On the long time behaviour of the Conical Kähler-Ricci flows

Xiuxiong Chen, Yuanqi Wang
February 27, 2014

Abstract

We prove that the conical Kähler-Ricci flows introduced in [17] exist for all time $t \in [0, +\infty)$. These immortal flows possess maximal regularity in the conical category. As an application, we show if the twisted first Chern class $C_{1,\beta}$ is negative or zero, the corresponding conical Kähler-Ricci flows converge to Kähler-Einstein metrics with conical singularities exponentially fast. To establish these results, one of our key steps is to prove a Liouville type theorem for Kähler-Ricci flat metrics (which are defined over $\mathbb{C}^n$) with conical singularities.

Contents

1 Introduction 2
2 Convention of notations, $C^0$ estimate, and $C^{1,1}$ estimate. 6
3 Hölder estimate for the second derivatives and proof of Theorem 1.1. 12
4 Poincare-Lelong equations. 19
5 A rigidity theorem. 23
6 Bounded weakly-subharmonic functions and weak maximum principle. 25
7 Dirichlet problem of conical elliptic equations. 28
8 Strong Maximum Principles and Trudinger’s estimate. 29
9 Proof of Theorem 1.14. 32
10 Bootstrapping of the conical Kähler-Ricci flow. 34
Exponential convergence when $C_{1,\beta} < 0$ or $= 0$.  

Appendix A: Liouville theorem when $\beta \leq \frac{1}{2}$.  

Appendix B: Trudinger’s Harnack inequality.

1 Introduction

Let $(M, [\omega_0])$ be a polarized Kähler manifold and $D$ is a smooth divisor of the anti-canonical line bundle. Suppose the “twisted” first Chern class ($\beta \in (0,1)$)

$$C_{1,\beta} = C_1(M) - (1 - \beta)C_1[D]$$

has a definite sign. One important question is to study the existence of the conical Kähler-Einstein metric in $(M, [\omega_0], (1 - \beta)[D])$

$$\text{Ric}(\omega) = \beta \omega + 2\pi(1 - \beta)[D].$$

This problem has been studied carefully by many authors, for instance, [4], [6], [28], [41], [32] etc. In particular, “conical Kähler-Einstein metric” is a key ingredient in the recent solution of existence problem for Kähler-Einstein metric with positive scalar curvature [12] [13] [14]. In light of these exciting developments, we introduce the notion of conical Kähler-Ricci flow in [17] to attack the existence problem of conical Kähler-Einstein metrics and conical Kähler-Ricci solitons. In [17], we establish short time existence for this flow initiated from any ($\alpha, \beta$) conical Kähler metric (see Section 2 for the definition of ($\alpha, \beta$) metrics while we follow the notations in [17] in general). This is the second paper in this series where we want to establish the long time existence of this flow.

**Theorem 1.1.** Suppose $g_0$ is an ($\alpha, \beta$)-conical Kähler metric in $(M, (1 - \beta)D)$ where $\alpha' \in (0, \min\{\frac{1}{\beta} - 1, 1\})$. Then the conical Kähler-Ricci flow equation admits a solution $\phi(t)$ (7) for $t \in [0, +\infty)$. Furthermore, we have

- for every $t \neq 0$, $g(t)$ is an ($\alpha, \beta$)-conical metric in $(M, (1 - \beta)D)$; for all $\alpha < \min\{\frac{1}{\beta} - 1, 1\}$;

- for all $N > 1$, over the time interval $[0, N]$, $g(t)$ is a $C^{\alpha, \frac{1}{2}, \beta}[0, N]$-family of conical metrics (for all $\alpha < \min\{\frac{1}{\beta} - 1, 1\}$).

**Remark 1.2.** The conical flow $\phi(t)$ in Theorem 1.1 possess $C^{2, \alpha, \beta}$-regularity, while the weak-flow $\phi(t)$ constructed in [48] only possess $C^{1,1}$-regularity.
apriorily (so the metric tensor is not $C^{\alpha,\beta}$ apriorily). This is the essential difference between (strong) conical flow and weak-conical flow. Theorem 1.1 actually implies the weak flow constructed in [48] is strong. Along the line of weak conical flows, in [34], Liu-Zhang also construct weak conical flows and obtain convergence results of their flows on Fano manifolds when $\beta \leq 1/2$.

**Remark 1.3.** For smooth Kähler-Ricci flow, the global existence of flow is proved by Cao [9]. For conical Kähler-Ricci flow, when $n = 1$, this is recently proved by Yin [55] and independently by Mazzeo-Rubinstein-Sesum [35] with different functional spaces.

**Remark 1.4.** For simplicity, we only present the case with one smooth divisor. Our proof certainly works with reducible smooth divisors with no self intersections and with possibly different angles along each component.

**Remark 1.5.** If the manifold is not Fano or the twisted first Chern has mixed sign, Theorem 1.1 still holds as long as the evolving Kähler class remains to be a Kähler class. In particular, the flow is immortal if it fixes the Kähler class.

**Remark 1.6.** This theorem may leads to some exciting, plausible future research: to “migrate” a network of important, fundamental results established in smooth Kähler-Ricci flow to our settings. A partial list of these works (which is far from complete) is given below and we refer interested readers to these papers and references therein for further readings: [54] [51] [16] [53] [44] [40] [42] [36] etc. A word of caution is, because of the presence of conical singularities, that this “migration” might not be at all straightforward!

As an almost direct application, the following is true.

**Theorem 1.7.** If $C_{1,\beta} < 0$ or $C_{1,\beta} = 0$, then the corresponding conical Kähler-Ricci flow converges exponentially fast to a conical Kähler-Einstein metric in the $C^{\alpha,\beta}_{1,1}$ topology of (1,1)-forms (in the sense of (117)).

**Remark 1.8.** When $C_{1,\beta} = 0$, $\beta \leq 1/2$, the existence of Ricci-Flat conical metrics is due to S. Brendle [6] via continuity method. When $C_{1,\beta} < 0$, the existence has been studied by via continuous methods by Jeffres-Mazzeo-Rubinstein [28], Campana-Guenancia-Paun [8], and Eyssidieux-Guedj-Zeriahi [20].

**Remark 1.9.** In the work of Li-Sun [52], they consider the log Calabi-Yau pair

$$(X, \sum_{i=1}^{N} (1 - \beta_i)D_i)$$

such that

$$C_1(X) - \sum_{i=1}^{N} (1 - \beta_i)C_1(D_i) = 0.$$
Then, Theorem 1.7 implies that the existence of Calabi-Yau metric with correct cone angle for any log Calabi-Yau pair. It seems that the existence result for log Calabi-Yau pair of this generality is new. For related topics, please see Song-Wang’s work [41].

Remark 1.10. In Cao’s proof [9] on the smooth case, the Li-Yau harnack inequality in [29] plays a key role when showing the limit is Kähler-Einstein when $C_{1,\beta} = 0$. In our conical case, it’s not clear to us whether the Li-Yau type estimates hold. In our case, the monotonicity of the K-energy directly implies the limit is Kähler-Einstein and the convergence of the metric tensor is exponential.

Going back to Theorem 1.1, much like the smooth counter part, we need to prove $C^0$-estimate of evolved potentials. First, one needs to reduce the flow into a scalar equation. Suppose $\omega_D$ is the model conical Kähler metric (defined in [18], also see the introduction of [17]) with cone angle $\beta$ over $D$ and $h_{\omega_D}$ denotes its Ricci potential. Then,

$$\frac{\partial \phi}{\partial t} = \log \left( \frac{\omega_D + \sqrt{-1} \partial \bar{\partial} \phi}{\omega_D^n} \right) + \beta \phi + h_{\omega_D}. \quad (2)$$

Routine calculation shows that $h_{\omega_D} \in C^{\alpha,\beta}$ for some $\alpha > 0$.

Proposition 1.11. Suppose the conical Kähler-Ricci flow exists up to time $T > 0$. Then, there exists a uniform constant $C_T$ such that

$$|\phi| + \left| \frac{\partial \phi}{\partial t} (t) \right| < C_T, \quad \text{forall } t \in [0, T).$$

Following [28] and [13] (elliptic case), we can use a parabolic type Chern-Lu inequality to obtain:

Proposition 1.12. Under the same assumptions as in Prop 1.11, we have

$$\frac{1}{K} \omega \leq \omega + \sqrt{-1} \partial \bar{\partial} \phi \leq K \omega.$$

Remark 1.13. Guenancia-Paun’s trick in [21] also works well for the $C^{1,1}$ estimate here. Actually, we have multiple choices here to prove the $C^{1,1}$ estimate.

To prove long time existence, we essentially need to prove a priori Holder estimate for the evolving conical Kähler forms. A critical step for this type estimate is to prove the following Liouville type theorem:

Theorem 1.14. (Liouville Theorem) Suppose $\omega$ is a $C^{\alpha,\beta}$ conical Kähler metric defined over $\mathbb{C}^n$. Suppose there is a constant $K$ such that

$$\omega^n = \omega^n_{\beta}, \quad \frac{1}{K} \omega_\beta \leq \omega \leq K \omega_\beta \text{ over } \mathbb{C} \times \mathbb{C}^{n-1} \setminus \{z = 0\}. \quad (3)$$

Then, there is a linear transformation $L$ which preserves $\{z = 0\}$ and

$$\omega = L^* \omega_\beta.$$
This plays a central role in the proof of long time existence theorem. When conical singularity is not presence, this is due to Riebesehl-Schulz [38] where higher derivatives are used heavily. The problem certainly goes back to the famous paper by E. Calabi [7] and Pogorelov [37]. Even in the smooth setting, this is considered an alternative approaches to the later famous Evans-Krylov Shauder estimate for Monge-Ampere equation (cf. [19] [23]).

To prove this Liouville type theorem, we need to extend the maximal principle to more general settings. In the literature, it seems to be a standard trick to use Jeffery’s trick whenever we need to apply Maximum principle. A standard feature of the Jeffery’s trick is to add a small copy of small power of $|S|$ where $S$ is the defining holomorphic section of divisor; and this will perturb the maximum point off from divisor, which allows us to use standard maximal principle. For this trick to work, an important pre-condition is that the function, which we applied maximum principle to, must be $C^{\alpha,\beta}$ for some $\alpha > 0$. This restricts severely how we can use maximum principle. In this paper, we are able to remove this restriction and are able to adapt both weak and strong maximal principle to our setting for function which is locally smooth away from divisor and $L^\infty$ globally. Indeed, we plan to apply maximum principle to $tr_{\omega_j,\omega_\phi}$ which can only be $L^\infty$ globally.

**Theorem 1.15.** Suppose $g(t)$, $t \in [0,T)$ is a solution to the CKRF in Theorem 1.1, $T < \infty$. Then there exists constant $K$ in the sense of Def 2.1 such that the potential $\phi$ satisfies the following bound

$$|\phi|_{2,\alpha,\beta} \leq K \text{ for all } t \in [0,T).$$

Consequently, the flow $g(t)$, $t \in [0,T)$ can be extended beyond $T$.

**Remark 1.16.** For conical Kähler Einstein metric, the corresponding a priori estimate is derived in [13] (c.f. discussions in [28]). This parabolic type Holder estimate should be able to extend as an a priori $C^{2,\alpha,\beta}$-estimate for continuity method for solving the Kähler-Einstein equations (as in [32] and [28]).

One of the key ingredients of Theorem 1.14 is the theory of weak solutions to the Laplace equation of a concial metric. Fortunately, in the polar coordinates, a cone metric is quasi-isometric to the Euclidean metric away from $D$ (by definition). Thus, though straightforward to observe, it’s suprising and amazing that the weak-solution theory in Chap III of [31] (De-Giorgi estimate), Chap 8 of [23], and Chap 4 of [26] are all directly applicable. Roughly speaking, this is because the weak-solution theory only involves $W^{1,2}$-quantities of the weak solution. Thus after integration by parts, we can transform the $W^{1,2}$-inequalities with respect to the cone metric $\omega$ to $W^{1,2}$-inequalities with respect to the Euclidean metric $g_E$ (in the polar coordinates)! Thus, all the classical tools can be applied. Though in
most of the place we directly use weak solution theory (Moser’s iteration, Weak harnack inequalities...), we still prove Trudinger’s Harnack inequality in detail in Appendix B, to show how to transform the $W^{1,2}$-inequalities with respect to the cone metric $\omega$ to $W^{1,2}$-inequalities with respect to the Euclidean metric, and then directly apply the results in [31], [23], and [26]. This proof, though straightforward, shows all the required estimates in weak solution theory are true in our situation.

Strategy of our work and organization of this article: In Section 2 we state some conventions of notations and prove the $C^0$ and $C^{1,1}$ estimate. Then we study the $C^{2,\alpha,\beta}$ regularity in Section 3—11. In Section 3 assuming Theorem 1.14 we prove Theorem 1.15 by showing the Hölder radius is uniformly bounded, thus settle down the $C^{2,\alpha,\beta}$ estimate and the proof of Theorem 1.1. In Section 4 we solve the Poincare-Lelong equation with the correct estimates, which is crucial when we perturb the rescaled limit back to get a contradiction. In Section 5—8 we establish the analytic tools for proving Theorem 1.14. In Section 9 we prove Theorem 1.14. In Section 10 we prove that the flow has maximal regularity immediately when $t > 0$, which is crucial when proving the convergence of the rescaled flows. In Section 11 we prove when $C_{1,\beta} < 0$ or $= 0$, the CKRF converges to conical Kähler-Einstein metric exponentially. In Appendix A, we present a short proof of the Liouville theorem in the case when $\beta \leq \frac{1}{2}$, where we use a regularity result due to Brendle [6]. In Appendix B, we prove Trudinger’s Harnack inequality in our case, by directly using the results from [23] and [26].

Acknowledgment: The first named author wish to thank Kai Zheng, Chengjiang Yao for helpful discussions on maximal principle over conical settings. He is also grateful to Xi Zhang for sharing his insight on weak conical-Kähler Ricci flow. The second named author wish to thank Prof. S.K Donaldson for communications on earlier versions of this work. The second author is grateful to Prof. Xianzhe Dai, Guofang Wei, and Rugang Ye for their interest in this work and their continuous support. He also would like to thank Yuan Yuan for related discussions on complex analysis. Both authors would like to thank Song Sun, Kai Zheng, and Haozhao Li for carefully reading earlier versions of this paper and related discussions.

2 Convention of notations, $C^0$ estimate, and $C^{1,1}$ estimate.

Definition 2.1. We would like to make following convention on the constants in this paper, similar to that of [17]: Without further notice, the ”$C$” in each estimate means a constant depending on the dimension $n$, the angle $\beta$, the background objects $(M, \omega_0, L, h, D, \omega_D)$, the $\alpha$ (and $\acute{\alpha}$ if any) in the same estimate or in the corresponding theorem (proposition, corollary,
lemma), the initial metric $\omega_0$, and finally the time $T$ (beyond which we want to extend the flow). We add index to the “C” if it depends on more factors than the above objects. Moreover, the “C” in different places might be different. The C never depends on $T' < T$ (unless it comes with another index, like $C(T')$).

Most of the notations in this article follow those of [17]. For the readers’ convenience, we introduce some key definitions from [17] here.

**Definition 2.2.** $(\alpha, \beta)$ conical Kähler metric: For any $\alpha \in (0, \min\{\frac{1}{\beta} - 1, 1\})$, a Kähler form $\omega$ is said to be an $(\alpha, \beta)$ conical Kähler metric on $(M, (1 - \beta)D)$ if it satisfies the following conditions.

1. $\omega$ is a closed positive $(1, 1)$-current over $M$.
2. For any point $p \in D$, there exists a holomorphic chart $\{z, u_i, i = 1, \ldots, n-1\}$ such that in this chart, $\omega$ is quasi-isometric to the standard cone metric
   $$\omega_{\beta} = \beta^2 |z|^{2\beta - 2} \frac{1}{2} dz \wedge d\bar{z} + \frac{1}{2} \sum_{j=2}^{n} du_j \wedge d\bar{u}_j.$$
3. There is a $\phi \in C^{2, \alpha, \beta}(M)$ such that
   $$\omega = \omega_0 + i \partial \bar{\partial} \phi.$$

**Remark 2.3.** If an $\omega$, defined either globally or locally, satisfies 1 and 2 in Definition 2.2 but only partially satisfies 3 in the sense that $\omega = \sqrt{-1} \partial \bar{\partial} \phi$ for some $\phi \in C^{\alpha, \beta}$, then we say $\omega$ is a weak conical metric. This definition can be found in Definition 1.2 in [48].

Near $D$, using the defining function $z$ of $D$, it’s easy to describe the function space $C^{2, \alpha, \beta}(M)$ near $D$. Namely, near $D$, let $\xi$ be singular coordinate, $z = |\xi|^{\beta - 1} \xi$. Let $z = re^{i\theta}$, and $s_i, i = 3, \ldots, 2n$ be real coordinates of $z_2, \ldots, z_n$ which are perpendicular to $z$. We define

**Definition 2.4.** ($C^{2, \alpha, \beta}$-functions).

1. $f(z, z_2, \ldots) \in C^{\alpha, \beta}$ iff $f(|\xi|^{\beta - 1}, z_2, \ldots) \in C^{\alpha}$ in terms of $\xi, z_2$...
2. $f(z, z_2, \ldots) \in C^{2, \alpha, \beta}$ iff
   $$|z|^{2-2\beta} \frac{\partial^2 f}{\partial z \partial \bar{z}} \in C^{\alpha, \beta},$$
   $$|z|^{-\beta} \frac{\partial^2 f}{\partial \theta \partial s_i} \in C^{\alpha, \beta},$$
   $$|z|^{-\beta} \frac{\partial^2 f}{\partial s_i \partial s_j} \in C^{\alpha, \beta}. $$
The full definition of the function space $C^{2,\alpha,\beta}(M)$ and the corresponding parabolic norm is in section 2 of [17].

The following model metric defined in [18] satisfies the above definition.

$$\omega_D = \omega_0 + \delta i \partial \bar{\partial} |S|^{2\beta},$$

where $\delta$ is a small enough number.

Let $r = |z|^{\frac{1}{3}}$ and $\theta$ be just the angle of $z$ from the positive real axis. In the polar coordinates $r, \theta, u_i, \ 2 \leq i \leq n$, $\omega_\beta$ can be written as

$$\omega_\beta = dr^2 + \beta^2 r^2 d\theta^2 + \sum_{j=2}^{n} du_j \otimes d\bar{u}_j.$$ (4)

From now on we will be using the polar coordinates in most of the sections, since there the conical metrics are quasi-isometric to the Euclidean metric.

Theorem 1.1 is a direct consequence of Proposition 2.5, 2.7, and Theorem 1.15. Now we prove the $C^0$ estimate.

Proposition 2.5. Suppose the flow exists over $[0, T']$. Then we have the following $C^0$ bound

$$|\phi|_0 \leq a + \frac{b}{\beta} e^{\beta t}, \text{ for all } (x, t) \in (M \setminus D) \times [0, T'],$$

where $a = \sup_x |\phi|(x, 0)$, and $b = \sup_x \left\{ |\log \left( \frac{\omega_D + \sqrt{-1} \partial \bar{\partial} \phi_0}{\omega_D} \right)^n + \beta \phi + h \omega_D \right\}(x, 0)$.

Over $M \setminus D$, denote $u = \frac{\partial \phi}{\partial t}$, we compute

$$\frac{\partial u}{\partial t} = \Delta_t u + \beta u.$$ (5)

We use the function $|S|^{2\tau}$ as barrier function such that $2\tau \leq \beta \alpha$. Suppose the flow is smooth over $[0, T']$. First we have the following lemma due to Jeffres [27].

Lemma 2.6. $u + \epsilon |S|^{2\tau}$ attains maximum in $M \setminus D$ when $t \in [0, T']$; $u - \epsilon |S|^{2\tau}$ attains minimum in $M \setminus D$.

For the reader’s convenience of include the proof here.

Proof. of Lemma 2.6 It suffices to prove $u + \epsilon |S|^{2\tau}$ attains maximum in $M \setminus D$ when $t \in [0, T']$, the other is similar. We argue by contradiction. If not, suppose $u + \epsilon |S|^{2\tau}$ attains maximum in $D$ at $(p, t_1)$. Let $\rho = |z|$, notice
that the integral curve of $\frac{\partial}{\partial \rho}$ is not necessarily the geodesic. Then let $q$ be on the integral curve $\gamma$ of $\frac{\partial}{\partial \rho}$ starting at $p$. We have

$$\frac{|u(q) - u(p)|}{\rho^{2\alpha}(q)} \leq |u|_\alpha < \infty; \quad \frac{|u(q) - u(p) + \epsilon |S|^{2\tau}(q)|}{\rho^{2\alpha}(q)} \leq 0. \quad (6)$$

Since $2\tau - \beta \alpha < 0$, when $\rho(q)$ is sufficiently small with respect to $|u|_\alpha$, we have

$$\frac{\epsilon |S|^{2\tau}}{\rho^{2\alpha}(q)} \geq C \epsilon \rho^{2\tau - \beta \alpha}(q) \geq 2|u|_\alpha + 1. \quad (7)$$

Therefore (7) contradicts (6).

**Proof.** of Proposition 2.5: We compute

$$\frac{\partial(u + \epsilon |S|^{2\tau})}{\partial t} = \Delta_t(u + \epsilon |S|^{2\tau}) + \beta(u + \epsilon |S|^{2\tau}) - \Delta_t \epsilon |S|^{2\tau} - \beta \epsilon |S|^{2\tau}.$$

Using (31) in [48], we know

$$\Delta_t \epsilon |S|^{2\tau} \geq -C(T') \epsilon, \quad (8)$$

then

$$\frac{\partial(u + \epsilon |S|^{2\tau})}{\partial t} \leq \Delta_t(u + \epsilon |S|^{2\tau}) + \beta(u + \epsilon |S|^{2\tau}) + C(T') \epsilon. \quad (9)$$

By Lemma 2.6, the maximum-principle applies to (9). Hence

$$(u + \epsilon |S|^{2\tau}) \leq e^{\beta t}(|u + \epsilon |S|^{2\tau}|_{(0,t=0)}) + C(T') \epsilon$$

Let $\epsilon \to 0$ we get $u \leq e^{\beta t}|u|_{(0,t=0)}$. Thus the upper bound is obtained. The lower bound $u \geq -e^{\beta t}|u|_{(0,t=0)}$ follows similarly. Then

$$|\frac{\partial \phi}{\partial t}| = |u| \leq e^{\beta t}|u|_{(0,t=0)}. \quad (10)$$

By integrating (10) and using

$$\frac{\partial \phi}{\partial t}|_{t=0} = (\log \frac{(\omega_D + \sqrt{-1} \partial \bar{\partial} \phi)^n}{\omega_D^n} + \beta \phi + h\omega_D)|_{t=0},$$

we obtain the desired bound in Proposition 2.5. \qed

Our next objective is to prove the following $C^{1,1}$ bound for conical KRF. We follow the approach in [28] and [13]. Notice that Guenancia-Paun’s trick in [21] also works for the $C^{1,1}$-estimate here.
Proposition 2.7. Under the same assumptions in Theorem 1.1, there exists a uniform constant $K$ in the sense of Definition 2.1 such that

$$\frac{1}{K} \omega_D \leq \omega_D + \sqrt{-1} \partial \bar{\partial} \phi \leq K \omega_D.$$ 

Let $u = g^{il} h_{jk} f^j_l f^k_l$, $f = id$ is treat as a harmonic map from $M$ to $M$ itself. Choose $z_p$ as normal coordinates of $g$ (with Kähler form $\omega$) at $x$, and let $l, i$, be normal coordinate indexes of $\omega$ also. Then we compute

$$\Delta_u \omega
= g^{il} h_{jk} f^j_l f^k_l + h_{jk, d\bar{m}} f^j_l f^k_l f^m_p f^\bar{m}_p + h_{jk} f^j_l f^k_l
+ h_{jk, d\bar{m}} f^j_l f^k_l f^m_p f^\bar{m}_p + h_{jk} f^j_l f^k_l + h_{jk} f^{d\bar{m}} f^j_l f^k_l f^m_p f^\bar{m}_p + h_{jk} d\bar{m} f^j_l f^k_l f^m_p f^\bar{m}_p.
$$

Choose $j, k, d, m$ as normal coordinate index of $h$, then

$$\Delta_u \omega
= g^{il} h_{jk} f^j_l f^k_l + h_{jk, d\bar{m}} f^j_l f^k_l f^m_p f^\bar{m}_p + h_{jk} f^j_l f^k_l
= R^{i\bar{m}} f^j_l f^k_l - R_{d\bar{m}j\bar{k}} f^j_l f^k_l f^m_p f^\bar{m}_p + f^j_l f^k_l.$$

Set $h = \omega_D$. Thus along the Kähler-Ricci flow, using $R^{h}_{j\bar{k}, d\bar{m}} \leq C_1 I$ (see Li-Rubinstein’s appendix in [28]), and

$$\frac{\partial}{\partial t} u = (R^{i\bar{m}} - \beta g^{i\bar{m}}) h_{jk} f^j_l f^k_l$$

over $M \setminus D$, we obtain

$$(\Delta_u - \frac{\partial}{\partial t}) u
= \beta f^j_l f^k_l - R^{i\bar{m}} f^j_l f^k_l f^m_p f^\bar{m}_p + f^j_l f^k_l
\geq -C_1 u^2 + \beta u + f^j_l f^k_l.$$

By adding the weight $e^{\lambda \phi} u$ we compute

$$(\Delta_u - \frac{\partial}{\partial t}) e^{\lambda \phi} u
\geq \lambda e^{\lambda \phi} (n - u) - C_1 e^{\lambda \phi} u^2 + \beta e^{\lambda \phi} u + e^{\lambda \phi} f^j_l f^k_l f^m_p f^\bar{m}_p + 2 \lambda e^{\lambda \phi} \nabla \omega \phi, \nabla \omega u
\geq -\lambda u \frac{\partial \phi}{\partial t} e^{\lambda \phi} + \lambda^2 u |\nabla \omega \phi|^2 e^{\lambda \phi}.$$

Using the inequality $(\Sigma_k a_k b_k)^2 \leq (\Sigma_k a_k^2)(\Sigma b_k^2)$ and the following estimate

$$|\nabla u|^2
= \Sigma_{i, k, p, s, t} f^i_k f^j_l f^m_p f^s_t
\leq \Sigma_p \{ (\Sigma_{i, k} |f^i_{kp}|^2)^{\frac{1}{2}} (\Sigma_{i, k} |f^k_l|^2)^{\frac{1}{2}} (\Sigma_{s, t} |f^s_t|^2)^{\frac{1}{2}} (\Sigma_{s, t} |f^t_{ls}|^2)^{\frac{1}{2}} \}$$

$$= u f^j_l f^k_l,$$
it’s easy to see
\[ e^{\lambda f} f^j_{ip} f^j_{ip} + 2\lambda e^{\lambda \phi} < \nabla \omega \phi, \nabla \omega u > + \lambda^2 u |\nabla \omega \phi|^2 e^{\lambda \phi} \geq 0. \]

Thus let \( C_2 = C_1 + 1 \) and \( \lambda = -C_2 \) we get

**Lemma 2.8.**

\[ (\Delta \omega - \frac{\partial}{\partial t})e^{-C_2 \phi} u \geq e^{-C_2 \phi} u^2 - C e^{-C_2 \phi} u + C_2 u \frac{\partial \phi}{\partial t} e^{-C_2 \phi}. \]

Now we are ready to prove the \( C^{1,1} \)-estimate.

**Proof.** of Proposition 2.7 From (2.8), we obtain

\[ (\Delta \phi - \frac{\partial}{\partial t})[e^{-C_2 \phi} u + \epsilon |S|^{2\tau}] \geq e^{-C_2 \phi} u + \epsilon |S|^{2\tau} - C [e^{-C_2 \phi} u + \epsilon |S|^{2\tau}] + C_2 e^{-C_2 \phi} u + \epsilon |S|^{2\tau} - C_2 \epsilon |S|^{2\tau} \frac{\partial \phi}{\partial t} - 2\epsilon |S|^{2\tau} - e^{C_2 \phi} (\epsilon |S|^{2\tau})^2 + \Delta \phi |S|^{2\tau}. \]

Again similar to the proof of Proposition 2.5, since \( e^{-C_2 \phi} u \in C^{\alpha,\beta}[0,T'] \), then \( \max (e^{-C_2 \phi} u + \epsilon |S|^{2\tau}) \) is attained in \( M \setminus D \) when \( \tau < \alpha \beta \). Using \( \Delta \epsilon |S|^{2\tau} \geq -C(T') \) (see formula (31) in [48], (10), Proposition 2.5 and maximum-principle, we have the following inequality

\[ \{e^{-C_2 \phi} u + \epsilon |S|^{2\tau}\}_p \leq C(T') + C\{e^{-C_2 \phi}\}_p, \]

where \( p \) is the maximum point of \( e^{-C_2 \phi} u + \epsilon |S|^{2\tau} \). Thus by taking \( \epsilon \to 0 \), we end up with

\[ u \leq C e^{C_2 \text{osc} \phi}. \]

(11) means the following. Suppose \( z_i, i \in (1,...n) \) are the normal coordinates of the background metric \( \omega \) at a general point \( p \) such that it also diagonalize \( \sqrt{-1\partial \bar{\partial} \phi} \) at \( p \), we have

\[ \Sigma_i \frac{1}{1 + \phi_{ii}} \leq C. \]

Since \( \phi \) satisfies the equation

\[ \frac{(\omega_D + \sqrt{-1\partial \bar{\partial} \phi})^n}{\omega_D^n} = e^{\frac{\partial \phi}{\partial \tau} - h - \phi} \]

and we have

\[ |\frac{\partial \phi}{\partial \tau}| + |\phi| \leq C, \]

11
we obtain
\[
\frac{1}{C}\omega_D \leq \omega_D + \sqrt{-1}d\bar{\partial}\phi \leq C\omega_D.
\]

At this point, actually we’ve arrived at a simple proof of the long time existence when the complex dimension is 1, with the help of the Harnack inequality.

**Proposition 2.9.** When \( n = 1 \), the long time existence (Theorem 1.1) follows from Proposition 2.6, 2.7, and Theorem 4.2 in [18], without involving the proof of Theorem 1.15 in the next section.

Proof. of Proposition 2.9 Proposition 2.6, Proposition 2.7, and equation \( 2 \) say that the assumptions in Theorem 4.2 in [18] are fulfilled. Thus from Theorem 4.2 in [18], there is a \( \alpha > 0 \) such that \( |\frac{\partial^2}{\partial t} \phi|_{2,\alpha,\beta} \leq C, \ t \in [0, T'] \) for any \( T' < T \). Since \( n = 1 \), from the potential equation \( 2 \) we get
\[
|\phi|_{2,\alpha,\beta} \leq C \text{ over } [0, T).
\]
By the discussions in Step 2 of the proof of Theorem 1.15 the flow can be extended beyond \( T \).

\[\square\]

3 **Hölder estimate for the second derivatives and proof of Theorem 1.1.**

Based on Theorem 1.14 we are able to prove Theorem 1.15 which in turn implies our main Theorem 1.1 in an obvious way. Let us first introduce a new notion of Hölder radius, which is motivated by the Harmonic Radius in Anderson’s work [2].

From now on in this section, we work in the singular polar coordinates, unless otherwise specified. For the reader’s convenience, we use the main definitions from [17]. Let \( w_j, j = 2 \cdots n \) be the tangential variables. We consider a basis of \((1, 0)\) vectors as
\[
a = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial r} - \sqrt{-1} \frac{\partial}{\partial \theta} \right), \ \frac{\partial}{\partial w_j}, j = 2 \cdots n.
\]
Set \( \xi = z^\beta = re^{i\beta\theta} \), notice that
\[
\frac{\partial^2}{\partial \xi \partial \bar{\xi}} = \frac{1}{4} \left[ \frac{\partial^2}{\partial r^2} + r^{-1} \frac{\partial}{\partial r} + \frac{1}{\beta^2} r^{-2} \frac{\partial^2}{\partial \theta^2} \right].
\]
In this singular polar coordinates, we define the polar \( \sqrt{-1}d\bar{\partial}\)-operator to be the operator with the following basis.
\[
\frac{\partial^2}{\partial \xi \partial \bar{\xi}} a_i \frac{\partial}{\partial w_i}, a_j \frac{\partial}{\partial w_i}, \frac{\partial^2}{\partial w_i \partial w_j}, 2 \leq i, j \leq n,
\]

12
By abuse of notation, the ”\( \sqrt{-1}\partial \bar{\partial} \)”s in the polar coordinates all mean the polar \( \sqrt{-1}\partial \bar{\partial} \)-operator defined above.

From now on, when we write ”\( [\cdot] \)” , we mean seminorm; when we write ”\( |\cdot| \)” , we mean norm (which contain lower order terms). These definitions can be found in section 2 of [17].

**Remark 3.1.** In the polar coordinates, under the above basis, we have 

\[
[\omega_\beta]_{\alpha,\beta} = 0.
\]

This means \( \omega_\beta \) is a constant tensor.

**Definition 3.2.** Hölder radius: Let \( K \) be as in Proposition 2.7, let \( \hat{K} \) and \( K \) be two constants large enough. Given a point \( p \in B_0(R) \) (in the polar coordinates), and a \( C^{\alpha,\beta} \)-metric \( \omega \) defined over \( B_0(R) \), we define the Hölder radius \( r_p \) of \( \omega \) at a point \( p \in B_0(R_0) \) to be the largest radius (with respect to the Euclidean metric in the singular polar coordinates), such that there exists a potential \( \phi \) in \( B_p(r_p) \) which satisfies

- \( \omega = \sqrt{-1}\partial \bar{\partial} \phi \) over \( B_p(r_p) \), \( r_p \leq d_{\beta,E}(p, \partial B_0(R)) \).
- \( \phi \in C^{2,\alpha,\beta} \), \( [\phi]_{2,\alpha,\beta} \leq \delta_0 r_p^{-\alpha} \), \( [\phi]_{2,\beta} \leq K \), \( |\phi| \leq \hat{K} r_p^2 \),

where \( \delta_0 \) is small enough with respect to the \( \delta \) in Proposition 5.2. For the second item, the norms are defined in the polar coordinates, as in section 2 in [17]. The balls are all with respect to \( d_{\beta,E} \), which is the distance with respect to the Euclidean metric \( g_E \) in the polar coordinates.

**Proof.** of Theorem 1.15 and 1.15

Step 1: By the \( C^{1,1} \)-estimate in Proposition 2.7, using Theorem 4.2 in [48] and equation (5), we deduce

\[
\left| \frac{\partial \phi}{\partial t} \right|_{\alpha', \alpha, \beta, [0,T]} \leq C, \text{ for some } \alpha' > 0.
\] (13)

Moreover, by Theorem 4.2 in [48], the \( C^0 \)-estimate in Proposition 2.5, and the \( C^{1,1} \)-estimate in Proposition 2.7 and (13), we obtain

\[
[\phi]_{\alpha', \alpha, \beta, [0,T]} \leq C.
\] (14)

by making \( \alpha' \) smaller if necessary.

Step 2. In this step we show \( [\phi]_{2,\alpha,\beta, [0,T]} \) is uniformly bounded, for any \( \alpha < \alpha' \). We follow the Anderson-type argument as in the proof of Lemma 2.2 in [2]. By abuse a notation, we still denote \( \phi \) as the potential of \( \omega \) near \( D \) i.e \( \omega = \sqrt{-1}\partial \bar{\partial} \phi \).

Denote \( \omega_i = \omega(t_i) \), \( t_i \in [0,T] \) is a time sequence. Denote

\[
F_i = \left( \frac{\partial \phi}{\partial t} - \beta \phi + f \right)|_{t_i},
\]

13
where $f$ is a function depending on $\omega_D$. By (13), we have

$$|F_i|_{\alpha', \beta} \leq C.$$  

Without loss of generality, it suffices to show in $B_0(R_0)$, $R_0$ sufficiently small with respect to the background geometry (so a local coordinate system is defined), $\frac{r_{p, \omega_i}}{d_{\beta, E}(p, \partial B_0(R_0))}$ is uniformly bounded away from 0 independent of $p$ and $i$.

We prove by contradiction. By Theorem 10.1 and Proposition 4.1 if $r_{p, \omega_i}$ is not uniformly bounded away from 0 independent of $p$ and $i$, then there exists a subsequence $(p_i, \omega_i)$, $i \to \infty$ such that

$$\frac{r_{p_i, \omega_i}}{d_{\beta, E}(p_i, \partial B_0(R_0))} \to 0, \ p_i \to D, \ and$$

$$0 < \frac{r_{p_i, \omega_i}}{d_{\beta, E}(p_i, \partial B_0(R_0))} \leq 2 \min_p \frac{r_{p, \omega_i}}{d_{\beta, E}(p, \partial B_0(R_0))}.$$  

Next we consider the rescaled metric $\hat{\omega}_i = r_{p_i, \omega_i}^{-2} \hat{T}_R \ast \omega_i$ at $p_i$, where $T_R$ is defined as

$$\hat{\omega}_i \circ T_R = R_{\hat{\omega}_i}^{\frac{1}{2}} z, \ \hat{\omega}_i \circ T_R = R \omega_i.$$  

The following properties of $\hat{\omega}_i$ are obvious from the rescaling hypothesis and Proposition 2.7 ($\hat{\omega}_\beta$ and $\hat{d}_{\beta, E}$ are the rescaled metric and distance in the rescaled coordinates).

- $\hat{\omega}_i^n = e^{\hat{F}_i} \hat{\omega}_\beta^n$.  

- $\hat{\omega}_i$ is defined on $B_0(\frac{R_0}{r_{p_i, \omega_i}})$, $\hat{F}_i$ is the pull back of $F_i$ via the rescaling map.

- $\hat{d}_{\beta, E}(p_i, \partial B_0(\frac{R_0}{r_{p_i, \omega_i}})) \to \infty$.

- For the same $K$ as in Proposition 2.7, we have

$$\frac{1}{K} \hat{\omega}_\beta \leq \hat{\omega}_i \leq K \hat{\omega}_\beta.$$  

- By definition, for any $p \in B_0(\frac{R_0}{r_{p_i, \omega_i}})$ and $i$, we have

$$r_{\hat{\omega}_i, p} \geq \frac{\hat{d}_{\beta, E}(p_i, \partial B_0(\frac{R_0}{r_{p_i, \omega_i}}))}{3d_{\beta, E}(p_i, \partial B_0(\frac{R_0}{r_{p_i, \omega_i}}))}.$$  

Notice $\hat{d}_{\beta, E}(p_i, \partial B_0(\frac{R_0}{r_{p_i, \omega_i}})) \to \infty$. Consequently, suppose $\hat{d}_{\beta, E}(p_i, p) < \lambda < \infty$ with respect to the rescaled Euclidean metric in polar coordinates, we have

$$\liminf_{i \to \infty} r_{\hat{\omega}_i, p} \geq \frac{1}{3}. \quad (17)$$
At $p_i$, we have $r_{\omega_i, p_i} = 1$.

Claim 3.3. For any $p$, when $i$ is large enough, the rescaled potential $\hat{\omega}_i$ satisfies

$$|\hat{\phi}_i|_{2, \alpha', \beta, B_p(\frac{1}{2})} \leq C.$$  

To prove the claim, without loss of generality we consider $p = 0$. In $B_0(\frac{1}{2})$, by (17), when $i$ is large enough, there exists a potential $\hat{\phi}_{p,i}$ such that

- $\hat{\omega}_i = \sqrt{-1} \partial \bar{\partial} \hat{\phi}_{p,i}$,
- $\hat{\phi}_{p,i} \in C^2, \alpha', \beta, [\hat{\phi}_{p,i}]_{2, \alpha, \beta, B_0(\frac{1}{2})} \leq 4\delta_0, [\hat{\phi}_{p,i}]_{2, \beta, B_0(\frac{1}{2})} \leq K$.

Since $\delta_0$ is small enough in the sense of Definition 3.2, the proof of Proposition 5.2 or the discussion of (37) in [13] directly imply the claim is true. For the reader’s convenience, we include the crucial step here. Without loss of generality, we assume $\hat{\omega}_i$ satisfies the normalization condition at the point $0: \hat{\omega}_i(0) = \omega_\beta$. By the small oscillation condition $([\hat{\phi}_{p,i}]_{2, \alpha, \beta, B_0(\frac{1}{2})}, \beta \leq 4\delta_0)$, we deduce

$$[\det(i\partial \bar{\partial} \hat{\phi}_{p,i}) - \Delta \hat{\phi}_{p,i}]^{(*)}_{\alpha', B_0(\frac{1}{2})} \leq \epsilon [i\partial \bar{\partial} \hat{\phi}_{p,i}]^{(*)}_{\alpha', B_0(\frac{1}{2})};$$  

where $\epsilon$ is small enough with respect to $\delta_0$. Since

$$det(i\partial \bar{\partial} \hat{\phi}_{p,i}) = e^{\hat{F}_i} \in C^{\alpha'}$$ in polar coordinates, $\alpha' > \alpha$, combining (14), we deduce

$$[\Delta \hat{\phi}_{p,i}]^{(*)}_{\alpha', B_0(\frac{1}{2})} \leq \epsilon [i\partial \bar{\partial} \hat{\phi}_{p,i}]^{(*)}_{\alpha', B_0(\frac{1}{2})} + [e^{\hat{F}_i}]^{(*)}_{\alpha', B_0(\frac{1}{2})} \leq \epsilon [i\partial \bar{\partial} \hat{\phi}_{p,i}]^{(*)}_{\alpha', B_0(\frac{1}{2})} + C.$$  

Then continuing as in the discussion after (37) in Chen-Donaldson-Sun’s work [13], or as (52)–(54) in the proof of Proposition 5.2, Claim 3.3 is proved.

By (14) and (13), the following crucial estimate is true.

$$\lim_{i \to \infty} e^{\hat{F}_i} = C_1$$ uniformly on compact sets over $\mathbb{C}^n$ in $C^{\alpha, \beta} - \text{topology}$, (20)

where $C_1$ is a positive constant.

Claim 3.3 implies $\hat{\omega}_i$ subconverge to a $\omega_\infty$ over $\mathbb{C}^n$ locally in $C^{\alpha, \beta} - \text{topology}$. Moreover,

- $\omega_\infty^n = C_1 \omega_\beta^n$,

$$\frac{1}{K} \hat{\omega}_\beta \leq \omega_\infty \leq K \hat{\omega}_\beta,$$  

(21)

- For any $p \in \mathbb{C}^n$, we have $r_{\omega_\infty, p} \geq \frac{1}{3}$.  

15
We show in the following two cases, the above all lead to contradictions. Case 1: Suppose $d_{\beta, E}(p_\infty, D) \leq \frac{1000}{\beta \sin \beta \pi}$. By translation along the tangential direction of $D$, we can assume $d_{\beta, E}(p_\infty, D) = d_{\beta, E}(p_\infty, 0)$. Under the translation along the tangential direction of $D$, the form of equation (16) is invariant, because $\omega_\beta$ is invariant under these tangential translations. This case is the main issue (while the other cases are easier to handle). From Theorem 1.14, for some linear transformation $L$ which preserves $D = (0) \times C^{n-1}$, we have

$$\omega_\infty = L^* \omega_\beta.$$  

By the proof of Proposition 25 in [13] and (21), we obtain

$$\frac{1}{K} \leq |a_{11}|^{2\beta} \leq K,$$

where $a_{11}$ is the $(1, 1)$ element of $L$.  

(22)  

Along the tangential direction of $D$, $L$ reduces to a $(n-1) \times (n-1)$ matrix $L_T$. By (21) again, we get

$$|L_T| \leq CK^{\frac{1}{2}}.$$  

(23)  

Claim 3.4. Suppose $\text{dist}_{\beta}(p_\infty, D) \leq \frac{1000}{\beta \sin \beta \pi}$. We can choose $\phi_\infty$ such that $\omega_\infty = \sqrt{-1} \partial \bar{\partial} \phi_\infty$ and

$$|\phi_\infty| \leq CK \text{ over } B_{p_\infty}(90).$$

The proof of Claim 3.4 is as follows. Consider the most natural potential function

$$\phi_\infty = L^*(|z|^{2\beta} + \sum_{j=2}^{n} |w_j|^2).$$

We obviously have $\omega_\infty = L^* \omega_\beta = \sqrt{-1} \partial \bar{\partial} \phi_\infty$. By (22), (23), and the proof of Proposition 25 in [13], we directly obtain

$$|\phi_\infty| \leq CK \text{ over } B_{p_\infty}(90).$$

Obviously, we also have

$$[\phi_\infty]_{2, \beta} \leq CK \text{ over } B_{p_\infty}(90), \quad [\phi_\infty]_{2, \alpha, \beta} = 0.$$

The proof of Claim 3.4 is completed.

Now, when $i$ is sufficiently large, we perturb $\phi_\infty$ to be a potential $\phi$ defined in $B_{p_i, \omega_i}(2)$ which satisfies the conditions in Definition 3.2, thus a contradiction will be obtained. We consider the equation

$$\sqrt{-1} \partial \bar{\partial} v_i = \omega_i - \omega_\infty.$$  

Notice $|\omega_i - \omega_\infty|_{\alpha, \beta} \to 0$, uniformly over compact subdomains of $C^m$. Using Proposition 4.1, we obtain a solution $v_i$ such that

$$|v_i|_{2, \alpha, \beta, B_{p_i}(2)} \leq C|\omega_i - \omega_\infty|_{\alpha, \beta, B_{p_i}(50)}.$$  

(24)
Thus the identity \( \omega_i = \omega_\infty + \sqrt{-1} \partial \bar{\partial} v_i \) holds in \( B_p(2) \). By Proposition 4.1 and (24), we have when \( i \) is sufficient large that
\[
[u_1]_{2,\alpha,\beta,B_p(2)} \leq \frac{\delta_0}{100}.
\]
Let \( \phi_i = \phi_\infty + v_i \). Notice the fact \( [\omega_\infty]_{\alpha,\beta} = [L^* \omega_\beta]_{\alpha,\beta} = 0 \) (as in Remark 3.1) is quite important to show the oscillation before rescaling is small. From Claim 3.4, (13), (14), (21), and (25), by making \( K \) and \( \hat{K} \) large enough, we obtain
\[
[u_i]_{2,\alpha,\beta,B_p(2)} \leq \frac{\delta_0}{100},
\]
This is a contradiction since we assumed that there is no such potential for \( \hat{\omega}_i \) in a ball (centered at \( p_i \)) of radius larger than 1!

Case 2: Suppose \( \infty > d_{\beta,E}(p_\infty,D) > \frac{100}{\sin \beta \pi} \). By translation along the tangential direction of \( D \), we can also assume \( d_{\beta,E}(p_\infty,D) = d_{\beta,E}(p_\infty,0) \). This case is easier, since before taking limit, the coordinate \( u = z^\beta \) is well defined in \( B_{p_1}(90) \). This is because \( B_{p_1}(90) \) does not cover a whole period \([0,2\pi]\) in this case, then we can choose the single-value branch of \( z^\beta \) over \([0,2\pi]\) in \( B_{p_1}(90) \). Denote \( p_i = (z_i^\beta, w_{1,i}, \ldots, w_{n-1,i}) \). Notice with respect to the coordinate \( u = z^\beta, w_2, \ldots, w_n \), we have
\[
\omega_\beta = \omega_{Euc},
\]
where \( \omega_{Euc} \) is the Euclidean metric in the coordinates \( u, w_2, \ldots, w_n \).

Hence, we still consider the origin 0 as our base point. By exactly the small oscillation argument in case 1, the rescaled limit \( \omega_\infty \) still equals \( L^* \omega_\beta \). Using (27) and Proposition 4.1 we perturb the following potential
\[
\phi_\infty = L^*([z^\beta - z^\beta_{p_\infty}]^2 + \sum_{j=2}^n |w_j - w_j_{p_\infty}|^2)
\]
to a potential before \( i \) goes to \( \infty \), in \( B_{p_1}(2) \) when \( i \) is large enough. Then we get the same contradiction as in Case 1 to the hypothesis that there is no such potential in ball (centered at \( p_i \)) with radius larger than 1!

Case 3. Suppose \( d_{\beta,E}(p_i,D) \to \infty \). By translation along the tangential direction of \( D \), we still assume \( d_{\beta,E}(p_i,D) = d_{\beta,E}(p_i,0) \). This case is actually easier than Case 1 and Case 2, because the almost smallest Hölder radius occurs far away from \( D \). The argument is similar to Case 2. The difference is that, since in Case 3 the distance from \( p_i \) to \( D \) goes to \( \infty \), we should choose \( p_i \) as the base point of our convergence, not 0 (as in case 1 and 2) anymore. Still suppose \( p_i = (z_i, w_{2,i}, \ldots, w_{n,i}) \), we denote the following coordinates as \( \Psi_i \):
\[
\hat{u} = z^\beta - z^\beta_i, \hat{u}_2 = w_2 - w_{2,i}, \ldots, \hat{u}_n = w_n - w_{n,i}.
\]
With respect to the coordinate \( \Psi_i \), We have
\[
\omega_\beta = \hat{\omega}_{Euc}.
\]

17
where $\hat{\omega}_{Euc}$ is the Euclidean metric in the coordinates $\hat{u}, \hat{u}_2, ..., \hat{u}_n$. Then with respect to $\Psi_i$, by the translation invariance of $\hat{\omega}_{Euc}$ along all directions (not only the tangential directions), the Monge-Ampere equation (16) is written as

$$\hat{\omega}_i^n = e^{F_i} \hat{\omega}_{Euc}^{n} \text{ in } B_0(\lambda_i), \quad (28)$$

where $\hat{F}_i$ is the translated Ricci potential, and

$$\lambda_i = \min\{\hat{d}_{\beta,E}(p_i, \partial \hat{B}_0(R_0, \omega_i)), \frac{\sin(\beta \pi) d_{\beta,E}(p_i, \omega_i)}{100}\}. \quad (29)$$

By Theorem 1.14 (in the case when $\beta = 1$), we still have

$$\omega_{\infty} = L^* \omega_\beta \text{ over } C^n. \quad (30)$$

Using (27) and Proposition 4.1, we perturb the following potential of $\hat{\omega}_{\infty}$

$$\tilde{\phi}_{\infty} = L^*(|\hat{u}|^2 + \sum_{j=2}^n |\hat{u}_j|^2) \text{ in terms of the coordinate } \Psi_i,$$

to a potential before $i$ goes to $\infty$, in $B_0(2)$. Then, we obtain a contradiction as in Case 1 and Case 2 again, to the hypothesis that $r_{p_i,\omega_i} d_{\beta,E}(p_i, \partial \hat{B}_0(R_0))$ goes to 0!

Thus, $r_{p_i,\omega_i} d_{\beta,E}(p_i, \partial \hat{B}_0(R_0))$ can not go to 0. This shows

$$[\tilde{\phi}]_{2,\alpha,\beta, T_{R_0}}(D) \leq C, \quad (31)$$

where $T_{R_0}(D)$ is the tubular neighborhood of $D$ with width $\frac{R_0}{2}$. By parabolic Evans-Krylov-Safanov Theorem (as in 45), we deduce the following estimate away from $D$

$$[\phi]_{2,\alpha,\beta, M \setminus T_{R_0}}(D) \leq C. \quad (32)$$

(30) and (31) imply

$$[\phi]_{2,\alpha,\beta, M} \leq K.$$ 

The proof of Theorem 1.15 is complete.

Step 3: To prove the long time existence part, notice that by the proof of Theorem 1.2 in 17, the short time $t_0$ such that the CKRF exists only depend on the background geometry $(M, (1 - \beta)D, \omega_0)$ and $|\phi_0|_{2,\alpha,\beta}$, where $\phi_0$ is the potential of the initial metric with respect to the reference metric $\omega_D$. Since $|\phi(t)|_{2,\alpha,\beta} \leq K$, which is independent of $t \in [0,T)$, we can start
the short-time solution for time period $t_0$ from $\phi(T - \frac{t_0}{2})$, thus end up with a flow for $t \in [0, T + \frac{t_0}{2}]$. The $t_0$ is the short existence time in Theorem 1.2 of [17], subject to the bound $\mathbb{K}$ and the background geometry. Then the flow can be extended beyond any finite $T > 0$.

The proof of the long time existence is completed.

Since $T \geq t_0$, where $t_0$ is the short existence time of the CKRF in Theorem 1.2 of [17], from the proof in Step 1, we conclude that $\mathbb{K}$ depends on the background geometry $(M, L, h, \omega_0)$, the $C^1$-bound on $\phi$, $|\partial \phi / \partial t|_0$, and the initial metric of the flow.

\[\square\]

Remark 3.5. We actually proved more: when the volume form (with respect to $\omega_\beta$) is $C^{\alpha', \beta}$, we can obtain $C^{\alpha, \beta}$ estimate on the second derivatives ($\alpha < \alpha'$), provided the $C^{1, 1}$-estimate is already obtained. This is interesting even in smooth case (when $\beta=1$), and we will discuss it in detail in a sequel of this paper.

4 Poincare-Lelong equations.

In this section we work in the holomorphic coordinates. Our main target is to prove Proposition 4.1. This is crucial in the proof of Theorem 1.15, when we perturb the potential of the rescaled limit metric back to a potential before taking limit to get a contradiction (as in [2], where the Laplace equation is the main interest). Let $A_R$ be the cylinder (centered at 0) with respect to the model cone metric $\omega_\beta$, as in [17]. Let $\omega_E$ be the Euclidean metric in the holomorphic coordinates.

**Proposition 4.1.** There exists a constant $C$ depending on $\beta$ and $n$ with the following properties. Given the equation

$$\sqrt{-1} \partial \bar{\partial} v = \eta \text{ over } A_{20}, \quad (32)$$

where $\eta \in C^{\alpha, \beta}_{1,1}$ is a closed $(1,1)$-form such that $\eta = \sqrt{-1} \partial \bar{\partial} \phi_\eta$ for some $\phi_\eta \in C^{2, \alpha, \beta}$. Then there exists a solution $v$ in $C^{2, \alpha, \beta}$ such that

1. $|v|_{2, \alpha, \beta, A_5} \leq C |\eta|_{\alpha, \beta, A_{20}}$.
2. $|v|_{0, A_5} \leq C |\eta|_{0, \beta, A_{20}}$.

**Remark 4.2.** By the assumptions, $(34)$ is already solved by $\phi_\eta$. The point is that we want a solution with the correct estimate.

**Proof.** of Proposition 4.1 We only need to find a solution $v \in W^{1, 2}_{\omega_\beta}(A_6) \cap C^0(A_6)$ such that

$$|v|_{W^{1, 2}_{\omega_\beta}(A_6)} \leq C |\eta|_{0, \beta, A_{20}},$$

19
so consequently $v$ is a weak solution to (34), by Lemma 2.5 in [47]. Then the Schauder regularity estimate in [18] or in [17] implies $v$ is in $C^{2,\alpha,\beta}$ and $v$ satisfies interior Schauder estimate.

With the help of Lemma 4.4, the $W^{1,2}_{\omega_{\beta}}$-estimate of $v$ is actually straightforward. It suffices to observe that

$$
\int_{A_{10}} \beta^2 |z|^2 \beta^{-2\beta} \left( \frac{\partial v}{\partial z} \right)^2 \omega_{\beta}^n = \int_{A_{10}} \frac{\partial v}{\partial z} \beta^2 \omega_{Euc}^n \leq C. \tag{33}
$$

Obviously we have $\sum_{i=1}^{n-1} |\frac{\partial}{\partial y_i}|_{\omega_{\beta}} \leq C |\eta|_{0,\beta,A_{20}}$, then by Lemma 2.5 in [47], $v$ is actually a weak solution to the following trace equation

$$
\Delta_{\beta} v = \eta \text{ over } A_{10}. \tag{34}
$$

Thus the Moser’s iteration trick works again, as in the proof of Lemma 13.1. Thus $v \in C^{0,\beta}(A_{10}) \cap W^{1,2}_{\omega_{\beta}}$ and

$$
|v|_{0,A_{10}} \leq C |\eta|_{0,\beta,A_{10}}.
$$

Item 2 is thus proved.

By the main Theorem in [18] and item 1 in Lemma 4.4, we conclude $v \in C^{2,\alpha,\beta}$ and

$$
|v|_{2,\alpha,\beta,A_{10}} \leq C (|\eta|_{\alpha,\beta,A_{10}} + |v|_{0,A_{10}}) \leq C |\eta|_{\alpha,\beta,A_{20}}. \tag{35}
$$

The proof of item 1 is also complete.

Consider the natural orbifold map

$$
\Sigma : \ A_{\frac{1}{N},r_1,r_2}^{20,\frac{1}{N},20} \rightarrow A_{\frac{1}{N},r_1,r_2}^{20,\frac{1}{N},20}
$$

$$
\Sigma(w) = w^N = z,
$$

where $A_{\frac{1}{N},r_1,r_2}$ means the cylinder (centered at 0) of normal radius $R_1$ and tangential radius $R_2$, with respect to the orbifold model metric $\omega_{\frac{1}{N}}$.

**Lemma 4.3.** Suppose $\beta > \beta_0$, $\alpha < 1$, then

$$
C^{\alpha,\beta}_{1,1} \subset C^{{\hat{\alpha}},\beta_0}_{1,1},
$$

where $\hat{\alpha} = \min\{\alpha, \frac{\beta}{\beta_0} - 1\}$ and $C^{\alpha,\beta}_{1,1}$ is the space of $C^{\alpha,\beta}(1,1)$-forms. Moreover, suppose $\beta > \frac{1}{N-1} > \frac{1}{N}$, and $\eta = \sqrt{-1 \partial \bar{\partial} \phi_{\eta}}$ for some $\phi_{\eta} \in C^{2,\alpha,\beta}$, then we can pull back $\eta$ by $\Sigma$ such that $\Sigma^* \eta \in C^{\hat{\alpha}}$ in the usual sense upstairs, where $\hat{\alpha} < \min\left(\frac{1}{2(N-1)}, \alpha\right)$.

20
Proof of Lemma 4.3

This Lemma is easy to prove as follows. With respect to $(1,1)$-derivative, \( \eta \in C^{\alpha,\beta}_{0,1} \) means
\[
|z|^{2-2\beta} \eta(z) \in C^{\alpha,\beta}.
\]
Thus we have
\[
|z|^{2-2\beta_0} \eta(z) = |(z|^{2-2\beta} \eta(z))|(\eta|^{2-2\beta_0}).
\]
Since \(|z|^{2-2\beta} \eta(z) \in C^{\alpha,\beta} \) and \(|z|^{2\beta-2\beta_0} \in C^{\min(\frac{2\beta}{\beta_0},-2,1),\beta_0} \), then
\[
|z|^{2-2\beta_0} \eta(z) \in C^{\min(\frac{2\beta}{\beta_0}-2,\alpha),\beta_0}.
\]
Notice for the mixed derivatives we have for any \( 1 \leq i \leq n-1 \) that
\[
|z|^{1-\beta_0} \eta_{\bar{u}_i} = |z|^{|\beta-\beta_0|} |z|^{1-\beta} \eta_{\bar{u}_i}.
\]
Using the assumption
\[
|z|^{1-\beta} \eta_{\bar{u}_i} \in C^{\alpha,\beta} \in C^{\alpha,\beta_0}.
\]
and the fact \(|z|^{\beta-\beta_0} \in C^{\min(\frac{\beta}{\beta_0}-1,\alpha);\beta_0} \) we get
\[
|z|^{1-\beta_0} \eta_{\bar{u}_i} \in C^{\min(\frac{\beta}{\beta_0}-1,1),\beta_0}.
\]
Usually we can not pull back a current \( \eta \). However, in case when \( \eta = \sqrt{-1} i \partial \bar{\partial} \phi \eta \) for some \( \phi \eta \in C^{2,\alpha,\beta} \), we can pull back \( \eta \) by defining
\[
\Upsilon^* \eta = \sqrt{-1} \partial \bar{\partial} \Upsilon^* \phi \eta.
\]
Then, for the last part in Lemma 4.3 without of generality we only consider the mixed term \((\Upsilon^* \eta)_{\bar{u}_i}\), the other terms are similar. Notice that
\[
(\Upsilon^* \eta)_{\bar{u}_i} = \sum_{\alpha,\beta} N w^{N-1} \Upsilon^* (|z|^{1-\beta} |z|^{1-\beta} \eta_{\bar{u}_i})
\]
\[
= \sum_{\alpha,\beta} N w^{N-1} |w|^{N\beta-1} e^{-iN \theta w} \Upsilon^* (|z|^{1-\beta} \eta_{\bar{u}_i} + \frac{\partial}{\partial \theta} \bar{u}_i)
\]
\[
= \sum_{\alpha,\beta} N |w|^{N\beta-1} e^{-iN \theta w} \Upsilon^* (|z|^{1-\beta} \eta_{\bar{u}_i} + \frac{\partial}{\partial \theta} \bar{u}_i).
\]
Since
\[
|z|^{1-\beta} \eta_{\bar{u}_i} \in C^{\alpha,\beta} \in C^{\alpha,\frac{1}{N}}.
\]
then \( \Upsilon^* (|z|^{1-\beta} \eta_{\bar{u}_i} + \frac{\partial}{\partial \theta} \bar{u}_i) \in C^{\alpha} \) in the regular sense upstairs. On the other hand, by (38) we have \( |w|^{N\beta-1} e^{-i\theta w} \in C^{\frac{1}{N-1}} \).

The proof of the Lemma is completed. \( \square \)
Lemma 4.4. Under the same hypothesis of Proposition 4.1. There exists a weak solution \( v \in W^{1,2}_{\omega_E} \) to equation (34) such that

- \( |v|_{0,A_{10}} \leq C|\eta|_{\alpha,\beta,A_{20}} \),
- \( |\partial v/\partial z|_{L^2(A_{10})\omega_E} + \sum_{i=1}^{n-1} |\partial v/\partial u|_{0,A_{10}} \leq C|\eta|_{\alpha,\beta,A_{20}} \).

Proof. of Lemma 4.4: We use the orbifold method, which should be counted as a geometric argument. Fix a \( 1 > \beta > 0 \), there exists an integer \( N \) such that

\[ \beta > \frac{1}{N-1} > \frac{1}{N}. \]  

The geometry of \( A_{20} \) is like an orbifold. Moreover, it’s obvious that

\[ A_{20} = A_{\frac{1}{N},20,\frac{1}{N},20}. \]

where \( A_{\frac{1}{N},20,\frac{1}{N},20} \) is the cylinder (centered at 0) of normal radius \( 20\pi^{1/N} \) and tangential radius 20, with respect to the orbifold model metric \( \omega_{\frac{1}{N}} \).

Therefore we could treat \( \eta \) as a form in \( C_{1,1}^{\frac{1}{N}}, \hat{\alpha} \leq \min(\frac{1}{2(N-1)},\alpha) \).

The we consider the \( i\partial \bar{\partial}v \) equation over the upstairs space

\[ i\partial \bar{\partial}v = \Sigma^{(*)} \eta. \]  

By Claim 4.3 we obtain

\[ |\Sigma^{(*)} \eta|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}} \leq C|\eta|_{\alpha,\beta,A_{20}}. \]

Using Hormander’s results in \([25]\) and the standard proof of the \( i\partial \bar{\partial} \)-lemma as in \([22]\), we can find a solution \( \hat{v} \) to equation (39) with the following properties.

\[ \frac{|\hat{v}|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}} + |\partial \hat{v}/\partial w|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}} + \sum_{i=1}^{n-1} |\partial \hat{v}/\partial u|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}}}{|\hat{v}|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}}} \leq C|\Sigma^{(*)} \eta|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}}. \]  

Denote \( a_N = e^{2\pi i} \) as the \( n \)-th unit root, we define the renormalized solution as

\[ \bar{v}(w, \cdot) = \frac{1}{N}[\hat{v}(w, \cdot) + \hat{v}(a_N w, \cdot) + ... + \hat{v}(a_N^{N-1} w, \cdot)]. \]  

Then \( \bar{v} \) is invariant under the deck transformation over \( A_{\frac{1}{N},20,\frac{1}{N},20} \) (by multiplying \( a_N \)). Moreover, \( \bar{v} \) still solves (39). By (40), \( \bar{v} \) satisfies

\[ \frac{|\bar{v}|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}} + |\partial \bar{v}/\partial w|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}} + \sum_{i=1}^{n-1} |\partial \bar{v}/\partial u|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}}}{|\bar{v}|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}}} \leq C|\Sigma^{(*)} \eta|_{0,A_{\frac{1}{N},20,\frac{1}{N},20}}. \]  

22
Now we show that \( v(w) \) can be descended to \( v(z) \) over \( A_{10} \). Suppose \( z = \rho e^{i\theta}, \theta \in [0, 2\pi) \). Define
\[
v(z) \triangleq v(z_{\frac{1}{N}}),
\]
where \( z_{\frac{1}{N}} \) represents the single-valued branch as \( re^{i\theta} \to r e^{i\theta} \).

It’s easy to check \( v(z) \) has the same limit when \( \theta \) approaches 0 and \( 2\pi \), therefore \( v(z) \) is well defined. This is obvious from the construction in (41).

It’s also easy to check \( \frac{\partial v}{\partial z}, \frac{\partial^2 v}{\partial z \partial \bar{z}}, \frac{\partial^2 v}{\partial z^2} \) all match up when \( \theta = 0 \) and \( 2\pi \), therefore they are well defined and at least continuous away from \( \{z = 0\} \).

By construction we directly have \( v \in C^{2,0, \frac{1}{N}} \). Moreover, the \( C^0 \) estimate upstairs trivially descends downstairs, namely we have
\[
|v|_{0,A_{10}} \leq C|\eta|_{0, A_{10}^{\frac{1}{N}}, A_{20}^{\frac{1}{N}}},
\]
and the second inequality in (43) implies
\[
N^2|z|^{2-\frac{2}{\beta}} |\frac{\partial v}{\partial z}|^2 \leq C|\eta|_{\alpha, \beta, A_{20}}^2 \text{ for all } z \in A_{10}.
\]

Then we have
\[
\int_{A_{10}} |\frac{\partial v}{\partial z}|^2 \omega_\beta^n = \int_{A_{10}} N^2|z|^{2-\frac{2}{\beta}} |\frac{\partial v}{\partial z}|^2 \omega_\beta^n \leq C|\eta|_{\alpha, \beta, A_{20}}^2.
\]
The tangential derivatives are obviously bounded in \( C^0 \)-norm. Thus the proof of Lemma 4.4 is complete.

5 A rigidity theorem.

In this section we prove Theorem 5.1. This theorem implies Theorem 1.14 if we can show \( \omega \) has a tangent cone which is isomorphic to \( \omega_\beta \).

**Theorem 5.1.** Suppose \( \omega \) is a \( C^{\alpha,\beta} \) conical Kähler metric defined over \( C^n \). Suppose
\[
\omega^n = \omega_\beta^n, \quad \frac{\omega_\beta}{K} \leq \omega \leq K\omega_\beta \text{ over } C^n.
\]

Suppose for some linear transformation \( L, L^*\omega_\beta \) is one of the tangent cones of \( \omega \). Then \( \omega = L^*\omega_\beta \).

23
Proof. of Theorem 5.1 Without loss of generality we assume $L = id$. Consider scaling $B(R)$ to $B(1)$ as

$$\hat{z} \to R^{-\frac{1}{2}} \hat{z} = z; \hat{u_i} \to R^{-1} u_i; \hat{\phi} \to R^{-2} \hat{\phi} = \phi.$$ (46)

Suppose the tangent cone along the sequence $R_i$ is $\omega^\beta$, which means $\omega_i = R_i^{-2} \omega \to \omega^\beta$ over $B(\lambda)$ for all $\lambda > 0$. Take $\lambda = 1$, we have from the proof of Proposition 2.5 in [12] that

$$\lim_{i} |\omega_i - \omega^\beta|_{L^2(B_0(1))} = 0.$$ 

By the Moser’s iteration trick in the proof of Proposition 26 in [13], since $\omega_i$ is also Ricci flat and quasi-isometric to $\omega^\beta$ in the scaled down coordinates, we have

$$\lim_{i} |\omega_i - \omega^\beta|_{L^\infty(B_0(\frac{1}{2}))} = 0.$$ 

Thus, when $i$ is large enough, $\omega_i$ satisfies the assumptions in Proposition 5.2 over $B(\frac{1}{2})$. Then we obtain when $i$ is large that

$$[\omega_i]_{\alpha, \beta, B(\frac{1}{2})} \leq C.$$ 

Rescale back, we get

$$[\omega]_{\alpha, \beta, B(\frac{1}{2})} \leq CR_i^{-\alpha}.$$ 

Let $i \to \infty$, we get

$$[\omega]_{\alpha, \beta, C^n} = 0.$$ 

Then $\omega = \omega^\beta$ over $C^n$. The proof is complete. 

Proof. of Proposition 5.2. Suppose $\omega$ is a $C^{\alpha, \beta}$ conical Kähler metric defined over $B_0(1)$. Suppose there is a small enough $\delta$ such that

$$\omega^n = \omega^\beta, \quad \frac{\omega^\beta}{1 + \delta} \leq \omega \leq (1 + \delta) \omega^\beta$$ over $B_0(1)$. (47)

Then the following estimate holds in $B(\frac{1}{4})$.

$$[\omega]_{\alpha, \beta, B(\frac{1}{4})} \leq C.$$ 

Proof. of Proposition 5.2. By the solution to the Poincare-Lelong equation, we obtain a potential $\phi$ such that

$$i \partial \bar{\partial} \phi = \omega, \quad |\phi|_{0, B(\frac{1}{4})} \leq C.$$ (48)

Under the singular coordinates and the basis $a, du_1, ... du_{n-1}$ for $T^{1,0}$, we consider $i \partial \bar{\partial} \phi$ under these basis, as in page 11 of [17].
Then note that, by letting $F(M) = \det M - tr M$, we consider
\[ |F(i\partial\bar{\partial}\phi(x)) - F(i\partial\bar{\partial}\phi(y))|. \]

Since $(1 - \delta)I \leq i\partial\bar{\partial}\phi \leq (1 + \delta)I$, we obtain
\[ |F(i\partial\bar{\partial}\phi(x)) - F(i\partial\bar{\partial}\phi(y))| \leq \epsilon|i\partial\bar{\partial}\phi(x) - i\partial\bar{\partial}\phi(y)|, \quad (49) \]
for some $\epsilon(\delta)$ such that $\lim_{\delta \to 0} \epsilon(\delta) = 0$. Hence
\[ [\det(i\partial\bar{\partial}\phi) - \Delta\phi]_{\alpha,B(1)}^{(\epsilon)} \leq \epsilon[i\partial\bar{\partial}\phi]_{\alpha,B(1)}^{(\epsilon)}. \quad (50) \]

Since $\det(i\partial\bar{\partial}\phi) = 1$ in polar coordinates, we deduce
\[ [\Delta\phi]_{\alpha,B(1)}^{(\epsilon)} \leq \epsilon[i\partial\bar{\partial}\phi]_{\alpha,B(1)}^{(\epsilon)}. \quad (51) \]

Combining (51) and the usual conic Schauder estimate
\[ [i\partial\bar{\partial}\phi]_{\alpha,\beta,B(1)}^{(\epsilon)} \leq C\{[\Delta\phi]_{\alpha,\beta,B(1)}^{(\epsilon)} + |\phi|_{0,B(1)}\}, \quad (52) \]
we end up with
\[ [i\partial\bar{\partial}\phi]_{\alpha,\beta,B(1)}^{(\epsilon)} \leq C\epsilon[i\partial\bar{\partial}\phi]_{\alpha,\beta,B(1)}^{(\epsilon)} + C|\phi|_{0,B(1)} \leq C. \quad (53) \]

Let $\delta$ be small enough such that $C\epsilon < \frac{1}{2}$, we deduce
\[ [i\partial\bar{\partial}\phi]_{\alpha,\beta,B(1)}^{(\epsilon)} \leq C|\phi|_{0,B(1)} \leq C. \quad (54) \]

The proof is complete. $\square$

6 Bounded weakly-subharmonic functions and weak maximum principle.

In this section, we work in the polar coordinates (the balls, domains are all with respect to the polar coordinates). We mainly show the Dirichlet boundary problem is solvable, in the sense of Theorem 7.4 and 7.3. These are important in the last part of the proof of Theorem 1.14 in section 9.

Following [13], the following Lemma is true on bounded weakly-harmonic and weakly-subharmonic functions.

**Lemma 6.1.** Suppose $u \in C^2(B(1) \setminus D) \cap L^\infty(B(1))$. Then

1. Suppose $\Delta_\omega u \geq 0$ over $B(1) \setminus D$, then $u$ is a weak subsolution to $\Delta_\omega u \geq 0$ in $B(1)$;

2. Suppose $\Delta_\omega u = 0$ over $B(1) \setminus D$, then $u$ is a weak solution to $\Delta_\omega u = 0$ in $B(1)$, and $u \in C^{\alpha,\beta}$ for some $\alpha > 0$. 

25
**Proof.** This is proved by cutting off. Let \( \eta_\epsilon = \Psi(\frac{1}{3} - d_\epsilon)\Psi(\frac{r}{\epsilon} - 1) \), where \( \Psi \) is the Lipshitz cutoff function

\[
\Psi(s) = \begin{cases} 
0, & s \leq 0; \\
6s, & 0 \leq s \leq \frac{1}{6}; \\
1, & \frac{1}{6} \leq s \leq 1.
\end{cases}
\]

Notice that \( \Psi'(s) \leq 6 \) almost everywhere, then since \( \omega \) is quasi isometric to the Euclidean metric in the polar coordinates, we have when \( \epsilon \leq \frac{1}{100} \) that

\[
|\nabla_\omega \eta_\epsilon| \leq \frac{C}{\epsilon}.
\]

(55)

\( \eta_\epsilon \) not only cutoff the boundary of \( B(1) \), but also cutoff the divisor \( D \). Since \( u \) is smooth away from divisor, we multiply both handsides of the harmonic equation (in item 2) by \( \eta_\epsilon^2 u \) and integrate by parts to get

\[
- \int \eta_\epsilon^2 |\nabla_\omega u|^2 - 2 \int \eta_\epsilon u < \nabla_\omega u, \nabla_\omega \eta_\epsilon >= 0.
\]

(56)

Then by Cauchy-Schawartz inequality, we get

\[
\frac{1}{2} \int \eta_\epsilon^2 |\nabla_\omega u|^2 \leq C \int |\nabla_\omega \eta_\epsilon|^2 u^2.
\]

(57)

Thus, by the condition \( |u|_{L^\infty} < \infty \), the bound (55), and the definition of \( \eta_\epsilon \), we obtain

\[
\int |\nabla_\omega \eta_\epsilon|^2 u^2 \leq C|u|_{L^\infty}^2 \int \frac{2}{\epsilon^2} \frac{1}{\epsilon^2} r dr \leq C|u|_{L^\infty}^2.
\]

(58)

Hence (57) and (58) imply

\[
\int \eta_\epsilon^2 |\nabla_\omega u|^2 \leq C,
\]

(59)

where \( C \) is independent of \( \epsilon \)!

Therefore let \( \epsilon \to 0 \), we get

\[
\int_{B(1/2) \setminus D} |\nabla_\omega u|^2 \leq C.
\]

(60)

By Lemma 2.5 of [47], \( u \) is a weak solution to (67). By Theorem 8.22 of [23] or [31], we deduce \( u \in C^{\alpha, \beta} \).

The statement on subharmonicity is proved in the same way, by considering \( u - \inf u_{B(1)} \), which is nonnegative.

Recall the classical *weak maximum principle* for the subharmonic function on Euclidean space. Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded open subset, if \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfies \( \Delta u \geq 0 \), then \( \sup_{\Omega} u = \sup_{\partial\Omega} u \).

Let \( \Omega \subset \mathbb{C}^n \) be a connected bounded open subset which intersects \( D \), let \( \omega \) be a weak conical Kähler metric i.e a smooth Kähler metric on \( \Omega \setminus D \) and satisfies \( C^{-1} \omega_\beta \leq \omega \leq C \omega_\beta \).

26
Theorem 6.2. (Weak Maximum Principle)
Let $u$ be a $D$-subharmonic function on $\Omega$ (in the sense of Definition 7.1), then
\[ \sup_{\Omega} u = \sup_{\partial \Omega} u. \]

Remark 6.3. The difference of our weak maximal principle from Jeffres’ trick in [27] is that our weak maximal principle applies to $L^\infty$-functions, while Jeffres’ trick requires the function to have some Hölder continuity property near $D$.

Proof. of Theorem 6.2
Notice that the auxiliary function $\log |z|$ is pluri-harmonic in $\mathbb{C}^n \setminus D$ under any Kähler metric, therefore $u_\epsilon = u + \epsilon \log |z|$ is also weak-subharmonic away from $D$ (smaller than harmonic lift on any ball with no intersection with $D$). However since $u$ is bounded, $u_\epsilon(p)$ goes to $-\infty$ as $p$ approaches $D \cap \Omega$. We show here that $u + \epsilon \log |z|$ can’t attain interior maximum.

If not, there exists $q \notin D$ such that
\[ u_\epsilon(q) = \sup_{\Omega} u_\epsilon. \]
Choose a ball $B_q$ with no intersection with $D$ and there exists some point $b \in \partial B_q$ such that
\[ u_\epsilon(b) < \sup_{\Omega} u_\epsilon. \tag{61} \]

Then we consider the harmonic lifting of $u_\epsilon$ over $B_q$ as $\bar{u}_\epsilon$. By definition, we have $\bar{u}_\epsilon \geq u_\epsilon$. By maximal principle on $B_q$, we deduce $\sup \bar{u}_\epsilon \leq \sup u_\epsilon |_{\partial B_q}$. Then we see that $\bar{u}_\epsilon$ attains interior maximum in $B_q$ at $q$. This means the harmonic function $\bar{u}_\epsilon$ is a constant over the whole $B_q$, which contradicts (61).

Thus $u_\epsilon$ attain maximum on $\partial \Omega \setminus D$. We compute for any $p \notin D$ that
\[ u(p) = u_\epsilon(p) - \epsilon \log |z|(p) \leq \sup_{\partial \Omega \setminus D} u_\epsilon - \epsilon \log |z|(p) \leq \sup_{\partial \Omega \setminus D} u + (\sup_{\partial \Omega \setminus D} \epsilon \log |z|) - \epsilon \log |z|(p) \leq \sup_{\partial \Omega \setminus D} \varphi + \epsilon C - \epsilon \log |z|(p). \]

Let $\epsilon \to 0$ we obtain
\[ u(p) \leq \sup_{\partial \Omega \setminus D} \varphi. \]
Since $u \in C^0(\bar{\Omega} \setminus D)$, the proof is completed. \qed
7 Dirichlet problem of conical elliptic equations.

In this section we work in the polar coordinates.

**Definition 7.1.** We say \( v \in C^0(\overline{\Omega} \setminus D) \cap L^\infty(\Omega) \) to be a \( D \)-subharmonic function if for any ball \( B \in \Omega \) and \( B \cap D = \emptyset \), the harmonic lifting \( \bar{v} \) satisfies \( \bar{v} \geq v \) in \( B \).

**Remark 7.2.** In this section we don’t require the target function to be in \( C^2(\overline{\Omega} \setminus D) \), this is because we want the set of all \( D \)-subharmonic functions under the boundary condition to be closed under harmonic lifting away from \( D \). This shows upper envelope is harmonic away from \( D \) and is in \( L^\infty(\overline{B}) \), then Lemma 6.1 can be applied. These are crucial in the proof of Theorem 1.14 in section 9.

**Theorem 7.3.** Suppose \( B \) is a ball. Let \( \varphi \in C^0(\partial B \setminus D) \cap L^\infty(\partial B) \) be a function defined on \( \partial B \setminus D \). Then, there exists a \( \omega \)-harmonic function \( u \) defined on \( B \) such that \( u \) attain the boundary value \( \varphi \) continuously away from \( D \). i.e

\[
\Delta_{\omega} u = 0 \quad \text{over } B \setminus D,
\]

and for any \( \xi \in \partial B \setminus D \), we have

\[
\lim_{x \to \xi} |u(x) - \varphi(\xi)| = 0.
\]

**Theorem 7.4.** Suppose \( B \) is a ball. Let \( \varphi \in C^{2,\alpha}(\partial B \setminus D) \cap L^\infty(\partial B) \) be a function defined on \( \partial B \setminus D \). Then, there exists a \( \omega \)-harmonic function \( u \) defined on \( B \) such that \( u \) attain the boundary value \( \varphi \) in Lipshitz sense away from \( D \). i.e

\[
\Delta_{\omega} u = 0 \quad \text{over } B \setminus D,
\]

and for any \( \xi \in \partial \Omega \setminus D \), there exists a postive constant \( r_\xi \) and \( K(\xi) \) such that

\[
|u(x) - \varphi(\xi)| \leq K(\xi)|x - \xi|
\]

for all \( x \in B_\xi(r_\xi) \cap B \). Moreover, \( u \in C^{2,\alpha}((\overline{B} \setminus D)) \).

**Proof.** of Theorem 7.3 and 7.4. Given \( \varphi \in L^\infty(\overline{B}) \cap C^0(\overline{B} \setminus D) \), we define the value of \( \varphi \) at \( p \in D \cap \partial B \) as

\[
\varphi(p) = \lim_{x \to p, \ p \notin D} \inf \varphi(x).
\]

Define

\[
S_\varphi = \{ u | u \text{ is } D\text{-subharmonic and } u \leq \varphi \text{ over } \partial B \setminus D \},
\]

and the upper-envelope as

\[
u(x) = \sup_{u \in S_\varphi} \{ u(x) | x \in B \setminus D \}.
\]

We now prove the claim
Claim 7.5. $\Delta \omega u = 0$ over $B \setminus D$.

This goes exactly as in [23], except the harmonicity holds only over $B \setminus D$ and the harmonic-lifting are performed away from $B \setminus D$. For the reader’s convenience we include the crucial detail here. Suppose $p \notin D$ and $\lim_{k \to \infty} v_k(p) = u(p)$, we choose $B_p(R)$ with no intersection with $D$. We consider the harmonic lifting of $v_k$ in $B_p(R)$ as $\bar{v}_k$. Then

$$\lim_{k \to \infty} \bar{v}_k = \bar{v} \text{ over } B_p \left( \frac{R}{2} \right), \quad u(p) = \bar{v}(p).$$

(62)

It suffices to show $\bar{v} \equiv u$ over $B_p \left( \frac{R}{2} \right)$. If not, there exists a $q \in B_p \left( \frac{R}{4} \right)$ such that $\bar{v}(q) \neq u(q)$. Then there exists a $\hat{u} \in S_\varphi$ such that

$$v(q) < \hat{u}(q) \leq u(q).$$

(63)

Now we refine the sequence $v_k$ by considering $\max(v_k, \hat{u})$ and denote their harmonic lifting over $B(R)$ as $w_k$. Then we have

$$v_k \leq w_k \leq u \text{ in } B_p \left( \frac{3R}{5} \right).$$

Then let $k \to \infty$, $w_k \to w_\infty$ over $B_p \left( \frac{R}{2} \right)$. Then by maximal principle we have:

$$\bar{v}_\infty \leq w_\infty \leq u \text{ in } B_p \left( \frac{R}{2} \right).$$

(64)

Both $\bar{v}_\infty$ and $w_\infty$ are harmonic. We have

$$\bar{v}_\infty(q) < \hat{u}(q), \text{ but } \bar{v}_\infty(p) = \hat{u}(p),$$

This is a contradiction since by strong maximal principle over $B_p(R)$ which does not intersect $D$, we have $\bar{v}_\infty \equiv w_\infty$ over $B_R$. The proof of Claim 7.5 is complete.

On the attainability of the boundary value, since the domain we consider is a ball, which is convex, we choose the barriers at those $p \in \partial B \setminus D$ exactly as in formula (6.45) of [23], with $\tau = 1$ and $R$ small enough such that $B_p(10R) \cap D = \emptyset$. Then boundary value $\varphi$ is then attained continuously away from $D$. Thus the proof of Theorem 7.3 is complete.

Suppose $\varphi \in C^{2,\alpha}(\partial B \setminus D) \cap L^\infty(\partial B)$. The boundary value $\varphi$ is attainable in Lipshitz-sense together with the fact that $u \in C^{2,\alpha}(\bar{B} \setminus D)$ are trivially implied by the proof of Theorem 6.14 in [23]. The proof Theorem 7.4 is complete.

8 Strong Maximum Principles and Trudinger’s estimate.

In this section we prove the strong maximum principle, which is crucial in the proof of Theorem 1.14 in section 9.
Let us first recall the classical strong maximum principle in Euclidean space, which states the subharmonic function which takes interior supremum would be a constant function. The following main theorem of this section is a generalization of the classical strong maximum principle.

**Theorem 8.1.** (Strong Maximum Principle) Let $u \in C^2(\Omega \setminus D) \cap L^\infty(\bar{\Omega})$ be a bounded real value function on $\Omega$ which satisfies $\Delta u \geq 0$ on $\Omega \setminus D$, suppose there exists a smaller subdomain $\Omega' \subset \Omega$ such that $\sup_{\Omega'} u = \sup_{\Omega} u$, then $u$ must be a constant.

**Proof.** of Theorem 8.1

We have two proofs for this fact. One is a barrier construction and the other one is also straightforward by Trudinger’s estimate. Since both proofs have their own interest, we include both of them here.

Proof 1: Barrier construction.

By localizing the supremum of $u$ in the interior $\Omega \cap D$, there is no loss of generality in assuming that we are in a situation where $\Omega = B_2 = \{ (z, z_2, \cdots, z_n) | |z|^2 + |z_2|^2 + \cdots + |z_n|^2 \leq 4 \}$, and $\sup_{\Omega} u = 1 = \lim_{i \to \infty} u(p_i)$ for a sequence of points $p_i \to 0$.

We prove by contradiction. Suppose $u$ is not constant function 1, by the classical strong maximum principle in the smooth case, we know $u < 1$ in $\{ \frac{1}{2} \leq |z| \leq 1 \}$. Then we suppose $u \leq \tau_0 < 1$ in the ring-shaped piece $R_\delta = \{ |z|^2 + |z_2|^2 + \cdots + |z_n|^2 = 1, |z_2|^2 + \cdots + |z_n|^2 \leq \delta^2 \} \subset \partial B_1$ for some definite number $\delta > 0$ (this could be done by continuity of $u$), and $\sup_{\partial B_1} u = 1$.

Let $\psi = \frac{\delta^2}{2}(1 - \delta^2)^{-\beta} |z|^\beta - (|z_2|^2 + \cdots + |z_n|^2)$, then:

- On $R_\delta$, $\psi \leq \frac{\delta^2}{2}(1 - \delta^2)^{-\beta}$;
- On $(\partial B_1) \setminus R_\delta$, $\psi \leq -\frac{\delta^2}{2} < 0$;
- On $B_1 \setminus D$, $\Delta_\omega \psi \geq -(n - 1) C$. Since $\psi$ is bounded function, so it is also a global weak subharmonic function by virtue of Lemma 1.1;
- On $B_1 \setminus D$, $|\nabla \psi|^2 \geq C^{-1} |\nabla \psi|^{2(\beta)} \geq C^{-1} \beta^{-2} |z|^2 - 2\beta |\nabla \psi|^{2} \cdot \delta^4 (1 - \delta^2)^{-\beta} = C^{-1} \delta^4 (1 - \delta^2)^{-\beta}$.

It follows that

$$\Delta_\omega e^{a \psi} = (a^2 |\nabla \psi|^2 + a \Delta_\omega \psi) e^{a \psi} \geq (a^2 C^{-1} \delta^4 (1 - \delta^2)^{-\beta} - a(n - 1) C) e^{a \psi} \geq 0,$$

for $a \geq 16\delta^{-4}(1 - \delta^2)\beta (n - 1)$.

Therefore, for the “bumped function” $u_\epsilon = u + \epsilon (e^{a \psi} - 1)$ is a bounded subharmonic function on $B_1 \setminus D$ since $\Delta_\omega u_\epsilon = \Delta_\omega u + \epsilon \Delta_\omega e^{a \psi} \geq 0$.

On the other hand, $\lim_{i \to \infty} u_\epsilon(p_i) = \lim_{i \to \infty} \{ u(p_i) + \epsilon (e^{a \psi(p_i)} - 1) \} = 1$, while on the boundary:
• \( \sup_{\partial B_1 \setminus R_0} u_\epsilon \leq \sup_{\partial B_1} \{ u + \epsilon (e^{-\delta^2/2} - 1) \} \leq 1 - \epsilon(1 - e^{-\delta^2/2}); \)

• \( \sup_{R_3} u_\epsilon \leq \sup_{R_3} u + \epsilon(e^{a/2\delta^2(1-\delta^2)} - \frac{\delta^2}{2}) - 1 \leq \tau_0 + \epsilon(e^{a/2\delta^2(1-\delta^2)} - \frac{\delta^2}{2} - 1). \)

Thus, since \( \tau_0 < 0 \), by taking \( \epsilon > 0 \) small enough, we can make \( \sup_{\partial B_1} u_\epsilon < 1 \leq \sup_{B_1} u_\epsilon \), which contradicts the weak maximal principle in Theorem 6.2.

Proof 2: Trudinger’s Harnack inequality. Without loss of generality, we can still assume \( u \) attains interior maximum at 0. Suppose \( u \) is not a constant, then there exists a ball \( B_0(r_0) \) such that \( B_0(2r_0) \subset \Omega \), and \( u \neq u(0) \) at some point in \( \partial B_0(r_0) \). Using Theorem 7.3 we can find a solution \( v \in C^0(\Omega \setminus D) \cap L^\infty(\Omega) \) to the following equation

\[
\Delta_\omega v = 0 \text{ in } B_0(r_0), \quad v|_{\partial \Omega} = u|_{\partial \Omega}. \tag{65}
\]

By the weak maximal principle in Theorem 6.2 we have

\[
v(p) \leq \sup_{\partial B_0(r_0)} v = \sup_{B_0(r_0)} u \leq u(0) \text{ and } v(p) \geq u(p), \text{ for all } p \in B_0(r_0). \tag{66}
\]

This means \( v \) also attains interior maximum at 0. Using the Trudinger’s maximal principle in Proposition 5.2 (actually we only need the Harnack inequality to be true for some \( \rho_0 > 0 \) for the proof the strong maximal principle), \( v \equiv v(0) = u(0) \) is a constant. This contradicts the hypothesis that \( v(q) = u(q) \neq u(0) \) at some point \( q \in \partial B_0(r_0) \).

We work in the polar coordinates to reformulate the De-Giorgi estimate in Theorem 8.18 of [23] in the following proposition.

**Proposition 8.2.** (Trudinger’s strong maximal principle) Suppose \( \omega \) is a weak-conical metric over \( B(1) \). Suppose \( u \in L^\infty(B(1)) \cap C^2(B(1 \setminus D) \) is a \( \omega \)-harmonic function i.e

\[
\Delta_\omega u = 0 \text{ in } B(1). \tag{67}
\]

Suppose there exists a ball \( B_p(r_0) \subset B(1) \), such that \( u(p) = \sup_{B_p(r_0)} u \) or \( u(p) = \inf_{B_p(r_0)} u \). Then \( u \) is a constant over \( B(1) \).

**Proof.** of Proposition 8.2 This is a directly corollary of Lemma 13.1. We just prove the case when \( u(p) = \sup_{B_p(r_0)} u \). Since \( u \in L^\infty(B(1 \setminus D) \), then by Lemma 13.1 \( u \) is a weak solution to (67). Let \( v = u(p) - u = (\sup_{B_p(r_0)} u) - u \), then \( v \geq 0 \) in \( B_p(r_0) \). Then using Lemma 13.1 for some \( \rho > 0 \) (this is all we need, though Lemma 13.1 says more than this), we have

\[
|v|_{L^q(B_p(\frac{\rho}{2\rho}))} \leq C \rho^{2n} \inf_{B_p(\frac{\rho}{2\rho})} v. \tag{68}
\]
Using \( v(p) = 0 \) and the \( C^\alpha \)-continuity of \( v \) from Lemma 6.1, we get \( \inf_{B_p(\frac{r_0}{10})} v = 0 \). Hence (68) implies 
\[ |v|_{L^q, B_p(\frac{r_0}{10})} = 0, \]
which means \( v = 0 \) in \( B_p(\frac{r_0}{10}) \). This implies \( v \equiv 0 \) over \( B(1) \), which means \( u \) is a constant. \( \square \)

9 Proof of Theorem 1.14.

Consider \( tr_{\omega \beta} \omega \). Given the \( \omega \) and \( \phi \) as in Theorem 1.14, we define the 3rd derivative as
\[ S = \omega^{ij} \omega^{\bar{s}t} \phi_{z_i \bar{z}_j, z_s \bar{z}_t, \bar{z}_q}, \]
as in [50] and [6]. The derivatives concerned are all covariant derivatives with respect to \( \omega_{\beta} \). Nevertheless, since the connection of \( \omega_{\beta} \) is holomorphic, we have
\[ \phi_{z_i \bar{z}_t} = \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_t} = \omega_{z_i \bar{z}_t}. \]
Thus \( S \) is actually defined over the whole \( C^n \), without assuming the existence of a global potential \( \phi \). By equation (2.7) in [50], we have
\[ \Delta_\omega tr_{\omega \beta} \omega \geq \frac{S}{K} \geq 0, S \text{ as in (69).} \]
Without loss of generality, we may assume that
\[ \frac{1}{C_0} \leq tr_{\omega \beta} \omega \leq C_0 \]
and
\[ \sup_{C \times C^{n-1}} tr_{\omega \beta} \omega = C_0. \]
Since \( tr_{\omega \beta} \omega \) is subharmoic, if this sup is achieved in some finite ball, then the strong maximal principle (Theorem 8.1) implies that
\[ tr_{\omega \beta} \omega = \text{constant}. \]
Going back to (36), we see that \( \omega \) is covariant constant with respect to \( \omega_{\beta} \). This easily implies that \( \omega \) is isometric to \( \omega_{\beta} \) by a complex linear transformation.

Unfortunately, a bounded function will usually not achieve maximum at an interior point. Suppose
\[ \sup_{C \times C^{n-1}} tr_{\omega \beta} \omega = C_0. \]
Suppose there exists a sequence of points \( p_i \) such that
\[ tr_{\omega \beta} \omega(p_i) \rightarrow C_0, \text{ dist}(p_i, 0) \rightarrow \infty, \quad \text{as } i \rightarrow \infty. \]
Consider the rescaled sequence \((C \times C^{n-1}, 0, \omega_i = R_i^{-2} \omega)\). It converges locally smoothly to \((C \times C^{n-1}, o, \omega_\infty)\). Denote
\[
v_i = tr_{\omega, \beta}\omega_i \in \left[\frac{1}{C_0}, C_0\right] \quad \text{in } C \times C^{n-1}.
\]
Then
\[
v_i(p_i) \to C_0, \quad \text{as } i \to \infty.
\]
It is easy to see that \((p_i, v_i)\) converges to \((p_\infty, v_\infty)\) locally smooth away from divisor such that
\[
v_\infty = tr_{\omega, \beta}\omega_\infty \in \left[\frac{1}{C_0}, C_0\right] \quad \text{in } C \times C^{n-1},
\]
and
\[
\Delta \omega_\infty v_\infty = S_\infty \geq 0, \quad \text{dist}_\beta(p_\infty, 0) = 1. \quad (71)
\]
If \(|v|_{\alpha, \beta} \leq C\) before taking limit, then \(v_\infty\) achieves interior maximum at \(p_\infty\), from Theorem 8.1 we obtain \(v_\infty\) is a constant. By (71) we deduce \(S_\infty = 0\). Therefore,
\[
\omega_\infty = L^* \omega_\beta,
\]
for some linear transformation \(L\). Then, by Theorem 5.1 we know that \(\omega = L^* \omega_\beta\).

So the difficulty is to show \(v_\infty\) is a constant even it might not be continuous apriorily. Fortunately, \(v_\infty\) is apprximated by the sequence \(v_i\). It is here we apply harmonic lifting before letting \(i \to \infty\).

In the singular polar coordinates, we consider the ball centered at 0 and with radius 2. By Theorem 7.4 we can find a \(\omega_i\)-harmonic function \(h_i\) such that
\[
\Delta \omega_i h_i = 0, \quad \text{in } B_2(o)
\]
and
\[
h_i = v_i \quad \text{at } \partial B_2(o).
\]
Using weak maximal principle, we have \(h_i \geq v_i \geq 0\). Moreover, since \(v_i\) is bounded above by \(C_0\) in the boundary, It follows by maximum principle again that \(h_i \leq C_0\) in \(B_2\). Thus \(0 \leq h_i \leq C_0\). It follows that
\[
h_i(p_i) \to C_0, \quad \text{as } i \to \infty.
\]
By Lemma 6.1 we know that \(h_i\) is uniformly \(C^{\alpha, \beta}\) in the interior and continuous up to all smooth points on \(\partial B_2(o) \setminus D\).

Now we take limit as \(i \to \infty\), and denote the limit of \(h_i\) as \(h_\infty\). The convergence is locally smooth away from divisor, uniformly \(C^{\alpha, \beta}\) across the divisor. Thus, we have
\[
\Delta \omega_\infty h_\infty = 0, \quad \text{in } B_2(o, \omega_\beta)
\]
and 
\[ h_{\infty}(p_{\infty}) = C_0. \]
Applying strong maximal principle theorem (Theorem 3.3), we have
\[ h_{\infty} \equiv C_0 \text{ in } B_2(a). \]
Moreover, \( v_i \to v_{\infty} \) smoothly away from \( D \) and consequently \( h_{\infty}|_{\partial B_2} = v_{\infty}|_{\partial B_2} \).
It follows that, on \( \partial B_2(a) \setminus D \), we have
\[ v_{\infty} = h_{\infty} \equiv C_0. \]
Then \( v_{\infty} \) attains maximum over \( \partial B_2(a) \setminus D \) ! Using the subharmoncity in (71) and strong maximal principle again, we deduce \( v_{\infty} \) is a constant and consequently
\[ S_{\infty} \equiv 0. \]
Hence \( \omega_{\infty} = L^*\omega_{\beta} \). Since \( \omega_{\infty} \) is a tangent cone of \( \omega \), using Theorem 5.1, we conclude \( \omega = L^*\omega_{\beta} \).
The proof of Theorem 1.14 is complete.

10 Bootstrapping of the conical Kähler-Ricci flow.

In this section we show the bootstrapping of conical Kähler-Ricci flow is true. This is important when we show the convergence of the rescaled sequence in the proof of Theorem 1.1.

**Theorem 10.1.** Suppose \( \alpha > 0 \) and \( \phi \) is a \( C^{2+\alpha,1+\frac{2}{\alpha},\beta} \) solution to the conical Kähler-Ricci flow over \([0,t_0]\), then \( \phi \in C^{2+\tilde{\alpha},1+\frac{2}{\tilde{\alpha}},\beta} \) for all \( \tilde{\alpha} < \min\{\frac{1}{\beta} - 1,1\} \) when \( t > 0 \). Moreover there exists a constant \( C(|\phi|_{2+\alpha,1+\frac{2}{\alpha},\beta,M \times [0,t_0]}) \) (depending on \( |\phi|_{2+\alpha,1+\frac{2}{\alpha},\beta}, \alpha, g_0 \), and the data in Definition 2.1) such that
\[ |\phi|_{2+\alpha,1+\frac{2}{\alpha},\beta,M \times [0,t_0]} \leq C(|\phi|_{2+\alpha,1+\frac{2}{\alpha},\beta,M \times [0,t_0]}). \]

**Proof.** of Theorem 10.1 Temporarily we denote \( |\phi|_{2+\alpha,1+\frac{2}{\alpha},\beta,M \times [0,t_0]} = k \). Let \( u_i \) be a tangential variable near \( D \). Differentiating the CKRF (2) with respect to \( u_i \) we get
\[ \frac{\partial \phi_{u_i}}{\partial t} = \Delta_{\phi} \phi_{u_i} + \beta \phi_{u_i} + \tilde{h} \text{ over } B_0(r_0), \quad (72) \]
where \( \tilde{h} \) is a \( C^{\alpha,\frac{2}{\alpha},\beta} \) function and \( r_0 \) is sufficient small such that a coordinate exists in \( B_0(r_0) \). Then exactly as in the proof of Theorem 1.13 in [48], by applying the interior parabolic Schauder estimate in the equation (21) in [17], we obtain
\[ |\phi_{u_i}|_{2+\alpha,1+\frac{2}{\alpha},\beta,B_0(r_0) \times [0,t_0]} \leq C \times (1 + |\phi_{u_i}|_{0,B_0(r_0) \times [0,t_0]}) = C(k). \quad (73) \]
First we bound the spatial $C^{2,\alpha,\beta}$ norm when $t > 0$. Using the interpolation inequalities in Lemma 11.3 in [17] for $\phi_{ui}$, we end up with

$$|\phi_{ui}|_{1,\alpha',\beta, B_0(r_0) \times [0,t_0]} \leq C$$

for any $\alpha < \min\{\frac{1}{\beta} - 1, 1\}$. (74)

Hence the mixed derivatives and tangential second order derivatives satisfy

$$|\phi_{\bar{u}i}|_{\alpha',\beta, B_0 \times \{t\}} \leq C$$

(75)

$$|\phi_{ui}\bar{u}|_{\alpha',\beta, B_0 \times \{\tau\}} \leq C$$

(76)

Similarly we have

$$|\phi_{\bar{a}i}|_{\alpha',\beta, B_0 \times \{\tau\}} \leq C$$

(77)

Thus to prove the bootstrapping estimate for $\partial \bar{\partial} \phi$, it suffices to prove it for $\phi_{\bar{a}a}$. The key thing is that the CKRF equation (2) directly implies the bound for $\phi_{\bar{a}a}$. Without loss of generality we assume $n = 2$. Then the CKRF equation reads as

$$(\omega_{D,a\bar{a}} + \phi_{a\bar{a}})(\omega_{D,a\bar{a}} + \phi_{u\bar{u}}) - (\omega_{D,a\bar{a}} + \phi_{a\bar{a}})(\omega_{D,u\bar{u}} + \phi_{u\bar{u}}) = e^{h - \beta \phi + \frac{\partial}{\partial t} \omega_{D,u\bar{u}} + \phi_{u\bar{u}} - \omega_{D,a\bar{a}}}.$$ (78)

By Theorem 1.13 in [48] and interpolation, we deduce

$$|\partial \bar{\partial} \phi|_{\alpha',\beta, M \times [0,t_0]} \leq C.$$ (79)

Then by (75), (76), (77), (78), and (79), we conclude

$$|\phi_{a\bar{a}}|_{\alpha',\beta, B_0 \times \{t\}} \leq C \frac{1}{t^{1+\frac{\alpha}{2}}}.$$ (80)

The estimates for the time derivatives and timewise Hölder norms are similar. To be simple, using (73) and timewise interpolation, we can get similar estimate as follows

$$|\phi_{a\bar{a}}|_{0,\alpha',\beta, B_0 \times \{t\}} + |\phi_{\bar{a}i}|_{0,\alpha',\beta, B_0 \times \{t\}} + |\phi_{ui}\bar{u}|_{0,\alpha',\beta, B_0 \times \{t\}} \leq \frac{C}{t^{1+\frac{\alpha}{2}}}.$$ (81)
Thus, using (78) we can bound $|\phi_{\text{tilt}}|_{0,\frac{\alpha}{2},\beta,B_0(r_0)\times[t,t_0]}$ exactly as we get (80).

The proof is complete. Actually what we proved is with better weight than what’s stated in Theorem 10.1.

In particular, with respect to the bootstrapping of conical Kähler-Einstein metrics, we’ve recovered a result of Chen-Donaldson-Sun in [13].

**Theorem 10.2.** (Chen-Donaldson-Sun): Suppose $\phi$ is a conical Kähler-Einstein metric and $\phi \in C^{2,\alpha,\beta}$ for some $\alpha > 0$. Then $\phi \in C^{2+\delta,1+\frac{4}{\beta},\beta}$ for all $\delta < \min\{\frac{\beta}{\alpha} - 1, 1\}$ and

$$|\phi|_{2,\delta,\beta,M} \leq C(|\phi|_{2,\alpha,\beta,M})$$

11 Exponential convergence when $C_{1,\beta} < 0$ or $= 0$.

In this section, we prove Theorem 1.7 on the convergence of CKRF. We follow the proof of Cao [9] and employ some modifications which are necessary in the conical case at this point.

We point out a convention of notations in this section: The $C$’s in this section are all time independent constants, the other dependence of the $C$’s in this section is as Definition 2.1.

**Proof.** of Theorem 1.7 We only prove the case when $C_{1,\beta} = 0$, since the case when $C_{1,\beta} < 0$ is much much easier and doesn’t require any other machinery except maximal principle of the heat equation and Theorem 1.8 in [17].

By Theorem 1.13 in [48], we know $Ric$ and $\sqrt{-1} \partial \bar{\partial} \frac{\partial h}{\partial t}$ are $C^{\alpha,\beta}$ (1,1)-forms. Moreover, the scalar curvature $s_{\phi}$ and $\nabla_{\bar{A}}^{\phi}$ are all in $C^{\alpha,\beta}$. Then, using regularity of lower order items established in [17], the identities in the following proof are all well defined.

In the Calabi-Yau case, there is a smooth function $h_{\omega_0}$ such that

$$Ric_{\omega_0} = i\partial \bar{\partial} \{h_{\omega_0} - (1 - \beta) \log h\}, \quad Ric_{\omega_0} - 2\pi (1 - \beta)[D] = i\partial \bar{\partial} H_{\beta}, \quad (82)$$

where

$$H_{\beta} = h_{\omega_0} - (1 - \beta) \log |S|^2$$

and $h$ is the metric of the line bundle $L_D$. The potential equation of the Calabi-Yau CKRF reads as

$$(\omega_D + i\partial \bar{\partial} \phi)^n = e^{-h_{\omega_D} \frac{\partial h}{\partial t}} \omega_D^n. \quad (83)$$
Step 1. The most important thing is to obtain a time-independent bound for $\text{osc}\phi$. This is achieved similarly as in [9], the difference is that we apply the Poincaré inequality here, while in [9] the lower bound on the Green function is applied. Notice $\frac{\partial \phi}{\partial t}$ satisfies
\[ \frac{\partial}{\partial t} \frac{\partial \phi}{\partial t} = \Delta \frac{\partial \phi}{\partial t}. \] (84)

By maximal principle we obtain
\[ |\frac{\partial \phi}{\partial t}|_{0, [0, \infty)} \leq C. \] (85)

The from the Calabi-Yau CKRF equation we get
\[ (\omega_D + i\partial\bar{\partial}\phi)^n = e^{F(t)}\omega_D^n \text{ for } |F(t)|_{[0, \infty)} \leq C. \] (86)

Hence, by considering $(\omega_D + i\partial\bar{\partial}\phi)^n - \omega_D^n$, we compute
\[ i\partial\bar{\partial}\phi \wedge (\omega_\phi^{n-1} + \ldots + \omega_D^{n-1}) = [e^{F(t)} - 1]\omega_D^n. \] (87)

Now we take $\phi_0 = \phi - \phi\bar{\phi}$ so that the average of $\phi_0$ with respect to $\omega_D$ is 0. Then we multiply (87) by $\phi_0$ and integrate over $M$ we get
\[ -\int_M \partial\phi_0 \wedge \bar{\partial}\phi_0 \wedge (\omega_\phi^{n-1} + \ldots + \omega_D^{n-1}) = \int_M \phi_0 [e^{F(t)} - 1]\omega_D^n \leq C. \] (88)

Notice that every form in the parenthesis on the left hand side is positive, we obtain
\[ \int_M |\nabla_{\omega_D} \phi_0|^2 \omega_D^n = n \int_M \partial\phi_0 \wedge \bar{\partial}\phi_0 \wedge \omega_D^{n-1} \leq C \int_M |\phi_0| \omega_D^n. \] (89)

By the Poincaré inequality for $\omega_D$ (stated in Remark 4.4 in [18]), and the assumption $\frac{1}{\text{vol}(M)} \int_M \phi_0 \omega_D = 0$, we obtain
\[ \int_M \phi_0^2 \omega_D^n \leq C \int_M |\nabla_{\omega_D} \phi_0|^2 \omega_D^n \leq C \int_M |\phi_0| \omega_D^n \leq C + \frac{1}{100} \int_M \phi_0^2 \omega_D^n. \] (90)

Therefore we obtain
\[ \int_M \phi_0^2 \omega_D^n \leq C, \] (91)

which is the necessary $L^2$-bound in the Moser iteration scheme.

Let
\[ \phi_{0,+} = \max\{\phi_0, 0\}, \quad \phi_{0,-} = -\min\{\phi_0, 0\}. \]

Notice that both $\phi_{0,+}$ and $\phi_{0,-}$ are nonnegative. Lemma 7.6 of [23] and the existence of singular coordinate near $D$ immediately implies both $\phi_{0,+}$ and
\( \phi_{0,-} \) are Lipshitz functions with respect to \( \omega_D \). Thus for any \( p > 1 \), we can also multiply equation (87) by \( \phi_{0,+}^p \) and apply Lemma 7.6 of [23] to get

\[-p \int_M \phi_{0,+}^{p-1} \partial \phi_{0,+} \wedge \bar{\partial} \phi_{0,+} \wedge (\omega^n_{\phi} + \cdots + \omega_{\phi}^{n-1}) = \int_M \phi_{0,+}^p [e^{F(t)} - 1] \omega^n_D. \quad (92)\]

Thus we obtain

\[ \int_M |\nabla_{\omega_D} \phi_{0,+}|^2 \omega^n_D \leq \frac{C(p+1)^2}{4p} \int_M \phi_{0,+}^p \omega^n_D. \quad (93)\]

By the Sobolev constant bound (see Remark 4.4 in [18]) and (91), the Moser’s iteration as in [9] works and we obtain the time-independent bound on \( \phi_{0,+} \):

\[ |\phi_{0,+}|_{0,\infty} \leq C. \quad (94)\]

In the same way we get \( |\phi_{0,-}|_{0,\infty} \leq C \). Thus finally we completed step 1 by obtaining

\[ \text{osc } \phi \leq C. \quad (95)\]

Step 2. By the proof of Proposition 2.7, the equation (85), and (95), we obtain

\[ \frac{C}{\omega_D} \leq \omega_{\phi} \leq C \omega_D. \quad (96)\]

Therefore by the last part of the proof of Theorem 1.15 (on the norm dependence, section 3), and equation (83) (which does not concern any 0th order term of \( \phi \) on the right hand side), we obtain

\[ |i \bar{\partial} \partial \phi|_{\alpha, \beta, 0, \infty} \leq C. \quad (97)\]

Thus the \( C^{\alpha, \beta} \) norm of \( \omega_{\phi} \) is bounded independent of time and any sequence \( \omega_{\phi_1} \) at least subconverges to a limit \( \omega_{CY, \infty} \). Furthermore, by (84), Theorem 1.18 in [17], and (85), we obtain

\[ \left| \frac{\partial \phi}{\partial t} \right|_{2, \alpha, \beta, t, \infty} \leq C(t), \quad C(t) < \infty \text{ when } t > 0. \quad (98)\]

Step 3. In this step we prove the flow subconverges to a Ricci-Flat metric to show the existence of such a critical metric. This is achieved by the K-energy in the Calabi-Yau setting.

We define the Calabi-Yau K-energy \( M_{\omega_0, \beta} \) as

\[ M_{\omega_0, \beta} = \int_M \log \left( \frac{\omega_0^n}{e^{H \phi_0}} \right) \omega_0^\beta n!. \]
Routine computation shows that
\[
\frac{dM_{\omega, \beta}}{dt} = -\frac{1}{n!} \left\{ \int_M \frac{\partial \phi}{\partial t} s_\phi \omega^n - 2n\pi (1 - \beta) \int_D \frac{\partial \phi}{\partial t} \omega_{\phi}^{n-1} \right\} = -\frac{1}{(n!)} \int_{M \setminus D} s_\phi \frac{\partial \phi}{\partial t} \omega_{\phi}^n,
\] (99)

where \(s_\phi\) is the scalar curvature of \(\omega_\phi\).

Along the Calabi-Yau CKRF, we have
\[
\Delta \phi \frac{\partial \phi}{\partial t} = -s_\phi \text{ over } M \setminus D.
\] (100)

Then (99) and (100) tell us
\[
\frac{dM_{\omega, \beta}}{dt} = -\frac{1}{(n!)} \int_M |\nabla \frac{\partial \phi}{\partial t}|^2 \omega_{\phi}^n \leq 0.
\] (101)

By (96), (97), and (95), we see
\[
|M_{\omega_0, \beta}(\omega_\phi)| \leq C \text{ over } [0, \infty).
\] (102)

Since \(\frac{dM_{\omega_0, \beta}}{dt} \leq 0\), then there exists a sequence \(t_k \to \infty\) such that
\[
\left| \frac{dM_{\omega_0, \beta}}{dt} \right|_{t_k} \to 0.
\] (103)

(101) and (103) imply
\[
\int_M |\nabla \frac{\partial \phi}{\partial t}|^2 \omega_{\phi_{t_k}}^n \to 0.
\] (104)

By the discussion at the end of Step 2 and (104), \(\omega_{\phi_{t_k}}\) subconverges in \(C^{\alpha, \beta}\) topology to a Ricci flat metric \(\omega_{KE}\). At this point, we have already shown the existence of a Ricci-flat metric.

Step 4: In this step we show the flow converges to the unique \(\omega_{KE}\) (obtained in the previous step) and the convergence is exponential, in the sense of (117). This is also straightforward by using the Calabi-Yau K-energy. Denote
\[
v = \frac{\partial \phi}{\partial t} - \frac{1}{Vol(M)} \int_M \frac{\partial \phi}{\partial t} \omega_{\phi}^n/n!.
\]

Obviously we have
\[
\frac{\partial v}{\partial t} = \Delta \phi v + \frac{1}{Vol(M)} \int_M |\nabla v|^2 \omega_{\phi}^n/n!.
\] (105)
Thus $v$ has zero average with respect to $\omega^n_{\phi}$ and Poincare inequality can be applied. By (98), we have

$$\int_M |\nabla v|^2 \omega^n_{\phi} \leq C. \quad (106)$$

From (102) and (101) on the K-energy, for any $\epsilon > 0$, there is a $T_0$ large enough such that

$$\int_{T_0}^{\infty} \int_M |\nabla v|^2 \omega^n_{\phi} dt < \epsilon. \quad (107)$$

Then using parabolic Moser’s iteration and (107), by letting $\epsilon$ be small enough, we deduce

$$|v|_{[0,T_0+1,\infty)} \leq \frac{1}{2}. \quad (108)$$

Therefore, as in [9], consider

$$E = \frac{1}{2} \int_M v^2 \omega^n_{\phi}. \quad (109)$$

Routine computation shows that

$$\frac{\partial E}{\partial t} = -\int_M (1 + v)|\nabla v|^2 \omega^n_{\phi}. \quad (109)$$

Combining (108) and (109) and the Poincare inequality in Remark 4.4 of [48], we compute

$$\frac{\partial E}{\partial t} \leq -\frac{1}{2} \int_M |\nabla v|^2 \omega^n_{\phi} \leq -\frac{CP}{2} \int_M v^2 \omega^n_{\phi} = -CP, \text{ over } [T_0 + 1, \infty).$$

Thus we obtain the exponential decay of the Dirichlet energy $E$:

$$E \leq Ce^{-CPt}. \quad (110)$$

Hence

$$\int_{t-1}^{\infty} \int_M v^2 \omega^n_{\phi} dt = 2 \int_{t-1}^{\infty} Edt \leq Ce^{-CPt}. \quad (111)$$

Using (110), by integrating $\frac{\partial E}{\partial t} \leq -\frac{1}{2} \int_M |\nabla v|^2 \omega^n_{\phi}$ from $t$ to $\infty$ we also end up with a better decay estimate than (107):

$$\int_{t}^{\infty} \int_M |\nabla v|^2 \omega^n_{\phi} dt < Ce^{-CPt}. \quad (112)$$

Therefore using (112) and the Poincare inequality to perform Moser’s iteration to (105), we get a better decay estimate than (108):

$$|v|_{[0,t,\infty)} \leq Ce^{-CPt}. \quad (113)$$
By (105), (106), (113), (112), and Theorem 1.18 in [17], we obtain

\[ |v|_{2,\alpha,\beta,[t,\infty]} \leq C e^{-CPT}. \]  

(114)

By the arguments in the proof of Proposition 2.2 in [9] and (114), we see

\[ \int_M |\phi_0 - \phi_{KE}| \omega^n_D |t| \leq C e^{-CPT}. \]

(115)

Here \( \phi_{KE} \) is normalized such that

\[ \frac{1}{Vol(M)} \int_M \phi_{KE} \omega^n_D n! = 0. \]

Next, subtract log of the equation

\[(\omega_D + i\partial \bar{\partial} \phi_{KE})^n = e^{-h_D} \omega^n_D \]

from log of (83), we get the following linear equation

\[ \Delta (\phi_0 - \phi_{KE}) = v + a_e. \]  

(116)

where

\[ \Delta = \int_0^1 g(b,0+(1-b)) \phi_{KE} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} db \]

and \( a_e \leq C e^{-CPT} \).

Then, finally, by (116), (113), (115), Theorem 1.18 in [17], and the Moser’s iteration, we obtain our desired estimate

\[ |\phi_0 - \phi_{KE}|_{2,\alpha,\beta,[t,\infty]} \leq C (|\phi_0 - \phi_{KE}|_{L^1(M),[t-1,\infty]} + |v + a_e|_{\alpha,\beta,[t-1,\infty]}) \leq C e^{-CPT}, \]

which means the metric \( \omega_{\phi} \) converges to \( \omega_{KE} \) in the following sense

\[ |\omega_\phi - \omega_{KE}|_{\alpha,\beta,[t,\infty]} \leq C e^{-CPT}. \]  

(117)

\[ \square \]

12 Appendix A: Liouville theorem when \( \beta \leq \frac{1}{2} \).

When \( \beta < \frac{1}{2} \), Calabi’s 3rd derivative estimate works in the conical case (see [6]). Though Theorem [1.14] already settles down the Liouville theorem for all \( \beta \in (0,1) \), it still might be interesting to present the following extremely short proof of the Liouville theorem when \( \beta \leq \frac{1}{2} \).
Theorem 12.1. (Weak Liouville Theorem) Suppose $\beta \leq \frac{1}{2}$ and $\omega$ is a $C^{\alpha,\beta}$ conical Kähler metric defined over $\mathbb{C}^n$. Suppose

$$\omega^n = \omega^0_{\beta}, \quad \frac{1}{C} \omega_{\beta} \leq \omega \leq C \omega_{\beta} \text{ over } \mathbb{C}^n. \quad (118)$$

Then, there is a linear transformation $L$ which preserves $D$, such that $\omega = L^* \omega_{\beta}$.

Proof. of Theorem 12.1: Case 1: When $\beta = \frac{1}{2}$, the situation is very easy. Just consider the orbifold map $T : \mathbb{C}^n \to \mathbb{C}^n :$

$$T(w, u_1, ..., u_{n-1}) = (w^2, u_1, ..., u_{n-1}).$$

Apparently, in this case, $T^* \omega$ satisfies

$$(T^* \omega)^n = \omega^n_{Euc}, \quad \frac{1}{C} \omega_{Euc} \leq \omega \leq C \omega_{Euc} \text{ over } \mathbb{C}^n \setminus \{z = 0\}, \quad (119)$$

where $\omega_{Euc}$ is the regular Euclidean metric. Then using Proposition 16 in [14], $T^* \omega$ extends to a smooth positive $(1,1)$-form over $\mathbb{C}^n$. Then $T^* \omega = L^* \omega_{Euc}$, and $T^* \omega$ is invariant under the deck transformation:

$$(z, ...) \to (-z, ...).$$

Thus downstairs, we have

$$\omega = L^* \omega_{\frac{1}{2}},$$

where $L$ is a linear transformation which preserves $\{z = 0\}$.

Case 2: $\beta < \frac{1}{2}$. This is the case where we can do the 3rd-order estimate as Calabi, Yau, and Brendle.

We consider the scaling down again as

$$\hat{\phi} = R^{-2} \phi, \quad \hat{\omega} = R^{-2} \omega, \quad \hat{\omega}_{\beta} = R^{-2} \omega_{\beta}. \quad (120)$$

$\hat{\omega}_{\beta}$ is the standard conical model metric under the new coordinates $\hat{z} = R^{-\frac{1}{\beta}} z$, $w_i = R^{-1} w_i$. Similarly we denote

$$\hat{S} = \hat{\omega}^{i\bar{j}} \hat{\omega}^{k\bar{l}} \hat{\omega}_{i\bar{j},k\bar{l}} \hat{\phi}_{\xi_i,\zeta_i,\xi_p,\zeta_p} \hat{\phi}_{\xi_j,\zeta_j,\xi_q,\zeta_q}.$$

By formula (2.7) in [50], we directly have

$$\Delta_{\hat{\omega}}(\Delta_{\hat{\omega}_{\beta}} \hat{\phi}) \geq \frac{\hat{S}}{K}, \quad K \geq 0. \quad (121)$$

Then we multiply (121) by a cutoff function $\eta^2$, and integrate integration by parts with respect to $\omega$, we have

$$\int_{\mathbb{C}^n} \eta^2 \hat{S} \hat{\omega}^n \leq K \int_{\mathbb{C}^n} (\Delta_{\hat{\omega}} \eta^2)(\Delta_{\hat{\omega}_{\beta}} \hat{\phi}) \hat{\omega}^n. \quad (122)$$
By the second order estimate, choose a proper cutoff function \( \eta^2 \) such that \( \eta = 1 \) in \( B(1) \) and vanishes outside \( B(2) \), we have

\[
\int_{B_1} \hat{S} \omega^n \leq C|\Delta \hat{\omega} \hat{\phi}|_{L^\infty B(1)} \leq C. \tag{123}
\]

Since our reference metric \( \omega_\beta \) is flat, by the formula below formula (16) in [6], we obtain

\[
\Delta \hat{\omega} \hat{S} \geq 0. \tag{124}
\]

By Proposition 6.6 in [6], we have \( \hat{S} \in L^\infty[ B(2) ] \). Thus, by Lemma 6.1, \( \hat{S} \) is a weak subsolution to (124). Then, the Moser’s iteration as in Theorem 1.1 of Chap 4 in [26] is applicable. We deduce

\[
|\hat{S}|_{L^\infty B(\frac{1}{2})} \leq \int_{B_1} \hat{S} \omega^n \leq C. \tag{125}
\]

Then by rescaling, we have for \( \omega \) that

\[
R^2 |S|_{L^\infty B(\frac{1}{4})} \leq C. \tag{126}
\]

Then divide both hand sides by \( R^2 \), let \( R \to \infty \), we have \( S = 0 \) over \( C^n \). \( S = 0 \) implies \( \omega \) is a covariant constant tensor with respect to \( \omega_\beta \), then \( \omega = L^* \omega_\beta \), for some linear transformation \( L \) preserving \( D \).

The proof of Theorem 12.1 is thus completed. \( \square \)

13 Appendix B: Trudinger’s Harnack inequality.

In this section we work in the polar coordinates.

**Lemma 13.1.** *(Trudinger’s Harnack inequality)* Suppose \( \omega \) is a weak conical metric. Suppose

\[
\Delta_\omega u \leq 0 \text{ in the weak sense in } B(R), \ u \in W^{1,2}[B(R)] \cap C^2[ B(R) \setminus D], \tag{127}
\]

and \( u \) is nonnegative almost everywhere. Then for all \( 0 < p < \frac{n}{n-1} \), we have

\[
R^{-\frac{2n}{p}} |u|_{L^p B(\frac{R}{2})} \leq C(p) \inf_{B(\frac{R}{4})} u,
\]

where the \( L^p \)-norm is with respect to the volume form of \( \omega \).

**Proof.** of Lemma: Without loss of generality, we assume \( R = 1 \). Consider \( \bar{u} = u + k \), \( k > 0 \). Later we will let \( k \to 0 \). Consider the test function \( \bar{u}^{-2} \varphi \).

Then we apply the weak supersolution condition to get

\[
- \int_{B(1)} \nabla_\omega \bar{u} \cdot \nabla_\omega (\varphi \bar{u}^{-2}) \leq 0.
\]
Hence
\[ \int_{B(1)} (\nabla_{\omega} \frac{1}{\tilde{u}}) \cdot \nabla \varphi + 2 \int_{B(1)} |\nabla_{\omega} \tilde{u}| \tilde{u}^{-3} \varphi \leq 0. \]
Let \( v = \frac{1}{\tilde{u}} \), since \( 2 \int_{B(1)} |\nabla_{\omega} \tilde{u}| \tilde{u}^{-3} \varphi \geq 0 \), we end up with
\[ \int_{B(1)} \nabla_{\omega} v \cdot \nabla \varphi \leq 0. \] (128)
This means \( v \) is a positive weak-subsolution to \( \Delta_{\omega} v \geq 0 \! \). Since \( \omega \) is a weak conical metric, the following holds by definition.
\[ \frac{g_{E}}{C} \leq \omega \leq Cg_{E} \quad \text{over} \quad B(1) \setminus D, \] (129)
where \( g_{E} \) is the Euclidean metric in the polar coordinates. Let \( \varphi = \eta^{2} v_{p} \), by using Cauchy-Schwartz inequality, we obtain
\[ \frac{2p}{(p+1)^{2}} \int_{B(1)} \eta^{2} |\nabla_{\omega} v_{p+1}| |\omega|^{n} \leq \frac{4}{p} \int_{B(1)} |\nabla_{\omega} \eta|^{2} v_{p+1}^{p} \omega^{n}. \] (130)
By (129) and (130)
\[ \frac{2p}{(p+1)^{2}} \int_{B(1)} \eta^{2} |\nabla_{E} v_{p+1}^{\frac{p+1}{2}}|^{2} dvol_{E} \leq \frac{C}{p} \int_{B(1)} |\nabla_{E} \eta|^{2} v_{p+1} dvol_{E}. \] (131)
This is precisely the inequality which the Moser’s iteration trick requires.
Then from [26] Theorem 1.1 Chapter 4, we deduce for any \( p > 0 \) that
\[ \sup_{B(\frac{3}{4})} v \leq C(p) |v|_{L^{p}(B(\frac{1}{2}))}. \]
Hence,
\[ \inf_{B(\frac{1}{2})} \tilde{u} \geq \left( \int_{B(\frac{1}{2})} \tilde{u}^{-p} \right)^{-\frac{1}{p}} = \frac{\left( \int_{B(\frac{1}{2})} \tilde{u}^{p} dvol_{E} \right)^{\frac{1}{p}}}{\left( \int_{B(\frac{1}{2})} \tilde{u} dvol_{E} \right)^{\frac{1}{p}} \left( \int_{B(\frac{1}{2})} \tilde{u}^{-p} dvol_{E} \right)^{\frac{1}{p}}}. \] (132)
To apply the John-Nirenberg inequality, we need to verify the condition (7.51) in [23], by the superharmonic equation in terms of \( \omega \). Namely, the following claim is true.

**Claim 13.2.** For any \( p \in B(\frac{3}{4}) \) and \( r \leq 0.01 \), we have
\[ \int_{B(r)} |\nabla_{E} \log \tilde{u}| dvol_{E} \leq Cr^{n}(\int_{B(r)} |\nabla_{E} \log \tilde{u}|^{2} dvol_{E})^{\frac{1}{2}} \leq C r^{2n-1}. \] (133)
To prove the claim, we apply equation (127) to the test function \( \eta^2 \bar{u}^{-1} \), we get

\[
- \int_{B(1)} 2\eta \nabla \omega \cdot \nabla_{\bar{u}} \omega - \int_{B(1)} \eta^2 \frac{|\nabla \omega \bar{u}|^2}{\bar{u}^2} \omega \leq 0. \tag{134}
\]

By Cauchy-Schwartz inequality, we end up with

\[
\int_{B(1)} \eta^2 |\nabla \omega \log \bar{u}|^2 \omega \leq 16 \int_{B(1)} |\nabla \omega \eta|^2 \omega. \tag{135}
\]

By (129), we can transform the \( W^{1,2} \)-inequality in terms of \( \omega \) to be in terms of \( g_E \) again! Namely we have

\[
\int_{B(1)} \eta^2 |\nabla_E \log \bar{u}|^2 dvol_E \leq C \int_{B(1)} |\nabla_E \eta|^2 dvol_E. \tag{136}
\]

For any \( p \) and \( r \), we choose \( \eta \) to be a cutoff function which is 1 over \( B_p(r) \), 0 over \( C_n \setminus B_p(4r) \), and \( |\nabla_E \eta| \leq \frac{1}{r} \). Therefore, (136) implies

\[
\int_{B(r)} |\nabla_E \log \bar{u}| dvol_E \leq C r^n \left( \int_{B(r)} |\nabla_E \log \bar{u}|^2 dvol_E \right)^{\frac{1}{2}} \leq C r^{2n-1}.
\]

Thus Claim 13.2 is proved.

Claim 13.2 means \( \log \bar{u} \) satisfies the hypothesis of the John-Nirenberg inequality in Theorem 7.21 in [23]. Then applying this theorem to \( \log \bar{u} \), we obtain the following by exactly the argument in the last part of the proof of Theorem 8.18 in [23].

\[
\left( \int_{B(\frac{1}{2})} \bar{u}^{p_0} dvol_E \right)^{\frac{1}{p_0}} \left( \int_{B(\frac{1}{2})} \bar{u}^{-p_0} dvol_E \right)^{\frac{1}{-p_0}} \leq C \quad \text{for some} \ p_0 > 0.
\]

Then by (132), we obtain

\[
\inf_{B(\frac{1}{4})} \bar{u} \geq \left( \int_{B(\frac{1}{2})} \bar{u}^{p_0} dvol_E \right)^{\frac{1}{p_0}}. \tag{137}
\]

Thus Lemma 13.1 is already true for this particular \( p_0 \), Proof 2 of Theorem 8.1 already goes through.

To show Lemma 13.1 is true for all \( 0 < p < \frac{n}{n-1} \), it suffices to show the following Claim holds.

**Claim 13.3.** For all \( \frac{n}{n-1} > p > p_0 \), we have

\[
|\bar{u}|_{L^p[B(\frac{1}{2})]} \geq C(p)|\bar{u}|_{L^p[B(\frac{1}{4})]},
\]

where the \( L^p \) is with respect to the volume form of \( g_E \).
To prove the Claim, we should appeal to the superharmonicity (127) again, and transform the $W^{1,2}$-type inequality with respect to $\omega$ to $W^{1,2}$-type inequality with respect to the Euclidean metric $g_E$. We start from applying (41) to the test function $\bar{u} - a\eta^2$. Then we end up with
\[
a \int_{B(1)} |\nabla_\omega \bar{u}|^2 \bar{u}^{-a-1}\eta^2 \omega^n \leq \int_{B(1)} 2\eta (\nabla_\omega \bar{u} \cdot \nabla_\omega \eta) (\bar{u}^{-a}). \tag{138}
\]

Using Cauchy-Schwartz inequality and standard management again, we deduce
\[
\int_{B(1)} \eta^2 |\nabla_\omega \bar{u}^{1-a} |^2 \omega^n \leq \frac{(1-a)^2}{a^2} \int_{B(1)} |\nabla_\omega \eta|^2 \bar{u}^{1-a} \omega^n.
\]

Then, by (129) again, we deduce
\[
\int_{B(1)} \eta^2 |\nabla_{E} \bar{u}^{1-a} |^2 dvol_E \leq \frac{C}{a^2} \int_{B(1)} |\nabla_{E} \eta|^2 \bar{u}^{1-a} dvol_E. \tag{139}
\]

Thus we get a reverse Hölder inequality with respect to $g_E$ from the reverse Hölder inequality with respect to $\omega$ again. Apply exactly Step II of the proof in [26], Claim 13.3 holds.

Notice the $L^p$-norm with respect to $g_E$ is equivalent to the $L^p$-norm with respect to $\omega$. Now let $k \to 0$, (137) and Claim 13.3 directly imply Lemma 13.1.

References

[1] L, Ahlfors. *Complex Analysis*. McGraw-Hill.

[2] M.T, Anderson. *Convergence and rigidity of manifolds under Ricci curvature bounds*. Invent.math.102,429-445 (1990).

[3] T, Aubin. *Nonlinear Analysis on Manifolds. Monge-Ampere Equations*. Grundlehren der mathematischen Wissenschaften. Volume 252, 1982.

[4] R, Berman. *A thermodynamic formalism for Monge-Ampere equations, Moser-Trudinger inequalities and Kahler-Einstein metrics*. Advances in Mathematics. Volume 248, 25. November 2013, Pages 1254-1297.

[5] R, Berman. *K-polystability of Q-Fano varieties admitting Kähler-Einstein metrics*. arxiv1205.6214.

[6] S, Brendle. *Ricci flat Kahler metrics with edge singularities*. International Mathematics Research Notices 24, 5727–5766 (2013).

[7] E, Calabi. *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*. Michigan Math. J. 1958.
[8] F. Campana; H. Guenancia; M. Paun. *Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields*. arXiv:1104.4879. To appear in Annales Scientifiques de l’ENS.

[9] H-D. Cao. *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*. Invent.Math.81(1985), no.2, 359-372.

[10] J. Cheeger; T.H, Colding. *Lower Bounds on Ricci Curvature and the Almost Rigidity of Warped Products*. Annals of Mathematics, 2nd Ser., Vol. 144, No. 1. (Jul., 1996), pp. 189-237.

[11] X-X, Chen; S, Donaldson; S, Sun. *Kähler-Einstein metrics and stability*. arXiv: 1210.7494. To appear in Int. Math. Res. Not (2013).

[12] X-X, Chen; S, Donaldson; S, Sun. *Kähler-Einstein metric on Fano manifolds, I: approximation of metrics with cone singularities*. arXiv1211.4566. To appear in JAMS.

[13] X-X, Chen; S, Donaldson; S, Sun. *Kähler-Einstein metric on Fano manifolds, II: limits with cone angle less than 2π*. arXiv1212.4714. To appear in JAMS.

[14] X-X, Chen; S, Donaldson; S, Sun. *Kähler-Einstein metric on Fano manifolds, III: limits with cone angle approaches 2π and completion of the main proof*. arXiv1302.0282. To appear in JAMS.

[15] X-X, Chen; B, Wang. *Kähler-Ricci flow on Fano manifolds (I)*. arXiv0909.2391. To appear in Journal of European Mathematical Society.

[16] X-X, Chen; B, Wang. *On the conditions to extend Ricci flow(III)*. Int Math Res Notices (2012), Vol.2012.

[17] X-X, Chen; Y,Q, Wang. *Bessel functions, heat kernel and the Conical Kähler-Ricci flow*. arXiv:1305.0255.

[18] S,K, Donaldson. *Kähler metrics with cone singularities along a divisor*. Essays in mathematics and its applications, 4979, Springer, Heidelberg, 2012.

[19] L,C, Evans. *Partial differential equations*. Graduate Studies in Mathematics, Vol 19. AMS.

[20] P, Eyssidieux; V, Guedj; A, Zeriahi. *Singular Kähler-Einstein metrics*. J. Amer. Math. Soc. 22 (2009), 607-639.

[21] Henri Guenancia, Mihai Paun. *Conic singularities metrics with prescribed Ricci curvature: the case of general cone angles along normal crossing divisors*. arXiv:1307.6375
[22] P, Griffith; J, Harris. *Principles of Algebraic Geometry*. Wiley. 1994.

[23] D, Gilbarg; N, S, Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer.

[24] R, Hamilton. *Three-Manifolds with Positive Ricci Curvature*. J. Diff. Geom. 1982. Volume 17, Number 2 (1982), 255-306.

[25] L, Hormander. *$L^2$-estimates and existence theorems for the $\bar{\partial}$-operators*. Acta. Math. 113. (1965). 89-152.

[26] Q, Han; F, H, Lin. *Elliptic Partial Differential Equations*. American Mathematical Soc. 2011.

[27] T, Jeffres. *Uniqueness of Kähler-Einstein cone metrics*. Publ. Math. 44 (2000).

[28] Jeffres, T; Mazzeo; R, Rubinstein. *Kähler-Einstein metrics with edge singularities*. arXiv:1105.5216. To appear in Annals of Math.

[29] P, Li; S, T, Yau. *On the parabolic kernel of the Schrödinger operator*. Acta. Mathematica. July. 1986. Volume 156, Issue 1, pp 153-201.

[30] S, Kolodziej. *Hölder continuity of solutions to the complex Monge-Ampere equation with the right-hand side in $L^p$: the case of compact Kähler manifolds*. Math Ann (2008), 379–386.

[31] Ladyzenskaja; Solonnikov; Ural’ceva. *Linear and quasi-linear equations of parabolic type*. Translations of Mathematical Monographs 23, Providence, RI: American Mathematical Society.

[32] C, Li; S, Sun. *Conical Kähler-Einstein metric revisited*. arXiv1207.5011.

[33] G, Lieberman. *Second order parabolic equations*. World Scientific, 1996.

[34] J, W, Liu; X, Zhang. *The conical Kähler-Ricci flow on Fano manifolds*. arXiv:1402.1832.

[35] R, Mazzeo; Y, Rubinstein; N, Sesum. *Ricci flow on surfaces with conic singularities*. arXiv:1306.6688.

[36] D, H, Phong; J, Sturmf. *On stability and the convergence of the Kähler-Ricci flow*. J. Differential Geom. Volume 72, Number 1 (2006), 149-168.

[37] A, V, Pogorelov. *The multidimensional Minkowski problem*. Whasinton D.C. Winston, 1978.

[38] D, Riebesehl; F, Schulz. *A priori estimates and a Liouville theorem for complex Monge-Ampère equations*. Math. Z. 186 (1984), no. 1, 5766.
[39] Y.T. Siu. *Lectures on Hermitian-Einstein Metrics for Stable Bundles and Kähler-Einstein Metrics*. Birkhauser. 1987.

[40] J. Song; G. Tian. *The Kähler-Ricci flow through singularities*. arXiv:0909.4898.

[41] J. Song; X. Wang. *The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality*. arXiv:1207.4839.

[42] J. Song; B. Weinkove. *Contracting exceptional divisors by the Kähler-Ricci flow*. arXiv:1003.0718. To appear in Duke Math. J.

[43] G. Tian; S-T. Yau. *Complete Kähler Manifolds with Zero Ricci Curvature. I*. Journal of the American Mathematical Society, Vol. 3, No. 3. (Jul., 1990), pp. 579-609.

[44] G. Tian; X.H. Zhu. *Convergence of Kähler-Ricci flow*. J. Amer. Math. Soc. 20 (2007), no. 3, 675–699.

[45] L.H. Wang. *On the regularity theory of fully nonlinear parabolic equations*. Bull. Amer. Math. Soc. (N.S.) 22 (1990), no. 1, 107114.

[46] Yu, Wang. *A remark on $C^{2,\alpha}$-regularity of the complex Monge-Ampere Equation*. arXiv:1111.0902.

[47] Y.Q, Wang. *Notes on the $L^2$-estimates and regularity of parabolic equations over conical manifolds*. Unpublished work.

[48] Y.Q, Wang. *Smooth approximations of the Conical Kähler-Ricci flows*. arXiv:1401.5040.

[49] C.J, Yao. *Existence of Weak Conical Kähler-Einstein Metrics Along Smooth Hypersurfaces*. arXiv:1308.4307.

[50] S-T, Yau. *On the Ricci curvature of a compact Kähler manifold and the Complex Monge Ampère equation I*. Comm. Pure Appl. Math. 31 (1978).

[51] R, Ye. *Sobolev Inequalities, Riesz Transforms and the Ricci Flow*. arXiv:0709.0512.

[52] H, Yin. *Ricci flow on surfaces with conical singularities*. Journal of Geometric Analysis. October 2010, Volume 20, Issue 4, pp 970-995.

[53] H, Yin. *Ricci flow on surfaces with conical singularities, II*. arXiv:1305.4355.

[54] Q, Zhang. *A uniform Sobolev inequality under Ricci flow*. Int. Math. Res. Not. IMRN 2007, no. 17.
[55] Q, Zhang. *Bounds on volume growth of geodesic balls under Ricci flow.* arXiv:1107.4262.

Xiuxiong Chen, Department of Mathematics, Stony Brook University, NY, USA; xiu@math.sunysb.edu.

Yuanqi Wang, Department of Mathematics, University of California at Santa Barbara, Santa Barbara, CA, USA; wangyuanqi@math.ucsb.edu.