Exact Schwarzschild-Like Solution for Yang-Mills Theories

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Abstract

Drawing on the parallel between general relativity and Yang-Mills theory we obtain an exact Schwarzschild-like solution for SU(2) gauge fields coupled to a massless scalar field. Pushing the analogy further we speculate that this classical solution to the Yang-Mills equations shows confinement in the same way that particles become confined once they pass the event horizon of the Schwarzschild solution. Two special cases of the solution are considered.

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I. INTRODUCTION

Exact solutions to non-linear field theories are notoriously difficult to find since there exists no general method for discovering them. The usual approach is to make some guess as to the form of the solution, and insert it into the field equations to see if it solves them. The Schwarzschild and the Kerr metrics are examples of such exact solutions for Einstein’s field equations. For Yang-Mills theories there are also some known exact solutions such as Coleman’s plane wave solution \([1]\) and the Prasad- Sommerfield solution \([2]\). One possible avenue for discovering new solutions for the Yang-Mills equations is to exploit the long known parallel \([3]\) that exists between Yang-Mills gauge theories and general relativity, and to see if some of the known solutions of general relativity can be used as a guide for finding Yang-Mills solutions. Of particular interest would be finding the Yang-Mills equivalent of the Schwarzschild solution, since it exhibits a property that has long been looked for in Yang-Mills theories, namely confinement. Once a particle crosses the event horizon of the Schwarzschild solution it becomes confined to the region inside the event horizon.

In this note we give the exact, Schwarzschild-like solutions for the Yang-Mills equations for an SU(2) gauge field coupled to a massless scalar. It is argued that this classical solution exhibits the property of confinement that has been looked for in non-Abelian gauge theories.

II. THE SCHWARZSCHILD-LIKE SOLUTION

The model which we consider here is an SU(2) gauge field coupled to a massless scalar triplet. The Lagrangian for this theory is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{\alpha} F_{\mu\nu}^{\alpha} + \frac{1}{2} D^{\mu}(\phi^{\alpha}) D_{\mu}(\phi^{\alpha})
\]

where

\[
F_{\mu\nu}^{\alpha} = \partial_{\mu}W_{\nu}^{\alpha} - \partial_{\nu}W_{\mu}^{\alpha} + g e^{abc} W_{\mu}^{b} W_{\nu}^{c}
\]
\[ D_\mu \phi^a = \partial_\mu \phi^a + ge^{abc} W^b_{\mu} \psi^c \]  \hspace{1cm} (3)

We are interested in static solutions so all the time derivatives will be zero. With this condition the gauge field equations from Eq. (3) are

\[ \partial_i F^{i\mu a} + g\epsilon^{abc} W^b_{\mu} F^{i\mu c} = g\epsilon^{abc} (D^\mu \phi^b) \phi^c \]  \hspace{1cm} (4)

For the scalar fields the equations are

\[ \partial_i (D^i \phi^a) + g\epsilon^{abc} W^b_{\mu} (D^\mu \phi^c) = 0 \]  \hspace{1cm} (5)

Further assuming that the gauge fields and scalar fields are radial we use the Wu-Yang ansatz \[ W^a_i = \epsilon_{aij} \frac{r^j}{gr^2} [1 - K(r)] \]
\[ W^a_o = \frac{r^a}{gr^2} J(r) \]
\[ \phi^a = \frac{r^a}{gr^2} H(r) \]  \hspace{1cm} (6)

Inserting this ansatz into the field equations of Eqs. (4), (5) yields three coupled non-linear differential equations \[ r^2 K'' = K(K^2 + H^2 - J^2 - 1) \]
\[ r^2 J'' = 2JK^2 \]
\[ r^2 H'' = 2HK^2 \]  \hspace{1cm} (7)

The scalar field function, \( H(r) \), and the time component of the gauge field function, \( J(r) \), enter the above equations in almost the same way except for a difference in sign in the first equation. The \( W^a_o \) components act like an isotriplet scalar field with a negative metric. The task of finding gauge and scalar fields that solve Eqs. (4) and (5) thus simplifies somewhat into the task of finding three functions, \( K(r) \), \( J(r) \) and \( H(r) \), which satisfy Eq. (7). One such solution was discovered by Prasad and Sommerfield \[ \text{in their investigations of 't Hooft-Polyakov monopoles and Julia-Zee dyons. Here we present another exact solution} \]

which was found by using the connection between Yang-Mills theory and general relativity.
Roughly speaking the objects in general relativity which correspond to the gauge fields, \( W_\mu^a \), are the connection coefficients, \( \Gamma^a_{\beta\gamma} \). (Ref. [6], which develops gravity as a gauge theory, shows in what respects this last statement is a bit of an oversimplification). Looking at some of the connection coefficients of the Schwarzschild solution from general relativity we find

\[
\begin{align*}
\Gamma^t_{rt} &= \frac{K}{2r(r-K)} \\
\Gamma^r_{rr} &= -\frac{K}{2r(r-K)}
\end{align*}
\]

(8)

where \( K = 2GM \). Using these connection coefficients as a guide, and taking into account that there is an explicit \( 1/r \) factor already in the ansatz of Eq. (6), we are led to try the following solution

\[
\begin{align*}
K(r) &= \frac{Cr}{1-Cr} \\
J(r) &= \frac{B}{1-Cr} \\
H(r) &= \frac{A}{1-Cr}
\end{align*}
\]

(9)

where \( A, B \) and \( C \) are arbitrary constants. It is a straightforward exercise to insert these expressions for \( K(r) \), \( J(r) \) and \( H(r) \) into the coupled differential equations of Eq. (7) and check that they are solutions. The only constraint imposed is that \( A^2 - B^2 = 1 \), so that the solution of Eq. (4) involves only two arbitrary constants. Inserting \( K(r), J(r) \) and \( H(r) \) into the expressions for the gauge and scalar fields of Eq. (6), it is seen that both the gauge and scalar fields become infinite at the radius

\[
r_0 = \frac{1}{C}
\]

(10)

Further, using these singular gauge potentials to calculate the “electric” and “magnetic” fields \( E^i_a = F^a_{i0} \) and \( B^i_a = -\frac{1}{2} e^{ijk} F^j_{ak} \) respectively) it is seen that these fields are also infinite at \( r_0 = 1/C \). Therefore a particle which carries an SU(2) gauge charge becomes permanently confined if it crosses into the region \( r < r_0 \). The non-Abelian gauge potentials and the scalar fields of Eq. (3) also become singular at \( r = 0 \), which is true as well for the Schwarzschild
solution and for the Coulomb potential of a point charge in classical electromagnetism. The singularity of all these solutions at \( r = 0 \) are of the same character in that they all imply a delta function point “charge” sitting at the origin (where the charge of general relativity is mass-energy, and the charge of our non-Abelian model is SU(2) color charge). The singularity in our solution at \( r = 1/C \) is has an essentually different character than the Schwarzschild horizon in general relativity. The Schwarzschild horizon is not a true singularity, but rather it is a coordinate singularity which arises because of the choice of coordinates. This can be seen when the Schwarzschild solution is given in Kruskal coordinates which only have a singularity at \( r = 0 \). For our SU(2) solution the singularity at \( r = 1/C \) can not be transformed away by choosing a different coordinate system, so it is a real singularity. Just as the singularity at the origin can be taken to be a point source of SU(2) charge, so the singularity at \( r = 1/C \) can be taken to be a spherical shell of SU(2) charge. This shell structure is a unique feature of our solution, and it points to a possible connection with the various phenomenological bag models of quark bound states.

There are two special cases which can be considered. First there is the case where the time component of the gauge field equals zero. This corresponds to taking \( B = 0 \) in Eq. (9). The condition \( A^2 - B^2 = 1 \) then implies that \( A = \pm 1 \) so that the solution becomes

\[
K(r) = \frac{Cr}{1 - Cr}, \\
H(r) = \frac{\pm 1}{1 - Cr} 
\tag{11}
\]

There is also the unusual pure gauge case where there is no scalar field. This corresponds to \( H(r) = 0 \) which implies \( A = 0 \). The condition \( A^2 - B^2 = 1 \) then requires that \( B = \pm i \) so that the solution becomes

\[
K(r) = \frac{Cr}{1 - Cr}, \\
J(r) = \frac{\pm i}{1 - Cr} 
\tag{12}
\]

It may seem strange to have a pure imaginary potential, however it does solve the field equations. When these gauge fields are used to calculate the energy in the fields one obtains
a real, although trivial answer (i.e. we will find that the field energy is zero). Therefore one should be wary of the physical significance of this pure gauge case.

The energy of the various gauge and scalar field configurations of this Schwarzschild-like solution can be obtained by taking the volume integral of the time-time component of the energy-momentum tensor

\[ T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial(L\sqrt{-g})}{\partial g^{\mu\nu}} = F^{\mu\rhoa} F^{\nu\rhoa} + D^\mu \phi^a D^\nu \phi^a + g^{\mu\nu} L \quad (13) \]

The energy in the fields is then

\[ E = \int T^{00} d^3x = \frac{4\pi}{g^2} \int_{r_c}^{\infty} \left( K'^2 + \frac{(K^2 - 1)^2}{2r^2} + \frac{J^2 K^2}{r^2} + \frac{(rJ' - J)^2}{2r^2} + \frac{H^2 K^2}{r^2} + \frac{(rH' - H)^2}{2r^2} \right) dr \quad (14) \]

Notice that the integral has been cut off from below at an arbitrary distance \( r_c \), which can be \( > r_0 \) or \( < r_0 \). This was done to avoid the singularity at \( r = 0 \), since integrating down to \( r = 0 \) would give an infinite field energy in the same way that the Coulomb potential of a point electric charge yields an infinite field energy when integrated down to zero. An additional argument for introducing the cutoff \( r_c \) is the fact that our classical solution does not exhibit asymptotic freedom. In this light \( r_c \) could be taken to delineate the boundary between the region where our classical field solution dominates and the region where the quantum effect of asymptotic freedom dominates. Letting the scalar field have a mass and a self coupling might smooth out the behaviour of the fields at the origin, as is the case with the 't Hooft-Polyakov monopole solution. However when the scalar field is allowed to have a mass and self coupling we can find no analytical solution, so numerical solutions must be used. Inserting \( K(r) \), \( J(r) \) and \( H(r) \) into Eq. (14) we find

\[ E = \frac{2\pi}{g^2} \frac{(A^2 + B^2 + 1)(1 - 2Cr_c)}{r_c(1 - Cr_c)^3} \]

\[ = \frac{4\pi A^2}{g^2} \frac{(1 - 2Cr_c)}{r_c(1 - Cr_c)^3} \quad (15) \]
where in the last expression the condition on the constants, \( A^2 - B^2 = 1 \), has been used. In the special case where there is only a scalar field and the space components of the gauge fields the energy in the fields is given by the above expression with \( A^2 = 1 \) and \( B^2 = 0 \). For the pure gauge case \( A^2 = 0 \) and \( B^2 = -1 \) so that the energy of Eq. (13) becomes zero. This together with the requirement that the \( W^a_0 \) components of this solution are pure imaginary raises doubts about the physical importance of this special case.

If this general Schwarzschild-like solution is responsible for the confinement mechanism in non-Abelian field theories, and if we discard the zero energy pure gauge case, then it is necessary for the Lagrangian to always include scalar fields in order for an acceptable solution to exist. Under these assumptions scalars fields become crucial to the confinement mechanism. This is in contrast to the conventional ideas about confinement, where scalar fields do not play a role.

One possible test for the physical importance of this solution to strong interaction physics would be to see if it could be used to calculate the constituent masses of the light quarks in their various bound states (mesons and baryons). For the light quark bound states (e.g. protons, pions) most of the mass is believed to reside in the gluon fields rather than in the current quark masses. Before any numerical results could be extracted it would first be necessary to specify the arbitrary constant \( C \). The equivalent object in the Schwarzschild solution of general relativity is \( 1/(2GM) \). Using the analogy between Yang-Mills and general relativity it can be argued that \( C \) should be related to the strength of the interaction and the magnitude of the “color” charge carried by the quark. Also one should be able to get an experimental estimate for this constant since \( 1/C \) should be roughly related to the radius of the bound state. Second, our present solution is for an SU(2) gauge theory while QCD is formulated in terms of the SU(3) gauge group. Thus one would need to generalize the present solution to SU(3) or if possible to SU(N). This should be possible by embedding the SU(2) solution in the higher rank gauge group. Finally the most serious obstacle to using this Schwarzschild-like solution to calculate QCD bound states is that the field equations are highly non-linear so that the superposition of two solutions will not necessarily be a solution.
However the present solution might provide a framework for a numerical calculation, or for some approximate phenomenological development for obtaining rough estimates for the constituent quark masses.

Finally, just like the Prasad-Sommerfield solution \[2\], our solution can be seen to carry a topological magnetic charge when the electromagnetic field is embedded into the SU(2) theory via ’t Hooft’s \[7\] generalized, gauge invariant, electromagnetic field strength tensor

\[
F_{\mu\nu} = \partial_\mu(\hat{\phi}^a W^\mu_a) - \partial_\nu(\hat{\phi}^a W^\mu_a) - \frac{1}{g} \epsilon^{abc} \hat{\phi}^a (\partial_\mu \hat{\phi}^b)(\partial_\nu \hat{\phi}^c) \tag{16}
\]

where \( \hat{\phi}^a = \phi^a (\phi^b \phi^b)^{-1/2} \). Using this generalized electromagnetic field strength tensor the magnetic field of our solution is

\[
B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} = -\frac{r_i}{g r^3} \tag{17}
\]

which is the magnetic field of a point monopole of strength \( -\frac{4\pi}{g} \) at the origin. Looking in detail at Eq. (16) it can be seen that only the last term contributes to the magnetic charge of the solution. The fact that the magnetic charge comes entirely from the scalar field, in this gauge, has been explained in terms of the topology of the scalar field \[8\]. The electric field of our solution is different from that of the Prasad-Sommerfield solution. Using the generalized electromagnetic field strength tensor we find that the electric field of our solution is

\[
E_i = -F_{0i} = \frac{r_i}{r} \frac{d}{dr} \left( \frac{J(r)}{r} \right) = \frac{B(2Cr - 1)r_i}{r^3(1 - Cr)^2} \tag{18}
\]

As \( r \to \infty \) this electric field falls off like \( 1/r^3 \), unlike the Prasad-Sommerfield solution which has a \( 1/r^2 \) behaviour for large \( r \). In the Prasad-Sommerfield case the electric field at large \( r \) indicated the presence of some net charge which was distributed in a cloud around the origin. For our solution the behaviour of the electric field at large \( r \) indicates that while there is no net charge, there is some distribution of charge which has nonzero higher order moments (the \( 1/r^3 \) behaviour of the electric field is like that of a dipole charge distribution, although the electric field of Eq. (18) is definitely not that of a point dipole). The charge density of our solution can easily be found by applying \( \rho(r) = \nabla \cdot E \) to the electric field of
Eq. (18). Notice that while the topological magnetic charge associated with our solution is similar to that of the Prasad-Sommerfield solution, the non-topological electric charge is significantly different.

III. CONCLUSIONS

Drawing on the parallels between general relativity and Yang-Mills theory we have discovered an exact Schwarzschild-like solution for an SU(2) gauge field coupled to a massless scalar field. The general solution involved both the time and space components of the gauge fields as well as scalar fields. There were two special cases which occurred: First the time component of the gauge fields could be zero leaving only the scalar fields and space component of the gauge field; second there was the pure gauge solution, where the scalar fields were absent, leaving only the time and space components of the gauge fields. These classical solutions exhibited a form of confinement. Any particle which carries an SU(2) gauge charge and enters the region $r < r_0 = 1/C$ would no longer be able to leave this region. This is analogous to what happens with the Schwarzschild solution in general relativity, where once a particle passes the event horizon it is permanently confined. This is still a long way from demonstrating confinement for QCD, since our entire argument has been for classical Yang-Mills fields, and the ansatz that we adopted is specific to the SU(2) group, depending on the fact that the number of SU(2) generators and the number of spatial dimensions are the same. Still it is encouraging that at the classical level an analytical solution, which seems to exhibit confinement, can be found for a non-Abelian gauge theory.

There are several open questions which the present letter does not discuss. First, are there other known solutions of general relativity which can be translated into Yang-Mills gauge theories? One interesting possibility would be to see if the axially symmetric, rotating mass solution (the Kerr metric) could be used to discover an analogous Yang-Mills solution. Second, just as a Schwarzschild black hole is thought to emit Hawking radiation, the Schwarzschild-like solution presented here might be conjectured to exhibit a similar
effect with respect to virtual particle pairs which carried the non-Abelian gauge charge. Before such a conjecture can be checked the present classical field solution would have to be quantized.

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