FRAMES OF MULTI-WINDOWED EXPONENTIALS ON SUBSETS OF $\mathbb{R}^d$

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Abstract. Given discrete subsets $\Lambda_j \subset \mathbb{R}^d$, $j = 1, \ldots, q$, consider the set of windowed exponentials $\bigcup_{j=1}^{q} \{g_j(x)e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda_j\}$ on $L^2(\Omega)$. We show that a necessary and sufficient condition for the windows $g_j$ to form a frame of windowed exponentials for $L^2(\Omega)$ with some $\Lambda_j$ is that $m \leq \max_{j \in J} |g_j| \leq M$ almost everywhere on $\Omega$ for some subset $J$ of $\{1, \ldots, q\}$. If $\Omega$ is unbounded, we show that there is no frame of windowed exponentials if the Lebesgue measure of $\Omega$ is infinite. If $\Omega$ is unbounded but of finite measure, we give a sufficient condition for the existence of Fourier frames on $L^2(\Omega)$. At the same time, we also construct examples of unbounded sets with finite measure that have no tight exponential frame.

1. Introduction

Let $\Omega$ be a Lebesgue measurable set on $\mathbb{R}^d$ and let $g_j \in L^2(\Omega) \setminus \{0\}$, $j = 1, \ldots, q$ and $q < \infty$. Let also $\Lambda_j$, $j = 1, \ldots, q$ be some countable sets on $\mathbb{R}^d$. The collection $\bigcup_{j=1}^{q} \mathcal{E}(g_j, \Lambda_j) = \bigcup_{j=1}^{q} \{g_j(x)e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda_j\}$ is called a set of windowed exponentials with windows $g_j$. Recall that $\bigcup_{j=1}^{q} \mathcal{E}(g_j, \Lambda_j)$ is a frame for $L^2(\Omega)$ if there exist $A, B > 0$ such that

$$A \|f\|_{L^2(\Omega)}^2 \leq \sum_{j=1}^{q} \sum_{\lambda \in \Lambda_j} \left| \int_{\Omega} f(x)\overline{g_j(x)}e^{-2\pi i \langle \lambda, x \rangle} dx \right|^2 \leq B \|f\|_{L^2(\Omega)}^2$$

(1.1)

for all $f \in L^2(\Omega)$, where $\|f\|_{L^2(\Omega)}^2 = \int_{\Omega} |f(x)|^2 dx$. If the second inequality in (1.1) is satisfied, then the set of the windowed exponentials is called a Bessel sequence. If the collection is generated by the single window $g = \chi_\Omega$, it is called a Fourier frame.

The study of Fourier frames was initiated by Duffin and Schaeffer in their work on non-harmonic Fourier series [DS]. The existence of Fourier frames $\{e^{2\pi i \langle \lambda, \cdot \rangle}\}_{\lambda \in \Lambda}$ on $L^2(\Omega)$ was also known to be equivalent to the sampling problems on the Paley-Wiener space $PW_\Omega$, which ask for the reconstruction of the band-limited functions $f$ by their sampled values $\{f(\lambda)\}$ (see [Y]). Nowadays, Fourier frames and, more
generally, windowed exponentials have a wide range of applications in different areas of mathematics, engineering and signal processing [AG, Chr].

Windowed exponentials also arise naturally in frame theory. Gröchenig and Razafinjatovo [GR] derived the famous necessary Beurling density condition of Landau [Lan] on Fourier frames by considering windowed exponentials. It is also known that the study of frame of translates and regular Gabor frames can be reduced to that of windowed exponentials via the Fourier transform and the Zak transform respectively [Chr, G]. Heil et al have recently made extensive studies in the basis properties, density conditions and different aspects of the windowed exponentials [HK, HY], and the reader can refer to [H] for a comprehensive introduction to the theory of windowed exponentials.

In this paper, we will give a complete characterization of the collections of windows $g_j$ with the property that $\bigcup_{j=1}^q \mathcal{E}(g_j, \Lambda_j)$ form a frame on $L^2(\Omega)$ for some discrete sets $\Lambda_j \subset \mathbb{R}^d$. This characterization is motivated by the recent work of the second named author on Fourier frames of absolutely continuous measures [Lai, DL]. In fact, we will see that Fourier frames of absolutely continuous measures are equivalent to frames of windowed exponentials generated by a single window (Proposition 5.2). Therefore, our result will be a further generalization. Denoting by $|\Omega|$ the Lebesgue measure of $\Omega$, we have the following result.

**Theorem 1.1.** If $|\Omega|$ is infinite, then there is no frame of windowed exponentials on $L^2(\Omega)$.

This statement is no longer true if we allow infinitely many windows. For example on $\mathbb{R}^d$, the system
\[
\bigcup_{n \in \mathbb{Z}^d} \{ \chi_{[0,1]^d+n}(\cdot) e^{2\pi i \langle m, \cdot \rangle} : m \in \mathbb{Z}^d \} = \{ e^{2\pi i \langle m, \cdot \rangle} \chi_{[0,1]^d}(\cdot - n) : m, n \in \mathbb{Z}^d \},
\]
is a standard example of a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$.

If the measure of $\Omega$ is finite, we need to separate our analysis into the case where $\Omega$ is bounded or unbounded.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lebesgue measurable set and let $g_j$, $j = 1, 2, \ldots, q$, be a finite set of functions in $L^2(\Omega)$. Let also
\[
J = \{ j : \|g_j\|_\infty < \infty \}.
\]
Then there exists $\Lambda_j$ such that $\bigcup_{j=1}^q \mathcal{E}(g_j, \Lambda_j)$ form a frame in $L^2(\Omega)$ if and only if there exists $m > 0$ such that
\[
\max_{j \in J} |g_j| \geq m
\]
almost everywhere on $\Omega$. 


An intermediate result of independent interest, Theorem 3.2, is needed in the proof of the above theorem, in which explicit upper and lower bounds for the quantities \( \max_{j \in J} |g_j| \) are given in terms of the frame bounds and upper Beurling densities of certain measures associated with the sets \( \Lambda_j \).

The characterization given in Theorem 1.2 implies that the unbounded functions in the original collection are actually not needed in producing frame of windowed exponentials and we just need to check whether the maximum of the moduli of the remaining bounded functions is bounded away from 0 a.e. on \( \Omega \). In particular, if the collection consists only of unbounded functions, it cannot form any frame of windowed exponentials.

One essential ingredient in our proofs is a surprising relationship between Beurling densities and the bounds in some convolution inequalities developed in [Ga2, Ga3]. Convolution inequalities arise naturally in the study of frame theory, tilings and spectral sets. Making use of this relationship has the advantage to simplify many technical calculations and to allow the theorems above to hold in the more general setting of *generalized frames of windowed exponentials* (Remark 3.4) without much additional work.

The situation unfortunately becomes vastly more complicated if the set \( \Omega \) is unbounded but still of finite Lebesgue measure. The necessary condition given in Theorem 1.2 for a system of windowed exponentials to form a frame for \( L^2(\Omega) \) still holds in the unbounded case, as the proof only uses the fact that \( \Omega \) has finite Lebesgue measure (see Theorem 3.2). However, we do not know of any unified argument to show that frames of windowed exponentials for \( L^2(\Omega) \) exist for every such \( \Omega \). In all the known examples, the construction of frames of windowed exponentials and Fourier frames is based on a tight frame defined on a larger set. We therefore examine the existence of tight Fourier frames for unbounded sets of finite measure. It is not hard to prove that *if there exists a lattice \( \Gamma \) such that \( \sum_{\gamma \in \Gamma} \chi_{\Omega}(\cdot + \gamma) \leq 1 \) (i.e. elements in \( \Omega \) are distinct residue class of \( \Gamma \)), then \( L^2(\Omega) \) will admit a tight Fourier frame.* (Proposition 4.1). We don’t know whether or not this condition is necessary but, on the other extreme, we can construct examples where no tight frames can exist using the following theorem.

**Theorem 1.3.** Suppose \( \Omega \) is a measurable set of finite Lebesgue measure such that \( |\Omega \cap \Omega + x| > 0 \) for all \( x \in \mathbb{R}^d \) with \( |x| > R \) for some \( R > 0 \), then \( L^2(\Omega) \) does not admit any tight Fourier frame (i.e. a Fourier frame with \( A = B \) in (1.1)).

Examples of sets satisfying the conditions in the above theorem are not difficult to obtain and the theorem shows that ordinary method of Fourier frame construction fails for these. However, we cannot prove whether or not Fourier frames always exist for such sets.
If the condition $|\Omega \cap (\Omega + x)| > 0$ is valid for all $x \in \mathbb{R}^d$, we can strengthen the conclusion of Theorem 1.3 to obtain that the only tight frame measures (see (4.3)) for $L^2(\Omega)$ are the positive multiples of the Lebesgue measure on $\mathbb{R}^d$ (Theorem 4.5).

We organize the paper as follows. We will give some preliminaries on convolution inequalities in Section 2. We then prove Theorem 1.1 and 1.2 in Section 3. After that, we discuss frames on unbounded sets of finite measure and prove Theorem 1.3 in Section 4. In the last section, we will apply our results on windowed exponentials to related systems: frames of translates, frames of absolutely continuous measures and Gabor frames. Although these results are known, we are able to recover them with a new approach and simpler proofs.

2. Preliminaries

Let $\mu$ be a positive Borel measure on $\mathbb{R}^d$. We define its associated upper and lower Beurling densities as

$$D^+(\mu) = \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\mu(x + Q_h)}{h^d}, \quad D^-(\mu) = \liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\mu(x + Q_h)}{h^d},$$

where $Q_h$ is the cube of side length $h$ centered at the origin. If $\Lambda$ is a countable set on $\mathbb{R}^d$ and we denote by $\delta_{\Lambda}$ the measure $\sum_{\lambda \in \Lambda} \delta_{\lambda}$, then the above definitions of Beurling densities become those of the usual Beurling densities of a discrete set $\Lambda$, which we denote by $D^+(\Lambda)$ and $D^-(\Lambda)$, respectively. We say that a measure $\mu$ is translation-bounded if for every compact set $K$ there exists a constant $C_K > 0$ such that $\mu(x + K) \leq C_K$ for all $x \in \mathbb{R}^d$.

From the definition above, we can easily obtain the following inequalities.

**Proposition 2.1.** Let $\mu$ and $\nu$ be positive Borel measures on $\mathbb{R}^d$. Then

$$D^+(\mu) \leq D^+(\mu + \nu) \leq D^+(\mu) + D^+(\nu).$$

In particular, if $D^+(\nu) = 0$, then $D^+(\mu + \nu) = D^+(\mu)$.

**Proof.** We note that from the definition, we immediately have

$$\sup_{x \in \mathbb{R}^d} \frac{\mu(x + Q_h)}{h^d} \leq \sup_{x \in \mathbb{R}^d} \frac{(\mu + \nu)(x + Q_h)}{h^d} \leq \sup_{x \in \mathbb{R}^d} \frac{\mu(x + Q_h)}{h^d} + \sup_{x \in \mathbb{R}^d} \frac{\nu(x + Q_h)}{h^d}.$$ 

Hence, passing to the limit, we have $D^+(\mu) \leq D^+(\mu + \nu) \leq D^+(\mu) + D^+(\nu)$. The second statement is clear from the inequalities.

**Remark 2.2.** It should be pointed out that Proposition 2.1 is not true for the lower Beurling density. To see this, we can let $\mu = \sum_{n=0}^{\infty} \delta_n$ and $\nu = \sum_{n=1}^{\infty} \delta_{-n}$. Then $D^-(\mu) = D^-(\nu) = 0$, but $D^-(\mu + \nu) = D^-(\mathbb{Z}) = 1$. 

For a positive Borel measure $\mu$ and an locally integrable function $f \geq 0$ on $\mathbb{R}^d$, we define the convolution $\mu * f$ using the formula
\[
\mu * f(x) := \int f(x - y) d\mu(y), \quad x \in \mathbb{R}^d.
\]

In our proofs, we need to exploit a relationship between the convolutions and the Beurling densities for several measures. The following theorem was proved in [Ga2, Corollary 6 and 7].

**Theorem 2.3.** For $i = 1, \cdots, m$, let $\mu_i$ be positive Borel measures on $\mathbb{R}^d$, let $h_i$ be non-negative functions in $L^1(\mathbb{R}^d)$ and write $\mu = \sum_{i=1}^m \left( \int_{\mathbb{R}^d} h_i(x) dx \right) \mu_i$.

(i) Suppose that there exists $B > 0$ such that $\sum_{i=1}^m \mu_i * h_i \leq B$ a.e. on $\mathbb{R}^d$, then $D^+(\mu) \leq B$.

(ii) Suppose that there exists $A > 0$ such that $A \leq \sum_{i=1}^m \mu_i * h_i$ a.e. on $\mathbb{R}^d$ and that all the $\mu_i$ are translation-bounded, then $A \leq D^-(\mu)$.

We also recall an important condition equivalent to the translation-boundedness of a measure $\mu$ [Ga2, Proposition 1].

**Proposition 2.4.** Let $\mu$ be a positive Borel measure measure on $\mathbb{R}^d$. Then $\mu$ is translation-bounded if and only if there exists $f \in L^1(\mathbb{R}^d)$ with $f \geq 0$ and a constant $C > 0$ such that $\mu * f \leq C$ a.e. on $\mathbb{R}^d$.

The Fourier transform a function $f \in L^1(\mathbb{R}^d)$ is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx, \quad \xi \in \mathbb{R}^d,
\]
and extended in the usual way as a unitary operator on $f \in L^2(\mathbb{R}^d)$. The following proposition illustrates how convolution inequalities appear naturally when dealing with Bessel systems or frames of windowed exponentials.

**Proposition 2.5.** (i) Let $\bigcup_{j=1}^q \mathcal{E}(g_j, \Lambda_j)$ be a Bessel sequence of windowed exponentials on $L^2(\Omega)$ (where $|\Omega|$ can be finite or infinite). Then, for any $f \in L^2(\Omega)$ such that $fg_j \in L^2(\Omega)$ for all $j$, we have
\[
D^+(\mu_f) \leq B \|f\|_{L^2(\Omega)}^2
\]
where $\mu_f = \sum_{j=1}^q \left( \int_{\Omega} |fg_j|^2 \right) \delta_{\Lambda_j}$ and all the measures $\delta_{\Lambda_j}, j = 1, \ldots, q$, are translation-bounded.

(ii) If, furthermore, the collection $\bigcup_{j=1}^q \mathcal{E}(g_j, \mu_j)$ is a frame of windowed exponentials for $L^2(\Omega)$, then
\[
A \|f\|_{L^2(\Omega)}^2 \leq D^-(\mu_f).
\]
Proof. (i) Replacing $f$ with the function $f e^{2\pi i \langle \xi, \cdot \rangle}$ in the definition of Bessel sequence of windowed exponentials in (1.1), we have
\[
\sum_{j=1}^q \sum_{\lambda \in \Lambda_j} \left| \hat{\chi}_\Omega f g_j (\xi - \lambda) \right|^2 \leq B \| f \|_{L^2(\Omega)}^2.
\]
In particular, we can write the inner sum as
\[
\sum_{\lambda \in \Lambda_j} \left| \hat{\chi}_\Omega f g_j (\xi - \lambda) \right|^2 = \delta_{\Lambda_j} * (|\hat{\chi}_\Omega f g_j|^2)(\xi)
\]
to obtain
\[
\sum_{j=1}^q \delta_{\Lambda_j} * (|\hat{\chi}_\Omega f g_j|^2)(\xi) \leq B \| f \|_{L^2(\Omega)}^2. \tag{2.1}
\]
As $\int |\hat{\chi}_\Omega f g_j|^2 = \int_{\Omega} |f g_j|^2 (< \infty)$ from the Plancherel identity and the assumption on $f$, it follows from Theorem 2.3 (i) that
\[
D^+(\mu_f) \leq B \| f \|_{L^2(\Omega)}^2,
\]
where $\mu_f = \sum_{j=1}^q \left( \int_{\Omega} |f g_j|^2 \right) \delta_{\Lambda_j}$. From (2.1), for all $j = 1, \ldots, q$, we have
\[
\delta_{\Lambda_j} * (|\hat{\chi}_\Omega f g_j|^2)(\xi) \leq B \| f \|_{L^2(\Omega)}^2, \quad \xi \in \mathbb{R}^d.
\]
and thus each measure $\delta_{\Lambda_j}$ is translation-bounded by Proposition 2.4.

(ii) By (i), all the measures $\delta_{\Lambda_j}$ are translation-bounded. By an argument similar to the one used in (i), we obtain
\[
A \| f \|_{L^2(\Omega)}^2 \leq \sum_{j=1}^q \delta_{\Lambda_j} * (|\hat{\chi}_\Omega f g_j|^2)(\xi).
\]
By Theorem 2.3 (ii), the conclusion follows. \qed

3. WINDOWED EXPONENTIALS

In this section, we will prove our main results concerning general frames of windowed exponentials. We first need a lemma.

Lemma 3.1. Let $\bigcup_{j=1}^q \mathcal{E}(g_j, \Lambda_j)$ be a Bessel sequence of windowed exponentials for $L^2(\Omega)$ with $g_j \neq 0$ for all $j$.

(i) If $|\Omega| < \infty$, then $D^+ \Lambda_j < \infty$ for all $j$ and at least one of the $\Lambda_j$ has positive upper Beurling density if the windowed exponentials form a frame for $L^2(\Omega)$.

(ii) If the windowed exponentials form a frame for $L^2(\Omega)$ and $|\Omega| = \infty$, then we have $D^+ \Lambda_j = \infty$ for all $j$. 

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Proof. (i) If $|\Omega|$ is finite, letting $f = \chi_\Omega$ in Proposition 2.5, we have
\[ D^+(\mu_f) \leq B|\Omega|, \]
where $\mu_f = \sum_{j=1}^{q} \|g_j\|^2_{L^2(\Omega)} \delta_{\Lambda_j}$. Letting $m = \min_{j} \|g_j\|^2_{L^2(\Omega)} > 0$, we have the inequality $\mu_f \geq m \sum_{j=1}^{q} \delta_{\Lambda_j}$. Hence, invoking Proposition 2.1, it follows that
\[ D^+(\Lambda_j) \leq D^+(\sum_{j=1}^{q} \delta_{\Lambda_j}) \leq \frac{1}{m} D^+(\mu_f) \leq \frac{B|\Omega|}{m} < \infty, \quad j = 1, \ldots, q. \]
In addition, if the set of the windowed exponentials is a frame, we have also the inequality $D^-(\mu_f) \geq A|\Omega|$. Letting $M = \max_{j} \|g_j\|^2_{L^2(\Omega)}$, we have
\[ 0 < \frac{A|\Omega|}{M} \leq \frac{1}{M} D^+(\mu_f) \leq D^+(\sum_{j=1}^{q} \delta_{\Lambda_j}) \leq \sum_{j=1}^{q} D^+(\Lambda_j), \]
showing that $D^+(\Lambda_j) > 0$ for some $j$.

(ii). If $|\Omega| = \infty$, let $\Omega_N = \Omega \cap Q_N$, where $Q_N$ is the cube of side length $N$ centered at origin. Then $L^2(\Omega_N) \subset L^2(\Omega)$ and it is easy to see that $\bigcup_{j=1}^{q} E(g_j, \Lambda_j)$ is still a frame of $L^2(\Omega_N)$ for all $N$ large enough so that $|\Omega_N| > 0$. Applying (3.1) to $\Omega_N$, we obtain
\[ \frac{A|\Omega_N|}{M} \leq \sum_{j=1}^{q} D^+(\Lambda_j) \]
and the result thus follows by taking $N \to \infty$. \qed

Using the previous lemma, we now prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that there exists a frame of windowed exponentials $\bigcup_{j=1}^{q} E(g_j, \Lambda_j)$ for $L^2(\Omega)$ with $|\Omega| = \infty$. By Lemma 3.1(ii), $D^+\Lambda_j = \infty$ for all $j$. On the other hand, we consider $\Omega_N$ as in the proof of Lemma 3.1(ii) with $N$ large enough so that $|\Omega_N| > 0$. The windowed exponentials continue still form a frame for $L^2(\Omega_N)$. By Lemma 3.1(i) applied to $\Omega_N$, $D^+\Lambda_j < \infty$ for all $j$. This leads us to a contradiction and hence there cannot be any frame of windowed exponentials for $L^2(\Omega)$ if $|\Omega| = \infty$. \qed

From now on, we assume $|\Omega| < \infty$. The estimates obtained in following theorem are the main technical tools used to characterizing when a system of windowed exponentials forms a Bessel sequence or a frame. It further gives us explicit relationships between the frame bounds, the Beurling densities, and the essential supremum and infimum of the moduli of the windows when dealing with a Bessel sequence or frame of windowed exponentials. Now, given a set $\bigcup_{j=1}^{q} E(g_j, \Lambda_j)$, we recall the definition of $J$ and define a related index set $J'$:
\[ J = \{ j : \|g_j\|_\infty < \infty \}, \quad J' = J \cap \{ j : D^+(\Lambda_j) > 0 \} \]
Theorem 3.2. Let $\Omega \subset \mathbb{R}^d$ such that $|\Omega| < \infty$ and let $\bigcup_{j=1}^{q} \mathcal{E}(g_j, \Lambda_j)$ be a set of windowed exponentials in $L^2(\Omega)$.

(i) Let the collection $\bigcup_{i=1}^{q} \mathcal{E}(g_i, \Lambda_i)$ form a Bessel sequence in $L^2(\Omega)$ with Bessel constant $B$ and suppose, furthermore, that $D^+(\Lambda_j) > 0$ for some $j \in \{1, \ldots, q\}$. Then, $|g_j| \leq \sqrt{B/D^+(\Lambda_j)}$ almost everywhere on $\Omega$.

(ii) If the collection $\bigcup_{i=1}^{q} \mathcal{E}(g_i, \Lambda_i)$ is a frame of windowed exponentials in $L^2(\Omega)$ with frame bound $A, B(A < B)$, then we have the inequalities

$$\sqrt{A/D^+(\sum_{j \in J} \delta_{\Lambda_j})} \leq \max_{j \in J} |g_j| \leq \max_{j \in J'} \sqrt{B/D^+(\Lambda_j)}$$

almost everywhere on $\Omega$.

Proof. (i) Suppose that $\bigcup_{i=1}^{q} \mathcal{E}(g_i, \Lambda_i)$ is a Bessel sequence in $L^2(\Omega)$, then $\mathcal{E}(g_i, \Lambda_i)$ are Bessel sequences in $L^2(\Omega)$ for all $i$. Now if $D^+(\Lambda_j) > 0$ for some $j$, consider the measurable set

$$E_M = \{x \in \Omega : M \leq |g_j(x)|\}.$$

Note that $|E_M| \leq \|g_j\|^2_{L^2(\Omega)}/M^2 < \infty$, so $f := \chi_{E_M} \in L^2(\Omega)$ and $f$ satisfies the assumption in Proposition 2.5(i). Hence,

$$D^+ \left( \left( \int_{E_M} |g_j|^2 \right)^{\frac{1}{2}} \right) \leq B|E_M|.$$

Since $\int_{E_M} |g_j|^2 \geq M^2|E_M|$, we obtain that

$$|E_M| \left( M^2 D^+(\Lambda_j) - B \right) \leq 0.$$

If $M > \sqrt{B/D^+(\Lambda_j)}$, we would have $\left( M^2 D^+(\Lambda_j) - B \right) > 0$ which would force $|E_M|$ to be zero. Hence, $|g_j| \leq \sqrt{B/D^+(\Lambda_j)}$ a.e. on $\Omega$. This establishes (i).

(ii) From (i), we have shown for those $j \in J'$, $|g_j|$ is essentially bounded above by $\sqrt{B/D^+(\Lambda_j)}$, from which the second inequality in (3.3) follows. It remains to establish the first inequality in (3.3). We consider, for $\epsilon > 0$, the set

$$F_\epsilon := \bigcap_{j \in J'} \{x \in \Omega : |g_j| < \epsilon\}.$$

Define $f = \chi_{F_\epsilon}$. Then $\int_{\Omega} |fg_j|^2 \leq \epsilon^2 |F_\epsilon| \leq \epsilon^2 |\Omega| < \infty$. By Proposition 2.5,

$$A|F_\epsilon| \leq D^- (\mu_f) \leq D^+ (\mu_f) \leq \sum_{j=1}^{q} \int_{\Omega} |g_j|^2 \delta_{\Lambda_j}$$

where $\mu_f = \sum_{j=1}^{q} \left( \int_{\Omega} |f g_j|^2 \right) \delta_{\Lambda_j}$. Note that if $g_j$ is not essentially bounded above, then $D^+(\Lambda_j) = 0$ by (i) above. If $j \in J \setminus J'$, then $D^+(\Lambda_j) = 0$ also by the definition of $J'$. We can now use Proposition 2.1 to conclude that

$$D^+(\mu_f) = D^+ \left( \sum_{j \in J'} \int_{\Omega} |f g_j|^2 \delta_{\Lambda_j} \right).$$
Note from the definition of $F_\epsilon$ that for those $j \in J'$, \( \int_{\Omega} |fg_j|^2 \leq \epsilon^2 |F_\epsilon| \) and
\[
D^+(\mu_f) \leq \epsilon^2 |F_\epsilon| D^+\left(\sum_{j \in J'} \delta_{\Lambda_j}\right).
\]

Using (3.4), we obtain that
\[
|F_\epsilon| \left( A - \epsilon^2 D^+\left(\sum_{j \in J'} \delta_{\Lambda_j}\right) \right) \leq 0.
\]

This shows that $|F_\epsilon| = 0$ and $\Omega \setminus F_\epsilon$ has full measure in $\Omega$ if $\epsilon < \sqrt{A/D^+\left(\sum_{j \in J'} \delta_{\Lambda_j}\right)}$. As $\Omega \setminus F_{\epsilon_0} = \bigcap_{n=1}^{\infty} \Omega \setminus F_{\epsilon_0-1/n}$ and $|\Omega| < \infty$, $\Omega \setminus F_{\epsilon_0}$ has full measure in $\Omega$ for $\epsilon_0 = \sqrt{A/D^+\left(\sum_{j \in J'} \delta_{\Lambda_j}\right)}$. Note that
\[
\left\{ x \in \Omega : \max_{j \in J'} |g_j(x)| \geq \epsilon_0 \right\} = \bigcup_{j \in J'} \left\{ x \in \Omega : |g_j(x)| \geq \epsilon_0 \right\} = \Omega \setminus F_{\epsilon_0}.
\]

This establishes the lower bound. \(\square\)

Now we prove Theorem 1.2.

**Proof of Theorem 1.2.** Suppose that there exists a frame of windowed exponentials $\bigcup_{j \in J} E(g_j, \Lambda_j)$ for $L^2(\Omega)$. By Lemma 3.1(i), $D^+(\Lambda_j) < \infty$ for all $j$ and $D^+(\Lambda_j) > 0$ for at least one such $j$. Using Theorem 3.2(i) the corresponding $g_j$ is essentially bounded above on $\Omega$. Hence, $J'$ and therefore $J$ in (3.2) are non-empty. By Theorem 3.2 (ii), $\max_{j \in J'} |g_j|$ is essentially bounded away from 0 on $\Omega$ and, since $J \supset J'$, so is $\max_{j \in J} |g_j|$. This shows the necessity of that condition.

Suppose now $m \leq \max_{j \in J} |g_j|$. Since $J$ is a finite set, the definition of $J$ shows that $\max_{j \in J} |g_j| \leq M$ for some $M < \infty$. As $\Omega$ is bounded, we can cover $\Omega$ by a cube $Q_R$. We know that $L^2(Q_R)$ has a Fourier frame (in fact an orthonormal basis), which we denote by $\{e^{2\pi i \langle \lambda, \cdot \rangle}\}_{\lambda \in \Lambda}$. Define
\[
\Lambda_j = \begin{cases} 
\Lambda, & j \in J; \\
\{0\}, & j \notin J.
\end{cases}
\]

To prove the upper bound in the frame inequality, we note that, for $j \in J$,
\[
\int_{\Omega} |fg_j|^2 \leq M^2 \int_{\Omega} |f|^2 < \infty, \quad f \in L^2(\Omega).
\]

Hence, denoting by $B$ the Bessel constant of the sequence of the exponentials associated with $\Lambda$ for $L^2(Q_R)$, we have
\[
\sum_{j \in J} \sum_{\lambda \in \Lambda} \left| \int_{\Omega} f(x)g_j(x)e^{-2\pi i \langle \lambda, x \rangle} dx \right|^2 \leq \sum_{j \in J} B \|fg_j\|^2_{L^2(Q_R)} \leq B q M^2 \|f\|^2_{L^2(\Omega)}
\]
for all $f \in L^2(\Omega)$ (we take $f = 0$ on $Q_R \setminus \Omega$). While for $j \not\in J$, we simply use Cauchy-Schwarz inequality to obtain

$$\sum_{j \not\in J} | \int_{\Omega} f(x) \overline{g_j(x)} dx |^2 \leq q |\Omega| \max_j \{ \|g_j\|_{L^2(\Omega)}^2 \} \|f\|_{L^2(\Omega)}^2.$$ 

Hence, combining these last two inequalities, we obtain

$$\sum_{j=1}^q \sum_{\lambda \in \Lambda} \left| \int_{\Omega} f(x) \overline{g_j(x)} e^{-2\pi i (\lambda, x)} dx \right|^2 \leq q(BM^2 + |\Omega| \max_j \{ \|g_j\|_{L^2(\Omega)}^2 \}) \|f\|_{L^2(\Omega)}^2,$$

which yields the upper bound in the frame inequality.

To establish the lower bound, we note that we can remove those $\Lambda_j$ with $j \not\in J$ in the sum appearing in the middle of the inequalities in (1.1). Now, using the fact that $m \leq \max_{j \in J} |g_j|$ a.e., we deduce that

$$\left| \Omega \setminus \bigcup_{j \in J} \{ |g_j| \geq m \} \right| = 0.$$

We can replace, if necessary, the sets $\{ |g_j| \geq m \}$ with $j \in J$ by subsets $T_j$, $j \in J$, which still cover $\Omega$ and are pairwise disjoint. Denoting by $A$ lower frame bound for the set of exponentials on $L^2(Q_R)$ with associated frequencies in $\Lambda$, we have thus

$$\sum_{j \in J} \sum_{\lambda \in \Lambda} \left| \int_{\Omega} f(x) \overline{g_j(x)} e^{-2\pi i (\lambda, x)} dx \right|^2 \geq A \sum_{j \in J} \|fg_j\|_{L^2(\Omega)}^2 \geq A \sum_{j \in J} \int_{T_j} |fg_j|^2 \geq A m^2 \sum_{j \in J} \int_{T_j} |f|^2 = Am^2 \int_{\Omega} |f|^2,$$

where the pairwise disjointness of the sets $T_j$, $j \in J$, and their covering property is used to obtain the last equality. This yields the lower bound and proves our claim.

\[ \square \]

**Example 3.3.** On the interval $[0,1]$, let $g_1(x) = x^\alpha$ and $g_2(x) = (1 - x)^\alpha$ with $\alpha \geq 0$. Then

$$\max\{g_1(x), g_2(x)\} = \begin{cases} (1 - x)^\alpha, & 0 \leq x \leq 1/2; \\ x^\alpha, & 1/2 < x \leq 1. \end{cases}$$

Hence, $1/2 \leq \max\{g_1, g_2\} \leq 1$ on $[0,1]$. We can produce a frame of exponentials on $[0,1]$ by taking $\Lambda_1 = \Lambda_2 = \mathbb{Z}$ for instance.

On the other hand, the functions $g_3(x) = x^\beta$ and $g_4(x) = (1 - x)^\beta$ are in $L^2([0,1])$ if $-1/2 < \beta < 0$. Since they are both unbounded, they cannot be used to produce windowed exponentials for $L^2([0,1])$. 

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If we now consider the collection \( \{g_1, g_2, g_3, g_4\} \). We note that the sub-collection \( \{g_1, g_2\} \) satisfies Theorem 1.2, so we can use this collection to form a frame of windowed exponentials and the unbounded functions \( g_3 \) and \( g_4 \) are redundant.

We end this section with a remark about generalized frames of windowed exponentials.

**Remark 3.4.** Let \( g_1, \ldots, g_q \in L^2(\Omega) \setminus \{0\} \), and \( \mu_1, \ldots, \mu_q \) be locally finite Borel measures on \( \mathbb{R}^d \), we say that the collections \( \bigcup_{j=1}^q \mathcal{E}(g_j, \mu_j) \) form a generalized frame of windowed exponentials for \( L^2(\Omega) \) if we can find \( 0 < A, B < \infty \) such that

\[
A \|f\|_{L^2(\Omega)}^2 \leq \sum_{j=1}^q \left| \int_{\Omega} f(x) \overline{g_j(x)} e^{-2\pi i \langle \lambda, x \rangle} \, dx \right|^2 \, d\mu_j(\lambda) \leq B \|f\|_{L^2(\Omega)}^2
\]

for all \( f \in L^2(\Omega) \). Similar generalized frames were also studied in [DHW]. In the case where \( \mu_j = \delta_{\Lambda_j} \), \( j = 1, \ldots, q \), we recover the system of the windowed exponentials defined in the introduction. As Theorem 2.3 and Proposition 2.4 are true for general measures \( \mu \), all the arguments in Proposition 2.5 and this section holds by directly replacing \( \delta_{\Lambda_j} \) with \( \mu_j \). Therefore, Theorem 1.1 and 1.2 actually holds for generalized frames of windowed exponentials.

4. UNBOUNDED SETS OF FINITE MEASURES

We know that a bounded set in \( \mathbb{R}^d \) can be covered by a hypercube and, in particular, the orthonormal bases of exponentials defined on the cube that we mentioned earlier will generate a tight frame for that set when restricted to it. When the set is unbounded but is of finite Lebesgue measure, such argument generally fails unless the set considered has some special properties such as in the following proposition. The result is known (see e.g. [GaL] for the one-dimensional case), but we provide here a simple proof for the reader’s convenience.

**Proposition 4.1.** Let \( \Omega \) be a set of finite Lebesgue measure (bounded or unbounded). Let \( \Gamma \) be a lattice in \( \mathbb{R}^d \) with \( \Gamma^* = \{ \lambda : \langle \lambda, \gamma \rangle \in \mathbb{Z} \} \) being its dual lattice. Then, the following are equivalent.

(i) \( \sum_{\gamma \in \Gamma} \chi_{\Omega}(x + \gamma) \leq 1 \) almost everywhere on \( \mathbb{R}^d \).

(ii) The collection \( \{e^{2\pi i \langle \lambda, \cdot \rangle}\}_{\lambda \in \Gamma^*} \) is a (tight) Fourier frame for \( L^2(\Omega) \).

*Proof.* (i) \( \implies \) (ii). Let \( Q \) be a fundamental domain of \( \Gamma \) and let \( f \in L^2(\Omega) \). We have

\[
\int_{\Omega} |f(x)|^2 \, dx = \int_Q \sum_{\gamma \in \Gamma} \chi_{\Omega}(x + \gamma) |f(x + \gamma)|^2 \, dx.
\]
From the assumption $\sum_{\gamma \in \Gamma} \chi_\Omega(x + \gamma) \leq 1$ almost everywhere, we have that almost every $x \in Q$, there exists at most one $\gamma_x \in \Gamma$ such that $\chi_\Omega(x + \gamma_x) = 1$ which implies that
\[
\sum_{\gamma \in \Gamma} \chi_\Omega(x + \gamma)|f(x + \gamma)|^2 = \left| \sum_{\gamma \in \Gamma} \chi_\Omega(x + \gamma)f(x + \gamma) \right|^2 \text{ a.e. on } Q.
\]
It is well known that the system $\{e^{2\pi i \langle \lambda, \cdot \rangle}\}_{\lambda \in \Gamma^*}$ forms an orthogonal basis for $L^2(Q)$ ([Fu]). Combining this fact with (4.1), we obtain
\[
\int_Q |f(x)|^2 \, dx = \int_Q \left| \sum_{\lambda \in \Gamma^*} \chi_\Omega(x + \gamma) f(x + \gamma)e^{-2\pi i \langle \lambda, x \rangle} \right|^2 dx
\]
which implies
\[
\int_\Omega |f(x)|^2 \, dx = |Q| \sum_{\lambda \in \Gamma^*} \left| \int_\Omega f(x) e^{-2\pi i \langle \lambda, x \rangle} \, dx \right|^2.
\]
(ii) $\implies$ (i). Let $Q$ be a fundamental domain of $\Gamma$. Proving the statement in (i) is equivalent to showing that
\[
\sum_{\gamma \in \Gamma} \chi_\Omega(x + \gamma) \leq 1 \text{ a.e. on } Q
\]
since the term on the left-hand side of the inequality above is $\Gamma$-periodic. This is in turn equivalent to showing that
\[
(|\Omega - \gamma) \cap (\Omega - \gamma') \cap Q| = 0 \text{ for all } \gamma, \gamma' \in \Gamma \text{ with } \gamma \neq \gamma'.
\]
We argue by contradiction and suppose that there exists $\gamma, \gamma' \in \Gamma$ such that $\gamma \neq \gamma'$ and $|(\Omega - \gamma) \cap (\Omega - \gamma') \cap Q| > 0$. Let $E = (\Omega - \gamma) \cap (\Omega - \gamma') \cap Q$ and consider $f = \chi_{E+\gamma} - \chi_{E+\gamma'} \in L^2(\Omega)$. Note that $f$ is a non-zero function in $L^2(\Omega)$ since $|E| > 0$. On the other hand, for all $\lambda \in \Gamma^*$,
\[
\int_\Omega f(x)e^{-2\pi i \langle \lambda, x \rangle} \, dx = \int_{E+\gamma} e^{-2\pi i \langle \lambda, x \rangle} \, dx - e^{2\pi i \langle \lambda, \gamma - \gamma' \rangle} \int_{E+\gamma} e^{-2\pi i \langle \lambda, x \rangle} \, dx = 0.
\]
This shows the system $\{e^{2\pi i \langle \lambda, \cdot \rangle}\}_{\lambda \in \Gamma^*}$ is incomplete in $L^2(\Omega)$ and hence cannot be a Fourier frame for $L^2(\Omega)$. This contradicts the assumption in (ii).
We will now proceed to prove Theorem 1.3. To this end, we need another approach using certain convolution identities due to Kolountzakis [K1, K2]. If a function $f \in L^1(\mathbb{R}^d)$ and a countable set $\Lambda \subset \mathbb{R}^d$ are such that 

$$\delta_\Lambda \ast f(x) = \sum_{\lambda \in \Lambda} f(x - \lambda) = w \text{ a.e. } x \in \mathbb{R}^d,$$

$f$ is said to tile by $\Lambda$ at level $w$ for some constant $w$ (see [K1, K2]). Kolountzakis [K1, Theorem 2] proved the following result in which $\hat{f} \in L^2(\mathbb{R}^d)$ is the Fourier transform of $f$.

**Proof of Theorem 1.3.** We argue by contradiction. Suppose that we are given a function $f$, set $\Omega$ of finite measure and $R > 0$ such that $\Omega \cap \Omega + x > 0$ for all $|x| > R$ and that $L^2(\Omega)$ admits a tight frame $\{e^{2\pi i (\cdot, \cdot)}\}_{\lambda \in \Lambda}$ with frame constant $A$. Consider $\Omega_N = \Omega \cap [-N, N]^d$. Clearly, when restricted to $\Omega_N$, this tight frame produces a tight frame for $L^2(\Omega_N)$. Hence, applying the definition of tight frame to the function $\chi_{\Omega_N} e^{2\pi i (\cdot, \cdot)}$, $\xi \in \mathbb{R}^d$, we obtain

$$(\delta_\Lambda \ast |\widehat{\chi_{\Omega_N}}|^2)(\xi) = \sum_{\lambda \in \Lambda} |\widehat{\chi_{\Omega_N}}(\xi - \lambda)|^2 = A |\Omega_N|, \quad \xi \in \mathbb{R}^d.$$

Letting $f_N = |\widehat{\chi_{\Omega_N}}|^2$, we have $\hat{f}_N = \chi_{\Omega_N} \ast \widehat{\chi_{\Omega_N}}$ where $\widehat{\chi_{\Omega_N}}(x) = \chi_{\Omega_N}(-x)$. A simple calculation shows that

$$\hat{f}_N(x) = |\Omega_N \cap \Omega_N + x|, \quad x \in \mathbb{R}^d.$$

Since $\Omega_N$ is bounded, $\hat{f}_N$ has compact support also and $\hat{f}_N \geq 0$. By Proposition 4.2, 

$$\text{supp } \delta_\Lambda \subset \{x : |\Omega_N \cap \Omega_N + x| = 0\} \cup \{0\} \text{ for all } N > 0.$$

Note that, since $\Omega_N$ is an increasing sequence of sets whose union is $\Omega$, $|\Omega_N \cap \Omega_N + x|$ converges pointwise to $|\Omega \cap \Omega + x|$ as $N \to \infty$. Hence, using our assumption on $\Omega$, for all $x$ such that $|x| > R$, there exists $N$ such that $|\Omega_N \cap \Omega_N + x| > 0$. This means that $\bigcap_{N} \{x : |\Omega_N \cap \Omega_N + x| = 0\} \subset \{x \in \mathbb{R}^d : |x| < R\}$. Therefore, 

$$\text{supp } \delta_\Lambda \subset \bigcap_{N=1}^{\infty} \{x : |\Omega_N \cap \Omega_N + x| = 0\} \cup \{0\} \subset \{x \in \mathbb{R}^d : |x| \leq R\}$$

showing that the support of $\delta_\Lambda$ is compact. This leads to a contradiction since, by the Paley-Wiener-Schwartz theorem ([R, p.199]), any tempered distribution whose Fourier transform is compactly supported must be the restriction to $\mathbb{R}^d$ of an entire analytic function, but here $\delta_\Lambda$ is a purely discrete measure on $\mathbb{R}^d$. \qed
We can strengthen Theorem 1.3 to a more general setting. We say that $L^2(\Omega)$ admits a \textit{generalized tight frame of exponentials} if there exists a locally finite Borel measure $\mu$ on $\mathbb{R}^d$ and a constant $A > 0$ such that

$$
\int_{\mathbb{R}^d} |\hat{f}(\lambda)|^2 d\mu(\lambda) = A \int_{\Omega} |f(x)|^2 dx, \quad f \in L^2(\Omega).
$$

(4.3)

In this situation, $\mu$ is called a \textit{tightly frame measure} for $L^2(\Omega)$ (see [DHW, DL]). It is clear that if $\mu$ is the Lebesgue measure on $\mathbb{R}^d$, then, by the Plancherel theorem, $\mu$ is a tight frame measure for $L^2(\Omega)$ for any $\Omega$. In Theorem 4.5, we give a necessary and sufficient condition for the Lebesgue measure on $\mathbb{R}^d$ to be the only tight frame measure for $L^2(\Omega)$. An analogous problem was also considered by the first named author in the setting of Gabor analysis ([Ga1]). In particular, we need a lemma in ([Ga1, Lemma 4.5]).

\textbf{Lemma 4.3.} Let $\mu$ be a positive translation-bounded measure on $\mathbb{R}^d$. Suppose that for some $r > 0$ and some $\tau \in \mathbb{R}^d,$

$$
\text{supp} \hat{\mu} \cap B_r(\tau) = \{\tau\},
$$

where $B_r(\tau)$ is the ball of radius $r$ centered at $\tau$. Then, there exists $a \in \mathbb{C}$ such that

$$
\hat{\mu} = a\delta_\tau \text{ on } B_r(\tau).
$$

\textbf{Remark 4.4.} If $\tau = 0$ in the previous lemma, then $a > 0.$ Indeed, if $\varphi \in C_0^\infty(\mathbb{R}^d)$ is supported on $B_{r/2}(0)$, then the support of $\varphi \ast \overline{\varphi}$ (recall that $\overline{\varphi}(x) = \varphi(-x)$) is contained in $B_r(0)$ and $a\|\varphi\|_2^2 = \langle \hat{\mu}, \varphi \ast \overline{\varphi} \rangle = \int |\overline{\varphi}(\xi)|^2 d\mu(\xi) > 0$. Hence, $a > 0.$

We also need to use Proposition 4.2 with $\delta_\Lambda$ replaced by $\mu$. This is possible by a simple modification of the argument in the proof given in [K1, Theorem 2]. We leave the details to the interested reader.

\textbf{Theorem 4.5.} Let $\Omega$ be a measurable subset of $\mathbb{R}^d$ with $|\Omega| < \infty$. Then the following are equivalent.

(i) The only tight frame measure for $L^2(\Omega)$ is the Lebesgue measure on $\mathbb{R}^d$, up to a positive constant multiple.

(ii) $|\Omega \cap (\Omega + x)| > 0$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

\textbf{Proof.} (ii) $\implies$ (i). Suppose that $\Omega$ is a set of finite measure satisfying (ii) and that $L^2(\Omega)$ admits a tight frame measure. Consider $\Omega_N = \Omega \cap [-N,N]^d$. By restriction, this tight frame measure continues to be a tight frame measure for $L^2(\Omega_N)$. Replacing $f$ by $\chi_{\Omega_N} e^{2\pi i \xi \cdot \cdot}$, $\xi \in \mathbb{R}^d$, in (4.3), we obtain

$$
\langle \mu \ast |\hat{\chi}_{\Omega_N}|^2 \rangle (\xi) = A|\Omega_N|, \quad \xi \in \mathbb{R}^d.
$$

Using a similar argument as in the proof of Theorem 1.3 with $\delta_\Lambda$ replaced by $\mu$, we deduce that $\text{supp} \hat{\mu} = \{0\}$ since $\bigcap_{N=1}^\infty \{x : |\Omega_N \cap \Omega_N + x| = 0\} = \{0\}$. By Lemma
4.3 and the remark following it, it follows that $\hat{\mu} = a \delta_0$. This shows that the only tight frame measures for $L^2(\Omega)$ are the positive multiples of the Lebesgue measure on $\mathbb{R}^d$.

(i) $\implies$ (ii). Suppose that there exists $x_0 \neq 0$ such that $|\Omega \cap (\Omega + x_0)| = 0$. Then it is easy to see that $|\Omega \cap (\Omega - x_0)| = 0$ also. Define the following locally finite measure

$$d\mu(\xi) = \left(1 + \frac{e^{2\pi i (x_0, \xi)} + e^{-2\pi i (x_0, \xi)}}{2}\right) d\xi.$$ 

It is a positive measure since the density is equal to $1 + \cos(2\pi \langle x_0, \xi \rangle) \geq 0$. We now claim that it is another tight frame measure for $L^2(\Omega)$ and this will prove our claim. Indeed, for any $f \in L^2(\Omega)$,

$$\int |\hat{f}(\xi)|^2 e^{2\pi i (x_0, x)} d\xi = \int \hat{f}(\xi) e^{2\pi i (x_0, x)} \overline{\hat{f}(\xi)} d\xi = \int f(x + x_0) f(x) dx.$$ 

As $f$ is supported on $\Omega$, the integrand $f(x + x_0) f(x)$ is supported on the intersection $\Omega \cap (\Omega - x_0)$ which has zero Lebesgue measure. This shows the integral above is 0. The same also applies to $\int |\hat{f}(\xi)|^2 e^{-2\pi i (x_0, x)} d\xi$. This shows that

$$\int |\hat{f}(\xi)|^2 d\mu(\xi) = \int |\hat{f}(\xi)|^2 d\xi = \int_\Omega |f(x)|^2 dx,$$

which completes the proof. \qed

We now give an example of a set $\Omega$ of finite measure such that any two translates of $\Omega$ always intersect on a set of positive measure. We only present an example on $\mathbb{R}$ as higher dimensional example can easily be constructed from it.

**Example 4.6.** The set $\Omega = \bigcup_{n \in \mathbb{Z}} \left(\frac{-1}{2|n|}, \frac{1}{2|n|}\right] + n$ is a set of finite Lebesgue measure satisfying $|\Omega \cap (\Omega + x)| > 0$ for all $x \in \mathbb{R}$.

If we let $\Omega_k = [-1, 1] \cup \bigcup_{|n| > k} \left(\frac{-1}{2|n|}, \frac{1}{2|n|}\right] + n$ for $k \geq 4$, then $\Omega_k$ has finite measure and $|\Omega_k \cap (\Omega_k + x)| > 0$ for all $|x| \geq k$, but the set of $x$ such that $\Omega_k \cap (\Omega_k + x) = \emptyset$ has positive measure.

**Proof.** The finiteness of the Lebesgue measure of $\Omega$ and $\Omega_k$ are clear. Let $x \in \mathbb{R}$ and let $n$ be the unique integer such that $n \leq x < n + 1$, then $\Omega + x \supset [x, x + 1] \cap [n, n + 1) = (n + 1 - \frac{1}{2|n+1|}, n + 1 + \frac{1}{2|n+1|}]$ intersects the interval $[x, x + 1]$ on a set of positive measure. This shows that $\Omega \cap \Omega + x$ has positive Lebesgue measure.

Using the same method above for $\Omega_k$, we can show $|\Omega_k \cap (\Omega_k + x)| > 0$ for all $|x| \geq k$. Now consider $x = 5/2$. Then $\Omega_k + x = [3/2, 7/2] \cup \bigcup_{|n| > k} \left(\frac{-1}{2|n|}, \frac{1}{2|n|}\right] + n + 5/2$. As $[3/2, 7/2]$ does not intersect $\Omega_k$ if $k \geq 4$ and the lengths of the remaining intervals centered at $n + 1/2$ are all less than $1/2^k$, $\Omega_k + 5/2$ is disjoint from all the intervals in $\Omega_k$. Moreover, $\Omega_k$ and $\Omega_k + 5/2$ are at a positive distance from each other. Therefore, for all $x$ close to $5/2$, $\Omega_k$ and $\Omega_k + x$ also has this property. Hence, this shows the set of $x$ such that $\Omega_k \cap (\Omega_k + x) = \emptyset$ has positive measure. \qed
We conclude this section with some remarks.

**Remark 4.7.** (1). It is unknown whether (non-tight) Fourier frames always exist for unbounded sets of finite measure. This problem was addressed earlier in [OU]. From all the approaches we tried in which we assume that any two translates intersect on a set of positive measure, we cannot formulate a definite conjecture to this problem. For instance,

(i) It is even possible construct sets of finite measure such that any finite number of translates intersect with positive measure. For these sets, we can show that if a Fourier frame exists with some frequency set $\Lambda$, then the set $\Lambda \pmod{\text{Gamma}}$ has to be dense in the fundamental domain of any lattice $\Gamma$.

(ii) On the other hand, Matei and Meyer [MM] recently constructed from simple quasicrystals a universal Fourier frame. This means that the frequency set $\Lambda$ will form a Fourier frame on any $L^2(K)$ such that $D^{-\Lambda} > |K|$ and $K$ is compact with boundary measure 0. Their method may be extendable to cover our sets.

(2). Another problem of a similar nature asks whether or not a Fourier frame exists on the singular one-third Cantor measure. In the existing methods, the construction of a Fourier frame is based on the existence of a singular measure for which there exists an orthonormal basis of exponentials ([HLL, DL]). While it is known that the one-third Cantor measure cannot admit any exponential orthogonal basis ([JP]), we are interested in the existence of Fourier frames for a measure which genuinely cannot be derived from some already existing tight frames.

5. SOME APPLICATIONS

We now give some application of our result to other well-known types of frames.

(I) **Frame of translates**

Given $g_1, \cdots, g_m \in L^2(\mathbb{R}^d)$ and associated countable sets $\mathcal{J}_i$ in $\mathbb{R}^d$, $i = 1, \cdots, m$, it was shown in [CDH] that $\bigcup_{j=1}^m \{g_j(x-t) : t \in \mathcal{J}_j\}$ cannot be a frame for $L^2(\mathbb{R}^d)$. We give a simple proof of this fact, based on our previous results concerning windowed exponentials.

**Theorem 5.1.** There is no frame of the form $\bigcup_{j=1}^m \{g_j(x-t) : t \in \mathcal{J}_j\}$ on $L^2(\mathbb{R}^d)$.

**Proof.** Suppose there is a frame of the given form. Then the Fourier transforms of the functions in the system will also form a frame for $L^2(\mathbb{R}^d)$. Since $\hat{g_j}(\cdot-t)(\xi) = \hat{g_j}(\xi)e^{2\pi i(t,\xi)}$, this new system will be in the form of windowed exponentials on $L^2(\mathbb{R}^d)$, generated by a finite number of windows, contradicting Theorem 1.1. □

(II) **Frames of absolutely continuous measures**
Let $\mu$ be an absolutely continuous measure with compact support. We write $d\mu(x) = \varphi(x)dx$, where $\varphi$ is its Radon-Nikodym derivative. In [Lai], the second named author completely characterized the kind of density such that the measure admits a Fourier frame. We say that a measure $\mu$ has an associated frame of windowed exponentials if we can find $\bigcup_{j=1}^q \mathcal{E}(g_j, \Lambda_j)$ with $g_j \in L^2(\mu)$ which forms a frame for $L^2(\mu)$, i.e.

$$A\|f\|_{L^2(\mu)}^2 \leq \sum_{j=1}^q \sum_{\lambda \in \Lambda_j} \left| \int f(x)g_j(x)e^{-2\pi i(\lambda, x)} \, d\mu(x) \right|^2 \leq B\|f\|_{L^2(\mu)}^2, \quad f \in L^2(\mu).$$

(5.1)

The following proposition shows that for an absolutely continuous measure, the notion of frame of windowed exponentials associated with the measure and the frame of exponentials on the support of the measure are equivalent.

**Proposition 5.2.** Let $\mu = \varphi(x)dx$ be an absolutely continuous measure and let $\Omega = \{\varphi \neq 0\}$. Then $\bigcup_{j=1}^q \mathcal{E}(g_j, \Lambda_j)$ is a frame of exponentials for $L^2(\varphi dx)$ if and only if $\bigcup_{j=1}^q \mathcal{E}(g_j, \sqrt{\varphi}, \Lambda_j)$ is a frame of exponentials for $L^2(\Omega)$.

**Proof.** Suppose $\bigcup_{j=1}^q \mathcal{E}(g_j, \Lambda_j)$ is a frame of exponentials of $L^2(\varphi dx)$, then for any $f \in L^2(\Omega)$, we have $\int_{\Omega} \left| \frac{f(x)}{\sqrt{\varphi(x)}} \right|^2 \varphi(x)dx = \int_{\Omega} |f(x)|^2 dx < \infty$. Hence, we can replace $f$ by $f/\sqrt{\varphi}$ in (5.1), we obtain (1.1).

Conversely, if $\bigcup_{j=1}^q \mathcal{E}(g_j, \sqrt{\varphi}, \Lambda_j)$ is a frame of exponentials of $L^2(\Omega)$. Then for any $f \in L^2(\varphi dx)$, we have $\int_{\Omega} |f(\sqrt{\varphi})|^2 = \int_{\Omega} |f|^2 \varphi dx < \infty$. Therefore, replacing $f$ by $f/\sqrt{\varphi}$ in (1.1) and the windows $g_j$ by $g_j/\sqrt{\varphi}$, we obtain (5.1), which proves our claim. □

This leads to the following characterization for the frame of windowed exponentials in $L^2(\varphi dx)$. The proof follows easily from Theorem 1.2 and Proposition 5.2.

**Theorem 5.3.** Let $\mu = \varphi(x)dx$ be an absolutely continuous measures with $\Omega = \{\varphi \neq 0\}$ and let $g_j, \, j = 1, 2, \ldots, q$ be a finite set of functions in $L^2(\varphi dx)$. Then there exists $\Lambda_j$ such that $\bigcup_{j=1}^q \mathcal{E}(g_j, \Lambda_j)$ form a frame in $L^2(\varphi dx)$ if and only if there is a sub-collection of functions $\{g_j\}_{j \in J}, \, J \subset \{1, \cdots q\}$ and constants $m, M$ with $0 < m \leq M < \infty$ such that

$$\frac{m}{\sqrt{\varphi}} \leq \max_{j \in J} |g_j| \leq \frac{M}{\sqrt{\varphi}}$$

almost everywhere on $\Omega$.

If there is only one window $g = \chi_\Omega$ on $L^2(\varphi(x)dx)$, Theorem 5.3 states that $\varphi$ must be bounded above and bounded away from 0 on $\Omega$, which recovers the result in [Lai].

(III) Gabor frames

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Let \( g \in L^2(\mathbb{R}^d) \) and consider the Gabor system with lattice time-frequency shifts defined as follows.

\[
G(g, a, b) = \{ e^{2\pi i m b} g(x - n a) : m, n \in \mathbb{Z}^d \}.
\]

It is well-known that if \( G(g, a, b) \) forms a frame for \( L^2(\mathbb{R}^d) \), then \( ab \leq 1 \). The converse is in general false and characterizing the kind of functions which form a Gabor frame is an important question. Since rescaling the function \( g \) does not affect the frame property, one can assume \( b = 1 \) and \( a \leq 1 \). One of the major tools in the theory of Gabor frames is the Zak transform. It is a unitary mapping from \( L^2(\mathbb{R}^d) \) to \( L^2([0, 1]^{2d}) \) defined by

\[
Zf(x, t) = \sum_{k \in \mathbb{Z}^d} f(x - k) e^{2\pi i \langle k, t \rangle}.
\]

If the previous definition of \( Zf \) is extended to all of \( \mathbb{R}^{2d} \), \( Zf \) is quasiperiodic in the following sense:

\[
Zf(x, t + n) = Zf(x, t), \quad Zf(x + n, t) = e^{2\pi i \langle n, t \rangle} Zf(x, t), \quad \forall \ n \in \mathbb{Z}^d.
\]

It is also well known that if \( a = 1 \), then \( G(g, 1, 1) \) is a Gabor frame if and only if

\[
0 < A \leq \max_{j \in \{0, \ldots, q-1\}^d} \|Zg_j\| \leq B < \infty \text{ a.e. on } [0, 1]^{2d}. \tag{5.2}
\]

If \( a = \frac{1}{q} \), the converse also holds.

**Theorem 5.4.** Let \( g \in L^2(\mathbb{R}^d) \) and \( a = \frac{p}{q} \) be a rational number with \( p < q \) and \( p, q \) are co-prime. Define \( g_j = Zg(x - \frac{p}{q} j, t) \) for \( j \in \{0, 1 \cdots, q-1\}^d \). If \( G(g, a, 1) \) is a Gabor frame of \( L^2(\mathbb{R}^d) \), then there exists \( A, B \) such that

\[
0 < A \leq \max_{j \in \{0, \ldots, q-1\}^d} \|Zg_j\| \leq B < \infty \text{ a.e. on } [0, 1]^{2d}.
\]

If \( a = \frac{1}{q} \), the converse also holds.

**Proof.** Note that \( Z \) is a unitary mapping between \( L^2(\mathbb{R}^d) \) and \( L^2([0, 1]^{2d}) \) and that \( G(g, a, 1) \) is a Gabor frame on \( L^2(\mathbb{R}^d) \) if and only if the image of the Gabor system under the Zak transform \( Z[G(g, a, 1)] \) is a frame on \( L^2([0, 1]^{2d}) \). Writing \( n = rq + j \) with \( r \in \mathbb{Z}^d \) and \( j \in \{0, \cdots, q-1\}^d \), we have

\[
Z \left[ e^{2\pi i \langle m, \cdot \rangle} g(\cdot - \frac{p}{q} j) \right] (x, t) = e^{2\pi i \langle m, x \rangle} \sum_{k \in \mathbb{Z}} g \left( x - k - pr - \frac{p}{q} j \right) e^{2\pi i \langle k, t \rangle} \\
= Zg(x - \frac{p}{q} j, t) e^{2\pi i \langle m, x \rangle} e^{2\pi i \langle rp, t \rangle}.
\]
From this, we see that

\[ Z[\mathcal{G}(g, a, 1)] = \bigcup_{j \in \{0, 1, \ldots, q-1\}^d} \bigcup_{m, r \in \mathbb{Z}^d} \{g_j(x)e^{2\pi i(m,x)}e^{2\pi i(rp,t)}\} \]

\[ = \bigcup_{j \in \{0, 1, \ldots, q-1\}^d} \mathcal{E}(g_j, \mathbb{Z}^d \times p\mathbb{Z}^d). \]

i.e. \( Z(\mathcal{G}(g, a, 1)) \) is a system of windowed exponentials on \([0, 1]^{2d}\). Therefore, if \( \mathcal{G}(g, a, 1) \) form a Gabor frame, then \( Z[\mathcal{G}(g, a, 1)] \) forms a frame of windowed exponentials. Moreover, \( \mathbb{Z}^d \times p\mathbb{Z}^d \) has positive upper Beurling density. By Theorem 3.2, (5.2) has to hold.

Conversely, if \( a = 1/q \), then the exponential frequency set becomes \( \mathbb{Z}^{2d} \) and the associated set of exponentials is an orthonormal basis for \( L^2([0, 1]^{2d}) \). According to the proof of Theorem 1.2, (5.2) implies that \( Z[\mathcal{G}(g, a, 1)] \) forms a frame of windowed exponentials on \( L^2([0, 1]^{2d}) \). Therefore, the original Gabor system forms a frame for \( L^2(\mathbb{R}^d) \). This completes the proof. \( \square \)

Zibulski and Zeevi [ZZ] showed when \( a = 1/q \), \( \mathcal{G}(g, a, 1) \) is a Gabor frame if and only if \( \sum_{j=0}^{q-1} |g_j|^2 \) is bounded above and bounded away from 0. Our result is consistent with their characterization since \( \ell^2 \)-norm and \( \ell^\infty \)-norm are equivalent on \( \mathbb{R}^q \). For \( a = p/q \), our condition gives a simple necessary condition. For a necessary and sufficient condition, we refer the reader to Zibulski and Zeevi [ZZ], who expressed it in terms of the boundedness of the eigenvalues of an associated positive-definite matrix.

References

[AG] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, SIAM Rev., 43 (2001), 585-620.

[Chr] O. Christensen, An Introduction to Frames and Riesz Bases, Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2003.

[CDH] O. Christensen, B.-Q Deng and C. Heil, Density of Gabor frames, Appl. Comput. Harmon. Anal., 7 (1999), 292-304.

[DHW] D. Dutkay, D.G. Han, and E. Weber, Continuous and discrete Fourier frames for Fractal measures, preprint.

[DS] R. Duffin, and A. Schaeffer, A class of nonharmonic Fourier series, Tran. Amer. Math. Soc., 72(1952), 341-366.

[DL] D. Dutkay and C-K, Lai, Uniformity of measures with Fourier frames, preprint.

[Fu] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, J. Funct. Anal., 16 (1974), 101-121.

[G] K. Gröchenig, Foundations of time-frequency analysis, Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, Basel, Berlin, 2001.

[Ga1] J.-P Gabardo, Weighted irregular Gabor tight frames and dual systems using windows in the Schwartz class, J. Funct. Anal., 256 (2009), 635-672.
J.-P Gabardo, Convolutional inequalities for positive Borel measures on \( \mathbb{R}^d \) and Beurling density, to appear in "Excursions in Harmonic Analysis, Volume 2: The February Fourier Talks at the Norbert Wiener Center", Andrews, T.D.; Balan, R.; Benedetto, J.J.; Czaja, W.; Okoudjou, K.A. (Eds.), Birkhauser Basel, 2013.

J.-P Gabardo, Convolutional inequalities in locally compact groups and unitary systems, Numer. Funct. Anal. Opt., 33 (2012), 1005-1030.

J.-P Gabardo and Y-Z Li, Density results for Gabor systems associated with periodic subsets of the real lines, J Approx. theory, 157 (2009), 172-192.

K. Gröchenig and H. Razafinjatovo, On Landau’s necessary density conditions for sampling and interpolation of band-limited functions, J. London Math. Soc. 54 (1996), 557-565.

C. Heil, A Basis Theory Primer, Expanded edition., Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2011.

C. Heil and G. Kutyniok, Density of frames and Schauder bases of windowed exponentials, Houston J. Math., 34 (2008), 565-600.

X.-G. He, C.-K. Lai and K.-S. Lau, Exponential spectra in \( L^2(\mu) \), Appl. Comput. Harmon. Anal., 34 (2013), 327-338.

C. Heil and G. J. Yoon, Duals of windowed exponential systems, Acta Appl. Math., 119 (2012), 97-112.

P. Jorgensen and S. Pedersen, Dense analytic subspaces in fractal \( L^2 \) spaces., J. Anal. Math., 75 (1998), 185-228.

M. Kolountzakis, Non-symmetric convex domains have no basis of exponentials, Illinois J. Math., 44(2000), 542-550.

M. Kolountzakis, The study of translational tiling with Fourier analysis, Fourier Analysis and Convexity, Appl. Numer. Harmon. Anal., Birkhauser Boston (2004), 131-187.

C.-K. Lai, On Fourier frame of absolutely continuous measures, J. Funct. Anal., 261 (2011), 2877-2889.

H. Landau, Necessary density conditions for sampling and interpolation of certain entrie functions, Acta Math., 117 (1967), 37-52.

B. Matei and Y. Meyer, Simple quasicrystals are sets of stable sampling, Complex Var. Elliptic Equ., 55 (2010), 947-964.

A. Olevskii and A. Ulanovskii, Uniqueness sets of unbounded spectra, C. R. Acad. Sci. Paris, Ser I., 349 (2011), 679-681.

W. Rudin, Functional Analysis, McGraw-Hill, New York, 1991.

R. Young, An Introduction to nonharmonic Fourier series, San Diego : Academic Press, Rev. 1st ed, c2001.

M. Zibulski and Y.-Y Zeevi, Analysis of multiwindow Gabor-type schemes by frame methods, Appl. Comput. Harmon. Anal., 4 (1997), 181-221.