THE VOLUME OF KÄHLER-EINSTEIN FANO VARIETIES 
AND CONVEX BODIES

ROBERT J. BERMAN, BO BERNDTSSON

Abstract. We show that the complex projective space $\mathbb{P}^n$ has maximal degree (volume) among all $n$-dimensional Kähler-Einstein Fano manifolds admitting a holomorphic $\mathbb{C}^*$-action with a finite number of fixed points. The toric version of this result, translated to the realm of convex geometry, thus confirms Ehrhart’s volume conjecture for a large class of rational polytopes, including duals of lattice polytopes. The case of spherical varieties/multiplicity free symplectic manifolds is also discussed. The proof uses Moser-Trudinger type inequalities for Stein domains and also leads to criticality results for mean field type equations in $\mathbb{C}^n$ of independent interest. The paper supersedes our previous preprint [5] concerning the case of toric Fano manifolds.

Contents

1. Introduction 1
2. The volume of Kähler-Einstein Fano manifolds 7
3. The volume of (singular) Toric varieties and convex bodies 18
4. Symplectic geometry and multiplicity free actions 22
References 25

1. Introduction

1.1. Complex geometry. Let $X$ be an $n$-dimensional complex manifold $X$ which is Fano, i.e. its first Chern class $c_1(X)$ is ample (positive) and in particular $X$ is a projective algebraic variety. For some time it was expected that the top-intersection number $c_1(X)^n$, also called the (anti-canonical) degree of $X$, is maximal for the $n$-dimensional complex projective space, i.e.

\[ c_1(X)^n \leq (n + 1)^n, \]

There are now counterexamples to this bound. For example, as shown by Debarre (see page 139 in [16]), even in the case when $X$ is toric (i.e. $X$ admits an effective holomorphic action of the complex torus $(\mathbb{C}^*)^n$ with an open dense orbit) there is no universal polynomial upper bound on the $n$-th root of the degree of $X$. A more recent conjecture says that the bound above holds for any Fano manifold whose Picard number is one [37]. Given the special role of Kähler-Einstein metrics in complex geometry - in particular in connection to Chern number inequalities [59] - it is also natural to ask if the bound above
holds for any Fano manifold admitting a Kähler-Einstein metric $\omega$. Then $c_1(X)^n/n!$ is the volume of $X$ in the metric $\omega$. One step in this direction was taken by Gauntlett-Martelli-Sparks-Yau [27], who showed, using Bishop’s volume inequality, that if $X$ is a Fano Kähler-Einstein manifold then the inequality (1.1) holds when the right hand side is multiplied by $(n+1)/I(X)$, where $I(X)$ is the Fano index of $X$, i.e. the largest positive integer $I$ such that $c_1(X)/I$ is an integral class in the Picard group of $X$. As is well-known $I(X) \leq n + 1$ with equality precisely for $X = \mathbb{P}^n$ and hence the latter result leaves the question of the maximization property of $\mathbb{P}^n$ open. The main result in this paper shows that the inequality (1.1) indeed holds for Kähler-Einstein Fano manifolds in the presence of a certain amount of symmetry:

**Theorem 1.1.** Let $X$ be a Fano manifold which admits a Kähler-Einstein metric and a holomorphic $\mathbb{C}^*-\text{action}$ with a finite number of fixed points. Then the first Chern class $c_1(X)$ satisfies the following upper bound

$$c_1(X)^n \leq (n+1)^n$$

In other words, the complex projective space $\mathbb{P}^n$ has maximal degree among all Fano manifolds $X$ as above.

The starting point of the proof of the theorem is the fact that, under the assumptions in the theorem, there is a holomorphic $S^1$–action on $X$, preserving the Kähler-Einstein metric and with an attractive fixed point $p$. The key point of the proof, which builds on our previous work, is then to study $S^1$–invariant Moser-Trudinger type inequalities in a sufficiently large $S^1$–invariant Stein domain $\Omega$ in $X$ containing the fixed point $p$ (compare section 1.3 below).

The simplest class of varieties in which the assumption in the previous theorem are satisfied is the class of (generalized) flag varieties, i.e. rational $G$–homogeneous spaces. As is well-known these are all Fano manifolds and by homogeneity they also carry Kähler-Einstein metrics, invariant under the maximal compact subgroup $K$ of $G$. In this case the bound in the previous theorem was first obtained by Snow [54], using representation theory and quite elaborate calculcations.

More generally, the previous theorem applies to any Fano manifold $X$ on which a reductive connected complex algebraic group $G$ (i.e. $G$ is the complexification of compact Lie group $K$) acts algebraically with finitely many orbits (see Remark 4.2). A particularly rich class of such $G$–varieties is given by spherical varieties (i.e. a Borel subgroup $B$ of $G$ has an open dense orbit in $X$) [43] [11] [10]. In case a spherical variety is Fano it may or may not admit a Kähler-Einstein metrics and the inequality in the previous theorem can hence be viewed as a new obstruction for the existence of a Kähler-Einstein metric on spherical Fano varieties. According to a formula of Brion [11] the top-intersection number $c_1(L)$ of a polarized spherical variety $(X, L)$ can be expressed as an explicit integral over a certain polytope $P$ naturally associated to $X$. We will recall the symplecto-geometric description
of Brion’s formula in section 4. Let us also point out that an interesting classical subclass of spherical varieties is offered by Schubert varieties and it is well-known that any smooth Schubert variety in a Grassmannian is Fano. Another rich subclass is given by $G$–equivariant compactifications of symmetric spaces and in particular the so called wonderful compactifications, which are often Fano. In fact, smooth wonderful compactifications are always weakly Fano, i.e. $-K_X$ is nef and big and, in fact, Theorem 1.1 is still valid when $X$ is merely weakly Fano if one uses the notion of (singular) Kähler-Einstein metrics introduced in [7] (see Remark 2.8).

Of course, in Theorem 1.1 it is enough to assume that $X$ can be deformed to a complex manifold satisfying the assumptions in the theorem. Moreover, in the absence of a Kähler-Einstein metric we show that the inequality in the Theorem 1.1 still holds when the right hand side is multiplied by $1/R(X)^n$, where $R(X)$ is the greatest lower bound on the Ricci curvature of $X$ (see Theorem 2.9).

Toric Fano varieties. In the special spherical case when $X$ is a toric manifold, i.e. the groups $G$ and $B$ both coincide with the complex torus $\mathbb{C}^n$, the inequality in Theorem 1.1 was conjectured to hold by Nill-Paffenholz [47]. We expect that the previous theorem can be extended to singular spherical Fano varieties admitting (singular) Kähler-Einstein metrics, but we will only show this for toric varieties. First recall that, by definition, $X$ is a Fano variety if $K_X$ is an ample $\mathbb{Q}$–line bundle and $\omega$ is a singular Kähler-Einstein metric on $X$ if it is a bona fide Kähler-Einstein metric on the regular locus of $X$ such that $\omega$ extends to a global current in $c_1(X) \in H^2(X, \mathbb{Q})$ with continuous local potentials (see [7]).

Theorem 1.2. Let $X$ be an $n$–dimensional toric Fano variety which admits a (singular) Kähler-Einstein metric. Then its first Chern class $c_1(X)$ satisfies the following upper bound which is attained when $X$ is the complex projective space $\mathbb{P}^n$:

$$c_1(X)^n \leq (n + 1)^n$$

The universal bound in the previous theorem should be contrasted with the well-known fact that the volume of a general Fano variety $X$ of a fixed dimension $n \geq 2$ can be arbitrarily large unless conditions on the singularities of $X$ are imposed (see [24] and references therein and example 4 in [17] for a simple toric example).

According to the fundamental Yau-Tian-Donaldson conjecture in Kähler geometry the existence of a Kähler-Einstein metric on a Fano manifold $X$ is equivalent to $X$ being $K$-stable (see the recent survey [50]). This notion of stability is of an algebro-geometric nature. The case of toric Fano manifolds was settled by Wang-Zhou [57], who more precisely showed that a toric Fano manifold $X$ admits a Kähler-Einstein metric precisely when 0 is the barycenter of the canonical lattice polytope $P_X$ associated to $X$ (see below).

In the paper [6] we extend the result of Wang-Zhou to the setting of general (possibly singular) Fano varieties:
Theorem 1.3. [6] Let $X$ be an $n$-dimensional toric Fano variety. Then the following is equivalent:

- $X$ admits a (singular) Kähler-Einstein metric
- The barycenter is the unique interior lattice point of the polytope $P_X$ associated to $X$.

More generally, the result is shown to hold in the setting of toric log Fano varieties $(X, \Delta)$ familiar from the Minimal Model Program (MMP), i.e. $X$ is a toric variety and $\Delta$ is a torus invariant $\mathbb{Q}$-divisor on $X$ with coefficients $< 1$ such that the anti-canonical divisor $-(K_X + \Delta)$ of $(X, \Delta)$ defines an ample $\mathbb{Q}$-line bundle on $X$. In this general setting Theorem 1.2 holds for the log first Chern class $c_1(-(K_X + \Delta))$ of $(X, \Delta)$ if the coefficients of are positive and Theorem 1.2 holds for any toric log Fano variety $(X, \Delta)$.

The barycenter condition for the existence of a Kähler-Einstein metric on a toric variety is the link to Ehrhart’s volume conjecture in convex geometry, to which we next turn.

1.2. Convex geometry. There is a well-known dictionary relating toric polarized varieties $(X, L)$ and rational polytopes $P$ [17, 21, 15]. In particular, the top intersection number $c_1(L)^n$ coincides with $n!$ times the volume of the corresponding polytope $P$. As pointed out in [17] one of the motivations for the bound (1.1) on $c_1(X)^n$ in the toric setting is another more general conjecture of Ehrhart in the realm of convex geometry, which can be seen as a variant of Minkowski’s first theorem for non-symmetric convex bodies:

Conjecture. (Ehrhart). Let $P$ be an $n$-dimensional convex body which contains precisely one interior lattice point. If the point coincides with the barycenter of $P$ then

$$\text{Vol}(P) \leq \frac{(n + 1)^n}{n!}$$

The case when $n = 2$ was settled by Ehrhart [22], as well as the special case of simplices in arbitrary dimensions [23]. As explained in the survey [28] the best upper bound in Ehrhart’s conjecture, to this date, is $\text{Vol}(P) \leq (n + 1)^n/n^n$. As we will next explain Theorem 1.2 (or rather its more general Log version) confirms Ehrhart’s conjecture for a large class of rational polytopes. First recall that any rational polytope in $\mathbb{R}^n$, containing zero in its interior, may be written uniquely as

$$(1.2) \quad P = \{ p \in \mathbb{R}^n : \langle l_F, p \rangle \geq -a_F \},$$

where the index $F$ ranges over the facets of $P$, $a_F$ is a positive rational number and the vector $l_F$ is a primitive lattice vector, i.e. it has integer coefficients with no common factors (geometrically, $l_F$ is the inward normal vector of the facet $F$ normalized with respect to the integer structure). As is well-known [17] toric Fano varieties with $-K_X$ an ample $\mathbb{Q}$-line bundle correspond to rational polytopes $P$ as above with $a_F = 1$. More generally, toric log Fano varieties $(X, \Delta)$ with $\Delta$ an effective $\mathbb{Q}$-divisor on $X$ correspond to polytopes
$P$ with $a_F \leq 1$ [15] [6]. Combining the Log version of Theorem 1.2 above with the existence result for Kähler-Einstein metrics on (possibly singular) toric varieties hence gives the following

**Corollary 1.4.** The bound in the Ehrhart conjecture holds for all rational polytopes satisfying $a_F \leq 1$ in the representation 1.2 and such that 0 is the barycenter of $P$.

In any polytope such that $a_F \leq 1$ the origin is indeed the unique lattice point (see the foot note on page 105 in [17]). The class of such rational polytopes $P$ is vast and appears naturally both in algebraic geometry and in combinatorics. For example, the dual (polar) $P = Q^*$ of any lattice polytope $Q$ containing 0 in its interior is in this class (see [26] for the combinatorics of such polytopes). In particular, this shows that the bound in the Ehrhart conjecture holds for any reflexive convex lattice polytope (i.e. a lattice polytope $P$ containing 0 such that its dual $Q$ is also a lattice polytope). Such polytopes correspond to Gorenstein toric Fano varieties and were introduced and studied by Batyrev [3] in connection to mirror symmetry of pairs of Calabi-Yau manifolds. In dimension $n \leq 8$ the Ehrhart conjecture up to $n \leq 8$ for reflexive Delzant polytopes (i.e. those corresponding to smooth toric Fano varieties) has previously been confirmed by computer assistance (as announced in [47]), using the classification of such polytopes for $n \leq 8$ [60].

As explained in section 3 the arguments in the proof of Theorem 1.2 and its Corollary above can be carried out directly in terms of convex analysis in $\mathbb{R}^n$ without any reference to the corresponding toric variety. In fact, it is enough to assume that $P$ is a convex body and one then obtains the following

**Theorem 1.5.** Let $P$ be a convex body contained in the positive octant with barycenter $b_P = (1,1,...,1)$. Then the volume of $P$ is maximal when $P$ is a regular simplex, i.e. for $(n+1)$ times the unit-simplex:

$$\text{Vol}(P) \leq \frac{(n+1)^n}{n!}$$

Applying the previous theorem to a suitable affine transformation of a given rational polytope gives us back 1.3.

After we had proved the Theorem 1.5 by the arguments outlined above, Bo'az Klartag showed us a short and elegant direct proof of this statement using tools from convex geometry. His argument, that we will reproduce below as Remark 3.2 is based on Grunbaum’s inequality [29]. Nevertheless, we have decided to keep here our original argument as well since it exemplifies the relation of this kind of inequalities in convex geometry to Kähler geometry. Also, our argument may also be useful when dealing with other singular varieties than toric ones, such as spherical varieties.

Interestingly, the proof of Grunbaum’s inequality is based on a clever application of the Brunn-Minkowski inequality for convex bodies and the latter inequality, or more precisely its functional form due to Prekopa [52]
also plays a key role in our proof (since it is used in the proof of the Moser-Trudinger type inequalities). In fact, using the positivity of the direct image bundles in \[8\] we will obtain a complex geometric generalization of Prekopa’s result in the presence of a suitable action of a compact Lie group on a Stein manifold (Theorem \[2.3\]). This also leads to a generalization of the Prekopa theorem in \(\mathbb{R}^n\) to non-compact real symmetric spaces of independent interest (Corollary \[2.5\]). Since such spaces typically have negatively sectional curve this latter result appears to be rather intriguing, when contrasted with the results in \[14\], which demand non-negative Ricci curvature.

1.3. Critical mean field type equations for \(S^1\)–invariant domains in \(\mathbb{C}^n\). As explained in section \[2.3\] the starting point of the proof of Theorem \[1.1\] is that an \(S^1\)–invariant Kähler-Einstein metric on \(X\) induces a solution \(\phi\) to a mean field type equation on a Stein domain \(\Omega\) admitting an \(S^1\)–action with an attractive fixed point. The proof is thus reduced to establishing a criticality result for solutions of such equations. For concreteness, here we will only state the result in the case when the domain \(\Omega\) is contained in \(\mathbb{C}^n\) with its standard action by \(S^1\) (or more generally any linear action of \(S^1\) with positive weights \(m_i\), i.e. defined by \((e^{i\theta}, (z_1, ..., z_n)) \mapsto (e^{im_1\theta}z_1, ..., e^{im_n\theta}z_n)\)).

Denote by \(dV\) the Euclidean volume element and write \(dd^c_\phi = i\partial\bar{\partial}/2\pi\), so that \((dd^c_\phi)^n\) is the Monge-Ampère measure of \(\phi\), whose density is equal to \((\frac{1}{2\pi})^n\) times the determinant of the complex Hessian \((\frac{\partial^2\phi}{\partial z_i \partial \bar{z_j}})\).

**Theorem 1.6.** Let \(\Omega\) be a connected smoothly bounded \(S^1\)–invariant pseudoconvex domain in \(\mathbb{C}^n\), containing 0. For any given positive real number \(\gamma\) a necessary condition for the existence of an \(S^1\)–symmetric plurisubharmonic solution \(\phi \in C^\infty(\bar{\Omega})\) to the equation

\[
(dd^c_\phi)^n = \frac{e^{-\gamma\phi}dV}{\int_{\Omega} e^{-\gamma\phi}dV} \quad \text{in } \Omega \quad \phi = 0 \text{ on } \partial\Omega
\]

is that \(\gamma \leq (n + 1)\).

More generally, the proof of the previous theorem shows that it is enough to assume that \(\phi\) is a continuous solution in the sense of pluripotential theory. As shown in \[4\] \(\gamma < (n + 1)\) is a sufficient condition for existence of such weak solutions for any pseudoconvex domain (by \[30\] any such continuous solution is in fact smooth in the interior of \(\Omega\)). Hence \(\gamma = n + 1\) appears to be a critical parameter for the equations \[1.3\] on \(\Omega\) in the presence of an \(S^1\)–action as above. Moreover, the condition in the previous theorem that the fixed point 0 be contained in \(\Omega\) is crucial. For example, if \(\Omega\) is an annulus in \(\mathbb{C}\) then it is well-known that there exist \(S^1\)–invariant solutions for \(\gamma\) arbitrarily large (see section 5 in \[13\]).

To prove the theorem we first show that any solution \(\phi\) as in the previous theorem is an extremal for a Moser-Trudinger type inequality on \(\Omega\), which becomes stronger as the parameter \(\gamma\) increases. We then go on to show that when \(\gamma > (n + 1)\), a suitable regularization of the pluricomplex Green
function \( g \) of \( \Omega \) with a logarithmic pole at 0 violates the corresponding Moser-Trudinger type inequality.

This approach should be compared with a result of Ding-Tian [20] saying that for any Fano manifold \( X \) admitting a Kähler-Einstein metric there is a corresponding Moser-Trudinger type inequality for positively curved metrics on the anti-canonical line bundle \( -K_X \) which has the metric on \( -K_X \) induced by the Kähler-Einstein metric as an extremal. However, it is well-known that there are global obstructions to the existence of metrics on \( -K_X \) with prescribed logarithmic poles and this is the reason that we need to replace \( X \) with a Stein subdomain \( \Omega \). The prize we have to pay is that an appropriate symmetry assumption is then needed to deduce the corresponding Moser-Trudinger type inequalities on \( \Omega \) (compare the discussion on symmetry breaking in section 2.3.1).

We conjecture that the inequality \( \gamma \leq n + 1 \) in the previous theorem is in fact a strict inequality and that if \( \phi_j \) is a sequence of \( S^1 \)-symmetric solutions \( \phi_j \) associated to a sequence of parameters \( \gamma_j \) converging to the critical value \( n + 1 \), then \( \phi_j \) converges weakly to the pluricomplex Green function \( g_\Omega \) of \( \Omega \) with a logarithmic pole at 0. This is easy to verify in the case when \( \Omega \) is the unit-ball and \( \phi_j \) is the radial solution (see [4]). The motivation for this conjecture comes from the concentration-compactness principles extensively studied in the one-dimensional situation (see [13] and references therein). As explained in [13] the equations above then appear as mean field equations for statistical mechanical models with \( \gamma \) playing the role of the inverse temperature and the critical value corresponding to a phase transition.

1.4. Organization. After having set up the complex geometric and group-theoretic frame work in the beginning of Section 2 we establish Prekopa type convexity inequalities and give the proof of Theorem 1.1 by reducing it to the proof of Theorem 1.6. Then in Section 3 the singular setting on a toric variety is considered and the proof of Theorem 1.2 is explained, using real convex analysis. Finally, in Section 4 Theorem 1.1 is applied to spherical varieties and rephrased in terms of Lie algebras and symplectic geometry.

1.5. Acknowledgments. We are grateful to Benjamin Nill for helpful comments on the toric setting, Michel Brion for his help with spherical varieties, Gabor Székelyhidi for encouraging us to consider the relation to the invariant \( R(X) \) and Bo'az Klartag for allowing us to include here his beautiful reduction to Grunbaum's inequality.

2. The volume of Kähler-Einstein Fano manifolds

2.1. Preliminaries.
2.1.1. **Kähler-Einstein metrics and Monge-Ampère equations.** Let $L \to X$ be a holomorphic line bundle over an $n$-dimensional compact complex manifold. We will denote by $H^0(X, L)$ the space of all global holomorphic sections with values in $L$. A Hermitian metric $\| \cdot \|$ on $L$ may be represented by a collection of local functions $\phi_U := \{ \phi_U \}$ defined as follows: given a local trivializing section $s$ of $L$ on an open subset $U \subset X$ we define the local weights $\phi_U := - \log \|s\|^2$ of the metric. Of course, $\phi_U$ depends on $s$, but the (normalized) curvature form $\ddc \phi_U := i \frac{\partial}{\partial \bar{\partial}} \phi$ is a globally well-defined two form on $X$ representing the first Chern class $c_1(L)$. The normalizations have been chosen so that $c_1(L)$ is an integral class.

Note that the metric on $L$ has semi-positive curvature form precisely when the local weights $\phi_U$ are plurisubharmonic (psh, for short). We recall that according to the Kodaira embedding theorem the line bundle $L$ is ample, i.e. the Kodaira map $X \to \mathbb{P}(H^0(X, L))^*$ is an embedding for $k$ sufficiently large, precisely when $L$ admits a metric with positive curvature. For any such $k$ the open manifold $S := \{ s = 0 \}$, for $s$ a given non-trivial element in $H^0(X, kL)$, is a Stein manifold, i.e. $S$ admits a smooth and strictly plurisubharmonic exhaustion-function $\phi_S$. Indeed, $\phi_S$ can be taken as $- \log \|s\|^2$ for any positively curved metric on $L$.

A Kähler metric $\omega$ on $X$ is said to be Kähler-Einstein if it has constant Ricci curvature, i.e.

$$\text{Ric } \omega = \Lambda \omega$$

for some constant $\Lambda$. We will be interested in the case when $\Lambda$ is positive and after a scaling we may as well assume that $\Lambda = 1$. Then $X$ is necessarily a Fano manifold, i.e. the dual $-K_X$ of the canonical line bundle $K_X := \Lambda^n(T^*X)$ is ample. Equivalently, $\omega$ is a Kähler-Einstein metric on the Fano manifold $X$ iff $\omega$ is the curvature of a positively curved metric $\| \cdot \|$ on $-K_X$ such that, if $s$ is a local trivialization of $-K_X$ over $U$, then there is a positive constant $C$ such that the following Monge-Ampère equation holds on $U$:

$$(dd^c \phi)^n = C e^{-\phi} dV, \quad dV := i^n \theta \wedge \bar{\theta}$$

where $\phi$ is the corresponding weight of the metric and $\theta$ is the holomorphic $n$-form which is dual to $s$. More generally, given a positive integer $k$ the local section $s$ can be replaced by a local non-vanishing holomorphic section $s_k$ of $-kK_X \to U$ and one then sets $\phi := - \frac{1}{k} \log \|s_k\|^2$, replacing the dual $\theta$ above with its $k$th root. We remark that this latter flexibility allows one to define Kähler-Einstein metrics on a singular (normal) variety $X$ (see [7]). Indeed, in general $K_X$ is then merely defined as a Weil divisor, but assuming that $X$ is a Fano variety, i.e. $-kK_X$ is an ample line bundle for some positive integer $k$ (in other words $K_X$ is an ample $\mathbb{Q}$-Cartier divisor) the previous definition makes sense on the regular locus of $X$ and one then adds the global
condition that the corresponding metric be continuous on all of $X$. Anyway, in this paper we will mainly stick to the case when $X$ is smooth.

2.1.2. Group actions. Let $G$ be a complex Lie group acting by holomorphisms on a compact complex manifold $X$. In other words, $X$ is a compact complex $G$–manifold. If $G$ acts linearly on a vector space $V$, we write $V^G$ for the subspace of $G$–invariant vectors and $V^{(G)}$ for the subspace of $G$–eigenvectors, i.e. $v \in V^{(G)}$ iff $g v = \chi(g)v$, where $\chi$ is a character, i.e. a homomorphism from $G$ to $\mathbb{C}^*$. In particular, if $L$ is a $G$–equivariant line bundle over $X$ then $G$ acts linearly on the vector space $H^0(X,L)$ by setting $(g \cdot s)(p) := g(s(g^{-1}p))$ for any $s \in H^0(X,L)$.

In the proof of Theorem 1.1 we will have great use for the following

**Lemma 2.1.** Let $X$ be a smooth projective variety admitting a holomorphic action by the circle $S^1$ with isolated fixed points. If the first Betti number of $X$ vanishes and $L$ is a given $S^1$–equivariant ample line bundle over $X$, then there exists a fixed point $p \in X$ and an $S^1$–eigenvector $s \in H^0(X,kL)$ for some $k > 0$ such that $s(p) \neq 0$ and

$$H^0(S)^r = \mathbb{C},$$

where $S$ is the Stein manifold $S := X - \{s = 0\}$ containing $p$.

**Proof.** Since $L$ is ample it admits a metric with positive curvature form $\omega$, defining a symplectic form on $X$. Moreover, averaging over the compact group $S^1$ we may assume that $\omega$ is $S^1$–invariant, i.e. the action is symplectic. Since the first Betti number of $X$ vanishes the action admits a Hamiltonian function $f$, i.e. $df = \omega(V, \cdot)$ where $V$ is the vector field on $X$ generating the $S^1$–action. It then follows from general principles that the action lifts to $L$. Anyway, in our setting $L$ will be equal to $-K_X$ which admits a canonical lift of the $S^1$–action on $X$. We let $p$ be a point where the minimum of $f$ is attained. Then $V$ vanishes at $f$ and hence $p$ is a fixed point. Next, we pick a positive number $k$ such that $kL$ is globally generated and decompose $H^0(X,kL) = \oplus V_m$ in the one-dimensional eigenspaces for the $S^1$–action. By the assumption of global generation there is a section $s \in H^0(X,kL)$ such that $s(p) \neq 0$ and by the previous decomposition we can thus take $s \in V_m$ for some $m$. Let now $S := X - \{s = 0\}$, which, as explained above, is a Stein manifold. To prove 2.1 we note that the action of $S^1$ on the tangent space at the fixed point $p$ has positive weights in the following sense (i.e. it is an attractive fixed point): by a general result for compact Lie group (see Satz 4.4 in [32]) we may linearize the action in an invariant neighborhood of $U$ of $p$ so that $e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{im_1 \theta}z_1, \ldots, e^{im_n \theta}z_n)$. The positivity referred to above then amounts to having $m_i > 0$ for all $i$. Indeed, after performing a linear change of coordinates we may assume that $f(z) = \sum m_i |z_i|^2 + o(|z|^2)$ and since $f$ has a minimum at $p$ (corresponding to $z = 0$) and the fixed point $p$ is isolated it must be that $m_i > 0$. Taylor expanding a given holomorphic function $g$ on $U$ wrt the variables $z_i$ then reveals that $g$ is $S^1$–invariant only
if it is constant on $U$. But since $S$ is connected this concludes the proof of the lemma. □

**Example 2.2.** Set $X = \mathbb{P}^1$ and $L = -K_X$ with its usual $S^1$-action obtained by identifying $X$ with the two-sphere and considering rotations around a fixed axes. Then $X$ has two fixed points (the north and the south pole) and fixing the standard affine chart $U_0 \cong \mathbb{C}$ containing the south pole, where the action is given by $(e^{i\theta}, z) \mapsto e^{i\theta}z$, we can write any section $s_i \in H^0(X, -K_X)$ over $U_0$ as $s = f(z) \frac{\partial}{\partial z}$ so that $(e^{i\theta} \cdot s)(z) = e^{i\theta}f(e^{-i\theta}z)dz$. Hence, we can take the section $s$ in the previous lemma as the one determined by $f(z) = 1$ (which has weight $m = 1$) so that $S = U_0$.

Even though if it will not - strictly speaking - be needed for the proof of Theorem 1.1, we make a brief digression to explain how, using the Bialynicki-Birula decomposition (see Theorem 4.4 in [9]), one may, essentially, take the Stein domain $S$ to be equivariantly isomorphic to $\mathbb{C}^n$ with a linear $\mathbb{C}^* -$ action. First recall that the Bialynicki-Birula decomposition says that any non-singular $n$-dimensional projective variety $X$ with a $\mathbb{C}^* -$ action having only isolated fixed points may be written as the disjoint union $X = \bigsqcup_p X_p$ where $p$ ranges over the fixed point $p$ in $X$ and where $X_p$ is $\mathbb{C}^* -$ invariant set equivariantly isomorphic to the affine space $T^*X|_p$, i.e. the direct sum of the positive weight spaces in $TX|_p$ with the induced linear $\mathbb{C}^* -$ action. Concretely, $X_p$ is the attracting set for $p$ under the $\mathbb{C}^* -$ action, i.e. $x \in X_p$ iff $\lim_{\lambda \to 0} \lambda \cdot x = p$. In particular, if all the weights at $p$ are all positive (as for $p$ in the previous lemma) then $X_p$ is a Zariski open subvariety of $X$ which is equivariantly isomorphic to $\mathbb{C}^n$ with a linear $\mathbb{C}^* -$ action. We note that

$$X_p \subset S,$$

where $S = \{ s \neq 0 \}$ is the Stein manifold appearing in the previous lemma. Moreover, given a positively curved $S^1-$ invariant metric on $L$ the function $\phi := -\log \| s \|^2$ is a psh exhaustion function of $X_p$. To see that $\phi$ is indeed proper we note that since $\phi$ is strictly convex along the $\mathbb{C}^* -$ orbits and (since $s(p) \neq 0$) is bounded from below close to $p$ it follows that $\phi \to \infty$ as $|\lambda| \to \infty$ along a given $\mathbb{C}^* -$ orbit in $X_p$, which implies properness by a basic compactness argument. As for the $S^1-$ invariance it follows from the fact that $s$ is an eigenvector (as explained in the beginning of section 2.3 below).

### 2.2. A Prekopa type convexity result on Stein manifolds under group actions.

One of the key ingredients in the proof of Theorem 1.1 is a convexity result of Prekopa type. Let us first recall the Prekopa inequality in its original form [52]: If $\phi_t(x)$ is a convex function on $I \times \mathbb{R}^n$, where $I$ is an open interval in $\mathbb{R}$ with coordinate $t$, then the function

$$t \mapsto -\log \int_{\mathbb{R}^n} e^{-\phi_t}dx$$

is convex. Here we will obtain a complex geometric generalization of this result. We let $S$ be an $n-$dimensional Stein manifold with trivial canonical
line bundle $K_S$, i.e. $S$ admits a non-vanishing holomorphic $n$–form $\theta$ (also called a holomorphic volume form).

**Theorem 2.3.** Let $K$ be a compact group acting on a bounded Stein domain $\Omega$ with a holomorphic volume form $\theta$ and such that all $K$–invariant holomorphic functions are constant, i.e.

$$H^0(\Omega)^K = \mathbb{C}. \tag{2.2}$$

If $\phi_t(z)$ is a $K$–invariant bounded psh function on $D \times \Omega$, where $D$ is the unit-disc in $\mathbb{C}$ then the function

$$t \mapsto -\log i^n \int_{\Omega} e^{-\phi_t} \theta \wedge \bar{\theta}, \tag{2.3}$$

is subharmonic in $t$. More generally, if $\Omega$ is unbounded and $e^{-\phi_t}$ is integrable on $\Omega$ for $t$ fixed, the same conclusion holds if the space $H^0(\Omega)$ is replaced by $H^0(\Omega) \cap L^2(e^{-\phi_t} \theta \wedge \bar{\theta})$.

**Proof.** Consider the (infinite dimensional) Hermitian holomorphic vector bundle $E \rightarrow D$ whose fiber $E_t$ is the Hilbert space of all holomorphic functions on $\Omega$ of finite $L^2$–norm

$$\|f\|_t^2 := \sqrt[2n]{\int_{\Omega} |f|^2 e^{-\phi_t} \theta \wedge \bar{\theta}},$$

As shown in [8] this bundle has positive curvature in the following sense: for any given holomorphic section $\Lambda$ of the dual bundle $E^*$ the function

$$t \mapsto -\log(\|\Lambda\|_{\phi_t}^2) := -\log(\sup_{f \in E_t} |\langle \Lambda, f \rangle|^2 / \|f\|_{\phi_t}^2)$$

is subharmonic. Strictly speaking the proof in [8] concerned the case when $\Omega$ is a pseudoconvex domain in $\mathbb{C}^n$, but the proof can be repeated word for word in the Stein case, using that Hörmander’s $L^2$–estimates for $\bar{\partial}$ are still valid. We now let $\sigma$ be the invariant probability measure on $K$ (i.e. the Haar measure) and set

$$\Lambda(f) := \int_K(k^*f)\sigma$$

Since the rhs above is a $K$–invariant holomorphic function on $\Omega$ it is, by assumption, constant (for $t$ fixed) and hence $\Lambda$ indeed defines a holomorphic section of $E^*$. Thus it will be enough to show that, under the condition $\|\Lambda\|_{\phi_t}^2 = 1/ \int_{\Omega} e^{-\phi_t} \theta \wedge \bar{\theta}$, i.e. the sup above is attained for constant functions. But this follows immediately from estimating

$$\int_{\Omega} e^{-\phi_t} \theta \wedge \bar{\theta} \|\Lambda\|_{\phi_t}^2 \leq \int_{\Omega} d\sigma \int_{K} e^{-\phi_t} |k^*f|^2 \theta \wedge \bar{\theta} = \int_{\Omega} e^{-\phi_t} |f|^2 \theta \wedge \bar{\theta},$$

using that $\phi_t$ is $K$–invariant in the last equality. \hfill \Box

For example, if $\Omega = S = \mathbb{C}^n$, $\theta = dz$ and $\phi_t$ grows as $(n+1) \log(|z|^2) + O(1)$ at infinity one can take the compact group $K$ in the theorem to be trivial, i.e. no symmetry assumption is needed. The point is that, viewing $\mathbb{C}^n$ as a
Zariski open set in $X := \mathbb{P}^n$ the space $H^0(\mathbb{C}^n) \cap L^2(e^{-\phi_t})$ can be identified with $H^{0,0}(X, -K_X) \cong H^0(\mathbb{P}^n, \mathbb{C})$ which is one-dimensional. However, for a general psh function $\phi_t$ in $\mathbb{C}^n$ one can construct counter-examples to the subharmonicity in formula (2.2) by adapting an example of Kiselman [34] to the present setting.

**Remark 2.4.** Replacing $\phi$ by $m\phi$ and letting $m \to \infty$ in the previous theorem shows that the function

$$t \mapsto \inf_{\Omega} \phi_t$$

is also subharmonic in $t$. This property is a well-known instance of the Kiselman minimum principle [34] extended to the setting of Lie groups by Loeb [42]. It should however be pointed out that the setting in [42] is more general as it applies to certain non-compact groups $K$.

**An application to symmetric spaces.** Before continuing we make a brief digression, showing how the previous theorem implies a Prekopa type convexity inequality for real symmetric spaces. More precisely, we consider a non-compact symmetric space with compact dual $K$, i.e. the symmetric space may be written as $G/K$ where $G$ is the complexification of the compact group $K$ (so that $G$ is a connected reductive complex Lie group and for simplicity we will assume that $G$ is semi-simple [25]). As is well-known $G$ is a Stein manifold. More precisely, by results of Chevalley $G$ is an affine algebraic variety (see the appendix in [46]) with a $G$-bi-invariant pseudo-Riemannian metric (induced by the Killing form on the Lie algebra of $G$). The corresponding $K$-principal fiber bundle

$$\pi : G \to G/K$$

induces a $G$-bi-invariant symmetric Riemannian metric on $G/K$. There is a $G$-bi-invariant volume form $\mu_G$ on $G$ and it can be written as $\mu_G = \theta \wedge \bar{\theta}$, where $\theta$ is a $G$-bi-invariant holomorphic top form on $G$. The push-forward of $\mu_G$ under the projection $\pi$ clearly coincides with the $G$-bi-invariant volume form $\mu_{G/K}$ on the symmetric space $G/K$.

**Corollary 2.5.** Let $\phi_t$ be a geodesically convex function on the Riemannian product $\mathbb{R} \times G/K$, where $\mathbb{R}$ is the real line with its Euclidean Riemannian metric. Then the function

$$t \mapsto -\log \int_{G/K} e^{-\phi_t} \mu_{G/K}$$

is convex on $\mathbb{R}$.

**Proof.** It is well-known that the convex functions on the symmetric space $G/K$ may be written as $\pi_* \psi$ where $\psi$ is a $K$-invariant psh function on $G$ (see for example Lemma 2, page 34 in [21]). In particular, we may identify $\phi_t$ with a $S^1 \times K$-invariant psh function $\psi_t$ on $\mathbb{C}^* \times G$. Hence, the corollary will follow from the previous theorem once we have checked that any $K$-invariant holomorphic function on $G$ is constant. But this follows immediately from
the basic fact that a holomorphic function on an $n$-dimensional connected complex manifold is uniquely determined by its restriction to a totally real submanifold (here $K$) of real dimension $n$. □

**Example 2.6.** If $G = \text{SL}(2, \mathbb{C})$ and $K = \text{SU}(2)$, then the symmetric space $G/K$ may be identified with the three-dimensional real hyperbolic space.

It is interesting to compare the previous corollary with the results in [14], where a so called Prekopa-Leindler inequality is obtained valid for any (possibly non-symmetric) Riemannian manifold $(M, g)$ with non-negative Ricci curvature. This inequality is in fact stronger than the Prekopa inequality on $(M, g)$, i.e. the analog of Corollary 2.5 for $(M, g)$. For example, in the case when $(M, g)$ is Euclidean space (which may be obtain as above by taking $G$ to be abelian) the Prekopa-Leindler inequality implies the Brunn-Minkowski inequality for general sets, while the Prekopa inequality a priori only implies the Brunn-Minkowski inequality for convex sets. Conversely, as shown in [14] the validity of the Prekopa-Leindler inequality on $(M, g)$ actually implies that the Ricci curvature of $(M, g)$ is non-negative and hence it cannot hold on a general symmetric space as above, since it is well-known that an irreducible symmetric space $G/K$ has negative sectional curvatures when $G$ is non-abelian.

### 2.3. Proofs of Theorems 1.1, 1.6

We will start by reducing the proof of Theorem 1.1 to proving Theorem 1.6. In the following we will use the notation $T = S^1$ for the one-dimensional real torus inbedded in $\mathbb{C}^*$ and hence acting on $X$. Since $X$ admits a Kähler-Einstein metric it also admits a $T$-invariant Kähler-Einstein metric $\omega$ [2], which is the curvature form of a $T$-invariant metric $\|\|$ on $-K_X$. To simplify the notation we assume that $-K_X$ is globally generated (otherwise just replace $-K_X$ with a large tensor power) and take $p$ to be the $T$-fix point and $s$ the holomorphic section of $-K_X$, furnished by Lemma 2.1. As explained above

$$\phi := -\log \|s\|^2$$

is then a strictly psh smooth exhaustion function of the Stein manifold $S := \{s \neq 0\}$ containing the fixed point $p$. Moreover, $\phi$ is $T$-invariant. Indeed, by construction $g \cdot s = \chi(g)s$ where $\chi$ is a character on the compact group $T$. Hence $\chi$ takes values in the unit-circle $S^1 \subset \mathbb{C}$, showing that $\phi$ is $T$-invariant, as desired.

As explained above the Kähler-Einstein equation for $\omega$ (and the correspond metric on $-K_X$) then translates to the Monge-Ampère equation on $S$

$$\tag{2.4} (dd^c \phi)^n = Ce^{-\phi}dV$$

for a positive constant $C$, where $dV$ is the volume form induced by the trivialization $s$. We may rewrite $C = V_X / \int_S e^{-\phi}dV$, where

$$V_X = \int_S (dd^c \phi)^n$$

13
which coincides with the top-intersection number $c_1(X)^n$. Let $\Omega_R$ be the set where $\phi < R$ and note that the sets $\Omega_R$ exhaust $\mathbb{C}^n$, since $\phi$ is proper. Assume now, to get a contradiction, that the bound in the theorem to be proved is not valid for $X$. Then we can fix $R$ sufficiently large so that

$$V_{\Omega_R} := \int_{\Omega_R} (dd^c\phi)^n > (n+1)^n.$$ 

Writing $\Omega := \Omega_R$ and replacing $\phi$ by $\phi - R$ we then obtain a $T$-invariant smooth plurisubharmonic function $\phi$ solving the following equation:

$$(dd^c\phi)^n = V_{\Omega} \frac{e^{-\phi}dV}{\int_{\Omega} e^{-\phi}dV}, \quad \phi = 0 \text{ on } \partial\Omega$$

on the $T$-invariant domain $\Omega$ which is a hyperconvex domain, i.e. it admits a negative continuous psh exhaustion function (namely $\phi$). More precisely, since $\phi$ is smooth and strictly psh the domain $\Omega$ is a Stein domain. Finally, rescaling, i.e. replacing $\phi$ with $V_{\Omega}^{1/n}\phi$ we have thus obtained a solution to the equation in Theorem 1.6 with a parameter $\gamma := V_{\Omega}^{1/n} > (n+1)$ and all that remains is thus establishing Theorem 1.6 in the slightly more general case when $\mathbb{C}^n$ is replaced with a Stein domain with a holomorphic $S^1$-action admitting an attractive fixed point.

**Proof of Theorem 1.6.** We will denote by $\mathcal{H}_0(\Omega)$ the space of all psh functions on $\Omega$ which are continuous up to the boundary, where they are assumed to vanish. Its $T$-invariant subspace will be denoted by $\mathcal{H}_0(\Omega)^T$. Let $\phi_0$ be a solution to equation 1.3 with a fixed parameter $\gamma$. The following Moser-Trudinger type inequality can then be established on $\Omega$ (which has $\phi_0$ as an extremal): there is a positive constant $C$ such that

$$\left(\frac{1}{\gamma} \log \int_{\Omega} e^{-\gamma\phi}dV \leq \frac{1}{(n+1)} \int_{\Omega} (-\phi)(dd^c\phi)^n + C$$

for any $\phi \in \mathcal{H}_0(\Omega)^T$. We also write the previous inequality as $G_\gamma(\phi) \leq C$ for the corresponding Moser-Trudinger type functional $G_\gamma$. Given the convexity property in Theorem 2.3 the proof may be obtained by repeating the proof of Theorem 1.4 in [4], but for completeness we have recalled the proof in section 2.4 below.

The desired contradiction will now be obtained by exhibiting a function violating the previous Moser-Trudinger type inequality. To this end we first recall the definition of the pluricomplex Green function $g_p$ of a pseudoconvex domain $\Omega$ with a pole at a given point $p$:

$$g_p := \sup \{ \phi : \phi \in (PSH)(\Omega) \cap c^0)(\Omega - \{p\}) : \phi \leq 0, \phi \leq \log|z|^2 + O(1) \}$$

where $z$ denotes fixed local holomorphic coordinates centered at $p$. We will often write $g = g_p$ to simplify the notation. As is well-known $g$ is continuous up to the boundary on $\Omega$ apart from a singularity at $p$ and satisfies

$$(dd^c g)^n = \delta_p \text{ on } \Omega - \{p\}, \quad g = \log|z|^2 + O(1)$$
In particular $\int (dd^c g)^n = 1$ and $\int_\Omega e^{-ng} dV = \infty$. We note that if $T$ acts, as above, on $\Omega$ and $p$ is taken as a fixed point for the action, then $g$ is $T$–invariant. Indeed, since $p$ is invariant under the action of $T$ so is the the convex class of functions where the sup in \ref{2.7} is taken and hence the sup $g$ must be $T$–invariant. The contradiction is now obtained by showing that there is a family of functions $g_t$ in $\mathcal{H}_0(\Omega)^T$ decreasing to $g$ such that, for $t$ sufficiently large $g_t$ violates the Moser-Trudinger type inequality \ref{2.6} if $\gamma > (n + 1)$. To this end we set

$$g_t := \log(e^{-2t} + e^g) - C_t, \quad \phi_t = (e^{-2t} + Ce^{\log |z|^2})$$

where $C$ is a constant such that $-g_t \leq -\phi_t$ and $C_t$ is the constant ensuring that $g_t$ vanishes on the boundary, i.e. $C_t = \log(e^{-2t} + 1) = o(1/t)$. In fact, this part of the argument does not require any symmetry assumptions. We fix local holomorphic coordinates $z$ centered at $p$ such that, locally around $p$, $\phi(z) = \log |z|^2 + O(1)$. Trivially we have, by replacing $\Omega$ with a coordinate ball $B$, that

$$-\frac{1}{\gamma} \log \int_{\Omega} e^{-\gamma g_t} dV \leq -\frac{1}{\gamma} \log \int_{B} e^{-\gamma \phi_t} dz \wedge d\bar{z} + C'$$

Denoting by $F_t$ the scaling map defined by $F_t(z) = e^t z$ we have $\phi_t = F_t^* \phi_0 - 2t + o(1/t)$. Hence, we may write

$$-\frac{1}{\gamma} \log \int_{B} e^{-\gamma \phi_t} dz \wedge d\bar{z} = -2t + o(1/t) + -\frac{1}{\gamma} \log \int_{B} e^{-\gamma F_t^* \phi_0} dz \wedge d\bar{z}$$

Rewriting $dz \wedge d\bar{z}$ as $e^{-2tn} F_t^* dz \wedge d\bar{z}$ gives

$$-\frac{1}{\gamma} \log \int_{B} e^{-\gamma F_t^* \phi_0} dz \wedge d\bar{z} = 2tn \frac{1}{\gamma} - I_t, \quad I_t := \frac{1}{\gamma} \log \int_{e^{1/t} B} e^{-\gamma \phi_0} dz \wedge d\bar{z}$$

Next, we consider the energy term: since $-g_t \leq -\log(e^{-2t} + 0) - C_t = 2t + o(1/t)$ we get

$$\int_{\Omega} (-g_t)(dd^c g_t)^n \leq (2t + o(1/t)) \int_{\Omega} (dd^c g_t)^n.$$

But since $g_t$ is a sequence of bounded psh functions tending to zero at the boundary of $\Omega$ and decreasing to $g$ it follows from well-known convergence properties that $\int_{\Omega}(dd^c g_t)^n \to \int_{\Omega}(dd^c g)^n = 1$ (in fact, in our case $g$ may be taken to be smooth close to the boundary and then the convergence follows immediately from Stokes theorem). All in all this means that

$$-\mathcal{G}(g_t) \leq \frac{1}{(n + 1)} 2t + (2t)(-1 + \frac{n}{\gamma}) - \log I_t + o(1/t),$$

Note that when $\gamma = (n + 1)$ we have $(-1 + \frac{n}{\gamma}) = \frac{1}{(n + 1)}$ and hence when $\gamma > (n + 1)$ first term above, which is linear in $t$, tends to $-\infty$ as $t \to \infty$. Moreover, since $I_t \geq -C$ it follows that $-\mathcal{G}(g_t) \to -\infty$ when $t \to \infty$ and hence $\mathcal{G}$ is not bounded from above, which contradicts the Moser-Trudinger
inequality. This completes the proof of Theorem 1.6 and thus of Theorem 1.1 as well.

Remark 2.7. There is an alternative way of obtaining a contradiction to the Moser-Trudinger inequality with parameter $\gamma > (n + 1)$ (see our previous preprint [5]). Indeed, as shown in [4] the inequality induces another inequality of Brezis-Merle-Demailly type on the $(n + 1)$-dimensional product $\Omega' = \Omega \times D$ of $\Omega$ with the unit-disc which in particular implies that

$$\int_{\Omega'} e^{-(n+1)\phi} dV < \infty,$$

for any $S^1$-invariant plurisubharmonic function on $\Omega'$, say with isolated singularities compactly contained in $\Omega'$, vanishing on the boundary and with unit Monge-Ampère mass on $\Omega'$. But this is immediately seen to be contradicted by the pluricomplex Green function of $\Omega'$ with a pole at the origin.

2.3.1. Symmetry breaking. The assumption in Theorem 1.6 that 0 be contained in $\Omega$ is crucial. For example, when $\Omega$ is an annulus, $r < |z| < 1$, in $\mathbb{C}$, it is well-known [13] that there exists a (uniquely determined) $S^1$-invariant solution $\phi_\gamma$ for any value of $\gamma$. Moreover, by the method of moving planes any solution of the equations is necessarily $S^1$-invariant [13] and thus coincides with $\phi_\gamma$. In the range $\gamma < 2$ the solution $\phi_\gamma$ is an extremal for the corresponding Moser-Trudinger inequalities which are known to hold for general functions $\phi$ in $H_0(\Omega)$. We note however that at the critical value $\gamma = 2$ an interesting instance of symmetry breaking appears. Indeed, since Theorem 2.3 applies for any $\gamma > 0$, we deduce, as before, that $\phi_\gamma$ is always an extremal for the corresponding Moser-Trudinger type inequality for $S^1$-invariant functions in $H_0(\Omega)$. But when $\gamma > 2$ the corresponding Moser-Trudinger inequality does not hold on all of $H_0(\Omega)$. Indeed, as before it is violated by a suitable regularization of a Green function with a pole in any given point in $\Omega$. This also shows that Theorem 2.3 cannot hold general if the invariance assumption is removed.

For completeness we will next recall the proof in [11] of the inequality used above. The arguments carry over verbatim to the setting of Stein domains, even if as explained above the $\mathbb{C}^n$-setting is adequate for our purposes.

2.4. Proof of the Moser-Trudinger type inequality 2.6. Let

$$G(\phi) := \frac{1}{\gamma} \log \int_{\Omega} e^{-\gamma \phi} dV + \frac{1}{(n+1)} \int_{\Omega} (dd^c \phi)^n,$$

whose Euler-Lagrange equation (i.e. the critical point equation $dG_{|\phi} = 0$) is precisely the complex Monge-Ampère equation 2.5. Given $\phi_0$ and $\phi_1$ in $H_0(\Omega)$ there is a unique geodesic $\phi_t$ connecting them in $H_0(\Omega)$. It may be defined as the following envelope: setting $\Phi(z,t) := \phi_t(z)$, where now $t$ has been extended to a complex strip $\mathcal{T}$ by imposing invariance in the imaginary $t$-direction, we set $M := \Omega \times \mathcal{T}$ the boundary data $\Phi_{\partial M}$ is determined by
\( \phi_0 \) and \( \phi_1 \) and we set
\[
(2.10) \quad \Phi(z,t) := \sup \{ \Psi(z,t) : \Psi \in C^0(\bar{M}) \cap PSH(M), \ \Psi_{\partial M} \leq \Phi_{\partial M} \}
\]
Since \( M \) is hyperconvex it follows that \( \Phi \in C^0(\bar{M}) \cap PSH(M) \) is the unique solution of the following Dirichlet problem:
\[
(dd^c \Phi)^{n+1} = 0, \quad \text{in} \ M
\]
and on \( \partial M \) the function \( \Phi \) coincides with the boundary data determined by \( \phi_i \) (see \cite{4} and references therein). We also note that if \( \phi_t \) is \( T \)-invariant for \( t = 0,1 \) then it is in fact \( T \)-invariant for any \( t \), as follows from its definition \(2.10 \) as an envelope (just as in the similar case of the Green function discussed above).

By Theorem \ref{2.3} the functional
\[
t \mapsto \frac{1}{\gamma} \log \int_{\Omega} e^{-\gamma \phi_t} dV
\]
is concave along any geodesic (indeed, the assumption \ref{2.2} follows immediately from Taylor expanding a holomorphic function on \( \Omega \) around the origin and using the positivity of the weights of the action). Next, we note that
\[
\mathcal{E}(\phi) := \int_{\Omega} \phi (dd^c \phi)^n
\]
is affine along a geodesic \( \phi_t \). Indeed, letting \( t \) be complex a direct calculation gives
\[
(2.11) \quad dd^c \mathcal{E}(\phi_t) = \int_{\Omega} (dd^c \phi)^{n+1}
\]
which, by definition, vanishes if \( \phi_t \) is a geodesic. All in all this means that \( \mathcal{G}(\phi_t) \) is concave along a geodesic. Letting now \( \phi \) be an arbitrary element in \( \mathcal{H}_0(\Omega)^T \) we take \( \phi_t \) to be the geodesic connecting the solution \( \phi_0 \) of equation \ref{2.5} (obtained from the invariant Kähler-Einstein metric on \( X \)) and \( \phi_1 = \phi \). Heuristically, \( \mathcal{G}(\phi_t) \) has a critical point at \( t = 0 \), i.e. its right derivative vanishes for \( t = 0 \) and hence by concavity \( \mathcal{G}(\phi_1) \leq \mathcal{G}(\phi_0) \). However, since \( \phi_t \) is not, a priori, smooth one has to be a bit careful when differentiating \( \mathcal{G}(\phi_t) \). However, by the affine concavity of \( \mathcal{E} \) it is not hard too see that
\[
\frac{1}{(n+1)} \frac{d}{dt} \bigg|_{t=0^+} \mathcal{E}(\phi_t) \leq \frac{1}{n!} \phi_0 (dd^c \phi_0)^n/n!
\]
(see Lemma 3.4 in \cite{4}) and hence, by concavity, \( \mathcal{G}(\phi_1) \leq 0 + \mathcal{G}(\phi_0) \), which concludes the proof of the M-T inequality with \( C = \mathcal{G}(\phi_0) \).

Remark 2.8. Theorem is still valid when \( X \) is merely weakly Fano, i.e. \( -K_X \) is nef and big, if one uses the notion of (singular) Kähler-Einstein metrics introduced in \cite{7}. Indeed, by \cite{7} such a metric is smooth on a Zariski open subset of \( X \) and hence, by the Kähler-Einstein equation, strictly positively curved there. Moreover, by the Kawamata-Shokurov basepoint free theorem (\cite{58}, Theorem 3.3), \( -mK_X \) is base point free for \( m \) sufficiently large and...
hence Lemma 2.1 still applies and together with the Bialynicki- Birula decomposition one gets an exhaustion by $T$–invariant Stein domains of a dense open embedding of $\mathbb{C}^n$ in $X$. The rest of the proof then proceeds exactly as before.

2.5. Bounds on the volume in the absence of a Kähler-Einstein metric. Recall that for a general Fano manifold $X$ the “greatest lower bound on the Ricci curvature” is the invariant $R(X) \in [0, 1]$ defined as the sup of all positive numbers $r$ such that $\text{Ric } \omega \geq r \omega$ (see [50]). A simple modification of the proof of Theorem 1.1 then gives the following more general statement:

**Theorem 2.9.** Let $X$ be a Fano manifold admitting a holomorphic $\mathbb{C}^* -$ action with a finite number of fix points. Then the first Chern class $c_1(X)$ satisfies the following upper bound

$$c_1(X)^n \leq \left( \frac{n+1}{R(X)} \right)^n$$

The point is that, as shown in [56], the invariant $R(X)$ coincides with the sup of all $r \in [0, 1]$ such that, for any given Kähler form $\eta$ there exists $\omega_r$ such that

$$\text{Ric } \omega_r = r \omega_r + (1 - r) \eta$$

As is well-known $\omega_r$ is uniquely determined. In particular, if $X$ admits a holomorphic action by $S^1$ then, by averaging over $S^1$, we may take $\eta$ to be $S^1$–invariant. By the uniqueness of solutions to the previous equation it then follows that $\omega_r$ is also $S^1$–invariant. For any fixed $r$ we can now repeat the proof of Theorem 1.1 and obtain an $S^1$–invariant psh solution $\phi$ to

$$\left( d\omega^c \right)^n = \frac{V_{\Omega} e^{-r \phi} e^{-(1-r) \phi_0} dV}{\int_{\Omega} e^{-r \phi} e^{-(1-r) \phi_0} dV}, \quad \phi = 0 \text{ on } \partial \Omega,$$

where $\eta = d\omega^c \phi_0$ on $\Omega$. The Moser-Trudinger inequality still applies when $dV$ is replaced with $e^{-(1-r) \phi_0} dV$. Indeed, we just apply Theorem 2.8 to the curve $(1-r) \phi_t + r \phi_0$, where $\phi_t$ is a geodesic as before. In fact, since $\phi_0$ is bounded this gives the same Moser-Trudinger inequality as before, but with the integrand $e^{-\phi}$ replaced with $e^{-r \phi}$. Hence, rescaling we obtain a contradiction if $r^n V(X) > (n+1)^n$, just as before. In particular, taking the sup over all $r < R(X)$ then concludes the proof of the previous theorem.

3. The volume of (singular) Toric varieties and convex bodies

We will next briefly explain how to carry out the proof of Theorem 1.2 possibly singular toric Fano varieties directly using convex analysis in $\mathbb{R}^n$. This approach has the advantage of bypassing some technical difficulties related to the singularities of the toric variety in question. The motivation comes from the well-known correspondence between $T$–invariant positively curved metrics on toric line bundles $L \to X_P$ and convex functions in $\mathbb{R}^n$ whose (sub-gradient) image is contained in the corresponding polytope $P$ (see [21] [8] and references therein) More precisely, a psh function $\Phi(z)$ on...
the complex torus $\mathbb{C}^n$, embedded in $X$, is the weight of $a$, possibly singular, positively curved metric on $L \to X_P$ iff $\Phi$ is the pull-back under the Log map

$$\log \mathbb{C}^n \to \mathbb{R}^n : z \mapsto x := (\log(|z_1|^2, \ldots, \log(|z_n|^2)),$$

of a convex function $\phi(x)$ on $\mathbb{R}^n$ such that the (sub-) gradient image $d\phi(\mathbb{R}^n)$ is contained in $P$. This correspondence has been mostly studied in the case when $X_P$ is smooth and the metric on $L$ is smooth and positively curved. Then the corresponding convex function has the property that the differential $(\text{gradient}) d\phi$ defines a diffeomorphism from $\mathbb{R}^n$ to the interior of $P$ and moreover its Legendre transform satisfies Guillemin’s boundary conditions (see [21] and references therein). However, we stress that these refined regularity properties will play no role here. Our normalizations are such that

$$(\text{Log})^* MA(\Phi) = MA(\phi),$$

where $MA(\phi)$ denotes $n!$ times the usual real Monge-Ampère measure of the convex function $\phi$, i.e. $MA(\phi)(E) := n! \int_{d\phi(E)} dp$, for any Borel measure $E$.

We also point out that in the case when $X_P$ is smooth one may assume that $P$ is contained in the positive octant with a vertex at 0 and the psh function $\Phi$ can then be taken to be defined on all of $\mathbb{C}^n$, as in the proof of Theorem 1.1. The technical difficulty in the general case when $X_P$ may be singular - when seen from the complex point of view - is that, even if we may still arrange that $P$ is in the positive octant with a vertex at 0, the function $\phi$ will a priori only be continuous on $\mathbb{C}^n$, i.e. away from the coordinate axes.

Anyway, from the real point of view the latter theorem can be rephrased entirely in terms of convex analysis on convex domains in $\mathbb{R}^n$ of the form

$$\Omega := \{\psi < 0\}$$

where $\psi$ is a convex function on $\mathbb{R}^n$ such that its (sub-) gradient image of a convex body $P$ in the positive octant. We propose to call such a domain monotone. Note that, since $\partial \psi / \partial x_j \geq 0$ any monotone convex domain $\Omega$ is invariant under the action of the additive semi-group $[-\infty, 0]^n$ by translations. In particular, $\Omega$ is unbounded in contrast to the standard setting of bounded convex domains in convex analysis. Still, it is not hard to generalize the usual properties valid for convex functions on bounded domains, as long as one works with bounded convex functions on $\Omega$. More precisely, a convenient function space to work with is the space $\mathcal{H}(\Omega, \psi)$ of all bounded convex functions $\phi$ on $\Omega$ such that $\phi \geq \delta \psi$ for some positive number $\delta$ (depending on $\phi$). For example, it is not hard to see that for any such $\phi$ we have $\phi = 0$ on $\partial \Omega$ and the total real Monge–Ampère mass of $\phi$ on $\Omega$ is finite. Now all the usual notions concerning psh functions on bounded pseudoconvex domains can be transported to the setting of monotone convex domains. For example, we can define the Green function $g$ of $\Omega$ (playing the role of the usual pluricomplex Green function for a domain in $\mathbb{C}^n$ with a pole at the origin)
by

\[(3.1) \quad g(x) := \sup \left\{ \phi(x) : \phi \in \mathcal{H}(\Omega, \psi) : \phi(x) \leq \log(\sum_{i=1}^{n} e^{x_i}) + O(1) \right\} \]

Then standard arguments show that

- \(g\) is convex and continuous on \(\bar{\Omega}\) and \(g = 0\) on \(\partial \Omega\)
- \(g(x) = \log(\sum_{i=1}^{n} e^{x_i}) + O(1)\)
- \(MA_{\mathbb{R}}(g) = 0\) and if \(g_t := \log(e^{-2t} + e^{g_0})\) then \(g_t \in \mathcal{H}(\Omega, \psi)\) with \(\lim_{t \to \infty} \int_{\Omega} MA_{\mathbb{R}}(g_t) = 1\) and

\[(3.2) \quad \lim_{t \to \infty} \int_{\Omega} e^{-ng_t} d\nu_n = \infty,\]

where

\[(3.3) \quad d\nu_n(x) = e^{\sum_{i=1}^{n} x_i} dx,\]

i.e. \(d\nu_n\) is a multiple of the push-forward under the Log map of the Lebesgue measure on \(\mathbb{C}^n\).

Moreover, if \(\Phi\) is a \(T^n\)-invariant psh function in \(\mathbb{C}^n\) satisfying the Kähler-Einstein equation

\[(2.4) \quad \nabla_{\mathbb{R}} \Phi + \nabla_{\mathbb{R}} \psi = \frac{1}{2} \text{det}(\nabla_{\mathbb{R}} \Phi)\]

then its push-forward \(\psi\) satisfies the following real Monge-Ampère equation

\[(3.4) \quad MA_{\mathbb{R}}(\psi) = e^{-\psi(x)} d\nu_n(x),\]

The following theorem, proved in [6], shows that the previous equation admits a solution iff \((1, ..., 1)\) is the barycenter of \(P\):

**Theorem 3.1.** [6] Let \(P\) be a convex body and fix an element \(p_0\) in \(P\). Then there is a smooth convex function \(\phi\) such that \(d\phi\) defines a diffeomorphism from \(\mathbb{R}^n\) to the interior of \(P\) and such that \(\phi\) solves the equation

\[(MA_{\mathbb{R}}(\phi) = e^{-\phi(x)} + (p_0, x)) dx\]

on \(\mathbb{R}^n\) iff \(p_0\) is the barycenter of \(P\).

One can now go on to obtain Moser-Trudinger type inequalities (and Brezis-Merle-Demailly type inequalities) essentially as before, using Prekopa’s convexity theorem in \(\mathbb{R}^n\). The only technical difficulty is to make sure that the corresponding energy type functional

\[\mathcal{E}_{\mathbb{R}}(\phi) := \frac{1}{(n+1)} \int_{\Omega} \phi MA_{\mathbb{R}}(\phi)\]

still has the appropriate properties (for its first and second derivatives along geodesics). To this end one can first establish that if \(M(\phi_1, ..., \phi_n)\) denotes the real mixed Monge-Ampère measure obtained by polarizing the usual real Monge-Ampère measure, then the pairing

\[(\phi_0, \phi_1, ..., \phi_n) \mapsto \int_{\Omega} \phi_0 M(\phi_1, ..., \phi_n),\]
is finite and symmetric on $H(\Omega, \psi)^{n+1}$. The new technical difficulty compared to the classical situation (see for example [35] and references therein) comes from the unboundedness of $\Omega$. But using that $\Omega$ is complete “at infinity” in $\Omega$ (wrt the Euclidean metric) one can use standard cut-off function argument to carry out the required integrations by parts. Concretely, given a smooth compactly supported function $f$ on $\Omega$ one can use the sequence $\chi_j(x) := f(|x|/j)^2$ as cut-off functions on $\Omega$ (this is sometimes referred to as “the Gaffney trick”).

The advantage of the $\mathbb{R}^n$–approach is that it applies equally well to the case when the corresponding toric variety $X_P$ is singular. In fact, one may as well replace $P$ with any convex body, even though there is then no corresponding toric variety. Mimicking the proof of Theorem 1.1, replacing $\phi$ with a solution of the equation in Theorem 3.1 one then obtains a proof of Theorem 1.5 essentially as above.

Next, we explain how to deduce Cor 1.4 from Theorem 1.5. Given a real polytope $P$ with non-empty interior we can write it as

$$P = \{ p \in \mathbb{R}^n : \alpha_F(p) \geq 0 \},$$

where $F$ is an index running over the facets $\{\alpha_F = 0\}$ of $P$. We fix a vertex $v$ and affine functions $\alpha_{F_1}, ..., \alpha_{F_n}$ cutting out $n$ faces of $P$ meeting $v$ and spanning a cone of maximal dimension. Then

$$P' = \alpha(P), \quad \alpha(p) := (\alpha_{F_1}(p)/\alpha_{F_1}(b_P), ..., \alpha_{F_n}(p)/\alpha_{F_n}(b_P))$$

is an $n$–dimensional polytope in the positive octant $[0, \infty]^n$ such that $b_P = (1, 1, ..., 1)$. Moreover, if $P$ is a rational polytope as in the statement of Cor 1.4 then

$$\text{Vol}(P') = \frac{d}{a_{F_1} \cdots a_{F_n}} \text{Vol}(P)$$

where $d$ is the determinant of the linear map $p \mapsto (l_{F_1}(p), ..., l_{F_n}(p))$. But the map is represented by an invertible matrix with integer coefficients and hence $d$ is a positive integer and in particular $d \geq 1$. Since, by assumption, $a_{F_i} \leq 1$ it follows that $\text{Vol}(P') \geq \text{Vol}(P)$ and hence it will be enough to prove Theorem 1.5.

Remark 3.2. As kindly pointed out to us by Bo’az Klartag Theorem 1.5 can also be deduced from Grunbaum’s inequality. We thank him for allowing us to reproduce his elegant argument here. First recall that the Grunbaum inequality says that if $P$ is a convex body, and $P^-$ denotes the intersection of $P$ with an affine half-space defined by one side of a hyperplane $H$ passing through the barycenter of $P$, then

$$\text{Vol}(P^-) \geq \left( \frac{n}{(n+1)} \right)^n \text{Vol}(P).$$

In particular, if $P$ is a convex body as in the statement of Theorem 1.5 we can take $P^-$ to be $n$ times the unit-simplex $\Delta$ in the positive octant with
one vertex at 0. Since, \( \text{Vol}(P_-) \leq \text{Vol}(n\Delta) = \frac{n^n}{n!} \) this gives the desired inequality.

4. Symplectic geometry and multiplicity free actions

4.1. The homogeneous case. Let us start by specializing Theorem 1.1 to the case when \( X \) is a rational homogeneous space. Even though this is the simples case the corresponding degree bound, when rephrased in terms of representation theory, is highly non-trivial and was first obtained by Snow [54].

Let us first recall some basic representation theory (see [54] and references therein). Let \( K \) be a compact complex semi-simple Lie group and denote by \( G \) its complexification. Under the adjoint action of a fixed maximal torus \( T \) the Lie algebra \( L(G) \) decomposes as

\[
L(G) = L(T_c) \oplus E_+ \oplus \overline{E}_+, \quad E_+ = \bigoplus_{\alpha \in R^+} E_\alpha,
\]

where \( R^+ \) is a consistent choice of positive roots \( \alpha \in L(T)^* \) and \( E_\alpha \) denote the corresponding weight spaces, i.e. \( E_\alpha \) is generated by a vector \( Z_\alpha \) such that \([t,Z_\alpha] = i \langle t, \alpha \rangle Z_\alpha \) for any \( t \in L(T) \). A consistent choice of positive roots \( R^+ \) corresponds to a choice of Borel group \( B \) with Lie algebra

\[
L(B) = L(T) \oplus \overline{E}_+.
\]

The corresponding (complete) flag variety is defined as the \( G \)-homogeneous compact complex manifold \( G/B(= K/T) \). As is well-known any rational \( G \)-homogeneous compact complex manifold \( X \) may be written as

\[
X = G/P,
\]

for some \( G \) as above and a parabolic subgroup \( P \) of \( G \) (i.e. a subgroup containing a Borel group which we may assume is \( B \)). We recall that any \( G \)-homogeneous line bundle \( L \rightarrow G/P \) is determined by a weight \( \lambda \), i.e. an element in the weight lattice of \( L(T_c)^* \). Indeed,

\[
L_\lambda = G \times_{(P, \rho_\lambda)} \mathbb{C},
\]

where \( \rho_\lambda \) is a homomorphism \( P \rightarrow \mathbb{C}^* \), which is uniquely determined by its restriction to the complex torus in \( P \) and hence determined by an element \( \lambda \) in the weight lattice of \( L(T_c)^* \). By construction, the tangent bundle \( TX \) is generated at the identity coset by root vectors \( Z_\alpha \in E_\alpha \) for \( \alpha \) in a subset \( R^+_X \) of the positive roots. In particular, the weight \( \lambda_X \) of the anti-canonical line bundle \( -K_X \) may be written as

\[
\lambda_X = \sum_{\alpha \in R^+_X} \alpha,
\]

\[(4.1)\]
defining an ample line bundle, so that $X$ is Fano. Moreover, by the general Weyl character formula,

$$c_1(L^\lambda)^n = n! \prod_{\alpha \in R^+_X} \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \rho \rangle},$$

with $\rho$ denoting, as usual, the half-sum of all the positive roots $\alpha$ and where $\langle \cdot, \cdot \rangle$ denotes the Killing form. Hence, Theorem 1.1 applied to $G/P$ translates to a Lie algebra statement first shown by Snow [54]. Indeed, as shown in [54] (Prop 1) formula 4.1 can be simplified using the Dynkin diagram of $G$. Using this latter formula and the Weyl character formula above together with classification theory for semi-simple Lie groups, Snow then shows, using quite elaborate calculations, how to deduce the bound in Theorem 1.1 for the Fano manifold $X = G/P$. It should also be pointed out that the results in [54] also give that $P^n$ is the unique maximizer of the degree among all partial flag manifolds.

4.2. The case of spherical varieties and multiplicity free actions. In this section we will reformulate Theorem 1.1 in the case when $X$ is spherical variety, using Brion’s formula for the volume [11] of a line bundle on $X$. We will use the symplecto-geometric formulation (see the end of [11] and [10] where further references can be found).

Let us start by recalling the definition of the Duistermaat-Heckman measure in symplectic geometry. Let $(X, \omega)$ be a symplectic manifold and $K$ a compact connected Lie group acting by symplectomorphisms on $X$ and fix a compact maximal torus $T$ in $K$. Assume for simplicity that the first Betti number of $X$ vanishes (which will be the case here since $X$ will be a Fano manifold). Then there is a moment map

$$\mu_T : X \to L(T)^*,$$

where $L(T)$ denotes the real vector space given by the Lie algebra of $T$. The image $P := \mu_T(X)$ is a convex rational polytope and the density $v(p)$ of the Duistermaat-Heckman measure $(\mu_T)_*\omega^n/n!$ on $\mu_T(X)$ is continuous and piecewise polynomial. The convex polytope obtained by intersecting $\mu_T(X)$ with a fixed positive Weyl chamber $\Lambda_+$ in $L(T)^*$ is called the moment polytope.

The action of $K$ is said to be multiplicity-free if the group of symplectomorphisms of $(X, \omega)$ which commute with $K$ is abelian (equivalently, all $K$-invariant functions on $X$ Poisson commute). As is well-known [10] any $G$-spherical non-singular complex algebraic variety $X$ with a $G$-equivariant ample line bundle $L \to X$ equipped with a fixed positive curvature form $\omega$ in $c_1(L)$ corresponds to a symplectic manifold $(X, \omega)$ with a symplectic action by the real form $K$ of $G$ which is multiplicity-free. Moreover, as shown by Brion [11], in the spherical (i.e. multiplicity free) case the density $v$ of the
Duistermaat-Heckman measure is explicitly given by
\[ v(p) = \prod_{\alpha} \frac{\langle \alpha, p \rangle}{\langle \alpha, \rho \rangle}, \]
where \( \rho \) denotes the half-sum of all the positive roots \( \alpha \) and the products runs over all positive roots \( \alpha \) such that \( \langle \alpha, p \rangle > 0 \). Moreover, the Lesbesgue measure \( dp \) has been normalized to give unit-volume to the fundamental domain of the lattice in \( P \) (see section 4.1). By the definition of \( v \) as the density of the Duistermaat-Heckman measure

\[ \int_X \omega^n / n! = \int_P v(p) dp \]

Hence, applying Theorem 1.1 to a spherical non-singular Fano variety gives the following

**Corollary 4.1.** Let \((X, \omega)\) be symplectic manifold with a symplectic action by a compact Lie group \( K \) which is multiplicity-free and denote by \( P \) the image of the moment map associated to a fixed maximal torus \( T \) in \( K \). If \( X \) admits an integrable \( \omega \)-compatible complex structure \( J \) preserved by \( K \) and such that the cohomology class \([\omega]\) contains a Kähler-Einstein metric on \((X, J)\) then

\[ \int_P v(p) dp \leq (n + 1)^n / n! \]

Equality hold when \((X, \omega)\) is complex projective space equipped and \(\omega\) is the standard suitably normalized \(SU(n)\)-invariant symplectic form (i.e. \(\omega\) is \((n + 1)\) times the Fubini-Study form).

We recall that the condition that a spherical variety \(X\) be Fano can be expressed rather explicitly in algebro-geometric terms [12].

**Remark 4.2.** It is well-known that any spherical variety \(X\) has a holomorphic \(\mathbb{C}^*\)-action such that the fixed point set. \(X^{\mathbb{C}^*}\) is finite, so that Theorem 1.1 indeed can be applied to \(X\). Let us briefly recall the reason that \(X^{\mathbb{C}^*}\) is finite (as kindly explained to us by Michel Brion). The starting point is the fundamental fact that any \(G\)-spherical variety \(X\) can be covered by a finite number of \(G\)-orbits [43]. Next one shows that if \(G\) is a reductive group acting transitively on a set \(Y\) (here an orbit of \(G\)) then the fixed point set \(Y^T_c\) is finite for any maximal complex torus \(T_c\) in \(G\). Indeed, the Weyl group \(N_G(T_c)/T_c\) is finite if \(G\) is reductive and its acts transitively on \(Y^T_c\), as follows from the definition of the normalizer \(N_G(T_c)\) (see Proposition 7.2 in [18]). Finally, it is a general fact that for a generic (regular) one-parameter subgroup \(\mathbb{C}^*\) in \(T_c\) one has that \(Y^T = Y^{\mathbb{C}^*}\) (as can be proved by reducing the problem to the linear action of \(T\) on a vector space using [55])

In particular, the previous corollary applies to horospherical Fano varieties. These are homogenous toric bundles over a rational homogenous variety [51 49]. Let us for simplicity consider the case of homogenous fibrations
X over a (complete) flag variety:

$$X \rightarrow G/B$$

where any fiber is biholomorphic to a given toric variety \( F \). Then the corresponding polytope \( P \) is contained in the interior of a positive Weyl chamber, coinciding with the moment polytope of \( F \) under the induced torus action. Moreover, \( X \) is Fano with a Kähler-Einstein metric iff \( F \) is Fano and the sum of the positive roots \( \sum \alpha \) coincides with the barycenter of the reflexive polytope \( P \) wrt the Duistermaat-Heckman measure \( vdp \) (this was first shown in [51], but see also the illuminating discussion in section 4.1 in [21]). Hence, we arrive at the following

**Corollary 4.3.** Let \( G \) be a semi-simple complex Lie group and fix a maximal torus \( T \) in \( G \) and a set \( R^+ \) of \( n \) positive roots \( \alpha_i \) for the Lie algebra of \( G \). Let \( P \) be a reflexive lattice polytope in the positive Weyl chamber of \( L(T)^* \) which is Delzant and such that \( \sum_{\alpha \in R^+} \alpha \) is the barycenter of \( P \) wrt the Duistermaat-Heckman measure \( vdp \). Then

$$\int_P vdp \leq (n + 1)^n / n!$$

We expect that the condition that \( P \) be Delzant, i.e. the corresponding toric variety is smooth, can be removed.

Finally, let us mention the connection to Okounkov bodies. As shown by Okounkov [18] one can associate another convex polytope \( \Delta \) to a polarized spherical variety \( X \), such that \( \Delta \) fibers over the moment polytope \( P \) (the fibers being the Gelfand-Cetlin string polytopes). The definition is made so that

$$c_1(L)^n / n! = \text{Vol}(\Delta)$$

More generally, to any polarized projective variety \((X, L)\) there is convex body \( \Delta \) associated (further depending on an auxiliary choice of flag in \( X \)) such that the previous formula for \( c_1(L)^n / n! \) holds [40, 33]. In the light of Theorem [11] and the toric case it would be interesting to know if the condition that \( X \) be Fano (and \( L = -K_X \)) with a Kähler-Einstein metric can be naturally expressed in terms of properties of \( \Delta \)? See also [1] for the case of reductive spherical varieties.

**References**

[1] Alexeev, V. and Katzarkov, L. On K-stability of reductive varieties. Geometric and Functional Analysis 15 2005 (297-310)

[2] Bando, S.; Mabuchi, T: Uniqueness of Einstein Kahler metrics modulo connected group actions, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 11-40.

[3] Batyrev, V.V.: Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. J. Algebr. Geom. 3, 493-535 (1994)

[4] Berman, R.J.; Berndtsson, B: Moser-Trudinger type inequalities for complex Monge-Ampère operators and Aubin’s “hypothèse fondamentale”. Preprint in 2011 at arXiv:1109.1263
[5] Berman, R.J.; Berndtsson, B: The projective space has maximal volume among all toric Kähler-Einstein manifolds. [arXiv:1112.4445]
[6] Berman, R.J.; Berndtsson, B: Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. Preprint.
[7] Berman; R.J: Eyssidieu, P; Boucksom, S; Guedj, V; Zeriahi, A: Convergence of the Kähler-Ricci flow and the Ricci iteration on Log-Fano varieties. [arXiv:1111.7158]
[8] Berndtsson, B: Curvature of vector bundles associated to holomorphic fibrations. Annals of Math. Vol. 169 (2009), 531-560
[9] Białynicki; Birula: Some theorems on actions of algebraic groups. Ann. Math. 98 (1973) 480-497
[10] M. Brion: Sur l'image de l'application moment, p. 177-192 in: S'émiaire d'alg'ebre Paul Dubreil et Marie-Paul Malliavin, Lecture Notes in Math. 1296, Springer, New York 1987
[11] Brion, M: Groupe de Picard et nombres caractéristiques des variétés sphériques. Duke Math. J., 58, 397-424 (1989).
[12] Brion, M: Curves and divisors in spherical varieties, Algebraic groups and Lie groups (G. Lehrer, ed.), Australian Math. Soc. Lecture Series, vol. 9, Cambridge Univ. Press, Cambridge, 1997, pp. 21–34.
[13] Caglioti.E; Lions, P-L; Marchioro.C; Pulvirenti.M: A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Communications in Mathematical Physics (1992) Volume 143, Number 3, 501-525
[14] D. Cordero -Érausquin, R.J. McCann and M. Schmuckenschläger, A Riemannian interpolation inequality 'a la Borell, Brascamp and Lieb, Invent. Math. 146 (2001), 219–257.
[15] Cox, David A.; Little, John B.; Schenck, Henry K. Toric varieties. Graduate Studies in Mathematics, 124. American Mathematical Society, Providence, RI, 2011
[16] Debarre, O: Higher-dimensional algebraic geometry, Universitext, New York, NY, Springer, 2001.
[17] Debarre, O: Fano varieties. Higher dimensional varieties and rational points (Budapest, 2001), 93–132, Bolyai Soc. Math. Stud., 12, Springer, Berlin, 2003.
[18] De Concini, C.; Procesi, C. Complete symmetric varieties. Invariant theory (Montecatini, 1982), 1–44, Lecture Notes in Math., 996, Springer, Berlin, 1983.
[19] Demailly, J-P: Estimates on Monge-Ampère operators derived from a local algebra inequality. Complex analysis and digital geometry, 131–143, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 86, Uppsala Universitet, Uppsala, 2009.
[20] Ding, W. and Tian, G.: The generalized Moser-Trudinger Inequality. Proceedings of Nankai International Conference on Nonlinear Analysis, 1993.
[21] Donaldson, Simon K. Kähler geometry on toric manifolds, and some other manifolds with large symmetry. Handbook of geometric analysis. No. 1, 29–75, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.
[22] Ehrhart, E: Une generalisation du théoreme de Minkowski, C. R. Acad. Sci. Paris 240 (1955), 483–485.
[23] Ehrhart, E: Volume réticulaire critique d’un simplexe, Journal für die reine und angewandte Mathematik (Crelles Journal). Volume 1979, Issue 305, Pages 218–220 (1979)
[24] Hacon, C; McKernan, J: Boundedness results in birational geometry. (English summary) Proceedings of the International Congress of Mathematicians. Volume II, 427–449, Hindustan Book Agency, New Delhi, 2010.
[25] Helgason, Sigurdur Differential Geometry, Lie groups, and symmetric spaces. Corrected reprint of the 1978 original. Graduate Studies in Mathematics, 34. American Mathematical Society, Providence, RI, 2001. xxvi+641 pp.
[26] Fiset, M. H. J.; Kasprzyk, A. M.: A note on palindromic $\delta$-vectors for certain rational polytopes. Electron. J. Combin. 15 (2008), no. 1, Note 18, 4 pp.
[27] J.P. Gauntlett, D. Martelli, J. Sparks, S.-T. Yau, Obstructions to the Existence of Sasaki-Einstein Metrics, Commun. Math. Phys. 273 (2007), 803–827.
[28] Gritzmann, P.; Wills, J.M: Lattice points, in: Handbook of convex geometry, 765–797, North-Holland, Amsterdam, 1993.
[29] Grunbaum, B: Partitions of mass-distributions and of convex bodies by hyperplanes. Pacific J. Math., Vol. 10, (1960), 1257-1261.
[30] Guedj, V; Kolev, B; Yeganefar, N: Kähler-Einstein fillings. arXiv:1111.5320
[31] Hwang, J-M: On the zeroes of holomorphic vector fields on algebraic manifolds. Trends in Mathematics Information Center for Mathematical Sciences Volume 2, December 1999, Pages 86{90
[32] Kaup, W.: Reelle Transformationsgruppen und invariante Metriken auf komplexen Räumen. Invent. Math. 3, 43-70 (1967)
[33] Kaveh, K; , Khovanskii, A.G: Convex bodies and algebraic equations on affine varieties. arXiv:0904.3350
[34] Kiselman, C. O. The partial Legendre transformation for plurisubharmonic functions. Invent. Math. 49 (1978), no. 2, 137–148.
[35] Klartag, B. Marginals of geometric inequalities. Geometric aspects of functional analysis, 133–166, Lecture Notes in Math., 1910, Springer, Berlin, 2007.
[36] Klimek, M: Pluripotential theory. London Mathematical Society Monographs. New Series, 6. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991. xiv+266 pp. ISBN:
[37] Kollar, J Rational curves on algebraic varieties, Springer, Berlin, 1996;
[38] Kollár; Mori, S; Birational Geometry of Algebraic Varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.
[39] Kreuzer, M.; Skarke, H.: PALP: A package for analyzing lattice polytopes with applications to toric geometry. Computer Phys. Comm., 157, 87-106 (2004)
[40] Lazarsfeld, R.; Mustata, M. Convex bodies associated to linear series. Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 5, 783–835.
[41] Li, C: Greatest lower bounds on Ricci curvature for toric Fano manifolds. Adv. Math. 226 (2011), no. 6, 4921–4932
[42] Loeb, J-J: Action d’une forme réelle d’un groupe de Lie complexe sur les fonctions plurisousharmoniques. Ann. Inst. Fourier (Grenoble) 35 (1985), no. 4, 59–97.
[43] Luna, D.; Vust, Th.: Plongements d’espaces homogènes. Comment. Math. Helv. 58 (1983), no. 2, 186–245.
[44] Mabuchi, T: Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties. Osaka J. Math. 24 (1987), no. 4, 705–737
[45] Matsushima, Y: Sur la structure du groupe d’homéomorphismes analytiques d’une certaine variété kählérienne. (French) Nagoya Math. J. 11 1957 145–150.
[46] Matsushima, Y: Espaces homogènes de Stein des groupes de Lie complexes. (French) Nagoya Math. J. 16 1960 205–218.
[47] Nill, B; Paffenholz, A: Examples of non-symmetric Kähler-Einstein toric Fano manifolds. Preprint in 2009 at arXiv:0909.2054
[48] Okounkov, A.Y: Note on the Hilbert polynomial of a spherical variety. Functional Analysis and Its Applications Volume 31, Number 2, 138-140
[49] Pasquier, Boris Variétés horosphériques de Fano. Bull. Soc. Math. France 136 (2008), no. 2, 195–225.
[50] Phong, D.H; Sturm, J: Lectures on Stability and Constant Scalar Curvature. Adv. Lect. Math. (ALM), 14, Int. Press, Somerville, MA, 2010.
[51] Podesta and Spiro, Kahler-Ricci solitons on homogeneous toric bundles I, II Arxiv DG/0604070/0604071
[52] Prekopa A., On logarithmically concave measures and functions, Acta Sci. Math. 34 (1973) 335-343.
[53] Ruzzi, A: Projectively normal complete symmetric varieties and Fano complete symmetric varieties. Ph D thesis, http://hal.inria.fr/docs/00/57/59/74/PDF/these-ruzzi.pdf
[54] Snow, D: Bounds for the anticanonical bundle of a homogeneous projective rational manifold. Doc. Math. 9 (2004), 251–263
[55] Sumihiro, H: Equivariant completion. J. Math. Kyoto Univ. 14 (1974), 1–28.
[56] Székelyhidi, Gábor Greatest lower bounds on the Ricci curvature of Fano manifolds. Compos. Math. 147 (2011), no. 1, 319–331
[57] Wang, X; Zhu, X: Kähler–Ricci solitons on toric manifolds with positive first Chern class. Advances in Mathematics 188 (2004), 87–103.
[58] Woo, A; Yong, A: When is a Schubert variety Gorenstein? Adv. Math. 207 (2006), no. 1, 205–220.
[59] S.T. Yau, Calabi’s conjecture and some new results in algebraic geometry. Proc. Natl. Acad. Sci. USA, 74 (1977), 1798-1799.
[60] M.Øbro, An algorithm for the classification of smooth Fano polytopes, arXiv:0704.0049 2007.

E-mail address: robertb@chalmers.se, bob@chalmers.se
Current address: Mathematical Sciences - Chalmers University of Technology and University of Gothenburg - SE-412 96 Gothenburg, Sweden