Conformastationary disk-haloes in Einstein-Maxwell gravity

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An Exact solution of the Einstein-Maxwell field equations for a conformastationary metric with magnetized disk-halos sources is worked out in full. The characterization of the nature of the energy momentum tensor of the source is discussed. All the expressions are presented in terms of a solution of the Laplace’s equation. A “generalization” of the Kuzmin solution of the Laplace’s equations is used as a particular example. The solution obtained is asymptotically flat in general and turns out to be free of singularities. All the relevant quantities show a reasonable physical behaviour.

I. INTRODUCTORY REMARKS

In a recent work \cite{1}, we presented a relativistic model describing a thin disk surrounded by a halo in presence of an electromagnetic field. The model was obtained by solving the Einstein-Maxwell equations on a particular conformastatic spacetime background and by using the distributional approach for the energy-momentum tensor. The class of solution corresponding to the model is asymptotically flat and singularity-free, and satisfies all the energy conditions. The purpose of the present work is to extend the above-mentioned study to the conformastationary case, and the Kuzmin solution of the Laplace's equation to include a “generalized” Kuzmin solution of the Laplace’s equation. The reason to undertake such an endeavour are easy to understand. Indeed, the issue of the exact solution of the Einstein and Einstein-Maxwell equations describing isolated sources self gravitating in a stationary axially symmetric spacetime appears to be of great interest both from a mathematical and physical point of view. For details of the astrophysics importance and the most relevant developments concerning the disks and disk-haloes sources the reader is referred to the works \cite{1,2} and references therein.

In this work we present a new exact solution of the Einstein-Maxwell field equations for a thin disk surrounded by a magnetized halo in a conformastationary background. This solution is notoriously simple in its mathematical form. Moreover, the interpretation of the energy-momentum tensor presented here generalises the commonly used pressure free models to a fluid with non-vanishing pressure, heat flux and anisotropic tensor. In Section II we present an exact general relativistic model describing a disk surrounded by an electromagnetized halo and we obtain a solution of the Einstein-Maxwell field equations in terms of a solution of the Laplace’s equation. In Section III we express the surface energy-momentum tensor of the disk in the canonical form and we present a physical interpretation of it in terms of a fluid with non-vanishing pressure and heat flux. In Section IV a particular family of conformastationary magnetized disk-haloes solutions is presented. We complete the paper with a discussion of the results in Section V.

II. GENERAL RELATIVISTIC MAGNETIZED HALOES SURROUNDING THIN DISKS

To obtain an exact general relativistic model describing a disk surrounded by an electromagnetized halo in a conformastationary spacetime, we solved the distributional Einstein-Maxwell field equations assuming axial symmetry and that the derivatives of the metric and electromagnetic potential across the disk space-like hyper-surface are discontinuous. Here, the energy-momentum tensor is taken to be the sum of two distributional components, the purely electromagnetic (trace-free) part and a “material” (trace) part. Accordingly, the Einstein-Maxwell equations,
in geometrized units such that $c = 8\pi G = \mu_0 = \epsilon_0 = 1$, are equivalent to the system of equations

\begin{align}
G^\pm_{\alpha\beta} &= R^\pm_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R^\pm = E^\pm_{\alpha\beta} + M^\pm_{\alpha\beta} \tag{1a} \\
H_{\alpha\beta} - \frac{1}{\sqrt{g_{\alpha\beta}}}H &= 0 \tag{1b} \\
F_{\alpha\beta}^{\pm} &= J_\alpha^{\pm}, \tag{1c} \\
[F_{\alpha\beta}]^\pm &\equiv \mathcal{T}^\pm, \tag{1d}
\end{align}

where $H = g^{\alpha\beta}H_{\alpha\beta}$ and $g$ the determinant of the metric tensor. Here, the square brackets in expressions such as $[F_{\alpha\beta}]$ denote the jump of $F_{\alpha\beta}$ across of the surface $z = 0$, $n_\alpha$ denotes a unitary vector in the direction normal to it and

\begin{equation}
H_{\alpha\beta} = \frac{1}{2} \left\{ \gamma_+^a \delta^\beta_\alpha - \gamma_-^a \delta^\alpha_\beta - \gamma^a_\beta \delta^\alpha_\beta - g^{zz} \gamma_{0\beta} \right\}, \tag{2}
\end{equation}

with $\gamma_{ab} = [g_{ab}, z]$ and all the quantities are evaluated at $z = 0^+$. In the appendix (A) we give the corresponding field equations and the energy-momentum of the halo and of the disk for a sufficiently general metric.

To obtain a solution of the distributional Einstein-Maxwell describing a system composed by a magnetized halo surrounding a thin disk in a conformastationary background, we shall restrict ourselves to the case where the electric potential $A_r = 0$. We also conveniently assume the existence of a function $\phi$ depending only on $r$ and $z$ in such a way that the metric (A1) can be written in the form

\begin{equation}
ds^2 = -e^{2\beta}(dt + \omega d\phi)^2 + e^{-2\beta}(dr^2 + dz^2 + r^2d\phi^2), \tag{3}
\end{equation}

with $\beta$ an arbitrary constant. Accordingly, for the non-zero components of the energy-momentum tensor of the halo we have

\begin{align}
M^+_{tt} &= -e^{2(1+\beta)}\phi \left\{ \beta^2 \nabla \phi \cdot \nabla \phi - 2\beta \nabla^2 \phi + \frac{1}{2} \nabla^2 \phi \cdot \nabla \phi + \frac{1}{4} r^2 e^{2(1+\beta)} \nabla \omega \cdot \nabla \omega \right\} \tag{4a} \\
M^+_{t\phi} &= e^{2(1+\beta)}\phi \left\{ \frac{\beta}{2} \nabla \omega \cdot \nabla \phi + \frac{3}{4} r^2 e^{2(1+\beta)} \nabla \omega \cdot \nabla \phi + 2\beta \omega \nabla^2 \phi + \frac{3}{2} \nabla \omega \cdot \nabla \phi \\
M^+_{r\phi} &= (1 - \beta) \nabla^2 \phi - (1 - \beta) \phi \nabla \phi + (\beta^2 - 2\beta) \phi^2 + \phi^2 - \frac{1}{2} r^2 e^{2(1+\beta)} A^2_{\phi,r} - A^2_{\phi,z} \\
M^+_{\phi\phi} &= \frac{1}{4} r^2 e^{2(1+\beta)} \left( \omega^2_{rr} - \omega^2_{zz} \right) \tag{4c} \\
M^+_{rz} &= \frac{1}{2} r^2 e^{2(1+\beta)} \phi \omega \omega_{rz} - (1 - \beta^2 + 2\beta) \phi \omega_{r,z} - (1 - \beta) \phi_{r,z} - r^2 e^{2(1+\beta)} A_{r\phi,r} A_{\phi,z}, \tag{4d} \\
M^+_{z\phi} &= -\frac{1}{4} r^2 e^{2(1+\beta)} \phi \left( \omega^2_{\phi\phi} + \phi^2 \right) - (1 - \beta) \phi_{zz} + (1 - \beta) \nabla^2 \phi + (\beta^2 - 2\beta) \phi_z^2 + \frac{1}{2} r^2 e^{2(1+\beta)} A^2_{\phi,r} - A^2_{\phi,z}, \tag{4e} \\
M^+_{\phi\phi} &= r^2 \nabla \phi \cdot \nabla \phi + (1 - \beta) r^2 \nabla^2 \phi - (1 - \beta) \nabla \phi \cdot \nabla r - \frac{1}{2} e^{2\beta} \phi \nabla A_{\phi} \cdot \nabla A_{\phi} + \mathcal{H}(r, z), \tag{4f} \\
\mathcal{H}(r, z) &= e^{2(1+\beta)} \phi \left\{ \frac{1}{4} (1 + 3 r^2 e^{2\beta} \omega^2) \nabla \omega \cdot \nabla \omega - \beta^2 \omega^2 \nabla \phi \cdot \nabla \phi + 2\beta \omega^2 \nabla^2 \phi + \omega \nabla^2 \omega \right. \\
&\quad - 2r^{-1} \omega \nabla \omega \cdot \nabla r + (3 + \beta) \omega \nabla \phi \cdot \nabla \phi - \frac{1}{2} r^2 e^{2\beta} \phi \nabla A_{\phi} \cdot \nabla A_{\phi} \right\}. \tag{4g}
\end{align}

Whereas for the non-zero components of the electric current density on the halo has the form

\begin{align}
J^+_{\phi\phi} &= e^{-(1-3\beta)} \nabla \cdot \{ \omega r^{-2} e^{2(1+\beta)} \phi \nabla A_{\phi,z} \}, \tag{5a} \\
J^+_{\phi r} &= e^{-(1-3\beta)} \nabla \cdot \{ r^{-2} e^{2(1+\beta)} \phi \nabla A_{\phi,z} \}. \tag{5b}
\end{align}

The non-zero components of the surface energy-momentum tensor (SEMT) and the non-zero components of the surface electric current density (SECD) of the disk are given by

\begin{align}
S_{tt} &= 4\beta e^{(2+\beta)} \phi \phi_{z,z}, \tag{6a} \\
S_{t\phi} &= e^{(2+\beta)} \phi (4\beta \omega \phi_{z,z} + \omega_{zz}), \tag{6b} \\
S_{rr} &= 2(1 - \beta) e^{-\beta} \phi_{z,z}, \tag{6c} \\
S_{\phi\phi} &= e^{(2+\beta)} \phi \left\{ (4\beta \omega^2 + 2(1 - \beta) r^2 e^{-2(1+\beta)} \phi) \phi_{z,z} + 2\omega \phi_{z,z} \right\}. \tag{6d}
\end{align}
and

\[ F^t = r^{-2}e^{3\beta\phi}\omega[A_{\varphi,z}], \]
\[ F^\varphi = -r^{-2}e^{3\beta\phi}[A_{\varphi,z}], \]

respectively. Note that all the quantities are evaluated on the surface of the disk. Next, we assume that there are not exist any electric currents on the halo by setting in (5)

\[ J^t_\pm = J^\varphi_\pm = 0. \]

Hence the system of equations (3) is equivalent to the very simple system

\[ \nabla \omega \cdot \nabla A_\varphi = 0, \]
\[ \nabla \cdot \{r^{-2}F\nabla A_\varphi\} = 0, \]

where \( F \equiv e^{(1+\beta)\phi} \). If \( \hat{e}_\varphi \) is a unit vector in azimuthal direction and \( \lambda \) is an arbitrary function independent of the azimuthal coordinate \( \varphi \), then one has the identity

\[ \nabla \cdot \{r^{-1}\hat{e}_\varphi \times \nabla \lambda\} = 0. \]

The identity (10) may be regarded as the integrability condition for the existence of the function \( \lambda \) defined by

\[ r^{-2}F\nabla A_\varphi = r^{-1}\hat{e}_\varphi \times \nabla \lambda. \]

Accordingly, the identity (10) implies the equation

\[ \nabla \cdot \{F^{-1}\nabla \lambda\} = 0 \]

for the “auxiliary” potential \( \lambda(r,z) \). In order to have an explicit form of the metric function \( \phi \) and magnetic potential \( A_\varphi \) we suppose that \( \phi \) and \( A_\varphi \) depend explicitly on \( \lambda \). Consequently the equation (12) implies

\[ -F^{-1}F^t\nabla \lambda \cdot \nabla \lambda + \nabla^2 \lambda = 0, \]

where

\[ F' = (1 + \beta)F \frac{d\phi}{d\lambda}. \]

Let us assume the very useful simplification

\[ F^{-1}F' = k, \]

with \( k \) an arbitrary constant. Then, we have \( F = k_3e^{k_\lambda} \) and

\[ -k\nabla \lambda \cdot \nabla \lambda + \nabla^2 \lambda = 0, \]

where \( k_3 \) is an arbitrary constant. Furthermore, if we assume the existence of a function

\[ U = k_4e^{-k_\lambda} + k_5, \]

with \( k_4 \) and \( k_5 \) arbitrary constants, then

\[ \nabla^2 U = -kk_4e^{-k_\lambda}\{-k\nabla \lambda \cdot \nabla \lambda + \nabla^2 \lambda\} = 0. \]

Accordingly, \( \lambda \) can be represented in terms of solutions of the Laplace’s equation:

\[ e^{k_\lambda} = \frac{k_4}{U - k_5}, \quad \nabla^2 U = 0. \]

Hence, the metric potential \( \phi \) can be written in terms of \( U \) as

\[ e^{(\beta+1)\phi} = \frac{k_3k_4}{U - k_5}. \]

To obtain the metric function \( \omega \) we first note that from (11) we have the relationship between \( A_\varphi \) and \( \lambda \):

\[ \nabla A_\varphi = \lambda A_{\varphi,r}\hat{e}_r + \lambda A_{\varphi,z}\hat{e}_z = rF^{-1}\hat{e}_\varphi \times (\lambda_{,r}\hat{e}_r + \lambda_{,z}\hat{e}_z). \]

Then we have, \( A_{\varphi,r} = -rF^{-1}\lambda_{,r} \) and \( A_{\varphi,z} = rF^{-1}\lambda_{,z} \), or, in terms of \( U \)

\[ A_{\varphi,r} = k_6tU_{,z}, \]
\[ A_{\varphi,z} = -k_6tU_{,r}, \]

where \( k_6 = 1/(kk_3k_4) \). Furthermore, with (22) into (22a) we arrive to

\[ \omega_{,z}U_{,z} - \omega_{,r}U_{,r} = 0, \]

which admits the solution \( \omega = k_\omega U + k_8 \), with \( k_\omega \) and \( k_8 \) arbitrary constants. With the aim to have an asymptotic behaviour of the metric potentials, without loss of generality, we take \( k_3k_4 = -k_5 = -1 \) and \( k_8 = 0 \).
III. THE SEMT OF THE DISK

We now assume that it is possible to express the SEMT of the disk in the canonical form

\[ S_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + P g_{\alpha\beta} + Q_\alpha V_\beta + Q_\beta V_\alpha + \Pi_{\alpha\beta}. \]  

(24)

Consequently, we can say that the disk is constituted by a mass-energy distribution described by the last surface energy-momentum tensor and \( V^\alpha \) is the four velocity of certain observer. Accordingly \( \mu, P, Q_\alpha \) and \( \Pi_{\alpha\beta} \) are then the energy density, the isotropic pressure, the heat flux and the anisotropic tensor on the surface of the disk, respectively. Thus, it is immediate to see that

\[ \mu = S_{\alpha\beta} V^\alpha V^\beta, \]  

(25a)

\[ P = \frac{1}{3} \mathcal{H}^{\alpha\beta} S_{\alpha\beta}, \]  

(25b)

\[ Q_\alpha = -\mu V_\alpha - S_{\alpha\beta} V^\beta, \]  

(25c)

\[ \Pi_{\alpha\beta} = \mathcal{H}_\alpha^\mu \mathcal{H}_\beta^\nu (S_{\mu\nu} - P g_{\mu\nu}), \]  

(25d)

with \( \mathcal{H}_{\mu\nu} \equiv g_{\mu\nu} + V_\mu V_\nu \) and \( \alpha = (t, r, \varphi) \). It is easy to note that by choosing the angular velocity to be zero in \( B_7 \) we have then a fluid comoving in our coordinates system. Accordingly, we may introduce a suitable reference frame in terms of the local observers tetrad \( B_3 \) and \( B_4 \) in the form

\[ \{V_\alpha, I_\alpha, K_\alpha, Y_\alpha\} \equiv \{h_\alpha(t), h_\alpha(r), h_\alpha(z), h_\alpha(\varphi)\}. \]  

Accordingly, by using (25a), (25b) and (6) we have for the surface energy density and the pressure of the disk

\[ \mu = 4\beta e^{\beta \phi} \delta_{\Phi,z}, \]  

(26)

and

\[ P = \frac{4}{3} (1 - \beta) e^{\beta \phi} \delta_{\Phi,z} = \frac{1 - \beta}{3\beta} \mu, \]  

(27)

respectively. By introduce (6) in (25c) we obtain for the non-zero components of the heat flux

\[ Q_\alpha = -e^{(\beta+1)\phi} \omega_s \delta^e_\alpha. \]  

(28)

Notice that \( Q_\alpha V^\alpha = Q^\alpha V_\alpha = 0 \). In the same way, by using \( 25a \) and \( 6 \) we have for the non-zero components of the anisotropic tensor

\[ \Pi_{rr} = \frac{2(1 - \beta)}{3} e^{-\beta \phi} \delta_{\phi,z}, \]  

(29a)

\[ \Pi_{\varphi\varphi} = \frac{2(1 - \beta)}{3} r^2 e^{-\beta \phi} \delta_{\phi,z} = r^2 \Pi_{rr}. \]  

(29b)

Analogously, the SECD of the disk \( J^\alpha \) can be also written in the canonical form

\[ J^\alpha = \sigma V^\alpha + j Y^\alpha, \]  

(30)

then \( \sigma \) can be interpreted as the surface electric charge density and \( j \) as the “current of magnetization” of the disk. A direct calculation shows that the surface electric charge density \( \sigma = -V_\alpha J^\alpha = 0 \), whereas the “current of magnetization” of the disk is given by

\[ j = Y_\alpha J^\alpha = -r^{-1} e^{2\beta \phi} [A_{\varphi,z}], \]  

(31)

where, as above, \( [A_{\varphi,z}] \) denotes the jump of the \( z \)-derivative of the magnetic potential across of the disk and, all quantities are evaluated on the disk. Thus, by using the results of the precedent section, we can write the surface energy density, the pressure, the heat flux, the non-zero components of the anisotropic tensor and the current of
magnetization on the surface of the disk, respectively, as
\[
\mu = \frac{4\beta U_z}{(\beta + 1)(1 - U)^{n+1}} \tag{32a}
\]
\[
P = \frac{(1 - \beta)}{3\beta} \mu, \tag{32b}
\]
\[
Q_\alpha = -\frac{k_\omega U_z}{1 - U} \delta_{\alpha \phi}, \tag{32c}
\]
\[
\Pi_{rr} = \frac{2(1 - \beta)U_z}{3(1 + \beta)(1 - U)^{n+1}}, \tag{32d}
\]
\[
P_{rr} = r^2 \Pi_{rr}, \tag{32e}
\]
\[
j = -\frac{[U_r]}{k(1 - U)^{3/2}}. \tag{32f}
\]
where, as we know, \(U(r, z)\) is an arbitrary suitable solution of the 2-dimensional Laplace’s equation in cylindrical coordinates and \([U, r]\) denotes the jump of the \(r\)-derivative of the \(U\) across of the disk. All the quantities are evaluated on the surface of the disk. It is important to note that \(k_\omega\) is a defining constant in \(32c\). Indeed, when \(k_\omega = 0\) the heat flux \(Q_\alpha\) vanishes, a feature of the static disk.

### IV. A PARTICULAR FAMILY OF CONFORMASTATIONARY MAGNETIZED DISK-HALOES SOLUTIONS

In precedent works \([1, 5]\) we have presented a model for a conformastatic relativistic thin disk surrounded by a material electromagnetized halo from the Kuzmin solution of the Laplace’s equation in the form
\[
U_K = \frac{m}{\sqrt{r^2 + (|z| + a)^2}}, \quad (a, m > 0). \tag{33}
\]
As it is well known, \(\nabla^2 U_K\) must vanish everywhere except on the plane \(z = 0\). At points with \(z < 0\), \(U_K\) is identical to the potential of a point mass \(m\) located at the point \((r, z) = (0, -a)\), and when \(z > 0\), \(U_K\) coincides with the potential generated by a point mass at \((0, a)\). Accordingly, it is clear that \(U_K\) is generated by the surface density of a Newtonian mass
\[
\rho_K(r, z = 0) = \frac{am}{2\pi(r^2 + a^2)^{3/2}}. \tag{34}
\]
In this work, we present a sort of generalization of the Kuzmin solution by considering a solution of the Laplace’s equation in the form \([6]\),
\[
U = -\sum_{n=0}^{N} \frac{b_n P_n(z/R)}{R^{n+1}}, \quad P_n(z/R) = (-1)^n \frac{R^{n+1}}{n!} \frac{\partial^n}{\partial z^n} \left(\frac{1}{R}\right), \tag{35}
\]
\(P_n = P_n(z/R)\) being the Legendre polynomials in cylindrical coordinates which has been derived in the present form by a direct comparison of the Legendre polynomial expansion of the generating function with a Taylor of \(1/r\) \([7]\). \(R^2 \equiv r^2 + z^2\) and \(b_n\) arbitrary constant coefficients. The corresponding magnetic potential, obtained from \([22]\), is
\[
A_\phi = -\frac{1}{k} \sum_{n=0}^{N} \frac{b_n (-1)^n}{n!} \frac{\partial^n}{\partial z^n} \left(\frac{z}{R}\right) \tag{36}
\]
where, we have imposed \(A_\phi(0, z) = 0\) in order to preserve the regularity on the axis of symmetry. Next, to introduce the corresponding discontinuity in the first-order derivatives of the metric potential and the magnetic potential required to define the disk we preform the transformation \(z \rightarrow |z| + a\). Thus, taking account of \([62]\), the surface energy density of the disk, the heat flux and the non-zero components of the anisotropic tensor are
\[
\mu(r) = \frac{4\beta \sum_{n=0}^{N} b_n (n + 1) P_{n+1}(a/R_a) R_a^{-(n+2)}}{\left(1 + \beta\right) \left(1 + \sum_{n=0}^{N} b_n P_n(a/R_a) R_a^{-(n+1)}\right)^{(2\beta + 1)/(\beta + 1)}}, \tag{37a}
\]
\[
Q_\alpha = \frac{k_\omega \sum_{n=0}^{N} b_n P_n(a/R_a) R_a^{-(n+1)}}{1 + \sum_{n=0}^{N} b_n P_n(a/R_a) R_a^{-(n+1)}}, \tag{37b}
\]
and
\[ \Pi_{rr} = \frac{2(1 - \beta) \sum_{n=0}^{N} b_n (n + 1) P_{n+1} (a/R_a) R_a^{-(n+2)}}{3(1 + \beta) \left( 1 + \sum_{n=0}^{N} b_n P_n (a/R_a) R_a^{-(n+1)} \right)^{1/(1+\beta)}}, \] (38a)
respectively, where \( R_a^2 = r^2 + a^2 \). As we know, the other quantities are \( P = (1 - \beta)\mu/(3\beta) \) and \( \Pi_{\varphi\varphi} = r^2 \Pi_{rr} \).

The current of magnetization is
\[ j = -\frac{2r \sum_{n=0}^{N} b_n P_{n+1} (a/R_a) R_a^{-(n+3)}}{k(1 + \sum_{n=0}^{N} b_n P_n (a/R_a) R_a^{-(n+1)})^{2\beta/(1+\beta)}}, \] (39)
where we have used (32f) and we first assumed that the \( z \)-derivative of the magnetic potential present a finite discontinuity through the disk. In fact, as we have said above, the derivatives of \( U \) and \( A_\varphi \) are continuous functions across of the surface of the disk. We artificially introduce the discontinuity through the transformation \( z \rightarrow |z| + a \).

In order to illustrate the last solution we consider particular solutions with \( N = 0 \) and \( N = 1 \). Then we have \( U_N \) for the two first members of the family of the solutions as follows,
\[ U_0 = -\frac{\tilde{b}_o}{\sqrt{\tilde{r}^2 + (|\tilde{z}| + 1)^2}}, \] (40a)
\[ U_1 = -\frac{\tilde{b}_o}{\sqrt{\tilde{r}^2 + (|\tilde{z}| + 1)^2}} \left\{ 1 + \frac{\tilde{b}_1 (|\tilde{z}| + 1)}{\tilde{b}_o (|\tilde{z}| + 1)^2 + \tilde{r}^2} \right\}, \] (40b)
where \( \tilde{b}_o = b_o / a \) and \( \tilde{b}_1 = b_1 / a^2 \) whereas \( \tilde{r} = r / a \) and \( \tilde{z} = z / a \). For the corresponding magnetic potentials we have then
\[ \tilde{A}_{\varphi0} = -\frac{\tilde{b}_o (|\tilde{z}|^2 + 1)}{k \sqrt{\tilde{r}^2 + |\tilde{z}|^2 + 1}}, \] (41a)
\[ \tilde{A}_{\varphi1} = -\frac{\tilde{b}_o (|\tilde{z}|^2 + 1)}{k \sqrt{\tilde{r}^2 + |\tilde{z}|^2 + 1}} \left\{ 1 - \frac{\tilde{b}_1 \tilde{r}^2}{\tilde{b}_o (|\tilde{z}| + 1)^2 + \tilde{r}^2} \right\}, \] (41b)
where \( \tilde{A}_\varphi = A_\varphi / a \). Whereas for the surface energy density we get
\[ \tilde{\mu}_0 = \frac{4\beta \tilde{b}_o}{(1 + \beta)(\tilde{r}^2 + 1)^{(\beta+2)/(2\beta+2)} \left( \tilde{b}_o + \sqrt{\tilde{r}^2 + 1} \right)^{(2\beta+1)/(\beta+1)}}, \] (42a)
\[ \tilde{\mu}_1 = \frac{4\beta \{ \tilde{b}_o - \tilde{b}_1 \tilde{r}^2 + \tilde{b}_o + 2\tilde{b}_1 \}}{(1 + \beta)(\tilde{r}^2 + 1)^{(2-\beta)/(2\beta+2)} \left\{ (\tilde{r}^2 + 1)^{3/2} + \tilde{b}_o (\tilde{r}^2 + 1) + \tilde{b}_1 \right\}^{(2\beta+1)/(\beta+1)}}. \] (42b)
For the two first members of the family we have the heat flux as
\[ Q_{\alpha0} = \frac{\tilde{b}_o k_w \delta^{\alpha}_o}{\sqrt{\tilde{r}^2 + 1 + \tilde{b}_o}}, \] (43a)
\[ Q_{\alpha1} = \frac{k_w \delta^{\alpha}_o \left\{ \tilde{b}_o (\tilde{r}^2 + 1) + \tilde{b}_1 \right\}}{(\tilde{r}^2 + 1)^{3/2} + \tilde{b}_o (\tilde{r}^2 + 1) + \tilde{b}_1}, \] (43b)
and the corresponding anisotropic tensor
\[ \tilde{\Pi}_{rr0} = \frac{2(1 - \beta) \tilde{b}_o}{3(1 + \beta) \left( \sqrt{\tilde{r}^2 + 1 + \tilde{b}_o} \right)^{1/(1+\beta)} \tilde{r}^{(3\beta+2)/(2+2\beta)}}, \] (44a)
\[ \tilde{\Pi}_{rr1} = \frac{2(1 - \beta) \left( \tilde{b}_o - \tilde{b}_1 \tilde{r}^2 + \tilde{b}_o + 2\tilde{b}_1 \right)}{3(1 + \beta) \left\{ (\tilde{r}^2 + 1)^{3/2} + \tilde{b}_o (\tilde{r}^2 + 1) + \tilde{b}_1 \right\}^{1/(1+\beta)} \tilde{r}^{(5\beta+2)/(2+2\beta)}}. \] (44b)
Finally, for the two first members of the family we have the current of magnetization as

\[ j_0 = -\frac{2\tilde{b}_0 \tilde{r}}{(\tilde{r}^2 + 1)^{(3+\beta)/(2+2\beta)}} \left( \tilde{b}_0 + \sqrt{\tilde{r}^2 + 1} \right)^{2\beta/(\beta+1)}. \]  
(45a)

\[ j_1 = -\frac{2\tilde{r} \left( \tilde{b}_0 (\tilde{r}^2 + 1) + 3\tilde{b}_1 \right)}{(\tilde{r}^2 + 1)^{(5-\beta)/(2+2\beta)}} \left( \tilde{r}^2 + 1 \right)^{3/2} + \tilde{b}_0 (\tilde{r}^2 + 1) + \tilde{b}_1 \right)^{2/(1+\beta)}. \]  
(45b)

In the last expressions we have used the dimensionless expressions \( \tilde{\mu} = a\mu, \tilde{\Pi}_{\phi\phi} = a\Pi_{rr}, \) and \( \tilde{j} = aj. \) In Fig. 1 we show the dimensionless surface energy densities \( \tilde{\mu} \) as a function of \( \tilde{r} \). In each case, we plot \( \tilde{\mu}_0(\tilde{r}) \) [Fig. 1(a)] and \( \tilde{\mu}_1(\tilde{r}) \) [Fig. 1(b)] for different values of the parameter \( \beta \) with \( \tilde{b}_0 = 1 \). It can be seen that the surface energy density is everywhere positive fulfilling the energy conditions. It can be observed that for all the values of \( \beta \) the maximum of the surface energy density occurs at the center of the disk and that it vanishes sufficiently fast as \( \tilde{r} \) increases. It can also be observed that the surface energy density in the central region of the disk increases as the values of the parameter \( \beta \) increase. We also computed the functions \( \tilde{\mu} \) and \( \tilde{j} \) for other values of the parameters and, in all the cases, we found the same behaviour. We do not plot the heat flux, it shows a similar behaviour to that of the surface energy density.

V. CONCLUDING REMARKS

We have used the formalism presented in [3] to obtain an exact relativistic model describing a system composed of a thin disk surrounded by a magnetized halo in a conformastationary space-time background. The model was obtained by solve the Einstein-Maxwell distributional field equations through the introduction of an auxiliary harmonic function that determines the functional dependence of the metric components and the electromagnetic potential under the
Fig. 2. Dimensionless current of magnetization $\tilde{\mu}$ as a function of $\tilde{r}$. In each case, we plot $\tilde{\mu}_0(\tilde{r})$ and $\tilde{\mu}_1(\tilde{r})$ for different values of the parameter $\beta$ with $\tilde{b}_0 = 1$ and $\tilde{b}_0 = 0.5$.

The energy-momentum tensor can be expressed as the sum of two distributional contributions, one due to the electromagnetic part and the other due to a material part. As we can see, due to the spacetime here considered is non-static (conformastationary) this distributional approach allows us a strongly non-linear partial equation system. We have considered for simplicity the astrophysical consistent case in that there is not electric charge on the halo region. Consequently, it obtained that the charge density on the disk region is zero.

The distributional approach model and the inverse method techniques allowed us to derive explicit expressions for the metric and the electromagnetic potential of the disk region and the halo as well. All the expressions are presented in terms of a solution of the Laplace’s equation. An important feature of the model is the introduction of the $\beta$ and $k_\omega$ parameters in the metric solution. Indeed, when $\beta$ is equal to one the usual conformastationary line element is obtained and then the pressure and the anisotropic tensor disappear. In a similar way, when $k_\omega$ is equal to zero, the heat flux $Q_\alpha$ vanishes, a feature of the static disk. Furthermore, when we take simultaneously $k_\omega = 0$ and $\beta = 1$, the results presented here reduce to those in [1] for the special case when the electric potential vanishes.

To analyse the physical content of the energy-momentum tensor of disk we expressed it in the canonical form and we projected it in a comoving frame defined through of the local observers tetrad. Accordingly, we found the explicit expressions for the surface energy density, pressure, electric current, heat flux, anisotropic tensor and electromagnetic potential of the disk in terms a solution of the Laplace’s equation. As a particular example, we used a “generalization” of the Kuzmin solution of the Laplace’s equations. The solution obtained is asymptotically flat in general and turns out to be free of singularities. Since all the relevant quantities show a physically reasonable behaviour, we conclude that the solution presented here can be useful to describe the gravitational and electromagnetic field of a conformastationary thin disk surrounded by a halo in the presence of an electromagnetic field. In a subsequent work we will present a detailed analysis of the energy-momentum tensor of the halo as well as a thermodynamic features of the system halo-disk.

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Appendix A: The Einstein-Maxwell equations and the thin-disk-halo system

Inspired by inverse method techniques, let us assume that the solution has the general form \[\text{(A1)}\]

\[
\begin{align*}
\text{ds}^2 &= -f^2(\text{dt} + \omega \text{d}\varphi)^2 + \Lambda^4(\text{dr}^2 + \text{dz}^2 + r^2 \text{d}\varphi^2),
\end{align*}
\]

where we have introduced the cylindrical coordinates \(x'^\alpha = (t, r, z, \varphi)\) in which the metric function \(f, \Lambda\) and \(\omega\) and the electromagnetic potential, \(A_\alpha = (A_t, 0, 0, A_\varphi)\) depend only on \(r\) and \(z\). Accordingly, the non-zero components of the energy-momentum tensor of the halo, \(M_{\alpha\beta}^\pm = G_{\alpha\beta}^\pm - E_{\alpha\beta}^\pm\), are given by

\[
\begin{align*}
M_{tt}^\pm &= \frac{1}{4r^2\Lambda^2}(3f^4\nabla \omega \cdot \nabla \omega - 16r^2f^2\Lambda^3\nabla^2 \Lambda - 2f^2\nabla A_\varphi \cdot \nabla A_\varphi + 4f^2\omega \nabla A_t \cdot \nabla A_\varphi - 2(r^2\Lambda^4 + f^2\omega^2)\nabla A_t \cdot \nabla A_t), \quad (A2a)

M_{t\varphi}^\pm &= \frac{1}{4r^2\Lambda^2}(-4r^2f^2\Lambda^3\nabla \omega \cdot \nabla \Lambda + 3f^4\nabla \omega \cdot \nabla \omega - 16r^2f^2\Lambda^3\omega \nabla^2 \Lambda + 6r^2\Lambda^4f \nabla \omega \cdot \nabla f
+ 2\Lambda^4f^2 \nabla^2 \omega - 4r^2f^2\Lambda^4\nabla \omega \cdot \nabla \omega - 2f^2\nabla A_\varphi \cdot \nabla A_\varphi + 2(\omega\Lambda^4r^2 - f^2\omega^3)\nabla A_t \cdot \nabla A_t
+ 4(f^2\omega^2 - r^2\Lambda^4\nabla \omega \cdot \nabla A_\varphi - \nabla A_\varphi) = M_{t\varphi}^\pm, \quad (A2b)

M_{rr}^\pm &= -\frac{1}{4r^2\Lambda^2f^2}(-f^4(\omega^2 - \omega_\varphi^2) - 4\Lambda^4r^2f^2\omega^2f + 4r^2f\Lambda^4frr - 8r^2f^2\Lambda^3\omega \nabla^2 \Lambda + 8r^2f^2\Lambda^3A_{rr}
- 16\Lambda^2f^2f_rA_r + 8\Lambda^2\Lambda^4\nabla \omega - 16f^2\nabla^2 \Lambda - 2f^2(\Lambda^2f_r - \Lambda^2f_\varphi, z) - 8(\Lambda^4r^2 - f^2\omega^2)(A_{rr}^2 - A_{t\varphi}^2)
- 4f^2(\Lambda^4r^2 - f^2\omega^2)(A_{rr}^2 - A_{t\varphi}^2)), \quad (A2c)

M_{rz}^\pm &= 2r^2\Lambda^2f^2(4r^2f^2\Lambda^3(\Lambda f_r f_z + A_z f_r) + 12r^2f^2\Lambda^2A_r A_z + f^4\omega_r \omega_z - 4r^2f^2\Lambda^3A_{rz}
- 2f^2\Lambda^4f f_r f_z - 2(f^2\omega^2 - \Lambda^4f^2)A_{t\varphi, r} + f^4\omega_r (A_{t\varphi, r} + A_{t\varphi, \varphi}) - 2f^2A_{t\varphi, r} A_{t\varphi, z}), \quad (A2d)

M_{zz}^\pm &= \frac{1}{4r^2\Lambda^2f^2}(f^4(\omega^2 - \omega_\varphi^2) + 4r^2\Lambda^4f^2\omega^2f - 4r^2\Lambda^4f_{f_z, z} + 8r^2f^2\Lambda^3\omega \nabla \omega - 8r^2f^2\Lambda^3A_{zz}
- 8r^2f^2\Lambda^2A_{zz} + 16\Lambda^2f^2f_r A_z + 16\Lambda^2f^2\Lambda^2A_{zz} + 2f^2(\Lambda^2f_r - \Lambda^2f_\varphi, z) - 4\omega f^2(A_{t\varphi, r} A_{\varphi, r} A_{t\varphi, z, z})
- 2(\Lambda^4f^2 - f^2\omega^2)(A_{t\varphi, r}^2 - A_{t\varphi, z}^2)), \quad (A2e)

M_{\varphi\varphi}^\pm &= \frac{1}{4r^2\Lambda^2f^2}(4r^2f^2\Lambda^3f^2 - 4r^3\Lambda^3f \nabla \omega \cdot \nabla f + f^4(4\Lambda^4f^2 + 3\omega^2f^2)\nabla \omega \cdot \nabla \omega - 16f^2\omega^2 f^2\Lambda^3\nabla^2 \Lambda
+ 4r^2\Lambda^4f\omega \nabla^2 \omega - 8r^4f^4\omega^2 \nabla \omega \cdot \nabla \omega + 12\Lambda^4f^2 \nabla \omega \cdot \nabla f - 8r^2\omega f^4\Lambda^3 \nabla \omega \cdot \nabla \Lambda
+ 8\Lambda^2f^2 f_r \nabla \Lambda - 8f^2\Lambda^2 \nabla \Lambda \cdot \nabla f - 8r^4f^2 \nabla \nabla \Lambda \cdot \nabla \Lambda - 2(f^2\Lambda^4f^2 + f^2\omega^2)\nabla A_\varphi \cdot \nabla A_\varphi
- 4(f^2\Lambda^4f^2 - \omega^2f^4)\nabla A_\varphi \cdot \nabla A_\varphi - 2(\Lambda^4r^4 - 2\Lambda^4f^2 f^2 - f^4\omega^4)\nabla A_t \cdot \nabla A_t). \quad (A2f)
\end{align*}
\]

Furthermore, the electric current density of the halo reads

\[
\begin{align*}
J_{t\varphi}^\pm &= \frac{1}{\Lambda^2 f^2} \nabla \cdot \{\Lambda^2f^{-1}\nabla A_t + r^{-2}\omega f\Lambda^{-2}(\nabla A_\varphi - \omega \nabla A_t)\}, \quad (A3a)

J_{t\varphi}^\pm &= \frac{1}{\Lambda^2 f^2} \nabla \cdot \{r^{-2}\omega f\Lambda^{-2}(\nabla A_\varphi - \omega \nabla A_t)\}. \quad (A3b)
\end{align*}
\]

The discontinuity in the \(z\)-direction of \(Q_{\alpha\beta}\) and \(T^\alpha\) defines, respectively, the surface energy-momentum tensor (SEMT) and the surface electric current density (SECD) of the disk \(S_{\alpha\beta}\), more precisely

\[
\begin{align*}
S_{\alpha\beta} &= \int Q_{\alpha\beta} \delta(z) ds_n = \sqrt{g_{zz}} Q_{\alpha\beta}, \quad (A4a)

J^\alpha &= \int T^\alpha \delta(z) ds_n = \sqrt{g_{zz}} T^\alpha. \quad (A4b)
\end{align*}
\]
where \( ds_n = \sqrt{g_{zz}} \, dz \) is the “physical measure” of length in the direction normal to the \( z = 0 \) surface. Accordingly, the non-zero components of the SEMT for the line element (A1) are given by

\[
S^t_t = \frac{1}{r^2 \Lambda^6} \{ -f^2 \omega \omega_z + 8 \nu^2 \Lambda^3 \Lambda_z \}, \tag{A5a}
\]
\[
S^t_\varphi = -\frac{1}{r^2 \Lambda^6 f} \{ -4 r^2 \omega f \Lambda^3 \Lambda_z + \omega^2 f^3 \omega_z + 2 r^2 \Lambda^4 \omega f_z + r^2 f \Lambda^4 \omega_z \}, \tag{A5b}
\]
\[
S^r_t = \frac{2}{\Lambda^3 f} \{ 2 f \Lambda_z + \Lambda f_z \}, \tag{A5c}
\]
\[
S^\varphi_t = \frac{f^2}{r^2 \Lambda^6 \omega_z}, \tag{A5d}
\]
\[
S^\varphi_\varphi = \frac{1}{r^2 \Lambda^6} \{ 2 r^2 \Lambda^4 f_z + 4 r^2 \Lambda^3 f \Lambda_z + \omega f^3 \omega_z \}, \tag{A5e}
\]

whereas the non-zero components of the (SECD) are

\[
J^t = \frac{1}{r^2 \Lambda^6 f^2} \{ \omega f^2 \Lambda \varphi_z + (r^2 \Lambda^4 - f^2 \omega^2) \Lambda t_z \}, \tag{A6a}
\]
\[
J^\varphi = \frac{1}{r^2 \Lambda^6} \{ \omega \Lambda t_z - \Lambda \varphi_z \}, \tag{A6b}
\]

where all the quantities are evaluated on the surface of the disk.

**Appendix B: The local observers**

We write the metric (3) in the form

\[
ds^2 = -F(dt + \omega d\varphi)^2 + F^{-\beta} [dr^2 + dz^2 + r^2 d\varphi^2], \tag{B1}
\]

where we have rewritten \( F = e^{2\phi} \). The tetrad of the local observers \( h^{(\alpha)}_\mu \), in which the metric has locally the form of Minkowskian metric

\[
ds^2 = \eta_{(\mu)(\nu)} h^{(\mu)} \otimes h^{(\nu)}, \tag{B2}
\]

is given by

\[
h^{(t)}_\alpha = F^{1/2} \{ 1, 0, 0, \omega \}, \tag{B3a}
\]
\[
h^{(r)}_\alpha = F^{-\beta/2} \{ 0, 1, 0, 0 \}, \tag{B3b}
\]
\[
h^{(z)}_\alpha = F^{-\beta/2} \{ 0, 0, 1, 0 \}, \tag{B3c}
\]
\[
h^{(\varphi)}_\alpha = F^{-\beta/2} \{ 0, 0, 0, r \}. \tag{B3d}
\]

The dual tetrad reads

\[
h^{(t)}_\alpha = F^{-1/2} \{ 1, 0, 0, 0 \}, \tag{B4a}
\]
\[
h^{(r)}_\alpha = F^{\beta/2} \{ 0, 1, 0, 0 \}, \tag{B4b}
\]
\[
h^{(z)}_\alpha = F^{\beta/2} \{ 0, 0, 1, 0 \}, \tag{B4c}
\]
\[
h^{(\varphi)}_\alpha = \frac{F^{\beta/2}}{r} \{ -\omega, 0, 0, 1 \}. \tag{B4d}
\]

The circular velocity of the system disk-halo can be modelled by a fluid space-time whose circular velocity \( V^\alpha \) can be written in terms of two Killing vectors \( t^\alpha \) and \( \varphi^\alpha \),

\[
V^\alpha = V^t (t^\alpha + \Omega^\alpha), \tag{B5}
\]
where

\[ \Omega \equiv \frac{\omega}{u^t} = \frac{du}{dt} \]

is the angular velocity of the fluid as seen by an observer at rest at infinity. The velocity satisfy the normalization \( V^\alpha V_\alpha = -1 \), accordingly for the metric (B1) we have

\[ (V^t)^2 = \frac{1}{-t^\alpha t_\alpha - 2\Omega t^\phi t_\phi - \Omega t^\phi t^\phi}, \]

with

\[ t^\alpha t_\alpha = g_{tt} = -F \] (B8a)
\[ t^\alpha \phi = g_{t\phi} = -\omega F \] (B8b)
\[ \phi^\alpha \phi = g_{\phi\phi} = r^{-2}F^{-\beta}(1 - F^{1+\beta}\Omega^2). \] (B8c)

consequently we write the velocity as

\[ V^t = \frac{1}{F^{1/2}(1 + \omega\Omega)\sqrt{1 - V_{LOC}^2}}, \]

where

\[ V_{LOC} \equiv \frac{r\Omega}{F^{(1+\beta)/2}(1 + \omega\Omega)}, \]

is the velocity as measured by the local observers.

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