COMPACT GAUDUCHON-FLAT HERMITIAN MANIFOLDS

RAMIRO A. LAFUENTE AND JAMES STANFIELD

Abstract. We complete the classification of compact Hermitian manifolds with a flat Gauduchon connection. In particular, we confirm a conjecture of Yang and Zheng, by proving that except for the cases of a flat Chern or Bismut connection, such manifolds are Kähler. We also treat the non-compact case.

1. INTRODUCTION

Let \((M, J, g)\) be a Hermitian manifold. Unless \(g\) is Kähler, the complex structure \(J\) is not parallel with respect to the Levi–Civita connection. It is therefore more natural to consider Hermitian connections: those for which \(g\) and \(J\) are parallel. In general, Hermitian connections form an infinite dimensional affine space. However, by imposing natural constraints on the torsion, Gauduchon identified in [4] a distinguished line of “canonical” Hermitian connections, given by

\[
\{ \nabla^s := (1 - \frac{s}{2}) \nabla^c + \frac{s}{2} \nabla^b : s \in \mathbb{R} \}.
\]

Here, \(\nabla^c = \nabla^0\) and \(\nabla^b = \nabla^2\) are respectively the Chern and Bismut connections of \((M, J, g)\). Recall that the Chern connection is the unique Hermitian connection for which the \((1,1)\)-part of the torsion vanishes, and the Bismut connection is the unique Hermitian connection with totally skew-symmetric torsion. For each \(s \in \mathbb{R}\), we henceforth refer to \(\nabla^s\) as the \(s\)-Gauduchon connection.

When \((M, J, g)\) is Kähler, all Gauduchon connections coincide with the Levi–Civita connection \(\nabla^{LC}\). On the other hand, they are pairwise distinct on non-Kähler Hermitian manifolds. Remarkably, Gauduchon’s line also includes two other distinguished connections: the so-called minimal connection \(\nabla^\frac{1}{2}\), whose torsion has minimal norm among all Hermitian connections; and the Lichnerowicz connection \(\nabla^1 = \frac{1}{2} (\nabla^{LC} - J \nabla^{LC} J)\) (sometimes also called the first canonical, or associated connection), the Hermitian part of the Levi–Civita connection.

A fundamental question in non-Kähler Hermitian geometry asked by S.T. Yau [10, Problem 87], is whether one can say something nontrivial about a compact Hermitian manifold whose holonomy is a proper subgroup of \(U(n)\). Yau remarks that the problem is that the connection need not be the Levi-Civita connection. It seems natural to consider connections on the Gauduchon line. In this direction, our main result answers Yau’s question in the case of a discrete holonomy group:

**Theorem A.** Any compact Hermitian manifold for which \(\nabla^s\) is flat must be Kähler, unless \(s = 0, 2\).

This also answers affirmatively a conjecture by Yang and Zheng ([7 Conjecture 1.1]). Recall that the cases \(s = 0, 2\) are well-understood. By a classical result of Boothby [2] a compact Chern-flat Hermitian manifold is covered by a simply-connected complex Lie group with left-invariant Hermitian metric. On the other hand, Wang, Yang and Zheng showed in [6] that the universal cover of a Bismut-flat Hermitian manifold is an open complex submanifold of a Samelson space, that is, a Lie group with left-invariant complex structure and bi-invariant metric.

Theorem A was previously known when \(n = 2\) ([7 Theorem 1.4]), and for arbitrary dimension under the additional assumption that \(s \notin (4 - 2\sqrt{3}, 1) \cup (1, 4 + 2\sqrt{3}) \setminus \{0\}\) ([7 Theorem 1.5] and [11 Theorem 1]). It was also known when \(g\) is assumed to be locally conformally Kähler, with two possible exceptions for \(s\) depending on \(n\) ([7 Theorem 1.7], or balanced [7 Proposition 1.8]. Hermitian manifolds with flat Lichnerowicz connection were previously studied in [11 3]).

It is well known that the result of Boothby [2] for Chern-flat manifolds does not extend to the non-compact case. Thus, it is natural to ask if there exist non-compact, non-Kähler, \(s\)-Gauduchon flat —that is, \(\nabla^s\) is flat— Hermitian manifolds for \(s \notin \{0, 2\}\). To this end, our methods allow us to solve the problem for (not necessarily compact) Hermitian manifolds admitting cocompact groups of symmetries.
Theorem B. Let \((M, J, g)\) be a Hermitian manifold admitting a cocompact group action by biholomorphic isometries. If \(\nabla^s\) is flat, then \((M, J, g)\) is Kähler unless \(s = 0, 2\).

Recall that a group action is called cocompact if its orbit space is compact. Thus, Theorem A follows from Theorem \([\text{A}]\) by considering the trivial action on a compact Hermitian manifold. We also yield an answer in the rich class of homogeneous Hermitian manifolds; those admitting transitive actions by biholomorphic isometries.

Corollary C. Let \((M^{2n}, J, g)\) be a homogeneous Hermitian manifold that is \(s\)-Gauduchon flat. If \(s \neq 0, 2\), then \((M, J, g)\) is Kähler.

This generalises results of Vezzoni, Yang and Zheng in \([\text{A}]\). They showed that Theorem \([\text{B}]\) holds when \((M^{2n} = G, J, g)\) is a Lie group with left-invariant Hermitian structure and either \(n = 2\) or \((G, J, g)\) admits a left-invariant \(\nabla^s\)-parallel frame (equivalently if \(\nabla^sX = 0\) for all \(X \in \mathfrak{g} = \text{Lie}(G)\)).

We now briefly summarise the proof of Theorem A, which captures the main ideas of our methods. Let \(\{e_i\}_{i=1}^n\) be a local unitary frame on \((M, J, g)\), a compact Hermitian manifold, and \(T \in \Gamma(M, TM \otimes \Lambda^2 T^* M)\) the torsion of the Chern connection. For \(s \neq \frac{2}{3}, 1\), we consider the norm of \(\eta := \sum_{i=1}^n \langle T(e_i, \cdot), e_i \rangle \in \Omega^{1,0}(M)\), the \((1, 0)\)-part of the Lee form of \((M, J, g)\). When \((M, J, g)\) is \(s\)-Gauduchon flat, we find several algebraic identities relating contractions of the tensor \(T^{\otimes 4}\) to show that
\[
\Delta |\eta|^2 + (d|\eta|^2, \alpha_s) \geq 0 \text{ (or } \leq 0),
\]
for some one-form \(\alpha_s \in \Omega^1(M)\) depending on \(s\), where equality holds if and only if \(T \equiv 0\) (i.e. \(g\) is Kähler). But since \(M\) is compact, the maximum principle implies that equality must in fact hold. For the minimal connection \(s = \frac{2}{3}\), we instead need to consider the Laplacian of the full norm of the torsion \(|T|^2\). A very similar argument then applies. The case \(s = 1\) is again treated separately and follows from \([\text{B}]\) Proposition 1 (in fact in this case, we may drop all symmetry assumptions, see Corollary \([\text{C}]\)).

We remark here that the tools in \([\text{B}]\) generalise directly to the Gauduchon-Kähler-like case (that is, when the curvature of \(\nabla^s\) has the same symmetries as the curvature tensor of a Kähler metric, see \([\text{A}]\)). Our methods however use flatness of \(\nabla^s\) in a crucial way (see e.g. Lemma \([\text{B}]\)). Thus, we do not yet offer insight into the Kähler-like case. A modification of our ideas applied to this setting will be the subject of a forthcoming article.

The rest of the article is structured as follows: In Section 2 we fix notation and lay out the most important formulas for our computations. In Section 3 we prove Theorem B for \(s \neq \frac{2}{3}\). In Section 4 we prove Theorem B for \(s = \frac{2}{3}\). Finally, in Section 5 we discuss the Chern-flat case and finish with a summary of some known results.

Acknowledgements. The first-named author was supported by an Australian Research Council DECRA fellowship (project ID DE190101063). The second named author was supported by an Australian Government Research Training Program (RTP) Scholarship.

2. Preliminaries

In this section we fix notation and recall some important results. Let \((M^{2n}, J, g)\) be a Hermitian manifold and for each \(s \in \mathbb{R}\), let \(\nabla^s\) be as in Equation (1). Denote by \(\omega := g(J \cdot, \cdot)\) the fundamental form associated to the Hermitian structure. Let \(T := \nabla^c(\cdot) - \nabla^c(\cdot) - [\cdot, \cdot] \) be the torsion of the Chern connection. Suppose \(U \subset M\) is open and \(\{e_i\}_{i=1}^n \subset \Gamma(U, T^{1,0} M)\) is a local unitary frame. Let \(\{e^i\}_{i=1}^n \subset \Omega^{1,0}(U)\) be the associated dual frame. In the rest of the article, repeated indices are always symmed. We then define the functions
\[
T_{ij}^k := \frac{1}{2} \langle T(e_i, e_j), e_k \rangle: U \to \mathbb{C}; \quad 1 \leq i, j, k \leq n.
\]
It follows that
\[
T = T_{ij}^k e_k \otimes e^i \wedge e^j + T_{ij}^k e_k \otimes e^i \wedge e^j,
\]
where all repeated indices imply a summation from 1 up to \(n\). We define also the one-form
\[
\eta := \eta_j e^j := T_{kj}^k e^j \in \Omega^{1,0}(U); \quad 1 \leq j \leq n.
\]
This is globally defined and, up to scaling, is the \((1,0)\)-part of the Lee form associated with \((M,J,g)\). Our first observation is that for certain unitary frames \(\{e_i\}^n_{i=1}\), the exterior derivatives of the dual frame take a particularly nice form.

**Lemma 2.1.** Let \(x \in M\) and suppose \(\{e_i\}^n_{i=1} \subset \Gamma(U,T^{1,0}U)\) is a local unitary frame in a neighbourhood \(U\) of \(x\) such that \(\nabla^* e_i = 0\) at \(x\) for all \(1 \leq i \leq n\). Then at \(x\),
\[
\partial e^i = (1 - s)T^i_{jk} e^j \land e^k, \quad \text{and} \quad \partial e^i = sT^i_{jk} e^j \land e^k,
\]
for any index \(1 \leq i \leq n\).

**Proof.** For each \(s \in \mathbb{R}\) and unitary frame \(\{e_i\}^n_{i=1}\), let \(\theta^s \in \Gamma(U,u(n) \otimes T^s U)\) be the local \(u(n)\)-valued connection form of \(\nabla^s\) which satisfies \(\nabla^s e_j = e_i \otimes (\theta^s)^i_j\) for all \(1 \leq j \leq n\). By [11, Lemma 2.1], \(\theta^s = \theta^0 + s\gamma\), where \(\gamma \in \Gamma(U,u(n) \otimes T^0 U)\) is defined by \(\gamma^i_j := T^i_{jk} e^k - T^j_{ik} e^k\) for \(1 \leq i, j \leq n\). If \(\nabla^s e_i = 0\) at some point \(x \in U\), then \(\theta^0 = -s\gamma\) at \(x\). By the first Cartan structure equation,
\[
dc^i = e^j \land (\theta^0)^i_j + T^i_{jk} e^j \land e^k = -s e^j \land \gamma^i_j + T^i_{jk} e^j \land e^k = (1 - s)T^i_{jk} e^j \land e^k + sT^i_{jk} e^j \land e^k.
\]
The result now follows by equating the \((1,0)\) and \((1,1)\) parts. \(\Box\)

**Remark 2.2.** It is well-known (see e.g. [5, Lemma 4]) that a unitary frame satisfying the conditions of 2.1 always exists.

The main machinery for computing our results comes from the following formulas:

**Lemma 2.3 ([11]).** Let \((M^{2n},J,g)\) be a Hermitian manifold. Suppose \(s \neq 0\) and \(\nabla^s\) is flat. Then in any local unitary frame \(\{e_i\}^n_{i=1} \subset \Gamma(U,T^{1,0}M), U \subset M\),

1. \(T^k_{ij,l} = -(s - 2) T^k_{ij,l} T^i_{kl}\)
2. \(0 = (s - 1) \left( T^i_{jk,l} T^j_{il} + T^j_{kl,i} T^l_{ij} + T^l_{ij,k} T^i_{kl} \right)\)
3. \(c T^k_{ij,l} = a T^k_{ij,l} + b \left( T^k_{ij,l} T^i_{kl} - T^i_{kl,j} T^k_{ij} \right) + s^3 \left( T^k_{ij,l} T^l_{jr} - T^l_{ij,k} T^k_{jr} \right)\)
4. \(2(s - 1) \eta_{ij,l} = s^2 |T|^2 + s(2 - 3s) |\eta|^2\)

for any indices \(1 \leq i, j, k, l \leq n\), where a comma denotes covariant differentiation with respect to \(\nabla^s\) and
\[
a = a(s) := -4s(s - 1)^2, \quad b = b(s) := -s(5s^2 - 10s + 4), \quad c = c(s) := 4(s - 1)(2s - 1).
\]

**Proof.** This is the content of [11], Lemmas 2, 3 and 5 (in that article, \(r = 1 - s\)). \(\Box\)

**Remark 2.4.** Lemma 2.3 actually holds in the case where \(\nabla^s\) is Kähler-like (see [11]). Most results in this article however use flatness of \(\nabla^s\) in another explicit way, so they do not generalise. A modification of our techniques applied to the s-Gauduchon-Kähler-like setting will be detailed in a forthcoming article.

Taking traces in Lemma 2.3 gives the following useful identities for \(\eta^s\):

**Corollary 2.5.** If \(\nabla^s\) is flat and \(s \notin \{0,1\}\),
\[
(1) \eta_{ik,l} = -(s - 2) T^k_{ik,l} T^i_{kl},
(2) T^i_{jk,l} \eta_{ij} = 0,
(3) c \eta_{ij,l} = (a - b) T^k_{kj,l} T^k_{il} + b \eta_{ij,l} T^k_{kl} + s^3 (T^k_{kp,l} T^k_{lp} - T^k_{kp,l} T^k_{lp}),
(4) c \eta_{ij,l} = (a - b) T^k_{kj,l} T^k_{il} \eta_{ij},
\]
for all indices \(1 \leq i, k, j, l \leq n\).

**Proof.** The first three items come from taking a trace of items (1)-(3) in Lemma 2.3. Item (4) comes from multiplying item (3) by \(\eta_{ij}\) and applying Item (1). \(\Box\)
The key tool for our proof is the real Laplacian acting on functions. It is defined as the operator $d^*d: C^\infty(M) \to C^\infty(M)$, where $d^* := -\ast d\ast$ and the real operator $\ast: \Omega^{p,q}(M) \to \Omega^{n-q,n-p}(M)$ is the Hodge star, satisfying $\alpha \wedge \ast \beta = (\alpha, \beta)\omega^n$ for all $\alpha \in \Omega^{q,p}(M)$ and $\beta \in \Omega^{p,q}(M)$. The exterior derivative decomposes as $d = \partial + \bar\partial$, where $\partial: \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar\partial: \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. Moreover, $d^* = \partial^* + \bar\partial^*$, where

$$\partial^* := -\ast \bar\partial\ast: \Omega^{p,q}(M) \to \Omega^{p-1,q}(M).$$

We can now find an expression for $\partial^*$ acting on one-forms.

**Corollary 2.6.** If $\alpha = \alpha_i e^i \in \Omega^{1,0}(U)$ is a local $(1,0)$-form on a neighbourhood $U \subset M$, then

$$\partial^*\alpha = -\alpha_i \ast + \langle \alpha, \eta \rangle$$

In particular, for $f: U \to \mathbb{R}$, its Laplacian is given by

$$d^* df = 2 \operatorname{Re}(\partial^* df) = -2 \operatorname{Re}(f \bar\eta) + \langle df, \eta + \bar\eta \rangle.$$

**Proof.** We prove the identity at a point $x \in U$. Since both sides are tensorial, we may assume that $\{e_i\}_{i=1}^n$ is a unitary frame satisfying $\nabla^i e_i = 0$ at $x$. First, it is easy to verify that

$$\ast e^i = -i^n e^1 \wedge \bar{e}^1 \wedge \cdots \wedge e^i \wedge e^{i+1} \wedge \bar{e}^{i+1} \wedge \cdots \wedge e^n \wedge \bar{e}^n \in \Omega^{n,n-1}(U).$$

Thus, by applying Lemma 2.1, we have

$$\partial^* e^i = -\ast \bar\partial \ast e^i$$

$$= -\ast \left( -i^n \sum_{k \leq i} e^1 \wedge e^1 \wedge \cdots \wedge e^i \wedge e^{i+1} \wedge \cdots \wedge e^n \wedge \bar{e}^n \right)$$

$$+ i^n \sum_{k > i} e^1 \wedge e^1 \wedge \cdots \wedge e^i \wedge e^{i+1} \wedge \cdots \wedge \bar{e}^k \wedge \bar{e}^k \wedge \cdots \wedge e^n \wedge \bar{e}^n$$

$$+ i^n \sum_{k < i} e^1 \wedge \bar{e}^1 \wedge \cdots \wedge e^k \wedge \bar{e}^k \wedge \cdots \wedge e^i \wedge e^{i+1} \wedge \cdots \wedge \bar{e}^n$$

$$- i^n \sum_{k > i} e^1 \wedge \cdots \wedge e^i \wedge e^{i+1} \wedge \cdots \wedge \bar{e}^k \wedge \cdots \wedge e^n \wedge \bar{e}^n)$$

$$= -\ast \left( -i^n \sum_{k \leq i} \bar{\partial} e^k(e_p, e_q) e^1 \wedge e^1 \wedge \cdots \wedge e^i \wedge e^{i+1} \wedge \cdots \wedge e^n \wedge \bar{e}^n \right)$$

$$+ i^n \sum_{k > i} \bar{\partial} e^k(e_p, e_q) e^1 \wedge e^1 \wedge \cdots \wedge e^i \wedge e^{i+1} \wedge \cdots \wedge e^n \wedge \bar{e}^n$$

$$+ \frac{1}{2} i^n \sum_{k < i} \bar{\partial} e^k(e_p, e_q) e^1 \wedge e^1 \wedge \cdots \wedge e^k \wedge \bar{e}^k \wedge e^1 \wedge \cdots \wedge \bar{e}^n$$

$$- \frac{1}{2} i^n \sum_{k > i} \bar{\partial} e^k(e_p, e_q) e^1 \wedge e^1 \wedge e^i \wedge e^{i+1} \wedge \cdots \wedge e^k \wedge e^{k+1} \wedge \cdots \wedge \bar{e}^n)$$

$$= -\ast \left( -\sum_{k \leq i} \bar{\partial} e^k(e_k, e_i) \omega^n - \sum_{k > i} \bar{\partial} e^k(e_k, e_i) \omega^n - \frac{1}{2} \sum_{k < i} \bar{\partial} e^k(e_k, e_i) \omega^n - \frac{1}{2} \sum_{k > i} \bar{\partial} e^k(e_k, e_i) \omega^n \right)$$

$$= \frac{1}{2} (2\bar{\partial} e^k(e_k, e_i) + \bar{\partial} e^k(e_k, e_i))$$

$$= \frac{1}{2} (2s \eta + 2(1-s) \bar{\eta})$$

$$= \eta.$$

4
Hence,
\[
\partial^* \alpha = - \star J \ast \partial \alpha = - \star \overline{J}(\partial \alpha) + \alpha_i e^i
\]
\[
= - \varepsilon^i \partial \alpha + \alpha^i e^i
\]
\[
= - e^k(\alpha_i) \ast (\overline{e^k} \wedge \ast e^i) + \alpha_i \partial^* e^i
\]
\[
= - e^i(\alpha_i) + \alpha_i \eta_i
\]
\[
= - \alpha_i^2 + \langle \alpha, \eta_i \rangle,
\]
as claimed. \hfill \Box

3. The case \(s \neq 2/3\)

In this section, we prove Theorem 3.1 for \(s \neq 2/3\). We first cover the case \(s = 1\) (Hermitian manifolds with flat Lichnerowicz connection). In fact, we can weaken the flatness assumption to yield:

**Proposition 3.1.** Let \((M, J, g)\) be a Hermitian manifold with Kähler-like Lichnerowicz connection (in the sense defined in \(\textbf{1.1}\)). Then \((M, J, g)\) is Kähler.

**Proof.** By \(\textbf{1.1}\) Proposition 1, \(\eta = 0\). As noted in Remark 2.3, Lemma 2.3 holds when the flatness assumption on \(\nabla^s\) is weakened to \(\nabla^s\) being Kähler-like. Thus, subbing \(\eta = 0\) and \(s = 1\) into Item (4) gives \(|T|^2 = 0\) and the result is immediate (see also \(\textbf{7}\) Proposition 1.8)). \hfill \Box

**Corollary 3.2.** Hermitian manifolds with flat Lichnerowicz connection are Kähler.

The remaining goal for this section is to consider \(s \neq 1\), and compute the laplacian of the function \(|\eta|^2\). We will then apply the maximum principle to prove Theorem 3.1 for \(s \neq 2/3\). By Corollary 2.6 if \(\{e_i\}_{i=1}^n\) is a local unitary frame, then
\[
-d^* d \frac{1}{2} |\eta|^2 = \text{Re}(\eta_j \overline{\eta_j} + \eta_j \overline{\eta_j}) + |\nabla^s \eta|^2 - \langle d \frac{1}{2} |\eta|^2, \eta + \overline{\eta} \rangle.
\]

The main difficulty will come in computing the first term on the right hand side. Once this is done, the rigidity stems from the following lemma:

**Lemma 3.3.** Let \((M, J, g)\) be a Hermitian manifold such that \(\nabla^s\) is flat. If \(s \notin \{0, 2\}\) and \(\nabla^s \eta = 0\), then \((M, J, g)\) is Kähler.

**Proof.** By Corollary 3.2, it suffices to assume \(s \neq 1\). Then by Corollary 2.6 Item (4),
\[
0 = \text{Re}(\eta_j \overline{\eta_j}) = (a - b) T^p_{kj} \overline{\eta_j} \overline{\eta_j} + s^2 (s - 2) T^p_{kj} \overline{\eta_j} \overline{\eta_j}.
\]

Since \(s \neq 0, 2\), we have \(T^p_{kj} \overline{\eta_j} = 0\) for all indices \(1 \leq p, k \leq n\). Tracing over \(p\) and \(k\) gives \(|\eta|^2 = 0\). The result now follows from Lemma 2.6 Item (4) since \(s \neq 0\). \hfill \Box

We define now three important (complex valued) functions on \((M, J, g)\) that will appear frequently in the sequel. In a local unitary frame \(\{e_i\}_{i=1}^n\), define
\[
A := T^p_{pj} T^r_{kl} \overline{\eta_j} \overline{\eta_j}, \quad B := T^p_{kj} T^r_{pr} \overline{\eta_j} \overline{\eta_r}, \quad C := T^p_{kj} \overline{\eta_j} \overline{T^r_{kl} \eta_r} = |T(\eta^2, \cdot)|^2.
\]

These are contractions of the tensor \(T \otimes T \otimes T \otimes T\) and are hence globally defined. Recall that for a co-vector \(\alpha \in T^*M, \alpha^2 \in TM\) is the vector satisfying \(\alpha = g(\cdot, \alpha^2)\). There will be one more term that appears in the upcoming computations. As an immediate application of Lemma 2.6 Item (2), this term is described in the next lemma.

**Lemma 3.4.** Suppose \(s \notin \{0, 1\}\). If \((M, J, g)\) is s-Gauduchon flat then, \(T^p_{pj} T^r_{kl} \overline{\eta_j} \overline{\eta_j} = 2A\).

We assume now and for the rest of this section that \((M^2, J, g)\) is s-Gauduchon flat for some \(s \notin \{0, 1\}\). In particular, Lemma 2.6 and Lemma 3.4 holds.

The real part of \(A\) can naturally be interpreted as the derivative of the norm of the torsion in the direction of the Lee form (up to scaling and taking a real part).
Lemma 3.5. \( c⟨\partial |T|^2, \eta⟩ = 2(a - b - c(s - 2))A = -2(s - 2)(7s^2 - 12s + 4)A \)

Proof. Choosing a local unitary frame \( \{e_i\}_{i=1}^n \), by a direct computation using Lemma 2.3 and Corollary 2.6

\[
c⟨\partial |T|^2, \eta⟩ = c(T^{k}_{ij} \overline{T^{k}_{ij}}) \eta_i \eta_j
= -c(s - 2)T^{k}_{ij} T^{k}_{ij} \eta_i \eta_j
+ T^{k}_{ij} \left( a \overline{T^{r}_{kl}} \overline{T^{r}_{kl}} + b \left( T^{k}_{kr} T^{r}_{ir} - \overline{T^{k}_{lr}} \overline{T^{r}_{jr}} \right) + s^3 \left( T^{l}_{ir} T^{l}_{kr} - \overline{T^{l}_{lr}} \overline{T^{l}_{kr}} \right) \right) \eta_i \eta_j
= -2c(s - 2)A + 2aA - bA - bA,
\]
as required. \( \square \)

We may also relate the functions \( B \) and \( C \) by a first-order term.

Lemma 3.6. \( c⟨\partial |η|^2, \eta⟩ = -(s - 2)B + (a - b)C \).

Proof. By a direct computation using Corollary 2.6 in a local unitary frame \( \{e_i\}_{i=1}^n \),

\[
c⟨\partial |η|^2, \eta⟩ = c(η_i \overline{η}_j) \eta_i \eta_j = cη_i j \overline{η}_j \eta_i + c\overline{η}_i j \η_ j
= -c(s - 2)T^{k}_{ij} T^{k}_{ij} \eta_i \eta_j + (a - b)η_j \overline{T^{k}_{jkl}} \eta_i \eta_j = -(s - 2)B + (a - b)C,
\]
as desired. \( \square \)

The next lemma deals with second derivatives of \( η \), which will aid us in computing the Laplacian.

Lemma 3.7. The following identities hold:

1. \( cη_i j \overline{η}_j = (s - 2) \left( (3b - a) - 2s^3 \right) A - bB + s^3 C \),
2. \( c^2 η_i j \overline{η}_j = ((a - b)(-c(s - 2) + a - 2b + 2s^3) - 2as^3) A + (2b - a)s^3 B - (b^2 + bs^3 + s^6) C \),
3. \( c(\nabla [e_i \overline{η}_j] \eta)_i \eta_j = -c(s - 2)B - s(a - b)C \).

Proof. Let \( \{e_i\}_{i=1}^n \) be a local unitary frame. We prove the first two parts using Corollary 2.6. First, we have

\[
cη_i j \overline{η}_j = -c(s - 2)(T^{p}_{kl} T^{k}_{pl} \overline{T^{p}_{kl}}) \eta_i \eta_j
= -(s - 2) \left( aT^{r}_{kl} \overline{T^{r}_{kl}} + b \left( T^{k}_{kr} T^{r}_{ir} - \overline{T^{k}_{lr}} \overline{T^{r}_{jr}} \right) + s^3 \left( T^{l}_{ir} T^{l}_{kr} - \overline{T^{l}_{lr}} \overline{T^{l}_{kr}} \right) \right) \eta_i \eta_j
= -c(s - 2)(2aA + bA - s^3 A)
- (aA - bB + 2bA - s^3 A + s^3 C)
= (s - 2) \left( (3b - a) - 2s^3 \right) A - bB + s^3 C,
\]
which proves the first part. The second part follows from a similar but more complicated computation:

\[
\begin{align*}
    c^2 \eta_{jl} \eta_{lj} &= c\left((a - b)T^i_{kj} \overline{T^i_{kl}} + b \eta_p T^p_{il} + s^3(T^i_{kp} T^i_{kl} - T^i_{jp} T^i_{lp})\right) \eta_{lj} \\
    &= - (a - b)c(s - 2)T^i_{kj} T^j_{pl} \overline{T^l_{kl}} \eta_{lj} \\
    &+ (a - b)T^i_{kj} \left(a \overline{T^i_{kl}} T^i_{pl} + b \left(T^p_{kr} T^i_{lr} - \overline{T^i_{lr}} T^k_{pr}\right) + s^3 \left(T^i_{kr} T^i_{pr} - \overline{T^i_{lr}} T^k_{pr}\right)\right) \eta_{lj} \\
    &- bc(s - 2)T^i_{kp} T^j_{l} \eta_{lj} \\
    &+ b \eta_p \left(a \overline{T^i_{kp}} T^r_{jl} + b \left(T^i_{kr} T^r_{lp} - \overline{T^r_{lp}} T^i_{lr}\right) + s^3 \left(T^i_{kr} T^r_{lp} - \overline{T^r_{lp}} T^i_{lr}\right)\right) \eta_{lj} \\
    &- cs^3(s - 2)T^i_{kp} T^j_{l} \eta_{lj} \\
    &+ s^3T^i_{kp} \left(a \overline{T^i_{kp}} T^r_{jl} + b \left(T^r_{kr} T^i_{lp} - \overline{T^i_{lp}} T^r_{lr}\right) + s^3 \left(T^r_{kr} T^i_{lp} - \overline{T^i_{lp}} T^r_{lr}\right)\right) \eta_{lj} \\
    &+ c(s - 2)T^i_{kp} T^j_{l} \eta_{lj} \\
    &= - (a - b)c(s - 2)A \\
    &+ a(a - b)A + b(a - b)C - 2b(a - b)A + s^3(a - b)A - s^3(a - b)B + 0 \\
    &- abC - bs^3C + 0 \\
    &- 2as^3A - s^6A - s^6A + 0 \\
    &+ 0 \\
    &+ s^3(a - b)A + bs^3B + 2s^6A - s^6C
\end{align*}
\]

For part 3, as both sides are tensors, we may assume at the point we are computing the derivative that local unitary frame satisfies \(\nabla^* e_i = 0\). Then Lemma 2.1 yields

\[
\begin{align*}
    c(\nabla_{[e_i, e_j]} \eta_{lj}) \eta_{lj} &= c \partial^k (e_i, e_j)(\eta_{lj}) \eta_{lj} + c \partial^k (e_i, \overline{e_j})(\eta_{lj}) \eta_{lj} \\
    &= - c \partial^k (e_i, e_j)(\eta_{lj}) \eta_{lj} - c \partial^k (e_i, \overline{e_j})(\eta_{lj}) \eta_{lj} \\
    &= - cT^k_{lj} \eta_{lj} + csT^k_{lj} \eta_{lj} \\
    &= - cs(s - 2)T^i_{kp} T^j_{l} \eta_{lj} - cs \eta_p \eta_{lj} \eta_{lj} \\
    &= - cs(s - 2)B - \eta_p \eta_{lj} \eta_{lj} \\
    &= - cs(s - 2)B - \eta_p \eta_{lj} \eta_{lj} \\
    &= - cs(s - 2)B - s(a - b)C,
\end{align*}
\]

as stated.

Since we now have formulas for the second covariant derivatives of \(\eta\), we can use flatness to derive new algebraic identities.

**Lemma 3.8.** If \(R^*\) denotes the curvature of \(\nabla^*\), then

\[
0 = c^2(R^*(\overline{e_i}, e_i)\eta^j, \overline{e_j}) = x_1 A + x_2 B + x_3 C,
\]

where

\[
\begin{align*}
x_1 &:= (b - a)(4c(s - 1) + a - 2b) + 2bs^3 = -4s^2(s - 1)(3s - 2)(4s^2 - 9s + 4), \\
x_2 &:= (a - b)s^3 - c(s - 2)(b + cs), \text{ and} \\
x_3 &:= b^2 + bs^3 + s^6.
\end{align*}
\]
Proof. By definition of $R^*$, we have
\begin{equation}
\langle R^* (\eta, e_i) \eta, \eta \rangle = \langle \eta_{j,i}, \eta - \nabla [\eta_{j,i}] \rangle.
\end{equation}
The result now follows from multiplying Equation (3) by $e^2$, applying Lemma 3.7, and observing that $(a - b) = s^2(s - 2)$.

\begin{lemma}
The curvature $R^*$ of $\nabla^*$ satisfies
\begin{equation}
0 = e^2 \langle R^* (\eta, e_i) \eta, \eta \rangle = y_1 A + y_2 B + y_3 C - (b - s^3)c \langle \partial \eta^2, \eta \rangle,
\end{equation}
where
\begin{align*}
y_1 &:= 2 \left( (a - b + c^3)(b - a) + cs^4 \right), \\
y_2 &:= - (bc(s - 2) + cs^4), \text{ and} \\
y_3 &:= acs.
\end{align*}
\end{lemma}

\begin{proof}
Similar to Lemma 3.8, we compute the right hand side of
\begin{equation}
\langle R^* (\eta, e_i) \eta, \eta \rangle = \langle \eta_{j,i}, \eta - \nabla \eta_{j,i} \rangle.
\end{equation}
By Corollary 2.5, $\eta_{j,i} = \eta_{i,j}$, the same computation as in the proof for Lemma 3.8 applies to the first term here:
\begin{equation}
c \eta_{j,i} = (s - 2) \left( (3(b - a) - 2s^3)A - bB + s^3 C \right).
\end{equation}
The second term is
\begin{align*}
c^2 \eta_{j,i} &:= c^2 \langle \partial \eta_{j,i}, \eta \rangle \\
&= c \langle \partial \left( (a - b)T_{kl}^+ + b \eta_{lp} - T_{lp}^+ + s^3 (T_{lp}^+ T_{kl}^+ - T_{lp}^+ \eta_{kl}^+) \right), \eta \rangle \\
&= c \langle (a - b + s^3) \partial |\eta|^2 + (b - s^3) \partial \eta^2, \eta \rangle \\
&= 2(a - b - c(s - 2)) (a - b + s^3) A + (b - s^3) c \langle \partial |\eta|^2, \eta \rangle,
\end{align*}
where in the last inequality we used Lemma 2.4. For the last term, we assume we are computing at a point $x \in M$ such that $\nabla^* e_i | x = 0, 1 \leq i \leq n$. Then, using Lemma 2.1, Lemma 2.3, and Corollary 2.5 we get
\begin{align*}
c \langle \nabla [\eta_{j,i}] \eta, \eta \rangle &:= c \eta_{j,k} \eta_{k,l} + c \eta_{j,l} \eta_{k,l} - c T_{kl}^\eta \eta_{j,i} \\
&= c s T_{kl} \eta_{j,k} \eta_{j,l} - cs T_{kl} \eta_{j,i} \\
&= - s T_{kl} \left( (a - b)T_{kl}^+ + b \eta_{lp} - T_{lp}^+ + s^3 (T_{lp}^+ T_{kl}^+ - T_{lp}^+ \eta_{kl}^+) \right) \eta_{j,i} \\
&= - s(a - b) A - bsC - 2s^4 A + s^4 B \\
&= - s(a - b + 2s^3) A + s^3 B - bsC.
\end{align*}
The conclusion now easily follows.
\end{proof}

We can combine Lemmas 3.9 and 3.10 to yield the following linear system:

\begin{lemma}
It holds that
\begin{equation}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix} = \begin{pmatrix}
\langle \partial |\eta|^2, \eta \rangle \\
(b - s^3)c
\end{pmatrix},
\end{equation}
where $\Lambda$ is a $3 \times 3$ matrix given by
\begin{equation}
\Lambda = \begin{pmatrix}
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
0 & - c(s - 2) & (a - b)
\end{pmatrix}.
\end{equation}
In particular,
\begin{equation}
\det \Lambda = 8s^4(s - 2)^2(s - 1)(3s - 2)Q(s)
\end{equation}
where $Q(s) := -96 + 864s - 3232s^2 + 6516s^3 - 7684s^4 + 5372s^5 - 2114s^6 + 373s^7$.
We are now able to deduce that \((M, J, g)\) is Kähler, so long as this system has a solution.

**Lemma 3.11.** If \((M, J, g)\) satisfies the assumptions of Theorem 12 and is s-Gauduchon-flat for some \(s \notin \{0, \frac{1}{3}, \frac{2}{3}, 2\} \cup Q^{-1}(\{0\})\), then \((M, J, g)\) is Kähler.

**Proof.** By the assumption on \(s\), and Lemma 3.10 \(\Lambda\) is invertible. Thus, solving the linear system gives \(A, B, C \in \mathbb{R}(\partial|\eta|^2, \overline{\eta})\), and so the real parts of \(A, B\) and \(C\) are each a constant multiple of \((\partial|\eta|^2, \eta + \overline{\eta})\). Multiplying Equation (2) by \(c \neq 0\), and applying Lemma 3.7 we get

\[-d^*d\left(\frac{1}{2}|\eta|^2\right) = c^2|\nabla s\eta|^2 + k(\frac{1}{2}|\eta|^2, \eta + \overline{\eta}),\]

for some \(k \in \mathbb{R}\). By the assumptions of Theorem 13 there is a group acting cocompactly on \((M, J, g)\) by biholomorphic isometries. Thus, the function \(|\eta|^2\) is constant on the orbits of this action. Since the orbit space is compact, \(|\eta|^2\) must achieve both its maximum and minimum on \(M\). The maximum principle then implies that \(|\eta|^2\) is constant, and therefore \(\nabla s\eta = 0\). The conclusion now follows from Lemma 3.3. \(\square\)

In the next theorem, we eliminate all but one of the possible exceptions for \(s\).

**Theorem 3.12.** If \((M, J, g)\) satisfies the assumptions of Theorem 12 and is s-Gauduchon-flat for some \(s \notin \{0, \frac{1}{3}, 2\}\), then \((M, J, g)\) is Kähler.

**Proof.** The cases \(s = \frac{1}{3} \) and \(s = 1\) are covered by [11] Lemma 3] and Corollary 3.2 respectively. Thus, by Lemma 3.11 we need only consider \(s \in Q^{-1}(\{0\})\). One can verify via polynomial root isolation methods (e.g. Sturm’s Theorem, see [1] Ch. VI) that \(Q\) has three real roots in \(\mathbb{R}\) and that the intervals \((0.47, 0.48), (0.75, 0.76), \) and \((1.1, 1.11)\) each contain exactly one of these roots. It therefore suffices to assume \(s \in (0.47, 0.48) \cup (0.75, 0.76) \cup (1.1, 1.11)\). Again, multiplying Equation (2) by \(c\) which is non-zero on \((0.47, 0.48) \cup (0.75, 0.76) \cup (1.1, 1.11)\) and applying Lemma 3.7 and flatness gives

\[d^*d\left(\frac{1}{2}|\eta|^2\right) - c|\nabla s\eta|^2 = c\text{Re}(\eta j_i \nabla \eta j_i + \eta \overline{\eta} \eta j_i) + \text{f.o.t.} \]

\[= c\text{Re}(2\eta j_i \overline{\eta} \eta j_i + (\nabla \eta j_i \eta j_i) \eta j_i) + \text{f.o.t.} \]

\[= 2(s - 2)(3(b - a) - 2s^3)\text{Re}(A) - b(s - 2)\text{Re}(B) + 2s(a - b)C - s(a - b)C + \text{f.o.t.} \]

\[= 2(s - 2)(3(b - a) - 2s^3)\text{Re}(A) - (s - 2)(b + cs)\text{Re}(B) + s(a - b)C + \text{f.o.t.} \]

where f.o.t. (“first order term”) means a constant multiple of \((d|\eta|^2, \eta + \overline{\eta})\). To eliminate the \(A\) term, we multiply by \(x_1\), which is non-zero on \((0.47, 0.48) \cup (0.75, 0.76) \cup (1.1, 1.11)\) and apply Lemma 3.8 to yield

\[-d^*d\left(x_1|\eta|^2\right) = cx_1\frac{1}{2}|\nabla s\eta|^2 + 2(s - 2)(3(b - a) - 2s^3)(-x_2\text{Re}(B) - x_3C) - (s - 2)(b + cs)\text{Re}(B) + s(a - b)C + \text{f.o.t.} \]

\[= cx_1|\nabla s\eta|^2 - (2x_2(3(b - a) - 2s^3) + b + cs)(s - 2)\text{Re}(B) + (s(a - b) - 2x_2(s - 2)(3(b - a) - 2s^3))C + \text{f.o.t.} \]

Now multiplying by \(c\) once more and applying Lemma 3.9 to eliminate the \(B\) term gives

\[-d^*d\left(c^2x_1\frac{1}{2}|\eta|^2\right) = c^2x_1\frac{1}{2}|\nabla s\eta|^2 - (a - b)(2x_2(3(b - a) - 2s^3) + b + cs)C + (sc(a - b) - 2cx_3(s - 2)(3(b - a) - 2s^3))C + \text{f.o.t.} \]

\[= c^2x_1\frac{1}{2}|\nabla s\eta|^2 \]

\[= \left((a - b)\left(2x_2(3(b - a) - 2s^3) + b + 2cx_3(s - 2)(3(b - a) - 2s^3)\right)C + \text{f.o.t.} \]

\[= \frac{c^2}{2}\lambda^2 - s^2(s - 2)(2(6 - 5s)(s^2x_2 + cx_3) + b)\lambda^2 + \text{f.o.t.} \]

If the coefficients of both norms are non-zero and share the same sign, then

\[\pm d^*d\left(|\eta|^2\right) + \text{f.o.t.} \geq 0,\]
and arguing as in the proof of Lemma 3.11 we conclude via the maximum principle that $|\eta|^2$ is constant in either case. It follows that $\nabla^* \eta = 0$ and thus $(M, J, g)$ is Kähler by Lemma 3.3. Now it remains to show that
\[
P(s) := -x_1 s^2 (s - 2) \left( 2(6 - 5s) \left(s^2 x_2 + c x_3 \right) + b \right) > 0,
\]
for all $s \in (0.47, 0.48) \cup (0.75, 0.76) \cup (1.1, 1.11)$. I.e., that the coefficients have the same sign. Again, via root isolation methods, one readily checks that no roots of $P$ occur in $(0.47, 0.48) \cup (0.75, 0.76) \cup (1.1, 1.11)$ and that $P(0.47), P(0.48), P(0.75), P(0.76), P(1.1), P(1.11) > 0$. Thus $P > 0$ on $(0.47, 0.48), (0.75, 0.76) \cup (1.1, 1.11)$ and the result follows as above.

4. The case $s = 2/3$

In this section we cover the remaining possible exception for Theorem 3.3 namely $s = 2/3$. Recall that $\nabla^{2/3}$ corresponds to the minimal connection. We consider now the Laplacian of the full norm of the torsion $|T|^2$. As before, if $(M, J, g)$ is $s$-Gauduchon flat and $\{e_i\}_{i=1}^n$ is a local unitary frame, we have by Corollary 2.6
\[
-d^* d |T|^2 = 2 \text{Re} \left( T_{ij}^k \overline{T_{jk}}^l \overline{T_{kl}}^l \right) + 2 |\nabla^* T|^2 - \langle d |T|^2, \eta + \bar{\eta} \rangle.
\]

Following analogously to the previous section, we define important functions that will appear in the forthcoming computations:
\[
D := T_{ij}^k \overline{T_{kl}}^l \overline{T_{jk}}^l = |\langle (\cdot, \cdot), \overline{\mathcal{T}(\cdot, \cdot)} \rangle|^2, \quad E := T_{ij}^k \overline{T_{kl}}^l \overline{T_{jk}}^l,
\]
\[
F := T_{ij}^k \overline{T_{jk}}^l \overline{T_{kl}}^l = |\langle (\epsilon_k, \cdot), \overline{\mathcal{T}(\epsilon_k, \cdot)} \rangle|^2, \quad G := T_{ik}^l T_{jk}^l \overline{T_{kl}}^l \overline{T_{jk}}^l.
\]

Note that all are non-negative. We again assume from now that $(M, J, g)$ is an $s$-Gauduchon flat Hermitian manifold for some $s \notin \{0, 1\}$. Again, following the previous section, by Lemma 2.3 Item (2) the following lemma holds:

Lemma 4.1. $E = 2 T_{il}^k T_{lj}^k \overline{T_{ij}}^l \overline{T_{lj}}^l$.

We can now compute the norm of $\nabla^* T$ in terms of the functions defined above:

Lemma 4.2. $c^2|\nabla^* T|^2 = c^2(s - 2) E + a^2 D + (2(a + b)s^3 - b^2 - s^6) E + 2(b^2 + s^6) F - 4b s^3 G$.

Proof. Let $\{e_i\}_{i=1}^n$ be a local unitary frame. Clearly by Lemma 2.3 $T_{ij}^k \overline{T_{kl}}^l = (s - 2)^2 E$. Moreover,
\[
c^2 T_{ij,l}^k \overline{T_{kl,l}}^l = \left( a T_{ij,l}^k \overline{T_{kl,l}}^l + b \left( T_{ij,l}^k \overline{T_{kl,l}}^l - T_{ij}^k \overline{T_{kl}}^l \right) + s^3 \left( T_{ij,l}^k \overline{T_{kl,l}}^l - T_{ij}^k \overline{T_{kl}}^l \right) \right) \times
\]
\[
\left( a T_{ij,l}^k \overline{T_{kl,l}}^l + b \left( T_{ij,l}^k \overline{T_{kl,l}}^l - T_{ij}^k \overline{T_{kl}}^l \right) + s^3 \left( T_{ij,l}^k \overline{T_{kl,l}}^l - T_{ij}^k \overline{T_{kl}}^l \right) \right)
\]
\[
= a^2 D + abE - abE + as^3 E + as^3 E
\]
\[
+ b^2 F - b^2 E + bs^3 E - 2b s^3 G
\]
\[
+ b^2 F - 2bs^3 G + bs^3 E
\]
\[
+ s^6 F - s^6 E
\]
\[
+ s^6 F,
\]
from which the statement follows since $|\nabla^* T|^2 = T_{ij,l}^k \overline{T_{kl,l}}^l + T_{ij,l}^k \overline{T_{kl,l}}^l$.

The next step is to expand out second covariant derivative formulas for the torsion.

Lemma 4.3. The following hold:

(1)
\[
c^2 T_{ij,l}^k \overline{T_{kl,l}}^l = - abD + (c(b - a)(s - 2) - b s^3 + a^2 + b^2 + s^6) E - 2(ab + bs^3 + s^6) F + 2b s^3 G
\]
\[
+ 2(a - b) \left( b A - s^3 A \right),
\]

(2)
\[
c(\nabla_{\epsilon_1, \epsilon_2} T_{ij}) \overline{T_{ij}}^k = 2c(s - 2) A + 2(a - b) A,
\]
when computed in any unitary frame \( \{e_i\}_{i=1}^{n} \).

**Proof.** The first part is a consequence of Lemma 2.3, which we compute as follows:

\[
c^2 T^{k}_{ij,j} \overline{T}^{k}_{ij} = c \left( a T^{T}_{ij,j} T^{k}_{i} + b \left( T^{k}_{ij} T^{l}_{j} - T^{l}_{j} T^{k}_{i} \right) + s^3 \left( T^{k}_{ij} \overline{T}^{l}_{k} - \overline{T}^{l}_{k} T^{k}_{j} \right) \right) \overline{T}^{k}_{ij}
\]

\[
= -ac(s - 2) T^{T}_{ij,j} T^{k}_{i} \overline{T}^{k}_{ij} + a \left( a T^{T}_{ij,j} T^{k}_{i} + b \left( T^{k}_{ij} T^{l}_{j} - T^{l}_{j} T^{k}_{i} \right) + s^3 \left( T^{k}_{ij} \overline{T}^{l}_{k} - \overline{T}^{l}_{k} T^{k}_{j} \right) \right) \overline{T}^{k}_{ij}
\]

\[
- bc(s - 2) T^{T}_{ij,j} T^{k}_{i} \overline{T}^{k}_{ij} + b \left( a T^{T}_{ij,j} T^{k}_{i} + b \left( T^{k}_{ij} T^{l}_{j} - T^{l}_{j} T^{k}_{i} \right) + s^3 \left( T^{k}_{ij} \overline{T}^{l}_{k} - \overline{T}^{l}_{k} T^{k}_{j} \right) \right) \overline{T}^{k}_{ij}
\]

\[
- cs^3(s - 2) T^{T}_{ij,j} T^{k}_{i} \overline{T}^{k}_{ij} + s^3 \left( a T^{T}_{ij,j} T^{k}_{i} + b \left( T^{k}_{ij} T^{l}_{j} - T^{l}_{j} T^{k}_{i} \right) + s^3 \left( T^{k}_{ij} \overline{T}^{l}_{k} - \overline{T}^{l}_{k} T^{k}_{j} \right) \right) \overline{T}^{k}_{ij}
\]

\[
= -ac(s - 2) E + a^2 E + 2abA - abD + as^3E - 2as^3A + \frac{1}{2} bc(s - 2) E - abF + b^2 E - b^2A + bs^3 A - bs^3 F
\]

\[
+ \frac{1}{2} bc(s - 2) E - abF + b^2 E - b^2A + bs^3 A - bs^3 F + 0 - \frac{1}{2} as^3E + bs^3 G - \frac{1}{2} bs^3 E + \frac{1}{2} s^6 E - s^6 F
\]

\[
+ 0 - \frac{1}{2} as^3E + bs^3 G - \frac{1}{2} bs^3 E + \frac{1}{2} s^6 E - s^6 F
\]

\[
= - abD + (c(b - a)(s - 2) - bs^3 + a^2 + 2b^2 + s^6) E - 2(ab + bs^3 + s^6) F + 2bs^3 G + 2(a - b) (bA - s^3 A),
\]

as claimed. For the second part, we assume without loss of generality that we are computing at a point \( x \in M \) for which \( \nabla^s e_i = 0 \). Applying Lemma 2.1 gives

\[
c(\nabla_{[e_i,e_i]} T^{k}_{ij}) \overline{T}^{k}_{ij} = c e^r \left( (e_i, e_i) T^{k}_{ij} \overline{T}^{k}_{ij} + c e^r (e_i, e_i) T^{k}_{ij} \overline{T}^{k}_{ij} \right) \overline{T}^{k}_{ij}
\]

\[
= c(s - 2) T^{T}_{ij,j} T^{k}_{i} \overline{T}^{k}_{ij} + c(T^{k}_{ij} T^{l}_{j} - T^{l}_{j} T^{k}_{i}) \overline{T}^{k}_{ij} + c T^{k}_{ij} \overline{T}^{k}_{ij} + c e^r (e_i, e_i) T^{k}_{ij} \overline{T}^{k}_{ij}
\]

\[
= c(s - 2) T^{T}_{ij,j} T^{k}_{i} \overline{T}^{k}_{ij} + c e^r (e_i, e_i) T^{k}_{ij} \overline{T}^{k}_{ij}
\]

\[
+ c T^{k}_{ij} \overline{T}^{k}_{ij} + b \overline{T}^{k}_{ij} \overline{T}^{k}_{ij} + c e^r (e_i, e_i) T^{k}_{ij} \overline{T}^{k}_{ij}
\]

\[
= 2c(s - 2)A - 2aA - 2bA,
\]

as required. \( \square \)

These formulae allow us to finally compute the Laplacian of \( |T|^2 \) and apply the maximum principle.
Theorem 4.4. Suppose \((M, J, g)\) satisfies the assumptions of Theorem 3 and \(\nabla^s \hat{\tau}\) is flat. Then, \((M, J, g)\) is Kähler.

Proof. For \(s \notin \{0, 1\}\), applying flatness, Lemma 4.2 and Lemma 4.3 to Equation (6) yields
\[
-d^* d|T|^2 = 4 \Re \left( T_{ij} T_{ij} + \frac{1}{2} \nabla_{[\eta, \tau]} T_{ij} T_{ij} \right) + 2|\nabla^s T|^2 - \langle d|T|^2, \eta + \bar{\eta} \rangle
\]
\[
= -4 abD + 4c(b-a)(s-2) - bs^3 + a^2 + 2b^2 + s^6)E - 8(ab + bs^3 + s^6)F + 8bs^3 G
\]
\[
+ 2c^2(s-2)^2E + 2a^2 D + 2(2(a+b)s^3 - b^2 - s^6)E + 4(b^2 + s^6)F - 8bs^3 G
\]
\[
+ 4(c^2(s-2) + c(a-b)) Re(A) - \langle d^2 \|T\|^2, \eta + \bar{\eta} \rangle
\]
\[
= 2a(a - 2b)D + 2(c^2(s-2)^2 + 2c(b-a)(s-2) + 2(a+b)s^3 - 2bs^3 + 2a^2 + 3b^2 + s^6) E
\]
\[
+ 4(b^2 - 2ab - 2bs^3 - s^6) F + 4(c^2(s-2) + c(a-b)) Re(A) - \langle d^2 |T|^2, \eta + \bar{\eta} \rangle.
\]
Substituting \(s = \frac{2}{3}\) into this gives
\[
d^*[d(9|T|^2)] = 24D + 248E - \langle d(9|T|^2), \eta + \bar{\eta} \rangle.
\]
As in Lemma 8.11 since there is a group action of biholomorphic isometries on \((M, J, g)\), the function \(|T|^2\) is constant on the orbits. As the action is cocompact, \(|T|^2\) achieves both its maximum and minimum. Thus, since \(D\) and \(E\) are non-negative functions, the maximum principle implies that \(|T|^2\) is constant. In particular, \(D = |(T(\cdot, \cdot), \overline{T(\cdot, \cdot)})|^2\) vanishes identically, so \((M, J, g)\) is Kähler.

Theorem 3 now follows from Theorem 3.12 and Theorem 4.4. As remarked in the introduction, Theorem A follows by taking the group action to be the trivial one on a compact Hermitian manifold.

5. Non-compact Gauduchon flat manifolds

We will now briefly treat the Chern-flat case for non-compact Hermitian manifolds and make some final remarks.

First, we make the following observation about Chern-flat manifolds with cocompact symmetry group to extend the result in [2]:

Proposition 5.1. Let \((M, J, g)\) be a complete Hermitian manifold admitting a cocompact group action by biholomorphic isometries. If \(\nabla^e = \nabla^0\) is flat, then \((M, J, g)\) is covered by a complex Lie group \((G, \hat{J}, \hat{g})\) (i.e. \(J\) is bi-invariant) with left-invariant Hermitian metric \(\hat{g}\).

Proof. By the computations in [2] §4, \(|T|^2_{\hat{\eta} = \nabla^e T|^2\). Thus, by Corollary 2.6
\[
d^*[d|T|^2] - \langle d|T|^2, \eta + \bar{\eta} \rangle = -2|\nabla^e T|^2 \leq 0.
\]

Arguing as in Lemma 8.11 and Theorem 4.4 we see that \(|T|^2\) achieves its maximum and minimum. Thus, the maximum principle implies that \(|T|^2\) is constant, so \(\nabla^e T = 0\). The result now follows from completeness as in the proof of Theorem 4 in [2] §4.

Let us finish by summarising some known results about flat Gauduchon connections:

Theorem 5.2. Let \((M^{2n}, J, g)\) be a Hermitian manifold and suppose \(\nabla^s \hat{\tau}\) is flat for some \(s \in \mathbb{R}\).

(1) If \(s = 1\), or \(s = \frac{1}{2}\), then \((M, J, g)\) is Kähler.
(2) If \(s \neq 0\) and \((M, J, g)\) is balanced (i.e. \(\eta = 0\)), then it is Kähler.
(3) If \(s \neq 1\) and \(n = 2\), then \((M, J, g)\) is Kähler.
(4) If \(s = 2\), then \((M, J, g)\) is covered by (an open complex submanifold of) a Lie group \((G, \hat{J}, \hat{g})\) with left-invariant complex structure \(\hat{J}\) and bi-invariant metric \(\hat{g}\) (i.e. a Samelson space).
(5) If \((M, J, g)\) admits a cocompact group action by biholomorphic isometries (e.g. if \(M\) is compact or homogeneous), and
   (a) \(s = 0\), and \((M, J, g)\) is complete, then it is covered by a complex Lie group \((G, \hat{J}, \hat{g})\) with left-invariant Hermitian metric \(\hat{g}\).
   (b) \(s \neq 0, 2\), then \((M, J, g)\) is Kähler.
Item (1) is by Corollary 3.2 and [7, Proposition 1.8] respectively. Item (2) is also covered by [7, Proposition 1.8]. Item (3) is given by [7, Theorem 1.2]. Item (4) is the content of [6, Theorem 5]. Finally, Item (5) is of course Proposition 5.1 and Theorem B.

References

[1] A. Andrada, M. L. Barberis, and I. G. Dotti. Abelian Hermitian geometry. Differential Geom. Appl., 30(5):509–519, 2012.
[2] William M. Boothby. Hermitian manifolds with zero curvature. Michigan Math. J., 5(2):229–233, 1958.
[3] G. Ganchev and O. Kassabov. Hermitian manifolds with flat associated connection. Kodai Math. J., 29(2):281–298, 2006.
[4] Paul Gauduchon. Hermitian connections and Dirac operators. Bull. Un. Mat. Ital. B (7), 11(2, suppl.):257–288, 1997.
[5] Luigi Vezzoni, Bo Yang, and Fangyang Zheng. Lie groups with flat Gauduchon connections. Math. Z., 293(1-2):597–608, 2019.
[6] Qingsong Wang, Bo Yang, and Fangyang Zheng. On Bismut flat manifolds. Trans. Amer. Math. Soc., 373(8):5747–5772, 2020.
[7] Bo Yang and Fangyang Zheng. On compact Hermitian manifolds with flat Gauduchon connections. Acta Math. Sin. (Engl. Ser.), 34(8):1259–1268, 2018.
[8] Bo Yang and Fangyang Zheng. On curvature tensors of Hermitian manifolds. Comm. Anal. Geom., 26(5):1195–1222, 2018.
[9] Chee Keng Yap. Fundamental problems of algorithmic algebra. Oxford University Press, New York, 2000.
[10] Shing-Tung Yau. Open problems in geometry. In Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), volume 54 of Proc. Sympos. Pure Math., pages 1–28. Amer. Math. Soc., Providence, RI, 1993.
[11] Quanting Zhao and Fangyang Zheng. On Gauduchon Kähler-like manifolds. J. Geom. Anal., 32(4):Paper No. 110, 27, 2022.