Marginal log-linear models and mediation analysis

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Abstract

We review some not well known results about marginal log-linear models, derive some new ones and show how they might be relevant in mediation analysis within logistic regression. In particular, we elaborate on the relation between interaction parameters defined within different marginal distributions and describe an algorithm for estimating the same interaction parameters within different marginals.

Keywords: Marginal Log-linear models, Direct effects, logistic regression

1. Introduction

Marginal log-linear models, Bergsma and Rudas (2002), were conceived to construct discrete multivariate distributions subject to restrictions imposed, simultaneously, on different marginals. Consider the simple context where \( X \) denotes a treatment, \( W \) one or more variables which might be affected by \( X \) and may influence the response \( Y \) which, for simplicity, we assume to be binary. In this context, we might be interested in the marginal distributions \( XW \) and \( XY \) in addition to the joint distribution \( XWY \).

1.1. Notations and preliminary results

A list of variables, say \((X, W, Y)\), shortened as \(XWY\) will be used to denote both a marginal distribution and the interaction among the variables in the list; let \( \mathcal{I}, \mathcal{M} \) denote two such lists with \( \mathcal{I} \subseteq \mathcal{M} \); \( \lambda_{\mathcal{I},\mathcal{M}} \) will denote the log-linear interactions \( \mathcal{I} \) defined within the marginal \( \mathcal{M} \), coded either as contrasts between adjacent categories (Ac) or with respect to a reference category (Rc) depending on the context; in both cases, variables in \( \mathcal{M}\setminus\mathcal{I} \) will be set to the initial or reference category coded as 0. When \( X \) and \( W \) quantitative, the linear logistic model including the \( XW \) interaction has the form

\[
\log \frac{P(Y = 1 \mid X = x, W = w)}{P(Y = 0 \mid X = x, W = w)} = \beta_0 + \beta_X x + \beta_W w + \beta_{XW} x w
\]

where the \( XY \) interaction is equal to \( \beta_X \) under Ac and to \( \beta_{Xx} \) under Rc.

To introduce the mixed parametrization, recall that in a general multi-way table with \( k \) cells, the saturated model may be parameterized as

\[
p = G\theta - 1_k \log[1_k \exp(G\theta)],
\]

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where $G$ is made of $k - 1$ linearly independent columns which do not span the unitary vector and $\theta$ is a vector of log-linear (canonical) parameters. Let $H$ be the left inverse of $G$ such that $H1_k = 0$, then (2) may be inverted as $\theta = H \log p$; note the one to one correspondence between the rows of $H$, the columns of $G$ and the log-linear parameters. Define the vector of mean parameters $\mu = G^p$; clearly there is a one to one correspondence between elements of $\mu$ and $\theta$. Let $G_I$ be the collection of columns of $G$ that correspond to the set of interactions in $I$, then the vector $\mu_I = G_I^p$ has the same size as $\theta_I$.

Given a partition of the collection of all possible interactions for the joint distribution into two disjoint sets $U$, $V$, the mixed parametrization, (Barndorff-Nielsen, 1978, pag. 121-22), is made of $(\mu_V, \theta_U)$ and has the following properties:

**Lemma 1.** (i) there is a one to one mapping between $p$ and $(\mu_V, \theta_U)$, (ii) the two components of the mixed parametrization are variation independent and (iii) the expected information matrix is block diagonal.

The following results on the differential properties of the mixed parametrization will be used later: let $I \subseteq M$ and $p_M$ denote the distribution within the marginal $M$; let $\Omega(p) = \text{diag}(p) - pp'$, then we have (see Forcina, 2012, Lemma 3, 4);

**Lemma 2.**

$$C_I = \frac{\partial \mu_I}{\partial \theta_I^T} = \frac{\partial \mu_I}{\partial \eta_{I,M}} = G_I^T \Omega(p) G_I;$$

in addition, $C_I$ is symmetric and positive definite if the elements of $p$ are strictly positive.

### 2. Main results

It is well known that the parameters in the marginal logistic models for $Y | W = w$, $Y | X = x$ and $W | X = x$ do not determine those in (1); the mixed parametrization allows to sharpen this result as follows:

**Proposition 1.** (i) The parameters of the three logistic regression models defined on the marginals $XW$, $XY$, $WY$ are variation independent from $\beta_{XW}$. (ii) If $\beta_{XW} = 0$, then the parameters of the three marginals determine uniquely the joint distribution.

**Proof:** the log linear parameters within the $XW$, $WY$, $XY$ marginals are uniquely determined by the set of mean parameters $\mu_X$, $\mu_W$, $\mu_Y$, $\mu_{XW}$, $\mu_{XY}$, $\mu_{WY}$ which are variation independent from $\theta_{XWY} = \beta_{XW}$. The above list of mean parameters together with $\theta_{XWY}$ constitute a mixed parametrization of the joint distribution, thus (ii) follows from Lemma 1. □

**Remark 1.** In principle, under (ii), the parameters in (1) could be written as functions of the mean parameters; the algorithm in Forcina (2012), A2, provides an efficient and accurate numerical alternative.
For the model in (1), Stanghellini and Doretti (2019) derived an expression for \( \delta_X = \beta_X - \beta^*_X \), where \( \beta^*_X \) is the regression coefficient of \( X \) in the linear logistic model defined within the marginal \( XY \) distribution. For the case of a multivariate discrete distribution on a set of binary random variables, an expression for the difference between the same interaction parameters defined within two different marginals, say \( \mathcal{N} \subset \mathcal{M} \), was derived by Evans (2015), Theorem 3.1. In the Appendix we rewrite the latter result in the case where \( X \) and \( W \) are discrete and show that, by setting \( \mathcal{N} = XYZ \) and \( \mathcal{M} = XWY \), they are essentially equivalent to those in Stanghellini and Doretti (2019).

The following provides some additional insights into the relation between interaction parameters defined within different marginals:

**Proposition 2.** Suppose that \( \theta_I \) has size \( d \), then
\[
\frac{\partial \lambda_{I,M}}{\partial \theta_I} = \frac{\partial \lambda_{I,M}}{\partial \mu_I} \frac{\partial \mu_I}{\partial \theta_I} = I_d. \tag{3}
\]

**Proof:** Follows from Lemma 2. \( \square \)

In the special case when \( d = 1 \), Proposition 2 simply says that \( \beta_X - \beta^*_X \) and \( \beta_X \) are variation independent which is somehow implied by the derivation in Stanghellini and Doretti (2019). Additional features of the result are clarified in the example below.

**Example 1.** Consider an \( XWY \) distribution where \( W, Y \) are binary and \( X \) has \( k \) categories; suppose we have two probability distributions \( p^1, p^2 \), with all log-linear parameters being equal, except for \( \theta^{1}_{XY} \neq \theta^{2}_{XY} \). Then the difference between corresponding pairs of marginal interactions \( \lambda^{1}_{XY,XY} - \lambda^{2}_{XY,XY} \) is equal to \( \theta^{1}_{XY} - \theta^{2}_{XY} \).

It is well known that we cannot impose log-linear restrictions on the \( XY \) interactions both in the \( XY \) marginal and in the joint distribution; for a formal argument see Bergsma and Rudas (2002). However, Colombi and Forcina (2014) proved a result that, within the Rc coding and assuming that \( W \) has \( m \) categories, may be stated as follows:

**Proposition 3.** Within \( XWY \), the marginal log-linear parametrization with elements
\[
(\lambda_{X,XY}, \lambda_{Y,XY}, \lambda_{XY,XY}, \lambda_{W,XW}, \lambda_{XW,XW}, \theta_{XY}, \theta_{WY}, \tilde{\theta}_{XWY})
\]
where \( \tilde{\theta}_{XWY} \) is obtained from \( \theta_{XWY} \) by deleting all elements with \( W = w \neq 0 \) is a smooth parametrization of the saturated model.

In words, if we want to define (and possibly constraint) the \( XY \) interactions both in the marginal and in the joint, we need to remove a subset of the \( XWY \) interactions corresponding to a fixed value of \( W \). This may be seen as an added flexibility in the modelling process: if we are interested in imposing constraints to the \( XY \) interaction both in the marginal \( XY \) and in the joint, the price to pay is that we cannot model a subset of the \( XWY \) interactions. The feature is illustrated in the next section.
3. Application

3.1. The data

The data come from the NCDS, a UK cohort study that included everybody born in
UK from March 3rd to March 9th 1958. several variables concerning the parents and the
child are recorded; a full description of the data set is available at [http://cls.ucl.ac.uk/cls-studies/1958].
In this simplified analysis, we consider the number of years of schooling for each parent,
parents’ concern about the education of the child shown at different stages (as recorded
by the teachers), the weekly income of parents and the academic qualification reached
by the child, an ordered categorical variable with four categories. The issue of interest is
the effect of parents’ education on that of the child. Intuitively, parents’ education might
affect income by which to offer better chances to the child. In addition, more educated
parents might show more concern being more aware of the importance of education. Di-
rect effects may work through the atmosphere inside the family, like having books and
meeting more educated friends.

For simplicity, the analysis below is restricted to the sample of 2161 daughters, the
response \( Y = 1 \) if the child got at least an high school degree; income and concern are
dichotomized at the median. The exposure \( X \) is a categorical variable with four levels
obtained by splitting at quantiles the following measure of parent’s education

\[
\tilde{X} = E_m + E_f - | E_m - E_f | / 3,
\]

where \( E_m, E_f \) denote the number of years of schooling for mother and father and \( | E_m - E_f | / 3 \) is a penalty for unequally educated parents. We also assume there are two
mediators: \( U \), the father weekly income (that of the mother was ignored, having a large
number of missing values) and \( V \), an average measure of the concern shown by parent at
different stages, as recorded by teachers. Finally define \( W = (U, V) \).

3.2. Two alternative models

We compare two alternative models, both parameterized with the adjacent coding;
because all variables except \( X \) are binary, assuming that, say, the \( Y X \) adjacent interactions
are constant in \( X \) is equivalent to assume that the logits of \( Y \mid X = x \) is a linear
functions of \( x \). However, because the evidence against linearity in \( X \) was rather strong,
the dependence on \( X \) was left unconstrained.

M1: Define the overall effect of \( X \) on \( Y \) in the corresponding marginal distribution, in
addition, model the effect of \( X \) on the mediators in the marginal \( XUV \). Define
all other interactions within the joint \( XUVY \), including the \( XY \) interactions; the
parameters already in the model determine the \( XUVY \) interactions which cannot be
modeled. Then we constrain to 0 the \( XUV \) interactions in \( XUV \) and the \( YUV \) and
\( XVY \) interactions within \( XUVY \); this model fits well with a deviance of 7.82 and 7
dof. Parameter estimates and standard errors for interaction parameters involving
the \( XY \) term are given in Table 1.
Table 1: Estimates of interactions containing the $XY$ in the M1 and M2 models.

|                  | Estimates under M1 |                  |                  |                  |
|------------------|--------------------|------------------|------------------|------------------|
|                  | $\lambda_{XY,XY}$ | $\theta_{XY}$    | $\theta_{XYU}$  | $\theta_{XYV}$  |
| $X$              | Est.   | s.e.  | Est.   | s.e.  | Est.   | s.e.  |
| $0 \rightarrow 1$ | -0.0066 | 0.1016 | -0.1948 | 0.2311 | 0.6587 | 0.1672 |
| $1 \rightarrow 2$ | 0.5990  | 0.3194 | 0.7854  | 0.1230 | -1.5142 | 0.2817 |
| $2 \rightarrow 3$ | 1.2045  | 0.1339 | 0.8004  | 0.1827 | 0.4604  | 0.1215 |

|                  | Estimates under M2 |                  |                  |                  |
|------------------|--------------------|------------------|------------------|------------------|
|                  | $\theta_{XY}$     | $\theta_{XYV}$  | $\theta_{XYU}$  |                  |
| $X$              | Est.   | s.e.  | Est.   | s.e.  | Est.   | s.e.  |
| $0 \rightarrow 1$ | -0.0441 | 0.1035 | -0.3048 | 0.1729 | 0.3282 | 0.2287 |
| $1 \rightarrow 2$ | 0.6216  | 0.2649 | 0.3186  | 0.2581 | -0.6978 | 0.1236 |
| $2 \rightarrow 3$ | 0.7761  | 0.1346 | 0.0240  | 0.1182 | 0.0964  | 0.1807 |

M2: Define the effects of $X$ on $U, V$ within the $XUV$ marginal as above and all other effects within the joint $XUVY$; next, constrain to 0 the $XUV$ interactions in the $XUV$ marginal as above and the $UVY$ and $XUVY$ interactions in the joint. This model, which is the closest analog to the one considered above, has a deviance of 13.04 with the same number of dof. Estimates and standard errors for the dependence of $Y$ on $X$ are displayed in Table 1.

Table 2: Model M1: dependence of income, $U$, and concern, $V$, on parents’ education.

|                  | $XU$           | $XV$           |
|------------------|----------------|----------------|
| $X$              | Est.   | s.e.  | Est.   | s.e.  |
| $0 \rightarrow 1$ | 0.4470 | 0.3175 | -0.0718 | 0.1591 |
| $1 \rightarrow 2$ | 0.2616 | 0.1298 | 0.4766  | 0.1478 |
| $2 \rightarrow 3$ | 0.9327 | 0.1762 | 1.2654  | 0.0928 |

The effect of $X$ on $U, V$ is strongest in going from 2 to 3; the same holds for the marginal effect of $X$ on $Y$. Within M2 the effects of $X$ conditional on $U = V = 0$ and $U = 1, V = 0$ are roughly similar the the corresponding ones under M1.

If we assume that there are no unobserved confounders, the estimated joint distribution under M1 allows to compute an estimate of the natural direct and indirect effect of parents’ education on academic qualification of the daughter, by changing $X$ from one category to the next (see VanderWeele et al., 2013, equations (1) and (2)). Results are in Table 3 with standard errors estimated by bootstrap; the direct effect is always the largest component of the total though going from 0 to 1 does not seem to matter.

Appendix

Rephrasing Robin Evans result

Let $\mathcal{N} \subset \mathcal{M}$ be two nested marginals and $\mathcal{R} = \mathcal{M} \setminus \mathcal{N}$; assume that we define interactions as contrasts relative to the reference category coded as 0; we also use the convention
Table 3: Natural direct and indirect effects, when changing parents’ education from one category to the next, on that of their daughters

|       | 0 → 1 |       | 1 → 2 |       | 2 → 3 |
|-------|-------|-------|-------|-------|-------|
| Est   | -0.0107 | s.e.  | 0.0196 | Est   | 0.0609 | s.e.  | 0.0255 | Est   | 0.1549 | s.e.  | 0.0298 |
| Ind.  | 0.0083  | s.e.  | 0.0069 | 0.0298 | 0.0100 | s.e.  | 0.0179 |
| Total | -0.0024 | s.e.  | 0.0210 | 0.0907 | 0.0257 | s.e.  | 0.2641 | 0.0299 |

that, when the value of the conditioning variables are not given, they are fixed to the reference value; the derivation below is, essentially, a re-writing of Evans (2015). Let $\lambda_{I,K}(x_I)$ denote the log-linear interaction among variables in $I$ computed within the marginal $K$ fixed at the value $x_I$.

**Lemma 3.**

$$\lambda_{I,M}(x_I) - \lambda_{I,N}(x_I) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \log p_{R|N}(0_R, x_J; 0_{N \setminus J}),$$  \hspace{1cm} (4)

where the conditional probabilities on the right-hand side are of the event $x_R = 0_R$ when the conditioning set is split into a component taking the original values and the remaining ones fixed to 0.

**Proof:** Start from the expansion of $\lambda_{I,M}(x_I)$, add and subtract $\lambda_{I,N}(x_I)$ and write the difference between the two in terms of conditional probabilities

$$\lambda_{I,M}(x_I) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \log p_{M}(x_J, 0_{M \setminus J})$$

$$= \sum_{J \subseteq I} (-1)^{|I \setminus J|} \log \frac{p_{M}(x_J, 0_{N \setminus J}, 0_R)}{p_{N}(x_J, 0_{N \setminus J})} + \lambda_{I,N}(x_I).$$

□

We now apply Lemma 3 to the special case where $M = XWY$, $N = XY$, $Y$ is binary and $X$, $W$ are discrete; to simplify notations, let $p_W(x, y) = P(W = 0 | X = x, Y = y)$; in addition, because $XWY$ is the joint distribution, replace $\lambda_{I,M}$ with $\theta_I$.

**Corollary 1.**

$$\lambda_{XY}(x, y) - \lambda_{XY;XY}(x, y) = \log \frac{p_W(0, 0) p_W(x, y)}{p_W(0, y) p_W(x, 0)}$$  \hspace{1cm} (5)

this may also be expressed in terms of log-linear parameters defined within $XWY$ as

$$\lambda_{XY}(x, y) - \lambda_{XY;XY}(x, y) = -\log \frac{1 + \sum_{w>0} \exp[\lambda_W(w)]}{1 + \sum_{w>0} \exp[\lambda_W(w) + \lambda_{WX}(w, x)]}$$

$$+ \log \frac{1 + \sum_{w>0} \exp[\lambda_W(w) + \lambda_{WY}(w, y)]}{1 + \sum_{w>0} \exp[\lambda_W(w) + \lambda_{WX}(w, x) + \lambda_{WY}(w, y) + \lambda_{WXY}(w, x, y)]]$$
Proof. The first part follows from Lemma 3 by noting that, because $I$ has just two elements, the expansion contains four elements which can be arranged into the form of a log odds ratio. For the second part, first write the conditional distribution of $W \mid X, Y$ as a multinomial and then apply (3) in Stanghellini and Doretti (2019) for expanding interactions conditional to $X, Y$ into a sum of higher order interactions.

Log-linear versus logistic parameterizations

For what follows, it might be useful to recall how, under the corner point coding, log-linear parameters may be mapped into the corresponding logistic parameters. When the dependent variable, like $Y$, is binary, we have

$$\log \frac{P(Y = 1 \mid X = x, W = w)}{P(Y = 0 \mid X = x, W = w)} = \lambda_Y + \lambda_{XY}(x) + \lambda_{WY}(w) + \lambda_{XWY}(x, w),$$

with the convention that the log-linear parameter is 0 whenever at least one of the arguments is 0. Having assumed that $W$ is multinomial with, possibly, more than two categories, its logits may be written as

$$\log \frac{P(W = w \mid X = x, Y = y)}{P(W = 0 \mid X = x, Y = y)} = \lambda_w(w) + \lambda_{Wx}(w, x) + \lambda_{WY}(w, y) + \lambda_{XWY}(x, w, y).$$

The results of Stanghellini and Doretti

As above, let $Y$ be binary and $X, W$ be discrete; equation (A2) in Stanghellini and Doretti (2019) may be written as

$$\log \frac{P(W = w \mid Y = 1, X = x)}{P(W = w \mid Y = 0, X = x)} = \log \frac{P(Y = 1 \mid X = x, W = w)}{P(Y = 0 \mid X = x, W = w)} - \log \frac{P(Y = 1 \mid X = x)}{P(Y = 0 \mid X = x)}$$

which follows by expanding the left-hand side as

$$\log \frac{P(W = w \mid Y = 1, X = x)}{P(W = w \mid Y = 0, X = x)} = \log \frac{P(W = w, Y = 1, X = x)}{P(W = w, Y = 0, X = x)} - \log \frac{P(Y = 1, X = x)}{P(Y = 0, X = x)}$$

and noting that logits may be computed equivalently either on the joint or conditional distribution.

To derive an extension of their (A3) to non binary $W$, first swap conditioning

$$\log \frac{P(W = w \mid X = x, Y = y)}{P(W = 0 \mid X = x, Y = y)} = \log \frac{P(Y = y \mid X = x, W = w)}{P(Y = y \mid X = x, W = 0)} + \log \frac{P(W = w \mid X = x)}{P(W = 0 \mid X = x)}$$

next expand the first term on the right hand-side by adding and subtracting $\log P(Y = 0 \mid X = x, W = w)$ and $\log P(Y = 0 \mid X = x, W = 0)$,

$$\log \frac{P(Y = y \mid X = x, W = w)}{P(Y = y \mid X = x, W = 0)} = \log \frac{P(Y = 0 \mid X = x, W = w)}{P(Y = 0 \mid X = x, W = 0)} + y \left[ \log \frac{P(Y = 1 \mid X = x, W = w)}{P(Y = 0 \mid X = x, W = w)} - \log \frac{P(Y = 1 \mid X = x, W = 0)}{P(Y = 0 \mid X = x, W = 0)} \right].$$
Thus the analog of the log-linear expansion in their (A3) is

\[
\log \frac{P(W = w \mid X = x, Y = y)}{P(W = 0 \mid X = x, Y = y)} = +y [\lambda_{WY}(w) + \lambda_{XWY}(x, w)] \\
+ \log \frac{1 + \exp(\lambda_Y + \lambda_{XY}(x))}{1 + \exp(\lambda_Y + \lambda_{XY}(x) + \lambda_{WY}(w) + \lambda_{XWY}(x, w))} + \lambda_W(w) + \lambda_{WX}(w, x).
\]

which is a equivalent to (1) in the special case when \( W, Y \) are both binary variables.

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