ON SMALL SETS OF INTEGERS

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Abstract. An upper quasi-density on \( H \) (the integers or the non-negative integers) is a real-valued subadditive function \( \mu^* \) defined on the whole power set of \( H \) such that \( \mu^*(X) \leq \mu^*(H) = 1 \) and \( \mu^*(k \cdot X + h) = \frac{1}{k} \mu^*(X) \) for all \( X \subseteq H \), \( k \in \mathbb{N}^+ \), and \( h \in \mathbb{N} \), where \( k \cdot X := \{ kx : x \in X \} \); and an upper density on \( H \) is an upper quasi-density on \( H \) that is non-decreasing with respect to inclusion. We say that a set \( X \subseteq H \) is small if \( \mu^*(X) = 0 \) for every upper quasi-density \( \mu^* \) on \( H \).

Main examples of upper densities are given by the upper analytic, upper Banach, upper Buck, and upper Pólya densities, along with the uncountable family of upper \( \alpha \)-densities, where \( \alpha \) is a real parameter \( \geq -1 \) (most notably, \( \alpha = -1 \) corresponds to the upper logarithmic density, and \( \alpha = 0 \) to the upper asymptotic density).

It turns out that a subset of \( H \) is small if and only if it belongs to the zero set of the upper Buck density on \( \mathbb{Z} \). This allows us to show that many interesting sets are small, including the integers with less than a fixed number of prime factors, counted with multiplicity; the numbers represented by a binary quadratic form with integer coefficients whose discriminant is not a perfect square; and the image of \( \mathbb{Z} \) through a non-linear integral polynomial in one variable.

1. Introduction

It is not infrequently the case in number theory and other fields that one is faced with the problem of determining whether a given set of integers is “small” in a suitable sense, driven by the idea that “largeness implies structure” and with the proviso that if a set is “small” then its complement is “large”. In such cases, one is not necessarily interested in “quantifying the largeness” of a set \( X \), but only in establishing whether \( X \) is large. A classic approach is to let the collection of all “small sets” be an ideal, i.e., a family \( \mathcal{I} \) of proper subsets of a certain “ambient set” such that \( \mathcal{I} \) is closed under taking subsets and finite unions. In fact, many natural ideals on \( \mathbb{N} \) can be represented as the inverse image of 0 through an appropriate “measure of largeness” \( f : \mathcal{P}(\mathbb{N}) \to \mathbb{R} \), cf. [16] and see § 1.1 for notation. This has eventually led to the study of a diversity of set functions that, while retaining fundamental features of measures, are better suited than measures to

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certain applications; some of these “surrogate measures”, recently considered in [10, 11] and called upper quasi-densities (Definition 2.1), are also the subject of the present work.

In detail, the plan of the paper is as follows. In §2, we introduce the notion of “small set” specifically used in the present work (Definition 2.3) and characterize small sets in terms of the zero set of the upper Buck density (Theorem 2.4), a little known “upper density”, first considered in [4], that happens to play a central role in the theory. Then, in §§3 and 4, we prove that the following sets are small: the integers with less than a fixed number of prime factors, counted with multiplicity (Corollary 3.4); the non-negative integers whose base-b representation does not contain a given non-empty string of digits (Corollary 3.7); the image of Z through a non-linear integer polynomial in one variable (Theorem 3.9); the inverse image of the primes under a non-constant integral polynomial in one variable (Theorem 3.10); the numbers represented by a binary quadratic form with integer coefficients whose discriminant is not a perfect square (Theorem 4.2). We conclude in §5 with some questions and remarks.

All in all, we can thus extend and unify several “independent results” so far only known for some of the classic “upper densities” encountered in the literature. In particular, Theorem 4.2 generalizes the well-known fact that the asymptotic density of the non-negative integers representable by a sum of two squares is zero.

1.1. Generalities. We refer to [10] for most notation, terminology, and conventions used through this paper. In particular, we write R, Z, and N, resp., for the sets of reals, integers, and non-negative integers; and unless noted otherwise, we reserve the letters h, i, and k (with or without subscripts) for non-negative integers, the letters m and n for positive integers, and the letter s for a real number.

We let P ⊆ Z be the set of (positive or negative) primes, and for all k ∈ Z and m ∈ N+ we denote by k mod m the smallest non-negative integer r such that k ≡ r mod m. For all a, b ∈ R ∪ {±∞}, we take [a, b] := [a, b] ∩ Z; and for every X ⊆ R and h, k ∈ R, we define X+ := X ∩ [0, ∞] and k · X + h := {kx + h : x ∈ X}.

We let H be either the integers or the non-negative integers, and we use P(H) for the power set of H. We regard H as a “parameter”; though it makes no big difference to stick to the assumption that H = N most of the time, some statements (e.g., Theorems 2.4 and 4.2) will be sensitive to the actual choice of H. Accordingly, we take an arithmetic progression of H to be a set of the form k · H + h with k ∈ N+ and h ∈ N; and we denote by H the family of all finite unions of arithmetic progressions of H.

For each k ∈ N+ and X ⊆ Z, we write rk(X) for the number of residues h ∈ [0, k − 1] such that X ∩ (k · Z + h) ≠ ∅. Note that, if X ⊆ N then rk(X) is also equal to the number of residues h ∈ [0, k − 1] such that X ∩ (k · N + h) ≠ ∅.
2. Upper quasi-densities and small sets

We start with recalling the notion of “density” that will be used through this work.

**Definition 2.1.** A function $\mu^* : \mathcal{P}(H) \to \mathbb{R}$ is an upper density (on $H$) if it is

1. **normalized**, i.e., $\mu^*(H) = 1$;
2. **monotone**, i.e., $\mu^*(X) \leq \mu^*(Y)$ for all $X, Y \subseteq H$ with $X \subseteq Y$;
3. **subadditive**, i.e., $\mu^*(X \cup Y) \leq \mu^*(X) + \mu^*(Y)$ for every $X, Y \subseteq H$;
4. **$(-1)$-homogeneous**, i.e., $\mu^*(k \cdot X) = \frac{1}{k} \mu^*(X)$ for all $X \subseteq H$ and $k \in \mathbb{N}^+$;
5. **shift-invariant**, i.e., $\mu^*(X + h) = \mu^*(X)$ for every $X \subseteq H$ and $h \in \mathbb{N}$.

In addition, we call $\mu^*$ an upper quasi-density (on $H$) if $\mu^*(X) \leq 1$ for all $X \subseteq H$ and $\mu^*$ satisfies (F1) and (F3)-(F5).

Every upper density is obviously an upper quasi-density, and the existence of non-monotone upper quasi-densities is guaranteed by [10, Theorem 1]. While it is arguable that non-monotone “densities” are not very interesting from the point of view of applications, it seems meaningful to establish if certain properties of a specific class of objects depend or not on a particular assumption (in the present case of interest, the axiom of monotonicity): This usually contributes to a better understanding of the objects under consideration and is basically our motivation for dealing with upper quasi-densities instead of restricting attention to upper densities.

**Remark 2.2.** It turns out that each of the following set functions is an upper density in the sense of Definition 2.1:

- the **upper $\alpha$-density** (on $H$), that is, the function
  $$\mathcal{P}(H) \to \mathbb{R} : X \mapsto \lim_{n \to \infty} \sup \frac{\sum_{i \in X \cap \llbracket 1, n \rrbracket} i^\alpha}{\sum_{i \in \llbracket 1, n \rrbracket} i^\alpha},$$
  where $\alpha$ is a real parameter $\geq -1$ (most notably, $\alpha = -1$ corresponds to the upper logarithmic density, and $\alpha = 0$ to the upper asymptotic density);
- the **upper Banach (or upper uniform) density**, that is, the function
  $$\mathcal{P}(H) \to \mathbb{R} : X \mapsto \lim_{n \to \infty} \max_{k \geq 0} \frac{|X \cap [k + 1, k + n]|}{n};$$
- the **upper analytic density**, that is, the function
  $$\mathcal{P}(H) \to \mathbb{R} : X \mapsto \lim_{s \to 1^+} \lim_{n \to \infty} \frac{1}{\zeta(s)} \sum_{i \in X^+} \frac{1}{i^s},$$
  where $\zeta$ is the restriction to the interval $[1, \infty[$ of the Riemann zeta function;
- the **upper Pólya density**, that is, the function
  $$\mathcal{P}(H) \to \mathbb{R} : X \mapsto \lim_{s \to 1^-} \lim_{n \to \infty} \frac{|X \cap [1, n]| - |X \cap [1, ns]|}{(1 - s)n};$$
• the upper Buck density, that is, the function

\[ b_\mathcal{H}^* : \mathcal{P} (\mathcal{H}) \to \mathbb{R} : X \mapsto \inf_{A \in \mathcal{A}_\mathcal{H} : X \subseteq A} d_\mathcal{H}^*(A), \]

where \( d_\mathcal{H}^* \) is the upper asymptotic density on \( \mathcal{H} \) (see the first bullet on this list).

We refer the reader to [7] for the existence of the limit in the above definition of the upper Banach density, and to [15, Satz III, p. 559] for the existence of the limit in the above definition of the upper Pólya density. For further details, see [10, § 2 and Examples 4–6 and 8].

As mentioned in the introduction, “densities” are mainly a technical device to formalize the idea that a set is, or is not, “small”. This leads straight to the next definition.

**Definition 2.3.** We say that a subset \( X \) of \( \mathcal{H} \) is small if \( \mu^*(X) = 0 \) for every upper quasi-density \( \mu^* \) on \( \mathcal{H} \).

Throughout, we will often use the following characterization of small sets, which is basically a corollary of some of the main results of [10, §§ 4 and 6].

**Theorem 2.4.** Let \( X \) be a subset of \( \mathcal{H} \). Then \( X \) is small (as per Definition 2.3) if and only if \( b_\mathcal{H}^*(X) = 0 \), if and only if \( b_\mathcal{Z}^*(X) = 0 \) (see Remark 2.2 for the notation).

**Proof.** We know from [10, Proposition 2(vi) and Theorem 3] that \( 0 \leq \mu^*(X) \leq b_\mathcal{H}^*(X) \) for every upper quasi-density \( \mu^* \) on \( \mathcal{H} \). Hence, \( X \) is small if and only if \( b_\mathcal{H}^*(X) = 0 \). To complete the proof, it is therefore enough to show that \( b_\mathcal{H}^*(X) = b_\mathcal{Z}^*(X) \).

To this end, note that \( d_\mathcal{Z}^*(S) = d_\mathcal{Z}^*(S \cap \mathbb{N}) = d_\mathbb{N}^*(S \cap \mathbb{N}) \) for every \( S \subseteq \mathbb{Z} \). On the other hand, a set \( A \in \mathcal{A}_\mathbb{N} \) containing a subset \( Y \) of \( \mathbb{N} \) extends, in an obvious way, to a set \( A' \in \mathcal{A}_Z \) with \( Y \subseteq A' \) and \( d_\mathbb{N}^*(A) = d_\mathcal{Z}^*(A') \); and conversely, if a set \( B \in \mathcal{A}_Z \) contains a subset \( Y \) of \( \mathbb{N} \), then it is clear that

\[ Y \subseteq B \cap \mathbb{N}, \quad B \cap \mathbb{N} \in \mathcal{A}_\mathbb{N}, \quad \text{and} \quad d_\mathcal{Z}^*(B) = d_\mathbb{N}^*(B \cap \mathbb{N}). \]

Stitching all the pieces together, we thus conclude that

\[ b_\mathcal{H}^*(X) = \inf_{A \in \mathcal{A}_\mathcal{H} : X \subseteq A} d_\mathcal{H}(A) = \inf_{B \in \mathcal{A}_\mathcal{Z} : X \subseteq B} d_\mathcal{Z}^*(B) = b_\mathcal{Z}^*(X). \]

And this finishes the proof. \( \blacksquare \)

As a consequence of Theorem 2.4, the property of being small is independent of the choice of \( \mathcal{H} \). In addition, since the upper Buck density on \( \mathbb{Z} \) is monotone and subadditive (as is true for any upper density), we obtain:

**Corollary 2.5.** The family of small subsets of \( \mathcal{H} \) is an ideal on \( \mathcal{H} \). In particular, every subset of a small set is small.
Notice that Corollary 2.5 is not obvious a priori, since it is unknown whether the zero set of an upper quasi-density on \( H \) is closed under taking subsets, cf. [10, Question 5].

With this said, we are going to derive an “explicit formula” for (a certain generalization of) the upper Buck density that is perhaps of independent interest in light of Theorem 2.4 and the role played by the upper Buck density in the present work (cf. [14, Theorem 1] for a weaker result along the same lines). The reader may want to review § 1.1 and [10, Example 5] before reading further.

**Proposition 2.6.** Let \( \mu^* \) be an upper quasi-density on \( H \) and \( b^*(\mathcal{A}_H; \mu^*) \) the function

\[
P(H) \to \mathbb{R} : X \mapsto \inf_{A \in \mathcal{A}_H : X \subseteq A} \mu^*(A).
\]

Moreover, let \( X \) be a subset of \( H \) and \( (k_n)_{n \geq 1} \) an increasing sequence of positive integers such that, for every \( m \in \mathbb{N}^+ \), \( k_n \) is divisible by \( m \) for all large \( n \in \mathbb{N}^+ \). Then

\[
b^*(\mathcal{A}_H; \mu^*)(X) = b^*_H(X) = \inf_{k \geq 1} \frac{r_k(X)}{k} = \lim_{n \to \infty} \frac{r_{k_n}(X)}{k_n}.
\]

**Proof.** To begin, let \( A \in \mathcal{A}_H \) and suppose \( X \subseteq A \). By definition, \( A = \bigcup_{h \in \mathcal{H}} (k \cdot H + h) \) for some \( k \in \mathbb{N}^+ \) and \( \mathcal{H} \subseteq [0, k - 1] \). Accordingly, set

\[
\mathcal{H}' := \{ h \in \mathcal{H} : X \cap (k \cdot H + h) \neq \emptyset \} \quad \text{and} \quad A' := \bigcup_{h \in \mathcal{H}'} (k \cdot H + h).
\]

Then \( X \subseteq A' \subseteq A \) and \( A' \in \mathcal{A}_H \), and it is clear that \( r_k(X) = |\mathcal{H}'| \). Hence, we obtain from [10, Propositions 7 and 9] that

\[
\frac{r_k(X)}{k} = \frac{|\mathcal{H}'|}{k} = \mu^*(A') \leq \mu^*(A).
\]

Conversely, let \( k \in \mathbb{N}^+ \), and take \( A := \bigcup_{h \in \mathcal{H}} (k \cdot H + h) \), where \( \mathcal{H} \) denotes the set of all residues \( h \in [0, k - 1] \) for which \( X \cap (k \cdot H + h) \) is non-empty. Then \( A \in \mathcal{A}_H \) and \( X \subseteq A \), and similarly as in the previous paragraph, \( r_k(X) = |\mathcal{H}| \) and \( \mu^*(A) = r_k(X)/k \).

So putting it all together, we can readily conclude from the above that

\[
b^*(\mathcal{A}_H; \mu^*)(X) = \inf_{k \geq 1} \frac{r_k(X)}{k}.
\]

On the other hand, it is straightforward that \( r_k(X)/k \leq r_h(X)/h \) for all \( h, k \in \mathbb{N}^+ \) such that \( h \mid k \). Therefore, it follows from our assumptions that, for every \( k \in \mathbb{N}^+ \), the inequality \( r_{k_n}(X)/k_n \leq r_k(X)/k \) holds true for all but finitely many \( n \in \mathbb{N}^+ \); and this implies by (1) that \( r_{k_n}(X)/k_n \to b^*(\mathcal{A}_H; \mu^*)(X) \) as \( n \to \infty \).

The proof is thus complete, because the preceding conclusions are independent from the choice of the upper quasi-density \( \mu^* \), and letting \( \mu^* \) be the upper asymptotic density on \( H \) yields by Theorem 2.4 that \( b^*(\mathcal{A}_H; \mu^*)(X) = b^*_H(X) \).
3. Criteria for smallness and examples

Given an upper [quasi]-density \( \mu^* \) on \( H \), we follow [10] and refer to the function

\[
\mu_* : \mathcal{P}(H) \to \mathbb{R} : X \mapsto 1 - \mu^*(H \setminus X)
\]

as the lower [quasi]-density on \( H \) conjugate to \( \mu^* \), or simply as the conjugate of \( \mu^* \). We list below some basic properties of upper and lower [quasi]-densities.

**Lemma 3.1.** Let \( \mu^* \) be an upper quasi-density on \( H \) with conjugate \( \mu_* \), and let \( X \) be a subset of \( H \). The following hold:

(i) \( \mu^*(\mathcal{P}(H)) = \mu_*(\mathcal{P}(H)) = [0, 1] \) and \( \mu_*(X) \leq \mu^*(X) \).

(ii) If \( X \) is finite, then \( \mu^*(X) = 0 \).

(iii) If \( k \cdot H + h \subseteq X \) for some \( k \in \mathbb{N}^+ \) and \( h \in \mathbb{N} \), then \( \mu^*(X) \geq \frac{1}{k} \). Symmetrically, if \( X \subseteq k \cdot H + h \) then \( \mu^*(X) \leq \frac{1}{k} \).

**Proof.** See [10, Theorem 2, Propositions 2(vi) and 6, and Corollary 2].

The next result extends a criterion used in [4, § 3, p. 563] to demonstrate that the upper Buck density of the set of squares is zero; see, in particular, the corollary to [4, Theorem 3].

**Proposition 3.2.** Let \( \mu^* \) be an upper density on \( H \) with conjugate \( \mu_* \), and for every \( k \in \mathbb{N}^+ \) and \( S \subseteq H \) denote by \( w_k(S) \) the cardinality of the set

\[
\mathcal{W}_k(S) := \{ h \in [0, k - 1] : \mu^*(S \cap (k \cdot H + h)) \neq 0 \}.
\]

Next, let \( X \) be a subset of \( H \), and let \( (k_n)_{n \geq 1} \) be a sequence of pairwise coprime positive integers. The following hold:

(i) \( w_k(X) \leq r_k(X) \) for all \( k \in \mathbb{N}^+ \), and for every \( n \in \mathbb{N}^+ \) we have

\[
\prod_{i=1}^{n} \left(1 - \frac{w_{k_i}(H \setminus X)}{k_i} \right) \leq \mu_*(X) \leq \mu^*(X) \leq \prod_{i=1}^{n} \frac{w_{k_i}(X)}{k_i}.
\]

(ii) If \( \sum_{n=1}^{\infty} (1 - w_{k_n}(X)/k_n) = \infty \), then \( \mu^*(X) = 0 \). In particular, if \( \sum_{n=1}^{\infty} k_n^{-1} = \infty \) and \( w_{k_n}(X) \leq k_n - 1 \) for all \( n \in \mathbb{N}^+ \), then \( \mu^*(X) = 0 \).

(iii) If \( \sum_{n=1}^{\infty} k_n^{-1} = \infty \) and \( r_{k_n}(X) \leq k_n - 1 \) for all \( n \in \mathbb{N}^+ \), or more generally if \( \sum_{n=1}^{\infty} (1 - r_{k_n}(X)/k_n) = \infty \), then \( X \) is small.

**Proof.** (i) The first inequality is obvious. For the other, notice that the function \( \mathbb{N}^+ \to \mathbb{N} : q \to w_q(X) \) is submultiplicative, that is, \( w_{mn}(X) \leq w_m(X)w_n(X) \) for all \( m, n \in \mathbb{N}^+ \) with \( \gcd(m, n) = 1 \): This is so because, for all \( m, n \in \mathbb{N}^+ \), the function

\[
\mathcal{W}_{mn}(X) \to \mathcal{W}_m(X) \times \mathcal{W}_n(X) : h \mapsto (h \mod m, h \mod n)
\]
is well defined (here is where we use that $\mu^*$ is monotone); and if, in addition, $m$ and $n$ are coprime, then the function is injective by the Chinese remainder theorem. Consequently, we get from [10, Proposition 11] that, for every $n \in \mathbb{N}^+$,

$$\mu^*(X) \leq \frac{w_{k_1 \cdots k_n}(X)}{k_1 \cdots k_n} \leq \prod_{i=1}^{n} \frac{w_{k_i}(X)}{k_i}$$

and

$$\mu_*(X) \geq 1 - \frac{w_{k_1 \cdots k_n}(\mathbb{H} \setminus X)}{k_1 \cdots k_n} \geq 1 - \prod_{i=1}^{n} \frac{w_{k_i}(\mathbb{H} \setminus X)}{k_i} \geq \prod_{i=1}^{n} \left(1 - \frac{w_{k_i}(\mathbb{H} \setminus X)}{k_i}\right),$$

where we have used that $1 - a_1 \cdots a_n \geq 1 - a_1 \geq (1 - a_1) \cdots (1 - a_n)$ for all $a_1, \ldots, a_n \in [0, 1]$. By Lemma 3.1(i), this suffices to finish the proof.

(ii) If $w_{k_n}(X) = 0$ for some $n \in \mathbb{N}^+$, the claim follows at once from (2). Otherwise, $1 \leq w_{k_n}(X) \leq k_n$ for all $n \in \mathbb{N}^+$. So, recalling that

$$\log x \leq -(1 - x), \quad \text{for } x \in [0, 1],$$

we find that, for every $n \in \mathbb{N}^+$,

$$\prod_{i=1}^{n} \frac{w_{k_i}(X)}{k_i} = \exp \left(\sum_{i=1}^{n} \log \frac{w_{k_i}(X)}{k_i}\right) \leq \exp \left(-\sum_{i=1}^{n} \frac{1 - w_{k_i}(X)}{k_i}\right). \quad (3)$$

But the right-most side of (3) tends to 0 as $n \to \infty$, since we are assuming $\sum_{n=1}^{\infty} (1 - w_{k_n}(X)/k_n) = \infty$; and this, together with part (i), implies $\mu^*(X) = 0$.

The rest is straightforward, because if $w_{k_n}(X) \leq k_n - 1$ for every $n \in \mathbb{N}^+$, then it is clear that $\sum_{n=1}^{\infty} (1 - w_{k_n}(X)/k_n) \geq \sum_{n=1}^{\infty} k_n^{-1} = \infty$.

(iii) This is immediate from parts (i) and (ii) and the arbitrariness of $\mu^*$.

We continue with a common generalization of [13, Corollary 2] and [3, Lemma 2].

**Proposition 3.3.** Let $X \subseteq \mathbb{H}$, and let $\mu^*$ be an upper density on $\mathbb{H}$. Moreover, assume that $(k_n)_{n \geq 1}$ is a sequence of pairwise coprime positive integers such that $\sum_{n=1}^{\infty} k_n^{-1} = \infty$, and for each $n \in \mathbb{N}^+$ set $X_n := \{x \in X : k_n \mid x\}$. If there are only finitely many $n \in \mathbb{N}^+$ for which $\mu^*(X_n) > 0$, then $\mu^*(X) = 0$; in particular, the set $\{k_n : n \in \mathbb{N}^+\}$ is small.

**Proof.** By hypothesis, the series $\sum_{n=1}^{\infty} k_n^{-1}$ diverges to $\infty$ and the set

$$I := \{0\} \cup \{n \in \mathbb{N}^+ : \mu^*(X_n) > 0\}$$

is finite (and non-empty). Hence, $\sum_{n=n_0}^{\infty} k_n^{-1} = \infty$ and $w_{k_n}(X) \leq k_n - 1$ for all $n \geq n_0$, where $n_0 := 1 + \max I$ and $w_{k_n}(X)$ is the number of residues $h \in [0, k_n - 1]$ such that $\mu^*(X \cap (k \cdot \mathbb{H} + h)) \neq 0$. By Proposition 3.2(ii), it follows that $\mu^*(X) = 0$.

In particular, if $X$ is the set $\{k_n : n \in \mathbb{N}^+\}$, then it is clear that $X_n = \{k_n\}$ for all $n \in \mathbb{N}^+$, and we conclude from Lemma 3.1(iii) and the above that $\mu^*(X) = 0$, which implies that $X$ is small since we may take $\mu^*$ to be the upper Buck density on $\mathbb{H}$. ■
We will repeatedly resort to Propositions 3.2 and 3.3 to show that various sets of integers are small. We begin with a generalization of [3, Theorems 1–3] and [4, Theorem 1]; then we proceed to prove a result on the “density” of a factorial-like sequence (cf. Remark 3.8).

**Corollary 3.4.** For every \( k \in \mathbb{N} \), the set \( X^{(k)} \) (resp., \( Y^{(k)} \)) of all integers \( x \in \mathbb{H} \) that factor into a product of exactly \( k \) (resp., at most \( k \)) primes (counted with multiplicity), is small.

**Proof.** Fix \( k \in \mathbb{N} \). By subadditivity, it suffices to show that \( X^{(k)} \) is small, since we have that \( Y^{(k)} = X^{(0)} \cup \cdots \cup X^{(k)} \). We argue by induction on \( k \).

In light of Lemma 3.1(ii), the claim is trivial if \( k = 0 \), because \( X^{(0)} = \mathbb{H} \cap \{ \pm 1 \} \).

Accordingly, assume that \( k \) is a positive integer and \( X^{(k-1)} \) is small, and let \( p \in \mathbb{P}^+ \).

Then it is clear that \( X^{(k)}_p := \{ x \in X^{(k)} : p \mid x \} = p \cdot X^{(k-1)} \), and this implies by \((f4)\) and the inductive hypothesis that \( \mu^*(X^{(k)}_p) = 0 \) for every upper quasi-density \( \mu^* \) on \( \mathbb{H} \). Therefore, we can conclude from Proposition 3.3, applied with \( k_n \) equal to the \( n \)-th prime of \( \mathbb{N}^+ \), that also \( X^{(k)} \) is small, since it is well known (see, e.g., [1, Theorem 1.13]) that \( \sum_{p \in \mathbb{P}^+} \frac{1}{p} = \infty \). ■

**Corollary 3.5.** Let \((x_n)_{n \geq 1}\) be a sequence in \( \mathbb{H} \) with the property that \( x_n \) divides \( x_{n+1} \) for each \( n \.

Then the set \( X := \{ x_n : n \in \mathbb{N}^+ \} \) is small.

**Proof.** By Lemma 3.1(ii), we can assume without loss of generality that the sequence \((x_n)_{n \geq 1}\) consists of pairwise distinct elements and \( |x_n| \leq |x_{n+1}| \) for all \( n \in \mathbb{N}^+ \). In particular, this ensures that \( x_1 \neq 0 \) and \( |x_{2n}| < |x_{2n+2}| \) for every \( n \).

Then \( r_{|x_{2n}|}(X) \leq 2n \) for every \( n \), since \( x_h \mid x_k \) for all \( h, k \in \mathbb{N}^+ \) such that \( h \mid k \). On the other hand, it is easy to verify (by induction) that \( |x_{2n}| \geq 2^{n-1} \) for all \( n \). So, we obtain from Proposition 3.2(i) that

\[
\mu^*(X) \leq \inf_{n \geq 1} \frac{r_{|x_{2n}|}(X)}{|x_{2n}|} \leq \liminf_{n \to \infty} \frac{2n}{2^{n-1}} = 0,
\]

for every upper quasi-density \( \mu^* \) on \( \mathbb{H} \). In other terms, \( X \) is small. ■

In particular, it follows from Corollaries 2.5 and 3.5 that a set \( X \subseteq \mathbb{H} \), whose elements are factorials, primorials, or numbers of the form \( a^k \) for some fixed base \( a \in \mathbb{H} \), is small. This is further strengthened by the next result, which is also a generalization of the unnumbered corollary after Theorem 3 in [4, p. 565].

**Corollary 3.6.** The set \( X := \bigcup_{n \geq 2} \{ a^n : a \in \mathbb{H} \} \) is small.
Proof. Let \( p \in \mathbb{P}^+ \) and pick an element \( x \in X \). It is clear that \( p \mid x \) if and only if \( p^2 \mid x \). Hence, \( r_{p^2}(X) \leq p^2 - p + 1 \leq p^2 - \frac{1}{2}p \). It follows (cf. Corollary 3.4) that
\[
\sum_{p \in \mathbb{P}^+} \left( 1 - \frac{r_{p^2}(X)}{p^2} \right) \geq \frac{1}{2} \sum_{p \in \mathbb{P}^+} \frac{1}{p} = \infty.
\]
So we can conclude from Proposition 3.2(iii), applied with \( k_n \) equal to the square of the \( n \)-th prime of \( \mathbb{N}^+ \), that \( X \) is small.

We conclude our series of corollaries with a result on “digit representations”, herein regarded as words in the free monoid over \([0, b - 1]\) for a fixed base \( b \geq 2 \).

**Corollary 3.7.** Given \( b \in \mathbb{N}_{\geq 2} \), let \( s = (s_1, \ldots, s_k) \) be a non-empty sequence of length \( k \in \mathbb{N}^+ \) with entries in \([0, b - 1]\). Then the set \( X \) of all \( x \in H \) which do not have the word \( s_1 \cdots s_k \) appearing in their base-\( b \) representation, is small.

**Proof.** Let \( n \in \mathbb{N}^+ \). Obviously, \( r_{p^{nk}}(X) \) is bounded above by the number of residues \( h \in [0, b^{nk} - 1] \) whose base-\( b \) representation does not contain the word \( s_1 \cdots s_k \), or equivalently by the number of sequences \((a_0, \ldots, a_{nk - 1}) \in [0, b - 1]^{nk}\) with \((a_i, \ldots, a_{i+k-1}) \neq s\) for every \( i \in [0, k - 1] \). It follows \( r_{p^{nk}} \leq (b^k - 1)^n \); whence \( r_{p^{nk}}(X) / b^{nk} \to 0 \) as \( n \to \infty \). So, by Proposition 3.2(i), \( X \) is small.

**Remark 3.8.** Based on the previous results, one might be drawn to think that every “sufficiently sparse” set of integers is small. However, we have from Proposition 2.6 that the upper Buck density of the set \( X := \{ h! + h : h \in \mathbb{N} \} \) equals 1 (no matter whether \( H = \mathbb{N} \) or \( H = \mathbb{Z} \)), as it is easily seen that \( r_k(X) = k \) for every \( k \in \mathbb{N}^+ \).

The next theorems are about integral polynomials in one variable; it could be interesting to extend them to more general classes of integer-valued functions (see § 4 for a first step in this direction).

**Theorem 3.9.** Let \( F : \mathbb{Z} \to \mathbb{Z} \) be a polynomial function with coefficients in \( H \). Then \( F(H) \) is small if and only if \( \deg F \neq 1 \).

**Proof.** If \( F \) is constant, then its image is small, by Lemma 3.1(ii). If, on the other hand, \( F \) is of degree 1, then there exist \( a, b \in H \) with \( a \neq 0 \) such that \( F(x) = ax + b \) for all \( x \in H \); and this in turn implies that, for every upper quasi-density \( \mu^* \) on \( H \),
\[
\mu^*(F(H)) = \mu^*(a \cdot H + b) = \frac{1}{|a|} > 0.
\]

Accordingly, assume hereafter that \( \deg F \geq 2 \). Then a well-known theorem of Frobenius (see, e.g., [17, p. 32]) ensures that the set \( P_F \) of primes \( p \in \mathbb{P}^+ \) for which \( F \) has at least two roots modulo \( p \), has non-zero Dirichlet density, meaning that the limit
\[
\lim_{s \to 1^+} \frac{\sum_{p \in P_F} 1/p^s}{\sum_{p \in \mathbb{P}^+} 1/p^s}
\]
exists and is positive. It follows (by the monotone convergence theorem for series) that
\[ \sum_{p \in P_F} 1/p = \infty, \]
because \( \sum_{p \in \mathbb{P}^+} 1/p = \infty \) (cf. Corollary 3.4); and since \( r_p(F(Z)) \leq p-1 \) for every \( p \in P_F \), we conclude from Proposition 3.2(iii) that \( F(H) \) is small. 

\[ \square \]

**Theorem 3.10.** Let \( F : \mathbb{Z} \to \mathbb{Z} \) be a non-constant polynomial function with integer coefficients. Then the set \( X := \{ k \in H : F(k) \in \mathbb{P} \} \) is small.

**Proof.** Let \( \mu^* \) be an upper quasi-density on \( H \), and for each \( n \in \mathbb{N}^+ \) denote by \( w_n(X) \) the number of residues \( h \in [0, k-1] \) such that \( \mu^*(X \cap (k \cdot H + h)) \neq 0 \).

Similarly as in the proof of Theorem 3.9, there is a set \( P_F \subseteq \mathbb{P}^+ \) such that \( \sum_{p \in P_F} 1/p = \infty \) and \( F \) has at least one zero modulo \( p \) for every \( p \in P_F \), that is, \( p \mid F(h_p) \) for some \( h_p \in [0, p-1] \). In particular, it follows that, for each \( p \in P_F \), the set \( X \cap (p \cdot H + h_p) \) is finite (otherwise, \( |F(pk + h_p)| = p \) for infinitely many \( k \in H \), in contradiction to the fact that \( F \) is non-constant), and hence, by Lemma 3.1(ii), \( w_p(X) \leq p - 1 \).

So, putting it all together, we get from Proposition 3.2(i) that \( \mu^*(X) = 0 \), and this is enough to show that \( X \) is small (since \( \mu^* \) was arbitrary). 

\[ \square \]

4. **Binary quadratic forms**

It is folklore that the asymptotic density of the set of integers that can be written as a sum of two squares is zero. In the present section, we generalize this to binary quadratic forms, while replacing the asymptotic density with an arbitrary upper quasi-density.

**Lemma 4.1.** Let \( d \) be an integer, but not a perfect square. Then there exist \( m \in \mathbb{N}^+ \) and \( r \in \mathbb{N} \) with \( \gcd(m, r) = 1 \) such that, for every prime \( p \in \mathbb{P}^+ \) with \( p \equiv r \mod m \), \( d \) is not a quadratic residue modulo \( p \).

**Proof.** Write \( d = 2^ku \varepsilon \), where \( t \) and \( u \) are odd positive integers, \( u \) is squarefree, \( k \) is a non-negative integer, and \( \varepsilon \) is the sign of \( d \) (i.e., \( \varepsilon = 1 \) if \( d \geq 1 \), and \( \varepsilon = -1 \) otherwise).

If \( u = 1 \), then it is sufficient to consider that, for every odd prime \( p \in \mathbb{P}^+ \), we have from [1, Theorems 9.9(a), 9.5, and 9.10] that

\[ \left( \frac{d}{p} \right) = \left( \frac{2^k}{p} \right) \left( \frac{\varepsilon}{p} \right) = (-1)^{\frac{k}{2}(p^2-1)} \varepsilon^{\frac{1}{2}(p-1)}, \]

where \( (\cdot) \) is a Jacobi symbol, see [1, § 9.7]; whence we can take \( p \equiv 5 \mod 8 \) if \( k \) is odd, and \( p \equiv 3 \mod 4 \) if \( k \) is even (note that, in the latter case, \( \varepsilon \) must be equal to \(-1\), or else \( d \) would be a perfect square).

Thus, assume from now on that \( u \geq 3 \). Then \( u = q_1 \cdots q_n \), where \( q_1, \ldots, q_n \in \mathbb{P}^+ \) are pairwise distinct odd prime numbers; and it follows by [1, Theorems 9.1] that there exist \( r_1, \ldots, r_n \in \mathbb{N} \) with

\[ \left( \frac{r_1}{q_1} \right) = -1 \quad \text{and} \quad \left( \frac{r_i}{q_i} \right) = 1 \quad \text{for each} \quad i \in [2, n]. \]
As a result, we conclude from the Chinese remainder theorem and [1, Theorem 9.9(c)] that there is \( r \in \mathbb{N} \) with the property that
\[
\left( \frac{r}{q_i} \right) = -1 \quad \text{and} \quad \left( \frac{r}{q_i} \right) = 1 \quad \text{for each} \quad i \in [2,n].
\]
So, letting \( s \in \mathbb{N} \) be such that \( 8s + 1 \equiv r \mod u \) (this is possible because \( u \) is odd), we have from [1, Theorem 9.9, parts (c) and (b)] that
\[
\left( \frac{8s + 1}{u} \right) = \left( \frac{r}{u} \right) = \prod_{i=1}^{n} \left( \frac{r}{q_i} \right) = -1,
\]
and this implies in particular that \( 8s + 1 \) and \( u \) are coprime.

Consequently, if \( p \equiv 8s + 1 \mod 8u \), then \( \gcd(2u,p) = 1 \) and \( p \equiv 1 \mod 8 \); and we get from [1, Theorems 9.5, 9.9(a), 9.9(d), 9.10 and 9.11] that
\[
\left( \frac{d}{p} \right) = \left( \frac{2k}{p} \right) \left( \frac{s}{p} \right) \left( \frac{u}{p} \right) = (-1)^{k(p^2-1)+s(p-1)(u-1)} \left( \frac{u}{p} \right) = \left( \frac{8s + 1}{u} \right),
\]
which, together with (4), yields \( \left( \frac{d}{p} \right) = -1 \) and completes the proof. \( \blacksquare \)

**Theorem 4.2.** Let \( \mu^* \) be an upper quasi-density on \( H \), and set \( X := \{ax^2 + bxy + cy^2 : x, y \in H\} \) and \( D := b^2 - 4ac \), where \( a, b, c \in H \) are fixed. The following hold:

(i) If \( D \) is not a perfect square or \( D = 0 \), then \( X \) is small.

(ii) If \( D \) is a non-zero perfect square and either \( ac = 0 \) or \( H = \mathbb{Z} \), then \( \mu^*(X) > 0 \).

**Proof.** Let \( w_k(\cdot) \) be defined as in Proposition 3.2. We have several cases.

**Case 1.** \( D \) is not a perfect square (and hence \( ac \neq 0 \)). In light of Lemma 4.1, there are \( m \in \mathbb{N}^+ \) and \( r \in \mathbb{N} \) such that, for every \( p \in \mathbb{P}^+ \) with \( p \equiv r \mod m \), \( D \) is not a quadratic residue modulo \( p \). Accordingly, let \( P \) be the set of all primes \( p \equiv r \mod m \) such that \( p \geq 2 + \max(|a|, |b|, |c|) \geq 3 \).

If \( p \in P \), \( z \in \mathbb{Z} \), and \( p \nmid z \), then \( ax^2 + bxy + cy^2 \neq pz \) for all \( x, y \in \mathbb{Z} \). Otherwise, \( (2ax + by)^2 \equiv Dy^2 \mod p \), which is only possible if \( p \mid y \) (since \( D \) is not a quadratic residue modulo \( p \)); consequently, we find that \( p \mid 2ax \), and hence \( p \mid x \) (because \( p \nmid 2a \)); this, however, means that \( p \mid ax^2 + bxy + cy^2 \), contradicting that \( p \nmid z \).

Thus, \( w_{p^2}(X) \leq r_{p^2}(X) \leq p^2 - p + 1 \leq p^2 - \frac{1}{4}p \) for every \( p \in P \), which implies, by Theorem 3.2(ii), that \( \mu^*(X) = 0 \), since we have from Dirichlet’s theorem on primes in arithmetic progressions (see, e.g., [12, Corollary 4.12(c)]) that
\[
\sum_{p \in P} \left( 1 - \frac{w_{p^2}(X)}{p^2} \right) \geq \frac{1}{2} \sum_{p \in P} \frac{1}{p} = \infty.
\]

**Case 2:** \( D = ac = 0 \). Note that \( b = 0 \), and assume by symmetry that \( c = 0 \). It follows that \( X = \{ax^2 : x \in \mathbb{H}\} \), and Corollaries 2.5 and 3.6 yield \( \mu^*(X) = 0 \).
Case 3: $D$ is a non-zero perfect square and $ac = 0$. We have $|b| = \sqrt{D} > 0$, and it is immediate that $X \supseteq |b| \cdot H + a + c$. Hence $\mu^*(X) \geq |b|^{-1} > 0$, by Lemma 3.1(iii).

Case 4: $D = q^2$ for some $q \in \mathbb{N}$ and $ac \neq 0$. Let $\varepsilon$ be the sign of $a$, and observe that $b - q \neq 0$ and, by axiom (F4), $\mu^*(X) = 4|a|\mu^*(4|a| \cdot X)$. Next, notice that

$$4|a| \cdot X = \{(2ax + (b-q)y)(2ax + (b+q)y)\varepsilon : x, y \in H\}. \quad (5)$$

Building on these premises, we distinguish two subcases.

Case 4.1: $q = 0$. It is clear from (5) that $4|a| \cdot X \subseteq \{x^2 \varepsilon : x \in H\}$, and we derive from Corollaries 2.5 and 3.6 that $\mu^*(X) = 4|a|\mu^*(4|a| \cdot X) = 0$.

Case 4.2: $q \neq 0$ and $H = \mathbb{Z}$. Denote by $Y$ the set

$$\{(-2a(b-q)z + 2a(b-q)(z+1)(-2a(b-q)z + 2a(b+q)(z+1))\varepsilon : z \in \mathbb{Z}\}.$$

By (5), $Y$ is a subset of $4|a| \cdot X$, and we see that

$$Y = 4a^2 \cdot \{2q(b-q)\varepsilon z + (b^2 - q^2)\varepsilon : z \in \mathbb{Z}\} = 8a^2q |b-q| : \mathbb{Z} + 4a^2(b^2 - q^2)\varepsilon.$$

In other words, $4|a| \cdot X$ contains an arithmetic progression of $\mathbb{Z}$, which, along with Lemma 3.1(iii) and the above, implies $\mu^*(X) = 4|a|\mu^*(4|a| \cdot X) > 0$. 

\textbf{Remark 4.3.} Set $X := \{ax^2 + bxy + cy^2 : x, y \in \mathbb{N}\}$, where $a, b, c \in \mathbb{N}$, $ac \neq 0$, and $b^2 - 4ac = q^2$ for some $q \in \mathbb{N}^+$. This case is not covered by Theorem 4.2, and it turns out that $\mu^*(X)$ is zero for some choices of the upper density $\mu^*$ and positive for others.

In fact, we will prove that $d_N^*(X) = 0 \neq b_N^*(X)$. To begin, it is easy to check that

$$4a \cdot X = \{(2ax + (b-q)y)^2 + 2qy : x, y \in \mathbb{N}\}.$$ 

Consequently, we have that $2a(b-q) \cdot A \subseteq 4a \cdot X \subseteq B$, where

$$A := \{2a(b-q)(u+v)^2 + 4aqv : u, v \in \mathbb{N}\} \subseteq \mathbb{N}$$

and

$$B := \{xy : x, y \in \mathbb{N} \text{ and } x \leq y \leq (2q+1)x\} \subseteq \mathbb{N}.$$ 

Since $d_N^*$ and $b_N^*$ are upper densities (and hence satisfy (F2) and (F4)), it follows that

$$b_N^*(X) = 4a \cdot b_N^*(4a \cdot X) \geq 2(b-q)^{-1}b_N^*(A) \quad \text{and} \quad d_N^*(X) \leq d_N^*(B).$$

Thus, it suffices to show that $b_N^*(A) \neq 0 = d_N^*(B)$. To this end, fix $k \in \mathbb{N}^+$. We have

$$\{2a(b-q)(ku + 1)^2 + 4aq((k-1)u + 1) : u \in \mathbb{N}\} \subseteq A,$$

which means that $r_k(A)$ is larger than or equal to the number of residues $h \in [0, k-1]$ such that $2a(b-q) + 4aq - 4aqu \equiv h \mod k$ for some $u \in \mathbb{N}$. Then $r_k(A) \geq [(4aq)^{-1} k]$, and we conclude from Proposition 2.6 that $b_N^*(A) \geq (4aq)^{-1} > 0$. 

\textbf{\hfill}
As for the rest, let \( n \in \mathbb{N}^+ \) and pick \( z \in B \cap [1, n] \). By construction, \( z = xy \) for some \( x, y \in \mathbb{N}^+ \) with \( x \leq y \leq 2q + 1 \). Therefore, we have \( y^2 \leq (2q + 1)xy \leq (2q + 1)n \), and hence \( x \leq y \leq \sqrt{(2q + 1)n} \). In other words, \( B \cap [1, n] \) is a subset of the multiplication table for positive integers \( \leq \sqrt{(2q + 1)n} \). So, we get from a classic result of Erdős [5, Part 3] that \( |B \cap [1, n]| = o(n) \) as \( n \to \infty \), which yields \( d_\mathcal{K}(B) = 0 \).

5. Closing remarks and open questions

We conclude the paper with a couple of open questions naturally stemming from the results of §§ 2 and 4 that we have not been able to settle.

**Question 5.1.** By Corollary 2.5 and Lemma 3.1(ii), the family \( \mathcal{S} \) of all small subsets of \( \mathbb{H} \) is an ideal containing all finite subsets of \( \mathbb{H} \); and by Theorem 2.4, it coincides with the intersection of \( \mathbb{H} \) and the zero set of the upper Buck density on \( \mathbb{Z} \). One may wonder if \( \mathcal{S} \) is also “closed under products”, meaning that

\[
XY := \{xy : (x, y) \in X \times Y\} \in \mathcal{S}, \quad \text{for all } X, Y \in \mathcal{S}.
\]

The answer is negative. Indeed, let \( \mu^* \) be an upper quasi-density on \( \mathbb{H} \), and let \( A \) (resp., \( B \)) be the set of all \( x \in \mathbb{H} \) whose positive prime divisors are all equal to 1 (resp., 3) modulo 4. Then \( \{x \in A : p \mid x\} \) is empty for every \( p \in \mathbb{P}^+ \) with \( p \equiv 3 \pmod{4} \); and in a similar way, \( \{y \in B : q \mid y\} \) is empty for every \( q \in \mathbb{P}^+ \) with \( q \equiv 1 \pmod{4} \) (observe that \( 0 \notin A \cup B \), because \( k \mid 0 \) for all \( k \in \mathbb{N} \)). Therefore, we get from Dirichlet’s theorem on primes in arithmetic progressions, Lemma 3.1(ii), and Proposition 3.3 that both \( A \) and \( B \) are small. But \( AB = 2 \cdot \mathbb{H} + 1 \) (note that \( \mathbb{H} \cap \{\pm 1\} \subseteq A \cap B \)), and we have by (F4) that \( \mu^*(2 \cdot \mathbb{H} + 1) = \frac{1}{2} \).

Likewise, \( \mathcal{S} \) is not “closed under sums” either, in the sense that there is \( X \in \mathcal{S} \) such that \( X + X := \{x + y : x, y \in X\} \notin \mathcal{S} \). In fact, the set \( \mathcal{Q} := \{x^2 + y^2 : x, y \in \mathbb{H}\} \subseteq \mathbb{H} \) is small by Theorem 4.2; but \( \mathcal{Q} + \mathcal{Q} = \mathbb{N} \) (by Lagrange’s four square theorem), and hence the upper Buck density of \( \mathcal{Q} + \mathcal{Q} \) is 1 (regardless of whether \( \mathbb{H} = \mathbb{N} \) or \( \mathbb{H} = \mathbb{Z} \)).

So, we may ask if, for a given \( \alpha \in [0, 1] \), there exists a set \( X \subseteq \mathcal{S} \) such that \( \mu_*(XX) = \mu^*(XX) = \alpha \) (resp., \( \mu_*(X + X) = \mu^*(X + X) = \alpha \)) for every upper quasi-density \( \mu^* \) on \( \mathbb{H} \), where \( \mu_* \) is the conjugate of \( \mu^* \) (as defined in the first lines of § 3). Questions in a similar vein have been recently addressed in [9].

Questions in the same vein have been recently addressed by various authors in the special case of the upper asymptotic density, see [6, Theorem 1.3 and Question Q4], [8, §§ 2 and 4], and [2, Theorems 1.1.a), 1.5, and 1.8].

**Question 5.2.** Let \( k \in \mathbb{N} \). By Corollary 3.4, the set \( \mathcal{Y}^{(k)} \) of all \( x \in \mathbb{H} \) that factor into a product of at most \( k \) primes (counted with multiplicity), is small. Does the same hold true for the inverse image of \( [0, k] \) under the function \( \omega : \mathbb{H} \setminus \{0\} \to \mathbb{N} \) that maps a non-zero integer \( x \in \mathbb{H} \) to the number of primes \( p \in \mathbb{P}^+ \) such that \( p \mid x \)? If yes, this
would be a stronger result than Corollary 3.4, because every subset of a small set is small (Corollary 2.5), and it is clear that $Y^{(k)} \subseteq \omega^{-1}([0, k])$.

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