A Randomized Block Coordinate Iterative Regularized Gradient Method for High-dimensional Ill-posed Convex Optimization

Harshal Kaushik\textsuperscript{1} and Farzad Yousefian\textsuperscript{2}

Abstract—Motivated by high-dimensional nonlinear optimization problems as well as ill-posed optimization problems arising in image processing, we consider a bilevel optimization model where we seek among the optimal solutions of the inner level problem, a solution that minimizes a secondary metric. Our goal is to address the high-dimensionality of the bilevel problem, and the nondifferentiability of the objective function. Minimal norm gradient, sequential averaging, and iterative regularization are some of the recent schemes developed for addressing the bilevel problem. But none of them address the high-dimensional structure and nondifferentiability. With this gap in the literature, we develop a randomized block coordinate iterative regularized gradient descent scheme (RB-IRG). We establish the convergence of the sequence generated by RB-IRG to the unique solution of the bilevel problem of interest. Furthermore, we derive a rate of convergence $O\left(\frac{1}{n}\right)$, with respect to the inner level objective function. We demonstrate the performance of RB-IRG in solving the ill-posed problems arising in image processing.

I. INTRODUCTION

In this work we are interested in solving a bilevel problem given as,

$$\begin{align*}
\text{minimize} & \quad g(x) \\
\text{s.t.} & \quad x \in \text{argmin}\{f(x) : x \in X\},
\end{align*}$$

(P\textsuperscript{P})

where functions $f$ and $g$ are defined as $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$. (P\textsuperscript{P}) is a high-dimensional structure in a sense that the dimensions of the solution space can be huge. This causes high computation efforts in taking the gradient at any iteration. Set $X$ is assumed to be having a block structure, i.e. it can be written as, $X = \prod_{i=1}^{d} X_i$, where $X_i \subseteq \mathbb{R}^{n_i}$ and $\sum_{i=1}^{d} n_i = n$. Precisely, the assumptions on (P\textsuperscript{P}) are provided next.

Assumption 1: Let the following hold:
(a) Any block $i$ of set $X (X_i \subseteq \mathbb{R}^{n_i})$ is assumed to be nonempty, closed, and convex for all $i = 1, \ldots, d$.
(b) $f : \mathbb{R}^n \to (-\infty, \infty]$ is a nondifferentiable, proper, and convex function.
(c) $g : \mathbb{R}^n \to (-\infty, \infty]$ is a nondifferentiable, proper, and $\mu$-strongly convex function ($\mu > 0$).
(d) $X \subseteq \text{int}(\text{dom}(f) \cap \text{dom}(g))$.

A. Motivating examples

Here we present two applications of the formulation (P\textsuperscript{P}).

(i) High-dimensional nonlinear constrained optimization: Consider the following problem with nonlinear constraints,

$$\begin{align*}
\text{minimize} & \quad g(x) \\
\text{s.t.} & \quad h_i(x) \leq 0 \quad \text{for } i = 1, \ldots, m \\
x & \in X \triangleq \prod_{i=1}^{d} X_i,
\end{align*}$$

(1)

The assumptions on (1) are: (i) function $h_i : \mathbb{R}^n \to \mathbb{R}$ is convex; (ii) function $g : \mathbb{R}^n \to \mathbb{R}$ is convex; (iii) $X \subseteq \mathbb{R}^n$ is convex, satisfying Assumption 1(a).

Provided that the feasible set of (1) is nonempty, this problem can be equivalently written in a bilevel structure as following,

$$\begin{align*}
\text{minimize} & \quad g(x) \\
\text{s.t.} & \quad x \in \text{argmin}\left\{ f(x) \triangleq \sum_{i=1}^{m} \max\{0, h_i(x)\} : x \in X \right\}.
\end{align*}$$

(ii) Ill-posed optimization: Linear inverse problems arising in image deblurring can be written as the following optimization problem,

$$\begin{align*}
\text{minimize} & \quad \|Ax - b\|^2 \\
\text{s.t.} & \quad x \in \mathbb{R}^n,
\end{align*}$$

(2)

where $A$ is a blurring operator ($A \in \mathbb{R}^{m \times n}$), $b$ is the given blurred image ($b \in \mathbb{R}^m$), and $x$ is a deblurred image ($x \in \mathbb{R}^n$). This is an ill-posed problem in a sense that there may be multiple solutions or the optimal solution $x$ may be very sensitive to the perturbation in the input $b$. To address the ill-posedness, problem (2) can be reformulated in a bilevel structure as following, (see [13]).

$$\begin{align*}
\text{minimize} & \quad \|x\|^2 \\
\text{s.t.} & \quad x \in \text{argmin}\left\{ \|Ax - b\|^2 : x \in \mathbb{R}^n \right\}.
\end{align*}$$

B. Existing methods

Sequential regularization, minimal norm gradient, sequential averaging, and iterative regularization are some of the recent schemes developed for addressing problem (P\textsuperscript{P}). One of the classical approaches to address the ill-posedness is the regularization technique.

$$\begin{align*}
\text{minimize} & \quad f(x) + \eta g(x) \\
\text{s.t.} & \quad x \in X.
\end{align*}$$

(P\eta)

Tikhonov in [25] showed that under some assumptions, the solution of regularized problem (P\eta) converges to the solution of the inner level problem of (P\textsuperscript{P}) as the regularization
parameter $\eta$ goes to zero. Later the threshold value of $\eta$, under which the solution of (P_k) is same as the solution of the inner level problem of (P_0) was studied under the area of exact regularization [12], [14], [17]. There have been numerous theoretical studies in the 80’s, 90’s [4], [5], [7], [11], [14], [17] and early 2000 [6], [9] on finding the suitable $\eta$, but in practice there is not much guidance on tuning this parameter. Finding a suitable $\eta$ necessitate solving a sequence of problem (P_k) for $\eta_k$, where $\eta_k \to 0$. This two loop scheme is highly inefficient, especially in high dimensional spaces.

In the past decade, interest has been shifted to solving the bilevel problem (P_0) using single loop schemes. Solodov in [24] showed that for both functions $g$ and $f$ in (P_0) with Lipschitz gradient, and $f$ to be a composite function with the indicator function, solutions to (P_0) can be found by iterative regularized gradient descent with sequence $\eta_k \to 0$ and $\sum_{k=1}^{\infty} \eta_k = \infty$. In (P_0), when $g$ is $\ell_2$ norm in variational inequality regimes, Yousefian et al showed that solution to (P_0) can be found by employing an iterative regularized gradient descent scheme (see [28]).

In 2014, minimal norm gradient (MNG) scheme was proposed [3]. This involves solving the projection (this itself is another optimization problem) for each iteration $k$, which makes MNG to be difficult to implement for the large scale problems. Later in [21] a sequential averaging scheme (BiG-SAM) was developed with a rate of convergence $O(1/k)$. Recently in [13] a general iterative regularized algorithm based on a primal-dual diagonal descent method was proposed to solve (P_0).

In these papers, the missing part is addressing the high-dimensional structure, which is common in the high resolution image processing problems. Our goal is to bridge this gap by developing a randomized block coordinate iterative regularized gradient descent scheme to solve the high-dimensional problems.

Coordinate descent methods have recently gained popularity due to their potential of solving the large-scale optimization problems. In [18], [20], block coordinate descent found to be effective when the size of solution space is of the order $10^8 - 10^{12}$. Therefore block strategy is effective when dealing with high dimensionality. Cyclic coordinate descent is a common strategy to make the selection of block. It is well studied in the past but recently the focus has been shifted to randomized strategy due to theoretical [18], [20], [23] and practical advantages it offers in solving the large scale machine learning problems [8], [15], [22], [23].

High-dimensional nonlinear constrained optimization is another problem we consider in this work. One of the popular primal-dual methods is Alternating Direction Method of Multipliers (ADMM) [1], [26], [27]. One of the underlying assumptions for ADMM is the linear constraints. In our work, bilevel problem addresses the nonlinearity in the constraints and the high-dimensionality of the space.

C. Main contributions

(I) We develop a single loop first order scheme with the mild requirements such as $f$ and $g$ can be nondifferentiable functions. (II) RB-IRG can handle the high-dimensional structure of bilevel problem (P_0). (III) We establish the convergence of the sequence generated from RB-IRG to the unique solution of (P_0). (IV) We derive the rate of convergence $O\left(\frac{1}{1/k}\right)$, with respect to the inner level function of the bilevel problem.

To highlight the contribution of our work and its distinction from the other methods, we provide a table (see TABLE I). In Section II, we propose RB-IRG scheme with preliminaries. Section III is for showing the convergence of RB-IRG to the solution of bilevel problem (P_0). In Section IV, we show the rate analysis of RB-IRG with respect to the inner level objective function of (P_0). In Section V, we apply RB-IRG to image deblurring application and discuss the computational effectiveness of our scheme. In Section VI, we highlight the main contribution and provide the concluding remarks.

**Notation:** Vector $x$ is assumed to be a column vector ($x \in \mathbb{R}^n$), $x^T$ is the transpose. $x^{(i)}$ denotes the $i$th block.
of dimensions for a vector $x$. $X_i$ denotes the $i$th block of dimensions for set $X$. $\|x\|$ denotes the Euclidean vector norm, i.e., $\|x\| = \sqrt{x^T x}$. $\mathcal{P}_S(s)$ is used for the Euclidean projection of vector $s$ on a set $S$, i.e., $\|s - \mathcal{P}_S(s)\| = \min_{y \in S} \|s - y\|$, a.s. used for 'almost surely'. $\nabla_i$ denotes the set of variables $\{0, \ldots, i-1\}$. For a random variable $i_k$, $\text{Prob}(i_k = i)$ is $\rho_{ik}$. $\nabla$ denotes the subgradient and $\delta$ denotes the subdifferential set. $\nabla_i f(x)$ is the $i$th block of $\nabla f(x)$. $\min_{1 \leq i \leq d} \{p_i\}$ is denoted by $p_{\text{min}}$ and $\max_{1 \leq i \leq d} \{p_i\}$ is denoted by $p_{\text{max}}$.

II. ALGORITHM OUTLINE

Here we explain algorithm RB-IRG the required preliminaries for convergence and rate analysis.

A. Proposed scheme RB-IRG

Here, a randomized block coordinate iterative regularized gradient descent scheme (RB-IRG) is proposed for solving (P*). In RB-IRG, both the sequences of regularization parameter $\eta_k$ and stepsize parameter $\gamma_k$ are in terms of iteration $k$. To address the high-dimensionality, at each iteration we update a random block of the iterate $x_k$. Selection of block $i_k$ at iteration $k$ is governed by Assumption 2. Finally, averaging is employed which will be helpful in deriving the rate statement.

Algorithm 1 Randomized block coordinate iterative regularized gradient descent (RB-IRG) algorithm

1: Initialization:
   - Set $k = 0$, select a point $x_0 \in X$, parameters $\gamma_0 > 0$, and $\eta_0 > 0, S_0 = \gamma_0^r$, and $\bar{x}_0 = x_0$.
2: for $k = 0, 1, \ldots, N-1$ do
   3: $i_k$ is generated by Assumption 2.
   4: Compute $\nabla_i f(x_k) \in \partial f(x_k)$ and $\nabla_i g(x_k) \in \partial g(x_k)$ for $x_k \in X_i$.
   5: Update $x_{k+1} = \frac{S_k x_k + \gamma_k r_{k+1} x_{k+1}}{S_k}$.
   6: Update $S_k$ as follows:
   7: end for

Assumption 2: (Random sample $i_k$) Random variable $i_k$ is generated at each iteration $k$ from an i.i.d. distribution governed by probability $\rho_{ik}$ where $\text{Prob}(i_k = i) = \rho_{ik} > 0$, and $\sum_{i=1}^d \rho_{ik} = 1$.

B. Preliminaries

Throughout the paper, we use $x_k^*$ and $x_{\eta_k}^*$ to denote the unique minimizers of (P*), and (P*), respectively.

Remark 1: From Assumptions 1b, c), the objective function of (P*), is closed and convex (from Assumption 1a). Therefore (P*) has a unique minimizer. (cf. Ch. 2 of [10]). Similarly, we can claim that (P*) has a unique minimizer.

Remark 2: in problem (P*), for any $x_1, x_2 \in X$, for a convex function $f$ and $\mu$-strongly convex function $g$, $\langle \nabla_i f(x_1) - \nabla_i f(x_2), x_1 - x_2 \rangle \geq 0$, $\langle \nabla_i g(x_1) - \nabla_i g(x_2), x_1 - x_2 \rangle \geq \mu \|x_1 - x_2\|^2$.

The following lemma is used in proving the convergence.

Lemma 1: (Lemma 10, pg. 49 of [19]): Let $\{v_k\}$ be a sequence of nonnegative random variables, where $E[v_0] < \infty$, and let $\{\alpha_k\}$ and $\{\beta_k\}$ be deterministic scalar sequences such that $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\sum_{k=0}^{\infty} \beta_k < \infty$, $\lim_{k\to\infty} \beta_k = 0$. Then, $v_k \to 0$, a.s., and $\lim_{k\to\infty} E[v_k] = 0$.

The next result will be used in our analysis.

Lemma 2: (Theorem 6, pg. 75 of [16]): Let $\{u_k\} \subset \mathbb{R}^n$ be a convergent sequence such that it has a limit point $\bar{u} \in \mathbb{R}^n$ and consider another sequence $\{v_k\}$ of positive numbers such that $\sum_{k=0}^{\infty} \alpha_k = \infty$. Suppose $u_k$ is given by $u_k = \sum_{i=0}^{k-1} \alpha_i$, for all $k \geq 1$. Then $\lim_{k\to\infty} u_k = \bar{u}$.

Remark 3: From Assumption 1b, c, d), for all $x \in X$, the set $\partial f(x)$ is nonempty and bounded (cf. Ch. 3 of [2]). Similarly $\partial g(x)$ is nonempty and bounded for all $x \in X$.

Remark 4: From Remark 3 let us say that for any $x^{(i)} \in X_i$, there is a scalar $C_{fi}$ such that $\|\nabla f(x)\| \leq C_{fi}$.

Let $C_f = \sqrt{\sum_{i=1}^d C_{fi}^2}$. Now we have, $\|\nabla f(x)\| \leq C_f$ for all $x \in X$.

In the following lemma, we provide the bound on sequence $\{x_k\}$, which is solution of (P*).

Lemma 3: (Bound on $x_k$), see Proposition 1 of [28]: Consider problem (P*) and (P*). Suppose Assumption 1 holds. Then for a sequence $\{x_{\eta_k}\}$, and $x^*$ for any $k \geq 1$, the following hold.

(a) $\|x_{\eta_k} - x_{\eta_{k-1}}\| \leq \frac{\bar{C}_p}{\beta_k} \frac{\eta_k^{\alpha_k}}{\eta_{k-1}^{\alpha_k}} - 1$. (b) When $\eta_k$ goes to zero, $\{x_{\eta_k}\}$ converges to $x^*$.

Our objective is to show $\|x_{k+1} - x_{\eta_k}\| \to 0$. From the triangle inequality, $\|x_{k+1} - x_{\eta_k}\| \to 0$ and $\|x_{\eta_k} - x^*\| \to 0$.

We know $\|x_{\eta_k} - x^*\| \to 0$ as $\eta_k \to 0$. Our main objective is to show $\|x_{k+1} - x_{\eta_k}\| \to 0$. Next we define an error function which will be used in the convergence analysis.

Definition 1: Let Assumption 1 hold. Then for any $x, y \in \mathbb{R}^n$, function $L(x, y) = \sum_{i=1}^d p_{\text{max}} \|x^{(i)} - y^{(i)}\|^2$. The following corollary holds from Definition 1.

Corollary 1: Consider Definition 1 $p_{\text{max}}$ and $p_{\text{min}}$ as defined in the notation, and let Assumption 2 hold. Then for any $x, y \in \mathbb{R}^n$, $p_{\text{max}} L(x, y) \leq \|x - y\|^2 \leq p_{\text{min}} L(x, y)$.

III. CONVERGENCE ANALYSIS OF RB-IRG SCHEME

Here we begin with deriving a recursive error bound, that will be used later to show the convergence.

Lemma 4: (Recursive relation for $L(x_{k+1}, x_{\eta_k}^*)$): Consider problem (P*), and (P*). Let Assumptions 1 and 2 hold. Let $\{x_k\}$ be the sequence generated from Algorithm 1. Let
positive sequences \( \{\gamma_k\} \), and \( \{\eta_k\} \) be non-increasing and \( \gamma_0 \eta_0 < 1/\mu \) min. Then the following relation holds,

\[
E[\mathcal{L}(x_{k+1}, x_{\eta_k}) | \mathcal{F}_k] \leq (1 - \mu \gamma_k \eta_k \mu_{min}) \mathcal{L}(x_{k}, x_{\eta_k}) + \frac{2 \gamma_k^2}{\mu^2 \min} C_f^2 \sum_{i=1}^{d} (\eta_i \eta_i - 1)^2 + 2 \gamma_k^2 (C_f^2 + \gamma_k^2 C_g^2).
\]

**Proof:** Consider \( \mathcal{L}(x_{k+1}, x_{\eta_k}) \). From the Definition [1],

\[
\mathcal{L}(x_{k+1}, x_{\eta_k}) = \sum_{i=1}^{d} p_{i}^{-1} \left\| x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right\|^2 + \frac{d}{d-1} p_{i}^{-1} \left\| x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right\|^2 + \frac{2 \gamma_k^2}{\mu^2 \min} C_f^2 \left\| x_{\eta_k}^{(i)} - x_{\eta_k}^{(i)} \right\|^2.
\]

Since \( x_{\eta_k}^{(i)} \in X \), we have \( x_{\eta_k}^{(i)} \in X_{i_k} \). Now from the non-expansive property of projection operator, term-1 becomes,

\[
\left\| x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right\|^2 \leq \left\| x_{(i)}^{(k)} - \eta_k \nabla_i f(x) + \eta_k \nabla_i g(x) - x_{\eta_k}^{(i)} \right\|^2.
\]

From the previous two preceding relations, we have,

\[
\mathcal{L}(x_{k+1}, x_{\eta_k}) = \sum_{i=1}^{d} \left( p_{i}^{-1} \left\| x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right\|^2 + \frac{d}{d-1} p_{i}^{-1} \left\| x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right\|^2 \right) - 2 \gamma_k \eta_k \left( x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right)^T (\nabla_i f(x) + \eta_k \nabla_i g(x)).
\]

Substituting the conditional expectation on both the sides, and taking into account \( \mathcal{L}(x, x_{\eta_k}) \) is \( \mathcal{F}_k \) measurable, we get,

\[
E[\mathcal{L}(x_{k+1}, x_{\eta_k}) | \mathcal{F}_k] \leq \mathcal{L}(x_k, x_{\eta_k}) + 2 \gamma_k \eta_k E[\nabla_i f(x) + \eta_k \nabla_i g(x)] - 2 \gamma_k \eta_k \left( x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right)^T (\nabla_i f(x) + \eta_k \nabla_i g(x)).
\]

From Assumptions [1] (d) and Remark [2], term-1 becomes,

\[
\gamma_k^2 \left\| \nabla_i f(x) + \eta_k \nabla_i g(x) \right\|^2 \leq 2 \gamma_k^2 C_f^2 \eta_i + 2 \gamma_k^2 \gamma_k^2 C_g^2.
\]

Thus from [1], and Definition [1], we obtain,

\[
\mathcal{L}(x_{k+1}, x_{\eta_k}) \leq \mathcal{L}(x_k, x_{\eta_k}) + 2 \gamma_k \eta_k \left( x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right)^T (\nabla_i f(x) + \eta_k \nabla_i g(x)).
\]

Substituting the conditional expectation on both the sides, and taking into account \( \mathcal{L}(x, x_{\eta_k}) \) is \( \mathcal{F}_k \) measurable, we get,

\[
E[\mathcal{L}(x_{k+1}, x_{\eta_k}) | \mathcal{F}_k] \leq \mathcal{L}(x_k, x_{\eta_k}) + 2 \gamma_k \eta_k \left( x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right)^T (\nabla_i f(x) + \eta_k \nabla_i g(x)).
\]

From Lemma [3] and Corollary [1], we obtain,

\[
\mathcal{L}(x_k, x_{\eta_k}) \leq (1 + \mu \min \mu \gamma_k \eta_k) \mathcal{L}(x_k, x_{\eta_k}) + \frac{1}{\mu \min \mu \gamma_k \eta_k} \left( x_{\eta_k}^{(i)} - x_{\eta_k}^{(i)} \right)^T (x_{\eta_k}^{(i)} - x_{\eta_k}^{(i)}).
\]

Now consider \( \gamma_k \eta_k \left( x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right)^T (\nabla_i f(x) + \eta_k \nabla_i g(x)). \) Substituting this in [4], we obtain the following,

\[
E[\mathcal{L}(x_{k+1}, x_{\eta_k}) | \mathcal{F}_k] \leq \mathcal{L}(x_k, x_{\eta_k}) + 2 \gamma_k \eta_k \left( x_{(i)}^{(k+1)} - x_{\eta_k}^{(i)} \right)^T (\nabla_i f(x) + \eta_k \nabla_i g(x)).
\]

Now from Remark [2] for \( x_1 = x_k \) and \( x_2 = x_{\eta_k} \), we have,

\[
\left( \nabla f(x_k) + \eta_k \nabla g(x_k) \right)^T (x_k - x_{\eta_k}) - \left( \nabla f(x_{\eta_k}) + \eta_k \nabla g(x_{\eta_k}) \right)^T (x_k - x_{\eta_k}) \geq \eta_k \mu \left\| x_k - x_{\eta_k} \right\|^2.
\]

From the optimality conditions on \( F_0 \), we have,

\[
\left( \nabla f(x_k) + \eta_k \nabla g(x_k) \right)^T (x_k - x_{\eta_k}) \geq 0. \]

Thus from [4] and the preceding inequality, we can write,

\[
E[\mathcal{L}(x_{k+1}, x_{\eta_k}) | \mathcal{F}_k] \leq \mathcal{L}(x_k, x_{\eta_k}) - 2 \gamma_k \eta_k \mu \left\| x_k - x_{\eta_k} \right\|^2
\]

Now, from the preceding inequality, we can write,

\[
E[\mathcal{L}(x_{k+1}, x_{\eta_k}) | \mathcal{F}_k] \leq \mathcal{L}(x_k, x_{\eta_k}) - 2 \gamma_k \eta_k \mu \left\| x_k - x_{\eta_k} \right\|^2 + 2 \gamma_k^2 C_f^2 + 2 \gamma_k^2 \gamma_k^2 C_g^2.
\]

From Corollary [1] bounding term-6, we have,

\[
E[\mathcal{L}(x_{k+1}, x_{\eta_k}) | \mathcal{F}_k] \leq (1 - 2 \gamma_k \eta_k \mu_{min}) \mathcal{L}(x_k, x_{\eta_k}) + 2 \gamma_k^2 C_f^2 + 2 \gamma_k^2 \gamma_k^2 C_g^2.
\]

Now consider \( \left\| x_k - x_{\eta_k} \right\|^2 \). It can be written as,

\[
\left\| x_k - x_{\eta_k} \right\|^2 = \left\| x_k - x_{\eta_k} - x_{\eta_k} \right\|^2 + \left\| x_{\eta_k} - x_{\eta_k} \right\|^2 + 2 \left( x_k - x_{\eta_k} \right)^T (x_{\eta_k} - x_{\eta_k}).
\]

Substituting above in equation [7], with \( c = \sqrt{\mu \min \mu \gamma_k \eta_k} \),

\[
\left\| x_k - x_{\eta_k} \right\|^2 \leq (1 + \mu \min \mu \gamma_k \eta_k) \left\| x_k - x_{\eta_k} - x_{\eta_k} \right\|^2 + \left(1 + \frac{1}{\mu \min \mu \gamma_k \eta_k} \right) \left\| x_{\eta_k} - x_{\eta_k} \right\|^2.
\]

Dividing both sides of previous inequality by \( \mu \min \), and substituting this in [4], we obtain the following,

\[
E[\mathcal{L}(x_{k+1}, x_{\eta_k}) | \mathcal{F}_k] \leq 12 \gamma_k^2 C_f^2 + 2 \gamma_k^2 \gamma_k^2 C_g^2.
\]
We have $\gamma_0 \eta_0 < \frac{d}{\rho_{\min} \mu^2}$. Bounding term-10, we have,
\[
E[\mathcal{L}(x_{k+1}, x_{\eta_k}) | x_k] \leq (1 - \mu \gamma_k \eta_k \rho_{\min}) \mathcal{L}(x_k, x_{\mu^{\gamma_k} \eta_k}^* \rho_{\min}) + \frac{C^2_g}{\rho_{\min} \mu^2} \left( \frac{2}{\rho_{\min} \mu^2 \gamma_k \eta_k^2} \right) \gamma_k \eta_k (\eta_k - 1)^2 + 2 \gamma_k^2 C_f^2 + 2 \gamma_k^2 \rho^2 C_g^2.
\]
Bounding non-increasing sequence, $\eta_k$ we get the result.

A. Convergence analysis

Remark 5: Throughout the analysis, we assume that blocks are randomly selected using a uniform distribution.

Assumption 3: Let the following hold:
(a) $\{\gamma_k\}$ and $\{\eta_k\}$ are positive sequences for $k \geq 0$ converging to zero such that $\gamma_0 \eta_0 < \frac{d}{\mu}$;
(b) $\sum_{k=0}^{\infty} \gamma_k \eta_k = \infty$;
(c) $\sum_{k=0}^{\infty} \eta_k = \infty$;
(d) $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$;
(e) $\lim_{k \to \infty} \left( \frac{\eta_k - 1}{\gamma_k \eta_k} \right) = 0$;
(f) $\lim_{k \to \infty} \frac{2 \gamma_k}{\gamma_k \eta_k} = 0$.

Next, we show the a.s. convergence of the sequence $\{x_k\}$.

Proposition 1: (a.s. convergence of $\{x_k\}$): Consider Problem $\text{(P)}$ and $\text{(P)}_g$. Let Assumption 3 hold. Consider the sequence $\{x_k\}$ is obtained by Algorithm 3 and the sequence $\{x_{\eta_k}\}$ suppose obtained by solving $\text{(P)}_g$. Then, $\mathcal{L}(x_k, x_{\eta_k}^*)$ goes to zero a.s. and $\lim_{k \to \infty} E[\mathcal{L}(x_k, x_{\eta_k})] = 0$.

Proof: We apply Lemma 9 to the result of Lemma 4 and set $\eta_k = \mathcal{L}(x_k, x_{\eta_k})$, $\alpha_k = \mu \gamma_k \eta_k \rho_{\min}$, $\beta_k = \left( \frac{2 \gamma_k^2}{\rho_{\min} \mu^2 \gamma_k \eta_k^2} \right) \gamma_k \eta_k (\eta_k - 1)^2 + 2 \gamma_k^2 (C_f^2 + \rho^2 C_g^2)$. Now, in order to claim the convergence of $\mathcal{L}(x_k)$, we show that all conditions of Lemma 9 hold. Note that $\rho_{\min} = 1/\mu$. From Assumption 3 (a), definition of $\{\gamma_k\}$, $\{\eta_k\}$, and from $\gamma_0 \eta_0 < \frac{d}{\mu}$, the first condition of Lemma 9 is satisfied. Now consider sequence $\beta_k$. From Assumption 3 (a), sequences $\{\eta_k\}$, $\{\eta_k\}$ and the constant $\mu$ are positive, so the second condition of Lemma 9 is satisfied. Now in $\sum_{k=0}^{\infty} \alpha_k$, i.e., $\sum_{k=0}^{\infty} \mu \gamma_k \eta_k \rho_{\min}$. From Assumption 3 (b), the third condition of Lemma 9 holds. Now from the definition of $\beta_k$ and from Assumption 3 (c) and (d), the fourth condition of Lemma 9 holds. Finally consider $\lim_{k \to \infty} \left( \frac{\beta_k}{\alpha_k} \right) = 0$. Using the definition of $\beta_k$ and Assumption 3 (e, f), condition 5 of Lemma 9 holds. Thus we get the required result.

Next in Lemma 9 we give the choice of sequences $\gamma_k$ and $\eta_k$ that satisfy Assumption 3.

Lemma 5: Let Assumption 3 hold. Then sequences $\{\gamma_k\}$ and $\{\eta_k\}$ given by $\gamma_k = \gamma_0 (k + 1)^{-a}$ and $\eta_k = \eta_0 (k + 1)^{-b}$ where $a$, and $b$ satisfy, $a > 0$, $b > 0$, $a + b < 1$, $b < a$, $a > 0.5$, where $\gamma_0 > 0$ and $\eta_0 > 0$. Then $\{\gamma_k\}$ and $\{\eta_k\}$ satisfy Assumption 3.

Proof: Similar to the proof of Lemma 5 in [28]. Omitted because of the space requirements.

Next, we show the a.s. convergence of the sequence $\{\tilde{x}_k\}$.

Theorem 1: (a.s. convergence of $\{\tilde{x}_k\}$): Consider problem $\text{(P)}$. Let $\gamma_k$ and $\eta_k$ be the sequences defined by Lemma 5 where $\gamma_0 > 0$, $\eta_0 > 0$, and $a < 1$. Then $\{\tilde{x}_k\}$ converges to the unique solution of $\text{(P)}$, $x^*_g$ a.s.

Proof: From $\lambda_{t,k} = \frac{\gamma^*}{\sum_{k=0}^{t} \gamma^*} \| \tilde{x}_k - x^*_g \| \leq \sum_{k=0}^{t} \lambda_{t,k} \| x_k - x^*_g \| = \sum_{k=0}^{t} \lambda_{t,k} \| (x_k - x^*_g) \|$. Using the triangle inequality, $\| \tilde{x}_k - x^*_g \| \leq \lambda_{t,k} \| x_k - x^*_g \|$. From definition of $\lambda_{t,k}$, $\| \tilde{x}_k - x^*_g \| \leq \sum_{k=0}^{t} \gamma^* \| x_k - x^*_g \|$. Comparing with Lemma 2, $\lambda_{t,k} \eta_k \leq \sum_{k=0}^{t} \gamma^* \| x_k - x^*_g \|$.

IV. RATE OF CONVERGENCE

In this section, first we derive the rate of convergence of RB-IRG with respect to the inner level problem in $\text{(P)}$.

Lemma 6: (Feasibility error bound for $\text{Algorithm 1}$) Consider problem $\text{(P)}_g$ and $\{\tilde{x}_k\}$, the sequence generated by Algorithm 1. Let Assumption 3 hold, $r \epsilon (0, 1)$ be an arbitrary scalar, and $\gamma_k$ be a non-increasing sequence. Let $\eta_k$ be a non-increasing sequence and $X$ to be bounded, i.e., $\|x\| \leq M$ for all $x \in X$ for some $M > 0$. Then for any $z \in X$, the following holds,
\[
E[f(\tilde{x}_N)] - f(z) \leq \left( \frac{\sum_{k=0}^{N-1} \gamma^*}{\sum_{k=0}^{N-1} \gamma^*} \right) \left( 2 M_g \sum_{k=0}^{N-1} \gamma_k \eta_k + 2 \rho_{\max} M^2 \left( \frac{\gamma^*}{\sum_{k=0}^{N-1} \gamma^*} + \frac{\gamma^*}{\sum_{k=0}^{N-1} \gamma^*} \right) \left( C_f^2 + C_g^2 \rho_{\max} \right) \right),
\]
where $M_g > 0$ is a scalar such that $g(x) \leq M_g$ for all $x \in X$.

Proof: Consider equation $\text{(1)}$ in step 6 of $\text{Algorithm 1}$ and assuming that $\tilde{x}_k = \sum_{i=0}^{k} \lambda_{t,k} x_{i,t}$, where $\tilde{x}_k = \tilde{x}_k$. Note that using induction, it can be shown that $\tilde{x}_k = \sum_{i=0}^{k} \lambda_{t,k} x_{i,t}$, where $\lambda_{t,k} = \frac{\gamma^*}{\sum_{k=0}^{N-1} \gamma^*}$.

Next, consider $\{x_k\}$ be the sequence generated from $\text{Algorithm 1}$ and $z \in X$. Then from Definition 4 we have,
\[
\mathcal{L}(x_{k+1}, z) = \sum_{i=1}^{d} \rho_i^{-1} \| x_{i,k} - z_i \|^2 + \rho_i^{-1} \| x_{i,k} - z_i \|^2.
\]
Consider term-1. From $\text{RB-IRG}$ substituting $x_{i,k}$ and using the non-expansiveness property of the projection operator,
\[
\| x_{i,k} - z_i \|^2 \leq \| x_{i,k} - z_i \|^2 + \| x_{i,k} - z_i \|^2 \leq 2 \gamma_k \| x_{i,k} - z_i \|^2 + \gamma_k \| x_{i,k} - z_i \|^2.
\]
Substituting the bound on term-1, we obtain,
\[
\mathcal{L}(x_{k+1}, z) = \sum_{i=1}^{d} \rho_i^{-1} \| \nabla_{i,k} f(x_k) + \eta \nabla_{i,k} g(x_k) \|^2 + \rho_i^{-1} \| \nabla_{i,k} f(x_k) + \eta \nabla_{i,k} g(x_k) \|^2 - 2 \gamma_k \| x_{i,k} - z_i \|^2.
\]
here we used Definition\textsuperscript{1} From Remark\textsuperscript{4}, bounding term-2, 
\[ \left\| \tilde{\nabla}_{ik} f(x_k) + \eta_k \tilde{\nabla}_{ik} g(x_k) \right\|^2 \leq 2C^2_{g,ik} + 2\eta_k^2 C^2_{g,ik}. \]
Substituting the bound of term-2, we get,
\[ \mathcal{L}(x_{k+1}, z) \leq \mathcal{L}(x_k, z) + 2p^{-1}_k C^2_{g,ik} + 2p^{-1}_k \eta_k^2 C^2_{g,ik} - 2p^{-1}_k \eta_k (z^{(ik)} - z^{(ik)})^T \left( \tilde{\nabla}_{ik} f(x_k) + \eta_k \tilde{\nabla}_{ik} g(x_k) \right). \]
By taking conditional expectation on the both sides of equation above, and since \( \mathcal{L}(x_k, z) \) is \( F_k \) measurable, 
\[ E[\mathcal{L}(x_{k+1}, z) | F_k] \leq \mathcal{L}(x_k, z) + 2\gamma_k^2 E[p^{-1}_k C^2_{g,ik} | F_k] - 2\gamma_k \]
\[ \left( \tilde{\nabla}_{ik} f(x_k) + \eta_k \tilde{\nabla}_{ik} g(x_k) \right) \mid F_k \] 
\[ + 2\eta_k^2 \gamma_k^2 E[p^{-1}_k C^2_{g,ik} | F_k]. \] (8)
Using definition of expectation, term-3 = \( C^2_{g} \), term-4 = \( C^2_{g} \), term-5 = \( (x_k - z)^T \left( \tilde{\nabla} f(x_k) + \eta_k \tilde{\nabla} g(x_k) \right) \). From (8), 
\[ E[\mathcal{L}(x_{k+1}, z) | F_k] \leq \mathcal{L}(x_k, z) + 2\gamma_k^2 C^2_{g} + 2\eta_k^2 \gamma_k^2 C^2_{g} + 2\gamma_k (z - x_k)^T \left( \tilde{\nabla} f(x_k) + \eta_k \tilde{\nabla} g(x_k) \right). \] (9)
Using the definition of subgradient at point \( x_k \), 
\[ \text{term-6} = (z - x_k)^T \left( \tilde{\nabla} f(x_k) + \eta_k (z - x_k) \right)^T \tilde{\nabla} g(x_k) \leq f(z) - f(x_k) + \eta_k g(z) - \eta_k g(x_k). \]
Bounding term-6, using conditional and total expectation, 
\[ E[\mathcal{L}(x_{k+1}, z)] \leq E[\mathcal{L}(x_k, z)] + 2\gamma_k^2 C^2_{g} + 2\eta_k^2 \gamma_k^2 C^2_{g} + 2\gamma_k (f(z) + \eta_k g(z) - E[f(x_k) + \eta_k g(x_k)]). \] (10)
Multiplying the both sides of equation (10) by \( \gamma_k^{-1} \), and adding, subtracting \( \gamma_k^{-1} E[\mathcal{L}(x_{k+1}, z)] \) on the left-hand side, 
\[ \gamma_k^{-1} E[\mathcal{L}(x_{k+1}, z)] - \left( \gamma_k^{-1} - \gamma_k^{-1} \right) E[\mathcal{L}(x_k, z)] - \gamma_k^{-1} E[\mathcal{L}(x_k, z)] - \gamma_k^{-1} \left( E[\mathcal{L}(x_k, z)] \right) \leq 2\gamma_k^{-1} C^2_{g} + 2\gamma_k^{-1} \eta_k^2 C^2_{g} + 2\gamma_k (f(z) + \eta_k g(z) - E[f(x_k) + \eta_k g(x_k)]). \] (11)
Since \( r < 1 \) and \( \gamma_k \) is a non-increasing, \( \gamma_k^{-1} \) is a non-negative sequence. From Lemma\textsuperscript{1}\( \mathcal{L}(x_k, z) \leq p_{\max} \| x_k - z \|^2 \leq 2p_{\max} (\| x_k \|^2 + \| z \|^2) \). From the boundedness of set \( X \), \( E[\mathcal{L}(x_k, z)] \leq 4p_{\max} M^2 \). Substituting bound on term-7 in (11) and summing up over \( k = 1, \ldots, N - 1 \), 
\[ -2\gamma_0^{-1} E[\mathcal{L}(x_1, z)] - 4\gamma_0^{-1} p_{\max} M^2 \leq 2C^2_{g} \sum_{k=1}^{N-1} \gamma_k^{-1} + 2\sum_{k=1}^{N-1} \gamma_k^{-1} (f(z) + \eta_k g(z)) - 2\sum_{k=1}^{N-1} \gamma_k^{-1} E[f(x_k) + \eta_k g(x_k)]. \] (12)
putting \( k = 0 \) in (10), 
\[ E[\mathcal{L}(x_1, z)] \leq E[\mathcal{L}(x_0, z)] + 2\gamma_0^2 C^2_{g} + 2\gamma_0^2 \eta_0^2 C^2_{g} + 2\gamma_0 (f(z) + \eta_0 g(z) - E[f(x_0) + \eta_0 g(x_0)]). \]
Now, term-8 \( \leq 4p_{\max} M^2 \) and \( \gamma_0^{-1} = \gamma_{N-1}^{-1} + \gamma_{N-1}^{-1} \). Multiplying the both sides of equation with \( \gamma_0^{-1} \), we get, 
\[ \gamma_0^{-1} E[\mathcal{L}(x_1, z)] - 4\gamma_0^{-1} p_{\max} M^2 \leq 2\gamma_0^{-1} C^2_{g} + 2\gamma_0^{-1} \gamma_{N-1}^{-1} + 2\gamma_0 (f(z) + \eta_0 g(z) - E[f(x_0) + \eta_0 g(x_0)]). \] (13)
Adding (12) and (13) together, and combining the terms, 
\[ -4p_{\max} M^2 \left( \gamma_0^{-1} + \gamma_{N-1}^{-1} \right) \leq 2C^2_{g} \left( \sum_{k=0}^{N-1} \gamma_k^{-1} \right) + 2 \left( \sum_{k=0}^{N-1} \gamma_k^{-1} (f(z) + \eta_k g(z)) \right) - 2 \left( \sum_{k=0}^{N-1} \gamma_k^{-1} E[f(x_k) + \eta_k g(x_k)] \right). \]
Dividing the both sides by \( N^{-1} \gamma_i \), and denoting \( \sum_{i=0}^{N-1} \gamma_i = \lambda_{k,N-1} \), we get, 
\[ \sum_{k=0}^{N-1} \lambda_{k,N-1} E[f(x_k) + \eta_k g(x_k)] - \sum_{k=0}^{N-1} \lambda_{k,N-1} (f(z) \leq \mathcal{L}(x_k, z) \leq 2C^2_{g} \left( \sum_{k=0}^{N-1} \gamma_k^{-1} \right) + 2 \left( \sum_{k=0}^{N-1} \gamma_k^{-1} (f(z) + \eta_k g(z)) \right) - 2 \left( \sum_{k=0}^{N-1} \gamma_k^{-1} E[f(x_k) + \eta_k g(x_k)] \right). \] (14)
Using the convexity of \( f \) and the definition of \( \lambda_{k,N-1} \), we have term-10 \( \leq \sum_{k=0}^{N-1} \lambda_{k,N-1} f(x_k) \), and term-11 = \( f(z) \). 
\[ E[f(x_N)] \leq \sum_{k=0}^{N-1} \lambda_{k,N-1} \eta_k g(z) - \sum_{k=0}^{N-1} \lambda_{k,N-1} \eta_k g(x_k) \] 
\[ + \left( \sum_{i=0}^{N-1} \gamma_i \right)^{-1} \left( 2p_{\max} M^2 (\gamma_0^{-1} + \gamma_{N-1}^{-1}) \right) + C^2_{g} \left( \sum_{k=0}^{N-1} \gamma_k^{-1} \right) + C^2_{g} \left( \sum_{k=0}^{N-1} \gamma_k^{-1} \right) \].
Using definition of $M_g$, we obtain,

$$\text{term-12} = E \left[ \sum_{k=0}^{N-1} \lambda_k, N-1 \eta_k g(z) - \sum_{k=0}^{N-1} \lambda_k, N-1 \eta_k g(x_k) \right]$$

$$\leq E \left[ \sum_{k=0}^{N-1} \lambda_k, N-1 \eta_k g(z) - g(x_k) \right]$$

$$\leq 2M_g \sum_{k=0}^{N-1} \lambda_k, N-1 \eta_k.$$

Bounding term-12 and using the definition of $\lambda_k, N-1$,

$$E[f(\bar{x}_N)] - f(z) \leq \left( \sum_{i=0}^{N-1} \gamma_i \right)^{-1} \left( 2M_g \sum_{k=0}^{N-1} \gamma_k \eta_k + C_g^2 \sum_{k=0}^{N-1} \eta_k \right) \left( \sum_{k=0}^{N-1} \gamma_k + 2p_{max} M^2 \gamma_{0}^{-1} \left( \sum_{i=0}^{N-1} \frac{1}{(k+1)^{ar}} \right)^{-1} \right)^{\alpha + 1} \left( \sum_{i=0}^{N-1} \frac{1}{(k+1)^{ar}} \right)^{-1} \left( 2M_g \eta_0 + \gamma_0 \left( C_g^2 + C_g^2 \eta_0^2 \right) \right).$$

The above equation can also be written as

$$E[f(\bar{x}_N)] - f^* \leq 2p_{max} M^2 \gamma_{0}^{-1} \left( \text{term-3 + term-2} \right) + \text{term-4} \left( 2M_g \eta_0 + \gamma_0 \left( C_g^2 + C_g^2 \eta_0^2 \right) \right).$$

From Lemma 7, we have, term-2 $\leq \frac{1-ar}{N^{1-ar}+1} = O \left( N^{(1-ar)} \right)$, term-3 $\leq \frac{1-ar}{N^{1-ar}+1} = O \left( N^{(1-ar)} \right)$, term-4 $\leq \frac{1-ar}{N^{1-ar}+1} = O \left( N^{(1-ar)} \right)$. Now, substituting bounds of terms-2, 3, and 4, we have,

$$E[f(\bar{x}_N)] - f^* \leq O \left( \max \left( N^{(1-ar)}, N^{(1-ar)} \right) \right) = O \left( N^{\min \{1-ar, 1-a, b \}} \right).$$

From definitions of $a, r, \text{ and } \delta$, we obtain the result.

**V. APPLICATION OF RB-IRG**

One of the ways to address the ill-posedness in image deblurring is employing the regularization. The ill-posed problem (2) is converted into the regularized problem (7). By substituting functions $f(x) = \|b - Ax\|^2$, and $g(x) = \|x\|_2^2$ in (7). As the value of regularization parameter $\eta$ changes, we solve a different optimization problem (7). The basic idea is, $\eta \in (0, +\infty)$ governs the way by which solutions of linear inverse problem (2) are approximated by (7).

We are provided with the blurred noisy image Fig. 1 (a), which is further converted into the column vector $b$. Our objective is to get the original image, Fig. 1 (a) using image deblurring. Here we compare two ways of deblurring: standard regularization, and RB-IRG.

**Inference:** Fig. 2 (a)–(e) show the deblurred images obtained by conventional regularization at different $\eta$ for 105 iterations. Fig. 2 (f)–(j) show the deblurred images using RB-IRG with stopping at different iteration. RB-IRG is computationally effective because unlike as the case of conventional regularization, in RB-IRG we solve the problem instance just once. The tricky part is at what iteration $k$ we should stop. Stopping at a suitable iteration $k$ is desired because that governs the deblurred image quality. Practically (using RB-IRG), this seems to be feasible because we could save images after a regular interval of iterations and would stop at any iteration $k$ when the deblurred picture is good enough.
In this work, we consider a bilevel optimization problem \( P \) with high dimensional solution space. Random block coordinate iterative regularized gradient descent (RB-IRG) scheme is developed to address problem \( P \). We establish the convergence of sequence generated from RB-IRG to the unique solution of \( P \). Furthermore, we derive the rate of convergence \( O(\frac{1}{k}) \), with respect to the inner level function of the bilevel problem. Our ground assumptions in the convergence proof and rate analysis are mild, such that \( f \) and \( g \) can be nondifferentiable functions. Demonstration of RB-IRG on image processing shows that our scheme computationally performs well compared to the conventional (two loop) regularization schemes.

VI. CONCLUSION

In this work, we consider a bilevel optimization problem \( P \) with high dimensional solution space. Random block coordinate iterative regularized gradient descent (RB-IRG) scheme is developed to address problem \( P \). We establish the convergence of sequence generated from RB-IRG to the unique solution of \( P \). Furthermore, we derive the rate of convergence \( O(\frac{1}{k}) \), with respect to the inner level function of the bilevel problem. Our ground assumptions in the convergence proof and rate analysis are mild, such that \( f \) and \( g \) can be nondifferentiable functions. Demonstration of RB-IRG on image processing shows that our scheme computationally performs well compared to the conventional (two loop) regularization schemes.

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