Some Aspects of $c=-2$ Theory

M. A. Rajabpour*, S. Rouhani † and A. A. Saberi ‡

Department of Physics, Sharif University of Technology, Tehran, P.O.Box: 11365-9161, Iran

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Abstract

We investigate some aspects of the $c=-2$ logarithmic conformal field theory. These include the various representations related to this theory, the structures which come out of the Zhu algebra and the W algebra related to this theory. We try to find the fermionic representations of all of the fields in the extended Kac table especially for the untwisted sector case. In addition, we calculate the various OPEs of the fields, especially the energy-momentum tensor. Moreover, we investigate the important role of the zero modes in this model. We close the paper by considering the perturbations of this theory and their relationship to integrable models and generalization of Zamolodchikov’s $c$–theorem.

Keywords: Logarithmic conformal field theory, $c=-2$ model, Integrable Models, $c$ theorem

1 Introduction

Conformal field theory (CFT) as a powerful tool for investigating critical systems initially appeared in the seminal paper by Belavin, Polyakov and Zamolodchikov [1]. Following this paper many other papers appeared expanding various aspects of CFT, one may look up the books [2, 3] for good introduction to the field. Conformal field theory investigates the quantum theory of models with conformal invariance. However it is most powerful in two dimensions where the conformal group is infinite dimensional. It also turns out that in two dimensions, systems which have scale invariance are also conformally invariant, though there is a counter example to this rule [4]. Soon interest was shown in theories with larger symmetries such as Wess, Zummino, Witten, Novikov models [5]. The other way to extend the conformal symmetry is the introduction of the $W$ algebra, firstly investigated by Zamolodchikov [6]. This closed algebra is a generalization of Virasoro algebra by introducing some primary operators, for a good review see [7].

The advent of CFT was in tune with another development namely the development of integrable models in field theory [8] and exactly solvable models in statistical mechanics [9]. In integrable models one also has an infinite number of conserved currents; but unlike CFT, an infinite dimensionality symmetry group is not involved. Therefore one may view CFT as a

* e-mail: rajabpour@physics.sharif.ir
† e-mail: rouhani@ipm.ir
‡ e-mail: a_saberi@physics.sharif.ir
special integrable model, in fact CFT describes fixed points in the renormalization flow of integrable models [13]. Therefore it is natural to investigate integrable models as perturbations of CFTs, this was first addressed by Zamolodchikov [10]. In this paper Zamolodchikov solved the two dimensional Ising model in the presence of a magnetic field at the critical point. After Zamolodchikov, many papers in this direction appeared, for example see [11, 12].

The other important version of CFT is Logarithmic CFT (LCFT) which contains unusual representations of Virasoro algebra. LCFT first appeared in the works of Kniznick [14] and then Saluer and Rozanskey [15, 16]. But LCFT as a new representation of Virasoro algebra with some special operator product expansion (OPE) was studied first by Gurarie [17]. He showed that LCFT contains an irreducible but indecomposable representation of Virasoro algebra and as an example he studied the \( c = -2 \) model. The \( c = -2 \) action was first introduced in [18] for studying dense polymers. After the work of Gurarie many works were done to investigate various aspects of \( c = -2 \) model. The \( W(2,3) \) algebra structure was first discovered by Kausch [19]. Some other \( W \) algebra structures of \( c = -2 \) model were established by other authors like [20]. In this paper all of the highest weight representations of the \( c = -2 \) model were introduced for the first time. This work was completed by the Gaberdiel and Kausch in the series of papers [21, 22, 23, 24, 25]. All of important results are gathered in Gaberdiel’s review paper [26]. The \( c = -2 \) model is also interesting for its many applications, some of which are dense polymers [15, 18], sandpile models [27, 28, 29], Quantum hall effect [30, 31] and dressing quantum gravity [32]. And recently new logarithmic integrable lattice models were introduced which is related to the critical dense polymers [33, 34]. Moreover the boundary \( c = -2 \) was studied widely by many authors [35] which many of the results are gathered in [36]. Because of its many applications together with being the simplest example of LCFTs, it is widely believed that \( c = -2 \) is a good lab to study LCFTs.

In this work we want to investigate many different aspects of \( c = -2 \) model. Our works are based on the free ghost action. We start by introducing the action and then identifying some of the concepts derived for LCFTs before in the context of this action and the fields inside the model.

The paper is organized as follows: In the second section we review most important properties of LCFTs, it is a brief review with some new results, for more details you can see the classic reviews [37, 38]. In the third section, by introducing the \( c = -2 \) action, we investigate some primary fields as well as the logarithmic partners. In this section we calculate some useful correlation functions and OPEs which are in agreement with the second section’s results. In addition we establish the algebraic method of Gaberdiel and Kausch for classification of \( c = -2 \) representations. This is based on the series of papers [21, 22, 23, 24, 25, 26]. We relate these representations to the fields which we investigate at the beginning of the third section. In the forth section we calculate correlation functions of logarithmic energy-momentum tensor with some other fields like stress-energy tensor and those fields which appear in the OPE of these two fields. In addition we establish the logarithmic Sugawara construction of Kogan and Nichols [39]. In the fifth section, we will study the extended Kac table of model and try to relate every elements of the grid to the known fields defined in the previous sections. In addition we present some facts about the \( W \) algebra structure of \( c = -2 \) model. This structure of \( c = -2 \) is not known well so far.

In the sixth section, we study some different ways to perturb the model and try to find some of perturbations which are related to integrable models. We concentrate on finding the conserved quantities, some of the results have appeared in our previous work [40].

Finally, we investigate some aspects of Zamolodchikov’s \( c \) theorem in the \( c = -2 \) model and see how the zero mode can affect the structure of this theorem. Moreover, we study the
generalized c theorem, which is definable in the integrable models, and find the generalized
renormalization group (RG) quantities which decrease under the RG flow.

2 Logarithmic CFT Theory

An indicative feature of LCFT models is that there are a number of primary fields with
the same conformal weights. Clearly the action of the stress energy tensor on such a set is
not diagonalisable. As an example let’s take a pair of operators as Φ and Ψ which have the
following OPE with the energy-momentum tensor:

\[ T(z)Φ(w) = \frac{hΦ(w)}{(z-w)^2} + \frac{∂Φ(w)}{z-w} + \cdots \] (2.1)
\[ T(z)Ψ(w) = \frac{hΨ + Φ(w)}{(z-w)^2} + \frac{∂Ψ(w)}{z-w} + \cdots. \] (2.2)

One can see from above that the action of \( L_0 \) on the pair Φ, Ψ is not diagonal rather it has a
Jordan form. This fact implies many interesting results, such as the appearance of logarithms
in correlation functions.

To investigate some of these results we begin by looking at the infinitesimal transforma-
tions consistent with the above:

\[ \delta_ε Φ(z) = (h∂_z ε + ε∂_z)Φ(z) \] (2.3)
\[ \delta_ε Ψ(z) = (h∂_z ε + ε∂_z)Ψ(z) + ∂_z εΦ(z). \] (2.4)

One can rewrite the above equations in a compact form by using nilpotent variables [38]

\[ \delta_ε Φ(z, λ) = ((h + λ)∂_z ε + ε∂_z)Φ(z, λ). \] (2.5)

where \( λ^2 = 0 \) and \( Φ(z, λ) = Φ(z) + λΨ(z) \), for the bigger Jordan cells it is sufficient to assume
\( λ^0 = 0 \), and expand \( Φ(z, λ) \) accordingly. In this article we work with just rank two cells,
husce we shall take \( λ^2 = 0 \). This implies that conformal weights have a nilpotent component
and we may derive the finite transformations under a conformal mapping of the complex
plane \( w \) as:

\[ Φ(z, λ) = \left( \frac{∂w}{∂z} \right)^{h+λ} Φ(w, λ) \] (2.6)

Now we can derive the two point functions of the pair of logarithmic fields by using the
invariance under the translation, rotation, scale and special conformal transformations

\[ ⟨Φ(z, λ_1)Φ(w, λ_2)⟩ = \frac{a(λ_1, λ_2)}{(z-w)^{2h+λ_1+λ_2}}, \] (2.7)

where \( a(λ_1, λ_2) = a_1(λ_1 + λ_2) + a_{12}λ_1λ_2 \). The interesting result is that the two point function
of the field \( Φ(z) \) is zero and there is logarithm in the two point function of the field Ψ(z)
which is named the logarithmic partner of the field Φ(z). Similar results can be derived for
the higher order correlators. For example the three point functions of the logarithmic fields can be derived by using equation (2.6) like two point functions; they have the following form:

\[
\langle \Phi(z_1, \lambda_1) \Phi(z_2, \lambda_2) \Phi(z_3, \lambda_3) \rangle = f(\lambda_1, \lambda_2, \lambda_3) z_{12}^{-a_{12}} z_{23}^{-a_{23}} z_{31}^{-a_{31}},
\]

(2.8)

where \(z_{ij} = (z_i - z_j)\), \(a_{ij} = h_i + h_j - h_k + (\lambda_i + \lambda_j - \lambda_k)\) and the \(f(\lambda_1, \lambda_2, \lambda_3)\) has the following form:

\[
f(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{3} C_i \lambda_i + \sum_{1 \leq i < j \leq 3} C_{ij} \lambda_i \lambda_j + C_{123} \lambda_1 \lambda_2 \lambda_3.
\]

(2.9)

Notice that the three point function of the field \(\Phi(z)\) is also zero like the two point function, one can prove this by using the Ward identity. In general the \(n\) point function of this field is zero. A well known example is the identity operators in the \(c = -2\) model which we investigate in the next section. Note that equation (2.8) actually contains four different correlations.

One can extend the OPE (2.2) by inserting another singular term on the rhs of the OPE of the energy-momentum tensor with \(\Psi\), for example suppose we have the following more singular OPE:

\[
T(z)\Psi(w) = \frac{\xi(w)}{(z - w)^3} + \frac{h\Psi(w) + \Phi(w)}{(z - w)^2} + \frac{\partial \Psi(w)}{z - w} + \ldots
\]

(2.10)

where \(\xi\) is an ordinary (non logarithmic) primary field with the conformal weight \(h - 1\). So one can derive the finite transformation of \(\Psi\) as:

\[
\Psi(z) = \left(\frac{\partial w}{\partial z}\right)^h \left(\Psi(w) + \log \left(\frac{\partial w}{\partial z}\right) \Phi(w) + \frac{\partial^2 w}{\partial z^2} \xi(w)\right).
\]

(2.11)

We can check that the above equation is consistent under sequence of successive analytic transformations \(z \rightarrow w \rightarrow w'\). One can now follow the familiar procedures to derive the modified two point functions resulting out of the presence of the last term \(\xi(w)\). This term is not important in the translation, rotation and scale transformation but the special conformal transformation has to be taken into consideration. We can see that the two point functions of \(\Phi\) and \(\Psi\) are similar to (2.7), the difference arises in the two point function of the \(\xi\) and the other fields. We observe that the two point function of \(\xi\) with \(\Phi\) is zero but the two point functions of \(\xi\) with itself and \(\Psi\) have the following forms:

\[
\langle \Psi(z) \xi(w) \rangle = \frac{b}{(z - w)^{2h-1}}
\]

(2.12)

\[
\langle \xi(z) \xi(w) \rangle = \frac{b}{(z - w)^{2h-2}}.
\]

(2.13)

Observe that \(\xi\) and \(\Psi\) have different conformal weights but the two point functions are non vanishing. One can repeat the above process for three point functions and observe that again the three point functions of \(\Phi\) and \(\Psi\) remain unchanged whereas the three point functions involving \(\xi\) have the following forms:
\begin{align}
\langle \Phi(z_1, \lambda_1)\Phi(z_2, \lambda_2)\xi(z_3) \rangle &= a\lambda_1\lambda_2 z_{12}^{h+1} z_{23}^{-1} z_{31}^{-1} \tag{2.14} \\
\langle \Phi(z_1, \lambda_1)\xi(z_2)\xi(z_3) \rangle &= c\lambda_1^{h-2} z_{23}^{h} z_{31}^{-1} \tag{2.15} \\
\langle \xi(z_1)\xi(z_2)\xi(z_3) \rangle &= d z_{12}^{h-1} z_{23}^{-1} z_{31}^{-1}. \tag{2.16}
\end{align}

In the next section we give examples of these operators in the \(c = -2\) model and we establish some representations by using the explicit fermionic action. It may happen that even more singular fields appear in the OPE of the field with the energy-momentum tensor. For such a case the correlators of these fields may also be calculated similar to the above.

3 The Logarithmic \(c = -2\) Theory

The conventional \(c = -2\) theory is constructed using a pair of free grassmanian scalar fields: \(\theta^\alpha = (\theta, \bar{\theta})\) with the action,

\[
S = \frac{1}{2\pi} \int \varepsilon_{\alpha\beta} \partial \theta^\alpha \bar{\partial} \theta^\beta = \frac{1}{\pi} \int \partial \theta \bar{\partial} \bar{\theta},
\]

where \(\varepsilon\) is the canonical symplectic form, \(\varepsilon_{12} = +1\). Note that \(\theta^1 = \theta\) and \(\theta^2 = \bar{\theta}\).

In order to calculate the correlators we need to be careful about the zero modes. Expanding in terms of modes we have:

\[
\theta^\alpha(z) = \sum_{n \neq 0} \theta^\alpha_n z^{-n} + \theta^\alpha_0 \log(z) + \xi^\alpha. \tag{3.2}
\]

Here \(n\) is an integer number for untwisted sector which is related to the periodic boundary condition and for the twisted sector we must choose \(n\) as half integer. The zero modes \(\xi\) (and \(\bar{\xi}\)) appear which do not enter the action \((3.1)\). So, to avoid the vanishing of any correlation function involving \(\theta^\alpha = (\theta, \bar{\theta})\), we have to insert these zero modes in the expectations. Therefore, we can compute the different nonzero correlation functions of \(\theta\) and \(\bar{\theta}\)

\[
\langle \theta^\alpha(z)\bar{\theta}^\beta(w)\xi\bar{\xi} \rangle = \varepsilon^{\alpha\beta} \log |z - w| \tag{3.3}
\]

where \(\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta}\). The correlation functions related to the derivatives of \(\theta\) and \(\bar{\theta}\) are simply obtained from the above relation for instance

\[
\langle \partial \theta^\alpha(z)\partial \theta^\beta(w)\bar{\xi}\bar{\xi} \rangle = \varepsilon^{\alpha\beta} \frac{1}{2(z - w)^2}. \tag{3.4}
\]

Moreover, in this theory we have:

\[
\langle \bar{\theta}(w)\theta(z) \rangle = \langle \xi\bar{\xi} \rangle = 1, \quad \langle 1 \rangle = 0. \tag{3.5}
\]

Let us now proceed to calculate the OPE of the energy-momentum tensor \(T = 2 : \partial \theta \partial \bar{\theta} :\) with itself. This results in the characteristic OPE of a CFT with central charge \(c = -2:\)

\[
T(z)T(w) = -\frac{1}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w} + \cdots. \tag{3.6}
\]
Thus if we expand the energy-momentum tensor by using the modes we have the following Virasoro algebra

\[ [L_n, L_m] = (n - m)L_{n+m} - \frac{1}{6}n(n^2 - 1)\delta_{n+m}. \tag{3.7} \]

The same result can be found for the conjugate algebra \( \bar{L}_n \) related to \( \bar{T} \).

We continue this section by establishing some famous representations of \( c = -2 \) model such as \( R_0, R_1 \) and \( R \). The two representations \( R_0, R_1 \) are the highest weight representations but \( R \) is a local representation whose amplitudes are local. In the end of this section we shall investigate briefly all of the highest weight representations of \( c = -2 \) model by using Zhu’s method \[26\].

### 3.1 \( R_0 \) Representation

The \( R_0 \) representation consists of the identity operator and the field \( \theta \bar{\theta} : \) which is constructed out of the elementary fields \( \theta \) and \( \bar{\theta} \). The fields \( \theta \) and \( \bar{\theta} \) are primary fields with conformal dimensions \((0,0)\), while the bosonic composit operator \( : \theta \bar{\theta} : \) has the following OPE with \( T \):

\[ T(z) : \theta \bar{\theta} : (w) = -\frac{I}{2(z-w)^2} + \frac{\partial : \theta \bar{\theta} : (w)}{z-w} + \cdots. \tag{3.8} \]

If one rescales the operator \( : \theta \bar{\theta} : \) as \( \tilde{I} = -2 : \theta \bar{\theta} : \) in the above OPE, the conventional logarithmic form is obtained. The field \( \tilde{I} \) is the logarithmic partner of the identity and it is not an ordinary primary field. A simple calculation shows that the fields \( \theta \) and \( \bar{\theta} \) have zero conformal dimensions too. The other zero dimension doublet is \( \partial \theta \) and \( \partial \bar{\theta} \).

In this way, representations of the \( c = -2 \) theory can be constructed where there are many primary fields. However in this subsection we investigate just the simplest primary fields and in the forthcoming sections we establish more complex primary fields and relate them to the extended Kac table.

So one can obtain the OPEs of the above set of operators as \[21\]:

\[ \theta^\alpha(z)\theta^\beta(w) = \varepsilon^{\alpha\beta} \left( \frac{I(w)}{2} + \log |z-w|I \right) + \cdots \tag{3.9} \]

\[ \theta^\alpha(z)\tilde{I}(w) = -\log |z-w|^2\theta^\alpha(w) \tag{3.10} \]

\[ \tilde{I}(z)\tilde{I}(w) = -\log |z-w|^2(\tilde{I}(w) + \log |z-w|^2I). \tag{3.11} \]

It is easy to check that the two point functions of the fields \( I \) and \( \tilde{I} \) satisfy the equation \[27\]. We finish this subsection with the statement that \( R_0 \) representation is generated from a logarithmic pair of operators \( I \) and \( \tilde{I} \) with a Jordan cell structure of rank 2.

### 3.2 \( R_1 \) Representation

The \( R_1 \) representation consists of the fields \( \varphi^\alpha = \partial \theta^\alpha \) and \( \psi^\alpha = : \partial \theta^\alpha \tilde{I} : \). The operators \( \varphi^\alpha \) have conformal weight \((1,0)\) and they are a doublet with trivial correlation functions however the fields \( \psi^\alpha \) have unusual properties. The OPE of these fields with \( T \) has the following form:

\[ T(z)\psi^\alpha(w) = \frac{\theta^\alpha(w)}{2(z-w)^3} + \frac{\varphi^\alpha(w) + \psi^\alpha(w)}{(z-w)^2} + \frac{\partial \psi^\alpha(w)}{z-w} + \cdots. \tag{3.12} \]
This is an example of \((2.10)\) with \(\xi(w) = -\frac{1}{2} \theta^\alpha(w)\). Many consequences follow, as an example consider the following relations for the fields \(\psi^\alpha\) and \(\varphi^\alpha\),

\[
L_0 \varphi^\alpha = \varphi^\alpha, \quad L_0 \psi^\alpha = \psi^\alpha + \varphi^\alpha, \quad L_1 \psi^\alpha = -\frac{1}{2} \theta^\alpha. \tag{3.13}
\]

One can see that \(\varphi^\alpha\) and \(\psi^\alpha\) form a Jordan cell with respect to \(L_0\), but \(L_1\) does not vanish on \(\psi^\alpha\), in fact this triplet of fields are related by the action of the Virasoro generators.

### 3.3 \(\mathcal{R}\) Representation

The representation \(\mathcal{R}\) is constructed out of \(I, \bar{I}\) and the following higher level operators

\[
\rho^{\alpha \bar{\alpha}} = \partial \theta^\alpha \bar{\theta}^{\bar{\alpha}}, \quad \bar{\rho}^{\alpha \bar{\alpha}} = -\partial \bar{\theta}^{\bar{\alpha}} \theta^\alpha, \quad \psi^{\alpha \bar{\alpha}} = \varphi^{\alpha \bar{\alpha}} = \varphi^{\alpha \bar{\alpha}} \bar{I} : \]

The \(\varphi^{\alpha \bar{\alpha}}\) are primary fields of weight \((1,1)\). The OPEs of \(\rho^{\alpha \bar{\alpha}}, \bar{\rho}^{\alpha \bar{\alpha}}\) and \(\psi^{\alpha \bar{\alpha}}\) with energy-momentum tensor \(T\) are given by

\[
T(z) \rho^{\alpha \bar{\alpha}}(w) = \frac{\varepsilon^{\alpha \bar{\alpha}} I}{2(z-w)^3} + \frac{\rho^{\alpha \bar{\alpha}}(w)}{(z-w)^2} + \frac{\partial \rho^{\alpha \bar{\alpha}}(w)}{z-w} + \cdots \tag{3.15}
\]

\[
T(z) \bar{\rho}^{\alpha \bar{\alpha}}(w) = \frac{\varphi^{\alpha \bar{\alpha}}(w)}{z-w} + \cdots \tag{3.16}
\]

\[
T(z) \psi^{\alpha \bar{\alpha}}(w) = -\frac{\bar{\rho}^{\alpha \bar{\alpha}}(w)}{(z-w)^3} + \frac{\varphi^{\alpha \bar{\alpha}}(w) + \psi^{\alpha \bar{\alpha}}(w)}{(z-w)^2} + \frac{\partial \psi^{\alpha \bar{\alpha}}(w)}{z-w} + \cdots. \tag{3.17}
\]

The OPE \((3.14)\) is similar to the mentioned form \((2.10)\), so one expects that the two point functions of \(\rho^{\alpha \bar{\alpha}}\) with itself and \(\psi^{\alpha \bar{\alpha}}\) have the following forms as \((2.7)\) and \((2.12)\):

\[
\langle \rho^{\alpha \bar{\alpha}}(z) \rho^{\beta \bar{\beta}}(w) \rangle = -\frac{\varepsilon^{\alpha \beta} \varepsilon^{\bar{\alpha} \bar{\beta}}}{2(z-w)^2}, \quad \langle \rho^{\alpha \bar{\alpha}}(z) \psi^{\beta \bar{\beta}}(w) \rangle = \frac{\varepsilon^{\alpha \beta} \varepsilon^{\bar{\alpha} \bar{\beta}}}{2(z-w)^2(z-w)}, \tag{3.18}
\]

\[
\langle \psi^{\alpha \bar{\alpha}}(z) \psi^{\beta \bar{\beta}}(w) \rangle = \frac{\varepsilon^{\alpha \beta} \varepsilon^{\bar{\alpha} \bar{\beta}}}{2|z-w|^4}(1 + \log |z-w|^2). \tag{3.18}
\]

In addition, the other correlations can be obtained as bellow:

\[
\langle \bar{I}(z) \rho^{\alpha \bar{\alpha}}(w) \rangle = \frac{\varepsilon^{\alpha \bar{\alpha}}}{z-w}, \quad \langle \bar{I}(z) \bar{\rho}^{\alpha \bar{\alpha}}(w) \rangle = \frac{\varepsilon^{\alpha \bar{\alpha}}}{z-w}, \tag{3.19}
\]

\[
\langle \bar{I}(z) \psi^{\alpha \bar{\alpha}}(w) \rangle = \frac{-\varepsilon^{\alpha \bar{\alpha}}}{|z-w|^2}, \quad \langle \rho^{\alpha \bar{\alpha}}(z) \rho^{\beta \bar{\beta}}(w) \rangle = \frac{-\varepsilon^{\alpha \beta} \varepsilon^{\bar{\alpha} \bar{\beta}}}{2(z-w)^2}, \tag{3.19}
\]

\[
\langle \rho^{\alpha \bar{\alpha}}(z) \bar{\rho}^{\beta \bar{\beta}}(w) \rangle = 0, \quad \langle \rho^{\alpha \bar{\alpha}}(z) \psi^{\beta \bar{\beta}}(w) \rangle = \frac{\varepsilon^{\alpha \beta} \varepsilon^{\bar{\alpha} \bar{\beta}}}{2(z-w)^2(z-w)}, \tag{3.19}
\]

\[
\langle \rho^{\alpha \bar{\alpha}}(z) \bar{\rho}^{\beta \bar{\beta}}(w) \rangle = 0, \quad \langle \psi^{\alpha \bar{\alpha}}(z) \psi^{\beta \bar{\beta}}(w) \rangle = \frac{-\varepsilon^{\alpha \beta} \varepsilon^{\bar{\alpha} \bar{\beta}}}{2|z-w|^4}. \tag{3.19}
\]

The fields \(\varphi^{\alpha \bar{\alpha}}\) for \(\alpha \neq \bar{\alpha}\) are the same as those which appear in the action \((6.18)\). In section \((6)\) we show that if one perturbs the action with the fields \(\varphi^{\alpha \bar{\alpha}}\) then the theory remains conformal.
3.4 Fields with unusual Logarithmic OPEs

One of the important fields in the \( c = -2 \) theory that is well known in sandpile model \[27, 40\] is \( \phi =: \partial \bar{\theta} \frac{\partial \theta}{\bar{\theta}} \) which is composed of the \( R \) representation’s fields. These fields are related to some height correlation functions in sandpile model playing a role in spanning trees and spanning forest \[41\]. Using the OPE of this field with energy-momentum tensor one can simply read off that \( \phi \) is a primary field with conformal dimensions \((1, 1)\)

\[
T(z)\phi(w) = \frac{\phi(w)}{(z-w)^2} + \frac{\partial \phi(w)}{z-w} + \cdots. \tag{3.20}
\]

Also \( \phi \) is a descendent of \( : \partial \bar{\theta} \frac{\partial \theta}{\bar{\theta}} : \) and the Virasoro algebra generators \( L_n \)'s operate on this field as

\[
L_{-1} \phi = \partial \phi, \quad L_0 \phi = \phi \quad \text{and} \quad L_n \phi = 0 \quad \text{for} \quad n \geq 1. \tag{3.21}
\]

The OPE of \( \phi \) with itself has the following form :

\[
\phi(z)\phi(w) = -\frac{I}{2|z-w|^4} + \frac{T(w)}{2(z-w)^2} + \frac{T(w)}{2(\bar{z}-\bar{w})^2} + \cdots. \tag{3.22}
\]

So we can write the closed algebra for The modes of \( \phi \). If we show the Modes as \( \phi_{m,n} \) then we can write the algebra as the following form by using the Virasoro modes

\[
[\phi_{m,n}, \phi_{p,q}] = \frac{nm}{4}\delta_{n+q,0}\delta_{m+p,0} + \frac{m}{2}\delta_{m+p,0}L_{n+q-2} + \frac{n}{2}\delta_{n+q,0}\tilde{L}_{m+p-2}. \tag{3.23}
\]

In addition the OPE of \( \phi \) with the \( : \partial \bar{\theta} \frac{\partial \theta}{\bar{\theta}} : \) is

\[
\phi(z) : \partial \bar{\theta} \frac{\partial \theta}{\bar{\theta}} : (w) = -\frac{I}{2|z-w|^4} + \frac{\tilde{T}(w)}{2(z-w)^2} + \frac{\tilde{T}(w)}{2(\bar{z}-\bar{w})^2} + \cdots. \tag{3.24}
\]

By using the above OPE one can write a closed algebra for the modes of \( \phi \) and \( \tilde{T} \) too. Insertion of \( : \partial \bar{\theta} \frac{\partial \theta}{\bar{\theta}} : \) aside \( \phi \) yields a logarithmic field \( \psi =: \partial \bar{\theta} \frac{\partial \theta}{\bar{\theta}} + \partial \bar{\theta} \frac{\partial \theta}{\bar{\theta}} : \)

It is believed that the field \( \psi \) is a logarithmic partner of \( \phi \), and our explicit calculation of OPE of \( \psi \) with the energy-momentum tensor shows that too

\[
T(z)\psi(w) = \frac{\tilde{T}(z) : \partial \bar{\theta} \frac{\partial \theta}{\bar{\theta}} : (w)}{2(z-w)^3} + \frac{\psi(w) - \frac{1}{2}\phi(w)}{(z-w)^2} + \frac{\partial \psi(w)}{(z-w)} + \cdots. \tag{3.25}
\]

In this expression the third order singular term appears whereas the standard form of OPEs in LCFT has no similar term. The field \( \tilde{T}(z) : \partial \bar{\theta} \frac{\partial \theta}{\bar{\theta}} : \) has conformal dimension of \((0,1)\) and it is similar to the field \( \xi \) in the equation \(2.10\). Ignoring the first term, up to a rescaling \( \tilde{\psi} = -2\psi \) the other terms have the general logarithmic form.

Operation of the Virasoro algebra generators \( L_n \)'s on the rescaled field \( \tilde{\psi} \) yields

\[
L_{-1} \tilde{\psi} = \partial \tilde{\psi}, \quad L_0 \tilde{\psi} = \tilde{\psi} + \phi, \quad L_1 \tilde{\psi} = \frac{1}{2} \tilde{\psi}, \quad L_n \tilde{\psi} = 0 \quad \text{for} \quad n > 1. \tag{3.26}
\]
Moreover, the OPEs of the field $\psi$ with $\phi$ is explicitly calculated using the Wick theorem

$$
\phi(z)\psi(w) = -\frac{\Theta\bar{\Theta} : (w)}{2|z-w|^4} + \frac{\phi(w)}{2|z-w|^2} + \frac{\overline{\partial} : \Theta\bar{\Theta} : (w)}{4|z-w|^2(z-w)} + \frac{\partial : \Theta\bar{\Theta} : (w)}{4|z-w|^2(z-w)} \\
+ \frac{i(w)}{2(z-w)^2} + \frac{t(w)}{2(\bar{z}-\bar{w})^2} - \frac{\partial t(w)}{4(z-w)} + \ldots. 
$$

(3.27)

In this OPE one can see that the two operators $t = 2 : \partial \Theta \partial \bar{\Theta} :$ and $\tilde{t} = 2 : \bar{\partial} \Theta \bar{\partial} \bar{\Theta} :$ appear which we name them logarithmic energy-momentum tensors. We cannot write a closed algebra for the modes of $\psi$ as we did in equation (3.23). Moreover the OPEs of $\psi$ with itself and $: \Theta \bar{\Theta} :$ are given in the appendix, these OPEs are useful in calculating the RG equations.

Note that the equations (3.27) and (A.1) don’t satisfy the ordinary logarithmic OPEs.

It is not difficult to calculate all of the two point correlation functions including the fields $\psi$ and $\phi$. One can obtain them by using the wick theorem and the original correlations

$$
\langle \phi(z)\phi(w) \rangle = 0, \\
\langle \psi(z)\phi(w) \rangle = \frac{1}{2|z-w|^4}, \\
\langle \psi(z)\psi(w) \rangle = \frac{1}{2|z-w|^4} \{1 + \log|z-w|^2\}, \\
\langle \phi(z)\overline{\partial} : \Theta\bar{\Theta} : (w) \rangle = 0, \\
\langle \psi(z)\overline{\partial} : \Theta\bar{\Theta} : (w) \rangle = \frac{1}{2(z-w)(\bar{z}-\bar{w})^2}.
$$

(3.28) (3.29) (3.30) (3.31) (3.32)

The above correlators are similar to ones which were investigated in the first section, here we have $h = 1$. It is interesting that the two point functions of the fields with different weights can be found by using just the OPEs of these fields with energy-momentum tensor.
3.5 W algebra and the Highest Weight Representations of 
c = −2 Model

We conclude this section by relating the above calculations to the algebraic approach of Gaberdiel and Kausch [26]. Using Zhu’s algebra, one can show that the representation which contains \( I \) and \( \tilde{I} \) is the only logarithmic one which is generated from a highest weight state, but the other logarithmic representations which may be constructed by other methods cannot be found in this way. An example is the representation generated from \( \phi \) and \( \tilde{\psi} \) with the conformal weight of one, and \( \xi \) with the weight of zero named \( \mathcal{R}_\infty \) (see [26] for more details). The method of Gaberdiel and Kausch is based on the \( W \) algebra structure of the \( c = -2 \) model. One may simply check that all of the following fields have conformal dimension of three

\[
W^+ = \partial^2 \theta \partial \theta \\
W^0 = \frac{1}{2} (\partial^2 \theta \partial \theta + \partial^2 \tilde{\theta} \partial \tilde{\theta}) \\
W^- = \partial^2 \tilde{\theta} \partial \theta.
\]

The above fields are isospin one fields and have the following OPEs with themselves [32].

\[
W^i(z)W^j(w) = g^{ij} \left( \frac{1}{(z-w)^6} - \frac{3}{2} \frac{T(w)}{(z-w)^3} - \frac{3}{2} \frac{T(w)}{(z-w)^3} + \frac{3}{2} \frac{\partial^2 T(w)}{(z-w)^2} \right) \\
- \frac{4}{z-w} \left( \frac{T^2(w)}{z-w} + \frac{1}{6} \frac{\partial^3 T(w)}{z-w} - \frac{4}{z-w} \frac{\partial^2 T(w)}{z-w} \right) - 5f^{ij}_k \left( \frac{W^k(w)}{(z-w)^3} \right) \\
+ \frac{1}{2} \frac{\partial W^k}{(z-w)^2} + \frac{1}{25} \frac{\partial^2 W^k}{z-w} + \frac{1}{25} \frac{(TW^k)(w)}{z-w},
\]

where \( g^{ij} \) is the metric on the isospin one representation, \( g^{+-} = g^{-+} = 2 \) and \( g^{00} = -1 \), and \( f^{ij}_k \) are the structure constants of \( SL(2) \).

By using the above OPEs, one can write the \( W \) algebra which was first investigated by Zamolodchikov [6]. Following Gaberdiel and Kausch [22] we have

\[
[L_m, W^i_n] = (2m - n)W^i_{m+n} \tag{3.35}
\]

\[
[W^i_m, W^j_n] = g^{ij}(2(m-n)\Lambda_{m+n} + \frac{1}{20}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\
- \frac{1}{120}m(m^2 - 1)(m^2 - 4)\delta_{m+n}) \\
+ f^{ij}_k \left( \frac{5}{14}(2m^2 + 2n^2 - 3mn - 4)W^k_{m+n} + \frac{12}{5}V^k_{m+n} \right) \tag{3.36}
\]

where \( \Lambda = :T^2: - 3/10 \partial^2 T \) and \( V^a = :TW^a: - 3/14 \partial^2 W^a \) are quasiprimary normal ordered fields. The above algebra is a little different from the original Zamolodchikov’s \( W \) algebra, the last term multiplied by the factor \( f^{ij}_k \) is different.

By using the above \( W \) algebra, Gaberdiel and Kausch found two following relevant nontrivial null vectors
\[ N^i = (2L_{-3}W^i_{-3} - \frac{4}{3}L_{-2}W^i_{-4} + W^i_{-6})I, \]  
\[ N^{ij} = W^i_{-3}W^j_{-3}I - g^{ij}\left(\frac{8}{9}L_{-2}^3 + \frac{19}{36}L_{-3}^2 + \frac{14}{9}L_{-4}L_{-2} - \frac{16}{9}L_{-6}\right)I - f^{ij}_k \left(-2L_{-2}W^k_{-4} + \frac{5}{4}W^k_{-6}\right)I. \]  
\[ (3.37) \]

Actually, one can simply check that the field \( W^0 \) is a level two descendent of some fields which will be investigated later. We use the above null vectors to determine the allowed highest weight representations of \( c = -2 \) model. Any correlator involving the states in a representation of CFT and null vectors must vanish. One can simplify this condition by stating that the operation of the zero mode of a null vector on every highest weight state must vanish. These modes are named Zhu’s modes. The zero modes of \( (3.37) \) enforce the following relation for any arbitrary highest weight field \( \varphi \):

\[ L_0^2(8L_0 + 1)(8L_0 - 3)(L_0 - 1)\varphi = 0. \]  
\[ (3.39) \]

The above equation implies that \( h \) must be \( h = 0, -1/8, 3/8, 1 \). In other words, by evaluating the constraints which come from the null vectors, we find that for irreducible representations, we have the highest weight representations just for the fields with the above weights. The representations are named \( V_h \).

The \( h = 0 \) representation is a logarithmic representation but the others are not, although the fusion products of some of them can produce logarithmic representations. Let’s first investigate \( V_0 \). This representation comes from \( L_0^3\varphi = 0 \) which can be satisfied by \( \bar{I} \) and \( I \) where \( L_0\bar{I} = I \) and \( L_0I = 0 \). This representation is the only logarithmic highest weight representation of \( c = -2 \) model. Although, there are many other logarithmic representations but they are not highest weight representations. In every logarithmic theory, we must have a term like \((L_0 - h)^n\) in the Zhu’s algebra for rank \( n \) Jordan cells with a logarithmic highest weight representation to exist. In \( c = -2 \) model, there are three other highest weight representations which two of them are related to the twisted sector and the other is nontwisted. To complete the classification of the highest weight representations, one can write the following algebra for the zero modes of \( W \) algebra by using the constraint coming from the zero modes of the null vectors:

\[ [W^i_0, W^j_0] = \frac{2}{9}(6h - 1)f^{ij}_k W^k_0. \]  
\[ (3.40) \]

This is similar to the \( SU(2) \) algebra, so we can use it to label the representations. Like the ordinary representations of \( SU(2) \), \( j \) and \( m \) label the representations. For \( c = -2 \) there is another constraint, \( W^i_0W^j_0 = W^j_0W^i_0 \). So we must have \( j(j + 1) = 3m^2 \). With this constraint, one can find that just representations \( j = 0, \frac{1}{2} \) exist. By using \( (3.40) \), we simply find that the representations \( V_0 \) and \( V_{-1/8} \) are related to \( j = 0 \) but the others are related to \( j = 1/2 \). So, we have two singlet representations and two doublet representations. These four irreducible highest weight representations can be multiplied together to produce the well known representations \[26\]. The fusion product of the above representations have the following forms.
Let us now calculate the finite transformation of the operators on \( \tilde{\phi} \) modes. First observe that \( \tilde{\phi} \) is a operator in the local theory \[26\]. The fields in \( \tilde{\phi} \) representation \[3.41\] be simply shown that

\[
W^i_0 \varphi^\alpha = 2t^i_\beta \varphi^\beta \\
W^i_{-1} \theta^\alpha = t^i_\beta \varphi^\beta \\
W^i_0 \theta^\alpha = 0 \\
W^i_1 \varphi^\alpha = t^i_\beta \theta^\beta
\]

where \( t^i_\beta \) is the spin \( 1/2 \) representation of \( S U(2) \) where just the elements \( t^0_\pm = \pm 1/2 \) and \( t^1_{\pm \mp} = 1 \) are not zero.

The other important property is that one can calculate the other fusion products and prove that the four representations \( \mathcal{V}_h \) with allowed weights and \( \mathcal{R}_0 \) and \( \mathcal{R}_1 \) are closed under fusion. It means that we have a rational logarithmic conformal field theory. Finally the representation \( \mathcal{R} \) is a local representation which is a direct sum of \( \mathcal{R}_0, \mathcal{R}_1 \) and \( \mathcal{V}_h \) with some subtractions. For \( \mathcal{R} \) to be local, we need to have \( h - \tilde{h} \in \mathbb{Z} \) and \( S\tilde{\phi} = (L^0_0 - L^0_0)\phi = 0 \) where \( \phi \) is a operator in the local theory \[26\]. The fields in \( \mathcal{R} \) satisfy these properties and it is a local triplet representation for \( c = -2 \) model.

4 Logarithmic Energy-Momentum Tensor and Logarithmic Sugawara Construction

The logarithmic partner of the energy-momentum tensor \( t \) can be constructed using the fields \( T \) and \( \theta \tilde{\theta} \), namely \( T\theta \tilde{\theta} \). Despite the fact that \( t \) has been known to exist for a while, its OPEs were not well known. The OPE of \( t \) with \( T \) takes on the form:

\[
T(z)t(w) = - \frac{\theta \tilde{\theta} (w)}{(z-w)^4} + \theta \tilde{\theta} (w) \frac{\partial}{(z-w)^3} + 2t(w) \frac{2i(w)}{(z-w)^2} - T(w) \frac{2(z-w)^2}{2(z-w)^2} + \frac{\partial t(w)}{z-w} + \cdots
\]

Let us rescale \( \tilde{t} = -2t \) in order to find familiar equations. We can see that the Virasoro operators on \( \tilde{t} \) take the form:

\[
L_{-2}\tilde{t} = \tilde{t}, \quad L_{-2}l = T \\
L_{-1}\tilde{t} = \partial \tilde{t}, \quad L_{0}l = 2\tilde{t} + T \\
L_{1}\tilde{t} = \partial \tilde{t}, \quad L_{2}l = -\tilde{l} \\
L_{n}\tilde{t} = 0 \quad \text{for} \quad n > 2.
\]

Let us now calculate the finite transformation of the operator \( t \) using the OPE \[4.1\]:

\[
t(z) = \left( \frac{\partial w}{\partial z} \right)^2 t(w) - \frac{1}{2} \log \left( \frac{\partial w}{\partial z} \right) \left( \frac{\partial w}{\partial z} \right)^2 T(w) - \frac{1}{6} \{w, z\} \\
+ \frac{1}{2} \frac{\partial^2 w}{\partial z^2} \left( \theta \tilde{\theta} (w) - \frac{\partial^2 w}{2 \partial z^2} \right) - \frac{1}{6} \{w, z\} \left( \theta \tilde{\theta} (w) - \frac{1}{2} \log \left( \frac{\partial w}{\partial z} \right) \right)
\]
where the \( \{w, z\} \) is the schwarzian derivative defined by
\[
\{w, z\} = \frac{\partial^3 w}{\partial z^3} - \frac{3}{2} \left( \frac{\partial^2 w}{\partial z^2} \right)^2.
\] (4.4)

To find equation (4.3) we used the general conformal transformation of the energy-momentum tensor
\[
T(z) = \left( \frac{\partial w}{\partial z} \right)^2 T(w) - \frac{1}{6} \{w, z\}.
\] (4.5)

Nevertheless, here the two point functions including the fields \( t \) and \( T \) have their standard logarithmic form:
\[
\langle T(z)T(w) \rangle = 0,
\] (4.6)
\[
\langle t(z)T(w) \rangle = \frac{1}{(z - w)^4},
\] (4.7)
\[
\langle t(z)t(w) \rangle = \frac{1}{(z - w)^4} \{1 + \log |z - w|^2\}.
\] (4.8)

The correlation functions of \( T \) with the other operators vanish but the one for \( t \) have nonzero values. For example, the two point function of \( t \) with \( :\bar{\theta}\theta: \) is:
\[
\langle t(z) :\bar{\theta}\theta: (w) \rangle = \frac{1}{2(z - w)^2}.
\] (4.9)

The OPEs of \( t \) with the fields \( :\bar{\theta}\theta:, \psi \) and \( \phi \) are given in the appendix.

The other important OPE is \( t \) with itself. as we can see from the above nontrivial correlation functions it does not have a simple form:
\[
t(z)t(w) = \frac{I}{4(z - w)^4} \{1 + 2 \log |z - w|^2 + \log^2 |z - w|^2\} - \frac{1}{(z - w)^4} \{1 + \log |z - w|^2\} :\bar{\theta}\theta: (w) + \frac{1}{2(z - w)^2} \log |z - w|^2 t(w)
\] - \frac{1}{2(z - w)^2} \log |z - w|^2 \{1 + \log |z - w|^2\} T(w) + \cdots.
\] (4.10)

The existence of logarithms do not allow derivation of a closed algebra for the modes of \( t \). One can write a different logarithmic energy-momentum tensor which is related to the Logarithmic Sugawara construction. Logarithmic Sugawara construction is defined with a little defect. Given the primary pre-logarithmic currents \( J^i \) then we can define the energy-momentum tensors as:
\[
J^i(z)J^i(0) \sim \cdots + (T \log z + \hat{t}) + \cdots.
\] (4.11)

For using the above construction for \( c = -2 \) model we need some currents. The action (3.1) is invariant under \( SL(2) \) transformations on the fields \( \theta^\alpha \) which their infinitesimal transformation is
\[
\delta \theta^\alpha = -i\Lambda(z, \bar{z})t^\alpha_\beta \theta^\beta.
\] (4.12)
So by using Noether’s theorem and some rescaling we can write the following currents

\[ J^+ = \frac{4}{3} \partial \bar{\theta}, \]
\[ J^0 = \frac{2}{3} (\partial \theta + \bar{\theta} \partial \bar{\theta}), \]
\[ J^- = \frac{4}{3} \bar{\theta} \partial \bar{\theta}. \]  

(4.13)

One can simply prove that all of these operators are primary fields with conformal dimensions equal to one. In fact these operators are related to the fields of \( \mathcal{R} \) representation. Like the previous sections by using fundamental correlations one can find the following OPE for the currents

\[ J^i(z)J^j(0) = g^{ij} \left( \frac{\log z I + \bar{I} + I}{z^2} + \frac{\partial \bar{I}}{2z} \right) + f^{ij}_{k} J^k \frac{1}{z} + \cdots, \]

(4.14)

which is similar to affine Lie algebra. By looking at equation (4.11), one can guess that the \( \hat{t} \) must have the following form

\[ \hat{t} = -2t - \frac{3}{2} y^2 \bar{I} - 5T. \]

(4.15)

\( \hat{t} \) is a descendent field of \( I \) and \( \bar{I} \), and may be written as follows:

\[ \hat{t} = (L_{-2} - \frac{5}{2} I_{-2}) \bar{I} - 4L_{-2} I. \]

(4.16)

It is not difficult to calculate the two point functions of this field with \( T \) and itself. These are similar to the two point functions of \( t \) case whereas its OPEs are different.

5 Kac Table in \( c = -2 \) Theory

Usually the operator content of minimal models \( \mathcal{M}(p,p') \) with \( p > p' \) is the \( \phi_{r,s} \) such that \( 0 < r < p' \), \( 0 < s < p \). For the \( c = -2 \) model the Kac table is the case with \( (p,p') = (2,1) \) and it is trivial. The solution is to extend the Kac table to find a nontrivial operator content. The Kac formula of conformal dimension \( h_{r,s} \) for minimal model \( \mathcal{M}(p,p') \) is given as

\[ h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'}. \]

(5.1)

The formula for \( h_{r,s} \) has an important symmetry \( h_{r,s} = h_{p-r,q-s} \). For \( c = -2 \) model we have the following formulas

\[ h_{r,s} = \frac{(2r - s)^2 - 1}{8}, \quad h_{r,s} = h_{1-r,2-s} = h_{r-1,s-2} \]

(5.2)

It is interesting to relate the primary fields which we investigated in the previous sections
the conformal weight of three fields. Another example is the fields equivalent to the $\phi$ result states that just the two lines on the boundary of grid a are important and produce new must notice that $\phi$, showed [37], is related to $\varphi$ primary fields to the logarithmic partners, the logarithmic partner of $\sigma$ at level two. The doublet of twist fields $\phi$ and $\phi$ is the twist operator $\mu$ with the null vector at level two. The doublet of twist fields $\sigma_\alpha = (\theta_\alpha - \frac{1}{2}) \mu$ of dimension $\frac{3}{8}$ is equivalent to $\phi_{2,2}$ and $\phi_{1,4}$ in the Kac table. The other interesting fields are the $W^i$ fields which are related to the $\phi_{3,4}$, the logarithmic partner of this field is $\phi_{1,7}$. So one can find an algorithm to relate the primary fields to the logarithmic partners, the logarithmic partner of $\phi_{r,1}$ is $\phi_{1,2r+1}$. One must notice that $\phi_{r,s} = \phi_{r+1,s+2}$ and so the fields $\phi_{2,2}$ and $\phi_{3,4}$ are equivalent. This important result states that just the two lines on the boundary of grid are important and produce new fields. Another example is the fields equivalent to the $\phi_{4,1}$ with null vector at level four and the conformal weight of three

$$
H^2 = \partial^3 \theta \partial^2 \theta \theta
$$

$$
H^3 = \frac{1}{3} (\partial^3 \theta \partial^2 \theta \theta + \partial^3 \theta \partial^2 \theta \theta + \partial^3 \theta \partial^2 \theta \theta)
$$

$$
H'^2 = \frac{1}{3} (\partial^3 \theta \partial^2 \theta \theta + \partial^3 \theta \partial^2 \theta \theta + \partial^3 \theta \partial^2 \theta \theta)
$$

$$
H'^3 = \partial^3 \theta \partial^2 \theta \theta.
$$

(5.3)

Generally, one can check that $\phi_{r,1}$ generates Isospin $\frac{r-1}{2}$ fields. So in level $(r, 1)$ we have $r$ fields with conformal dimension of $\frac{r^2}{2} - 1$. For example in level $(5, 1)$ we have the following primary fields

$$
M^2 = \partial^4 \theta \partial^3 \theta \partial^2 \theta \theta
$$

$$
M^1 = \frac{1}{4} (\partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta)
$$

$$
M^0 = \frac{1}{6} (\partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta)
$$

$$
+ \partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta)
$$

$$
M^{-1} = \frac{1}{4} (\partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta + \partial^4 \partial^3 \partial^2 \theta \theta)
$$

$$
M^{-2} = \partial^4 \partial^3 \partial^2 \theta \theta.
$$

(5.4)
Can be checked that all of the above fields have null vectors at level five. The higher primary conformal fields can be found in the same way.

One can also find the fermionic representation of other primary fields in the Kac table. The column \((r, 2)\) is related to the twisted sector. Suppose in (3.2) we take \(n = m - \tau\) with integer \(m\) and half integer \(\tau\); then the fields \(\nu_\alpha = (\theta_\alpha)_{-\tau} \mu\) have conformal weight \(h_\tau = \frac{\tau(\tau - 1)}{2}\). They form the second column in the Kac table, for example \(\phi_{3,2}\) and \(\phi_{1,6}\) are related to \(\tau = \frac{5}{2}\).

Generally, the fields \(\phi_{r,2}\) and \(\phi_{1,2r}\) have the same conformal weights.

5.1 Hierarchy of W algebras

In section (3) we saw that there exists a \(W(2,3,3,3)\) algebra with the primary operators \(W^i\) and conformal dimension of three. These fields are descendants of the \(J^i\) currents:

\[
(L_{-2} - \frac{1}{2}L_{-1}^2)J^i = 2W^i. \tag{5.5}
\]

Because of the absence of zero modes, the two point correlation functions of all these fields with themselves vanish. Notice that the existence of zero modes is not sufficient for nontrivial correlation functions, the best examples are the logarithmic partners of \(W^i\) fields which are produced by insertion of \(\tilde{I}\) aside them. Their two point functions have the following forms

\[
\langle W^i(z)\tilde{W}^j(w) \rangle = g^{ij}(z - w)^6, \tag{5.6}
\]

\[
\langle \tilde{W}^i(z)\tilde{W}^j(w) \rangle = g^{ij}(z - w)^6(1 + \log z). \tag{5.7}
\]

The fields \(\tilde{W}^i\) have the following OPE with energy-momentum tensor

\[
T(z)\tilde{W}^i(w) = \frac{-3J^i}{2(z - w)^4} + \frac{K^i}{2(z - w)^3} + \frac{3\tilde{W} + W}{(z - w)^2} + \frac{\partial \tilde{W}}{z - w}, \tag{5.8}
\]

where \(K^i\)'s are non-primary fields with the following explicit forms:

\[
K^+ = \theta \partial^2 \theta, \quad K^0 = \theta \partial^2 \bar{\theta} + \bar{\theta} \partial^2 \theta, \quad K^- = \bar{\theta} \partial^2 \bar{\theta}. \tag{5.9}
\]

Whereas the operator \(W^0\) has a null vector at level three, the operators \(W^+\theta, W^-\bar{\theta}\) and \(W^0\bar{\theta}^\alpha\) do not have a null vector at level three. Operating the following level three operator on these fields, one can find the \(\frac{1}{4}H^i\) fields

\[
(L_{-3} - \frac{2}{5}L_{-1}L_{-2} + \frac{1}{20}L_{-1}^3). \tag{5.10}
\]

This is similar to equation (5.5), in fact the same relations exist for higher level primary fields. For example, by operating the level four null operator on the \(H^7\theta\), one can obtain the operators proportional to the \(M^2\).

The above argument comes from the relation \(h_{r,1} + r = h_{r+1,1}\) in which \(r\) is the level of the null vector of \(\phi_{r,1}\). It does not indicate that \(W(2, 6, 6, 6), W(2, 6), W(2, 10, 10, 10, 10, 10)\) or \(W(2, 10)\) algebras in \(c = -2\) model exist. [43, 44] show that \(W(2, 10)\) algebra is consistent with \(c = -2\) whereas the \(W(2, 6)\) is inconsistent. In [20] the authors used the \(W(2, 10)\) algebra
and found some of the results which we have established in section (3). Our calculations are consistent with the existence of some other \( W \) algebras. However this is not quite clear yet and work in this direction is under progress.

These \( W \) algebras are not the only algebras which appear in the \( c = -2 \) model. One of the other closed complex algebras is the \((3.23)\) algebra which mixes the holomorphic and antiholomorphic parts of the Virasoro algebra; a property which cannot be seen in the ordinary \( W \) algebras.

All of these expressions show that the action \((3.1)\) is a good representation for \( c = -2 \) conformal field theory. This action has suitable representations for all of the fields which appear in the Kac table.

In the next sections we will use some of the fields introduced before to perturb the \( c = -2 \) theory. In this way, we will investigate some off-critical models which are not conformally invariant but their contents are similar to the ordinary \( c = -2 \) model.

### 6 Perturbation of the \( c = -2 \) Theory

Conformal field theories describe the behavior of a system at its critical point. Also in two dimensions CFT’s are integrable since the conformal algebra is infinite dimensional. Therefore it is natural to expect that perturbation of a CFT by an operator may lead to an integrable model in two dimensions \([10]\). However not all perturbations may lead to integrable models. The perturbing operator has to be chosen carefully so that an infinite number of currents remain conserved, therefore the structure of the CFT becomes important. The case for unitary models with central charge \( c = 1 - \frac{6}{p(p+1)} \), \( p = 3, 4, 5, ... \), and perturbing fields being \( \phi_{1,2}, \phi_{2,1}, \phi_{1,3} \) was analyzed in \([10]\). The usefulness of this approach lies in the fact that one may use the structure of CFT to investigate the integrable models. In fact, using this device, Zamolodchikov solved the Ising model in two dimensions in presence of magnetic field \([6, 12]\). In this section, we investigate theories which are obtained from perturbation of \( c = -2 \) theory by fields such as : \( \theta \bar{\theta} \) : and : \( \partial \theta \bar{\partial} \theta \) : or powers of the energy-momentum tensor \( T_{2n} \).

Moreover we briefly discuss the integrability of these theories. We just focus on the conserved currents; the exact study needs the consistency of \( S \) matrix.

The perturbed action is obtained from the critical action \( S^* \) by the addition of the operator \( \Phi(z, \bar{z}) \),

\[
S = S^* + \alpha \int d^2z \Phi(z, \bar{z}).
\]  

(6.1)

For clarification, let us investigate the conservation of energy-momentum tensor in a typical theory perturbed with an arbitrary scaling field \( \Phi \) with conformal dimension of \( h \). The correlation functions of a particular operator \( J(z, \bar{z}) \) are given by the following equation:

\[
\langle J(z, \bar{z}) \cdots \rangle = \langle J(z, \bar{z}) \cdots \rangle_{s^*} + \alpha \int d^2z \langle J(z, \bar{z}) \Phi(z_1, \bar{z}_1) \cdots \rangle_{s^*} + \mathcal{O}(\alpha^2).
\]  

(6.2)

Generally, the \( \bar{z} \) dependence of the finite case of this integral emerges from the singularities in the neighborhood of \( z_1 \). So, in this limit i.e. \( z \to z_1 \), we can use the following OPE in the above integral

\[
J(z, \bar{z})\Phi(z_1, \bar{z}_1) = \sum_i a_i \frac{\phi_i(z_1, \bar{z}_1)}{|z - z_1|^{\Delta + \Delta_j - \Delta_i}},
\]  

(6.3)
where \( \Delta = 2h \) and \( \Delta_J \) and \( \Delta_i \) are the scaling dimension of the fields \( J \) and \( \phi_i \) respectively. These singularities are integrable for \( \Delta + \Delta_J - \Delta_i < 2 \). Since in a unitary theory all dimensions are positive, only a finite number of operators \( \phi_i \) contribute in the correlation expansion up to the first order in \( \alpha \).

In particular for the energy-momentum tensor, the OPE is:

\[
T(z)\Phi(z_1, \bar{z}_1) = \frac{h}{(z-z_1)^2} \Phi(z_1, \bar{z}_1) + \frac{1}{z-z_1} \partial_1 \Phi(z_1, \bar{z}_1),
\]

(6.4)

where \( \partial_1 \) denotes \( \partial_{z_1} \).

Using the equation (6.2) and regularizing the second term by cutting out a small section \( |z-z_1|^2 \leq a^2 \), where \( a \) is a microscopic length scale, one can immediately read off the conservation law for the energy-momentum tensor as:

\[
\bar{\partial} T + \partial U = 0,
\]

(6.5)

Let us now consider particular examples of \( \Phi(z, \bar{z}) \) for the \( c = -2 \) model.

### 6.1 Perturbation with \( \theta \bar{\theta} \):

The action of the off-critical, massive theory can be obtained by perturbation of \( c = -2 \) theory \( S^* \), with the logarithmic partner of the identity \( \theta \bar{\theta} \):

\[
S = S^* + \frac{m^2}{4} \int : \theta \bar{\theta} : \]

(6.6)

The correlation functions can be obtained from the equation of motion

\[
\langle \theta(z) \bar{\theta}(w) \rangle = K_0(m|z-w|), \quad \langle \theta(z) \theta(w) \rangle = \langle \bar{\theta}(z) \bar{\theta}(w) \rangle = 0 \quad (6.7)
\]

\[
\langle \bar{\partial} \theta(z) \bar{\partial} \bar{\theta}(0) \rangle = -\frac{m^2}{4} \frac{2K_0''(m|z|) - K_0(m|z|)}{z} \quad (6.8)
\]

where the function \( K_0 \), is the modified Bessel function. The massless limit of these correlations is exactly what was obtained before in (3.3) and (3.4) which contain the zero modes. To investigate the integrability of a theory, we need to look for conserved currents which can be obtained from the OPE of the perturbing field (in this case : \( \theta \bar{\theta} : \)) and the quantity under consideration.

Using the equations of motion, we can show that a class of operators; \( \tilde{T}_{2n}(z) = 2^n : \bar{\partial}^n \theta \bar{\partial}^n \theta : (z) \) for \( n=1,2,...,\infty \) satisfy the continuity equation. To show this, we look at the OPE of \( \tilde{T}_{2n} \) with the perturbing field : \( \theta \bar{\theta} \):

\[
\tilde{T}_{2n}(z, \bar{z}) : \theta \bar{\theta} : (w, \bar{w}) = -\frac{2^{n-1}(n-1)!^2}{(z-w)^{2n}} + \cdots + \frac{1}{z-w} \tilde{T}_{2(n-1)}(z, \bar{z}) + \cdots ,
\]

(6.9)

where the dots denote less singular terms. If we apply the above method, the following continuity equation can be obtained for \( \tilde{T}_{2n} \):

\[
\bar{\partial} \tilde{T}_{2n}(z, \bar{z}) = \frac{m^2}{4} \tilde{T}_{2(n-1)}(z, \bar{z}) \quad n = 1, 2, \ldots, \infty.
\]

(6.10)
Since there are an infinite number of conserved quantities, this theory is integrable. In this theory there are many other conserved quantities which are more familiar in the context of Virasoro algebra such as
\[ T_{2n} = L_n - 2I \]
for \( n = 1, 2, 3, \ldots, \infty \). With the exception of \( T_2 \), the first three conserved quantities have the following explicit shapes in terms of fundamental fields:

\[
T_4 = \partial^3 \bar{\theta} \partial \theta + \partial \theta \partial^3 \bar{\theta} \tag{6.11}
\]

\[
T_6 = \frac{1}{4} \left( \partial^5 \bar{\theta} \partial \theta + \partial \theta \partial^5 \bar{\theta} \right) + \frac{1}{16} T_6 \tag{6.12}
\]

\[
T_8 = \frac{1}{4} \left( \partial^5 \bar{\theta} \partial^3 \theta + \partial^3 \theta \partial^5 \bar{\theta} \right) + \frac{1}{24} \left( \partial^7 \theta \partial \theta + \partial \theta \partial^7 \bar{\theta} \right) \tag{6.13}
\]

Of course one can simply check that all of the fields \( \partial^n \theta \partial^m \bar{\theta} \) are conserved. The most important examples are the \( W^i \) fields which are related to the \( W \) algebras investigated before.

### 6.2 Perturbation with \( \tilde{T}_{2n} \)

Perturbing the action of \( c = -2 \) theory with the generalized energy-momentum tensor results in:

\[
S = S^* + \frac{\alpha}{\pi} \int \tilde{T}_{2n} \quad n = 1, 2, \ldots, \infty. \tag{6.14}
\]

This theory is integrable only for \( n = 1 \), since infinite number of \( \tilde{T}_{2n} \)'s are conserved. To investigate the integrability of the theory, according to the mentioned prescription in the last subsection, we need the OPE of \( \tilde{T}_{2n} \) with \( T \equiv T_2 \)

\[
\tilde{T}_{2n}(z, \bar{z}) T(w, \bar{w}) = \frac{(n!)^2}{(z-w)^{2n+2}} + \cdots + \frac{1}{z-w} \partial \tilde{T}_{2n}(z, \bar{z}) + \cdots \tag{6.15}
\]

The continuity equation becomes

\[
\partial \tilde{T}_{2n}(z, \bar{z}) = \alpha \partial \tilde{T}_{2n}(z, \bar{z}) \quad n = 1, 2, \ldots, \infty. \tag{6.16}
\]

This result was predictable, since any conformal field theory perturbed with energy-momentum tensor has an infinite number of conserved quantities.

Application of the equation of motion for \( n = 1 \) yields the correlation functions as:

\[
\langle \theta(z, \bar{z}) \theta(w, \bar{w}) \rangle = \langle \bar{\theta}(z, \bar{z}) \bar{\theta}(w, \bar{w}) \rangle = 0, \]

\[
\langle \theta(z, \bar{z}) \bar{\theta}(w, \bar{w}) \rangle = -\log(\bar{z} - \bar{w}). \tag{6.17}
\]

Note that these correlations depend only on the anti-holomorphic terms.

### 6.3 Perturbation with \( \partial \bar{\theta} \partial \theta \) :

The action of \( c = -2 \) theory is constructed using a primary field \( \varphi^{\alpha \bar{\alpha}} \) with \( \alpha = \bar{\alpha} \) with conformal dimension of (1,1) which is a member of the \( \mathcal{R} \) representation. It is interesting to add the other two \( \varphi^{\alpha \bar{\alpha}} \)'s with \( \alpha = \bar{\alpha} \) to the action and see what happens. The action is
\[ S = S^* + \frac{\alpha}{\pi} \int : \partial \theta \bar{\partial} \theta : \]  

(6.18)

The OPE of perturbing field with the energy-momentum tensor is:

\[ T(z) : \partial \theta \bar{\partial} \theta : (w, \bar{w}) = \frac{1}{(z - w)^2} : \partial \theta \bar{\partial} \theta : (z, \bar{z}) + \cdots. \]  

(6.19)

To obtain this result, we utilized the equation of motion i.e. \( \partial \bar{\partial} \theta(z, \bar{z}) = 0 \). Moreover, the continuity equation is \( \bar{\partial}T(z) = 0 \). This means that the trace of the energy-momentum tensor is zero i.e. the perturbed theory is still conformal. One can also obtain this result explicitly by calculating the elements of the energy-momentum tensor as bellow:

\[ T_{zz} = 0, \quad \bar{T}_{\bar{z}z} = 0, \quad \bar{T}(\bar{z}) = 2 : \partial \theta \bar{\partial} \theta : (\bar{z}). \]  

(6.20)

The correlation functions in the presence of the perturbing term do not change and the theory remains conformal. All of these results hold in a theory perturbed by \( : \partial \bar{\partial} \theta \): as well.

7 \ Zamolodchikov’s c-Theorem and \( c = -2 \) Theory

Zamolodchikov’s c-theorem concerns the behavior of a two dimensional conformal field theory under the renormalization group (RG) flow. This theorem is correct just for unitary, renormalizable quantum field theories, and implies that there is a function \( c(g) \) of the coupling constants \( g \) which decreases monotonically under the RG flow [13]. This function has constant values only at fixed points, the fixed points being conformally invariant, and at these points Zamalodchikov’s function takes on the value of the central charge of the corresponding CFT.

Suppose we have an integrable theory with an infinite number of conserved quantities:

\[ \bar{\partial}T_{2n} + \partial U_{2(n-1)} = 0, \]  

(7.1)

for any conserved current \( T_{2n} \) there is a function \( c_{2n}(g) \) which decreases under the RG flow and takes it’s constant value at the fixed point, i.e. the generalized central charge. Consider the following correlation functions:

\[ \langle T_{2n}(z, \bar{z})T_{2n}(0, 0) \rangle = F(|z\bar{z}|) \frac{1}{z^{4n}}, \]  

\[ \langle T_{2n}(z, \bar{z})U_{2(n-1)}(0, 0) \rangle = G(|z\bar{z}|) \frac{1}{z^{4n-1}}, \]  

(7.2)

\[ \langle U_{2(n-1)}(z, \bar{z})U_{2(n-1)}(0, 0) \rangle = H(|z\bar{z}|) \frac{1}{z^{2(2n-1)}}. \]

Applying the conservation law (7.1), one obtains:

\[ |x| \frac{dc_{2n}}{d|x|} = -8(2n - 1)(4n - 1)H \]  

(7.3)
where

\[ c_{2n}(g) = 2 \{ F - 2(2n-1)G - (4n-1)H \}. \]  

(7.4)

Since for a unitary field theory we have \( H \geq 0 \), one concludes that \( \frac{dc_{2n}}{d|x|} \leq 0 \).

By integrating eq. (7.3) we get:

\[ \Delta c_{2n} = \frac{1}{\pi} \frac{1}{4(2n-1)(4n-1)} \int d^2z z^{4n-3} \bar{z} \langle U_2(n-1)(z, \bar{z})U_2(n-1)(0, 0) \rangle, \]  

(7.5)

where \( c \equiv c_2 \) is the central charge of the corresponding CFT. This relation is true for any theory irrespective of being unitary or not.

Since \( c = -2 \) theory is nonunitary, Zamolodchikov’s \( c \)-theorem could not be applied for it. Rather, a function does exist but it may not be monotonically decreasing. But as mentioned at the end of the last section, since eq. (7.5) is usable for both unitary and nonunitary theories, we can apply it for \( c = -2 \) theory. In this way, we utilize the perturbed massive theory (6.6) as introduced before in section (6). Using both eq. (7.5) for \( n = 1 \) and correlation functions (6.7), the central charge of the theory can be exactly obtained equal to \(-2\), whereas according to (4.6) the central charge is incorrectly zero. This disagreement emerges from the important role of the zero modes in \( c = -2 \) theory, because the repeat of the calculation of two point function (4.6) containing the zero modes yields:

\[ \langle T(z)T(w)\xi\xi \rangle = -\frac{1}{(z-w)^4}, \]  

(7.6)

giving \( c = -2 \).

Now, if we repeat the procedure for another conserved quantity \( \bar{T}_{2n} \), we obtain a generalized central charge \( c_{2n} \) for each \( n \) as mentioned in [45]:

\[ c_{2n} = -2^{2n-1}\{(2n-1)!\}^2. \]  

(7.7)

In this case, one should again take zero modes into account too. The correlation function is explicitly obtained using the Wick theorem:

\[ \langle \bar{T}_{2n}(z)\bar{T}_{2n}(w)\xi\xi \rangle = -\frac{2^{2n-2}\{(2n-1)!\}^2}{(z-w)^{4n}}. \]  

(7.8)

that as compared with \( \langle \bar{T}_{2n}(z)\bar{T}_{2n}(0)\xi\xi \rangle = \frac{c_{2n}}{2z^4} \), concludes the previous result (7.7).

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A Some OPEs and Correlations

In this appendix, we calculate some of OPEs mentioned in sections 3 and 4. These explicit forms may be used in calculating RG equations. Moreover, using these expressions one can simply read off the two point functions of the fields or their derivatives.

The OPE of field $\psi$ with itself is rigorously obtained using Wick theorem as follows:

$$\psi(z)\psi(w) = \frac{\partial : \theta \bar{\theta} : (w)}{2|z - w|^4}(1 + \log |z - w|^2) + \frac{I}{8|z - w|^4}(2(1 + \log |z - w|^2) + \log^2 |z - w|^2)$$

Moreover, the OPE of $\psi$ with the logarithmic partner of identity i.e. $:\theta \bar{\theta} :$, turns out to be:

$$: \theta \bar{\theta} : (z) \psi(w) = \frac{\partial : \theta \bar{\theta} : (w)}{2|z - w|^2} - \frac{\partial : \theta \bar{\theta} : (w)}{4(z - w)} \log |z - w|^2 - \frac{\partial : \theta \bar{\theta} : (w)}{4(\bar{z} - \bar{w})} \log |z - w|^2$$

$$+ \log |z - w|^2 \psi(w) - \frac{1}{4} \log^2 |z - w|^2 \phi(w) + \cdots.$$ (A.2)

Since all fields have zero expectations except $:\theta \bar{\theta} :$, correlation of $\psi$ with itself and $:\theta \bar{\theta} :$ can be simply obtained:

$$\langle \psi(z)\psi(w) \rangle = \frac{1}{2|z - w|^4} \{1 + \log |z - w|^2\},$$

$$\langle \psi(z) : \theta \bar{\theta} : (w) \rangle = \frac{1}{2|z - w|^2}. $$ (A.3)

Which are consistent with the equations 3.30 and 3.32.

We have also calculated the OPEs of the logarithmic partner of energy-momentum tensor $t$ with the fields $:\theta \bar{\theta} :$, $\phi$ and $\psi$.

The first OPE is:

$$: \theta \bar{\theta} : (z) : t(w) = \frac{-\partial : \theta \bar{\theta} : (w)}{2(z - w)^2} - \frac{1}{2(z - w)} \log |z - w|^2 \partial : \theta \bar{\theta} : (w)$$

$$- \frac{1}{4} \log^2 |z - w|^2 T(w) + \log |z - w|^2 t(w) + \cdots.$$ (A.4)
According to this OPE, the equation (4.9) can be clearly obtained. The OPE of $t$ with $\phi$ has the following form:

$$
t(z)\phi(w) = \frac{\partial : \theta \bar{\theta} : (w)}{2(z-w)^2(z-w)} - \frac{\partial(\theta \bar{\theta} + \bar{\theta} \theta)}{2|z-w|^2}
$$

$$+ \frac{2\psi(w) - \phi(w)}{2(z-w)^2} - \frac{\partial\phi(w) - 2\partial\psi(w) + \bar{\partial}t(w)}{2(z-w)} - \frac{7}{8} \frac{\partial T(w)}{z-w} + \cdots. \quad (A.5)
$$

Note that the corresponding correlation vanishes.

Finally, the most singular terms appearing in the OPE of $t$ with $\psi$ are:

$$
t(z)\psi(w) = \frac{1}{4(z-w)^3(z-w)} \left\{ I + \log |z-w|^2 I - 2 : \theta \bar{\theta} : (w) \right\}
$$

$$+ \frac{1}{4(z-w)^3} \left\{ 1 + \log |z-w|^2 \right\} \bar{\partial} : \theta \bar{\theta} : (w)
$$

$$- \frac{1}{4(z-w)^2(z-w)} \left\{ 1 + \log |z-w|^2 \right\} \partial : \theta \bar{\theta} : (w) + \cdots. \quad (A.6)
$$

Which leads to the two point function

$$\langle t(z)\psi(w) \rangle = \frac{1}{2(z-w)^3(z-w)}. \quad (A.7)
$$

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