A CATEGORICAL APPROACH TO CYCLIC DUALITY

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Abstract. The aim of this paper is to provide a unifying categorical framework for the many examples of para-(co)cyclic modules arising from Hopf cyclic theory. Functoriality of the coefficients is immediate in this approach. A functor corresponding to Connes’s cyclic duality is constructed. Our methods allow, in particular, to extend Hopf cyclic theory to (Hopf) bialgebroids.

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Introduction

Cyclic cohomology extends and unifies cohomology theories like de Rham cohomology and Lie algebra cohomology of matrices. It has applications e.g. in homological algebra, algebraic topology, Lie algebras, algebraic K-theory and so non-commutative differential geometry.

The foot-stone in cyclic cohomology theory is a so called cocyclic object, i.e. a cosimplicial object equipped with an isomorphism at each grade $n$, roughly implementing a cyclic permutation of the coface and codegeneracy morphisms. In particular, the $n + 1$st power of this cocyclic morphism is required to be the identity. The study of cocyclic objects, or a quest of their examples, can be divided to two steps. First one can deal with a more general structure, called a para-cocyclic object, obtained by relaxing the requirement about the $n + 1$st power of the cocyclic morphism at grade $n$ to be trivial. Truly cocyclic subobjects or quotients of para-cocyclic objects are then studied as a subsequent step.

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There are many known examples of (para-)cocyclic objects, relevant for various purposes. A large family of examples, occurring as symmetries in non-commutative differential geometry, is associated to (co)module (co)algebras of bialgebras. The first example of this kind appeared in [9] where it was used by Connes and Moscovici to give a geometrical interpretation of the non-commutative Chern-character. Further examples of para-cocyclic objects, associated to (co)module (co)algebras of Hopf algebras were constructed by Hajac at al. in [12], where also non-trivial coefficients provided by (co)modules of the Hopf algebra were introduced. As a most important achievement, also criteria (on the coefficients) for the existence of truly cocyclic subobjects and quotients were found. These constructions were extended to bialgebras (extending Hopf algebras) by Kaygun in [16] and [17]. A new type of coefficients, so-called contramodules, was proposed by Brzeziński in [5]. In this way, currently there are known eight families of para-cocyclic objects associated to bialgebras: A cosimplicial object can be constructed from a module algebra or a comodule algebra, or from a module coalgebra or a comodule coalgebra \(A\) (yielding four possibilities), using either a functor of the form \(A \otimes (-)\) or a functor of the form \(\text{Hom}(A, -)\) (doubling the number of examples). In each case there turns out to be an appropriate choice of the coefficients resulting in a para-cocyclic structure.

Dually to (para-)cocyclic objects, one may consider (para-)cyclic objects, i.e. (para-)cocyclic objects in the opposite category. Using bialgebras, there can be constructed again eight families of examples.

As it was observed by Connes in [8], the category of cyclic objects and the category of cocyclic objects in a given category are isomorphic. This isomorphism, called cyclic duality, does not extend to the categories of para-cyclic and para-cocyclic objects only to their appropriate subcategories. These (full) subcategories have those objects whose para-(co)cyclic morphisms are isomorphisms at each grade, cf. Khalkhali and Rangipour’s work [18].

For para-(co)cyclic objects associated to (co)module (co)algebras of bialgebras, the para-(co)cyclic morphisms are not isomorphisms in general. They are isomorphisms, however, if the bialgebra in question is a Hopf algebra with an invertible antipode. In this case the eight families of associated para-cocyclic objects and the eight families of para-cyclic objects turn out to be pairwise related by cyclic duality.

The aim of this paper is to provide a general construction of para-(co)cyclic objects, including in particular existing constructions in terms of bialgebras together with their generalizations to bialgebroids, cf. [10]. We do not investigate here, however, the existence of truly (co)cyclic subobjects or quotients.

An important antecedent work of similar aims is Kaygun’s paper [17], where a universal construction of para-(co)cyclic objects, including examples from bialgebras, was presented. The construction in this work is built on monoids and comonoids in symmetric monoidal categories. Therefore, while it is perfectly suitable to describe (co)module (co)algebras of bialgebras, it has to be generalized in order to be able to cope with bialgebroids over non-commutative base algebras. Such a generalization (under the names admissible septuple and its transposition map) was proposed in our previous work [4]. In Section 2 of the current paper we introduce a category in which admissible septuples and their transposition maps are special objects. This newly introduced category \(\mathcal{A}\) comes
equipped with a functor \( Z^* \) from \( \mathcal{A} \) to the category of para-cocyclic objects in the category of functors. As a consequence, any object in \( \mathcal{A} \) induces a functor from a category (of coefficients) to the category of para-(co)cyclic objects in another category (usually the category of modules over a commutative ring). Thus the resulting construction of para-(co)cyclic modules is functorial for the choice of coefficients.

Behind the construction of the above category \( \mathcal{A} \) there are some 2-categorical considerations. Consider an abstract 2-category \( T \) with a single 0-cell \( o \), a 1-cell \( t : o \to o \) and 2-cells \( \eta : o \Rightarrow t \) and \( \mu : tt \Rightarrow t \), such that \( \mu \circ \eta t = t = \mu \circ t \eta \) and \( \mu \circ \mu t = \mu \circ t \mu \) (i.e. such that \( (t, \mu, \eta) \) is a monad in \( T \)). A monad \( (T, m, u) \) on a category \( \mathcal{M} \) can be described then as a 2-functor \( F \) from \( T \) to an appropriate 2-category \( \text{Cat} \) of (some) categories, functors and natural transformations, such that \( F o = \mathcal{M}, Ft = T, F \mu = m \) and \( F \eta = u \). A lax natural transformation between 2-functors \( F, F' : T \to \text{Cat} \) is precisely the same as a monad morphism \( (Ft, F \mu, F \eta) \to (F't, F' \mu, F' \eta) \) in the sense of [23] (for a review see Section 1 below). Extending this picture, we may consider an abstract 2-category \( S \) of three 0-cells \( o, d \) and \( c \), with two monads on \( o \) related by a distributive law \( \phi \), together with a so-called \( S(\phi, c) \)-algebra in \( S(o, c) \) and a \( S(d, \phi) \)-algebra in \( S(d, o) \). (For a review of algebras over distributive laws in [4] see Section 1 below.) The objects in our category \( \mathcal{A} \) are 2-functors \( S \to \text{Cat} \) and the morphisms are lax natural transformations between them. In Section 3 we collect eight families of examples of objects in \( \mathcal{A} \) associated to bialgebroids.

A dual construction in Section 4 provides us with another category \( \mathcal{B} \), admitting a functor \( Z^* \) from \( \mathcal{B} \) to the category of para-cyclic objects in the category of functors. In Section 5 we construct eight families of examples of objects in \( \mathcal{B} \) in terms of bialgebroids.

In Section 6 we investigate the natural question how Connes’ cyclic duality functor (more precisely its extension \( \sim \) to the categories of para-(co)cyclic objects with invertible para-(co)cyclic morphisms at each grade) lifts to a functor \( \sim \) between appropriate subcategories of \( \mathcal{A} \) and \( \mathcal{B} \), such that \( Z^* \circ \sim \cong \sim \circ Z^* \). Behind this lifting there are liftings of (co)monads.

In Section 7 we show that the examples associated to (co)module (co)algebras of a Hopf algebroid with a bijective antipode, belong to the subcategories of \( \mathcal{A} \) and \( \mathcal{B} \) on which the lifted cyclic duality functor \( \sim \) is defined. We also check that these examples are pairwise related by the functor \( \sim \).

The paper contains an Appendix, summarizing some facts about modules, comodules and contramodules of bialgebroids and Hopf algebroids, used to construct the examples in the paper.

**Notation.** Composition of functors and the corresponding composition of natural transformations is denoted by juxtaposition. That is, for consecutive functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{E} \), the composite is denoted by \( GF \), with object map \( X \mapsto GX \). For endofunctors \( T : \mathcal{C} \to \mathcal{C} \), we also write \( TT = T^2 \). For natural transformations \( v : F \to F' \) and \( w : G \to G' \), the value of \( v \) at an object \( X \) is denoted by \( vX \). Moreover, \( Gv \) is a natural transformation \( GF \to GF' \) whose value at \( X \) is given by \( G(vX) \) — written simply as \( GvX \). Similarly, \( wF \) is a natural transformation \( GF \to GF' \), with value at \( X \) given by \( w(FX) = wFX \). The composition of consecutive natural transformations is denoted by \( \circ \). The identity morphism of any object \( X \) is denoted by the same symbol \( X \). In order to simplify some computations, we shall use the string representation of
functors and their natural transformations. Throughout this paper, the composition of functors is represented by horizontal juxtaposition of strings, the functor acting first being represented by the rightmost string. A natural transformation $F_1 \cdots F_n \rightarrow G_1 \cdots G_m$ will be represented as a ‘microchip’ with $n$ inputs $F_1, \ldots, F_n$ at its top and $m$ outputs $G_1, \ldots, G_m$ at its bottom. The identity natural transformation of a functor $F$ will be represented just as a string, without any box, as in the first diagram. For a natural transformation $v : F \rightarrow F'$, we draw $Gv : GF \rightarrow GF'$ and $vH : FH \rightarrow F'H$ as in the second and the third pictures.

The composition of natural transformations is represented by vertical juxtaposition of the corresponding ‘layers’. The natural transformation acting first is represented by the top layer. Hence if translating diagrams to formulae, we have to read our diagrams from the bottom to the top and from left to right: For example, let us consider natural transformations $u_1 : K_1H_2 \rightarrow L_1L_2$, $u_2 : G_2H_1 \rightarrow K_1$, $u_3 : G_3F_2F_3 \rightarrow H_1H_2$, $u_4 : F_1 \rightarrow G_1G_2G_3$.

Then the fourth diagram of the above picture is the string representation of the natural transformation

$$G_1u_1 \circ G_1u_2H_2 \circ G_1G_2u_3 \circ u_4F_2F_3.$$  

In this notation, naturality of morphisms is visualized by their behaviour as ‘pearls’ on the strings. That is, those boxes which do not have common ingoing or outgoing strings, can be freely moved above or below each other, cf. the last equality in the above figure.

The above diagrammatic notation is used more generally in any 2-category: 1-cells are represented by vertical strings, their domains corresponding to the surfaces on their right and codomains corresponding to the surfaces on their left. 2-cells are represented by boxes, with domains represented by ‘incoming legs’ at their top and codomains represented by ‘outgoing legs’ at their bottom. Horizontal and vertical compositions in a 2-category are represented by horizontal and vertical juxtapositions of such diagrams. Diagrams like the rightmost one above, come from the middle four interchange law.

Note that in the literature the dual diagrammatic notation is used equally frequently. In that case, our strings representing 1-cells are replaced by the orthogonal lines – hence surfaces on the sides of the original lines are replaced by source and target points of the new orthogonal lines; and source and target points of the original lines are replaced by surfaces on the sides of the the new orthogonal lines. In this notation 2-cells are represented by labels of the faces surrounded by their domain and codomain 1-cells. E.g.
the 2-cell in the second figure above is represented as

\[
\begin{array}{c}
\text{G} \\
\text{F} \\
\text{F}'
\end{array}
\]

Though both diagrammatic notations contain precisely the same information, in this paper we prefer to work with string diagrams.

1. Φ-module functors and their morphisms

In this section we recall some notions from category theory, the constructions of the later sections are built on.

The following notions are introduced in [23].

**Definition 1.1.** A *monad* on a category \(\mathcal{M}\) is a triple \((T, m, u)\), where \(T : \mathcal{M} \to \mathcal{M}\) is a functor and \(m : T^2 \to T\) and \(u : \mathcal{M} \to T\) are natural transformations, called the *multiplication* and *unit*, respectively, which satisfy the last two (associativity and unitality) conditions in the following figure. (Our string representations for the multiplication and the unit of a monad are introduced in the first two equalities in the same figure.)

\[
\begin{array}{ccc}
T & T & T \\
\downarrow & \downarrow & \downarrow \\
T & T & T \\
\end{array}
\quad \quad
\begin{array}{c}
m = \\
T
\end{array}
\quad \quad
\begin{array}{cc}
T & T \\
\downarrow & \downarrow \\
T & T \\
\end{array}
\quad \quad
\begin{array}{c}
u = \\
T
\end{array}
\quad \quad
\begin{array}{c}
m = \\
T
\end{array}
\quad \quad
\begin{array}{c}
T
\end{array}
\]

A *morphism* from a monad \((T', m', u')\) on \(\mathcal{M}'\) to a monad \((T, m, u)\) on \(\mathcal{M}\) is a pair \((F, f)\), where \(F : \mathcal{M}' \to \mathcal{M}\) is a functor and \(f : TF \to FT'\) is a natural transformation which satisfies the following two relations.

\[
\begin{array}{ccc}
T & T & F \\
\downarrow & \downarrow & \downarrow \\
F & T' & F \\
\end{array}
\quad \quad
\begin{array}{ccc}
T & T & F \\
\downarrow & \downarrow & \downarrow \\
F & T' & F \\
\end{array}
\quad \quad
\begin{array}{c}
f = \\
F
\end{array}
\quad \quad
\begin{array}{c}
f = \\
F
\end{array}
\]

Monads and their morphisms constitute a category, with identity morphism \((\mathcal{M}, T) : (T, m, u) \to (T, m, u)\) and composition law \((G, g) \circ (F, f) = (GF, Gf \circ gF)\), for monad morphisms \((F, f) : (T'', m'', u'') \to (T', m', u')\) and \((G, g) : (T', m', u') \to (T, m, u)\).

The category of monads and their morphisms is in fact isomorphic to a full subcategory in the category of 2-functors and lax natural transformations. Let us denote in any 2-category the horizontal composition by juxtaposition and the vertical composition by \(\circ\). Consider the 2-category \(\textbf{Cat}\) whose 0-cells are some categories (whose monads we are aiming to describe), 1-cells are functors and 2-cells are natural transformations. (In order to avoid set theoretical problems arising from the paradox of "the category of all categories", some restrictions on the 0-cells are needed. Since our most important examples in Sections 3 and 5 are (co)module categories, for our purposes allowing only for small categories would be too restrictive. All of our examples are included, for example, if 0-cells are all admissible or all locally presentable categories, cf. [21]. Readers interested in other
examples might choose other classes (or in some cases even finite sets) of 0-cells to define an appropriate (large) 2-category \( \text{Cat} \).) On the other hand, consider the 2-category freely generated by a monad. That is, the 2-category \( \mathcal{T} \) with a single 0-cell \( o \), a non-identity 1-cell \( t : o \to o \) and its iterated horizontal composites, and 2-cells given by composites of the non-identity 2-cells \( \eta : o \Rightarrow t \) and \( \mu : tt \Rightarrow t \), modulo the relations \( \mu \circ \eta t = t = \mu \circ t \eta \) and \( \mu \circ \mu t = \mu \circ t \mu \) (for a diagrammatic representation see Definition 1.1); that is, such that \((t, \mu, \eta)\) is a monad in \( \mathcal{T} \). A 2-functor \( K : \mathcal{T} \to \text{Cat} \) is precisely the same as a monad \((Kt, K\mu, K\eta)\) on the category \( Ko \).

A lax natural transformation \( K' \to K \), for 2-functors \( K, K' : \mathcal{C} \to \mathcal{D} \) between any 2-categories \( \mathcal{C} \) and \( \mathcal{D} \), is given by 1-cells \( F_C : K'C \to KC \) in \( \mathcal{D} \), labelled by the 0-cells \( C \) of \( \mathcal{C} \), and 2-cells \( f_C : (K\gamma)F_{C'} \to FC(K'\gamma) \) in \( \mathcal{D} \), labelled by the 1-cells \( \gamma : C' \to C \) in \( \mathcal{C} \). These data obey the following conditions.

(i) Naturality of \( f \); that is, for any 2-cell \( \Gamma : \gamma \to \delta \) in \( \mathcal{C} \), the following diagram commutes.

\[
\begin{array}{ccc}
(K\gamma)F_{C'} & \xrightarrow{f_{\gamma}} & FC(K'\gamma) \\
(K\Gamma)F_{C'} & \downarrow & \downarrow F_C(K'\Gamma) \\
(K\delta)F_{C'} & \xrightarrow{f_{\delta}} & FC(K'\delta). \\
\end{array}
\]

(ii) Compatibility of \( f \) with the horizontal composition; that is, for any 1-cells \( C'' \xleftarrow{\gamma''} C' \xleftarrow{\gamma} C \), the identity \( f_{\gamma''} \gamma = f_{\gamma'}(K'\gamma') \circ (K\gamma) f_{\gamma'} \) holds.

(iii) Compatibility of \( f \) with the identity 1-cells; i.e., for any identity 1-cell \( C \xrightarrow{c} C \), \( f_C \) is equal to the identity 2-cell \( F_C \xrightarrow{c} F_C \).

A lax natural transformation between 2-functors \( K, K' : \mathcal{T} \to \text{Cat} \) is given then by a functor \( F = F_o : K'o \to Ko \) and a natural transformation \( f = f_t : (Kt)F \to F(K't) \). (The value of \( f \) on the other 1-cells \( t^n \) for \( n \neq 1 \) is determined by the compatibility conditions (ii) and (iii) with the horizontal composition and with the identity 1-cell.) The naturality condition (i), applied to \( \Gamma = \mu \) and \( \Gamma = \eta \), respectively, yields precisely the same conditions in Definition 1.1 on a monad morphism \((F, f) : (K't, K'\mu, K'\eta) \to (Kt, K\mu, K\eta)\).

**Definition 1.2.** An algebra for a monad \((T, m, u)\) on a category \( \mathcal{M} \), is a monad morphism from the identity monad on the terminal category (of a single object and its identity morphism) to \((T, m, u)\). That is, a pair \((M, \varrho)\), where \( M \) is an object in \( \mathcal{M} \) and \( \varrho : TM \to M \) is a morphism in \( \mathcal{M} \) such that the first two diagrams in

\[
\begin{array}{ccc}
T^2M & \xrightarrow{T\varrho} & TM \\
\downarrow mM & & \downarrow \varrho \\
TM & \xrightarrow{\varrho} & M
\end{array}
\quad
\begin{array}{ccc}
M & \xrightarrow{uM} & TM \\
\downarrow \varrho & & \downarrow \varrho \\
TM & \xrightarrow{\varrho} & M
\end{array}
\quad
\begin{array}{ccc}
TM' & \xrightarrow{T\varrho'} & TM \\
\downarrow \varrho' & & \downarrow \varrho \\
M' & \xrightarrow{\varphi} & M
\end{array}
\]

commute. A morphism of \( T \)-algebras \((M', \varrho') \to (M, \varrho)\) is a morphism \( \varphi : M' \to M \) in \( \mathcal{M} \) such that the third diagram above commutes. Algebras of a monad \( T \) and their morphisms constitute the so-called Eilenberg-Moore category \( \mathcal{M}^T \).

Note that via composition on the right, a monad \( T : \mathcal{M} \to \mathcal{M} \) induces a monad \( \text{Cat}(T, -) \) on the category \( \text{Cat}(\mathcal{M}, -) \), whose objects are functors of domain \( \mathcal{M} \) and
whose morphisms are natural transformations. Symmetrically, there is a monad \( \text{Cat}(-, T) \) acting by composition on the left on the category \( \text{Cat}(-, \mathcal{M}) \), whose objects are functors of codomain \( \mathcal{M} \) and whose morphisms are natural transformations. In order to distinguish algebras of these induced monads from \( T \)-algebras, we call an algebra of the monad \( \text{Cat}(T, -) \) a right \( T \)-module functor and we term an algebra of the monad \( \text{Cat}(-, T) \) a left \( T \)-module functor.

**Definition 1.3.** Consider two monads \( (T_l, m_l, u_l) \) and \( (T_r, m_r, u_r) \) on the same category \( \mathcal{M} \). A (monad) distributive law is a natural transformation \( \Phi : T_r T_l \to T_l T_r \) such that the following four relations hold true.

\[
\begin{align*}
T_r T_r T_l T_t &= T_r T_r T_l T_t, \\
T_l T_r T_l T_t &= T_l T_r T_l T_t, \\
T_r T_l T_l T_t &= T_r T_l T_l T_t, \\
T_l T_l T_r T_t &= T_l T_l T_r T_t.
\end{align*}
\]

The last two defining relations are equivalent to the fact that \( (T_l, \Phi) \) is a monad endomorphism of \( (T_r, m_r, u_r) \).

Note that in the representation of \( \Phi \), the string corresponding to \( T_r \) crosses over the other one. If \( \Phi \) is an isomorphism, then the string representation of \( \Phi^{-1} \) is obtained from that of \( \Phi \) by an up-down reflection.

As it was proven in [1], a distributive law \( \Phi : T_r T_l \to T_l T_r \) as in Definition 1.3 induces a monad structure on the composite functor \( T_l T_r \), with the multiplication \( m \) and unit \( u \), whose string representations are

\[
\begin{align*}
m &= T_l T_r T_l T_r, \\
u &= T_l T_r T_l T_r.
\end{align*}
\]

The following well-known lemma describes morphisms between such composite monads.

**Lemma 1.4.** Consider two monads \( (T_l, m_l, u_l) \) and \( (T_r, m_r, u_r) \) on the same category \( \mathcal{M} \) and two monads \( (T_l', m_l', u_l') \) and \( (T_r', m_r', u_r') \) on \( \mathcal{M}' \). Let \( \Phi : T_r T_l \to T_l T_r \) and \( \Phi' : T_r' T_l' \to T_l' T_r' \) be distributive laws. For monad morphisms \( (G, q_l) : (T_l', m_l', u_l') \to (T_l, m_l, u_l) \) and \( (G, q_r) : (T_r', m_r', u_r') \to (T_r, m_r, u_r) \), the following assertions are equivalent.

(i) \( (G, q_l T_l' \circ T_l q_r) \) is a monad morphism \( T_l' T_r' \to T_l T_r \), cf. (1.1);

(ii) The following identity holds true.

\[
\begin{align*}
T_r T_l G &\quad T_l T_r G \\
\left(\begin{array}{c}
q_l \\
q_r
\end{array}\right) &\quad \left(\begin{array}{c}
q_l \\
q_r
\end{array}\right) \\
G T_l' T_r' &\quad G T_l' T_r'
\end{align*}
\]
Distributive laws $(T_l, T_r, \Phi)$ as objects, with triples $(G, q, q_r)$ satisfying the equivalent conditions in Lemma 1.4 as morphisms between them, constitute a category which can be regarded again as a full subcategory in the category of 2-functors and lax natural transformations. To this end, consider now a 2-category $\mathcal{L}$ of a single 0-cell $\circ$ generated by two monads $(t_l, \mu_l, \eta_l)$ and $(t_r, \mu_r, \eta_r)$ on $\circ$ and a 2-cell $\phi : t_r t_l \to t_l t_r$ such that

\[
\begin{align*}
\phi \circ \mu_r t_l &= t_l \mu_r \circ \phi t_r \circ t_r \phi \\
\phi \circ t_r \mu_l &= \mu_l t_r \circ t_l \phi \circ \phi t_l \\
\phi \circ t_r \eta_l &= \eta_l t_r
\end{align*}
\]

(for a diagrammatic representation see Definition 1.3) i.e. such that $\phi$ is a distributive law. A 2-functor $K : \mathcal{L} \to \text{Cat}$ is the same as a pair of monads on the category $K \circ$ related by a distributive law $K \Phi$. A lax natural transformation between such 2-functors is the same as a pair of monad morphisms as in Lemma 1.4.

The following definition is quoted from [7].

**Definition 1.5.** Consider two monads $(T_l, m_l, u_l)$ and $(T_r, m_r, u_r)$ on the same category $\mathcal{M}$ and a distributive law $\Phi : T_r T_l \to T_l T_r$. A $\Phi$-algebra is a pair, consisting of an object $X$ of $\mathcal{M}$ and a morphism $\xi : T_r X \to T_l X$, such that the following diagrams commute.

\[
\begin{array}{ccc}
T^2_l X & \xrightarrow{T_l \xi} & T_l T_l X \\
\downarrow m_r X & & \downarrow \xi \\
T_r X & \xrightarrow{\xi} & T_l X
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{u_r X} & \xi \\
\downarrow u_l X & & \downarrow T_l X
\end{array}
\]

If $T_l$ is equal to the identity functor $\mathcal{M}$ then there is a trivial distributive law $\Phi = T_r$. In this case $\Phi$-algebras are the same as $T_r$-algebras.

A distributive law $\Phi : T_r T_l \to T_l T_r$, between monads on $\mathcal{M}$, induces a distributive law $\text{Cat}(\Phi, -) : \text{Cat}(T_l, -) \text{Cat}(T_r, -) \to \text{Cat}(T_r, -) \text{Cat}(T_l, -)$ between monads on $\text{Cat}(\mathcal{M}, -)$ and a distributive law $\text{Cat}(-, \Phi) : \text{Cat}(-, T_r) \text{Cat}(-, T_l) \to \text{Cat}(-, T_l) \text{Cat}(-, T_r)$ between monads on $\text{Cat}(-, \mathcal{M})$. Algebras for these induced distributive laws are called right and left $\Phi$-module functors, respectively. Explicitly, a right $\Phi$-module functor is a pair, consisting of a functor $\sqcap : \mathcal{M} \to \mathcal{C}$ (where $\mathcal{C}$ is any category) and a natural transformation $i : \sqcap T_l \to \sqcap T_r$, such that the following relations hold true.

\[
\begin{array}{ccc}
\sqcap T_l T_l & \xrightarrow{i} & \sqcap T_l T_l \\
\downarrow i & & \downarrow i \\
\sqcap T_l & \xrightarrow{=} & \sqcap T_l
\end{array}
\]

A left $\Phi$-module functor is a pair, consisting of a functor $\sqcup : \mathcal{D} \to \mathcal{M}$ (where $\mathcal{D}$ is any category) and a natural transformation $w : T_l \sqcup \to T_l \sqcup$, such that the following relations

\[
\begin{array}{ccc}
\sqcup T_l & \xrightarrow{=} & \sqcup T_l \\
\downarrow w & & \downarrow w \\
\sqcup T_l & \xrightarrow{=} & \sqcup T_l
\end{array}
\]

(1.3)
hold true.

\[ T_r \triangleleft T_r' \sqsupseteq T_r \triangleleft T_r' \]

(1.4)

Once again, right or left \( \Phi \)-module functors can be described as 2-functors from an appropriately chosen 2-category to \( \text{Cat} \). Consider a 2-category \( \mathcal{R} \), with two 0-cells \( o \) and \( c \), generated by two monads \((t_l, \mu_l, \eta_l)\) and \((t_r, \mu_r, \eta_r)\) on \( o \) related by a distributive law \( \phi : t_r t_l \to t_l t_r \), and a further 1-cell \( p : o \to c \) together with a 2-cell \( \iota : p t_l \to p t_r \) such that

\[ \iota \circ p \mu_l = p \mu_r \circ \iota t_r \circ p \phi \circ \iota t_l \quad \text{and} \quad \iota \circ p \eta_l = p \eta_r, \]

(for a diagrammatic representation see (1.3)) i.e. such that \((p, \iota)\) is an \( \mathcal{R}(\phi, c) \)-algebra in \( \mathcal{R}(o, c) \). A 2-functor \( K \) from \( \mathcal{R} \) to \( \text{Cat} \) can be described then as a pair of monads on the category \( Ko \), related by a distributive law \( K\phi \), together with a right \( K\phi \)-module functor \((Kp, K\iota)\). A 2-functor from the horizontal opposite of \( \mathcal{R} \) to \( \text{Cat} \) corresponds to a pair of monads related by a distributive law and a left module functor for it.

Making use of the above observations, we can define morphisms between right or left \( \Phi \)-module functors as lax natural transformations between the corresponding 2-functors. Explicitly, this yields the following.

**Definition 1.6.** Consider two monads \((T_l, m_l, u_l)\) and \((T_r, m_r, u_r)\) on the same category \( \mathcal{M} \) and two monads \((T'_l, m'_l, u'_l)\) and \((T'_r, m'_r, u'_r)\) on \( \mathcal{M}' \). Let \( \Phi : T_r T_l \to T_l T_r \) and \( \Phi' : T'_r T'_l \to T'_l T'_r \) be distributive laws.

A **morphism** from a right \( \Phi' \)-module functor \((\sqcap') : \mathcal{M}' \to \mathcal{C}', i' : \sqcap' T'_l \to \sqcap' T'_r\) to a right \( \Phi \)-module functor \((\sqcap : \mathcal{M} \to \mathcal{C}, i : \sqcap T_l \to \sqcap T_r)\) is a quintuple \((G, q_l, q_r, \lambda, \pi)\), where \((G, q_l) : (T'_l, m'_l, u'_l) \to (T_l, m_l, u_l)\) and \((G, q_r) : (T'_r, m'_r, u'_r) \to (T_r, m_r, u_r)\) are monad morphisms, \(\lambda : \mathcal{C}' \to \mathcal{C}\) is a functor and \(\pi : \sqcap G \to \sqcap' G\) is a natural transformation, such that (1.2) and the following relation hold.

\[ \begin{array}{c}
\sqcap T_l G \\
\pi \end{array} = \begin{array}{c}
\sqcap' T_r G \\
\pi'
\end{array} \]

Similarly, a **morphism** from a left \( \Phi' \)-module functor \((\sqcup' : \mathcal{D}' \to \mathcal{M}', w' : T'_l \sqcup' \to T'_l \sqcup')\) to a left \( \Phi \)-module functor \((\sqcup : \mathcal{D} \to \mathcal{M}, w : T_l \sqcup \to T_l \sqcup)\) is a quintuple \((G, q_l, q_r, \vee, \omega)\), where \((G, q_l) : (T'_l, m'_l, u'_l) \to (T_l, m_l, u_l)\) and \((G, q_r) : (T'_r, m'_r, u'_r) \to (T_r, m_r, u_r)\) are monad morphisms, \(\vee : \mathcal{D}' \to \mathcal{D}\) is a functor and \(\omega : \sqcup \vee \to G \sqcup'\) is a natural transformation,
such that (1.2) and the following relation hold.

\[
\begin{array}{c}
T_r \sqcup \lor = T_r \sqcup \lor \\
\omega \qquad \omega \\
\rho \qquad \rho \\
G T'_l \sqcup' = G T'_l \sqcup'
\end{array}
\]  

(1.6)

It is immediately clear by their definition as lax natural transformations that morphisms of (left or right) \( \Phi \)-module functors can be composed in the appropriate sense.

2. PARA-COCYCLIC OBJECTS AND \( \Phi \)-MODULE FUNCTORS

In this section we construct a category \( \mathcal{A} \), which comes equipped with a functor to a category of para-cocyclic objects in the category of functors. This implies that any object of \( \mathcal{A} \) induces a functor from a category \( \mathcal{D} \) (of coefficients) to the category of para-cocyclic objects in some category \( \mathcal{C} \).

Motivated by the constructions in Section 1, consider a 2-category \( \mathcal{S} \) with three 0-cells \( o, c \) and \( d \), generated by two monads on \( o \) related by a distributive law \( \phi \) and an \( \mathcal{S}(\phi, c) \)-algebra in \( \mathcal{S}(o, c) \) and an \( \mathcal{S}(d, \phi) \)-algebra in \( \mathcal{S}(d, o) \). Explicitly, \( \mathcal{S} \) has 1-cells depicted in

\[
\begin{array}{c}
\begin{array}{c}
d \quad q \quad o \\
\circ \quad \circ \\
t_l \quad t_r
\end{array}
\end{array}
\]

2-cells are horizontal and vertical composites of identity 2-cells and the following 2-cells.

\[
\begin{align*}
\mu_l : t_l^2 \Rightarrow t_l & \quad \eta_l : o \Rightarrow t_l \\
\mu_r : t_r^2 \Rightarrow t_r & \quad \eta_r : o \Rightarrow t_r \\
\phi : t_r t_l \Rightarrow t_l t_r & \quad \varpi : p t_l \Rightarrow p t_r
\end{align*}
\]

On these generating 2-cells one imposes three types of relations. The first one

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
t_l \quad t_l \quad t_l \quad t_l \\
t_l
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
t_l \quad t_l \quad t_l \quad t_l \\
t_l
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
t_l \quad t_l \quad t_l \quad t_l \\
t_l
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
t_l \quad t_l \quad t_l \quad t_l \\
t_l
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

means that \((t_l, \mu_l, \eta_l)\) and \((t_r, \mu_t, \eta_t)\) are monads in \( \mathcal{S} \). One also imposes the relations

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
t_l \quad t_l \quad t_l \quad t_l \\
t_l
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
t_l \quad t_l \quad t_l \quad t_l \\
t_l
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]
so that \( \phi \) is a distributive law. Finally, the relations

\[
\begin{align*}
(p, t_t) &= (p, t_{1}) \\
(p, t_r) &= (p, t_{r})
\end{align*}
\]

mean that \( (p, i) \) is \( S(\phi, c) \)-algebra in \( S(o, c) \) and \( (q, \varpi) \) is an \( S(d, \phi) \)-algebra in \( S(d, o) \).

We define a category \( \mathcal{A} \) as the opposite of the category of 2-functors from \( \mathcal{S} \) to \( \text{Cat} \) and lax natural transformations between them. Explicitly, this means the following.

**Definition 2.1.** The category \( \mathcal{A} \) is defined to have objects \( (T_l, T_r, \Phi, \sqcap, i, \sqcup, w) \), where

- \( T_l \) and \( T_r \) are monads on the same category \( \mathcal{M} \);
- \( \Phi : T_l T_l \to T_l T_r \) is a distributive law;
- \((\sqcap : \mathcal{M} \to \mathcal{C}, i : \sqcap T_l \to \sqcap T_r \) is a right \( \Phi \)-module functor;
- \((\sqcup : \mathcal{D} \to \mathcal{M}, w : T_l \sqcup \to T_l \sqcup) \) is a left \( \Phi \)-module functor.

A morphism \( (T_l, T_r, \Phi, \sqcap, i, \sqcup, w) \to (T'_l, T'_r, \Phi', \sqcap', \sqcup', w') \) is a datum \((G, q_l, q_r, \land, \pi, \lor, \omega)\), such that

- \((G, q_l, q_r, \land, \pi)\) is a morphism from the right \( \Phi' \)-module functor \((\sqcap', i')\) to the right \( \Phi \)-module functor \((\sqcap, i)\) (cf. Definition 1.6);
- \((G, q_l, q_r, \lor, \omega)\) is a morphism from the left \( \Phi' \)-module functor \((\sqcup', w')\) to the left \( \Phi \)-module functor \((\sqcup, w)\) (cf. Definition 1.6).

Recall that a para-cocyclic object in a category \( \mathcal{C} \) consists of a family \( \{Z^n\}_n \) of objects in \( \mathcal{C} \), for all non-negative integers \( n \), and morphisms

\[
d^k : Z^{n-1} \to Z^n, \quad s^k : Z^{n+1} \to Z^n, \quad \text{for } k = 0, \ldots, n,
\]

called coface and codegeneracy morphisms, respectively, satisfying cosimplicial relations, together with so called para-cocyclic morphisms \( t^n : Z^n \to Z^n \), for all \( n \geq 0 \), which satisfy, for all \( k = 1, \ldots, n \),

\[
t^n \circ d^0 = d^n, \quad t^n \circ d^k = d^{k-1} \circ t^{n-1}, \quad t^n \circ s^0 = s^n \circ t^{n+1} \circ t^{n+1}, \quad t^n \circ s^k = s^{k-1} \circ t^{n+1}.
\]

A morphism \((Z^*, d^*, s^*, t^*) \to (Z'^*, d'^*, s'^*, t'^*)\) is a family of morphisms \( \{f^n : Z^n \to Z'^n\}_{n \geq 0} \) in \( \mathcal{C} \), compatible with the coface, codegeneracy and para-cocyclic morphisms in the evident sense.

**Definition 2.2.** The category \( \mathcal{P} \) is defined to have objects that are para-cocyclic objects in the category of functors. That is, for any non-negative integer \( n \), a functor \( Z^n : \mathcal{D} \to \mathcal{C} \) together with natural transformations \( d^k : Z^{n-1} \to Z^n, \quad s^k : Z^{n+1} \to Z^n, \quad t^n : Z^n \to Z^n \), for \( 0 \leq k \leq n \), satisfying the defining relations of a para-cocyclic object.

Morphisms from \((Z^* : \mathcal{D} \to \mathcal{C}, d^*, s^*, t^*)\) to \((Z'^* : \mathcal{D}' \to \mathcal{C}', d'^*, s'^*, t'^*)\) are triples \((\land, \lor, \xi^*)\), where \( \land : \mathcal{C} \to \mathcal{C} \) and \( \lor : \mathcal{D}' \to \mathcal{D} \) are functors and \( \xi^* : (Z^* \lor, d^* \lor, s^* \lor, t^* \lor) \to (\land Z^*, \land d^*, \land s^*, \land t^*) \) is a morphism of para-cocyclic objects.

In terms of \( \Phi : T_l T_l \to T_l T_r \) and \( q_l : T_l G \to G T_l \), we define inductively some new natural transformations: Let \( q^0_l \) be the identity natural transformation \( G \to G \) and \( \Phi^0 \) be the identity natural transformation \( T_r \to T_r \). Put \( q^1_l := q_l \) and \( \Phi^1 := \Phi \). For every
To see this, we now define \( \Phi^n : T_r T^n \to T^n r \) and \( q^n : T^n_l G \to G T^n_l \) by \( \Phi^n := T^n_{l-1} \Phi \circ T^n_{l-1} \Phi \) and \( q^n := q^n \circ T^n_{l-1} \Phi \), respectively. For these natural transformations we will use the string diagrams in the figure below.

\[
\Phi^n = \begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (0,2);
  \draw (0,0) -- (1,1);
  \draw (1,1) -- (2,2);
  \draw (0,2) -- (1,1);
  \draw (1,1) -- (2,2);
\end{tikzpicture}
\end{array}
\quad q^n = \begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (0,2);
  \draw (0,0) -- (1,1);
  \draw (1,1) -- (2,2);
  \draw (0,2) -- (1,1);
  \draw (1,1) -- (2,2);
\end{tikzpicture}
\end{array}
\]

Note that in this figure each stripe represents a bunch of \( n \) strings, each one representing the functor \( T_l \).

**Theorem 2.3.** There is a functor \( \mathcal{Z}^* : \mathcal{A} \to \overline{\mathcal{P}} \), with object map

\[
(T_l, T_r, \Phi, \sqcap : \mathcal{M} \to \mathcal{C}, i, \sqcup : \mathcal{D} \to \mathcal{M}, w) \mapsto (\sqcap T^{l+1}_l \sqcup : \mathcal{D} \to \mathcal{C}, d^s, s^s, t^s).
\]

The functor \( \mathcal{Z}^* \) takes a morphism

\[
(G, q_l, q_r, \land, \lor, \omega) : (T_l, T_r, \Phi, \sqcap, i, \sqcup, w) \to (T'_l, T'_r, \Phi', \sqcap', i', \sqcup', w')
\]

to the triple \((\land, \lor, \xi^*)\). At every degree \( n \geq 0 \) and for \( 0 \leq k \leq n \), the coface morphisms \( d^k \), the codegeneracy morphisms \( s^k \), the para-cyclic morphism \( t^n \) and the morphism \( \xi^n \) are given by the natural transformations below.

\[
d^k = \begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (0,2);
  \draw (0,0) -- (1,1);
  \draw (1,1) -- (2,2);
  \draw (0,2) -- (1,1);
  \draw (1,1) -- (2,2);
\end{tikzpicture}
\end{array}
\quad s^k = \begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (0,2);
  \draw (0,0) -- (1,1);
  \draw (1,1) -- (2,2);
  \draw (0,2) -- (1,1);
  \draw (1,1) -- (2,2);
\end{tikzpicture}
\end{array}
\quad t^n = \begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (0,2);
  \draw (0,0) -- (1,1);
  \draw (1,1) -- (2,2);
  \draw (0,2) -- (1,1);
  \draw (1,1) -- (2,2);
\end{tikzpicture}
\end{array}
\quad \xi^n = \begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (0,2);
  \draw (0,0) -- (1,1);
  \draw (1,1) -- (2,2);
  \draw (0,2) -- (1,1);
  \draw (1,1) -- (2,2);
\end{tikzpicture}
\end{array}
\]

**Proof.** The datum \((\sqcap T^{l+1}_l \sqcup, d^s, s^s)\) is obviously a cosimplex in the category of functors, cf. [25]. Its para-cyclicity is checked with the same steps in [4, Theorem 1.10]. It remains to show that \( \xi^* \) is a morphism of para-cyclic objects. Its compatibility with the coface and codegeneracy morphisms follows by naturality and monad morphism property of \( q_l \), see the following diagrammatic computations.
Compatibility with the para-cocyclic morphisms is proved in the figure below.

It follows by using naturality and (1.3) in the first equality, then applying (1.2) repeatedly in the second equality, and using in the last equality (1.6) together with naturality. □

**Corollary 2.4.** Any object \((T_1, T_r, \Phi, \sqcap : \mathcal{M} \to \mathcal{C}, i, \sqcup : \mathcal{D} \to \mathcal{M}, w)\) of the category \(\mathcal{A}\) determines a functor from \(\mathcal{D}\) to the category of para-cocyclic objects in \(\mathcal{C}\). The objects of \(\mathcal{D}\) play the role of coefficients for the resulting para-cocyclic object in \(\mathcal{C}\).

### 3. Examples from Hopf Cyclic Theory

In this section we list some examples of objects in the category \(\mathcal{A}\) in Definition 2.1 arising from Hopf cyclic theory (of bialgebroids, hence in particular of bialgebras). They give rise to families of para-cocyclic objects in the category \(\text{Mod}-k\) of modules over a commutative ring \(k\). They extend examples in [13], [12], [16] and [5].

Regarding the regular \(R\)-bimodule \(R\) as a right \(R^e := R \otimes R^{op}\)-module, and regarding any \(R\)-bimodule as a left \(R^e\)-module, we can define a functor \(R \otimes_{R^e} (-) : R\text{-Mod}-R \to \text{Mod}-k\). Applying it to the \(R\)-module tensor product of two \(R\)-bimodules \(M\) and \(N\), it yields the so called *cyclic \(R\)-module tensor product*. Throughout the paper, it will be denoted by

\[
M \hat{\otimes}_RN := R \otimes_{R^e} (M \otimes_R N) \cong M \otimes_{R^e} N.
\]

For finitely many \(R\)-bimodules \(\{M_i\}_{i=1,...,n}\), we put

\[
M_1 \hat{\otimes}_R M_2 \hat{\otimes}_R \ldots \hat{\otimes}_R M_n := (M_1 \otimes_R \ldots \otimes_R M_{i-1}) \otimes_{R^e} (M_i \otimes_R \ldots \otimes_R M_n),
\]

where the right hand side yields the same \(k\)-module for any \(i = 1, \ldots, n\) (defining the 0-fold tensor product to be equal to \(R\)).

For a short review of modules, comodules and contramodules of a bialgebroid, we refer to the Appendix. Throughout, actions in modules are denoted by juxtaposition and for coactions in comodules we use a Sweedler type index notation, with implicit summation understood.

The first example of an object in the category \(\mathcal{A}\) in Definition 2.1 arises from [4] Theorem 2.4.

**Example 3.1.** Let \(B\) be a left bialgebroid over a \(k\)-algebra \(L\) and \(A\) be a left \(B\)-module algebra. Then \(A\) is in particular an \(L\)-ring, with multiplication \(\mu : A \otimes_L A \to A\) and unit \(\eta : L \to A\). An object in \(\mathcal{A}\) is given by the following data.
• The monads \( T_l = A \otimes_L (-) \) and \( T_r = (-) \otimes_R A \) on \( L\text{-Mod}-L \), with monad structures
\[
\mu \otimes_L (-) : T_l^2 \rightarrow T_l, \quad \eta \otimes_L (-) : L\text{-Mod}-L \rightarrow T_l \quad \text{and} \quad (-) \otimes_L \mu : T_r^2 \rightarrow T_r, \quad (-) \otimes_L \eta : L\text{-Mod}-L \rightarrow T_r,
\]
respectively;

• The trivial distributive law \( \Phi = A \otimes_L (-) \otimes_L A : T_rT_l \rightarrow T_lT_r \);

• The right \( \Phi \)-module functor \((\sqcap, i)\), where \( \sqcap = L \otimes_{L^e} (-) : L\text{-Mod}-L \rightarrow \text{Mod}-k \) and for any \( L\)-bimodule \( P \),
\[
i_P : A \otimes_LP \rightarrow P \otimes_LA, \quad a \otimes_LP \mapsto p \otimes_la;
\]

• The left \( \Phi \)-module functor \((\sqcup, w)\), where \( \sqcup : B\text{-Comod} \rightarrow L\text{-Mod}-L \) is the forgetful functor and for any left \( B\)-comodule \( M \), with coaction \( m \mapsto m_{[-1]} \otimes_L m_{[0]} \),
\[
w_M : M \otimes_LA \rightarrow A \otimes_L M, \quad m \otimes_LA \mapsto m_{[-1]}a \otimes_L m_{[0]}.
\]

Applying the functor \( Z^* \) in Theorem 2.3 we obtain a para-cocyclic object in \( \text{Mod}-k \), for any left \( B\)-comodule \( M \). At degree \( n \), it is the \( k\)-module \( A^{\otimes_L n+1} \otimes_L M \). Coface and codegeneracy maps are
\[
d^k(a_0 \otimes_L a_1 \otimes_L \cdots \otimes_L a_{n-1} \otimes_L m) = a_0 \otimes_L \cdots \otimes_L a_{k-1} \otimes_L a_k \otimes_L \cdots \otimes_L a_{n-1} \otimes_L m,
\]
\[
s^k(a_0 \otimes_L a_1 \otimes_L \cdots \otimes_L a_{n+1} \otimes_L m) = a_0 \otimes_L \cdots \otimes_L a_{k-1} \otimes_L a_k a_{k+1} \otimes_L a_{k+2} \otimes_L \cdots \otimes_L a_{n+1} \otimes_L m,
\]
for \( k = 0, \ldots, n \). The para-cocyclic operator comes out as
\[
t^n(a_0 \otimes_LA a_1 \otimes_L \cdots \otimesLA_n \otimes_L m_{[-1]}a_0 \otimes_L m_{[0]}).
\]

The next example is obtained from [4, Theorem 2.7].

**Example 3.2.** Let \( B \) be a right bialgebroid over a \( k\)-algebra \( R \) and \((A, \mu, \eta)\) be a right \( B\)-comodule algebra (hence in particular an \( R\)-ring). For the \( B\)-coaction on \( A \), use the notation \( a \mapsto a[0] \otimes_R a[1] \). An object in \( A \) is given by the following data.

• The same monads \( T_l = A \otimes_R (-) \) and \( T_r = (-) \otimes_R A \) on \( R\text{-Mod}-R \), introduced in Example 3.1 (replacing \( L \) by \( R \));

• The trivial distributive law \( \Phi = A \otimes_R (-) \otimes_R A : T_rT_l \rightarrow T_lT_r \);

• The same right \( \Phi \)-module functor \((\sqcap, i)\), introduced in Example 3.1 (replacing \( L \) by \( R \));

• The left \( \Phi \)-module functor \((\sqcup, w)\), where \( \sqcup : \text{Mod}-B \rightarrow R\text{-Mod}-R \) is the forgetful functor and for any right \( B\)-module \( N \),
\[
w_N : N \otimes_RA \rightarrow A \otimes_R N, \quad m \otimes_RA \mapsto a[0] \otimes_RAma[1].
\]

For any right \( B\)-module \( N \), the corresponding para-cocyclic object in \( \text{Mod}-k \) is, at degree \( n \), \( A^{\otimes_R n+1} \otimes_R N \). Coface and codegeneracy maps are given by the same formulae in Example 3.1 (replacing \( L \) by \( R \) and \( M \) by \( N \)). The para-cocyclic operator has the form
\[
n^n(a_0 \otimes_RA a_1 \otimes_R \cdots \otimes_RAa_n \otimes_R m) = a_1 \otimes_RAa_2 \otimes_RAa_3 \otimes_RA \cdots \otimes_RAa_{n+1} \otimes_RAma[0] \otimes_RAma[1].
\]

In the following example, for a left \( R\)-module \( P \), a right \( R\)-module \( Q \) and \( R\)-bimodules \( C \) and \( D \), the canonical isomorphisms
\[
\Hom_{-R}(C, \Hom_{-R}(D, Q)) \cong \Hom_{-R}(C \otimes_R D, Q) \quad \text{and} \quad \Hom_{R-}(C, \Hom_{R-}(D, P)) \cong \Hom_{R-}(D \otimes_R C, P)
\]
are suppressed.

**Example 3.3.** Let $B$ be a right bialgebroid over a $k$-algebra $R$ and $(C, \Delta, \epsilon)$ be a right $B$-module coring (hence in particular an $R$-coring). An object in $\mathcal{A}$ is given by the following data.

- The monads $T_l := \text{Hom}_{-R}(C, -)$ and $T_r := \text{Hom}_{R,-}(C, -)$ on $R\text{-Mod}-R$, with monad structures
  \[
  \text{Hom}_{-R}(\Delta, -): T_l^2 \to T_l \quad \text{and} \quad \text{Hom}_{R,-}(\Delta, -): T_r^2 \to T_r,
  \]
  respectively;
- The distributive law $\Phi : \text{Hom}_{R,-}(C, \text{Hom}_{-R}(C, -)) \cong \text{Hom}_{R,R}(C \otimes_k C, -) \cong \text{Hom}_{-R}(C, \text{Hom}_{R,-}(C, -))$, given by switching the arguments;
- The right $\Phi$-module functor $(\sqcap, i)$, where $\sqcap = \text{Hom}_{R,R}(R, -) : R\text{-Mod}-R \to \text{Mod}-k$ and
  \[
  i_P : \text{Hom}_{R,R}(R, \text{Hom}_{-R}(C, P)) \cong \text{Hom}_{R,R}(C, P) \cong \text{Hom}_{R,R}(R, \text{Hom}_{R,-}(C, P))
  \]
  is the hom-tensor adjunction natural isomorphism, for any $R$-bimodule $P$;
- The left $\Phi$-module functor $(\sqcup, w)$, where $\sqcup : B\text{-Ctmod} \to R\text{-Mod}-R$ is the forgetful functor and for a left $B$-contramodule $(Q, \alpha)$,
  \[
  w_Q : \text{Hom}_{R,-}(C, Q) \to \text{Hom}_{-R}(C, Q), \quad f \mapsto \big( c \mapsto \alpha(f(c^-)) \big).
  \]

For any left $B$-contramodule $(Q, \alpha)$, this yields a para-cocyclic object in $\text{Mod}-k$. It is given by $\text{Hom}_{R,R}(C \otimes^R n+1, Q)$, at degree $n$. Coface and codegeneracy maps are

\[
\begin{align*}
(d^k \varphi^{(n-1)})(c_0 \otimes_R c_1 \otimes_R \ldots \otimes_R c_n) &= \varphi^{(n-1)}(c_0 \otimes_R \ldots \otimes_R c_{k-1} \epsilon(c_k) \otimes_R c_{k+1} \otimes_R \ldots \otimes_R c_n), \\
(s^k \varphi^{(n+1)})(c_0 \otimes_R c_1 \otimes_R \ldots \otimes_R c_n) &= \varphi^{(n+1)}(c_0 \otimes_R \ldots \otimes_R c_{k-1} \Delta(c_k) \otimes_R c_{k+1} \otimes_R \ldots \otimes_R c_n),
\end{align*}
\]

for $\varphi^{(j)} \in \text{Hom}_{R,R}(C \otimes^R j+1, Q)$ and $k = 0, \ldots, n$. The para-cocyclic map is equal to

\[
(t^n \varphi^{(n)})(c_0 \otimes_R c_1 \otimes_R \ldots \otimes_R c_n) = \alpha\big( \varphi^{(n)}(c_n(-) \otimes_R c_0 \otimes_R c_1 \otimes_R \ldots \otimes_R c_{n-1}) \big).
\]

**Example 3.4.** Let $B$ be a left bialgebroid over a $k$-algebra $L$ and $(C, \Delta, \epsilon)$ be a left $B$-comodule coring (hence in particular an $L$-coring), with $B$-coaction $c \mapsto c_{[-1]} \otimes_L c_{[0]}$. An object in $\mathcal{A}$ is given by the following data.

- The same monads $T_l := \text{Hom}_{-L}(C, -)$ and $T_r := \text{Hom}_{L,-}(C, -)$ on $L\text{-Mod}-L$ introduced in Example 3.3 (replacing $R$ by $L$);
- The same distributive law $\Phi$ introduced in Example 3.3 (replacing $R$ by $L$);
- The same right $\Phi$-module functor $(\sqcap, i)$ introduced in Example 3.3 (replacing $R$ by $L$);
- The left $\Phi$-module functor $(\sqcup, w)$, where $\sqcup : \text{Mod}-B \to L\text{-Mod}-L$ is the forgetful functor and for any right $B$-module $N$,
  \[
  w_N : \text{Hom}_{L,-}(C, N) \to \text{Hom}_{-L}(C, N), \quad f \mapsto \big( c \mapsto f(c_{[0]}c_{[-1]}) \big).
  \]
The cosimplicial structure of the para-cocyclic object in \( \mathbf{Mod}-k \), corresponding to a right \( B \)-module \( N \), is the same in Example 3.3 (replacing \( R \) by \( L \) and \( Q \) by \( N \)). The para-cocyclic map comes out as

\[
(t^n \varphi^{(n)})(c_0 \otimes_L c_1 \otimes_L \cdots \otimes_L c_n) = \varphi^{(n)}(c_{n[0]} \otimes_L c_0 \otimes_L c_1 \otimes_L \cdots \otimes_L c_{n-1})c_{n[-1]}.
\]

Specializing the above four examples to bialgebras instead of bialgebroids, in all of them the functors \( \sqcap \) become identity functors.

For an \( R \)-coring \( C \), a \( C \)-bicodule is an \( R \)-bimodule \( M \), together with a right \( C \)-coaction \( \varrho^M : M \to M \otimes_R C \) and a left \( C \)-coaction \( \varrho^M : M \to C \otimes_R M \), such that \( \varrho^M \) is a left \( R \)-module map, \( \varrho^M \) is a right \( R \)-module map and \( \varrho^M \circ \varrho^M = (C \otimes_R \varrho^M) \circ \varrho^M \). Morphisms of bicomodules are right \( C \)-comodule maps as well as left \( C \)-comodule maps. The category of \( C \)-bicomodules is denoted by \( C \text{-Comod-} C \).

**Example 3.5.** Let \( B \) be a right bialgebroid over a \( k \)-algebra \( R \) and \( (C, \Delta, \varepsilon) \) be a left \( B \)-comodule coring, hence in particular an \( L := R^{op} \)-coring. An object in \( \mathcal{A} \) is given by the following data.

- The monads \( T_i = C \otimes_L (-) \) and \( T_r = (-) \otimes_R C \) on \( C \text{-Comod-} C \). For a \( C \)-bicodule \( (M, \varrho^M, \varrho^M) \), \( T_iM = C \otimes_L M \) is a \( C \)-bicodule via the left and right coactions

\[
c \otimes_L m \mapsto \Delta(c) \otimes_L m \quad \text{and} \quad c \otimes_L m \mapsto c \otimes_L \varrho^M(m).
\]

The monad structure of \( T_i \) is given by the multiplication and unit

\[
C \otimes_L \varepsilon \otimes_L M : T^2_iM \to T_iM \quad \text{and} \quad \varrho^M : M \to T_iM.
\]

Symmetrically, \( T_r \), \( M \otimes_L C \) is a \( C \)-bicodule via the left and right coactions

\[
m \otimes_L c \mapsto \varrho^M(m) \otimes_L c \quad \text{and} \quad m \otimes_L c \mapsto m \otimes_L \Delta(c).
\]

The monad structure of \( T_r \) is given by the multiplication and unit

\[
M \otimes_L \varepsilon \otimes_L C : T^2_rM \to T_rM \quad \text{and} \quad \varrho^M : M \to T_rM.
\]

- The trivial distributive law \( \Phi = C \otimes_L (-) \otimes_L C \);
- The right \( \Phi \)-module functor \( \sqcap : C \text{-Comod-} C \to \mathbf{Mod}-k \), given by the equalizer

\[
\sqcap M \longrightarrow L \hat{\otimes}_LM \xrightarrow{L \varrho^M} C \otimes_L M \cong M \otimes_L C
\]

for any \( C \)-bicodule \( M \). The natural transformation \( i \) is given by the isomorphism

\[
i : \sqcap T_i \cong L \hat{\otimes}_L(-) \cong (-) \otimes_L L \cong \sqcap T_r.
\]

- The left \( \Phi \)-module functor \( \sqcup = (-) \otimes_L C : \mathbf{Mod}-B \to C \text{-Comod-} C \). For the left \( B \)-coaction on \( C \), introduce the index notation \( c \mapsto c^{[-1]} \otimes_R c^{[0]} \) and for the comultiplication in \( C \) write \( \Delta(c) = c^{(1)} \otimes_L c^{(2)} \). For any right \( B \)-module \( N \), \( \sqcup N = N \otimes_L C \) is a \( C \)-bicodule via the left and right coactions

\[
m \otimes_L c \mapsto c^{(1)} \otimes L mc^{(1)}^{[-1]} \otimes_L c^{(2)} \quad \text{and} \quad m \otimes_L c \mapsto m \otimes_L c^{(1)} \otimes_L c^{(2)}.
\]

The natural transformation \( w : (-) \otimes_L C \otimes_L C \to C \otimes_L (-) \otimes_L C \) is given by

\[
w_N(m \otimes_L c \otimes_L d) = c^{[0]} \otimes_L mc^{[-1]} \otimes_L d.
\]
For any right $B$-module $N$, this yields a para-cocyclic object in $\text{Mod-}k$. At degree $n$, it is given by $\cap(C^\otimes Ln^{n+1} \otimes LN) \cong C^\otimes Ln^{n+1} \otimes LN$. For every $0 \leq k \leq n - 1$, the corresponding coface map is:

$$d^k(c_0 \otimes_L \ldots \otimes_L c_{n-1} \otimes_L m) = c_0 \otimes_L \ldots c_{k-1} \otimes_L \Delta(c_k) \otimes_L c_{k+1} \otimes_L \ldots \otimes_L c_{n-1} \otimes_L m,$$

while

$$d^n(c_0 \otimes_L \ldots \otimes_L c_{n-1} \otimes_L m) = c_{0/2} \otimes_L c_1 \otimes_L \ldots \otimes_L c_{n-1} \otimes_L c_{0(1)}^{[0]} \otimes_L m c_{0(1)}^{[-1]}.$$  

If $0 \leq k \leq n$, then the codegeneracy map $s^k$ is given by

$$s^k(c_0 \otimes_L \ldots \otimes_L c_{n+1} \otimes_L m) = c_0 \otimes_L \ldots c_{k-1} \otimes_L c_k \epsilon(c_{k+1}) \otimes_L c_{k+2} \otimes_L \ldots \otimes_L c_{n+1} \otimes_L m.$$  

The para-cocyclic map is

$$t^n(c_0 \otimes_L \ldots \otimes_L c_n \otimes_L m) = c_1 \otimes_L \ldots \otimes_L c_n \otimes_L c_0^{[0]} \otimes_L m c_0^{[-1]}.$$  

**Example 3.6.** Let $B$ be a right bialgebroid over a $k$-algebra $R$ and $(C, \Delta, \epsilon)$ be a right $B$-module coring, hence in particular an $R$-coring. An object in $\mathcal{A}$ is given by the following data.

- The same monads $T_l = C \otimes_R (-)$ and $T_r = (-) \otimes_R C$ on $C\text{-Comod-}C$, introduced in Example 3.3 (replacing $L$ by $R$);
- The trivial distributive law $\Phi = C \otimes_R (-) \otimes_R C$;
- The same right $\Phi$-module functor $(\cap, i)$, introduced in Example 3.3 (replacing $L$ by $R$);
- The left $\Phi$-module functor $\sqcup = (-) \otimes_R C : B\text{-Comod} \rightarrow C\text{-Comod-}C$. For a left $B$-comodule $M$, with coaction $m \mapsto m_{[-1]} \otimes_R m^{[0]}$, using the notation $\Delta(c) = c_{(1)} \otimes_R c_{(2)}$, $\sqcup M = M \otimes_R C$ is a $C$-bicomodule with left and right coactions

$$m \otimes_R c \mapsto c_{(1)} m_{[-1]} \otimes_R m^{[0]} \otimes_R c_{(2)}, \text{ and } m \otimes_R c \mapsto m \otimes_R c_{(1)} \otimes_R c_{(2)}.$$  

The natural transformation $w : (-) \otimes_R C \otimes_R C \rightarrow C \otimes_R (-) \otimes_R C$ is given by

$$w_N(m \otimes_R c \otimes_R d) = cm_{[-1]} \otimes_R m^{[0]} \otimes_R d.$$  

For any left $B$-comodule $M$ this determines a para-cocyclic object in $\text{Mod-}k$. At degree $n$, it is given by $\cap(C^\otimes Rn^{n+1} \otimes_R M \otimes_R C) \cong C^\otimes Rn^{n+1} \otimes_R M$. For every $0 \leq k \leq n - 1$, the corresponding coface map is:

$$d^k(c_0 \otimes_R \ldots \otimes_R c_{n-1} \otimes_R m) = c_0 \otimes_R \ldots c_{k-1} \otimes_R \Delta(c_k) \otimes_R c_{k+1} \otimes_R \ldots \otimes_R c_{n-1} \otimes_R m,$$

while

$$d^n(c_0 \otimes_R \ldots \otimes_R c_{n-1} \otimes_R m) = c_{0(2)} \otimes_R c_1 \otimes_R \ldots \otimes_R c_{n-1} \otimes_R c_{0(1)}^{m[-1]} \otimes_R m^{[0]}.$$  

If $0 \leq k \leq n$, then the codegeneracy map $s^k$ is given by

$$s^k(c_0 \otimes_R \ldots \otimes_R c_{n-1} \otimes_R m) = c_0 \otimes_R \ldots c_{k-1} \otimes_R c_k \epsilon(c_{k+1}) \otimes_R c_{k+2} \otimes_R \ldots \otimes_R c_{n-1} \otimes_R m.$$  

The para-cocyclic map is

$$t^n(c_0 \otimes_R \ldots \otimes_R c_n \otimes_R m) = c_1 \otimes_R \ldots \otimes_R c_n \otimes_R c_0^{m[-1]} \otimes_R m^{[0]}.$$  

Restricting to the case when $B$ is a bialgebra over $k$, this para-cocyclic module yields a symmetrical version of [12] (2.1)-(2.4) (note the minor difference of using a left or a right module coalgebra). In [12] additional assumptions are made on the left comodule $M$ under which a truly cocyclic quotient exists.
Example 3.7. Let $B$ be a left bialgebroid over a $k$-algebra $L$ and $(A, \mu, \eta)$ be a left $B$-module algebra, hence in particular an $L$-ring. The following data define an object in $A$.

- The monads $T_1 = \text{Hom}_{-L}(A, -)$ and $T_2 = \text{Hom}_{L,-}(A, -)$ on $A$-$\text{Mod}$-$A$. For any $A$-bimodule $X$, $T_1 X = \text{Hom}_{-L}(A, X)$ is an $A$-bimodule via
  $$(a_1 f a_2)(a) = a_1 f(a_2 a), \quad \text{for } a, a_1, a_2 \in A, \; f \in \text{Hom}_{-L}(A, X).$$

  The monad structure is given by
  $$\text{Hom}_{-L}(A \otimes_L \eta, X) : T_1^2 X \to T_1 X \text{ and } X \to T_1 X, \; x \mapsto [a \mapsto xa].$$

  Symmetrically, $T_2 X = \text{Hom}_{L,-}(A, X)$ is an $A$-bimodule via
  $$(a_1 g a_2)(a) = g(aa_1)a_2, \quad \text{for } a, a_1, a_2 \in A, \; g \in \text{Hom}_{L,-}(A, X).$$

  The monad structure is given by
  $$\text{Hom}_{L,-}(\eta \otimes_L A, X) : T_2^2 X \to T_2 X \text{ and } X \to T_2 X, \; x \mapsto [a \mapsto ax].$$

- The distributive law
  $$\Phi : \text{Hom}_{-L}(A, \text{Hom}_{L,-}(A, -)) \cong \text{Hom}_{L,L}(A \otimes_k A, -) \cong \text{Hom}_{L,-}(A, \text{Hom}_{-L}(A, -)).$$

- The right $\Phi$-module functor $\cap : A$-$\text{Mod}$-$A \to A$-$\text{Mod}$-$k$, given for any $A$-bimodule $X$ by the equalizer
  $$\cap X \longrightarrow \text{Hom}_{L,L}(L, X) \longrightarrow \frac{f \to [a \mapsto a f(1)]}{f \to [a \mapsto f(1) a]} \text{Hom}_{L,L}(A, X).$$

  That is, $\cap X$ is the center of the $A$-bimodule $X$. The natural transformation $i$ is given by the isomorphism
  $$\cap \text{Hom}_{-L}(A, -) \cong \text{Hom}_{L,L}(L, -) \cong \cap \text{Hom}_{L,-}(A, -).$$

- The left $\Phi$-module functor $\sqcup = \text{Hom}_{L,-}(A, -) : B$-$\text{Ctrmod} \to A$-$\text{Mod}$-$A$. For any left $B$-contramodule $(Q, \alpha)$, $\sqcup Q = \text{Hom}_{L,-}(A, Q)$ is an $A$-bimodule, via
  $$(a_1 g a_2)(a) = \alpha(g(\langle -a_2 \rangle a a_1)), \quad \text{for } a, a_1, a_2 \in A, \; g \in \text{Hom}_{L,-}(A, Q).$$

  $w : \text{Hom}_{L,-}(A, \text{Hom}_{L,-}(A, -)) \to \text{Hom}_{-L}(A, \text{Hom}_{L,-}(A, -))$ is given by
  $$(w_Q(h))(a)(b) = \alpha(h(b)(-a)), \quad \text{for } a, b \in A, \; h \in \text{Hom}_{L,-}(A, \text{Hom}_{L,-}(A, Q)).$$

  For any left $B$-contramodule $(Q, \alpha)$, this determines a para-cocyclic object in $A$-$\text{Mod}$-$k$. At degree $n$, it is given by $\cap \text{Hom}_{-L}(A \otimes_k^{n+1}, \text{Hom}_{L,-}(A, Q)) \cong \text{Hom}_{L,L}(A \otimes^{n+1}_L Q)$. For every $0 \leq k \leq n - 1$, the corresponding coface map is
  $$(d^k \varphi^{(n-1)})(a_0 \otimes_L \cdots \otimes_L a_n) = \varphi^{(n-1)}(a_0 \otimes_L \cdots \otimes_L a_{k-1} \otimes_L a_{k+1} \otimes_L a_{k+2} \otimes_L \cdots \otimes_L a_n),$$

Meanwhile,
  $$(d^n \varphi^{(n-1)})(a_0 \otimes_L \cdots \otimes_L a_n) = \alpha\left(\varphi^{(n-1)}((-a_n) a_0 \otimes_L a_1 \otimes_L \cdots \otimes_L a_{n-1})\right).$$

If $0 \leq k \leq n$, then the codegeneracy map $s^k$ is given by
  $$(s^k \varphi^{(n+1)})(a_0 \otimes_L \cdots \otimes_L a_n) = \varphi^{(n+1)}(a_0 \otimes_L \cdots \otimes_L a_{k} \otimes_L a_{k+1} \otimes_L \cdots \otimes_L a_n).$$

The para-cocyclic map is
  $$(t^n \varphi^{(n)})(a_0 \otimes_L \cdots \otimes_L a_n) = \alpha(\varphi^{(n)}(-a_n \otimes_L a_0 \otimes_L \cdots \otimes_L a_{n-1})).$$
where \( \varphi^{(j)} \in \text{Hom}_{L,L}(A^{\otimes j+1}, Q) \).

This yields a non-commutative base version of the para-cocyclic module in [5, p. 6]. (In [5] additional assumptions are made on the left contramodule \((Q, \alpha)\) under which a truly cocyclic submodule exists.)

**Example 3.8.** Let \( B \) be a left bialgebroid over a \( k \)-algebra \( L \) and \((A, \mu, \eta)\) be a right \( B \)-comodule algebra with \( B \)-coaction \( a \mapsto a_{[0]} \otimes_L a_{[1]} \). Then \((A, \mu, \eta)\) is in particular an \( R := L^{op}\)-ring. The following data determine an object in \( \mathcal{A} \).

- The same monads \( T_l = \text{Hom}_{-,(A, -)} \) and \( T_r = \text{Hom}_{R,-}(A, -) \) on \( A\text{-Mod-}A \) introduced in Example 3.7 (replacing \( L \) by \( R \));
- The same distributive law \( \Phi \) introduced in Example 3.7 (replacing \( L \) by \( R \));
- The same right \( \Phi \)-module functor \((\land, i)\) introduced in Example 3.7 (replacing \( L \) by \( R \));
- The left \( \Phi \)-module functor \( \sqcup = \text{Hom}_{R-,}(A, -) : \text{Mod-}B \to A\text{-Mod-A} \). For any right \( B \)-module \( N, \sqcup N = \text{Hom}_{R-,}(A, N) \) is an \( A \)-bimodule via

\[
(a_1g a_2)(a) = g(a_{[0]} a a_{[1]} a_{[1]}), \quad \text{for } a, a_1, a_2 \in A, \ g \in \text{Hom}_{R-,}(A, N).
\]

\( w : \text{Hom}_{R-,}(A, \text{Hom}_{R-,}(A, -)) \to \text{Hom}_{R,R}(A, \text{Hom}_{R-,}(A, -)) \) is given by

\[
(w_N(h))(a)(b) = h(b)(a_{[0]} a_{[1]}).
\]

For any right \( B \)-module \( N \), this determines a para-cocyclic object in \( \text{Mod-}k \). At degree \( n \), it is given by \( \land \text{Hom}_{R-,}(A^{\otimes n+1}, \text{Hom}_{R-,}(A, N)) \cong \text{Hom}_{R,R}(A^{\otimes n+1}, N) \). For every \( 0 \leq k \leq n-1 \), the corresponding coface map is

\[
(d^k \varphi^{(n-1)})(a_0 \otimes_R \ldots \otimes_R a_n) = \varphi^{(n-1)}(a_0 \otimes_R \ldots \otimes_R a_{k-1} \otimes_R a_{k+1} \otimes_R a_{k+2} \otimes R \ldots \otimes_R a_n),
\]

while

\[
(d^n \varphi^{(n-1)})(a_0 \otimes_R \ldots \otimes_R a_n) = \varphi^{(n-1)}(a_{n[0]} a_0 \otimes_R a_1 \otimes_R \ldots \otimes_R a_{n-1})a_{n[1]}.
\]

If \( 0 \leq k \leq n \), then the codegeneracy map \( s^k \) is given by

\[
(s^k \varphi^{(n+1)})(a_0 \otimes_R \ldots \otimes_R a_n) = \varphi^{(n+1)}(a_0 \otimes_R \ldots \otimes_R a_{k-1} \otimes_R a_k \otimes_R a_{k+1} \otimes_R \ldots \otimes_R a_n).
\]

The para-cocyclic map is

\[
(l^n \varphi^{(n)})(a_0 \otimes_R \ldots \otimes_R a_n) = \varphi^{(n)}(a_{n[0]} \otimes_R a_0 \otimes_R \ldots \otimes_R a_{n-1} a_{n[1]}),
\]

where \( \varphi^{(j)} \in \text{Hom}_{R,R}(A^{\otimes j+1}, N) \).

This yields a non-commutative base version of the para-cocyclic module in [12, (3.1)–(3.4)]. (In [12] additional assumptions are made on the right module \( N \) under which a truly cocyclic submodule exists.)

### 4. Para-cyclic objects

Symmetrically to the considerations in Section 1 and Section 2 one can obtain another category \( \mathcal{B} \) together with a functor \( \mathcal{Z}_* \) from \( \mathcal{B} \) to a category of para-cyclic objects in the category of functors. Without repeating the details, in this section we summarize the main steps.
Definition 4.1. A comonad on a category \( \mathcal{M} \) is a triple \((S, d, e)\), where \( S : \mathcal{M} \to \mathcal{M} \) is a functor and \( d : S \to S^2 \) and \( e : S \to \mathcal{M} \) are natural transformations called the comultiplication and counit, respectively. Their string representations are shown in the first two pictures of the figure below. They satisfy the coassociativity and counitality constraints expressed by the third and the fourth equalities in the same figure.

A morphism from a comonad \((S', d', e')\) on \( \mathcal{M}' \) to a comonad \((S, d, e)\) on \( \mathcal{M} \) is a pair \((F, f)\), where \( F : \mathcal{M}' \to \mathcal{M} \) is a functor and \( f : SF \\to F S' \) is a natural transformation which is compatible with the comultiplications and the counits in the sense of the last two relations of the following figure.

Comonads and their morphisms constitute a category which is isomorphic to the category of 2-functors and lax natural transformations, from the vertical opposite of the 2-category \( \mathcal{T} \) in Section 1 to \( \mathbf{Cat} \).

Definition 4.2. A coalgebra for a comonad \( S \) on a category \( \mathcal{M} \) is a pair \((M, \varrho)\), where \( M \) is an object in \( \mathcal{M} \) and \( \varrho : M \to SM \) is a morphism in \( \mathcal{M} \) which is coassociative and counital in the evident sense.

A morphism of \( S \)-coalgebras \((M', \varrho') \to (M, \varrho)\) is a morphism \( \varphi : M' \to M \) in \( \mathcal{M} \), such that \( S\varphi \circ \varrho' = \varrho \circ \varphi \). Coalgebras of a comonad \( S \) and their morphisms constitute the so-called Eilenberg-Moore category \( \mathcal{M}^S \).

Via composition on the right, a comonad \( S : \mathcal{M} \to \mathcal{M} \) induces a comonad \( \mathbf{Cat}(S, -) \) on the category \( \mathbf{Cat}(\mathcal{M}, -) \). Symmetrically, there is a comonad \( \mathbf{Cat}(-, S) \) on the category \( \mathbf{Cat}(-, \mathcal{M}) \). We call a coalgebra of the comonad \( \mathbf{Cat}(S, -) \) a right \( S \)-comodule functor and we term a coalgebra of the comonad \( \mathbf{Cat}(-, S) \) a left \( S \)-comodule functor.

Definition 4.3. Consider two comonads \((S_l, d_l, e_l)\) and \((S_r, d_r, e_r)\) on the same category \( \mathcal{M} \). A comonad distributive law is a natural transformation \( \Psi : S_lS_r \to S_rS_l \), such that the following equalities hold.

A comonad distributive law \( \Psi : S_lS_r \to S_rS_l \) as in Definition 4.3 induces a comonad structure on the composite functor \( S_lS_r \), with comultiplication \( d \) and counit \( e \) whose string
representations are given in the figure below.

\[
d = \begin{array}{ccc}
S_l & S_r & S_l \\
S_l & S_r & S_l \\
S_l & S_r & S_l
\end{array} = \begin{array}{ccc}
S_l & S_r & S_l \\
S_l & S_r & S_l \\
S_l & S_r & S_l
\end{array}
\]

\[d = \begin{array}{ccc}
S_l & S_r & S_l \\
S_l & S_r & S_l \\
S_l & S_r & S_l
\end{array} = \begin{array}{ccc}
S_l & S_r & S_l \\
S_l & S_r & S_l \\
S_l & S_r & S_l
\end{array}
\]

**Definition 4.4.** Consider two comonads \((S_l, d_l, e_l)\) and \((S_r, d_r, e_r)\) on the same category \(\mathcal{M}\) and a comonad distributive law \(\Psi : S_lS_r \to S_rS_l\). A **\(\Psi\)-coalgebra** is a pair consisting of an object \(X\) in \(\mathcal{M}\) and a morphism \(\xi : S_lX \to S_rX\) rendering commutative the following diagrams.

\[
\begin{array}{ccc}
S_lX & \xrightarrow{\xi} & S_rX \\
\downarrow d_lX & & \downarrow d_rX \\
S_l^2X & \xrightarrow{s_l \xi} & S_r^2X \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
S_lX & \xrightarrow{\xi} & S_rX \\
\downarrow e_lX & & \downarrow e_rX \\
X & \cong & X
\end{array}
\]

Coalgebras of the comonad distributive law \(\text{Cat}(\Psi, -)\) are called **right \(\Psi\)-comodule functors**. Consider two comonads \((S_l, d_l, e_l)\) and \((S_r, d_r, e_r)\) on the same category \(\mathcal{M}\) and two comonads \((S'_l, d'_l, e'_l)\) and \((S'_r, d'_r, e'_r)\) on \(\mathcal{M}'\). Let \(\Psi : S_lS_r \to S_rS_l\) and \(\Psi' : S'_lS'_r \to S'_rS'_l\) be comonad distributive laws. A **morphism** from a right \(\Psi\)-comodule functor \((\sqcap : \mathcal{M} \to \mathcal{C}', \iota' : \sqcap S_r' \to \sqcap S_l')\) to a right \(\Psi\)-module functor \((\sqcap : \mathcal{M} \to \mathcal{C}, i : \sqcap S_r \to \sqcap S_l)\) is a quintuple \((G, q_l, q_r, \wedge, \pi)\), where \(\wedge : \mathcal{C}' \to \mathcal{C}\) is a functor, \(\pi : \sqcap G \to \sqcap \sqcap\) is a natural transformations, \((G, q_l) : (S'_l, d'_l, e'_l) \to (S_l, d_l, e_l)\) and \((G, q_r) : (S'_r, d'_r, e'_r) \to (S_r, d_r, e_r)\) are comonad morphisms. These data are, in addition, subject to the following two conditions.

\[
\begin{array}{ccc}
S_l & S_r & G \\
\downarrow q_l & \downarrow q_r & \downarrow \pi \\
G & S_r' & S'_r \\
\wedge \quad \iota' \\
\end{array} = \begin{array}{ccc}
S_l & S_r & G \\
\downarrow q_l & \downarrow q_r & \downarrow \pi \\
G & S_r' & S'_r \\
\wedge \quad \iota' \\
\end{array}
\]

\[\Xi = \begin{array}{ccc}
\sqcup S_r & G \\
\downarrow q_r & \downarrow \pi \\
G & \sqcup S'_r \\
\wedge \quad \iota' \\
\end{array} = \begin{array}{ccc}
\sqcup S_r & G \\
\downarrow q_r & \downarrow \pi \\
G & \sqcup S'_r \\
\wedge \quad \iota' \\
\end{array}
\]

Right \(\Psi\)-comodule functors and their morphisms constitute a category which is isomorphic to the category of 2-functors and lax natural transformations, from the vertical opposite of the 2-category \(\mathcal{R}\) in Section 1 to \(\text{Cat}\).

Symmetrically, a **left \(\Psi\)-comodule functor** is a coalgebra for the comonad distributive law \(\text{Cat}(-, \Psi)\). A **morphism** from a left \(\Psi\)-module functor \((\sqcup : \mathcal{D}' \to \mathcal{M}', w' : S_l\sqcup \to S_l\sqcup')\) to a left \(\Psi\)-module functor \((\sqcup : \mathcal{D} \to \mathcal{M}, w : S_l\sqcup \to S_l\sqcup)\) is a quintuple \((G, q_l, q_r, \vee, \omega)\), where \((G, q_l) : (S'_l, d'_l, e'_l) \to (S_l, d_l, e_l)\) and \((G, q_r) : (S'_r, d'_r, e'_r) \to (S_r, d_r, e_r)\) are comonad morphisms such that the first condition in (4.2) holds, \(\vee : \mathcal{D}' \to \mathcal{D}\) is a functor and

\[
\vee = \begin{array}{ccc}
\sqcup S_r & G \\
\downarrow q_r & \downarrow \pi \\
G & \sqcup S'_r \\
\wedge \quad \iota' \\
\end{array} = \begin{array}{ccc}
\sqcup S_r & G \\
\downarrow q_r & \downarrow \pi \\
G & \sqcup S'_r \\
\wedge \quad \iota' \\
\end{array}
\]

\[\Xi = \begin{array}{ccc}
\sqcup S_r & G \\
\downarrow q_r & \downarrow \pi \\
G & \sqcup S'_r \\
\wedge \quad \iota' \\
\end{array} = \begin{array}{ccc}
\sqcup S_r & G \\
\downarrow q_r & \downarrow \pi \\
G & \sqcup S'_r \\
\wedge \quad \iota' \\
\end{array}
\]
$\omega : \sqcup \mathcal{V} \to G \sqcup \mathcal{V}$ is a natural transformation such that the following relation holds.

\[
\begin{array}{c}
S_l \sqcup \mathcal{V} \\
| \omega \\
\downarrow \\
\leftarrow \\
G \mathcal{S}_r' \sqcup \mathcal{V}'
\end{array}
\begin{array}{c}
S_l \sqcup \mathcal{V} \\
| \omega' \\
\downarrow \\
\rightarrow \\
G \mathcal{S}_r' \sqcup \mathcal{V}'
\end{array}
\]

(4.3)

Left $\Psi$-comodule functors and their morphisms constitute a category which is isomorphic to the category of 2-functors and lax natural transformations, from the horizontal and vertical opposite of the 2-category $\mathcal{R}$ in Section 1 to $\mathbf{Cat}$.

We can define a category $\mathcal{B}$ as the opposite of the category of 2-functors and lax natural transformations, from the vertical opposite of the 2-category $\mathcal{S}$ in Section 1 to $\mathbf{Cat}$:

**Definition 4.5.** The category $\mathcal{B}$ is defined to have objects $(S_l, S_r, \Psi, \sqcap, i, \sqcup, w)$, where

- $S_l$ and $S_r$ are comonads on the same category $\mathcal{M}$;
- $\Psi : S_l S_r \to S_r S_l$ is a comonad distributive law;
- $(\sqcap : \mathcal{M} \to \mathcal{C}, i : \sqcap S_r \to \sqcap S_l)$ is a right $\Psi$-comodule functor;
- $(\sqcup : \mathcal{D} \to \mathcal{M}, w : S_l \sqcup \to S_r \sqcup)$ is a left $\Psi$-comodule functor.

A morphism $(S_l, S_r, \Psi, \sqcap, i, \sqcup, w) \to (S'_l, S'_r, \Psi', \sqcap', i', \sqcup', w')$ is a datum $(G, q_l, q_r, \wedge, \pi, \vee, \omega)$, such that

- $(G, q_l, q_r, \wedge, \pi)$ is a morphism from the right $\Psi'$-comodule functor $(\sqcap', i')$ to the right $\Psi$-comodule functor $(\sqcap, i)$;
- $(G, q_l, q_r, \vee, \omega)$ is a morphism from the left $\Psi'$-comodule functor $(\sqcup', w')$ to the left $\Psi$-comodule functor $(\sqcup, w)$.

Recall that the opposite $\mathcal{C}^{op}$ of a category $\mathcal{C}$ has the same objects and morphisms as $\mathcal{C}$, but composition of morphisms is opposite to that in $\mathcal{C}$. A para-cyclic object in a category $\mathcal{C}$ is, by definition, a para-cocyclic object in $\mathcal{C}^{op}$.

**Definition 4.6.** Objects of the category $\mathcal{P}$ are para-cyclic objects in the category of functors. Morphisms from $(Z_* : \mathcal{D} \to \mathcal{C}, d_*, s_*, t_*)$ to $(Z'_* : \mathcal{D}' \to \mathcal{C}', d'_*, s'_*, t'_*)$ are triples $(\wedge, \vee, \xi_*)$, where $\wedge : \mathcal{C}' \to \mathcal{C}$ and $\vee : \mathcal{D}' \to \mathcal{D}$ are functors and $\xi_* : (Z_* \vee, d_* \vee, s_* \vee, t_* \vee) \to (\wedge Z'_*, \wedge d'_*, \wedge s'_*, \wedge t'_*)$ is a morphism of para-cyclic objects.

Dually to Theorem 2.3, the following holds.

**Theorem 4.7.** There is a functor $\mathcal{Z}_* : \mathcal{B} \to \mathcal{P}$, with object map

$(S_l, S_r, \Psi, \sqcap : \mathcal{M} \to \mathcal{C}, i, \sqcup : \mathcal{D} \to \mathcal{M}, w) \mapsto (\sqcap S_l^{*+1} \sqcup : \mathcal{D} \to \mathcal{C}, d_*, s_*, t_*)$.

The functor $\mathcal{Z}_*$ takes a morphism

$(G, q_l, q_r, \wedge, \pi, \vee, \omega) : (S_l, S_r, \Psi, \sqcap, i, \sqcup, w) \to (S'_l, S'_r, \Psi', \sqcap', i', \sqcup', w')$

to the triple $(\wedge, \vee, \xi_*)$. At every degree $n \geq 0$ and for $0 \leq k \leq n$, the face morphisms $d_k$, the degeneracy morphisms $s_k$, the para-cyclic morphism $t_n$ and the morphism $\xi_n$ are
given by the natural transformations below.

\[
d_k = \begin{array}{c}
\Bigoplus_{j=0}^{k-1} s^{\otimes j}_L \otimes s^{\otimes k-j-1}_L \\
\Bigoplus_{j=0}^{k-1} s^{\otimes j}_L \\
\Bigoplus_{j=0}^{k-1} s^{\otimes j}_L \\
\Bigoplus_{j=0}^{k-1} s^{\otimes j}_L \\
\Bigoplus_{j=0}^{k-1} s^{\otimes j}_L \\
\Bigoplus_{j=0}^{k-1} s^{\otimes j}_L \\
\end{array}
\]

\[
s_k = \begin{array}{c}
\bigoplus_{i=0}^{k-1} t^{\otimes i}_L \\
\bigoplus_{i=0}^{k-1} t^{\otimes i}_L \\
\bigoplus_{i=0}^{k-1} t^{\otimes i}_L \\
\bigoplus_{i=0}^{k-1} t^{\otimes i}_L \\
\bigoplus_{i=0}^{k-1} t^{\otimes i}_L \\
\bigoplus_{i=0}^{k-1} t^{\otimes i}_L \\
\end{array}
\]

\[
t_n = \begin{array}{c}
\bigoplus_{i=0}^{n-1} \xi^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \xi^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \xi^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \xi^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \xi^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \xi^{\otimes i}_L \\
\end{array}
\]

\[
\xi_n = \begin{array}{c}
\bigoplus_{i=0}^{n-1} \tau^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \tau^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \tau^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \tau^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \tau^{\otimes i}_L \\
\bigoplus_{i=0}^{n-1} \tau^{\otimes i}_L \\
\end{array}
\]

**Corollary 4.8.** Any object \((S_l, S_r, \Psi, \Box : \mathcal{M} \to \mathcal{C}, i, \sqcap : \mathcal{D} \to \mathcal{M}, w)\) of the category \(\mathcal{B}\) determines a functor from \(\mathcal{D}\) to the category of para-cyclic objects in \(\mathcal{C}\). The objects of \(\mathcal{D}\) play the role of coefficients for the resulting para-cyclic object in \(\mathcal{C}\).

### 5. Examples from Hopf Cyclic Theory

In this section we present several examples of objects in the category \(\mathcal{B}\) in Definition 4.5 similar to those we have seen in Section 3. Throughout this section, the same notational conventions are used as in Section 3.

The first example is obtained from [4, Theorem 2.9] (for the case of a bialgebra see [17, Sec 5.3]).

**Example 5.1.** Let \(B\) be a left bialgebroid over a \(k\)-algebra \(L\) and \((C, \Delta, \epsilon)\) be a left \(B\)-comodule coring (hence in particular an \(L\)-coring), with \(B\)-coaction \(c \mapsto c_{[-1]} \otimes_L c_{[0]}\). An object in the category \(\mathcal{B}\) is given by the following data.

- The comonads \(S_l = C \otimes_L (-)\) and \(S_r = (-) \otimes_L C\) on \(L\)-Mod-\(L\), with comonad structures

\[
\Delta \otimes_L (-) : S_l \to S_l^2 \quad \epsilon \otimes_L (-) : S_l \to L\text{-Mod-}L \quad \text{and}
\]

\[
(-) \otimes_L \Delta : S_r \to S_r^2 \quad (-) \otimes_L \epsilon : S_r \to L\text{-Mod-}L,
\]

respectively;

- The trivial comonad distributive law \(\Psi = C \otimes_L (-) \otimes_L C\);

- The right \(\Psi\)-comodule functor \((\Box, i)\), where \(\Box = L \otimes_{L'} (-) : L\text{-Mod-}L \to \text{Mod-}k\) and for any \(L\)-bimodule \(P\),

\[
i_P : P \otimes_L C \to C \otimes_L P, \quad p \otimes_L c \mapsto c \otimes_L p;
\]

- The left \(\Psi\)-comodule functor \((\sqcap, w)\), where \(\sqcap : B\text{-Mod} \to L\text{-Mod-}L\) is the forgetful functor and for any left \(B\)-module \(N\),

\[
w_N : C \otimes_L N \to N \otimes_L C, \quad c \otimes_L m \mapsto c_{[-1]} m \otimes_L c_{[0]}.
\]

For any left \(B\)-module \(N\), the corresponding para-cyclic object in \(\text{Mod-}k\) is given by \(C \otimes_L m \otimes_L N\) at degree \(n\). Face and degeneracy maps are, for \(k = 0, \ldots, n\),

\[
d_k(c_0 \otimes_L c_1 \otimes_L \cdots \otimes L c_n \otimes_L m) = c_0 \otimes_L c_1 \otimes_L \cdots \otimes L c_{k-1} \otimes_L \epsilon(c_k) \otimes_L c_{k+1} \otimes_L \cdots \otimes L c_n \otimes_L m
\]

\[
s_k(c_0 \otimes_L c_1 \otimes_L \cdots \otimes L c_n \otimes_L m) = c_0 \otimes_L c_1 \otimes_L \cdots \otimes L c_{k-1} \otimes_L \Delta(c_k) \otimes_L c_{k+1} \otimes_L \cdots \otimes L c_n \otimes_L m.
\]

The para-cyclic map comes out as

\[
t_n(c_0 \otimes_L c_1 \otimes_L \cdots \otimes L c_n \otimes_L m) = c_{[n]} \otimes_L c_0 \otimes_L \cdots \otimes L c_{n-1} \otimes_L c_{[-1]} m.
\]
A next example arises from \[4\] Theorem 2.11. Restricting it to the case of a bialgebra, it yields a symmetrical version of the para-cocyclic module in \[17\] Sec 5.1.

**Example 5.2.** Let \( B \) be a right bialgebroid over a \( k \)-algebra \( R \) and \((C, \Delta, \epsilon)\) be a right \( B \)-module coring (so in particular an \( R \)-coring). An object in \( B \) is given by the following data.

- The same comonads \( S_l = C \otimes_R (-) \) and \( S_r = (-) \otimes_R C \) on \( R \text{-Mod}-R \) introduced in Example 5.1 (replacing \( L \) by \( R \));
- The trivial comonad distributive law \( \Psi = C \otimes_R (-) \otimes_R C \);
- The same right \( \Psi \)-comodule functor \((\sqcap, i)\) introduced in Example 5.1 (replacing \( L \) by \( R \));
- The left \( \Psi \)-comodule functor \((\sqcup, w)\), where \( \sqcup : \text{Comod}-B \to R \text{-Mod}-R \) is the forgetful functor and for any right \( B \)-comodule \( M \), with coaction \( m \mapsto m^{[0]} \otimes_R m^{[1]} \),
  \[
  w_M : C \otimes_R M \to M \otimes_R C, \quad c \otimes_R m \mapsto m^{[0]} \otimes_R cm^{[1]}.
  \]

For any right \( B \)-comodule \( M \), the corresponding para-cyclic module has the same simplicial structure in Example 5.1 (replacing \( L \) by \( R \) and \( N \) by \( M \)). The para-cyclic map is

\[
t_n(c_0 \otimes_R c_1 \otimes_R \cdots \otimes_R c_n \otimes_R m) = c_n m^{[1]} \otimes_R c_0 \otimes_R c_1 \otimes_R \cdots \otimes_R c_{n-1} \otimes_R m^{[0]}.
\]

**Example 5.3.** Let \( B \) be a left bialgebroid over a \( k \)-algebra \( L \) and \((A, \mu, \eta)\) be a left \( B \)-module algebra (so in particular an \( L \)-ring). An object in \( B \) is given by the following data.

- The comonads \( S_l = \text{Hom}_{-,L}(A, -) \) and \( S_r = \text{Hom}_{L,-}(A, -) \) on \( L \text{-Mod}-L \), with comonad structures
  \[
  \text{Hom}_{-,L}(\mu, -) : S_l \to S_l^2 \quad \text{Hom}_{-,L}(\eta, -) : S_l \to L \text{-Mod}-L \quad \text{and}
  \text{Hom}_{L,-}(\mu, -) : S_r \to S_r^2 \quad \text{Hom}_{L,-}(\eta, -) : S_r \to L \text{-Mod}-L,
  \]
  respectively;
- The comonad distributive law
  \( \Psi : \text{Hom}_{-,-L}(A, \text{Hom}_{L,-}(A, -)) \cong \text{Hom}_{L,-L}(A \otimes_k A, -) \cong \text{Hom}_{L,-}(A, \text{Hom}_{-,-L}(A, -)) \),
  given by switching the arguments;
- The right \( \Psi \)-comodule functor \((\sqcap, i)\), where \( \sqcap = \text{Hom}_{L,L}(L, -) : L \text{-Mod}-L \to \text{Mod}-k \) and the natural transformation
  \[
  i : \text{Hom}_{L,L}(L, \text{Hom}_{-,-L}(A, -)) \cong \text{Hom}_{L,L}(A, -) \cong \text{Hom}_{L,L}(L, \text{Hom}_{L,-}(A, -))
  \]
  is given by the hom-tensor adjunction isomorphisms;
- The left \( \Psi \)-comodule functor \((\sqcup, w)\), where \( \sqcup : \text{Ctrmod}-B \to L \text{-Mod}-L \) is the forgetful functor and for any right \( B \)-contramodule \((Q, \alpha)\),
  \[
  w_Q : \text{Hom}_{-,-L}(A, Q) \to \text{Hom}_{L,-}(A, Q), \quad f \mapsto \left( a \mapsto \alpha(f(-a)) \right).
  \]

The para-cyclic module corresponding to a right \( B \)-contramodule \((Q, \alpha)\) is given at degree \( n \) by the \( k \)-module \( \text{Hom}_{L,L}(A \otimes_k \n, Q) \). Face and degeneracy maps are, for \( k = 0, \ldots, n \),

\[
(d_k \varphi^{(n)})(a_0 \otimes L a_1 \otimes L \cdots \otimes L a_{n-1}) = \varphi^{(n)}(a_0 \otimes L \cdots \otimes L a_{k-1} \otimes L A \otimes L a_k \otimes L \cdots \otimes L a_{n-1})
\]
\[
(s_k \varphi^{(n)})(a_0 \otimes L a_1 \otimes L \cdots \otimes L a_{n+1}) = \varphi^{(n)}(a_0 \otimes L \cdots \otimes L a_{k-1} \otimes L a_k a_{k+1} \otimes L a_{k+2} \otimes L \cdots \otimes L a_{n+1}).
\]
The para-cyclic map is

\[(i_n \varphi^{(n)})(a_0 \otimes_L a_1 \otimes_L \ldots \otimes_L a_n) = \alpha \left( \varphi^{(n)}(a_1 \otimes_L \ldots \otimes_L a_n \otimes_L (-)a_0) \right),\]

for \( \varphi^{(n)} \in \text{Hom}_{L,L}(A \otimes_L n+1, Q) \).

**Example 5.4.** Let \( B \) be a right bialgebroid over a \( k \)-algebra \( R \) and \( (A, \mu, \eta) \) be a right \( B \)-comodule algebra (so in particular an \( R \)-ring), with \( B \)-coaction \( a \mapsto a^{[0]} \otimes R a^{[1]} \). An object in \( \mathcal{B} \) is given by the following data.

- The same comonads \( S_l = \text{Hom}_{-R}(A, -) \) and \( S_r = \text{Hom}_{R,-}(A, -) \) on \( R \text{-Mod-} R \) constructed in Example 5.3 (replacing \( L \) by \( R \));
- The same comonad distributive law \( \Psi \) constructed in Example 5.3 (replacing \( L \) by \( R \));
- The same right \( \Psi \)-comodule functor \( (\sqcup, i) \) constructed in Example 5.3 (replacing \( L \) by \( R \));
- The left \( \Psi \)-comodule functor \( (\sqcap, w) \), where \( \sqcap : B \text{-Mod} \rightarrow R \text{-Mod-} R \) is the forgetful functor and for any left \( B \)-module \( N \),

\[ w_N : \text{Hom}_{-R}(A, N) \rightarrow \text{Hom}_{R,-}(A, N), \quad f \mapsto (a \mapsto a^{[1]} f(a^{[0]})) \] .

The para-cyclic module corresponding to a left \( B \)-module \( N \) has the same simplicial structure in Example 5.3 (replacing \( L \) by \( R \) and \( Q \) by \( N \)). The para-cyclic map is

\[(t_n \varphi^{(n)})(a_0 \otimes_R a_1 \otimes_R \ldots \otimes_R a_n) = a_0^{[0]} \varphi^{(n)}(a_1 \otimes_R \ldots \otimes_R a_n \otimes_R a_0^{[0]}).\]

**Example 5.5.** Let \( B \) be a left bialgebroid over a \( k \)-algebra \( L \) and \( (A, \mu, \eta) \) be a left \( B \)-module algebra (so in particular an \( L \)-ring). An object in \( \mathcal{B} \) is given by the following data.

- The comonads \( S_l = A \otimes_L (-) \) and \( S_r = (-) \otimes_L A \) on \( A \text{-Mod-} A \). For any \( A \)-bimodule \( X \), \( S_l X = A \otimes_L X \) is an \( A \)-bimodule via

\[ a_1 (a \otimes_L x) a_2 = a_1 a \otimes_L x a_2, \quad \text{for } a, a_1, a_2 \in A, \; x \in X. \]

The comonad structure is given by

\[ A \otimes_L \eta \otimes_L X : S_l X \rightarrow S_l^2 X \quad \text{and} \quad S_l X \rightarrow X, \; a \otimes_L x \mapsto ax. \]

Symmetrically, \( S_r X = X \otimes_L A \) is an \( A \)-bimodule via

\[ a_1 (x \otimes_L a) a_2 = a_1 x \otimes_L a a_2, \quad \text{for } a, a_1, a_2 \in A, \; x \in X. \]

The comonad structure is given by

\[ X \otimes_L \eta \otimes_L A : S_r X \rightarrow S_r^2 X \quad \text{and} \quad S_r X \rightarrow X, \; x \otimes_L a \mapsto xa. \]

- The trivial comonad distributive law \( \Psi = A \otimes_L (-) \otimes_L A \);
- The right \( \Psi \)-comodule functor \( \sqcap : A \otimes_A (-) : A \text{-Mod-} A \rightarrow \text{Mod-} k \). The natural transformation \( i \) is given by the isomorphism

\[ i : A \otimes_A (-) \otimes_L A \cong L \otimes_L (-) \cong A \otimes_A (A \otimes_L (-)); \]
- The left \( \Psi \)-comodule functor \( \sqcup = (-) \otimes_L A : \text{Comod-} B \rightarrow A \text{-Mod-} A \). For any right \( B \)-comodule \( M \), with coaction \( m \mapsto m^{[0]} \otimes_L m^{[1]} \), \( \sqcup M = M \otimes_L A \) is an \( A \)-bimodule via

\[ a_1 (m \otimes_L a) a_2 = m^{[0]} \otimes_L (m^{[1]} a_1 a a_2), \quad \text{for } a, a_1, a_2 \in A, \; m \in M. \]
The natural transformation \( w : A \otimes_L (-) \otimes_L A \rightarrow (-) \otimes_L A \otimes_L A \) is given by

\[
w_M(a \otimes_L m \otimes_L b) = m_{[0]} \otimes_L m_{[1]} a \otimes_L b.
\]

For any right \( B \)-comodule \( M \), this determines a para-cyclic object in \( \text{Mod}_{-k} \). At degree \( n \) it is given by \( A \otimes_A (A \otimes_{-k} M \otimes_L A) \cong A \otimes_{-k} M \). For every \( 0 \leq k \leq n-1 \), the corresponding face map is

\[
d_k(a_0 \otimes_L \ldots \otimes_L a_n \otimes_L m) = a_0 \otimes_L \ldots \otimes_L a_{k-1} \otimes_L a_k \otimes_L a_{k+1} \otimes_L \ldots \otimes_L a_{n-1} \otimes_L m,
\]

while

\[
d_n(a_0 \otimes_L \ldots \otimes_L a_n \otimes_L m) = (m_{[1]} a_n) a_0 \otimes_L a_1 \otimes_L \ldots \otimes_L a_{n-1} \otimes_L m_{[0]}.
\]

If \( 0 \leq k \leq n \), then the degeneracy map \( s_k \) is given by

\[
s_k(a_0 \otimes_L \ldots \otimes_L a_n \otimes_L m) = a_0 \otimes_L \ldots \otimes_L a_k \otimes_L \ldots \otimes_L a_{n-1} \otimes_L \otimes_L a_n \otimes_L m.
\]

The para-cyclic map is

\[
t_n(a_0 \otimes_L \ldots \otimes_L a_n \otimes_L m) = m_{[1]} a_n \otimes_L a_0 \otimes_L \ldots \otimes_L a_{n-1} \otimes_L m_{[0]}.
\]

If the left bialgebroid \( B \) above is a constituent left bialgebroid in a Hopf algebroid with a bijective antipode \( S \), then there is a bijective correspondence between right \( B \)-comodules \( M \), with coaction \( m \mapsto m_{[0]} \otimes_L m_{[1]} \), and left coactions of the constituent right bialgebroid of the Hopf algebroid on \( M \), \( m \mapsto S(m_{[1]}) \otimes_L m_{[0]} \), cf. [A.10] Expressing in the above formulae the right \( B \)-coaction on \( M \) in terms of this left coaction, we obtain a para-cyclic module. Restricting to the case of a Hopf algebra (instead of a Hopf algebroid) and applying to the obtained para-cyclic module the functor \( \text{Hom}_k(-, k) \), we obtain the para-cyclic module in [12] (2.6)–(2.9)]. (Note that in [12] further properties of the comodule \( M \) are assumed which ensure that the associated para-cyclic module has a truly cyclic subobject.)

**Example 5.6.** Let \( B \) be a left bialgebroid over a \( k \)-algebra \( L \) and \( (A, \mu, \eta) \) be a right \( B \)-comodule algebra with \( B \)-coaction \( a \mapsto a_{[0]} \otimes_L a_{[1]} \). Then \( (A, \mu, \eta) \) is in particular an \( R := L^{op} \)-ring. An object in \( \mathcal{B} \) is given by the following data.

- The same comonads \( S_l = A \otimes_R (-) \) and \( S_r = (-) \otimes_R A \) on \( A-\text{Mod}-A \) constructed in Example 5.5 (replacing \( L \) by \( R \));
- The same comonad distributive law \( \Psi \) in Example 5.5 (replacing \( L \) by \( R \));
- The same right \( \Psi \)-comodule functor \((\cap, i)\) in Example 5.5 (replacing \( L \) by \( R \));
- The left \( \Psi \)-comodule functor \( \sqcup = (-) \otimes_R A : B-\text{Mod} \rightarrow A-\text{Mod}-A \). For any left \( B \)-module \( N \), \( \sqcup N = N \otimes_R A \) is an \( A \)-bimodule via

\[
a_1(m \otimes_R a) a_2 = a_{[1]} m \otimes_R a_{[0]} a a_2, \quad \text{for } a, a_1, a_2 \in A, \ m \in N.
\]

The natural transformation \( w : A \otimes_R (-) \otimes_R A \rightarrow (-) \otimes_R A \otimes_R A \) is given by

\[
w_N(a \otimes_R m \otimes_R b) = a_{[1]} m \otimes_R a_{[0]} \otimes_R b.
\]

For any left \( B \)-module \( N \), this determines a para-cyclic object in \( \text{Mod}_{-k} \). At degree \( n \) it is given by \( A \otimes_A (A \otimes_{-k} N \otimes_R A) \cong A \otimes_{-k} N \). For every \( 0 \leq k \leq n-1 \), the corresponding face map is

\[
d_k(a_0 \otimes_R \ldots \otimes_R a_n \otimes_R m) = a_0 \otimes_R \ldots \otimes_R a_{k-1} \otimes_R a_k a_{k+1} \otimes_R a_{k+2} \otimes_R \ldots \otimes_R a_n \otimes_R m,
\]

while

\[
d_n(a_0 \otimes_R \ldots \otimes_R a_n \otimes_R m) = (m_{[1]} a_n) a_0 \otimes_R a_1 \otimes_R \ldots \otimes_R a_{n-1} \otimes_R m_{[0]}.
\]
while
\[ d_n(a_0 \otimes_R \ldots \otimes_R a_n \otimes_R m) = a_{n[0]}a_0 \otimes_R a_1 \otimes_R \ldots \otimes_R a_{n-1} \otimes_R a_{n[1]} m. \]

If \( 0 \leq k \leq n \), then the degeneracy map \( s_k \) is given by
\[ s_k(a_0 \otimes_R \ldots \otimes_R a_n \otimes_R m) = a_0 \otimes_R \ldots \otimes_R a_k \otimes_R 1_A \otimes_R a_{k+1} \otimes_R \ldots \otimes_R a_n \otimes_R m. \]

The para-cyclic map is
\[ t_n(a_0 \otimes_R \ldots \otimes_R a_n \otimes_R m) = a_{n[0]} \otimes_R a_0 \otimes_R \ldots \otimes_R a_{n-1} \otimes_R a_{n[1]} m. \]

This is a non-commutative base version of the para-cyclic module in [12, (3.5)-(3.8)]. Note that in [12] further properties of the left module \( N \) are assumed which ensure that the associated para-cyclic module has a truly cyclic subobject.

A bicontramodule of an \( R \)-coring \( C \) is an \( R \)-bimodule \( Y \), together with a right \( C \)-contramodule structure \( \beta_r : \text{Hom}_{-, R}(C, Y) \to Y \) and a left \( C \)-contramodule structure \( \beta_l : \text{Hom}_{R, -}(C, Y) \to Y \), such that \( \beta_r \) is a left \( R \)-module map, \( \beta_l \) is a right \( R \)-module map and
\[ \beta_l \circ \text{Hom}_{R, -}(C, \beta_r) = \beta_r \circ \text{Hom}_{-, R}(C, \beta_l), \]
up to the (suppressed) canonical isomorphism
\[ \text{Hom}_{R, -}(C, \text{Hom}_{-, R}(C, Y)) \cong \text{Hom}_{-, R}(C \otimes_k C, Y) \cong \text{Hom}_{-, R}(C, \text{Hom}_{R, -}(C, Y)). \]

A morphism of bicontramodules is a right contramodule map as well as a left contramodule map. The category of \( C \)-bicontramodules is denoted by \( C\text{-Ctrmod}-C \).

**Example 5.7.** Let \( B \) be a right bialgebroid over a \( k \)-algebra \( R \) and \((C, \Delta, \epsilon)\) be a right \( B \)-module coring (hence in particular an \( R \)-coring). An object in \( B \) is given by the following data.

- The comonads \( S_l = \text{Hom}_{-, R}(C, -) \) and \( S_r = \text{Hom}_{R, -}(C, -) \) on \( C\text{-Ctrmod}-C \). For any \( C \)-bicontramodule \((Y, \beta_l, \beta_r)\), \( S_l Y = \text{Hom}_{-, R}(C, Y) \) is a bicontramodule, via the structure maps
\[ \text{Hom}_{R, -}(C, \text{Hom}_{-, R}(C, Y)) \cong \text{Hom}_{-, R}(C, \text{Hom}_{R, -}(C, Y)) \xrightarrow{\text{Hom}_{-, R}(C, \beta_l)} \text{Hom}_{-, R}(C, Y) \]
\[ \text{Hom}_{-, R}(C, \text{Hom}_{R, -}(C, Y)) \cong \text{Hom}_{-, R}(C \otimes_k C, Y) \xrightarrow{\text{Hom}_{-, R}(\Delta, Y)} \text{Hom}_{-, R}(C, Y). \]

The comonad structure is given by
\[ \text{Hom}_{-, R}(C \otimes_R \epsilon, Y) : S_l Y \to S_l^2 Y \quad \text{and} \quad \beta_r : S_l Y \to Y. \]
Symmetrically, \( S_r Y = \text{Hom}_{R, -}(C, Y) \) is a bicontramodule, via the structure maps
\[ \text{Hom}_{R, -}(C, \text{Hom}_{R, -}(C, Y)) \cong \text{Hom}_{R, -}(C \otimes_k C, Y) \xrightarrow{\text{Hom}_{R, -}(\Delta, Y)} \text{Hom}_{R, -}(C, Y) \]
\[ \text{Hom}_{R, -}(C, \text{Hom}_{R, -}(C, Y)) \cong \text{Hom}_{R, -}(C, \text{Hom}_{-, R}(C, Y)) \xrightarrow{\text{Hom}_{R, -}(C, \beta_r)} \text{Hom}_{R, -}(C, Y). \]

The comonad structure is given by
\[ \text{Hom}_{R, -}(\epsilon \otimes_R C, Y) : S_r Y \to S_r^2 Y \quad \text{and} \quad \beta_l : S_r Y \to Y. \]

- The comonad distributive law
\[ \Psi : \text{Hom}_{-, R}(C, \text{Hom}_{R, -}(C, -)) \cong \text{Hom}_{R, R}(C \otimes_k C, -) \cong \text{Hom}_{R, -}(C, \text{Hom}_{R, -}(C, -)), \]
given by switching the arguments;
• The right $\Psi$-comodule functor $\sqcap: C\text{-Ctrmod}-C \to \text{Mod}-k$, given by the coequalizer

$$\text{Hom}_{R,R}(C,Y) \xrightarrow{\text{Hom}_{R,R}(R,\alpha)} \text{Hom}_{R,R}(R,Y) \longrightarrow \sqcap Y.$$ 

The natural transformation $i$ is given by the isomorphism

$$i: \sqcap \text{Hom}_{R,-}(C,-) \cong \text{Hom}_{R,R}(R,-) \cong \sqcap \text{Hom}_{-,-}(C,-).$$

• The left $\Psi$-comodule functor $\sqcup = \text{Hom}_{R,-}(C,-): \text{Ctrmod}-B \to C\text{-Ctrmod}-C$. For any right $B$-contramodule $(Q,\alpha)$, $\sqcup Q = \text{Hom}_{R,-}(C,Q)$ is a $C$-bicontramodule, via the structure maps

$$\text{Hom}_{R,-}(C,\text{Hom}_{R,-}(C,Q)) \xrightarrow{\cong} \text{Hom}_{R,-}(C \otimes R C, Q) \xrightarrow{\text{Hom}_{R,-}(\Delta,Q)} \text{Hom}_{R,-}(C, Q);$$

$$\text{Hom}_{-,R}(C,\text{Hom}_{R,-}(C,Q)) \xrightarrow{\text{Hom}_{-,R}(C,Q)} \text{Hom}_{R,-}(C \otimes R C, Q) \xrightarrow{\cong} \text{Hom}_{R,-}(C, Q),$$

where $w: \text{Hom}_{-,R}(C,\text{Hom}_{R,-}(C,-)) \to \text{Hom}_{R,-}(C,\text{Hom}_{R,-}(C,-))$ is given by

$$(w_Q(h))(c)(d) = \alpha(h(d-))(c).$$

For any right $B$-contramodule $(Q,\alpha)$, this determines a para-cyclic object in $\text{Mod}-k$. At degree $n$, it is given by $\sqcup \text{Hom}_{R,-}(C^{\otimes R n+1},\text{Hom}_{R,-}(C,Q)) \cong \text{Hom}_{R,R}(C^{\otimes R n+1},Q)$. Denote $\Delta(c) = c_{(1)} \otimes_R c_{(2)}$. For every $0 \leq k \leq n - 1$, the corresponding face map is

$$(d_k\varphi^{(n)})(c_0 \otimes_R \cdots \otimes_R c_{n-1}) = \varphi^{(n)}(c_0 \otimes_R \cdots \otimes_R c_{k-1} \otimes_R \Delta(c_k) \otimes_R c_{k+1} \otimes_R \cdots \otimes_R c_{n-1}),$$

while

$$(d_n\varphi^{(n)})(c_0 \otimes_R \cdots \otimes_R c_{n-1}) = \alpha\big(\varphi^{(n)}(c_0(2) \otimes_R c_1 \otimes_R \cdots \otimes_R c_{n-1} \otimes_R c_0(1))\big).$$

If $0 \leq k \leq n$, then the degeneracy map $s_k$ is given by

$$(s_k\varphi^{(n)})(c_0 \otimes_R \cdots \otimes_R c_{n+1}) = \varphi^{(n)}(c_0 \otimes_R \cdots \otimes_R c_k \epsilon(c_{k+1}) \otimes_R c_{k+2} \otimes_R \cdots \otimes_R c_{n+1}).$$

The para-cyclic map is

$$(t_n\varphi^{(n)})(c_0 \otimes_R \cdots \otimes_R c_n) = \alpha\big(\varphi^{(n)}(c_1 \otimes_R \cdots \otimes_R c_n \otimes_R c_0)\big).$$

for $\varphi^{(n)} \in \text{Hom}_{R,R}(C^{\otimes R n+1},Q)$.

This is a non-commutative base version of the para-cyclic module in [5, p 4] (though note the minor difference of using a left or a right module coalgebra $C$). In [5] additional properties of the contramodule $Q$ are assumed so that the associated para-cyclic module has a cyclic subobject.

**Example 5.8.** Let $B$ be a right bialgebroid over a $k$-algebra $R$ and $(C,\Delta,\epsilon)$ be a left $B$-comodule coring with coaction $c \mapsto c^{(-1)} \otimes_R c^0$. Then $(C,\Delta,\epsilon)$ is in particular an $L := R^{op}$-coring. An object in $\mathcal{B}$ is the following data:

• The same comonads $S_l = \text{Hom}_{R,L}(C,-)$ and $S_r = \text{Hom}_{L,-}(C,-)$ on $C\text{-Ctrmod}-C$ constructed in Example [5.7] (replacing $R$ by $L$);

• The same distributive law $\Psi$ in Example [5.7] (replacing $R$ by $L$);

• The same right $\Psi$-comodule functor $(\sqcap, i)$ in Example [5.7] (replacing $R$ by $L$);
The left \( \Psi \)-comodule functor \( \sqcup = \text{Hom}_{L,-}(C, -) : B\text{-Mod} \to C\text{-Ctrmod}_C \). For any left \( B \)-module \( N \), \( \sqcup N = \text{Hom}_{L,-}(C, N) \) is a bicontramodule via the structure maps

\[
\text{Hom}_{L,-}(C, \text{Hom}_{L,-}(C, N)) \xrightarrow{\cong} \text{Hom}_{L,-}(C \otimes_L C, N) \xrightarrow{\text{Hom}_{L,-}(\Delta, N)} \text{Hom}_{L,-}(C, N)
\]

\[
\text{Hom}_{-L}(C, \text{Hom}_{L,-}(C, N)) \xrightarrow{w_N} \text{Hom}_{L,-}(C, \text{Hom}_{L,-}(C, N)) \xrightarrow{\cong} \text{Hom}_{L,-}(C \otimes_L C, N) \xrightarrow{\text{Hom}_{L,-}(\Delta, N)} \text{Hom}_{L,-}(C, N),
\]

where \( w : \text{Hom}_{-L}(C, \text{Hom}_{L,-}(C, -)) \to \text{Hom}_{L,-}(C, \text{Hom}_{L,-}(C, -)) \) is given by

\[
(w_N(h))(c)(d) = d^{i-1}h(d^{[0]})(c).
\]

For any left \( B \)-module \( N \), this determines a para-cyclic object in \( \text{Mod}_k \). At degree \( n \), it is given by \( \sqcup \text{Hom}_{-L}(C^{\otimes_{L,n+1}}, \text{Hom}_{L,-}(C, N)) \cong \text{Hom}_{L,L}(C^{\otimes_{L,n+1}}, N) \). Denote \( \Delta(c) = c_{(1)} \otimes_L c_{(2)} \). For every \( 0 \leq k \leq n-1 \), the corresponding face map is

\[
(d_k \varphi^{(n)})(c_0 \otimes_L \cdots \otimes_L c_{n-1}) = \varphi^{(n)}(c_0 \otimes_L \cdots \otimes_L c_{k-1} \otimes_L \Delta(c_k) \otimes_L c_{k+1} \otimes_L \cdots \otimes_L c_{n-1}),
\]

while

\[
(d_n \varphi^{(n)})(c_0 \otimes_L \cdots \otimes_L c_{n-1}) = c_{0(1)}^{(0)} \varphi^{(n)}(c_{0(2)} \otimes_L c_{1} \otimes_L \cdots \otimes_L c_{n-1} \otimes_L c_{0(1)}^{(0)}).
\]

If \( 0 \leq k \leq n \), then the degeneracy map \( s_k \) is given by

\[
(s_k \varphi^{(n)})(c_0 \otimes_L \cdots \otimes_L c_{n+1}) = \varphi^{(n)}(c_0 \otimes_L \cdots \otimes_L c_{k-1} \otimes_L c_k \epsilon(c_{k+1}) \otimes_L c_{k+2} \otimes_L \cdots \otimes_L c_{n+1}).
\]

The para-cyclic map is

\[
(t_n \varphi^{(n)})(c_0 \otimes_L \cdots \otimes_L c_n) = c_{0}^{(0)} \varphi^{(n)}(c_{1} \otimes_L \cdots \otimes_L c_{n} \otimes_L c_{0}^{(0)}),
\]

for \( \varphi^{(n)} \in \text{Hom}_{L,L}(C^{\otimes_{L,n+1}}, N) \).

6. The cyclic duality functor

The functor recently known as the cyclic duality functor, appeared first in Connes’ work [8]. In its original form, it is an isomorphism between the category of cyclic objects and the category of cocyclic objects in a given category. It was extended in [18] to an isomorphism between certain full subcategories of the categories of para-cyclic, and of para-cocyclic objects. The objects of these full subcategories are those para-(co)cyclic objects whose para-(co)cyclic morphisms are isomorphisms at all degrees. The aim of the current section is to extend cyclic duality to a functor between appropriate subcategories of \( \mathcal{A} \) and \( \mathcal{B} \) in Definitions 2.1 and 4.5 respectively.

Connes’s cyclic duality functor (in the extended form in [18]) and also its dual version (from a subcategory of the category of para-cyclic objects to a subcategory of the category of para-cocyclic objects) both will be denoted by \( (\neg) \).

Denote by \( \mathcal{A}^x \) the full subcategory of \( \mathcal{A} \) in Definition 2.1 whose objects \( (T_t, T_r, \Phi, \sqcup, i, \sqcap, w) \) obey the property that \( \Phi, i \) and \( w \) are natural isomorphisms. In the category \( \mathcal{P} \) in Definition 2.2, introduce the full subcategory \( \mathcal{P}^x \) whose objects have para-cocyclic morphisms which are natural isomorphisms at all degrees. Clearly, the functor \( \mathcal{Z}^* \) in Theorem 2.3 induces a functor \( \mathcal{Z}^{*x} : \mathcal{A}^x \to \mathcal{P}^x \). Symmetrically, introduce the full subcategory \( \mathcal{B}^x \) of the category \( \mathcal{B} \) in Definition 4.3 for whose objects \( (S_t, S_r, \Psi, \sqcap, i, \sqcup, w) \)
the natural transformations $\Psi$, $i$ and $w$ are isomorphisms. By Theorem 4.7, there is an induced functor $Z^\times: B^\times \to P^\times$, where $P^\times$ is the full subcategory of $P$, for whose objects the para-cyclic morphisms are natural isomorphisms at all degrees. Finally, denote by $A^\times$ the full subcategory of $A^\times$, for whose objects $(T, T, \Phi, \Box, i, \Box, w)$ the codomain category $C$ of the right $\Phi$-module functor $\Box$ possesses coequalizers. Symmetrically, denote by $B^\times$ the full subcategory of $B^\times$, for whose objects $(S, S, \Psi, \Box, i, \Box, w)$ the codomain category $C$ of the right $\Psi$-comodule functor $\Box$ possesses equalizers. Restrictions of the functor $Z^{\times\times}$ to $A^\times$ and $Z^\times$ to $B^\times$ are denoted by the same symbols $Z^{\times\times}$ and $Z^\times$.

**Theorem 6.1.** Using the notation in the paragraph preceding the theorem, there exists a functor $(\widetilde{-}): A^\times \to B^\times$, such that the diagram

$$
\begin{array}{ccc}
A^\times & \xrightarrow{(\widetilde{-})} & B^\times \\
\downarrow Z^{\times\times} & & \downarrow Z^\times \\
P^\times & \xrightarrow{(\widetilde{-})} & P^\times
\end{array}
$$

commutes up to a natural isomorphism.

Symmetrically, there exists a functor $(\widetilde{-}): B^\times \to A^\times$, such that the functors $Z^{\times\times}(\widetilde{-})$ and $Z^\times(\widetilde{-})$ are naturally isomorphic.

The proof of Theorem 6.1 goes through a series of lemmata.

Recall that for the Eilenberg-Moore category $M^T$ of a monad $T: M \to M$, there is a forgetful functor $U: M^T \to M$, with object map $(M, g) \mapsto M$ and acting on the morphisms as the identity map. The forgetful functor $U$ has a left adjoint $F$, with object map $M \mapsto (TM, mM)$ and morphism map $f \mapsto Tf$.

**Lemma 6.2.** Let $(T_l, m_l, u_l)$ and $(T_r, m_r, u_r)$ be monads on the same category $M$ and $\Phi: T_rT_l \to T_lT_r$ be a distributive law which is a natural isomorphism. Consider the induced monad $(\text{1.1})$ and the forgetful functor $U: M^{T_lT_r} \to M$.

1. There is a comonad $(\tilde{T}_l, d_l, e_l)$ on $M^{T_lT_r}$, such that $U\tilde{T}_l = T_lU$, $Ud_l = T_lTu_lU$ and $Ue_l(M, g) = g \circ T_lu_rM$, for any $T_lT_r$-algebra $(M, g)$.

2. There is a comonad $(\tilde{T}_r, d_r, e_r)$ on $M^{T_lT_r}$, such that $U\tilde{T}_r = T_rU$, $Ud_r = T_rTu_rU$ and $Ue_r(M, g) = g \circ u_lT_rM$, for any $T_lT_r$-algebra $(M, g)$.

3. There is a comonad distributive law $\tilde{\Phi}: \tilde{T}_l\tilde{T}_r \to \tilde{T}_r\tilde{T}_l$, such that $U\tilde{\Phi} = \Phi^{-1}U$.

**Proof.** (1) By Beck’s classical theorem [11, p 122], the distributive law $\Phi$ induces a monad $(\tilde{T}_l, \tilde{m}_l, \tilde{u}_l)$ on the category of $T_r$-algebras, such that the forgetful functor $U_r: M^{T_r} \to M$ satisfies

$$
U_r\tilde{T}_l = T_lU_r, \quad U_r\tilde{m}_l = m_lU_r, \quad U_r\tilde{u}_l = u_lU_r.
$$

Moreover, the category $(M^{T_r})^{\tilde{T}_l}$ of $\tilde{T}_l$-algebras is isomorphic to the category of $T_lT_r$-algebras. Consider the forgetful functor $\tilde{U}_l: (M^{T_r})^{\tilde{T}_l} \to M^{T_r}$ and its left adjoint $\tilde{F}_l$. The composite functor $\tilde{U}_l\tilde{F}_l$ is equal to $\tilde{T}_l$ while $\tilde{T}_l := \tilde{F}_l\tilde{U}_l$ is a comonad on $(M^{T_r})^{\tilde{T}_l} \cong M^{T_lT_r}$. Its comultiplication is given by $\tilde{F}_l\tilde{u}_l\tilde{U}_l: \tilde{F}_l\tilde{U}_l \to \tilde{F}_l\tilde{T}_l\tilde{U}_l = \tilde{F}_l\tilde{T}_l\tilde{F}_l\tilde{U}_l$ and the counit is $\tilde{\cdot}_l: \tilde{F}_l\tilde{U}_l(M, \tilde{\cdot}_l) = (\tilde{T}_lM, \tilde{m}_lM) \to (M, \tilde{\cdot}_l)$, for any $\tilde{T}_l$-algebra $(M, \tilde{\cdot}_l)$. Since the composite
functor $U, \tilde{U}_l$ differs from the forgetful functor $U : \mathcal{M}^{\tilde{T}_l \tilde{T}_r} \to \mathcal{M}$ by the isomorphism $(\mathcal{M}^{T_r})^{\tilde{T}_l} \cong \mathcal{M}^{\tilde{T}_l T_r}$, the comonad $\tilde{T}_l$ obeys the required properties.

Part (2) follows by applying the same reasoning as in part (1) to the distributive law $\Phi^{-1}$.

For any $T_l T_r$-algebra $(M, \varrho)$, the $T_l T_r$-actions on $\tilde{T}_l(M, \varrho)$ and $\tilde{T}_r(M, \varrho)$ are given by the respective morphisms $\tilde{\rho}_l : T_l T_r T_l M \to T_l M$ and $\tilde{\rho}_r : T_l T_r T_r M \to T_r M$, where

$$\tilde{\rho}_l := T_l \varrho \circ T_l u_{T_r} M \circ m_{T_r} M \circ T_l \Phi M \quad (6.1)$$

$$\tilde{\rho}_r := T_r \varrho \circ T_r u_{T_l} M \circ \Phi^{-1} M \circ \Phi T_l M. \quad (6.2)$$

(3) The composite monad $T_l T_r$ in (1.) induces a monad $\text{Cat}(\mathcal{M}, T_l T_r)$ on the category $\text{Cat}(\mathcal{M}, \mathcal{M})$ of functors $\mathcal{M} \to \mathcal{M}$. The natural transformation $\Phi^{-1}$ yields a morphism of its algebras $(T_l T_r, T_l m_r \circ m_{T_r} T_r \circ T_l \Phi T_r) \to (T_l T_r, T_r m_l \circ \Phi^{-1} T_l \circ T_l m_r T_l)$. Indeed, using string computation, we have

$$T_l T_r = T_l T_r T_l = T_l T_r = T_l T_r$$

where for the first and the third equations one uses the definition of distributive laws and the second relation follows by $\Phi^{-1} \circ \Phi = T_l T_l$. Comparing these actions with (6.1) and (6.2), this implies the existence of a natural transformation $\tilde{\Phi} : \tilde{T}_l \tilde{T}_r \to \tilde{T}_l \tilde{T}_r$ such that $U \tilde{\Phi} = \Phi^{-1} U$. Since the forgetful functor $U$ reflects isomorphisms, $\tilde{\Phi}$ is a natural isomorphism. Using that $U$ is faithful, it is easy to check that since $\Phi$ is a monad distributive law, $\tilde{\Phi}$ is a comonad distributive law. \(\square\)

For an object $(T_l, T_r, \Phi, \sqcap : \mathcal{M} \to \mathcal{C}, i, \sqcup : \mathcal{D} \to \mathcal{M}, w)$ of $\mathcal{A}_c^x$, consider the forgetful functor $U : \mathcal{M}^{T_l T_r} \to \mathcal{M}$ and the natural transformations $\xi_l : T_l U \to U$ and $\xi_r : T_r U \to U$, given for any $T_l T_r$-algebra $(M, \varrho)$ by the morphisms

$$\xi_l(M, \varrho) := \varrho \circ u_{T_r} M \quad \text{and} \quad \xi_r(M, \varrho) := \varrho \circ u_{T_l} M. \quad (6.3)$$

Since coequalizers in $\mathcal{C}$ exist by assumption, we can define a functor $\sqcap \cdot : \mathcal{M}^{T_l T_r} \to \mathcal{C}$ via the coequalizer

$$\sqcap T_l U \xrightarrow{\sqcap \xi_l} \sqcap U \xrightarrow{p} \sqcap \,$$

in the category of functors. For any $T_l T_r$-algebra $(M, \varrho)$, $p(M, \varrho) : \sqcap M \to \sqcap (M, \varrho)$ is the coequalizer of $\sqcap \varrho \circ u_{T_l} M$ and $\sqcap \varrho \circ u_{T_r} M \circ m M$. For any morphism $f : (M, \varrho) \to (M', \varrho')$ in $\mathcal{M}^{T_l T_r}$, the composite $p(M', \varrho') \circ \sqcap f$ coequalizes the parallel morphisms in (6.4) (evaluated at $(M, \varrho)$). Hence we can define $\sqcap f$ as the unique morphism for which $\sqcap f \circ p(M, \varrho) = p(M', \varrho') \circ \sqcap f$.

**Lemma 6.3.** Consider an object $(T_l, T_r, \Phi, \sqcap, i, \sqcup, w)$ of $\mathcal{A}_c^x$. For the forgetful functor $U : \mathcal{M}^{T_l T_r} \to \mathcal{M}$, the monads $\tilde{T}_l$ and $\tilde{T}_r$ in Lemma 6.2 and the functor $\sqcap \cdot$ in (6.4), there
are natural isomorphisms $\theta_l : \hat{T}_l \to \nabla U$ and $\theta_r : \hat{T}_r \to \nabla U$ such that
\[
\theta_l \circ p^T_l \circ i^{-1} U = \nabla \xi_r \quad \text{and} \quad \theta_r \circ p^T_r \circ i U = \nabla \xi_l.
\]

Proof. By definition, $\hat{T}_r$ is the coequalizer of the natural transformations $\nabla T_r \xi_l \circ \nabla \Phi^{-1} U$ and $\nabla m_r U \circ i T_r U$. Since $\Phi$ is an isomorphism, this is equivalent to the coequalizer of $\nabla T_r \xi_l$ and $\nabla m_r U \circ i T_r U \circ \nabla \Phi U = i U \circ \nabla m_r U \circ i^{-1} T_r U$, where we used that $(\nabla, i)$ is a right $\Phi$-module functor, cf. first condition in (6.3). Using that $i$ is an isomorphism, we conclude that $\hat{T}_r$ is the coequalizer of $i^{-1} U \circ \nabla T_r \xi_l \circ i T_r U = \nabla T_l \xi_l$ and $\nabla m_l U$. The coequalizer
\[
\begin{array}{ccc}
T_l T_l U & \xrightarrow{T_l \xi_l} & T_l U \\
\downarrow m_l & & \downarrow \xi_l \\
& U & 
\end{array}
\]
is split, as $(M, \xi_l(M, \varrho))$ is a $T_l$-algebra for any $T_l T_r$-module $(M, \varrho)$. Hence it is preserved by composing with $\nabla$ on the left. Thus, by the universal property of coequalizers, there is a unique natural isomorphism $\theta_r : \hat{T}_r \to \nabla U$ such that $\theta_r \circ p^T_r \circ i U = \nabla \xi_l$. The existence of the isomorphism $\theta_l : \hat{T}_l \to \nabla U$ is proven similarly. $\square$

Lemma 6.4. Consider an object $(T_l, T_r, \Phi, \nabla, i, \sqcup, w)$ of $A_{c}^\times$. Then $T_r \sqcup$ is a left module functor for the composite monad $T_l T_r$ in (6.1). Hence there is a functor $\hat{\sqcup} : D \to M^{T_l T_r}$, with object map and morphism map
\[
\hat{\sqcup} Y := (T_r \sqcup Y, w^{-1} Y \circ m_l \sqcup Y \circ T_l w Y \circ T_l m_r \sqcup Y) \quad \text{and} \quad \hat{\sqcup} f := T_r \sqcup f.
\]

Proof. Unitality of the given $T_l T_r$-action is immediate by unitality of the multiplications of $T_l$ and $T_r$. Associativity is checked as follows.

The first and the fourth equalities follow by using that $(\sqcup, w)$ is a left $\Phi$-module functor, cf. first identity in (6.4). The second equality follows by associativity of $m_r$. Naturality is used in the third equality, and also in the last two ones, in the case of the last equality together with the associativity of $m_l$ and of $m_r$. The fifth equality is a consequence of the fact that $\Phi$ is a distributive law. $\square$

Proof of Theorem 6.1. Let $(T_l, T_r, \Phi, \nabla, i, \sqcup, w)$ be an object in $A_{c}^\times$, with $\nabla : M \to C$ and $\sqcup : D \to M$. First we show that the septuple $(\hat{T}_l, \hat{T}_r, \hat{\Phi}, \hat{\nabla}, \hat{i}, \hat{\sqcup}, \hat{w})$ is an object of $B$, where $U : M^{T_l T_r} \to M$ is the forgetful functor, the comonads $\hat{T}_l$ and $\hat{T}_r$ and the comonad distributive law $\hat{\Phi}$ are constructed in Lemma 6.2 the functor $\hat{\nabla}$ is defined by (6.4), the functor $\hat{\sqcup}$ is constructed in Lemma 6.4 while $\hat{i}$ and $\hat{w}$ are given by the relations
\[
\hat{i} := \theta_l^{-1} \circ \theta_r \quad \text{and} \quad U \hat{w} := T_r w^{-1} \circ \Phi^{-1} \sqcup.
\]
(6.5)
The following string computation proves that $\hat{w}Y : \overline{T}_I Y \to \overline{T}_r Y$ is a morphism in $\mathcal{M}^{T_j T_r}$, for any object $Y$ in $\mathcal{D}$.

Here we used the form of the $T_i T_r$-actions on $\overline{T}_I(M, p)$, $\overline{T}_r(M, p)$ and $\overline{Y}$, for any $T_i T_r$-algebra $(M, p)$ and for any object $Y$ of $\mathcal{D}$; see relations (6.1), (6.2) and Lemma 6.4. For the first, second and third equalities we also used that $\hat{\Phi}$ is a distributive law and that it holds true if and only if the outer square in $\mathcal{D}$.

By naturality of $p$ in (6.4) and by Lemma (6.2)

$$p\hat{T}_r^2 \circ \cap T_r u U = \overline{\cap d_r} \circ p\hat{T}_r, \quad p\hat{T}_r \circ \cap T_i u U = \overline{\cap d_l} \circ p\hat{T}_l, \quad p\hat{T}_r \hat{T}_r \circ \cap \hat{\Phi}^{-1} U = \overline{\cap \hat{\Phi}} \circ p\hat{T}_r \hat{T}_r.$$

Using Lemma (6.3) (in the first and third equalities), the second relation in (1.3) on $i$ and unitarity of $\xi_r$ (in the second equality) and the first equality in (6.5) defining $\hat{\iota}$ (in the fourth equality), we see that

$$p\hat{T}_r \circ \cap u U \circ \cap \xi_l \circ i^{-1} U = \theta_i^{-1} \circ \cap \xi_r \circ i U \circ \cap u U \circ \cap \xi_l \circ i^{-1} U$$

$$= \theta_i^{-1} \circ \cap \xi_l \circ i^{-1} U = \theta_i^{-1} \circ \theta_r \circ p\hat{T}_r = \hat{\iota} \circ p\hat{T}_r. \quad (6.6)$$

The first condition in (4.1) for $(\overline{\cap}, \hat{\iota})$ can be expressed as commutativity of the inner square in the diagram below. Since $p\hat{T}_r$ is a natural epimorphism, the above considerations imply that it holds true if and only if the outer square in
commutes, what follows by the compatibility of $\Phi$ and $i$ with the units $u_l$ and $u_r$ of both monads and unitality of the multiplication $m_l$, cf. the following string computation.

Similarly, by $(6.6)$, by naturality of $p$ and Lemma 6.2 the second condition in $(4.1)$ for $(\tilde{\Pi}, \tilde{i})$, i.e. the identity $\tilde{\Pi}e_l \circ \tilde{i} = \tilde{\Pi}e_r$, holds true if and only of the outer square in $(6.4)$.

Next we prove that $(\tilde{\square}, \tilde{w})$ satisfies the conditions in $(4.1)$. Since $U$ is faithful, applying it to the first relation in $(4.1)$ we obtain an equivalent condition. In view of Lemma 6.2 and the construction of $\tilde{w}$ via the second equality in $(6.3)$, it takes the form

$$T_r^2 w^{-1} \circ T_r \Phi^{-1} \sqcup \Phi^{-1} T_r \sqcup \circ T_l T_r w^{-1} \circ T_l \Phi^{-1} \sqcup \circ T_l u_l T_r \sqcup = T_l u_l T_r \sqcup \circ T_r w^{-1} \circ \Phi^{-1} \sqcup.$$ 

This holds true by the computation below, where we use the compatibility between the unit of $T_l$ with $\Phi$ and $w$, and the fact that $u_l$ is a natural transformation.

The second relation in $(4.1)$ may be proved analogously, showing that it is equivalent to the fact that $w$ is compatible with the multiplications of $T_l$ and $T_r$. In conclusion, $(\tilde{T_l}, \tilde{T_r}, \tilde{\Phi}, \tilde{\Pi}, \tilde{i}, \tilde{w})$ is an object of $\mathcal{B}$.

Next we define the functor $(\tilde{\cdot})$ on morphisms. Let $(G, q_l, q_r, \land, \pi, \lor, \omega)$ be a morphism in $\mathcal{A}^{\times}_L$. Its image in $\mathcal{B}^{\times}$ under $(\tilde{\cdot})$ will be denoted by $(\tilde{G}, \tilde{q}_l, \tilde{q}_r, \land, \tilde{\pi}, \lor, \tilde{\omega})$. The functor $\tilde{G} : \mathcal{M}^{T_l T_r'} \rightarrow \mathcal{M}^{T_l T_r}$ is defined, for $(M, \rho)$ in $\mathcal{M}^{T_l T_r'}$, by $\tilde{G}(M, \rho) = (GM, \tilde{\rho})$, where

$$\tilde{\rho} = G \rho \circ q_l T_r' M \circ T_l q_r M.$$ 

(6.7)
On a morphism $f$ in $\mathcal{M}', \mathcal{T}' \mathcal{M}'$, we put $\widehat{G} f = G f$. In fact, $\widehat{G}$ is the lifting of $G$ (in the sense that $U \widehat{G} = G U'$), which is induced by the monad morphism $q_r T_r' \circ T_l q_r$, cf. Lemma [1.4] and [14] Lemma 1. Since

$$
\widehat{T}_l \widehat{G}(M, \rho) = (T_l GM, T_l G \rho \circ T_l q_T M \circ T_l q_f M \circ T_l u_l T_r G M \circ m_l T_r GM \circ T_f \Phi GM),
$$

$$
\widehat{G} T_l'(M, \rho) = (G T_l' M, G T_l' G \rho \circ G T_l' u_l' T_r' M \circ G m_l T_r' M \circ G T_l' \Phi' M \circ q_l T_l' T_r' M \circ T_q q_r T_l' M),
$$

we can check easily that $q_l M : T_l GM \to GT_l M$ is a morphism of $T_l T_r$-algebras with respect to the above actions. Hence one may define $\widehat{q}_l : \widehat{T}_l \widehat{G} \to \widehat{G} T_l'$ by $U \widehat{q}_l (M, \rho) := q_l M$ and, proceeding similarly, one may take $\widehat{q}_r : \widehat{T}_r \widehat{G} \to \widehat{G} T_r'$ to be defined by $U \widehat{q}_r (M, \rho) := q_r M$.

The left square in the following diagram is commutative by naturality, if choosing the upper ones of the parallel arrows. It is commutative also choosing the lower ones of the parallel arrows, by (135) and naturality. Therefore, universality of the coequalizer in the top row implies the existence of a unique natural transformation $\widehat{\pi}$, rendering commutative the diagram.

![Diagram](https://example.com/diagram.png)

Finally, we put $U \widehat{\omega} := q_r T_r' \circ U \omega$. We need to show that this defines indeed a natural transformation $\widehat{\omega} : \widehat{\Pi} \to \widehat{G} \Pi'$; that is, that $q_r T_r' \circ U \omega$ commutes with the $T_l T_r$-actions on $U \widehat{\Pi} = T_r \Pi$ and $U \widehat{G} \Pi' = GT_r \Pi'$. For a diagrammatic proof see the following computation, where for the first and third equality one uses that $(G, q_l)$ and $(G, q_r)$ are morphisms of monads. The second and the fourth relations follow by (160). In the first three equalities we also use that all maps are natural transformations.

![Diagram](https://example.com/diagram.png)

We turn to proving that the so constructed datum $(\widehat{G}, \widehat{q}_l, \widehat{q}_r, \wedge, \widehat{\pi}, \forall, \widehat{\omega})$ yields a morphism in $\mathcal{B}$. In order to check that $(G, \widehat{q}_r)$ is a comonad morphism, it suffices to show that

$$U \widehat{q}_l T_l' \circ U T_l' \widehat{q}_r \circ U d_r \widehat{G} = U \widehat{G} d_r' \circ U \widehat{q}_r \quad \text{and} \quad U e_r \widehat{G} = U \widehat{G} e_r' \circ U \widehat{q}_r,
$$

since the forgetful functor $U$ is faithful. By Lemma (6.2) the first condition is equivalent to

$$q_r T_r' U' \circ T_r q_r U' \circ T_r u_r GU' = G T_r' u_r' U' \circ q_r U',$$
which holds true since \((G, q_r)\) is a monad morphism and by naturality. The second condition holds true by construction of the functor \(\tilde{G}\) (cf. (6.7)) and the relations in Lemma 5.2 on \(e_r\) and \(e'_r\). Symmetrically, also \((G, \hat{q}_i)\) is a comonad morphism. By faithfulness of \(U\), the first condition in (1.2) is equivalent to

\[ q_r T'_l U' \circ T_r q_i U' \circ \Phi^{-1} GU'' = G \Phi^{-1} U' \circ q_i T'_l U' \circ T_i q_r U', \]

which holds true by (1.2). The second condition in (1.2) is equivalent to commutativity of the inner square in the following diagram. Since \(p \tilde{T}_l \tilde{G}\) is a natural epimorphism, it follows by the constructions of the morphisms \(%_r, \hat{q}_l\) and \(\hat{\pi}\) any by the equality (6.6) that the second condition in (1.2) holds true if and only if the outer square in

commutes. This follows by the compatibility of \(q_i\) with \(u_l\), naturality and (1.5). Finally, by faithfulness of \(U\), (1.3) is equivalent to

\[ G T'_r w^{-1} \circ G \Phi^{-1} \circ q_i T'_r \bigcup \circ T_r q_r \bigcup \circ T_i \bigcup \omega = q_r T'_r \bigcup \circ T_r q_r \bigcup \circ T_r \bigcup \omega \circ T_i w^{-1} \bigcup \bigcup \bigcup. \]

This holds true by naturality, (1.2) and (1.6). This finishes the construction of the functor \((\hat{\pi})\). It is straightforward to see that it is a functor indeed, i.e. it preserves identity morphisms and composition.

It remains to construct a natural isomorphism \(\tau : Z^*_\chi (\hat{\pi}) \rightarrow Z^{*\chi} (\hat{\pi})\). For a given object \((T_l, T_r, \Phi, \cap, i, \cup, w)\) in \(A^\chi_c\), let \((\hat{T}_l, \hat{T}_r, \hat{\Phi}, \hat{\cap}, \hat{i}, \hat{\cup}, \hat{w})\) denote the object in \(B^\chi\), constructed as in the first part of the proof. The para-cyclic object \(\hat{Z}^*_\chi := Z^*_\chi (\hat{T}_l, \hat{T}_r, \hat{\Phi}, \hat{\cap}, \hat{i}, \hat{\cup}, \hat{w})\), associated to the latter object as in Theorem 4.7, is given, at any non-negative degree \(n\), by the functor \(\hat{n} \tilde{T}_l \hat{n} \cup\). The para-cocyclic object \(Z^{*\chi} := Z^{*\chi} (T_l, T_r, \Phi, \cap, i, \cup, w)\) in Theorem 2.3 (and thus also its cyclic dual) is given at degree \(n\) by \(\cap T_i^{n+1} \cup\). The desired natural transformation \(\tau_n\) is defined as the composition of the following two morphisms

\[
\hat{n} \tilde{T}_l \hat{n} \cup \circ \tilde{T}_l \tilde{T}_l^{n+1} \cup \circ \tilde{T}_l^{n+1} \cup = \tilde{T}_l^{n+1} \cup \tilde{T}_l \cap \tilde{T}_l^{-n} w \circ \tilde{T}_l^{-n} w \tilde{T}_l^{n+1} \cup . \quad (6.8)
\]

Clearly \(\tau_n\) is a natural isomorphism. We claim that \(\tau_n\) is also an isomorphism of para-cyclic objects between \(\hat{Z}^*_\chi\) and the cyclic dual \(\hat{Z}^{*\chi}\) of \(Z^{*\chi}\). Applying Connes’s cyclic duality
functor (in the form it can be found in [13]) to $Z^\times$, for every degree $n$ the para-cyclic morphism $t_n : \cap T_i^{n+1} \cap \rightarrow \cap T_i^n \cap \cap T_i^w$ of $Z^\times$ comes out as

$$t_n = i^{-1}T_i^n \cap \cap \Phi^{-1}T_i^{n-1} \cap \cap \cdots \cap \cap T_i^{n-1} \cap \Phi^{-1} \cap \cap T_i^w \cap.$$  

On the other hand, by Theorem 4.7, the para-cyclic morphism $\hat{t}_n : \cap \hat{T}_i^{n+1} \cap \rightarrow \cap \hat{T}_i^n \cap \cap \hat{T}_i^w$, of $\hat{Z}^\times$ is given by

$$\hat{t}_n = i\hat{T}_i^n \cap \cap \phi \hat{T}_i^{n-1} \cap \cap \phi \hat{T}_i^{n-2} \cap \cap \phi \hat{T}_i^{n-3} \cap \Phi \cap \cap \hat{T}_i^w,$$

for every $n$. Since $p\hat{T}_i^{n+1}$ is an epimorphism, $\tau_n$ commutes with these para-cyclic morphisms if and only if

$$\hat{t}_n \circ \tau_n \circ p\hat{T}_i^{n+1} = \tau_n \circ \hat{t}_n \circ p\hat{T}_i^{n+1}, \quad (6.9)$$

For the proof of this relation, using string computations, see the following sequence of equations.

Taking into account the formulae that give $\hat{t}_n$ and $\tau_n$, on one hand, and the defining relation of $\theta_i$ in Lemma 6.3 together with the definition of $\xi_r$ in (6.3), on the other hand, one can see that the first diagram represents the left hand side of (6.9). In view of the definition of the maps $\hat{t}_n$, $\hat{i}_n$, $\hat{\omega}_n$, $\hat{\Phi}$ and $\theta_r$, the last diagram corresponds to the right hand side of (6.9). To deduce the first equality we use $n$ times the compatibility relation between the multiplication of $T_r$ and $\Phi$. The second relation is a consequence of the fact that the string and the stripe may be unknotted, by using repeatedly the relation $\Phi^{-1} \circ \Phi = T_r \cap T_i$. For the third equation one uses that $(\cap, i)$ is right $\Phi$-module functor, i.e. the first condition in (13). The last relation follows by using that $w$ is natural.

For $0 \leq k \leq n$, let $d_k := \cap \hat{T}_i^k \cap$ be the face maps of $\hat{Z}^\times$. For $0 < k \leq n$, the face maps of $\hat{Z}^\times$ are $d_k := \cap T_i^{k-1} \cap T_i^{-k} \cap$ while $d_0 := \cap T_i^{n-1} \cap \cap \hat{T}_i^w \cap$. We prove next that the operators $\tau_n$ are compatible with these face maps, i.e. for $0 \leq k \leq n$,

$$\hat{d}_k \circ \tau_n \circ p\hat{T}_i^{n+1} = \tau_{n-1} \circ \hat{d}_k \circ p\hat{T}_i^{n+1}, \quad (6.10)$$
If \( 1 \leq k \leq n \), then proceeding as in the case of the para-cyclic operator, one can see that the string representation of (6.10) is the following.

\[
\begin{array}{c}
\cap T_l T_r \downarrow \cap T_l T_r \downarrow \cap T_l T_r \downarrow \\
\cap T_l T_r \downarrow \cap T_l T_r \downarrow \cap T_l T_r \downarrow \\
\cap T_l T_r \downarrow \cap T_l T_r \downarrow \cap T_l T_r \downarrow 
\end{array}
\]

Visibly this equality holds, as the multiplication of a monad is a natural transformation and \( \Phi \) is a distributive law. Finally, by the definition of para-cyclic objects, \( \hat{d}_0 = \hat{d}_n \circ \hat{r}_n^{-1} \) and \( \hat{d}_0 = \hat{d}_n \circ \hat{r}_n^{-1} \). Thus, in view of relation (6.9) and of the fact that (6.10) holds for \( k = n \), we conclude that (6.10) also holds for \( k = 0 \). One proves in a similar way that the operators \( \tau_* \) are compatible with the codegeneracy maps.

Our final task is to prove naturality of this isomorphism \( \tau \). For an arbitrary morphism

\[
\zeta := (G, q_t, q_r, \wedge, \pi, \vee, \omega) : (T_l, T_r, \Phi, \cap, i, \cup, w) \rightarrow (T_l', T_r', \Phi', \cap', i', \cup', w')
\]

in \( \mathcal{A}_c^x \), let \( \hat{\zeta} := (\hat{G}, \hat{q}_t, \hat{q}_r, \wedge, \hat{\pi}, \vee, \hat{\omega}) \) denote the corresponding morphism in \( \mathcal{B} \). Since \( p \) is a natural epimorphism, it is sufficient to prove that

\[
\tau_*' \circ \hat{\zeta}_* \circ p\hat{T}_l^{n+1} \square \vee = \hat{\zeta}_* \circ \tau_* \circ p\hat{T}_l^{n+1} \square \vee,
\]

(6.11)

where \( \hat{\zeta}_* := Z^x(\hat{\zeta}) \) (cf. Theorem 4.7), \( \hat{\zeta}_* := Z^x(\zeta) \) (cf. Theorem 2.3); and \( \tau_* \) and \( \tau_*' \) are morphisms constructed as in (6.8), corresponding to the domain and the codomain of \( \zeta \), respectively. The string representations of both sides of (6.11) are given in the first and last terms of the following sequence of equations.

\[
\begin{array}{c}
\cap T_l T_r \downarrow \cap T_l T_r \downarrow \cap T_l T_r \downarrow \\
\cap T_l T_r \downarrow \cap T_l T_r \downarrow \cap T_l T_r \downarrow \\
\cap T_l T_r \downarrow \cap T_l T_r \downarrow \cap T_l T_r \downarrow 
\end{array}
\]

Recall that \( q_t^n : T_l^n G \rightarrow GT_l^n \) is defined inductively by \( q_t^1 = q_t \) and \( q_t^{n+1} := q_t T_l^n \circ T_l q_t^n \).

The first identity follows by applying (1.5), while the second one is obtained by using repeatedly (1.2) and naturality. The third equation follows as \( q_1 \) is a morphism of monads and the last equality is a consequence of (1.6).

The statements about the functor \( (-) : \mathcal{B}_c^x \rightarrow \mathcal{A}_c^x \) are proven symmetrically. \( \Box \)
7. Back to the examples

In this section we return to the examples in Section 6 and Section 5. We show that they all belong to \( \mathcal{A}_c^\times \) and \( \mathcal{B}_c^\times \), respectively, whenever the occurring bialgebroids correspond to a Hopf algebroid with a bijective antipode and the occurring \((\text{co(ntra)})\)modules are \((\text{co(ntra)})\)modules of the Hopf algebroid (cf. Appendix). In light of Theorem 6.1 in this case we may apply the functor \( \sim \) to any of these examples. The examples in Section 6 and Section 5 turn out to be pairwise related via the functor \( \sim \).

Example 7.1. Let \( H \) be a Hopf algebroid over base algebras \( L \) and \( R \), with a bijective antipode \( S \), and \( A \) be a left \( H \)-module algebra. This means that \( A \) is a left module algebra of the constituent left bialgebroid \( H_L \), so there is a corresponding object \((T_i, T_r, \Phi, \sqcap, i, \sqcup, w)\) of \( \mathcal{A} \) in Example 3.1. Obviously, using the forgetful functor \( F : H-\text{Comod} \to H_L-\text{Comod} \), we can construct another object
\[
(T_i, T_r, \Phi, \sqcap, i, \sqcup, F, wF)
\]
of \( \mathcal{A} \). We claim that (7.1) belongs to \( \mathcal{A}_c^\times \) (hence to \( \mathcal{A}_c^\times \), since \( \text{Mod-}k \) possesses coequalizers). Indeed, \( \Phi \) and \( i \) are obviously isomorphisms. The inverse of \( w_{FM} \) is given, for a left \( H \)-comodule \( M \) with coaction \( m \mapsto m^{-1} \otimes_R m^0 \) of the constituent right bialgebroid, as
\[
A \otimes_L M \to M \otimes_L A, \quad a \otimes_L m \mapsto m^{0} \otimes_L S^{-1}(m^{-1})a.
\]
Thus we can apply to (7.1) the functor \( \sim \). The resulting object of \( \mathcal{B}_c^\times \) can be obtained from the object in Example 5.5 by composing on the right the left comodule functor \( \sqcup : \text{Comod-}H_L \to A-\text{Mod-}A \) in Example 5.5 with the forgetful functor \( \text{Comod-}H \to \text{Comod-}H_L \) and with the isomorphism \( I_S : H-\text{Comod} \to \text{Comod-}H \), induced by the bijective antipode \( S \), cf. A.10.

Example 7.2. Let \( H \) be a Hopf algebroid over base algebras \( L \) and \( R \), with a bijective antipode \( S \), and \( A \) be a right \( H \)-comodule algebra. Then \( A \) is in particular a right comodule algebra of the constituent right bialgebroid. Hence there is a corresponding object of \( \mathcal{A} \) as in Example 3.2. We claim that it belongs to \( \mathcal{A}_c^\times \) (hence to \( \mathcal{A}_c^\times \)). Indeed, \( \Phi \) and \( i \) are obviously isomorphisms. The inverse of \( w_N \) is given, for any right \( H \)-module \( N \), by
\[
A \otimes_R N \to N \otimes_R A, \quad a \otimes_R m \mapsto mS^{-1}(a_{[1]}) \otimes_R a_{[0]},
\]
where \( a \mapsto a_{[0]} \otimes_L a_{[1]} \) denotes the coaction of the constituent left bialgebroid. Thus we can apply the functor \( \sim \). The resulting object of \( \mathcal{B}_c^\times \) can be obtained from the object in Example 5.6 by composing on the right the left comodule functor \( \sqcup : \text{H-Mod} \equiv H_L-\text{Mod} \to A-\text{Mod-}A \) in Example 5.6 with the isomorphism \( I_{S^{-1}} : \text{Mod-}H \to \text{H-Mod} \), induced by the bijective antipode \( S \), cf. A.9.

Example 7.3. Let \( H \) be a Hopf algebroid over base algebras \( L \) and \( R \), with a bijective antipode \( S \), and \( C \) be a right \( H \)-module coring. This means that \( C \) is a right module coring of the constituent right bialgebroid \( H_R \), so there is a corresponding object \((T_i, T_r, \Phi, \sqcap, i, \sqcup, w)\) of \( \mathcal{A} \) in Example 3.3. Using the forgetful functor \( F : H-\text{Ctrmod} \to H_R-\text{Ctrmod} \), we can construct another object \((T_i, T_r, \Phi, \sqcap, i, \sqcup, F, wF)\) of \( \mathcal{A} \). We claim that the modified object belongs to \( \mathcal{A}_c^\times \) (hence to \( \mathcal{A}_c^\times \)). Indeed, \( \Phi \) and \( i \) are obviously isomorphisms. The inverse of \( w_{FQ} \) is given, for a left \( H \)-contramodule \( Q \) with structure maps
\( \alpha_L : \text{Hom}_{L,-}(H, Q) \rightarrow Q \) and \( \alpha_R : \text{Hom}_{R,-}(H, Q) \rightarrow Q \), by

\[
\text{Hom}_{-R}(C, Q) \rightarrow \text{Hom}_{R,-}(C, Q), \quad f \mapsto (c \mapsto \alpha_L(f(cS^{-1}(-)))).
\]

Thus we can apply the functor \( \simeq \). The resulting object of \( B^X \) can be obtained from the object in Example 5.7 by composing on the right the left comodule functor \( \sqcup : \text{Ctrmod}-H_R \rightarrow C-\text{Ctrmod}-C \) in Example 5.7 with the forgetful functor \( \text{Ctrmod}-H \rightarrow \text{Ctrmod}-H_R \) and with the isomorphism \( I_S : H-\text{Ctrmod} \rightarrow \text{Ctrmod}-H \), induced by the bijective antipode \( S \), cf. A.11.

**Example 7.4.** Let \( H \) be a Hopf algebroid over base algebras \( L \) and \( R \), with a bijective antipode \( S \), and \( C \) be a left \( H \)-comodule coring. Then \( C \) is in particular a left comodule coring of the constituent left bialgebroid. Hence there is a corresponding object of \( A \) as in Example 3.4. We claim that it belongs to \( A^X \) (hence to \( A^X_\simeq \)). Indeed, \( \Phi \) and \( i \) are obviously isomorphisms. The inverse of \( w_N \) is given, for any right \( H \)-module \( N \), by

\[
\text{Hom}_{-L}(C, N) \rightarrow \text{Hom}_{-L,L}(C, N), \quad f \mapsto (c \mapsto f(c[0]S^{-1}(c[-1]))),
\]

where \( c \mapsto c[-1] \otimes_R c[0] \) denotes the coaction of the constituent right bialgebroid. Thus we can apply the functor \( \simeq \). The resulting object of \( B^X \) can be obtained from the object in Example 5.8 by composing on the right the left comodule functor \( \sqcup : \text{Mod}-H \equiv H_R^R \rightarrow C-\text{Comod}-C \) in Example 5.8 with the isomorphism \( I^{-1}_S : \text{Mod}-H \rightarrow H-\text{Mod} \), induced by the bijective antipode \( S \), cf. A.9.

**Example 7.5.** Let \( H \) be a Hopf algebroid over base algebras \( L \) and \( R \), with a bijective antipode \( S \), and \( C \) be a left \( H \)-comodule coring. Then \( C \) is in particular a left comodule coring of the constituent left bialgebroid. Hence there is a corresponding object of \( B \) as in Example 5.1. We claim that it belongs to \( B^X \) (hence to \( B^X_\simeq \)). Indeed, \( \Phi \) and \( i \) are obviously isomorphisms. The inverse of \( w_N \) is given, for any left \( H \)-module \( N \), by

\[
N \otimes_L C \rightarrow C \otimes_L N, \quad m \otimes_L c \mapsto c[0] \otimes_L S^{-1}(c[-1])m,
\]

where \( c \mapsto c[-1] \otimes_R c[0] \) denotes the coaction of the constituent right bialgebroid. Thus we can apply the functor \( \simeq \). The resulting object of \( A^X \) can be obtained from the object in Example 5.5 by composing on the right the left comodule functor \( \sqcup : \text{Mod}-H \equiv \text{Mod}-H_R \rightarrow C-\text{Comod}-C \) in Example 5.5 with the isomorphism \( I^{-1}_S : \text{Mod}-H \rightarrow H-\text{Mod} \), induced by the bijective antipode \( S \), cf. A.9.

**Example 7.6.** Let \( H \) be a Hopf algebroid over base algebras \( L \) and \( R \), with a bijective antipode \( S \), and \( C \) be a right \( H \)-module coring. This means that \( C \) is a right module coring of the constituent right bialgebroid \( H_R \), so there is a corresponding object \( (T, T_r, \Phi, \sqcap, i, \sqcup, w) \) of \( B \) in Example 5.2. Using the forgetful functor \( F : \text{Comod}-H \rightarrow \text{Comod}-H_R \), we can construct another object \( (T, T_r, \Phi, \sqcap, i, \sqcup F, wF) \) of \( B \). We claim that the modified object belongs to \( B^X \) (hence to \( B^X_\simeq \)). Indeed, \( \Phi \) and \( i \) are obviously isomorphisms. The inverse of \( w_{FM} \) is given, for a right \( H \)-comodule \( M \) with coaction \( m \mapsto m[0] \otimes_L m[1] \) of the constituent left bialgebroid, by

\[
M \otimes_R C \rightarrow C \otimes_R M, \quad m \otimes_R c \mapsto cS^{-1}(m[1]) \otimes_R m[0].
\]

Thus we can apply the functor \( \simeq \). The resulting object of \( A^X \) can be obtained from the object in Example 5.6 by composing on the right the left comodule functor \( \sqcup : H_R^R \rightarrow \text{Comod} \rightarrow C-\text{Comod}-C \) in Example 5.6 with the forgetful functor \( H-\text{Comod} \rightarrow H_R^R-\text{Comod} \).
and with the isomorphism \( I_{S}^{-1} : \text{Comod}-H \rightarrow H-\text{Comod} \), induced by the bijective antipode \( S \), cf. [A.10].

**Example 7.7.** Let \( H \) be a Hopf algebroid over base algebras \( L \) and \( R \), with a bijective antipode \( S \), and \( A \) be a left \( H \)-module algebra. This means that \( A \) is a left module algebra of the constituent left bialgebroid \( H_{L} \), so there is a corresponding object \((T_{L}, T_{r}, \Phi, i, \square, w)\) of \( \mathcal{B} \) in Example 5.3. Using the forgetful functor \( F : \text{Ctrmod}-H \rightarrow \text{Ctrmod}-H_{L} \), we can construct another object \((T_{L}, T_{r}, \Phi, i, \square, F, wF)\) of \( \mathcal{B} \). We claim that the modified object belongs to \( \mathcal{B}^{\times} \) (hence to \( \mathcal{B}_{\varepsilon}^{\times} \)). Indeed, \( \Phi \) and \( i \) are obviously isomorphisms. The inverse of \( w_{F} \) is given, for a right \( H \)-contramodule \((Q, \alpha_{L}, \alpha_{R})\) by

\[
\text{Hom}_{L,-}(A, Q) \rightarrow \text{Hom}_{-L}(A, Q), \quad g \mapsto (a \mapsto \alpha_{R}(g(S^{-1}(-a)))).
\]

Thus we can apply the functor \( ^{\wedge} \). The resulting object of \( \mathcal{A}^{\times} \) can be obtained from the object in Example 3.7 by composing on the right the left comodule functor \( \Box : H_{L}-\text{Ctrmod} \rightarrow A-\text{Mod}-A \) in Example 3.7 with the forgetful functor \( H-\text{Ctrmod} \rightarrow H_{L}-\text{Ctrmod} \) and with the isomorphism \( I_{S}^{-1} : \text{Ctrmod}-H \rightarrow H-\text{Ctrmod} \), induced by the bijective antipode \( S \), cf. [A.11].

**Example 7.8.** Let \( H \) be a Hopf algebroid over base algebras \( L \) and \( R \), with a bijective antipode \( S \), and \( A \) be a right \( H \)-comodule algebra. Then \( A \) is in particular a right comodule algebra of the constituent right bialgebroid. Hence there is a corresponding object of \( \mathcal{B} \) as in Example 5.3. We claim that it belongs to \( \mathcal{B}^{\times} \) (hence to \( \mathcal{B}_{\varepsilon}^{\times} \)). Indeed, \( \Phi \) and \( i \) are obviously isomorphisms. The inverse of \( w_{N} \) is given, for any left \( H \)-module \( N \), by

\[
\text{Hom}_{R,-}(A, N) \rightarrow \text{Hom}_{-R}(A, N), \quad g \mapsto (a \mapsto S^{-1}(a_{[1]})g(a_{[0]})),
\]

where \( a \mapsto a_{[0]} \otimes_{L} a_{[1]} \) denotes the coaction of the constituent left bialgebroid. Thus we can apply the functor \( ^{\wedge} \). The resulting object of \( \mathcal{A}^{\times} \) can be obtained from the object in Example 3.8 by composing on the right the left comodule functor \( \Box : \text{Mod}-H \equiv \text{Mod}-H_{L} \rightarrow A-\text{Mod}-A \) in Example 3.8 with the isomorphism \( I_{S} : H-\text{Mod} \rightarrow \text{Mod}-H \), induced by the bijective antipode \( S \), cf. [A.9].

**Appendix A. Modules, comodules and contramodules of Hopf algebroids**

In this appendix we shortly review algebraic structures over non-commutative base algebras, which are used to construct the examples in the paper. For more information on them we refer to [2]. Structures as \( R \)-rings, \( R \)-corings, bialgebroids and Hopf algebroids below, generalize the notions of an algebra, a coalgebra, a bialgebra and a Hopf algebra over a commutative ring, respectively.

Throughout, let \( k \) be a commutative, associative and unital ring. By an *algebra* \( R \) we mean an associative and unital algebra over \( k \). The *enveloping algebra* \( R \otimes_{k} R^{\text{op}} \) is denoted by \( R^{e} \). We tacitly identify \( R^{e} \)-modules with \( R \)-bimodules.

**A.1.** An *\( R \)-ring* is a monoid in the monoidal category of \( R \)-bimodules. In fact, an \( R \)-ring \( A \) is equivalent to a \( k \)-algebra \( A \), together with an algebra map \( \iota : R \rightarrow A \). Denoting the multiplication in an \( R \)-ring \( A \) by \( \mu : A \otimes_{R} A \rightarrow A \), there is an induced monad

\[
((-) \otimes_{R} A, (-) \otimes_{R} \mu, (-) \otimes_{R} \iota)
\]
on the category \(\text{Mod}-R\) of right \(R\)-modules. Algebras of this monad are equivalent to right modules of the \(k\)-algebra \(A\). Symmetrically, algebras for the monad

\[
(A \otimes_R (-), \mu \otimes_R (-), t \otimes_R (-)) \tag{A.2}
\]
on the category \(R\)-\text{Mod} of left \(R\)-modules are equivalent to left modules of the \(k\)-algebra \(A\). Note that the same formulæ (A.1) and (A.2) define monads also on the category \(R\)-\text{-comod}\ of \(R\)-bimodules, with respect to the \(R\)-actions

\[
r(p \otimes_R a)r' = rp \otimes_R ar' \quad \text{and} \quad r(a \otimes_R p)r' = ra \otimes_R pr', \tag{A.3}
\]
for \(r, r' \in R, a \in A, p \in P\) and any \(R\)-bimodule \(P\).

A.2. For our considerations, \(R^e\)-rings are of special interest. Note that an \(R^e\)-ring is equivalent to an algebra \(B\), together with algebra maps \(s : R \to B\) and \(t : R^{\text{op}} \to B\), such that \(s(r)t(r') = t(r')s(r), \) for all \(r, r' \in R\). The maps \(s\) and \(t\) are known as the source and target maps, respectively. An immediate example of an \(R^e\)-ring is the algebra \(\text{End}_k(R)\) of \(k\)-linear endomorphisms of \(R\). It is an algebra via composition of endomorphisms and source and target maps are

\[
R \to \text{End}_k(R), \quad r \mapsto r(-) \quad R^{\text{op}} \to \text{End}_k(R), \quad r \mapsto (-)r.
\]

Any \(R^e\)-ring \(B\) carries four commuting \(R\)-actions:

\[
r \triangleright b = s(r)b, \quad b \triangleright r = t(r)b, \quad r \triangleright b = bt(r), \quad b \triangleleft r = bs(r).
\]
In terms of these actions, the following construction can be performed. Take first the \(R\)-module tensor product

\[
B \otimes_R B := B \otimes_k B/\{ b \triangleleft r - r \triangleright b, \quad \forall \ b \in B, \ r \in R \}
\]
and then the \(k\)-submodule

\[
B \times_R B := \{ \sum b_i \otimes_R b'_i \in B \otimes_R B \mid \sum r \triangleright b_i \otimes_R b'_i = \sum b_i \otimes_R b'_i \triangleleft r, \quad \forall \ r \in R \}.
\]
\(B \times_R B\) is known as the Takeuchi product, and it is easily checked to be an \(R^e\)-ring with factorwise multiplication and source and target maps

\[
R \to B \times_R B, \quad r \mapsto s(r) \otimes_R 1_B, \quad R^{\text{op}} \to B \times_R B, \quad r \mapsto 1_B \otimes_R t(r).
\]

A.3. An \(R\)-coring is a comonoid in the monoidal category of \(R\)-bimodules. That is, an \(R\)-coring is an \(R\)-bimodule \(C\), equipped with an \(R\)-bilinear coassociative comultiplication \(\Delta : C \to C \otimes_R C\), possessing an \(R\)-bilinear counit \(\epsilon : C \to R\). For the comultiplication we use the index notation \(c \mapsto c_{(1)} \otimes_R c_{(2)}\), where implicit summation is understood.

Any \(R\)-coring \(C\) induces a comonad

\[
((-) \otimes_R C, (-) \otimes_R \Delta, (-) \otimes_R \epsilon) \tag{A.4}
\]
on \(\text{Mod}-R\). Coalgebras of this comonad are called right \(C\)-comodules. Explicitly, this means right \(R\)-modules \(M\), equipped with a right \(R\)-linear coaction \(M \to M \otimes_R C\), subject to coassociativity and counitality constraints. For a right coaction, the index notation \(m \mapsto m_{[0]} \otimes_R m_{[1]}\) is used (with lower, or with upper indeces), where implicit summation is understood. Symmetrically, coalgebras for the comonad

\[
(C \otimes_R (-), \Delta \otimes_R (-), \epsilon \otimes_R (-)) \tag{A.5}
\]
on \(R\)-\text{Mod} are called left \(C\)-comodules. For the coaction on a left \(C\)-comodule, we use the index notation \(m \mapsto m_{[-1]} \otimes_R m_{[0]}\) (with lower, or with upper indeces), where implicit
summation is understood. Morphisms of (right or left) \(C\)-comodules are morphisms of coalgebras for the appropriate comonad (\(\mathcal{A.4}\) or \(\mathcal{A.5}\)). That is, (right or left) \(R\)-module maps which are compatible with the \(C\)-coaction.

Note that the same formulae \(\mathcal{A.4}\) and \(\mathcal{A.5}\) define comonads also on \(R\)-Mod-\(R\), with respect to the \(R\)-bimodule structures as in \(\mathcal{A.3}\).

To an \(R\)-coring \(C\), one can associate also monads. The triple

\[
(\text{Hom}_{\mathcal{R}}(-, C), \text{Hom}_{\mathcal{R}}(\Delta, -), \text{Hom}_{\mathcal{R}}(\epsilon, -))
\]

is a monad on \(\text{Mod}-\mathcal{R}\) (where we used standard hom-tensor identities to identify 
\(\text{Hom}_{\mathcal{R}}(C \otimes \mathcal{R} C, -) \cong \text{Hom}_{\mathcal{R}}(C, \text{Hom}_{\mathcal{R}}(C, -))\) and \(\text{Hom}_{\mathcal{R}}(\mathcal{R}, -) \cong \text{Mod}-\mathcal{R}\). Algebras of this monad are called right \(C\)-contramodules, cf. \(\mathcal{A.4}\), \(\mathcal{A.5}\). Symmetrically, left \(C\)-contramodules are algebras of the monad

\[
(\text{Hom}_{\mathcal{R}, -}(C, -), \text{Hom}_{\mathcal{R}, -}(\Delta, -), \text{Hom}_{\mathcal{R}, -}(\epsilon, -))
\]

on \(R\)-Mod. Morphisms of (right or left) \(C\)-contramodules are morphisms of algebras for the appropriate monad (\(\mathcal{A.6}\) or \(\mathcal{A.7}\)). That is, (right or left) \(R\)-module maps which are compatible with the contramodule structure.

Note that the same formulae \(\mathcal{A.6}\) and \(\mathcal{A.7}\) define monads also on \(R\)-Mod-\(R\), with respect to the \(R\)-bimodule structures

\[
(\text{r}f\text{r}', c) = rf(c) \quad \text{and} \quad (\text{r}g\text{r}', c) = g(c)r',
\]

for \(r, r' \in \mathcal{R}, c \in C, f \in \text{Hom}_{\mathcal{R}}(C, P), g \in \text{Hom}_{\mathcal{R}, -}(C, P)\) and any \(R\)-bimodule \(P\).

\(\mathcal{A.4}\). A left \(R\)-bialgebroid \([24], [20]\) is an \(R^e\)-ring \((B, s, t)\), possessing also an \(R\)-coring structure \((B, \triangleright, \triangleleft, \Delta, \epsilon)\), subject to the following compatibility axioms.

- The comultiplication \(\Delta : B \to B \otimes \mathcal{R} B\) factorizes through \(B \times \mathcal{R} B\);
- Corestriction of \(\Delta\) is a homomorphism of \(R^e\)-rings \(B \to B \times \mathcal{R} B\);
- The map \(B \to \text{End}_k(\mathcal{R}), b \mapsto \epsilon(b(\mathcal{R}))\) is a homomorphism of \(R^e\)-rings.

Some equivalent forms of the definition can be found e.g. in \(\mathcal{A.5}\). The notion of a right \(R\)-bialgebroid is obtained symmetrically, by interchanging the roles of the \(R\)-actions \((\triangleright, \triangleleft)\) and \((\triangleleft, \triangleright)\) in an \(R^e\)-ring, given by multiplication on the right, and on the left, respectively. For more details we refer to \([15]\) or \([2]\).

\(\mathcal{A.5}\). Modules of an \(R\)-bialgebroid \(B\) are modules of the underlying \(k\)-algebra \(B\). Since \(B\) is an \(R^e\)-ring, there is a forgetful functor from the category of (left or right) \(B\)-modules to the category of (left or right) \(R^e\)-modules, equivalently, to the category of \(R\)-bimodules. By \([22]\) Theorem 5.1, the category of left (resp. right) modules of a left (resp. right) bialgebroid is a monoidal category, with monoidal product given by the \(R\)-module tensor product. Left (resp. right) \(module\ algebras\) of a left (resp. right) bialgebroid \(B\) are defined as monoids in the monoidal category of left (resp. right) \(B\)-modules. \(B\)-module algebras are thus in particular \(R\)-rings. By the same principle, left (resp. right) \(B\)-\(module\ corings\) are comonoids in the monoidal category of left (resp. right) \(B\)-modules. They are in particular \(R\)-corings.

\(\mathcal{A.6}\). Comodules of a (left or right) \(R\)-bialgebroid are comodules of the constituent \(R\)-coring. As a consequence of the bialgebroid axioms, any right comodule \(M\) of a right \(R\)-bialgebroid \(B\) can be equipped also with a unique left \(R\)-action, such that the range of the coaction \(m \mapsto m^{(0)} \otimes \mathcal{R} m^{(1)}\) lies within the center of the \(R\)-bimodule \(M \otimes \mathcal{R} B\). That
is, for any $m \in M$ and $r \in R$, $rm^0 \otimes_R m^1 = m^0 \otimes_R t(r)m^1$ (where $t : R^{op} \to B$ is the target map). This equips any $B$-comodule with an $R$-bimodule structure and the category of right $B$-comodules becomes monoidal with respect to the $R$-module tensor product (cf. [2] Theorem 3.18). In other words, there is a strict monoidal ‘forgetful’ functor from the category $\text{Comod}_B$ of right $B$-comodules to $R\text{-Mod}_R$. Symmetrically, also the category of left comodules of a right $R$-bialgebroid is monoidal, via $\otimes_{R^{op}}$. In the same way, categories of left and right comodules of a left $R$-bialgebroid are monoidal, with respect to the $R$-module tensor product and the $R^{op}$-module tensor product, respectively. Left (resp. right) comodule algebras of a left or right bialgebroid $B$ are defined as monoids in the monoidal category of left (resp. right) $B$-comodules. $B$-comodule algebras are in particular $R$-rings or $R^{op}$-rings (depending on the monoidal product of the appropriate comodule category). By the same principle, left (resp. right) $B$-comodule corings are comonoids in the monoidal category of left (resp. right) $B$-comodules (hence they are $R$, or $R^{op}$-corings).

A.7. Contramodules of a (left or right) $R$-bialgebroid are contramodules of the constituent $R$-coring. As a consequence of the bialgebroid axioms, any left contramodule $(Q, \alpha : \text{Hom}_{R^{op}}(B, Q) \to Q)$ of a right $R$-bialgebroid $B$ can be equipped also with a right $R$-action

$$qr := \alpha(\varepsilon(s(r))q),$$

for $q \in Q, r \in R$, such that $Q$ becomes an $R$-bimodule. This construction yields a ‘forgetful’ functor from the category $B\text{-Ctrmod}$ of left $B$-contramodules to $R\text{-Mod}_R$. Symmetrically, also right contramodules of a right $R$-bialgebroid and left and right contramodules of a left $R$-bialgebroid possess canonical $R$-$R$-bimodule structures. Note, however, that the category of contramodules of an arbitrary bialgebroid is not known to be monoidal.

A.8. A Hopf algebroid $H$ consists of a left bialgebroid structure $(H, s_L, t_L, \Delta_L, \varepsilon_L)$ over a base algebra $L$, and a right bialgebroid structure $(H, s_R, t_R, \Delta_R, \varepsilon_R)$ over a base algebra $R$, on the same $k$-algebra $H$, together with a $k$-module map $S : H \to H$, called the antipode. These structures are subject to the following axioms.

- $s_R \circ \varepsilon_R \circ t_L = t_L, t_R \circ \varepsilon_R \circ s_L = s_L, s_L \circ \varepsilon_L \circ t_R = t_R, t_L \circ \varepsilon_L \circ s_R = s_R$.
- $(\Delta_R \otimes L H) \circ \Delta_L = (H \otimes_R \Delta_L) \circ \Delta_R$ and $(\Delta_L \otimes_R H) \circ \Delta_R = (H \otimes_L \Delta_R) \circ \Delta_L$.
- For all $h \in H, l \in L$ and $r \in R, S(t_L(l)ht_R(r)) = s_R(r)S(h)S_L(l)$.
- $\mu_R \circ (H \otimes_R S) \circ \Delta_R = s_L \circ \varepsilon_L$ and $\mu_L \circ (S \otimes_L H) \circ \Delta_L = s_R \circ \varepsilon_R$,

where $\mu_R : H \otimes_R H \to H$ denotes multiplication in the $R$-ring $s_R : R \to H$ and $\mu_L : H \otimes_L H \to H$ denotes multiplication in the $L$-ring $s_L : L \to H$. Note that the second axiom is meaningful because of the first axiom and the fourth axiom is meaningful because of the third one.

These axioms imply that the algebras $L$ and $R$ are anti-isomorphic, and the antipode is a bialgebroid morphism from the constituent left bialgebroid to the opposite-coopposite of the right bialgebroid, and also from the constituent right bialgebroid to the opposite-coopposite of the left bialgebroid.

A.9. Modules of a Hopf algebroid $H$ are by definition modules of the underlying $k$-algebra. In this way the category of (left or right) $H$-modules coincides with the (left or right) module category of any of the constituent bialgebroids. Hence both categories of left and right $H$-modules are monoidal. A (left or right) module algebra of a Hopf algebroid $H$
is defined as a monoid in the monoidal category of (left or right) $H$-modules. Similarly, a (left or right) module coring of a Hopf algebroid $H$ is defined as a comonoid in the monoidal category of (left or right) $H$-modules.

If the antipode $S$ of a Hopf algebroid $H$ is bijective, then it induces an isomorphism $I_S : H-\text{Mod} \rightarrow \text{Mod}-H$ between the categories of left and right $H$-modules. This isomorphism takes a left $H$-module $N$ to $N$ as a right $H$-module with action $n \cdot h = S^{-1}(h)n$. On the morphisms $I_S$ acts as the identity map. A similar isomorphism $I_{S^{-1}} : H-\text{Mod} \rightarrow \text{Mod}-H$ is obtained by replacing $S$ by $S^{-1}$.

**A.10.** Right comodules of a Hopf algebroid $H$ over base algebras $L$ and $R$ are triples $(M, \varrho_L, \varrho_R)$, where $M$ is a right $L$-module and a right $R$-module, $(M, \varrho_L)$ is a right comodule of the constituent left bialgebroid, $(M, \varrho_R)$ is a right comodule of the constituent right bialgebroid, such that both structure maps are comodule maps for the other bialgebroid as well. That is, $\varrho_L$ is a right $R$-module map, $\varrho_R$ is a right $L$-module map and the compatibility conditions

$$(M \otimes_L \Delta_R) \circ \varrho_L = (\varrho_L \otimes_R H) \circ \varrho_R \quad \text{and} \quad (M \otimes_R \Delta_L) \circ \varrho_R = (\varrho_R \otimes_L H) \circ \varrho_L$$

hold. It follows that the right $R$-, and $L$-actions on $M$ commute, i.e. $M$ is a right $R \otimes L$-module. Morphisms of $H$-comodules are defined as comodule maps for both constituent bialgebroids. Right comodules of a Hopf algebroid $H$ and their morphisms constitute the category $\text{Comod}-H$. The category $H-\text{Comod}$ of left $H$-comodules is defined symmetrically.

The category of (left or right) comodules of a Hopf algebroid is monoidal and the forgetful functors to the comodule categories of the constituent bialgebroids are strict monoidal [2, Theorem 4.9]. A (left or right) comodule algebra of a Hopf algebroid $H$ is defined as a monoid in the monoidal category of (left or right) $H$-comodules. Similarly, a (left or right) comodule coring of a Hopf algebroid $H$ is defined as a comonoid in the monoidal category of (left or right) $H$-comodules.

If the antipode $S$ of a Hopf algebroid $H$ is bijective, then it induces an isomorphism $I_S : H-\text{Comod} \rightarrow \text{Comod}-H$. Take a left $H$-comodule $M$, with coaction $m \mapsto m_{[-1]} \otimes_L m_{[0]}$ of the constituent left $L$-bialgebroid and coaction $m \mapsto m_{[-1]} \otimes_R m_{[0]}$ of the constituent right $R$-bialgebroid. The isomorphism $I_S$ takes it to $M$ as a right $H$-comodule, with right $R$, and $L$-actions induced by the algebra isomorphism $R \cong L^\text{op}$ and with coaction $m \mapsto m_{[0]} \otimes_L S^{-1}(m_{[-1]})$ of the constituent left $L$-bialgebroid and coaction $m \mapsto m_{[0]} \otimes_R S^{-1}(m_{[-1]})$ of the constituent right $R$-bialgebroid. On the morphisms $I_S$ acts as the identity map. A similar isomorphism $I_{S^{-1}} : H-\text{Comod} \rightarrow \text{Comod}-H$ is obtained by replacing $S$ by $S^{-1}$.

**A.11.** Right contramodules of a Hopf algebroid $H$ over base algebras $L$ and $R$ are triples $(Q, \alpha_L, \alpha_R)$, where $Q$ is a right $L$-module and a right $R$-module, $(M, \alpha_L)$ is a right contramodule of the constituent left bialgebroid, $(M, \alpha_R)$ is a right contramodule of the constituent right bialgebroid, such that both structure maps $\alpha_L$ and $\alpha_R$ are contramodule maps for the other bialgebroid as well. That is, $\alpha_L$ is a right $R$-module map, $\alpha_R$ is a right $L$-module map and the compatibility conditions

$$\alpha_L \circ \text{Hom}_{-,-}(H, \alpha_R) = \alpha_R \circ \text{Hom}_{-,-}(\Delta_L, H) \quad \text{and} \quad \alpha_R \circ \text{Hom}_{-,-}(H, \alpha_L) = \alpha_L \circ \text{Hom}_{-,-}(\Delta_R, H)$$

hold. It follows that the right $R$-, and $L$-actions on $Q$ commute, i.e. $Q$ is a right $R \otimes L$-module. Morphisms of $H$-contramodules are defined as contramodule maps for both
constituent bialgebroids. Right contramodules of a Hopf algebroid $H$ and their morphisms constitute the category $\text{Ctrmod}-H$. The category $H\text{-Ctrmod}$ of left $H$-contramodules is defined symmetrically.

If the antipode $S$ of a Hopf algebroid $H$ is bijective, then it induces an isomorphism $I_S : H\text{-Ctrmod} \to \text{Ctrmod}-H$. Take a left $H$-contramodule $Q$, with structure map $\alpha_L : \text{Hom}_L(H, Q) \to Q$ as a contramodule of the constituent left $L$-bialgebroid and structure map $\alpha_R : \text{Hom}_R(H, Q) \to Q$ as a contramodule of the constituent right $R$-bialgebroid. The isomorphism $I_S$ takes it to $Q$ as a right $H$-contramodule, with right $R$, and $L$-actions induced by the algebra isomorphism $R \cong L^{op}$ and with structure maps $\text{Hom}_{-L}(H, Q) \to Q, \ f \mapsto \alpha_R(f \circ S^{-1})$ and $\text{Hom}_{-R}(H, Q) \to Q, \ g \mapsto \alpha_L(g \circ S^{-1})$.

On the morphisms it acts as the identity map. A similar isomorphism $I_{S^{-1}} : H\text{-Ctrmod} \to \text{Ctrmod}-H$ is obtained by replacing $S$ by $S^{-1}$.

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