REGULARITY BOOTSTRAPPING FOR FOURTH ORDER NONLINEAR ELLIPTIC EQUATIONS

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Abstract. We consider nonlinear fourth order elliptic equations of double divergence type. We show that for a certain class of equations where the nonlinearity is in the Hessian, solutions that are $C^{2,\alpha}$ enjoy interior estimates on all derivatives.

1. Introduction

In this paper, we develop Schauder and bootstrapping theory for solutions to fourth order nonlinear elliptic equations of the following double divergence form

$$\int_{\Omega} a^{ij,kl}(D^2 u) u_{ij} \eta_{kl} dx = 0, \quad \forall \eta \in C^\infty_0(\Omega)$$

in $B_1 = B_1(0)$. For the Schauder theory, we require the standard Legendre-Hadamard ellipticity condition,

$$a^{ij,kl}(D^2 u(x)) \xi_{ij} \xi_{kl} \geq \Lambda |\xi_{rs}|^2$$

while in order to bootstrap, we will require the following condition:

$$b^{ij,kl}(D^2 u(x)) = a^{ij,kl}(D^2 u(x)) + \frac{\partial a^{pq,kl}}{\partial u_{ij}}(D^2 u(x)) u_{pq}(x)$$

satisfies

$$b^{ij,kl}(D^2 u(x)) \xi_{ij} \xi_{kl} \geq \Lambda_1 \|\xi\|^2.$$

Our main result is the following: Suppose that conditions (1.1) and (1.3) are met on some open set $U \subseteq S^{n \times n}$ (space of symmetric matrices). If $u$ is a $C^{2,\alpha}$ solution with $D^2 u(B_1) \subset U$, then $u$ is smooth on the interior of the domain $B_1$.

One example of such an equation is the Hamiltonian Stationary Lagrangian equation, which governs Lagrangian surfaces that minimize the area functional

$$\int_{\Omega} \sqrt{\det(I + (D^2 u)^T D^2 u)} dx.$$
among potential functions $u$. (cf. [Oh93], [SW03, Proposition 2.2]). The minimizer satisfies a fourth order equation, that, when smooth, can be factored into a a Laplace type operator on a nonlinear quantity. Recently in [CW16], it is shown that a $C^2$ solution is smooth. The results in [CW16] are the combination of an initial regularity boost, followed by applications of the second order Schauder theory as in [CC95].

More generally, for a functional $F$ on the space of matrices, one may consider a functional of the form

$$\int_M F(D^2 u) dx.$$ 

The Euler-Lagrange equation will generically be of the following double-divergence type:

$$\frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{\partial F}{\partial u_{ij}}(D^2 u) \right) = 0. \tag{1.6}$$

Equation (1.6) need not factor into second order operators, so it may be genuinely a fourth order double-divergence elliptic type equation. It should be noted that in general, (1.6) need not take the form of (1.1). It does when $F(D^2 u)$ can be written as a function of $D^2 u^T D^2 u$ (as for example (1.5)). Our results in this paper apply to a class of Euler-Lagrange equations arising from such functionals. In particular, we will show that if $F$ is a convex function of $D^2 u$ and a function of $D^2 u^T D^2 u$ (such as 1.5 when $|D^2 u| \leq 1$) then $C^{2,\alpha}$ solutions will be smooth.

The Schauder Theory for second order divergence and non-divergence type elliptic equations is by now well-developed, see [HL11], [GT01] and [CC95]. For higher order non-divergence equations, Schauder theory is available, see [Sim97]. However, for higher order equations in divergence form, much less is known. One expects the results to be different: For second order equations, solutions to divergence type equations with $C^\alpha$ coefficients are known to be $C^{1,\alpha}$, [HL11, Theorem 3.13], whereas for non-divergence equations, solutions will be $C^{2,\alpha}$ [GT01, Chapter 6]. Recently, Dong and Zhang [DZ15] have obtained general Schauder theory results for parabolic equations (of order $2m$) in divergence form, where the time coefficients are allowed to be merely measurable. Their proof (like ours) is in the spirit of Campanato techniques, but requires smooth initial conditions. Our result is aimed at showing that weak solutions are in fact smooth. Classical Schauder theory for general systems has been developed, [MJ09, Chapter 5, 6]. However, it is non-trivial to apply the general classical results to obtain
the result we are after. Even so, it is useful to focus on a specific class
of fourth order double-divergence operators, and offer random access to
the non-linear Schauder theory for these cases. Regularity for fourth
order equations remains an important developing area of geometric
analysis.

Our proof goes as follows: We start with a $C^{2,\alpha}$ solution of (1.1)
whose coefficient matrix is a smooth function of the Hessian of $u$. We
first prove that $u \in W^{3,2}$ by taking a difference quotient of (1.1) and
give a $W^{3,2}$ estimate of $u$ in terms of its $C^{2,\alpha}$ norm. Again by taking
difference a quotient and using the fact that now $u \in W^{3,2}$, we prove
that $u \in C^{3,\alpha}$.

Next, we make a more general proposition where we prove a $W^{3,2}$
estimate for $u \in W^{2,2}$ satisfying a uniformly elliptic equation of the
form
\[
\int (c^{ij,kl} u_{ik} + h^{jl}) \eta_{jl} dx = 0
\]
in $B_1(0)$, where $c^{ij,kl}, h^{kl} \in W^{1,2}(B_1)$ and $\eta$ is a test function in $B_1$.
Using the fact that $u \in W^{3,2}$, we prove that $u \in C^{3,\alpha}$ and also derive a
$C^{3,\alpha}$ estimate of $u$ in terms of its $W^{3,2}$ norm. Finally, using difference
quotients and dominated convergence, we achieve all higher orders of
regularity.

**Definition 1.1.** We say an equation of the form (1.1) is **regular on**
$U \subseteq S^{n \times n}$ when the coefficients of the equation satisfy the following
conditions on $U$:

1. The coefficients $a^{ij,kl}$ depend smoothly on $D^2u$.
2. The coefficients $a^{ij,kl}$ satisfy (1.2).
3. Either $b^{ijkl}$ or $-b^{ijkl}$ (given by (1.3)) satisfy (1.4).

The following is our main result.

**Theorem 1.2.** Suppose that $u \in C^{2,\alpha}(B_1)$ satisfies the following
fourth order equation
\[
\int_{B_1(0)} a^{ij,kl}(D^2u(x))u_{ij}(x)\eta_{kl}(x)dx = 0
\]
$\forall \eta \in C_0^\infty(B_1(0))$

If $a^{ij,kl}$ is regular on an open set containing $D^2u(B_1(0))$, then $u$ is
smooth on $B_r(0)$ for $r < 1$.

To prove this, we will need the following two Schauder type esti-
mates.
Proposition 1.3. Suppose $u \in W^{2,\infty}(B_1)$ satisfies the following

$$\int_{B_1(0)} \left[ c^{ij,kl}(x)u_{ij}(x) + f^{kl}(x) \right] \eta_{kl}(x)dx = 0$$

$\forall \eta \in C_0^\infty(B_1(0))$

where $c^{ij,kl}, f^{kl} \in W^{1,2}(B_1)$, and $c^{ij,kl}$ satisfies (1.2). Then $u \in W^{3,2}(B_{1/2})$ and

$$\|D^3u\|_{L^2(B_{1/2})} \leq C(\|u\|_{W^{2,\infty}(B_1)}, \|f^{kl}\|_{W^{1,2}(B_1)}, \|c^{ij,kl}\|_{W^{1,2}(\Lambda_1)}).$$

Proposition 1.4. Suppose $u \in C^{2,\alpha}(B_1)$ satisfies (1.7) in $B_1$ where $c^{ij,kl}, f^{kl} \in C^{1,\alpha}(B_1)$ and $c^{ij,kl}$ satisfies (1.2). Then we have $u \in C^{3,\alpha}(B_{1/2})$ with

$$\|D^3u\|_{C^{0,\alpha}(B_{1/4})} \leq C(1 + \|D^3u\|_{L^2(B_{3/4})})$$

and $C = C(\|c^{ij,kl}\|_{C^{\alpha}(B_1)}, \Lambda_1, \alpha)$ is a positive constant.

We note that the above estimates are appropriately scaling invariant: Thus we can use these to obtain interior estimates for a solution in the interior of any sized domain.

2. Preliminaries

We begin by considering a constant coefficient double divergence equation.

Theorem 2.1. Suppose $w \in H^2(B_r)$ satisfies the constant coefficient equation

$$\int c^{ik,jl}_0 w_{ik} \eta_{jl} dx = 0$$

$\forall \eta \in C_0^\infty(B_r(0)).$

Then for any $0 < \rho \leq r$ there holds

$$\int_{B_\rho} |D^2w|^2 \leq C_1(\rho/r)^n \|D^2w\|^2_{L^2(B_r)}$$

$$\int_{B_\rho} |D^2w - (D^2w)_\rho|^2 \leq C_2(\rho/r)^{n+2} \int_{B_r} |D^2w - (D^2w)_\rho|^2.$$

Here $(D^2w)_\rho$ is the average value of $D^2w$ on a ball of radius $\rho$.

Proof. By dilation we may consider $r = 1$. We restrict our consideration to the range $\rho \in (0, a]$ noting that the statement is trivial for $\rho \in [a, 1]$ where $a$ is some constant in $(0, 1/2)$. 
First, we note that $w$ is smooth \[Dri03, \text{Theorem 33.10}\]. Recall \[DK11, \text{Lemma 2, Section 4, applied to elliptic case}\]: For an elliptic 4th order $L_0$

$$L_0u = 0 \text{ on } B_R \implies \|Du\|_{L^\infty(B_R/4)} \leq C_3(\Lambda, n) \|u\|_{L^2(B_R)}.$$ 

We may apply this to the second derivatives of $w$ to conclude that

$$\|D^3w\|^2_{L^\infty(B_a)} \leq C_4(\Lambda, n) \int_{B_1} \|D^2w\|^2. \tag{2.2}$$

For small enough $a < 1$. Now

$$\int_{B_\rho} |D^2w|^2 \leq C_5(n) \rho^n \|D^2w\|^2_{L^\infty(B_a)}$$

$$= C_5\rho^n \inf_{x \in B_a} \sup_{y \in B_a} |D^2w(x) + D^2w(y) - D^2w(x)|^2$$

$$\leq C_5\rho^n \inf_{x \in B_a} \left[ D^2w(x) + 2a \|D^3w\|_{L^\infty(B_a)} \right]^2$$

$$\leq 2C_5\rho^n \left[ \inf_{x \in B_a} \|D^2w\|^2 + 4a^2 \|D^3w\|_{L^\infty(B_a)} \right]$$

$$\leq 2C_5\rho^n \left[ \frac{1}{|B_a|} \|D^2w\|^2_{L^2(B_a)} + 4a^2C_4 \|D^2w\|^2_{L^2(B_a)} \right]$$

$$\leq C_6(a, n) \rho^n \|D^2w\|^2_{L^2(B_1)}.$$ 

Similarly

$$\int_{B_\rho} |D^2w - (D^2w)_{x_0}|^2 \leq \int_{B_\rho} |D^2w - D^2w(x_0)|^2$$

$$\leq \int_{S^{n-1}} \int_0^\rho r^2 |D^3w|^2 r^{n-1} dr d\phi$$

$$= C_7\rho^{n+2} \|D^3w\|^2_{L^\infty(B_a)}. \tag{2.3}$$

Next, observe that (2.1) is purely fourth order, so the equation still holds when a second order polynomial is added to the solution. In particular, we may choose

$$D^2\tilde{w} = D^2w - (D^2w)_1$$

for $\tilde{w}$ also satisfying the equation. Then

$$D^3\tilde{w} = D^3w.$$
so

\begin{equation}
\|D^3w\|_{L^\infty(B_\rho)}^2 = \|D^3\bar{w}\|_{L^\infty(B_\rho)}^2 \\
\leq C_4 \int_{B_1} \|D^2\bar{w}\|^2 = C_4 \int_{B_1} \|D^2w - (D^2w)_1\|^2.
\end{equation}

We conclude from (2.4) and (2.3)

\[ \int_{B_\rho} |D^2w - (D^2w)_\rho|^2 \leq C_7 \rho^{n+2} C_4 \int_{B_1} \|D^2w - (D^2w)_1\|^2. \]

Next, we have a corollary to the above theorem.

**Corollary 2.2.** Suppose $w$ is as in the Theorem 2.1. Then for any $u \in H^2(B_r)$, and for any $0 < \rho \leq r$, there holds

\begin{equation}
\int_{B_\rho} |D^2u|^2 \leq 4C_1(\rho/r)^n \|D^2u\|_{L^2(B_r)}^2 + (2 + 8C_1) \|D^2v\|_{L^2(B_r)}^2.
\end{equation}

and

\begin{equation}
\int_{B_\rho} |D^2u - (D^2u)_\rho|^2 \leq 4C_2(\rho/r)^n \int_{B_r} |D^2u - (D^2u)_\rho|^2 \\
+ (8 + 16C_2) \int_{B_r} |D^2v|^2.
\end{equation}

**Proof.** Let $v = u - w$. Then (2.5) follows from direct computation:

\[ \int_{B_\rho} |D^2u|^2 \leq 2 \int_{B_\rho} |D^2w|^2 + 2 \int_{B_\rho} |D^2v|^2. \]

\[ \leq 2C_1(\rho/r)^n \|D^2w\|_{L^2(B_r)}^2 + 2 \int_{B_r} |D^2v|^2 \\
\leq 4C_1(\rho/r)^n \left[ \|D^2v\|_{L^2(B_r)}^2 + \|D^2u\|_{L^2(B_r)}^2 \right] + 2 \int_{B_r} |D^2v|^2 \\
= 4C_1(\rho/r)^n \|D^2u\|_{L^2(B_r)}^2 + 2[1 + 4C_1(\rho/r)^n] \|D^2v\|_{L^2(B_r)}^2. \]
Similarly
\[
\int_{B_\rho} |D^2 u - (D^2 u)_\rho|^2 \leq 2 \int_{B_\rho} |D^2 w - (D^2 w)_\rho|^2 + 2 \int_{B_\rho} |D^2 v - (D^2 v)_\rho|^2
\]
\[
\leq 2 \int_{B_\rho} |D^2 w - (D^2 w)_\rho|^2 + 8 \int_{B_\rho} |D^2 v|^2
\]
\[
\leq 2C_2(\rho/r)^{n+2} \int_{B_r} |D^2 w - (D^2 w)_r|^2 + 8 \int_{B_\rho} |D^2 v|^2
\]
\[
\leq 2C_2(\rho/r)^{n+2} \left\{ 2 \int_{B_r} |D^2 u - (D^2 u)_r|^2 + 2 \int_{B_r} |D^2 w - (D^2 w)_r|^2 \right\} + 8 \int_{B_r} |D^2 v|^2
\]
\[
\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2 u - (D^2 u)_r|^2
\]
\[
+ (8 + 16C_2(\rho/r)^{n+2}) \int_{B_r} |D^2 v|^2.
\]

The statement follows, noting that \( \rho/r \leq 1 \).

We will be using the following Lemma frequently, so we state it here for the reader’s convenience.

**Lemma 2.3.** [HL11, Lemma 3.4]. Let \( \phi \) be a nonnegative and non-decreasing function on \([0, R]\). Suppose that
\[
\phi(\rho) \leq A \left( \left( \frac{\rho}{r} \right)^\alpha + \varepsilon \right) \phi(r) + Br^\beta
\]
for any \( 0 < \rho \leq r \leq R \), with \( A, B, \alpha, \beta \) nonnegative constants and \( \beta < \alpha \). Then for any \( \gamma \in (\beta, \alpha) \), there exists a constant \( \varepsilon_0 = \varepsilon_0(A, \alpha, \beta, \gamma) \) such that if \( \varepsilon < \varepsilon_0 \) we have for all \( 0 < \rho \leq r \leq R \)
\[
\phi(\rho) \leq c \left[ \left( \frac{\rho}{r} \right)^\gamma \phi(r) + Br^\beta \right]
\]
where \( c \) is a positive constant depending on \( A, \alpha, \beta, \gamma \). In particular, we have for any \( 0 < r \leq R \)
\[
\phi(r) \leq c \left[ \left( \frac{\phi(R)}{R} \right)^{\gamma} + Br^\beta \right].
\]

### 3. Proofs of the propositions

We begin by proving Proposition 1.3.

**Proof.** By approximation, (1.7) holds for \( \eta \in W_0^{2,2} \). We are assuming that \( u \in W^{2,\infty} \), so (1.7) must hold for the test function
\[
\eta = -[\tau^4 u h_v]^{-h_v}.
\]
where $\tau \in C_c^\infty$ is a cutoff function in $B_1$ that is 1 on $B_{1/2}$, and the subscript $h_p$ refers to taking difference quotient in the $e_p$ direction. We choose $h$ small enough after having fixed $\tau$, so that $\eta$ is well defined. We have

$$
\int_{B_1} (c^{ij,kl} u_{ij} + f^{kl}) \tau^4 u^{h_p}_{kl} dx = 0
$$

For $h$ small we can integrate by parts with respect to the difference quotient to get

$$
\int_{B_1} (c^{ij,kl} u_{ij} + f^{kl}) h_p \tau^4 u_{kl}^{h_p} dx = 0.
$$

Using the product rule for difference quotients we get

$$
\int_{B_1} [((c^{ij,kl}(x)) h_p) u_{ij}(x) + c^{ij,kl}(x + h e_p) u_{ij}^{h_p} + (f^{kl}) h_p] \tau^4 u_{kl}^{h_p} dx = 0
$$

Letting $v = u^{h_p}$, differentiating the second factor gives:

$$
\int_{B_1} [(c^{ij,kl}(x)) h_p) u_{ij}(x) + c^{ij,kl}(x + h e_p) v_{ij}(x) + (f^{kl}) h_p(x)] \tau^4 v_{kl} dx = 0
$$

from which

$$
\int_{B_1} \tau^4 c^{ij,kl} (x + h e_p) v_{ij} v_{kl} dx =
$$

$$
- \int_{B_1} [(c^{ij,kl}(x)) h_p) u_{ij}(x) + c^{ij,kl}(x + h e_p) v_{ij}(x) + (f^{kl}) h_p(x)]
$$

$$
\times \left[ 4\tau^3 \tau_k v_l + 4\tau^3 \tau_l v_k 
+ 4v (\tau^3 \tau_k + 3\tau^2 \tau_l \tau) \right] dx
$$

$$
- \int_{B_1} [(c^{ij,kl}(x)) h_p) u_{ij}(x) + (f^{kl}) h_p(x)] \tau^4 v_{kl} dx
$$

First we bound the terms on the right side of (3.1). Starting at the top:

$$
\int_{B_1} [(c^{ij,kl}(x)) h_p) u_{ij}(x) + (f^{kl}) h_p(x)] \times \left[ 4\tau^3 \tau_k v_l + 4\tau^3 \tau_l v_k 
+ 4v (\tau^3 \tau_k + 3\tau^2 \tau_l \tau) \right] dx
$$

$$
\leq \left[ \|u\|_{W^{2,\infty}(B_1)}^2 + 1 \right] \int_{B_1} \left( |(c^{ij,kl}(x)) h_p|^2 + |(f^{kl}) h_p(x)|^2 \right) dx
$$

$$
+ C_8(\tau, D\tau, D^2\tau) \int_{B_1} (|Dv|^2 + |v|^2) dx.
$$
Next, by Young’s inequality we have:
\[
\int_{B_1} c^{ij,kl}(x + h e_p) v_{ij}(x) \times \left[ 4\tau^3 \tau_j v_l + 4\tau^3 \tau_l v_j + 4v(\tau^3 \tau_{jl} + 3\tau_2^2 \tau_j \tau_l) \right] dx
\]
\[
\leq \frac{C_9(\tau, D\tau, D^2\tau, c^{ij,kl})}{\varepsilon} \int_{B_1} (|Dv|^2 + v^2) \, dx + \varepsilon \int_{B_1} \tau^4 |D^2v|^2 \, dx
\]
(3.3)

and also
\[
\int_{B_1} \left[ (c^{ij,kl}(x))^{h_p} u_{ij}(x) + (f^{kl})^{h_p}(x) \right] \tau^4 v_{kl} \, dx
\]
\[
\leq \varepsilon \int_{B_1} \tau^4 \|D^2v\|^2 \, dx
\]
(3.4)

Now by uniform ellipticity (1.2), the left hand side of (3.1) is bounded below by
\[
\Lambda \int_{B_1} \tau^4 \|D^2v\|^2 \, dx \leq \int_{B_1} \tau^4 c^{ij,kl}(x + h e_p) v_{ik}(x) v_{kl}(x) \, dx
\]
(3.5)

Combining all (3.1), (3.2), (3.3), (3.4) and (3.5) and choosing \( \varepsilon \) appropriately, we get
\[
\frac{\Lambda}{2} \int_{B_1} \tau^4 \|D^2v\|^2 \, dx
\]
\[
\leq C_{11}(|\tau|_{W^{2,\infty}(B_1)}, ||u||_{W^{2,\infty}(B_1)}) \left( \int_{B_1} |(f^{kl})^{h_p}|^2 + |c^{ij,kl}|^2 + |(c^{ij,kl})^{h_p}|^2 \right)
\]
\[
\leq C_{12}(|\tau|_{W^{2,\infty}(B_1)}, ||u||_{W^{2,\infty}(B_1)}, ||f^{kl}||_{W^{1,2}(B_1)}, ||c^{ij,kl}||_{W^{1,2}(B_1)}^2, \Lambda).
\]

Now this estimate is uniform in \( h \) and direction \( e_p \) so we conclude that the difference quotients of \( u \) are uniformly bounded in \( W^{2,2}(B_{1/2}) \). Hence \( u \in W^{3,2}(B_{1/2}) \) and
\[
||D^3f||_{L^2(B_{1/2})}
\]
\[
\leq \frac{2C_{12}}{\Lambda} (||\tau||_{W^{2,\infty}(B_1)}, ||u||_{W^{2,\infty}(B_1)}, ||f^{kl}||_{W^{1,2}(B_1)}, ||c^{ij,kl}||_{W^{1,2}(B_1)}^2, \Lambda).
\]

We now prove Proposition 1.4
Proof. We begin by taking a difference quotient of the equation
\[ \int (c^{ij,kl} u_{ij} + f^{kl}) \eta_{kl} dx = 0 \]
along the direction \( h_m \). This gives
\[ \int [(c^{ij,kl}(x))^{h_m} u_{ij}(x) + c^{ij,kl}(x + h e_m) u_{ij}^{h_m}(x) + (f^{kl})^{h_m}] \eta_{kl}(x) dx = 0 \]
which gives us the following PDE in \( u_{ij}^{h_m} \):
\[ \int c^{ij,kl}(x + h e_m) u_{ij}^{h_m}(x) \eta_{kl}(x) dx = \int q(x) \eta_{kl}(x) dx \]
where
\[ q(x) = -(f^{kl})^{h_m}(x) - (c^{ij,kl}(x))^{h_m} u_{ij}(x) \]
Note that \( q \in C^\alpha(B_1) \) and \( c^{ij,kl}(x + h e_m) \) is still an elliptic term for all \( x \) in \( B_1 \). For compactness of notation we denote
(3.6) \[ g = u^{h_m} \]
and replace \( c^{ij,kl}(x + h e_m) \) with \( c^{ij,kl} \), as the difference is immaterial. Our equation reduces to
(3.7) \[ \int c^{ij,kl} g_{ij} \eta_{kl} dx = \int q \eta_{kl} dx \]
Using integration by parts we have
\[ \int c^{ij,kl} g_{ij} \eta_{kl} dx = - \int q \eta_k dx \]
\[ = - \int (q - q(0)) \eta_k dx \]
\[ = \int (q - q(0)) \eta_{kl} dx \]
Now for each fixed \( r < 1 \) we write \( g = v + w \) where \( w \) satisfies the following constant coefficient PDE on \( B_r \subseteq B_1 \):
(3.8) \[ \int_{B_1(0)} c^{ij,kl}(0) w_{ij} \eta_{kl} dx = 0 \]
\[ \forall \eta \in C^\infty_0(B_r(0)) \]
\[ w = g \text{ on } \partial B_r \]
\[ \nabla w = \nabla g \text{ on } \partial B_r. \]
By the Lax Milgram Theorem the above PDE with the given boundary condition has a unique solution. By combining (3.7) and (3.8) we conclude

\[(3.9) \int_{B_r} c^{ij,kl}(0)v_{ij}\eta_{kl}dx = \int_{B_r} (c^{ij,kl}(0) - c^{ij,kl}(x))g_{ij}\eta_{kl}dx + \int_{B_r} q\eta_{kl}dx\]

Now \(w\) is smooth (again see [Dri03, Theorem 33.10]), and \(g = u^{\text{hom}}\) is \(C^{2,\alpha}\), so \(v = g - w\) is \(C^{2,\alpha}\) and can be well approximated by smooth test functions in \(H^2_0(B_r)\). It follows that \(v\) can be used as a test function in (3.9): On the left hand side we have by (1.2)

\[\left(\int_{B_r} c^{ij,kl}(0)v_{ij}v_{kl}dx\right)^2 \geq \Lambda^2 \left(\int_{B_r} |D^2v|^2dx\right)^2.\]

Defining

\[(3.10) \, \zeta(r) = \sup\{ |c^{ij,kl}(x) - c^{ij,kl}(y)| : x, y \in B_r \}\]

and using the Cauchy-Schwarz inequality we get

\[\left(\int_{B_r} (c^{ij,kl}(0) - c^{ij,kl}(x))g_{ij}v_{kl}dx\right)^2 \leq \zeta^2(r) \int_{B_r} |D^2g|^2dx \int_{B_r} |D^2v|^2dx.\]

Using Holder’s inequality

\[\left(\int_{B_r} |(q(x) - q(0))v_{kl}(x)| dx\right)^2 \leq \int_{B_r} |q(x) - q(0)|^2 dx \int_{B_r} |D^2v|^2dx\]

This gives us

\[\Lambda^2 \left(\int_{B_r} |D^2v|^2dx\right)^2 \leq \zeta^2(r) \int_{B_r} |D^2g|^2dx \int_{B_r} |D^2v|^2dx + \int_{B_r} |q(x) - q(0)|^2 dx \int_{B_r} |D^2v|^2dx\]

which implies

\[(3.11) \, \Lambda^2 \int_{B_r} |D^2v|^2dx \leq \zeta^2(r) \int_{B_r} |D^2g|^2dx + \int_{B_r} |q(x) - q(0)|^2 dx.\]

Using corollary 2.2 for any \(0 < \rho \leq r\) we get

\[(3.12) \, \int_{B_{\rho}} |D^2g|^2 dx \leq 4C_1(\rho/r)^n \|D^2g\|^2_{L^2(B_{\rho})} + (2 + 8C_1) \|D^2v\|^2_{L^2(B_{\rho})}\]
Now combining (3.12) and (3.11) we get
\[
\int_{B_r} |D^2 g|^2 dx \leq 4C_1 (\rho/r)^n \|D^2 g\|_{L^2(B_r)}^2
\]
\[
+ \frac{(2 + 8C_1)}{\Lambda^2} \left[ \zeta^2(r) \int_{B_r} |D^2 g|^2 dx + \int_{B_r} |q(x) - q(0)|^2 dx \right]
\]
\[
= \left[ \frac{(2 + 8C_1)}{\Lambda^2} \zeta^2(r) + 4C_1 (\rho/r)^n \right] \int_{B_r} |D^2 g|^2 dx
\]
\[
+ \frac{(2 + 8C_1)}{\Lambda^2} \int_{B_r} |q(x) - q(0)|^2 dx.
\]
(3.13)

Also from Corollary 2.2
\[
\int_{B_r} |D^2 g - (D^2 g)_\rho|^2 dx \leq 4C_2 (\rho/r)^{n+2} \int_{B_r} |D^2 g - (D^2 g)_\rho|^2 dx
\]
\[
+ (8 + 16C_2) \int_{B_r} |D^2 v|^2 dx
\]
\[
\leq 4C_2 (\rho/r)^{n+2} \int_{B_r} |D^2 g - (D^2 g)_\rho|^2 dx
\]
\[
+ \frac{(8 + 16C_2)}{\Lambda^2} \left[ \zeta^2(r) \int_{B_r} |D^2 g|^2 dx + \int_{B_r} |q(x) - q(0)|^2 dx \right].
\]

Because \(c^{ij,kl} \in C^{1,\alpha}\) we have from (3.10) that
\[
\zeta(r)^2 \leq C_{13} r^{2\alpha}
\]
Again \(q\) is a \(C^\alpha\) function which implies
\[
|q(x) - q(0)| \leq \|q\|_{C^\alpha(B_1)} |x - 0|^{\alpha}
\]
and
\[
\int_{B_r} |q - q(0)|^2 dx \leq C_{14} \|q\|_{C^\alpha(B_1)} r^{n+2\alpha}
\]
So we have
\[
\int_{B_r} |D^2 g - (D^2 g)_\rho|^2
\]
\[
\leq 4C_2 (\rho/r)^{n+2} \int_{B_r} |D^2 g - (D^2 g)_\rho|^2
\]
\[
+ \frac{(8 + 16C_2)}{\Lambda^2} C_{13} r^{2\alpha} \int_{B_r} |D^2 g|^2
\]
\[
+ \frac{(8 + 16C_2)}{\Lambda^2} C_{14} \|q\|_{C^\alpha(B_1)} r^{n+2\alpha}.
\]
(3.15)
For $r < r_0 < 1/4$ to be determined, we have (3.13)

$$
\int_{B_r} |D^2 g|^2 \leq C_{15} \left\{ [(\rho/r)^n + r^{2\alpha}] \int_{B_r} |D^2 g|^2 + r_0^{2\alpha + 2\delta} r^{n-2\delta} \right\}.
$$

Where $\delta$ is some positive number. Now we apply [HL11, Lemma 3.4]. In particular, take

$$
\phi(\rho) = \int_{B_r} |D^2 g|^2
$$

$$
A = C_{15}
$$

$$
B = r_0^{2\alpha + 2\delta}
$$

$$
\alpha = n
$$

$$
\beta = n - 2\delta
$$

$$
\gamma = n - \delta.
$$

There exists $\varepsilon_0(A, \alpha, \beta, \gamma)$ such that if

(3.16) \quad r_0^{2\alpha} \leq \varepsilon_0

we have

$$
\phi(\rho) \leq C_{15} \left\{ [(\rho/r)^n + \varepsilon_0] \phi(r) + r_0^{2\alpha + 2\delta} r^{n-2\delta} \right\}
$$

and the conclusion of [HL11, Lemma 3.4] says that for $\rho < r_0$

$$
\phi(\rho) \leq C_{16} \left\{ [(\rho/r)^\gamma] \phi(r) + r_0^{2\alpha + 2\delta} \rho^{n-2\delta} \right\}
$$

$$
\leq C_{16} \frac{1}{r_0^{n-\delta}} \rho^{n-\delta} \| D^2 g \|_{L^2(B_{r_0})} + r_0^{2\alpha + 2\delta} \rho^{n-2\delta}
$$

$$
\leq C_{17} \rho^{n-\delta}
$$

This $C_{17}$ depends on $r_0$ which is chosen by (3.16) and $\| D^2 g \|_{L^2(B_{3/4})}$. So there is a positive uniform radius upon which this holds for points well in the interior. In particular, we choose $r_0 \in (0, 1/4)$ so that the estimate can be applied uniformly at points centered in $B_{1/2}(0)$ whose balls remain in $B_{3/4}(0)$. Turning back to (3.15), we now have,

$$
\int_{B_r} |D^2 g - (D^2 g)_\rho|^2 \leq 4C_2 (\rho/r)^{n+2} \int_{B_r} |D^2 g - (D^2 g)_\rho|^2 + C_{18} r^{2\alpha} \rho^{n-\delta}
$$

$$
+ C_{19} \| q \|_{C^\alpha(B_1)} r^{n+2\alpha}
$$

$$
\leq 4C_2 (\rho/r)^{n+2} \int_{B_r} |D^2 g - (D^2 g)_\rho|^2 + C_{20} r^{n+2\alpha-\delta}
$$
Again we apply [HL11, Lemma 3.4]: This time, take

\[
\phi(\rho) = \int_{B_\rho} |D^2 g - (D^2 g)_\rho|^2
\]

\[
A = 4C_2
\]

\[
B = C_{20}
\]

\[
\alpha = n + 2
\]

\[
\beta = n + 2\alpha - \delta
\]

\[
\gamma = n + 2\alpha
\]

and conclude that for any \( r < r_0 \)

\[
\int_{B_r} |D^2 g - (D^2 g)_\rho|^2 \leq C_{21} \left\{ \frac{1}{r_0^{n+2\alpha}} \int_{B_{r_0}} |D^2 g - (D^2 g)_{r_0}|^2 r^{n+2\alpha} + C_{20} r^{n+2\alpha-\delta} \right\}
\]

\[
\leq C_{22} r^{n+2\alpha-\delta}
\]

with \( C_{22} \) depending on \( r_0, \|D^2 g\|_{L^2(B_{3/4})}, \|q\|_{C^\alpha(B_1)} \) etc. It follows by [HL11, Theorem 3.1] that \( D^2 g \in C^{(2\alpha-\delta)/2}(B_{1/4}) \), in particular, must be bounded locally:

\[
(3.17) \quad \|D^2 g\|_{L^\infty(B_{1/4})} \leq C_{23} \left\{ 1 + \|D^2 g\|_{L^2(B_{1/2})} \right\}.
\]

This allows us to bound

\[
\int_{B_r} |D^2 g|^2 \leq C_{24} r^n
\]

which we can plug back in to (3.15):

\[
\int_{B_\rho} |D^2 g - (D^2 g)_\rho|^2 \leq 4C_2 (\rho/r)^{n+2} \int_{B_\rho} |D^2 g - (D^2 g)_\rho|^2 + C_{25} r^{2\alpha} C_{24} r^n
\]

\[
+ C_{19} \|q\|_{C^\alpha(B_1)} r^{n+2\alpha}
\]

\[
\leq C_{26} r^{n+2\alpha}
\]

This is precisely the hypothesis in [HL11, Theorem 3.1]. We conclude that

\[
\|D^2 g\|_{C^\alpha(B_{1/4})} \leq C_{27} \left\{ \sqrt{C_{26}} + \|D^2 g\|_{L^2(B_{1/2})} \right\}.
\]

Recalling (3.6) we see that \( u \) must enjoy uniform \( C^{3,\alpha} \) estimates on the interior, and the result follows. □
4. Proof of the Theorem

The propositions in the previous section allow us to prove the following Corollary, from which the Main Theorem will follow.

**Corollary 4.1.** Suppose \( u \in C^{N,\alpha}(B_1) , \ N \geq 2,\) and satisfies the following regular (recall (1.3)) fourth order equation

\[
\int_{\Omega} a^{ij,kl}(D^2 u) u_{ij} \eta_{kl} \, dx = 0, \ \forall \eta \in C_{0}^{\infty}(\Omega).
\]

Then

\[
\|u\|_{C^{N+1,\alpha}(B_r)} \leq C(n, b, \|u\|_{W^{N,\infty}(B_1)}).
\]

In particular

\[ u \in C^{N,\alpha}(B_1) \implies u \in C^{N+1,\alpha}(B_r) \]

**Case 1 \( N = 2.\)** The function \( u \in C^{2,\alpha}(B_1) \) and hence also in \( W^{2,\infty}(B_1) .\) By approximation (1.1) holds for \( \eta \in W^{2,\infty}_0, \) in particular, for

\[
\eta = -[\tau^4 u^h_m]^h_m
\]

where \( \tau \in C_0^\infty(B_1) \) is a cut off function in \( B_1 \) that is 1 on \( B_{1/2}, \) and superscript \( h_m \) refers to the difference quotient. As before, we have chosen \( h \) small enough (depending on \( \tau \)) so that \( \eta \) is well defined. We have

\[
\int_{\Omega} a^{ij,kl}(D^2 u) u_{ij} [\tau^4 f^h_m]_{kl} \, dx = 0.
\]

Integrating by parts as before with respect to the difference quotient, we get

\[
\int_{B_1} [a^{ij,kl}(D^2 f) u_{ij}]^{h_m}[\tau^4 u^h_m]_{kl} \, dx = 0
\]

Let \( v = u^h_m .\) Observe that the first difference quotient can be expressed as

(4.1)

\[
[a^{ij,kl}(D^2 f) u_{ij}]^{h_m}(x) = a^{ij,kl}(D^2 u(x + he_m)) \frac{u_{ij}(x + he_m) - u_{ij}(x)}{h}
\]

\[
+ \frac{1}{h} [a^{ij,kl}(D^2 u(x + he_m)) - a^{ij,kl}(D^2 u(x))] u_{ij}(x)
\]

\[
= a^{ij,kl}(D^2 u(x + he_m)) v_{ij}(x)
\]

\[
+ \left[ \int_0^1 \frac{\partial a^{ij,kl}}{\partial u_{pq}} (t D^2 u(x + he_m) + (1-t) D^2 u(x)) dt \right] v_{pq}(x) u_{ij}(x).
\]

We get

(4.2)

\[
\int_{B_1} b^{ij,kl} v_{ij} [\tau^4 v]_{kl} \, dx = 0
\]
where
\begin{equation}
\tilde{b}^{ij,kl}(x) = a^{ij,kl}(D^2u(x+he_m)) + \int_0^1 \frac{\partial a^{pq,kl}}{\partial u_{ij}}(tD^2u(x+he_m) + (1-t)D^2u(x))dt \ u_{pq}(x).
\end{equation}

Expanding derivatives of the second factor in (4.2) and collecting terms gives us
\begin{align*}
\int_{B_1} \tilde{b}^{ij,kl} v_{ij} \tau^4 v_{kl} \, dx &\leq \int_{B_1} \left| \tilde{b}^{ij,kl} \right| \left| v_{ij} \right| \tau^2 C_{28}(\tau, D\tau, D^2\tau) (1 + |v| + |Dv|) \, dx \\
&\leq C_{29} \sup_{B_1} \left\| D^3v \right\|_{L^2(B_1)} \left( \int_{B_1} (1 + |v| + |Dv|)^2 \, dx \right).
\end{align*}

Now for \( h \) small, \( \tilde{b}^{ij,kl} \) very closely approximates \( b^{ij,kl} \), so we may assume \( h \) is small. Applying (1.4) and Young’s inequality
\begin{equation}
\int_{B_1} \tau^4 \Lambda_1 |D^2v|^2 \leq C_{28} \sup_{B_1} \tilde{b}^{ij,kl} \int_{B_1} \left( \varepsilon \tau^4 |D^2v|^2 + C_{32} \frac{1}{\varepsilon} (1 + |v| + |Dv|)^2 \right) \, dx.
\end{equation}

That is
\begin{equation}
\int_{B_{1/2}} |D^2v|^2 \leq C_{29} \int_{B_1} (1 + |v| + |Dv|)^2 \, dx.
\end{equation}

Now this estimate is uniform in \( h \) (for \( h \) small enough) and direction \( e_m \), so we conclude that the derivatives are in \( W^{2,2}(B_{1/2}) \). This also shows that
\begin{equation}
\left\| D^3u \right\|_{L^2(B_{1/2})} \leq C_{30} \left( \left\| D^2u \right\|_{L^2(B_1)}, \left\| D^2u \right\|_{L^2(B_1)} \right).
\end{equation}

Remark: We only used uniform continuity of \( D^2u \) to allow us to take the limit, but we did require the precise modulus of continuity.

For the next step, we are not quite able to use Proposition 1.4 because the coefficients \( a^{ij,kl} \) are only known to be \( W^{1,2} \). So we proceed by hand.

We begin by taking a single difference quotient
\begin{equation}
\int_{B_1} [a^{ij,kl}(D^2f)u_{ij}]^{hm} \eta_{kl} \, dx = 0
\end{equation}

and arriving at the equation in the same fashion as to (4.2) above (this time letting \( g = u^{hm} \)) we have
\begin{equation}
\int_{B_1} \tilde{b}^{ij,kl} g_{ij}(x) \eta_{kl} \, dx = 0.
\end{equation}

Inspecting (4.3) we see that \( \tilde{b}^{ij,kl} \) is \( C^\alpha \):
\begin{equation}
\left\| \tilde{b}^{ij,kl}(x) - \tilde{b}^{ij,kl}(y) \right\| \leq C_{31} |x - y|^\alpha
\end{equation}
where $C_{31}$ depends on $\|D^2u\|_{C^\alpha}$ and on bounds of $Da^{ij,kl}$ and $D^2a^{ij,kl}$.

As in the proof of Proposition 1.4, for a fixed $r < 1$ we let $w$ solve the boundary value problem
\[
\int \tilde{b}^{ij,kl}(0) u_{ij} \eta_{kl} \, dx = 0, \forall \eta \in C_0^\infty(B_r)
\]
where $\tilde{b}^{ij,kl}(x) = 0$ on $\partial B_r$.

Let $v = g - w$. Note that
\[
\int \tilde{b}^{ij,kl}(0) v_{ij} \eta_{kl} \, dx = \int \left( \tilde{b}^{ij,kl}(0) - \tilde{b}^{ij,kl}(x) \right) g_{ij} \eta_{kl} \, dx.
\]

Now $v$ vanishes to second order on the boundary, and we may use $v$ as a test function. We get
\[
\int \tilde{b}^{ij,kl}(0) v_{ij} v_{kl} \, dx = \int \left( \tilde{b}^{ij,kl}(0) - \tilde{b}^{ij,kl}(x) \right) g_{ij} v_{kl} \, dx.
\]

As before,
\[
\left( \Lambda \int_{B_r} |D^2v|^2 \, dx \right)^2 \leq \left[ \sup_{x \in B_r} \left| \tilde{b}^{ij,kl}(0) - \tilde{b}^{ij,kl}(x) \right| \right]^2 \int_{B_r} |D^2g|^2 \, dx \int_{B_r} |D^2v|^2 \, dx.
\]

Defining
\[
(4.4) \quad \zeta(r) = \sup\{ \left| \tilde{b}^{ij,kl}(x) - \tilde{b}^{ij,kl}(y) \right| \mid x, y \in B_r \}
\]
\[
\leq 4^\alpha C_{31} r^{2\alpha}
\]
then
\[
\int_{B_r} (\tilde{b}^{ij,kl}(0) - \tilde{b}^{ij,kl}(x)) g_{ij} v_{kl} \, dx)^2 \leq \zeta^2(r) \int_{B_r} |D^2g|^2 \int_{B_r} |D^2v|^2.
\]

So now we have :
\[
\int_{B_r} |D^2v|^2 \leq \frac{\zeta^2(r)}{\Lambda^2} \int_{B_r} |D^2g|^2.
\]

Using Corollary 2.2, for any $0 < \rho \leq r$ we get
\[
\int_{B_\rho} |D^2g - (D^2g)_\rho|^2 \leq 4C_2 (\rho/r)^{n+2} \int_{B_\rho} |D^2g - (D^2g)_r|^2
\]
\[
+ (8 + 16C_2) \int_{B_\rho} |D^2v|^2
\]
\[
(4.5) \quad \leq 4C_2 (\rho/r)^{n+2} \int_{B_\rho} |D^2g - (D^2g)_r|^2 + \frac{(8 + 16C_2) \zeta^2(r)}{\Lambda^2} \left\| D^2g \right\|_{L^2(B_\rho)}^2.
\]
Also by Corollary 2.2

\[ \int_{B_r} |D^2 g|^2 \leq 4C_1 (\rho/r)^n \|D^2 g\|^2_{L^2(B_r)} + (2 + 8C_1) \|D^2 v\|^2_{L^2(B_r)} \]

\[ \leq 4C_1 (\rho/r)^n \|D^2 g\|^2_{L^2(B_r)} + (2 + 8C_1) \frac{C^2(r)}{\Lambda^2} \|D^2 g\|^2_{L^2(B_r)}. \]

This implies

\[ \int_{B_{r_0}} |D^2 g|^2 \leq (4C_1 (\rho/r)^n + (2 + 8C_1) 4^{2\alpha} C_{31}^2 r^{2\alpha}) \|D^2 g\|^2_{L^2(B_{r_0})}. \]

Now we can apply [HL11, Lemma 3.4] again, this time with

\[ \phi(\rho) = \int_{B_{r_0}} |D^2 g|^2 \]

\[ A = 4C_1 \]

\[ \alpha = n \]

\[ B, \beta = 0 \]

\[ \gamma = n - 2\delta \]

\[ \varepsilon = (2 + 8C_1) 4^{2\alpha} C_{31}^2 r^{2\alpha}. \]

There exists a constant \( \varepsilon_0(A, \alpha, \gamma) \) such that by choosing

\[ r_0^{2\alpha} \leq \frac{\varepsilon_0}{(2 + 8C_1) 4^{2\alpha} C_{31}^2} < \frac{1}{4} \]

we may conclude that for \( 0 < r \leq r_0 \)

\[ (4.6) \quad \int_{B_r} |D^2 g|^2 \leq C_{32} r^{n-2\delta} \int_{B_{r_0}} |D^2 g|^2 \frac{r^{n-2\delta}}{r_0^{n-2\delta}}. \]

Next, for small \( \rho < r < r_0 \) we have combining (4.4) (4.5) and (4.6)

\[ (4.7) \quad \int_{B_{r_0}} |D^2 g - (D^2 g)_\rho|^2 \leq 4C_2 (\rho/r)^{n+2} \int_{B_r} |D^2 g - (D^2 g)_r|^2 \]

\[ + \frac{(8 + 16C_2) 4^\alpha}{\Lambda^2} \frac{1}{r_0^{n-2\delta}} C_{31} C_{32} r^{n-2\delta} \frac{r^{n-2\delta}}{r_0^{n-2\delta}} \]

\[ \leq C_{33} r^{n+2\alpha-\delta} \]

with \( C_{33} \) depending on \( \|D^2 g\|_{L^2(B_{3/4})}, r_0, \varepsilon_0 \). Again, we apply [HL11, Theorem 3.1] to \( D^2 g \in C^{(2\alpha-\delta)/2}(B_{1/4}) \). From here, the argument is
identical to the argument following \(3.17\). We conclude that

\[
\|D^2 g\|_{C^\alpha(B_{1/4})} \leq C_{34} \left\{ 1 + \|D^2 g\|_{L^2(B_{3/4})} \right\}.
\]

Substituting \(g = u^{h_m}\) we see that \(u\) must enjoy uniform \(C^{3,\alpha}\) estimates on the interior, and the result follows.

Case 2 \(N = 3\). We may take a difference quotient of \(1.1\) directly.

\[
\int_{B_1} \left[ a^{ij,kl}(D^2 u) u_{ij} \right]^{h_m} \eta_{kl} dx = 0, \quad \forall \eta \in C_0^\infty(\Omega).
\]

(To be more clear we, are using a slightly offset test function \(\eta(x + he_m)\) and then using a change of variables, subtracting, and dividing by \(h\).

We get

\[
\int_{B_1} \left[ a^{ij,kl}(D^2 u(x + he_m)) u_{ij}^{h_m}(x) + \frac{\partial a^{ij,kl}}{\partial u_{pq}} (M^*(x)) u_{pq}^{h_m}(x) u_{ij}(x) \right] \eta_{kl} = 0.
\]

where \(M^*(x) = t^* D^2 u(x + he_m) + (1 - t^*) D^2 u(x)\) and \(t^* \in [0, 1]\). Now we are assuming that \(u \in C^{3,\alpha}(B_1)\), so the first and second derivatives of the difference quotient will converge to the second and third derivatives, uniformly. We can then apply dominated convergence, passing the limit as \(h \to 0\) inside the integral and recalling \(u_m = v\) as before, we obtain

\[
\int_{B_1} \left[ a^{ij,kl}(D^2 u(x)) v_{ij}(x) + \frac{\partial a^{pq,kl}}{\partial u_{pq}} (D^2 u(x)) v_{ij}(x) u_{pq}(x) \right] \eta_{kl} = 0
\]

that is

\[
(4.8) \quad \int_{B_1} b^{ij,kl}(D^2 u(x)) v_{ij}(x) \eta_{kl}(x) = 0, \quad \forall \eta \in C_0^\infty(\Omega).
\]

It follows that \(v \in C^{2,\alpha}\) satisfies a fourth order double divergence equation, with coefficients in \(C^{1,\alpha}\). First, we apply Proposition \(1.3\):

\[
\|D^3 v\|_{L^2(B_{1/2})} \leq C_{35} \left( \|v\|_{W^{2,\infty}(B_{1})} \right) (1 + \|b^{ij,kl}\|_{W^{1,2}(B_{1})}).
\]

In particular, \(u \in W^{4,2}(B_{1/2})\). Next, we apply \(1.4\)

\[
\|D^3 v\|_{C^{0,\alpha}(B_{1/4})} \leq C(1 + \|D^3 v\|_{L^2(B_{1/2})}) \leq C(\|u\|_{W^{2,\infty}(B_{1})}, \|b^{ij,kl}\|_{W^{1,2}(B_{1})}) \leq C_{36}(n, b, \|u\|_{C^{3,\alpha}(B_{1})}).
\]

We conclude that \(u \in C^{4,\alpha}(B_r)\) for any \(r < 1\).

Case 3 \(N \geq 4\). Let \(v = D^\alpha u\) for some multindex \(\alpha\) with \(|\alpha| = N - 2\). Observe that taking the first difference quotient and then taking a limit yields \(4.8\), when \(u \in C^{3,\alpha}\). Now if \(u \in C^{4,\alpha}\) we may take a difference quotient and limit of \(4.8\) to obtain

\[
\int_{B_1} \left[ b^{ij,kl}(D^2 u(x)) u_{ij m_1 m_2} + \frac{\partial b^{ij,kl}}{\partial u_{pq}} (D^2 u(x)) u_{pq m_2} u_{ij} \right] \eta_{kl}(x) = 0, \quad \forall \eta \in C_0^\infty(\Omega).
\]
and if \( u \in C^{N,\alpha} \), then \( v \in C^{2,\alpha} \), so we may take \( N - 2 \) difference quotients to obtain

\[
\int_{B_1} \left[ b^{ij,kl}(D^2 u(x))v_{ij}(x) + f^{kl}(x) \right] \eta_{kl}(x) = 0, \quad \forall \eta \in C_0^\infty(\Omega).
\]

where

\[
f^{kl} = D^\alpha \left( b^{ij,kl}(D^2 u(x))u_{ij} \right) - b^{ij,kl}(D^2 u(x))D^\alpha u_{ij}.
\]

One can check by applying the chain rule repeatedly that \( f^{kl} \) is \( C^{1,\alpha} \).

So we may apply Proposition 1.3 to (4.9) and obtain that

\[
||D^3 v||_{L^2(B_{1/2})} \leq C_{37}(||v||_{W^{2,\infty}(B_1)})(1 + ||b^{ij,kl}||_{W^{1,2}(B_1)})
\]

that is

\[
||u||_{W^{N+1,2}(B_r)} \leq C_{38}(n, b, ||u||_{W^{N,\infty}(B_1)}).
\]

Now apply Proposition 1.4

\[
||D^3 v||_{C^{0,\alpha}(B_{1/4})} \leq C_{39}(1 + ||D^3 v||_{L^2(B_{3/4})})
\]

that is

\[
||u||_{C^{N+1,\alpha}(B_r)} \leq C_{40}(n, b, ||u||_{W^{N,\infty}(B_1)}).
\]

The Main Theorem follows.

5. Critical Points of Convex Functions of the Hessian

Suppose that \( F(D^2 u) \) is either a convex or a concave function of \( D^2 u \), and we have found a critical point of

\[
(5.1) \quad \int_{\Omega} F(D^2 u)dx
\]

for some \( \Omega \subset \mathbb{R}^n \), where we are restricting to compactly supported variations, so that the Euler-Lagrange equation is (1.6). If we suppose that \( F \) also has the additional structure condition,

\[
(5.2) \quad \frac{\partial F(D^2 u)}{\partial u_{ij}} = a^{pq,ij}(D^2 u)u_{pq}
\]

for a some \( a^{ij,kl} \) satisfying (1.2), then we can derive smoothness from \( C^{2,\alpha} \), as follows.

**Corollary 5.1.** Suppose \( u \in C^{2,\alpha}(B_1) \) is critical point of (5.1), where \( F \) is a smooth function satisfying (5.2) with \( a^{ij,kl} \) satisfying (1.2) and \( F \) is uniformly convex or uniformly concave on \( U \subseteq S_n \times S_n \) where \( U \) is the range of \( D^2 u(B_1) \) in the Hessian space.

Then \( u \in C^\infty(B_r) \), for all \( r < 1 \).
Proof. If \( u \) is a critical point of (5.1), then it satisfies the weak equation (1.1), for \( a_{ij}^{kl} \) in (5.2). To apply the main Theorem, all we need to show is that
\[
b_{ij}^{kl}(D^2u(x)) = a_{ij}^{kl}(D^2u(x)) + \frac{\partial a_{pq,kl}^{ij}}{\partial u_{ij}}(D^2u(x))u_{pq}(x)
\]
satisfies (1.2). From (5.2):
\[
(5.3) \quad \frac{\partial}{\partial u_{kl}} \left( \frac{\partial F(D^2u)}{\partial u_{ij}} \right) = a_{kl,ij}^{ij} + \frac{\partial a_{pq,ij}^{pq}}{\partial u_{kl}} u_{pq}.
\]
So
\[
b_{ij}^{kl}(D^2u(x))\xi_{ij}\xi_{kl} = \frac{\partial}{\partial u_{kl}} \left( \frac{\partial F(D^2u)}{\partial u_{ij}} \right) \xi_{ij}\xi_{kl} \geq \Lambda |\xi|^2
\]
for some \( \Lambda > 0 \), because \( F \) is convex. If \( F \) is concave, \( u \) is still a critical point of \( -F \) and the same argument holds.

We mention one special case.

**Lemma 5.2.** Suppose \( F(D^2u) = f(w) \) where \( w = (D^2u)^T(D^2u) \). Then
\[
(5.4) \quad \frac{\partial F(D^2u)}{\partial u_{ij}} = a_{ij}^{kl}(D^2u)u_{kl}
\]

**Proof.** Let
\[
w_{kl} = u_{ka}\delta^{ab}u_{bl}.
\]
Then
\[
\frac{\partial F(D^2u)}{\partial u_{ij}} = \frac{\partial f(w)}{\partial w_{kl}} \frac{\partial w_{kl}}{\partial u_{ij}}
\]
\[
= \frac{\partial f(w)}{\partial w_{kl}} \left( \delta_{ka,ij}\delta^{ab}u_{kl} + u_{ka}\delta^{ab}\delta_{bl,ij} \right)
\]
\[
= \frac{\partial f(w)}{\partial w_{kl}} \left( \delta_{ki}u_{jl} + u_{kl}\delta_{ij} \right)
\]
\[
= \frac{\partial f(w)}{\partial w_{il}} \delta_{jm}u_{ml} + \frac{\partial f(w)}{\partial w_{kj}} u_{km}\delta_{im}
\]
\[
= \frac{\partial f(w)}{\partial w_{il}} \delta_{jk}u_{kl} + \frac{\partial f(w)}{\partial w_{kj}} u_{kl}\delta_{il}.
\]
This shows (5.4) for
\[
a_{ij}^{kl} = \frac{\partial f(w)}{\partial w_{il}} \delta_{jk} + \frac{\partial f(w)}{\partial w_{kj}} \delta_{il}.
\]
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