Revisiting the plasma sheath — dust in plasma sheath

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Abstract

In this work, we have considered the formation of warm plasma sheath in the vicinity of a wall in a plasma with considerable presence of dust particles. As an example, we have used the parameters relevant in case of lunar plasma sheath, though the results obtained in this work can be applied to any other physical situation such as laboratory plasma. In the ion-acoustic time scale, we neglect the dust dynamics. The dust particles affect the sheath dynamics by affecting the Poisson equation which determines the plasma potential in the sheath region. We have assumed the current to a dust particle to be balanced throughout the analysis. This makes the grain potential dependent on plasma potential, which is then incorporated into the Poisson equation. The resultant numerical model becomes an initial value problem, which is described by a 1-D integro-differential equation, which is then solved self-consistently by incorporating the change in plasma potential caused by inclusion of the dust potential in the Poisson equation.
I. INTRODUCTION

The physics of sheath in plasmas is an intriguing subject in fundamental plasma physics and has been widely studied. Nevertheless, by no means the study can be considered to be complete. The basic reason behind this is the hugely varied categories of space and laboratory plasmas with disparate physical conditions, where plasma sheaths form. These phenomena range from basic laboratory plasmas (gas discharge devices)\textsuperscript{1-4} to fusion plasma devices (viz. tokamaks)\textsuperscript{5} and from ionospheric plasmas in planetary atmospheres\textsuperscript{6} to astrophysical plasmas such as planetary nebulae\textsuperscript{7,8}. The plasma solitons can also be considered as close cousins of stationary plasma sheaths, which occur in laboratory as well as space plasmas and can be considered as moving sheaths\textsuperscript{9} in a plasma. Insight into sheath dynamics requires the Poisson’s equation to be solved with certain assumptions for the ion and electron densities and only if the relevant parameters are known. Tonks and Langmuir\textsuperscript{10} were among the firsts to be able to reduce the complete sheath equation to a simpler integral equation. Their solution gives the potential distribution within the plasma. Auer\textsuperscript{11}, Caruso and Cavaliere\textsuperscript{12}, and Self\textsuperscript{13} have analyzed the phenomena to find out the potential distribution solution throughout the plasma and in the sheath region. It was Riemann\textsuperscript{14}, who presented a review regarding the sheath formation and the basic features of sheath and its relation to Bohm sheath criterion. As the study of plasma sheath has gained attention due to its practical importance, researchers have been able to establish a very good correlation between the theory and the experiment\textsuperscript{15-18}.

Though by plasmas we usually mean electron-ion plasmas, practically they are never so pure and always have contaminations in the form of dust. In extreme cases, these become dusty plasmas. In almost all plasma sheaths, which usually form due to plasma-wall interactions, some kind of impurities do intrude into the sheath. We in this work, have studied such a phenomena which we refer to as dust in plasma sheath. As we know, dust particles are abundant in all kinds of plasmas and they modify the plasma dynamics by being constituent plasma component which happens as electrons (usually) settle down on the dust surfaces due to their large electrical capacitance. In most cases, the charging of the dust particles is treated in collisionless plasma physics, theoretically with the well known Orbit Motion Limited (OML) theory\textsuperscript{19-21}. As the dust particles are considerably heavier ($m_d \sim 10^{15} m_i$), their dynamical time scale is quite larger than the slowest electron-ion time scale i.e. ion-acoustic time scale. However they do affect the ion-acoustic dynamics by creating a charge imbalance in the electron-ion population. Usually this is through electron depletion, although one can fabricate different physical scenarios in laboratory plasma\textsuperscript{22}, where positively
charged dust particles are present.

Among different space plasma environments, the Moon provides a convenient natural laboratory to study plasma-wall interactions in the space environment. From various satellite observations, we now know that a thin layer of dust (∼ 30 cm) exists around the lunar surface. Lunar Prospector (LP) and Apollo-era missions gave the first hint that there must be a complex and coupled system of plasmas and dust over the Moon’s surface. The ambient plasma and solar ultraviolet (UV) radiation incident on the lunar surface are the main causes that the surface of the Moon becomes electrically charged. On the dayside, photoemission typically dominates and the lunar surface charges to a positive potential, while on the nightside the plasma electrons usually dominate and the surface of the Moon charges to a negative potential. As the lunar surface (and other bodies without any protective environments) is exposed to solar UV radiation, cosmic rays, and constant bombardment by interplanetary bodies, a plasma sheath forms immediately above the lunar surface into which dust particles are injected from the regolith. This provides a very natural environment to study the complex plasma sheath dynamics.

In this work, we have revisited the phenomena of plasma sheath with a considerable presence of dust particles. In the ion-acoustic time-scale, the the dust dynamics can be neglected and the dust particles are considered to be stationary. The dust particles affect the sheath dynamics by affecting the plasma potential. We have self-consistently considered the dust-effects in the Poisson equation, which is then solved to obtain the plasma and dust potentials. As we consider a stationary sheath, we consider the current to the dust particles to be balanced throughout the formation of the plasma sheath. As a result, different currents to the surface of a dust particle become dependent on the grain potential, which is a function of the particle size. The resultant steady-state model is a 4-dimensional differential model, which we have reduced to a one-dimensional integro-differential model, owing to the integrability of the ion continuity and ion momentum equations. Our results show that the dust-effect in the Poisson equation considerably changes the structure of the dust layer, which might form as a result of levitation due to oppositely directed electrostatic and gravitational forces. As a probable application, we have used parameters for lunar plasma environment, though the results obtained in this work could be used in other plasma-wall interactions as well, with appropriate parameters. In Sec.II, we consider our plasma model. In Sec.III, we develop the sheath model. In Sec.IV, we consider the dust-effects on the plasma sheath. In Sec.V, we consider contributions of various forces on a dust particle inside a plasma sheath. Finally, in Sec.VI, we summarise our results.
II. PLASMA MODEL

Our basic equations are due to a collisionless electron-ion plasma with a considerable presence of dust grains. The equations (in 1-D) are ion continuity and momentum equations and electrons are assumed to be Boltzmannian owing to their negligible mass,

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \tag{1}
\]

\[
\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{1}{m_i} \frac{\partial p_i}{\partial x} - \frac{e}{m_i} \frac{\partial \phi}{\partial x}, \tag{2}
\]

\[
n_e = n_0 e^{\phi/T_e}, \tag{3}
\]

where \(\phi\) is the plasma potential and the temperature is expressed in energy unit. We use the ion equation of state as,

\[
p_i \propto n_i^\gamma, \tag{4}
\]

The model is closed by the Poisson equation,

\[
\frac{\epsilon_0}{\epsilon} \frac{\partial^2 \phi}{\partial x^2} = e(n_e - n_i + z_d n_d). \tag{5}
\]

All the symbols have their usual meanings. Note that the presence of dust grains are incorporated into the model through the Poisson equation. The number of charge that resides on the surface of a dust grain is a function of the grain potential \(\phi_g\). We have chosen to normalize the plasma potential with \(T_e/e\), length with Debye length \(\lambda_D\), velocity with ion-thermal velocity \(u_s = \sqrt{T_e/m_i}\), time with \(\lambda_D/u_s\), electron, ion, and dust densities with the respective equilibrium densities \(n_{i0,e0}\), and ion pressure \(p_i\) with \(n_{i0} T_i\). We also denote the ion to electron temperature ratio by \(\sigma = T_i/T_e\). The normalized equations now read as,

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \tag{6}
\]

\[
\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + \frac{\sigma}{n_i} \frac{\partial p_i}{\partial x} = -\frac{\partial \phi}{\partial x}, \tag{7}
\]

\[
n_e = e^{\phi}, \tag{8}
\]

\[
p_i = n_i^\gamma, \tag{9}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = n_e - \delta_i n_i + \delta_d z_d, \tag{10}
\]

where \(\delta_i = n_{i0}/n_{e0}\) and \(\delta_d = z_{d0} n_{d0}/n_{e0}\) are the equilibrium density ratios.
III. SHEATH EQUATIONS

Consider now a stationary plasma sheath\(^{22}\). Far away from the sheath, the plasma potential vanishes and other plasma parameters approaches their bulk (equilibrium) values i.e. as \( x \to \infty \), \( \phi \to 0, u_i \to u_0 \equiv M, p_i \to 1, n_i \to 1, z_d = z_d/z_d0 \to 1, M \) is the Mach number and \( z_d0 = z_d(\phi)|_{\phi=0} \) is the dust charge number in the bulk plasma. The steady state dynamical equations are,

\[
\frac{\partial}{\partial x} (n_i u_i) = 0, \quad (11)
\]

\[
-u_i \frac{\partial u_i}{\partial x} + \sigma \frac{\partial p_i}{\partial x} = -\frac{\partial \phi}{\partial x}. \quad (12)
\]

From the continuity equation, we have

\[
n_i = M/u_i. \quad (13)
\]

Integration of Eq.\((12)\) results the conservation of total energy flux which is a combination of the kinetic flux, enthalpy flux, and electrostatic flux,

\[
\phi = \frac{1}{2} n_i^2 M^2 \left( n_i^2 - 1 \right) + \frac{\gamma \sigma}{(\gamma - 1)} \left( 1 - n_i^{-1} \right). \quad (14)
\]

An expression for \( n_i \) as a function of \( \phi \) can be found from Eqs.\((13,14)\), which must be solved numerically for arbitrary \( \gamma \). For \( \gamma = 3 \) however, we can find an analytical expression for \( n_i(\phi) \) as,

\[
n_i = \frac{1}{2\sqrt{3\sigma}} \left[ \left\{ (M + \sqrt{3\sigma})^2 - 2\phi \right\}^{1/2} - \left\{ (M - \sqrt{3\sigma})^2 - 2\phi \right\}^{1/2} \right]. \quad (15)
\]

The signs in front of the square roots are fixed through the boundary condition on \( n_i \). Expressing ion density as a function of the plasma potential, \( n_i \equiv n_i(\phi) \), the Poisson’s equation can be integrated to get,

\[
\frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 + V(\phi, M, \sigma, \gamma) = 0, \quad (16)
\]

where \( V(\phi, M, \sigma, \gamma) \) is the equivalent Sagdeev potential or pseudo potential\(^{26}\) for a sheath, given by,

\[
V(\phi, M, \sigma, \gamma) = 1 - e^{\phi} + \delta_i \int_0^\phi n_i(\phi) d\phi - \delta_d \int_0^\phi z_d(\phi) d\phi, \quad (17)
\]

where we have neglected the variation in the dust density keeping in view that dust particles can be considered to be stationary in the ion-acoustic time scale. For real solution, \( V(\phi, M, \sigma, \gamma) < 0 \) for all values of \( \phi \). This also determines the minimum velocity for the ions \( (u_0 = M) \) at the sheath boundary (the Bohm condition\(^{14,27}\)). The boundary condition on \( V \) is: at \( \phi = 0, V(\phi) = 0 \). In Fig.1, the pseudo potential for various Mach numbers are shown for \( \phi < 0 \) with \( \delta_d = 0 \) (refer to Sec.3.3) and \( \gamma = 5/3 \).
The shape of the pseudo potential for different Mach number for $\sigma = 0.1$ when dust effects are neglected. Note that we must have $M > 1$ for $\sigma > 0$ in a warm sheath with $\gamma = 5/3$ for real solution of Eq. (16). The allowed region is given by $V < 0$.

A. The modified Bohm condition

The condition for ions to have a minimum velocity at the sheath boundary for which $V(\phi, M, \sigma, \gamma) < 0$, gives rise to the so called Bohm condition\cite{14,27}. In general, for warm ions, we must find out the Bohm condition numerically. For warm ions with $\gamma = 3$ and $\delta_d = 0$, we can find out the Bohm condition by demanding a local maxima for $V_{\gamma=3}(\phi, M, \sigma)|_{\phi \to 0}$.

$$M^2 \geq 1 + 3\sigma,$$  \hspace{1cm} (18)

which reduces to the classical Bohm condition, $M^2 \geq 1$ for $\sigma = 0$\cite{3}.

IV. DUST IN PLASMA SHEATH

A. Wall potential

The electron distribution far away from the wall (or sheath) is given by the usual Maxwell’s distribution\cite{3,22},

$$f_e(v_e) = n_0 \left(\frac{m_e}{2\pi T_e}\right)^{3/2} \exp \left(-\frac{m_e v_e^2}{2 T_e}\right).$$  \hspace{1cm} (19)
The electron current at the wall is due to those electrons which can reach the wall with a minimum velocity $v_{\text{min}}$ to overcome the negative potential at the wall $\phi_w$,

$$v_{\text{min}} = \left( -\frac{2e\phi_w}{m_e} \right)^{1/2},$$

from which we get the electron current density at the wall as,

$$j_e = -e \int_{v_{\text{min}}}^{\infty} \int_{-\infty}^{\infty} v f_e(v) \, dv,$$

$$= -n_0 e \left( \frac{T_e}{2\pi m_e} \right)^{1/2} \exp \left( \frac{e\phi_w}{T_e} \right).$$

The ion current is given by,

$$j_i = en_i u_i \left( \frac{T_e}{m_i} \right)^{1/2} \equiv en_0 u_0 \left( \frac{T_e}{m_i} \right)^{1/2},$$

where $u_0$ is the ion velocity (dimensional) at the sheath boundary (the Mach number). For a stationary sheath, we must have $j_e + j_i = 0$, which determines the wall potential (normalized)\[^{16} \]

$$\phi_w = -2.84 + \ln M,$$

where $M$ must have value for which $V(\phi, M, \sigma, \gamma) < 0$.

### B. Current balance

We assume that the current to the surface of dust particles remain balanced all throughout the plasma including the sheath\[^{22} \]. So at any instant, we must have,

$$I_e + I_i = 0.$$  \hspace{1cm} (24)

We further assume that the dust particles are negatively charged. The ion current to the dust particles (dimensional) can be written as\[^{22} \],

$$I_i = en_i u_i \sigma_i,$$  \hspace{1cm} (25)

where $\sigma_i = \pi b^2$ is the collision cross-section for ions with the dust particles, $b$ being the impact parameter. Conservation of angular momentum in an ion-dust collision leads to the condition,

$$u_i b = u_g r_d,$$  \hspace{1cm} (26)
where \( u_g \) is the grazing velocity for the ions with respect to a collision-event with dust particles and \( r_d \) is the radius of a dust particle. From energy consideration, we further have

\[
\frac{1}{2} m_i u_i^2 = \frac{1}{2} m_i u_g^2 + e \phi_d,
\]

where \( \phi_d = \phi_g - \phi \), \( \phi_g \) being the grain potential. Using Eqs.(25-27) we can write the normalized expression for \( I_i \) as,

\[
I_i = \pi r_d^2 n_i u_i \left( 1 - \frac{2 \phi_d}{u_i^2} \right),
\]

where \( r_d \) is the dust radius measured in terms of Debye length. Through Eq.(13), the above equation becomes

\[
I_i = \pi r_d^2 M \left( 1 - \frac{2 \phi_d}{M^2 n_i^2} \right)
\]

in which we need to substitute \( n_i \) as a function of \( \phi \) by solving Eq.(14). We note that for cold ions, the ion current becomes

\[
I_{icold} = \pi r_d^2 M \left( 1 - \frac{2 \phi_d}{M^2 - 2 \phi} \right),
\]

which can be obtained by substituting \( \sigma = 0 \) in Eq.(14). For negatively charged dust particles, the normalized electron current takes the form

\[
I_e = -\sqrt{8} \pi r_d^2 \delta_m e^{\phi+\phi_d},
\]

where \( \delta_m = \sqrt{m_i/m_e} \approx 43 \). The current balance equation, Eq.(24) is now given by,

\[
M \left( 1 - \frac{2 \phi_d}{M^2 n_i^2} \right) - \sqrt{8} \delta_m e^{\phi+\phi_d} = 0,
\]

Note that for spherical dust particles, the total amount of charge \( Q_d \) on the surface of a dust particle can be written as,

\[
Q_d = C \Delta V = 4\pi \epsilon_0 r_d \phi_d,
\]

where \( C \) is the grain capacitance. The dust potential \( \phi_d \) is the principal solution of Eq.(32) and is shown in Fig.2 (right panel) against the normalized distance \( x \) for cold and warm sheath. The finite ion temperature \( (\sigma > 0) \) affects the dust potential through the ion density \( n_i \). In this figure, the plasma potential \( \phi \) is a solution of the Poisson equation with \( \delta_d = 0 \). Warm ions causes the sheath to expand much like a Debye shielding expanding with temperature. This can be seen from the behaviour of the potentials for warm sheaths, which approaches the bulk value at a larger \( x \) than in cold sheath.
FIG. 2. The plasma potential $\phi$ and dust potential $\phi_d$ as a function of the normalized distance $x$ for cold ($\sigma = 0$) and warm ($\sigma > 0$) sheath with $\gamma = 5/3$. The wall is at $x = 0$.

1. Other contributions to $I_{e,i}$

We note that apart from the electron and ion currents to the dust particles, there are other sources\cite{28,29}. For example in case of dust particles near lunar surface, photoelectron current $I_{ph}$ emitted by the lunar surface and net photoemission current $I_{h\nu}$ emitted by the dust particles may become significant\cite{29}. For $\phi_d < 0$, which usually is the case, we have

$$I_{ph} = \frac{1}{4} J_{ph} \pi r_d^2,$$

$$I_{h\nu} = J_{h\nu} \pi r_d^2 \exp \left( \frac{e\phi_d}{T_{ph}} \right),$$

where

$$J_{h\nu} = -en_{ph} \left( \frac{T_{ph}}{2\pi m_e} \right),$$

is the photoemission current density and $J_{ph} \sim 4.5 \mu A/m$ is the photoelectron current density. However, in this work for the sake of simplicity we have omitted these currents as contributions from both these currents can be incorporated into the ion and electron currents.

C. Negligible dust

When number of dust particles are considerably less than that of ions and electrons ($n_d \ll n_{i,e}$), $\delta_d$ can be neglected and the pseudo potential given by Eq.\cite{17} can be evaluated without any effect from the dust,

$$V(\phi, M, \sigma, \gamma) = 1 - e^\phi + \delta_i \int_0^\phi n_i(\phi) \, d\phi. \quad (34)$$
In such case, for $\gamma = 3$, we can analytically evaluate the above integration in,

$$V_{\gamma=3}(\phi, M, \sigma, \delta_d = 0) = \left(1 - e^\phi\right) - \frac{1}{6\sqrt{3}\sigma} \left[ \left( (M + \sqrt{3}\sigma)^2 - 2\phi \right)^{3/2} - (M + \sqrt{3}\sigma)^3 \right] - \left[ \left( (M - \sqrt{3}\sigma)^2 - 2\phi \right)^{3/2} + (M - \sqrt{3}\sigma)^3 \right].$$

When the ion temperature is also neglected ($\sigma = 0$), we get the classical sheath equation

$$V_{\sigma=0}(\phi, M, \delta_d = 0) = M^2 \left[ \left( 1 - \frac{2\phi}{M^2} \right)^{1/2} - 1 \right] + e^\phi - 1.$$ (36)

The above expression can also be obtained by a series expansion of Eq.(35) or Eq.(17) in the limit $\sigma \to 0$ when $\delta_d = 0$ (equivalently $\delta_i = 1$).

D. With dust effects

When there are considerable presence of dust particles, they begin to affect the pseudo potential through the Poisson equation and we need to use the full form of the Poisson equation, Eq.(10). This procedure requires numerical solution of the plasma model. From Eqs.(10,15,32), the complete numerical model can be summarised as,

$$\frac{d^2\phi}{dx^2} = e^\phi - \delta_i N(\phi) + \delta_d \frac{F(\phi)}{F(0)},$$ (37)

$$2\phi N(\phi)^2 - M^2 \left[N(\phi)^2 - 1\right] + \frac{2\gamma \sigma}{(\gamma - 1)} N(\phi)^2 \left[1 - N(\phi)^{\gamma - 1}\right],$$ (38)

$$\sqrt{8}\delta_m e^\phi + F(\phi) = M \left[ 1 - \frac{2}{M^2} F(\phi) N(\phi)^2 \right],$$ (39)

where $N(\phi) \equiv n_i$, $F(\phi) \equiv \phi_d$ are numerical functions of $\phi$ corresponding to the ion density and dust potential. The function $N(\phi)$ can be evaluated by solving Eq.(38) which when substituted in Eq.(39), can be solved for $F(\phi)$. Poisson equation Eq.(37) then, must be solved self-consistently to evaluate $\phi(x)$, which can be used to generate $\phi_d(x)$ and $n_i(x)$.

Note that the above system of equations can also be cast as a fully differentiable system,

$$\frac{d^2\phi}{dx^2} = e^\phi - \delta_i N(\phi) + \delta_d \frac{F(\phi)}{F(0)},$$ (40)

$$\frac{dN}{dx} = -\frac{N^3}{M^2 + \gamma \sigma N^{\gamma + 1}} \frac{d\phi}{dx},$$ (41)

$$\frac{dF}{dx} = -\frac{d\phi}{dx} + \frac{N}{N^2 + \sqrt{2}\delta_m M e^\phi + F} \left[N \frac{d\phi}{dx} - 2F \frac{dN}{dx}\right].$$ (42)

Many authors use the system as given by Eqs.(40-42) to solve the sheath problem with some initial conditions, usually starting from the bulk plasma. However, we note that starting with initial
conditions in the bulk plasma (i.e. starting from the beginning of the sheath region and integrating toward the wall) is not suitable in many cases as all the derivatives in the above equations vanishes in the bulk plasma, thereby numerically rendering it impossible to evolve. To circumvent this, the standard procedure is to resort to a series expansion at the sheath boundary, which is essentially like pushing the solution through its desired track and one may often arrive at erroneous values of the variables by the time the solution reaches the wall, depending on accuracy of the series expansion and the position of the sheath boundary. Besides, as Eqs.(40-42) is a four-dimensional system, in certain parameter regime the system may become highly sensitive to the initial conditions and one may arrive at a completely different solution from what it was intended to. In our opinion, the differential system is best solved as a multi-dimensional boundary value problem (BVP). However solving a multi-dimensional BVP has its own demerits.

In this work, we choose to solve the system as given by Eqs.(37-39) by self consistently solving Eq.(37) along with Eqs.(38,39). This is done by writing the Poisson equation in terms of the pseudo potential as an integro-differential equation,

$$\frac{1}{\sqrt{2}} \frac{d\phi}{dx} = \left[ e^\phi + (\delta_i - 1) \int_0^\phi F(\psi) d\psi - \delta_i \int_0^\phi N(\psi) d\psi - 1 \right]^{1/2}, \quad (43)$$

where we have substituted $\delta_d = \delta_i - 1$ (from the quasi-neutrality condition). We have solved the above equation with the help of a modified 4th order Runge-Kutta method with adaptive step-size control and evaluating the integrations numerically in each step. The differential equation is solved from the wall ($x = 0$) toward the bulk plasma ($x \to \infty$) with the single initial condition $\phi(x = 0) = \phi_w$ as given by Eq.(23). The evaluation of the integrations requires Eqs.(38,39) to be solved self-consistently, which is done with the help of Newton’s method. As we have the exact initial condition at our disposal, the solution can be built very accurately into the bulk plasma, as far as required. Practically we solve from $x = 0$ (wall) to a large number, say $x \sim 50$ (bulk plasma).

The results are summarised in Fig.3 where we have shown the potentials for both cold ($\sigma = 0$) and warm ($\sigma > 0$) sheaths, with ($\delta_i > 1$) and without ($\delta_i = 1$) dust effects. The results are shown for two sets of values, each for the Mach number $M$, $\sigma$, and $\delta_i$ (for details, see figure caption). Note that a value of $\delta_i = 2$ indicates 50% depletion of the electrons in comparison to ions and a value of $\delta_i = 5$ indicates 80% depletion. We note that in the ion-acoustic time scale, the dust particles does not contribute to the wall potential but they modify the potential in the sheath by just being there as a huge collections of negative charges, requiring more number of ions to neutralise in the sheath away from the wall as compared to a sheath, devoid of dust. As a result, the sheath must
FIG. 3. The variation of plasma and dust potentials with normalized distance from the wall ($x$) with ($\delta_i > 1$) and without dust effects ($\delta_i = 1$), which are indicated in the figures. The blue ($\sigma = 0$) and red ($\sigma > 0$) colored curves are for $\delta_i = 1$. The black ($\sigma = 0$) and orange ($\sigma > 0$) colored curves are for $\delta_i > 1$. The values of $(M, \sigma, \delta_i)$ for the left column are $M = 1.5, \delta_i = (1, 2), \sigma = (0, 0.2)$ and the right column are for $M = 2, \delta_i = (1, 5), \sigma = (0, 0.5)$.

get thicker as we move away from the wall. This effect can be seen in Fig. 3.

V. FORCES ON DUST PARTICLES

There may be several forces which act simultaneously on a dust particle. The most significant of these are electrostatic, gravity, polarisation, ion drag, and neutral drag force. The total force $\mathbf{F}$ on the dust particles (dimensional) can be written as,

$$\mathbf{F} = \mathbf{F}_E + \mathbf{F}_{pol} + \mathbf{F}_g + \mathbf{F}_{ion} + \mathbf{F}_n,$$

(44)
where the forces are respectively electrostatic, polarisation, gravity, ion drag, and neutral drag force. The electrostatic force is given by \( F = Q_d E \). The polarisation force \( F_{\text{pol}} = \nabla(\mathbf{P} \cdot \mathbf{E}) \) can be important when shielding cloud around a dust particle becomes distorted due to polarisation, \( \mathbf{P} \) being the dipole moment. The polarisation force can be neglected except for very dense dusty plasmas\(^{30}\) where there is a possibility of distortion of the charged cloud around a dust particle. The force due to gravity is given by \( m_d \mathbf{g} \), where \( m_d \) is the mass of a dust particle and \( \mathbf{g} \) is the acceleration due to gravity.

The ion drag force is given by\(^{31-33}\)

\[
F_{\text{ion}} = F_{\text{coll}} + F_{\text{Coul}},
\]

where

\[
F_{\text{coll}} = \pi r_d^2 m_i n_i \bar{u} \left( 1 - \frac{2 e \phi_d}{m_i \bar{u}^2} \right), \quad \bar{u} = \left( u_i^2 + u_s^2 \right)^{1/2},
\]

(46)
takes care of the momentum transfer due to ion-dust collision (collection force). The Coulomb scattering part of ion-dust collision is given by

\[
F_{\text{Coul}} = 4 \pi b_\perp^2 m_i n_i \bar{u} \ln \left( \frac{\lambda_D^2 + b_\perp^2}{b_{\text{max}}^2 + b_\perp^2} \right)^{1/2}.
\]

(47)

In the above expression,

\[
b_{\text{max}} = r_d \left( 1 - \frac{2 e \phi_d}{m_i \bar{u}^2} \right)^{1/2},
\]

(48)

\[
b_\perp = \frac{e Q_d}{4 \pi \epsilon_0 m_i \bar{u}^2},
\]

(49)

are the maximum impact parameter and impact parameter for 90° scattering for an ion-dust collision.

The neutral drag force is given by\(^{34}\)

\[
F_n = -\beta m_d \mathbf{u}_d, \quad \beta = \frac{8 \rho}{\pi r_d m_d n_d u_n},
\]

(50)

where \( u_n \) is the thermal velocity of the neutrals, \((\mathbf{u}, m, n)_d\) are the dust velocity, mass, density, \( p \equiv p_i \) is the gas pressure, and \( \beta \) plays the role of friction coefficient. The parameter \( \delta \) depends on details of the dust-neutral collision and estimated as \( \delta = 1.26 \pm 0.13 \)\(^{35}\).

### A. Simplified force model

As we shall see that except the electrostatic force and force due to gravity (assumed to be acting perpendicular to the sheath downward), the other forces are negligibly smaller for the dust in plasma model considered in this paper and can be safely neglected (see below).
We note that in view of the ion-acoustic time scale, the dust particles can be assumed to be stationary and we can assume that $u_d \ll u_i$ and so, the neutral drag force can be neglected. The remaining force is the ion drag force and can be written as (normalized),

$$F_{\text{ion}} = 3R\bar{u}M \left[ \frac{1}{4} r_d^2 + b_\perp^2 \ln \left( \frac{1 + b_\perp^2}{b_{\text{max}}^2 + b_\perp^2} \right)^{1/2} \right],$$

(51)

where the first term inside the ‘[ ]’ is due to $F_{\text{coll}}$ and the second one is due to $F_{\text{Coul}}$ and the normalization factor for $F$ is $(4/3)\pi \lambda_D^3 \rho_d g$. The factor $R$ is a dimensionless parameter representing the ratio of thermal force on the electrons ($F_p = n_0 T_e / \lambda_D$) to the gravitational force on the dust particles ($F_g = \rho_d g$)

$$R = \frac{F_p}{F_g} = \frac{n_0 T_e / \lambda_D}{\rho_d g},$$

(52)

$\rho_d$ being the matter density of the dust particles. In the above expressions, all parameters are normalized as before and the normalized expressions for the impact parameters are

$$b_{\text{max}} = r_d \left( 1 - \frac{2\phi_d}{\bar{u}^2} \right),$$

(53)

$$b_\perp = r_d \frac{\phi_d}{\bar{u}^2},$$

(54)

where

$$\bar{u} = \left( \frac{M^2}{n_i^2} + 1 \right)^{1/2}.$$  

(55)

With $F_{E,g,\text{ion}}$, the normalized expression for total force can be written as,

$$F = -3r_d R\phi_d \left( \frac{d\phi}{dx} \right) - r_d^3 + 3R\bar{u}M \left[ \frac{1}{4} r_d^2 + b_\perp^2 \ln \left( \frac{1 + b_\perp^2}{b_{\text{max}}^2 + b_\perp^2} \right)^{1/2} \right],$$

which is a highly nonlinear function of $r_d$. If we now assume that $F_{\text{ion}}$ is negligible, the total force perpendicular to the plasma sheath can be written as,

$$F = r_d \left( r^2_{\text{bal}} - r_d^2 \right),$$

(56)

where

$$r_{\text{bal}} = \left[ -3R\phi_d \left( \frac{d\phi}{d\xi} \right) \right]^{1/2}.$$  

(57)

The quantity $r_{\text{bal}}$ indicates the normalized radius of the dust particles for which the total force on the dust particles become zero, which leads to levitation of the dust particles in the plasma sheath. A plot of $r_{\text{bal}}$ is shown in Fig.4. As a prototype case, we have considered plasma sheath over lunar surface and have taken various parameters as $n_{i0} = 5 \times 10^6 \text{ m}^{-3}$, $\rho_d = 1000 \text{ kg/m}^3$, $T_e =$
FIG. 4. Distribution of size of levitated dust particles ($r_{bal}$) (top panel) and ion density (bottom panel) with normalized distance ($x$) from the wall in the sheath region. The parameters and color-codes are same as in Fig. 3.

50 eV, $g = 1.6 \text{ m/s}^2$. If we now calculate the magnitudes of different forces on a dust particle for these parameters assuming an average radius for dust particle to be $\sim 10^{-6} \mu m$, at $x = 0$, the forces (normalized) are $F_E \sim 2.8 \times 10^{-22}$, $F_g \sim -7.9 \times 10^{-23}$, and $F_{ion} \sim 2.2 \times 10^{-29}$ and we can see that

$$F_{ion} \ll F_{E,g},$$

so that the total force on a dust particle can be approximated as given by Eq.(56). We note that although we have calculated the forces for a particular set of parameters, the above condition remains valid for any values of parameters for this particular model of dust in plasma. As the size of a dust particle increases it acquires more negative charges on its surface as well as also get heavier. While more number charges (negative in this case) causes the dust particle to repelled from the
FIG. 5. The net force $F$ on the dust particles are shown in the top panels for various parameters as indicated in the figure. The corresponding locus of the point $F = 0$ in the $x$-$r_d$ plane is shown in the lower panels. Column-wise they correspond to same sets of parameters. For details, see text.

wall, the increasing weight causes it to be attracted toward the wall, assuming the gravity to be acting perpendicular to the wall downward. So, the exact position where the dust particle remain levitated depends on its weight. In this particular case, the gravity overtakes the electrostatic force and larger dust particles of size $r_d \gtrsim 1.75 \mu\text{m}$ (for $\delta_i = 2$) and $r_d \gtrsim 2 \mu\text{m}$ (for $\delta_i = 5$) always settles down on the wall (see Fig.4). At the same time, as dust particles gets more laden with negative charges, the increasing electrostatic force expels the dust particle away from the wall and the levitation distance from the wall increases. As can be seen from Fig.4, the levitation distance from the wall increases from $2.8\lambda_D$ to $3.6\lambda_D$ by about 30% for a dust particle of 1 \mu m size when $\delta_i$ increases to 5 from 2. The ion density also follows the dust distribution.

The net force $F$ on a dust particle is shown in Fig.5 (top panel). All parameters are indicated in the figure. As can be seen, the force passes through the zero axis indicating the position in
FIG. 6. Variation of $r_{d\text{min}}^m$ with $M$ for cold and warm sheath with and without dust effects. All parameters are same as in earlier figures.

the sheath where a particular dust particle ($r_d = 1 \mu\text{m}$) levitates. The Mach numbers are 1.5 and 5. The top panel on the left is for $\sigma = 0$ and 0.2 and on the right is for $\sigma = 1$ (equal ion and electron temperature). The blue curves on both panels are for $\delta_i = 1$ and the orange curves are for $\delta_i = 10$ signifying 90% depletion of electron on the surface of dust particles. The lower panels correspond to the respective parameters of the top panels but show the locus of the point where the net force $F = 0$ in the $x$-$r_d$ plane. From this figure, the effect of higher dust concentration in the plasma sheath can be clearly seen. For both low and high Mach numbers, the behaviour of $F$ with $x$, remains almost same when the dust effect is not considered. For lower Mach numbers (left panes), hot ions ($\sigma = 0.2$) causes the sheath to expand which pushes the dust particles away from the wall which is well understood. However for high Mach numbers (right panels), increase of dust particles inside the sheath prevents this push and they tend to settle down closer to the wall, even when the ions are now as hot as the electrons ($\sigma = 1$).

Levitation of dust particles in the sheath is due to the balance of the electrostatic force and the force due to gravity. Heavier dust particles always settle down on the wall irrespective of the repulsive electrostatic force on them which gives rise to a minimum size $r_{d\text{min}}^m$, for the dust particles for levitation. This can be calculated by following the locus of the maxima of the the net force $F(x)$ as a function of $x$ in the $x$-$r_d$ plane. From Eq.(56), the maximum of $F(x)$ is given by,

$$\frac{d}{dx}r_{\text{bal}} = 0,$$

(59)
which translates to the condition,
\[
\frac{d}{dx} \left( \phi_d \frac{d\phi}{dx} \right) = \phi_d' \phi' + \phi_d \phi'' = 0,
\]
where the prime denotes derivative with respect to \(x\). The above equation can be solved for \(x\) resulting \(x_{\text{max}}\) such that \(F'(x_{\text{max}}) = 0\). We then need to solve the equation
\[
F(r_d)|_{x_{\text{max}}} = 0
\]
for \(r_d\), which determines \(r_d^{\text{min}}\). In Fig. 6 we have shown the numerically calculated \(r_d^{\text{min}}\) as a function of the Mach number. Though more number of dust particles in the sheath restricts the size of dust particles nearer the wall, it nevertheless pushes up the minimum size of the dust particles, which can be levitated in the sheath. As a result, due to the dust effecting the plasma potential inside the sheath region, we can expect a significant reduction of the distance away from the wall up to which we have dust levitation, but the levitated dust particles should now have an extended distribution of size as \(r_d^{\text{min}}\) increases.

VI. CONCLUSIONS

In this work, we have considered the formation of warm plasma sheath in the vicinity of a wall in an environment with a considerable presence of dust particles. As an example, we have used the parameters relevant in case of lunar plasma sheath, though the results obtained in this work could be applied to any other physical situation such as laboratory plasma. In the ion-acoustic time scale, we have neglected the dust dynamics and thus the dust particle provides a stationary background to the electron-ion plasma. However, the dust particles do affect the sheath dynamics by affecting the Poisson equation which determines the plasma potential in the sheath region. We have assumed that the current to a dust particle remains balanced throughout the sheath formation process. This makes the grain potential dependent on plasma potential, which is then incorporated into the Poisson equation. The resultant numerical model becomes an initial value problem, which is described by a 1-D integro-differential equation, which is then solved self-consistently by incorporating the change in plasma potential caused by inclusion of the dust potential in the Poisson equation. The initial value is given by the plasma potential at the wall determined by the Mach number (equilibrium ion thermal velocity).

We have shown that the presence of massive dust particles inside the sheath region considerably affects the plasma sheath. As dust particles are introduced into the sheath, which may happen
naturally in case of space plasma environments and through plasma-wall interactions in laboratory plasmas, they become sources of huge collection of negative charges (in this case) requiring more number of ions to neutralise the negative charges while the wall potential remain same. As a result, the sheath becomes thicker. The warm ions helps this process of thickening much like a Debye shielding expanding due to temperature.

We have also considered the phenomenon of levitation of dust particles inside the plasma sheath, which happens due to counteracting electrostatic force and force due to gravity (assuming gravity to be acting perpendicularly downward tom the sheath). Though there may be many other sources of force on a dust particle, they can be neglected in most cases. Bigger dust particles acquire more charges on their surfaces though at the same time get heavier. As a result they may get levitated inside the sheath although the levitation distance (as measured in terms of distance from the wall) may differ depending on the constituent of dust particles. For low Mach numbers, hot ions causes the sheath to expand which pushes the dust particles away from the wall. This effect is like the expansion of Debye shielding cloud expanding due to temperature. For high Mach numbers, however, increase of dust particles inside the sheath prevents this push and they tend to settle down closer to the wall, even when the ions are as hot as electrons. Heavier dust particles always settle down on the wall, which gives rise to a minimum dust size which is required for levitation. Increasing dust density in the plasma sheath restricts the size of levitating dust particles nearer the wall. However, on the average, it pushes up the minimum levitation size. As a result, while due to the effect of dust potential affecting the floating plasma potential in the sheath causes a significant reduction of the levitation distance, the dust distribution becomes broader.

In summary, our analysis provides a generic framework for dust in plasma sheath which could be paramerized to suit laboratory experiments as well as different space and astrophysical environments. To analyse the detailed distribution of levitating dust particles in a plasma sheath in a particular physical scenario, we need to input the respective physical parameters into our analysis. The levitating dust particles in plasma sheath may very well affect results of various plasma processing experiments and the knowledge of detailed dust distribution in plasma sheath is desired to tailor particular needs of experiments.
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