CLASSIFICATION OF A SUBCLASS OF QUASILINEAR TWO-DIMENSIONAL LATTICES BY MEANS OF CHARACTERISTIC ALGEBRAS

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Abstract. We consider a classification problem of integrable cases of the Toda type two-dimensional lattices $u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y})$. The function $f = f(x_1, x_2, \cdots, x_5)$ is assumed to be analytic in a domain $D \subset \mathbb{C}^5$. The sought function $u_n = u_n(x, y)$ depends on real $x$, $y$ and integer $n$. Equations with three independent variables are complicated objects for study and especially for classification. It is commonly accepted that for a given equation, the existence of a large class of integrable reductions indicates integrability. Our classification algorithm is based on this observation. We say that a constraint $u_0 = \varphi(x, y)$ defines a degenerate cutting off condition for the lattice if it divides this lattice into two independent semi-infinite lattices defined on the intervals $-\infty < n < 0$ and $0 < n < +\infty$, respectively. We call a lattice integrable if there exist cutting off boundary conditions allowing us to reduce the lattice to an infinite number of hyperbolic type systems integrable in the sense of Darboux. Namely, we require that lattice is reduced to a finite system of such kind by imposing degenerate cutting off conditions at two different points $n = N_1$, $n = N_2$ for arbitrary pair of integers $N_1$, $N_2$. Recall that a system of hyperbolic equations is called Darboux integrable if it admits a complete set of integrals in both characteristic directions. An effective criterion of the Darboux integrability of the system is connected with properties of an associated algebraic structures. More precisely, the characteristic Lie-Rinehart algebras assigned to both characteristic directions have to be of a finite dimension. Since the obtained hyperbolic system is of a very specific form, the characteristic algebras are effectively studied. Here we focus on a subclass of quasilinear lattices of the form

$$u_{n,xy} = p(u_{n-1}, u_n, u_{n+1})u_{n,x} + r(u_{n-1}, u_n, u_{n+1})u_{n,y} + q(u_{n-1}, u_n, u_{n+1}).$$

Keywords: two-dimensional lattice, integrable reduction, characteristic Lie algebra, degenerate cutting off condition, Darboux integrable system, $x$-integral.

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1. INTRODUCTION

The problem of classifying integrable two-dimensional chains of the form

$$u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}), \quad -\infty < n < \infty,$$

is topical and currently remains open. The function $f = f(x_1, x_2, \cdots, x_5)$ is assumed to be analytic in a domain $D \subset \mathbb{C}^5$, and the sought function $u_n = u_n(x, y)$ depends on real $x$, $y$ and integer $n$.

In this paper we focus on the following subclass of quasilinear lattices (1.1):

$$u_{n,xy} = p(u_{n+1}, u_n, u_{n-1})u_{n,x} + r(u_{n+1}, u_n, u_{n-1})u_{n,y} + q(u_{n+1}, u_n, u_{n-1}), \quad -\infty < n < \infty.$$

(1.2)
Here functions \( p(x_1, x_2, x_3), r(x_1, x_2, x_3), q(x_1, x_2, x_3) \) are assumed to be analytic in a domain \( D \subset \mathbb{C}^3 \).

Equations with three independent variables are complicated objects for study and especially for classification. Currently, there are different approaches to studying integrable multidimensional equations \([1]–[10]\). The presence of a wide class of integrable reductions indicates the integrability of the equation. This fact is often used in the study of multidimensional equations, see \([1]–[3]\), where the existence of integrable reductions of a hydrodynamic type is taken to determine the integrability. Here we use a similar idea by treating integrability as the presence of an infinite sequence of Darboux integrable hyperbolic systems.

In describing Darboux integrable systems of hyperbolic equations of a special type, the concept of the characteristic Lie algebra \([11]–[13]\) was used a lot. The transition to a more general characteristic Lie-Rinehart algebra opens up new possibilities \([14]–[18]\).

The characteristic Lie algebra for two-dimensional lattices was introduced in \([19]\). Namely, the structure of this algebra was described for two-dimensional Toda lattice. It was observed in paper \([16]\) that any integrable lattice of the form (1.1) admits a so-called degenerate cutting off boundary conditions. When such kind boundary conditions are imposed at two different points \( n = N_1 \) and \( n = N_2 \), the lattice reduces to a Darboux integrable system of the hyperbolic type equations. In our works \([16], [17], [18]\), we suggested and developed a classification algorithm based on this observation. Let us briefly discuss the essence of the method.

We say that a constraint

\[
u_0 = \varphi(x, y)
\]

defines a degenerate cutting off condition for lattice (1.1) if it divides (1.1) into two independent semi-infinite lattices defined on the intervals \(-\infty < n < 0 \) and \( 0 < n < +\infty \), respectively.

**Definition 1.1.** Lattice (1.1) is called integrable if there exist functions \( \varphi_1 \) and \( \varphi_2 \) such that for any pair of integers \( N_1, N_2 \), where \( N_1 < N_2 - 1 \), the hyperbolic system

\[
\begin{align*}
u_{N_1} &= \varphi_1(x, y), \\
u_{n,xy} &= f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}), \quad N_1 < n < N_2, \\
u_{N_2} &= \varphi_2(x, y)
\end{align*}
\]

obtained from lattice (1.1) by imposing degenerate boundary conditions is integrable in the sense of Darboux.

Recall that a system of hyperbolic equations is called Darboux integrable if it admits a complete set of integrals in both characteristic directions, see \([14], [15]\). An effective criterion of the Darboux integrability of the system is connected with properties of an associated algebraic structures. More precisely, the characteristic Lie-Rinehart algebras \([20], [21]\) assigned to both characteristic directions have to be of a finite dimension. Since the obtained hyperbolic system is of a very specific form, this allows us to study effectively the characteristic algebras. The method was shown to be effective in our articles \([17], [18]\). A large class of the integrable lattices of form (1.1) was represented in \([22]\), where they were studied in the framework of the symmetry approach. It is remarkable that all equations of this class turned out to be integrable in the sense of Definition 1.1. Another argument in favor of our definition is that the resulting hyperbolic systems admit explicit solutions, which are extended to solutions of the original nonlinear chain. So, the chains integrable in our sense have a very wide class of explicit solutions.

In this article we continue the study initiated in \([17], [18]\), where the integrable in the sense of Definition 1.1 cases of the two-dimensional quasilinear lattices of the form

\[
u_{n,xy} = \alpha_n u_{n,x} u_{n,y} + p_n u_n + r_n u_{n,x} + q_n,
\]

(1.3)
were described under the non-degeneracy condition $\frac{\partial \alpha_n}{\partial u_{n+1}} \neq 0$. Here the coefficients depend on three successive variables

$$\begin{align*}
\alpha_n &= \alpha(u_{n+1}, u_n, u_{n-1}), \\
p_n &= p(u_{n+1}, u_n, u_{n-1}), \\
q_n &= q(u_{n+1}, u_n, u_{n-1}).
\end{align*}$$

We mention review [23], where a complete classification of lattices of the form

$$u_{n,xx} = f(u_{n-1}, u_n, u_{n+1}, u_{n,x}), \quad \frac{\partial f}{\partial u_{n+1}} \frac{\partial f}{\partial u_{n-1}} \neq 0,$$

was presented. In our paper [18], we found two new equations of form [1.3], which were integrable in the sense of Definition 1.1. We note that these equations were two-dimensional generalizations of the equations from the list in paper [23].

Now we focus on a particular case (1.2) of lattice (1.3), as $\alpha_n$ vanishes identically. We suppose that the following conditions are satisfied: at least one of the following derivatives is non-zero:

$$\begin{align*}
\frac{\partial r_n}{\partial u_{n+1}} \neq 0 & \quad \text{or} \quad \frac{\partial r_n}{\partial u_{n-1}} \neq 0, \\[1.4] \frac{\partial p_n}{\partial u_{n+1}} \neq 0 & \quad \text{or} \quad \frac{\partial p_n}{\partial u_{n-1}} \neq 0. \tag{1.5}
\end{align*}$$

The main result of this paper is as follows.

**Theorem 1.1.** If chain (1.2), (1.4) is integrable in the sense of Definition 1.1, then by point transformations it is reduced to one of the following forms:

$$\begin{align*}
u_{n,xy} &= (e^{u_n-u_{n-1}} - e^{u_{n+1}-u_n}) u_{n,y}, \\[1.6] u_{n,xy} &= (-u_{n+1} + 2u_n - u_{n-1}) u_{n,y}. \tag{1.7}
\end{align*}$$

Lattices (1.6), (1.7) were known before [22]. Condition (1.5) implies that lattices obtained under classification procedure coincide with these lattices up to the change $x \leftrightarrow y$.

In the next section we describe briefly a theoretical base of the main research method; a detailed explanation was presented in [17] [18].

2. **Preliminaries**

According Definition 1.1 there exist cutting off conditions at two points that reduce (1.2) to the finite hyperbolic type system:

$$\begin{align*}
u_{-1} &= \varphi_1, \\
u_{n,xy} &= p_n u_{n,x} + r_n u_{n,y} + q_n, \quad 0 \leq n \leq N, \\
u_{N+1} &= \varphi_2.
\end{align*}$$

Here $p_n = p(u_{n-1}, u_n, u_{n+1}), \ r_n = r(u_{n-1}, u_n, u_{n+1}), \ q_n = q(u_{n-1}, u_n, u_{n+1})$.

We recall that a hyperbolic system of partial differential equations (2.1) is integrable in the sense of Darboux if it admits a complete set of functionally independent $x$- and $y$-integrals (see [14]). A function $I$ depending on finitely many dynamical variables $u, u_x, u_{xx}, \ldots$ is called $y$-integral if it solves the equation $D_y I = 0$ (see [14]), where $D_y$ is the operator of total derivative with respect to variable $y$ and $u$ is a vector with coordinates $u_0, u_1, \ldots, u_N$. Since system (2.1) is autonomous, we consider autonomous $y$-integrals depending at least on one of the dynamical variables $u, u_x, u_{xx}, \ldots$.

We suppose that system (2.1) is Darboux integrable and denote by $I(u, u_x, \ldots)$ its nontrivial $y$-integral. By definition, $I$ solves the equation $D_y I = 0$. Let us calculate an action of the
operator $D_y$ on functions of the form $I(u, u_x, \ldots)$. It is determined by the rule $D_y I = YI$, where

$$Y = \sum_{i=0}^{N} \left( u_{i,y} \frac{\partial}{\partial u_i} + f_i \frac{\partial}{\partial u_{i,x}} + f_{i,x} \frac{\partial}{\partial u_{i,xx}} + \cdots \right).$$

Here $f_i = p_i u_{i,x} + r_i u_{i,y} + q_i$ is the right hand side of lattice (1.2). Therefore, the function $I$ satisfies the equation $YI = 0$. Coefficients of the equation $YI = 0$ depend on the variables $u_{i,y}$, whereas a solution $I$ is independent of $u_{i,y}$. Hence, $I$ satisfies the system of linear equations:

$$YI = 0, \quad X_j I = 0, \quad j = 0, \ldots, N, \quad (2.2)$$

where $X_i = \frac{\partial}{\partial u_{i,y}}$. It follows from (2.2) that the commutator $Y_i = [X_i, Y] = X_i Y - Y X_i$ of operators $Y$ and $X_i$, $i = 0, 1, \ldots, N$ also annihilates $I$. In the case of lattice (1.2) operator $Y$ can be represented as:

$$Y = \sum_{i=0}^{N} u_{i,y} Y_i + R, \quad (2.3)$$

where $Y_i$ and $R$ are defined as

$$Y_i = \frac{\partial}{\partial u_i} + X_i (f_i) \frac{\partial}{\partial u_{i,x}} + X_i (D_x f_i) \frac{\partial}{\partial u_{i,xx}} + \cdots$$

$$= \frac{\partial}{\partial u_i} + r_i \frac{\partial}{\partial u_{i,x}} + (D_x (r_i)) \frac{\partial}{\partial u_{i,xx}} + \cdots \quad (2.4)$$

$$R = \sum_{i=0}^{N} (u_{i,x} p_i + q_i) \frac{\partial}{\partial u_i} + (D_x (u_{i,x} p_i + q_i)) \frac{\partial}{\partial u_{i,xx}} + \cdots$$

Let $\mathcal{F}$ be a ring of locally analytical functions of the dynamical variables $u, u_x, u_{xx}, \ldots$. We consider the Lie-Rinehart algebra $\mathcal{L}(y, N)$ over the ring $\mathcal{F}$ generated by differential operators $Y, Y_0, Y_1, \ldots, Y_N$. We call this algebra the characteristic Lie algebra of system (2.1) along the $y$-direction. We shall show that we can multiply the elements in the algebra by functions depending on finitely many dynamical variables; this fact distinguishes our algebra from an ordinary Lie algebra. The characteristic Lie algebra of system (2.1) along the $x$-direction is defined in the same way.

Now we shall work with the operators in the algebra $\mathcal{L}(y, N)$. Algebra $\mathcal{L}(y, N)$ is of a finite dimension if there exist a finite basis $Z_1, Z_2, \ldots, Z_k$ consisting of linearly independent operators such that each element $Z \in \mathcal{L}(y, N)$ is represented as a linear combination $Z = a_1 Z_1 + a_2 Z_2 + \ldots + a_k Z_k$, where the coefficients $a_1, a_2, \ldots, a_k$ are analytic functions depending on the dynamical variables defined in an open domain. Then the identity $a_1 Z_1 + a_2 Z_2 + \ldots + a_k Z_k = 0$ implies that $a_1 = a_2 = \ldots = a_k = 0$. System (2.1) is integrable in the sense of Darboux if and only if the characteristic Lie algebras in both directions are of a finite dimension [14].

In our study, we shall apply the operator $D_x$ to smooth functions depending on the dynamical variables $u, u_x, u_{xx}, \ldots$. On this class of functions, we obtain the following commutation relations for the operators $Y_i, R$:

$$[D_x, Y_i] = -r_i Y_i, \quad (2.5)$$

$$[D_x, R] = - \sum_{i=0}^{N} (u_{i,x} p_i + q_i) Y_i,$$

The following statement holds [13] [19] [14].
Lemma 2.1. If a vector field
\[
Z = \sum_i z_{1i} \frac{\partial}{\partial u_{ix}} + z_{2i} \frac{\partial}{\partial u_{ixx}} + \cdots
\]
solves the equation \([D_x, Z] = 0\), then \(Z = 0\).

We shall also use the standard notation \(\text{ad}_X(Z) := [X, Z]\).

The key method, on which the classification algorithm is based, is a test sequence method. We call a sequence of operators \(W_0, W_1, W_2, \ldots\) in the algebra \(\mathcal{L}(y, N)\) a test sequence if
\[
[D_x, W_m] = \sum_{j=0}^{m} w_{j,m} W_j
\]
holds true for all \(m\). The test sequence allows us to derive integrability conditions for hyperbolic type system (2.1), see [24, 14, 15].

The first step of our study is to define the functions \(p_n, r_n\). Let us note that when we search the function \(r_n\) we study the subalgebra Lie generated by the operators \(Y_i\), see (2.4). It follows from (2.3), (2.4), (2.5) that this subalgebra coincides with the Lie algebra of a hyperbolic type system corresponding to the lattice
\[
\begin{align*}
  u_{n,xy} &= r_n(u_{n+1}, u_n, u_{n-1}) u_{n,y}, \quad (2.6) \\
\end{align*}
\]

The following statement holds true for this lattice.

Lemma 2.2. If lattice (2.6) is integrable in the sense of Definition 1.1 then it is reduced by point transformations to one of the following forms:
\[
\begin{align*}
  u_{n,xy} &= (e^{u_n-u_{n-1}} - e^{u_{n+1}-u_n}) u_{n,y}, \quad (2.7) \\
  u_{n,xy} &= (-u_{n+1} + 2u_n - u_{n-1}) u_{n,y}. \quad (2.8)
\end{align*}
\]

Proof of Lemma (2.2) is given in Section 3.

Remark 2.1. If the function \(r_n\) depends only on the variable \(u_n\), that is \(r_n = r_n(u_n)\), then
\[
[Y_k, Y_j] = 0
\]
for all \(k, j\) and system (2.6) splits into the system of independent equations \(u_{n,xy} = r_n(u_n) u_{n,y}\). This system has integrals in the direction we consider. We mention that a wide class of scalar equations of the form \(u_{x,y} = f(u, u_x, u_x)\) was studied in [14] within the characteristic Lie algebras approach. But the case \(r_n = r_n(u_n)\) or \(p_n = p_n(u_n)\) holds for lattice (1.2) and is to be studied, see Section 4.

3. Integrability conditions

3.1. The first test sequence. Let us define a sequence of operators in the characteristic algebra \(\mathcal{L}(y, N)\) by the reccurent formula:
\[
Y_0, \quad Y_1, \quad W_1 = [Y_0, Y_1], \quad W_2 = [Y_0, W_1], \quad \ldots \quad W_{k+1} = [Y_0, W_k], \quad \ldots \quad (3.1)
\]
The following commutation relations are valid for the first elements of the sequence (3.1), see formula (2.5):
\[
[D_x, Y_0] = -r_0 Y_0, \quad [D_x, Y_1] = -r_1 Y_1. \quad (3.2)
\]
By using the Jacobi identity we get the formulae
\[
[D_x, W_1] = -(r_1 + r_0)W_1 - Y_0(r_1)Y_1 + Y_1(r_0)Y_0, \quad (3.3)
\]
\[
[D_x, W_2] = -(r_1 + 2r_0)W_2 - Y_0(2r_1 + r_0)W_1 - Y_0^2(r_1)Y_1 + Y_0Y_1(r_0)Y_0, \quad (3.4)
\]
\[
[D_x, W_3] = -(r_1 + 3r_0)W_3 - Y_0(3r_1 + 3r_0)W_2 - Y_0^2(3r_1 + r_0)W_1
- Y_0^3(r_1)Y_1 + Y_0^2Y_1(r_0)Y_0, \quad (3.5)
\]
\[
[D_x, W_4] = -(r_1 + 4r_0)W_4 - Y_0(4r_1 + 6r_0)W_3 - Y_0^2(6r_1 + 4r_0)W_2
- Y_0^3(4r_1 + r_0)W_1 - Y_0^4(r_1)Y_1 + Y_0^3Y_1(r_0)Y_0. \quad (3.6)
\]

It can be proved by induction that (3.1) is a test sequence. Moreover, for \( k \geq 4 \)
\[
[D_x, W_k] = a_k W_k + b_k W_{k-1} + s_k W_{k-2} + t_k W_{k-3} + \cdots, \quad (3.7)
\]
where
\[
a_k = -(r_1 + kr_0), \quad b_k = \frac{k - k^2}{2}Y_0(r_0) - Y_0(r_1)k, \quad (3.8)
\]
\[
s_k = -Y_0^2(3r_1 + r_0) + \frac{1}{2}(k - 3)Y_0(q_3 + q_{k-1}),
\]
\[
t_k = -Y_0^3(4r_1 + r_0) + \frac{1}{2}(k - 4)Y_0(s_4 + s_{k-1}).
\]

By assumption, in the algebra \( \mathcal{L}(y, N) \) there are finitely many linearly independent elements of sequence (3.1). Therefore, there exists \( M \) such that
\[
W_M = \lambda W_{M-1} + \cdots, \quad (3.9)
\]
the operators \( Y_0, Y_1, W_1, \ldots, W_{M-1} \) are linearly independent, the dots stand for a linear combination of the operators \( Y_0, Y_1, W_1, \ldots, W_{M-2} \).

Let us consider the first three elements.

**Lemma 3.1.** If condition (1.4) holds, then the operators \( Y_0, Y_1, W_1 \) are linear independent. Otherwise, if \( r_0 = r_0(u_0) \) depends only on the variable \( u_0 \), then \( W_1 = 0 \).

**Proof.** Let \( r_0 \) depend on at least one of the variables \( u_{-1}, u_1 \). We are going to prove that \( Y_0, Y_1, W_1 \) are linear independent in this case. We argue by contradiction assuming that
\[
\lambda_1 W_1 + \mu_1 Y_1 + \mu_0 Y_0 = 0.
\]

The operators \( Y_0, Y_1 \) are of the form
\[
Y_0 = \frac{\partial}{\partial u_0} + \cdots, \quad Y_1 = \frac{\partial}{\partial u_1} + \cdots,
\]
while \( W_1 \) contains terms of the form \( \frac{\partial}{\partial u_0} \) and \( \frac{\partial}{\partial u_1} \). Hence, the coefficients \( \mu_1, \mu_0 \) are equal to zero. If \( \lambda_1 \neq 0 \), then \( W_1 = 0 \). We apply the operator \( \text{ad}_{D_x} \) to both sides of the last identity, then by virtue of (3.2) we obtain the equation
\[
Y_0(r_1)Y_1 - Y_1(r_0)Y_0 = 0.
\]
It implies that \( Y_0(r_1) = r_{1,u_0} = 0 \) and \( Y_1(r_0) = r_{0,u_1} = 0 \). This is equivalent to \( r_{0,u_{-1}} = 0, \) \( r_{0,u_1} = 0 \) and we arrive at a contradiction to condition (1.4).

By direct calculation of the operator
\[
W_1 = [Y_0, Y_1] = Y_0Y_1 - Y_1Y_0
\]
and using formula (2.4), we prove the second part of the lemma. The proof is complete.

In what follows we assume that \( M \geq 2 \) and condition (1.4) holds.
Lemma 3.2. If relation (3.9) holds true for $M \geq 2$, then the function $r_0$ has one of the following forms:
i) if $\lambda = 0$, then
$$r_0(u_1, u_0, u_{-1}) = \alpha(u_{-1}) - \frac{2}{M - 1} \alpha(u_0) + \delta(u_1); \quad (3.10)$$
ii) if $\lambda \neq 0$, then
$$r_0(u_1, u_0, u_{-1}) = \beta(u_{-1}) e^{-\frac{2}{M(M - 1)} \lambda u_0} + \psi(u_0, u_1), \quad (3.11)$$
where functions $\beta$ and $\psi$ satisfy the equation
$$\lambda \psi(u_0, u_1) + \frac{1}{2} M(M - 1) \psi_{u_0}(u_0, u_1) + M e^{-\frac{2}{M(M - 1)} \lambda u_1} \beta'(u_0) = 0. \quad (3.12)$$

Proof. We apply the operator $\text{ad}_D$ to both sides of identity (3.9). Combining the coefficients before $W_{M-1}$, we get the equation:
$$D_x(\lambda) = \lambda(a_M - a_{M-1}) + b_M. \quad (3.13)$$
We substitute formulae (3.8) into (3.13):
$$D_x(\lambda) = -r_0 \lambda - \frac{M(M - 1)}{2} r_{0,u_0} - Mr_{1,u_0}. \quad (3.14)$$
From identity (3.14) it follows that $\lambda$ is a constant and
$$r_0 \lambda + \frac{M(M - 1)}{2} r_{0,u_0} + Mr_{1,u_0} = 0. \quad (3.15)$$
Let us apply the operator $\frac{\partial}{\partial u_2}$ to (3.15):
$$Mr_{1,u_0,u_2} = 0.$$
This is equivalent to $r_0,u_{-1},_1 = 0$ and, hence,
$$r_0(u_1, u_0, u_{-1}) = \varphi(u_{-1}, u_0) + \psi(u_0, u_1). \quad (3.16)$$
We substitute function (3.15) into (3.14):
$$\lambda \varphi_{u_{-1}} + \frac{M(M - 1)}{2} \varphi_{u_{0u_{-1}}} = 0. \quad (3.17)$$
We consider two different cases:
i) $\lambda = 0$;
ii) $\lambda \neq 0$.
If i) holds, then $\varphi_{u_{0u_{-1}}} = 0$, so that $\varphi(u_{-1}, u_0) = \alpha(u_{-1}) + \beta(u_0)$ and
$$r_0(u_1, u_0, u_{-1}) = \alpha(u_{-1}) + \beta(u_0) + \psi(u_0, u_1).$$
We re-denote $\beta + \psi \rightarrow \psi$ and we get
$$r_0(u_1, u_0, u_{-1}) = \alpha(u_{-1}) + \psi(u_0, u_1). \quad (3.18)$$
We substitute (3.18) and $\lambda = 0$ into (3.15):
$$\frac{M(M - 1)}{2} \psi_{u_0}(u_0, u_1) + Ma'(u_0) = 0. \quad (3.19)$$
Applying the operator $\frac{\partial}{\partial u_2}$ to identity (3.19), we obtain $\psi_{u_0u_1} = 0$ and, hence,
$$\psi(u_0, u_1) = \gamma(u_0) + \delta(u_1).$$
We substitute $\psi$ into (3.19) and we find
$$\gamma(u_0) = -\frac{2}{M - 1} \alpha(u_0) + c_1.$$
and, then
\[ r_0(u_1, u_0, u_{-1}) = \alpha(u_{-1}) - \frac{2}{M-1} \alpha(u_0) + \delta(u_1). \]

Let us consider case ii). Solution of equation (3.17) is the function
\[ \varphi(u_{-1}, u_0) = \alpha(u_0) + e^{-\frac{2}{M(M-1)}\lambda u_0} \beta(u_{-1}). \]
Then function (3.16) becomes
\[ r_0(u_1, u_0, u_{-1}) = \alpha(u_0) + e^{-\frac{2}{M(M-1)}\lambda u_0} \beta(u_{-1}) + \psi(u_0, u_1). \]
We redenote \( \alpha + \psi \rightarrow \psi \) and we get
\[ r_0(u_1, u_0, u_{-1}) = e^{-\frac{2}{M(M-1)}\lambda u_0} \beta(u_{-1}) + \psi(u_0, u_1). \]
Substituting \( r_0 \) into (3.15), we obtain (3.12). The proof is complete. \( \square \)

3.2. Second test sequence. We construct the test sequence containing operators \( Y_0, Y_1, \)
\( Y_2 \) and their multiple commutators:
\begin{align*}
Z_0 &= Y_0, & Z_1 &= Y_1, & Z_2 &= Y_2, & Z_3 &= [Y_1, Y_0], & Z_4 &= [Y_2, Y_1], \\
Z_5 &= [Y_2, Z_3], & Z_6 &= [Y_1, Z_3], & Z_7 &= [Y_1, Z_4], & Z_8 &= [Y_1, Z_3].
\end{align*}
The elements \( Z_m, m > 8 \) are defined by the recurrent formula \( Z_m = [Y_1, Z_{m-3}] \).
The following commutation relations hold:
\begin{align*}
[D_x, Y_0] &= -r_0 Y_0, & [D_x, Y_1] &= -r_1 Y_1, & [D_x, Y_2] &= -r_2 Y_2, \quad (3.20) \\
[D_x, Z_3] &= -(r_1 + r_0) Z_3 + Y_0(r_1) Y_1 - Y_1(r_0) Y_0, \quad (3.21) \\
[D_x, Z_4] &= -(r_2 + r_1) Z_4 + Y_1(r_2) Y_2 - Y_2(r_1) Y_1, \quad (3.22) \\
[D_x, Z_5] &= -(r_0 + r_1 + r_2) Z_5 - Y_2(r_1 + r_0) Z_3 + Y_0(r_1) Z_4 \\
&\quad + Y_2 Y_0(r_1) Y_1 - Y_2 Y_1(r_0) Y_0, \quad (3.23) \\
[D_x, Z_6] &= -(r_0 + 2r_1 + r_2) Z_6 - Y_1(2r_0 + r_1) Z_3 + Y_0 Y_0(r_1) Y_1 - Y_1^2(r_0) Y_0, \\
[D_x, Z_7] &= -(2r_1 + r_2) Z_7 - Y_1(r_1 + 2r_2) Z_4 + Y_1^2(r_2) Y_2 - Y_1 Y_2(r_1) Y_1, \\
[D_x, Z_8] &= -(r_0 + 2r_1 + r_2) Z_8 + Y_0(r_1) Z_7 - Y_2(r_0 + r_1) Z_6 - Y_1(r_0 + r_1 + r_2) Z_5 \\
&\quad + Y_1 Y_0(r_1) Z_4 - Y_1 Y_2(r_1) Z_3 + Y_1 Y_2 Y_0(r_1) Y_1 - Y_1 Y_2 Y_1(r_0) Y_0. \quad (3.24)
\end{align*}
We recall that we assume condition (1.4), otherwise, starting with \( Z_3 \), all elements of the
sequence vanish.

**Lemma 3.3.** The operators \( Z_0, Z_1, \ldots, Z_5 \) are linearly independent.

**Proof.** It is easy to show that the operators \( Z_0, Z_1, \ldots, Z_4 \) are linearly independent; this is
similar to the proof of Lemma 3.1. We prove Lemma 3.3 by arguing by contradiction. We suppose that
\[ Z_5 = \sum_{j=0}^{4} \lambda_j Z_j. \quad (3.25) \]
We apply the operator $\text{ad}_{D_x}$ to both sides of identity (3.25), and we use formulae (3.21)–(3.23) to simplify the obtained identity:

$$-(r_0 + r_1 + r_2) \sum_{j=0}^{4} \lambda_j Z_j + Y_0(r_1)Z_4 - Y_2(r_1)Z_3 + Y_2Y_0(r_1)Y_1 - Y_2Y_1(r_0)Y_0$$

$$= \sum_{j=0}^{4} D_x(\lambda_j)Z_j + \lambda_4(-(r_2 + r_1)Z_4 + Y_1(r_2)Y_2 - Y_2(r_1)Y_1)$$

$$+ \lambda_3(-(r_1 + r_0)Z_3 + Y_0(r_1)Y_1 - Y_1(r_0)Y_0)$$

$$- \lambda_2r_2Y - 2 - \lambda_1r_1Y - \lambda_0r_0Y_0.$$  

Combining the coefficients at $Z_4$ in (3.26), we get the equation:

$$D_x(\lambda_4) = -r_0\lambda_4 + r_{1,u_0}.$$  

This identity implies that $\lambda_4$ is a constant and

$$-r_0\lambda_4 + r_{1,u_0} = 0.$$  

(3.27)

We shall study this equation in two different cases i) and ii):

i) If $r_0$ is defined by formula (3.10), then (3.27) becomes

$$-\left(\alpha(u_{-1}) - \frac{2}{M-1}\alpha(u_0) + \delta(u_1)\right)\lambda_4 + \alpha'(u_0) = 0.$$  

(3.28)

If $\lambda_4 \neq 0$, then (3.28) implies that functions $\alpha, \delta$ are constants since the variables $u_1$, $u_0$, $u_{-1}$ are independent. Then we get that $r_0$ is a constant that contradicts condition (1.4). If $\lambda_4 = 0$, then it follows from (3.28) that $\alpha'(u_0) = 0$ and

$$r_0(u_1, u_0, u_{-1}) = \delta(u_1).$$  

(3.29)

ii) If $r_0$ is defined by (3.11), then identity (3.27) becomes

$$-\left(\beta(u_{-1})e^{-\frac{2}{M-1}u_0} + \psi(u_0, u_1)\right)\lambda_4 + \beta'(u_0)e^{-\frac{2}{M-1}\lambda u_1} = 0.$$  

(3.30)

We apply the operator $\frac{\partial}{\partial u_{-1}}$ to (3.30):

$$\beta'(u_{-1})e^{-\frac{2}{M-1}\lambda u_0}\lambda_4 = 0.$$  

If $\lambda_4 = 0$, then it follows from (3.30) that $\beta'(u_0) = 0$ and, hence, $\beta = c_4$, where $c_4$ is a constant. If $\lambda_4 \neq 0$, then it follows from (3.30) that

$$\beta(u_{-1})e^{-\frac{2}{M-1}u_0} + \psi(u_0, u_1) = 0.$$  

The expression in the left hand side of the last identity coincides exactly with $r_0(u_1, u_0, u_{-1})$. Therefore, the last identity contradicts condition (1.4).

Thus, we obtain that $\lambda_4 = 0$ and $r_0$ is defined by the formula

$$r_0(u_1, u_0, u_{-1}) = c_4e^{-\frac{2}{M-1}u_0} + \psi(u_0, u_1).$$  

(3.31)

We collect the coefficients at $Z_3$ in (3.26), take into consideration that $\lambda_4 = 0$, and we obtain the equation

$$D_x(\lambda_3) = -r_2\lambda_3 - r_{1,u_2}.$$  

Hence, $\lambda_3$ is a constant, and

$$r_2\lambda_3 + r_{1,u_2} = 0.$$  

Applying the shift operator, we get the equation

$$r_1\lambda_3 + r_{0,u_1} = 0.$$  

(3.32)
i) Let us substitute the function \( r_0 \) defined by formula (3.29) into (3.32):

\[
\delta(u_2)\lambda_3 + \delta'(u_1) = 0.
\]

A simple analysis of the last equation gives the contradiction to condition (1.4).

ii) Let us substitute the function \( r_0 \) defined by formula (3.31) into (3.32):

\[
\left( c_1 e^{-\frac{\beta(\delta - \gamma)}{\nu_1} + \psi(u_1, u_2)} \right) \lambda_3 + \psi(u_0, u_1) = 0.
\]  \hspace{1cm} (3.33)

We apply the operator \( \frac{\partial}{\partial u_2} \) to both sides of identity (3.33) \( \psi(u_1, u_2)\lambda_3 = 0 \). Studying (3.33) in this case, we arrive to a contradiction to condition (1.4).

Otherwise, if \( \lambda_3 \neq 0 \), then the expression in the left hand side of identity (3.33), coinciding with \( r_1 \), is equal to zero. Thus, we obtain the contradiction to condition (1.4). The proof is complete.

For further purposes, it is convenient to divide sequence (3.26) into three subsequences \( \{ Z_{3m+1} \} \), \( \{ Z_{3m+2} \} \)

**Lemma 3.4.** Operator \( \text{ad}_{\mathcal{D}_x} \) acts on sequence (3.26) according the following formulae:

\[
[D_x, Z_{3m}] = -(r_0 + mr_1)Z_{3m} + \left(\frac{m-m^2}{2}Y_1(r_1) - mY_1(r_0)\right)Z_{3m-3} + \cdots,
\]

\[
[D_x, Z_{3m+1}] = -(r_2 + mr_1)Z_{3m+1} + \left(\frac{m-m^2}{2}Y_1(r_1) - mY_1(r_2)\right)Z_{3m-2} + \cdots,
\]

\[
[D_x, Z_{3m+2}] = -(r_0 + mr_1 + r_2)Z_{3m+2} + Y_0(r_1)Z_{3m+1} - Y_2(r_1)Z_{3m} - (m-1)\left(\frac{m}{2}Y_1(r_1) + Y_1(r_0 + r_2)\right)Z_{3m-1} + \cdots.
\]

Lemma 3.4 can be easily proved by induction.

**Theorem 3.1.** Assume that \( Z_{3k+2} \) is a linear combination

\[
Z_{3k+2} = \lambda_k Z_{3k+1} + \mu_k Z_{3k} + \nu_k Z_{3k-1} + \cdots
\]  \hspace{1cm} (3.34)

of the previous terms in sequence (3.26) and none of the operators \( Z_{3j+2} \) for \( j < k \) is a linear combination of operators \( Z_s \) with \( s < 3j + 2 \). Then the coefficient \( \nu_k \) satisfies the equation

\[
D_x(\nu_k) = -r_1\nu_k - \frac{k(k-1)}{2}Y_1(r_1) - (k-1)Y_1(r_0 + r_2).
\]  \hspace{1cm} (3.35)

**Lemma 3.5.** Suppose that the assumptions of Theorem 3.1 are satisfied and the operator \( Z_{3k} \) (the operator \( Z_{3k+1} \)) is linearly expressed in terms of the operators \( Z_i \), \( i < 3k \). Then in this decomposition the coefficient at \( Z_{3k-1} \) vanishes.

**Proof.** We argue by contradiction. Suppose that

\[
Z_{3k} = \lambda Z_{3k-1} + \cdots
\]  \hspace{1cm} (3.36)

and \( \lambda \neq 0 \). We apply the operator \( \text{ad}_{\mathcal{D}_x} \) to both sides of identity (3.36). Using formulae from Lemma 3.4 we get

\[
-(r_0 + kr_1)\lambda Z_{3k-1} + \cdots = D_x(\lambda)Z_{3k-1} - \lambda(r_0 + (k-1)r_1 + r_2)Z_{3k-1} + \cdots
\]

Collecting coefficients at \( Z_{3k-1} \), we obtain

\[
D_x(\lambda) = \lambda(r_2 - r_1)k.
\]

This equation implies that \( \lambda \) is a constant and \( \lambda(r_2 - r_1)k = 0 \). Then \( r_2 = r_1 = \text{const} \) that contradictions condition (1.4). The proof is complete.
In order to prove Theorem 3.1 we apply the operator $\text{ad}_{D_x}$ to both sides of the identity (3.34). Then we simplify a obtained identity using formulae from Lemma 3.4. Collecting coefficients at $Z_{3k-1}$, we obtain equation (3.35).

The next step of our work is studying equation (3.35) as $r_0$ is defined by formulae (3.10) or (3.11) under condition (1.4) and for $M \geq 2$, $k \geq 2$.

We find exact values of coefficients in equation (3.35) and substitute them into (3.35):

$$D_x(\nu_k) = -r_1\nu_k - \frac{k(k-1)}{2} r_{1,u_1} - (k-1)(r_{u_1} + r_{2,u_1}).$$

This equation implies that $\nu_k$ is a constant and, hence,

$$\nu_k r_1 + \frac{k(k-1)}{2} r_{1,u_1} + (k-1)(r_{u_1} + r_{2,u_1}) = 0. \tag{3.37}$$

**Lemma 3.6.** If relations (3.39), (3.34) hold true for some $M \geq 2$, $k \geq 2$, and condition (1.4) holds true, then

i) if $\lambda = 0$, $\nu_k = 0$, then

$$r_n(u_{n+1}, u_n, u_{n-1}) = \alpha(u_{n-1}) - \frac{2}{M-1} \alpha(u_n) + \left(\frac{k}{M-1} - 1\right) \alpha(u_{n+1}) + c_1; \tag{3.38}$$

ii) if $\lambda \neq 0$, $\nu_k = 0$, $k = 2$, $M \neq 3$, then

$$r_n(u_{n+1}, u_n, u_{n-1}) = e^{-hu_{n-1}} + ce^{u_{n+1}-hu_n}; \tag{3.39}$$

iii) if $\lambda \neq 0$, $\nu_k \neq 0$, then $r_n$ is defined by formula (3.39). The proof of this lemma is rather complicated and is presented in Appendix.

We proceed to relations (3.9), (3.34).

We need another one test sequence:

$$W_0, Y_1, W_1 = [Y_1, W_0], W_2 = [Y_1, W_1], \ldots W_{k+1} = [Y_1, W_k]. \ldots$$

The following commutation relation hold:

$$[D_x, W_1] = -(r_1 + r_0)W_1 + Y_0(r_1)Y_1 - Y_1(r_0)Y_0, \tag{3.40}$$

$$[D_x, W_2] = -(2r_1 + r_0)W_2 - Y_1(r_1 + 2r_0)W_1 + Y_1Y_0(r_1)Y_1 - Y_1^2(r_0)Y_0, \tag{3.41}$$

$$[D_x, W_3] = -3(r_1 + r_0)W_3 - Y_1(3r_1 + 3r_0)W_2 - Y_1^2(r_1 + 3r_0)W_1$$

$$+ Y_1^3Y_0(r_1)Y_1 - Y_1^3(r_0)Y_0, \tag{3.42}$$

$$[D_x, W_4] = -(4r_1 + r_0)W_4 - Y_1(6r_1 + 4r_0)W_3 - Y_1^2(4r_1 + 6r_0)W_2$$

$$- Y_1^3(r_1 + 4r_0)W_1 + Y_1^3Y_0(r_1)Y_1 - Y_1^4(r_0)Y_0. \tag{3.43}$$

It is easy to prove that

$$[D_x, W_k] = a_kW_k + b_kW_{k-1} + c_kW_{k-2} + \ldots, \tag{3.44}$$

for $k \geq 3$, where

$$a_k = -(kr_1 + r_0), \quad b_k = \frac{k-k^2}{2} Y_1(r_1) - Y_1(r_0)k,$$

$$c_k = -Y_1^2(r_1 + 3r_0) + \frac{1}{2}(k-3)Y_1(\bar{q}_{k+2} + \bar{q}_{k-1}).$$

We observe that the first terms $Y_0, Y_1, W_1 = -W_1$ obey Lemma 3.1.

We suppose that $\mathcal{L}(y, N)$ is finitely-dimensional, that is, each sequence of its elements terminates at some step. Consequently, there exists $N$ such that:

$$W_N = \Lambda W_{N-1} + \ldots, \tag{3.45}$$

where the operators $Y_0, Y_1, W_1, \ldots, W_{N-1}$ are linearly independent, and the dots stand for linear combination of the operators $Y_0, Y_1, W_1, \ldots, W_{N-2}$.
3.3. Case $M = 2$. Suppose that relation (3.9) holds true for $M = 2$:
\[ W_2 = \lambda W_1 + \varepsilon Y_1 + \eta Y_0. \] (3.46)
We apply the operator $\text{ad}_{D_x}$ to both sides of identity (3.46) and we get:
\[-(r_1 + 2r_0)(\lambda W_1 + \varepsilon Y_1 + \eta Y_0) - Y_0(2r_1 + r_0)W_1 - Y_0^2(r_1)Y_1 + Y_0Y_1(r_0)Y_0 = \lambda( -(r_1 + r_0)W_1 - Y_0(r_1)Y_1 + Y_1(r_0)Y_0) - \varepsilon r_1 Y_1 - \eta r_0 Y_0.\]
Collecting the coefficients at independent operators $W_1$, $Y_1$, $Y_0$, we obtain the system
\[ r_0 \lambda + 2r_1u_0 + r_{0,u_0} = 0, \] (3.47)
\[ 2r_0 \varepsilon + r_1u_0u_0 - \lambda r_1u_0 = 0, \] (3.48)
\[ -(r_1 + r_0)\eta + r_{0,u_0}u_1 - \lambda r_{0,u_1} = 0. \] (3.49)
3.3.i) Let us consider the case when the function $r_n$ is described by formula (3.38) and $\lambda = 0$. We substitute function (3.38) and $\lambda = 0$ into system (3.47)–(3.49). Then equation (3.47) becomes identity and we arrive to the system:
\[ 2(\alpha(u_{-1}) - 2\alpha(u_0) + (k - 1)\alpha(u_1) + c_1)\varepsilon + \frac{d^2\alpha(u_0)}{du_0^2} = 0, \]
\[ (\alpha(u_0) + 2\alpha(u_1) - (k - 1)\alpha(u_2) + 2c_1 + \alpha(u_{-1}) + (k - 1)\alpha(u_1))\eta = 0. \]
This system yields that
\[ \varepsilon = \eta = 0, \quad \alpha(u_0) = C_1u_0 + C_2, \]
\[ r_0(u_1, u_0, u_{-1}) = (k - 1)C_1u_1 - 2C_1u_0 + C_1u_{-1} + C_3, \]
where $C_3 = -2C_2 + kC_2 + c_1$. We will study the lattice corresponding to this function, in Section 3.5.i, see (3.62).
3.3.ii) Let us consider the case when the function $r_n$ is described by formula (3.39) and $\lambda \neq 0$. System (3.47)–(3.49) casts into the form:
\[ (\lambda + 1)e^{u_0-hu_{-1}} + (-ch + \lambda c - 2h)e^{u_1-hu_0} = 0, \]
\[ 2\varepsilon e^{u_0-hu_{-1}} + (2\varepsilon c + h^2 + \lambda h)e^{u_1-hu_0} = 0, \]
\[ -\eta e^{u_0-hu_{-1}} + (-\eta - \varepsilon c - ch - \lambda c)e^{u_1-hu_0} - \eta \varepsilon e^{u_2-hu_1} = 0. \]
A simple analysis of the last system leads us to the identities $\lambda = -1$, $h = 1$, $c = -1$, $\varepsilon = \eta = 0$. We get that $r_n$ has the following form:
\[ r_n(u_{n+1}, u_n, u_{n-1}) = e^{u_n-u_{n-1}} - e^{u_{n+1}-u_n}. \] (3.50)
And $W_2 = -W_1$, $W_2 = W_{-1}$.
Now let us substitute (3.50) into (3.37):
\[ \left( \nu_k + \frac{1}{2}k^2 - \frac{3}{2}k + 1 \right) e^{u_1-u_0} + \left( -\nu_k + \frac{1}{2}k^2 - \frac{3}{2}k + 1 \right) e^{u_2-u_1} = 0, \]
which implies $\nu_k = 0$, $k = 2$.
Thus, relation (3.34) is of the form
\[ Z_8 = \rho Z_4 + \sigma Z_3 + \tau Z_2 + \phi Z_1 + \pi Z_0, \] (3.51)
and
\[ Z_6 = [Y_1, [Y_1, Y_0]] = W_2 = W_{-1} = [Y_1, Y_0] = Z_3, \]
\[ Z_7 = [Y_1, [Y_2, Y_1]] = D_n[Y_0, [Y_1, Y_0]] = -D_nW_2 = D_nW_1 = -Z_4. \]
Commutation relation (3.24) become
\[ [D_x, Z_8] = -(r_0 + 2r_1 + r_2)Z_4 - Y_1(r_0 + r_1 + r_2)Z_5 + (Y_1Y_0(r_1) - Y_0(r_1))Z_4 \]
\[ - (Y_2(r_0 + r_1) + Y_1Y_2(r_1))Z_3 + Y_1Y_2Y_0(r_1)Y_1 - Y_1Y_2Y_1(r_0)Y_0. \] (3.52)

We apply the operator \( \text{ad}_{D_x} \) to both sides of identity (3.51) and take into consideration the formulae (3.20)–(3.23), (3.52), then we collect coefficients at independent operators \( Z_4, Z_3, Z_2, Z_1, Z_0 \):

\[
- (e^{u_0 - u_1} - e^{u_2 - u_1})\rho = 0, \quad (-e^{u_3 - u_2} + e^{u_1 - u_0})\sigma = 0, \\
- (e^{u_0 - u_1} + e^{u_1 - u_0} - 2e^{u_2 - u_1})\tau + \rho e^{u_2 - u_1} = 0, \\
- (e^{u_0 - u_1} + e^{u_1 - u_0} - e^{u_3 - u_2})\phi - \rho e^{u_2 - u_1} + \sigma e^{u_1 - u_0} = 0, \\
- (2e^{u_1 - u_0} - e^{u_2 - u_1} - e^{u_3 - u_2})\pi - \sigma e^{u_1 - u_0} = 0.
\]

It is clear that \( \rho = \sigma = \tau = \phi = \pi = 0 \). Hence, \( Z_8 = 0 \).

3.4. Case \( M = 3 \). Suppose that relation (3.9) holds true for \( M = 3 \):
\[
W_3 = \lambda W_2 + \rho W_1 + \varepsilon Y_1 + \eta Y_0.
\] (3.53)

We apply the operator \( \text{ad}_{D_x} \) to both sides of identity (3.53) and use formulae (3.2), (3.3), (3.4), (3.5). Collecting coefficients at the independent operators, we obtain the system
\[
r_0\lambda + 3r_{1,u_0} + 3r_{0,u_0} = 0, \quad \lambda(2r_1,u_0 + r_{0,u_0}) - 3r_{1,u_0,u_0} - r_{0,u_0,u_0} = 0, \quad 3r_0\varepsilon + \lambda r_{1,u_0,u_0} + pr_{0,u_0} - r_{1,u_0,u_0} = 0, \\
- (r_1 + 2r_0)\eta - \lambda r_{0,u_0,u_1} - pr_{0,u_1} + r_{0,u_0,u_1} = 0.
\] (3.54)–(3.57)

3.4.i) Let us consider case (3.38), \( \lambda = 0 \). It follows from equations (3.54)–(3.57) that
\[
\alpha(u_n) = C_1u_n + C_2, \quad r_n(u_{n+1}, u_n, u_{n-1}) = \frac{k-2}{2} C_1 u_{n+1} - C_1 u_n + C_1 u_{n-1} + C_3,
\]
where \( C_3 = \frac{1}{2} C_2 k - C_2 + c_1 \). Further study of the lattice with \( r_n \) defined by this formula is provided in 3.5.1, see (3.62).

3.4.ii) Let us consider case (3.39) \( \lambda \neq 0 \). We substitute \( r_n \) into (3.47)–(3.49). Studying this system, we obtain that \( \lambda = -3, \rho = -2, \varepsilon = \eta = 0, h = 1, c = -\frac{1}{2} \). The function (3.39) becomes
\[
r_n(u_{n+1}, u_n, u_{n-1}) = e^{u_{n+1} - u_{n-1}} + \frac{1}{2} e^{u_{n+1} - u_n}.
\]

We substitute this function into equation (3.37) and we get \( k = 1 \) or \( k = \frac{5}{2} \). These identities contradict condition \( k \geq 2 \).

3.5. Case \( M > 3 \). Let the following relation be true for \( M > 3 \)
\[
W_M = \lambda W_{M-1} + \rho W_{M-2} + \kappa W_{M-3} + \cdots
\] (3.58)

Taking into account formula (3.7), we apply the operator \( \text{ad}_{D_x} \) to both sides of the above identity:
\[
a_M(\lambda W_{M-1} + \rho W_{M-2} + \kappa W_{M-3} + \cdots) + b_M W_{M-1} + s_M W_{M-2} + t_M W_{M-3} + \cdots
\]
\[
= \lambda(a_M - 1)W_{M-1} + b_{M-1} W_{M-2} + s_{M-1} W_{M-3} + \cdots \\
+ \rho(a_M - 2) W_{M-2} + b_{M-2} W_{M-3} + \cdots + \kappa(a_M - 3) W_{M-3} + \cdots
\]

We collect coefficients at the independent operators \( W_{M-1}, W_{M-2}, W_{M-3} \):
\[
\lambda(a_M - a_{M-1}) + b_M = 0, \quad \rho(a_M - a_{M-2}) + s_M - \lambda b_{M-1} = 0, \\
\kappa(a_M - a_{M-3}) + t_M - \lambda s_{M-1} - \rho b_{M-2} = 0.
\] (3.59)–(3.61)
3.5.1) By system (3.59)–(3.61) we obtain that $\alpha(u_n) = C_1u_n + C_2$ and
\[ r_n(u_{n+1}, u_n, u_{n-1}) = \frac{k - (M - 1)}{M - 1} C_1 u_{n+1} - \frac{2C_1}{M - 1} u_n + C_1 u_{n-1} + C_3, \]
where
\[ C_3 = \frac{c_1 M - c_1 - 2C_2 + kC_2}{M - 1}. \]

Now consider the function
\[ r_n(u_{n+1}, u_n, u_{n-1}) = c_1 u_{n+1} + c_2 u_n + c_3 u_{n-1} + c_4. \] (3.62)

Commutation relations (3.7), (3.44) become
\[ [D_x, W_k] = a_k W_k + b_k W_{k-1}, \quad [D_x, \overline{W}_k] = \overline{a}_k \overline{W}_k + \overline{b}_k \overline{W}_{k-1}, \] (3.63)
where
\[ a_k = -(r_1 + kr_0), \quad b_k = \frac{k - k^2}{2} c_2 - c_3 k, \] (3.64)
\[ \overline{a}_k = -(kr_1 + r_0), \quad \overline{b}_k = \frac{k - k^2}{2} c_2 - c_1 k. \] (3.65)

Assume that sequence $\{W_n\}$ is terminated at the step $M$:
\[ W_M = \sum_{k=1}^{M-1} \Lambda_{M-k} W_{M-k} + \phi_1 Y_1 + \phi_0 Y_0. \] (3.66)

We apply the operator $ad_{D_x}$ to both sides of identity (3.66)
\[ a_M \left( \sum_{k=1}^{M-1} \Lambda_{M-k} W_{M-k} + \phi_1 Y_1 + \phi_0 Y_0 \right) + b_M W_{M-1} \]
\[ = \sum_{k=1}^{M-2} \Lambda_{M-k}(a_{M-k} W_{M-k} + b_{M-k} W_{M-k-1}) \]
\[ + \Lambda_1 (-r_1 + r_0) W_1 - c_3 Y_1 + c_1 Y_0 - \phi_1 r_1 Y_1 - \phi_0 r_0 Y_0. \]

We collect the coefficients at $W_{M-1}$ in this identity:
\[ \Lambda_{M-1}(a_M - a_{M-1}) + b_M = 0. \]

We substitute formulae (3.64), (3.65) into the last equation:
\[ -\Lambda_{M-1}(c_1 u_1 + c_2 u_0 + c_3 u_{-1} + c_4) + \frac{M - M^2}{2} c_2 - c_3 M = 0. \]

A simple analysis of this equation shows that $\Lambda_{M-1} = 0$ and $c_3 = \frac{1-M}{M} c_2$. Then, collecting coefficients before $W_{M-k}$, $k = 2, \ldots, M - 2$, we arrive at the equations
\[ \Lambda_{M-k}(a_M - a_{M-k}) = 0, \quad k = 2, \ldots, M - 2, \]
which implies $\Lambda_{M-k} = 0$, $k = 2, \ldots, M - 2$. The coefficient at $W_1$ is $\Lambda_1(a_M + r_1 + r_0) = 0$. Then $\Lambda_1 = 0$. The coefficients at $Y_1$ and $Y_0$ read as $(a_M + r_1)\phi_1 = 0$, $(a_M + r_0)\phi_0 = 0$ and hence, $\phi_1 = \phi_0 = 0$. Thus, $W_M = 0$.

Similarly, if sequence $\{\overline{W}_k\}$ is terminated at step $N$, then $c_1 = \frac{1-N}{N} c_2$ and $\overline{W}_N = 0$. As a result, we obtain:
\[ r_n(u_{n+1}, u_n, u_{n-1}) = \frac{1 - N}{2} c_2 u_{n+1} + c_2 u_n + \frac{1 - M}{2} c_2 u_{n-1} + c_4. \]

By rescaling $\frac{a}{2} u_i \to v_i$, the original lattice is reduced to a lattice of the same form with function $r_n$ defined by the formula
\[ r_n(u_{n+1}, u_n, u_{n-1}) = (1 - N) u_{n+1} + 2u_n + (1 - M) u_{n-1} + c, \]
where $c$ is an arbitrary constant. If $4 - M - N \neq 0$, then we exclude constant $c$ by the shift transformation $u \rightarrow u - \frac{c}{4 - M - N}$. If $M + N = 4$, then $M = N = 2$, and $c$ is excluded by the transformation $u_n \rightarrow u_n + \frac{c}{2}n^2$. Thus, the function $r_n$ becomes:

$$r_n(u_{n+1}, u_n, u_{n-1}) = (1 - N)u_{n+1} + 2u_n + (1 - M)u_{n-1}$$

(3.67)

and, in particular,

$$r_n(u_{n+1}, u_n, u_{n-1}) = -u_{n+1} + 2u_n - u_{n-1}.$$  

(3.68)

We substitute (3.67) into (3.37), and we get that $k = M + N - 2$. We substitute (3.68) into (3.37), and we get that $k = 2$.

Let us consider lattice (2.6) when $r_n$ is defined by (3.67). We impose cut-off conditions $u_0 = 0$, $u_{L+1} = 0$ and we reduce this lattice to the following hyperbolic system:

$$
\begin{align*}
    u_{1,xy} &= (2u_1 + pu_2)u_{1,y}, \\
    u_{k,xy} &= (qu_{k-1} + 2u_k + pu_{k+1})u_{k,y}, \quad 2 \leq k \leq L - 1, \\
    u_{L,xy} &= (qu_{L-1} + 2u_L)u_{L,y},
\end{align*}
$$

(3.69)

where $p = 1 - N$, $q = 1 - M$; we recall that $N > 1$, $M > 1$. This system is reduced by differential substitution $v_i = \ln u_{i,y}$ to the exponential system:

$$
\begin{align*}
    v_{1,xy} &= 2e^{v_1} + pe^{v_2}, \\
    v_{k,xy} &= qe^{v_{k-1}} + 2e^{v_k} + pe^{v_{k+1}}, \quad 2 \leq k \leq L - 1, \\
    v_{L,xy} &= qe^{v_{L-1}} + 2e^{v_L}.
\end{align*}
$$

(3.70)

We denote by $A$ the matrix of coefficients before exponents in the right hand side of the system and we denote by $\mathbf{v} = (v_1, v_2, \ldots, v_K)^T$, $e^{\mathbf{v}} = (e^{v_1}, e^{v_2}, \ldots, e^{v_K})^T$ the column vectors. System (3.70) is related with the system

$$
\begin{align*}
    w_{1,xy} &= e^{2w_1 + pw_2}, \\
    w_{k,xy} &= e^{qw_{k-1} + 2w_k + pw_{k+1}}, \quad 2 \leq k \leq L - 1, \\
    w_{L,xy} &= e^{qw_{L-1} + 2w_L}.
\end{align*}
$$

(3.71)

by the following point change of variables

$$
\begin{align*}
    v_1 &= 2w_1 + pw_2, \\
    v_k &= qw_{k-1} + 2w_k + pw_{k+1}, \quad 2 \leq k \leq L - 1, \\
    v_L &= qw_{L-1} + 2w_L.
\end{align*}
$$

(3.72)

System (3.71) is reduced to system (3.69) by differential substitution

$$u_i = w_{i,x}.$$  

(3.73)

It is shown in [11, 25] (see also [14]) that if $A$ is the Cartan matrix of a simple Lie algebra, then the system (3.70) is integrated in quadratures. Comparing the Cartan matrix and matrix $A$, one can see that $p = q = -1$. Thus, we have that $M = N = 2$. In this case we find that $W_2 = 0$, $W_2 = 0$, $Z_\delta = 0$.

Let us show that if systems (3.70), (3.71) is integrable in the sense of Darboux then system (3.69) is integrable in the sense of Darboux, too. Suppose that $T(w_x, w_{xx}, \cdots)$ is an $y$-integral of system (3.71). We change variables by the rule $w_{i,x} = u_i$, $w_{i,xx} = u_{i,x}$ and so on, due to
We shall determine the function \( I(v_y, v_{yy}, \ldots) \). Assume that \( I(v_y, v_{yy}, \ldots) \) is an \( x \)-integral of system (3.69). Using (3.73) and (3.72), we derive:

\[
\mathbf{u} = \mathbf{w}_x = A^{-1}\mathbf{v}_x.
\]

Hence, by virtue (3.70)

\[
\mathbf{u}_y = A^{-1}\mathbf{v}_{xy} = A^{-1}Ae^v = e^v.
\]

We change variables in the function \( I(v_y, v_{yy}, \ldots) \) by the rule \( v_i = \ln u_i, y, v_{i,y} = (\ln u_i)_y \) and so on. Thus, we get an \( x \)-integral \( I((\ln u_{i,y})_y, (\ln u_{i,y})_{yy}, \ldots) \) of system (3.69).

3.5.ii Let us consider case (3.39) and \( \lambda \neq 0 \). We substitute \( r_n \) into system (3.59)–(3.61) and into equation (3.37), we get the following system:

\[
\begin{align*}
A_1 e^{u_0 - hu_1} + B_1 e^{u_1 - hu_0} &= 0, \\
A_2 e^{u_0 - hu_1} + B_2 e^{u_1 - hu_0} &= 0, \\
A_3 e^{u_0 - hu_1} + B_3 e^{u_1 - hu_0} &= 0, \\
A_4 e^{u_1 - hu_0} + B_4 e^{u_2 - hu_1} &= 0.
\end{align*}
\]

Obviously, the coefficients \( A_i, B_i \) at independent exponent functions have to be equal to zero. Thus, we obtain a system of 8 algebraic equations in 8 unknowns \( c, h, \lambda, \rho, \kappa, k, \nu_k \). Studying this system, we get the following possible variants:

\[
M = 4, \quad k = \frac{10}{3}; \quad M = 5, \quad k = \frac{17}{4}; \quad M = 2, \quad k = 2.
\]

All of these variants contradict our assumptions about values of \( k, M \).

Thus, we have proved the following statement.

**Lemma 3.7.** If relations (3.39), (3.34), (3.45) hold true for some \( M \geq 2, k \geq 2, N \geq 2 \), then the function \( r_n \) casts into one of the forms (3.50) or (3.68) up to point transformations.

Lemma 2.2 is implied immediately by Lemma 3.7.

Summarizing the results of this section, we observe that we have lattice (1.2) for further study, where the function \( r_n \) is defined by one of the formulae \( r_n = r_n(u_n) \), (3.50), (3.68). Similarly, function \( p_n \) is defined by one of the following formulae:

\[
\begin{align*}
p_n &= p_n(u_n), \\
p_n(u_{n+1}, u_n, u_{n-1}) &= e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}, \\
p_n(u_{n+1}, u_n, u_{n-1}) &= -u_{n+1} + 2u_n - u_{n-1}.
\end{align*}
\]

4. **Function \( q_n \)**

We recall that the operator \( Y \) can be represented as follows, see formula (2.3):

\[
Y = \sum_i u_{i,y}Y_i + R,
\]

where

\[
\begin{align*}
Y_i &= \frac{\partial}{\partial u_i} + r_i \frac{\partial}{\partial u_{i,x}} + \left( D_x (r_i) + r_i^2 \right) \frac{\partial}{\partial u_{i,xx}} + \cdots \\
R &= \sum_i (u_{i,x}p_i + q_i) \frac{\partial}{\partial u_{i,x}} + \left( D_x (u_{i,x}p_i + q_i) + (u_{i,x}p_i + q_i) r_i \right) \frac{\partial}{\partial u_{i,xx}} + \cdots
\end{align*}
\]

We shall determine the function \( q_n \) by using the operator \( R \). We define a sequence of operators in the characteristic algebra \( \mathcal{L}(y, N) \) by the following recurrent formula:

\[
\begin{align*}
Y_{-1}, \quad Y_0, \quad Y_1, \quad Y_{0,-1} = [Y_0, Y_{-1}], \quad Y_{1,0} = [Y_1, Y_0], \quad (4.1) \\
R_0 = [Y_0, R], \quad R_1 = [Y_0, R_0], \quad R_2 = [Y_0, R_1], \quad \ldots \quad R_{k+1} = [Y_0, R_k].
\end{align*}
\]
For elements of the sequence the following commutation relations hold:

\[
[D_x, Y_{-1}] = -r_- Y_{-1}, \quad [D_x, Y_0] = -r_0 Y_0, \quad [D_x, Y_1] = -r_1 Y_1,
\]

\[
[D_x, Y_{n-1}] = -(r_- + r_0)Y_{n-1} - Y_0(r_-)Y_{-1} + Y_{-1}(r_0)Y_0,
\]

\[
[D_x, Y_{1,0}] = -(r_0 + r_1)Y_{1,0} - Y_1(r_0)Y_0 + Y_0(r_1)Y_1,
\]

\[
[D_x, R] = -\sum_i h_i Y_i, \quad h_i = p_i u_{i,x} + q_i,
\]

\[
[D_x, R_0] = -r_0 R_0 + h_1 Y_{1,0} - h_{-1} Y_{n-1}
\]

\[- Y_0(h_1)Y_1 + (R(r_0) - Y_0(h_0))Y_0 - Y_0(h_{-1})Y_{-1},
\]

\[
[D_x, R_1] = -2r_0 R_1 - Y_0(r_0)R_0 + \cdots ,
\]

\[
[D_x, R_2] = -3r_0 R_2 - 3Y_0(r_0)R_1 - Y_0^2(r_0)R_0 + \cdots,
\]

\[
[D_x, R_3] = -4r_0 R_3 - 6Y_0(r_0)R_2 - 4Y_0^2(r_0)R_1 - Y_0^3(r_0)R_0 + \cdots,
\]

where the dots stand for a linear combinations of the operators $Y_{1,0}, Y_{0,-1}, Y_1, y_0, Y_{-1}$. By induction we prove that the following formula holds for all $n \geq 2$:

\[
[D_x, R_n] = a_n R_n + b_n R_{n-1} + \cdots,
\]

where

\[
a_n = -(n+1)r_0, \quad b_n = -\frac{n^2 + n}{2} Y_0(r_0),
\]

and the dots stand for a linear combination of the operators $R_k, k < n-1, Y_{1,0}, Y_{0,-1}, Y_1, Y_0, Y_{-1}$.

**Lemma 4.1.** If the operator $R_0$ is linearly expressed in terms of operators (4.1)

\[
R_0 = \mu Y_{1,0} + \tilde{\mu} Y_{0,-1} + \nu Y_1 + \eta Y_0 + \varepsilon Y_{-1},
\]

then chain (2.2) is reduced to one of forms (2.7), (2.8) by point transformations.

**Proof.** We apply the operator $D_x$ to both sides of identity (4.2). Collecting the coefficients at independent operators $Y_{1,0}, Y_{0,-1}, Y_1, Y_0, Y_{-1}$, we get the system of equations

\[
D_x(\mu) = r_1 \mu + h_1,
\]

\[
D_x(\tilde{\mu}) = r_{-1} \tilde{\mu} - h_{-1},
\]

\[
D_x(\nu) = (r_1 - r_0) \nu - Y_0(h_1) - \mu Y_0(r_1),
\]

\[
D_x(\eta) = R(r_0) - Y_0(h_0) + \mu Y_1(r_0) - \tilde{\mu} Y_{-1}(r_0),
\]

\[
D_x(\varepsilon) = (r_{-1} - r_0) \varepsilon - Y_0(h_{-1}) + \tilde{\mu} Y_0(r_{-1}.)
\]

We consider equation (4.3):

\[
r_1(u_2, u_1, u_0) \mu + p_1(u_2, u_1, u_0) u_{1,x} + q_1(u_2, u_1, u_0) = D_x(\mu).
\]

A simple analysis of this equation shows that $\mu = \mu(u_1)$ and, hence, this equation splits into two equations

\[
\mu'(u_1) = p_1(u_2, u_1, u_0), \quad r_1(u_2, u_1, u_0) \mu(u_1) + q_1(u_2, u_1, u_0) = 0.
\]

Hence,

\[
p_n(u_{n+1}, u_n, u_{n-1}) = \mu'(u_n), \quad q_n(u_{n+1}, u_n, u_{n-1}) = -r_n(u_{n+1}, u_n, u_{n-1}) \mu(u_n).
\]

Using equation (4.4), we obtain that $\tilde{\mu} = \tilde{\mu}(u_{-1}), \tilde{\mu}(\nu) = -\mu(\nu)$.

We simplify identity (4.5) using (4.8) and we get

\[
D_x(\nu) = (r_1 - r_0) \nu.
\]

It easy to see that $\nu = 0$. Similarly, it follows from (4.7) that $\varepsilon = 0$.

We simplify identity (4.6) as follows:

\[
D_x(\eta) = -p_0 u_0 u_{0,x} - q_0 u_0 - r_0 p_0 + \mu r_0 u_{1} - \tilde{\mu} r_{0,u_{-1}}.
\]
A simple analysis of this equation shows that \( \eta = \eta(u_0) \) and, hence, this equation splits into two equations

\[
\eta'(u_0) = -p_0 u_0, \quad -q_0 u_0 - r_0 p_0 + \mu r_0 u_1 - \tilde{\mu} r_0 u_{-1} = 0. \tag{4.9}
\]

We substitute formulae (4.8) into identities (4.9) and we obtain \( \eta'(u_0) = -\mu''(u_0) \),

\[
r_0 u_0 \mu(u_0) + \mu(u_1) r_0 u_1 + \mu(u_{-1}) r_0 u_{-1} = 0. \tag{4.10}
\]

We substitute the function \( r_n \) defined by formula (3.50) into (4.10), and we get that \( \mu = c \) is an arbitrary constant. Therefore, \( p_n = 0, q_n = -cr_n \), and lattice (1.2) becomes

\[
u_n,\eta = (e^{u_n - u_{-1}} - e^{u_{n+1} - u_n}) u_{n,y} - c(e^{u_n - u_{-1}} - e^{u_{n+1} - u_n}).
\]

The transformation \( u_n - cy \to u_n \) reduces this lattice to (2.7).

If \( r_n \) is defined by (3.68), then (4.10) implies \( \mu = c \), where \( c \) is an arbitrary constant. Hence, \( p_n = 0, q_n = -cr_n \), and lattice (1.2) takes the following form:

\[
u_n,\eta = (-u_{n+1} + 2u_n - u_{n-1}) u_{n,y} + c(-u_{n+1} + 2u_n - u_{n-1}).
\]

The transformation \( u_n - cy \to u_n \) reduces this lattice to (2.8).

If \( r_n = r_n(u_n) \), then it follows from (4.10) that \( \mu = 0 \) or \( r_0 u_0 = 0 \). In the first case formulae (4.8) imply \( p_n = 0, q_n = 0 \). Then chain (1.2) becomes \( u_{n,\eta} = r_n(u_n) u_{n,y} \). In the second case \( r_0 = c_1 \), where \( c_1 \) is an arbitrary constant, hence, by (4.8), \( p_n = \mu'(u_n), q_n = -c_1 \mu(u_n) \), and chain (1.2) casts into the form \( u_{n,\eta} = \mu'(u_n) u_{n,x} + c_1 u_{n,y} - c_1 \mu(u_n) \). The proof is complete.

Suppose that \( R_n \) depends linearly on \( R_k, k < n, Y_{1,0}, Y_{0,-1}, Y_1, Y_0, Y_{-1} \) for some \( n \): \( R_n = \lambda R_{n-1} + \cdots, \quad n > 0 \). \tag{4.11}

Lemma 4.2. If function \( r_n \) has one of forms (3.50), (3.68), then case (4.11) is not realized.

Proof. We apply the operator \( \text{ad}_{D_x} \) to both sides of identity (4.11). Collecting coefficients at \( R_{n-1} \) in obtained relation, we get the equation:

\[
D_x(\lambda) = -r_0 \lambda - \frac{n^2 + n}{2} r_0 u_0.
\]

A simple analysis of this equation shows that \( \lambda \) is a constant, hence

\[
r_0 \lambda + \frac{n^2 + n}{2} r_0 u_0 = 0. \tag{4.12}
\]

Substituting formulae (3.50), (3.68) into (4.12), we get that \( \lambda = 0 \) and \( n^2 + n = 0 \). Hence, \( n = 0 \) or \( n = -1 \). Both solutions contradict the assumption \( n > 0 \). The proof is complete.

Theorem 1.1 is implied Lemma 4.1, 4.2.

5. Appendix. Proof of Lemma 3.6

The proof is a study of equation (3.37):

\[
\nu_k r_1 + \frac{k(k-1)}{2} r_1 u_1 + (k-1)(r_0 u_1 + r_2 u_1) = 0 \tag{5.1}
\]

in different cases (3.10) and (3.11) under conditions (1.4), \( M \geq 2, k \geq 2 \). We denote \( \nu_k = \nu \).

i) We substitute function \( r_0 \) defined by formula (3.10) into (5.1)

\[
\nu \left( \alpha(u_0) - \frac{2}{M-1} \alpha(u_1) + \delta(u_2) \right) - \frac{k(k-1)}{M-1} \alpha'(u_1) + (k-1) (\delta'(u_1) + \alpha'(u_1)) = 0. \tag{5.2}
\]

We apply the operator \( \frac{\partial}{\partial u_2} \) to this identity, and we get \( \nu \delta'(u_2) = 0 \). It is easy to show that the case \( \nu \neq 0 \) leads us to a contradiction to (1.4). Assume that \( \nu = 0 \), then from (5.2) we obtain that the function \( r_0 \) becomes

\[
r_0(u_1, u_0, u_{-1}) = \alpha(u_{-1}) - \frac{2}{M-1} \alpha(u_0) + \left( \frac{k}{M-1} - 1 \right) \alpha(u_1) + c_1.
\]
ii) We substitute the function \( r_0 \) defined by formula (3.11) into equation (5.1):

\[
\beta(u_0)(\nu M^2 - \nu M + k\lambda - k^2\lambda) e^{-\frac{2\lambda u_0}{M(M-1)}} + (k - 1)\beta'(u_1) e^{-\frac{2\lambda u_1}{M(M-1)}} + \nu\psi(u_1, u_2) + (k - 1)\frac{\partial\psi(u_1, u_2)}{\partial u_1} + \frac{1}{2}(k - 1)k\frac{\partial^2\psi(u_1, u_2)}{\partial u_1\partial u_2} = 0. \tag{5.3}
\]

We apply the operator \( \frac{\partial}{\partial u_0} \) to both sides of identity (5.3):

\[
\beta'(u_0)(\nu M^2 - \nu M + k\lambda - k^2\lambda) e^{-\frac{2\lambda u_0}{M(M-1)}} + (k - 1)\frac{\partial\beta'(u_1)}{\partial u_1} = 0. \tag{5.4}
\]

This equation has the following solution:

\[
\psi(u_0, u_1) = \beta(u_0)(\nu M^2 - \nu M + k\lambda - k^2\lambda) e^{-\frac{2\lambda u_0}{M(M-1)}} + F_1(u_0) + F_2(u_1). \tag{5.5}
\]

We substitute function (5.4) into equation (5.3), then we differentiate an obtained identity with respect to \( u_2 \), and we multiple both sides of the obtained identity by \( e^{\frac{2\lambda u_2}{M(M-1)}} \):

\[
-\frac{\nu(\nu M^2 - \nu M - k^2\lambda + k\lambda)\beta(u_1)}{M(k - 1)(M - 1)} - \frac{1}{2} \left( \frac{\lambda k^3 + k^2\lambda + kM^2\nu - kM\nu + 4k\lambda - 4\lambda)\beta'(u_1)}{(M - 1)M} + F_2(u_2)e^{\frac{2\lambda u_2}{M(M-1)}}\nu = 0. \tag{5.6}
\]

Let us consider two different cases \( \nu = 0 \) and \( \nu \neq 0 \).

ii.1) If \( \nu = 0 \), then (5.5) becomes

\[
\frac{1}{2}\lambda(k - 1)(k - 2)(k + 2)\beta'(u_1) = 0. \tag{5.7}
\]

It follows from this identity that \( k = 2 \) or \( \beta'(u_1) = 0 \).

ii.1.1) If \( k = 2 \), then equation (5.3) casts into the form \( F_1'(u_1) + F_2'(u_1) = 0 \). It is clear that \( F_2(u_1) = -F_1(u_1) + c_1 \). Equation (3.12) becomes

\[
\left( -\lambda\beta(u_0) - \frac{1}{2}\beta'(u_0)M^2 + \frac{3}{2}M\beta'(u_0) \right) e^{-\frac{2\lambda u_0}{M(M-1)}} + \frac{1}{2}M(M - 1)F_1'(u_0) + \lambda(F_1(u_0) - F_1(u_1) + c_1) = 0. \tag{5.8}
\]

We apply the operator \( \frac{\partial^2}{\partial u_1\partial u_0} \) to both sides of identity (5.7):

\[
\lambda e^{-\frac{2\lambda u_0}{M(M-1)}} \frac{2\lambda\beta'(u_0) + M(M - 3)\beta''(u_0)}{M(M - 1)} = 0. \tag{5.9}
\]

By the condition \( \lambda \neq 0 \) we see that

\[
2\lambda\beta'(u_0) + M(M - 3)\beta''(u_0) = 0. \tag{5.10}
\]

ii.1.1) If \( M = 3 \), then \( \beta(u_0) = c_0 \), where \( c_0 \) is an arbitrary constant. The function \( r_0 \) defined by formula (3.11) becomes

\[
r_0(u_1, u_0, u_{-1}) = c_0 e^{-\frac{1}{4}\lambda u_0} - c_0 e^{-\frac{1}{4}\lambda u_1} + F_1(u_0) - F_1(u_1) + c_1, \tag{5.11}
\]

and equation (3.12) reads as

\[
-\lambda c_0 e^{-\frac{1}{4}\lambda u_1} - \lambda F_1(u_1) + \lambda F_1(u_0) + \lambda c_1 + 3F_1'(u_0) = 0. \tag{5.12}
\]

We apply the operator \( \frac{\partial}{\partial u_1} \) to identity (5.10):

\[
\frac{1}{3}\lambda^2 c_0 e^{-\frac{1}{4}\lambda u_1} - \lambda F_1(u_1) = 0, \tag{5.13}
\]
We denote:

\[ F_1(u_1) = -c_0 e^{-\frac{1}{3}\lambda u_1} + C_1. \]

We substitute \( F_1 \) into (5.9) and we get \( r_0 = C_1 \). This contradicts condition (1.4).

ii.1.1.2) If \( M \neq 3 \), then equation (5.8) has the solution

\[ \beta(u_0) = C_1 + C_2 e^{-\frac{2\lambda u_0}{M(M-3)}}. \]

We differentiate equation (3.12) with respect to \( u_4 \) and, since \( \lambda \neq 0 \), this equation gives

\[ F_1(u_1) = -C_1 e^{-\frac{2\lambda u_1}{M(M-1)}} + C_2. \]

Equation (3.12) becomes \( \lambda c_1 = 0 \), hence, \( c_1 = 0 \), and, finally,

\[ r_0(u_1, u_0, u_{-1}) = C_2 e^{-\frac{2\lambda u_0}{M(M-1)}u_0 - \frac{2\lambda u_{-1}}{M(M-1)}u_{-1}} - C_2 e^{-\frac{2\lambda u_0}{M(M-1)}u_0 - \frac{2\lambda u_1}{M(M-1)}u_1}. \]

We return back to equation (5.12) and consider the following case.

ii.1.2) If \( \beta'(u_1) = 0 \), then \( \beta(u_1) = c_3 \), where \( c_3 \) is an arbitrary constant. By equation (5.1) we find

\[ F_2(u_1) = -\frac{1}{2} k F_1(u_1) + c_4. \]

Equation (3.12) is transformed as

\[ \lambda F_1(u_0) + \frac{1}{2} M(M-1) F_1'(u_0) = 0. \]

We apply the operator \( \frac{\partial}{\partial u_1} \) to both sides of identity (5.12)

\[ -\frac{1}{2} \frac{k \lambda}{M(M-1)} \left( -2c_3 \lambda e^{-\frac{2\lambda u_1}{M(M-1)}} + M(M-1)F_1'(u_1) \right) = 0. \]

This equation has the solution

\[ F_1(u_1) = -c_3 e^{-\frac{2\lambda u_1}{M(M-1)}} + c_5. \]

We substitute \( F_1 \) into (5.12), and we find \( c_4 \): \( c_4 = \frac{1}{2} c_5 (k-2) \). We substitute the found functions and constants into (3.11) and we get \( r_0(u_1, u_0, u_{-1}) = 0 \), which contradicts condition (1.4).

We return back to equation (5.5).

ii.2) If \( \nu \neq 0 \), then \( F_2'(u_2) e^{\frac{2\lambda u_2}{M(M-1)}} = c_1 \) and, hence,

\[ F_2(u_2) = -\frac{1}{2} M(M-1)c_1 e^{\frac{2\lambda u_2}{M(M-1)}} + c_2. \]

Equation (5.5) reads as

\[ -\frac{\nu(M^2 - \nu M - M^2 \lambda + k \lambda)}{M(k-1)(M-1)} \beta(u_1) = \frac{1}{2} \frac{(\lambda k^3 + k^2 \lambda + k M^2 \nu - k M \nu + 4 k \lambda - 4 \lambda) \beta'(u_1)}{(M-1)M} + c_1 \nu = 0. \]

We denote:

\[ A = \nu M^2 - \nu M - M^2 \lambda + k \lambda, \]
\[ B = \lambda k^3 + k^2 \lambda + k M^2 \nu - k M \nu + 4 k \lambda - 4 \lambda. \]

We shall consider the following different cases:

ii.2.1) \( A = 0 \), \( B = 0 \);
ii.2.2) \( A = 0 \), \( B \neq 0 \);
ii.2.3) \( A \neq 0 \), \( B = 0 \);
ii.2.4) \( A \neq 0 \), \( B \neq 0 \).
In case ii.2.1), that is, as
\[\nu M^2 - \nu M - k^2 \lambda + k \lambda = 0, \quad \lambda k^3 + k^2 \lambda + kM^2 \nu - kM \nu + 4k \lambda - 4 \lambda = 0.\]
Then we express \(\nu\) from the first equation and we substitute this function into the second equation, and we get \(4(k - 1) \lambda = 0\), which contradicts to \(k \geq 2, \lambda \neq 0\).

ii.2.2) Assume that
\[A = \nu M^2 - \nu M - k^2 \lambda + k \lambda = 0.\]
We express \(\nu\) from this identity and we substitute \(\nu\) into (5.13). This equation has the solution \(\beta(u_1) = \frac{1}{2} k c_1 u_1 + c_3\). Equation (5.3) becomes
\[\frac{1}{2} k(k - 1) \frac{dF_1(u_1)}{du_1} + \frac{k(k - 1) \lambda}{M(M - 1)} F_1(u_1) + c_1(k - 1)e^{-\frac{2\lambda u_1}{M(M - 1)}} + k(k - 1)c_2 \lambda \frac{M}{M(M - 1)} = 0.\]
This equation has the solution
\[F_1(u_1) = -\frac{(2c_1 u_1 - c_1 k)}{k} e^{-\frac{2\lambda u_1}{M(M - 1)}} - c_2.\]
Equation (3.12) casts into the form
\[-\frac{c_1 M(M - 1)}{k} e^{-\frac{2\lambda u_1}{M(M - 1)}} - \frac{c_1 M(M - k - 1)}{2} e^{-\frac{2\lambda u_1}{M(M - 1)}} = 0.\]
It is clear that this identity holds true only if \(c_1 = 0\) (we are working under the condition \(M \geq 2\)). Hence, we have
\[r_0(u_1, u_0, u_{-1}) = (c_3 + c_4) e^{-\frac{2\lambda u_0}{M(M - 1)}},\]
which contradicts condition (1.4).

ii.2.3) Suppose that
\[B = \lambda k^3 + k^2 \lambda + kM^2 \nu - kM \nu + 4k \lambda - 4 \lambda = 0.\]
We express \(\nu\):
\[\nu = \frac{\lambda(k - 1)(k - 2)(k + 2)}{kM(M - 1)}.\]
Since \(\nu \neq 0\), then \(k \neq 2\). Equation (5.13) becomes
\[\frac{\lambda(k - 1)(k - 2)(k + 2)}{k^2 M^2(M - 1)^2} (c_1 kM^2 - c_1 kM + 4\lambda \beta(u_1)) = 0.\]
We find the function \(\beta\):
\[\beta(u_1) = -\frac{1}{4} \frac{c_1 k M(M - 1)}{k^2 \lambda}. \quad (5.16)\]
Taking into consideration the obtained function, we simplify equation (5.3):
\[\frac{1}{2} k(k - 1) \frac{dF_1(u_1)}{du_1} + \frac{\lambda(k - 1)(k - 2)(k + 2)}{kM(M - 1)} F_1(u_1) + (k - 1)c_1 e^{-\frac{2\lambda u_1}{M(M - 1)}} + c_2 \frac{2\lambda(k - 2)(k - 1)(k + 2)}{kM(M - 1)} = 0.\]
This equation has the solution:
\[F_1(u_1) = \frac{c_1 k M(M - 1)}{4\lambda} e^{-\frac{2\lambda u_1}{M(M - 1)}} + C_1 e^{-\frac{2\lambda(k - 2)(k + 2)}{k^2 M(M - 1)} u_1} - c_2.\]
Let us transform equation (3.12)
\[\frac{4\lambda C_1}{k^2} e^{-\frac{2\lambda(k - 2)(k + 2)}{k^2 M(M - 1)} u_1} = 0.\]
It follows from this identity that \(C_1 = 0\). Substitution found functions and constants into (3.11), we obtain that \(r_0 = 0\), that contradicts condition (1.4).
ii.2.4) If $A \neq 0$ and $B \neq 0$, then equation (5.13) has the solution
\[
\beta(u_1) = c_3 e^{- 2\nu \frac{M^2 - M - k^2 \lambda}{M - 1} k} + \frac{M c_1 (k - 1)(M - 1)}{\nu M^2 - \nu M - k^2 \lambda + k \lambda}.
\] (5.17)

We substitute (5.17) into (5.3) and we obtain
\[
\frac{1}{2} k(k - 1) \frac{dF_1(u_1)}{du_1} + \nu F_1(u_1) + c_1 (k - 1) e^{- \frac{2\nu u_1}{M(M - 1)}} + \nu c_2.
\] (5.18)

Equation (5.18) has the solution
\[
F_1(u_1) = -c_2 + c_4 e^{\frac{2\nu u_1}{k(k - 1)}} - \frac{c_1 M(M - 1)(k - 1)}{\nu M^2 - \nu M - k^2 \lambda + k \lambda} e^{- \frac{2\nu u_1}{M(M - 1)}}.
\]

Function (3.11) becomes
\[
r_0(u_1, u_0, u_0) = c_4 e^{\frac{2\nu u_0}{k(k - 1)}} + c_3 e^{\frac{2\nu u_{0-1}}{M(M - 1)}} + \frac{A c_3}{2(k - 1) \lambda} e^{\frac{-2\nu u_1}{M(M - 1)}} + \frac{2\nu a_0}{M(M - 1)} = 0.
\]

Here $A$, $B$ are defined by formulae (5.14), (5.15). We substitute these functions into (3.12)
\[
- \frac{A c_4}{k(k - 1)} e^{\frac{-2\nu u_1}{k(k - 1)}} - \frac{A c_3 (\lambda B + \nu M^2 A - \nu MA - k B - 4M \nu \lambda + 4 M \nu k \lambda)}{2 B k \lambda (k - 1)^2} e^{- \frac{2\nu u_1}{M(M - 1)}}.
\]

Since $A \neq 0$, $\nu \neq 0$, it follows from the last identity that $c_1 = 0$ and
\[
\lambda B + \nu M^2 A - \nu MA - k B - 4M \nu \lambda + 4 M \nu k \lambda = 0.
\]

Thus, we have specified the function $r_0$:
\[
r_0(u_1, u_0, u_{0-1}) = c_4 e^{\frac{2\nu u_0}{k(k - 1)}} + c_3 e^{\frac{2\nu u_{0-1}}{M(M - 1)}} + \frac{A c_3}{2(k - 1) \lambda} e^{\frac{-2\nu u_1}{M(M - 1)}} + \frac{2\nu a_0}{M(M - 1)}.
\]

We can rewrite $r_0$ in the following form:
\[
r_0(u_1, u_0, u_{0-1}) = C_1 e^{h_1 u_0 - h_2 u_{0-1}} + C_2 e^{h_1 u_{0-1} - h_2 u_0},
\]

where $C_1 C_2 \neq 0$, $h_1 h_2 \neq 0$ are some constants.

Lattice (1.2) is reduced to one with $r_n$ of the following form
\[
r_n(u_{n+1}, u_n, u_{n-1}) = e^{u_n - h_{n-1}} + c e^{u_{n+1} - h_n}
\]
by rescaling $h_1 u_n \rightarrow u_n$, $c_1 h_1 x \rightarrow x$. Similarly transformations one can apply to the lattice in case ii.1.1.2 (see (5.11)). The proof of Lemma 3.6 is complete.

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