Research Article

Indiscernibility and Discernibility Relations Attribute Reduction with Variable Precision

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1. Introduction

The rough set (RS) theory [1, 2], proposed by Polish mathematician Zdzisław Pawlak, is a useful data processing method for dealing with incomplete and inconsistent problems. The investigation of indiscernibility and discernibility relations between objects is an important task in RS theory. In classical RS theory, an indiscernibility relation between objects exists when all of the objects have the same attribute values. A discernibility relation exists between the objects if and only if not every attribute value of the objects is the same. Existing research has studied in-depth indiscernibility and discernibility relations in RSs [3–5] in certain research fields. For instance, in political sociology, if one side emphasizes the difference between countries, this can cause conflict to expand; on the contrary, if one side emphasizes the commonality between countries, this can provide better conditions for negotiation [6]. Because classical RS theory is sensitive to classification error, its application is quite limited, and its classification ability is poor. To overcome this limitation, the variable precision (VP) model [7, 8] is constructed using precision, which processes the data more effectively, thus advancing the development of RS theory and broadening its application to other fields.

Even more in-depth research has been performed on attribute reduction, which forms the minimum subset of knowledge classification by deleting redundant or unrelated attributes according to specific rules. Attribute reduction has been studied in reference [9–14] and has been applied to fault diagnosis, risk assessment, and other fields [15]. In reference [16], a VP model that is more fault-tolerant than the RS model was proposed, and seven forms of reduction in the VP model were discussed. In reference [17–19], the concept of attribute reduction in the VP model was...
proposed, and through a strict proof of the corresponding discernibility matrix, a reduction algorithm based on that matrix was given. Beynon [20] studied the VP reduction by comparing different thresholds. The influence of parameters on classification was studied in the VP model [21]. Chen et al. [22] defined a fuzzy relation on condition attributes, and then, a local attribute reduction algorithm was proposed using the corresponding discernibility matrix. In addition, fuzzy relations [23–25] have been studied in [26–28]. kQhe aim of most research on attribute reduction is to solve practical problems in different scenarios and has led to various forms of attribute reduction, such as covering reduction [29, 30], cost-sensitive reduction [31–33], local attribute reduction [34–38], and information entropy reduction [39, 40].

Skowron et al. [41, 42] were the first to propose using the discernibility matrix to reduce attributes in information systems. This method can determine all the reductions by transforming the function in conjunctive normal form (CNF) into a function in disjunctive normal form (DNF). This method is called the discernibility matrix-based reduction method and is supported by mathematical proofs. Although its calculation complexity is high, it remains the main method to obtain all reductions.

In this paper, the relative β-indiscernibility relation (β–IR) and relative β-discernibility relation (β–DR) of the VP model are proposed, in which VP model, probabilistic model [43], and decision-theoretic model [44] all extend RS. However, the differences among them lie in the selection of parameters. Because all the results can be obtained using the discernibility matrix, this paper is based on the discernibility matrix. Moreover, the relationships among relative β–IR reduction, relative β–DR reduction, and positive region reduction when the precision is greater than 0.5 are investigated after modifying the decision values of some objects.

The structure of this paper is divided into six parts. In Section 2, there are reviews of upper approximation, lower approximation, and β-approximation. In Section 3, the definition and reduction of the relative β–IR are proposed, and then, an algorithm for relative β–IR reduction is given. In Section 4, the definition and reduction of the relative β–DR are proposed, and then, an algorithm for relative β–DR relation reduction is also given. In Section 5, it is shown that when the precision is greater than 0.5, the relative β–IR (relative β–DR) relation in a decision table is equivalent to the positive region reduction in a new decision table in which some objects’ decision values have been updated. Then, an optimization algorithm for converting the relative β–IR (relative β–DR) reduction to a positive region reduction is proposed. In Section 6, we used a naive Bayes (NB) classifier, support vector machine (SVM), and decision tree (DT) on UCI datasets to compare the accuracy after reduction and evaluate the effectiveness of the proposed algorithms. Section 7 concludes the paper.

2. Preliminary

An information table [1, 2, 5] is also called an information system. It is assumed that the knowledge in an information table is described and represented by a set of rows and a set of columns, in which rows and columns denote objects and attributes, respectively. The information table is a tuple $S = (U, At, [V_a | a \in At], \{I_a | a \in At\})$, where $U$ is a universal set, $At$ is a set of attributes, $V_a$ is a nonempty set of values for an attribute $a$, and $I_a: U \rightarrow V_a$ is a function such that each $x \in U$ takes a value $I_a(x)$ on attribute $a$. Given $A \subseteq At$, the equivalence relation is defined by $R_A = \{(x, y) | x, y \in U \times U, I_a(x) = I_a(y)\}$ for each $a \in A$. Moreover, an equivalence class for $x$ on a set of attribute $A$ is denoted as $[x]_A = \{y | (x, y) \in R_A\}$. It can be easily seen that this equivalence relation is reflexive, symmetric, and transitive. For $A \subseteq At$, the quotient set is denoted by $U/A = \{A_1, A_2, \ldots, A_k\}$, where $[x]_A \in U/A$.

In $S = (U, At)$, if $C$ is a set of condition attributes and $D$ is a set of decision attributes, where $At = C \cup D$ and $C \cap D = \emptyset$, then an information table is called a decision table. In this paper, a decision table $S$ is written as $(U, C \cup D)$, where $C = \{a_1, a_2, \ldots, a_m\}$, $D = \{d_1, d_2, \ldots, d_l\}$, and $R_d = \cap_{i=1}^{l} R_{d_i}$. For the convenience of proof, a set of decision attributes is written as a singleton set, namely $D = \{d\}$.

Definition 1 (see [1, 2]). Let $(U, C)$ be an information system and $X$ be a subset of $U$, $B \subseteq C$. The lower and upper approximations of $X$ are as follows:

$$R_L(X) = \{x | [x]_B \subseteq X\},$$
$$R_U(X) = \{x | [x]_B \cap X \neq \emptyset\}. \quad (1)$$

Definition 2 (see [1, 2]). Let $(U, C \cup D)$ be a decision table in which $U/D$ is a quotient set induced by $D$. The positive region is defined as follows:

$$\text{Pos}_C(D) = \bigcup_{i=1}^{l} (R_C(D_i)), \quad (2)$$

where $D_i \in U/D$.

That is, for each $[x]_C$, if $[x]_C \subseteq \text{Pos}_C(D)$, then $[x]_C$ is contained in $[x]_D$. Conversely, if $[x]_C$ is contained in $[x]_D$, then $[x]_C \subseteq \text{Pos}_C(D)$.

Definition 3 (see [19]). Let $(U, C \cup D)$ be a decision table, where $B \subseteq C$. If it satisfies the following conditions,

$$\text{Pos}_C(D) = \text{Pos}_B(D), \quad (3)$$

for any $B \subseteq B, \text{Pos}_B(D) \neq \text{Pos}_B(D)$, then $B$ is called the positive region reduction of $C$.

Given a decision table, its corresponding discernibility matrix [11] of positive region reduction $MP = (mp_{ij})_{x\alpha}$ is as follows:
In a matrix $F$, $s$ is equal to $|\text{Pos}_C(D)|$, $n$ is the cardinality of $U$, and $|\text{Pos}_C(D)|$ is the cardinality of the positive region.

Definition 4. (see [34]). Given $X \subseteq U$, for each $x \in U$, the characteristic function $\lambda_X(x)$ is defined as follows:

$$\lambda_X(x) = \begin{cases} 1, & x \in X, \\ 0, & x \notin X. \end{cases}$$

Lemma 1 (see [7, 34]). For positive integers $i \in \{1, 2, \ldots, n\}$, where $[x_i]_R$ is an equivalence class on relation $R$ and $X \subseteq U$, $W^R\lambda_X$ is expressed as follows:

$$W^R\lambda_X = \left[ \begin{array}{c} |[x_1]_R \cap X|/|[x_1]_R| \\ |[x_2]_R \cap X|/|[x_2]_R| \\ \vdots \\ |[x_n]_R \cap X|/|[x_n]_R| \end{array} \right]^T,$$

where $T$ denotes the transpose.

For $x_i \in U$, $|[x_i]_R \cap X|/|[x_i]_R|$ is the characteristic function $\lambda(x_i)$, $\lambda(x_i)$ is defined as follows:

$$\lambda(x_i) = \begin{cases} 1, & x_i \in X, \\ 0, & x_i \notin X. \end{cases}$$

Lemma 2 (see [3, 4]). For a decision table, suppose that $\mu_{CD} = (W^R\lambda_1, W^R\lambda_2, \ldots, W^R\lambda_D)$, where $D$ is a decision class of objects with a decision value that is equal to $f_d(i)$. Then, $(b_{ij})_{D \times n}$ is a fuzzy matrix denoted as

$$\begin{pmatrix} \mu_{CD}(x_1) \\ \mu_{CD}(x_2) \\ \vdots \\ \mu_{CD}(x_n) \end{pmatrix} = \begin{pmatrix} |[x_1]_C \cap D_1|/|[x_1]_C| & |[x_1]_C \cap D_2|/|[x_1]_C| & \cdots & |[x_1]_C \cap D_D|/|[x_1]_C| \\ |[x_2]_C \cap D_1|/|[x_2]_C| & |[x_2]_C \cap D_2|/|[x_2]_C| & \cdots & |[x_2]_C \cap D_D|/|[x_2]_C| \\ \vdots & \vdots & \ddots & \vdots \\ |[x_n]_C \cap D_1|/|[x_n]_C| & |[x_n]_C \cap D_2|/|[x_n]_C| & \cdots & |[x_n]_C \cap D_D|/|[x_n]_C| \end{pmatrix},$$

where $b_{ij} = |[x_i]_C \cap D_j|/|[x_i]_C| = P(D_j|[x_i]_C)$, $i$ is a positive integer, and $j$ is a positive integer.

Theorem 2 (see [7, 34]). Given $(U, C \cup D)$, $\forall x \in U$, for $\beta \in [0, 1)$,

$$\left( \mu_{CD} \right)_\beta = \lambda(x_1)^{\beta} (D_1)(x) \lambda(x_2)^{\beta} (D_2)(x) \ldots \lambda(x_n)^{\beta} (D_n)(x).$$

The proof of Theorem 2 is shown in reference [34]. In this section, $(\mu_{CD})_\beta$ was introduced using the $\beta$-cut set of fuzzy set $X$. In Section 3, the relative $\beta$-indiscernibility relation is proposed for decision tables, and then, the corresponding discernibility matrix is proposed.

3. Relative $\beta$-Indiscernibility

Relation Reduction

Given $(U, C \cup D)$, with the quotient set $U/D = \{D_1, D_2, \ldots, D_D\}$ induced by the equivalence relation $R_D$, we first define the relative $\beta$-indiscernibility relation for $\beta \in (0, 1)$.

Definition 6. Given a set $B$, the relative $\beta$-indiscernibility relation is defined as follows:

$$\text{Ind}_B^\beta(D) = \{(x, y) | x, y \in U, (\mu_{BD} (x))_\beta = (\mu_{BD} (y))_\beta\},$$

where $\mu_{BD} (x)$ is the characteristic function $\lambda(x)$, $\lambda(x)$ is defined as follows:

$$\lambda(x) = \begin{cases} 1, & x \in X, \\ 0, & x \notin X. \end{cases}$$

The condition set $C$ contains $a_1, a_2, a_3$, and $D$ is a decision attribute set. Here, $U/C = \{x_1, x_2\}$, $\{x_3, x_4, x_5\}$, $\{x_6, \{x_7\}\}$ and $U/D = \{D_1, D_2, D_3, D_4\}$, where $D_1 = \{x_1, x_3\}$, $D_2 = \{x_2, x_4, x_5, x_6\}$, and $D_4 = \{x_7\}$. For different values of the threshold $\beta$, $\text{Ind}_B^\beta(D)$ may be different.

If $\beta = 0.3$, then by Theorem 2, we have $(\mu_{CD}(x_1))_{0.3} = (\mu_{CD}(x_2))_{0.3} = (1, 1, 0)$. 

(\mu_{CD}(x_3))_{0.3}
\[ \forall, (\ ) \]

\[ \begin{array}{cccc}
\text{U} & a_1 & a_2 & a_3 \\
\hline
x_1 & 1 & 1 & 1 & 0 \\
x_2 & 1 & 1 & 1 & 0 \\
x_3 & 0 & 1 & 0 & 0 \\
x_4 & 0 & 1 & 0 & 1 \\
x_5 & 0 & 1 & 0 & 1 \\
x_6 & 1 & 0 & 0 & 1 \\
x_7 & 0 & 1 & 1 & 2 \\
\end{array} \]

\[ (\mu_{CD}(x_i))_{0.3} = (\mu_{CD}(x_j))_{0.3} = (1, 1, 0), \quad (\mu_{CD}(x_k))_{0.3} = (0, 1, 0), \quad \text{and} \quad (\mu_{CD}(x_l))_{0.3} = (0, 0, 1). \]

If \( \beta = 0.6 \), then by Corollary 2, we have \((\mu_{CD}(x_i))_{0.6} = (\mu_{CD}(x_j))_{0.6} = (0, 0, 0), \quad (\mu_{CD}(x_k))_{0.6} = (0, 1, 0), \quad \text{and} \quad (\mu_{CD}(x_l))_{0.6} = (0, 0, 1). \]

**Definition 7.** Given \((U, C \cup D)\), if \(B \subseteq C\) satisfies the following two conditions:

(a) \( \text{Ind}^C_B(D) = \emptyset \)

(b) for any \( B' \subseteq B \), \( \text{Ind}^C_{B'}(D) \neq \emptyset \)

then \( B \) is the reduction of the relative \( \beta \)-IR.

Given a decision table, a corresponding discernibility matrix \( M^\beta = (m^\beta_{ij})_{nm} \) for the relative \( \beta \)-IR is as follows:

\[
m^\beta_{ij} = \begin{cases} 
\{a | a \in C, (x_i, x_j) \notin R \}, & \text{if } (x_i, x_j) \notin \text{Ind}^C_{B}(D), \quad \text{if } (x_i, x_j) \notin \text{Ind}^C_{B}(D), \\
\emptyset, & \text{otherwise}
\end{cases}
\]

where \( n \) is \( |U| \).

**Lemma 3.** Let \((U, C \cup D)\) be a decision table and \( \forall (x, y) \in U \times U \). If \( (x, y) \notin \text{Ind}^C_{B}(D) \), then \( m^\beta_{ij} \neq \emptyset \).

**Proof.** If \( (x, y) \notin \text{Ind}^C_{B}(D) \), then \( \exists D_k \in U/D \) s.t. \( (\text{Ind}^C_{B}(D))^\beta \neq \emptyset \). Thus, \( (x_i, x_j) \notin R_B \). Hence, there exists \( a_s \in C \) such that \( (x_i, x_j) \notin R_B \). Then, \( m^\beta_{ij} \neq \emptyset \).

**Theorem 3.** Let \((U, C \cup D)\) be a decision table. For \( B \subseteq C \), the following three conditions are equivalent:

(a) \( \text{Ind}^C_B(D) = \emptyset \)

(b) \( m^\beta_{ij} \neq \emptyset \), \( m^\beta_{ij} \cap B \neq \emptyset \)

(c) \( (x, y) \in R_B \), then \( (x, y) \in \text{Ind}^C_B(D) \)

**Proof.** (a) \( \Rightarrow \) (b) Let \( m^\beta_{ij} \neq \emptyset \). Then, there exists \( (x_i, x_j) \notin \text{Ind}^C_B(D) \). Using proof by contradiction, suppose \( m^\beta_{ij} \cap B = \emptyset \). Then, \( (x_i, x_j) \notin R_B \). Thus, by Definition 6, \( (x_i, x_j) \in \text{Ind}^C_B(D) \), which leads to a contradiction.

(b) \( \Rightarrow \) (c) Using proof by contradiction, assume \( (x_i, x_j) \notin \text{Ind}^C_B(D) \). By Lemma 3, \( m^\beta_{ij} \neq \emptyset \). From condition (b), \( m^\beta_{ij} \cap B \neq \emptyset \), and hence, \( a_s \in m^\beta_{ij} \cap B \), namely, \( (x_i, x_j) \notin R_B \), which is a contradiction.

(c) \( \Rightarrow \) (a) In this case, \( (x_i, x_j) \in \text{Ind}^C_B(D) \) holds if and only if \( (\mu_{CD}(x_i))_{B} = (\mu_{CD}(x_j))_{B} \), namely, for each \( D_k \in U/D \), \( (\text{Ind}^C_{B}(D))^\beta \neq \emptyset \). By Lemma 3, \( m^\beta_{ij} \neq \emptyset \). From condition (b), \( m^\beta_{ij} \cap B \neq \emptyset \), and hence, \( a_s \in m^\beta_{ij} \cap B \), namely, \( (x_i, x_j) \notin R_B \), which is a contradiction.

(i) If \( |x_i| \cap D_k \geq \beta \), then \( |x_j| \cap D_k \geq \beta \).

In contrast to the discernibility

\[ \sum_{i=1}^{n} |x_i| \cap D_k \geq \beta \sum_{i=1}^{n} |x_i| \cap |x_i| \geq \beta \]

\[ \text{Table 1: Decision table.} \]

\[
\begin{array}{cccc}
\text{U} & a_1 & a_2 & a_3 \\
\hline
x_1 & 1 & 1 & 1 & 0 \\
x_2 & 1 & 1 & 1 & 0 \\
x_3 & 0 & 1 & 0 & 0 \\
x_4 & 0 & 1 & 0 & 1 \\
x_5 & 0 & 1 & 0 & 1 \\
x_6 & 1 & 0 & 0 & 1 \\
x_7 & 0 & 1 & 1 & 2 \\
\end{array}
\]

4. **Relative \( \beta \)-Discernibility Relation Reduction**

The relative \( \beta \)-indiscernibility relation was proposed in Section 3, and the concept of its complementary relation is proposed in this section. In contrast to the discernibility
matrix of the relative \( \beta \)-IR(\( \beta \)-DR) reduction, in \((U, C \cup D)\), when the precision is greater than 0.5, the decision values of some objects can be modified, and then, the relative \( \beta \)-discernibility relation is calculated using a positive region reduction.

**Definition 8.** Let \((U, C \cup D)\) be a decision table and \(B\) be a subset of \(C\). The relative \( \beta \)-discernibility relation is defined as follows:

\[
\text{Dis}_B^\beta(D) = \{(x, y) \mid x, y \in U, (\mu_B(x))_\beta \neq (\mu_B(y))_\beta \},
\]

where \((\mu_B(x))_\beta = (\lfloor x \rfloor_B \cap D_1)/\lfloor x \rfloor_B, \lfloor x \rfloor_B \cap D_2)/\lfloor x \rfloor_B, \ldots, \lfloor x \rfloor_B \cap D_n)/\lfloor x \rfloor_B\).

For \((U, C \cup D)\), \(\beta \in (0, 1]\), relation \(\text{Ind}_C^\beta(D)\) and relation \(\text{Dis}_C^\beta(D)\) are complementary to each other. For a binary relation \(\text{Dis}_C^\beta(D)\), if \((x, y) \in \text{Dis}_C^\beta(D)\), then \((y, x) \in \text{Dis}_C^\beta(D)\), and hence, it is symmetric. It is not reflexive because \((\mu_{CD}(x))_\beta = (\mu_{CD}(x))_\beta\) for all \(x \in U\). Moreover, it is not necessarily transitive.

For different \(\beta\), the set of \(\text{Dis}_C^\beta(D)\) may also be different. In Table 1, for example, let \(\beta = 0.3\). Then, \((x_2, x_3) \notin \text{Dis}_C^{0.3}(D)\) and \((x_3, x_4) \notin \text{Dis}_C^{0.3}(D)\). However, let \(\beta = 0.6\). We then have \((x_2, x_3) \in \text{Dis}_C^{0.6}(D)\), and \((x_3, x_6) \notin \text{Dis}_C^{0.6}(D)\).

**Definition 9.** Let \((U, C \cup D)\) be a decision table, \(B \subseteq C\). If \(B\) satisfies the following conditions:

1. \(\text{Dis}_B^{0.3}(D) = \text{Dis}_C^{0.3}(D)\)
2. For any \(B' \subseteq B\), \(\text{Dis}_B^{\beta}(D) \neq \text{Dis}_C^{\beta}(D)\)

then \(B\) is the reduction of the relative \( \beta \)-DR.

Given a decision table, its corresponding discernibility matrix is \(M' = (m_{ij}^{\beta})_{n \times n}\) for the relative \( \beta \)-DR, where \(n\) is the cardinality of \(U\). Each element \(m_{ij}^{\beta}\) is defined as follows:

\[
m_{ij}^{\beta} = \begin{cases} \{a | a \in C, (x_i, x_j) \notin R_{\beta} \}, & \text{if } (x_i, x_j) \in \text{Dis}_C^{\beta}(D), \\ \emptyset, & \text{otherwise}. \end{cases}
\]

**Lemma 4.** Let \((U, C \cup D)\) be a decision table. \(\forall (x_i, x_j) \in U \times U\), if \((x_i, x_j) \in \text{Dis}_C^{\beta}(D)\), then \(m_{ij}^{\beta} \neq \emptyset\).

**Proof.** The proof is similar to that of Lemma 3 in Section 3.

**Theorem 4.** Let \((U, C \cup D)\) be a decision table. If \(B \subseteq C\), then the following conditions \((a)\), \((b)\), and \((c)\) are equivalent:

1. \(\text{Dis}_B^{\beta}(D) = \text{Dis}_C^{\beta}(D)\)
2. If \(m_{ij}^{\beta} \neq \emptyset\), \(m_{ij}^{\beta} \cap B \neq \emptyset\)

**Proof.** The proof is similar to that of Theorem 3 in Section 3.

**Lemma 5.** Let \((U, C \cup D)\) be a decision table. If \((x_i, x_j) \in U \times U\), then \(m_{ij}^{\beta} = m_{ij}^{\beta}\).

**Proof.** For all \((x_i, x_j) \in U \times U\), if \((x_i, x_j) \in \text{Dis}_C^{\beta}(D)\) if and only if \((x_i, x_j) \notin \text{Ind}_C^{\beta}(D)\), and hence, \(m_{ij}^{\beta} = m_{ij}^{\beta}\).
Because \((x_i, x_j) \in \text{Diag}_C^\beta(D)\) is the complement of a set \(\text{Ind}_C^\beta(D)\), it is known from equations (12) and (14) that their discernibility matrices are the same.

**Corollary 2.** Let \((U, C \cup D)\) be a decision table. If and precision \(\beta \in [0, 1]\), then \(B\) is the reduction of the relative \(\beta\)-DR if and only if it is a minimal subset, which satisfies \(m_{ij}^\beta \cap B \neq \emptyset\) for any \(m_{ij}^\beta \neq \emptyset\).

Using Corollary 2, the algorithm of the relative \(\beta\)-DR reduction for \((U, C \cup D)\) is proposed as follows.

Considering the decision table given in Table 1, letting \(\beta = 0.3\), the discernibility matrix is constructed as follows:

\[
\begin{array}{cccccc}
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & [a_2, a_3, a_1] \\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & [a_2, a_3, a_1] \\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & [a_1, a_3, a_3] \\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & [a_1, a_3, a_3] \\
[a_2, a_3] & [a_2, a_3] & [a_1, a_2] & [a_1, a_2] & [a_1, a_2] & \emptyset & C \\
[a_1] & [a_1] & [a_3] & [a_3] & [a_3] & C & \emptyset
\end{array}
\]

(15)

The discernibility function in CNF is \(f = (a_1 + a_2)(a_2 + a_3)a_1a_3\). From CNF to DNF, it is \(f = a_1a_3\). Thus, according to Definition 9, \([a_1, a_3]\) is the unique result in the decision table.

**5. Optimization of the Reduction Algorithm**

Let \((U, C \cup D)\) be a decision table with \(D_k \in U/D\). For all \(x \in D_k\) and value \(d_k \in V_d\), let \(I_{d_k}(x) = d_k\), which means that the object \(x\) takes the value \(d_k\) in \(V_d\). Obviously, every \(x\) in \(D_k\) has the same decision value \(d_k\) for the decision attribute. Given precision \(\beta \in (0.5, 1]\), for any \([x_c]\) satisfying \(\beta \leq [x_c] \cap D_k/|[x_c]| < 1\), suppose \(x_c \in [x_c]\) and \(I_{d_k}(x) \neq d_k\). Then, change the values that objects \(x_i\) have for decision attribute \(d\) such that \(I_{d_k}(x_i) = d_k\), that is, \([x_c] \subseteq D_k\). This constructs a new quotient set \(U/D'\), denoted by \(U/D' = \{D_1', D_2', \ldots, D_l'\}\). The decision table that changes the decision values of some objects using this method is called the new decision table and is denoted by \((U, C \cup D')\). Indeed, we note that \(D_k'\) may be the empty set for \(\exists D_k' \in U/D'\).

**Theorem 5.** Let \((U, C \cup D)\) be a decision table with quotient set \(U/D\). Given precision \(\beta \in (0.5, 1]\), an updated new decision table \((U, C \cup D)\) is constructed with a new quotient set \(U/D' = \{D_1', D_2', \ldots, D_l'\}\). Then, \(\mu_{\text{CD}}(x)\beta = (\mu_{\text{CD}}(x))\).

**Proposition 1.** Given a new decision table \((U, C \cup D')\), where \(U/D'\) is a new quotient set by \(D'\) and \(B \subseteq C\), \(B\) is the positive region reduction of \(C\) if and only if \(B\) is the reduction of the relative \(\beta\)-IR (\(\beta\)-DR) for \(\beta = 1\).

**Proof.** \((\Rightarrow)\) If \(B\) is the attribute reduction of \(C\), and \(\text{Pos}_C(D') = \text{Pos}_B(D')\), then \([x_c] \subseteq D_k\) if and only if \([x_c] \subseteq D_k\) for \(\forall D_k' \in U/D'\), that is, \([x_c] \cap D_k/|[x_c]| = [x_c] \cap D_k/|[x_c]|\) because \(\mu_{\text{CD}}(x)\beta = (\mu_{\text{CD}}(x))\). Hence, \(\text{Ind}^\beta(C) = \text{Ind}^\beta(D')\). Suppose \(B' \subseteq B\), \(\text{Pos}_C(D') = \text{Pos}_B(D')\). Then, there exists \(D_k' \in U/D'\), s.t. \([x_c] \subseteq D_k'\) and \([x_c] \subseteq D_k\). Therefore, \(\text{Ind}^\beta(C) = \text{Ind}^\beta(D')\).

**Corollary 3.** Given a decision table \((U, C \cup D)\) with precision \(\beta \in (0.5, 1]\), a new decision table \((U, C \cup D')\) is constructed. Then, \(B\) is the reduction of the relative \(\beta\)-IR (\(\beta\)-DR) with respect to \((U, C \cup D)\) if and only if \(B\) is the positive region for \((U, C \cup D')\).

**Proof.** By Theorem 5 and Proposition 1, Corollary 3 clearly holds.

The above shows that the relative \(\beta\)-IR (\(\beta\)-DR) reduction with a precision greater than 0.5 is equivalent to positive region reduction, after modifying the decision values of some objects that satisfy the condition. When the precision of \(\beta\)-IR (\(\beta\)-DR) reduction is greater than 0.5, the decision values of some objects can be modified, and a new decision table can be constructed. In the new decision table, many heuristic algorithms can be used to calculate the positive region (Definition 3) reduction.

According to Corollary 3, an optimization algorithm that converts the relative \(\beta\)-IR (\(\beta\)-DR) reduction to a positive region reduction (CRRPRR) is as follows.

When comparing the relative \(\beta\)-IR (\(\beta\)-DR) reduction algorithm with the Algorithm 3, the time complexity of both algorithms is \(O(|C| + |D| \times |U|)\), where \(C\) represents the number of condition attributes and \(D\) represents the number of decision attributes. For Algorithm 1 and 2, the time complexity of constructing the discernibility matrix is \(O(|D| \times |C|)\) and the space complexity of constructing discernibility matrix is \(O(|U| \times |U|)\) where \(|U|/|D\) is the number of equivalent classifications induced by decision set \(D\). For the Algorithm 3, the time complexity of calculating the positive region in the new decision table is \(O(|U| \times |U| \times (|(|[x_c]| \times |D_k|) + |[x_c]|)))\), where \(|U/D|\) is the number of equivalent classifications induced by decision set \(D\) and \(|D|\) is the basis of the equivalent class \(D_j\).

Moreover, the time complexity of constructing the discernibility matrix is \(O(|\text{Pos}_B(D') \times |U| \times |C|)\), and the
For the original algorithm. The discernibility function transformation of both algorithms are the same. Obviously, the CRRPRR algorithm is better than the CRRRR algorithm in terms of precision.

A new decision table is constructed as follows:

\[
\begin{array}{c|c|c|c|c|c}
U & C_1 & C_2 & C_3 & C_4 \\
\hline
x_1 & 1 & 0 & 1 & 0 \\
\hline
x_2 & 0 & 1 & 0 & 1 \\
\hline
\end{array}
\]

Input: (\(U, C \cup D\))
Output: reduction results of \(C\)
(1) \(m_D^0 = \emptyset, B_1 = \emptyset\);
(2) for \(x\) in \(U\) do
(3) compute \((\mu_D(x))_\beta\);
(4) endfor
for \(x\) in \(U\) do
(5) if \((x_i, x_j) \in D_\beta^\delta(D)\) then
\(m_{ij}^\beta = m_{ij}^\beta \cup \{a_i, a_j \in C\} \)
endif
(7) endfor
(8) endfor
(9) execute CNF to DNF function
(10) return all \(B_i\) // \(B_i (i = 1, 2, \ldots, k)\) is one of the attribute reductions

Algorithm 2: The algorithm of the relative \(\beta\)-DR reduction.

Input: decision table \((U, C \cup D)\), given \(\beta \in (0, 1]\)
Output: reduction results of \(C\)
(1) \(m_{DP} = \emptyset, B_1 = \emptyset\);
(2) for \(x\) in \(U\) do
(3) if \(\beta \leq |\{x\} \cap D_k|/|\{x\}| < 1\) then
(4) \(m_P(x)\) is modified;
(5) endif
(6) endfor
new decision table \((U, C \cup D')\) is constructed //a
for \(x\) in \(D_P(C)\)
for \(x\) in \(U\) do
(7) if \(x \in D_P(C), (x_i, x_j) \notin R_\beta\) then
\(m_D^\beta = m_D^\beta \cup \{a_i, a_j \in C\} \)
endif
(8) endfor
(10) endfor
(11) execute CNF to DNF function
(12) return all \(B_i\) // \(B_i (i = 1, 2, \ldots, k)\) is one of the attribute reductions

Algorithm 3: An optimization algorithm CRRP RR.

In Table 1, \([x_1]_{C_1} = \{x_1, x_2\}, [x_3]_{C_1} = \{x_3, x_4, x_5\}\). Given precision \(\beta = 0.6\), \(|[x_3]_{C_1} \cap D_1|/|[x_3]_{C_1}| \geq \beta\), with \(D_2 = \{x_2, x_4, x_5\}\). Thus, \(I_D(x_3) = 1\) in a new decision table. For \(|[x_1]_{C_1} \cap D_1|/|[x_1]_{C_1}| < \beta\), with \(D_1 = \{x_1, x_3\}\), the decision values of any objects in \([x_1]_{C_1}\) are not changed. Hence, the new decision table \((U, C \cup D')\) is shown in Table 2.

According to the CRRP RR optimization algorithm, \(PosC(D') = \{x_3, x_4, x_5, x_6, x_7\}\) by Definition 2. Then, the discernibility matrix is constructed as follows:

\[
\begin{bmatrix}
\{a_1, a_3\} & \{a_1, a_3\} & \emptyset & \emptyset & \emptyset & \{a_1, a_2\} & \{a_3\} \\
\{a_1, a_3\} & \{a_1, a_3\} & \emptyset & \emptyset & \emptyset & \{a_1, a_2\} & \{a_3\} \\
\{a_1, a_3\} & \{a_1, a_3\} & \emptyset & \emptyset & \emptyset & \{a_1, a_2\} & \{a_3\} \\
\{a_2, a_3\} & \{a_2, a_3\} & \{a_1, a_2\} & \{a_1, a_2\} & \emptyset & \emptyset & \{a_3\} \\
\{a_1\} & \{a_1\} & \{a_3\} & \{a_3\} & \{a_3\} & \emptyset & \emptyset \\
\end{bmatrix}
\]

The discernibility function in CNF is \(f = (a_1 + a_3)(a_1 + a_2)(a_1 + a_3)a_1a_3\). From CNF to DNF, we have \(f = a_1a_3\). Thus, \(\{a_1, a_3\}\) is the unique attribute reduction of \(C\) in \((U, C \cup D')\).

It must be explained that the basic condition for the conversion of the relative \(\beta\)-IR (\(\beta\)-DR) reduction into...
positive region reduction is that the precision must be greater than 0.5. When the precision is greater than 0.5, this not only improves the fault tolerance of the model but also ensures its credibility. If \( \beta \geq 0.5 \) is given (e.g., as in Table 1), for \([x_1, x_2] \), because \( I_q(x_1) = 0, I_d(x_2) = 1 \), the decision values of any objects in \([x_1, x_2] \) cannot be modified according to the conversion method in this paper. Another case occurs when \( \beta = 0.8 \). In this case, none of the decision values of any objects can be modified in Table 1, and the above conversion method is not feasible.

\[\] 6. Experimental Analysis

In this section, we evaluate the performances of the proposed algorithms through some comparison experiments. In our experiments, twenty datasets from the UCI were used. All the information is shown in Table 3, in which \(|U|\) is the cardinality of \( U \), and \(|C|\) and \(|U/D|\) are the numbers of condition attributes and decision classes, respectively. Using tenfold cross-validation, three classifiers (kernel NB, fine Gaussian SVM, and a DT) were used to test the classification accuracies of the results after reduction. Kernel NB is a classifier that uses estimated kernel densities. The classification accuracy of fine Gaussian SVM is higher than that of SVM after the Gaussian kernel is introduced. DT is a supervised learning model that learns decision rules. The algorithms were implemented on a MacBook Pro (early 2015) with an Intel(R) Core(TM) i5 CPU at 2.7 GHz and Intel Iris Graphics 6100 GPU. The algorithms in this paper were coded in Python 3.6.8 using scikit-learn 0.20.3.

Because of the requirements of Algorithm 3, the accuracy of the tables in this experiment must be greater than 0.5. The runtime results for different precisions (0.7, 0.8, and 0.9) are shown in Figure 1. In this figure, the yellow line represents the average time for Algorithm 3 to run for each precision. The experimental results show that Algorithm 1, Algorithm 2, and Algorithm 3 obtain the same results, but the runtime of the latter is less than that of the former. Although the cost of converting CNF to DNF is high, the advantage of the Algorithm 3 is obvious when the number of objects is large and the number of attributes is relatively small.

Because the proposed algorithms can obtain all the results, the mean classification accuracy is reported. For all precision values, the classification accuracies obtained by kernel NB, fine Gaussian SVM, and DT are shown in Figures 2–4, respectively.

In Figure 2, in most cases, the classification accuracy of the kernel NB classifier is low for the reduction results obtained from a low precision value. For example, for the HS
dataset, given precision values of 0.7, 0.8, and 0.9, the accuracy results are 81.25%, 85.66%, and 92.11%, respectively.

For all data sets except for Wine, SH, and TAE, a higher precision value leads to a higher classification accuracy for the fine Gaussian SVM classifier in Figure 3. However, the classification accuracy is relatively low. In most cases, when the precision is high, the reduction results lead to higher classification accuracies for the DT classifier, but the classification accuracy is low. For example, on the HR dataset, given precision values of 0.7, 0.8, and 0.9, the accuracy results are 62.33%, 70.15%, and 76.01%, respectively.

This experiment demonstrates the feasibility of the concept proposed in this paper. When the precision is less than 0.5, its reliability is not high. Moreover, the CRRPRR optimization algorithm and the relative $\beta$–IR ($\beta$–DR) reduction algorithm could not be linked. Therefore, the precision of the algorithms must be greater than 0.5. Figures 2–4 show that when the precision is larger, the classification accuracy is higher when the results are obtained by the proposed algorithms. On the contrary, it is lower when the precision is lower.

7. Conclusions

This study extends the work of [5, 22, 34, 41] to investigate the relative $\beta$–IR ($\beta$–DR) for the first time using a discernibility matrix-based method. Under certain conditions, precision $\beta > 0.5$, and the relationship between the relative $\beta$–IR ($\beta$–DR) reduction and positive region reduction was found by modifying some decision values, which is of a certain significance to the study of VP RSs. The corresponding optimization algorithm was then proposed. The discernibility matrix corresponding to the relative $\beta$–IR ($\beta$–DR) reduction has a high time complexity of $O(|U|^2 \times |C|)$, whereas the time complexity for constructing a positive region reduction is much less. Therefore, when the precision $\beta > 0.5$, although the reduction results are the same, the optimization CRRPRR algorithm reduces the computational complexity. In addition, because the attribute importance...
cannot be calculated in the relative $\beta$–IR ($\beta$–DR) reduction to obtain the results, we modified some of the decision values that satisfy the condition so that the attribute importance can also be calculated in the new decision table. The feasibility of the algorithm proposed in this paper was demonstrated by experimental analysis. In future, we will attempt to remove the restriction of equivalence relation and further study problems such as precision reduction and the relative $\beta$–IR($\beta$–DR) reduction.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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