MENGER REMAINDERS OF TOPOLOGICAL GROUPS

ANGELO BELLA, SECİL TOKGÖZ, AND LYUBOMYR ZDOMSKYY

Abstract. In this paper we discuss what kind of constrains combinatorial covering properties of Menger, Scheepers, and Hurewicz impose on remainders of topological groups. For instance, we show that such a remainder is Hurewicz if and only if it is $\sigma$-compact. Also, the existence of a Scheepers non-$\sigma$-compact remainder of a topological group follows from CH and yields a $P$-point, and hence is independent of ZFC. We also make an attempt to prove a dichotomy for the Menger property of remainders of topological groups in the style of Arhangel’skii.

1. Introduction

All topological spaces are assumed to be completely regular. All undefined topological notions can be found in [12]. For a space $X$ and its compactification $bX$ the complement $bX \setminus X$ is called a remainder of $X$. The interplay between the properties of spaces and their remainders has been studied since more than 50 years and resulted in a number of duality results describing properties of $X$ in terms of those of their remainders. A typical example of such a duality is the celebrated result of Henriksen and Isbell stating that a topological space $X$ is Lindelöf if and only if all (equivalently any) of its remainders is of countable type, that is, any compact subspace can be enlarged to another compact subspace with countable outer base.

In the last years, remainders in compactifications of topological groups have been a popular topic. This is basically due to the fact that topological groups are much more sensitive to the properties of their remainders than topological spaces in general. A major role in this study was played by Arhangel’skii, who initiated a systematic study of this topic. Among many other things, he discovered two elegant dichotomies.

Theorem 1.1 ([5]). Let $G$ be a topological group. If $bG$ is a compactification of $G$, then $bG \setminus G$ is either Lindelöf or pseudocompact.

Theorem 1.2 ([4]). Let $G$ be a topological group. If $bG$ is a compactification of $G$, then $bG \setminus G$ is either $\sigma$-compact or Baire.
We recall that a topological space $X$ is Baire if the intersection of countably many open dense subsets is dense.

In this paper we will focus our attention on topological properties which are strictly in between $\sigma$-compact and Lindelöf. Recall from [20] that a space $X$ is Menger (or has the Menger property) if for any sequence $(U_n)_{n \in \omega}$ of open covers of $X$ one may pick finite sets $V_n \subset U_n$ in such a way that $\bigcup V_n : n \in \omega$ is a cover of $X$. A family $\{W_n : n \in \omega\}$ of subsets of $X$ is called an $\omega$-cover (resp. $\gamma$-cover) of $X$, if for every $F \in [X]^{<\omega}$ the set $\{n \in \omega : F \subset W_n\}$ is infinite (resp. co-finite). The properties of Scheepers and Hurewicz are defined in the same way as the Menger property, the only difference being that we additionally demand that $\bigcup V_n : n \in \omega$ is a $\omega$-cover (resp. $\gamma$-cover) of $X$. It is immediate that

$$\sigma\text{-compact} \Rightarrow \text{Hurewicz} \Rightarrow \text{Scheepers} \Rightarrow \text{Menger} \Rightarrow \text{Lindelöf}.$$ 

The properties mentioned above have recently received great attention, mainly because of their combinatorial nature and game-theoretic characterizations. One of the most striking results about this property is due to Aurichi who proved [6] that any Menger space is a $D$-space.

Our initial idea was to find counterparts of the properties of Menger, Scheepers, and Hurewicz in the style of Theorems 1.1 and 1.2. It turned out that the counterpart of the Hurewicz property is already given by Theorem 1.1 because of the following result, see Section 2 for its proof.

**Theorem 1.3.** Let $G$ be a topological group. If $\beta G \setminus G$ is Hurewicz, then it is $\sigma$-compact.

Let us note that there are ZFC examples of Hurewicz sets of reals which are not $\sigma$-compact (see [15, Theorem 5.1] or [23, Theorem 2.12]), and thus Theorem 1.3 is specific for remainders of topological groups.

As it follows from the theorems below, which are the main results of this paper, for the properties of Scheepers and Menger the situation depends on the ambient set-theoretic universe. Each subspace of $\mathcal{P}(\omega)$ (e.g., an ultrafilter) is considered with the subspace topology. Let us recall from [15, Theorem 3.9] that if all finite powers of a topological space $X$ are Menger then $X$ is Scheepers. The converse of this statement fails consistently: under CH there exists a Hurewicz subspace of $\mathcal{P}(\omega)$ whose square is not Menger, see [15, Theorem 3.7].

**Theorem 1.4.** There exists a Scheepers ultrafilter iff there exists a topological group $G$ such that $\beta G \setminus G$ is Scheepers and not $\sigma$-compact iff there exists a topological group $G$ such that all finite powers of $\beta G \setminus G$ are Menger and not $\sigma$-compact.

**Corollary 1.5.** The existence of a topological group $G$ such that $\beta G \setminus G$ is Scheepers (resp. has all finite powers Menger) and not $\sigma$-compact is independent from ZFC. More precisely, such a group exists under $\mathfrak{d} = \mathfrak{c}$, and its existence yields $\mathcal{P}$-points.

Theorem 1.4 and Corollary 1.5 are proved in Section 3. Let us note that there exists a ZFC example of a dense Baire subspace $X$ of $[\omega]^{\omega}$ all of whose
finite powers are Menger (and thus also Scheepers), and hence it is indeed essential in Theorem 1.4 and Corollary 1.5 that we consider remainders of topological groups. In fact, such a subspace $X$ can be chosen to be a filter, see [10, Claim 5.5] and the proof of [18, Theorem 1].

Regarding the Menger property, we have the following partial result established in Section 4. Note that the assumption on the remainder we make in it is formally weaker than that made in Theorem 1.4, see the last equivalent statement there.

**Theorem 1.6.** *It is consistent that for any topological group $G$ and compactification $bG$, if $(bG \setminus G)^2$ is Menger, then it is $\sigma$-compact.*

In light of Theorems 1.6 and 1.4 it is natural to ask the following questions.

**Question 1.7.** Is there a ZFC example of a topological group with a Menger non-$\sigma$-compact remainder?

**Question 1.8.** Is it consistent that there exists a topological group $G$ such that $\beta G \setminus G$ is Menger and not Scheepers? Does CH imply the existence of such a group?

Since we do not have an analogous statement to Theorem 1.4 for the Menger property (in Theorem 1.6 we make a somewhat unpleasant assumption that the square of the remainder is Menger), it may still be the case that for the Menger property there exists a dichotomy similar to Theorems 1.1 and 1.2. In Section 5 we analyze some properties which might be counterparts of the Menger one for remainders of topological groups.

2. **Hurewicz remainders**

According to the definition on [3, p. 235], a topological group $G$ is *feathered* if it contains a non-empty compact subspace with countable outer base. By [3, Lemma 4.3.10] every feathered group has a compact subgroup with countable outer base. The following fact is probably well-known.

**Lemma 2.1.** *For a feathered group $G$ the following conditions are equivalent:*

1. $G$ is $\check{C}$ech-complete;
2. Each closed subgroup $G_0$ of $G$ admitting a dense $\sigma$-compact subspace is $\check{C}$ech-complete;
3. There exists a compact subgroup $H$ of $G$ with countable outer base such that $\langle QH \rangle$ is $\check{C}$ech-complete for every countable $Q \subset G$, where for $X \subset G$ we denote by $\langle X \rangle$ the smallest subgroup of $G$ containing $X$.

**Proof.** The implication (1) $\rightarrow$ (2) is straightforward, and (2) $\rightarrow$ (3) is a direct consequence of [3, Prop. 4.3.11]. The proof of (3) $\rightarrow$ (1) will be obtained by a tiny modification of that of [3, Theorem 4.3.15]. Here we shall use the notation from the latter theorem and its proof and explain which changes should be made.
The definitions of the (pseudo)metrics $\rho, \rho^*$ there do not use that $G$ is Čech-complete. The latter property is used only in the proof of the fact that $\rho^*$ is complete. However, the argument on [3, p. 238] works also if one only assumes that the subgroup $G_0 := \bigcup \{x_nH : n \in \omega\}$ of $G$ is Raikov complete, i.e., the condition (3) suffices. Thus $\rho^*$ is complete, and therefore $G$ is Čech-complete being a perfect preimage of a complete metric space, see the last paragraph of the proof of [3, Theorem 4.3.15]. □

We are in a position now to present the

**Proof of Theorem 1.3.** Let $G$ be a topological group such that $\beta G \setminus G$ is Hurewicz. Then $\beta G \setminus G$ is Lindelöf, hence $G$ is of countable type [13], and therefore it is feathered. By Lemma 2.1 it is enough to show that each subgroup $G_0$ of $G$ admitting a dense $\sigma$-compact subspace is Čech-complete. Let $G_0$ be as above, $F$ be a dense $\sigma$-compact subspace of $G_0$, and $X = G_0 \setminus G_0$, where the closure is taken in $\beta G$. Since $X$ is closed in $\beta G \setminus G$, it is Hurewicz. Applying [8, Theorem 27] to the Hurewicz space $X$ and Čech-complete space $G_0 \setminus F$ containing it, we conclude that there exists a $\sigma$-compact space $F'$ such that $X \subset F' \subset G_0 \setminus F$, which implies that $G_0 \setminus F'$ is a dense (because it contains $F$) Čech-complete subspace of $G_0$. Thus $G_0$ is Čech complete by [5, Theorem 1.2], which completes our proof. □

It is well-known [25, Lemma 22] that if player II has a winning strategy in the Menger game on a space $X$ (see Section 4 for its definition) then $X$ is Hurewicz. Therefore Theorem 1.3 generalizes [7, Corollary 3.5].

### 3. Scheepers remainders

In the proof of Theorem 1.4, which is the main goal of this section, we shall need set-valued maps, see [17] for more information on them. By a **set-valued map** $\Phi$ from a set $X$ into a set $Y$ we understand a map from $X$ into $\mathcal{P}(Y)$ and write $\Phi : X \Rightarrow Y$ (here $\mathcal{P}(Y)$ denotes the set of all subsets of $Y$). For a subset $A$ of $X$ we set $\Phi(A) = \bigcup_{x \in A} \Phi(x) \subset Y$. A set-valued map $\Phi$ from a topological space $X$ to a topological space $Y$ is said to be

- **compact-valued**, if $\Phi(x)$ is compact for every $x \in X$;
- **upper semicontinuous**, if for every open subset $V$ of $Y$ the set $\Phi^{-1}(V) = \{x \in X : \Phi(x) \subset V\}$ is open in $X$.

To abuse terminology, we shall call compact-valued upper semicontinuous maps **cvusc maps**. It is known [26, Lemma 1] that all combinatorial covering properties considered in this paper are preserved by cvusc maps. Also, if $f : X \to Y$ is a perfect map, then $f^{-1} : Y \Rightarrow X$ assigning to $y \in Y$ the subset $f^{-1}(y)$ of $X$, is a cvusc maps. Therefore the properties of Menger, Scheepers, Hurewicz, having Menger square, etc., are preserved by perfect maps in both directions. That is, if $f$ is perfect and $Z \subset X$ (resp. $Z \subset Y$) has one of these properties, then so does $f(Z)$ (resp. $f^{-1}(Z)$). In particular, this implies that if one of the remainders of a space $X$ has one
of these covering properties, then all others also have it. In addition, all these properties are preserved by product with ω equipped with the discrete topology.\footnote{For the Scheepers property this fact is slightly non-trivial and follows from \cite{24} Proposition 4.7.}

We shall also need some additional notation. For $X \subset P(\omega)$ we shall denote by $\sim X$ the set $\{\omega \setminus x : x \in X\}$. Note that $\sim X$ is homeomorphic to $X$ because $x \mapsto \omega \setminus x$ is a homeomorphism from $P(\omega)$ to itself. For subsets $a, b$ of $\omega$ (resp. $a, b \in \omega^\omega$) $a \subset^* b$ (resp. $a \leq^* b$) means $|a \setminus b| < \omega$ (resp. $|\{n : a(n) > b(n)\}| < \omega$). A collection $\mathcal{F}$ of infinite subsets of $\omega$ is called a semifilter if for any $a \in \mathcal{F}$ and $a \subset^* b$ we have $b \in \mathcal{F}$. For a semifilter $\mathcal{F}$ we set $\mathcal{F}^+ = \{x \subset \omega : \forall a \in \mathcal{F}(a \cap x \neq \emptyset)\}$. Note that $\mathcal{F}^+ = P(\omega) \setminus \sim \mathcal{F}$. $\mathfrak{F}$ denotes the minimal with respect to inclusion semifilter which consists of all co-finite sets. For the other notions used in the proof of the following statement we refer the reader to \cite{3}.

**Lemma 3.1.** Suppose that $G$ is a topological group, $K$ is a compact subgroup of $G$ with countable outer base in $G$, and $QK$ is dense in $G$ for some countable $Q \subset G$. Let $P$ be a property of topological spaces preserved by images under cvusc maps and product with $\omega$ equipped with the discrete topology.

If $\beta G \setminus G$ has $P$ and is not $\sigma$-compact, then there exists a semifilter $\mathcal{F}$ such that $\mathcal{F}^+ \subset \mathcal{F}$ and $\mathcal{F}$ has $P$. If, moreover, $(\beta G \setminus G)^2$ is Menger, then there exists a semifilter $\mathcal{F}$ such that $\mathcal{F} = \mathcal{F}^+$ and $\mathcal{F}^2$ is Menger.

**Proof.** Observe that $G$ is not locally compact because otherwise its remainders would be compact. Since $G$ is feathered, there exists a Čech-complete group $\hat{G}$ containing $G$ as a dense subgroup, see \cite{3} Theorem 4.3.16. Let $\beta \hat{G}$ be the Stone-Čech compactification of $\hat{G}$. It follows that $\beta \hat{G} \setminus G$ is not $\sigma$-compact, and hence $G \neq \hat{G}$. Fix $g \in \hat{G} \setminus G$ and note that $gG$ is dense in $\beta \hat{G}$ and $gG \cap G = \emptyset$. Therefore both $\beta \hat{G} \setminus G$ and $\beta \hat{G} \setminus gG$ have property $P$ being remainders of spaces homeomorphic to $G$, and $\beta \hat{G} = (\beta \hat{G} \setminus G) \cup (\beta \hat{G} \setminus gG)$.

Note that $K$ has a countable outer base also in $\hat{G}$, and hence the quotient space $X := \hat{G}/K = \{zK : z \in \hat{G}\}$ is metrizable. It is also separable by our assumption on $G$, so there exists a metrizable compactification $bX$ of $X$. In addition, the quotient map $\pi_K : \hat{G} \to X$, $\pi_K(z) = zK$, is perfect by \cite{3} Theorem 1.5.7, and hence by \cite{12} Theorem 3.7.16 it can be extended to a (perfect) map $\pi : \beta \hat{G} \to bX$ such that $\pi(\beta \hat{G} \setminus G) = bX \setminus X$. $\pi \upharpoonright G = \pi_K$, hence $G = \pi^{-1}(\pi(G))$, $gG = \pi^{-1}(\pi(gG))$, and consequently $A := \pi(\beta \hat{G} \setminus gG)$ and $B := \pi(\beta \hat{G} \setminus G)$ are both co-dense subsets of $bX$ with property $P$ covering $bX$.

Note that $bX$ has no isolated points because both $G = \pi^{-1}(\pi(G))$ and $\beta \hat{G} \setminus G = \pi^{-1}(\pi(\beta \hat{G} \setminus G))$ are nowhere locally compact. Since $bX$ is a metrizable compact, there exists a continuous surjective map $f : P(\omega) \to X$. Applying \cite{12}, 3.1.C(a)) we can find a closed subspace $T$ of $P(\omega)$ such that $f \upharpoonright T \to bX$ is surjective and irreducible, i.e., $f[T] \neq bX$ for any closed $T \subsetneq T$. $T$ has no isolated points: if $t \in T$ were isolated then the
irreducibility of $f \upharpoonright T$ would give that $t = (f \upharpoonright T)^{-1}[f(t)]$, which infers that $f(t)$ is isolated in $bX$ and this leads to a contradiction. Therefore $T$ is homeomorphic to $\mathcal{P}(\omega)$, and hence there exists a continuous surjective irreducible $h : \mathcal{P}(\omega) \to bX$. Since $h$ is irreducible, both $C := h^{-1}(A)$ and $D := h^{-1}(B)$ are co-dense. Since $h$ is perfect, they also have $\mathcal{P}$: both of them are cvusc images of $\beta \hat{G} \setminus G$. Note also that $\mathcal{P}(\omega) = C \cup D$.

The Cantor set $\mathcal{P}(\omega)$ has the following fundamental property (see [2] and references therein) which can be directly proved by Cantor’s celebrated back-and-forth argument: for any countable dense subsets $I_0, I_1, J_0, J_1$ of $\mathcal{P}(\omega)$ such that $I_0 \cap I_1 = \emptyset$ and $J_0 \cap J_1 = \emptyset$ there exists a homeomorphism $i : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ with the property $i(I_0) = J_0$ and $i(I_1) = J_1$. Therefore there is no loss of generality to assume that $[\omega]<\omega \subset \mathcal{P}(\omega) \setminus C$ and $\exists r \subset \mathcal{P}(\omega) \setminus D$. Set

$$\mathcal{F}_0 = \{ x \subset \omega : \exists c \in C, u \in [\omega]<\omega, v \subset \omega : (x = \{c \setminus u \} \cup v) \},$$

$$\mathcal{I}_0 = \{ x \subset \omega : \exists d \in D, u \in [\omega]<\omega, v \subset \omega : (x = \{d \cup u \} \cup v) \},$$

and note that both $\mathcal{F}_0$ and $\sim \mathcal{I}_0$ are semifilters. It follows that both $\mathcal{F}_0$ and $\mathcal{I}_0$ are countable unions of continuous images of $C \times \mathcal{P}(\omega)$ and $D \times \mathcal{P}(\omega)$, respectively, and consequently they are cvusc images of $(\beta \hat{G} \setminus G) \times \omega$, where $\omega$ is considered with the discrete topology. Since the property $\mathcal{P}$ is preserved by product with $\omega$, we conclude that both $\mathcal{F}_0$ and $\mathcal{I}_0$ have it.

Set $\mathcal{F} = \mathcal{F}_0 \cup \sim \mathcal{I}_0$ and note that it has property $\mathcal{P}$ for the same reason as $\mathcal{F}_0$, $\mathcal{I}_0$ do. Since $C \subset \mathcal{F}_0 \subset \mathcal{F}$ and $D \subset \mathcal{I}_0 = \sim (\sim \mathcal{I}_0) \subset \mathcal{F}$, we have that $\mathcal{P}(\omega) = \mathcal{F} \cup \sim \mathcal{F}$. Therefore $\mathcal{F}^+ \subset \mathcal{F}$ because $\mathcal{F}^+ = \mathcal{P}(\omega) \setminus \sim \mathcal{F}$.

To prove the “moreover” part assume that $\mathcal{F}^2$ is Menger and consider the following map $\phi : (\mathcal{F} \cap \sim \mathcal{F}) \to [\omega]<\omega$:

$$\phi(a) = (a \cup \{n + 1 : n \in a\}) \setminus a.$$  

Note that $\mathcal{F} \cap \sim \mathcal{F}$ is homeomorphic to the closed subset $\{ (x, x) : x \in \mathcal{P}(\omega) \} \cap (\mathcal{F} \times \sim \mathcal{F})$ of the Menger space $\mathcal{F} \times \sim \mathcal{F}$ and thus is Menger itself. Therefore $\phi(\mathcal{F} \cap \sim \mathcal{F})$ is not dominating.

Every strictly increasing sequence $\bar{k} = (k_n)_{n \in \omega}$ of integers such that $k_0 = 0$ generates a monotone surjection $\psi_{\bar{k}} : \omega \to \omega$ by letting $\psi_{\bar{k}}^{-1}(n) = [k_n, k_{n+1})$. We claim that there exists $\bar{k}$ as above such that $\psi_{\bar{k}}(\mathcal{F})^+ = \psi_{\bar{k}}(\mathcal{F})$. Suppose to the contrary that for every $\bar{k}$ there exists $a_{\bar{k}} \in \mathcal{F}$ such that $\psi_{\bar{k}}(a_{\bar{k}}) \in \psi_{\bar{k}}(\mathcal{F}) \setminus \psi_{\bar{k}}(\mathcal{F})^+$, i.e., $\omega \setminus \psi_{\bar{k}}(a_{\bar{k}}) \in \psi_{\bar{k}}(\mathcal{F})$. Then both $b_{\bar{k}} := \psi_{\bar{k}}^{-1}(\psi_{\bar{k}}(a_{\bar{k}}))$ and $\omega \setminus b_{\bar{k}}$ are in $\mathcal{F}$, and therefore $b_{\bar{k}} \in \mathcal{F} \cap \sim \mathcal{F}$. Note, however, that $\phi(b_{\bar{k}}) \subset \{k_n : n \in \omega\}$, which means that $\bar{k} \leq^* \phi(b_{\bar{k}})$. Since $\bar{k}$ was chosen arbitrarily we get that $\phi(\mathcal{F} \cap \sim \mathcal{F})$ is dominating, which is impossible. This contradiction implies that $\psi(\mathcal{F})^+ = \psi(\mathcal{F})$ for some monotone surjection $\psi : \omega \to \omega$, and then $\psi(\mathcal{F})$ is the semifilter with Menger square we were looking for.

Recall that a family $\mathcal{X} \subset \mathcal{P}(\omega)$ is centered if $\bigcap \mathcal{X}'$ is infinite for every $\mathcal{X}' \in [\mathcal{X}]<\omega$. We are in a position now to present the

**Proof of Theorem 1.4.** If $\mathcal{F}$ is a Scheepers ultrafilter, then $\sim \mathcal{F}$ is a subgroup of $(\mathcal{P}(\omega), \Delta)$ and $\mathcal{F} \cup \sim \mathcal{F} = \mathcal{P}(\omega)$. Thus $\mathcal{F}$ is a Scheepers non-$\sigma$-compact
Claim 3.2. 

Proof. Given any \( x \in \mathcal{F} \), then \( x \in \mathcal{F} \) and not \( \sigma \)-compact. In the same way as at the beginning of the proof of Theorem 1.3 we conclude that \( G \) is feathered. By Lemma 2.1 we may assume without loss of generality that \( G \) satisfies the premises of Lemma 3.1. Applying this lemma for \( \mathcal{P} \) being the Scheepers property, we conclude that there exists a Scheepers semifilter \( \mathcal{F} \) such that \( \mathcal{F}^+ \subset \mathcal{F} \). For every \( n \in \omega \) let us denote by \( O_n \) the open subset \( \{ x \in \omega : n \in x \} \) of \( \mathcal{P}(\omega) \) and note that each \( x \in \mathcal{F} \) belongs to infinitely many members of \( \mathcal{U}_0 = \{ O_n : n \in \omega \} \). Applying [19] Theorem 21] (namely the implication (1) \( \rightarrow \) (2) there) we conclude that there exists an increasing number sequence \( (n_k)_{k \in \omega} \) such that \( n_0 = 0 \) and

\[
\{ \bigcup O_n : n \in [n_k, n_{k+1}) \} : k \in \omega \}
\]

is an \( \omega \)-cover of \( \mathcal{F} \). The latter means that for any family \( \{ x_0, \ldots, x_l \} \subset \mathcal{F} \) there exist infinitely many \( k \in \omega \) such that \( x_i \cap [n_k, n_{k+1}) \neq \emptyset \) for all \( i \leq l \).

Let us define \( \phi : \omega \rightarrow \omega \) by letting \( \phi^{-1}(k) = [n_k, n_{k+1}) \) for all \( k \) and set

\[
S = \{ s \subset \omega : \phi^{-1}(s) \subset \mathcal{F} \} = \{ \phi(x) : x \in \mathcal{F} \}.
\]

Then \( S \) is a Scheepers semifilter being a continuous image of \( \mathcal{F} \).

Claim 3.3. \( \mathcal{S}^+ \subset \mathcal{S} \).

Proof. Take any \( x \in \mathcal{S}^+ \) and set \( y = \phi^{-1}(x) \). Then \( y \in \mathcal{F}^+ \): given \( u \in \mathcal{F} \), note that \( \phi(u) \in \mathcal{S} \), and hence \( |\phi(u) \cap x| = \omega \), which implies that \( |u \cap y| = \omega \) and thus \( y \) meets all elements of \( \mathcal{F} \). Since \( \mathcal{F}^+ \subset \mathcal{F} \), we have that \( y \in \mathcal{F} \), and consequently \( x = \phi(y) \in \mathcal{S} \).

Claim 3.2 implies that \( |s \cap \bigcap_{i \leq l} s_i| = \omega \), hence \( \bigcap_{i \leq l} s_i \in \mathcal{S}^+ \), and therefore \( \bigcap_{i \leq l} s_i \in \mathcal{S} \) by Claim 3.3. Thus \( \mathcal{S} \) is a filter, and consequently it is an ultrafilter by Claim 3.3.

This completes our proof.

We call a semifilter \( \mathcal{F} \) a \( P \)-semifilter if for every sequence \( (F_n)_{n \in \omega} \in \mathcal{F}^\omega \) there exists a sequence \( (A_n)_{n \in \omega} \) such that \( A_n \in [F_n]^\omega \) and \( \bigcup_{n \in \omega} A_n \in \mathcal{F} \). Note that if \( \mathcal{F} \) is a filter then we get a standard definition of a \( P \)-filter. \( P \)-filters which are ultrafilters are nothing else but \( P \)-points.

Recall that for every \( n \in \omega \) we denote by \( O_n \) the clopen subset \( \{ x \in \omega : n \in x \} \) of \( \mathcal{P}(\omega) \). The following fact is straightforward.

Observation 3.4. Let \( A \subset \omega \) and \( \mathcal{F} \) be a semifilter. Then \( \{ O_n : n \in A \} \) covers \( \mathcal{F}^+ \) iff \( A \in \mathcal{F} \). Consequently, if \( \mathcal{F}^+ \) is Menger, then \( \mathcal{F} \) is a \( P \)-semifilter.
Proof of Corollary 1.3. It is known [9] that under \( d = \mathfrak{c} \) there exists an ultrafilter \( \mathcal{F} \) on \( \omega \) such that the Mathias forcing \( \mathbb{M}(\mathcal{F}) \) does not add dominating reals, see [9] for corresponding definitions. Applying [10, Theorem 1] we conclude that \( \mathcal{F} \) is Menger when considered with the topology inherited from \( \mathcal{P}(\omega) \). By [10, Claim 5.5] we have that all finite powers of \( \mathcal{F} \) are Menger, and hence \( \mathcal{F} \) is Scheepers by [13, Theorem 3.9].

Now suppose that \( \mathcal{F} \) is a Menger ultrafilter. By the maximality of \( \mathcal{F} \) we have \( \mathcal{F} = \mathcal{F}^+ \). Now it suffices to apply Observation 3.4. \( \square \)

4. Menger remainders

This section is devoted to the proof of Theorem 1.6 which is divided into a sequence of lemmata. In the proof of the next lemma we shall need the following game of length \( \omega \) on a topological space \( X \): In the \( n \)th move player \( I \) chooses an open cover \( \mathcal{U}_n \) of \( X \), and player \( II \) responds by choosing a finite \( \mathcal{V}_n \subset \mathcal{U}_n \). Player \( II \) wins the game if \( \bigcup_{n \in \omega} \bigcup \mathcal{V}_n = X \). Otherwise, player \( I \) wins. We shall call this game the Menger game on \( X \). It is well-known that \( X \) is Menger if and only if player \( I \) has no winning strategy in the Menger game on \( X \), see [14] or [20, Theorem 13].

Formally, a strategy for player \( I \) is a map \( \delta : \tau^{< \omega} \to \mathcal{O}(X) \), where \( \tau \) is the topology of \( X \) and \( \mathcal{O}(X) \) is the family of all open covers of \( X \). The strategy \( \delta \) is winning if \( \bigcup_{n \in \omega} \mathcal{U}_n \neq X \) for any sequence \( (\mathcal{U}_n)_{n \in \omega} \in \tau^\omega \) such that \( \mathcal{U}_n \) is a union of a finite subset of \( \delta(U_0, \ldots, U_{n-1}) \) for all \( n \in \omega \).

Lemma 4.1. Suppose that \( \mathcal{F} \) is a Menger semifilter. Then for every sequence \( \{B_i : i \in \omega\} \in (\mathcal{F}^+)^\omega \) and increasing \( h \in \omega^\omega \) there exists increasing \( \delta \in \omega^\omega \) such that

\[
\bigcup_{i \in \omega} B_i \cap [h(2\delta(i)), h(2\delta(i + 1))] \in \mathcal{F}^+.
\]

Proof. For every \( n \in \omega \) let us denote by \( O_n \) the subset \( \{x \subset \omega : n \in x\} \) of \( \mathcal{P}(\omega) \) and note that \( O_n \) is clopen. It is easy to see that for \( B \subset \omega \) the collection \( \mathcal{U}_B := \{O_n : n \in B\} \) is an open cover of \( \mathcal{F} \) if and only if \( B \in \mathcal{F}^+ \). Set \( \delta(0) = 0 \) and consider the following strategy for player \( I \) in the Menger game on \( \mathcal{F} \): In the 0th move he chooses \( \mathcal{U}_{B_0 \setminus h(0)} = \mathcal{U}_{B_0 \setminus h(\delta(0))} \). Suppose that for some \( i \in \omega \) we have already defined \( \delta(i) \). Then player \( I \) chooses \( \mathcal{U}_{B_i \setminus h(2\delta(i))} \). If player \( II \) responds by choosing \( \mathcal{V}_i \in [\mathcal{U}_{B_i \setminus h(2\delta(i))}]^{< \omega} \), then we define \( \delta(i + 1) \) to be so that \( \mathcal{V}_i \subset \{O_n : n \in [h(2\delta(i)), h(2\delta(i + 1))] \cap B_i\} \), and the next move of player \( I \) is \( \mathcal{U}_{B_{i+1} \setminus h(2\delta(i + 1))} \).

The strategy for player \( I \) we described above is not winning, so there exists a run in the Menger game in which he uses this strategy and loses. Let \( \delta \) be the function defined in the course of this run. It follows that \( \bigcup \{\mathcal{V}_i : i \in \omega\} \supset \mathcal{F} \), where the \( \mathcal{V}_i \)'s are the moves of player \( II \), and hence

\[
\bigcup_{i \in \omega} B_i \cap [h(2\delta(i)), h(2\delta(i + 1))] \in \mathcal{F}^+.
\]

because \( \mathcal{V}_i \subset \{O_n : n \in [h(2\delta(i)), h(2\delta(i + 1))] \cap B_i\} \). \( \square \)
For a semifilter $F$ we denote by $P_F$ the poset consisting of all partial maps $p$ from $\omega \times \omega$ to $2$ such that for every $n \in \omega$ the domain of $p_n : k \mapsto p(n, k)$ is an element of $\sim F$. If, moreover, we assume that and $\text{dom}(p_n) \subset \text{dom}(p_{n+1})$ for all $n$, the corresponding poset will be denoted by $P^*_F$. A condition $q$ is stronger than $p$ (in this case we write $q \leq p$) if $p \subset q$. For filters $F$ the poset $P^*_F$ is obviously dense in $P_F$, and the latter is proper and $\omega^\omega$-bounding if $F$ is a non-meager $P$-filter [21, Fact VI.4.3, Lemma VI.4.4]. In light of Observation [3,4] the following lemma may be thought of as a topological counterpart of [21, Fact VI.4.3, Lemma VI.4.4].

**Lemma 4.2.** If $F^+$ is a Menger semifilter, then both $P_F$ and $P^*_F$ are proper and $\omega^\omega$-bounding.

**Proof.** We shall present the proof for $P_F$. The one for the poset $P^*_F$ could be obtained from the presented proof simply by replacing $P_F$ with $P^*_F$.

To prove the properness let us fix a countable elementary submodel $M \supseteq P_F$ of $H(\theta)$ for $\theta$ big enough, a condition $p \in P_F \cap M$, and list all open dense subsets of $P_F$ which are elements of $M$ as $\{D_i : i \in \omega\}$. Let us denote by $\tau$ the collection of all open subsets of $P(\omega)$. For every $s \in [\omega]^\omega$ we shall denote by $O_s$ the set $\{x \in \omega : x \cap s \neq \emptyset\}$. $O_s$ is clearly a clopen subset of $P(\omega)$.

In what follows we shall define a strategy $\xi : \tau^\omega \rightarrow O(F^+)$ of player I in the Menger game on $F^+$ as well as a map $\xi_0 : \tau^\omega \cap M \rightarrow P_F \cap M$. Set $p^0 = p$, $\xi_0(\emptyset) = p^0$, and

$$\xi(\emptyset) = \{O_s : \exists l \in \omega [s = (\omega \setminus \text{dom}(p^0_0)) \cap l]\}.$$ 

Now suppose that for some $n \in \omega$ and all sequences $(U_k)_{k \leq n}$ of open subsets of $P(\omega)$ we have defined $p^n = \xi_0((U_k)_{k \leq n})$ and $\xi((U_k)_{k \leq n})$, and fix such a sequence $(U_k)_{k \leq n}$ of length $n$. If $U_n$ is not of the form $\bigcap_{i \leq n} O_{s_i}$, where $s_i^n = (\omega \setminus \text{dom}(p^n_i)) \cap l$ for some $l \in \omega$, then $\xi_0((U_k)_{k \leq n})$ and $\xi((U_k)_{k \leq n})$ are irrelevant. Otherwise write $\prod_{i \leq n} 2^{l_i \times s_i^n}$ in the form $\{(t_{i,j})_{i \leq n} : j \leq N\}$, set $p^{n-1} = p^n$, and by induction on $j \leq N$ define a decreasing sequence $(p^{n,j})_{j \leq N}$ of conditions in $P_F \cap M$ with the following properties:

(i) $\text{dom}(p^{n,j}_i) \cap s_i^n = \emptyset$ for all $j \leq N$ and $i \leq n$;

(ii) $p^{n,j} \cup \bigcup_{i \leq n} t^{n,j}_i \in D_n$ for all $j \leq N$.

Then we let $p^{n+1} = p^{n,N}, \xi_0((U_k)_{k \leq n}) = p^{n+1}$ and

$$\xi((U_k)_{k \leq n}) = \left\{ \bigcap_{i \leq n+1} O_{s_i^{n+1}} : \exists l \in \omega \forall i \leq n+1 [s_i^{n+1} = (\omega \setminus \text{dom}(p_i^{n+1})) \cap l] \right\}.$$

Since $F^+$ is Menger, $\xi$ cannot be a winning strategy for player I, and hence there exists a sequence $(U_n)_{n \in \omega}$ of open subsets of $P(\omega)$ with the following properties:

(iii) $p^n := \xi_0((U_i)_{i \leq n}) \in P_F \cap M$ for all $n \in \omega$;

(iv) For every $n \in \omega$ there exists $l_n \in \omega$ such that $U_n = \bigcap_{i \leq n} O_{s_i^n}$, where $s_i^n = (\omega \setminus \text{dom}(p_i^n)) \cap l_n$;

(v) $l_n \leq l_{n+1}$ for all $n \in \omega$;

(vi) $\text{dom}(p_i^{n+1}) \cap s_i^n = \emptyset$ and...
Proof. Let $V$ contain $F$ be a $\mathbb{P}_x$-generic over $V$. Then we can inductively construct an increasing sequence $\langle y_i \rangle_{i \in \omega}$ such that $\text{dom}(p_i) = \emptyset$. Since $i \in \omega$ was chosen arbitrarily, we conclude that $q := \bigcup_{n \in \omega} p_n \in \mathbb{P}_x$.

We claim that $q$ is $(M, \mathbb{P}_x)$-generic. Indeed, pick $q' \leq q$, $n \in \omega$, and $r \leq q'$ such that $\text{dom}(r_i) \supset s_i^n$ for all $i \leq n$. Then there exists $j \leq N$ such that $r_i \upharpoonright (\{i\} \times s_i^n) = t_i^{n,j}$ for all $i \leq n$, and consequently

$$r \leq q \cup \bigcup_{i \leq n} t_i^{n,j} \leq p^{n,j} \cup \bigcup_{i \leq n} t_i^{n,j} \in D_n$$

by (ii). This implies that $r$ is compatible with an element of $D_n \cap M$ and thus completes our proof of the properness.

Note that for every $n$ we have found a finite subset $A_n$ (namely $\{p_i^n : j \leq N\}$) of $D_n \cap M$ such that any extension of $q$ is compatible with some element of $A_n$. If $\dot{f} \in M$ is a $\mathbb{P}_x$-name for a real, then the open dense subset of $\mathbb{P}_x$ consisting of those conditions which determine $\dot{f}(k)$ equals $D_{n_k}$ for some $n_k \in \omega$. It follows from the above that $q$ forces that $\dot{f}(k)$ cannot exceed $\max\{\ell : \exists u \in A_{n_k}(u \Vdash \dot{f}(k) = \dot{l})\}$, and therefore $\mathbb{P}_x$ is $\omega^\omega$-bounding.

For a relation $R$ on $\omega$ and $x, y \in \omega$ we denote by $[x \, R \, y]$ the set $\{n : x(n) R y(n)\}$.

Lemma 4.3. Suppose that $\mathcal{F} = \mathcal{F}^+$ is a semifilter with Menger square. Let $\mathfrak{A} : \omega \times \omega \to 2$ be a $\mathbb{P}^*_x$-generic, $Q \in V[\mathfrak{A}]$ be an $\omega^\omega$-bounding poset, and $H$ be a $Q$-generic over $V[\mathfrak{A}]$. Then in $V[G * H]$ there is no semifilter $\mathcal{G} = \mathcal{G}^+ \subset \mathcal{F}$ containing $\mathcal{F}$ such that $\mathcal{G}^2$ is Menger.

Proof. Suppose to the contrary that such a $\mathcal{G}$ exists. Set $x_j(n) = x(j, n)$. In $V[\mathfrak{A} * H]$, the following 2 cases are possible.

a). For every $m \in \omega$ there exists $k > m$ such that $\bigcup_{j \in \omega} [x_j = x_m] \in \mathcal{G}$. Then we can inductively construct an increasing sequence $\langle m_k : k \in \omega \rangle$ such that

$$\bigcup_{j \in \omega} [x_j = x_{m_k}] \in \mathcal{G} \text{ for all } k. \tag{1}$$

Since $\mathbb{P}^*_x \ast Q$ is $\omega^\omega$-bounding, we may additionally assume that this sequence is in $V$.

b). There exists $m$ such that $\bigcup_{j \in \omega} [x_j = x_m] \in \mathcal{G}$ for all $k > m$. This means that $\bigcap_{j \in \omega} [x_j \neq x_m] \in \mathcal{G}$ for all $k > m$. Then

$$[x_i = x_{i+1}] \supset [x_i \neq x_{i+1}] \cap [x_{i+1} \neq x_m] \supset \bigcap_{j \in \omega} [x_j \neq x_m] \in \mathcal{G}$$

for all $i > m$. Thus the sequence $m_k = m + 1 + 2k$ satisfies (1), and hence there always exists a sequence $\langle m_k : k \in \omega \rangle \in V$ satisfying (1).
Set $A_k = \bigcup_{j \in [m_k, m_{k+1}]} [x_j = x_{m_k}] \in \mathcal{G}$ and $U_k = \{ U_n^k : n \in \omega \}$, where

$U_n^k = \{ (X, Y) \in \mathcal{P}(\omega)^2 : \forall i \leq k ((X \cap A_i \cap [k, n] \neq \emptyset) \land (Y \cap A_i \cap [k, n] \neq \emptyset)) \}$. 

Since $A_k \in \mathcal{G} = \mathcal{G}^+$ for all $k$, $U_k$ is easily seen to be an open cover of $\mathcal{G}^2$. The Menger property of $\mathcal{G}^2$ yields a strictly increasing $f \in \omega^\omega \cap \mathcal{V}[x \ast H]$ such that $\{ U_n^k : k \in \omega \}$ covers $\mathcal{G}^2$. Since $\mathbb{P}_x^\ast \mathbb{Q}$ is $\omega^\omega$-bounding, we could additionally assume that $f \in \mathcal{V}$. Set $h(0) = f(0) + 1$ and $h(l + 1) = f(h(l)) + 1$ for all $l$. Therefore there exists $\epsilon \in 2$ such that

$$W_n^k = \{ X \in \mathcal{P}(\omega) : \forall i \leq k (X \cap A_i \cap [k, n] \neq \emptyset) \}. $$

Then

$$O_\epsilon := \bigcup_{\epsilon \in \omega} \{ W_{f(k)}^k : k \in \bigcup_{l \in \omega} [h(2l + \epsilon), h(2l + \epsilon + 1)) \} \supset \mathcal{G} :$$

If there were $X_\epsilon \in \mathcal{G} \setminus O_\epsilon$, for all $\epsilon \in 2$, then $\langle X_0, X_1 \rangle$ could not be an element of $U_{f(k)}^k$ for any $k$ thus contradicting the choice of $f$. Without loss of generality $\epsilon = 0$ is as above.

**Claim 4.4.** Let $\delta \in \omega^\omega$ be strictly increasing. Then

$$A_\delta := \bigcup_{i \in \omega} A_i \cap [h(2\delta(i)), h(2\delta(i + 1))] \in \mathcal{G}. $$

**Proof.** Given any $X \in \mathcal{G}$, find $l \in \omega$ and $k \in [h(2l), h(2l + 1))$ such that $X \in W_{f(k)}^k$. Let $i \in \omega$ be such that $l \in [\delta(i), \delta(i + 1))$. Note that $i \leq l \leq k$, hence $X \in W_{f(k)}^k$ implies $X \cap A_i \cap [k, f(k)) \neq \emptyset$. It follows that

$$(k, f(k)) \subset [h(2l), f(h(2l + 1))) \subset [h(2l), h(2l + 2)) \subset [h(2\delta(i)), h(2\delta(i + 1))),$$

consequently $X \cap A_i \cap [h(2\delta(i)), h(2\delta(i + 1))] \neq \emptyset$, which implies $\bigcup_{i \in \omega} A_i \cap [h(2\delta(i)), h(2\delta(i + 1))] \in \mathcal{G}^+$. $\square$

Let us fix any $p \in \mathbb{P}_x^\ast$ and set $B_i = \omega \setminus \text{supp}(p_{m_i+1}) \in \mathcal{F}^+$. By Lemma 4.3 used in $V$ there exists an increasing $\delta$ such that $B := \bigcup_{i \in \omega} B_i \cap [h(2\delta(i)), h(2\delta(i + 1))] \in \mathcal{F}^+ = \mathcal{F}$. For every $m \in \omega$ find $i$ such that $m \in [m_i, m_{i+1})$ and set

$$q_m = p_m \cup (B_i \cap [h(2\delta(i)), h(2\delta(i + 1))) \times \{ 0 \})$$

if $m = m_i$ and

$$q_m = p_m \cup (B_i \cap [h(2\delta(i)), h(2\delta(i + 1))] \times \{ 1 \})$$

otherwise. This $q$ obviously forces (i.e., any condition in $\mathbb{P}_x^\ast \mathbb{Q}$ whose first coordinate is $q$ forces) that $B_i \cap A_i \cap [h(2\delta(i)), h(2\delta(i + 1))] = \emptyset$ for all $i$, and hence it also forces $B \cap A_\delta = \emptyset$. Thus the set of those $q \in \mathbb{P}_x^\ast$ which force $B \cap A_\delta = \emptyset$ is dense, which means that $B \cap A_\delta = \emptyset$ (here $A_\delta = A_{\delta^G + H}$). However, $B \in \mathcal{F} \subset \mathcal{G}$ by the choice of $\delta$ and $A_\delta \in \mathcal{G}$ by Claim 4.4 and therefore $B \cap A_\delta = \emptyset$ contradicts $\mathcal{G} = \mathcal{G}^+$. This contradiction completes our proof. $\square$
Proof of Theorem 1.6. Suppose that there exists a topological group $G$ and
a compactification $bG$, such that $(bG \setminus G)^2$ is Menger but not $\sigma$-compact.
Then in the same way as at the beginning of the proof of Theorem 1.3, we
conclude that $G$ is feathered. By Lemma 2.1 we may assume without loss of
generality that $G$ satisfies the premises of Lemma 3.1. Applying this lemma
for $P$ being the property of having the Menger square we get a semifilter
$F = F^+$ such that $F^2$ is Menger. Thus the theorem will be proved as soon
as we construct a model of ZFC in which there are no semifilters $F = F^+$
with Menger square.

To this end let us assume that GCH holds in $V$ and consider a function
$B : \omega_2 \to H(\omega_2)$, the family of all sets whose transitive closure has size
$< \omega_2$, such that for each $x \in H(\omega_2)$ the family $\{\alpha : B(\alpha) = x\}$ is $\omega_1$-
stationary. Let $(P_\alpha, Q_\beta : \beta < \alpha \leq \omega_2)$ be the following iteration with at
most countable supports: If $B(\alpha)$ is a $P_\alpha$-name for $P^{+}_\alpha$ for some semifilter
$F$ such that $P_\alpha \vdash F = F^+$ and $F^2$ is Menger, then $Q_\alpha = P^{+}_\alpha$. Otherwise
we let $Q_\alpha$ to be the $P_\alpha$-name for the trivial forcing. Then $P_{\omega_2}$ is $\omega_2$-
bounding forcing notion with $\omega_2$-c.c. being a countable support iteration of length $\omega_2
of proper $\omega^\omega$-bounding posets of size $\omega_1$ over a model of CH.

Let $G$ be a $P_{\omega_2}$-generic over $V$ and suppose that $F \subseteq V[G]$ is a semifilter
such that $F = F^+$ and $F^2$ is Menger. Then the set $\{\alpha : F_\alpha := (F \cap V[G \cap
P_\alpha]) \subseteq V[G \cap P_\alpha], F_\alpha = F_\alpha^+ \text{ and } F_\alpha^2 \text{ is Menger in } V[G \cap P_\alpha]\}$
contains an $\omega_1$-club subset of $\omega_2$, and hence for one of these $\alpha$ we have that
$Q_\alpha = P^{+}_\alpha$, where $F_\alpha$ is a $P_\alpha$-name such that $F_\alpha^{G \cap P_\alpha} = F \cap V[G \cap P_\alpha]$.
Now, a direct application of Lemmata 4.3 and 4.2 implies that $F_\alpha \subseteq F$ cannot be enlarged
to any semifilter $U \subseteq V[G]$ such that $U^2$ is Menger and $U^+ = U$, which
contradicts our choice of $F$. □

5. ON A POSSIBLE DICHTOMY FOR THE Menger PROPERTY

Our first attempt to find a counterpart of the Menger property is based
on its game characterization we have exploited in Section 3. As the Menger
game produces a strengthening of the Lindelöf property, we should consider
a game which produces a strengthening of the Baire property.

There is an obvious candidate for this purpose: the Banach-Mazur game,
see for instance [16] for more information. This game BM($X$) is played on
the space $X$ in $\omega$-many innings between two players $\alpha$ and $\beta$ as follows. $\beta$
makes the first move by choosing a non-empty open set $U_0$ and $\alpha$ responds
by taking a non-empty open set $V_0 \subseteq U_0$. In general, at the $n$-th inning
$\beta$ chooses a non-empty open set $U_n \subseteq V_{n-1}$ and $\alpha$ responds by taking
a non-empty open set $V_n \subseteq U_n$. The rule is that $\alpha$ wins if and only if
$\bigcap \{V_n : n < \omega\} \neq \emptyset$. The relationship of the Banach-Mazur game with
Baire spaces is given by the following [16, Theorem 8.11].

Theorem 5.1. A space $X$ is Baire if and only if player $\beta$ does not have a
winning strategy in BM($X$).

Consequently, if $\alpha$ has a winning strategy, then the space is Baire.
Definition. A Space $X$ is weakly $\alpha$-favorable if player $\alpha$ has a winning strategy in the Banach-Mazur game. $X$ is said to be $\alpha$-favorable if player $\alpha$ has a winning tactic, i.e. a winning strategy depending only on the last move of $\beta$. \hfill $\Box$

Every pseudocompact space is $\alpha$-favorable: player $\alpha$ has an easy winning tactic by choosing for any $U_n$ a non-empty open set $V_n$ such that $\overline{V_n} \subseteq U_n$. Of course, every weakly $\alpha$-favorable space is Baire. Moreover, the following observation shows that being weakly $\alpha$-favorable often contradicts the Menger property.

Observation 5.2. No nowhere locally compact weakly $\alpha$-favorable subset $X$ of the real line is Menger.

Proof. Since $X$ is nowhere locally compact, we may assume that $X \subseteq \mathbb{R} \setminus \mathbb{Q}$, and the latter we shall identify with $\omega^n$. By [16, Theorem 8.17(1)] $X \supset Y$ for some dense $G_\delta$ subset $Y$ of $\omega^n$. By the Baire category theorem $Y$ cannot be contained in a $\sigma$-compact subspace of $\omega^n$, and hence it contains a copy $Z$ of $\omega^n$ which is closed in $\omega^n$ according to [16, Corollary 21.23]. Therefore $Z$ is a closed in $X$ copy of $\omega^n$, which implies that $X$ is not Menger as the Menger property is inherited by closed subspaces. \hfill $\Box$

Therefore, weak $\alpha$-favorability seems to be the right candidate to be the counterpart of the Menger property. However, this is not the case by Theorem 5.6 below. Let us recall that a set $S \subseteq \mathbb{R}$ is a Bernstein set provided that both $S$ and $\mathbb{R} \setminus S$ meet every closed uncountable subset of $\mathbb{R}$.

Lemma 5.3. There is a subgroup $G$ of the real line $\mathbb{R}$ which is a Bernstein set. Moreover, $\mathbb{Q} \subseteq G$.

Proof. Let $\{C_\alpha : \alpha < \omega\}$ be the collection of all closed uncountable subsets of $\mathbb{R}$. Here, we will consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Choose a point $x_0 \in C_0$ and denote by $G_0$ the vector subspace of $\mathbb{R}$ generated by $\{1, x_0\}$. Obviously, we have $|G_0| = \omega$. Then, pick a point $y_0 \in C_0 \setminus G_0$. We proceed by transfinite induction, by assuming to have already constructed a non decreasing family of vector subspaces $\{G_\beta : \beta < \alpha\}$ of $\mathbb{R}$ satisfying $|G_\beta| \leq |\beta| + \omega$ for each $\beta$ and points $x_\beta, y_\beta \in C_\beta$ in such a way that $x_\beta \in G_\beta$ and $\{y_\beta : \beta < \alpha\} \cap \bigcup\{G_\beta : \beta < \alpha\} = \emptyset$. The set $H_\alpha = \bigcup\{G_\beta : \beta < \alpha\}$ has cardinality not exceeding $|\alpha| + \omega$ and therefore even the vector subspace $K_\alpha$ generated by the set $H_\alpha \cup \{y_\beta : \beta < \alpha\}$ has cardinality less than $\omega$. So we may pick a point $x_\alpha \in C_\alpha \setminus K_\alpha$. Then, let $G_\alpha$ be the vector subspace generated by $H_\alpha \cup \{x_\alpha\}$ and finally pick a point $y_\alpha \in C_\alpha \setminus G_\alpha$. It is clear that $|G_\alpha| \leq |\alpha| + \omega$. To complete the induction, we need to show $y_\beta \notin G_\alpha$ for each $\beta < \alpha$. Indeed, if we had $y_\beta \in G_\alpha$ for some $\beta$, then $y_\beta = z + qx_\alpha$, where $z \in H_\alpha$ and $q \in \mathbb{Q} \setminus \{0\}$. But, this would imply $x_\alpha = q^{-1}z - q^{-1}z \in K_\alpha$, in contrast with the way $x_\alpha$ was chosen.

Now, we let $G = \bigcup\{G_\alpha : \alpha < \omega\}$. It is clear that $G$ is a $\mathbb{Q}$-vector subspace, and hence a subgroup, of $\mathbb{R}$ which is also a Bernstein set. \hfill $\Box$

Lemma 5.4. A Bernstein set $X \subseteq \omega^n$ does not have the Menger property.
Proof. For any \( f \in \omega^\omega \) there exists some \( g \in X \) such that \( f(n) < g(n) \) for each \( n \in \omega \). This comes from the fact that \( X \) must meet the Cantor set \( \prod_{n<\omega} \{ f(n) + 1, f(n) + 2 \} \). To finish, recall that a dominating subset of \( \omega^\omega \) is never Menger. Indeed, for any \( n < \omega \), let \( \pi_n : \omega^\omega \to \omega \) be the projection onto the \( n \)-th factor and put \( U_n = \{ \pi_n^{-1}(k) \cap X : k \in \omega \} \). Each \( U_n \) is an open cover of \( X \). For any choice of a finite set \( V_n \subseteq U_n \), we may define a function \( g : \omega \to \omega \) by letting \( g(n) = \max \pi_n(\bigcup V_n) \), if \( V_n \neq \emptyset \), and \( g(n) = 0 \) otherwise. Since \( X \) is dominating, there is some \( f \in X \) such that \( g(n) < f(n) \) for each \( n \). Clearly, \( f \notin \bigcup \{ \bigcup V_n : n < \omega \} \) and so \( X \) is not Menger. \( \square \)

Lemma 5.5. A Bernstein set \( X \subseteq \mathbb{R} \) is not weakly \( \alpha \)-favorable.

Proof. By [16, Theorem 8.17(1)] any weakly \( \alpha \)-favorable subspace of \( \mathbb{R} \) is comeager, while no Bernstein set can be comeager because any comeager subspace of \( \mathbb{R} \) contains homeomorphic copies of the Cantor set. \( \square \)

These three lemmas imply:

**Theorem 5.6.** There exists a topological group \( G \) and a compactification \( bG \) such that the remainder \( bG \setminus G \) is neither Menger nor weakly \( \alpha \)-favorable.

Proof. Recall that the set of irrationals \( \mathbb{R} \setminus \mathbb{Q} \) is homeomorphic to \( \omega^\omega \). Let \( G \) be such as in Lemma 5.3. By Lemma 5.4 \( \mathbb{R} \setminus \mathbb{Q} \subseteq \omega^\omega \) is not Menger, and by Lemma 5.5 \( \mathbb{R} \setminus \mathbb{Q} \) is not weakly \( \alpha \)-favorable. Now, it suffices to take as \( bG \) the compactification of \( \mathbb{R} \) obtained by adding two end-points. \( \square \)

Theorem 5.6 implies that the counterpart of the Menger property should be in between of weakly \( \alpha \)-favorable and Baire.

6. Miscellanea

A very important example of a topological group is \( C_p(X) \), the subspace of \( \mathbb{R}^X \) with the Tychonoff product topology consisting of all continuous functions. We expect that the remainder of \( C_p(X) \) cannot distinguish between being Menger and \( \sigma \)-compact, but we cannot prove this.

**Question 6.1.** Is it true that a remainder of \( C_p(X) \) is Menger if and only if it is \( \sigma \)-compact?

Below we present some results giving a partial solution of Question 6.1.

**Proposition 6.2.** Let \( Z \) be a compactification of \( C_p(X) \). If \( Z \setminus C_p(X) \) is Menger, then \( C_p(X) \) is first countable and hereditarily Baire.

Proof. Since Menger spaces are Lindelöf, by Henriksen-Isbell’s theorem [13], \( C_p(X) \) is of countable type, and therefore it contains a compact subgroup with countable outer base according to [3, Lemma 4.3.10]. It is easy to see that there is no compact subgroup of \( C_p(X) \) except for \( \{0\} \): for any \( f \in C_p(X) \setminus \{0\} \), the set \( \{nf : n \in \omega\} \) is not contained in any compact \( K \subset C_p(X) \) because \( \{nf(x) : n \in \omega\} \) is unbounded in \( \mathbb{R} \) if \( f(x) \neq 0 \). Therefore \( C_p(X) \) is first-countable, and hence \( X \) is countable.
Since \( Z \setminus C_p(X) \) is Menger, it follows that \( C_p(X) \) contains no closed copy of \( \mathbb{Q} \). Now, a theorem of Debs \cite{Debs} implies that \( C_p(X) \) is hereditarily Baire. \( \square \)

The following fact together with Proposition 6.2 gives the positive answer to Question 6.1 for spaces containing non-trivial convergent sequences.

**Observation 6.3.** If \( X \) contains a non-trivial convergent sequence then \( C_p(X) \) is not Baire.

**Proof.** There is nice characterization of the Bairness for spaces of the form \( C_p(X) \) due to Tkachuk. However, we shall present here a direct elementary proof. Suppose that \( (x_n)_{n \in \omega} \) is an injective sequence converging to \( x \). Set \( F_n = \{ f \in C_p(X) : \forall m \geq n \ (|f(x) - f(x_m)| \leq 1) \} \).

It is easy to check that each \( F_n \) is closed nowhere dense in \( C_p(X) \) and \( C_p(X) = \bigcup_{n \in \omega} F_n \). \( \square \)

By a theorem of Lutzer (see Problem 265 in \cite{Lutzer}), \( C_p(X) \) is \( \check{\text{C}} \)-complete if and only if \( X \) is countable and discrete. So to answer Question 6.1 in the affirmative we need to show that \( C_p(X) \) has a Menger remainder only if \( X \) is countable and discrete.

Note that in the proof of the “if” part of Theorem 1.4 the compactification was a topological group itself, namely \( (\mathcal{P}(\omega), \Delta) \). We do not know whether complements to Menger subspaces in other Polish groups (e.g., \( \mathbb{R} \)) may consistently be subgroups. The next proposition imposes some restrictions.

**Proposition 6.4.** Let \( G \) be an analytic topological group and \( M \) be a non-empty Menger subspace of \( G \). If \( G \setminus M \) is a subgroup of \( G \), then \( G \) is \( \sigma \)-compact and \( M \) contains a topological copy of \( \mathcal{P}(\omega) \).

**Proof.** Suppose that \( H = G \setminus M \) is a subgroup of \( G \) and fix \( g \in M \). Then \( H \subset g^{-1} * M \), where \(*\) is the underlying operation on \( G \). Therefore \( G = M \cup g^{-1} * M \) is Menger, and hence it is \( \sigma \)-compact, see \cite{Menger}.

Now suppose that \( M \) contains no topological copy of \( \mathcal{P}(\omega) \) and let \( X \subset G \) be homeomorphic to \( \mathcal{P}(\omega) \). If \( X \subset H \) then \( g * X \subset g * H \subset M \) which is impossible by our assumption above. Thus \( X \cap M \neq \emptyset \). Since \( M \) contains no copy of \( \mathcal{P}(\omega) \), \( X \setminus M \) is dense in \( X \), and hence there exists a countable dense subset \( Q \) of \( X \) disjoint from \( M \). Then \( X \cap M = (X \setminus Q) \cap M \) is a closed subset of \( M \). Note that \( (X \setminus Q) \) is a copy of \( \omega^\omega \) and \( M \cap (X \setminus Q) \) is a Bernstein set in \( (X \setminus Q) \). To finish, it suffices to apply Lemma 5.4. \( \square \)

The following statement shows that the celebrated Cantor-Bendixon inductive procedure does not have any variant allowing to separate a “nowhere perfect” core of a Menger space from its “\( \sigma \)-compact part”.

**Proposition 6.5.** There exists a Baire dense nowhere locally compact subgroup \( \mathcal{I} \) of \( \mathcal{P}(\omega) \) with the Menger property such that for every \( \sigma \)-compact subspace \( S \) of \( \mathcal{I} \) there exists \( K \subset \mathcal{I} \) homeomorphic to \( \mathcal{P}(\omega) \) such that \( K \cap S = \emptyset \).
Proof. It is well-known that there exists a non-meager Menger filter $\mathcal{F}$ on $\omega$, see, e.g., the proof of Theorem 1 in [IS]. Let $\mathcal{I}$ be the dual ideal of $\mathcal{F}$. Then $\mathcal{I}$ is Menger, nowhere locally compact, and non-meager being homeomorphic to $\mathcal{F}$. Also, $\mathcal{I}$ is a subgroup of $\mathcal{P}(\omega)$, and hence it is Baire because each non-meager topological group is so. Note that $\mathcal{I}$ contains copies of $\mathcal{P}(\omega)$: for every infinite $I \in \mathcal{I}$ the set $\mathcal{P}(I) \subset \mathcal{I}$ is such a copy. Let us fix $\mathcal{X} \subset \mathcal{I} \text{ homeomorphic to } \mathcal{P}(\omega)$ and a $\sigma$-compact $S \subset I$. Then there exists $I \in \mathcal{I} \setminus (S + \mathcal{X})$ because $S + \mathcal{X}$ is $\sigma$-compact and $\mathcal{I}$ is not. It follows that $\mathcal{K} := \{I\} - \mathcal{X}$ is a copy of $\mathcal{P}(\omega)$ disjoint from $S$. □

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Department of Mathematics and Computer Science, University of Catania, Città universitaria, viale A. Doria 6, 95125 Catania, Italy.

E-mail address: bella@dmi.unict.it

Hacettepe University, Faculty of Science, Department of Mathematics, 06800 Beytepe–Ankara, Turkey.

E-mail address: secil@hacettepe.edu.tr

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Straße 25, A-1090 Wien, Austria.

E-mail address: lzdomsky@gmail.com

URL: http://www.logic.univie.ac.at/~lzdomsky/