Exploring dynamical phase transitions and prethermalization with quantum noise of excitations

Pietro Smacchia, Michael Knap, Eugene Demler, and Alessandro Silva

1SISSA, International School for Advanced Studies, via Bonomea 265, 34136 Trieste, Italy
2Department of Physics, Harvard University, Cambridge MA 02138, USA
3ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138, USA
4Abdus Salam ICTP, Strada Costiera 11, 34100 Trieste, Italy

Dynamical phase transitions can occur in isolated quantum systems following a sudden change of the system parameters. We discuss the characterization of such dynamical phase transitions based on the statistics of excitations produced in a quantum quench. We consider a specific case of an $O(N)$ model in the large $N$ limit and show that the transition manifests itself most dramatically not in the average number of defects but in the higher moments. We argue that the growth of the second moment of the defect density in time following a quench is a signature of a prethermal state. Our theoretical results should be relevant to quench experiments with ultracold bosonic atoms in optical lattices from the Mott insulating to the superfluid phase.

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The dynamics of quantum many-body systems is a subject of interest in many areas of physics involving cold atomic gases [1], solid state pump and probe experiments [2], quantum optics [3], heavy ions collisions and cosmology. A particularly intriguing question in this context is the possible emergence of new dynamical critical phenomena appearing in the steady or quasi-steady states of such systems (see for example Ref. [2]) and their detection. In this work, we will discuss possible experimental consequences of dynamical phase transitions occurring after an abrupt change of one of the parameters of an isolated system (a quantum quench). At long times after a quantum quench a many-body system is expected to either thermalize [4, 5], or in the presence of integrability to relax to a Generalized Gibbs Ensemble (GGE) [6]. However, even when thermalization occurs, its dynamics can be highly non-trivial, requiring a two step process through a pre-thermal state [7, 8], a phenomenon expected both in low dimensional systems close to integrability [9–12] as well as for high dimensional systems close to the mean field limit. These intermediate states, as well as the GGE, have the intriguing feature of supporting non-thermal behavior [13, 14] as well as, in certain instances, genuine dynamical critical effects, i.e. critical phenomena in the steady state attained after the quench [15–20]. Examples of pre-thermalization and dynamical critical behavior were first observed studying the dynamics of the Hubbard model [15, 16], in a variety of mean field models [17, 18], and field theories [19] such as the three dimensional quantum $O(N)$ model in the infinite $N$ limit [20]. However, the nature of these transitions and how to distinguish them from thermal critical phenomena both theoretically and, most importantly, experimentally, is elusive thus far.

In this Letter, we address these issues and discuss a characterization of dynamical critical phenomena in bosonic systems based on the full statistics of excitations generated in a quench (see Fig. 1). In particular, we will argue that dynamical phase transitions can be detected qualitatively by studying how the fluctuations in the number of excitations grow as a function of the time, and as well as quantitatively by studying the non analytic behavior of the stationary number of excitations (or higher moments) as a function of the quench parameters (see Fig. 2). Experiments of this type are feasible with cold atomic gases, where recently developed high resolution optical imaging techniques give a unique opportunity to study the dynamics of cold atoms in optical lattices with single site resolution [21–23], as shown by recent measurements of the defects produced by ramping a system across a quantum critical point [22], or the first direct measure of a string order parameter [24], as well as the
detection of light-cone spreading of correlations [25], and the study of the dynamics of a mobile spin impurity [26].

In order to corroborate our claims we will work out in detail the example of the quantum $O(N)$ model (which at equilibrium and for $N = 2$ is in the Bose-Hubbard universality class), where a genuine dynamical phase transition can be observed for large $N$ and dimensions $d > 2$. While this transition induces strong qualitative effects on the second or higher moments of the excitations, a characterization in terms of traditional critical exponents would suggest that the dynamical transition has the same universality as the corresponding thermal phase transition. Specifically, while the fluctuations in the number of defects produced in the quench saturate as a function of the time spent above the dynamical critical point (i.e. in the dynamically disordered phase), it grows indefinitely in time for quenches to or below the dynamical critical point (i.e. to the dynamically ordered phase), see Fig. 2a. Furthermore, observables that saturate as a function of time show quantitative signatures of the dynamical transition in the form of kinks in the derivative with respect to the quench parameter, as shown in Fig. 2b. Finite $N$ corrections are expected to eventually lead to a saturation of this indefinite growth and to a smearing of the kinks at times $\propto N$, so that the above description can be thought to be valid for the intermediate time regime in experimental systems.

The quantum $O(N)$ model consists of an $N$ component real scalar field in $d$ spatial dimensions with quartic interaction, whose Hamiltonian reads

$$H = \frac{1}{2} \int d^d x \left[ (\Pi \phi)^2 + (\nabla \phi)^2 + r_0 (\phi)^2 + \frac{\lambda}{12 N} (\phi)^4 \right],$$ (1)

where $[\phi_i(x), \Pi_j(x')] = i \delta^d(x-x') \delta_{ij}$, with $i$ and $j$ denoting different components. Below we will consider the $N \rightarrow \infty$ limit (see [27] for an introduction) where the model is soluble. In the disordered phase, where $\langle \phi \rangle = 0$, it can be described by a quadratic theory with an effective mass parameter

$$r = r_0 + \frac{\lambda}{12} \int_k \frac{1}{\sqrt{|\bar{k}|^2 + r}},$$ (2)

where from now on $\int_k = \int^{\Lambda} \frac{d^d k}{(2\pi)^d}$, and $\Lambda$ is the ultraviolet cutoff. The equilibrium critical point is identified by the condition $r = 0$, giving $r_{0,c} = -\frac{\lambda}{12} \int_k \frac{1}{|k|}$, which is finite for $d > 1$. From Eq. (2) it is also possible to compute the critical exponent $\nu$, since $\xi^{-1} \sim \sqrt{r}$, obtaining $\nu = 1/2$ (mean field) for $d \geq 3$, and $\nu = \frac{1}{d-1}$ for $1 < d < 3$.

Let us now imagine to prepare the system in the ground state for $r_0 = r_{0,i}$ corresponding to an effective mass $r_i$ and perform a quench to $r_{0,f}$. Numerical evidence for a dynamical transition following a quench of $r_0$ starting within the ordered phase has been found in this model when $d = 3$ [20]. Below we will instead consider quenches starting in the disordered phase [28] and look for the dynamical critical point, i.e. the point at which the asymptotic effective mass vanishes.

The dynamics of the system can still be described by an effective quadratic model, but the self-consistently determined effective mass becomes time-dependent and is given by

$$r(t) = r_{0,f} + \frac{\lambda}{6} \int_k \langle \phi_k(t) \phi_{-k}(t) \rangle,$$ (3)

where $\phi$ now represents one of the components of the field. From now on we will focus on a single component, since everything is symmetric in the component space. Expanding the field in terms of the operators $a_k$ and $a_k^\dagger$ diagonalizing the initial Hamiltonian, i.e. $H_0 = \int_k (|\bar{k}|^2 + r)^{1/2} (a_k^\dagger a_k + 1/2)$, as

$$\phi_k(t) = f_k(t) a_k + f_k(t) a_k^\dagger,$$ (4)

and imposing the Heisenberg equation of motions we find that the functions $f_k(t)$ have to satisfy the equation

$$\frac{d^2 f_k(t)}{dt^2} + \left(|\bar{k}|^2 + r(t)\right) f_k(t) = 0,$$ (5a)
that this ansatz gives the correct asymptotic value as long as $r_{0,f}$ is above or at the dynamical transition, identified by the condition $r^* = 0$. When $r_{0,f} < r^*_{0,f}$, it predicts a negative value, while the numerical solution for the asymptotic value is always zero. Using Eq. (6) one obtains

$$r^* = r_{0,f} + \frac{\lambda}{24} \int_k \frac{2|\vec{k}|^2 + r_i + r^*}{(|\vec{k}|^2 + r_i)^{1/2}}.$$  

(6)

A comparison with the exact integration of Eq. (5) shows that this ansatz gives the correct asymptotic value as long as $r_{0,f}$ is above or at the dynamical transition, identified by the condition $r^* = 0$. When $r_{0,f} < r^*_{0,f}$, it predicts a negative value, while the numerical solution for the asymptotic value is always zero. Using Eq. (6) one obtains

$$r^*_{0,f} = -\frac{\lambda}{24} \int_k \frac{2|\vec{k}|^2 + r_i}{|\vec{k}|^2}.$$  

(7)

We can notice that $r^*_{0,f}$ is finite for $d > 2$, which is thus the lower critical dimension of the transition, and it is always less than $r^*_{0,c}$, i.e. always in the ordered phase. From Eq. (7) we can also derive that for quenches above the phase, but near to the critical point $r^* \sim (\delta r_{0,f})^{2/(d-2)}$ for $2 < d < 4$, while $r^* \sim \delta r_{0,f}$ for $d \geq 4$, where we introduced $\delta r_{0,f} = r^*_{0,f} - r_{0,f}$. This translates to the behavior of the correlation length in the stationary state $\xi^*$, since $(\xi^*)^{-1} \sim \sqrt{r^*}$. Thus defining the exponent $\nu^*$ as $(\xi^*)^{-1} \sim (\delta r_{0,f})^{\nu^*}$, we have $\nu^* = \frac{1}{d-2}$ for $2 < d < 4$ and $\nu^* = 1/2$ for $d \geq 4$, with $d = 4$ playing the role of an upper critical dimension.

As anticipated, the critical properties described above are similar to that of the finite temperature transition, i.e. critical dimensions and exponents are obtained by a shift up of one dimension as compared to the corresponding quantum phase transition. However, we will now show that, contrary to the thermal case, the dynamical transition leaves strong signatures on the statistics of the of excitations produced in a quench. As shown in Fig. 1 we imagine to start in the disordered phase and perform a first quench of $r_0$ at, or close to, the dynamical critical point, then let the system evolve for a time $t$ and finally return to $r_{0,i}$ in order to count the number of excitations generated, which in the present case is described by the operator $\hat{N} = \int_k a_k^\dagger a_k$. This number is a fluctuating quantity characterized by a probability distribution $P(N,t)$, which can equivalently be described in terms of the moment generating function $G(s,t) = \langle e^{-s\hat{N}} \rangle_t$. For the $O(N)$ model in the large N limit, this quantity can be computed exactly (see Supplementary Material) and has the form $G(s,t) = \exp(-L^d f(s,t))$, with

$$f(s,t) = \frac{1}{2} \int_k \log \left[ 1 + \rho_k(t) \left( 1 - e^{-2s} \right) \right],$$  

(8)

defined for $s > -\bar{s} = \frac{1}{2} \sup_k \log \frac{\rho_k(t)}{1 + \rho_k(t)}$, where $L$ is the linear size of the system and $\rho_k(t) = |\hat{f}_k(t)|^2 \xi_k^2 + |\hat{f}_k(t)|^2 - 1/2$, with $k = |\vec{k}|$ from now on. The function $\rho_k$ that fully determines the statistics of the excitations can be obtained from the integration of Eq. (5) and represents the average number of excitations for each mode.

Let us now characterize the dynamical critical behavior of the system by studying all the cumulants $C_n$’s of the distribution of excitations, using the formula $C_n(t) = (-1)^n \frac{d^n}{ds^n} \log G(s,t)|_{s=0}$. Below, we will start by focusing on the first two, i.e. the average $N(t)$ and the variance $\sigma^2(t)$, and discuss their dependence on the waiting time $t$ between the two quenches if the intermediate value $r_{0,f}$ of the bare mass is above, below or at the dynamical critical point in $d = 3$.

It is first of all important to notice that the time evolution of the average and of the variance are totally different. The former does not display striking features and tends for all $r_{0,f}$ to a stationary value as a function of $t$. However its asymptotic value as a function of $r_{0,f}$ displays non-analytic behavior at the dynamical critical point, see Fig. 3. In contrast, as discussed in the introduction the variance has three qualitatively different behaviors for quenches above, to, below the dynamical
Critical point. If, indeed, the first quench is at the dynamical critical point, i.e. $r_{0,f} = r_{0,f}^c$, the variance per unit volume appears to grow in a logarithmic fashion as a function of $t$ (see Fig. 4a). This should be contrasted with what one would expect for such scaling in a free field theory (see Supplementary Material), where in $d = 3$ the variance grows linearly. A totally different behavior is observed for quenches below the dynamical critical point, i.e. $r_{0,f} < r_{0,f}^c$: in this case the variance grows as a power law $t^\alpha$ with $\alpha = 1$ in $d = 3$, Fig. 4b. Finally in the case of an intermediate value of the bare mass above the dynamical transition, i.e. $r_{0,f} > r_{0,f}^c$, the variance saturates to a finite value as a function of $t$. For $r^* \neq 0$ the curves describing the variance follow the critical line until a certain time that is the longer the smaller $r^*$, and then deviate and saturate, Fig. 4a. We note that also in $d = 4$ these three different qualitative behaviors have been observed, with the difference that the power law growth for quenches with $r_{0,f} < r_{0,f}^c$ has the exponent $\alpha = 2$. For further details see the Supplemental Material.

The physical motivation to explore higher moments of the excitations is that they expose the overabundance of the small momentum modes which inevitably characterize the dynamical criticality. More specifically, the statistics of the excitations and the scaling of all the cumulants for large $t$ is fully determined by the scaling for small $k$ of $\rho_k(t)$. Indeed $\rho_k(t)$ behaves singularly as $1/k^{\gamma_n}$ up to an infrared cutoff shrinking to zero as $1/t$. Therefore, noticing that the $n$-th cumulant is given by a weighted sum of the integrals over $k$ of all the integers powers of $\rho_k$ up to $n$, we can infer its asymptotic behavior in $t$, given by $C_n \sim \int_{1/l} \, dk \, k^{d-1-\gamma_n} \sim t^{\gamma_n-d}$. Numerical results in $d = 3$ confirm that, as expected from the behavior of the variance $\gamma = 3/2$ for quenches at the critical point and $\gamma = 2$ for quenches below the critical point, while in $d = 4$ we have $\gamma = 2$ and $\gamma = 3$ respectively.

Our study of the statistics of dynamically generated excitation in the $O(N)$ field theory should provide insight into the dynamics of the Bose-Hubbard model in higher dimension. At equilibrium the lattice Bose-Hubbard model falls into the universality class of the $O(N)$ field theory with $N = 2$ with the upper critical dimension $d \geq 3$. To demonstrate that in the case of lattice bosons the emergence of prethermalization is not generic for all dimensions, we also study double quench protocol in the limit of the nonintegrable one-dimensional Bose-Hubbard model using exact numerical techniques. In that case no dynamical phase transition is expected. Rather we find that for quenches deep in the ordered phase, all moments of the generated defects rapidly thermalize and saturate which unveils the corresponding dynamical crossover diagram, see Supplemental Material for details.

In conclusion, we observed that the large $N$ limit of an $O(N)$ model possesses a well defined dynamical phase transition that leaves a strong imprint in the excitations generated in a quench. Whether signatures of such dynamical transition can be observed in experimental systems, e.g. the Bose-Hubbard model, is an interesting question: indeed $1/N$ corrections are expected to drive the systems toward full thermalization, implying that the dynamical criticality occurs in an unstable prethermal state and the divergence of the cumulants will be cut off by thermalization. However, even though the excitations will not grow indefinitely in a real experimental system, the dynamical phase transition will still leave a fingerprint on the statistics of excitations in the intermediate pre-thermal state, before full thermalization occurs. Furthermore, we observe that when the parity projection of excitations is measured, as it was done in the experiments of Refs. [21, 23], the fluctuations will saturate for all quench parameters, but their asymptotic value exhibits a nonanalyticity at the dynamical phase transition, similarly to the density in Fig. 3, see Supplemental Material. Experimental studies with ultracold atoms might therefore be able to shed light on this chal-
lenging question.

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Supplemental Material for
Exploring dynamical phase transitions and prethermalization with quantum noise of excitations

Supplemental signatures of the dynamical transition

As we discussed in the main text, finite $N$ correction are expected to lead to full thermalization of the system, implying that the growth in time of the variance and higher order cumulants will eventually saturate and that the sharp kinks in observables that saturate in the large $N$ limit will be smeared out at asymptotically long times $t \gtrsim N$. The infinite $N$ results reported in the main text are expected to at most describe the intermediate time regime.

Let us now focus on observables that saturate for all quench parameters upon reaching the pre-thermal state described by the limit $N \to +\infty$. One example of such an observable is the average number of excitations, see Fig. 3 in the main text, which shows a pronounced kink in its derivative as function of the quench parameter $r_{0,f}$. Here, we will show additional examples of this kind. First of all, we considered the effect of having a finite (but still large) volume $V = L^3$, which can be understood by putting an infrared cutoff $\sim \pi/L$ in the integral over the momentum. In that case, the variance always saturates as function of time, but there are still signatures of the dynamical transition, as we can see from Fig. 5b.

Moreover, experiments with quantum gas microscopes [21, 23] typically measure the parity projected density ($n \mod 2$). As discussed in the main text, fluctuations and higher order cumulants of the parity projected density do not diverge as a function of time regardless of the quench parameter. However, we show in Fig. 5a that the parity cumulants exhibit a non-analyticity at the dynamical critical point and thus allow for its unambiguous characterization.

Free theory and stationary state

In this section we will first of all consider the dynamics of the systems for a quench of $r_0$ from $r_{0,i}$ to $r_{0,f}$, when there is no quartic interaction, i.e, $\lambda = 0$. The first obvious consequences are that there is no renormalization of the initial mass, and that there is no dynamics of the mass after the quench. The equation for the functions $f_k(t)$, which are the coefficients of the expansion of the field $\phi_k(t)$ in the basis of the initial Hamiltonian, becomes

$$\frac{d^2 f_k(t)}{dt^2} + \left( |\vec{k}|^2 + r_{0,f} \right) f_k(t) = 0,$$

with initial conditions $f_k(0) = 1/\sqrt{2\omega_{k,i}}$, $\dot{f}_k(0) = -i\sqrt{\omega_{k,i}}/2$, $\omega_{k,i} = \sqrt{|\vec{k}|^2 + r_{0,i}}$, fixed by the requirement that $a_k$ and $a_k^\dagger$ diagonalize the initial Hamiltonian.

![Figure 5](image-url)

Figure 5: (a) Long-time saturation value of the first three cumulants generated by measurements of parity projected density. Inset: the first derivative of the cumulants emphasizes the nonanalyticity at the dynamical phase transition, indicated by the tick gray solid line. (b) Long-time saturation value of the first two cumulants for a finite volume as a function of the quench parameter $r_{0,f}$. They show signatures of the dynamical phase transition, indicated by the tick gray solid line.
The solution of the previous equation is readily found to be 

\[ f_\delta(t) = \frac{1}{\sqrt{2\omega_0}} \cos \left( t\sqrt{|k|^2 + r_{0,f}} \right) - \frac{t}{\sqrt{|k|^2 + r_{0,f}}} \sin \left( t\sqrt{|k|^2 + r_{0,f}} \right). \]

From this expression one can compute all the quantities of interest, including the equal time two-point correlator of the field \( \langle \phi_\delta(t)\phi_{-\delta}(t) \rangle = |f_\delta(t)|^2 \):

\[
\langle \phi_\delta(t)\phi_{-\delta}(t) \rangle = \frac{2|\vec{k}|^2 + r_{0,i} + r_{0,f}}{4(|\vec{k}|^2 + r_{0,f})\sqrt{|\vec{k}|^2 + r_{0,i}}} + \frac{r_{0,f} - r_{0,i}}{4(|\vec{k}|^2 + r_{0,f})\sqrt{|\vec{k}|^2 + r_{0,i}}} \cos \left( 2t\sqrt{|\vec{k}|^2 + r_{0,f}} \right). \tag{10}
\]

Instead in the case of the interacting theory with \( \lambda \neq 0 \), the time dependent effective mass is given by

\[ r(t) = r_{0,f} + \frac{\lambda}{6} \int_k \langle \phi_\delta(t)\phi_{-\delta}(t) \rangle. \tag{11} \]

The numerical integration of the equation of motions shows that for large \( t \) this relaxes toward a stationary value. To predict this stationary values we make the ansatz that the stationary part of the equal time Green function \( \langle \phi_\delta(t)\phi_{-\delta}(t) \rangle \) is the same as the free theory but with renormalized masses. Namely, we take Eq. (10), disregard the part multiplying the cosine, and make the substitutions \( r_{0,i} \rightarrow r_i \) and \( r_{0,f} \rightarrow r^* \), with \( r^* \) denoting the stationary value of the mass to be self-consistently determined from Eq. (11). In this way we obtain the self-consistent equation for \( r^* \) written in the main text, that is

\[ r^* = r_{0,f} + \frac{\lambda}{24} \int_k \frac{2|\vec{k}|^2 + r_i + r^*}{(|\vec{k}|^2 + r^*)\sqrt{|\vec{k}|^2 + r_i}}. \tag{12} \]

Fig. 6 shows how well this equation predicts the stationary value until the dynamical critical point, identified by the condition \( r^* = 0 \). The figure shows only the cases of \( d = 3 \) or \( d = 4 \), since we focused on these cases in more detail, but we checked Eq. (12) also in lower and higher dimensions.

From Eq. (12) it is also possible to derive the behavior of the asymptotic mass \( r^* \) for small distances of \( r_{0,f} \) from the critical point \( r_{0,c}^* \). Let us define \( \delta r_{0,f} = r_{0,f} - r_{0,c}^* \), then, for \( \delta r_{0,f} > 0 \) we will have,

\[ r^* = \delta r_{0,f} - \frac{\lambda}{6} r^* \int_k \frac{\sqrt{|\vec{k}|^2 + r_i}}{4|\vec{k}|^2(|\vec{k}|^2 + r^*)}. \tag{13} \]

For \( d > 4 \) the integral is convergent in the limit \( r^* \rightarrow 0 \), so that \( r^* \sim \delta r_{0,f} \), while for \( 2 < d < 4 \) the integral is the dominant term implying \( r^* \sim (\delta r_{0,f})^{2/(d-2)} \). This behavior of \( r^* \) as a function of \( \delta r_{0,f} \) is reflected in the behavior of
the correlation length $\xi^*$ in the stationary state. We saw that the stationary behavior of the two point function is well described by the stationary part of Eq. (10), from which we derive $(\xi^*)^{-1} \sim \sqrt{r^s}$, so that we recover the result of the main text.

Generating function of the statistics of excitations

In this section we will describe in more detail how to derive the moment generating function of the statistics of excitations

$$G(s, t) = \langle e^{-s\hat{N}} \rangle_t,$$

where $\hat{N} = \int k a_k^\dagger a_k^\ast$, with $a_k^\dagger$ and $a_k^\ast$ diagonalizing the initial Hamiltonian, and the average is taken over the state $|\psi(t)\rangle = U(t)|0\rangle$, that is the evolved state at time $t$. Since the theory is effectively quadratic and the different $k$-modes interacts only through the renormalization of the mass $r(t)$, we can focus on a single mode $k$ and we will have $G(s, t) = \prod_k G_k(s, t)$, with $G_k(s, t)$ representing the generating function for a single mode.

In order to compute $G_k(s, t)$ we will first express the evolved state $|\psi(t)\rangle_k$ as a function of $a_k^\dagger$ and $a_k^\ast$. The starting point is the expansion of the evolved field $\phi_k(t)$ in the same basis, which can be translated from Heisenberg to Schrödinger picture by writing

$$\phi_k(0) = f_k(t)\tilde{a}_k(t) + f_k^*(t)a_{-k}^\dagger(t),$$

$$\Pi_k(0) = \dot{f}_k(t)\tilde{a}_k(t) + f_k^*(t)a_{-k}^\dagger(t),$$

with the operators $\tilde{a}_k$ and $a_{-k}^\dagger$ defined by the relation $\tilde{a}_k(t)|\psi(t)\rangle = 0$. At the same time, we know that

$$\phi_k(0) = \frac{1}{\sqrt{2\omega_{k,i}}}(a_k^\dagger + a_{-k}^\ast),$$

$$\Pi_k(0) = i\sqrt{\frac{\omega_{k,i}}{2}}(a_{-k}^\dagger - a_k^\ast).$$

By inverting Eq. (15), taking into account that $f_k(t)f_k^*(t) - \dot{f}_k(t)f_k^*(t) = i$, and inserting the result into Eq. (16), one obtains

$$\tilde{a}_k(t) = \alpha_k^*(t)a_k^\dagger - \beta_k^*(t)a_{-k}^\dagger,$$

with

$$\alpha_k(t) = f_k(t)\sqrt{\frac{\omega_{k,i}}{2}} + i\dot{f}_k(t)\frac{1}{\sqrt{2\omega_{k,i}}},$$

$$\beta_k(t) = f_k(t)\sqrt{\frac{\omega_{k,i}}{2}} - i\dot{f}_k(t)\frac{1}{\sqrt{2\omega_{k,i}}}.$$

From Eq. (17) and the requirement that $\tilde{a}_k(t)$ annihilates the evolved state, one finally finds

$$|\psi(t)\rangle = \frac{1}{\sqrt{|\alpha_k^*(t)|}} \exp\left(\frac{\beta_k^*(t)}{2\alpha_k^*(t)} a_{k}^\dagger a_{-k}^\dagger\right) |0\rangle,$$

with $a_k^\dagger|0\rangle = 0$. Having the expression of the state in terms of $a_k^\dagger$ and $a_{-k}^\dagger$, the computation of $G_k(s, t)$ can be straightforwardly done, for example using coherent states, obtaining

$$G_k(s, t) = \frac{1}{\sqrt{1 + |\rho_k(t)|^2(1 - e^{-2s})}},$$
with

\[ \rho_k(t) = |\beta_k|^2 = |f_k(t)|^2 \frac{\omega k_i}{2} + \frac{|f_k(t)|^2}{2\omega k_i} - 1/2. \]  

(21)

Finally using the relation \( \log G(s, t)/L^d = \int_k \log G_k(s, t) \) one recovers the result of the main text.

Using the solution of the Eq. (9) when \( \lambda = 0 \) and Eq. (21), one can find the function \( \rho_k(t) \), and so determining the full statistics of excitations, for the free case. The result of such a procedure is

\[ \rho_k(t) = \frac{(r_{0,f} - r_{0,i})^2}{4(|k|^2 + r_{0,f})(|k|^2 + r_{0,i})} \sin \left( t\sqrt{|k|^2 + r_{0,f}} \right)^2. \]  

(22)

From this expression, and in particular from its low-\( k \) behavior, one can find the behavior of all the cumulants, as explained in the main text. We see that apart from the sine that provides an infrared cutoff evolving as \( 1/t \), when \( r_{0,f} \neq 0 \), \( \rho_k \) is regular at low \( k \), while for \( r_{0,f} = 0 \), which is the critical point of the free theory, \( \rho_k \sim 1/k^2 \), implying \( k_n \sim t^{2n-d} \), with \( k_n \) denoting the \( n \)-th cumulant and \( t^0 \) indicating a logarithmic growth.

**Supplemental results**

A systematic study of the time dependent fluctuations in \( d = 3 \) for various parameters of our model is shown in Fig. 7.

Instead, in Fig 8 the result obtained for the variance in \( d = 4 \) are shown. Here it is possible to notice that, as mentioned in the main text, the variance has still three different qualitative behavior: saturation if \( r_{0,f} \) is above the dynamical critical point, logarithmic growth if \( r_{0,f} \) is at the dynamical critical point and power law growth, with exponent equal to two if \( r_{0,f} \) is below the dynamical critical point.

![Figure 7](image-url)  
Figure 7: Fluctuations of excitations for quenches (a) above, (b) to, and (c) below the dynamic phase transition for various parameters as indicated in the figure caption and legend.
Insights into the nonequilibrium crossover diagram taking into account the full many-body interactions. In one-dimension no dynamical criticality is expected, however, our simulations provide integrability, we also study the quench dynamics in the nonintegrable, one-dimensional Bose-Hubbard model using exact numerical techniques. In our simulations all cumulants saturate. We attribute this to the fact that the nonlinearities are fully treated in the exact simulations and therefore the unbounded growth observed in the field theory gets regularized. In Fig. 9 we show the saturation value of the first and second cumulant (average and variance, respectively) normalized by the volume of the system for various quench parameters.

The statistics of excitations evaluated in the thermal state seem to be largest. Note, however, that the deviations from the thermal results are vanishingly small for all $J_f/U_f$ at which the equilibrium gap corresponds to the equilibrium phase transition, gray thick line, small deviations between the thermal and the long time average can be observed. However, we study rather small systems of $L = 8$ sites and in order to make a conclusive statement in that regime a proper finite size scaling needs to be done. The thick dashed line shows the ratio of $J_f/U_f$ at which the equilibrium gap corresponds to the energy density of the quantum quench. Around this coupling the deviations of the global defect statistics from the thermalized state seem to be largest. Note, however, that the deviations from the thermal results are vanishingly small for all $J_f/U_f$ when we consider the statistics of local (instead of global) defects, not shown. In the right column of Fig. 9(a) we show the ratio of the cumulants. The trend here is that the larger $J_f/U_f$, the larger the ratios $C_1/C_2$ and $C_2/C_3$, which is opposite to the prediction of the field theory for higher dimension, see inset of Fig. 4(a) in the

Nonintegrable one-dimensional Bose-Hubbard model

In order to demonstrate that in the case of lattice bosons the emergence of prethermalization is a consequence of integrability, we also study the quench dynamics in the nonintegrable, one-dimensional Bose-Hubbard model using exact numerical techniques. In one-dimension no dynamical criticality is expected, however, our simulations provide insights into the nonequilibrium crossover diagram taking into account the full many-body interactions.

We introduce the Bose-Hubbard model on a lattice

$$\hat{H}_{BH} = -J \sum_{\langle i,j \rangle} (\hat{b}_i^\dagger \hat{b}_j + \text{h.c.}) + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i,$$

where $J$ is the kinetic energy, $U$ the interaction energy, and $\mu$ the chemical potential. The boson creation and annihilation operators are $\hat{b}_i^\dagger$ and $\hat{b}_i$, respectively, which define the density operator $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$. We follow the protocol of the double quench introduced in the main text, by starting out deep in the disordered phase at commensurate filling with $J/U = 0.01$, where the eigenstate is close to a product state. The dynamics is initialized by quenching the kinetic energy to $J_f/U_f$. Consequently the system evolves for the wait time $t$ at which the statistics of global defects $\bar{D} = \sum \vert \hat{n}_i - n \vert$ is measured, where $n$ is the density of bosons. Higher cumulants can be obtained from the generating function in the usual way. In our simulations all cumulants saturate. We attribute this to the fact that the nonlinearities are fully treated in the exact simulations and therefore the unbounded growth observed in the field theory gets regularized. In Fig. 9 we show the saturation value of the first and second cumulant (average and variance, respectively) normalized by the volume of the system for various quench parameters $J_f/U_f$ and commensurate density $n = 1$ and $n = 2$. We find that the more energy is pumped into the system by the quantum quench, i.e., the larger the final kinetic energy $J_f/U_f$ is, the larger is the saturation value of the global defect density $D$, and its higher order statistics.

Since the Bose-Hubbard model is not integrable, one expects that it thermalizes rather quickly. To study this effect, we perform finite temperature simulations in which the temperature is self-consistently determined by the energy density pumped into the system: $E^* = \exp[-\hat{H}_{BH}(J_f,U_f)/T^*]$. The statistics of excitations evaluated in the thermal state are indicated by the thin solid lines in Fig. 9(a) and support thermalization for large values of $J_f/U_f$ already after a few inverse hopping times. At low values of $J_f/U_f \lesssim 0.3$, which marks the equilibrium phase transition, gray thick line, small deviations between the thermal and the long time average can be observed. However, we study rather small systems of $L = 8$ sites and in order to make a conclusive statement in that regime a proper finite size scaling needs to be done. The thick dashed line shows the ratio of $J_f/U_f$ at which the equilibrium gap corresponds to the energy density of the quantum quench. Around this coupling the deviations of the global defect statistics from the thermalized state seem to be largest. Note, however, that the deviations from the thermal results are vanishingly small for all $J_f/U_f$ when we consider the statistics of local (instead of global) defects, not shown. In the right column of Fig. 9(a) we show the ratio of the cumulants. The trend here is that the larger $J_f/U_f$, the larger the ratios $C_1/C_2$ and $C_2/C_3$, which is opposite to the prediction of the field theory for higher dimension, see inset of Fig. 4(a) in the
An important difference between the Bose-Hubbard model at low filling and the field theory is that excitations of particles and holes are asymmetric in the former: While infinitely many particle excitations can be created in the Bose-Hubbard model on top of a certain state with commensurate filling \( n \), only \( n \) holes can be created locally. Therefore, one could expect, that for dynamics the agreement between field theory and exact simulations improves at higher filling. In Fig. 9(b) we thus show the saturation values when starting out at filling \( n = 2 \). The main difference here is that the ratio \( C_1/C_2 \) decreases for larger \( J_f/U_f \) similarly to the results obtained from the field theory in higher dimension.

Figure 9: Statistics of the global defect density in the nonintegrable one-dimensional Bose-Hubbard model. First and second cumulant as a function of the quench parameter \( J_f/U_f \), left, and their ratios, right, at filling (a) \( n = 1 \) and (b) \( n = 2 \). Vertical lines indicate the equilibrium phase transition, solid gray line, and the ratio \( J_f/U_f \) at which the equilibrium gap in the Mott phase corresponds to the energy density pumped into the system, dashed line. The thin solid lines show the thermal value of the cumulants at the self-consistently determined effective temperature.