Newton maps for quintic polynomials

Francisco Balibrea, J. Orlando Freitas, and J. Sousa Ramos

Abstract. The purpose of this paper is to study some properties of the Newton maps associated to real quintic polynomials. First using the Tschirnhaus transformation, we reduce the study of Newton’s method for general quintic polynomials to the case \( f(x) = x^5 - c x + 1 \). Then we use symbolic dynamics to consider this last case and construct a kneading sequences tree for Newton maps. Finally, we prove that the topological entropy is a monotonic non-decreasing map with respect to the parameter \( c \).

1. Introduction and motivation

The classical problem of solving equations has substantially influenced the development of mathematics throughout the centuries and still has several important applications to the theory and practice of present-day computing. “Solving the quintic” is one of the few topics in mathematics which has been of enduring and widespread interest for centuries.

We cannot solve the general polynomial equation of fifth degree or higher using radicals. Consequently, methods for estimating numerical solutions of equations as simple as polynomials are necessary. On a mundane level, numerical methods can be used to find the zeros (real or complex) to any required degree of accuracy. This is a useful practical method.

Many mathematical problems can be reduced to compute the solutions of \( f(x) = 0 \), and Newton’s method

\[
x_{n+1} = N_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, ...
\]

is the most common algorithm to solve this problem. The geometric interpretation of the Newton’s method is well known. In such a case \( x_{n+1} \) is the point where the tangent line \( y - f(x_n) = f'(x_n)(x - x_n) \) to the graph of \( f(x) \) at the point \( (x_n, f(x_n)) \) intersects the \( x \)-axis.

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The fundamental property of Newton’s method is that it transforms the problem of finding roots of \( f(x) \) into the problem of finding attracting fixed points of the associate Newton’s map \( N_f(x) \).

We may ask however for the set of all points \( x_0 \) from which the Newton’s method is converging to a solution.

In 1829, Cauchy \[3\] first proved a convergence theorem which does not assume any existence of a solution. Under standard assumptions, the Newton’s method is locally convergent in a suitable disk centered at the solution. The possibility that a small change in \( x_0 \) can cause a drastic change in convergence indicates the nasty nature of the convergence problem.

A key notion in the study of discrete dynamical systems is that of chaos and sensitive dependence on initial conditions. There have been several definitions of chaos, for example Devaney’s definition \[5\]. In one-dimensional case Devaney’s definition is equivalent to the existence of a dense orbit and another criterion is considering the system as chaotic whether the entropy is strictly positive.

A detailed treatment of the cubic polynomial case can be seen in \[11\] and a complete description of its combinatorics is given in \[18\].

We study the quintic function \( f(x) = x^5 + c_1 x^4 + c_2 x^3 + c_3 x^2 + c_4 x + c_5 \).

In the first place we reduce the number of parameters. We now discuss the necessary algebraic aspects of this reduction. As it is well-known, the equation

\[
x^5 + c_1 x^4 + c_2 x^3 + c_3 x^2 + c_4 x + c_5 = 0,
\]

with arbitrary coefficients \( c_j \), can be transformed to the *Bring-Jerrard* type

\[
x^5 + a x + b = 0,
\]

by a Tschirnhaus transformation \[6\] pp. 212-214].

Tschirnhaus’s transformation reduces the \( n^{th} \) degree polynomial equation

\[
c_0 x^n + c_1 x^{n-1} + ... + c_{n-1} x + c_n = 0
\]

to one with up to three fewer terms

\[
x^n + b_1 x^{n-4} + ... + b_{n-1} x + b_n = 0
\]

by transforming the root as follows

\[
x_j = \gamma_4 x_j^4 + \gamma_3 x_j^3 + \gamma_2 x_j^2 + \gamma_1 x_j + \gamma_0, \quad (j = 1, ..., n)
\]

where the \( \gamma_j \) can be expressed in radicals in terms of the \( a_j \). Thus every quintic can be transformed into one of the form

\[
x^5 + b_4 x + b_5 = 0.
\]

The \( b_j \) can ultimately be expressed in radicals in terms of the \( a_j \) \[20\].

**Remark 1.** The resulting expressions are really complicated. For a general quintic with symbolic coefficients they require a lot of computation and storage. However this is not a problem of present-day computer.
The outline of the paper is as follows. In section 2 we use topological conjugacy to study the Newton map for quintic polynomials of the form $f(x) = x^5 + a x + b$. Apparently, the idea of conjugation is important to understand the iteration of $N_f(x)$. Indeed, it was E. Schröder who observed the importance of conjugations, mainly to obtain a convenient form for calculations, (see Peitgen and Haeseler [16]). The concept of conjugations has proven extremely useful in the modern theory of iteration and we use it. With this idea we reduce the general case to most interesting having the form $f_c(x) = x^5 - c x + 1$ whose picture is shown in Figure 1.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{newton_map}
\caption{Typical graph of $N_{f_c}$ with one fixed point.}
\end{figure}

In section 3, using standard symbolic dynamics, we introduce the admissibility rules of the sequences associated to Newton maps $N_f$. Then we study the structure of the set of admissible sequences. The techniques of symbolic dynamics are based on the notions of the kneading theory for one-dimensional multimodal maps, (see Milnor and Thurston [14]). We construct a kneading sequences tree for $N_{f_c}$.

In section 4 we are devoted to consider the topological entropy. An immediate consequence of these results is the exact computation of the topological entropy which is made in this section. Our main result is the last theorem of the section.

The connection between kneading theory and subshifts of finite type is shown by using a commutative diagram derived from the topological configurations associated with $m$-modal maps, (see Lampreia and Sousa Ramos [13]).

2. Newton maps for quintics

We study the polynomial function $f(x) = x^5 + a x + b$. As we said before we use topological conjugacy. It is well known that $f$ and $g$ are topologically conjugate provided there is an homeomorphisms $\tau$ such that $f \circ \tau = \tau \circ g$. 
In such case for $f^n$ and $g^n$ we have the same relationship. So $f$ and $g$ are holomorphically the "same" dynamical systems. Indeed topological conjugacy is a very efficient concept to carry over difficult dynamical problems in simpler ones. It plays an important role in the investigation of the dynamics of general one-dimensional maps \[4\] p. 122.

**Proposition 1.** Let $g(x) = x^5 + ax + b^5$ where $b \neq 0$ and define $f(x) = x^5 + cx + 1$ where $c = a/b^4$. Then $N_g$ and $N_f$ are topologically conjugate via the homeomorphism $\tau(x) = x/b$.

**Proof.** First we calculate

$$N_g(x) = x - \frac{g(x)}{g'(x)} = \frac{4x^5 - b^5}{5x^4 + a}$$

and

$$N_f(x) = x - \frac{f(x)}{f'(x)} = \frac{4x^5 - 1}{5x^4 + c}.$$  

We have $\tau \circ N_g(x) = \frac{4x^5 - b^5}{5bx^4 + ab}$ and $N_f \circ \tau(x) = \frac{4x^5 - b^5}{5bx^4 + ab}$, so we have

$$N_g = \tau^{-1} N_f \circ \tau,$$

i.e., $N_g$ and $N_f$ are topologically conjugate.

Let us see what happens when $b = 0$.

**Proposition 2.** Let $g(x) = x^5 + a^4x$, $\tau(x) = x/a$, with $a \neq 0$, and $p_+(x) = x^5 + x$. Under such conditions $N_g$ and $N_{p_+}$ are topologically conjugate by $\tau$. By other hand, if $g(x) = x^5 - a^4x$ and $p_-(x) = x^5 - x$ then $N_g$ and $N_{p_-}$ are topologically conjugate by $\tau$.

**Proof.** We can calculate

$$N_g(x) = x - \frac{g(x)}{g'(x)} = \frac{4x^5}{5x^4 + a^2}$$

and

$$N_{p_+}(x) = x - \frac{p_+(x)}{p'_+(x)} = \frac{4x^5}{5x^4 + 1}.$$  

We have

$$\tau \circ N_g(x) = \frac{4x^5}{5ax^4 + a^5}$$

and

$$N_{p_+} \circ \tau(x) = \frac{4x^5}{5ax^4 + a^5},$$

so

$$N_g = \tau^{-1} N_{p_+} \circ \tau.$$  

It is analogous for the second case and we have

$$N_g = \tau^{-1} N_{p_-} \circ \tau,$$

i.e., $N_g$ and $N_{p_-}$ are topologically conjugate.
We must consider now the case $a = 0$ which leaves $f(x) = x^5$.

**Remark 2.** Last two propositions imply that either the dynamics of Newton’s map for the quintic $g(x) = x^5 + a x + b$ are equivalent to the dynamics of Newton’s map for the polynomial family $f_c(x) = x^5 + c x + 1$ or to $g_a(x) = x^5 + a x$. Moreover, the Newton’s map for function $g_a(x)$ is topologically conjugate to Newton map for one of the three polynomials:

$$p_-(x) = x(x^4 - 1), \quad p_+(x) = x(x^4 + 1), \quad \text{or} \quad p_0(x) = x^5.$$

Therefore the study of Newton map for quintic polynomials is reduced to the case $f_c(x) = x^5 + c x + 1$. Indeed, as $c \in \mathbb{R}$ we use $f_c(x) = x^5 - c x + 1$ instead of $f_c(x) = x^5 + c x + 1$. In this case we have $f_c'(x) = 5x^4 - c$.

![Figure 2](image-url)

**Figure 2.** Map $f_c$ and typical map $N_{f_c}$ for $c < 0$.

- When $c < 0$, it is easy to verify that $N_{f_c}$ has exactly one real root and that its stable set (the set of points which are forward asymptotic to it) contains all $\mathbb{R}$ as we see in Fig 2.
- When $c = 0$, there is also one real root and its stable set contains all real numbers except 0.
- When $c > 0$ we have three interesting cases.

The polynomial $f(x)$ has a relative maximum at $d_1 = -\sqrt[4]{c/5}$ and a relative minimum at $d_3 = \sqrt[4]{c/5}$.

We note that when $c$ increases, the relatives minimum of $f$ decreases and the relative maximum increases. When $c = 5 \times 2^{-8/5} = 1.64938\ldots$ the relative minimum is 0, see Figure 3. We denote the parameter $c = 5 \times 2^{-8/5}$ by $c_0$.

Note that when $c$ is bigger than $c_0$, $f$ has three real roots as showed in Figure 4.

In last case we can use the following result:

**Theorem 1** (Rényi [17]). Let $f : \mathbb{R} \to \mathbb{R}$ be defined on $(-\infty, +\infty)$. Let us suppose that $f''(x)$ is monotone increasing for all $x \in \mathbb{R}$ and that $f(x) = 0$ has exactly three real roots $a_i$ ($i = 1, 2, 3$).
Then the sequence \( x_{n+1} = x_n - f(x_n)/f'(x_n) \) converges to one of the roots for every choice of \( x_0 \) except for \( x_0 \) belonging to a countable set \( E \) of singular points, which can be explicitly given. For any \( \varepsilon > 0 \) there exists an interval \((x, x + \varepsilon)\) containing three points \( y_i \) \( (x < y_i < x + \varepsilon, \ i = 1, 2, 3) \) having the property that if \( x_0 = y_i \), then \((x_n)_{n=0}^{\infty} \) converges to \( a_i \) \( (i = 1, 2, 3) \).

The polynomial \( f(x) = x^5 - c x + 1 \) has three real roots when \( c > c_0 \) and \( f''(x) = 20x^3 \) is monotone increasing for all \( x \in \mathbb{R} \), so we are in the conditions of last theorem.

Finally we have the most interesting case, showed in Figure 5, when \( c \) is between 0 and \( c_0 \), in this case \( f(x) \) has only one real root and we denote it by \( d_0 \).

Much of the motivation for the material to be presented comes from the following theorem due to Fatou [8]:

**Theorem 2 (Fatou).** Let \( R : \mathbb{C} \to \mathbb{C} \) be a rational function with a stable periodic orbit \((R^n(z))_{n=0}^{\infty} \) \((|R'(z)| < 1)\), then the orbit of at least one critical point \( \omega \) \( (R(\omega) = 0) \) converges to \( z \).

**Proof.** A complete and detailed proof of this fact can be found in [2].
This theorem has an important implication for the family of mappings $N_{f_c}(x)$, because there is not another stable periodic orbit except that of the critical point of $f_c(x)$.

Now we consider $f_c(x) = x^5 - cx + 1$, so

$$N'_{f_c}(x) = \frac{f''_c(x)f_c(x)}{(f'_c(x))^2} = \frac{20x^3f_c(x)}{(f'_c(x))^2}.$$

If $N'_{f_c}(x) = 0$ we have $x = 0$ or $f_c(x) = 0$.

As the roots of $f_c(x)$ are super-stable fixed points ($f'(x) = 0$) the only interesting critical point of $N_{f_c}$ is 0 and we denote it by $d_2$, so for the study of the iteration of $N_{f_c}$ we will start at $x_0 = d_2$.

Let us now describe the numerical experiments which were performed in the $c$-parameter plane, see Figure 6.

To investigate this behavior further, we compute the bifurcation diagram for $N_{f_c}$ with $f_c(x) = x^5 - cx + 1$, varying $c$ from 0 to $c_0$.

**Remark 3.** In Figure 6 we see that there is a sequence of “windows” where $N_{f_c}(d_2)$ converges to a stable periodic orbit with period $n$ ($n \in \mathbb{N}$), intercalated with intervals where the critical point $d_2$ converges to the fixed point $d_0$.

Until now we have studied the case with only two coefficients in the quintic polynomials (in such a case we have at most three roots).

Now we refer to the other two cases, i.e.:

1. the quintic polynomial with four distinct real roots, one of them double;
2. the quintic polynomial with five distinct real roots.

Now we recall the follow result:

**Theorem 3** (Barna [11]). *If $f$ is a real polynomial having all real roots and at least four distinct ones, then the set of initial values for which Newton’s method does not yield to a root of $f$ is homeomorphic to a Cantor set. The set of exceptional initial values $J(f)$ is of Lebesgue measure zero.*
Proof. For a proof we refer to Hurley and Martin [12]. They all give modern proof of Barna’s result [1]. The underlying idea is to show that the set $J(f)$ arises in a manner which is very similar to the usual Cantor set construction. Wong proves this result using symbolic dynamics [21].

In the next sections we concentrate in the most interesting case $f_c(x) = x^5 - cx + 1$ for $c \in ]0, c_0[.$

3. Symbolic dynamics

Kneading theory is an appropriate tool to classify topologically the dynamics of maps.

First we introduce the symbolic dynamics for the map $N_{f_c}$ where $f_c(x) = x^5 - cx + 1$, and $0 < c < c_0$.

We consider the alphabet $A = \{A, B, L, C, M, R\}$, and the set $Ω = A^{\mathbb{N}_0}$ of symbolic sequences on the elements of $A$. Now we introduce the map

$$i_c : \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}_0} N_{f_c}^{-n}(\{d_1, d_3\}) \to Ω$$

defined by
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$$i_c(x)_m = \begin{cases} 
A & \text{if } N_{f_c}^m(x) < d_0 \\
B & \text{if } d_0 < N_{f_c}^m(x) < d_1 \\
L & \text{if } d_1 < N_{f_c}^m(x) < d_2 \\
C & \text{if } N_{f_c}^m(x) = d_2 \\
M & \text{if } d_2 < N_{f_c}^m(x) < d_3 \\
R & \text{if } N_{f_c}^m(x) > d_3 
\end{cases}$$

as we can see in Figure 7.

Figure 7. Symbolic dynamic for $N_{f_c}$ with $f_c = x^5 - cx + 1$ and $0 < c < c_0$.

If we now consider the shift operator $\sigma : \Omega \to \Omega$, $\sigma(X_0X_1X_2...) = X_1X_2X_3..$, we have the commutative diagram

\[
\begin{array}{c}
\Lambda \\
N_{f_c} \\
i_c \\
\downarrow \\
\Sigma \\
\downarrow \\
\Omega \\
\sigma
\end{array}
\]

where

$$\Lambda = \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} N_{f_c}^{-n}(\{d_1, d_3\}).$$

We introduce in $\Omega$ an order, induced lexicographically by the order in $\mathbb{R}$, with parity introduced by the subintervals where the function is decreasing, $A < B < L < C < M < R$ when it is even and $-R < -M < C < -L < -B < -A$ when it is odd.
DEFINITION 1. We say that \( X \prec Y \) for \( X, Y \in \Omega \), iff:

\[
\exists k : X_i = Y_i, \forall 0 \leq i < k \quad \text{and} \quad (-1)^{n_{BL}(X_1...X_{k-1})}X_k < (-1)^{n_{BL}(X_1...X_{k-1})}Y_k
\]

where \( n_{BL}(X_1...X_{k-1}) \) is equal to the number of times that the symbols \( B \) or \( L \) appear in \( X_1...X_{k-1} \).

EXAMPLE 1. \( MRRM... \prec MRRR... \) and \( RLRA... \succ RLRR... \)

PROPOSITION 3. Let \( x, y \in \Lambda \). Then

(i) \( x < y \Rightarrow i_c(x) \preceq i_c(y) \)

(ii) \( i_c(x) \prec i_c(y) \Rightarrow x < y \).

PROOF. It is sufficient to adapt the proof given to this end in Milnor and Thurston \[14\].

We define the kneading sequence of the orbit of the critical point \( x = d_2 \) by

\[
i : J \to \Omega \quad \text{and} \quad c \mapsto \sigma(i_c(d_2)).
\]

with

\[
J = \{ c : c \in [0, c_0[ \quad \text{and} \quad c \text{ is such that } \bigcup_{n \in \mathbb{N}_0} N_{f_c}^p(d_2) \cap \{d_1, d_3\} = \emptyset \}.
\]

Also we define the kneading sequences of the orbits of the discontinuous point \( d_1 \) (respectively \( d_3 \)) by \( \sigma(i_c(d_1)) \) (respectively \( \sigma(i_c(d_3)) \)). We denote by \( (U, X, Y, Z) \) the kneading data \( (\sigma(i_c(d_0)), \sigma(i_c(d_1)), \sigma(i_c(d_2)), \sigma(i_c(d_3))) \).

Now we characterize the admissible sequences looking at the typical graph of \( N_{f_c}^p \) (see Figure 7). We get the following transition matrix \( T \) where rows and columns are labeled by the elements of \( A \).

\[
T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

(3.1)

Then as the critical point \( d_2 \) is a local minimum, we get the following admissibility

\[
\begin{align*}
\sigma^i(Y)_1 &= A \Rightarrow \sigma^{i+1}(Y) = A^\infty \\
\sigma^i(Y)_1 &= B \Rightarrow \sigma^{i+1}(Y) = A^\infty \\
\sigma^i(Y)_1 &= L \Rightarrow \sigma^{i+1}(Y) \geq Y \\
\sigma^i(Y)_1 &= M \Rightarrow \sigma^{i+1}(Y) \geq Y
\end{align*}
\]

(3.2)

Let the set \( \Omega^+ = \{ Y \in A^\infty : Y \text{ verifies } T_{Y_i, Y_{i+1}} = 1 \} \). We call \( \Omega^+ \) the set of admissible sequences.
Example 2. To see the admissibility we must pay attention to the fact that the critical point \( d_2 \) is a local minimum and, in such case, if we have \( \sigma^i(Y)_1 = L \) or \( \sigma^i(Y)_1 = M \) (where \( Y \) is a periodic sequence with period \( n \) of the critical point \( d_2 \), \( 1 \leq i < n \)) then we must have \( \sigma(Y) > Y \). So the sequence \( (RLRC)^\infty \) is admissible – its occurrence can be seen in Figure 7 near \( c = 1.3346... \) – while the sequences \( (LMAC)^\infty \) and \( (RMRC)^\infty \) are not admissible for the same reason.

![Tree T_Y of symbolic sequences corresponding to d_2 in the inverse order. The kneading sequences are marked with a circle when the last symbol is C and a square when the last symbol is A.](image)

In a similar way to what is made by Milnor and Thurston in [14] we define the kneading increments associated to the kneading data by

\[
\nu_{d_i} = \theta(d_i^+) - \theta(d_i^-) \quad \text{with} \quad i = 0, 1, 2, 3
\]

where \( \theta(x) \) is the invariant coordinate of each symbolic sequence associated to the itinerary of each point \( d_i \), see [14].

Using this we define the kneading matrix \( N(t) \) and the kneading determinant

\[
D(t) = \frac{(-1)^i + 1 D_i(t)}{(1 - \varepsilon_i t)} = \frac{d_Y(t)}{(1 - t)(1 - t^k)}
\]

where \( D_i(t) \) is the determinant of \( N(t) \) without the column \( i \) and the cyclotomic polynomials in the denominator correspond to the stable periodic orbits of \( d_0 \) and \( d_2 \), see [14].

Example 3. We exemplify the kneading increment for the sequence \( RLRC \). We have

\[
\theta(d_0^+) = B - At - At^2 - ... = B - At(1 + t + t^2 + ...) = B - \frac{At}{1 - t}
\]
and
\[ \theta(d^{-}_0) = A + At + At^2 + ... = A(1 + t + t^2 + ...) = \frac{A}{1 - t}, \]
consequently
\[ \nu_d = \theta(d^+_0) - \theta(d^-_0) = B - A \frac{1 + t}{1 - t}. \]

Similarly, we get
\[ \theta(d^+_1) = L - Rt - Rt^2 - ... = L - (1 + t + t^2 + t^3 + ...) \quad \frac{Rt}{1 - t} \]
and
\[ \theta(d^-_1) = B - At - At^2 - ... = B - (1 + t + t^2 + t^3 + ...) \quad \frac{At}{1 - t}. \]
consequently
\[ \nu_{d_1} = \theta(d^+_1) - \theta(d^-_1) = L - B + \frac{(A - R)t}{1 - t}. \]

Similarly, we get
\begin{align*}
\theta(d^+_2) &= M + Rt + Lt^2 - Rt^3 - Lt^4 + ... \\
&= M + (1 + t^4 + t^8 + ...)(Rt + Lt^2 - Rt^3 - Lt^4) \\
&= M + \frac{Rt + Lt^2 - Rt^3 - Lt^4}{1 - t^4}
\end{align*}
and
\begin{align*}
\theta(d^-_2) &= L - Rt - Lt^2 + Rt^3 + Lt^4 - ... \\
&= L - (1 + t^4 + t^8 + ...)(Rt + Lt^2 - Rt^3 - Lt^4) \\
&= L - \frac{Rt + Lt^2 - Rt^3 - Lt^4}{1 - t^4},
\end{align*}
consequently
\[ \nu_{d_2} = \theta(d^+_2) - \theta(d^-_2) = M - L + \frac{2(Rt + Lt^2 - Rt^3 - Lt^4)}{1 - t^4}. \]

Similarly, we get
\[ \theta(d^+_3) = R + At + At^2 + ... = R + (1 + t + t^2 + t^3 + ...) \quad \frac{At}{1 - t} \]
and
\[ \theta(d^-_3) = M + Rt + Rt^2 + ... = M + (1 + t + t^2 + t^3 + ...) \quad \frac{Rt}{1 - t}. \]
consequently
\[ \nu_{d_3} = \theta(d^+_3) - \theta(d^-_3) = R - M + \frac{(A - R)t}{1 - t}. \]
So we have the kneading matrix

\[
N(t) = \begin{bmatrix}
-\frac{1+t}{t-1} & 1 & 0 & 0 & 0 \\
\frac{1}{t-1} & -1 & 1 & 0 & -\frac{t}{t-1} \\
0 & 0 & -1 + \frac{2t^2 - 2t^4}{1-t^2} & 1 & \frac{2t^2 - 2t^3}{1-t^4} \\
\frac{t}{1-t} & 0 & 0 & -1 & 1 - \frac{t}{t-1}
\end{bmatrix}.
\]

With \( i = 2 \) we have \( \varepsilon_2 = -1 \) (because \( N'_f(x)|_{[d_0,d_1]} < 0 \))

\[
D(t) = \frac{(-1) D_2(t)}{1+t} = \frac{(1+t)(1-t-t^2-t^3)}{(1-t)(1-t^4)}.
\]

Next we denote by \( d_Y(t) \) the numerator of \( D(t) \) given by \( D(t) (1-t) (1-t^k) \), where \( k \) is the period of the critical point \( d_2 \) and \( Y \) is the kneading sequence associated to \( d_2 \). Each kneading data determines a kneading determinant but the most significant factor of the numerator is determined by the kneading sequence \( Y \).

It is easy to see the following result

**Proposition 4.** To the set \( \Omega^+ \) of the ordered kneading sequences can be associated the tree \( T_Y \), where in each \( k \)-level of the tree are localized kneading sequence of \( k \) length.

It is easy to see the following result

**Corollary 1.** To \( T_Y \) we associate a tree \( T_{d_Y(t)} \) of the numerators of kneading determinant.

To proof this corollary we need the following lemma.

**Lemma 1.** Let \( Y \) be an admissible periodic sequence corresponding to orbit of the critical point \( d_2 \) of period \( k \) whose the numerator of the kneading determinant \( d_Y(t) \), now we designate by \( d_k(t) \). Then \( d_k(t) \) has degree \( n = k \) and the polynomials correspondent to the periodic sequences of period \( k+1 \) (\( k+1 \)-level of the tree) follow the rule of construction:

\[
(1-t)d_k(t) = 1 - t + a_2 t^2 + a_3 t^3 + \ldots + a_k t^{k-1} - \delta t^k - \delta t^{k+1} = p(t) + q(t)
\]

\[
A \quad \downarrow \quad L \quad \downarrow \quad M \quad \downarrow \quad R
\]

\[
(1-t)d_{k+1}(t) = p(t) - 2\delta t^k; \quad p(t) + 2\delta t^k; \quad p(t) - 2\delta t^{k+1}; \quad p(t) + tq(t)
\]

with \( a_k \in \{-2,0,2\} \) and \( \delta = (-1)^{n_L} \) where \( n_L \) is equal to the number of times that the symbol \( L \) appears in the symbolic sequence \( Y \).

**Proof.** Computing the kneading determinants of the sequences in each level \( k \) of the set \( \Omega^+ \) of the ordered kneading sequences and analyzing the passage from level \( k \) to level \( k+1 \) and using the induction it goes out the rule of the indicated construction. \( \blacksquare \)
Remark 4. If we want to see the symbolic dynamics in the tree for the converging points in the Newton's method, it is of ramifications ending with $A^\infty$. In this case $d_Y(t)$ stays constant after reaching the first symbol $A$ and we note when it reaches the symbol $A$ it is $A$ forever.

Theorem 4. Let $P$ and $Q$ be kneading sequences in $\Omega^+$ with the lexicographic order $\prec$. If $P \prec Q$ then $c_P > c_Q$, where $c_P$ (respectively $c_Q$) is the parameter value corresponding to the kneading sequence $P$ (respectively $Q$). If $c_1 > c_2$ then there are $P, Q \in \Omega^+$ with $P \preceq Q$, where $P$ (respectively $Q$) is the kneading sequence realized by the parameter value $c_1$ (respectively $c_2$).

Proof. It is sufficient to extend the Tsujii results on the quadratic map to the Newton map. Then we prove that the kneading sequence $P$ associated to the orbit of the critical point $d_2$ (minimum) is monotone decreasing with respect to parameter $c$.

4. Topological entropy

In the known paper by Misiurewicz and Szlenk [15] the topological entropy is determined by

$$h_{top}(N_{f_c}) = \log s(N_{f_c}),$$

where $s(N_{f_c}) = \lim_{k \to \infty} (L_k)^{1/k}$, $L_k$ is the number of laps of $N_{f_c}^k$, i.e., the numbers of sub-intervals where $N_{f_c}^k(x)$ is monotone (see also [14]).

When the orbit of the critical point $d_2$ of $N_{f_c}(x)$ is periodic we have a Markov partition which is determined by the itineraries of the critical point. Once we have the Markov partition, a subshift of finite type is determined by the transition matrix. Given a Markov partition $P = \{I_j\}_{j=1}^m$, the transition matrix $M = (a_{ij})$ of the type $(n \times n)$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \text{int}(N_{f_c}(I_i) \cap I_j) \neq \emptyset \\ 0 & \text{if } \text{int}(N_{f_c}(I_i) \cap I_j) = \emptyset \end{cases}.$$ 

Like in [13] the topological entropy $h_{top}(N_{f_c})$ is obtained from the smallest real root $t^*$ of $d_M(t)$, $t^* \in [\sqrt{2} - 1, 1]$, where $d_M(t) = \det(I-tM)$ is the characteristic polynomial of the transition matrix $M$. The value $t^* = \sqrt{2} - 1$ corresponds to $c = c_0$, which occurs to the kneading sequence $M^\infty$.

Also we can compute $h_{top}(N_{f_c}) = \log 1/t^*$, where $t^*$ is the minimal solution of $D(t^*) = 0$, with

$$D(t) = \frac{d_Y(t)}{(1-t)(1-t^k)} = \frac{d_M(t)}{(1-t)(1-t^k)(1-t)^2}$$

the kneading determinant, see [14], [13].

Remark 5. The tree $T_{d_Y(t)}$ shows the relation between the symbolic sequences, namely the characteristic polynomials of the transition matrix $M$, and the topological entropy for each parameter $c$ of the family of Newton map $N_{f_c}$ for the quintic $f_c(x) = x^5 - c x + 1$, with $c$ between 0 and $c_0$. 

If we magnify the bifurcation diagram near \( c = 1.3342 \ldots \) we can see in Figure 9 the cascade beginning with period two on left and period duplication.

\[ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \]

and by formula (4.1) we calculate \( h_{\text{top}}(\mathcal{N}_{j_{1,3342\ldots}}) = \log 1.83929 \ldots \)

We have computed more examples and we plot in Figure 11 the graph of the entropy for \( c \in ]0, c_0[. \)
Corollary 2. Let $N_{f_c}$ be the Newton map associated to the quintic map $f_c(x) = x^5 - c x + 1$, with $c \in ]0, c_0]$. The topological entropy of $N_{f_c}$ is a non-decreasing function with respect to the parameter $c$. The topological entropy varies between 0 and $\log(1 + \sqrt{2})$.

Proof. This result is a consequence of the order and admissibility defined before in the kneading sequence set $\Omega^+$ and of Theorem [4]. In the extreme is the kneading sequence $M^\infty$, where $s(N_{f_c}) = 1 + \sqrt{2}$.

A next step for the future would be to study the types of bifurcation and the Hausdorff dimension of the set of badly initial points, that is the set of points not convergent to any root.

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Facultad de Matemáticas, Universidad de Murcia Campus de Espinardo, 30100 Murcia, Spain
E-mail address: balibrea@um.es

DEPT. OF MATHEMATICS AND ENGINEERING, UNIVERSITY OF MADEIRA, CAMPUS UNIVERSITÁRIO DA PENTEADA, 9000-390 FUNCHAL, PORTUGAL
E-mail address: orlando@uma.pt

Department of Mathematics, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisbon, Portugal
E-mail address: sramos@math.ist.utl.pt
URL: http://www.math.ist.utl.pt/~sramos