Sofic profiles of $S(\omega)$ and computability

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Abstract. We realize sofic profiles in the group of computable permutations of $\mathbb{N}$ so that the approximating morphisms can be viewed as restrictions of permutations to finite subsets of $\mathbb{N}$. We also study some relevant effectivity conditions.

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1 Introduction

We remind the reader that an abstract group $G$ is called sofic if $G$ embeds into a metric ultraproduct of finite symmetric groups with the Hamming distance $d_H$, where the latter is defined as follows

$$d_H(g, h) = 1 - \frac{|\text{Fix}(g^{-1}h)|}{n}, \text{ for } g, h \in S_n$$

(see [1], [14]). It is an open question of M. Gromov whether every countable group is sofic.

A more detailed approach to this notion was suggested by Arzhantseva, Cherix and de Cornulier, see for example [5]. Firstly as in [9] we call a finite subset $E$ of a group $G$ with 1 a chunk of $G$. In the definition below we assume that $\inf \emptyset = +\infty$.

Definition 1.1 Let $E$ be a chunk of $G$ and let $n \in \mathbb{N}$, $0 < \varepsilon < 1$. 
• An \( \varepsilon \)-morphism from \( E \) to \( (S_n, d_H) \) is a mapping \( f : E \to S_n \) such that \( f(1) = 1 \in S_n \) and \( d_H(f(xy), (f(x)f(y))) \leq \varepsilon \) for all \( x, y \in E \) with \( xy \in E \).

• A mapping \( f : E \to S_n \) is said to be \((1-\varepsilon)\)-expansive if \( d_H(f(x), f(y)) \geq (1-\varepsilon) \) whenever \( x, y \) are distinct points of \( E \).

• Define the sofic profile of the chunk \( E \) as the function:

\[
\text{prof}_E(r) = \inf\{ n \in \mathbb{N} : \text{there exists a } (1-r^{-1})\text{-expansive } r^{-1}\text{-morphism } E \to (S_n, d_H) \} , \ r > 1.
\]

• We say that the chunk \( E \) is sofic if for any \( r > 1 \), \( \text{prof}_E(r) < \infty \).

• A group \( G \) is sofic if each chunk of \( G \) is sofic.

As a result we have an ultraproduct-free definition of soficity. We will consider chunks as partial groups. In particular we say that a map \( f : E_1 \to E_2 \) between two chunks is a homomorphism if it preserves the multiplication and the unit. When \( f \) is bijective we say that \( E_1 \) is a realization of \( E_2 \). Note that in this case \( f \) is not necessarily an isomorphism.

Since permutation groups are involved into the definition of soficity, they often arise in this context, see [3], [16]. In our case the idea is as follows. When \( E_1 \) is a chunk of \( S(\omega) \), the group of all permutations of the set of natural numbers, and \( f : E_1 \to E_2 \) is a bijective homomorphism to a sofic chunk, then restricting elements of \( E_1 \) to some finite subsets \( D \) of \( \mathbb{N} \), one may obtain approximations of \( E_2 \) in appropriate \( S_n \) as in the definition above. These restrictions are not necessarily permutations of \( D \), but under some natural conditions one can often find \( D \) where the corresponding restrictions are almost permutations. In this paper we consider such a condition, which roughly says that the elements of \( E_1 \) do not grow very fast.

By Theorem 3.11 each sofic chunk can be so realized. This gives a possibility of study of some questions concerning sofic chunks in versions available for analysis and evaluate their complexity. We mainly concentrate on sofic profiles of these chunks, on their growth and computability of sofic profiles of groups (see Definition 1.2). This develops some issues of M. Cavaleri’s investigations in [2], [3], [4].

It is worth noting that sofic chunks always have computable profiles. Using this we show that they can be realized by computable permutations. This connects our approach with previous work in computability theory concerning algebraic and computably structure of the group of computable permutations, see [11] and [12].

We finish this introduction by the following definitions (see [5]). If \( u, v : [1, \infty] \to [0, \infty] \) are non-decreasing functions, we write \( u \preceq_{pf} v \) if there are
positive real numbers $C, C', C''$ such that

$$(\forall r \geq 1)(u(r) \leq Cv(C'r) + C''),$$

and we write $u \simeq_{pf} v$ if $u \preceq_{pf} v \preceq_{pf} u$.

**Definition 1.2** Let $G$ be an abstract group. The sofic profile of $G$ is the family of $\simeq_{pf}$-equivalence classes of the functions $\text{prof}_E$ for all chunks $E$ of $G$.

In fact in this paper we are interested in sofic profiles of computably enumerable groups which can be embedded into a subgroup of $S(\omega)$ consisting of computable permutations which do not grow very fast.

### 2 Computability and computable profiles

We use standard material from the computability theory (see [13] and [15]). In particular we assume that the reader knows the definition of arithmetical hierarchy. From now on we identify each finite set $F \subset \mathbb{N}$ with its Gödel number.

#### 2.1 Computability

Let $G$ be a countable group generated by some $X \subseteq G$. The group $G$ is called **recursively presented** (see [10]) if $X$ can be identified with $\mathbb{N}$ (or with some $\{0, \ldots, n\}$) so that $G$ has a recursively enumerable set of relators in $X$. Below we give an equivalent definition, see Definition 2.3. It is justified by a possibility identification of the whole $G$ with $\mathbb{N}$. We follow the approach of [9].

**Definition 2.1** Let $G$ be a group and $\nu : \mathbb{N} \to G$ be a surjective function. We call the pair $(G, \nu)$ a numbered group. The function $\nu$ is called a numbering of $G$. If $g \in G$ and $\nu(n) = g$, then $n$ is called a number of $g$.

**Definition 2.2** A numbered group $(G, \nu)$ is a **computable presentation** if $\nu$ is a bijection and the set

$$\text{MultT} := \{(i, j, k) : \nu(i)\nu(j) = \nu(k)\}$$

is computable (= recursive).

Any finitely generated group with decidable word problem obviously has a computable presentation. This also holds in the case of the free group $F_\omega$ with the free basis $\{x_0, \ldots, x_i, \ldots\}$. If we fix a computable presentation $(F_\omega, \nu_F)$ then for every recursively presented group $G = \langle X \rangle$ and a natural homomorphism $\rho : F_\omega \to G$ (taking $\omega$ onto $X$) we obtain a numbering $\nu = \rho \circ \nu_F$ which satisfies the following definition.
Definition 2.3 A numbered group \((G, \nu)\) is computably enumerable if the set
\[
\text{MultT} := \{(i, j, k) : \nu(i) \nu(j) = \nu(k)\}
\]
is computably enumerable.

Remark 2.4 Let \((G, \nu)\) be a computably enumerable group.

- There exists recursive function \(\Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}\) such that for all \(x, y \in \mathbb{N}\) the equality \(\nu(x)\nu(y) = \nu(\Phi(x, y))\) holds.
- For every \(x \in \mathbb{N}\) we can effectively find \(y \in \mathbb{N}\) with \(\nu(x)\nu(y) = 1\).
- The sets \(\{n : \nu(n) = 1\}\) and \(\{(n_1, n_2) : \nu(n_1) = \nu(n_2)\}\) are computably enumerable.

Remark 2.5 If \((G, \nu)\) is a numbered group and the set MultT from Definition 2.3 is computable, then \(G\) has a computable presentation (possibly under another numbering). In this case we also have

- the set \(\{(n_1, n_2) : \nu(n_1) = \nu(n_2)\}\) is computable.

Groups \((G, \nu)\) as in this remark are called computable groups. They corresponds to groups with solvable word problem. In this case the numbering \(\nu\) is called a constructivization.

2.2 Effective soficity

Let \(G\) be a countable group and \(E\) be a chunk of \(G\). Consider the profile \(\text{prof}_E\) as a function \(\mathbb{N} \to \mathbb{N}\) (taking the restriction to \(\mathbb{N}\)). Then we easily see the following lemma.

Lemma 2.6 If \(E\) is sofic then the function \(\text{prof}_E\) is computable.

Proof. The statement follows from the assumption that \(\text{prof}_E\) is total and the observation that set of pairs \(\{(n, m) : \text{prof}_E(n) < m\}\) is decidable. □

- Thus the \(\simeq_{pf}\)-class of total \(\text{prof}_E\) consists of subrecursive functions.

Let \(\varphi(x, y)\) be the universal computable function and \(\varphi_k(x) = \varphi(k, x), k \in \mathbb{N}\). The family of sofic chunks can be viewed as the family

\[
\Sigma-SC = \{(E, k) : E \text{ is a chunk, } \varphi_k \text{ is total non-decreasing and } \forall n(\text{prof}_E(n) \leq \varphi_k(n))\}.
\]

Indeed, if \(\text{prof}_E\) is bounded above by a computable function then \(\text{prof}_E\) is computable. We note the following observations.
The set of triples \( \{(E, n, m) : \text{prof}_E(n) < m \} \) is decidable, where each \( E \) is a finite set with a partial binary operation.

If \( \varphi_k \) is a fixed total and non-decreasing function then

\[
\{ E : E \text{ is a chunk, and } \forall n (\text{prof}_E(n) \leq \varphi_k(n)) \} \in \Pi^0_1.
\]

Since the graph of the universal computable function \( \varphi(x, y) \) is a computably enumerable set,

\[\text{TND} := \{ k : \varphi_k \text{ is total and non-decreasing } \} \in \Pi^0_2.\]

Now it is easy to see that \( \Sigma-SC \) can be presented as a \( \Pi^0_2 \)-set of natural numbers. Moreover \( \Sigma-SC \) is \( \Pi^0_2 \)-complete, i.e. the problem of recognition of its members belongs to \( \Pi^0_2 \) and any other problem of \( \Pi^0_2 \) is reducible to it.

**Proposition 2.7** \( \Sigma-SC \) is a \( \Pi^0_2 \)-complete set.

**Proof.** Note that \( \text{TND} \) is reducible to \( \Sigma-SC \). Indeed, let \( ss(k) \) be the computable function which finds the number of \( \varphi_k + 2 \). Considering \( \mathbb{Z}/2\mathbb{Z} \) as a chunk we see

\[ k \in \text{TND} \Leftrightarrow (\mathbb{Z}/2\mathbb{Z}, ss(k)) \in \Sigma-SC. \]

Thus to see the statement of the proposition it remains to notice that \( \text{TND} \) is \( \Pi^0_2 \)-complete. Let \( \text{INF} = \{ e \mid W_e \text{ is infinite} \} \). This set is \( \Pi^0_2 \)-complete, \[13\], \[15\]. It is easy to see that \( \text{INF} \) is reducible to \( \text{TND} \). \( \square \)

**Remark 2.8** Since each countable group embeds into \( S(\omega) \) it is clear that \( E \) can be realized in \( S(\omega) \). Theorem 3.11 below can be interpreted as an effective version of this observation. It roughly states that every sofic chunk \( E \) can be realized by a bijective homomorphism \( E' \to E \) where the realization \( E' \) is a finite family of \( \varphi_k \in S(\omega) \), such that the corresponding sofic approximations of \( E \) are restrictions of \( E' \) to initial segments of \( \omega \). This will give another \( \Pi^0_2 \)-parametrization of sofic chunks. We will see in Section 3.3 that this parametrizations is also \( \Pi^0_2 \)-complete.

**Definition 2.9** Let \( (G, \nu) \) be a computably enumerable group. We say that \( (G, \nu) \) is effectively sofic if there is a uniform algorithm which for every finite \( D \subset \mathbb{N} \) and every \( n \) finds the value \( m = \text{prof}_E(n) \) for \( E = \nu(D) \) (and the corresponding \( (1 - n^{-1}) \)-expansive \( n^{-1} \)-morphism into \( S_m \)).

In fact effective soficity was introduced by M. Cavaleri in \[2\] in the case of finitely generated groups. We now formulate a version of a theorem from \[2\].
Proposition 2.10 A computably enumerable group \((G, \nu)\) is effectively sofic if and only if \((G, \nu)\) is sofic and computable.

Proof. It is clear that any effectively sofic group is sofic. To verify the equality \(\nu(i) \cdot \nu(j) = \nu(k)\) let us apply effective soficity to \(D = \{i, j, k\}\) and \(n = 3\). Computing in the corresponding \(S_m\) the distance between \(\nu(i) \cdot \nu(j)\) and \(\nu(k)\) we check if it is \(< \frac{1}{3}\) or \(\geq \frac{2}{3}\).

If \((G, \nu)\) is sofic and computable, \(n \in \mathbb{N}\) and \(D\) is a finite subset of \(\mathbb{N}\) we start the procedure verifying for natural numbers \(1, \ldots, m, \ldots\) if there is a \((1 - n^{-1})\)-expansive \(n^{-1}\)-morphism into \(S_m\). This gives an algorithm for effective soficity. □

3 Function growth and the corresponding subgroups of \(S(\omega)\)

3.1 Growths and permutation groups

We will denote \(\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}\) and assume that \(n < \infty\) for all \(n \in \mathbb{N}\).

- Let \(\Omega_\infty\) be the semigroup of all functions \(g : \mathbb{N}_\infty \to \mathbb{N}_\infty\) such that \(g(n + 1) \geq g(n) > n\) for all \(n \in \mathbb{N}\) and \(g(\infty) = \infty\).

Clearly, the semigroup \(\Omega_\infty\) has no identity element.

We order \(\Omega_\infty\) as follows:

\[ f < g \iff \text{there exists } n_0 \in \mathbb{N} \text{ such that } \forall n > n_0 (f(n) < g(n)). \]

We now define

\[ f \ll g \iff (\forall k \in \mathbb{N})(f^k < g), \text{ and} \]

\[ f \sim g \iff (\exists k \in \mathbb{N})(f < g^k \land g < f^k), \]

where \(f^k\) is the \(k\)-th power of \(f\) with respect to composition. The equivalence classes of \(\sim\) will be called growths. The following easy observation is proved in [8]:

- each growth \([f]_\sim\) is a subsemigroup of \(\Omega_\infty\).

Definition 3.1 Define the profile of \(g \in \Omega_\infty\) as

\[ \text{prof}_g(r) = \inf\{n \in \mathbb{N} : (\exists m \in \mathbb{N})(g(m) \leq n) \land (\frac{n - m}{n} < r^{-1})\}, r \geq 1. \]

Note that when \(f < g\) then \(\text{prof}_f < \text{prof}_g\). Moreover when \(\text{prof}_g(r) = n\), then \(n\) is of the form \(g(m)\).
Definition 3.2 We say that \( g \in \Omega_\infty \) is slow if it satisfies

\[
\lim_{n \to \infty} \frac{n}{g(n)} = 1.
\]

It is clear that the profile of a slow function is a total function.

Lemma 3.3 (a) If a growth \([g]_\sim\) contains a slow function then all functions of the growth are slow.
(b) If \( g \) and \( h \) are slow and have the same growth, then \( \text{prof}_g \) and \( \text{prof}_h \) are \( \simeq_{pf} \)-equivalent.

Proof. (a) The first statement follows from the observation that given functions \( g, h \in \Omega_\infty \) satisfying

\[
\lim_{n \to \infty} \frac{n}{g(n)} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{n}{h(n)} = 1.
\]

we also have:

\[
\lim_{n \to \infty} \frac{n}{(gh)(n)} = \lim_{n \to \infty} \frac{n}{h(n)} \cdot \frac{h(n)}{g(n)} = 1.
\]

(b) It is enough to show that \( \text{prof}_g \) and \( \text{prof}_{g^2} \) are \( \simeq_{pf} \)-equivalent. It is clear that for any \( C > 1 \), \( g(n) \prec C \cdot n \). In particular, if for sufficiently large \( n \) (we may additionally assume that \( g(n) < C \cdot n \)) we have a number \( m \) with \( g(m) \leq n \) and \( \left| \frac{n}{m} - 1 \right| < \varepsilon \), then we find two numbers \( m_1 \) and \( m_2 \) such that \( g(m_1) \leq C \cdot n \), \( g(m_2) \leq m_1 \),

\[
\left| \frac{m_1}{C \cdot n} - 1 \right| < \varepsilon \quad \text{and} \quad \left| \frac{m_2}{m_1} - 1 \right| < \varepsilon.
\]

Thus

\[
\left| \frac{m_1}{C \cdot n} \cdot \frac{m_2}{m_1} - \frac{m_1}{C \cdot n} \right| < \varepsilon
\]

and \( \left| \frac{m_2}{C \cdot n} - 1 \right| < 2\varepsilon \).

We conclude that

\[
(\forall r \geq 1)(\text{prof}_{g^2}(r) \leq C \text{prof}_g(\frac{1}{2}r) + C''),
\]

where \( C'' \) is chosen so that \( g(n) < C \cdot n \) for all \( n > C'' \). □

Note that the growth represented by \( f(n) = n + 1 \) consists of slow functions (by statement (a) of the lemma).

Growths of slow functions will usually appear in the following context.

Definition 3.4 A permutation \( \rho \in S(\omega) \) is bounded by \( g \in \Omega_\infty \) if for every \( n \in \mathbb{N} \) and \( m \leq n \) we have \( \rho(m) \leq g(n) \).
It is obvious that:

- if $\rho_1$ is bounded by $g_1$ and $\rho_2$ is bounded by $g_2$ then $\rho_1 \cdot \rho_2$ is bounded by $g_1 \circ g_2$.

**Definition 3.5** Let $\alpha = [f]_{\sim}$ be a growth. We define

$$S_\alpha(\omega) = \{ \rho \in S(\omega) : \rho \text{ and } \rho^{-1} \text{ are bounded by some } g \in \alpha \}.$$  

The chunks of $S_\alpha(\omega)$ and their corresponding profiles are the main objects of our further investigations. Let us mention the following remarks.

- $S_\alpha(\omega)$ is a subgroup of $S(\omega)$ containing $SF(\omega)$, the subgroup of all finitary permutations.
- If $\alpha$ is represented by $f = \infty$, then $S_\alpha(\omega) = S(\omega)$.

The following statement is a corollary of Lemma [3.3].

**Corollary 3.6** Let $\alpha$ be a growth which contains a slow function. Then for any chunk $E \subseteq S_\alpha(\omega)$ there is a slow $g \in \alpha$ such that all elements of $E$ are bounded by $g$.

### 3.2 Profiles of g-chunks

**Definition 3.7** Let $g \in \Omega_{\infty}$. A chunk $E \subset S(\omega)$ is called a **g-chunk** if all elements of $E$ are bounded by $g$.

**Definition 3.8** Let $g \in \Omega_{\infty}$ and $E \subseteq S(\omega)$ be a g-chunk. Given $n \in \mathbb{N}$ a mapping $\sigma : E \to S_n$ is called a **supp-morphism** if $\sigma(1) = 1$ and

$$\forall \rho \in E \forall m \leq n(\rho(m) \leq n \to \rho(m) = \sigma(\rho)(m)).$$

The idea of supp-morphisms is to associate to each $\rho \in E$ a permutation from $S_n$ which can be considered as a restriction of $\rho$ to $\{1, \ldots, n\}$. The existence of supp-morphic images in $S_n$ is obvious.

The following proposition compares $\text{Prof}_E$ and $\text{Prof}_g$.

**Proposition 3.9** Let $g : \mathbb{N} \rightarrow \mathbb{N}$ have total $\text{prof}_g$ and let $E \subset S(\omega)$ be a g-chunk. For every $n \in \omega$ choose a supp-morphism $\sigma_n : E \rightarrow S_n$.

Then the following statements hold.

1. For any $\varepsilon$ there is $n$ such that $\sigma_n$ is an $\varepsilon$-morphism.
2. The profile of the property that the mapping $\sigma_n$ is an $r^{-1}$-morphism is bounded above by $\text{prof}_g(2r)$.
3. If for any \( r \) and \( n \in \mathbb{N} \) with \( \text{prof}_g(r) = n \) and any distinct \( \rho_1, \rho_2 \in E \) we have

\[
g(n - |\text{Fix}(\sigma_n(\rho_1)(\sigma_n(\rho_2)^{-1}))|) \geq n,
\]

then the sofic profile of \( E \) is bounded above by \( \text{prof}_g(2r) \).

**Proof.** 1. Since \( E \) is a \( g \)-chunk it is easy to see that for any \( \rho \in E \) the \( \sigma_n \)-image of \( \rho \) can differ from the partial map which is the restriction \( \rho|_n \), only inside \( \{m, \ldots, n\} \) where \( m \) is maximal with \( g(m) \leq n \). Thus \( \sigma_n(\rho)\sigma_n(\rho') \) can differ from \( (\rho\rho')|_n \) only inside \( g^{-1}(\{m, \ldots, n\}) \cup \{m, \ldots, n\} \). So by the definition of the Hamming metric,

\[
d_H(\sigma_n(\rho)\sigma_n(\rho'), \sigma_n(\rho\rho')) \leq 2\left(\frac{n-m}{n}\right).
\]

Now statement 1 follows from the definition of \( \text{prof}_g \).

Statement 2 follows from the proof of statement 1.

To see statement 3 note that the inequality

\[
n - |\text{Fix}(\sigma_n(\rho_1)(\sigma_n(\rho_2)^{-1}))| \geq m
\]

implies that \( d_H(\sigma_n(\rho_1), \sigma_n(\rho_2)) \geq 1 - \frac{n-m}{n} \). If \( \frac{n-m}{n} \leq (2r)^{-1} \), then

\[
d_H(\sigma_n(\rho_1), \sigma_n(\rho_2)) \geq 1 - (2r)^{-1}.
\]

The rest follows from statement 2. \( \square \)

The following observation is an easy corollary of the definition of \( S_\alpha(\omega) \), Lemma 3.3 and Proposition 3.9. In statements 2 and 3 observe that the profile of a slow computable function is computable.

**Corollary 3.10** Let \( \alpha \) be a growth which contains a slow function.

- Then for any chunk \( E \subseteq S_\alpha(\omega) \) there is \( g \in \alpha \) such that \( E \) is a \( g \)-chunk and for any choice of supp-morphisms \( \sigma_n : E \rightarrow S_n \) the profile of the property that the mapping \( \sigma_n \) is an \( r^{-1} \)-morphism is bounded above by \( \text{prof}_g(2r) \).

- Moreover if in the previous statement \( \alpha \) is represented by a computable function, then for any choice of supp-morphisms \( \sigma_n : E \rightarrow S_n \) the profile of the property that the mapping \( \sigma_n \) is an \( r^{-1} \)-morphism is a computable function.

The following theorem is the main statement of this section. It justifies \( g \)-chunks of slow functions \( g \).
Theorem 3.11 A chunk $E$ is sofic if and only if there is a slow $g \in \Omega_\infty$ and a $g$-chunk $E' \subset S(\omega)$ realizing $E$ under a bijective homomorphism $h : E' \to E$ such that there is a family of supp-morphisms $\delta_n$ which provide sofic approximations of $E$ under the identification $h : E' \to E$.

Moreover for a sofic chunk $E$ the function $g$, the $g$-chunk $E'$ as above and the profile of the property that $\delta_n$ are sofic approximations of $E$ can be realized by computable functions.

Proof. For every $n > 1$ fix $m_n = \text{prof}_{E}(n)$ and an appropriate $(1 - \frac{1}{n})$-expansive $\frac{1}{n}$-morphism $\sigma_n : E \to (S_{m_n}, d_H)$. Let $f$ be a function $\mathbb{N} \to \mathbb{N}$. The direct sum $\bigoplus_{1<i\leq n} S_{m_i}^{f(i)}$ is naturally considered as a permutation group on the set of natural numbers less than $\sum_{1<i\leq n} f(i)m_i$. We associate to every $e \in E$ and $n \in \omega$ the permutation

$$\sigma_{\bowtie, n}(e) = \bigoplus_{1<i\leq n} \sigma_{f(i)}^i(e) \in \bigoplus_{1<i\leq n} S_{m_i}^{f(i)}.$$

Let $\sigma(e)$ be the limit of these elements in the natural direct limit of all $\bigoplus_{1<i\leq n} S_{m_i}^{f(i)}$, $n \to \infty$. Then $\sigma(e) \in S(\omega)$, the permutation $\sigma_{\bowtie, n}(e)$ is the restriction of $\sigma(e)$ to the initial segment of $\sum_{1<i\leq n} f(i)m_i$ elements of $\omega$ and the distance between $\sigma_{\bowtie, n}(e)$ and $\sigma_{\bowtie, n}(e')$ in $S_{\sum_{1<i\leq n} f(i)m_i}$ is equal to

$$\frac{\sum_{1<i\leq n} f(i)(m_i - |\text{Fix}(\sigma_i(e)\sigma_i(e')^{-1})|)}{\sum_{1<i\leq n} f(i)m_i}.$$  

To satisfy the conditions of the theorem, i.e. to realize $\delta_n$ as $\sigma_{\bowtie, n}$, we need an appropriate function $f$. Given $f(2), \ldots, f(n - 1)$ one can choose $f(n)$ so that $\sigma_{\bowtie, n}$ is an $(1 - \frac{1}{n})$-expansive $\frac{1}{n-1}$-morphism into $S_{\sum_{1<i\leq n} f(i)m_i}$.

We define the function $g$ as follows. Let $g(j) = j + m_2 + \ldots + m_n$ for each $j$ with $\sum_{1<i\leq n} f(i)m_i \leq j < \sum_{1<i\leq n} f(i)m_i$ (where the empty sum is 0). Then each $\sigma(e)$ with $e \in E$ is bounded by $g$. Thus $E' = \{ \sigma(e) : e \in E \}$ is a $g$-chunk.

Given $f(2), \ldots, f(n - 1)$ one can correct $f(n)$ so that

$$\frac{\sum_{1<i\leq n} m_i}{(\sum_{1<i\leq n} f(i)m_i) - 1 + \sum_{1<i\leq n} m_i} < \frac{1}{n}.$$  

Since this exactly means

$$1 - \frac{(\sum_{1<i\leq n} f(i)m_i) - 1}{g((\sum_{1<i\leq n} f(i)m_i) - 1)} < \frac{1}{n},$$

we conclude that taking $f$ sufficiently fast we guarantee that $g$ is slow and the maps $\sigma_{\bowtie, n}$ form a family of supp-morphisms which are $(1 - \frac{1}{n-1})$-expansive $\frac{1}{n-1}$-morphisms into $S_{\sum_{1<i\leq n} f(i)m_i}$.  


Since $E$ is sofic, the sequences $m_2, m_3, \ldots, m_n, \ldots$ is computable. In particular we can define an algorithm of computation of $f$ so that the inequalities guaranteeing slowness of $g$ and the property that $\sigma_{\otimes, n}$ is an $(1 - \frac{1}{n-1})$-expansive $\frac{1}{n-1}$-morphism into $S_{\sum_{i \leq n} f(i)m_i}$, are satisfied. Computability of permutations $\sigma(e)$, $e \in E$, follows from computability of the sequence $\sigma_2, \sigma_3, \ldots, \sigma_n, \ldots$ in the beginning of the proof. □

**Remark 3.12** It is interesting to compare Theorem 3.11 with the theorem of A. Morozov [11] that there is a finitely generated group with the word problem in $\Pi^0_1$ which is not embeddable into the group of computable permutations (it is obvious that any finitely generated group of computable permutations has the word problem in $\Pi^0_1$). In our case soficity of a chunk is equivalent to possibility of its computable realization.

The following example is slightly connected with Problem 3.18 from [5] which asks if there is a group for which the sofic profile is unbounded and not linear. Indeed assuming below that for the function $g$ the profile $\text{prof}_g(2r)$ is non-linear, we can choose a family of supp-morphisms $\sigma_n$ so that the $g$-chunk $E$ below has the property that linearity of its sofic profile cannot be realized by these $\sigma_n$.

**Example 3.13** For any slow function $g \in \alpha$ there is a $g$-chunk $E \subseteq S_\alpha(\omega)$ and a family of supp-morphisms $\sigma_n : E \to S_n$ such that the profile of the property that $\sigma_n$ is an $r^{-1}$-morphism is equivalent to $\text{prof}_g(2r)$.

We build a 3-element chunk as follows. Let $h$ be a disjoint union of 3-element cycles:

for each $k \in \mathbb{N}$ let $h(3k) = 3k + 1$, $h(3k + 1) = 3k + 2$, and $h(3k + 2) = 3k$.

Let $E$ consist of $h, h^2$ and $\text{id} = h^3$. Let $g \in \alpha$ be sufficiently fast, for example $g(n) > 30 + n$. For $n = 3m + i$ with $i \in \{0, 1\}$ let $\sigma_n(h)$ coincide with $h$ for all $l < 3m$ and $\sigma_n(h^2)$ coincide with $h^2$ for all $l$ with $g(l) \leq n$. We additionally assume that for all $l$ with $g(l - 2) > n$ the permutation $\sigma_n(h^2)$ coincides with $\text{id}$. Then assuming that $m$ is the maximal number with $g(m) \leq n$ we have $|m - |\text{Fix}(((\sigma_n(h))^{-2}\sigma_n(h^2)))|| \leq 5$. This is enough for the second statement. □

### 3.3 $g$-Chunks of computable functions

In this subsection we show that Theorem 3.11 provides a parametrization of sofic chunks which is different from the one given in Section 2.2, but which is still $\Pi^0_2$-complete.
Theorem 3.14 The family of all triples \((D, \circ, n)\) with conditions given below is \(\Pi^0_2\)-complete:

- \(n \in \mathbb{N}\) such that the profile of \(\varphi_n\) is total,
- \(D\) is a finite subset of \(\mathbb{N}\), \(\circ\) is a partial binary operation on \(D\) and the functions realizable by \(\varphi_k(x)\) for \(k \in D\) form a \(\varphi_n\)-chunk of \(S(\omega)\) which realizes \((D, \circ)\) with respect to the map \(\varphi_k \rightarrow k\), and
- the chunk \((D, \circ)\) is sofic under approximations by \(\text{supp}\)-morphisms.

Proof. To see that the set of triples of the statement belongs to the class \(\Pi^0_2\) let us list the following family of statements where the next statement strengthens the previous one. The following families belong to the class \(\Pi^0_2\):

1. The family of all pairs \((D, \circ)\), where \(D\) is a finite subsets \(\mathbb{N}\) such that functions realizable by \(\varphi_k(x)\) for \(k \in D\) form a chunk of \(S(\omega)\) which realizes the chunk \((D, \circ)\) under the map \(\varphi_k \rightarrow k\).

2. The family of all triples \((D, \circ, m)\), where \(D \subseteq \mathbb{N}\) and \(\circ\) are as in the previous statement and the chunk \(\{\varphi_k : k \in D\}\) forms a \(g\)-chunk where the profile of \(g = \varphi_m\) is total.

3. The family of all tuples \((D, \circ, l, m, n)\), where \((D, \circ, m)\) are as in the previous statement and there is a \(\text{supp}\)-morphism \(\sigma_n\) to \(S_n\) which is a \((1 - \frac{1}{l})\)-expansive \(\frac{1}{l}\)-morphism.

4. The family of all triples \((D, \circ, m)\), where \((D, \circ, m)\) are as in the previous statement and the chunk \((D, \circ)\) is sofic under approximations by \(\text{supp}\)-morphisms.

Concerning the first case note that there is a computable enumeration of all pairs \((D, \circ)\) satisfying the implication: "if \(\{\varphi_k(x) : k \in D\}\) form a chunk of \(S(\omega)\) then the map \(\varphi_k \rightarrow k\) is a 1-1-homomorphism”.

Concerning the third case note that given \(\sigma : D \rightarrow S_n\) we can express that it realizes an \((1 - \frac{1}{l})\)-expansive \(\frac{1}{l}\)-morphism from \(E\) by a satatement concerning \(\sigma(D) \subseteq S_m\) and a family of conditions of the form \(\varphi_i \circ \varphi_j \neq \varphi_k\), where \(i, j, k \in D\). Under the \(\Pi^0_2\)-requirement that all \(\varphi_k\) are permutations for \(k \in D\), the inequalities above distinguish a computably enumerable set of triples \(i, j, k\).

To show that \(\text{INF} = \{e|W_e\text{ is infinite}\}\) is reducible to the family of triples \((D, \circ, n)\) as in the statement we define an algorithm which finds triples \((D_e, \circ_e, n_e)\), where \((D_e, \circ_e) \cong (\mathbb{Z}/2\mathbb{Z}, +)\), the number \(n_e\) is a computability number of the function \(x + 3\) (i.e all \(n_e\) are the same) and the members of \(D_e\) are two elements \(d_e, d'_e\) which satisfy the following property:
$d'_e$ is a fixed number of the identity permutation of $\mathbb{N}$ and for all $e \in \mathbb{INF}$ the number $d_e$ is a number of a permutation of order 2 with support $\mathbb{N} \setminus 3\mathbb{N}$.

At Stage 0 we initialize the procedure by setting $f_{e,0}$ the function which for every natural $l$ takes the elements $3l, 3l+1$ to $3l+2$ and the elements $3l+2$ to $3l+1$. At Stage $s+1$ having input $(e, s+1)$ we verify if the algorithm enumerating $W_e$ adds a new element to the previously computed part of $W_e$, i.e. if $W_{e,s+1} \setminus W_{e,s} \neq \emptyset$. If this is not the case we set $f_{e,s+1} = f_{e,s}$. When $W_{e,s+1} \setminus W_{e,s} \neq \emptyset$ we change the definition of $f_{e,s}$ by $f_{e,s+1}(k) = k$, where $k$ is the first number of the form $3l$ which is taken by $f_{e,s}$ to $3l+2$.

Let $f_e$ be the limit $\lim_{s \to \infty} f_{e,s}$. It is clear that $f_e$ is computable and is the same for all $e$ with infinite $W_e$. Let $d_e$ be a number $t$ provided by the algorithm for $W_e$ such that $\varphi_t = f_e$. When $e \in \mathbb{INF}$ the function $f_e$ is a permutation of order 2. Moreover in this case any restriction of $f_e$ to an initial segment of the form $\{0,1,\ldots,3l\}$ is a permutation of order 2. Thus supp-morphisms of this form give sofic approximations. When $e \notin \mathbb{INF}$ the function $f_e$ is not a permutation, thus the triple $(D_e, o_e, n_e)$ does not satisfy the condition of the statement of the theorem. □

4 The computable part of $S_\alpha(\omega)$

Let $\alpha$ be a growth which does not contain $\infty$ and which is represented by a computable function. Let

$$S_\alpha^{\text{comp}}(\omega) = \{ \rho \in S(\omega) : \rho \text{ and } \rho^{-1} \text{ are computable and bounded above} \}$$

by some $g \in \alpha$.

It is clear that

$$SF(\omega) < S_\alpha^{\text{comp}}(\omega) < S^{\text{comp}}(\omega),$$

where $S^{\text{comp}}(\omega)$ consists of all computable permutations.

Is it possible to get a version of Theorem 3.14 where the numbering of computable functions $\varphi_k$ is replaced by a numbering $\nu$ of $S_\alpha^{\text{comp}}(\omega)$ which makes it a computably enumerable group? We will see below that $S_\alpha^{\text{comp}}(\omega)$ even is not computably enumerable.

4.1 Computably enumerable actions

Let $G$ be a computably enumerable subgroup of $S^{\text{comp}}(\omega)$ and let $\nu$ be the corresponding numbering. Let $\text{MultT}$ be the graph of the multiplication:

$$\text{MultT} = \{(l, m, n) : \nu(l) \cdot \nu(m) = \nu(n)\}.$$
The natural action of $G$ on $\mathbb{N}$ defines the following ternary relation on $\mathbb{N}$:

$$R_{act} = \{(l, m, n) : (\nu(l))(m) = n\}.$$  

Assuming that $R_{act}$ is computably enumerable we arrive at the situation of a **computably enumerable action**. The following proposition demonstrates that this condition makes some families of chunks slightly less complicated than in the case of the numbering $\varphi_k$, $k \in \omega$. Note that condition 3 together with the conjunction of condition 2 for all $m$ and $n$ gives the family of all sofic $g$-chunks of $G$ as in the second part of Theorem 3.11.

**Proposition 4.1** Let $(G, \nu)$ be a computably enumerable subgroup of $S^{\text{comp}}(\omega)$ with the computably enumerable action and let $n_0$ be the number of the identity function on $\mathbb{N}$. Let $g : \mathbb{N} \to \mathbb{N}$ be a slow computable total function.

1. The following family is computable:
   - all pairs $(D, \circ)$, where $D$ is a finite subset of $\mathbb{N}$, $n_0 \in D$ and $\circ$ is a partial binary operation such that all $\nu(k)$ for $k \in D$ form a chunk $E$ of $G$ which realizes $(D, \circ)$ with respect to the map $\nu(k) \to k$.

2. The following family is computably enumerable:
   - all triples $(D, \circ, n)$, where the pair $(D, \circ)$ is as in statement 1 and there is $m \in \mathbb{N}$ and a $(1 - \frac{1}{n})$-expansive $\frac{1}{n}$-morphism $\sigma_m : E \to S_m$ such that$
   \forall k \in D \forall l \leq m(\nu(k)(l) \leq m \to \nu(k)(l) = \sigma_m(\nu(k))(l)).$

3. The following family belongs to $\Pi^0_1$:
   - all pairs $(D, \circ)$ as in statement 1, where all $\nu(k)$ for $k \in D$ form a

   **Proof.** Concerning statement 1 notice that computable enumerability of $(G, \nu)$ gives a computable enumeration of all pairs $(D, \circ)$ where the map $\nu(k) \to k$ is not a homomorphism. In particular pairs of statement 1 belongs to $\Pi^0_1$. On the other hand since the action of $(G, \nu)$ is computably enumerable there is a computable enumeration of all pairs $(D, \circ)$ where the map $\nu(k) \to k$ is a 1-1-homomorphism.

   Note that given $m$ and $n$ statement 2 is a decidable property. The remaining arguments are standard and are omitted. $\square$

We will see in the next subsection that in these proposition $G$ cannot be $S^{\text{comp}}_\alpha(\omega)$. On the other hand for $G = S^{\text{comp}}_\alpha(\omega)$ relativized statements of this proposition still make sense: one can fix a Turing degree $d$ and consider computably enumerable groups and computably enumerable actions with respect to $d$.
4.2 The computable part of $S_{\alpha}(\omega)$

The following theorem says that $S_{\alpha}^{\text{comp}}(\omega)$ is not computably enumerable.

**Theorem 4.2** For any growth $\alpha$ the group $S_{\alpha}^{\text{comp}}(\omega)$ is not computably enumerable.

**Proof.** This statement is known for the group $S^{\text{comp}}(\omega)$ (i.e. when $\alpha$ is represented by $\infty$), see p. 301, Section 6.1 in [7]. We will adapt the corresponding proof. Since [7] is not easily available we include some details.

We firstly note that any finitary permutation belongs to $S_{\alpha}^{\text{comp}}(\omega)$ for any $\alpha$. We will concentrate on transpositions. We need the following property of them:

- the supports of transpositions $\gamma$ and $\gamma'$ have a common point if and only if $(\gamma \cdot \gamma')^3 = 1$.

We will use the transpositions $\delta_1 = (0, 1)$, $\delta_2 = (0, 2)$. The conjugacy class of this pair consists of all pairs of transpositions which supports have exactly one common point. If pairs $(\gamma_1, \gamma_2)$ and $(\gamma'_1, \gamma'_2)$ belong to this class then they have the same intersections of supports iff

$$(\gamma_1 \cdot \gamma'_1)^3 = (\gamma_1 \cdot \gamma'_2)^3 = (\gamma_2 \cdot \gamma'_1)^3 = (\gamma_2 \cdot \gamma'_2)^3 = 1.$$ 

The following permutations $\delta$ belong to $S_{\alpha}^{\text{comp}}(\omega)$ for any $\alpha$:

$$\delta(1) = 0, \delta(2n) = 2(n + 1) \text{ for } n \in \mathbb{N}, \text{ and } \delta(2n + 3) = 2n + 1 \text{ for } n \in \mathbb{N}.$$ 

Indeed both $\delta$ and $\delta^{-1}$ are bounded by $n + 3$, which is the third iteration of $n + 1$.

For each $i$ let $(\gamma_{2i}, \gamma'_{2i})$ be $\delta^{-i}(\delta_1, \delta_2)\delta^i$ and $(\gamma_{2i+1}, \gamma'_{2i+1})$ be $\delta^{i+1}(\delta_1, \delta_2)\delta^{-i-1}$. Then for any permutation $\rho \in S_{\alpha}^{\text{comp}}(\omega)$ we have the following equivalence:

$$\rho(k) = n \Leftrightarrow (\gamma_n \cdot \gamma'_k)^3 = (\gamma_n \cdot \gamma'_k)^3 = (\gamma_n \cdot (\gamma'_k)^\rho)^3 = (\gamma_n \cdot (\gamma'_k)^\rho)^3 = 1.$$ 

Assuming that $S_{\alpha}^{\text{comp}}(\omega)$ is computably enumerable under a numbering $\nu$ we define an element of $S_{\alpha}^{\text{comp}}(\omega)$, say $\tau$, as follows. For every natural $n$ the permutation $\tau$ realizes a 3-element cycle on $\{3n, 3n+1, 3n+2\}$. If the permutation $\nu(n)$ takes $3n$ to $3n+1$ we put $\tau(3n) = 3n+2$, $\tau(3n+2) = 3n+1, \tau(3n+1) = 3n$. In the contrary case we put $\tau(3n) = 3n+1, \tau(3n+1) = 3n+2, \tau(3n+2) = 3n$. As a result $\tau$ belongs to $S_{\alpha}(\omega)$ and does not have any number under $\nu$.

To get a contradiction let us notice that $\tau$ is a computable function. Indeed, since $(S_{\alpha}^{\text{comp}}(\omega), \nu)$ is computably enumerable there is an algorithm which enumerates all triples $k, m, n$ such that for $\rho = \nu(m)$ the following equalities are satisfied:

$$(\gamma_n \cdot \gamma'_k)^3 = (\gamma_n \cdot \gamma'_k)^3 = (\gamma_n \cdot (\gamma'_k)^\rho)^3 = (\gamma_n \cdot (\gamma'_k)^\rho)^3 = 1.$$
As we already know these equalities say that \((\nu(m))(k) = n\). As a result it is decidable whether \((\nu(n))(3n) = 3n + 1\). In particular \(\tau\) is computable. □

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