TRAVELING WAVES FOR A DIFFUSIVE SEIR EPIDEMIC MODEL

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In Memory of My Parents

Abstract. In this paper, we propose a diffusive SEIR epidemic model with saturating incidence rate. We first study the well posedness of the model, and give the explicit formula of the basic reproduction number $R_0$. And hence, we show that if $R_0 > 1$, then there exists a positive constant $c^* > 0$ such that for each $c > c^*$, the model admits a nontrivial traveling wave solution, and if $R_0 \leq 1$ and $c \geq 0$ (or, $R_0 > 1$ and $c \in [0, c^*)$), then the model has no nontrivial traveling wave solutions. Consequently, we confirm that the constant $c^*$ is indeed the minimal wave speed. The proof of the main results is mainly based on Schauder fixed theorem and Laplace transform.

1. Introduction. In order to describe the transmission of communicable diseases, Kermack and McKendrick [10] proposed the classic deterministic Susceptible-Infected-Removed (SIR) model

$$
\frac{d}{dt}S(t) = -\beta S(t)I(t),
$$
$$
\frac{d}{dt}I(t) = \beta S(t)I(t) - \gamma I(t),
$$
$$
\frac{d}{dt}R(t) = \gamma I(t),
$$

where $S(t), I(t)$ and $R(t)$ denote the number of the susceptible, infected and removed individuals, respectively. The constant $\beta$ is the transmission coefficient, and $\gamma$ is the recovery rate. Given $S(0) = S^0 > 0$, $I(0) > 0$ and $R(0) = 0$. It is well known that, by the so-called basic reproduction number $R_0 := \frac{\beta S^0}{\gamma}$, Kermack and McKendrick[10] obtained full information about the transmission dynamics and epidemic potential: if $R_0 > 1$, then $I(t)$ first increases to its maximum and then decreases to zero and hence an epidemic occurs; if $R_0 < 1$, then $I(t)$ decreases to zero and epidemic does not happen.
When individuals move randomly, a modified version of model (1.1) will be studied, which takes into the following diffusive SIR model

\[
\begin{align*}
\frac{\partial}{\partial t} S(t, x) &= d_1 \Delta S(t, x) - \beta S(t, x) I(x, t), \\
\frac{\partial}{\partial t} I(t, x) &= d_2 \Delta I(t, x) + \beta S(t, x) I(x, t) - \gamma I(t, x).
\end{align*}
\]

This model was first considered by Hosono and Ilyas [8], where it is showed that if \( S(0, x) = S^0 > 0 \) and \( \beta S^0 > \gamma \), then for each \( c \geq c^* := 2 \sqrt{d_2 (\beta S^0 - \gamma)} \) there exists a positive constant \( \varepsilon < S_0 \) such that model (1.2) has a traveling wave solution \((S(x + ct), I(x + ct))\) satisfying \( S(-\infty) = S^0, \ S(+\infty) = \varepsilon, \ I(\pm \infty) = 0 \). On the other hand, there is no traveling wave solution for (1.2) when \( \beta S^0 \leq \gamma \).

In recent years, many researchers have attention to study the existence and non-existence of traveling wave solutions for some diffusive epidemic models (see, e.g. [1, 12, 20, 22, 23, 24, 25, 30, 33]). In particular, Wang and Wu [23] investigated the existence and non-existence of traveling wave solutions for a diffusive Kermack-McKendrick epidemic model with non-local delayed transmission. In [22], Wang, Wang and Wu studied traveling waves of reaction-diffusion equations for a diffusive SIR model. In a very recent paper, for a three-dimensional reaction-diffusion systems, Zhang and Wang [33] established the existence of traveling wave solutions for influenza model with treatment.

In present paper, we will formulate a simple diffusive epidemic model incorporating exposed individuals into model (1.1). In the model considered, the total individuals at time \( t \geq 0 \) and position \( x \in \mathbb{R} \), denoted by a constant \( N \), is subdivided into four mutually exclusive compartments of susceptible \( S(x, t) \), exposed \( E(x, t) \), infectious \( I(x, t) \) and recovered \( R(x, t) \) individuals, so that \( S(x, t) + E(x, t) + I(x, t) + R(x, t) = N \). Here the model is called an SEIR model [2, 17] where individuals are susceptible, then exposed (i.e., in the latent period), then infectious, then recovered from infectious individuals. In addition, we also assume that both the exposed and infected individuals are infected, they are distinguished by the absence/presence of their ability to infect other population [6, 7]. On the other hand, it is believed that, in the modeling of infectious diseases, the incidence function plays a very important role, it can determine the rise and fall of epidemics [3, 6, 7]. Here we will use the saturating incidence rate denoted as \( \frac{\beta SI}{1 + aI} \) \((a > 0)\) [3] other than the bilinear incidence rate \( \beta SI \). As reported in the literature, the saturating incidence rate is more realistic than the bilinear rate and can cause some interesting dynamic behaviors of infectious diseases, such as limit cycle, heteroclinic orbit, saddle-node bifurcation, transcritical bifurcation and Hopf bifurcation, see, [7, 14, 18, 28, 29, 31], for example.

In view of the above discussion, the proposed model is given by the following diffusive Susceptible-Exposed-Infected-Removed (SEIR) model with a saturating incidence rate

\[
\begin{align*}
\frac{\partial}{\partial t} S(t, x) &= d_1 \Delta S(t, x) - \beta S(t, x) g(I(t, x)), \\
\frac{\partial}{\partial t} E(t, x) &= d_2 \Delta E(t, x) + \beta S(t, x) g(I(t, x)) - \alpha E(t, x), \\
\frac{\partial}{\partial t} I(t, x) &= d_3 \Delta I(t, x) + \alpha E(x, t) - \gamma I(t, x), \\
\frac{\partial}{\partial t} R(t, x) &= d_4 \Delta R(t, x) + \gamma I(t, x),
\end{align*}
\]
in which $g(I) = \frac{I}{1 + au}$ ($a > 0$), $\Delta = \frac{\partial^2}{\partial x^2}$, $d_i > 0$ ($i = 1, 2, 3, 4$) are the diffusion rates. The constant $\alpha > 0$ is the rate constant for exposed individuals becoming infectious, $\beta > 0$ is the transmission coefficient, $\gamma > 0$ is the rate constant that the infective individuals become recovered. In the sequel, we always assume that the initial free equilibrium is $(S^0, 0, 0, 0)$ with $S^0 > 0$. In contrast, for the detailed epidemiological description of the corresponding ODE model, we refer to [2, 6, 7, 17].

The purpose of the current paper is to study the existence and non-existence of traveling wave solutions of (2.1). We will show that there exists a positive constant $c^* > 0$ such that (2.1) has a traveling wave solution if $c > c^*$ and $\beta S^0 > \gamma$. To prove the existence theorem (see Theorem 3.1), we will employ Schauder fixed point theorem [9, 11, 15] to the non-monotone operator used in a suitable invariant convex set. In order to construct the appropriate invariant convex set, we also use the idea of the iteration process [20, 22, 23, 24, 25, 30] to construct the upper-lower solutions. One important feature of our method, which is different from the ones, is that we need to construct the vector-value upper-lower solutions for (2.3) (see Section 2.3) since system (2.3) consists of three equations. The ideas of such a construction is motivated by Weng and Zhao [26], see also [5, 19] for the related works. Further, we establish that (2.1) has no traveling wave for any $c \geq 0$ and $\beta S_0 < \gamma$ (see, Theorem 3.2). For $c \in (0, c^*)$ and $\beta S^0 > \gamma$, we conclude the non-existence of traveling wave solutions for (2.1) (see, Theorem 3.3). Here the critical method of the proof of the non-existence is based on the two-side Laplace transform. As we know that the application of the Laplace transform requires the prior estimate of the exponential decay of the traveling wave solutions [4, 20, 22, 23, 30]. However, it seems that the analytical method in [4, 20, 22, 23, 30] cannot give the prior estimate due to the decay of the traveling wave solutions [4, 20, 22, 23, 30]. Instead, we approve the approach recently introduced by [33] to get the prior estimate.

This paper is organized as follows. In Section 2, we first consider the well posedness of system (2.1) in $[0, +\infty)$ and then derive basic properties of the waves. Sections 3 is devoted to the study of the existence and non-existence of traveling waves for system (2.1). Finally, a brief discussion section completes the paper.

2. Preliminaries. In this section, we should give some preliminary results such as the basic reproduction ratio and the well posedness of system (2.1), the eigenvalue problems for the wave profile (2.3), constructing a pair of upper and lower solutions for system (2.3) and verifying the conditions of the Schauder fixed point theorem.

2.1. The basic reproduction ratio and the well-posedness. Because $R(t, x)$ does not appear in the first three equations of (1.3), and for the simplicity of notation, let $(S, E, I) = (u_1, u_2, u_3)$, we consider the following new system

$$
\begin{align*}
\frac{\partial}{\partial t} u_1(t, x) &= d_1 \Delta u_1(t, x) - \beta u_1(t, x) g(u_3(t, x)), \\
\frac{\partial}{\partial t} u_2(t, x) &= d_2 \Delta u_2(t, x) + \beta u_1(t, x) g(u_3(t, x)) - \alpha u_2(t, x), \\
\frac{\partial}{\partial t} u_3(t, x) &= d_3 \Delta u_3(t, x) + \alpha u_2(t, x) - \gamma u_3(t, x),
\end{align*}
$$

(2.1)

where $g(u) = \frac{u}{1 + au}$. Accompanied with (2.1), we consider the initial value conditions

$$
u_i(0, x) = \varphi_i(x) \geq 0, \quad x \in \mathbb{R}, \quad i = 1, 2, 3,$$

but not identically zero. (2.2)

In this subsection, we focus on the existence, uniqueness, invariance of solutions for the Cauchy problem to system (2.1) in $\mathbb{R}$ (see, Theorem 2.1). Firstly, we define
the basic reproduction ratio $R_0$ for system (2.1). By similar arguments to those in [21, Theorem 2.3], we can show that the basic reproduction ratio $R_0$ equals the spectral radius of the following $2 \times 2$ matrix

$$ B := \begin{pmatrix} 0 & \frac{\beta S^0}{\gamma} \\ 1 & 0 \end{pmatrix}. $$

Hence, $R_0 = \frac{3S^0}{\gamma}$. For the definition of the basic reproduction ratio $R_0$ for the reaction-diffusion models and its biological interpretation, we refer the readers to [21] for details. In the following, we always assume that $R_0 > 1$.

Throughout this paper, the cone $\mathbb{R}^+_1$ denotes the subset of $\mathbb{R}^3$ with vectors $x \geq 0$. Let $\mathbb{X} = C(\mathbb{R}, \mathbb{R}^3)$ be Banach space with the supremum norm $\| \cdot \|_{\mathbb{X}}$. Define $\mathbb{X}^+ = C(\mathbb{R}, \mathbb{R}^+_1)$, then $(\mathbb{X}, \mathbb{X}^+)$ is a strongly ordered space. Moreover, we shall use the standard partial order in $\mathbb{R}^3$ or $\mathbb{X}$. Let

$$ \mathbb{X}_M = \{ \varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{X}^+: 0 \leq \varphi(x) \leq M, \forall x \in \mathbb{R} \}, $$

in which $0 := (0, 0, 0)$ and

$$ M := \left( S^0, \frac{\gamma}{a\alpha} \left( \frac{\beta S^0}{\gamma} - 1 \right), \frac{1}{a} \left( \frac{\beta S^0}{\gamma} - 1 \right) \right). $$

For any $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{X}_M$ and $x \in \mathbb{R}$, we define $F = (F_1, F_2, F_3): \mathbb{X}_M \rightarrow \mathbb{X}$ by

$$ F_1(\varphi)(x) = -\beta \varphi_1(x) g(\varphi_3(x)), $$

$$ F_2(\varphi)(x) = \beta \varphi_1(x) g(\varphi_3(x)) - \alpha \varphi_2(x), $$

$$ F_3(\varphi)(x) = \alpha \varphi_2(x) - \gamma \varphi_3(x). $$

Then $F$ is locally Lipschitz continuous in any bounded subset of $\mathbb{X}_M$. We now reformulate (2.1)-(2.2) as the following abstract differential equation

$$ \frac{du}{dt} = Au + F(u), \quad t > 0, $$

$$ u_0 = \varphi \in \mathbb{X}_M, $$

where $u = (u_1, u_2, u_3)$, $A := (d_1 \Delta u_1, d_2 \Delta u_2, d_3 \Delta u_3)^T$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3)$.

**Theorem 2.1.** For any given initial data $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{X}_M$, system (2.1)-(2.2) admits a unique nonnegative solution $u(t, x, \varphi)$ which exists globally on $[0, +\infty)$ such that $u(0, \cdot, \varphi) = \varphi$ and $u(t, \cdot, \varphi) \in \mathbb{X}_M$ for all $t \geq 0$.

**Proof.** For any $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{X}_M$ and any sufficient small $k \geq 0$, we get

$$ \varphi(0, x) + kF(\varphi)(x) = \begin{pmatrix} \varphi_1(x) \left( 1 - k \beta g(\varphi_3(x)) \right) \\ (1 - k\alpha)\varphi_2(x) + k\beta \varphi_1(x) g(\varphi_3(x)) \\ (1 - k\gamma)\varphi_3(x) + k\alpha \varphi_2(x) \end{pmatrix} \geq \begin{pmatrix} (1 - \frac{k\beta}{a})\varphi_1(x) \\ (1 - k\alpha)\varphi_2(x) \\ (1 - k\gamma)\varphi_3(x) \end{pmatrix} \geq 0. $$

Note that $g(x)$ is increasing for $x \geq 0$. Then

$$ 0 \leq g(\varphi_3(x)) \leq \frac{\gamma}{a\beta S^0} \left( \frac{\beta S^0}{\gamma} - 1 \right). $$
Consequently,

$$\varphi(0, x) + kF(\varphi)(x) \leq \begin{pmatrix} S_0 \\ \frac{\gamma}{\alpha} \left( \frac{\beta S^0}{\gamma} - 1 \right) \\ \frac{1}{a} (\frac{\beta S^0}{\gamma} - 1) \end{pmatrix} = M.$$ 

Thus, it follows \( \varphi(x) + kF(\varphi)(x) \in M. \) This in turn implies that

$$\lim_{k \to 0^+} \frac{1}{k} \text{dist}(\varphi(x) + kF(\varphi)(x), M) = 0, \quad \forall \varphi \in M.$$ 

It then follows from [16, Corollary 4] (taking the delay as zero) that (2.1)-(2.2) admit a unique nonnegative mild solution \( u(t, x, \varphi) \) on \([0, \infty) \times \mathbb{R} \) with \( u(0, \cdot, \varphi) = \varphi, \) and \( u(t, \cdot, \varphi) \in \mathcal{X}_M \) for \( t \geq 0. \) Moreover, by [27, Corollary 2.2.5], the mild solution is classic on \([0, \infty). \]

In view of Theorem 2.1, the dynamics of system (2.1) can be analyzed in the bounded feasible region \( \mathcal{X}_M. \) Furthermore, the region \( \mathcal{X}_M \) is positively invariant with respect to model (2.1) and the model is well posedness.

### 2.2. Eigenvalue problem.

This subsection deals with the eigenvalue problems for the wave profile, which is obtained by substituting \( u_i(t, x) = U_i(x + ct) \) \( (i = 1, 2, 3) \) in (2.1). Here \( (U_1, U_2, U_3) \) is called the wave profile, \( \xi := x + ct \) the wave coordinate and \( c \) the speed. For the sake of convenience, we still use \( u_i \) instead of \( U_i, \xi. \) And for technical reasons, we let \( d_2 = d_3 := d, \) and then consider the following wave profile equation

\[
\begin{align*}
\frac{d}{dt} u_1''(t) - cu_1'(t) - \beta u_1(t)g(u_3(t)) &= 0, \\
\frac{d}{dt} u_2''(t) - cu_2'(t) - \alpha u_2(t) + \beta u_1(t)g(u_3(t)) &= 0, \\
\frac{d}{dt} u_3''(t) - cu_3'(t) - \gamma u_3(t) + \alpha u_2(t) &= 0.
\end{align*}
\]

We now consider the eigenvalue problem at \((S^0, 0, 0).\) Linearizing of the last two equations of (2.3) at \((S^0, 0, 0)\) implies that

\[
\begin{align*}
\frac{d}{dt} u_2''(t) - cu_2'(t) - \alpha u_2(t) + \beta S^0 u_3(t) &= 0, \\
\frac{d}{dt} u_3''(t) - cu_3'(t) - \gamma u_3(t) + \alpha u_2(t) &= 0.
\end{align*}
\]

Plugging \( u_i(t) = v_i e^{\lambda t}, \ i = 2, 3, \) into the above equations, we get the following eigenvalue problem

\[
\det A(\lambda) = 0, \quad A(\lambda) \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = 0,
\]

where

\[
A(\lambda) = \begin{pmatrix} h_1(\lambda) & \beta S^0 \\ \alpha & h_2(\lambda) \end{pmatrix}
\]

with \( h_1(\lambda) = d\lambda^2 - c\lambda - \alpha \) and \( h_2(\lambda) = d\lambda^2 - c\lambda - \gamma. \)

Clearly, the fact \( R_0 > 1 \) implies \( \det A(0) = \alpha \gamma (1 - R_0^2) < 0. \) Then the equation \( \det A(\lambda) = 0 \) has at least one positive root. When \( R_0 > 1, \) we set

\[
e^* := \sqrt{2d \left( (\alpha + \gamma)^2 + 4\alpha \gamma (R_0^2 - 1) - (\alpha + \gamma) \right)}.
\]
About the distribution of eigenvalue and their eigenvectors, we have the following results.

**Lemma 2.1.** Assume that \( R_0 > 1 \). Then we have the following statements.

1. If \( c > c^* \), then the equation \( \det A(\lambda) = 0 \) has three positive roots \( 0 < \lambda_1 < \lambda_2 < \lambda_3 \), and a negative root \( \lambda_4 \). Moreover, for any \( \varepsilon \in (0, \frac{c}{2c^*} - \frac{1}{\alpha}) \),
   \[
   h_1(\lambda_1 + \varepsilon) < h_2(\lambda_1) < 0, \quad h_2(\lambda_1 + \varepsilon) < h_2(\lambda_2) < 0.
   \]
   And, the eigenvector \((v^{(1)}_2, v^{(1)}_3)\) corresponding to the smallest eigenvalue \( \lambda_1 \) is strongly positive, i.e., \((v^{(1)}_2, v^{(1)}_3) \gg 0\).

2. If \( c = c^* \), then the equation \( \det A(\lambda) = 0 \) has two different positive roots, and a negative root.

3. If \( 0 < c < c^* \), then the equation \( \det A(\lambda) = 0 \) has a positive root, a negative root, and a pair of complex roots with positive real parts.

**Proof.** Let \( \rho = d\lambda^2 - c\lambda \). Then
\[
\det A(\lambda) = \rho^2 - (\alpha + \gamma)\rho - \alpha\gamma(R_0^2 - 1) =: H(\rho).
\]
Consequently, it follows from the fact \( R_0 > 1 \) that the equation \( H(\rho) = 0 \) has a positive root \( \rho_+ \) and a negative root \( \rho_- \), where
\[
\rho_+ = \alpha + \gamma + \frac{(c^*)^2}{4d}, \quad \rho_- = -\frac{(c^*)^2}{4d}.
\]
Clearly, the equation \( d\lambda^2 - c\lambda - \rho_+ = 0 \) has a positive root \( \lambda_3 \) and a negative root \( \lambda_4 \), here
\[
\lambda_3 = \frac{c + \sqrt{c^2 + (c^*)^2 + 4d(\alpha + \gamma)}}{2d} > 0, \quad \lambda_4 = \frac{c - \sqrt{c^2 + (c^*)^2 + 4d(\alpha + \gamma)}}{2d} < 0.
\]
If \( c > c^* \), then the roots of the equation \( d\lambda^2 - c\lambda - \rho_- = 0 \) are real and positive, i.e.,
\[
\lambda_1 = \frac{c - \sqrt{c^2 - (c^*)^2}}{2d} > 0, \quad \lambda_2 = \frac{c + \sqrt{c^2 - (c^*)^2}}{2d} > 0.
\]
Obviously, \( 0 < \lambda_1 < \lambda_2 < \lambda_3 \) and \( 0 < \lambda_1 < \frac{c^*}{2c^*} \). Note that \( h_1(\lambda) \) is decreasing in \((0, \frac{c^*}{2c^*})\), and \( h_1(\frac{c^*}{2c^*}) = \frac{c^*}{2c^*} - \frac{1}{\alpha} < 0 \). Therefore, \( h_1(\lambda_1 + \varepsilon) < h_1(\lambda_1) < 0 \) for any \( \varepsilon \in (0, \frac{c}{2c^*} - \frac{1}{\alpha}) \). By the similar discussions, we also get \( h_2(\lambda_1 + \varepsilon) < h_2(\lambda_1) < 0 \).

On the other hand, in view of the fact that \( h_1(\lambda_1) < 0 \) and \( h_2(\lambda_1) < 0 \), then there exists a pair of \((v^{(1)}_2, v^{(1)}_3) \gg 0\) such that \( A(\lambda_1)(v^{(1)}_2, v^{(1)}_3)^T = 0 \). This shows that the conclusion (1) holds.

The proofs of the conclusions (2) and (3) are easy and omitted the details. \( \square \)

### 2.3. Construction of the upper and lower solutions

To establish existence of traveling wave solutions of (2.1), we will construct a convex invariant set. For this, we will use the iteration process to construct a pair of vector-value upper-lower solutions for (2.3). Namely, we first choose the upper solution \( \varpi_1(t) = S^0 \) for the \( u_1 \)-component (Lemma 2.2), which is then used to build the upper solutions \( \varpi_2(t) \) and \( \varpi_3(t) \) for the \( u_2 \) and \( u_3 \)-components (Lemma 2.3). The upper solutions \( \varpi_4(t) \) is in turn employed to produce the lower solutions \( \underline{u}_1(t) \) and \( \underline{u}_3(t) \) for the \( u_1 \)-component (Lemma 2.4). Finally, the lower solutions \( \underline{u}_1(t) \) and \( \underline{u}_3(t) \) are further built to construct the lower solution \( \underline{u}_2(t) \) for the \( u_3 \)-components (Lemma 2.5). Here we also would like to stress that, different from that in [23], the pair of upper-lower solutions constructed in this paper are vector-value type and bounded on \( \mathbb{R} \).
In the following, we always assume that $\mathcal{R}_0 > 1$, i.e., $\beta S^0 > \gamma$, and $c > c^*$. Let $\lambda_1$ be the smallest eigenvalue defined as in Lemma 2.1(1) and $(v_2^{(1)}, v_3^{(1)}) \gg 0$ its associating eigenvector. For practical reasons, it will be convenient to choose the vector $(v_2^{(1)}, v_3^{(1)}) = (v_2^{(1)}, 1) \gg 0$, where $v_2^{(1)}$ satisfies

$$h_1(\lambda_1)v_2^{(1)} + \beta S^0 = 0 \quad \text{or} \quad \alpha v_2^{(1)} + h_2(\lambda_1) = 0,$$

for the calculations in the proofs of Lemmas 2.3 and 2.5.

For $t \in \mathbb{R}$, we define six continuous functions as follows

$$\varpi_1(t) = S^0,$$

$$\varpi_2(t) = \min \left\{ v_2^{(1)} e^{\lambda_1 t}, \frac{\gamma}{a\alpha} \left( \frac{\beta S^0}{\gamma} - 1 \right) \right\},$$

$$\varpi_3(t) = \min \left\{ e^{\lambda_1 t}, \frac{1}{a} \left( \frac{\beta S^0}{\gamma} - 1 \right) \right\},$$

and

$$\varpi_4(t) = \max \left\{ S^0 - \frac{1}{\sigma} e^{\sigma t}, 0 \right\},$$

$$\varpi_5(t) = \max \left\{ v_2^{(1)} e^{\lambda_1 t}(1 - Me^{\epsilon t}), 0 \right\},$$

$$\varpi_6(t) = \max \left\{ e^{\lambda_1 t}(1 - Me^{\epsilon t}), 0 \right\},$$

in which $\sigma, M, \epsilon$ are positive constants determined in the following lemmas.

It is easy to see the following Lemma 2.2 holds.

**Lemma 2.2.** The function $\varpi_1(t)$ satisfies the inequality

$$d_1 \varpi_1'(t) - c \varpi_1'(t) - \beta \varpi_1(t) g(\varpi_4(t)) \leq 0,$$

for all $t \in \mathbb{R}$.

**Lemma 2.3.** The functions $\varpi_2(t)$ and $\varpi_3(t)$ satisfy the inequalities

$$d \varpi_2'(t) - c \varpi_2'(t) - \alpha \varpi_2(t) + \beta S^0 g(\varpi_3(t)) \leq 0, \quad t \neq t_1 := \frac{1}{\lambda_1} \ln \frac{\gamma}{a\alpha v_2^{(1)}} \left( \frac{\beta S^0}{\gamma} - 1 \right),$$

$$d \varpi_3'(t) - c \varpi_3'(t) - \gamma \varpi_3(t) + \alpha \varpi_3(t) \leq 0, \quad t \neq t_2 := \frac{1}{\lambda_1} \ln \frac{1}{a} \left( \frac{\beta S^0}{\gamma} - 1 \right).$$

**Proof.** We only show the inequality (2.4) holds, since the proof of (2.5) is similar to that of (2.4). Indeed, when $t < t_1$, $\varpi_2(t) = v_2^{(1)} e^{\lambda_1 t}$. Note that $\varpi_3(t) \leq e^{\lambda_1 t}$ for all $t \in \mathbb{R}$ and $g(x) \leq x$ for all $x \geq 0$. Then

$$d \varpi_2'(t) - c \varpi_2'(t) - \alpha \varpi_2(t) + \beta S^0 g(\varpi_3(t)) \leq d \varpi_2'(t) - c \varpi_2'(t) - \alpha \varpi_2(t) + \beta S^0 \varpi_3(t) \leq e^{\lambda_1 t} (h_1(\lambda_1)v_2^{(1)} + \beta S^0) = 0.$$

When $t > t_1$, $\varpi_2(t) = \varpi_2^{(1)}(\frac{\beta S^0}{\gamma} - 1)$. It follows from the fact that $\varpi_3(t) \leq \frac{1}{a} \left( \frac{\beta S^0}{\gamma} - 1 \right)$ for all $t \in \mathbb{R}$, and the function $g(x)$ is increasing in $x > 0$. Then

$$d \varpi_2'(t) - c \varpi_2'(t) - \alpha \varpi_2(t) + \beta S^0 g(\varpi_3(t)) \leq d \varpi_2'(t) - c \varpi_2'(t) - \alpha \varpi_2(t) + \frac{\gamma}{a} \left( \frac{\beta S^0}{\gamma} - 1 \right) = -\frac{\gamma}{a} \left( \frac{\beta S^0}{\gamma} - 1 \right) + \frac{\gamma}{a} \left( \frac{\beta S^0}{\gamma} - 1 \right) = 0.$$
Thus, the inequality (2.4) holds. This completes the proof. □

**Lemma 2.4.** Let

\[
\sigma < \min \left\{ \frac{\lambda_1}{2}, \frac{c}{d_1 + \beta(S^0)^2} \right\}. \tag{2.6}
\]

Then the functions \( \underline{u}_1(t) \) and \( \underline{u}_2(t) \) satisfy the inequality

\[
d_1 \underline{u}_1''(t) - c \underline{u}_1'(t) - \beta \underline{u}_1(t)g(\overline{u}_3(t)) \geq 0, \quad t \neq t_3 := \frac{1}{\sigma} \ln(\sigma S_0). \tag{2.7}
\]

**Proof.** If \( t > t_3 \), then the inequality (2.7) holds immediately since \( \underline{u}_1(t) = 0 \) on \([t_3, \infty)\). If \( t < t_3 \), then \( \underline{u}_1(t) = S_0 - \frac{1}{\sigma} e^{\sigma t} \). Note that \( \overline{u}_3(t) \leq e^{\lambda_1 t} \) for all \( t \in \mathbb{R} \). One gets

\[
d_1 \underline{u}_1''(t) - c \underline{u}_1'(t) - \beta \underline{u}_1(t)g(\overline{u}_3(t)) \geq d_1 \underline{u}_1''(t) - c \underline{u}_1'(t) - \beta \underline{u}_1(t)\overline{u}_3(t)
\]

\[
= -d_1 \sigma e^{\sigma t} + ce^{\sigma t} - \beta(S^0 - \frac{1}{\sigma} e^{\sigma t})e^{\lambda_1 t}
\]

\[
\geq e^{\sigma t} \left( -d_1 \sigma + c - \beta S_0 e^{(\lambda_1 - \sigma) t} \right),
\]

which follows from the fact \( e^{(\lambda_1 - \sigma) t} < (\sigma S^0)^{(\lambda_1 - \sigma)/\sigma} \) for \( t < t_3 \) that

\[
d_1 \underline{u}_1''(t) - c \underline{u}_1'(t) - \beta \underline{u}_1(t)g(\overline{u}_3(t)) \geq e^{\sigma t} \left( -d_1 \sigma + c - \beta S_0 e^{(\lambda_1 - \sigma) t} \right).
\]

Noting that (2.6), we see \( (\sigma S^0)^{(\lambda_1 - \sigma)/\sigma} \leq \sigma S^0 \). Then, by (2.6) again,

\[
d_1 \underline{u}_1''(t) - c \underline{u}_1'(t) - \beta \underline{u}_1(t)g(\overline{u}_3(t)) \geq e^{\sigma t} \left( -d_1 \sigma + c - \beta S_0 e^{(\lambda_1 - \sigma) t} \right) > 0, \quad t < t_3,
\]

which completes the proof. □

**Lemma 2.5.** Let

\[
0 < \epsilon < \min \left\{ \sigma, \lambda_1, \frac{c}{2d} - \lambda_1 \right\}.
\]

Then, for

\[
M > \max \left\{ 1, -\frac{\beta(1 + a\sigma S^0)}{\sigma(h_1(1 + \epsilon)v_{2}^{(1)} + \beta S^0)} \right\},
\]

the functions \( \underline{u}_1(t) \), \( \underline{u}_2(t) \) and \( \underline{u}_3(t) \) satisfy the inequalities

\[
d_2 \underline{u}_2''(t) - c \underline{u}_2'(t) - \alpha \underline{u}_2(t) + \beta \underline{u}_1(t)g(\underline{u}_3(t)) \geq 0, \quad t \neq t_4 := \frac{1}{\epsilon} \ln \frac{1}{M}, \tag{2.8}
\]

\[
d_3 \underline{u}_3''(t) - c \underline{u}_3'(t) - \gamma \underline{u}_3(t) + \alpha \underline{u}_2(t) \geq 0, \quad t \neq t_4. \tag{2.9}
\]

**Proof.** By Lemma 2.1(1), \( h_1(1 + \epsilon) < h_1(1) < 0 \) and \( h_2(1 + \epsilon) < h_2(1) < 0 \). Then

\[
h_1(1 + \epsilon)v_{2}^{(1)} + \beta S^0 < h_1(1)v_{2}^{(1)} + \beta S^0 = 0, \tag{2.10}
\]

\[
\alpha v_{2}^{(1)} + h_2(1 + \epsilon) < \alpha v_{2}^{(1)} + h_2(1) = 0. \tag{2.11}
\]

Now we first show (2.8) holds. In fact, the inequality (2.8) holds immediately since \( \underline{u}_2(t) = 0 \) on \([t_4, \infty)\). For \( t < t_4 < 0 \), \( \underline{u}_2(t) = v_{2}^{(1)} e^{\lambda_1 t}(1 - Me^{t}) \). In view of the fact \( g(x) \geq x(1 - ax) \) for all \( x \geq 0 \), and

\[
S_0 - \frac{1}{\sigma} e^{\sigma t} \leq \underline{u}_1(t) \leq S^0, \quad e^{\lambda_1 t}(1 - Me^{t}) \leq \underline{u}_3(t) \leq e^{\lambda_1 t}, \quad \forall t \in \mathbb{R},
\]
we get
\[\begin{align*}
&\frac{d}{dt}u_2''(t) - cu_2'(t) - \alpha u_2(t) + \beta u_1(t)g(u_3(t)) \\
&\quad \geq \frac{d}{dt}u_2''(t) - cu_2'(t) - \alpha u_2(t) + \beta u_1(t)\left(1 - a u_3(t)\right) \\
&\quad \geq \frac{d}{dt}u_2''(t) - cu_2'(t) - \alpha u_2(t) \\
&\quad \quad + \beta \left(S^0 - \frac{1}{\sigma}e^{\sigma t}\right)\left(e^{\lambda_1 t} - Me^{(\lambda_1 + \epsilon) t}\right) - a \beta S^0 e^{2 \lambda_1 t} \\
&= -e^{(\lambda_1 + \epsilon)t}\left(Mh_1(\lambda_1 + \epsilon)v_2^{(1)} + \beta S^0\right) + \frac{\beta}{\sigma} e^{(\sigma - \epsilon)t} + a \beta S^0 e^{(\lambda_1 - \epsilon)t}.
\end{align*}\]

Note that \(\epsilon < \min\{\sigma, \lambda_1\}\). Then \(e^{(\sigma - \epsilon)t} < 1\) and \(e^{(\lambda_1 - \epsilon)t} < 1\) for \(t < t_4 < 0\). Therefore, by (2.10), for \(t < t_4 < 0\),
\[\begin{align*}
&\frac{d}{dt}u_2''(t) - cu_2'(t) - \alpha u_2(t) + \beta u_1(t)g(u_3(t)) \\
&\quad \geq -e^{(\lambda_1 + \epsilon)t}\left(Mh_1(\lambda_1 + \epsilon)v_2^{(1)} + \beta S^0\right) + \frac{\beta}{\sigma} e^{(\sigma - \epsilon)t} + a \beta S^0 > 0.
\end{align*}\]
Consequently, (2.8) holds.

Next, we show (2.9) holds. Clearly, (2.9) holds since \(u_3(t) = 0\) for \(t > t_4\). For \(t < t_4\), \(u_3(t) = e^{\lambda_1 t}(1 - Me^{\epsilon t})\). Note that \(u_3(t) \geq v_2^{(1)}e^{\lambda_1 t}(1 - Me^{\epsilon t})\) for all \(t \in \mathbb{R}\). Hence, by (2.11),
\[\frac{d}{dt}u_3''(t) - cu_3'(t) - \gamma u_3(t) + \alpha u_2(t) \geq -Me^{(\lambda_1 + \epsilon)t}(\alpha v_2^{(1)} + h_2(\lambda_1 + \epsilon)) > 0,
\]
which follows (2.9) holds for \(t < t_4 < 0\). Hence, we complete the proof. \(\square\)

### 2.4. The verification of Schauder fixed point theorem

In this subsection, we will use the upper and lower solutions \((\overline{u}_1(t), \overline{u}_2(t), \overline{u}_3(t))\) and \((\underline{u}_1(t), \underline{u}_2(t), \underline{u}_3(t))\) to verify that the conditions of Schauder fixed point theorem hold. Here, we use the usual Banach space \(B = C(\mathbb{R}, \mathbb{R}^3)\) of bounded continuous functions endowed with the maximum norm, see [32],
\[\|u(t)\| = \sup_{t \in \mathbb{R}}|u_1(t)| + |u_2(t)| + |u_3(t)|.\]

For \(c > c^*\), let
\[\Gamma = \{ (u_1, u_2, u_3) \in B : \underline{u}_i(t) \leq u_i(t) \leq \overline{u}_i(t), i = 1, 2, 3 \}.\]

Define the operator \(H = (H_1, H_2, H_3) : \Gamma \rightarrow B\) by
\[\begin{align*}
H_1(u_1, u_2, u_3)(t) &:= \alpha_1 u_1(t) - \beta u_1(t)g(u_3(t)), \\
H_2(u_1, u_2, u_3)(t) &:= (\alpha_2 - \alpha)u_2(t) + \beta u_1(t)g(u_3(t)), \\
H_3(u_1, u_2, u_3)(t) &:= \alpha u_2(t) + (\alpha_3 - \gamma)u_3(t),
\end{align*}\]
where each \(\alpha_i\) is a large positive number such that \(\alpha_1 > \frac{\beta}{\alpha}, \alpha_2 > \alpha\) and \(\alpha_3 > \gamma\). Then system (2.3) can be rewritten as
\[\begin{align*}
d_1 u_1''(t) - cu_1'(t) - \alpha_1 u_1(t) + H_1(u_1, u_2, u_3)(t) &= 0, \\
d_2 u_2''(t) - cu_2'(t) - \alpha_2 u_2(t) + H_2(u_1, u_2, u_3)(t) &= 0, \\
d_3 u_3''(t) - cu_3'(t) - \alpha_3 u_3(t) + H_3(u_1, u_2, u_3)(t) &= 0.
\end{align*}\]

### 2.5. Conclusion
Let
\[
\lambda_{11} = \frac{c - \sqrt{c^2 + 4d_1 \alpha_1}}{2d_1}, \quad \lambda_{12} = \frac{c + \sqrt{c^2 + 4d_1 \alpha_1}}{2d_1},
\]
\[
\lambda_{i1} = \frac{c - \sqrt{c^2 + 4d \alpha_i}}{2d}, \quad \lambda_{i2} = \frac{c + \sqrt{c^2 + 4d \alpha_i}}{2d}, \quad i = 2, 3.
\]

Define the operator \( F : \Gamma \to \mathcal{B} \) by
\[
F(u)(t) = (F_1(u), F_2(u), F_3(u))(t),
\]
where \( u = (u_1, u_2, u_3) \), and
\[
F_1(u)(t) = \frac{1}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) H_1(u)(s)ds,
\]
\[
F_2(u)(t) = \frac{1}{\rho_2} \left( \int_{-\infty}^{t} e^{\lambda_{21}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{22}(t-s)} \right) H_2(u)(s)ds,
\]
\[
F_3(u)(t) = \frac{1}{\rho_3} \left( \int_{-\infty}^{t} e^{\lambda_{31}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{32}(t-s)} \right) H_3(u)(s)ds,
\]
in which
\[
\rho_1 = d_1(\lambda_{12} - \lambda_{11}) = \sqrt{c^2 + 2d_1 \alpha_1}, \quad \rho_i = d(\lambda_{i2} - \lambda_{i1}) = \sqrt{c^2 + 2d \alpha_i}, \quad i = 2, 3.
\]
It is easily verified that the operator \( F \) is well defined for \( u = (u_1, u_2, u_3) \in \Gamma \), and satisfies
\[
d_1 F_1''(u)(t) - cF_1'(u)(t) - \alpha_1 F_1(u)(t) + H_1(u)(t) = 0,
\]
\[
d_2 F_2''(u)(t) - cF_2'(u)(t) - \alpha_2 F_2(u)(t) + H_2(u)(t) = 0,
\]
\[
d_3 F_3''(u)(t) - cF_3'(u)(t) - \alpha_3 F_3(u)(t) + H_3(u)(t) = 0.
\]
Thus the fixed point of \( F \) is the solution of (2.12), which is a traveling wave solution of (2.1). Hence the existence of solution of (2.12) is reduced to verify that the operator \( F \) satisfies the conditions of Schauder fixed point theorem. Here we divide the proof into the following two lemmas.

**Lemma 2.6.** The operator \( F \) maps \( \Gamma \) into \( \Gamma \).

**Proof.** Given \( u = (u_1, u_2, u_3) \in \Gamma \). It is obvious that we only need to show that
\[
u_i(t) \leq F_i(u)(t) \leq \overline{\nu}_i(t), \quad \forall t \in \mathbb{R}, \quad i = 1, 2, 3.
\]
Based on the choice of the constants \( \alpha_1, \alpha_2 \) and \( \alpha_3 \), it suffices to prove that, for any \( t \in \mathbb{R} \),
\[
\underline{\nu}_1(t) \leq F_1(u_1, u_2, u_3)(t) \leq F_1(u_1, u_2, u_3)(t) \leq \overline{\nu}_1(t), \quad (2.13)
\]
\[
\underline{\nu}_2(t) \leq F_2(u_1, u_2, u_3)(t) \leq F_2(u_1, u_2, u_3)(t) \leq \overline{\nu}_2(t), \quad (2.14)
\]
\[
\underline{\nu}_3(t) \leq F_3(u_1, u_2, u_3)(t) \leq F_3(u_1, u_2, u_3)(t) \leq \overline{\nu}_3(t). \quad (2.15)
\]
Firstly, we prove (2.13) holds. In fact, for \( t > t_3 \), then, by (2.7),
\[
F_1(\overline{u}_1, \overline{u}_2, \overline{u}_3)(t)
\]
\[
= \frac{1}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) H_1(\overline{u}_1, \overline{u}_2, \overline{u}_3)(s)ds
\]
\[
\geq \frac{1}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) (\alpha_3 \overline{u}_1(s) + c_u'(s) - d_3 \overline{u}_1''(s))ds
\]
\[
= \overline{u}_1(t) + \frac{d_1}{\rho_1} e^{\lambda_{11}(t-t_3)} (\overline{u}_1'(t_3) + 0) - \overline{u}_1'(t_3 - 0))
\]
\[
\geq \overline{u}_1(t).
\]
Similarly, for \( t \leq t_3 \), we also get
\[
F_1(\overline{u}_1, \overline{u}_2, \overline{u}_3)(t) \geq \overline{u}_1(t).
\]
Note that \( H_1(\overline{u}_1, \overline{u}_2, \overline{u}_3)(t) \leq \alpha_1 \overline{u}_1(t) \) for \( t \in \mathbb{R} \). Then
\[
F_1(\overline{u}_1, \overline{u}_2, \overline{u}_3)(t) = \frac{1}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) H_1(\overline{u}_1, \overline{u}_2, \overline{u}_3)(s)ds
\]
\[
\leq \frac{1}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) (\alpha_3 \overline{u}_1(s))ds
\]
\[
\leq \frac{\alpha_1 S^0}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} ds + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} ds \right)
\]
\[
= \frac{\alpha_1 S^0}{\rho_1} \left( \frac{1}{\lambda_{12}} - \frac{1}{\lambda_{11}} \right) = S^0.
\]
Therefore, (2.13) holds.

The proofs of (2.14) and (2.15) are similar to that of (2.13) and are omitted. Consequently, we complete the proof. \( \square \)

**Lemma 2.7.** The operator \( F = (F_1, F_2, F_3) : \Gamma \to \Gamma \) is continuous and compact with respect to the norm \( \| \cdot \| \).

**Proof.** We first show that \( F = (F_1, F_2, F_3) : \Gamma \to \Gamma \) is continuous with respect to the norm \( \| \cdot \| \). Indeed, for any \( u = (u_1, u_2, u_3), \tilde{u} = (\overline{u}_1, \overline{u}_2, \overline{u}_3) \in \Gamma \), it is easy to see that there exists \( L > 0 \) such that
\[
|H_i(u)(t) - H_i(\tilde{u})(t)| \leq \frac{L}{3} \sum_{j=1}^{3} |u_j(t) - \tilde{u}_j(t)| \leq L \| u - \tilde{u} \|, \forall t \in \mathbb{R}, \ i = 1, 2, 3.
\]

Therefore,
\[
|F_1(u)(t) - F_1(\tilde{u})(t)|
\]
\[
\leq \frac{1}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) |H_1(u)(s) - H_1(\tilde{u})(s)|ds
\]
\[
\leq \frac{L}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} \right) ds \| u - \tilde{u} \|
\]
\[
= \frac{L}{\rho_1} \left( \frac{1}{\lambda_{12}} - \frac{1}{\lambda_{11}} \right) \| u - \tilde{u} \|
\]
\[
= \frac{L}{\alpha_1} \| u - \tilde{u} \|,
\]
which indicates that the operator $F_1$ is continuous with respect to the norm $\| \cdot \|$. Similarly, we also can show that operator $F_i: \Gamma \to \Gamma$, $i = 2, 3$, is continuous with respect to the norm $\| \cdot \|$.

Next, we show that $F$ is compact with respect to the norm $\| \cdot \|$. In fact, note that there exists $H_0 > 0$ such that $|H_i(u(t))| \leq H_0 (i = 1, 2, 3)$ for $u = (u_1, u_2, u_3) \in \Gamma$. Consequently, for any given $t_1, t_2 \in \mathbb{R}$ ($t_2 > t_1$), and $u = (u_1, u_2, u_3) \in \Gamma$, keeping in mind that $\lambda_{11} < \lambda_{12}$, then

$$
\left| F_1(u)(t_2) - F_1(u)(t_1) \right| = \frac{1}{\rho_1} \left| \left( \int_{-\infty}^{t_2} e^{\lambda_{11}(t_2-s)} - \int_{-\infty}^{t_1} e^{\lambda_{11}(t_1-s)} \right) H_1(u)(s)ds \right|
\leq \frac{H_0}{\rho_1} \left( 1 - e^{\lambda_{11}(t_2-t_1)} \right) \int_{-\infty}^{t_1} e^{\lambda_{11}(t_1-s)}ds + \int_{t_1}^{t_2} e^{\lambda_{11}(t_2-s)}ds
+ (e^{\lambda_{12}(t_2-t_1)} - 1) \int_{t_2}^{+\infty} e^{\lambda_{12}(t_1-s)}ds + \int_{t_1}^{t_2} e^{\lambda_{12}(t_1-s)}ds
= \frac{2H_0}{\rho_1} \left( \frac{e^{\lambda_{11}(t_2-t_1)} - 1}{\lambda_{11}} + \frac{1 - e^{\lambda_{12}(t_2-t_1)}}{\lambda_{12}} \right)
\leq \frac{4H_0}{\rho_1} |t_2 - t_1|,
$$

since $e^t \geq 1 + t$ for all $t \in \mathbb{R}$. Similarly, one also gets

$$
\left| F_2(u)(t_2) - F_2(u)(t_1) \right| \leq \frac{4H_0}{\rho_2} |t_2 - t_1|,
$$

and

$$
\left| F_3(u)(t_2) - F_3(u)(t_1) \right| \leq \frac{4H_0}{\rho_3} |t_2 - t_1|.
$$

It follows that $F(\Gamma)$ is equicontinuous with respect to the norm $\| \cdot \|$. Recall that $F(\Gamma)$ is uniformly bounded (by Lemma 2.6). Then the Arzelà-Ascoli theorem implies that $F: \Gamma \to \Gamma$ is compact with respect to the norm $\| \cdot \|$. Therefore, we complete the proof.

\[ \square \]

3. Existence and non-existence of traveling wave solutions. In this section, we shall state precisely and prove the main results of this paper.

3.1. Existence of traveling wave solutions. Here we establish the existence of traveling waves for system (2.1) with $d_2 = d_3 =: d$. To begin with, we first give some propositions.

**Proposition 3.1.** Assume that $R_0 > 1$. For any $c > c^*$, system (2.1) with $d_2 = d_3 =: d$ admits a nontrivial traveling wave solution $(u_1(x+ct), u_2(x+ct), u_3(x+ct))$ satisfying

$$
0 < u_1(t) < S_0, \ 0 < u_2(t) < \frac{\gamma}{\alpha a} \left( \frac{\beta S^0}{\gamma} - 1 \right), \ 0 < u_3(t) < \frac{\gamma}{\alpha} \left( \frac{\beta S^0}{\gamma} - 1 \right), \ \forall t \in \mathbb{R},
$$

and it holds

$$
\lim_{t \to -\infty} (u_1(t), u_2(t), u_3(t)) = (S^0, 0, 0), \ \lim_{t \to -\infty} (u'_1(t), u'_2(t), u'_3(t)) = (0, 0, 0).
$$

(3.1)
Furthermore,
\[
\lim_{t \to -\infty} \frac{u_2'(t)}{u_2(t)} = \lambda_1, \quad \lim_{t \to -\infty} \frac{u_3'(t)}{u_3(t)} = \lambda_1.
\]

Proof. In view of Lemmas 2.6 and 2.7, it follows from Schauder fixed point theorem that there exists a pair of \( u = (u_1, u_2, u_3) \in \Gamma \), which is a fixed point of the operator \( F \). Consequently, the solution \((u_1(x + ct), u_2(x + ct), u_3(x + ct))\) is a traveling wave solution of system (2.1), and

\[
0 \leq u_1(t) \leq S^0, \quad 0 \leq u_2(t) \leq \frac{\gamma}{a\alpha} \left( \frac{\beta S^0}{\gamma} - 1 \right), \quad 0 \leq u_3(t) \leq \frac{1}{a} \left( \frac{\beta S^0}{\gamma} - 1 \right), \quad \forall t \in \mathbb{R}.
\]

We claim that the strict inequalities hold. Indeed, note that \( u = (u_1, u_2, u_3) \in \Gamma \) is a fixed point of the operator \( F \), then \( u_1(t) = F_1(u_1, u_2, u_3)(t) \). Consequently,

\[
\begin{align*}
    u_1(t) &= F_1(u_1, u_2, u_3)(t) \geq F_1(u_1, \bar{u}_2, \bar{u}_3)(t) \\
    &= \frac{1}{\rho_1} \left( \int_{-\infty}^t e^{\lambda_1(t-s)} + \int_t^{+\infty} e^{\lambda_2(t-s)} \right) H_1(u_1, \bar{u}_2, \bar{u}_3)(s)ds \\
    &\geq \frac{1}{\rho_1} (\alpha_1 - \frac{\beta}{a}) \left( \int_{-\infty}^t e^{\lambda_1(t-s)} + \int_t^{+\infty} e^{\lambda_2(t-s)} \right) u_1(s)ds \\\n    &> 0,
\end{align*}
\]

since \( u_1(t) \) is continuous and is not identically zero, and

\[
H_1(u_1, \bar{u}_2, \bar{u}_3)(t) = \alpha_1 u_1(t) - \beta \bar{u}_1(t) g(\bar{u}_3(\bar{t})) \geq (\alpha_1 - \frac{\beta}{a}) u_1(t), \quad \forall t \in \mathbb{R}.
\]

Similarly, we can obtain that other inequalities are also strict ones. Thus, (3.1) holds.

On the other hand, in view of the following inequalities

\[
S^0 - \frac{1}{\sigma} e^{\sigma t} \leq u_1(t) < S^0, \quad v_2^{(1)} e^{\lambda_1 t} (1 - Me^{ct}) \leq u_2(t) \leq v_2^{(1)} e^{\lambda_1 t},
\]

and

\[
e^{\lambda_1 t} (1 - Me^{ct}) \leq u_3(t) \leq e^{\lambda_1 t}, \quad \forall t \in \mathbb{R},
\]

which follows that

\[
\lim_{t \to -\infty} u_1(t) = S^0, \quad \lim_{t \to -\infty} u_2(t) = 0, \quad \lim_{t \to -\infty} u_3(t) = 0,
\]

and

\[
\lim_{t \to -\infty} e^{-\lambda_1 t} u_2(t) = v_2^{(1)}, \quad \lim_{t \to -\infty} e^{-\lambda_1 t} u_3(t) = 1. \tag{3.3}
\]

Note that \( u = (u_1, u_2, u_3) \in \Gamma \) is a fixed point of the operator of \( F \). Applying L'Hôpital rule to the maps \( F_1, F_2 \) and \( F_3 \), it is easy to see that

\[
\lim_{t \to -\infty} (u_1'(t), u_2'(t), u_3'(t)) = 0.
\]

Hence, we have shown that (3.2) holds.

Now, integrating both sides of the second equation of (2.3) from \(-\infty\) to \( t \) gives

\[
du_2'(t) = cu_2(t) - \beta \int_{-\infty}^t u_1(s)g(u_3(s))ds + \alpha \int_{-\infty}^t u_2(s)ds. \tag{3.4}
\]
Thus, recall that (3.3), by L’Hôpital rule again, we get
\[
\lim_{t \to -\infty} e^{-\lambda t} u_2'(t) = \lim_{t \to -\infty} \left( \frac{c}{d} e^{-\lambda t} u_2(t) - \frac{\beta}{d} e^{-\lambda t} \int_{-\infty}^{t} u_1(s) g(u_3(s)) ds + \frac{\alpha}{d} e^{-\lambda t} \int_{-\infty}^{t} u_2(s) ds \right) = \frac{(c\lambda + \alpha) v_2(1) - \beta S^0}{\lambda_1 d} = \lambda_1 v_2(1),
\]
which implies, by the first one of (3.3), \(\lim_{t \to -\infty} u_2'(t) = \lambda_1\). Similarly, we also show \(\lim_{t \to -\infty} u_3'(t) = \lambda_1\). Hence, we complete the proof. \(\Box\)

**Proposition 3.2.** Assume that \(R_0 > 1\). Then, for any \(c > c^*\), let \((u_1(x+ct), u_2(x+ct), u_3(x+ct))\) be a nontrivial traveling wave solution of system (2.1) with \(d_2 = d_3 =: d\) satisfying (3.2). Then

1. \(u_1(t)\) is monotonically decreasing in \(t \in \mathbb{R}\);
2. \(\lim_{t \to +\infty} (u_1(t), u_2(t), u_3(t)) = (S_0, 0, 0), \lim_{t \to +\infty} (u_1'(t), u_2'(t), u_3'(t)) = (0, 0, 0)\).
3. It holds
\[
\int_{-\infty}^{+\infty} u_1(s) g(u_3(s)) ds = \frac{c}{\beta}(S^0 - S_0),
\]
and
\[
\int_{-\infty}^{+\infty} u_2(t) dt = \frac{c}{\alpha}(S^0 - S_0), \quad \int_{-\infty}^{+\infty} u_3(t) dt = \frac{c}{\gamma}(S^0 - S_0).
\]

**Proof.** We first show the conclusion (1) holds. Indeed, in view of the fact (3.2), integrating the two sides of the first equation of (2.3) from \(-\infty\) to \(t\) follows
\[
d_1 u_1'(t) = c(u_1(t) - S^0) + \beta \int_{-\infty}^{t} u_1(s) g(u_3(s)) ds. \tag{3.5}
\]
We now claim that the integral
\[
\int_{-\infty}^{+\infty} u_1(s) g(u_3(s)) ds < +\infty. \tag{3.6}
\]
Indeed, if not, note that the fact \(0 < u_1(t) < S^0\) for all \(t \in \mathbb{R}\), by (3.5), we then conclude that there exists \(\delta_0 > 0\) such that \(u_1'(t) > \delta_0\) for all large \(t > 0\), which implies that \(u_1(t) \to +\infty\) as \(t \to +\infty\), this is contradiction. Hence, the improper integral \(\int_{-\infty}^{+\infty} u_1(s) g(u_3(s)) ds\) converges, i.e., (3.6) holds. As a result that it follows \(u_1'(t)\) is uniformly bounded for all \(t \in \mathbb{R}\). Here it is clear from the first equation of (2.3) that
\[
(e^{-\frac{\beta}{d_1} t} u_1'(t))' = \frac{\beta}{d_1} e^{-\frac{\beta}{d_1} t} u_1(t) g(u_3(t)), \quad \forall t \in \mathbb{R}.
\]
Integrating the last equality from \(t\) to \(+\infty\) yields
\[
u_1'(t) = -\frac{\beta}{d_1} e^{-\frac{\beta}{d_1} t} \int_{t}^{+\infty} e^{-\frac{\beta}{d_1} s} u_1(s) g(u_3(s)) ds < 0, \quad \forall t \in \mathbb{R},
\]
which, together with the fact \(u_1(t) > 0\) and \(u_3(t) > 0\) are continuous in \(t \in \mathbb{R}\), implies \(u_1'(t) < 0\) for all \(t \in \mathbb{R}\). Thus, \(u_1(t)\) is monotonically decreasing in \(t \in \mathbb{R}\), that
is, the conclusion (1) holds. And let \( S_0 := \lim_{t \to +\infty} u_1(t) \), consequently, \( S^0 > S_0 \geq 0 \).

Note that \( u_2(t) \) satisfies the second equation of (2.3). Then

\[
\begin{align*}
  u_2(t) &= \frac{\beta}{\rho_2^2} \left( \int_{-\infty}^{t} e^{\lambda_2'(t-s)} + \int_{t}^{+\infty} e^{\lambda_2'(t-s)} \right) u_1(s) g(u_3(s)) ds, \quad \forall t \in \mathbb{R}. \tag{3.7}
\end{align*}
\]

here,

\[
\lambda_2' = \frac{c - \sqrt{c^2 + 4d\alpha}}{2d}, \quad \lambda_2' = \frac{c + \sqrt{c^2 + 4d\alpha}}{2d}, \quad \rho' = d_2(\lambda_2' - \lambda_2').
\]

By (3.6), let \( \int_{-\infty}^{+\infty} u_1(t)g(u_3(t))dt =: A_0 < \infty \), it follows from Fubini’s theorem that

\[
\begin{align*}
  \int_{-\infty}^{+\infty} u_2(t) dt &= \frac{\beta}{\rho_2^2} \left( \int_{-\infty}^{t} e^{\lambda_2'(t-s)} + \int_{t}^{+\infty} e^{\lambda_2'(t-s)} \right) u_1(s) g(u_3(s)) dsdt \\
  &= \frac{\beta}{\rho_2^2} \left( -\frac{1}{\lambda_2'} + \frac{1}{\lambda_2''} \right) A_0 = \frac{\beta}{\alpha} A_0. \tag{3.8}
\end{align*}
\]

Note that for \((u_1, u_2, u_3) \in \Gamma\), it is clear that

\[
0 < u_1(t)g(u_3(t)) \leq \frac{S^0}{a}, \quad \forall t \in \mathbb{R}.
\]

Then, for any \( t \in \mathbb{R} \),

\[
|u_2'(t)| \leq \frac{\beta}{\rho_2^2} \left( |\lambda_2'| \int_{-\infty}^{t} e^{\lambda_2'(t-s)} + \lambda_2' \int_{t}^{+\infty} e^{\lambda_2'(t-s)} \right) |u_1(s) g(u_3(s))| ds \\
\leq \frac{\beta S^0}{a \rho_2^2} \left( |\lambda_2'| \int_{-\infty}^{t} e^{\lambda_2'(t-s)} ds + \lambda_2' \int_{t}^{+\infty} e^{\lambda_2'(t-s)} ds \right) \\
= \frac{2\beta S^0}{a \rho_2^2}.
\]

So that, \( u_2'(t) \) is uniformly bounded, which, together with that \( u_2(t) > 0 \) is integrable on \( \mathbb{R} \) (by (3.8)), implies \( \lim_{t \to +\infty} u_2(t) = 0 \). In fact, we can find a number \( \varepsilon_0 > 0 \), a sequence \( t_n \to +\infty \) as \( n \to \infty \) and a number \( \delta > 0 \) such that \( u_1(t) \geq \varepsilon_0 \) for all \( |t - t_n| < \delta \), which contradicts the integrability of \( u_2(t) \) on \( \mathbb{R} \). Furthermore, recall that \( \lim_{t \to +\infty} u_1(t) = S_0 \) exists, letting \( t \to +\infty \) in (3.5), and note that (3.6), we know \( \lim_{t \to +\infty} u_1'(t) \) exists. It follows from the fact \( u_1'(t) < 0 \) for all \( t \in \mathbb{R} \) that \( \lim_{t \to +\infty} u_1'(t) \leq 0 \). Consequently, \( \lim_{t \to +\infty} u_1'(t) = 0 \). Otherwise, \( \lim_{t \to +\infty} u_1'(t) < 0 \), which implies that \( \lim_{t \to +\infty} u_1(t) = -\infty \), a contradiction to the fact \( u_1(t) > 0 \) for all \( t \in \mathbb{R} \).

Recall that \( \lim_{t \to +\infty} u_1'(t) = 0 \) and \( \lim_{t \to +\infty} u_1(t) = S_0 \). Letting \( t \to +\infty \) in (3.5), we have

\[
A_0 = \int_{-\infty}^{+\infty} u_1(t)g(u_3(t))dt = \frac{c}{\beta}(S^0 - S_0). \tag{3.9}
\]

Combining (3.8) with (3.9), we derive that

\[
\int_{-\infty}^{+\infty} u_2(t) dt = \frac{c}{\alpha}(S^0 - S_0). \tag{3.10}
\]

Letting \( t \to +\infty \) in (3.4), by (3.9) and (3.10), we get \( \lim_{t \to +\infty} u_2'(t) = 0 \).
On the other hand, note that $u_3(t)$ satisfies the third equation of (2.3). Then
\[ u_3(t) = \frac{\alpha}{\rho_3'} \left( \int_{-\infty}^{t} e^{\lambda_{31}'(t-s)} + \int_{t}^{+\infty} e^{\lambda_{32}'(t-s)} \right) u_2(s) ds, \forall t \in \mathbb{R}. \]  
\[
(3.11)
\]
here,
\[ \lambda_{31}' = \frac{c - \sqrt{c^2 + 4d\gamma}}{2d}, \quad \lambda_{32}' = \frac{c + \sqrt{c^2 + 4d\gamma}}{2d}, \quad \rho_3' = d(\lambda_{32}' - \lambda_{31}'). \]
Then, by Fubini’s theorem again, by (3.11),
\[
\int_{-\infty}^{+\infty} u_3(t) dt = \frac{\alpha}{\rho_3'} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{t} e^{\lambda_{31}'(t-s)} + \int_{t}^{+\infty} e^{\lambda_{32}'(t-s)} \right) u_2(s) ds dt
\]
\[
\quad = \frac{\alpha}{\rho_3'} \left( \frac{1}{\lambda_{31}'} + \frac{1}{\lambda_{32}'} \right) c(S^0 - S_0)
\]
\[
\quad = \frac{c}{\gamma} (S^0 - S_0).
\]
\[
(3.12)
\]
As the same as the proof of the before, we also get $u_3'(t)$ is uniformly bounded, together with (3.12), we get $\lim_{t \to -\infty} u_3(t) = 0$. Integrating the last equality of (2.3) from $-\infty$ to $t$ gives
\[
du_3'(t) = cu_3(t) + \gamma \int_{-\infty}^{t} u_3(s) ds - \alpha \int_{-\infty}^{t} u_2(s) ds.
\]
Letting $t \to +\infty$ in the above, which, together with (3.10) and (3.12), implies $\lim_{t \to +\infty} u_3(t) = 0$. Consequently, we have shown the conclusion (2) holds. Obviously, it follows from (3.9), (3.10) and (3.12) that the conclusion (3) holds. The proof is completed.

In view of Propositions 3.1 and 3.2, we are in a position to state and prove the existence of traveling wave solutions of system (2.1) with $d_2 = d_3 =: d$.

**Theorem 3.1.** Assume that $R_0 > 1$. For any $c > c^*$, system (2.1) with $d_2 = d_3$ admits a nontrivial traveling wave solution $(u_1(x + ct), u_2(x + ct), u_3(x + ct))$ such that

1. The following boundary conditions
\[
\lim_{t \to -\infty} (u_1(t), u_2(t), u_3(t)) = (S^0, 0, 0), \quad \lim_{t \to +\infty} (u_1(t), u_2(t), u_3(t)) = (S_0, 0, 0)
\]
\[
(3.13)
\]
hold. Moreover, $S_0 \leq \frac{\gamma}{\beta}$.
2. $u_1(t)$ is monotonically decreasing in $t \in \mathbb{R}$, and
\[
0 < u_i(t) < S^0 - S_0, \forall t \in \mathbb{R}, i = 2, 3.
\]

**Proof.** To prove Theorem 3.1, by Propositions 3.1 and 3.2, it suffices to show that
\[
S_0 \leq \frac{\gamma}{\beta} \quad \text{and} \quad 0 < u_i(t) < S^0 - S_0, \forall t \in \mathbb{R}, i = 2, 3.
\]
To this end, we first show $S_0 \leq \frac{\gamma}{\beta}$ holds. If not, in the other words, $S_0 > \frac{\gamma}{\beta}$, note that $\lim_{t \to +\infty} \frac{u_1(t)}{1 + au_3(t)} = S_0$, then there exists $t_1 > 0$ such that $\frac{u_1(t)}{1 + au_3(t)} > \frac{1}{2}(S_0 + \frac{\gamma}{\beta})$ for all $t > t_1$. Consequently, $\frac{\beta u_1(t)}{1 + au_3(t)} - \gamma > \frac{1}{2}(\beta S_0 - \gamma) > 0$ for $t > t_1$. Adding the second equation and the third one of (2.3) gives
\[
d(u_2(t) + u_3(t))' - c(u_2(t) + u_3(t))' = -\left( \frac{\beta u_1(t)}{1 + au_3(t)} - \gamma \right) u_3(t) < 0.
\]
\[
(3.14)
\]
Recall that \( \lim_{t \to +\infty} u_i(t) = 0 \) and \( \lim_{t \to +\infty} u_i'(t) = 0 \) (by Proposition 3.2(2)), \( i = 2, 3 \), integrating (3.14) from \( t \) to \( +\infty \), \( t > t_1 \), we obtain
\[
d(u_2(t) + u_3(t))' - c(u_2(t) + u_3(t)) = \int_t^{+\infty} \left( \frac{\beta u_1(s)}{1 + au_3(s)} - \gamma \right) u_3(s)ds > 0, \quad t > t_1,
\]
which yields \( e^{-\frac{\beta}{\gamma} t} (u_1(t) + u_2(t))' > 0 \) for \( t > t_1 \). Thus,
\[
e^{-\frac{\beta}{\gamma} t} (u_1(t) + u_2(t)) < \lim_{t \to +\infty} e^{-\frac{\beta}{\gamma} t} (u_1(t) + u_2(t)) = 0
\]
for \( t > t_1 \), a contradiction. Hence the assertion that \( S_0 \leq \frac{d}{a} \) holds.

Next, we show \( 0 < u_i(t) < S^0 - S_0 \), \( \forall \, t \in \mathbb{R} \), \( i = 2, 3 \). In fact, we let
\[
N_2(t) := u_2(t) + \frac{\alpha}{c} \int_t^{+\infty} u_2(s)ds + \frac{\alpha}{c} \int_t^{+\infty} e^{\frac{\beta}{d} (t-s)} u_2(s)ds, \quad \forall \, t \in \mathbb{R},
\]
which, by L'Hôpital's rule, together with (3.2), implies that
\[
\lim_{t \to -\infty} N_2(t) = 0, \quad \lim_{t \to +\infty} N_2(t) = S^0 - S_0.
\]
Similarly, we obtain, by differentiating (3.15) once,
\[
N_2'(t) = u_2'(t) + \frac{\gamma}{d} \int_t^{+\infty} e^{\frac{\beta}{d} (t-s)} u_2(s)ds,
\]
which yields
\[
\lim_{t \to -\infty} N_2'(t) = 0, \quad \lim_{t \to +\infty} N_2'(t) = 0.
\]
Furthermore, by differentiating (3.17) once, and noting that \( u_2(t) \) satisfies the second equation of (2.3), one gets
\[
cN_2''(t) = dN_2''(t) + \beta u_1(t)g(u_3(t)), \quad \forall \, t \in \mathbb{R},
\]
which follows that
\[
N_2''(t) = \frac{\beta}{d} e^{\frac{\beta}{d} t} \int_t^{+\infty} e^{\frac{\beta}{d} s} u_2(s)g(u_3(s))ds > 0, \quad \forall \, t \in \mathbb{R}.
\]
Consequently, \( N_2(t) \) is increasing in \( t \in \mathbb{R} \). Further, by (3.16),
\[
0 < u_2(t) < N_2(t) \leq \lim_{t \to +\infty} N_2(t) = S^0 - S_0, \quad \forall \, t \in \mathbb{R}.
\]
Similarly, we also get
\[
0 < u_3(t) < S^0 - S_0, \quad \forall \, t \in \mathbb{R}.
\]
This completes the proof.

3.2. Non-existence of traveling wave solutions. In this subsection, we will establish the nonexistence of traveling waves for system (2.1) either \( \mathcal{R}_0 < 1 \) and \( c \geq 0 \), or \( \mathcal{R}_0 > 1 \) and \( c \in (0, c^*) \).

Firstly, it is easily see that a traveling wave solution \( (u_1(x + ct), u_2(x + ct), u_3(x + ct)) \) of (2.1) satisfying (3.13). Then \( u_2(t) \) and \( u_3(t) \) satisfies (3.7) and (3.11), respectively. Furthermore, an application of L'Hôpital rule to \( u_2'(t) \) and \( u_3'(t) \), we can easily get the following asymptotic behavior of \( u_2(t) \) and \( u_3(t) \) as follows
\[
\lim_{t \to +\infty} u_i(t) = 0, \quad \lim_{t \to +\infty} u_i'(t) = 0, \quad i = 2, 3.
\]

**Theorem 3.2.** Assume that \( \mathcal{R}_0 \leq 1 \). Then, for any \( c \geq 0 \), system (2.1) with \( d_2 = d_3 \) has no nontrivial and nonnegative traveling wave \( (u_1(x + ct), u_2(x + ct), u_3(x + ct)) \) satisfying (3.13).
Proof. When \( R_0 \leq 1 \), i.e., \( \beta S^0 \leq \gamma \), then \( \frac{\beta u_1(t)}{1 + au_3(t)} \leq \beta S^0 \) for all \( t \in \mathbb{R} \) since \( u_1(t) \leq S^0 \) for all \( t \in \mathbb{R} \) (by Theorem 2.1). Adding the second equation and third one of (2.3), we get (3.14) holds. Consequently, 

\[
\left( e^{-\hat{\alpha}t}(u_2(t) + u_3(t)) \right)' = -\frac{1}{d} \left( \frac{\beta u_1(t)}{1 + au_3(t)} - \gamma \right) u_3(t) \geq -\frac{1}{d} (\beta S^0 - \gamma) u_3(t) \geq 0,
\]

which implies that the function \( e^{-\hat{\alpha}t}(u_2(t) + u_3(t))' \) is nondecreasing in \( t \in \mathbb{R} \). It follows from the fact (3.18) and \( \lim_{t \to +\infty} e^{-\hat{\alpha}t} = 0 \) that \( (u_2(t) + u_3(t))' \leq 0 \) for all \( t \in \mathbb{R} \). Again by (3.18), we know that \( u_2(t) + u_3(t) = 0 \) for all \( t \in \mathbb{R} \), which is a contradiction. This ends the proof.

Theorem 3.3. Assume that \( R_0 > 1 \) and any \( c \in (0, c^*) \). Then system (2.1) with \( d_2 = d_3 \) has no nontrivial and nonnegative traveling wave solution \( (u_1(x + ct), u_2(x + ct), u_3(x + ct)) \) satisfying (3.13).

Proof. Assume that there exists a nontrivial and nonnegative traveling wave solution \( (u_1(x + ct), u_2(x + ct), u_3(x + ct)) \) of system (2.1) satisfying (3.2). Then (3.19) can be rewritten as 

\[
y' = By + f(t, y),
\]

where \( y = (u_1, u_2, u_3, u_4)^T \), and

\[
B = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{\alpha}{d} & \frac{c}{d} & -\frac{\beta S^0}{d} & 0 \\
0 & 0 & 0 & \frac{1}{d} \\
-\frac{\alpha}{d} & 0 & \frac{\gamma}{d} & \frac{c}{d}
\end{pmatrix}, \quad f(t, y) = \begin{pmatrix}
0 \\
\frac{\beta S^0}{d} - u_3 - \frac{\beta}{d} u_1 g(u_3) \\
0 \\
0
\end{pmatrix}.
\]

Since \( (u_1(t), u_2(t), u_3(t)) \) satisfies (3.13), we have \( \lim_{t \to -\infty} y(t) = 0 \). It is easy to show that the characteristic equation for \( B \) is given by \( \det A(\lambda) = 0 \). Since \( \lim_{t \to -\infty} u_1(t) = S^0 \), for any small constant \( \epsilon > 0 \) there exists \( t_0 \in \mathbb{R} \) such that

\[
\|f(t, y_1) - f(t, y_2)\| \leq \epsilon \|y_1 - y_2\|, \quad \forall t < t_0.
\]

Recall that the translation invariance of the traveling wave solution, without loss of generality, we take \( t_0 = 0 \). It follows from Lemma 2.1(3) that the equation \( \det A(\lambda) = 0 \) have one positive eigenvalue, one negative eigenvalue and a pair of complex conjugate eigenvalue with positive real parts. Hence we get the similar required version as Case (I) in [33, page 488]. Then following the steps in the proof of [33, Lemma 3.1], we can obtain

\[
\sup_{t \in \mathbb{R}} \{u_i(t)e^{-\alpha_0 t}\} < +\infty, \quad \sup_{t \in \mathbb{R}} \{u_i'(t)e^{-\alpha_0 t}\} < +\infty, \quad i = 2, 3, \quad (3.20)
\]

for some \( \alpha_0 > 0 \). Also, by the second equation and the third one of (2.3), we get

\[
\sup_{t \in \mathbb{R}} \{u_i''(t)e^{-\alpha_0 t}\} < +\infty, \quad i = 2, 3. \quad (3.21)
\]
Next we consider $S^0u_3(t) - u_1(t)g(u_3(t))$. Clearly,
\[ S^0u_3(t) - u_1(t)g(u_3(t)) = \frac{S^0u_3^2(t)}{1 + \alpha u_3(t)} + (S^0 - u_1(t)) \frac{S^0u_3(t)}{1 + \alpha u_3(t)}. \]

Then, note that $\frac{u_3^2(t)}{1 + \alpha u_3(t)} < u_3^2(t)$ for all $t \in \mathbb{R}$, by (3.20),
\[ \sup_{t \in \mathbb{R}} \left\{ \frac{u_3^2(t)}{1 + \alpha u_3} e^{-2\alpha_0 t} \right\} < +\infty. \tag{3.22} \]

Let $U_1(t) = S^0 - u_1(t) \geq 0$ for all $t \in \mathbb{R}$. Integrating the first equation of (2.3) from $-\infty$ to $t$ yields
\[ d_1U_1'(t) - cU_1(t) = -f(t), \tag{3.23} \]
in which
\[ f(t) = \beta \int_{-\infty}^t u_1(s)g(u_3(s))ds \leq C_0 e^{\alpha_0 t}, \forall t \in \mathbb{R} \]
for some constant $C_0 > 0$. That is, $f(t) = O(e^{\alpha_0 t})$ as $t \to -\infty$. Solving (3.23) follows
\[ U_1(t) = U_1(0)e^{\frac{\beta}{1 + \alpha}} + \frac{1}{d_1} e^{\frac{\beta}{1 + \alpha}} \int_0^t e^{-\frac{\beta}{1 + \alpha} s} f(s)ds, \forall t \in \mathbb{R}, \]
which, together with the fact $f(t) = O(e^{\alpha_0 t})$ as $t \to -\infty$, follows that $U_1(t) = O(e^{\alpha_0 t})$ as $t \to -\infty$, where $\alpha_0' = \min\{\alpha_0, \frac{\beta}{1 + \alpha}\}$. In view of the fact $0 \leq U_1(t) \leq S^0$ for all $t \in \mathbb{R}$, we have
\[ \sup_{t \in \mathbb{R}} \{U_1(t)e^{-\alpha_0 t}\} < +\infty. \tag{3.24} \]

Thus, by (3.22) and (3.24), we get
\[ \sup_{t \in \mathbb{R}} \{|u_1(t)g(u_3(t)) - S^0u_3(t)|e^{-(\alpha_0 + \alpha) t}\} < +\infty. \tag{3.25} \]

Now we rewrite system (3.19) as follows
\[ du_2'(t) - cu_2(t) - \alpha u_2(t) + \beta S^0u_3(t) = -\beta \{u_1(t)g(u_3(t)) - S^0u_3(t)\}, \]
\[ du_3'(t) - cu_3(t) - \gamma u_3(t) + \alpha u_2(t) = 0. \]

Then, we introduce two-side Laplace transform on the above system and get
\[ h_1(\lambda)L_2(\lambda) + \beta S^0L_3(\lambda) = -\beta \int_{-\infty}^{+\infty} e^{-\lambda t} \{u_1(t)g(u_3(t)) - S^0u_3(t)\} dt, \]
\[ h_2(\lambda)L_3(\lambda) + \alpha L_2(\lambda) = 0, \]
where $L_i(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda t} \phi_i(t)dt$, $\lambda > 0$. Multiplying the first equation and the second one of the above system yields
\[ \det A(\lambda)L_3(\lambda) = \alpha \beta \int_{-\infty}^{+\infty} e^{-\lambda t} \{u_1(t)g(u_3(t)) - S^0u_3(t)\} dt. \tag{3.26} \]

Note that both sides of (3.26) is defined for $\lambda \in \mathbb{C}$ with $0 < \text{Re}\lambda < \alpha_0$. Furthermore, the two Laplace integrals can be analytically continued to the whole right half line; otherwise, the integral on the left of (3.26) has singularity at $\lambda = \lambda^{**}_0$ and it is analytic for all $\lambda < \lambda^{**}_0$, cf. [4, 20, 22, 23]. But, by (3.25), the integral on the right of (3.26) is actually analytic for all $\lambda \leq \lambda^{**}_0 + \alpha'$, a contradiction. Thus, (3.26) holds for all $\lambda > 0$, and can rewritten as
\[ \int_{-\infty}^{+\infty} e^{-\lambda t} u_3(t) \left( \det A(\lambda) + \frac{\alpha \beta u_1(t)}{1 + \alpha u_3(t)} - S^0 \right) dt = 0. \tag{3.27} \]
Here, in view of the fact that $\det A(\lambda) = 0$ is always positive for all $\lambda > \lambda_0$, here $\lambda_0$ is the unique positive root of $\det A(\lambda) = 0$ when $c \in (0, c^*)$ (by Lemma 2.1(3)). Then
\[
\det A(\lambda) + \frac{\alpha \beta u_1(t)}{1 + au_3(t)} - S^0 \to \infty \text{ as } \lambda \to +\infty,
\]
which lead to a contradiction to (3.27). Finally, we conclude that if $R_0 > 1$ and $c \in (0, c^*)$, then there does not exist a non-trivial and non-negative traveling wave solutions satisfying the boundary condition (3.13). Thus we conclude the proof.

4. Discussion. There have been intensive studies about the existence and non-existence of traveling waves for some diffusive SIR model (see, for instance, [1, 8, 12, 20, 22, 23, 30]). In this paper, we have incorporated the exposed individuals and the population diffusions into the classic SIR model (1.1), and then presented a diffusive SEIR epidemic model (1.3) with a saturating incidence rate. By an abstract treatment, we studied the well posedness of system (2.1). Also, applying [21, Theorem 2.3], we gave the explicit formula of the basic reproduction number $R_0$ for system (2.1).

For the model under consideration, the traveling wave solutions describes the disease propagation into the susceptible individuals from an initial disease-free equilibrium to the final, also disease-free equilibrium. In this paper, we first use the iteration process to construct the vector-value upper-lower solutions for (2.3). Together with the Schauder fixed theorem, we can establish existence of such a traveling wave solution. Second, we use the two-sides Laplace transform to the non-existence of such a traveling wave solution. These results could formulate the possible propagation modes of the disease.

Theorem 3.1 gives some asymptotic behaviors of the traveling wave solution $(u_1(x + ct), u_2(x + ct), u_3(x + ct))$ of system (2.1) with speed $c > c^*$. Note that, in the condition, $S^0 > 0$ is a constant representing the number of the susceptible individuals before being infected. Clearly, at any fixed $x \in \mathbb{R}$, Theorem 3.1(1) and (2) describe that all the individuals were susceptible a long time ago ($t \to -\infty$) and all the susceptible individuals will be decreasing to $S_0$ after a long time ($t \to +\infty$). In particular, if $S_0 = 0$ (due to $S_0$ may be zero), then all the susceptible individuals will become the removed individuals also after a long time ($t \to +\infty$). Hence, the natural question arises. Can we know the value of $S(+\infty) = S_0$? As pointed in [22], for general system such as (1.2) with nonzero diffusion terms, it seems impossible to obtain the value of $S_0$, see [22] for a brief discussion on related works for this problem. We conjecture $S_0 = 0$, but unfortunately, we do not know to prove it. Here it follows from Theorem 3.1(1) we only provide an upper bound for $S_0$, that is, $S_0 \leq \frac{\gamma}{\eta}$.

Further, by Theorems 3.1 and 3.3, we see that the constant $c^*$ defined by Lemma 2.1 is indeed the minimal wave speed $c^*$. Hence, the natural question arises, is the constant $c^*$ the asymptotic speed of propagation? However, we have no way to prove the $c^*$ is the asymptotic speeds of spread for system (2.1). We have noticed that Liang and Zhao [13] obtained some general results on the asymptotic speeds of spread for monotone semiflows, which two speeds indeed coincide. Since system (2.1) cannot generate monotone semiflows, the theory [13] is invalid for the system (2.1). How to overcome technical problems that prevent a full analysis of relations of them should be a challenging work and leave it as our further project.
Theorems 3.1 and 3.3 combined establish a threshold condition for the existence and non-existence of traveling wave solutions in terms of the basic reproduction number $R_0$ of system (2.1). Namely, $R_0 > 1$ implies $c^* > 0$. It follows from Theorems 3.1 that the propagation of the pathogen as a wave with a fixed shape and a fixed speed for any $c > c^*$ from the initial disease-free equilibrium $(S^0, 0, 0)$ to another disease-free equilibrium $(S^0, 0, 0)$. For every $c < c^*$, Theorem 3.3 implies that system (2.1) has no nontrivial and nonnegative traveling wave connecting the two equilibria $(S^0, 0, 0)$ and $(S^0, 0, 0)$. The study of traveling wave solutions provides important insight into the spatial patterns of invading diseases.

As observed, the proofs of Theorem 3.1(1) and Theorem 3.2 depend heavily on the choice of the same diffusion rate $d$ for exposed individuals and infected ones, i.e., the last two equations of system (2.1) have the same diffusion rate $d$. Hence, some results obtained here can not be extrapolated for a more general class of population models. However, in realistic communities, the infectious disease transmits in the spatial movement of the region by the different diffusion rates for exposed and infected individuals. From the view of mathematical biology, it is important problem to deal with this case and we will consider the topic in our further study.

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