Damping of Hydrodynamic Modes in a Trapped Bose Gas above the Bose-Einstein Transition Temperature

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We calculate the damping of low-lying collective modes of a trapped Bose gas in the hydrodynamic regime, and show that this comes solely from the shear viscosity, since the contributions from bulk viscosity and thermal conduction vanish. The hydrodynamic expression for the damping diverges due to the failure of hydrodynamics in the outer parts of the cloud, and we take this into account by a physically motivated cutoff procedure. Our analysis of available experimental data indicates that higher densities than have yet been achieved are necessary for investigating hydrodynamic modes above the Bose-Einstein transition temperature.

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In recent experiments on magnetically-trapped atomic vapors, alkali atoms [1–3] have been cooled to temperatures at which they are degenerate and indeed Bose-Einstein condensation has been observed in them. Frequencies and damping rates of collective modes in these systems have been investigated, both above and below the Bose-Einstein transition temperature, T_c [4–6]. In this Letter we shall focus on properties above T_c. We calculate the first term on the right hand side of Eq. (2) by using the energy conservation condition [8], and find that

\[ \frac{\partial (\rho c)}{\partial t} = -\nabla \cdot (\rho w \mathbf{v}) + \rho \mathbf{v} \cdot \mathbf{f}, \]  (3)

where \( \rho \) is the mass density and \( \epsilon \) and \( w \) are, respectively, the internal energy and the enthalpy of the fluid per unit mass. Since we assume local thermodynamic equilibrium and neglect contributions to the energy due to interparticle interactions, we may use the results \( p = \rho (w - \epsilon) \) and \( \rho c = 3p/2 \), and find that

\[ \frac{\partial p(r,t)}{\partial t} = -\frac{5}{3} \nabla \cdot [p_0(r)\mathbf{v}] + \frac{2}{3} n_0(r) \mathbf{v} \cdot \mathbf{f}. \]  (4)

Combining Eqs. (3) and (4), and using the fact that \( \nabla n_0(r) \) is proportional to \( \nabla U_0(r) \), we obtain for the equation of motion for \( \mathbf{v}(r,t) \),

\[ \frac{\partial^2 \mathbf{v}}{\partial t^2} = \frac{5}{3 m n_0(r)} \nabla [\nabla \cdot \mathbf{v}] + \nabla [\mathbf{v} \cdot \mathbf{f}] + \frac{2}{3} [\nabla \cdot \mathbf{v}] \mathbf{f}. \]  (5)

Equation (5) has previously been derived by Griffin et al. [7] using kinetic theory.

In our present discussion we assume that the potential is axially symmetric,

\[ U_0(r) = \frac{1}{2} m \omega_0^2 (x^2 + y^2 + \lambda z^2), \]  (6)

where \( \omega_0 \) is the frequency of the trap in the \( x - y \) plane; \( \lambda \equiv \omega_z^2/\omega_0^2 \) — where \( \omega_z \) is the frequency along the \( z \)-axis — expresses the anisotropy of the trap. Most experiments on oscillations have been done in such traps, but our results may be generalized to the case of traps with no axis of symmetry. As shown in Ref. [6], in the lowest modes, which in a spherical trap correspond to monopole and quadrupole vibrations, the velocity field has the form \( \mathbf{v}(r,t) = \mathbf{v}(r) \cos(\omega t) \), where \( \mathbf{v}(r) = a(\hat{x} r + \hat{y} y) + b \hat{z} \). Here \( a \) and \( b \) are constants, and the frequencies, \( \omega \), of the two modes are

\[ m n_0(r) \frac{\partial^2 \mathbf{v}}{\partial t^2} = -\nabla \frac{\partial p(r,t)}{\partial t} - \nabla \cdot [n_0(r) \mathbf{v}] \mathbf{f}. \]  (2)

We calculate the first term on the right hand side of Eq. (2) by using the energy conservation condition [8].
\[
\left( \frac{\omega}{\omega_0} \right)^2 = \frac{1}{3} \left[ 4\lambda + 5 \pm (16\lambda^2 - 32\lambda + 25)^{1/2} \right], \quad (7)
\]
and \( b/a = 3\omega^2/2\omega_0^2 - 5 \).

We turn now to the damping of these modes. We adopt the standard approach of evaluating the rate of change of the mechanical energy, \( \dot{E}_{\text{mech}} \), associated with the mode, which is given by

\[
\dot{E}_{\text{mech}} = - \int \left( \frac{\kappa}{T} |\nabla T|^2 + \int \zeta (\nabla \cdot \mathbf{v})^2 \right) dr - \int \left( \frac{\eta}{2} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{i,k} \nabla \cdot \mathbf{v} \right)^2 \right) dr, \quad (8)
\]
where \( \kappa \) is the thermal conductivity, \( \eta \) is the first, or shear, viscosity, \( \zeta \) is the second, or bulk, viscosity and \( T \) is the temperature. Because the system is inhomogeneous, the transport coefficients are generally spatially dependent. Next we show that the shear viscosity is the only source of damping of the low-lying modes described above. The contribution from thermal conduction vanishes because there are no temperature gradients for these modes, and the contribution from the bulk viscosity vanishes because \( \zeta \) vanishes. To demonstrate the absence of temperature gradients, we observe that for the modes under consideration, which have the velocity field given above, \( \nabla \cdot \mathbf{v} = (2a + b) \cos(\omega t) \) is independent of position. Thus, from the continuity equation, \( dn/dt + n \nabla \cdot \mathbf{v} = 0 \), it follows that \( (dn/dt)/n \) is also constant.

A simple relaxation-time approximation, with a scattering rate equal to \( n_0(r) \sigma v \), where \( v = (2e/m)^{1/2} \) is the particle velocity and \( \epsilon \) is the single-particle energy, leads to the result \( C_\eta = 2\pi^2/(15\pi^{1/2}) \approx 0.426 \), while a variational calculation gives \( C_\eta = 5\pi^{1/2}/2^4 \approx 0.554 \).

An important feature of the expression for the viscosity is its dependence of the particle density. Consequently, the integral in Eq. \( \text{(1)} \) formally diverges at large distances from the center of the trap. The reason for this is that hydrodynamics fails, because in the outer parts of the cloud, particle mean free paths are too long for hydrodynamics to be applicable. To solve this problem, one should treat the outer parts of the cloud using kinetic theory, rather than hydrodynamics, but to obtain a first estimate of the effects we shall assume that the hydrodynamic description holds out to a distance such that an atom incident from outside the cloud has a probability of no more than \( 1/e \) of not suffering a collision with another atom. In the parlance of radiative transport, this corresponds to an optical depth of unity. Mathematically, this condition is

\[
1 \approx \int_{r_0}^{\infty} \frac{ds}{\tilde{l}(r)}. \quad (11)
\]
where the local mean free path is \( l(r) = [n_0(r) \sigma]^{-1} \), and \( r_0 \) is the cutoff, which depends on direction. The integral in Eq. \( \text{(12)} \) is to be performed along the path for which the density gradient is steepest, that is along \( \nabla U_0 \), and \( ds \) is the corresponding line element. For a classical distribution, \( n_0(r) = n(0)e^{-U_0(r)/k_B T} \), where \( n(0) \) is the density at the center. Since \( dU_0 = |\nabla U_0| ds \), Eq. \( \text{(11)} \) can be written as

\[
1 \approx n(0) \sigma k_B T \left[ \frac{1}{\nabla U_0(r_0)} e^{-U_0(r_0)/k_B T} \right]. \quad (12)
\]
For large values of the dimensionless parameter \( \lambda^{1/2} N a_{\text{scat}}^2 m \omega_0^2 / k_B T \), Eq. \( \text{(12)} \) gives

\[
r_{0,s} \approx 2\frac{k_B T}{m \omega_0^2} \frac{1}{\sin^2 \theta + 2\lambda \cos^2 \theta} \ln \tau_s(\theta), \quad (13)
\]
where \( \theta \) is the angle of \( r_{0,s} \) with respect to the z-axis. The dimensionless quantity \( \tau_s(\theta) \) is essentially the total optical depth at the center of the cloud

\[
\tau_s(\theta) \equiv \tau_0 \lambda^{1/2} \left( \frac{\sin^2 \theta + 2\lambda \cos^2 \theta}{\sin^2 \theta + 2\lambda \cos^2 \theta} \right)^{1/2}, \quad (14)
\]
where

\[
\tau_0 = \sigma n(0) \left( \frac{k_B T}{2m \lambda \omega_0^2} \right)^{1/2}. \quad (15)
\]

The density at the center of the cloud is \( n(0) = N\lambda^{1/2}(m \omega_0^2 / 2\pi k_B T)^{3/2} \). The volume of the atomic cloud

\[
\eta = C_\eta \frac{(m k T)^{1/2}}{\sigma}. \quad (10)
\]
is given by integrating $r_{0,s}^3/3$, Eq. (13), over the solid angle. Equation (10) with the variational estimate for the viscosity ($C_f = 5\pi^{1/2}/2^4$) can then be written as

$$\langle E_{\text{mech}} \rangle \approx -\frac{5\pi^{1/2}}{2^{7/2}3^2} (a-b)^2 \frac{(k_B T)^2}{m \omega_0^2 \sigma_{\text{scat}}} f(\lambda, \tau_0).$$

(16)

The function $f(\lambda, \tau_0)$ is defined as

$$f(\lambda, \tau_0) = \int_{-1}^{1} dx \times \left[ \ln \left( \tau_0 \lambda^{1/2} \left( \frac{(\lambda-1)x^2+1}{(\lambda-2)x^2+1} \right)^{1/2} \right) \right]^{3/2}.$$  

(17)

We expect the leading term ($\propto [\ln \tau_0]^{3/2}$) to be asymptotically exact for large $\tau_0$, but the value of the cutoff in the logarithm will depend on the detailed kinetic-theory solution to the boundary-layer problem.

**FIG. 1.** We plot $f(\lambda, \tau_0)$ from Eq. (14) as function of the number of particles $N$, for $\lambda = 8$, $a_{\text{scat}} = 53 \text{ Å}$, $\nu_0 = 129 \text{ Hz}$, and $T = 1.3 T_c$, which are the parameters in the experiment of Ref. [6]. Asymptotically, for $\tau_0 \gg 1$, $f(\lambda = 8, \tau_0) \approx 2\lambda^{-1/2}[\ln(\tau_0 \lambda^{1/2})]^{3/2} - 0.53[\ln(\tau_0 \lambda^{1/2})]^{1/2}$.

To calculate the damping rate we need to evaluate the mechanical energy of the cloud, $E_{\text{mech}}$. In any oscillator the time-average of $E_{\text{mech}}$ is equal to the maximum kinetic energy in the mode, so

$$\langle E_{\text{mech}} \rangle = \frac{1}{2} \int m v_0^2(r) dr,$$

and therefore,

$$\langle E_{\text{mech}} \rangle = N \frac{k_B T}{\omega_0} \left( a^2 + b^2/2\lambda \right).$$

(19)

We can now introduce the amplitude damping rate $\tau_{\text{damp}}^{-1}$ (as opposed to the energy damping rate, which is twice the amplitude damping rate), given by the absolute value of the ratio $\langle E_{\text{mech}} \rangle / 2 \langle E_{\text{mech}} \rangle$,

$$\tau_{\text{damp}}^{-1} \approx \frac{5\pi^{1/2}}{2^{9/2}3^2} \frac{k_B T}{m\omega_0^2 \sigma_{\text{scat}}} \frac{1}{N} \left( a-b \right)^2 \frac{(a-b)^2}{(a^2 + b^2/2\lambda)} f(\lambda, \tau_0).$$

(20)

In order to elucidate the origin of the damping of the modes given by Eq. (20), we remark that when the viscosity is taken to be constant, the non-equilibrium stress tensor is constant in space, and therefore does not give rise to any force on a fluid element. The damping is caused by the entropy generated by the non-equilibrium energy current density, which has a constant, non-zero divergence.

Let us now turn to the experiments on sound propagation that have been performed to date. A necessary condition for hydrodynamics to be applicable is that the mean free path be small compared with the characteristic length scale of the mode. For the low modes we are studying here, this is equivalent to the requirement that the optical depth $\tau_0(\theta)$ be large compared with unity in all directions. For the MIT experiment [3], for which $\lambda < 1$, this condition implies $\lambda^{1/2} \tau_0 > 1$. In terms of the characteristic lengths, $R_\perp = (2k_B T/m\omega_0^2)^{1/2}$ and $R_\parallel = (2k_B T/m\omega_0^2)^{1/2}$, which measure the spatial extent of the cloud along the z-axis and perpendicular to it, the condition $\lambda^{1/2} \tau_0 > 1$ is seen to be equivalent to the requirement $R_\perp \gg l(0)$, where $l(0)$ is the mean free path in the center of the cloud. For the parameters of this experiment [3], the minimum number of particles that is required for this condition to be satisfied is $\approx 3 \times 10^9$, whereas the number of particles at $T = 2 T_c$ is found to be $N \approx 5 \times 10^7$ [11]. For the parameters in the JILA experiment [14], the minimum value of $\tau_0(\theta)$ is $\tau_0$, since $\lambda > 1$. In this case the condition $\tau_0 \gg 1$ is equivalent to $R_\perp \gg l(0)$. Therefore, for the JILA experiment, the minimum number of particles required to attain hydrodynamic conditions is $\approx 1 \times 10^7$ at $T = 1.3 T_c$, which is to be compared with the experimental value $N \approx 8 \times 10^4$ at $T = 1.3 T_c$ [3]. Figure 1 shows $f(\lambda, \tau_0)$ as function of $N$, with all the other parameters equal to the ones of Ref. [6] and $T = 1.3 T_c$. The lowest value of $N$ is chosen to be $N = 1.1 \times 10^7$, i.e., the one for which $\tau_0 = 1$.

Now we estimate for the MIT and JILA experiments the magnitude of the characteristic lengths $R_\perp$ and $R_\parallel$, and compare them with the mean free path in the center of the cloud. The transition temperature is obtained from $k_B T_c = 0.94 N^{1/3} h(\omega_0/\omega_c)^{1/3}$. For the MIT experiment $N \approx 2.5 \times 10^7$ at $T = 1.1 T_c$ [11], while $\nu_0 \approx 2 \times 250 \text{ Hz}$ and $\lambda \approx 5.8 \times 10^{-3}$ [3]. At $T = 2 T_c$ the particle number is $\approx 5 \times 10^7$, resulting in $R_\perp \approx 380 \mu m$ and $R_\parallel \approx 29 \mu m$. The corresponding central density $n(0) = N/(\pi^{3/2} R_\perp R_\parallel^2)$ is $\approx 2.9 \times 10^{13} \text{ cm}^{-3}$. With the scattering cross section $\sigma \approx 1.9 \times 10^{-12} \text{ cm}^2 (a_{\text{scat}} \approx 28 \text{ Å})$ [12], we obtain $l(0) = [n(0) \sigma]^{-1} \approx 180 \mu m$, which is less than $R_\perp$, but much larger than $R_\parallel$. The other condition for hydrodynamic behaviour is that the frequency of the mode be small compared with an average particle scattering rate. A single particle undergoes collisions at a rate $n_0 v(r) \sigma_{\text{th}}$, where $v_{\text{th}} = (8k_B T/m)^{1/2}$ is the average particle velocity. The average scattering rate in the cloud is thus
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