Splitting of quantum information using $N$-qubit linear cluster states

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Abstract

We provide a number of schemes for the splitting up of quantum information among \( k \) parties, such that the original information can be reconstructed only if all the parties cooperate using a \( N \)-qubit linear cluster state as a quantum channel. Explicit circuits are provided for these schemes, which are based on the concept of measurement based locking and unlocking of quantum information. These are experimentally feasible as they require measurements to be performed only on product basis.

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1 Introduction

Entangled photons [1] have been used for several quantum communicational protocols like teleportation [2], cryptography [3], secret sharing [4] and superdense coding [5]. Quantum teleportation [2] is the disembodied transfer of an unknown quantum state from one location to another, with the sender neither knowing the information to be sent nor the location of the receiver. This acts as an important ingredient for different types of quantum communicational networks and quantum computers as it helps performing certain quantum gate operations [6]. Teleportation has been experimentally realized in various systems [7, 8, 9, 10, 11, 12]. The main difficulty in its experimental realization is measuring the states in an entangled basis. To overcome this difficulty, one often prefers to convert entangled basis to product basis measurements by applying suitable quantum gates, which is relatively easier for experimental realization [13]. Teleportation also opens up the possibility of achieving several tasks like secret sharing [4, 14] (of both classical and quantum information) and one way quantum computing [15].

The technique of splitting and sharing quantum information among two or more parties, such that none of them can retrieve the information fully by operating on their own qubits, is usually referred to as Quantum information splitting (QIS). QIS of \(|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle, (\alpha, \beta \in C, |\alpha|^2 + |\beta|^2 = 1)\) has been proposed using GHZ [14, 16] and asymmetric W states [17]. Later, QIS of \(|\psi_1\rangle\) was experimentally demonstrated using single
Recently, the search for genuinely entangled channels, which can be used for the deterministic QIS of an arbitrary two qubit state $|\psi_2\rangle_{12} = \alpha|00\rangle + \mu|10\rangle + \gamma|01\rangle + \beta|11\rangle$, where $|\alpha|^2 + |\mu|^2 + |\gamma|^2 + |\beta|^2=1$ and $\alpha, \mu, \gamma, \beta \in C$ has attracted much attention. It is worth mentioning that the GHZ and the asymmetric W states cannot be used for the QIS of arbitrary two qubit state $|\psi_2\rangle_{12}$, because they do not possess the required entangled structure to carry out the task [19]. However, a few specifically entangled multiqubit states [20, 21, 22, 23, 24] have been found useful for splitting $|\psi_2\rangle_{12}$ only among three parties. Further, these protocols required the parties to perform entangled measurements which are extremely difficult to realize in laboratory conditions. It is worth mentioning that one would require atleast five qubits for splitting of an arbitrary two qubit state [21].

This motivates us to devise protocols for the splitting up of $|\psi_2\rangle_{12}$ among $k$ parties, using a $N$ qubit linear cluster state [25, 26, 27, 28] by utilizing only product basis measurements. These states can be generated from graph states consisting of nodes and edges, wherein the nodes stand for the Hadamard state and the edges denote a control phase shift operation between the connecting nodes. Using the same notation, linear $N$-qubit cluster states can be represented as a chain of $N$ nodes arranged in a linear manner with the edges connecting the adjacent nodes. In general, an $N$-qubit linear cluster state can be represented as [25]

$$|C_N\rangle = \frac{1}{2^{N/2}} \otimes^N_{a=1} (|0\rangle_a \sigma^+_a + |1\rangle_a). \quad (1)$$

These states have attracted recent attention due to their wide applications in various quantum computation protocols like one way quantum computation and for quantum error correction [29] and their experimental feasibility [30, 31, 32]. Further, they have also been found to be robust against decoherence [33]. Interestingly, there exists two entangled bits between many subsystems of a $N$-qubit cluster state, making them a prospective resource for teleportation and state sharing of $|\psi_2\rangle$. Recently, two of the present authors have shown that the five qubit cluster state can be used for the QIS of $|\psi_2\rangle$ among three parties [22]. However, to the best of authors’ knowledge, the usefulness of generalized $N$-qubit $|C_N\rangle$ has not yet been studied in the context of QIS among $k$ parties. This inspires us to examine the same.
The paper has been organized as follows. We first describe explicit circuits for the generation of $N$-qubit linear cluster states. In the next section, we study the splitting of arbitrary two qubit quantum information $|\psi_2\rangle_{12}$ among $k$ different parties. Explicit circuits for the same have been constructed, wherein the measurements have been performed on the product basis. In the last section, we explain the protocol further by giving illustrations of QIS using five and six qubit cluster states.

In general, any $N$-qubit linear cluster state can be generated from $|000...0\rangle_{123..N}$ by implementing the circuit diagram shown in Fig. 1.

![Figure 1: Circuit diagram for the generation of $|C_N\rangle$](image)

\[ |C_N\rangle \rightarrow |C'_N\rangle \]

\[ |C_N\rangle \rightarrow |C'_N\rangle \]

2 QIS of $|\psi_2\rangle_{12}$ among $k$ parties

The protocol for the splitting of an arbitrary two qubit secret $|\psi_2\rangle_{12}$ among $k$ different parties using $|C_N\rangle$ can be divided into two major steps: "Locking" and "unlocking" of quantum secret. We label the participants Alice, $Bob_1$, $Bob_2$, ..., $Bob_{k-1}$ and Charlie, where Charlie is designated to get the final state. Before distributing the qubits among the parties, the qubits of $|C_N\rangle$ are swapped in the following manner,

\[ |C_{N}\rangle \xrightarrow{\text{Swap}(N-2,N)\ldots\text{Swap}(3,5)\text{, Swap}(1,3)} (|C'_{N}\rangle) \text{, if } N \text{ is odd} \] \( (2) \)

\[ |C_{N}\rangle \xrightarrow{\text{Swap}(N/2,N)\text{, Swap}(N/2+1)\ldots\text{Swap}(2,4)} (|C'_{N}\rangle) \text{, if } N \text{ is even} \] \( (3) \)

where $\text{Swap}(i,j)$ represents the swapping of "$i$"th and "$j$"th qubits respectively. We now distribute the qubits such that $|c_1\rangle$ and $|c_2\rangle$ belong to Alice, $|c_3\rangle$ and $|c_4\rangle$ belong to $Bob_1$, etc.
qubit $|c_5\rangle$ to Bob$_2$, .. and the qubits $|c_{(N-1)}\rangle$ and $|c_N\rangle$ to Charlie, where the "$i"th qubit ($i \geq N$) of $|C_N\rangle$ is denoted by $|c_i\rangle$. The QIS scheme for $N = 5$ and 6 will be explicated below. For $N \geq 6$, we let Alice, Bob$_1$, Charlie possess two qubits each and each of the remaining $(N - 5)$ participants possess one qubit.

### 2.1 Locking the quantum secret

In order to lock $|\psi\rangle_{12}$ among the other participants, she initially swap the qubits of $|C_N\rangle$ as per the rule discussed above and swaps the qubit $|\psi_2\rangle_2$ and $|C_N\rangle_2$, as is explicitly shown in Fig.2. This is followed by a $CNOT$ gate between $|\psi_1\rangle$ and $|\psi_2\rangle$, a Hadamard on $|\psi_2\rangle$ in order to "break" the entangled measurements into product measurements. She measures each of her four qubits individually in the basis ($|0\rangle, |1\rangle$) and conveys the outcome of the measurement to Charlie via four classical bits. The information is thus locked amongst the parties Bob$_1$, Bob$_2$ ... Bob$_{(N-5)}$, $(N > 5)$ and Charlie such that none of them can obtain the quantum secret by operating on their own qubits. The circuits explicitly constructed for this protocol are shown in Fig.2 or Fig.3 for $N$ even or odd respectively.

### 2.2 Unlocking the quantum secret

For unlocking $|\psi_2\rangle_{12}$, the parties should act as follows. Initially, Bob$_1$ performs a $CNOT_{3,4}$ operation on the two qubits $|c_3\rangle$ and $|c_4\rangle$, projects the two qubits on the computational basis given by, ($|00\rangle, |01\rangle, |10\rangle, |11\rangle$) and conveys the outcome of the measurement to Charlie via two cbits. The other participants, Bob$_i, i \in 2,...,(N-5)$ perform measurements in the Hadamard basis $\frac{|0\pm i|}{\sqrt{2}}$ and conveys the outcome to Charlie via cbits. Once Charlie obtains all the $(N - 4)$ measurement results (including Alice's measurement outcome), he can perform a suitable set of operations on his two qubits and deterministically obtain $|\psi_2\rangle_{12}$. Thus, Alice's quantum secret $|\psi_2\rangle_{12}$ which was initially split among $(N - 5)$ intermediate parties was sent to Charlie by performing only product basis measurements. This completes the proposed QIS scheme. The quantum circuits in Fig. 2 and Fig. 3, depending on whether $N$ is even or odd, show these steps clearly.
3 Illustrations

We shall now illustrate the above proposed protocol explicitly for $N = 5$ and $N = 6$ respectively. We shall also provide relations between the classical bits received by Charlie by the different parties and the local operations to be performed by him in order to deterministically obtain $|\psi_2\rangle_{12}$.

3.1 QIS of $|\psi_2\rangle_{12}$ using five qubit cluster state $|C_5\rangle$

The five qubit cluster state

$$|C_5\rangle = \frac{1}{2}(|00101\rangle - |00010\rangle - |11001\rangle + |11110\rangle),$$

(4)

can be generated using the circuit shown in Fig. 1. After performing the required SWAP operations between qubits 1 and 3 and the qubits 3 and 5, the resultant state is given by,

$$|C'_5\rangle = \frac{1}{2}(|00010\rangle + |01101\rangle - |10100\rangle - |11011\rangle).$$

(5)
\( |C'_N\) forms an important resource for QIS among three parties. The qubits are distributed such that Alice possesses the qubits 1 and 2 of \( |C'_N\) along with \( |\psi_2\rangle_{12} \), which is to be split among the two parties, \( Bob_1 \) and Charlie. We let \( Bob_1 \) possess qubit 3 and Charlie possess qubits 4 and 5. In the next step, Alice performs a measurement on each of her four qubits individually in the basis \(|0\rangle, |1\rangle\), thereby locking the quantum secret in the Bob-Charlie system. She then conveys the outcome of her measurement to Charlie via four classical bits. It is worth mentioning that, at this stage Charlie cannot decipher \( |\psi_2\rangle_{12} \), with Alice’s measurement outcome alone. In order to unlock \( |\psi_2\rangle_{12} \), \( Bob_1 \) performs a Hadamard measurement on his qubit (since no entangling operation is performed for \( N \leq 5 \)) and sends the result to Charlie via one classical bit. Having obtained the outcomes of both Alice and \( Bob_1 \), Charlie can now deterministically reconstruct \( |\psi_2\rangle_{12} \) by applying suitable unitary operations on his qubits.

We denote the 4 classical bits sent by Alice to Charlie as, "," \( a_1a_2a_3a_4 \)" and the single classical bit sent by \( Bob_1 \) as "\( b_1 \)". The unitary local operation \( U \) to be performed by
Charlie in order to obtain $|\psi_2\rangle_{12}$ is then given by,

$$U = (a_4 \cdot a_2 (\sigma_x \otimes I) + a_4 \cdot a_2 (I \otimes \sigma_x) + a_4 \cdot a_2 (I \otimes I) + a_4 \cdot a_2 (\sigma_x \otimes \sigma_x)).$$

(6)

Here, $\oplus$ and $\overline{a_i}$ denote the classical XOR and NOT respectively.

For instance, let us suppose that the cbits sent by Alice, $a_1a_2a_3a_4$ be 1110, and that by Bob be "1". The unitary operation $U$ that Charlie should apply on his two qubits is then given by, $U = (\sigma_x \otimes I).CNOT_{2,1}.Swap_{1,2}.(I \otimes \sigma_z)$, $CNOT_{2,1}.Swap_{1,2}.(I \otimes \sigma_z)$. The local operations $U$ for other classical messages can be obtained in a similar manner.

An explicit circuit showing the two stages, locking and unlocking of the $|\psi_2\rangle_{12}$ is given in Fig. 4. This protocol assumes significance, since five is the threshold number of qubits that is required for the QIS of an arbitrary two qubit state $|\psi_2\rangle_{12}$ in the case where both the parties involved need not meet. Further, this protocol is easier for experimental implementation than the previous protocol as it involves only product measurements [21].

**Figure 4:** Circuit diagram for the locking and unlocking in QIS using $|C_5\rangle$

### 3.2 QIS of $|\psi_2\rangle_{12}$ using six qubit cluster state

The six qubit linear cluster state $|C_6\rangle$ can be generated using the circuit shown in Fig.1. We then perform swap operations ($Swap(1, 4), Swap(3, 6)$) on $|C_6\rangle$ and the resultant clus-
The state is given by,
\[
|C_6'\rangle = \frac{1}{2\sqrt{2}}(|010101\rangle - |010010\rangle - |001001\rangle + |001110\rangle + |100101\rangle - |100010\rangle - |111001\rangle - |111110\rangle).
\]

This state can be used to establish the QIS protocol among \((N-3) = 3\) parties namely, Alice, \(Bob_1\), and Charlie. To initialize the protocol, we let Alice possess the qubits 1 and 2 (along with \(|\psi_2\rangle\)), \(Bob_1\) possess qubits 3 and 4 and Charlie possess qubits 5 and 6, as stated in the generalized scheme discussed in section 3.1. Next, Alice performs a four particle computational basis measurement, and conveys the outcome to Charlie using four classical bits, thereby locking the quantum secret between \(Bob_1\) and Charlie. In the next step, in order to unlock \(|\psi_2\rangle_{12}\), \(Bob_1\) performs a \(CNOT_{3,4}\) operation on his two qubits. He measures the outcome in the computational basis as in the previous case after applying Hadamard states and sends the results to Charlie via two classical bits. Having known the outcomes of the measurement of Alice and Bob, Charlie can apply suitable unitary operation \(U\) and deterministically retrieve \(|\psi_2\rangle_{12}\).

If \(a_1a_2a_3a_4\) denotes the four cbits sent by Alice and \(b_1b_2\) denote the ones sent by Bob, then the local unitary operation to be performed by Charlie, corresponding to the different messages, is given by,
\[
U = \begin{bmatrix} a_4 \cdot ((\sigma_x \otimes I).((a_1 \oplus a_2 \oplus b_1) \oplus (a_3 \oplus b_2)) + (I \otimes I).((a_1 \oplus a_2 \oplus b_1) \oplus (a_3 \oplus b_2))) \\
+ a_4 \cdot ((I \otimes \sigma_x).((a_1 \oplus a_2 \oplus b_1) \oplus (a_3 \oplus b_2))) \\
+ (\sigma_x \otimes \sigma_x).((a_1 \oplus a_2 \oplus b_1) \oplus (a_3 \oplus b_2))) \cdot \text{CNOT}_{2,1}.((a_1 \oplus a_3).((\sigma_z \otimes I).((a_3 \oplus b_2))) \\
+ (I \otimes \sigma_z).(a_3 \oplus b_2)) + (a_1 \oplus a_3).((\sigma_z \otimes \sigma_z).(a_3 \oplus b_2))) + (I \otimes I).(a_3 \oplus b_2)) \end{bmatrix}.
\]

For instance, if the four cbits sent by Alice, \(a_1a_2a_3a_4\) are 0100 and that by Bob \(b_1b_2\) are 01, then the unitary operation \(U\) is given by, \(U = (I \otimes \sigma_x).\text{CNOT}_{2,1}.(I \otimes \sigma_z)\). Appropriate local operations corresponding to other messages can be obtained in a similar way.

An explicit circuit showing locking and unlocking of the \(|\psi_2\rangle_{12}\) for the case of \(|C_6\rangle\) is
shown below in Fig. 5.

Figure 5: Circuit diagram for the locking and unlocking of two qubit quantum secret using $|C_6\rangle$

4 Conclusion

In this paper, we have explicated the creation and the use of $N$-qubit cluster states for the generalization of quantum information splitting (QIS) protocol among $k$ different parties. Explicit circuit diagrams involving only experimentally realizable quantum gates have been described. Unlike the presented scheme, most of the schemes that deal with the QIS of an arbitrary two qubit state in the literature involve splitting of quantum information only among limited number of parties. However, using the protocol proposed in this paper, one can have a QIS scheme that involves any number of parties. Secondly, the schemes developed so far in literature, either involve highly correlated multipartite measurements, which are extremely difficult to implement in experimental conditions or they use realizable Bell type measurements but with larger number of entangled photons. However, the illustrated protocol uses $N$-qubit linear cluster states $|C_N\rangle$ which employs only computational basis measurements thereby making it feasible for experimental realization. Further, the initial resource used, i.e., the cluster states are shown to be robust against decoherence [34]. We hope that this will lead to experimental realization of QIS of an arbitrary two qubit state among any number of involved parties, which has long
been a challenge to the experimentalists.

References

[1] M. A. Nielsen, I. L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge Univ. Press, 2002).

[2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, W. K. Wootters, Phys. Rev. Lett. 70 (1993) 1895.

[3] A. K. Ekert, Phys. Rev. Lett. 67 (1991) 661.

[4] D. Gottesman, Phys. Rev. A 61 (2000) 042311.

[5] C. H. Bennett, S. J. Wiesner, Phys. Rev. Lett. 69 (1992) 2881.

[6] D. Gottesman, I. L. Chuang, Nature 402 (1999) 390.

[7] D. Bouwmeester, J. W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature 390 (1997) 575.

[8] M. Riebe, H. Hffner, C. F. Roos, W. Hnsel, J. Benhelm, G. P. T. Lancaster, T. W. Krber, C. Becher, F. S. Kaler, D. F. V. James, R. Blatt, Nature 429 (2004) 734.

[9] M. D. Barrett, J. Chiaverini, T. Schaetz, J. Britton, W. M. Itano, J. D. Jost, E. Knill, C. Langer, D. Leibfried, R. Ozeri, D. J. Wineland, Nature 429 (2004) 737.

[10] I. Marcikic, H. de Riedmatten, W. Tittel, H. Zbinden, N. Gisin, Nature 421 (2003) 509.

[11] R. Ursin, T. Jennewein, M. Aspelmeyer, R. Kaltenbaek, M. Lindenthal, P. Walther, A. Zeilinger, Nature 430 (2004) 849.

[12] S. B. Zheng, G. C. Guo, Phys. Rev. Lett. 85 (2000) 5188.

[13] M. A. Nielsen, E. Knill, R. Laflamme, Nature 96 (1998) 52.

[14] M. Hillery, V. Buzek, A. Berthiaume, Phys. Rev. A 59 (1999) 1829.

[15] R. Raussendorf, H. J. Briegel, Phys. Rev. Lett. 86 (2001) 5188.

[16] S. Bandyopadhyay, Phys. Rev. A 62 (2000) 012308.

[17] S. B. Zheng, Phys. Rev. A 74 (2006) 054303.
[18] C. Schmid, P. Trojek, M. Bourennane, C. Kurtsiefer, M. Zukowski, H. Weinfurter, Phys. Rev. Lett. 95 (2005) 230505.

[19] S. Muralidharan, S. Karumanchi, R. Srikanth, P. K. Panigrahi, eprint quant-ph/0907.3532.

[20] S. Muralidharan, P. K. Panigrahi, Phys. Rev. A 77 (2008) 032321.

[21] S. Muralidharan, P. K. Panigrahi, Phys. Rev. A 78 (2008) 062333.

[22] S. Choudhury, S. Muralidharan, P. K Panigrahi, J. Phys. A 42 (2009) 115303.

[23] X. W Wang, Z.H. Peng, C. X. Jia, Y. H Wang, X. J. Liu, Opt. Commun. 282 (2009) 670.

[24] S. Muralidharan, S. Karumanchi, P. K. Panigrahi, eprint quant-ph/0804.4206v2.

[25] H. J. Briegel, R. Raussendorf, Phys. Rev. Lett. 86 (2001) 910.

[26] C. Y. Lu, X. Q. Zhou, O. Ghne, W. B. Gao, J. Zhang, Z. S. Yuan, A. Goebel, T. Yang, J. W. Pan, Nature 3 (2007) 91.

[27] P. J. Blythe, B. T. H Varcoe, New J. Phys. 8 (2006) 231.

[28] Y. Soudagar, F. Bussires, G. Berln, S. Lacroix, J. M. Fernandez, N. Godbout, Jour. Opt. Soc. Amer. B 24 (2007) 226.

[29] D. Schlingemann, R. F. Werner, Phys. Rev. A 65 (2001) 012308.

[30] P. Walther, K. J. Resch, T. Rudolph, E. Schenck, H. Weinfurter, V. Vedral, M. Aspelmeyer, A. Zeilinger, Nature 86 (2005) 434.

[31] T. Tanamoto, Y. X. Liu, S. Fujita, X. Hu, F. Nori, Phys. Rev. Lett. 97 (2006) 230501.

[32] X. Zou, W. Mathis, Phys. Rev. A 72 (2005) 013809.

[33] M. Hein, W. Dur, H. J. Briegel, Phys. Rev. A 71 (2005) 032350.

[34] W. Dur, H. J. Briegel, Phys. Rev. A 92 (2004) 180403.