Minimum Potential Energy Principle on Slip Calculation of Composite Beams

Xu Chen*, Pingjia Xiong and Jianghong Ma
Department of Urban and Rural Construction and Engineering Management, Kunming University, Kunming, Yunnan, 650500, China
* Corresponding author’s e-mail: Xuchen_22@163.com

Abstract. Energy method has the mathematical convenience because it is scalar. The work potential is work done by external forces. After it is transferred to the work done by the internal force, the classical second-order differential equation of slip effects is deduced according to Minimum Potential Energy Principal. In the derivation, integration by parts is used to transfer the higher order variation into the first order according to Euler Equation. For simply-supported composite beam, the slip-induced stresses are self-equilibrium and the redistributed internal forces are significant. The slip effects is important for composite beams in service time, like the compression stress of the top flange of steel beam may become the 11.5 times, which arises the problem of local buckling.

1. Introduction
The mechanical behavior of composite beams is linear-elastic in service time. Minimum Potential Energy Principal can be used to calculate the interface slip of shear connectors. It provides mathematical convenience because energy is a scalar quantity. For composite beams, the slip reduces flexural stiffness and increases deflection [1-9]. Slip effects are significant. Therefore, it is meaningful to find out what is the critical results due to slip individually before more complicate analysis is carried out, like the coupling of the slip and the creep and shrinkage.

2. Derivation of the differential equation
The energy method is more extensive than the static method. Total potential energy ($\Pi$) includes the deformation energy ($U$) and the work potential ($V$). In the formula of energy, deformation (strain and displacement) should be considered.

For composite beams including three individual parts (concrete slab, steel beam and shear connector), axial deformation pure bending and are assumed for the former two (shear is neglected in this paper), strain ($\varepsilon$) and curvature ($\kappa$) are strain quantities, and slip ($s$) is the shear deformation of connector. Then the deformation energy of composite beam with a length $L$ becomes

$$U = \frac{1}{2} \int_0^L \left[ E_c A_c \varepsilon_c^2 + E_s A_s \varepsilon_s^2 + (E_s I_s + E_c I_c) \kappa^2 + ks^2 \right] dx$$

(1)

where $E$, $A$ and $I$ are elastic modular, area and the second moment of area respectively, the subscript c and s denote concrete and steel respectively, and $k$ is the shear stiffness of interface.

The relationships of strain and displacement (in Figure 1) are established by kinematic and geometric compatibility,
where the differential is with respect to longitudinal coordinate $x$, $u_s$ and $u_c$ are axial displacements of steel beam and concrete slab respectively, $w$ is the lateral displacement of the beam, $d$ is the distance between centre of gravity of concrete slab and steel beam.

Substituting Equation 2 into Equation 1, one obtains

$$U = \frac{1}{2} \int_0^L \left[ E_c A_c u_c'^2 + E_s A_s u_s'^2 + k \left( u_s - u_c - \kappa d \right)^2 \right] \, dx$$  \hfill (3)

Because work done by external forces (like $P_1$, $P_i$, $q$ in Figure 1) is equal to that by the internal force work, the latter can be used to formulate the work potential. For composite beams subjected to pure bending moment ($M$), it becomes

$$V = -\int_0^L Mw'' \, dx$$ \hfill (4)

Then the total potential energy becomes

$$\Pi = U + V = \int_0^L \left[ E_c A_c u_c'^2 + E_s A_s u_s'^2 + k \left( u_s - u_c - \kappa d \right)^2 \right] \, dx + \int_0^L \left[ E_c A_c u_c'' + E_s A_s u_s'' + k \left( u_s - u_c - \kappa d \right) \right] \, dx$$ \hfill (5)

It can be observed from Equation 5 that the integrand ($F$) includes three unknown quantities and high high order variation of them. The high order variation must be transferred into the first order, therefore integration by parts is used according to Euler Equation [10]. Finally the following variation of the total potential energy can be deduced,

$$\delta \Pi = \int_0^L \left[ \frac{\partial F}{\partial u_s} \frac{d}{dx} \left( \frac{\partial F}{\partial u_s'} \right) + \frac{\partial F}{\partial u_c} \frac{d}{dx} \left( \frac{\partial F}{\partial u_c'} \right) + \frac{\partial F}{\partial \kappa} \frac{d}{dx} \left( \frac{\partial F}{\partial \kappa'} \right) \right] \, dx$$ \hfill (6)

According to Minimum Potential Energy Principal, the actual displacements make $\delta \Pi = 0$, furthermore the nature of $\delta u_s$, $\delta u_c$ and $\delta w$ is arbitrary, then one obtains
\[ \left\{ \begin{array}{c} \frac{\partial F}{\partial u_c} - \frac{d}{dx} \left( \frac{\partial F}{\partial u_c'} \right) = 0 \\ \frac{\partial F}{\partial u_s} - \frac{d}{dx} \left( \frac{\partial F}{\partial u_s'} \right) = 0 \\ \frac{d}{dx^2} \left( \frac{\partial F}{\partial \omega^s} \right) - \frac{d}{dx} \left( \frac{\partial F}{\partial \omega^s'} \right) = 0 \end{array} \right. \]  

Substituting function \( F \) into Equation 7, one obtains

\[ \begin{cases} k(u_c - u_s - w'd) - E_A u_s' = 0 \\ -k(u_c - u_s - w'd) - E_A u_s' = 0 \\ EI \omega^{(4)} - M^* + kd(u_c' - u_s' - w'd) = 0 \end{cases} \]  

Taking internal axial force \( N_i \) as independent variable, the relationships between strains and \( N_i \) are

\[ \left\{ \begin{array}{l} \varepsilon_c = -\frac{N_i}{E_c A_c} \\ \varepsilon_s = \frac{N_i}{E_s A_s} \end{array} \right. \]  

Considering the equilibrium condition, one obtains

\[ \begin{align*} w^* &= \frac{M - N_i d}{EI} \\ EI \omega^{(4)} &= M^* - N^* d \\ u_c' &= \frac{N_i}{E_c A_c} \\ u_s' &= -\frac{N_i}{E_s A_s} \end{align*} \]  

Finally the classical differential equation is obtained,

\[ N_i'' = -\frac{1}{EA + \frac{d^2}{EI}} \left( k N_i - \frac{k d}{EI} M \right) \]  

### 3. Example

The composite beam \(( L = 8 \text{m})\) is subjected to uniformly distributed load \(( q = 20 \text{kN} \cdot \text{m})\). The section parameters are given as follows.

\[ E_c = 28 \text{Gpa}, A_c = 1330 \text{cm}^2, I_c = 11080 \text{cm}^4, E_s = 210 \text{Gpa}, A_s = 67.25 \text{cm}^2, I_s = 9400 \text{cm}^4 \]

\[ d = 20 \text{cm}, EA = 1.02 \times 10^8 \text{kN}, EI = 2.28 \times 10^5 \text{kN} \cdot \text{cm}^2, k = 34.72 \text{kN/cm}^2, \omega = 0.973 \text{m}^{-1}, \omega L = 7.78 \]

For simply-supported composite beams, the internal forces of individual sections are redistributed because of slip, while the internal force of the whole section is not changed because the slip-induced stresses are eigen stresses. For example, in midspan the internal forces are

Without slip: \( M = 160 \text{kN} \cdot \text{m}, N_i = 513.58 \text{kN}, M_c = 2.79 \text{kN} \cdot \text{m}, M_s = 17.72 \text{kN} \cdot \text{m} \)

With slip: \( M = 160 \text{kN} \cdot \text{m}, N_i = 448.55 \text{kN}, M_c = 9.55 \text{kN} \cdot \text{m}, M_s = 60.74 \text{kN} \cdot \text{m} \)

The reduced internal axial force means that the reduction of composite action between the two individual sections, which is unwilling to see by the designer. On the other side, the shear force of...
connectors is unloaded because of the slip, and therefore in the design of connector the benefit can be considered.

As seen the moments of the individual sections increase about three times, which arises the change of stresses at the interface in midspan as followings.

Without slip: \( \sigma_c = -0.35 \text{Mpa} \), \( \sigma_s = -2.63 \text{Mpa} \)

With slip: \( \sigma_c = 0.94 \text{Mpa} \), \( \sigma_s = -30.23 \text{Mpa} \)

For the bottom concrete slab, the stress becomes tension due to slip and the risk of crack may be considered. For the top flange of steel beam, the stress becomes 11.49 times. It is amazing. Therefore, the risk of local buckling must be considered.

For the deflection, it is 16.72mm without slip and 20.87 with slip, which means that the deflection magnification and stiffness reduction ratio are 1.25 and 0.8 respectively.

4. Conclusions
Deformation energy is formulated by the strains of concrete slab, shear connector and steel beam, and the work potential by the work done of the internal force. Integration by parts is used to transfer the higher order variation into the first order according to Euler Equation. The classical second-order differential equation of slip effects is deduced according to Minimum Potential Energy Principal. For simply-supported beams, the slip-induced stresses are eigen stress and significant redistribution of the internal forces of individual sections can be observed, as well as the increase of the deflection. The risk of local buckling due to slip is very high for the top flange of steel beam.

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