Positive Tropical Flags and the Positive Tropical Dressian

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November 25, 2021

Abstract

We study the totally non-negative part of the complete flag variety and of its tropicalization. We start by showing that Lusztig’s notion of non-negative complete flag variety coincides with the flags in the complete flag variety which have non-negative Plücker coordinates. This mirrors the characterization of the totally non-negative Grassmannian as those points in the Grassmannian with all non-negative Plücker coordinates. We then study the tropical complete flag variety and complete flag Dressian, which are two tropical versions of the complete flag variety, capturing realizable and abstract flags of tropical linear spaces, respectively. The complete flag Dressian properly contains the tropical complete flag variety. However, we show that the totally non-negative parts of these spaces coincide.

1 Introduction

The Grassmannian of \(k\) planes in \(n\) space describes \(k\) dimensional linear subspaces in \(n\) dimensional space. It is an algebraic variety cut out by the Plücker relations. We can tropicalize these relations to obtain the tropical Plücker relations. The set of points satisfying the tropical Plücker relations, called the Dressian, is the parameter space of abstract tropical linear spaces [20]. The set of points satisfying the tropicalizations of all polynomials in the ideal generated by the Plücker relations, called the tropical Grassmannian, is the parameter space of realizable tropical linear spaces [5]. In general, the Dressian properly contains the tropical Grassmannian (see, for instance, [6]). However, in [19], it is shown that if we restrict to positive solutions, for an appropriate notion of positivity, the situation is simpler: the positive Dressian equals the positive tropical Grassmannian. More explicitly, this means that a positive solution to the tropicalizations of the Plücker relations is also a positive solution to the tropicalization of any polynomial in the ideal generated by the Plücker relations. Our goal is to generalize this fact to the setting of the complete flag variety.

The complete flag variety, \(F\)\(l_n\), is the set of complete flags of linear subspaces \(\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{R}^n\). Any point of this variety is determined by a set of coordinates called its Plücker coordinates. These are cut out by the incidence-Plücker relations, a set of polynomials which extends the Plücker relations, which generate an ideal called the incidence-Plücker ideal. We consider the set of points satisfying the tropicalizations of the incidence-Plücker relations, called the complete flag Dressian, \(F\)\(l\)\(D\)\(r_n\), and the set of points satisfying the tropicalizations of all polynomials in the incidence-Plücker ideal, called the tropical complete flag variety, \(T\)\(r\)\(F\)\(l_n\). These parameterize abstract flags of tropical linear spaces and realizable flags of tropical linear spaces, respectively [3].

The tropical spaces \(F\)\(l\)\(D\)\(r_n\) and \(T\)\(r\)\(F\)\(l_n\) are generally different [3]. Motivated by the example of the tropical Grassmannian, we will investigate the totally non-negative (TNN) parts of these spaces. We define the totally non-negative complete flag Dressian to be the set of simultaneous positive solutions to the tropicalizations of the incidence-Plücker relations and the totally non-negative tropical complete flag variety to be the set of simultaneous positive solutions to the tropicalizations of all the polynomials in the incidence-Plücker ideal. Our main result, Theorem 4.11, says the following:

**Theorem.** The TNN tropical complete flag variety, \(T\)\(r\)\(F\)\(l_n^{\geq 0}\), equals the TNN complete flag Dressian, \(F\)\(l\)\(D\)\(r_n^{\geq 0}\).

*We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [funding reference number 557353-2021]. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [numéro de référence 557353-2021].*
A number of authors, among them [16], [21], [8] and [11], have proven that the TNN Grassmannian, in the sense of Lusztig [9], consists precisely of points in the Grassmannian where each Plücker coordinate is non-negative. We extend this result to the setting of the complete flag variety. Specifically, in proving theorem Theorem 4.11, we will need to carefully study the totally non-negative complete flag variety, denoted $F^0_l n$. A construction based on the parameterization of $F^0_l n$ by Marsh and Rietsch [12] will allow us to understand explicitly the Plücker coordinates $\{P_I(F)\}_{I \subseteq [n]}$ of an arbitrary flag $F$ in $F^0_l n$. In Theorem 3.14, we show:

**Theorem.** The TNN complete flag variety $F^0_l n$ equals the set $\{F \in F_l n | P_I(F) \geq 0 \forall I \subseteq [n]\}$.

We have learned recently that this result has been independently proven in [1], where they show moreover that the only partial flag variety for which this theorem holds are those where the dimensions of the constituent subspaces are consecutive integers. This includes $FL^0_n$, with constituent dimensions $\{1, 2, \cdots, n\}$, and the TNN Grassmannian of $k$ planes in $n$ space, with constituent dimension $k$.

We also introduce an alternative description of $F(DR) n$, as the set of points which are common solutions to the three term tropical incidence-Plücker relations and whose supports form flag matroids. This is analogous to the description of the Dressian as the set of common solutions of all the three-term tropical Plücker relations [14, Theorem 5.2.25]. While this result is independent of the rest of this abstract, it motivates the introduction of a related TNN tropical space which is used in the proof of Theorem 4.11.

The structure of this extended abstract is as follows: In section 2, we introduce the TNN complete flag variety. In section 3, we give a parametrization of this space and study its Plücker coordinates. In section 4, we introduce three tropicalizations of the complete flag variety and demonstrate that two of them are in fact the same. We then focus in on the TNN parts of these three tropical spaces and demonstrate that they are all equal.

**Acknowledgements:** I would like to thank my supervisor Lauren Williams for introducing me to the tropical flag variety and for many helpful conversations as this work developed. I would also like to thank both Melissa Sherman-Bennett and Mario Sanchez for helpful conversations, examples and references.

## 2 The Totally Non-Negative Complete Flag Variety

**Definition 2.1.** The complete flag variety $F_l n$ is the collection of all complete flags in $\mathbb{R}^n$, which are collections $(V_i)_{i=0}^n$ of linear subspaces satisfying $\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{R}^n$.

We first observe that $F_l n$ is a multi-projective variety. We can represent a flag $(V_i)_{i=1}^n$ by a full rank $n$ by $n$ matrix $M$ such that $V_i$ equals the span of the topmost $i$ rows of $M$. Let $GL_n$ be the group of invertible $n$ by $n$ matrices and $SL(n, \mathbb{R})$ be the special linear group of real matrices with determinant $1$. Let $B_-$ be the Borel subgroup of $GL_n$ consisting of lower triangular matrices. One can check that two matrices $M$ and $M'$ represent the same flag if and only if they are related by left multiplication by some $B \in B_-$. Thus, we can think of the complete flag variety as $F_l n = \{B_- u | u \in SL(n, \mathbb{R})\}$, where a flag in $F_l n$ represented by a matrix $u$ is identified with the set $B_- u$.

For $I \subseteq [n] = \{1, \cdots, n\}$ and $M$ an $n$ by $n$ matrix, the Plücker coordinate (or, alternatively, flag minor) $P_I(M)$ is the determinant of the submatrix of $M$ in rows $\{1, 2, \cdots, |I|\}$ and columns $I$.

For any flag $F$, associate the collection of Plücker coordinates $\{P_I(F)\}_{I \subseteq [n]}$, defined to be the Plücker coordinates of any matrix representative of that flag. By [13, Proposition 14.2], this is an embedding of $F_l n$ in $\mathbb{R}P(\binom{n}{\cdot} - 1) \times \cdots \times \mathbb{R}P(\binom{n}{\cdot} - 1)-1$. The Plücker coordinates of flags in $F_l n$ are cut out by multi-homogeneous polynomials, as shown in the following definition and theorem. Note that we will often use shorthand notation such as $(S \setminus \{a, b\}) \cup \{c, d\}$.

**Definition 2.2** ([4]). Consider $\mathbb{R}P(\binom{n}{r} - 1) \times \cdots \times \mathbb{R}P(\binom{n}{s} - 1)-1$, with coordinates indexed by proper subsets of $[n]$. For $1 \leq r \leq s \leq n$, the **Incidence-Plücker relations** for indices of size $r$ and $s$ are

$$\mathcal{P}_{r,s;n} = \left\{ \sum_{j \in J \setminus I} \text{sign}(j, I, J) P_I \cup J | I \in \binom{n}{r-1}, J \in \binom{n}{s+1} \right\},$$

(1)
where \(\text{sign}(j, I, J) = (-1)^{|\{k \in J | k < j\}| + |\{i \in I | j < i\}|}\).

The full set of incidence-Plücker relations is given by \(\mathcal{P}_{IP; n} = \bigcup_{1 \leq r \leq s \leq n} \mathcal{P}_{r, s; n}\). The ideal generated by \(\mathcal{P}_{IP; n}\), denoted \(I_{IP; n}\), is called the incidence-Plücker ideal.

Remark: Note that the above definition allows for the option of \(r = s\). The incidence-Plücker relations for which \(r = s\) are called the (Grassmann) Plücker relations.

**Theorem 2.3** ([4, Section 9, Proposition 1] and discussion following its proof). Let \(P \in \mathbb{RP}^{-1} \times \cdots \times \mathbb{RP}^{-1}\). Then \(P = P(F)\) for some \(F \in Fl_n\) if and only if \(P\) satisfies the incidence-Plücker relations \(\mathcal{P}_{IP; n}\).

In particular, this means the incidence-Plücker relations are precisely the relations between the topmost minors of a generic full rank matrix.

Lusztig introduced the notion of non-negativity for flag varieties. We outline here the definition of the totally non-negative complete flag variety, following [10]. We work in type A and so the appropriate simplifications will be made in presenting the definition. Let \(s_i\) be the transposition \((i, i+1)\) in the symmetric group \(S_n\) and let \(w_0\) be the longest permutation in \(S_n\). For \(1 \leq k < n\), let \(x_k(a)\) be the \(n \times n\) matrix which is the identity matrix with an \(a\) added in row \(k\) of column \(k+1\). Explicitly,

\[
x_k(a) = \begin{pmatrix}
1 & \cdots & k & k+1 \\
\vdots & & \ddots & \iddots \\
k & \cdots & 1 & a \\
0 & \cdots & \ddots & 1
\end{pmatrix},
\]

where unmarked off-diagonal matrix entries are 0.

**Definition 2.4** ([9]). Let \(N = \binom{n}{2}\). Pick \((i_1, i_2, \ldots, i_N)\) such that \(s_{i_1} \cdots s_{i_N} = w_0\). Then let

\[
U_{>0}^+ = \{ x_{i_1}(a_1) \cdots x_{i_N}(a_N) | a_i \in \mathbb{R}_{>0} \forall i \}
\]

This definition is independent of the choice of sequence \((i_1, \ldots, i_N)\).

**Definition 2.5** ([9]). Let \(B_{>0} = \{ B_r | r \in U_{>0}^+ \} \subset Fl_n\). The totally non-negative complete flag variety (of type A), \(\text{Fl}^{>0}_n\), is the closure of \(B_{>0}\).

### 3 Parametrization of the TNN Complete Flag Variety

#### 3.1 The Marsh-Rietsch Parametrization

As shown by Rietsch [16], \(\text{Fl}^{>0}_n\) is a cell complex, whose cells \(\mathcal{R}^{>0}_v\) are indexed by pairs of permutations \(v \leq w\) in the Bruhat order on \(S_n\). Each such \(\mathcal{R}^{>0}_v\) is given an explicit parameterization in [12]. We will describe this parameterization here, making some choices that in principle are arbitrary but will be convenient for our purposes, and invite the reader to look at the above references for full generalities.

Any permutation \(w\) in \(S_n\) can be written as a product of simple transpositions \(s_i\), called an expression for \(w\). The length of \(w\), \(\ell(w)\), is the fewest number of transpositions in any expression for \(w\). An expression for \(w\) consisting of \(\ell(w)\) transpositions is called reduced. Let \(w = s_{i_1} s_{i_2} \cdots s_{i_k}\) be a reduced expression for \(w\). If \(v \leq w\) in the Bruhat order, then there is a reduced subexpression \(v = s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_m}}\) for \(v\) in \(w\), where \(1 \leq j_1 < j_2 < \cdots < j_m \leq k\). We will be interested in a special choice of subexpression which is called the positive distinguished subexpression. Intuitively, this can be thought of as the lefmost subexpression.

**Definition 3.1.** Let \(v \leq w\). Choose a a reduced expression \(w = s_{i_1} s_{i_2} \cdots s_{i_k}\) for \(w\) and a subexpression \(v = s_{i_{j_1}} \cdots s_{i_{j_m}}\) for \(v\) in \(w\). Then \(v\) is a positive distinguished subexpression if:
1. It is a reduced expression for $v$.

2. Whenever $\ell(s_{p}s_{i_{j_{p}}} \cdots s_{i_{j_{m}}}) < \ell(s_{i_{j_{r}}} \cdots s_{i_{j_{m}}})$ for $j_{r-1} \leq p < j_{r}$, we have $p = j_{r-1}$.

Lemma 3.2 ([12, Lemma 3.5]). For every $v \leq w$, and every reduced expression $w$ of $w$, there is a unique positive distinguished subexpression for $v$ in $w$.

Example 3.3. Let $n = 4$. Set $w = s_{1}s_{2}s_{3}s_{1}s_{2}s_{1}$ and $v = s_{1}s_{2}s_{1}$. The leftmost subexpression for $v$ in $w$ is $j_{1} = 1$, $j_{2} = 2$ and $j_{3} = 4$. Indeed, one can verify that this choice satisfies the definition.

For $1 \leq k < n$, let $\dot{s}_{k}$ be the $n$ by $n$ identity matrix with the $2 \times 2$ submatrix in rows $\{k, k + 1\}$ and columns $\{k, k + 1\}$ replaced by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Explicitly,

$$\dot{s}_{k} = \begin{pmatrix} \cdot \cdot \cdot & 1 \\ \cdot \cdot \cdot & 0 \\ k & k + 1 \\ k + 1 & 1 \end{pmatrix},$$

where unmarked off-diagonal matrix entries are 0.

We will describe each cell of $F^{\geq 0}_{n}$ as a product of matrices of the form $x_{k}$ and $\dot{s}_{k}$.

Definition 3.4. Fix $v \leq w$ in the Bruhat order. Fix a vector $a \in \mathbb{R}^{(\ell(w) - \ell(v))}$. Consider the reduced expression $w_{0} = (s_{1}s_{2} \cdots s_{n-1})(s_{1}s_{2} \cdots s_{n-2})(\cdots)(s_{1}s_{2})(s_{1})$ for $w_{0}$, the longest permutation in the Bruhat order in $S_{n}$

Choose the positive distinguished subexpression $w$ for $v$ in $w_{0}$, and the positive distinguished subexpression $v$ for $v$ in $w$, and write them as $w = s_{i_{1}} \cdots s_{i_{k}}$ and $v = s_{i_{j_{1}}} \cdots s_{i_{j_{m}}}$, respectively. Let $J = \{j \mid j = j_{t}$ for some $t\}$. In other words, $J$ are those indices which correspond to transpositions that are used in $v$. Then set

$$M_{v, w}(a) := M_{1} \cdots M_{k}, \quad \text{where} \quad M_{j} = \begin{cases} \dot{s}_{i_{j}}, & j \in J \\ x_{i_{j}}(a_{j}), & j \notin J. \end{cases}$$

Theorem 3.5 (Marsh-Rietsch Parametrization, [12]). Each cell $R^{> 0}_{v, w}$ of $F^{\geq 0}_{n}$ can be parameterized as

$$R^{> 0}_{v, w} = \left\{ M_{v, w}(a) \mid a \in \mathbb{R}^{(\ell(w) - \ell(v))} \right\}.$$

In particular, each flag $F \in F^{\geq 0}_{n}$ is uniquely represented in some unique $R^{> 0}_{v, w}$. Moreover, each $R^{> 0}_{v, w}$ is a cell, meaning it is homeomorphic to an open ball.

Example 3.6. Let $n = 4$, $w = s_{1}s_{3}s_{2}s_{1}$ and $v = s_{2}$. The positive distinguished subexpression for $v$ in $w$ is the subexpression where $j_{1} = 3$, so $J = \{3\}$. Thus, $M_{1} = x_{1}(a_{1})$, $M_{2} = x_{3}(a_{2})$, $M_{3} = \dot{s}_{2}$ and $M_{4} = x_{1}(a_{3})$.

The cell of the non-negative flag variety corresponding to $v \leq w$ is represented by matrices of the form

$$M = M_{1}M_{2}M_{3}M_{4} = \begin{pmatrix} 1 & a_{3} & a_{1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & a_{2} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the $a_{i}$ range over all positive real numbers. Note that the Plücker coordinates of $M$ are all non-negative, as Theorem 3.14 will show must be true for any cell of $F^{\geq 0}_{n}$.

We now give a useful property of the cells $R^{> 0}_{v, w}$.

Lemma 3.7. Each cell $R^{> 0}_{v, w}$ of $F^{\geq 0}_{n}$ consists entirely of flags for which some fixed collection of Plücker coordinates is strictly positive and the rest are 0.

\[1\] This choice of expression is arbitrary in the context of the Marsh-Rietsch parameterization, but plays an important role in the proofs underlying later results in this abstract.
3.2 Extremal Non-Zero Plücker Coordinates

We define a special subset of the Plücker coordinates of a flag which we call extremal non-zero Plücker coordinates. The set of indices of the extremal non-zero Plücker coordinates of a flag in $\mathbb{R}^0_{\geq 0}$ will depend only on which cell $\mathcal{R}_{v,w}$ that flag lies in. Further, we will show that in any given cell of $\mathbb{R}^0_{\geq 0}$, the extremal non-zero Plücker coordinates determine all of the other Plücker coordinates.

For any $1 \leq k < n$ and any $P \in \mathbb{R}P^{(k)}_{-1} \times \cdots \times \mathbb{R}P^{(n-k)}_{-1}$, we define a map $\Xi_P : \binom{[n]}{k} \rightarrow \binom{[n]}{k}$. Intuitively, when applied to the index of a non-zero Plücker coordinate $I$, this map finds the largest member of $I$ that can be increased without making the corresponding Plücker coordinate 0 and increases it maximally. Explicitly, for $I$ such that $P_I \neq 0$, define $b = \max_{i \in I} \left\{ i \mid \exists j, \ i < j \notin I, \ P_{(I \setminus \{j\}) \cup \{i\}} \neq 0 \right\}$, if that set is non-empty. Otherwise, say $b$ does not exist. If $b$ exists, define $a = \max_{j \notin I} \left\{ j \mid P_{(I \cup \{j\}) \setminus \{b\}} \neq 0 \right\}$. Then,

$$\Xi_P(I) = \begin{cases} (I \setminus \{b\}) \cup \{a\} & \text{if } I \text{ is the index of a non-zero Plücker coordinate and } b \text{ exists,} \\ I & \text{otherwise.} \end{cases}$$

Note that the indices of non-zero Plücker coordinates with index of some fixed size can be seen as the bases of a matroid. In this light, $\Xi_P$ acts by basis exchange. Also note that, by Lemma 3.7, for a TNN flag $F$, the map $\Xi_{P(F)}$ depends only on the cell $\mathcal{R}_{v,w}$ in which $F$ lies.

The extremal non-zero Plücker coordinates will be indexed by certain $\Xi$ orbits. To properly define them, we first need a preliminary result on matroids:

**Definition 3.8.** The Gale order on subsets of $[n]$ of size $k$ is a partial order such that, if $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$, then we say $I \leq J$ if $i_r \leq j_r$ for every $r \in [k]$.

**Lemma 3.9 ([2, Theorem 1.3.1])** Any matroid has a unique Gale minimal basis and a unique Gale maximal basis.

Note that the Gale minimal and maximal bases referenced in the previous lemma must simply be the lexicographically minimal and maximal bases, respectively.

**Definition 3.10.** Given a set of Plücker coordinates $\{P_I\}$ of a flag, let $I_k$ be the Gale minimal index of size $k$ such that $P_{I_k} \neq 0$. The set of indices of the extremal non-zero Plücker coordinates (referred to as extremal indices) of a point $P$ in $\mathbb{R}P^{(k)}_{-1} \times \cdots \times \mathbb{R}P^{(n-k)}_{-1}$ is the set consisting of those indices which are in the $\Xi_P$ orbit of $I_k$ for some $k \in [n-1]$.

If $F$ is a TNN flag, the extremal indices of the Plücker coordinates $P(F)$ depend only on the cell $\mathcal{R}_{v,w}$ in which $F$ lies, since $\Xi_{P(F)}$ depends only on the cell in which $F$ lies.

If we have a collection of Plücker coordinates $\{P_I\}$ such that $P_I \geq 0$ for all $I \subset \binom{[n]}{k}$, the indices of the non-zero Plücker coordinates of $P$ of fixed size form not just the bases of a matroid, but the bases of a positroid. For background on positroids, see [15]. Many of the useful properties of the extremal Plücker coordinates only hold in this context of positroids.

**Example 3.11.** Let $a, b, c, d, e, f, g \in \mathbb{R}_{>0}$ and consider

$$M = \begin{pmatrix} 1 & a + e + g & ab + af + ef & abc & abcd \\ 0 & 1 & b + f & bc & bcd \\ 0 & 0 & 1 & c & cd \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

The minors of this matrix are non-negative and specifically, focusing on indices of size 2, we can see that they are all positive except for $P_{15} = 0$. Thus, the non-zero Plücker coordinate with Gale minimal index of size 2 is $P_{12}$. Then, $\Xi_{P(M)}(12) = 15$, replacing the 2 with a 5. Next, $\Xi_{P(M)}(15) = 35$, replacing the 1 with a 3. Thus, $P_{12} = 1$, $P_{15} = bcd$ and $P_{35} = bcdef$ are the extremal non-zero Plücker coordinates of size 2 of this flag.

The next theorem highlights the importance of the extremal non-zero Plücker coordinates.

**Theorem 3.12.** For any flag $F$ with non-negative Plücker coordinates, the extremal non-zero Plücker coordinates of $F$ uniquely determine the other non-zero Plücker coordinates of $F$ by three-term incidence-Plücker relations.
3.3 Plücker Coordinates of the TNN Flag Variety

Now, given a set of extremal non-zero Plücker coordinates for a flag lying in \( R_{v,w}^{>0} \), we want to understand how to construct a set of parameters \( a_i \) for which Theorem 3.5 yields a matrix agreeing with those coordinates.

**Theorem 3.13.** For any \( v \leq w \) with \( r = \ell(w) - \ell(v) \), let \( \Psi_{v,w} : R_{v,w}^{>0} \rightarrow \mathbb{R}^r \) be the map \( M_{v,w}(a) \mapsto a \), in the notation of Theorem 3.5. The map \( \Psi_{v,w} \) consists of Laurent monomials in the extremal Plücker coordinates.

We can use this theorem to prove the following, which is one of our main results:

**Theorem 3.14.** The TNN flag variety defined in Definition 2.5 is precisely the set of flags with non-negative Plücker coordinates. In other words, \( Fl_n^{>0} = \{ F \in Fl_n | P_I(F) \geq 0 \ \forall \ I \subset [n] \} \).

It is shown in [7, Lemma 3.10] that any flag in \( Fl_n^{>0} \) has non-negative Plücker coordinates. We now outline the strategy used to obtain the converse.

**Definition 3.15.** A (complete) flag matroid on a ground set \( E \) of size \( n \) is a sequence of matroids \( \mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_{n-1}) \) on the ground set \( E \) with the rank of \( \mathcal{M}_i \) equal to \( i \), called constituent matroids, such that for any \( j < k \),

- each basis of \( \mathcal{M}_j \) is contained in some basis of \( \mathcal{M}_k \).
- each basis of \( \mathcal{M}_k \) contains some basis of \( \mathcal{M}_j \).

We identify a flag matroid with the collection of bases of its constituent matroids, collectively referred to as the bases of the flag matroid. For more details about flag matroids, as well as cryptomorphic definitions, see [2]. Note that the indices of non-zero Plücker coordinates of an invertible square matrix are easily seen to form a flag matroid.

**Definition 3.16.** A flag matroid on \([n]\) is realizable if its bases are the non-zero Plücker coordinates of some \( F \in Fl_n \).

We now define two types of flag positroid. The difference between their definitions mirrors the apparent difference between a flag lying in \( Fl_n^{>0} \) according to Definition 2.5 and a flag with non-negative Plücker coordinates.

**Definition 3.17.** A realizable flag positroid on \([n]\) is the set of indices of non-zero Plücker coordinates of a flag \( F \in Fl_n^{>0} \) (as defined in Definition 2.5). A synthetic flag positroid on \([n]\) is the set of indices of non-zero Plücker coordinates of a flag \( F \) satisfying \( P_I(F) \geq 0 \) for all \( I \subset [n] \).

A priori, one may expect that there could be more synthetic flag positroids than realizable flag positroids, but this is not the case.

**Theorem 3.18.** The set of synthetic flag positroids on \([n]\) equals the set of realizable flag positroids on \([n]\).

Note that by Lemma 3.7, the realizable flag positroid arising from the non-zero Plücker coordinates of a TNN flag only depends on which cell \( R_{v,w}^{>0} \) that flag lies in. Thus, we can associate a cell \( R_{v,w}^{>0} \) to any realizable flag positroid. Let \( F \) be a flag whose Plücker coordinates \( P_I \) are all non-negative. Let \( \mathcal{M} \) be the synthetic (equivalently, realizable) flag positroid which has \( I \subset [n] \) as a basis if and only if \( P_I > 0 \). As above, let \( R_{v,w}^{>0} \) be the cell associated to \( \mathcal{M} \). To prove Theorem 3.14, we are left to show that \( F \in R_{v,w}^{>0} \). Extending the domain of the map \( \Psi_{v,w} \) defined in Theorem 3.13, one can apply it to \( F \) and show that \( \mathcal{M}_{v,w}(\Psi_{v,w}(F)) \) is a flag in \( R_{v,w}^{>0} \) which has the same extremal Plücker coordinates as \( F \). Then, using Theorem 3.12, one may conclude that \( F \) itself lies in \( R_{v,w}^{>0} \), completing the proof of Theorem 3.14.

4 Tropicalizing the Complete Flag Variety

We now discuss tropical varieties and introduce the precise definitions of the TNN tropical complete flag variety and the TNN complete flag Dressian.
Definition 4.1. Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( b = (b_1, \ldots, b_n) \in \mathbb{N}^n \). We will use the notation \( x^b = x_1^{b_1} \cdots x_n^{b_n} \). Let \( p = \sum_i a_i x^b_i \) be a polynomial, where each \( a_i > 0 \) and each \( b_i \in \mathbb{N}^n \). We define the tropicalization of \( p \) by \( \text{trop}(p) = \min \{ a_i + x \cdot b_i \} \). We say that a point \( y \in \mathbb{T}^n := (\mathbb{R} \cup \infty)^n \) is a solution of the tropicalization of \( p \) if

\[
\min_i \{ a_i + y \cdot b_i \} = \min_i \{ a_i + y_1(b_1)_1 + \cdots + y_n(b_n)_n \}
\]

is achieved at least twice. We further say that a point in \( \mathbb{T}^n \) is a positive solution of the tropicalization of \( p \) if additionally, at least one of the minima comes from a term with a \(+\) sign, and at least one of the minima comes from a term with a \(-\) sign. Equivalently, if we rewrite \( p = 0 \) in the form \( \sum_j c_j x^{b_j} = \sum_i a_i x^{b_i} \) with all \( c_j \) and \( a_i \) positive, then we want at least one minimum to occur in a term coming from each side of the equality.

The tropical objects we are interested in will live in projective tropical spaces, which are spaces that interact nicely with homogeneous polynomials.

Definition 4.2. Projective tropical space. denoted \( \mathbb{TP}^n \) is given by \( (\mathbb{T}^{n+1} \setminus (\infty, \cdots, \infty)) / \sim \) where the equivalence relation is \( x \sim y \) if there exists \( c \in \mathbb{R} \) such that \( x_i = y_i + c \) for all \( i \in [n] \).

The following is immediate from the definition:

Proposition 4.3. If \( p \) is a homogeneous polynomial, then \( x \) is a (positive) solution of \( \text{trop}(p) \) if and only if \( y \) is a (positive) solution of \( \text{trop}(p) \) for all \( y \sim x \).

Definition 4.4. Given a set of multi-homogeneous polynomials \( P \), each of which is homogeneous with respect to sets of variables of sizes \( \{n_i\}_{i=1}^t \), and the ideal \( I \) which they generate, we define the following sets in \( \mathbb{TP}^{n_1-1} \times \cdots \times \mathbb{TP}^{n_t-1} \):

- The tropical prevariety \( \text{trop}(P) \) or \( \text{trop}(I) \) is the set of simultaneous solutions to the tropicalizations of all the polynomials in \( P \) or in \( I \), respectively.

- The non-negative tropical prevariety, \( \overline{\text{trop}^\geq 0}(P) \) or \( \overline{\text{trop}^\geq 0}(I) \), is the set of simultaneous positive solutions of the tropicalizations of all the polynomials in \( P \) or in \( I \), respectively.

Solutions of tropicalizations of polynomials can alternatively be described in a way that more clearly explains the term “positive solution”. Let \( \mathcal{C} = \bigcup_{n=1}^\infty \mathbb{C}(t^{1/n}) \) be the field of Puiseux series over \( \mathbb{C} \). A Puiseux series \( p(t) \in \mathcal{C} \) has a term with a lowest exponent, say \( at^u \) with \( a \in \mathbb{C}^* \) and \( u \in \mathbb{Q} \). In this case, we define \( \text{val}(p(t)) = u \). Also, we will define the semifield \( \mathcal{R}^+ \) to be the set of \( p(t) \) in \( \mathcal{C} \) where the coefficient of \( t^{\text{val}(p(t))} \) is in \( \mathbb{R}^+ \). In fact, \( \mathcal{R}^+ \) and \( \mathcal{C} \) can be thought of as analogous to \( \mathbb{R}^+ \) and \( \mathbb{C} \), respectively. Given an ideal \( I \subseteq \mathbb{C}[x_1, \ldots, x_n] \), let \( V(I) \subseteq \mathbb{C}^n \) be the variety where all polynomials in \( I \) vanish. We define the positive part of this variety to be \( V^+(I) = V(I) \cap (\mathcal{R}^+)^n \).

Proposition 4.5 ([17] Theorem 2.1 and [18] Proposition 2.2). Let \( I \) be an ideal of \( \mathbb{C}[x_1, \ldots, x_n] \). Then \( \text{trop}(I) = \text{val}(V(I)) \) and \( \overline{\text{trop}^\geq 0}(I) = \text{val}(V^+(I)) \), where \( \text{val}(V(I)) \) and \( \text{val}(V^+(I)) \) are the closures of \( \text{val}(V(I)) \) and \( \text{val}(V^+(I)) \), respectively.

Having introduced \( F_{\mathbb{T}} \), we now define two tropical analogues of this space along with their totally non-negative parts. Recall that \( \mathcal{I}_{IP,n} \) is the set of incidence-Plücker relations and \( I_{IP,n} \) is the ideal generated by those relations.

Definition 4.6. We define the tropical complete flag variety to be \( \text{trFl}_n = \overline{\text{trop}(I_{IP,n})} \) and the totally non-negative tropical complete flag variety to be \( \text{trFl}^\geq 0_n = \overline{\text{trop}^\geq 0(I_{IP,n})} \). We define the complete flag Dressian to be \( \text{FIDr}_n = \overline{\text{trop}(\mathcal{I}_{IP,n})} \) and the totally non-negative complete flag Dressian to be \( \text{FIDr}^\geq 0_n = \overline{\text{trop}^\geq 0(\mathcal{I}_{IP,n})} \).

Theorem 4.11 will show that \( \text{trFl}^\geq 0_n \) and \( \text{FIDr}^\geq 0_n \) coincide. Note that this is not obvious, since a point in \( \text{trFl}^\geq 0_n \) satisfies more relations than a point in \( \text{FIDr}^\geq 0_n \). In fact, in general, the tropical prevariety of a collection of polynomials will properly contain the tropical prevariety of the ideal those polynomials generate. In the specific case of the complete flag variety, it is shown in [3, Example 5.2.4] that for \( n \geq 6 \), \( \text{FIDr}_n \) properly contains \( \text{trFl}_n \). Before getting to Theorem 4.11, we need to discuss one other type of Dressian.
Definition 4.7. The support of \( P \in \mathbb{TP}_{1}^{(1)} \times \cdots \times \mathbb{TP}_{n-1}^{(n-1)} \) is the set \( \{ I \subset [n] | P_I \neq \infty \} \).

Definition 4.8. Let \( P_{1,3,\text{term}}^{n} \) be the set of incidence-Plücker relations with precisely three terms. We denote by \( \text{FLDr}^{3M} \) the subset of \( \text{trop}(P_{1,3,\text{term}}^{n}) \) consisting of points whose supports form a flag matroid. We call this the \textbf{three-term complete flag Dressian}. Similarly, we denote by \( (\text{FLDr}^{3M}_{n})^{>0} \) the subset of \( \text{trop}^{>0}(P_{1,3,\text{term}}^{n}) \) consisting of points whose supports form a flag matroid, which we call the \textbf{totally non-negative three-term complete flag Dressian}.

We can write down the following complete flag version of the fact that the Dressian is cut out by tropical three term Plücker relations [14, Theorem 5.2.25].

Theorem 4.9. The sets \( \text{FLDr}^{3M} \) and \( \text{FLDr}_{n} \) are equal.

We now shift our attention to the non-negative parts of the tropical varieties we have introduced. For \( v \leq w \) in the Bruhat order with \( r = \ell(w) - \ell(v) \), let \( \Phi_{v,w} : \mathbb{R}_{>0} \times \mathbb{R}^{r}_{>0} \to \mathbb{RP}_{1}^{(1)} \times \cdots \times \mathbb{RP}_{n-1}^{(n-1)} \) be the map which takes a collection of \( a \in \mathbb{R}_{>0}^{r} \) to the Plücker coordinates of the matrix \( M_{v,w}(a) \), in the notation of Theorem 3.5. Note that by construction, this map consists of a collection of polynomials in the \( a_{ij} \), and so we can tropicalize this map, obtaining a map \( \text{Trop} \Phi : \mathbb{R}^{r} \to \mathbb{TP}_{1}^{(1)} \times \cdots \times \mathbb{TP}_{n-1}^{(n-1)} \). The following corollary can be deduced from Theorem 3.14 with little extra work, and then put to use to help prove our second main result.

Corollary 4.10. Every point in \( (\text{FLDr}^{3M}_{n})^{>0} \) lies in the image of \( \text{Trop} \Phi_{v,w} \) for some \( v \leq w \).

Theorem 4.11. The following sets are equal:

1. The TNN topical flag variety \( \text{trFL}^{>0}_{n} \),
2. The TNN three-term complete flag Dressian \( (\text{FLDr}^{3M}_{n})^{>0} \),
3. The TNN complete flag Dressian \( \text{FLDr}^{>0}_{n} \).

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