Phase separation in a gravity field

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Abstract. We prove here well-posedness and convergence to equilibria for the solution trajectories associated to a model for solidification of a liquid content of a rigid container in a gravity field. We observe that the gravity effects, which can be neglected without considerable changes of the process on finite time intervals, have a substantial influence on the long time behavior of the evolution system. Without gravity, we find a temperature interval, in which all phase distributions with a prescribed total liquid contents are admissible equilibria, while, under the influence of gravity, the only equilibrium distribution in a connected container consists in two pure phases separated by one plane interface perpendicular to the gravity force.

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1 Introduction

In this paper we continue the discussion started in [7], extending the model to the case in which also gravity effects are considered during the phase transition process. We derive a model for solid-liquid phase transition of a medium inside a rigid container. The main goal is to give a qualitative and quantitative description of the interaction between volume, pressure, phase, and temperature changes in the situation that the specific volume of the solid phase exceeds the specific volume of the liquid phase. We observe, in particular, that the solidification may take place at a temperature slightly above the critical temperature $\theta_c$. The overheating is due to the fact that the pressure decreases from the bottom to the top. A quantitative description of this phenomenon is given in here by means of the so-called Clausius-Clapeyron formula (cf. (2.40)).

There is an abundant classical literature on the study of phase transition processes, see e.g. the monographs [1], [3], [9] and the references therein. In particular, in [1], the authors proposed to interpret a phase transition process in terms of a balance equation for macroscopic motions, and to include the possibility of voids, while the microscopic approach has been pursued in [3] in the case of two different densities $\rho_1$.
and \( \rho_2 \) for the two substances undergoing phase transitions. Let us, however, refer

to the Introduction of [7] for a more detail description of the previous works in the

literature on this topic.

The forces occurring as a result of solid-liquid phase transitions in small contain-

ers are very strong, much stronger than gravity forces. In a bottle of water of less than

one meter height for example, they differ by at least four orders of magnitude. From

the quantitative viewpoint, the gravity effects can thus be neglected without consider-

able changes of the process on finite time intervals. They have, however, a substantial

influence on the long time behavior of the evolution system. Without gravity, we ob-

serve a temperature interval, in which all phase distributions with a prescribed total

liquid contents are admissible equilibria (cf. also [7]). If even a weak gravity field is

assumed to be present, then the only equilibrium distribution in a connected con-

tainer consists in two pure phases separated by one plane interface perpendicular to

the gravity force.

Here we proceed as follows: in Section 2, we derive a model describing the evo-

lution of the process which is driven by an energy balance, a quasistatic momentum

balance, and a phase dynamics equation. Still in Section 2, we verify the thermody-

namic consistency of the model, and we study the equilibria.

The well-posedness of the corresponding system of evolution equations is proved

in Section 3, while the study of the long-time behavior of solutions and convergence
to equilibria is proved in the last Section 4.

2 The model

As reference state, we consider a liquid substance contained in a bounded connected

bottle \( \Omega \subset \mathbb{R}^3 \) with boundary of class \( C^{1,1} \). The state variables are the absolute

temperature \( \theta > 0 \), the displacement \( u \in \mathbb{R}^3 \), and the phase variable \( \chi \in [0,1] \). The

value \( \chi = 0 \) means solid, \( \chi = 1 \) means liquid, \( \chi \in (0,1) \) is a mixture of the two.

We make the following modeling hypotheses.

(A1) The displacements are small. Therefore, we state the problem in Lagrangian

coordinates, in which the mass conservation is equivalent to the condition of a

constant mass density \( \rho_0 > 0 \).

(A2) The substance is compressible, and the speed of sound does not depend on the

phase.

(A3) The evolution is slow, and we neglect shear viscosity and inertia effects.

(A4) We neglect shear stresses.

In agreement with (A1), we define the strain \( \varepsilon \) as an element of the space \( T^{3 \times 3}_{\text{sym}} \) of

symmetric tensors by the formula

\[
\varepsilon = \nabla_s u := \frac{1}{2}(\nabla u + (\nabla u)^T).
\]
Let $\delta \in T_{3 \times 3}^{3 \times 3}$ denote the Kronecker tensor. By (A4), the elasticity matrix $A$ has the form

$$A\varepsilon = \lambda(\varepsilon : \delta) \delta,$$

where “$:$” is the canonical scalar product in $T_{3 \times 3}^{3 \times 3}$, and $\lambda > 0$ is the Lamé constant (or bulk elasticity modulus), which we assume to be independent of $\chi$ by virtue of (A2). Note that $\lambda$ is related to the speed of sound $v_0$ by the formula $v_0 = \sqrt{\lambda/\varrho_0}$.

We want to model the situation where the specific volume $V_{solid}$ of the solid phase is larger than the specific volume $V_{liquid}$ of the liquid phase. Considering the liquid phase as the reference state, we introduce the dimensionless phase expansion coefficient $\alpha = (V_{solid} - V_{liquid}) / V_{liquid} > 0$, and we define the phase expansion strain $\tilde{\varepsilon}$ by

$$\tilde{\varepsilon}(\chi) = \frac{\alpha}{3}(1 - \chi)\delta.$$  \hspace{1cm} (2.3)

We fix positive constants $c_0$ (specific heat), $L_0$ (latent heat), $\theta_c$ (freezing point at standard atmospheric pressure), $\beta$ (thermal expansion coefficient), and consider the specific free energy $f$ in the form

$$f = c_0 \theta \left(1 - \log \left(\frac{\theta}{\theta_c}\right)\right) + \frac{\lambda}{2\varrho_0}((\varepsilon - \tilde{\varepsilon}(\chi)) : \delta)^2 - \frac{\beta}{\varrho_0}(\theta - \theta_c)\varepsilon : \delta$$

$$+ L_0 \left(\chi \left(1 - \frac{\theta}{\theta_c}\right) + I(\chi)\right),$$  \hspace{1cm} (2.4)

where $I$ is the indicator function of the interval $[0, 1]$. The stress tensor $\sigma$ is decomposed into the sum $\sigma^v + \sigma^e$ of the viscous component $\sigma^v$ and elastic component $\sigma^e$. The state functions $\sigma^v, \sigma^e, s$ (specific entropy), and $e$ (specific internal energy) are given by the formulas

$$\sigma^v = \nu(\varepsilon_t : \delta)\delta$$

$$\sigma^e = \varrho_0 \frac{\partial f}{\partial \varepsilon} = (\lambda(\varepsilon : \delta - \alpha(1 - \chi)) - \beta(\theta - \theta_c))\delta,$$

$$s = -\frac{\partial f}{\partial \theta} = c_0 \log \left(\frac{\theta}{\theta_c}\right) + \frac{L_0}{\theta_c}\chi + \frac{\beta}{\varrho_0}\varepsilon : \delta,$$

$$e = f + \theta \cdot s = c_0 \theta + \frac{\lambda}{2\varrho_0}(\varepsilon : \delta - \alpha(1 - \chi))^2 + \frac{\beta}{\varrho_0}\theta_c\varepsilon : \delta + L_0(\chi + I(\chi)),$$  \hspace{1cm} (2.8)

where $\nu > 0$ is the volume viscosity coefficient. The scalar quantity

$$p := -\nu \varepsilon_t : \delta - \lambda(\varepsilon : \delta - \alpha(1 - \chi)) + \beta(\theta - \theta_c)$$

is the pressure, and the stress has the form $\sigma = -p\delta$. The process is governed by the balance equations

$$-\text{div } \sigma = f_{vol} \quad \text{(mechanical equilibrium)}$$

$$\varrho_0 \varepsilon_t + \text{div } q = \sigma : \varepsilon_t \quad \text{(energy balance)}$$

$$-\gamma_0 \chi_t \in \partial_\chi f \quad \text{(phase relaxation law)}$$  \hspace{1cm} (2.12)
where $\gamma_0$ is the phase relaxation coefficient, $\partial_\chi$ is the partial subdifferential with respect to $\chi$, $f_{vol}$ is a given volume force density (the gravity force)

$$f_{vol} = -\gamma_0 g \delta_3,$$

(2.13)

with standard gravity $g$ and vector $\delta_3 = (0, 0, 1)$, and $q$ is the heat flux vector that we assume in the form

$$q = -\kappa \nabla \theta$$

(2.14)

with a constant heat conductivity $\kappa > 0$. The equilibrium equation (2.10) can be rewritten in the form $\nabla p = -\gamma_0 g$, hence

$$p(x, t) = P(t) - \gamma_0 g x_3$$

(2.15)

with a function $P$ of time only, which is to be determined. On $\partial \Omega$, we assume boundary conditions in the form

$$u = 0$$

(2.16)

$$q \cdot n = h(x)(\theta - \theta_T),$$

(2.17)

with a given positive measurable function $h$ (heat transfer coefficient), and a constant $\theta_T > 0$ (external temperature). Identity (2.16) means that the boundary is rigid. Other possibilities (elastic or elastoplastic boundary response) have been considered in another context ([7, 8]).

By Gauss’ Theorem, we have

$$\int_\Omega \text{div} u(x, t) \, dx = 0$$

(2.18)

We have $\varepsilon : \delta = \text{div} u$. Using (2.9), we write the mechanical equilibrium equation (2.15) as

$$\nu \text{div} u + \lambda(\text{div} u - \alpha(1 - \chi)) - \beta(\theta - \theta_c) + P(t) - \gamma_0 g x_3 = 0. $$

(2.19)

Integrating over $\Omega$ and using (2.18) we obtain

$$P(t) = \frac{\alpha \lambda}{|\Omega|} \int_\Omega (1 - \chi) \, dx' + \frac{\beta}{|\Omega|} \int_\Omega (\theta - \theta_c) \, dx' + \frac{\gamma_0 g}{|\Omega|} \int_\Omega x_3' \, dx'.$$

(2.20)

We see that in liquid ($\chi = 1$) and at temperature $\theta = \theta_c$, the pressure $p(x, t)$ vanishes on the “midsurface” of $\Omega$ given by the equation $x_3|\Omega| = \int_\Omega x_3' \, dx'$. Hence, $p(x, t)$ can be interpreted as the difference between the absolute pressure and the standard pressure. This difference is higher below and lower above the midsurface.

Eq. (2.20) enables us to eliminate $P(t)$ and rewrite (2.19) in the form

$$\nu \text{div} u + \lambda(\text{div} u - \alpha(1 - \chi)) - \beta(\theta - \theta_c) + \left(1 - \frac{1}{|\Omega|} \int_\Omega x_3' \, dx'\right)$$

$$= \gamma_0 g \left(x_3 - \frac{1}{|\Omega|} \int_\Omega x_3' \, dx'\right).$$

(2.21)
As a consequence of (2.4), the energy balance and the phase relaxation equation in (2.11)–(2.12) have the form

\[\rho_0 c_0 \partial_t \theta - \kappa \Delta \theta = \nu (\text{div } \mathbf{u}_t)^2 - \beta \theta \text{div } \mathbf{u}_t - (\alpha \lambda (\text{div } \mathbf{u} - \alpha (1 - \chi)) + \rho_0 L_0) \chi_t, \tag{2.22}\]

\[-\rho_0 \gamma_0 \chi_t \in \alpha \lambda (\text{div } \mathbf{u} - \alpha (1 - \chi)) + \rho_0 L_0 \left(1 - \frac{\theta}{\theta_c} + \partial I(\chi)\right), \tag{2.23}\]

where \(\partial\) denotes the subdifferential. For simplicity, we now set

\[c := \rho_0 c_0, \quad \gamma := \rho_0 \gamma_0, \quad L := \rho_0 L_0. \tag{2.24}\]

The system now completely decouples. For the unknown absolute temperature \(\theta\), local relative volume increment \(U = \text{div } \mathbf{u}\), and liquid fraction \(\chi\), we have the evolution system (note that mathematically, \(\partial I(\chi)\) is the same as \(L \partial I(\chi)\))

\[c \partial_t \theta - \kappa \Delta \theta = \nu U_t^2 - \beta \theta U_t - (\alpha \lambda (U - \alpha (1 - \chi)) + L) \chi_t, \tag{2.25}\]

\[\nu U_t + \lambda U = \alpha \lambda (1 - \chi) + \beta (\theta - \theta_c) + \rho_0 g \left(x_3 - \frac{1}{|\Omega|} \int_{\Omega} x_3' \, dx'\right) \tag{2.26}\]

\[-\frac{1}{|\Omega|} \int_{\Omega} (\alpha \lambda (1 - \chi) + \beta (\theta - \theta_c)) \, dx'\]

\[-\gamma \chi_t \in \alpha \lambda (U - \alpha (1 - \chi)) + L \left(1 - \frac{\theta}{\theta_c}\right) + \partial I(\chi), \tag{2.27}\]

with boundary condition (2.17), (2.14), that is,

\[\kappa \nabla \theta \cdot \mathbf{n} + h(x)(\theta - \theta_\Gamma) = 0. \tag{2.28}\]

We then find \(\mathbf{u}\) as a solution to the equation \(\text{div } \mathbf{u} = U\) in \(\Omega\), \(\mathbf{u} = 0\) on \(\partial \Omega\). It is indeed not unique, and due to our hypotheses (A3), (A4), we lose any control on possible volume preserving turbulences. This, however, has no influence on the system (2.25)–(2.27), which is the subject of our interest here.

Let us describe the set of all possible stationary states. It follows from (2.25) and (2.28) that the only temperature equilibrium is \(\theta = \theta_\Gamma\). The equilibrium values \(\chi_\infty\) and \(U_\infty\) satisfy the system

\[\lambda (U_\infty - \alpha (1 - \chi_\infty)) = -\frac{\alpha \lambda}{|\Omega|} \int_{\Omega} (1 - \chi_\infty) \, dx' + \rho_0 g \left(x_3 - \frac{1}{|\Omega|} \int_{\Omega} x_3' \, dx'\right), \tag{2.29}\]

\[-\lambda (U_\infty - \alpha (1 - \chi_\infty)) \in \frac{L}{\alpha} \left(1 - \frac{\theta_\Gamma}{\theta_c}\right) + \partial I(\chi_\infty), \tag{2.30}\]

almost everywhere in \(\Omega\), that is,

\[\frac{L}{\alpha} \left(\frac{\theta_\Gamma}{\theta_c} - 1\right) + \frac{\alpha \lambda}{|\Omega|} \int_{\Omega} (1 - \chi_\infty) \, dx' - \rho_0 g \left(x_3 - \frac{1}{|\Omega|} \int_{\Omega} x_3' \, dx'\right) \in \partial I(\chi_\infty(x)). \tag{2.31}\]
We claim that unlike in the case without gravity, (2.31) determines the equilibria uniquely. The set Ω is connected. We can therefore define \((a, b) \subset \mathbb{R}\) as the maximal interval such that \(Ω \cap (\mathbb{R}^2 \times \{x_3\}) \neq \emptyset\) for \(x_3 \in (a, b)\), and set

\[
m = \frac{1}{|Ω|} \int_Ω x_3' \, dx' \in (a, b), \quad ℓ = b - a.
\]

(2.32)

We introduce two dimensionless constants

\[
d = \frac{α^2λ}{L}, \quad G_0 = \frac{αλ}{ρ_0gℓ}.
\]

For water, we have for instance \(d \approx 0.055, G_0 \approx 2 \cdot 10^4/ℓ\). Eq. (2.31) then reads

\[
\frac{1}{d} \left( \frac{θ_r}{θ_c} - 1 \right) + \frac{1}{|Ω|} \int_Ω (1 - χ_∞) \, dx' + \frac{1}{G_0ℓ}(m - x_3) \in ∂I(χ_∞(x)).
\]

(2.33)

The quantity

\[
Z := \frac{1}{d} \left( \frac{θ_r}{θ_c} - 1 \right) + \frac{1}{|Ω|} \int_Ω (1 - χ_∞) \, dx'
\]

(2.34)

is independent of \(x\), so that the left hand side of (2.33) is positive for \(x_3 < m + G_0ℓZ\) and negative for \(x_3 > m + G_0ℓZ\). By definition of the subdifferential, we necessarily have

\[
χ_∞(x) = \begin{cases} 
1 & \text{if } x_3 < m + G_0ℓZ, \\
0 & \text{if } x_3 > m + G_0ℓZ.
\end{cases}
\]

(2.35)

Let Ω\((r)\) denote the set \(\{x \in Ω : x_3 > r\}\) for \(r \in \mathbb{R}\). Eq. (2.35) states that the set \(Ω(m + G_0ℓZ)\) corresponds to the solid domain. We have \(|Ω(r)| = 0\) for \(r \geq b\), \(|Ω(r)| = |Ω|\) for \(r \leq a\), and

\[
\frac{1}{|Ω|} \int_Ω (1 - χ_∞) \, dx' = F(Z) := \frac{|Ω(m + G_0ℓZ)|}{|Ω|}.
\]

(2.36)

We easily identify \(Z\) as the only solution to the equation

\[
Z = \frac{1}{d} \left( \frac{θ_r}{θ_c} - 1 \right) + F(Z),
\]

(2.37)

since \(F\) is nonincreasing. We see that one of the following three cases necessarily occurs:

(i) \(Z \leq (a - m)/(G_0ℓ)\), \(F(Z) = 1\). Then \(χ_∞(x) = 0\) a.e. in \(Ω\) and we have pure solid with temperatures

\[
θ_r = θ_c(1 + d(Z - 1)) \leq θ_c \left( 1 - d \left( 1 + \frac{m - a}{G_0ℓ} \right) \right);
\]

(ii) \(Z \geq (b - m)/(G_0ℓ)\), \(F(Z) = 0\). Then \(χ_∞(x) = 1\) a.e. in \(Ω\) and we have pure liquid with temperatures

\[
θ_r = θ_c(1 + dZ) \geq θ_c \left( 1 + d \frac{b - m}{G_0ℓ} \right);
\]
(iii) \( (a-m)/(G_0\ell) < Z < (b-m)/(G_0\ell) \), \( 0 < F(Z) < 1 \). Then \( \chi_\infty(x) = 0 \) a.e. in \( \Omega(m + G_0\ell Z) \), \( \chi_\infty(x) = 1 \) a.e. in \( \Omega \setminus \Omega(m + G_0\ell Z) \), and

\[
\theta_c \left( 1 - d \left( 1 + \frac{m-a}{G_0\ell} \right) \right) < \theta_T < \theta_c \left( 1 + d \frac{b-m}{G_0\ell} \right).
\]

We observe that solidification may take place at temperatures slightly above \( \theta_c \). For water in a container of \( \ell = 50 \text{ cm} \) height, the relative size of the “overheated ice temperature domain” is smaller than \( d/G_0 \approx 1.4 \cdot 10^{-6} \), hence it is far beyond the standard measurement accuracy. The overheating is due to the fact the pressure decreases from the bottom to the top, as pointed out after formula (2.20). A quantitative characterization of this phenomenon is given by the so-called Clausius-Clapeyron equation, which relates the freezing temperature with the pressure. It can be derived here as follows. The equilibrium relative pressure \( p_\infty \) depends only on \( x_3 \), and is given, by virtue of (2.15) and (2.20), by the formula

\[
p_\infty(x_3) = \frac{\alpha \lambda}{|\Omega|} \int_\Omega (1 - \chi_\infty) dx' + \beta (\theta_T - \theta_c) + \frac{\rho g}{|\Omega|} \int_\Omega x_3 dx' - \rho g x_3. \tag{2.38}
\]

Using (2.31), we obtain

\[
\frac{L}{\alpha} \left( \frac{\theta_T}{\theta_c} - 1 \right) - \beta (\theta_T - \theta_c) + p_\infty(x_3) \in \partial I(\chi_\infty(x)). \tag{2.39}
\]

The phase interface at temperature \( \theta_T \) is located at level \( x_3 \) if the right hand side of (2.39) vanishes. Setting \( L_\beta = L_0 - \alpha \theta_c / \rho_0 \), we thus obtain the Clausius-Clapeyron condition for phase transition in the form of \([10, \text{Equation (288)}]\), namely

\[
\frac{p_\infty(x_3)}{\theta_T - \theta_c} = -\frac{\rho_0 L_\beta}{\alpha \theta_c} = \frac{L_\beta}{(V_{\text{liquid}} - V_{\text{solid}})\theta_c}. \tag{2.40}
\]

In terms of the new variables \( \theta, U, \chi \), the energy \( e \) and entropy \( s \) can be written as

\[
e = c_0 \theta + \frac{\lambda}{2 \rho_0} (U - \alpha (1 - \chi))^2 + \frac{\beta}{\rho_0} \theta_c U + L_0 (\chi + I(\chi)), \tag{2.41}
\]

\[
s = c_0 \log \left( \frac{\theta}{\theta_c} \right) + \frac{L_0}{\theta_c} \chi + \frac{\beta}{\rho_0} U, \tag{2.42}
\]

and the energy and entropy balance equations now read

\[
\frac{d}{dt} \int_\Omega (\rho_0 e(x,t) - \rho_0 g x_3 U(x,t)) dx = \int_\Omega h(x)(\theta_T - \theta) ds(x), \tag{2.43}
\]

\[
\rho_0 s_t + \text{div} \left( \frac{\partial}{\partial \theta} \right) = \frac{\kappa |\nabla \theta|^2}{\theta^2} + \frac{\gamma}{\theta} \dot{\chi}^2 + \frac{\nu}{\theta} U_t^2 \geq 0, \tag{2.44}
\]

\[
\frac{d}{dt} \int_\Omega \rho_0 s(x,t) dx = \int_\Omega \frac{h(x)}{\theta} (\theta_T - \theta) ds(x) \tag{2.45}
\]

\[
+ \int_\Omega \left( \frac{\kappa |\nabla \theta|^2}{\theta^2} + \frac{\gamma}{\theta} \dot{\chi}^2 + \frac{\nu}{\theta} U_t^2 \right) dx.
\]
The entropy balance (2.44) says that the entropy production on the right hand side is nonnegative in agreement with the second principle of thermodynamics. The system is not closed, and the energy supply through the boundary is given by the right hand side of (2.43).

We prescribe the initial conditions

\[
\theta(x,0) = \theta^0(x) \quad (2.46)
\]
\[
U(x,0) = U^0(x) \quad (2.47)
\]
\[
\chi(x,0) = \chi^0(x) \quad (2.48)
\]

for \(x \in \Omega\), such that \(\int_\Omega U^0(x) \, dx = 0\), and compute from (2.41)–(2.42) the corresponding initial values \(e^0\) and \(s^0\) for specific energy and entropy, respectively. Let \(E^0 = \int_\Omega \varrho e^0 \, dx\), \(S^0 = \int_\Omega \varrho s^0 \, dx\) denote the total initial energy and entropy, respectively. From the energy end entropy balance equations (2.43), (2.45) and using the condition \(\int_\Omega U \, dx = 0\), we derive the following crucial (formal for the moment) balance equation for the “extended” energy \(\varrho_0 (e - \theta \Gamma s)\):

\[
\int_\Omega \left( c\theta + \frac{\lambda}{2} (U - \alpha (1 - \chi))^2 + L \chi - \varrho_0 g x_3 U \right) (x,t) \, dx \\
+ \theta_\Gamma \int_0^t \int_\Omega \left( \frac{\kappa \left| \nabla \theta \right|^2}{\theta^2} + \frac{\gamma \chi^2}{\theta} + \frac{\nu U^2}{\theta} \right) (x,\tau) \, dx \, d\tau \\
+ \int_0^t \int_{\partial \Omega} h(x) (\theta_\Gamma - \theta)^2 (x,\tau) \, ds(x) \, d\tau
\]

\[
= E^0 - \theta_\Gamma S^0 - \varrho_0 g \int_\Omega x_3 U^0(x) \, dx + \theta_\Gamma \int_\Omega \left( c \log \left( \frac{\theta}{\theta_c} \right) + \frac{L}{\theta_c} \chi \right) (x,t) \, dx.
\]

We have \(\log (\theta/\theta_c) = \log (\theta/2\theta_\Gamma) - \log (\theta_c/2\theta_\Gamma) \leq (\theta/2\theta_\Gamma) - 1 - \log (\theta_c/2\theta_\Gamma)\), hence there exists a constant \(C > 0\) independent of \(t\) such that for all \(t > 0\) we have

\[
\int_\Omega (\theta + U^2) (x,t) \, dx + \int_0^t \int_\Omega \left( \frac{\left| \nabla \theta \right|^2}{\theta^2} + \frac{\chi^2}{\theta^2} + \frac{U^2}{\theta^2} \right) (x,\tau) \, dx \, d\tau \\
+ \int_0^t \int_{\partial \Omega} \frac{h(x)}{\theta} (\theta_\Gamma - \theta)^2 (x,\tau) \, ds(x) \, d\tau \leq C.
\]

3 Existence and uniqueness of solutions

We construct the solution of (2.26)–(2.27) by the Banach contraction argument. The method of proof is independent of the actual values of the material constants, and we choose for simplicity

\[
L = 2, \quad g = c = \varrho_0 = \theta_c = \alpha = \beta = \gamma = \kappa = \lambda = \nu = 1.
\]
System (2.25)–(2.27) with boundary condition (2.17) then reads

\[ \int_{\Omega} \theta_t w(x) \, dx + \int_{\Omega} \nabla \theta \cdot \nabla w(x) \, dx = \int_{\Omega} \left( U^2_t - \theta U_t - (U + \chi + 1) \chi_t \right) w(x) \, dx \tag{3.2} \]

\[- \int_{\partial \Omega} h(x)(\theta - \theta_{\Gamma}) w(x) \, ds(x), \]

\[ U_t + U = -\chi + \theta + \left( x_3 - \frac{1}{|\Omega|} \int_{\Omega} x'_3 \, dx' \right) \tag{3.3} \]

\[ \chi_t + U + \chi + \partial I(\chi) \ni 2\theta - 1, \tag{3.4} \]

where (3.2) is to be satisfied for all test functions \( w \in W^{1,2}(\Omega) \) and a.e. \( t > 0 \), while (3.3)–(3.4) are supposed to hold a.e. in \( \Omega_\infty := \Omega \times (0, \infty) \).

In this section we prove the following existence and uniqueness result.

**Theorem 3.1** Let \( 0 < \theta_* \leq \theta_{\Gamma} \leq \theta^* \) and \( p_0 \in \mathbb{R} \) be given constants, and let the data satisfy the conditions

\[ \theta^0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega), \quad \theta_0(x) \leq \theta^0(x) \leq \theta^*, \quad \text{a.e.}, \]

\[ U^0, \chi^0 \in L^\infty(\Omega), \quad \int_{\Omega} U^0(x) \, dx = 0, \quad 0 \leq \chi^0(x) \leq 1 \quad \text{a.e.} \]

Then there exists a unique solution \((\theta, U, \chi)\) to (3.2)–(3.4), (2.46)–(2.48), such that \( \theta > 0 \) a.e., \( \chi \in [0, 1] \) a.e., \( U, U_t, \chi_t, 1/\theta \in L^\infty(\Omega_\infty), \theta_t, \Delta \theta \in L^2(\Omega_\infty), \) and \( \nabla \theta \in L^\infty(0, T; L^2(\Omega)) \cap L^2(\Omega_\infty) \).

**Remark 3.2** For existence and uniqueness alone, we might allow the external temperature \( \theta_{\Gamma} \) to depend on \( x \) and \( t \), and assume only that it belongs to the space \( W^{1,2}_{\text{loc}}(0, \infty; L^2(\partial \Omega)) \cap L^\infty_{\text{loc}}(\partial \Omega \times (0, \infty)) \). For the global bounds, the assumption that \( \theta_{\Gamma} \) be constant plays a substantial role.

The proof of Theorem 3.1 will be carried out in the following subsections. Notice first that the term \( U^2_t - \theta U_t - (U + \chi + 1) \chi_t \) on the right hand side of (3.2) can be rewritten alternatively, using (3.4) and (3.3), as

\[ U^2_t - \theta U_t - (U + \chi + 1) \chi_t = U^2_t - \theta U_t + \chi^2_t - 2\theta \chi_t \tag{3.5} \]

\[ = -(\chi + U) U_t + \left( x_3 - \frac{1}{|\Omega|} \int_{\Omega} x'_3 \, dx' \right) U_t \]

\[- \left( \frac{1}{|\Omega|} \int_{\Omega} (- \chi + \theta) \, dx' \right) U_t - (U + \chi + 1) \chi_t, \]
We now fix some constant $R > 0$ and construct the solution for the truncated system

\[
\int_{\Omega} \theta_t w(x) \, dx + \int_{\Omega} \nabla \theta \cdot \nabla w(x) \, dx = \int_{\Omega} (U_t^2 + \chi_t^2 - Q_R(\theta)(U_t + 2\chi_t)) w(x) \, dx + \int_{\partial \Omega} h(x)(\theta - \theta_T)w(x) \, ds(x) \quad \forall w \in W^{1,2}(\Omega),
\]

\[
U_t + U = -\chi + Q_R(\theta) + \left( x_3 - \frac{1}{|\Omega|} \int_{\Omega} x_3' \, dx' \right) - \frac{1}{|\Omega|} \int_{\Omega} (-\chi + Q_R(\theta)) \, dx',
\]

\[
\chi_t + U + \chi + \partial I(\chi) \geq 2Q_R(\theta) - 1
\]

first in a bounded domain $\Omega_T := \Omega \times (0, T)$ for any given $T > 0$, where $Q_R$ is the cutoff function $Q_R(z) = \min\{z^+, R\}$. We then derive upper and lower bounds for $\theta$ independent of $R$ and $T$, so that the local solution of (3.6)–(3.8) is also a global solution of (3.2)–(3.4) if $R$ is sufficiently large.

### 3.1 A gradient flow

System (2.26)–(2.27) can be considered as a gradient flow similarly as in the case without gravity and with an elastic boundary, see [7, Section 4.1]. Set

\[
v = \begin{pmatrix} U \\ \chi \end{pmatrix},
\]

\[
\psi(v) = \int_{\Omega} \left( \frac{1}{2}(U - (1 - \chi))^2 + 2\chi(1 - \theta_T) - x_3U + I(\chi) \right) \, dx + \frac{1}{|\Omega|} \int_{\Omega} U \, dx \int_{\Omega} (1 - \chi + x_3) \, dx + C_\psi,
\]

\[
f = \begin{pmatrix} (\theta - \theta_T) - 1/|\Omega| \int_{\Omega}(\theta - \theta_T) \, dx \\ 2(\theta - \theta_T) \end{pmatrix}
\]

with a constant $C_\psi \geq 0$ to ensure that $\psi(v) \geq 0$. Every solution $U$ of (2.26) necessarily satisfies the condition $\int_{\Omega} U \, dx = 0$, hence (2.27) can be written in the form

\[
\chi_t + U + \chi + 1 - 2\theta - \frac{1}{|\Omega|} \int_{\Omega} U \, dx + \partial I(\chi) \geq 0,
\]

and (2.26), (3.12) are in turn equivalent to the gradient flow

\[
\dot{v} + \partial \psi(v) \geq f
\]

in $L^2(\Omega) \times L^2(\Omega)$. We have used the obvious identity $\theta - 1 - 1/|\Omega| \int_{\Omega}(\theta - 1) \, dx = (\theta - \theta_T) - 1/|\Omega| \int_{\Omega}(\theta - \theta_T) \, dx$. We state here the following Lemma, whose proof can be found in [7, Lemma 4.3].
Lemma 3.3 Let \( f, \dot{f} \) belong to \( L^2(0, \infty; H) \). Then \( \lim_{t \to \infty} \dot{v}(t) = 0 \).

We apply the above result to the case \( H = L^2(\Omega) \times L^2(\Omega) \), and \( v, f, \psi \) as above and we see that Eqs. (3.1) with \( \theta \) replaced by \( \hat{\theta} \) can be equivalently written as a gradient flow (3.9)–(3.13). For its solutions, we prove the following result.

Proposition 3.4 Let the hypotheses of Theorem 3.1 hold, and let a function \( \hat{\theta} \in L^2_{\text{loc}}(0, \infty; L^2(\Omega)) \) be given. Let \( (U, \chi) \) be the solution of (3.9)–(3.13). Then there exists a constant \( C_0 \) independent of \( x, t \) and \( R \), such that a.e. in \( \Omega \infty \) we have

\[
|U(x, t)| + |U_t(x, t)| + |\chi(x, t)| \leq C_0(1 + R). \tag{3.14}
\]

Let furthermore \( \hat{\theta}_1, \hat{\theta}_2 \in L^2_{\text{loc}}(0, \infty; L^2(\Omega)) \) be two functions, and let \( (U_1, \chi_1), (U_2, \chi_2) \) be the corresponding solutions of (3.9)–(3.13). Then the differences \( \hat{\theta}_d = \hat{\theta}_1 - \hat{\theta}_2 \), \( U_d = U_1 - U_2 \), \( \chi_d = \chi_1 - \chi_2 \) satisfy for every \( t \geq 0 \) and a.e. \( x \in \Omega \) the inequality

\[
\int_0^t (|(U_d)_t| + |(\chi_d)_t|)(x, \tau) \, d\tau \leq C_0(1 + t) \int_0^t \left( |\hat{\theta}_d(x, \tau)| + t|\hat{\theta}_d|_2 \right) \, d\tau, \tag{3.15}
\]

where the symbol \( | \cdot |_2 \) stands for the norm in \( L^2(\Omega) \).

In what follows, we denote by \( C_1, C_2, \ldots \) any constant independent of \( x, t \) and \( R \).

Proof. We rewrite (3.9)–(3.13) as two scalar gradient flows

\[
U_t + \partial \psi_1(U) = a, \tag{3.16}
\]
\[
\chi_t + \partial \psi_2(\chi) \geq b, \tag{3.17}
\]
where \( \psi_1(U) = \frac{1}{2}U^2 \), \( \psi_2 = \frac{1}{2} \chi^2 + I(\chi) \), \( a = Q_R(\hat{\theta}) - \chi - \frac{1}{|\Omega|} \int_\Omega (Q_R(\hat{\theta}) - \chi) \, dx + \left(x_3 - \frac{1}{|\Omega|} \int_\Omega x_3 \, dx'\right) \), \( b = 2Q_R(\hat{\theta}) - 1 - U \). The bounds (3.14) are obvious. To prove (3.15), we consider two different inputs. As above, we denote the differences \( \{\}_1 - \{\}_2 \) by \( \{\} d \) for all symbols \( \{\} \). By [6, Theorem 1.12], we have for all \( t > 0 \) and a.e. \( x \in \Omega \) that

\[
\int_0^t (|(U_d)_t| + |(\chi_d)_t|)(x, \tau) \, d\tau \leq 2 \int_0^t (|a_d| + |b_d|)(x, \tau) \, d\tau. \tag{3.18}
\]

We multiply the difference of (3.16) by \( U_d \), the difference of (3.17) by \( \chi_d \), and sum them up to obtain that

\[
(U_d)_t U_d + (\chi_d)_t \chi_d + (U_d + \chi_d)^2 \leq |\hat{\theta}_d|(|U_d| + 2|\chi_d|) - \left( \frac{1}{|\Omega|} \int_\Omega (Q_R(\hat{\theta})_d - \chi_d) \, dx \right) U_d \quad \text{a.e.} \tag{3.19}
\]

We first integrate (3.19) over \( \Omega \). Using the symbol \( | \cdot |_2 \) for the norm in \( L^2(\Omega) \), we get for a.e. \( t > 0 \) that

\[
\frac{1}{2} \frac{d}{dt} \left(|U_d|^2 + |\chi_d|^2\right) \leq |\hat{\theta}_d|_2 (|U_d|_2 + 2|\chi_d|_2) \leq \sqrt{5}|\hat{\theta}_d|_2 \left(|U_d|_2^2 + |\chi_d|_2^2\right)^{1/2}. \tag{3.20}
\]
Hence, \( \frac{4}{\pi} (|U_d|^2 + |\chi_d|^2)^{1/2} \leq \sqrt{5} |\hat{\theta}_d|_2 \) a.e., and integrating over \( t \), we find that
\[
(|U_d|^2 + |\chi_d|^2)^{1/2} (t) \leq \sqrt{5} \int_0^t |\hat{\theta}_d(\tau)|_2 \, d\tau.
\] (3.21)

Using again [3.19], we find for a.e. \((x, t) \in \Omega_\infty\) the inequality
\[
\frac{1}{2} \frac{\partial}{\partial t} (U_d^2 + \chi_d^2) (x, t) \leq |\hat{\theta}_d|(x, t)(|U_d| + 2|\chi_d|)(x, t) + C_1|U_d|(x, t) \left( |\hat{\theta}_d(t)|_2 + \int_0^t |\hat{\theta}_d(\tau)|_2 \, d\tau \right).
\] (3.22)

Hence, we get
\[
\frac{1}{2} \frac{\partial}{\partial t} (U_d^2 + \chi_d^2) (x, t) \leq C_2(U_d^2 + \chi_d^2)^{1/2}(x, t) \left( |\hat{\theta}_d|(x, t) + |\hat{\theta}_d(t)|_2 + \int_0^t |\hat{\theta}_d(\tau)|_2 \, d\tau \right),
\] (3.23)

which in turn implies that
\[
(|U_d|^2 + |\chi_d|^2)^{1/2} (x, t) \leq C_3 \left( \int_0^t |\hat{\theta}_d(x, \tau)| \, d\tau + (1 + t) \int_0^t |\hat{\theta}_d(\tau)|_2 \, d\tau \right) \text{ a.e.} (3.24)
\]

This enables us to estimate the right hand side of (3.18) and obtain the bound
\[
\int_0^t (|U_d| + |\chi_d|)(x, \tau) \, d\tau \leq C_4 \int_0^t \left( (1 + t)|\hat{\theta}_d|(x, \tau) + t(1 + t)|\hat{\theta}_d(\tau)|_2 \right) \, d\tau (3.25)
\]

for a.e. \( x \in \Omega \) and all \( t \geq 0 \). This completes the proof. \( \square \)

3.2 Existence of solutions for the truncated problem

We construct the solution of (3.6)–(3.8) for every \( R > 0 \) by the Banach contraction argument on a fixed time interval \((0, T)\).

Lemma 3.5 Let the hypotheses of Theorem 3.1 hold, and let \( T > 0 \) and \( R > 0 \) be given. Then there exists a unique solution \( (\theta, U, \chi) \) to (3.4)–(3.8), (2.40)–(2.48), such that \( U \in W^{1,\infty}(\Omega_T), \theta > 0 \) a.e., \( \chi, \tilde{\theta}, \tilde{\theta}/\theta \in L^{\infty}(\Omega_T), \tilde{\theta}, \Delta \theta \in L^2(\Omega_T), \) and \( \nabla \theta \in L^\infty(0, T; L^2(\Omega)) \).

Proof. Let \( \hat{\theta} \in L^2(\Omega_T) \) be a given function, and consider the system
\[
\int_{\Omega} \theta_1 w(x) \, dx + \int_{\Omega} \nabla \theta \cdot \nabla w(x) \, dx = \int_{\Omega} \left( U_t^2 + \chi_t^2 - Q_R(\hat{\theta})(U_t + 2\chi_t) \right) w(x) \, dx (3.26)
\]
\[
- \int_{\partial \Omega} h(x)(\vartheta - \theta_T)w(x) \, ds(x) \quad \forall w \in W^{1,2}(\Omega),
\]
\[
U_t + U = -\chi + Q_R(\hat{\theta}) + \left( x_3 - \frac{1}{|\Omega|} \int_{\Omega} x'_3 \, dx' \right) (3.27)
\]
\[
- \frac{1}{|\Omega|} \int_{\Omega} (\varphi + Q_R(\hat{\theta})) \, dx',
\]
\[
\chi_t + U + \chi + \partial I(\chi) \geq 2Q_R(\hat{\theta}) - 1 \quad (3.28)
\]
Equations (3.27)–(3.28) are solved as a gradient flow problem from Subsection 3.1 while (3.26) is a simple linear parabolic equation for \( \theta \). Testing (3.26) by \( \theta_t \), we obtain by Proposition 3.4 that
\[
\int_0^T \int_\Omega \theta_t^2 \, dx \, dt + \sup_{t \in (0,T)} \left( \int_\Omega |\nabla \theta|^2 \, dx + \int_{\partial \Omega} h(x)(\theta - \theta_T)^2 \, ds(x) \right) \leq T|\Omega| \left( C_0(1 + R)(2C_0(1 + R) + 3R) \right)^2 =: M_R.
\]
Hence, we can define the mapping that with \( \hat{\theta} \) associates the solution \( \theta \) of (3.26)–(3.28) with initial conditions (2.46)–(2.48). We now show that it is a contraction on the set
\[
\Xi_{T,R} := \{ \hat{\theta} \in L^2(\Omega_T) : \text{conditions (3.31)–(3.34) hold} \},
\]
where
\[
\hat{\theta}_t \in L^2(\Omega_T); \quad \nabla \hat{\theta} \in L^\infty(0,T; L^2(\Omega));
\]
\[
\int_0^T \int_\Omega \hat{\theta}_t^2 \, dx \, dt + \sup_{t \in (0,T)} \left( \int_\Omega |\nabla \hat{\theta}|^2 \, dx + \int_{\partial \Omega} h(x)(\hat{\theta} - \theta_T)^2 \, ds(x) \right) \leq M_R;
\]
\[
\hat{\theta}(x, 0) = \theta^0(x) \text{ a.e. (3.34)}.
\]
Let \( \hat{\theta}_1, \hat{\theta}_2 \) be two functions in \( \Xi_{T,R} \), and let \( (\theta_1, U_1, \chi_1), (\theta_2, U_2, \chi_2) \), be the corresponding solutions to (3.26)–(3.28) with the same initial conditions \( \theta^0, U^0, \chi^0 \). We see from (3.29) that \( \theta_1, \theta_2 \) belong to \( \Xi_{T,R} \). Integrating Eq. (3.26) for \( \theta_1 \) and \( \theta_2 \) with respect to time and testing their difference by \( w = \theta_d := \theta_1 - \theta_2 \), we obtain, using Proposition 3.4 that
\[
\int_\Omega \theta_d^2(x, t) \, dx + \frac{d}{dt} \left( \int_\Omega |\nabla \int_0^t \theta_d(x, \tau) \, d\tau|^2 \, dx + \int_{\partial \Omega} h(x) \left| \int_0^t \theta_d(x, \tau) \, d\tau \right|^2 \, ds(x) \right)
\leq C_5(1 + R) \int_\Omega \left( \int_0^t \left( |(U_d)_t| + |(\chi_d)_t| + |\theta_d| \right) (x, \tau) \, d\tau \right) \theta_d(x, t) \, dx \text{ a.e. (3.35)}.
\]
From (3.15) and Minkowski’s inequality, it follows that
\[
\left| \int_0^t \left( |(U_d)_t| + |(\chi_d)_t| \right) (x, \tau) \, d\tau \right|_2 \leq C_6(1 + t)^2 \int_0^t |\theta_d(\tau)|_2 \, d\tau
\leq C_6(1 + t)^2 \left( t \int_0^t |\theta_d(\tau)|^2_2 \, d\tau \right)^{1/2}.
\]
By Young’s inequality, we rewrite (3.35) as
\[
\int_\Omega \theta_d^2(x, t) \, dx + \frac{d}{dt} \left( \int_\Omega |\nabla \int_0^t \theta_d(x, \tau) \, d\tau|^2 \, dx + \int_{\partial \Omega} h(x) \left| \int_0^t \theta_d(x, \tau) \, d\tau \right|^2 \, ds(x) \right)
\leq C_7(1 + R^2)(1 + t)^5 \int_0^t |\theta_d(\tau)|^2_2 \, d\tau \text{ a.e. (3.36)}.
\]
Set $\Theta^2(t) = \int_0^t |\theta_d(\tau)|^2 \, d\tau$, $\hat{\Theta}^2(t) = \int_0^t |\hat{\theta}_d(\tau)|^2 \, d\tau$. Integrating (3.36) with respect to time, we obtain

$$\Theta^2(t) \leq C_T (1 + R^2) \int_0^t (1 + \tau)^5 \hat{\Theta}^2(\tau) \, d\tau. \quad (3.37)$$

We set $C_R := (C_T (1 + R^2)/6)$ and introduce in $L^\infty(0,T)$ the norm

$$||w||_C := \sup_{\tau \in [0,T]} e^{-C_R (1+\tau)^6} |w(\tau)|.$$  

Then $||\Theta||_C^2 \leq \frac{1}{2} ||\hat{\Theta}||_C^2$, and hence the mapping $\hat{\theta} \mapsto \theta$ is a contraction in $L^2(\Omega_T)$ with respect to the norm induced by $\| \cdot \|_C$. The set $\Xi_{T,R}$ is a closed subset of $L^2(\Omega_T)$. This implies the existence of a fixed point $\theta \in \Xi_{T,R}$, which is indeed a solution to (3.6)–(3.8). The positive upper and lower bounds for $\theta$ follow from the maximum principle (for the proof cf. [7, Section 4.2]). This completes the proof of Lemma 3.5.

### 3.3 Proof of Theorem 3.1

The unique solution $(\theta, U, \chi)$ to (3.6)–(3.8), (2.46)–(2.48) exists globally in the whole domain $\Omega_\infty$. We now derive uniform bounds independent of $t$ and $R$. Take first for instance any $R > 2\theta^*$. We know that the solution component $\theta$ of (3.6)–(3.8) remains smaller than $R$ in a nondegenerate interval $(0,T)$ with $T > \theta^*/(C_6(1+R)^2)$. Let $(0,T_0)$ be the maximal interval in which $\theta$ is bounded by $R$. Then, in $(0,T_0)$, the solution given by Lemma 3.5 is also a solution of the original problem (3.2)–(3.4). Moreover, due to estimate (2.50), we know that $\theta$ admits a bound in $L^\infty(0,T_0; L^1(\Omega))$ independent of $R$. In order to prove that $T_0 = +\infty$ if $R$ is sufficiently large, we need the following variant of the Moser iteration lemma, whose proof can be found in [7, Prop. 4.6].

**Proposition 3.6** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitzian boundary. Given nonnegative functions $h \in L^1(\partial\Omega)$ and $r \in L^\infty(0,\infty; L^q(\Omega))$ with a fixed $q > N/2$, $|r|_{L^\infty(0,\infty; L^q(\Omega))} =: r^*$, an initial condition $v^0 \in L^\infty(\Omega)$, and a boundary datum $v_T \in L^\infty(\partial\Omega \times (0,\infty))$, consider the problem

$$v_t - \Delta v + v = r(x,t)\mathcal{H}[v] \quad \text{a.e. in } \Omega \times (0,\infty), \quad (3.38)$$

$$\nabla v \cdot n = -h(x) (f(x,t,v(x,t)) - v_T(x,t)) \quad \text{a.e. on } \partial\Omega \times (0,\infty), \quad (3.39)$$

$$v(x,0) = v^0 \quad \text{a.e. in } \Omega, \quad (3.40)$$

under the assumption that there exist positive constants $m, H_0, C_f, V, V_T, E_0$ such that the following holds:

1. The mapping $\mathcal{H} : L^\infty_{\text{loc}}(\Omega \times (0,\infty)) \to L^\infty_{\text{loc}}(\Omega \times (0,\infty))$ satisfies for every $v \in L^\infty_{\text{loc}}(\Omega \times (0,\infty))$ and a.e. $(x,t) \in \Omega \times (0,\infty)$ the inequality

$$v(x,t)\mathcal{H}[v](x,t) \leq H_0 |v(x,t)| \left(1 + |v(x,t)| + \int_0^t \xi(t^{-}\tau) |v(x,\tau)| \, d\tau \right),$$

where $H_0$ and $\xi$ are constants.
where $\xi \in W^{1,1}(0, \infty)$ is a given nonnegative function such that 
\[ \dot{\xi}(t) \leq -\xi(0) \xi(t) \quad \text{a.e.} \] (3.41)

(ii) $f$ is a Carathéodory function on $\Omega \times (0, \infty) \times \mathbb{R}$ such that $f(x, t, v) v \geq C_f v^2$ a.e. for all $v \in \mathbb{R}$.

(iii) $|v^0(x)| \leq V$ a.e. in $\Omega$.

(iv) $|v_T(x, t)| \leq V_T$ a.e. on $\partial \Omega \times (0, \infty)$.

(v) System (3.38)–(3.40) admits a solution $v \in W^{1,2}_{\text{loc}}(0, \infty; (W^{1,2})'(\Omega)) \cap L^2_{\text{loc}}(0, \infty; W^{1,2}(\Omega)) \cap L^\infty(\Omega \times (0, \infty))$ satisfying the estimate 
\[ \int_\Omega |v(x, t)| \, dx \leq E_0 \quad \text{a.e. in } (0, \infty). \]

Then there exists a positive constant $C^*$ depending only on $|h|_{L^1(\partial \Omega)}$, $C_f$, $H_0$ such that 
\[ |v(t)|_{L^\infty(\Omega)} \leq C^* \max \{1, V, V_T, E_0\} \quad \text{for a.e. } t > 0. \] (3.42)

We now finish the proof of Theorem 3.1 by showing that $T_0$ introduced at the beginning of this subsection is $+\infty$ if $R$ is sufficiently large. Using (3.3), we obtain that 
\[ |U(x, t)| \leq C_8 \left( 1 + \int_0^t e^{r-t} \theta(x, \tau) \, d\tau \right) \quad \text{a.e.}, \] (3.43)
\[ |U_t(x, t)| \leq C_9 \left( 1 + \theta(x, t) + \int_0^t e^{r-t} \theta(x, \tau) \, d\tau \right) \quad \text{a.e.}, \] (3.44)

hence also (cf. (3.4)) 
\[ |\chi_t(x, t)| \leq C_{10} \left( 1 + \theta(x, t) + \int_0^t e^{r-t} \theta(x, \tau) \, d\tau \right) \quad \text{a.e.} \] (3.45)

By (2.50), the function $U$ is in $L^\infty(0, \infty; L^2(\Omega))$ and the bound does not depend on $R$. Eq. (3.2), with $\theta$ added to both the left and the right hand side, thus satisfies the hypotheses of Proposition 3.6 for $N = 3$ and $q = 2$. This enables us to conclude that $\theta(x, t)$ is uniformly bounded from above by a constant, independently of $R$, so that $\theta$ never reaches the value $R$ if $R$ is sufficiently large, which we wanted to prove. By (3.43)–(3.45), also $U$, $U_t$, and $\chi_t$ are uniformly bounded by a constant.

We proceed similarly to prove a uniform positive lower bound for $\theta$. Set $R_0 := \sup \theta$, and in Eq. (3.6) with $R > R_0$ put $w = -\tilde{\omega}/\theta$, $\tilde{\omega} \in W^{1,2}(\Omega)$. For a new (nonnegative) variable $v(x, t) := \log R_0 - \log \theta(x, t)$ we obtain the equation 
\[ \int_\Omega v_t \tilde{\omega}(x) \, dx + \int_\Omega \nabla v \cdot \nabla \tilde{\omega}(x) \, dx + \int_{\partial \Omega} h(x) \left( \frac{\theta_T}{\theta} - 1 \right) \tilde{\omega}(x) \, ds(x) \] (3.46)
\[ = \int_\Omega \left( -\frac{U_t^2}{\theta} + \frac{\chi_t^2}{\theta} - \frac{|
abla \theta|^2}{\theta^2} + U_t + 2\chi_t \right) \tilde{\omega}(x) \, dx. \]
We now set
\[ H[v] = \text{sign}(v) \left( \frac{-U_t^2 + \chi_t^2}{\theta} - \frac{\nabla \theta^2}{\theta^2} + U_t + 2\chi_t \right) \]
and check that the hypotheses of Proposition 3.6 are satisfied with \( f(v) = (\theta_T/R_0)(e^v - 1) \), \( v_T = (R_0 - \theta_T)/R_0 \), \( r \equiv 1 \), and \( vH[v] \leq 2C|v| \), where \( C \) is a common upper bound for \( U_t \) and \( \chi_t \). Hence, \( v \) is bounded above by some \( v^* \), which entails \( \theta \geq R_0 e^{-v^*} \). This concludes the proof of Theorem 3.1.

4 Long time behavior

We have the following statement.

**Proposition 4.1** Let the hypotheses of Theorem 3.1 hold. Then we have
\[
\int_0^\infty \left( \int_\Omega \left( \theta_t^2 + U_t^2 + \chi_t^2 + \frac{\nabla \theta^2}{\theta^2} \right) \, dx + \int_{\partial \Omega} h(x)(\theta - \theta_T)^2 \, ds(x) \right) \, dt < \infty, \quad (4.1)
\]
\[
\lim_{t \to \infty} \left( \int_\Omega \left( U_t^2 + \chi_t^2 + \frac{\nabla \theta^2}{\theta^2} \right) (x,t) \, dx + \int_{\partial \Omega} h(x)(\theta - \theta_T)^2(x,t) \, ds(x) \right) = 0. \quad (4.2)
\]
Furthermore, letting \( t \) tend to \( \infty \), the temperature \( \theta \) converges strongly in \( W^{1,2}(\Omega) \) to its equilibrium value \( \theta_T \), and also both \( \chi(x,t) \) and \( U(x,t) \) converge strongly in \( L^1(\Omega) \) (hence, strongly in every \( L^p(\Omega) \) for \( p < \infty \)) to their respective unique equilibrium values \( \chi_\infty \) and \( U_\infty \) defined in Section 2.

**Proof.** The proof of the relations (4.1)–(4.2) exactly follows the argument of [7, Proposition 5.1]. For the convergence of the whole trajectory, it seems easier to prove it directly without referring to the general theory of dynamical systems with a unique equilibrium (see, e.g., [2, Lemma 2.4]).

We eliminate from (2.26)–(2.27) the term \( U - \alpha(1 - \chi) \) and obtain, using the notation (2.32)–(2.34), the inclusion
\[
A(x,t) + \frac{\alpha \lambda}{|\Omega|} \int_\Omega (\chi_\infty - \chi) \, dx' + g_0 g(m + G_0 \ell Z - x_3) \in \partial I(\chi), \quad (4.3)
\]
where we set
\[
A(x,t) := \nu U_t - \frac{\gamma}{\alpha} \chi_t + \left( \frac{L}{\alpha \theta c} - \beta \right)(\theta - \theta_T) + \frac{\beta}{|\Omega|} \int_\Omega (\theta - \theta_T) \, dx'.
\]
From (4.3) it follows that
\[
\frac{\alpha \lambda}{|\Omega|} (\chi_\infty - \chi) \int_\Omega (\chi_\infty - \chi) \, dx' + g_0 g(\chi_\infty - \chi)(m + G_0 \ell Z - x_3) \leq (\chi_\infty - \chi) A(x,t) \quad (4.4)
\]
for a.e. \((x, t) \in \Omega_\infty\). Integrating (4.4) over \(\Omega\) and using (2.35) we obtain

\[
\alpha \frac{\lambda}{|\Omega|} \left| \int_{\Omega} (\chi_\infty - \chi) \, dx' \right|^2 + g_0 g \int_{\Omega} |\chi_\infty - \chi| \, |m + G_0 \ell Z - x_3| \, dx' \leq \int_{\Omega} |\chi_\infty - \chi| |A(x', t)| \, dx'.
\]

(4.5)

By virtue of (4.2), we have \(\lim_{t \to \infty} \int_{\Omega} |A(x', t)| \, dx' = 0\). Hence, the right hand side of (4.5) converges to 0 as \(t \to \infty\). Since \(\chi\) is a priori bounded, the weighted \(L^1\) convergence of \(\chi_\infty - \chi\) with weight \(|m + G_0 \ell Z - x_3|\) implies that \(\chi \to \chi_\infty\) strongly in \(L^1(\Omega)\) as \(t \to \infty\). The convergence \(U \to U_\infty\) follows directly from (2.26), (2.29), and (4.2).

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