Intrinsic area near the origin for self-similar growth-fragmentations and related random surfaces

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Abstract

We study the behaviour of a natural measure defined on the leaves of the genealogical tree of some branching processes, namely self-similar growth-fragmentation processes. Each particle, or cell, is attributed a positive mass that evolves in continuous time according to a positive self-similar Markov process and gives birth to children at negative jumps events. We are interested in the asymptotics of the mass of the ball centered at the root, as its radius decreases to 0. We obtain the almost sure behaviour of this mass when the Eve cell starts with a strictly positive size. This differs from the situation where the Eve cell grows indefinitely from size 0. In this case, we show that, when properly rescaled, the mass of the ball converges in distribution towards a non-degenerate random variable. We then derive bounds describing the almost sure behaviour of the rescaled mass. Those results are applied to certain random surfaces, exploiting the connection between growth-fragmentations and random planar maps obtained in [6]. This allows us to extend a result of Le Gall [24] on the volume of a free Brownian disk close to its boundary, to a larger family of stable disks. The upper bound of the mass of a typical ball in the Brownian map is refined, and we obtain a lower bound as well.

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1 Introduction

Growth-fragmentation processes form a family of continuous time branching processes that have been introduced by Bertoin [5]. They model particle systems without interaction where each particle is described by a positive real number that corresponds to its mass (or size), evolving by growing and splitting, with rates that can depend on its current mass. Note that these processes differ from (pure) fragmentation processes [3, 4], for which growth is not allowed. The fragmentations are binary and when a particle splits, its mass is instantaneously randomly distributed among the two resulting fragments. A self-similar growth-fragmentation \((X(t))_{t \geq 0}\) is a growth-fragmentation where every particle evolves according to a driving self-similar Markov process. Self-similarity refers to the property that the evolution of a particle of size \(x > 0\) is a scaling transformation of that of a particle of unit size, depending on some index \(\alpha \in \mathbb{R}\). The law of \(X\) is characterized by \(\alpha\) and its cumulant function \(\kappa : \mathbb{R}((-\infty, \infty], \), both depending on the driving process.

In this paper we are interested in the cases where \(\alpha < 0\) and \(\kappa\) has two positive root \(\omega_- < \omega_+\). Under some further conditions on \(\alpha\) and \(\kappa\), the growth-fragmentation yields a natural measure on the genealogical tree of the branching process seen as a metric space, namely the intrinsic area measure. Denoting \(A(t)\) the intrinsic area of the ball of radius \(t\) centered at the root of the tree, one can investigate the regularity of the Stieltjes measure \(dA(t)\). This has been studied in [19], in which Theorem 1 shows that if \(\alpha > -\omega_-\), then \(dA(t)\) is absolutely continuous whereas it is singular when \(\alpha \leq -\omega_-\). In this paper, we consider the absolutely continuous case. A noteworthy fact is that the density is null at \(t = 0\), meaning that the dissipation of the area occurs as small particles are about to vanish. We study further properties of \(A\).

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Our first result is Theorem 1 that determines the almost sure behaviour of $A(\epsilon)$ as $\epsilon \to 0+$, when $X$ starts from a single particle of positive size $x$. Namely, there exists a regularly varying function $f: \mathbb{R}_+ \to \mathbb{R}_+$ explicitly given in terms of the characteristics of the driving process, such that $A(\epsilon)/f(\epsilon) \to 1$ as $\epsilon \to 0+$.

It is possible to tilt the law of $X$ such that the Eve cell starts from size 0. Indeed, in [3], the authors introduced a new probability measure, under which $X$ initiates from a distinguished particle that grows indefinitely from the initial size 0, and evolves differently from the others which eventually die out. Let denote $A^+$ the analog of $A$ under this new measure. In this case, Theorem 1 does not apply. However, Proposition 2 shows that, $t \mapsto e^{\omega t} A^+(\epsilon t)$ defines a stationary process. In particular, $t^{\omega-\alpha} A^+(t)$ has a limit in distribution as $t \to 0+$, but has no almost sure limit. We prove in Propositions 2 and 3 that, almost surely, $A^+(t)$ deviates from $t^{\omega-1/\alpha}$ by at most a power of $|\log(t)|$.

One of the motivations of the present work is that these branching processes turn out to be geometrically connected to some random surfaces. The Brownian map is a random surface homeomorphic to the two-dimensional sphere that appears as the Gromov-Hausdorff scaling limit when $n \to \infty$ of uniformly distributed $q$-angulations with $n$ faces of the sphere [29, 31]. Similarly, Brownian disks are random compact metric spaces homeomorphic to the unit disk of $\mathbb{R}^2$, obtained as scaling limits of random planar maps with a boundary [11]. In [17], the Brownian plane, locally isomorphic to the Brownian map, is obtained as scaling limits of the UIPQ and uniform quadrangulations. Moreover it can also be seen as the Gromov-Hausdorff tangent cone of the Brownian map at its root.

In [7], it was shown that the collection of perimeters of the holes observed when slicing Boltzmann triangulations at all heights converges, when properly rescaled, towards a particular self-similar growth-fragmentation. We also mention Theorems 3 and 23 of Le Gall and Riera [27], which show that, when slicing directly the free Brownian disk, the holes’ perimeters are described by the same growth-fragmentation as in [6] (see also Section 4). When $X$ starts from a single cell of size 0 that grows indefinitely, the geometrical connection corresponds this time to the holes in a sliced discrete approximation of the Brownian plane.

There is actually a broader family of continuum random surfaces, with different scaling exponents, that arise as limiting objects of Boltzmann random planar maps. They are known as stable maps and were first obtained by Le Gall and Miermont [26, 25, and references therein], by considering very specific distributions for the degree of a typical face, that have infinite variance (see also [29] for the same scaling limits under relaxed hypotheses). However, only the Brownian map has been characterized yet, in the sense that for other stable maps, the uniqueness of the scaling limit is not yet proven.

Similarly as in the Brownian case, the authors in [6] were able to extend the geometrical connection previously mentioned, to holes’ perimeters in discrete approximations of stable disks and plane (in the so-called dilute case) and a specific family of self-similar growth-fragmentations.

There is a natural way of measuring the "size" of these stable surfaces, namely, the so-called intrinsic volume measure, which can be constructed as the scaling limit of the number of vertices (or faces) in the approximation by discrete random maps. This measure corresponds to the intrinsic area measure in the related growth-fragmentation; we shall rather use this name since we consider planar objects.

Besides being aesthetic, this connection has already been fruitful in both directions: for instance, it allowed the authors in [6] to use results on discrete random planar maps to determine the law of the total intrinsic area of the related growth-fragmentations. On the other hand, it was known that the intrinsic area of the Brownian map cannot be derived as the length of the perimeters of the holes and the height, as for smooth surfaces, since the latter defines a measure that is not locally finite. From the analysis of growth-fragmentations, it has been shown in [19] that the intrinsic area of the Brownian map can be written as the integral against the Lebesgue measure of some function of the perimeters of the holes.

Our results in this paper apply to stable surfaces. In particular, Thanks to Theorem 1, we retrieve Theorem 3 of Le Gall [24], which shows that, in the case of the free Brownian disk with
boundary length \( x > 0 \), it holds that \( \epsilon^{-2}A(\epsilon) \to x \) almost surely, as \( \epsilon \to 0+ \) (see (25)). More generally, we obtain the analogue for other stable disks, with different exponents.

Applying Proposition 2 improves the upper bound that was known for the almost sure behaviour of the area of the ball of radius \( \epsilon \) around the root of the Brownian map (i.e. \( A^+(\epsilon) \)): the previous bound was \( \epsilon^{\delta-\delta} \) for \( \delta > 0 \) arbitrary small, we obtain \( \epsilon^{\delta}|\log(\epsilon)|^{\delta+\delta} \). We moreover obtain the lower bound \( \epsilon^{\delta}|\log(\epsilon)|^{-\delta} \) for any \( \delta > 6 \), thanks to Proposition 3. Again, the analogue holds for stable maps.

The organisation of the paper is the following. In Section 2, we formally introduce the growth-fragmentations setting. This includes definitions of positive self-similar Markov processes and Lamperti’s transformation, the intrinsic area measure of a growth-fragmentation, as well as the two spinal decompositions introduced in [6] that are central throughout this work. We also establish an important Markov-branching property of \( A \) in Lemma which, roughly speaking, reduces the study of \( A \) to that of a single self-similar Markov process.

We state our results in Section 3. Theorem 1 concerns the almost sure behaviour of \( A(\epsilon) \) as \( \epsilon \to 0^+ \), when \( X \) starts from a typical cell with positive initial size. Our second result is Proposition 2 which provides an upper bound for the almost sure behaviour of \( A^+(\epsilon) \) when the initial cell starts from 0 and behaves differently from the others, with indefinite growth. A lower bound is then given in Proposition 3.

We prove Theorem 1 in Section 4. The first subsection looks at the expectation of \( A(\epsilon) \). In the second subsection, we introduce a useful martingale to control the fluctuations of \( A(\epsilon) \) around its expectation and prove Theorem 1.

Section 5 is devoted to the proof of Propositions 2 and 3, that are respectively upper bound and lower bound for \( A^+(\epsilon) \) as \( \epsilon \to 0^+ \). After having established Proposition 1, we prove Proposition 2 in the first subsection. We then turn our attention to Proposition 3 in the second subsection. The arguments are different and the proof is more involved.

We conclude this paper with Section 6, in which we apply our results to stable surfaces.

2 Self-similar growth-fragmentations

The law of a typical cell. For all \( x > 0 \), let \( \mathbb{P}_x \) denote the law of a positive self-similar Markov process \( X = (X(t))_{t \geq 0} \) starting from \( x \) and absorbed at 0. Self-similarity refers to the property that under \( \mathbb{P}_x \), the law of \( (X(t))_{t \geq 0} \) is the same as that of \( (xX(tx^\alpha))_{t \geq 0} \) under \( \mathbb{P}_1 \), for some real index \( \alpha \). We shall always assume here that \( \alpha < 0 \). For all \( t \geq 0 \), let introduce the time-change

\[
\tau_t := \int_0^t X(s)^\alpha ds.
\]

The well-known Lamperti’s transformation states that there exists a unique Lévy process \( \xi = (\xi(t))_{t \geq 0} \) such that

\[
X(t) = \exp (\xi(\tau_t)), \quad \forall t \geq 0.
\]

Equivalently to (1), one can write

\[
\tau_t = \inf \left\{ s \geq 0 : \int_0^s \exp(-\alpha \xi(u))du \geq t \right\}.
\]

Let \( \Lambda \) be the Lévy measure of \( \xi \) and assume that \( \int_{(1,\infty)} e^{uy} \Lambda(dy) < \infty \). We thus have, at least for \( q \in [0,1] \), that \( \mathbb{E}(\exp(q\xi(t))) = \exp(t\psi(q)) \) for all \( t \geq 0 \), with

\[
\psi : q \mapsto bq + \frac{\sigma^2}{2}q^2 + \int_{\mathbb{R}} \left( e^{qy} - 1 + q(1-e^y) \right) \Lambda(dy), \quad q \geq 0,
\]
where \(b, \sigma^2 \geq 0\). We point out that this is not the traditional Lévy-Khintchine formula for the Laplace exponent \(\psi\) of a Lévy process, in which \((1 - e^y)\) above is usually replaced by \(y\mathbb{1}_{\{|y|<1\}}\), but (2) is more convenient for our purposes. We assume throughout this paper that \(\psi'(0) < 0\). It is known that this entails that \(\xi\) drifts to \(-\infty\) almost surely, and in particular that the absorption time in 0 of \(X\), namely \(\int_0^\infty \exp(-\alpha \xi(t)) dt\), is finite almost surely.

The cell-system. We briefly recall Bertoin’s construction of the cell-system as in [5]. We use the classical Ulam-Harris-Neveu notation to label the particles of the branching process. Let \(\mathbb{U} := \bigcup_{n \geq 0} \mathbb{N}^n\) and let \(\partial \mathbb{U}\) be the set of infinite sequences of natural integers. We shall also denote \(\overline{\mathbb{U}} := \mathbb{U} \cup \partial \mathbb{U}\). If \(u \in \mathbb{N}^n\) for some \(n \in \mathbb{N}\), we write \(u(k) \in \mathbb{N}^k\) its ancestor at generation \(k \leq n\) and \(|u| = n\) its generation.

The process starts with a single cell that we call the Eve cell, indexed by \(\emptyset\) and size at any time \(t \geq 0\) denoted by \(\chi_\emptyset(t)\). It evolves in time, starting from its birthtime \(b_\emptyset := 0\) according to \(\mathbb{P}_1\) until its absorption time \(\zeta_\emptyset\) at 0. In this context, \(\zeta_\emptyset\) is called the lifetime of \(\chi_\emptyset\).

Since \(\chi_\emptyset\) converges to 0 almost surely, it is possible to rank all its negative jumps in the decreasing order of the absolute values of their sizes. If \(\{ (b_i, \Delta_i) : i \geq 1 \}\) is the collection of times and sizes of these negative jumps ranked in this way, then for each \(i \geq 1\), we start at time \(b_i\) a new positive self-similar Markov process \(\chi_i\) under the law \(\mathbb{P}_{\Delta_i}\), independently of every other cell. We denote its lifetime \(\zeta_i\), that is \(\chi_i(t)\) is the size of the cell labelled \(i\) if \(b_i \leq t < b_i + \zeta_i\), and is sent to some cemetery state \(\partial\) otherwise. In this manner, we construct recursively the whole cell-system indexed by \(\overline{\mathbb{U}}\), and we denote \(\mathbb{P}_1\) its law. More generally, we can construct such a cell-system with the Eve cell starting from a size \(x > 0\), its law is then denoted \(\mathbb{P}_x\).

The growth-fragmentation. The growth-fragmentation process \((X(t))_{t \geq 0}\) induced by the cell-system is the process following the collection of particles’ sizes in time, forgetting about their genealogy, that is

\[
X(t) := \{ \chi_u(t - b_u) : u \in \mathbb{U}, b_u \leq t < b_u + \zeta_u \},
\]

where the elements are repeated according to their multiplicity.

The law of \(X\) is characterized by \(\alpha\) and a particular function \(\kappa : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}\) called the cumulant function of \(X\) (see [39]), which is defined as

\[
\kappa(q) := \psi(q) + \int_{(-\infty, 0)} (1 - e^y)^q \Lambda(dy).
\]

The importance of \(\kappa\) for the study of self-similar growth-fragmentations comes from the fact, taken from [5] Lemma 3, that \(\mathbb{E}_x \left( \sum_{i \geq 1} \chi_i(0)^q \right) = x(1 - \kappa(q)/\psi(q))\), when it makes sense. This means that when \(\kappa(q) = 0\), the initial size of a cell is equal to the expectation of the sum of the sizes raised to the power \(q\) of its first generation children at their birthtime. For this reason, we assume throughout this work that the Cramér hypothesis holds, that is there exists \(\omega_- > 0\) such that

\[
\kappa(\omega_-) = 0, \quad -\infty < \kappa'(\omega_-) < 0.
\]  \hspace{1cm} (3)

We suppose that \(\kappa\) has a second root \(\omega_+ > \omega_-\) and is finite in a right neighbourhood of \(\omega_+\), which by convexity of \(\kappa\) implies \(0 < \kappa'(\omega_+) < \infty\).

Intrinsic area measure. Thanks to [43], the process

\[
\mathcal{M}(n) := \sum_{|u| = n} \chi_u(0)^{\omega_-}, \quad n \geq 0
\]
is a uniformly integrable martingale, see Lemma 2.4 in [6]. Its terminal value \( M := \lim_{n \to \infty} \mathcal{M}(n) \) is called the intrinsic area of \( X \) and we point out that, by construction, it does not depend on \( \alpha \).

Endowed with the distance \( d(\ell, \ell') := \exp(-\sup\{n \geq 0 : \ell(n) = \ell'(n)\}) \), \( \partial U \) is a complete metric space. The intrinsic area measure \( \mathcal{A} \) is then the unique measure on \( \partial U \) such that the mass of the subsets of leaves having ancestor \( u \in \mathbb{N}^n \) at generation \( n \geq 0 \) satisfies

\[
\mathcal{A} (\{\ell \in \partial U : \ell(n) = u\}) = \lim_{k \to \infty} \sum_{|v| = k} \chi_{uv}(0)^{\omega^{-}}.
\]

Note that the total mass of \( \mathcal{A} \) is \( M \). Denoting by \( \zeta_\ell \) the height of \( \ell \in \partial U \), that is the time at which the lineage of this leaf goes extinct in terms of the growth-fragmentation’s time, we define the area of the ball with radius \( t \in \mathbb{R}^+ \) centered at the origin by

\[
A : t \mapsto \mathcal{A} (\{\ell \in \partial U : \zeta_\ell \leq t\}).
\]

The increasing function \( A \) will be the object of interest of the rest of the paper. It satisfies the following Markov-branching type property that we shall use all along this work.

**Lemma 1.** For all \( t \geq 0 \), it holds that

\[
A(t) = \sum_{s \leq t} |\Delta \chi_0(s)|^{\omega^{-}} A_s ((t-s)|\Delta \chi_0(s)|^{\alpha}),
\]

where \( \Delta \chi_0(s) := \min \{0, \chi_0(s) - \chi_0(s-)\} \) (i.e. we only consider the negative jumps) and the \( A_s \)'s are i.i.d. copies of \( A \) under \( \mathcal{P}_1 \), independent from \( \chi_0 \).

**Proof.** We first introduce a notation. If \( \Delta \chi_0(s) \neq 0 \) for some \( s > 0 \), i.e. if the Eve cell gives birth at time \( s \), we denote \( U_s \) the sub-tree generated by the newborn cell. We write

\[
A(t) = \mathcal{A} (\{\ell \in \partial U : \zeta_\ell \leq t\}) = \sum_{s \leq t} \mathbb{1}_{\{\Delta \chi_0(s) \neq 0\}} \mathcal{A} (\{\ell \in \partial U_s : \zeta_\ell \leq t\}).
\]

Thanks to [3], Lemma 3.2 of [6] entails that the areas in the above sum are independent and that for all \( s > 0 \) the conditional distribution of \( \mathcal{A} (\{\ell \in \partial U_s : \zeta_\ell \leq t\}) \) given \( \chi_0 \) is that of \( A(t-s) \) under \( \mathcal{P}_{\Delta \chi_0(s)} \), which by self-similarity is identical to that of \( |\Delta \chi_0(s)|^{\omega^{-}} A((t-s)|\Delta \chi_0(s)|^{\alpha}) \) under \( \mathcal{P}_1 \). (The time shift \(-s \) comes from the fact that the root of \( U_s \) is at height \( s \) in \( U \).)

We shall sometimes use Lemma 1 replacing \( \chi_0(s) \) by \( X(s) \), or equivalently \( \exp(\xi_x) \).

**Choosing a spine according to the intrinsic area.** A classical tool in the study of branching processes is the so called spinal decomposition (see [37]). In [6], the authors introduced one related to the intrinsic area (we refer to their work for proofs and more details). Let \( \mathcal{P}_x \) be the joint law of a cell system \( (\chi_u : u \in U) \) starting from an Eve cell \( \chi_0 \) with initial size \( x > 0 \) and a distinguished leaf \( \ell \in \partial U \), such that the law of \( (\chi_u : u \in U) \) under \( \mathcal{P}_x \) is absolutely continuous with respect to its law under \( \mathcal{P}_1 \), with density \( x^{-\omega^{-}} \mathcal{M} \), and \( \ell \) has conditional law \( \mathcal{A}(\cdot)/\mathcal{M} \) given the cell system.

The spine is the process following in time the size of the ancestor of \( \ell \). Up to modifying the genealogy, we can consider the spine to be the Eve cell of the cell-system, the law of the growth-fragmentation remains unchanged (see [6] Section 4.2). Whereas the spine has a tilted distribution, its daughters generate, given the spine, independent growth-fragmentations with laws \( \left( \mathcal{P}_x \right)_{x > 0} \) according to their initial sizes. Its law is that of a positive self-similar Markov process \( Y^- = (Y^-(t))_{t \geq 0} \) with same index \( \alpha \). Let \( \mathcal{P}_x^- \) be the law of \( Y^- \) starting from \( x > 0 \). As for \( X \), we have that \( (Y^-(t))_{t \geq 0} = (\exp(\eta^- (\tau_i))\mathbb{1}_{t \geq 0} \eta^- \) is a Lévy process and \( (\tau_i)_{i \geq 0} \) the associated time-change from the Lamperti’s transformation. (It will always be clear from
the forthcoming lemma 2 that states an explicit and fruitful connection between random variables has been extensively studied (see e.g. [34] and references therein).

Since $\phi'(0) < 0$ by [3], we know that $Y^-$ is absorbed at 0 in finite time almost surely. By definition of Lamperti’s time-change, its absorption time $I$ is given by

$$ I = \int_0^\infty \exp(-\alpha \eta^-(t)) \, dt. \tag{6} $$

Thus written, $I$ is known as an exponential functional of the Lévy process $\eta^-$, and this kind of random variables has been extensively studied (see e.g. [34] and references therein).

One of the interests of this spinal decomposition is that it allows us to study $A$, e.g. through the forthcoming lemma [2] that states an explicit and fruitful connection between $I$ and $A$.

Conditioning the spine to grow indefinitely. A second spinal decomposition was also introduced in [6]. Under assumption [3], the process $n \mapsto \sum_{|u|=n} \chi_u(0)^{\omega_u}$ indexed by generation is also a martingale, but now with terminal value 0 almost surely. One can define the joint law $\mathcal{P}_x^+$ of $(\chi_u : u \in U)$ starting from an Eve cell of initial size $x > 0$ and a leaf $\ell$, such that if $\Gamma_n$ is an event measurable with respect to the sigma-field generated by the cells at generations at most $n \geq 0$, then

$$ \mathcal{P}_x^+(\Gamma_n) = x^{-\omega_\ell} \mathcal{E}_x \left( \mathbb{1}_{\Gamma_n} \sum_{|u|=n+1} \chi_u(0)^{\omega_u} \right). $$

In [6], the ancestors of $\ell$ are selected as follows: let $\ell(n + 1) \in \mathbb{N}^{n+1}$ be the parent of $\ell$ at generation $n + 1$, then its conditional law is

$$ \mathcal{P}_x^+ (\ell(n + 1) = v \left| (\chi_u)_{|u| \leq n} \right) = \frac{\chi_v(0)^{\omega_v}}{\sum_{|u|=n+1} \chi_u(0)^{\omega_u}}, \quad \forall v \in \mathbb{N}^{n+1}. $$

The law of the spine is again that of a positive self-similar Markov process $(Y^+(t))_{t \geq 0} = (\exp(\eta^+(\tau_u)))_{t \geq 0}$ with index $\alpha$, that we denote by $\mathbb{P}_x^+$ when $Y^+$ starts from $x > 0$. Assigning the role of the Eve cell to the spine instead of $\chi_0$ reorders the genealogy of the cell system, but leaves the law of $X$ unchanged, see Section 4.2 in [6]. We shall henceforth write $\mathbb{P}_x^+$ for the distribution of the reordered cell system where $\chi_0$ has law $\mathbb{P}_x^+$, and its daughters generate independent growth-fragmentations given the spine, with laws $(\mathcal{P}_y^+)$ for according to their initial sizes.

The Lévy measure $\Pi^+$ of the Lévy process $\eta^+$ is given by

$$ \Pi^+(dy) = e^{\omega+y}(\Lambda + \tilde{\Lambda})(dy), \tag{7} $$

and the Laplace exponent of $\eta^+$ is

$$ \phi^+(q) = \kappa(\omega_+ + q), \quad q \geq 0. \tag{8} $$

Since $\kappa'(\omega_+) > 0$, the process $\eta^+$ diverges to $\infty$ almost surely, which is therefore also the case of $Y^+$ (but does not explode in finite time since $\alpha < 0$).

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\[ ^{1}\text{See the argument of [6] Lemma 2.1 for the fact that } \phi_- \text{ defines indeed a Laplace exponent of a Lévy process.} \]
It is possible to make sense of $Y^+$ starting from 0 as the limit of $\mathbb{P}_x^+$ as $x \to 0^+$. We can thus define $\mathcal{P}_0^+$, as the law of the growth-fragmentation whose Eve cell has law $\mathbb{P}_x^+$, see Corollary 4.4 in [3]. Note that in this latter result, the convergence of $\mathcal{P}_x^+$ towards $\mathcal{P}_0^+$ as $x \to 0$ is proved only in the sense of finite dimensional distributions. This does not allow us, for instance, to study $A$ with index $\omega$ under $\mathbb{P}_x^+$ in [6]. Note that in this latter result, the convergence of $\Lambda$ denotes the Lévy measure of $\xi$.

Before stating the results of this paper, we make a last assumption on the parameters. Recall that $\Lambda$ is regularly varying at $0^+$ with index $\beta$, and that $\Lambda$ is regularly varying at $0^+$ with index $\omega_\beta$. Indeed, suppose (9) holds and recall that by (4), we have that we see that

$$\Pi^\omega((\infty, \log(x))) = \int_{(\log(1-x), 0)} e^{\omega-y} \Lambda(dy) + \int_{(\log(1-x), 0)} (1 - e^y)^{\omega-y} \Lambda(dy).$$

The converse also follows from the same theorem, that is if $x \to \Pi^\omega((\infty, \log(x)))$ has regular variation at $0^+$ with index $\omega_\beta - \rho$ then it is also the case for $\Lambda$ with index $\beta$. Indeed, coming back to the first part of (10) and using $1 - e^{-y} \sim y$ and $\log(1-y) \sim y$ as $y \to 0^+$, we see that

$$\Pi^\omega((\infty, \log(x))) \sim \int_{(\log(1-x), 0)} |y|^{\omega_\beta - \beta} \Lambda(dy).$$

This is enough to conclude using [12] Theorem 1.6.5 (see the discussion following the proof of the theorem).

Similarly, one can show that regular variation with index $-\rho$ of $\Lambda$ at $0^+$ is equivalent to regular variation with index $\omega_\beta - \rho$ of $x \to \Pi^\omega((\infty, \log(x)))$ as $x \to 0^+$.

Recall the definition of $I$ from [3].

**Theorem 1.** Consider a growth-fragmentation $\mathbf{X}$ such that (3) and (4) are satisfied. Then, for every $x > 0$, it holds that

$$\mathbb{E}^{-\beta / |\alpha|} \Lambda \left( e^{-\beta / |\alpha|} A(e) \to \int_{(\omega_\beta - \rho) (\omega_\beta + |\alpha| - \rho)} |y|^{\omega_\beta - \beta} \Lambda(dy) \right) I^{-\beta / |\alpha|} x^{\omega_\beta - \rho},$$

$\mathcal{P}_x$-almost surely and in $L^1$, where the expectation is finite.
Since \( \alpha < 0 \), we know from [6] Proposition 4.1 that for any \( x > 0 \), \( P_x^+ \) is absolutely continuous with respect to the restriction of \( P_x \) to the natural filtration of the growth-fragmentation and the spine up to time \( t > 0 \). Therefore the above convergence also holds \( P_x^+ \)-a.s.

In Section 6, we shall see that the following interesting result easily follows from self-similarity.

**Proposition 1.** Suppose that (3) holds. Then under \( P_0^+ \), the process

\[
\left( e^{-\frac{\omega}{\alpha}} A(e^{\omega}) \right)_{u \in \mathbb{R}}
\]

is stationary. In particular, the law of \( t^{-\frac{\omega}{\alpha}} A(t) \) does not depend on \( t > 0 \).

The next two propositions provide some bound about the almost sure behaviour of this area and shows that almost surely, \( A(t) \) does not deviate from \( t^{\omega - |\alpha|} \) by more than a power of \( |\log(t)| \):

**Proposition 2.** Suppose that (3) and \( \kappa (\omega_+ + \omega_- + \alpha) < \infty \) hold. For all \( \delta > 0 \), we have that

\[
\limsup_{t \to 0^+} |\log(t)|^{-1-\delta} t^{-\frac{\omega}{\alpha}} A(t) = \mathcal{P}_0^- - \text{a.s.}
\]

**Proposition 3.** Suppose that (3) and (9) hold. Then, there exists \( q_0 > 0 \) such that for all \( q > q_0 \), we have that

\[
\liminf_{t \to 0^+} |\log(t)|^q t^{-\frac{\omega}{\alpha}} A(t) = \infty, \quad \mathcal{P}_0^- - \text{a.s.}
\]

The proof of Proposition 3 provides an explicit \( q_0 \), with a rather complicated expression given in Remark 1 in Section 5. We do not claim however that the bounds of the two above Propositions are optimal.

In Section 6 we shall recall the connection between growth-fragmentations and a family of random surfaces, and apply the above results to the latter, refining in particular some results for the Brownian map.

## 4 Area near the origin

We assume in this section that (3) and (9) hold. Our goal is to prove Theorem 1. We first look at the expectation of \( A(\epsilon) \) as \( \epsilon \to 0^+ \).

### 4.1 Behaviour of the expectation

Under \( \mathcal{P}_1^- \), the spine is chosen according to \( \mathcal{A} \). We can hence study \( A \) through the lifetime of the spine \( I \), thanks to the following relation.

**Lemma 2.** For all \( t \geq 0 \), it holds that

\[
\mathcal{E}_1(A(t)) = \mathbb{P}_1^- (I \leq t).
\]

This is a straightforward consequence of Lemmas 2.1 and 2.2 in [19]. Thanks to Lemma 2 we can deduce

**Lemma 3.** It holds that

\[
\mathcal{E}_1(A(\epsilon)) \sim_{\epsilon \to 0} \frac{|\alpha| \rho}{(\omega_- - \rho)(\omega_- + |\alpha| - \rho)} \mathbb{E}_1^- (I^{-\frac{\omega}{\alpha}})^{1 + \frac{\omega}{\alpha}} \Lambda(\epsilon^{1/|\alpha|}),
\]

where the expectation on the right-hand side is finite.
Proof. Let $\Pi_\alpha^-$ be the Lévy measure of $|\alpha|\gamma^-$. The behaviour of $\mathbb{P}^-_1(I \leq t)$ as $t \to 0$ is given in Theorem 7 of [1]. More precisely, provided that $\Pi_\alpha^-((-\infty, -(x+y))/\Pi_\alpha^-((-\infty, -x)) \sim e^{-\gamma y}$ as $x \to \infty$ with $\mathbb{E}^-_1(\exp(|\alpha|\gamma^-)) < 1$ (corresponding to $|\alpha|\gamma \in (0, \omega_+ - \omega_-)$ by [3]), it holds that

$$\mathbb{P}^-_1(I \leq t) \sim \frac{\mathbb{E}^-_1(I^{-\gamma})}{1+\gamma} t\Pi_\alpha^-((-\infty, \log(1/t))).$$

By [10], we have that

$$\Pi_\alpha^-((-\infty, -x)) \sim \frac{1}{x} \mathcal{A} \left( \frac{e^{\omega x/\alpha}}{\omega - \rho} \right),$$

We thus see that $\Pi_\alpha^-((-\infty, -x))$ is bounded from above by $\exp(-\frac{\omega - \rho}{|\alpha|} y)$ as $x \to \infty$. It remains to check that $\omega - \rho < \omega_+ - \omega_-$, which is true since $2\omega_+ - \omega_+ < \rho < \omega_-$ by [9]. We therefore get by [1] Theorem 7 that

$$\mathbb{P}^-_1(I \leq \epsilon) \sim \epsilon^{-p+1} \left( 1 + \frac{\omega - \rho}{|\alpha|} \right)^{-1} \mathbb{E}^-_1 \left( I^{-\omega/|\alpha|} \right) \epsilon \Pi_\alpha^-((-\infty, \log(\epsilon))) \sim \epsilon^{-p+1} \left( \frac{1}{\omega - \rho} - \frac{\omega}{|\alpha|} \right) \mathbb{E}^-_1 \left( I^{-\omega/|\alpha|} \right) \epsilon \Pi_\alpha^-((-\infty, \log(\epsilon))),$$

as claimed, where the expectation is finite.

We shall need a bit more than the speed of convergence to 0 of the first moment.

Lemma 4. For $p = p(\epsilon) := 1 + 1/\sqrt{|\log(\epsilon)|}$, for all $0 < q < 1 + \frac{\omega - \rho}{|\alpha|}$ it holds that

$$\mathcal{E}_1(A(\epsilon)^p) = o(\epsilon^q), \quad \text{as} \quad \epsilon \to 0.$$ 

Proof. Let $n \geq 1$ be arbitrary large. We write

$$\mathcal{E}_1(A(\epsilon)^p) \leq \mathcal{E}_1(A(\epsilon)^1) \mathbb{1}_{\{A(\epsilon) \leq \epsilon^{-n} \}} \times \epsilon^{-(p-1)} + \mathcal{E}_1(A(\epsilon)^p) \mathbb{1}_{\{A(\epsilon) > \epsilon^{-n} \}}.$$

By Lemma 3 the first term is of order $\epsilon^{1+\frac{n}{|\alpha|}(p-1)} \mathcal{A}(\epsilon^{1/|\alpha|})$, with $p - 1 = 1/\sqrt{-\log(\epsilon)}$ going to 0 with $\epsilon$. It remains to bound the second term. Take $q,q' > 1$ such that $1/q + 1/q' = 1$ and $pq < \omega_+ / \omega_-$, Hölder’s Inequality then yields that

$$\mathcal{E}_1(A(\epsilon)^p) \mathbb{1}_{\{A(\epsilon) > \epsilon^{-n} \}} \leq \mathcal{E}_1(A(\epsilon)^p) \mathbb{1}_{\{A(\epsilon) > \epsilon^{-n} \}} 1/q \times \mathcal{P}(A(\epsilon) > \epsilon^{-n})^{1/q'}.$$ 

Since $pq < \omega_+ / \omega_-$ we know that $\mathcal{E}_1(A(\epsilon)^p) < \infty$ by Lemma 2.3 in [9]. Markov’s Inequality shows that the probability on the right-hand side is bounded from above by

$$\mathcal{E}_1(A(\epsilon))^{1/q'} \times \epsilon^{n/q'}.$$

Since $n$ was chosen arbitrary large, this concludes the proof.

The proof above would work for any $p(\epsilon)$ converging to 1 from above as $\epsilon \to 0$, but will shall use it with this specific choice of $p(\epsilon)$ in order to have a technical result, that we record now for later use.

Lemma 5. For $p = p(\epsilon)$ as in Lemma 4, it holds that

$$\epsilon^{1+\frac{n}{|\alpha|}\mathcal{A}(\epsilon^{1/|\alpha|})} = O\left( \epsilon^{p(1+\frac{n}{|\alpha|})\mathcal{A}(\epsilon^{1/|\alpha|})} \right), \quad \text{as} \quad \epsilon \to 0 +.$$
Proof. By (9) and Theorem 1.4.1(iii) in [12], we know that there exists a function \( \ell : (0, \infty) \to (0, \infty) \) with slow variation at 0 such that \( \Lambda(\ell) \sim \epsilon^{-\rho}\ell(\epsilon) \). We write
\[
\log \left( \frac{e^{1 + \frac{x}{\epsilon^{1/|\alpha|} \Lambda(\epsilon^{1/|\alpha|})}}}{\epsilon^{(1 + \frac{x}{\epsilon^{1/|\alpha|} \Lambda(\epsilon^{1/|\alpha|})})^p}} \right) = (1 - p) \log(\epsilon) + (1 - p) \log \left( \frac{\Lambda(\epsilon^{1/|\alpha|})}{\epsilon^{1/|\alpha|}} \right)
\]
\[
\sim (1 - p) \log(\epsilon) + \frac{(1 - p)p}{\alpha} \log(\epsilon) + (1 - p) \log(\ell(\epsilon^{1/|\alpha|})).
\]
Since \( \ell \) has slow variation, the terms with \( \log(\epsilon) \) dominates. By definition of \( p \) and since \( |\alpha| < \rho \) by (9), we see that \((1 - p)(1 + \rho/\alpha) \log(\epsilon) = (1 + \rho/\alpha) \sqrt{\log(\epsilon)} \to -\infty\) as \( \epsilon \to 0^+ \), which entails the claim. \( \blacksquare \)

4.2 The almost sure behaviour

Throughout this subsection, we shall only work under \( \mathcal{P}_1 \), since the other cases follow from the self-similarity. Indeed, under \( \mathcal{P}_x \), we have \( (A(t))_{t \geq 0} \overset{d}{=} (x^{\alpha} - A(tx^{\alpha}))_{t \geq 0} \) under \( \mathcal{P}_1 \). It means that if Theorem [10] is true under \( \mathcal{P}_1 \), we easily deduce that under \( \mathcal{P}_x \),
\[
\frac{e^{(1 + \frac{x}{\epsilon^{1/|\alpha|} \Lambda(\epsilon^{1/|\alpha|})})}}{\Lambda(\epsilon^{1/|\alpha|})} A(\epsilon) \overset{d}{=} x^{\alpha} \frac{\Lambda((x^{\alpha})^{1/|\alpha|})}{\Lambda(\epsilon^{1/|\alpha|})} \cdot \frac{\Lambda((x^{\alpha})^{1/|\alpha|})}{\Lambda((x^{\alpha})^{1/|\alpha|})} A(x^{\alpha}), \quad \text{under } \mathcal{P}_1
\]
\[
\sim x^{\rho+\alpha} \lim_{\epsilon \to 0^+} \frac{\epsilon^{(1 + \frac{x}{\epsilon^{1/|\alpha|} \Lambda(\epsilon^{1/|\alpha|})})^{1/|\alpha|}}}{\Lambda((x^{\alpha})^{1/|\alpha|})} A(x^{\alpha}), \quad \text{under } \mathcal{P}_1,
\]
where we used the assumption of regular variation of \( \Lambda \) at 0+, as written in (9).

Recall that \( \xi \) is the Lévy process associated with \( X \) by Lamperti’s transformation, and that under \( \mathcal{P}_1 \), \( \chi_0 \) is distributed as \( X \). Let \( \sigma : s \mapsto \int_0^s e^{-\alpha u^{1/|\alpha|}} du \) be the inverse change time of \((\tau_t)_{t \geq 0}\).

Define the compensated process \((M_t)_{t \geq 0}\) as
\[
M_t := \sum_{s \leq t} (\Delta_+ e^{\xi(s)})^{\omega_+} A_s \left( (\sigma \tau_s - \sigma_s)(\Delta_+ e^{\xi(s)})^\alpha \right) - S_t,
\]
where the \( A_s \)'s are i.i.d. copies of \( A \) under \( \mathcal{P}_1 \), independent from \( \xi \), with
\[
S_t := \int_0^t dse^{-\omega_-^{\xi(s)}} \int_{(\infty,0]} \Lambda(dy)(1 - e^y)^{\omega_-} \mathbb{P} \left( I \leq (1 - e^y)^\alpha e^{\alpha \xi(s)}(\sigma_t - \sigma_s)|F_s \right),
\]
where \( I \) is independent from \( \xi \) and \( (F_s)_{s \geq 0} \) is the natural filtration of \( \xi \).

Lemma 6. The process \((M_t)_{t \geq 0}\) is a martingale. Moreover, identifying \( \chi_0 \) with \( X = \exp(\xi_\tau) \), we have that \( M_{\tau_t} = A(t) - S_{\tau_t} \) for all \( t \geq 0 \).

Proof. Thanks to Lemma 2 and independence of the \( A_s \)'s with \( \xi \), one sees that \((S_t)_{t \geq 0}\) is the predictable compensator of the series in \((M_t)_{t \geq 0}\), so that the latter is a martingale. We then write
\[
M_{\tau_t} = \sum_{s \leq \tau_t} (\Delta_- e^{\xi(s)})^{\omega_-} A_s \left( (t - \sigma_s)(\Delta_- e^{\xi(s)})^\alpha \right) - S_{\tau_t}
\]
\[
= \sum_{s \leq t} (\Delta_- e^{\xi(\tau_s)})^{\omega_-} A_{\tau_s} \left( (t - s)(\Delta_- e^{\xi(\tau_s)})^\alpha \right) - S_{\tau_t}
\]
\[
= A(t) - S_{\tau_t},
\]
by Lemma 1 (where we reindexed the \( A_s \)'s), as claimed. \( \blacksquare \)

The main idea of the current section is to use Lemma 6, namely the martingale property of \( M \) and its relation to \( A \), to show that \( A(\epsilon) \sim S_{\tau_\epsilon} \). In this direction, we first establish the following:
Lemma 7. $\mathcal{P}_1$-almost surely, it holds that

$$S_{\tau_\epsilon} \sim_{\epsilon \to 0} \frac{|\alpha|}{(\omega - \rho)(\omega - |\alpha| - \rho)} \mathbb{E}_\tau \left( \int_{-\infty}^{(1-\epsilon^2) \mathbb{E}_\tau \left( e^{1+\frac{\omega - \rho}{|\alpha|}} \right) \right) .$$

Proof. Recall (1). After a change of variables and replacing $\exp(\xi(\tau_\epsilon))$ by $X(s)$, we have that

$$S_{\tau_\epsilon} = \int_0^\epsilon ds X(s) \omega (\omega - \rho) \int_{-\infty}^{0} \Lambda(dy)(1 - e^y) \omega (\omega - \rho) \mathbb{P}_\tau \left( I \leq (1 - e^{-\rho}) X(s)^\alpha (\epsilon - s) \mathcal{F}_{\tau_s} \right) \mathcal{P}_1.$$

It is well known that the law of $I$ is absolutely continuous, see e.g. [8] Theorem 3.9. Hence, since $X(s) \sim 1$ as $s \to 0^+$, one sees that

$$S_{\tau_\epsilon} \sim_{\epsilon \to 0} \int_0^\epsilon ds \int_{-\infty}^{0} \Lambda(dy)(1 - e^y) \omega \mathbb{P}_\tau \left( I \leq (\epsilon - s)(1 - e^{-\rho}) \mathcal{P}_1 \right),$$

we express the right-hand side in the form

$$\int_{-\infty}^{0} \Lambda(dy)(1 - e^y) \omega \mathbb{P}_\tau \left( I \leq s(1 - e^{-\rho}) \right)$$

$$= \int_{-\infty}^{0} \Lambda(dy)(1 - e^y) \omega \mathbb{P}_\tau \left( I \leq s \right)$$

$$= C_1(\epsilon) + C_2(\epsilon),$$

where $C_1(\epsilon)$ is the part of the first integral with domain restricted to $E_1 := (-\infty, \log \left( 1 - e^{-\rho} \right))$ for $\delta > 0$ arbitrary small, and $C_2(\epsilon)$ is that restricted to $E_2 := (0, \log (1 - e^{-\rho}))$. We first address $C_1(\epsilon)$. Note that for $y \in E_1$, we have that $s \leq \epsilon(1 - e^{-\rho}) \leq e^\delta$. Appealing to lemmas [2] and [3] there exists a constant $C$ such that

$$\mathbb{P}_\tau \left( I \leq s \right) \leq Cs^{1+\frac{\omega - \rho}{|\alpha|}} \mathbb{E}_\tau \left( s^{1/|\alpha|} \right), \quad \forall s \leq e^\delta.$$

In the following, we shall keep writing $C$ for any positive and finite constant that does not depend on $\epsilon$ and that may change from line to line. We have

$$C_1(\epsilon) \leq C \int_{E_1} \Lambda(dy)(1 - e^y) \omega \mathbb{P}_\tau \left( s^{1+\frac{\omega - \rho}{|\alpha|}} \right) ds$$

$$\leq C \int_{E_1} \Lambda(dy)(1 - e^y) \omega \mathbb{P}_\tau \left( s^{1+\frac{\omega - \rho}{|\alpha|}} \right) \times e^{\delta (2+\frac{\omega - \rho}{|\alpha|}) \mathbb{E}_\tau \left( \mathbb{E}_\tau \left( \mathbb{P}_\tau \left( I \leq s \right) \right) \right),$$

where we used Theorem 1.5.11(ii) of [12]. Thanks to Theorem 1.6.4 of the same book, we conclude that

$$C_1(\epsilon) = O \left( e^{\delta (1+\frac{\omega - \rho}{|\alpha|}) \mathbb{E}_\tau \left( \mathbb{P}_\tau \left( I \leq s \right) \right) \times e^{\delta (2+\frac{\omega - \rho}{|\alpha|}) \mathbb{E}_\tau \left( \mathbb{P}_\tau \left( I \leq s \right) \right) \right)$$

$$= O \left( e^{\delta \times \epsilon^{1+\frac{\omega - \rho}{|\alpha|}} \mathbb{E}_\tau \left( \mathbb{P}_\tau \left( I \leq s \right) \right) \times \mathbb{E}_\tau \left( \mathbb{P}_\tau \left( I \leq s \right) \right) \right)$$

$$= O \left( e^{\epsilon^{1+\frac{\omega - \rho}{|\alpha|}} \mathbb{E}_\tau \left( \mathbb{P}_\tau \left( I \leq s \right) \right) \times \mathbb{E}_\tau \left( \mathbb{P}_\tau \left( I \leq s \right) \right) \right),$$

where the last line follows straightforwardly from the existence of a slowly varying function $\ell : (0, \infty) \to (0, \infty)$ at the origin such that $\mathbb{E}_\tau \left( X \right) = x^{-\ell(x)}$, see [12] Theorem 1.4.1(iii).

We now investigate $C_2(\epsilon)$. Let $k$ denote the density of $I$ under $\mathbb{P}_\tau$. After applying Tonelli’s Theorem, we have that

$$C_2(\epsilon) = \int_0^{\epsilon^\delta} k(x) dx \int_{E_2} \Lambda(dy)(1 - e^y)(\epsilon - x(1 - e^{-\rho})) + \int_{\epsilon^\delta}^\infty k(x)xf(\epsilon/x) dx, \quad (11)$$
where 

\[ f(u) := \int_{\log(1-u^{-\alpha})}^{0} \Lambda(dy)(1 - e^{y})^{\omega}(u - (1 - e^{y})^{\alpha}). \]

Theorem 1.6.5 of [12] allows us to bound the first term in the right-hand side of (11) by

\[
\mathbb{P}_{1}^{\epsilon} \left( I \leq \epsilon^{\delta} \right) \times \epsilon \int_{E_{\epsilon}} \Lambda(dy)(1 - e^{y})^{\omega} \sim C_{\epsilon} \mathbb{P}_{1}^{\epsilon} \left( I \leq \epsilon^{\delta} \right) \epsilon^{\frac{1 - \delta}{1 + \alpha/|\alpha|}} \epsilon^{1 + \frac{\alpha}{|\alpha|}} \Lambda(\epsilon^{1/|\alpha|})
\]

with the same argument as for \( C_{1}(\epsilon) \).

We turn our attention to the second term in (11). First, with the help of Theorem 1.6.5 of [12], it can be shown that

\[
f(u) \sim_{u \to 0+} \left( \frac{\rho}{\omega - \rho} - \frac{\rho}{\omega + |\alpha| - \rho} \right) u^{\frac{\rho}{|\omega| + |\alpha|}} \Lambda(u^{1/|\alpha|})
\]

\[
= \frac{|\alpha|\rho}{(\omega - \rho)(\omega + |\alpha| - \rho)} u^{\frac{\rho}{|\omega| + |\alpha|}} \Lambda(u^{1/|\alpha|})
\]

Note that \( \epsilon/x \leq \epsilon^{1-\delta} \) for all \( x > \epsilon^{\delta} \). The estimate above yields

\[
\int_{\epsilon^{\delta}}^{\infty} k(x)x f(\epsilon/x) dx \sim_{\epsilon \to 0+} \frac{|\alpha|\rho}{(\omega - \rho)(\omega + |\alpha| - \rho)} \int_{\epsilon^{\delta}}^{\infty} k(x)x \frac{\rho}{|\omega| + |\alpha|} \Lambda((\epsilon/x)^{1/|\alpha|}) dx.
\]

By dominated convergence, we then see that

\[
\left( \epsilon^{1 + \frac{\alpha}{|\omega| + |\alpha|}} \Lambda(\epsilon^{1/|\alpha|}) \right)^{-1} \int_{\epsilon^{\delta}}^{\infty} k(x)x f(\epsilon/x) dx \sim \frac{|\alpha|\rho}{(\omega - \rho)(\omega + |\alpha| - \rho)} \mathbb{E}_{1}^{\left( \int \frac{\rho}{|\omega| + |\alpha|} \right)},
\]

as claimed.

We are ready to provide the proof of Theorem 4

**Proof of Theorem 4.** Provided that \( \epsilon \mapsto M_{\epsilon} \) has regular variation at 0, one has that \( M_{\epsilon} \sim M_{\epsilon} \) as \( \epsilon \to 0^{+}, \) as a consequence of \( \tau_{\epsilon} \sim \epsilon \). Thus, thanks to Lemmas 6 and 7, it suffices to show that

\[
\lim_{\epsilon \to 0} \left( \epsilon^{1 + \frac{\alpha}{|\omega| + |\alpha|}} \Lambda(\epsilon^{1/|\alpha|}) \right)^{-1} \epsilon_{1} = 0, \quad \mathcal{P}_{1}\text{-a.s.}
\]

Define the stopping time

\[
T := \inf \left\{ s > 0 : |\xi(s)| > 1 \right\}.
\]

We proceed as follows: we know that \( M \) has finite moment of order \( p \) as soon as \( p < \omega_{+}/\omega_{-} \) by [6] Lemma 2.3. Let \( p = p(\epsilon) = 1 + \frac{1}{\sqrt{\log(\epsilon)}} \) as in Lemma 4 and let \( a > 0 \) be arbitrary small, we have by Markov’s Inequality that

\[
\mathbb{P}_{1} \left( \sup_{s \leq \epsilon \wedge T} \frac{|M_{s}|}{\epsilon^{1 + \frac{\alpha}{|\omega| + |\alpha|}} \Lambda(\epsilon^{1/|\alpha|})} \geq a \right) \leq \left( a \epsilon^{1 + \frac{\alpha}{|\omega| + |\alpha|}} \Lambda(\epsilon^{1/|\alpha|}) \right)^{-p} \mathbb{E}_{1} \left( \sup_{s \leq \epsilon \wedge T} |M_{s}|^{p} \right)
\]

\[
\leq (6p)^{p} \left( a \epsilon^{1 + \frac{\alpha}{|\omega| + |\alpha|}} \Lambda(\epsilon^{1/|\alpha|}) \right)^{-p} \mathbb{E}_{1} (|M|^{p/2}),
\]

thanks to Burkholder-Davis-Gundy Inequality, where \( |M| \) denotes the quadratic variation of \( M \) (see Theorem 92 p.304 in [18] for the constant \((6p)^{p}\)). In particular, since \( M \) is a purely
discontinuous martingale, appealing to [28] Section 3(c) we get that the previous quantity is bounded from above by

\[
(6p)^p \left( e^{1 + \frac{\omega}{w}} \Lambda(e^{1/|\alpha|}) \right)^{-p} \mathcal{E}_1 \left( \sum_{s \leq \epsilon \wedge T} (\Delta e^{\xi(s)})^{p_{\omega - \alpha}} A_s \left( (\epsilon \wedge T - s)(\Delta e^{\xi(s)})^\alpha \right)^p \right),
\]

Since \( T > 0 \) almost surely by right-continuity of \( \xi \), in order to conclude, it is sufficient to show that the expectation above is \( o \left( e^{p(1 + \frac{\omega}{w}) \Lambda(e^{1/|\alpha|})} \right) \), as \( \epsilon \to 0 \). We write

\[
\mathcal{E}_1 \left( \sum_{s \leq \epsilon \wedge T} (\Delta e^{\xi(s)})^{p_{\omega - \alpha}} A_s \left( (\epsilon \wedge T - s)(\Delta e^{\xi(s)})^\alpha \right)^p \right)
= \mathcal{E}_1 \left( \sum_{s \leq \epsilon \wedge T} e^{p_{\omega - \alpha}(s^-)} (1 - e^{\Delta \xi(s)})^{p_{\omega - \alpha}} A_s \left( (\epsilon \wedge T - s)e^{\alpha \xi(s^-)}(1 - e^{\Delta \xi(s)})^\alpha \right)^p \right)
\leq e^{p_{\omega - \alpha}} \mathcal{E}_1 \left( \sum_{s \leq \epsilon} (1 - e^{\Delta \xi(s)})^{p_{\omega - \alpha}} A_s \left( e^{\alpha}(\epsilon - s)(1 - e^{\Delta \xi(s)})^\alpha \right)^p \right).
\]

(Recall that the \( A_s \)'s are non-decreasing and non-negative functions.) Applying the compensation formula, the upper bound above becomes, after an implicit change of variables,

\[
e^{p_{\omega - \alpha}} \int_0^\epsilon ds \int_{(-\infty,0)} \Lambda(dy)(1 - e^y)^{p_{\omega - \alpha}} \mathcal{E}_1 \left( A \left( e^{\alpha} s (1 - e^y)^\alpha \right)^p \right)
= e^{p_{\omega - \alpha} + |\alpha|} \int_{(-\infty,0)} \Lambda(dy)(1 - e^y)^{p_{\omega - \alpha}} \int_0^\epsilon ds \mathcal{E}_1 \left( A \left( e^{\alpha} s (1 - e^y)^\alpha \right)^p \right)
= e^{p_{\omega - \alpha} + |\alpha|} \int_{(-\infty,0)} \Lambda(dy)(1 - e^y)^{p_{\omega - \alpha} + |\alpha|} \int_0^{e(1 - e^y)^\alpha} ds \mathcal{E}_1 \left( A \left( e^{\alpha} s \right)^p \right)
= e^{p_{\omega - \alpha} + |\alpha|} \int_0^\epsilon ds \mathcal{E}_1 \left( A \left( e^{\alpha} s \right)^p \right) \int_{(-\infty,0)} \Lambda(dy)(1 - e^y)^{p_{\omega - \alpha} + |\alpha|}
+ e^{p_{\omega - \alpha} + |\alpha|} \int_0^\epsilon ds \mathcal{E}_1 \left( A \left( e^{\alpha} s \right)^p \right) \int_{\log(1 - e^y)\frac{s_{\omega}}{m_{\alpha}}.0} \Lambda(dy)(1 - e^y)^{p_{\omega - \alpha} + |\alpha|}
\leq e^{p_{\omega - \alpha} + |\alpha|} \int_0^{K\epsilon} ds \mathcal{E}_1 \left( A \left( e^{\alpha} s \right)^p \right) \int_{(-\infty,0)} \Lambda(dy)(1 - e^y)^{p_{\omega - \alpha} + |\alpha|}
+ e^{p_{\omega - \alpha} + |\alpha|} \int_0^\epsilon ds \mathcal{E}_1 \left( A \left( e^{\alpha} s \right)^p \right) \int_{\log(1 - e^y)\frac{s_{\omega}}{m_{\alpha}}.0} \Lambda(dy)(1 - e^y)^{p_{\omega - \alpha} + |\alpha|},
\]

where \( K \) is a constant intended to be large. The first term is negligible since by Lemma [3] for \( \delta > 0 \) small enough, we have that

\[
\int_0^\epsilon \mathcal{E}_1 \left( A \left( e^{\alpha} s \right)^p \right) ds = o(e^{2 + \frac{\omega}{w} \frac{\epsilon}{m_{\alpha}} - \delta}) = o(e^{p(1 + \frac{\omega}{w} \frac{\epsilon}{m_{\alpha}} - \delta)}).
\]

(Recall that \( p = 1 + 1/\sqrt{\log(|\alpha|)} \).) To bound the second term, we first note that for all \( s > K\epsilon \), we have \( \epsilon/s \leq 1/K \), so that Theorem 1.6.5 of [12] entails the existence of a constant \( C \) such that

\[
\int_{\log(1 - e^y)\frac{s_{\omega}}{m_{\alpha}}.0} \Lambda(dy)(1 - e^y)^{p_{\omega - \alpha} + |\alpha|} \leq C \left( \frac{\epsilon}{s} \right)^{\frac{p_{\omega - \alpha} + |\alpha|}{|\alpha|}} \Lambda \left( (\epsilon/s)^{1/|\alpha|} \right)
\leq C e^{1 + \frac{p_{\omega - \alpha} - p}{|\alpha|}} \left( \frac{1}{s} \right)^{1 + \frac{p_{\omega - \alpha} - p}{|\alpha|}} \ell \left( (\epsilon/s)^{1/|\alpha|} \right),
\]

13
Lemma 8. Suppose that \( E^{\ell} \) where \( \ell \).

In this subsection, we do not assume regular variation of \( E^{\ell} \) holds. The convergence in \( L^1 \) is just a consequence of Lemma 5 and Scheffé's Lemma and the proof is complete. ■

5 Area near the root starting from 0

We start this section by proving Proposition 1.

Proof of Proposition 1. Let \( Y^+ \) have law \( P_0^+ \). We identify \( \chi_0 \) with \( Y^+ \) and similarly to Lemma 4 under \( P_0^+ \), we can write

\[
\{ t \rightarrow A(t); t \geq 0 \} = \{ t \rightarrow \sum_{s \leq t} |\Delta_+ Y^+(s)|^{\omega-} A_s \left( (t-s)|\Delta_+ Y^+(s)|^{\alpha} \right); t \geq 0 \},
\]

where \( (A_s)_{s \in \mathbb{R}_+} \) is a family of i.i.d. copies of \( A \) under \( P_1 \), mutually independent from \( Y^+ \). The fact that for all \( t \geq 0 \), \( A(t) < \infty \) \( P_0^+ \)-a.s. follows from \( \sum_{s \leq t} |\Delta_+ Y^+(s)|^{\omega-} < \infty \) \( P_0^+ \)-a.s. by Lemma 4.3 in [6] and that each \( A_s(t) \) is bounded for all \( t \geq 0 \) by its total mass \( M_s \), having expectation 1.

Recall the self-similarity property \((Y^+(t))_{t \geq 0} \overset{d}{=} (xY^+(tx^\alpha))_{t \geq 0}\) for any \( x > 0 \). Combining this fact and the above equation yields, fixing \( u \in \mathbb{R} \), that

\[
\left\{ e^{\frac{\omega}{\alpha}(u+t)} A(e^{u+t}); t \geq 0 \right\} = \left\{ e^{\frac{\omega}{\alpha}(u+t)} \sum_{s \leq e^{u+t}} |\Delta_+ Y^+(s)|^{\omega-} A_s \left( (e^{u+t} - s)|\Delta_+ Y^+(s)|^{\alpha} \right); t \geq 0 \right\}
\]

\[
= \left\{ e^{\frac{\omega}{\alpha}(u+t)} \sum_{s \leq e^t} |\Delta_+ Y^+(se^\alpha)|^{\omega-} A_{se^\alpha} \left( (e^{u+t} - se^\alpha)|\Delta_+ Y^+(se^\alpha)|^{\alpha} \right); t \geq 0 \right\}
\]

\[
= \left\{ e^{\frac{\omega}{\alpha}t} \sum_{s \leq e^t} |\Delta_+ Y^+(s)|^{\omega-} A_{se^\alpha} \left( (e^t - s)|\Delta_+ Y^+(s)|^{\alpha} \right); t \geq 0 \right\}
\]

\[
= \left\{ e^{\frac{\omega}{\alpha}t} A(e^t); t \geq 0 \right\},
\]

which shows the claim. ■

Our goal in the rest of this section is to prove Propositions 2 and 3.

5.1 The upper bound

In this subsection, we do not assume regular variation of \( A \) as in \( 13 \), but only that \( \kappa(\omega_+ + \omega_- + \alpha) < \infty \). We aim at proving Proposition 2. Thanks to Proposition 1, we can restrict ourselves without loss of generality to the case where \( t \to 0+ \). We shall need the finiteness of \( E_0^+(t^{\omega-}/\alpha A(t)) \) for all \( t > 0 \), in order to apply Markov’s Inequality on some well chosen sequence of events. Thanks to Proposition 1, it suffices to look at \( E_0^+(A(1)) \).

Lemma 8. Suppose that \( \kappa(\omega_+ + \omega_- + \alpha) < \infty \). It holds that \( E_0^+(A(1)) < \infty \).
Proof. Thanks to \([14]\), under \(\mathcal{P}_0^+\), we have that
\[
A(1) = \sum_{s \leq 1} |\Delta_- Y^+(s)|^{\omega-} A_s \left( (1 - s)|\Delta_- Y^+(s)|^{\alpha} \right) \leq \sum_{s \leq 1} |\Delta_- Y^+(s)|^{\omega-} \mathcal{M}_s, 
\]
where each \(\mathcal{M}_s\) is distributed as \(\mathcal{M}\) under \(\mathcal{P}_1\) and is independent from \(Y^+\). We can see the above upper bound as the stochastic integral of \(s \mapsto \mathcal{M}_s\) with respect to the non-decreasing process \(t \mapsto \sum_{s \leq t} |\Delta_- Y^+(s)|^{\omega-}\) until time 1. In particular, we can use the optional projection theorem from \([8]\) Theorem 57 (see Theorem 43 of the same book for the definition of the optional projection), which states that the expectation of the stochastic integral is equal to the same expectation where each \(\mathcal{M}_s\) has been replaced by its conditional expectation given the natural filtration of \(t \mapsto \sum_{s \leq t} |\Delta_- Y^+(s)|^{\omega-}\). In particular, for every negative jump time \(s > 0\), \(\mathcal{E}_0^+(\mathcal{M}_s|(Y^+(u))_{u \leq s}) = 1\). Therefore,
\[
\mathcal{E}_0^+(A(1)) \leq \mathcal{E}_0^+ \left( \sum_{s \leq 1} |\Delta_- Y^+(s)|^{\omega-} \right).
\]
The expected value of the above sum is equal to the expectation of its predictable compensator, given in the proof of Lemma 4.3 in \([6]\). We thus obtain that the upper bound is equal to
\[
\int_{(-\infty,0)} \Pi^+(dy)(1 - e^y)^{\omega-} e^{\omega+ y} \mathcal{E}_0^+ \left( \int_0^1 Y^+(s)^{\omega- + \alpha} ds \right).
\]
The first integral is finite by \([7]\) and \([3]\). After using Tonelli’s Theorem and self-similarity, the other part of the expression becomes
\[
\int_0^1 s^{\omega- + 1} ds \mathcal{E}_0^+ \left( Y^+(1)^{\omega- + \alpha} \right),
\]
Hence, it only remains to check that the expectation is finite. If \(|\alpha| \leq \omega_-\), then this is the case by Proposition 1(ii) in \([10]\), which ensures that the positive moments of \(Y^+(1)\) are finite.

Suppose now that \(\alpha < -\omega_-\). Theorem 1(iii) in \([9]\) shows that
\[
\mathcal{E}_0^+ \left( Y^+(1)^{\omega- + \alpha} \right) = C \mathbb{E}_1^+ \left( I_{\eta^+}^{\omega- + \alpha} \right),
\]
where \(C\) is an explicit constant and
\[
I_{\eta^+} := \int_0^\infty \exp(\alpha \eta^+(t)) dt
\]
(recall that \(\eta^+(0) = 0\) under \(\mathbb{E}_1^+\)). Proposition 2 in \([10]\) gives the finiteness and an expression for the negative moments of \(I_{\eta^+}\), under the assumption that \(e^{\eta^+(1)}\) admits positive moments of all order. However, the proof straightforwardly adapts to \(\mathbb{E}_1^+ (I_{\eta^+}^{\omega- + \alpha})\) as soon as \(\mathbb{E}_1^+ (\exp((\omega_- + \alpha)\eta^+(1))) < \infty\), or equivalently \(\kappa(\omega_+ + \omega_- + \alpha) < \infty\). ■

Proof of Proposition \([2]\). Let \(\delta > 0\). For all \(n \geq 1\), we have that
\[
\mathcal{P}_0^+ \left( 2^{(n+1)\omega-} A(2^{-n}) > \log(2^{n+1})^{1+\delta} \right) = \mathcal{P}_0^+ \left( A(1) > 2^{\omega-} \log(2^{n+1})^{1+\delta} \right) \leq \frac{1}{n^{1+\delta} \log(2^{n+1})^{1+\delta}} \mathcal{E}_0^+(A(1)).
\]
The latter is finite by Lemma \([5]\) and therefore summable over \(n\). Borel-Cantelli’s Lemma ensures that \(\mathcal{P}_0^+\)-a.s. for all \(n\) sufficiently large, \(A(2^{-n}) \leq 2^{(n+1)\omega-} \log(2^{n+1})^{1+\delta}\). Since \(A\) is non-decreasing, it means that \(\mathcal{P}_0^+\)-almost surely, for all \(n\) large enough and all \(t \in [2^{-n}, 2^{-n-1}]\), it holds that
\[
A(t) \leq t^{\omega-} \log(t)^{1+\delta},
\]
which concludes the proof. ■
5.2 The lower bound

Our purpose now is to show Proposition \[14\]. We hence assume that \[14\] holds. Similarly as for the upper bound, we can study only the asymptotic behaviour as \(t \to \infty\) and easily deduce the one as \(t \to 0^+\), thanks to Proposition \[11\].

The strategy is to decompose \(A(t)\) over the jumps of \(Y^+\) as in \[14\], then to find two functions such that, \(\mathcal{P}_0^+\)-almost surely, the motion of \(Y^+(s)\) is circumscribed in between them for all \(s\) large enough.

5.2.1 Upper and lower envelopes of the Eve cell

The so-called lower and upper envelopes of positive self-similar Markov processes are described respectively in \[16\] and \[33\]. We need some preparations to apply the results of these two papers.

Let \(U(x)\) be the last passage time of \(Y^+\) below \(x > 0\), that is \(U(x) := \sup \{ t \geq 0: Y^+(t) \leq x \}\). Define \(\nu := Y^+(U(x)-)/x\) (note that if \(\Lambda((0,\infty)) = 0\) then \(\nu = 1\) almost surely). When the process admits positive jumps, the law of \(\nu\) is given in \[16\] Lemma 1: it has its support included in \([0,1]\) and satisfies

\[
\mathbb{P}(\nu \leq u) = \mathbb{E}(H(1)^{-1}) \int_0^1 dv \int_{(-\log(v)/|\alpha|, \infty)} y\Pi_H(dy/|\alpha|), \quad u \in (0,1),
\]

where the subordinator \((H(t))_{t \geq 0}\) is the ascending ladder height process associated with \(\eta^+\) and \(\Pi_H\) is its Lévy measure (see Chapter VI of \[2\] for background).

The next lemma will be needed to apply the results describing the envelopes of \(U\) and \(Y^+\). Recall the definition of \(I_{\eta^+}\) in \[15\].

**Lemma 9.** (i) For all \(0 < p < (\omega_+ - \omega_-)/|\alpha|\), it holds that

\[
\mathbb{P}_1(I_{\eta^+} \leq t) = o(t^p) \quad \text{and} \quad \mathbb{P}_1(I_{\eta^+} > 1/t) = O(t), \quad \text{as} \quad t \to 0^+.
\]

(ii) Let \(q^* := \min\{\omega_+ - \omega_-, \sup\{p \geq 0: \kappa(\omega_+ + p) < \infty\}\}\). For all \(q < q^*\), it holds that

\[
\mathbb{P}_1(\nu I_{\eta^+} \leq t) = o(t^{q/|\alpha|}), \quad \text{as} \quad t \to 0^+,
\]

where \(\nu\) and \(I_{\eta^+}\) are independent.

**Proof.** (i) It essentially follows from \[35\] Lemma 3 that provides finiteness of some positive and negative moments of \(I_{\eta^+}\). (Note that Lemma 3 in \[35\] gives \(\mathbb{E}_1(I_{\eta^+}^{-1}) = |\alpha'\omega_+|\), which is indeed finite under our assumption \[35\].)

(ii) For all \(u \in (0,1)\), equation \[14\] entails that

\[
\mathbb{P}(\nu \leq u) \leq \mathbb{E}(H(1)^{-1})|\alpha| \int_{(-\log(u)/|\alpha|, \infty)} y\Pi_H(dy)
\]

\[
= \mathbb{E}(H(1)^{-1})|\alpha| \left(-\log(u)/|\alpha|\right)\Pi_H(-\log(u)/|\alpha|) + \int_{-\log(u)/|\alpha|}^{\infty} \Pi_H(z)dz,
\]

by Tonelli’s Theorem. Theorem 7.8 in \[21\] shows that the right tail of \(\Pi_H\) is described for all \(y > 0\) by

\[
\Pi_H(y) = \int_{[0,\infty)} \mathbb{U}(dz)\Pi^+(z + y),
\]

where \(\mathbb{U}\) is the renewal measure of the descending ladder height process of \(\eta^+\). Note that the integral is finite, see e.g. \[21\] Corollary 5.3. By assumption \[35\], there exists \(q > 0\) such that \(q < \sup\{p \geq 0: \kappa(\omega_+ + p) < \infty\}\). In particular, \(\kappa(\omega_+ + q) < \infty\), which is equivalent to

\[
\int_{(1,\infty)} e^{\nu y}\Pi^+(dy) = \Pi^+(1) + q \int_{0}^{\infty} e^{\nu x}\Pi^+(x)dx < \infty,
\]
so in particular $\Pi^+(x) = o(e^{-qx})$ as $x \to \infty$. This leads to
\[
\Pi_H(y) \leq e^{-qy} \int_{[0, \infty)} \hat{U}(dz)e^{-qz} = O(e^{-qy})
\]
Coming back to (17), we see that
\[
\mathbb{P}(\nu \leq u) = O \left( |\log(u)|u^{q/|\alpha|} + \int_0^u \Pi_H(-\log(y)/|\alpha|) \frac{du}{y} \right) = O \left( |\log(u)|u^{q/|\alpha|} + \int_0^u y^{q/|\alpha| - 1} dy \right)
\]
\[
= O \left( |\log(u)|u^{q/|\alpha|} \right).
\]
For $\delta > 0$ small enough, the same reasoning holds with $q$ replaced by $q + \delta$, so that $\mathbb{P}(\nu \leq u) = o(u^{q/|\alpha|})$.

To conclude, suppose that $X_1$ and $X_2$ are independent positive random variables such that for $i = 1, 2$, $\mathbb{P}(X_i \leq t) = o(t^{p_i})$ as $t \to 0+$, for some $q_1 > q_2 > 0$. Note that there exists a constant $C$ such that $\mathbb{P}(X_2 \leq t) \leq Ct^{q_2}$ for all $t \geq 0$. Then, we write
\[
\mathbb{P}_1(X_1X_2 \leq t) \leq \int_{\mathbb{R}^+} \mathbb{P}(X_1 \leq tx)\mathbb{P}(X_2 \leq t/x) \leq \int_{\mathbb{R}^+} \mathbb{P}(X_1 \leq dx)C(t/x)^{q_2} = O(t^{q_2}),
\]
since $q_1 > q_2$. The bound for $\mathbb{P}_1(\nu I_{1+} \leq t)$ as $t \to 0+$ thus follows from this fact and part (i).

Let $p > 1/|\alpha|$ and $\overline{p} > 1/q^*$ where $q^*$ is defined in Lemma 9. Define on $(1, \infty)$ the functions
\[
g^+ : t \mapsto t^{1/|\alpha|} \log(t)^{\overline{p}}, \quad g^i : t \mapsto t^{1/|\alpha|} \log(t)^{-\overline{p}}.
\]
Note that $g^+$ and $g^i$ are regularly varying at $\infty$ with index $1/|\alpha|$. Let $\overline{g}(t) := \sup_{s \leq t} g^+(s)$ and similarly, let $\underline{g}(t) := \inf_{s \geq t} g^i(s)$. These functions have the nice property to be strictly increasing and by Theorem 1.5.3. in [12] we have that
\[
\overline{g}(t) \sim t^{\overline{g}^+}(t) \quad \text{and} \quad \underline{g}(t) \sim t^{\underline{g}^i}(t).
\]

**Lemma 10.** $\mathbb{P}_0^+$-almost surely, there exists $s > 0$ such that
\[
g(t) < Y^+(t) < \overline{g}(t), \quad \forall t \in (s, \infty).
\]

**Proof.** First, note that $Y^+(t) < x$ implies $U(x) > t$. Let $T(1) := \inf\{t \geq 0 : Y^+(t) \geq 1\}$ and note that $T(1) \leq U(1)$. Proceeding as in the proof of Proposition 6 in [33], one can show that
\[
\mathbb{P}(T(1) < t) \leq C\mathbb{P}(\nu I_{1+} \leq t).
\]

Lemma 10(ii) in this paper and Proposition 4 in [33] then show the claim for \(\overline{g}\). The statement involving \(\underline{g}\) is shown using Theorem 1(i) in [13] and our Lemma 9(i).

5.2.2 Proof of Proposition 3

For clarity purpose, we prove two Lemmas that will make the proof of Proposition 3 straightforward. We need some notation. Define
\[
f(t) := t \log(t)^{\overline{p}}.
\]
We note for later use that (18) and (19) implies that
\[
\overline{g}(f(t)) \sim t^{\overline{g}^+(t)} \log(t)^{-\overline{p}}, \quad \underline{g}(f(t)) \sim t^{\underline{g}^i(t)}.
\]
To control the fluctuations of $Y^+$, we work on the event

$$E_t := \{ \forall s \geq f(t) : g(s) < Y^+(s) < \mathcal{G}(s) \}.$$  

Note that $f$ being eventually increasing, Lemma 11 ensures that $P^+_0 \{ \lim \inf t \to \infty E_t \} = 1$. Moreover, noting that $f(t)/t \to 0$ as $t \to \infty$, we see that $J_t := [f(t), f(2t)] \subset [0, t]$.

**Lemma 11.** For all $a > 0$, $\mathcal{P}^+_0$-almost surely, it holds on the event $E_t$ that

$$A(t) \geq t^\frac{3}{2\alpha} \log(t)^{-\omega-(a+\mathcal{P}_+\mathcal{P})} \sum_{s \in J_t} A_s(1/3) 1_{\{ \Delta_-(\tau_s) \leq \log(1-\log(t)^{-a}) \}},$$

for all $t$ large enough.

**Proof.** We need the lower envelope of $Y^+$ to ensure that the jumps are large enough, and the upper envelope of $Y^+$ for the area generated from it to be not too small. On the event $E_t$, we have

$$A(t) = \sum_{s \leq t} |\Delta_- Y^+(s)|^{\omega-} A_s ( (t-s) |\Delta_- Y^+(s)|^\alpha )$$

$$\geq \sum_{s \in J_t} |\Delta_- Y^+(s)|^{\omega-} A_s ( (t-s) |\Delta_- Y^+(s)|^\alpha )$$

$$\geq \sum_{s \in J_t} g(s)^{\omega-} (1 - e^{\Delta_- \eta^+(\tau_s)})^{\omega-} A_s \left( (t-s) \mathcal{G}(s)^{\alpha} (1 - e^{\Delta_- \eta^+(\tau_s)})^\alpha \right)$$

$$\geq \sum_{s \in J_t} g(f(t))^{\omega-} (1 - e^{\Delta_- \eta^+(\tau_s)})^{\omega-} A_s \left( (t-f(2t)) \mathcal{G}(f(2t))^{\alpha} (1 - e^{\Delta_- \eta^+(\tau_s)})^\alpha \right).$$

Note that $f(2t)/t \to 0$ and (21) entails that $t \mathcal{G}(f(2t))^\alpha \to 1/2$ as $t \to \infty$. Hence, taking $t$ large enough, we see for each $s \in J_t$,

$$A_s \left( (t-f(2t)) \mathcal{G}(f(2t))^{\alpha} (1 - e^{\Delta_- \eta^+(\tau_s)})^\alpha \right) \geq A_s(1/3).$$

We write

$$A(t) \geq g(f(t))^{\omega-} \sum_{s \in J_t} (1 - e^{\Delta_- \eta^+(\tau_s)})^{\omega-} A_s (1/3) 1_{\{ \Delta_- \eta^+(\tau_s) < \log(1-\log(t)^{-a}) \}}$$

$$\geq g(f(t))^{\omega-} \log(t)^{-\omega-} \sum_{s \in J_t} A_s (1/3) 1_{\{ \Delta_- \eta^+(\tau_s) < \log(1-\log(t)^{-a}) \}}$$

$$\sim t^\frac{3}{2\alpha} \log(t)^{-\omega-(a+\mathcal{P}_+\mathcal{P})} \sum_{s \in J_t} A_s(1/3) 1_{\{ \Delta_- \eta^+(\tau_s) < \log(1-\log(t)^{-a}) \}},$$

where we used (21).

For the lower bound of Lemma 11 to be useful, we need to make sure that the sum contains at least one non negligible term. We shall use the following:

**Lemma 12.** For all $a > \max\{0, \frac{|\alpha|}{\rho} (\mathcal{G} - \mathcal{P}) \}$, $\mathcal{P}^+_0$-almost surely, for all $t$ large enough, the number of elements in $\{ s \in J_t : \Delta_- \eta^+(\tau_s) \leq \log(1-\log(t)^{-a}) \}$ is stochastically bounded by a Poisson random variable with parameter

$$\frac{1}{2} \log(t)^{|\alpha|/\rho + a},$$

where

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Proof. By definition of the Lamperti time-change $s \mapsto \tau_s$, we have that
\[
f(2t) - f(t) = \int_{\tau_f(2t)}^{\tau_f(t)} e^{\alpha \eta^+(s)} ds,
\]
so in particular, on the event $E_t$ such that $\eta^+(\tau_s) < \log(\eta_f(2t))$ for all $s \in J_t$, we get that
\[
(f(2t) - f(t)) \eta_f(2t)^\alpha \leq \tau_f(2t) - \tau_f(t)
\]
Therefore, changing the variables, the domain of the sum with respect to $u = \tau_s$ becomes at least of length
\[
\frac{f(2t) - f(t)}{\eta_f(2t)^\alpha} \sim 2t|\log(\eta_f(2t)|^{\alpha \eta_f(2t)}| t \to \infty \sim \frac{1}{2} |\log(t)|^{\alpha |\eta_f(2t)|^{-2}}
\]
Hence, $\eta^+$ being a Lévy process, the number of $s \in J_t$ such that $\mathbb{I}_{\{\Delta \eta^+(\tau_s) < \log(1 - \log(t)^{-a})\}} = 1$ is greater than a Poisson random variable with parameter
\[
\frac{1}{2} |\log(t)|^{\alpha |\eta_f(2t)|^{-2}} \Pi^+((-\infty, 1 - \log(t)^{-a}))
\]
\[
\sim \frac{1}{2} |\log(t)|^{\alpha |\eta_f(2t)|^{-2}} \Pi^+((-\infty, -\log(t)^{-a})) \geq \frac{1}{2} |\log(t)|^{\alpha |\eta_f(2t)|^{-2} + \alpha p},
\]
where we used (7) to write
\[
\Pi^+((-\infty, -\log(t)^{-a})) \geq \int_{(\infty, -\log(t)^{-a})} e^{\log(y) A(dy)} = O(\log(t)^{\alpha p}),
\]
as $t \to \infty$, by (9) and Theorem 1.6.4 of [13].

Proof of Proposition 3. Recall that Lemma [13] ensures that $\mathbb{P}^+_0(\liminf_{t \to \infty} E_t) = 1$. In view of Lemma [11], it suffices to establish that the sum in its lower bound is almost surely bounded away from 0 for all $t$ large enough.

Choose $a > \max\{0, \frac{\alpha}{p}(p - p)\}$ and, to ease the notation, let $\delta := \alpha(p - p) + \alpha p$ and note that $\delta > 0$. Let $n \geq 1$. Thanks to Lemma [12] we can write
\[
\mathcal{P}_0^+ \left( \sum_{s \in J_{2^n}} A_s (1/3) \mathbb{I}_{\{\Delta \eta^+(\tau_s) < \log(1 - \log(2^n)^{-a})\}} \right) \leq 1
\]
\[
= O \left( e^{-\log(2^n)\delta} \sum_{m \geq 0} \frac{\log(2^n)\delta}{m!} \mathcal{P}_1(A(1/3) \leq 1)^m \right)
\]
\[
= \exp \left(-n^\delta \log(2^\delta) \mathcal{P}_1(A(1) > 1/3))\right).
\]
The latter being summable, Borel-Cantelli’s Lemma shows that $\mathcal{P}_0^+$-almost surely, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, it holds that
\[
\sum_{s \in J_{2^n}} A_s (1/3) \mathbb{I}_{\{\Delta \eta^+(\tau_s) < \log(1 - \log(2^n)^{-a})\}} > 1.
\]
This, the facts that $\bigcup_{n \geq n_0} J_{2^n} = [f(2^{n_0}), \infty)$, that $A$ is non-decreasing and Lemma [11] together imply that on the event $E_t$, $\mathcal{P}_0^+$-almost surely it holds that
\[
A(t) \geq t^{-\omega(1)} \log(t)^{-\omega(1)a + \omega(p)},
\]
which ensures the claim.

Remark 1. Although we do not claim that it is optimal, Equation (22) shows that Proposition 3 applies for all $q > \omega(1/|\alpha| + 1/q^* + \max(0, \frac{\log(1-q^* - 1/|\alpha|)))$, where $q^*$ is defined in Lemma 4. Indeed, we have respectively chosen $\frac{p}{p}$ arbitrary close from above to $1/\alpha$ and $\omega(p)$ arbitrary close from above to $1/q^*$.
6 Application to random maps

A specific family of growth-fragmentations. We start this section by recalling the connection between growth-fragmentations and random surfaces that has been observed in [7] for Boltzmann triangulations approximating Brownian disks, and that has been generalised in [6] to a broader family of Boltzmann maps approximating stable disks and plane.

More details on what follows can be found in [6]. Let \( \theta \in (1, 3/2) \) and for all \( q \in (\theta, 2\theta + 1) \), let

\[
\kappa_\theta(q) := \frac{\cos(\pi(q - \theta))}{\sin(\pi(q - 2\theta))} \cdot \frac{\Gamma(q - \theta)}{\Gamma(q - 2\theta)}.
\]

Thus defined, \( \kappa_\theta \) is the cumulant of a specific self-similar growth-fragmentation \( X_\theta \). Let its index of self-similarity be \( \alpha = 1 - \theta \).

Informally, the collection of cycles’ lengths observed at heights in some discrete random maps with large boundary converges, when properly rescaled, towards \( X_\theta \), where \( \theta \) depends on the tail of the distribution of the degree of a typical face (see Theorem 6.8 in [6]). The Brownian case corresponds to \( \theta = 3/2 \). This means that we obtain \( X_{3/2} \) under \( \mathcal{P}_1 \) (respectively under \( \mathcal{P}_0^+ \)) in the scaling limit of the sliced approximation of the free \({}\) Brownian disk (respectively plane); As noted in introduction, \( X_{3/2} \) also appears when directly slicing the Brownian disk or the Brownian plane (see [23] Theorem 3 and 23).

In the case of \( X_\theta \), \( \eta^+ \) and \( \eta^- \) belong to the class of hypergeometric Lévy processes, see Proposition 5.2 in [6] for this fact and [20] for a definition and references on hypergeometric Lévy processes. This yields an explicit expression of the densities of \( \Pi^+ \) and \( \Pi^- \).

Lemma 13. Let \( c_- := \frac{\Gamma(\theta + 1)}{\pi} \) and \( c_+ := \frac{\Gamma(\theta + 1)}{\pi} \sin(\pi(\theta - 1/2)) \). The densities of \( \Pi^+ \) and \( \Pi^- \) are given by

\[
\Pi^+(dy)/dy = c_- e^{3y/2} (1 - e^y)^{\theta+1} \mathbb{1}_{\{y<0\}} + c_+ e^{3y/2} (e^y - 1)^{\theta+1} \mathbb{1}_{\{y>0\}},
\]

\[
\Pi^-(dy)/dy = c_- e^{y/2} (1 - e^y)^{\theta+1} \mathbb{1}_{\{y<0\}} + c_+ e^{y/2} (e^y - 1)^{\theta+1} \mathbb{1}_{\{y>0\}}.
\]

Proof. In the proof of Proposition 5.2 in [6], it is shown that the density \( h \) of the image of \( \Pi^+ \) by \( x \mapsto e^x \) is given by

\[
h(z) = \frac{\Gamma(\theta + 1)}{\pi} \frac{z^{1/2}}{(1 - z)^{\theta+1}} \mathbb{1}_{\{0 < z < 1\}} + \frac{\Gamma(\theta + 1) \sin(\pi(\theta - 1/2))}{\pi} \frac{z^{1/2}}{(z - 1)^{\theta+1}} \mathbb{1}_{\{z > 1\}}.
\]

The expression of \( \Pi^+(dy)/dy \) follows from a straightforward change of variables. One then gets the expression of \( \Pi^-(dy)/dy \) from [13] Section 2 by identifying \( c_- \) and \( c_+ \).

From (23), one sees that (3) is satisfied with \( \omega_- = \theta + 1/2 \) and \( \omega_+ = \theta + 3/2 \). Moreover, one can check that the assumption (9) is also satisfied with \( \rho = \theta \), by looking at the behaviour of \( \Pi^-((-\infty, \log(x))) \), as explained after (10). This means that our results apply in particular to \( X_\theta \). To sum up, the parameters in terms of \( \theta \) are

\[
\begin{cases}
\alpha = 1 - \theta \\
\omega_- = \theta + 1/2 \\
\omega_+ = \theta + 3/2 \\
\rho = \theta
\end{cases}
\] (24)

\(^2\)free refers to the fact that the total intrinsic area of the surface is not fixed but random, with law described in [13] Proposition 4.
Area of a small annular in Brownian and stable disks. Le Gall showed the following in [24] Theorem 3: denote $\nu$ the intrinsic area measure of the free Brownian disk with boundary size $x > 0$, and let $B_\epsilon$ be the annular of width $\epsilon$ from the boundary (the set of points at distance smaller than $\epsilon$ from the boundary). Then it holds that

$$\lim_{\epsilon \to 0^+} \epsilon^{-2} \nu(B_\epsilon) = x, \quad \text{almost surely.} \quad (25)$$

As explained in the beginning of the section, the above convergence can be translated in terms of $X_{3/2}$ under $\mathbb{P}_x$, where the area of the annular corresponds to $A(\epsilon)$. We check whether we retrieve the same result using our theorem [1].

Consider $X_\theta$ for $\theta \in (1,3/2]$. The computations before Theorem [1] show that

$$\overline{X}(\epsilon) = \Lambda((\infty,-\epsilon)) \sim \frac{1/2}{\theta} \epsilon^{-\theta-1/2} \Pi^-(((-\infty, \log(\epsilon)),$$

which thanks to Lemma [13] can be written as $\overline{X}(\epsilon) \sim \frac{c}{\theta} \epsilon^\theta$. Recalling [24], Theorem [1] thus reads as follows: For all $x > 0$, it holds that

$$\lim_{\epsilon \to 0^+} \epsilon^{-\theta-1/2} A(\epsilon) = \frac{2(\theta-1)}{(\theta-1/2)} \cdot c \cdot E_1^-(I^{-1})(1) \cdot x, \quad \mathbb{P}_x\text{-a.s. and in } L^1.$$  

Note that for $\theta = 3/2$, we get

$$\lim_{\epsilon \to 0^+} \epsilon^{-2} A(\epsilon) = c \cdot E_1^-(I^{-1})(1) \cdot x = \frac{\Gamma(5/2)}{2\pi} |\kappa_{3/2}^\prime(2)| x,$$

where the value of the expectation is given in [14] Proposition 3.1(iv). (Note that since $I = \int_0^\infty \exp(\eta^{-}(t)/2) dt$, we get $E_1^-(I^{-1}) = |E_1^-(\eta^{-}(1)/2)|$.) Because $\kappa_{3/2}(2) = 0$ and using the explicit expression of $\kappa_{3/2}$ given in [23], we have that

$$\kappa_{3/2}^\prime(2) = \lim_{q \to 2} \frac{\Gamma(q-3/2)}{(q-2)\Gamma(q-3)} \cdot \frac{\Gamma(1/2)}{\text{Res}(-1)} = -\sqrt{\pi},$$

where we used that the residue of $\Gamma$ at $-1$ is $-1$ and $\Gamma(1/2) = \sqrt{\pi}$. Well known properties of the gamma function entails that $\Gamma(5/2) = 3\Gamma(1/2)/4 = 3\sqrt{\pi}/4$. We then obtain

$$\epsilon^{-2} A(\epsilon) \sim \epsilon^3 \frac{3}{8} x.$$  

Note that we get an additional factor $3/8$ compare to [25]. The cumulant function $\kappa$ in [27] Section 11.1 is equal to $\sqrt{8/3} \cdot \kappa_{3/2}$, which simply corresponds to multiplying the distance in the Brownian map by a constant. This means in particular, considering this $\kappa$ instead of ours, that $\eta^-$ becomes $\sqrt{8/3} \cdot \eta^-$, and the constant $c_-$ of Lemma [14] becomes $\sqrt{8/3} \cdot c_-$. The above is therefore consistent with [27].

Area of a small ball in Brownian and stable maps. If we read Proposition [1] in terms of $X_\theta$, we get that the law of $t^{(\theta+1/2)/(1-\theta)} A(t)$ is the same for all $t > 0$. This coincides with the volume growth exponent of infinite Boltzmann planar maps in the so-called dilute phase, meaning that the number of vertices at height $n$ as $n \to \infty$ in such maps grows at a speed $n^{(\theta+1/2)/(1-\theta)}$, see equation (4.3) in Theorem 4.2 of [13].

When $\theta = 3/2$, this exponent is equal to $-4$. In [22] Lemma 6.2 (see also [30] Lemma 4.4.4), it is shown that in this case, for any $\delta > 0$, it holds that

$$\limsup_{t \to 0^+} t^{-4+\delta} A(t) = 0, \quad \mathbb{P}_0^+\text{-a.s.}$$  

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The assumption $\kappa(\omega_+ + \omega_- + \alpha) < \infty$ in Proposition \ref{prop:main} reads as $\kappa_{3/2}(5/2) < \infty$, which is indeed the case by \cite{Bertoin1996}. In particular, our propositions \ref{prop:main} and \ref{prop:improve} thus improve the above, replacing $t^\theta$ by a power of $|\log(t)|$ and considering the lim inf as well. The exponent $q_0$ on the log of the lower bound is given in Remark \ref{rem:q0} in terms of some constant $q^*$ defined in Lemma \ref{lem:q0}. One can check that for any $\theta \in (1, 3/2]$, one has that $q^* = \min\{1, \theta - 1/2\} = \theta - 1/2$, and therefore $q_0 = (\theta + 1/2)((\theta - 1)^{-1} + (\theta - 1/2)^{-1})$. In the Brownian case $\theta = 3/2$, we obtain $q_0 = 6$.

Thus, for $X_\theta$ with any $\theta \in (1, 3/2]$, the following holds: for all $\delta > 0$ and $q > q_0$ as above, we have that

$$
\limsup_{t \to 0 \text{ or } \infty} |\log(t)|^{-1-\delta} t^{\theta+1/2} A(t) = 0, \quad \mathbb{P}_0^+-\text{a.s.}
$$

$$
\liminf_{t \to 0 \text{ or } \infty} |\log(t)|^q t^{\theta+1/2} A(t) = \infty, \quad \mathbb{P}_0^+-\text{a.s.}
$$

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