GEOMETRIC AUTOMORPHISM GROUPS OF SYMPLECTIC 4-MANIFOLDS

BO DAI, CHUNG-I HO, AND TIAN-JUN LI

Abstract. Let $M$ be a closed, oriented, smooth 4–manifold with intersection form $\Gamma$, $A(\Gamma)$ the automorphism group of $\Gamma$ and $D(M)$ the subgroup induced by orientation-preserving diffeomorphisms of $M$. In this note we study the question when $D(M)$ is of infinite index in $A(\Gamma)$ for a symplectic 4–manifold.

Contents

1. Introduction 1
2. Infinite $A(\Gamma)$ 3
  2.1. Quadratic forms 3
  2.2. Infinite $A(\Gamma)$ and criterion for subgroups of infinite index 4
3. $D(M)$ with infinite index 7
  3.1. $\kappa = -\infty$ 8
  3.2. $\kappa = 0$ with $\Gamma$ odd, and $\kappa \geq 1$ 8
4. Symplectic Calabi-Yau surfaces 9
  4.1. Homological classification 9
  4.2. Kähler CY surfaces 10
  4.3. Known non-Kähler CY surfaces 12
References 13

1. Introduction

For a given unimodular symmetric bilinear form $\Gamma$, let $A(\Gamma)$ be its automorphism group. Given a closed, oriented, topological 4–manifold $M$, let $\Lambda_M$ be the free abelian group obtained from $H^2(M; \mathbb{Z})$ by modulo torsion, and $\Gamma_M$ the associated unimodular symmetric bilinear form, namely, the intersection form on $\Lambda_M$. By a celebrated result of Freedman, any unimodular symmetric bilinear form is realized as the intersection form of an oriented, simply connected topological 4–manifold. Moreover, for such a topological manifold $M$, the natural map from the group of orientation-preserving homeomorphisms to $A(\Gamma_M)$ is surjective.
For a smooth, closed, oriented 4–manifold \( M \) with intersection form \( \Gamma \), there is a natural map from the group of orientation-preserving diffeomorphisms \( \text{Diff}^+(M) \) to the automorphism group of \( \Gamma \), \( A(\Gamma) \). Let \( D(M) \) be the image of this natural map. In other words, an automorphism is in \( D(M) \) if it is realized by an orientation-preserving diffeomorphism. \( D(M) \) is called the geometric automorphism group. The group \( D(M) \), both as an abstract group and as a subgroup of \( A(\Gamma) \), is a powerful smooth invariant, which is nonetheless hard to compute in general.

Wall initiated the comparison of \( D(M) \) and \( A(\Gamma) \) in a series of papers [19], [20], [21]. In particular, he proved in [21] the following beautiful result: for any simply connected smooth manifold with \( \Gamma \) strongly indefinite or of rank at most 10, if there is an \( S^2 \times S^2 \) summand in its connected sum decomposition, then \( D(M) = A(\Gamma) \). For Kähler surfaces, especially elliptic surfaces, rational surfaces, ruled surfaces, we have a rather good understanding of \( D(M) \) due to Friedman, Morgan, Donaldson, Lönne [3], [5], [2], [14] (see also [9], [10]).

In this note we will focus on the question when \( D(M) \) is of infinite index in \( A(\Gamma_M) \) if \( A(\Gamma_M) \) is an infinite group. We first observe in Theorem 2.3 that \( A(\Gamma) \) is infinite if \( \Gamma \) is indefinite of rank at least 3. Moreover, we offer a simple criterion for a subgroup to have infinite index. We apply this criterion to symplectic manifolds and obtain an almost complete answer.

To state our result, let us first recall some definitions. For a smooth 4–manifold \( M \) with a symplectic form \( \omega \), let \( K_\omega \) denote the symplectic canonical class. A symplectic 4–manifold is said to be minimal if it does not contain any embedded symplectic sphere with self-intersection \(-1\). A general symplectic 4–manifold \( (M, \omega) \) can be symplectically blown down to a minimal one, which is called a minimal model.

The Kodaira dimension of a symplectic 4–manifold \( (M, \omega) \) is defined below.

**Definition 1.1.** If \( (M, \omega) \) is minimal, the Kodaira dimension of \( (M, \omega) \) is defined in the following way:

\[
\kappa(M, \omega) = \begin{cases} 
-\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } K_\omega \cdot K_\omega < 0, \\
0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } K_\omega \cdot K_\omega = 0, \\
1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega = 0, \\
2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega > 0.
\end{cases}
\]

For a general \( (M, \omega) \), \( \kappa(M, \omega) \) is defined to be that of any of its minimal model.
It is shown in [7] that $\kappa(M, \omega)$ is well-defined and agrees with the holomorphic Kodaira dimension if $(M, \omega)$ is Kähler. Moreover, it turns out that $\kappa(M, \omega)$ only depends on $M$ so we will denote it by $\kappa(M)$.

**Theorem 1.2.** Suppose $M$ has symplectic structures and $A(\Gamma_M)$ is infinite. Then $D(M)$ is of infinite index if

1. $\kappa(M) = -\infty$, and $M = \mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ with $n \geq 10$ or $(\Sigma \times S^2) \# n\overline{\mathbb{CP}}^2$ with $n \geq 1$, where $\Sigma$ is a closed Riemann surface of positive genus.
2. $\kappa(M) = 0$ and $\Gamma_M$ is odd.
3. $\kappa(M) \geq 1$.

This result follows from Propositions 3.2, 3.4, 3.3.

$M$ is called a symplectic Calabi-Yau surface if there is a symplectic form $\omega$ on $M$ such that $K_{\omega}$ vanishes in the real cohomology. The third author showed in [7] that $M$ is a symplectic Calabi-Yau surface exactly when $\kappa(M) = 0$ and $\Gamma_M$ is even. With this understood, Theorem 1.2 can be restated as: When $M$ is symplectic and $A(\Gamma_M)$ is infinite, $D(M)$ is of finite index only when $M$ is a symplectic Calabi-Yau surface, or $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ with $2 \leq n \leq 9$.

We define Kähler Calabi-Yau surfaces in the same way. There are three Kähler Calabi-Yau surfaces with infinite $A(\Gamma)$: K3 surface, Enriques surface, $T^4$. All of them have finite index geometric automorphism group. The only known non-Kähler Calabi-Yau surfaces with infinite $A(\Gamma)$ are the so-called Kodaira-Thurston manifolds. We will show in the last section that they have infinite index geometric automorphism group. Thus we further make the following conjecture.

**Conjecture 1.3.** Suppose $M$ has symplectic structures and $A(\Gamma_M)$ is infinite. Then $D(M)$ is of finite index if and only if $M$ is

1. a Kähler Calabi-Yau surface, or
2. $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ with $2 \leq n \leq 9$.

2. **Infinite $A(\Gamma)$**

2.1. **Quadratic forms.** Let $\Lambda$ be a finitely generated free abelian group, and $\Gamma : \Lambda \times \Lambda \to \mathbb{Z}$ a unimodular symmetric bilinear form on $\Lambda$. Sometimes we abbreviate $\Gamma(x, y)$ as $x \cdot y$. It induces a quadratic form $Q : \Lambda \to \mathbb{Z}$ as $Q(x) = \Gamma(x, x)$. $Q(x)$ is called the norm of $x$. $\Gamma$ is of even type if $Q(x)$ is even for any vector $x \in \Lambda$. Otherwise, it is called of odd type. The rank $r(\Gamma)$ of $\Gamma$ is the rank of $\Lambda$. Let $b^+, b^-$ be the number of 1, -1 respectively, on a diagonal matrix over $\mathbb{R}$ representing $\Gamma$. The signature $\sigma(\Gamma)$ of $\Gamma$ is the difference $b^+ - b^-$. 
Definition 2.1. \( \Gamma \) is called \textit{definite, nearly definite} or \textit{strongly indefinite} if \( \min\{b^+, b^-\} = 0, 1 \) or \( \geq 2 \) respectively.

The following classification is well known, see e.g. [19].

**Theorem 2.2.** The classification of indefinite unimodular symmetric forms is given by their rank, signature and type.

Let \( U, E \) be respectively the hyperbolic lattice and the (positive definite) \( E_8 \) lattice. The list of indefinite unimodular symmetric forms are

\[ m\langle 1 \rangle \oplus n\langle -1 \rangle, \quad pU \oplus qE, \quad \text{where } m, n, p \in \mathbb{N}, q \in \mathbb{Z} \]

2.2. Infinite \( A(\Gamma) \) and criterion for subgroups of infinite index.

Let \( A(\Gamma) \) be the automorphism group of \( \Gamma \). In this subsection we establish the following criterion for subgroups of \( A(\Gamma) \) to have infinite index.

**Theorem 2.3.** Let \( A(\Gamma) \) be the automorphism group of \( \Gamma \).

- \( A(\Gamma) \) is finite if and only if it is definite or indefinite of rank 2.
- Suppose \( A(\Gamma) \) is infinite, i.e. \( \Gamma \) is indefinite of rank \( \geq 3 \). If there are finitely many nonzero characteristic classes invariant under a subgroup \( D \), then \( D \) is of infinite index.

2.2.1. Infinite transitive actions for strongly indefinite \( \Gamma \). A vector \( x \in \Lambda \) is called primitive if it cannot be divided by any integer except \( \pm 1 \). A vector \( x \) is called characteristic if for all vectors \( y \) in \( \Lambda \), \( x \cdot y \equiv y \cdot y \) (mod 2). Otherwise, it is called ordinary. It is clear that \( A(\Gamma) \) preserves the norm and type of each vector. When \( \Gamma \) is strongly indefinite, \( A(\Gamma) \) often acts transitively with infinite orbits.

**Proposition 2.4.** Assume \( \Gamma \) is strongly indefinite.

1. \( A(\Gamma) \) acts transitively on primitive vectors of given norm and type.
2. For any \( k \in \mathbb{Z} \), there are infinitely many nonzero characteristic classes of norm \( \sigma(\Gamma) + 8k \).

**Proof.**

(1) See Theorem 6 in [19].

(2) It is enough to consider the case \( r(\Gamma) = 4 \). General case then follows by extension.

If \( \Gamma = 2U \) and \( x_0, y_0, x_1, y_1 \) is a basis of \( \Lambda \) such that \( x_0 \cdot y_0 = x_1 \cdot y_1 = 1 \), we know that any characteristic class is of the form \( x = 2ax_0 + 2by_0 + 2cx_1 + 2dy_1 \) for some \( a, b, c, d \in \mathbb{Z} \). We have \( Q(x) = 8(ab + cd) \) and it is clear that there are infinitely many quadruple \((a, b, c, d)\) satisfying \( ab + cd = k \). For instance, \( ak + (1 - a)k = k \) for any \( a \).
If $\Gamma$ is even, then it is of the form $2U \oplus L$. Any characteristic vector $c$ for $2U$ gives rise to a characteristic vector $(c, 0)$ for $\Gamma$ of the same norm.

If $\Gamma = 2^{<1>} + 2^{<−1>}$ and $\Lambda$ has a basis $p_1, p_2, q_1, q_2$ with $p_1^2 = p_2^2 = -q_1^2 = -q_2^2 = 1$, any characteristic class is of the form $x = ap_1 + bp_1 + cq_1 + dq_2$ for some odd integers $a, b, c, d$. If $k = 2^{t_r}$ and $r$ is odd, we can set $a = 2^{t_r+1} + r$ and $c = 2^{t_r+1} - r$. Then $Q(ax_1 + cy_1) = 8k$. So $Q(ap_1 + bp_1 + cq_1 + bq_2) = 8k$ for any $b$.

If $\Gamma$ is odd, then it is of the form $(2^{<1>} + 2^{<−1>}) \oplus L$. Any characteristic vector $c$ for $2^{<1>} + 2^{<−1>}$ gives rise to a characteristic vector $(c, (1))$ for $\Gamma$ whose norm differs from that of $c$ by a fixed constant.

In some cases, higher dimensional subspaces also have such transitive property. A subgroup $W \subset \Lambda$ is called full if $(W \otimes \mathbb{Q}) \cap \Lambda = W$. We define

$$U(\Gamma) = \{W \subset \Lambda|W : \text{full, isotropic, of dimension 2}\}$$

**Lemma 2.5.** $A(2U)$ acts transitively on $U(2U)$ and $|U(2U)| = \infty$.

**Proof.** Using the notations in the proof of Proposition 2.4 if $Y \in U(2U)$ and $x, y \in Y$ are linearly independent primitive vectors, Proposition 2.4 (1) implies that $\alpha x = x_0$ for some $\alpha \in A(2U)$. So $\alpha y = ax_0 + bx_1$ or $ax_0 + by_1$ for some $a, b$. Hence $\alpha(Y) = \langle x_0, x_1 \rangle$ or $< x_0, y_1 >$ and $U(2U)$ is transitive. Moreover, we can show that

$$U(2U) = \{< ax_0 + bx_1, by_0 - ay_1 >, < ax_0 + by_1, by_0 - ax_1 > | gcd(a, b) = 1\}.$$

So $|U(2U)| = \infty$. □

**2.2.2. Infinite orbits for nearly definite $\Gamma$.** Now consider the case that $\Gamma$ is nearly definite, i.e. $b^+ = 1$ or $b^- = 1$. In this case, we cannot always establish transitivity of actions. Instead we show the infiniteness of orbits.

**Lemma 2.6.** Let $\Gamma$ be nearly definite of rank at least 3. For any nonzero $x \in \Lambda$, the orbit of $x$ under $A(\Gamma)$ is infinite.

**Proof.** We only consider nearly positive definite case. First consider odd type case. Let $H_1, \cdots, H_n, F$, $n \geq 2$, be an orthogonal basis of $\Lambda$ such that $H_i^2 = -F^2 = 1$. Let $x = a_1H_1 + \cdots + a_nH_n + bF$. Without loss of generality, we may assume $|a_1| \geq |a_2| \geq \cdots \geq |a_n|$. Consider
the reflection $R_\gamma$ of $\gamma = \varepsilon_1 H_1 + \varepsilon_2 H_2 + F$ where $\varepsilon_i = \pm 1$. It is easy to see that

$$R_\gamma(F) = 2\varepsilon_1 H_1 + 2\varepsilon_2 H_2 + 3F$$
$$R_\gamma(H_1) = -H_1 - 2\varepsilon_1\varepsilon_2 H_2 - 2\varepsilon_1 F$$
$$R_\gamma(H_2) = -2\varepsilon_1\varepsilon_2 H_1 - H_2 - 2\varepsilon_2 F$$

So

$$R_\gamma(x) = (-a_1 - 2\varepsilon_1\varepsilon_2 a_2 + 2\varepsilon_1 b) H_1 + (-2\varepsilon_1\varepsilon_2 a_1 - a_2 + 2\varepsilon_2 b) H_2
\quad + a_3 H_3 + \cdots + a_n H_n + (-2a_1\varepsilon_1 - 2a_2\varepsilon_2 + 3b) F$$

Choosing $\varepsilon_1, \varepsilon_2$ appropriately, the coefficient of $F$ in $R_\gamma(x)$ is monotone.

Now consider even type case. $\Gamma$ is equivalent to $U + lE$ with $l > 0$. Let $x, y$ be a standard basis of $U$, i.e. $x^2 = y^2 = 0$, $x \cdot y = 1$. Then any (nonzero) class in $\Gamma$ has a unique decomposition $\eta + ax + by$, where $\eta \in lE$. There are three cases.

(1) $\eta \neq 0$, $ax + by \neq 0$. Without loss of generality, assume $b \neq 0$.

Since $lE$ has a basis such that each vector has square 2, there exists an $\omega \in lE$ such that $\omega^2 = 2$, and $\omega \cdot \eta \neq 0$. For any $k \in \mathbb{Z}$, $\omega + kx)^2 = 2$. Consider

$$R_{\omega+kx}(\eta + ax + by) = \eta + ax + by - (\omega \cdot \eta + kb)(\omega + kx)$$
$$= \eta - (\omega \cdot \eta + kb)\omega + (a - (\omega \cdot \eta + kb)k)x + by.$$

We can choose $k \in \mathbb{Z}$ such that $\eta - (\omega \cdot \eta + kb)\omega \neq 0$, and the coefficient of $x$ is monotone (decreases if $b > 0$, and increases if $b < 0$). Repeating this process, we see that the orbit is infinite.

(2) $0 \neq \eta \in lE$, $ax + by = 0$. Choose $\omega \in lE$ such that $\omega^2 = 2$.

Consider

$$R_{\omega+y}(\eta) = \eta - (\omega \cdot \eta)(\omega + y) = \eta - (\omega \cdot \eta)\omega - (\omega \cdot \eta)y.$$

By properties of $E$, we can choose $\omega$ such that $\omega \cdot \eta \neq 0$, and $\eta - (\omega \cdot \eta)\omega \neq 0$. Then we are back to case (1).

(3) $\eta = 0$, $ax + by \neq 0$. We may assume $b \neq 0$. Choose $\omega \in lE$ such that $\omega^2 = 2$, and $k \neq 0$. Consider

$$R_{\omega+kx}(ax + by) = ax + by - kb(\omega + kx) = -(kb)\omega + (a - k^2 b)x + by.$$

Then we are back to case (1) again.

$\square$
2.2.3. Proof of Theorem 2.3

Proof. Let us first show that $A(\Gamma)$ is finite if and only if $\Gamma$ is definite or indefinite of rank 2. The if part is known, namely, if $\Gamma$ is definite or indefinite of rank 2, then $A(\Gamma)$ is finite. See the remarks after conclusion of [19].

For the only if part, when $\Gamma$ is strongly indefinite, it follows from Proposition 2.4 (1) and (2).

When $\Gamma$ is nearly definite of rank $\geq 3$, it follows from Lemma 2.6.

Notice that exactly the same argument proves the statement in the second bullet. □

3. $D(M)$ WITH INFINITE INDEX

Let $M$ be a closed, oriented, smooth 4-manifold. A symplectic form on $M$ is a closed 2-form $\omega$ on $M$ such that $\omega \wedge \omega$ is a volume form inducing the given orientation of $M$. Given $\omega$, it comes with a contractible set of almost complex structures tamed by $\omega$. Suppose an almost complex structure $J$ is from this contractible set. The canonical class of $\omega$ is then defined to be $-c_1(M, J)$, and denoted by $K_\omega$. We call $K_\omega \in H^2(M; \mathbb{Z})$ a symplectic canonical class of $M$. It is a characteristic class in $\Gamma_M$ with norm $2\chi(M) + 3\sigma(\Gamma_M)$, where $\chi(M)$ is the Euler number of $M$.

Let $K_M = \{K_\omega | \omega$ a symplectic form on $M\}$

be the set of symplectic canonical classes of $M$. Clearly, $K_M$ is nonempty if and only if $M$ has symplectic structures. Let $\overline{K}_M$ be the image of $K_M$ in $\Gamma_M$.

Theorem 3.1. $\overline{K}_M$ has the following properties.

• $\overline{K}_M$ is preserved by $D(M)$.
• Suppose $M$ has symplectic structures and $\kappa(M) \geq 0$. Then $\overline{K}_M$ is a finite set.
• $\overline{K}_M$ contains 0 if and only if $M$ is a symplectic Calabi-Yau surface.

Proof. The first statement follows from the following simple observation: For any symplectic form $\omega$ and orientation preserving diffeomorphism $\phi$, $\phi^* \omega$ is still a symplectic form, and $K_{\phi^* \omega} = \phi^* K_\omega$.

The second statement in the case $b^+ \geq 2$ follows from Taubes’s fundamental results in [17] and [18], and in the case $b^+ = 1$ it is established in [11].

The last statement is noted in [7]. □
3.1. \( \kappa = -\infty \). There is a classification of \( \kappa = -\infty \) manifolds: \( M \) is either rational or ruled ([13], [15]).

For a rational 4-manifold \( M = \mathbb{C}P^2 \# n\mathbb{C}P^2 \) with \( n \leq 9 \) or \( S^2 \times S^2 \), a classical result of Wall [21] says that \( D(M) \) coincides with \( A(\Gamma) \).

For \( M = \mathbb{C}P^2 \# n\mathbb{C}P^2 \) with \( n \geq 10 \), Friedman and Morgan [3] showed that \( D(M) \) is a subgroup of \( A(\Gamma) \) with infinite index, and characterized it in terms of super \( P \)-cells. Another proof of these results appeared in [10] by presenting an explicit and finite generating set of \( D(M) \).

The case of irrational ruled 4-manifolds has been studied in [5] and [9]. Let \( \Sigma \) be a closed Riemann surface of positive genus, and \( \Sigma \times S^2 \) be the nontrivial \( S^2 \)-bundle over \( \Sigma \). Then any minimal irrational ruled manifold \( M \) is diffeomorphic to \( \Sigma \times S^2 \) or \( \Sigma \times S^2 \) for some \( \Sigma \). For such manifolds, it is known that \( A(\Gamma) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), and \( D(M) \cong \mathbb{Z}_2 \) [8]. So \( D(M) \) is a proper subgroup of \( A(\Gamma) \), and both are finite groups.

Any non-minimal irrational ruled 4-manifold is diffeomorphic to \((\Sigma \times S^2) \# n\mathbb{C}P^2\) with \( n \geq 1 \). There is a unique spherical class \( f \) (up to sign) of square zero, namely the class represented by the \( S^2 \) factor in the \( \Sigma \times S^2 \) summand. In the case, Friedman and Morgan proved that an automorphism \( \tau \in A(\Gamma) \) is in \( D(M) \) if and only if \( \tau(f) = \pm f \). By presenting an explicit and finite generating set of \( D(M) \), it was proved in [10] that for \( M = (\Sigma \times S^2) \# n\mathbb{C}P^2 \) with \( n \geq 1 \), \( D(M) \) is a subgroup of \( A(\Gamma) \) with infinite index. Alternatively, this also follows from Lemma 2.6.

We summarize the discussion in the following proposition.

**Proposition 3.2.** Suppose \( \kappa(M) = -\infty \). Then

- \( D(M) = A(\Gamma) \) if \( M = S^2 \times S^2 \) or \( M = \mathbb{C}P^2 \# n\mathbb{C}P^2 \) for \( n \leq 9 \).
- \( A(\Gamma) \) is finite and \( D(M) \) is a subgroup of index 2 if \( M \) is an \( S^2 \)-bundle over a positive genus surface.
- \( D(M) \) is of infinite index in all other cases.

3.2. \( \kappa = 0 \) with \( \Gamma \) odd, and \( \kappa \geq 1 \).

3.2.1. \( \kappa = 0 \) with \( \Gamma \) odd.

**Proposition 3.3.** When \( \kappa(M) = 0 \) and \( \Gamma \) is odd, \( A(\Gamma) \) is infinite and \( D(M) \) is of infinite index in \( A(\Gamma) \).

**Proof.** When \( \kappa(M) = 0 \) and \( \Gamma \) is odd, \( M \) is non-minimal. Let \( M' \) be a minimal model. Then \( M' \) is a symplectic Calabi-Yau surface. From the table in the next section, \( b^-(M') \geq 1 \). Thus \( b^-(M) \geq 2 \). Since \( M \)
admits a symplectic structure we have \( b^+(M) \geq 1 \). Thus by the first statement of Theorem 2.3, \( A(\Gamma) \) is infinite.

Since \( M \) is non-minimal, the set \( \mathfrak{K}_M \) is finite, consists of nonzero classes and is invariant under \( D(M) \) by Theorem 3.1. Now apply Theorem 2.3.

3.2.2. \( \kappa \geq 1 \).

**Proposition 3.4.** If \( \kappa(M) \geq 1 \) and \( A(\Gamma) \) is infinite, then \( D(M) \) is of infinite index in \( A(\Gamma) \).

**Proof.** Since it is assumed that \( A(\Gamma) \) is infinite, the conclusion follows directly from Theorem 3.1 and the second statement of Theorem 2.3.

3.2.3. **Proof of Theorem 1.2.**

**Proof.** It follows from Propositions 3.2, 3.4, 3.3.

4. **Symplectic Calabi-Yau surfaces**

In this section we will focus on symplectic CY surfaces. Specifically we will provide evidences for Conjecture 1.3 by showing that

1. for any Kähler CY surface, \( D \) is of finite index.
2. for any known non-Kähler CY surface, if \( A(\Gamma) \) is infinite, then \( D \) is of infinite index.

4.1. **Homological classification.** A symplectic Calabi-Yau surface is a minimal manifold with \( \kappa = 0 \).

There is a homological classification of symplectic CY surfaces in [6] and [11].

**Theorem 4.1.** A symplectic CY surface is a \( \mathbb{Z} \)-homology K3 surface, a \( \mathbb{Z} \)-homology Enriques surface or a \( \mathbb{Q} \)-homology \( T^2 \)-bundle over \( T^2 \).

The following table list possible homological invariants of symplectic CY surfaces [6]:

| \( b_1 \) | \( b_2 \) | \( b^+ \) | \( \chi \) | \( \sigma \) | known manifolds                      |
|--------|--------|--------|--------|--------|-------------------------------------|
| 0      | 22     | 3      | 24     | -16    | K3                                  |
| 0      | 10     | 1      | 12     | -8     | Enriques surface                    |
| 4      | 6      | 3      | 0      | 0      | 4-torus                             |
| 3      | 4      | 2      | 0      | 0      | \( T^2 \)-bundles over \( T^2 \)    |
| 2      | 2      | 1      | 0      | 0      | \( T^2 \)-bundles over \( T^2 \)    |

It is also speculated that in fact a symplectic CY surface is actually the K3 surface, Enriques surface or a \( T^2 \)-bundle over \( T^2 \).
4.2. Kähler CY surfaces.

**Proposition 4.2.** \( D(M) \) is of index at most 4 if \( M \) is a Kähler CY surface.

**Proof.** According to the Kodaira classification, a Kähler CY surface is either a hyperelliptic surface, the Enriques surface, the K3 surface, or \( T^4 \).

For hyperelliptic surfaces, \( \Gamma = U \), so \( A(\Gamma) \) is the order 4 group \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). For the K3 surface, it was shown by Donaldson in [2] that \( D \) is of index 2. For the Enriques surface it was shown by Lönne in [14] that \( D = A \). It remains to deal with \( T^4 \). Our claim is that \( [A(3U) : D(T^4)] = 4 \).

Use the standard model \( \mathbb{R}^4/\mathbb{Z}^4 \) for \( T^4 \), with coordinates \( t_1, \ldots, t_4 \). Then \( H^1 \) is generated by \( \{dt_1, \ldots, dt_4\} \), and \( H^2 \) is generated by \( \{dt_i \wedge dt_j, \ i < j\} \). Let

\[
\begin{align*}
x_1 &= dt_1 \wedge dt_2, & y_1 &= dt_3 \wedge dt_4, \\
x_2 &= dt_1 \wedge dt_3, & y_2 &= dt_4 \wedge dt_2, \\
x_3 &= dt_1 \wedge dt_4, & y_3 &= dt_2 \wedge dt_3.
\end{align*}
\]

Then the nonzero relations are \( x_i \cdot y_i = y_i \cdot x_i = 1 \), and \( \Gamma = 3U \).

In the following, we define certain automorphisms of \( \Gamma \) by listing the non-invariant terms

\[
\begin{align*}
n_i : & \quad x_i \mapsto -x_i, \quad y_i \mapsto -y_i, \\
s_i : & \quad x_i \mapsto y_i, \quad y_i \mapsto x_i, \\
p_{ij} (= p_{ji}) : & \quad x_i \mapsto x_j, \quad y_i \mapsto y_j, \quad x_j \mapsto x_i, \quad y_j \mapsto y_i, \\
a_{ij} : & \quad x_i \mapsto x_i + x_j, \quad y_j \mapsto y_j - y_i
\end{align*}
\]

Wall showed in [20] that \( A(3U) \) is generated by \( n_i, s_i, p_{ij} \) and \( \alpha_{ij} \).

If \( A = (a_{ij}) \in SL(4, \mathbb{Z}) \) which induces automorphism \( A(dt_j) = \sum a_{ij} dt_i \) of \( H^1 \), then

\[
\Lambda^2 A(dt_i \wedge dt_j) = \sum_{k,l} a_{ki}a_{lj} dt_k \wedge dt_l = \sum_{k<l} (a_{ki}a_{lj} - a_{li}a_{kj}) dt_k \wedge dt_l.
\]

Let \( C = \Lambda^2 A = (p_{kl,ij}) \in A(3U) \). Then we have relations

\[
P_{kl,ij} : p_{kl,ij} = a_{ki}a_{lj} - a_{li}a_{kj}.
\]

Note that \( p_{kl,ij} = -p_{lk,ij} = -p_{kl,jl} \). If we choose

\[
A = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
the corresponding $\Lambda^2 A$ are $n_1 n_2, s_1 s_2, p_{12} n_1 (= n_2 p_{12})$ and $\alpha_{12}$ respectively. By symmetry, $n_1 n_2, s_1 s_2, p_{ij} n_i, \alpha_{ij}, s_i \alpha_{ij} s_i, s_j \alpha_{ij} s_j$ are also in the image of $\Lambda^2$. Let $N$ be the subgroup of $A(3U)$ generated by these elements. So $N \subset D(T^4)$.

Using the following relations (i, j, k distinct)

\[ n_i^2 = s_i^2 = p_{ij}^2 = 1, \quad s_i s_j = s_j s_i, \quad n_i n_j = n_j n_i, \quad p_{ik} p_{ij} = p_{jk} p_{ik}, \]

\[ n_i s_t = s_t n_i, \quad n_i p_{ij} = p_{ij} n_j, \quad n_k p_{ij} = p_{ij} n_k, \quad s_i p_{ij} = p_{ij} s_j, \quad s_k p_{ij} = p_{ij} s_k, \]

\[ n_i \alpha_{ij} = \alpha_{ij}^{-1} n_i, \quad n_j \alpha_{ij} = \alpha_{ij}^{-1} n_j, \quad n_k \alpha_{ij} = \alpha_{ij} n_k, \]

\[ s_k \alpha_{ij} = \alpha_{ij} s_k, \quad p_{ij} \alpha_{ij} p_{ij} = \alpha_{ji}, \quad p_{ik} \alpha_{ij} p_{ik} = \alpha_{kj}, \]

we know that $N$ is a normal subgroup of $A(3U)$ of index 4. Hence $[A(3U) : D(T^4)] \leq 4$.

To finish our proof, we only need to show that the automorphisms $n_3, s_3, n_3 s_3$, which give different cosets of $N$, are not in $D(T^4)$. They have the same coefficients in $C(x_1), C(y_1), C(x_2), C(y_2)$, which are $p_{kl,ij} = \delta_{ki} \delta_{lj} - \delta_{kj} \delta_{li}, \quad j \neq \tau(i), \quad \tau = (14)(23) \in S_4$. \hfill (4.1)

We want to use them and relations $P_{kl,ij}$ to give constraints on $a_{ij}$ and show that $C(x_3), C(y_3)$ are also determined. The linear combination $a_{sj} P_{kl,ij} - a_{kj} P_{sl,ij}$ gives new relation

\[ R_{k,l,s,i,j} : a_{lj} p_{ks,ij} - a_{sj} p_{kl,ij} + a_{kj} p_{sl,ij} = 0. \]

(4.1) implies $a_{ij} = 0$ if $l \neq j, \tau(j)$. $P_{ij,ij}, j \neq \tau(i)$ becomes $a_{ii} a_{jj} = 1$. So $a_{ii} = a_{11} = \pm 1$ for any $i$. Now $P_{li,\tau(l)\tau(i)}$ implies $a_{i\tau(i)} = 0$. So $(a_{ij}) = \pm I$. This shows that $C = I$ and $n_3, s_3, n_3 s_3$ are not in the image of $\Lambda^2$.

The following two remarks provide some Lie group insights in the K3 case and the $T^4$ case respectively.

**Remark 4.3.** If $\Gamma$ is indefinite, $\text{Aut}(\Gamma \otimes \mathbb{R})$ has two components and the identity component $\text{Aut}^0(\Gamma \otimes \mathbb{R})$ consists of the automorphisms of spinor norm 1 [3] p.397). For the K3 surface, $D$ is exactly the intersection of $A(\Gamma)$ with $\text{Aut}^0(\Gamma \otimes \mathbb{R})$.

**Remark 4.4.** Here we explain that $D(T^4)$ has finite index in $A(3U)$ via algebraic group theory, which was communicated to us by S. Adams. We refer to the book of Platonov-Rapinchuk [16] for relevant definitions and theorems. Let $\Lambda^2 : SL(4, \mathbb{C}) \rightarrow O(3U, \mathbb{C})$ be the $\mathbb{Q}$-morphism of algebraic groups defined as in the proof of Proposition 4.2 $A = (a_{ij}) \mapsto \Lambda^2 A = (p_{kl,ij})$, where $p_{kl,ij} = a_{ki} a_{ij} - a_{kj} a_{li}$. It is easy to see that the kernel consists of $\pm \text{id}$ (Section 2 in [12]). As $SL(4, \mathbb{C})$
and $O(3U, \mathbb{C})$ have same dimension, the image of $\Lambda^2$ contains a neighborhood of the identity of $O(3U, \mathbb{C})$. The connectedness of $SL(4, \mathbb{C})$ then implies that image of $\Lambda^2$ is $O^0(3U, \mathbb{C})$, the identity component of $O(3U, \mathbb{C})$. By Theorem 4.13 in [16], p. 213 (or Theorem 4.14, p. 220), $SL(4, \mathbb{Z})$ is a lattice in $SL(4, \mathbb{R})$ (i.e., a discrete subgroup such that $SL(4, \mathbb{R})/SL(4, \mathbb{Z})$ has finite invariant volume). Theorem 4.1 on page 204 of [16] then implies that $\Lambda^2(SL(4, \mathbb{Z}))$ is an arithmetic subgroup of $O(3U, \mathbb{C})$, i.e., $\Lambda^2(SL(4, \mathbb{Z})) \cap A(3U) = \Lambda^2(SL(4, \mathbb{Z}))$ has finite index in $A(3U)$.

4.3. Known non-Kähler CY surfaces. The only known examples of non-Kähler CY are $T^2$–bundles over $T^2$.

If further, $A(\Gamma)$ is infinite, then according to the table above, $M$ is a $T^2$–bundles over $T^2$ with $b^+ = b^- = 2$. Such a manifold is a so-called Kodaira-Thurston manifold.

As a $T^2$ bundle over $T^2$, $M$ is described by a triple

$$\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, I, (0, 0) \},$$

where the first two matrices in $SL(2, \mathbb{Z})$ are monodromies and the third term denotes Euler numbers.

**Proposition 4.5.** If $M$ is a $T^2$–bundle over $T^2$ with $b^+ = 2$, then $D(M)$ is of infinite index.

**Proof.** It is convenient to use the following description of $M$. Let $G = Nil^3 \times E^1$ act on $X = \mathbb{R}^4$ from left as

$$(x_0, y_0, z_0, t_0)(x, y, z, t) = (x_0 + x, y_0 + y, z_0 + z + \lambda x_0 y, t_0 + t)$$

and $L$ is a discrete subgroup of $G$ generated by

$$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1).$$

$M$ is then the quotient $L \backslash X$. $G$–invariant 1–forms are generated by

$$dx, dy, dz - \lambda y dx, dt,$$

and $H^1 = \langle dx, dy, dt \rangle$. $H^2$ is generated by

$$F_1 = dx \wedge dt, \quad F_2 = dy \wedge (dz - \lambda y dx), \quad F_3 = dy \wedge dt, \quad F_4 = dz \wedge dx$$

with $F_1 \cdot F_2 = F_3 \cdot F_4 = 1$. Hence $\Gamma = 2U$.

Observe that $\text{Im}(H^1 \times H^1 \overset{\wedge}{\to} H^2) = \langle F_1, F_3 \rangle$ is 2–dimensional isotropic subspace of $H^2$ invariant under $D(M)$. By Lemma 2.5, $D(M)$ has infinite index.

$\square$
We can also show that $D(M)$ is infinite. Let $\alpha = (x, y)$. For any $T \in GL(2, \mathbb{Z})$, there exists a matrix $B \in M_{2 \times 2}(\mathbb{Q})$ such that the map $\phi_{T} : \mathbb{R}^{4} \to \mathbb{R}^{4}$ defined as

$$\phi_{T}(\alpha, z, t) = (\alpha T, \det(T)z + \alpha B\alpha^{t}, t)$$

preserves $L$. So $D(M)$ is infinite for any $\lambda$.

Acknowledgment. The authors appreciate useful discussions with Scot Adams. The research for the first named author is partially supported by NSFC Grant 10990013. The research for the second named author is partially supported by NCTS postdoctoral fellowship. The research for the third named author is partially supported by NSF.

References

[1] S. Bauer, Almost complex 4-manifolds with vanishing first Chern class, J. Diff. Geom., 79 (2008), no. 1, 25–32.
[2] S. Donaldson, Polynomial invariants for smooth four-manifolds, Topology 29 (1990), 257-315.
[3] R. Friedman and J. Morgan, On the diffeomorphism types of certain algebraic surfaces I, J. Diff. Geom., 27 (1988), 297–369.
[4] R. Friedman and J. Morgan, Smooth four-manifolds and complex surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 27, Springer-Verlag, Berlin, 1994.
[5] R. Friedman and J. Morgan, Algebraic surfaces and Seiberg-Witten invariants, J. Algebraic Geom., 6 (1997), 445–479.
[6] T. J. Li, Symplectic Calabi-Yau surfaces. Handbook of geometric analysis, No. 3, 231–356, Adv. Lect. Math., 14, Int. Press, Somerville, MA, 2010.
[7] T. J. Li, Symplectic 4-manifolds with Kodaira dimension zero, J. Differential Geometry 74 (2006) 321-352.
[8] B. H. Li and T. J. Li, Minimal genus smooth embeddings in $S^{2} \times S^{2}$ and $\mathbb{C}P^{2} \# n\overline{\mathbb{C}P^{2}}$ with $n \leq 8$, Topology, 37 (1998), no. 3, 575–594.
[9] B. H. Li and T. J. Li, Symplectic genus, minimal genus and diffeomorphisms, Asian J. Math., 6 (2002), no. 1, 123–144.
[10] B. H. Li and T. J. Li, On the diffeomorphism groups of rational and ruled 4-manifolds, J. Math. Kyoto. Univ., 46 (2006), no. 3, 583–593.
[11] T. J. Li and A. K. Liu, Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with $b^{+} = 1$, J. Diff. Geom., 58 (2001), no. 2, 331–370.
[12] T. J. Li and A. Tomassini, Almost Kähler structures on four dimensional uni-modular Lie algebras, J. Geom. Phys., 62 (2012), no. 7, 1714–1731.
[13] A. K. Liu, Some new applications of the general wall crossing formula, Math. Res. Letters 3 (1996), 569-585.
[14] M. Lönne, On the diffeomorphism groups of elliptic surfaces, Math. Ann. 310 (1998), no. 1, 103–117.
[15] H. Ohta and K. Ono, Notes on symplectic 4-manifolds with $b^{+}_{2} = 1$, II. Internat. J. Math. 7 (1996), no. 6, 755-770.
[16] V. Platonov and A. Rapinchuk, Algebraic Groups and Number Theory, Translated by R. Rowen, Academic Press, Inc., 1994.

[17] C. Taubes, The Seiberg-Witten invariants and symplectic forms, Math. Res. Lett. 1 (1994), no. 6, 809-822.

[18] C. Taubes, SW ⇒ Gr: from the Seiberg-Witten equations to pseudoholomorphic curves, J. Amer. Math. Soc. 9 (1996), no. 3, 845–918.

[19] C. T. C. Wall, On the orthogonal groups of unimodular quadratic forms, Math. Ann., 147 (1962), 328-338.

[20] C. T. C. Wall, On the orthogonal groups of unimodular quadratic forms. II, J. reine angewandte math., 213 (1963), 122-136.

[21] C. T. C. Wall, Diffeomorphisms of 4-manifolds, J. London Math. Soc., 39 (1964), 131–140.

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China  
E-mail address: daibo@math.pku.edu.cn

National Center for Theoretical Sciences, Math. Division, Hsinchu, Taiwan  
E-mail address: cihoh@math.cts.nthu.edu.tw

Department of Mathematics, University of Minnesota, Minneapolis, MN 55455  
E-mail address: tjli@math.umn.edu