Divisibility of quantum dynamical maps and collision models

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Divisibility of dynamical maps is visualized by trajectories in the parameter space and analyzed within the framework of collision models. We introduce ultimate completely positive (CP) divisible processes, which lose CP divisibility under infinitesimal perturbations, and characterize Pauli dynamical semigroups exhibiting such a property. We construct collision models with factorized environment particles, which realize additivity and multiplicativity of generators of CP divisible maps. A mixture of dynamical maps is obtained with the help of correlated environment. Mixture of ultimate CP divisible processes is shown to result in a new class of eternal CP indivisible evolutions. We explicitly find collision models leading to weakly and essentially non-Markovian Pauli dynamical maps.

I. INTRODUCTION

Theory of open quantum systems studies dynamical maps \( \Phi_t \) that naturally occur when the system in question interacts with its environment. Dynamical maps are the key objects in the analysis of quantum information transmission through noisy channels [1] and quantum information processing in real systems [2]. The effect of open system dynamics on quantum entanglement and entanglement-based information protocols is reviewed, e.g., in Ref. [3]. Over the last decade quantum dynamical maps were intensively studied with respect to characterization of their non-Markovian behavior [4–6] and its experimental observation [7–13]. Quantitative approaches to non-Markovianity include non-monotonic distinguishability of states [14, 15], different divisibilities of dynamical maps [16, 17], monitoring the volume of accessible states [18] and others [19–22]. In particular, the divisibility approach is based on the decomposition property \( \Phi_{t+s} = \Theta_{t, t+s} \Phi_t \) and explores features of the intermediate map \( \Theta_{t, t+s} \) [23]. Various types of divisibility induce alternative measures to quantify non-Markovianity, however, one should be careful with the physical interpretation of memory effects [24].

From mathematical viewpoint, the open system dynamics in the Schrödinger picture is given by the transformation \( \rho(t) = \Phi_t[\rho(0)] \), where \( \Phi_t \) is the dynamical map (process) that is a one-parameter family of completely positive trace preserving (CPT) maps, \( t \geq 0 \) is the evolution time, \( \Phi_0 = \text{Id} \), the identity transformation. The dilation of the dynamical map is

\[
\Phi_t[\rho] = \text{tr}_{\text{env}} \left\{ U_t (\rho \otimes \xi) U_t^\dagger \right\},
\]

where \( U_t \) is the unitary evolution of the system and the environment, \( \xi \) is the initial state of environment.

Complete positivity (CP) of \( \Phi_t \) means that the map \( \Phi_t \otimes \text{Id}_k \) is positive for all identity transformations \( \text{Id}_k \) of \( k \)-level ancillary systems, which can be potentially entangled with the system in question. If \( \Theta_{t, t+s} \) is CP for all \( t \) and \( s \neq t \), then the process \( \Phi_t \) is called CP divisible. Such a definition of CP divisibility is a global-in-time property of the whole family \( \{\Phi_t\}_{t \geq 0} \). In contrast, to underline the time-local behavior, we will refer to a process \( \Phi_t \) as CP divisible at time \( t_0 \) if there exists \( s_0 > 0 \) such that \( \Theta_{t_0, t_0 + s} \) is CP for all \( s \in (0, s_0) \). If the dynamical map \( \Phi_t \) is not CP divisible for all time moments \( t \geq 0 \), then \( \Phi_t \) is called eternal CP indivisible [25]. CP divisibility of a bijective dynamical map was shown to be equivalent to the distinguishability of states in the extended Hilbert space [26].

Replacing CP by any other property [viz. positivity (P), k-positivity, volume of accessible states, etc.] we obtain definitions of the global and time-local divisibility properties of the dynamical map \( \Phi_t \). Processes, which are not CP divisible but are P divisible, are also called weakly non-Markovian, whereas P indivisible processes are called essentially non-Markovian [27].

Since any linear map \( \Phi \) between finite dimensional spaces can be defined by a set of real parameters \( \lambda = \lambda_1, \ldots, \lambda_n \), any smooth process \( \Phi_t \) is then determined by a continuous trajectory \( \lambda(t) \) in the parameter space. Such a trajectory provides a pictorial representation of the dynamical map \( \Phi_t \) in \( \mathbb{R}^n \), which is particularly visual in the case of qubit Pauli maps given by 3 parameters (see, e.g., [28, 29]). Analyzing the process trajectory in the parameter space, one can not only get an intuition about the quantum dynamics (for instance, by observing the Bloch ball transformation for qubit dynamics) but also reveal its divisibility properties. The first goal of this paper is to describe different forms of Markovian and non-Markovian Pauli dynamical maps in terms of trajectories in the parameter space.

Pictorial representation of some dynamical map \( \Phi_t \) in the form of trajectory \( \lambda(t) \) raises a question of stability of the process with respect to a continuous (infinitely differentiable) trajectory perturbation \( \lambda(t) \to \lambda(t) + \delta \lambda(t) \), with the perturbed map \( \Phi_t + \delta \Phi_t \) being a valid quantum dynamical evolution. A process \( \Phi_t \), which is originally CP divisible at time \( t_0 \), may lose the property of being
CP divisible at this time due to a time-local perturbation $\delta \lambda(t)$ such that $\delta \lambda(t) = 0$ if $t \leq t_0$. If this is the case, $\Phi_t$ is called ultimate CP divisible at time $t_0$. There exist processes $\Phi_t$ that are ultimate CP divisible for all time moments $t \geq 0$. We fully characterize Pauli dynamical semigroups exhibiting such a property.

In this paper, we show that the mathematical concepts of divisibility are closely related with the underlying physical models of quantum dynamical maps. From a physical viewpoint, any dynamical map $\Phi_t$ can be seen as a simplified description of the system-environment evolution with no regard to the environment structure and particular microscopic interactions between environment quanta and the system. Many dynamical maps can be derived under some assumptions (weak coupling, low density, etc.) from a microscopic system-environment Hamiltonian and particular state of the environment $|\psi\rangle$. In our analysis, we will resort to so-called collision models in which the motional and internal degrees of freedom can be considered separately: the motion Hamiltonian determines a sequence of collisions with environment particles, and the system-environment interaction Hamiltonian becomes significant during collisions and affects internal degrees of freedom of the system and an impacted environment particle. Relaxation mechanism via such a "stirring" process was first considered in Ref. [35]. Thermalization, homogenization of the system to a particular state, and pure dephasing were simulated via a collision model with identical uncorrelated environment particles in [33, 50]. Even if environment particles are uncorrelated originally, they become partially correlated (entangled) with the system during collisions, so such an environment exhibits memory effects for further systems interacting with it [36, 40]. Moreover, environment particles may be initially correlated (quantumly or classically) due to interactions between each other as it takes place in solids and quantum gases, and such correlations may result in non-Markovian dynamics [12, 41, 43]. Non-Markovian effects also appear in collision models, where the system can interact with the same environment particle several times [15, 45], or an environment particle impacted by a system collides with another environment particle, which later collides with the system [17, 50]. The latter scheme is equivalent to a scenario, when the quantum system in question is coherently coupled to an auxiliary system interacting with Markovian bath via collisions [51, 52]. Collision models adequately describe a particle in semi-quantal spin gases [53, 54], a micromaser [55], a two-level system that interacts with spatial-temporal modes passing through it only once [56], and more complex systems with involved interaction graphs [57, 59] as well as experiments with an engineered environment in nuclear magnetic resonance [12] and in photonic systems [11, 50, 61]. Collision models were also exploited in the microscopic description of Landauer’s principle [62].

In the appropriate continuous limit of infinitesimal interaction time $\tau \rightarrow 0$, the collision model describes a smooth dynamical map $\Phi_t$ [57, 63]. Even if we consider simple interaction graphs, when the system interacts with each environment particle only once, collision models successfully simulate dynamical processes $\Phi_t$ with different divisibility properties [61]. So we resort to a collision model with generally correlated environment particles, the correlations being attributed to prior interactions among environment constituents.

The second goal of this paper is to demonstrate that the divisibility property of the dynamical map $\Phi_t$ is closely related with the collision model describing it. Clearly, any CP divisible dynamics can be obtained with uncorrelated (factorized) environment states. We fully characterize ultimate CP divisible Pauli dynamical semigroups and corresponding collision models. CP indivisible dynamics necessarily involves correlations among environment particles.

Surprisingly, a convex sum $p_1 \Phi_t^{(1)} + p_2 \Phi_t^{(2)} + \ldots$ of CP divisible processes $\Phi_t^{(1)}, \Phi_t^{(2)}, \ldots$ can exhibit eternal CP indivisibility, for instance, this takes place for the convex sum of two dephasing dynamical maps [23, 25]. We provide new families of eternal CP indivisible processes and construct a collision model with correlated environment, which simulates them.

In contrast to a convex sum of dynamical maps, a conical (weighted) combination $\alpha L_t^{(1)} + \beta L_t^{(2)}$ of time-dependent generators $L_t^{(1)}$ and $L_t^{(2)}$ does not necessarily represent a valid map [64, 65] unless master equations $\frac{\partial \rho}{\partial t} = L_t^{(1)}[\rho]$ and $\frac{\partial \rho}{\partial t} = L_t^{(2)}[\rho]$ both define CP divisible processes. When the latter condition is fulfilled, we demonstrate a collision model realizing the master equation $\frac{\partial \rho}{\partial t} = \alpha L_t^{(1)}[\rho] + \beta L_t^{(2)}[\rho]$ for arbitrary non-negative weight coefficients $\alpha$ and $\beta$.

The paper is organized as follows. In Sec. II, we review divisibility properties of Pauli dynamical maps in pictorial representation. In Sec. III, ultimate CP divisible semigroups are studied. In Sec. IV, we provide a general collision model for ultimate CP divisible Pauli processes. In Sec. V, we demonstrate collision models that realize multiplicativity and additivity of time-local generators for CP divisible processes. In Sec. VI, we construct a correlated environment which leads to a mixture of CP divisible processes. In Sec. VII, a new two-parameter family of eternal CP indivisible Pauli maps is presented. In Secs. VII and VIII, we review P divisible and P indivisible processes, respectively, as well as the physics of underlying collision models. In Secs. IX and X, we provide a constructive collision model for an arbitrary Pauli dynamical map $\Phi_t$. In Sec. XI, we discuss dynamical maps which continuously shrink the volume of accessible states, however, are not P divisible. In Sec. XII, brief conclusions are given.

II. DIVISIBILITY OF PAULI MAPS IN PICTORIAL REPRESENTATION

A trajectory $\lambda(t)$ becomes particularly visual for Pauli qubit processes $\Phi_t : B(H_2) \rightarrow B(H_2)$ that are characterized by three real parameters $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ as follows:

$$\Phi_t[\sigma] = \frac{1}{2} \left( \text{tr} [\sigma I] + \sum_{j=1}^3 \lambda_j(t) \text{tr} [\sigma_j \sigma] \right),$$ (2)

where $\sigma_1, \sigma_2, \sigma_3$ is a conventional set of Pauli operators. The map $\Phi_t$ is known to be positive if $-1 \leq \lambda_j(t) \leq 1, \forall t \geq 0$. The divisibility property $\Phi_t$ is defined as divisibility $\Phi_t$ if the generator $L_t$ can be represented as a sum of two dephasing dynamical maps [24, 25]. We consider the following master equations:

1. $\frac{\partial \rho}{\partial t} = L_t^{(1)}[\rho]$
2. $\frac{\partial \rho}{\partial t} = L_t^{(2)}[\rho]$

where $L_t^{(1)}$ and $L_t^{(2)}$ are the master equations of two dephasing processes, respectively.

In Sec. II, we demonstrate collision models that realize multiplicativity and additivity of time-local generators for CP divisible processes. In Sec. VI, we construct a correlated environment which leads to a mixture of CP divisible processes. In Sec. VII, a new two-parameter family of eternal CP indivisible Pauli maps is presented. In Secs. VII and VIII, we review P divisible and P indivisible processes, respectively, as well as the physics of underlying collision models. In Secs. IX and X, we provide a constructive collision model for an arbitrary Pauli dynamical map $\Phi_t$. In Sec. XI, we discuss dynamical maps which continuously shrink the volume of accessible states, however, are not P divisible. In Sec. XII, brief conclusions are given.
\[\lambda_1(t), \lambda_2(t), \lambda_3(t) \leq 1 \text{ (cube in the parameter space) and completely positive if } 1 \pm \lambda_3(t) \geq |\lambda_1(t) \pm \lambda_2(t)| \text{ (tetrahedron in the parameter space)} \text{ [45, 67].} \]

In the case of a general physical evolution \( \Phi_t \) with initially factorized system and environment, the trajectory \( \lambda(t) \) can be an arbitrary smooth curve inside the tetrahedron \( 1 \pm \lambda_3(t) \geq |\lambda_1(t) \pm \lambda_2(t)| \) (see Fig. 4).

Suppose the map \( \Phi_t \) is invertible and \( s \) tends to zero, then

\[\Phi_t[s(0)] = \Phi_t[s(0)] + s \Phi_t[s(0)] = \phi(t) + s \Phi_t \circ \Phi_t^{-1}[\phi(t)] = \Theta_{t,t+s}[\phi(t)],\]

where \( \Phi_t = \frac{d}{dt} \Phi_t \). From Eq. (3) it follows that

\[\Theta_{t,t+s} = \text{Id} + s \Phi_t \circ \Phi_t^{-1}\]

as \( s \to 0 \).

The map \( \Phi_t \circ \Phi_t^{-1} \) defines a direction in the parameter space in which the process progresses. Using the explicit form of Eq. (2) we get

\[\Phi_t \circ \Phi_t^{-1}[X] = \frac{1}{2} \sum_{j=1}^{3} \lambda_j(t) \lambda_j(t) \lambda_j(t) X \sigma_j X,\]

which identifies the vector

\[\kappa(t) = \left( \begin{array}{c}
\lambda_1(t) \\
\lambda_2(t) \\
\lambda_3(t)
\end{array} \right)\]

representing the dynamical map in the parameter space of qubit unital channels. Let us stress that \( \kappa(t) \) is not a tangent line to the trajectory \( \lambda(t) \). Such vector \( \kappa(t) \) can be drawn at any time moment \( t \) for a sufficiently smooth trajectory \( \lambda(t) \) in the parameter space, making the divisibility property more apparent.

In particular, if \( s \to 0 \), then the map \( \Theta_{t,t+s} \) is completely positive if and only if the vector \( \kappa(t) \) drawn from the corner \( (1,1,1) \) of the parameter space points inside the cube of positive maps (see Fig. 4), i.e., the scalar products of \( \kappa(t) \) with vectors \((-1,1,1), \) \((1,-1,1), \) and \((1,1,-1) \) are all non-positive:

\[-\kappa_1(t) + \kappa_2(t) + \kappa_3(t) \leq 0,\]

\[\kappa_1(t) - \kappa_2(t) + \kappa_3(t) \leq 0,\]

\[\kappa_1(t) + \kappa_2(t) - \kappa_3(t) \leq 0.\]

Analogously, if \( s \to 0 \), then the map \( \Theta_{s,t+s} \) is positive if and only if the vector \( \kappa(t) \) drawn from the corner \( (1,1,1) \) of the parameter space points inside the cube of positive maps in Fig. 4, i.e., the scalar products of \( \kappa(t) \) with vectors \((1,0,0), \) \((0,1,0), \) and \((0,0,1) \) are all non-positive:

\[\kappa_1(t) \leq 0, \quad \kappa_2(t) \leq 0, \quad \kappa_3(t) \leq 0.\]

For the uniform measure of qubit states inside the Bloch ball (metric induced by Hilbert–Schmidt distance [67]), the volume of accessible states for the Pauli map is \( V(t) = |\lambda_1(t)\lambda_2(t)|\lambda_3(t)| \). The map \( \Theta_{s,t+s} \) shrinks the volume of accessible states if and only if \( \prod_{t=1}^{3}(1+s\kappa_3(t)) \leq 1 \), which in the limit \( s \to 0 \) transforms into requirement

\[\kappa_1(t) + \kappa_2(t) + \kappa_3(t) \leq 0.\]

![FIG. 1](a) Physical Pauli processes correspond to curves inside the set of completely positive maps. The direction of the vector \( \kappa \) given by Eq. (6) defines the properties of Pauli dynamical maps: (b) CP divisibility, (c) P divisibility, (d) monotonic shrink of the volume of accessible states.

Here we have taken into account that if \( \kappa_1 + \kappa_2 + \kappa_3 = 0 \) and at least one \( \kappa_i \neq 0 \), then \( \kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1 = -\frac{1}{2}(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) < 0 \), which implies \( \prod_{i=1}^{3}(1+s\kappa_i) < 1 \). Geometrically, the vector \( \kappa(t) \) has non-positive scalar product with the vector \((1,1,1) \), i.e., the vector \( \kappa(t) \) drawn from the corner \((1,1,1) \) in parameter space points to a specific half-space separated by the plane \( \lambda_1 + \lambda_2 + \lambda_3 = 3 \) (see Fig. 4).

### III. ULTIMATE CP DIVISIBILITY OF SEMIGROUP DYNAMICS

Consider a semigroup dynamics \( \Phi_t = e^{Lt} \), where \( L : B(H_2) \to B(H_2) \) is a time-independent generating map of the form [45, 67]

\[L[\rho] = -i[H, \rho] + \sum_k \gamma_k \left( A_k \rho A_k^\dagger - \frac{1}{2} \{A_k, A_k^\dagger \} \right),\]

where \( H \) is Hermitian and \( \gamma_k \geq 0 \). It follows that for semigroup dynamics the identity \( \Phi_{t+s} = \Phi_t \circ \Phi_s \), holds for all \( t, s \geq 0 \). Consequently \( \Theta_{t,t+s} = \Phi_s = e^{Ls} \), hence, the semigroup dynamics is always CP divisible.

The time evolution of the density operator is given by equation

\[\frac{\partial \rho}{\partial t} = L[\rho].\]

Consider now an infinitesimal perturbations of Eq. (13):

\[\frac{\partial \rho}{\partial t} = (L + \delta L_t)[\rho],\]
where $\delta L_0 = 0$. The term $\delta L_1$ describes an infinitely differentiable deviation from dynamics (13), and can be attributed to, e.g., a slightly modified environment or a fluctuating interaction between system and environment. By definition we say that a semigroup dynamics is ultimate CP divisible if it becomes CP indivisible under some perturbation $\delta L$.

For qubit unital semigroup processes (12) we have $\lambda_i(t) = e^{-\Gamma_i t}$ and, consequently, the vector $\kappa(t) = -\Gamma$ is time-independent and we used $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$. By definition the perturbations $\delta L_i$ are introducing only minor changes and the deviated vector $\kappa + \delta \kappa$ would satisfy Eqs. (7)–(9) whenever these inequalities for the unperturbed case $\kappa$ are strict. It turns out that qubit unital semigroup dynamics can be ultimately CP only if $\kappa_i + \kappa_j - \kappa_k = 0$ for some permutation of indexes $i, j, k \in \{1, 2, 3\}$. In fact, in such case there exists an infinitesimal perturbation $\delta L_i$ resulting in a dynamical map $\Phi_t + \delta \Phi_t$, violating Eqs. (7)–(9). Taking into account the definition of $\kappa$ we find that the condition $\kappa_i + \kappa_j - \kappa_k = 0$ translates into differential equation $\frac{d}{dt} \ln(\lambda_i\lambda_j) = \frac{d}{dt} \ln(\lambda_k)$ with the solution $\lambda_i(\lambda_j)(\lambda_k)(t) = c\lambda_k(t)$, where the constant $c$ can be found from the initial condition $\lambda_1(0) = \lambda_2(0) = \lambda_3(0) = 1$. In conclusion, $c = 1$ and ultimate CP divisible unital processes satisfy the identity

$$\lambda_i(t)\lambda_j(t) = \lambda_k(t).$$

(15)

A general qubit unital semigroup evolution (up to unitary freedom) takes the form

$$\mathcal{L}[\rho] = -\frac{1}{2} \sum_{j=1}^{3} \Gamma_j \rho [\sigma_j \rho \sigma_j] \sigma_j = \frac{1}{2} \sum_{j=1}^{3} \gamma_j \rho [\sigma_j \rho \sigma_j - \rho],$$

(16)

where $\Gamma_j^{-1}$ are experimentally measurable timescales of decoherence processes and $\gamma_j$ are dissipator rates given by formula

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix}.$$

(17)

The conditions of ultimate CP divisibility on $\kappa$ implies $\Gamma_i + \Gamma_j - \Gamma_k = \gamma_k = 0$ (for some permutation of indexes $i, j, k$). It follows that the generator for the ultimate CP divisible Pauli semigroup contains at most two terms:

$$\mathcal{L}[\rho] = \frac{\gamma_i}{2} [\sigma_i \rho \sigma_i - \rho] + \frac{\gamma_j}{2} [\sigma_j \rho \sigma_j - \rho],$$

(18)

and the trajectory in the parameter space is $\lambda_i = e^{-\gamma_i t}$, $\lambda_j = e^{-\gamma_j t}$, $\lambda_k = e^{-(\gamma_i + \gamma_j) t}$. The class of time evolutions for ultimate CP divisible Pauli semigroups is illustrated in Fig. 2. In the next section, we will provide a physical realization of the generator [18].

Physical examples of ultimate CP divisible processes include

- pure phase damping process, when $\lambda_i(t) = 1$, $\lambda_j(t) = \lambda_k(t) = e^{-\gamma_1 t}$ and corresponding to the choice of dissipation rates $\gamma_j = \gamma_k = 0$ (green lines in Fig. 2).

IV. COLLISION MODELS OF ULTIMATE CP DIVISIBLE SEMIGROUPS

Physically, the evolution $\frac{d}{dt} \rho = \mathcal{L}[\rho]$ with dissipator (18) is achievable as a result of sequential interactions of the
system qubit with environment qubits (collision model, Fig. 3). Let all environment qubits be in the same state \( \xi = \frac{1}{2} I \), Fig. 4. The system qubit and the \( n \)-th environment qubit interact pairwise during the time period \( \tau \), with the interaction Hamiltonian being

\[
H_{\text{int}} = \frac{1}{2} (g_1 \sigma_x \otimes \sigma_x + g_2 \sigma_y \otimes \sigma_y).
\]  

(20)

The system qubit and the \( n \)-th environment qubit experience the unitary transformation

\[
U_\tau = \exp(-i H_{\text{int}} \tau) = \cos \left( \frac{g_1 \tau}{2} \right) \cos \left( \frac{g_2 \tau}{2} I \otimes I - i \sin \left( \frac{g_1 \tau}{2} \right) \cos \left( \frac{g_2 \tau}{2} \right) \sigma_x \otimes \sigma_x \\
- i \cos \left( \frac{g_1 \tau}{2} \right) \sin \left( \frac{g_2 \tau}{2} \right) \sigma_y \otimes \sigma_y + \sin \left( \frac{g_1 \tau}{2} \right) \sin \left( \frac{g_2 \tau}{2} \right) \sigma_z \otimes \sigma_z.
\]

(21)

As a result of such an interaction, the system state \( \varrho \) transforms as follows:

\[
\varrho \rightarrow \Phi_\tau [\varrho] = \text{tr}_n \left\{ U_\tau \left( \varrho \otimes \frac{1}{2} I \right) U_\tau^\dagger \right\},
\]

(22)

where \( \text{tr}_n \) denotes the partial trace over \( n \)-th environment qubit. Some algebra yields the single interaction elementary map \( \Phi_\tau \), which is unital, does not depend on \( n \) and reads

\[
\Phi_\tau [\varrho] = \frac{1}{2} \left( \text{tr}[\varrho] I + \cos(g_2 \tau) \text{tr}[\sigma_x \varrho] \sigma_x \\
+ \cos(g_1 \tau) \text{tr}[\sigma_y \varrho] \sigma_y + \cos(g_1 \tau) \cos(g_2 \tau) \text{tr}[\sigma_z \varrho] \sigma_z \right).
\]

(23)

Since the system qubit always interacts with a fresh environmental particle, after \( \frac{\tau}{\tau} \) interactions we get the dynamical map

\[
\Phi_t = (\Phi_\tau)^{t/\tau}
\]

(24)

with parameters \( \lambda_1(t) = [\cos(g_2 \tau)]^{t/\tau} \), \( \lambda_2(t) = [\cos(g_1 \tau)]^{t/\tau} \), and \( \lambda_3(t) = \lambda_1(t) \lambda_2(t) \). In the stroboscopic limit \( \tau \rightarrow 0 \), \( g_2 \tau \rightarrow 2\gamma_1 \), \( g_2 \tau \rightarrow 2\gamma_2 \) we get the continuous dynamics \( \lambda_1(t) = e^{-\gamma_1 t} \), \( \lambda_2(t) = e^{-\gamma_2 t} \), and \( \lambda_3(t) = e^{-(\gamma_1+\gamma_2) t} \). Thus, parameters \( \lambda_1(t), \lambda_2(t), \) and \( \lambda_3(t) \) satisfy condition (15) and the induced dynamics is CP divisible. This proves that ultimate CP divisible dynamics with the dissipator (18) can be realized in the stroboscopic limit of the collision model with the elementary pairwise Hamiltonian \( H = g_1 \sigma_1 \otimes \sigma_1 + g_2 \sigma_2 \otimes \sigma_2 \), where the coefficients \( g_1 \) and \( g_2 \) satisfy \( \frac{g_1}{g_2} = \sqrt{\frac{\gamma_2}{\gamma_1}} \). Trajectories of ultimate CP divisible Pauli semigroups are depicted in Fig. 5.

![FIG. 3: Physics of collision model.](image)

![FIG. 4: Collision model for Pauli dynamical maps with ultimate CP divisible semigroup property.](image)

V. MULTIPlicativity AND ADDITIVITY OF GENERATORS IN COLLISION MODELS

Any CP divisible process \( \Phi_t \) can be realized stroboscopically via a collision model with the arbitrary chosen precision. In fact, since \( \Theta_{t+t_s} \) is a valid dynamical map for all \( t \) and \( s \), its dilation (unitary operator \( V_{t+t_s} \) and environment state \( \xi_{t+t_s} \)) is continuous with respect to \( t \) and \( s \) [24]. Fixing \( s = \tau \), we get a sequence of environment states

\[
\xi_{0, \tau}, \xi_{\tau, 2\tau}, \ldots, \xi_{(n-1)\tau, n\tau}, \ldots
\]

(25)

and a sequence of unitary operators acting on the system and \( n \)-th environment particle

\[
V_{0, \tau}, V_{\tau, 2\tau}, \ldots, V_{(n-1)\tau, n\tau}, \ldots
\]

(26)

such that the dynamics \( \Phi_t [\varrho] \) coincides with the simulation \( \Phi^{\text{sim}}_{n\tau} [\varrho] = \text{tr}_n [V_{n\tau} (\varrho \otimes \xi_{0, \tau} \otimes \xi_{\tau, 2\tau} \otimes \ldots \otimes \xi_{(n-1)\tau, n\tau}) V_{0, \tau}^\dagger V_{\tau, 2\tau}^\dagger \ldots V_{(n-1)\tau, n\tau}^\dagger] \) at time moments \( t = n\tau \). Thus, there exists a collision model with factorized environment which simulates master equation \( \frac{d}{dt} \varrho = \mathcal{L}_t [\varrho] \) for the generator \( \mathcal{L}_t = \Phi_t \circ \Phi_t^{-1} \), if \( \Phi_t \) is CP divisible.

Analogously, if we replace the generator \( \mathcal{L}_t \) by \( \alpha \mathcal{L}_t \) with some positive \( \alpha \), then the resulting evolution is still CP divisible and can be realized stroboscopically at the same time moments \( t = n\tau \) (each collision increments time by \( \tau \)) with a modified sequence of environment states

\[
\xi_{0, \alpha\tau}, \xi_{\tau, (1+\alpha)\tau}, \ldots, \xi_{(n-1)\tau, (n-1+\alpha)\tau}, \ldots
\]

(27)

and a sequence of unitary operators acting on the system and \( n \)-th environment particle

\[
V_{0, \alpha\tau}, V_{\tau, (1+\alpha)\tau}, \ldots, V_{(n-1)\tau, (n-1+\alpha)\tau}, \ldots
\]

(28)

Note that such an apparent construction of collision model for multiplicative generator \( \alpha \mathcal{L}_t \) is valid only if the original process \( \Phi_t \) is CP divisible. If this is not the case, the modified master equation \( \frac{d}{dt} \varrho = \alpha \mathcal{L}_t [\varrho] \) may lead to nonphysical solutions, with the example being presented in Ref. [65].

Consider two CP divisible processes \( \Phi^{(1)}_t \) and \( \Phi^{(2)}_t \) defined via master equations \( \frac{d}{dt} \varrho = \mathcal{L}^{(1)}_t [\varrho] \) and \( \frac{d}{dt} \varrho = \mathcal{L}^{(2)}_t [\varrho] \), respectively. Each dynamical map \( \Phi^{(i)}_t \) can be simulated stroboscopically with a sequence of environment states \( \{ \xi^{(i)}_{(n-1)\tau, n\tau} \} \) and unitary operators \( \{ V^{(i)}_{(n-1)\tau, n\tau} \}, i = 1, 2. \)
If the system interacts during time $\tau$ alternatively with particles from the first and second sequences, i.e. with particles from the first environment at odd collisions and with particles from the second environment at even collisions (Fig. 5), then the resulting dynamics simulates the master equation \[
\frac{\partial \rho}{\partial t} = \frac{1}{2} \left( \mathcal{L}_{1}^{(1)} + \mathcal{L}_{2}^{(2)} \right) [\rho] \]
at times $t = 2n\tau$. In a more general physical situation, when the system interacts independently with two types of environments, the effective generator reads $p_1 L_1^{(1)} + p_2 L_2^{(2)}$, where $p_1$ and $p_2$ are the probabilities of encountering a particle from the first and second environment, respectively. Therefore, additivity of generators can be realized in a stroboscopic model if those generators lead to CP divisible dynamics. When the latter condition is violated, addition of generators may also lead to nonphysical solutions [64].

VI. MIXTURES OF CP DIVISIBLE PROCESSES

Consider a dynamical map which is a mixture of CP divisible processes:

\[
\Phi_t = \sum_{m=1}^{M} p_m \Phi_t^{(m)},
\]

where \( \{p_m\} \) are the probabilities with which CP divisible dynamical maps \( \{\Phi_t^{(m)}\} \) contribute to the map \( \Phi_t \), $p_m \geq 0$ and $\sum_{m=1}^{M} p_m = 1$. Note that this situation is substantially different from the weighted sum of generators since $\mathcal{L}_t = \Phi_t \circ \Phi_t^{-1} \neq \sum_{m=1}^{M} w_m \Phi_t^{(m)} \circ \left( \Phi_t^{(m)} \right)^{-1} = \sum_{m=1}^{M} w_m \mathcal{L}_t^{(m)}$ in general.

Surprisingly, even if all the processes $\Phi_t^{(m)}$ are CP divisible, $\Phi_t$ can still be CP indivisible. The prominent example is the mixture

\[
\Phi_t^{\text{mix}} = p_1 e^{\mathcal{L}_1 t} + p_2 e^{\mathcal{L}_2 t} + p_3 e^{\mathcal{L}_3 t}
\]
of purely dephasing maps $e^{\mathcal{L}_i t}$ with $\mathcal{L}_i[g] = \gamma (\sigma_i g \sigma_i - g)$. In Ref. [24] the region of simplex $(p_1, p_2, p_3)$ is found, for which $\Phi_t^{\text{mix}}$ is not CP divisible for all $t > t_\ast$. If only one of probabilities $p_1, p_2, p_3$ equals zero, then $\Phi_t^{\text{mix}}$ is eternal CP indivisible.

In what follows we will design a collision model simulation of a general mixture $\Phi_t = \sum_{m=1}^{M} p_m \Phi_t^{(m)}$ of CP divisible dynamical maps. Suppose each $\Phi_t^{(m)}$ is realized by a collision model with environment states $\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_n^{(m)}$ and elementary unitary transformations $U_1^{(m)} = \exp(-iH_1^{(m)} \tau), U_2^{(m)} = \exp(-iH_2^{(m)} \tau), \ldots, U_n^{(m)} = \exp(-iH_n^{(m)} \tau)$. The whole environment of $m$-th process reads $\Xi^{(m)} = \xi_1^{(m)} \otimes \xi_2^{(m)} \otimes \cdots \otimes \xi_n^{(m)} = \otimes_{k=1}^{n} \xi_k^{(m)}$.

In a probabilistic sense the mixture can be realized as a mixture of collision models for each individual $\Phi_t^{(m)}$, however, such implementation is not “operationally faithful”, because in each run of the experiment a randomly chosen but different CP divisible process $\Phi_t^{(m)}$ is realized. We will present an alternative realization of such mixtures and design a collision model with a correlated state of the environment implementing the desired mixture in each individual run of the experiment.

In particular, consider the following initial state of the environment

\[
\Xi = \bigoplus_{m=1}^{M} p_m \Xi^{(m)} = \begin{pmatrix} p_1 \Xi^{(1)} \\ \cdots \\ p_M \Xi^{(M)} \end{pmatrix} = \begin{pmatrix} p_1 \otimes_{k=1}^{n} \xi_k^{(1)} \\ \cdots \\ p_M \otimes_{k=1}^{n} \xi_k^{(M)} \end{pmatrix},
\]

which does not have the tensor product structure with respect to collisions, i.e. $\Xi \neq \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$. Let us note that this state is correlated, but not entangled. Also, note that the Hermitian operator $H_k^{(m)}$ is an interaction Hamiltonian between the system and the $k$-th particle of $m$-th environment, so $H_k^{(m)}$ acts non-trivially on vectors in the subspace $\mathcal{H}_\text{sys} \otimes H_k^{(m)}$ only. In other words, $H_k^{(m)}$ involves degrees of freedom of the system and the $k$-th particle of

\[
\rho(0) \xrightarrow{U} \rho(t)
\]
$m$-th block of matrix $\xi$. Consequently, $H^{(m)}_k H^{(m')}_{k'} = 0$ if $m \neq m'$. The combined Hamiltonian $H_k = \sum_{m=1}^M H^{(m)}_k$ generates the unitary evolution operator

$$U_k = \exp \left( -i H_k \tau \right) = \sum_{l=0}^{\infty} \frac{(-i \tau)^l}{l!} (H_k)^l = \sum_{l=0}^{\infty} \frac{(-i \tau)^l}{l!}$$

$$\times \sum_{m=1}^M \left( H^{(m)}_k \right)^l = \sum_{m=1}^M \exp \left( -i H^{(m)}_k \tau \right) = \sum_{m=1}^M U^{(m)}_k,$$

where the support of $U^{(m)}_k = \exp \left( -i H^{(m)}_k \tau \right)$ is $\mathcal{H}_{\text{sys}} \otimes \mathcal{H}^{(m)}_k$, so $U^{(m)}_k U^{(m')}_{k'} = 0$ if $m \neq m'$. Sequence of $n$ collisions results in the evolution operator

$$U_n \cdots U_2 U_1 = \sum_{m=1}^M U^{(m)}_n \cdots U^{(m)}_2 U^{(m)}_1,$$

where $U^{(m)}_n \cdots U^{(m)}_2 U^{(m)}_1$ does not vanish on vectors involving the system and the $m$-th block of matrix $\xi$.

The dynamical map after $n$ collisions reads

$$\Phi[\rho] = \text{tr}_{\text{env}} \left[ U_n \cdots U_2 U_1 \rho \otimes \Xi U_1^\dagger \cdots U_n^\dagger \right]$$

$$= \sum_{m=1}^M p_m \text{tr}_{\text{env}} \left[ \tau^{(m)}_u \cdots \tau^{(m)}_1 \rho \otimes \Xi^{(m)} \tau^{(m)}_1 \cdots \tau^{(m)}_n \right]$$

$$= \sum_{m=1}^M p_m \Phi^{(m)}[\rho].$$

Therefore, the correlated environment $\xi$ enables realization of the mixture of dynamical maps $\Phi^{(m)}$.

**Example 1.** Consider a mixture of pure dephasing qubit channels, Eq. (30), $M = 3$. Deterministic collision model of such a dynamics is achieved with the environment composed of $n$ 6-level systems, Fig. 6. The classically correlated state of $n$ environment particles is

$$\Xi = \left( p_1 \otimes \frac{1}{2} I^{(1)} + p_2 \otimes \frac{1}{2} I^{(2)} + p_3 \otimes \frac{1}{2} I^{(3)} \right),$$

which assigns probability $p_1$ ($p_2, p_3$) to the occurrence of collision with the first (second, third) pair of levels within the 6-level system.

Elementary unitary transformations $\tau^{(m)}_k$ coincide for all collisions $k = 1, \ldots, n$ and represent a generalization of a controlled-unitary operation, where the system is a controlled qubit, and $m$-th qubit within the triple serves as a controlling qubit:

$$U^{(m)}_k = \exp \left( -i \gamma \sigma_m \otimes [0_m \l 0_m] + e^{-i \gamma \sigma_m} \otimes [1_m \l 1_m] \right).$$

In the stroboscopic limit $l \tau \rightarrow 0$ and $\gamma^2 \tau \rightarrow 2 \gamma$, Eq. (34) leads to the dynamical map $\Phi_t^{\text{mix}} = p_1 e^{\xi_1 t} + p_2 e^{\xi_2 t} + p_3 e^{\xi_3 t}$ with $\xi_t[\rho] = \gamma (\sigma_1 \rho \sigma_1 - \rho)$.

**VII. ETERNAL CP INDIVISIBILITY**

 Eternal CP indivisible dynamical maps $\Phi_t$ are those that are not CP divisible for any time $t > 0$. It was shown recently that eternal CP indivisibility is quite a general property for spin-boson systems [23]. Known examples of eternal CP indivisible Pauli dynamical maps include non-trivial convex combinations $p_1 \Phi_1^{\text{pd}} + p_j \Phi_j^{\text{pd}}$ of pure dephasing processes $\Phi_1^{\text{pd}}$ and $\Phi_j^{\text{pd}}$ (in the basis of eigenstates of operators $\sigma_i$ and $\sigma_j$, respectively), $i, j = x, y, z, i \neq j$. In what follows, we extend this one-parameter family (since $p_1 + p_j = 1$) to a wider class, namely, a two-parameter family of eternal CP indivisible maps. The underlying idea is to consider such smooth trajectories $\lambda(t)$ in the parameter space $\lambda_1, \lambda_2, \lambda_3$ that do not belong to the geometrical body in Fig. 2. These trajectories are beyond ultimate CP divisible processes, as a result $\kappa$-vector always points beyond the tetrahedron in Fig. 6.

To start with, focus on three ultimate CP divisible semi-group processes:

- $\Phi_1^{(1)} = e^{L_1^{(1)} t}$ with the dissipator $L_1^{(1)}[\rho] = \frac{\gamma}{2} (\sigma_i \rho \sigma_i - \rho) + \frac{\gamma}{2} (\sigma_j \rho \sigma_j - \rho), i \neq j, \gamma_{ij} > 0$, which describes a “skewed” amplitude damping process towards a completely mixed state $\frac{1}{2} I$ via contact with high-temperature environment;

- $\Phi_1^{(2)} = e^{L_1^{(2)} t}$ with the dissipator $L_1^{(2)}[\rho] = \frac{\gamma}{2} (\sigma_i \rho \sigma_i - \rho)$, which is a pure phase damping process in the basis of eigenstates of $\sigma_i$;
• \( \Phi^{(3)} = e^{L^{(3)}t} \) with the dissipator \( L^{(3)}[\rho] = \frac{1}{2} (\sigma_j \rho \sigma_j - \rho) \), which is a pure phase damping process in the basis of eigenstates of \( \sigma_j \).

Let us demonstrate that any non-trivial mixture \( \Phi_t = p_1 \Phi_t^{(1)} + p_2 \Phi_t^{(2)} + p_3 \Phi_t^{(3)} \) with 0 < \( p_{1,2,3} > 0 \) is CP-indivisible. In fact, parameters of the unital map \( \Phi_t \) read

\[
\lambda_j(t) = (p_1 + p_3)e^{-\gamma_j t} + p_2, \quad \lambda_2(t) = (p_1 + p_2)e^{-\gamma_1 t} + p_3, \quad \lambda_3(t) = p_1e^{-\gamma_1 t} + p_2e^{-\gamma_1 t} + p_3e^{-\gamma_1 t}.
\]

Therefore, the probability of confusing two states \( \rho \) and \( \sigma \) monotonically increases in P divisible processes \( [S(\rho)\|\sigma) \) monotonically decreases].

### B. Distinguishability

The trace distance \( D(\rho, \sigma) = \frac{1}{2} \| \rho - \sigma \|_1 \) between qubit states \( \rho \) and \( \sigma \) is a monotone under qubit positive maps \( \Theta \), i.e.

\[
D(\Theta[\rho], \Theta[\sigma]) \leq D(\rho, \sigma).
\]

On the other hand, the trace distance quantifies the probability of successful discrimination of quantum states \( \rho \) and \( \sigma \) in a single-shot measurement. For P divisible processes this probability monotonically decreases.

### C. Classical capacity

If the process \( \Phi_t \) is unital, then the map \( \Theta_{t+s} \) is also unital. Classical capacity \( C \) of a qubit unital channel reads

\[
C(\Phi_t) = 1 - h_2 \left[ \frac{1}{2} (1 - \max(\lambda_1(t), |\lambda_2(t)|, |\lambda_3(t)|)) \right],
\]

where \( h_2(x) = -x \log_2 x - (1-x) \log_2 (1-x) \). It is not hard to see, that all \( |\lambda_j(t)| \), \( i = 1, 2, 3 \), monotonically decrease if \( \Theta_{t+s} \) is positive for all \( t,s \). Therefore, if the qubit unital process \( \Phi_t \) is P divisible, then its classical capacity \( C(\Phi_t) \) monotonically decreases with time \( t \).

### D. Separability

If a positive map \( \Theta \) is applied to a part of separable state \( R = \sum \pi_i \rho_i \otimes \sigma_i \), then its separability is preserved since \( (\Theta \otimes \text{Id})[R] = \sum \pi_i \Theta[\rho_i] \otimes \sigma_i \) is a valid density operator. Thus, if the process \( \Phi_t \) is P divisible, then its action on a part of a composite system cannot result in the revival of entanglement.

Suppose that by time \( t = t_{EB} \) the process \( \Phi_t \) becomes entanglement breaking \( [78, 79] \), i.e. \( \Phi_{t_{EB}} \) is effectively a measure-and-prepare procedure (quantum-classical-quantum channel) of the Holevo form \( \Phi_{t_{EB}}[\rho] = \sum \pi_k \text{tr}[\rho E_k] g_k \), where \( \{E_k\} \) is a positive operator-valued measure.

Suppose that by time \( t = t_{EB} \) the process \( \Phi_t \) becomes entanglement-breaking \( \Phi_{t_{EB}} \) by time \( t = t_{BA} \), and \( \Phi_t \) is P divisible for \( t > t_{EB} \). Then \( \Phi_{t_{FB}} \) and \( \Phi_{t_{BA}} \) are separable and \( \Phi_{t_{FB}} \) remains separable for \( t > t_{EB} \).

Suppose the channel \( \Phi_t \) becomes entanglement-annihilating \( \Phi_{t_{BA}} \) by time \( t = t_{EA} \), and \( \Phi_t \) is P divisible for \( t > t_{EA} \). Then \( \Phi_{t_{BA}} \) is separable and \( \Phi_{t_{BA}} \) remains separable for \( t > t_{BA} \).

For instance, the Pauli channel \( \Phi \) with parameters \( \lambda_1, \lambda_2, \lambda_3 \) results in entanglement-annihilating channel \( \Phi \otimes \Phi \) if and only if \( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 1 \). The process \( \Phi_t = p_1 e^{t A_1} + p_2 e^{t A_2} + p_3 e^{t A_3} \) with dissipators \( L_i[\gamma] = \gamma (\sigma_i \rho \sigma_i - \rho) \) becomes entanglement annihilating if

\[
p_1^2 + p_2^2 + p_3^2 = 1 - e^{-\gamma t_{EA}} - e^{-2\gamma t_{EA}} \leq 1 - e^{-\gamma t_{EA}} - e^{-2\gamma t_{EA}}.
\]

Positive divisibility of the map \( \Phi_t \) guarantees separability of \( \Phi_{t_{BA}} \) for all \( t > t_{BA} \).

### VIII. P DIVISIBILITY

In this section, we review elementary operational features of P divisible dynamical processes.

#### A. Probability of confusion

Consider a positive map \( \Theta \) [82], then the quantum relative entropy \( S(\rho)\|\sigma) = \text{tr}[\rho (\ln \rho - \ln \sigma)] \) is a monotone under positive maps \([74]\), i.e.

\[
S(\Theta[\rho]\|\Theta[\sigma]) \leq S(\rho\|\sigma)
\]

for all density matrices \( \rho \) and \( \sigma \). On the other hand, quantum analogue of Sanov’s theorem \([75]\) states that the probability of confusing two quantum states \( \rho \) and \( \sigma \) after performing \( n \) measurements on \( \sigma \) equals

\[
P_n(\rho \rightarrow \sigma) = e^{-n S(\rho\|\sigma)} \text{ if } n \gg 1.
\]

Therefore, the probability of confusing two states \( \rho \) and \( \sigma \) monotonically increases in P divisible processes \( [S(\rho)\|\sigma) \) monotonically decreases].
E. Tensor power

Clearly, a map $\Theta \otimes \Theta$ can be non-positive even if $\Theta$ is positive [80, 81]. Thus, even if $\Phi_t$ is $P$ divisible, $\Phi_t \otimes \Phi_t$ can still be P indivisible. However, if $\Phi_t \otimes \Phi_t$ is P divisible then $\Phi_t$ is CP divisible [65].

IX. COLLISION MODELS FOR P INDIVISIBLE DYNAMICAL MAPS

$P$ indivisible (essentially non-Markovian) dynamical maps $\Phi_t$ can exhibit properties opposite to those described in the previous section, namely, the probability of confusion of two states, distinguishability of states, and classical capacity can be non-monotonic functions of time. In following subsections, we construct collision models of specific and general $P$ indivisible processes and present an example of the dynamical map, which monotonically shrinks the volume of accessible states but is not $P$ divisible.

A. Essentially non-Markovian dephasing process

As an example of $P$ indivisible dynamics, consider a correlated environment. The evolution of the system and $k$-th environment qubit. Suppose $U_k = e^{ig^2} \otimes |0\rangle_k \langle 0| + e^{-ig^2} \otimes |1\rangle_k \langle 1|$, then after $n = \frac{t}{T}$ collisions we get

$$
\Phi^p_t |\psi\rangle = \text{tr}_{\text{env}} \left[ U_n \cdots U_2 U_1 |\psi\rangle \otimes U_1^\dagger U_2^\dagger \cdots U_n^\dagger \right]
$$

$$
= \cos^2 (nt) |\psi\rangle + \sin^2 (nt) \sigma_\sigma |\psi\rangle.
$$

Clearly, the resulting dephasing dynamics $\Phi_t$ is $P$ divisible if $0 < t < \frac{T}{2}$ and $P$ indivisible if $\frac{T}{2} < t < \frac{T}{2}$. Then the periods of $P$ divisibility and $P$ indivisibility alternate. The information about the initial system state $|\psi\rangle$ is stored in the environment when the process is $P$ divisible, and the back-flow of information occurs when the process is $P$ indivisible.

B. Mixture of essentially non-Markovian dephasing processes

Similarly to the previous subsection, $P$ indivisible dephasing processes $\Phi_t^p$ and $\Phi_t^p$ along $x$ and $y$ axes of the Bloch ball can be achieved by collision models. The mixture $\Phi_t = p_1 \Phi_t^x + p_2 \Phi_t^y + p_3 \Phi_t^z$ can be realized with a correlated environment made of $n$ 6-level systems in the state

$$
\Xi = \sum_{m=1,2,3} \frac{p_m}{2} \left( |0\rangle_m \langle 0|_m + |1\rangle_m \langle 1|_m \right),
$$

where $m$ labels pairs of levels (effective qubit states $|0_m\rangle$ and $|1_m\rangle$), and elementary unitary transformations

$$
U_k = \sum_{m=1,2,3} e^{i\theta m} \otimes |0_m\rangle_k \langle 0_m| + e^{-i\theta m} \otimes |1_m\rangle_k (1_m|.
$$

Pictorial representation of the resulting dynamical map

$$
\Phi_t = p_1 \Phi_t^x + p_2 \Phi_t^y + p_3 \Phi_t^z
$$

is a straight line in the parameter space $\lambda_1, \lambda_2, \lambda_3$.

For instance, in the case $p_1 = p_2 = p_3 = \frac{1}{3}$, we obtain a depolarizing map $\mathcal{D}_{p(t)}[\rho] = p(t)\rho + (1 - p(t))\text{tr}[\rho] \frac{1}{2} I$ with $p(t) = \frac{1}{3} + 2\cot^2 t$. In such a process, the Bloch ball gradually shrinks to a point, then extends in inverted form unless its radius equals $\frac{1}{3}$ (the best approximation of universal NOT operation), and then the process goes in opposite direction until the region of accessible states occupies the whole Bloch ball again, after that the process continuous from the very beginning.

C. Arbitrary pure dephasing process

In subsection IXA we considered a essentially non-Markovian pure dephasing process with the coherence function $\cos(2gt)$. In this subsection, we construct a collision model which results in a pure dephasing process with the arbitrary continuous real coherence function $f(t)$ that is bounded ($|f(t)| \leq 1$) and $f(0) = 1$.

We start with a dephasing process in the basis of eigenvectors of $\sigma_z$, i.e. the density matrix transformation

$$
\begin{pmatrix}
\theta_{11} & \theta_{12} \\
\theta_{21} & \theta_{22}
\end{pmatrix} \to \begin{pmatrix}
\theta_{11} & f(t) \theta_{12} \\
\theta_{21} & \theta_{22}
\end{pmatrix},
$$

which corresponds to a trajectory $\lambda(t) = (f(t), f(t), 1)$ in the parameter space.

Consider a correlated environment in the state

$$
\Xi = \sum_k \frac{1}{2} \sum_k \left( |i_k\rangle \langle i_k| \otimes |\overline{i}_k\rangle \langle \overline{i}_k| \otimes \cdots \right.
$$

$$
+ \left. |\overline{i}_k\rangle \langle i_k| \otimes |\overline{i}_k\rangle \langle \overline{i}_k| \otimes \cdots \right)
$$

$$
= \frac{1}{2} \bigotimes_k |i_k\rangle \langle i_k| + \frac{1}{2} \bigotimes_k |\overline{i}_k\rangle \langle \overline{i}_k|,
$$

where $n$ labels pairs of levels (effective qubit states $|0_m\rangle$ and $|1_m\rangle$), and elementary unitary transformations

$$
U_k = \sum_{m=1,2,3} e^{i\theta m} \otimes |0_m\rangle_k \langle 0_m| + e^{-i\theta m} \otimes |1_m\rangle_k (1_m|.
$$

Pictorial representation of the resulting dynamical map

$$
\Phi_t = p_1 \Phi_t^x + p_2 \Phi_t^y + p_3 \Phi_t^z
$$

is a straight line in the parameter space $\lambda_1, \lambda_2, \lambda_3$.

For instance, in the case $p_1 = p_2 = p_3 = \frac{1}{3}$, we obtain a depolarizing map $\mathcal{D}_{p(t)}[\rho] = p(t)\rho + (1 - p(t))\text{tr}[\rho] \frac{1}{2} I$ with $p(t) = \frac{1}{3} + 2\cot^2 t$. In such a process, the Bloch ball gradually shrinks to a point, then extends in inverted form unless its radius equals $\frac{1}{3}$ (the best approximation of universal NOT operation), and then the process goes in opposite direction until the region of accessible states occupies the whole Bloch ball again, after that the process continuous from the very beginning.
where either \( i_k = 0 \) and \( \overline{k} = 1 \), or \( i_k = 1 \) and \( \overline{k} = 0 \). Elementary unitary transformations \( U_k = e^{i g r \sigma_z} \otimes |0\rangle_k \langle 0| + e^{-i g r \sigma_z} \otimes |1\rangle_k \langle 1| \) result in the following dynamical map after \( n = \frac{t}{\tau} \) collisions:

\[
\Phi_{1}[g] = \text{tr}_{\text{env}} \left[ U_n \cdots U_2 U_1 \rho \otimes \Xi U_1^\dagger U_2^\dagger \cdots U_n^\dagger \right] = \cos^2 \left\{ [n_0(t) - n_1(t)] \sigma_z \right\} g + \sin^2 \left\{ [n_0(t) - n_1(t)] \sigma_z \right\} \sigma_x \otimes \sigma_z,
\]

(56)

where \( n_0(t) = \sum_{k=1}^{n} \delta_{i_k \overline{k}, 0} \) and \( n_1(t) = \sum_{k=1}^{n} \delta_{i_k \overline{k}, 1} \) \( n = n_0 \). Apparently, \( [n_0(t) - n_1(t)] \tau = 2n_0(t) \tau - t \) and \( f(t) = \cos \{ 2g \{ 2n_0(t) \tau - t \} \} \).

Therefore, to get the desired dynamics one needs to arrange the number \( n_0(t) \) of 0’s in indices \( i_k \) of environment state (55) in accordance with the formula

\[
n_0(t) \tau = \frac{\arccos f(t)}{4g} + \frac{t}{2}.
\]

(58)

In the usual continuous limit \( \tau \to 0 \), \( g \tau \to \text{const} \), the left hand side of Eq. (58) has the meaning of the integral \( n_0(t) \tau = \int_{t_0}^{t} w_0(t') dt' \), where \( w_0(t) \) is the probability of encountering 0 at every collision in the first line of the environment state (55). Finally,

\[
w_0(t) = -\frac{f'(t)}{4g \sqrt{1-f^2(t)}} + \frac{1}{2}.
\]

(59)

If \( f'(t) = 0 \) when \( f(t) = 1 \), then the right hand side can be made non-negative and bounded from above by 1 for sufficiently large \( g \). If \( f'(t) \neq 0 \) when \( f(t) = 1 \) (as it takes place, e.g., in Markov approximation), one has to resort to the stroboscopic limit and replace \( g \) by \( \frac{1}{\tau} \), which enables to meet the requirement \( 0 < w_0(t) \leq 1 \). Similarly, one can construct the processes of arbitrary dephasing in the bases of eigenstates of operators \( \sigma_x \) and \( \sigma_y \).

D. Arbitrary Pauli dynamical maps

In this subsection, we construct a collision model which is able to reproduce any dynamics \( \Lambda(t) \) satisfying the condition of complete positivity of the corresponding Pauli dynamical map \( \Phi_{1} \). In other words, given a trajectory in the parameter space (Fig. 1), we construct a collision model leading to such a trajectory.

The requirement of complete positivity is automatically fulfilled if the functions \( q_j(t) \) defined through

\[
\begin{bmatrix}
q_0(t) \\
q_1(t) \\
q_2(t) \\
q_3(t)
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix} \begin{bmatrix}
\lambda_1(t) \\
\lambda_2(t) \\
\lambda_3(t)
\end{bmatrix}
\]

(60)

satisfy \( q_j(t) \geq 0 \) for all \( j = 0, 1, 2, 3 \).

To get an arbitrary Pauli dynamical map with non-negative functions \( q_0(t), q_1(t), q_2(t), q_3(t) \), one needs to combine three (essentially non-Markovian) dephasing processes considered in the previous section. This is achieved with the environment composed of 3 types of qubits. Denote these types \( x, y, z \), then \( k \)-th collision of system qubit with \( m \)-th type of environment qubits is described by the elementary unitary transformation

\[
U_k^{(m)} = e^{i g r \sigma_m} \otimes |0\rangle_k \langle 0| + e^{-i g r \sigma_m} \otimes |1\rangle_k \langle 1|.
\]

(61)

Qubits of the same kind are correlated, so the total environment state reads

\[
\Xi = \bigotimes_{m=x,y,z} \left( \frac{1}{2} \sum_{k \in \{k_m\}} |i_k\rangle \langle i_k| + \frac{1}{2} \sum_{k \in \{k_m\}} |i_k\rangle \langle i_k| \right),
\]

(62)

where \( \{k_x\}, \{k_y\}, \{k_z\} \) are subsequences of collision numbers \( k \in \mathbb{N} \) such that \( \{k_m\} \cap \{k_m'\} = \emptyset \) if \( m \neq m' \) and \( \{k_x\} \cup \{k_y\} \cup \{k_z\} = \mathbb{N} \). Physics of such collisions is depicted in Fig. 1.

The resulting map is

\[
\Phi_{1}[g] = \text{tr}_{\text{env}} \left[ U_n \cdots U_2 U_1 \rho \otimes \Xi U_1^\dagger U_2^\dagger \cdots U_n^\dagger \right],
\]

(63)

where \( U_k = U_k^{(m)} \otimes \Phi_{1}^{(m)} \). The intermediate map \( \Theta_{t,t+\tau} = \Phi_{1}^{(m)} \otimes \Phi_{1}^{(m)} \) between collisions realizes one of the infinitesimal maps \( \Theta_{t,t+\tau}^{(x)} \), \( \Theta_{t,t+\tau}^{(y)} \) and \( \Theta_{t,t+\tau}^{(z)} \). Collision with \( m \)-th particle results in the map \( \Theta_{t,t+\tau}^{(m)} \). Clearly, for a fixed \( m \) the product \( \prod_{t=0}^{\tau} \Theta_{t,t+\tau}^{(m)} \) is nothing else but the dephasing map in the eigenbasis of operator \( \sigma_m \) with dephasing function \( f_m(t) \) given by a modification of Eq. (57):

\[
f_m(t) = \cos \left\{ 2 \left[ n_0^{(m)}(t) - n_1^{(m)}(t) \right] \right\} \sigma_m \tau,
\]

(64)

where \( n_0^{(m)}(t) = \sum_{k_m \leq n} \delta_{i_{k_m}0} \) and \( n_1^{(m)}(t) = \sum_{k_m \leq n} \delta_{i_{k_m}1} \). All physical functions \( f_m(t), m = 1, 2, 3 \), can be realized in the usual continuous or stroboscopic limit as it was demonstrated for a single dephasing map. Then a sequence of collisions with different types of qubits during a short time \( dt \) \( (dt \gg \tau) \) results in the product

\[
\Theta_{t,t+dt} = \Theta_{t,t+dt}^{(x)} \Theta_{t,t+dt}^{(y)} \Theta_{t,t+dt}^{(z)}.
\]

(65)

Note that all \( \Theta_{t,t+dt}^{(m)} \) commute. Consequently, the parameters \( \lambda_1(t), \lambda_2(t), \lambda_3(t) \) of the map \( \Phi_{1} \) satisfy differential equations \( \lambda_1(t) = f_1^2(t) + f_1^3(t), \lambda_2(t) = f_2^2(t) + f_2^3(t) \), \( \lambda_3(t) = \lambda_3^2(t) + \lambda_3^3(t) \).
the explicit form of the functions $\lambda_0 \text{ Eq. (60)}$ and $E$ of accessible states of a qubit dynamical map by Hilbert-Schmidt distance for qubit states, the volume of accessible states [18]. Using the metric induced Markovianity is based on the quantification of the volume of accessible states. Arrows represent directions of the $\kappa$-vector at particular time moments.

$$f_1(t) + f_2(t), \lambda_1(t) = f_1(t) + f_2(t),$$

from which it follows that $\lambda_1(t) = f_2(t) + f_2(t) - 1, \lambda_2(t) = f_1(t) + f_2(t) - 1, \lambda_3(t) = f_1(t) + f_2(t) - 1$. Finally, using Eq. (60), we find the explicit form of the functions $f_m(t) = 1 - 2q_m(t)$.

The algorithm for producing arbitrary dynamics $\lambda(t)$ in parameter space is the following. Calculate $q_m(t)$ by Eq. (60) and $f_m(t) = 1 - 2q_m(t)$. For each $m$ distribute 0’s and 1’s in accordance with formula (61). Create the correlated state (2) with corresponding distributions of 0’s and 1’s in $m$th branch. Let the system qubit interact with environment qubits of type $m$ according to the elementary evolution operator (61).

E. Dynamical maps shrinking the volume of accessible states

One more approach to characteristic of non-Markovianity is based on the quantification of the volume of accessible states [18]. Using the metric induced by Hilbert-Schmidt distance for qubit states, the volume of accessible states of a qubit dynamical map $\Phi_t$ is simply the volume of the ellipsoid in the Bloch ball picture, which corresponds to the domain of $\Phi_t$. For Pauli dynamical maps $\Phi_t$ of the form (2) the volume of accessible states $V(t) = |\lambda_1(t)\lambda_2(t)\lambda_3(t)|$. A process $\Phi_t$ monotonically shrinks the volume of accessible states if any intermediate map $\Theta_{t,t+\epsilon}$ does so.

Let us present an example of the Pauli dynamical map $\Phi_t$ which monotonically shrinks the volume of accessible states but is not P divisible. Let $\lambda_1(t) = e^{-2t}(1 - \frac{1}{10}[1 - \cos 40t]), \lambda_2(t) = e^{-2t}(1 - \frac{1}{10}\sin 40t), \lambda_3(t) = e^{-4t}$, then $\Phi_t$ is a physical process indeed since all $q_i(t) \geq 0$ and can be realized via collision model with correlated environment (see the preceding subsection). It is not hard to see that the volume of accessible states $V(t)$ monotonically decreases, whereas both $\lambda_1(t)$ and $\lambda_2(t)$ are not monotonic, i.e. the process is not P divisible (see the trajectory and the corresponding $\kappa$-vector in Fig. 10).

FIG. 10: Trajectory of the Pauli dynamical map, which is not P divisible but monotonically shrinks the volume of accessible states. Arrows represent directions of the $\kappa$-vector at particular time moments.

X. CONCLUSIONS

We have studied the relation between different forms of divisibility of dynamical maps and collision models that stroboscopically simulate such dynamical maps. Our findings are illustrated by Pauli dynamical maps, which allow a particularly visual pictorial representation of process trajectories in the parameter space.

A concept of ultimate CP divisible maps has been introduced: ultimate CP divisible processes can be understood as ultimate dynamical maps still simulable by collision models with factorized environment. Ultimate CP divisible semigroups of Pauli maps are fully characterized, with the interaction Hamiltonian being specified.

Within the framework of collision models, we have demonstrated additivity and multiplicativity of time-dependent generators of CP divisible processes. The environment remains factorized in this case. Roughly speaking, to realize a weighted sum of generators of CP divisible maps one has to shuffle individual environments using tensor product.

Using correlated environment states, we have explicitly constructed a collision model realizing the mixture of CP divisible maps. The latter technique was used to simulate a new two-parameter family of eternal CP indivisible maps. This family represents a mixture of two pure dephasing processes and a skewed version of the generalized amplitude damping process. Continuing the rough analogy, a mixture of dynamical maps corresponds to uniting individual environments via direct sum operation.

Also, we have reviewed general properties of P divisible dynamical maps. In particular, using a quantum analogue of Sanov’s theorem we have noticed that the probability of confusing two states monotonically increases in P divisible processes. As far as P indivisible processes are concerned, we have explicitly constructed collision models simulating arbitrary Pauli dynamical maps.

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