Viscosity gradient-driven instability of ‘shear mode’ in a strongly coupled plasma

D Banerjee\textsuperscript{1,3}, M S Janaki\textsuperscript{1}, N Chakrabarti\textsuperscript{1} and M Chaudhuri\textsuperscript{2}

\textsuperscript{1} Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata-700 064, India
\textsuperscript{2} Max-Planck-Institut für Extraterrestrische Physik, 85741 Garching, Germany

E-mail: debabrata.banerjee@saha.ac.in

New Journal of Physics \textbf{12} (2010) 123031 (8pp)
Received 20 August 2010
Published 23 December 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/12/123031

Abstract. The influence of the viscosity gradient (due to shear flow) on low-frequency collective modes in a strongly coupled dusty plasma is analyzed. It is shown that for a well-known viscoelastic plasma model, the velocity shear-dependent viscosity leads to instability of the shear mode. The inhomogeneous viscous force and velocity shear coupling supply the free energy for the instability. The combined strength of the shear flow and viscosity gradient dominates over any stabilizing force and makes the shear mode unstable. The implications of this novel instability and its applications are briefly described.

Contents

1. Introduction 2
2. Basic equations and equilibrium 3
3. Stability analysis 5
4. Summary 7
References 7

\textsuperscript{3} Author to whom any correspondence should be addressed.
1. Introduction

In complex physical systems, for example multispecies charged fluid (dusty plasma), various physical processes exist and these interact simultaneously. The stability properties of such systems are complicated due to the presence of various free energy sources that ultimately lead to instabilities. Despite a large amount of experimental and theoretical effort, it has not been possible to identify all of the sources of free energy available in such a complex system. In this paper, we have identified a free energy source (viscosity gradient due to velocity shear) in a complex dusty plasma system and demonstrate a novel instability of the ‘shear mode’ due to the combined effect of the viscosity gradient and velocity shear. The collective modes in dusty plasmas have been the object of serious study in recent years due to their novel character and wide applications. Normally in a three-component plasma, in addition to the electrons and the most abundant ion species, there is an additional species with a different mass and charge whose abundance is not negligible compared to other constituents. This additional heavy micron-size species, having a wide range of values for the mass-to-charge ratio, is referred to as ‘dust’ in the dusty plasma literature [1]. The presence of the new species is expected to result in new effects on the collective-mode behavior in the plasma [2]. This is because the various species are mutually coupled through electromagnetic forces. In such a scenario, the plasma is able to support many new modes as compared to those in a simple electron–ion plasma [3]. Due to the large amount of charge on a single dust particle, the dust fluid can also exhibit strong coupling behavior, which shows the strong viscous properties of the medium even leading to viscoelastic behavior [3]. The strongly coupled complex plasma has been realized in different experiments [4]–[7]. The strength of the coupling is characterized by the Coulomb coupling parameter \( \Gamma = q_d^2 / (k_B T_d a) \), where \( q_d \) is the charge on the dust grains, \( a \approx n_d^{-1/3} \) is the average distance between them for density \( n_d \), \( T_d \) is the temperature of the dust component and \( k_B \) is the Boltzmann constant [8]. In the regime of \( \Gamma \) from 1 to \( \Gamma_c \) (a critical value beyond which the system becomes crystalline), both viscosity and elasticity are equally important and these properties together are known as viscoelasticity. When \( \Gamma > \Gamma_c \), viscosity disappears and only elasticity dominates over the system. Experiments [6] have also shown that as \( \Gamma \) increases, the dust components become strongly coupled, and for large \( \Gamma \) values, the dust component becomes crystalline. This phenomenon, plasma condensation, is useful for studying phase transitions [7, 9] and low-frequency wave propagation [10, 11]. It has been shown that the strong correlations make the dusty plasma system rather rigid so that it can support a transverse ‘shear mode’ [3]. This shear mode has also been found experimentally [12] and its variants theoretically [13].

An interesting property observed in the case of complex dusty plasmas is the strong density dependence of the viscosity parameter [14], and this owes its existence to the large amount of charge on each dust particle. Recent experiments [15] reveal that complex-plasma fluid has the signature of a non-Newtonian property similar to other non-Newtonian fluids. Beyond some critical value of the velocity shear rate, the medium shows shear-thinning property, which means that the coefficient of viscosity decreases with an increase in shear rate. Based on experimental input, Ivlev et al [15] have shown the power-law dependence of viscosity on velocity shear. The experiment has been performed with gas-induced shear flow for different discharge currents and by applying laser beams of different power. Hence, Ivlev et al have measured the shear viscosity for a wide range of velocity shear rates and confirmed the shear thinning property.
over a considerable range. Very recently, a similar experiment has been reported by Gavrikov et al [18] in a dusty plasma liquid. Simulation work in this direction has also been reported [19].

Motivated by these experimental results, we have investigated the effect of the equilibrium viscosity gradient on a shear mode in a strongly coupled plasma. The shear thinning property with the generalized Oldroyd-B model [16, 17] has been studied in a neutral viscoelastic fluid. In this paper, we have demonstrated that indeed a novel instability exists due to the coupling of shear flow to the velocity fluctuations via the velocity shear-induced viscosity gradient.

2. Basic equations and equilibrium

In a standard fluid description of dusty plasma for studying low-frequency (\(\omega \ll kv_{\text{th,e}}, kv_{\text{th,i}}\)) phenomena, normally we treat electrons and ions as a light fluid which can be modeled by a Boltzmann distribution, neglecting their inertial effects in the momentum equations. This is justified because due to higher temperature and smaller electric charge compared to dust, they can easily thermalize and give rise to a Boltzmann distribution. The ion and electron densities can be written in this way, \(n_{\text{e(i)}} = n_{0\text{e(i)}} \exp[\pm e\phi/T_{\text{e(i)}}]\), where \(n_{0\text{e(i)}}\), \(T_{\text{e(i)}}\) correspond to the equilibrium densities and temperatures for the electrons (ions). Here, \(\phi\) is the electrostatic potential.

The dust component, on the other hand, can be described by the generalized hydrodynamic (GH) equation described in Frenkel’s book [21]. We follow the same procedure and write the generalized equation of motion of dust fluid in a viscoelastic medium, 

\[
\left(1 + \tau \frac{\partial}{\partial t}\right) \rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \mathbf{v} - n_d q \mathbf{E} + \nabla p_d = \frac{\partial \sigma_{ij}}{\partial x_j},
\]  

(1)

where \(\mathbf{v}\) is the dust fluid velocity, \(\rho = m_d n_d\) is the mass density of dust fluid, \(n_d\) is the corresponding number density, \(q\) is the charge on a dust particle, and \(p_d (= n_d T_d)\) is the dust pressure, where \(T_d\) is the dust temperature. The parameter \(\tau\) is the relaxation time of the medium [21] and viscosity tensor \(\sigma_{ij}\) is given for an incompressible medium by

\[
\sigma_{ij} = \eta(S) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).
\]

Here, \(\eta\) is the coefficient of shear viscosity. For a Newtonian fluid, \(\eta\) is a constant. However, for a non-Newtonian fluid, \(\eta\) depends on the scalar invariants of the strain tensor. For an incompressible fluid, it has been shown [20, 22] that the scalar invariant can be written as 

\[
I = \sum_i \sum_j \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).
\]

In this paper, we consider an incompressible plasma with constant mass density, and since the medium is non-Newtonian, the viscosity coefficient can be considered to be a function of the scalar invariant. Hence, the viscosity parameter is taken to be of the form \(\eta(S)\), where \(S = \sqrt{I/2}\).

It has been shown that in the kinetic limit \(\tau \partial/\partial t \gg 1\), the linearized equation (1) gives rise to a ‘shear wave’ whose velocity is given by \(V_s = \sqrt{\eta/\rho \tau}\), where \(\eta\) is a constant [3].
Figure 1. Dependence of shear viscosity on velocity shear rate. In the low shear rate region it is Newtonian (I), and in the high shear rate shear thickening (III). Our interest is in studying region II, where $\eta$ decreases with the velocity shear rate.

We would like to investigate the dynamics of this mode in the presence of the velocity shear-dependent viscosity coefficient. In the kinetic limit, 1 can be neglected with respect to $\tau \partial / \partial t$ in equation (1). We assume that the equilibrium velocity is directed along the $y$-direction and has variation in the $x$-direction, i.e. $v_0 = v_{0y}(x) \hat{e}_y$, where $\hat{e}_y$ is a unit vector along the $y$-direction. It is clear that the left-hand side of equation (1) will not contribute in an equilibrium situation, so that the equilibrium is described by the equation

$$\frac{d}{dx} \left[ \eta(S_0) \frac{dv_{0y}}{dx} \right] = 0,$$

where $S_0$ is the equilibrium value of the shear parameter. For small velocity fluctuations, we can write $S = S_0 + S_1$, and it can be shown that $S_0 = d v_{0y}/dx$ and $S_1 = (\partial v_{1x}/\partial y + \partial v_{1y}/\partial x)$. Recently, Ivlev et al [15] proposed a power-law model for the functional dependence of $\eta(S_0)$ and performed an experiment to show the shear thinning property of dusty plasmas. In that paper, they showed that $\eta$ remains constant for low shear rate, and after some critical value, $\eta$ decreases with an increase of $S_0$. Shear thinning behavior exists for a wide range of velocity shear, and for some very high shear rate, $\eta$ increases with $S_0$. The schematic diagram in figure 1 presents the variation of shear viscosity with velocity shear rate. Here, we concentrate on the shear thinning region (where $\eta$ decreases with $S_0$) and use the same model consistent with the experiment. The functional form can be written as

$$\eta(S_0) = \tilde{\eta}_0 \left( \frac{S_0}{S_c} \right)^{-2\delta/1+\delta},$$

where $\delta$ is a positive exponent and $\tilde{\eta}_0$ is a constant having the dimension of the viscosity coefficient. If we define $2\delta/(1+\delta) = \alpha$, then the parameter $\alpha$ is a positive nonzero constant $\alpha < 2$ and $S_c$ is of the order of unity. The power-law model can only depict the shear thinning region (region II) of figure 1 and it can explain neither the constant behavior of viscosity in low shear region (I) nor the shear thickening property for very high shear rate (region III). In low shear region (I), the system behaves as a Newtonian fluid. In high shear region (III), another power-law model like $\eta \propto (S_0)^\delta$ with a different positive exponent $\delta$ describes the shear...
thickening behavior. When the equilibrium profile of \( \eta(S_0) \) from equation (3) is substituted in equation (2), we find that \( \mathrm{d}v_0/\mathrm{d}x \) is a constant and hence we can write the equilibrium velocity as \( v_{10}(x) = v_0^\prime x \), where \( v_0^\prime \) is a constant having the dimension of frequency.

### 3. Stability analysis

We restrict our attention to two-dimensional incompressible perturbations such that all variations are in the \( x-y \) plane. The incompressibility condition given by \( \nabla \cdot \mathbf{v} = 0 \) is consistent with the equilibrium flow. Now we perturb the system around this equilibrium flow writing \( \mathbf{v} = v_{10}(x) \hat{e}_x + \mathbf{v}_1(x, y, t) \), and after straightforward algebra we find that \( v_{1x} \) satisfies a differential equation that is given by

\[
\frac{\partial}{\partial t} \left[ \frac{\partial}{\partial t} + v_0^\prime \frac{\partial}{\partial y} \right] \nabla_\perp^2 v_{1x} = \left( \frac{\eta_0}{\tau \rho} \right) \nabla_\perp^2 v_{1x} + \left( \frac{\eta_0^\prime v_0^\prime}{\tau \rho} \right) \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right)^2 v_{1x}, \tag{4}
\]

where \( \nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \). We note here that if the velocity shear is absent in equations (4), we get back the shear mode whose dispersion relation can be written as \( \omega^2 = (k_x^2 + k_y^2) V_s^2 \).

For an inhomogeneous plasma, the general solution of equation (4) including viscosity gradient can be obtained in various ways. The traditional starting point of an investigation of linear plasma stability is the eigenvalue analysis in which we assume a solution of the form \( v_{1x} = v_1(x) \exp(ik_y y - i\omega t) \), where \( k_y \) is the wave vector in the \( y \)-direction and \( \omega \) is the frequency of the mode. We note here that Fourier-type solutions have been considered only in the \( y \)-direction because inhomogeneity is present in the \( x \)-direction through velocity shear. The perturbed variable \( v_1 \), in general, will satisfy a differential equation, which is given by

\[
\frac{\omega^2}{k_y^2} \left( 1 + \frac{\eta_0^\prime v_0^\prime}{\eta_0} \right) \frac{\mathrm{d}^4}{\mathrm{d}x^4} + \left[ \omega^2 - \omega k_y v_0^\prime x - 2\omega_x^2 \left( 1 - \frac{\eta_0^\prime v_0^\prime}{\eta_0} \right) \right] \frac{\mathrm{d}^2}{\mathrm{d}x^2} v_1 + \left[ k_x^2 \omega_x^2 \left( 1 + \frac{\eta_0^\prime v_0^\prime}{\eta_0} \right) - k_y^2 (\omega^2 - \omega k_y v_0^\prime x) \right] v_1 = 0, \tag{5}
\]

where \( \omega_x^2 = k_x^2 \eta_0/\rho \tau \) is the frequency of the shear wave in an inhomogeneous plasma. From the above differential equation, first we can carry out the local analysis in which one uses the approximation \( kL \gg 1 \). This implies that the perturbation wavelength \( k^{-1} \) is much smaller than the inhomogeneity scale length \( L = v_0/v_0^\prime \). For the present problem, the local analysis is carried out by considering that the perturbed quantity \( v_1 \) has also an exponential variation in \( x \), i.e. \( v_1 \sim \exp(ik_x x) \), where \( k_x \) is the wave vector in the \( x \)-direction. Substituting in equation (5), the dispersion equation is obtained as

\[
\omega^2 = \frac{\omega_x^2 (k_x^2 + k_y^2)}{k_x^2} \left[ 1 + \frac{\eta_0^\prime v_0^\prime}{\eta_0} \left( \frac{k_x^2 - k_y^2}{k_y^2 + k_x^2} \right)^2 \right], \tag{6}
\]

where \( \eta_0^\prime = \mathrm{d}\eta_0/\mathrm{d}v_0^\prime \) and \( \omega \gg k_x v_0^\prime \). Now, recalling the form of viscosity \( \eta_0 \) from equation (3), we can write \( \eta_0^\prime v_0^\prime/\eta_0 = -\alpha \). It is clear that the shear mode will be unstable if \( \alpha > (k_x^2 + k_y^2)^2/(k_x^2 - k_y^2)^2 \) and the growth rate is of the order of shear frequency.
Next, we consider the nonlocal analysis of equation (5) in which this eigenvalue equation may be solved to obtain well-behaved solutions corresponding to unstable eigenvalues. Here, we are looking for a long radial ($r$) scale solution for the differential equation and therefore the fourth-order derivative is subdominant compared with the second. Ignoring the fourth derivative in equation (5), we can reproduce the character of the mode with very little change (this is apparent from the dispersion relation). This assumption simplifies the algebra without taking away the essential physics. The desired eigenvalue equation can be written as

$$\frac{d^2 v_1}{dx^2} - k_y^2 \left[ \frac{\omega^2 - \omega_s^2(1 - \alpha) - \omega k_y v_0^\prime x}{\omega^2 - 2\omega_s^2(1 + \alpha) - \omega k_y v_0^\prime x} \right] v_1 = 0. \quad (7)$$

For the condition $\omega k_y v_0^\prime / [\omega^2 - 2\omega_s^2(1 + \alpha)] \ll 1$, which implies that when the shear rate is small compared to the frequency of the mode, the above equation can be written in terms of the well-known Weber equation, which is given by

$$\frac{d^2 v_1}{d\xi^2} - (\xi^2 - K) v_1 = 0, \quad (8)$$

where

$$\xi = \left[ k_y^2 \beta_2^2 \left( \frac{\beta_2 - \beta_1}{\beta_1} \right) \right]^{1/4} \left( x + \frac{1}{2\beta_2} \right), \quad K = -\frac{k_y}{4} \left( \frac{1}{\beta_2} + \frac{3}{\beta_1} \right) \sqrt{\frac{\beta_1}{\beta_2 - \beta_1}},$$

and

$$\beta_1 = \frac{k_y v_0^\prime \omega}{\omega^2 - \omega_s^2(1 - \alpha)}, \quad \beta_2 = \frac{k_y v_0^\prime \omega}{\omega^2 - 2\omega_s^2(1 + \alpha)}.$$

The solution of equation (8) for the lowest-order eigenmode is given by

$$v_1 \sim \exp \left[ -\frac{1}{2} k_y \beta_2 \sqrt{\left( \frac{\beta_2 - \beta_1}{\beta_1} \right)} \left( x + \frac{1}{2\beta_2} \right)^2 \right], \quad (9)$$

representing the existence of an unstable eigenmode. The condition for the bounded solution is $\text{Re} (\beta_2 [(\beta_2/\beta_1) - 1])^{1/2} > 0$. The behavior of the eigenfunction $v_1$ at $x \to \pm\infty$ is bounded and the typical mode width

$$\Delta \sim \left[ \frac{1}{k_y \beta_2} \sqrt{\frac{\beta_1}{\beta_2 - \beta_1}} \right]^{1/2}.$$

The corresponding dispersion relation is given by

$$\omega^2 - \frac{\omega_s^2}{4} (5 - \alpha) = -v_0^\prime \omega_0 \sqrt{\frac{1 + 3\alpha}{\omega^2 - 2\omega_s^2(1 + \alpha)}}, \quad (10)$$

where $\alpha = -\eta_0 v_0^\prime / \eta_0$ and $\omega_s^2 = k_y^2 \eta_0 / \rho \tau$. In a homogeneous plasma, i.e. when $v_0^\prime = \alpha = 0$, we get back the shear mode, i.e. $\omega^2 \sim \omega_s^2$. In the presence of velocity shear and velocity shear-induced viscosity gradient, we have solved equation (10) and found that for $\alpha < 2$ there is one unstable root for real $\omega > 0$. The growth rate for instability for the given range of $\alpha$ can be seen in figure 2. The shear mode is more unstable when the velocity shear is stronger.
Figure 2. Normalized growth rate $\gamma/\omega_s$ as a function of $\alpha = |\eta_0' v_0/\eta_0|$, showing the growth rate of shear mode for different velocity shear parameters.

4. Summary

We have studied the effect of the velocity shear-induced viscosity gradient of low-frequency shear waves in a viscoelastic dusty plasma. The dust dynamics have been modeled by including velocity shear-dependent viscosity, which is the main ingredient to drive a new instability in a complex plasma. The principal effect on the generation of the novel instability is the velocity dependence of viscosity that leads to a coupling between velocity fluctuations and equilibrium flow. The variation in the velocity is responsible for viscosity modulation, which provides feedback to the velocity through the momentum equation. For positive feedback of the velocity, an instability is triggered. This novel low-frequency instability disappears when viscosity is uniform and we are left with a shear wave. We would like to point out that the instability theoretically investigated in this paper has not yet been observed in real experiment; detailed experimental investigation of it is therefore of great interest. It would be of interest, therefore, to look for shear-wave-driven instabilities discussed in this model calculation which is based on the experimental finding for a shear thinning region.

References

[1] Rao N N, Shukla P K and Yu M Y 1990 Planet. Space Sci. 38 543–6
[2] Ganguli G and Rudakov L 2004 Phys. Rev. Lett. 93 135001
[3] Kaw P K and Sen A 1998 Phys. Plasmas 5 3552–9
[4] Chu J H and Lin I 1994 Phys. Rev. Lett. 72 4009
[5] Hayashi Y and Tachibana K 1994 Japan J. Appl. Phys. 33 L804–6
[6] Thomas H, Morfill G E, Demmel V, Goree J and Möhlmann B F D 1994 Phys. Rev. Lett. 73 652–5
[7] Thomas H M and Morfill G E 1996 Nature 379 806–9

New Journal of Physics 12 (2010) 123031 (http://www.njp.org/)
[8] Ikeji H 1986 Phys. Fluids 29 1764–6
[9] Morfill G E, Thomas H M, Konopka U and Zuzic M 1999 Phys. Plasmas 6 1769–80
[10] Pieper J and Goree J 1996 Phys. Rev. Lett. 77 3137–40
[11] Melzer A, Homann A and Piel A 1996 Phys. Rev. E 53 2757–66
[12] Pramanik J, Prasad G, Sen A and Kaw P K 2002 Phys. Rev. Lett. 88 175001
[13] Sorasio G, Shukla P K and Resendes D P 2003 New J. Phys. 5 81
[14] Steinberg V, Ivlev A V, Kompaneets R and Morfill G E 2008 Phys. Rev. Lett. 100 254502
[15] Ivlev A V, Steinberg V, Kompaneets R, Hofner H, Sidorenko I and Morfill G E 2007 Phys. Rev. Lett. 98 145003
[16] Arada N and Sequeira A 2003 Math. Models Methods Appl. Sci. 13 1303–23
[17] Nadau L and Sequeira A 2007 Comput. Math. Appl. 53 547–67
[18] Gavrikov A V, Goranskaya D N, Ivanov A S, Petrov O F, Timirkhanov R A, Vorona N A and Fortov V E 2010 J. Plasma Phys. 76 579–92
[19] Donko Z, Goree J, Hartmann P and Kutasi K 2006 Phys. Rev. Lett. 96 145003
[20] Byron Bird R, Armstrong R C and Hassger O 1987 Dynamics of Polymeric Liquids vol 1 (New York: Wiley)
[21] Frenkel Y I 1946 Kinetic Theory of Liquids (Oxford: Clarendon)
[22] Malkin A Y and Isayev A I 2006 Rheology, Concepts, Methods, and Applications (Toronto, ON: ChemTec)