Uniqueness of the representation for $G$-martingales with finite variation

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Abstract

Our purpose is to prove the uniqueness of the representation for $G$-martingales with finite variation.

Key words: uniqueness; representation theorem; $G$-martingale; finite variation; $G$-expectation

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1 Introduction

In [P07b], processes in form of $\int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s)ds$, $\eta \in M^1_G(0, T)$ are proved to be $G$-martingales. However, the uniqueness of the representation remains unresolved. In order to prove the uniqueness, we must find ways to distinguish the two classes of processes in forms of $\int_0^t \eta_s d\langle B \rangle_s$ and $\int_0^t \zeta_s ds$, $\eta, \zeta \in M^1_G(0, T)$.

For a process $\{K_t\}$ with finite variation, motivated by [Song10], we define

$$d(K) := \limsup_{n \to \infty} \hat{E}[\int_0^T \delta_n(s)dK_s],$$

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where, for $n \in \mathbb{N}$, $\delta_n(s)$ is defined in the following way:

$$\delta_n(s) = \sum_{i=0}^{n-1} (-1)^i 1_{\left[\frac{i}{n}, \frac{i+1}{n}\right]}(s), \text{ for all } s \in [0, T].$$

We prove that $d(K) = 0$ if $K_t = \int_0^t \zeta_s \, ds$ for some $\zeta \in M^1_G(0, T)$ and that $d(K) > 0$ if $K_t = \int_0^t \eta_s \, dB_s$ for some $\eta \in M^1_G(0, T)$ such that $\hat{E}[\int_0^T |\eta_s| \, ds] > 0$. By this, we distinguish these two classes of processes completely:

If $\int_0^t \eta_s \, dB_s = \int_0^t \zeta_s \, ds$, for some $\eta, \zeta \in M^1_G(0, T)$, then we have

$$\hat{E}[\int_0^T |\eta_s| \, ds] = \hat{E}[\int_0^T |\zeta_s| \, ds] = 0.$$

As an application, we prove the uniqueness of the representation for $G$-martingales with finite variation.

This article is organized as follows: In section 2, we recall some basic notions and results of $G$-expectation and the related space of random variables. In section 3, we present the main results and some corollaries. In section 4, we give the proofs to the main results.

## 2 Preliminaries

We recall some basic notions and results of $G$-expectation and the related space of random variables. More details of this section can be found in [P07a, P07b, P08, P10].

**Definition 2.1** Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ with $c \in \mathcal{H}$ for all constants $c$. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.

(b) Constant preserving: $\hat{E}(c) = c$.

(c) Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$.

(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X)$, $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

**Definition 2.2** Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$.
\[ \hat{E}_2[\varphi(X_2)], \forall \varphi \in C_{l,Lip}(R^n), \text{ where } C_{l,Lip}(R^n) \text{ is the space of real continuous functions defined on } R^n \text{ such that} \]
\[ |\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \forall x, y \in R^n, \]
where \( k \) depends only on \( \varphi \).

**Definition 2.3** In a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) a random vector \( Y = (Y_1, \cdots, Y_n) \), \( Y_i \in \mathcal{H} \) is said to be independent to another random vector \( X = (X_1, \cdots, X_m) \), \( X_i \in \mathcal{H} \) under \( \hat{E}(\cdot) \), denoted by \( Y \perp X \), if for each test function \( \varphi \in C_{l,Lip}(R^m \times R^n) \) we have \( \hat{E}[\varphi(X,Y)] = \hat{E}[\varphi(x,y)]_{x=y} \).

**Definition 2.4** (G-normal distribution) A \( d \)-dimensional random vector \( X = (X_1, \cdots, X_d) \) in a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called G-normal distributed if for each \( a, b \in R \) we have
\[ aX + b\hat{X} \sim \sqrt{a^2 + b^2}X, \]
where \( \hat{X} \) is an independent copy of \( X \). Here the letter \( G \) denotes the function
\[ G(A) := \frac{1}{2} \hat{E}[(AX, X)] : S_d \to R, \]
where \( S_d \) denotes the collection of \( d \times d \) symmetric matrices.

The function \( G(\cdot) : S_d \to R \) is a monotonic, sublinear mapping on \( S_d \) and \( G(A) = \frac{1}{2} \hat{E}[(AX, X)] \leq \frac{1}{2}|A|\hat{E}[|X|^2] = : \frac{1}{2}|A|\sigma^2 \) implies that there exists a bounded, convex and closed subset \( \Gamma \subset S_d^+ \) such that
\[ G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{Tr}(\gamma A). \tag{2.0.1} \]

If there exists some \( \beta > 0 \) such that \( G(A) - G(B) \geq \beta \text{Tr}(A - B) \) for any \( A \geq B \), we call the G-normal distribution is non-degenerate.

**Definition 2.5** i) Let \( \Omega_T = C_0([0, T]; R^d) \) with the supremum norm, \( \mathcal{H}_T^0 := \{ \varphi(B_{t_1}, \cdots, B_{t_n})| \forall n \geq 1, t_1, \cdots, t_n \in [0, T], \forall \varphi \in C_{l,Lip}(R^{d \times n}) \} \), G-expectation is a sublinear expectation defined by
\[ \hat{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})] \]
\[ = \hat{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)], \]
for all \( X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}}) \), where \( \xi_1, \cdots, \xi_n \) are identically distributed \( d \)-dimensional G-normal distributed random vectors in a sublinear expectation space \((\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\hat{E}})\) such that \( \xi_{i+1} \) is independent to \( (\xi_1, \cdots, \xi_i) \) for each \( i = 1, \cdots, m \). \((\Omega_T, \mathcal{H}_T^0, \hat{E})\) is called a G-expectation space.
ii) For \( t \in [0, T] \) and \( \xi = \varphi(B_{t_1}, \ldots, B_{t_m}) \in \mathcal{H}_T^0 \), the conditional expectation defined by (there is no loss of generality, we assume \( t = t_i \))

\[
\hat{E}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})] = \hat{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}),
\]

where

\[
\hat{\varphi}(x_1, \ldots, x_i) = \hat{E}[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_m} - B_{t_{m-1}})].
\]

Define \( \|\xi\|_{p,G} = [\hat{E}(|\xi|^p)]^{1/p} \) for \( \xi \in \mathcal{H}_T^0 \) and \( p \geq 1 \). Then \( \forall t \in [0, T] \), \( \hat{E}_t(\cdot) \) is a continuous mapping on \( \mathcal{H}_T^0 \) with norm \( \|\cdot\|_{1,G} \) and therefore can be extended continuously to the completion \( L^1_G(\Omega_T) \) of \( \mathcal{H}_T^0 \) under norm \( \|\cdot\|_{1,G} \).

Let \( L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_m})|n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in C_{b,Lip}(R^{d \times n})\} \), where \( C_{b,Lip}(R^{d \times n}) \) denotes the set of bounded Lipschitz functions on \( R^{d \times n} \).

[DHP08] proved that the completions of \( C_b(\Omega_T), \mathcal{H}_T^0 \) and \( L_{ip}(\Omega_T) \) under \( \|\cdot\|_{p,G} \) are the same and we denote them by \( L^p_G(\Omega_T) \).

**Definition 2.5** Let \( M^0_T(0, T) \) be the collection of processes in the following form: for a given partition \( \{t_0, \ldots, t_N\} = \pi_T \) of \([0, T]\),

\[
\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),
\]

where \( \xi_i \in L_{ip}(\Omega_{t_i}), i = 0, 1, 2, \ldots, N - 1 \). For \( p \geq 1 \) and \( \eta \in M^p_T(0, T) \), let \( \|\eta\|_{M^p_T} = \{\hat{E}(\int_0^T |\eta_s|^p ds)\}^{1/p} \) and denote by \( M^p_T(0, T) \) the completion of \( M^p_T(0, T) \) under the norm \( \|\cdot\|_{M^p_T} \).

**Theorem 2.6** ([DHP08]) There exists a tight subset \( \mathcal{P} \subset \mathcal{M}_1(\Omega_T) \) such that

\[
\hat{E}(\xi) = \max_{p \in \mathcal{P}} E_p(\xi) \quad \text{for all } \xi \in \mathcal{H}_T^0.
\]

\( \mathcal{P} \) is called a set that represents \( \hat{E} \).

**Remark 2.7** Let \( (\Omega^0, \{\mathcal{F}^0_t\}, \mathcal{F}^0, P^0) \) be a filtered probability space and \( \{W_t\} \) be a \( d \)-dimensional Brownian motion under \( P^0 \). [DHP08] proved that

\[
\mathcal{P}_M := \{P_0 \circ X^{-1}|X_t = \int_0^t h_s dW_s, h \in L^2_T([0, T]; \Gamma^{1/2})\}
\]

is a set that represents \( \hat{E} \), where \( \Gamma^{1/2} := \{\gamma^{1/2}|\gamma \in \Gamma\} \) and \( \Gamma \) is the set in the representation of \( G(\cdot) \) in the formula (2.0.1).


## 3 Main results

In the sequel, we only consider the $G$-expectation space $(\Omega_T, L^1_G(\Omega_T, \hat{\mathcal{E}}))$ with $\Omega_T = C_0([0, T], R)$ and $\hat{\sigma}^2 = \hat{\mathcal{E}}(B^2_t) > -\hat{\mathcal{E}}(-B^2_t) = \sigma^2 \geq 0$.

**Proposition 3.1** For each $\eta \in M^1_G(0, T)$, let

$$d(\eta) = \limsup_{n \to \infty} \hat{\mathcal{E}}[\int_0^T \delta_n(s) \eta_s d\langle B \rangle_s].$$

Then

$$-\frac{\hat{\sigma}^2 - \sigma^2}{2} \hat{\mathcal{E}}[- \int_0^T |\eta_s| ds] \leq d(\eta) \leq \frac{\hat{\sigma}^2 - \sigma^2}{2} \hat{\mathcal{E}}[\int_0^T |\eta_s| ds]. \quad (3.0.2)$$

**Proof.** It suffices to prove the conclusion for $\eta \in M^0_G(0, T)$. Let $\eta_s = \sum_{i=0}^{m-1} \xi_{t_i} 1_{[t_i, t_{i+1})}(s)$, $\xi_{t_i} \in L^1_G(\Omega_{t_i})$, $i = 0, \cdots, m - 1$.

\[
\hat{\mathcal{E}}[\int_0^T \delta_n(s) \eta_s d\langle B \rangle_s] - \frac{\hat{\sigma}^2 - \sigma^2}{2} \hat{\mathcal{E}}[- \int_0^T |\eta_s| ds] \\
= \hat{\mathcal{E}}[\sum_{i=0}^{m-1} |\xi_{t_i}| \int_{t_i}^{t_{i+1}} \delta_n(s) \text{sgn}(\xi_{t_i}) d\langle B \rangle_s] - \hat{\mathcal{E}}[\sum_{i=0}^{m-1} |\xi_{t_i}| \int_{t_i}^{t_{i+1}} \frac{\hat{\sigma}^2 - \sigma^2}{2} ds] \\
\leq \sum_{i=0}^{m-1} \hat{\mathcal{E}}[|\xi_{t_i}| (\int_{t_i}^{t_{i+1}} \delta_n(s) \text{sgn}(\xi_{t_i}) d\langle B \rangle_s - \int_{t_i}^{t_{i+1}} \frac{\hat{\sigma}^2 - \sigma^2}{2} ds)] \to 0
\]

as $n$ goes to infinity. So

$$d(\eta) \leq \frac{\hat{\sigma}^2 - \sigma^2}{2} \hat{\mathcal{E}}[\int_0^T |\eta_s| ds].$$

On the other hand,

\[
\hat{\mathcal{E}}[\int_0^T \delta_n(s) \eta_s d\langle B \rangle_s] + \frac{\hat{\sigma}^2 - \sigma^2}{2} \hat{\mathcal{E}}[- \int_0^T |\eta_s| ds] \\
= \hat{\mathcal{E}}[\sum_{i=0}^{m-1} |\xi_{t_i}| \int_{t_i}^{t_{i+1}} \delta_n(s) \text{sgn}(\xi_{t_i}) d\langle B \rangle_s] + \hat{\mathcal{E}}[\sum_{i=0}^{m-1} (|\xi_{t_i}|) \int_{t_i}^{t_{i+1}} \frac{\hat{\sigma}^2 - \sigma^2}{2} ds] \\
\geq \hat{\mathcal{E}}[\sum_{i=0}^{m-1} |\xi_{t_i}| (\int_{t_i}^{t_{i+1}} \delta_n(s) \text{sgn}(\xi_{t_i}) d\langle B \rangle_s - \int_{t_i}^{t_{i+1}} \frac{\hat{\sigma}^2 - \sigma^2}{2} ds)] \\
\geq \sum_{i=0}^{m-1} [-\hat{\mathcal{E}}(|\xi_{t_i}|) a_i(n)],
\]
where \( a_i(n) = \max\{|\hat{\mathcal{E}}(\int_{t_i}^{t_{i+1}} \delta_n(s)d\langle B \rangle_s - \int_{t_i}^{t_{i+1}} \frac{\sigma^2}{2} ds)|, |\hat{\mathcal{E}}(-\int_{t_i}^{t_{i+1}} \delta_n(s)d\langle B \rangle_s - \int_{t_i}^{t_{i+1}} \frac{\sigma^2}{2} ds)|\} \to 0 \) as \( n \) goes to infinity. So

\[
-\frac{\sigma^2 - \sigma^2}{2} \hat{\mathcal{E}}[-\int_0^T |\eta_s|ds] \leq d(\eta).
\]

\[\square\]

**Remark 3.2**

(i) A straightforward corollary of Proposition 3.1 is that if \( \int_0^T |\eta_s|ds \) is symmetric (i.e., \( \hat{\mathcal{E}}[\int_0^T |\eta_s|ds] = -\hat{\mathcal{E}}[-\int_0^T |\eta_s|ds] \)), the equality \( d(\eta) = \frac{\sigma^2 - \sigma^2}{2} \hat{\mathcal{E}}[\int_0^T |\eta_s|ds] \) holds.

(ii) By Lemma 3.1, we could not conclude that \( d(\eta) > 0 \) whenever \( \hat{\mathcal{E}}[\int_0^T |\eta_s|ds] > 0 \), which is the conclusion of Theorem 3.3 below.

(iii) The inequalities in (3.0.2) may be strict:

Let \( \eta_s = \langle B \rangle_{T/2}1_{T/2}(s) + a1_{[0,T/2]}(s), a = T(\sigma^2 - \sigma^2)/4 \).

Then

\[
d(\eta) = \lim_{n \to \infty} \hat{\mathcal{E}}[\int_0^T \delta_{2n}(s)\eta_sd\langle B \rangle_s] = a\sigma^2 T/2,
\]

\[
\frac{\sigma^2 - \sigma^2}{2} \hat{\mathcal{E}}[\int_0^T |\eta_s|ds] = a^2 + a\sigma^2 T/2,
\]

\[-\frac{\sigma^2 - \sigma^2}{2} \hat{\mathcal{E}}[-\int_0^T |\eta_s|ds] = -a^2 + a\sigma^2 T/2.
\]

\[\square\]

Now, we shall state the main result of this article, whose proof is postponed to Section 4.

**Theorem 3.3** For \( \eta \in M_G^1(0, T) \) with \( \hat{\mathcal{E}}[\int_0^T |\eta_s|ds] > 0 \), we have

\[
d(\eta) = \limsup_{n \to \infty} \hat{\mathcal{E}}[\int_0^T \delta_n(s)\eta_sd\langle B \rangle_s] > 0.
\]

**Theorem 3.4** Let \( \eta \in M_G^1(0, T) \). Then \( \lim_{n \to \infty} \hat{\mathcal{E}}[\int_0^T \delta_n(s)\eta_sd\langle B \rangle_s] = 0 \).

**Proof.** For \( \eta \in M_G^0(0, T) \), the claim is obvious. For \( \eta \in M_G^1(0, T) \), there exists a sequence of \( \{\eta^m\} \subset M_G^0(0, T) \) such that \( \hat{\mathcal{E}}[\int_0^T |\eta^m - \eta|ds] \to 0 \). Then

\[
|\hat{\mathcal{E}}[\int_0^T \delta_n(s)\eta^m_sds] - |\hat{\mathcal{E}}[\int_0^T \delta_n(s)\eta^m_sds]| \leq \hat{\mathcal{E}}[\int_0^T \delta_n(s)\eta^m_sds] + \hat{\mathcal{E}}[\int_0^T |\eta^m - \eta|ds].
\]

First let \( n \to \infty \), then let \( m \to \infty \), and we get the desired result. \[\square\]

**Remark 3.5** Let \( (\Omega, F, \mathcal{F}, P) \) be a filtered probability space. We recall that for any progressively measurable process \( \eta \) such that \( \hat{\mathcal{E}}[\int_0^T |\eta_s|ds] < \infty \), we
have
\[ \lim_{n \to \infty} \hat{E} \left[ \int_0^T \delta_n(s) \eta_s ds \right] = 0. \]

Therefore, Theorem 3.3 presents a particular property of $G$-expectation space relative to probability space.

**Corollary 3.6** Let $\zeta, \eta \in M^1_G(0, T)$. If $\int_0^t \eta_s d\langle B \rangle_s = \int_0^t \zeta_s ds$ for all $t \in [0, T]$, then $E[\int_0^T \eta_s ds] = \hat{E}[\int_0^T \zeta_s ds] = 0$.

**Proof.** By Theorem 3.4, we have
\[ \lim_{n \to \infty} \hat{E} \left[ \int_0^T \delta_n(s) \eta_s d\langle B \rangle_s \right] = \lim_{n \to \infty} \hat{E} \left[ \int_0^T \delta_n(s) \zeta_s ds \right] = 0. \]

By Theorem 3.3, we have $\hat{E}[\int_0^T \eta_s ds] = 0$, which leads to $\hat{E}[\int_0^T \zeta_s ds] = 0$. □

The following corollary is about the uniqueness of representation for $G$-martingales with finite variation.

**Corollary 3.7** Let $\zeta, \eta \in M^1_G(0, T)$. If for all $t \in [0, T]$,
\[ \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds = \int_0^t \zeta_s d\langle B \rangle_s - \int_0^t 2G(\zeta_s) ds, \quad (3.0.3) \]
we have $\hat{E}[\int_0^T |\eta_s - \zeta_s| ds] = 0$.

**Proof.** By the assumption, we have
\[ \int_0^t (\eta_s - \zeta_s) d\langle B \rangle_s = \int_0^t 2[G(\eta_s) - G(\zeta_s)] ds, \quad \text{for all } t \in [0, T]. \]

Since $\eta - \zeta, 2[G(\eta) - G(\zeta)] \in M^1_G(0, T)$, we have $\hat{E}[\int_0^T |\eta_s - \zeta_s| ds] = 0$ by Corollary 3.6. □

**Remark 3.8** (i) In the setting considered in this article, $G(a) = \frac{1}{2}(\sigma^2 a^+ - \sigma^2 a^-)$. For $\varepsilon \in (0, \frac{\sigma^2 \sigma^2}{2})$, [HuP10] defined $G_\varepsilon$ in the following way:
\[ G_\varepsilon(a) = G(a) - \frac{\varepsilon}{2} |a|, \quad \text{for all } a \in \mathbb{R}. \]

Indeed, Proof to Theorem 3.3 in the next section leads to the following conclusion:
\[ d(\eta) \geq \varepsilon \hat{E}_{G_\varepsilon} [\int_0^T |\eta_s| ds]. \quad (3.0.4) \]
(ii) For \( \eta \in M^1_G(0, T) \), let \( K_t = \int_0^T \eta_s d\langle B \rangle_s - \int_0^T 2G(\eta_s) ds \). Then, by Theorem 3.4, we have
\[
\hat{E}(-K_T) \geq \limsup_{n \to \infty} \hat{E}\left( \int_0^T \delta_n(s) dK_s \right) = d(\eta). 
\]
This, combined with (3.0.4), leads to the following estimate:
\[
\hat{E}\left( -K_T \right) \geq \varepsilon \hat{E}\left( \int_0^T |\eta_s| ds \right),
\]
which was already proved in [HuP10]. Then, for \( \eta, \zeta \in M^1_G(0, T) \) such that (3.0.5) and
\[
\int_0^t 2[G(\eta_s) - G(\zeta_s)] ds = \int_0^t 2[G(\eta_s - \zeta_s)] ds \text{ for all } t \in [0, T] \tag{3.0.6}
\]
hold, we have \( \hat{E}\left( \int_0^T |\eta_s - \zeta_s| ds \right) = 0 \). However, (3.0.6) does not hold generally since the nonlinearity of \( G \), which is the main difficulty to deal with such questions. □

4 Proof to Theorem 3.3

In order to prove Theorem 3.3, we first introduce two lemmas.

Let \( \Omega_T = C_b([0, T]; R) \) be endowed with the supremum norm and let \( \sigma : [0, T] \times \Omega_T \to R \) be a measurable mapping satisfying
\begin{enumerate}[i)]
\item \( \sigma \) is bounded;
\item There exists \( C > 0 \) such that \( |\sigma(s, x) - \sigma(s, y)| \leq C \|x - y\| \) for any \( s \in [0, T] \) and \( x, y \in C_b([0, T]; R) \);
\item For \( t \in [0, T] \), \( \sigma(t, \cdot) \) is \( \mathcal{B}_t(\Omega_T) \) measurable.
\end{enumerate}

Then the following lemma is easy.

**Lemma 4.1** Let \( (\Omega, F, F, P) \) be a filtered probability space and let \( M \) be a continuous \( F \)-martingale with \( \langle M \rangle_t - \langle M \rangle_s \leq C(t - s) \) for some \( C > 0 \) and any \( 0 \leq s < t \leq T \). Let \( F^X \) be the augmented filtration generated by \( X \). Then for any \( Y_0 \in F_0^X \), there exists a unique \( F \)-adapted continuous process with \( E[\sup_{t \in [0, T]} |Y_t|^2] < \infty \) such that \( Y_t = Y_0 + \int_0^t \sigma(s, Y) dX_s \). Moreover, \( Y \) is \( F^X \)-adapted. □

Let \( (\Omega, F, P) \) be a probability space and let \( \{W_t\} \) be a standard 1-dimensional Brownian motion on \( (\Omega, F, P) \). Let \( F^W \) be the augmented filtration generated by \( W \).
Denote by $A^0([c, C])$, for some $0 < c \leq C < \infty$, the collection of $F^W$ adapted processes in the following form

$$h_s = \sum_{i=0}^{m-1} \xi_i \sqrt{\frac{1}{m}} (i+1)T_m(s),$$

where $\xi_i = \psi_i(\int_{\frac{iT_m}{m}}^{\frac{(i+1)T_m}{m}} h_s dW_s, \cdots, \int_{\frac{(i-1)T_m}{m}}^{\frac{iT_m}{m}} h_s dW_s)$, $\psi_i \in C_{b, \text{lip}}(R^i)$, $c \leq |\psi_i| \leq C$.

Denote by $A([c, C])$ the collection of $F^W$ adapted processes such that $c \leq |h_s| \leq C$.

**Lemma 4.2** $A^0([c, C])$ is dense in $A([c, C])$ under the norm

$$\|h\|_2 = [E(\int_0^T |h_s|^2 ds)]^{1/2}.$$

**Proof.** Let $h_s = \sum_{i=0}^{m-1} \xi_i \sqrt{\frac{1}{m}} (i+1)T_m(s)$, where

$$\xi_i = \varphi_i(\frac{W_{(i+1)T_m}}{m} - \frac{W_{iT_m}}{m}, \cdots, \frac{W_{iT_m}}{m} - \frac{W_{0}}{m}),$$

$$\varphi_i \in C_{b, \text{lip}}(R^i), c \leq |\varphi_i| \leq C.$$

Then $\sigma(s, x) = h_s^{-1}(x)$ is a bounded Lipschitz function. Let $X_t := \int_0^t h_s dW_s$.

Since $W_t = \int_0^t \sigma(s, W)dX_s$, we conclude, by Lemma 4.1, that $W$ is $F^X$ adapted.

For a process $\{X_t\}$, we denote the vector $(X_T - X_{\frac{(m-1)T}{m}}, \cdots, X_{\frac{T}{m}} - X_0)$ by $X_{\frac{m}{0,T}}^m$.

For arbitrary $\varepsilon_i > 0, i = 0, \cdots, m - 1$, there exists $\psi_i \in C_{b, \text{lip}}(R^{n_i})$ with the Lipschitz constant $L_i$ such that $E[|\xi_i - \tilde{\xi}_i|^2] < \varepsilon_i^2$. Here $\tilde{\xi}_i = \psi_i(\frac{X_{\frac{m}{0,T}}^{n_i}}{\frac{T}{m}})$, $c \leq |\psi_i| \leq C$. Without loss of generality, we assume that there exists $K_{ji} \in N$ such that $n_j = K_{ji}n_i$ for $m - 1 \geq i > j \geq 0$.

Define $\tilde{\xi}_i$ in the following way:

$\tilde{\xi}_0 = \tilde{\xi}_0$,

For $s \in [0, \frac{T}{m}], \tilde{h}_s = \tilde{\xi}_0$.

Assume that we have defined $\tilde{h}_s$ for all $s \in [0, \frac{T}{m}], 0 \leq i \leq m - 1$.

Define $\tilde{X}_t := \int_0^t \tilde{h}_s dW_s$, for $t \in [0, \frac{T}{m}]$,

$\tilde{\xi}_i = \psi_i(\tilde{X}_{\frac{m}{0,T}}^{n_i})$,

For $s \in [\frac{iT}{m}, \frac{(i+1)T}{m}], \tilde{h}_s = \tilde{\xi}_i$.  

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We claim that for any \( m - 1 \geq i \geq 1 \),
\[
\hat{E}[|\hat{\xi}_i - \tilde{\xi}_i|^2] \leq \sum_{j=0}^{i-1} A_j^i \varepsilon_j^2,
\]
(4.0.7)

where \( A_j^i = 2TL_j^2 (\sum_{k=j+1}^{i-1} A_k^j + 1) \), for \( i \geq j + 2 \), \( A_{i-1}^i = 2TL_i^2 \), which shows that \( A_j^i \) depends only on \( L_{j+1}, \ldots, L_i \) and \( T \).

Indeed, \( E[|\xi_1 - \hat{\xi}_1|^2] \leq L_1^2 E[|\hat{\xi}_0 - \xi_0|^2] E[|W_{m,1}^{n_1,0}|^2] = \frac{T}{m} L_1^2 \varepsilon_0^2 \leq A_1^0 \varepsilon_0^2 \).

Assume (4.0.7) holds for \( 1 \leq i \leq l \). For \( i = l + 1 \),
\[
E[|\hat{\xi}_{l+1} - \tilde{\xi}_{l+1}|^2]
\leq L_{l+1}^2 \sum_{i=0}^{l} E[|\hat{\xi}_i - \xi_i|^2] E[|W_{m,1}^{n_{l+1},l+1}|^2]
\leq 2TL_{l+1}^2 \sum_{i=0}^{l} E[(|\hat{\xi}_i - \tilde{\xi}_i|^2 + |\tilde{\xi}_i - \xi_i|^2)]
\leq 2TL_{l+1}^2 \sum_{i=0}^{l} \left( \varepsilon_i^2 + \sum_{j=0}^{i-1} A_j^i \varepsilon_j^2 \right)
\leq 2TL_{l+1}^2 \sum_{j=0}^{l} \left( A_j^i + 1 \right) \varepsilon_j^2 + \varepsilon_l^2
\leq \sum_{j=0}^{l} A_j^{l+1} \varepsilon_j^2.
\]

Then
\[
E[|\hat{\xi}_i - \xi_i|^2]
\leq 2(E[|\hat{\xi}_i - \tilde{\xi}_i|^2] + E[|\tilde{\xi}_i - \xi_i|^2])
\leq 2 \varepsilon_i^2 + 2 \sum_{j=0}^{i-1} A_j^i \varepsilon_j^2
\leq \sum_{j=0}^{i} B_j^i \varepsilon_j^2,
\]

which shows that \( B_j^i \) depends only on \( L_{j+1}, \ldots, L_i \) and \( T \). So for any \( \varepsilon > 0 \), we can choose \( \hat{\xi}_i, i = 0, \ldots, m - 1 \) defined above such that \( E[|\hat{\xi}_i - \xi_i|^2] < \varepsilon \) for all \( i = 0, \ldots, m - 1 \). Then
\[
E[\int_0^T |h_s - \hat{h}_s|^2] < T \varepsilon.
\]
Proof to Theorem 3.3. For \( \eta \in M^1_G(0, T) \) with \( E[\int_0^T |\eta_s| ds] > 0 \), by Theorem 2.6 and Remark 2.7, there exists \( \varepsilon > 0 \) and \( P \in \mathcal{P}_M \) such that \( E_P[\int_0^T |\eta_s| ds] =: A > 0 \) and for any \( 0 \leq s < t \leq T \)
\[
(\sigma^2 + \varepsilon)(t - s) \leq \langle B \rangle_t - \langle B \rangle_s \leq (\sigma^2 - \varepsilon)(t - s), \quad P\hbox{-a.s.}
\]
For any \( \frac{A\varepsilon}{(\sigma^2 + \varepsilon)} > \delta > 0 \), there exists \( \zeta \in M^0_G(0, T) \) such that
\[
E[\int_0^T |\eta_s - \zeta_s| ds] < \delta.
\]

Let \( (\Omega^0, F = \{\mathcal{F}^0_t\}, \mathcal{F}^0, P^0) \) be a filtered probability space, and \( \{W_t\} \) be a \( d \)-dimensional Brownian motion under \( P^0 \). By Remark 2.7, there exists an \( F \) adapted process \( h \) with \( \sigma^2 + \varepsilon \leq h^2_s \leq \sigma^2 - \varepsilon \) such that \( P = P^0 \circ (\int_0^T h_s dW_s)^{-1} \).

Without loss of generality, by Lemma 4.2, we assume that there exists \( m \in \mathbb{N} \) such that
\[
\zeta_s = \sum_{i=0}^{m-1} \xi_{\frac{i}{m}}^T 1_{[\frac{i}{m}, \frac{i+1}{m})} (s)
\]
where \( \xi_{\frac{i}{m}} = \varphi_i(B_{\frac{i}{m}} - B_{\frac{i-1}{m}}, \cdots, B_{\frac{i}{m}} - B_0), \varphi_i \in C_{b, lip}(\mathbb{R}^d) \), for all \( 0 \leq i \leq m - 1 \);
\[
h_s = \sum_{i=0}^{m-1} a_{\frac{i}{m}}^T 1_{[\frac{i}{m}, \frac{i+1}{m})} (s)
\]
where \( a_{\frac{i}{m}} = \psi_i(\int_{\frac{i}{m}}^{\frac{i+1}{m}} h_s dW_s, \cdots, \int_{\frac{i}{m}}^{\frac{i+1}{m}} h_s dW_s) \), \( \sigma^2 + \varepsilon \leq |\psi_i|^2 \leq \sigma^2 - \varepsilon \), \( \psi_i \in C_{b, lip}(\mathbb{R}^d) \), for all \( 0 \leq i \leq m - 1 \).

1. Define \( H^i : [\sigma^2 + \varepsilon, \sigma^2 - \varepsilon] \rightarrow [\sigma^2, \sigma^2] \), \( i = 1, -1 \) in the following way:
\[
H^1(x)^2 = \sigma^2 1_{[x \geq \frac{\sigma^2 - \varepsilon + \varepsilon}{2}]} + (2x - \sigma^2) 1_{[x < \frac{\sigma^2 - \varepsilon + \varepsilon}{2}]};
\]
\[
H^{-1}(x)^2 = (2x - \sigma^2) 1_{[x \geq \frac{\sigma^2 - \varepsilon + \varepsilon}{2}]} + \sigma^2 1_{[x < \frac{\sigma^2 - \varepsilon + \varepsilon}{2}]}.
\]
It’s easily seen that \( H^1(x)^2 + H^{-1}(x)^2 = 2x \) and \( H^1(x)^2 - H^{-1}(x)^2 \geq 2\varepsilon \).

For \( n \in \mathbb{N} \), define \( H^i_n : [0, 1/m] \times [\sigma^2 + \varepsilon, \sigma^2 - \varepsilon] \rightarrow [\sigma^2, \sigma^2], i = 1, -1 \) by
\[
H^i_n(s, x) = \sum_{j=0}^{2n-1} 1_{\left[\frac{j}{2mn}, \frac{(j+1)}{2mn}\right]}(s) H^{(-1)^i}(x).
\]
2. Fix \( n \in \mathbb{N} \).
\[ a_0^n = a_0, \xi_0^n = \xi_0. \]

For \( s \in [0, \frac{T}{m}] \), \( h_s^n = H^{s_{\text{sgn}}(\xi_0^n)}_n(s, (a_0^n)^2) \);

Assume that we have defined \( h_s^n \) for all \( s \in [0, \frac{T}{m}] \), \( 0 \leq i \leq m - 1 \).

\[ a_{\frac{i}{m}}^n = \psi_i(\int_{\frac{i}{m}}^{\frac{i+1}{m}} h_s^n dW_s, \ldots, \int_{0}^{\frac{i}{m}} h_s^n dW_s), \]
\[ \xi_{\frac{i}{m}}^n = \varphi_i(\int_{\frac{i}{m}}^{\frac{i+1}{m}} h_s^n dW_s, \ldots, \int_{0}^{\frac{i}{m}} h_s^n dW_s), \]

For \( s \in \left[ \frac{T}{m}, \frac{(i+1)T}{m} \right] \), \( h_s^n = H_n^{s_{\text{sgn}}(\xi_{\frac{i}{m}}^n)}(s - \frac{iT}{m}, (a_{\frac{i}{m}}^n)^2) \).

3. \( E_P[\int_0^T |\zeta_s| ds] = E_{P_0}[\int_0^T |\zeta_s| ds]. \)

In fact,

\[
E_P[\int_0^T |\zeta_s| ds] = \frac{T}{m} E_P \sum_{i=0}^{m-1} |\varphi_i(\int_{\frac{i}{m}}^{\frac{i+1}{m}} h_s dW_s, \ldots, \int_{0}^{\frac{i}{m}} h_s dW_s)|
\]
\[
= E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{\frac{T}{m}} h_s dW_s, \ldots, \int_{0}^{\frac{T}{m}} h_s dW_s)]
\]

and

\[
E_{P_0}[\int_0^T |\zeta_s| ds] = E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{\frac{T}{m}} h_s dW_s, \ldots, \int_{0}^{\frac{T}{m}} h_s dW_s)].
\]

Let \( x = (x_{m-1}, \ldots, x_1) \). Noting that

\[
\Phi_{m-1}(x) := E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{\frac{T}{m}} H_n^{s_{\text{sgn}}(\varphi_{m-1}(x))}(s - \frac{(m-1)T}{m}, \psi_{m-1}(x)^2 dW_s, x))]
\]
\[
= E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{\frac{T}{m}} \psi_{m-1}(x) dW_s, x)],
\]

we have

\[
E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{\frac{T}{m}} h_s dW_s, \ldots, \int_{0}^{\frac{T}{m}} h_s dW_s)] = E_{P_0}[\Phi_{m-1}(\int_{\frac{(m-2)T}{m}}^{\frac{(m-1)T}{m}} h_s dW_s, \ldots, \int_{0}^{\frac{T}{m}} h_s dW_s)]
\]
\[
E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{\frac{T}{m}} h_s^n dW_s, \ldots, \int_{0}^{\frac{T}{m}} h_s^n dW_s)] = E_{P_0}[\Phi_{m-1}(\int_{\frac{(m-2)T}{m}}^{\frac{(m-1)T}{m}} h_s^n dW_s, \ldots, \int_{0}^{\frac{T}{m}} h_s^n dW_s)].
\]
By induction on $m$, we get the desired result.

4.

\[
\hat{E}\left[\int_0^T \delta_{2mn}(s)\eta_s d\langle B\rangle_s\right] \\
\geq \hat{E}\left[\int_0^T \delta_{2mn}(s)\zeta_s d\langle B\rangle_s\right] - \hat{E}\left[\int_0^T |\eta_s - \zeta_s|d\langle B\rangle_s\right] \\
\geq E_{P_{\bar{\sigma}^2}}\left[\int_0^T \delta_{2mn}(s)\zeta_s d\langle B\rangle_s\right] - \bar{\sigma}^2\delta \\
= E_{P_{\bar{\sigma}^2}}\left[\sum_{i=0}^{m-1} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} \delta_{2mn}(s)\zeta_s d\langle B\rangle_s\right] - \bar{\sigma}^2\delta \\
= E_{P_{\bar{\sigma}^2}}\left[\sum_{i=0}^{m-1} \xi_{\frac{mT}{i+1}} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} \delta_{2mn}(s)d\langle B\rangle_s\right] - \bar{\sigma}^2\delta \\
\geq \frac{T}{m} \varepsilon E_{P_{\bar{\sigma}^2}}\left[\sum_{i=0}^{m-1} |\xi_{\frac{mT}{i+1}}|\right] - \bar{\sigma}^2\delta \\
= \varepsilon E_{P_{\bar{\sigma}^2}}\left[\int_0^T |\zeta_s|ds\right] - \bar{\sigma}^2\delta \\
= \varepsilon E_P\left[\int_0^T |\zeta_s|ds\right] - \bar{\sigma}^2\delta \\
\geq \varepsilon E_P\left[\int_0^T |\eta_s|ds\right] - \varepsilon\delta - \bar{\sigma}^2\delta \\
\geq A\varepsilon - \varepsilon\delta - \bar{\sigma}^2\delta > 0.
\]

Since $A, \varepsilon, \delta$ do not depend on $n$, we have $d(\eta) \geq A\varepsilon - \varepsilon\delta - \bar{\sigma}^2\delta > 0$. The proof is completed. □

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