Cardinalities of countably based topologies

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ABSTRACT

Let $T$ be the family of open subsets of a topological space (not necessarily Hausdorff or even $T_0$). We prove that if $T$ has a countable base and is not countable, then $T$ has cardinality at least continuum.

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Topological spaces are not assumed to be Hausdorff, or even $T_0$.

Theorem 1 Let $T$ be the set of open subsets of a topological space, and suppose that $T$ has a countable base $B$ (more precisely, $B$ is a countable subset of $T$ which is closed under finite intersections, and the sets in $T$ are the unions of subsets of $B$). Then the cardinality of $T$ is either $2^{|B|}$ or $\leq \aleph_0$.

This answers a question of Kishor Kale. We thank Wilfrid Hodges for telling us the question and for writing up the proof from notes. In a subsequent

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work we shall deal with the case $\lambda \leq |B| < 2^\lambda$, $|T| > |B|$, $\lambda$ strong limit of cofinality $\aleph_0$ and prove that $|T| \geq 2^\lambda$.

Our proof begins with some notation. A set $\Omega$ is given, together with a countable family $B$ of subsets of $\Omega$; $\Omega = \bigcup B$ and $B$ is closed under finite intersections. We write $T$ for the set of all unions of subsets of $B$. Thus $T$ is a topology on $\Omega$ and $B$ is a base for this topology.

We write $X, Y$ etc. for subsets of $\Omega$. We write $T(X)$ for the set $\{X \cap Y : Y \in T\}$, and likewise $B(X)$ with $B$ in place of $T$. We say $X$ is small if $|T(X)| \leq \omega$, and large otherwise.

**Lemma 2** If $|\Omega| = \aleph_0$ and $|T| > \aleph_0$ then $|T| = 2^{\aleph_0}$.

**PROOF:** Identify $\Omega$ with the ordinal $\omega$, and list the set $B$ by a function $\rho$ with domain $\omega$, so that $B = \{\rho(m) : m < \omega\}$. Then a set $X$ is in $T$ if and only if

$$(\exists Y \subseteq \omega)(\forall n \in \omega) (n \in X \leftrightarrow \exists m (m \in Y \land n \in \rho(m))).$$

Thus $B$ is an analytic set, and so its cardinality must be either $2^{\aleph_0}$ or $\leq \aleph_0$ (cf. Mansfield & Weitkamp [1] Theorem 6.3). $\blacksquare$

**Lemma 3** Suppose $\Omega$ is linearly ordered by some ordering $\preceq$ in such a way that the sets in $T$ are initial segments of $\Omega$ and any initial segment of the form $(-\infty, x)$ is open. If $|T| > \aleph_0$ then $|T| = 2^{\aleph_0}$.

**PROOF:** As $B$ is countable, the linear order has a countable dense subset $D$, but as $T$ is countable, the rationals are not embeddable in $D$, i.e. $D$ is scattered. By Hausdorff’s structure theorem for scattered linear orderings, $D$ has at most countably many initial segments (cf. Mansfield & Weitkamp [1] Theorem 9.21). $\blacksquare$

Henceforth we assume that $\Omega$ is uncountable and large, and that $|T| < 2^{\aleph_0}$, and we aim for a contradiction. Replacing $\Omega$ by a suitable subset if necessary, we can also assume:

**Hypothesis** The cardinality of $\Omega$ is $\aleph_1$.

Finally we can assume without loss that if $x, y$ are any two distinct elements of $\Omega$ then there is a set in $B$ which contains one but not the other. (Define $x$
Lemma 4 If for each \( n < \omega \), \( X_n \) is a small subset of \( \Omega \), then \( \bigcup_{n<\omega} X_n \) is small.

PROOF: Each \( X_n \) has a countable subset \( Y_n \) such that if \( V, W \) are elements of \( T \) with \( V \cap X_n \neq W \cap X_n \) then there is some element \( y \in Y_n \) which is in exactly one of \( V \) and \( W \). Now if \( V, W \) are elements of \( T \) which differ on \( \bigcup_{n<\omega} X_n \), then they already differ on some \( X_n \) and hence they differ on \( Y = \bigcup_{n<\omega} Y_n \). But \( Y \) is countable; so Lemma 2 implies that either \( Y \) is small or \( |T(Y)| = 2^{\aleph_0} \). The latter is impossible since \( |T| < 2^{\aleph_0} \), and so \( Y \) is small, hence \( \bigcup_{n<\omega} X_n \) is small. \( \blacksquare \)

Our main argument lies in the next lemma, which needs some further notation. Let \( Z \) be a subset of \( \Omega \). The \( Z \)-closure of a subset \( X \) of \( Z \) is the set \( \text{cl}_Z(X) \) of all elements \( y \) of \( Z \) such that every set in \( B \) which contains \( y \) meets \( X \). Given an element \( x \) of \( Z \) and a subset \( X \) of \( Z \), we write \( \text{back}_Z(x, X) \) for the set \( \{ y \in Z : y \not\in X \cup \text{cl}_Z\{x\} \} \).

Lemma 5 Suppose \( Z \) is a large subset of \( \Omega \). Then there are an element \( x \) of \( Z \) and a set \( X \in B \) such that \( x \in X \) and \( \text{back}_Z(x, X) \) is large.

PROOF: Assume \( Z \) is a counterexample; we shall reach a contradiction. By a \( Z \)-rich set we mean a subset \( N \) of \( Z \cup \mathcal{P}Z \) such that

- \( N \) is countable.
- If \( x \in N \) and \( X \in B \) then \( \text{back}_Z(x, X) \in N \).
- If \( U \) is a subset of \( Z \) which is a member of \( N \) and is small, and \( V, W \) are elements of \( T \) such that \( V \cap U \neq W \cap U \), then there is some element of \( N \cap U \) which lies in exactly one of \( V \) and \( W \).

Since \( Z \) has cardinality at most \( \omega_1 \), we can construct a strictly increasing continuous chain \( \langle N_i : i < \omega_1 \rangle \) of \( Z \)-rich sets, such that \( Z \subseteq \bigcup_{i<\omega_1} N_i \).

Let us say that an element \( x \) of \( Z \) is pertinent if there is some \( i < \omega_1 \) such that \( x \in N_{i+1} \setminus N_i \), and \( x \) lies in some small subset of \( Z \) which is in \( N_i \). If \( z \) is not pertinent, we say it is impertinent.
We claim that if $V, W$ are any two distinct members of $T(Z)$ then some impertinent element is in exactly one of $V$ and $W$. For this, consider the least $i < \omega_1$ such that some element $z$ of $N_{i+1} \setminus N_i$ is in the symmetric difference of $V$ and $W$. If $z$ is pertinent, then by the last clause in the definition of $Z$-rich sets, some element of $N_i$ already distinguishes $V$ and $W$, contradicting the choice of $i$. This proves the claim.

Now let $I$ be the set of all impertinent elements of $Z$. Since $Z$ is large and $N_0 \cap Z$ is small, the claim implies that $I$ is large. Thinning the chain if necessary, we can arrange that for each $i < \omega_1$, $N_{i+1} \setminus N_i$ contains infinitely many elements of $I$.

We can partition $I$ into countably many sets, so that for every $i < \omega_1$, each set meets $I \cap (N_{i+1} \setminus N_i)$ in exactly one element. By Lemma 3 above, since $I$ is large, at least one of these partition sets must be large. Let $J$ be a large partition set. We define a binary relation $\preceq$ on $J$ by:

\[ x \preceq y \iff \text{ for all } U \in B, \text{ if } y \in U \text{ then } x \in U. \]

We shall reach a contradiction with Lemma 3 by showing that $\preceq$ is a linear ordering and $T(J)$ is a set of initial segments of $J$ under $\preceq$ which contains all the initial segments of the form $\{x : x \prec y\}$.

The relation $\preceq$ is clearly reflexive and transitive. We made it antisymmetric by assuming that no two distinct elements of $\Omega$ lie in exactly the same sets in $B$. We must show that if $x$ and $y$ are distinct elements of $Z$ then either $x \preceq y$ or $y \preceq x$.

Let $x, y$ be a counterexample, so that there are sets $X, Y \in B$ with $x \in X \setminus Y$ and $y \in Y \setminus X$. By symmetry and the choice of $J$ we can assume that for some $i < \omega_1$, $x \in N_i$ and $y \in N_{i+1} \setminus N_i$. Since $y$ is impertinent, no small set containing $y$ is in $N_i$. In particular $\text{back}_Z(x, X)$ contains $y$ and hence is not both small and in $N_i$. But since $N_i$ is $Z$-rich, it contains $\text{back}_Z(x, X)$. Also we assumed that $Z$ is a counterexample to the lemma; this implies that $\text{back}_Z(x, X)$ is small. We have a contradiction.

Thus it follows that $\preceq$ is a linear ordering of $J$, and the definition of $\preceq$ then implies that $T(J)$ is a set of initial segments of $\preceq$. As $B$ separates points, every set $\{x : x \prec y\}$ is open. This contradicts Lemma 3 and so proves the present lemma. $\square$

PROOF OF THEOREM 1: Now we can finish the proof of the theorem. We shall find elements $x_n$ of $\Omega$ and sets $X_n \in T (n < \omega)$ such that $x_m \in X_n$
if and only if \( m = n \). By taking arbitrary unions of the sets \( X_n \) it clearly follows that \(|T| = 2^\omega\).

We define \( x_n \) and \( X_n \) by induction on \( n \). Writing \( Z_{-1} \) for \( \Omega \) and \( Z_n \) for \( \text{back}_{Z_{n-1}}(x_n, X_n) \), we require that \( x_{n+1} \in Z_n \) and each set \( Z_n \) is large. Since \( \Omega \) is large, Lemma 5 tells us that we can begin by choosing \( x_0 \) and \( X_0 \) so that \( \text{back}_{\Omega}(x_0, X_0) \) is large.

After \( x_n \) and \( X_n \) have been chosen, we use Lemma 5 again to choose \( x_{n+1} \) in \( Z_n \) and \( Y_{n+1} \) in \( B \) so that \( x_{n+1} \in Y_{n+1} \) and \( \text{back}_{Z_n}(x_{n+1}, Y_{n+1}) \) is large. Since \( \Omega \) is large, we can begin by choosing \( x_0 \) and \( X_0 \) so that \( \text{back}_{\Omega}(x_0, X_0) \) is large.

The following theorem has a similar proof. We omit details, except to say that (i) “countable” is replaced by “of cardinality at most \(|B|\)”, and \( \omega_1 \) by \(|B|^+\), and (ii) a more complicated analogue of Lemma 2 is needed.

**Theorem 6** Let \( T \) be the set of open subsets of a topological space \( \Omega \) (not necessarily Hausdorff, nor even \( T_0 \)), and suppose that \( T \) has a base \( B \) which is closed under finite intersections, and \(|T| > |B| + \aleph_0 \). Then

1. there are \( x_n \in \Omega \) and \( X_n \in B \) for \( n < \omega \) such that for all \( m, n < \omega \), \( x_n \in X_m \) iff \( m = n \), and
2. \(|T| \geq 2^\aleph_0 \).

One naturally asks whether we can let \( B \) in Theorem 4 be any set such that \( T \) is the set of unions of sets in \( B \), without the requirement that \( B \) is closed under finite intersections. The answer is no, for the following reason.

**Lemma 7** Suppose there is a tree \( S \) with \( \delta \) levels, \( \mu \) nodes and at least \( \lambda \) branches of length \( \delta \), where \( \lambda \geq \mu \); suppose also that \( S \) is normal (i.e. at each limit level there are never two or more nodes with the same predecessors). Then there are a set \( \Omega \) of cardinality \( \lambda \) and a family of \( \mu \) subsets of \( \Omega \) which has exactly \( \lambda \) unions.
CONSTRUCTION: Let $\Omega$ be a set of $\lambda$ branches of length $\delta$; for each $s \in S$ let $U_s$ be $\{x \in \Omega : s \notin x\}$. Lastly let $B$ be the family of sets $\{U_s : s \in S\}$, so that $|B| = \mu$. Now the sets in $T$ are: members of $B$, $\Omega$ itself and complements of singletons; so $|T| = \lambda$. $\square$

Thus by starting with the full binary tree of height $\omega$, we can build examples where $B$ is countable and $T$ is any cardinal between $\omega$ and $2^\omega$.

References

[1] Richard Mansfield and Galen Weitkamp, Recursive aspects of descriptive set theory, Oxford Univ. Press 1985.
Cardinalities of topologies with small base*

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Abstract

Let $T$ be the family of open subsets of a topological space (not necessarily Hausdorff or even $T_0$). We prove that if $T$ has a base of cardinality $\leq \mu$, $\lambda \leq \mu < 2^\lambda$, $\lambda$ strong limit of cofinality $\aleph_0$, then $T$ has cardinality $\leq \mu$ or $\geq 2^\lambda$. This is our main conclusion (21). In Theorem 2 we prove it under some set theoretic assumption, which is clear when $\lambda = \mu$; then we eliminate the assumption by a theorem on pcf from [Sh 460] motivated originally by this. Next we prove that the simplest examples are the basic ones; they occur in every example (for $\lambda = \aleph_0$ this fulfill a promise from [Sh 454]). The main result for the case $\lambda = \aleph_0$ was proved in [Sh 454].

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Why does we deal with $\lambda$ strong limit of cofinality $\aleph_0$? Essentially as other cases are closed.

**Example 1** If $I$ is a linear order of cardinality $\mu$ with $\lambda$ Dedekind cuts then there is a topology $T$ of cardinality $\lambda > \mu$ with a base $B$ of cardinality $\mu$.

**CONSTRUCTION:** Let $B = \{[-\infty, x)_I : x \in I\}$ where $[-\infty, x)_I = \{y \in I : I \models y < x\}$

Remarks: as it is well known, if $\mu = \mu^{<\mu}, \mu < \lambda \leq \chi = \chi^\mu$ then there is a $\mu^+$-c.c. $\mu$-complete forcing notion $Q$, of cardinality $\chi$ such that in $V^Q$ we have $2^\mu = \chi$, there is a $\lambda$-tree with exactly $\mu$ $\lambda$-branches (and $\leq \mu$ other branches) hence a linear order of cardinality $\mu$ with exactly $\lambda$ Dedekind cuts. As possibly $\lambda^{\aleph_0} > \lambda$, this limits possible generalizations of our main Theorem. Also there are results guaranteeing the existence of such trees and linear orders, e.g. if $\mu$ is strong limit singular of uncountable cofinality, $\mu < \lambda \leq 2^\mu$ (see [Sh 262], [Sh 355, 3.5 +§ 5]) and more (see [Sh 430]).

So we naturally concentrate on strong limit cardinals of countable cofinality. We do not try to “save” in the natural numbers like $n(*) + 6$ used during the proof.

**Theorem 2** (Main) Assume

(a) $\lambda_n$ for $n < \omega$ are regular or finite cardinals, $2^{\lambda_n} < \lambda_{n+1}$ and $\lambda = \Sigma_{n<\omega} \lambda_n (\geq \aleph_0)$.

(b) $\lambda = \Sigma_{n<\omega} \mu_n$ (even $\mu_{n+1} \geq \lambda_n$) and $\Sigma_\beta (\mu_n) < \lambda_n$, $\lambda \leq \mu < \lambda^{\aleph_0}(= 2^\lambda)$ and $\text{cov} (\mu, \lambda_+, \lambda^+_n, \mu^+_n) \leq \mu$ (see Definition below, trivial when $\lambda = \aleph_0$ and easy when $\mu = \lambda$)

(c) Let $T$ be the family of open subsets of a topological space (not necessarily Hausdorff or even $T_0$), and suppose that $T$ has a base $B$ of cardinality $\leq \mu$ (i.e. $B$ is a subset of $T$ which is closed under finite intersections, and the sets in $T$ are the unions of subfamilies of $B$).

Then

1. The cardinality of $T$ is either at least $\lambda^{\aleph_0}(= 2^\lambda)$ or at most $\mu$.  

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2. In fact, if $|T| > \mu$ then for some set $X_0$ of $\lambda$ points, \{ $U \cap X_0 : U \in T$ \} has cardinality $2^\lambda$. Moreover, for some $B' \subseteq T$ of cardinality $\lambda$, \{ $X_0 \cap U : U$ is the union of a subfamily of $B'$ \} has cardinality $2^\lambda$.

**Definition 3** ([Sh 355, 5.1]) \( \text{cov}(\mu, \lambda^+, \lambda^+, \kappa) = \min\{|P| : P \text{ a family of subsets of } \mu \text{ each of cardinality } \leq \lambda, \text{ such that if } a \subseteq \mu, |a| \leq \lambda \text{ then for some } \alpha < \kappa \text{ and } a_i \in P (\text{for } i < \alpha) \text{ we have } a \subseteq \bigcup_{i<\alpha} a_i \} \)

**Proof:** Suppose we have a counterexample $T$ to 2(2) (as 2(1) follows from 2(2)) with a base $B$ and let $\Omega$ be the set of points of the space, so w.l.o.g. $\lambda \leq \mu = |B| < 2^\lambda$. Our result, as explained in the abstract, for the case $\lambda = \aleph_0$ was proved in [Sh 454], and see background there; the proof as written here applies to this case too but we usually do not mention when things trivialize for the case $\lambda = \aleph_0$; wlog $\Omega = \bigcup B$, $\emptyset \in B$ and $B$ is closed under finite intersections and unions. So $T$ is the set of all unions of subfamilies of $B$.

We prove first that:

**Observation 4** For each $n$ there is a family $R$ of cardinality $\leq \mu$ of partial functions from $\lambda_n$ to $\mu$ such that: for every function $f$ from $\lambda_n$ to $\mu$ there is a partition $\langle r_\zeta | \zeta < \mu_n \rangle$ of $\lambda_n$ (i.e. pairwise disjoint subsets of $\lambda_n$ with union $\lambda_n$) for which $\bigwedge_{\zeta<\mu_n} f \upharpoonright r_\zeta \in R$.

**Proof:** By assumption (b) and $2^{\lambda_n} < \mu \leq \mu$ and $\lambda$ is strong limit of cofinality $\aleph_0 \leq \mu_n$. $$\blacksquare$$

**Claim 5** Assume $Z^*$ is a subset of $\Omega$ of cardinality at most $\mu$ and $T'$ is a subfamily of $T$ satisfying

\[(*) \quad (\forall U_1, U_2 \in T')[U_1 = U_2 \iff U_1 \cap Z^* = U_2 \cap Z^*],\]

$|T'| > \mu$ and $n < \omega$.

Then we can find a subset $Z$ of $Z^*$ of cardinality $\mu_n$, subsets $Z_\alpha$ of $Z$ and members $U_\alpha$ of $T$ and subfamilies $T_\alpha$ of $T'$ of cardinality $> \mu$ for $\alpha < \mu_n$ such that:

(a) the sets $Z_\alpha$ for $\alpha < \mu_n$ are pairwise distinct
(b) for $\alpha < \mu_n$ and $V \in T'$ we have: $V \in T_{\alpha}$ iff $V \cap Z = Z_{\alpha} \subseteq U_{\alpha} \subseteq V$.

PROOF: We shall use (*) freely. Define an equivalence relation $E$ on $Z^*$:

$$x Ey \text{ iff } |\{U \in T' : x \in U \leftrightarrow y \notin U\}| \leq \mu$$

(check that $E$ is indeed an equivalence relation).

Let $Z^\circ \subseteq Z^*$ be a set of representatives. Now for $V \in T'$ we have:

$$(*) \quad \{U \in T' : U \cap Z^\circ = V \cap Z^\circ\} \subseteq \bigcup_{x \in Ez, \{x, z\} \subseteq Z^*} \{U \in T' : z \in U \equiv x \notin U \text{ but } U \cap Z^\circ = V \cap Z^\circ\} \cup \{V^*\}$$

where $V^* = \{y \in Z^* : \text{for the } x \in Z^\circ \text{ such that } yEx \text{ we have } x \in V\}$. [Why? assume $U$ is in the left side i.e. $U \in T'$ and $U \cap Z^\circ = V \cap Z^\circ$; now we shall prove that $U$ is in the right side; if $U = V^*$ this is straight, otherwise for some $x \in Z^*$, $x \in U \equiv x \notin V^*$; as $Z^\circ$ is a set of representatives for $E$ for some $z \in Z^\circ$, we have $zEx$ so by the definition of $V^*$, $x \in V^* \iff z \in V$. But as $U \cap Z^\circ = V \cap Z^\circ$ we have $z \in V \iff z \notin U$. Together $x \in U \iff z \notin U$ and we are done.]

Now the right side of $(*)$ is the union of $|Z^*|^2$ sets, each of cardinality $\leq \mu$ (by the definition of $xEz$). Hence the left side in $(*)$ has cardinality $\leq |Z^*|^2 \times \mu \leq \mu$. Let $\{V_i : i < i^*\} \subseteq T'$ be maximal such that: $V_i \cap Z^\circ$ are pairwise distinct and $V_i \in T'$. So clearly $|T'| = |\bigcup_{i < i^*} \{U \in T' : U \cap Z^\circ = V_i \cap Z^\circ\}| \leq \sum_{i < i^*} \mu = \mu|i^*|$, but $|T'| > \mu$ hence $i^* = |\{U \cap Z^\circ : U \in T'\}| > \mu$. Hence (as $\lambda$ is strong limit) necessarily $|Z^\circ| \geq \lambda$, so we can let $z_\beta \in Z^\circ$ for $\beta < \lambda_n$ be distinct. For $\alpha < \beta < \lambda_n$ we know that $z_{\alpha}Ez_\beta$ hence for some truth value $t_{\alpha,\beta}$ we have $|\{U \in T' : z_{\alpha} \in U \equiv z_\beta \notin U \equiv t_{\alpha,\beta}\}| > \mu$. But $B$ is a base of $T$ of cardinality $\leq \mu$, hence for some $V_{\alpha,\beta} \subseteq B$ the set

$$S_{\alpha,\beta} = \{U \in T' : z_{\alpha} \in U \equiv z_\beta \notin U \equiv t_{\alpha,\beta}, \text{ and } \{z_{\alpha}, z_\beta\} \cap U \subseteq V_{\alpha,\beta} \subseteq U\}$$

has cardinality $> \mu$.

Choose $U_{\alpha}\beta^1 \in S_{\alpha,\beta}$ such that $\mu < |S_{\alpha,\beta}^1|$ where

$$S_{\alpha,\beta}^1 \overset{\text{def}}{=} \{U \in S_{\alpha,\beta} : U \cap \{z_\zeta : \zeta < \lambda_n\} = U_{\alpha,\beta} \cap \{z_\zeta : \zeta < \lambda_n\}\},$$

note that $U_{\alpha,\beta}^1$ exists as $2^{\lambda_n} < \lambda \leq \mu < |S_{\alpha,\beta}|$.

By observation 4 we can find a family $R$ of cardinality $\leq \mu$, members of $R$ has the form $\overline{u} = \langle u_\alpha : \alpha \in r \rangle$, where $r \subseteq \lambda_n$, $u_\alpha \in B$ such that for every
sequence $\pi = \langle u_\alpha : \alpha < \lambda_n \rangle$ of members of $B$, there is a partition $\langle r_\zeta : \zeta < \mu_n \rangle$ of $\lambda_n$ (so $r_\zeta = r_\zeta[\pi] \subseteq \lambda_n$ for $\zeta < \mu_n$) such that $\pi[r_\zeta] \in R$ (remember $\emptyset \in B$). Wlog if $\pi' = \langle u'_\alpha : \alpha \in r^1 \rangle \in R$ for $\ell = 1, 2$ then $\pi = \langle u_\alpha : \alpha \in r \rangle \in R$ where $r = r^1 \cup r^2$ and $u_\alpha = \{ u^1_\alpha : \alpha \in r^1 \}, u^2_\alpha : \alpha \in r^2 \setminus r^1 \}.$

For each $V \in T'$ we can find $\pi[V] = \langle u_\gamma[V] : \gamma < \lambda_n \rangle$, such that (remember $\emptyset \in B$):

$$u_\gamma[V] \in B,$$

$$z_\gamma \in V \Rightarrow u_\gamma[V] \subseteq V,$$

$$z_\gamma \notin V \Rightarrow u_\gamma[V] = \emptyset.$$ 

Clearly there is $U^2_{\alpha, \beta} \in S^1_{\alpha, \beta}$ such that:

$$(**):$$

for any finite subset $w \subseteq \mu_n$ and $\alpha < \beta < \lambda_n$, the following family has cardinality $> \mu$:

$$S^2_{\alpha, \beta, w} \overset{\text{def}}{=} \{ U \in S^1_{\alpha, \beta} : (\forall \zeta < \mu_n)(r_\zeta[\pi[U]] = r_\zeta[\pi[U^2_{\alpha, \beta}]] \text{ and } (\forall \zeta \in w)(\pi[U] \setminus r_\zeta = \pi[U^2_{\alpha, \beta}][r_\zeta]).\}$$

By the Erdős Rado theorem for some set $M \in [\lambda_n]^\mu$:

(a) for every $\alpha < \beta$ from $M$, $t_{\alpha, \beta}$ are the same

(b) for every $\alpha < \beta \in M, \gamma, \varepsilon \in M$ the truth values of “$\pi[V_{\alpha, \beta}^1]$”, “$\pi[V_{\alpha, \beta}^2]$” and the value of “Min$\{\zeta < \mu_n : \gamma \in r_\zeta[\pi[U^2_{\alpha, \beta}]]\}$”

depend just on the order and equalities between $\alpha, \beta, \gamma$ and $\varepsilon$.

Let $M = \{ \alpha(i) : i < \mu^+_n \}$ where $[i < j \Rightarrow \alpha(i) < \alpha(j)], \text{ let } t$ be 0 if

$i < j \Rightarrow t_{\alpha(i), \alpha(j)} =$truth and 1 if $i < j \Rightarrow t_{\alpha(i), \alpha(j)} =$false.

Case 1 If $i < j < \mu^+_n$ and $\varepsilon < i \land \varepsilon > j$ then $\pi(z_\varepsilon) \notin U^2_{\alpha(i), \alpha(j)}.$

So for every $i < \mu^+_n, \zeta_1 = \min\{ \zeta : \alpha(i) + t \in r_\zeta[\pi[U^2_{\alpha(i), \alpha(i+1)}]]\};$

We let $Z = \{ \pi_\zeta(i) : i < \mu_n \}, Z_i = \{ \pi_\zeta(i+t) \}, U_i = u_\pi(i+t[\pi[U^2_{\alpha(2i), \alpha(2i+1)}]]$.

Clearly $U_i \cap Z \subseteq U^2_{\alpha(2i), (2i+1)}$ and $U_i \cap Z = Z_i,$ lastly let $T_i = S^2_{\alpha(2i), \alpha(2i+1), \zeta_1};$

now $Z, Z_i, U_i, T_i$ are as required.
Case 2: If $i < j < \mu^+_n$ then

\[\varepsilon < i \Rightarrow z_\alpha(\varepsilon) \in U^2_{\alpha(i),\alpha(j)}\]

\[\varepsilon > j \Rightarrow z_\alpha(\varepsilon) \notin U^2_{\alpha(i),\alpha(j)}\]

So for some $\zeta_1 < \mu_n$, $\zeta_2 < \mu_n$

⊗ (a) for $\varepsilon < i < j < \mu^+_n$

\[\zeta_1 = \min\{\zeta : \alpha(\varepsilon) \in r_\zeta[U^2_{\alpha(i),\alpha(j)}]\}\]

(b) for $i < \mu^+_n$

\[\zeta_2 = \min\{\zeta : \alpha(i + t) \in r_\zeta[U^2_{\alpha(i),\alpha(i+1)}]\}\]

Let $Z = \{z_\alpha(i) : i < \mu_n\}$, $Z_i = \{z_\alpha(\varepsilon) : \varepsilon < 2i\} \cup \{z_\alpha(2i+t)\}$, $U_i = \cup\{u_\alpha(\varepsilon)[U^2_{\alpha(2i),\alpha(2i+1)}] : \varepsilon \leq 2i + 1\}$ and $T_i = S^2_{\alpha(2i),\alpha(2i+1),\{\zeta_1,\zeta_2\}}$.

Case 3: If $i < \mu^+_n$ then

\[\varepsilon < i \Rightarrow z_\alpha(\varepsilon) \notin U^2_{\alpha(i),\alpha(j)}\]

\[\mu_n^+ > \varepsilon > j \Rightarrow z_\alpha(\varepsilon) \in U^2_{\alpha(i),\alpha(j)}\]

So for some $\zeta_1 < \mu_n$, $\zeta_2 < \mu_n$

⊗ (a) for $i < \mu^+_n$, $\zeta_1 = \min\{\zeta : \alpha(\varepsilon) \in r_\zeta[U^2_{\alpha(i),\alpha(i+1)}]\}\]

(b) for $i < j < \varepsilon < \mu^+_n$, $\zeta_2 = \min\{\zeta : \alpha(\varepsilon) \in r_\zeta[U^2_{\alpha(i),\alpha(j)}]\}\]

Let $Z = \{z_\alpha(i) : i < \mu_n\}$, $Z_i = \{z_\alpha(\varepsilon) : \varepsilon = 2i + t\ or\ 2i + 1 < \varepsilon < \mu_n\}$

\[U_i = \cup\{u_\alpha(\varepsilon)[U^2_{\alpha(2i),\alpha(2i+1)}] : \varepsilon = 2i + t\ or\ 2i + 1 < \varepsilon < \mu_n\}\]

and $T_i = S^2_{\alpha(2i),\alpha(2i+1),\{\zeta_1,\zeta_2\}}$.

Case 4: If $i < j < \mu^+_n$ then

\[\varepsilon < i \Rightarrow z_\alpha(\varepsilon) \in U^2_{\alpha(i),\alpha(j)}\]

\[\varepsilon > j \Rightarrow z_\alpha(\varepsilon) \in U^2_{\alpha(i),\alpha(j)}\]

So for some $\zeta_1, \zeta_2, \zeta_3 < \mu_n$
Claim 6 If \( Z^* \subseteq \Omega \), \( |Z^*| \leq \mu \), then \( \{ U \cap Z^* : U \in T \} \) has cardinality \( \leq \mu \).

PROOF: Assume not. We can find \( T' \subseteq T \) such that:

(a) for \( U_1, U_2 \in T' \) we have \( U_1 = U_2 \iff U_1 \cap Z^* = U_2 \cap Z^* \).

(b) \( |T'| > \mu \).

By induction on \( n \) we define \( (T_n, Z^1_n, Z^2_n, U_\eta : \eta \in \prod_{\ell < n} \mu_\ell) \) such that:

(a) \( T_n \) is a subset of \( T' \) of cardinality \( > \mu \)

(b) if \( \nu \prec \eta \) then \( T_\eta \subseteq T_\nu \)

(c) if \( \eta = \langle \rangle \) then \( T_\eta = T', Z^1_\eta = Z^2_\eta = \emptyset \), \( U_\eta = \emptyset \)

(d) \( Z^1_\eta \subseteq Z^2_\eta \subseteq Z^* \) and \( |Z^2_\eta| \leq \mu_{\lg \eta}, Z^2_\eta \) disjoint to \( \cup \{ U_{\eta \ell} : \ell \prec \lg \eta \} \)

(e) \( U_\eta \in T \)

(f) if \( V \in T_\eta \) then \( U_\eta \subseteq V \) and \( V \cap Z^2_\eta = U_\eta \cap Z^2_\eta = Z^1_\eta \)

(g) if \( \lg(\eta) = \lg(\nu) = n + 1 \) and \( \eta \upharpoonright n = \nu \upharpoonright n \) then \( Z^2_\eta = Z^2_\nu \) but

(h) if \( \lg(\eta) = \lg(\nu) = n + 1 \), \( \eta \upharpoonright n = \nu \upharpoonright n \) but \( \eta \neq \nu \) then \( Z^1_\eta \neq Z^1_\nu \).

Why this is sufficient? Let \( Z \overset{\text{def}}{=} \cup \{ Z^2_\eta : \eta \in \bigcup_n \prod_{\ell < n} \mu_\ell \} \). It is a subset of \( Z^* \) of cardinality \( \leq \lambda \). The set \( B' \overset{\text{def}}{=} \{ U_\eta : \eta \in \bigcup_{n<\omega} \prod_{\ell < n} \mu_\ell \} \) is included in \( T \) and has cardinality \( \leq \lambda \). For \( \eta \in \bigcup_n \mu_\eta \) we let \( U_\eta = \bigcup_{n<\omega} U_{\eta \langle n \rangle} \). Now as \( U_{\eta \langle n \rangle} \in T \) (by clause (e)), clearly \( U_\eta \in T \). Now suppose \( \eta \neq \nu \) are in
\(\prod_{n<\omega} \mu_n\) and we shall prove that \(U_\eta \cap Z \neq U_\nu \cap Z\), as \(|\prod_n \mu_n| = 2^\lambda\) this suffices (giving (1) + (2) from Theorem 2). Let \(n\) be minimal such that \(\eta(n) \neq \nu(n)\), so \(\eta \upharpoonright n = \nu \upharpoonright n\). By clause (g), \(Z_{\eta_1(n+1)}^2 = Z_{\nu_1(n+1)}^2\). So (by clause (h)) \(Z_{\eta_1(n+1)}^1\), \(Z_{\nu_1(n+1)}^1\) are distinct subsets of \(Z_{\eta_1(n+1)}^2 = Z_{\nu_1(n+1)}^2 \subseteq Z\). So it suffices to show \(U_\eta \cap Z_{\eta_1(n+1)}^2 = Z_{\eta_1(n+1)}^1\) and \(U_\nu \cap Z_{\nu_1(n+1)}^2 = Z_{\nu_1(n+1)}^1\) and by symmetry it suffices to prove the first. Now \(Z_{\eta_1(n+1)}^1 \subseteq U_\eta\) by clause (f), hence \(Z_{\eta_1(n+1)}^1 \subseteq U_\eta\) so it suffices to prove that \(U_\eta \cap Z_{\eta_1(n+1)}^2 \subseteq Z_{\eta_1(n+1)}^1\); for this it suffices to prove that for \(\ell < \omega\)

\[
(*) \quad U_{\eta|\ell} \cap Z_{\eta_1(n+1)}^2 \subseteq Z_{\eta_1(n+1)}^1.
\]

**Case 1:** \(\ell = n + 1\). This holds by clause (f).

**Case 2:** \(\ell > n + 1\). Then choose any \(V \in T_{\eta|\ell}\), so we know \(U_{\eta|\ell} \subseteq V\) (by clause (f)) and \(V \in T_{\eta(n+1)}\) (by clause (b)), and \(V \cap Z_{\eta_1(n+1)}^2 = Z_{\eta_1(n+1)}^1\) (by clause (h)), together finishing.

**Case 3:** \(\ell \leq n\). By clause (d), \(Z_{\eta_1(n+1)}^2\) is disjoint from \(U_{\eta|\ell}\).

So we have finished to prove sufficiency, but we still have to carry the induction. For \(n = 0\) try to apply (c), the main point being \(|T_0| > \mu\) which holds by the choice of \(T'\) (which was possible by the assumption that the claim fails). Suppose we have defined for \(n\) and let \(\eta \in \prod_{\ell < n} \mu_{\ell}\). We apply claim \((\Box)\) with \(T_\eta\), \(Z^* \setminus \bigcup_{\ell < n} U_{\eta|\ell}\) and \(n\) here standing for \(T', Z^*, n\) there.

We get there \(Z, Z_\alpha, T_\alpha, U_\alpha (\alpha < \mu_n)\) satisfying (a)+ (b) there. We choose \(T_{\eta^*(\alpha)}\) to be \(T_\alpha\), \(U_{\eta^*(\alpha)}\) to be \(U_\alpha\), \(Z_{\eta^*(\alpha)}^2\) to be \(Z\) and \(Z_{\eta^*(\alpha)}^1\) to be \(Z_\alpha\). You can check the induction hypotheses, so we have finished. \(\square\)

**Definition 7** \(X \subseteq \Omega\) is small if \(\{X \cap U : U \in T\}\) has cardinality \(\leq \mu\). The family of small \(X \subseteq \Omega\) will be denoted by \(\mathcal{I} = \mathcal{I}_T\) (or more exactly, \(\mathcal{I}_{T,\Omega}\))

**Claim 8** The family of small sets, \(\mathcal{I}\), is a \(\mu^+\)-complete ideal (on \(\Omega\), including all singletons of course).

**PROOF:** Clearly \(\mathcal{I}\) is a family of subsets of \(\Omega\), and it is trivial to check that \(X \in \mathcal{I}\) and \(Y \subseteq X \Rightarrow Y \in \mathcal{I}\). So assume \(X_\alpha \in \mathcal{I}\) for \(\alpha < \alpha(*)\), \(\alpha(*) \leq \mu\) and we shall prove that \(X = \bigcup_\alpha X_\alpha \in \mathcal{I}\). Each \(X_\alpha\) has a subset \(Y_\alpha\) such that
(a) $|Y_\alpha| \leq \mu$ and

(b) if $V, W$ are elements of $T$ with $V \cap X_\alpha \neq W \cap X_\alpha$ then there is some
element $y \in Y_\alpha$ which is in exactly one of $V, W$ (possible as $X_\alpha \in \mathcal{I}$).

Now if $V, W$ are elements of $T$ which differ on $X = \bigcup_{\alpha < \alpha(*)} X_\alpha$, then they
already differ on some $X_\alpha$ and hence they differ on some $Y_\alpha$ hence on $Y \equiv \bigcup_{\alpha < \alpha(*)} Y_\alpha$. So $|\{U \cap X : U \in T\}| = |\{U \cap Y : U \in T\}|$, so it suffice to prove
that $Y$ is small. But $Y$ has cardinality $\leq |\bigcup_{\alpha} Y_\alpha| \leq \sum_{\alpha} |Y_\alpha| \leq \mu \times \mu = \mu$; so
claim 6 implies that $Y$ is small and hence $X$ is small. \hfill \Box

### Conclusion 9

Wlog $\text{card}(\Omega) = \mu^+$

**Proof:** As obviously $\{x\} \in \mathcal{I}$ for $x \in \Omega$, by claim 8 we know $|\Omega| > \mu$. Let $T' \subseteq T$ be of cardinality $\mu^+$ and let $\Omega' \subseteq \Omega$ be of cardinality $\mu^+$ such that: if $U \neq V$ are from $T'$ then $U \cap \Omega' \neq V \cap \Omega'$. Let $T'' = \{U \cap \Omega' : U \in T\}$ and $B' = \{U \cap \Omega' : U \in B\}$. Now $T'', B', \Omega'$ are also a counterexample to the
main theorem and satisfies the additional demand. \hfill \Box

### Claim 10

Wlog for some $n(*)$, for no $Z \subseteq \Omega$ of cardinality $\mu_{n(*)}$ and $U_\alpha$, $T_\alpha$, $Z_\alpha (\alpha < \mu_{n(*)})$ does the conclusion of claim 8 (with $\Omega, T$ here standing for $Z^*, T'$ there) holds.

**Proof:** Repeat the proof of claim 8. I.e. we let $Z^* \equiv \Omega$, and add the demand

$$(i) \quad T_\eta = \{U \in T : U_{\eta | l} \subseteq U \text{ and } U \cap Z_{\eta | l}^2 \subseteq U_{\eta | l} \text{ for } l < \lg \eta\}.$$ 

The only change is in the end of the paragraph before the last one where we have used claim 8, now instead we say that if we fail then for our $n$, replacing $T, \Omega$ by $T_\eta, Z^* \setminus \bigcup_{l < n} U_{\eta | l}$ resp. gives the desired conclusion (note $T_\eta$ has a
basis of cardinality $\leq \mu$:

$$B_\eta \equiv \{U \cup \bigcup_{l < \lg \eta} U_{\eta | l} : U \in B \text{ and } U \cap Z_{\eta | l}^2 \subseteq U_{\eta | l} \text{ for } l < \lg \eta\}$$

which is included in $T_\eta$). \hfill \Box
Observation 11 Suppose $\lambda$ is strong limit of cofinality $\aleph_0$, $I$ is a linear order of cardinality $\leq \mu$, $\lambda \leq \mu < \lambda^{\aleph_0}$, and $I$ has $> \mu$ Dedekind cuts, then it has $\geq \mu^{\aleph_0}(= \lambda^{\aleph_0})$ Dedekind cuts.

Remark: This observation does not relay on the assumptions of Theorem 2.

PROOF: We define by induction on $\alpha$ when does $rk(I, (x, y)) = \alpha$ for $x < y$ in $I$.

for $\alpha = 0$ $rk(I, (x, y)) = \alpha$ iff $(x, y)_I = \{ z \in I : x < z < y \}$ has cardinality $< \lambda$.

for $\alpha > 0$ $rk(I, (x, y)) = \alpha$ if: for $\beta < \alpha$, $\neg [rk(I, (x, y)) = \beta]$ but for any $(x_i, y_i)$ $(i < \lambda)$, pairwise disjoint subintervals of $(x, y)$, there is $i$ such that $\bigvee_{\beta < \alpha} rk(I, (x_i, y_i)) = \beta$.

(*)$_1$ Note that by thinning the family, without loss of generality, $[x_i, y_i]$ are pairwise disjoint,

[why? e.g. as for every $j$ the set $\{ i : [x_i, y_i] \cap [x_j, y_j] \neq \emptyset \}$ has at most three members].

(*)$_2$ for $\alpha > 0$ and $x < y$ from $I$, $rk(I, (x, y)) = \alpha$ iff for $\beta < \alpha$, $\neg [rk(I, (x, y)) = \beta]$ and for some $\lambda' < \lambda$ for any $(x_i, y_i)$ $(i < \lambda')$, pairwise disjoint subintervals of $(x, y)$ there are $i < \lambda'$ and $\beta < \alpha$ such that $rk(I, (x_i, y_i)) = \beta$.

[Why? the demand in (*)$_2$ certainly implies the demand in the definition, for the other direction assume that the definition holds but the demand in (*)$_2$ fails, and we shall derive a contradiction. So for each $n < \omega$ there are pairwise disjoint subintervals $(x^n_i, y^n_i)$ of $(x, y)$, for $i < \lambda_n$ such that $\neg [rk(I, (x^n_i, y^n_i)) = \beta]$ (when $\beta < \alpha$ and $i < \lambda_n$). As we can successively replace $\{(x^n_i, y^n_i) : i < \lambda_n \}$ by any subfamily of the same cardinality (when the $\lambda_n$’s are finite - by a subfamily of cardinality $\lambda_{n-1}$) wlog: for each $n$, all members of $\{x^n_i : i < \lambda_n \}$ realize the same Dedekind cut of $\{x^m_j, y^m_j : m < n, j < \lambda_m \}$ and similarly for all members of $\{y^n_i : i < \lambda_n \}$. So for $m < n, i < \lambda_n$, the interval $(x^n_i, y^n_i)$ cannot contain a point from $\{x^m_j, y^m_j : j < \lambda_m \}$ (as then the same occurs for all such $i$’s, for the same point contradicting the “pairwise disjoint”) so either our interval $(x^n_i, y^n_i)$ is disjoint to all the intervals $(x^m_j, y^m_j)$ for $j < \lambda_m$ or it is contained in one of the intervals $(x^m_j, y^m_j)$; as $j$ does not depend on $i$]
we denote it by \( j(m, n) \); if \( \lambda = \aleph_\alpha \), by the Ramsey theorem wlog for \( m < n \) \( j(m, n) \) does not depend on \( n \); now the family \( \{(x^m_i, y^m_i) : m < \omega, i < \lambda_m \) and for every \( n < \omega \) which is \( > m \) we have \( i \neq j(m, n) \)} contradicts the definition]

If \( rk_I(x, y) \) is not equal to any ordinal let it be \( \infty \). Let \( \alpha^* = \sup \{rk_I(x, y) + 1 : x < y \text{ in } I \text{ and } rk_I(x, y) < \infty \} \). Clearly \( rk_I(x, y) \in \alpha^* \cup \{ \infty \} \) for every \( x < y \) in \( I \) (and in fact \( \alpha^* < \mu^+ \)). As we can add to \( I \) the first and the last elements it suffices to prove:

(A) if \( rk_I(x, y) = \alpha < \infty \) then \( (x, y)_I \) has \( \leq \mu \) Dedekind cuts and

(B) if \( rk_I(x, y) = \infty \) then it has \( \geq \lambda^{\aleph_0} \) Dedekind cuts

(B) is straightforward.

Proof of (A): We prove this by induction on \( \alpha \). If \( \alpha \) is zero this is trivial. So assume that \( \alpha > 0 \), hence by \((*)_2\) for some \( \lambda' < \lambda \) there are no pairwise disjoint subintervals \( (x_i, y_i) \) for \( i < \lambda' \) such that \( \beta < \alpha \) implies \( \neg [rk_I(x_i, y_i) = \beta] \). Let \( J \) be the completion of \( I \), so each member of \( J \) realizes on \( I \) a Dedekind cut with no last element in the lower half and no first element in the upper half, and \( |J| > \mu \geq |I| \). Let \( J^+ \overset{\text{def}}{=} \{ z \in J : z \not\in I \text{ and if } x \in I, y \in I \text{ and } x <_J z <_J y \text{ and } \beta < \alpha \text{ then } \neg [rk_I(x, y) = \beta] \} \). By the induction hypothesis, easily \( |J \setminus J^+| \leq \mu \) hence the cardinality of \( J^+ \) is \( > \mu \).

By Erdös-Rado theorem, (remembering \( \lambda \) is strong limit and \( \lambda' < \lambda \)) there is a monotonic (by \( <_J \)) sequence \( \langle z_i : i < \lambda' \rangle \) of members of \( J^+ \); by symmetry wlog \( \langle z_i : i < \lambda' \rangle \) is \( <_J \)-increasing. Now for each \( i < \lambda' \) as \( z_i <_J z_{i+1} \) both in \( J^+ \) necessarily there is a member \( x_i \) of \( I \) such that \( z_i <_J x_i <_J z_{i+1} \). So \( x_i <_J z_{i+1} <_J z_{i+1} \) and \( x_i \in I, x_{i+1} \in I \) and \( z_{i+1} \in J^+ \) hence by the definition of \( J^+ \) we know that for no \( \beta < \alpha \) is \( rk_I(x_i, x_{i+1}) = \beta \). So finally the family \( \{(x_i, x_{i+1}) : i < \lambda' \} \) of subintervals of \( (x, y) \) gives the desired contradiction to \((*)_2\).

\[ \square \]

Definition 12 We define an equivalence relation \( E \) on \( \Omega \): \( xEy \) iff \( \{ U \in T : x \in U \equiv y \not\in U \} \) has cardinality \( \leq \mu \).

Conclusion 13 (0) The equivalence relation \( E \) has \( < \lambda_{n(*)} \) \( < \lambda \) equivalence classes (for some \( n(*) < \omega \), which wlog is as required in claim \( [10] \) too).

(1) wlog for each \( x \in \Omega \) one of the following sets has cardinality \( \leq \mu \):

(a) \( \{ U \in T : x \in U \} \)
(b) \( \{ U \in T : x \notin U \} \)

(2) \textit{wlog for all } \( x \in \Omega \) \textit{we get the same case above, in fact it is case (b).}

(3) \textit{wlog for any two distinct members } \( x, y \) \textit{of } \( \Omega \) \textit{for some } \( U \in B \) \textit{we have } \( x \in U \iff y \notin U \).

PROOF: (0) By claim 10 and the proof of claim 5 (if \( E \) has \( \geq \lambda \) equivalence classes we can repeat the proof of claim 5 and get contradiction to claim 10).

(1), (2), (3) Let \( \langle X \xi : \xi < \xi^* \rangle \) list the \( E \)-equivalence classes, so \( \xi^* < \lambda_n(\ast) \).

As \( \Omega \notin I \), and \( I \) is \( \mu^+ \)-complete (claim 8) for some \( \zeta \), \( X \zeta \notin I \). Let \( \Omega' = X \zeta \), \( T' = \{ U \cap \Omega' : U \in T \} \), \( B' = \{ U \cap \Omega' : U \in B \} \); so \( \Omega', B', T' \) has all the properties we attribute to \( \Omega, B, T \) and in addition now \( E \) has one equivalence class. So we assume this.

Fix any \( x_0 \in \Omega \), let \( B^0 = \{ U \in B : x_0 \notin U \} \), \( T^0 = \{ U \in T : x_0 \notin U \} \cup \{ \Omega \} \), \( B^1 = \{ U \in B : x_0 \in U \} \), \( T^1 = \{ U \in T : x_0 \in U \} \cup \{ \emptyset \} \) For some \( \ell \in \{ 0, 1 \} \), \( |T^\ell| > \mu \), and then \( \Omega, B^\ell, T^\ell \) satisfies the earlier requirements and the demands in (1) and (2). For (3) define an equivalence relation \( E' \) on \( \Omega \): \( xE'y \iff (\forall U \in B) [x \in U \equiv y \in U] \), let \( \Omega' \subseteq \Omega \) be a set of representatives, \( B' = \{ U \cap \Omega' : U \in B \} \) and finish as before. The only thing that is left is the second phrase in (2). But if it fails then for every \( U \in T \setminus \{ \emptyset \} \) choose a nonempty subset \( V[U] \) from \( B \). As the number of possible \( V[U] \) is \( \leq |B| \leq \mu \), for some \( V \in B \setminus \{ \emptyset \} \), for \( > \mu \) members \( U \) of \( T \), \( V = V[U] \) and hence \( V \subseteq U \). Choose \( x \in V \); so for \( x \) clause (a) of (2) fails and hence for all \( y \in \Omega \) clause (b) of (2) holds, as required.

PROOF 14 (of Theorem 2 (MAIN)):

Consider for \( n = n(\ast) \) (from claim 13(0) and as in claim 10) the following:

(*) \textit{there are an open set } \( V \) \textit{and a subset } \( Z \) \textit{of } \( V \) \textit{such that:}

(a) for \( \alpha < \beta < \lambda_n \) the sets \( V_\alpha \cap Z, V_\beta \cap Z \) are distinct

(b) \( U_\alpha \cap Z = Z_\alpha \)

(c) the number of sets \( U \in T \) satisfying \( U \cap Z = V_\alpha \cap Z \) and \( U_\alpha \subseteq U \) is \( > \mu \)

So by claim 10 we know that this fails for \( n \).
Let $\chi$ be large enough and let $\bar{N} = \langle N_i : i < \mu^+ \rangle$ be an elementary chain of submodels of $(H(\chi), \in)$ of cardinality $\mu$ (and $B, \Omega, T$ belong to $N_0$ of course) increasing fast enough hence e.g.: if $X \in N_i$ is a small set, $U \in T$ then there is $U' \in N_i \cap T$ with $U \cap X = U' \cap X$ (you can avoid the name ”elementary submodel ” if you agree to list the closure properties actually used; as done in [Sh 454]). For $x \in \Omega$ let $i(x)$ be the unique $i$ such that $x$ belongs to $N_{i+1} \setminus N_i$ or $i = -1$ if $x \in N_0$ (remember $|\Omega| = \mu^+$).

**Definition 15** We define: $x \in \Omega$ is $N$-pertinent if it belongs to some small subset of $\Omega$ which belongs to $N_{i(x)}$ (and $i(x) \geq 0$) and $N$-impertinent otherwise.

**Observation 16** $\Omega_{ip} = \{ x \in \Omega : x$ is $N$-impertinent $\}$ is not small (see Definition 2).

**Proof:** As $N_0 \cap \Omega$ is small by claim 8, for some $U^*, T' \overset{\text{def}}{=} \{ U \in T : U \cap N_0 \cap \Omega = U^* \cap N_0 \cap \Omega \}$ has cardinality $> \mu$. So it suffices to prove:

$(\ast)$ $U_1 \neq U_2 \in T' \Rightarrow U_1 \cap \Omega_{ip} \neq U_2 \cap \Omega_{ip}$.

Choose $x \in (U_1 \setminus U_2) \cup (U_2 \setminus U_1)$ with $i(x)$ minimal. As $U_1, U_2 \in T'$, $i(x) = -1$ (i.e. $x \in N_0$) is impossible, so $x \in (N_{i+1} \setminus N_i) \cap \Omega$ for $i = i(x)$. If $x \in \Omega_{ip}$ we succeed so assume not i.e. $x$ is $N$-pertinent, so for some small $X \in N_i$ $x \in X$. Hence by the choice of $\bar{N}$: for some $U'_1, U'_2 \in N_i \cap T$ we have: $U'_1 \cap X = U_1 \cap X, U'_2 \cap X = U_2 \cap X$ so $U'_1 \cap X, U'_2 \cap X \in N_i$ are distinct (as $x$ witness) so there is $x' \in N_i \cap X, x' \in U'_1 \equiv x' \notin U'_2$; but this implies $x' \in U_1 \equiv x' \notin U_2$, contradicting $i(x)$’s minimality. □

We define a binary relation $\preceq$ on $\Omega_{ip}$ by:

$$x \preceq y \iff \text{ for all } U \in B, \text{ if } y \in U \text{ then } x \in U.$$

**Claim 17** The relation $\preceq$ is clearly reflexive and transitive. It is antisymmetric [why antisymmetric? by claim [13(3)]].

**Observation 18** If $J \subseteq \Omega_{ip}$ is linearly ordered by $\preceq$ then $J$ is small.
PROOF: For each $U_1, U_2 \in B$ such that $U_1 \cap J \not\subseteq U_2 \cap J$ choose $y_{U_1, U_2} \in J \cap (U_1 \setminus U_2)$. Let $I = \{y_{U_1, U_2} : U_1, U_2 \in B \& U_1 \cap J \not\subseteq U_2 \cap J\}$. Clearly $|I| \leq \mu$. We claim that $I$ is dense in $J$ (with respect to $\preceq$, i.e. $I$ has a member in every non empty interval of $J$). Suppose that $x, y, z \in J$, $x \prec y \prec z$. By $\lfloor \text{3} \rfloor$ we find $U_1, U_2 \in B$ such that $x \in U_1, y \notin U_1$, and $y \in U_2, z \notin U_2$. Consider $y_{U_2, U_1} \in I$. Easily $x \prec y_{U_2, U_1} \prec z$. Thus if $(x, z) \neq \emptyset$ then $(x, z) \cap I \neq \emptyset$.

Now note that each Dedekind cut of $I$ is an restriction of at most 3 Dedekind cuts of $J$ (and the restriction of a Dedekind cut of $J$ to $I$ is a Dedekind cut of $I$). For this suppose that $Y_1, Y_2, Y_3, Y_4$ are lower parts of distinct Dedekind cuts of $J$ with the same restriction to $I$, wlog $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq Y_4$. For $i = 2, 3, 4$ choose $y_i \in Y_i$ such that $Y_1 \prec y_2, Y_2 \prec y_3$ and $Y_3 \prec y_4$. As $(y_2, y_4) \neq \emptyset$ we find $x \in (y_2, y_4) \cap I$. Since $y_2 \prec x$ we get $x \notin Y_1$ and since $x \prec y_4$ we obtain $x \in Y_4$. Consequently $x$ distinguishes the restrictions of cuts determined by $Y_1$ and $Y_4$ to $I$.

To finish the proof of the observation apply observation $\lfloor \text{1} \rfloor$ to $I$ (which has essentially the same number of Dedekind cuts as $J$).

Continuation 19 (of the proof of theorem $\lfloor \text{2} \rfloor$)

Now it suffices to prove that for each $x \in \Omega_{ip}, i = i(x) > 0$ there is no member $y$ of $\Omega_{ip} \cap N_i$ such that $x, y$ are $\preceq$-incomparable.

[Why? then we can divide $\Omega_{ip}$ to $\mu$ sets such that any two in the same part are $\preceq$-comparable contradicting $\lfloor \text{16,18} \rfloor$ and $\lfloor \text{8} \rfloor$; How? By defining a function $h : \Omega_{ip} \to \mu$ such that $h(x) = h(y) \Rightarrow x \preceq y \lor y \preceq x$. We define $h_i(\Omega_{ip} \cap N_i)$ by induction on $i$, in the induction step let $N_{i+1} \cap N_i = \{x_{i, \varepsilon} : \varepsilon < \mu\}$. Choose $h(x_{i, \varepsilon})$ by induction on $\varepsilon$: for each $\varepsilon$ there are $\preceq |\varepsilon| < \mu$ forbidden values so we can carry the definition.]

So assume this fails, so we have: for some $x \in \Omega_{ip}, i = i(x) > 0$ there is $y_0 \in N_i \cap \Omega_{ip}$ which is $\preceq$-incomparable with $x$; so there are $U_0, V_0 \in B$ such that $x \in V_0, x \notin U_0, y_0 \in U_0, y_0 \notin V_0$. Now $U^* = \bigcup\{U \in T : y_0 \notin U\}$ is in $T \cap N_i$ and $x \in U^*$ (as $V_0$ witnesses it) but by $\lfloor \text{13} \rfloor(2)$ we know that $U^*$ is small, so it contradicts “$x \in \Omega_{ip}$”. This finishes the proof of theorem $\lfloor \text{2} \rfloor$. $\square$

Concluding Remarks 20 Condition (b) of Theorem $\lfloor \text{2} \rfloor$ holds easily for $\mu = \lambda$. Still it may look restrictive, and the author was tempted to try to eliminate it (on such set theoretic conditions see [Sh 420, §6]). But instead of working “honesty” on this the author for this purpose proved (see [Sh 460]) that it follows from ZFC, and therefore can be omitted, hence
Conclusion 21 (Main) If \( \lambda \) is strong limit, cf \( \lambda = \aleph_0 \), and \( T \) a topology with base \( B \), \( |T| > |B| \geq \lambda \) then \( |T| \geq 2^\lambda \) and the conclusion of \( \mathbb{F}(2) \) holds.

Theorem 22  
1. Under the assumptions of Theorem \( \mathbb{F} \), if the topology \( T \) is of the size \( \geq 2^\lambda \) then there are distinct \( x_\eta \in \Omega \) for \( \eta \in \bigcup_{n<\omega} \Pi_{l<n} \mu_l \) such that letting \( Z = \{x_\eta : \eta \in \bigcup_{n<\omega} \Pi_{l<n} \mu_l \} \) one of the following occurs:

   (a) there are \( U_\eta \in T \) (i.e. open) for \( \eta \in \Pi_{l<\omega} \mu_l \) such that:
   
   \[
   U_\eta \cap Z = \{x_\nu \in Z : (\exists n < lg(\nu))(\nu|n = \eta|n & \nu(n) < \eta(n))\}
   \]

   (b) there are \( U_\eta \in T \) for \( \eta \in \Pi_{l<\omega} \mu_l \) such that:
   
   \[
   U_\eta \cap Z = \{x_\nu \in Z : (\exists n < lg(\nu))(\nu|n = \eta|n & \nu(n) > \eta(n))\}
   \]

   (c) there are \( U_\eta \in T \) for \( \eta \in \Pi_{l<\omega} \mu_l \) such that:
   
   \[
   U_\eta \cap Z = \{x_\nu \in Z : \sim \nu \triangleleft \eta\}
   \]

2. If in addition \( \lambda = \aleph_0 \) then we get

\( \oplus \) there are distinct \( x_\eta \in \Omega \) for \( \eta \in \mathbb{Q} \) (the rationals) such that for every real \( r \), for some (open) set \( U \in T \)

\[
U \cap \{x_\eta : \eta \in \mathbb{Q}\} = \{x_\eta : \eta \in \mathbb{Q}, \eta < r\}.
\]

Observation 23 Suppose that there are distinct \( x_\eta \in \Omega \) (for \( \eta \in \bigcup_{n<\omega} \Pi_{l<n} \mu_l \)) such that one of the following occurs:

(d) there are \( U_\eta \in T \) for \( \eta \in \Pi_{l<\omega} \mu_l \) such that:
   
   \[
   U_\eta \cap Z = \{x_\nu \in Z : \nu = \rho^{*}(\zeta) \& \sim \rho \triangleleft \eta \text{ or } \rho \triangleleft \eta \text{ and } \eta(lg(\rho)) = \zeta\}
   \]

(e) there are \( U_\eta \in T \) for \( \eta \in \Pi_{l<\omega} \mu_l \) such that:
   
   \[
   U_\eta \cap Z = \{x_\nu \in Z : \nu = \rho^{*}(\zeta) \& \sim \rho \triangleleft \eta \text{ or } \rho \triangleleft \eta \text{ and } \eta(lg(\rho)) < \zeta\}
   \]

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(f) there are \( U_\eta \in T \) for \( \eta \in \prod_{l<\omega} \mu_l \) such that:

\[
U_\eta \cap Z = \{ x_\nu \in Z : \nu = \rho^\ast \langle \zeta \rangle \land [\neg \rho < \eta \lor \rho < \eta \land \eta(\lg(\rho)) > \zeta] \}
\]

Then for some distinct \( x'_\nu \in \Omega \ (\nu \in \bigcup_{n<\omega} \) the clause (c) of theorem 22 holds.

PROOF Let \( U_\eta \) (for \( \eta \in \prod_{l<\omega} \mu_l \)) be given by one of the clauses. For \( \nu \in \prod_{l<n} \mu_l \), \( n \in \omega \) let \( g(\nu) \in \prod_{l<2n} \mu_l \) be such that \( g(\nu)(2l) = 0, g(\nu)(2l + 1) = \nu(l) \) and for \( \eta \in \prod_{l<\omega} \mu_l \) let \( g(\eta) = \bigcup_{l<\omega} g(\eta \upharpoonright l) \) (we assume that \( \mu_l < \mu_{l+1} \)). Next define points \( x'_\nu \in \Omega \) and open sets \( U'_\eta \) as

\[
U'_\eta = U_{g(\eta)}, \quad x'_\nu = \begin{cases} x_{g(\nu)^\ast (0)} & \text{if we are in clause (d)} \\ x_{g(\nu)^\ast (1)} & \text{if we are in clauses (e), (f)} \end{cases}
\]

Then \( x'_\nu, U'_\eta \) exemplify clause (c) of theorem 22.

PROOF 24 of 22 for the case \( \lambda = \aleph_0 \)

It suffices to prove 22(2), as \( \oplus \) implies (a). Let \( \mu = \lambda^+ \). By Theorem 2(2) and 21 wlog \( |\Omega| = \lambda, |\mathcal{I}| \leq \lambda \). Let \( \mathcal{I} = \{ Z \subseteq \Omega : |\{ U \cap Z : U \in T \}| < \mu \} \), again it is a proper ideal on \( \Omega \) (but not necessarily even \( \aleph_1 \)-complete). Let \( P = \{ (U, V) : U \subseteq V \text{ are from } T, V \setminus U \notin \mathcal{I} \} \). Clearly \( P \neq \emptyset \) (as \( (\emptyset, \Omega) \in P \)), if for every \( (U_0, U_1) \in P \) there is \( U \) such that \( (U_0, U), (U, U_1) \) are in \( P \) then we can easily get clause \( \oplus \). So by renaming wlog

\[(*)_1 \quad (\forall V \in T)(V \in \mathcal{I} \lor \Omega \setminus V \in \mathcal{I}). \]

We try to choose by the induction on \( n < \omega \), \( (x_n, U_n) \) such that

(a) \( x_n \in U_n \in T \)

(b) \( x_n \notin \bigcup_{l<n} U_l \)

(c) \( U_n \in \mathcal{I} \) and \( x_l \notin U_n \) for \( l < n \)

(d) \( |\{ V \in T : (\forall l \leq n)(x_l \notin V) \}| \geq \mu \).
If we succeed, \( \{U \cap \{x_n : n < \omega\} : U \in T\} \) includes all subsets of the infinite set \( \{x_n : n < \omega\} \), which is much more than required (in particular \( \oplus \) holds).

Suppose we have defined \((x_n, U_n)\) for \(n < m\) and that there is no \((x_m, U_m)\) satisfying (a)–(d). This means that if \(x \in U \in T \cap I, (\forall n < m)(x_n \notin U)\) and \(x \notin \bigcup_{n<m} U_n\) then

\[
(*)_2 \quad |\{V \in T : (\forall n < m)(x_n \notin V) \text{ and } x \notin V\}| < \mu.
\]

Let \(U^* = \bigcup\{U \in T \cap I : (\forall n < m)(x_n \notin U)\}. \) As \(|\Omega| < \mu = \text{cf} \mu\) we get

\[
(*)_3 \quad |\{V \in T : (\forall n < m)(x_n \notin V) \land U^* \setminus (V \cup \bigcup_{n<m} U_n) \neq \emptyset\}| < \mu.
\]

Suppose that \(U^* \notin I\). Then, by \((*)_1\), \(\Omega \setminus U^* \in I\) (as \(U^*\) is open). Since (by clause (c)) \(\bigcup_{n<m} U_n \in I\) we find an open set \(U\) such that \((\forall n < m)(x_n \notin U)\) and

\[
\mu \leq |\{V \in T : V \cap (\bigcup_{n<m} U_n \cup (\Omega \setminus U^*)) = U \cap (\bigcup_{n<m} U_n \cup (\Omega \setminus U^*))\}|
\]

(this is possible by (d)). But if \(V \cap (\bigcup_{n<m} U_n \cup (\Omega \setminus U^*)) = U \cap (\bigcup_{n<m} U_n \cup (\Omega \setminus U^*))\), \(V \neq U \cup U^*\) then \(U^* \setminus (V \cup \bigcup_{n<m} U_n) \neq \emptyset, (\forall n < m)(x_n \notin V)\). This contradicts to \((*)_3\). Thus \(U^* \in I\). Hence (by (d)) we have

\[
(*)_4 \quad \mu \leq |\{V \in T : V \setminus U^* \neq \emptyset \land (\forall n < m)(x_n \notin V)\}|.
\]

Since \(|B| < \mu\) we find \(V_0 \in B\) such that \(V_0 \setminus U^* \neq \emptyset, (\forall n < m)(x_n \notin V_0)\) and \(\mu \leq |\{V \in T : V_0 \subseteq V\}|\). The last condition implies that \(\Omega \setminus V_0 \notin I\) and hence \(V_0 \in I\) (by \((*)_1\)). By the definition of \(U^*\) we conclude \(V_0 \subseteq U^*\) - a contradiction, thus proving \(22\) (when \(\lambda = \aleph_0\)).

\[
\square
\]

**PROOF 25** of \(22\) when \(\lambda > \aleph_0\).

By Theorem \(\square\) \(\text{wlog} |\Omega| = |B| = \lambda\). Let \(I = \{A \subseteq \Omega : |\{U \cap A : U \in T\}| \leq \lambda\}\), it is an ideal. Let \(I^+ = \mathcal{P}(\Omega) \setminus I\).

**Observation 26** It is enough to prove

\(\otimes_1\) for every \(Y \in I^+\) and \(n\) we can find a sequence \(\bar{U} = (U_\zeta : \zeta < \mu_n)\) of open subsets of \(\Omega\) such that one of the following occurs:
(a) \(U\) increasing, \(Y \cap U_{\zeta+1} \setminus U_{\zeta} \in \mathcal{I}^+\)

(b) \(\bar{U}\) decreasing, \(Y \cap U_{\zeta} \setminus U_{\zeta+1} \in \mathcal{I}^+\)

(c) \(Y \cap U_{\zeta} \setminus U_{\varepsilon \neq \zeta} \in \mathcal{I}^+\)

(d) for some \((V_{\zeta}, y_{\zeta} : \zeta < \mu_n)\) we have \(Y \cap (\bigcap_{\zeta < \mu_n} U_{\zeta} \setminus \bigcup_{\zeta<\mu_n} V_{\zeta}) = \mathcal{I}^+\), \(V_{\zeta}\)'s and \(U_{\zeta}\)'s are open, \(V_{\zeta} \subseteq U_{\zeta}\), \(y_{\zeta} \in Y\) are pairwise distinct and

\[(*)\quad U_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = V_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = \{y_{\varepsilon} : \varepsilon \leq \zeta\}\]

(e) like (d) but

\[(*)'\quad U_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = V_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = \{y_{\varepsilon} : \zeta \leq \varepsilon < \mu_n\}\]

(f) like (d) but

\[(*)''\quad U_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = V_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = \{y_{\varepsilon}\}\]

(g) like (d) but

\[(*)'''\quad U_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = V_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = \{y_{\varepsilon} : \varepsilon < \mu_n, \varepsilon \neq \zeta\}\]

(h) there are \(V_{\zeta}, y_{\zeta}\) for \(\zeta < \mu_n\) such that \(V_{\zeta} \subseteq U_{\zeta}\) are open, \(y_{\zeta} \in Y\) are pairwise distinct, \((U_{\zeta} \setminus V_{\zeta}) \cap \bigcap_{\zeta \neq \zeta} V_{\varepsilon} \in \mathcal{I}^+\) and

\[(**)\quad U_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = V_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = \{y_{\varepsilon} : \varepsilon < \zeta\}\]

(i) like (h) but

\[(**)'\quad U_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = V_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = \{y_{\varepsilon} : \zeta \leq \varepsilon < \mu_n\}\]

(j) like (h) but

\[(**)''\quad U_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = V_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = \{y_{\varepsilon}\}\]

(k) like (h) but

\[(**)'''\quad U_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = V_{\zeta} \cap \{y_{\varepsilon} : \varepsilon < \mu_n\} = \{y_{\varepsilon} : \zeta \neq \varepsilon, \varepsilon < \mu_n\}\]
PROOF: First note that if \( n < m < \omega \), \( Y_1 \subseteq Y_0 \), \( Y_1 \subseteq Y_0 \subseteq \mathcal{I}^+ \) and one of the cases (a)–(k) of \( \otimes_1 \) occurs for \( Y_1 \), \( m \) then the same case holds for \( Y_0 \), \( n \). Consequently, \( \otimes_1 \) implies that for each \( Y \subseteq \mathcal{I}^+ \) one of (a)–(k) occurs for \( Y \), \( n \) for every \( n \in \omega \). Moreover, if \( \otimes_1 \) then for some \( x \in \{a, b, c, d, e, f, g, h, i, j, k\} \) and \( Y_0 \subseteq \mathcal{I}^+ \) we have

\[ \ast \quad \text{for every} \ Y_1 \subseteq Y_0 \text{ from} \mathcal{I}^+ \text{ and} \ n \in \omega \text{ case (x) holds.} \]

If \( x = a \), clause (a) of \( 22(1) \) holds. For this we inductively define open sets \( V_\eta, \eta^- \) for \( \eta \in \bigcup_{n \in \omega} \prod_{l<n} \mu_l \) such that for \( \eta \in \prod_{l<n} \), \( \zeta < \mu_n \):

1. \( \eta^- \subseteq V_\eta \), \( (V_\eta \setminus V^-_\eta) \cap Y_0 \subseteq \mathcal{I}^+ \), \( (V^-_{\eta^-(\zeta+1)} \setminus V^-_{\eta^-(\zeta)}) \cap Y_0 \subseteq \mathcal{I}^+ \)

2. if \( \xi < \mu_{n+1} \) then \( V_\eta \subseteq V_{\eta^-(\zeta)} \subseteq V_{\eta^-(\zeta,\xi)} \subseteq V^-_{\eta^-(\zeta+1)} \).

Let \( \langle U_\zeta : \zeta < \mu_0 \rangle \) be the increasing sequence of open sets given by (a) for \( Y_0, n = 0 \). Put \( V_{\zeta} = U_{2\zeta+1}, V^-_{\zeta} = U_{2\zeta} \) for \( \zeta < \mu_0 \). Suppose we have defined \( V^-, V^- \) for \( \lg(\eta) \leq m \). Given \( \eta \in \prod_{l<m} \mu_l \), \( \zeta < \mu_{m-1} \). Apply (a) for \( (V^-_{\eta^-(\zeta+1)} \setminus V^-_{\eta^-(\zeta)}) \cap Y_0 \) and \( n = m \) to get a sequence \( \langle U_\xi : \xi < \mu_m \rangle \). Put

\[
V^-_{\eta^-(\zeta,\xi)} = (U_{2\xi+1} \cap V^-_{\eta^-(\zeta+1)}) \cup V^-_{\eta^-(\zeta)}
\]

Next for each \( \eta^-(\zeta) \in \bigcup_{n \in \omega} \prod_{l<n} \mu_l \) choose \( x_{\eta^-(\zeta)} \in (V^-_{\eta^-(\zeta+1)} \setminus V^-_{\eta^-(\zeta+1)}) \cap Y_0 \)

As the last sets are pairwise disjoint we get that \( x_{\eta^-(\zeta)} \) are pairwise distinct.

Moreover, if we put \( U_\eta = \bigcup_{n \in \omega} V_{\eta|n} \) (for \( \eta \in \prod_{n \in \omega} \mu_l \)) then we have

\[
U_\eta \cap \{x_\nu : \nu \in \bigcup_{n \in \omega} \prod_{l<n} \mu_l \} = \{x_\nu : (\exists n < \lg(\nu))(\nu|n = \eta|n & \nu(n) = \eta(n))\}.
\]

Similarly one can show that if \( x = b \), clause (b) of \( 22(1) \) holds and if \( x = c \) then we can get a discrete set of cardinality \( \lambda \) hence all clauses \( 22(1) \) hold.

Suppose now that \( x = d \). By the induction on \( n \) we choose \( Y_n, \langle U_n,\zeta, V_n,\zeta, y_n,\zeta : \zeta < \mu_n \rangle \):

\[
Y_0 = Y (\in \mathcal{I}^+)
\]

\[
U_{n,\zeta}, V_{n,\zeta}, y_{n,\zeta} \text{ (for} \zeta < \mu_n \text{) are given by (d) for} Y_n,
\]

\[
Y_{n+1} = Y_n \cap \bigcap_{\zeta < \mu_n} U_{n,\zeta} \cup \bigcup_{\zeta < \mu_n} V_{n,\zeta} \in \mathcal{I}^+.
\]
For \( \eta \in \prod_{l \leq n} \mu_l \) \((n \in \omega)\) we let

\[ W'_\eta = V_{n,\eta(n)} \cap \bigcap_{m < n} U_{m,\eta(m)}. \]

As \( V_{n,\eta(n)} \cap \{ y_{n,\zeta} : \zeta < \mu_n \} = \{ y_{n,\zeta} : \zeta \leq \eta(n) \} \) and \( \{ y_{n,\zeta} : \zeta < \mu_n \} \subseteq Y_n \subseteq Y_{m+1} \subseteq U_{m,\eta(m)} \) (for \( m < n \)) we get

\[ W'_\eta \cap \{ y_{n,\zeta} : \zeta < \mu_n \} = \{ y_{n,\zeta} : \zeta \leq \eta(n) \}, \]

\( W'_\eta \cap \{ y_{m,\zeta} : \zeta < \mu_m \} \subseteq U_{m,\eta(m)} \cap \{ y_{m,\zeta} : \zeta < \mu_m \} \subseteq \{ y_{m,\zeta} : \zeta \leq \eta(m) \} \) (for \( m < n \)).

Now for \( \eta \in \Pi_{n<\omega} \mu_n \) we define \( W_\eta = \bigcup_{l<\omega} W'_\eta \). Then for each \( n \), \( W_\eta \cap \{ y_{n,\zeta} : \zeta < \mu_n \} = \{ y_{n,\zeta} : \zeta \leq \eta(n) \} \). By renaming this implies clause (a) of \( \ref{1} \).

For \( \eta \in \Pi_{l<\omega} \mu_l \) let \( x_\eta = y_{n+1,\gamma(\eta)+1} \), where \( \gamma(\eta) = \mu^0_n \times \eta(0) + \mu^{n-1}_n \times \eta(1) + \cdots + \mu^1_n \times \eta(n-1) + \eta(n) \). Note: \( \mu^s_n \) is the \( l \)-th ordinal power of \( \mu_n \). For \( \eta \in \Pi_{l<\omega} \mu_l \) let \( \gamma(\eta) = 0^\gamma(\eta) \rangle \gamma(\eta) \rangle \gamma(\eta) \rangle \cdots \) and let \( U_\eta = W_\gamma(\eta) \).

For \( x = e \) we similarly get clause (b) of \( \ref{1} \). For \( x = f \) we similarly get a discrete set of cardinality \( \lambda \) so all clauses of \( \ref{1} \) hold. The case \( x = g \) corresponds to the clause (c) of \( \ref{1} \).

Suppose now that \( x = h \). By induction on \( n \) we define \( Y_\eta, U_\eta, V_\eta \) and \( x_\eta \) for \( \eta \in \Pi_{l<\omega} \mu_l \):

\[ Y_\eta = Y, \]

\[ U_\eta = V_\eta \setminus V_{\eta(n)}, \]

\[ x_\eta = x \]

given by the clause (h) for \( Y_\eta, \mu_{n+1}, V_{\eta}\).

For \( \eta \in \prod_{n<\omega} \mu_n \) put \( U'_\eta = \bigcup_{l<\omega} V_{\eta(l)} \). Then

\[ U'_\eta \cap \{ y_\nu : \nu \in \bigcup_{n<\omega} \prod_{l \leq n} \mu_l \} = \{ y_\nu : \nu = \rho \upharpoonright \zeta \rangle \langle \rho \prec \eta \text{ or } \rho \prec \eta \text{ and } \eta(\log(\rho)) < \zeta \} \]

witnessing case (e) of \( \ref{1} \).

If \( x = i \) then we similarly get case (f) and if \( x = j \) we get (d). Lastly \( x = k \) implies the case (c) of \( \ref{1} \). \( \square \)

**Claim 27** If \( \kappa < \lambda \), \( \{ Z_\zeta : \zeta < \kappa \} \) is a partition of \( \Omega \), then for some countable \( w^* \subseteq \kappa \), for every infinite \( w \subseteq w^* \), \( \cup_{\zeta \in w} Z_\zeta \notin \mathcal{I} \).

**Proof:** Otherwise there are \( \mathcal{P} \subseteq [\kappa]^{|\Omega|} \) and \( \langle T_w : w \in \mathcal{P} \rangle, T_w \subseteq T, |T_w| \leq \lambda \) such that for every \( w^* \in [\kappa]^{|\Omega|} \) and \( U \in T \), for some \( w \subseteq w^* \), \( w \in \mathcal{P} \) and \( V \in T_w \) we have \( U \cap (\cup_{\zeta \in w} Z_\zeta) = V \cap (\cup_{\zeta \in w} Z_\zeta) \). Let \( \{ U_\zeta : \zeta < \kappa \} \) list \( \cup_{T_w : w \in \mathcal{P}} \) (note that since \( \kappa < \lambda \) also \( |[\kappa]^{|\Omega|}| = \kappa^{|\Omega|} < \lambda \)). We claim that there is \( U \in T \) such that for every \( \xi < \lambda \) there are \( \alpha, \beta \in \Omega \) for which:
Indeed, to find such $U$ consider equivalence relations $E_\xi$ (for $\xi < \lambda$) determined by (b) and (c), i.e. for $\alpha, \beta \in \Omega$:

$$\alpha \equiv_{E_\xi} \beta \text{ if and only if } (\forall \zeta < \xi)(\alpha \in U_\zeta \iff \beta \in U_\zeta) \text{ and } (\forall \varepsilon < \kappa)(\alpha \in Z_\varepsilon \iff \beta \in Z_\varepsilon).$$

The relation $E_\xi$ has $\leq 2^{\xi+\kappa} < \lambda$ equivalence classes. Consequently for each $\xi < \lambda$

$$|\{V \in T : V \text{ is a union of } E_\xi\text{-equivalence classes}\}| < \lambda.$$ 

As $|T| > \lambda$ we find a nonempty open set $U$ which for no $\xi < \lambda$ is a union of $E_\xi$-equivalence classes. This $U$ is as needed.

Now let $(\alpha_n, \beta_n)$ be a pair $(\alpha, \beta)$ satisfying (a)–(c) for $\xi = \lambda_n$ and let $\{\alpha_n, \beta_n\} \subseteq Z_{\zeta_n}$. Then $w^* = \{\zeta_n : n < \omega\}$, $U$ contradict the choice of $P$ and $(T_w : w \in P)$. 

PROOF 28 of $\otimes_1$:

For the notational simplicity we assume that $Y = \Omega$. Let $B = \bigcup_{n<\omega} B_n$, $|B_n| < \lambda, \emptyset \in B_0$.

As in the proof of claim $\mathfrak{3}$ wlog for every $x \neq y$ from $\Omega$ we have

$$|\{U \in T : x \in U \iff y \notin U\}| > \lambda.$$ 

Let $y_\zeta \in \Omega$ for $\zeta < \mu_{n+6}$ be pairwise distinct. For each $\zeta < \xi < \mu_{n+6}$ there is $\varepsilon = \varepsilon(\zeta, \xi) \in \{\zeta, \xi\}$ such that $T_{\zeta, \xi}^0 \overset{\text{def}}{=} \{U \in T : \{y_\zeta, y_\xi\} \cap U = \{y_\varepsilon\}\}$ has cardinality $> \lambda$. For each $U \in T_{\zeta, \xi}^0$ there is $V[U] \in B$, $y_\varepsilon \in V[U] \subseteq U$. As $|B| \leq \lambda$ for some $V^*_{\zeta, \xi} \in B$ we have that the set

$$T_{\zeta, \xi}^1 = \{U \in T : \{y_\zeta, y_\xi\} \cap U = \{y_\varepsilon\} \text{ and } y_\varepsilon \in V^*_{\zeta, \xi} \subseteq U\}$$

has cardinality $> \lambda$. For $U \in T$ let $f_U, g_U$ be functions such that:
1. \( f_U : \mu_{n+6} \to \omega, \ g_U : \mu_{n+6} \to B \),
2. \( g_U(\varepsilon) = 0 \) iff \( y_\varepsilon \notin U \),
3. if \( y_\varepsilon \in U \) then \( y_\varepsilon \in g_U(\varepsilon) \subseteq U \),
4. \( f_U(\varepsilon) = \min\{ n \in \omega : g_U(\varepsilon) \in B_n \} \).

For each \( \zeta < \xi < \mu_{n+6} \) we find \( f_{\zeta, \xi} : \mu_{n+6} \to \omega \) such that the set 
\[
T^2_{\zeta, \xi} = \{ U \in T^1_{\zeta, \xi} : f_U = f_{\zeta, \xi} \}
\]
has the cardinality \( > \lambda \). By Erdős-Rado theorem we may assume that for each \( \zeta < \xi < \mu_{n+5}, \varepsilon < \mu_{n+5} \) the value of \( f_{\zeta, \xi}(\varepsilon) \) depends on relations between \( \zeta, \xi \) and \( \varepsilon \) only. Consequently for some \( n^* < \omega \), if \( \varepsilon < \mu_{n+5}, U \in T^2_{\zeta, \xi}, \zeta < \xi < \mu_{n+5} \) then \( g_U(\varepsilon) \in B_{n^*} \). As \( |B_{n^*}| < \lambda \) we find (for each \( \zeta < \xi < \mu_{n+5} \)) a function \( g_{\zeta, \xi} : \mu_{n+6} \to B_{n^*} \) such that the set 
\[
T^3_{\zeta, \xi} = \{ U \in T^2_{\zeta, \xi} : g_U = g_{\zeta, \xi} \}
\]
is of the size \( > \lambda \). Let 
\[
U_{\zeta, \xi} = \bigcup T^3_{\zeta, \xi}, \quad V_{\zeta, \xi} = \bigcup_{\varepsilon < \mu_{n+5}} g_{\zeta, \xi}(\varepsilon).
\]
Clearly 
\[ (*) \quad V_{\zeta, \xi} \subseteq U_{\zeta, \xi}, U_{\zeta, \xi} \setminus V_{\zeta, \xi} \notin \mathcal{I}, U_{\zeta, \xi} \cap \{ y_\zeta, y_\xi \} = V_{\zeta, \xi} \cap \{ y_\zeta, y_\xi \} = \{ y_\varepsilon(\zeta, \xi) \} \]
and 
\[ (**) \quad U_{\zeta, \xi} \cap \{ y_\delta : \delta < \mu_{n+5} \} = V_{\zeta, \xi} \cap \{ y_\delta : \delta < \mu_{n+5} \}. \]

Let \( T_1 = \{ V_{\zeta, \xi} : \zeta < \xi < \mu_{n+5} \} \), so \( |T_1| < \lambda \). Define a two place relation \( E_{T_1} \) on \( \Omega \):
\[
x E_{T_1} y \iff (\forall U \in T_1)(x \in U \leftrightarrow y \in U).
\]
Clearly \( E_{T_1} \) is an equivalence relation with \( \leq 2^{|T_1|} < \lambda \) equivalence classes. Hence by claim 27 for each \( \zeta < \xi < \mu_{n+5} \), for some \( \omega \)-sequence of \( E_{T_1} \)-equivalence classes \( \langle A_{\zeta, \xi, n} : n < \omega \rangle \) we have:
\[
A_{\zeta, \xi, n} \subseteq U_{\zeta, \xi} \setminus V_{\zeta, \xi} \quad \text{and for each infinite } w \subseteq \omega, \bigcup_{n \in w} A_{\zeta, \xi, n} \notin \mathcal{I}.
\]
By Erdös-Rado theorem, wlog for \( \zeta_1 < \zeta_2 < \mu_{n+4} \), \( \zeta_1, \zeta_2 < \mu_{n+4} \) the truth values of “\( \zeta(\zeta_1, \zeta_2) = \zeta_1 \)”, “\( y_1 \in V_{\zeta_1, \zeta_2} \)”, “\( y_\xi \in U_{\zeta_1, \zeta_2} \)”, “\( A_{\zeta_1, \zeta_2,n} \subseteq U_{\zeta_1, \zeta_2} \)”, “\( A_{\zeta_1, \zeta_2,n} \subseteq V_{\zeta_1, \zeta_2} \)”, “\( A_{\zeta_1, \zeta_2,n} = A_{\zeta_1, \zeta_2,m} \)”, “\( A_{\zeta_1, \zeta_2,n} = A_{\zeta_1, \zeta_2,m} \)” depend just on the order and equalities among \( \zeta_1, \zeta_2, \xi, \zeta \) (and of course \( n, m \)).

As each infinite union \( \bigcup_{n \in \omega} A_{\zeta, \xi, n} \) is large, wlog those truth values also does not depend on \( n \) (for the last one we mean “\( A_{\zeta_1, \zeta_2,n} = A_{\zeta_1, \zeta_2,n} \)”). Note: if \( A_{1,2,n} = A_{3,4,m} \) then \( A_{1,2,n} = A_{3,4,n} = A_{1,2,m} \).

Now, \( A_{\zeta, \xi, n} \) is either included in \( U_{\xi, \zeta} \) or is disjoint from it (uniformly for \( n \)); similarly for \( V_{\xi, \zeta} \).

**Case A:** \( A_{3,4,n} \cap U_{1,2} = \emptyset \)

Let \( U'_{\zeta} = \bigcup_{\xi \leq \zeta} U_{2\xi, 2\xi+1} \). Then \( \langle U'_{\zeta} : \zeta < \mu_n \rangle \) is an increasing sequence of open sets and \( \bigcup_{n \in \omega} A_{2\xi + 2, 2\xi + 3, n} \subseteq U'_{\zeta + 1} \setminus U'_\zeta \), which witnesses that the last set is in \( T^+ \). Thus we get clause (a).

**Case B:** \( A_{1,2,n} \cap U_{3,4} = \emptyset \)

Let \( U'_{\zeta} = \bigcup_{\xi \leq \zeta \leq \mu_n} U_{2\xi, 2\xi+1} \). Then \( \langle U'_{\zeta} : \zeta < \mu_n \rangle \) is a decreasing sequence of open sets and \( \bigcup_{n \in \omega} A_{2\zeta, 2\zeta + 1, n} \subseteq U'_\zeta \setminus U'_{\zeta + 1} \). Consequently we get clause (b).

Thus we have to consider the case

\[
A_{1,2,n} \subseteq U_{3,4} \quad \text{and} \quad A_{3,4,n} \subseteq U_{1,2}
\]

only. So we assume this.

**Case C:** \( A_{1,2,n} \cap V_{3,4} = \emptyset, A_{3,4,n} \cap V_{1,2} = \emptyset \)

Let \( U'_{\zeta} = U_{2\zeta, 2\zeta+1}, V'_{\zeta} = V_{2\zeta, 2\zeta+1} \).

**subcase C1:** \( y_1 \in U_{3,4}, y_5 \in U_{3,4} \)

Then let \( y'_{\zeta} \) is the unique member of \( \{ y_{2\zeta}, y_{2\zeta+1} \} \setminus \{ y_{\xi(2\zeta, 2\zeta+1)} \} \).

By (***) we easily get that \( \langle U'_{\zeta}, V'_{\zeta}, y'_{\zeta} : \zeta < \mu_n \rangle \) witnesses the clause (g).

**subcase C2:** either \( y_1 \notin U_{3,4} \) or \( y_5 \notin U_{3,4} \)

Then we put \( y'_{\zeta} = y_{\xi(2\zeta, 2\zeta+1)} \) and we get one of the cases (d), (e) or (f).

**Case D:** \( A_{1,2,n} \subseteq V_{3,4}, A_{3,4,n} \cap V_{1,2} = \emptyset \)

We let \( U'_{\zeta} = \bigcup \{ V_{2\xi, 2\xi+1} : \xi \leq \zeta \} \). Thus \( U'_{\zeta} \) increases with \( \zeta \) and \( U'_{\zeta + 1} \setminus U'_\zeta \) includes \( \bigcup_{n \in \omega} A_{2\zeta, 2\zeta+1, n} \). Thus clause (a) holds.

**Case E:** \( A_{1,2,n} \cap V_{3,4} = \emptyset, A_{3,4,n} \subseteq V_{1,2} \)

Let \( U'_{\zeta} = \bigcup \{ V_{2\xi, 2\xi+1} : \xi \geq \zeta \} \). Then \( U'_{\zeta} \) decrease with \( \zeta \) and the clause (b) holds.

**Case F:** \( A_{1,2,n} \subseteq V_{3,4}, A_{3,4,n} \subseteq V_{1,2} \)

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Let $U_{\zeta}' = U_{2\zeta, 2\zeta+1}$, $V_{\zeta}' = V_{2\zeta, 2\zeta+1}$. If $y_1, y_5 \in U_{3, 4}$ then we put $y_1' = \{y_{2\zeta}, y_{2\zeta+1}\} \setminus \{y_{3(2\zeta, 2\zeta+1)}\}$ and we get case (k). Otherwise we put $y_1' = y_{3(2\zeta, 2\zeta+1)}$ and we obtain one of the cases (h), (i) or (j).

**Concluding Remarks 29**

1. Assume that a topology $T$ on $\Omega$ with a base $B$ and $\lambda$, $\langle \mu_n : n \in \omega \rangle$ are as before ($\mu_n$ regular for simplicity). If

(*)& 
$x_\nu \in \Omega$ for $\nu \in \bigcup_{n<\omega} \prod_{l<n} \mu_l$ and $U_\eta \in T$ for $\eta \in \bigcap_{n<\omega} \mu_n$ and

(**) if $n < \omega$, $\nu \in \prod_{l<n} \mu_l$ and $\eta \in \prod_{l<n} \mu_l$ then for some $k$,

$\forall \eta' \in \prod_{l<\omega} \mu_l \& \eta' \restriction k = \eta \restriction k \Rightarrow U_{\eta'} \cap \{x_\nu\} = U_{\eta} \cap \{x_\nu\}$.

Then we can find $S \subseteq \bigcup_{n<\omega} \prod_{l<n} \mu_l$ and $\langle U_{\eta, \nu} : \eta, \nu \in \prod_{l<n} \mu_l \cap S \rangle$ for some $n$) and $\langle U^*_{\eta} : \eta \in \lim S \rangle$ (where $\lim S = \{ \eta \in \prod_{l<\omega} \mu_l : (\forall l < \omega)(\eta \restriction l \in S) \}$) such that

(a) $\langle \rangle \in S$, $S$ is closed under initial segments and

$\eta \in S \& n = \lg \eta \Rightarrow (\exists \alpha)(\eta^\cdot(\alpha) \in S)$

and for some infinite $w \subseteq \omega$, for every $n < \omega$ and $\eta \in \lim S$ we have:

$n \in w \iff (\exists^{\geq 2} \alpha < \mu_n)(\eta^\cdot(\alpha) \in S) \iff (\exists^{m^*} \alpha < \mu_n)(\eta^\cdot(\alpha) \in S)$.

(b) if $\rho, \nu \in \prod_{l<n} \mu_l \cap S$ and $\nu \triangleleft \eta \in S \cap \prod_{l<\omega} \mu_l$ then $U^*_{\eta} \cap \{x_\rho\} = U^*_{\nu, \eta} \cap \{x_\rho\}$,

(c) for $\eta \in \lim S$, $U^*_{\eta} \cap \{x_\rho : \rho \in S\} = U_{\eta} \cap \{x_\rho : \rho \in S\}$.

2. So in Theorem 22, the case (c) can be further described.

3. We can consider basic forms for any analytic families of subsets of $\lambda$ (then we have more cases; as in 23 and $\otimes_1$ of 26).
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