the case of the previous lemma. Now we shall use variations of metric and connection to prove that

\[ D_{(a, \phi)} w|a=\delta \phi = 0 = \Omega^2_+ (pu(E)). \]

Condition

\[ (d_a^+ (\delta a) + r^* F^{-}_a + (\delta t (\phi \otimes \overline{\phi}))_0, \Phi) = 0 \quad (3.12) \]

for self dual non zero \(\Phi \in \Omega^2_+ (u(E))\) is equivalent to

\[ d_a (\Phi) = 0 = d_a^* (\Phi) \quad (3.13) \]

and images of \(F^{-}_a\) and \(\Phi\) considered as map from \(\Lambda^2, \Lambda^2_+\) resp. to \(pu(E)\) are orthogonal as well as those of \((\phi \otimes \overline{\phi})\) and \(\Phi\) (cf. [F-U], Lemma 3.7). That is \(\Phi\) does not vanish on an open set. Thus maximal possible rank of the image of \(F^{-}_a\) is 2. In that case \(\Phi\) has rank one in generic point and therefore may be written as \(\Phi = \lambda \otimes \nu\).
where $\chi \in \Omega^2_\chi, u \in \Omega^0(u(E))$ and $|u| = 1$. Standard computation as in [F-U, Thm 3.4] shows that $d\chi = d_a u = 0$. Thus image of $F_a^-$ is perpendicular to $u$. The same reference gives $(F_a^-, u) = 0 \Rightarrow [F_a^-, u] \neq 0$. So $[F_a, u] = [F_a^+, u] \oplus [F_a^-, u] \neq 0$

$$d_a u = [F_a, u] = [F_a^-, u] \oplus [-\phi \otimes \bar{\phi}, u] \neq 0$$

which gives a contradiction. This means that $F_a^-$ and $(-\phi \otimes \bar{\phi})$ has rank at most one and it is exactly one at generic point and their images are parallel. This is possible if only $\phi$ considered as a map $W^+ \to E$ has rank one. In this case by [PT, Prop. 1.3.5] $a$ is reducible as well and we are done.

Results of this section can be formulated as follows:

**Theorem 3.2.** For generic choice of parameter in $T \times \text{Metr} \times \Omega^1 \times \text{moduli space}$ 

$$\widehat{\mathcal{M}}^g,\omega_B(c + c_1(E), p_1(E))$$

is a smooth manifold of dimension given by (2.16) with the boundary

$$\mathcal{M}^g,\omega_B(c + c_1(E), p_1(E)) \cup \bigcup_{\beta|(f-\beta)^2=p_1} \text{Link of } \mathcal{M}_{SW}(\omega^+, t, \beta)$$

**Remark.** In the case $b_2^+(X) = 1$ one has chamber structure for both invariants. Specifying chambers for Seiberg-Witten invariants related to some fixed chamber for Donaldson polynomial one has to take in account shift of the period by $\omega^+$ which may a priory move it out of the positive cone (i.e. period space of the Donaldson theory).

§4. Compactification.

The Weitzenböck formula

$$D_a^* D_a \phi = \nabla_a^* \nabla_a \phi - (F_a^+ + F_{\text{det}}) \phi + \frac{s}{4} \phi$$

provides universal estimates:

$$|\phi| \leq \text{const}_1$$
$$|F_a^+|^2 \leq \text{const}_2$$

in the same way it does for $U(1)$ case (compare (1.10)) although in this case it depends not only on the scale curvature of the manifold but also on the maximum of the curvature $F_{\text{det}}$

$$0 \leq \Delta |\phi| \leq \frac{-s}{4} |\phi|^2 + \langle (t-1)(\phi \otimes \bar{\phi}) \rangle_{00} (\phi, \phi) + \langle (F_{\text{det}}(\phi), \phi) =$$

$$= \frac{-s}{4} |\phi|^2 - \frac{1}{2} |\phi|^4 + \langle (t(\phi \otimes \bar{\phi}) \rangle_{00} (\phi, \phi) + \langle (F_{\text{det}}(\phi), \phi) \leq$$

$$\leq \frac{-s}{4} + |F_{\text{det}}||\phi|^2 - (\frac{1}{2} - |t|)|\phi|^4 \leq \frac{-s}{4} + |F_{\text{det}}||\phi|^2 - \frac{1}{4} |\phi|^4$$

provided $|t| \leq \frac{1}{4}$ (i.e. $t \in T_{\frac{1}{4}}$). Therefore

$$|\phi| \leq -s + 4 |F_{\text{det}}|. \quad (4.1')$$

Weil formula for $c_2(E)$ gives

$$8\pi^2 c_2(E) \leq \|F_a^-\|_{L^2} \leq 8\pi^2 c_2(E) + \text{const}_2 \cdot \text{vol}_X. \quad (4.2)$$

This inequality provides following...
Lemma 4.1. Let \((a_i, \phi_i)\) be a sequence of solutions of our system and \(\epsilon\) - a real positive number. Then there is a subsequence \(i_j\) and a finite set \(x_1, \ldots, x_p \in X\) such that

\[
\forall y \in X - \{x_i\} \exists D_y \ni y \text{ such that } \forall j \|F_{a_{i_j}}\|_{L^2}^2 \leq \epsilon.
\]

By this lemma and Theorem 2.3.7 of [DK] for each such \(D_y\) there exist a gauge in which one has an estimate

\[
\|a_{|D_y}\|_{L^2} \leq M\|F_{a_{|D_y}}\|_{L^2}
\]

for some constant \(M\).

To get \(C^\infty\)-convergence on the disk \(D_y\) (or in fact on the smaller disk \(\hat{D}_y \subset D_y\)) for any \(y \in X - \{x_i\}\), and therefore on a punctured manifold \(X - \{x_i\}\), one uses approach of [DK], Ch 2, for the proof of Uhlenbeck theorem.

The only place one needs antiselfduality for the proof of the Theorem 2.3.8 of [DK] is getting from \(L^2_1\) bound on connection \(a\) to each of these and say that a sequence of points \((y_{i,m})\) one has to be replaced by an estimate:

\[
\|a_{|\hat{D}_y}\|_{L^2} \leq \text{const} \cdot (\|a_{|D_y}\|_{L^2} + \|\phi\|_{L^2}) \leq \text{const} \cdot \|a_{|D_y}\|_{L^2} + \text{const}_2 \cdot \text{vol}_{D_y}(4.3')
\]

where \(\hat{D}_y \subset D_y\) which gives at the end an estimate

\[
\|a_{|D_y}\|_{L^2} \leq M_i,\text{vol}_y \|F_a\|_{L^2(D_y)} + \text{const} \cdot \text{vol}(D_y)
\]

while the rest of the scheme applied is unchanged. The estimate (4.3') provides \(C^\infty\)-convergence on any compact subset of \(X - \{x_i\}\) together with an estimate of the norm of curvature of the limit \((a_{\infty}, \phi_{\infty})\):

\[
\|F_{a_{\infty}}\|_{L^2} \leq 8\pi^2 c_2(E) + \text{const}_2 \cdot \text{vol}_X
\]

Removing of singularities is also a slight modification of that for asd-connections. Since harmonic spinor is smooth (provided we coupled the Dirac operator with a smooth connection \(a\)) one has regularity of \(L^2_1\) solutions of our system. Cutting off the singularity \((\psi^2 a_0, \psi \phi)\) in a small disk \(D_x\) centred in a singularity with a smooth cut-off function \(\psi\) vanishing in a \(r\)-disk and equal to one outside of \(2r\)-disk gives a small error to the first equation of the system:

\[
F_{\psi^2 a_0} + (\psi \phi \otimes \psi \phi)_{00} = \psi^2 (F_{a_0} + (\phi \otimes \phi)_{00}) + d(\psi^2) \cdot a_0 + (\psi^4 - \psi^2)[a_0, a_0] + \\
\|F_{\psi^2 a_0} + (\psi \phi \otimes \psi \phi)_{00}\|_{L^2(D_x)} \leq \|d(\psi^2)\|_{L^4(D_x)} \|a_0\|_{L^4(D_x)} + \|a_0\|_{L^4(D_x)} \leq \text{const} \|F_{a_0}\|_{L^2(D_x)}
\]

with a similar estimate for \(\|D_{\psi \phi}(\psi \phi)\|_{L^2}\). Taking weak \(L^2_1\) limits in a Coulomb gauge when \(r\) tends to zero gives a limit pair \((a', \phi')\) with smooth connection and spinor provided by the above mentioned regularity of \(L^2_1\) solutions.

Let us add ideal points to our moduli space. These are triples:

\[
((a, \phi), (x_1, ..., x_i)), \text{ where } (a, \phi) \in M_B(p_1 + 4l), x_i \in X.
\]

We shall assign a (curvature) density

\[
|F_a|^2 + 8\pi \sum_{i=1}^l \delta_{x_i}
\]

to each of these and say that a sequence of points \((a_j, \phi_j)\) converges to the ideal point if curvature densities converges as measures and there is a \(C^\infty\) convergence on compact subsets in \(X \setminus \{x_i\}\). Let us omit for simplicity some indexes:

\[
M_B^\infty(\varrho, \rho(F)) = M_B^\infty(\varrho, \rho(F))
\]
**Definition 4.1.** Let $\overline{\mathcal{M}}_B(p_1)$ be a closure of the moduli space in the union

$$\mathcal{M}_B(p_1) \cup \mathcal{M}_B(p_1 + 4) \times X \cup \ldots \cup \mathcal{M}_B(p_1 + 4k) \times s^k X \cup \ldots,$$

endowed with the convergence topology.

As a result one has following

**Theorem 4.1.** The space $\overline{\mathcal{M}}_B$ is compact

Dimension counting shows that for generic parametra only a subvariety of codimension 2 of the stratum

$$\mathcal{M}_B(p_1 + 4k) \times s^k X$$

is in the closure of $\mathcal{M}_B$.

**Corollary.** If $m_i \in \mathcal{M}_B(p_1) \cap V_\Sigma$ is a sequence converging to a point $(m_\infty, \{x_1, \ldots, x_k\})$ of the compactification, then either $x_i \in \Sigma$ or $m_\infty \in \mathcal{M}_B(p_1 + 4k) \cap V_\Sigma$.

§5. **Localisation of the Donaldson polynomial.**

Now one can look at our moduli spaces as at bordism connecting links of reducibles points. Reducibles of the first type, moduli space $\mathcal{M}_P$, is a boundary of $\mathcal{M}_B$ and its link is a copy of $\mathcal{M}_P$ itself (or relevant compactification if considering $\overline{\mathcal{M}}_B$). The set of reducibles of the second type is union of all Seiberg-Witten moduli spaces $\mathcal{M}_{SW}(\beta)$ such, that $(-\beta + f)^2 = p_1$ or $(-\beta + f)^2 \geq p_1$ if one takes in account compactification $\overline{\mathcal{M}}_B$.

We shall take a Poincare dual to certain cohomology classes on our moduli space $\mathcal{M}_B$ and on its compactification $\overline{\mathcal{M}}_B$. If one takes Poincare dual on $\overline{\mathcal{M}}_B$ to the cohomology class of degree $\dim \mathcal{M}_B - 1$ one has a 1-dimensional singular manifold $\mathcal{I}$ with singularities diffeomorphic to the real cone over finite number of points. We shall assume that this manifold intersect lower strata of the compactification only at reducible points of the second type. Indeed dimension of the moduli space drops by 6:

$$\dim \mathcal{M}_B(p_1) - \dim \mathcal{M}_B(p_1 + 4) = 6.$$  

(5.1)

Therefore if one bubbling point “kills” not more then two degree 2 classes (like $\mu_\Sigma$ or $t$) or one degree 4 class ( $\nu$ -class) then $\mathcal{I}$ intersects lower strata of the compactification in reducible points only. This works since one takes representatives “localised” over Riemannian surfaces (for $\mu_\Sigma$) or a point (for $\nu$ or $t$) unless there is a trivial connection in the compactification. Since all $\mu$-classes are lifted from the moduli of asd-connections one can use standard dimension counting to show that $\mathcal{I}$ avoids lower strata in the compactification of reducibles of the first type $\overline{\mathcal{M}}_P$ and the number $\sharp \partial \mathcal{I} \cap \mathcal{M}_P$ is the value of the relevant Donaldson polynomial (or Spin polynomial more generally). Thus manifold with boundary $\mathcal{I}$ is relating certain intersection number on $\mathcal{M}_P$ with certain integrals over links of singularities of the second type in $\overline{\mathcal{M}}_B$. If one takes, say, class $\prod_1^d \mu_\Sigma \times t^n$, $2d = v.\dim \mathcal{M}_{asd}(p_1) = -2p_1 - 3(1 + b^+_2)$ one has following equality:

$$\gamma_{p_1, fmod2-w_2(X)(\Sigma)} = \prod_{p_1, fmod2-w_2(X)(\Sigma)} \int \text{link of } \mathcal{M}_{SW}(\beta) \prod_1^d \mu_\Sigma u^n.$$  

(5.2)
Of course the summand in the r.h.s is nontrivial only if the moduli space $\mathcal{M}_{SW}(\beta)$ is nonempty.

Now to describe a value of the Donaldson polynomial we have to describe for any Seiberg-Witten class $\beta$ the link of

$$\mathcal{M}_{SW}(\beta) \subset Sing_2 \subset \mathcal{M}_B(f, p_1)$$

(We omit the upper indexes of the moduli space because it doesn’t depend on the continuous parametra) and the integral of the type (5.2) over this link. The point is that this integral is the standard polynomial of the linear forms

$$\langle \beta, \rangle, \langle f, \rangle$$

and the quadratic intersection form $q_X$. Let call it the local polynomial and denote it as $loc\gamma$.

Now in the localisation formula (5.2) the left side doesn’t depend on $f$ and we can consider one as the relations between the local polynomials. Using this relations we will compute the local polynomials precisely in the next paper.

For this computation we will construct pure geometrical finite space $MH^1(\beta, r)$, where $4r = (-\beta + f)^2 - p_1$ is number of points on $X$, as the space of ”gluing parameters”, (which doesn’t depend on 2-cohomology class $f$) and the virtual vector bundle on $MH^1(\beta, r)$ such that the intersections of $\mu$-classes with ”top” Chern class of this bundle give the local polynomials. These constructions are pure geometrical and here we can say ”goodbye” to the gauge theory, connections, moduli spaces and all standard stuff of the Donaldson’s Theory.

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