Yang–Mills Instantons vs. type IIB D-instantons

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Abstract

We report on the computation of the one-instanton contribution to the 16-point Green function of fermionic composite operators in $\mathcal{N}=4$ Super YM theory. The remarkable agreement, initially found in the case of an $SU(2)$ gauge group, with the one D-instanton contribution to the corresponding type IIB superstring amplitude on $AdS_5 \times S^5$ is reviewed here. The recent extension by other authors of this result to any $SU(N)$ gauge group and to multi-instantons in the large $N$ limit is briefly discussed. We also argue that for the AdS/SCFT correspondence under consideration to work for any $N \geq 2$, at least for some protected interactions, string effects should require a truncation of the Kaluza–Klein spectrum for finite $S^5$ radius much in the same way as worldsheet unitarity restricts the allowed isospins in $SU(2)$ WZW models. Finally, we briefly comment on the logarithmic behavior of some four-point Green functions of scalar composite operators.

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1 Introduction

This talk will report on the correspondence between instanton effects in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in $D = 4$ and D-instanton effects in type IIB superstring on anti de Sitter space. Most of the discussion here will rely on the work [1], for ease of presentation we will drop many numerical coefficients that have been carefully taken into account in that paper.

Many of the recent non-perturbative insights in string as well as Yang–Mills theories are related to the properties of D-branes [2]. D-branes are BPS $p$-brane solutions of the type II (A or B) supergravity equations of motion that couple to antisymmetric tensors belonging to the Ramond-Ramond (R–R) sector. The massless R–R spectrum of the type IIB superstring consists of a scalar, a two-form tensor and a self-dual four-form. These fields and their magnetic duals naturally couple to $p$-branes with odd $p$ ($p = -1, 1, 3, 5, 7, 9$). In the rest of the talk we will be particularly interested in the type IIB D3- and D(-1)-branes. The latter are localized in space and (Euclidean) time and deserve the name of D-instantons [3]. A microscopic description of D$p$-branes is available in terms of $p$-dimensional hypersurfaces on which open-strings can end [2]. The $9-p$ transverse superstring coordinates satisfy Dirichlet boundary conditions whence the name. Since the lowest lying modes of the open strings comprise a vector supermultiplet, the world-volume dynamics of a $p$-brane is governed by the dimensional reduction of $\mathcal{N}=1$ SYM theory from $D = 10$ to $D = p + 1$. For a collection of $N$ coincident $p$-branes the world-volume gauge symmetry is enhanced to $U(N)$. The $U(1)$ factor describes the center-of-mass motion and decouples from the rest [4].

According to the proposal made by Maldacena in [5] $SU(N) \mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory in the large-$N$ limit is equivalent to type IIB supergravity on $AdS_5 \times S^5$. The SYM theory is located on the four-dimensional boundary of the five-dimensional anti de Sitter space and is interpreted as the world-volume theory for $N$ coincident D3-branes [2]. The classical D3-brane soliton solution with $N$ units of R–R 5-form charge is endowed with a constant dilaton, that determines the string coupling $g_s = e^{\phi_0}$, and a metric

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} dx \cdot dx + \left(1 + \frac{L^4}{r^4}\right)^{1/2} dy \cdot dy, \quad (1)$$

where $(x^\mu, y^i)$ ($\mu = 0, \ldots, 3$ and $i = 1, \ldots, 6$) are Cartesian coordinates (with $r^2 = y \cdot y$ the transverse distance) and $L^4 = 4\pi g_s N\alpha'^2$. In the near-horizon limit $r \to 0$ or, better, for $L^4 \gg \alpha'^2$ (1) tends to the metric on $AdS_5 \times S^5$ described by

$$ds^2 = \frac{L^2}{\rho^2}(dx \cdot dx + d\rho^2) + d\omega_5^2, \quad (2)$$

where $\rho^2 = L^4/r^4$ and $d\omega_5^2$ is the spherically-symmetric constant curvature metric on $S^5$.

The $SU(4) \times SO(4,2)$ isometry of the type IIB background (2) is the bosonic part of the exact superconformal symmetry of the $\mathcal{N} = 4$ SYM theory. The (conjectured) exact $SL(2,\mathbb{Z})$ “S-duality” of the type IIB theory corresponds to the (conjectured) exact electromagnetic duality [8] of four-dimensional $\mathcal{N}=4$ SYM theory [7]. The former is
connected to the existence of an infinite set of stable dyonic BPS string solitons [8], the latter to the existence an infinite set of stable dyonic BPS states [9]. The abelian Coulomb phase of the SYM theory corresponds to separating the D3-branes or, equivalently, to giving non-vanishing vacuum expectations values (vev’s) to the scalar fields in the Cartan subalgebra. The resulting theory is free and the superconformal invariance is broken. The phase in which all scalar fields have vanishing vev’s describes a highly nontrivial superconformal field theory [10, 5, 11, 12]. Superconformal invariance is protected against quantum corrections by the exact vanishing of the renormalization group β-function [13].

The complexified coupling constant of the \( \mathcal{N} = 4 \) SYM theory is related to the one of the type IIB superstring on \( AdS_5 \times S^5 \) by

\[
g_s = g_{YM}^2 / 4\pi, \quad \chi_0 = \theta_{YM} / 2\pi,
\]

where \( \chi_0 \) is the constant R–R axionic background, \( g_{YM} \) is the SYM coupling and \( \theta_{YM} \) is the SYM vacuum angle.

Long time before any D-brane argument was put forward, the maximally supersymmetric compactification of the type IIB superstring on the \( AdS_5 \times S^5 \) background was considered by Schwarz [14] and the spectrum of excitations was studied in great detail [15]. It is thus natural to make Maldacena’s proposal more precise and suggest that the boundary values of the fields of the bulk type IIB superstring are sources that couple to gauge-invariant composite operators of the four-dimensional boundary \( \mathcal{N} = 4 \) superconformal field theory (SCFT) [11, 12, 14, 17]. The lowest lying states, assembled into the \( \mathcal{N}=8 \) gauged supergravity multiplet, couple to the supermultiplet of SYM currents that are “bilinear” in the elementary SYM fields. Higher \( \ell \) spherical harmonics, \( i.e. \) Kaluza–Klein (KK) descendants of the supergravity fields, couple to other chiral primary fields, that correspond to “order \( \ell \)” gauge-invariant polynomials in the elementary SYM fields. For finite \( N \), \( i.e. \) for finite radius, one expects a truncation to \( \ell \leq N \) that should result by taking into account full-fledged stringy geometry as in any string compactifications. For instance, in the compactification on the \( SU(2) \) group manifold unitarity of the world-sheet theory restricts the allowed isospins of the primary fields to the range \( j = 0, \ldots k/2 \). The level \( k \) of the current algebra is related to the radius of \( S^3 \). Similarly the number \( N \) of D3-branes is related to the radius of \( S^5 \). In the case at hand, \( AdS_5 \times S^5 \), despite some interesting progress [18], a useful world-sheet description is still lacking. Nevertheless the truncation of the KK spectrum seems to be a robust result that matches with the naive expectation on the SYM side. Moreover, stable string excitations that couple to non-chiral primary fields are needed for the closure of the operator algebra [19].

In this picture, the effective action of the type IIB superstring is the generating functional of connected gauge-invariant correlation functions in the SYM theory. The semiclassical supergravity approximation is valid at length scales much larger than the string scale or, equivalently, in the limit \( g_s N >> 1 \). From the SYM viewpoint, this coincides with a subcase of the large-\( N \) limit introduced by ’t Hooft [20] in which the coupling \( \hat{g}^2 = g_{YM}^2 N \) is fixed at a large value. By taking into account string corrections to the low energy supergravity approximation, the AdS/SCFT correspondence under consideration is expected to work for finite values of \( N \) as well. Indeed a surprising agreement between SYM instanton computations for \( N = 2 \) and some D-instanton corrections to the type
IIB superstring effective action has been shown in [1].

The explicit connection between the bulk theory and the boundary theory can be expressed symbolically as [12, 11, 21]

\[
\int [D\Phi] \exp(-S_{IIB}[\Phi]) = \int [DA] \exp(-S_{YM}[A] + O[A]J),
\]

where \( S_{IIB} \) is the effective action of the type IIB superstring which is evaluated in terms of the ‘massless’ supergravity fields, their Kaluza–Klein descendents, and possibly the string excitations. We have collectively denoted the bulk fields by \( \Phi(z; \omega) \), where \( \omega \) are the coordinates on \( S^5 \) and \( z^M \equiv (x^\mu, \rho) \) \((M = 0, 1, 2, 3, 5 \text{ and } \mu = 0, 1, 2, 3)\) the coordinates on \( AdS_5 \) \((\rho \equiv z_5 \text{ is the coordinate transverse to the boundary})\). The functional integral depends on the boundary values, \( J(x) \), of the bulk fields. Elementary fields of the \( \mathcal{N} = 4 \) SYM theory on the boundary are denoted by \( A \) and \( O(A) \) are gauge-invariant composite operators to which \( J(x) \)'s couple. The conformal dimension \( \Delta \) of an operator \( O \) is related to the AdS ‘mass’ \( m \) of the corresponding bulk field \( \Phi \). For scalars one has \((mL)^2 = \Delta(\Delta - 4)\). Only the positive branch of \( \Delta_\pm = 2 \pm \sqrt{4 + (mL)^2} \) is relevant for the lowest-‘mass’ supergravity multiplet.

The correct recipe for computing correlation functions amounts to computing the truncated Green functions in the bulk and attaching bulk-to-boundary propagators to the external legs [11, 12, 22]. The precise forms of the latter depend on the spin and ‘mass’ of the field. For instance, the bulk-to-boundary Green function for a bulk scalar field \( \Phi_m \) associated to a dimension \( \Delta \) operator \( O_\Delta \) on the boundary reads

\[
K_\Delta(x^\mu, \rho; x'^\mu, 0) = \frac{\rho^\Delta}{(\rho^2 + (x - x')^2)^\Delta},
\]

up to an overall normalization constant. For simplicity we have suppressed the \( \omega \) dependence so that the expression (3) is appropriate for an ‘S-wave’ process in which there are no excitations in the directions of the five-sphere, \( S^5 \). In terms of \( K_\Delta \) the bulk field

\[
\Phi_m(z; J) = \int d^4x' K_\Delta(x, \rho; x', 0)J_\Delta(x')
\]

satisfies the boundary condition \( \Phi_m(x, \rho; J) \approx \rho^{4-\Delta} J_\Delta(x) \) as \( \rho \to 0 \), since \( \rho^{\Delta-4} K_\Delta \) reduces to a \( \delta \)-function on the boundary.

For our considerations it will prove crucial that the expression (3) in the case \( \Delta = 4 \) has exactly the same form as the contribution of a Yang–Mills (YM) instanton to \( \text{Tr}(F_{\mu\nu}^-)^2 \) (where \( F_{\mu\nu}^- \) is the non-abelian self-dual field strength) when the fifth coordinate \( \rho \) is identified with the instanton scale. At the same time, it has been shown in [1] that in this case (3) has precisely the same form as the profile of a five-dimensional D-instanton centered at the point \( z^M \) in \( AdS_5 \) and evaluated at the boundary point \((x^\mu, 0)\). This is a key observation in identifying D-instanton effects of the bulk theory with YM instanton effects in the boundary theory. It is related to the fact that the moduli space of a YM instanton contains an \( AdS_5 \) factor.

Two-point and three-point correlation functions of superconformal currents are not renormalized from their free-field values due to the \( \mathcal{N} = 4 \) superconformal invariance
so they do not get interesting interaction corrections \[22, 23, 24, 25\]. However, higher-point correlation functions do receive nontrivial interaction corrections. In this talk we will be concerned with calculations of processes in which the one-instanton contributions can be evaluated exactly to lowest order in perturbation theory. The contributions of a one-instanton configuration to some correlation functions in the $SU(2)$ SYM theory have been evaluated in \([1]\). The result for the 16-point function, $\langle \hat{\Lambda}(x_1) \ldots \hat{\Lambda}(x_{16}) \rangle$, of gauge-invariant fermionic composite operators $\hat{\Lambda} = F^- - \lambda$, where $F^-$ is the self-dual field strength and $\lambda$ is the spin-1/2 gaugino, has been later generalized to any $N$ \([26]\). More importantly, the $K$-instanton contributions have been evaluated in the large $N$ limit by means of a saddle-point approximation of the integral over the relevant moduli space \([27]\).

The common feature of all the correlation functions considered in \([1, 26, 27]\), as well as many others that are related to them by supersymmetry, is that they provide precisely the sixteen ‘geometric’ fermionic zero modes that are needed to give a nonzero result in the instanton background.

In \([1]\) it has been shown that the form of the leading charge-one D-instanton contribution to the $\Lambda^{16}$ amplitude in the type IIB superstring matches with the expression for the sixteen-$\hat{\Lambda}$ correlator calculated in the $\mathcal{N} = 4$ $SU(2)$ SYM theory. Given this agreement, that have been further confirmed in \([26]\) and definitively established in \([27]\), supersymmetry guarantees that all of the related correlators must also agree with their AdS counterparts. Although a detailed comparison for all such correlation functions has not been completed, the correspondence has been outlined in \([1, 27]\).

In order to further clarify the correspondence between YM instantons and D-instantons, the classical D-instanton solution of the type IIB supergravity equations in the $AdS_5 \times S^5$ background has been explicitly determined in \([1]\). The solution is particularly simple to establish by using the fact that $AdS_5 \times S^5$ is conformally flat, which relates the solution to the flat-space solution of \([28]\). Once the D-instanton solution in $AdS_5 \times S^5$ is known one can directly compute the semi-classical approximation to the $\Lambda^{16}$ correlation function and check the agreement with the previous computations.

## 2 $\mathcal{N}=4$ SYM in $D=4$

The field content of the maximally supersymmetric ($\mathcal{N}=4$) four-dimensional Yang–Mills \([7, 13, 29]\) theory is unique apart from the choice of the gauge group. As mentioned above, the theory is classically invariant under superconformal transformations as well as under global $SU(4)$ transformations, which form the R-symmetry group of internal automorphisms of the $\mathcal{N}=4$ superconformal algebra. There is also an external $U(1)_B$ automorphism under which the Poincaré and conformal left supercharges transform with opposite charge \([30]\). The field content consists of six real scalars, four Weyl spinors and one vector which are all in the adjoint representation of the gauge group. More precisely, the scalars $\varphi^{AB} = -\varphi^{BA}$ (with $\mathcal{T}_{AB} = \frac{1}{2}\varepsilon_{ABCD}\varphi^{CD}$) are in the $6$ of the $SU(4)$ R-symmetry group\([4]\). In the following we will also use the notation $\varphi^i = \frac{1}{2}\mathcal{T}_{AB}\varphi^{AB}$ ($i = 1, \ldots, 6$) where $\varphi^i = \varphi^{i*}$ and $\mathcal{T}_{AB}$ are Clebsch–Gordan coefficients that couple two $4$’s to a $6$ (these are six-dimensional generalizations of the four-dimensional $\sigma^{\mu\nu\alpha\beta}$ matrices). The spinors

\[1\] The superscripts $A, B = 1, \ldots, 4$ label $4$’s of $SU(4)$ while subscripts label the $4^*$’s.
\( \lambda^A, \chi_B \) transform as 4 and 4*, respectively, and the vector \( A_\mu \) is a singlet of \( SU(4) \). All couplings are proportional to \( g_{YM} \) and to \( f_{abc} \).

The extended supersymmetry transformations with parameters \( \eta_\alpha^A \) and \( \overline{\eta}^i_A \) are given in component-field notation by
\[
\begin{align*}
\delta \varphi^i & = \frac{1}{2} ( \overline{\eta}^i_A \lambda^A_\alpha \eta_\alpha^B + t^{iAB} \eta_\alpha^A \overline{\eta}^i_B ) \\
\delta \lambda^A_\alpha & = -\frac{1}{2} F^-_{\mu \nu} \sigma^{\mu \nu} \eta_\beta^B + 2i ( \partial_\alpha \varphi^i ) t^i_A \eta^\alpha_B + 2 g_{YM} [ \varphi^i, \varphi^j ] t^A_i A^j_B \eta_\alpha^B \\
\delta A_\mu & = -i \lambda^A_\alpha \sigma^\mu {\alpha \overline{\alpha}} \overline{\eta}^i_A - i \eta^A_\alpha \sigma^\mu {\alpha \overline{\alpha}} \lambda^i_A ,
\end{align*}
\]
where as usual \( t_{i_1 \ldots i_p} \) denote \( p \)-fold antisymmetric products of \( t_i \) (and \( \overline{t}_i \)) matrices with strength-one. The classical theory is superconformally invariant \[29\] and this property is believed to be preserved at the quantum level thanks to the exact vanishing of the \( \beta \)-function \[13\]. The Noether currents associated with the superconformal transformations, together with those corresponding to chiral \( SU(4) \) transformations, constitute a supermultiplet that is bilinear in the elementary fields in the abelian case. The current supermultiplet includes the energy-momentum tensor, \( T_{\mu \nu} \), the supersymmetry currents, \( \Sigma^{\mu , A} \), and the \( SU(4) \) R-symmetry currents, \( J^{\mu}{}^A_B \). The remaining components are obtained by on-shell supersymmetry transformations. They consist of three scalar components (\( C, \mathcal{E}^{(AB)}, Q^{ij} \)), two fermionic spin-1/2 components (\( \overline{\lambda}^{\alpha} C_{AB} \) and \( \Lambda^A \)) and one antisymmetric tensor (\( B^{[\mu AB]} \)). In the non-abelian case, aside from the obvious trace over the colour indices that enters, for instance, the definition of
\[
Q^{ij} = \text{Tr} \left( \varphi^i \varphi^j - \frac{1}{6} \delta^{ij} \varphi^k \varphi^k \right) ,
\]
there are additional terms that are not always exhausted, as in the definition
\[
\hat{\Lambda}^A_\alpha = \frac{1}{2} \text{Tr} \left( F^-_{\mu \nu} \sigma^{\mu \nu} \lambda^\beta_B \right) ,
\]
by the covariantization of all the derivatives. In particular, the complete non-abelian expression for \( \mathcal{E}^{(AB)} \) \[11\],
\[
\mathcal{E}^{(AB)} = \text{Tr} \left( \lambda^{\alpha A} \lambda_\alpha^B \right) + g_{YM} t^{(AB)}_{[ijk]} \text{Tr} \left( \varphi^i \varphi^j \varphi^k \right) ,
\]
has played a crucial rôle in showing a one-loop non-renormalization theorem for the three point functions of descendants of the chiral primary fields \( Q \) \[31\].

It is often useful to decompose the \( \mathcal{N}=4 \) multiplet in terms of either \( \mathcal{N}=1 \) or \( \mathcal{N}=2 \) multiplets. We will need the \( \mathcal{N}=2 \) description in the following. The global symmetry that is manifest in this description is \( SU(2)_Y \times SU(2)_H \times U(1) \) and the \( \mathcal{N}=4 \) elementary supermultiplet decomposes into a \( \mathcal{N}=2 \) vector multiplet \( \mathcal{V} \) and a hypermultiplet, \( \mathcal{H} \). We will denote the representations of \( SU(2)_Y \times SU(2)_H \times U(1) \) by \( (r_Y, r_H)_q \) with the subscript \( q \) referring to the \( U(1) \) charge and will make use of the following notation for the component fields,
\[
\begin{align*}
\mathcal{V} & \rightarrow \lambda^u \in (2, 1)_{+1}, \quad \varphi \in (1, 1)_{+2}, \quad A_\mu \in (1, 1)_{0} \\
\mathcal{H} & \rightarrow \psi^\dot{u} \in (1, 2)_{-1}, \quad \eta_{u\dot{u}} \in (2, 2)_{0}
\end{align*}
\]
where \( u, \dot{u} = 1, 2 \).

In calculating the correlation functions it is important to understand the systematics of the fermion zero modes associated with the YM instanton. These can either be determined directly from the supersymmetry transformations (7) or by making use of the explicit Yukawa couplings given in [1]. In the following we will review the computation of correlation functions of currents in the \( \mathcal{N}=4 \) SYM theory which receive contributions in the one-instanton semi-classical approximation. As in [1] we will mainly restrict our discussion to the case of an \( SU(2) \) gauge group although we will make some qualitative comments about the \( SU(N) \) case following [26] and briefly mention the steps required for the computation of \( K \)-instanton contributions in the large \( N \) limit following [27]. The special feature shared by the particular observables that we will consider is that they can saturate the 16 supersymmetric plus superconformal gaugino zero modes that arise in the instanton background.

### 3 Type IIB superstring in \( D=10 \)

We now turn to consider the other side of the correspondence, i.e. the type IIB superstring. In \( D=10 \), this theory enjoys chiral \( \mathcal{N}=(2,0) \) supersymmetry associated to the presence of a complex gravitino of positive chirality, that transforms with charge \(+1/2\) under an anomalous \( U(1)_B \) R-symmetry. The massless bosonic spectrum includes the metric, a scalar dilaton, \( \phi \), and a two-form potential in the Neveu-Schwarz – Neveu-Schwarz (NS–NS) sector and a pseudo-scalar, \( \chi \), another antisymmetric tensor and a four-form potential whose field-strength is constrained to be self-dual, in the R–R sector. The classical theory is invariant under \( SL(2,\mathbb{R}) \) transformations that act projectively on the complex scalar field \( \tau = \chi + i e^{-\phi} \). The two antisymmetric tensors form a doublet, while the metric (in Einstein frame!) and the self-dual four-form potential are inert. At the quantum level the classical continuous symmetry is expected to be broken to \( SL(2,\mathbb{Z}) \). Due to the self-duality constraint finding the full action including the four-form potential is far from obvious [32]. For our purposes we only need to display the two leading terms in the derivative expansion of the type IIB effective action that involve the (derivatives of the) metric and some overall function of the complex scalar \( \tau \). In string frame they are given by [3],

\[
(\alpha')^{-4} \int d^{10}X \sqrt{g} \left( e^{-2\phi} R + \kappa (\alpha')^3 f_4(\tau, \bar{\tau}) e^{-\phi/2} \mathcal{R}^4 \right) = L^{-8} \int d^{10}X \sqrt{g} \left( (4\pi N)^2 R + \kappa L^6 (4\pi N)^{1/2} f_4(\tau, \bar{\tau}) \mathcal{R}^4 \right),
\]

where \( \kappa \) is a known constant and the relation \( L^4 = 4\pi g_s \alpha'^2 \) has been substituted in the second line.

The Riemann curvature enters the \( \mathcal{R}^4 \) term in a manner that only involves the Weyl tensor. In [33] it was pointed out that the \( \mathcal{R}^4 \) term vanishes in the \( AdS_5 \times S^5 \) background because it involves a fourth power of the (vanishing) Weyl tensor. The first, second and third functional derivatives of \( \mathcal{R}^4 \) vanish as well, and as a result one finds no corrections to zero, one, two and three-point amplitudes. However, there is a non-zero four-graviton amplitude arising from this term. The boundary values of these gravitons are sources
for various bosonic components of the SYM current supermultiplet. For example, the components of the metric in the AdS$_5$ directions couple to the stress tensor, $\mathcal{T}^{\mu\nu}$, whereas the traceless components polarized in the $S^5$ directions couple to massive Kaluza-Klein states. A linear combination of the “quadrupole moments” of the trace of the metric on $S^5$ and the fluctuations of the R–R four-form potential couples to the scalar composite operator, $Q^{ij}$.

We will be particularly concerned with a sixteen-fermion interaction [3 4], that is related to $\mathcal{R}^4$ by supersymmetry. It appears at the same (subleading) order in the derivative expansion and reads

$$ (\alpha')^{-1} \int d^{10}X \sqrt{g}e^{-\phi/2}f_{16}(\tau, \bar{\tau})\Lambda^{16} + \text{c.c.} , $$

(13)

where $\Lambda$ is a complex spin 1/2 dilatino of negative chirality which transforms under the $U(1)_B$ R-symmetry with charge $+3/2$ and the interaction is antisymmetric in the sixteen spinor indices. Supersymmetry arguments [35, 36] imply that the functions $f_{16}$ and $f_4$ are related by the action of $SL(2, \mathbb{Z})$ modular covariant derivatives. Whereas $f_4$ transforms with modular weight $(0, 0)$, the function $f_{16}$ has weight $(12, -12)$ and therefore transforms with a non-trivial phase under $SL(2, \mathbb{Z})$. This is precisely the phase required to compensate for the anomalous $U(1)_B$ transformation of the 16 $\Lambda$’s so that the full expression (13) is invariant under the $SL(2, \mathbb{Z})$ S-duality group. In the small coupling limit, all of these terms can be expanded in the form

$$ e^{-\phi/2}f_n = a_n \zeta(3)e^{-2\phi} + b_n + \sum_{K=1}^{\infty} \mu_K (Ke^{-\phi})^{n-7/2}e^{2\pi i K \tau} \left( 1 + \sum_{k=1}^{\infty} c_{K,n}^K (Ke^{-\phi})^{-k} \right) . $$

(14)

Whereas the first two terms have the form of string tree-level and one-loop contributions, the last one has the appropriate weight of $e^{2\pi i K \tau}$ to be associated with a charge-$K$ D-instanton effect. Anti-D-instanton contributions have not been displayed. The coefficients $c_{K,n}^K$ as well as the ‘D-instanton partition functions’ $\mu_K$ are explicitly given in [34]. The coefficient of the leading term in the series of perturbative fluctuations around a charge-$K$ D-instanton is independent of which particular interaction term is being discussed and reads

$$ Z_K = \mu_K (Ke^{-\phi})^{-7/2}e^{2\pi i K \tau} . $$

(15)

It should be identified with the contribution of a charge-$K$ D-instanton to the measure in string frame which, up to an overall numerical factor, reads

$$ d\Omega_K^{(s)} = (\alpha')^{-1}d^{10}Xd^{16}\Theta Z_K . $$

(16)

In doing this we are being cavalier about the fact that the full series is not convergent (it is actually an asymptotic approximation to a Bessel function). In the end, consistency of the full theory, and in particular modular invariance, should require considering the complete expression for the instanton contribution.

As observed in [33] the charge-$K$ D-instanton action that appears in the exponent in (15) coincides with the action of a charge-$K$ YM instanton in the boundary theory. This indicates a correspondence between these sources of non-perturbative effects, that is reinforced by the correspondence between other factors. For example, after substituting
\( e^\phi = g_{YM}^2 / 4\pi \) and \( \alpha' = g_{YM}^{-1} L^2 N^{-1/2} \), the measure (16) contains an overall factor of the coupling constant in the form \( g_{YM}^8 \). Indeed, this is exactly the power expected on the basis of the \( AdS_5 \times S^5 / SYM \) correspondence since the one-instanton contribution to the Green functions in the \( \mathcal{N}=4 \) SYM theory also has a factor of \( g_{YM}^8 \) arising from the combination of the bosonic and fermionic zero modes norms. Following [1] we will pursue this issue further in the next section by comparing the leading instanton contributions to type IIB superstring amplitudes with the corresponding \( \mathcal{N} = 4 \) current correlators.

4 Testing the AdS/SCFT correspondence

The initial tests of Maldacena’s conjecture have amounted to the computation of two-point and three-point correlations of primary fields [37]. Up to an overall constant these are fixed by superconformal invariance and the tests amount to checking the consistency of the relative normalizations and the (non)-vanishing of some three-point couplings. Four-point functions, on the contrary, display some interesting dynamical issues that we will briefly address in the last part of the talk.

Momentarily, we review the computation of the one-instanton contribution to the SYM correlation function \( G_{16} = \langle \hat{\Lambda}(x_1) \ldots \hat{\Lambda}(x_{16}) \rangle \) for an \( SU(2) \) gauge group and its comparison with the correspondent D-instanton contribution to the \( \Lambda^{16} \) amplitude in the type IIB superstring. In the case of the bulk type IIB theory, D-instanton effects may be either extracted directly from the exactly known terms in the effective action or deduced from the integration over the semi-classical fluctuations around the \( AdS_5 \times S^5 \) D-instanton solution.

4.1 Super Yang–Mills description

The contribution of the YM one-instanton sector to the correlation function

\[
G_{16}(x_p) = \langle \prod_{p=1}^{16} g_{YM}^2 \hat{\Lambda}^{A_p}_{\alpha_p}(x_p) \rangle_{K=1},
\]

of sixteen fermionic superconformal currents \( \hat{\Lambda}^A_{\alpha} \), defined in (9), is particularly simple to analyze. Since each \( \hat{\Lambda} \) insertion provides a single fermion zero mode it is necessary to consider the product of sixteen \( \hat{\Lambda} \)'s in order to saturate the sixteen grassmannian integrals. To leading order in \( g_{YM} \), \( G_{16} \) does not receive contribution from anti-instantons. The leading term is simply obtained by replacing each \( F_{\mu\nu} \) with the \( SU(2) \) instanton solution

\[
F_{(0)\mu\nu} = -4 \frac{\eta_{\mu\nu}^a \rho_0^2}{g_{YM} (\rho_0^2 + (x - x_0)^2)^2},
\]

where \( \eta_{\mu\nu}^a \) is the 't Hooft symbol [38], and each \( \lambda_{\mu}^A \) with the corresponding zero mode, \( \lambda_{(0)\alpha}^A \). The latter can be deduced from the action of the supersymmetry transformations that are broken in the instanton background. The relevant transformation may be found in the second line of [9] and gives

\[
\lambda_{(0)\alpha}^A = \frac{1}{2} F_{(0)\mu\nu} \sigma^\mu_{\alpha} \frac{1}{\sqrt{\rho_0}} \left( \rho_0 \eta_{\beta}^A + (x - x_0)_{\mu} \sigma_{\beta}^\mu \tilde{\epsilon}^\beta A \right).
\]

9
Up to an overall numerical coefficient, see \cite{1}, the resulting correlation function has thus the form
\[
G_{16}(x_p) = g_{YM}^8 e^{-\frac{8s^2}{g_{YM}^2} + i\theta_{YM}} \int \frac{d^4 x_0}{\rho_0^4} \int d^8 \eta d^8 \xi \prod_{p=1}^{16} \left[ \frac{\rho_0^4}{[\rho_0^2 + (x_p - x_0)^2]^4} \right]^\frac{1}{4} \left[ \rho_0 \eta \mu_p \sigma_{\alpha_p \beta_p} \xi_{\alpha_p \beta_p} \right].
\] (20)

The integration over the fermionic collective coordinates leads to an $SU(4)$ and Lorentz invariant sixteen-index tensor, $t_{16}$. We will leave the expression in the unintegrated form (20) which will be momentarily compared with the corresponding expression obtained in type IIB superstring theory on $AdS_5 \times S^5$. Again, the resemblance of the instanton form factor to the AdS Green function will be of significance here.

The above computation has been recently extended to generic $N$ \cite{26} and to higher instanton number in the large $N$ limit \cite{27}. We can not help briefly reviewing the main steps of these extensions here. Since for a gauge group $SU(N)$ there are $2N \cdot N$ gaugino zero modes in the one-instanton background it might appear from a superficial analysis that the $\hat{\Lambda}$-correlation function should vanish for $N \geq 2$. However it has been argued in \cite{1} and then clearly shown in \cite{26} that only the sixteen ‘geometric’ (8 supersymmetric + 8 superconformal) zero-modes are actually relevant in the general case of $SU(N)$, since all the remaining ones, $\mu_i^A$ are ‘lifted’ by a four-fermion term that is generated in the one-instanton action. The instanton action computed in \cite{26} has the form
\[
S_{\text{inst}} = \frac{8\pi^2 K}{g_{YM}^2} + S_{4F},
\] (21)

where $S_{4F}$ contains a quartic fermionic interaction built up with the $8N - 2$ additional fermionic collective coordinates. The exact form of $S_{4F}$ in the $K = 1$ sector is
\[
S_{4F} = \frac{\pi^2}{16\rho^2 g_{YM}^2} \varepsilon_{ABCD} \left[ \sum_{i=1}^{N-2} \mathcal{P}_i^A \mu_i^B \right] \left[ \sum_{j=1}^{N-2} \mathcal{P}_j^C \mu_j^D \right].
\] (22)

$S_{4F}$ is supersymmetric and lifts all but the 16 ‘geometric’ gaugino zero modes. In computing the functional integral for the Green function it is convenient to perform a Hubbard-Stratonovich transformation of the fermion bilinears and represent $S_{4F}$ as a gaussian integral over auxiliary bosonic ‘collective coordinates’ $\chi_{AB}$. As a result the expression for the $\hat{\Lambda}$-correlator reviewed above for the case $N = 2$ is actually true for all $N$, including an overall $N$-dependent factor that takes into account the scaling with $\alpha'$ in the type IIB description. The auxiliary bosonic ‘collective coordinates’ $\chi_{AB}$ play a crucial role in the evaluation of $K$-instanton contributions. In \cite{27} it is shown that after a saddle-point approximation the $\chi$’s can be viewed as coordinates on $K$ copies of $S^5$ thus completing the identification of the YM-instanton moduli space (for gauge-invariant observables) in the large $N$ limit with the type IIB D-instanton moduli space!
4.2 Type IIB description

The first method for obtaining the correspondent $\Lambda^{16}$-amplitude in the type IIB description is to expand the function $f_{16}$ in (13) to extract the one-instanton term. In order to compare with the SYM sixteen-point correlator we need to consider the situation in which all sixteen fermions propagate to prescribed configurations on the boundary. The Dirac operator acting on spin-1/2 fields in AdS was given in [39] by

$$D\Lambda = e^{\hat{L}}_{M} \gamma^{L} \left( \partial_{M} + \frac{1}{4} \omega^{\hat{M} \hat{N}}_{M} \gamma_{\hat{M} \hat{N}} \right) \Lambda = \left( \rho \gamma_{5} + \rho^\mu \partial_{\mu} - 2 \gamma_{5} \right) \Lambda,$$

where $e_{L}^{M}$ is the vielbein and $\omega^{\hat{M} \hat{N}}_{M}$ the spin connection for the AdS part of the metric (hatted indices refer to the tangent space), and $\gamma^\mu$ are the four-dimensional Dirac matrices. Equation (23) leads to the normalized bulk-to-boundary propagator of the fermionic field $\Lambda$ of AdS ‘mass’ $m = -3/2L$, associated to the composite operator $\hat{\Lambda}$ of dimension $\Delta = \frac{7}{2}$,

$$K^F_{7/2}(\rho_0, x_0; x) = K_4(\rho_0, x_0; x) \frac{1}{\rho_0} \left( \rho_0 \gamma_{5} + (x_0 - x)^\mu \gamma_{\mu} \right),$$

which, suppressing all spinor indices, leads to

$$\Lambda_{J}(x_0, \rho_0) = \int d^4x K^F_{7/2}(\rho_0, x_0; x) J_{\Lambda}(x),$$

where $J_{\Lambda}(x)$ is a left-handed boundary value of $\Lambda$ and acts as the source for the composite operator $\hat{\Lambda}$ in the boundary $\mathcal{N} = 4$ SYM theory. As a result, the classical action for the operator $(\Lambda)^{16}$ in the AdS$_5 \times S^5$ supergravity action is

$$S_{\Lambda}[J] = e^{-2s(\frac{1}{g_s} + i\chi_0)} g_s^{-12} V_{S^5} \int \frac{d^4x_0 d\rho_0}{\rho_0} \left[ \prod_{p=1}^{16} \left( K_4(\rho_0, x_0; x_p) \frac{1}{\rho_0} \left( \rho_0 \gamma_{5} + (x_0 - x_p)^\mu \gamma_{\mu} \right) J_{\Lambda}(x_p) \right) \right],$$

where we have set $e^{\phi} = g_s$ and $\chi = \chi_0$ (since the scalar fields are taken to be constant in the AdS$_5 \times S^5$ background) and $V_{S^5} = \pi^3$ is the $S^5$ volume. The 16-index invariant tensor $t_{16}$ is the same as the one defined after (20). The overall power of the coupling constant comes from the factor of $g_s^2$ in the measure (16) and the factor of $g^{-16}_s$ from the leading term in (14). Using the dictionary (3) and differentiating with respect to the chiral sources this result agrees with the expression (20) obtained in the Yang Mills calculation, including the power of $g_{YM}$.

The remarkable agreement of the one D-instanton contribution to the $\Lambda^{16}$ amplitude in type IIB superstring on AdS$_5 \times S^5$ with the corresponding one-instanton contribution to the sixteen-current correlation function in SYM theory, found in [4] in the case of an $SU(2)$ gauge group, has been beautifully extended to any $SU(N)$ [26] and to $K$-instanton contributions [27]. In the last paper, in particular, it is neatly clarified how the relevant YM-instanton moduli space reduces to a $K$-fold symmetric product of AdS$_5 \times S^5$ in the
large $N$ limit and then collapses to a single copy of $AdS_5 \times S^5$ due to an effective attraction among the various instantons. This is precisely what was expected for the moduli space of type IIB D-instantons on $AdS_5 \times S^5$. A non-renormalization theorem may protect the 16-fermion amplitude and all other type IIB superstring amplitudes related to this by supersymmetry from $\alpha'$ corrections so that they should agree with their SYM counterparts for finite $N$ as well. Supersymmetry arguments and the precise agreement for the 16-fermion correlator in SYM with the correspondent 16-fermion amplitude in type IIB superstring guarantee the agreement for certain four-point correlators whose correspondence with type IIB amplitudes is less straightforward to see [1]. We will not pursue this line of investigation any further if not for the purposes of showing the peculiar logarithmic behavior of some four-point functions of scalar composite operators. Before doing that, we will review to what extent the information about the charge-one D-instanton term can be directly extracted from a semi-classical type IIB supergravity computation around a D-instanton background.

5 The D-instanton solution in $AdS_5 \times S^5$

In flat ten-dimensional Euclidean space the charge-$K$ D-instanton solution is a finite-action Euclidean supersymmetric (BPS–saturated) solution in which the metric is flat in the Einstein frame but the complex scalar $\tau = \chi + ie^{-\phi}$ has a nontrivial profile with a singularity at the position of the D-instanton. The (Euclidean) R–R scalar is related to the dilaton by the BPS condition $\partial_{\Sigma} \chi = \pm i \partial_{\Sigma} e^{-\phi}$. The solution for the dilaton profile, found in [28] and corrected in [1], is

$$e^{\hat{\phi}(10)} = g_s + \frac{3K\alpha'^4}{\pi^4 |X - X_0|^8}$$

and satisfies the ten-dimensional Laplace equation, $\partial^2 e^\phi = 0$, outside an infinitesimal sphere centered on the point $X_0^\Lambda$ (where $X^\Lambda$ is the ten-dimensional coordinate and $X_0^\Lambda$ is the location of the D-instanton). In (27) $g_s$ is the asymptotic value of the string coupling and the second term has a quantized coefficient in virtue of the Dirac–Nepomechie–Teitelboim condition that quantizes the charge of an electrically charged $p$-brane and of its magnetically charged $p'$-brane dual [11]. In solving the equations of motion of the type IIB theory for a D-instanton in Euclidean $AdS_5 \times S^5$, the BPS condition requires

$$e^{-\phi} g^{\Lambda \Sigma} \nabla_\Lambda \nabla_\Sigma e^\phi = 0 \ .$$

In the Einstein frame, the Einstein equations are unaltered by the presence of the D-instanton (because the associated Euclidean stress energy tensor vanishes) so that the metric of $AdS_5 \times S^5$ remains a solution. The D-instanton equation (28) is easy to solve using the conformal flatness of the $AdS_5 \times S^5$ metric (3). In Cartesian coordinates $X^\Lambda = (x^\mu, y^i)$, (3) reads $ds^2 = L^2 \rho^{-2} dX \cdot dX$, where $\rho^2 = y \cdot y$ and the solution of (28) with a constant asymptotic behavior is of the form

$$e^{\hat{\phi}} = g_s + \frac{\rho^4 \rho'^4}{L^8} \left( e^{\hat{\phi}(10)} - g_s \right),$$

2The $y^i$ here are related to the $y^i$ in (4) by an inversion.
where $\rho^2 = y_0 \cdot y_0$ and $e^{\hat{\phi}(10)}$ is the same harmonic function that appears in the flat ten-dimensional case, \((27)\). In evaluating D-instanton dominated amplitudes we are interested in the case in which the point $X = (x, y)$ approaches the boundary ($\rho \to 0$). It is then necessary to rescale the dilaton profile (just as it is necessary to rescale the scalar bulk-to-bulk propagator, \([11, 12]\)) so that the combination

$$
\rho^{-4} \left( e^{\hat{\phi}} - g_s \right) = \frac{3K(\alpha')^4}{L^8 \pi^4 \left( (x - x_0)^2 + \rho_0^2 \right)^4},
$$

will be of relevance in this limit.

As mentioned earlier, the correspondence with the YM instanton follows from the fact that $\rho_0^4 / ((x - x_0)^2 + \rho_0^2)^4 = K_4$ is proportional to the instanton number density, $(F_{(0)})^2$, in the $\mathcal{N} = 4$ SYM theory. Strikingly, the scale size of the YM instanton is replaced by the distance $\rho_0$ of the D-instanton from the boundary. This is another indication of how the geometry of the SYM theory is encoded in the type IIB superstring. Note, in particular, that as the D-instanton approaches the boundary $\rho_0 \to 0$, the expression for $\rho^{-4} e^{\hat{\phi}}$ reduces to a $\delta$-function that corresponds to a zero-size YM instanton.

Using the BPS condition, the action for a D-instanton of charge $K$ can be written as

$$
S_K = \frac{L^{10}}{\alpha' L} \int \frac{d\rho d\phi \omega}{\rho^5} g^{\Lambda \Sigma} \left( \partial_\Lambda e^{\hat{\phi}} \right) \left( \partial_\Sigma e^{\hat{\phi}} \right),
$$

which reduces to an integral over the boundary of $AdS_5 \times S^5$ and the surface of an infinitesimal sphere centered on the D-instanton at $x = x_0$, $y = y_0$. Substituting for $\hat{\phi}$ from \((29)\) gives

$$
S_K = \frac{2\pi |K|}{g_s},
$$

which is the same answer as in the flat ten-dimensional case. Remarkably, the boundary integrand is identical to the action density of the standard four-dimensional YM instanton.

The D-instanton contribution to the amplitude with sixteen external dilatinos may then be obtained directly by semi-classical quantization around the classical D-instanton solution in $AdS_5 \times S^5$. The leading instanton contribution can be determined by applying supersymmetry transformations to the classical scalar field background \((30)\). Since the D-instanton background breaks half the supersymmetries the relevant transformations are those in which the supersymmetry parameter corresponds to the Killing spinors for the sixteen broken supersymmetries.

Up to the overall constant factor, the amplitude computed in \([1]\) agrees with \((26)\) and therefore with \((20)\). In similar manner the instanton profiles of all the fields in the supergravity multiplet follow by applying the broken supersymmetries to the D-instanton solution just as they do in the flat ten-dimensional case \([3]\). The one D-instanton contributions to any correlation function can then be determined in principle.

6 Logarithmic behavior of four-point functions

In this last part of the talk we would like to show some peculiar behaviors of the exact non-perturbative correlators of superconformal currents. To this aim we shall consider
four-point functions. The prototype correlation function of four superconformal currents is the correlation function of four stress tensors. However, due to its complicated tensorial structure even the free-field expression for this correlator is awkward to compute and it is much simpler to consider correlations of four gauge-invariant composite scalar operators, such as the 20 scalars $Q^{ij}$ defined in [8] or the scalar $SU(4)$ singlet $C(x) = \text{Tr}(F^-)^2$. After calculating correlation functions of these currents one can derive those of any other currents in the $\mathcal{N}=4$ supercurrent multiplet by making use of the superconformal symmetry [24]. Instead of considering the most general correlation function,

$$\langle Q^{i_1j_1}(x_1)Q^{i_2j_2}(x_2)Q^{i_3j_3}(x_3)Q^{i_4j_4}(x_4) \rangle,$$

in order to evaluate one-instanton contributions in the most convenient fashion, it is useful to exploit the $SU(4)$ symmetry and only compute the correlation function of two $\varphi^2$ and two $\varphi^2$. It will then prove useful to work in the manifest $\mathcal{N}=2$ supersymmetric description of (11). After performing the necessary Wick contractions, the free-field expression for this particular correlator is

$$\langle \varphi^2(x_1)\varphi^2(x_2)\varphi^2(x_3)\varphi^2(x_4) \rangle_{\text{free}} = \frac{1}{(4\pi^2)^4} \left( \frac{4N^2}{x_{14}^2x_{24}^2} + \frac{4N^2}{x_{14}^2x_{23}^2} + \frac{16N}{x_{11}^2x_{13}^2x_{23}^2x_{24}^2} \right).$$

The computation of the one-instanton contribution to this Green function is straightforward using standard techniques [11]. The semi-classical approximation obtains after replacing the fields $\varphi$ with the expression

$$\varphi(x) \rightarrow \varphi(0)(x) = \frac{1}{2\sqrt{2}} \varepsilon_{uv} \xi_+^{\mu\nu} \zeta_+^u F_{(0)\mu\nu},$$

and a similar one for $\varphi$. These are the leading nonvanishing terms that result from Wick contractions with Yukawa-coupling terms lowered from the exponential of the action until a sufficient number of fermion fields are present to saturate the fermionic integrals. Of course, these expressions can also be obtained directly from the supersymmetry transformations (7) by acting twice on $F_{(0)\mu\nu}$ with the broken supersymmetry generators. After some elementary Fierz rearrangements on the fermionic collective coordinates the fermionic integrations can be performed in the standard manner to yield

$$G_4(x_p) = g^8_{YM} e^{-\frac{8g^2}{g^2_{YM}} + i\theta_{YM}} \int \frac{d\rho_0 d^4 x_0}{\rho_0^5} x_{12}^4 x_{34}^4 \prod_{p=1}^4 \left( \frac{\rho_0}{\rho_0^2 + (x_p - x_0)^2} \right)^4.$$

As remarked earlier, the fact that the instanton form factor, $\rho_0^4/[(\rho_0^2 + (x_p - x_0)^2)]^4$, that enters this expression is identical to $K_4$ in (3) is of the outmost significance in establishing of the AdS/Yang–Mills correspondence. The integration in (36) resembles that of a standard Feynman diagram with momenta replaced by position differences and can be performed by introducing the appropriate Feynman parametrization. Up to an overall numerical constant, the five-dimensional integral then yields

$$G_4(x_p) = g^8_{YM} \prod_p \alpha_p^2 d\alpha_p \delta \left( 1 - \sum_q \alpha_q \right) \frac{x_{12}^4 x_{34}^4}{(\sum_p \alpha_p \alpha_q x_{pq}^2)^8}.$$
This integral can be obtained by acting with six derivatives on the ‘box-integral’ with four massless external particles. Indeed

\[ G_4(x_p) = g_{YM}^8 e^{-\frac{x_p^2}{g_{YM}^2}+i\theta_{YM}} x_{12}^A x_{34}^A \prod_{p < q} \frac{\partial}{\partial x_{pq}^2} B(x_{pq}) , \]

where the box integral \( B(x_{pq}) \) turns out to be a combination of logarithms and dilogarithms \[42\]

\[ B(x_{pq}) = \frac{1}{\sqrt{\Delta}} \left[ -\frac{1}{2} \log \left( \frac{u_+ u_-}{(1-u_+)^2(1-u_-)^2} \right) \log \left( \frac{u_+}{u_-} \right) \right. \]
\[ \left. - \text{Li}_2(1-u_+) + \text{Li}_2(1-u_-) - \text{Li}_2 \left( 1 - \frac{1}{u_-} \right) + \text{Li}_2 \left( 1 - \frac{1}{u_+} \right) \right] , \]

where

\[ \Delta = \det_{4 \times 4}((x_{pq}^2)) = X^2 + Y^2 + Z^2 - 2XY - 2YZ - 2ZX \]

and

\[ u_\pm = \frac{Y + X - Z \pm \sqrt{\Delta}}{2Y} , \]

with \( X = x_{12}^2 x_{34}^2, Y = x_{13}^2 x_{24}^2 \) and \( Z = x_{14}^2 x_{23}^2 \).

Up to the overall factor \( x_{12}^4 x_{34}^4 \) needed for the correct scaling, \( G_4 \) turns out to be a function of the two independent superconformally invariant cross-ratios \( X/Z \) and \( Y/Z \). Although not immediately apparent, the expression for \( B(x_{pq}) \) is symmetric under any permutation of the external legs.

Unlike correlation functions of elementary fields that are infra-red problematic and gauge-dependent \[43\], the above correlator is well defined at non-coincident points. Up to the derivatives acting on the box integral, the result is exactly the one expected for the correlator of four scalar operators of dimension \( \Delta = 2 \) each. As observed in \[1\] it is not completely straightforward to compare the detailed expression for this correlator with the one D-instanton contribution to the \( \mathcal{R}^4 \) term in the type IIB effective action around the AdS background. In this respect, it is easier and perhaps more interesting from a ‘phenomenological’ viewpoint to consider the correlator of four scalar ‘glueball’ fields \( \mathcal{G} = \text{Re} \mathcal{C} = \text{Tr}(F^2) \). To leading order and up to an overall numerical constant, the one-instanton contribution to this four-point correlation function is expected to be

\[ \langle \mathcal{G}(x_1) \mathcal{G}(x_2) \mathcal{G}(x_3) \mathcal{G}(x_4) \rangle_{K=1} = g_{YM}^8 e^{-\frac{x_p^2}{g_{YM}^2}+i\theta_{YM}} \int \frac{dp_0 d^4 x_0}{\rho_0^5} \prod_{p=1}^4 \left( \frac{\rho_0}{\rho_0^2 + (x_p - x_0)^2} \right)^4 . \]

and coincides with \[36\] after removing the factor \( x_{12}^4 x_{34}^4 \), consistently with superconformal invariance. Differently from \[17\], the correlators \[12\] and \[36\] respect the \( U(1)_B \) symmetry \[30\] and receive an analogous (complex conjugate) contribution from an anti-instanton. As observed by other groups \[17\], this expression shows a logarithmic singularity in the limit \( x_{ij} \to 0 \) for each pair of external points. This may be either a drawback of the lowest-order approximation or an intrinsic pathology of the theory. The logarithms by themselves do not violate the superconformal symmetry of the theory that is manifest in
the final expression of the four-point functions given in [1] and reviewed above. In some
respect, logarithmic behaviors in four-point functions should not sound unexpected in a
superconformal theory such as $\mathcal{N}=4$ SYM theory that contains a large number of primary
fields all of whose (protected) dimensions are integer. Rather they may signal a violation
of the Operator Product Expansion. The latter may be due to the presence of an infinite
tower of stable BPS dyons that collapse to vanishing mass and size in the superconformal
phase. From the Euclidean viewpoint this effect may be due to the presence of a gas of
instantons of vanishing size. This problem is under active investigation [45].

7 Discussion

The fact that YM-instanton and type IIB D-instanton effects appear to match so closely
should be interpreted as a strong support to the conjectured correspondence put forward
by Maldacena [5]. Most notably, we have seen that the YM instanton scale size has a
natural interpretation as the position of the D-instanton in the extra dimension transverse
to the four-dimensional boundary space-time. This reflects the fact that the one-instanton
moduli spaces in both cases contain an $AdS_5$ factor. The additional $S^5$-directions, which
are not at all apparent in standard $\mathcal{N}=4$ SYM perturbation theory, are associated to
auxiliary bosonic ‘collective coordinates’ [27].

The remarkable agreement that has been found in [1] for the lowest non-trivial value
of $N$ ($N=2$) and its extension to any $N$ in [26] neatly show the computational power
of instanton calculus. Indeed it would have been hopeless to try a matching of lower
derivative terms in the type IIB effective action with the planar series to corresponding
SYM correlation functions. On the contrary, the agreement between non-perturbative
and thus highly suppressed contributions to some protected interactions may even lead
one to suggest that the original conjecture be valid for finite, indeed any, $N \geq 2$. For
consistency one should expect a truncation to $\ell \leq N$ of the KK spectrum in the type IIB
superstring compactification on $S^5$ for finite radius. Worldsheet unitarity requirements
analogous to the ones familiar in $SU(2)$ WZW models may underlie these stringy effects.
Indeed this seems to be the case in the correspondence between two-dimensional SCFT’s
and type IIB superstring on $AdS_3 \times S^3$ [46]. However, despite some interesting work [18]
the quantization of the GS-type action governing the dynamics of the type IIB superstring
around $AdS_5 \times S^5$ does not seem to be at reach in the near future.

For this reason, the evaluation of $K$-instanton contributions in $\mathcal{N}=4$ SYM theory in
the large $N$ limit and its agreement with type IIB D-instanton effects [27] may be consid-
ered as the strongest support to the correspondence found so far. A detailed analysis of
the contribution of multiply-charged ($K > 1$) instanton configurations for finite $N$ should
clarify what survives of the correspondence for finite $N$. The quantitative agreement of
the charge-$K$ D-instanton coefficients in the type IIB modular functions $f_n$ and the cor-
responding coefficients in SYM theory for large $N$ is a promising step in this direction
[27].

An alternative approach to the non-perturbative computations performed so far [1,
20, 27] consists in viewing the instanton-dominated correlators as a topological subsector
of the $\mathcal{N}=4$ SYM theory. In this respect, it is worth noticing that the modular-covariant
non-holomorphic functions of the complexified couplings, $f_n(\tau, \bar{\tau})$, that appear in the type IIB D-instanton description are amazingly similar to some expressions found in the computation of the Witten index for topologically twisted $\mathcal{N}=4$ theories on curved manifolds [17].

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