A note on numerical ranges of tensors

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**ABSTRACT**
Theory of numerical range and numerical radius for tensors is not studied much in the literature. Ke et al. [Linear Algebra Appl. 508 (2016), 100-132: MR3542984] introduced first the notion of the numerical range of a tensor via the $k$-mode product. However, the convexity of the numerical range via the $k$-mode product was not proved by them. In this paper, the notion of numerical range and numerical radius for even-order square tensors via the Einstein product are introduced first. Using the notion of the numerical radius of a tensor, we provide some sufficient conditions for a tensor to be unitary. The convexity of the numerical range is also proved. We also provide an algorithm to plot the numerical range of a tensor. Furthermore, some properties of the numerical range for the Moore–Penrose inverse of a tensor are discussed.

**1. Introduction**

The concepts of numerical range and numerical radius of matrices and operators have been studied extensively over the last few decades as they are useful in studying and understanding the role of matrices and operators [1–4] in applications such as numerical analysis and differential equations [5–12]. The numerical radius is frequently employed as a more reliable indicator of the rate of convergence of iterative methods than the spectral radius [5,7]. Recently, Ke et al. [13] introduced tensor numerical ranges using tensor inner products and tensor norms via $k$-mode product. These have the same properties as those of the numerical ranges of matrices, except the normality, projection, and unitary invariance properties.

The numerical range of an $n \times n$ matrix $A$ is the set of complex numbers defined as follows:

$$W(A) = \{ \langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1 \},$$

where $\langle x, y \rangle = y^*x$ for $x, y \in \mathbb{C}^n$ and $\|x\| = \langle x, x \rangle^{1/2}$. Note that the notion of the numerical range of a matrix is applicable for square matrices, and it uses the conjugate transpose. So, to extend the notion of the numerical range of a matrix to tensor case, we need a square tensor and the notion of tensor transpose.

Tensors are generalizations of scalars (that have no index), vectors (that have exactly one index), and matrices (that have exactly two indices) to an arbitrary number of indices.

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An $N^{th}$-order tensor is an element of $\mathbb{F}^{I_1 \times \cdots \times I_N}$, which is the set of order $N$ tensors. Here $I_1, I_2, \ldots, I_N$ are dimensions of the first, second, ..., $N^{th}$-mode/way, respectively. The order of a tensor is the number of modes present in it. Thus, a zero-order tensor is a scalar, a first-order tensor is a vector while a second-order tensor is a matrix. Higher-order tensors are tensors of order three or higher. If $N$ is even, then it is an even-order tensor wise it is an odd-order tensor. Further, if $N = m$ and $I_1 = I_2 = \ldots = I_m = n$, then the tensor is said to be $m^{th}$-order $n$-dimensional tensor.

Higher-order tensors are denoted by calligraphic letters like $\mathcal{A}$. In particular, $a_{ijk}$ denotes an $(i, j, k)^{th}$ element of a third-order tensor $\mathcal{A}$. Different parts of a third-order tensor are shown in Figure 1. In particular, consider a third-order tensor of dimension $3 \times 3 \times 3$ as in Figure 2. Then, there are 3 number of frontal slices, 3 number of horizontal slices, 3 number of lateral slices and 27 number of tuber fibres.

For simplicity, let us denote $I_{1 \ldots N} := I_1 \times I_2 \times \cdots \times I_N$. The notation $a_{i_1 \ldots i_N}$ (with $1 \leq i_j \leq I_j, j = 1, \ldots, N$) represents an $(i_1, \ldots, i_N)^{th}$ element of an $N^{th}$-order tensor $\mathcal{A} \in \mathbb{F}^{I_{1 \ldots N}}$. For a tensor $\mathcal{A} \in \mathbb{F}^{I_{1 \ldots N}}$, the notation $\mathcal{A}(\cdot, \cdot, \ldots, ; k), k = 1, 2, \ldots, I_N$ represents a $(N - 1)^{th}$-order tensor in $\mathbb{F}^{I_{1 \ldots (N-1)}}$ which is extracted when the last index is fixed and is called frontal slice. A fibre is identified by fixing each index except one. For a tensor $\mathcal{A} \in \mathbb{F}^{I_{1 \ldots N}}$, the notation $\mathcal{A}(i_1, i_2, \ldots, i_{N-1}, :)$ represents a $1^{st}$-order tensor in $\mathbb{F}^{I_N}$ which is extracted by fixing each index except the $N^{th}$-index, and is called mode-$N$ fibre. The higher-order analogue of matrix rows and columns is the fibres.

Figure 1. Different parts of a third-order tensor. (a) Frontal slice. (b) Horizontal slice. (c) Lateral slice. (d) Tube fibre.

Figure 2. A third-order tensor of dimension $3 \times 3 \times 3$.  

|                  | $\mathcal{A}(\cdot, \cdot, 1)$ | $\mathcal{A}(\cdot, \cdot, 2)$ | $\mathcal{A}(\cdot, \cdot, 3)$ |
|------------------|-------------------------------|-------------------------------|-------------------------------|
| $a_{111}$        | $a_{121}$                     | $a_{131}$                     | $a_{113}$                     |
| $a_{211}$        | $a_{221}$                     | $a_{231}$                     | $a_{213}$                     |
| $a_{311}$        | $a_{321}$                     | $a_{331}$                     | $a_{313}$                     |
|                  | $a_{122}$                     | $a_{122}$                     | $a_{123}$                     |
|                  | $a_{222}$                     | $a_{222}$                     | $a_{223}$                     |
|                  | $a_{322}$                     | $a_{322}$                     | $a_{323}$                     |
|                  | $a_{132}$                     | $a_{132}$                     | $a_{133}$                     |
|                  | $a_{232}$                     | $a_{232}$                     | $a_{233}$                     |
|                  | $a_{332}$                     | $a_{332}$                     | $a_{333}$                     |
Let \( \mathbf{A} \in \mathbb{F}^{I_1 \times I_2 \times \cdots \times I_M} \) be a tensor, and let \( \pi \) be a permutation in \( S_M \) except the identity permutation, where \( S_M \) represents the permutation group over the set \( \{1, 2, \ldots, M\} \). Then the \( \pi \)-transpose of the tensor \( \mathbf{A} \) is defined as

\[
\mathbf{A}^{T\pi} = (a_{i_1 \pi(1)j_1 \pi(2) \cdots i_{\pi(M)}j_{\pi(M)}}) \in \mathbb{F}^{I_{\pi(1)} \times I_{\pi(2)} \times \cdots \times I_{\pi(M)}}.
\]  

(2)

Thus, there are \( M! - 1 \) possible transposes associated with the tensor \( \mathbf{A} \in \mathbb{F}^{I_1 \times I_2 \times \cdots \times I_M} \).

In particular, for \( \mathbf{A} \in \mathbb{F}^{I_1 \times I_2 \times \cdots \times I_M \times I_1 \times I_2 \times \cdots \times I_N} \) and \( \pi \in S_{M+N} \) such that \( \mathbf{A}^{T\pi} = (a_{i_1j_2 \cdots j_Ni_1j_2 \cdots i_M}) \in \mathbb{F}^{I_1 \times I_2 \times \cdots \times I_N \times I_1 \times I_2 \times \cdots \times I_M} \), then it is simply written as \( \mathbf{A}^T \). Similarly, for a complex tensor \( \mathbf{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_M \times I_1 \times I_2 \times \cdots \times I_N} \), we denote the conjugate transpose of \( \mathbf{A} \) by \( \mathbf{A}^H \), and defined as \( \mathbf{A}^H = (\overline{a}_{i_1j_2 \cdots j_Ni_1j_2 \cdots i_M}) \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N \times I_1 \times I_2 \times \cdots \times I_M} \). In this paper, whenever we write \( \mathbf{A}^H \) or \( \mathbf{A}^T \) for \( M + N \) or \( 2M \) order tensor, then it is always with respect to partition after \( M \)-modes. Furthermore, if \( \mathbf{A} \in \mathbb{F}^{I_1 \times I_2 \times \cdots \times I_M} \), then \( \mathbf{A}^T = (a_{i_1i_2 \cdots i_M}) \in \mathbb{F}^{I_1 \times I_2 \times \cdots \times I_M} \).

There are two ways to define a square tensor. One when each modes are of equal size, i.e. \( n \times n \times \cdots \times n \), and another when the first \( N \) modes are repeated in the same order, i.e. \( I_1 \times I_2 \times \cdots \times I_N \times I_1 \times I_2 \times \cdots \times I_N \). Recently, Ke et al. [13] extended the notion of the numerical range of a matrix to the former type of square tensor case. They considered tensor numerical ranges based on inner products via \( k \)-mode product which may not be convex in general (see Example 1, [13]). Pakmanesh and Afshin [14], continued the same study for even-order tensors. Note that, a square tensor \( \mathcal{D} = (d_{i_1 \cdots i_Nj_1 \cdots j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) is said to be a diagonal tensor if \( d_{i_1 \cdots i_Nj_1 \cdots j_N} = 0 \) whenever \( (i_1, \ldots, i_N) \neq (j_1, \ldots, j_N) \). A diagonal tensor \( \mathcal{D} = (d_{i_1 \cdots i_Nj_1 \cdots j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) such that \( d_{i_1 \cdots i_Nj_1 \cdots j_N} = \alpha_{i_1 \cdots i_N} \) whenever \( (i_1, \ldots, i_N) = (j_1, \ldots, j_N) \) and \( d_{i_1 \cdots i_Nj_1 \cdots j_N} = 0 \) otherwise, is denoted by \( \mathcal{D} = \text{diag}(\alpha_{i_1 \cdots i_N}, \ldots, \alpha_{i_1 \cdots i_N}, \ldots, \alpha_{i_1 \cdots i_N}) \), where \( \alpha_{i_1 \cdots i_N} \in \mathbb{C} \), \( 1 \leq i_k \leq I_k \), \( k = 1, 2, \ldots, N \).

The first objective of this paper is to study the numerical range of a tensor based on inner product, and its convexity, via Einstein product.

The Einstein product [15] \( \mathbf{A} \ast_N \mathbf{B} \in \mathbb{C}^{I_1 \times \cdots \times I_{N-1} \times K_1 \times \cdots \times K_N} \) of tensors \( \mathbf{A} \in \mathbb{C}^{I_1 \times \cdots \times K_1} \) and \( \mathbf{B} \in \mathbb{C}^{K_1 \times \cdots \times I_{N-1}} \) is defined by the operation \( \ast_N \) via

\[
(\mathbf{A} \ast_N \mathbf{B})_{i_1 \cdots i_Mj_1 \cdots j_L} = \sum_{k_1 \cdots k_N} a_{i_1 \cdots i_Mk_1 \cdots k_N} b_{k_1 \cdots k_Nj_1 \cdots j_L}.
\]

The associative law for the Einstein product holds. In the above formula, if \( \mathbf{B} \in \mathbb{C}^{K_1 \times \cdots \times N} \), then \( \mathbf{A} \ast_N \mathbf{B} \in \mathbb{C}^{I_1 \times \cdots \times M} \) and

\[
(\mathbf{A} \ast_N \mathbf{B})_{i_1 \cdots i_M} = \sum_{k_1 \cdots k_N} a_{i_1 \cdots i_Mk_1 \cdots k_N} b_{k_1 \cdots k_N}.
\]

This product is used in the study of the theory of relativity [15] and in the area of continuum mechanics [16]. Let \( \mathbf{A} \in \mathbb{R}^{m \times n} \) and \( \mathbf{B} \in \mathbb{R}^{n \times l} \). Then the Einstein product \( \ast_1 \) reduces to the standard matrix multiplication as

\[
(\mathbf{A} \ast_1 \mathbf{B})_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.
\]
A natural question arises now which is ‘why do we study the notion of the numerical range of a tensor based on the inner product via the Einstein product?’. We next point out some of the reasons for this.

- The set of invertible tensors [17] forms a group under the Einstein product.
- The tensor formulation via the Einstein product preserves the low-rank structure in the solution and the right-hand side. Such as, in high-dimensional Poisson problem, the solution and the right-hand side, both represented as \( n \times n \times \cdots \times n \) data arrays.
- The matrix unfolding of a tensor may give rise to larger bandwidths than the original tensor which increases the number of operations and storage locations. For example, the Laplacian matrices in high dimensions have larger bandwidths than the Laplacian tensors.
- The convexity of the numerical range of a tensor via the Einstein product can be proved.

We refer to [18] for further advantages of studying theory of tensors via the Einstein product.

We define the numerical range of a tensor via the Einstein product by intending that it will contain the tensor eigenvalues defined in the sense of Definition 2.3 of [19]. In 2019, Liang and Zheng [19] introduced the notion of eigenvalue of a tensor via the Einstein product as following.

**Definition 1.1 (Definition 2.3, [19]):** Let \( A \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N} \) be a given tensor. If a complex number \( \lambda \) and a non-zero tensor \( \mathbf{X} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N} \) satisfy

\[
A \ast_N \mathbf{X} = \lambda \mathbf{X},
\]

then we say that \( \lambda \) is an eigenvalue of \( A \), and \( \mathbf{X} \) is the eigentensor with respect to \( \lambda \).

The set of all the eigenvalues of \( A \) is denoted by \( \sigma(A) \). The spectral radius of the tensor \( A \), denoted by \( \rho(A) \), is defined as \( \rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} \). Furthermore, the positive square roots of eigenvalues of \( A^H \ast_N A \) are called singular values of \( A \).

It is well known that the numerical range of a matrix contains its eigenvalues. So, the study of numerical ranges is useful in designing fast algorithms for the calculation of its eigenvalues. The reason for this is revealed in the following well-known theorem and corollary (for proofs and discussions, we refer an interested reader to [3,4,9,20,21]).

**Theorem 1.2:** Let \( M_n(\mathbb{C}) \) be the set of all square matrices of order \( n \) over \( \mathbb{C} \) and let \( w(A) = \max\{|z| : z \in W(A)\} \) for a matrix \( A \in M_n(\mathbb{C}) \). Then,

1. \( (1/2)\|A\|_2 \leq w(A) \leq \|A\|_2 \), where \( \| \cdot \|_2 \) denotes the spectral (operator) norm induced on \( M_n(\mathbb{C}) \).
2. For every positive integer \( m \), \( w(A^m) \leq w(A)^m \).

**Corollary 1.3:** Let \( w(A) = \max\{|z| : z \in W(A)\} \) for a matrix \( A \in M_n(\mathbb{C}) \). Then, for any positive integer \( m \),

\[
w(A^m)^{1/m} \leq \|A^m\|_2^{1/m} \leq 2^{1/m} w(A^m)^{1/m} \leq 2^{1/m} w(A).
\]
The second objective of this paper is to develop algorithms to compute the numerical ranges of tensors, which will be useful in designing faster algorithms for the calculation of its eigenvalues.

In 2013, Brazell et al. [17] first introduced the notion of the ordinary inverse of a tensor via the Einstein product. For $A \in \mathbb{C}^{I_1 \ldots I_N \times I_1 \ldots I_N}$, if there exists a tensor $\mathcal{X} \in \mathbb{C}^{I_1 \ldots I_N \times I_1 \ldots I_N}$ such that $A *_N \mathcal{X} = I = \mathcal{X} *_N A$, then the tensor $\mathcal{X}$ is called inverse of the tensor $A$, and is denoted by $A^{-1}$. In 2016, Sun et al. [22] introduced formally a generalized inverse called the Moore–Penrose inverse of an even-order tensor via the Einstein product. The authors [22] then used the Moore–Penrose inverse to find the minimum-norm least-squares solution of some multilinear systems. Panigrahy and Mishra [23], Stanimirovic et al. [24], and Liang and Zheng [19] independently improved the definition of the Moore–Penrose inverse of an even-order tensor to a tensor of any order via the same product. The definition of the Moore–Penrose inverse of an arbitrary order tensor is recalled below.

**Definition 1.4 (Definition 1.1, [23]):** Let $A \in \mathbb{R}^{I_1 \ldots I_N \times J_1 \ldots J_M}$. The tensor $\mathcal{X} \in \mathbb{R}^{J_1 \ldots J_M \times I_1 \ldots I_N}$ satisfying the following four tensor equations:

\begin{align*}
A *_M \mathcal{X} *_N A &= A; \quad (4) \\
\mathcal{X} *_N A *_M \mathcal{X} &= \mathcal{X}; \quad (5) \\
(A *_M \mathcal{X})^H &= A *_M \mathcal{X}; \quad (6) \\
(\mathcal{X} *_N A)^H &= \mathcal{X} *_N A, \quad (7)
\end{align*}

is defined as the Moore–Penrose inverse of $A$, and is denoted by $A^\dagger$.

The third objective of this paper is to investigate some properties of the numerical range of the Moore–Penrose inverse of a tensor.

The rest of this paper is structured as follows. In Section 2, the notion of the numerical range of a tensor is introduced based on inner product via the Einstein product. Also, the convexity of numerical range is verified. In Section 3, an algorithm to plot the boundary of the numerical range of a tensor is derived. In Section 4, the notion of the numerical radius is used to verify the unitary property of a tensor. Finally, Section 5 gathers some properties of the numerical range of the Moore–Penrose inverse of a tensor.

### 2. Numerical range of a tensor

In this section, the numerical range of a tensor via the Einstein product is introduced. Also, the spectral containment and the convexity of the numerical range of a tensor are shown.

For two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1 \ldots N}$, we define an inner product $\langle \mathcal{X}, \mathcal{Y} \rangle = \mathcal{Y}^H *_N \mathcal{X}$, and a norm induced by this inner product as $\| \mathcal{X} \| = (\mathcal{X}, \mathcal{X})^{1/2}$. A tensor $\mathcal{X} \in \mathbb{C}^{I_1 \ldots N}$ is said to be a unit tensor if $\| \mathcal{X} \| = 1$. According to (1), it is natural to consider the following generalization.

**Definition 2.1:** The numerical range of an even-order square tensor $A \in \mathbb{C}^{I_1 \ldots N \times I_1 \ldots N}$ that we denote it by $W(A)$, is defined as

\[ W(A) = \{ \langle A *_N \mathcal{X}, \mathcal{X} \rangle : \mathcal{X} \text{ is a unit tensor in } \mathbb{C}^{I_1 \ldots N} \}. \]
With some elementary calculations, it can be shown that
\[
W(\mathcal{A}) = \left\{ \frac{\langle \mathcal{A} \ast_N \mathcal{X}, \mathcal{X} \rangle}{\|\mathcal{X}\|} : \mathcal{O} \neq \mathcal{X} \in \mathbb{C}^{I_{1...N}} \right\},
\]
(9)
where $\mathcal{O}$ is the null tensor having all the entries zero.

Note that, in the above Definition 2.1 when $N = 1$, it coincides with the matrix numerical range defined in (1). The spectrum of a tensor is always contained in its numerical range, and is shown in the next theorem.

**Theorem 2.2:** The spectrum of $\mathcal{A}$ always lies in $W(\mathcal{A})$, i.e. $\sigma(\mathcal{A}) \subseteq W(\mathcal{A})$.

**Proof:** Let $\mathcal{X}$ be a unit eigentensor corresponding to an eigenvalue $\lambda$ of $\mathcal{A}$, i.e. $\mathcal{A} \ast_N \mathcal{X} = \lambda \mathcal{X}$ with $\|\mathcal{X}\| = 1$. Thus, $\lambda = \langle \lambda \mathcal{X}, \mathcal{X} \rangle = \langle \mathcal{A} \ast_N \mathcal{X}, \mathcal{X} \rangle \in W(\mathcal{A})$. Since $\lambda$ is an arbitrary eigenvalue of $\mathcal{A}$, so $\sigma(\mathcal{A}) \subseteq W(\mathcal{A})$. \hfill $\blacksquare$

Note that when $\mathcal{A} = \alpha \mathcal{I}$, where $\mathcal{I} \in \mathbb{C}^{I_{1...N} \times I_{1...N}}$ is the identity tensor whose entries are 1 when $(i_1, \ldots, i_N) = (j_1, \ldots, j_N)$, otherwise 0, we have $\sigma(\mathcal{A}) = \{ \alpha \} = W(\mathcal{A})$. The following example shows that $W(\mathcal{A})$ contains some elements which are not in $\sigma(\mathcal{A})$.

**Example 2.3:** Consider a tensor $\mathcal{A} \in \mathbb{C}^{3 \times 2 \times 3 \times 2}$ such that

| $\mathcal{A}(\cdot; 1,1)$ | $\mathcal{A}(\cdot; 1,2)$ | $\mathcal{A}(\cdot; 2,1)$ | $\mathcal{A}(\cdot; 2,2)$ | $\mathcal{A}(\cdot; 3,1)$ | $\mathcal{A}(\cdot; 3,2)$ |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1                        | 0                        | 0                        | 0                        | 0                        | 0                        |
| 0                        | 0                        | 0                        | 0                        | 0                        | 0                        |
| 0                        | 0                        | 0                        | 0                        | 0                        | 0                        |

Here, $\sigma(\mathcal{A}) = \{1, 2, 5, 8, 9, 11\}$. Let $\mathcal{X} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \in \mathbb{R}^{3 \times 2}$. Then, $\|\mathcal{X}\| = 1$ and hence $\langle \mathcal{A} \ast_N \mathcal{X}, \mathcal{X} \rangle = 6 \in W(\mathcal{A})$. But, $6 \notin \sigma(\mathcal{A})$.

**Theorem 2.4:** Let $\mathcal{A} \in \mathbb{C}^{I_{1...N} \times I_{1...N}}$ and $\alpha, \beta \in \mathbb{C}$. Then $W(\alpha \mathcal{A} + \beta \mathcal{I}) = \alpha W(\mathcal{A}) + \beta$.

**Proof:** From the Definition 2.1, we have
\[
W(\alpha \mathcal{A} + \beta \mathcal{I}) = \left\{ \langle (\alpha \mathcal{A} + \beta \mathcal{I}) \ast_N \mathcal{X}, \mathcal{X} \rangle : \mathcal{X} \text{ is a unit tensor in } \mathbb{C}^{I_{1...N}} \right\} = \alpha \langle \mathcal{A} \ast_N \mathcal{X}, \mathcal{X} \rangle + \beta = \alpha W(\mathcal{A}) + \beta. \hfill \blacksquare
\]

Next result shows that the numerical range of the sum of two tensors $\mathcal{A}$ and $\mathcal{B}$ is always contained in the sum of the numerical ranges of the individual sets, i.e. $W(\mathcal{A} + \mathcal{B}) \subseteq W(\mathcal{A}) + W(\mathcal{B})$.

**Theorem 2.5:** Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_{1...N} \times I_{1...N}}$. Then $W(\mathcal{A} + \mathcal{B}) \subseteq W(\mathcal{A}) + W(\mathcal{B})$. 

**Proof:** Let \( \alpha \in W(\mathcal{A} + \mathcal{B}) \). Then, there exists a unit tensor \( \mathcal{X} \in \mathbb{C}^{I_1 \ldots I_N} \) such that \( \alpha = \langle (\mathcal{A} + \mathcal{B}) *_{N} \mathcal{X}, \mathcal{X} \rangle = \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle + \langle \mathcal{B} *_{N} \mathcal{X}, \mathcal{X} \rangle \). Since \( \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle \in W(\mathcal{A}) \) and \( \langle \mathcal{B} *_{N} \mathcal{X}, \mathcal{X} \rangle \in W(\mathcal{B}) \), so \( \alpha \in W(\mathcal{A}) + W(\mathcal{B}) \). Thus, \( W(\mathcal{A} + \mathcal{B}) \subseteq W(\mathcal{A}) + W(\mathcal{B}) \). \( \blacksquare \)

In the above theorem, the equality holds when one of the two tensors is scalar multiple of the identity tensor, i.e. \( \mathcal{A} = \alpha \mathcal{I} \) or \( \mathcal{B} = \beta \mathcal{I} \) where \( \alpha, \beta \in \mathbb{C} \). The relation between real part and imaginary part of the numerical range of a tensor with the numerical range of the Hermitian part and skew-Hermitian part of that tensor, respectively, are presented in the next theorem. ‘Re’ and ‘Im’ are used to denote the real and imaginary parts of a set, respectively.

**Theorem 2.6:** If \( H(\mathcal{A}) = (\mathcal{A} + \mathcal{A}^H)/2 \) and \( S(\mathcal{A}) = (\mathcal{A} - \mathcal{A}^H)/2 \) are the Hermitian part and the skew-Hermitian part of \( \mathcal{A} \in \mathbb{C}^{I_1 \ldots \times I_1 \ldots I_N} \), respectively, then \( \text{Re} W(\mathcal{A}) = W(H(\mathcal{A})) \) and \( \text{Im} W(\mathcal{A}) = W(S(\mathcal{A})) \).

**Proof:** Using Definition 2.1, we have

\[
W(H(\mathcal{A})) = \left\{ \left( \frac{(\mathcal{A} + \mathcal{A}^H)/2 *_{N} \mathcal{X}, \mathcal{X} \rangle}{: \mathcal{X} \text{ is a unit tensor in } \mathbb{C}^{I_1 \ldots I_N}} \right) \right. \\
= \left\{ \frac{1}{2} \left( \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle + \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle \right) : \mathcal{X} \text{ is a unit tensor in } \mathbb{C}^{I_1 \ldots I_N} \right\} \\
= \text{Re} \left\{ \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle : \mathcal{X} \text{ is a unit tensor in } \mathbb{C}^{I_1 \ldots I_N} \right\} \\
= \text{Re} \ W(\mathcal{A}).
\]

Analogously, one can easily verify that \( \text{Im} W(\mathcal{A}) = W(S(\mathcal{A})) \). \( \blacksquare \)

The numerical range of a tensor remains unaltered after taking its transpose. However, the numerical range of the conjugate transpose of a tensor is equal to that of conjugate of the tensor, which is further equal to the conjugate of the numerical range of the tensor. Next, we prove this as a theorem.

**Theorem 2.7:** Let \( \mathcal{A} \in \mathbb{C}^{I_1 \ldots \times I_1 \ldots I_N} \). Then \( W(\mathcal{A}^T) = W(\mathcal{A}) \) and \( W(\mathcal{A}^H) = W(\overline{\mathcal{A}}) = W(\mathcal{A}) \).

**Proof:** Let \( z \in W(\mathcal{A}) \) for \( \mathcal{A} = (a_{i_1i_2\ldots i_Nj_1\ldots j_N}) \in \mathbb{C}^{I_1 \ldots \times I_1 \ldots I_N} \). Then, there exists a unit tensor \( \mathcal{X} = (x_{i_1i_2\ldots i_N}) \in \mathbb{C}^{I_1 \ldots I_N} \) such that

\[
z = \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle = \sum_{i_1j_2\ldots i_N} \bar{x}_{i_1i_2\ldots i_N} \left( \sum_{j_1j_2\ldots j_N} a_{i_1i_2\ldots i_Nj_1j_2\ldots j_N} x_{j_1j_2\ldots j_N} \right) \\
= \sum_{i_1j_2\ldots i_N} \left( \sum_{j_1j_2\ldots j_N} a_{i_1i_2\ldots i_Nj_1j_2\ldots j_N} x_{j_1j_2\ldots j_N} \right) \bar{x}_{i_1i_2\ldots i_N} \sum_{j_1j_2\ldots j_N} \bar{x}_{i_1i_2\ldots i_N} x_{j_1j_2\ldots j_N}.
\]
For $\mathcal{Y} = \mathcal{X}$, we have
\[
\langle A^T \ast_N \mathcal{Y}, \mathcal{Y} \rangle = \sum_{i_1, j_2, \ldots, i_N, j_1, j_2, \ldots, j_N} a_{ij_1 \ldots i_N j_1 \ldots j_N} y_{i_1 \ldots i_N, j_1 \ldots j_N}.
\]

Also, we have
\[
\langle A^T, A \rangle = \sum_{i_1, j_2, \ldots, i_N, j_1, j_2, \ldots, j_N} a_{ij_1 \ldots i_N j_1 \ldots j_N} x_{i_1 \ldots i_N, j_1 \ldots j_N}.
\]

So, $z \in W(A^T)$. Thus, $W(A) \subseteq W(A^T)$. Similarly, it can be shown that $W(A^T) \subseteq W(A)$. Therefore, $W(A) = W(A^T)$.

Recall that a tensor $A \in \mathbb{C}^{I_1 \ldots \times I_1 \ldots \times I_N}$ is said to be unitary tensor if $A \ast_N A^H = I = A^H \ast_N A$, where $A^H$ denotes the conjugate transpose of the tensor $A$, and $I$ denotes the identity tensor of suitable size.

**Theorem 2.8:** Let $B \in \mathbb{C}^{I_1 \ldots \times I_1 \ldots \times I_N}$ such that $B^H \ast_M B = I$. Then, for any $A \in \mathbb{C}^{I_1 \ldots \times I_1 \ldots \times I_M}$, $W(B^H \ast_M A \ast_M B) \subseteq W(A)$. Equality holds, if $M = N$ and $(I_1, \ldots, I_M) = (J_1, \ldots, J_N)$, i.e. $B$ is unitary.

**Proof:** Let $z \in W(B^H \ast_M A \ast_M B)$. Then, there exists a unit tensor $X \in \mathbb{C}^{I_1 \ldots \times I_N}$ such that $z = \langle (B^H \ast_M A \ast_M B) \ast_N X, X \rangle = \langle A \ast_M V, V \rangle$, where $V = B \ast_N X$ and $\|V\| = 1$. Thus, $z \in W(A)$. Hence, $W(B^H \ast_M A \ast_M B) \subseteq W(A)$.

Suppose that $B$ is unitary (i.e. $B^H = B^{-1}$, where $B^{-1}$ denotes the inverse of $B$). Let $z \in W(A)$. Then, there exists a unit tensor $X \in \mathbb{C}^{I_1 \ldots \times I_N}$ such that $z = \langle A \ast_M X, X \rangle = \langle B^H \ast_M A \ast_M V, V \rangle$ where $V = B \ast_M X$ and $\|V\| = 1$. Thus, $z \in W(B^H \ast_M A \ast_M B)$. Hence, $W(A) \subseteq W(B^H \ast_M A \ast_M B)$. Therefore, when $B$ is unitary $W(A) = W(B^H \ast_M A \ast_M B)$.

Recall the following well-known result for a continuous function.

**Theorem 2.9:** Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Then $f$ is continuous if, and only if, the functions $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.

Next, we recall the notion of a path and path connected in a space.
Definition 2.10: Given points $x$ and $y$ of the space $X$, a path in $X$ from $x$ to $y$ is a continuous map $f : [a, b] \rightarrow X$ of some closed interval in the real line into $X$, such that $f(a) = x$ and $f(b) = y$. A space $X$ is said to be path connected if every pair of points of $X$ can be joined by a path in $X$.

The next lemma is about the construction of a path-connected set associated with an element of the numerical range of a tensor.

Lemma 2.11: Let $A \in \mathbb{C}^{I_1 \cdots I_N \times I_1 \cdots I_N}$ be a Hermitian tensor, i.e. $A^H = A$. Also, let $T_A(\alpha) = \{X \in \mathbb{C}^{I_1 \cdots I_N} : \langle A *_N X, X \rangle = \alpha \}$. Then $T_A(\alpha)$ is path connected for $\alpha \in W(A)$.

Proof: Suppose that $A$ is Hermitian. Then, $A$ can be expressed as $A = U^H *_N D *_N U$, where $U$ is a unitary tensor and $D$ is a real diagonal tensor. Now, using Theorem 2.8, we obtain $W(A) = W(U^H *_N D *_N U) = W(D)$. Thus, without loss of generality, we can assume that $A$ is a real diagonal tensor. Now, keeping in mind that $A$ is a diagonal tensor, we have

$$W(A) = \{ \langle A *_N X, X \rangle : X \text{ is a unit tensor in } \mathbb{C}^{I_1 \cdots I_N} \}$$

$$= \left\{ \sum_{i_1, i_2, \cdots, i_N} a_{i_1 i_2 \cdots i_N} x_{i_1 i_2 \cdots i_N} \bar{x}_{i_1 i_2 \cdots i_N} : x_{i_1 i_2 \cdots i_N} \in \mathbb{C}, \sum_{i_1, i_2, \cdots, i_N} |x_{i_1 i_2 \cdots i_N}|^2 = 1 \right\}.$$

Let $\alpha \in W(A)$, and $X = (x_{i_1 \cdots i_N})$, $Y = (y_{i_1 \cdots i_N}) \in T_A(\alpha)$. Then, $\sum_{i_1, i_2, \cdots, i_N} a_{i_1 i_2 \cdots i_N} x_{i_1 i_2 \cdots i_N} \bar{x}_{i_1 i_2 \cdots i_N} = \alpha$ and $\sum_{i_1, i_2, \cdots, i_N} a_{i_1 i_2 \cdots i_N} x_{i_1 i_2 \cdots i_N} \bar{y}_{i_1 i_2 \cdots i_N} = \alpha$. Consider the continuous map $f : [0, 1] \rightarrow \mathbb{C}^{I_1 \cdots I_N}$ defined by

$$f(t) = C = (c_{i_1 i_2 \cdots i_N}), \text{ where } c_{i_1 i_2 \cdots i_N} = \sqrt{(1 - t)|x_{i_1 i_2 \cdots i_N}|^2 + t|y_{i_1 i_2 \cdots i_N}|^2}. \quad (10)$$

Thus, $f(0) = X$ and $f(1) = Y$. Also, it can be easily verified that $f(t) \in T_A(\alpha)$. Therefore, there exists a path from $X$ to $Y$. Thus, the set $T_A(\alpha)$ is path connected.

The next result verifies that the numerical range of a tensor defined in Definition 2.1 is a convex set.

Theorem 2.12: For a tensor $A \in \mathbb{C}^{I_1 \cdots I_N \times I_1 \cdots I_N}$, the numerical range $W(A)$ is convex.

Proof: Let $\alpha, \beta \in W(A)$. We can choose $\alpha = 0$ and $\beta = 1$ without loss of generality due to Theorem 2.4. To show $W(A)$ is convex, it is sufficient to show that the line segment $[0, 1] \subseteq W(A)$. We have, $A = H(A) + S(A)$, where $H(A) = (A + A^H)/2$ and $S(A) = (A - A^H)/2$ are the Hermitian and the skew-Hermitian part of $A$. Using Lemma 2.11, it is clear that the set $T_{S(A)}(0) = \{X \text{ is a unit tensor in } \mathbb{C}^{I_1 \cdots I_N} : \langle S(A) *_N X, X \rangle = 0 \}$ is path connected. Let $X$ and $Y$ be two unit tensors in $\mathbb{C}^{I_1 \cdots I_N}$ such that $\langle A *_N X, X \rangle = 0$ and $\langle A *_N Y, Y \rangle = 1$. Thus, $X, Y \in T_{S(A)}(0)$. Since $T_{S(A)}(0)$ is path connected, there exists a continuous map $Z : [0, 1] \rightarrow T_{S(A)}(0)$ with $Z(0) = X$ and $Z(1) = Y$. Now, consider a map $U : [0, 1] \rightarrow W(A)$ defined by $U(t) = (A *_N Z(t), Z(t))$. Since $A =$
\[ H(\mathcal{A}) + S(\mathcal{A}) \] and \( Z(t) \in T_{S(\mathcal{A})}(0) \), so \( \langle \mathcal{A} *_{N} Z(t), Z(t) \rangle = \langle H(\mathcal{A}) *_{N} Z(t), Z(t) \rangle \). Thus, the map \( U \) defined above is a real-valued continuous map in the variable \( t \). Further, \( U(0) = \langle \mathcal{A} *_{N} Z(0), Z(0) \rangle = \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle = 0 \) and \( U(1) = \langle \mathcal{A} *_{N} Z(1), Z(1) \rangle = \langle \mathcal{A} *_{N} \mathcal{Y}, \mathcal{Y} \rangle = 1 \). Therefore, there exists a path \( U \) in \( W(\mathcal{A}) \) from 0 to 1. Thus, \([0, 1] \subseteq W(\mathcal{A})\), which implies \( W(\mathcal{A}) \) is convex.

3. An algorithm for finding tensor numerical range

In this section, we provide an algorithm to plot the boundary of the numerical range of a tensor. The set of tensors \( \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\} \), where \( \mathcal{U}_i \in \mathbb{C}^{l_i \times \ldots \times l_N} \), is called orthogonal if for \( i \neq j, \langle \mathcal{U}_i, \mathcal{U}_j \rangle = 0 \), and further if it satisfies \( \langle \mathcal{U}_i, \mathcal{U}_i \rangle = 1 \) then it is called orthonormal.

**Theorem 3.1:** Let \( \mathcal{A} \in \mathbb{C}^{l_1 \times \ldots \times l_N} \) and \( \mathcal{X} \in \mathbb{C}^{l_1 \times \ldots \times l_N} \). Then the followings are equivalent.

(i) \( \text{Re} \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle = \max \{ \text{Re}(z) : z \in W(\mathcal{A}) \} \).

(ii) \( \langle H(\mathcal{A}) *_{N} \mathcal{X}, \mathcal{X} \rangle = \max \{ r : r \in W(H(\mathcal{A})) \} \).

(iii) \( H(\mathcal{A}) *_{N} \mathcal{X} = \lambda_{\text{max}} \mathcal{X} \), where \( \lambda_{\text{max}} = \max \{ \lambda : \lambda \in \sigma(H(\mathcal{A})) \} \).

**Proof:** Suppose that (i) holds. Since

\[
\text{Re} \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle = \frac{\langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle + \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle}{2} = \frac{\langle (\mathcal{A} + \mathcal{A}^{H}) *_{N} \mathcal{X}, \mathcal{X} \rangle}{2} = \langle H(\mathcal{A}) *_{N} \mathcal{X}, \mathcal{X} \rangle,
\]

so \( \text{Re} \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle = \langle H(\mathcal{A}) *_{N} \mathcal{X}, \mathcal{X} \rangle \). Also,

\[
\max \{ \text{Re}(z) : z \in W(\mathcal{A}) \} = \max \left\{ \frac{z + \overline{z}}{2} : z \in W(\mathcal{A}) \right\} = \max \left\{ \frac{\langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle + \langle \mathcal{A} *_{N} \mathcal{X}, \mathcal{X} \rangle}{2} : \mathcal{X} \text{ is a unit tensor in } \mathbb{C}^{l_1 \times \ldots \times l_N} \right\} = \max \{ \langle H(\mathcal{A}) *_{N} \mathcal{X}, \mathcal{X} \rangle : \mathcal{X} \text{ is a unit tensor in } \mathbb{C}^{l_1 \times \ldots \times l_N} \}.
\]

Thus, (i) is equivalent to (ii).

Let \( \{\mathcal{X}_1, \ldots, \mathcal{X}_{l_1 \times \ldots \times l_N}\} \) be the orthonormal basis of eigentensors of \( H(\mathcal{A}) \), where \( \mathcal{X}_i \) is an eigentensor corresponding to the eigenvalue \( \lambda_i \). Then, for a unit tensor \( \mathcal{Y} \in \mathbb{C}^{l_1 \times \ldots \times l_N} \), there exists a set of scalars \( \{\alpha_1, \ldots, \alpha_{l_1 \times \ldots \times l_N}\} \) such that \( \mathcal{Y} = \sum_{i=1}^{l_1 \times \ldots \times l_N} \alpha_i \mathcal{X}_i \). Since \( \mathcal{Y} \) is a unit tensor, so we have \( \sum \alpha_i \lambda_i = 1 \). Thus, \( \langle H(\mathcal{A}) *_{N} \mathcal{Y}, \mathcal{Y} \rangle = \sum \overline{\alpha}_i \alpha_i \lambda_i \leq \lambda_{\text{max}} \). Also, as \( \langle H(\mathcal{A}) *_{N} \mathcal{X}, \mathcal{X} \rangle \in W(H(\mathcal{A})) \) and \( \sigma(H(\mathcal{A})) \subset W(H(\mathcal{A})) \), so \( \langle H(\mathcal{A}) *_{N} \mathcal{X}, \mathcal{X} \rangle \geq \lambda_{\text{max}} \). Thus, \( \langle H(\mathcal{A}) *_{N} \mathcal{X}, \mathcal{X} \rangle = \lambda_{\text{max}} \). Suppose that \( H(\mathcal{A}) *_{N} \mathcal{X} \neq \lambda_{\text{max}} \).
\( \lambda_{\max} \mathcal{X} \), then this implies \( \langle H(A) \ast_N \mathcal{X}, \mathcal{X} \rangle \neq \lambda_{\max} \langle \mathcal{X}, \mathcal{X} \rangle = \lambda_{\max} \). This is a contradiction. Thus, \( H(A) \ast_N \mathcal{X} = \lambda_{\max} \mathcal{X} \). Conversely, suppose that \( H(A) \ast_N \mathcal{X} = \lambda_{\max} \mathcal{X} \). Then,

\[
\langle H(A) \ast_N \mathcal{X}, \mathcal{X} \rangle = \lambda_{\max}. \tag{12}
\]

But, we have shown earlier that for any unit tensor \( \mathcal{Y} \in \mathbb{C}^{I_1 \ldots I_N} \), \( \langle H(A) \ast_N \mathcal{Y}, \mathcal{Y} \rangle \leq \lambda_{\max} \) and hence \( \max\{r : r \in W(H(A))\} \leq \lambda_{\max} \). Also, since \( \sigma(H(A)) \subseteq W(H(A)) \), so \( \max\{r : r \in W(H(A))\} \geq \lambda_{\max} \). Therefore,

\[
\max\{r : r \in W(H(A))\} = \lambda_{\max}. \tag{13}
\]

From (12) and (13), we have (ii). Thus, (ii) is equivalent to (iii). This completes the proof. \( \blacksquare \)

Next, an immediate consequence of the above theorem is presented as a corollary without proof.

**Corollary 3.2:** Let \( A \in \mathbb{C}^{I_1 \ldots I_N \times I_1 \ldots N} \). Then,

\[
\max\{\text{Re}(z) : z \in W(A)\} = \max\{r : r \in W(H(A))\} = \max\{\lambda : \lambda \in \sigma(H(A))\}.
\]

Note that, according to Theorem 2.4, we have \( e^{-i\theta} W(e^{i\theta} A) = W(A) \) for all \( 0 \leq \theta \leq 2\pi \).

**Theorem 3.3:** Let \( A \in \mathbb{C}^{I_1 \ldots I_N \times I_1 \ldots N} \) and \( X_{\theta} \) be the normalized eigentensor corresponding to the maximum eigenvalue of \( H(e^{i\theta} A) \) for some \( \theta \in [0, 2\pi] \). Then, the complex number \( X_{\theta} \ast_H A \ast_N X_{\theta} = \langle A \ast_N X_{\theta}, X_{\theta} \rangle \) is a boundary point of \( W(A) \).

**Proof:** Let \( \lambda_{\max} = \max\{\lambda : \lambda \in \sigma(H(e^{i\theta} A))\} \). By Theorem 3.1 and Corollary 3.2,

\[
\max\{\text{Re}(z) : z \in W(e^{i\theta} A)\} = \lambda_{\max} = \text{Re}\{e^{i\theta} A \ast_N X_{\theta} \ast N X_{\theta}\}.
\]

So, the line \( \lambda_{\max} + it \), for \( t \in \mathbb{R} \) is a tangent to \( W(A) \). This implies that \( \langle e^{i\theta} A \ast_N X_{\theta}, X_{\theta} \rangle \) is a boundary point of \( W(e^{i\theta} A) \). Hence \( \langle A \ast_N X_{\theta}, X_{\theta} \rangle = \langle e^{i\theta} A \ast_N X_{\theta}, X_{\theta} \rangle \) is a boundary point of \( W(A) \). \( \blacksquare \)

Based on the above theory, we next present an algorithm to plot the boundary of the numerical range of a tensor.

**Algorithm 3.1:**

1. Choose \( \theta \in [0, 2\pi] \)
2. Calculate \( T = e^{i\theta} A \), for a given \( A \in \mathbb{C}^{I_1 \ldots I_N \times I_1 \ldots N} \)
3. \( \lambda_{\max} = \max\{\lambda : \lambda \in \sigma(H(T))\} \)
4. Calculate the normalized eigentensor corresponding to \( \lambda_{\max} \), \( X_{\max} \) (say)
5. Calculate \( z = \langle A \ast_N X_{\max}, X_{\max} \rangle \)
6. Plot \( z \)
7. Repeat **Step 1.–Step 6**.

Next, we present a few numerical examples to illustrate the algorithm introduced above.
Figure 3. Numerical range of the tensor $\mathcal{A}$ (the ‘∗’ symbols in red represent the eigenvalues of $\mathcal{A}$).

Example 3.4: Consider a tensor $\mathcal{A} \in \mathbb{C}^{3 \times 2 \times 3 \times 2}$ such that

Now, applying Algorithm 3.1 to the tensor $\mathcal{A}$ for 500 different choices of $\theta$, we obtain Figure 3. Also, the eigenvalues of the tensor are plotted (highlighted by ‘∗’), all of which lies inside the boundary of the numerical range of the tensor.

Example 3.5: Consider a tensor $\mathcal{A} \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$ such that

Now, applying Algorithm 3.1 to the tensor $\mathcal{A}$ for 500 different choices of $\theta$, we obtain Figure 4. Each eigenvalue of the tensor (highlighted by ‘∗’) lies inside the boundary of the numerical range of the tensor.

Example 3.6: Consider a tensor $\mathcal{A} \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$ such that

Now, applying Algorithm 3.1 to the tensor $\mathcal{A}$ for 500 different choices of $\theta$, we obtain Figure 5. All the eigenvalues (highlighted by ‘∗’) of the tensor are on the boundary of the numerical range of the tensor.
**Figure 4.** Numerical range of the tensor $\mathcal{A}$ (the ‘*’ symbols in red represent the eigenvalues of $\mathcal{A}$).

**Figure 5.** Numerical range of the tensor $\mathcal{A}$ (the ‘*’ symbols in red represent the eigenvalues of $\mathcal{A}$).

| $\mathcal{A}(:, :, 1, 1)$ | $\mathcal{A}(:, :, 2, 1)$ | $\mathcal{A}(:, :, 1, 2)$ | $\mathcal{A}(:, :, 2, 2)$ |
|--------------------------|--------------------------|--------------------------|--------------------------|
| 1                        | -3i                      | 1-i                      | 3i                       |
| -i                       | 2-5i                     | 1                        | 3+i                      |
| 1                        | 3-i                      | 1+i                      | 7+i                      |
| 1-i                      | -3+i                     | 1+1                      | 3+1                      |
| 3-i                      | 1-i                      | 7-i                      | 0                        |

**Example 3.7:** Consider a Hermitian tensor $\mathcal{A} \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$ such that

Now, applying Algorithm 3.1 to the tensor $\mathcal{A}$ for 500 different choices of $\theta$, we obtain the Figure 6. Each eigenvalue of the tensor (highlighted by ‘*’) lies inside the boundary of numerical range of the tensor.

4. **Numerical radius of a tensor**

In this section, we introduce the notion of numerical radius of a tensor, and investigate its properties.
**Figure 6.** Numerical range of the tensor $\mathcal{A}$ (the ‘*’ symbols in red represent the eigenvalues of $\mathcal{A}$).

**Definition 4.1:** The numerical radius of an even-order square tensor $\mathcal{A} \in \mathbb{C}^{I_1 \ldots I_N \times I_1 \ldots I_N}$ is denoted as $w(\mathcal{A})$, and is defined as

$$w(\mathcal{A}) = \max\{|z| : z \in W(\mathcal{A})\}.$$ (14)

Tensors satisfy the Cauchy–Schwarz inequality. Next, we state this without proof. One can follow the steps of the existing proof for matrices in the literature to verify the inequality.

**Theorem 4.2:** Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \ldots I_N}$. Then $|\langle \mathcal{A}, \mathcal{B} \rangle| \leq \|\mathcal{A}\| \|\mathcal{B}\|$.

Now, we state a very popular theorem for a numerical radius inequality of a tensor which may contribute to develop new theories of numerical radius of a tensor.

For $\mathcal{A} \in \mathbb{C}^{I_1 \ldots I_N \times I_1 \ldots I_N}$ and $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1 \ldots I_N}$, we define $\|\mathcal{A}\| = \sup\{\|\mathcal{A} * N \mathcal{X}\| : \|\mathcal{X}\| = 1\} = \sup\{|\langle \mathcal{A} * N \mathcal{X}, \mathcal{Y} \rangle| : \|\mathcal{X}\| = \|\mathcal{Y}\| = 1\}$.

**Theorem 4.3:** Let $\mathcal{A} \in \mathbb{C}^{I_1 \ldots I_N \times I_1 \ldots I_N}$. Then $\frac{1}{2} \|\mathcal{A}\| \leq w(\mathcal{A}) \leq \|\mathcal{A}\|$.

**Proof:** For any unit tensor $\mathcal{X} \in \mathbb{C}^{I_1 \ldots I_N}$, we have

$$|\langle \mathcal{A} * N \mathcal{X}, \mathcal{X}' \rangle| \leq \|\mathcal{A} * N \mathcal{X}\| \|\mathcal{X}'\| \leq \|\mathcal{A}\|. \quad (15)$$

Taking supremum over $\|\mathcal{X}\| = 1$ both sides of the inequality (15), we obtain $w(\mathcal{A}) \leq \|\mathcal{A}\|$. Again, suppose that $\|\mathcal{X}\| = 1$ and $\|\mathcal{Y}\| = 1$. Then

$$4 |\langle \mathcal{A} * N \mathcal{X}, \mathcal{Y} \rangle| \leq |\langle \mathcal{A} * N (\mathcal{X} + \mathcal{Y}), \mathcal{X} + \mathcal{Y} \rangle - \langle \mathcal{A} * N (\mathcal{X} - \mathcal{Y}), \mathcal{X} - \mathcal{Y} \rangle + i\langle \mathcal{A} * N (\mathcal{X} + i\mathcal{Y}), \mathcal{X} + i\mathcal{Y} \rangle - i\langle \mathcal{A} * N (\mathcal{X} - i\mathcal{Y}), \mathcal{X} - i\mathcal{Y} \rangle|$$

$$\leq |w(\mathcal{A})\|\mathcal{X} + \mathcal{Y}\|^2 - w(\mathcal{A})\|\mathcal{X} - \mathcal{Y}\|^2 + iw(\mathcal{A})\|\mathcal{X} + i\mathcal{Y}\|^2 - iw(\mathcal{A})\|\mathcal{X} - i\mathcal{Y}\|^2|$$
\[
\leq w(A) \left[ \|X + Y\|^2 + \|X - Y\|^2 + \|X + iY\|^2 + \|X - iY\|^2 \right]
= 4w(A) \left[ \|X\|^2 + \|Y\|^2 \right]
= 8w(A).
\]

Thus, \(|\langle A \ast_N X, Y \rangle| \leq 2w(A)\). Hence, \(|A| \leq 2w(A)\). This completes the proof. ■

Next, we present a corollary as an immediate consequence of Theorem 4.3 without proof.

**Corollary 4.4:** Let \(A, B \in C^{I_1 \ldots N \times I_1 \ldots N}\). Then \(w(A \ast_N B) \leq 4w(A)w(B)\).

Next, we recall two results on the notion of the determinant of a tensor due to Liang et al. [25].

**Theorem 4.5 (Theorem 3.16, [25]):** Let \(A, B \in R^{I_1 \ldots N \times I_1 \ldots N}\) be two tensors. Then \(\det(A \ast_N B) = \det(A) \det(B)\).

**Theorem 4.6 (Theorem 4.10, [25]):** Let \(A = (a_{i_1 \ldots i_N j_1 \ldots j_N}) \in C^{I_1 \ldots N \times I_1 \ldots N}\) be a given tensor. Then \(\det(A) = \prod_{i_1, \ldots, i_N} \lambda_{i_1 \ldots i_N}\).

For two tensors \(A, B \in C^{I_1 \ldots N \times I_1 \ldots N}\), define \(|A| := (A^H \ast_N A)^{1/2}\) and \(A \geq B\) means \(A - B\) is a positive definite tensor. The following result provides a sufficient condition for unitarity of a tensor.

**Lemma 4.7:** Let \(A \in C^{I_1 \ldots N \times I_1 \ldots N}\) be an invertible tensor such that \(w(A) \leq 1\) and \(|A| \geq I\). Then \(A\) is unitary.

**Proof:** Since \(w(A) \leq 1\), so \(|\lambda| \leq 1\) for all \(\lambda \in \sigma(A)\) due to Theorem 2.2. Again, \(|A| \geq I\) implies \(A^H \ast_N A \geq I\). Thus, \(X^H \ast_N A^H \ast_N A \ast_N X \geq 1\), for all \(X \in C^{I_1 \ldots N}\) such that \(\|X\| = 1\). Therefore, the absolute value of each eigenvalues of \(A^H \ast_N A\) are greater than or equal to 1, i.e. \(\alpha_i^2 \geq 1\), where \(\alpha_i\)'s are the singular values of \(A\). Now, since \(\det(A^H \ast_N A) = \det(A^H) \det(A)\) by Theorem 3.16 of [25] and Theorem 4.10 of [25], we have \(1 \leq \prod |\alpha_i|^2 = \prod |\lambda_i|^2 \leq 1\). Since \(|\lambda_i| \leq 1\) and \(\alpha_i \geq 1\), so \(\alpha_i = 1\) for all \(i\). Thus, \(A\) is unitary. ■

Now, we recall the singular value decomposition (SVD) of a tensor which was first introduced in [17] for a real tensor and was then for a complex tensor in [22].

**Lemma 4.8 (Lemma 3.1, [22]):** A tensor \(A \in C^{I_1 \ldots N \times I_1 \ldots N}\) can be decomposed as

\[
A = U \ast_N B \ast_N V^H,
\]

where \(U \in C^{I_1 \ldots N \times I_1 \ldots N}\) and \(V \in C^{I_1 \ldots N \times I_1 \ldots N}\) are unitary tensors, and \(B \in C^{I_1 \ldots N \times I_1 \ldots N}\) is a tensor such that \((B)_{i_1 \ldots I_N j_1 \ldots j_N} = 0\), if \((i_1, \ldots, i_N) \neq (j_1, \ldots, j_N)\).

Based on the notion of singular value decomposition [22] of a tensor, we next propose a new decomposition of a tensor.
Lemma 4.9: Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N}$. Then, there exists a unitary tensor $M$ such that $A = M \ast_N |A|$, where $|A| = (A \ast_N A^H)^{1/2}$.

Proof: Let $A = U \ast_N D \ast_N V^H$ be a singular value decomposition of $A \in \mathbb{C}^{I_1 \times \cdots \times I_N}$. Also, let $M = U \ast_N V$ and $T = V \ast_N D \ast_N V^H$. Then, $M \ast_N T = A$. Furthermore, $M$ is a unitary tensor and $|A| = (A \ast_N A^H)^{1/2} = T$. Thus, the claim.

We call the decomposition in Lemma 4.9 as the polar decomposition of an even-order square tensor $A$. The next result provides a characterization of a unitary tensor.

Theorem 4.10: $A \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is an invertible tensor such that $w(A) \leq 1$ and $w(A^{-1}) \leq 1$ if, and only if, $A$ is unitary.

Proof: Let $A = U \ast_N |A|$ be the polar decomposition of $A$. Then, $(A^{-1})^H = (U \ast_N |A|^{-1}U^{-1})^H = U \ast_N (|A|^{-1})^H = U \ast_N |A|^{-1}$. Since $w(A^{-1}) \leq 1$, so for any unit tensor $X \in \mathbb{C}^{I_1}$, we have $|\langle U \ast_N |A|^{-1} \ast_N X, X \rangle| = |\langle (A^{-1})^H \ast_N X, X \rangle| \leq 1$. Let $B := U \ast_N \frac{|A| + |A|^{-1}}{2}$. Here $|B| = \frac{|A| + |A|^{-1}}{2}$.

5. Numerical range of the Moore–Penrose inverse of an even-order square tensor

In this section, we concentrate on the numerical range of the Moore–Penrose inverse of an even-order square tensor and investigate how it relates with the numerical range of the original tensor. The first result of this section confirms that both the tensors $\tilde{A}$ and its Moore–Penrose inverse $\tilde{A}^\dagger$ are Hermitian or normal, simultaneously.

Theorem 5.1: Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N}$. Then $A$ is normal (resp. Hermitian) if, and only if, $A^\dagger$ is normal (resp. Hermitian).

Proof: Suppose that $A$ is normal, i.e. $A \ast_N A^H = A^H \ast_N A$. Since $A \ast_N A^H$ and $A^H \ast_N A$ satisfy the reverse-order law [26,27], taking the Moore–Penrose inverse on both sides of $A \ast_N A^H = A^H \ast_N A$, we get $A^{H\dagger} \ast_N A^{\dagger} = A^{\dagger} \ast_N A^{H\dagger}$, i.e. $A^{\dagger H} \ast_N A^{\dagger} = A^{\dagger} \ast_N A^{\dagger H}$. Thus, $A^\dagger$ is normal.
Conversely, suppose that \( A^\dagger \) is normal, then \( A^\dagger H \ast_N A^\dagger = A^\dagger \ast_N A^\dagger H \), i.e. \((A \ast_N A^H)^\dagger = (A^H \ast_N A)^\dagger \). Now, taking the Moore–Penrose inverse on both sides, we get \( A \ast_N A^H = A^H \ast_N A \). Hence, the result follows.

In general, if \( \lambda \neq 0 \) is an eigenvalue of a tensor \( A \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N} \), then \( 1/\lambda \) may not be an eigenvalue of the tensor \( A^\dagger \). However, if \( A \) is normal, then \( \lambda \neq 0 \) is an eigenvalue of a tensor \( A \) implies \( 1/\lambda \) is an eigenvalue of the tensor \( A^\dagger \). While if 0 is an eigenvalue of a tensor \( A \), then 0 is always an eigenvalue of \( A^\dagger \) for any tensor \( A \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N} \). This is shown in the next result.

**Theorem 5.2:** Let \( A \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N} \). Then

(i) \( 0 \in \sigma (A) \) if, and only if, \( 0 \in \sigma (A^\dagger) \);

(ii) If \( A \) is normal and \( \lambda \neq 0 \), then \( \lambda \in \sigma (A) \) if, and only if, \( 1/\lambda \in \sigma (A^\dagger) \).

**Proof:** (i) From Definition 1.4, we have \( A \ast_N A^\dagger \ast_N A = A \) and \( A^\dagger \ast_N A \ast_N A^\dagger = A^\dagger \). Taking the determinant on both sides of \( A \ast_N A^\dagger \ast_N A = A \), we get \( \det^2(A) \det(A^\dagger) = \det(A) \), and taking the determinant on both sides of \( A^\dagger \ast_N A \ast_N A^\dagger = A^\dagger \) gives \( \det(A) \det^2(A^\dagger) = \det(A^\dagger) \). Thus, if \( \det(A) = 0 \), then \( \det(A^\dagger) \) yields \( \det(A) = 0 \), and if \( \det(A^\dagger) = 0 \), then \( \det^2(A) \) yields \( \det(A) = 0 \). Hence, the claim.

(ii) Suppose that \( A \) is normal. Then, \( A \) can be decomposed as \( A = U \ast_N D \ast_N U^H \), where \( D = \text{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) \) with \( \lambda_i \neq 0 \) for \( i \in \{1, 2, \ldots, k\} \). Now, \( A^\dagger = U \ast_N D^\dagger \ast_N U^H \), where \( D^\dagger = \text{diag}(1/\lambda_1, \ldots, 1/\lambda_k, 0, \ldots, 0) \). Thus, the claim.

Note that Theorem 5.2 (ii) does not hold if \( A \) is not normal, and is shown next.

**Example 5.3:** Consider the tensor \( A \in \mathbb{C}^{3 \times 2 \times 3 \times 2} \) as below

| \( A(i,1,1) \) | \( A(i,2,1) \) | \( A(i,3,1) \) | \( A(i,1,2) \) | \( A(i,2,2) \) | \( A(i,3,2) \) |
|-------------|-------------|-------------|-------------|-------------|-------------|
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Then its Moore–Penrose inverse \( A^\dagger \in \mathbb{C}^{3 \times 2 \times 3 \times 2} \) becomes

| \( A^\dagger(i,1,1) \) | \( A^\dagger(i,2,1) \) | \( A^\dagger(i,3,1) \) | \( A^\dagger(i,1,2) \) | \( A^\dagger(i,2,2) \) | \( A^\dagger(i,3,2) \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.1667 | 0.1667 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1667 | 0.1667 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1667 | 0.1667 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Now, \( \sigma (A) = \{0, 1\} \) and \( \sigma (A^\dagger) = \{0, 0.1667\} \). Observe that here \( A \ast_N A^\dagger \neq A^\dagger \ast_N A \) and 1 is an eigenvalue of \( A \) while 1 is not an eigenvalue of \( A^\dagger \).

Next result confirms that 0 belongs to the numerical range of a tensor if, and only if, it is in the numerical range of its Moore–Penrose inverse.
Theorem 5.4: Let $A \in \mathbb{C}^{I_1 \ldots N \times I_1 \ldots N}$. Then $0 \in W(A)$ if, and only if, $0 \in W(A^\dagger)$.

Proof: Here two cases are possible.

Case 1: Suppose that $A$ is singular. Then $0 \in \sigma(A) \subseteq W(A)$ and it is possible if, and only if, $0 \in \sigma(A^\dagger) \subseteq W(A^\dagger)$.

Case 2: Suppose that $A$ is non-singular. Then, $A^\dagger = A^{-1}$. Now using (9), we have

$$W(A) = \left\{ \frac{(A \ast_N \mathcal{Y}, \mathcal{Y})}{\|\mathcal{Y}\|_2^2} : \mathcal{O} \neq \mathcal{Y} \in \mathbb{C}^{I_1 \ldots N} \right\};$$

$$W(A^{-1}) = \left\{ \frac{(A^{-1} \ast_N \mathcal{X}, \mathcal{X})}{\|\mathcal{X}\|_2^2} : \mathcal{O} \neq \mathcal{X} \in \mathbb{C}^{I_1 \ldots N} \right\} = \left\{ \frac{(A^{-1} \ast_N A \ast_N \mathcal{Y}, A \ast_N \mathcal{Y})}{\|A \ast_N \mathcal{Y}\|_2^2} : \mathcal{X} \in \mathbb{C}^{I_1 \ldots N}, A \ast_N \mathcal{Y} = \mathcal{X} \right\} = \left\{ \frac{(A^H \ast_N A^{-1} \ast_N A \ast_N \mathcal{Y}, \mathcal{Y})}{\|A \ast_N \mathcal{Y}\|_2^2} : \mathcal{X} \in \mathbb{C}^{I_1 \ldots N}, A \ast_N \mathcal{Y} = \mathcal{X} \right\} = \left\{ \frac{(A^H \ast_N \mathcal{Y}, \mathcal{Y})}{\|A \ast_N \mathcal{Y}\|_2^2} : \mathcal{O} \neq \mathcal{Y} \in \mathbb{C}^{I_1 \ldots N} \right\}. \quad (17)$$

Suppose that, $0 \in W(A)$. Then, there exists a non-zero tensor $\mathcal{Y} \in \mathbb{C}^{I_1 \ldots N}$ such that $\langle A \ast_N \mathcal{Y}, \mathcal{Y} \rangle = 0$ by (16), which implies $\langle A^H \ast_N \mathcal{Y}, \mathcal{Y} \rangle = 0$. Thus, $0 \in W(A^\dagger)$ due to (17).

Conversely, suppose that $0 \in W(A^\dagger)$. Then, there exists a non-zero tensor $\mathcal{Y} \in \mathbb{C}^{I_1 \ldots N}$ such that $\langle A^H \ast_N \mathcal{Y}, \mathcal{Y} \rangle = 0$, which gives $\langle A \ast_N \mathcal{Y}, \mathcal{Y} \rangle = 0$. Thus, $0 \in W(A)$. This completes the proof.

The next result shows that $W(A) = W(A^H)$ is sufficient to confirm that the sets $W(A)$ and $\alpha^2 W(A^\dagger)$ are disjoint, where ‘$\alpha$’ is a singular value of $A$.

Theorem 5.5: Let $A \in \mathbb{C}^{I_1 \ldots N \times I_1 \ldots N}$ such that $W(A) = W(A^H)$. Then

$$W(A) \cap \alpha^2 W(A^\dagger) \neq \emptyset$$

where ‘$\alpha$’ is a singular value of $A$.

Proof: Let $A = U \ast_N \Sigma \ast_N \mathcal{Y}^H$ be an SVD of $A$, where $\Sigma = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a diagonal tensor such that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$, where $\alpha_i$’s are the singular values of $A$ and $n = I_1 \cdot I_2 \ldots I_N$. If $A$ is singular, then $0 \in \sigma(A) \subseteq W(A).$ Then, by Theorem 5.4, we have $0 \in W(A^\dagger)$. Hence $W(A) \cap \alpha^2 W(A^\dagger) \neq \emptyset$, for every singular value $\alpha$ of $A$.

Suppose that $A$ is non-singular, and let $\alpha = \alpha_1 \neq 0$ be a singular value of $A$. Then, $1$ is a singular value of $A/\alpha$. Let $\mathcal{X} \in \mathbb{C}^{I_1 \ldots N}$ be a unit tensor such that $\mathcal{X} = ((\mathcal{X}^H \ast_N \mathcal{X})_1, 0, 0, \ldots, 0)^T$. Now, $\langle (A/\alpha) \ast_N \mathcal{X}, \mathcal{X} \rangle = \mathcal{X}^H \ast_N (A/\alpha) \ast_N \mathcal{X} = (U^H \ast_N \mathcal{X}_1)(\mathcal{X}^H \ast_N \mathcal{X}_1)^T$. Since $W(A/\alpha) = W((A/\alpha)^H)$, so $((U^H \ast_N \mathcal{X}_1)(\mathcal{X}^H \ast_N \mathcal{X}_1)^T) \in W(A/\alpha)$. Also,

$$\langle (\alpha A^{-1}) \ast_N \mathcal{X}, \mathcal{X} \rangle = \mathcal{X}^H \ast_N (\alpha A^{-1}) \ast_N \mathcal{X}$$
\[ = \mathcal{X}^H \ast_N \mathcal{V} \ast_N (\alpha \Sigma^{-1}) \ast_N \mathcal{U}^H \ast_N \mathcal{X} \]

\[ = (\mathcal{V}^H \ast_N \mathcal{X})_1 (\mathcal{U}^H \ast_N \mathcal{X})_1. \]

Thus, \((\mathcal{V}^H \ast_N \mathcal{X})_1 (\mathcal{U}^H \ast_N \mathcal{X})_1 \in W(\alpha \mathcal{A}^{-1})\) and hence \(W(A/\alpha) \cap W(\alpha \mathcal{A}^{-1}) \neq \emptyset\), which is equivalent to \(W(A) \cap \alpha^2 W(\mathcal{A}^{-1}) \neq \emptyset\), for every singular value \(\alpha\) of \(A\). \hfill \(\blacksquare\)

We want to bring the readers attention to the fact that if \(W(A) = W(\mathcal{A}^H)\) is omitted from the above result, then the result may not hold.

**Example 5.6:** Consider a tensor \(A \in \mathbb{C}^{3 \times 2 \times 3 \times 2}\) such that

| \(A(i,1,1)\) | \(A(i,2,1)\) | \(A(i,3,1)\) | \(A(i,1,2)\) | \(A(i,2,2)\) | \(A(i,3,2)\) |
|---|---|---|---|---|---|
| 1+i | 0 | 0 | 0 | 4 | 0 | 0 |
| 0 | 0 | i | 0 | 0 | 0 | 5+i |
| 0 | 0 | 0 | 3+i | 0 | 0 | 6+i |

Then, the conjugate transpose of \(A\), \(A^H \in \mathbb{C}^{3 \times 2 \times 3 \times 2}\), is

| \(A^H(i,1,1)\) | \(A^H(i,2,1)\) | \(A^H(i,3,1)\) | \(A^H(i,1,2)\) | \(A^H(i,2,2)\) | \(A^H(i,3,2)\) |
|---|---|---|---|---|---|
| 1-i | 0 | 0 | 0 | 4 | 0 | 0 |
| 0 | 0 | -i | 0 | 0 | 0 | 5-i |
| 0 | 0 | 0 | 3-i | 0 | 0 | 6-i |

Thus, \(W(A) \neq W(\mathcal{A}^H)\). The set of singular values of the tensor \(A\) is \(\{1, \sqrt{2}, \sqrt{10}, 4, \sqrt{26}, \sqrt{37}\}\). Now, the Moore–Penrose inverse of \(A\), \(A^\dagger \in \mathbb{C}^{3 \times 2 \times 3 \times 2}\), is

| \(A^\dagger(i,1,1)\) | \(A^\dagger(i,2,1)\) | \(A^\dagger(i,3,1)\) | \(A^\dagger(i,1,2)\) | \(A^\dagger(i,2,2)\) | \(A^\dagger(i,3,2)\) |
|---|---|---|---|---|---|
| 0.5-0.5i | 0 | i | 0 | 0.25 | 0 | 0 |
| 0 | 0 | -i | 0 | 0 | 0 | (5-i)/26 |
| 0 | 0 | 0 | 0.3-0.1i | 0 | 0 | 0 | (6-i)/37 |

From Figure 7, it is clear that \(W(A) \cap \alpha^2 W(A^\dagger) = \emptyset\) for \(\alpha \in \{1, \sqrt{2}, \sqrt{10}, \sqrt{26}, \sqrt{37}\}\) and when \(\alpha = 4\) we have \(W(A) \cap \alpha^2 W(A^\dagger) = \{4\} \neq \emptyset\).

A tensor \(A \in \mathbb{C}^{I_1 \times \ldots \times I_N}\) is called an **EP-tensor**, if it satisfies \(A \ast_N A^\dagger = A^\dagger \ast_N A\). Thus, the invertible tensors are examples of EP-tensor. The following result can be easily verified using Definition 1.4.

**Lemma 5.7:** Let \(A \in \mathbb{C}^{I_1 \times \ldots \times I_N}\). Then,

(i) \((A^H)^\dagger = (A^\dagger)^H\)

(ii) \((\mathcal{U} \ast_M A \ast_N \mathcal{V})^\dagger = \mathcal{V}^H \ast_N A^\dagger \ast_M \mathcal{U}^H\), where \(\mathcal{U} \in \mathbb{C}^{I_1 \times \ldots \times I_1 \times M}\) and \(\mathcal{V} \in \mathbb{C}^{I_1 \times \ldots \times I_N}\) are unitary tensors.

Next, we recall the definition of a row block tensor proposed by Sun *et al.* [22].
Figure 7. Boundaries of numerical ranges of $W(A)$ and $\alpha^2 W(A^\dagger)$. (a) $\alpha^2 = 1$. (b) $\alpha^2 = 2$. (c) $\alpha^2 = 10$. (d) $\alpha^2 = 16$. (e) $\alpha^2 = 26$. (f) $\alpha^2 = 37$.

**Definition 5.8:** For the given tensors $A = (a_{i_1...i_N j_1...j_M}) \in \mathbb{C}^{I_1...N \times J_1...M}$ and 

$$B = (b_{i_1...i_N k_1...k_M}) \in \mathbb{C}^{I_1...N \times K_1...M},$$

the 'row block tensor' consisted of $A$ and $B$ is denoted by 

$$[A \ B] \in \mathbb{C}^{\alpha^N \times \beta_i \times \ldots \times \beta_M}, \quad (18)$$

where $\alpha^N = I_1...N$, $\beta_i = J_i + K_i$, $i = 1, \ldots, M$ and 

$$[A \ B]_{i_1...i_N l_1...l_M} = \begin{cases} 
    a_{i_1...i_N l_1...l_M}, & \text{if } (i_1, \ldots, i_N) \in [I_1] \times \ldots \times [I_N], (l_1, \ldots, l_M) \\
    b_{i_1...i_N (l_1-f_1)...(l_M-f_M)}, & \text{if } (i_1, \ldots, i_N) \in [I_1] \times \ldots \times [I_N], (l_1, \ldots, l_M) \\
    0, & \text{otherwise,}
\end{cases} \begin{array}{c}
\in [J_1] \times \ldots \times [J_M]; \\
\in [\Gamma_1] \times \ldots \times [\Gamma_M]; \\
\end{array}$$

where $\Gamma_i = [J_i + 1, \ldots, J_i + K_i], i = 1, \ldots, M.$
\textbf{Definition 5.9:} For the given tensors $A = (a_{j_1...j_{M_i}...j_N}) \in \mathbb{C}^{I_1...I_N}$ and $B = (b_{k_1...k_{M_i}...k_N}) \in \mathbb{C}^{K_1...K_N}$, the ‘column block tensor’ consisted of $A$ and $B$ is denoted by
\[
\begin{bmatrix} A & B \end{bmatrix} = [A^T \; B^T]^T \in \mathbb{C}^{\beta_1 \times \cdots \times \beta_M \times \alpha_N},
\]
where $\alpha_N = I_1...N$, $\beta_i = J_i + K_i$, $i = 1, \ldots, M$.

Let $T_1 = [A_1 \; B_1]$ and $T_2 = [A_2 \; B_2]$ be the ‘row block tensors’, where $A_1 \in \mathbb{C}^{I_1...I_N \times I_1...I_M}$, $B_1 \in \mathbb{C}^{I_1...I_N \times K_1...K_M}$, $A_2 \in \mathbb{C}^{I_1...I_N \times I_1...I_M}$ and $B_2 \in \mathbb{C}^{I_1...I_N \times K_1...K_M}$. Then the ‘column block tensor’ $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ can be written as
\[
\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \in \mathbb{C}^{\rho_1 \cdots \rho_N \times \beta_1 \cdots \beta_M},
\]
where $\rho_i = I_i + L_i$, $1 \leq i \leq N$ and $\beta_j = J_j + K_j$, $1 \leq j \leq M$. For more details on block tensors, one may refer to [22,28]. The next result can be proved easily with the help of Proposition 2.4 [22] and Definition 1.4.

\textbf{Theorem 5.10:} Let $A \in \mathbb{C}^{I_1...I_N \times I_1...I_N}$ be an invertible tensor and let a block tensor $B$ be defined by
\[
B = \begin{bmatrix} A & O \\ O & O \end{bmatrix}.
\]
Then,
\[
B^\dagger = \begin{bmatrix} A^{-1} & O \\ O & O \end{bmatrix}.
\]
We define a $k$-mode null space of a tensor $A \in \mathbb{C}^{I_1 \times \cdots \times I_{k-1} \times I_k \times I_{k+1} \times \cdots \times I_N}$ as the collection of tensors $\mathcal{X} \in \mathbb{C}^{I_{k+1} \times \cdots \times I_N}$ such that $A *_{(N-k)} \mathcal{X} = O$, and we denote it by $N_k(A)$, i.e. $N_k(A) = \{ \mathcal{X} \in \mathbb{C}^{I_{k+1} \times \cdots \times I_N} : A *_{(N-k)} \mathcal{X} = O \}$.

\textbf{Lemma 5.11:} Let $A \in \mathbb{C}^{I_1...I_M \times I_1...I_N}$. Then, $N_N(A^\dagger *_M A) = N_M(A)$ and $N_M(A *_M A^\dagger) = N_N(A^H)$.

\textbf{Proof:} Let $\mathcal{X} \in N_M(A)$. Then
\[
A *_N \mathcal{X} = O. \tag{21}
\]
Pre-multiplication of $A^\dagger$ on both sides of (21) results $A^\dagger *_M A *_N \mathcal{X} = O$. Thus, $\mathcal{X} \in N_N(A^\dagger *_M A)$. Conversely, suppose that $\mathcal{X} \in N_N(A^\dagger *_M A)$. Then
\[
A *_M A *_N \mathcal{X} = O. \tag{22}
\]
Pre-multiplication of $A$ on both sides of (22) yields $A *_N \mathcal{X} = O$. Thus, $\mathcal{X} \in N_M(A)$. Therefore, $N_N(A^\dagger *_M A) = N_M(A)$. Analogously, the other part can be shown. ■
Using Lemma 5.11 to an EP-tensor, one can easily conclude the following corollary.

**Corollary 5.12:** Let \( \mathcal{A} \in \mathbb{C}^{I_1 \times \ldots \times I_K} \) be an EP-tensor. Then, \( N(\mathcal{A}^H) = N(\mathcal{A}) \).

The following result helps to construct an EP-tensor.

**Theorem 5.13:** Let \( \mathcal{B} \in \mathbb{C}^{I_1 \times \ldots \times I_K} \) be an invertible tensor and let a block tensor \( \mathcal{A} \) be defined by
\[
\mathcal{A} = \mathcal{U} *_N \left[ \begin{array}{cc} \mathcal{B} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{array} \right] *_N \mathcal{U}^H,
\]
where \( \mathcal{U} \) is a unitary tensor. Then, \( \mathcal{A} \) is an EP-tensor.

**Proof:** Applying Lemma 5.7 and Theorem 5.10 to (23), we get
\[
\mathcal{A}^\dagger = \mathcal{U} *_N \left[ \begin{array}{cc} \mathcal{B}^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{array} \right] *_N \mathcal{U}^H.
\]
Now, a simple calculation leads to \( \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A}^\dagger *_N \mathcal{A} \), i.e. \( \mathcal{A} \) is an EP-tensor. \( \blacksquare \)

Hermitian, skew-Hermitian, and unitary tensors are also EP-tensors as these are normal. As every normal tensor is unitarily diagonalizable and so is EP by the above theorem. Another interesting property of this class of tensors is that the Moore–Penrose inverse of an EP-tensor coincides with the Drazin inverse [29] of a tensor. Next, we establish a relation between \( \sigma(\mathcal{A}) \), \( W(\mathcal{A}) \) and \( \frac{1}{W(\mathcal{A}^\dagger)} \) for an EP-tensor.

**Theorem 5.14:** Let \( \mathcal{A} \in \mathbb{C}^{I_1 \times \ldots \times I_K} \) be an EP-tensor. Then,
\[
\sigma(\mathcal{A}) \subseteq W(\mathcal{A}) \cap \frac{1}{W(\mathcal{A}^\dagger)}.
\]

**Proof:** Let \( \lambda \) be an eigenvalue of \( \mathcal{A} \). Suppose that \( \lambda = 0 \). Then, \( 0 \in \sigma(\mathcal{A}) \subseteq W(\mathcal{A}) \), hence \( 0 \in W(\mathcal{A}^\dagger) \). Thus, \( 0 \in \frac{1}{W(\mathcal{A}^\dagger)} \). Therefore, \( 0 \in W(\mathcal{A}) \cap \frac{1}{W(\mathcal{A}^\dagger)} \). Suppose that \( \lambda \neq 0 \). Let \( \mathcal{X} \in \mathbb{C}^{I_1 \times \ldots \times I_K} \) be a unit eigentensor corresponding to the eigenvalue \( \lambda \).

Now,
\[
\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{X} = \lambda \mathcal{A}^\dagger *_N \mathcal{X},
\]
which implies that \( \mathcal{A} *_N \mathcal{X} = \lambda \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{X} \). Thus, \( \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{X} = \mathcal{X} \) and hence \( \mathcal{X}^H *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{X}^H \). Again, (24) yields
\[
\mathcal{A}^\dagger *_N \mathcal{X} = \frac{1}{\lambda} \mathcal{A} *_N \mathcal{A} *_N \mathcal{X}
\]
\[
= \frac{1}{\lambda} \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{X} \quad (\because \mathcal{A} \text{ is EP})
\]
\[
= \frac{1}{\lambda} \mathcal{X}.
\]
Thus, \( (A^\dagger \ast_N X, X') = \frac{1}{\lambda} \in W(A^\dagger) \).

Hence, \( \lambda \in W(A) \cap \frac{1}{W(A^\dagger)} \). Thus, the claim.

Let \( A, B \in \mathbb{C}^{I_{1} \times \ldots \times I_{n}} \), then we define a block tensor [28]

\[
A \oplus B = \begin{bmatrix} A & O \\ O & B \end{bmatrix} \in \mathbb{C}^{I_{1} \times \ldots \times I_{n}},
\]

where \( O \in \mathbb{C}^{I_{1} \times \ldots \times I_{n}} \) and \( J_i = 2I_i \) where \( i = 1, 2, \ldots, N \). The next result provides a procedure to calculate the Moore–Penrose inverse of a special tensor.

**Theorem 5.15:** Let \( \{U_1, U_2, \ldots, U_r\} \) and \( \{V_1, V_2, \ldots, V_r\} \) be two orthonormal subsets of \( \mathbb{C}^{I_{1} \times \ldots \times I_{n}} \). If \( A = U_1 \ast_N V_1^H + U_2 \ast_N V_2^H + \ldots + U_r \ast_N V_r^H \), then \( A^\dagger = V_1 \ast_N U_1^H + V_2 \ast_N U_2^H + \ldots + V_r \ast_N U_r^H \), and \( W(A^\dagger) = W(A^H) \).

**Proof:** Let \( X = V_1 \ast_N U_1^H + V_2 \ast_N U_2^H + \ldots + V_r \ast_N U_r^H \). It can be easily seen that \( X \) satisfies all the Moore–Penrose equations for \( A \). Since Moore–Penrose inverse of \( A \) is unique, \( A^\dagger = X \). Form the notion of conjugate transpose it is clear that \( A^\dagger = A^H \). Hence, the result.

The following result gives an inequality between the product of spectral norm of a tensor with its Moore–Penrose inverse and their product of numerical radii.

**Theorem 5.16:** Let \( O \neq A \in \mathbb{C}^{I_{1} \times \ldots \times I_{n}} \). Then, for the spectral norm \( \| \cdot \| \),

\[
1 \leq \| A \| \| A^\dagger \| \leq 4w(A)w(A^\dagger).
\]

**Proof:** Consider \( O \neq A \in \mathbb{C}^{I_{1} \times \ldots \times I_{n}} \). Then, there exists \( X \) such that \( A \ast_N X \neq O \). So, \( A \ast_N A^\dagger \ast_N A \ast_N X = A \ast_N X \), which implies \( 1 \in \sigma(A \ast_N A^\dagger) \). Since \( A \ast_N A^\dagger \) is idempotent and Hermitian, so \( W(A \ast_N A^\dagger) = [0, 1] \) and hence \( w(A \ast_N A^\dagger) = 1 \). Now, using Theorem 4.3, we get

\[
1 = w(A \ast_N A^\dagger) \leq \| A \ast_N A^\dagger \| \leq \| A \| \| A^\dagger \| \leq 4w(A)w(A^\dagger).
\]

Next, we provide an example to verify the above inequality.

**Example 5.17:** Consider a tensor \( A \in \mathbb{R}^{2 \times 2 \times 2} \) such that

| \( A(\cdot, \cdot, 1, 1) \) | \( A(\cdot, \cdot, 2, 1) \) | \( A(\cdot, \cdot, 1, 2) \) | \( A(\cdot, \cdot, 2, 2) \) |
|---|---|---|---|
| 2 | 5 | 7 | 9 |
| 0 | 11 | 1 | -1 |

Then, the Moore–Penrose inverse of \( A \), \( A^\dagger \in \mathbb{R}^{2 \times 2 \times 2} \), is

Here, \( \| A \| = 19.9331, \| A^\dagger \| = 1.0076, w(A) = 18.9853 \) and \( w(A^\dagger) = 0.8253 \). Thus, \( 1 \leq \| A \| \| A^\dagger \| \leq 4w(A)w(A^\dagger) \) holds.
6. Conclusions

In this paper, we have introduced the notion of the numerical range and numerical radius of an even-order square tensor via the Einstein product to prove the convexity of the numerical range of a tensor. As an application, we have established some sufficient conditions involving these notions for a tensor to be unitary. Several examples are considered for plotting the boundary of the numerical range of different types of tensors. We have provided a characterization of a normal tensor in terms of normality of its Moore–Penrose inverse. The connection between the spectrum of an EP-tensor, and the numerical ranges of a tensor and the numerical ranges of the Moore–Penrose inverse of the same tensor, has been illustrated. Finally, we have presented an upper bound for the product of the numerical radii of a tensor and its Moore–Penrose inverse.

Acknowledgements

The authors are thankful to Dr. Nachiketa Mishra for giving his precious time to a fruitful discussion. The authors also thank the anonymous referee for his/her comments that have significantly improved the quality of the paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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