APPLICATIONS OF GEOMETRIC ALGEBRA TO BLACK HOLES AND HAWKING RADIATION

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Abstract

This paper is about the application of geometric algebra to black holes. Geometric algebra is the natural extension of the concept of multiplication from real numbers to geometric objects. It is a Clifford algebra with a geometric interpretation. The most important tool within geometric algebra used here is the Gauge Theory of Gravity (GTG). The theory uses gauge fields to ensure that all relations between physical quantities are independent of the position and orientation of the matter fields. These are position-, rotation-, and phase-gauge fields.

Applications begin with a preliminary discussion on cosmological Hawking radiation and the Unruh effect. The result for the ‘Hawking temperature’ of the de Sitter cosmological horizon is quite interesting. If a thermodynamic system has a negative temperature then that system should be unstable, and tend to expand or collapse. But, if we instead abandon the concept of particles as being observer-independent in this case, we find the usual positive ‘Hawking temperature’. Besides, geometric algebra was also successfully applied to treat the Unruh effect.

The discussion then moves on to consider relativistic wave equations in spherically symmetric static black hole backgrounds. The Klein-Gordon and Dirac equations are treated on Schwarzschild and Reissner-Nordström black hole metrics. The analysis is generalised to allow for the presence of magnetic monopoles. An interesting situation concerning electromagnetism in the Schwarzschild black hole background is treated. Then it is repeated on the axially symmetric stationary (i.e., Kerr) black hole background. The Hawking temperatures were derived for all cases and compared with the results of ‘standard’ calculations. Remarkably, they agreed.

It was also found that the absolute values of the real part of the non-zero roots in the Klein-Gordon, Dirac, and electromagnetic field cases are the values of the spin of the corresponding particles yet electromagnetism, at least, is completely classical. It seems that geometric algebra can tell that Maxwell’s purely classical theory concerns spin-1 particles (photons) even though the picture of bosonic particles carrying forces is a quantum description! In some sense, the quantal ‘geometric’ nature of the theory is recognised by geometric algebra!
1 Introduction

The study of black holes began with the classic paper of Oppenheimer and Snyder [52]. They considered the collapse of a homogeneous spherical gas without pressure in Einstein’s General Relativity (GR). They found that the sphere eventually becomes cut off from all communication with the rest of the universe, and at the end the matter was crushed to infinite density at the centre. Black holes were largely ignored until Wheeler and his collaborators revived the subject in the 1960’s and triggered the golden era of black hole research.

GR clearly predicts the existence of black holes, which are the most exciting predicted compact objects in the universe (see, for example, Kramer, Stephani, MacCallum, and Herlt [40], Misner, Thorne, and Wheeler [47], Hawking and Ellis [31], Christodoulou [10], and de Felice and Clarke [12]; for popular accounts see, for example, Einstein [22], Thorne [59], Begelman and Rees [4], Wheeler and Ford [66] and Gribbin [28]). The Gauge Theory of Gravity (GTG) employed in this dissertation also predicts black holes as in GR but with some modifications as found in Lasenby, Doran, and Gull [41]. These solutions have spacetime singularities from which nothing can emerge to infinity. There is also a null hypersurface called the event horizon or ‘surface’ of the black hole, which separates the spacetime within it from the rest of the universe. In Newtonian language, the escape velocity from inside the surface exceeds the speed of light, so that nothing can escape. The concept of the event horizon is purely classical. If the singularities are hidden behind the event horizon, nothing goes wrong with GR or GTG, because anything which comes in cannot go out, so we can argue that the ingoing information is stored behind the event horizon even if we cannot access it. But, if Quantum Mechanics (QM) is involved, it seems that GR or GTG will break down because smooth initial data (pure states) are able to evolve into singular field configurations (mixed states). Hawking showed semiclassically that QM implies the emission of particles from black holes, which is classically impossible [30] (see also [10] and [64]: see [55] for the interpretation of Hawking radiation as tunneling). The radiation which escapes is purely thermal, as if a black hole is a black body with a temperature proportional to its surface gravity and carries no information at all. When a black hole evaporates completely, all the information of the collapsed matter is lost. This phenomenon, which is usually called information loss or quantum decoherence, would imply non-unitary evolution in contradiction to one of the
basic laws of QM. Instead of discussing the information loss phenomenon, we shall consider below how the Hawking radiation appears naturally in GTG within the language of geometric algebra.

This paper is mainly about the applications of geometric algebra to black holes. Geometric algebra is firstly introduced. This powerful mathematical tool extends the concept of multiplication from real numbers to geometric objects. It is a Clifford algebra with a geometric interpretation. Its power to unify different languages in mathematical physics has been rediscovered time after time. It promises one common mathematical language throughout physics. In the Dirac equation, for instance, we can replace the usual imaginary scalar $i$ with a quantity interpretable as rotation in a particular plane. This permits the disentanglement of geometry and quantum effects, clarifying the latter.

The language of geometric algebra is used to express gravity as a gauge theory (Gauge Theory of Gravity, or GTG). The theory uses gauge fields to ensure that all relations between physical quantities are independent of the position and orientation of the matter fields. The discussion of GTG begins with the gauge fields. These are position-, rotation-, and phase-gauge fields.

An interesting example on the cosmological background, i.e., the de Sitter metric, that relates to cosmological Hawking radiation and the Unruh effect is given.

The relativistic wave equations on the spherically symmetric static black hole backgrounds are then discussed. The Klein-Gordon and Dirac equations are treated on the Schwarzschild and Reissner-Nordström black hole metrics. The Hawking temperature for each case is derived and compared with the result from the ‘standard’ calculations. Generalisation to allow for the presence of magnetic monopoles is also presented. An interesting case concerning electromagnetism in the Schwarzschild black hole background is examined as well.

The relativistic wave equations on the axially symmetric stationary black hole backgrounds are investigated. The Klein-Gordon and Dirac equations are treated on the Kerr black hole metric. The Hawking temperature for each case is again derived and compared with the result from the ‘standard’ calculations.

Finally, there are two appendices which comprise short notes on spherical monogenic functions and vector spherical harmonics. The computations in this paper is performed with the help of Maple package [57].
2 Geometric Algebra

This brief introduction to geometric algebra is intended to establish out notation and conventions. More complete introduction may be found in [41] and [32]. The basic idea is to extend the algebra of scalars to that of vectors. We do this by introducing an associate (Clifford) product over a graded linear space. We identify scalars with the grade 0 elements of this space, and vectors with the grade 1 elements. Under this product scalars commute with all elements, and vectors square to give scalars. If \( a \) and \( b \) are two vectors, then we write the geometric (Clifford) product as the juxtaposition \( ab \). This product decomposes into a symmetric and an antisymmetric part, which define the inner and outer products between vectors, denoted by a dot and a wedge, respectively:

\[
\begin{align*}
  a \cdot b & \equiv \frac{1}{2}(ab + ba) \\
  a \wedge b & \equiv \frac{1}{2}(ab - ba)
\end{align*}
\]

It is simple to show that \( a \cdot b \) is a scalar, but \( a \wedge b \) is neither a scalar or a vector. It defines a new geometric element called a bivector (grade 2). This may be regarded as a directed plane segment, which specifies the plane containing \( a \) and \( b \). Note that if \( a \) and \( b \) are parallel, then \( ab = ba \), whilst \( ab = -ba \) for \( a \) and \( b \) perpendicular. This process may be repeated to generate higher grade elements, and hence a basis for the linear space.

2.1 The Spacetime Algebra (STA)

The Spacetime Algebra is the geometric algebra of spacetime. This is familiar to physicists as the algebra if the Dirac \( \gamma \)-matrices. The STA is generated by four orthogonal vectors \( \{ \gamma_\mu \}, \mu = 0 \ldots 3 \), satisfying

\[
\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu} = \text{diag}(+---).
\]

A full basis for the STA is given by the set

\[
\begin{array}{cccc}
  \text{grade} & \{ \gamma_\mu \} & \{ \sigma_k, i\sigma_k \} & \{ i\gamma_\mu \} & i \\
 0 & 1 \text{ scalar} & 4 \text{ vectors} & 6 \text{ bivectors} & 4 \text{ trivectors} & 1 \text{ pseudoscalar}
\end{array}
\]

where \( \sigma_k \equiv \gamma_k \gamma_0, k = 1 \ldots 3 \), and \( i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3 \). The pseudoscalar \( i \) squares to \(-1\) and anticommutes with all odd-grade elements. The \( \{ \sigma_k \} \)
generate the geometric algebra of Euclidean 3-space, and are isomorphic to the Pauli matrices. They represent a frame of ‘relative vectors’ (‘relative’ to the timelike vector $\gamma_0$ used in their definition). The $\{\sigma_k\}$ are bivectors in four-dimensional spacetime, but 3-vectors in the relative 3-space orthogonal to $\gamma_0$.

An arbitrary real superposition of the basis elements (3) is called a ‘multivector’, and these inherit the associative Clifford product of the $\{\gamma_\mu\}$ generators. For a grade-$r$ multivector $A_r$ and a grade-$s$ multivector $B_s$ we define the inner and outer products with

$$A_r \cdot B_s \equiv \langle A_r B_s \rangle_{|r-s|}, \quad A_r \wedge B_s \equiv \langle A_r B_s \rangle_{|r+s|},$$

where $\langle M \rangle_r$ denotes the grade-$r$ part of $M$. We will also use the commutator product

$$A \times B \equiv \frac{1}{2} (AB - BA).$$

The operation of reversion, denoted by a tilde, is defined by

$$(AB)^\sim \equiv \tilde{B} \tilde{A}$$

and the rule that vectors are unchanged under reversion. We adopt the convention that in the absence of brackets, inner, outer and commutator products take precedence over Clifford products.

Vectors are usually denoted in lower case Latin, $a$, or Greek for basis frame vectors. Introducing coordinates $\{x^\mu(x)\}$ gives rise to a (coordinate) frame of vectors $\{e^\mu\}$, satisfies $e_\mu \cdot e^\nu = \delta^\nu_\mu$. The vector derivative $\nabla (\equiv \partial_x)$ is then defined by

$$\nabla \equiv e^\mu \partial_\mu$$

where $\partial_\mu \equiv \partial/\partial x^\mu$.

Linear functions mapping vectors to vectors are usually denoted with an underbar, $\underline{f}(a)$ (where $a$ is the vector argument), with the adjoint denoted with an overbar, $\overline{f}(a)$. Linear functions extend to act on multivectors via the rule

$$\overline{f}(a \wedge b \wedge \cdots \wedge c) \equiv \overline{f}(a) \wedge \underline{f}(b) \wedge \cdots \wedge \underline{f}(c).$$
which defines a grade-preserving linear operation. In the STA, tensor objects are represented by linear functions, and all manipulations can be carried out in a coordinate-free manner.

All Lorentz boosts or spatial rotations are performed with rotors. These are even-grade elements $R$, satisfying $R \tilde{R} = 1$. Any element of the algebra, $M$, transforms as

$$M \mapsto RM \tilde{R}. \quad (9)$$

A general rotor may be written as $R = \exp(B/2)$ where $B$ is a bivector in the plane of rotation.

### 2.2 Gauge Theory of Gravity (GTG)

Physical equations, when written in the STA, always take the form

$$A(x) = B(x) \quad (10)$$

where $A(x)$ and $B(x)$ are multivector fields, and $x$ is the four-dimensional position vector in the (background) Minkowski spacetime. We demand that the physical content of field equations be invariant under arbitrary local displacements of the fields in the background spacetime,

$$A(x) \mapsto A(x'), \quad x' = f(x), \quad (11)$$

where $f(x)$ a non-singular function of $x$. We further demand that the physical content of the field equations be invariant under an arbitrary local rotation

$$A(x) \mapsto RA(x) \tilde{R}, \quad (12)$$

with $R$ a non-singular rotor-valued function of $x$. These demands are clearly equivalent to requiring covariance (form-invariance under the above transformation) of the field equations. These requirements are automatically satisfied for non-derivative relations, but to ensure covariance in the presence of derivatives we must gauge the derivative in the background spacetime. The gauge fields must transform suitably under (local) displacements and rotations, to ensure covariance of the field equations. This leads to the introduction of two gauge fields: $\overline{h}(a)$ and $\Omega(a)$. The first of these, $\overline{h}(a)$, is a position-dependent linear function mapping the vector argument $a$ to vectors. The position dependence is usually left implicit. Its gauge-theoretic purpose
Table 1: Symmetry transformations of the gravitational gauge fields

| gauge transformation | transformed fields | 
|----------------------|--------------------|
|                      | \( \psi'(x) \)    | \( \eta'(a, x) \) |
| displacements        | \( \psi(x') \)    | \( \eta(f^{-1}(a), x') \) |
| spacetime rotations  | \( R\psi(x) \)   | \( R\eta(a, x) \) |
| phase rotations      | \( \psi(x)e^{i\phi} \) | \( \eta(a, x) \) |

is to ensure covariance of the equations under arbitrary local displacements of the matter fields in the background spacetime \[41\]. The second gauge field, \( \Omega(a) \), is a position-dependent linear function which maps the vector \( a \) to bivectors. Its introduction ensures covariance of the equation under local rotations of vector and tensor fields, at a point, in the background spacetime.

Once this gauging has been carried out, and a suitable lagrangian for the matter fields and gauge fields has been constructed, we find that gravity has been introduced. Despite this, we are still parameterising spacetime points by vectors in a flat background Minkowski spacetime. The covariance of the field equations ensures that the particular parameterisation we choose has no physical significance. The feature that is particularly relevant to this is that we still have all the features of the flat-space STA at our disposal. A particular choice of parameterisation is called a gauge. Under gauge transformations, the physical fields and the gauge fields will change, but this does not alter physical predictions if we demand that such predictions be extracted in a gauge-invariant manner. List of symmetry transformations which make the action invariant is given in Table 1, while the conventions of GTG are shown in Table 2 \[41\].

The covariant Riemann tensor \( \mathcal{R}(a \wedge b) \) is a linear function mapping bivectors to bivectors. It is defined via the field strength of the \( \Omega(a) \) gauge field:

\[
\mathcal{R}h^{-1}(a \wedge b) \equiv a \cdot \nabla \Omega(b) - b \cdot \nabla \Omega(a) + \Omega(a) \times \Omega(b),
\]

(13)

The Ricci tensor, Ricci scalar and Einstein tensor are formed from contractions of the Riemann tensor \[41\].
Table 2: Definitions and conventions

|                                |                                          |
|--------------------------------|------------------------------------------|
| displacement gauge fields      | $\mathbf{h}(a)$                         |
| rotation gauge fields          | $\Omega(a)$, $\omega(a) = \Omega_h(a)$ |
| spinor derivatives             | $D_\alpha \psi = a \cdot \nabla \psi + \frac{1}{2} \Omega(a) \psi$ |
|                                | $a \cdot D \psi = a \cdot h(\nabla) \psi + \frac{1}{2} \omega(a) \psi$ |
| observable derivatives         | $\mathcal{D}_a M = a \cdot \nabla M + \Omega(a) \times M$ |
|                                | $a \cdot \mathcal{D} M = a \cdot \mathbf{h}(\nabla) M + \omega(a) \times M$ |
|                                | $\mathcal{D} M = \partial_a a \cdot \mathcal{D} M = \mathcal{D} \cdot M + \mathcal{D} \wedge M$ |
| vector derivative              | $\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu}$ |
| multivector derivative         | $\partial_X = \sum_{i<j} e^i \wedge \cdots \wedge e^j (e^j \wedge \cdots \wedge e_i) \ast \partial_X$ |

Ricci Tensor: $\mathcal{R}(b) = \partial_a \cdot \mathcal{R}(a \wedge b)$
Ricci Scalar: $\mathcal{R} = \partial_a \cdot \mathcal{R}(a)$
Einstein Tensor: $\mathcal{G}(a) = \mathcal{R}(a) - \frac{1}{2} a \mathcal{R}$.  \hspace{1cm} (14)

The Einstein equation may then be written as

$$\mathcal{G}(a) = \kappa \mathcal{T}(a),$$ \hspace{1cm} (15)

where $\mathcal{T}(a)$ is the covariant, matter stress-energy tensor. The remaining field equation gives the $\Omega$-function in terms of the $\mathbf{h}$-function, and the spin of the matter field [41].

3 Cosmological Hawking Radiation and the Unruh Effect

3.1 Dirac equation on the de Sitter metric

In this section we discuss the Dirac equation on the de Sitter background [31,27]. We employ the minimally coupled Dirac equation from [41,18],
which is in this case

$$D\psi I\sigma_3 = m_p\psi\gamma_0.$$  \hfill (16)

We use the Kerr-Schild gauge:

$$\vec{h}(a) = a + \lambda(r)a \cdot e_\perp e_\perp$$  \hfill (17)

where

$$\lambda(r) \equiv \Lambda r^2,$$  \hfill (18)

and $e_\perp \equiv \gamma_0 - e_r$. The Dirac equation now becomes

$$\nabla \psi I\sigma_3 - [\{\lambda(r)(-\gamma_0 + \gamma_r)(\partial_t - \partial_r)\}\psi I\sigma_3 + [(-\gamma_0 + \gamma_r)(\frac{\Lambda}{3} r)]\psi i\sigma_3 = m_p\psi\gamma_0.$$  \hfill (19)

Upon multiplying this equation by $\gamma_0$ from the left, and using the symbol $j$ to represent right-sided multiplication by $I\sigma_3$, i.e., $j\psi \equiv \psi I\sigma_3$, we have

$$j\partial_t \psi = -j\nabla \psi - j\lambda(r)(1 + \sigma_r)(\partial_t - \partial_r)\psi + j(\frac{\Lambda}{3} r)(1 + \sigma_r)\psi + m_p\overline{\psi}$$  \hfill (20)

where $\overline{\psi} \equiv \gamma_0\psi\gamma_0$. By making a trial separation of variables

$$\psi(x) = \alpha(t)\psi(x)$$  \hfill (21)

we find that

$$\alpha(t) = \exp(-jEt),$$  \hfill (22)

and

$$\nabla \psi(x) - \lambda(r)(1 + \sigma_r)\partial_r \psi(x) - (\frac{\Lambda}{3} r)(1 + \sigma_r)\psi(x) + \lambda(r)(1 + \sigma_r)(-jE)\psi(x) - jE\psi(x) + jm_p\overline{\psi}(x) = 0.$$  \hfill (23)

Next, we need to separate out the angular dependence. This can be done by using the spherical monogenics [20]. Note that, from equation (24) to equation (27) below, we temporarily employ $I\sigma_3$ instead of $j$; these after we use $j$ for $I\sigma_3$ again. The (unnormalized) monogenic is defined by
\[ \psi_l^m = [(l + m + 1)P_l^m(\cos \theta) - P_l^{m+1}(\cos \theta)]e^{im\phi}I_{m\phi} \]  

(24)

where \( l \geq 0, -(l + 1) \leq m \leq l \), and \( P_l^m \) are the associated Legendre polynomials (see, for example, [1]). Two important properties of the \( \psi_l^m \) are

\[ \nabla \psi_l^m = -(l/r)\sigma_r \psi_l^m \]  

(25)

and

\[ \nabla (\sigma_r \psi_l^m) = (l + 2)/r \psi_l^m. \]  

(26)

After some manipulation [20], the solutions are as follows:

\[
\psi(x, \kappa) = \begin{cases} 
\psi_l^m u(r) + \sigma_r \psi_l^m v(r)I_{m3}, & \kappa = l + 1, \\
\sigma_r \psi_l^m u(r)\sigma_3 + \psi_l^m I_{v3}, & \kappa = -(l + 1),
\end{cases}
\]  

(27)

where \( \kappa \) is a non-zero integer and \( u(r) \) and \( v(r) \) are complex functions of \( r \). By substituting equation (27) into equation (23) and employing the properties of the spherical monogenics above, we obtain the coupled equations (in matrix form)

\[
A \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]  

(28)

where

\[
A \equiv \begin{pmatrix} 1 - \lambda(r) & -\lambda(r) \\ -\lambda(r) & 1 - \lambda(r) \end{pmatrix}
\]  

(29)

and

\[
B \equiv \begin{pmatrix} \frac{\kappa}{r} + \zeta(r) & j(E + m_p) + \zeta(r) \\ j(E - m_p) + \zeta(r) & -\frac{\kappa}{r} + \zeta(r) \end{pmatrix}
\]  

(30)

Here \( \lambda(r) \) is given by equation (18), and

\[
\zeta(r) \equiv -\frac{\lambda(r)}{r} + \frac{\Lambda}{3r} + jE \lambda(r)
\]  

(31)

where \( u_1 \) and \( u_2 \) are the reduced functions defined as in the Schwarzschild case:
\[ u_1 = ru, \quad u_2 = jrv \]  

and the primes in (28) denote differentiation with respect to \( r \). Equation (28) can also be written as

\[ (1 - \frac{\Lambda}{3} r^2) \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = A' B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \]  

with

\[ A' \equiv \begin{pmatrix} 1 - \lambda(r) & \lambda(r) \\ \lambda(r) & 1 - \lambda(r) \end{pmatrix}. \]  

We now look for power series with a limited radius of convergence around the horizon, by introducing the series

\[ u_1 = \eta^s \sum_{k=0}^{\infty} a_k \eta^k, \quad u_2 = \eta^s \sum_{k=0}^{\infty} b_k \eta^k, \]  

where

\[ \eta = r - r_H, \]  

and

\[ r_H \equiv \sqrt{\frac{3}{\Lambda}}. \]  

As in the Schwarzschild case [41], the index \( s \) controls the radial dependence of \( \psi \) at the horizon, so that it represents a physical quantity. To find the values of \( s \) we substitute (35) into (33) and set \( \eta = 0 \). The resulting indicial equation is

\[ \det \left[ A'B + \left( \frac{2}{3} \sqrt{3\Lambda} sI \right) \right]_{r=r_H} = 0, \]  

from which we find the solutions for the two indicial roots:

\[ s = 0, -\frac{1}{2} - jE\sqrt{\frac{3}{\Lambda}}. \]  

Since \( s = 0 \) is always a solution of the indicial equation, solutions that are analytic at the horizon always exist [41]. In [41] it is claimed that one can
calculate reflection coefficients and scattering amplitudes by determining the split between ingoing and outgoing states of these solutions. However, the problem here is whether the second root can be physically significant. We now consider the ‘Hawking temperature’, which will be denoted by $T_H$. In these calculations we shall need the non-covariant conserved current (which generates the streamlines whose curves are timelike) that can be expressed as:

$$J = h(\psi \gamma_0 \tilde{\psi}) \det(h)^{-1}. \quad (40)$$

This satisfies the flatspace conservation equation $\nabla \cdot J = 0$. We express $\eta^s$ as

$$\eta^s = \exp \left\{ \left( -\frac{1}{2} - jE \sqrt{3} \Lambda \right) \ln(r - \sqrt{3} \Lambda) \right\}. \quad (41)$$

We can now write

$$\ln(r - \sqrt{3} \Lambda) = \ln |(r - \sqrt{3} \Lambda)| \quad (42)$$

with the choice of argument

$$\arg (r - \sqrt{3} \Lambda) = \begin{cases} 0, & r > r_H \\ -\pi, & r < r_H. \end{cases} \quad (43)$$

We can split $E$ into real and imaginary parts as

$$E = E_r + j\epsilon. \quad (44)$$

If we now take the limit $r \to \sqrt{3} \Lambda$ from above and below we find that the $\gamma_0$ component of $J$ is given by

$$\gamma_0 J = B_1(\theta, \phi) e^{-2\epsilon t} \left| (r - \sqrt{3} \Lambda) \right|^{(-1+2) \sqrt{3} \Lambda} \times \begin{cases} 1, & r > r_H, \\ \exp \left[-2\pi \sqrt{3} \Lambda E_r\right], & r < r_H, \end{cases} \quad (45)$$

where $B_1(\theta, \phi)$ is a positive-definite finite term.
We now calculate the radial component of $J$:

$$
e^r \cdot J = B_2(\theta, \phi) e^{-2 \epsilon t} |(r - \sqrt{\frac{3}{\Lambda}})|^{(2\sqrt{\frac{3}{\Lambda}})} \times$$

$$
\begin{cases}
1, & r > r_H, \\
-\exp \left[-2\pi \sqrt{\frac{3}{\Lambda}} E_r\right], & r < r_H,
\end{cases}
$$

(46)

where $B_2(\theta, \phi)$ is a positive-definite finite term.

From these results we can derive the ‘Hawking temperature’. By taking the ratio of the inward flux to the total flux, which is

$$\left| \frac{e^r \cdot J_+}{e^r \cdot J_+ - e^r \cdot J_-} \right| = \frac{1}{\exp \left[2\pi \sqrt{\frac{3}{\Lambda}} E_r\right] + 1},$$

(47)

we identify a Fermi-Dirac distribution with ‘Hawking temperature’ given by

$$T_H = \frac{1}{2\pi k_B} \sqrt{\frac{\Lambda}{3}}.$$  (48)

In the calculations above we used the inward flux, not the outward flux, because we are concerned with an observer inside the de Sitter universe. We therefore obtain the same result as by the standard approach, but much more economically.

Consider now an observer outside the de Sitter horizon (i.e., in region II of Figure 2 in [27]). In this case we take the ratio of the outward flux to the total flux,

$$\left| \frac{e^r \cdot J_-}{e^r \cdot J_+ - e^r \cdot J_-} \right| = \frac{1}{\exp \left[-2\pi \sqrt{\frac{3}{\Lambda}} E_r\right] + 1},$$

(49)

to obtain a Fermi-Dirac distribution with a negative ‘Hawking temperature’

$$T_H = -\frac{1}{2\pi k_B} \sqrt{\frac{\Lambda}{3}}.$$  (50)

This is a remarkable result. An observer inside the de Sitter horizon (i.e., in region I of Figure 2 in [27]) will detect a positive Hawking temperature, while an observer outside it (i.e., in region II of Figure 2 in [27]) will detect a negative Hawking temperature! It seems that the Hawking temperature is
observer-dependent. In fact this is because our concept of particle is observer-dependent, as explained below.

We know that the de Sitter universe is unstable and forever expanding\cite{17}. The inflationary era which is believed to have taken place very early in the evolution of the universe has a similar nature to the de Sitter universe (the difference is that inflationary era took place for a very short period of time, not endlessly). The negative temperature above may reflect this instability with respect to an observer outside the de Sitter universe. If negative temperature is really an indication that a thermodynamical system is unstable, then the Cauchy horizons (i.e., inner horizons) in black holes should be unstable and should not therefore exist (from the point of view of an observer outside the black hole). Instead of expanding like the cosmological horizon in the de Sitter case, we find that the Cauchy horizons actually tend to collapse\cite{21}.

We also know that inside a black hole there are particle states which have negative energy with respect to an external stationary observer. As discussed by Gibbons and Hawking\cite{27}, a cosmological event horizon has many similarities with a black hole event horizon. Hence, if we abandon the concept of particles as being observer-independent and apply this view to our case, in which we are concerned with the observer inside the event horizon of a de Sitter universe, then we get the positive ‘Hawking temperature’ as in the calculations above.

3.2 The Unruh Effect

Suppose a detector is being uniformly accelerated in Minkowski spacetime. According to Unruh\cite{61}, this detector behaves as if it were placed in a thermal bath of ‘real’ particles with temperature (i.e., Unruh temperature) given by

\[
T_U = \frac{a}{2\pi k_B},
\]

where \(a\) is the acceleration of the detector. The ordinary vacuum state in Minkowski spacetime from the viewpoint of an accelerating observer has thermal properties similar to those due to true particle creation by a black hole as seen by a stationary observer at some constant distance from the black hole. Hence, accelerated systems find themselves in a thermal bath. This is called the Unruh effect. Both Hawking and Unruh temperatures can
be related to information loss, associated with real and accelerated observer horizons respectively (see, for example, [62, 63, 46, 54]).

In an interesting paper [14], a connection was sought between surface gravity and the Unruh temperature, with the aim of establishing the principle of equivalence between constant acceleration and ‘true’ gravity effects by globally embedding curved spaces in higher dimensional flat spaces. The relevant acceleration in the flat spaces gives the correct Hawking temperature in the curved ones.

Consider now a constant electric field $E_c$ produced by two oppositely charged plates facing each other. If the electric field is strong enough then the vacuum located between the plates produces real charged particles, as near to the event horizon of the black hole. Each particle (with mass $M$ and charge $e$) propagates with a uniform acceleration given by

$$a = \frac{eE_c}{M}.$$  \hfill (52)

In the Unruh effect, the propagation of the detector is fully described by a given classical trajectory; only the internal transitions accompanied by the emission or absorption of quanta of the radiation fields are treated quantum-mechanically. The analysis is in some sense semiclassical, like Hawking’s treatment of black hole evaporation [30]. However, Unruh used second-quantised quantum field theory. In this paper only first-quantised quantum theory is used.

Unruh found that an accelerated observer in Minkowski spacetime (in the normal vacuum state) can detect and absorb particles. According to an inertial observer this absorption appears to be emission from the accelerated observer’s detector. Similarly, an observer at a constant distance from a black hole detects a steady flux of particles coming out from the black hole, with a thermal spectrum, while an observer who falls freely into the black hole does not see particles radiating. This is a consequence of the principle of equivalence.

We now discuss the Dirac equation on the Rindler background to study the Unruh effect [63]. We still employ the minimally coupled Dirac equation [41, 18], which is in this case

$$D\psi I\sigma_3 = m_p \psi \gamma_0.$$  \hfill (53)

We use the Kerr-Schild gauge

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\[ \bar{T}(b) = b + \lambda(z)b \cdot e_\pm e_\pm \]  
(54)

with \( e_- \equiv \gamma_0 - \gamma_3 \), and \( \lambda(z) \equiv az \), where \( a \) is the acceleration. With the ansatz

\[ \psi(x) = (u(z) + j\sigma_3 v(z))e^{-jEt}, \]  
(55)

the Dirac equation becomes (in matrix form):

\[ A \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \]  
(56)

where

\[ A \equiv \begin{pmatrix} 1 - \lambda(z) & -\lambda(z) \\ -\lambda(z) & 1 - \lambda(z) \end{pmatrix} \]  
(57)

and

\[ B \equiv \begin{pmatrix} \frac{1}{z} + \zeta(z) & j(E + m_p) + \zeta(z) \\ j(E - m_p) + \zeta(z) & \frac{1}{z} + \zeta(z) \end{pmatrix}. \]  
(58)

Here \( \zeta(z) \) is given by

\[ \zeta(z) \equiv -\frac{a}{2} + jE\lambda(z) \]  
(59)

and \( u_1 \) and \( u_2 \) are the reduced functions defined as in the Schwarzschild case:

\[ u_1 = zu, \quad u_2 = jzv, \]  
(60)

and the primes in (56) denote differentiation with respect to \( z \). Equation (56) can also be written as

\[ (1 - 2az) \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = A'B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \]  
(61)

where

\[ A' \equiv \begin{pmatrix} 1 - \lambda(z) & \lambda(z) \\ \lambda(z) & 1 - \lambda(z) \end{pmatrix}. \]  
(62)

As usual, we introduce the series
\[ u_1 = \eta^s \sum_{k=0}^{\infty} a_k \eta^k, \quad u_2 = \eta^s \sum_{k=0}^{\infty} b_k \eta^k, \quad (63) \]

with

\[ \eta = z - z_H, \quad (64) \]

and

\[ z_H \equiv \frac{1}{2a}, \quad (65) \]

to generate the indicial equation

\[ \det \left[ A'B - 2asf \right]_{z = z_H} = 0, \quad (66) \]

The solutions for the two indicial roots are

\[ s = 0, -\frac{1}{2} + \frac{jE}{a}. \quad (67) \]

As in the previous section, the fact that \( s = 0 \) is always a solution of the indicial equation implies that analytic solutions always exist at \( z_H \), which in this case acts as an event horizon. (An interesting account of this is given in Chapter 4 of Birrell and Davies \[6\].) As before, we wish to investigate whether the second root can be physically significant. We again express \( \eta^s \) as

\[ \eta^s = \exp \left\{ \left( -\frac{1}{2} + \frac{jE}{a} \right) \ln \left( z - \frac{1}{2a} \right) \right\}. \quad (68) \]

We can now write

\[ \ln \left( z - \frac{1}{2a} \right) = \ln | (z - \frac{1}{2a}) | + j \arg (z - \frac{1}{2a}), \quad (69) \]

with the choice of argument

\[ \arg \left( z - \frac{1}{2a} \right) = \begin{cases} 0, & z > z_H \\ -\pi, & z < z_H \end{cases}. \quad (70) \]

Before proceeding further, we split \( E \) into real and imaginary parts as

\[ E = E_r - j\epsilon. \quad (71) \]
If we now take the limit \( z \rightarrow \frac{1}{2a} \) from above and below we find that the \( \gamma_0 \) component of \( J \) is given by

\[
\gamma_0 \cdot J = B_1 e^{-2\epsilon t} \left| \left( z - \frac{1}{2a} \right) \right|^{(-1+\frac{2\epsilon}{a})} \times \begin{cases} 1, & z > z_H, \\ \exp \left[ \frac{2\pi E_r}{a} \right], & z < z_H, \end{cases}
\]  

(72)

where \( B_1 \) is a finite term.

We now calculate the \( z \)-component of \( J \):

\[
\gamma_3 \cdot J = B_2 e^{-2\epsilon t} \left| \left( z - \frac{1}{2a} \right) \right|^{\left(\frac{2\epsilon}{a}\right)} \times \begin{cases} 1, & z > z_H, \\ -\exp \left[ \frac{2\pi E_r}{a} \right], & z < z_H, \end{cases}
\]  

(73)

where \( B_2 \) is finite.

From the above equation we are able to derive the ‘Unruh temperature’. The ratio of the ‘outward flux’ to the total flux is

\[
\left| \frac{\gamma_3 \cdot J_+}{\gamma_3 \cdot J_+ - \gamma_3 \cdot J_-} \right| = \frac{1}{\exp \left[ \frac{2\pi E_r}{a} \right] + 1},
\]  

(74)

which is, remarkably, a Fermi-Dirac distribution with temperature

\[
T_U = \frac{a}{2\pi k_B}
\]  

(75)

as we found in the standard approach. Thus, the ordinary Minkowskian vacuum state is seen by an accelerating observer to have thermal properties very similar to the thermal effects resulting from true particle creation by a black hole. The temperature (in cgs units) is

\[
T_U \approx 4 \times 10^{-23} \text{ a K},
\]  

(76)

so this effect is too small to be perceived by an ordinary laboratory detector.

In regard to the alternative explanation at the end of the previous section, if we naively take \( T_U = T_H \) as suggested in [14], then we have

\[
a = \sqrt{\frac{\Lambda}{3}},
\]  

(77)
From the discussion above we may conclude that an accelerating observer with acceleration \( \sqrt{\Lambda}/3 \) in Minkowski spacetime perceives a thermal bath with temperature \( \sqrt{\Lambda}/(2\pi k_B) \), while an inertial observer in the de Sitter universe also detects the same temperature. In the former case the acceleration causes the same effect as that seen by an inertial observer immersed in a thermal bath with corresponding temperature, while in the latter case the creation of particles is described as a consequence of cosmic expansion.

4 Spherically Symmetric Black Holes

4.1 Dirac equation on Reissner-Nordström black hole metric

In this section we discuss the Dirac equation on the Reissner-Nordström black hole background \([24, 23]\); see also \([26]\). A similar discussion for the Schwarzschild background has been given by Lasenby, Doran, and Gull \([41]\) and is a special case of the present analysis for which the charge \( q \) is zero. We use the minimally coupled Dirac equation from \([41, 18]\) (see also \([34, 33, 35]\)), which is

\[
D \psi \sigma_3 - eA \psi = m_p \psi \gamma_0, \tag{78}
\]

and the Kerr-Schild gauge \([13]\) (see also \([25, 38, 39, 49, 50]\))

\[
\bar{h}(a) = a + \lambda(r)a \cdot e_+ e_- \tag{79}
\]

where

\[
\lambda(r) \equiv \frac{M}{r} - \frac{q^2}{2r^2}, \tag{80}
\]

and \( e_- \equiv \gamma_0 - e_r \).

With the gravitational field of the Reissner-Nordström black hole background expressed in spherical polar coordinates, the Dirac equation becomes

\[
\nabla \psi \sigma_3 - eA \psi - [(\lambda(r)(-\gamma_0 + \gamma_r)(\partial_t - \partial_r))\psi I \sigma_3 + [(-\gamma_0 + \gamma_r)(M/2r^2)]\psi \sigma_3 = m_p \psi \gamma_0. \tag{81}
\]
By multiplying this equation by $\gamma_0$ from the left and using the symbol $j$ to represent right-sided multiplication by $I\sigma_3$, i.e., $j\psi \equiv \psi I\sigma_3$, we find

$$j\partial_t \psi = e\Phi \psi - j\nabla \psi - j\lambda(r)(1 + \sigma_r)(\partial_t - \partial_r + je\Phi)\psi + j\left(\frac{M}{2r^2}\right)(1 + \sigma_r)\psi + m_p\overline{\psi}$$

(82)

where $\overline{\psi} \equiv \gamma_0\gamma\psi_0$, and $\Phi$ is the potential of the black hole event horizon. We make a trial separation of variables

$$\psi(x) = \alpha(t)\psi(x)$$

(83)

which gives as the solution of the $t$-equation

$$\alpha(t) = \exp(-jEt).$$

(84)

For the spatial dependence we obtain

$$\nabla \psi(x) - \lambda(r)(1 + \sigma_r)\partial_r \psi(x) - \left(\frac{M}{2r^2}\right)(1 + \sigma_r)\psi(x) + \lambda(r)(1 + \sigma_r)(-jE + je\Phi)\psi(x) + jE\psi(x) + je\Phi + jm_p\overline{\psi}(x) = 0.$$  

(85)

Next, we need to separate out the angular dependence. This can be done by using the spherical monogenics [20] as in the de Sitter case. As before, we get the coupled equations (in matrix form):

$$A \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

(86)

with

$$A \equiv \begin{pmatrix} 1 - \lambda(r) & -\lambda(r) \\ -\lambda(r) & 1 - \lambda(r) \end{pmatrix}$$

(87)

and

$$B \equiv \begin{pmatrix} \frac{\kappa}{r} + \zeta(r) & j(Ee\Phi + m) + \zeta(r) \\ j(Ee\Phi - m) + \zeta(r) & -\frac{\kappa}{r} + \zeta(r) \end{pmatrix}$$

(88)

where $\lambda(r)$ is given by equation (80), and
\[ \zeta(r) \equiv -\frac{\lambda(r)}{r} + \frac{M}{2r^2} + j(E - e\Phi)\lambda(r). \]  

(89)

In (86) \( u_1 \) and \( u_2 \) are the reduced functions defined as in the Schwarzschild case:

\[ u_1 = ru, \quad u_2 = jrv, \]  

(90)

and the primes denote differentiation with respect to \( r \). Equation (86) can also be written as

\[ (1 - \frac{2M}{r} + \frac{q^2}{r^2}) \left( \frac{u'_1}{u_1} \right) = A'B \left( \frac{u_1}{u_2} \right) \]  

(91)

where

\[ A' \equiv \begin{pmatrix} 1 - \lambda(r) & \lambda(r) \\ \lambda(r) & 1 - \lambda(r) \end{pmatrix}. \]  

(92)

We now look for power series solutions with a definite radius of convergence around the horizon, by introducing the series

\[ u_1 = \eta^s \sum_{k=0}^{\infty} a_k \eta^k, \quad u_2 = \eta^s \sum_{k=0}^{\infty} b_k \eta^k, \]  

(93)

where there are now two values of \( \eta \), which are

\[ \eta_\pm = r - r_\pm, \]  

(94)

with

\[ r_\pm \equiv M \pm \sqrt{M^2 - q^2}. \]  

(95)

In the Reissner-Nordström case there are two event horizons, the outer \( (r_+) \) and the inner \( (r_-) \), not just one horizon as in the Schwarzschild case.

We first consider \( \eta_+ \). As in the Schwarzschild case, the index \( s \) controls the radial dependence of \( \psi \) at the horizon, which means that it represents a physical quantity. To find the values of \( s \), we substitute (93) into (91) and set \( \eta_+ = 0 \). This gives

\[ \left[ 2\sqrt{M^2 - q^2} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = r^2 A'B \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \right]_{r=r_+} \]  

(96)
where we substitute

$$r_+ = M + \sqrt{M^2 - q^2}$$

for $r$ in equation (96). By rewriting equation (96) in terms of a single matrix, setting $\eta_+ = 0$, and setting the resulting determinant to zero, the indicial equation is found to be

$$\det \left[ A'B - \left(2\sqrt{M^2 - q^2}\frac{s}{r^2}f\right) \right]_{r=r_+} = 0,$$

from which we find the solutions for the two indicial roots:

$$s = 0, -\frac{1}{2} + j(E - e\Phi) \left[ \left(\frac{M + \sqrt{M^2 - q^2}}{\sqrt{M^2 - q^2}}\right)^2 \right].$$

By repeating these steps for $\eta_-$, the indicial roots which are now

$$s = 0, -\frac{1}{2} - j(E - e\Phi) \left[ \left(\frac{M - \sqrt{M^2 - q^2}}{\sqrt{M^2 - q^2}}\right)^2 \right].$$

We now derive the Hawking temperature $T_H$. We shall need the non-covariant conserved current (which generates the streamlines whose curves are timelike), which can be expressed as [41]:

$$J = h(\psi\gamma_0\tilde{\psi}) \det(h)^{-1}.$$ (101)

This satisfies the flatspace conservation equation $\nabla \cdot J = 0$.

Consider first the case with $\eta_+$ (and $r_+$). We express $\eta^s$ as

$$\eta^s = \exp \left\{ \left( -\frac{1}{2} + j(E - e\Phi) \left[ \left(\frac{M + \sqrt{M^2 - q^2}}{\sqrt{M^2 - q^2}}\right)^2 \right] \right) \ln (r - (M + \sqrt{M^2 - q^2})) \right\}.$$ (102)

We are now able to write

$$\ln (r - (M + \sqrt{M^2 - q^2})) = \ln \left| (r - (M + \sqrt{M^2 - q^2})) \right| + j \arg (r - (M + \sqrt{M^2 - q^2}))$$

(103)
with the choice of argument
\[
\arg (r - (M + \sqrt{M^2 - q^2})) = \begin{cases} 
0, & r > r_+ \\
-\pi, & r_- < r < r_+.
\end{cases}
\tag{104}
\]

Before proceeding further, we split \( E \) into real and imaginary parts as
\[
E = E_r - j\epsilon.
\tag{105}
\]

If we now take the limit \( r \to (M + \sqrt{M^2 - q^2}) \) from above and below, we find that the \( \gamma_0 \) component of \( J \) is given by
\[
\gamma_0 \cdot J = B_1(\theta, \phi) e^{-2\xi} \left| (r - (M + \sqrt{M^2 - q^2})) \right|^{(-1 + 2 \frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2}})} \times \\
\begin{cases} 
1, & r > r_+, \\
\exp \left[ 2\pi \frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2}} (E_r - e\Phi) \right], & r_- < r < r_+,
\end{cases}
\tag{106}
\]

where \( B_1(\theta, \phi) \) is a positive-definite finite term.

Next, we calculate the radial component of \( J \) as
\[
e^r \cdot J = B_2(\theta, \phi) e^{-2\xi} \left| (r - (M + \sqrt{M^2 - q^2})) \right|^{(2 \frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2}})} \times \\
\begin{cases} 
1, & r > r_+, \\
-\exp \left[ 2\pi \frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2}} (E_r - e\Phi) \right], & r_- < r < r_+,
\end{cases}
\tag{107}
\]

where \( B_2(\theta, \phi) \) is a positive-definite finite term.

From equation (107) we are now able to derive the Hawking temperature. By taking the ratio of the outward flux to the total flux,
\[
\frac{e^r \cdot J_+}{e^r \cdot J_+ - e^r \cdot J_-} = \frac{1}{\exp \left[ 2\pi \frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2}} (E_r - e\Phi) \right] + 1},
\tag{108}
\]

we identify a Fermi-Dirac distribution with temperature.
\[ T_H = \frac{1}{2\pi k_B} \left[ \frac{\sqrt{M^2 - q^2}}{(M + \sqrt{M^2 - q^2})^2} \right]. \] 

(109)

As a comparison, see, for example, [51]. When \( q = 0 \), we reproduce the Hawking temperature as that of the Schwarzschild case [41].

By repeating the calculations for \( \eta_- \) (and \( r_- \)), and with the following choice of argument (see also, for example, [45])

\[
\arg (r - (M - \sqrt{M^2 - q^2})) = \begin{cases} 0, & r_- < r < r_+, \\ -\pi, & r < r_- , \end{cases}
\]

(110)

the Hawking temperature emerges as

\[ T_H = \frac{1}{2\pi k_B} \left[ \frac{\sqrt{M^2 - q^2}}{(M + \sqrt{M^2 - q^2})^2} \right], \]

(111)

which is negative! This is in fact impossible (see, for example, [30, 5, 3]). It is actually the case as far as an observer outside the black hole is concerned. But, from the viewpoint of an observer inside the Cauchy (i.e., inner) horizon, the Hawking temperature of this horizon is positive! In this sense, the Cauchy horizon behaves like the event horizon of the de Sitter universe. So it is also unstable. But, in this case, instead of expanding, it tends to collapse [21]. If it were expanding, it would collide with the outer event horizon and the black hole could explode; black holes would then be very rare if exist at all.

4.2 Klein-Gordon equation on Schwarzschild black hole metric

We now discuss another application, the Klein-Gordon equation in the Schwarzschild black hole background (see also [56]). We still employ the symbol \( j \) to represent right-sided multiplication by \( I\sigma_3 \) and work in the Kerr-Schild gauge.

The Klein-Gordon equation is [58, 7, 37, 65]

\[ (D^2 + m_p^2) \psi = 0 \] 

(112)
with $D \equiv \overline{h}(e^\mu)\partial_\mu + e^\mu \omega(e_\mu) \times$, and we use the separability ansatz

$$\psi = u(r) \Theta(\theta) e^{im\phi} e^{-jEt}.$$  \hfill (113)

By substituting $\psi$ into equation (112) and separating out the equations for $\Theta(\theta)$, we get

$$u'' + \left[\frac{-4jEMr + 2r - 2M^2}{r(r - 2M)}\right] u' + \left[C_1 + \frac{C_2}{r}\right] u = 0,$$  \hfill (114)

where the primes denote differentiation with respect to $r$, and $C_1$ and $C_2$ are separation constants. In this case we have $2MC_1 = -2ME^2 + C_2 = -2ME^2 + jE + \frac{C}{2M}$, where the further constant $C$ appears in the spherical harmonic equation for $\Theta$,

$$\Theta'' + \frac{\cos \theta}{\sin \theta} \Theta' + (C - \frac{m^2}{\sin^2 \theta}) \Theta = 0,$$  \hfill (115)

by which we get $C = l(l + 1)$. Note that we have used $\lambda(r) \equiv M/r$ here.

Again, we introduce the series

$$u = \eta^s \sum_{k=0}^{\infty} a_k \eta^k,$$  \hfill (116)

to find the power-series solutions around the horizon, where $\eta = r - 2M$. By putting this series into equation (114), we find the two indicial roots

$$s = 0, 4jME.$$  \hfill (117)

The non-covariant conserved current in the Klein-Gordon case (with $\hbar = 1$) is

$$J = \frac{\hbar \left( \text{Im}(-\psi^* \overline{h}(\nabla)\psi) \right)}{m_p} \det \overline{h}^{-1}.$$  \hfill (118)

Note that $\det \hbar = 1$ in the Kerr-Schild gauge. We will discuss the case for $s = 4jME$ ($s = 0$ is trivial). We first write $\eta^s$ as

$$\eta^s = \exp[(4jME) \ln(r - 2M)].$$  \hfill (119)

Write now

$$\ln(r - 2M) = \ln |r - 2M| + j \arg(r - 2M)$$  \hfill (120)

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with the choice of argument
\[
\arg (r - 2M) = \begin{cases} 
0, & r > 2M, \\
-\pi, & r < 2M.
\end{cases}
\] (121)

As before, we split \(E\) into real and imaginary parts as \(E = E_r - j\epsilon\).

By taking the limit \(r \to 2M\) from above and below we find that the \(\gamma_0\) component of \(J\) is given by
\[
\gamma_0 \cdot J = A(\theta, \phi) (\frac{E_r}{m_p}) e^{-2\epsilon t} (4M) | r - 2M |^{-1+8\delta} \times \begin{cases} 
1, & r > 2M, \\
\exp[8\pi ME_r], & r < 2M,
\end{cases}
\] (122)

where \(A(\theta, \phi)\) is a positive-definite finite term. The radial component of \(J\) is
\[
e^r \cdot J = B(\theta, \phi) (\frac{E_r}{m_p}) e^{-2\epsilon t} | r - 2M |^{8\delta} \times \begin{cases} 
1, & r > 2M, \\
\exp[8\pi ME_r], & r < 2M,
\end{cases}
\] (123)

where \(B(\theta, \phi)\) is a positive-definite finite term. Upon taking the ratio of the outward flux to the total flux
\[
\left| \frac{e^r \cdot J_+}{e^r \cdot J_+ - e^r \cdot J_-} \right| = \frac{1}{e^{8\pi ME_r} - 1},
\] (124)

we identify a Bose-Einstein distribution with Hawking temperature
\[
T_H = \frac{1}{8\pi Mk_B}.
\] (125)

**4.3 Klein-Gordon equation on Reissner-Nordström black hole metric**

We now study the Klein-Gordon equation with a Reissner-Nordström background. Following the same steps, we find

\[
u'' + \left[\frac{-4j(E - e\Phi)Mr + 2r - 2M + 2j(E - e\Phi)q^2}{r^2 - 2Mr + q^2}\right]u' + Cu = 0,
\] (126)

where \(C\) is a constant and defined via the spherical harmonic equation as above.
Introducing series as in equation (116), we get two values of $\eta$ as before. We first consider $\eta + (r + M + \sqrt{M^2 + q^2})$. By substituting the series into equation (126), we find the two indicial roots

$$s = 0, j(E - e\Phi) \left[\frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2}}\right].$$

(127)

From equation (118) together with equations (103), (104), and (105), by taking the limit $r \to (M + \sqrt{M^2 - q^2})$ from above and below we find that the $\gamma_0$ component of $J$ is given by

$$\gamma_0 \cdot J = B_1(\theta, \phi)\left(\frac{E_r}{m_p}\right)e^{-2\epsilon t} \mid (r - (M + \sqrt{M^2 - q^2})) \mid (1 + 2\frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2} \epsilon}) \times
\begin{cases} 
1, & r > r_+,
\exp \left[2\pi \frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2}} (E_r - e\Phi)\right], & r_- < r < r_+,
\end{cases}$$

(128)

where $B_1(\theta, \phi)$ is a positive-definite finite term.

The radial component of $J$ is

$$e^r \cdot J = B_2(\theta, \phi)\left(\frac{E_r}{m_p}\right)e^{-2\epsilon t} \mid (r - (M + \sqrt{M^2 - q^2})) \mid (2\frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2} \epsilon}) \times
\begin{cases} 
1, & r > r_+,
\exp \left[2\pi \frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2}} (E_r - e\Phi)\right], & r_- < r < r_+,
\end{cases}$$

(129)

where $B_2(\theta, \phi)$ is a positive-definite finite term. Upon taking the ratio of the outward flux to the total flux,

$$\frac{e^r \cdot J_+}{e^r \cdot J_+ - e^r \cdot J_-} = \frac{1}{\exp \left[2\pi \frac{(M + \sqrt{M^2 - q^2})^2}{\sqrt{M^2 - q^2}} (E_r - e\Phi)\right] - 1},$$

(130)

we identify a Bose-Einstein distribution with Hawking temperature
By performing similar calculations for \( \eta_- \) (for \( r_- = M - \sqrt{M^2 - q^2} \)) with the choice of argument of (110), we find the Hawking temperature

\[
T_H = -\frac{1}{2\pi k_B} \left[ \sqrt{M^2 - q^2} \left( M + \sqrt{M^2 - q^2} \right)^2 \right],
\]

(132)

which is negative as in equation (111). Further discussion can be found in [6].

4.4 Magnetic Monopoles

In this paper we add magnetic monopoles to the discussion of the previous section to generalise the corresponding expression for the Hawking temperature [48, 2, 29, 42, 43, 44, 53, 67]. The magnetic monopole, which is a point magnetic charge, was introduced by Dirac to remove the asymmetry in Maxwell’s electromagnetic equations. Dirac derived a quantization condition for the magnetic charge: if magnetic charge exists, then it will take discrete values. In Dirac’s proposal, the existence of any magnetic monopole is responsible for the quantization of electric charge. With \( \hbar = c = 1 \), the Dirac quantization condition for electric charge \( e \) and magnetic charge \( q_M \) (such that \( B = q_M/r^2\sigma_r \) for a point charge) is

\[
2\epsilon q_M = N,
\]

(133)

where \( N \) is an integer.

We can express the vector potential \( A \) for a magnetic monopole by the following equation:

\[
A = q_M (1 + \cos \theta) \nabla \phi.
\]

(134)

The ‘string’ along \( \theta = 0 \) is unobservable provided that \( N \) is an integer. The additional term due to monopoles causes the change \( q^2 \to q^2 + q_M^2 \) (see [47] for details), so the \( \lambda(r) \) changes slightly to
\[
\lambda(r) \equiv \frac{M}{r} - \frac{q^2 + q_M^2}{2r^2}
\]

All other terms stay the same. We now make a different ansatz:

\[
\psi = u(r)\psi^+ + Iv(r)\psi^-
\]

where \(\psi^+\) and \(\psi^-\) are given by

\[
\psi^+ = \frac{1}{2\sqrt{l+1}} \left( (\sqrt{l+1+k} + \sqrt{l+1-k}) + \Phi^{mk}_l \right)
\]

\[
+ (\sqrt{l+1+k} - \sqrt{l+1-k}) - \Phi^{mk}_{l+1} \right),
\]

and

\[
\psi^- = \frac{1}{2\sqrt{l+1}} \left( (\sqrt{l+1+k} + \sqrt{l+1-k}) + \Phi^{mk}_l \right)
\]

\[
-(\sqrt{l+1+k} - \sqrt{l+1-k}) - \Phi^{mk}_{l+1} \right),
\]

with eigenvalues \(((l + 1)^2 - k^2) \pm \sqrt{(l + 1)^2 - k^2} \equiv \kappa(k \pm 1)\) respectively, where \(\Phi^{mk}_l\) and \(\Phi^{mk}_{l+1}\) are

\[
\Phi^{mk}_l \propto \pm \sqrt{l(l+1) - m(m-1)}Y^{mk}_l - I\sigma_2 l(m - 1)Y^{m+1k}_l,
\]

and

\[
\Phi^{mk}_{l+1} \propto \pm (l - m)Y^{mk}_l + I\sigma_2 \sqrt{l(l+1) - m(m-1)}Y^{m+1k}_l.
\]

Details of the above monogenics are given in Appendix A. After some algebra, we get the coupled equations as usual as

\[
A \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

with

\[
A \equiv \begin{pmatrix} 1 - \lambda(r) & -\lambda(r) \\ -\lambda(r) & 1 - \lambda(r) \end{pmatrix}
\]

and

28
\[ B \equiv \begin{pmatrix} \frac{\kappa}{T} + \zeta(r) & j(E - e\Phi + mp) + \zeta(r) \\ j(E - e\Phi - mp) + \zeta(r) & -\frac{\kappa}{T} + \zeta(r) \end{pmatrix} \] (143)

where

\[ \zeta(r) \equiv -\frac{\lambda(r)}{r} + \frac{M}{2r^2} + j(E - e\Phi)\lambda(r) \] (144)

and \( u_1 \) and \( u_2 \) are the reduced functions defined as in the Schwarzschild case:

\[ u_1 = r u, \quad u_2 = j rv. \] (145)

Here the primes denote the differentiation to \( r \), so that

\[ (1 - \frac{2M}{r} + \frac{q^2 + q_m^2}{r^2}) \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = A'B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \] (146)

where

\[ A' \equiv \begin{pmatrix} 1 - \lambda(r) & \lambda(r) \\ \lambda(r) & 1 - \lambda(r) \end{pmatrix}. \] (147)

We look for power series solutions of the form

\[ u_1 = \eta^s \sum_{k=0}^{\infty} a_k \eta^k, \quad u_2 = \eta^s \sum_{k=0}^{\infty} b_k \eta^k, \] (148)

where we have two values of \( \eta \), which are

\[ \eta_{\pm} = r - r_{\pm}, \] (149)

with

\[ r_{\pm} \equiv M \pm \sqrt{M^2 - (q^2 + q_m^2)}. \] (150)

We now consider \( \eta_+ \). As in the Schwarzschild case [41], the index \( s \) controls the radial dependence of \( \psi \) at the horizon, so that it represents a physical quantity. After some algebra we get the following indicial equation

\[ \det \left[ A'B - \left( 2\sqrt{M^2 - (q^2 + q_m^2)} \right) \frac{\zeta}{r^2} \right]_{r=r_+} = 0, \] (151)
from which the solutions for the two indicial roots are

\[ s = 0, -\frac{1}{2} + j(E - e\Phi) \left[ \frac{(M + \sqrt{M^2 - (q^2 + q_M^2)})^2}{\sqrt{M^2 - (q^2 + q_M^2)}} \right]. \tag{152} \]

In deriving the Hawking temperature \( T_H \) we shall need the non-covariant conserved current (which generates the streamlines whose curves are timelike)

\[ J = \hbar(\psi\gamma_0\tilde{\psi}) \det(h)^{-1} \tag{153} \]

which satisfies the flatspace conservation equation \( \nabla \cdot J = 0 \).

For the solution \( \eta_+ \) (with \( r_+ \)), we can express \( \eta^s \) as

\[ \eta^s = \exp \left\{ \left( -\frac{1}{2} + j(E - e\Phi) \left[ \frac{(M + \sqrt{M^2 - (q^2 + q_M^2)})^2}{\sqrt{M^2 - (q^2 + q_M^2)}} \right] \right) \times \ln(r - (M + \sqrt{M^2 - (q^2 + q_M^2)})) \right\}. \tag{154} \]

We can now write

\[ \ln(r - (M + \sqrt{M^2 - (q^2 + q_M^2)})) = \ln |(r - (M + \sqrt{M^2 - (q^2 + q_M^2)}))| + j \arg(r - (M + \sqrt{M^2 - (q^2 + q_M^2)})) \]

with the choice of argument

\[ \arg(r - (M + \sqrt{M^2 - (q^2 + q_M^2)})) = \begin{cases} 0, & r > r_+ \\ -\pi, & r_- < r < r_+. \end{cases} \tag{155} \]

We split \( E \) into real and imaginary parts as

\[ E = E_r - j\epsilon. \tag{157} \]

If we now take the limit \( r \to (M + \sqrt{M^2 - (q^2 + q_M^2)}) \) from above and below we find that the \( \gamma_0 \) component of \( J \) is given by
\[ \gamma_0 \cdot J = B_1(\theta, \phi)e^{-2et} \frac{(r - (M + \sqrt{M^2 - (q^2 + q_M^2)})\right)^{(-\frac{1+2(M+\sqrt{M^2-(q^2+q_M^2)})^2}{\sqrt{M^2-(q^2+q_M^2)}})}}{\bigg| (r - (M + \sqrt{M^2 - (q^2 + q_M^2)})\right)^{(-\frac{1+2(M+\sqrt{M^2-(q^2+q_M^2)})^2}{\sqrt{M^2-(q^2+q_M^2)}})}}\times \begin{cases} 1, & r > r_+, \\ \exp \left[ 2\pi \frac{M+\sqrt{M^2-(q^2+q_M^2)}}{\sqrt{M^2-(q^2+q_M^2)}}(E_r - e\Phi) \right], & r_- < r < r_+ \end{cases}, \] (158)

where \( B_1(\theta, \phi) \) is a positive-definite finite term.

The radial component of \( J \) is

\[ e^r \cdot J = B_2(\theta, \phi)e^{-2et} \frac{(r - (M + \sqrt{M^2 - (q^2 + q_M^2)})\right)^{(-\frac{1+2(M+\sqrt{M^2-(q^2+q_M^2)})^2}{\sqrt{M^2-(q^2+q_M^2)}})}}{\bigg| (r - (M + \sqrt{M^2 - (q^2 + q_M^2)})\right)^{(-\frac{1+2(M+\sqrt{M^2-(q^2+q_M^2)})^2}{\sqrt{M^2-(q^2+q_M^2)}})}}\times \begin{cases} 1, & r > r_+, \\ -\exp \left[ 2\pi \frac{M+\sqrt{M^2-(q^2+q_M^2)}}{\sqrt{M^2-(q^2+q_M^2)}}(E_r - e\Phi) \right], & r_- < r < r_+ \end{cases}, \] (159)

where \( B_2(\theta, \phi) \) is a positive-definite finite term.

We are now able to derive the Hawking temperature. By taking the ratio of the outward flux to the total flux,

\[ \frac{e^r \cdot J_+}{e^r \cdot J_+ - e^r \cdot J_-} = \frac{1}{\exp \left[ 2\pi \frac{M+\sqrt{M^2-(q^2+q_M^2)}}{\sqrt{M^2-(q^2+q_M^2)}}(E_r - e\Phi) \right] + 1}, \] (160)

we identify a Fermi-Dirac distribution with the revised Hawking temperature (for \( r_+ \))

\[ T_H = \frac{1}{2\pi k_B} \left[ \frac{\sqrt{M^2 - (q^2 + q_M^2)}}{\left( M + \sqrt{M^2 - (q^2 + q_M^2)} \right)^2} \right]. \] (161)

### 4.5 Electromagnetism in the Schwarzschild black hole background

We first find a covariant Maxwell equation in the presence of a gravitational field. We extend the 'wedge' equation
\[ D \wedge \hat{h}(a) = \kappa S \cdot \hat{h}(a), \quad (162) \]

to include higher-grade terms \cite{11}. To keep the derivation general we include the spin term, so that

\[ \dot{D} \wedge \hat{h}(a \wedge b) = [\dot{D} \wedge \hat{h}(a)] \wedge \hat{h}(b) - \hat{h}(a) \wedge [\dot{D} \wedge \hat{h}(b)] \]
\[ = \kappa [\hat{h}(a) \cdot S] \wedge \hat{h}(b) - \kappa \hat{h}(a) \wedge [\hat{h}(b) \cdot S] \quad (163) \]
\[ = \kappa S \times \hat{h}(a \wedge b). \]

More generally, we can write

\[ D \wedge \hat{h}(A_r) = \hat{h}(\nabla \wedge A_r) + \kappa \langle S \hat{h}(A_r) \rangle_{r+1}. \quad (164) \]

We can therefore replace \( \nabla \wedge F = 0 \) by

\[ D \wedge F - \kappa S \times F = 0. \quad (165) \]

Now we use the rearrangement

\[ \nabla \cdot (\hat{h}(F) \det(\hat{h})^{-1}) = I \nabla \wedge (I \hat{h}(F) \det(\hat{h})^{-1}) \]
\[ = I \nabla \wedge (\hat{h}^{-1}(I F)) \quad (166) \]
\[ = I \hat{h}^{-1}[D \wedge (I F) + \kappa (I F) \times S], \]

to write

\[ \nabla \cdot [\hat{h} h(F) \det(\hat{h})^{-1}] = J, \quad (167) \]

with \( F \equiv \nabla \wedge A \), where \( A \) is the vector potential and \( J \) is the charge current, as

\[ D \cdot F - \kappa S \cdot F = I \hat{h}(JJ) = J. \quad (168) \]

By combining the equations above, we get

\[ DF - \kappa SF = J. \quad (169) \]
To find the free-field stress-energy tensor, we apply the definition
\[
\det(h) \partial_{\mu(a)} (\mathcal{L}_m \det(h))^{-1} = T h^{-1}(a),
\]
(170)
to get
\[
\mathcal{T}_{em} h^{-1}(a) = \frac{1}{2} \det(h) \partial_{\mu(a)} (\bar{h}(F) h(F) \det(h))^{-1}
\]
(171)
\[
= \bar{h}(a \cdot F) \cdot F - \frac{1}{2} h^{-1}(a) F \cdot F.
\]
Hence,
\[
\mathcal{T}_{em}(a) = \bar{h}(h(a) \cdot F) \cdot F - \frac{1}{2} a F \cdot F = (a \cdot F) \cdot F - \frac{1}{2} a F \cdot F = -\frac{1}{2} F a F,
\]
(172)
which is the natural covariant extension of the gravitation-free form \(-\frac{1}{2} F a F\). This tensor is symmetric, as expected for fields with vanishing spin density.

We are now ready to investigate electromagnetism in a Schwarzschild black hole background. Here we use the Kerr-Schild gauge
\[
\bar{h}(a) = \lambda(r) a \cdot e_\gamma e_\gamma,
\]
(173)
where
\[
\lambda(r) \equiv \frac{M}{r},
\]
(174)
and \(e_- \equiv \gamma_0 - e_r\). We begin by writing the electromagnetic field \(\mathcal{F}\) (see Appendix B for details of the vector spherical harmonics) as
\[
\mathcal{F} = F(r,t) \frac{1}{2} (1 + \sigma_r) X_l^m + G(r,t) \frac{1}{2} (1 - \sigma_r) X_l^m + H(r,t) \sigma_r Y_l^m
\]
(175)
where the functions \(F, G,\) and \(H\) are complex, and
\[
\nabla X_l^m = \frac{1}{r} \sigma_r X_l^m - \frac{\sqrt{l(l+1)}}{r} \sigma_r Y_l^m,
\]
(176)
\[
\nabla (\sigma_r X_l^m) = \frac{1}{r} X_l^m + \frac{\sqrt{l(l+1)}}{r} \sigma_r Y_l^m,
\]
(177)
\[
\nabla (\sigma_r Y_l^m) = \frac{1}{r} X_l^m + \frac{\sqrt{l(l+1)}}{r} Y_l^m.
\]
(178)
Here we have defined the normalised harmonic
\[
X_l^m(\theta, \phi) \equiv \frac{1}{\sqrt{l(l+1)}} \mathbf{L} Y_l^m(\theta, \phi),
\]
(179)
where
\[
\mathbf{L} \equiv -\mathbf{x} \times \nabla.
\]
(180)
We also know that
\[
Y_l^m(\theta, \phi) \equiv \sqrt{\frac{2l + 1}{4\pi}} \frac{(l-m)!}{(l+m)!} P^m_l(\cos \theta) e^{j m \phi},
\]
(181)
with \( Y_{l}^{-m} = (-1)^{m} (Y_{l}^{m})^* \). By forming \( \mathcal{D} \mathcal{F} = 0 \), we get
\[
\begin{pmatrix}
1 + \frac{2M}{r} & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\dot{F} \\
\dot{G} \\
\dot{H}
\end{pmatrix}
+ \begin{pmatrix}
1 - \frac{2M}{r} & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
F' \\
G' \\
H'
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{r} & 0 & \sqrt{l(l+1)} \\
0 & -\frac{1}{r} & \sqrt{l(l+1)} \\
(1 - \frac{2M}{r}) \frac{\sqrt{l(l+1)}}{2r} & -\frac{\sqrt{l(l+1)}}{2r} & 0 \\
(1 + \frac{2M}{r}) \frac{\sqrt{l(l+1)}}{2r} & -\frac{\sqrt{l(l+1)}}{2r} & \frac{2}{r}
\end{pmatrix}
\begin{pmatrix}
F \\
G \\
H
\end{pmatrix}.
\]
(182)
Upon separating of variables as \( F(r)e^{-jEt}, G(r)e^{-jEt}, \) and \( H(r)e^{-jEt} \), the radial part of the field emerges in terms of the transverse components as
\[
H = -\frac{\sqrt{l(l+1)}}{2jrE} \left( (1 - \frac{2M}{r}) F + G \right).
\]
(183)
The radial equations then become

34
\[
\left(1 - \frac{2M}{r}\right) F' + \frac{F}{r} - \left(1 + \frac{2M}{r}\right) jEF - \frac{l(l+1)}{2jEr^2} \left(\left(1 - \frac{2M}{r}\right) F + G\right) = 0,
\]
(184)

\[
G' + \frac{G}{r} + jEG + \frac{l(l+1)}{2jEr^2} \left(\left(1 - \frac{2M}{r}\right) F + G\right) = 0.
\]
(185)

Next, series expansion of these equations, as before at the horizon using \(\eta \equiv r - 2M\), yields

\[
\begin{pmatrix}
F' \\
G'
\end{pmatrix} = \begin{pmatrix}
-1 + 4jEM & -\frac{j(l+1)}{4EM} \\
0 & 0
\end{pmatrix} \begin{pmatrix}
F \\
G
\end{pmatrix} + O(1).
\]
(186)

The roots of the indicial equation are

\[s = 0, \ -1 + 4jEM.\]
(187)

The regular solution at the horizon satisfies

\[
\left((-1 + 4jEM) F - \frac{j(l+1)}{4EM} G\right)|_{r=2M} = 0
\]
(188)

As before, we now proceed to calculate the Hawking temperature. Since the imaginary part of the non-zero root in this case is the same as in the Dirac and Klein-Gordon cases, we expect to recover the same Hawking temperature.

If we look at the absolute values of the real part of the non-zero roots in the Klein-Gordon, Dirac, and electromagnetic cases discussed in former chapters, we find that they are equal to 0, \(\frac{1}{2}\), and 1, respectively. These are also the values of the spin of the particles involved. Moreover electromagnetism, as discussed here, is completely classical. It seems that geometric algebra can detect that Maxwell’s theory concerns spin-1 particles (i.e., photons) even though the picture of bosonic particles carrying forces is a quantum description! The quantal ‘geometric’ nature inherent in the theory can in some sense be recognised by geometric algebra!

We now calculate the Hawking temperature. We use the stress-energy tensor

35
\[ T_{em}(a) = -\frac{1}{2} \mathcal{F} a \mathcal{F}. \]  

(189)

We first write \( \eta^* \) as

\[ \eta^* = \exp[(-1 + 4 j EM) \ln(r - 2M)], \]  

(190)

and then write

\[ \ln(r - 2M) = \ln |r - 2M| + j \arg(r - 2M) \]  

(191)

with the choice of argument

\[ \arg(r - 2M) = \begin{cases} 0, & r > 2M, \\ -\pi, & r < 2M. \end{cases} \]  

(192)

As before, we split \( E \) into real and imaginary parts as \( E = E_r - j \epsilon \). Then we calculate \( T_{em}(e^r) \), which reads

\[ T_{em}(e^r) = Z(\theta, \phi)e^{-2\epsilon t} |r - 2M|^{8Me} \times \begin{cases} 1, & r > 2M, \\ \exp[8\pi ME_r], & r < 2M. \end{cases} \]  

(193)

where \( Z(\theta, \phi) \) is a finite term. By taking the ratio of the outward stress-energy tensor to its total:

\[ \left| \frac{T_{em+} - T_{em-}}{T_{em+}} \right| = \frac{1}{e^{8\pi ME_r} - 1}, \]  

(194)

we identify a Bose-Einstein distribution with Hawking temperature

\[ T_H = \frac{1}{8\pi M k_B}. \]  

(195)

5 Axially Symmetric Black Holes

5.1 Klein-Gordon equation on Kerr black hole metric

We now begin the calculations using GTG [41, 18]. We shall discuss the Klein-Gordon equation in the Kerr black hole background [15]. As usual, we will employ the symbol \( j \) to represent right-sided multiplication by \( I \sigma_3 \), and work in the Kerr-Schild gauge. The Klein-Gordon equation is [58]
\((\mathcal{D}^2 + m_p^2) \psi = 0\) \hspace{1cm} (196)

with \(\mathcal{D} \equiv \overline{h}(e^\mu) \partial_\mu + e^\mu \omega(e_\mu) \times,\) and using the following Kerr-Schild gauge [13] (as a comparison, see also [13, 8]):

\[
\overline{h}(a) = a + \vartheta_1(r, \theta) a \cdot e_- + \left(\frac{L}{\rho} - 1\right) a \wedge (e_r e_t) e_r e_t - \frac{L}{\rho^2} a \cdot e_t e_\phi + \frac{L^2}{2 \rho^2} \sin^2 \theta a \cdot e_+ e_- + \frac{L}{\rho^2} a \cdot e_\phi e_- \hspace{1cm} (197)
\]

where

\[
\vartheta_1(r, \theta) \equiv \frac{Mr}{\rho^2}, \hspace{1cm} (198)
\]

and

\[
\rho^2 \equiv r^2 + L^2 \cos^2 \theta, \hspace{1cm} (199)
\]

with

\[
e_- \equiv \gamma_0 - e_r, \hspace{1cm} e_+ \equiv \gamma_0 + e_r. \hspace{1cm} (200)
\]

We now employ the separation ansatz

\[
\psi = u(r) \Theta(\theta) e^{im\phi} e^{-jEt}. \hspace{1cm} (201)
\]

By substituting \(\psi\) into the Klein-Gordon equation and separating the equations for \(\Theta(\theta)\), we get

\[
(r^2 - 2Mr + L^2)u'' + (2r - 2M + 2jmL - 4jEMr)u' + (F_r(r) - \frac{C}{r^2})u = 0, \hspace{1cm} (202)
\]

where \(C\) is a constant, \(m\) is the azimuthal quantum number, the primes denote differentiation with respect to \(r\), and \(F_r(r)\) is given by

\[
F_r(r) = -2jEM + 2rME^2 + r^2(E^2 - m_p^2). \hspace{1cm} (203)
\]

The equation for \(\Theta(\theta)\) is given by

\[
\sin^2 \theta \Theta'' - 2 \cos \theta \Theta' + (F_\theta(\theta) + C)\Theta = 0, \hspace{1cm} (204)
\]
where \( \dot{} \equiv \frac{d}{d\theta} \). Solutions are spheroidal harmonics in the standard form \( S_{mn}(Lp, \cos \theta) \) with \( p^2 = E^2 - m_p^2 \), and \( C \) is the same constant as in the previous equation \( (C = \lambda_{mn}) \) (see [11] for details), and \( F_\theta(\theta) \) is given by

\[
F_\theta(\theta) = (E^2 - m_p^2)L^2\cos^2 \theta - \frac{m^2}{\sin^2 \theta}.
\]  

(205)

In this case we need the equation for \( u(r) \). Again, we introduce the series

\[
u = \eta^s \sum_{k=0}^{\infty} a_k \eta^k,
\]

(206)
to study the solutions around the horizon. We find two values of \( \eta \) as in the Dirac case. First we consider the case of \( \eta^+ \) (for \( r^+ = M + \sqrt{M^2 - L^2} \)). By substituting the series into the equation, (202), we obtain the two indicial roots

\[
s = 0, j\left[ E \left( (M + \sqrt{M^2 - L^2})^2 + L^2 \right) - mL \right] \sqrt{M^2 - L^2}. \]

(207)

From [16] we know that \( \Omega_H \equiv L/(r_+^2 + L^2) \), where \( \Omega_H \) is the angular velocity of the horizon of the black hole. So we can rewrite the two indicial roots above as

\[
s = 0, j(E - m\Omega_H) \left[ \frac{(M + \sqrt{M^2 - L^2})^2 + L^2 \sqrt{M^2 - L^2}}{M^2 - L^2} \right]. \]

(208)

The non-covariant conserved current in the Klein-Gordon case (with \( \hbar = 1 \)) is given by

\[
J = \frac{\hbar \left( \text{Im}(-\psi^* \overrightarrow{\nabla} \psi) \right)}{m_p \det \hbar^{-1}}. \]

(209)

Consider first the case for \( \eta^+ \) (with \( r^+ \)). We express \( \eta^s \) as

\[
\eta^s = \exp \left\{ \left( j(E - m\Omega_H) \left[ \frac{(M + \sqrt{M^2 - L^2})^2}{\sqrt{M^2 - L^2}} \right] \right) \times \right. \ln(r - (M + \sqrt{M^2 - L^2})) \}. \]

(210)

We can now write
\[
\ln(r - (M + \sqrt{M^2 - L^2})) = \ln |r - (M + \sqrt{M^2 - L^2})| + j \arg(r - (M + \sqrt{M^2 - L^2}))
\] (211)

with the choice of argument

\[
\arg(r - (M + \sqrt{M^2 - L^2})) = \begin{cases} 
0 & r > r_+ \\
-\pi & r_- < r < r_+.
\end{cases}
\] (212)

Before proceeding further, we split \(E\) into real and imaginary parts as

\[
E = E_r - j\epsilon.
\] (213)

If we now take the limit \(r \to (M + \sqrt{M^2 - L^2})\) from above and below, we find that the \(\gamma_0\) component of \(J\) is given by

\[
\gamma_0 \cdot J = K_1(\theta, \phi) \left( \frac{E_r}{m_p} \right) e^{-2\epsilon t} \times
\]

\[
\begin{cases} 
1, & r > r_+ \\
\exp \left[ 2\pi \left( \frac{(M + \sqrt{M^2 - L^2})^2 + L^2}{\sqrt{M^2 - L^2}} \right)(E_r - m\Omega_H) \right], & r_- < r < r_+.
\end{cases}
\] (214)

where \(K_1(\theta, \phi)\) is a positive-definite finite term.

The radial component of \(J\) is

\[
e^r \cdot J = K_2(\theta, \phi) \left( \frac{E_r}{m_p} \right) e^{-2\epsilon t} \times
\]

\[
\begin{cases} 
1, & r > r_+ \\
\exp \left[ 2\pi \left( \frac{(M + \sqrt{M^2 - L^2})^2 + L^2}{\sqrt{M^2 - L^2}} \right)(E_r - m\Omega_H) \right], & r_- < r < r_+.
\end{cases}
\] (215)

where \(K_2(\theta, \phi)\) is a positive-definite finite term.

Upon taking the ratio of the outward flux to the total flux,
\[ \frac{e^r \cdot J_+}{e^r \cdot J_+ - e^r \cdot J_-} = \frac{1}{\exp \left[ 2\pi \left( \frac{(M+\sqrt{M^2-L^2})^2+L^2}{\sqrt{M^2-L^2}}(E_r-m\Omega_H)\right) \right] - 1}, \] (216)

we identify a Bose-Einstein distribution with Hawking temperature

\[ T_H = \frac{1}{2\pi k_B} \left[ \frac{\sqrt{M^2-L^2}}{(M + \sqrt{M^2-L^2})^2 + L^2} \right]. \] (217)

The same results can also be found in [60], [6], [16], [36], [9], and [11].

By running through the calculations for \( \eta_- \) (for \( r_- = M - \sqrt{M^2-L^2} \)), and taking the choice of argument as in the Dirac case, with indicial roots

\[ s = 0, -j \left[ \left( \frac{(M + \sqrt{M^2-L^2})^2 + L^2}{\sqrt{M^2-L^2}}(E - m\Omega_H) \right) \right], \] (218)

we find the Hawking temperature is

\[ T_H = -\frac{1}{2\pi k_B} \left[ \frac{\sqrt{M^2-L^2}}{(M + \sqrt{M^2-L^2})^2 + L^2} \right]. \] (219)

We can also do the analysis for the Kerr solution in oblate spheroidal coordinates [19]. We take the gauge fields in Kerr-Schild form [18]:

\[ \bar{h} = a + a \cdot l l, \] (220)

where \( l = \sqrt{\alpha}(1 + \mathbf{n})\gamma_0 \). In oblate spheroidal coordinates

\[ \{x, y, z\} = \{ L \cosh U \cos V \cos \phi, L \cosh U \cos V \sin \phi, L \sinh U \sin V \}, \] (221)

we have

\[ \mathbf{n} = \frac{1}{L \cosh U} (eU\gamma_0 + L \cos V \sigma_\phi), \] (222)
and
\[ \alpha = \frac{M \sinh U}{L(\cosh^2 U - \cos^2 V)}. \]  \tag{223} 

The horizons of this solution are given by
\[ L \sinh U = M \pm \sqrt{M^2 - L^2}. \]  \tag{224} 

With the separation ansatz
\[ \psi = \mathcal{U}(U)\mathcal{V}(V)e^{jm\phi}e^{-jEt}, \]  \tag{225} 

we successfully obtain separated equations for \( \mathcal{V}(V) \) and \( \mathcal{U}(U) \). These are
\[ (1 - \mu^2)\mathcal{V}'' - 2\mu\mathcal{V}' + \left( C + E^2L^2(1 - \mu^2) + \frac{m^2}{1 - \mu^2} \right) \mathcal{V} = 0, \]  \tag{226} 

where \( C \) is a constant, \( \mu = \sin V \), and

\[
\begin{align*}
&\left( 2M \sinh U - L \cosh^2 u \right) \mathcal{U}'' \\
&+ \left( (2M - L \sinh U \cosh^2 U) - 4jM \sinh U(EL \cosh^2 U + m) \right) \frac{\mathcal{U}'}{\cosh U} \\
&+ \left( 2M \sinh U(m + EL \cosh^2 U)^2 + m^2L \cosh^2 U + E^2L^2 \cosh^4 U \right) \mathcal{U} = 0.
\end{align*}
\]  \tag{227} 

The roots of the indicial equations are the same as those above. Note that \( r = L \sinh U \).

### 5.2 Dirac equation on Kerr black hole metric

We do not discuss Dirac equation on Kerr black hole metric in this paper. We only noted that the roots of the indicial equation obtained for that case are given by
\[ s = 0, -\frac{1}{2} + j(E - m'\Omega_H) \left[ \frac{(M + \sqrt{M^2 - L^2})^2 + L^2}{\sqrt{M^2 - L^2}} \right], \]  \tag{228}
where \( m' \equiv (m + \frac{1}{2}) \) and \( \Omega_H \equiv L/(r_+^2 + L^2) \). This gives the same Hawking temperature as in the Klein-Gordon case. The temperature of the black hole is determined by the imaginary part of the non-zero root. In this case, the root is the same as in the Klein-Gordon case apart from the difference between \( m' \) and \( m \), which does not affect the temperature.

We do not discuss the Kerr-Newman case in this paper since it is straightforward. The only change is to send \( M \to M - \frac{q^2}{2r} \).

6 Conclusion

The results in this paper have shown us that geometric algebra, and specifically spacetime algebra, substantially simplifies the study of the Dirac equation and its consequences. It provides a powerful and flexible tool even for the analysis of strong gravitational backgrounds for which the Gauge Theory of Gravity (GTG) was also employed.

We have studied how geometric algebra and GTG can be applied to black holes. We have considered the Klein-Gordon (spin-0), Dirac (spin-1/2), electromagnetic field (spin-1) equations in the Schwarzschild, Reissner-Nordström, and Kerr black hole backgrounds. The Hawking temperatures were also derived for all cases and compared with the results of ‘standard’ calculations. Remarkably, they agreed. We draw the conclusion that we can use the first-quantised quantum theory instead of the second-quantised one originally used by Hawking \[30\] in treating the black hole evaporation.

In this work it was found that the absolute values of the real part of the non-zero roots in the Klein-Gordon, Dirac, and electromagnetic field cases are the values of the spin of the corresponding particles yet electromagnetism, at least, is completely classical. It seems that geometric algebra can tell that Maxwell’s purely classical theory concerns spin-1 particles (photons) even though the picture of bosonic particles carrying forces is a quantum description! In some sense, the quantal ‘geometric’ nature of the theory is recognised by geometric algebra!

The result for the ‘Hawking temperature’ of the de Sitter cosmological horizon is also interesting. If a thermodynamic system has a negative temperature then that system should be unstable, and tend to expand or collapse. But, if we instead abandon the concept of particles as being observer-independent in this case, we find the usual positive ‘Hawking temperature’. Geometric algebra was also successfully applied to treat the Unruh effect.
In the separation $E = E_r - j\epsilon$ (or $E = E_r + j\epsilon$), the $\epsilon$ term is never equal to zero unless the potential of the event horizon is infinite. In the cases discussed here the potential is finite, so that $\epsilon$ is non-zero. There is not then any problem with the density, which is not normalizable at the event horizon if $\epsilon$ is equal to zero; it is normalizable even at the event horizon. Moreover, even if $\psi$ is singular at the event horizon, the normalization integral is finite because the density is normalizable there. We may conclude that there is no problem with the normalizability.

We compare this situation with the tunnelling effect through a finite potential barrier in quantum mechanics. The particles in a box defined by such a potential will leak out from the box. This means that the imaginary component of $E$ is non-zero, because the flux of particles tunnelling through the barrier is non-zero. Recall that the event horizon of a black hole is the boundary which the particles which come from inside the black hole cannot pass through unless they have superluminal velocity. But this does not mean that the potential of the event horizon itself is infinite. The potential remains finite, and from the viewpoint of quantum mechanics (which also supports the idea of non-locality) it can be penetrated. That is why $\epsilon$ should be non-zero in this case.

According to second-quantised theory, the source of particles in Hawking radiation comes from somewhere very near to, but outside, the event horizon of a black hole, and is due to vacuum polarization. Particles with positive energies escape to infinity and those with negative energies fall into the black hole. The black hole therefore ‘evaporates’. This argument circumvents superluminal effects (the breakdown of causality) at the expense of introducing the vacuum polarization near the event horizon. If we accept the non-locality inherent in Quantum Mechanics and the possibility that causality breaks down in very strong gravitational fields (in which we should employ a quantum gravity approach), we can use the first-quantised theory instead of second-quantised theory when dealing with problems involving sources with very strong gravitational fields, such as black holes, as in the evaporation process studied here.

A hard problem in black hole physics concerns the endpoint of black hole evaporation. The difficulty is that at this late stage the effect of quantum gravity can no longer be ignored. This is a future topic for research within the GTG framework, which has proved very powerful in this work. It will be interesting to see what the application of GTG will say about black holes as well as quantum gravity in the future.
7 Appendix A

Assuming that spherical monogenic \( \psi_l^m \) is an eigenstate of the \( \mathbf{x} \wedge \nabla \) and \( J_3 \) operators (see [20] for details), we find after some calculation that

\[
- \mathbf{x} \wedge \nabla \psi_l^m = l \psi_l^m \quad l \geq 0 \\
J_3 \psi_l^m = (m + \frac{1}{2}) \psi_l^m \quad -1 - l \leq m \leq l
\]  

(229)  

(230)

The highest-\( m \) case satisfies

\[ J_+ \psi_l^m = 0, \]  

(231)

and, after introducing a convenient factor, this is solved by

\[ \psi_l^m = (2l + 1) P_l^m(\cos \theta) e^{i \sigma_3}, \]  

(232)

where the associated Legendre polynomials \( P_l^m(x) \) are given by

\[ P_l^m(x) = \frac{(-1)^m}{2^l l!} (1 - x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l, \]  

(233)

which satisfy the following recursion relations:

\[ (1 - x^2) \frac{dP_l^m(x)}{dx} + mx P_l^m(x) = -(1 - x^2)^{1/2} P_{l+1}^m(x), \]  

(234)

\[ (1 - x^2) \frac{dP_l^m(x)}{dx} - mx P_l^m(x) = (1 - x^2)^{1/2} (l + m)(l - m + 1) P_{l-1}^m(x). \]  

(235)

The lowering operator \( J_- \) acting on \( \psi \) produces

\[ J_- \psi = [-\partial_\theta \psi + \cot \theta \partial_\phi \psi I \sigma_3] e^{-i \sigma_3} - I \sigma_2 (\psi + \sigma_3 \psi \sigma_3), \]  

(236)

and the derivatives applying on \( \psi_l^m \) give

\[
[-\partial_\theta \psi_l^m + \cot \theta \partial_\phi \psi_l^m I \sigma_3] e^{-i \sigma_3} \\
= (2l + 1) [\partial_\theta P_l^m(\cos \theta) - l \cot \theta P_l^m(\cos \theta)] e^{(l-1) i \sigma_3} \\
= (2l + 1) 2l P_{l-1}^m(\cos \theta) e^{(l-1) i \sigma_3},
\]  

(237)
Since
\[ \sigma_\phi = \sigma_2 e^{\phi I_3}, \] (238)
we get after some manipulation the following spherical monogenics:
\[ \psi^m_l = \left[ (l + m + 1)P^m_l (\cos \theta) - P^{m+1}_{l+1} (\cos \theta) I_\sigma_\phi \right] e^{m \phi I_3}, \] (239)
where \( l \) is a non-negative integer, \( m \) runs from \(-(l + 1)\) to \( l \) and \( P^m_l(x) \) are taken to be zero if \(|m| > l\).

In order to get generalise the spherical monogenic functions suitably, we need to generalise the spherical harmonics by including dependence on the third Euler angle \( \chi \). We begin by considering the quantum treatment of rotating bodies, which has an obvious relation to the generalised spherical monogenics. We express the space axis angular momentum operators in terms of the usual Euler angles \( \theta, \phi, \chi \):
\[ j \hat{L}_s^x / \hbar = -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi + \csc \theta \cos \phi \partial_\chi, \]
\[ j \hat{L}_s^y / \hbar = \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi + \csc \theta \sin \phi \partial_\chi, \]
\[ j \hat{L}_s^z / \hbar = \partial_\phi. \] (240)
The corresponding form for the components with respect to the body axis is given by
\[ j \hat{L}_b^1 / \hbar = \sin \chi \partial_\theta - \csc \theta \cos \chi \partial_\phi + \cot \theta \cos \chi \partial_\chi, \]
\[ j \hat{L}_b^2 / \hbar = \cos \chi \partial_\theta + \csc \theta \sin \chi \partial_\phi - \cot \theta \sin \chi \partial_\chi, \]
\[ j \hat{L}_b^3 / \hbar = \partial_\chi. \] (241)
These operators satisfy the commutation relations
\[ [\hat{L}_s^x, \hat{L}_s^y] = j \hbar \hat{L}_s^z; \ [\hat{L}_s^b, \hat{L}_b^b] = -j \hbar \hat{L}_s^b; \ [\hat{L}_s^b, \hat{L}_b^s] = 0, \text{ etc.} \] (242)
so that the space and body operators commute with each other, but their commutators amongst themselves differ in sign.

In geometric algebra, we need a coordinate-free translation of the angular momentum operators in terms of the generators of an active rotation, which will be described by a rotor \( R (R \tilde{R} = 1) \). The effect of \( R \) on any multivector \( M \) is
\[ M \mapsto RM \tilde{R}. \] (243)
The configuration of a rotating body is described by a rotor $R$ giving the directions of the body axes $e_k$ in terms of the space axes $\sigma_k$:

$$e_k = R\sigma_k \tilde{R}. \quad (244)$$

Let us imagine a molecule as our rotating body. We can set up a wavefunction $\psi(R)$ on this configuration space. Note that, under an infinitesimal space rotation in the plane of bivector $B$,

$$\psi(R) \mapsto \psi(e^{\epsilon B/2}R) \approx \psi(R) + \frac{\epsilon}{2} \langle BR\partial_R \rangle \psi \quad (245)$$

where $\partial_R$ is the multivector derivative with respect to $R$. The ‘space’ component of the angular momentum operator for rotations in the plane of the bivector $B$ is therefore

$$\hat{L}^s_B \equiv \frac{1}{2} j \hbar \langle BR\partial_R \rangle. \quad (246)$$

It is easily verified that

$$[\hat{L}^s_A, \hat{L}^s_B] = \left(\frac{1}{2} j \hbar\right)^2 \left(\langle AR\partial_R \rangle \langle BR\partial_R \rangle - \langle B\partial_R \rangle \langle A\partial_R \rangle \right)$$

$$= \left(\frac{1}{2} j \hbar\right)^2 \left(\langle AR\partial_R B \rangle - \langle BR\partial_R A \rangle \right) \quad (247)$$

$$= -\frac{1}{2} \hbar^2 \left(\frac{1}{2} BA - AB\right) R\partial_R = -j \hbar \hat{L}_{A\times B}.$$

The sign of these commutators reflects the fact that we have chosen to work with bivector components rather than the usual vector components.

The ‘body’ components are fixed in the rotating frame, so we send $B \mapsto RB\tilde{R}$ in the previous formula, and define

$$\hat{L}^b_B \equiv \frac{1}{2} j \hbar \langle BR\partial_R \rangle. \quad (248)$$

These operators commute with the space operators:

$$[\hat{L}^s_A, \hat{L}^b_B] = \left(\frac{1}{2} j \hbar\right)^2 \left(\langle AR\partial_R \rangle \langle RB\partial_R \rangle - \langle RB\partial_R \rangle \langle A\partial_R \rangle \right)$$

$$= \left(\frac{1}{2} j \hbar\right)^2 \left(\langle AR\partial_R B \rangle - \langle BR\partial_R A \rangle \right) = 0. \quad (249)$$

A similar calculation yields

$$[\hat{L}^b_A, \hat{L}^b_B] = j \hbar \hat{L}_{A\times B}, \quad (250)$$

46
as expected.

We are now ready to resume seeking the generalised spherical monogenic functions. The unnormalised generalised spherical harmonics related to the functions we need will be written

\[
Y_{l}^{mk}(\mu, \phi, \chi) \propto P_{l}^{mk}(\mu) \exp I(m\phi + k\chi),
\]

where \( P_{l}^{mk}(\mu) \) are the generalised associated Legendre functions. The \( Y_{l}^{mk} \) are eigenfunctions of the space axis and body axis angular momentum operators given by

\[
\hat{L}_{3}^{s}Y_{l}^{mk} = mY_{l}^{mk}, \quad \hat{L}_{k}^{h}Y_{l}^{mk} = kY_{l}^{mk}, \quad \hat{L}_{s}^{\lambda} \hat{L}_{s}^{\lambda}Y_{l}^{mk} = l(l + 1)Y_{l}^{mk}.
\]

The Clifford versions of these operators are

\[
\hat{L}_{3}^{s} \leftrightarrow -\frac{j}{2} (I \sigma_{\lambda} R \partial_{R}), \quad \hat{L}_{k}^{h} \leftrightarrow -\frac{j}{2} (RI \sigma_{\lambda} \partial_{R}),
\]

where \( j \) is the Clifford version of the bivector imaginary operator. The solution is standard and \( P_{l}^{mk}(\mu) \) is given by

\[
P_{l}^{mk}(\mu) = \frac{(-1)^{m}}{2^{m}} (1 - \mu)^{|\alpha|/2} (1 + \mu)^{|\beta|/2} P_{n}^{(\alpha, \beta)}(\mu),
\]

where \( \alpha = |m - k|, \beta = |m + k|, n = l - \alpha/2 - \beta/2, \) and \( P_{n}^{(\alpha, \beta)} \) are the Jacobi polynomials

\[
P_{n}^{(\alpha, \beta)}(x) = \frac{(-1)^{n}}{2^{n} n!} \frac{1}{(1 - x)^{\alpha}(1 + x)^{\beta}} \frac{d^{n}}{dx^{n}} \left\{ (1 - x)^{\alpha+n} (1 + x)^{\beta+n} \right\},
\]

The allowed values of \( l, m, k \) are

\[
l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots; \quad -l \leq m, k \leq l,
\]

which reflects the fact that \( \alpha, \beta, n \) must be non-negative integers. The half-integral angular momentum states have appeared already; the simplest is

\[
Y_{l}^{\pm \frac{1}{2}} = \cos(\theta/2) \exp(I(\phi + \chi)/2).
\]
The normalising constant in this case is

\[
\left( \frac{n!(\alpha + \beta + n)(2n + \alpha + \beta + 1)}{2^{\alpha + \beta + 1}(n + \alpha)!(n + \beta)!} \right)^{1/2},
\]

and the action of the ladder operators is then

\[
\hat{L}_\pm Y^{mk}_l = \pm \sqrt{l(l+1) - m(m \mp 1)} Y^{m-1k}_l
\]

\[
\hat{L}_\pm^b Y^{mk}_l = \pm \sqrt{l(l+1) - k(k \mp 1)} Y^{m k-1}_l
\]

(259)

where the overall sign depends on \(m\) and \(k\).

The first set of generalised monogenic functions \(\Phi^{\alpha\beta \gamma}_{mk}\) we need are the eigenfunctions of \(\hat{J}_s^s, \hat{J}_s^a, \) and \(\hat{L}_b^b,\) with eigenvalues \((l + \frac{1}{2})(l + \frac{3}{2}), m,\) and \(k,\) respectively. These are the \(j = l + \frac{1}{2}\) states; we will find \(j = l - \frac{1}{2}\) afterwards. Since \(Y^{lk}_l\) is already a generalised monogenic, we develop those for \(m < l\) using the \(\hat{J}_-\) operator, and find for \(m < l\) that

\[
\Phi^{\alpha\beta \gamma}_{l,\frac{1}{2}} \propto \pm \sqrt{l(l+1) - m(m - 1)} Y^{m^*}_{l} - I_{\sigma 2}(l - m)Y^{m+k}_{l}
\]

(260)

where the + sign is taken if \(m < k\). The normalisation is obvious (note that \(I_{\sigma 2} e^{I_{\sigma 3}\phi} = I_{\sigma \phi}\)). The range of \(l, m, k\) is

\[
l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots; \quad -l - 1 \leq m \leq l; \quad -l \leq k \leq l,
\]

(261)

so that the total number of states for a given \(j = l + \frac{1}{2}\) is \((2j + 1)(2k + 1)\).

The simplest example of a half-integral state is

\[
\Phi^{1/21/2} = \cos(\theta/2) \exp(I_{\sigma 3}(\phi + \chi)/2).
\]

(262)

Conventional Quantum Mechanics leads us to expect another set of \(j = l - \frac{1}{2}\) monogenics provided that \(l > 0\). The other set of monogenics is

\[
\Phi^{\alpha\beta \gamma}_{l,\frac{1}{2}} \propto \pm (l - m)Y^{m^*}_{l} + I_{\sigma 2} \sqrt{l(l+1) - m(m - 1)} Y^{m+k}_{l}
\]

(263)

The range of \(l, m, k\) is

\[
l = \frac{1}{2}, 1, \frac{3}{2}, \ldots; \quad -l \leq m \leq l - 1; \quad -l \leq k \leq l,
\]

(264)
so that the total number of states for a given $j = l - \frac{1}{2}$ is still $(2j+1)(2k+1)$. The simplest example of a half-integral state is the $j = 0$ doublet $l = \frac{1}{2}$, $m = -\frac{1}{2}$, $k = \pm \frac{1}{2}$.

$$-\Phi_{1/2}^{-1/2} = \sin(\theta/2) \exp(I\sigma_3(-\phi + \chi)/2) + I\sigma_2 \cos(\theta/2) \exp(I\sigma_3(\phi + \chi)/2).$$

From the generalised spherical harmonics we can define the monogenics $\pm \Phi_{l}^{mk}$, which are the eigenfunctions of the space angular momentum operator $\hat{J}^s$ with eigenvalues $j = l \pm \frac{1}{2}$. For the case $k = 0$, there is a simple relation

$$-\Phi_{l+1}^{m} = \pm I\sigma_r + \Phi_{l}^{m} I\sigma_3$$

between the normalised monogenics. The sign in this relation depends on the sign of $(m + \frac{1}{2})$.

The monogenics developed above can handle problems involving the magnetic moment, which involve strong spin-orbit coupling. The conventional wisdom of adding $L + S = \hat{J}$ should then be used. We therefore need to find suitable combinations of $\pm \Phi$ states that are eigenstates of the $(\hat{p} - e\hat{A})^2$ operator. Since $j, m, k$ are still good quantum numbers, we expect to use a combination of $+\Phi_{l}^{mk}$ and $-\Phi_{l+1}^{mk}$ to treat the magnetic monopole case using the form of $\psi^+$ and $\psi^-$ given by

$$\psi^+ = \frac{1}{2\sqrt{l+1}} \left( (\sqrt{l+1+k} + \sqrt{l+1-k}) + \Phi_{l}^{mk} + (\sqrt{l+1+k} - \sqrt{l+1-k}) - \Phi_{l+1}^{mk} \right),$$

and

$$\psi^- = \frac{1}{2\sqrt{l+1}} \left( (\sqrt{l+1+k} + \sqrt{l+1-k}) + \Phi_{l}^{mk} - (\sqrt{l+1+k} - \sqrt{l+1-k}) - \Phi_{l+1}^{mk} \right),$$

which have eigenvalues $(l+1)^2 - k^2 \pm \sqrt{(l+1)^2 - k^2} \equiv \kappa(\kappa \pm 1)$ respectively.

Care is necessary with the notations employed because it can be difficult to avoid overlapping use of the same symbols. However, it is always clear from the context what is meant.
8 Appendix B

We construct the vector spherical harmonics from the scalar spherical harmonics given by

\[ Y_l^m(\theta, \phi) \equiv \sqrt{\frac{2l + 1 (l - m)!}{4\pi (l + m)!}} P_l^m(\cos \theta) e^{im\phi}, \]  

(269)

with \( Y_{l}^{-m} = (-1)^{m}(Y_{l}^{m})^{*} \). The vector harmonics arise from the action of the quantum angular momentum operator

\[ L \equiv -I \mathbf{\times} \nabla = -\mathbf{\times} \nabla, \]  

(270)

where \( \mathbf{\times} \) denotes the Gibbs vector product and \( \mathbf{\times} \) is the STA commutator product. We define the normalised harmonic

\[ X_l^m(\theta, \phi) \equiv \frac{1}{\sqrt{l(l+1)}} L Y_l^m(\theta, \phi), \]  

(271)

whose ‘Cliffordised’ versions of the interrelations are given by

\[ \nabla X_l^m = \frac{1}{r} \sigma_r X_l^m - \frac{\sqrt{l(l+1)}}{r} \sigma_r Y_l^m, \]  

(272)

\[ \nabla(\sigma_r X_l^m) = \frac{1}{r} X_l^m + \frac{\sqrt{l(l+1)}}{r} \sigma_r Y_l^m, \]  

(273)

\[ \nabla(\sigma_r Y_l^m) = \frac{1}{r} X_l^m + \frac{\sqrt{l(l+1)}}{r} Y_l^m. \]  

(274)

The other harmonics we need are related to \( X_l^m \), the full set being

\[ X_l^m, \quad \sigma_r \mathbf{x} X_l^m \text{(transverse)}, \quad \sigma_r Y_l^m \text{(longitudinal)}. \]  

(275)

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