ON SINGULAR FINSLER FOLIATION

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Abstract. In this paper we introduce the concept of singular Finsler foliation, which generalizes the concepts of Finsler actions, Finsler submersions and (regular) Finsler foliations. We show that if $F$ is a singular Finsler foliation with closed leaves on a Randers manifold $(M, Z)$ with Zermelo data $(h, W)$, then $F$ is a singular Riemannian foliation on the Riemannian manifold $(M, h)$. As a direct consequence we infer that the regular leaves are equifocal submanifolds (a generalization of isoparametric submanifolds) when the wind $W$ is an infinitesimal homothety of $h$ (e.g. when $W$ is killing vector field or $M$ has constant Finsler curvature).

We also present a slice theorem that locally relates singular Finsler foliations on Finsler manifolds with singular Finsler foliations on Minkowski spaces.

1. Introduction

Let $(M, F)$ be a Finsler manifold; recall definition in Section 2. A partition $\mathcal{F} = \{L\}$ of $M$ into connected immersed smooth submanifolds (the leaves) is called a singular Finsler foliation if it satisfies the following two conditions:

(a) $\mathcal{F}$ is a singular foliation, i.e., for each $v \in T_p L$ there exists a smooth vector field $X$ tangent to the leaves so that $X(p) = v$;

(b) $\mathcal{F}$ is Finsler, i.e., if a geodesic $\gamma : (a, b) \to M$, with $0 \in (a, b) \in \mathbb{R}$, is orthogonal to the leaf $L_{\gamma(0)}$; (i.e., $g_{\gamma(0)}(\dot{\gamma}(0), v) = 0$ for each $v \in T_{\gamma(0)} L$), then $\gamma$ is horizontal, i.e., orthogonal to each leaf it meets. Here, $g_{\gamma}$ denotes the fundamental tensor associated with the Finsler metric $F$; see Section 2.1.

As we will see in Lemma 3.7, part (b) above is equivalent to saying that the leaves are locally equidistant, where the distance between the plaque $P_x$ and the plaque $P_y$ does not need to be the same as the distance between the plaque $P_y$ and $P_x$.

A typical example of a singular Finsler foliation is the partition of $M$ into orbits of a Finsler action. Recall that an action $\mu : G \times M \to M$ on a Finsler manifold $(M, F)$ is called a Finsler action if $F \circ d\mu^g(\cdot) = F(\cdot)$ for every $g \in G$, i.e., if the action preserves the Finsler metric. For example, an isometric action $G \times M \to M$ on a Riemannian manifold $(M, h)$ is also a Finsler action for the Zermelo metric with initial data $(h, W)$, where $W$ is a $G$-invariant vector field, recall Lemma 2.7.

For more examples and results on Finsler actions, see [16] and [17].

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Another class of examples is given by (regular) Finsler foliations. In particular the foliations given by the fibers of Finsler submersions; see [9] and [16], section 2.4. Recall that a submersion \( \pi : (M, F^1) \to (B, F^2) \) between Finsler manifolds is a **Finsler submersion** if for all \( p \in M \) we have \( \pi_p(B_p(0,1)) = B_{\pi(p)}(0,1) \), where \( B_p(0,1) \) and \( B_{\pi(p)}(0,1) \) are the unit balls of \( (T_p M, F^1_p) \) and \( (T_{\pi(p)} B, F^2_{\pi(p)}) \), respectively; see also Corollary 3.13.

Recently Radeschi [30] constructed infinitely many examples of non-homogeneous singular Riemannian foliations on spheres using polynomial maps. More precisely he generalized the construction of Ferus, Karcher and Münzner [18] constructing Clifford foliations \( F_C \) on spheres with metric \( h_1 \) (the round metric of constant sectional curvature equal to 1). It is not difficult to produce Randers metrics \( Z \) with Zermelo data \( (h_1, W) \) so that the foliations \( F_C \) turn to be non-homogeneous singular Finsler foliations with respect to \( Z \), see Example 2.13. More generally, as briefly explained in Remark 2.14 if we start with a singular Riemannian foliation \( F \) on \( (M, h) \) and an \( F \)-foliated vector field \( W \) (i.e., its flow takes leaves to leaves), the foliation \( F \) turns to be a singular Finsler foliation on the Randers space \( (M, Z) \), with Zermelo data \( (h, W) \).

This leads us to the following question: **may one find a Riemannian metric \( g \) for each singular Finsler foliation \( F \) on \( (M, F) \) so that \( F \) turns to be a singular Riemannian foliation on \( (M, g) \)?** For regular foliations, this question has already been answered positively, see [26] [27] [29]. Very roughly speaking the idea in the regular case was to locally describe the regular foliations by submersions and to produce the smooth metric by averaging the fundamental tensors on the base spaces of these submersions. In the singular case one should be quite more careful to produce a smooth metric, because the local description of singular foliations is not so trivial as in the regular case. Our first main result gives a positive answer to the above question in the case of Randers spaces.

**Theorem 1.1.** Let \( F = \{L\} \) be a singular Finsler foliation with closed leaves on a Randers manifold \( (M, Z) \) with Zermelo data \( (h, W) \). Then \( F \) is a singular Riemannian foliation on the Riemannian manifold \( (M, h) \). In addition the wind \( W \) is an \( F \)-foliated vector field.

We say that the wind \( W \) of a Randers space \( (M, Z) \) with Zermelo data \( (h, W) \), is an infinitesimal homothety of \( h \) if \( \mathcal{L}_W h = -\sigma h \), where \( \mathcal{L}_W \) is the Lie derivative in the direction of \( W \) and \( \sigma \) is a constant. In this particular, but important class of Randers space one knows relations between geodesics of \( h \) and the geodesics of \( Z \), see e.g. [31]. This already suggests that the above theorem could be useful to infer a property (namely equifocality) whose Riemannian counterpart has been playing a fundamental role in the theory of Singular Riemannian Foliations.

Given a singular foliation \( F \) on a complete Finsler space \( M \), we say that a regular leaf \( L \) of \( F \) is an **equifocal submanifold**, if for each \( p \in L \), there exists a neighborhood \( U \subset L \) of \( p \) so that for each basic vector field \( \xi \) along \( U \) the **(future) endpoint map** \( \eta_{\xi} : U \to M \), defined as \( \eta_{\xi}(x) = \exp_x(\xi) \), has derivative with constant rank. In addition \( \eta_{\xi}(U) \subset L_q \) where \( q = \xi p \).

The concept of equifocal submanifold was introduced by Terng and Thorbergsson [32] to generalize the concept of isoparametric submanifolds. As proved in [8] each regular leaf of a singular Riemannian foliation on a complete Riemannian manifold is equifocal. The equifocality has been used in the study of topological properties (see e.g. [1] [3]), metric properties (see e.g. [2] [6]) and semi-local dynamical behavior.
(see e.g. [7]) of singular Riemannian foliations. It also plays a relevant role in the Wilking’s proof of the smoothness of Sharafutdinov projection, i.e., the metric projection into the soul of an open non negative curved space; see [19, 33].

**Corollary 1.2.** Assume that the wind $W$ of a Randers space $(M, Z)$ with Zermelo data $(h, W)$ is an infinitesimal homothety of $h$. Also assume that $h$ and $W$ are complete. Let $F$ be a singular Finsler foliation with closed leaves on $(M, Z)$. Then the regular leaves of $F$ are equifocal.

In order to prove Theorem 1.1 we present in Propositions 3.15 and 3.18 a *slice reduction* that locally relates singular Finsler foliations on Finsler manifolds with singular Finsler foliations on Minkowski spaces. More precisely we prove the next result:

**Theorem 1.3** (*Slice theorem*). Let $F = \{L\}$ be a singular Finsler foliation with closed leaves on a Finsler manifold $(M, F)$. Then, given a point $q \in M$, there exists a slice $S_q$ transversal to $L_q$ (i.e. $T_qM = T_qS_q \oplus T_qL_q$) and a Finsler metric $\hat{F}$ on $S_q$ with the following properties:

(a) the leaves of the slice foliation $F_q = F \cap S_q$ endow $(S_q, \hat{F})$ of a structure of singular Finsler foliation;

(b) the distance between the leaves of $F_q$ (with respect to $\hat{F}$) and the distance between the leaves of $F$ (with respect to $F$) coincide locally;

(c) the slice foliation $F_q$ on $(S_q, \hat{F})$ is foliated-diffeomorphic to a singular Finsler foliation on an open subset of Minkowski space.

**Remark 1.4.** We stress that both theorems and the corollary are proved under a hypothesis weaker than having closed leaves, namely, the leaves are locally closed (see Definition 3.17).

This paper is organized as follows. In Section 2 we fix some notations and briefly review a few facts about Finsler Geometry. In section 3 we sum up a few results on singular Finsler foliations that are analogous to the classical results on singular Riemannian foliations. In particular we prove Propositions 3.15 and 3.18 which are collected above in Theorem 1.3. In Section 4 we prove Theorem 1.1 and Corollary 1.2.

This paper establishes a connection between the theory of singular foliations and Finsler Geometry. This leads us to the natural question about how much the reader should know about these two topics. In this presentation we tried to balance brevity of explanations with the amount of prerequisites that the reader needs. In this way we hope to have made the main ideas accessible to at least two kinds of readers, those who know Finsler geometry but may not have previous experience with singular foliations and those with experience in the theory of singular foliations but without previous contact with Finsler geometry.

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2. Preliminaries

In this section we fix some notations and briefly review a few facts about Finsler Geometry which will be used in this paper. For more details see [9, 12, 22, 34].

2.1. Finsler metrics, fundamental tensor and orthogonal cone. Let us begin by introducing the concepts at the level of vector spaces. Let $V$ be a vector space and $F : V \to [0, +\infty)$ a function. We say that $F$ is a Minkowski norm if

1. $F$ is smooth on $V \setminus \{0\}$,
2. $F$ is positive homogeneous of degree 1, that is $F(\lambda v) = \lambda F(v)$ for every $v \in V$ and $\lambda > 0$,
3. for every $v \in V \setminus \{0\}$, the fundamental tensor $g$ of $F$ defined as

$$g_v(u, w) = \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(v + tu + sw)|_{t=s=0}$$

for any $u, w \in V$ is a positive-definite bilinear symmetric form. We will say that $(V, F)$ is a Minkowski space.

Let us consider a manifold $M$. We say that a function $F : TM \to [0, +\infty)$ is a Finsler metric if

1. $F$ is smooth on $TM \setminus \{0\}$,
2. for every $p \in M$, $F_p = F|_{T_p M}$ is a Minkowski norm on $T_p M$.

In such a case, if $\tau : TM \to M$ is the natural projection and $v \in TM \setminus \{0\}$, we will say that the fundamental tensor $g_v$ of $F$ is the one defined in (2.1) for $F_{\tau(v)}$. Let us now sum up some properties of the fundamental tensor.

**Lemma 2.1.** Let $(M, F)$ be a Finsler manifold and, given $v \in TM \setminus \{0\}$, let $g_v$ be the fundamental tensor defined in (2.1). Then

(a) $g_{\lambda v} = g_v$ for $\lambda > 0$,
(b) $g_v(v, v) = F^2(v)$,
(c) $g_v(v, u) = \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(v + zu)|_{t=s=0}$.

**Remark 2.2.** Part (c) above implies in particular that the Legendre transformation $\mathcal{L} : TM \setminus \{0\} \mapsto TM^* \setminus \{0\}$ associated with $\frac{1}{2} F^2$ can be computed as $\mathcal{L}(v) = g_v(v, \cdot)$. Also recall that $\mathcal{L} : TM \setminus \{0\} \mapsto TM^* \setminus \{0\}$ is a diffeomorphism.

We will also need the concept of orthogonal cone. Given a submanifold $P$ on a Finsler manifold $(M, F)$ we say that a vector $v \in T_q M$ is orthogonal to $P$ at $q$ if $g_v(v, u) = 0$ for all $u \in T_q P$. The set of all (non zero) normal vectors to $P$ at $q$, denoted as $\nu_q P$ is called the orthogonal cone and, as the name suggests, it is not always a subspace but a cone. In particular, it is a submanifold of $T_q M$ of dimension $n - \dim P$. Moreover, $\nu(P)$ will denote the space of all orthogonal vectors to $P$ and $\nu^1(P)$, the space of unit orthogonal vectors.

**Proposition 2.3.** Let $(M, F)$ be a Finsler manifold and $P$ a submanifold of $M$. Then $\nu(P)$ and $\nu^1(P)$ are smooth submanifolds and the restrictions of the canonical projection $\tau : \nu(P) \to P$ and $\tau : \nu^1(P) \to P$ are submersions.

**Proof.** Let $\mathcal{N}(P)$ be the submanifold of $TM^* \setminus \{0\}$ of the annihilators of $P$ and $\tau^* : TM^* \to M$ the canonical projection. Then $\nu(P) = (\mathcal{L}^* - 1)(\mathcal{N}(P))$ (recall that the Legendre transformation is a diffeomorphism, Remark 2.2) and then it is a submanifold. Moreover, $\nu^1(P) = \nu(P) \cap T^1 P$, where $T^1 P$ is the $F$-unit bundle of $P$. As $\nu(P)$ and $T^1 P$ are transversal, we conclude that $\nu^1(P)$ is a submanifold.
Moreover, it is straightforward to check that $\tau^*|_{\mathcal{N}(P)} : \mathcal{N}(P) \to P$ is a submersion and $\tau : \nu(P) \to P$ is the composition $\tau^*|_{\mathcal{N}(P)} \circ \mathcal{L}|_{\nu(P)}$, and hence, $\tau^*|_{\nu(P)}$ is a submersion. That $\tau : \nu^1(P) \to P$ is a submersion follows using that $\tau : \nu(P) \to P$ is a submersion and the fact that the tangent space to $\nu(P)$ can be decomposed as the sum of the tangent space to $\nu^1(P)$ and a vertical part. 

**Proposition 2.4.** Let $(V, F)$ be a Minkowski space, $W \subseteq V$ a linear subspace of $V$. Then $T_{\nu_0}(W) = \{ u \in V : g_v(u, w) = 0 \forall w \in W \}$, where $g$ is the fundamental tensor of $F$.

**Proof.** Recall that the Cartan tensor of $F$ can be defined as

$$C_v(w_1, w_2, w_3) = \frac{1}{2} \frac{\partial}{\partial z} g_{v + zw}(w_2, w_3)$$

for $v \in V \setminus 0$ and $w_1, w_2, w_3 \in V$. Given $u \in T_{\nu_0}(W)$, there exists a smooth curve $\alpha : (-\epsilon, \epsilon) \to \nu_0(W)$ for some $\epsilon > 0$, such that $u = \alpha'(0)$ and $v = \alpha(0)$. In particular, we have that $g_{\alpha(t)}(\alpha(t), w) = 0$ for all $t \in (-\epsilon, \epsilon)$ and $w \in W$. Moreover,

$$\frac{d}{dt} g_{\alpha(t)}(\alpha(t), w) = 2C_{\alpha(t)}(\alpha'(t), \alpha(t), w) + g_{\alpha(t)}(\alpha'(t), w)$$

and by homogeneity, the first term on the right side is zero (last formula is a very particular case of the compatibility of the Chern connection with $g$, but as we are in a tangent space, Chern connection can be indentified with a standard derivative). Therefore, it follows that $g_v(u, w) = 0$ for all $w \in W$. This implies the inclusion $" \subseteq "$ and as the dimensions are the same, they coincide. 

### 2.2. Geodesics and the exponential map

Given a Finsler manifold $(M, F)$, geodesics can be defined as critical points of the energy functional

$$E_F(\gamma) = \frac{1}{2} \int_a^b F(\dot{\gamma})^2 ds$$

defined for piecewise smooth curves $\gamma : [a, b] \to M$ with fixed endpoints. Alternatively, they can also be defined with one of the connections associated with Finsler metrics, [12] §2.4. In particular, observe that the Chern connection can be interpreted as a family of affine connections (see for example [22, 25]). As a matter of fact, for every vector $v \in TM$, there exists a unique maximal geodesic $\gamma_v : (a, b) \to M$ such that $\gamma_v(0) = v$ and one can define the exponential map in an open subset $\mathcal{U} \subset TM$ for those vectors $v$ such that the maximal interval of definition $(a, b)$ of $\gamma_v$ includes the value 1. Then $\exp : \mathcal{U} \to M$ is defined as $\exp(v) = \gamma_v(1)$ and it is smooth away from the zero section and $C^1$ on the zero section [12] §5.3. In particular, this allows one to show that geodesics minimize the distance associated with Finsler metrics in some interval. More generally, geodesics minimize (in some interval) the distance with any submanifold $P$ whenever they are orthogonal to $P$.

### 2.3. Zermelo and Randers metric

Let us now recall a central example of Finsler metrics that appears naturally in several applications; see e.g., [14, 15]. These metrics can be described in several ways. Classically, a Randers metric in a manifold $M$, $R : TM \to [0, +\infty)$, is defined as $R(v) = \sqrt{a(v, v)} + \beta(v)$, where $a$ is a Riemannian metric in $M$ and $\beta$, a one-form such that $|\beta|_a < 1$ at every point. Here the norm $\| \cdot \|_a$ is computed using the norm of the Riemannian metric $a$. On the other hand, a Finsler metric $Z : TM \to [0, +\infty)$ is said to be a Zermelo metric.
with Zermelo Data $(h, W)$, for a Riemannian metric $h$ and smooth vector field $W$ with $h(W, W) < 1$ on the whole $M$ (the wind), if $Z$ is the solution of

$$h\left(\frac{v}{Z(v)} - W, \frac{u}{Z(v)} - W\right) = 1, v \in TM \setminus 0.$$ 

Denote $I^Z_p$ the indicatrix of $Z$ at $p \in M$, i.e. $I^Z_p = \{v \in T_p M; Z(v) = 1\}$. The above equation is equivalent to saying that $I^Z_p = I^h + W$. It turns out that both types of metrics, Randers and Zermelo, are different ways of describing the same family (see [11 §1.3] and also [13 Prop. 3.1]).

**Lemma 2.5.** Let $Z: TM \to [0, +\infty)$ be a Finsler metric in a manifold $M$. Then the following claims are equivalent:

(a) $Z$ is a Zermelo metric with Zermelo data $(h, W)$ of a Riemannian metric $h$ and a wind $W$ with $h(W, W) < 1$ at every point;

(b) $Z$ is a Randers metric, $Z(v) = \sqrt{a(v, v) + \beta(v)}$, for a Riemannian metric $a$ and a 1-form $\beta$ with $\|\beta\|_a < 1$ at every point.

In addition, given a Zermelo data $(h, W)$, the pair $(a, \beta)$ is obtained as follows:

$$a(u, v) = \frac{\lambda h(u, w) + h(u, W)h(w, W)}{\lambda^2}, \quad \beta(w) = -\frac{h(w, W)}{\lambda},$$

where $\lambda := 1 - h(W, W)$. Moreover, if $Z$ is a Randers metric as in (b), its Zermelo data is given by

$$h(w, u) = \mu(a(w, u) - a(\beta, w)a(\beta, u)), \quad W = -\frac{\beta}{\mu},$$

where $\beta(\cdot) = a(\cdot, \beta)$ and $\mu := 1 - a(\beta, \beta)$.

Once we have recalled the equivalence between the definitions of Randers and Zermelo metrics, we can present the explicit calculation of the fundamental tensor [24, Cor. 4.17].

**Lemma 2.6.** Consider a Randers metric $Z = a + \beta = \sqrt{a} + \beta$ with Zermelo data $(h, W)$. Then its fundamental tensor is given by:

$$g_\alpha(w, u) = \frac{Z(v)}{\alpha(v)} \left[ a(w, u) - \frac{a(v, w)a(v, u)}{\alpha^2(v)} \right] + \left( \frac{\alpha(v)}{\alpha(v)} + \beta(u) \right) \frac{a(v, u)}{\alpha(v)} - \beta(u),$$

for $v, w \in T_p M$ and $v \neq 0$. In particular

$$g_\alpha(v, u) = Z(v) \left( \frac{a(v, u)}{\alpha(v)} + \beta(u) \right) = \frac{Z(v)}{\mu \alpha(v)} \left( h(v - Z(v)W, u) \right).$$

We stress that the calculation of second equation above, follows easily from Lemma 2.1. Furthermore,

**Lemma 2.7.** Let $\psi: (M_1, Z_1) \to (M_2, Z_2)$ be a diffeomorphism between two Randers spaces with $Z_i = \sqrt{a_i} + \beta_i$, with Zermelo data $(h_i, W_i)$, $i = 1, 2$. Then the following three statements are equivalent:

(a) $\psi$ lifts to an isometry between $Z_1$ and $Z_2$,

(b) $\psi^*a_2 = a_1$ and $\psi^*\beta_2 = \beta_1$,

(c) $\psi^*h_2 = h_1$ and $\psi^*W_1 = W_2$.

Finally, given a vector space $V$ and a Minkowski norm $Z$ of Randers type on it, we will say that $(V, Z)$ is a Randers-Minkowski space.
2.4. Finsler Submersions. As recalled in the introduction, a submersion between Finsler manifolds \( \pi : (M, F_1) \to (B, F_2) \) is a Finsler submersion if \( d\pi_p(B_p(0,1)) = B_{\pi(p)}(0,1) \), for every \( p \in M \), where \( B_p(0,1) \) and \( B_{\pi(p)}(0,1) \) are the unit balls of the Minkowski spaces \( (T_p M, (F_1)_p) \) and \( (T_{\pi(p)} B, (F_2)_{\pi(p)}) \), respectively. The next characterization was given in [13] Prop. 2.1.

**Lemma 2.8.** If \( \pi : (M, F_1) \to (B, F_2) \) is a Finsler submersion, then for every \( w \in T_{\pi(p)} B \) we have \( F_2(w) = \inf\{F_1(v) ; v \in T_p M \text{ and } d\pi_p(v) = w\} \).

It is clear that \( F_2(d\pi(v)) \leq F_1(v) \). The vectors for which the equality holds are said to be horizontal and form the horizontal cone of \( \pi \). The above lemma, the convexity of \( F^2 \) and Lemma 2.4 imply that to be horizontal with respect to the Finsler submersion \( \pi \) or to be orthogonal to the fibers of \( \pi \) (in the sense of Section 2.4) are equivalent concepts. Therefore the horizontal cones of \( \pi \) are equal to the orthogonal cones to the fibers of \( \pi \).

**Lemma 2.9.** Let \( \pi : V_1 \to V_2 \) be a linear submersion, with \( V_1, V_2 \) vector spaces. If \( F_1 \) is a Minkowski norm on \( V_1 \) and \( \nu(\pi)_0 \) is the subset of the \( F_1 \)-orthogonal vectors to \( \pi^{-1}(0) \), then

(a) the map \( \varphi := \pi|_{\nu(\pi)_0} : \nu(\pi)_0 \to V_2 \setminus \{0\} \) is a diffeomorphism which can be extended continuously to zero and it is positive homogeneous,

(b) the function \( F_2(v) = F_1(\varphi^{-1}(v)) \) is the unique Minkowski norm on \( V_2 \) that makes \( \pi : (V_1, F_1) \to (V_2, F_2) \) a linear Finsler submersion.

**Proof.** For part (a), observe that

\[
(2.3) \quad v \in \nu(\pi)_0 \text{ if and only if } \pi^{-1}(\pi(v)) \text{ is tangent to } \mathcal{I}_1(F(v))
\]

where \( \mathcal{I}_1(r) = \{ v \in V_1 : F_1(v) = r \} \). This is a consequence of part (c) of Lemma 2.4. Given \( w \in V_2 \setminus \{0\} \), let \( r = \inf\{F_1(v) : v \in \pi^{-1}(w)\} \). Then by the strong convexity of \( \mathcal{I}_1(r) \), there exists a unique \( v_r \in V_1 \) such that \( \pi(v_r) = w \) and \( F_1(v_r) = r \), which turns out to be the unique element in \( \nu(\pi)_0 \) that projects into \( w \), as \( \pi^{-1}(w) \) is transverse to every \( \mathcal{I}_1(v') \) with \( r' > r \). This proves that \( \varphi \) is one-to-one. As it is the restriction of a smooth map to a smooth submanifold (see Proposition 2.3), it is smooth. Moreover, observe that, using Proposition 2.4, it follows that the tangent space to \( \nu(\pi)_0 \) at \( v \) is given by the \( g_v \)-orthogonal space to \( \pi^{-1}(w) \), and then \( d_{\nu(\pi)_0}(w) = 0 \) for \( u \in T_v \nu(\pi)_0 \) if and only if \( u = 0 \). Applying the inverse function theorem, it follows that \( \varphi \) is a diffeomorphism. For part (b), it is enough to see that \( F_2 \) is strongly convex, which is equivalent to prove that the second fundamental form of its indicatrix is positive definite with respect to the opposite to the position vector \( \varphi^{-1} \circ \rho \) is a curve with image in \( \mathcal{I}_1 \) and

\[
(\pi \circ \tilde{\rho})''(0) = \pi(\tilde{\rho}''(0)) = \pi(\nabla^0_{\tilde{\rho}'(0)} \tilde{\rho}'(0)) - \tilde{\sigma}(\tilde{\rho}'(0), \tilde{\rho}'(0)) \rho(0),
\]

where \( \nabla^0 \) is the induced connection on \( \mathcal{I}_1 \) by the affine connection of \( V_1 \) and \( \tilde{\sigma} \) is the second fundamental form of \( \mathcal{I}_1 \) with respect to the opposite to the position vector \( \xi \). Observe that \( (2.3) \) implies that the tangent space to \( \mathcal{I}_1 \) in a horizontal vector projects into the tangent space to \( \mathcal{I}_2 \), and then \( \pi(\nabla^0_{\tilde{\rho}'(0)} \tilde{\rho}'(0)) \) is tangent to \( \mathcal{I}_2 \). Therefore we conclude that the second fundamental form of \( \mathcal{I}_2 \) with respect to the opposite of the position vector \( \xi = -\rho(0) \) is given by \( \sigma^\xi(\rho'(0), \rho'(0)) = \tilde{\sigma}(\tilde{\rho}'(0), \tilde{\rho}'(0)) \) and then \( \sigma^\xi \) is positive definite.
Lemma 2.10. Let \( \pi : M \to B \) be a submersion, \( F_1 \) a Finsler metric on \( M \) and \( S \) a submanifold of \( M \) transversal to the fibers of \( \pi \) and with the same dimension of \( B \) in such a way that \( \pi|_S : S \to B \) is a diffeomorphism. Then

(a) the subset \( \nu(\pi) \) of non zero \( F_1 \)-orthogonal vectors to the fibers of \( \pi \) is a smooth submanifold of dimension \( \dim M + \dim B \) and the restriction of the natural projection \( \tau|_{\nu(\pi)} : \nu(\pi) \to M \) is a submersion,

(b) considering at each \( x \in S \) the Minkowski norm \( (F_2)_{\pi(x)} \) on \( T\pi(x)B \) obtained in part (b) of Lemma 2.9, for \( dx_x : (T_2M, (F_1)_x) \to T\pi(x)B \), we get a Finsler metric \( \tilde{F} \) in \( S \) given at each \( x \in S \) by \( \tilde{F}_x = (F_2)_{\pi(x)} \circ dx_x \).

Proof. Let \( \mathcal{N} \) be the submanifold of \( TM^* \setminus 0 \) of the annihilators of the fibers of \( \pi \); see \cite{9} Proposition 5.2]. Then \( \nu(\pi) = \mathcal{L}^{-1}(\mathcal{N}) \), where \( \mathcal{L}(v) = g_v(v) \) is the Legendre transformation \( \mathcal{L} : TM \setminus 0 \to TM^* \setminus 0 \) associated with \( \frac{1}{2} F_1^2 \) and therefore it is a submanifold of \( TM \) of dimension \( \dim M + \dim B \) (the same as the annihilator); see \cite{9} Proposition 2.4] and Remark 2.2. Moreover, as the Legendre transformation is a global diffeomorphism that preserves the fibers and \( \tau^* : TM^* \to M \) is a submersion, being \( \tau^* : TM^* \to M \) the canonical projection, it follows that \( \tau|_{\nu(\pi)} : \nu(\pi) \to M \) is also a submersion.

In order to prove part (b), it is enough to prove that \( \tilde{F} \) is smooth. Let \( \nu(\pi)_S = \{ v \in \nu(\tilde{F}) : \tau(v) \in S \} \). Note that \( \nu(\pi)_S = \mathcal{L}^{-1}(\mathcal{N} \cap (\tau^*)^{-1}(S)) \). Therefore \( \nu(\pi)_S \) is a submanifold and in fact a subbundle of \( TS = \tau^{-1}(S) \). As in part (a), \( \tau|_{\nu(\pi)_S} : \nu(\pi)_S \to S \) is a submersion. This implies that \( \dim \nu(\pi)_S = 2 \dim(B) \) and then the only vertical vectors which are tangent to \( \nu(\pi)_S \) are those tangent to \( \nu(\pi)_S \cap T_p M \) for some \( p \in M \). Now observe that from Lemma 2.9 it follows that \( \varphi_S := d\pi|_{\nu(\pi)_S} : \nu(\pi)_S \to TB \setminus 0 \) is one-to-one and we only have to prove that it is in fact a diffeomorphism, or equivalently, that \( d\varphi_S(u) = 0 \) if only if \( u = 0 \). Observe that if \( d\varphi_S(u) = 0 \), then \( u \) is vertical, but by the above observation, it is tangent to \( \nu(\pi)_S \cap T_p M \) for some \( p \in M \). Lemma 2.9 implies that \( u = 0 \) as required.

We end this section with some remarks about Randers submersions. For the next results let us denote by \( F_W \) the Finsler metric whose indicatrix is the translation of the indicatrix of a Finsler metric \( F \) with a vector field \( W \) such that \( F(-W) < 1 \) on the whole manifold.

Lemma 2.11. Let \( \pi : (M, \tilde{F}) \to (B, F) \) be a Finsler submersion, \( W \) a vector field in \( B \) and \( \tilde{W} \) a vector field in \( M \) that projects onto \( W \), i.e., \( d\pi(\tilde{W}) = W \). Then \( \pi : (M, \tilde{F}_W) \to (B, F_W) \) is also a Finsler submersion.

Proof. As \( \pi : (M, \tilde{F}) \to (B, F) \) is a Finsler submersion,

\[
\begin{align*}
(2.4) & \quad d\pi(p)(B_{\tilde{F}_p}(0, 1)) = B_{F_p(p)}(0, 1).
\end{align*}
\]

Observe that \( B_{(\tilde{F}_W)_p}(0, 1) = B_{\tilde{F}_p}(0, 1) + \tilde{W} \) and \( B_{(F_W)_{\pi(p)}}(0, 1) = B_{F_{\pi(p)}}(0, 1) + W \). By the linearity of \( d\pi \), equation (2.4) and the relation \( d\pi(\tilde{W}) = W \), we conclude that \( d\pi(B_{(\tilde{F}_W)_p}(0, 1)) = B_{(F_W)_{\pi(p)}}(0, 1) \) as required.

We will say that the Finsler submersion \( \pi : (M, \tilde{F}_W) \to (B, F_W) \) is a translation of \( \pi : (M, \tilde{F}) \to (B, F) \).
Proposition 2.12. Let \( \pi : (M, Z_1) \to (B, Z_2) \) be a submersion between Finsler manifolds with \( Z_1 \) a Randers metric. Then \( \pi \) is Finsler if and only if \( Z_2 \) is a Randers metric and \( \pi \) is the translation of a Riemannian submersion.

Proof. Let us first show that if \( \pi : (M, Z_1) \to (B, Z_2) \) is Finsler, then \( Z_2 \) is Randers. Let \((h_1, W_1)\) be the Zermelo data of \( Z_1 \). Given \( p \in M \), let \( Q \) be the \( h_1\)-orthogonal subspace to the fibers in \( T_p M \). Then \( \psi = d\pi|_Q : Q \to T_{\pi(p)} B \) is a linear isomorphism. Consider on \( T_{\pi(p)} B \) the metric \( h_2 = (\psi^{-1})^*(h_1|_{Q \times Q}) \). Then \( d\pi_p : (T_p M, h_1) \to (T_{\pi(p)} B, h_2) \) is a linear Riemannian submersion. From Lemma 2.11 \( d\pi_p : (T_p M, Z_1) \to (T_{\pi(p)} B, Z_2) \) is a linear Finsler submersion, where \( Z_2 \) is the Randers metric with Zermelo data \((h_2, W_2)\), for \( W_2 = d\pi_p(W_1) \). Now the unicity of part (b) of Lemma 2.11 implies that \( Z_2 \) is \((Z_2)_{\pi(p)}\), and hence \( Z_2 \) is the Randers metric with Zermelo data \((h_2, W_2)\). Therefore \( W_1 \) is projectable. Applying Lemma 2.11 with \(-W_1 \) and \(-W_2 \), it follows that \( \pi : (M, h_1) \to (B, h_2) \) is a Riemannian submersion as required. The converse follows immediately from Lemma 2.11.

Example 2.13. Construction of (non-Riemannian) singular Finsler foliations which are non-homogeneous: in \([30]\) Radeschi constructed a polynomial map \( \pi_C : S^{2l-1} \to \mathbb{R}^{m+1} \) so that, for \( l > m + 1 \) and \( m \neq 1, 2, 4 \), the preimage of \( \pi_C \) are (non-homogeneous) leaves of a singular Riemannian foliation on \((S^{2l-1}, F_C)\) whose leaf space is the disk \( \mathbb{D}_C \) (i.e., the image of \( \pi_C \)). Consider a small smooth radial vector field \( W_d \) that is zero at the center and near \( \partial \mathbb{D}_C \). Let \( W \) be a basic vector field on \( S^{2l-1} \) that projects to \( W_d \). Set \( Z \) the Zermelo metric on \( S^{2l-1} \) with Zermelo data \((h_1, W)\), where \( h_1 \) is the round metric of constant sectional curvature 1. Then it follows from Proposition 2.12 that \( F_C \) turns out to be a singular Finsler foliation on \((S^{2l-1}, Z)\), which, by Lemma 2.11, has non-homogeneous leaves.

Remark 2.14. As remarked in Proposition 3.21, if \( F \) is a singular Riemannian foliation on a Riemannian manifold \((M, g)\), and if there exists a Finsler metric \( \tilde{F} \) such that \( \tilde{F} \) restricted to each stratum is a (regular) Finsler foliation, then \( \tilde{F} \) is a singular Finsler foliation on \((M, F)\). This gives us another way to construct examples. In particular if we start with a singular Riemannian foliation \( F \) on \((M, h)\) and an \( F \)-foliated vector field \( W \), then \( F \) turns to be a singular Finsler foliation on the Randers space \((M, Z)\), with Zermelo data \((h, W)\). Also Proposition 3.21 and suspension of homomorphism allow us to construct examples of singular Finsler foliations with non closed leaves; see \([4\), Chapter 5]\).

3. SOME PROPERTIES OF SINGULAR FINSLER FOLIATIONS

In this section we sum up a few properties about singular Finsler foliations. Their Riemannian counterparts can be found in \([5\) and \([28]\).

Definition 3.1. Let \((M, F = \{L\})\) be a singular foliation and \( F \) a Finsler metric on \( M \). We will say that an open subset \( P_q \) of a leaf \( L_q \) is a plaque if it is relatively compact and connected. In this case, it is possible to consider a tubular neighborhood \( U \) of \( P_q \) which we will assume to be relatively compact \([10]\). Recall that \( U \) is called a tubular neighborhood (of radius \( \epsilon \)) if exp sends \( \nu(P_q) \cap F^{-1}((0, \epsilon)) \) diffeomorphically to \( U \setminus P_q \), and all the orthogonal unit geodesics from the plaque minimize the distance from the plaque at least in the interval \([0, \epsilon]\). When necessary, we will denote it as \( O(P_q, \epsilon) \). Observe that if \( O(P_q, \epsilon) \) of \( P_q \) is a tubular neighborhood then
Remark 3.2. Note that the plaque $P_q$ does not have to be a “small” submanifold, but it could be even a compact submanifold. We will need to reduce the plaque when we stress the differential structure, see for example part (iv) of Proposition 3.12.

Remark 3.3. For every point $x \in U$ in a tubular neighborhood $U$ (resp. a reverse tubular neighborhood $\tilde{U}$), there exists a unique geodesic $\gamma^+_x : [0, r_1] \to U$ from $P_q$ to $x$ (resp. $\gamma^-_x : [-r_2, 0] \to \tilde{U}$ from $x$ to $P_q$) which is orthogonal to $P_q$. It is also possible to consider a reverse tubular neighborhood $\bar{U}$ of $P_q$, namely, a tubular neighborhood for the reverse metric $\bar{F}$, defined as $\bar{F}(v) = F(-v)$ for $v \in TM$, with analogous properties. Then convenient, we will denote it by $\bar{O}(P_x, \epsilon)$ in analogy with the straight case.

Definition 3.4. Let $\mathcal{F} = \{L\}$ be a singular foliation on a Finsler manifold $(M, F)$, $P_q$ be a plaque, and $U$ a tubular neighborhood of $P_q$. Then the function $f^+ : U \to [0, +\infty)$ given by $f^+(x) \equiv d(P_q, x)$ is continuous on its domain and smooth on $U \setminus P_q$. Analogously, if $\bar{U}$ is a reverse tubular neighborhood of $P_q$, $f^- : \bar{U} \to [0, +\infty)$, given by $f^-(x) = d(x, P_q)$, is continuous on its domain and smooth on $\bar{U} \setminus P_q$.

Moreover, define the future (resp. past) cylinder $C^+_x(P_q)$ (resp. $C^-_x(P_q)$) with axis $P_q$ as the level set of $f^+$ (resp. $f^-$) for $r \in \text{Im}(f^+) \setminus \{0\}$ (resp. $r \in \text{Im}(f^-) \setminus \{0\}$). In other words $C^+_x(P_q) = f^+_x(r)$ and $C^-_x(P_q) = f^-_{x}(r)$. As $U$ and $\bar{U}$ are tubular neighborhoods, $C^+_x(P_q)$ and $C^-_x(P_q)$ are smooth hypersurfaces.

Lemma 3.5. With the above notation, given a plaque $P_q$ and a point $x$ in a future cylinder $C^+_x(P_q)$ (resp. past cylinder $C^-_x(P_q)$), there exists a unique segment of geodesic $\gamma^+_x : [0, r_1] \to U$ (resp. $\gamma^-_x : [-r_2, 0] \to \bar{U}$) from $P_q$ to $x$ (resp. from $x$ to $P_q$) orthogonal to $P_q$ and $C^+_x(P_q)$ (resp. $C^-_x(P_q)$).

Proof. Let us prove the result for future cylinders as for past cylinders the proof is analogous using the reverse metric. If $x \in C^+_x(P_q)$, in particular $x$ belongs to the tubular neighborhood $U$, and then there exists a unique minimizing geodesic $\gamma^+_x$ from $P_q$ to $x$ of length equal to $r_1$. By the definition of $C^+_x(P_q)$, $\gamma^+_x$ also minimizes the distance from $P_q$ to $C^+_x(P_q)$. Moreover, by [14, Remark 2.1 (ii)], it follows that $\gamma^+_x$ is orthogonal to $P_q$ and $C^+_x(P_q)$.

Definition 3.6. We will say that a singular foliation is locally forward (resp. backward) equidistant if given a plaque $P_q$, a tubular neighborhood $U$ (resp. a reverse tubular neighborhood $\bar{U}$) of $P_q$ and a point $x \in U$ (resp. $x \in \bar{U}$) which belongs to the future cylinder $C^+_x(P_q)$ (resp. the past cylinder $C^-_x(P_q)$), then the plaque $P_x \subset U$ (resp. $P_x \subset \bar{U}$) is contained in $C^+_x(P_q)$ (resp. $C^-_x(P_q)$).

Lemma 3.7. A singular foliation $\mathcal{F}$ is Finsler if and only if its leaves are locally forward and backward equidistant.

Proof. Lemma 3.5 allows us to check that if the leaves of $\mathcal{F}$ are locally equidistant, then $\mathcal{F}$ is a singular Finsler foliation.

Now assume that $\mathcal{F}$ is a singular Finsler foliation, $P_q$ a plaque and $U$, a tubular neighborhood of $P_q$. Given $x_0 \in U$, since $\mathcal{F}$ is a singular Finsler foliation, for each $x \in P_{x_0}$, $\gamma^+_x$ is orthogonal to $P_{x_0}$ at $x$. On the other hand, by Lemma 3.5 $\gamma^+_x$ is also...
orthogonal to \( C^+_r(P_q) \) at \( x = \gamma^+_x(r) \). Therefore \( T_x P_{x_0} \subset T_x C^+_r(P_q) \). Since this holds for each \( x \in P_{x_0} \), we conclude that \( P_{x_0} \) must be contained in a future cylinder. Therefore the leaves of \( \mathcal{F} \) are locally forward equidistant. Analogously, one can check that it is locally backward equidistant using a reverse tubular neighborhood.

\[ \square \]

**Definition 3.8** (Homothetic transformations). Consider a plaque \( P_q \) of a singular Finsler foliation \( \mathcal{F} \) and \( O(P_q, \varepsilon) \) a tubular neighborhood of \( P_q \). Then for each \( \lambda \in (0, 1) \) (resp. \( \lambda > 1 \)) we can define the *future homothetic transformation* \( h^+_\lambda : O(P_q, \varepsilon) \to O(P_q, \lambda \varepsilon) \) (resp. \( h^+_\lambda : O(P_q, \lambda \varepsilon) \to O(P_q, \varepsilon) \)) as \( h^+_\lambda(x) = \gamma^+_x(\lambda r) \) where \( \gamma^+_x(r) = x \). In particular \( h^+_\lambda : C^+_r(P_q) \to C^+_r(P_q) \). In a similar way, given a reverse homothetic transformation \( h^-_\lambda : \hat{O}(P_q, \varepsilon) \to \hat{O}(P_q, \lambda \varepsilon) \) (resp. \( h^-_\lambda : \hat{O}(P_q, \lambda \varepsilon) \to \hat{O}(P_q, \varepsilon) \)) as \( h^-_\lambda(x) = \gamma^-_x(-\lambda r) \), where \( x = \gamma^-_x(-r) \). In particular \( h^-_\lambda : C^-_r(P_q) \to C^-_r(P_q) \).

**Lemma 3.9** (Homothetic transformation Lemma). The future and past homothetic transformations \( h^+_\lambda \) and \( h^-_\lambda \) send plaques to plaques of a singular Finsler foliation.

**Proof.** We will prove the lemma for the future homothetic transformation \( h^+_\lambda \). A similar proof is valid for the past homothetic transformation.

Let \( x_0 \) be a point in the future cylinder \( C^+_{r_0}(P_q) \) with \( r_0 < \varepsilon \). For each \( x \in P_{x_0} \) consider \( \gamma^+_x : [0, r_0] \to U \) the segment of geodesic defined in Lemma 3.5 minimizing the distance from \( P_q \) to \( x \).

First we will consider \( \lambda < 1 \) close enough to 1. In particular, we will assume that there exists a reverse tubular neighborhood \( \hat{V} \) of \( P_{x_0} \) (recall Remark 3.3) and the past cylinder \( C^-_{r_2}(P_{x_0}) \subset \hat{V} \) for \( r_2 = (1 - \lambda)r_0 \). Therefore, \( h^+_\lambda(x_0) \in \hat{V} \) and let \( \hat{P}_{h^+_\lambda(x_0)} \) be the plaque of \( h^+_\lambda(x_0) \) in \( \hat{V} \). Then, by Lemma 3.7

\[ (3.1) \quad \hat{P}_{h^+_\lambda(x_0)} \subset C^-_{r_2}(P_{x_0}), \]

\[ (3.2) \quad P_{h^+_\lambda(x_0)} \subset C^+_{r_1}(P_q), \]

where \( r_0 = r_1 + r_2 \). In other words \( r_2 = (1 - \lambda)r_0 \) and \( r_1 = \lambda r_0 \). For \( x_0 \in \hat{P}_{h^+_\lambda(x_0)} \cap P_{h^+_\lambda(x_0)} \), there exists a segment of geodesic \( \gamma^+_x \) with unit velocity so that \( \gamma^+_2(r_1) = x_0 \), \( \gamma^+_2(r_0) \in P_{x_0} \) and \( \gamma^+_2(r_0) \) is orthogonal to \( P_{x_0} \); see (3.1) and recall Lemma 3.3. Let \( \gamma^+_1 \) be the segment of geodesic with \( \gamma^+_1(0) \in P_q \), \( \gamma^+_1(r_1) = x_0 \) and \( \gamma^+_1(0) \) orthogonal to \( P_q \); see (3.2) and Lemma 3.3. Let \( \hat{\gamma} \) be the (possible broken) geodesic constructed as the concatenation of \( \gamma^+_1 \) with \( \gamma^+_2 \). Then we have that \( \hat{\gamma} \) joins \( P_q \) to \( \hat{\gamma}(r_1 + r_2) = \gamma^+_2(r_0) \in P_{x_0} \subset C^+_{r_0}(P_q) \), it is orthogonal to \( P_q \) and has length \( r_0 = r_1 + r_2 \). Therefore \( \hat{\gamma} \) must be the geodesic \( \gamma^+_x \), where \( x = \gamma^+_x(r_1 + r_2) \) and \( h^+_\lambda(x) = x \). This proves that \( h^+_\lambda(P_{h^+_\lambda(x_0)} \cap P_{h^+_\lambda(x_0)}) \subset P_{x_0} \). In particular,

\[ (3.3) \quad \dim P_{h^+_\lambda(x_0)} \leq \dim P_{x_0} \]

(recall that \( h^+_\lambda \) and \( h^+_\lambda \) are inverse diffeomorphisms). Proceeding analogously, one can prove that \( h^-_\lambda(\hat{P}_{h^-_\lambda(x_0)} \cap P_{h^-_\lambda(x_0)}) \subset P_{x_0} \) where \( \hat{P}_{h^-_\lambda(x_0)} \) is the plaque of \( h^-_\lambda(x_0) \) in a tubular neighborhood of \( P_{x_0} \) and (3.3) holds for \( \lambda > 1 \) small enough, obtaining that \( \dim P_{x_0} \) is a local maximum for the curve \( \lambda \to \dim P_{h^+_\lambda(x_0)} \). As \( x_0 \) is arbitrary, choosing \( h\lambda(x_0) \) as departing point, applying (3.3), and taking into account that
holds for every $x \neq h$ we know that

$$\text{Let } 12. M. M. \text{ ALEXANDRINO, B. ALVES, AND M. A. JAVALOYES}$$

while the plaque connected, it follows that the closure of this subset, property (P) shows that it is in the interior and then

$$\text{Lemma 3.10.}$$

Let us now review a useful and standard lemma.

Lemma 3.10. Let $F$ be a singular foliation on $M$. Then for $q \in M$ there exists a neighborhood $U$ of $q$, a regular foliation $F^2$ on $U$ (i.e., all the leaves of $F^2$ have the same dimension) and an embedded submanifold $S_q$ (a slice) so that:

(a) $F^2$ is a subfoliation of $F$, i.e., for each $x \in U$, the leaf $L^2_x$ of $F^2$ through $x$ is contained in the leaf $L_x$ of $F$;

(b) $L^2_q$ is a relatively compact open subset of $L_q$ (and in particular it has the same dimension as $L_q$)

c) $S_q$ is transverse to $F^2$, i.e., $T^*_xM = T^*_xS_q \oplus T^*_xL^2_x$ for each $x \in S_q$, $q \in S_q$ and $S_q$ meets all the plaques of $F^2$ in $U$.

Definition 3.11. Given a tubular neighborhood $O(P_q, \epsilon)$ of a plaque $P_q$, in a singular Finsler foliation $(M, F, F)$, we can define the projection map $\rho : O(P_q, \epsilon) \to P_q$ as $\rho := h^+_{\lambda_0}$, namely, the homothetic transformation $h^+_\lambda$ in the limit case of $\lambda = 0$, or, more precisely, $\rho^+(x) = \gamma^+_\lambda(0)$ for $x \in O(P_q, \epsilon) \setminus P_q$ and $h^+(x) = x$ if $x \in P_q$. Moreover, given $p \in P_q$, we define the Finslerian slice $\Lambda_p$ as the image by the exponential map of $\nu(P_q) \cap T_pM \cap F^{-1}(0, \epsilon)$.

Proposition 3.12. Given a point $q \in M$, a plaque $P_q$ of $q$ and a tubular neighborhood $O(P_q, \epsilon)$, the following properties hold:

(i) for every $p \in P_q$, the Finslerian slice $\Lambda_p$ is transversal to all the plaques that meets,

(ii) the projection map $\rho : O(P_q, \epsilon) \to P_q$ is a surjective submersion,

(iii) given any plaque $P_x$ in $O(P_q, \epsilon)$, the restriction $\rho|_{P_x} : P_x \to P_q$ is a submersion and, when the leaves of $F$ are closed, $\rho|_{P_x}$ is surjective, and
(iv) by reducing $P_q$ and $\epsilon$ if necessary, we can assume that $\rho|_{P_\epsilon}$ is surjective.

**Proof.** For part (i), given $y \in \Lambda_p$, consider the unique $v \in \nu(y)(P)$ with $F(v) < \epsilon$ such that $\exp(v) = y$. Consider a basis $\{e_1, \ldots, e_r\}$ of $T_{\rho(y)}P_q$ and a regular subfoliation $F^2$ of the foliation $F$ in a neighborhood $U$ of $\rho(y)$ as in Lemma 3.10. Then by reducing $U$ if necessary, we can extend $\{e_1, \ldots, e_r\}$ to a frame $\{X_1, \ldots, X_r\}$ of the leaves of $F^2$. Consider now a basis $\{Y_1, \ldots, Y_n\}$ of the $g_\rho$-orthogonal space to $T_{\rho(y)}P_q$. Observe that defining $Z_i(t) = d\exp_{tv}[Y_i]$, for $t \in (0, 1]$ and $i = 1, \ldots, n - r$ and using Proposition 2.34 and that $g_\rho = g_{tv}$ for $t > 0$, it follows that $\{Z_1(t), \ldots, Z_{n-r}(t)\}$ is a basis of $T_{\exp(tv)}\Lambda_p$. Consider the system

$$\{X_1(\exp(tv)), \ldots, X_r(\exp(tv)), Z_1(t), \ldots, Z_{n-1}(t)\}$$

for $t \in [0, 1]$. As in $t = 0$ this system is linearly independent, by continuity, it will be linearly independent for small $t$, which implies that $\Lambda_p$ is transversal to the plaques of $F$ in $\exp(tv)$ for such a small $t$. As the slice $\Lambda_p$ is invariant by the homothetic transformation, applying the homothetic transformation lemma, it follows that $\Lambda_p$ is also transversal in $y = \exp(v)$, proving part (i). Part (ii) follows from the observation that $\rho$ is the composition of a diffeomorphism with $\tau : \nu(P_q) \cap F^{-1}(e) \to P_q$, which is a submersion by Proposition 2.3. The surjectivity of $\rho$ follows from definition. For part (iii), observe that the kernel of $\rho$ is the tangent space to the slices $\Lambda_p$. As $\Lambda_p$ is transversal to the plaques, $d(\rho|_{P_\epsilon})$ has to be surjective by a counting of dimensions.

In order to see that $\rho|_{P_\epsilon}$ is surjective, observe that its image $\rho(P_\epsilon)$ is open and closed in $P_q$. Open because $\rho|_{P_\epsilon}$ is a submersion. Let us see that it is closed. If $\{u_n\}_{n \in \mathbb{N}} \subseteq \rho(P_\epsilon)$ is a sequence that converges to $u \in P_q$, let $\gamma^{u_n} : [0, b] \to M$ be a unit minimizing geodesic from $u_n$ to $P_\epsilon$ ($b \in \mathbb{R}$ does not depend on $n$). Then $\{\gamma^{u_n}(0)\}_{n \in \mathbb{N}}$ admits a subsequence in $\nu\epsilon(P_q)$ converging to $v \in \nu\epsilon(P_q)$. Let $\gamma_v : [0, b] \to M$ be the geodesic such that $\dot{\gamma}_v(0) = v$. It turns out that $\gamma_v([0, b]) \subset \mathcal{O}(P_q, e)$, $\gamma_v(b) \in P_\epsilon$, because $P_\epsilon$ is closed, and $u = \rho(\gamma_v(b))$, as required. Part (iv) follows from part (iii) observing that it is possible to reduce $P_q$ and $\epsilon$ in such a way that there exists a regular subfoliation $F^2$ with closed leaves, and then, analogously to part (iii), the restriction of $\rho$ to the plaques of $F^2$ is also a surjective submersion.

The above results can be used to prove the next proposition.

**Corollary 3.13.** Let $(M, F, \mathcal{F})$ be a (regular) Finsler foliation with connected fibers given by the leaves of a submersion $\pi : M \to N$. Then there exists a unique Finsler metric $\hat{F}$ on $N$ such that $\pi : (M, F) \to (N, \hat{F})$ is a Finsler submersion. As a consequence, every regular Finsler foliation is described locally by a Finsler submersion.

**Proof.** Given $p \in M$, consider the map $\hat{F}_{\pi(p)} : T_{\pi(p)}N \to \mathbb{R}$ given by

$$(3.4) \quad \hat{F}_{\pi(p)} := F_p \circ \varphi_p^{-1},$$

where $\varphi_p^{-1}$ is the horizontal lift of $d\pi$ (see part (a) of Lemma 2.9). Observe that, for every $p \in M$, $\hat{F}_p = F_p \circ \varphi_p^{-1}$ is a Minkowski norm such that $d\pi_p : (T_pM, F_p) \to (T_{\pi(p)}N, \hat{F}_{\pi(p)})$ is a linear Finsler submersion (see part (b) of Lemma 2.9). Let us see that this map is well-defined, namely, $F \circ \varphi_p^{-1} = F \circ \varphi_p^{-1}$, for every $p_1, p_2 \in \pi^{-1}(q)$.

As the fibers are connected, it is enough to prove that this property is open. Therefore we can assume that $p_1$ and $p_2$ belong to the same plaque, say $P_1 :=$
$P_1$. Given $v \in T_1N$, let $v_i = \varphi^{-1}_i(v) \in T_1M$ with $i = 1, 2$. We will show that $F(v_1) = F(v_2)$. Consider $O(P_1, \varepsilon)$ a tubular neighborhood of $P_1$ and fix $P$ a plaque in $O(P_1, \varepsilon)$ which contains $\gamma_{v_i}(s)$ for some $s$. Let $w \in \nu_P(P_1)$ be such that $\gamma_{w}|[0,s]$ attains the distance $d_{F}(P_1, P)$ (recall part (iii) of Proposition 3.12. In particular, $F(v_1) = F(w)$. By the homothetic lemma 3.9, we get that $d_{\pi}(w) = d_{\pi}(v_1) = d_{\pi}(v_2)$. Therefore $w = v_2$ by the injectivity of $\varphi_2$, and consequently $F(v_1) = F(v_2)$. The smoothness and unicity of $\hat{F}$ follow from Lemmas 2.9 and 2.10.

\[\square\]

**Remark 3.14.** Recall that given a Finsler submersion $\pi : (M, F) \to (N, \hat{F})$, each segment of geodesic in $(N, \hat{F})$ can be lifted to a horizontal segment of geodesic in $(M, F)$, recall [9] Theorem 3.1. This fact together with the above corollary allow us to transport horizontal segments of geodesics along regular leaves. More precisely, let $F$ be a singular Finsler foliation on a complete Finsler manifold $(M, F)$, $\gamma : [0,1] \to M$ be a horizontal segment of geodesic such that $\gamma|_{[0,1]}$ has only regular points and $\beta : [0,1] \to L_{\gamma(1)}$ a curve with $\beta(0) = \gamma(1)$. Then there exists a variation of horizontal segments of geodesics $s \to \gamma^s$ such that $\gamma^0 = \gamma$, $\gamma^s(1) = \beta(s) \in L_{\gamma(1)}$, $\gamma^s|_{[0,1]}$ has only regular points and $\pi(\gamma^s) = \pi(\gamma)$, where $\pi : M \to M/F$ is the canonical projection. Due to the homothetic lemma we can also infer that $\gamma^*(0) \in L_{\gamma(0)}$ even if $L_{\gamma(0)}$ is a singular leaf. Analogously, the transport can be done from the initial leaf $L_{\gamma(0)}$ to $L_{\gamma(1)}$ assuming that $\gamma|_{[0,1]}$ has only regular points.

Recall that a transnormal function $f : M \to \mathbb{R}$ of a Finsler manifold $(M, F)$ is a function such that there exists another real function $b : f(M) \to \mathbb{R}$ such that $F(\nabla f)^2 = b(f)$, where $\nabla f$ is the gradient of $f$, namely, $\nabla f = L^{-1}(df)$.

**Proposition 3.15.** Let $F = \{L\}$ be a singular Finsler foliation on a Finsler manifold $(M, F)$. Given a point $q \in M$, there exists a slice $S_q$ transversal to $L_q$ and a Finsler metric $\hat{F}$ on $S_q$ with the following properties:

(a) the leaves of the slice foliation $F_q = F \cap S_q$ endow $(S_q, \hat{F})$ of a structure of singular Finsler foliation,

(b) the distance between the leaves of $F_q$ (with respect to $\hat{F}$) and the distance between the leaves of $F$ (with respect to $F$) coincide locally.

**Proof.** The proof is adapted from the one of the Riemannian version [3] Proposition 2.2. Let us start by constructing the desired metric $\hat{F}$.

Let $F^2$ be the subfoliation of $F$ with plaque $P_q = L_q^2$ in a neighborhood $U$ and $S_q$ be the slice presented in Lemma 3.10.

Using Lemma 3.10 and reducing $U$ if necessary, we can define a submersion $\pi : U \to S_q$ whose fibers are the leaves of $F^2$. By part (b) of Lemma 2.10 there exists a Finsler metric $\hat{F}$ on $S_q$ such that

\[(3.5) \quad d\pi_x : (T_xM, F_x) \to (T_xS_q, \hat{F}_x)\]

is a linear Finsler submersion for every $x \in S_q$. Fix $p \in S_q$ and let $P_p$ be the plaque at $p$ in $U$. Consider now a tubular neighborhood $\bar{U}$ of the plaque $P_p$ and define $f_+ : \bar{U} \setminus P_p \to \mathbb{R}$ as $f_+(x) = d^+(P_p, x)$, which is a transnormal function, and let $X$ be the $F$-gradient of $f_+$. Consider also the restriction $f^+ = f_+|_{\bar{U} \cap S_q \setminus P_p \cap S_q}$ and the vector field along $S_q$, $\hat{X} = d\pi(X)$. Observe that the fibers of the linear submersions
In [3.15] are contained in the tangent spaces of the cylinders with axis the plaque $P_\pi$ because these cylinders contain the plaques of the foliation $F$ and also those of $F^2$ (see Lemma 3.17). This implies that $X$ is $F$-orthogonal to the fibers of $\pi$.

As a consequence, $F(X) = \hat{F}(\hat{X})$ and $d\pi_x : (T_xM, g_X) \to (T_xS_q, \hat{g}_X)$ is a linear Riemannian submersion by [9, Prop. 2.2], where $g$ is the fundamental tensor of $F$ and $\hat{g}$ the fundamental tensor of $\hat{F}$. In particular, $g_X(X, v) = \hat{g}_X(\hat{X}, d\pi_x(v))$ for all $v \in T_xM$. Given $v \in T_xS_q$, we have $df_+(v) = df_+(\hat{v}) = g_X(X, v) = \hat{g}_X(\hat{X}, v)$ and consequently $\hat{X}$ is the $\hat{F}$-gradient of $f_+$; recall Remark 2.2. Moreover, $\hat{F}(\hat{X}) = F(X) = 1$, which implies that $\hat{f}_+$ is a transnormal function of $(\hat{U} \cap S_q, \hat{F})$, which extends to zero on $P_\pi \cap S_q$. Observe that part (2) of [20, Proposition 4.1] does not require any kind of completeness. Then applying this result to $f_+$, it follows that the integral lines of the $\hat{F}$-gradient of $f_+$ are segments of geodesics, and it is not difficult to check that these geodesics meet $P_\pi \cap S_q$ orthogonally. Considering a tubular neighborhood $\hat{V}$ of $P_\pi \cap S_q$ for $(\hat{U} \cap S_q, \hat{F})$, it follows that the level sets of $\hat{f}_+|\hat{V}$ coincide with the future cylinders of $P_\pi \cap S_q$ with respect to $\hat{F}$ and then the plaques of $F \cap S_q$ in $\hat{V}$ are contained in these cylinders as the plaques in $\hat{U}$ of the foliation $F$ are contained in the level sets of $f_+$. Proceeding analogously with the reverse metrics of $F$ and $\hat{F}$, we deduce that the leaves of $F \cap S_q$ are contained locally in the past cylinders of $\hat{F}$, concluding that $(F \cap S_q, \hat{F})$ is a singular Finsler foliation. The fact that $\hat{F}(\hat{X}) = F(X) = 1$ also allows us to conclude that locally the distances between the leaves coincide with those of $(M, F, F)$.

Remark 3.16. Observe that the meaning of locally in part (b) of Proposition 3.15 as it can be checked in the last proof is the following: given a plaque $P_\pi$ of $U$ with $p \in S_q$ as in the last proof, there exists a tubular neighborhood $U$ of $P_\pi$ for $(M, F)$ and another tubular neighborhood $\hat{V}$ of $P_\pi \cap S_q$ for $(\hat{U} \cap S_q, \hat{F})$ such that given $x \in \hat{V}$, if $\hat{P}_x$ is the plaque of $x$ in $\hat{V}$, the $\hat{F}$-distance from $P_\pi \cap S_q$ to $\hat{P}_x$ coincides with the $F$-distance from $P_\pi$ to $P_\pi$.

Definition 3.17. We will say that a singular Finsler foliation $(M, F, F)$ has locally closed leaves if, for every point $q \in M$, there exists a slice $S_q$ endowed with a structure of singular Finsler foliation as in Proposition 3.15 with closed (and then compact) leaves.

Proposition 3.18 (Slice reduction). Let $F = \{L\}$ be a singular Finsler foliation with locally closed leaves on a Finsler manifold $(M, F)$. Then for every point $q \in M$, there exists a slice $S_q$ and a Finsler metric $\hat{F}$ in $S_q$ as in Proposition 3.15 such that $F_q = S_q \cap F$ with the metric $\hat{F}$ is foliated diffeomorphic to an open subset of a Minkowski space endowed with a structure of singular Finsler foliation.

Proof. Let $\hat{F}$ be the Finsler metric on $S_q$ defined in Proposition 3.15. Then the proof turns out to be similar to the proof of the analogous result for singular Riemannian foliations; see [8, Proposition 2.3]. In fact, on the Finsler space $(S_q, \hat{F})$ and reducing $S_q$ if necessary, consider the homothetic transformation $h_\lambda : S_q \to S_q$ defined as $h_\lambda(x) = \gamma(\lambda r)$, where $\gamma$ is the radial (Finsler) geodesic on $(S_q, \hat{F})$ starting at $q = \gamma(0)$ and $x = \gamma(r)$. Set $\hat{F}^\lambda := \frac{1}{\lambda} h_\lambda^* \hat{F}$. Note that $\hat{F}^\lambda$ converges (uniformly)
to $\hat{F}^0$ when $\lambda \to 0$, where $\hat{F}^0$ is the Finsler metric on $S_q$ so that $\hat{F}_q = \exp_q^* \hat{F}^0$ for the Minkowski norm $\hat{F}_q$ on $T_q S_q$.

Since $\mathcal{F}_q$ is a singular Finsler foliation on $(S_q, \hat{F})$, it follows from Lemma 3.6 that $\mathcal{F}_q$ is a singular Finsler foliation on $(S_q, \hat{F}^\lambda)$.

We claim that $\mathcal{F}_q$ is a singular Finsler foliation on $(S_q, \hat{F}^0)$, reducing the slice $S_q$ if necessary. This will imply that $\mathcal{F}_q$ is diffeomorphic to an open subset of a singular Finsler foliation on the Minkowski space $(T_q S_q, \hat{F}_q)$ concluding the proof. Observe that even if, in principle, the singular Finsler foliation is defined using the exponential map in an open subset $U$ of $T_q S_q$ of the zero vector, we can extend this foliation to the whole $T_q S_q$ by using the homothetic lemma, namely, the leaves of the extended foliation are determined by the images by the homothetic transformations of the leaves in $U$, and this singular foliation is Finsler because the orthogonality is invariant by homotheties and geodesics are always straight lines.

In order to prove the claim we only need to check that, given leaves $L, \tilde{L} \subset S_q$, and $x, y \in \tilde{L}$, there exists $\lambda_0$ so that

$$d_\lambda(L, x) = d_\lambda(L, y)$$

for $\lambda < \lambda_0$, where $d_\lambda$ is the distance in $S_q$ computed with the metric $\hat{F}^\lambda$.

Once we have proved (3.6), we can take the limit and conclude that

$$d_0(L, x) = d_0(L, y).$$

The claim will follow from (3.6), its analogous for the metric $v \to F^0(\nu)$ and from Lemma 3.7

Since $\mathcal{F}_q$ is a singular Finsler foliation with respect to $\hat{F}^\lambda$, (3.6) is clearly true if $x$ and $y$ are in a tubular neighborhood $U_\lambda$ of $L$. The problem is that the neighborhood $U_\lambda$ could be arbitrarily small when $\lambda$ goes to zero. The underlying reason for the apparent lack of control on $U_\lambda$ in the Finslerian setting is that it is not clear if $\{\hat{F}^\lambda\}$ converges smoothly to $\hat{F}^0$ (recall, unlike what happens in the Riemannian geometry setting, $\exp$ is just $C^1$ at zero in Finsler geometry). We aim to prove (3.6) for all $x$ and $y$ in $S_q$ (not only for those contained in a tubular neighborhood of $L$). Here we will use compactness of the leaves of $\mathcal{F}_q$. In fact this is the only place in this paper where we have used this assumption.

First note that there exists $R_2, 0 < R_1 < R_2$ and $\lambda_0 > 0$ with the following properties: the ball $B_{R_2}(q) \subset S_q$ of center $q$ and radius $R_2$ with respect to $\hat{F}$ is precompact and, if $\tilde{x}$ and $\tilde{y}$ are in $B_{R_1}(q)$ and $\lambda < \lambda_0$, then there exists a segment of geodesic $\gamma^\lambda$ (with respect to $\hat{F}^\lambda$) joining $\tilde{x}$ and $\tilde{y}$ such that $\ell_\lambda(\gamma^\lambda) = d_\lambda(\tilde{x}, \tilde{y})$, where $\ell_\lambda(\gamma^\lambda)$ is the length of $\gamma^\lambda$ with respect to $\hat{F}^\lambda$. One can check this observation using the fact that $\hat{F}^\lambda$ converges to $\hat{F}^0$ uniformly.

From now on, we will assume that $L, \tilde{L} \subset B_{R_1}(q)$. Thus proving (3.6) for this data, we will conclude by reducing the slice $S_q$ to $B_{R_1}(q)$, since (3.7) implies that all the leaves which are close enough to $L$ remain in an $\hat{F}^0$-tubular neighborhood of $L$ contained in $B_{R_1}(q)$. First we consider the case in that $\tilde{L}$ is regular. Let $x_0 \in L$ so that $d_\lambda(L, \tilde{L}) = d_\lambda(L, x_0)$ and let $\gamma^{x_0}$ be a segment of geodesic so that $\gamma^{x_0}(0) \in L$, $\gamma^{x_0}(1) = x_0$ and $\ell_\lambda(\gamma^{x_0}) = d_\lambda(L, x_0)$. Note that $\gamma^{x_0}|_{(0, 1)}$ has only regular points. In fact assume by contradiction that there exists $0 < t_1 < 1$ so that $\gamma^{x_0}(t_1)$ is singular and $\gamma^{x_0}(t)$ is regular for $t > t_1$. Note that, by the homothetic lemma and Proposition 3.12 there exists a segment of geodesic $\beta|_{[t_1, t_1+c]}$ such
that $\beta(t_1) = \gamma^{x_0}(t_1)$, $\beta(t_1 + \epsilon) \in P_{\gamma^{x_0}(t_1+\epsilon)}$, for a small $\epsilon > 0$, which has the same projection on the regular stratum as $\gamma^{x_0}|_{[t_1, t_1+\epsilon]}$. Since they have the same projection and $\gamma^{x_0}|_{[t_1, t_1]}$ lies in the regular stratum, then $\beta(1) \in P_{\gamma(t_1)} \subset \tilde{L}$ and $\ell_\lambda(\beta) = \ell_\lambda(\gamma^{x_0}|_{[t_1, t_1]}).$ Indeed, as the points of $\gamma^{x_0}|_{[t_1+\epsilon, 1]}$ are regular, $\beta|_{[t_1+\epsilon, 1]}$ will be the transport the segment of geodesic $\gamma = \gamma^{x_0}$ desic, producing segments which is absurd; compare with Kleiner’s lemma (recall Remark 3.17). Therefore $\tilde{\gamma} = \beta \ast \gamma^{x_0}|_{[0, t_1]}$ is a minimal broken geodesic joining $L$ to $\tilde{L}$ so that $\ell_\lambda(\tilde{\gamma}) = \ell_\lambda(\gamma^{x_0})$, which is absurd; compare with Kleiner’s lemma (see [4, Lemma 3.70]).

Once $\gamma^{x_0}|_{(0, 1)}$ has only regular points, one can “transport” this segment of geodesic, producing segments $\gamma^x$ and $\gamma^y$ joining $L$ to $x$ and $y$ so that $\ell_\lambda(\gamma^x) = \ell_\lambda(\gamma^y) = \ell_\lambda(\gamma^{x_0}) = d_\lambda(L, \tilde{L})$. This last equation implies (3.6) when $\tilde{L}$ is regular. The case in that $\tilde{L}$ is singular follows from the regular case and from fact that the set of regular points is dense on $S_q$ (which can be proved again using the Kleiner’s Lemma type argument discussed above) and using (3.7) for the sequence of regular leaves approximating $\tilde{L}$.

For the next proposition we will need the following useful lemma.

**Lemma 3.19.** Let $\mathcal{F}$ be a singular Finsler foliation on a Minkowski space $V$. Assume that $(0) = L_0$, i.e., zero is a leaf. Then the minimal stratum $\Sigma$ (i.e., the union of leaves with dimension zero) is a subspace of $V$.

**Proof.** Observe that given a point in the minimal strata, it is possible to choose the whole $V$ as a simple neighborhood. Then the result follows easily by repeatedly applying Lemma 3.9. □

**Proposition 3.20** (Stratification). Let $\mathcal{F} = \{L\}$ be a singular Finsler foliation with locally closed leaves on a complete Finsler manifold $(M, F)$. Then the union of the leaves with the same dimension is a disjoint union of embedded submanifolds (called strata). The collection of all the strata is a stratification in the usual sense. In addition the induced foliation $\mathcal{F}|_\Sigma$ restricted to a stratum $\Sigma$ is a (regular) Finsler foliation.

**Proof.** The fact that each connected component of a stratum is an embedded submanifold follows from Proposition 3.18 and Lemma 3.19. The same results imply that the collection of all strata is a stratification; recall [4, Definition 3.100]. Alternatively this can be proved in an analogous way as it was proved in [28, Proposition 6.3] for singular Riemannian foliations.

Let us see why $\mathcal{F}|_\Sigma$ is a regular Finsler foliation. Let $y \in \Sigma$ and consider a plaque $P_y$ which contains $y$ and a tubular neighborhood $U$ of $P_y$. Then, for every $x \in U \cap \Sigma$, there exists a minimal segment of horizontal geodesic $\gamma^x_\Sigma$ (contained in $U$) joining the plaque $P_y$ to $x$. By Lemma 3.9 $\gamma^x_\Sigma$ is contained in $\Sigma$. This implies that the transverse geometry of $\Sigma$ coincides with the transverse geometry of $M$. In particular the future and past cylinders in $\Sigma$ coincide with the intersection of $\Sigma$ with the future and past cylinders with axis contained in $\Sigma$. The result now follows from Lemma 3.7. □

The next proposition can be adapted from [28, Proposition 6.4]; see also [2, Proposition 2.14].
Proposition 3.21. Let $F = \{L\}$ be a singular Finsler foliation with locally closed leaves on a complete Finsler manifold $(M, F_1)$. Assume that there exists a complete Finsler metric $F_2$ such that the foliation restricted to each stratum is a (regular) Finsler foliation with respect to $F_2$. Then $F$ is a singular Finsler foliation on $(M, F_2)$.

We stress that we don’t need the above proposition in the proof of the main results, just the Riemannian case proved in [2, Proposition 2.14].

4. Singular Finsler foliations on Randers spaces

In this section we prove Theorem 1.1. For this purpose, we apply the slice reduction presented in Proposition 3.18 to relate local singular Finsler foliations on Randers spaces with singular Finsler foliations on Randers-Minkowski spaces.

4.1. Singular Finsler foliations on Randers-Minkowski spaces. In this section we present some facts about singular Finsler foliations $F = \{L\}$ on a Randers-Minkowski space $(\mathbb{R}^n, Z)$, where $Z(v) = \sqrt{\langle v, v \rangle + \beta(v)}$ with $\langle \cdot, \cdot \rangle$ denoting the Euclidean product and $\beta(v) = \langle v, \beta \rangle$ for a constant vector field $\beta$ with length smaller than 1. Recall that $\beta$ is multiple of the associated wind; see Lemma 2.5. We will assume that the minimal leaves (i.e., the leaves with the lowest dimension) have dimension zero and that $\{0\} = L_0$, i.e., the zero is one of these minimal leaves.

The main goal of this section is to prove Lemma 4.2, which shows that the constant vector field $\beta$ (and hence the wind) is tangent to the minimal stratum $\Sigma$ (i.e., the union of minimal leaves).

Lemma 4.1. For each leaf $L$ of $F$ we have that the tangent space $TL$ is perpendicular to the constant vector field $\beta$ with respect to the Euclidean metric. In particular we infer that $L \subset V$, where $V = \{y \in \mathbb{R}^n, \langle y, \beta \rangle = c\}$ for some $c \in \mathbb{R}$.

Proof. From Lemma 2.6 we have:

$$g_\nu(v, w) = Z(v) \left( \frac{\langle v, w \rangle}{\|v\|} + \langle w, \beta \rangle \right).$$

Since $F$ is a singular Finsler foliation then the straight lines $\mathbb{R} \ni t \to tv \in \mathbb{R}^n$ and $\mathbb{R} \ni t \to (1 - t)v \in \mathbb{R}^n$ are (Finsler) orthogonal to the leaf $L_v$. Hence $g_\nu(v, w) = 0$ and $g_{-v}(-v, w) = 0$ for every $w \in T_vL$, which, together with (4.1), implies that $\langle w, \beta \rangle = 0$, for every $w \in T_{\gamma(t_0)}L$.

Lemma 4.2. The vector $\beta$ (and hence the wind) is tangent to the minimal stratum $\Sigma$.

Proof. Recall that the tangent space to the future sphere $S^+(0) = \{v \in \mathbb{R}^n : Z(v) = 1\}$ is given by the vectors $w \in \mathbb{R}^n$ such that $g_\nu(v, w) = 0$ as $w(Z) = 2g_\nu(v, w)$. Let $\delta > 0$ be such that $p = \delta \beta \in S^+(0)$. From the last observation and (4.1), it follows that $\beta$ is orthogonal (with the Euclidean metric) to the tangent space to $S^+(0)$ in $p$. Alternatively this fact can be directly checked through calculations using Lemma 2.5. Since $F$ is a singular Finsler foliation, the leaf $L_p$ is contained in $S^+(0)$. On the other hand, it follows from Lemma 1.1 that $L_p$ is contained in $V = \{y \in \mathbb{R}^n, \langle y, \beta \rangle = c\}$ for some $c \in \mathbb{R}$, which is tangent to $S^+(0)$ as $\beta$ is orthogonal to $S^+(0)$. By the strong convexity of $S^+(0)$, $V \cap S^+(0) = \{p\}$ and...
therefore \( L_p \) must be just the point \( p \). In other words, we conclude that \( 0 \) and \( p = \delta \vec{\beta} \) (for some \( \delta > 0 \)) are contained in the minimal stratum \( \Sigma \) and Lemma 3.19 concludes.

\[\square\]

Remark 4.3. Let \( \Sigma^\perp \) be the orthogonal space to \( \Sigma \) with respect to the Euclidean metric \( \langle \cdot, \cdot \rangle \) at some point \( p \in \Sigma \). Then the above lemma implies that the vector \( \vec{\beta} \) is orthogonal to \( \Sigma^\perp \) and hence \( \mathcal{F} \cap \Sigma^\perp \) is an S.R.F. Using the homothetic transformation lemma, it is possible to prove that \( \mathcal{F} \) is a singular Riemannian foliation on \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\).

4.2. Proof of Theorem 1.1. The strategy of the proof is simple. In order to prove that a singular Finsler foliation \( \mathcal{F} \) on a Randers space \( M \) with Zermelo data \((h, W)\) is a singular Riemannian foliation with respect to the metric \( h \), we start by showing that the wind \( W \) is tangent to each singular stratum, see Proposition 4.4. This is done by applying the slice foliation; recall Proposition 3.18.

Once we have shown that the wind is tangent to each stratum \( \Sigma \), we have that \( \mathcal{F}|_{\Sigma} \) is a (regular) Finsler foliation on the Randers space \( \Sigma \) with Zermelo data \((h, W)\). From Proposition 4.5 we will conclude that \( \mathcal{F}|_{\Sigma} \) is a Riemannian foliation with respect to \( h \).

Finally we will apply [2, Proposition 2.15] to conclude that \( \mathcal{F} \) is an S.R.F. on \((M, h)\), because \( \mathcal{F}|_{\Sigma} \) is a Riemannian foliation on each stratum \((\Sigma, h)\). Alternatively one can use Proposition 3.21.

Proposition 4.4. Let \( \mathcal{F} = \{L\} \) be a singular Finsler foliation with locally closed leaves on a Randers space \((M, Z)\) with Zermelo data \((h, W)\). Then the wind \( W \) is always tangent to the strata of \( \mathcal{F} \).

Proof. Let \((S_q, \mathcal{F})\) be the singular Finsler foliation obtained in Proposition 3.15 for \( q \in M \). Let \( d\pi_q : T_q M \to T_q S_q \) be the map defined in (3.5). Observe that \( \hat{F}_q \) is a Zermelo metric with wind \( d\pi_q(W) \) (see Proposition 2.12). Recall that the fiber \((d\pi_q)^{-1}(0)\) coincides with \( T_q L_q \), which is contained in \( T_q \Sigma \), i.e., in the tangent space of the stratum containing the point \( q \). Note that \( W_q = d\pi_q(W) + W_q^\perp \), where \( W_q^\perp \in T_q L_q \subset T_q \Sigma \). Therefore, in order to prove that \( W_q \in T_q \Sigma \), it suffices to check that \( d\pi_q(W_q) \in T_q \Sigma \). But, on the other hand, this follows from Lemma 4.2 and Proposition 3.18.

\[\square\]

Proposition 4.5. Let \( \mathcal{F} = \{L\} \) be a (regular) Finsler foliation on a Randers space \((\Sigma, Z)\) with Zermelo data \((h, W)\). Then \( \mathcal{F} \) is a (regular) Riemannian foliation with respect to \( h \).

Proof. As a regular Finsler foliation is a Finsler submersion locally (see Corollary 3.13), the result follows from Proposition 2.12.

\[\square\]

Remark 4.6. Consider a singular Finsler foliation \( \mathcal{F} \) with locally closed leaves on a Randers manifold \((M, Z)\) with Zermelo data \((h, W)\). Then Propositions 4.3, 5.20 and 2.12 imply that the wind \( W \) is an \( \mathcal{F} \)-foliated vector field.
4.3. Proof of Corollary 1.2. In this section we assume that the wind $W$ of a Randers space $(M,Z)$ with Zermelo data $(h,W)$ is an infinitesimal homothety of $h$, i.e., $L_wh = -\sigma h$. We also assume that $h$ and the wind $W$ are complete. This implies that the metric $Z$ is also complete; recall [23, Theorem 1.2] and [21].

Since $F$ is a singular Finsler foliation on $(M,Z)$, Theorem 1.1 implies that $F$ is a singular Riemannian foliation on $(M,h)$. Therefore, it follows from [8] that each regular leaf of $F$ is equifocal with respect to $h$. In other words, we have the following result:

Lemma 4.7 ([8]). For each $p$ in a regular leaf $L$, there exists a neighborhood $U \subset L$ of $p$ such that for each (unit) basic vector field $\xi$ on $U$ (with respect to $h$) the endpoint map $\eta_{\xi} : U \to M$, defined as $\eta_{\xi}(x) = \exp_{x}(s\xi)$, with $s > 0$ fulfills the following properties:

- the derivative of $\eta_{\xi}$ has constant rank,
- $\eta_{\xi}(U)$ is an open set of $L_q$, where $q = \eta_{\xi}(p)$.

Our goal in this section is to prove that $L$ is equifocal with respect to $Z$. More precisely, we are going to check that the above properties hold for each $t\xi$ where $\xi$ is a (unit) normal vector field (with respect to $Z$) on the neighborhood $U$ defined above and $t > 0$.

In order to prove this, we need to recall the next result.

Lemma 4.8 ([23] [21] [31]). A curve $\gamma : \mathbb{R} \to M$ is a geodesic with unit velocity on $(M,Z)$ if and only if $\gamma'(t) = \varphi_t(\tilde{\gamma}(t))$ where $\tilde{\gamma} : \mathbb{R} \to M$ is a reparametrization of geodesic with respect to $h$ so that $h(\tilde{\gamma}'(t), \tilde{\gamma}'(t)) = e^{-\sigma t}$, where $\varphi$ is the flow of $W$. In particular, by deriving at $t = 0$, we have that $\gamma'(0) = \tilde{\gamma}'(0) + W(\gamma(0))$.

From Lemma 4.8, we can also conclude that:

Lemma 4.9. If $\xi$ is a unit vector field on $U$ orthogonal to $L$ with respect to $Z$ then $\xi - W$ is orthogonal to $L$ (with respect to $h$).

We are now ready to prove the corollary. Let $\xi$ be a unit basic vector field on $U$ with respect to $Z$. For $x \in U$ set $t \to \gamma_x(t) = \eta_{\xi}(x)$. Let $\tilde{\gamma}_x$ be the reparametrization of geodesic defined in Lemma 4.8, i.e., $\gamma_x(t) = \varphi_t(\tilde{\gamma}_x(t))$. Set $\xi := \xi - W$. Note that $\tilde{\gamma}'_x(0) = \xi(x)$ (recall Lemma 4.8). Due to Lemma 4.9, the vector field $\xi$ is orthogonal to $L$. Also note that $\xi$ is projectable and hence basic, because $W$ is a foliated vector field (recall Theorem 1.1) and that $h(\xi, \xi) = 1$. We can now infer that

\begin{equation}
\eta_{\xi}(x) = \gamma_x(t) = \varphi_t(\tilde{\gamma}_x(t)) = \varphi_t \circ \eta_{\xi}(x)
\end{equation}

where $s = \frac{-2}{\sigma} \left( \exp(-\frac{\sigma}{2}t) - 1 \right)$, if $\sigma \neq 0$ and $s = t$, if $\sigma = 0$. Since $t$ and $s$ are fixed and $W$ is foliated (i.e., the flow $\varphi_t$ sends leaves to leaves) we conclude from the above equation and Lemma 4.7 that $L$ is equifocal, i.e., the derivative of $\eta_{\xi}$ has constant rank and $\eta_{\xi}(U)$ is an open set of $L_q$, where $q = \eta_{\xi}(p)$.

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