Out-of-equilibrium clock model at the verge of criticality

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We consider an out-of-equilibrium lattice model consisting of 2D discrete rotators, in contact with heat reservoirs at different temperatures. The equilibrium counterpart of such model, the clock-model, exhibits three phases; a low-temperature ordered phase, a quasi-liquid phase, and a high-temperature disordered phase, with two corresponding phase transitions. In the out-of-equilibrium model the simultaneous breaking of spatial symmetry and thermal equilibrium give rise to directed rotation of the spin variables. In this regime the system behaves as a thermal machine converting heat currents into motion. In order to quantify the susceptibility of the machine to the thermodynamic force driving it out-of-equilibrium, we introduce and study a dynamical response function. We show that the optimal operational regime for such a thermal machine occurs when the out-of-equilibrium disturbance is applied around the critical temperature at the boundary between the first two phases, namely where the system is mostly susceptible to external thermodynamic forces and exhibits a sharper transition. We thus argue that critical fluctuations in a system of interacting motors can be exploited to enhance the machine overall dynamic and thermodynamic performances.

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The critical behaviour of equilibrium systems is well understood and supported by innumerable experimental observations. As a system approaches a critical point larger fluctuations in the local order parameter take longer time to emerge and longer time to decay. Also the response to a change in an external thermodynamic force can become significantly large. For example, in a gas at equilibrium with its liquid phase at the critical point a small change in the pressure produces a large density change, corresponding to a diverging compressibility. Other well known examples of diverging responses are the specific heat and the susceptibility of a critical 3D Ising model. Thus, criticality is characterized by wide fluctuations with diverging characteristic lengths, susceptibilities, and relaxation times.

One may wonder what is the fate of the equilibrium critical fluctuations if a disturbance is applied, that drives the system out of equilibrium. The system must be certainly removed from criticality, but at least for small nonequilibrium perturbation one may expect that fluctuations in the system persist and are still quite large. An even more interesting question is what happens to the diverging response of a critical system when such a nonequilibrium disturbance is applied. Is the response of a system close to its critical points enhanced with respect to the case where the system is non critical?

This question is particularly relevant in the context of stochastic and quantum thermodynamics, where several groups have addressed the issue of performance optimization in microscopic machines operating as engines and motors. In particular, a few publications have shown that collective effects can enhance the efficiency of interacting microscopic machines. In models of interacting molecular motors the efficiency at maximum power of the many-motor system can be larger than in the single motor case. Models of interacting work-to-work transducers with all-to-all interaction exhibit a similar behaviour. Furthermore, in ref. the authors showed that the Carnot efficiency can be approached at finite output power in a system of microscopic interacting Otto engines when the working substances operate at the verge of a second order phase transition and the critical exponents satisfy some given constraints.

In this paper we investigate the role of criticality in the performance of thermal machines driven by temperature gradients: in particular we show that the out-of-equilibrium dynamic response of a system is maximal in the case in which the driving perturbation is applied at the verge of one of its critical points, i.e. when the system is most susceptible to equilibrium thermodynamic forces. We also demonstrate that phase transitions with different strengths are characterized by different dynamical responses, with the smoother equilibrium phase transition exhibiting the feeble out-of-equilibrium response.

Specifically we consider a model of interacting autonomous thermal motors, that converts steady state heat currents into mechanical motion. Autonomous motors are systems which continuously and cyclically convert one form of energy into another: such machines are characterized by the absence of one or more external agents that change their Hamiltonian or temperature. As such they are different from reciprocating engines that perform thermodynamic cycles, such as the Carnot, the Stirling and the Otto cycle, which have been widely investigated in the context of stochastic and quantum thermodynamics.

Our working tool will be the 2D clock model, consisting of \(L \times L\) discrete rotors on a discrete 2D lattice, interacting through the Hamiltonian

\[
H(\{\theta_i\}) = -\frac{k}{2} \sum_{(i,j)} \cos [\theta_i - \theta_j + \varphi]. \tag{1}
\]
has been shown to work as an autonomous motor: when different temperatures the spins rotate with a given average frequency, thus converting heat currents into motes. We subdivide the lattice into two sublattices, each in contact with a heat bath at a different temperature $T \pm T$. This model has the advantage that i) a single dimer consisting of only two spins interacting with the Hamiltonian \[ H = \frac{\pi}{2} (J_{ij} + J_{kl}) \] ($i = 1, j = 2$) has been shown to work as an autonomous motor: when different temperatures the spins rotate with a given average frequency, thus converting heat currents into motion and eventually into work \[ 14, 15; \] and ii) its non-chiral equilibrium counterpart ($\varphi = 0$, $\Delta T = 0$) exhibits a rich phase diagram with two second-order phase transitions \[ 16].

In the following we use Monte Carlo (MC) simulations with Metropolis algorithm to first investigate the equilibrium properties of the model \[ 11 \] ($\Delta T = 0$) and then its out-of-equilibrium dynamic properties, in particular the spin rotation rate for $\Delta T \neq 0$, and the heat rates exchanged by the system with the two reservoirs $\mathbb{Q}_\pm$. We define the angular velocity per spin as $\omega = (\omega_+ + \omega_-)/2$, where $\omega_\pm$ is the angular velocity per spin in the hot (cold) sublattice. The heat currents $Q_\pm$ along a single MC trajectory can be evaluated by sampling the total energy exchanged by the system with each of the two reservoirs along that trajectory.

**Equilibrium properties of the model**

We first study the equilibrium behaviour of the model \[ 11 \], and we benchmark our results with those discussed by Lapilli et al. in \[ 16 \], where the equilibrium properties of the non–chiral clock model ($\varphi = 0$) were extensively studied, and the model phase diagram was established by using MC simulations. In particular, as discussed by the authors, the model \[ 11 \] with $4 < N_s < \infty$ and $\varphi = 0$ exhibits three phases: a low-temperature ordered and a high-temperature disordered phase, as in the Ising model, along with a quasi-liquid intermediate phase. As $N_s \to \infty$ the ordered phase is suppressed, and the model becomes equivalent to the continuous $XY$ model. We notice that, because of the additional $1/2$ prefactor in our model \[ 11 \], the temperature scale in the present paper is halved with respect to \[ 16 \]. We label $T_1^eq$ the equilibrium transition temperature between the low-temperature ordered and the quasi-liquid phase, and $T_2^eq$ the equilibrium transition temperature between the quasi-liquid and the disordered phase. By using MC simulations, Lapilli et al. \[ 16 \] provided an expression for the transition temperature $T_1^eq$ as a function of the number of states for $N_s \geq 6$. They also reported an extended universality, which implies that the transition $T_2^eq$ must be of the Berezinskii–Kosterlitz–Thouless (BKT) class, thus indistinguishable from $N_s = \infty$, for $N_s \geq 8$. Such equilibrium properties are reported in fig. \[ 11 \] as lines, together with the results from our MC simulations.

Similarly, we run equilibrium MC simulations ($\Delta T = 0$) for the chiral model $\varphi = \bar{\varphi}$ \[ 22 \], where we fix $\bar{\varphi} = \pi/(2N_s)$, and evaluate the equilibrium transition temperatures $T_1^eq$ and $T_2^eq$, for different values of $N_s$, by following the approach discussed in Appendix \[ A \] and in \[ 16 \]. These results are also shown in fig. \[ 11 \] and the agreement with the findings of \[ 16 \] are excellent for $N_s \geq 8$. This can be understood by noticing that the chosen phase $\varphi = \bar{\varphi}$ decreases with $N_s$, so the equilibrium properties of the two models become equivalent in the limit of large $N_s$.

![FIG. 1: Circles, squares, lines: equilibrium transition temperatures for the non-chiral ($\varphi = 0$) and the chiral clock model ($\varphi \neq 0$) as given by eq. (1). Lines: the expressions for $T_1^eq$ (full) and $T_2^eq$ (dotted) are taken from ref. \[ 16 \]. Open circles: $T_1^eq$ for $N_s \leq 6$, $\varphi = 0$, $\Delta T = 0$ \[ 16 \], the dashed line is a guide to the eye for the results with $N_s \leq 6$. Full circles: numerical results for the first transition temperature $T_1^eq$ as a function of $N_s$ ($N_s \leq 40$) for the equilibrium model with $\varphi = \bar{\varphi}$. Squares: numerical results for the second transition temperature $T_2^eq$ as a function of $N_s$, for the equilibrium model with $\varphi = \bar{\varphi}$. Triangles: temperature $T$ that maximizes the dynamic susceptibility $\chi$ in the out-of-equilibrium clock model with $\varphi = \bar{\varphi}$, see text and figure \[ 3 \]. Inset: Zoom of the small $N_s$ region.](image)
equilibrium system with $T_\pm = T_m \pm \Delta T$ is plotted in fig. 2 as a function of $T_m$ for $N_s = 16$ and for different values of $L$ and $\Delta T$. We find that at large $L$ the curve $\omega(T_m)$ becomes independent of $L$, see fig. 2 (a).

\[
\chi_\omega = \frac{\partial \omega}{\partial \Delta T} \bigg|_{\Delta T = 0},
\]

i.e., the response of the angular velocity with respect to the thermodynamic force that generates it. In the following we argue that this is the relevant quantity to characterize the out-of-equilibrium response of a system close to its critical points.

Given that $\omega = 0$ when $\Delta T = 0$ we can thus evaluate the response function $\chi_\omega$ by numerical derivation $\chi_\omega \approx \omega/\Delta T$. Unless otherwise stated, in the following we take $\Delta T \approx T_1^{eq}/100$, for any $N_s$. Indeed when $\Delta T$ is as small as $T_1^{eq}/100$ the curves $\omega(T_m)/\Delta T$ collapse into a single curve, see fig. 2 (b).

The results for $\chi_\omega$ as a function of $T_m$ and for different values of $N_s$ are shown in fig. 4 (a). We find that each curve $\chi_\omega(T_m, N_s)$ exhibits a peak at a temperature that depends on $N_s$. We indicate such a temperature $T^*$: it decreases as $N_s$ increases. We refer to the value of the dynamic response function at its maximum as $\chi^{*}_{\omega} = \chi_\omega(T^*, N_s)$.

For any number of states $N_s$, we then compare the temperature $T^*$, as obtained for two values of $\Delta T$, with the model equilibrium properties. In particular we compare the peak temperature with the two transition temperatures $T_1^{eq}$ and $T_2^{eq}$. Interestingly, the temperature $T^*$ (triangles in fig. 1) where $\chi_\omega(T_m, N_s)$ has its maximum $\chi^{*}_{\omega}$ follows very closely the equilibrium transition temperature $T_1^{eq}$ for $N_s \geq 4$ (full circles in fig. 1) with a relative difference $(T^* - T_{1}^{eq})/T_{1}^{eq} < 0.15$. This in turn implies that the points $T^*$ with $N_s \geq 6$ lie near the curve given in 16) for the equilibrium temperature $T_1^{eq}$ of the non-chiral model (full line in fig. 1). Accordingly, we conclude that the response of the system to the out-of-equilibrium perturbation $\Delta T$ is maximal when the perturbation is applied on the system in the proximity of one of its critical points: namely the transition point between the ordered and the quasiliquid phase.

A more precise analysis on the correspondence between $T^*$ and $T_1^{eq}$ can be found in Appendix B. It is worth to note that the equilibrium transition temperatures $T_1^{eq}$ reported in 17) for the non-chiral system are somewhat larger than those found in 16), and confirmed by our simulations. The method to evaluate $T_1^{eq}$ in 17) is however different from the one used in 16), and that we employ in the present manuscript. Furthermore, in 17) a smaller range of $N_s$ was considered, $N_s \leq 16$. For completeness sake we report the equilibrium temperature $T_1^{eq}$ as given in 17) in fig. 2.

Unlike $T_1^{eq}$, the temperature $T_2^{eq}$ does not give rise to any nonvanishing response, for any value of $N_s$. This might be seen as a consequence of the fact that the critical behaviour of the $XY$ model at the BKT transition exhibits very weak singularities 18, 19). For example the specific heat at $T_{BKT}$ displays a singularity which is essentially unobservable 19).

![Fig. 2:](image)

FIG. 2: (a) Average angular velocity as function of the mean lattice temperature $T_m$, for $N_s = 16$ and $\Delta T = 4 \cdot 10^{-4}$. (b) Ratio of the average angular velocity to the temperature gradient as a function of the mean lattice temperature $T_m$, for $N_s = 16$ and $L = 64$. The vertical dashed lines correspond to the equilibrium transition temperature $T_1^{eq}$ for the clock model ($\varphi = 0$, light gray) 16, and the chiral clock model ($\varphi \neq 0$, dark gray), see fig. 1. The light gray dashed-dotted vertical line marks $T_1^{eq}$ for the equilibrium clock model ($\varphi = 0$) as obtained in 17). In both panels we considered trajectories consisting of 1000 $L^2$ MC steps. In (a) the results are obtained by averaging over $(10^6, 2 \cdot 10^5, 5 \cdot 10^4, 8000, 8000, 1000)$ independent trajectories for the different system sizes ($L = 4, 8, 16, 32, 64, 128$). In (b) the results are obtained by averaging over $(10^4, 8000, 200, 100, 100)$ trajectories for increasing $\Delta T$.

In order to investigate the role of the critical fluctuations in a non-equilibrium system we introduce a dynamic response to a non-equilibrium perturbation, akin to the susceptibility in equilibrium systems. In equilibrium statistical mechanics the susceptibility of an order parameter $\phi$ with respect to its conjugate thermodynamic force $f_\phi$ reads $\chi_{\phi} = \partial \phi / \partial f_\phi |_{f_\phi = 0}$. Similarly we introduce the susceptibility

\[
\chi_\omega = \frac{\partial \omega}{\partial \Delta T} \bigg|_{\Delta T = 0},
\]
in order to quantify the performance of the system, and confirmed by our simulations (data shown in Appendix C).

Interestingly, such a figure increases linearly with the number of states, although with a slow rate. However, it is worth noticing that the velocity goes to zero as $N_s \to \infty$, as can be seen in the lower inset of fig. 4. The fact that the two-temperature continuous XY model exhibits a vanishing rotational velocity has already been discussed in [14]: in this limit the energy $\mathcal{H}$ exhibits an $O_2$ symmetry, the group of rotations in a two-dimensional plane, and thus the model does not break the spatial symmetry discussed above.

Interestingly, the increase of $\chi_\omega^\ast$ is accompanied by the decrease of the order parameter critical exponent for the corresponding equilibrium system at $T_s^\text{eq}$, see Appendix A. We thus conclude that as the equilibrium transition becomes sharper, so does the corresponding out-of-equilibrium response.

![Fig. 3](image1.png)

**Fig. 3:** (a) Out-of-equilibrium dynamical susceptibility $\chi_\omega$ as a function of the mean lattice temperature $T_m$, for different number of states $N_s$, $L = 64$ and $\Delta T \approx T_s^\text{eq}/100$. We consider 8000 independent trajectory, each consisting of 1000 $L^2$ MC steps, after the system has reached its steady state. For the system with $N_s = 6$ we indicate the temperature $T^\ast$ for which $\chi_\omega$ is maximum. The BKT temperature $T_s^\text{eq}$, indicated in panel (a), is independent of the number of states $N_s$, see fig. 4. Inset: $\chi_\omega$ as a function of $T_m$ for larger values of $N_s$. (b) Ratio of the average angular velocity to the heat current $z = \langle \omega \rangle/\langle Q_+ \rangle$, as a function of the mean temperature. Inset: $z$ as a function of $T_m$ for larger values of $N_s$.

The model here considered entails a tight coupling between the thermal ($Q_\pm$) and the mechanical currents ($\omega$). Indeed, since the transition rates for the single rotator obey a local detailed balance condition, the change of orientation of one rotator is associated with a concomitant exchange of heat with the corresponding bath. Thus one expects the heat current to peak around $T^\ast$. This result confirms that the optimal operational regime is achieved at $T_m \approx T_s^\text{eq}$, see fig. 3(b).

Inspection of fig. 4 suggests that the height of the peak in both the dynamic susceptibility and the figure of merit is an increasing function of the number of states $N_s$. Thus as next point we investigate the dependency of $\chi_\omega^\ast$ on $N_s$; the results are plotted in fig. 4. Such a quantity increases linearly with the number of states, although with a slow rate. However, it is worth noticing that the velocity goes to zero as $N_s \to \infty$, as can be seen in the lower inset of fig. 4. The fact that the two-temperature continuous XY model exhibits a vanishing rotational velocity has already been discussed in [14]: in this limit the energy $\mathcal{H}$ exhibits an $O_2$ symmetry, the group of rotations in a two-dimensional plane, and thus the model does not break the spatial symmetry discussed above.

Interestingly, the increase of $\chi_\omega^\ast$ is accompanied by the decrease of the order parameter critical exponent for the corresponding equilibrium system at $T_s^\text{eq}$, see Appendix A. We thus conclude that as the equilibrium transition becomes sharper, so does the corresponding out-of-equilibrium response.

![Fig. 4](image2.png)

**Fig. 4:** Points: maximum value of the dynamical susceptibility $\chi_\omega^\ast$ as a function of the number of states $N_s$, from the data in fig. 3(a), as obtained with MC simulations for different number of states $N_s$, $L = 64$, and $\Delta T \approx T_s^\text{eq}/100$. The full line is a linear fit to the data, with slope $b = (941.0 \pm 1.6) \cdot 10^{-6}$. Upper inset: zoom of the small $N_s$ region. Lower inset: average angular velocity $\langle \omega^\ast \rangle$ at $T^\ast$ as a function of the number of states.

**Conclusions**

To summarize, we have used the 2D chiral clock model to show that the optimal operational regime of a system of interacting autonomous motors occurs at the verge of a phase transition, in the limit of small driving thermodynamic forces. While the out-of-equilibrium response around the ordered-quasiliquid phase transition is noticeably sharp, the system response is vanishing in the other temperature ranges, in particular at the quasiliquid-disordered phase transition temperature. The same behaviour is found if one considers a figure of merit akin to
a miles per gallon equivalent: the motors’ “consumption” is most favorable around the strong phase transition. Interestingly, we find also that the response of the system increases, although slowly, with the number of states (directions) $N_s$ in the clock model. This increase occurs while the corresponding phase transition for the equilibrium system becomes sharper. Our results provide strong evidence for the following protocol that one has to follow in order to find the optimal working regime in systems of microscopic interacting motors: one has to first identify the system critical points, if any, and then systematically perturb the system at the verge of criticality.

Experimentally, a 2D system of interacting molecular rotors deposited on an hexagonal lattice have been reported in [20]. That artificial molecular system represents a possible setup for an experimental realization of the model we have studied here.

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Appendix A: Critical points for the chiral clock model at equilibrium

The Hamiltonian for the clock model in two dimensions $\varphi = 0$ with a global symmetry $Z_N_s$ ($H(\theta_i)$ is invariant under the transformations $\theta_i \rightarrow \theta_i + 2\pi n/N_s$). Thus one introduces the order parameter

$$\langle m \rangle = \frac{\langle M \rangle}{L^2} = \frac{\langle \sum_{j=1}^{N_s} \exp(i\varphi_j) \rangle}{L^2},$$

(A1)

that is invariant under the same symmetry group as the Hamiltonian in the high-temperature disordered phase. Besides this phase, the system exhibits two other phases for $N_s \geq 4$ (low-temperature ordered and quasi-liquid intermediate phase) and thus two critical points, as discussed in [10]: an ordered to quasi-liquid second-order phase transition and a quasi-liquid to disordered phase transition, that turns out to be of the Berezinskii-Kosterlitz-Thouless class for $N_s \geq 8$.

In the present paper we have considered the chiral model with $\varphi = \varphi$: through equilibrium MC simulations of such a model we have evaluated the order parameter (A1) as a function of the temperature. Such a quantity is shown in fig. 5 for $N_s = 6$. Inspection of this figure indicates that the chiral model also exhibits two phase transitions.
The order parameter for the standard clock model \((\varphi = 0)\) is included for comparison.

The order parameter curves display two abrupt drops as the temperature is increased, from the low-temperature ordered phase \((\langle m \rangle \approx 1)\) to the high-temperature disordered phase \((\langle m \rangle \approx 0)\). In order to evaluate the corresponding transition temperatures we follow the approach used in \[16\]. To obtain the low-temperature transition temperature \(T_{1}^{\text{eq}}\) of the non-chiral \((\varphi = 0)\) clock model obtained by Lapilli et al. \[10\]. Inset: Temperature derivative (full lines) of the equilibrium order parameter around the first transition temperature for \(L = 160\) for both \(\varphi = 0\) and \(\varphi \neq 0\). We consider 100 independent trajectories each consisting of 1000 \(L^2\) MC steps following an equilibration run of 1000 \(L^2\) MC steps.

The critical temperatures as computed employing the aforementioned approaches for the equilibrium chiral clock model are depicted in fig.\[11\] The second equilibrium transition temperature is the same for both the chiral version \((\varphi \neq 0)\) and the standard clock model \((\varphi = 0)\), as it can be anticipated by the collapse of the corresponding order parameters curves in fig.\[5\]. However, the order parameter curves deviate as they approach the lower-temperature critical point, thus the chiral clock model exhibits the first transition at a lower temperature. This deviation decreases with \(N_s\) as the two models become equivalent in this limit.

For different values \(N_s\), and different sizes \(L\), we evaluate the Binder’s fourth order cumulant for the energy

\[
V_L = 1 - \frac{\langle E^4 \rangle}{3 \langle E^2 \rangle^2}. \tag{A3}
\]

The critical temperatures are computed employing the following analysis \[21\] to obtain

\[
U_L = 1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2}. \tag{A2}
\]

For \(T > T_{2}^{\text{eq}}\) and assuming \(L \gg \xi\), where \(\xi\) is the correlation length of the order parameter fluctuations, \(U_L \propto L^{-d}\) \((d\) is the dimension of the system). On the other hand, \(U_L \to U_\infty = 2/3\) when \(T < T_{2}^{\text{eq}}\) and \(L \gg \xi\); and \(U_L\) varies weakly with \(T\) and \(L\) for \(L \ll \xi\). Thus, the cumulant \[A2\] is scale invariant at the transition temperature, and therefore the transition temperature is given by the intersection between ratios of cumulants for different system sizes \[22\].

The critical temperatures as computed employing the aforementioned approaches for the equilibrium chiral
Appendix B: Correspondence between $T^*$ and $T_{1}^{eq}$

The temperature at which the susceptibility eq. (2) is maximal $T^*$ has been shown to depend on $\Delta T$, see fig. 2(b). Its behavior when varying the number of states $N_s$ is depicted in fig. 1, where two different temperature gradients are considered $\Delta T \simeq T_{1}^{eq}/10$ and $\Delta T \simeq T_{1}^{eq}/100$. The maximum temperature $T^*$ follows closely the equilibrium clock model transition temperature $T_{1}^{eq}$ for both $\varphi = 0, \bar{\varphi}$, yet some discrepancy is already anticipated in fig. 1. In order to compare $T^*$ and $T_{1}^{eq}$ a more convenient representation of the data in fig. 1 is to plot the relative deviation of the maximum susceptibility’s temperature $T^*$ with respect to the equilibrium transition temperature $T_{1}^{eq}$ as a function of the number of states, see fig. 7.

When decreasing the number of states, and hence when the standard clock model begins to deviate from the chiral one, the maximal susceptibility temperature $T^*$ tends to approach the corresponding equilibrium temperature $T_{1}^{eq}$. Yet a deviation persists, so that $T^*$ is always above $T_{1}^{eq}$ when the temperature gradient is small. On the other hand, the difference between $T^*$ and $T_{1}^{eq}$ is larger for the smaller $\Delta T$ considered, in agreement with the results plotted in fig. 2(b).

Appendix C: Heat current

The heat currents $\dot{Q}_\pm$ along a single MC trajectory can be retrieved by exploiting the conservation of energy: the energy rate exchanged by each of the two sublattices with the corresponding heat reservoir must amount to the energy variation due to the sublattice’s spin transitions, given by eq. (1). Accordingly, the sum of the two heat currents should sum up to zero on average.

At the microscopic level the local detailed balance condition is obeyed by the transition rates of the single rotators: this ensures tight coupling between the thermal ($\dot{Q}_\pm$) and the mechanical currents ($\omega_\pm$). Therefore the heat currents for different $N_s$ peak in proximity of $T^*$, see fig. S(a).

That peak at $T_m \simeq T^*$ is nonetheless not unique; a second maximum (possibly larger than the one discussed above) is found above $T_{1}^{eq}$, see fig. S(b), although only for small $N_s$.

The maxima displayed by the curves $\langle \dot{Q}_+ \rangle/\Delta T$ close to the transition temperature $T_{1}^{eq}$ is constant with respect to the number of states $N_s$, see fig. S(b), at variance with the mechanical current response function as defined in eq. (2) see also fig. S(a) in the main text. A rescaling of the $x$-axis thus leads to a collapse of the heat current curves around the transition temperature $T_{1}^{eq}$, see fig. S(c). Both the mechanical and thermal currents are expected to be linear with respect to the thermodynamic force when $\Delta T$ is small, as it is in our case, according to the Onsager reciprocal relations. However, we find a different scaling behavior of the two currents with respect to $N_s$, as suggested when comparing fig. S(a) and fig. S(b).
FIG. 8: (a) Average steady-state heat current entering the system from the heat reservoir \( \langle \dot{Q}_+ \rangle \) as a function of the mean lattice temperature \( T_m \) for different number of states \( N_s \), \( L = 64 \) and \( \Delta T \approx T_1^{eq}/100 \), corresponding to the simulations in fig. 3. (b,c) Ratio of the average heat current entering the system from the heat reservoir to the out-of-equilibrium perturbation \( \Delta T \) as a function of the mean temperature \( T_m \) (b) and as function of \( T_m/T_1^{eq} \) (c) for different number of states \( N_s \), \( L = 64 \) and \( \Delta T \approx T_1^{eq}/10 \). The coloured dashed lines and the full grey line in (b) label respectively the equilibrium transition temperatures \( T_1^{eq}(N_s) \) and \( T_2^{eq}(N_s \geq 8) = T_{BKT} \) for the standard clock model (\( \phi = 0 \)) as obtained in [16].