THE COX RING OF A DEL PEZZO SURFACE

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Abstract. Let \( X_r \) be a smooth Del Pezzo surface obtained from \( \mathbb{P}^2 \) by blowing up \( r \leq 8 \) points in general position. It is well known that for \( r \in \{3, 4, 5, 6, 7, 8\} \) the Picard group \( \text{Pic}(X_r) \) contains a canonical root system \( R_r \in \{A_2 \times A_1, A_4, D_5, E_6, E_7, E_8\} \). We prove some general properties of the Cox ring of \( X_r \) (\( r \geq 4 \)) and show its similarity to the homogeneous coordinate ring of the orbit of the highest weight vector in some irreducible representation of the algebraic group \( G \) associated with the root system \( R_r \).

1. Introduction

Let \( X \) be a projective algebraic variety over a field \( \k \). Assume that the Picard group \( \text{Pic}(X) \) is a finitely generated abelian group. Consider the vector space

\[
\Gamma(X) := \bigoplus_{[D] \in \text{Pic}(X)} H^0(X, \mathcal{O}(D)).
\]

One wants to make it a \( \k \)-algebra which is graded by the monoid of effective classes in \( \text{Pic}(X) \) such that the algebra structure will be compatible with the natural bilinear map

\[
b_{D_1, D_2} : H^0(X, \mathcal{O}(D_1)) \times H^0(X, \mathcal{O}(D_2)) \to H^0(X, \mathcal{O}(D_1 + D_2)).
\]

However, there exist some problems in the realization of this idea. First of all there is no any natural isomorphism between \( H^0(X, \mathcal{O}(D)) \) and \( H^0(X, \mathcal{O}(D')) \) if \( [D] = [D'] \). There exists only a canonical bijection between the linear systems \( |D| \cong |D'| \) (where \( |D| \) is the projectivization of the \( \k \)-vector space \( H^0(X, \mathcal{O}(D)) \)). As a consequence, the bilinear map \( b_{D_1, D_2} \) depends not only on the classes \( [D_1], [D_2], [D_1 + D_2] \in \text{Pic}(X) \), but also on their particular representatives. One can easily see that only the morphism

\[
s_{[D_1], [D_2]} : |D_1| \times |D_2| \to |D_1 + D_2|
\]

of the product of two projective spaces \( |D_1| \times |D_2| \) to another projective space \( |D_1 + D_2| \) is well-defined. For this reason, it is much more natural to consider the graded set of projective spaces

\[
|\Gamma(X)| := \bigsqcup_{[D] \in \text{Pic}(X)} |D|
\]

together with all possible morphisms \( s_{[D_1], [D_2]} \) any two effective classes \( [D_1], [D_2] \in \text{Pic}(X) \).

Key words and phrases. Del Pezzo surfaces, torsors, homogeneous spaces, algebraic groups.
Inspired by the paper of Cox on the homogeneous ring of a toric variety \([\text{Cox}]\), Hu and Keel \([\text{H-K}]\) suggested a definition of a Cox ring
\[
\text{Cox}(X) = R(X, L_1, \ldots, L_r) := \bigoplus_{(m_1, \ldots, m_r) \in \mathbb{Z}^r} H^0(X, \mathcal{O}(m_1 L_1 + \cdots + m_r L_r))
\]
which uses a choice of some \(\mathbb{Z}\)-basis \(L_1, \ldots, L_r\) in \(\text{Pic}(X)\) (e.g. if \(\text{Pic}(X) \cong \mathbb{Z}^r\) is a free abelian group). Using such a \(\mathbb{Z}\)-basis, one obtains a particular representative for each class in \(\text{Pic}(X)\) together with a well-defined multiplication so \(R(X, L_1, \ldots, L_r)\) becomes a well-defined \(k\)-algebra. If \(L'_1, \ldots, L'_r\) is another \(\mathbb{Z}\)-basis of \(\text{Pic}(X)\), then the corresponding Cox algebra \(R(X, L'_1, \ldots, L'_r)\) is isomorphic to \(R(X, L_1, \ldots, L_r)\). Unfortunately, we can not expect to choose a \(\mathbb{Z}\)-basis of \(\text{Pic}(X)\) in a natural canonical way. More often one can choose in a natural way some effective divisors \(D_1, \ldots, D_n\) on \(X\) such that \(\text{Pic}(X)\) is generated by \([D_1], \ldots, [D_n]\). If we set
\[
U := X \setminus (D_1 \cup \cdots \cup D_n)
\]
and assume that \(X\) is smooth, then \(\text{Pic}(U) = 0\) and we obtain the exact sequence
\[
1 \to \mathbb{k}^* \to \mathbb{k}[U]^* \to \bigoplus_{i=1}^n \mathbb{Z}[D_i] \to \text{Pic}(X) \to 0.
\]
Choosing a \(\mathbb{k}\)-rational point \(p\) in \(U\), we can split the monomorphism \(\mathbb{k}^* \to \mathbb{k}[U]^*\), so that one has an isomorphism
\[
\mathbb{k}[U]^* \cong \mathbb{k}^* \oplus G,
\]
where \(G \subset \mathbb{k}[U]^*\) is a free abelian group of rank \(n - r\). The choice of a \(\mathbb{k}\)-rational point \(p \in U\) allows to give another approach to the graded space \(\Gamma(X)\) and to the Cox algebra:

**Definition 1.1.** Let \(X, U, p, D_1, \ldots, D_n\) be as above. We consider the graded \(\mathbb{k}\)-algebra
\[
\Gamma(X, U, p) := \bigoplus_{(m_1, \ldots, m_n) \in \mathbb{Z}^n} H^0(X, \mathcal{O}(m_1 D_1 + \cdots + m_n D_n))
\]
and define
\[
\text{Cox}(X, U, p) := \Gamma(X, U, p)_G
\]
as the quotient of the \(\Gamma(X, U, p)\) modulo the ideal generated by
\[
\{ x - gx \mid x \in \Gamma(X, U, p), g \in G \}.
\]
Since \(\text{Pic}(X) \cong \mathbb{Z}^n/G\), we obtain a natural \(\text{Pic}(X)\)-grading on \(\text{Cox}(X, U, p)\).

We expect that the algebra \(\text{Cox}(X, U, p)\) can be applied to some arithmetic questions on \(\mathbb{k}\)-rational points in \(U \subset X\).

**Remark 1.2.** The above definition of the ring \(\text{Cox}(X, U, p)\) depends on the choice of an open subset \(U \subset X\) and a \(\mathbb{k}\)-rational point \(p \in U\). A similar idea was used by Colliot-Thélène and Sansuc in \([\text{C-S}]\) for constructing universal torsors and deriving explicit equations for them. The lack of a canonical construction is precisely what makes descending the universal torsor an interesting problem. Some applications of the universal torsor for Del Pezzo surfaces of degree 5 was considered by Skorobogatov in \([\text{S1}]\) (see also \([\text{S2}]\)). Recently, Hassett and Tschinkel have investigated the Cox rings and the universal torsors for some interesting special cubic surfaces \([\text{H-T}]\).
Remark 1.3. If $X$ is a smooth projective toric variety and $U \subset X$ is the open dense torus orbit, then the choice of a point $p \in U$ defines an isomorphism of $U$ with the algebraic torus $T$, so that the subgroup $G \subset k[U]^*$ can be identified with the character group of $T$. In this way, one can show that $\text{Cox}(X, U, p)$ is isomorphic to a polynomial ring in $n$ variables ($n$ is the number of irreducible components of $X \setminus U$, cf. [Cox]).

Remark 1.4. The field of fractions of the ring $\text{Cox}(X, U, p)$ is a pure transcendental extension of degree $r$ of the field of rational functions on $X$. Therefore, $\dim \text{Spec } \Gamma(X, U, p) = \dim X + r$, if $\Gamma(X, U, p)$ is a finitely generated $k$-algebra.

Let $X_r$ be a smooth Del Pezzo surface obtained from $\mathbb{P}^2$ by blow-up of $r \leq 8$ points in general position. It is well known that for $r \in \{3, 4, 5, 6, 7, 8\}$ the Picard group $\text{Pic}(X_r)$ contains a canonical root system $R_r \in \{A_2 \times A_1, A_4, D_5, E_6, E_7, E_8\}$. Moreover, the natural embedding $\text{Pic}(X_{r-1}) \hookrightarrow \text{Pic}(X_r)$ induces the inclusion of root systems $R_{r-1} \hookrightarrow R_r$. If $G(R_r)$ is a connected algebraic group corresponding to the root system $R_r$, then the embedding $R_{r-1} \hookrightarrow R_r$ defines a maximal parabolic subgroup $P(R_{r-1}) \subset G(R_r)$ [Hu]. We expect that for $r \geq 4$ there should be some relation between a Del Pezzo surface $X_r$ and the GIT-quotient of the homogeneous space $G(R_r)/P(R_{r-1})$ modulo the action of a maximal torus $T_r$ of $G(R_r)$.

Our starting observation is the well-known isomorphism $X_4 \cong G(3, 5)/T_4$ which follows from an isomorphism between the homogeneous coordinate ring of the Grassmannian $G(3, 5) = G(A_4)/P(A_2 \times A_1) \subset \mathbb{P}^9$ and the Cox ring of $X_4$ (see [P]). Another proof of this fact follows form the identification of $X_4$ with the moduli space $\overline{M}_{0,5}$ of stable rational curves with 5 marked points [K].

In this paper, we start an investigation of the Cox ring of Del Pezzo surfaces $X_r$ ($r \geq 4$). It is natural to choose the classes of all exceptional curves $E_1, \ldots, E_{N_r} \subset X_r$ as a generating set for the Picard group $\text{Pic}(X_r)$. There is a natural $\mathbb{Z}_{\geq 0}$-grading on $\text{Pic}(X_r)$ defined by the intersection with the anticanonical divisor $-K$.

We prove some general properties of the Cox rings of a Del Pezzo surface $X_r$ ($r \geq 4$) and show their similarity to the homogeneous coordinate ring of $G(R_r)/P(R_{r-1})$. We remark that the homogeneous space $G(R_r)/P(R_{r-1})$ can be interpreted as the orbit of the highest weight vector in some natural irreducible representation of $G(R_r)$.

Remark 1.5. Some other connections between Del Pezzo surfaces and the corresponding algebraic groups were considered also by Friedman and Morgan in [F-M]. A similar topic was considered by Leung in [Le].

In this paper, we show that the Cox ring of a Del Pezzo surface $X_r$ is generated by elements of degree 1. This implies that the homogeneous coordinate ring of $G(R_r)/P(R_{r-1})$ is naturally graded by the monoid of effective divisor classes on the surface $X_r$ (the same monoid defines the multigrading of the Cox ring of $X_r$). Moreover, we obtain some results of the quadratic relations between the generators of the Cox ring of $X_r$.

The authors would like to thank Yu. Tschinkel, A. Skorobogatov, E. S. Golod, S. M. Lvovski and E. B. Vinberg for useful discussions and encouraging remarks.
2. Del Pezzo Surfaces

Let us summarize briefly some well-known classical results on Del Pezzo surfaces which can be found in [Ma, Dem, Na].

One says that \( r \) \((r \leq 8)\) points \( p_1, \ldots, p_r \) in \( \mathbb{P}^2 \) are in general position if there are no 3 points on a line, no 6 points on a conic \((r \geq 6)\) and a cubic having seven points and one of them double does not have the eighth one \((r = 8)\).

Denote by \( X_r \) \((r \geq 3)\) the Del Pezzo surfaces obtained from \( \mathbb{P}^2 \) by blowing up of \( r \) points \( p_1, \ldots, p_r \) in general position. If \( \pi : X_r \to \mathbb{P}^2 \) the corresponding projective morphism, then the Picard group \( \text{Pic}(X_r) \cong \mathbb{Z}^r \) contains a \( \mathbb{Z} \)-basis \( l_i, (0 \leq i \leq r), \ l_0 = [\pi^*\mathcal{O}(1)] \) and \( l_i : = [\pi^{-1}(p_i)], i = 1, \ldots, r \). The intersection form \( (\ast, \ast) \) on \( \text{Pic}(X_r) \) is determined in the chosen basis by the diagonal matrix: \((l_0, l_0) = 1, (l_i, l_i) = -1 \) for \( i \geq 1 \), \((l_i, l_j) = 0 \) for \( i \neq j \). The anticanonical class of \( X_r \) equals \(-K = 3l_0 - l_1 - \cdots - l_r \). The number \( d : = (K, K) = 9 - r \) is called the degree of \( X_r \). The anticanonical system \(|-K| \) of a Del Pezzo surface \( X_r \) is very ample if \( r \leq 6 \), it determines a two-fold covering of \( \mathbb{P}^2 \) if \( r = 7 \), and it has one base point, determining a rational map to \( \mathbb{P}^1 \) if \( r = 8 \). Smooth rational curves \( E \subset X_r \) such that \(|E, E| = -1 \) and \(|E, -K| = 1 \) are called exceptional curves.

**Theorem 2.1.** [Ma] The exceptional curves on \( X_r \) are the following:

1. blown-up points \( p_1, \ldots, p_r \);
2. lines through pairs of points \( p_i, p_j \);
3. conics through 5 points from \( \{p_1, \ldots, p_r\} \) \((r \geq 5)\);
4. cubics, containing 7 points and 1 of them double \((r \geq 7)\);
5. quartics, containing 8 points and 3 of them double \((r = 8)\);
6. sextics, containing 8 of point and 6 of them double \((r = 8)\);
7. septic, containing 8 of those points, 7 of them double and 1 triple \((r = 8)\).

The number \( N_r \) of exceptional curves on \( X_r \) is given by the following table:

| \( r \) | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|
| \( N_r \) | 6 | 10 | 16 | 27 | 56 | 240 |

The root system \( R_r \subset \text{Pic}(X_r) \) is defined as

\[
R_r := \{ \alpha \in \text{Pic}(X_r) : (\alpha, \alpha) = -2, (\alpha, -K) = 0 \}.
\]

It is easy to show that \( R_r \) is exactly the set of all classes \( \alpha = [E_i] - [E_j] \) where \( E_i \) and \( E_j \) are two exceptional curves on \( X_r \) such that \( E_i \cap E_j = \emptyset \).

The corresponding Weyl group \( W_r \) is generated by the reflections \( \sigma : x \mapsto x + (x, \alpha)\alpha \) for \( \alpha \in R_r \). There are so called simple roots \( \alpha_1, \ldots, \alpha_r \) such that the corresponding reflexions \( \sigma_1, \ldots, \sigma_r \) form a minimal generating subset of \( W_r \). The set of simple roots can be chosen as

\[
\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = l_0 - l_1 - l_2 - l_3, \quad \alpha_i = l_{i-1} - l_i, \quad i \geq 4.
\]

The blow up morphism \( X_r \to X_{r-1} \) determines an isometric embedding of the Picard lattices \( \text{Pic}(X_{r-1}) \hookrightarrow \text{Pic}(X_r) \). This induces the embeddings for root systems, simple roots and Weyl groups \( W_r \). For \( r \geq 3 \), the Dynkin diagram of \( R_r \) can be considered as the subgraph on the vertices \( \alpha_i \ (i \leq r) \) of the following graph:
In particular, we obtain $R_3 = A_2 \times A_1, R_4 = A_4, R_5 = D_5, R_6 = E_6, R_7 = E_7, R_8 = E_8$.

Denote by $\varpi_1, \ldots, \varpi_r$ the dual basis to the $\mathbb{Z}$-basis $-\alpha_1, \ldots, -\alpha_r$. Each $\varpi_i$ is the highest weight of an irreducible representation of $G(R_r)$ which is called a fundamental representation. We shall denote by $V(\varpi)$ the representation space of $G(R_r)$ with the highest weight $\varpi$.

**Definition 2.2.** A dominant weight $\varpi$ is called minuscule if all weights of $V(\varpi)$ are nonzero and the $W_r$-orbit of the highest weight vector is a $k$-basis of $V(\varpi)$ [G/P-I]. A dominant weight $\varpi$ is called quasiminuscule [G/P-III], if all nonzero weights of $V(\varpi)$ have multiplicity 1 and form an $W_r$-orbit of $\varpi$ (the zero weight of $V(\varpi)$ may have some positive multiplicity).

One can see from the explicit description of the root systems $R_r$ that $\varpi_r$ is minuscule for $3 \leq r \leq 7$, and $\varpi_8$ is quasiminuscule.

The dimension $d_r$ of of the irreducible representation $V(\varpi_r)$ of $G(R_r)$ is given by the following table:

| $r$ | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|
| $d_r$ | 10 | 16 | 27 | 56 | 248 |

We will need the following statement:

**Proposition 2.3.** Let $D$ be a divisor on a Del Pezzo surface $X_r$ ($2 \leq r \leq 8$) such that $(D, E) \geq 0$ for every exceptional curve $E \subset X_r$. Then the following statements hold:

(i) the linear system $|D|$ has no base points on any exceptional curve $E \subset X_r$;
(ii) if $r \leq 7$, then the linear system $|D|$ has no base points on $X_r$ at all.

**Proof.** Induction on $r$. If $r = 2$, then there exists exactly 3 exceptional curves $E_0, E_1, E_2$, whose classes in the standard basis are $l_0 - l_1 - l_2, l_1, l_2$. Moreover $[E_0], [E_1]$ and $[E_3]$ form a basis of the Picard lattice $Pic(X_2)$. The dual basis w.r.t. the intersection form is $l_0, l_0 - l_1, l_0 - l_2$. Therefore the above conditions on $D$ imply that

$$[D] = n_0 l_0 + n_1 (l_0 - l_1) + n_2 (l_0 - l_2), \quad n_0, n_1, n_2 \in \mathbb{Z}_{\geq 0}$$

So it is sufficient to check that the linear systems with the classes $l_0, l_0 - l_1, l_0 - l_2$ have no base points. The latter immediately follows from the fact that the first system defines the birational morphism $X_2 \to \mathbb{P}^2$ contracting $E_1$ and $E_2$, the second and third linear systems define conic bundle fibrations over $\mathbb{P}^1$.

For $r > 2$, we consider a second induction on $\deg D = (D, -K)$.

If there is an exceptional curve $E \subset X_r$ with $(D, E) = 0$, then the invertible sheaf $\mathcal{O}(D)$ is the inverse image of an invertible sheaf $\mathcal{O}(D')$ on the Del Pezzo surface $X_{r-1}$ obtained by the contraction of $E$. Since the pull-back of any exceptional curve on $X_{r-1}$ under the birational morphism $\pi_E : X_r \to X_{r-1}$ is again an exceptional curve on $X_r$,
we obtain that $D'$ satisfy all conditions of the proposition on $X_{r-1}$. By the induction assumption $(r - 1 \leq 7)$, $|D'|$ has no base points on $X_{r-1}$. Therefore, $|D| = |\pi_E^* D'|$ has no base points on $X_r$.

If there is no exceptional curve $E \subset X_r$ with $(D, E) = 0$, then we denote by $m$ the minimal intersection number $(D, E)$ where $E$ runs over all exceptional curves. Since we have $(E, -K) = 1$ for all exceptional curves, the divisor $D' := D + mK$ has nonnegative intersections with all exceptional curves and there exists an exceptional curve $E \subset X_r$ with $(D', E) = 0$. Since $\deg D' = (D', -K) = (D, -K) - m(K, K) < (D, -K) = \deg D$, by the induction assumption, we obtain that $|D'|$ is base point free. If $r \leq 7$, then the anticanonical linear system $|-K|$ has no base points. Therefore, $|D| = |D' + m(-K)|$ is also base point free. In the case $r = 8$, $|-K|$ does have a base point $p \in X_8$. However, $p$ cannot lie on an exceptional curve $E$, because the short exact sequence

$$0 \to H^0(X_8, \mathcal{O}(-K - E)) \to H^0(X_8, \mathcal{O}(-K)) \to H^0(E, \mathcal{O}(-K)|_E) \to 0$$

induces an isomorphism $H^0(X_8, \mathcal{O}(-K)) \cong H^0(E, \mathcal{O}(-K)|_E)$ (since $\deg(-K - E) = 0$ and $H^0(X_8, \mathcal{O}(-K - E)) = 0$).

\section{Generators of $\text{Cox}(X_r)$}

Let $\{E_1, \ldots, E_N\}$ be the set of all exceptional curves on a Del Pezzo surface $X_r$. We choose a $k$-rational point $p \in U := X_r \setminus \bigcup_{i=1}^N E_i$ and denote the ring $\text{Cox}(X_r, U, p)$ (see \cite{11}) simply by $\text{Cox}(X_r)$.

The ring

$$\text{Cox}(X_r) = \bigoplus_{[D] \in M_{\text{eff}}(X_r)} \text{Cox}(X_r)^{[D]}$$

is graded by the semigroup $M_{\text{eff}}(X_r) \subset \text{Pic}(X_r)$ of classes $[D]$ of effective divisors $D$ on $X_r$. There is a coarser grading on $\text{Cox}(X_r)$ given by

$$\text{Cox}(X_r)^d := \bigoplus_{\deg[D] = d} \text{Cox}(X_r)^{[D]}$$

where $\deg[D] := (D, -K)$.

\begin{proposition}
The graded ring $\text{Cox}(X_3)$ is isomorphic to a polynomial ring in 6 variables $k[x_1, \ldots, x_6]$, where $x_i$ are sections defining all 6 exceptional curves on $X_3$.
\end{proposition}

\begin{proof}
The Del Pezzo surface $X_3$ is a toric variety which can be described as the blow-up of 3 torus invariant points $(1:0:0), (0:1:0)$ and $(0:0:1)$ in $\mathbb{P}^2$. So we can apply a general result of Cox on toric varieties \cite{14} (see also \cite{13}).
\end{proof}

\begin{theorem}
For $3 \leq r \leq 8$, the ring $\text{Cox}(X_r)$ is generated by elements of degree 1. If $r \leq 7$, then the generators of $\text{Cox}(X_r)$ are global sections of invertible sheaves defining the exceptional curves. If $r = 8$, then we should add to the above set of generators two linearly independent global sections of the anticanonical sheaf on $X_8$.
\end{theorem}

\begin{proof}
Induction on $r$. The case $r = 3$ is settled by the previous proposition.

For $r > 3$ we choose an effective divisor $D$ on $X_r$. We call a section $s \in H^0(X_r, \mathcal{O}(D))$ a \textit{distinguished global section} if its support is contained in the union of exceptional curves
of $X_r$ ($r \leq 7$), or if its support is contained in the union of exceptional curves of $X_8$ and some anticanonical curves on $X_8$. Our purpose is to show that the vector space $H^0(X_r, \mathcal{O}(D))$ is spanned by all distinguished global sections.

This will be proved by induction on $\deg D := (D, -K) > 0$.

We consider several cases:

- If there exists an exceptional curve $E$ such that $(D, E) < 0$, then $H^0(X_r, \mathcal{O}(D)|_E) = 0$ and it follows from the exact sequence

$$H^1(X_r, \mathcal{O}(D)|_E) \to H^0(X_r, \mathcal{O}(D - E)) \to H^0(X_r, \mathcal{O}(D)) \to 0$$

that the multiplication by a non-zero distinguished global section of $\mathcal{O}(E)$ induces an epimorphism $H^0(X_r, \mathcal{O}(D - E)) \to H^0(X_r, \mathcal{O}(D))$. Since $\deg (D - E) = \deg D - 1$, using the induction assumption for $D' = D - E$, we obtain the required statement for $D$.

- If there exists an exceptional curve $E$ such that $(D, E) = 0$, then $\mathcal{O}(D)$ is the inverse image of a sheaf $\mathcal{O}(D')$ on the Del Pezzo surface $X_{r-1}$ obtained by the contraction of $E$. Therefore we have an isomorphism $H^0(X_r, \mathcal{O}(D)) \cong H^0(X_{r-1}, \mathcal{O}(D'))$ and, by the induction assumption for $r - 1$, we obtain the required statement for $D$, because distinguished global sections of $\mathcal{O}(D')$ lift to distinguished global sections of $\mathcal{O}(D)$.

- If $D = -K$, (or, equivalently, if $(D, E) = 1$ for every exceptional curve $E$), then $\mathcal{O}(D)|_E$ is isomorphic to $\mathcal{O}_E(1)$ and we have $H^1(X_r, \mathcal{O}(D)|_E) = 0$ together with the exact sequence

$$0 \to H^0(X_r, \mathcal{O}(D - E)) \to H^0(X_r, \mathcal{O}(D)) \to H^0(X_r, \mathcal{O}(D)|_E) \to 0,$$

where $H^0(X_r, \mathcal{O}(D)|_E)$ is 2-dimensional. Since $\deg (D - E) = \deg D - 1 < \deg D$, we can apply the induction assumption for $D' = D - E$. It remains show that there exists two linearly independent distinguished global sections of $\mathcal{O}(D)$ such that their restriction to $E$ are two linearly independent global sections of $\mathcal{O}(D)|_E$. We describe these two distinguished sections explicitly for each value of $r \in \{4, 5, 6, 7, 8\}$. Without loss of generality we can assume that $[E] = l_1$.

If $r = 4$, then we write the anticanonical class $-K = 3l_0 - l_1 - \cdots - l_4$ in the following two ways:

$$-K = (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_2 - l_3) + l_2 + l_3 = (l_0 - l_1 - l_3) + (l_0 - l_2 - l_4) + (l_0 - l_2 - l_3) + l_2 + l_3.$$

These two decompositions of $-K$ determine two distinguished global sections of $\mathcal{O}(-K)$ with support on 5 exceptional curves. The projections of these sections under the morphism $X_4 \to \mathbb{P}^2$ are shown below in Figure 1.

The restriction of the first section to $E$ vanishes at the intersection point $q_1$ of $E$ with the exceptional curve with the class $l_0 - l_1 - l_2$. The restriction of the second section to $E$ vanishes at the intersection point $q_2$ of $E$ with the exceptional curve with the class $l_0 - l_1 - l_3$. It is clear that $q_1 \neq q_2$. So the distinguished anticanonical sections are linearly independent.
If $r = 5$, then we write the anticanonical class as

$$-K = 3l_0 - l_1 - \cdots - l_5$$

$$= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_4 - l_5) + l_4$$

$$= (l_0 - l_1 - l_5) + (l_0 - l_2 - l_3) + (l_0 - l_3 - l_4) + l_3.$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of $E$ with the exceptional curves belonging to the classes $l_0 - l_1 - l_2$ and $l_0 - l_1 - l_5$.

If $r = 6$, then we write the anticanonical class as

$$-K = 3l_0 - l_1 - \cdots - l_6$$

$$= (l_0 - l_1 - l_2) + (l_0 - l_3 - l_4) + (l_0 - l_5 - l_6)$$

$$= (l_0 - l_1 - l_6) + (l_0 - l_4 - l_5) + (l_0 - l_3 - l_6).$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of $E$ with the exceptional curves belonging to the classes $l_0 - l_1 - l_2$ and $l_0 - l_1 - l_6$.

If $r = 7$, then we write the anticanonical class as

$$-K = 3l_0 - l_1 - \cdots - l_7$$

$$= (2l_0 - l_1 - l_2 - l_3 - l_4 - l_5) + (l_0 - l_6 - l_7)$$

$$= (2l_0 - l_7 - l_6 - l_5 - l_4 - l_3) + (l_0 - l_2 - l_1).$$

The corresponding distinguished anticanonical sections vanish at two different intersection points of $E$ with the exceptional curves belonging to the classes $2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$ and $l_0 - l_2 - l_1$.

If $r = 8$, then $\deg D - E = 0$. Therefore, $H^0(X_8, \mathcal{O}(D - E)) = 0$ (see the proof of [2.8]) and we have an isomorphism

$$H^0(X_8, \mathcal{O}(D)) \cong H^0(X_8, \mathcal{O}(D)|_E).$$

So $H^0(X_8, \mathcal{O}(D)|_E)$ is generated by the restrictions of the anticanonical sections and we’re done.
• If \((D, E) \geq 1\) for all exceptional curves \(E\) and \(D \neq -K\), then we denote by \(m\) the minimum of the numbers \((D, E)\) for all exceptional curves. Let \(E_0\) be an exceptional curve such that \((D, E_0) = m \geq 1\). We define \(D' = D - E_0\) and \(D'' := D + mK\). By 2.3, \(|D'|\) and \(|D''|\) have no base points (if \(r \leq 7\)). In particular, \(D''\) is represented by an effective divisor. Since \(\deg D'' = \deg D - m(K, K) < \deg D\), \(D''\) can be seen as zero of a distinguished global section \(s \in H^0(X_r, \mathcal{O}(D + mK))\) whose support does not contain the exceptional curve \(E_0\) (if \(r \leq 8\)). We have the short exact sequence

\[
0 \to H^0(X_r, \mathcal{O}(D')) \to H^0(X_r, \mathcal{O}(D)) \to H^0(X_r, \mathcal{O}(D)|_{E_0}) \to 0.
\]

By the induction assumption, the space \(H^0(X_r, \mathcal{O}(D'))\) is generated by distinguished global sections. It remains to show that there exist distinguished global sections of \(\mathcal{O}(D)\) such that their restriction to \(E_0\) generate the space \(H^0(X_r, \mathcal{O}(D)|_{E_0})\). Since \((-mK, E_0) = (D, E_0) = m\), the space \(H^0(X_r, \mathcal{O}(D)|_{E_0})\) is isomorphic to \(H^0(X_r, \mathcal{O}(-mK)|_{E_0})\). Since \((D'', E_0) = 0\) the distinguished global section \(s \in H^0(X_r, \mathcal{O}(D + mK))\) is nonzero at any point of \(E_0\). Therefore the multiplication by the distinguished global section \(s\) defines a homomorphism

\[
H^0(X_r, \mathcal{O}(-mK)) \to H^0(X_r, \mathcal{O}(D))
\]

whose restriction to \(E_0\) is an isomorphism

\[
H^0(X_r, \mathcal{O}(-mK)|_{E_0}) \cong H^0(X_r, \mathcal{O}(D)|_{E_0}).
\]

Therefore, it is enough to show that restrictions of the distinguished global sections of \(\mathcal{O}(-mK)\) to \(E_0\) generate the space \(H^0(X_r, \mathcal{O}(-mK)|_{E_0})\). Our previous considerations have shown this for \(m = 1\). The general case \(m \geq 1\) follows now immediately from the fact that the homomorphism \(H^0(X_r, \mathcal{O}(-K)) \to H^0(E_0, \mathcal{O}_{E_0}(1))\) is surjective and the space \(H^0(E_0, \mathcal{O}_{E_0}(m))\) is spanned by tensor products of \(m\) elements from \(H^0(E_0, \mathcal{O}_{E_0}(1))\).

**Corollary 3.3.** The semigroup \(M_{\text{eff}}(X_r) \subset \text{Pic}(X_r)\) of classes of effective divisors on a Del Pezzo surfaces \(X_r\) \((2 \leq r \leq X_r)\) is generated by elements of degree 1. These elements are exactly the classes of exceptional curves if \(r \leq 7\) and the classes of exceptional curves together with the anticanonical class for \(r = 8\).

**Proposition 3.4.** If \(D\) is an effective divisor of degree \(\geq 2\) on \(X_8\), then the vector space \(H^0(X_8, \mathcal{O}(D))\) is spanned by distinguished global sections of \(\mathcal{O}(D)\) with supports only on exceptional curves.

**Proof.** By 3.2 it is sufficient to check the statement for \(D = -2K\) and for \(D = -K + E\) for any exceptional curve. The latter case immediately follows from 3.2 because \(D = -K + E\) is the pull back of the anticanonical sheaf on \(X_7\) obtained by the contraction of \(E\). In the case \(D = -2K\), we obtain 120 distinguished global sections of \(\mathcal{O}(D)\) from 120 pairs of exceptional curves \(E_i, E_j\) such that \((E_i, E_j) = 3:\)

\[
-2K = 6l_0 - 2l_1 - \ldots - 2l_8 = l_1 + (6l_0 - 3l_1 - 2l_2 \ldots - 2l_8).
\]

It is well-known (see e.g. [Dem]) that \(X_8\) can be realized as a hypersurface of degree 6 in the weighted projective space \(\mathbb{P}(3, 2, 1, 1)\). In particular, the linear system \(|-2K|\) defines
a double covering of $X_8$ over a singular quadratic cone $Q \cong \mathbb{P}(2,1,1) \subset \mathbb{P}^3$. The single
singular point $p \in Q$ is the image of the base-point $b \in X_8$ of $[-K]$ on $X_8$. Let $C \subset Q$ be
the ramification locus ($C$ is a curve of degree 6 in $\mathbb{P}(2,1,1)$). Then 120 pairs of exceptional
curves $E_i, E_j$ on $X_8$ such that $[E_i] + [E_j] = 2[-K]$ one-to-one correspond to conics in
$\mathbb{P}(2,1,1)$ which are 3-tangent to the ramification curve $C$. Since every such conic in $Q$
is uniquely determined as $Q \cap H$ for some plane $H \subset \mathbb{P}^3$. Therefore, the distinguished
sections in $H^0(X_8, \mathcal{O}(-2K))$ can be identified (up to a scalar multiple) with the above
planes $H \subset \mathbb{P}^3$. It remains to show that all these 120 planes $H$ cannot pass through the
some common point $x \in \mathbb{P}^3$ for a generic choice of the sextic $C \subset \mathbb{P}(2,1,1)$. The later
can be checked by standard dimension arguments. □

Remark 3.5. Since $H^0(X_r, \mathcal{O}(E))$ is 1-dimensional for each exceptional curve $E \subset X_r$, we
can choose a nonzero section $x_E \in H^0(X_r, \mathcal{O}(E))$ which is determined up to multiplication
by a nonzero scalar. Therefore the affine algebraic variety $A(X_r) := \text{Spec} \text{Cox}(X_r)$
is canonically embedded into the projective space $\mathbb{P}(\mathcal{I}(X_r), \mathcal{O}(E))$ which is determined up to
multiplication by a nonzero scalar. Therefore the affine algebraic variety $A(X_r) := \text{Spec} \text{Cox}(X_r)$
is embedded into the affine space $\mathbb{A}^{N_r}$ on which the maximal torus $T_r \subset G(R_r)$ acts in a
canonical way such that the space $A^{N_r}$ can identified with the representation space $V(\pi_r)$
of the algebraic group $G(R_r)$ (if $r \leq 7$). In the case $r = 8$, all 240 exceptional curves
on $X_8$ can be similarly identified with all non-zero weights of the adjoint representation
of $G(E_8)$ in $V(\pi_r)$. The space $V(\pi_r)$ contains the weight-0 subspace of dimension 8, but the ring $\text{Cox}(X_r)$ has only 2-dimensional space of anticanonical sections. Thus, we
cannot identify the degree-1 homogeneous component of $\text{Cox}(X_8)$ with the representation
space $V(\pi_8)$ of $G(E_8)$.

Since the kernel of the surjective homomorphism

$$\text{deg} : \text{Pic}(X_r) \to \mathbb{Z},$$

can be identified with the character group $\chi(T_r)$ of a maximal torus $T_r \subset G(R_r)$ and the
torus $T_r$ acts on the homogeneous space $G(R_r)/P(R_{r-1})$ embedded into the projective
space $\mathbb{P}V(\pi_r)$ we obtain a natural $\text{Pic}(X_r)$-grading of the homogeneous coordinate ring
of the projective variety $G(R_r)/P(R_{r-1})$.

Theorem 3.6. Let $\lambda$ be an element in $\text{Pic}(X_r)$. The weight-$\lambda$ subspace in the homo-
geeous coordinate ring of the projective variety $G(R_r)/P(R_{r-1})$ is nonzero if and only if
$\lambda$ is represented by an effective divisor on $X_r$ (i.e., $\lambda \in M_{\text{eff}}(X_r)$).

Proof. It is known that the projective variety $G(R_r)/P(R_{r-1})$ is arithmetically normal and
Cohen-Macaulay [D-L], [G/P-V]. In particular, the homogeneous coordinate ring of
$G(R_r)/P(R_{r-1})$ is generated by elements of degree 1. Therefore, the weight-$\lambda$ subspace in
the coordinate ring is nonzero if and only if $\lambda$ is a nonnegative integral linear combination
of $\text{Pic}(X_r)$-weights having positive multiplicity in $V(\pi_r)$. By 3.3 and 3.5 the latter is
equivalent to $\lambda \in M_{\text{eff}}(X_r)$.

4. Quadratic relations in $\text{Cox}(X_r)$

Let us denote $P(X_r) := \text{Proj} \text{Cox}(X_r)$. If $4 \leq r \leq 7$, then the projective variety $P(X_r)$
is canonically embedded into the projective space $\mathbb{P}^{N_r-1}$ ($N_r$ is the number of exceptional
curves on $X_r$). The affine variety $A(X_r) \subset A^{N_r}$ is the affine cone over $P(X_r)$.
Proposition 4.1. The ring \( \text{Cox}(X_4) \) is isomorphic to the subring of all \( 3 \times 3 \)-minors of a generic \( 3 \times 5 \)-matrix. In particular, the projective variety \( \mathbb{P}(X_4) \subset \mathbb{P}^9 \) is isomorphic to the Plücker embedding of the Grassmannian \( Gr(3,5) \).

Proof. In order to describe the multiplication in \( \text{Cox}(X_4) \), one needs to choose a basis in \( \text{Pic}(X_4) \).

Let \( x : y : z \) be the homogeneous coordinates on \( \mathbb{P}^2 \). We choose the basis \( l_0, \ldots, l_4 \), as in Section 2, i.e., \( l_0 \) is the preimage of the line \( z = 0 \) at infinity, \( l_1, l_2, l_3, l_4 \) are classes of the exceptional fibers over 4 points \( p_1, \ldots, p_4 \in \mathbb{P}^2 \). We identify the representatives of each class in \( \text{Pic}(X_4) \) with the subsheaves \( \mathcal{O}(\sum_{i=0}^4 m_i l_i) \) of the constant sheaf \( \mathcal{K}(X_4) \) of rational functions on \( X_4 \). Then the ring multiplication in \( \text{Cox}(X_4) \) is just the multiplication of the corresponding rational functions in \( \mathcal{K}(X_4) \).

Let \( (x_i : y_i : z_i) \) be the coordinates of the blown-up point \( p_i \in \mathbb{P}^2 \) \((i = 1, \ldots, 4)\). Consider the \( 3 \times 5 \)-matrix

\[
M = \begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x/z \\
y_1 & y_2 & y_3 & y_4 & y/z \\
z_1 & z_2 & z_3 & z_4 & 1
\end{pmatrix}.
\]

For any 3-element subset \( I = \{i,j,k\} \subset \{1, \ldots, 5\} \), we denote by \( M_I \) the maximal minor of \( M \) consisting of the columns with numbers in \( I \) taken in the natural order.

We choose the rational functions in \( \mathcal{K}(X_4) \) representing the generators \( x_E \) of \( \text{Cox}(X_4) \) as follows:

\[
x_{l_1} := M_{\{2,3,4\}}, \quad x_{l_2} := M_{\{1,3,4\}}, \quad x_{l_3} := M_{\{1,2,4\}}, \quad x_{l_4} := M_{\{1,2,3\}},
\]

\[
x_{l_0-l_i-l_j} := M_{\{i,j,5\}}, \quad 1 \leq i < j \leq 4.
\]

All these functions are non-zero because the points \( p_1, \ldots, p_4 \) are in general position.

It is known that the generators of the homogeneous coordinate ring of \( G(3,5) \) are naturally identified with the maximal minors of a generic \( 3 \times 5 \)-matrix. Consider the homomorphism \( \varphi \) of the homogeneous coordinate ring of \( G(3,5) \) to \( \text{Cox}(X_4) \), which sends these generic minors into the corresponding minors of the matrix \( M \) above. Since \( \text{Cox}(X_4) \) is generated by \( \{x_E\} \), this homomorphism is surjective. By \[ \text{3.0} \] \( \varphi \) respects the \( \text{Pic}(X_4) \)-grading (in particular, \( \varphi \) respects the \( \mathbb{Z}_{\geq 0} \)-grading as well). The surjectivity of \( \varphi \) induces a closed embedding of \( \mathbb{P}(X_4) \) into \( G(3,5) \). Since both varieties are irreducible of dimension 6 (see \[ \text{1.4} \]), we obtain an isomorphism of \( \mathbb{P}(X_4) \) and \( G(3,5) \) as subvarieties of \( \mathbb{P}^9 \). Therefore \( \varphi \) is an isomorphism of the homogeneous coordinate ring of \( G(3,5) \) and \( \text{Cox}(X_4) \). In particular, \( \text{Cox}(X_4) \) is defined by 5 quadratic Plücker relations. One of these relations is

\[
M_{\{1,2,5\}} M_{\{3,4,5\}} - M_{\{1,3,5\}} M_{\{2,4,5\}} + M_{\{1,4,5\}} M_{\{2,3,5\}} = 0.
\]

The article \[ \text{G/P-I} \] describes a \( k \)-basis for the homogeneous coordinate ring of \( G/P \) in the case, when \( P \) is a maximal parabolic subgroup containing a Borel subgroup \( B \) such that the fundamental weight \( \varpi \) corresponding to \( P \) is minuscule (see \[ \text{2.2} \]). It also shows that this ring is always defined by quadratic relations.
A way to write explicitly the quadratic relations for the orbit of the highest weight vector for any representation of a semisimple Lie group is given in \[\text{[L-T]}\]. A more geometric approach to these quadratic equations is contained in the proof of Theorem 1.1 in \[\text{[L-T]}\]:

**Proposition 4.2.** The orbit \(G/P_\pi\) of the highest weight vector in the projective space \(\mathbb{P}V(\pi)\) is the intersection of the second Veronese embedding of \(\mathbb{P}V(\pi)\) with the subrepresentation \(V(2\pi)\) of the symmetric square \(S^2V(\pi)\). Moreover, these quadratic relations generate the ideal of \(G/P_\pi \subset \mathbb{P}V(\pi)\).

We expect that the following general statement is true:

**Conjecture 4.3.** The ideal of relations between the degree-1 generators of \(\text{Cox}(X_r)\) is generated by quadrics for \(4 \leq r \leq 8\).

For any exceptional curve \(E \subset X\), we consider the open chart \(U_E \subset \mathbb{P}^{N_r-1}\) defined by the condition \(x_E \neq 0\). Thus, we obtain an open covering of \(\mathbb{P}(X_r)\) by \(N_r\) affine subsets \(U_E \cap \mathbb{P}(X_r)\).

**Proposition 4.4.** Let \(X_{r-1}\) the Del Pezzo surface obtained by the contraction of \(E\) on \(X_r\). Then there exist a natural isomorphism

\[
U_E \cap \mathbb{P}(X_r) \cong \mathbb{A}(X_{r-1}).
\]

**Proof.** Let \(\pi : X_r \to X_{r-1}\) be the contraction of \(E\). Then we obtain the ring homomorphism \(\pi^* : \text{Cox}(X_{r-1}) \to \text{Cox}(X_r)\). We shall show that the localization \(\text{Cox}(X_r)_{x_E}\) of the ring \(\text{Cox}(X_r)\) by the element \(x_E\) can be identified with the Laurent polynomial extension of \(\pi^* \text{Cox}(X_{r-1})\) by \(x_E\), i.e. there exist a ring isomorphism

\[
\text{Cox}(X_r)_{x_E} \cong \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].
\]

For simplicity, we assume that \([E] = l_r\) and \(\{l_0, \ldots, l_{r-1}\}\) is the pull-back of the standard basis in \(\text{Pic}(X_{r-1})\). We remark that any divisor class

\[
[D] = m_r l_r + \sum_{i=0}^{r-1} m_i l_i \in \text{Pic}(X_r)
\]

is uniquely represented as sum \(m_r[E] + [D']\) where \([D'] = \sum_{i=0}^{r-1} m_i l_i \in \pi^*(\text{Pic}(X_{r-1}))\).

Using \(\pi^*\), we identify two fields of rational functions \(\mathcal{K}(X_{r-1})\) and \(\mathcal{K}(X_r)\). This identification allows us to consider the vector space

\[
H^0(X_r, \mathcal{O}(m_r l_r + \sum_{i=0}^{r-1} m_i l_i))
\]

as a subspace of

\[
H^0(X_{r-1}, \mathcal{O}(\sum_{i=0}^{r-1} m_i l_i))x_E^{m_r} \subset \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].
\]

For fixed integers \(m_0, m_1, \ldots, m_{r-1}\), the embedding of the vector spaces

\[
H^0(X_r, \mathcal{O}(m_r l_r + \sum_{i=0}^{r-1} m_i l_i)) \hookrightarrow H^0(X_{r-1}, \mathcal{O}(\sum_{i=0}^{r-1} m_i l_i))x_E^{m_r}
\]
is an isomorphism for sufficiently large $m_r$. Moreover, this embedding of vector spaces respects the multiplications in $\text{Cox}(X_r)$ and in $\pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}]$. Thus, we obtain an embedding of rings

$$\text{Cox}(X_r) \hookrightarrow \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

On the other hand, it is clear that $\pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}]$ is a subring of the localization $\text{Cox}(X_r)_E$. Thus, we get an isomorphism

$$\text{Cox}(X_r)_E \cong \pi^* \text{Cox}(X_{r-1})[x_E, x_E^{-1}].$$

Now we remark that the coordinate ring of the affine variety $U_E \cap \mathbb{P}(X_r)$ is degree-0 component of $\text{Cox}(X_r)_E$. By the above isomorphism, this component is isomorphic to $\text{Cox}(X_{r-1})$.

**Corollary 4.5.** The singular locus of the algebraic varieties $\mathbb{P}(X_r)$ and $\mathbb{A}(X_r)$ has codimension 7.

**Proof.** Since $\mathbb{A}(X_3) \cong \mathbb{A}^6$, we obtain that $\mathbb{P}(X_4)$ is a smooth variety covered by 10 affine charts which are isomorphic to $\mathbb{A}^6$. Using the isomorphism $\mathbb{P}(X_4) \cong G(3, 5)$ (see [4]), we obtain that $\mathbb{A}(X_4)$ has an isolated singularity at 0. Therefore, the singular locus of $\mathbb{P}(X_5)$ consists of 16 isolated points. The singular locus of $\mathbb{P}(X_6)$ is 1-dimensional and the singular locus of $\mathbb{P}(X_7)$ is 2-dimensional etc. \qed

**Definition 4.6.** A divisor class $[D]$ is called a ruling if it can be written as a sum of two classes of exceptional curves $[E_i] + [E_j]$ such that $(E_i, E_j) = 1$, or, equivalently, if $D$ satisfies the conditions $(D, D) = 0$, $(D, -K) = 2$. The invertible sheaf corresponding to a ruling determines a conic bundle morphism $X_r \rightarrow \mathbb{P}^1$.

**Remark 4.7.** Lemma 5.3 of [F-M] says that the Weyl group acts transitively on rulings.

Each ruling $[D]$ can be represented by $r - 1$ different ways as a sum of two classes of exceptional curves corresponding to degenerate fibers of the conic bundle $X_r \rightarrow \mathbb{P}^1$. Thus, we obtain $r - 1$ distinguished sections in the 2-dimensional space $H^0(X_r, O(D))$. If $r \geq 4$, then for each ruling $[D]$, we obtain in this way $r - 3$ linearly independent quadratic relations between generators of $\text{Cox}(X_r)$.

**Remark 4.8.** We note that $\text{Pic}(X_4)$ has exactly 5 rulings. Each such a ruling defines a Plücker quadric (see the proof of [4]).

We cannot expect in general that all quadratic relations among generators are coming from rulings. However, the following statement is true:

**Theorem 4.9.** For $4 \leq r \leq 6$, the ring $\text{Cox}(X_r)$ is defined by the radical of the ideal generated by the quadratic relations corresponding to rulings.

**Proof.** Let $Z_r \subset \mathbb{A}^{N_r}$ is the affine subvariety defined by the quadratic relations coming from rulings. We want to show that $Z_r = \mathbb{A}(X_r)$ ($4 \leq r \leq 6$).

For $r = 4$, the statement follows from [4].

Obviously, the zero $0 \in \mathbb{A}^{N_r}$ is common point of $Z_r$ and $\mathbb{A}(X_r)$ for all $r$. Consider the affine open coverings of $Z_r \setminus \{0\}$ and $\mathbb{A}(X_r) \setminus \{0\}$ defined by affine open subsets $x_E \neq 0$, where $E$ runs over all exceptional curves of $X_r$. Using the induction on $r$ and Proposition
we want to show that $Z_r \cap U_E = \mathbb{A}(X_r) \cap U_E$ for each exceptional curve. For this purpose, it is important to remark that the affine coordinate ring of $Z_r \cap U_E$ is generated by all elements $x_F/x_E$ such that $(E, F) = 0$. For $r = 5, 6$, the last property follows from the fact that if $(E, E') > 0$ for two exceptional curves $E, E'$ on $X_r$, then $(E, E') = 1$, i.e., $[E] + [E']$ is a ruling and there exists a ruling quadratic relation

$$x_E x_{E'} = \sum_i a_i X_{E_i} X_{E'_i},$$

where all exceptional curves $E_i, E'_i$ do not intersect $E$. The last property shows that $Z_r \cap U_E \cong Z_{r-1} \times (\mathbb{A}^1 \setminus \{0\})$. It follows from the proof of 4.4 that

$$\mathbb{A}(X_r) \cap U_E \cong \mathbb{A}(X_{r-1}) \times (\mathbb{A}^1 \setminus \{0\}).$$

By induction, we have the equality $Z_{r-1} = \mathbb{A}(X_{r-1})$. This implies the equality $Z_r \cap U_E = \mathbb{A}(X_r) \cap U_E$ for each exceptional curve. Thus, $Z_r = \mathbb{A}(X_r)$.

\[\square\]

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