Echoes of a Squeezed Oscillator

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Pulses applied to an inhomogeneously broadened set of harmonic oscillators, previously prepared in squeezed states, can lead to a recovery of coherence, manifesting itself as echoes, similar to those exhibited by an ensemble of spins when excited by properly designed electromagnetic pulses. Such echoes, of classical or quantum nature, are expected to arise in the squeezing of linear systems of various sorts and, in particular, light and vibrational modes.
In quantum mechanics, squeezing is used to describe non-stationary states for which the uncertainty of a particular operator can become smaller than that for the vacuum [1]. Although the term was originally introduced in quantum electrodynamics [2,3] and, more generally, to describe bosonic systems such as lattice modes [4,5], magnons [6] and localized vibrations [7,8,9], it has also been applied to spin states [10,11] as well as to ensembles of classical oscillators [12]. The reduction of noise below the thermal- or shot-noise limit is the main motivation for numerous proposals and experimental realizations of squeezing [1,2,4,6,7,10,12,13,14,15], including the use of squeezed light in the detection of gravitational waves [16].

Unrelated to squeezing, spin echoes refers to the refocusing of the magnetization that results from the application of an electromagnetic pulse to a heterogeneous ensemble of spins [17,18] or few-level systems [19]. This effect is central to magnetic resonance imaging [20] and widely used in quantum optics and quantum information science [21]. In this letter, we show that the impulsive excitation of a dephased, heterogeneous set of squeezed oscillators can lead to a spontaneous resurgence of coherence in the form of echoes, similar to those exhibited by a collection of spins.

Squeezed states can be produced using different methods, which all exploit either a particular nonlinear property of the system one wishes to squeeze or, more commonly, of its interaction with an ancilla [4,7,10,15,22,23]. Here, we focus on squeezing resulting from frequency control, for which the relevant Hamiltonian is that of a frequency-driven parametric oscillator

$$H = P^2 / 2 + \left[ \Omega^2 - 2F(t) \right] Q^2 / 2$$

(1)

where $Q$ is the displacement from equilibrium, $P$ is the canonical momentum and $\Omega$ is the frequency. We are interested in the case where the frequency is modified by a sequence of two pulses applied at $t = 0$ and $t = \Delta$. Provided the duration of the pulses is small compared with the period of oscillations, we approximate
\[ F = \lambda \delta(t) + \mu \delta(t - \Delta); \] (2)

\( \lambda \) and \( \mu \) are constants.

We discuss first the classical problem. Let \( Q = Q_0 \) and \( \dot{Q} = P_0 \) be the position and momentum of the oscillator just before the first pulse is applied. The pulses do not change the displacement, but introduce a sudden change in the momentum. For \( 0 < t < \Delta \), \( Q(t) = U(t) \) where

\[ U(t) = Q_0 \cos \Omega t + \frac{(P_0 + 2\lambda Q_0)}{\Omega} \sin \Omega t \] (3)

and, for \( t > \Delta \),

\[ Q(t) = U(t) + \frac{2\mu U(\Delta)}{\Omega} \sin \Omega(t - \Delta). \] (4)

Consider now the behavior of \( \langle Q \rangle \), where the angle brackets denote the mean over a particular ensemble. Results for a set of \( N_o \) oscillators with Lorentzian-distributed frequencies are shown in Fig. 1 (their number per unit of frequency is given by \( dN / d\omega = \frac{\gamma N_o / \pi}{(\Omega - \Omega_o) + \gamma^2 / 4} \)). The oscillators are assumed to have all the same displacement and momentum at \( t = 0 \). Following the initial excitation pulse, the amplitude of the oscillations decays with time as different oscillators move at different periods. Mirroring the spin-echo problem [17], the second pulse, at \( t = \Delta \), partially removes the inhomogeneous dephasing, and the evolution rephases coherently to produce the \( \langle Q \rangle \)-echo at time \( 2\Delta \). The echo manifests itself as a dip in the data for the variance \( \langle Q^2 \rangle - \langle Q \rangle^2 \), shown in the inset. We note that the evolution of the classical ensemble after the first pulse mimics that of a coherent-squeezed quantum state [24,25].

The results in Fig. 1 can be explained simply by rewriting the second term of Eq. (4) as the sum of two terms of the form \( \sin(\Omega t + \alpha) \) and \( \sin[\Omega(t - 2\Delta) + \beta] \), where the phases depend on the
frequency. This explains the absence of $\langle Q \rangle$-oscillations at $t \approx \Delta$. Similarly, it can be shown that $\langle Q^2 \rangle$ exhibits three sets of oscillations of frequency $2\Omega$ starting at $t = 0$ and $t = \Delta$, and peaking at $t = 2\Delta$, which originate, respectively, from the terms $\sin^2(\Omega t + \alpha)$, $\sin(\Omega t + \alpha)\sin[\Omega(t - 2\Delta) + \beta]$ and $\sin^2[\Omega(t - 2\Delta) + \beta]$. The tail end of the oscillations induced by the second pulse can be seen in the inset of Fig. 1, just before the echo.

For a thermal distribution of identical oscillators, $\langle Q \rangle_T \equiv 0$. A simple calculation gives the variance for $t < \Delta$

$$\langle Q^2 \rangle_T = \frac{k_B T}{\Omega^2} \left[ 1 + 2\lambda^2 / \Omega^2 \right] + \frac{2\lambda}{\Omega} \sin 2\Omega t - \frac{2\lambda^2}{\Omega^2} \cos 2\Omega t$$

and the echo signal at $t \approx 2\Delta$

$$\langle Q^2_{\text{echo}} \rangle_T = \frac{2\mu^2 \lambda}{\Omega^2} k_B T \left[ \sin 2\Omega(t - 2\Delta) + \frac{\lambda}{\Omega} \cos 2\Omega(t - 2\Delta) \right].$$

Here, $k_B$ is Boltzmann’s constant and $T$ is the temperature. According to Eq. (5), the duty cycle for classical squeezing, that is, the fraction of the period for which $\langle Q^2 \rangle_T < k_B T / \Omega^2$ is largest for $\lambda << \Omega$ and goes to zero when $\lambda \to \infty$. As shown below, this also applies to quantum squeezing.

For the quantum problem, define $\sigma(t) = \langle \Psi | Q^2 | \Psi \rangle$ where $H\Psi = i\hbar \partial \Psi / \partial t$; see Eq. (1). Then, we get $\dot{\sigma} = -i\hbar + 2\langle \Psi | QP | \Psi \rangle$, $\ddot{\sigma} = 2\langle \Psi | P^2 | \Psi \rangle - 2\left[ \Omega^2 - 2F(t) \right] \sigma$ and, finally,

$$\dddot{\sigma} + 4\dot{\sigma} \left[ \Omega^2 - 2F(t) \right] - 4\dot{F}(t) \sigma = 0.$$
Free of forces, $\sigma$ oscillates with frequency $2\Omega$. Impulsive excitation leads to discontinuities in $\dot{\sigma}$ and $\ddot{\sigma}$, but not in $\sigma$. If $\sigma = \zeta$, $\dot{\sigma} = \dot{\zeta}$ and $\ddot{\sigma} = \ddot{\zeta}$ just before, say, the $\lambda$-pulse is applied, immediately after the pulse we have $\sigma = \zeta$, $\dot{\sigma} = \dot{\zeta} + 4\lambda \zeta$ and $\ddot{\sigma} = \ddot{\zeta} + 4\lambda \ddot{\zeta} + 8\lambda^2 \dddot{\zeta}$. Assume that the oscillator is in its ground state, $\Psi_0$, for $t < 0$. Then, for $0 < t < \Delta$, $\sigma(t) = \Sigma(t)$ where

$$\Sigma(t) / \sigma_0 = \left(1 + \frac{2\lambda^2}{\Omega^2}\right) + \frac{2\lambda}{\Omega} \sin 2\Omega t - \frac{2\lambda^2}{\Omega^2} \cos 2\Omega t ;$$

(8)

$$\sigma_0 = \langle \Psi_0 | Q^2 | \Psi_0 \rangle = \hbar / 2\Omega$$ is the variance of the zero-point motion. It is worth to point out that, after the pulse, the oscillator finds itself instantly in the squeezed state $e^{-\Delta Q^2 / \hbar} \Psi_0$. Also, note that $\langle Q \rangle = 0$ at all times, so that $\sigma$ is always equal to the variance.

For $t > \Delta$, we get

$$\sigma(t) = \Sigma(t) + \left(\frac{\mu \Sigma(\Delta) + 2\mu^2 \Sigma(\Delta)}{\Omega^2}\right) + \frac{2\mu \Sigma(\Delta)}{\Omega} \sin 2\Omega(t - \Delta) - \left(\frac{\mu \Sigma(\Delta) + 2\mu^2 \Sigma(\Delta)}{\Omega^2}\right) \cos 2\Omega(t - \Delta)$$

(9)

Given that $\Sigma(\Delta)$ is of the form $A \sin(2\Omega \Delta + \varphi) + B$, $\sigma(t)$ will exhibit additional oscillations from $t = \Delta$ due to the second pulse and, in addition, an echo at $t = 2\Delta$. Explicitly, the echo signal is

$$\sigma_{\text{echo}}(t) = \frac{2\lambda \mu^2}{\Omega^3} \sigma_0 \left[\sin 2\Omega(t - 2\Delta) + \frac{\lambda}{\Omega} \cos 2\Omega(t - 2\Delta)\right] .$$

(10)

Other than for the constant factors, Eq. (8) and the above expression are identical, respectively, to Eq. (5) and Eq. (6) (thus, the previous comment about the duty cycle applies also to the quantum case). This is not an accident, but a manifestation of the fact that classical oscillators also obey Eq. (7). The results in Fig. 2 are for an ensemble with a Lorentzian distribution of frequencies. The
data shows the decaying oscillations in the variance induced by the two pulses and, as anticipated, rephasing leading to the coherent echo at \( t = 2\Delta \).

In summary, we have shown that an inhomogenous set of parametrically driven harmonic oscillators, when impulsively excited, behave in a manner that is very similar to that of spins or few-level systems under resonant pulsed electromagnetic excitation. As for the latter, the occurrence of squeezed echoes hold promise for the development of techniques to distinguish homogeneous from inhomogeneous broadening in the decay of molecular vibrations, phonons, magnons and other bosons, as well as imaging methods similar to those that rely on magnetic resonance.
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FIGURE CAPTIONS

Figure 1 – Classical oscillators. Time-dependence of $\langle Q \rangle$ for a Lorentzian distribution of frequencies, with parameters $\Omega_0=7.5$ and $\gamma = 0.5$. The force acting on the (undamped) oscillators is $F = \lambda \delta(t) + \mu \delta(t - \Delta)$ ($\Delta = 10$, $\lambda=5$ and $\mu=2.5$) and the common initial conditions are $Q_0 = 25$ and $P_0 = 10$. The echo is at $t \approx 2\Delta$. Inset: $\langle Q^2 \rangle - \langle Q \rangle^2$ (arbitrary units) vs. time. Parameters are the same as in the main figure, except for $\lambda=2$ and $\mu=5$.

Figure 2 – Time dependence of the variance for a set of quantum oscillators with Lorentzian distributed frequencies; parameters are $\Omega_0=7.5$ and $\gamma = 0.5$. The oscillators are initially in the ground state and are acted upon by the two-pulse force $F = \lambda \delta(t) + \mu \delta(t - \Delta)$, with $\Delta = 10$, $\lambda=5$ and $\mu=2.5$. 
FIGURE 1
FIGURE 2