Report on GR16, Session A3:
Mathematical Studies of the Field Equations

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Abstract

In this report, which is an extended version of that appearing in the Proceedings of GR16, I will give a summary of the main topics covered in Session A.3. on mathematical relativity at GR16, Durban. The summary is mainly based on extended abstracts submitted by the speakers. I would like to thank all participants for their contributions and help with this summary.

1 The Horowitz-Myers Soliton

According to a conjecture by Maldacena[1], supported by a growing body of evidence, there is a correspondence between string theory in Anti–deSitter (AdS) spacetime and a conformal field theory (CFT) on the boundary of AdS. Stability considerations motivated by the AdS/CFT conjecture led to the conjecture that the boundary of asymptotically AdS space–times should be connected. The Riemannian version of this was proved by Witten and Yau[2] and generalized by Cai and Galloway[3]. The non–supersymmetric version of the AdS/CFT correspondence, leads to consideration of asymptotically locally AdS spacetimes with conformal boundary such that one spatial direction is compactified on a circle. If we allow nontrivial topology at infinity, the positive mass theorem for general relativity is not valid in general[4]. In case of negative cosmological constant, $\Lambda < 0$, a partial Cauchy surface in an asymptotically locally AdS $n+1$-dimensional space–time may have any orientable $n – 1$–manifold as boundary at infinity. The Kottler and Nariai metrics give explicit examples of static AdS metrics on manifolds with nontrivial topology at infinity[5]. These examples all have horizons.

Horowitz and Myers[6] have found a static solution of the Einstein equations with negative cosmological constant, $\Lambda < 0$, which is asymptotically locally AdS

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and contains no horizon. I will refer to this spacetime as the HM soliton. The HM soliton in \( n + 1 \) dimensions is given by the metric
\[
 ds^2 = -r^2 dt^2 + \frac{1}{V(r)} dr^2 + V(r) d\phi^2 + r^2 \sum_{i=1}^{n-2} (dy^i)^2
\]  
(1)
where \( V(r) = r^2 \left( 1 - \frac{r_0^n}{r^n} \right) \), \( \ell^2 = -\frac{n(n-1)}{2\Lambda} \), and \( r_0 \) is a constant. In order to avoid an orbifold singularity, \( \phi \) must be identified with period \( \beta_0 = \frac{4\pi\ell^2}{nr_0} \). After making the \( y^i \) coordinates periodic as well (with arbitrary period), the resulting spacetime has conformally compact Cauchy surfaces with flat torus conformal boundary.

Computing the mass of the soliton using a standard method gives the negative answer \( E = -\frac{\beta r_0^2}{4G\ell^2} \). The mass at infinity of metrics asymptotic to the soliton is related via the AdS/CFT correspondence to the negative Casimir energy of a non–supersymmetric gauge theory on the boundary. If a non–supersymmetric version of the AdS–CFT conjecture holds, then stability considerations imply that the soliton must be the lowest energy state with the given asymptotic conditions. This leads to a new positive mass conjecture,

**Conjecture 1 (Horowitz and Myers)** Spacetimes satisfying a dominant energy condition with \( \Lambda < 0 \), and with the asymptotic behavior of the soliton (1), have mass bounded from below by \( E \). The soliton is the unique lowest mass solution for all spacetimes in its class.

This conjecture is supported by perturbation calculations up to second order[4, 6]. A further piece of supporting evidence was provided by the recent work of Galloway, Surya and Woolgar[7], described by Greg Galloway in his talk at session A.3. Galloway et al. proved that the HM soliton is the unique static metric in a class of metrics with the given asymptotic conditions. In order to formulate the main result of Galloway et al., let \( (\Sigma, h, N) \) be a solution of the static vacuum field equations \( R_{ab} = N^{-1} \nabla_a \nabla_b N + \frac{2\Lambda}{n-1} h_{ab}, \Delta N = -\frac{2\Lambda}{n-1} N \), which is conformal to a compact manifold with boundary \( (\Sigma, \hat{h}) \) with conformal factor \( \hat{N} = 1/N \) (with \( \hat{N} = 0 \) on \( \partial\Sigma \)). Then \( (\Sigma, h, N) \) is said to satisfy condition (S) if the level sets \( \hat{N} = c \) near infinity are weakly convex in the sense that the principal curvatures are either all positive or all negative. The mass is computed as an integral over the conformal boundary of the mass aspect of \( (\Sigma, h, N) \), defined as (specializing the Ashtekar–Magnon conformal mass to the static case) \( \mu = -\partial^{n-2} R/\partial x^{n-2} \big|_{x=0} \) (where \( x = \hat{N} \) and \( R \) is the scalar curvature function of \( (\Sigma, \hat{h}) \)).

**Theorem 1.1** Let \( (\Sigma, h, N) \) be a static spacetime as above, satisfying condition (S). Suppose in addition, that (a) The conformal boundary of \( (\Sigma, h, N) \) is the same as that of the H&M soliton, (b) The mass aspect \( \mu \) of \( (\Sigma, h, N) \) is pointwise negative, and (c) The kernel of the inclusion map \( i_* : \Pi_1(\partial\Sigma) \to \Pi_1(\Sigma) \) is generated by the \( S^1 \) factor. Then the spacetime \( M^{n+1} = \mathbb{R} \times \Sigma \), \( g = -N^2 dt^2 \oplus h \), determined by \( (\Sigma, h, N) \) is isometric to the AdS soliton [4].
The proof uses complete null lines constructed using the convexity of the boundary together with a null splitting theorem of Galloway\cite{8} to prove that the universal cover of the conformal compactification $(\tilde{\Sigma}, \tilde{h})$ splits as a metric product $\mathbb{R}^{n-2} \times W$ with the metric on $W$ determined uniquely by the field equations.

2 Dynamical systems in general relativity

Claes Uggla reviewed some applications of dynamical systems methods in general relativity. Dynamical systems formulations arise naturally in the study of Einstein’s field equations for special classes of spacetimes, such as hypersurface–homogenous and hypersurface-self-similar spacetimes\cite{9}. Further examples are provided by the geodesic equations of certain classes of spacetimes, where dynamical systems techniques can be used to e.g., calculate the null geodesics and the corresponding cosmic microwave background radiation pattern in a cosmological model\cite{10, 11}, and the density perturbations of spatially homogenous models. The classical example of application of dynamical systems methods to the Einstein equations is the study of spatially homogenous (Bianchi) spacetimes, with or without matter. The Bianchi family of spacetimes contains the Friedman spacetimes as a special case, and gives a rich set of examples of the influence of matter and geometry on the early and late time evolution of spacetimes. It is a natural problem to analyze completely the behavior of the resulting family of dynamical systems.

The book on dynamical systems in cosmology edited by Ellis and Wainwright\cite{12} gives a good overview of the state of the art of this approach, some years ago. As an outgrowth of that work, substantial progress has been made towards understanding the Bianchi systems both near the initial singularity and in the asymptotically expanding direction. Hans Ringström\cite{13, 14} proved that Bianchi VIII and IX have oscillating singularity, which is an important step in understanding the full BKL picture for these models. An important step in the analysis was to prove that Bianchi II is an attractor for Bianchi IX. This sort of hierarchical behavior in the Bianchi models, is an important organizing principle also for further work on systems with lower dimensional isometry groups. Ringström has also, as he described in his talk, been able to analyze in detail\cite{15} the asymptotic behavior of the expanding vacuum Bianchi class A spacetimes (this excludes Bianchi IX). For the most complicated of these, Bianchi VIII, the result is that the geometry is asymptotically Taub, with precise control on the asymptotic behavior of the commutator variables. For Bianchi VII$_0$, the system approaches (in terms of scale free variables), a flat VII$_0$ spacetime. The Cauchy surfaces of these spacetimes are all Seifert fibered. The results show that the scale free geometry of the spatial sections in these spacetimes collapse along the fiber direction. The result is consistent with the picture for the limiting behavior in the asymptotic expanding direction developed by M. T. Anderson\cite{17}. Analyzing the hierarchical structure of systems with small symmetry group, such as Gowdy models, or without symmetry, is likely to be a crucial step in understanding the global dynamics of the Einstein equations.
3 Well-posed forms of the 3+1 trace–decoupled Einstein equations

The standard ADM form of the Einstein evolution equations fails to be well posed both from the mathematical and numerical points of view. Due to the demand for stable numerical procedures for solving the Einstein evolution equations, a great deal of interest has recently been focussed on well posed hyperbolic formulations of the Einstein equations. This is of particular importance in solving boundary value problems. Oscar Reula discussed various well–posed reformulations of the Einstein equations. It is possible to derive a well–posed form of the standard 3+1 Einstein equations\cite{17} by reducing to first order form, densitizing the lapse function and adding combinations of the constraints to the evolution equations\cite{18, 19}. An alternative approach, which recently has been shown to possess striking computational advantages over the standard form\cite{20}, is to decompose the fundamental variables in order to extract the trace of the extrinsic curvature and the determinant of the 3-metric with the purpose of evolving them independently from the remaining components. This trace–decoupled form of the Einstein equation turns out, oddly enough, to be ill–posed. Reula and Frittelli have used techniques similar to those used for the standard evolution equations, to obtain versions of the 3+1 Einstein equations which are both trace-decoupled and well posed. This well posed version propagates the constraints in a stable manner, which is relevant to unconstrained evolution. It uses essentially the same variables as the original system, but requires a gauge condition, namely that the lapse to be proportional to the determinant of the intrinsic geometry of the surfaces.

4 Critical phenomena

Critical phenomena in general relativity have been extensively studied since the discovery by Choptuik of universal behavior in the collapse of a self–gravitating scalar field. Since then a number of matter models, such as charged self–gravitating scalar fields, Yang–Mills and dilaton fields have been studied and a systematic understanding of these phenomena has been gained, in particular through the study of the stability of models with continuous or discrete self–similarity. We now have a good qualitative understanding of criticality in gravitational collapse: there is an intermediate (codimension-1) attractor of the evolution flow in Phase Space, sitting on the threshold of black hole formation. The symmetries of that attractive solution determine the type of criticality: ‘type I’ if the solution is static, ‘type II’ when it is self-similar.

The Einstein–Vlasov system describes the evolution of a statistical ensemble of non-interacting particles coupled to gravity through their average properties. Jose M. Martin–Garcia described joint work with Carsten Gundlach on critical phenomena in the Einstein–Vlasov system, which is important in Critical Phenomena Theory because it is the only one where type II critical phenomena have not yet been found.
There are two different ways of checking for the existence of criticality. On the one hand, it is possible to evolve numerically families of initial conditions and study the collapse threshold, looking for intermediate attractors and universality. On the other hand, one can construct static or self-similar solutions and then investigate whether they are codimension-1 linearly stable.

There are two relevant works in this problem, both using the method of evolution of initial conditions: Rendall, Rein and Schaeffer\cite{21} have not found criticality and suggest that, if there is any, it must be type I. Olabarrieta and Choptuik\cite{22} confirm the apparent nonexistence of type II phenomena and have found some signs of type I criticality, though not conclusive because there is not universality in the critical exponents. Gundlach and Martin–Garcia look for type II criticality in the massless Einstein-Vlasov system using the second method.

In his talk Martin–Garcia reported on the first step: the numerical construction of self-similar solutions, which is an interesting problem in itself. The situation resembles what is already known analytically for static solutions of the same system\cite{23}. There is a free function of two variables, representing different distributions of the energy-momentum among the particles of the system, but all giving rise to the same gravitational field. Therefore a candidate critical solution can not be isolated. It is likely that a similar result in the massive case could explain the observed non-universality.

Recent numerical investigations\cite{24, 25} of near-critical scalar field collapse in 2+1 dimensional AdS spacetime show a continuous self-similar (CSS) behaviour near the central singularity and power law scaling for the black hole mass. Garfinkle\cite{26} found that this behaviour is well approximated, at intermediate times, by a member of a one-parameter family of exact CSS solutions to the equations with vanishing cosmological constant $\Lambda$. However the extension of these solutions to $\Lambda \neq 0$, necessary to explain black hole formation (as vacuum black holes exist only for $\Lambda < 0$) is a hard problem which has not been solved up to now.

Gérard Clément in his talk discussed joint work with Alessandro Fabbri\cite{27} on quasi-CSS 2+1 dimensional scalar field spacetimes. They have derived by a limiting process a new class of $\Lambda = 0$ CSS solutions

$$\begin{align*}
ds^2 &= du dv - (-u)^{2(1+c^2)} d\theta^2, \\
\phi &= -\frac{c}{1+c^2} \ln(-u)
\end{align*}$$

(2)

These present for $c^2 \leq 1$ a point singularity ($u = 0, v = +\infty$), and for $c^2 < 1$ a null line singularity $u = 0$. They can be extended to exact solutions of the $\Lambda = -l^{-2} < 0$ equations by means of the ansatz

$$\begin{align*}
ds^2 &= e^{2\sigma(x)} du dv - (-u)^{2(1+c^2)} \rho^2(x) d\theta^2, \\
\phi &= -\frac{c}{c^2 + 1} \ln |u| + \psi(x),
\end{align*}$$

(3)

where $x = uv$, with the initial conditions $\rho(0) = 1, \sigma(0) = \psi(0) = 0$. These extended solutions inherit the CSS behaviour close to the central singularity, and have the correct AdS behaviour at spatial infinity.
To show that these quasi-CSS solutions are indeed threshold solutions for black hole formation, one studies their linear perturbations. The case $c = 0$ is at the same time simple and interesting. In this case the scalar field decouples, $\phi = 0$, and the corresponding quasi-CSS solution is the vacuum BTZ solution. The linear perturbation equations with appropriate boundary conditions lead to a unique solution, with eigenvalue $k = 2$. This solution is found to be an exact linearization in $M$ of the BTZ black hole. In the case of genuine scalar perturbations, one finds two possible modes $k_a = c^2 + 3/2$ and $k_b = c^2 + 2$. For the $a$ mode the singularity and the apparent horizon appear simultaneously on the null line $u = 0$ at the time $v = 0$ and evolve in the region $v > 0$ in a physically meaningful way, while for the $b$ mode the singularity still appears for $v = 0$, but the apparent horizon seems to be eternal, as in the case of the static BTZ black hole. The $b$ mode, which cannot describe actual gravitational collapse with regular initial conditions, appears to be unphysical. The critical exponent is related to the eigenvalue $k$ by $\gamma = 1/k$. In the BTZ case $c = 0$ Clément and Fabbri obtain $\gamma = 1/2$. In the scalar field case, choosing the critical value $c^2 = 1$ they obtain for the $a$ mode $\gamma = 0.4$. This value differs significantly from those found in the numerical analysis, i.e. $\gamma \sim 1.2$ and $\gamma \sim 0.81$, showing that further work is needed in order to clarify this issue.

5 Isolated systems

It is an open problem to construct a nonflat vacuum spacetime, which has a regular conformal infinity in the sense of Penrose. The conformal compactification of a nonflat spacetime must have a singularity at spatial infinity $i_0$, due to the slow falloff of the Weyl tensor at $i_0$. Sergio Dain described joint work with Helmut Friedrich, which constructs a large class of asymptotically flat initial data with non-vanishing mass and angular momentum for which the metric and the extrinsic curvature have asymptotic expansions at space-like infinity in terms of powers of a radial coordinate. These asymptotic expansions are of the form $\tilde{h}_{ij} \sim (1 + 2m/\tilde{r}) \delta_{ij} + \sum_{k \geq 2} \tilde{h}^k_{ij}$, $\tilde{\Psi}_{ij} \sim \sum_{k \geq 2} \tilde{\Psi}^k_{ij}$, where $\tilde{h}^k_{ij}$ and $\tilde{\Psi}^k_{ij}$ are smooth function on the unit 2-sphere (thought as being pulled back to the spheres $\tilde{r} = const.$ under the map $\tilde{x}^l \rightarrow \tilde{x}^l/\tilde{r}$). In earlier work stationary data were shown to admit expansions of this type.

In the study of isolated system, an important and difficult problem is to give an unambiguous definition of angular momentum. The difficulty lies in the fact that without introducing extra structure, there is no unambiguously defined Poincaré subgroup of the asymptotic isometries of an isolated system. This is known as the supertranslation ambiguity. In order to provide with an unambiguous notion of angular momentum in radiative spacetimes, it is essential to make use of a reference frame system, which embodies the notion of rest frame. Osvaldo Moreschì presented a definition of angular momentum for radiative spacetimes which does not suffer from supertranslation ambiguity. The definition uses the construction of so called ‘nice sections’ at $I^+$. The defining equations for these
have recently been proved\cite{33,34} to have solutions in terms of a 4-parameter family of translations. The solutions have the expected physical properties of a rest frame. The nice section construction singles out a Poincaré structure from the infinite dimensional BMS group. In particular, given a fixed observational point \( p \) at \( \mathcal{I}^+ \), there is precisely a 3-degree of freedom of spacelike translations which generate all the nice sections that contain \( p \). In contrast, without this constructions there is an infinite dimensional family of general sections that contain \( p \), one for each supertranslation. The condition that the spatial part of the Bondi 4–momentum is zero singles out a one parameter family of nonintersecting\cite{34} sections \( S_{\text{cm}} \) of \( \mathcal{I}^+ \), which define a center of mass frame and can be used to describe the detailed asymptotic structure of the spacetime. The angular momentum is defined in terms of a charge integral over the \( S_{\text{cm}} \).

Detectors of gravitational waves can be considered as observers at future null infinity. The construction of center of mass sections, provides for each point \( p \) at scri a prescription that singles out the unique center of mass section containing \( p \), together with the appropriate rest frame to calculate intrinsic quantities like angular momentum and multipole moments, which are important for the description of the gravitational waves that one wishes to detect.

John L. Friedman described in his talk joint work with Kōji Uryū and Masaru Shibata. They consider compact binary systems, modeled as vacuum or perfect-fluid spacetimes with a helical Killing vector \( k^a \), heuristically, the generator of time-translations in a corotating frame. Systems that are stationary in this sense are not asymptotically flat, but have asymptotic behavior corresponding to equal constant fluxes of ingoing and outgoing radiation. For black-hole binaries, a rigidity theorem implies that the Killing vector lies along the horizon’s generators, and from this one can deduce the zeroth law (constant surface gravity of the horizon). Remarkably, although the mass and angular momentum of such a system are not defined, a finite Noether charge is defined on any sphere enclosing the matter and black holes, its value the same on each such sphere. An exact first law relates the change in this charge to the changes in the vorticity, baryon mass and entropy of the fluid and in the area of black holes. Modeling a binary system by a spacetime with a helical Killing vector is accurate if the energy emitted in a dynamical time is small compared to the kinetic energy of the system.

Other approaches use a post–Newtonian approximation, or conformally flat spacelike slices that satisfy a truncated set of field equations. Like the post-Newtonian spacetimes, these conformally flat spacetimes are nonradiative and asymptotically flat. In the isolated horizon framework, for a horizon with a single Killing vector, one shows the existence of a charge \( E \) defined on an isolated horizon for which \( \delta E = \kappa \delta A \). The first law for black holes with helical symmetry, in contrast, relates this change in the black-hole charges to the changes in the Noether charge of a sphere surrounding all black holes and all matter and to the changes in the entropy, baryon number, and circulation of the fluid. The existence of such a first law depends precisely on what is not assumed in the isolated horizon framework: a globally defined Killing vector.
5.1 Isolated horizons

A closely related topic, isolated horizons (IH) was discussed by G. Date. Here again the work is motivated by the question if it is possible to have analogues of the usual laws of BH mechanics in spacetimes more general than the usual stationary asymptotically flat black holes.

G. Date stated the following results\[35, 36\]: A Killing horizon (KH) is always an IH, and an IH is a KH provided the IH admits a neighbourhood which is either stationary or has two commuting Killing vectors. Further, a vacuum, non-rotating IH necessarily admits an EH and for a rotating IH , an EH exists if \( \int E + \pi \bar{\pi} - 2\pi < 0 \). Otherwise, the IH is accessible from infinity. An EH could exist but the IH will be ‘outside’ of it.

5.2 Stars

The Einstein equations for stationary axisymmetric perfect fluid spacetimes reduces to an elliptic system, known as the Ernst equations. These spacetimes are commonly studied as models of isolated stars in general relativity.

Christian Klein presented joint work with Jörg Frauendiener on exact solutions of the stationary axisymmetric Einstein equations for perfect fluid matter. A boundary value problem for the Ernst equations is solved using Riemann–Hilbert techniques. As an application he presented the solution for a family of counter-rotating dust disks\[37, 38\] which contains the rigidly rotating dust disk\[39\] as a limiting case.

H. Pfister considered existence and non–existence results proved using Banach space fixed point techniques\[40\].

6 Variational formulation

Naresh Dadhich and Niall Ó Murchadha gave talks relating to the variational formulation of general relativity. Dadhich considered the conditions under which one can deduce the field equations from the equations of motion for particles. In the context of electromagnetic field, this question was, according to Dyson\[41\], first considered by Feynman. By considering commutation relations between position and velocity, he could obtain the Bianchi set of the equations. This gave rise to considerable activity in this direction involving rederivations, generalizations and extensions. However the attention centred on the Bianchi set. Singh and Dadhich\[42\] derived the full set of the Maxwell equations by demanding the law of motion to be linear in velocity and derivable from a Lagrangian. A similar procedure has been applied to the Einstein equations\[43\]. In this case, the fact that the particle law in general relativity is quadratic in the 4–velocity leads to the Einstein equations. This is a novel way of deducing the field equations from the particle law of motion.

Niall Ó Murchadha described joint work with Julian Barbour and Brendan Foster\[44\], where a derivation of General Relativity was given which does
not depend on the relativity principle. The fundamental idea is to choose super-space (the space of all riemannian three geometries) as the configuration space and to consider a parametrised curve in this configuration space, with parameter $\lambda$. Ó Murchadha et al. consider a generalized Jacobi-type action of the form $\int d\lambda \int dx^3 \sqrt{qP}$ on such curves where one first integrates on each given three manifold and then integrates along the curve. ‘$P$’ is the potential which is a scalar on the three manifold but otherwise arbitrary, while ‘$T$’ is the generalized kinetic energy which means that it is a quadratic function of $\partial g_{ij}/\partial \lambda$. It turns out that in order for the constraints of the theory to be propagated, it is necessary to choose the potential energy term to be the three scalar curvature and the kinetic energy to be defined with the DeWitt supermetric. This recovers the Baierlein, Sharp, Wheeler parametrised action for G.R. as the unique self-consistent Jacobi-type action on superspace.

7 Singularities

There were several talks concerned with the structure of cosmological and black hole singularities.

An important development has been the application of Fuchsian techniques to construct families of asymptotically velocity term dominated (AVTD) cosmological singularities. M. Narita discussed his work, applying the Fuchsian method to Gowdy spacetimes with stringy matter, showing that these have AVTD singularities. Stringy gravity has received a great deal of attention recently due to the work of Damour and Henneaux, who showed that for stringy gravity in of no symmetries in spacetime dimension 11 and higher, one expects mixmaster type behavior.

Brien Nolan has studied the gravitational collapse of spherically symmetric thick shells admitting a homothetic Killing vector field under the assumption that the energy momentum tensor corresponds to the absence of a pure outgoing component. The stability of the Cauchy horizon against linear perturbations is investigated. The behaviour of a massless scalar field $\phi$ propagating on a background space-time is studied. The following boundary conditions are imposed: (i) finiteness of the flux of $\phi$ as measured by any time-like observer crossing $N$ and (ii) finiteness of the flux of $\phi$ in the ingoing null direction measured at $J^-$. Imposing these boundary conditions, it is found that the flux of $\phi$ as measured by any time-like (geodesic) observer crossing the Cauchy horizon is always finite. This is in contrast to the case of Cauchy horizons inside black holes. Those arising here always lie outside any possible event horizon. This indicates that the Cauchy horizon may be stable against linear and possibly non-linear gravitational perturbations, and so members of this class of space-times may provide physically realisable examples of naked singularities.

A. Beesham also discussed naked singularities. In joint work with S.G. Ghosh and R.V. Saraykar he has studied gravitational collapse of radiation shells in a non self-similar higher dimensional spherically symmetric spacetime. Strong curvature naked singularities were found to form for a highly inhomogeneous
collapse, violating the cosmic censorship conjecture for this class of models.

Deborah Konkowski reviewed the status of two instability conjectures for Cauchy horizons [51].

8 Miscellaneous

8.1 Lanczos and Bel–Robinson

In his talk, Brian Edgar reviewed his work with F. Andersson on the Lanczos potential for the Weyl tensor [52, 53]. The existence proof for the Lanczos potential is now on firm foundation, and the construction of the potential in several situations with symmetry is understood. Unfortunately, a really interesting application of this intriguing object is still lacking.

Jose M. Senovilla discussed his work on Bel–Robinson squares [54]. This work generalizes and unifies the earlier literature on Bel–Robinson and Bel tensors. The Bel–Robinson tensor was introduced as a candidate for a “stress–energy” tensor for the gravitational field, and in 3+1 dimensional vacuum spacetimes, this symmetric, trace–free 4–tensor is divergence free and positive definite in a certain sense. The Bel tensor plays a similar role for Einstein equations with matter. The work of Senovilla gives a systematic procedure for constructing similar objects in higher dimensional spacetimes.

8.2 Distributional Geometry

James Vickers and Michael Kunzinger both in their talks discussed distributional geometry and its relation to the cosmic censorship hypothesis. Kunzinger concentrated on applications of ideas from Colombeau theory in distributional geometry, while Vickers concentrated on some examples related to cosmic censorship.

The study of singular spacetimes by distributional methods faces the fundamental obstacle of the inherent nonlinearity of the field equations. Staying strictly within the distributional (in particular: linear) regime, excludes a number of physically interesting examples such as strings and point particles [55].

In recent years, several authors have therefore employed nonlinear theories of generalized functions, in particular Colombeau’s theory of generalized functions in order to derive a suitable mathematical framework for a general “nonlinear distributional geometry” adapted to the needs of general relativity [56, 57, 58, 59].

Under the influence of these applications in general relativity the nonlinear theory of generalized functions itself has undergone a rapid development lately, resulting in a diffeomorphism invariant global theory of nonlinear generalized functions on manifolds [60, 61, 62]. In particular, a generalized pseudo-Riemannian geometry allowing for a rigorous treatment of generalized (distributional) spacetime metrics has been developed.
According to the Cosmic Censorship hypothesis realistic singularities should be hidden by an event horizon. However there are many examples of physically realistic spacetimes which are geodesically incomplete, and hence singular according to the usual definition, which are not inside an event horizon.

Many of these counterexamples to the cosmic censorship conjecture have a curvature tensor which is reasonably behaved (for example bounded or integrable) as one approaches the singularity. James Vickers gave examples of a class of weak singularities which may be described as having distributional curvature.

The propagation of test fields on spacetimes with weak singularities has also been investigated. A class of singularities which do not disrupt the Cauchy development of test fields and result in spacetimes which satisfy Clarke’s criterion of ‘generalised hyperbolicity’ has been found. Vickers argued that points which are well behaved in this way, and where Einstein’s equations make sense distributionally, should be regarded as interior points of the spacetime rather than counterexamples to cosmic censorship.

8.3 The a–boundary

Susan M. Scott discussed the application of the a–boundary construction to cosmic censorship. The singularity theorems of Hawking and Penrose prove the existence of incomplete causal geodesics in space-times which satisfy quite general physical conditions. It has long been conjectured that these incomplete causal geodesics would terminate at curvature singularities, but this has never been proven. Recently established a theorem which goes a considerable distance towards closing this gap left by the singularity theorems. By application of the abstract boundary (a-boundary) construction of Scott and Szekeres, to strongly causal space-times, Ashley and Scott have shown that strongly causal space-times satisfying the conditions of the singularity theorems will always have an a-boundary essential singularity.

8.4 The Ehlers group

Marc Mars discussed his work on the Ehlers group. The Ehlers group, which is a symmetry of the Einstein vacuum field equations for strictly stationary spacetimes, is considered by Mars from a spacetime perspective. In this setting, the Ehlers group becomes a subgroup of the infinite dimensional group of transformations that maps Lorentz metrics into Lorentz metrics. Mars has analyzed the global conditions which are required on the spacetime for the existence of the Ehlers group and has found the transformation law for the Weyl tensor under Ehlers transformations. This allows one to study where, and under which circumstances, curvature singularities in the transformed spacetime will arise. As an application, one finds a local characterization of the Kerr-NUT metric.
8.5 Composite spacetimes

Katsuhito Yasuno considered the dynamics of spatially compact ‘composite’ spacetimes\[67\]. Motivated by Thurston’s geometrization conjecture, it is interesting to look for examples of spacetimes with more complicated topology than is allowed by the spatially homogenous models. The composite spacetimes are built from the spatially compact locally homogeneous vacuum spacetimes which have two commuting local Killing vector fields and are homeomorphic to torus bundles over the circle.

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