SPHERE THEOREM FOR MANIFOLDS WITH POSITIVE CURVATURE

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Abstract. In this paper, we prove that, for any integer \( n \geq 2 \), and any \( \delta > 0 \) there exists an \( \epsilon(n, \delta) \geq 0 \) such that if \( M \) is an \( n \)-dimensional complete manifold with sectional curvature \( K_M \geq 1 \) and if \( M \) has conjugate radius \( \rho \geq \pi/2 + \delta \) and contains a geodesic loop of length \( 2(\pi - \epsilon(n, \delta)) \) then \( M \) is diffeomorphic to the Euclidian unit sphere \( \mathbb{S}^n \).

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1. Introduction. One of the fundamental problems in Riemannian geometry is to determine the relation between the topology and the geometry of a Riemannian manifold. In this way the Toponogov’s theorem and the critical point theory play an important role. Let \( M \) be a complete Riemannian manifold and fix a point \( p \) in \( M \) and define \( d_p(x) = d(p, x) \). A point \( q \neq p \) is called a critical point of \( d_p \) or simply of the point \( p \) if, for any nonzero vector \( v \in T_qM \), there exists a minimal geodesic \( \gamma \) joining \( q \) to \( p \) such that the angle \( \angle(v, \gamma'(0)) \leq \pi/2 \). Suppose \( M \) is an \( n \)-dimensional complete Riemannian manifold with sectional curvature \( K_M \geq 1 \). By Myers’ theorem the diameter of \( M \) is bounded from above by \( \pi \). In [4] Cheng showed that the maximal value \( \pi \) is attained if and only if \( M \) is isometric to the standard sphere. It was proved by Grove and Shiohama [5] that if \( K_M \geq 1 \) and the diameter of \( M \) \( \text{diam}(M) > \pi/2 \) then \( M \) is homeomorphic to a sphere.

Hence the problem of removing homeomorphism to diffeomorphism or finding conditions to guarantee the diffeomorphism is of particular interest. In [13] C. Xia showed that if \( K_M \geq 1 \) and the conjugate radius \( \rho(M) \) of \( M \) is greater than \( \pi/2 \) and if \( M \) contains a geodesic loop of length \( 2\pi \), then \( M \) is isometric to \( \mathbb{S}^n \).

Definition 1.1. Let \( M \) be an \( n \)-dimensional Riemannian manifold and \( p \) be a point in \( M \). Let \( \text{Conj}(p) \) denote the set of first conjugate points to \( p \) on all geodesics issuing from \( p \). The conjugate radius \( \rho(p) \) of \( M \) at \( p \) in the sense of Xia [13] is defined as

\[
\rho(p) = d(p, \text{Conj}(p)) \quad \text{if} \ \text{Conj}(p) \neq \emptyset
\]

and

\[
\rho(p) = +\infty \quad \text{if} \ \text{Conj}(p) = \emptyset
\]

Then the conjugate radius of \( M \) is given by

\[
\rho(M) = \inf_{x \in M} \rho(x).
\]
Many interesting results have been proved by using the critical points theory and Toponogov’s theorem [3], [5], [7], [8], [10], [11], [12], [13], etc...

The purpose of this paper is to prove the following result.

**Theorem 1.2.** For any \( n \geq 2 \) and any \( \delta > 0 \), there exists a positive constant \( \epsilon(n, \delta) \) depending only on \( n \) and \( \delta \) such that for any \( \epsilon \leq \epsilon(n) \), if \( M \) is an \( n \)-dimensional complete connected Riemannian manifold with sectional curvature \( K_M \geq 1 \) and conjugate radius \( \rho(M) > \frac{\pi}{2} + \delta \) and if \( M \) contains a geodesic loop of length \( 2(\pi - \epsilon) \) then \( M \) is diffeomorphic to an \( n \)-dimensional unit sphere \( \mathbb{S}^n \) and the metric \( g \) of \( M \) is \( \epsilon' = \epsilon'(\epsilon, n, \delta, \alpha) \) close in the \( C^\alpha \) topology to the canonical metric of curvature 1 of \( \mathbb{S}^n \) for any \( \alpha \in ]0, 1[ \).

**Proof.** Let \( i(M) \) denote the injectivity radius of \( M \). By definition we have

\[
i(M) = \inf_{x \in M} d(x, C(x)),
\]

where \( C(x) \) is the set of cut points of \( x \). \( \square \)

A classical result due to Klingenberg (see for instance corollary 4.14 of [9]) asserts that if \( M \) is compact then \( i(M) = \min\{l_0, \frac{\pi}{2}\} \), where \( l_0 \) is the minimum of the length of non trivial closed geodesics of \( M \) and \( l_0 \) is the minimum over unit vector \( u \) of \( TM \) of the first conjugate value \( t_0(u) \) along the geodesic \( \gamma_0(t) = \exp(tu) \).

**Lemma 2.1.** Let \( M \) be an \( n \)-dimensional complete, connected Riemannian manifold with sectional curvature \( K_M \geq 1 \). With Xia’s convention on the conjugate radius we have \( i(M) \geq \rho(M) \).

The proof is a direct application of the Klingenberg’s result: by the definition above of the conjugate radius we have \( t_0 \geq \rho(M) \) and, since \( K_M \geq 1 \), every geodesic \( \gamma \) issued from a point \( p \) hits \( \mathrm{Conj}(p) \) at a point \( q \) (by the Rauch comparison theorem). Consequently, the length of every non trivial closed geodesic issued from \( p \) is bounded below by \( 2d(p, q) \geq 2\rho(M) \).

**Lemma 2.2.** For any \( \delta > 0 \), there exists a function \( \tau_{\delta} \) which satisfies \( \lim_{\epsilon \to 0} \tau_{\delta}(\epsilon) = 0 \) and such that if \( M \) is a complete manifold with \( K_M \geq 1 \), injectivity radius \( i(M) \geq \frac{\pi}{2} + \delta \) and which contains a geodesic loop of length \( 2(\pi - \epsilon) \) then we have \( \mathrm{diam}(M) \geq \pi - \tau_{\delta}(\epsilon) \).

**Proof.** Let \( \gamma \) be a loop with length \( 2\pi - 2\epsilon \). Let \( x = \gamma(0) = \gamma(2\pi - 2\epsilon), y = \gamma(\pi/2 + \delta), m = \gamma(\pi - \epsilon) \) and \( z = \gamma(3(\pi - \epsilon)/2 - \delta) \).

Let

\[
\gamma_1 = \gamma \left[ 0, \frac{\pi}{2} + \delta \right], \quad \gamma_2 = \gamma \left[ \frac{\pi}{2} + \delta, \pi - \epsilon \right], \quad \gamma_3 = \gamma \left[ \pi - \epsilon, \frac{3(\pi - \epsilon)}{2} - \delta \right],
\]

and

\[
\gamma_4 = \gamma \left[ \frac{3(\pi - \epsilon)}{2} - \delta, 2\pi - 2\epsilon \right].
\]

Then the geodesics \( \gamma_1 \) are minimal. Let \( \sigma \) be a minimal geodesic joining \( m \) and \( x \).

Set \( \alpha = \angle(\sigma'(0), -\gamma'(\pi - \epsilon)) \) and \( \beta = \angle(\sigma'(0), \gamma'(\pi - \epsilon)) \).

We have \( \alpha \leq \pi/2 \) or \( \beta \leq \pi/2 \). Suppose, without loss of generality, that \( \alpha \leq \pi/2 \).

Applying the Toponogov comparison theorem on length to the hinge formed by \( \gamma_2 \)
and $\sigma$ at $\gamma(\pi - \epsilon)$ we have

$$\cos\left(\frac{\pi}{2} + \delta\right) \geq \cos L(\sigma) \cos\left(\frac{\pi}{2} - \epsilon - \delta\right) + \cos \alpha \sin L(\sigma) \sin\left(\frac{\pi}{2} - \epsilon - \delta\right)$$

so that

$$\cos L(\sigma) \leq -\frac{\sin \delta}{\sin (\delta + \epsilon)} \Rightarrow L(\sigma) \geq \pi - \tau_0(\epsilon)$$

and the conclusion follows. \-box

Note that Anderson [1] and Otsu [6] constructed, for $n \geq 4$-dimensional closed manifolds with $\text{Ric} \geq n - 1$ and diameter arbitrarily close to $\pi$ but whose homotopy type is distinct from that of the sphere. Thus additional assumptions are needed.

In [2] G. Pacelli Bessa proved the following theorem from which we deduce Theorem 1.2.

**Theorem 2.3.** Given $n \geq 2$ and $i_0 > 0$ there exists an $\epsilon = \epsilon(n, i_0)$ such that if $M$ admits a metric $g$ satisfying

$$\text{Ric} \geq n - 1, \quad i(M) \geq i_0, \quad \text{Diam}(M) \geq \pi - \epsilon$$

then, for any $\alpha \in [0, 1]$, $M$ is diffeomorphic to $\mathbb{S}^n$ and the metric $g$ of $M$ is $\epsilon' = \epsilon'(\epsilon, n, \alpha)$ close in the $C^\alpha$ topology to the canonical metric of curvature 1 of $\mathbb{S}^n$, where $\epsilon'$ tends to 0 with $\epsilon$.

**Remark.** The complex projective space shows that theorem 1.2 is false under the weaker hypothesis $\rho \geq \frac{\pi}{2}$.

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