Abstract. In this paper, we present a refined framework for the large time behavior estimates for the pressureless Euler–Navier–Stokes system. Specifically, under a suitable assumption on the density of the pressureless Euler fluid flow, we show that the decay rate of the higher-order derivatives of fluid velocities, whose order is smaller than the dimension, is faster than that of lower-order derivatives. As a byproduct, we establish the global-in-time existence and uniqueness of classical solutions to our main system in the two-dimensional case.

1. Introduction

In this work, we revisit the large time behavior estimates for a coupled hydrodynamic system in the whole space. Our main system consists of the pressureless Euler equations and incompressible Navier–Stokes equations (in short, pressureless ENS system), which are coupled through the drag force, also often called friction force. More precisely, let \( \rho = \rho(x, t) \) and \( u = u(x, t) \) be the density and velocity for the pressureless Euler fluid flow, respectively, and let \( v = v(x, t) \) be the velocity for the incompressible Navier–Stokes fluid flow. Then our main system reads as

\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) &= -\rho(u - v), \\
\partial_t v + (v \cdot \nabla_x) v + \nabla_x p - \Delta_x v &= \rho(u - v), \\
\nabla_x \cdot v &= 0,
\end{align*}
\]  

subject to initial data:

\[
(\rho(x, 0), u(x, 0), v(x, 0)) = (\rho_0(x), u_0(x), v_0(x)), \quad x \in \mathbb{R}^d.
\]  

The system (1.1) can be formally derived from the Vlasov–Navier–Stokes system, which describes the behavior of a large cloud of particles interacting with the incompressible fluid [11, 12], in the case of mono-kinetic particle distributions [6]. On the other hand, if the pressureless Euler equations in (1.1) are replaced by the isothermal Euler equations, i.e., the pressure term \( \nabla_x \rho \) is added, then the resulting system, Euler–Navier–Stokes system, can be rigorously derived from Vlasov–Fokker–Planck–Navier–Stokes system by considering a singular parameter in the nonlinear Fokker–Planck operator [1]. For the interactions with compressible fluid, the derivation of Euler–Navier–Stokes system, can also be rigorously derived [5]. We also refer to [7, 8, 9, 13] for the hydrodynamic limits for Vlasov–Navier–Stokes system in various regimes.

The main purpose of the current work is to improve the previous work [6], where the global Cauchy problem for the system (1.1) is discussed. Under smallness and regularity assumptions on the initial data, the global-in-time existence and uniqueness of classical solutions to the system (1.1) are established in [6] when \( d \geq 3 \). Since the pressureless Euler equations in (1.1) may develop a finite-time formation of singularities, in order to prevent the possible finite-time breakdown of smoothness of solutions, a priori large time behavior of solutions is discussed in [6], see also [10, 4] for dealing with the periodic domain. In particular, this gives the time-integrability of \( \| (\nabla_x u)(\cdot, t) \|_{H^s} \), and thus this combined with classical energy estimates provides the uniform-in-time bound on \( \| \rho(\cdot, t) \|_{H^s} \). To be more specific, the following decay rate of convergence of
Moreover, for every \(k\) of \(d/2\) from 2 of system (1.1) in the two-dimensional case, and furthermore, relax the condition for the lower bound for \(d\) from this improved large time behavior estimate of solutions, we can also cover the global-in-time existence of solutions is obtained in [6]:

\[
\rho_0(x) > 0 \text{ for every } x \in \mathbb{R}^d \text{ and }\]

\[
\|\phi\|_{H^{s-1}} + \|u_0\|_{H^{s+2}} + \|v_0\|_{H^{s+1}} + \|v_0\|_{L^1} < \varepsilon_0
\]

for \(\varepsilon_0 > 0\) sufficiently small, the Cauchy problem (1.1)–(1.2) has a unique global classical solution \((\rho, u, v)\) in \(C([0, +\infty); H^s(\mathbb{R}^d)) \times C([0, +\infty); H^{s+2}(\mathbb{R}^d)) \times C([0, +\infty); H^{s+1}(\mathbb{R}^d))\) satisfying \(\rho(x, t) > 0\) for all \((x, t)\) in \(\mathbb{R}^d \times [0, +\infty)\) and

\[
\sup_{t \geq 0} (\|\rho(\cdot, t)\|_{H^s} + \|\rho(\cdot, t)\|_{H^{s-1}} + \|u(\cdot, t)\|_{H^{s+2}} + \|v(\cdot, t)\|_{H^{s+1}}) < \infty.
\]

Moreover, for every \(k = 0, \ldots, \min\{d-1, s\}\) and \(\alpha \in (0, d/2 + k)\), there exists a constant \(C > 0\) independent of \(t\) such that

\[
\|\nabla^k u(\cdot, t)\|_{H^{s-k}} + \|\nabla^k v(\cdot, t)\|_{H^{s-k}} \leq \frac{C}{(1 + t)^\alpha} \quad \forall t \geq 0.
\]
Remark 1.1. When \( k = 0 \), the decay estimates \([1, 4]\) becomes \([1, 3]\) which is obtained in [6].

1.3. Organization of paper. The rest of this paper is organized as follows. In Section 2 we present several inequalities used throughout this paper, large time behavior estimates of the total energy, and local-in-time existence and uniqueness of solutions for system \([1, 1]\) in our desired Sobolev spaces. In Section 3 we present the \( a \) \( p \) riori estimates for the higher-order derivatives of the newly defined quantity \( \phi \) and fluid velocities. In Section 4 we use the estimates from the previous section to show the desired time decay estimates for fluid velocities. As a byproduct, we also establish the global existence of classical solutions in the two-dimensional case.

2. Preliminaries

In this section, we present several useful inequalities and time decay estimate of the total energy for the pressureless ENS system \([1, 1]\) that will be significantly and frequently used throughout the paper. We also state the local-in-time existence and uniqueness theorem.

2.1. Useful inequalities. We first list some classical inequalities in the lemma below.

Lemma 2.1. (i) For any pair of functions \( f, g \in (H^k \cap L^\infty)(\mathbb{R}^d) \), we obtain

\[
\|\nabla^k (fg)\|_{L^2} \leq C (\|f\|_{L^\infty}\|\nabla^k g\|_{L^2} + \|\nabla^k f\|_{L^2}\|g\|_{L^\infty}).
\]

Furthermore, if \( \nabla f \in L^\infty(\mathbb{R}^d) \), we have

\[
\|\nabla^k (fg) - f\nabla^k g\|_{L^2} \leq C (\|\nabla f\|_{L^\infty}\|\nabla^{k-1} g\|_{L^2} + \|g\|_{L^\infty}\|\nabla^k f\|_{L^2}).
\]

Here \( C > 0 \) only depends on \( k \) and \( d \).

(ii) For \( f \in H^{|d/2|+1}(\mathbb{R}^d) \) with \( d \geq 3 \), we have

\[
\|f\|_{L^\infty} \leq C\|\nabla f\|_{H^{|d/2|}}.
\]

(iii) For \( \ell = 0, \ldots, [(d-1)/p] \) and \( f \in W^{\ell, (d-1)}(\mathbb{R}^d) \), we have

\[
\|f\|_{L^\infty} \leq C\|\nabla^\ell f\|_{L^p}.
\]

(iv) For \( s, s_1 \) and \( s_2 \) satisfying \( s \leq \min\{s_1, s_2, s_1 + s_2 - d/2\} \) and \( (f, g) \in H^{s_1}(\mathbb{R}^d) \times H^{s_2}(\mathbb{R}^d) \),

\[
\|fg\|_{H^s} \leq C\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}.
\]

(v) Let \( q, r \) be any numbers satisfying \( 1 \leq q, r \leq \infty \), and let integers \( j, m \) satisfy \( 0 \leq j \leq m \). If \( f \in (W^{m, r} \cap L^q)(\mathbb{R}^d) \), then

\[
\|\nabla^j f\|_{L^p} \leq C\|\nabla^m f\|_{L^q}^\alpha\|f\|_{L^r}^{1-\alpha},
\]

where

\[
\frac{1}{p} = \frac{j}{d} + \alpha \left( \frac{1}{r} - \frac{m}{d} \right) + (1 - \alpha) \frac{1}{q}
\]

for all \( \alpha \) with \( \frac{j}{m} \leq \alpha \leq 1 \).

Here \( C > 0 \) is independent of \( f \).

We next provide a technical lemma which is a type of weighted Gagliardo–Nirenberg–Sobolev inequality.

Lemma 2.2. Let \( d \geq 2, m \in \mathbb{N} \), and \( r \geq 0 \). Let \( f \) and \( \rho \) be functions satisfying \( \rho^\alpha \nabla^m f \in L^p(\mathbb{R}^d) \) with \( d > pm \). If \( \phi = \nabla (\log \rho) \in L^q(\mathbb{R}^d) \) and \( \|\phi\|_{L^q} \leq \varepsilon_1 \ll 1 \), then \( \rho^\alpha f \in L^p(\mathbb{R}^d) \) with the estimate:

\[
\|\rho^\alpha f\|_{L^p} \leq C\|\rho^\alpha \nabla^m f\|_{L^p},
\]

where \( 1/p^* := 1/p - m/d \). Here \( C > 0 \) depends only on \( \varepsilon_1 > 0 \) and \( r \), but independent of \( f \) and \( \rho \).
Proof. For the proof, we use the inductive argument on $m$. First, we readily check that for $m = 1$

\[
\|\rho^\ell f\|_{L^{p^\ell}} \leq \|\nabla (\rho^\ell f)\|_{L^p}
\]

\[
\leq r \|\rho^\ell \phi f\|_{L^p} + \|\rho^\ell \nabla f\|_{L^p}
\]

\[
\leq r \|\phi\|_{L^d} \|\rho^\ell f\|_{L^{p^\ell}} + \|\rho^\ell \nabla f\|_{L^p}
\]

\[
\leq C \varepsilon^1 \|\rho^\ell f\|_{L^{p^\ell}} + \|\rho^\ell \nabla f\|_{L^p}
\]

due to Hölder’s inequality. Here $C > 0$ depends only on $\varepsilon_1 > 0$ and $r$. Thus, we obtain

\[
\|\rho^\ell f\|_{L^{p^\ell}} \leq C \|\rho^\ell \nabla f\|_{L^p}.
\]

Now, we assume that (2.1) holds for $1 < \ell < m$. Then for $p^\ell$ with

\[
\frac{1}{p^\ell} := \frac{1}{p} - \frac{m-1}{d} = \left(\frac{1}{p} - \frac{1}{d}\right) - \frac{m-1}{d},
\]

by induction hypothesis, we have

\[
\|\rho^\ell f\|_{L^{p^\ell}} \leq C\|\rho^\ell \nabla f\|_{L^p} \leq C \|\rho^\ell \nabla f\|_{L^p}.
\]

This completes the proof. □

2.2. Energy estimates & local well-posedness. The total energy for the system (1.1) is given by

\[
E(t) := \frac{1}{2} \int_{\mathbb{R}^d} \rho(x,t)|u(x,t)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |v(x,t)|^2 \, dx.
\]

Then it can be easily check that $E$ satisfies

\[
\frac{d}{dt} E(t) + D(t) = 0,
\]

under the sufficient regularity assumptions on the solutions, where $D$ is the dissipation rate given by

\[
D(t) := \int_{\mathbb{R}^d} |\nabla v(x,t)|^2 \, dx + \int_{\mathbb{R}^d} \rho(x,t)|(u - v)(x,t)|^2 \, dx.
\]

The above estimate only gives that the total energy is not increasing in time, however, by using almost the same argument as in [6, Proposition 3.1], we can have the following estimate of the large time behavior of classical solutions to the system (1.1).

**Proposition 2.1.** For $T > 0$ and $d \geq 2$, let $(\rho, u, v)$ be a classical solution to the pressureless ENS system (1.1) on the time interval $[0, T]$ satisfying $\|\rho\|_{L^\infty([\mathbb{R}^d \times (0,T)])} < \infty$. Then, there exists a constant $C > 0$ independent of $T$ such that for every $\alpha \in (0, d/2)$,

\[
E(t)(1 + t)\alpha + \int_0^t (1 + \tau)^\alpha D(\tau) \, d\tau \leq C(E(0) + \|v_0\|_{L^1}^2) \quad \forall t \in [0, T].
\]

Finally, in the theorem below, we give the local-in-time existence and uniqueness of classical solutions to the system (1.1). For the proof, we refer to [4, 10] for the readers who are interested in it.

**Theorem 2.1.** Let $d \geq 2$ and $s \geq [d/2] + 1$. Suppose that the initial data $(\rho_0, u_0, v_0)$ satisfy the assumptions (i) and (ii) in Theorem 1.3. Then for any positive constants $\varepsilon_0 < \delta_0$, there exists a positive constant $T_0$ depending only on $\varepsilon_0$ and $\delta_0$ such that if

\[
\|u_0\|_{H^{s+2}} + \|v_0\|_{H^{s+1}} + \|v_0\|_{L^1} < \varepsilon_0,
\]

then the pressureless ENS system (1.1) - (1.2) admits a unique solution

\[
(\rho, u, v) \in C([0,T_0]; H^s(\mathbb{R}^d)) \times C([0,T_0]; H^{s+2}(\mathbb{R}^d)) \times C([0,T_0]; H^{s+1}(\mathbb{R}^d))
\]

satisfying $\rho(x,t) > 0$ for all $(x, t) \in \mathbb{R}^d \times [0, T_0]$ and

\[
\sup_{0 \leq t \leq T_0} (\|u(\cdot, t)\|_{H^{s+2}} + \|v(\cdot, t)\|_{H^{s+1}}) \leq \delta_0, \quad \sup_{0 \leq t \leq T_0} \|\rho(\cdot, t)\|_{H^s} < \infty.
\]
3. A priori estimates

In this part, we provide the a priori estimates for the derivatives of \( \phi, u \) and \( v \), which will be used to derive our desired large time behavior estimates. Let \( T > 0 \), \( d \geq 2 \), and \( s \geq \lceil d/2 \rceil + 2 \). Throughout this section, we assume that for \( \varepsilon_1 > 0 \) small enough,

\[
X(s; T) := \sup_{0 \leq t \leq T} \left( \| \phi(\cdot, t) \|_{H^{s-1}}^2 + \| u(\cdot, t) \|_{H^{s+2}}^2 + \| v(\cdot, t) \|_{H^{s+1}}^2 \right) < \varepsilon_1^2 \ll 1.
\]

We denote by

\[
X_0(s) := \| \phi_0 \|_{H^{s-1}}^2 + \| u_0 \|_{H^{s+2}}^2 + \| v_0 \|_{H^{s+1}}^2.
\]

First, we begin with estimates related to \( \phi \).

**Lemma 3.1.** For \( k, \ell \in \mathbb{N} \cup \{0\} \), we have

\[
\left\| \frac{\partial^{k+1} \rho}{\rho} \right\|_{H^{t}} \leq C \left( 1 + \| \phi \|_{H^{k+\ell}} + \| \phi \|_{H^{[k]+1}} \right)^k \| \phi \|_{H^{k+\ell}},
\]

where \( C \) is a positive constant only depending on \( k, \ell \), and \( d \).

**Proof.** We proceed by the induction argument on \( k \), \( \ell \), and \( d \).

First, the case \( k = 0 \) is just obvious. Now, assume that the argument holds for all \( \ell \in \mathbb{N} \cup \{0\} \) and \( 0 \leq k < m \). Here, repetitive computation yields

\[
\begin{align*}
\frac{\partial^{m+1} \rho}{\rho} &= \partial \left( \frac{\partial^m \rho}{\rho} \right) + \frac{\partial^m \rho}{\rho} \frac{\partial \rho}{\rho}, \\
\partial \left( \frac{\partial^m \rho}{\rho} \right) &= \partial^2 \left( \frac{\partial^{m-1} \rho}{\rho} \right) + \partial \left( \frac{\partial^{m-1} \rho}{\rho} \frac{\partial \rho}{\rho} \right), \\
&\vdots \\
\partial^{m-1} \left( \frac{\partial^2 \rho}{\rho} \right) &= \partial^m \left( \frac{\partial \rho}{\rho} \right) + \partial^{m-1} \left( \frac{\partial \rho}{\rho} \frac{\partial \rho}{\rho} \right).
\end{align*}
\]

Thus, we sum all the above identities to get

\[
\left\| \frac{\partial^{m+1} \rho}{\rho} \right\|_{H^{t}} \leq \| \phi \|_{H^{m+\ell}} + C \sum_{n=1}^{m} \left\| \partial^{m-n} \left( \frac{\partial^n \rho}{\rho} \frac{\partial \rho}{\rho} \right) \right\|_{H^{t}}
\leq \| \phi \|_{H^{m+\ell}} + C \sum_{n=1}^{m} \left\| \left( \frac{\partial^n \rho}{\rho} \right) \right\|_{H^{m-n+\ell}}.
\]

Here, we use Lemma 2.1 (iv) to get

\[
\left\| \frac{\partial^n \rho}{\partial \rho} \right\|_{H^{m-n+\ell}} \leq C \left\{ \begin{array}{cl}
\left\| \frac{\partial^n \rho}{\rho} \right\|_{H^{m-n+\ell}} & \text{if } m - n + \ell \leq d/2, \\
\left\| \frac{\partial^n \rho}{\rho} \right\|_{H^{m-n+\ell}} & \text{if } m - n + \ell > d/2.
\end{array} \right.
\]
Thus, we obtain
\[
\left\| \frac{\partial^{m+1} L}{\rho} \right\|_{H^t} \leq C \sum_{1 \leq n \leq m} \left\| \frac{\partial^n \rho \partial \phi}{\rho} \right\|_{H^{m-n+t}} + C \sum_{1 \leq n \leq m} \left\| \frac{\partial^n \rho}{\rho} \right\|_{H^{m-n+t}}
\]
\[
\leq \left\| \phi \right\|_{H^{m+1}} + C \left\{ \left\| \phi \right\|_{H^{[\frac{3}{2}]}+1} + \left\| \phi \right\|_{H^{m+x}} \right\} \left\| \frac{\partial^n \rho}{\rho} \right\|_{H^{m-n+t}}
\]
\[
\leq \left\| \phi \right\|_{H^{m+1}} + C \left( \left\| \phi \right\|_{H^{[\frac{3}{2}]}+1} + \left\| \phi \right\|_{H^{m+x}} \right) \left( \sum_{n=1}^{\infty} \left( 1 + \left\| \phi \right\|_{H^{m+1}} + \left\| \phi \right\|_{H^{[\frac{3}{2}]}+1} \right) \right)^{n-1} \left\| \phi \right\|_{H^{m+1}}
\]

This completes the induction and here we get the desired estimate. \( \square \)

Now, we proceed to the estimates for the derivatives of \( u \) and \( v \). First, we investigate the first-order derivatives.

**Lemma 3.2.** For \( T > 0 \), there exists \( C > 0 \) independent of \( T > 0 \) such that
\[
\frac{d}{dt} \left( \| \sqrt{\rho} \nabla u \|_{L^2}^2 + \| \nabla v \|_{L^2}^2 \right) + \| \sqrt{\rho} \nabla (u - v) \|_{L^2}^2 + \| \nabla^2 v \|_{L^2}^2 \leq C \varepsilon_1 \left( \| \sqrt{\rho} \nabla (u - v) \|_{L^2}^2 + \| \nabla^2 v \|_{L^2}^2 \right)
\]
for all \( t \in (0, T) \).

**Proof.** We separately estimate \( u \) and \( v \) as follows.

\( \bullet \) (Step A: Estimates for \( v \)): For \( v \), we have
\[
\frac{1}{2} \frac{d}{dt} \| \partial v \|_{L^2}^2 + \| \nabla \partial v \|_{L^2}^2 = - \int \partial v \cdot (\partial v \cdot \nabla v) dx + \int \rho (\partial u - \partial v) \cdot \partial v dx + \int \partial \rho (u - v) \cdot \partial v dx
\]
due to the incompressibility of \( v \).

\( \diamond \) (Estimates for \( I_1 \)): If \( d \leq 5 \), we see that
\[
\| \nabla v \|_{L^3} \leq C \| \nabla^2 v \|_{L^2}^{\frac{d+6}{2}} \| v \|_{L^2}^{\frac{d+6}{2}}
\]
This yields
\[
I_1 \leq \| \nabla v \|_{L^3}^{\frac{3}{2}} \leq C \| \nabla^2 v \|_{L^2}^{\frac{d+6}{2}} \| v \|_{L^2}^{\frac{d+6}{2}} \leq C \varepsilon_1 \| \nabla^2 v \|_{L^2}^2.
\]
On the other hand, when \( d \geq 6 \), it is possible to get
\[
I_1 \leq C \| \partial v \|_{L^6} \| \nabla v \|_{L^{\frac{\gamma}}}, \quad \| \nabla v \|_{L^{\frac{\gamma}}} \leq C \varepsilon_1 \| \nabla v \|_{L^2}^2,
\]
where we used
\[
\| \nabla v \|_{L^{\frac{\gamma}}} \leq C \| \nabla^{[\frac{3}{2}]} v \|_{L^2} \| v \|_{L^2}^{\frac{\gamma}{2}}, \quad \gamma = \frac{d+1}{2} - 1.
\]
Thus, in either case, we have
\[
I_1 \leq C \varepsilon_1 \| \nabla^2 v \|_{L^2}^2.
\]

\( \diamond \) (Estimates for \( I_3 \)): For \( d = 2 \), we use (2.1) with \( m = 1 \) and
\[
\| f \|_{L^{\frac{1}{2}}} \leq C \| \nabla f \|_{L^{1/3}}, \quad \| f \|_{L^{\frac{1}{2}}} \leq C \| \nabla f \|_{L^{8/5}}
\]
to obtain
\[ \int_{\mathbb{R}^d} \rho \partial(u - v) \cdot \partial v \, dx = \int_{\mathbb{R}^d} \frac{\partial \rho}{\rho} \rho^{1/8 + 3/4 + (5/8 - 1/2)} (u - v) \partial v \, dx \]
\[ \leq \| \phi \|_{L^{8/5}} \| \rho^{3/4} (u - v) \|_{L^4} \| \rho^{1/8} \partial v \|_{L^8} \]
\[ \leq C \| \rho \|_{L^{8/5}} \| \phi \|_{L^2} \| \rho^{3/4} \nabla (u - v) \|_{L^{8/3}} \| \rho^{1/8} \nabla \partial v \|_{L^{8/5}} \]
\[ \leq C \| \rho \|_{L^{8/5}} \| \phi \|_{L^2} \| \sqrt{\rho} (u - v) \|_{L^2} \| \rho^{1/4} \|_{L^4} \| \nabla \partial v \|_{L^2} \| \rho^{1/8} \|_{L^8} \]
\[ \leq C \varepsilon_1 \| \sqrt{\rho} (u - v) \|_{L^2} \| \nabla \partial v \|_{L^2}. \]

When \( d \geq 3 \), we again use (3.2) with \( m = 1 \) to find
\[ \int_{\mathbb{R}^d} \rho \partial(u - v) \cdot \partial v \, dx = \int_{\mathbb{R}^d} \frac{\partial \rho}{\rho} \rho^{1/2 + 1/2} (u - v) \partial v \, dx \]
\[ \leq \| \phi \|_{L^{8/5}} \| \sqrt{\rho} (u - v) \|_{L^{8/3}} \| \partial v \|_{L^{8/3}} \]
\[ \leq C \| \rho \|_{L^{8/5}} \| \phi \|_{L^2} \| \sqrt{\rho} (u - v) \|_{L^2} \| \nabla \partial v \|_{L^2} \]
\[ \leq C \| \rho \|_{L^{8/5}} \| \phi \|_{L^2} \| \sqrt{\rho} (u - v) \|_{L^2} \| \nabla \partial v \|_{L^2} \]

Here we used
\[ \| \rho \|_{L^{8/5}} \leq C \| \nabla \|_{L^2} \| \rho \|_{L^2} \| \rho \|_{L^8} \leq C \| \nabla \|_{L^2} \| \rho \|_{L^2} \| \rho \|_{L^8} \]

Thus, we collect the estimates for \( I_i \)'s and use Young’s inequality and the smallness assumption on \( \varepsilon_1 > 0 \) to deduce
\[ \frac{1}{2} \frac{d}{dt} \| \nabla v \|_{L^2}^2 + \| \nabla^2 v \|_{L^2}^2 \leq C \varepsilon_1 \left( \int_{\mathbb{R}^d} \rho \nabla (u - v) \|_{L^2}^2 \right) + \int_{\mathbb{R}^d} \rho (\nabla u - \nabla v) \cdot \nabla v \, dx, \]

where \( C > 0 \) is independent of \( \varepsilon_1 \) and \( t \).

- (Step B: Estimates for \( u \)): For \( u \), direct computation gives
\[ \frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} \partial u \|_{L^2}^2 = - \int_{\mathbb{R}^d} \rho \partial u \cdot (\partial u \cdot \nabla u) \, dx - \int_{\mathbb{R}^d} \rho (\partial u \cdot \nabla u) \cdot \partial u \, dx, \]
where
\[ \left| \int_{\mathbb{R}^d} \rho \partial u \cdot (\partial u \cdot \nabla u) \, dx \right| \leq \| \nabla u \|_{L^\infty} \| \sqrt{\rho} \partial u \|_{L^2}^2 \leq C \varepsilon_1 \left( \| \sqrt{\rho} \partial (u - v) \|_{L^2}^2 + \| \sqrt{\rho} \partial v \|_{L^2}^2 \right). \]

On the other hand, by using (2.1) and Hölder’s inequality, we get
\[ \| \sqrt{\rho} \partial v \|_{L^2} \leq \begin{cases} \| \rho \|_{L^{8/5}}^{1/4} \| \rho^{1/4} \partial v \|_{L^{8/3}} \leq C \| \rho \|_{L^{8/5}}^{1/4} \| \partial v \|_{L^{8/3}} \leq C \| \rho \|_{L^{8/5}}^{1/4} \| \nabla \partial v \|_{L^2}, & \text{if } d = 2, \\ \| \rho \|_{L^{\infty}}^{1/2} \| \partial v \|_{L^{8/5}}^{1/2} \leq C \| \nabla \partial v \|_{L^2}, & \text{if } d \geq 3. \end{cases} \]

This gives
\[ \frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} \nabla u \|_{L^2}^2 \leq C \varepsilon_1 \left( \int_{\mathbb{R}^d} \rho \| \nabla (u - v) \|_{L^2} \| \nabla^2 v \|_{L^2} \right) - \int_{\mathbb{R}^d} \rho (\nabla u - \nabla v) \cdot \nabla v \, dx. \]

Then we combine (3.3) with (3.1) and use the smallness of \( \varepsilon_1 > 0 \) to conclude the desired result. \( \square \)

We next proceed to the higher-order estimates. We start with the estimates for \( u \) in the lemma below.

**Lemma 3.3.** For \( 2 \leq m \leq s + 1 \), there exists \( C > 0 \) independent of \( t \) such that for any \( t > 0 \),
\[ \frac{d}{dt} \| \sqrt{\rho} \partial^m u \|_{L^2} + 2 \int_{\mathbb{R}^d} \rho \partial^m (u - v) \cdot \partial^m u \, dx \]
\[ \leq C \begin{cases} \varepsilon_1 (\| \sqrt{\rho} \partial^m (u - v) \|_{L^2}^2 + \| \nabla^{m+1} v \|_{L^2}^2), & \text{if } d \geq 3 \text{ and } 2 \leq m \leq d - 1, \\ \varepsilon_1 \sum_{\ell = d - 1}^m (\| \sqrt{\rho} \partial^\ell (u - v) \|_{L^2}^2 + \| \nabla^{\ell+1} v \|_{L^2}^2), & \text{if } d \leq m \leq s + 1. \end{cases} \]
Proof. We split the proof into two cases.

• (Case A: $d \geq 3$ and $2 \leq m \leq d - 1$): Similarly as in the zeroth-order estimate, we find

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} \partial^m u\|_{L^2}^2 \leq C \sum_{\ell=1}^{m} \int_{\mathbb{R}^d} \rho \partial^m u \cdot (\partial^\ell u \cdot \nabla \partial^{m-\ell} u) \, dx - \int_{\mathbb{R}^d} \rho \partial^m (u-v) \cdot \partial^m u \, dx.$$  

We then separately estimate the term $\int_{\mathbb{R}^d} \rho \partial^m u \cdot (\partial^\ell u \cdot \nabla \partial^{m-\ell} u) \, dx$ as follows.

○ (Case A-1: $m - \ell < [d/2]$): Here, we have $\frac{1}{2} - \frac{m-\ell}{d} > 0$ and use (2.1) and (3.2) to get

$$\int_{\mathbb{R}^d} \rho \partial^m u \cdot (\partial^\ell u \cdot \nabla \partial^{m-\ell} u) \, dx \leq \|\sqrt{\rho} \partial^m u\|_{L^2} \|\sqrt{\rho} \partial^\ell u\|_{L^2} \|\nabla \partial^{m-\ell+1} u\|_{L^\infty} \leq C \|\sqrt{\rho} \nabla^m u\|_{L^2} \|\nabla u\|_{H^{[\frac{d}{2}]}} \leq C \varepsilon |1 + \frac{\|\sqrt{\rho} \nabla^m u\|_{L^2}^2 + \|\sqrt{\rho} \nabla^m v\|_{L^2}^2}{\|\nabla^{m+1} v\|_{L^2}^2} \},$$

where we also used, for $0 \leq j < d/2,$

$$\|\nabla^{j+1} f\|_{L^2} \leq C \|\nabla^{[\frac{d}{2}] + j} f\|_{L^2} \|f\|_{L^2}^{1-\gamma} \leq C \|\nabla f\|_{H^{[\frac{d}{2}]}} \leq \gamma = \frac{d/2}{[d/2] + 1}. \quad (3.4)$$

○ (Case A-2: $m - \ell \geq [d/2]$) In this case, since $\ell - 1 \leq m - [d/2] - 1 < d/2,$ i.e. $d/2 > \ell - 1,$ we obtain

$$\int_{\mathbb{R}^d} \rho \partial^m u \cdot (\partial^\ell u \cdot \nabla \partial^{m-\ell} u) \, dx \leq C \|\nabla^m u\|_{L^2} \|\nabla^\ell u\|_{L^2} \|\nabla \partial^{m-\ell+1} u\|_{L^\infty} \leq C \|\nabla u\|_{H^{[\frac{d}{2}]}} \|\nabla^m u\|_{L^2} \leq C \varepsilon \left(\|\nabla^m (u-v)\|_{L^2}^2 + \|\nabla^{m+1} v\|_{L^2}^2 \right),$$

due to (2.1), (3.2), and (3.4).

Thus, we combine all the estimates to obtain

$$\frac{d}{dt} \|\sqrt{\rho} \partial^m u\|_{L^2}^2 \leq C \varepsilon \left(\|\sqrt{\rho} \nabla^m (u-v)\|_{L^2}^2 + \|\nabla^{m+1} v\|_{L^2}^2 \right)$$

for some $C > 0$ independent of $t.$

• (Case B: $d \leq m \leq s + 1$): We first observe

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} \partial^m u\|_{L^2}^2 = - \int_{\mathbb{R}^d} \rho \partial^m u \cdot (\partial^m u \cdot \nabla u) \, dx - \int_{\mathbb{R}^d} \rho \partial^m (u-v) \cdot \partial^m u \, dx \leq C \sum_{\ell=1}^{m} \int_{\mathbb{R}^d} \rho \partial^m u \cdot (\partial^\ell u \cdot \nabla \partial^{m-\ell} u) \, dx - \int_{\mathbb{R}^d} \rho \partial^m (u-v) \cdot \partial^m u \, dx \leq C \|\nabla u\|_{L^\infty} \|\sqrt{\rho} \nabla^m u\|_{L^2}^2 + C \sum_{\ell=1}^{m-1} \int_{\mathbb{R}^d} \rho \partial^m u \cdot (\partial^\ell u \cdot \nabla \partial^{m-\ell} u) \, dx \leq \frac{d}{dt} \|\sqrt{\rho} \partial^m u\|_{L^2}^2 - \int_{\mathbb{R}^d} \rho \partial^m (u-v) \cdot \partial^m u \, dx.$$
Here we separately estimate the second term in the last inequality as
\[
\sum_{\ell=2}^{m-1} \left| \int_{\mathbb{R}^d} \rho \partial^m u \cdot (\partial^\ell u \cdot \nabla \partial^{m-\ell} u) \, dx \right|
\]
\[
= \left( \sum_{\ell=2}^{d-2} + \sum_{\ell=\left[\frac{d-1}{2}\right]+1}^{m-1} \right) \left| \int_{\mathbb{R}^d} \rho \partial^m u \cdot (\partial^\ell u \cdot \nabla \partial^{m-\ell} u) \, dx \right|
\]
\[
= : J_1 + J_2 + J_3.
\]
\(\diamond\) (Estimates for \(J_1\)): We use \((2.1)\) and \((3.4)\) to get
\[
J_1 \leq \sum_{\ell=2}^{\left[\frac{d-1}{2}\right]} \| \sqrt{\rho} \partial^m u \|_{L^2} \| \partial^\ell u \|_{L^{\frac{d}{d-2}}} \| \sqrt{\nabla} \partial^{m-\ell+1} u \|_{L^{\frac{d}{d-4}}} \leq C \| \nabla u \|_{H^{\left[\frac{d}{2}\right]+1}} \| \sqrt{\rho^d} \partial^m u \|_{L^2}.
\]
\(\diamond\) (Estimates for \(J_2\)): In this case, we use
\[
\| f \|_{L^2} \leq C \| \nabla^{\left[\frac{d}{2}\right]+1-\ell} f \|_{L^2} \| f \|_{L^2}^{1-\gamma} \leq C \| f \|_{H^{\left[\frac{d}{2}\right]+1-\ell}}, \quad \gamma = \frac{d/2 - \ell}{d/2 + 1 - \ell}, \quad 0 \leq \ell \leq \lfloor d/2 \rfloor, \tag{3.5}
\]
and combine this with \((2.1)\) to yield
\[
J_2 \leq \sum_{\ell=\left[\frac{d-1}{2}\right]+1}^{d-2} \| \sqrt{\rho} \partial^m u \|_{L^2} \| \sqrt{\rho^d} \partial^\ell u \|_{L^{\frac{d}{d-2}}} \| \nabla^{m-\ell+1} u \|_{L^{\frac{d}{d-4}}} \leq C \| \nabla u \|_{H^{\left[\frac{d}{2}\right]+1}} \| \sqrt{\rho^d} \partial^m u \|_{L^2} \| \sqrt{\rho^d} \partial^{d-1} u \|_{L^2}.
\]
\(\diamond\) (Estimates for \(J_3\)): We simply use the Hölder’s inequality to estimate
\[
J_3 \leq \sum_{\ell=d-1}^{m-1} \| \sqrt{\rho} \partial^m u \|_{L^2} \| \sqrt{\rho^d} \partial^\ell u \|_{L^2} \| \nabla^{m-\ell+1} u \|_{L^2}
\]
\[
\leq C \| \nabla u \|_{H^{\left[\frac{d}{2}\right]+1}} \| \sqrt{\rho^d} \partial^m u \|_{L^2} \sum_{\ell=d-1}^{m-1} \| \sqrt{\rho^d} \partial^\ell u \|_{L^2}.
\]
We gather the estimates for \(J_i\)'s to have
\[
\frac{d}{dt} \| \sqrt{\rho^d} \partial^m u \|_{L^2}^2 + 2 \int_{\mathbb{R}^d} \rho \partial^m (u - v) \cdot \partial^m u \, dx \leq C \varepsilon_1 \sum_{\ell=d-1}^{m} \| \sqrt{\rho^d} \partial^\ell u \|_{L^2}^2
\]
\[
\leq C \varepsilon_1 \sum_{\ell=d-1}^{m} \left( \| \sqrt{\rho^d} \partial^\ell (u - v) \|_{L^2}^2 + \| \nabla^{\ell+1} v \|_{L^2}^2 \right),
\]
where we used \((3.5)\), and this gives the desired result.
\(\square\)

We finally show the higher-order estimates for \(v\).

**Lemma 3.4.** For \(2 \leq m \leq s + 1\), there exists \(C > 0\) independent of \(t\) such that for every \(t > 0\),
\[
\frac{d}{dt} \| \partial^m v \|_{L^2} + 2 \| \nabla \partial^m v \|_{L^2} + 2 \int_{\mathbb{R}^d} \rho \partial^m (v - u) \cdot \partial^m v \, dx
\]
\[
\leq C \| \nabla v \|_{L^\infty} \| \nabla^m v \|_{L^2}^2
\]
\[
+ C \varepsilon_1 \left\{ \begin{array}{ll}
\| \sqrt{\rho^d} \partial^m (u - v) \|_{L^2}^2 + \| \nabla^{m+1} v \|_{L^2}^2 & \text{if } d \geq 3 \text{ and } 2 \leq m \leq d - 1, \\
\| \nabla^{m+1} v \|_{L^2}^2 + \sum_{\ell=d-1}^{m} \| \sqrt{\rho^d} \partial^\ell (u - v) \|_{L^2}^2 & \text{if } d \leq m \leq s + 1.
\end{array} \right.
\]
Proof. Similarly to the proof of Lemma 3.3, we split the proof into two cases.

- (Case A: \( d \geq 3 \) and \( 2 \leq m \leq d-1 \)): First, we use Lemma 3.4 to yield

\[
\frac{1}{2} \int_{\mathbb{R}^d} \rho \frac{\partial \rho}{\partial t} \rho^m \partial^m v \cdot \nabla \partial^m v \, dx = - \int_{\mathbb{R}^d} \rho \frac{\partial \rho}{\partial t} \rho^m \partial^m v \cdot (\nabla v - v \cdot \nabla \rho \partial^m v) \, dx + \int_{\mathbb{R}^d} \rho \partial^m (\rho (u-v)) \cdot \partial^m v \, dx
\]

\[
\leq C \| \nabla v \|_{L^\infty} \| \rho \rho^m (u-v) \|_{L^\infty} \| \partial^m v \|_{L^2} + \int_{\mathbb{R}^d} \rho \partial^m (\rho (u-v)) \cdot \partial^m v \, dx \leq C \| \nabla v \|_{L^\infty} \| \rho \rho^m (u-v) \|_{L^\infty} \| \partial^m v \|_{L^2} + \int_{\mathbb{R}^d} \rho \partial^m (\rho (u-v)) \cdot \partial^m v \, dx
\]

\[
= C \| \nabla v \|_{L^\infty} \| \rho \rho^m (u-v) \|_{L^\infty} \| \partial^m v \|_{L^2} + \int_{\mathbb{R}^d} \rho \partial^m (u-v) \cdot \partial^m v \, dx + K_1 + K_2 + K_3.
\]

- (Estimates for \( K_1 \)): If \( 1 \leq \ell < [d/2] \), we use Lemma 3.4 and the estimates (2.41), (3.2), and (3.3) to get

\[
\left| \int_{\mathbb{R}^d} \rho \partial^m \partial^m \rho (u-v) \cdot \partial^m v \, dx \right| \leq \left| \int_{\mathbb{R}^d} \rho \partial \rho \partial^m \rho \partial^m \rho (u-v) \cdot \partial^m v \, dx \right| \leq C \| \nabla v \|_{L^\infty} \| \rho \rho^m (u-v) \|_{L^\infty} \| \partial^m v \|_{L^2} + \int_{\mathbb{R}^d} \rho \partial^m (\rho (u-v)) \cdot \partial^m v \, dx
\]

This implies

\[
K_1 \leq C \| \nabla v \|_{L^\infty} \| \rho \rho^m (u-v) \|_{L^\infty} \| \partial^m v \|_{L^2}.
\]

- (Estimates for \( K_2 \)): If \([d/2] \leq \ell \leq m-1 \leq d-2 \) (in this case, we naturally assume \([d/2] + 1 \leq m \)), we choose \( p \in (1, 2) \) so that

\[
\max \left\{ \frac{d}{d-1}, \frac{1}{2} \right\} < p < \frac{\ell+1}{d}.
\]

Then we estimate

\[
\left| \int_{\mathbb{R}^d} \rho \partial^m \partial^m \rho (u-v) \cdot \partial^m v \, dx \right| \leq \left| \int_{\mathbb{R}^d} \rho \partial \rho \partial^m \rho \partial^m \rho (u-v) \cdot \partial^m v \, dx \right| \leq C \| \nabla v \|_{L^\infty} \| \rho \rho^m (u-v) \|_{L^\infty} \| \partial^m v \|_{L^2} + \int_{\mathbb{R}^d} \rho \partial^m (\rho (u-v)) \cdot \partial^m v \, dx
\]

where we used

\[
\left\{ \begin{array}{ll} \frac{1}{d-1} \leq \ell \leq d-1 & \implies |\rho|_{L^\infty} = \frac{1}{d} \leq \frac{1}{d-1} \leq \\
\ell-1 \leq m-2 \leq d-3 \leq s-1 & \implies |\rho|_{H^{\ell-1}} \leq |\rho|_{H^{s-1}} \leq C \varepsilon_1.
\end{array} \right.
\]

This shows

\[
K_2 \leq C \varepsilon_1 \| \nabla \rho \rho^m (u-v) \|_{L^\infty} \| \partial^m v \|_{L^2}.
\]
\(\diamond\) (Estimates for \(K_3\)): Similarly as before, we estimate

\[
\left| \int_{\mathbb{R}^d} \partial^{m-1} \rho (u - v) \cdot \partial^m v \, dx \right| \leq \frac{1}{\rho} \left| \frac{\partial^m v}{v} \right| L^\frac{1}{2} + \frac{\rho}{\rho} \left| \frac{\partial^m v}{v} \right| L^\frac{1}{2} \| \partial^m v \|_{L^2}
\]

\[
\leq C \left| \frac{1}{\rho} \left| \frac{\partial^m v}{v} \right| \right| L^\frac{1}{2} + \frac{\rho}{\rho} \left| \frac{\partial^m v}{v} \right| L^\frac{1}{2} \| \partial^m v \|_{L^2}
\]

\[
\leq C \left| \frac{1}{\rho} \left| \frac{\partial^m v}{v} \right| \right| L^\frac{1}{2} + \frac{\rho}{\rho} \left| \frac{\partial^m v}{v} \right| L^\frac{1}{2} \| \sqrt{\rho \nabla^m (u - v)} \|_{L^2} \| \partial^m v \|_{L^2}
\]

\[
\leq C \| \partial_0 \|_{H^{m-1}} \| \sqrt{\rho \nabla^m (u - v)} \|_{L^2} \| \partial^m v \|_{L^2}
\]

\[
\leq C \varepsilon_1 \| \sqrt{\rho \nabla^m (u - v)} \|_{L^2} \| \partial^m v \|_{L^2},
\]

where \(p\) in this case satisfies

\[
\frac{m}{d} \leq \frac{1}{p} \leq \frac{m + 1}{d}.
\]

We now combine all the estimates for \(K_i\)'s to have

\[
\frac{d}{dt} \| \partial^m v \|_{L^2} + 2 \| \nabla \partial^m v \|_{L^2} + 2 \int_{\mathbb{R}^d} \rho \partial^m (v - u) \cdot \partial^m v \, dx
\]

\[
\leq C \| \partial_0 \|_{L^\infty} \| \nabla \partial^m v \|_{L^2} + C \varepsilon_1 \left( \| \sqrt{\rho \nabla^m (u - v)} \|_{L^2}^2 + \| \nabla^m v \|_{L^2}^2 \right).
\]

\(\bullet\) (Case B: \(d \leq m \leq s + 1\)): Similarly to the previous case, we estimate

\[
\frac{1}{2} \frac{d}{dt} \| \partial^m v \|_{L^2}^2 + \| \nabla \partial^m v \|_{L^2}^2
\]

\[
= - \int_{\mathbb{R}^d} \partial^m v \cdot \partial^m (v - u) \cdot \nabla \partial^m v \, dx + \int_{\mathbb{R}^d} \partial^m (\rho (u - v)) \cdot \partial^m v \, dx
\]

\[
\leq C \| \nabla v \|_{L^\infty} \| \nabla^m v \|_{L^2} + \int_{\mathbb{R}^d} \rho \partial^m (u - v) \cdot \partial^m v \, dx
\]

\[
+ \int_{\mathbb{R}^d} \| \partial^m (\rho (u - v)) - \rho \partial^m (u - v) \| \cdot \partial^m v \, dx
\]

\[
\leq C \varepsilon_1 \| \nabla^m v \|_{L^2}^2 + \int_{\mathbb{R}^d} \rho \partial^m (u - v) \cdot \partial^m v \, dx + C \sum_{\ell = 1}^{[\frac{d + 1}{2}]} \int_{\mathbb{R}^d} \partial^\ell \rho \partial^m \partial^\ell (u - v) \, \partial^m v \, dx
\]

\[
+ C \sum_{\ell = \frac{d + 1}{2} + 1}^{[\frac{d - 1}{2}]} \int_{\mathbb{R}^d} \partial^\ell \rho \partial^m \partial^\ell (u - v) \, \partial^m v \, dx + C \sum_{\ell = d}^{d - 1} \int_{\mathbb{R}^d} \partial^\ell \rho \partial^m \partial^\ell (u - v) \, \partial^m v \, dx
\]

\[
+ C \int_{\mathbb{R}^d} \partial^{m-1} \rho (u - v) \partial^m v \, dx
\]

\[
\leq C \varepsilon_1 \| \nabla^m v \|_{L^2}^2 + \int_{\mathbb{R}^d} \rho \partial^m (u - v) \cdot \partial^m v \, dx + L_1 + L_2 + L_3 + L_4.
\]

\(\diamond\) (Estimates for \(L_1\)): Similarly to the estimate of \(K_1\), we find

\[
L_1 \leq C \sum_{\ell = 1}^{[\frac{d + 1}{2}]} \| \partial^\ell \rho \|_{L^\frac{1}{2}} \| \sqrt{\rho \partial^m \partial^\ell (u - v)} \|_{L^\frac{1}{2}} \| \sqrt{\rho \partial^m v} \|_{L^2}
\]

\[
\leq C \sum_{\ell = 1}^{[\frac{d + 1}{2}]} \| \partial^\ell \rho \|_{L^\frac{1}{2}} \| \rho \partial^m (u - v) \|_{L^\frac{1}{2}} \| \sqrt{\rho \nabla^m (u - v)} \|_{L^2} \| \nabla^m v \|_{L^2}
\]

\[
\leq C \varepsilon_1 \| \sqrt{\rho \nabla^m (u - v)} \|_{L^2} \| \nabla^m v \|_{L^2}.
\]

\(\diamond\) (Estimates for \(L_2\)): For \([[(d - 1)/2] + 1 \leq \ell \leq d - 1\), we choose \(p \in (1, 2)\) satisfying

\[
\frac{\ell}{d} \leq \frac{1}{p} \leq \frac{\ell + 1}{d}
\]
to get
\[
L_2 \leq C \sum_{\ell=\lceil \frac{d}{d+1} \rceil + 1}^{d-1} \left\| \rho^{1-\frac{1}{p}} \frac{\partial^\ell \rho}{\rho} \right\|_{L^{\frac{1}{1-\frac{1}{p}}}} \left\| \rho^{\frac{1}{p}} \partial^m \ell (u - v) \right\|_{L^{\frac{1}{1-\frac{1}{p}}}} \left\| \partial^\ell v \right\|_{L^{\frac{1}{1-\frac{1}{p}}}}.
\]

\[
\leq C \sum_{\ell=\lceil \frac{d}{d+1} \rceil + 1}^{d-1} \left\| \rho^{1-\frac{1}{p}} \frac{\partial^\ell \rho}{\rho} \right\|_{L^{2}} \left\| \rho^{\frac{1}{p}} \partial^m \ell (u - v) \right\|_{L^{p}} \left\| \nabla \partial^m v \right\|_{L^{2}}
\]

\[
\leq C \sum_{\ell=\lceil \frac{d}{d+1} \rceil + 1}^{d-1} \left\| \partial^\ell \rho \right\|_{L^{2}} \left\| \sqrt{\rho} \nabla^m (u - v) \right\|_{L^{2}} \left\| \nabla \partial^m v \right\|_{L^{2}}
\]

\[
\leq C \sum_{\ell=\lceil \frac{d}{d+1} \rceil + 1}^{d-1} \left\| \phi \right\|_{H^{\ell-1}} \left\| \sqrt{\rho} \nabla^m (u - v) \right\|_{L^{2}} \left\| \nabla \partial^m v \right\|_{L^{2}}
\]

\[
\leq C \| \phi \|_{H^{d-2}} \left\| \sqrt{\rho} \nabla^m (u - v) \right\|_{L^{2}} \left\| \nabla \partial^m v \right\|_{L^{2}}.
\]

\diamond (Estimates for $L_3$): We choose $p$ satisfying
\[
\frac{d-1}{d} < \frac{1}{p} < 1
\]
to obtain
\[
L_3 \leq C \sum_{\ell=d}^{m-1} \left\| \rho^{1-\frac{1}{p}} \frac{\partial^\ell \rho}{\rho} \right\|_{L^{\frac{1}{1-\frac{1}{p}}}} \left\| \rho^{\frac{1}{p}} \partial^m \ell (u - v) \right\|_{L^{\frac{1}{1-\frac{1}{p}}}} \left\| \partial^m v \right\|_{L^{\frac{1}{1-\frac{1}{p}}}}.
\]

\[
\leq C \sum_{\ell=d}^{m-1} \left\| \rho \right\|_{L^{1}}^{1-\frac{1}{p}} \left\| \partial^\ell \rho \right\|_{L^{2}} \left\| \rho^{\frac{1}{p}} \partial^m \ell + d-1 (u - v) \right\|_{L^{p}} \left\| \nabla \partial^m v \right\|_{L^{2}}
\]

\[
\leq C \sum_{\ell=d}^{m-1} \left\| \partial^\ell \rho \right\|_{L^{2}} \left\| \sqrt{\rho} \nabla^m \ell + d-1 (u - v) \right\|_{L^{2}} \left\| \nabla \partial^m v \right\|_{L^{2}}
\]

\[
\leq C \| \phi \|_{H^{m-2}} \left\| \nabla \partial^m v \right\|_{L^{2}} \sum_{\ell=d}^{m-1} \left\| \sqrt{\rho} \nabla^\ell (u - v) \right\|_{L^{2}}.
\]

\diamond (Estimates for $L_4$): We again choose $p$ satisfying $(d-1)/d < 1/p < 1$ and use
\[
\left\| f \right\|_{L^{\infty}} \leq C \| \nabla f \|_{L^{\frac{d}{1-\frac{1}{p}}}} \left\| f \right\|_{L^{\frac{d(1-\frac{1}{p})}{1-\frac{1}{p}}}}
\]
and combine this with (2.1) to yield
\[
L_4 \leq C \left\| \frac{\partial^{m-1} \rho}{\rho} \right\|_{L^{2}} \left\| \rho (u - v) \right\|_{L^{\infty}} \left\| \partial^m v \right\|_{L^{2}}
\]

\[
\leq C \| \phi \|_{H^{m-2}} \left( \left\| \rho (u - v) \right\|_{L^{\frac{d}{1-\frac{1}{p}}}} + \left\| \sqrt{\rho}(u - v) \right\|_{L^{\frac{d}{1-\frac{1}{p}}}} \right) \left\| \partial^m v \right\|_{L^{2}}
\]

\[
\leq C \| \phi \|_{H^{m-2}} \left( \| \rho \nabla^{d-1}(u - v) \|_{L^{p}} + \| \rho \nabla^d(u - v) \|_{L^{p}} \right) \left\| \partial^m v \right\|_{L^{2}}
\]

\[
\leq C \| \phi \|_{H^{m-2}} \left( \| \sqrt{\rho} \nabla^{d-1}(u - v) \|_{L^{2}} + \| \sqrt{\rho} \nabla^d(u - v) \|_{L^{2}} \right) \left\| \partial^m v \right\|_{L^{2}}.
\]

Thus we collect all the estimates for $L_i$’s to obtain
\[
\frac{d}{dt} \left\| \partial^m v \right\|_{L^{2}} + 2 \left\| \nabla \partial^m v \right\|_{L^{2}} + 2 \int_{\mathbb{R}^{d}} \rho \partial^m (v - u) \cdot \partial^m v \, dx
\]

\[
\leq C \left\| \nabla v \right\|_{L^{\infty}} \left\| \nabla^m v \right\|_{L^{2}}^{2} + C \varepsilon_1 \left( \left\| \nabla^{m+1} v \right\|_{L^{2}}^{2} + \sum_{\ell=d-1}^{m} \left\| \sqrt{\rho} \nabla^\ell (u - v) \right\|_{L^{2}}^{2} \right).
\]

This completes the proof. \qed
4. Large time behavior estimates

In this section, we present the improved large time behavior estimates for system (1.1) by using the a priori estimates obtained in the previous section. We first slightly improve the previous large time behavior estimates in [6] on the fluid velocities $u$ and $v$. We then show the uniform-in-time bound estimate on $\|\phi(\cdot, t)\|_{H^{s-1}}$. These estimates enable us to establish the global-in-time existence and uniqueness of solutions for our main system (1.1) in the desired Sobolev space. We finally provide the detail on the proof for Theorem 1.1.

Let us start with the estimates on $\|\sqrt{\rho} \nabla^\ell u(\cdot, t)\|_{L^2}$ for $1 \leq \ell \leq s + 1$ and $\|\nabla v(\cdot, t)\|_{H^2}$.

**Proposition 4.1.** Let $T > 0$ be given. For $\alpha \in (0, d/2 + 1)$, there exists a constant $C > 0$ independent of $T$ such that

\[
(1 + t)^\alpha \sum_{\ell=1}^{s+1} \|\sqrt{\rho} \nabla^\ell u(\cdot, t)\|_{L^2}^2 + (1 + t)^\alpha \|\nabla v(\cdot, t)\|_{H^s}^2 \\
+ \sum_{\ell=1}^{s} \int_0^t (1 + \tau)^\alpha \|\sqrt{\rho} \nabla^\ell (u - v)(\cdot, \tau)\|_{L^2}^2 d\tau + \int_0^t (1 + \tau)^\alpha \|\nabla^2 v(\cdot, \tau)\|_{H^s}^2 d\tau \leq C
\]

for all $t \in [0, T]$.

**Remark 4.1.** In [6], the above large time behavior estimates were obtained for $\alpha \in (0, d/2)$.

**Proof of Proposition 4.1.** We combine Lemmas 3.2, 3.3, 3.4 to get

\[
\frac{d}{dt} \left[ \sum_{\ell=1}^{s+1} \|\sqrt{\rho} \nabla^\ell u\|_{L^2}^2 + \|\nabla^\ell v\|_{L^2}^2 \right] + 2 \sum_{\ell=1}^{s+1} \left( \|\sqrt{\rho} \nabla^\ell (u - v)\|_{L^2}^2 + \|\nabla^{\ell+1} v\|_{L^2}^2 \right) \\
\leq C \varepsilon_1 \sum_{\ell=1}^{s+1} \left( \|\sqrt{\rho} \nabla^\ell (u - v)\|_{L^2}^2 + \|\nabla^{\ell+1} v\|_{L^2}^2 \right)
\]

for all $t \in [0, T]$, where $C > 0$ is independent of $T$. We then choose $\varepsilon_1 > 0$ small enough to obtain

\[
\frac{d}{dt} \left[ \sum_{\ell=1}^{s+1} \|\sqrt{\rho} \nabla^\ell u\|_{L^2}^2 + \|\nabla^\ell v\|_{L^2}^2 \right] + \sum_{\ell=1}^{s+1} \left( \|\sqrt{\rho} \nabla^\ell (u - v)\|_{L^2}^2 + \|\nabla^{\ell+1} v\|_{L^2}^2 \right) \leq 0. \quad (4.1)
\]

For simplicity, we let

\[
E_1(t) := \sum_{\ell=1}^{s+1} \left( \|\sqrt{\rho} \nabla^\ell u\|_{L^2}^2 + \|\nabla^\ell v\|_{L^2}^2 \right) \quad \text{and} \quad D_1(t) := \sum_{\ell=1}^{s+1} \left( \|\sqrt{\rho} \nabla^\ell (u - v)\|_{L^2}^2 + \|\nabla^{\ell+1} v\|_{L^2}^2 \right).
\]

Here, we note that

\[
\|\rho\|_{L^{\infty}} \leq C \|\nabla^{[\frac{d}{2}]+1} \rho\|_{L^{\infty}}^{\frac{1}{2}} \|\rho\|^{1-\gamma}_{L^1} \leq C \left\| \frac{\nabla^{[\frac{d}{2}]+1} \rho}{\rho} \right\|_{L^2}^{\gamma} \|\rho\|_{L^{\infty}}, \quad \gamma = \frac{d}{[d/2]+d/2+1},
\]

which gives

\[
\|\rho\|_{L^{\infty}} \leq C \left\| \frac{\nabla^{[\frac{d}{2}]+1} \rho}{\rho} \right\|_{L^2}^{\frac{1}{\gamma}} \leq C \left( 1 + \|\phi\|_{H^{[\frac{d}{2}]+1}} \right) \left\| \frac{\nabla^{[\frac{d}{2}]+1} \phi}{\rho} \right\|_{L^2}^{\frac{1}{\gamma}} \leq C \varepsilon_1 \frac{1}{\gamma}.
\]

Thus,

\[
\int_{\mathbb{R}^d} \rho |\phi'(u - v)|^2 \, dx \geq \int_{\mathbb{R}^d} \rho |\phi' u|^2 \, dx - C \varepsilon_1 \int_{\mathbb{R}^d} |\phi' v|^2 \, dx.
\]

Now, we define

\[
g^2(t) := \frac{4\alpha}{4d+8+t} \leq \frac{1}{2} \quad \text{and} \quad \tilde{g}^2(t) := \frac{g^2(t)}{4} = \frac{\alpha}{4d+8+t}.
\]
Then we estimate
\[
\|\nabla^2 v\|_{L^2} = \int_{\mathbb{R}^d} |\xi|^2 |\nabla v|^2 d\xi \\
\geq \int_{(1|x| \leq \epsilon(t))} |\xi|^2 |\nabla v|^2 d\xi \\
\geq g^2(t)\|\nabla v\|_{L^2}^2 - g^2 \int_{(1|x| \leq \epsilon(t))} |\xi|^2 |\nabla v|^2 d\xi \geq g^2(t)\|\nabla v\|_{L^2}^2 - g^4\|v\|_{L^2}^2.
\]

This implies
\[
D_1(t) \geq \frac{1}{2}D_1(t) + g^2 \sum_{\ell=1}^{s+1} \|\sqrt{\rho} \nabla^\ell (u - v)\|_{L^2}^2 + \frac{1}{2}\|\nabla^2 v\|_{L^2}^2 \\
\geq \frac{1}{2}D_1(t) + g^2 \sum_{\ell=1}^{s+1} \|\sqrt{\rho} \nabla^\ell (u - v)\|_{L^2}^2 - C\varepsilon_1 g^2 \sum_{\ell=2}^{s+1} \|\nabla^\ell v\|_{L^2}^2 + g^2 (1/2 - C\varepsilon_1)\|\nabla v\|_{L^2}^2 - \frac{g^4}{2}\|v\|_{L^2}^2 \\
\geq \frac{1}{4}D_1(t) + \frac{g^2}{4}E_1(t) - \frac{g^4}{2}\|v\|_{L^2}^2,
\]

where \(\varepsilon_1 > 0\) is chosen small enough. We combine the above with (4.11) to deduce
\[
\frac{d}{dt}E_1(t) + \alpha(4d + 8 + t)^{-1}E_1(t) + \frac{1}{4}D_1(t) \leq \frac{g^4}{2}\|v\|_{L^2}^2 \leq C(4d + 8 + t)^{-2}\|v\|_{L^2}^2.
\]

We choose \(\delta \in (0, 1)\) so that \(\alpha - 1 + \delta \in (0, d/2)\). Since there exists a constant \(C > 0\) independent of \(T\) such that \(\|v(t)\|_{L^2}^2 \leq C(1 + t)^{-\beta}\) for every \(\beta \in (0, d/2)\) due to Proposition 2.1, we get
\[
\frac{d}{dt}E_1(t) + \alpha(4d + 8 + t)^{-1}E_1(t) + \frac{1}{4}D_1(t) \leq C(4d + 8 + t)^{-1-\alpha-\delta}.
\]

Then the above yields
\[
\frac{d}{dt}[(4d + 8 + t)^\alpha E_1(t)] + \frac{(4d + 8 + t)^\alpha}{4}D_1(t) \leq C(4d + 8 + t)^{-1-\delta},
\]

and integration with respect to \(t\) gives the desired result.

We next show the time decay estimates on \(\|\nabla u(t)\|_{H^{s+1}}\).

**Lemma 4.1.** Let \(T > 0\) be given. For each \(\alpha \in (0, d/2 + 1)\), there exists a constant \(C > 0\) independent of \(T\) such that
\[
(1 + t)^\alpha \|\nabla u(t)\|_{H^{s+1}} + \int_0^t (1 + \tau)^\alpha \|\nabla^2 u(\tau)\|_{H^s}^2 d\tau \leq C(X_0(s) + \|v_0\|_{L^1}^2)
\]
for all \(t \in [0, T]\).

**Proof.** From the momentum equations in (1.1), we obtain that for \(1 \leq m \leq s + 2\),
\[
\frac{1}{2} \frac{d}{dt}\|\partial^m u\|_{L^2}^2 = -\int_{\mathbb{R}^d} (u \cdot \nabla \partial^m u) \partial^m u dx - \int_{\mathbb{R}^d} (\partial^m (u \cdot \nabla u) - u \cdot \nabla \partial^m u) \partial^m u dx \\
+ \int_{\mathbb{R}^d} \partial^m (u - v) \partial^m u dx \\
\leq C\|\nabla u\|_{L^\infty} \|\nabla^m u\|_{L^2}^2 - \frac{1}{2}\|\partial^m u\|_{L^2}^2 + \frac{1}{2}\|\nabla^m v\|_{L^2}^2,
\]

where \(C\) is independent of \(T\). Since \(\varepsilon_1 > 0\) is sufficiently small, we get
\[
\frac{d}{dt}\|\nabla^m u\|_{L^2}^2 + \frac{1}{2}\|\nabla^m u\|_{L^2}^2 \leq \|\nabla^m v\|_{L^2}^2.
\]

If \(m = 1\), since Proposition 4.1 implies
\[
\|\nabla v\|_{L^2}^2 \leq \frac{C(X_0(s) + \|v_0\|_{L^1}^2)}{(1 + t)^\alpha}
\]
for every $\alpha \in (0, d/2 + 1)$, we get
\[
\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|\nabla u\|_{L^2}^2 \leq \|\nabla v\|_{L^2}^2 \leq \frac{C (X_0(s) + \|v_0\|_{L^2})}{(1 + t)^\alpha}.
\]
Then applying the Grönwall’s lemma to the above gives
\[
\|\nabla u(\cdot, t)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 e^{-t/2} + C \left( X_0(s) + \|v_0\|_{L^2} \right) e^{-t/2} \int_0^t \frac{e^{s/2}}{(1 + s)^\alpha} ds.
\]
Here, as in [6] Section 4.3, we find
\[
\int_0^t \frac{e^{s/2}}{(1 + s)^\alpha} ds = \int_0^{t/2} \frac{e^{s/2}}{(1 + s)^\alpha} ds + \int_{t/2}^t \frac{e^{s/2}}{(1 + s)^\alpha} ds \leq e^{t/4} - 1 + \frac{e^{t/2} - e^{t/4}}{(1 + t/2)^\alpha},
\]
and subsequently this yields
\[
\|\nabla u(\cdot, t)\|_{L^2}^2 \leq \frac{C \left( X_0(s) + \|v_0\|_{L^2} \right)}{(1 + t)^\alpha} \quad \forall t \geq 0
\]
for some $C > 0$ independent of $T$.

If $2 \leq m \leq s + 1$, for every $\alpha \in (0, d/2 + 1)$, we can deduce from (1.1) that
\[
\frac{d}{dt} \|\nabla^m u\|_{L^2}^2 + \alpha (2d + 4 + t)^{-1} \|\nabla^m u\|_{L^2}^2 + \frac{1}{4} \|\nabla^m u\|_{L^2}^2 \leq \|\nabla^m v\|_{L^2}^2,
\]
which reduces to
\[
\frac{d}{dt} \left( (2d + 4 + t)^\alpha \|\nabla^m u\|_{L^2}^2 + \frac{1}{4} (2d + 4 + t)^\alpha \|\nabla^m u\|_{L^2}^2 \right) \leq (2d + 4 + t)^\alpha \|\nabla^m v\|_{L^2}^2.
\]
Then, by Proposition 4.1, integrating the above relation with respect to $t$ concludes the desired result.

We now provide the uniform estimates for $\phi$.

**Lemma 4.2.** Let $T > 0$ be given. There exists a constant $C > 0$ independent of $T$ such that
\[
\|\phi(\cdot, t)\|_{H^{s-1}} \leq C \left( \sqrt{X_0(s)} + \|v_0\|_{L^1} \right)
\]
for all $t \in [0, T]$.

**Proof.** It follows from the continuity equation in (1.1) that $\phi$ satisfies
\[
\partial_t \phi + \nabla (\phi \cdot u) + \nabla (\nabla \cdot u) = 0.
\]
Then straightforward computations yield
\[
\frac{1}{2} \frac{d}{dt} \|\phi\|_{L^2}^2 = - \int_{\mathbb{R}^d} (u \cdot \nabla \phi + \phi \cdot \nabla u) \cdot \phi \, dx - \int_{\mathbb{R}^d} \phi \cdot \nabla (\nabla \cdot u) \, dx
\]
\[
\leq C \|\nabla u\|_{L^\infty} \|\phi\|_{L^2}^2 + \|\phi\|_{L^2} \|\nabla^2 u\|_{L^2}
\]
and
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t \phi\|_{L^2}^2 = - \int_{\mathbb{R}^d} \partial_t (u \cdot \nabla \phi + \phi \cdot \nabla u) \cdot \partial_t \phi \, dx - \int_{\mathbb{R}^d} \partial_t \phi \cdot \nabla (\nabla \cdot u) \, dx
\]
\[
\leq C \left( \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty} \right) \|\phi\|_{H^s}^2 + \|\partial_t \phi\|_{L^2} \|\nabla^2 (\nabla \cdot u)\|_{L^2}.
\]
For the higher-order estimates, we find that for $2 \leq k \leq s - 1$,
\[
\frac{1}{2} \frac{d}{dt} \|\partial^k \phi\|_{L^2}^2 = - \int_{\mathbb{R}^d} (u \cdot \nabla \partial^k \phi) \cdot \partial^k \phi \, dx - \int_{\mathbb{R}^d} \left[ \partial^k (u \cdot \nabla \phi) - u \cdot \nabla \partial^k \phi \right] \cdot \partial^k \phi \, dx
\]
\[
- \int_{\mathbb{R}^d} \partial^k (\phi \cdot \nabla u) \cdot \partial^k \phi \, dx - \int_{\mathbb{R}^d} \partial^k (\nabla (\nabla \cdot u)) \cdot \partial^k \phi \, dx
\]
\[
\leq \frac{\|\nabla^k u\|_{L^\infty}}{2} \|\partial^k \phi\|_{L^2}^2 + C \|\partial^k \phi\|_{L^2} \left( \|\nabla^k \phi\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla \phi\|_{L^\infty} \|\partial^k u\|_{L^2} \right)
\]
\[
+ C \|\partial^k \phi\|_{L^2} \left( \|\phi\|_{L^s} + \|\nabla^{k+1} u\|_{L^2} + \|\partial^k \phi\|_{L^2} \|\nabla u\|_{L^\infty} \right) + \|\partial^k \phi\|_{L^2} \|\partial^k \nabla (\nabla \cdot u)\|_{L^2},
\]
and hence, we gather all the above estimates to obtain
\[
\frac{d}{dt} \|\phi\|_{H^{s-1}} \leq C \left( \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{H^{s-1}} \right) \|\phi\|_{H^{s-1}} + \|\nabla^2 u\|_{H^{s-1}}.
\]
Applying Grönwall’s lemma to the above yields
\[
\|\phi(\cdot, t)\|_{H^{s-1}} \leq \|\phi_0\|_{H^{s-1}} \exp \left( C \int_0^t \|\nabla u(\cdot, \tau)\|_{L^\infty} + \|\nabla^2 u(\cdot, \tau)\|_{H^{s-1}} \, d\tau \right) \\
+ \int_0^t \|\nabla^2 u(\cdot, \tau)\|_{H^{s-1}} \exp \left( C \int_\tau^t \|\nabla u(\cdot, r)\|_{L^\infty} + \|\nabla^2 u(\cdot, r)\|_{H^{s-1}} \, dr \right) \, d\tau.
\]

Now we split the proof into two cases: \(d = 2\) and \(d \geq 3\).

• (Case A: \(d = 2\)): For the two-dimensional case, we use Gagliardo–Nirenberg interpolation and Ladyzhenskaya’s inequalities to get
\[
\|\nabla u\|_{L^\infty} \leq C \|\nabla^2 u\|_{L^4}^{1/2} \|\nabla u\|_{L^4}^{1/2} \leq C \|\nabla^2 u\|_{H^1}^{3/4} \|\nabla u\|_{L^2}^{1/4}.
\] (4.3)

Now, we choose \(\beta \in (5/8, 7/8)\) and \(\varepsilon > 0\) so that \(8\beta/3 + \varepsilon < 7/3\). We use Lemma 4.11, 4.3, \(8\beta/5 > 1\), and \(8\beta/3 - 1/3 + \varepsilon < 2\) yields
\[
\int_0^t \|\nabla u(\cdot, \tau)\|_{L^\infty} \, d\tau \leq C \left( \int_0^t (1 + \tau)^{8\beta/3} \|\nabla^2 u(\cdot, \tau)\|_{H^1}^2 \|\nabla u(\cdot, \tau)\|_{L^2}^{2/3} \, d\tau \right)^{3/8} \left( \int_0^t (1 + \tau)^{-8\beta/5} \, d\tau \right)^{5/8}
\leq C \left( \mathcal{X}_0(s) + \|v_0\|_{L^1}^2 \right)^{1/8} \left( \int_0^t (1 + \tau)^{8\beta/3 - 1/3 + \varepsilon} \|\nabla^2 u(\cdot, \tau)\|_{H^1}^2 \, d\tau \right)^{3/8}
\leq C \left( \mathcal{X}_0(s) + \|v_0\|_{L^1}^2 \right)^{1/2},
\]
where we used from Lemma 4.11 that
\[
\|\nabla u\|_{L^2}^2 \leq C(\mathcal{X}_0(s) + \|v_0\|_{L^1}^2)(1 + t)^{-1+3\varepsilon}
\]
for some \(C > 0\) independent of \(T\). We also find from Lemma 4.11 that
\[
\int_0^t \|\nabla^2 u(\cdot, \tau)\|_{H^{s-1}} \, d\tau \leq \left( \int_0^t (1 + \tau)^{2\beta} \|\nabla^2 u(\cdot, \tau)\|_{H^{s-1}}^2 \, d\tau \right)^{1/2} \left( \int_0^t (1 + \tau)^{-2\beta} \, d\tau \right)^{1/2}
\leq C(\sqrt{\mathcal{X}_0(s)} + \|v_0\|_{L^1}),
\]
due to \(2\beta \in (5/4, 7/4)\). These two estimates yield the desired result.

• (Case B: \(d \geq 3\)): In this case, recall from [6] that we already have
\[
\int_0^t (1 + \tau)^{r+1} \|\nabla u(\cdot, \tau)\|_{H^r}^2 \, d\tau \leq C(\mathcal{X}_0(s) + \|v_0\|_{L^1})^2, \quad r \in (0, d/2 - 1).
\]

Hence, we use the fact \(s \geq [d/2] + 2\) and
\[
\int_0^t \|\nabla u(\cdot, \tau)\|_{L^\infty} \, d\tau \leq \int_0^t \|\nabla u(\cdot, \tau)\|_{H^s} \, d\tau
\leq \left( \int_0^t (1 + \tau)^{r+1} \|\nabla u(\cdot, \tau)\|_{H^r}^2 \, d\tau \right)^{1/2} \left( \int_0^t (1 + \tau)^{-(r+1)} \, d\tau \right)^{1/2}
\leq C \left( \sqrt{\mathcal{X}_0(s)} + \|v_0\|_{L^1} \right)
\]
to deduce
\[
\|\phi(\cdot, t)\|_{H^{s-1}} \leq \|\phi_0\|_{H^{s-1}} \exp \left( C \int_0^t \|\nabla u(\cdot, \tau)\|_{H^s} \, d\tau \right) \\
+ \int_0^t \|\nabla u(\cdot, \tau)\|_{H^s} \exp \left( C \int_\tau^t \|\nabla u(\cdot, r)\|_{H^s} \, dr \right) \, d\tau
\leq C \left( \|\phi_0\|_{H^{s-1}} + \sqrt{\mathcal{X}_0(s)} + \|v_0\|_{L^1} \right)
\leq C \left( \sqrt{\mathcal{X}_0(s)} + \|v_0\|_{L^1} \right).
\]
This completes the proof. \(\square\)
Remark 4.2. For the estimates of $\rho$, we already observed that
\[ \|\rho\|_{L^\infty} \leq C \left\| \frac{\nabla \rho}{\rho} \right\|_{L^2} \leq C \left[ (1 + \|\phi\|_{H^{[\frac{d}{2}]+1}}) \|\phi\|_{H^{[\frac{d}{2}]+1}} \right] \leq \infty, \]
and for $1 \leq m \leq s$,
\[ \|\nabla^m \rho\|_{L^2} \leq \|\rho\|_{L^\infty} \left\| \frac{\nabla^m \rho}{\rho} \right\|_{L^2} \leq C \|\rho\|_{L^\infty} (1 + \|\phi\|_{H^{m-1}} + \|\phi\|_{H^{[\frac{d}{2}]+1}})^{m-1} \|\phi\|_{H^{m-1}}. \]
Thus the uniform estimate for $\|\phi\|_{H^{m-1}}$ implies the uniform upper bound for $\|\rho\|_{H^r}$.

For the lower bound estimate of $\rho$, we define a backward characteristic flow $\eta = \eta(x, t)$ by
\[ \partial_\eta \rho(x, t) = u(\eta(x, s), s) \quad \text{with} \quad \eta(x, 0) = x. \]
Then similarly to [6] Lemma 4.7, we have the following positivity of the density.

Lemma 4.3. Let $T > 0$ be given. There exists a constant $C > 0$ independent of $T$ such that
\[ \rho(x, t) \geq \rho_0(\eta(x, 0)) \exp \left( -C \left( X_0(s) + \|v_0\|_{L^1}^2 \right)^{3/8} \right) > 0 \]
for all $t \in [0, T]$.

Proof. Since the case $d \geq 3$ is already shown in [9], we only consider the case $d = 2$. It follows from the continuity equation in (1.1) that
\[ \partial_\tau \rho(\eta(x, s), s) = - (\nabla \cdot u)(\eta(x, s), s) \rho(\eta(x, s), s). \]
This gives
\[ \rho(x, t) = \rho_0(\eta(x, 0)) \exp \left( - \int_0^t (\nabla \cdot u)(\eta(x, \tau), \tau) \, d\tau \right). \]
Analogously to that of Lemma 4.2, we find
\[ \left| \int_0^t (\nabla \cdot u)(\eta(x, \tau), \tau) \, d\tau \right| \leq \int_0^t \|\nabla u(\cdot, \tau)\|_{L^\infty} \, d\tau \leq C \left( X_0(s) + \|v_0\|_{L^1}^2 \right)^{1/2}. \]
This completes the proof. \qed

4.1. Proof of Theorem 1.1: global-in-time existence and uniqueness. Before we proceed to the estimates for the large time behavior with the desired time decay rate, we first prove the existence theory in Theorem 1.1. Based on the previous estimates, we prove the following uniform-in-time estimates on the solutions.

Proposition 4.2. For $T > 0$, we can find a sufficiently small $0 < \varepsilon_1 \ll 1$ such that if
\[ X(s; T) + \|v_0\|_{L^1}^2 < \varepsilon_1^2, \]
then there exists a constant $C^* > 0$ independent of $T$ satisfying
\[ X(s; T) \leq C^* \left( X_0(s) + \|v_0\|_{L^1}^2 \right). \]

Proof. Note that Lemma 4.2 directly implies
\[ \sup_{0 \leq t \leq T} \|\phi(\cdot, t)\|_{H^{m-1}}^2 \leq C \|\phi_0\|_{H^{m-1}}^2 \leq C \left( X_0(s) + \|v_0\|_{L^1}^2 \right). \] (4.4)
Next, Proposition 2.1 together with Lemma 4.1 gives
\[ \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^{m+2}} \leq \sup_{0 \leq t \leq T} \left( \|u(\cdot, t)\|_{L^2}^2 + \|\nabla u(\cdot, t)\|_{H^{m+1}}^2 \right) \]
\[ \leq C \left( X_0(s) + \|v_0\|_{L^1}^2 \right). \] (4.5)
Finally, Proposition 2.1 combined with Proposition 4.1 yields
\[ \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^{m+1}}^2 \leq C \left( X_0(s) + \|v_0\|_{L^1}^2 \right). \] (4.6)
Thus, we collect (4.4), (4.5), and (4.6) to conclude the desired result. \qed
We choose a positive constant \( \varepsilon_1 \ll 1 \) sufficiently small so that it satisfies the required smallness condition given in Lemmas 3.2, 3.3, Proposition 4.1, and Lemma 4.1. Then, assume that
\[
\mathcal{X}_0(s) + \|v_0\|_L^2 \leq \frac{\varepsilon_1^2}{2(1 + C^*)},
\]
where \( C^* > 0 \) appeared in Proposition 4.2. Then we set
\[
\mathcal{S} := \{T \geq 0 \mid \mathcal{X}(s; T) < \varepsilon_1^2\}.
\]
By the local-in-time well-posedness theory in Theorem 2.1, the set \( \mathcal{S} \) is non-empty. Now, we argue by contradiction to show \( \sup \mathcal{S} = \infty \). Assume that \( T^* := \sup \mathcal{S} < \infty \). Then we have
\[
\varepsilon_1^2 = \lim_{t \to T^*} \mathcal{X}(s; t) \leq C^* (\mathcal{X}_0(s) + \|v_0\|_L^2) \leq \frac{C^*}{2(1 + C^*)} \varepsilon_1^2 \leq \frac{\varepsilon_1^2}{2},
\]
which leads to a contradiction. This implies \( T^* = \infty \), and hence the unique classical solution obtained in Theorem 2.1 globally exists.

### 4.2. **Proof of Theorem 1.1 large time behavior.**
In this part, we refine the argument used in the proof of Proposition 4.1 to have better decay estimates for the higher-order derivatives of solutions \( u \) and \( v \) which appeared in Theorem 1.1. Thanks to the previous results, we can assume that for a sufficiently small \( \varepsilon_1 > 0 \),
\[
\mathcal{X}(s; \infty) := \sup_{t \geq 0} \left( \|\phi(\cdot, t)\|_H^{2s+1} + \|u(\cdot, t)\|_H^{2s+2} + \|v(\cdot, t)\|_H^{2s+3} \right) < \varepsilon_1 \ll 1
\]
and
\[
(1 + t)^{\alpha} \|\nabla u(\cdot, t)\|_H^{2s+1} + \|\nabla v(\cdot, t)\|_H^{2s+1} + \int_0^t (1 + \tau)^{\alpha} \left( \|\nabla u(\cdot, \tau)\|_H^{2s} + \|\nabla^2 v(\cdot, \tau)\|_H^{2s} \right) d\tau \leq C(\mathcal{X}_0(s) + \|v_0\|_L^2)
\]
for any \( t > 0 \) and \( \alpha \in (0, d/2 + 1) \), where \( C > 0 \) depends only on \( \alpha \), \( d \), and \( s \). Thus, we only need to show that for every \( k = 1, \ldots, d-1 \) and \( \alpha \in (0, d/2 + k) \), there exists a constant \( C > 0 \) independent of \( t \) such that
\[
\|\nabla^k u(\cdot, t)\|_H^{2s+1} + \|\nabla^k v(\cdot, t)\|_H^{2s+1} \leq \frac{C}{(1 + t)^{\alpha}} \quad \forall t > 0.
\]

**Lemma 4.4.** For any \( 1 \leq k \leq \min\{d-1, s\} \) and \( \alpha \in (0, d/2 + k) \), there exists a constant \( C > 0 \) independent of \( t \) such that
\[
(1 + t)^{\alpha} \left( \|\nabla^k u(\cdot, t)\|_L^2 + \|\nabla^k v(\cdot, t)\|_L^2 \right) + \int_0^t (1 + \tau)^{\alpha} \left( \|\nabla^k (u - v)(\cdot, \tau)\|_L^2 + \|\nabla^{k+1} v(\cdot, \tau)\|_L^2 \right) d\tau \leq C \quad \forall t > 0.
\]
Moreover, we have
\[
\|\nabla^k u(\cdot, t)\|_L^2 \leq C(1 + t)^{-\alpha} \quad \forall t > 0.
\]

**Proof.** We proceed by induction on \( k \). Since we already showed the case \( k = 1 \), it suffices to show the induction step. For this, we assume that for any \( 1 \leq \ell < k \) and \( \alpha \in (0, d/2 + \ell) \), the following holds:
\[
(1 + t)^{\alpha} \left( \|\nabla^\ell u(\cdot, t)\|_L^2 + \|\nabla^\ell v(\cdot, t)\|_L^2 \right) + \int_0^t (1 + \tau)^{\alpha} \left( \|\nabla^\ell (u - v)(\cdot, \tau)\|_L^2 + \|\nabla^{\ell+1} v(\cdot, \tau)\|_L^2 \right) d\tau \leq C \quad \forall t > 0.
\]
for some \( C > 0 \) independent of \( t \). Here, for given \( \alpha \in (0, d/2 + k) \), we choose a small \( \delta \in (0, 1) \) so that \( \gamma := \alpha - 1 + \delta < d/2 + k - 1 \). We use the above induction hypothesis to have that there exists \( C > 0 \), independent of \( t \), such that
\[
(1 + t)^{\alpha} \left( \|\nabla^{k-1} u(\cdot, t)\|_L^2 + \|\nabla^{k-1} v(\cdot, t)\|_L^2 \right) + \int_0^t (1 + \tau)^{\alpha} \left( \|\nabla^{k-1} (u - v)(\cdot, \tau)\|_L^2 + \|\nabla^{k} v(\cdot, \tau)\|_L^2 \right) d\tau \leq C \quad \forall t > 0.
\]
We set
\[
E_k(t) := \|\nabla^{k} u(\cdot, t)\|_L^2 + \|\nabla^{k} v(\cdot, t)\|_L^2
\]
and
\[ D_k(t) := \|\nabla^{k+1} v(\cdot, t)\|_{L^2}^2 + \int_{\mathbb{R}^d} \rho(\cdot, t)|\nabla^k (u - v)(\cdot, t)|^2 \, dx. \]

Then we gather the results in Lemmas 3.3 and 3.4 to get
\[ \frac{d}{dt} E_k(t) + \frac{3}{2} D_k(t) \leq C \|\nabla v(\cdot, t)\|_{L^\infty} \|\nabla^k v(\cdot, t)\|_{L^2}^2. \]

From the previous results on \(\nabla v\), since \(d \geq 3\), we can find a constant \(C_k > 0\) and \(\eta > 0\) such that
\[ \frac{d}{dt} E_k(t) + \frac{3}{2} D_k(t) \leq C_k (4d + 8k + t)^{-1-\eta} \|\nabla^k v(\cdot, t)\|_{L^2}^2. \quad (4.11) \]

Now, we choose \(c_0\) so that
\[ \frac{\|\rho\|_{L^\infty}}{1 + c_0} \leq \frac{1}{2}, \]

and then define
\[ g^2(t) := \frac{4(\alpha + C_k)(1 + c_0)}{4d + 8k + t} \quad \text{and} \quad \tilde{g}^2(t) := \frac{g^2(t)}{4(1 + c_0)} = \frac{\alpha + C_k}{4d + 8k + t}. \]

Note that our setting above implies
\[ \frac{\alpha}{\alpha + C_k} \frac{g^2(t)}{2} \leq \frac{1 + c_0}{2}. \]

On the other hand, by using
\[ \int_{\mathbb{R}^d} \rho |\nabla^k (u - v)|^2 \, dx \geq \int_{\mathbb{R}^d} \rho |\nabla^k u|^2 \, dx - \|\rho\|_{L^\infty} \int_{\mathbb{R}^d} |\nabla^k v|^2 \, dx, \]

we estimate
\[ \|\nabla^{k+1} v(\cdot, t)\|_{L^2}^2 = \int_{\mathbb{R}^d} |\xi|^2 |\widehat{\nabla^k v}(\xi, t)|^2 \, d\xi \]
\[ \geq \int_{\{||\xi|| \geq g(t)\}} |\xi|^2 |\widehat{\nabla^k v}(\xi, t)|^2 \, d\xi \]
\[ \geq g^2(t) \|\nabla^k v(\cdot, t)\|_{L^2}^2 - \tilde{g}^4(t) \int_{\{||\xi|| \leq g(t)\}} |\widehat{\nabla^{k+1} v}(\xi, t)|^2 \, d\xi \]
\[ \geq g^2(t) \|\nabla^k v(\cdot, t)\|_{L^2}^2 - \tilde{g}^4(t) \|\nabla^{k-1} v(\cdot, t)\|_{L^2}^2. \quad (4.12) \]

This yields
\[ D_k(t) \geq \frac{1}{2} D_k(t) + \frac{\alpha}{\alpha + C_k} \frac{g^2(t)}{2(1 + c_0)} \int_{\mathbb{R}^d} \rho(\cdot, t)|\nabla^k (u - v)(\cdot, t)|^2 \, dx + \frac{1}{2} \|\nabla^{k+1} v(\cdot, t)\|_{L^2}^2 \]
\[ \geq \frac{1}{2} D_k(t) + \frac{\alpha}{\alpha + C_k} \frac{g^2(t)}{2(1 + c_0)} \|\sqrt{\rho} \nabla u(\cdot, t)\|_{L^2}^2 + \frac{\alpha}{\alpha + C_k} \tilde{g}^2(t) \left(1 - \frac{\|\rho\|_{L^\infty}}{1 + c_0} \frac{\alpha}{\alpha + C_k}\right) \|\nabla^k v(\cdot, t)\|_{L^2}^2 \]
\[ - \frac{g^4(t)}{2} \|\nabla^{k-1} v(\cdot, t)\|_{L^2}^2 \]
\[ \geq \frac{1}{2} D_k(t) + \frac{g^2(t)}{4} \|\nabla^k v(\cdot, t)\|_{L^2}^2 + \frac{\alpha}{\alpha + C_k} \tilde{g}^2(t) \|\sqrt{\rho} \nabla^k u(\cdot, t)\|_{L^2}^2 \]
\[ \geq \frac{1}{2} D_k(t) + \frac{\alpha + C_k}{4d + 8k + t} \|\nabla^k v(\cdot, t)\|_{L^2}^2 + \frac{\alpha}{4d + 8k + t} \|\sqrt{\rho} \nabla^k u(\cdot, t)\|_{L^2}^2 \]
\[ \geq \frac{1}{2} D_k(t) + \frac{\alpha + C_k}{4d + 8k + t} \|\nabla^k v(\cdot, t)\|_{L^2}^2 + \frac{\alpha}{4d + 8k + t} \|\sqrt{\rho} \nabla^k u(\cdot, t)\|_{L^2}^2 \]
\[ - \frac{g^4(t)}{2} \|\nabla^{k-1} v(\cdot, t)\|_{L^2}^2. \]

From (4.11), we can see that \(\|\nabla^{k-1} v(\cdot, t)\|_{L^2}^2 \leq C(1 + t)^{-\alpha - \delta + 1}\) for some constant \(C > 0\). Together with this, we apply (4.12) to (4.11) and then obtain
\[ \frac{d}{dt} E_k(t) + \alpha (4d + 8k + t)^{-1} E_k(t) + D_k(t) \leq C(4d + 8k + t)^{-\alpha - \delta - 1}. \]

This implies
\[ \frac{d}{dt} \left[(4d + 8k + t)^\alpha E_k(t) + (4d + 8k + t)^\alpha D_k(t) \right] \leq C(4d + 8k + t)^{-\alpha - \delta - 1}, \]
and integrate with respect to $t$ to complete the induction. Hence we have the first assertion (4.8).

For the second one, we first notice from (4.7) that the inequality (4.9) holds for $k = 1$. For $k \geq 2$, by Lemma 2.1 we find

$$\frac{1}{2} \frac{d}{dt} \| \partial^k u \|_{L^2}^2 = - \int_{\mathbb{R}^d} (u \cdot \nabla (\partial^k u)) \partial^k u \, dx - \int_{\mathbb{R}^d} (\partial^k (u \cdot \nabla u) - u \cdot \nabla (\partial^k u)) \partial^k u \, dx$$

$$- \| \partial^k u \|_{L^2}^2 + \int_{\mathbb{R}^d} \partial^k u \cdot \Delta v \, dx$$

$$\leq C \| \nabla u \|_{L^2} \| \partial^k u \|_{L^2}^2 - \frac{3}{4} \| \partial^k u \|_{L^2}^2 + 2 \| \partial^k v \|_{L^2}^2$$

$$\leq C \varepsilon_1 \| \partial^k u \|_{L^2}^2 - \| \partial^k u \|_{L^2}^2 + 2 \| \partial^k v \|_{L^2}^2,$$

which deduces

$$\frac{d}{dt} \| \partial^k u \|_{L^2}^2 + \| \partial^k u \|_{L^2}^2 \leq 2 \| \partial^k v \|_{L^2}^2.$$

Then, by using a similar argument as in the proof of Lemma 4.1, we get

$$\| \partial^k u \|_{L^2}^2 \leq C (1 + t)^{-\alpha}.$$

For $\dot{H}^1$-estimate, we use $\frac{\alpha}{\alpha + 1} \leq 1$ to have

$$\frac{d}{dt} (\alpha + t)^\alpha \| \partial^k+1 u(\cdot, t) \|_{L^2}^2 \leq 2 (\alpha + t)^\alpha \| \partial^k+1 v(\cdot, t) \|_{L^2}^2.$$

Then we integrate the above relation and use (4.8) to conclude the desired result. \hfill \Box

To complete the proof for Theorem 1.1, it suffices to show that if $d - 1 < s$, there exists a constant $C > 0$ independent of $t$ such that for $d - 1 \leq \ell \leq s + 1$ and $\alpha \in (0, d/2 + d - 1)$,

$$(1 + t)\alpha \left( \| \sqrt{\rho} \nabla^\ell u(\cdot, t) \|_{L^2}^2 + \| \nabla^\ell v(\cdot, t) \|_{L^2}^2 \right)$$

$$+ \int_0^t (1 + \tau)^\alpha \left( \| \sqrt{\rho} \nabla^\ell (u - v)(\cdot, \tau) \|_{L^2}^2 + \| \sqrt{\rho} \nabla^{\ell+1} v(\cdot, \tau) \|_{L^2}^2 \right) \, d\tau \leq C \quad \forall t > 0. \tag{4.13}$$

For the proof, we again proceed by induction. Note that we already proved the case $\ell = d - 1$ in Lemma 3.4.

For the induction step, assume that (4.13) holds for all $d - 1 \leq \ell < m$. We combine Lemma 3.3 with Lemma 3.4 to yield

$$\frac{d}{dt} \left( \| \sqrt{\rho} \nabla^m u \|_{L^2}^2 + \| \nabla^m v \|_{L^2}^2 \right) + \left( \| \sqrt{\rho} \nabla^m (u - v) \|_{L^2}^2 + \| \nabla^{m+1} v \|_{L^2}^2 \right)$$

$$\leq C \varepsilon_1 \| \sqrt{\rho} \nabla^m u \|_{L^2}^2 + C \varepsilon_1 \sum_{\ell = d - 1}^{m-1} \left( \| \sqrt{\rho} \nabla^\ell (u - v) \|_{L^2}^2 + \| \nabla^{\ell+1} v \|_{L^2}^2 \right).$$

Here, we use

$$\| \sqrt{\rho} \nabla^m (u - v) \|_{L^2}^2 \geq \| \sqrt{\rho} \nabla^m u \|_{L^2}^2 - \| \rho \nabla^m v \|_{L^2}^2$$

to deduce

$$\frac{d}{dt} \left( \| \sqrt{\rho} \nabla^m u \|_{L^2}^2 + \| \nabla^m v \|_{L^2}^2 \right) + \frac{1}{4} \| \sqrt{\rho} \nabla^m u \|_{L^2}^2 + \frac{1}{2} \left( \| \sqrt{\rho} \nabla^m (u - v) \|_{L^2}^2 + \| \nabla^{m+1} v \|_{L^2}^2 \right)$$

$$\leq C \sum_{\ell = d - 1}^{m-1} \left( \| \sqrt{\rho} \nabla^\ell (u - v) \|_{L^2}^2 + \| \nabla^{\ell+1} v \|_{L^2}^2 \right).$$

We also use

$$\frac{\alpha}{2d + 4(d - 1) + t} = \frac{\alpha}{6d - 4 + t} \leq \frac{1}{4} \quad \forall t > 0.$$
to obtain
\[
\frac{d}{dt} \left( \| \sqrt{\rho} \nabla^m u \|^2_{L^2} + \| \nabla^m v \|^2_{L^2} \right) + \alpha (6d - 4 + t)^{-1} \left( \| \sqrt{\rho} \nabla^m u \|^2_{L^2} + \| \nabla^m v \|^2_{L^2} \right) \\
+ \frac{1}{2} \left( \| \sqrt{\rho} \nabla^m (u - v) \|^2_{L^2} + \| \nabla^{m+1} v \|^2_{L^2} \right) \\
\leq C \sum_{\ell = d-1}^{m-1} \left( \| \sqrt{\rho} \nabla^\ell (u - v) \|^2_{L^2} + \| \nabla^{\ell+1} v \|^2_{L^2} \right).
\]

This gives
\[
\frac{d}{dt} \left[ (6d - 4 + t)^\alpha \left( \| \sqrt{\rho} \nabla^m u \|^2_{L^2} + \| \nabla^m v \|^2_{L^2} \right) \right] + \frac{(6d - 4 + t)^\alpha}{2} \| \sqrt{\rho} \nabla^m (u - v) \|^2_{L^2} + \| \nabla^{m+1} v \|^2_{L^2} \\
\leq C(6d - 4 + t)^\alpha \sum_{\ell = d-1}^{m-1} \left( \| \sqrt{\rho} \nabla^\ell (u - v) \|^2_{L^2} + \| \nabla^{\ell+1} v \|^2_{L^2} \right).
\]

Finally, we integrate the above with respect to \( t \) and use the induction hypothesis \((4.13)\) to complete the inductive argument. For the decay rate of \( u \), we can use the proof of Lemma \((4.3)\) together with the result \((4.13)\) to get the desired result.

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