HIGHER SIGNATURES FOR THE DERIVED WITT GROUPS

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Abstract. The signature of a quadratic form is used to construct “higher” global signatures from the derived Witt groups to the real cohomology with \( \mathbb{Z} \)-coefficients. We use these signatures to study torsion in the Witt group of a scheme, answering a question of M. Karoubi on bounding the torsion in the Witt group of a real variety, proving an assertion of J. Fasel on the \( I_j \)-cohomology, and describing explicitly the kernel of the total signature in terms of singular cohomology.

Introduction

Let \( F \) be a field of characteristic different from 2. If \( F \) has an ordering \( P \) (See Section 1.1), then any non-degenerate quadratic form \( \phi \) over \( k \) may be split as an orthogonal sum \( \phi \cong \phi_+ \perp \phi_- \), where the form \( \phi_+ \) is positive definite with respect to the ordering and the form \( \phi_- \) is negative definite with respect to the ordering (i.e. \( -\phi_- \) is positive definite). Sylvester’s inertia theorem states that the numbers \( n_+ := \dim \phi_+ \) and \( n_- := \dim \phi_- \) have “inertia” in the sense that they do not change under an isometry of \( \phi \) [KS89, Chapter 1, Section 2, Satz 2]. The number \( n_+ - n_- \in \mathbb{Z} \) is called the signature of \( \phi \) with respect to \( P \) and is denoted by \( \text{sign}_P(\phi) \).

Now let \( X \) be a scheme. The real spectrum of \( X \) is the topological space formed by glueing the real spectra \( \text{sper} \ A \alpha \) of an open affine cover \( \{ A \alpha \} \). The real spectrum of a ring \( A \) has underlying set consisting of pairs \( (p, P) \), where \( p \in \text{spec} \ A \) and \( P \) is an ordering on the residue field \( k(p) \). The global signature is a ring homomorphism

\[
\text{sign} : W(X) \to H^0(X_r, \mathbb{Z})
\]

from the Witt ring of symmetric bilinear forms over \( X \) (c.f. [Kne77]) to the ring of locally constant integer valued functions on \( X_r \). It is defined as follows: for any \( \phi \in W(X) \), \( \text{sign}(\phi) \) is the function which assigns \( (x, P) \), where \( P \) is an ordering on \( k(x) \) and \( i_x : x \to X \) is any point, to \( \text{sign}_P(i_x^*\phi) \). That sign \( (\phi) \) is locally constant, i.e. belongs to \( H^0(X_r, \mathbb{Z}) \), follows from the fact that it is so whenever \( X \) is affine [Mah82, Section 1.3]. The most important result on the global signature was proved by L. Mahé, who showed that if \( X = \text{spec} \ A \) is an affine scheme, then the cokernel is 2-primary torsion [Mah82, Theoreme 3.2]. Equivalently, after inverting 2 (i.e. localization of \( \mathbb{Z} \)-modules with respect to multiplicative set \( S = \{1, 2, 2^2, 2^4, \ldots \} \)), the global signature induces a surjection

\[
\text{sign} : W(A)[1/2] \to H^0(\text{sper} A, \mathbb{Z})[1/2]
\]

of rings. For \( A \) a field this was well-known (c.f. [Lam77, p.34, Theorem 3.4])

Regarding the kernel of the global signature, when \( A \) is a connected ring and \( \text{sper} A \neq \emptyset \), it is the nilradical in the Witt ring [Mah82, Section 1.3]. It follows that when \( A \) is a connected local ring: if \( \text{sper} A \neq \emptyset \), then the kernel is 2-primary torsion; if \( \text{sper} A = \emptyset \), then the Witt group \( W(A) \) is 2-primary torsion (c.f. [Kne81, Theorem 1.2] for the fact that either the
nilradical is 2-primary torsion or the entire Witt ring $W(A)$ is 2-primary torsion). When $A$ is a field, these statements on the kernel are Pfister’s local-global principle [Pfi66, Satz 22].

**Global signature as a morphism of sheaves.**

Let $X$ be a scheme. Let $\mathcal{W}$ denote the Zariski sheaf on $X$ associated to the presheaf $U \mapsto W(U)$. The presheaf $U \mapsto H^0(U_r, \mathbb{Z})$ is a Zariski sheaf isomorphic to the sheaf $\operatorname{supp}_* \mathbb{Z}$ (c.f. Lemma 2.7). When $X$ is noetherian, for any Zariski sheaf $\mathcal{F}$ on $X$, we define $\mathcal{F}^{[1/2]}$ to be the sheaf $U \mapsto \mathcal{F}(U)[1/2]$ (c.f. Corollary (2.5) for the fact that it is a sheaf). The sheaf $\operatorname{supp}_* \mathbb{Z}^{[1/2]}$ on $X$ can be identified with the sheaf $U \mapsto H^p(U_r, \mathbb{Z})^{[1/2]}$ for $p \geq 0$ (c.f. Lemmas 2.7 and Corollary 2.5).

It follows that the global signature induces a morphism of Zariski sheaves

$$\operatorname{Sign} : \mathcal{W} \to \operatorname{supp}_* \mathbb{Z}$$

which we call the *global signature morphism of sheaves*. While the global signature is not in general an isomorphism after inverting 2 (for example, see Proposition 0.10), the global signature morphism of sheaves is!

**0.1. Theorem.** Let $X$ be a regular, noetherian scheme. After inverting 2, the global signature morphism of sheaves $\operatorname{Sign}$ induces an isomorphism

$$\operatorname{Sign} : \mathcal{W}^{[1/2]} \cong \operatorname{supp}_* \mathbb{Z}^{[1/2]}$$

of Zariski sheaves on $X$.

Indeed, for any point $x \in X$, $\operatorname{Sign}$ induces on stalks

$$\operatorname{Sign}_x : W(\mathcal{O}_{X,x}) \to H^0(\operatorname{sper} \mathcal{O}_{X,x}, \mathbb{Z})$$

the global signature. Since $\mathcal{O}_{X,x}$ is local and connected whenever $\mathcal{O}_{X,x}$ is regular, this theorem follows from the facts on the kernel and cokernel of the global signature recalled earlier.

**On an assertion of J. Fasel.**

Let $F$ be a field of characteristic different from 2. Let $X$ be an integral, noetherian, regular, separated $F$-scheme. In the case that $F$ has no ordering, it follows that $X_r$ is empty, and that the Witt groups $W^i(X)$ are 2-primary torsion, bounded if $\operatorname{vcd}_2(X)$ is finite. In this case, our results produce statements which are either trivial or already known. Nevertheless they remain true statements, so we refrain from making the restriction to fields $F$ with an ordering.

The sections of the Witt sheaf over any open subscheme $U$ in $X$ is a subgroup of $W(K)$, where $K$ is the function field of $X$. For $j \geq 0$, define $\mathcal{I}^j$ to be the Zariski presheaf that assigns to every open $U$ the pullback

$$\mathcal{W}(U) \longrightarrow W(K)$$

$$\mathcal{I}^j(U) \longrightarrow I^j(K)$$

In fact, $\mathcal{I}^j$ is a sheaf. This follows by using the definition of $\mathcal{I}^j$ and the fact that $\mathcal{W}$ is a sheaf. In Theorem 0.2 (4) below, we use the restriction to $\mathcal{I}^j$ of the signature sheaf morphism $\operatorname{Sign}$ in order to answer a question about the so-called $I^j$-cohomology, that is, the cohomology groups $H^*_\text{Zar}(X, \mathcal{I}^j)$. In the introduction to his paper [Fas13] J. Fasel has
said that for a smooth variety $X$ over the reals the “$I^j$-cohomology of $X$ is the analogue of the singular cohomology groups $H^*_\text{sing}(X(\mathbb{R}), \mathbb{Z})$”. He proved they are isomorphic when $X$ is affine and $j \geq d + 1$ [Fas11]. The next theorem proves this analogy over any base field using real cohomology. Before stating it, recall that the virtual cohomological 2-dimension of $X$, denoted $vcd_2(X)$, is defined to be $vcd_2(X) = \sup\{vcd_2(\eta) \mid \eta \text{ runs through the generic points of } X\}$. For example, if $X$ is an algebraic variety over $F$ with function field $K$ and $vcd_2(F) < \infty$, then $vcd_2(X) = vcd_2(K) = \dim X + vcd_2(F)$. In particular, if $X$ is a variety over $\mathbb{R}$, then $vcd_2(X) = \dim X$.

0.2. Theorem. Let $X$ be an excellent, integral, noetherian, regular, separated $F$-scheme, $F$ a field of characteristic different from 2. If $vcd_2(X)$ is finite, say $vcd_2(X) = s$, then:

1. For $j \geq s + 1$, $\text{Sign}$ restricts to the subsheaf $\mathcal{I}^j$ to give an isomorphism of sheaves

$$\text{Sign}_j : \mathcal{I}^j \xrightarrow{\sim} \text{supp}_* 2^j \mathbb{Z}$$

is isomorphic via $\text{Sign}_j$ to

$$\text{supp}_* 2^{j+1} \mathbb{Z} \rightarrow \text{supp}_* 2^j \mathbb{Z} \rightarrow \text{supp}_* 2^j / \text{supp}_* 2^{j+1} \mathbb{Z} \cong \text{supp}_* \mathbb{Z} / 2\mathbb{Z}$$

2. Let $\mathcal{H}^j$ denote the Zariski sheaf on $X$ obtained by sheafifying the presheaf $U \mapsto H^j_{\text{ét}}(U, \mathbb{Z}_2)$ on $X$. For $j \geq s + 1$, the short exact sequence of sheaves

$$\mathcal{I}^{j+1} \rightarrow \mathcal{I}^j \rightarrow \mathcal{H}^j$$

3. for $j \geq s + 1$, $\text{Sign}_j$ induces an isomorphism of cohomology groups

$$H^p_{\text{Zar}}(X, \mathcal{I}^j) \cong H^p(X_r, 2^j \mathbb{Z})$$

for all $p \geq 0$.

4. for $j \geq s + 1$, when $F = \mathbb{R}$ and $X$ is a smooth, quasi-projective, algebraic variety, the isomorphism of (3) determines an isomorphism of cohomology groups

$$H^p_{\text{Zar}}(X, \mathcal{I}^j) \cong H^p_{\text{sing}}(X(\mathbb{R}), 2^j \mathbb{Z}) \cong H^p_{\text{sing}}(X(\mathbb{R}), \mathbb{Z})$$

for all $p \geq 0$.

Additionally, from Theorem 0.2 (2) above, it follows that there is an isomorphism $\mathcal{H}^j \xrightarrow{\text{sign}} \text{supp}_* \mathbb{Z} / 2\mathbb{Z}$ for $j \geq s + 1$. For smooth $\mathbb{R}$-varieties this is a classic theorem of R. Parimala and J. Colliot-Thelene [CTP90] and is due to C. Scheiderer in the general case of schemes with 2 invertible and $vcd_2(X) < \infty$ [Sch94, Corollary (19.5.1)]. The proof of Theorem 0.2 appears in Section 2.

Higher signatures.

Now let $X$ be a regular, noetherian, separated $F$-scheme. A main idea of the article is to use the group homomorphism

$$(0.3) \quad H^1_{\text{Zar}}(X, \mathcal{W}) \rightarrow H^1(X_r, \mathbb{Z})$$

induced by $\text{Sign}$ to define “higher” signature maps.

Recall that the derived Witt groups $W^i(X)$ are 4-periodic, $W^i(X) \cong W^{4+i}(X)$, so we need only define signatures for $W^0(X), W^1(X), W^2(X)$ and $W^3(X)$. For $0 \leq i \leq 3$, edge maps in the coniveau spectral sequence for the derived Witt groups (see Section 3.3) define group homomorphisms

$$W^i(X) \rightarrow H^1_{\text{Zar}}(X, \mathcal{W})$$
and by composing them with the maps (0.3) we obtain a group homomorphism
\[ \text{sign}^i : W^i(X) \to H^i(X_r, \mathbb{Z}) \]
that we call the \textit{i-th global signature}. We will also refer to these maps as the \textit{higher global signatures} of \(X\). See Section 3.6 for details.

\textbf{Atiyah-Hirzebruch type Spectral sequence for Witt groups.}
Recall that for any group \(M\), the real cohomology \(H^*(X_r, M)\) of a quasi-projective \(\mathbb{R}\)-variety is the singular cohomology \(H^*(X(\mathbb{R}), M)\) of the real points \(X(\mathbb{R})\). Hence, one might think of Theorem 0.5 below as giving an Atiyah-Hirzebruch type spectral sequence for Witt groups. In Section 3 we recall the construction of this spectral sequence for readers not familiar with the coniveau spectral sequence and explain the details of the proof, although it is almost an immediate consequence of Theorem 0.1.

\textbf{0.5. Theorem.} Let \(X\) be a noetherian, regular, separated \(F\)-scheme, \(F\) a field of characteristic different from 2. The coniveau spectral sequence for the derived Witt groups, after inverting 2, determines a spectral sequence
\[ E_2^{p,q} = \begin{cases} H^p(X_r, \mathbb{Z}[1/2]) & \text{if } q \equiv 0 \mod 4 \\ 0 & \text{otherwise} \end{cases} \implies W^{p+q}(X)[1/2] \]
abutting to the derived Witt groups with 2 inverted. For \(r \geq 1\), the differentials \(d_r\) are of bidegree \((r, r-1)\). The groups \(H^p(X_r, \mathbb{Z}[1/2])\) are the cohomology of the real spectrum with coefficients in \(\mathbb{Z}[1/2]\). For integers \(0 \leq i \leq 3\), the higher global signatures \(\text{sign}^i\), after inverting 2, are edge maps in this spectral sequence. When the dimension of \(X\) is finite, this spectral sequence strongly converges.

The next corollary immediately follows.

\textbf{0.6. Corollary.} If \(\dim X \leq 3\), then the spectral sequence of Theorem 0.5 collapses on the \(E_2\)-page and the higher signature maps induce isomorphisms of groups after inverting the integer 2
\[
\begin{align*}
W^0(X)[1/2] & \xrightarrow{\text{sign}^0} H^0(X_r, \mathbb{Z}[1/2]) \\
W^1(X)[1/2] & \xrightarrow{\text{sign}^1} H^1(X_r, \mathbb{Z}[1/2]) \\
W^2(X)[1/2] & \xrightarrow{\text{sign}^2} H^2(X_r, \mathbb{Z}[1/2]) \\
W^3(X)[1/2] & \xrightarrow{\text{sign}^3} H^3(X_r, \mathbb{Z}[1/2])
\end{align*}
\]

In Section 3, we explain how Mahé’s result on the cokernel of the total signature being 2-primary is equivalent to the differentials leaving \(H^0(X_r, \mathbb{Z}[1/2])\) in this spectral sequence being zero. So when \(X\) is affine, we obtain the following corollary.

\textbf{0.7. Corollary.} If \(X\) is affine, then the differentials leaving \(H^0(X, \mathbb{Z}[1/2])\) in the spectral sequence of Theorem 0.5 are zero. Equivalently, the images of the differentials leaving \(H^0(X, W)\) in the coniveau spectral sequence for Witt groups are 2-primary torsion.

\textbf{0.8. Question.} Does the spectral sequence of Theorem 0.5 collapse on the \(E_2\)-page?

For fields and local rings \(F\), it is classical that torsion in the Witt group \(W(F)\) is always 2-primary. The next corollary generalizes this to a wide class of schemes.
0.9. Corollary. Let $X$ be a regular, noetherian, separated $F$-scheme, $F$ a field of characteristic different from 2. If $\dim X \leq 3$, then torsion in $W(X)$ is 2-primary.

It is known that the Witt group of a ring can have odd torsion.

In [Mah82, start of Section 2], Mahé remarked that one might ask if the kernel of the global signature is torsion in general. He stated that Knebusch has answered this question negatively by providing the example of the ring of real-valued continuous functions on the real sphere, which has $W(C(S^4))$ torsion-free [Kne77, p.189].

Using the spectral sequence of Theorem 0.5, we are able to describe the kernel of the global signature in terms of real cohomology. The following proposition is an example of how, for real varieties, torsion in the Witt group and the kernel of the signature can be understood in terms of singular cohomology.

0.10. Proposition. If $X$ is a smooth and quasi-projective algebraic variety over $\mathbb{R}$ and $4 \leq \dim X \leq 7$, then there is an exact sequence of groups

\begin{equation}
0 \to H^{4}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}[1/2]) \to W(X)[1/2] \xrightarrow{\text{sign}} H^{0}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}[1/2]) \to 0
\end{equation}

In particular:

(1) if the singular cohomology group $H^{4}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z})$ contains an element which is torsion of odd order, then so does $W(X);$ 

(2) if $H^{4}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z})$ contains an element which is not torsion, then the kernel of the global signature is not torsion.

This proposition is a particular case of a more general theorem proving the same for derived Witt groups, namely Theorem 3.6, which is proved in Section 3.

Finally, for real varieties, one can also bound the ranks of the Witt groups using the spectral sequence of Theorem 0.5.

0.12. Theorem. Let $X$ be a smooth and quasi-projective algebraic variety over $\mathbb{R}$ of dimension $d = \dim X$. Then $X$ satisfies the hypotheses of Theorem 0.13. Let $b_i$ denote the $i$th Betti number of $X(\mathbb{R})$. For each integer $0 \leq j \leq 3$, let $N_j = \left\lfloor \frac{d-j}{4} \right\rfloor$, where we set $N_j = 0$ if $d-j \leq 0$. Then: the ranks of the Witt groups of $X$ are bounded in terms of the Betti numbers:

\[
\begin{align*}
\text{rank}W^0(X) &\leq \sum_{i=0}^{N_0} b_{4i} \\
\text{rank}W^1(X) &\leq \sum_{i=0}^{N_1} b_{4i+1} \\
\text{rank}W^2(X) &\leq \sum_{i=0}^{N_2} b_{4i+2} \\
\text{rank}W^3(X) &\leq \sum_{i=0}^{N_3} b_{4i+3}
\end{align*}
\]

The proof appears in Section 3.
On a question of Karoubi on bounded torsion.

Another application is to a question of M. Karoubi asking if the order of the torsion elements in the Witt group of a real variety is bounded. M. Karoubi and C. Weibel have a work in progress which answers this question affirmatively with precise bounds. The following theorem answers this question by giving sufficient conditions for torsion to be bounded over a general base field $F$.

0.13. Theorem. Let $F$ be a field of characteristic different from 2. Let $X$ be an excellent, integral, noetherian, regular, separated $F$-scheme of finite Krull dimension of $X$ and of finite virtual cohomological 2-dimension. If the real cohomology groups $H^p(X_r, \mathbb{Z})$ are finitely generated, then:

1. the torsion in the Witt groups $W^i(X)$ is of bounded order;
2. each group $W^i(X)[\frac{1}{2}]$ is finitely generated;
3. the rank of the Witt group $W^i(X)$ is finite;
4. each group $W^i(X)$ splits as a direct sum $W^i(X) \simeq W^i_{tor}(X) \oplus \mathbb{Z}^r$ where $W^i_{tor}(X)$ denotes the subgroup of torsion elements in $W^i(X)$ and $r$ is the rank of $W^i(X)$;
5. $W^i_{tor}(X)$ is a (possibly infinite) direct sum of cyclic groups, only finitely many of which have exponent a power of an odd-prime.

This theorem is proved in Section 4.

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Notation. Throughout the article, $F$ will always denote a field of characteristic different from 2. An algebraic variety over $F$, or $F$-variety will be an integral, separated scheme of finite type over $F$.

1. Reminder on the signature

In this section we recall some classical facts about the signature which will be “globalized” to sheaves in Section 2 and used throughout the paper. We begin by recalling the definition of the signature following the excellent text of Knebusch and Scheiderer on the subject of the real spectrum [KS89].

1.1. Let $F$ be a field. An ordering on $F$ is a subset $P \subset F$ satisfying the following:

1. $P + P \subset P$, $PP \subset P$;
2. $P \cap (-P) = 0$;
3. $P \cup (-P) = F$.

The pair $(F, P)$ is called an ordered field. Given any such ordering $P$, one defines $a \leq b$ if $b - a \in P$. It follows from the axioms that if $F$ is nontrivial, then $1 > 0$. Given an ordering $P$ on $F$, we define a function $\text{sign}_P : F^* \to \{1, -1\}$ by defining $\text{sign}_P(a)$ to be 1 if $a \in P$ and $\text{sign}_P(a)$ to be $-1$ if $a \in -P$.

1.2. If $F$ has an ordering $P$, then any non-degenerate quadratic form $\phi$ over $F$ splits as an orthogonal sum $\phi \simeq \phi_+ \perp \phi_-$, where the form $\phi_+$ is positive definite with respect to the ordering (for all $0 \neq v, q(v) > 0$ with respect to $P$) and the form $\phi_-$ is negative definite with respect to the ordering (i.e. $-\phi_-$ is positive definite). The numbers $n_+ := \dim \phi_+$ and
n_ := \text{dim } \phi_ do not change under an isometry of } \phi \text{ [KS89, Chapter 1, Section 2, Satz 2]. The integer sign}_P(\phi) := n_ + n_- is defined to be \text{signature of } [\phi] \text{ with respect to } P. \text{ As the signature of the hyperbolic form is trivial, assigning an isometry class } [\phi] \text{ to its signature sign}_P(\phi) \text{ defines a map }

\text{sign}_P : W(F) \rightarrow \mathbb{Z}

which is clearly a homomorphism of rings.

Let \( H^0(\text{sper } F, \mathbb{Z}) \) (often written elsewhere \( C(X_F, \mathbb{Z}) \)) denote the set of maps which are continuous with respect to the constructible topology on sper \( F \), that is, the weakest topology for which the constructible functions (functions which are a finite sum \( \sum m_i S_i \) where the sets \( S_i \) are constructible subsets of sper \( F \)) are continuous. For every isometry class \([\phi]\), assigning an ordering \( P \) to sign\(_P(\phi) \) determines a map 

\text{sign} : W(F) \rightarrow H^0(\text{sper } F, \mathbb{Z})

since for any \( a \neq 0 \), sign\((a) = 1_{\{a>0\}} - 1_{\{a<0\}} \) which is a constructible function. The map \text{sign} is a morphism of rings because sign\(_P(\phi) \) is and it is called the total signature. If \( F \) has no ordering, then sign is trivial.

1.3. This definition of signature agrees with the definition often used elsewhere. Indeed, if \( F_P \) is a real closure of \( F \) with respect to \( P \), then the composition \( W(F) \rightarrow W(F_P) \rightarrow \mathbb{Z} \) agrees with \( W(F) \rightarrow \mathbb{Z} \) on generators of \( W(F) \): for any \( a \in F^* \), by definition \( \text{sign}_P(1_{\{F^P\}}) = \text{dim}(\{F^P, +\}) - \text{dim}(\{F^P, -\}) = \text{dim}(\{+, \} - \text{dim}(\{-, \}) = \text{sign}_P(\langle a \rangle) \).

1.4. As hyperbolic forms have even rank, assigning a quadratic form to its rank modulo 2 determines a ring homomorphism \( e : W(F) \rightarrow \mathbb{Z}/2\mathbb{Z} \). The kernel of \( e \) is denoted \( I(F) \) and is called the fundamental ideal of \( F \). Since sign\(_P(\phi) = \text{dim } \phi_ - \text{dim } \phi_ \) (see 1.2.) and \( \text{dim } \phi_ + \text{dim } \phi_ \mod 2 = \text{dim } V \), the diagram below is commutative

\[
\begin{array}{ccc}
W(F) & \xrightarrow{\text{sign}} & \mathbb{Z} \\
\downarrow{\text{mod } 2} & & \downarrow{\text{mod } 2} \\
\mathbb{Z}/2\mathbb{Z} & & \\
\end{array}
\]

For \( a \in F^* \), using the facts from (1.1.) and (1.2.) we find that the signature of the Pfister form \( \langle a \rangle \) is the function

\[
\text{sign}(\langle a \rangle) = \text{sign}(\langle 1, -a \rangle) = \text{sign}(\langle 1 \rangle) - \text{sign}(\langle a \rangle) = (1_{\{1>0\}} - 1_{\{1<0\}}) - (1_{\{a>0\}} - 1_{\{a<0\}}) = 1 - 1_{\{a>0\}} + 1_{\{a<0\}} = \begin{cases} 0 & \text{if } a > 0 \\ 2 & \text{if } a < 0 \end{cases} = 21_{\{a<0\}}.
\]

Since \text{sign} is a ring homomorphism we find that

\[
\text{sign}(\langle a_1, \ldots, a_j \rangle) = 2^j 1_{\{a_1<0, \ldots, a_j<0\}}
\]
The powers of the fundamental ideal $I^j(F)$ are generated by Pfister forms $\langle a_1, \cdots, a_j \rangle$, so it follows from what was just shown that the signature induces a homomorphism of rings without unit

$$\text{sign} : I^j(F) \to H^0(\text{sper } F, 2^j\mathbb{Z})$$

Since $2 = \langle 1, 1 \rangle$ in $W(F)$ and sign is a ring homomorphism, it follows that the diagram below commutes

$$
\begin{align*}
I^j(F) & \xrightarrow{\text{sign}} H^0(\text{sper } F, 2^j\mathbb{Z}) \\
\downarrow^2 & \quad \downarrow^2 \\
I^{j+1}(F) & \xrightarrow{\text{sign}} H^0(\text{sper } F, 2^{j+1}\mathbb{Z})
\end{align*}
$$

and hence one obtains a morphism of filtered colimits.

(1.1) \[
\lim_{\to}(W(F) \xrightarrow{2} I(F) \xrightarrow{2} I^2(F) \xrightarrow{2} \cdots) \xrightarrow{\text{sign}} H^0(\text{sper } F, \mathbb{Z})
\]

The following is a theorem due to Arason and Knebusch.

1.2. Lemma. The morphism (1.1) is an isomorphism of groups.

1.5. Virtual cohomological 2-dimension and signature.

Since the cokernel of sign is 2-primary torsion, one defines the reduced stability index $\text{st}_v(F)$ to be the integer $n$ if the cokernel of sign : $W(F) \to H^0(\text{sper } F, \mathbb{Z})$ has exponent $2^n$ and we write $\text{st}_v(F) = \infty$ otherwise. Recall that for any field $F$, the virtual cohomological 2-dimension is defined to be

$$\text{vcd}_2(F) = \text{cd}_2(F(\sqrt{-1}))$$

where $\text{cd}_2(F(\sqrt{-1}))$ denotes the cohomological 2-dimension of the field $F(\sqrt{-1})$.

The following Lemma collects in a way we find convenient important known results.

1.3. Lemma. Let $F$ be a formally real field and $s \geq 0$ an integer. Then, the following are equivalent:

(i) $\text{vcd}_2(F) \leq s$;
(ii) for $j \geq s + 1$, $I^j(F(\sqrt{-1})) = 0$;
(iii) for $j \geq s + 1$, $I^j(F)$ is torsion free and $2I^{j-1}(F) = I^j(F)$;
(iv) for $j \geq s + 1$, $I^j(F)$ is torsion free and $\text{st}_v(F) \leq s$.
(v) for $j \geq s + 1$, $\text{sign}_j : I^j(F) \to H^0(\text{sper } F, 2^j\mathbb{Z})$ is an isomorphism of rings.

Proof. For lack of reference, we prove the equivalence of (i) and (ii). To prove that (i) implies (ii), use the Milnor conjecture, as proved in [OVV07, Theorem 4.1], to obtain that for every integer $n \geq 0$ there is a short exact sequence

$$0 \to I^{j+1}(F(\sqrt{-1})) \to I^j(F(\sqrt{-1})) \to H^j(\text{Gal}(F(\sqrt{-1}), \mathbb{Z}/2\mathbb{Z})$$

So (i) implies that $H^j(\text{Gal}(F(\sqrt{-1}), \mathbb{Z}/2\mathbb{Z})$ is trivial for $j \geq s + 1$. Then $I^j(F(\sqrt{-1})) = \bigcap_{k \geq j} I^k(F(\sqrt{-1}))$, and the intersection is trivial by the Arason-Pfister Haupsatz. We have that (ii) implies (i) since $I^j(F(\sqrt{-1}))$ surjects onto $H^j(\text{Gal}(F(\sqrt{-1}), \mathbb{Z}/2\mathbb{Z})$. For the proof of the equivalence of (ii) and (iii), see c.f. [EKM08, Corollary 35.27 (1) and (4)]. The equivalence of (iii) and (iv) was proved by L. Bröcker [Brö74, ZurTheoreiDerquad, Satz 3.17]. The equivalence of equivalence of (iv) and (v) by J. Arason and M. Knebusch [AK78, p.184].
2. ON THE TOTAL SIGNATURE AS A MORPHISM OF SHEAVES

2.1. The Witt sheaf.
Since the Gersten conjecture is known for all local rings of any regular $F$-scheme $X$, the next lemma follows from a result of P. Balmer and C. Walter on the coniveau spectral sequence [BW02, Lemma 8.4 (a)].

2.1. Lemma. Let $X$ be an integral, noetherian, regular, separated, $F$-scheme. For each $x \in X^1$ choose a uniformizing parameter $\pi$. For any open subscheme $U$ in $X$, the sections of the Witt sheaf over $U$ are

$$W(U) := \ker(W(K) \xrightarrow{\otimes \partial_x} \bigoplus_{x \in U^1} W(k(x)))$$

where $\partial_x$ is, for every $x \in U^1$, the second residue morphism for $\mathcal{O}_{X,x}$ and $K$ is the function field of $X$.

The following lemma restates well-known facts on the second residue (c.f. [MH73, Chap. IV (1.2)-(1.3)].

2.2. Lemma. Let $A$ be a discrete valuation ring with fraction field $K$. Choose a uniformizing parameter $\pi$ for $A$. Then,

1. every rank one quadratic form over $K$ is isometric to $\langle c \rangle$, where $c = b\pi^n$, $b$ is a unit in $A$, and either $n = 0$ or $n = 1$;
2. the second residue $\partial_x$ has the following description on rank one forms $\langle c \rangle$ as in (1):

$$\partial(\langle c \rangle) = \begin{cases} \langle b \rangle & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases}$$

2.2. Cohomology of the real spectrum.
In [Sch94], C. Scheiderer developed a theory of real cohomology for schemes. It “globalizes” to all schemes the singular cohomology of the real points of a real variety in the same way that étale cohomology globalizes the singular cohomology of the complex points of a complex variety. We recall the definition and some properties we will need following [Sch94].

2.3. The real spectrum.
We briefly recall some of the notation and facts we will use following C. Scheiderer. For the facts below see [Sch94, (0.4)]. The real spectrum of a ring $A$ is denoted by sper $A$. As a set it consists of all pairs $\xi = (p, P)$ with $p \in \text{spec } A$ and $P$ an ordering of the residue field $k(p)$. For any points $\xi \in \text{ sper } A$, let $k(\xi)$ denote the real closure of the ordered field $k(p)$ with respect to $P$. For $a \in A$, write $a(\xi) > 0$ to indicate that the image of $a$ in $k(\xi)$ is positive. The sets of the form $D(a) := \{\xi \in \text{ sper } A : a(\xi) > 0\}$, $a \in A$, form a subbasis of open sets for the topology on sper $A$. The support map $\text{ supp } : \text{ sper } A \to \text{ spec } A$, $(p, P) \mapsto p$ is continuous, in fact, spectral. The real spectrum of a scheme $X$ is the topological space $X_r$ formed by glueing the real spectra of its open affine subschemes. This does not depend on the open cover of $X$ that was chosen. The support map for affine schemes induces, for any scheme $X$, a support map $\text{ supp } : X_r \to X$. For a point $x \in X$ of a scheme, we will often denote sper $k(x)$ by $x_r$. 
2.4. The real site.
Let $X$ be a scheme. First we recall the definition of the real site of $X$, which we will also denote by $X_r$. It is the pair consisting of the category $O(X_r)$ of open subsets of $X_r$ equipped with the "usual" coverings, i.e. a family of open subspaces $\{U_\lambda \to U\}$ is a covering of $U \in O(X_r)$ if $U = \cup U_\lambda$. The category of abelian sheaves, i.e. sheaves of abelian groups, on $X_r$ is denoted $\text{Ab}(X_r)$. If $X$ is quasi-compact, then $X_r$ is quasi-compact. It follows that when $X$ is noetherian, the real site is also noetherian, i.e. every open $U \in O(X_r)$ is quasi-compact. For any abelian group $A$, we will also denote by $A$ the sheaf on $X_r$ obtained by sheafifying the presheaf $U \mapsto A$, $U$ any open in $X_r$. Such a sheaf is called a constant sheaf.

2.5. Real cohomology.
For a scheme $X$ and a sheaf $F$ on $X_r$, the real cohomology of $X$, or cohomology of the real spectrum of $X$, is the derived functors $R^p\Gamma F$ of the global sections functor $\Gamma : \text{Ab}(X_r) \to \text{Ab}$.

The following is well-known.

2.3. Lemma. Let $R$ be a ring and $M$ an $R$-module. For any element $r \in R$, the localization $A[1/r]$ of the $R$-module $M$ with respect to the multiplicative set $S = \{1, r, r^2, r^3, \ldots\}$ is equal as an $R$-module to the direct limit $\lim_{\to}(M \to M \to \cdots)$.

The following Lemma recalls well-known facts on Grothendieck topologies, c.f. [Tam94, 3.2.3 i)] for (1) below and [Tam94, Theorem (3.11.1)] for (2) below.

2.4. Lemma. Let $X$ be a scheme and let $X_t$ be either the real site $X_r$ or the Zariski site $X_{\text{Zar}}$. Then:

1. direct limits of sheaves $\lim_{\to}(F_i)$ exist in $\text{Ab}(X_t)$ and equal the sheaf associated to the presheaf $U \mapsto \lim_{\to} F_i(U)$;
2. if $X$ is noetherian, then for any $p \geq 0$ and any open $U$ in $X_t$, the canonical morphisms $F_i \to \lim_{\to} F_i$ induce an isomorphism $\lim_{\to} H^p(U_t, F_i) \to H^p(U_t, \lim_{\to} F)$ of groups.

2.5. Corollary. Let $X$ be a scheme, let $X_t$ be either the real site $X_r$ or the Zariski site $X_{\text{Zar}}$, and let $F \in \text{Ab}(X_t)$. Then:

1. the sheafification of the presheaf $U \mapsto F(U)[1/2]$ equals $\lim_{\to} (F \to F \to \cdots)$;
2. if $X$ is noetherian, then for any $p \geq 0$ and any open $U$ in $X_t$, the canonical morphisms $F \to \lim_{\to} (F \to F \to \cdots)F$ induce an isomorphism $H^p(U_t, F)[1/2] \to H^p(U_t, F[1/2])$ of groups.

2.6. Definition. Let $X$ be a scheme, let $X_t$ be either the real site $X_r$ or the Zariski site $X_{\text{Zar}}$, and let $F \in \text{Ab}(X_t)$. We denote by $F[1/2]$ the sheaf $\lim_{\to} (F \to F \to \cdots)$. If $X$ is noetherian, then it follows from corollary (2) immediately above that the sections $F[1/2](U)$ can be identified with the sections $F(U)[1/2]$ over any open $U$ in $X_t$.

2.7. Lemma. Let $X$ be a scheme. For any sheaf $F \in \text{Ab}(X_r)$, $H^p(X_{\text{Zar}}, \text{supp}_xF) \simeq H^p(X_r, F)$.
Proof. Note $\text{supp}_s$ sends injective sheaves to sheaves which are acyclic with respect to global sections because $X_s$ is equivalent to the real étale site $X_{\text{ret}}$ and the $X_{\text{Zar}}$ is coarser than $X_{\text{ret}}$ [Sch94, Section 19.1]. Hence one may use the Grothendieck spectral sequence for the composition of functors $\text{supp}_s$ and the global sections functor. This spectral sequence has $E_2^{p,q} = H^p_{\text{Zar}}(X, R^q\text{supp}_s F)$ and abuts to $H^{p+q}(X, F)$ [Tam94, Theorem (2.3.5)]. The functor $\text{supp}_s : \text{Ab}(X) \to \text{Ab}(X_{\text{Zar}})$ is exact [Sch94, Theorem 19.2], hence for $q > 0$ the sheaves $R^q\text{supp}_s F$ vanish. Therefore the edge maps in this spectral sequence determine isomorphisms $H^p_{\text{Zar}}(X, \text{supp}_s Z) \xrightarrow{\cong} H^p(X, Z)$ for $p \geq 0$. \hfill \qed

2.6. Gersten type resolution for real cohomology. Next we recall the work of C. Scheiderer [Sch95] in which he constructs a “Bloch-Ogus” style complex that computes the real cohomology. The codimension of support filtration on $X$ determines a coniveau spectral sequence for real cohomology. Scheiderer shows that for regular excellent schemes $X$ this spectral sequence has $E_1$ page zero except for the complex $C^\bullet(W, G) := E_1^{0,0}$ and hence obtains the result below.

2.8. Proposition. [Sch95, 2.1 Theorem] Let $X$ be a regular excellent scheme. Let $W$ be an open constructible subset of $X$, and let $F$ be a locally constant sheaf on $W$. Then there is a complex $C^\bullet(W, F)$ of abelian groups

\begin{equation}
0 \rightarrow \bigoplus_{x \in X^0} H^0_W(W, F) \rightarrow \bigoplus_{x \in X^1} H^1_W(W, F) \rightarrow \bigoplus_{x \in X^2} H^2_W(W, F) \rightarrow \cdots,
\end{equation}

natural in $W$ and $F$, whose $q$th cohomology group is canonically isomorphic to $H^q(W, F)$, $q \geq 0$. Here $H^q_W(W, F) := H^q_W(\text{sper} \mathcal{O}_{X,x} \cap W, F)$. This complex is contravariantly functorial for flat morphisms of schemes.

2.10. Definition. Let $(A, m)$ be a discrete valuation ring such that the residue field has an ordering. Let $K$ denote the fraction field of $A$ and let $\pi$ be a uniformizing parameter for $A$. Let $M$ be an abelian group. It is well-known that for any ordering $\zeta \in \text{sper} A/m$, there are exactly two orderings on $K$ which restrict to $A/m$, say $\eta_+, \eta_-$, where $\text{sign}_{\eta_+}(\pi) = 1$ and $\text{sign}_{\eta_-}(\pi) = -1$. Assigning $s \in H^0(\text{sper} K, M)$ to the map $\zeta \mapsto \beta_\pi(s)(\zeta) := s(\eta_+) - s(\eta_-)$ defines a map $\beta_\pi : H^0(\text{sper} K, M) \rightarrow H^0(\text{sper} A/m, M)$ which is a group homomorphism. For the proof that $\beta_\pi(s)$ is locally constant, that is, $\beta_\pi(s) \in H^0(\text{sper} A/m, M)$ see [Sch95, Proof of Lemma (1.3), it follows from the hypothesis on the residue field of $A$ that it is excellent].

2.11. Lemma. Let $A$ be an excellent discrete valuation ring. The morphism $\beta_\pi$ of Definition 2.10 has the following description on elements $s = \text{sign}(\langle c \rangle)$, where $c = b\pi^n$, $b$ is a unit in $\mathcal{O}_{X,x}$, and either $n = 0$ or $n = 1$:

$$
\beta_\pi(\text{sign}(\langle c \rangle))(\zeta) = \begin{cases} 
2\text{sign}(\langle b \rangle) & \text{if } n = 1 \\
0 & \text{if } n = 0
\end{cases}
$$
Proof. Let \( c = b\pi^n \), where \( b \) is a unit in \( \mathcal{O}_{X,x} \), and either \( n = 0 \) or \( n = 1 \). Using the facts established in Section 1 we have the following equalities which will prove the lemma:

\[
\partial_x (\text{sign} (\langle c \rangle ))(\zeta) = \text{sign}_{\eta_+}(\langle c \rangle ) - \text{sign}_{\eta_-}(\langle c \rangle )
\]

\[
= \text{sign}_{\eta_+}(c) - \text{sign}_{\eta_-}(c)
\]

\[
= \begin{cases} 
\text{sign}_{\zeta}(\overline{c}) - \text{sign}_{\zeta}(\overline{c}) & \text{if } n = 0 \text{ (both orderings restrict to } \zeta) \\
\text{sign}_{\eta_+}(\overline{b\pi}) - \text{sign}_{\eta_-}(\overline{b\pi}) & \text{if } n = 1 
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } n = 0 \\
\text{sign}_{\eta_+}(\overline{b})\text{sign}_{\eta_+}(\pi) - \text{sign}_{\eta_-}(\overline{b})\text{sign}_{\eta_+}(\pi) & \text{if } n = 1 
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } n = 0 \\
\text{sign}_{\eta_+}(\overline{b}) + \text{sign}_{\eta_-}(\overline{b}) & \text{if } n = 1 \text{ (by definition of } \eta_+ \text{ and } \eta_-) 
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } n = 0 \\
\text{sign}_{\zeta}(\overline{b}) + \text{sign}_{\zeta}(\overline{b}) & \text{if } n = 1 \text{ (both orderings restrict to } \zeta) 
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } n = 0 \\
2\text{sign}_{\zeta}(\overline{b}) & \text{if } n = 1 
\end{cases}
\]

\[\qed\]

The following Lemma is based on the proof of [Sch95, 2.6 Proposition] where \( F = \mathbb{Z}/2\mathbb{Z} \).

2.12. Lemma. Let \( X \) be a regular excellent scheme which is integral with function field \( K \). Let \( x \in X^1 \), let \( \pi \) be a choice of uniformizing parameter for \( \mathcal{O}_{X,x} \), and let \( A \) be a constant sheaf. Denote by

\[\partial : H^0(\text{sper } K, A) \to H^1_{X_r}(X_r, A)\]

the map induced by the differential \( C^0(X_r, A) \to C^1(X_r, A) \). Then, there is an isomorphism \( \iota_\pi : H^1_{X_r}(X_r, \mathbb{Z}) \to H^0(x_r, \mathbb{Z}) \) for which \( \iota_\pi(\partial) = \beta_\pi \).

Proof. Let \( X' = \text{sper } \mathcal{O}_{X,x} \), \( Z' = x_r \), let \( i : Z' \to X' \) denote the inclusion of the closed point of \( X' \), and let \( j : \text{sper } K \to X' \) denote the inclusion of the open complement to \( Z' \). For all abelian sheaves \( F \) on \( X' \) the sequence

\[F \to j_*j^*F \to i_\ast i^\ast F \to 0\]

is exact (c.f. [Tam94, 8.2.4 Corollary, Chapter II]. By [Sch95, Lemma 1.3], for any locally constant sheaf \( F \) on \( X' \) the sequence

\[F \to j_*j^*F \xrightarrow{\beta_\pi} i_\ast i^\ast F \to 0\]

is exact, where \( \beta_\pi \) is defined on stalks \( \zeta \) as \( (\beta s)_\zeta = s(\eta_+) - s(\eta_-) \in F_\zeta \). Note that since \( X' \) is local, this induces on global sections \( X' \) the same as \( \beta_\pi \) as defined earlier. Hence we get an isomorphism \( \iota_{pi} \) of cokernels and a commutative diagram

\[
\begin{array}{ccc}
\text{j}_*\text{j}^*A(X') & \xrightarrow{\partial} & \text{i}_*\text{R}^1\text{i}^*A(X') \\
\downarrow{\beta_\pi} & & \downarrow{\iota_\pi} \\
\text{i}_*\text{i}^*A(X') & \xrightarrow{\iota_\pi} & \text{i}_*\text{i}^*A(X')
\end{array}
\]
Tracking down all the definitions, one finds that this diagram is exactly equal to the diagram below.

$\begin{align*}
H^0(X' - Z', A) \xrightarrow{\partial} H^1_{X'}(X', A) \\
\downarrow^{\beta_n} \downarrow^{\iota_n} \\
H^0(Z', A)
\end{align*}$

where the vertical map is the isomorphism $\iota_n$ chosen, and sper $K$ equals $X' - Z'$. This finishes the proof of the Lemma. □

2.14. Corollary. Let $X$ be a regular excellent scheme which is integral with function field $K$. Let $x \in X^1$, let $\pi$ be a choice of uniformizing parameter for $O_{X,x}$, and let $A$ be a constant sheaf. For any open subscheme $U$ in $X$,

$H^0(U_r, Z) \cong \ker(H^0(\text{sper } K, A) \oplus \partial x \to \bigoplus_{x \in U^1} H^0(\text{sper } k(x), A))$

where $\beta_n : H^0(\text{sper } K, A) \to H^0(\text{sper } k(x), A)$ is the map of Definition 2.10.

Proof. We may choose isomorphisms $\iota_n$ for each $x \in X^1$ as in Lemma (2). Then, for every open $U$ we get an equality of kernels from $H^0(U_r, Z)$ to $\ker(H^0(\text{sper } K, Z) \oplus \partial x \to \bigoplus_{x \in U^1} H^0(\text{sper } k(x), Z))$. □

2.7. Restriction of $\text{Sign}$ to $I^j$.

The following proposition uses the definition of $I^j$ from the introduction.

2.15. Lemma. Let $X$ be an integral, regular, noetherian, separated, excellent $F$-scheme with 2 invertible and $n \geq 0$ an integer. For every open subscheme $U$ in $X$,

$I^j(U) = \ker(I^j(K) \oplus \partial x \to \bigoplus_{x \in U^1} I^{j-1}(k(x)))$

Proof. It is well-known that by restricting the second residue from $W(K)$ to $I^j(K)$ one obtains a map $I^j(K) \to \bigoplus_{x \in U^1} I^{j-1}(k(x))$. This is because the second residue maps respect the powers of the fundamental ideal. The Gersten conjecture is known for regular local rings which contain a field, hence for the local rings of $X$ [BGPW02, Theorem 6.1]. It follows that

$\ker(W(K) \oplus \partial x \to \bigoplus_{x \in U^1} W(k(x))$

equals $W(U)$. Then we have that $I^j(U)$ equals the restriction of the kernel of $\oplus \partial_x$ to $I^j(K)$ since, by definition, $I^j(U)$ is the pullback of $W(U)$ over $I^j(K)$. □

2.16. Proposition. Let $X$ be an integral, regular, noetherian, separated, excellent $F$-scheme with 2 invertible and $n \geq 0$ an integer. For every open subscheme $U$ in $X$:
(1) the diagram below commutes

\[
\begin{array}{ccc}
I^j(K) & \oplus \partial_x & \bigoplus_{x \in U^1} I^{j-1}(k(x)) \\
\downarrow \text{sign} & & \downarrow 2\text{sign} \\
H^0(\text{sper } K, 2^j \mathbb{Z}) & \oplus \beta_x & \bigoplus_{x \in U^1} H^0(x_r, 2^j \mathbb{Z})
\end{array}
\]

where the maps $\oplus \partial_x$ and $\oplus \beta_x$ are the maps of Lemmas 2.15 and 2.14.

(2) the signature $\text{sign} : I^j(K) \to H^0(\text{sper } K, 2^j \mathbb{Z})$ induces a map of kernels

$\mathcal{T}^j(U) \to \text{supp}_* 2^j \mathbb{Z}(U)$

Proof. To prove (1), recall (c.f. Lemma 2.2 (1)) that $W(K)$ is generated by rank one forms $\langle c \rangle$, where $c = b \pi^n$, $b$ is a unit in $O_{X,x}$, and either $n = 0$ or $n = 1$. Hence, it suffices to check commutativity on such forms. Using Lemma 2.11 we find that for each $x \in X^1$,

$\beta_x(\text{sign} (\langle c \rangle)) = \begin{cases} 
2\text{sign} (\langle b \rangle) & \text{if } n = 1 \\
0 & \text{if } n = 0
\end{cases}$

On the other hand, using Lemma 2.2 (2) we find that

$\partial(\langle c \rangle) = \begin{cases} 
\langle b \rangle & \text{if } n = 1 \\
0 & \text{if } n = 0
\end{cases}$

which finishes the proof of the (1).

To prove (2), note that we may identify the kernel over $U$ with $\mathcal{T}^j(U)$ using Lemma 2.15, and we may identify $\text{supp}_* 2^j \mathbb{Z}(U)$ with the kernel over $U$ using Corollary 2.14 together with Lemma 2.7.

2.17. Definition. Let $X$ be an integral, regular, noetherian, separated, excellent $F$-scheme with 2 invertible and $n \geq 0$ an integer. For $j \geq 1$, it follows from Proposition 2.16 that $\text{sign} : I^j(K) \to H^0(\text{sper } K, 2^j \mathbb{Z})$ induces a morphism of Zariski sheaves

$\text{Sign}_j \mathcal{T}^j \to \text{supp}_* 2^j \mathbb{Z}$

that we call the restriction of the global signature morphism of sheaves to $\mathcal{T}^j$.

2.18. Theorem. Let $X$ be an integral, regular, noetherian, separated, excellent $F$-scheme with 2 invertible and $n \geq 0$ an integer. Then:

(1) The restriction $\text{Sign}_j$ of the global signature morphism of sheaves to $\mathcal{T}^j$ determines a morphism of directed systems and taking colimits one obtains an isomorphism of sheaves

$\colim(W \xrightarrow{\partial} \mathcal{T} \xrightarrow{\beta} \mathcal{T} \xrightarrow{\beta} \cdots) \xrightarrow{\text{sign}} \text{supp}_* \mathbb{Z}$

(2) Define $W_2 := \colim(W/\mathcal{T} \xrightarrow{\beta} W/\mathcal{T} \xrightarrow{\beta} W/\mathcal{T} \xrightarrow{\beta} \cdots)$. The short exact sequence of Zariski sheaves on $X$

$\colim(W \xrightarrow{\partial} \mathcal{T} \xrightarrow{\beta} \mathcal{T} \xrightarrow{\beta} \cdots) \to W[1/2] \to W_2$

is isomorphic via $\text{Sign}$ to the short exact sequence of Zariski sheaves on $X$

$\text{supp}_* \mathbb{Z} \to \text{supp}_* \mathbb{Z}[1/2] \to \text{supp}_* \mathbb{Z}_2$
where \( \mathbb{Z}_2 \) is the cokernel of the inclusion \( \mathbb{Z} \rightarrow \mathbb{Z}[1/2] \).

(3) For \( j \geq 0 \), let \( \mathcal{H} \) denote the Zariski sheaf of \( X \) obtained by sheafifying the presheaf \( U \mapsto H^j_{\text{ét}}(U, \mu_2) \). Taking the colimit of the directed system below, starting with \( j \geq 0 \)

\[
\begin{array}{ccc}
\mathcal{I}^{j+1} & \longrightarrow & \mathcal{I}^j \\
\downarrow & & \downarrow \\
\mathcal{I}^{j+2} & \longrightarrow & \mathcal{I}^{j+1}
\end{array}
\]


determines a short exact sequence of Zariski sheaves on \( X \) isomorphic via \( \text{Sign} \) to the short exact sequence of Zariski sheaves on \( X \)

\[
\text{supp} \ast 2\mathbb{Z} \rightarrow \text{supp} \ast \mathbb{Z} \rightarrow \text{supp} \ast \mathbb{Z}/2
\]

**Proof.** We prove (1), the others follow. For every \( U \), it follows from Proposition 2.16 that the diagram below commutes

\[
\begin{array}{ccc}
I^j(K) & \oplus_{x \in U^1} & I^{j-1}(k(x)) \\
\downarrow \text{sign} & & \downarrow 2\text{sign} \\
H^0(\text{sper } K, 2^j \mathbb{Z}) & \oplus_{x \in U^1} & H^0(x_r, 2^j \mathbb{Z})
\end{array}
\]

Taking colimits, the left vertical arrow becomes an isomorphism by Lemma 1.2, and the right vertical arrow is also an isomorphism because it is the composition

\[
I^{j-1}(k(x)) \xrightarrow{\text{sign}} H^0(x_r, 2^{j-1} \mathbb{Z}) \xrightarrow{2} H^0(x_r, 2^j \mathbb{Z})
\]

and each of the maps in the composition are isomorphisms after taking colimits. Therefore, we get an isomorphism of kernels, and hence an isomorphism of sheaves which proves (1). \( \square \)

2.8. **Proof of Theorem 0.2.** We prove (1). The proof of (2) is an easy consequence of (1), and (3) follows immediately from (1). When \( \nu \text{cd}_2(X) = s \), for any \( x \in X^p \) we have that \( \text{cd}_2(k(x)[\sqrt{-1}]) \leq \text{cd}_2(X[\sqrt{-1}]) - p \) [Kah02, Proposition 4.1 (a)]. Therefore, \( \nu \text{cd}_2(k(x)) \leq s - p \). From this it follows that, for every open subscheme \( U \) of \( X \), in the commutative diagram from Proposition 2.16

\[
\begin{array}{ccc}
I^j(K) & \oplus_{x \in U^1} & I^{j-1}(k(x)) \\
\downarrow \text{sign} & & \downarrow 2\text{sign} \\
H^0(\text{sper } K, 2^j \mathbb{Z}) & \oplus_{x \in U^1} & H^0(x_r, 2^j \mathbb{Z})
\end{array}
\]

the vertical maps are isomorphisms for \( j \geq s + 1 \). Here, we use the fact that the right vertical map \( I^{j-1}(k(x)) \xrightarrow{2\text{sign}} H^0(x_r, 2^j \mathbb{Z}) \) is the composition of the isomorphism \( I^{j-1}(k(x)) \xrightarrow{\text{sign}} H^0(x_r, 2^{j-1} \mathbb{Z}) \xrightarrow{2} H^0(x_r, 2^j \mathbb{Z}) \).
3. Higher signatures

3.1. Witt groups.
The Witt group $W(X)$ is a part of a cohomology theory $W^n(X)$ for schemes. When 2 is invertible on $X$, each derived Witt group $W^n(X)$ can be constructed as the “triangular” Witt group [Bal00, Bal01] of the triangulated category $D^b(Vect(X))$ equipped with the shifted duality $Hom(-, \mathcal{O}_X[n])$, where $D^b(Vect(X))$ denotes the bounded derived category of vector bundles $Vect(X)$ on $X$. Further, there is 4-periodicity $W^n(X) \simeq W^{n+4}(X)$ and that M. Knebusch’s Witt group of a scheme is $W(X) \simeq W^0(X)$. Next, we briefly recall the definition of the derived Witt groups that we use throughout.

3.2. Definition of derived Witt groups.
Let $(\mathcal{E}, \sharp)$ be an exact category with duality $\sharp$. The homotopy category $K^b(\mathcal{E})$ of bounded chain complexes in $\mathcal{E}$ is a category having as objects bounded chain complexes in $\mathcal{E}$ and as morphisms the chain maps up to chain homotopy. The bounded derived category $D^b(\mathcal{E})$ is obtained from the homotopy category by formally inverting quasi-isomorphisms. The duality $\sharp$ on $\mathcal{E}$ induces a duality on the homotopy category $K^b(\mathcal{E})$ and on the derived category $D^b(\mathcal{E})$. Let $\varpi$ denote the isomorphism to the double dual $\mathcal{E} \overset{\varpi}{\rightarrow} \mathcal{E}^{\sharp\sharp}$ in $D^b(\mathcal{E})$ that is induced from the canonical one in $\mathcal{E}$. Then, $(D^b(\mathcal{E}), \sharp, \varpi, 1)$ is a triangulated category with duality. For a reference for these facts see [Bal01, Section 2.6].

Let $X$ be a scheme with 2 invertible in the global sections $\mathcal{O}_X(X)$. Let $Vect(X)$ denote the exact category of vector bundles on $X$, that is, the category of $\mathcal{O}_X$-modules which are locally free and of finite rank. For any vector bundle $\mathcal{E}$ on $X$, the usual duality $\mathcal{E}^\sharp := Hom_{ Vect(X)}(\mathcal{E}, \mathcal{O}_X)$ defines a duality on $Vect(X)$, making $(Vect(X), \sharp)$ an exact category with duality. The (derived) Witt groups of $X$ are the triangulated Witt groups $W^n(D^b(Vect(X)))$ of the triangulated category with duality $(D^b(Vect(X)), \sharp, \varpi, 1)$. They will be denoted by $W^n(X)$.

3.3. Coniveau Spectral Sequence.
Let $X$ be a noetherian, separated, regular $\mathbb{Z}^{[1/2]}$-scheme. In order to make clear what is the filtration and construction of the spectral sequence of Theorem 0.5, in this section we recall the construction of the coniveau spectral sequence for the Witt groups due to P. Balmer and C. Walter [BW02].

3.4. Filtration by codimension of support.
Let $D^b(X) := D^b(Vect(X))$. We recall the filtration on $D^b(X)$ by codimension of support. For any $\mathcal{O}_X$-module $M$, we denote by $\text{supp}M$ the set of points $x \in X$ for which the localization $M_x$ is non-zero. For any complex $M_\bullet \in D^b(X)$, $\text{supp}H_i(M_\bullet)$ is a closed subspace of $X$. The support of a complex $M_\bullet$ is defined to be

$$\text{supp}M_\bullet := \bigcup_{i \in \mathbb{Z}} \text{supp}H_i(M_\bullet)$$

and it is also a closed subspace of $X$ as it is a finite union, since complexes in $D^b(X)$ are bounded, of closed subspaces. Let $D^b_Z(X) \subset D^b(X)$ consist of those complexes $M_\bullet$ having $\text{supp}M_\bullet \subset Z$. Recall that the codimension of a closed subspace $Z$ of $X$ is defined as

$$\text{codim}(Z) := \min_{\eta \in Z} \dim \mathcal{O}_{X, \eta}$$

where the minimum runs over the finitely many generic points $\eta \in Z$ of $Z$ (as $X$ is noetherian, so is $Z$, hence $Z$ has finitely many irreducible components).
For any integer \( q \geq 0 \), let \( D^q(X) \subset D^b(X) \), or simply \( D^q \), consist of those complexes having codimension of support greater than or equal to \( q \), that is
\[
D^q(X) = \{ M_* \in D^b(X) \mid \text{codim}_X(\text{supp}M_*) \geq q \}
\]

For \( q \geq 0 \), we have short exact sequences [BW02, Theorem 3.1 and proof] of triangulated categories with duality
\[
D^{q+1} \rightarrow D^q \rightarrow D^q/D^{q+1}
\]
so we obtain from Balmer’s localization theorem [BW02, Theorem 3.1 and proof] the long exact sequence of groups
\[
\cdots \rightarrow W^p(D^q) \rightarrow W^p(D^{q+1}) \rightarrow \cdots \rightarrow W^p(D^q/D^{q+1}) \rightarrow \cdots
\]
where we index the maps based on the indices of their respective domain. The map \( i \) is induced by the inclusion \( D^{q+1} \rightarrow D^q \), \( j \) by the quotient \( D^q \rightarrow D^q/D^{q+1} \), and \( k \) is the connecting morphism.

From the long exact sequences 3.2 we obtain an exact couple by setting \( E^p_{1,q} := W^{p+q}(D^p/D^{p+1}) \), \( D^p_{1,q} := W^{p+q}(D^p) \), and taking the differential to be \( d^{p,q} := j_{p+q+1,p+1} \circ k_{p+q,p} \). By the well-known method of Massey’s exact couples, this exact couple determines the spectral sequence below
\[
E^p_{1,q} := W^{p+q}(D^p/D^{p+1}) \Rightarrow W^{p+q}(X)
\]
with abutment the derived Witt groups. The differential \( d_r \) on the \( r \)-th page of this spectral sequence has bidegree \((r, 1 - r)\). By saying that the abutment is the derived Witt groups, we mean that there is a filtration on the \( n \)-th derived Witt group \( W^n(X) \)
\[
\cdots \subset F^{d+1,n-d-1} \subset F^{d,n-d} \subset \cdots \subset F^{3,n-3} \subset F^{2,n-2} \subset F^{1,n-1} \subset W^n(X)
\]
where
\[
F^{p,q} := \text{im}(W^{p+q}(D^p) \rightarrow W^{p+q}(X)).
\]
When \( \dim X \) is finite, the spectral sequence strongly converges, by which we mean that the filtration on the abutment is a finite filtration, and that the terms \( F^{p,q} \) form exact sequences of groups
\[
0 \rightarrow F^{p+1,q-1} \rightarrow F^{p,q} \rightarrow F^{p,q}_{\infty} \rightarrow \cdots
\]
where \( F^{p,q}_{\infty} \) is the stable term, that is to say, for some \( r \) sufficiently large, \( E^r_{p,q} = E^\infty_{p,q} \) for \( n \geq 0 \) (because the dimension of \( X \) is finite and thus the differentials \( d_r \) are eventually zero for \( r \) sufficiently large) and this group is labeled \( E^\infty_{p,q} \).

\[ \textbf{3.5. The coniveau spectral sequence after inverting 2.} \]
Since localization of \( \mathbb{Z} \)-modules is exact, we obtain from the exact couple of the coniveau spectral sequence an exact couple \( HE^p_{1,q} := W^{p+q}(D^p/D^{p+1})[1/2] \), \( HD^p_{1,q} := W^{p+q}(D^p)[1/2] \). This determines the spectral sequence
\[
HE^p_{1,q} := W^{p+q}(D^p/D^{p+1})[1/2] \Rightarrow W^{p+q}(X)[1/2]
\]
with abutment the derived Witt groups with \( 2 \) inverted. Again the differential \( d_r \) on the \( r \)-th page of this spectral sequence has bidegree \((r, 1 - r)\). The filtration on the abutment \( W^n(X)[1/2] \) is
\[
\text{HF}^{d+1,n-d-1} \subset \text{HF}^{d,n-d} \subset \cdots \subset \text{HF}^{3,n-3} \subset \text{HF}^{2,n-2} \subset \text{HF}^{1,n-1} \subset W^n(X)[1/2]
\]
where
\[
\text{HF}^{p,q} := \text{im}(W^{p+q}(D^p)[1/2] \rightarrow W^{p+q}(X)[1/2]).
\]
The following Lemma recalls facts about the coniveau spectral sequence due to P. Balmer and C. Walter. We only give a proof to address the case with $2$ inverted.

3.4. **Lemma.** Let $X$ be a regular, noetherian, separated $F$-scheme. For $q$ not congruent to $0$ module $4$, the $E_2^{p,q}$-terms of the coniveau spectral sequence and the coniveau spectral sequence after inverting $2$ are zero. For $q$ congruent to $0$ module $4$, in the coniveau spectral sequence $E_2^{p,q} = H^p_{\text{Zar}}(X, W)$ and in the coniveau spectral sequence after inverting $2$, $HE_2^{p,q} = H^p_{\text{Zar}}(X, W[1/2])$.

**Proof.** Recall that the Gersten conjecture is known for any regular local ring containing a field $F$ (of characteristic different from $2$) [BGPW02, Theorem 6.1]. Hence, the description of the $E_2^{p,q}$-terms of the coniveau spectral sequence is [BW02, Theorem 7.2] together with [BGPW02, Lemma 4.2].

In order to verify our claim about inverting $2$, for each open $U$ in $X$, let $\mathcal{GW}C^p$ denote the sheaf associated to the presheaf $U \mapsto E^p_{1,0}(U)$ (in fact one can show this is already a sheaf). It is known (c.f. [BGPW02, Proof of Lemma 4.2]) that the sheaves $\mathcal{GW}C^p$ form a complex of sheaves over $X$ which is a flasque resolution of the Witt sheaf $W$. The sheaf associated to the presheaf $U \mapsto HE^p_{1,0}(U)$ is exactly $\mathcal{GW}C^p[1/2]$ by Corollary 2.5 (1) and Definition 2.6. By definition, $\mathcal{GW}C^p[1/2]$ is a filtered colimit of sheaves. Since filtered colimits are exact, the sheaves $\mathcal{GW}C^p[1/2]$ form a complex which is a flasque resolution of the Witt sheaf $W[1/2]$. Hence $HE_2^{p,q} = H^p_{\text{Zar}}(X, W[1/2])$ as claimed. \hfill \Box

3.6. **Definition of the higher signatures.**
If $X$ is finite dimensional and $i$ is any integer $0 \leq i \leq 3$, it follows from the description of the $E_2$-page of the coniveau spectral sequence in Lemma (3.4) that the stable terms $E_\infty^{i,0}$ are subgroups of $H^i_{\text{Zar}}(X, W)$ since they equal the kernel of a differential leaving $H^i_{\text{Zar}}(X, W)$. Let $E_\infty^{i,0} \hookrightarrow H^i_{\text{Zar}}(X, W)$ denote the inclusion map. The composition $W^i(X) \hookrightarrow E_\infty^{i,0} \hookrightarrow H^i_{\text{Zar}}(X, W)$ is the edge map in the coniveau spectral sequence.

3.5. **Definition.** Let $X$ be a regular, noetherian, separated $F$-scheme. The composition

$$W^i(X) \hookrightarrow E_\infty^{i,0} \hookrightarrow H^i(X, W) \xrightarrow{\text{sign}} H^i(X_r, \mathbb{Z})$$

determines a group homomorphism

$$\text{sign}^i : W^i(X) \rightarrow H^i(X_r, \mathbb{Z})$$

When $i = 0$, the map $\text{sign}^0$ agrees with the total signature $: W(X) \rightarrow H^0(X_r, \mathbb{Z})$ in the introduction because sign factors through $\mathcal{W}(X)$.

The following result gives sufficient conditions on when non-torsion elements or odd-torsion elements can be present in the kernel of the higher total signatures.

3.6. **Theorem.** Fix an integer $0 \leq i \leq 3$. If $X$ is a smooth and quasi-projective algebraic variety over $\mathbb{R}$ and $4 + i \leq \dim X \leq 7 + i$, then there is an exact sequence of groups

$$0 \rightarrow H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}[1/2]) \rightarrow W(X)^{i[1/2]} \xrightarrow{\text{sign}^i} H^i_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}[1/2])$$

In particular:

1. if the singular cohomology group $H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z})$ contains an element which is torsion of odd order, then so does $W^i(X)$;
(2) if $H^4_{\text{sing}}(X(\mathbb{R}), \mathbb{Z})$ contains an element which is not torsion, then the kernel of the global signature is not a torsion group.

Proof. Fix an integer $0 \leq i \leq 3$. In the spectral sequence of the Theorem 0.5, it follows from the hypothesis on the dimension of $X$ that: the differentials leaving $H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}[1/2])$ are all trivial, so $H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}[1/2]) = E^4_{\infty}[i-4][1/2]$; the group $E^4_{\infty}[i-4][1/2] = H^{4+i-4}[1/2]$. Hence the filtration of the spectral sequence gives a short exact sequence

$$0 \to H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}[1/2]) \to W^i(X)[1/2] \to E^4_{\infty}[1/2] \to 0 \quad (3.8)$$

Since $E^4_{\infty}[1/2]$ is the kernel of a differential leaving $H^i(X, \mathbb{Z}[1/2])$, it is a subgroup of $H^i(X, \mathbb{Z}[1/2])$, hence by composing the exact sequence above with the inclusion $H^i(X, \mathbb{Z}[1/2]) \to H^i(X, \mathbb{Z}[1/2])$, we obtain the exact sequence in the statement of the theorem. To prove the statement on torsion, note that since the kernel of $H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}) \to H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}[1/2])$ is the subgroup of 2-primary torsion $H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z})[2^\infty]$, it follows from the exact sequence (3.8) that there is an injection $H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z})/H^{4+i}_{\text{sing}}(X(\mathbb{R}), \mathbb{Z})[2^\infty] \to W^i(X)/W^i(X)[2^\infty]$. The two statements on torsion in the theorem follow from this injection. \hfill $\square$

3.7. Proof of Theorem 0.5.
Let $X$ be a regular, noetherian, separated $F$-scheme. As described in Section 3.5 and Lemma 3.4, after inverting 2 the coniveau spectral sequence determines a spectral sequence

$$\text{HE}^p_q = \begin{cases} H^p_{\text{Zar}}(X, \mathcal{W}[1/2]) & \text{if } q \equiv 0 \text{ mod } 4 \\ 0 & \text{otherwise} \end{cases} \Longrightarrow W^{p+q}(X)[1/2]$$

abutting to the derived Witt groups with 2 inverted. Using Theorem 0.1, we have that $\text{Sign}$, after inverting 2, determines an isomorphism $\mathcal{W}[1/2] \to H^0(X_r, \mathbb{Z}[1/2])$, which finishes the proof.

3.8. Proof of Corollary 0.7.
Let $X = \text{spec } A$ be an affine, regular, noetherian, separated $F$-scheme. From the definition of Mahé’s total signature it follows that Mahé’s total signature factors through $\mathcal{W}(X)$, hence the diagram below commutes

$$\begin{array}{ccc}
W(X)[1/2] & \longrightarrow & H^0(X_r, \mathbb{Z}[1/2]) \\
\downarrow & & \downarrow \\
E^0_{\infty}[1/2] & \longrightarrow & H^0_{\text{Zar}}(X, \mathcal{W}[1/2])
\end{array}$$

The rightmost vertical map is an isomorphism by Theorem 0.1, and the bottom horizontal map is injective (See Section 3.6). From Mahé’s Theorem [Mah82, Theoreme 3.2], which states that the cokernel of the total signature of an affine scheme is 2-primary, we have that $\text{sign} : W(X)[1/2] \to H^0_{\text{Zar}}(X_r, \mathbb{Z})[1/2]$ is a surjection. Hence the $E^0_{\infty}[1/2] \to H^0(X, \mathcal{W}[1/2])$ is an isomorphism. Since $E^0_{\infty}[1/2]$ is a subgroup of the kernel of every differential leaving $H^0_{\text{Zar}}(X, \mathcal{W}[1/2])$, it follows that all such differentials are zero. Therefore the images of the differentials leaving $H^0_{\text{Zar}}(X, \mathcal{W})$ in the coniveau spectral sequence must be 2-primary torsion groups. This finishes the proof of the corollary.
3.9. Proof of Theorem 0.12.
Let $X$ be a smooth and quasi-projective variety over $\mathbb{R}$ of finite dimension $d$. For the sake of clarity we explain only how to find the bound on the rank of $W^0(X)$. Finding the bound on $W^1(X)$, $W^2(X)$, and $W^3(X)$ is identical. From the construction of the spectral sequence of Theorem 0.5, the Witt group $W^0(X)[1/2]$ lives in the middle of a short exact sequence

$$0 \to \text{HF}^{1,-1}[1/2] \to W^0(X)[1/2] \to \text{HE}^{0,0}[1/2] \to 0$$

where $\text{HF}^{1,-1}[1/2]$ is the first group in a finite filtration on $W^0(X)[1/2]$

$$= \text{HF}^{d+1,-d-1} \subset \text{HF}^{d,-d} \subset \cdots \subset \text{HF}^{3,-3} \subset \text{HF}^{2,-2} \subset \text{HF}^{1,-1} \subset W^0(X)[1/2]$$

The other groups belong to short exact sequences, for every pair of integers $p, q$,

$$\emptyset \to \text{HF}^{p+1,q-1} \to \text{HF}^{p,q} \to \text{HE}^{p,q} \to$$

After tensoring these short exact sequences with $\mathbb{Q}$, they all split, and so we have that

$$\text{rank}W^0(X) = \sum_{k=0}^{d} \text{rank}\text{HE}^{k,-k}_{\infty}$$

As $\text{HE}^{k,-k}_{\infty} = 0$ when $k$ is not congruent to 0 mod 4, we have, letting $N_0 = \lfloor \frac{d}{4} \rfloor$,

$$\text{rank}W^0(X) = \sum_{k=0}^{N_0} \text{rank}\text{HE}^{4k,-4k}_{\infty} \leq \sum_{k=0}^{N_0} b_{4k}$$

where the inequality is obtained using the fact that the groups $\text{HE}^{4k,-4k}_{\infty}$ are subquotients of $H^4_{\text{sing}}(X(\mathbb{R}), \mathbb{Z}[1/2])$.

4. Bounding the order of torsion in the Witt group

To prove that the derived Witt groups have bounded torsion, it is not sufficient to prove bounded torsion for the $E_2$-terms of the coniveau spectral sequence (see Section 3.3). This is because a complex of abelian groups with bounded torsion may have cohomology groups that possess torsion elements of any order.

4.1. Serre subcategory.
Recall that a subcategory $C$ of the category of abelian groups is a Serre subcategory if for any short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0$$

we have that $B$ is in $C$ if and only if $A$ and $C$ are in $C$.

4.2. Lemma. In the category of abelian groups, consider the subcategory $C$ which consists of all abelian groups $G$ having the following two properties:

1. (Bounded torsion) There exists a nonzero integer $n$ such that $nG_{\text{tors}}$ is trivial, where $G_{\text{tors}}$ denotes the subgroup of torsion elements in $G$;
2. The quotient $G_{\text{red}} := G/G_{\text{tors}}$ is free and of finite rank, that is, $G_{\text{red}}$ is either trivial or is isomorphic to $\mathbb{Z}^m$ for some positive integer $m$.

Any group $G$ in $C$ has the following properties: the torsion part $G_{\text{tors}}$ is a direct sum (possibly infinite) of cyclic groups; $G$ splits $G \simeq G_{\text{tors}} \oplus G_{\text{red}}$. 

Proof. The two statements about the properties of a group \(G\) in \(C\) are well-known facts from group theory. To prove that \(C\) is a Serre subcategory, let \(A, B\) and \(C\) be abelian groups forming a short exact sequence as in (4.1). In the one direction, suppose that \(A\) and \(C\) are in \(C\) with \(mA_{\text{tors}} = 0, nC_{\text{tors}} = 0, A_{\text{red}} \simeq \mathbb{Z}^j\), and \(C_{\text{red}} \simeq \mathbb{Z}^k\). As elements of \(A_{\text{tors}}\) are of order at most \(m\), and elements of \(C_{\text{tors}}\) are of order at most \(n\), then it follows that elements of \(B_{\text{tors}}\) are of order at most \(mn\), hence \(mnB_{\text{tors}} = 0\). To prove that \(B_{\text{red}}\) if free and of finite rank, consider the commutative diagram below

\[
\begin{array}{c}
B_{\text{tors}} \longrightarrow B \longrightarrow B_{\text{red}} \\
\downarrow \quad \downarrow \quad \downarrow \\
C_{\text{tors}} \longrightarrow C \longrightarrow \mathbb{Z}^k \simeq C_{\text{red}}
\end{array}
\]

and apply the snake lemma to obtain the exact sequence below,

\[
0 \to A_{\text{tors}} \to A \to \ker(B_{\text{red}} \to \mathbb{Z}^k) \to \cok(B_{\text{tors}} \to C_{\text{tors}}) \to 0
\]

Recall \(A/A_{\text{tors}} \simeq \mathbb{Z}^j\), so from the previous exact sequence we obtain the short exact sequence below.

\[
\mathbb{Z}^j \to \ker(B_{\text{red}} \to \mathbb{Z}^k) \to \cok(B_{\text{tors}} \to C_{\text{tors}})
\]

Multiplying the previous exact sequence by \(n\) we obtain the commutative diagram below.

\[
\begin{array}{c}
\mathbb{Z}^j \longrightarrow \ker(B_{\text{red}} \to \mathbb{Z}^k) \longrightarrow \cok(B_{\text{tors}} \to C_{\text{tors}}) \\
\downarrow n \quad \downarrow n \quad \downarrow n \\
\mathbb{Z}^j \longrightarrow \ker(B_{\text{red}} \to \mathbb{Z}^k) \longrightarrow \cok(B_{\text{tors}} \to C_{\text{tors}})
\end{array}
\]

From \(nC_{\text{tors}} = 0\) we have \(ncok(B_{\text{tors}} \to C_{\text{tors}}) = 0\), so when we apply the snake lemma we obtain the injection

\[
0 \to \cok(B_{\text{tors}} \to C_{\text{tors}}) \to (\mathbb{Z}/n)^j
\]

so \(\cok(B_{\text{tors}} \to C_{\text{tors}})\) is finite. Then, from short exact sequence 4.4, we conclude that the torsion free group \(B_{\text{red}}\) is finitely generated, hence is free and of finite rank.

In the other direction, suppose that \(B\) is in \(C\). Then, \(A_{\text{tors}}\) has bounded order since \(A_{\text{tors}}\) injects into \(B_{\text{tors}}\), and that \(A_{\text{red}}\) is free and of finite rank since \(A_{\text{red}}\) injects into \(B_{\text{red}}\) and every subgroup of a free abelian group is free. Furthermore, as \(B\) surjects onto \(C\), it follows that \(B_{\text{red}} \simeq \mathbb{Z}^m\) surjects onto \(C_{\text{red}}\), hence \(C_{\text{red}}\) finitely generated and torsion free. As such, \(C_{\text{red}}\) is free and of finite rank.

To prove that \(C_{\text{tors}}\) is of bounded order, apply the snake lemma to diagram (4.3) to obtain that the cokernel of \(B_{\text{tors}} \to C_{\text{tors}}\) is finite because the finitely generated group \(B_{\text{red}}\) surjects onto it, hence \(C_{\text{tors}}\) lives in the middle of a short exact sequence with two groups that are both torsion of bounded order. As discussed earlier, it follows that \(C_{\text{tors}}\) is torsion of bounded order. This completes the proof. \(\square\)

4.5. Theorem. Let \(X\) be an integral, regular, noetherian, separated \(F\)-scheme. Assume that the real cohomology groups \(H^p(X, \mathbb{Z})\) are finitely generated for all \(p \geq 0\) and that \(\text{vcd}_2(X)\) is finite, say \(\text{vcd}_2(X) = s\). Then the cohomology groups \(H^p_{\text{Zar}}(X, W)\) are in the Serre subcategory of Lemma 4.2.
Proof. Let $\text{vcd}_2(X) = s$. For every $j \geq 0$ there is a long exact sequence in cohomology

\[ \cdots \to H^i_{\text{Zar}}(X, \mathcal{I}^j) \to H^i_{\text{Zar}}(X, \mathcal{W}) \to H^i_{\text{Zar}}(X, \mathcal{W}/\mathcal{I}^j) \to \cdots \]

(4.6)

The groups $H^i_{\text{Zar}}(X, \mathcal{W}/\mathcal{I}^j)$ are $2$-primary torsion, of order less than or equal to $2^j$: the map $\mathcal{W}/\mathcal{I}^j \to \mathcal{W}/\mathcal{I}^j$ is zero, hence the map it induces in cohomology is zero. For $j \geq s$, using Theorem 0.2 and the hypothesis on finite generation of real cohomology, we have that the groups $H^i_{\text{Zar}}(X, \mathcal{I}^j)$ are finitely generated. Therefore, both $H^i_{\text{Zar}}(X, \mathcal{I}^j)$ and $H^i_{\text{Zar}}(X, \mathcal{W}/\mathcal{I}^j)$ lie in the Serre subcategory of the Lemma, hence so too does $H^i_{\text{Zar}}(X, \mathcal{W})$. □

4.2. Proof of Theorem 0.13.

Let $X$ be an integral, regular, noetherian, separated $F$-scheme having finite Krull dimension. Assume that the real cohomology groups $H^p(X, \mathbb{Z})$ are finitely generated for all $p \geq 0$ and that $\text{vcd}_2(X)$ is finite. To prove Theorem 0.2, recall that if a spectral sequence converges strongly and the groups on the $E_2$-page lie in a Serre subcategory (4.1), then the $E_\infty$-terms, and hence the abutment, lie in the same subcategory. From Lemma 3.4, we have that the entries on the $E_2$-page of the coniveau spectral sequence (Section 3.3) are either 0 or equal to $H^p_{\text{Zar}}(X, \mathcal{W})$ for some $p \geq 0$. By Theorem 4.5, the groups $H^p_{\text{Zar}}(X, \mathcal{W})$ lie in the Serre subcategory of Lemma 4.2. Therefore, the abutment $W^i(X)$ does as well. This finishes the proof.

References

[AK78] Jón Kr. Arason and Manfred Knebusch, Über die Grade quadratischer Formen, Math. Ann. 234 (1978), no. 2, 167–192. MR 0506027 (58 #21933)

[Bal00] Paul Balmer, Triangular Witt groups. I. The 12-term localization exact sequence, K-Theory 19 (2000), no. 4, 311–363. MR 1763933 (2002h:19002)

[Bal01] ———, Triangular Witt groups. II. From usual to derived, Math. Z. 236 (2001), no. 2, 351–382. MR 1815833 (2002h:19003)

[BGPW02] Paul Balmer, Stefan Gille, Ivan Panin, and Charles Walter, The Gersten conjecture for Witt groups in the equicharacteristic case, Doc. Math. 7 (2002), 203–217 (electronic). MR 1934649 (2003j:19002)

[Brö74] Ludwig Bröcker, Zur Theorie der quadratischen Formen über formal reellen Körpere, Math. Ann. 210 (1974), 233–256. MR 0354549 (50 #7027)

[BW02] Paul Balmer and Charles Walter, A Gersten-Witt spectral sequence for regular schemes, Ann. Sci. École Norm. Sup. (4) 35 (2002), no. 1, 127–152. MR 1886007 (2003d:19005)

[CTP90] J.-L. Colliot-Thélène and R. Parimala, Real components of algebraic varieties and étale cohomology, Invent. Math. 101 (1990), no. 1, 81–99. MR 1055712 (91j:14015)

[EKM08] Richard Elman, Nikita Karpenko, and Alexander Merkurjev, The algebraic and geometric theory of quadratic forms, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008. MR 2427530 (2009d:11062)

[Fas11] Jean Fasel, Some remarks on orbit sets of unimodular rows, Comment. Math. Helv. 86 (2011), no. 1, 13–39. MR 2745274 (2012d:13017)

[Fas13] ———, The projective bundle theorem for $\mathcal{V}$-cohomology, J. K-Theory 11 (2013), no. 2, 413–464. MR 3061003

[Kah02] Bruno Kahn, $K$-theory of semi-local rings with finite coefficients and étale cohomology, K-Theory 25 (2002), no. 2, 99–138. MR 1906669 (2003d:19003)

[Kne77] Manfred Knebusch, Symmetric bilinear forms over algebraic varieties, Conference on Quadratic Forms—1976 (Proc. Conf., Queen’s Univ., Kingston, Ont., 1976), Queen’s Univ., Kingston, Ont., 1977, pp. 103–283. Queen’s Papers in Pure and Appl. Math., No. 46. MR 0498378 (58 #16506)

[Kne81] ———, On the local theory of signatures and reduced quadratic forms, Abh. Math. Sem. Univ. Hamburg 51 (1981), 149–195. MR 629142 (83b:10022)
[KS89] Manfred Knebusch and Claus Scheiderer, *Einführung in die reelle Algebra*, Vieweg Studium: Aufbaukurs Mathematik [Vieweg Studies: Mathematics Course], vol. 63, Friedr. Vieweg & Sohn, Braunschweig, 1989. MR 1029278 (90m:12005)

[Lam77] T. Y. Lam, *Ten lectures on quadratic forms over fields*, Conference on Quadratic Forms—1976 (Proc. Conf., Queen's Univ., Kingston, Ont., 1976), Queen's Univ., Kingston, Ont., 1977, pp. 1–102. Queen’s Papers in Pure and Appl. Math., No. 46. MR 0498380 (58 #16508)

[Mah82] Louis Mahé, *Signatures et composantes connexes*, Math. Ann. 260 (1982), no. 2, 191–210. MR 664376 (84a:14011)

[MH73] John Milnor and Dale Husemoller, *Symmetric bilinear forms*, Springer-Verlag, New York-Heidelberg, 1973, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73. MR 0506372 (58 #22129)

[OVV07] D. Orlov, A. Vishik, and V. Voevodsky, *An exact sequence for \( K_2^M/2 \) with applications to quadratic forms*, Ann. of Math. (2) 165 (2007), no. 1, 1–13. MR 2276765 (2008c:19001)

[Pfi66] Albrecht Pfister, *Quadratische Formen in beliebigen Körpren*, Invent. Math. 1 (1966), 116–132. MR 0200270 (34 #169)

[Sch94] Claus Scheiderer, *Real and étale cohomology*, Lecture Notes in Mathematics, vol. 1588, Springer-Verlag, Berlin, 1994. MR 1321819 (96c:14018)

[Sch95] ———, *Purity theorems for real spectra and applications*, Real analytic and algebraic geometry (Trento, 1992), de Gruyter, Berlin, 1995, pp. 229–250. MR 1320322 (96h:14076)

[Tam94] Günter Tamme, *Introduction to étale cohomology*, Universitext, Springer-Verlag, Berlin, 1994, Translated from the German by Manfred Kolster. MR 1317816 (95k:14033)

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