The space of free coherent states is isomorphic to the space of distributions on $p$-adic numbers

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Abstract

Free coherent states for a system with $p$ degrees of freedom are defined. An isomorphism of the space of free coherent states to the space of distributions on $p$-adic disk is constructed.

1 Construction of free (or Boltzmannian) coherent states

Free (or Boltzmannian) Fock space has been considered in some recent works on quantum chromodynamics [1], [2], [3] and noncommutative probability [4], [5], [6]. The subject of this work is free coherent states. We will introduce free coherent states and investigate the space of coherent states corresponding to a fixed eigenvalue of the operator of annihilation.

In this paper the connection between the theory of free Fock space and $p$-adic analysis will be established. The main result of the present paper is a construction of an isomorphism of the space of coherent states to the space of distributions on the ring of integer $p$-adic numbers.

We will consider the system with $p$ degrees of freedom. The system with one degree of freedom was investigated in [7].

The free commutation relations are particular case of $q$-deformed relations

$$A_i A_j^\dagger - q A_j^\dagger A_i = \delta_{ij}$$

with $q = 0$. A correspondence of $q$-deformed commutation relations and non-archimedean (ultrametric) geometry was discussed in [8]. Non-archimedean mathematical physics was studied in [10].

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Free coherent states lie in the free Fock space. Free (or Boltzmannian) Fock space $F$ over a Hilbert space $H$ is the completion of the tensor algebra

$$F = \bigoplus_{n=0}^{\infty} H^\otimes n.$$ 

Creation and annihilation operators are defined in the following way:

$$A^\dagger(f)f_1 \otimes \ldots \otimes f_n = f \otimes f_1 \otimes \ldots \otimes f_n$$

$$A(f)f_1 \otimes \ldots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \ldots \otimes f_n$$

where $\langle f, g \rangle$ is the scalar product in the Hilbert space $H$. Scalar product in the free Fock space is defined by the standard construction of the direct sum of tensor products of Euclidean spaces.

We consider the case when $H$ is $p$-dimensional Euclidean space. In this case we have $p$ creation operators $A^\dagger_i$, $i = 0, \ldots, p - 1$ and $p$ annihilation operators $A_i$, $i = 0, \ldots, p - 1$ with commutation relations

$$A_i A_j^\dagger = \delta_{ij}. \quad (1)$$

The vacuum vector $\Omega$ in the free Fock space satisfies

$$A_i \Omega = 0. \quad (2)$$

We introduce the linear space of free coherent states as the space of formal eigenvectors of the annihilation operator $A = \sum_{i=0}^{p-1} A_i$ for the eigenvalue $\lambda$,

$$A \Psi = \lambda \Psi.$$ 

It is easy to see that the formal solution of this equation is

$$\Psi = \sum_I \lambda^{|I|} \Psi_I A_I^\dagger \Omega. \quad (3)$$

Here the multiindex $I = i_0 \ldots i_{k-1}$, $i_j \in \{0, \ldots, p - 1\}$ and $A_I^\dagger = A_{i_{k-1}}^\dagger \ldots A_{i_0}^\dagger$, $\Psi_I$ are complex numbers that satisfy

$$\Psi_I = \sum_{i=0}^{p-1} \Psi_I i_i.$$ 

Summation in the formula (3) runs on all sequences $I$ with the finite length. The length of a sequence $I$ is denoted by $|I|$. This formal series define a functional with a dense domain in the free Fock space.

An arbitrary free coherent state $\Psi$ can be constructed in the following way. $\Psi$ is defined by the function $\Psi_I$. Let us construct this function. Let us define $\Psi_\emptyset$ as an arbitrary complex number. Then we use the inductive procedure: if we know $\Psi_I$ we define $\Psi_{Ii}$ as arbitrary complex numbers that satisfy the formula $\Psi_I = \sum_{i=0}^{p-1} \Psi_{Ii}$. It is easy to see that the function $\Psi_I$ satisfies the following formula

$$\Psi_I = \sum_{|J|=j} \Psi_{IJ} \quad (4)$$

for arbitrary $j$.

We will denote the linear space of free coherent states as $X'$ because this space is isomorphic to the space $D'(Z_p)$ of distributions on the $p$-adic disk $Z_p$. 

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2 A brief review of $p$-adic analysis

Let us make a brief review of $p$-adic analysis. The field $\mathbb{Q}_p$ of $p$-adic numbers is the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$-adic norm on $\mathbb{Q}$. This norm is defined in the following way. An arbitrary rational number $x$ can be written in the form $x = p^\gamma \frac{m}{n}$ with $m$ and $n$ that are not divisible by $p$. The $p$-adic norm of the rational number $x = p^\gamma \frac{m}{n}$ is equal to $||x||_p = p^{-\gamma}$.

The most interesting property of the field of $p$-adic numbers is ultrametricity. This means that $\mathbb{Q}_p$ obeys the strong triangle inequality

\[ ||x - y||_p \leq \max(||x - z||_p, ||z - y||_p). \]

We will consider disks in $\mathbb{Q}_p$ of the form $\{x \in \mathbb{Q}_p : ||x - x_0||_p \leq p^{-k}\}$. For example, the ring $\mathbb{Z}_p$ of integer $p$-adic numbers is the disk $\{x \in \mathbb{Q}_p : ||x||_p \leq 1\}$. The main properties of disks in arbitrary ultrametric space are the following:

1. Every point of a disk is the center of this disk.
2. Arbitrary two disks either do not intersect or one of these disks contains another.

The main result of the present paper is the isomorphism of the space $X'$ of free coherent states and the space $D'(\mathbb{Z}_p)$ of distributions on the $p$-adic disk. The space $D'(\mathbb{Z}_p)$ of distributions on the $p$-adic disk is the space of linear functionals on the linear space of locally constant complex valued functions. The space of locally constant functions is the linear span of indicators of $p$-adic disks with the radius that less or equal to 1. Here indicator of a set $S$ is the function that equals to 1 on this set and equals to 0 outside the set $S$.

For further reading on the subject of $p$-adic analysis see [9], [10].

3 Construction of an isomorphism of the space of coherent states to the space of distributions on the $p$-adic disk

In the present section the isomorphism $\phi$ of the space $X'$ of free coherent states to the space of distributions on the $p$-adic disk $\mathbb{Z}_p$ of integer $p$-adic numbers will be constructed. In the present section we take the eigenvalue $\lambda$, see (3), from the interval $(0, \sqrt{p})$.

Distributions on the $p$-adic disk are linear functionals on the space of locally constant functions [4], [10]. Therefore we have to construct the coherent state that corresponds to a locally constant function. It is sufficient to find coherent states that correspond to locally constant functions of the type

\[ \theta_k(x - x_0) = \theta(p^k ||x - x_0||_p); \quad \theta(t) = 0, t > 1; \quad \theta(t) = 1, t \leq 1. \]

Here $x, x_0 \in \mathbb{Z}_p$ lie in the ring of integer $p$-adic numbers and the function $\theta_k(x - x_0)$ equals to 1 on the disk $D(x_0, p^{-k})$ of radius $p^{-k}$ with the center in $x_0$ and equals to 0 outside this disk.
Let us consider the multiindex $I = i_0 \ldots i_{k-1}$, $i_j = 0, \ldots, p - 1$. Let us introduce the free coherent state $X_I$ of the form

$$X_I = \sum_{k=0}^{\infty} \lambda^k \left( \sum_{i=0}^{p-1} A_i^\dagger \right)^k \lambda^{|I|} A_I^\dagger \Omega + \sum_{l=1}^{\infty} \lambda^{-l} \left( \sum_{i=0}^{p-1} A_i \right)^l \lambda^{|I|} A_I^\dagger \Omega. \quad (5)$$

The sum on $l$ in fact contains $|I|$ terms. Under the isomorphism $\phi$ mentioned above the coherent state $X_I$ will correspond to the locally constant function $\phi_{\theta,I}(x-I)$, where $||\theta_I(x-I)||$ is $L_2$-norm. Here we identify the sequence $I = i_0 \ldots i_{k-1}$ and the $p$-adic number $I = \sum_{b=0}^{k-1} i_j p^j$.

For $\lambda \in (0, \sqrt{p})$ the coherent state $X_I$ lies in the Hilbert space (the correspondent functional is bounded).

We will prove that an arbitrary locally constant function corresponds to the linear combination of coherent states $X_I$. The linear span of the coherent states $X_I$ we will denote by $X$.

Every vector in $X$ is a function of $\lambda$. We will investigate the properties of the space $X$ with the scalar product

$$\langle X_I, X_J \rangle = \lim_{\lambda \to \sqrt{p}-0} \left( 1 - \frac{\lambda^2}{p} \right) (X_I, X_J). \quad (6)$$

Here $(X_I, X_J)$ is the scalar product in the free Fock space.

We will prove the following lemma.

**Lemma 1.** The limit of the scalar product of coherent states $X_I$, $X_J$ equals to the integral on $p$-adic disk with respect to the Haar measure

$$\lim_{\lambda \to \sqrt{p}-0} \left( 1 - \frac{\lambda^2}{p} \right) (X_I, X_J) =$$

$$= \frac{1}{\mu(D(I, p^{-|I|})) \mu(D(J, p^{-|J|}))} \int_{Z_p} \phi_{\theta,I}(x-I) \phi_{\theta,J}(x-J) dx =$$

$$= \left( \frac{\theta_I(x-I)}{||\theta_I(x-I)||^2}, \frac{\theta_J(x-J)}{||\theta_J(x-J)||^2} \right)_{L_2}.$$

Here $\mu(D)$ is the Haar measure of the disk $D$.

**Proof**

Let us calculate the scalar product of coherent states $X_I$ and $X_J$. Let $|I| \leq |J|$. If the the sequence $I$ coincides with the first $|I|$ indices of the sequence $J$ then the scalar product has the following form

$$(X_I, X_J) = \sum_{i=0}^{[I]-1} \lambda^{2i} + \sum_{i=|I|}^{[J]-1} \lambda^{2i} \left( \frac{1}{p} \right)^{i-|I|} + \sum_{i=|J|}^{\infty} \lambda^{2i} \left( \frac{1}{p} \right)^{i-|I|}.$$

If the sequence $I$ does not coincide with the first $|I|$ indices of the sequence $J$ then the series for scalar product $(X_I, X_J)$ contains only the finite number of terms. Therefore the limit $\lim_{\lambda \to \sqrt{p}-0} \left( 1 - \frac{\lambda^2}{p} \right) (X_I, X_J)$ equals to $\min(p^{|I|}, p^{|J|})$ if one of the
disks $D(I, p^{-|I|})$ and $D(J, p^{-|J|})$ contains another and equals to 0 if these disks do not intersect.

Therefore the limit of the scalar product of vectors $X_I$, $X_J$ in the free Fock space equals to the integral on $p$-adic disk with respect to the Haar measure. Coherent state $X_I$ corresponds to the locally constant function $\frac{1}{\mu(D(I, p^{-|I|}))} \theta_I(x - I) = \frac{\theta_I(x - I)}{\|\theta_I(x - I)\|^2}$.

Let us investigate functionals on the space $X$ of locally constant functions. Let us consider an infinite sequence $I = i_0 \ldots i_k \ldots$, $i_j = 0, \ldots, p - 1$ that corresponds to the $p$-adic number $I = \sum_{k=0}^{\infty} i_k p^k$. Let us denote $I_k = i_0 \ldots i_{k-1}$. Let us introduce the free coherent state $X_I$ of the form

$$X_I = \sum_{k=0}^{\infty} \lambda^{|I_k|} A_{I_k}^\dagger \Omega.$$ Let us consider the coherent state $X_J$, $J = j_0 \ldots j_{|J|-1}$.

**Lemma 2.** The limit of the action of the functional $X_I$ on the vector $X_J$ has the following form

$$\lim_{\lambda \to \sqrt{p}-0} \left( 1 - \frac{\lambda^2}{p} \right) (X_I, X_J) = \frac{1}{\mu(D(J, p^{-|J|}))} \int_{Z_p} \delta(x - I) \theta_J(x - J) dx.$$ Therefore the coherent state $X_I$ corresponds to the $\delta$-function $\delta(x - I)$.

The next lemma allows us to identify the space of free coherent states and the space of distributions on a $p$-adic disk.

**Lemma 3.** Vectors $X_I \in X$ lie in the domain of the functional $\Psi$ for an arbitrary free coherent state $\Psi$.

**Proof**

The coherent state $X_I$ has the following form

$$X_I = \sum_{k=0}^{\infty} \lambda^k \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^\dagger \right)^k \lambda^{|I|} A_I^\dagger \Omega + \sum_{l=1}^{\infty} \lambda^{-l} \left( \sum_{i=0}^{p-1} A_i \right)^l \lambda^{|I|} A_I^\dagger \Omega.$$ The action of the functional $\Psi$ is defined by the following formal series

$$(\Psi, X_I) = \sum_{k=0}^{\infty} \lambda^{2k} (\Psi^k, X_I^k).$$ (7)

Here $\Psi^k$ and $X_I^k$ are coefficients of $\lambda^k$ in series for $\Psi$ and $X_I$. $\Psi^k$ is defined by the formula

$$\Psi^k = \sum_{|J|=k} \Psi_J A_J^\dagger \Omega;$$ and $X_I^k$ for $k > |I|$ have a form

$$X_I^k = \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^\dagger \right)^{k-|I|} A_I^\dagger \Omega.$$
Let us calculate
\[
(\Psi^{[I]+k}, X^{[I]+k}) = (X^{[I]+k}, \Psi^{[I]+k})^* = (X^{[I]+k-1}, \frac{1}{p} \sum_{i=0}^{p-1} A_i \Psi^{[I]+k})^* =
\]
\[
= (X^{[I]+k-1}, \frac{1}{p} \sum_{i=0}^{p-1} A_i \sum_{|J|=|I|+k} \Psi_J A_J^\dagger \Omega)^* =
\]
for \(k > 0\). We have
\[
\frac{1}{p} \sum_{i=0}^{p-1} A_i \sum_{|J|=|I|+k} \Psi_J A_J^\dagger \Omega = \sum_{|J|=|I|+k-1} \frac{1}{p} \sum_{i=0}^{p-1} \Psi_J A_J^\dagger \Omega = \frac{1}{p} \Psi^{[I]+k-1}.
\]
Therefore
\[
(\Psi^{[I]+k}, X^{[I]+k}) = \frac{1}{p} (\Psi^{[I]+k-1}, X^{[I]+k-1}).
\]
We have \(\frac{\lambda^2}{p} < 1\) and the series \((\mathbb{F})\) converges.

Therefore an arbitrary coherent state \(\Psi\) corresponds to a distribution on the \(p\)-adic disk. We get the following theorem.

**Theorem.** The map \(\phi\)
\[
\phi : X \rightarrow D(Z_p);
\]
\[
X_I \mapsto \frac{1}{\mu(D(I, p^{-|I|}))} \theta_{|I|}(x - I);
\]
extends to the isomorphism of the space \(X\) of coherent states of a type \((\mathbb{F})\) onto the space \(D(Z_p)\) of locally constant functions on the ring of integer \(p\)-adic numbers with the scalar product defined with respect to the Haar measure.

The scalar product of locally constant functions in \(L_2\) with respect to the Haar measure equals to the limit of the scalar product in the free Fock space of corresponding coherent states
\[
\lim_{\lambda \rightarrow \sqrt{p^{-1}}} \left( 1 - \frac{\lambda^2}{p} \right) (X_I, X_J) =
\]
\[
= \frac{1}{\mu(D(I, 2^{-|I|})) \mu(D(J, 2^{-|J|}))} \int_{Z_p} \theta_{|I|}(x - I) \theta_{|J|}(x - J) dx.
\]

The map \(\phi\) defines an isomorphism of the space of free coherent states onto the space \(D'(Z_p)\) of distributions on the \(p\)-adic disk:
\[
\phi : X' \rightarrow D'(Z_p).
\]
The coherent state \(\Psi\) corresponds to a distribution on the ring of integer 2-adic numbers with the action on locally constant functions defined by the formula
\[
(\phi(\Psi), \phi(X_I)) = \lim_{\lambda \rightarrow \sqrt{p^{-1}}} \left( 1 - \frac{\lambda^2}{p} \right) (\Psi, X_I).
\]
Proof
We have to prove that an arbitrary distribution on $Z_p$ has a form $\phi(\Psi)$. A
distribution $f$ on $Z_p$ is defined by the action of $f$ on locally constant functions
$\theta_I(x - I)$ that correspond to coherent states $\mu(D(I, p^{-|I|}))X_I$. We have

$$\lim_{\lambda \to \sqrt{p-0}} \left(1 - \frac{\lambda^2}{p}\right) (\Psi, \mu(D(I, p^{-|I|}))X_I) = \Psi_I.$$ 

Here $\Psi_I$ can be an arbitrary function that satisfies the formula (4). Therefore $\phi(\Psi)$
can be an arbitrary distribution on the $p$-adic disk.

The formula (4) corresponds to the linearity of the action of the distribution
$\phi(\Psi)$ on the sum of indicators of $p$-adic disks.

The theorem is proved.

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