Density-based group testing

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Abstract. In this paper we study a new, generalized version of the well-known group testing problem. In the classical model of group testing we are given $n$ objects, some of which are considered to be defective. We can test certain subsets of the objects whether they contain at least one defective element. The goal is usually to find all defectives using as few tests as possible. In our model the presence of defective elements in a test set $Q$ can be recognized if and only if their number is large enough compared to the size of $Q$. More precisely for a test $Q$ the answer is yes if and only if there are at least $\alpha |Q|$ defective elements in $Q$ for some fixed $\alpha$.

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1 Introduction

The concept of group testing was developed in the middle of the previous century. Dorfman, a Swiss physician intended to test blood samples of millions of soldiers during World War II in order to find those who were infected by syphilis. His key idea was to test more blood samples at the same time and learn whether at least one of them are infected [3]. Some fifteen years later Rényi developed a theory of search in order to find which electrical part of his car went wrong. In his model – contrary to Dorfman’s one – not all of the subsets of the possible defectives (electric parts) could be tested [5].

Group testing has now a wide variety of applications in areas like DNA screening, mobile networks, software and hardware testing.

In the classical model we have an underlying set $[n] = \{1, \ldots, n\}$ and we suppose that there may be some defective elements in this set. We can test all

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subsets of \([n]\) whether they contain at least one defective element. The goal is to
find all defectives using as few tests as possible. One can easily see that in this
generality the best solution is to test every set of size 1. Usually we have some
additional information like the exact number of defectives (or some bounds on
this number) and it is also frequent that we do not have to find all defectives
just some of them or even just to tell something about them.

In the case when we have to find a single defective it is well-known that the
information theoretic lower bound is sharp: the number of questions needed in
the worst case is \([\log n]\), which can be achieved by binary search.

Another well-known version of the problem is when the maximum size of a
test is bounded. (Motivated by the idea that too large tests are not supposed to
be reliable, because a small number of defectives may not be recognized there).
This version can be solved easily in the adaptive case, but is much more difficult
in the non-adaptive case. This latter version was first posed by Rényi, Katona
[5] gave an algorithm to find the exact solution to Rényi’s problem and he also
proved the best known lower bound on the number of queries needed. The best
known upper bound is due to Wegener [7].

In this paper we assume that the presence of defective elements in a test set
\(Q\) can be recognized if and only if their number is large enough compared to the
size of \(Q\). More precisely for a test \(Q \subseteq [n]\) the answer is \textit{yes} if and only if there
are at least \(\alpha |Q|\) defective elements in \(Q\). Our goal is to find at least \(m\) defective
elements using tests of this kind.

**Definition 1.** Let \(g(n, k, \alpha, m)\) be the least number of questions needed in this
setting, i.e. to find \(m\) defective elements in an underlying set of size \(n\) which
contains at least \(k\) defective elements, where the answer is \textit{yes} for a question
\(Q \subseteq [n]\) if and only if there are at least \(\alpha |Q|\) defective elements in \(Q\).

We suppose throughout the whole paper that \(1 \leq m \leq k\) and \(0 < \alpha < 1\). Let
\(a = [\frac{k}{\alpha}]\), that is, \(a\) is the largest size of a set where the answer \textit{no} has the usual
meaning, namely that there are no defective elements in the set. It is obvious
that if a set of size greater than \(k/\alpha\) is asked then the answer is automatically
\textit{no}, so we will suppose that question sets has size at most \(k/\alpha\). All logarithms
appearing in the paper are binary.

It is worth mentioning that a similar idea appears in a paper by Damaschke
[1] and a follow-up paper by De Bonis, Gargano, and Vaccaro [2]. Since their
motivation is to study the concentration of liquids, their model deals with many
specific properties arising in this special case and they are interested in the
number of merging operations or the number of tubes needed in addition to the
number of tests.

If \(k = m = 1\), then the problem is basically the same as the usual setting
with the additional property that the question sets can have size at most \(\alpha\) this
is the above mentioned problem of Rényi. As we have mentioned, finding the
optimal non-adaptive algorithm, or even just good bounds is really hard even in
this simplest case of our model, thus in this paper we deal only with adaptive
algorithms.
In the next section we give some upper and lower bounds as well as some conjectures depending on the choices of \( n, k, \alpha, \) and \( m \). In the third section we prove our main theorem, which gives a general lower and a general upper bound, differing only by a constant depending only on \( k \). In the fourth section we consider some related questions and open problems.

## 2 Upper and lower bounds

First of all it is worth examining how binary search, the most basic algorithm of search theory works in our setting. It is easy to see that it does not work in general, not even for \( m = 1 \). If (say) \( k = 2 \) and \( \alpha = 0.1 \), then question sets have at most 20 elements (recall that we supposed that there are no queries containing more than \( k/\alpha \) elements, since they give no information at all, because the answer for them is always \textit{no}), thus if \( n \) is big, we cannot perform a binary search.

However, if \( k \geq n\alpha \), then binary search can be used.

**Theorem 1.** If \( \alpha \leq k/n \), then \( g(n, k, \alpha, m) \leq \lceil \log n \rceil + c \), where \( c \) depends only on \( \alpha \) and \( m \), moreover if \( m = 1 \), then \( c = 0 \).

**Proof.** We show that binary search can be used to find \( m \) defectives. That is, first we ask a set \( F \) of size \( \lfloor n/2 \rfloor \) and then the underlying set is substituted by \( F \) if the answer is \textit{yes} and by \( \overline{F} \) if the answer is \textit{no}. We iterate this process until the size of the underlying set is at most \( 2m/\alpha \). Now we check that the condition \( \alpha \leq k/n \) remains true after each step. Let \( n' = \lfloor n/2 \rfloor \) be the size of the new underlying set and \( k' \) be the number of defectives there. If the answer was \textit{yes}, then \( k' \geq \alpha n' \), thus \( \alpha \leq k'/n' \). If the answer was \textit{no}, then there are at least \( k - \lfloor \alpha n' \rfloor + 1 \) defectives in the new underlying set, that is \( k' \geq k - \lfloor \alpha n' \rfloor + 1 \geq \alpha n - \lfloor \alpha n' \rfloor + 1 \geq \alpha n' \), thus \( \alpha \leq k'/n' \) again.

Now if \( m = 1 \) we simply continue the binary search until we find a defective element, altogether using at most \( \lceil \log n \rceil \) questions.

If \( m > 1 \), then we can find \( m \) defectives in the last underlying set using at most \( c := \max_{n' \leq 2m/\alpha} g(n', m, \alpha, m) \) further queries.

(Notice that since the size of the last underlying set is greater than \( m/\alpha \), it contains at least \( m \) defectives.) This number \( c \) does not depend on \( k \), just on \( \alpha \) and \( m \) and it is obvious that we used at most \( \lceil \log n \rceil + c \) queries altogether. \( \square \)

This theorem has an easy, yet very important corollary. If the answer for a question \( A \) is \textit{yes}, then there are at least \( \alpha|A| \) defective elements in \( A \). If \( \alpha|A| \geq m \), then we can find \( m \) of these defectives using \( g(|A|, \alpha|A|, \alpha, m) \leq \log |A| + c \) questions, where \( c \) depends only on \( \alpha \) and \( m \). Basically it means that whenever we obtain a \textit{yes} answer, we can finish the algorithm quickly.

The proof of Theorem 1 is based on the fact that if the ratio of the defective elements \( k/n \) is at least \( \alpha \), then this condition always remains true during binary search. If \( k/n < \alpha \), then this trick does not work, however if the difference between \( k/n \) and \( \alpha \) is small, a similar result can be proved for \( m = 1 \). Recall that \( \alpha = \lfloor 1/\alpha \rfloor \).
**Theorem 2.** If \( k \geq \frac{a}{e} - \lfloor \log \frac{a}{e} \rfloor - 1 \) and \( k \geq 1 \), then \( g(n, k, \alpha, 1) \leq \lfloor \log n \rfloor + 1 \).

The proof of the theorem is based on the following lemmas.

**Lemma 1.** Let \( t \geq 0 \) be an integer. Then \( g(2^t a, 2^t - t, \alpha, 1) \leq t + \lfloor \log a \rfloor \).

**Proof.** We use induction on \( t \). For \( t = 0 \) and \( t = 1 \) the proposition is true, since we can perform a binary search on \( a \) or \( 2a \) elements (by asking sets of size \( a \) we learn whether they contain a defective element). Suppose now that the proposition holds for \( t \), we have to prove it for \( t + 1 \). That is, we have an underlying set of size \( 2^{t+1} a \) containing at least \( 2^{t+1} - t - 1 \) defectives. Our first query is a set \( A \) of size \( 2^t a \). If the answer is yes, then we can continue with binary search. If the answer is no, then there are less than \( 2^t a \leq 2^t \) defectives in \( A \), therefore there are at least \( 2^{t+1} - t - 1 - 2^t + 1 = 2^t - t \) defectives in \( A \). By the induction hypothesis \( g(2^t a, 2^t - t, \alpha, 1) \leq t + \lfloor \log a \rfloor \), thus \( g(2^{t+1} a, 2^{t+1} - t - 1, \alpha, 1) \leq t + 1 + \lfloor \log a \rfloor \) follows, finishing the proof of the lemma. \( \square \)

**Lemma 2.** Let \( t \geq 2 \) be an integer. Then \( g(2^t a, 2^t - t - 1, \alpha, 1) \leq t + \lfloor \log a \rfloor + 1 \).

**Proof.** Let us start with asking three disjoint sets, each of cardinality \( 2^{t-2} a \). If the answer to any of these is yes, then we can continue with binary search, using \( t - 2 + \lfloor \log a \rfloor \) additional questions. If all three answers are no, then there are at least \( 2^t - t - 1 - 3(2^{t-2} - 1) = 2^{t-2} - (t - 2) \) defectives among the remaining \( 2^{t-2} a \) elements, hence we can apply Lemma 1. \( \square \)

**Proof (of Theorem 2).** Let us suppose \( n > 2a \) (otherwise binary search works) and let \( t = \lfloor \log \frac{n}{a} \rfloor, r = n - 2^t a \). We have an underlying set of size \( n = 2^t a + r \) containing at least \( \frac{n}{a} - \lfloor \log \frac{n}{a} \rfloor - 1 \) defectives. If \( r = 0 \), then by Lemma 2 we are done. Otherwise let the first query \( A \) contain \( r \) elements. A positive answer allows us to find a defective element by binary search on \( A \) using altogether at most \( \lfloor \log n \rfloor + 1 \) questions (actually, at most \( \lfloor \log n \rfloor \) questions, because \( r \leq n/2 \)). If the answer is negative then the new underlying set contains \( 2^t a \) elements, of which more than \( \frac{n}{a} - \lfloor \log \frac{n}{a} \rfloor - a r - 1 = 2^t r/a - a r - \lfloor \log \frac{n}{a} \rfloor - 1 \) are defective. Since \( \lfloor \log \frac{n}{a} \rfloor = t \), the number of defectives is at least \( 2^t - t \), thus by Lemma 1 we need at most \( t + \lfloor \log a \rfloor \) more queries to find a defective element, thus altogether we used at most \( t + 1 + \lfloor \log a \rfloor \leq \lfloor \log n \rfloor + 1 \) queries, from which the theorem follows. \( \square \)

One might think that binary search is the best algorithm to find one defective if it can be used (i.e. for \( k \geq na \)). A counterexample for \( k \) really big is easy to give: if \( k = n \) then we do not need any queries and for \( m = 1, k = n - 1 \) we need just one query. It is somewhat more surprising that \( g(n, an, \alpha, 1) \geq \lceil \log n \rceil \) is not necessarily true.

For example, the case \( n = 10, k = 4, \alpha = 0.4, m = 1 \) can be solved using 3 queries: first we ask a set \( A \) of size 4. If the answer is yes, we can perform a binary search on \( A \), if the answer is no then there are at least 3 defectives among
the remaining 6 elements and now we ask a set \( B \) of size 2. If the answer is yes
then we perform a binary search on \( B \), otherwise there are at least 3 defectives
among the remaining 4 elements, so one query (of size 1) is sufficient to find a
defective. However, a somewhat weaker lower bound can be proved:

**Theorem 3.** \( g(n, k, \alpha, m) \geq \lceil \log(n - k + 1) \rceil \).

We prove the stronger statement that even if one can use any kind of yes-no
questions, still at least \( \lceil \log(n - k + 1) \rceil \) questions are needed. This is a slight
generalization of the information theoretic lower bound.

**Theorem 4.** To find one of \( k \) defective elements from a set of size \( n \), one needs
\( \lceil \log(n - k + 1) \rceil \) yes-no questions in the worst case and this is sharp.

**Proof.** Suppose there is an algorithm that uses at most \( q \) questions. The number
of sequences of answers obtained is at most \( 2^q \), thus the number of different
elements selected by the algorithm as the output is also at most \( 2^q \). This means
that \( n - 2^q \leq k - 1 \), otherwise it would be possible that all \( k \) defective elements
are among those ones that were not selected. Thus \( q \geq \lceil \log(n - k + 1) \rceil \) indeed.

Sharpness follows easily from the simple algorithm that puts \( k - 1 \) elements
aside and runs a binary search on the rest. \( \square \)

Theorem 3 is an immediate consequence of Theorem 4, but this is not true
for the sharpness of the result. However, Theorem 3 is also sharp: if \( \alpha \leq \frac{2}{n-k+1} \),
then we can run a binary search on any \( n - k + 1 \) of the elements to find a
defective.

We have seen in Theorem 1 that if \( n \leq k/\alpha \), then binary search works (with
some additional constant number of questions if \( m > 1 \)). On the other hand, if
\( n \) goes to infinity (with \( k \) and \( \alpha \) fixed), then the best algorithm is linear.

**Theorem 5.** For any \( k, \alpha, m \)

\[
\frac{n}{\alpha} + c_1 \leq g(n, k, \alpha, m) \leq \frac{n}{\alpha} + c_2,
\]

where \( c_1 \) and \( c_2 \) depend only on \( k, \alpha, \) and \( m \).

**Proof.** Upper bound: first we partition the underlying set into \( \lfloor \frac{n}{\alpha} \rfloor \) \( \alpha \)-element
sets and possibly one additional set of less than \( \alpha \) elements. We ask each of these
sets (at most \( \lfloor \frac{n}{\alpha} \rfloor + 1 \) questions). Then we choose \( m \) sets for which we obtained
a yes answer (or if there are less than \( m \) such sets, then we choose all of them).
We ask every element one by one in these sets (at most \( ma \) questions). One can
easily see that we find at least \( m \) defective elements, using at most \( \lfloor \frac{n}{\alpha} \rfloor + ma + 1 \)
questions.

Lower bound: We use a simple adversary’s strategy: suppose all the answers
are no and there are \( m \) elements identified as defectives. Let us denote the family
of sets that were asked by \( F \). It is obvious that those sets of \( F \) that have size
at most \( \alpha \) contain no defective elements. Suppose there are \( i \) such sets. We use
induction on \( i \). There are \( n' \geq n - ia \) elements not contained in these sets and
we should prove that at least $\frac{n}{a} + c_1 - i \leq \frac{n}{a'} + c_1$ other questions are needed. Hence by the induction it is enough to prove the case $i = 0$.

Suppose $i = 0$. If there is a set $A$ of size $k + 1$, such that $|A \cap F| \leq 1$ for all $F \in \mathcal{F}$, then any $k$-element subset of $|A|$ can be the set of the defective elements. In this case any element can be non-defective, a contradiction. Thus for every set $A$ of size $k + 1$ there exists a set $F \in \mathcal{F}$, such that $|A \cap F| \geq 2$.

Let $b = [\frac{n}{a}]$. We know that every set of $\mathcal{F}$ has size at most $b$. Then a given $F \in \mathcal{F}$ intersects at most $\sum_{j=2}^{k+1} \binom{n}{j} \binom{n-b}{k+1-j}$ $(k+1)$-element sets in at least two points. This number is $O(n^{k-1})$, and there are $\Omega(n^{k+1})$ sets of size $k + 1$, hence $|\mathcal{F}| = \Omega(n^2)$ is needed.

It follows easily that there is an $n_0$, such that if $n > n_0$, then $|\mathcal{F}| \geq \frac{n}{a}$. Now let $c_1 = -n_0/a$. If $n > n_0$ then $|\mathcal{F}| \geq \frac{n}{a} \geq \frac{n}{a} + c_1$, while if $n \leq n_0$ then $|\mathcal{F}| \geq 0 \geq \frac{n}{a} + c_1$, thus the number of queries is at least $\frac{n}{a} + c_1$, finishing the proof.

Remark. The theorem easily follows from Theorem 7, it is included here because of the much simpler proof.

It is easy to give a better upper bound for $m = 1$.

**Theorem 6.** Suppose $k + \log k + 1 \leq [\frac{n}{a}]$. Then

$$g(n, k, \alpha, 1) \leq [\frac{n}{a}] - k + \lceil \log a \rceil.$$  

**Proof.** First we ask a set $X$ of size $ka$. If the answer is YES, then we can find a defective element in $\lceil \log ka \rceil$ steps by Theorem 1. In this case the number of questions used is at most $1 + \lceil \log ka \rceil = 1 + \lceil \log k + \log a \rceil \leq 1 + \lceil \log k \rceil + \lceil \log a \rceil \leq \left[ \frac{n}{a} \right] - k + \log a$, where the last inequality follows from the condition of the theorem.

If the answer is NO, then we know that there are at most $k - 1$ defectives in $X$, so we have at least one defective in $X$. Continue the algorithm by asking disjoint subsets of $X$ of size $a$, until the answer is YES or we have at most $2a$ elements not yet asked. In these cases using at most $\lceil \log 2a \rceil$ questions we can easily find a defective element, thus the total number of questions used is at most $1 + \lceil \frac{n-k-a-2a}{a} \rceil + \lceil \log 2a \rceil = 1 + \lceil \frac{n-a}{a} \rceil - k - 2 + \lceil \log a \rceil + 1 = \left[ \frac{n}{a} \right] - k + \lceil \log a \rceil$, finishing the proof. \hfill \Box

Note that if the condition of Theorem 6 does not hold (that is, $k + \log k + 1 \leq [\frac{n}{a}]$), then $k \geq \frac{n}{a} - \lceil \log a \rceil - 1$, hence $\lceil \log n \rceil + 1$ questions are enough by Theorem 2.

The exact values of $g(n, k, \alpha, m)$ is hard to find, even for $m = 1$. The algorithm used in the proof of Theorem 6 seems to be optimal for $m = 1$ if $k + \log k + 1 \leq [\frac{n}{a}]$. However, counterexamples with $1/a$ not an integer are easy to find (consider i.e. $n = 24$, $k = 2$, $\alpha = \frac{1}{11}$).

**Conjecture 1** If $\frac{n}{a}$ is an integer and $k + \log k + 1 \leq [\frac{n}{a}]$, then the algorithm used in the proof of Theorem 6 is optimal for $m = 1$.  


It is easy to see that Conjecture 1 is true for \( k = 1 \). For other values of \( k \) it would follow from the next, more general conjecture.

**Conjecture 2** If \( \frac{1}{\alpha} \) is an integer, then \( g(n, k, \alpha, 1) \leq g(n, k + 1, \alpha, 1) + 1 \).

Obviously, Conjecture 2 also fails if \( 1/\alpha \) is not an integer. One can see for example that \( g(24, 1, 2/11, 1) = 7 \) and \( g(24, 2, 2/11, 1) = 5 \).

### 3 The main theorem

In this section we prove a lower and an upper bound differing only by a constant depending only on \( k \). For the lower bound we need the following simple generalization of the information theoretic lower bound.

**Proposition 1.** Suppose we are given \( p \) sets \( A_1, \ldots, A_p \) of size at least \( n, \) each one containing at least one defective and an additional set \( A_0 \) of arbitrary size containing no defectives. Let \( m \leq p \). Then the number of questions needed to find at least \( m \) defectives is at least \( [m \log n] \).

**Proof.** Suppose that we are given the additional information that every set \( A_i \) \((i \geq 1)\) contains exactly one defective element. Now we use the information theoretic lower bound: there are \( \prod_{i=1}^{p} |A_i| \) possibilities for the distribution of the defective elements at the beginning, and at most \( \prod_{i=m}^{p} |A_i| \) at the end (suppose we have found defective elements in every set \( A_i \) except in \( A_{j_1}, \ldots, A_{j_{p-m}} \)), thus if we used \( t \) queries, then \( 2^t \geq n^m \), from which the proposition follows. \( \square \)

Now we formulate the main theorem of the paper.

**Theorem 7.** For any \( k, \alpha, m \)

\[
\frac{n}{\alpha} + m \log \alpha - c_1(k) \leq g(n, k, \alpha, m) \leq \frac{n}{\alpha} + m \log \alpha + c_2(k),
\]

where \( c_1(k) \) and \( c_2(k) \) depend only on \( k \).

**Proof.** First we give an algorithm that uses at most \( \frac{n}{\alpha} + m \log \alpha + c_2(k) \) queries, proving the upper bound. In the first part of the procedure we ask disjoint sets \( A_1, A_2, \ldots, A_r \) of size \( a \) until either there were \( m \) \textit{yes} answers or there are no more elements left. In this way we ask at most \( \lfloor \frac{n}{\alpha} \rfloor \) questions.

Suppose we obtained \textit{yes} answers for the sets \( A_1, A_2, \ldots, A_m \), and \textit{no} answers for the sets \( A_{m1+1}, \ldots, A_r \). If \( m1 \geq m \), then in the second part of the procedure we use binary search in the sets \( A_1, A_2, \ldots, A_m \) in order to find one defective element in each of them. For this we need \( m \lfloor \log a \rfloor \) more questions.

If \( m1 < m \), then first we use binary search in the sets \( A_1, A_2, \ldots, A_m \) in order to find defective elements \( a_1 \in A_1, a_2 \in A_2, \ldots, a_{m1} \in A_m \). Then we iterate the whole process using \( S_1 = \cup_{i=1}^{m1} A_i \setminus \{a_i\} \) as an underlying set, that is we ask disjoint sets \( B_1, B_2, \ldots, B_t \) of size \( a \) until either we obtain \( m - m1 \) \textit{yes} answers or there are no more elements left. Suppose we obtained \textit{yes} answers.
for the sets $B_1, B_2, \ldots, B_{m_2}$ and no answers for the sets $A_{m_2+1}, \ldots, A_l$. If $m_2 \geq m - m_1$, then in the second part of the procedure we use binary search in the sets $B_1, B_2, \ldots, B_{m_1}$ in order to find one defective element in each of them, while if $m_2 < m - m_1$, then first we use binary search in the sets $B_1, B_2, \ldots, B_{m_2}$ in order to find defective elements $b_1 \in B_1, b_2 \in B_2, \ldots, b_{m_2} \in A_{m_2}$ and continue the process using $S_2 = \bigcup_{i=1}^{m_2} B_i \setminus \{ b_i \}$ as an underlying set, and so on, until we find $m = m_1 + m_2 + \ldots + m_j$ defective elements. Note that $m_i \geq 1$, $\forall i \leq j$, since $k \geq m$. We have two types of queries: queries of size $a$ and queries of size less than $a$ (used in the binary searches). The number of questions of size $a$ is at most $\lceil \frac{a}{2} \rceil$ in the first part and at most $m_1 + m_2 + \ldots + m_j < m \leq k$ in the second part. The total number of queries of size less than $a$ is at most $m \log a$, thus the total number of queries is at most $\lceil \frac{a}{2} \rceil + m \log a + k$, proving the upper bound.

To prove the lower bound we need the following purely set-theoretic lemma.

**Lemma 3.** Let $k, l, a$ be arbitrary positive integers and $\beta > 1$. Let now $\mathcal{H}$ be a set system on an underlying set $S$ of size $c(k, l, \beta) \cdot a = k\beta(2^{kl} - 1)a$, such that every set of $\mathcal{H}$ has size at most $\beta a$ and every element of $S$ is contained in at most $l$ sets of $\mathcal{H}$. Then we can select $k$ disjoint subsets of $S$ (called heaps) $K_1, K_2, \ldots, K_k$ of size $\beta a$, such that every set of $\mathcal{H}$ intersects at most one heap.

**Proof.** Let us partition the underlying set into $k$ heaps of size $\beta a(2^{kl} - 1)$ in an arbitrary way. Now we execute the following procedure at most $kl - 1$ times, eventually obtaining $k$ heaps satisfying the required conditions. In each iteration we make sure that the members of a subfamily $\mathcal{H}'$ of $\mathcal{H}$ will intersect at most one heap at the end.

In each iteration we do the following. We build the subfamily $\mathcal{H}' \subseteq \mathcal{H}$ by starting from the empty subfamily and adding an arbitrary set of $\mathcal{H}$ to our subfamily until there exists a heap $K_i$ such that $|K_i \cap \bigcup_{H \in \mathcal{H}} H| \geq |K_i|/2$, that is $K_i$ is at least half covered by $\mathcal{H}'$. We call $K_i$ the selected heap. If the half of several heaps gets covered in the same step, then we select one where the difference of the number of covered elements and the half of the size of the heap is maximum.

Now we keep the covered part of the selected heap and keep the uncovered part of the other heaps and throw away the other elements. We also throw away the sets of the subfamily $\mathcal{H}'$ from our family $\mathcal{H}$, as we already made sure that the members of $\mathcal{H}'$ will not intersect more than one heap at the end. In this way we obtain smaller heaps but we only have to deal with the family $\mathcal{H} \setminus \mathcal{H}'$.

We prove by induction that after $s$ iterations all heaps have size at least $\beta a(2^{kl - s} - 1)$. This trivially holds for $s = 0$. By the induction hypothesis, the heaps had size at least $\beta a(2^{kl - s + 1} - 1)$ before the $s$th iteration step. After the $s$th step the new size of the selected heap $K$ is at least $|K|/2 \geq \beta a(2^{kl - s + 1} - 1)/2 \geq \beta a(2^{kl - s} - 1)$. Now we turn our attention to the unselected heaps. Suppose the set we added last to $\mathcal{H}'$ is the set $I$. Clearly, $|K_j \cap \bigcup_{H \in \mathcal{H} \setminus \{ I \}} H| \leq |K_j|/2$ for all $j$. Let $K$ be the selected heap and $K_i$ be an arbitrary unselected heap. Now by the choice of $K$ we have $|K_i \cap \bigcup_{H \in \mathcal{H} \setminus \mathcal{H}'} H| \leq |K_i|/2 + |I|/2$, otherwise
\[ |K_i \cap \cup_{H \in \mathcal{H}} H| + |K \cap \cup_{H \in \mathcal{H}} H| > |K_i|/2 + |K|/2 + |I|, \] which is impossible, since
\[ |K_i \cap \cup_{H \in \mathcal{H}} H| + |K \cap \cup_{H \in \mathcal{H}} H| = |(K_i \cup K) \cap \cup_{H \in \mathcal{H}} H| \cup ((K_i \cup K) \cap I) \leq |K_i|/2 + |K|/2 + |I|. \]

Now, since \(|I| \leq \beta a\), the new size of the unselected heap \(K_i\) is
\[ |K_i'| = |K_i \setminus \cup_{H \in \mathcal{H}} H| \geq |K_i|/2 - \beta a/2 \geq \beta a(2^{k-l-1} - 1)/2 - \beta a/2 \geq \beta a(2^{k-1} - 1), \]
finishing the proof by induction.

Now in each iteration we delete a family that covers the selected heap, thus any heap can be selected at most \(l\) times, since every element is contained in at most \(l\) sets. After \(kl - 1\) iterations the size of an arbitrary heap will be at least \(\beta a\). Furthermore, all but one heaps were selected exactly \(l\) times, thus any remaining set of \(\mathcal{H}\) can only intersect the last heap. That is, heaps at this point satisfy the required condition for all sets of \(\mathcal{H}\).

If we can iterate the process at most \(kl - 2\) times, then after the last possible iteration more than half of any heap is not covered by the union of the remaining sets. Deleting the covered elements from each heap we obtain heaps of size at least \(\beta a\) that satisfy the condition. \(\square\)

Now we are in a position to prove the lower bound of Theorem 7. We use the adversary method, i.e. we give a strategy to the adversary that forces the questioner to ask at least \(\frac{a}{a} + m \log a - c_1(k)\) questions to find \(m\) defective elements.

Recall that all questions have size at most \(\lfloor k/\alpha \rfloor\) and now the adversary gives the additional information that there are exactly \(k\) defective elements.

During the procedure, the adversary maintains weights on the elements. At the beginning all elements have weight 0. Let us denote the set of the possible defective elements by \(S'\). At the beginning \(S' = S\). At each question \(A\) the strategy determines the answer and also assigns appropriate weights to the elements of \(A\). If a question \(A\) is of size at most \(a = \lfloor 1/\alpha \rfloor\), then the answer is no and weight 1 is given to all elements of \(A\). If \(|A| > a\), the answer is still no and weight \(a/[k/\alpha]\) is given to the elements of \(A\). Thus after some \(r\) questions the sum of the weights is at most \(ra\). If an element reaches weight 1, then the adversary says that it is not defective, and the element is deleted from \(S'\). The adversary does that until there are still \(ca\) elements in \(S'\) but in the next step \(S'\) would become smaller than this threshold (the exact value of \(c\) will be determined later). Up to this point the number of elements thrown away is at least \(n - ca - \lfloor k/\alpha \rfloor\), thus the number of queries is at least \(n - c - \lfloor k/\alpha \rfloor/a \geq \frac{n}{k} - c - k\).

Let the set system \(F\) consist of the sets that were asked up to this point and let \(\mathcal{F}' = \{F \cap S' \mid F \in \mathcal{F}, |F| > a\}\).

The following observations are easy to check.

**Lemma 4.**

- \(|S'| \geq ca\).
- Every set \(F \in \mathcal{F}'\) has size at most \(\lfloor k/\alpha \rfloor \leq k(a + 1) \leq 2ka\).
- Every element of \(S'\) is contained in at most \(\lfloor k/\alpha \rfloor/a \leq k(1 + 1/a) \leq 2k\) sets of \(\mathcal{F}'\).
– Every k-set that intersects each $F \in F'$ in at most one element is a possible set of defective elements.

Now let $l := 2k$, $\beta := 2k$, and $c := c(k, l, \beta) = k\beta(2^{kl} - 1) = 2k^2(2^{2k^2} - 1)$. By the observations above, we can apply Lemma 3 with $H = F'$. The lemma guarantees the existence of heaps $K_1, K_2, \ldots, K_k$ of size $\beta a \geq a$, such that every transversal of the $K_i$’s is a possible $k$-set of defective elements. Now by applying Proposition 1 with $A_i = K_i$ and $A_0 = S \setminus S'$, we obtain that the questioner needs to ask at least $\lceil m \log a \rceil$ more queries to find $m$ defective elements.

Altogether the questioner had to use at least $\frac{n}{\alpha} - c - k + m \log a$ queries, which proves the lower bound, since the number $c$ depends only on $k$ (the constant in the theorem is $c(k) = e + k$).

The constant in the lower bound is quite large, by a more careful analysis one might obtain a better one. For example, we could redefine the weights, such that we give weight $a/|A|$ to the elements of $A$, thus still distributing weight at most $a$ per asked set.

It is also worth observing that if $1/\alpha$ is an integer, then we can use Lemma 3 with $l = \beta = k$, instead of $l = \beta = 2k$. This way one can prove stronger results for small values of $k$ and $m$ if $1/\alpha$ is an integer. We demonstrate it for $k = 2$ in the next section. The following claim is easy to check.

Claim. Let $H$ be a set system on an underlying set $S$ of size $3a$, consisting of disjoint sets of size at most $2a$. Then we can select 2 disjoint subsets of $S$ (called heaps) $K_1, K_2$ of size at least $a$, such that every set of $\mathcal{H}$ intersects at most one heap.

4 The case $k = 2$

In this section we determine the exact value $g(n, 2, \alpha, 1)$. Let $\delta = \lfloor 2\left(\frac{1}{\alpha}\right) \rfloor$, where $\lfloor x \rfloor$ denotes the fractional part of $x$.

Consider the following algorithm $\mathcal{W}$, where $n$ denotes the number of remaining elements:

If $n \leq 2^{[\log a]} + 1$, we ask a question of size $\lfloor n/2 \rfloor \leq a$, then depending on the answer we continue in the part that contains at least one defective element, and find that with binary search.

If $2^{[\log a]} + 2 \leq n \leq 2^{[\log a]} + 1$, then we ask a question of size $2^{[\log a]} + 1$ (this falls between $a$ and $2a + 1$). If the answer is yes, we put an element aside and continue with the remaining elements of the set we asked, otherwise we continue with the elements not in the set we asked. This way independent of whether we got a yes or no answer, we have at most $2^{[\log a]}$ elements with at least one defective, hence we can apply binary search.

If $2^{[\log a]} + 1 \leq n \leq 3a + \delta + 2^{[\log a]}$, then first we ask a question of size $2a + \delta$. If the answer is yes, we put an element aside and continue with the remaining elements of the set we asked, otherwise we continue with the elements not in the set we asked. This way independent of whether we got a yes or no
answer, we have at most $2^{\log a} + a$ elements with at least one defective. We continue with a set of size $a$, and after that we can finish with binary search.

If $n \geq 3a + \delta + 2^{\log a} + 1$, then we ask a question of size $a$. If the answer is NO, we proceed as above. If the answer is YES, we can find a defective element with at most $\lceil \log a \rceil$ further questions.

Counting the number of questions used in each case, we can conclude.

Claim. If $n \leq 3a + \delta + 2^{\log a}$, then algorithm $W$ takes only $\lceil \log(n-1) \rceil$ questions, thus according to Theorem 4 it is optimal.

In fact a stronger statement is true. Note that the following theorem does not contradict to Conjecture 1, as the algorithm mentioned there uses the same number of steps as algorithm $W$ in case $k = 2$, $1/a$ is an integer and $\lfloor n/a \rfloor \geq 4$.

**Theorem 8.** Algorithm $W$ is optimal for any $n$.

**Proof.** We prove a slightly stronger statement, that algorithm $W$ is optimal even among those algorithms that have access to an unlimited number of extra non-defective elements. This is crucial as we use induction on the number of elements, $n$.

It is easy to check that the answer for a set that is greater than $2a + \delta$ is always NO, while if both defective elements are in a set of size $2a + \delta$, then the answer is YES. We say that a question is small if its size is at most $a$, and big if its size is between $a + 1$ and $2a + \delta$. Note that small questions test if there is at least one defective element in the set, while big questions test if both defective elements are in the set. Suppose by contradiction that there exists an algorithm $Z$ that is better than $W$, i.e. there is a set of elements for which $Z$ is faster than $W$. Denote by $n$ the size of the smallest such set and by $z(n)$ the number of steps in algorithm $Z$. We will establish through a series of claims that such an $n$ cannot exist. It already follows from Claim 4 that $n$ has to be at least $3a + \delta + 2^{\log a} + 1$.

Note that for $n = 3a + \delta + 2^{\log a}$ algorithm $W$ uses $\lceil \log(n-1) \rceil = \lceil \log(2a + \delta - 1) \rceil + 1$ questions. An important tool is the following lemma.

**Lemma 5.** If $n \geq 3a + \delta + 2^{\log a} + 1$, then algorithm $Z$ has to start with a big question. Moreover, it can ask a small question among the first $z(n) - \lceil \log(2a + \delta - 1) \rceil$ questions only if one of the previous answers was YES.

**Proof.** First we prove that algorithm $Z$ has to start with a big question. Suppose it starts with a small question. We show that in case the answer is NO, it cannot be faster than algorithm $W$. In this case after the first answer there are at least $n - a$ (and at most $n - 1$) elements which can be defective, and an unlimited number of non-defective elements, including those which are elements of the first question. By induction algorithm $W$ is optimal in this case, and one can easily see that it cannot be faster if there are more elements, hence algorithm $Z$ cannot be faster than algorithm $W$ on $n - a$ elements plus one more question. On the other hand algorithm $W$ clearly uses this many questions (as it starts with a question of size $a$), hence it cannot be slower than algorithm $Z$. 

Similarly, to prove the moreover part, suppose that the first \( z(n) - \lceil \log(2a + \delta - 1) \rceil \) answers are no and one of these questions, \( A \) is small. Let us delete every element of \( A \). By induction algorithm \( W \) is optimal on the remaining at least \( n - a \) elements, hence similarly to the previous case, algorithm \( Z \) uses more questions than algorithm \( W \) on \( n - a \) elements, hence cannot be faster than algorithm \( W \). More precisely, we can define algorithm \( W' \), which starts with asking \( A \), and after that proceeds as algorithm \( W \). One can easily see that algorithm \( W' \) cannot be slower than algorithm \( Z \) or faster than algorithm \( W \). \( \Box \)

Note that a yes answer would mean that \( \lceil \log(2a + \delta - 1) \rceil \) further questions would be enough to find a defective with binary search, hence in the worst case, (when the most steps are needed) no such answer occurs among the first \( z(n) - \lceil \log(2a + \delta - 1) \rceil \) questions anyway. Now we can finish the proof of the theorem with the following claim.

**Claim.** If \( n > 3a + \delta + 2^{\lceil \log a \rceil} \), then algorithm \( W \) is optimal.

**Proof.** If not, then the smallest \( n \) for which \( W \) is not optimal must be of the form 
\[ 2a + \delta + 2^{\lceil \log a \rceil} + za + 1, \]
where \( z \geq 1 \) integer. (This follows from the fact that the number of required questions is monotone in \( n \) if we allow the algorithm to have access to an unlimited number of extra non-defective elements.) By contradiction, suppose that algorithm \( Z \) uses only \( \lceil \log(2a + \delta - 1) \rceil + z \) questions. Suppose the answer to the first \( z \) questions are no. Then by to Lemma 5, these questions are big. Suppose that the \( z + 1 \)st answer is also no. We distinguish two cases depending on the size of the \( z + 1 \)st question \( A \). In both cases we will use reasoning similar to the one in Theorem 5.

Case 1. The \( z + 1 \)st question is small. After the answer there are \( \lceil \log(2a + \delta - 1) \rceil - 1 \) questions left, so depending on the answers given to them, any deterministic algorithm can choose at most \( 2^{\lceil \log(2a + \delta - 1) \rceil - 1} \) elements. Hence algorithm \( Z \) gives us after the \( z + 1 \)st answer a set \( B \) of at most \( 2^{\lceil \log(2a + \delta - 1) \rceil - 1} \) elements, which contains a defective.

Before starting the algorithm, all the \( \binom{n}{2} \) pairs are possible candidates to be the set of defective elements. However, after the \( z + 1 \)st question (knowing the algorithm) the only candidates are those which intersect \( B \). The \( z + 1 \)st question shows at most a non-defective elements, but all the pairs which intersect neither \( A \) nor \( B \) have to be excluded by the first \( z \) questions. Thus 
\[
\binom{n - |A| - |B|}{2} \leq \binom{2a + \delta + 2^{\lceil \log a \rceil} + za + 1 - a - 2^{\lceil \log(2a + \delta - 1) \rceil - 1}}{2} \geq \binom{(z+1)a + \delta + 1}{2}
\]

pairs should be excluded, but \( z \) questions can exclude at most \( z\binom{2a+\delta}{2} \) pairs, which is less if \( z \geq 1 \).

Case 2. The \( z + 1 \)st question is big. After it we have \( \lceil \log(2a + \delta - 1) \rceil - 1 \) questions left, so depending on the answers given to them, any deterministic algorithm can choose at most \( 2^{\lceil \log(2a + \delta - 1) \rceil - 1} \) elements. This means that we have to exclude with the first \( z + 1 \) questions at least 
\[
\binom{2a + \delta + 2^{\lceil \log a \rceil} + za + 1 - 2^{\lceil \log(2a + \delta - 1) \rceil - 1}}{2} \geq \binom{(z+2)a + \delta + 1}{2}
\]

pairs. But they can exclude at most \( (z + 1)\binom{2a+\delta}{2} \) pairs, which is less if \( z \geq 1 \). \( \Box \)

This finishes the proof of the theorem. \( \Box \)
5 Open problems

It is quite natural to think that \( g(n, k, \alpha, m) \) is increasing in \( n \) but we did not manage to prove that. The monotonicity in \( k \) and \( m \) is obvious from the definition. On the other hand, we could have defined \( g(n, k, \alpha, m) \) as the smallest number of questions needed to find \( m \) defectives assuming there are exactly \( k \) defectives (instead of at least \( k \) defectives) among the \( n \) elements, in which case the monotonicity in \( k \) is far from trivial. We conjecture that this definition gives the same function as the original one.

It might seem strange to look for monotonicity in \( \alpha \), but we have seen that for \( m = 1 \) we can reach the information theoretic lower bound (which is \( \lceil \log(n-k+1) \rceil \) in this setting) for \( \alpha \leq 2/(n-k+1) \). All the theorems from Section 2 also suggest that the smaller \( \alpha \) is, the faster the best algorithm is even for general \( m \). Basically in case of a no answer it is better if \( \alpha \) is small, and in case of a yes answer the size of \( \alpha \) does not matter very much, since the process can be finished fast. However, we could only prove Theorem 7 concerning this matter.

Another interesting question is if we can choose \( \alpha \). If \( m = 1 \) then we should choose \( \alpha \leq 1/(n-k+1) \), and as we have mentioned in the previous paragraph, we believe that a small enough \( \alpha \) is the best choice.

Another possibility would be if we were allowed to choose a new \( \alpha \) for every question. Again, we believe that the best solution is to choose the same, small enough \( \alpha \) every time. This would obviously imply the previous conjecture.

Finally, a more general model to study is the following. We are given two parameters \( \alpha \geq \beta \). If at least an \( \alpha \) fraction of the set is defective, then the answer is yes. If at most a \( \beta \) fraction, then it is no, while in between the answer is arbitrary. With these parameters, this paper studied the case \( \alpha = \beta \). This model is somewhat similar to the threshold testing model of [1], where instead of ratios \( \alpha \) and \( \beta \) they have fixed values \( a \) and \( b \) as thresholds.

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