The ’t Hooft coupling and baryon mass splitting in the large-$N$ chiral model

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April 22, 2015

Abstract

We study the ’t Hooft coupling $g_t$ and the mass splitting of the ground-state baryons in a chiral quark model inspired by the large $N_c$ QCD. Depending on $g_t$ the Hartree wavefunction of light quark in baryons is used to map the mass hyperfine splittings for the octet and decuplet baryons. The ’t Hooft coupling $g_t$ that reproduces the data of baryon masses is determined to be around 1.57, which is small in the sense that $g_t^2/(4\pi N_c) \ll 1$.

PACS number(s):12.20.Ds, 11.15.Tk, 14.20.-c,11.27.+d

Key Words: ’t Hooft coupling, Large N, Mass splitting, Baryon

1 Introduction

The ’t Hooft’s idea [1] that Quantum Chromodynamics (QCD) simplifies in the limit of large number of colors, $N_c$, had proven to be a valuable tool for exploring strong interaction[2, 3]. With the $1/N_c$ expansion of QCD, the qualitative explanations can be given for, for instance, the Okubo-Zweig-Iizuka rule, the Regge phenomenology, the soliton picture of baryon[2], the contracted $SU(2n_f)$ spin-flavor symmetry[4] of QCD, and the spin-flavor structure of many baryon properties[5, 6, 7, 8, 9]. The existences of the smooth limit of QCD at $N_c \to \infty$, with $g^2 N_c$ fixed($g$ is the QCD coupling...
constant), has been studied and confirmed via lattice simulation of $SU(N_c)$
gauge theory\[10, 11, 12\] (see also \[13\] for a recent review).

It is hard, however, to say on theoretical grounds whether $1/N_c = 1/3$
can be considered small in real world where $N_c = 3$ without actually solving
QCD for finite $N_c$. This is so because the answer depends crucially on how
large are the coefficients companying with the expanding parameter $1/N_c$,
the ’t Hooft’s coupling $g_t \equiv g^2 N_c$, which is assumed to be fixed to ensure the
$SU(N_c)$ QCD (the one-loop gluon vacuum polarization, for instance) to have
a smooth limit as $N_c$ tends to infinity. As argued recently by Weinberg\[14,\]
the ’t Hooft’s coupling $g_t$ may not be small at moderate energies such as the
scale of $m_\rho$ since the masses of the $\rho$ mesons $m_\rho$ is of order of $\Lambda_{QCD}$
in the chiral limit. Combining the $1/N_c$ expansion of QCD with the early idea of
the chiral theory \[15, 16\], Weinberg proposed a large-$N_c$ chiral theory\[14\] of
quarks, gluons, and pions which is effectively renormalizable to leading order
of $1/N_c$. He suggests that such an effective field theory, with a small value
of the ’t Hooft coupling $g_t$ at moderate energies, may explain why the naive
quark model are so successful. It remains unknown how small $g_t$ could be in
the effective theory.

The purpose of this Letter is to estimate the value of the ’t Hooft coupling
$g_t$ in the quark picture of baryons with pion included. We determine, in
a large-$N_c$ chiral quark model that is inspired by the Weinberg’s large-$N_c$
effective theory, the ’t Hooft coupling by computing the mass splittings of the
ground-state octet and decuplet baryons with the help of Witten’s Hartree
picture and mapping them to the data of baryon masses. The ’t Hooft
coupling $g_t$ is found to be about $g_t \simeq 1.57$. This infer that when pion enters
in the quark picture $g_t$ can be considered small in the sense that $g_t^2/(4\pi N_c) \ll
1$.

The model we utilized here is a potential version of the large-$N_c$ effective
theory\[14\] where the gluon interaction is effectively described by the Hartree
potentials with the form of the coulomb plus string potential. Conceptually,
it resembles the (topological) chiral bag model \[17\] in the sense that the
role of bag is played by the string-like confining potential, with the chiral
symmetry implemented via the quark-pion interaction.

2 Confined quarks in baryon at large $N_c$

Inspired by the large-$N_c$ QCD and the chiral theories\[14\], we use a large-$N_c$
chiral model of baryon\[15, 16\] in which each of quark $q_i (i = 1, \cdots, N_c)$ in
baryon moves in the confining Hartree potential of all $N_c - 1$ others, coupling
with the pion in an invariant way under chiral symmetry. The model is

\[ \mathcal{L}_{\text{Baryon}} = \sum_{i=1}^{N_c} \bar{q}_i [i \gamma^\mu \partial_\mu - (m_q + S) U_5 - \gamma^0 V_G] q_i + \mathcal{L}^x, \]

where \( m_q \) is the effective mass of the light (up or down) quark \( q_i \) and \( (S, V_G, U_5) \) are the Hartree fields experienced by \( q_i \), created by all other quarks in baryon. The scalar and vector interactions, \( S \) and \( V_G \), stand for the effective QCD gluon interaction, which can be approximated by the string potential and the one-gluon-exchange (OGE) interaction, while

\[ U_5 \equiv \exp(2i \gamma^5 T \cdot \pi/f_\pi), \]

is the chiral interaction that \( q_i \) is involved, with \( f_\pi \) the pion decay constant. The chiral dynamics \( \mathcal{L}^x \) of the pion mesons \( \pi \) is chosen, for simplicity, be the Skyrme Lagrangian \[18\],

\[ \mathcal{L}^x = \frac{f_\pi^2}{4} tr(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e_s^2} tr[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2, \]

with the chiral field \( U = \exp(2i T \cdot \pi/f_\pi) \) being the nonlinear representation of the Goldstone bosons, \( U^\dagger U = 1 \). Here, \( f_\pi \) and the self-coupling \( 1/e_s \) of the chiral field scales like \( \sqrt{N_c} \), while \( U \) (thereby \( U_5 \)) \( \sim \mathcal{O}(1) \) in the \( 1/N_c \) expansion. We note that the adding\[19\] of the term \( \bar{q} U_5 g \) in (1) is used to restore chiral symmetry \( q \rightarrow q e^{2i \gamma^5 T \cdot \theta} \) under which \( U_5 \rightarrow e^{-2i \gamma^5 T \cdot \theta} U_5 e^{-2i \gamma^5 T \cdot \theta} \).

The model (1) is invariant under chiral symmetry and can be viewed, in a sense, as a potential version of the large-\( N_c \) effective theory\[14\] in that instead of using gluon interaction, we use the Hartree potentials \( S \) and \( V_G \) to effectively describe the flavor-independent part of the color confining interactions between the valence quarks in baryon, which are assumed to arise from the gluon interaction at long and short distance limit, respectively. For the explicit form of \( S \) and \( V_G \) in (1), we choose them to be of the string-like and one-gluon-exchange

\[ S = \sigma_{\text{string}} r, V_G = f_c \alpha_s(r)/r, \]

with \( \sigma_{\text{string}} \) the string tension, \( \alpha_s(r \gg 1) = g^2/4\pi \sim 1/N_c \). Here, \( f_c = \frac{1}{2}(N_c - 1/N_c) \) is the color factor for the quark-antiquark interaction in its color singlet configuration. It comes from the formula

\[ f_c = \frac{1}{4} tr \left( \frac{\lambda^a}{\sqrt{N_c}} \cdot \frac{\lambda^a}{\sqrt{N_c}} \right) = \frac{N_c^2 - 1}{2N_c}, \]
which becomes 4/3 when $N_c = 3$. Obviously, $f_c \alpha_s \sim 1 + \mathcal{O}(1/N_c^2)$ for large $N_c$.

For the string tension $\sigma_{\text{string}}$ in (3), one known from the $SU(N_c)$ gauge theories on lattice[11] that $\sqrt{\sigma_{\text{string}}/g_t} = \mathcal{O}(1)$, up to a $1/N_c^2$ correction. Thus, the Hartree potentials (3) scale as $\mathcal{O}(1)$, being consistent with the argument in [2]. Note that $m_q \sim M_B/N_c \sim 1$ since the baryon mass grows linearly with $N_c$[2].

We describe our treatment of the strong coupling $\alpha_s$ in (3) as follows. With $n_f$ quark flavors and the quark masses much less than the momentum transfer $Q^2$, the strong coupling $\alpha_s$ is running with $Q$, in lowest-order QCD, like

$$\alpha_s(Q^2) = \frac{12\pi}{(11N_c - 2n_f) \ln(Q^2/\Lambda_{\text{QCD}}^2)},$$

Since $\Lambda_{\text{QCD}} \simeq 250MeV$, $\alpha_s(Q^2)$ is small for $Q^2 > 1GeV^2$, but for any $N_c \geq 2$ the above perturbative formula diverges as $Q \to \Lambda_{\text{QCD}}$, which signals the onset of confinement. Since in the soft regime we are working we cannot avoid this divergence, we need to regularize $\alpha_s(Q^2)$, by assuming[20], for instance, that it saturates at some critical value $\alpha_{\text{critical}}(Q^2)$ as $Q \to 0$. In the work [20], the running behavior of $\alpha_s(Q^2)$ has the form of $\sum_{k=1}^{3} \alpha_k \text{erf}(\gamma_k r)$ in the coordinate space through the Fourier transformation, with erf($x$) the error function. We parametrize this behavior in a simple form

$$\alpha_s(r) = \alpha_{\text{critical}} \tanh \left((r/d)^2\right), \quad (5)$$

$$\alpha_{\text{critical}} = \frac{g^2}{4\pi} = \frac{g_t^2}{4\pi N_c}, \quad (6)$$

with $d$ the infrared cutoff ($\sim 1/\Lambda_{\text{QCD}} \simeq 4GeV^{-1}$). The potential $V_G$ in (3), with $\alpha_s(r)$ specified by (5), is shown in FIG.1.

FIG.1. The chiral angle profile given by (14) and the effective potential $V_G(r)$ as a function of radial distance $r$ in fm.
Inclusion of the second (Skyrme) term in (2) may raise issues of double counting of baryons\(^1\) since the purely pionic part of the Lagrangian can have Skyrmion solutions, with masses of order \(N_c^2\), in addition to the \(N_c\)-quark states described above. Here, the nontrivial solution of the Skyrme Lagrangian (2) carries a topological charge\(^{18, 21}\) \(B_\pi\) known as baryon number, which may violate the baryon number \(B = 1\). This is the case if one ignores the nontrivial vacuum effect that may arise from the spectral asymmetry of the quark states. However, as Rho et al.\(^{22, 23}\) noted, the quark spectrum is not symmetric about zero energy and therefore that the quark vacuum can carry nonzero baryon number,

\[
B_{\text{vac}} = -\frac{1}{2} \lim_{t \to +0} \sum_n \text{sgn}(E_n) e^{-t|E_n|},
\]

(7)

where the sum is taken over all positive- and negative energy single-particle eigenstates of quark with the level \(E_n\).

While no general proof is available, we argue, by analogy with the proof\(^{24}\) in chiral bag model, that when applying (1) to a baryon the contribution \(B_{\text{vac}}\) to baryon number from the quark vacuum polarization (spectrum asymmetry) due to the nontrivial boundary conditions and the contribution \(B_{Sk}\) from the Skyrmon conceals exactly, \(B_{\text{vac}} = -B_{Sk}\), leaving a unity baryon number in total, \(B = B_{\text{vac}} + B_{Sk} + B_{3q} = B_{3q} = 1\). Here, the baryon numbers carried by pion are defined\(^{18, 23, 21}\) as

\[
B_\pi = \int_{V_\pi} \frac{d^3x}{24\pi^2} \epsilon^{\mu\nu\lambda} tr[U\partial_\mu UU\partial_\nu UUU\partial_\lambda U],
\]

(8)

where \(V_\pi\) is the region pion field exists. The grounds for the analogy with the chiral bag comes from the observation that the numbers in (7) and (8) are topological in their nature in the sense that they are invariant upon smoothly varying of the solutions to the model (1) insofar as the boundary condition for \(q\) and \(U\) remain intact. The cancellation identity \(B_{\text{vac}} = -B_\pi\), which is valid for the chiral bag\(^{24}\), remains intact as one smoothly varies the solution to (1) to that of the chiral bag, corresponding to varying of the confining potentials \(S\) and \(V_G\) into that of bag.

\(^1\)This has already been discussed in Ref.\(^{16}\), where a way is shown that binding a second pion can be avoided. In Ref.\(^{14}\), Weinberg argued that the contribution from quark loops in addition to the tree approximation for purely pionic interactions has to be taken into account at moderate energies of order \(\Lambda_{\text{QCD}}\) probed in the structure of Skyrmions, and this contribution may invalidate the argument of baryon number double counting.
3 Hartree wavefunction of quark in ground state baryon

The Hartree picture for a baryon suggests a Hamiltonian being the sum of \(N_c\) identical quark Hamiltonians, \(H_{\text{Baryon}} = N_c \mathcal{H}^q\), and the baryon wavefunction \(\Psi(x_1, \cdots, x_N)\) being the product of the single-quark wavefunctions \(q(x_i)\) (the antisymmetric color part is not included here),

\[
\Psi(x_1, \cdots, x_N) = \prod_{i=1}^{N_c} q(x_i),
\]

which becomes exact at \(N_c \to \infty\) limit. Here, \(\mathcal{H}^q\) is the effective Hamiltonian of a single quark in Hartree potential, and is taken, in this work, to be the corresponding Hamiltonian associated with (1).

To determine \(\Psi\), one has to make stationary the variational functional \(\langle \Psi | H_{\text{Baryon}} - E | \Psi \rangle\), or equivalently, \(\langle \Psi | H_{\text{Baryon}} - N_c \epsilon | \Psi \rangle\) where \(\epsilon\) is the energy of a single quark in the Hartree potential. If we denote the Hamiltonian associated with (1) by \(H^0 + H^\chi\), with \(H^0\) the Lagrangian of a single quark in baryon, the Hartree picture, to the leading order of \(N_c\), suggests a Hamiltonian of baryon

\[
H_{\text{Baryon}} = \sum_{i=1}^{N_c} H^0(x_i) + H^\chi,
\]

with \(S(x)\) and \(V_G(x)\) given by (3). Here, \(H^\chi\) is the Hamiltonian associated with Lagrangian (2), \(x_i\) is the coordinate of the \(i\)-th quark, while the collective chiral field \(U = U(x)\) depends only on the coordinate \(x\) of the Goldstone boson. Keeping the \(N_c\)-dependence of \(f_\pi\) and \(e_s\) in mind, one readily sees that \(H^\chi \sim O(N_c)\), so that \(H_{\text{Baryon}} \sim N_c\), as argued by Witten[2].

We consider the two-flavor case of baryons with nontrivial configuration of chiral field of pions. For this, we use the hedgehog ansatz for the chiral field in (10), \(U(r) = \exp[iY(r) \hat{\tau} \cdot \hat{r}]\), where \(r\) originated at the mass center of baryons. Writing the quark wavefunction (with the antisymmetric part of color ignored) as \(q = \sqrt{D} (G(r), -iF(r) \sigma \cdot \hat{r}) y_{ljm}(\theta \phi) \chi_f\), with \(D \equiv 1/[\int dr (G^2 + F^2)]\), \(y_{ljm}\) the Pauli spinor and \(\chi_f\) the flavor wavefunction, and using (10) and (9), one can rewrite the functional \(\langle \Psi | H_{\text{Baryon}} - N_c \epsilon | \Psi \rangle\) in terms of the single quark wavefunction \(q\). The result is

\[
H_{\text{Baryon}} = N_c [M^0 + M^k / N_c],
\]
where

\[ M^0 \equiv D^2 \int dr \left\{ \frac{dG}{dr} - \frac{dF}{dr} + \frac{2\kappa}{r} GF + (m_q + S) \cos Y (G^2 - F^2) \right. \]
\[ + V_G(G^2 + F^2) \left. \right\} , \]

\[ M^{Sk} = \frac{2\kappa f_s}{e_s} \int dz \left[ z \left( \left( \frac{dY}{dz} \right)^2 + 2 \sin^2 Y \cdot \left( 1 + \left( \frac{dY}{dz} \right)^2 \right) + \frac{\sin^4 Y}{z^2} \right) \right] \]

with \( z \equiv r/L \equiv e f_s r \) and \(-\kappa\) the eigenvalue of the grand spin operator \( K = \gamma^0 [\Sigma \cdot (r \times p) + 1] \) corresponding to the eigenstate \( y_{ljm} \).

The equation of motion of (11) for the S-state \((l = 0)\) is

\[ \frac{dG}{dz} + \kappa z G = \left[ \varepsilon q + (Lm_q + LS) \cos Y - LV_G \right] F, \]

\[ - \frac{dF}{dz} + \kappa z F = \left[ \varepsilon q - (Lm_q + LS) \cos Y - LV_G \right] G, \]

with \( \varepsilon_q = L e, LS = z/(e_s f_s a)^2, LV_G = f_s a_s / z, \) and \( a \equiv 1/\sqrt{\sigma_{\text{string}}} \).

Before solving (13) one has to know \( Y(z) \), which is coupled to \((G, F)\) through the term \( g U_5 g \sim (G^2 - F^2) \cos Y \) in (12). We use adiabatic approximations to determine the static Skyrmion profile first and then solve (13) for \((G, F)\). A good fit of the hedgehog Skyrmion [25] was given by that of the kink-like [26, 27]

\[ Y(z) \simeq 4 \arctan(e^{-1.002z}), \]

which is obtained by minimizing the energy functional \( M^{Sk}[Y(r)] \) in (12) to the value \( M^{Sk} = 73.62 f_s / e_s \), and does not depend upon the parameters \( f_\pi \) and \( e_s \). This is comparable to the result \( M^{Sk} = 73 f_\pi / e_s \) in Ref. [25]. The plot of the profile (14) against \( r = z/(e_s f_\pi) \) is presented in FIG.1.

Given the coupling (5) and the profile (14), one can solve the equations numerically in the case \( N_c = 3 \) for the parameters in Table I. The result for the spinor wavefunction \((G, F)\) is given in FIG.2.

**TABLE I.** The parameters of the chiral model. The parameters marked by asterisk are inputs from the data, and \( \alpha_{\text{critical}} = g_t^2/(8\pi) \).

| Parameters       | This work | CI [20] | ChBM [17] | SK [25] |
|------------------|-----------|---------|-----------|---------|
| \( N_c \)       | 3*        |         |           |         |
| \( e_s \)       | 3.80      | 4.51    | 5.45      |         |
| \( g_t \)       | 1.57      |         |           |         |
| \( m_q = m_{ud}(MeV) \) | 282.8    | 220     | 330       |         |
| \( m_s (MeV) \) | 471.3     | 419     |           |         |
| Zero-point (GeV) | -2.1989   | -0.615  | 0.170 (bag const.) |         |
| Smearing \( \sigma (GeV) \) | 1.0382   | 1.943   |           |         |
| cutoff \( d \) (GeV^{-1}) | 1.77      | 5.00 (1/\Lambda_{QCD}) |         |
| \( f_\pi (MeV) \) | 93.0*     | 93*     | 64.5      |         |
| \( \sigma_{\text{string}} (GeV^2) \) | 0.0655   | 0.15    |           |         |
FIG. 2. The spinor wavefunction \((G, F)\) against the radial distance \(r\) (fm).

FIG. 3. The nonrelativistic wavefunction of the light quark against \(r\) (fm).

It is useful to reduce the spinor wavefunction of quark into its nonrelativistic form. It can be given by

\[
\phi(x) = \sqrt{DG(r)} \left[ 1 + F(r)^2 / G(r)^2 \right]^{1/2},
\]

which satisfies the normalization \(\int |\phi(r)|^2 |y_{ljm}|^2 d^3x = 1\), and is plotted in FIG. 3.

We note that the isospin symmetry is implicitly assumed here so that the masses of the up and down quarks equal, \(m_q = m_{ud}\). The parameters are compared to the corresponding quantities or the counterparts that used in the other calculations. The solution (15) gives a leading order approximation to relativistic quark wavefunction. It will be used to compute the mass correction arising from the strong hyperfine interaction\((\sim 1/N_c)\) in the perturbative framework in the section 4 and 5.
4 Mass splitting from hyperfine interaction at order $1/N_c$

Being of the order $N_c$, the Hartree interactions in (10) and (11) discussed in the section 2 and 3 is not sufficient to map the baryon mass splitting in the real world ($1/N_c = 1/3$) which has to be considered as a subleading correction to the Hartree approximation of baryon. In fact, with the help of the large $N_c$ consistency conditions[4, 28], Jenkins [29] shows that baryon mass splittings are proportional to $J^2(J$ is spin quantum number of baryon) and first allowed at order $1/N_c$. One of striking illustrations for this is the Skyrme model, where the mass splittings, due to rigid rotation of soliton, are proportional to $J^2/N_c$[25].

Following the non-relativistic quark models[30, 20], we consider the spin-dependent pairwise interaction between quarks in baryon which is inspired by the Breit-Fermi interaction of QED[31], and thereby study the hyperfine mass splitting of baryons. The hyperfine splitting part of the Breit-Fermi interaction inspired by QED reads[30]

$$H_{hyp}^{ss} = \sum_{i<j} \frac{2\alpha_s}{3m_im_j} \left\{ \frac{8\pi}{3} \delta^3(r_{ij}) s_i \cdot s_j + \frac{1}{r_{ij}^3} \left( 3(s_i \cdot \mathbf{r}_{ij})(s_j \cdot \mathbf{r}_{ij}) - s_i \cdot s_j \right) \right\}.$$  (16)

To the Hamilton (10), which scales like $N_c$, we add the following two-body hyperfine interactions,

$$\mathcal{H}^{ss} = \frac{1}{N_c - 1} \sum_{i<j=1}^{N_c} V_{ij}^{ss},$$  (17)

with

$$V_{ij}^{ss} = \frac{f_c\alpha_s(r_{ij})}{m_im_j} \left\{ \frac{8\pi}{3} \delta^3(r_{ij}) s_i \cdot s_j + \frac{1}{r_{ij}^3} \left( 3(s_i \cdot \mathbf{r}_{ij})(s_j \cdot \mathbf{r}_{ij}) - s_i \cdot s_j \right) \right\}.$$  (18)

where $m_i, s_i$ are the mass and spin of the $i$-th quark in the baryon center-of-momentum frame, and $r_{ij} = r_i - r_j$ are the relative coordinate of the quark $i$ and $j$. From (17) and (18), one can check that $\mathcal{H}^{ss} \sim 1/N_c$ for the low-spin states with the baryon spin $J \ll N_c/2$, if we notice that the sum in (17) (put $m_u \approx m_d$),

$$\sum_{i<j} \frac{s_i \cdot s_j}{m_im_j} \simeq \frac{1}{2m_u^2} \left[ J(J + 1) - \frac{3N_c}{4} \right]$$  (19)

becomes $[j(j + 1) - 3N_c]/(8m_u^2)$ for $J = j/2 \ll N_c/2$ and $[N_c(N_c - 1)]/(8m_u^2)$ for $J = N_c/2$, which grows like $N_c^2$ at large $N_c$. However, we emphasize that
the later case does not indicate that the perturbative method fully fails for $J \sim N_c/2$ with $N_c = 3$ because in that situation (19) is $9/(2m_u)^2 \sim 1$.

The factor $1/(N_c - 1)$ in (17) comes from the generalization of the "1/2" rule([32],[33]) for pairwise interaction of quarks in a baryon to the $N_c$-body cases. The "1/2" rule claims that the pair potential between two quarks in a baryon is half the quark-antiquark potential in a meson. Intuitively, it says that two quarks in a baryon can be seen, when they come closer together, as a localized $\bar{3}$ source which is equivalent to an antiquark as in a meson. Sharing the $\bar{3}$ source by the color charge of each quark, the "1/2" rule follows.

For the same reason, $N_c - 1$ quarks in a baryon consist of $N_c$ quarks appear as a $\bar{N}_c$ source which is equivalent to an antiquark, and the factor $1/(N_c - 1)$ in (17) follows when the $\bar{N}_c$ source is shared by each of the $N_c - 1$ quarks in the $\bar{N}_c$ source.

The delta function in (18) is to be smeared([20]) by

$$\rho_{ij}(r - r') = \frac{\sigma_{ij}^3}{\pi^{3/2}} e^{-\sigma_{ij}^2(r-r')^2}. \quad (20)$$

While $\sigma_{ij}$ are sensitive to the flavors of quarks for meson, it is not for baryon([20]). Thus, for baryon, one can replace $\sigma_{ij}$ by a single parameter $\sigma$: $\rho_{ij} \rightarrow \rho_{ij} \equiv \frac{\sigma^3}{\pi^{3/2}} e^{-\sigma^2(r-r')^2}$.

By adding (17) to (11), one has

$$M^{Baryon} = N_c \left[ M^0 + \frac{M^{Sk}}{N_c} \right] + M^{ss}, \quad (21)$$

$$M^{ss} = \langle B|\mathcal{H}^{ss}|B \rangle. \quad (22)$$

The expectation value is taken over the nonrelativistic state $|B\rangle$ of baryon because the operator (17), analogous to the Breit-Fermi term of QED, is defined over the subspace which consists of the nonrelativistic wavefunction (denoted as $\Psi_B$). As a nonrelativistic approximation, $\Psi_B$ can be given by

$$\Psi_B \approx \prod_{i=1}^{N_c} \phi(x_i) \chi_{sf}, \quad (23)$$

with $\chi_{sf}$ the SU(6) spin-flavor wavefunction of baryon. Given $\Psi_B$ with the one-body orbital wavefunction $\phi$ solved form (13), one has

$$M^{ss} = \langle \Psi_B|\mathcal{H}^{ss}|\Psi_B \rangle. \quad (24)$$

For the S-wave of baryons, the second term in (18) vanishes. Before computing (24) we smear the delta function in (18), considering simultaneously the $r$-dependence of $\alpha_s(r)$ in (5), according to the Breit-Fermi-like
interaction\textsuperscript{[20]}, \( \frac{7}{3m_{m_j}} s_i \cdot s_j \nabla^2 V_{\text{int}} \). This can be done by simply rewriting \( \alpha_s \delta(r_{ij}) \) in the form of the Laplace expression \( \nabla^2 (-\alpha_s r_{ij})/4\pi r_{ij} \) at first and then smear \( \delta(r_{ij}) \) with \( \rho_\sigma \). Using \textsuperscript{[18]} one has for the mass splitting \textsuperscript{[24]}

\[
M^{ss} = \sum_{i<j} \frac{8\pi f_c \alpha_{\text{critical}}}{3(N_c - 1)m_im_j} \langle \Psi_B | \alpha(r_{ij}) \delta^3(r_{ij}) s_i \cdot s_j | \Psi_B \rangle,
\]

\[
= \sum_{i<j} A_{ij} \frac{s_i \cdot s_j}{m_im_j},
\]

(25)

where \( \alpha_s(r) \equiv \alpha_{\text{critical}} \cdot \alpha(r) \) is given by \textsuperscript{[20]}, with

\[
\alpha(r) \equiv \tanh((r/d)^2),
\]

(26)

and

\[
A_{ij} = \frac{g_l^2 (1 + 1/N_c)}{3N_c} \left| \frac{\nabla^2 \left( \frac{\alpha(r_{ij})}{4\pi r_{ij}} \right)}{ij} \right|.
\]

(27)

Here, \( \left| ij \right| \equiv |\phi(x_i) g_{ijm}^{(i)}|/|\phi(x_j) g_{ijm}^{(j)}| \) with \( |\phi(x_i) g_{ijm}^{(i)}| \) the spatial and spin wavefunction of the \( i \)-th quark, and the relations \textsuperscript{[23]}, \textsuperscript{[5]} and \( f_c \alpha_{\text{critical}} = g_l^2/(8\pi)(1 - 1/N_c^2) \) are used. Noticing the mathematical identity

\[
\nabla^2 \left( \frac{\alpha(r)}{4\pi r} \right) = \alpha(r) \delta^3(r) - \frac{d^2 \alpha(r)/dr^2}{4\pi r},
\]

(28)

and making replacement \( \delta^3(r) \to \rho_\sigma(r) \) subsequently, one has the smeared results for \textsuperscript{[27]},

\[
A_{ij} = \frac{g_l^2 (1 + 1/N_c)}{6N_c} [A_{ij}^{(1)} - A_{ij}^{(2)}],
\]

(29)

where (see appendix A).

\[
A_{ij}^{(1)} = \int \int dr_1 r_1^2 dr_2 r_2^2 |\phi_i(r_1)|^2 |\phi_j(r_2)|^2 \frac{\sigma^3 e^{-\sigma^2(r_1^2 + r_2^2)}}{\pi^{3/2}} U_1(r_1, r_2),
\]

(30)

\[
A_{ij}^{(2)} = \int \int dr_1 r_1^2 dr_2 r_2^2 |\phi_i(r_1)|^2 |\phi_j(r_2)|^2 \frac{1}{4\pi} U_2(r_1, r_2),
\]

(31)

and

\[
U_1(r_1, r_2) \equiv \int_{-1}^{+1} du e^{2r_1 r_2 \sigma^2 u} \alpha \left( (r_1^2 + r_2^2 - 2r_1 r_2 u)/d^2 \right),
\]

\[
U_2(r_1, r_2) \equiv \int_{-1}^{+1} du \alpha'' (r_1^2 + r_2^2 - 2r_1 r_2 u)/\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 u},
\]

(32)

Here, \( \alpha(r) \) is given by \textsuperscript{[20]} and \( \alpha'' = d^2 \alpha(r)/dr^2 \) its second-order derivative, and \( \phi_i \) stands for the solution \textsuperscript{[15]}, solved with the mass \( m_i(i = u, d) \).
Ignoring the isospin breaking, one can replace $A^{(m)}_{ij} (m = 1, 2)$ by a single parameter $A^{(m)}_{ij} \sim A^{(m)}$, and thereby finds

$$A = g_r^2 \frac{(1 + 1/N_c)}{6N_c} [A^{(1)} - A^{(2)}]. \quad (33)$$

with $A^{(1,2)}_{ij}$ given by (30) and (31). From the equations (26), (30), (31) and (32), one can easily see that $A \sim 1/N_c$ at large $N_c$.

Putting (25) into (11), we obtain

$$M_{\text{Baryon}} = N_c M^0 + M^{Sk} + A \sum_{i<j=1}^{N_c} \frac{s_i \cdot s_j}{m_i m_j}, \quad (34)$$

with $M^0$, $M^{Sk}$ given by (12) and $A$ by (33). Notice that (34) has the form of expansion $M_{\text{Baryon}} = a_0 N_c + a_1/N_c$ since $M^{Sk} \sim N_c$ and $A \sim 1/N_c$. This is exactly what the large-$N_c$ QCD predicts[29] for the $SU(3)_F$ symmetry limit.

5 Extension to three flavor case

In order to determine the 't Hooft coupling, we need more informations about the baryons than the two-flavor case considered in preceding sections to fix the parameters in (11). For this purpose, one has to extend the formula (34) into the three flavor case which includes the effects of the flavor $SU(3)$ breaking. One convenient way to do this is to attribute the $SU(3)_F$ breaking simply, as the naive quark model, to the mass differences of the quarks between strange and non-strange flavors, including the resulted differences of the single-particle mean-field energy $M^{(i)}$ ($i = u, d, s$) for quarks, defined by (12).

We introduce the strange quark mass $m_s$ (bigger than $m_{ud}$) so that the mass difference $m_s - m_{ud}$ breaks the flavor $SU(3)$ breaking explicitly. Instead of computing the strange quark mass $M^{(s)}$ directly using the bound state approach to strangeness[34], for instance, one can determine it by simply fitting the data of the mass difference between $\Lambda$ and nucleon. We then generalize the mass formula (34) into

$$M_{\text{Baryon}} = \sum_{i=1}^{N_c} M^{(i)} + M^{Sk} + A \sum_{i<j=1}^{N_c} \frac{s_i \cdot s_j}{m_i m_j}, \quad (35)$$

where $A$ and $M^{Sk}$ are taken to be same for three flavors ($i = u, d, s$) including the strange quark, $M^{(u)} = M^{(u,d)} = M^0$ is defined in (12). By applying (35)
to $\Lambda = uds$ and nucleon and eliminating the mass splitting, one has $M^{(s)} = M^{(u)} + \delta M_{A,N}$, which gives (with the experimental data $\delta M_{A,N} = 175 \text{MeV}$)

$$M^{(s)} \simeq M^{(u)} + 175 \text{MeV}. \quad (36)$$

The relation (35) is to be compared with the ground-state mass formula of the nonrelativistic quark model\[30\] \[35\]

$$M = \sum_{i=1}^{3} m_i + A' \sum_{i<j=1}^{3} \frac{s_i \cdot s_j}{m_i m_j}, \quad (37)$$

with $A' = (2m_{ud})^2 50 \text{MeV}$, and that of the Skyrme model\[36\]

$$M = M^{\text{soliton}} + \left( \frac{1}{2I_1} - \frac{1}{2I_2} \right) \mathbf{J}^2 + \frac{1}{2I_2} \left[ C^{p,q} - \frac{N_s^2}{12} \right], \quad (38)$$

with $C^{p,q}$ the Casimir operator and $I_1, I_2 \sim N_c$ the moments of inertia of soliton ($I_1 = 53/(f_\pi e^3)$ and $I_2 = 19.5/(f_\pi e^3)$, approximately).

We will show that (35) and (37) can be understood in the context of large-$N_c$ QCD, which claims \[4\] \[28\] for the $SU(3)_F$ breaking case, \[35\]

$$M = a_0 N_c + a_1 \frac{\mathbf{J}^2}{N_c} + b_1 \epsilon_F T^8 + \epsilon_F b_2 \frac{J^i G_{i8}}{N_c} + \cdots. \quad (39)$$

Here, $\epsilon_F$ is the $SU(3)_F$ breaking, $T^8 = (N_c - N_s)/\sqrt{12}$, $G_i^8 = (J^i - 3J^i_s)/\sqrt{12}$, $J^i$ and $J^i_s$ are spins of the baryon and strange quark, respectively, and $N_s$ the number of the strange quark. This can be seen if we notice the relation (19) for two flavor case and its generalization of the three-flavor case.

The three-flavor generalization of the relation (19) can be given, for the strangeness $-N_s$ baryons, by

$$\sum_{i<j=1}^{N_c} \frac{s_i \cdot s_j}{m_i m_j} = \left( \frac{m_s}{m_q} - 1 \right) \sum_{k<k'=1}^{N_c-N_s} \frac{s_k \cdot s_{k'}}{m_k m_{k'}} + \left( 1 - \frac{m_s}{m_q} \right) \sum_{i<i'=1}^{N_s} \frac{s_i \cdot s_{i'}}{m_s^2} + \frac{1}{m_q m_s} \left[ \frac{J(J+1)}{2} - \frac{3N_s}{8} \right],$$

then, when adding of the extra contribution $N_s(M^{(s)} - M^{(u)})$ to the order-$N_c$ mass $N_c M^0$ stemming from the strange-up energy difference $\delta M \equiv M^{(s)} - M^{(u)}$, one finds for the baryon mass formula (35) with three flavors

$$M^{\text{Baryon}} = N_c M^0 + \left[ M^{Sk} - \frac{3(A N_c)}{8 m_s m_q} \right] + A \frac{J(J+1)}{2 m_q m_s} + N_s \delta M$$

$$+ \left( \frac{m_s}{m_q} - 1 \right) A \left[ \sum_{k<k'=1}^{N_c-N_s} \frac{s_k \cdot s_{k'}}{m_k m_{k'}} - \sum_{i<i'=1}^{N_s} \frac{s_i \cdot s_{i'}}{m_s^2} \right] \quad (40)$$

13
where the small indices $k$ and $k'$ refer to the non-strange quarks ($u$ and $d$), and the capital indices to the strange quark, $m_q \equiv m_{ud}$, $\delta M \propto \epsilon_F \propto \left( \frac{m_s}{m_q} - 1 \right)$.

From (40), we see that the third term scales like $1/N_c$ the fourth like $\epsilon_F$, and the last scales as $\epsilon_F/N_c$, corresponding to the second, the third and the fourth terms in (39), respectively. We note here that $\delta M$ is of order $O(1)$ and $A \sim O(1/N_c)$. The individual spin product $s_i \cdot s_j$ ($i, j = u, d, s$) in (35) or (37) is $3/4$ if the spins of the quark($i$ and $j$) are parallel and $-3/4$ if they are anti-parallel.

Knowing the wavefunction (15) solved in the section 3 and the resulted energies $M^{(u)} = M^0$ and $M^{sk}$ given by (12), and using (36) and (33), one can map the masses of the octet and decuplet baryons using (35). Our results for the optimal fit of $g_t$ and $A$ in (33) are

$$g_t = 1.571, \quad A = 0.000243,$$

with the corresponding masses (35) calculated for the parameters in Table I listed in the Table II.

**TABLE II.** The calculated baryon masses are compared with experiment (in MeV/$c^2$) when $N_c = 3$. The values for ’t Hooft coupling and mass splitting coefficients in our work are $g_t = 1.571$ and $A = 0.000243$, respectively.

| Baryon | This work | NQM[20] | ChBM[17] | Instance[37] | Exp. [38] |
|--------|-----------|---------|---------|--------------|---------|
| $N$    | 939.0     | 960     | 938     | 938          | 938.9   |
| $\Lambda$ | 1114.0 | 1115   | 1149   | 1116         | 1115.7  |
| $\Sigma$ | 1117.4 | 1190   | 1211   | 1184         | 1193.1  |
| $\Xi$  | 1291.0    | 1305   | 1369   | 1329         | 1318.1  |
| $\Delta$ | 1232.0 | 1230   | 1205   | 1239         | 1232.0  |
| $\Sigma^*$ | 1405.3 | 1370   | 1370   | 1383         | 1384.6  |
| $\Xi^*$ | 1580.3   | 1505   | 1531   | 1528         | 1533.4  |
| $\Omega$ | 1752.9 | 1635   | 1683   | 1672         | 1672.5  |

6 Summary and concluding remark

We studied the ’t Hooft coupling and the baryon masses hyperfine splittings of the ground state baryons in a chiral quark model inspired by the large-$N_c$ QCD. The Hartree picture of baryon at large $N_c$ is used to map the masses of octet and decuplet baryons in the ground states. The ’t Hooft coupling $g_t$ that reproduces data of masses is found to be about 1.57. We believe that this gives an indication that in Weinberg’s effective field theory, the ’t Hooft coupling $g_t$ becomes small in the sense that $g_t^2/4\pi N_c \ll 1$ at moderate energies.
The fact that the quantum numbers and the group theoretic structure of baryons in the Skyrme model is identical to that of the naive quark model at large $N_c$ [39] implies some yet unknown connection between the two. We hope that the large-$N_c$ chiral model considered in this paper will enhance this connection, as indicated by (35), (37) and (38), in the sense that both fit well into the large $N$ formula (39).

Acknowledgements

D. J thanks X.Q Li, D.Y Chen and X. Liu for useful discussions. This work is supported by the Natural Science Foundation of China(No.11265014).

Appendix A

We write the smeared matrix element $\langle ij | \alpha(r_{ij}) \rho_\sigma(r_{ij}) | ij \rangle$ in (27) as

$$
\int \int d^3r_1 d^3r_2 |\phi_1(r_1)|^2 |\phi_2(r_2)|^2 |\varphi_{ijm}\rangle^{(i)} |\varphi_{ijm}\rangle^{(j)} \alpha(r_{ij}) \rho_\sigma(r_{ij})
= \frac{(2\pi)^2}{(4\pi)^2} \int_0^\pi d\theta_1 \sin(\theta_1) d\theta_2 \sin(\theta_2) \int dr_1 dr_2 (1/3 \sigma r_1^2) |\phi_1(r_1)|^2 |\phi_2(r_2)|^2 \times \alpha(|r_1 - r_2|) \rho_\sigma(|r_1 - r_2|),
$$

where $r_{ij}^2 = |r_1 - r_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$ when one choose the frame $(r_1, \theta_1, \varphi_1)$ and $(r_1, \theta_1, \varphi_1)$ so that the three vector $r_1, r_2, z$ in the same plane. Make a change of variables from $(r_1, \theta_1, \varphi_1)$ and $(r_1, \theta_1, \varphi_1)$ to $(r_1, \theta_1, \varphi_1)$ and $(r_1, \theta_2 - \theta_1, \varphi_2)$, and performing the integration, one gets (30), with $U_1(r_1, r_2)$ given by (32). Similarly, one can get (31) for $(ij | \alpha(r_{ij}) \rho_\sigma(r_{ij}) | ij)$. Here, we give the alternative expressions for (30), (31) and (32), with the integration over the dimensionless variable $z = r/L$, which are convenient for the numerical computation of mass splittings.

$$
A_{ij}^{(1)} = \frac{(DL)^2 \sigma^3}{\pi \beta^2} \int \int dz_1 dz_2 |\phi_1(z_1)|^2 |\phi_2(z_2)|^2 e^{-L^2 \sigma^2(z_1^2 + z_2^2)} U_1(z_1, z_2)
$$

$$
A_{ij}^{(2)} = \frac{(DL)^2}{4\pi} \int \int dz_1 dz_2 |\phi_1(z_1)|^2 |\phi_2(z_2)|^2 U_2(z_1, z_2)
$$

$$
U_1(z_1, z_2) = \int_{-1}^{+1} du e^{2z_1z_2L^2 \sigma^2 u} \tanh \left( \frac{(z_1^2 + z_2^2 - 2z_1z_2u)}{(L/d)^2} \right),
$$

$$
U_2(z_1, z_2) = \frac{1}{L^3} \int_{-1}^{+1} du \tanh^\alpha \left( \frac{(z_1^2 + z_2^2 - 2z_1z_2u)}{(L)^2} \right) \sqrt{z_1^2 + z_2^2 - 2z_1z_2u}.
$$
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Nonrelativistic wavefunction
The chiral angle $Y(r)$

The OGE potential $40V_G(r)$
