Static Solution of the General Relativistic Nonlinear $\sigma$-Model Equation

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The nonlinear $\sigma$-model is considered to be useful in describing hadrons (Skyrmions) in low energy hadron physics and the approximate behavior of the global texture. Here we investigate the properties of the static solution of the nonlinear $\sigma$-model equation coupled with gravity. As in the case where gravity is ignored, there is still no scale parameter that determines the size of the static solution and the winding number of the solution is $1/2$. The geometry of the spatial hyperspace in the asymptotic region of large $r$ is explicitly shown to be that of a flat space with some missing solid angle.

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When a certain symmetry of a system is broken spontaneously, a massless scalar particle naturally emerges as a Goldstone boson \([1]\) for each broken generator of the symmetry. The simplest lagrangian density, which has been used to demonstrate the above formal argument, can be written as

\[
\mathcal{L} = \frac{1}{2} \partial^\mu \phi^\alpha \partial_\mu \phi^\alpha - \lambda (|\phi|^2 - v^2)^2
\]

(1)

where \(\alpha = 0, 1, 2, 3\) are chosen as an example in this paper. The physical vacuum of the system can be defined as a field configuration which satisfies \(|\phi|^2 = v^2\). If we choose the \(\phi^0\)-direction as the one that acquires the non-vanishing vacuum expectation value such that

\[
\langle \phi^0 \rangle = v \quad \quad \langle \phi^i \rangle = 0, \quad i = 1, 2, 3
\]

(2)

then the symmetry of the system, \(SO(4)\), is reduced to \(SO(3)\) and \(\phi^i\)’s become massless Goldstone bosons. In the limit, \(\lambda \to \infty\), the massive mode in the \(\phi^0\)-direction becomes infinitely heavy and decoupled from the system. The symmetry now can be realized in terms of \(\phi^i\)’s only: the unbroken symmetry is realized linearly but the transformation in the direction of broken generators nonlinearly.

In terms of Goldstone bosons, the simplest lagrangian density in which the whole symmetry of the system is properly incorporated as discussed above is the nonlinear \(\sigma\)-model lagrangian density

\[
\mathcal{L} = -\frac{f_\phi^2}{4} \text{Tr}(\partial^\mu U \partial_\mu U^\dagger)
\]

(3)

where

\[
U = \exp(i \vec{\tau} \cdot \vec{\phi}/f_\phi)
\]

(4)

The classical soliton of Eq.(3) has been found to be very important because of its usefulness in describing hadrons (Skyrmions) \([2]\) in low energy hadron physics as well as the approximate behavior of the global texture which is attracting interest as a potential source
of density perturbation for large scale structure of the universe [3]. Since the simplest form of Eq.(3) cannot support a stable soliton with finite energy [4], higher derivative terms are needed in Eq.(1) or Eq.(3) to describe the system with finite energy. They are the Skyrme terms used in low energy hadron physics.

It is observed [5] that there is a set of solutions which extremize the action with the simplest lagrangian density, Eq.(3), using the hedgehog ansatz:

\[ U = \exp(i\vec{r} \cdot \hat{r}F(r)) \]  

(5)

The profile function F(r) satisfies the following equation;

\[ \frac{d^2F}{dr^2} + 2 \frac{dF}{dr} = \frac{1}{r^2} \sin 2F \]  

(6)

In this equation, one can see that there is no scale parameter which can determine the size of the soliton. This leads to the conclusion that, due to Derrick’s theorem, there is no stable solution with finite energy. Fig.1 shows a numerical solution of Eq.(6) with the boundary condition

\[ F(0) = \pi. \]  

(7)

It was analytically proved in Ref. [5] that there are no other solutions except for those solutions that can be obtained through the following transformations;

\[ F(r) \rightarrow F(\lambda r) \quad (\lambda = constant) \]

\[ \rightarrow F(r) + n\pi \quad (n = integer) \]

\[ \rightarrow -F(r) \]  

(8)

The profile function F(r) in Fig.1 approaches π/2 as r → ∞ as can be seen in the asymptotic form:

\[ F(r) \rightarrow \frac{\pi}{2} + \sqrt{\frac{\pi}{2}} \cos \left( \frac{\sqrt{2}}{2} \ln \left( \frac{r_0}{r} \right) + \alpha \right) \]  

(9)

The winding number defined by
\[ N = \frac{F(0) - F(\infty)}{\pi} - \frac{\sin 2F(0) - \sin 2F(\infty)}{2\pi} \]  

(10)

can be calculated to be \( \frac{1}{2} \). Although the energy of this solution is infinite, it cannot be excluded mathematically as one of the solutions that extremize the action. It might be useful in describing the physical situations where a very large number of particles are involved in heavy ion collisions or where the solution has to be truncated due to the finite size of the correlated volume which is restricted by the horizon in the early universe. In Ref. [6], the time evolution of the textures whose initial configurations are given by small perturbations from the solution in Fig.1 was investigated. Within the one parameter set of initial configurations considered there, it was found that textures with winding number larger (smaller) than 1/2 collapse (expand). The critical nature of the winding number 1/2 was also discussed in Ref. [7]. The author presented the analytical predictions about the effect of the winding number on behaviors (time evolutions) of the scalar field and showed configurations of the predictions by numerically solving the field equations with a couple of particular forms of initial configurations. Although the above findings do not provide the full general proof, they lead one to suspect that the critical property that determines the collapse is whether the winding number is larger than 1/2. Ref. [8] presents numerical investigations of the collapse and unwinding of global textures without recourse to the nonlinear \( \sigma \)-model approximation.

As was explicitly shown in Ref. [6], the static solution shown in Fig.1 is not a stable one. However, being a unique static solution that extremizes the action, it may represent the most probable initial configuration of texture formation. A small perturbation will eventually cause it to collapse or expand, but its collapse or expansion will be delayed longer than that of any other initial configuration.

When we consider the situation with very large number of particles or with very large amount of energy, the effects of gravity and/or quantum corrections which can be put into the lagrangian in the form of higher derivative terms may become important. Hence it is interesting to investigate those effects on the winding number one-half solution. Here we consider the effects of gravity starting with the following action;
\[ I = \int d^4x \sqrt{-g} \left( -\frac{1}{16\pi G} R - \frac{f_\phi^2}{4} Tr(\partial^\mu U \partial_\mu U^\dagger) \right) \] (11)

with \( U \) given by Eq.(9). For earlier works concerning the system of texture coupled with gravity, see for example Ref.’s [9], [10] and [11].

For a static and spherically symmetric solution, the metric can be written as

\[ ds^2 = e^{2q(r)}dt^2 - e^{2p(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \] (12)

The matter field equation obtained from \( \delta I/\delta F = 0 \) is

\[ \frac{d}{dr} \left( \frac{r^2 dF}{dr} \right) + \left( \frac{dq}{dr} - \frac{dp}{dr} \right) r^2 \frac{dF}{dr} - e^{2p} \sin(2F) = 0 \] (13)

The equations for \( p(r) \) and \( q(r) \) obtained from the Einstein’s equations, \( G^{\mu\nu} = -8\pi G T^{\mu\nu} \), are

\[ \frac{dp}{dr} = \frac{1}{2r} \left( 1 - e^{2p} \right) + \frac{\kappa r}{4} \left( \frac{dF}{dr} \right)^2 + e^{2p} \sin^2 F \] (14)

\[ \frac{dq}{dr} = -\frac{1}{2r} \left( 1 - e^{2p} \right) + \frac{\kappa r}{4} \left( \frac{dF}{dr} \right)^2 - e^{2p} \sin^2 F \] (15)

where \( \kappa = 8\pi G f_\phi^2 \). From Eq’s.(13), (14) and (15), we can see there is no scale parameter, and hence no scale to determine the size of the solution. It is then an interesting question whether there exists a static solution which extremizes the action, Eq.(11), which corresponds to Eq.(9) without gravity. In our calculations, we first put Eq.(15) into Eq.(13) to eliminate \( q(r) \), couple the resulting equation with Eq.(14) and numerically integrate them to obtain \( F(r) \) and \( p(r) \) with the boundary condition

\[ F(0) = \pi, \quad p(0) = 0. \] (16)

The only other input in the calculation is the value of \( dF/dr \) at \( r = 0 \). Changing the value induces only the change of the overall scale of the solution. The result of the calculation is given in Fig.2. Here also a series of solutions can be generated from the solution in Fig.2 through the transformations similar to Eq.(8),
\[ F(r), p(r) \to F(\lambda r), p(\lambda r) \quad (\lambda = \text{constant}) \]
\[ \to F(r) + n\pi, p(r) \quad (n = \text{integer}) \]
\[ \to -F(r), p(r) \quad (17) \]

Our numerical analysis indicates that
\[ |F(0) - F(\infty)| = \frac{\pi}{2}, \quad (18) \]
although we still lack the analytic proof. The asymptotic forms at large \( r \) can be calculated explicitly to be

\[ F \to \frac{\pi}{2} + \sqrt{\frac{r_0}{r}} \cos \left( \frac{\beta}{2} \ln \left( \frac{r_0}{r} \right) + \gamma \right), \quad (19) \]

\[ p \to a + b \frac{r_0}{r} \sin \left( \beta \ln \left( \frac{r_0}{r} \right) + \gamma + \delta \right), \quad (20) \]

\[ q \to c - (1 + \frac{\beta^2}{4}) \frac{\kappa r_0}{4r} - \frac{1}{1 + \beta^2} \frac{\kappa r_0}{4r} \cos \left( \beta \ln \left( \frac{r_0}{r} \right) + 2\gamma \right), \]
\[ + \frac{\beta^3}{4(1 + \beta^2)} \frac{\kappa r_0}{4r} \sin \left( \beta \ln \left( \frac{r_0}{r} \right) + 2\gamma \right) - b \frac{r_0}{r} \sin \left( \beta \ln \left( \frac{r_0}{r} \right) + \gamma + \delta \right) \quad (21) \]

where

\[ a = \frac{1}{2} \ln \left( \frac{1}{1 - \kappa} \right), \quad (22) \]

\[ \beta^2 = \frac{7 + \kappa}{1 - \kappa}, \quad (23) \]

\[ b^2 = \frac{1}{256} \frac{\kappa^2}{1 - \kappa}, \quad (24) \]

\[ \tan \delta = -\beta, \quad (25) \]

and \( r_0, \gamma \) and \( c \) are integration constants. We can put \( c \) to be zero by adjusting the scale of the coordinate \( t \) so that it coincides with the scale of proper time in the asymptotic region of \( r \to \infty \).

Let us now consider the geometry of the hypersurface of \( t = \text{constant} \) in the asymptotic region of large \( r \). The asymptotic form of the metric is given by

\[ ds^2 = \frac{1}{1 - \kappa}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (26) \]
or with the coordinate transformation, $\tilde{r} = \sqrt{\frac{1}{1-\kappa}} r$,

$$ds^2 = d\tilde{r}^2 + (1 - \kappa)\tilde{r}^2(d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (27)

This metric represents a flat space in the asymptotic region with the deficit solid angle

$$\delta\Omega = 4\pi\kappa$$  \hspace{1cm} (28)

To see that the metric represents a flat space in the asymptotic region, we calculate all components of the Riemann curvature tensor in the orthonormal basis which we take as

$$e^1 = \sqrt{\frac{1}{1-\kappa}} dr, \quad e^2 = r d\theta, \quad e^3 = r \sin \theta d\phi$$  \hspace{1cm} (29)

The only nonzero components are

$$R_{323}^2 = -\frac{\kappa}{r^2}$$  \hspace{1cm} (30)

and those related to it by algebraic symmetry or antisymmetry. We see that all components of the Riemann curvature tensor in the orthonormal basis converge to zero as $r \to \infty$. It has been shown in Ref. [12] that the self-similar texture solution also produces a flat spatial hypersurface with some missing solid angle in the asymptotic region of large $r$.

We next consider the energetics of our solution. With the use of the hedgehog ansatz (Eq.(3)) and the metric form of Eq.(12), the $tt$-component of the energy-momentum tensor $T^{\mu\nu}$ is calculated to be

$$T^{tt} = \frac{f^2}{4} \left\{ e^{-2q-2p} \left( \frac{dF}{dr} \right)^2 + e^{-2q} \frac{2\sin^2 F}{r^2} \right\}.$$  \hspace{1cm} (31)

To investigate the behavior($r$ - dependence) of the energy density in the asymptotic flat region of $r \to \infty$, we put the asymptotic solutions, Eq.'s (19), (20) and (21), into Eq. (31) and obtain the result

$$T^{tt} \sim \frac{1}{r^2}$$  \hspace{1cm} (32)

as $r \to \infty$. This shows that the total energy contained in a volume diverges linearly in $r$ as the volume is increased.
In conclusion, we see that the basic structure of the classical static solution is not changed by gravity: there is still no scale parameter that determines the size of the field configuration and the solution represents the knot of windung number $1/2$. The geometry of the spatial hypersurface in the asymptotic region of large $r$ is that of a flat space with some missing solid angle.

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FIGURES

FIG. 1. The profile function $F(r)$ without gravity.

FIG. 2. The profile function $F(r)$ and the function $p(r)$ with gravity ($\kappa = 0.865$).
This figure "fig1-1.png" is available in "png" format from:

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