C*-ALGEBRAS GENERATED BY MULTIPLICATION OPERATORS AND COMPOSITION OPERATORS BY FUNCTIONS WITH SELF-SIMILAR BRANCHES II

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Abstract. Let $K$ be a compact metric space and let $\varphi : K \to K$ be continuous. We study a $C^*$-algebra $MC_\varphi$ generated by all multiplication operators by continuous functions on $K$ and a composition operator $C_\varphi$ induced by $\varphi$ on a certain $L^2$ space. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a system of proper contractions on $K$. Suppose that $\gamma_1, \ldots, \gamma_n$ are inverse branches of $\varphi$ and $K$ is self-similar. We consider the Hutchinson measure $\mu_H$ of $\gamma$ and the $L^2$ space $L^2(K, \mu_H)$. Then we show that the $C^*$-algebra $MC_\varphi$ is isomorphic to the $C^*$-algebra $O_{\gamma}(K)$ associated with $\gamma$ under the open set condition and the measure separation condition. This is a generalization of our previous work, in which we studied the case where $\gamma$ satisfied the finite branch condition.

1. Introduction

Several authors considered $C^*$-algebras generated by composition operators (and Toeplitz operators) on the Hardy space $H^2(\mathbb{D})$ on the open unit disk $\mathbb{D}$ ([2], [5], [7], [8], [14], [15], [16], [18], [20], [21], [22]). On the other hand, there are some studies on $C^*$-algebras generated by composition operators on $L^2$ spaces, for example [3], [4], [17]. Matsumoto [17] introduced some $C^*$-algebras associated with cellular automata generated by composition operators and multiplication operators.

Let $R$ be a rational function of degree at least two, let $J_R$ be the Julia set of $R$ and let $\mu^L$ be the Lyubich measure of $R$. In [3], we studied the $C^*$-algebra $MC_R$ generated by all multiplication operators by continuous functions in $C(J_R)$ and the composition operator $C_R$ induced by $R$ on $L^2(J_R, \mu^L)$. We showed that the $C^*$-algebra $MC_R$ is isomorphic to the $C^*$-algebra $O_R(J_R)$ associated with the complex dynamical system $\{R^n\}_{n=1}^\infty$ introduced in [10].

Let $(K, d)$ be a compact metric space, let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a system of proper contractions on $K$ and let $\varphi : K \to K$ be continuous. Suppose that $\gamma_1, \ldots, \gamma_n$ are inverse branches of $\varphi$ and $K$ is self-similar. Let $\mu_H$ be the Hutchinson measure of $\gamma$. In [4], we studied the $C^*$-algebra $MC_\varphi$ generated by all multiplication operators by continuous functions in $C(K)$ and the composition operator $C_\varphi$ induced by $\varphi$ on $L^2(K, \mu_H)$. Assume that the system $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfies the open set condition, the finite branch condition and the measure separation condition in $K$. We showed that the $C^*$-algebra $MC_\varphi$ is isomorphic to the $C^*$-algebra $O_{\gamma}(K)$ associated with $\gamma$ introduced in [11].

In this paper we consider a generalization of [4]. We can remove the second condition that the system $\gamma$ satisfy the finite branch condition. We also show that

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$\mathcal{MC}_\varphi$ is isomorphic to $\mathcal{O}_\gamma(K)$ associated with $\gamma$. In this proof, we do not use a countable basis of a Hilbert bimodule.

There is an applications of the main theorem. Let $\tau$ be the tent map $\tau : [0, 1] \to [0, 1]$ defined by
\[
\tau(x) = \begin{cases} 
2x & 0 \leq x \leq \frac{1}{2}, \\
-2x + 2 & \frac{1}{2} < x \leq 1 
\end{cases}
\]
and let $\varphi$ be the map $\varphi : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ defined by $\varphi(x, y) = (\tau(x), \tau(y))$ for $x, y \in [0, 1]$. Suppose that $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are inverse branches of $\varphi$. Since the system $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ does not satisfy the finite branch condition, we cannot adapt [4] Theorem 4.4. However we can adapt this main theorem in this paper and the C*-algebra $\mathcal{MC}_\varphi$ is isomorphic to the Cuntz algebra $\mathcal{O}_\infty$.

2. Covariant relations

Let $(K, d)$ be a compact metric space. A continuous map $\gamma : K \to K$ is called a proper contraction if there exists constants $0 < c_1 \leq c_2 < 1$ such that
\[
c_1d(x, y) \leq d(\gamma(x), \gamma(y)) \leq c_2d(x, y), \quad x, y \in K.
\]

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a family of proper contractions on $(K, d)$. We say that $K$ is called self-similar with respect to $\gamma$ if $K = \bigcup_{i=1}^n \gamma_i(K)$. See [1] and [13] for more on fractal sets. We say that $\gamma$ satisfies the open set condition in $K$ if there exists a non-empty open set $V \subset K$ such that
\[
\bigcup_{i=1}^n \gamma_i(V) \subset V \quad \text{and} \quad \gamma_i(V) \cap \gamma_j(V) = \emptyset \quad \text{for} \quad i \neq j.
\]

For a system $\gamma$ of proper contractions on a compact metric space $K$, we introduce the following subsets of $K$.
\[
B_\gamma = \{ y \in K \mid y = \gamma_i(x) = \gamma_j(x) \text{ for some } x \in K \text{ and } i \neq j \}, \\
C_\gamma = \{ x \in K \mid \gamma_i(x) = \gamma_j(x) \text{ for some } i \neq j \}.
\]

We say that $\gamma$ satisfies the finite branch condition if $C_\gamma$ is finite set.

Let us denote by $\mathcal{B}(K)$ the Borel $\sigma$-algebra on $K$.

Lemma 2.1 ([6]). Let $K$ be a compact metric space and let $\gamma$ be a system of proper contractions. If $p_1, \ldots, p_n \in \mathbb{R}$ satisfy $\sum_{i=1}^n p_i = 1$ and $p_i > 0$ for $i$, then there exists a unique probability measure $\mu$ on $K$ such that
\[
\mu(E) = \sum_{i=1}^n p_i \mu(\gamma_i^{-1}(E))
\]
for $E \in \mathcal{B}(K)$.

We call the measure $\mu$ given by Lemma 2.1 the self-similar measure on $K$ with $\{p_i\}_{i=1}^n$. In particular, we denote by $\mu^H$ the self-similar measure with $p_i = \frac{1}{n}$ for $i$ and call this measure the Hutchinson measure. We say that $\gamma$ satisfies the measure separation condition in $K$ if $\mu(\gamma_i(K) \cap \gamma_j(K)) = 0$ for any self-similar measure $\mu$ and $i \neq j$.

For $a \in L^\infty(K, \mathcal{B}(K), \mu^H)$, we define the multiplication operator $M_a$ on $L^2(K, \mathcal{B}(K), \mu^H)$ by $M_af = af$ for $f \in L^2(K, \mathcal{B}(K), \mu^H)$. Let $\varphi : K \to K$ be measurable. Suppose that $\gamma_1, \ldots, \gamma_n$ are inverse branches of $\varphi$, that is, $\varphi(\gamma_i(x)) = x$ for $x \in K$ and
functions.

For $f \in C(K)$, we can easily see that $L_\varphi f \in C(K)$ since $\gamma_1, \ldots, \gamma_n$ are continuous functions.

**Proposition 2.2** ([1] Proposition 2.5). Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a system of proper contractions. Assume that $K$ is self-similar and the system $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfies the measure separation condition in $K$. Then $C_\varphi$ is an isometry on $L^2(K, \mathcal{B}(K), \mu^H)$, $L_\varphi$ is bounded on $L^\infty(K, \mathcal{B}(K), \mu^H)$, and

$$C_\varphi^* M_a C_\varphi = M_{L_\varphi(a)}$$

for $a \in L^\infty(K, \mathcal{B}(K), \mu^H)$.

The operator $C_\varphi$ is called the *composition operator* on $L^2(K, \mathcal{B}(K), \mu^H)$ induced by $\varphi$.

3. **C*-algebras associated with self-similar sets**

We recall the construction of Cuntz-Pimsner algebras [19] (see also [12]). Let $A$ be a C*-algebra and let $X$ be a right Hilbert $A$-module. We denote by $L(X)$ the C*-algebra of the adjointable bounded operators on $X$. For $\xi, \eta \in X$, the operator $\theta_{\xi, \eta}$ is defined by $\theta_{\xi, \eta}(\zeta) = \xi(\eta, \zeta)A$ for $\zeta \in X$. The closure of the linear span of these operators is denoted by $K(X)$. We say that $X$ is a *Hilbert bimodule* (or C*-correspondence) over $A$ if $X$ is a right Hilbert $A$-module with a *-homomorphism $\phi : A \to L(X)$.* We always assume that $\phi$ is injective.

A *representation* of the Hilbert bimodule $X$ over $A$ on a C*-algebra $D$ is a pair $(\rho, V)$ constituted by a *-homomorphism $\rho : A \to D$ and a linear map $V : X \to D$ satisfying

$$\rho(a)V_\xi = V_{\phi(a)\xi}, \quad V_\xi^* V_\eta = \rho(\langle \xi, \eta \rangle A)$$

for $a \in A$ and $\xi, \eta \in X$. It is known that $V_\xi \rho(b) = V_{\xi b}$ follows automatically (see for example [12]). We define a *-homomorphism $\psi_V : K(X) \to D$ by $\psi_V(\theta_{\xi, \eta}) = V_\xi V_\eta^*$ for $\xi, \eta \in X$ (see for example [19 Lemma 2.2]). A representation $(\rho, V)$ is said to be *covariant* if $\rho(a) = \psi_V(\phi(a))$ for all $a \in J(X) := \phi^{-1}(K(X))$.

Let $(i, S)$ be the representation of $X$ which is universal for all covariant representations. The *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the C*-algebra generated by $i(a)$ with $a \in A$ and $S_\xi$ with $\xi \in X$. We note that $i$ is known to be injective [19] (see also [12 Proposition 4.11]). We usually identify $i(a)$ with $a$ in $A$.

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a system of proper contractions on a compact metric space $K$. Assume that $K$ is self-similar. Let $A = C(K)$ and $Y = C(C)$, where $C = \bigcup_{i=1}^n \{ (\gamma_i(y), y) \mid y \in K \}$ is the cograph of $\gamma_i$. Then $Y$ is an $A$-$A$ bimodule over $A$ by

$$(a \cdot f \cdot b)(\gamma_i(y), y) = a(\gamma_i(y))f(\gamma_i(y), y)b(y), \quad a, b \in A, f \in Y.$$ 

We define an $A$-valued inner product $\langle \cdot, \cdot \rangle_A$ on $Y$ by

$$\langle f, g \rangle_A(y) = \sum_{i=1}^n f(\gamma_i(y), y) g(\gamma_i(y), y), \quad f, g \in Y, y \in K.$$
Then $Y$ is a Hilbert bimodule over $A$. The $C^*$-algebra $O_\gamma(K)$ is defined as the Cuntz-Pimsner algebra of the Hilbert bimodule $Y = C(C)$ over $A = C(K)$.

4. Main Theorem

Definition. Let $\varphi : K \to K$ be continuous. Suppose that composition operator $C_\varphi$ on $L^2(K, B(K), \mu^K)$ is bounded. We denote by $MC_\varphi$ the $C^*$-algebra generated by all multiplication operators by continuous functions in $C(K)$ and the composition operator $C_\varphi$ on $L^2(K, B(K), \mu^K)$.

Let $\varphi : K \to K$ be continuous. Let $A = C(K)$ and $X = C(K)$. Then $X$ is an $A$-$A$ bimodule over $A$ by

$$(a \cdot \xi \cdot b)(x) = a(x)\xi(x)b(\varphi(x)) \quad a, b \in A, \xi \in X.$$ We define an $A$-valued inner product $\langle \ , \ \rangle_A$ on $X$ by

$$\langle \xi, \eta \rangle_A(x) = \frac{1}{n} \sum_{i=1}^{\infty} \xi(\gamma_i(x)) \eta(\gamma_i(x)) \quad (= (L_\varphi(\xi \eta))(x)), \quad \xi, \eta \in X.$$ Then $X$ is a Hilbert bimodule over $A$. The left multiplication of $A$ on $X$ gives the left action $\phi : A \to \mathcal{L}(X)$ such that $\phi(a)\xi(x) = a(x)\xi(x)$ for $a \in A$ and $\xi \in X$. Let $\Phi : Y \to X$ be defined by $(\Phi(f))(x) = \sqrt{n}f(x, \varphi(x))$ for $f \in Y$. It is easy to see that $\Phi$ is an isomorphism and $X$ is isomorphic to $Y$ as Hilbert bimodules over $A$. Hence the $C^*$-algebra $O_\gamma(K)$ is isomorphic to the Cuntz-Pimsner algebra $O_X$ constructed from $X$.

For $x \in K$, we define

$$I(x) = \{ i \in \{1, \ldots, n\} \mid \text{there exists } y \in K \text{ such that } x = \gamma_i(y) \}.$$

Lemma 4.1 ([11] Lemma 2.2). Let $x \in K \setminus B_\gamma$. Then there exists an open neighbourhood $U_x$ of $x$ the following

1. $U_x \cap B(\gamma_1, \ldots, \gamma_n) = \emptyset,$
2. If $i \in I(x)$, then $\gamma_i^{-1}(U_x) \cap U_x = \emptyset$ for $j \neq i$.
3. If $i \notin I(x)$, then $U_x \cap \gamma_i(K) = \emptyset$.

We now recall a description of the ideal $J(X)$ of $A$. Assume that $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfies the open set condition in $K$. By [11] Proposition 2.6, we can write $J(X) = \{ a \in A \mid a \text{ vanishes on } B_\gamma \}$. We define a subset $J(X)^0$ of $J(X)$ by $J(X)^0 = \{ a \in A \mid a \text{ vanishes on } B_\gamma \text{ and has compact support on } K \setminus B_\gamma \}$. Then $J(X)^0$ is dense in $J(X)$.

Lemma 4.2. Assume that $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfies the open set condition in $K$. Then, for $a \in J(X)^0$, there exists $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_m \in X$ such that

$$\sum_{i=1}^{m} \theta_{\xi_i, \eta_i} = \phi(a).$$

Proof. For $x \in \text{supp}(a)$, choose an open neighbourhood $U_x$ of $x$ as in Lemma 4.1. By the same argument in the proof of [11] Proposition 2.4, we can choose $\{ f_i \}_{i=1}^{m+1}$ in $A$ such that $0 \leq f_i \leq 1$, supp($f_i$) $\subset U_x$, for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} f_i(x) = 1$ for $x \in \text{supp}(a)$. Define $\xi_i, \eta_i \in X$ by $\xi_i(x) = na(x)\sqrt{f_i(x)}$ and $\eta_i(x) = \sqrt{f_i(x)}$. By a similar argument in the proof of [11] Proposition 2.4, we have $\sum_{i=1}^{m} \theta_{\xi_i, \eta_i} = \phi(a).$
We regard the following lemma as an operator version of Lemma 4.2.

**Lemma 4.3.** Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a system of proper contractions. Assume that $K$ is self-similar and the system $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfies the open set condition and the measure separation condition in $K$. Suppose that $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_m \in X$ are in Lemma 4.2. Then, for $a \in J(X)^0$, we have

$$\sum_{i=1}^m M_{\xi_i} C_\varphi C_\varphi^* M_{\eta_i}^* = M_a.$$ 

**Proof.** Let $b \in C(K)$. By Lemma 4.2 there exists $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_m \in X$ such that

$$\sum_{i=1}^m \xi_i \cdot \langle \eta_i, b \rangle_A = ab.$$ 

Since $b = M_b C_\varphi 1$, we have

$$\sum_{i=1}^m M_{\xi_i} C_\varphi C_\varphi^* M_{\eta_i}^* b = \sum_{i=1}^m M_{\xi_i} C_\varphi C_\varphi^* M_{\eta_i}^* M_b C_\varphi 1$$

$$= \sum_{i=1}^m M_{\xi_i} C_\varphi C_\varphi^* M_{\eta_i}^* b$$

$$= \sum_{i=1}^m M_{\xi_i} C_\varphi M_{L_\varphi(\eta_i) 1}$$ by Proposition 2.2

$$= \sum_{i=1}^m M_{\xi_i} M_{L_\varphi(\eta_i) 1} \circ \varphi$$

$$= \sum_{i=1}^m \xi_i \cdot \langle \eta_i, b \rangle_A$$

$$= M_a b.$$ 

Since the Hutchinson measure $\mu^H$ on $K$ is regular, $C(K)$ is dense in $L^2(K, B(K), \mu^H)$, the proof is complete. 

The following theorem is the main result of the paper. This is a generalization of [4, Theorem 4.4].

**Theorem 4.4.** Let $(K, d)$ be a compact metric space, let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a system of proper contractions on $K$ and let $\varphi : K \to K$ be continuous. Suppose that $\gamma_1, \ldots, \gamma_n$ are inverse branches of $\varphi$. Assume that $K$ is self-similar and the system $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfies the open set condition and the measure separation condition in $K$. Then $MC_\varphi$ is isomorphic to $O_\gamma(K)$.

**Proof.** Put $\rho(a) = M_a$ and $V_\xi = M_\xi C_\varphi$ for $a \in A$ and $\xi \in X$. Then we can show $\rho(a) V_\xi V_\eta = V_a \xi$ and $V_\xi V_\eta = \rho(\langle \xi, \eta \rangle_A)$ for $a \in A$ and $\xi, \eta \in X$ as in the proof of [4, Theorem 4.4].
Let $a \in J(X)^0$. By Lemma 4.2 there exits $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_m \in X$ such that 

$$\phi(a) = \sum_{i=1}^{m} \theta_{\xi_i, \eta_i}.$$ 

From Lemma 4.3 it follows that 

$$\psi_V(\phi(a)) = \psi_V \left( \sum_{i=1}^{m} \theta_{\xi_i, \eta_i} \right) = \sum_{i=1}^{m} V_{\xi_i}V^*_{\eta_i} = \sum_{i=1}^{m} M_{\xi_i}C_\varphi C^*_\varphi M^*_{\eta_i} = M_a = \rho(a).$$

Since $J(X)^0$ is dense in $J(X)$, we have $\psi_V(\phi(a)) = \rho(a)$ for $a \in J(X)$.

By the universality and the simplicity of $O_\gamma(K)$ ([1, Theorem 3.8]), the C$^*$-algebra $MC_\varphi$ is isomorphic to $O_\gamma(K)$.

\[\square\]

5. Examples

We give some examples for C$^*$-algebras generated by a composition operator $C_\varphi$ and multiplication operators.

**Example.** A tent map $\tau : [0, 1] \to [0, 1]$ is defined by

$$\tau(x) = \begin{cases} 
2x & 0 \leq x \leq \frac{1}{2}, \\
-2x + 2 & \frac{1}{2} \leq x \leq 1.
\end{cases}$$

Let $\varphi : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ be given by $\varphi(x, y) = (\tau(x), \tau(y))$ for $x, y \in [0, 1]$. Then inverse branches of $\varphi$ are $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ such that

$$\gamma_1(x, y) = \left( \frac{1}{2}x, \frac{1}{2}y \right), \quad \gamma_2(x, y) = \left( \frac{1}{2}x, -\frac{1}{2}y + 1 \right)$$

and

$$\gamma_3(x, y) = \left( -\frac{1}{2}x + 1, \frac{1}{2}y \right), \quad \gamma_4(x, y) = \left( -\frac{1}{2}x + 1, -\frac{1}{2}y + 1 \right)$$

for $x, y \in [0, 1]$. The maps $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are proper contractions and $K = [0, 1] \times [0, 1]$ is the self-similar set with respect to $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. The system $\gamma$ satisfies the open set condition in $K$. Since $C_\gamma = ([0, 1] \times \{1\}) \cup (\{1\} \times [0, 1])$, the system $\gamma$ does not satisfy the finite branch condition. The Hutchinson measure $\mu^K$ on $K$ coincides with the Lebesgue measure $m$ on $K$. By [23], the system $\gamma$ satisfies the measure separation condition in $K$. We consider the composition operator $C_\varphi$ on $L^2(K, \mathcal{B}(K), m)$. By Theorem 4.4, the C$^*$-algebra $MC_\varphi$ is isomorphic to $O_\gamma(K)$.

Since

$$B_\gamma = \left( [0, 1] \times \left\{ \frac{1}{2} \right\} \right) \cup \left( \left\{ \frac{1}{2} \right\} \times [0, 1] \right),$$

we have $K_0(C(B_\gamma)) \cong \mathbb{Z}$ and $K_1(C(B_\gamma)) \cong 0$. Since $K_0(C(K)) \cong \mathbb{Z}$ and $K_1(C(K)) \cong 0$, it follows that $K_0(J(X)) \cong 0$ and $K_1(J(X)) \cong 0$. By the six-term exact sequence of the Cuntz-Pimsner algebra $O_\gamma(K)$ due to [19], we have $K_0(MC_\varphi) \cong \mathbb{Z}$, $K_1(MC_\varphi) \cong 0$ and the unit $[I]_0$ in $K_0$ maps to 1 in $\mathbb{Z}$. Hence $MC_\varphi$ is isomorphic to the Cuntz algebra $O_\infty$.

**Example.** A map $\sigma : [0, 1] \to [0, 1]$ is defined by

$$\sigma(x) = \begin{cases} 
3x & 0 \leq x \leq \frac{1}{3}, \\
-3x + 2 & \frac{1}{3} \leq x \leq \frac{4}{3}, \\
3x - 2 & \frac{4}{3} \leq x \leq 1.
\end{cases}$$
Let \( \varphi : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1] \) be given by \( \varphi(x, y) = (\tau(x), \sigma(y)) \) for \( x, y \in [0, 1] \).

Then inverse branches of \( \varphi \) are \( \gamma_1, \ldots, \gamma_6 \) such that

\[
\begin{align*}
\gamma_1(x, y) &= \left( \frac{1}{2} x, \frac{1}{3} y \right), \\
\gamma_2(x, y) &= \left( \frac{1}{2} x, -\frac{1}{3} y + \frac{2}{3} \right), \\
\gamma_3(x, y) &= \left( \frac{1}{2} x, \frac{1}{3} y + \frac{2}{3} \right), \\
\gamma_4(x, y) &= \left( -\frac{1}{2} x + 1, \frac{1}{3} y \right), \\
\gamma_5(x, y) &= \left( -\frac{1}{2} x + 1, -\frac{1}{3} y + \frac{2}{3} \right), \\
\gamma_6(x, y) &= \left( -\frac{1}{2} x + 1, \frac{1}{3} y + \frac{2}{3} \right),
\end{align*}
\]

for \( x, y \in [0, 1] \). The maps \( \gamma_1, \ldots, \gamma_6 \) are proper contractions and \( K = [0, 1] \times [0, 1] \) is the self-similar set with respect to \( \gamma = (\gamma_1, \ldots, \gamma_6) \). The Hutchinson measure \( \mu_H \) on \( K \) coincides with the Lebesgue measure \( m \) on \( K \). The system \( \gamma \) satisfies the open set condition and the measure separation condition in \( K \). Similar to the previous example, we have \( (\mathcal{K}_0(\mathcal{M}_\varphi), [I]_0, \mathcal{K}_1(\mathcal{M}_\varphi)) \cong (\mathbb{Z}, 1, 0) \). Hence the \( C^* \)-algebra \( \mathcal{M}_\varphi \) is isomorphic to the Cuntz algebra \( \mathcal{O}_\infty \).

Remark. Let \( \varphi : [0, 1] \to [0, 1] \) be given by \( \varphi(x) = \tau(x) \) for \( x \in [0, 1] \). By [4] Example, the \( C^* \)-algebra \( \mathcal{M}_\psi \) is isomorphic to the Cuntz algebra \( \mathcal{O}_\infty \). On the other hand, let \( \psi : [0, 1] \to [0, 1] \) be given by \( \psi(x) = \sigma(x) \) for \( x \in [0, 1] \). We have \( \mathcal{K}_0(\mathcal{M}_\psi) \cong \mathbb{Z}^2 \) and \( \mathcal{K}_1(\mathcal{M}_\psi) \cong \mathbb{Z} \). Hence the \( C^* \)-algebra \( \mathcal{M}_\psi \) is not isomorphic to the Cuntz algebra \( \mathcal{O}_\infty \).

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