On choice of connection in loop quantum gravity

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Abstract

We investigate the quantum area operator in the loop approach based on the Lorentz covariant hamiltonian formulation of general relativity. We show that there exists a two-parameter family of Lorentz connections giving rise to Wilson lines which are eigenstates of the area operator. For each connection the area spectrum is evaluated. In particular, the results of the su(2) approach turn out to be included in the formalism. However, only one connection from the family is a spacetime connection ensuring that the 4d diffeomorphism invariance is preserved under quantization. It leads to the area spectrum independent of the Immirzi parameter. As a consequence, we conclude that the su(2) approach must be modified accordingly to the results obtained since it breaks one of the classical symmetries.

1 Introduction

Recently, a new Lorentz covariant approach to loop quantum gravity was proposed [1, 2]. In its framework the Immirzi parameter problem [3] arising in the standard su(2) approach can be solved. Namely, it has been shown that i) the path integral [1], ii) the area spectrum derived from a modified loop approach [2] are independent of the Immirzi parameter. The key point of the latter derivation was the possibility to define Wilson line operators by use of a connection different from the canonical one. This is a general ambiguity of the loop approach to quantum gravity independently on the gauge group used. In fact, in the case of the Lorentz gauge group we have to use this ambiguity since the canonical Lorentz connection leads to the area operator nondiagonal on the states created by the Wilson lines. Therefore, to obtain the area spectrum one should search for a shifted connection for that the Wilson lines would be eigenstates of the area. In principle, one should take into account all possible choices of the connection and investigate the arbitrariness arising in the spectrum. In [2] it has been shown that if we require that the shift vanishes on the constraint surface there is a unique Lorentz connection diagonalizing the area operator. The resulting spectrum does not depend on the Immirzi parameter \( \beta \) and it is given by the values of two Casimir operators

\[
S = 8\pi\hbar G \sqrt{C(so(3)) - C_1(so(3,1))}.
\]
However, one can ask: what will we obtain if we do not impose the condition of coinciding of the canonical and shifted connections on the constraint surface? Is the found connection unique in this case? Or, if not, what spectra do another connections give?

The present paper is aimed to answer all these questions. In the next section we show that, indeed, there exists a two-parameter family of Lorentz connections leading to the area operator diagonal on the Wilson lines. In Sec. III we derive the corresponding spectra of the area operator. In Sec. IV we impose an additional physical condition on the connection to be used: it should be a spacetime connection, i.e. it should properly transform under all 4d diffeomorphisms. As a result, we arrive to a unique connection, which has been already found in the previous paper \cite{2} and, accordingly, to the area spectrum \cite{1} independent of the Immirzi parameter. In Sec. V we find that the standard results of the su(2) approach (for review, see \cite{4}) have a Lorentz covariant extension and are included in the present formalism. We discuss the meaning of the results obtained in the previous section for the su(2) approach and argue that it breaks one of the symmetries of the classical theory. The last section concludes the paper with some remarks. In appendix A one can find the basic ingredients of the covariant hamiltonian formulation. Appendix B is devoted to the investigation of the time diffeomorphisms in the canonical approach.

We use the following notations for indices. The indices \(i, j, \ldots\) from the middle of the alphabet label the space coordinates. The latin indices \(a, b, \ldots\) from the beginning of the alphabet are the \(so(3)\) indices, whereas the capital letters \(X, Y, \ldots\) from the end of the alphabet are the \(so(3, 1)\) indices.

\section{Lorentz connections in the covariant approach}

The aim of this section is to find all Lorentz connections \(A_i^X\) such that the states created by the corresponding Wilson lines

\[
U_\alpha[A] = \mathcal{P} \exp \left( \int_a^b dx^i A_i^X T_X \right)
\]

are eigenstates of the area operator. \(T_X\) is a Lorentz generator in an irreducible representation of \(so(3, 1)\). (For all details concerning the Lorentz covariant formalism we refer to the paper \cite{2} and appendix A.)

We suppose that, as in the su(2) case, the states forming the physical Hilbert space are some kind of spin network states such that their links are associated with the Wilson lines \(2\). Then our requirement means that these new “spin networks” should be eigenstates of the area. This goes again along the usual approach.

The quantum area operator is defined in terms of the so called smeared triad operators in the following way

\[
S = \lim_{\rho \to \infty} \sum_n \sqrt{g(S_n)}, \quad g(\Sigma) = g^{XY} \tilde{P}_X(\Sigma) \tilde{P}_Y(\Sigma),
\]

\[
\tilde{P}_X(\Sigma) = \int_\Sigma d^2 \sigma n_i(\sigma) \tilde{P}_X^i(\sigma),
\]

where we used a partition of a measured surface \(S = \bigcup_n S_n\). \(n_i = \varepsilon_{ijk} \frac{\partial x^j}{\partial \sigma} \frac{\partial x^k}{\partial \sigma}\) is the normal to the surface. The necessary condition for the operator \(3\) to be diagonal on the states \(2\) requires that the commutator of the connection \(A_i^X\) and the triad multiplet \(\tilde{P}_X^i\) is proportional to \(\delta^i_j\) \(\tilde{P}_Y^j\). This gives the first condition on the connection to be found. Other conditions arise from the fact that to be a Lorentz connection it must transform in a proper way under the
gauge as well as diffeomorphism transformations. Thus we arrive to the following list of requirements:

\[ (5) \quad \{ \mathcal{G}(n), \mathcal{A}^X_i \}_D = \partial_i n^X - f_{YZ}^X n^Y \mathcal{A}^Z_i, \]

\[ (6) \quad \{ \mathcal{D}(\bar{N}), \mathcal{A}^X_i \}_D = \mathcal{A}^X_j \partial_i N^j + N^j \partial_j \mathcal{A}^X_i, \]

\[ (7) \quad \{ \mathcal{A}^X_i, \bar{P}^j_Y \}_D \sim \delta^j_i. \]

Let us find all quantities satisfying the conditions (5)-(7). They can be constructed from both the initial canonical connection \( A^X_i \) and the triad multiplets. First of all, by dimensional reasons we can restrict ourselves to the quantities which are linear in the connection \( A^X_i \) or in the spatial derivative \( \partial_i \). Then one can show that it is impossible to construct a connection or a vector under the gauge transformations also satisfying the condition (6) from terms of the second type only. This means that terms linear in the canonical connection uniquely define the shifted connection. (Pure triad terms commute with \( \bar{P}^j_Y \) and would not contribute to Eq. (7).)

If to write down all possible structures linear in \( A^X_i \) one obtains a 12-parameter family of such terms. One can note that all of them satisfy Eq. (5). Then the condition (6) reduces the number of independent parameters to 4. However, the resulting quantity has anomalous contributions in the gauge transformations. To cancel them we should add some terms constructed only from the triad multiplets such that they themselves satisfy Eq. (5). The cancellation turns out to be possible and imposes additional two restrictions on the parameters. As a result, we end up with a two-parameter family of Lorentz connections diagonalizing the area operator

\[ A^X_i = A^X_i + \frac{1}{2} \left( 1 + \frac{a}{\beta} \right) g^{XX'} - \frac{1}{\beta} (1 - b) \Pi^{XX'} \right) I_{(Q)X'}^T \frac{R_Z^Y}{1 + \frac{1}{\beta} f_{ZW}^Y P_i^W \mathcal{G}_Y}

+ \left( a g^{XX'} + b \Pi^{XX'} \right) \left( I_{(Q)X'}^T \mathcal{A}^W_i - I_{(Q)X'}^{R W Y Z} Q_i^Y \partial_i Q_L^Z + \frac{1}{2} f_{W Y Z}^{R K L} Q_i^K \partial_i Q_L^Z \right) \Pi_{W R}. \]

What can we say about these connections? First of all, one can calculate the following Dirac brackets which turn out to be extremely simple:

\[ \{ \mathcal{A}^X_i, \bar{P}^j_Y \}_D = \delta^j_i \left( 1 + b \right) \delta_{X'}^X + a \Pi_{X'}^X \right) I_{(P)Y}^X \]

\[ \{ \mathcal{A}^X_i, P^j_Y \}_D = - \left( 1 + b \right) \delta_{X'}^X + a \Pi_{X'}^X \right) P^X_j P^Y_i \]

\[ \{ \mathcal{A}^X_i, I_{(P)}^Z \}_D = 0. \]

The remarkable feature of these relations is that the projectors \( I_{(P)} \) and \( I_{(Q)} \) behave like \( c \)-numbers with the point of view of the bracket algebra. Due to this behavior from the Jacobi identity it follows

\[ \{ \{ \mathcal{A}^X_i, A^Y_j \}_D, \bar{P}^k_Z \}_D = 0. \]

This means that the Dirac bracket of two shifted connections does not depend on the connection itself and is a function of the triad multiplet and its derivatives only. Therefore, if we quantize the system replacing the Dirac brackets by the quantum commutators, there will be no ordering ambiguity in this relation since in the right hand side there are only commuting objects.
3 Area spectrum

Let us consider the action of the area operator on a Wilson line (2) where the connection \( \mathcal{A}_i^X \) is taken from Eq. (8). Due to the simplicity of the new quantization rules (9) the action does not depend on the embedding and is given by a matrix operator:

\[
S \hat{U}_\alpha[A] = \hbar \hat{U}_\alpha[A] \sqrt{\Delta_S \hat{U}_\alpha[A]},
\]

\[
\Delta_S = \left( a^2 I^{XY}_{(Q)} - (1 - b)^2 I^{XY}_{(P)} - a(1 - b)\Pi^{XY} \right) T_X T_Y.
\]

(13)

One can show [2]

\[
g^{XY} T_X T_Y = C_1(\text{so}(3,1)),
\]

(14)

\[
\Pi^{XY} T_X T_Y = C_2(\text{so}(3,1)),
\]

(15)

\[
I^{XY}_{(Q)} T_X T_Y = C(\text{so}(3)).
\]

(16)

Thus, the quantity in the square root is nothing else but a linear combination of these Casimir operators:

\[
\Delta_S = (a^2 + (1 - b)^2)C(\text{so}(3)) - (1 - b)^2C_1(\text{so}(3,1)) - a(1 - b)C_2(\text{so}(3,1)).
\]

(17)

As a result, the area operator turns out to be diagonalizable and its spectrum is given by the eigenvalues of the Casimir operators (14)-(16) [5]:

\[
C_1(\text{so}(3,1)) = n^2 + \mu^2 - 1,
\]

(18)

\[
C_2(\text{so}(3,1)) = -2i\mu n, \quad 2n \in \mathbb{N}, \ \mu \in \mathbb{C},
\]

(19)

\[
C(\text{so}(3)) = j(j + 1), \quad (j - n) \in \mathbb{N}.
\]

(20)

Let us restrict ourselves to the unitary representations of the Lorentz group. It is natural since, due to the Plancherel theorem, they are sufficient to span all square-integrable functions on the group [8]. Therefore, we expect that only these representations will appear in the basis elements of the Hilbert space which is still to be found. There are two series of such representations. The **principal series** is described by purely imaginary \( \mu = i\rho, \ \rho \in \mathbb{R} \) and the **supplementary series** corresponds to the case of \( n = 0, \ \mu = \rho, \ -1 < \rho < 1 \). Therefore, if we use the unitary representations to generate physical states, the area spectrum is given by

\[
S^{pr} \sim \hbar \sqrt{(a^2 + (1 - b)^2)j(j + 1) + (1 - b)^2(\rho^2 - n^2 + 1) - 2a(1 - b)n\rho},
\]

(21)

\[
S^{sup} \sim \hbar \sqrt{(a^2 + (1 - b)^2)j(j + 1) + (1 - b)^2(1 - \rho^2)}.
\]

(22)

The remarkable property of the obtained spectra is that due to the conditions \( j \geq n \) and \( |\rho| < 1 \) the expressions in the square root in Eqs. (21) and (22) are strictly positive and the both spectra are always real. This is a different situation from that in Lorentzian spin foam models where an imaginary spectrum appears [8, 7]. The reason what makes the area operator well defined can be traced out to the appearance of the Casimir of the \( \text{SO}(3) \) subgroup in the operator (17). It cancels the negative contribution from the Lorentz Casimir. Therefore, one can think that its presence is essential. And, indeed, it is impossible to obtain the spectrum given by the Casimirs of the Lorentz group only varying the parameters \( a \) and \( b \).
4 Spacetime properties

We still have the two-parameter family of Lorentz connections (8) and results for the area spectrum (17). Therefore, we should impose an additional physical condition to select the correct one. Our previous analysis was based on symmetries: we required that the symmetries of the classical theory are preserved under quantization. This requirement was expressed in terms of transformation properties of the connection used in the definition of the Wilson line. However, if we look at the conditions (5) and (6) we can see that one symmetry was missing so far. Namely, we did not consider transformation properties of our connection under the time diffeomorphisms. In fact, only the connection with the right transformation properties under all 4d diffeomorphisms can lead to quantum theory independent on the foliation, since only in this case the Wilson line can be generalized to the line in 4d space.

Thus we arrive to the necessity to impose the following condition on the connections (8):

\[ iv \) \delta(\xi^0) A^X_0 = \xi^0 \partial_0 A^X_0 + A^X_0 \partial_0 \xi^0, \]

where \( A^X_0 \) is the corresponding generalization of the time component of the initial connection \( A^i_0 \). We will call the connections satisfying Eq. (23) spacetime connections.

First of all, one should find the generator of the time diffeomorphisms expressed in terms of the constraints. It is given by the full Hamiltonian:

\[ D_0(\xi^0) = \int d^3 x \xi^0 \left( N^X_G\mathcal{G}_X + N^i_D \mathcal{D}_i + \sim \mathcal{N} H \right), \]

where

\[ N^X_G = A^X_0 - N^i A^i_X, \]

\[ \mathcal{D}_i = H_i + A^i_X \mathcal{G}_X = \partial_j \left( A^X_i \tilde{P}_{(\beta)}^j \right) - \tilde{P}_{(\beta)}^j \partial_i A^X_i. \]

This expression was found in [8] in the case of the Ashtekar gravity. In the given case it is still valid, but there appear additional subtleties. They are discussed in appendix B where the action of the generator (24) is investigated. It is shown that on the surface of equations of motion and the Gauss constraint the family (8) contains only one Lorentz spacetime connection. It is obtained at vanishing parameters \( a \) and \( b \) and it is given by the following expression:

\[ A^X_i = A^i_X + \frac{1}{2 (1 + \frac{1}{\beta^2})} R^X_i R^{ST}_{(q)} R^Z_{T Z W} \bar{P}^W_i \mathcal{G}_Y. \]

It is quite expected that it coincides with the initial canonical connection on the constraint surface since \( A^X \) is definitely spacetime connection. From this it is clear that all other connections from the family (8) are not spacetime since they do not coincide with \( A^X \) on the constraint surface.

Remarkably, the connection (27) coincides with the one found in [2]. As a result, the corresponding spectrum of the area operator is given by Eq. (1) and does not depend on the Immirzi parameter.

The origin of this independence can be traced out to the independence of \( \beta \) of the commutator of the triad multiplet with the shifted connection

\[ [A^X_i, \tilde{P}^j_Y] = i\hbar \delta^X_i I_{(p)}^{X Y}. \]

This allows to hope that no dependence on the Immirzi parameter will appear in all physical quantities.
5 Results for the su(2) approach

Note, that if we choose the parameters $a$ and $b$ in Eqs. (21), (22) to be as follows

$$a = -\beta, \quad b = 1 \quad (29)$$

we reproduce the area spectrum found in loop quantum gravity [9, 10]

$$\mathcal{S} \sim \hbar \beta \sqrt{j(j+1)}. \quad (30)$$

Moreover, the connection used in this case is a Lorentz generalization of the standard su(2) connection. Indeed, from Eq. (8) with (29) we find

$$A^X_i = I^X_{(Q)Y} (\delta^Y_Z - \beta \Pi^Y_Z) A^Z_i$$

$$- \beta R^{XX'} \Pi_{WR} \left( f^{WYZ} Q^Y_i \tilde{Q}^Z_i - \frac{1}{2} f^{WYZ} \tilde{Q}^Y_i Q^Z_i \partial_k Q^k R \right). \quad (31)$$

It is easy to see that the second term is proportional to the field $\chi$ which vanishes in the time gauge and in the rest one can recognize the Ashtekar-Barbero connection [11]

$$A^X_i = \left( 0, \frac{1}{2} \varepsilon^{abc} \omega^i_c - \beta \omega^0_i \right). \quad (32)$$

Thus the standard results of the su(2) approach are included in the developed formalism. This fact allows to make some conclusions about the validity of the su(2) approach with the point of view of the covariant quantization.

First of all, it is clear that to restore the full Lorentz symmetry we have to work with the Lorentz extension (31) of the Ashtekar-Barbero connection. Otherwise the symmetry is explicitly broken. The connection (31) possesses a remarkable property. In the time gauge it turns out to be commutative like its gauge fixed counterpart in the su(2) approach. Indeed, the first term in Eq. (31) can be rewritten as

$$- \beta \left( 1 + \frac{1}{\beta^2} \right) \Pi^X_{YZ} \left( R^{-1} \right)^Z_W A^W_i, \quad (33)$$

where $A^X_i$ is the connection from Eq. (27). One can show by explicit calculations that the quantity (33) is commutative. Then due to Eq. (11), which means $\{A^X_i, \chi^a\}_D = 0$, the commutator $\{A^X_i, A^Y_j\}_D$ is proportional to $\chi$ and vanishes in the time gauge.

However, despite this attractive property the analysis of the previous section tells us that the quantization based on the connection (31) spoils the 4d diffeomorphism invariance. The Immirzi parameter problem arising in this case can be viewed just as a reflection of the break of this symmetry. Therefore, the su(2) approach cannot be considered as a correct quantization of gravity. There is only one way in the framework of loop approach to quantize general relativity preserving all symmetries of the classical theory and it is given by Eqs. (27) and (28).

1First, the observation that the Ashtekar-Barbero connection is not a spacetime connection has been made by J. Samuel [12]. This work is partially inspired by his ideas.
6 Conclusion

In this paper we analyzed arbitrariness in the loop quantization of general relativity in a manifestly Lorentz covariant formalism. This arbitrariness arises due to the possibility to use different connections in the definition of the Wilson line operators which are supposed to create physical states. Imposing the requirements that the used connection is a Lorentz connection and diagonalizes the area operator, we found a two-parameter family of such connections and the corresponding area spectra. Then we further restricted admissible quantities requiring them to be spacetime connections, i.e. to transform in a proper way under all 4d diffeomorphisms. This condition is sufficient to select a unique connection satisfying all requirements. We argue that the loop quantization should be based on the Wilson lines defined by this spacetime Lorentz connection, since only in this way one can preserve all classical symmetries.

An important byproduct of our analysis is that the standard su(2) approach cannot represent a correct quantization of gravity. We can make such a conclusion since we found a Lorentz covariant extension of the results of this approach. In particular, the Ashtekar-Barbero connection can be extended to a Lorentz connection which turns out to be in the mentioned two-parameter family. The corresponding area spectrum coincides with the usual one. However, this connection is not a spacetime connection and breaks the diffeomorphism invariance in quantum theory. The appearance of the Immirzi parameter in the spectrum of the area operator is a reflection of this phenomenon.

In this sense the spectrum we arrived from the found spacetime connection is remarkable. It cures all problems arising in different approaches to quantum gravity: 1) it does not suffer from the Immirzi parameter problem, 2) it is always real (see Eqs. (21), (22)) in contrast to the results in Lorentzian spin foam models, 3) even the trivial representation gives a nonzero value of area (cf. Eq. (30)).

We should emphasize that the presented analysis is essentially quantum despite of the lack of any information about the structure of the Hilbert space. It repeats step by step what has been done in for the case of the su(2) approach. The only made assumption is that the Wilson lines correspond to physical states and they are eigenstates of the area operator. All other is a consequence of the requirement to preserve the diffeomorphism and local Lorentz symmetries under quantization.

Of course, a lot of problems have to be solved yet. First of all, one should modify the Hilbert space structure found in the su(2) approach to the case of the Lorentz gauge group. May be the most important question at the moment is which representations should be taken into account. The answer on this question will allow to conclude what is the final form of the area spectrum which is derived in terms of Casimir operators only. The related problem is whether the spectrum is discrete or continuous. And, of course, it is necessary to find it to start the derivation of the black hole entropy in the new setup. Nevertheless, the obtained results can give us a guess how the Hilbert space should look like. For example, they imply that the Wilson lines should be somehow restricted to a definite representation of SO(3) subgroup. Moreover, the restriction must do not break the Lorentz invariance. All this can give important consequences for the structure of the resulting Hilbert space and it will be the subject of the subsequent work. Also, it would be interesting to understand the role of the Gauss constraint in recovery of spacetime transformation properties from the canonical formulation. May be, the most interesting application this analysis can find in spin foam models which also claim for Lorentz covariant quantization of general relativity.
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A Covariant Hamiltonian formulation

In this appendix we give a short summary of the Lorentz covariant canonical formulation. Its complete description can be found in [1].

The canonical formulation is obtained by means of the 3 + 1 decomposition:

\[ e^0 = N dt + \chi_a E^a_i dx^i \quad e^a = E^a_i dx^i + E^a_i N^i dt \]  

(34)

The field \( \chi_a \) describes the deviation of the normal to the spacelike hypersurface \( \{ t = 0 \} \) from the time direction. The usually used time gauge corresponds to \( \chi = 0 \).

The decomposed action is given by

\[ S(\beta) = \int dt d^3 x (\hat{P}_{(\beta)}^i X \partial_0 A^X_i + A^X_i G_X + N^i_b H_i + \tilde{N} H), \]

\[ G_X = \partial_i \hat{P}_{(\beta)}^i X + f_\beta^Z Y A^Y_i \hat{P}_{(\beta)}^i Z, \]

\[ H_i = -\hat{P}_{(\beta)}^j X F^X_{ij}, \]

\[ H = -\frac{1}{2 \left(1 + \frac{1}{\beta^2}\right)} \hat{P}_{(\beta)}^i X \hat{P}_{(\beta)}^j Y f_\beta^Z X Y R^Z_{XY} F^W F_{ij}, \]

\[ F^X_{ij} = \partial_i A^X_j - \partial_j A^X_i + f_\beta^X Y A^Y_i A^Z_j, \]

where the following 3 x 6 matrix fields are introduced:

\[ A^X_i = (\omega_i^0 a, \frac{1}{2} \varepsilon_{abc} \omega_i^a) \quad \text{connection multiplet}, \]

\[ \hat{P}_{(\beta)}^i X = (\hat{P}_{\beta}^i a, \varepsilon_{abc} \hat{P}_{\beta}^i c) \quad \text{first triad multiplet}, \]

\[ \hat{Q}_X = (-\varepsilon_{abc} \hat{E}_b^a X, \hat{E}_a^i X) \quad \text{second triad multiplet}, \]

\[ \hat{P}_{(\beta)}^i X = \hat{P}_{X}^i + \frac{1}{\beta} \hat{Q}_X \quad \text{canonical triad multiplet}, \]

\[ X, Y, \ldots = 1, \ldots, 6 \quad i, j, \ldots = 1, \ldots, 3. \]

The fields \( \hat{Q} \), \( \hat{P} \) and \( \hat{P}_{(\beta)} \) form multiplets in the adjoint representation of the Lorentz algebra and \( A \) is the true Lorentz connection. The triad multiplets have a clear interpretation. In the first triad multiplet one can recognize the spatial components of the so called \( B \)-field, \( B = e \wedge e \) (see, e.g. [1]), whereas the second triad multiplet is composed of its Hodge dual \( \star B \). The canonical multiplet is introduced since just this linear combination of \( \hat{Q} \) and \( \hat{P} \) plays the role of the momentum conjugated to \( A \). It is clear that all triad multiplets are related by simple numerical matrices:

\[ \hat{P}_{X}^i = \Pi^Y_X \hat{Q}_Y^i, \quad \Pi_Y^X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^b_a, \]  

(37)

\[ \hat{P}_{X}^i = \frac{R_{X}^Y}{1 + \frac{1}{\beta^2}} \hat{P}_{(\beta)}^i Y, \quad R_X^Y = \begin{pmatrix} 1 & -1 \\ \frac{1}{\beta} & 1 \end{pmatrix} \delta^b_a. \]  

(38)
Therefore, the use of this or that multiplet is the matter of convenience. The metric in Lorentz
indices is given by the Killing form:

\[
g_{XY} = \frac{1}{4} f_{XZ} f_{Y}^{Z}, \quad g^{XY} = (g^{-1})^{XY}, \quad g_{XY} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & -\delta_{ab} \end{pmatrix}.
\] (39)

Besides the first class constraints \(G_{X}, H_{i}\) and \(H\), in the theory \([12]\) there are the second
class constraints:

\[
\begin{align*}
\phi^{ij} &= \Pi^{XY} \tilde{Q}_{X}^{j} \tilde{Q}_{Y}^{i}, \\
\psi^{ij} &= 2 f_{XYZ} \tilde{Q}_{X}^{i} \tilde{Q}_{Y}^{j} \tilde{Q}_{Z} \partial_{i} \tilde{Q}_{Z}^{j} - 2(\tilde{Q} \tilde{Q})^{ij} \tilde{Q}_{Z} A_{i}^{Z} + 2(\tilde{Q} \tilde{Q})^{ij} \tilde{Q}_{Z} A_{i}^{Z} = 0,
\end{align*}
\] (40)

They give rise to a nontrivial Dirac bracket. This bracket has an important property: when
one of its arguments is a first class constraint it coincides with the usual Poisson bracket.
The only exception is the case of the Hamiltonian constraint. Then if the second argument
depends on the connection there is no coincidence.

The Dirac brackets of the canonical variables are

\[
\begin{align*}
\{\tilde{P}_{(b)}^{i}, \tilde{P}_{(b)}^{j}\}_{D} &= 0, \\
\{A_{i}^{X}, \tilde{P}_{(b)}^{j}\}_{D} &= 2 \partial_{j} \tilde{Q}_{X}^{j} - \frac{1}{2} R^{XY} \left( \tilde{Q}_{Z}^{i} \tilde{Q}_{W}^{j} + \tilde{Q}_{Z}^{j} \tilde{Q}_{W}^{i} \right) g_{WY}, \\
\{A_{i}^{X}, A_{j}^{Y}\}_{D} &= -\{A_{i}^{X}, \phi^{kl}\} (D^{-1})_{(kl)(mn)} \{\psi^{mn}, A_{r}^{Z}\} \{\tilde{P}_{(b)}^{r}, A_{j}^{Y}\}_{D} \\
&\quad - \{A_{i}^{X}, \tilde{P}_{(b)}^{r}\}_{D} \{A_{j}^{Y}, \psi^{mn}\} (D^{-1})_{(mn)(kl)} \{\phi^{kl}, A_{j}^{Y}\},
\end{align*}
\] (41)

where we introduced the so called inverse triad multiplets \(Q_{i}^{X}\) and projectors \(I_{(\bar{Q})}^{XY}\) (see \([4]\)).

## B Time diffeomorphisms in canonical formulation

Before to investigate transformation properties of fields under the time diffeomorphisms, one
should check whether the quantity \([24]\), indeed, generates these transformations. To be the
generator, it should satisfy the algebra of the 4d diffeomorphisms \((\mathcal{D} = (\mathcal{D}_{0}, \mathcal{D}_{i})\)):

\[
[\mathcal{D}(\xi), \mathcal{D}(\eta)] = -\mathcal{D}([\xi, \eta]).
\] (42)

This relation can be checked by using the transformation laws of the Lagrange multipliers \(\mathcal{N}^{\alpha} = (\mathcal{N}_{i}^{X}, \mathcal{N}_{i}^{0}, \mathcal{N})\),

\[
\xi^{\beta} \delta_{\beta} \mathcal{N}^{\alpha} = \partial_{\beta} \xi^{\alpha} - C_{\beta\gamma}^{\alpha} \xi^{\gamma}
\] (43)

and the equations of motion

\[
\begin{align*}
\partial_{0} A_{i}^{X} &= \{\mathcal{D}_{0}(1), A_{i}^{X}\}_{D}, \\
\partial_{0} \tilde{P}_{(b)}^{i} &= \{\mathcal{D}_{0}(1), \tilde{P}_{(b)}^{i}\}_{D}.
\end{align*}
\] (44, 45)

Since the constraints \(\mathcal{G}_{\alpha} = (\mathcal{G}_{X}, \mathcal{D}_{i}, \mathcal{H})\) form the usual algebra (coinciding, for example, with
the constraint algebra of the Ashtekar gravity) the result of \([8]\) is still valid and the relation
\([12]\) is fulfilled.

At the same time, one can note that if we take another equivalent set of constraints
\(\tilde{\mathcal{G}}_{\alpha} = (\mathcal{G}_{X}, \mathcal{H}_{i}, \mathcal{H})\) and the corresponding Lagrange multipliers \(\tilde{\mathcal{N}}^{\alpha} = (\tilde{A}_{i}^{X}, \mathcal{N}_{b}^{1}, \mathcal{N})\), we obtain
an anomalous term in the algebra which is proportional to the square of the Gauss constraint. The reason is that due to the noncommutativity of the connection different choices of the Lagrange multipliers lead to inequivalent quantizations. Fixing a Lagrange multiplier we fix the quantity which has vanishing brackets with the canonical variables. So if its redefinition depends on a canonical variable, it can give inequivalent results. In the given case it does happen and the situation turns out to be quite different from that in the Ashtekar gravity, for example.

Now it is straightforward to obtain transformation laws of any field. It is enough to note the simple formula:

\[
\{ D_0(\xi^0), \varphi(A_0, \tilde{P}_{(\beta)}) \}_D = \xi^0 \partial_0 \varphi + d_{\xi^0}[D_0, \varphi],
\]

\[
d_{\xi^0}[D_0, \varphi] = \{ D_0(\xi^0), \varphi \}_D - \xi^0 \{ D_0(1), \varphi \}_D.
\]

It means that only terms with derivatives of \( \xi^0 \) should be taken into account. Then one can obtain

\[
\{ D_0(\xi^0, A^X_0) \}_D = \xi^0 \partial_0 A^X_0 + A^X_0 \partial_0 \xi^0 + d_{\xi^0}[\mathcal{N}H, A^X_0],
\]

\[
d_{\xi^0}[\mathcal{N}H, A^X_0] = -\frac{N \partial_j \xi^0}{2(1+\frac{1}{\beta^2})} R^X (\delta^j_i I^S_{(Q)} + Q^T_i Q^j W^W - Q^S_i Q^j W^T) R^T G^Z.
\]

The appearance of the anomalous term is little bit strange. One can try to cancel it by a redefinition of the Lagrange multiplier \( \mathcal{N}^X_0 \) so that the Hamiltonian constraint get an additional term proportional to \( \mathcal{G}_X \). Then according to the reasoning in the previous paragraph it could change transformation properties. However, it turns out that only the diagonal terms \( \sim \delta^j_i I^S_{(Q)} \) can be canceled in this way. Thus we conclude that the Gauss constraint should be used to recover the spacetime form of the time diffeomorphisms.

Then it is clear that only the connection obtained at \( a = b = 0 \) from the family \( \mathcal{F} \) satisfy the condition \( (23) \). Indeed, the time diffeomorphisms of all other connections contain a lot of additional terms nonvanishing on the constraint surface. (They contain even terms with the second derivative of \( \xi^0 \).) We do not give explicit expressions since they are not used.

**References**

[1] S. Alexandrov, Class. Quantum Grav. 17, 4255 (2000) [gr-qc/0005085].

[2] S. Alexandrov and D. Vassilevich, Phys. Rev. D 64, 044023 (2001) [gr-qc/0103105].

[3] G. Immirzi, Nucl. Phys. Proc. Suppl. 57, 65 (1997) [gr-qc/9701052].

[4] M. Gaul and C. Rovelli, *Loop Quantum Gravity and the Meaning of Diffeomorphism Invariance* Lectures given at the 35th Karpacz Winter School on Theoretical Physics: From Cosmology to Quantum Gravity (1999) [gr-qc/9910073].

[5] W. Ruhl, *The Lorentz Group and Harmonic Analysis* (WA Benjamin Inc, New York, 1970).

[6] A. Perez and C. Rovelli, Phys. Rev. D 63, 041501 (2001) [gr-qc/0009021]; 3+1 spinfoam model of quantum gravity with spacelike and timelike components, [gr-qc/0011034].
[7] R. E. Livine and D. Oriti, Barrett-Crane spin foam model from generalized BF type action for gravity, gr-qc/0104043.

[8] M. Reisenberger, Nucl. Phys. B 457, 643 (1995) gr-qc/9505044.

[9] C. Rovelli and L. Smolin, Nucl. Phys. B442, 593 (1995); J. Lewandowski, The operators of quantum gravity lecture at the workshop on canonical and quantum gravity (Warsaw, 1995); S. Frittelli, L. Lehner, and C. Rovelli, Class Quantum Grav. 13, 2921 (1996).

[10] A. Ashtekar and J. Lewandowski, Class. Quantum Grav. 14, A55 (1997).

[11] J. F. Barbero, Phys. Rev. D 49, 6935 (1994); Phys. Rev. D 51, 5507 (1995); Phys. Rev. D 51, 5498 (1995); Phys. Rev. D 54, 1492 (1996).

[12] J. Samuel, Class. Quantum Grav. 17, L141 (2000) gr-qc/0005095; Phys. Rev. D 63, 068501 (2001).

[13] A. Ashtekar and J. Lewandowski, Representation theory of analytic holonomy $C^*$ algebras in Knots and quantum gravity ed. J. Baez (Oxford: Oxford University Press, 1994); J. Geom. Phys. 17, 191 (1995); J. Math. Phys. 36, 2170 (1995).

[14] S. Alexandrov, R. Livine and D. Vassilevich, in progress.