REMARKS ON THE ASYMMPTOTIC HECKE ALGEBRA

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Abstract. Let $G$ be a split reductive $p$-adic group, $I \subset G$ be an Iwahori subgroup, $\mathcal{H}(G)$ be the Hecke algebra and $\mathcal{C}(G) \supset \mathcal{H}(G)$ be the Harish-Chandra Schwartz algebra. The purpose of this note is to define (in spectral terms) a subalgebra $\mathcal{J}(G)$ of $\mathcal{C}(G)$, containing $\mathcal{H}(G)$, which we consider as an algebraic version of $\mathcal{C}(G)$. We show that the subalgebra $\mathcal{J}(G)$ is isomorphic to the Lusztig’s asymptotic Hecke algebra $\mathcal{J}$ and explain a relation between the algebra $\mathcal{J}(G)$ and the Schwartz space of the basic affine space studied in [2].

1. Introduction and statement of the results

1.1. Notation. Let $F$ be a non-archimedian local field with ring of integers $\mathcal{O}$; we shall choose a generator $\pi$ of the maximal ideal of $\mathcal{O}$. Typically, we shall denote algebraic varieties over $F$ with boldface letters (e.g. $G$, $X$ etc.) and the corresponding sets of $F$-points – with corresponding Roman letters (e.g. $G$, $X$ etc.).

In what follows we fix a connected split reductive group $G$ over $F$ with a Borel subgroup $B$ it unipotent radical $U$, maximal split torus $T = B/U$. Let $\Lambda$ be the lattice of cocharacters of $T$ and $\Lambda^\vee$ be the lattice of characters of $T$. We write $K_0 = G(\mathcal{O})$ and denote by $I \subset K_0$ an Iwahori subgroup of $G$. We denote by $\mathcal{H}(G)$ the full Hecke algebra of $G$ and by $\mathcal{H}(G, I)$ the Iwahori-Hecke subalgebra; we shall also denote by $\mathcal{H}_{\text{aff}}$ the corresponding algebra over $\mathbb{C}[v, v^{-1}]$ (thus $\mathcal{H}(G, I)$ is obtained from $\mathcal{H}_{\text{aff}}$ by specializing $v$ to $q^{1/2}$. We denote by $\mathcal{M}(G)$ the category of smooth representations of $G$. This is the same as the category of locally unital modules over $\mathcal{H}(G)$. For any smooth representation $(\pi, V)$ we denote by $(\pi^\vee, V^\vee)$ the subrepresentation of smooth vectors in the representation of $G$ on the space of linear functionals on $V$. For any $v \in V, \lambda \in V^\vee$ we denote by $m_{v, \lambda}: G \to \mathbb{C}$ the matrix coefficient $m_{v, \lambda}(g) := \lambda(\pi(g)(v))$.

1.2. Matrix Paley-Wiener theorem. Let $P$ be a parabolic subgroup of $G$ with a Levi group $M$. The set $X_M$ of unramified characters of $M$ is equal to $\Lambda^\vee_M \otimes \mathbb{C}^\times$ where $\Lambda^\vee_M \subset \Lambda^\vee$ is the subgroup of characters of $T$ trivial on $T \cap [M, M]$. So $X_M$ has a structure of a complex algebraic variety; the algebra of polynomial functions on $X_M$ is equal to $\mathbb{C}[\Lambda_M]$ where $\Lambda_M$ is the lattice dual to $\Lambda^\vee_M$. We denote by $X_{M, I} \subset X_M$ the subset of unitary characters.

For any $(\sigma, V) \in \mathcal{M}(M)$ we denote by $i_{GP}(\sigma)$ the corresponding unitarily induced object of $\mathcal{M}(G)$. As a representation of $K_0$ this representation is equal to $\text{ind}_{P \cap K_0}^{K_0}(\sigma)$. So for any unramified character $\chi: M \to \mathbb{C}^\times$ the space of the representation $i_{GP}(\sigma \otimes \chi)$ is isomorphic to the space $V_{\chi}$ of the representation $i_{GP}(\sigma)$ and is independent on a choice of $\chi$. Since
$X_M$ has a structure of an algebraic variety over $\mathbb{C}$ it makes sense to say that a family $\eta_{\chi} \in \text{End}(V, \chi \in T^\vee)$ is a regular (or a smooth) function of $\chi$.

We denote by $\text{Forg} : \mathcal{M}(G) \to \text{Vec}$ the forgetful functor, by $\hat{\mathcal{E}}(G) = \{e(\pi)\}$ the ring of endomorphisms of $\text{Forg}$ and define $\mathcal{E}(G) \subset \hat{\mathcal{E}}(G)$ as the subring of endomorphisms $\eta_\pi$ such that

1) For any Levi subgroup $M$ of $G$ and $\sigma \in \text{Ob}(\mathcal{M}(M))$, the endomorphisms $\eta_{G_M(\sigma \otimes \chi)}$ are regular functions of $\chi$.

2) There exists an open compact subgroup $K$ of $G$ such that $\eta_\pi$ is $K \times K$-invariant for every $\pi$.

By definition, we have a homomorphism $\text{PW} : \mathcal{H}(G) \to \mathcal{E}(G)$, $f \mapsto \pi(f)$.

The following is usually called "the matrix Paley-Wiener theorem" (cf. [1], Theorem 25):

**Theorem 1.3.** The map $\text{PW}$ is an isomorphism.

The group $G \times G$ acts on $\mathcal{E}(G)$ in the obvious way. We denote by $\mathcal{E}^I(G) \subset \mathcal{E}(G)$ the subring $I \times I$-invariant elements. It is clear that we can interpret $\mathcal{E}^I(G)$ as a subring of the ring of endomorphisms of the forgetful functor $\text{Rep}(\mathcal{H}(G, I)) \to \text{Vec}$.

1.4. **Harish-Chandra algebra.** Recall that for any $g \in G$, there exists a unique dominant coweight $\lambda(g)$ of $T$ such that $g \in G(O)\pi^{\lambda(g)}G(O)$. Let us set $\Delta(g) = q^{\langle \lambda, \rho \rangle}$. Then we say that a function $f : G \to \mathbb{C}$ is a Schwartz function if

(a) There exists an open compact subgroup $K$ of $G$ such that $f$ is both left and right $K$-invariant.

(b) For any polynomial function $p : G \to F$ and $n > 0$, there exists a constant $C = C_{p,n} \in \mathbb{R}_{>0}$ such that

$$\Delta(g)|f(g)| \leq C \ln^{-n}(1 + |p(g)|)$$

for all $g \in G$.

We denote by $\mathcal{C}(G)$ the space of all Schwartz functions. It is known that $\mathcal{C}(G)$ has an algebra structure with respect to convolution (cf. [7], Chapter 4 or [8]).

Obviously we have $\mathcal{H}(G) \subset \mathcal{C}(G)$.

For an open compact subgroup $K$ of $G$ we denote by $\mathcal{C}(G, K)$ the space of $K \times K$-invariants in $\mathcal{C}(G)$.

Below we recall the spectral description of $\mathcal{C}(G)$.

1.5. **Tempered representations.** Let $(\pi, V)$ be a representation of $G$ of finite length with central character $\eta : Z(G) \to \mathbb{C}^*$. Recall that $\pi$ is called tempered if

1) $\pi$ is unitary. In particular, $\eta$ is unitary (i.e., it takes values in $S^1 \subset \mathbb{C}^*$). In this case the absolute values $|m_{v,\lambda}|$ of matrix coefficients of $V$ are functions on $G/Z(G)$.

2) For any $\varepsilon > 0$ and any matrix coefficient $m_{v,\lambda}$ of $\pi$ we have

$$|m_{v,\lambda}| \in L^{2+\varepsilon}(G/Z(G)).$$

The following facts are well-known (cf. [7], [8]):
(F1) Let $\pi$ be a tempered representation of $G$. Then the action of $H(G)$ extends naturally to an action of $C(G)$.

(F2) Let $P$ be a parabolic subgroup of $G$ with a Levi group $M$. Let $\sigma$ be a tempered irreducible representation of $M$. Then the representation $i_{GP}(\sigma)$ is tempered.

(F3) For a generic unitary character $\chi : M \to S^1$ the representation $i_{GP}(\sigma \otimes \chi)$ (which is tempered by F2) is irreducible.

We denote by $M_t(G) \subset M(G)$ the subcategory of tempered representations. As follows from (F2), for any tempered representation $\sigma$ of $M$ and a unitary character $\chi$ of $M$ the representations $i_{GP}(\sigma \otimes \chi)$ belong to $M_t(G)$.

Let $E_t(G)$ be the subring of endomorphisms $\{\eta\}$ of the forgetful functor $For_g : M_t(G) \to Vect$ such that

(1) $\eta_{iGP(\sigma \otimes \chi)}$ is a smooth function of $\chi \in X_M, M$ for any Levi subgroup $M$ of $G$ and $\sigma \in \text{Ob}(M_t(M))$.

(2) $\eta$ is $K \times K$-invariant for some open compact subgroup of $G$.

The fact F1 can be upgraded to the following version of the matrix Paley-Wiener theorem (cf. [8]):

**Theorem 1.6.** The map $f \mapsto \pi(f)$ defines an isomorphism between $C(G)$ and $E_t(G)$.

We denote by $E'_t(G)$ the subring of $I \times I$-invariant elements of $E_t(G)$.

1.7. Asymptotic Hecke algebra. Recall that we denote by $H_{aff}$ the "algebraic" version of $H(G, I)$ which is an algebra over $\mathbb{C}[v, v^{-1}]$. Let us assume that $G$ is of adjoint type. In [4], G. Lusztig defined the so-called asymptotic Hecke algebra $J$ (we are going to recall the definition in Section 2). This is an algebra over $\mathbb{C}[[v, v^{-1}]]$ and there is a canonical embedding $H(G, I) \to J \otimes \mathbb{C}[v, v^{-1}]$ which becomes an isomorphism after some completion. Moreover, one can show that the specialization of this embedding to any $q \in \mathbb{C}^*$ is also injective. Hence we get an embedding $H(G, I) \hookrightarrow J$.

One of the main purposes of this paper is to formulate and prove a version of matrix Paley-Wiener theorem for $J$.

Let $P$ be a parabolic subgroup with Levi group $M$. We say that an unramified character $\chi : M \to \mathbb{C}^*$ is (non-strictly) positive if for any coroot $\alpha$ of $G$, such that the corresponding root subgroup lies in the unipotent radical $U_P$ of $P$ (which in particular defines a homomorphism $\alpha : F^* \to Z(M)$), we have $|\chi(\alpha(x))| \geq 1$ for $|x| \geq 1$.

Let $E_J(G)$ be ring of collections $\{\eta_{\pi} \in \text{End}_C(V)\}$ for tempered irreducible $(\pi, V)$ which extend to a rational function $E_{iGP(\sigma \otimes \chi)} \in \text{End}_C(\sigma \otimes \chi)$ for every tempered irreducible representation $\sigma$ of $M$ and which are

a) regular on the set of characters $\chi$ such that $\chi^{-1}$ is (non-strictly) positive.

b) $K$-invariant for some open compact subgroup $K$ of $G$.

As follows from the definition, we have an embedding $E'_J(G) \to E'_t(G)$.

**Theorem 1.8.**

1. Let $(\pi, V)$ be an irreducible tempered representation of $G$. Then the action of $H(G, I)$ on $V^I$ extends uniquely to $J$.

2. Let $P$ be a parabolic subgroup of $G$ with Levi group $M$, let $\sigma$ be an irreducible tempered representation of $M$, and let $\chi$ be a (non-strictly) positive character of $M$.
and \((\pi, V) = i_{GP}(\sigma \otimes \chi^{-1})\). Then the action of \(H(G, I)\) on \(V^I\) extends to an action of \(J\). This extension is unique up to isomorphism.

(3) The map \(f \mapsto \pi(f)\) defines an isomorphism between \(J\) and \(E^I_J(G)\).

It follows immediately from Theorem 1.3, Theorem 1.6 and Theorem 1.8 that we have inclusions \(H(G, I) \subset J \subset C(G, I)\). Theorem 1.8 allows giving the following

**Definition 1.9.** We define \(J(G)\) to be the preimage of \(E_J(G)\) in \(C(G)\). Note that we have natural embeddings \(H(G) \subset J(G) \subset C(G)\).

The algebra \(J(G)\) can be thought of as a ”beyond Iwahori” version of \(J(G, I)\). It follows again from Theorem 1.3 and Theorem 1.6 that we have the embeddings \(H(G) \subset J(G) \subset C(G)\). Also, this definition makes sense for any reductive \(G\).

The proof of Theorem 1.8 is given in the next Section; it is essentially an exercise on manipulating the results from [4], [5] and [6]. In Section 3 we also explain a connection between the algebra \(J(G)\) and the Schwartz space of the basic affine space studied in [2]. In the appendix we give some examples of elements of \(J\) (viewed as functions on \(G\)) for \(G = SL(2, F)\).

1.10. **An algebraic version.** Let us explain a version of Theorem 1.8 which is algebraic in the sense that it doesn’t use the notion of ”positive real number”. To emphasize this we are going to work not over \(\mathbb{C}\) but over an arbitrary algebraically closed field \(K\) of characteristic 0. First, let us make the following definitions:

1. A character \(\chi : F^* \to K^*\) is called special if \(\chi(a) = q^{2r \cdot \text{val}(a)}\) with \(r \in \mathbb{Z}_{>0}\).
2. A representation of \(F^*\) on a \(K\)-vector space \(V\) is called special if there is a non-zero vector in \(V\) on which \(F^*\) acts by a special character.
3. Let \(G, P, M\) be as above. Then a representation \(\sigma\) of \(M\) is called special if there exists a positive coroot \(\alpha : F^* \to Z(M)\) whose composition with \(\sigma\) is special.
4. A representation \(\pi\) of \(G\) is called quasi-tempered if for any parabolic \(P\) with Levi decomposition \(P = MU\), the Jacquet functor \(r_{GP}(\pi)\) is not a special representation of \(M\).
5. Let \(E_J, K\) be the algebra \(\{\eta_\pi \in \text{End}_C(V)| \text{ quasi-tempered irreducible } (\pi, V)\}\) which extend to a rational function \(E_{iGP}(\sigma) \in \text{End}_C(\sigma)\) for every non-special irreducible representation \(\sigma\) of \(M\). For an open compact subgroup \(K\) of \(G\) we denote by \(E_{J, K}(G, K)\) the algebra of \(K \times K\)-invariant elements in \(E_{J, K}\).

With these definitions in mind we have

**Theorem 1.11.** Let \(J_K = J_{\mathbb{Z}} \otimes K\). Then

1. Let \((\pi, V)\) be an irreducible quasi-tempered representation of \(G\). Then the action of \(H(G, I)\) on \(V\) extends uniquely to \(J_K\).
2. Let \(P\) be a parabolic subgroup of \(G\) with Levi group \(M\). Let \(\sigma\) be a non-special representation of \(M\). Let \((\pi, V) = i_{GP}(\sigma)\). Then the action of \(H(G, I)\) on \(V^I\) extends uniquely to \(J_K\).
3. The map \(f \mapsto \pi(f)\) defines an isomorphism between \(J_K\) and \(E_{J, K}(G, I)\).

It is not difficult to show that \(E_{J, K}\) is naturally isomorphic to some algebra of locally constant functions on \(G\) (where the algebra structure is given by convolution) which contains \(H_K(G)\); we shall denote this algebra by \(J_K(G)\). We shall not pursue the details in this paper.
We shall denote $K$ the representation of $H_s, u, \rho$; we shall say that the triple $(s, u, \rho)$ may be 0; we shall say that the triple $(s, u, \rho)$ is admissible if $\rho(s, u, \rho) \neq 0$. This representation may be 0; we shall say that the triple $(s, u, \rho)$ is admissible if $K(s, u, \rho) \neq 0$. We shall denote $K(s, u, \rho, q)$ the specialization of $K(s, u, \rho)$ to $v = q^{1/2}$ viewed as a representation of $H(G, I)$. Moreover, if $\rho$ is some (not necessarily irreducible) representation of 2.2. Representations of $H_{aff}$ and $H(G, I)$. Set $J_A = J \otimes A, J_k = J \otimes k, H_{aff,k} = H_{aff} \otimes k$.

It follows from the above that $H_{aff,k}$ can be regarded as a subalgebra of $J_k$. In what follows we denote by $G^\vee$ the Langlands dual group of $G$ (over $\mathbb{C}$).

Let $(s, u, \rho)$ be a triple where
(a) $s \in G^\vee(\mathbb{C})$ is a semi-simple element,
(b) $u \in G^\vee(\mathbb{C})$ is a unipotent element such that $su = us$,
(c) $\rho$ is an irreducible representation of the group of components of the centralizer $Z_{G^\vee}(s, u)$ of the pair $(s, u)$.

Recall that in [3] Kazhdan and Lusztig define a representation $K(s, u, \rho)$ of $H_{aff}$. This representation may be 0; we shall say that the triple $(s, u, \rho)$ is admissible if $K(s, u, \rho) \neq 0$. We shall denote $K(s, u, \rho, q)$ the specialization of $K(s, u, \rho)$ to $v = q^{1/2}$ viewed as a representation of $H(G, I)$. Moreover, if $\rho$ is some (not necessarily irreducible) representation of
the group of components of \( Z_{G^\vee}(s, u) \), we denote by \( K(s, u, \rho, q) \) the direct sum of \( K(s, u, \rho_i) \) where \( \rho \) is the direct sum of irreducible representations \( \rho_i \). Then the following facts are true:

1. (cf. Theorem 7.12 in [3]) \( K(s, u, \rho, q) \) has a unique simple quotient which we shall denote by \( L(u, s, \rho, q) \).

2. (cf. Theorem 8.2 in [3]) Assume that \( s \) is compact. Then the corresponding representation \( K(s, u, \rho, q) \) of \( H(G, I) \) is tempered and irreducible. Moreover, every irreducible tempered representation of \( H(G, I) \) is isomorphic to \( K(s, u, \rho, q) \) for a unique admissible triple \((s, u, \rho)\) with compact \( s \).

3. (cf. Theorem 6.2 in [3]) Let \( P \) be a parabolic subgroup of \( G \) with Levi group \( M \) and let \( P^\vee \) and \( M^\vee \) be the corresponding parabolic and Levi subgroups in \( G^\vee \). Let also \( p^\vee, m^\vee \) be their Lie algebras. Assume that \( s, u \in M^\vee \). Let \( Z_{M^\vee}(s, u) \) be the centralizer of \((s, u)\) in \( M^\vee \) and let \( Z_{G^\vee}(s, u) \) be the centralizer of \((s, u)\) in \( G^\vee \).

Now let \((g^\vee/p^\vee)_u\) be the kernel of \( 1 - u \) on \( g^\vee/p^\vee \). Assume that \( q^{-i} \) is not an eigenvalue of \( s \) on \((g^\vee/p^\vee)_u \) for every \( i > 0 \). Let \( K_M(s, u, \rho_j, q) \) denote the corresponding representation of \( M \). Then \( \iota_G(K_M(s, u, \rho, q)) \) is isomorphic to \( K(s, u, \tilde{\rho}, q) \) where \( \tilde{\rho} = \text{Ind}_{Z_{G^\vee}(s, u)/Z_{M^\vee}(s, u)}^{Z_{G^\vee}(s, u)} Z_{M^\vee}(s, u) \rho \). In particular, this is true if \( s \) is of the form \( s' \cdot \chi^{-1} \) where \( s' \) is compact and \( \chi \) is a (non-strictly) positive character of \( M \).

In [6] Lusztig proves the following result:

**Theorem 2.3.** Let \( E \) be an irreducible representation of \( J \) and let \( E_k = \rho \otimes k \). Then there exists a unique triple \((s, u, \rho)\) such that \( E_k \mid H_{aff,k} \) is isomorphic to \( K(s, u, \rho) \otimes k \) (we shall denote the latter \( H_{aff,k} \)-module by \( K(s, u, \rho)_k \)). Moreover, every admissible triple \((s, u, \rho)\) arises in this way.

Theorem 2.3 implies that we have a bijection between irreducible representations of \( J \) and admissible triples \((s, u, \rho)\). For any such triple \((s, u, \rho)\) we shall denote by \( E(s, u, \rho) \) the corresponding irreducible representation of \( J \).

Note that by specializing the embedding \( H_{aff} \hookrightarrow J \otimes A \) to \( v = q^{1/2} \) we get a homomorphism \( H(G, I) \to J \) which is injective by Proposition 1.7 of [3]. We now claim the following:

**Theorem 2.4.**

1. Let \( \pi \) be an irreducible tempered representation of \( H(G, I) \). Then \( \pi \) extends uniquely to an irreducible representation of \( J \).

2. Any module of the form \( \iota_G(\sigma \otimes \chi^{-1}) \), where \( \sigma \) is an irreducible tempered representation of the Levi group \( M \) and \( \chi \) is a (non-strictly) positive character of \( M \), extends to an irreducible module over \( J \). This extension is unique up to isomorphism.

3. Let \( M \) be a Levi subgroup of \( G \) and let \((s, u, \rho)\) be an admissible triple for \( M^\vee \) with compact \( s \). Then there exists a \( J \otimes \mathbb{C}[A_M] \)-module \( M(s, u, \rho) \) whose fiber at any non-strictly positive \( \chi \) is isomorphic to the \( J \)-module from (2).

In order to prove Theorem 2.4 we shall use the following result of N. Xi [9]:

**Theorem 2.5.** Let \((s, u, \rho)\) be an admissible triple. Then the restriction of \( E(s, u, \rho) \) to \( H(G, I) \) has a unique irreducible quotient, which is isomorphic to \( L(s, u, \rho, q) \). Moreover,
any irreducible subquotient of the kernel of the map \( E(s, u, \rho) |_{\mathcal{H}(G, I)} \to L(s, u, \rho, q) \) is not isomorphic to \( L(s, u, \rho, q) \).

This result implies the following:

**Corollary 2.6.** The representation \( E(s, u, \rho) |_{\mathcal{H}(G, I)} \) of \( \mathcal{H}(G, I) \) is isomorphic to \( K(s, u, \rho, q) \).

**Proof.** By definition we have an isomorphism \( E(s, u, \rho) |_{\mathcal{H}(G, I)} \simeq K(s, u, \rho, q) \). This obviously implies that

(a) \( \dim E = \dim_k K(s, u, \rho) = \dim C K(s, u, \rho, q) \).

(b) There exists a non-zero homomorphism \( K(s, u, \rho, q) \to E(s, u, \rho) |_{\mathcal{H}(G, I)} \).

According to [3], Theorem 7.12, the representation \( K(s, u, \rho, q) \) has a unique simple quotient \( L(s, u, \rho, q) \). It now follows from Theorem 2.5 that the map \( K(s, u, \rho, q) \to E(s, u, \rho) |_{\mathcal{H}(G, I)} \) is surjective (indeed, otherwise it would land inside the maximal proper submodule of \( E(s, u, \rho) |_{\mathcal{H}(G, I)} \) which doesn’t contain \( L(s, u, \rho, q) \) as a subquotient. On the other hand, any non-zero image of \( K(s, u, \rho, q) \) has a quotient isomorphic to \( L(s, u, \rho, q) \). Since the dimensions of these two modules are equal, it follows that the map \( K(s, u, \rho, q) \to E(s, u, \rho) |_{\mathcal{H}(G, I)} \) is an isomorphism.

### 2.7. Proof of Theorem 2.4

Let \( \pi \) be an irreducible tempered representation of \( \mathcal{H}(G, I) \). Then \( \pi \) is isomorphic to \( K(s, u, \rho, q) \) for some admissible triple \((s, u, \rho)\) with compact \( s \). Now Corollary 2.6 implies that \( \pi \) extends to an irreducible representation of \( J \) and any two such extensions are isomorphic as abstract \( J \)-modules. In other words, any two such extensions are conjugate by means of some \( \text{Aut}_{\mathcal{H}(G, I)}(\pi) = C^\times \). Hence any two such extensions are equal. This proves assertion (1).

Assertion (2) follows in a similar way from statement (3) above Theorem 2.3. Finally, the module \( \mathcal{M}(s, u, \rho) \) is constructed in the following way. Let \( B^\vee \) denote the flag variety of \( G^\vee \) and let \( B^\vee_{s,u} \) denote the variety of \((s, u)\)-fixed points on \( B^\vee \). Then it follows from [6] that \( J \) acts on the equivariant \( K \)-theory \( K_Z(M^\vee)(B^\vee_{s,u}) \) (here \( Z(M^\vee) \) denotes the center of \( M^\vee \)). Moreover, this \( K \)-theory has a natural action of the centralizer \( Z_{M^\vee}(s, u) \) and we let \( \mathcal{M}(s, u, \rho) \) denote its \( \rho \)-isotypic component with respect to \( Z_{M^\vee}(s, u) \). It easily follows from the above that \( \mathcal{M}(s, u, \rho) \) satisfies the requirements of (3).

### 2.8. Proof of Theorem 1.8

The first two assertions of Theorem 1.8 are exactly the two assertions of Theorem 2.3. So, it remains to prove the 3rd assertion.

It follows from part (3) of Theorem 2.4 that the map \( J \to \mathcal{E}_J^L(G) \) is well defined. We now need to prove that it is an isomorphism. First, given \( h \in J \) for any \( P, M, \sigma \) as above, we can define \( \eta_h(\chi) \in \text{End}_C(i_{GP}(\sigma \otimes \chi^{-1})) \) where \( \chi \) is a positive unramified character of \( M \); the fact that \( \eta_h(\chi) \) depends rationally on \( \chi \) follows immediately from the fact that the embedding \( \mathcal{H}(G, I) \to J \) is an isomorphism over the generic point of the center of \( \mathcal{H}(G, I) \). Thus we get an injective map \( J \to \mathcal{E}_J^L(G) \). We now want to prove that this map is also surjective.

For a unipotent element \( u \) in \( G^\vee \) let \( Z_u \) denote the algebra of ad-invariant polynomial functions on the centralizer \( Z_{G^\vee}(u) \) of \( u \) in \( G^\vee \). Let \( Z \) denote the direct sum of all the \( Z_u \) (where the sum is taken over conjugacy classes of unipotent elements in \( G^\vee \)). Then \( Z \) maps to the center of both \( J \) and \( \mathcal{E}_J^L(G) \) and both algebras are finitely generated modules over \( Z \).
To prove that the desired surjectivity holds, it is enough to prove that it holds modulo every maximal ideal of \( \mathcal{Z} \). Let \( m \) be such an ideal. Set \( J_m = J/mJ, \mathcal{E}_m = \mathcal{E}_{J_f}(G)/m\mathcal{E}_{J_f}(G) \). It is enough to prove that the map \( J_m \rightarrow \mathcal{E}_m \) is surjective for every \( m \). Let \((\pi_1, V_1), \cdots, (\pi_n, V_n)\) be all the different (non-isomorphic) representations of \( G \) which have the form \( i_{GP}(\sigma \otimes \chi^{-1}) \) with tempered \( \sigma \) and positive \( \chi \) such that \( V_i^f \neq 0 \) and \( \mathcal{Z} \) acts on \( V_i^f \) through the quotient by \( m \). Then by definition \( \mathcal{E}_m \) embeds into \( \bigoplus_i \text{End}_\mathbb{C}(V_i^f) \). On the other hand, \( V_1^f, \cdots, V_n^f \) are non-isomorphic irreducible representations of \( J_m \) and hence the map \( J_m \rightarrow \bigoplus_i \text{End}_\mathbb{C}(V_i^f) \) is surjective.

3. Connection to the Schwartz space of \( G/U \)

3.1. Digression on \([2]\). Let \( U \) be a maximal unipotent subgroup of \( G \) defined over \( F \), \( U = U(F) \). Set \( X = G/U \); it is endowed with a natural action of \( G \times T \). Let us denote by \( \mathcal{S}_c(X) \) the space of locally constant compactly supported functions on \( X \), also let \( C^\infty(X) \) denote just the space functions \( f : X \rightarrow \mathbb{C} \), such that there exists an open compact subgroup \( K \) of \( G \) such that \( f \) is \( K \)-invariant. Let \( G \times T \) acts on these spaces is such a way that the action of \( G \) comes from the right action of \( G \) on \( X \) and the action of \( T \) comes from the right action of \( T \) on \( X \) twisted by the character \( t \mapsto q^{(\text{val}(t), \rho)} \) of \( T \) where \( \text{val} : T \rightarrow \Lambda \) denotes the natural homomorphism. In \([2]\) we have defined the Schwartz space \( \mathcal{S}(G/U) \) of functions on the basic affine space \( G/U \) which contains \( \mathcal{S}_c(G/U) \) and it is contained in \( C^\infty(X) \) in the case when \( G \) is simply connected. Let us recall this definition.

The space \( X \) has unique up to scalar \( G \)-invariant measure and we denote by \( L^2(X) \) the \( L^2 \)-space with respect to this measure. When \( G \) is simply connected one can construct a natural action of the Weyl group \( W \) on \( L^2(X) \) by unitary operators \( \Phi_w \) which commute with \( G \times T \). In order to define these operators it is enough to consider the case when \( w = s_\alpha \) – a simple reflection (here \( \alpha \) is a simple root of \( G \)). Let us recall this definition as it will be used in the future.

For a simple root \( \alpha \) let \( P_\alpha \subset G \) be the minimal parabolic of type \( \alpha \) containing \( B \). Let \( B_\alpha \) be the commutator subgroup of \( P_\alpha \), and denote \( X_\alpha := G/B_\alpha \). We have an obvious projection of homogeneous spaces \( \pi_\alpha : X \rightarrow X_\alpha \). It is a fibration with the fiber \( B_\alpha/U = \mathbb{A}^2 - \{0\} \).

Let \( \overline{\pi}_\alpha : \overline{X}^\alpha \rightarrow X_\alpha \) be the relative affine completion of the morphism \( \pi_\alpha \). (So \( \overline{\pi}_\alpha \) is the affine morphism corresponding to the sheaf of algebras \( \pi_\alpha^*(\mathcal{O}_X) \) on \( X_\alpha \).) Then \( \overline{\pi}_\alpha \) has the structure of a 2-dimensional vector bundle; \( X \) is identified with the complement to the zero-section in \( \overline{X}^\alpha \). The \( G \)-action on \( X \) obviously extends to \( \overline{X}^\alpha \); moreover, it is easy to see that the determinant of the vector bundle \( \overline{\pi}_\alpha \) admits a canonical (up to a constant) \( G \)-invariant trivialization, i.e. \( \overline{\pi}_\alpha \) admits unique up to a constant \( G \)-invariant fiberwise symplectic form \( \omega_\alpha \). We will fix such a form for every \( \alpha \).

Obviously \( L^2(X) = L^2(\overline{X}^\alpha) \). Thus we define \( \Phi_\alpha = \Phi_{s_\alpha} \) to be equal to the Fourier transform in the fibers of \( \overline{\pi}_\alpha \), corresponding to the identification of \( \overline{X}^\alpha \) with the dual bundle by means of \( \omega_\alpha \).

Then

\[
\mathcal{S}(X) = \sum_{w \in W} \Phi_w(\mathcal{S}_c(X)).
\]
We can extend the above definition to the case when $G$ is not necessarily simply connected. First the definition of $[2]$ works without any change in the case when $[G, G]$ is simply connected. Now, given any connected reductive $G$ there always exists an algebraic reductive group $\tilde{G}$ and a central torus $Z$ in $\tilde{G}$ so that $G = \tilde{G}/Z$. We now denote by $\tilde{X}$ the basic affine space for $\tilde{G} = \tilde{G}(F)$ and we set $S(X) = S(\tilde{X})^T$. With this definition most results of $[2]$ extend word-by-word to any $G$.

3.2. **Action of $J(G)$ on $L^2(X)$**. By definition we have $S(X) \subset L^2(X)$. We claim that $C(G)$ acts on $L^2(X)$. Indeed, we have

$$L^2(X) = \bigoplus_{\theta: T(O) \rightarrow S^1} L^2(X)_{\theta},$$

where $L^2(X)_{\theta}$ denotes the subspace of $L^2(X)$ on which $T(O)$ acts by $\theta$. Now, each $L^2(X)_{\theta}$ is a direct integral of $G$-representations of the form $i_{G,B}(\chi)$ where $\chi$ is a unitary character of $T$ over a compact base (isomorphic to $(S^1)^{\dim T}$) and therefore it acquires a natural action of $C(G)$ (since it acts on each $i_{G,B}(\chi)$ with unitary $\chi$).

In particular, the algebra $J(G)$ acts on $L^2(X)$. The following conjecture provides an alternative definition of $S(X)$.

**Conjecture 3.3.** We have $S(X) = J(G) \cdot S_c(X)$.

**Remark.** We claim that Conjecture 3.3 is equivalent to saying that $S(X) = J(G)_U$ where the latter means $U$-coinvariants with respect to the right action of $U$ on $J(G)$ (note that $J(G)$ is a $G$-bimodule, since it contains $H(G)$ as a subalgebra). Indeed, let us assume Conjecture 3.3. Then we can define a map $\zeta : J(G) \rightarrow S(X)$ by sending every $f \in J(G)$ to $\int_U f(gu) du$. The fact that the action of $J(G)$ on $S_c(X)$ is well-defined guarantees that this integral is convergent; in fact we have $\zeta(f) = f * \delta_{K/K \cap U}$ for a sufficiently small open compact subgroup $K$ of $G$ (here $\delta_{K/K \cap U}$ denotes the multiple of the characteristic function of $K/K \cap U \subset G/U$ normalized by the condition that its integral over $G/U$ is equal to 1; it is easy to see that the result is independent of the choice of $K$ if we require that $f$ is $K$-invariant). Also Conjecture 3.3 guarantees that $f * \delta_{K/K \cap U} \in S(X)$. It is clear that $\zeta$ factorizes through $J(U)$ and the resulting map $J(U) \rightarrow S(X)$ is injective. On the other hand, the restriction of $\zeta$ to $H(G)$ defines a surjective map $H(G) \rightarrow S_c(X)$. Hence, by definition we $\zeta$ is a surjective map from $J(G)$ to $J(G) \cdot S_c(X) = S(X)$. Thus we have proved that Conjecture 3.3 implies that $S(X) = J(G)_U$. The converse statement obvious, since as a $J(G)$-module the space $J(G)_U$ is clearly generated by $H(G)_U = S_c(X)$.

We can prove the following weaker version of Conjecture 3.3

**Theorem 3.4.** We have $(S(X))_\theta = (J(G) \cdot S_c(X))_\theta$ where the character $\theta : T(O) \rightarrow S^1$ is either trivial or if the composition of $\theta$ with any coroot is non-trivial.

**Proof.** Let $S_c = S_c(X), S = S(X), S' = J(G) \cdot S_c(X)$. We want to show the equality $S_\theta = S'_\theta$ for $\theta$ as above.

**Step 1.** Let us first show that $S^I = (S')^I$. As before, let $G^\vee$ denote the Langlands dual group of $G$ over $\mathbb{C}$ and let $B$ be its flag variety. According to $[4]$ the algebra $J$ decomposes as a direct sum of subalgebras $J_0$ numbered by unipotent elements $u \in G^\vee$ up to conjugacy. We denote by $J_0$ the summand corresponding to the unit conjugacy class. We claim the action
on $J$ on $L^2(X)^I$ factorizes through the projection on $J_0$. This is obvious since $L^2(X)^I$ is a torsion-free module over the center of $\mathcal{H}(G, I)$ and every $J_u$ with non-trivial $u$ is annihilated by a non-zero ideal of the center.

According to [10] the algebra $J_0$ is naturally isomorphic to the $K_{G^\vee}(B \times B)$ (here $K_{G^\vee}(?)$ stands for the complexified Grothendieck group of $G^\vee$-equivariant coherent sheaves on $?)$. On the other hand, let $K = K_{T^\vee \times C^\vee}(B)$ and let $K_q$ be its specialization at $v = q^{1/2}$ where $K_{C^\vee}(pt) = \mathbb{C}[v, v^{-1}]$ (the action of $C^\vee$ on $B$ trivial). According to Section 5 of [2] the space $K$ has a natural action of $\mathcal{H}_{aff}$ and hence $K_q$ has a natural action of $\mathcal{H}(G, I)$. Moreover, we have an isomorphism $S^I \simeq K_q$ which identifies $S^I$ with the submodule generated by the skyscraper $\kappa$ at some $T^\vee$-invariant point $e \in B^\vee$. On the other hand $J_0 \otimes A = K_{G^\vee \times C^\vee}(B \times B)$ clearly acts on $K$ (and this action is compatible with the $\mathcal{H}_{aff}$-action with respect to the homomorphism $\mathcal{H}_{aff} \to J_0$ – this is proved in [10]); moreover the action of $K_{G^\vee \times C^\vee}(B \times B)$ on $\kappa$ defines an isomorphism $K_{G^\vee \times C^\vee}(B \times B) \simeq K$. Hence the same is true after specialization to $v = q^{1/2}$. We see that $J_0$ acts on $K_q = S^I$ and the latter is generated as a module by an element of $S^I$. This implies the equality $S^I = (S')^I$.

Step 2. Let $S_0, S_0'$ denote the $G$-module of coinvariants of $S$ (resp. $S'$) with respect to $T(O)$. Then both are subrepresentations of $C^\infty(X)_0$. Recall that if a $G$-module $V$ is generated by $I$-fixed vectors then two $G$-submodules $W_1$ and $W_2$ of $V$ coincide if and only if $W_1^I = W_2^I$. Applying this to $C^\infty(X)_0$, $W_1 = S_0$, $W_2 = S_0'$ and using Step 1 we get the equality $S_0 = S_0'$.

Step 3. Let $\theta : T(O) \to S^I$ be a character. For a $T$-module $V$ let $V_{\theta}$ denote the corresponding space of $(T(O), \theta)$-coinvariants. Let us prove that $S_{\theta} = S_{\theta}'$ assuming that the composition of $\theta$ with any coroot is a non-trivial character of $O^*$. We shall refer to such $\theta$ as ”regular”.

In this case it is obvious from the definition of $\Phi_w$ that for any simple coroot $\alpha$ the operator $\Phi_{s_{\alpha}}$ defines an isomorphism between $S_{c, \theta}$ and $S_{c, s_{\alpha}(\theta)}$. Indeed, any $(T(O), \theta)$-equivariant function automatically vanishes on $X \setminus \{X\}$ (recall the notation of Section 3.1) and the same is true for $\theta$ replaced with $s_{\alpha}(\theta)$. Since the notion of regularity is $W$-invariant it follows that for any $w \in W$ the operator $\Phi_w$ defines an isomorphism between $S_{c, \theta}$ and $S_{c, w(\theta)}$. On the other hand, we claim that $S_{\theta}' = S_{c, \theta}$. For this it is enough to prove that $S_{c, \theta}$ is $J(G)$-invariant. This would follow if we knew that for any character $\chi : T \to \mathbb{C}^*$ such that $\chi|_{T(O)} = \theta$ the action of $\mathcal{H}(G)$ on the space $S_{\chi, \theta}$ of $(T, \chi)$-coinvariants on $S_c$ extends to $J(G)$. For any $\chi$ as above we can find an element $w \in W$ such that $w(\chi)$ is non-negative. Hence by definition $J(G)$ acts on $S_{c, w(\chi)} = i_{GB}(w(\chi)) \simeq i_{GB}(\chi)$.

\[\square\]

3.5. The parabolic case. Let $P$ be a parabolic subgroup of $G$ with a Levi subgroup $M$ and unipotent radical $U_P$. Let $X_P = G/U_P$. This space has a natural $G \times M$ action. Therefore the space $S_c(X)$ of locally constant compactly supported functions on $X_P$ becomes a $G \times M$ module; for convenience we are going to twist the $M$ action by the square root of the absolute value of the determinant of the $M$-action on the Lie algebra $u_P$ of $U_P$.

As before, we can define the space $L^2(X_P)$. For the same reason as before it has an action of $\mathcal{C}(G)$. We now define $S(X_P) := J(G) \cdot S_c(X_P) \subset L^2(X_P)$. Equivalently $S(X_P) = J(G)_{U_P}$. 

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Conjecture 3.6. Let $P$ and $Q$ be two associate parabolics, i.e. two parabolics with the same Levi subgroup $M$. Then there exists a $G \times M$-equivariant unitary isomorphism $\Phi_{PQ} : L^2(X_P) \cong L^2(X_Q)$ whose restriction to $S(X_P)$ defines an isomorphism between $S(X_P)$ and $S(X_Q)$.

4. Appendix: an $SL(2)$-example

4.1. The purpose of this appendix is to show how the algebra $J$ gets realized inside the locally constant functions on $G$ for the case when $G = SL(2, F)$. Let $St$ denote the Steinberg representation of $G$. Let also $S$ denote the space of locally constant compactly supported functions on $F^2$. This space has an action of $G$ and a commuting action of the Fourier transform $\Phi$ and of the torus $T = F^\times$. Similarly, we let $S_c$ denote the space of locally constant compactly supported functions on $F^2\setminus\{0\}$. It follows from the above results that

$$J = \text{End} \left( St^I \right) \oplus J_0, \quad \text{where } J_0 = \text{End}_{\Phi, F^\times}(S^I)$$  

(4.1)

The embedding $\mathcal{H}(G, I) \hookrightarrow J$ is given by the action of $\mathcal{H}(G, I)$ on $St^I$ and on $S^I$. Since the first summand in (4.1) is a one-dimensional subspace that does not lie in $\mathcal{H}(G, I)$, it follows that the projection of $\mathcal{H}(G, I)$ to $J_0$ is an embedding. Moreover, it is clear that the codimension of $\mathcal{H}(G, I)$ in $J_0$ is at most 1. Indeed, an element of $J_0$ comes from an element of $\mathcal{H}(G, I)$ if and only if it sends $S^I_c$ to $S^I_c$. The quotient $S/S_c$ is naturally isomorphic to $\mathbb{C}$ (the map is given by evaluating a function at 0) with $F^\times$-action given by the character $x \mapsto q^{\frac{|x|}}$ (recall the convention about the torus action from Section 3.1). Denote this character by $\chi$ and let $S_{\chi}$ (resp. $(S_c)_{\chi}$) denote the space of $(F^\times, \chi)$-coinvariants on $S$ (resp. on $S_c$). Let $U$ denote the image of $(S_c)_{\chi}$ in $S_{\chi}$. Then it is clear that the quotient $J_0/\mathcal{H}(G, I)$ embeds into $\text{Hom}_T(U^I, \mathbb{C})$. However, $U$ is actually isomorphic to $St$, hence dim $U^I = 1$. This shows that dim $J_0/\mathcal{H}(G, I)$ is either 0 or 1 and in the latter case it is isomorphic to $\mathbb{C} \otimes St^I$ as an $\mathcal{H}(G, I)$-bimodule.

We would like to show that $J_0/\mathcal{H}(G, I)$ is indeed one-dimensional. Note that this will imply that dim $J/\mathcal{H}(G, I) = 2$. For this it is enough to find one $I \times I$-invariant function $f$ on $G$ such that

(a) $f$ does not have compact support but belongs to $C(G, I)$,

(b) The action of $f$ on $S^I$ is well-defined and it is given by a non-zero operator.

We are going to find such a function explicitly. In fact, the function $f$ will be $K \times I$-invariant, where $K = SL(2, O)$. We begin by some generalities.

4.2. General remarks and volume calculation. Let $G$ be a totally disconnected group, $x \in G$, $K_1, K_2$ - open compact subgroups of $G$; set $X = K_1 x K_2$. Let $V$ be a smooth representation of $G$, and $v \in V$. Fix a Haar measure on $G$. Then we have an element $e_X$ in the Hecke algebra of $G$. By definition, we have

$$e_X(v) = \int_{g \in X} g(v) dg.$$  

This formula can be rewritten in the following way, which will be crucial in the future:

$$e_X(v) = \frac{\text{vol}(X)}{\text{vol}(K_1) \cdot \text{vol}(K_2)} \int_{g_1 \in K_1, g_2 \in K_2} (g_1 x g_2)(v) dg_1 dg_2.$$  

(4.2)
In particular, let us assume that the Haar measure is chosen is such a way that \(\text{vol}(K_2) = 1\) and that \(v\) is \(K_2\)-invariant. Then \([4.2]\) takes the form

\[
e_X(v) = \frac{\text{vol}(X)}{\text{vol}(K)} \int_{g \in K} g(x(v))dg.
\]

We now want to apply this formula to the case when \(G = SL(2,F), K = K_1 = SL(2,\mathcal{O}), K_2 = I\) (so, in particular, we fix the Haar measure so that the volume of \(I\) is 1). Let \(t\) be a uniformizer of \(F\) and let \(x_n = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}\) where \(n \in \mathbb{Z}\) and let \(X_n = Kx_nI\).

**Lemma 4.3.** We have

\[
\frac{\text{vol}(X_n)}{\text{vol}(K)} = \begin{cases} q^{2n-1} & \text{if } n \geq 0 \\ q^{-2n} & \text{if } n < 0 \end{cases}
\]

**Proof.** Let \(H_n \subset K\) be the subgroup of all \(h \in K\) such that \(x_n^{-1}hx_n \in I\). Then \(\text{vol}(X_n) = \#(K/H_n)\). Recall that \(\text{vol}(K) = \#(K/I) = (q+1)\).

If \(h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) then \(x_n^{-1}hx_n = \begin{pmatrix} a & t^{-2n}b \\ c & d \end{pmatrix}\). Hence, if \(n > 0\), then \(x_n^{-1}hx_n \in I\) iff \(b \in t^{2n}\mathcal{O}\) and the cardinality of \(K/H_n\) is \(q^{2n-1}(q+1)\). If \(n \leq 0\) then the condition is \(c \in t^{-2n+1}\mathcal{O}\) and the cardinality of \(K/H_n\) is \(q^{-2n}(q+1)\). \(\Box\)

We now pass to the function \(f\).

**Proposition 4.4.** Let

\[
\gamma_n = \begin{cases} q^{2n} & \text{if } n \leq 0 \\ -q^{-2n+1} & \text{if } n > 0 \end{cases}
\]

Let \(f = \sum_{n \in \mathbb{Z}} \gamma_n \cdot \chi_{X_n}\) where \(\chi_{X_n}\) denotes the characteristic function of \(X_n\). Then \(f\) belongs to \(J_0\) (and it obviously does not belong to \(\mathcal{H}(G,I)\)).

**Corollary 4.5.** As a bimodule over \(\mathcal{H}(G,I)\) the quotient \(J/\mathcal{H}(G,I)\) is isomorphic to \(V \otimes St^{I}\) where \(V\) is a 2-dimensional representation of \(\mathcal{H}(G,I)\) which is a non-trivial extension of \(\mathbb{C}\) (trivial representation) by \(St^{I}\).

**Proof.** We have already explained above that Proposition 4.4 implies that \(\dim J/\mathcal{H}(G,I) = 2\). Moreover, it follows from \([4.1]\) that this quotient contains \(St^{I} \otimes St^{I}\) as an \(\mathcal{H}(G,I)\)-submodule and we have explained at the end of Section 4.6 that the quotient of \(J/\mathcal{H}(G,I)\) by \(St^{I} \otimes St^{I}\) is isomorphic to \(\mathbb{C} \otimes St^{I}\) as an \(\mathcal{H}(G,I)\)-bimodule. Hence we see that \(J/\mathcal{H}(G,I)\) is isomorphic to \(V \otimes St^{I}\) where \(V\) is a 2-dimensional representation of \(\mathcal{H}(G,I)\) which is an extension of \(\mathbb{C}\) by \(St^{I}\). If that extension were trivial, then \(f\) would have been \(G\)-invariant on the left modulo functions with compact support. This would imply that \(\gamma_n\) is constant for \(|n| >> 0\), which is obviously not the case (and it is also clear that no element of \(C(G)\) could satisfy this). \(\Box\)

Let us now turn to the proof of Proposition 4.4. It is easy to see that \(f\) belongs to \(C(G)\). Also since \(f\) is \(K\)-invariant on the left, its projection to the first summand of \([4.1]\) is 0. Hence to show that it belongs to \(J_0\) it is enough to show that its action on \(S^{I}\) is well-defined. For this, we are going to take \(f\) of the form \(\sum_{n} \gamma_n \chi_{X_n}\) with arbitrary coefficients \(\gamma_n\) and see
what conditions we need to impose on the coefficients so the action of $f$ on $S^r$ is well-defined. To check the latter, it is enough to verify that $f \ast \chi_{O \oplus O}$ and $f \ast \chi_{O \oplus tO}$ are well-defined.

4.5.1. The case of $\chi_{O \oplus O}$. Let us first compute the action of $f$ on $\chi_{O \oplus O}$. It is enough to compute the value of the action of $\chi_{X_n}$ on $\chi_{O \oplus O}$ at $(t^{-r}, 0)$ where $r \in \mathbb{Z}$. It is equal to

$$\frac{\text{vol}(X_n)}{\text{vol}(K)} \cdot \text{vol}(K_{n,r}) \text{ where } K_{n,r} = \{ h \in K | h(t^{-r}, 0) \in t^n\mathcal{O} \oplus t^{-r}\mathcal{O} \}.$$

Assume first that $n > 0$. Then we have the following cases:

1. $r > n$. Then the result is 0.
2. $r \leq -n$. Then $K_{n,r} = K$ and the result is $(q + 1)q^{2n-1}$.
3. $-n < r \leq n$. In this case if $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $h \in K_{n,r}$ iff $a \in t^{n+r}\mathcal{O}$ and thus

$$\text{vol}(K_{n,r}) = q^{-n-r+1}. \text{ Hence the value is } q^{n-r}.$$

On the other hand, assume that $n = -m, m > 0$. Then we have the following cases:

1. $r > m$ – the value is 0
2. $r \leq -m$. Then again $K_{n,r} = K$ and we get $(q + 1)q^2m$.
3. $-m < r \leq m$. Then $h \in K_{n,r}$ iff $c \in t^{r+m}\mathcal{O}$ and its volume is $q^{-r-m+1}$. Hence the value is $q^{m-r+1}$.

If $n = 0$ then we just get $(q + 1)\chi_{O \oplus O}$. Altogether, the value of $f \ast \chi_{O \oplus O}$ at $(t^{-r}, 0)$ for $r > 0$ is

$$\sum_{n \geq r} \gamma_n q^{n-r} + \sum_{m \geq r} \gamma_{-m} q^{m-r+1}.$$

Hence if we want this to be 0 (which we definitely want at least for $r$ large enough) we need

$$\gamma_r + q\gamma_{-r} = 0 \quad (4.5)$$

4.5.2. The case of $\chi_{O \oplus tO}$. In this case we define $K_{n,r}' = \{ h \in K | h(t^{-r}, 0) \in t^n\mathcal{O} \oplus t^{-n+1}\mathcal{O} \}$ and we need to repeat the above calculation with $K_{n,r}$ replaced by $K_{n,r}'$.

Let us first assume that $n > 0$. Then we have the following cases:

1. $r < n - 1 - \text{ get 0}$
2. $r \leq -n - \text{ get } (q + 1)\frac{\text{vol}(X_n)}{\text{vol}(K)} \gamma_n = (q + 1)q^{2n-1} \gamma_n$
3. $n < r \leq n - 1$

In this case $K_{n,r}'$ is given by the condition $a \in t^{n+r}\mathcal{O}$ and the value is $q^{-n-r+1}q^{2n-1} \gamma_n = q^{n-r}\gamma_n$.

On the other hand, assume that $n \leq 0$ and set $n = -m$. Then we have the following cases:

1. $r > m - \text{ get 0}$
2. $r \leq -m - 1 - \text{ get } (q + 1)q^2m \gamma_{-m}$.
3. $-m \leq r \leq m$. Then $K_{n,r}'$ is given by the condition $c \in t^{r+m+1}\mathcal{O}$ and the volume of $K_{n,r}'$ is $q^{-r-m}$. Thus we get $q^{-r-m}q^2m \gamma_{-m} = q^{m-r}\gamma_{-m}$.

Thus for $r \geq 0$ we get that the value of $f \ast \chi_{O \oplus tO}$ at $(t^{-r}, 0)$ is equal to

$$\sum_{n \geq r+1} q^{n-r} \gamma_n + \sum_{m \geq r} q^{m-r} \gamma_{-m}.$$
This going to be 0 for all \( r \geq 0 \) if
\[
q^{\gamma_{r+1}} + q^{-r} = 0.
\]
(4.6)

4.6. The function \( f \). Altogether it is now clear that if \( \gamma_n \) is given by (4.4) then \( f \) satisfies both (4.5) and (4.6).

Let us now compute the action of the resulting function \( f \) on both \( \chi_{O \oplus O} \) and \( \chi_{O \oplus tO} \). In the first case, we already know that \( f \star \chi_{O \oplus O}(t-r,0) = 0 \) if \( r > 0 \). On the other hand, for \( r \leq 0 \) we have
\[
f \star \chi_{O \oplus O}(t-r,0) =
(q + 1) + \sum_{0 < n \leq -r} (q + 1)q^{2n-1}(-q^{-2n+1}) + \sum_{n > -r} q^{n-r}(-q^{-2n+1}) +
\sum_{0 < m \leq -r} (q + 1)q^{2m-2m} + \sum_{m > -r} q^{m-r+1}q^{-2m} = q + 1,
\]
which shows that \( f \star \chi_{O \oplus O} = (q + 1)\chi_{O \oplus O} \). Similar calculation shows that \( f \star \chi_{O \oplus tO} = 0 \).

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