On Spherically Symmetric String Solutions in Four Dimensions

C.P. Burgess, R.C. Myers, and F. Quevedo

a Physics Department, McGill University
3600 University St., Montréal, Québec, Canada, H3A 2T8.

b Institut de Physique, Université de Neuchâtel
CH-2000 Neuchâtel, Switzerland.

Abstract

We reconsider here the problem of finding the general 4D spherically symmetric, asymptotically flat and time-independent solutions to the lowest-order string equations in the $\alpha'$ expansion. Our construction includes earlier work, but differs from it in three ways. (1) We work with general background metric, dilaton, axion and $U(1)$ gauge fields. (2) Much of the original solutions were required to be nonsingular at the apparent horizon, motivated by an interest in finding string corrections to black hole spacetimes. We relax this condition throughout, motivated by the realization that string theory has a less restrictive notion of what constitutes a singular field configuration than do point particle theories. (3) We can construct the general solution from a particularly simple one, by generating it from successive applications of the noncommuting $SL(2, \mathbb{R})$ and $O(1, 1)$ symmetries of the low-energy string equations containing $S$ and target–space dualities respectively. This allows its construction using relatively simple, purely algebraic, techniques. The general solution is determined by the asymptotic behaviour of the various fields: i.e. by the mass, dilaton charge, axion charge, electric charge, magnetic charge, and Taub-NUT parameter.
1. Introduction

Understanding the ultimate fate of a runaway gravitational collapse has been a long-standing problem ever since its discovery as a prediction of General Relativity (GR) many years ago. String theory is perhaps the only presently-known theory which has pretensions to describe physics at the Planck scale, and so potentially to provide some insight into this problem. The challenge has been to reliably compute string behaviour in the presence of very strong gravitational fields.

Since gravitational collapse is a classical phenomenon, the simplest approach is to investigate the corresponding solutions to the classical string equations. Classical string theory modifies classical GR in at least two ways. Firstly, the string field equations for the metric only reproduce those of GR in the limit of weak curvatures in comparison to the natural string scale (typically parameterized by $\alpha'$).\(^1\) In situations of strong curvature, higher derivative terms in the effective field equations will become significant. Secondly, string theory introduces additional light degrees of freedom, beyond the metric, which typically cannot vanish in nontrivial solutions to the full string equations.

In fact, there has been real progress in the understanding of the properties of strings in the presence of more complicated background fields over the last ten years. This progress has included (i) the construction [1] of strongly-curved field configurations which are known to be solutions to the full string equations; (ii) the discovery of ‘duality’ transformations [2], which relate superficially very different, but often actually physically identical, string configurations; and (iii) the application of these two tools to the detailed exploration of black-hole configurations in two spacetime dimensions [3], [4], and to black-$p$-brane configurations in higher dimensions [5], [6], which are known as exact conformal field theories.

One of the surprising features to emerge from these developments has been the realization that string theory may be quite forgiving in its notion of what constitutes a physically unacceptable singularity. What appears to be a malignantly singular field configuration from the point of view of point-particle theory, can be completely benign as a background for string propagation. The duality transformation constructed using a rotation symmetry of flat space furnishes a particularly striking example of this, since it produces a curved manifold with a curvature singularity at the rotation axis. Similarly, for two-dimensional

\(^1\) We use fundamental units, for which $\hbar=c=1$, and so $\alpha'$ is of order the Planck length squared.
black holes the nonsingular horizon is mapped by duality into the curvature singularity at \( r = 0 \), and for three-dimensional black strings the singularity is mapped to a regular surface in the asymptotically-flat region.

Taking seriously this broader perspective concerning the potential acceptability of singular field configurations has some immediate implications for the study of classical string configurations. In particular, the point of departure for studies of string propagation through complicated backgrounds has usually been the construction of solutions to the approximate string equations, to lowest order in \( \alpha' \). Interestingly, the string corrections are typically singular at the apparent horizon of the lowest-order black hole solutions of GR, although these singularities can be avoided by making an appropriate choice for the boundary conditions for the new fields, such as the dilaton. This observation led early workers [7], [8] to discard those solutions for which this adjustment was not made.

The purpose of the present paper is to re-examine the solutions to the low-energy string equations in four (and higher) dimensions. Keeping in mind the observation that, in string theory, curvature singularities need not be all that they seem, we construct the general time-independent, spherically-symmetric and asymptotically flat solution to the lowest-order string equations, and do not exclude those configurations in which singularities are not hidden by an event horizon. The nontrivial fields which we will consider are the metric, the dilaton, the Kalb-Ramond field and an abelian vector potential. The most general solutions would then be characterized by five independent parameters corresponding to the configuration’s mass, dilaton charge, axion charge, electric charge, and magnetic charge. Our construction will also naturally introduce a sixth parameter, namely the Taub-NUT parameter. All but the mass vanish in the usual Schwarzschild solution. (We do not consider nonvanishing topological charges such as the ‘axion hair’ considered in Ref. [9].) The final six-parameter family includes, but extends, many of the solutions that have been considered heretofore [10], [11], [12], [13], [14].

Rather than facing the daunting task of explicitly constructing the solutions to the relevant coupled nonlinear PDE’s, we instead construct these solutions by exploiting some of the extraordinary symmetries of the low-energy string equations. In particular, starting with the general spherically symmetric, static and asymptotically flat solution to the dilaton-metric system, we generate the others by successively applying the noncommuting \( SL(2, \mathbb{R}) \) and \( O(1,1) \) symmetries of the low-energy string equations containing \( S \) duality and target space duality respectively. This has the labour-saving advantage of only
requiring algebraic techniques.

The paper is organized as follows. In the next section we display our starting two-parameter family of static, spherically-symmetric dilaton-metric configurations in four dimensions. We follow this, in section 3, by the extension of these results to the more general solution of the metric–dilaton–axion system, which is the generic case for the closed bosonic string. Starting from the solutions of section 2, we generate solutions with nonvanishing axion field — i.e. the antisymmetric tensor field, $B_{\mu\nu}$ — by performing an $SL(2, \mathbb{R})$ transformation which is a symmetry of the low-energy field equations. Applying a target-space duality transformation to this result then produces new solutions with nonvanishing Taub–NUT parameter but zero axion field. These new solutions differ from our original ansatz in that they are stationary, as opposed to being static [15]. Further, they are only spherically symmetric in the generalized sense of being invariant under rotations that are combined with a simultaneous position-dependent time translation. A further $SL(2, \mathbb{R})$ transformation then generates a more general class of solutions with both a nonvanishing Taub–NUT parameter and a nonzero axion field. This underlines the fact that these two symmetries — standard duality and $SL(2, \mathbb{R})$ invariance — do not commute. Performing further duality transformations to this general solution does not yield any new backgrounds. In section 4, we extend this procedure to also include a nonzero electromagnetic gauge potential. We do so by using successive applications of the continuous $O(1, 1)$ symmetry (which contains ordinary duality as a special case) together with the $SL(2, \mathbb{R})$ symmetry. We obtain in this way two more parameters in our family of solutions, which can be identified with their electric and magnetic charges. These results are summarized in our concluding section. We include a (partial) generalization of these solutions to higher dimensions as an Appendix.

2. Spherically Symmetric Dilaton–Metric Solutions

The massless bosonic fields which always (in string perturbation theory) appear in the spectrum of a generic string theory consist of the metric, $G_{\mu\nu}$, the dilaton, $\phi$, an antisymmetric Kalb-Ramond field, $B_{\mu\nu}$. In heterotic strings these can also accompanied by one or more gauge potentials, $A_\mu$. The Lagrangian density which governs these fields at low energies is given by [16]:

$$\mathcal{L} = \frac{1}{8\pi} \left( \frac{1}{\alpha'} \right)^{(d-2)/2} \sqrt{-G} e^\phi \left[ R(G) + (\nabla \phi)^2 - \frac{1}{12} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - \frac{1}{8} F^{\mu\nu} F_{\mu\nu} \right] + \cdots, \quad (1)$$

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where \( H = dB + \) (Chern-Simons terms) and \( F = dA \) are, respectively, the field strengths for the Kalb-Ramond and electromagnetic fields, while \( R(G) \) is the Ricci scalar for the so-called ‘sigma-model’ metric, \( G_{\mu\nu} \), and \( \sqrt{-G} = \sqrt{-\det G_{\mu\nu}} \). The ellipses in eq. (1) represent terms which involve other massless fields and/or terms involving more derivatives that arise at higher orders in the \( \alpha' \) expansion. We do not consider a cosmological constant in (1) since we assume that the solutions we find are complemented by a corresponding conformal field theory (such as a toroidal or Calabi–Yau compactification) that saturates the central charge to produce a full solution with conformal invariance on the world-sheet.

It is sometimes useful to rescale the sigma-model metric in order to ensure that the coefficient of the scalar curvature is independent of \( \phi \). In \( d \) spacetime dimensions this is accomplished by transforming to the ‘Einstein’ metric,

\[
g_{\mu\nu} \equiv e^{2\phi/(d-2)} G_{\mu\nu}. \tag{2}
\]

In the following, we will denote the line element for the Einstein metric by \( ds^2 \), while \( dS^2 \) will be reserved for that of the sigma-model metric.

In the present section we set the antisymmetric tensor, \( B_{\mu\nu} \), and the gauge potential, \( A_\mu \), to zero and solve for the general dilaton–metric configuration. (We use the resulting solution to generate more general axion- and gauge-potential-dependent field configurations in the next two sections.) The relevant equations of motion then are

\[
R_{\mu\nu}(g) = \frac{1}{d-2} \nabla_\mu \phi \nabla_\nu \phi \\
\nabla^2 \phi = 0 \tag{3}
\]

2.1) Lowest-Order Four-Dimensional Solutions

We now turn to the solutions to these leading-order low-energy string equations. We specialize to field configurations which are explicitly static, spherically symmetric and asymptotically flat. That is, we take:

\[
\phi = \phi(r), \tag{4}
\]
in coordinates for which the Einstein metric (in $d = 4$ dimensions) takes the form:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + h^2(r)(d\theta^2 + \sin^2 \theta d\phi^2).$$

(5)

At large radius, asymptotic flatness implies that $f$ approaches unity while $h$ approaches $r$. In the same limit the dilaton asymptotically approaches a constant, $\phi_0$. It is convenient in what follows if we absorb this constant into the definition of Newton’s constant, $G_N$. For instance, for $d = 4$, comparison of the action of eq. (1) with its standard form gives $G_N = \frac{1}{2} e^{-\phi_0} \alpha'$. With this choice in mind we can choose $\phi \to 0$ at infinity, in which case the asymptotically flat solutions to these equations become [17]

$$f = \left(1 - \frac{\ell}{r}\right)^\delta$$

$$h^2 = r^2 \left(1 - \frac{\ell}{r}\right)^{1-\delta}$$

$$e^\phi = \left(1 - \frac{\ell}{r}\right)^\gamma$$

(6)

where $\ell$, $\delta$ and $\gamma$ are arbitrary constants, subject to the one condition $\delta^2 + \gamma^2 = 1$. (In the following, we will also assume that $\ell > 0$ for simplicity.) The choice $(\delta, \gamma) = (1, 0)$ yields the standard Schwarzschild solution with $\ell$ related to the black hole mass, $M$, according to $\ell = 2G_N M$. Up to a coordinate transformation, $(\delta, \gamma) = (-1, 0)$ also corresponds to a Schwarzschild black hole, albeit with $\ell = -2G_N M$.

Quite generally the two free parameters in this solution correspond to the two quantities which label static, spherically-symmetric and asymptotically flat dilaton–metric configurations: the mass, $M$, and dilaton charge, $Q_D$. We define the mass to be the conserved (ADM) energy [18], which emerges in a calculation of the energy using a gravitational stress-energy pseudo-tensor [19], using the Einstein metric. This is equivalent to defining $2G_N M$ to be the coefficient of $-(1/r)$ in the large-$r$ expansion of the function $f(r)$ which appears in the Einstein metric ansatz, eq. (5). We similarly define the dilaton charge, $Q_D$, as the coefficient of $-(1/r)$ in the large-$r$ expansion of $\phi$. For the solution of eq. (6) we therefore have: $M = (\delta\ell/2G_N)$ and $Q_D = \gamma\ell$.

When $\gamma$ is not zero, the solutions contain a curvature singularity at $r = \ell$, as can be
seen from the equations of motion:

\[
R(g) = \frac{1}{2} \left( \nabla \phi \right)^2 = \frac{\tilde{\gamma}^2 \ell^2}{2r^4} \left( 1 - \frac{\ell}{r} \right)^{\delta-2}.
\]  

(7)

Notice that \(|\delta| \leq 1\), and it is only at \(\delta = \pm 1\) that \(\gamma = 0\) and hence the above singularity vanishes. Of course, the latter solutions are still singular at \(r = 0\) (even though \(R = 0\) there). Other solutions with nonsingular horizons are also known when more string fields are present, such as with gauge fields [11], and antisymmetric Kalb-Ramond fields [11], [9]. We include those fields in the following sections.

3. Axionic and Taub–NUT extensions

We next turn to the generalization of the (leading order) four-dimensional solutions presented in section 2.1. We generate these new solutions by repeated applications of \(SL(2, \mathbb{R})\) [20], [12], [21] and discrete target-space duality transformations. We restrict ourselves here to spherically-symmetric, asymptotically flat, time independent configurations involving the metric, dilaton and axion fields, where the axion here represents the antisymmetric Kalb-Ramond tensor, \(B_{\mu\nu}\). We treat the case of background gauge fields in the next section. (An overview of our final construction is illustrated in the Figure.)

One of the interesting features of our construction is that, besides introducing a non-trivial axion background, repeated duality and \(SL(2, \mathbb{R})\) transformations also force us to generalize our disposition on the character of the solutions of interest. Up to this point we have taken a static metric ansatz [15], for which the metric is independent of a time coordinate, \(t\), and curves along which only \(t\) varies are orthogonal to the hypersurfaces of constant \(t\). Repeated symmetry transformations, however, take us to a metric which is only stationary, in that it is still \(t\) independent although it is impossible to choose \(t\) in a ‘hypersurface orthogonal’ way. The stationary metrics that we find are reminiscent of the ‘Taub-NUT’ metric [22], [23]. Thus the solutions also become spherically symmetric only in the modified sense that \(SO(3)\) transformations are only symmetries when they are compensated by an appropriate position-dependent time translation.

In four dimensions, the antisymmetric tensor is dual to a pseudoscalar field, \(a(x)\),
defined by

\[ H_{\mu \nu \rho} = -e^{-2\phi} \epsilon_{\mu \nu \rho \kappa} \nabla^\kappa a \]  

(8)

where \( H_{\mu \nu \rho} = \partial_\mu B_{\nu \rho} + \partial_\nu B_{\rho \mu} + \partial_\rho B_{\mu \nu} \) (Chern–Simons terms). In eq. (8) indices are raised and lowered with the Einstein metric \( g_{\mu \nu} \) and \( \epsilon_{\tau \tau \phi \phi} = \sqrt{-g} \).

3.1) The Static Solution

To find the axionic backgrounds we use the fact that the field equations are invariant under a continuous \( SL(2, \mathbb{R}) \) symmetry [20], [12], [21] acting on the complex field \( S = a + i e^\phi \) as:

\[ S \rightarrow \frac{a S + b}{c S + d} \]  

(9)

where \( a, b, c \) and \( d \) are real number satisfying \( ad - bc = 1 \). The Einstein metric is invariant under these transformations. The discrete subgroup, \( SL(2, \mathbb{Z}) \), of these transformations which contains strong–weak coupling duality (i.e., \( S \)-duality), is also conjectured to be a symmetry of the full quantum string theory, as well as the leading order low-energy field equations [24].

Given eq. (9), there is a three parameter family of backgrounds that can be generated by applying these transformations to any given four-dimensional solution. We start with a vanishing axion configuration, \( a = 0 \), together with the dilaton and (Einstein) metric backgrounds given in (4), (5) and (6). We repeat these here for ease of reference:

\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + h^2(r)(d\theta^2 + \sin^2 \theta d\phi^2) \]  

(10)

with \( \phi(r) \), \( f(r) \) and \( h(r) \) given as the following powers of the quantity, \( \Lambda(r) \equiv 1 - \ell/r \):

\[ e^{\phi(r)} = \Lambda^\gamma, \quad f(r) = \Lambda^\delta, \quad \text{and} \quad h^2(r) = r^2 \Lambda^{1-\delta}. \]  

(11)

Recall also that the parameters, \( \delta \) and \( \gamma \), are related by \( \delta^2 + \gamma^2 = 1 \).

Performing the \( SL(2, \mathbb{R}) \) transformation of eq. (9), we obtain new dilaton and axion fields, \( \hat{\phi} \) and \( \hat{a} \):

\[ e^{\hat{\phi}} = \frac{\Lambda^\gamma}{c^2 \Lambda^{2\gamma} + d^2}, \quad \hat{a} = \frac{ac \Lambda^{2\gamma} + bd}{c^2 \Lambda^{2\gamma} + d^2} \]  

(12)
Since the low-energy equations of motion are invariant under constant shifts of the axion field, $a$, we may choose the axion field to vanish asymptotically. We also continue to impose the vanishing of the dilaton at infinity. Only a one-parameter family of the $SL(2,\mathbb{R})$ transformations respects these conditions, however, given by $a = d = \sqrt{1 - c^2}$, $b = -c$). It is convenient to use the ratio $\omega \equiv c/d$ as the independent parameter, in which case the dilaton and axion configurations of eq. (12) become:

$$e^{\hat{\phi}} = (1 + \omega^2) \frac{\Lambda^\gamma}{\omega^2 \Lambda^{2\gamma} + 1}, \quad \hat{a} = \frac{\omega (\Lambda^{2\gamma} - 1)}{\omega^2 \Lambda^{2\gamma} + 1}.$$  \hspace{1cm} (13)

It is also convenient to follow our practice for the dilaton, and so to define the axion charge, $Q_A$, as the coefficient of $-1/r$ in the large-$r$ expansion of the axion field, $a(r)$. With this definition the dilaton and axion charges of the solutions of eq. (13) are

$$Q_D = \frac{1 - \omega^2}{1 + \omega^2} \gamma \ell \quad \text{and} \quad Q_A = \frac{2\omega \gamma \ell}{1 + \omega^2},$$  \hspace{1cm} (14)

respectively.

The original antisymmetric field, $B_{\mu\nu}$, which corresponds, using eq. (8), to the axion configuration, $\hat{a}$, has a particularly simple expression in terms of the charge, $Q_A$. It is:

$$\hat{B}_{\varphi t} = Q_A \cos \theta,$$  \hspace{1cm} (15)

which has no dependence on $r$.

The two-parameter family of dilaton–metric solutions of section 2 are easily verified to form the particular case $\omega = 0$ of the class of solutions of the present section. Similarly, the limit $\omega \to \infty$ — which corresponds to a pure $S$–duality transformation: $S \to -1/S$ — also reproduces the solutions of section 2, but with the opposite sign for $\gamma$. All other choices of $\omega$ lead to a nonzero axion configuration, and are therefore new solutions.

We also see in this way how the solutions $(\delta, \gamma)$ and $(\delta, -\gamma)$ of the pure dilaton–metric system are continuously connected by a one-parameter new class of solutions with a nonvanishing axion, corresponding to varying $\omega$ from zero to infinity. Notice that all of these solutions are still singular at $r = \ell$, except for the special case $\gamma = 0$. Also notice that, even though $Q_D$ vanishes for the parameter values $\omega = \pm 1$, the presence of
the axion field still induces a nonvanishing dilaton background. A final point of interest is that, for all values of $\omega$, the combination $Q_D^2 + Q_A^2 = \gamma^2 \ell^2$ remains fixed (a similar observation was made in Ref. [25]). We see that our $SL(2,\mathbb{R})$ transformations can be characterized as a rotation in the space of these two scalar charges — explicitly, if we set $\omega = (1 - \sin \Theta)/\cos \Theta$, eq. (14) reduces to $Q_D = \gamma \ell \sin \Theta$ and $Q_A = \gamma \ell \cos \Theta$.

The solutions we have obtained are not quite the most general solutions of the axion–dilaton–metric system in four dimensions that are static, spherically symmetric and asymptotically flat. This is because, following ref. [9], one can add a purely topological contribution to the antisymmetric tensor: $B_{\theta \phi} = Q_{\text{top}} \sin \theta$. This potential is spherically symmetric since it yields a field strength, $H = dB$, which completely vanishes. It nevertheless cannot be gauged away provided that the second homotopy group of the background spacetime is nontrivial [9]. Even though its field strength vanishes, such a topological configuration can have real physical effects for macroscopic strings in both the bosonic and heterotic string theories. Further note that one cannot introduce this topological charge when working with the pseudoscalar representation of the axion.

We therefore arrive at a three-parameter family of metric–dilaton–axion configurations, which precisely corresponds to the three physical quantities, $M$, $Q_D$, and $Q_A$, we expect to describe the asymptotic falloff of our three kinds of fields. Therefore (putting aside the topological exception just discussed) the solution we have obtained — viz the metric of eqs. (10) and (11), together with the dilaton and axion of eq. (13) — are the most general such static, spherically symmetric, and asymptotically flat low-energy string configuration.

3.2) The ‘Taub–NUT’ Case

We next generate a slightly more general class of solutions, for which the metric is not static, but is stationary. To do so we perform a target-space duality transformation based on the timelike isometry of time translation. The action of such a duality transformations for a nontrivial configuration involving the metric, dilaton and antisymmetric tensors is given by [26]:

$$\tilde{G}_{tt} = 1/G_{tt}, \quad \tilde{G}_{ti} = -B_{ti}/G_{tt}, \quad \tilde{G}_{ij} = G_{ij} - \frac{G_{ti} G_{tj} - B_{ti} B_{tj}}{G_{tt}}$$

$$\tilde{B}_{ti} = -G_{ti}/G_{tt}, \quad \tilde{B}_{ij} = B_{ij} + \frac{G_{ti} B_{tj} - G_{tj} B_{ti}}{G_{tt}}, \quad e^{\tilde{\phi}} = e^{\phi} \left(\frac{\det G}{\det \tilde{G}}\right)^{1/2} \quad (16)$$
where, in our case ‘$t$’ denotes the time direction. As for earlier sections, $G_{\mu\nu} = e^{-\phi}g_{\mu\nu}$ here represents the sigma-model metric.\footnote{The sign we give here for the transformation of the ‘$t-i$’ components of the fields is the opposite of what is often found in the literature [26], but this can be corrected by performing the coordinate transformation $t \rightarrow -i$ in the dual solution. A similar result was found in Ref.’s [27], [28]. As presented, the duality transformation is closely related to the $O(d,d+p)$ transformations applied in the next section.}

Applying these transformations to our general metric–dilaton–axion backgrounds of the previous section, eqs. (10) and (11) for the metric, together with the dilaton and axion configurations of eq. (13), we are led to a set of dual solutions to the low-energy string equations, which we denote by a tilde. The dual Einstein metric is given by:

$$d\tilde{s}^2 = -e^{\tilde{\phi}} (dt + Q_A \cos \theta \, d\varphi)^2 + e^{-\tilde{\phi}} \left[ dr^2 + \Lambda r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2) \right]$$  \hspace{1cm} (17)$$

where $e^{-\tilde{\phi}}$ represents the quantity given in eq. (13), and

$$e^{\tilde{\phi}} = f = \Lambda \delta, \hspace{1cm} \tilde{B}_{\mu\nu} = 0. \hspace{1cm} (18)$$

This new solution is reminiscent of the Taub–NUT solution [22] of the vacuum Einstein equations because of the appearance of $(dt + Q_A \cos \theta \, d\varphi)^2$ in the line element. In fact, eqs. (17) and (18) are an extension of the Taub–NUT solution to low-energy string theory which includes an arbitrary dilaton charge. In this solution, the dilaton charge is $\tilde{Q}_D = \delta \ell$ while the mass is given by $2G_N \tilde{M} = \frac{1-\omega^2}{1+\omega^2} \gamma \ell$. For $\delta = 0$, the dilaton vanishes and we recover precisely the Taub–NUT metric, of which the standard form can be achieved by a change of variables: $\tilde{r} = r - \frac{\omega^2 \ell}{\sqrt{1+\omega^2}}$.

We define the NUT parameter, $N$, in terms of the coefficient appearing in the time differential by, $dt + 2N \cos \theta \, d\varphi$. Thus the NUT parameter, $\tilde{N}$, of the dual solution is given in terms of the axion charge, $Q_A$, of the original configuration by $\tilde{N} = Q_A/2 = \frac{\omega \gamma \ell}{\sqrt{1+\omega^2}}$.

When $\delta$ is not zero, the surface $r = \ell$ contains a real curvature singularity. There are also conical singularities at the axes, $\theta = 0$ and $\theta = \pi$, unless the time coordinate happens to be periodically identified with period $8\pi N$ [23]. This metric (17) is both time-translation invariant and spherically symmetric, but these symmetries act more subtly than they did on our previous examples. In particular, the rotational symmetries act in the usual way
on the angular coordinates, but also involve time translations in order to preserve the
differential $dt + 2N \cos \theta \, d\varphi$. Thus we have lost spherical symmetry, in the conventional
sense. One finds then that surfaces of constant radius have the topology of a three-sphere,
in which there is a Hopf fibration of the $S^1$ of time over the spatial $S^2$ [23]. Note that
this interesting topology was created in the duality transformation, by the exchange of the
axion charge for the NUT parameter.

One can reobtain the general case with nonvanishing axion field by performing a
further $SL(2, \mathbb{R})$ transformation on the solution, (18), just constructed. This lead us in
the present case to the same Einstein metric as in eq. (17), but with new dilaton and axion
fields, $\phi'$ and $a'$. These are given explicitly by:

$$e^{\phi'} = (1 + \epsilon^2) \frac{\Lambda^\delta}{\epsilon^2 \Lambda^{2\delta} + 1} \quad \text{and} \quad a' = \frac{\epsilon (\Lambda^{2\delta} - 1)}{\epsilon^2 \Lambda^{2\delta} + 1},$$

(19)

where $\epsilon$ is the parameter of the new $SL(2, \mathbb{R})$ transformation.

Using eq. (8), we see

$$B'_{\varphi t}(\theta) = Q'_A \cos \theta$$

(20)

where new axion charge is $Q'_A = \frac{2\delta \ell}{1 + \epsilon^2}$. An asymptotic expansion of the new dilaton field
shows that this solution’s dilaton charge is given by $Q'_D = \frac{1 - \epsilon^2}{1 + \epsilon^2} \delta \ell$. Since the Einstein metric
is unchanged by the $SL(2, \mathbb{R})$ transformation, so are the mass and NUT parameters, which
remain as given above.

Clearly, since repeated applications of $SL(2, \mathbb{R})$ and duality transformations have
generated new classes of solutions, these two transformations do not commute. One might
wonder at this point if their repeated application would continue to generate new classes
of solutions. Fortunately, applying (16) to our latest class of solutions simply gives back
more solutions within the same class, and so no further solutions are generated in this way.
This is as would be expected since the four parameters of these solutions exhaust the four
quantities which define the asymptotic falloff of the fields we consider.

The three-dimensional ‘moduli’ space of the solutions that we have obtained, for a
fixed value of $\ell$, turns out to be compact. This is because (i) $\delta$ and $\gamma$ are restricted by
$\delta^2 + \gamma^2 = 1$; and (ii) changing $\omega \rightarrow 1/\omega$ leaves the solutions invariant provided that $\gamma$
is taken to $-\gamma$ at the same time. Thus we can restrict to values $\omega \leq 1$. Finally, (iii)
the parameters $\epsilon$ and $\delta$ are identified in precisely the same way as are $\omega$ and $\gamma$, and so it suffices to consider $\epsilon \leq 1$.

We next discuss the singularities in our four-parameter class of solutions. Singularities occur at $r = 0$ when $\ell < 0$, and at $r = \ell$ when $\ell > 0$. The latter becomes a nonsingular surface for $\delta = 0$ and $\gamma = 1$. (c.f. eq. (19), which shows that the parameter $\epsilon$ is irrelevant whenever $\delta = 0$.) If both $\omega = \delta = 0$, then we recover the Schwarzschild solution, for which $r = \ell > 0$ is the event horizon, and $r = 0$ is a curvature singularity. (As before, the case $\delta = 0, \gamma = -1$ corresponds to a negative mass Schwarzschild solution, without a horizon.)

Nonvanishing $\omega$ (with $\delta = 0$) yields the Taub–NUT solution, where again $r = \ell$ is only a coordinate singularity. If the time were chosen to be periodic in this solution, as discussed above, this surface would not be a global event horizon, although it would still be an apparent horizon. In this case, the geometry is entirely free of singularities [23], and one may extend the radial coordinate to $-\infty$.

Summarizing this section, we have found the most general static asymptotically flat, spherically symmetric background for the axion–dilaton–metric system in four-dimensions (apart from the possibility of a topological$^3$ contribution $B_{\theta \varphi} = Q_{\text{top}} \sin \theta$). Duality and $SL(2, \mathbb{R})$ transformations naturally extend this solution to one including a non-trivial NUT parameter as well, given by the Einstein metric (17), the dilaton (19) and the antisymmetric tensor (20). The Killing symmetries of this solution are still time translations, and $SO(3)$ rotations. Spherical symmetry in a conventional sense is lost, though, when the NUT parameter is nonzero, since the rotations act on the time coordinate as well. The final solution depends on four arbitrary parameters ($\ell, \delta, \omega$ and $\epsilon$ with $\delta^2 + \gamma^2 = 1$), which correspond to the four ‘physical charges’ which define the asymptotic behaviour of the fields involved: i.e., the mass $M$, dilaton charge $Q_D$, axion charge $Q_A$ and Taub-NUT parameter $N$.

### 4. Gauge Field Backgrounds

We next generalize the solutions of the previous two sections to include a nonvanishing (abelian) gauge field configuration, such as can appear in the heterotic string. We are led

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$^3$ We note in passing that in the larger class of solutions, for which the NUT parameter is nonzero and the time coordinate is periodically identified, the spacetime’s second homotopy group is trivial, and so this topological field configuration is pure gauge.
in this section to a six parameter family of axion–dilaton–metric–electromagnetic field configurations, for which the previous four parameters are supplemented by the solution’s electric and magnetic charges. This is a much broader class of solutions than has been obtained previously in the literature, which has considered either (i) an arbitrary mass and arbitrary electric and magnetic charges [12], [13], or (ii) arbitrary mass, dilaton charge, and one of either electric or magnetic charge [14].

Our starting point is the metric, dilaton and antisymmetric tensor fields, respectively given by eqs. (17), (19), and (20) of the previous section. The key to generalizing these configurations to the electromagnetic case, using only algebraic manipulations, is to use the continuous extension of the discrete duality transformation we have been using to this point.

It has be shown [29] that whenever the string background is independent of \( d \) of the spacetime coordinates, there exists an \( O(d, d) \) symmetry which acts in the space of solutions of the low energy field equations. The same results were extended to the heterotic string in ref. [30]. In the heterotic case, if the solutions are independent of \( d \) of the spacetime coordinates, and also have background gauge fields which lie in a commuting subgroup which has \( p \, U(1) \) generators, then the low-energy string equations admit an \( O(d, d + p) \) symmetry. Provided that the spacetime has a Minkowski-signature metric and that time translation is one of the symmetry directions — certainly the case of interest here — one can show, for infinitesimal transformations, that the generators in an \( O(d-1, 1) \times O(d+p-1, 1) \) subgroup of this group actually relate distinct solutions, while the remainder generate pure gauge transformations [31].

In the present instance, we consider only time translation symmetry and a single \( U(1) \) gauge field — i.e., \( d = p = 1 \). We therefore expect to be able to generalize our existing solutions using a one-parameter family of \( O(1, 1) \) transformations. The action of these transformations is most easily written when the background fields are written as the following \( 9 \times 9 \) matrix [30]

\[
\mathcal{M} = \begin{pmatrix}
K_T G^{-1} K_- & K_T G^{-1} K_+ & -K_T G^{-1} A \\
K_T G^{-1} K_- & K_T G^{-1} K_+ & -K_T G^{-1} A \\
-A^T G^{-1} K_- & -A^T G^{-1} K_+ & A^T G^{-1} A
\end{pmatrix}
\]

(21)

where

\[
(K_{\pm})_{\mu\nu} = -B_{\mu\nu} - G_{\mu\nu} - \frac{1}{4} A_\mu A_\nu \pm \eta_{\mu\nu}
\]

(22)
and $\eta_{\mu\nu}$ is the flat Minkowski metric in four dimensions. In order to make contact with the previous literature, we adopt here the convention that time is the fourth component — e.g., $\eta_{\mu\nu} = \text{diag}(1, 1, 1, -1)$.

The $O(1, 1)$ symmetry can be expressed as the invariance of the low-energy string equations under the transformation $\mathcal{M} \to \mathcal{M}' = \Omega \mathcal{M} \Omega^T$, where the $O(1, 1)$ transformation matrix is given by

$$\Omega = \begin{pmatrix} I_7 & 0 & 0 \\ 0 & x & \sqrt{x^2 - 1} \\ 0 & \sqrt{x^2 - 1} & x \end{pmatrix}. \quad (23)$$

Here $I_7$ represents the $7 \times 7$ unit matrix, and $x$ is a parameter which satisfies $x^2 \geq 1$. The dual fields, which we denote by $\mathcal{G}_{\mu\nu}$, $\mathcal{B}_{\mu\nu}$ and $\mathcal{A}_\mu$ can then be found by re-expressing $\mathcal{M}$ in the form (21). The symmetry also acts on the dilaton field according to the rule

$$e^{\phi'} = \left( \frac{\det G}{\det G'} \right)^{\frac{1}{2}} e^{\phi}. \quad (24)$$

The $O(1, 1)$ transformations we have just defined fall into two disconnected classes that are characterized by the sign of $x$, since either $x \geq 1$ or $x \leq -1$. Ordinary discrete duality as it has been used so far in the text, simply interchanges these two classes. For example, $\Omega(x = 1)$ is the identity transformation, while $\Omega(x = -1)$ generates the dual background.

Actually the result of composing two transformations $\Omega(x)\Omega(-1)$ gives a transformation in which one reverses both the sign of $x$, and the sign of the off-diagonal terms in eq. (23). The effect of the latter sign change is simply to reverse the sign of the electromagnetic fields. This leads us to decompose $\Omega(-1)$ in terms of two commuting matrices: $\Omega(-1) = \Omega_D \Omega_q$ where $\Omega_q = \text{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1)$ and $\Omega_D = \text{diag}(1, 1, 1, 1, 1, 1, -1, 1)$. Applying $\Omega_q$ to transform $\mathcal{M}$ changes the sign of the gauge field, while $\Omega_D$ generates the duality transformation of eq. (16).\(^4\)

We now apply these transformations to the dilaton–metric–axion configurations of the previous section. We start by converting the Einstein metric of eq. (17) to the sigma-model

\(^4\) Notice that $\Omega'_D = \text{diag}(1, 1, 1, -1, 1, 1, 1, 1, 1, 1)$ generates the duality transformation with the conventional signs [26].
metric, finding:

\[ dS^2 = -F(r) \left( dt + Q_\omega \cos \theta d\phi \right)^2 + G(r) \, dr^2 + H^2(r) (d\theta^2 + \sin^2 \theta \, d\phi^2), \quad (25) \]

with

\[ F(r) \equiv e^{\dot{\phi} - \phi'} = \left( \frac{1 + \omega^2}{1 + \epsilon^2} \right) \left( \frac{e^2 \Lambda^{2\delta} + 1}{\omega^2 \Lambda^{2\gamma} + 1} \right) \Lambda^{-\delta} \]

\[ G(r) \equiv e^{-(\dot{\phi} + \phi')} = \left( \frac{e^2 \Lambda^{2\delta} + 1}{(1 + \epsilon^2) (1 + \omega^2)} \right) \Lambda^{-\gamma - \delta}. \quad (26) \]

\[ H^2(r) \equiv r^2 \Lambda e^{-(\dot{\phi} + \phi')} = r^2 \Lambda G(r) \]

In these expressions \( \Lambda(r) = 1 - \ell/r \), as in previous sections, and \( Q_\omega = \frac{2\omega \gamma \ell}{1 + \omega^2}. \) We take the dilaton and antisymmetric tensor fields from eqs. (19) and (20), respectively (dropping the primes)

\[ e^\phi = (1 + \epsilon^2) \frac{\Lambda^\delta}{\epsilon^2 \Lambda^{2\delta} + 1} \]

\[ B_{\phi t} = Q_\epsilon \cos \theta \]

where \( Q_\epsilon = \frac{2\epsilon \delta \ell}{1 + \epsilon^2}. \)

Applying the \( O(1,1) \) transformation to these solutions, we generate a new class of solutions which depends on an additional parameter, \( x \). The sigma-model metric of this new class is given by:

\[ d\bar{S}^2 = \frac{F(r)}{J(r)^2} d\xi^2 + G(r) \, dr^2 + H(r)^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]

\[ \bar{B}_{\phi t} = \left( \frac{(1 + x) Q_\epsilon + (1 - x) Q_\omega F(r)}{2 J(r)} \right) \cos \theta, \]

\[ \overline{A}_t = \sqrt{x^2 - 1} \left( \frac{1 - F(r)}{J(r)} \right), \]

\[ \overline{A}_\phi = \sqrt{x^2 - 1} \left( \frac{Q_\epsilon - Q_\omega F(r)}{J(r)} \right) \cos \theta, \]

\[ e^{\overline{\phi}} = e^\phi J(r) = (1 + \epsilon^2) \frac{\Lambda^\delta}{\epsilon^2 \Lambda^{2\delta} + 1} J(r) \]

where

\[ d\xi \equiv dt + \left( \frac{1 - x}{2} Q_\epsilon + \frac{1 + x}{2} Q_\omega \right) \cos \theta \, d\phi \equiv dt + 2N \cos \theta \, d\phi \]

(29)
and
\[ J(r) \equiv \frac{1}{2}[(1 + x) + (1 - x) F(r)]. \] (30)

All the other components of \( \bar{G}_{\mu\nu}, \bar{B}_{\mu\nu} \) and \( \bar{A}_\mu \) turn out to vanish. Eq. (29) defines the NUT parameter, \( \bar{N} \), for this new metric.

For future use, we record here the dilaton and axion charges for the above solution:
\[ Q_D = \frac{\ell}{2} \left[ (1 + x) \frac{1 - \epsilon^2}{1 + \epsilon^2} \delta + (1 - x) \frac{1 - \omega^2}{1 + \omega^2} \gamma \right], \]
\[ Q_A = \ell \left[ (1 + x) \frac{\epsilon \delta}{1 + \epsilon^2} + (1 - x) \frac{\omega^2}{1 + \omega^2} \gamma \right]. \] (31)

These are extracted from the asymptotic expansions of the dilaton and antisymmetric tensor fields, where for the latter, \( \bar{B}_{\phi t}(r \to \infty) \to Q_A \cos \theta \) as \( r \to \infty \).

Next, we perform an \( SL(2, \mathbb{R}) \) transformation, thereby introducing another free parameter into our class of solutions. Since this symmetry is defined to act on the dilaton and axion fields, \( \phi \) and \( a \), it is first necessary to determine \( a \) from the given expression for \( B_{\mu\nu} \), using eq. (8). This requires knowledge of the antisymmetric field strength tensor, \( H_{\mu\nu\rho} \), for which we not only need the curl of \( B_{\mu\nu} \), but also the corresponding gauge-field Chern–Simons terms, since these no longer vanish for the configurations we are considering.

\[ H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{4} \bar{A}_\mu F_{\nu\rho} + \text{cyclic permutations} \] (32)

Now, inspection of eqs. (28) shows that the nonvanishing components of the gauge field strength are \( F_{tr}, F_{r\phi}, \) and \( F_{\theta\phi} \). Using these, as well as the expression for \( \bar{B}_{\mu\nu} \) from eq. (28), in eq. (32), we see that \( \bar{H}_{rt\phi} \) vanishes even though \( \bar{B}_{\phi t} \) is a function of \( r \). This provides a nontrivial check of our results, since a nonvanishing \( \bar{H}_{rt\phi} \) would have implied a \( \theta \) dependence for the scalar axion field, in contrast with the requirements of the \( SO(3) \) rotational symmetry. The only nonvanishing component of \( \bar{H}_{\mu\nu\lambda} \) turns out to be \( \bar{H}_{\theta\phi t} \), from which we obtain
\[ \bar{a}(r) = \left( \frac{1 - x}{2} \right) \frac{\omega \left( \Lambda^2 - 1 \right)}{\omega^2 \Lambda^2 + 1} + \left( \frac{1 + x}{2} \right) \frac{\epsilon \left( \Lambda^2 - 1 \right)}{\epsilon^2 \Lambda^2 + 1}. \] (33)

We choose here an arbitrary integration constant to ensure that \( \bar{a}(r) \) vanishes at infinity.
We now wish to perform the $SL(2, \mathbb{R})$ transformation to these configurations. In this case, since the gauge field background does not vanish, we must use a more general transformation rule. Not only must the dilaton and axion fields in $S = \tau + i e^{\phi}$ be transformed as in eq. (9), but we must also transform the gauge fields [12], [21]. The total transformation becomes

$$S \rightarrow \frac{aS + b}{cS + d}$$

$$(F_+)_{\mu\nu} \rightarrow (cS + d) (F_+)_{\mu\nu}$$

$$(F-)_{\mu\nu} \rightarrow (cS^* + d) (F-)_{\mu\nu}$$

where $(F_\pm)_{\mu\nu} \equiv F_{\mu\nu} \pm \frac{i}{2} \epsilon_{\mu\nu\rho\kappa} F^{\rho\kappa}$ are respectively the (Hodge) self-dual and the antiself-dual parts of the electromagnetic field strength, $S^*$ is the complex conjugate of $S$. Again, the Einstein metric, which is required to define the volume form and the contractions in $(F_\pm)_{\mu\nu}$, is left invariant under this transformation. As in the previous section, all but one of the three $SL(2, \mathbb{R})$ parameters are eliminated by the requirement that the dilaton and axion fields must vanish at infinity. We denote the single extra parameter which remains by $\rho \equiv c/d$.

Applying $SL(2, \mathbb{R})$ to the solutions (28), we obtain the new dilaton and axion fields, $\hat{\phi}$ and $\hat{a}$:

$$e^{\hat{\phi}} = \frac{(\rho^2 + 1) e^{\phi}}{\rho^2 e^{2\phi} + (\rho \tau + 1)^2}$$

$$\hat{a} = \frac{\rho (\tau^2 + e^{2\phi} - 1) - (\rho^2 - 1) \tau}{\rho^2 e^{2\phi} + (\rho \tau + 1)^2}$$

From which we find the corresponding charges

$$\hat{Q}_D = \frac{1 - \rho^2}{1 + \rho^2} \overline{Q}_D - \frac{2\rho}{1 + \rho^2} \overline{Q}_A$$

$$\hat{Q}_A = \frac{1 - \rho^2}{1 + \rho^2} \overline{Q}_A + \frac{2\rho}{1 + \rho^2} \overline{Q}_D$$

where $\overline{Q}_D$ and $\overline{Q}_A$ are given in eq. (31). Notice that for $\rho \to 0$ both reduce to their previous values, and that again the $SL(2, \mathbb{R})$ transformation acts here to rotate the charges preserving: $\overline{Q}_D^2 + \overline{Q}_A^2 = \hat{Q}_D^2 + \hat{Q}_A^2$.
For the gauge fields, eq. (34) also implies the following new field strength tensor:

$$
\hat{F}_{\mu\nu} = \frac{1}{\sqrt{1 + \rho^2}} \left[ (1 + \rho \bar{\varpi}) \hat{F}_{\mu\nu} - \frac{1}{2} \rho e^a \epsilon_{\mu\nu\rho\sigma} \hat{F}^{\rho\sigma} \right].
$$

(37)

Once again, the only nonvanishing components are $\hat{F}_{\theta\varphi}$, $\hat{F}_{tr}$ and $\hat{F}_{r\varphi}$. A gauge potential which produces this field strength, is given by:

$$
\hat{A}_\varphi = \Psi(r) \cos \theta \\
\hat{A}_t = \frac{\left( \Psi(r) + \hat{Q}_M \right)}{2\bar{N}},
$$

(38)

where

$$
\Psi(r) = \sqrt{\frac{x^2 - 1}{\rho^2 + 1}} \left[ (1 + \rho \bar{\alpha}) \frac{Q_\epsilon - Q_\omega \hat{F}}{J} + \rho \ell \left( \frac{\omega^2 \Lambda^2 \gamma - 1}{\omega^2 \Lambda^2 \gamma + 1} - \frac{\epsilon^2 \Lambda^2 \delta - 1}{\epsilon^2 \Lambda^2 \delta + 1} \right) \right]
$$

(39)

and $\bar{N}$ is the NUT parameter defined in eq. (29).

This electromagnetic field configuration has the following magnetic charge

$$
\hat{Q}_M = \sqrt{\frac{x^2 - 1}{\rho^2 + 1}} \ell \left[ \frac{\rho(1 - \omega^2) + 2\omega}{1 + \omega^2} \gamma - \frac{\rho(1 - \epsilon^2) + 2\epsilon}{1 + \epsilon^2} \delta \right],
$$

(40)

which can be determined by comparing to an asymptotic behavior of the form: $\hat{F}_{\theta\varphi} \simeq \hat{Q}_M \sin \theta$, or $\hat{A}_\varphi \simeq -\hat{Q}_M \cos \theta$. In eq. (38), we have used an arbitrary constant that appears in solving for $\hat{A}_t$ by requiring that $\hat{A}_t$ vanish as $r \to \infty$.

The electric charge of this configuration is similarly given by:

$$
\hat{Q}_E = \sqrt{\frac{x^2 - 1}{\rho^2 + 1}} \ell \left[ \frac{1 - \omega^2 - 2\rho\omega}{1 + \omega^2} \gamma - \frac{1 - \epsilon^2 - 2\rho\epsilon}{1 + \epsilon^2} \delta \right]
$$

(41)

as may be determined from the asymptotic behaviour: $\hat{F}_{tr} \simeq \hat{Q}_E/r^2$ or $\hat{A}_t \simeq \hat{Q}_E/r$. One may verify that as expected the $SL(2,\mathbb{R})$ transformation rotates the electric and magnetic charges amongst each other, preserving $\hat{Q}_E^2 + \hat{Q}_M^2$. 

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Finally, we can determine the antisymmetric tensor field, $\hat{B}_{\mu\nu}$ from our expression for $\hat{a}$ and $\hat{A}_\mu$, by using eqs. (8) and (32). The only component which can be nonvanishing is $\hat{B}_{\varphi t}$, and this is given by

$$\hat{B}_{\varphi t} = \left[\frac{\hat{Q}_M(\Psi(r) + \hat{Q}_M)}{8N} + \hat{Q}_A\right] \cos \theta .$$

(42)

Notice that, asymptotically, $B_{\varphi t} \to \hat{Q}_A \cos \theta$ as expected. It also reduces to its previous expression in the limit $\rho \to 0$.

Since the Einstein metric is left untouched by $SL(2, \mathbb{R})$ transformations, it can be read directly from eq. (28):

$$d\hat{s}^2 = e^{\hat{\phi}} \left( -\frac{F(r)}{J(r)^2} (dt + 2N \cos \theta d\varphi)^2 + G(r) \, dr^2 + H(r)^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right),$$

(43)

and the sigma-model metric is obtained from this by using $\hat{g}_{\mu\nu} = e^{-\hat{\phi}} \bar{g}_{\mu\nu}$. As was determined earlier, c.f. eq. (29), the Taub–NUT parameter for this solution is $N = \frac{1+x^2}{4} Q_\epsilon + \frac{1+x}{4} Q_\omega$. Finally, an asymptotic expansion of the Einstein metric gives the masses of these solutions to be

$$\hat{M} = \frac{\ell}{4G_N} \left[ (x + 1) \frac{1-\omega^2}{1+\omega^2} \gamma - (x - 1) \frac{1-\epsilon^2}{1+\epsilon^2} \delta \right].$$

(44)

Notice that these metrics have new singularities, that are associated with the radius $r = r_s$, for which $J(r_s) = 0$, in addition to the previous ones that are located at $r = 0$ and $r = \ell$.

5. Conclusions

We have finally arrived at a six-parameter family of backgrounds — the six parameters being $\ell, \delta, \omega, \epsilon, x$ and $\rho$. These six parameters can be traded for six physical constants which characterize the asymptotic form of the solutions: the mass, Taub-NUT parameter, axion charge, dilaton charge, and electric and magnetic charges. Within this family we find the most general class of four-dimensional solutions of the leading order string field equations, which are spherically symmetric, static and asymptotically flat. This subclass
is obtained by setting the Taub-NUT parameter to zero. The figure gives an overview of our construction, and shows the origin of each of the independent parameters.

Many of these solutions have singularities, and for this reason they have not appeared in many discussions of string corrections to black hole spacetimes. We have kept them here since we regard it as an open question whether the nominally singular solutions provide legitimate backgrounds for nonsingular string propagation. In any event, in many cases (such as at \( r = \ell \) for the \( d = 4 \) dilaton–metric solution, with \( 0 < \delta < 1 \) the singularity is lightlike, and so no asymptotic observer can see them. Thus they need not be regarded as ‘naked’ in the strictest sense. The same is not true when \( -1 < \delta \leq 0 \), since in this case the singularity at \( r = \ell \) is timelike and without a horizon.

If it should turn out that the dilaton were to be light enough to be of interest for systems of astrophysical size, then the existence of this new class of scalar-metric solutions to the low-energy string equations could have interesting consequences. This is because they do not appear to have been included amongst the alternatives to general relativity that are traditionally considered when theories of gravitation are confronted with experiment [32]. We hope to generalize the usual treatment in a future publication.

These solutions were obtained by performing a succession of \( SL(2, \mathbb{R}) \) and \( O(1, 1) \) symmetry transformations of the low-energy string equations, starting from the much simpler two-parameter family of dilaton–metric solutions of section 2. This type of algebraic manipulation is much easier to perform than would be a direct assault on the integration of the corresponding string field equations. Notice that it was the failure of the \( SL(2, \mathbb{R}) \) and \( O(1, 1) \) transformations to commute with one another which allowed us to build up the full six-parameter family from the original two-parameter class of solutions. Note that this is generically true for \( O(d, d + p) \) and \( SL(2, \mathbb{R}) \) transformations, and in fact both sets of transformations are subsumed within a larger group of transformations [33].

We claim that further \( SL(2, \mathbb{R}) \) transformations or \( O(1, 1) \) boosts (in the time-gauge directions as in eq. (23)) will not introduce any new solutions, but only map these solutions amongst themselves. Our reason for making this assertion is that since we have generated all of the possible charges which can describe the asymptotic behaviour of our fields at infinity, we have exhausted the space of solutions to the low-energy string equations which satisfy our stated symmetry ansatz for the fields. But since all of the \( SL(2, \mathbb{R}) \) and \( O(1, 1) \) transformations preserve these symmetries, as well as the asymptotic flatness, they must
lead to configurations that lie within the existing class.

It is instructive to consider some of the limits of our most general solution.

1. Setting $x = \pm 1$ leads to vanishing gauge fields, and gives back the general dilaton–axion–metric solutions of section 3. Notice that $\rho$ becomes a redundant parameter when $x = \pm 1$.

2. If we instead choose $\delta = 0$ (and $\gamma = 1$), then the parameter $\epsilon$ vanishes from the solutions. In this limit we recover the Taub–NUT dyons recently constructed in Ref.’s [34] and [35].

3. If, in addition to $\delta = 0$ and $\gamma = 1$, we also set $\omega = 0$, then we obtain the electrically and magnetically charged dilatonic black holes obtained in Ref. [12], [13].

4. Finally, the magnetically charged dilatonic black holes of Ref. [11] can be produced with the limit $\rho \to \infty$. Alternatively, $\rho = 0$ yields black holes with a purely electric charge.

We expect the techniques we have used to also have useful applications for the construction of more complicated string solutions, as for instance using the $\varphi$–independence of these backgrounds to generate more general axially symmetric solutions. Also an analysis of the supersymmetric nature of our solutions on the lines of Ref. [36], would be very interesting. Recently, an investigation of the non-commuting character of discrete $S$– and $T$–duality transformations has also appeared in Ref. [37].

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Appendix A. Higher-Dimensional Solutions

We begin by generalizing the dilaton–metric solutions of section 2 to spacetime dimensions $d \geq 4$. Following Callan et.al. [7], we write our static and spherically symmetric
metric (in the Einstein frame) in a slightly different way

\[ ds^2 = -U^2 dt^2 + V^2 (dr^2 + r^2 d\Omega_{d-2}) \]  

(45)

where \( d\Omega_{d-2} \) is the standard line element on a unit \((d-2)\)-sphere. The dilaton is chosen as above, \( \phi = \phi(r) \). Again allowing for arbitrary dilaton charge, the solutions may be written as

\[ U^2 = \left( \frac{\beta}{\alpha} \right)^{2H} \]

\[ V^2 = (\alpha \beta)^{2/(d-3)} \left( \frac{\alpha}{\beta} \right)^{2H/(d-3)} \]

\[ e^\phi = \left( \frac{\beta}{\alpha} \right)^K, \]

(46)

where

\[ \alpha = 1 + \left( \frac{\ell}{4r} \right)^{d-3}, \quad \beta = 1 - \left( \frac{\ell}{4r} \right)^{d-3}. \]  

(47)

The constants \( H \) and \( K \) are restricted to satisfy \( H^2 + K^2 (d-3)/(d-2)^2 = 1 \). \((H, K) = (1, 0)\) yields the standard Schwarzschild geometry in isotropic coordinates (as does \((-1, 0)\), although with a negative mass). The coefficients in eq. (47) have been chosen so that with \( d = 4 \), the constant \( \ell \) coincides with the same physical length appearing in eq. (6)—i.e., for the Schwarzschild solution, \( \ell = 2G_N M \) but note that in the present coordinates the horizon occurs at \( r = 4\ell \). These solutions have a nonvanishing dilaton charge for \( K \neq 0 \) and, in this case, \( r = 4\ell \) is a curvature singularity as is easily verified by evaluating the Ricci scalar using the equations of motion.

It is straightforward to apply an \( O(1,1) \) transformation as in section 4, to generate a gauge field for these solutions. The final result is an Einstein metric of the form

\[ ds^2 = \frac{U^2}{W^2/(d-3)/(d-2)} \ dt^2 + W^{2/(d-2)} V^2 (dr^2 + r^2 d\Omega_{d-2}) \]

(48)

where we have applied eq. (2), and

\[ W = \frac{1}{2} \left( 1 + x + (1 - x)U^2 e^{-2\phi/(d-2)} \right). \]

(49)
The final field configuration also includes the following dilaton and gauge fields

\[ e^{\hat{\phi}} = W e^{\phi} \]

\[ \hat{A}_t = \sqrt{x^2 - 1} \left( 1 - U^2 e^{-2\phi/(d-2)} \right) \frac{1}{W} . \]  

Now asymptotically, at large radius, one has:

\[ \hat{g}_{tt} \simeq -1 + \left[ 4H \left( 1 + (x - 1)\frac{d-3}{d-2} \right) - 4K(x - 1)\frac{d-3}{(d-2)^2} \right] \left( \frac{\ell}{4r} \right)^{d-3} + \cdots \]

\[ \hat{g}_{ij} \simeq \delta_{ij} \left[ 1 + \left[ 4H \left( \frac{1}{d-3} + \frac{x - 1}{d-2} \right) - 4K\frac{x - 1}{(d-2)^2} \right] \left( \frac{\ell}{4r} \right)^{d-3} \right] + \cdots \]  

\[ e^{\hat{\phi}} \simeq 1 + \left[ 2H(x - 1) - 2K \left( 1 + \frac{x - 1}{d-2} \right) \right] \left( \frac{\ell}{4r} \right)^{d-3} + \cdots \]

\[ \hat{A}_t \simeq \sqrt{x^2 - 1} \left( 4H - \frac{4K}{d-2} \right) \left( \frac{\ell}{4r} \right)^{d-3} + \cdots . \]  

As was the case in four dimensions, the asymptotic form of \( g_{ij} \) can be used to define the masses for these solutions. We find it to be given by [38]:

\[ 2G_N M = \frac{A_{d-2}}{2\pi} \left[ H(d - 2 + (x - 1)(d - 3)) - K(x - 1)\frac{d-3}{(d-2)^2} \right] \left( \frac{\ell}{4r} \right)^{d-3} , \]  

where \( A_{d-2} \) is the area of the unit \((d - 2)\)-sphere. Defining the dilaton charge of these solutions as the coefficient of \(-(1/r)^{d-3}\) in the asymptotic expansion of \( e^{\phi} \), we similarly obtain

\[ Q_D = \left[ 2K \left( 1 + \frac{x - 1}{d-2} \right) - 2H(x - 1) \right] \left( \frac{\ell}{4} \right)^{d-3} . \]  

Finally, using \( \hat{F}_{tr} \simeq \hat{Q}_E/r^{d-2} \), we find the electric charge to be

\[ Q_E = \sqrt{x^2 - 1} \left( 4H - \frac{4K}{d-2} \right) \left( \frac{\ell}{4} \right)^{d-3} . \]  

For \( d > 5 \) dimensions, these three physical charges completely characterize the solutions which are restricted to be static, asymptotically flat, and spherically symmetric.
(i.e., $SO(d - 2)$ invariant). For higher dimensions, there can be no magnetic charge associated with the gauge field, nor are there any spherically symmetric configurations of the Kalb-Ramond field. The one exception to the latter statement is for $d = 5$. In that case, solutions may be found with an extra magnetic-like charge from the $H$ field [11][39]. Presumably the known solutions, which have arbitrary masses and (magnetic) axion charges, could be generalized to a four parameter family of solutions including arbitrary electric gauge charges and dilaton charges, as well.
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7. Figure Captions

(1) The sequence of transformations with which we generate the six-parameter family of solutions. The second column shows the origins of each of the six parameters, $\ell$, $\delta$, $\omega$, $\epsilon$, $x$ and $\rho$. 
\[
\begin{array}{c}
\text{DILATON - STATIC METRIC} \\
\downarrow \\
\text{SL(2,R)} \\
\downarrow \\
\text{AXION - DILATON - STATIC METRIC} \\
\downarrow \\
\text{DUALITY} \\
\downarrow \\
\text{DILATON - TAUB-NUT METRIC} \\
\downarrow \\
\text{SL(2,R)} \\
\downarrow \\
\text{AXION - DILATON - TAUB-NUT METRIC} \\
\downarrow \\
\text{O(1,1)} \\
\downarrow \\
\text{E & M - AXION - DILATON - TAUB-NUT METRIC} \\
\downarrow \\
\text{SL(2,R)} \\
\downarrow \\
\text{E & M - AXION - DILATON - TAUB-NUT METRIC}
\end{array}
\]

\[
\ell, \delta, (\gamma) \\
\omega \\
\epsilon \\
x \\
\rho
\]

Figure 1
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9410142v2