Disruption of the three-body gravitational systems:
Lifetime statistics

V. V. Orlov\textsuperscript{1*}, A. V. Rubinov\textsuperscript{1} and I. I. Shevchenko\textsuperscript{2}

\textsuperscript{1}Sobolev Astronomical Institute, St.Petersburg State University, Universitetsk pr. 28, Stary Peterhoff, St.Petersburg 198504, Russia
\textsuperscript{2}Pulkovo Observatory of the Russian Academy of Sciences, Pulkovskoje ave. 65, St.Petersburg 196140, Russia

Accepted . Received ; in original form 2010 March 10

ABSTRACT
We investigate statistics of the decay process in the equal-mass three-body problem with randomized initial conditions. Contrary to earlier expectations of similarity with “radioactive decay”, the lifetime distributions obtained in our numerical experiments turn out to be heavy-tailed, i.e. the tails are not exponential, but algebraic. The computed power-law index for the differential distribution is within the narrow range, approximately from $-1.7$ to $-1.4$, depending on the virial coefficient. Possible applications of our results to studies of the dynamics of triple stars known to be at the edge of disruption are considered.

Key words: stellar dynamics – celestial mechanics – three-body problem – lifetime

1 INTRODUCTION
Disruption of a three-body gravitational system is an enigmatic dynamical process, statistics of which is mostly unexplored, at long timescales especially. Valtonen (1988) supposed that the lifetime distribution for a three-body system is an exponentially decaying function, in analogy with “radioactive decay”. Recent statistical numerical studies by Mikkola & Tanikawa (2007) of this process in the equal-mass problem revealed new important data on the statistics of the disruption times and raised new questions on the nature of this process. Mikkola & Tanikawa (2007) have explored a statistics of the disruption times
$T_d$ in the equal-mass three-body problem. (The disruption time $T_d$ is the system lifetime as a bound system.) They computed the disruption times for equal-mass systems with randomized initial conditions, and fitted the found disruption time distributions by exponential functions.

However, exponential decay, in the asymptotically long time scale, is in conflict with the theoretical result of Agekian et al. (1983) that the average lifetime of a general isolated three-body system is infinite. What is more, recently it was shown by Shevchenko (2009) on the basis of the Kepler map theory, applied to a hierarchical three-body problem, that at the edge of disruption the orbital periods and the size of the orbit of the escaping body behave as Lévy flights. Due to them, the time decay of the survival probability (the integral distribution of lifetimes) is heavy-tailed with the power-law index equal to $-2/3$. Combining the Kepler map theory and earlier theoretical findings of Hut (1993), Shevchenko (2009) made a conclusion that the $T_d^{-2/3}$ law is expected to be quite universal.

Here we explore the problem of the lifetime statistics in the three-body problem by means of numerical simulations. We pay particular attention to analysis of the tails of the lifetime distributions, and find the algebraic behaviour. We compare our results with previous numerical work and discuss why the algebraic tails had not been identified earlier. Possible applications of our results to studies of the dynamics of triple stars known to be at the edge of disruption are considered.

2 SETTING OF THE PROBLEM AND NUMERICAL EXPERIMENTS

To investigate disruption of triple systems we use numerical integration of equations of motion in the gravitational three-body problem. The equations are as follows:

$$m_k \frac{d^2 r_k}{dt^2} = \frac{\partial U}{\partial r_k} \quad (k = 1, 2, 3),$$

(1)

where $m_k$ are the masses of the bodies, $r_k$ are the coordinate vectors in the barycentric orthogonal coordinate system, $t$ is time, and $U$ is the potential:

$$U = G \left( \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right),$$

(2)

where the quantities $r_{ij}$ are the distances between bodies $i$ and $j$, $G$ is the gravitational constant.

We use the chain-regularization algorithm of Mikkola & Aarseth (1993) to treat behaviour of the system in the vicinity of close double and triple approaches accurately. Nu-
Numerical integration is performed by the Dormand–Prince integrator (Hairer et al. 1987), which realizes an explicit Runge–Kutta method of 8-th order with step size control.

We restrict our present study to the equal-mass three-body problem. The masses of all three bodies \( m_k, k = 1, 2, 3 \) are set to 1. We also choose \( G = 1 \). Total energy of the triple system \( E = -1 \).

To set up the initial conditions for the integration, we use an approach similar to that used in Mikkola & Tanikawa (2007). Namely, we produce random initial data by selecting the initial coordinate and velocity components of the bodies from a uniform distribution inside a 9-dimensional cube with size of the edge equal to 2. Then we transform these initial data to the barycentric system.

Using scale coefficients, the coordinates and velocities of bodies are transformed to satisfy fixed the virial ratio \( k = T/U \), where \( T \) and \( U \) are the kinetic energy and the potential, respectively, of the triple system. The total energy is \( E = T - U = -1 \). The following values of the initial virial ratio have been chosen: \( k = 0, 0.1, 0.3, 0.5, 0.7, 0.9 \). For each value of the virial ratio we study a set of \( 10^5 \) triple systems.

We follow the dynamical evolution of each triple system during the time interval of \( 10^5 \) time units, or until the escape criterion is satisfied. The escape criterion is the same as that used in (Mikkola & Tanikawa 2007); namely, we stop our integration, when a single body is moving away on a hyperbolic relative orbit at a distance \( d > 50 \) times the current semi-major axis of the final binary. The condition for hyperbolicity is

\[
E_{\text{out}} = \frac{M_{\text{in}} V_{\text{cm}}^2}{2} + \frac{m_{\text{out}} V_{\text{out}}^2}{2} - G \frac{M_{\text{in}} m_{\text{out}}}{d} > 0, \tag{3}
\]

where \( E_{\text{out}} \) is the total energy of the outer binary, \( M_{\text{in}} \) is the mass of the inner binary, \( m_{\text{out}} \) is the mass of the outer component, \( V_{\text{cm}} \) is the velocity of centre-of-mass of the inner binary, \( V_{\text{out}} \) is the velocity of the outer component, \( d \) is the distance between centre-of-mass of the inner binary and the outer component.

Using orbital elements we compute the formal time of pericentre passage for this hyperbolic orbit. This time \( T_d \) is considered to be the moment of the triple system disruption. The disruption time \( T_d \) is counted from the beginning of computation up to this moment.

3 RESULTS OF THE NUMERICAL EXPERIMENTS

The results of integration are qualitatively similar for all values of \( k \) in our set, that is why the graphs with distributions are presented here only for a single case of \( k \), namely, for \( k = 0 \).
In Fig. 1, we show the integral distribution $F_a(T_d)$ of the disruption times in decimal logarithmic scales. The quantity $F_a(T_d)$ is the fraction of the lifetimes greater than $T_d$. An initial exponential drop (a “bump” in logarithmic scales) and a subsequent power-law decay (a straight-line dependence in logarithmic scales) are prominent. In the very tail a smooth drop is evident, which is due to the presence of the upper limit on the time of integration (equal to $10^5$ time units). Owing to this final drop, the $F_a(T_d)$ dependence cannot be used straightforwardly (by linear fitting of the tail in the logarithmic scales) for reliable estimation of the index of the power-law decay.

We separate the initial exponential drop from the subsequent power-law decay by choosing the value of $T_d$, at which the distribution becomes algebraic. From Fig. 1 we see that this transition value lies between $10^3$ and $10^4$. We choose the latter value, so that to be completely sure that any contribution of the initial “bump” is minimal.

Thus the transition value of $T_d$ is found from the analysis of the bimodal structure of the integral distribution of $T_d$ in logarithmic scales. In the subsequent treatment the scales are no longer logarithmic. In Fig. 2, the differential distribution of the disruption times is shown for the same $k = 0$. Only the tail ($T_d > 10^4$) is presented. The quantity $f(T_d)$ is the fraction of the lifetimes in a pocket $(T_d, T_d + \Delta)$, where $\Delta = 10^3$. As soon as the distribution is differential, the upper limit on the time of integration does not inflict the form of the distribution, and so the distribution can be used for fitting. The solid line, drawn in the Figure, represents the algebraic fitting. Details of the fitting are given in Table 1. For all kinds of fittings in this paper, we use a non-linear least-squares method (Levenberg 1944; Marquardt 1963) to minimize $\chi^2$, thereby finding the best-fit parameter values and their standard errors.

The fitting here is two-parametric: we use the fitting function

$$f(T_d) = AT_d^{-\beta},$$

where $A$ and $\beta$ are free parameters. Only the value of $\beta$ is important for us and is therefore reported in Table 1. The correlation coefficient $R^2$ is also reported. Note that it is very close to 1, i.e. the fitting is very good. As one can see, it turns out to be close to the $T_d^{-5/3}$ law.

In Fig. 3, the same data are presented as in Fig. 2 but the distribution is integral. The quantity $F(T_d)$ is the fraction of the lifetimes less than $T_d$. (Note that the quantity $F_a(T_d)$ in Fig. 1 is the fraction of the lifetimes greater than $T_d$). The solid line, drawn in the Figure, represents the algebraic fitting. A two-parametric fitting cannot be employed in the integral
Table 1. The power law index $\beta$ for the tail of the differential distribution of disruption times

| $k$ | 0   | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|-----|-----|-----|-----|-----|-----|-----|
| $\beta$ | 1.695 | 1.639 | 1.522 | 1.448 | 1.586 | 1.684 |
| $\beta$ | $\pm 0.041$ | $\pm 0.032$ | $\pm 0.031$ | $\pm 0.022$ | $\pm 0.028$ | $\pm 0.057$ |
| $R^2$ | 0.967 | 0.977 | 0.972 | 0.984 | 0.981 | 0.935 |

Table 2. The power law index $\alpha$ for the tail of the integral distribution of disruption times

| $k$ | 0   | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|-----|-----|-----|-----|-----|-----|-----|
| $\alpha$ | 0.7073 | 0.6704 | 0.4244 | 0.4555 | 0.6783 | 0.6332 |
| $\alpha$ | $\pm 0.0016$ | $\pm 0.0013$ | $\pm 0.0022$ | $\pm 0.0018$ | $\pm 0.0014$ | $\pm 0.0041$ |
| $R^2$ | 0.9998 | 0.9998 | 0.9996 | 0.9997 | 0.9998 | 0.9985 |

case, because we do not know beforehand the fraction of the disruption times longer than the limiting time of integration (for the differential distribution, this fraction is absorbed in the coefficient $A$). So, the fitting here is three-parametric: we use the fitting function

$$F(T_d) = A - BT_d^{-\alpha},$$  \hspace{1cm} (5)$$

where $A$, $B$ and $\alpha$ are free parameters. Only the value of $\alpha$ is important for us and is therefore reported in Table 2. The fitting turns out to be close to $T_d^{-2/3}$ law. This law is expected from our results on the differential distribution, because the equality $\alpha = \beta - 1$ should hold. In practice this equality is not exact, mostly due to the difference in fitting schemes.

The correlation coefficient in the case of the integral distribution is much closer to 1, than in the case of the differential distribution. This means that the estimates of $\alpha$ are more reliable than those of $\beta$. This is natural, because the fitting results for the differential distribution are sensitive to the choice of the pocket width.

The results of implementing the fitting procedures at all values of $k$ are collected in Tables 1 and 2. As it can be seen from the data in Tables 1 and 2, the tails of distributions in all cases are algebraic. This is our first main result, following from the fact that the correlation coefficient $R^2$ in all cases is very close to 1, i.e. the fitting is very good.

Our second main result is that the computed power-law index for the differential distributions is in the narrow range, approximately from $-1.7$ to $-1.4$, depending on the virial coefficient $k$. The consequences are discussed below.
What is the nature of the observed algebraic decay? One may recall that a similar algebraic decay was observed by Shevchenko & Scholl (1996, 1997) in numerical experiments on asteroid dynamics. They showed that the tail of the integral distribution of the time intervals $T_s$ between jumps of the orbital eccentricity of asteroids in the $3/1$ mean motion resonance with Jupiter is not exponential, but algebraic: $F_a \propto T_s^{-\alpha}$ with $\alpha \sim 1.5-1.7$. This decay was due to sticking of orbits to chaos borders in the phase space of motion. The stickiness effect determines the character of the distribution of Poincaré recurrences on large timescales: it is algebraic (Chirikov & Shepelyansky 1981, 1984). Starting with the pioneering work by Chirikov & Shepelyansky (1981), the algebraic decay in the recurrence statistics in Hamiltonian systems with divided phase space was considered, in particular, in (Chirikov & Shepelyansky 1984, Chirikov 1990, 1999; Cristadoro & Ketzmerick 2008). Chirikov (1990), using his resonant theory of critical phenomena in Hamiltonian dynamics, justified a value of $3/2$ for the critical exponent $\alpha$.

However, the values of $\alpha$ observed in our numerical experiments ($\sim 0.4-0.7$) differ radically from those expected for the stickiness phenomenon ($\sim 1.5$). Therefore, though the decay is algebraic, it is not related to the stickiness effect. Then, what is the origin of the algebraic decay here? On the basis of the Kepler map theory, Shevchenko (2009) considered statistics of the disruption and Lyapunov times in a hierarchical three-body problem. It was shown that at the edge of disruption the orbital periods and the size of the orbit of the escaping body exhibit Lévy flights. Due to them, the time decay of the survival probability is heavy-tailed with the slope power-law index $\alpha$ equal to $2/3$ (while the relation between the Lyapunov and disruption times is quasilinear). Combining the Kepler map theory and earlier theoretical findings of Hut (1993), Shevchenko (2009) concluded that the $T_d^{-2/3}$ law was expected to be quite universal.

However, neither the $T_d^{-2/3}$ law, nor even any other algebraic law, were reported in the numerical studies by Mikkola & Tanikawa (2007). We believe that the reasons are as follows: solely the initial part of the distribution was considered by Mikkola & Tanikawa (2007). Besides, the algebraic fitting functions were not used anyway. As noted by Shevchenko (2009), the tail of the disruption time distribution in the given problem should be considered separately from the initial part, because it corresponds to a different dynamical situation: in the beginning, the regime of decay might be exponential; see dynamical analogues in
Disruption of the three-body systems

(Shevchenko & Scholl 1996, 1997; Shevchenko 1999) and the theory for the exponential decay in (Chirikov 1999).

As follows from Tables 1 and 2, for some values of \( k \), namely intermediate ones \( k = 0.3 \) and 0.5, the power-law index \( \beta \) of the differential distribution is equal to 1.45–1.52, while the power-law index \( \alpha \) of the integral distribution is equal to 0.42–0.46. Thus the values of \( \beta \) and \( \alpha \) are close to 3/2 and 1/2, respectively. This might testify for the presence of the inverse square root law \( F_a(T_d) \propto T_d^{-1/2} \), which is inherent to free diffusion in the central part of a chaotic layer until the finite width of the layer becomes important (Chirikov & Shepelyansky 1981, p. 9), see also (Shevchenko 1999).

Concluding, we see that the Kepler map statistics is valid at \( k \sim 0 \) and \( k \sim 1 \), while, hypothetically, the free diffusion seems to dominate at \( k \sim 0.5 \). As for the case \( k \sim 0.5 \), we believe that the free diffusion predominance is merely an effect of insufficient times of integration, and expect that an extension of the integration time to greater limits should make apparent the asymptotic behaviour typical for the Lévy flights in the given problem, when appropriately long timescales are achieved; in other words, we expect that the asymptotic behaviour with \( \alpha \sim 2/3 \) is universal for all values of \( k \).

5 MULTIPLE STARS: DYNAMICS AT THE EDGE OF STABILITY

In application to actual multiple stars, our results on the statistics of chaotic decay are appropriate to be used in studies of the dynamics of multiple stars known nowadays to be near the edge of stability.

A list of such multiple stars comprises: HD 40887, HD 76644 (ADS 7114= \( \iota \) Uma), HD 217675 (\( \omicron \) And), HD 222326 (ADS 16904) (Orlov & Zhuchkov 2005). Probably all these systems are quadruple. However, they contain close binary stars, therefore in computer simulations we can consider them as triples. The mass ratios in these systems are: 1) 0.65 : 0.52 : (0.69+0.2); 2) 0.41 : 0.42 : (1.94+0.82); 3) 4.2 : 3.4 : (6.8+2.3); 4) 1.2 : 1.3 : (0.7+1.0) solar masses, respectively. The ratios of the orbital periods of the outer and inner binaries are approximately 6, 21, 20, 14, respectively. If considered as triples, these systems have a weak hierarchy (Orlov & Zhuchkov 2005): the distant component is not very much far from the inner pair (with respect to the size of the latter), and its perturbation on the motion of the latter is not negligible. The mutual perturbations can be strong enough to modify the hierarchy and, finally, lead to disruption of the triple. Thus the property of weak hierarchy
implies that these systems are near the edge of the stability region in the phase space of the motion; see (Orlov & Zhuchkov 2005) for details. Though different from the equal-mass case, the dynamical behaviour of these systems should be as well dominated by Lévy flights, if they are close enough to disruption. This proximity should be further analyzed and verified.

Of course, a direct observation of Lévy flights in the dynamics of such systems would require unrealistic long times of observation. However, when the current coordinates and velocities of the components are determined observationally with sufficient accuracy, the Lévy flights can be studied and analyzed in numerical simulations of the future dynamical history of the triples, and estimates of their lifetimes can be obtained.

6 CONCLUSIONS

We have investigated statistics of the disruption times $T_d$ in the equal-mass three-body problem. The statistics has been explored in massive numerical simulations with randomized initial conditions. The distributions of the disruption times have turned out to be heavy-tailed with the power-law index being within narrow range near the value of $-5/3$ (for the differential distribution). The observed range is from $-1.7$ to $-1.4$; the value of the slope index slightly depends on the virial coefficient.

Our result is in conflict with earlier expectations and analogies with “radioactive decay”, which implied exponential tails, and also in conflict with numerical results of Mikkola & Tanikawa (2007), where exponential decay was found. We explain the divergence with the latter results by the fact that Mikkola & Tanikawa (2007) fitted the distribution as a whole (and what is more, taken at insufficiently long timescale), while the behaviour of the tails should be treated separately from the initial exponential drop.

On the other hand, our finding of the algebraic decay is in agreement with the theoretical result of Agekian et al. (1983), stating that the average lifetime of a general isolated three-body system is infinite. Moreover, it is in agreement with the Kepler map theory (Shevchenko 2009), predicting just the same value of the slope index ($\sim -5/3$) that we have observed in the majority of our numerical experiments. This makes evident the existence of Lévy flights in the process of decay.

At intermediate values of the virial coefficient, $k \sim 0.5$, we have observed the slope index $\sim -1.5$ (for the differential distribution). This is typical for free diffusion in the central part of a chaotic layer until the finite width of the layer becomes important. We believe
that the free diffusion predominance is merely an effect of insufficient times of integration. If one extends the integration time to sufficient limits, the asymptotic behaviour with the slope index \( \sim -5/3 \), characteristic of the Lévy flights in the given problem, should become apparent.

Several actual multiple stars are known nowadays to be near the edge of the stability region in the phase space of the motion, in particular, HD 40887, HD 76644 (ADS 7114 = \( \iota \) Uma), HD 217675 (o And), HD 222326 (ADS 16904) \[Orlov & Zhuchkov, 2005\]. Numerical integrations, based on initial data from further high-precision astrometric observations of these systems, would allow one to reveal long-term evolution features, in particular, the theoretically predicted Lévy flights in the orbital periods and the geometrical sizes of the systems.

ACKNOWLEDGMENTS

This work was partially supported by the Russian Foundation for Basic Research (project # 09-02-00267-a), the Programme of Fundamental Research of the Russian Academy of Sciences “Fundamental Problems in Nonlinear Dynamics”, and the President Programme for Support the Leading Scientific Schools (project # 3290.2010.2).

REFERENCES

Agekian T.A., Anosova Zh.P., Orlov V.V., 1983, Astrophysics, 19, 66 [Astrofizika, 19, 111]
Chirikov B.V., 1990, INP Preprint 90–109. Institute of Nuclear Physics, Novosibirsk
Chirikov B.V., 1996, JETP, 83, 646 [Zh. Eksp. Teor. Fiz., 110, 1174]
Chirikov B.V., 1999, INP Preprint 99–7. Institute of Nuclear Physics, Novosibirsk [eprint arXiv:nlin/0006013].
Chirikov B.V., Shepelyansky D.L., 1981, INP Preprint 81–69. Institute of Nuclear Physics, Novosibirsk (in Russian).
Chirikov B.V., Shepelyansky D.L., 1984, Physica, D 13, 395
Cristadoro G., Ketzmerick R., 2008, Phys. Rev. Lett., 100, 184101
Hairer E., Nørsett S.P., Wanner G., 1987, Solving Ordinary Differential Equations I. Non-stiff Problems. Springer-Verlag, Berlin
Hut P., 1993, Astrophys. J., 403, 256
Levenberg K., 1944, Quarterly Applied Math., 2, 164
Marquardt D.W., 1963, SIAM Journal Applied Math., 11, 431
Mikkola S., Aarseth S.J., 1993, Celest. Mech. Dyn. Astron., 57, 439
Mikkola S., Tanikawa K., 2007, Mon. Not. R. Astron. Soc., 379, L21
Orlov V.V., Zhuchkov R.Ya., 2005, Astron. Rep., 49, 201
Shevchenko I.I., 1999, in Henrard J., Ferraz-Mello S., eds, Proc. IAU Coll. 172, Impact of Modern Dynamics in Astronomy. Kluwer, Dordrecht, p. 383
Shevchenko I.I., 2009, Lévy flights in the three-body problem, [arXiv:0907.1773](http://arxiv.org/abs/0907.1773v3)
Shevchenko I.I., Scholl H., 1996, in Ferraz-Mello S., Morando B., Arlot J.-E., eds, Proc. IAU Symp. 172, Dynamics, Ephemerides and Astrometry of the Solar System. Kluwer, Dordrecht, p. 183
Shevchenko I.I., Scholl H., 1997, Celest. Mech. Dyn. Astron., 68, 163
Valtonen M.J., 1988, Vistas Astron., 32, 23
Disruption of the three-body systems

Figure 1. The integral distribution of the disruption times in logarithmic scales, $k = 0$. 
Figure 2. The differential distribution of the disruption times for $k = 0$. The solid line represents the algebraic fitting, which turns out to be close to $T_d^{-3/2}$ law.
Figure 3. The same as in Fig. 2 but the distribution is integral. The solid line represents the algebraic fitting.