Abstract Harmonic Analysis On the non Singular Matrix Lie Group

Kahar El-Hussein
Department of Mathematics, Faculty of Science and Arts at Al Qurayt,
Al Jouf University, KSA &
Department of Mathematics, Faculty of Science,
Al Furat University, Dear El Zore, Syria
E-mail: kumath@ju.edu.sa , kumath@hotmail.com
November 12, 2018

Abstract
As well known that it is no way to do the abstract harmonic analysis on the non connected Lie groups. The goal of this paper is to draw the attention of Mathematicians to solve this problem. Therefore let $\mathbb{R}^*$ be the group of nonzero real numbers with multiplication and let $H$ be the 3–dimensional Heisenberg group. I denote by $G = H \ltimes (\mathbb{R}^*_\rho)^3$ the 6-dimensional non connected solvable Lie group, which is isomorphic onto the 6– dimensional non singular triangular matrix Lie group. I define the Fourier-Mellin transform and establish the Plancherel theorem on the group $G$. Besides I prove the solvability of an invariant differential operator on the group $G_+ = H \ltimes (\mathbb{R}^*_\rho)^3$, which is the identity component of the group $G$. Finally, I give a classification of all left ideals of $L^1(G_+)$

Keywords: Non Connected Solvable Lie Group, Semi-Direct Product, Fourier Transform and Plancherel Theorem, Existence Theorem, Ideals of the Group Algebra

AMS 2000 Subject Classification: 43A30&35D 05


1 Background

1.1. The classical Fourier transform is one of the most widely used mathematical tools in engineering. However, few engineers know that extensions Fourier analysis on noncommutative Lie groups holds great potential for solving problems in robotics, image analysis, mechanics. Engineering applications of noncommutative harmonic analysis brings this powerful tool to the engineering world. In mathematics, Abstract harmonic analysis is a generalization of non commutative harmonic analysis in which results from Fourier analysis are extended to topological groups which are not commutative. Abstract harmonic analysis, having its roots in the mid-twentieth century. Its major business is the extension of the theory to all groups $G$ that are non commutative and locally compact. The results of its applications are used in the theory of dynamical systems, in the theory of group representations theory, in the theory of Banach algebra and in many other mathematical physics theories.

1.2. The best developed branch of abstract harmonic analysis have been obtained for compact groups by the Peter-Weyl theory for the representations of compact Lie groups. In the case of non-compact non-commutative groups the general theory is still far from complete and it is a difficult task due to the nature of the group representations. There is still no general theory for approaching the harmonic analysis of an arbitrary locally compact group.

1.3. In the second half of twenty century, two points of view were adopted by the community of the mathematics. The first one is the theory of representations of Lie groups. Unfortunately If the group $G$ is no longer assumed to be abelian, it is not possible anymore to consider its dual (i.e the set of all equivalence classes of unitary irreducible representations). For a long time, people have tried to construct objects in order to generalize Fourier transform to the non abelian case. However, with the dual object not being a group, it is not possible to define the Fourier transform and the inverse Fourier transform between the group $G$ and its dual $G$. These difficulties of Fourier analysis on noncommutative groups makes the noncommutative version of the problem very challenging. It was necessary to find a subgroup or at least a subset of locally compact groups which were not ”pathological”, or ”wild” as Kirillov calls them [14]

The second is the quantum groups (Hopf Algebra), which was introduced by Vladimir Drinfeld and Michio Jimbo. some little results were obtained
by this theory. Still now neither the theory of quantum groups nor the
representations theory have done to reach this goal.

1.4. Since 2006, and far away of theory of representations of Lie groups
and the of theory of quantum groups, I have opened a new way in abstract
harmonic analysis that no one has opened before.

Therefore, I would like to draw the attention of Scientists in Mathematics
and Physics on the ideas of my way which focus on two approaches:

1- The First one focuses on Fourier transform and Partial differential
equations with variable coefficients on Lie groups. By this way, I have solved
the Lewy and Mizohata operators. I believe that is will be the business
of the expertise in the theory of partial differential equations with variable
coefficients and their applications.

2- By the second way, we can solve the most major problems in Fourier
analysis on many Lie groups. In this paper, I will introduce the abstract
harmonic analysis on the group $G = H \rtimes (\mathbb{R}^* )^3$, which is not only non
commutative locally compact group but is not connected. So I open other
new way for Mathematicians and physicists in the theory of Fourier analysis
on a non connected Lie groups

2 Introduction and Results

2.1. In my book [10], I have proved the set $(\mathbb{R}^+ )^3 = \{ (x, y, z) \in (\mathbb{R}^* )^3; \n x \not< 0, y \not< 0, z \not< 0 \}$ is group and isomorphic onto the group $(\mathbb{R}^+ )^3 = \{ (x, y, z) \in (\mathbb{R}^* )^3; x \not> 0, y \not> 0, z \not> 0 \}$. I consider the non connected
solvable Lie group, which is consisted of all matrices

$$G = \left[\begin{array}{ccc}
a_1 & n_1 & n_3 \\
0 & a_1 & n_2 \\
0 & 0 & a_1
\end{array}\right], (a_1, a_2, a_3 ) \in (\mathbb{R}^* )^3$$ (1)

The 3-dimensional Heisenberg group $H$, which consists of all matrices

$$H = \left[\begin{array}{ccc}
1 & n_1 & n_3 \\
0 & 1 & n_2 \\
0 & 0 & 1
\end{array}\right], (a_1, a_2, a_3 ) \in (\mathbb{R}^* )^3$$ (2)
is normal sub-group of \( G \). The multiplication of two element \( n = (n_3, n_2, n_1) \) and \( m = (m_3, m_2, m_1) \) is given by

\[
n.m = (n_3, n_2, n_1)(m_3, m_2, m_1) = (n_3 + m_3 + n_1m_2, n_2 + m_2, n_1 + m_1)
\]  

(3)

So the group \( G \) can be identified with the group \( H \rtimes_{\rho} (\mathbb{R}^*)^3 \), which is the semi-direct of the group \( H \) with group \((\mathbb{R}^*)^3\). Since the set \( \mathbb{R}^*_+ \) is group isomorphic onto the group \( \mathbb{R}^*_+ \) see [10], then the group \( \mathbb{R}^*_+ \) is a two copies of the group \( \mathbb{R}^*_+ \). And so the group \((\mathbb{R}^*)^3\) becomes two copies of the group \((\mathbb{R}^*_+)^3\). It is enough to restrict my study on the group \( G_+ \) of all matrices

\[
G_+ = \left[ \begin{array}{ccc}
    a_1 & n_1 & n_3 \\
    0 & a_1 & n_2 \\
    0 & 0 & a_1
\end{array} \right], (a_1, a_2, a_3) \in (\mathbb{R}^*_+)^3
\]  

(4)

The group \( G_+ = H \rtimes_{\rho} (\mathbb{R}^*_+)^3 \) is the semi-direct product of \( H \) with \((\mathbb{R}^*_+)^3\), where \( \rho \) is the group homomorphism \( \rho : (\mathbb{R}^*_+)^3 \rightarrow Aut(H) \) defined by:

\[
\rho(a)(n_3, n_2, n_1) = \rho(a_1, a_2, a_3)(n_3, n_2, n_1) = (a_1a_3^{-1}n_3, a_2a_3^{-1}n_2, a_1a_2^{-1}n_1)
\]  

(5)

\[
\rho(a^{-1})(n_3, n_2, n_1) = \rho(a_1^{-1}, a_2^{-1}, a_3^{-1})(n_3, n_2, n_1) = (a_1^{-1}a_3n_3, a_2^{-1}a_3n_2, a_1^{-1}a_2n_1)
\]  

(6)

for any \( a = (a_1, a_2, a_3) \in (\mathbb{R}^*_+)^3 \) and \( (n_3, n_2, n_1) \in H \), where \( Aut(H) \) is the group of all automorphisms of \( H \). The multiplication of two elements \( X = (n_3, n_2, n_1, a_1, a_2, a_3) \) and \( Y = (m_3, m_2, m_1, b_1, b_2, b_3) \) in \( G_+ \) is given by

\[
X \cdot Y = (n_3, n_2, n_1, a_1, a_2, a_3)(m_3, m_2, m_1, b_1, b_2, b_3)
\]

\[
= ((n_3, n_2, n_1), \rho(a_1, a_2, a_3)(m_3, m_2, m_1), (a_1b_1, a_2b_2, a_3b_3))
\]

\[
= ((n_3, n_2, n_1), \rho(a_1, a_2, a_3)(m_3, m_2, m_1), (a_1b_1, a_2b_2, a_3b_3))
\]

\[
= ((n_3, n_2, n_1), (a_1a_3^{-1}m_3, a_2a_3^{-1}m_2, a_1a_2^{-1}m_1), (a_1b_1, a_2b_2, a_3b_3))
\]

\[
= ((n_3 + a_1a_3^{-1}m_3 + n_1a_2a_3^{-1}m_2, n_2 + a_2a_3^{-1}m_2, n_1 + a_1a_2^{-1}m_1), (a_1b_1, a_2b_2, a_3b_3))
\]  

(7)
The inverse of an element \( X = (n_3, n_2, n_1, a_1, a_2, a_3) \) in \( G_+ \) is

\[
X^{-1} = (n_3, n_2, n_1, a_1, a_2, a_3)^{-1} = (\rho((a_1, a_2, a_3)^{-1})((n_3, n_2, n_1))^{-1}, (a_1, a_2, a_3)^{-1})
\]

\[
= ((a_1^{-1}a_3(-n_3 + n_1n_2), -a_2^{-1}a_3n_2, -a_1^{-1}a_2n_1), (a_1^{-1}, a_2^{-1}, a_3^{-1}))
\] (8)

We denote by \( C^\infty(G), D(G), D'(G), E'(G) \) respectively the space of \( C^\infty \)-functions, \( C^\infty \)-functions with compact support, distributions and distributions with compact support. We denote by \( L^1(G) \) the Banach algebra that consists of all complex valued functions on the group \( G \), which are integrable with respect to the Haar measure of \( G \) and multiplication is defined by convolution on \( G \).

2.2. Let \( \mathcal{U} \) be the complexified universal enveloping algebra of the real Lie algebra \( g_+ \) of \( G_+ \); which is canonically isomorphic to the algebra of all distributions on \( G_+ \) supported by the identity element \((0, 0, 0, 1, 1, 1)\) of \( G_+ \). For any \( u \in \mathcal{U} \) one can define a differential operator \( P_u \) on \( G_+ \) as follows:

\[
P_u f(X) = u \ast f(X) = \int_G f(Y^{-1}X)u(Y) dY
\] (9)

for any \( f \in C^\infty(G_+) \), where \( Y = (m_3, m_2, m_1, b_1, b_2, b_3), dY = dm_3 dm_2 dm_1 db_1 db_2 db_3 \) is the right Haar measure on \( G_+ \), \( X = (n_3, n_2, n_1, a_1, a_2, a_3) \) and \( \ast \) denotes the convolution product on \( G_+ \). The mapping \( u \rightarrow P_u \) is an algebra isomorphism of \( \mathcal{U} \) onto the algebra of all right invariant differential operators on \( G_+ \).

2.3. Let \( K = H \times (\mathbb{R}_+^*)^3 \) be the group of the direct product of \( H \) and \( (\mathbb{R}_+^*)^3 \). We denote also by \( \mathcal{U} \) the complexified enveloping algebra of the real Lie algebra \( k \) of \( K \). For every \( u \in \mathcal{U} \), we can associate a differential operator \( Q_u \) on \( K \) as follows

\[
Q_u f(X) = u \ast f(X) = \int_K f(Y^{-1}X)u(Y) dY
\] (10)

for any \( f \in C^\infty(K), X \in K, Y \in K \), where \( \ast \) signify the convolution product on the commutative group \( K \) and. The mapping \( u \rightarrow Q_u \) is an algebra isomorphism of \( \mathcal{U} \) onto the algebra of all invariant differential operators on \( K \). For more details see[7, 13]
3 Fourier Transform and Plancherel Theorem on $G_+$

3.1. Let $L = H \times (\mathbb{R}_+^*)^3 \times (\mathbb{R}_+^*)^3$ be the group with multiplication

$$X \cdot Y = (n, x, a)(m, y, b)$$

$$= (n. \rho(a)m, xy, ab)$$

(11)

for all $X = (n, x, a) \in L$ and $Y = (m, y, b) \in L$, where $n = (n_3, n_2, n_1)$, and $m = (m_3, m_2, m_1)$ The inverse of an element $X = (n, x, a)$ in $L$ is given by:

$$X^{-1} = (n, x, a)^{-1}$$

$$= (\rho(a^{-1})n^{-1}, x^{-1}, a^{-1})$$

(12)

where $n^{-1} = (-n_3 + n_1 n_2, -n_2, -n_1)$, $\rho(a^{-1})n^{-1} = \rho((a_1, a_2, a_3)^{-1})((n_3, n_2, n_1))^{-1} = (a_1^{-1} a_3(-n_3 + n_1 n_2), -a_2^{-1} a_3 n_2, -a_1^{-1} a_2 n_1)$, $x^{-1} = (x_1^{-1}, x_2^{-1}, x_3^{-1})$, and $a^{-1} = (a_1^{-1}, a_2^{-1}, a_3^{-1})$. In this case, we can identify $G_+$ with the closed subgroup $H \times \{1\} \rtimes \rho(\mathbb{R}_+^*)^3$ of $L$ and $K$ with $H \times (\mathbb{R}_+^*)^3 \times \{1\}$ of $L$. From now on, I denote by an instead of $\rho(a)n$

**Definition 3.1.** For every $\phi \in C^\infty(G_+)$, one can define a function $\tilde{\phi} \in C^\infty(L)$ as follows:

$$\tilde{\phi}(n, a, x) = \phi(an, ax)$$

(13)

for all $(n, a, x) \in L$.

**Remark 3.1.** The function $\tilde{\phi}$ is invariant in the following sense

$$\tilde{\phi}(bn, ab^{-1}, xb) = \tilde{\phi}(n, a, x)$$

(14)

for any $(n, a, x) \in L$ and $b \in (\mathbb{R}_+^*)^3$. So every function $\psi(n, a)$ on $G_+$ extends uniquely as an invariant function $\tilde{\psi}(n, a, x)$ on $L$.

**Definition 3.2.** For every $F \in L^1(L)$ one can define two convolutions
product on the group $L$ as:

\[
g \ast F(n, a, x) = \int_{G_+} F \left[ (m, b)^{-1}(n, a, x) \right] g(m, b) db/b
\]

\[
= \int_{G_+} F \left[ ((b^{-1}m^{-1}), b^{-1})(n, a, x) \right] g(m, b) db/b
\]

\[
= \int_{G_+} F \left[ (b^{-1}(m^{-1}n), ax^{-1}) \right] g(m, b) db/b
\]

and

\[
g \ast F(n, a, x_1) = \int_{K} F \left[ ((m^{-1}, b^{-1})(n, a, x) \right] g(m, b) db/b
\]

\[
= \int_{K} F \left[ (m^{-1}n, b^{-1}a, x) \right] g(m, b) db/b
\]

for any $F \in L^1(L)$, where $a = (a_1, a_2, a_3) \in (\mathbb{R}_+^*)^3$, $x = (x_1, x_2, x_3) \in (\mathbb{R}_+^*)^3$, $b = (b_1, b_2, b_3) \in (\mathbb{R}_+^*)^3$, $dm/db/b = dm_3dm_2dm_1db_3/db_2/db_1$ is the right Haar measure on $G_+$, $\ast$ is the convolution product on $G_+$ and $\ast$ is the convolution product on $K$.

It results immediately if $F$ is invariant in sense (14), I get the following equality

\[
g \ast F(n, a, x) = u \ast F(n, a, x)
\] (15)

As in [10], we will define the Fourier-Mellin transform on $G_+$. Therefore let $\mathcal{S}(G_+)$ be the Schwartz space of $G_+$ which can be considered as the Schwartz space of $\mathcal{S}(H \times (\mathbb{R}_+^*)^3)$, and let $\mathcal{S}'(G_+)$ be the space of all tempered distributions on $G_+$.

**Definition 3.3.** If $f \in \mathcal{S}(G_+)$, we define the Fourier transform of its
invariant \( \tilde{f} \) as follows

\[
\mathcal{F}_H \mathcal{F} \tilde{f} (\xi, \lambda, 1) = \int_{\mathbb{H}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(n, a, b) e^{-i \xi^T a - i \lambda^T b - i \mu^T} \frac{da}{a} \frac{db}{b} d\mu
\]

\[
= \int_{\mathbb{R}^3} \mathcal{F}_H \mathcal{F} \tilde{f}(\xi, \lambda, \mu) d\mu = \mathcal{F}_H \mathcal{F} \tilde{f}(\xi, \lambda, 1) \quad (16)
\]

where \( \mathcal{F}_H \) is the Fourier transform on the 3-dimensional Heisenberg group, \( a = (a_1, a_2, a_3) \), \( b = (b_1, b_2, b_3) \), \( \frac{da}{a} = \frac{da_1}{a_1} \frac{da_2}{a_2} \frac{da_3}{a_3} \), \( \frac{db}{b} = \frac{db_1}{b_1} \frac{db_2}{b_2} \frac{db_3}{b_3} \), \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \), \( 1 = (1, 1, 1) \), \( \mu = (\mu_1, \mu_2, \mu_3) \) and \( d\mu = d\mu_1 d\mu_2 d\mu_3 \).

**Proposition 3.1** For every \( g \in \mathcal{S}(G) \), and \( f \in \mathcal{S}(G) \), we have

\[
\int_{\mathbb{R}^3} \mathcal{F}_H \mathcal{F}(g \ast \tilde{f})(\xi, \lambda, \mu) d\mu = \mathcal{F}_H \mathcal{F}(\tilde{f})(\xi, \lambda, 1) \mathcal{F}_H \mathcal{F}(g)(\xi, \lambda) \quad (17)
\]

for any \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{H} \), and \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \).

**Proof**: First, we have

\[
g \ast \tilde{f}(m, b, x)
= \int_{G^+} \tilde{f}((n, a)^{-1}(m, b, x))g(n, a)dn \frac{da}{a}
= \int_{G^+} \tilde{f}(a^{-1}(mn^{-1}), ba^{-1})g(n, a)dn \frac{da}{a}
= \int_{K} \tilde{f}(mn^{-1}, ba^{-1}, x)g(n, a)dn \frac{da}{a} = g \ast \tilde{f}(m, b, x) \quad (18)
\]

Secondly:

\[
\int_{\mathbb{R}^3} \mathcal{F}_H \mathcal{F}(\check{u} \ast \tilde{f})(\xi, \lambda, \mu) d\mu
= \int_{\mathbb{R}^3} \mathcal{F}_H \mathcal{F}(\check{u} \ast \tilde{f})(\xi, \lambda, \mu) d\mu = \mathcal{F}_H \mathcal{F}(\tilde{f})(\xi, \lambda, 1) \mathcal{F}(\check{u})(\xi, \lambda) \quad (19)
\]
Theorem 3.1. (Plancheral’s Theorem). For any \( f \in L^1(G_+) \cap L^2(G_+) \), we get

\[
\int_{G_+} f(n,a) \, dn \, da = \int_{\mathbb{R}^6} |Ff(\xi,\lambda)|^2 \, d\xi d\lambda
\]

where \( \xi = (\xi_1, \xi_2, \xi_3), (\lambda_1, \lambda_2, \lambda_3), d\xi = d\xi_1 d\xi_2 d\xi_3, d\lambda = d\lambda_1 d\lambda_2 d\lambda_3 \), and

\[
f(n, a, b) = f(an, ab) = f((an, ab)^{-1})
\]

Proof: First, we have

\[
\int_{G_+} f(n,a) \, dn \, da = \int_{\mathbb{R}^6} |Ff(\xi,\lambda)|^2 \, d\xi d\lambda
\]

Secondly by (14), we get

\[
\int_{G_+} f(n,a) \, dn \, da = \int_{\mathbb{R}^6} |Ff(\xi,\lambda)|^2 \, d\xi d\lambda
\]
So the Plancherl theorem on $G_+$. 

**Corollary 3.1.** For any $f \in L^2(G = H \rtimes \rho(\mathbb{R}^*)^3)$, I get

$$
\int_G |f(n, (a_1, a_2, a_3))|^2 \frac{d\alpha_1}{1} \frac{d\alpha_2}{2} \frac{d\alpha_3}{3} = 2 \int_{\mathbb{R}^6} |\mathcal{F}_H \mathcal{F}(f)((\lambda_1, \lambda_2, \lambda_3))|^2 d\lambda_1 d\lambda_2 d\lambda_3
$$

(23)

where $n = (n_3, n_2, n_1)$, $\xi = (\xi_3, \xi_2, \xi_1)$, and $d\xi = d\xi_3 d\xi_2 d\xi_1$

**Proof:** Since the group $G = H \rtimes \rho(\mathbb{R}^*)^3$ is a two copies of the group $H \rtimes \rho(\mathbb{R}^*)^3$. That means $H \rtimes \rho(\mathbb{R}^*)^3 = H \rtimes \rho(\mathbb{R}^*)^3 \cup H \rtimes \rho(\mathbb{R}_+^*)^2$, so I obtain

$$
\int_{\mathcal{C}^2} f \ast f(0, 1, 1) = \int_G |f(n, (a_1, a_2, a_3))|^2 \frac{d\alpha_1}{1} \frac{d\alpha_2}{2} \frac{d\alpha_3}{3} = 2 \int_{\mathbb{R}^6} |\mathcal{F}_H \mathcal{F}(f)((\lambda_1, \lambda_2, \lambda_3))|^2 d\lambda_1 d\lambda_2 d\lambda_3
$$

(24)

Hence the proof of the corollary

### 4 Left Ideals of the Group Algebra $L^1(G_+)$

First, I will prove the solvability of any invariant differential operator on the connected solvable group $G_+$. Therefore denote by $\mathcal{C}^\infty(G_+)$ (resp. $\mathcal{C}^\infty(K)$) the image of $\mathcal{C}^\infty(G_+)$ (resp. $\mathcal{C}^\infty(K)$) then we have

$$
\mathcal{C}^\infty(G_+)|_{G_+} = \mathcal{C}^\infty(G_+)
$$

$$
\mathcal{C}^\infty(K)|_{K} = \mathcal{C}^\infty(K)
$$

(25)

**Definition 4.1.** Let $\chi$ be the mapping : $\mathcal{C}^\infty(K)|_{K} \rightarrow \mathcal{C}^\infty(G_+)|_{G_+}$ defined by

$$
\tilde{f}|_K (n, a, 1) \rightarrow \tilde{f}|_{G_+}(n, 1, a)
$$

(26)
is topological isomorphism and its inverse is nothing but $\chi^{-1}$ defined by

$$\tilde{f}|_{G_+}(n, 1, a) \to \tilde{f}|_{K}(n, a, 1) \quad (27)$$

My main result is

**Theorem 4.1.** If $P_u$ any invariant differential operator on $G_+$ associated to the distribution $u \in \mathcal{U}$, then, we have

$$P_u C^\infty(G_+) = C^\infty(G_+) \quad (28)$$

**Proof:** Let $Q_u$ be the invariant differential operator on $K$ associated to $u$, then by the theory of my book for the invariant differential operators on the Heisenberg group $H$ and the theory of partial differential operators with constant coefficients [23], we get

$$Q_u C^\infty(K)|_K = C^\infty(K)|_K = C^\infty(K) \quad (29)$$

That means for any $\psi(n, a) \in C^\infty(K)$, there exist a function $\varphi(n, a, x) \in \widehat{C^\infty(K)}$, such that

$$Q_u \varphi(n, a, 1) = u \ast \varphi(n, a, 1) = \psi(n, a) \quad (30)$$

The function $\psi(n, a)$ can be transformed as an invariant function $\tilde{\psi} \in \widehat{C^\infty(K)}$ as follows

$$\psi(n, a) = \tilde{\psi}((a^{-1}n), a, 1) \quad (31)$$

In other side, we have

$$\chi Q_u \varphi(n, a, 1)
\quad = \quad Q_u \varphi(n, 1, a) = u \ast \varphi(n, 1, a)
\quad = \quad u \ast \varphi(n, 1, a) = P_u \varphi(n, 1, a)
\quad = \quad \chi \tilde{\psi}(a^{-1}n, a, 1) = \tilde{\psi}(a^{-1}n, 1, a)
\quad = \quad \psi(n, a) \quad (32)$$

where

$$u \ast \varphi(n, 1, a) = \left\{ \int_K \varphi(m^{-1}n, b^{-1}, a) \frac{d b}{b} u(m, b) dm, \varphi \in \widehat{C^\infty(K)} \right\} \quad (33)$$
and

\[ u \ast \varphi(n, 1, a) = \left\{ \int_{G_+} \varphi \left( (b^{-1}(m^{-1}n), 1, ab^{-1}) \right) g(m, b) dm \frac{db}{b}, \varphi \in C^\infty(G_+) \right\} \]

(34)

So the proof of the solvability of any right invariant differential operator on \( G_+ \).

If \( J \) is a subspace of \( L^1(G_+) \), we denote \( \tilde{I} \) its image by the mapping \( \sim \) let \( J = \tilde{I} \mid_K \). My main result is:

**Theorem 4.2.** Let \( I \) be a subspace of \( L^1(G_+) \), then the following conditions are equivalents.

(i) \( J = \tilde{I} \mid_K \) is a left ideal in the Banach algebra \( L^1(K) \).

(ii) \( I \) is a left ideal in the Banach algebra \( L^1(G_+) \).

**Proof:** (i) implies (ii) Let \( I \) be a subspace of the space \( L^1(L) \) such that \( J = \tilde{I} \mid_K \) is a left ideal in \( L^1(K) \), then we have:

\[ u \ast \tilde{I} \mid_K(n, a, 1) \subseteq \tilde{I} \mid_K(n, a, 1) \]

(35)

for any \( u \in L^1(K) \) and \( (n, a) \in K \), where

\[ u \ast \tilde{I} \mid_K(n, a, 1) = \left\{ \int_K \tilde{f} \mid_K \left[ (m^{-1}n, ab^{-1}, 1) \right] u(m, b) dm \frac{db}{b}, f \in I \right\} \]

(36)

It shows that

\[ u \ast \tilde{f} \mid_K(n, a, 1) \in \tilde{I} \mid_K(n, a, 1) \]

(37)

for any \( \tilde{f} \in \tilde{I} \). Then we get

\[ \chi(u \ast \tilde{f} \mid_K)(n, a, 1) = u \ast \tilde{f}(n, 1, a) \in \tilde{I} \mid_{G_+}(n, 1, a) = I \]

(38)

(ii) implies (i), if \( I \) is an ideal in \( L^1(G_+) \), then we get

\[ u \ast \tilde{I} \mid_{G_+}(n, 1, a) \subseteq u \ast I(n, a) \subseteq \tilde{I} \mid_{G_+}(n, 1, a) = I(n, a) \]

(39)
where
\[ u \ast \tilde{I} \big|_{G_+}(n, 1, a) = \left\{ \int_{G_+} \tilde{f} \big|_{G_+} (\cdot) (m, b) dm \frac{db}{b}, \ f \in I \right\} \]  

(40)

So, we obtain
\[ \chi^{-1}(u \ast \tilde{f} |_{G_+})(n, 1, a) \in \chi^{-1}(\tilde{I} |_{G_+})(n, 1, a) \]

(41)

and
\[ \chi^{-1}(u \ast \tilde{f} |_{G_+})(n, 1, a) = u \ast \tilde{f} |_{K}(n, a, 1) \in \tilde{I} |_{K}(n, a, 1) \]  

(42)

**Corollary 4.1.** Let \( I \) be a subspace of the space \( L^1(G_+) \) and \( \tilde{I} \) its image by the mapping \( \sim \) such that \( J = \tilde{I}|_K \) is an ideal in \( L^1(K) \), then the following conditions are verified.

1. \( J \) is a closed left ideal in the algebra \( L^1(K) \) if and only if \( I \) is a closed left ideal in the algebra \( L^1(G_+) \).
2. \( J \) is a prime left ideal in the algebra \( L^1(K) \) if and only if \( I \) is a prime left ideal in the algebra \( L^1(G_+) \).
3. \( J \) is a maximal left ideal in the algebra \( L^1(K) \) if and only if \( I \) is a maximal left ideal in the algebra \( L^1(G_+) \).
4. \( J \) is a left dense ideal in the algebra \( L^1(K) \) if and only if \( I \) is a dense left ideal in the algebra \( L^1(G_+) \).

The proof of this corollary results immediately from theorem 3.2.

The Heisenberg group \( H \) is the semi-direct product of the two vector Lie group \( \mathbb{R}^2 \rtimes_{\sigma} \mathbb{R} \), where \( \sigma : \mathbb{R} \rightarrow Aut(\mathbb{R}^2) \) is the homomorphism group defined. I extend the group \( K = H \times (\mathbb{R}^+_+)^3 \) by considering the new group \( S = \mathbb{R}^2 \times \mathbb{R} \times (\mathbb{R}^+_+) \times (\mathbb{R}^+_+)^3 \) with the following law

\[ X \cdot Y = (n_3, n_2, n_1, n_4, a_1, a_2, a_3) (m_3, m_2, m_1, m_4, b_1, b_2, b_3) \]

\[ = ((n_3 + \sigma(n_4)(m_3, m_2), n_2 + m_2, n_1 + m_1, n_4 + m_4), (a_1 b_1, a_2 b_2, a_3 b_3)) \]

\[ = ((n_3 + m_3 + n_4 m_2, n_2 + m_2, n_1 + m_1, n_4 + m_4), (a_1 b_1, a_2 b_2, a_3 b_3)) \]  

(43)

Denote by \( B = \mathbb{R}^2 \times \mathbb{R} \times (\mathbb{R}^+_+)^3 \) the commutative Lie group of the direct product of three Lie groups \( \mathbb{R}^2, \mathbb{R}^+, \) and \( (\mathbb{R}^+_+)^3 \). In this case the group \( K = \)
$H \times (\mathbb{R}_+)^3$ can be identified with the sub-group $\mathbb{R}^2 \times \{0\} \times \mathbb{R} \times (\mathbb{R}_+)^3$ and the group $B = \mathbb{R}^2 \times \mathbb{R} \times (\mathbb{R}_+)^3$ can be identified with the sub-group $\mathbb{R}^2 \times \mathbb{R} \times \{0\} \times (\mathbb{R}_+)^3$

**Definition 4.2.** Any function $\psi \in C^\infty(K)$ can be extended to a unique function $\Upsilon \psi$ belongs to $C^\infty(S)$, as follows

$$\Upsilon \psi((n_3, n_2, n_1, n_4), (x_1, x_2, x_3))$$

$$= \psi((\sigma(n_1)(n_3, n_2), n_1 + n_4), (x_1, x_2, x_3))$$

$$= \psi((n_1(n_3, n_2), n_1 + n_4), (x_1, x_2, x_3))$$

$$= \psi((n_3 + n_1n_2, n_2, n_1 + n_4), (x_1, x_2, x_3)) \quad (44)$$

for any $(n_3, n_2, n_1, n_4) \in H \times \mathbb{R}, x = (x_1, x_2, x_3) \in (\mathbb{R}_+)^3, n_1(n_3, n_2) = (n_3 + n_1n_2, n_2) = \sigma(n_1)(n_3, n_2)$. Note here the function $\Upsilon \psi$ is invariant in the following sense

$$\Upsilon \psi((n_3, n_2, n_1, n_4), (x_1, x_2, x_3))$$

$$= \Upsilon \psi((\sigma(m)(n_3, n_2), m^{-1}n_1, mn_4), (x_1, x_2, x_3)) \quad (45)$$

If $I$ is a subspace of $L^1(K)$, we denote $\Upsilon I$ its image by the mapping $\Upsilon$. Let $J = \Upsilon I \mid_B$.

My main results are:

**Theorem 4.3.** Let $I$ be a subspace of $L^1(K)$, then the following conditions are equivalents.

(i) $J = \Upsilon I \mid_B$ is an ideal in the commutative Banach algebra $L^1(B)$.

(ii) $I$ is a left ideal in the Banach algebra $L^1(K)$.

For the proof of this theorem, I refer to my book [10, Chap.I, theorem 3.1.]

**Corollary 4.2.** Let $I$ be a subspace of the space $L^1(K)$ and $\Upsilon I$ its image by the mapping $\Upsilon$ such that $J = \Upsilon I \mid_B$ is an ideal in $L^1(K)$, then the following conditions are verified.

(1) $J$ is an ideal in the commutative algebra $L^1(B)$ if and only if $I$ is a closed left ideal in the algebra $L^1(K)$ if and only if $I$ is a closed left ideal in the algebra $L^1(G_+)$.

(2) $J$ is a prime ideal in the commutative algebra $L^1(B)$ if and only if $I$ is a prime left ideal in the algebra $L^1(K)$ if and only if $I$ is a prime left ideal in the algebra $L^1(G_+)$.

(3) $J$ is a maximal ideal in the commutative algebra $L^1(B)$ if and only if $I$ is a maximal left ideal in the algebra $L^1(K)$ if and only if $I$ is a left maximal ideal in the algebra $L^1(G_+)$.
(4) \( J \) is a dense ideal in the commutative algebra \( L^1(B) \) if and only if \( I \) is a dense left ideal in the algebra \( L^1(K) \) if and only if \( I \) is a left dense ideal in the algebra \( L^1(G_+) \).

The proof of this corollary results as a consequence from theorems 4.3. and 4.2.

5 Discussion

5.1. In this paper I have proved that any invariant differential operator has the form

\[
P = \sum_{\alpha,\beta} a_{\alpha\beta} X^\alpha Y^\beta \tag{46}
\]

on the Lie group \( G_+ \), where \( X^\alpha = (X_1^{\alpha_1}, X_2^{\alpha_2}, X_3^{\alpha_3}), Y^\beta = (Y_1^{\beta_1}, Y_2^{\beta_2}, Y_3^{\beta_3}), \)
\( \alpha_i \in \mathbb{N}, \beta_i \in \mathbb{N} \ (1 \leq i \leq 3), \) and \( X = (X_1, X_2, X_3), Y = (Y_1, Y_2, Y_3) \) are the invariant vectors field on \( G_+ \), which are the basis of the Lie algebra \( g_+ \) of \( G_+ \) and \( a_{\alpha,\beta} \in \mathbb{C} \). As in theorem 5.1. any non zero invariant partial differential equation on the 6–dimensional group \( G_+ \) is solvable.

5.2. If we take the special case the 3–dimensional Heisenberg group. Any invariant differential operator has the form

\[
P = \sum_{\alpha,\beta,\gamma} a_{\alpha\beta\gamma} X^\alpha Y^\beta Z^\gamma \tag{47}
\]

on the Lie group \( H = \mathbb{R}^2 \times_{\rho} \mathbb{R} \), where \( \alpha \in \mathbb{N}, \beta \in \mathbb{N}, \gamma \in \mathbb{N} \), and \( X, Y, \) and \( Z \) are the invariant vectors on \( H \), which are the basis of the Lie algebra \( h \) of \( H \) and \( a_{\alpha,\beta,\gamma} \in \mathbb{C} \). Hence \( P \) is solvable. In particular the Lewy operator \( X + iY \) is solvable. For Mathematicians, there are many non solved problems in Fourier analysis on this group.

6 Conclusion.

6.1. It is well known the solvability of the invariant differential operators and the ideals of the group algebra of a Lie group play an important role in mathematical analysis and mathematical physics. In this paper I have
defined the Fourier transform and established the Plancherel theorem on the non connected group $G$. A classification of all left ideals in Banach algebra $L^1(G_+)$ is obtained and the solvability of any non zero invariant differential operator on the group $G_+$.

7 Competing Interests:

7.1. All Mathematicians interest in the theory of abstract harmonic analysis (non commutative harmonic analysis) to solve the major problems in analysis on non connected and non commutative Lie groups. As the Plancherel theorem, the solvability of the invariant differential operators on Lie groups. Also the study of the non commutative group algebra of the Lie groups and their ideals.

8 Author Contributions

8.1. Far a way from the representations theory, my contributions is: I have opened a new way in abstract harmonic analysis in order to do the Fourier analysis on Lie groups. The first one of my goals is the solvability and hypoellipticity of the invariant partial differential equations on many Lie groups as Nilpotent, Semi-simple, Lorentz group, Poincare group, ... ect. This leads us to study the non commutative group algebra and their ideals of the Lie groups. My discovery of new groups leads me to my second goal which is the Fourier analysis on non connected Lie groups. According to this goal I open other new way, which will be the business of the expertise in Mathematics and Mathematical Physics.

References

[1] E. Barletta, S. dragomir, On Lewy’s Unsolvability Phenomenon, in Complex variables and Elliptic Equations- January 2011, Publisher Francis & Taylor
[2] U. N. Bassey and M. E. Egwe, “Non Solvability of Heisenberg Laplacian by Factorization,” Journal of Mathematical Sciences, Vol. 21, No. 1, 2010, pp. 11-15.

[3] S. Berceanu, A. Gheorghe, Application of the Jacobi Group to Quantum, National Institute for Nuclear Physics and Engineering, P.O. Box MG-6, RO-077125 Bucharest-Magurele, Romania, 2008

[4] M. Bramanti, An Invitation to Hypoelliptic Operators and Hormander’s Vector Fields, Series: Springer Briefs in Mathematics, 2014.

[5] L. Corwin, L.P. Rothschild, Necessary Conditions for Local Solvability of Homogeneous Left Invariant Operators on Nilpotent Lie Groups, Acta Math., 147 (1981), pp. 265–288.

[6] L. Ehrenpreis, Solution of Some Problem Division, (I,II, III) Am. J. Math , vol 76,78,82.

[7] K. El- Hussein, Opérateurs Différentiels Invariants sur les Groupes de Deplacements, Bull. Sc. Math. 2e series 113,1989. p. 89-117.

[8] K. El- Hussein, Note on the Solvability of the Lewy Operator, International Mathematical Forum, 4, 2009, no. 26, 1301 - 1304.

[9] K. El- Hussein, Note on the Solvability of the Mizohata Operator, International Mathematical Forum, 5, 2010, no. 37, 1833 - 1838.

[10] K. El- Hussein, Abstract Harmonic Analysis on Poincare Space-time, Book, Verlag Publisher, June 2015.

[11] M. Frentz, Topics on Subelliptic Parabolic Equations Structured on Hormander Vector Field, Mathematics, Ume University, 2012, Doctoral Dissertation.

[12] L. Hormander, Differential Equations Without Solutions. Math. Ann., 140 :169–173, 1960.

[13] S. Helgason, Groups and Geometric Analysis, Academic Press, 1984.

[14] A. A. Kirillov, ed, Representation Theory and Noncommutative Harmonic Analysis I, Springer- Verlag, Berlin. (1994)
[15] Y. Kannai, An Unsolvable Hypoelliptic Differential Operator, Israel Journal of Mathematics. September 1971, Volume 9, Issue 3, pp 306-315.

[16] H. Lewy, An Example of a Smooth Linear Partial Differential Operator without Solution, Annals of Mathematics, Vol. 66, No. 2, 1957, pp. 155-158.

[17] N. Lerner, A Tribute to Lars Hormander, Matapli100, 2013.

[18] D. Müller and M. Peloso, Non-Solvability for a Class of Left-Invariant Second-Order Differential Operators on the Heisenberg Group, Transactions of the American Mathematical Society Volume 355, Number 5, Pages 2047-2064 S 0002-9947(02)03232-4 Article electronically published on December 18, 2002.

[19] B. Malgrange, Existence and Approximation des Solutions des équations aux Derivées Partielles et des équations de Convolutions, Ann. Inst. Fourier Grenoble, 6, 271, 1955.

[20] A. Mater, On Solvability of PDEs Studiorum Universita di Bologna, Anno Accademico 2011/2012.

[21] L. Nirenberg, F. Treves, Solvability of a first order partial Differential Equations, Communication on Pure and Applied Mathematics, VOL. XVI, 331-336, 1963.

[22] L. P. Rothschild, Local Solvability of Left Invariant Differential Operators on the Heisenberg Group, Proceedings of the American Mathematical Society, Vol. 74, No. 2, 1979, pp. 383-388.

[23] F. Treves, Linear Partial Differential Equations with Constant Coefficients, Garden and Breach, 1966.

[24] L. Venieri, Hypoelliptic Differential Operators in Heisenberg Group, Università di Bologna, Anno Accademico 2012/2013.