Properties of cosh-weighted Finite Hilbert Transform

Jiangsheng You

Cubic Imaging LLC, Andover, MA 01810
jshyou@gmail.com

Abstract

Several identities of the cosh-weighted finite Hilbert Transform and the Bertola-Katsevich-Tovbis inversion formulas are rederived by the Sokhotski-Plemelj formula and the Poincaré-Bertrand formula. The explicit formulas are derived for the cosh-weighted Hilbert transform of the exponential Chebyshev functions. Numerical experiments are performed to study the computational effects of the inversion formulas.

I. Mathematical preliminary

The finite Hilbert transform (FHT) over the open unit interval \( I = (-1, 1) \) is defined by

\[
F(s) = Hf(s) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(t)}{s-t} dt(s).
\]

(1.1)

For a complex constant \( \mu \in C \), the cosh-weighted FHT is defined as

\[
F_\mu(s) = H_\mu f(s) = \frac{1}{\pi} \int_{-1}^{1} \frac{\cosh[\mu(s-t)]}{s-t} f(t) dt.
\]

(1.2)

Throughout this paper we concern \( F(s) \) and \( F_\mu(s) \) in \( I \) though they can be defined on \( R^1 \). The inversion of FHT has been well established in history \([1-4, 8, 12, 14]\). The applications of (1.1) is referenced to \([6-7, 10, 21]\). The study of (1.2) was initiated by \([13]\) in which the reconstruction of half scan data from exponential Radon transform is converted into the inversion of (1.2). The early numerical inversions of (1.2) were studied in \([5, 11, 14]\) in which the analysis of numerical stability was very limited. From 2009 to 2012, the author investigated (1.2) in weighted \( L^2 \) space \([18-19]\) but left the research community after 2012. In the beginning of 2020, the author came back to revisit \([18-19]\) to complete some unfinished computations. It is very exciting to learn the arguments and results of \([2]\) in which the range condition and explicit inversions of (1.2) were derived by using vector Riemann-Hilbert problem (RHP). In this short paper, the author derives five identities and the Bertola-Katsevich-Tovbis inversion formulas by only using the Sokhotski-Plemelj formula and the Poincaré-Bertrand formula without resorting to RHP.

For \( 1 \leq p < \infty \), \( L^p(I) \) is the \( p \)-integrable Banach space. The FHT \( F(s) \) is continuous in \( L^p(I) \), \( 1 < p < \infty \) \([15]\). In \( L^2(I) \), the FHT is injective but not surjective \([12]\), the similar results were obtained for cosh-weighted FHT \([2]\). The author studied the properties of cosh-weighted FHT in the weighted \( L^2(I) \) in \([18-19]\). Define two weighted spaces

\[
L^2_\omega(I) = \{ f(t); \int_{-1}^{1} |f(t)|^2 \sqrt{1-t^2} dt < \infty \},
\]

(1.3)

1 This revision includes three major updates: 1) consider the jump when using Sokhotski-Plemelj formula though it does not appear to affect the final results since these jump terms cancel out eventually; 2) derive the explicit cosh-weighted Hilbert transform of the exponential Chebyshev functions; and 3) add numerical experiments to show the computational characteristics of (2.38).
\[ L_d^2(I) = \{ f(t) : \left| f(t) \right|^2 \frac{1}{\sqrt{1-t^2}} dt < \infty \}. \] (1.4)

Subscriptions \( m \) and \( d \) stand for the multiplication and division of \( \sqrt{1-t^2} \), respectively. Notice that \( L_d^2(I) \subset L_2(I) \subset L_m^2(I) \) and \( C_0^\infty(I) \) is dense in these linear spaces. Weighted spaces \( L_m^2(I) \) and \( L_d^2(I) \) are Hilbert spaces with the following inner products
\[
< f, g >_m = \frac{1}{\pi} \int f(t)g^*(t)\sqrt{1-t^2} dt,
< f, g >_d = \frac{1}{\pi} \int \frac{f(t)g^*(t)}{\sqrt{1-t^2}} dt.
\] (1.5)

Here the star stands for the complex conjugate. Notice that \( \| f \sqrt{1-t^2} \|_{L_m^2} = 1 \) but \( \| f \sqrt{1-t^2} \|_{L_d^2} \neq 1 \).

In this paper, motivated by [2] and other earlier works, the author presents alternative arguments to study the range and inversion of (1.2) in \( L_m^2(I) \) and \( L_d^2(I) \).

There are two expressions of the Poincaré-Bertrand formula, we use the format from [8] to handle the term \( \cosh[\mu(s-t)] \) more conveniently. For function \( \varphi(u,s) \), the formula reads
\[
\frac{1}{\pi} \int \frac{1}{s-u} \int \frac{1}{s-t} \varphi(u,s) du ds = \varphi(t,u) + \frac{1}{\pi} \int \frac{1}{t-u} - \frac{1}{u-s} \varphi(u,s) ds.
\] (1.6)

We choose a directional expression of the Sokhotski-Plemelj formula to compute the singular integrals with jumps cross the boundary. The Sokhotski-Plemelj formula reads
\[
\lim_{\epsilon \to 0^+} \frac{1}{\pi} \int \frac{1}{s-t} \varphi(s \pm i\epsilon) ds = \pm i \varphi^\dagger(t) + \frac{1}{\pi} \int \frac{1}{t-s} \varphi^\dagger(s) ds
\] (1.7)

Here \( \varphi^\dagger(s) = \lim_{\epsilon \to 0} \varphi(s \pm i\epsilon) \). More detailed exposition of the Sokhotski-Plemelj formula can be found in [8]. Formulas (1.6) and (1.7) were derived for holder continuous functions and extended to \( L^p(I) \) under proper conditions. In this paper we take the arguments from applying (1.6) and (1.7) to functions in \( C_0^\infty(I) \) and passing to the limit in \( L_m^2(I) \) and \( L_d^2(I) \).

II. Identities and Inversions of \( \cosh \)-weighted FHT

The arguments of [2] are to cast the inversion of (1.2) to a vector RHP to derive the left and right inverses, and then use two lemmas to compute the inversion formulas. In this paper, we derive five identities by complex analysis and (1.7), and prove the Bertola-Katsevich-Toebis pseudo inverses are indeed the inversion of (1.2) by using (1.6) and these identities.

**Lemma 1.** For \( t,u \in (-1, 1) \) and \( \mu \in C \), we have the following identities
\[
\frac{1}{\pi} \int \frac{1}{s-t} \cosh[\mu(u-s)] \frac{\cosh(\sqrt{1-s^2})}{\sqrt{1-s^2}} ds = \cosh(\mu u),
\] (2.1)
\[
\frac{1}{\pi} \int \frac{1}{s-t} \cosh[\mu(u-s)] \frac{\cosh(\sqrt{1-s^2})}{\sqrt{1-s^2}} ds = -\frac{\mu}{2} \sinh(\mu u),
\] (2.2)
\[
\frac{1}{\pi} \int \frac{1}{s-t} \cosh[\mu(u-s)] \sinh(\sqrt{1-s^2}) ds = i [\sinh(\mu(t-u)) \cosh(\mu \sqrt{1-t^2}) + \sinh(\mu t)] ,
\] (2.3)
\[
\frac{1}{\pi} \int \frac{1}{s-t} \cosh[\mu(u-s)] \frac{\cosh(\sqrt{1-s^2})}{\sqrt{1-s^2}} ds = i \sinh(\mu(t-u)) \sinh(\sqrt{1-t^2}) \frac{1}{\sqrt{1-t^2}},
\] (2.4)
\[
\frac{1}{\pi} \int \cosh[\mu(u-s)] \cosh(\sqrt{1-s^2}) \sqrt{1-s^2} ds
\] (2.5)
\[
= i \sinh(\mu(t-u)) \sinh(\sqrt{1-t^2}) \sqrt{1-t^2} + t \cosh(\mu u) - \frac{\mu}{2} \sinh(\mu u).
\]
Proof. Consider a complex function
\[ \Phi_{\mu}(u, z) = \frac{1}{2}e^{\mu(u - z\sqrt{1 - s^2})}, \quad u \in (-1, 1). \] (2.6)
Choose the cuts of \( \sqrt{z^2 - 1} \) satisfying
\[ \lim_{|z| \to \infty} \frac{\sqrt{z^2 - 1}}{z} = 1, \quad \lim_{|z| \to 0, \epsilon > 0} \sqrt{(s \pm i\epsilon)^2 - 1} = \pm i\sqrt{1 - s^2} \quad \text{for} \quad s \in (-1, 1). \] (2.7)
With such selection of cuts, \( \Phi_{\mu}(u, z) \) is analytical in \( C \setminus [-1, 1] \) and
\[ \Phi_{\mu}(u, \infty) = \lim_{|z| \to \infty} \Phi_{\mu}(u, z) = \frac{1}{2}e^{\mu u}, \quad \lim_{|z| \to 0, \epsilon > 0} [\Phi_{\mu}(u, z) - \Phi_{\mu}(u, \infty)]z = -\frac{1}{4}\mu e^{\mu u}, \] (2.8)
\[ \Phi_{\mu}^+(u, s) = \lim_{\epsilon \to 0_+} \Phi_{\mu}(u, s \pm i\epsilon) = \frac{1}{2}e^{\mu(u - z\sqrt{1 - s^2})}, \quad s \in (-1, 1). \] (2.9)
We mention the following identities for future computations:
\[ \frac{1}{2} \left[ \Phi_{\mu}^+(u, s) + \Phi_{\mu}^-(u, s) + \Phi_{\mu}^+(u, s) + \Phi_{\mu}^-(u, s) \right] = \cosh(\mu(u - s)) \cosh(\mu \sqrt{1 - s^2}), \] (2.10)
\[ \frac{1}{2} \left[ \Phi_{\mu}^+(u, t) + \Phi_{\mu}^+(u, t) - \Phi_{\mu}^-(u, t) - \Phi_{\mu}^-(u, t) \right] = \sinh(\mu(u - t)) \sinh(\mu \sqrt{1 - t^2}), \] (2.11)
\[ \frac{1}{2} \left[ \Phi_{\mu}^+(u, t) - \Phi_{\mu}^+(u, t) + \Phi_{\mu}^-(u, t) - \Phi_{\mu}^-(u, t) \right] = \sinh(\mu(u - t)) \cosh(\mu \sqrt{1 - t^2}), \] (2.12)
\[ \frac{1}{2} \left[ \Phi_{\mu}^+(u, s) - \Phi_{\mu}^+(u, s) - \Phi_{\mu}^-(u, s) + \Phi_{\mu}^-(u, s) \right] = \cosh(\mu(u - s)) \sinh(\mu \sqrt{1 - s^2}). \] (2.13)
To match the integrals around \([-1, 1]\], we take the contour integrals clockwise on large circles, for \( |z| \to \infty \), it is straightforward to derive
\[ \frac{1}{2\pi i} \int_{|z|=d, d>1} \frac{\Phi_{\mu}(u, z) + \Phi_{\mu}(u, z)}{z} d z = \lim_{d \to \infty} \frac{1}{2\pi i} \int_{|z|=d} \frac{\Phi_{\mu}(u, z) + \Phi_{\mu}(u, z)}{z} d z = \cosh(\mu t), \] (2.14)
\[ \frac{1}{2\pi i} \int_{|z|=d} \frac{\Phi_{\mu}(u, z) - \Phi_{\mu}(u, \infty)}{z} z d z = \lim_{d \to \infty} \frac{1}{2\pi i} \int_{|z|=d} \frac{\Phi_{\mu}(u, z) - \Phi_{\mu}(u, \infty)}{z} z d z = \mp \frac{1}{4}\mu e^{\mu u}, \] (2.15)
\[ \frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - z} [\Phi_{\mu}(u, z) - \Phi_{\mu}(u, z)] d z = \lim_{d \to \infty} \frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - z} [\Phi_{\mu}(u, z) - \Phi_{\mu}(u, z)] d z = 0. \] (2.16)
On the other hand, we compute the limits of these integrals around \([-1, 1]\) as follows.
\[ \frac{1}{2\pi i} \int_{|z|=d, d>1} \frac{\Phi_{\mu}(u, z) + \Phi_{\mu}(u, z)}{z} d z = \lim_{d \to \infty} \frac{1}{2\pi i} \int_{|z|=1} \frac{\Phi_{\mu}(u, s + i\epsilon) + \Phi_{\mu}(u, s + i\epsilon)}{\sqrt{(s + i\epsilon)^2 - 1}} d s \] (2.17)
\[ - \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|z|=1} \frac{\Phi_{\mu}(u, s - i\epsilon) + \Phi_{\mu}(u, s - i\epsilon)}{\sqrt{(s - i\epsilon)^2 - 1}} d s \]
\[ = \frac{1}{\pi} \int_{1} \frac{\cosh(\mu(1 - s^2))}{\sqrt{1 - s^2}} d s. \] (2.18)
The last step of (2.18) is due to (2.10), then (2.1) is derived by (2.14) and (2.18).
\[
\frac{1}{2\pi i} \int_{|z|=d} \frac{\Phi_\mu(u, z) + \Phi_{-\mu}(u, z)}{\sqrt{z^2 - 1}} \, dz = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z|=d} \frac{\Phi_\mu(u, s + i\varepsilon) + \Phi_{-\mu}(u, s + i\varepsilon)}{\sqrt{(s + i\varepsilon)^2 - 1}} \, ds
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z|=d} \frac{\Phi_\mu(u, s - i\varepsilon) + \Phi_{-\mu}(u, s - i\varepsilon)}{\sqrt{(s - i\varepsilon)^2 - 1}} \, ds
\]
\[
= \frac{1}{\pi} \int \cosh[\mu(s - u)] \cosh(i\mu\sqrt{1 - s^2}) \, ds.
\]

The last step of (2.19) is due to (2.10), then (2.2) is derived by (2.15), (2.19) and the following identity
\[
\int_{|z|=d} \frac{\Phi_\mu(u, z) - \Phi_\mu(u, \infty) + \Phi_{-\mu}(u, z) - \Phi_{-\mu}(u, \infty)}{\sqrt{z^2 - 1}} \, dz = \int_{|z|=d} \frac{\Phi_\mu(u, z) + \Phi_{-\mu}(u, z)}{\sqrt{z^2 - 1}} \, dz.
\]

To compute (2.3-2.4), we need (1.7) to calculate the jumps cross [-1, 1]. For (2.3) we have
\[
\frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - z} \left[ \Phi_\mu(u, z) - \Phi_{-\mu}(u, z) \right] \, dz
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - (s + i\varepsilon)} \left[ \Phi_\mu(u, s + i\varepsilon) - \Phi_{-\mu}(u, s + i\varepsilon) \right] \, ds
\]
\[
- \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - (s - i\varepsilon)} \left[ \Phi_\mu(u, s - i\varepsilon) - \Phi_{-\mu}(u, s - i\varepsilon) \right] \, ds
\]
\[
= \sinh[\mu(u - t)] \cosh(i\mu\sqrt{1 - t^2}) - \frac{1}{\pi i} \int \frac{\cosh[\mu(s - u)]}{t - s} \sinh(i\mu\sqrt{1 - s^2}) \, ds.
\]

By (1.7) the first limit of (2.21) is
\[
\frac{1}{2} \left[ \Phi_\mu^+(u, t) - \Phi_{-\mu}^+(u, t) \right] - \frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - s} \left[ \Phi_\mu^+(u, s) - \Phi_{-\mu}^+(u, s) \right] \, ds.
\]

By (1.7) the second limit of (2.21) is
\[
- \frac{1}{2} \left[ \Phi_\mu^- (u, t) - \Phi_{-\mu}^- (u, t) \right] - \frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - s} \left[ \Phi_\mu^- (u, s) - \Phi_{-\mu}^- (u, s) \right] \, ds.
\]

The last step of (2.21) is due to (2.12) and (2.13), then (2.3) is derived by (2.16) and (2.21).

For (2.4) we have
\[
\frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - z} \frac{\Phi_\mu(u, z) + \Phi_{-\mu}(u, z)}{\sqrt{z^2 - 1}} \, dz
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - (s + i\varepsilon)} \frac{\Phi_\mu(u, s + i\varepsilon) + \Phi_{-\mu}(u, s + i\varepsilon)}{\sqrt{(s + i\varepsilon)^2 - 1}} \, ds
\]
\[
- \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - (s - i\varepsilon)} \frac{\Phi_\mu(u, s - i\varepsilon) + \Phi_{-\mu}(u, s - i\varepsilon)}{\sqrt{(s - i\varepsilon)^2 - 1}} \, ds
\]
\[
= \sinh[\mu(u - t)] \sinh(i\mu\sqrt{1 - t^2}) + \frac{1}{\pi i} \int \frac{\cosh[\mu(s - u)] \cosh(i\mu\sqrt{1 - s^2})}{t - s} i\sqrt{1 - s^2} \, ds.
\]

By (1.7) the first limit of (2.24) is
\[
\frac{1}{2} \left( \Phi_\mu^+(u, t) + \Phi_{-\mu}^+(u, t) \right) + \frac{1}{2\pi i} \int_{|z|=d} \frac{1}{t - s} \left( \Phi_\mu^+(u, s) + \Phi_{-\mu}^+(u, s) \right) \, ds.
\]
By (1.7) the second limit of (2.24) is
\[
\frac{1}{2} \frac{\Phi_{\mu}(u,t) + \Phi_{-\mu}(u,t)}{i\sqrt{1-t^2}} + \frac{1}{2\pi i} \int_{t-s}^{t} \frac{1}{i\sqrt{1-s^2}} \Phi_{\mu}(u,s) + \Phi_{-\mu}(u,s) \, ds.
\] (2.26)

The last step of (2.24) is due to (2.10) and (2.11), then (2.4) is derived by (2.17) and (2.24). Identity (2.5) is derived by using (2.1), (2.2) and (2.4) as follows

\[
\frac{1}{\pi} \int_{-t}^{t} \frac{\cosh[\mu(u-s)]}{t-s} \cosh(i\mu\sqrt{1-s^2})\sqrt{1-s^2} \, ds = \frac{1}{\pi} \int_{-t}^{t} \frac{\cosh[\mu(u-s)]}{t-s} \cosh(i\mu\sqrt{1-s^2}) \, (1-s^2) \, ds
\]
\[
= \frac{1}{\pi} \int_{-t}^{t} \frac{\cosh[\mu(u-s)]}{t-s} \cosh(i\mu\sqrt{1-s^2}) \, (1-t^2 + t^2 - s^2) \, ds
\]
\[
= \frac{i}{\pi} \sinh[\mu(t-u)] \sinh(i\mu\sqrt{1-t^2}) \sqrt{1-t^2} + \frac{1}{\pi} \int_{-t}^{t} \frac{\cosh[\mu(u-s)]}{\sqrt{1-s^2}} \cosh(i\mu\sqrt{1-s^2}) \, (t+s) \, ds
\]
\[
= \frac{i}{\pi} \sinh[\mu(t-u)] \sinh(i\mu\sqrt{1-t^2}) \sqrt{1-t^2} + t \cosh(\mu t) - \frac{\mu}{2} \sinh(\mu t).
\] (2.27)

The proof of (2.1-2.5) is complete. □

**Remark 1.** Identity (2.3) is equivalent to Lemma A.2 of [2] and (2.5) is equivalent to Lemma A.1 of [2]. Take \( u = t \) in (2.4), \( \cosh(i\mu\sqrt{1-t^2})\sqrt{1-t^2} \) is the non-trivial function in the null space of \( H_\mu \) in \( L^2_m(I) \).

The author likes to mention that \( \Phi_{\mu}(s,z) \) of (2.6) is from [2]. Around 2011, the author tried to find such function but did not get the expression correct to meet the requirements (2.14-2.17). In the early February of 2020, after learning the arguments of [2] by Google search, the author quickly uses \( \Phi_{\mu}(s,z) \) to carry through the lengthy computations and then write this short paper. Now we prove the Bertola-Katsevich-Tovbis inversion formulas.

**Theorem 1.** For \( \mu \in C \), we rephrase the Bertola-Katsevich-Tovbis inversion formulas of cosh-weighted FHT in two identities. For \( f(t) \in L^2_j(I) \), we have

\[
f(t) = \cosh(i\mu\sqrt{1-t^2})\sqrt{1-t^2} \left[ \frac{1}{\pi} \int_{-t}^{t} \frac{F_{\mu}(s)}{s-t} \cosh(i\mu\sqrt{1-s^2}) \, ds \right]
\]
\[
- \sinh(i\mu\sqrt{1-t^2}) \left[ \frac{1}{\pi} \int_{-t}^{t} \frac{F_{\mu}(s)}{s-t} \sinh(i\mu\sqrt{1-s^2}) \, ds \right].
\] (2.28)

For \( f(t) \in L^2_m(I) \), we have

\[
f(t) = \frac{\cosh(i\mu\sqrt{1-t^2})}{\sqrt{1-t^2}} \left[ \frac{1}{\pi} \int_{-t}^{t} \frac{F_{\mu}(s)}{s-t} \cosh(i\mu\sqrt{1-s^2}) \sqrt{1-s^2} \, ds + \frac{1}{\pi} \int_{-t}^{t} \cosh(\mu t) f(t) \, dt \right]
\]
\[
- \sinh(i\mu\sqrt{1-t^2}) \left[ \frac{1}{\pi} \int_{-t}^{t} \frac{F_{\mu}(s)}{s-t} \sinh(i\mu\sqrt{1-s^2}) \, ds \right].
\] (2.29)

**Proof.** In [17], it was proved that (1.6) holds for functions in \( L^2_j(I) \) and \( L^2_m(I) \), here we adopt the argument to prove (2.28) and (2.29) for \( f(t) \in C_0^\infty(I) \) and pass to the limit in \( L^2_j(I) \) and \( L^2_m(I) \).
We compute \( \sinh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} F_\mu(s) \sinh(i\mu \sqrt{1-s^2}) ds \right) \) as follows

\[
\sinh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} F_\mu(s) \sinh(i\mu \sqrt{1-s^2}) ds \right)
= \sinh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \sinh(i\mu \sqrt{1-s^2}) \left[ \frac{1}{\pi} \int_{-1}^{1} \cosh[\mu(s-u)] f(u) du \right] ds \right)
= \sinh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\pi} \int_{-1}^{1} \left[ \cosh[\mu(s-u)] \sinh(i\mu \sqrt{1-s^2}) f(u) \right] du \right] ds.
\] (2.30)

Apply (1.6) to (2.30) with \( \varphi(u,s) = \cosh[\mu(u-s)] \sinh(i\mu \sqrt{1-s^2}) f(u) \), (2.3) yields

\[
\sinh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} F_\mu(s) \sinh(i\mu \sqrt{1-s^2}) ds \right)
= \sinh^2(i\mu \sqrt{1-t^2}) f(t) + \sinh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\pi} \int_{-1}^{1} \left( \frac{1}{t-s} - \frac{1}{u-s} \right) \varphi(u,s) ds \right) du
= \sinh^2(i\mu \sqrt{1-t^2}) f(t) + i \sinh(i\mu \sqrt{1-t^2}) \cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \left[ \sinh[\mu(u-t)] \right] du \right) f(u).
\] (2.31)

We compute \( \cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} F_\mu(s) \cosh(i\mu \sqrt{1-s^2}) ds \right) \) as follows

\[
\cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} F_\mu(s) \cosh(i\mu \sqrt{1-s^2}) ds \right)
= \cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \cosh(i\mu \sqrt{1-s^2}) \left[ \frac{1}{\pi} \int_{-1}^{1} \cosh[\mu(s-u)] f(u) du \right] ds \right)
= \cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \left[ \cosh[\mu(s-u)] \cosh(i\mu \sqrt{1-s^2}) f(u) \right] du \right] ds.
\] (2.32)

Apply (1.6) to (2.32) with \( \varphi(u,s) = \cosh[\mu(u-s)] \cosh(i\mu \sqrt{1-s^2}) f(u) \), (2.4) yields

\[
\cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} F_\mu(s) \cosh(i\mu \sqrt{1-s^2}) ds \right)
= \cosh^2(i\mu \sqrt{1-t^2}) f(t) + \cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\pi} \int_{-1}^{1} \left( \frac{1}{t-s} - \frac{1}{u-s} \right) \varphi(u,s) ds \right) du
= \cosh^2(i\mu \sqrt{1-t^2}) f(t) + i \sinh(i\mu \sqrt{1-t^2}) \cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \left[ \sinh[\mu(u-t)] \right] du \right) f(u).
\] (2.33)

We compute \( \frac{\cosh(i\mu \sqrt{1-t^2}) \left( 1 \int_{-1}^{1} F_\mu(s) \cosh(i\mu \sqrt{1-s^2}) \right) ds}{\sqrt{1-t^2}} \) as follows

\[
\frac{\cosh(i\mu \sqrt{1-t^2}) \left( 1 \int_{-1}^{1} F_\mu(s) \cosh(i\mu \sqrt{1-s^2}) \right) ds}{\sqrt{1-t^2}}
= \cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \cosh(i\mu \sqrt{1-s^2}) \left[ \frac{1}{\pi} \int_{-1}^{1} \cosh[\mu(s-u)] f(u) du \right] ds \right)
= \cosh(i\mu \sqrt{1-t^2}) \left( \frac{1}{\pi} \int_{-1}^{1} \left[ \cosh[\mu(u-s)] \cosh(i\mu \sqrt{1-s^2}) f(u) \right] du \right] ds.
\] (2.34)
Apply (1.6) to (2.34) with $\varphi(u,s) = \cosh(\mu (u-s)) \cosh(\mu \sqrt{1-s^2}) \sqrt{1-s^2} f(u)$, (2.5) yields

$$
\frac{\cosh(i \mu \sqrt{1-t^2})}{\sqrt{1-t^2}} \frac{1}{\pi \sqrt{1-t^2}} \int s-t ds F_\mu(s) \cosh(i \mu \sqrt{1-s^2}) \sqrt{1-s^2} ds
$$

$$
= \cosh^2(i \mu \sqrt{1-t^2}) f(t) + \frac{\cosh(i \mu \sqrt{1-t^2})}{\sqrt{1-t^2}} \frac{1}{\pi \sqrt{1-t^2}} \frac{1}{t-u} \left[ \frac{1}{t-s} - \frac{1}{u-s} \right] \varphi(u,s) ds du
$$

$$
= \cosh^2(i \mu \sqrt{1-t^2}) f(t) + \frac{\cosh(i \mu \sqrt{1-t^2})}{\sqrt{1-t^2}} \frac{1}{\pi \sqrt{1-t^2}} \frac{1}{t-u} f(u) \cosh(\mu \mu (u-t)) du
$$

$$
+ i \sinh(i \mu \sqrt{1-t^2}) \cosh(i \mu \sqrt{1-t^2}) \frac{1}{\pi \sqrt{1-t^2}} \sinh(\mu (u-t)) f(u) du
$$

$$
= \cosh^2(i \mu \sqrt{1-t^2}) f(t) + \frac{\cosh(i \mu \sqrt{1-t^2})}{\sqrt{1-t^2}} \frac{1}{\pi \sqrt{1-t^2}} \cosh(\mu \mu f(u)) dt
$$

$$
+ i \sinh(i \mu \sqrt{1-t^2}) \cosh(i \mu \sqrt{1-t^2}) \frac{1}{\pi \sqrt{1-t^2}} \sinh(\mu (u-t)) f(u) du.
$$

(2.35)

Recall that $\cosh^2 z - \sinh^2 z = 1$ holds for all $z \in \mathbb{C}$, subtracting (2.31) from (2.33) and (2.35), we obtain (2.28) and (2.29). □

**Remark 2.** Formula (2.29) is the same as Theorem 6.2 of [2]. Formula (2.28) was stated as part of Theorem 5.1 in [2] for $f \in L_p^p(I)$, $p > 2$. When $\mu = 0$, (2.28) and (2.29) degenerate to the inversion of FHT. In Theorem 2, we rephrase that two pseudo-inverses given in Theorem 5.1 of [2] are the true inverses in $L_2^2(I)$ and $L_2^2(I)$ from the range to the domain modulo the null space.

**Theorem 2.** We assume $\mu \in \mathbb{C}$ and define two subspaces $\mathcal{E}_m^2(I)$ and $\mathcal{E}_a^2(I)$ as

$$
\mathcal{E}_m^2(I) = \{ f(t) \in L_m^2(I) : \frac{1}{\pi \sqrt{1-t^2}} \cosh(\mu \mu f(t)) dt = 0 \},
$$

(2.36)

$$
\mathcal{E}_a^2(I) = \{ f(t) \in L_a^2(I) : \frac{1}{\pi \sqrt{1-t^2}} \cosh(i \mu \sqrt{1-t^2}) f(t) dt = 0 \}.
$$

(2.37)

The null space of $H_\mu$ in $L_a^2(I)$ is trivial and the null space of $H_\mu$ in $L_m^2(I)$ is defined by $\{ c \cosh(i \mu \sqrt{1-t^2}) / \sqrt{1-t^2} \}, c \in \mathbb{C}$. $H_\mu$ is continuous from $\mathcal{E}_a^2(I)$ to $L_m^2(I)$ and from $L_a^2(I)$ to $\mathcal{E}_a^2(I)$, respectively.

For $F_\mu(s) \in \mathcal{E}_a^2(I)$, the inverse $^d H_\mu^{-1}$ is given by

$$
^d H_\mu^{-1} F_\mu(t) = \cosh(i \mu \sqrt{1-t^2}) \sqrt{1-t^2} \frac{1}{\pi \sqrt{1-t^2}} \frac{1}{t-s} \cosh(i \mu \sqrt{1-s^2}) F_\mu(s) ds
$$

$$
- \sinh(i \mu \sqrt{1-t^2}) \frac{1}{\pi \sqrt{1-t^2}} \sinh(i \mu \sqrt{1-s^2}) F_\mu(s) ds.
$$

(2.38)

For $F_\mu(s) \in L_m^2(I)$, the inverse $^a H_\mu^{-1}$ is given by

$$
^a H_\mu^{-1} F_\mu(t) = \frac{\cosh(i \mu \sqrt{1-t^2})}{\sqrt{1-t^2}} \frac{1}{\pi \sqrt{1-t^2}} \cosh(i \mu \sqrt{1-s^2}) \sqrt{1-s^2} F_\mu(s) ds
$$

$$
- \sinh(i \mu \sqrt{1-t^2}) \frac{1}{\pi \sqrt{1-t^2}} \sinh(i \mu \sqrt{1-s^2}) F_\mu(s) ds.
$$

(2.39)
Then \(^{\mu}H_{\mu}^{-1}\) is continuous from \(L_m^2(I)\) to \(E_\mu(I)\) and \(^{\mu}H_{\mu}^{-1}F_{\mu}\) is continuous from \(E_\mu(I)\) to \(L_m^2(I)\).

The range of \(H_{\mu}\) in \(E_\mu(I)\) is \(L_m^2(I)\) and the range of \(H_{\mu}\) in \(L_m^2(I)\) is \(E_\mu(I)\).

**Proof.** Theorem 2 by large rephrases Theorem 5.1 of [2] to \(L_m^2(I)\) and \(E_\mu(I)\) for complex \(\mu \in \mathbb{C}\).

We omit the detailed proof except confirming that \(^{\mu}H_{\mu}^{-1}(L_m^2(I)) = E_\mu(I)\). Let \(g(t) = ^{\mu}H_{\mu}^{-1}F_{\mu}(t)\), we compute \(\int_{-1}^{1} \cosh(\mu t) g(t) dt\). Take \(u = 0\) in (2.4), the first term of (2.39) is

\[
\int_{-1}^{1} \cosh(\mu t) \left( \frac{\cosh(i \mu \sqrt{1 - t^2})}{\sqrt{1 - t^2}} \right) \frac{1}{\pi} \frac{1}{s - t} \cosh(i \mu \sqrt{1 - s^2}) \sqrt{1 - s^2} F_{\mu}(s) ds dt
\]

(2.40)

Take \(u = 0\) in (2.3), the second term of (2.39) is

\[
\int_{-1}^{1} \sinh(\mu \sqrt{1 - t^2}) \frac{1}{\pi} \frac{1}{s - t} \sinh(i \mu \sqrt{1 - s^2}) F_{\mu}(s) ds dt
\]

(2.41)

Subtracting (2.41) from (2.40) yields \(\int_{-1}^{1} \cosh(\mu t) f(t) dt = 0\). This completes the proof. \(\square\)

**Remark 3.** A family of two-parameter inversions are defined by (B.3) in [2]. Here we point out that a similar family of inversions can be derived through the following identity:

\[
\frac{1}{\pi} \left( \frac{1}{\cosh(\mu(u - s))} \right) \frac{\alpha s^2 + \beta s + \gamma}{\sqrt{1 - s^2}} \cosh(i \mu \sqrt{1 - s^2}) ds
\]

(2.42)

The proof of (2.42) is straightforward by combining identities from (2.2) to (2.5). Assume that \(\alpha t^2 + \beta t + \gamma \neq 0\) in \((-1, 1)\), lengthy computations yield

\[
f(t) = \frac{\sqrt{1 - t^2}}{\alpha t^2 + \beta t + \gamma} \cosh(i \mu \sqrt{1 - t^2}) \left[ \frac{1}{\pi} \frac{1}{s - t} \cosh(i \mu \sqrt{1 - s^2}) \frac{\alpha s^2 + \beta s + \gamma}{\sqrt{1 - s^2}} ds \right]
\]

(2.43)

If \(\alpha \neq 0\), \(\int_{-1}^{1} \cosh(\mu t) g(t) dt\) is required to be known for an exact inversion by (2.43). For \(\alpha = 0\), \(\beta = \pm 1\) and \(\gamma = 1\), we have
\[ f(t) = \frac{1\pm t}{\sqrt{1-t^2}} \cosh(i\mu\sqrt{1-t^2}) \left[ \frac{1}{\pi} \frac{F_\nu(s)}{s-t} \cosh(i\mu\sqrt{1-s^2}) \sqrt{1-s^2} \right] ds \]

\[-\sinh(i\mu\sqrt{1-t^2}) \left[ \frac{1}{\pi} \frac{F_\nu(s)}{s-t} \sinh(i\mu\sqrt{1-s^2}) ds \right]. \]  

(2.44)

### III. Exponential Chebyshev functions

It is well known that the Chebyshev polynomials satisfy the following relations

\[ \frac{1}{\pi} \int_1^1 \frac{T_n(s)}{t-s} \sqrt{1-s^2} ds = -U_{n-1}(t), \]  

(3.1)

\[ \frac{1}{\pi} \int_1^1 \frac{U_n(s)}{t-s} \sqrt{1-s^2} ds = T_{n+1}(t), \]  

(3.2)

Hereafter \( T_n(t) \) and \( U_n(t) \) stand for the Chebyshev polynomials of the first and second kind, respectively. In this section we introduce the so-called exponential Chebyshev functions studied in [13, 20] for the inversion of the exponential Radon transform and prove that the exponential Chebyshev functions meet the similar identities for the cosh-weighted Hilbert transform. For \( n \geq 0 \), the exponential Chebyshev functions are defined as

\[ T_{\mu,n}(s) = \frac{1}{2} \left[ e^{-\mu t\sqrt{1-s^2}} (s+i\sqrt{1-s^2})^n + e^{\mu t\sqrt{1-s^2}} (s-i\sqrt{1-s^2})^n \right] \]  

(3.3)

\[ U_{\mu,n}(s) = \frac{1}{2i\sqrt{1-s^2}} \left[ e^{\mu t\sqrt{1-s^2}} (s+i\sqrt{1-s^2})^{n+1} - e^{-\mu t\sqrt{1-s^2}} (s-i\sqrt{1-s^2})^{n+1} \right] \]  

(3.4)

If \( \mu = 0 \), (3.3-3.4) become \( T_n(t) \) and \( U_n(t) \), respectively. For the cuts of (2.7), we consider

\[ \Psi_{\mu,n}(u,z) = e^{\mu (u-z\sqrt{1-z^2})} (z - \sqrt{z^2-1})^n, \quad n \geq 0. \]  

(3.5)

For \( s \in (-1,1) \), we have

\[ \Psi_{\mu,n}^+(u,s) = \lim_{\epsilon \to 0, \epsilon > 0} \Psi_{\mu,n}(u,s \pm i\epsilon) = e^{\mu (u-z\sqrt{1-z^2})} (s ^{1 \pm i\sqrt{1-s^2})}^n \]  

(3.6)

Follow the computations in the proof of Lemma 1 we obtain

\[ 0 = \frac{1}{2\pi i} \int_{|z|=1,t=x+iy} \frac{1}{z-1} \frac{1}{\sqrt{z^2-1}} \frac{\Psi_{\mu,n}(u,z)}{dz} \]

\[ = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{t=s+\epsilon i} \frac{1}{t-(s+i\epsilon)} \frac{1}{\sqrt{(s+i\epsilon)^2-1}} ds - \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{t=s-\epsilon i} \frac{1}{t-(s-i\epsilon)} \frac{1}{\sqrt{(s-i\epsilon)^2-1}} ds \]

\[ = \frac{1}{2i\sqrt{1-t^2}} \int_{t=s} \frac{1}{\sqrt{1-s^2}} \frac{\Psi_{\mu,n}(u,s)}{i\sqrt{1-s^2}} ds. \]

(3.7)

Reorganizing the terms in (3.7), for \( n \geq 1 \) we obtain

\[ \frac{1}{\pi} \int_{t=0} \frac{1}{1-t^2} \frac{\exp[\mu (t-s)] T_{\mu,n}(s)}{\sqrt{1-s^2}} ds = -U_{\mu,n-1}(t). \]  

(3.8)

Notice that \( \Psi_{\mu,n}(u,\infty) = 0 \) for \( n \geq 1 \), we have
\[
0 = \frac{1}{2\pi i} \int_{|z|=1, dz=t-z} \frac{1}{z} \Psi_{\mu, n}(u, z) dz \\
= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{t - z} \Psi_{\mu, n}(u, s + i\epsilon) ds - \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{t - z} \Psi_{\mu, n}(u, s - i\epsilon) ds .
\]
(3.9)
\[
= \frac{1}{2} [\Psi_{\mu, n}^+(u, t) + \Psi_{\mu, n}^-(u, t)] + \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{t - s} [\Psi_{\mu, n}^+(u, s) - \Psi_{\mu, n}^-(u, s)] ds.
\]

It follows that for \( n \geq 1 \)
\[
\frac{1}{\pi} \int_{|z|=1} \frac{\exp[-\mu(t-s)]}{t-s} U_{\mu, n-1}(s) \sqrt{1 - s^2} ds = T_{\mu, n}(t) .
\]
(3.10)

To find the other half of (3.8) and (3.10), we consider the following complex function
\[
\Psi_{\mu, n}(u, z) = \frac{1}{2} e^{-\mu(u-z\sqrt{z^2-1})} (z - \sqrt{z^2-1})^n , \quad n \geq 0 .
\]
(3.11)

If \( z \to \infty \), \((z - \sqrt{z^2-1})^n\) tends to infinity for \( n \geq 1 \) so that the direct computation of the integrals is difficult for \( z \to \infty \). Let \( x = z - \sqrt{z^2-1} \), then \( z = \frac{1}{2}(x + \frac{1}{x}) \) is analytic in \( C \setminus \{0\} \) and we have
\[
\frac{1}{2\pi i} \int_{|z|=\rho, dz=t-z} \frac{1}{z} \Psi_{\mu, n}(u, z) dz \\
= \frac{1}{2\pi i} \int_{|z|=\rho, dz=t-z} \frac{2x}{2xt-x^2-1} \frac{2x}{1-1-x^2} \left[ \frac{1}{2} e^{-\mu(u-x)} x^n \right] \frac{1}{2x^2} (x^2-1) dx \\
= \frac{1}{2\pi i} \int_{|z|=\rho, dz=t-z} \frac{e^{-\mu} e^{\mu x}}{1-2xt+x^2} \frac{1}{x^n} dx \\
= \frac{1}{2\pi i} e^{-\mu} \int_{|z|=\rho, dz=t-z} \left( \sum_{m=0}^{\infty} \frac{\mu^m U_{m-1}(t)}{m!} \right) \frac{1}{x^n} dx.
\]
(3.12)

Here we use the generating function \( 1/(1-2xt+x^2) = \sum_{k=0} U_k(t) x^k \). It follows that
\[
\frac{1}{2\pi i} \int_{|z|=\rho, dz=t-z} \frac{1}{z} \Psi_{\mu, n}(u, z) dz = e^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k U_{k}(t)}{k!} .
\]
(3.13)

Around \((-1, 1)\) we have the following identities
\[
\frac{1}{2\pi i} \int_{|z|=\rho, dz=t-z} \frac{1}{z} \Psi_{\mu, n}(u, z) dz \\
= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|z|=\rho, dz=t-z} \frac{\Psi_{\mu, n}(u, s + i\epsilon)}{\sqrt{(s+i\epsilon)^2-1}} ds - \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|z|=\rho, dz=t-z} \frac{\Psi_{\mu, n}(u, s - i\epsilon)}{\sqrt{(s-i\epsilon)^2-1}} ds
\]
(3.14)
\[
= \frac{1}{2} [\Psi_{\mu, n}^+(u, t) - \Psi_{\mu, n}^-(u, t)] + \frac{1}{2\pi i} \int_{|z|=\rho, dz=t-s} \frac{\Psi_{\mu, n}^+(u, s) - \Psi_{\mu, n}^-(u, s)}{i\sqrt{1-s^2}} ds.
\]

Reorganizing the terms in (3.16), for \( n \geq 1 \) we obtain
\[
\frac{1}{\pi} \int_{|z|=1} \frac{\exp[-\mu(t-s)]}{t-s} T_{\mu, n}(s) \sqrt{1 - s^2} ds = U_{\mu, n-1}(t) - 2e^{-\mu} \sum_{k=0}^{\infty} \frac{1}{k!} \mu^k U_{k-1}(t) .
\]
(3.15)

Similarly, for \( n \geq 1 \) we have
\[
\frac{1}{2\pi i} \int_{|z| = r} \frac{1}{t-z} \Psi_{\mu,n}(u,z)dz \\
= \frac{1}{2\pi i} \int_{|z| = r} \frac{2x}{2xt - x^2} - \frac{1}{2} \left( \frac{1}{x^n} \right) \frac{1}{2x^n} (x^2 - 1)dx \\
= \frac{1}{2\pi i} \int_{|z| = r} \frac{e^{-\mu} e^{\mu}}{1 - 2xt + x^2} dx \\
= \frac{1}{2\pi i} \int_{|z| = r} \left( \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} e^{\mu} U_{m,k}(t) \right) \frac{1}{x^n} dx \\
= \frac{1}{2\pi i} \int_{|z| = r} \frac{1}{2} \left( \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \mu^k U_{m,k}(t) \right) \frac{1}{x^n} dx \\
\]

By residue theorem for \( n \geq 0 \) we have
\[
\frac{1}{2\pi i} \int_{|z| = r} \frac{1}{t-z} \Psi_{\mu,n}(u,z)dz = \frac{e^{-\mu} \left( U_1(t) + \mu \right)}{2} \left( \sum_{k=0}^{\infty} \frac{1}{k!} \mu^k U_{n-k}(t) - \sum_{k=0}^{\infty} \frac{1}{k!} \mu^k U_{n-2-k}(t) \right) \quad n \geq 1 \quad (3.17)
\]

Then for \( n \geq 1 \), we obtain
\[
\frac{1}{\pi i} \left[ \exp[-\mu(t-s)] \right] U_{\mu,n+1}(s) \sqrt{1-s^2} ds \\
= -T_{\mu,n}(t) + e^{-\mu} \left( \sum_{k=0}^{\infty} \frac{1}{k!} \mu^k U_{n-k}(t) - \sum_{k=0}^{\infty} \frac{1}{k!} \mu^k U_{n-2-k}(t) \right) \quad n \geq 2 \quad (3.18)
\]

Summarizing the preceding computations, for \( n \geq 1 \) we have
\[
\frac{1}{\pi i} \left[ \cosh[\mu(t-s)] \right] T_{\mu,n}(s) \sqrt{1-s^2} ds = -e^{-\mu} \sum_{k=0}^{\infty} \frac{1}{k!} \mu^k U_{n-k}(t) \quad n \geq 2 \quad (3.19)
\]

It is quite remarkable that (3.19) is almost identical (3.38) of [20] after combining the positive and negative exponential transforms. Notice that if \( \mu = 0 \), (3.19) and (3.20) indeed degenerate to (3.1) and (3.2) due to the identity \( T_n(t) = 0.5(U_n(t) - U_{n-2}(t)) \), respectively. We mention that the trigonometric expression of the Chebyshev functions is as follows
\[
T_{\mu,n}(\cos \theta) = \cos(n \theta - \mu \sin \theta), \quad (3.21)
\]
\[
U_{\mu,n+1}(\cos \theta) = \sin(n \theta - \mu \sin \theta) / \sin \theta. \quad (3.22)
\]

Seemingly complicated (3.1-3.2) are actually quite simple in the trigonometric representation. For the case of \( n = 1, 2 \) in (3.20), the numerical experiments are performed to verify the stability.
IV. Numerical experiments

In practical imaging applications, the object function to be reconstructed is always bounded, thus \( f(t) \in L^2(I) \) and (2.38) is preferred to avoid using \( \int_1^\infty \cosh(\mu t)f(t)dt \). The discretization of (2.38) can be carried out by using the Chebyshev series investigated in [18-19] for fast and stable numerical implementation. Here we describe the idea to convert the singular integrals of (2.38) to normal integrals. Define the coordinate transformations

\[
s = \cos \theta, \quad t = \cos \varphi, \quad \theta, \varphi \in (0, \pi).
\]

Under angular coordinates (4.1), \( L^2(I) \) is equivalent to \( L^2[0, \pi] \), and we use \( \| dH^{-1}_\mu F_\mu - f \|_{L^2[0,\pi]} \) to measure the error between \( f \) and \( dH^{-1}_\mu F_\mu \). \( \| dH^{-1}_\mu F_\mu - f \|_{L^2[0,\pi]} \) is also called the mean square error (MSE) between the ground truth data and inverted or noise-contaminated data. In this paper the noise-signal-ratio (nsr) will be used to measure the noise level and response in the inverse. With (4.1), the inverse \( dH^{-1}_\mu F_\mu \) of (2.38) becomes

\[
dH^{-1}_\mu F_\mu (\cos \varphi) = \cos(\mu \sin \varphi) \sin \varphi \frac{1}{\pi} \int_0^\pi \cos(\mu \sin \theta) \sin \theta \cosh(\mu \cos \theta) d\theta
\]

\[
+ \sin(\mu \sin \varphi) \frac{1}{\pi} \int_0^\pi \sin(\mu \sin \theta) \cos \theta \cosh(\mu \cos \theta) d\theta.
\]

On the Chebyshev sampling points, (4.2) can be carried out by the Sine and Cosine transforms. Sine and Cosine transforms are orthogonal so that the numerical realizations are stable as shown from the early results in [18, 19] and the results in this paper. Chebyshev nodes are

\[
t_m = s_m = \cos\left(\frac{m+0.5}{N} \pi\right), \quad m = 0, \ldots, N-1.
\]

Define two matrices \( C = \{c(m,n)\} \) and \( S = \{s(m,n)\} \), \( m,n = 0, \ldots, N-1 \)

\[
c(m,n) = \sqrt{\frac{2}{N}} \cos\left(\frac{m+0.5}{N} n\pi\right), \quad s(m,n) = \sqrt{\frac{2}{N}} \sin\left(\frac{m+0.5}{N} n\pi\right).
\]

The inverse Hilbert transform in (4.2) is realized by \( H^{-1} = SC^T \) in the numerical experiments.

Identities (2.3) and (2.5) construct two non-trivial pairs of functions and their cosh-weighted FHTs. We use the pair given by (2.5) in our numerical experiments.

\[
\begin{align*}
f(t) &= \cos(\mu \sqrt{1-t^2}) \sqrt{1-t^2} \\
F_\mu(s) &= s \cosh(\mu s) - \frac{\mu}{2} \sinh(\mu s).
\end{align*}
\]

Under the coordinate transforms of (4.1), the pair of (4.5) becomes

\[
\begin{align*}
f(\cos \varphi) &= \cos(\mu \sin \varphi) \sin \varphi \\
F_\mu(\cos \theta) &= \cos \theta \cosh(\mu \cos \theta) - \frac{\mu}{2} \sinh(\mu \cos \theta).
\end{align*}
\]

The numerical experiments are designed to compute \( dH^{-1}_\mu F_\mu \) by (4.2) and then compare it against the source \( f(t) \). All the results are computed by the popular Python Numpy library. Total 1000 angular points are evenly sampled and displayed in \([0, \pi]\). The display of nsr is restricted to the precision with 2 decimal digits in terms of percentage. The plots of the inverses for two real and two imaginary \( \mu = 4\pi, 8\pi, 4\pi i, 8\pi i \) are shown below.
In all the plots, the right side is the overlay display of the source function $f$ and numerical inverse $H^{-1}_\mu F_\mu$. One surprising observation is that the inverse accuracy for large $\mu$ seems to be very high as shown in the plots and nsr. This is something that we were not able to achieve with the FBP algorithm in [17] and the Fourier analysis in [9].

Next we add Gaussian random noises to $F_\mu(s)$. Let $\tilde{F}_\mu(s) = F_\mu(s) + n(s, \sigma)$, $n(s, \sigma)$ is the Gaussian random noise with zero mean and standard deviation of $\sigma$. Denote by $f(t) = H^{-1}_\mu \tilde{F}_\mu(t)$ the inverse from noisy $\tilde{F}_\mu(s)$. For different level of standard deviations, we show the inverse results and their errors for $\sigma = 0.1$. The constants are $\mu = 4 \pi, 8 \pi, \pi i, 2 \pi i$. 

\[\begin{align*}
\mu &= 4 \pi \\
\mu &= 8 \pi \\
\mu &= 4 \pi i \\
\mu &= 8 \pi i
\end{align*}\]
For the same noise level, the inverse results look visually good for large real $\mu$ while the results become much worse for imaginary $\mu$. For curiosity, we perform two experiments with complex constants $12.0 + 12.0i$ and $24.0 - 24.0i$. The results are shown below.
For $\mu = 12.0 + 12.0i$, the inverse results are visually very accurate and the inverse results remains reasonably good even for large $\mu = 24.0 - 24.0i$. However increasing the number of sampling points from 1000 to 4000, the inverse results for $\mu = 24.0 - 24.0i$ did not become better. Please keep in mind that the value ranges of $f$ and $\tilde{F}_\mu(s)$ for $\mu = 24.0 - 24.0i$ are much higher as shown in the plots. With the same level of noise, the results for $\mu = 3.0 + 3.0i, 6.0 - 6.0i$ are shown below.
The Python implementation of (2.38) is straightforward and has been posted on GitHub for interesting readers to study the numerical characteristics of (2.38). For example, it is interesting to observe the different numerical characteristics from other pairs by (2.3) and (3.20)

\[
\begin{align*}
    f(\cos \varphi) &= \sin(\mu \sin \varphi) \\
    F_\mu(\cos \theta) &= \sinh(\mu \cos \theta).
\end{align*}
\] (4.7)

\[
\begin{align*}
    f(\cos \varphi) &= \sin(2\varphi - \mu \sin \varphi) \\
    F_\mu(\cos \theta) &= 0.5 \exp(\mu \cos \theta)[4 \cos^2 \theta + 2 \mu \cos \theta + 0.5 \mu^2 - 2.0].
\end{align*}
\] (4.8)

Here (4.8) is for \( n = 2 \) in (3.20). All the numerical experiments don’t consider any application related data characteristics and it is interesting to study the reconstruction in SPECT and other type of tomographic imaging with imaginary constant \( \mu \).
V. Discussions

The study on the inversion of cosh-weighted FHT was initiated by [14]. With the Bertola-Katsevich-Tovbis inversion formulas [2], it seems that the theoretical study reaches a point at which the half scan formulated in [14] is solved with the explicit inversion formulas (2.38) and (2.39). The constant $\mu$ can be complex in the formulas (2.38) and (2.39), thus (1.2) is useful to ultrasonic tomography and Doppler tomography imaging described in [9, 13, 17].

Initial numerical results show the feasibility of (2.38) for practical use but more quantitative numerical analysis from complicated data pertinent to particular applications is desirable. Here the author would like to give more comments on the numerical advantage of (2.38) over (2.39). The weighted integral $\int_{-1}^{1} \cosh(\mu t) f(t) dt$ is required in (2.39) so that any noise or perturbation to that weighted integral could lead to deviations to $f(t)$ on each point. From application point of view, the confidence of using (2.39) is unknown. The term $1/\sqrt{1-\tau^2}$ could significantly increase the errors around two end points so that (2.39) is not robust to errors. Fortunately, formula (4.2), angular version of (2.38), can void the uncertainty of the weighted integral and the error amplification at two ends.

Another important numerical characteristic is that the Chebyshev nodes (4.3) and the transformation (4.4) generates remarkably high accuracy of (2.38) from four test functions.

Python implementation by Jupyter Notebook can be found from GitHub:

https://github.com/jshyou/Medical-Imaging/blob/master/complexCoshFHT.html
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