Equiform Differential Geometry of Curves in Minkowski Space-Time

H. S. Abdel-Aziz, M. Khalifa Saad* and A. A. Abdel-Salam

Dept. of Math., Faculty of Science, Sohag Univ., 82524 Sohag, Egypt

Abstract. In this paper, we establish equiform differential geometry of space and timelike curves in 4-dimensional Minkowski space $E^4_1$. We obtain some conditions for these curves. Also, general helices with respect to their equiform curvatures are characterized.

Keywords. Minkowski 4-space, Equiform differential geometry, General helices.

MSC(2010): 51B20, 53A35.

1 Introduction

In the case of a differentiable curve, at each point a tetrad of mutually orthogonal unit vectors (called tangent, normal, first binormal and second binormal) was defined and constructed, and the rates of change of these vectors along the curve define the curvatures of the curve in Minkowski space-time [3]. The corresponding Frenet’s equations for an arbitrary curve in the Minkowski space $E^4_1$ are given in [5,6]. Although the authors wrote these important equations for the spacelike curves in Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold, there was not a method to calculate all Frenet apparatus of an arbitrary spacelike curve. In the light of this, analogy with the method in [4,9], they have a method to calculate Frenet apparatus of the spacelike curves with non-null frame vectors according to signature (−, +, +, +). Among all curves, space and timelike curves have special emplacement regarding their properties and applicability in other sciences. Because of this they deserve especial attention in Euclidean as well as in other geometries. Besides the Euclidean geometry, a palette of new geometries has been developed over the last two centuries and some properties of curves and surfaces are more emphasized in newly developed non-Euclidean geometries than in the Euclidean. Among these non-Euclidean geometries there is also the Minkowski geometry which represent ambient geometries in which we investigate the geometry of space and timelike curves. We define equiform spacelike and timelike curves and give a characterization of these curves in Minkowski space-time. We found motivation for this work in [1,2,8], where

* E-mail address: mohamed_khalifa77@science.sohag.edu.eg
the authors considered characterizations of general helices in the Minkowski space-time, pseudo-Galilean, double isotropic and Galilean space. We examine a similar problem in the equiform differential geometry of the Minkowski space-time for space and timelike curves and these curves will be called equiform spacelike and timelike curves. The main goal of this article is to define, describe and characterize equiform spacelike and timelike curves in Minkowski 4-space $E^4_1$.

### 2 Preliminaries

Let $E^4 = \{(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \in \mathcal{R}\}$ be a 4-dimensional vector space. For any two vectors $x = (x_1, x_2, x_3, x_4), \ y = (y_1, y_2, y_3, y_4)$ in $E^4$, the pseudo scalar product of $x$ and $y$ is defined by $\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. We call $(E^4, \langle \cdot, \cdot \rangle)$ a Minkowski 4-space and denote it by $E^4_1$. We say that a vector $x$ in $E^4_1 \setminus \{0\}$ is a spacelike vector, a lightlike vector or a timelike vector if $\langle x, y \rangle$ is positive, zero, negative respectively. The norm of a vector $x \in E^4_1$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. For any two vectors $a, b$ in $E^4_1$, we say that $a$ is pseudo-perpendicular to $b$ if $\langle a, b \rangle = 0$. Let $\alpha : I \subset \mathcal{R} \to E^4_1$ be an arbitrary curve in $E^4_1$, we say that a curve $\alpha$ is a spacelike curve if $\langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle > 0$ for any $t \in I$.

The arclength of a spacelike curve $\gamma$ measured from $\alpha(t_0)$ ($t_0 \in I$) is

$$s(t) = \int_{t_0}^{t} \|\dot{\alpha}(t)\| \ dt.$$  \hspace{1cm} (2.1)

Then a parameter $s$ is determined such that $\|\alpha'(s)\| = 1$, where $\alpha'(s) = d\alpha/ds$. Consequently, we say that a spacelike curve $\alpha$ is parameterized by arclength if $\|\alpha'(s)\| = 1$.

Throughout the rest of this paper $s$ is assumed arclength parameter.

For any $x_1, x_2, x_3 \in E^4_1$, we define a vector $x_1 \times x_2 \times x_3$ by

$$x_1 \times x_2 \times x_3 = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1' & x_2' & x_3' & x_4' \\ x_3 & x_2 & x_1 & x_4 \end{vmatrix},$$  \hspace{1cm} (2.2)

where $x_i = (x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})$. Let $\alpha : I \to E^4_1$ be a spacelike curve in $E^4_1$. Then we can construct a pseudo-orthogonal frame $\{t(s), n(s), b_1(s), b_2(s)\}$, which satisfies the following Frenet-Serret type formulae of $E^4_1$ along $\alpha$.

$$\begin{bmatrix} t' \\ n \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & 0 & 0 \\ \mu_1 \kappa_1 & 0 & \mu_2 \kappa_2 & 0 \\ 0 & \mu_3 \kappa_2 & 0 & \mu_4 \kappa_3 \\ 0 & 0 & \mu_5 \kappa_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix},$$  \hspace{1cm} (2.3)
where \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) are respectively, first, second and third curvature of the spacelike curve \( \alpha \), and we have [8]

\[
\kappa_1(s) = \|\alpha''(s)\|, \\
n(s) = \frac{\alpha''(s)}{\kappa_1(s)} , \\
\mathbf{b}_1(s) = \frac{n'(s) + \mu_1 \kappa_1(s) t(s)}{\|n'(s) + \mu_1 \kappa_1(s) t(s)\|}, \\
\mathbf{b}_2(s) = t(s) \times n(s) \times \mathbf{b}_1(s).
\]

(2.4)

Denote by \( \{t(s), n(s), \mathbf{b}_1(s), \mathbf{b}_2(s)\} \) the moving Frenet frame along the spacelike vector \( \alpha \), where \( s \) is a pseudo arclength parameter. Then \( t(s) \) is a spacelike vector and due to the causal character of the principal normal vector \( n \) and the binormal vector \( \mathbf{b}_1 \), we have the following Frenet formulas [5]:

**Case 2.1** If \( n \) is spacelike vector, then \( \mathbf{b}_1 \) can have two causal characters.

**Case 2.1.1:** if \( \mathbf{b}_1 \) is spacelike vector, then \( \mu_i \ (1 \leq i \leq 5) \) read

\[ \mu_1 = \mu_3 = -1, \mu_2 = \mu_4 = \mu_5 = 1. \]

And \( t, n, \mathbf{b}_1 \) and \( \mathbf{b}_2 \) are mutually orthogonal vectors satisfying equations:

\[ \langle t, t \rangle = \langle n, n \rangle = \langle \mathbf{b}_1, \mathbf{b}_1 \rangle = 1, \ \langle \mathbf{b}_2, \mathbf{b}_2 \rangle = -1. \]

**Case 2.1.2:** if \( \mathbf{b}_1 \) is timelike vector, then \( \mu_i \ (1 \leq i \leq 5) \) read

\[ \mu_1 = -1, \mu_2 = \mu_3 = \mu_4 = \mu_5 = 1. \]

The vectors \( t, n, \mathbf{b}_1 \) and \( \mathbf{b}_2 \) satisfy the conditions:

\[ \langle t, t \rangle = \langle n, n \rangle = \langle \mathbf{b}_2, \mathbf{b}_2 \rangle = 1, \ \langle \mathbf{b}_1, \mathbf{b}_1 \rangle = -1. \]

**Case 2.2** If \( n \) is timelike vector. Then \( \mu_i \ (1 \leq i \leq 5) \) read

\[ \mu_1 = \mu_2 = \mu_3 = \mu_4 = 1, \mu_5 = -1. \]

For the vectors \( t, n, \mathbf{b}_1 \) and \( \mathbf{b}_2 \), we have

\[ \langle t, t \rangle = \langle \mathbf{b}_1, \mathbf{b}_1 \rangle = \langle \mathbf{b}_2, \mathbf{b}_2 \rangle = 1, \ \langle n, n \rangle = -1. \]

Now, let \( \gamma \) be a timelike curve. Then \( T \) is timelike vector and the Frenet equations have the form

\[
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix}' =
\begin{bmatrix}
0 & \bar{\kappa}_1 & 0 & 0 \\
\bar{\kappa}_1 & 0 & \bar{\kappa}_2 & 0 \\
0 & -\bar{\kappa}_2 & 0 & \bar{\kappa}_3 \\
0 & 0 & -\bar{\kappa}_3 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix},
\]

(2.5)

where \( T, N, B_1 \) and \( B_2 \) satisfy the following equations

\[ \langle N, N \rangle = \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = 1, \langle T, T \rangle = -1. \]

The functions \( \bar{\kappa}_1, \bar{\kappa}_2 \) and \( \bar{\kappa}_3 \) are the curvatures of \( \gamma \) [10].
3 Equiform differential geometry of space and timelike curves

Spacelike curves:
Let \( \alpha : I \rightarrow E^4 \) be a **spacelike curve**. We define the equiform parameter of \( \alpha (s) \) by

\[
\sigma = \int \frac{ds}{\rho} = \int \kappa_1 ds
\]

(3.1)

where \( \rho = \frac{1}{\kappa_1} \), is the radius of curvature of the curve \( \alpha \). It follows

\[
\frac{ds}{d\sigma} = \rho.
\]

(3.2)

Let \( h \) is a homothety with the center in the origin and the coefficient \( \lambda \). If we put \( \alpha^* = h(\alpha) \), then it follows

\[
s^* = \lambda s, \quad \rho^* = \lambda \rho,
\]

(3.3)

where \( s^* \) is the arclength parameter of \( \alpha^* \) and \( \rho^* \) the radius of curvature of \( \alpha^* \). Hence \( \sigma \) is an equiform invariant parameter of \( \alpha \).

**Notation 3.1** Let us note that \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) are not invariants of the homothety group, it follows \( \kappa_1^* = \frac{1}{\lambda} \kappa_1, \quad \kappa_2^* = \frac{1}{\lambda} \kappa_2 \) and \( \kappa_3^* = \frac{1}{\lambda} \kappa_3 \).

The vector

\[
V_1 = \frac{d\alpha (s)}{d\sigma},
\]

(3.4)

is called a tangent vector of the curve \( \alpha \) in the equiform geometry. From (3.2) and (3.4), we get

\[
V_1 = \frac{d\alpha (s)}{d\sigma} = \rho \frac{d\alpha (s)}{ds} = \rho t.
\]

(3.5)

Furthermore, we define the tri-normals by

\[
V_2 = \rho n, \quad V_3 = \rho b_1, \quad V_4 = \rho b_2.
\]

(3.6)

It is easy to check that the tetrahedron \( \{V_1, V_2, V_3, V_4\} \) is an equiform invariant tetrahedron of the curve \( \alpha \) [4]. Now, we will find the derivatives of these vectors with respect to \( \sigma \) using by (3.2), (3.4) and (3.6) as follows:

\[
V'_1 = \frac{d}{d\sigma} (V_1) = \rho \frac{d}{ds} (\rho t) = \dot{\rho} V_1 + \rho V_2,
\]

where the derivative with respect to the arclength \( s \) is denoted by a dot and respect to \( \sigma \) by a dash. Similarly, we obtain

\[
V'_2 = \frac{d}{d\sigma} (V_2) = \rho \frac{d}{ds} (\rho n) = \mu_1 V_1 + \dot{\rho} V_2 + \mu_2 \left( \frac{\kappa_2}{\kappa_1} \right) V_3,
\]

\[
V'_3 = \frac{d}{d\sigma} (V_3) = \rho \frac{d}{ds} (\rho b_1) = \mu_3 \left( \frac{\kappa_2}{\kappa_1} \right) V_2 + \dot{\rho} V_3 + \mu_4 \left( \frac{\kappa_3}{\kappa_1} \right) V_4,
\]

\[
V'_4 = \frac{d}{d\sigma} (V_4) = \rho \frac{d}{ds} (\rho b_2) = \mu_5 \left( \frac{\kappa_3}{\kappa_1} \right) V_3 + \dot{\rho} V_4,
\]

(3.7)
Definition 3.1 The functions $K_i : I \rightarrow \mathbb{R}$ ($i = 1, 2, 3$) defined by

$$K_1 = \dot{\rho}, \quad K_2 = \frac{\kappa_2}{\kappa_1}, \quad K_3 = \frac{\kappa_3}{\kappa_1}$$  \hspace{1cm} (3.8)

are called $i^{th}$ equiform curvatures of the curve $\alpha$. These functions $K_i$ are differential invariant of the group of equiform transformations, too.

Thus, the formulas analogous to famous the Frenet formulas in the equiform geometry of the Minkowski space $E^4_1$ have the following form:

$$
\begin{align*}
V_1' &= K_1 V_1 + V_2, \\
V_2' &= \mu_1 V_1 + K_1 V_2 + \mu_2 K_2 V_3, \\
V_3' &= \mu_3 K_2 V_2 + K_1 V_3 + \mu_4 K_3 V_4, \\
V_4' &= \mu_5 K_3 V_3 + K_1 V_4.
\end{align*}
$$  \hspace{1cm} (3.9)

Notation 3.2 The equiform parameter $\sigma = \int \kappa_1(s) ds$ for closed curves is called the total curvature, and it plays an important role in global differential geometry of the Euclidean space. Also, the functions $\frac{\kappa_2}{\kappa_1}$ and $\frac{\kappa_3}{\kappa_1}$ have been already known as conical curvatures and they also have interesting geometric interpretation.

Because of the equiform Frenet formulas (3.9), the following equalities regarding equiform curvatures can be given

$$
\begin{align*}
K_1 &= \frac{1}{\rho^2} \langle V_j', V_j \rangle; \quad (j = 1, 2, 3, 4), \\
K_2 &= \frac{1}{\mu_2 \rho^2} \langle V_2', V_3 \rangle = \frac{1}{\mu_3 \rho^2} \langle V_3', V_2 \rangle, \\
K_3 &= \frac{1}{\mu_4 \rho^2} \langle V_3', V_4 \rangle = \frac{1}{\mu_5 \rho^2} \langle V_4', V_3 \rangle.
\end{align*}
$$  \hspace{1cm} (3.10)

**Timelike curves:**

Now, if we consider $\gamma : I \rightarrow E^4_1$ be a timelike curve parametrized by arclength $s$. Then as above, we can write

$$
\begin{align*}
U_1 &= \rho T, \\
U_2 &= \rho N, \\
U_3 &= \rho B_1, \\
U_4 &= \rho B_2.
\end{align*}
$$  \hspace{1cm} (3.11)

Thus, $\{U_1, U_2, U_3, U_4\}$ is an equiform invariant tetrahedron of the curve $\gamma$. [7]
The derivatives of these vectors with respect to $\sigma$ are as follows:

$$
\begin{align*}
U'_1 &= \frac{d}{d\sigma} (U_1) = \rho \frac{d}{ds} (\rho T) = \dot{\rho} U_1 + U_2, \\
U'_2 &= \frac{d}{d\sigma} (U_2) = \rho \frac{d}{ds} (\rho N) = U_1 + \dot{\rho} U_2 + \left( \frac{\kappa_2}{\kappa_1} \right) U_3, \\
U'_3 &= \frac{d}{d\sigma} (U_3) = \rho \frac{d}{ds} (\rho B_1) = - \left( \frac{\kappa_2}{\kappa_1} \right) U_2 + \dot{\rho} U_3 + \left( \frac{\kappa_3}{\kappa_1} \right) U_4, \\
U'_4 &= \frac{d}{d\sigma} (U_4) = \rho \frac{d}{ds} (\rho B_2) = - \left( \frac{\kappa_3}{\kappa_1} \right) U_3 + \dot{\rho} U_4.
\end{align*}
$$

(3.12)

Hence, the Frenet formulas in the equiform geometry of the Minkowski 4-space can be written as

$$
\begin{align*}
U'_1 &= \tilde{\kappa}_1 U_1 + U_2, \\
U'_2 &= U_1 + \tilde{\kappa}_1 U_2 + \tilde{\kappa}_2 U_3, \\
U'_3 &= -\tilde{\kappa}_2 U_2 + \tilde{\kappa}_1 U_3 + \tilde{\kappa}_3 U_4, \\
U'_4 &= -\tilde{\kappa}_3 U_3 + \tilde{\kappa}_1 U_4.
\end{align*}
$$

(3.13)

The functions $\tilde{\kappa}_1$, $\tilde{\kappa}_2$, $\tilde{\kappa}_3$ are the equiform curvatures. In a matrix form, we have

$$
\begin{bmatrix}
U'_1 \\
U'_2 \\
U'_3 \\
U'_4
\end{bmatrix} =
\begin{bmatrix}
\tilde{\kappa}_1 & 1 & 0 & 0 \\
1 & \tilde{\kappa}_1 & \tilde{\kappa}_2 & 0 \\
0 & -\tilde{\kappa}_2 & \tilde{\kappa}_1 & \tilde{\kappa}_3 \\
0 & 0 & -\tilde{\kappa}_3 & \tilde{\kappa}_1
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4
\end{bmatrix},
$$

(3.14)

where

$$
\begin{align*}
\tilde{\kappa}_1 &= \frac{1}{\rho^2} \langle U'_j, U_j \rangle; \quad (j = 1, 2, 3, 4), \\
\tilde{\kappa}_2 &= \frac{1}{\rho^2} \langle U'_2, U_3 \rangle = -\frac{1}{\rho^2} \langle U'_3, U_2 \rangle, \\
\tilde{\kappa}_3 &= \frac{1}{\rho^2} \langle U'_3, U_4 \rangle = -\frac{1}{\rho^2} \langle U'_4, U_3 \rangle.
\end{align*}
$$

(3.15)

4 The characterizations of space and timelike curves in $E^4_1$

In this section, we characterize the space and timelike curves using their equiform curvatures $\kappa_i, \tilde{\kappa}_i$ ($i = 1, 2, 3$) in $E^4_1$ which have important geometric interpretation, as follows:

1. If $\kappa_2 = \text{const.}, \kappa_3 = \text{const.}$, of a spacelike curve, then the curve is a general helix and vice versa. Here, we do not set conditions on $\kappa_1$.

2. If the above condition holds and $\kappa_1$ is identically zero, then the spacelike curve is a $W$-curve (Since all three curvatures $\kappa_1, \kappa_2, \kappa_3$ are constant. For more details, see [8]).
**Theorem 4.1** Let \( \alpha \) be a spacelike curve in \( E^4_1 \) with the equiform invariant tetrahedron \( \{ V_1, V_2, V_3, V_4 \} \) and equiform curvature \( K_1 \neq 0 \). Then \( \alpha \) has \( K_2 \equiv 0 \) if and only if \( \alpha \) lies fully in a 2-dimensional subspace of \( E^4_1 \).

**Theorem 4.2** Let \( \alpha \) be a spacelike curve in \( E^4_1 \) and \( \{ V_1, V_2, V_3, V_4 \} \) is the equiform invariant tetrahedron of it. When \( K_1, K_2 \neq 0 \), then \( \alpha \) has \( K_3 \equiv 0 \) if and only if \( \alpha \) lies fully in a hyperplane of \( E^4_1 \).

**Proof.** If \( \alpha \) has \( K_3 \equiv 0 \), then from (3.9), we have

\[
\begin{align*}
\alpha' &= V_1, \\
\alpha'' &= K_1 V_1 + V_2, \\
\alpha''' &= (\rho \dot{K}_1 + K_2^2 + \mu_1) V_1 + 2K_1 V_2 + \mu_2 K_2 V_3, \\
\alpha^{(4)} &= \frac{d}{d\sigma} \left( \rho \dot{K}_1 + K_2^2 + \mu_1 \right) + K_1 \left( \rho \dot{K} + K_2^2 + \mu_1 \right) + 2\mu_1 K_1 V_1 \\
&+ \left( 3\rho \dot{K}_1 + 3K_2^2 + \mu_2 \mu_3 K_2^2 + \mu_1 \right) V_2 \\
&+ \left( 3\mu_2 K_1 K_2 + \mu_2 \rho \dot{K}_2 \right) V_3 + (\mu_2 \mu_4 K_2 K_3) V_4.
\end{align*}
\]

Hence, by using Maclaurin expansion for \( \alpha \), given by

\[
\alpha(\sigma) = \alpha(0) + \alpha'(0)\sigma + \alpha''(0)\frac{\sigma^2}{2!} + \alpha'''(0)\frac{\sigma^3}{3!} + \alpha^{(4)}(0)\frac{\sigma^4}{4!} + ..., \]

we obtain that \( \alpha \) lies fully in a spacelike hyperplane of the space \( E^4_1 \) by spanned

\[ \{ \alpha'(0), \alpha''(0), \alpha'''(0) \} . \]

Conversely, assume that \( \alpha \) satisfies the assumptions of the theorem and lies fully in a spacelike hyperplane \( \Pi \) of the space \( E^4_1 \). Then, there exist the points \( p, q \in E^4_1 \), such that \( \alpha \) satisfies the equation of \( \Pi \) given by

\[
\langle \alpha(\sigma) - p, q \rangle = 0, \tag{4.1}
\]

where \( q \in \Pi^\perp \) is a timelike vector. Differentiation of the last equation yields

\[
\langle \alpha', q \rangle = \langle \alpha'', q \rangle = \langle \alpha''', q \rangle = 0. 
\]

Therefore, \( \alpha', \alpha'', \alpha''' \in \Pi \). Now, since

\[
\alpha' = V_1 \text{ and } \alpha'' = K_1 V_1 + V_2,
\]

it follows that

\[
\langle V_1, q \rangle = \langle V_2, q \rangle = 0. \tag{4.2}
\]

Next, differentiation (4.2) gives

\[
\langle V'_2, q \rangle = 0. \tag{4.3}
\]
From the equiform Frenet equations, we obtain
\[
\langle V_3, q \rangle = 0. \tag{4.4}
\]
Again, differentiating (4.4) leads to
\[
\langle \mu_3 K_2 V_2 + K_1 V_3 + \mu_4 K_3 V_4, q \rangle = K_3 \langle V_4, q \rangle = 0.
\]
Because \( V_4 \) is the only vector perpendicular to \( \{ V_1, V_2, V_3 \} \), we obtain
\[
K_3 = 0.
\]
This completes the proof. \( \blacksquare \)

**Theorem 4.3** Let \( \alpha \) be a spacelike curve with equiform invariant vector \( V_3 \) in the equiform geometry of \( E_4^1 \). Then, the curve \( \alpha \) is a general helix if and only if
\[
V_3'' + \psi_1 V_3 = \psi_2 V_1 + \psi_3 V_2 + \psi_4 V_4, \tag{4.5}
\]
where
\[
\psi_1 = -\left( \rho \dot{K}_1 + K_1^2 + \mu_2 \mu_3 K_2^2 + \mu_4 \mu_5 K_3^2 \right),
\psi_2 = \mu_1 \mu_3 K_2,
\psi_3 = 2\mu_3 K_1 K_2,
\psi_4 = 2\mu_4 K_1 K_3.
\]

**Proof.** Suppose that the curve \( \alpha \) is a general helix. Thus, we have
\[
K_2 = \text{const.} \quad \text{and} \quad K_3 = \text{const.} \tag{4.6}
\]
From (3.9) and (4.6), it is easy to prove that the equation (4.5) is satisfied.
Conversely, we assume that the equation (4.5) holds. Then from (3.9), it follows that
\[
V_3' = \mu_3 K_2 V_2 + K_1 V_3 + \mu_4 K_3 V_4. \tag{4.7}
\]
Differentiating (4.7) with respect to \( \sigma \), we get
\[
V_3'' = (\mu_1 \mu_3 K_2) V_1 + \left( \mu_3 \rho \dot{K}_2 + 2\mu_3 K_1 K_2 \right) V_2 + \left( \rho \dot{K}_1 + K_1^2 + \mu_2 \mu_3 K_2^2 + \mu_4 \mu_5 K_3^2 \right) V_3 + \left( \mu_4 \rho \dot{K}_3 + 2\mu_4 K_1 K_3 \right) V_4,
\]
so, we obtain
\[
\dot{K}_2 = 0 \quad \text{and} \quad \dot{K}_3 = 0,
\]
which completes the proof. \( \blacksquare \)
Theorem 4.4 Let $\gamma$ be a timelike curve in $E_4^1$ with the equiform invariant tetrahedron $\{U_1, U_2, U_3, U_4\}$ and with equiform curvatures $\bar{K}_1 \neq 0$. Then $\gamma$ has $\bar{K}_2 \equiv 0$ if and only if $\gamma$ lies fully in a 2-dimensional subspace of $E_4^1$.

Theorem 4.5 Let $\gamma$ be a timelike curve in $E_4^1$ with the equiform invariant tetrahedron $\{U_1, U_2, U_3, U_4\}$ and with equiform curvatures $\bar{K}_1, \bar{K}_2 \neq 0$. Then $\gamma$ has $\bar{K}_3 \equiv 0$ if and only if $\gamma$ lies fully in a hyperplane of $E_4^1$.

Proof. If $\gamma$ has $\bar{K}_3 \equiv 0$, then by using the equiform Frenet equations (3.13) we obtain

$$
\gamma' = U_1,
\gamma'' = \bar{K}_1 U_1 + U_2,
\gamma''' = \left( \rho \bar{K}_1 + \bar{K}_2^2 + 1 \right) U_1 + 2 \bar{K}_1 U_2 + \bar{K}_2 U_3,
\gamma^{(4)} = \left( \frac{d}{d\sigma} \left( \rho \bar{K}_1 + \bar{K}_2^2 + 1 \right) \right) U_1 + \bar{K}_1 \left( \rho \bar{K}_1 + \bar{K}_2^2 + 1 \right) U_1 + 2 \bar{K}_1 U_2
+ \left( 3 \rho \bar{K}_1 + 3 \bar{K}_2^2 - \bar{K}_2^2 + 1 \right) U_2
+ \left( 3 \bar{K}_1 \bar{K}_2 + \rho \bar{K}_2 \right) U_3 + \bar{K}_2 \bar{K}_3 U_4.
$$

Next, all higher order derivatives of $\gamma$ are linear combinations of vectors $\gamma', \gamma'', \gamma'''$, so by using Maclaurin expansion for $\gamma$ given by

$$
\gamma(\sigma) = \gamma(0) + \gamma'(0) \sigma + \gamma''(0) \frac{\sigma^2}{2!} + \gamma'''(0) \frac{\sigma^3}{3!} + \gamma^{(4)}(0) \frac{\sigma^4}{4!} + ..., $$

we conclude that $\gamma$ lies fully in a timelike hyperplane of the space $E_4^1$, spanned by

$$
\left\{ \gamma'(0), \gamma''(0), \gamma'''(0) \right\}.
$$

Conversely, we suppose that $\gamma$ lies fully in a timelike hyperplane $\bar{\Pi}$ of the space $E_4^1$. Then, there exist the points $p, q \in E_4^1$ such that $\gamma$ satisfies the equation of $\bar{\Pi}$ given by

$$
\langle \gamma(\sigma) - p, q \rangle = 0,
$$

where $q \in \bar{\Pi}$. By differentiating (4.8) with respect to $\sigma$, we can write

$$
\langle \gamma', q \rangle = \langle \gamma'', q \rangle = \langle \gamma''', q \rangle = 0.
$$

Since

$$
\gamma' = U_1 \quad \text{and} \quad \gamma'' = \bar{K}_1 U_1 + U_2,
$$

it follows that

$$
\langle U_1, q \rangle = \langle U_2, q \rangle = 0.
$$
Similarly, we have

\[ \langle U_3, q \rangle = 0. \]  \hspace{1cm} (4.10)

Again, differentiation (4.10) gives

\[ \langle -\bar{\mathcal{K}}_2 U_2 + \bar{\mathcal{K}}_1 U_3 + \bar{\mathcal{K}}_3 U_4, q \rangle = \bar{\mathcal{K}}_3 \langle U_4, q \rangle = 0, \]

because \( U_4 \) is the only vector perpendicular to \( \{ U_1, U_2, U_3 \} \), we obtain

\[ \bar{\mathcal{K}}_3 = 0, \]

this completes the proof. \( \blacksquare \)

**Theorem 4.6** Let \( \gamma \) be a timelike curve with equiform invariant vector \( U_3 \) in the equiform geometry of \( E_4^1 \). Then \( \gamma \) is a general helix if and only if

\[ U_3'' + \varphi_1 U_3 = \varphi_2 U_1 + \varphi_3 U_2 + \varphi_4 U_4, \]  \hspace{1cm} (4.11)

where

\[ \begin{align*}
\varphi_1 &= -\rho \bar{\mathcal{K}}_1 - \bar{\mathcal{K}}_2 + \bar{\mathcal{K}}_2^2 + \bar{\mathcal{K}}_3^2, \\
\varphi_2 &= -\bar{\mathcal{K}}_2, \\
\varphi_3 &= -2 \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_2, \\
\varphi_4 &= 2 \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_3.
\end{align*} \]

**Proof.** Suppose that the curve \( \gamma \) is a general helix. Thus, we have

\[ \bar{\mathcal{K}}_2 = \text{const.} \text{ and } \bar{\mathcal{K}}_3 = \text{const.} \]  \hspace{1cm} (4.12)

From (3.13) and (4.9), it is easy to prove that the equation (4.11) is satisfied. Conversely, we assume that the equation (4.11) holds. Then from (3.13), it follows that

\[ U_3' = -\bar{\mathcal{K}}_2 U_2 + \bar{\mathcal{K}}_1 U_3 + \bar{\mathcal{K}}_3 U_4, \]  \hspace{1cm} (4.13)

By differentiating (4.9) with respect to \( \sigma \), we get

\[ \begin{align*}
U_3'' &= -\bar{\mathcal{K}}_2 U_1 - (\rho \bar{\mathcal{K}}_2 + 2 \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_2) U_2 \\
&\quad + (\rho \bar{\mathcal{K}}_1 + \bar{\mathcal{K}}_2^2 - \bar{\mathcal{K}}_3^2 - \bar{\mathcal{K}}_3^3) U_3 \\
&\quad + (\rho \bar{\mathcal{K}}_3 + 2 \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_3) U_4,
\end{align*} \]

so, we obtain

\[ \dot{\bar{\mathcal{K}}}_2 = 0 \text{ and } \dot{\bar{\mathcal{K}}}_3 = 0. \]

It completes the proof. \( \blacksquare \)
5 Conclusion

In the 4-dimensional Minkowski space, equiform differential geometry of space and time-like curves are investigated. Frenet formulas in the equiform differential geometry of the Minkowski 4- space $E^4_1$ for these curves are obtained. Moreover, we have characterized these curves using their equiform curvatures $K_i, \overline{K}_i$ ($i = 1, 2, 3$) in $E^4_1$ which have important geometric interpretation.

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