Analytical Solution of Homogeneous One-Dimensional Heat Equation with Neumann Boundary Conditions

Norazlina Subani*, Faizzuddin Jamaluddin, Muhammad Arif Hannan Mohamed, Ahmad Danial Hidayatullah Badrolhisam

Kolej GENIUS Insan, Universiti Sains Islam Malaysia, Bandar Baru Nilai, 71800, Nilai, Negeri Sembilan, Malaysia

*Corresponding author: norazlina.subani@usim.edu.my

Abstract. A partial differential equation is an equation which includes derivatives of an unknown function with respect to two or more independent variables. The analytical solution is needed to obtain the exact solution of partial differential equation. To solve these partial differential equations, the appropriate boundary and initial conditions are needed. The general solution is dependent not only on the equation, but also on the boundary conditions. In other words, these partial differential equations will have different general solution when paired with different sets of boundary conditions. In the present study, the homogeneous one-dimensional heat equation will be solved analytically by using separation of variables method. Our main objective is to determine the general and specific solution of heat equation based on analytical solution. To verify our objective, the heat equation will be solved based on the different functions of initial conditions on Neumann boundary conditions. The results have been compared with different values of initial conditions but the boundary condition remain the same. Based on the results obtained, it can be concluded that increase the number of \( n \) will reduce the heat temperature and the time taken. For short length of the rod, the heat temperature quickly converges to zero and take less time to release or reduced the heat temperature when compared to the long length of the rod.

Keywords: Heat equation, homogeneous one-dimensional, Neumann boundary condition, analytical solution, separation of variables
1. Introduction

Most of mathematical physics are described by partial differential equations. Typically, a given partial differential equation will be solved by using numerical solution [1-3] and analytical solution [4]. However, it is vital to understand the general theory of partial differential equations to ensure the numerical solution is valid. Thus, the analytical solution is needed to obtain the exact solution of partial differential equation.

A partial differential equation is an equation which includes derivatives of an unknown function with respect to two or more independent variables. The partial differential equation can be classified into three types, which are parabolic [5-6], hyperbolic and elliptic [7-8]. A parabolic partial differential equation describing a large family of problems in science such as ocean acoustic propagation and heat diffusion. Hyperbolic partial differential equation describing the wave transformation and vibrations of an elastic string, while elliptic partial differential equation describing the Laplace equation.

To solve these partial differential equations, the appropriate boundary and initial conditions are needed. The general solution is dependent not only on the equation, but also on the boundary conditions. In other words, these partial differential equations will have different general solution when paired with different sets of boundary conditions.

Heat equation propagates energy at infinite speed, which is strongly non-physical. However, the validity of the heat equation as a model of temperature evolution is still extremely good for all classical physics and engineering applications. One of the major effects of heat transfer is temperature change, where the heating process will increase the temperature, while cooling process decrease the temperature [9]. In this process, here is assume that no phase change and that no work is done on or by the system [10]. Javed [11] studies about dry or moist heat sources. Dry applications include hot water bottles, radiant heat and electric pads. Moist heat is considered more penetrating than dry heat, but this is due more to the fact that water-soaked materials lose heat slower than dry ones.

In Islamic perspective, Sabaeian et al. [12] stated that the temperature distribution function is essential in calculation, simulation, and prediction of thermal effects. Temperature are specific to heat capacity, or the amount of energy required to change the temperature of a substance. The measurement of changes in heat as a result of physical or chemical changes [13].

According to As-Suyuti [14] and Al-Mahalli [15], the verse in Quran (Surah Yassin: 80) tells us about the production of fire from green trees. On the other words, the fire can be produced by using green plants. In that life, fire is generated from the friction of two surface objects [16]. The heat will flow from one area of high heat to low. The rate of heat velocity depends on the degree of friction speed between the two objects. In this research, the velocity of heat from a high heat area to a low heat area would be calculated.

In the present study, the homogeneous one-dimensional heat equation will be solved analytically by using separation of variables method. Our main objective is to determine the general and specific solution of heat equation based on analytical solution. To verify our
objective, the heat equation will be solved based on the Neumann boundary conditions by using this separation of variables method.

2. Mathematical Formulation

The mathematical models are used to describe the one-dimensional homogeneous heat boundary value problems with Neumann boundary conditions are presented below. The heat equation is used to determine the change in the function of temperature, $u$ over time, $t$. The simplified diagram of a physical model of the heat equation problem is shown in Fig. 1.

$$u(x,0) = f(x)$$

(Initial temperature distribution)

$$u(x,0) = f(x) \quad u(L,0) = 0$$

Fig. 1 Simplified diagram of physical model of heat equation problem.

2.1. Boundary Value Problem

The partial differential equation of one-dimensional homogeneous heat conduction equation is given by:

$$u_x(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0$$

(1)

where $u$ is defined as heat temperature, $x$ is space and $t$ is time.

2.2. Boundary Conditions

The Neumann boundary conditions at the initial point, $x = 0$ and at the end point $x = 1$ are given by:

$$u_x(0,t) = 0, \quad t > 0 \quad \text{and} \quad u_x(1,t) = 0, \quad t > 0$$

(2)

2.3. Initial Conditions

The initial conditions at $t = 0$ is:

$$u(x,0) = x, \quad 0 < x < 1$$

(3)
These mathematical models of equations (1)-(3) described the heat conduction in a one-dimensional uniform rod of length one unit with no internal heat sources, thermal diffusivity one, perfect lateral insulation and initial condition, $x$ when $0 < x < 1$. Both left end and right end is insulated and kept at $0^\circ$.

3. Analytical Solution

Most of the heat equation will be solved numerically by using Crank-Nicolson [17], finite different method [18-20] or finite element method [21-22]. However, the analytical solution is needed to obtain the exact solution of partial differential equation. To solve analytically the partial differential equation (1), Separation of Variables (SOV) is used. The single partial differential equation can be separated into two ordinary differential equations, where there is only one independent variable for each equation.

3.1. Transform the Partial Differential Equation into Separable Method

The partial differential equation (1) can be written in the form:

$$u(x,t) = X(x)T(t)$$  \hspace{1cm} (4)

where

$$u_x(x,t) = X(x)T'(t)$$ \hspace{1cm} (5a)

$$u_{xx}(x,t) = X^*(x)T(t)$$ \hspace{1cm} (5b)

Then, substituting equations (5a) and (5b) into equation (1) yields:

$$X(x)T''(t) = X^*(x)T(t)$$

$$\frac{X^*(x)}{X(x)} = \frac{T'(t)}{T(t)} = k$$ \hspace{1cm} (6)

Now, the two ordinary differential equations become:

$X$-problem: $X^*(x) = kX(x)$ \hspace{1cm} (7a)

$T$-problem: $T'(t) = kT(t)$ \hspace{1cm} (7b)
3.2. Solve the X-Problem by Strum-Liouville

To solve the heat equation (1), the boundary conditions (2) will be used. From equation (4), we have:

\[ u(x,t) = X(x)T(t) \quad \text{and} \quad u_x(x,t) = X'(x)T(t) \]

By using boundary conditions (2), we get:

\[ u_x(0,t) = X'(0)T(t) \]
\[ X'(0) = 0, \quad T(t) \neq 0 \]  \hspace{1cm} (8a)

and

\[ u_x(1,t) = X'(1)T(t) \]
\[ X'(1) = 0, \quad T(t) \neq 0 \]  \hspace{1cm} (8b)

Then, solve the X-problem. To solve the boundary value problem (BVP), there are three cases need to be considered:

\[ \lambda = 0 \]
\[ \lambda > 0 \] (trivial solution)
\[ \lambda < 0, \quad \lambda = -\lambda^2 \quad \text{and} \quad \lambda \neq 0 \] (non-trivial solution)

The constant zero solution is called the trivial solution of the equation. Thus, the nonzero (non-trivial) solutions need to be consider. In this problem, assume that \( k = -\lambda \) as a constant of separation. The negative sign is an arbitrary number and it could be either positive or negative or even zero.

From equation (7a), we have \( X'(x) = -\lambda^2 X(x) \). Find the eigenvalues and Eigen functions of the two-point boundary value problem. By solving this equation, the characteristic equation becomes \( m^2 + \lambda^2 = 0 \), which has conjugate complex roots \( m = \pm \lambda i \). Thus, the solution becomes:

\[ X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x) \]  \hspace{1cm} (9)

By applying the boundary conditions (8a) and (8b) into equation (9) yields:

\[ X_n(x) = C_n \cos(n \pi x) \]  \hspace{1cm} (10)

where \( X_0(x) = C_0, \quad \lambda_n = n \pi \) and \( n = 1, 2, 3, \ldots \).
3.3. Solve the T-Problem

An integrating the T-problem we have:

\[ T'(t) = -\lambda^2 T(t) \]
\[ \int \frac{dT}{T(t)} = -\int \lambda^2 dt \]
\[ T(t) = Be^{-\lambda^2 t} \]
\[ T_n(t) = B_n e^{-(n\pi)^2 t} \] (11)

where \( n = 1, 2, 3, \ldots \).

3.4. Find the Fundamental Solution

An integrating the T-problem we have:

\[ T'(t) = -\lambda^2 T(t) \]
\[ \int \frac{dT}{T(t)} = -\int \lambda^2 dt \]
\[ T(t) = Be^{-\lambda^2 t} \]

By substituting equations (10) and (11) into equation (4), the fundamental solution can be written as:

\[ u_n(x, t) = X_n(x)I_n(t) \]
\[ u_n(x, t) = A_n \cos(n\pi x) e^{-(n\pi)^2 t} \]

where \( u_0(x, t) = A_0 \) and \( n = 1, 2, 3, \ldots \).

Therefore, the general solution of homogeneous one-dimensional heat equation with Neumann boundary condition can be written as:

\[ u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) e^{-(n\pi)^2 t} \] (12)

3.5. Find the Specific Solution

An integrating the T-problem we have:

\[ T'(t) = -\lambda^2 T(t) \]
\[ \int \frac{dT}{T(t)} = -\int \lambda^2 dt \]
By applying the initial condition (3) into equation (12), we have:

\[ u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n \pi x) = f(x) \]  

where

\[ A_0 = \frac{1}{L} \int_0^L f(x) \, dx \]

\[ A_n = \frac{1}{L} \int_0^L x \, dx = \left[ \frac{x^2}{2} \right]_0^1 = \frac{1}{2} \]  

and

\[ A_n = \frac{2}{L} \int_0^L x \cos(n \pi x) \, dx = \frac{2}{n^2 \pi^2} \left[ 1 - (-1)^n \right] \]

where \( n = 1, 2, 3, \ldots \).

Therefore, the complete solution of homogeneous one-dimensional heat equation (1) can be written as:

\[ u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[ 1 - (-1)^n \right] \cos(n \pi x) e^{-(n \pi)^2 t} \]

4. Results and Discussion

Figs. 2, 3 and 4 show the heat profile along the rod for \( t > 0 \). The times are varies from 1.0s, 5.0s, 10.0s, 15.0s and 20.0s. Based on these figures, different length of the rod are used. Fig. 2 shows the heat profile along the rod for \( 0 < x < 1 \)m, while the heat profile along the rod for \( 0 < x < 10 \)m and \( 0 < x < 100 \)m are shown in Fig. 3 and Fig. 4, respectively. Figs. 2 and 3 are clearly show that there are different patterns of graph. For short length of rod, the heat temperature quickly reduced when \( x = 0.4 \)m. However, for long length of rod, the heat temperature is remain high at 1.0°C for the length \( 2 < x < 8 \)m, then the heat temperature start to reduce when \( L > 8 \)m. For the rod with length \( 0 < x < 100 \)m, the pattern of heat temperature profile is remain same as Fig. 4.
Fig. 2 Temperature profile $u(x,t)$ with different $n=1,3,5$ based on the equation (16).

Fig. 3 Temperature profile $u(x,t)$ with different $n=1,3,5$ based on the equation (16).
For $n=2$, the values of heat temperature is remain same as $n=1$, while the values of heat temperature at $n=4$ is remain same as $n=3$. The results are remaining same because of the equation (16) has the power of $n$. For even number of $n=2, 4, 6, 8, \ldots$, the values of heat temperature at that values tends to zero. Increase the number of $n$ will reduce the heat temperature and the time taken.

From Figs. 2, 3 and 4 the graph clearly show that the maximum heat temperature reach to $0.97^\circ$C. However, when the time, $t$ start to increase, $t=5.0s$, $t=10.0s$, $t=15.0s$ and $t=20.0s$, the heat temperature are remain constant at $0.5^\circ$C. Based on the Fig. 2, the heat temperature clearly show that it is quickly converge to zero when compared to Figs. 3 and 4. In other word, the short length of rod take less time to release or reduce the heat temperature, compared with the long rod.

Table 1 shows the analytical solution of different functions of initial conditions on Neumann boundary conditions. The heat equation (1) and the boundary conditions (2) are remain the same. The results show that there have different solutions although there used the same boundary conditions.
Table 1 Analytical solution of different functions of initial conditions on Neumann boundary conditions.

| No. | Partial differential equation | Neumann Boundary condition | Initial condition | Solution |
|-----|-------------------------------|----------------------------|------------------|---------|
| 1   | \( u_t(x,t) = u_{xx}(x,t) \), where \( 0 < x < 1, \ t > 0 \) | \( u_x(0,t) = 0, \ t > 0 \) \( u_x(1,t) = 0, \ t > 0 \) | \( u(x,0) = x, \ 0 < x < 1 \) | \( u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} \left[ 1 - (-1)^n \right] \cos(n \pi x) e^{-n^2 \pi^2 t} \) |
| 2   | \( u_t(x,t) = u_{xx}(x,t) \), where \( 0 < x < 1, \ t > 0 \) | \( u_x(0,t) = 0, \ t > 0 \) \( u_x(1,t) = 0, \ t > 0 \) | \( u(x,0) = 3 - 2 \cos(4 \pi x), \ 0 < x < 1 \) | \( u(x,t) = 3 - 2 \cos(4 \pi x) e^{-(4 \pi^2 t)} \) |
| 3   | \( u_t(x,t) = u_{xx}(x,t) \), where \( 0 < x < 1, \ t > 0 \) | \( u_x(0,t) = 0, \ t > 0 \) \( u_x(1,t) = 0, \ t > 0 \) | \( u(x,0) = f(x) \) \( f(x) = \begin{cases} 0, & 0 < x \leq \frac{1}{2} \\ 2x, & \frac{1}{2} < x < 1 \end{cases} \) | \( u(x,t) = \sum_{n=1}^{\infty} \left[ \frac{4}{n^2 \pi^2} \left( (-1)^n \cos\left(\frac{n \pi}{2}\right) \right) \right] \cos(n \pi x) e^{-n^2 \pi^2 t} + \frac{3}{4} \) |

5. Conclusion

The solution of heat equation is dependent not only on the equation, but also on the boundary conditions or initial conditions. These partial differential equations will have different general solution when paired with different sets of boundary conditions or initial conditions. In the present study, the homogeneous one-dimensional heat equation will be solved analytically by using separation of variables method. To verify our objective, the heat equation will be solved based on the different function of initial conditions on Neumann boundary conditions. For short length of rod, the heat temperature quickly reduced compared to the long length of rod. For \( n = 2 \), the values of heat temperature is remain same as \( n = 1 \), while the values of heat temperature at \( n = 4 \) is remain same as \( n = 3 \). The results are remaining same because of the equation (16) has the power of \( n \). For even number of \( n = 2, 4, 6, 8, \ldots \), the values of heat temperature at that values tends to zero. Increase the number of \( n \) will reduce the heat temperature and the time taken. For short length of the rod, the heat temperature quickly converges to zero and take less time to release or reduced the heat temperature when compared to the long length of the rod.
Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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