RELATIVE ENTROPY FOR MAXIMAL ABELIAN SUBALGEBRAS OF MATRICES AND THE ENTROPY OF UNISTOCHASTIC MATRICES

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Dedicated to the memory of Professor Masahiro Nakamura

Abstract. Let $A$ and $B$ be two maximal abelian $*$-subalgebras of the $n \times n$ complex matrices $M_n(\mathbb{C})$. To study the movement of the inner automorphisms of $M_n(\mathbb{C})$, we modify the Connes-Størmer relative entropy $H(A|B)$ and the Connes relative entropy $H_\phi(A|B)$ with respect to a state $\phi$, and introduce the two kinds of the constant $h(A|B)$ and $h_\phi(A|B)$. For the unistochastic matrix $b(u)$ defined by a unitary $u$ with $B = uAu^*$, we show that $h(A|B)$ is the entropy $H(b(u))$ of $b(u)$. This is obtained by our computation of $h_\phi(A|B)$. The $h(A|B)$ attains to the maximal value $\log n$ if and only if the pair $\{A, B\}$ is orthogonal in the sense of Popa.

1. Introduction

In a step to introduce the notion of the entropy for automorphisms on operator algebras, Connes and Størmer defined in [2] the relative entropy $H(A|B)$ for finite dimensional von Neumann subalgebras $A$ and $B$ of a finite von Neumann algebra $M$ with a fixed normal normalized trace $\tau$.

After then, the study about the $H(A|B)$ is extended to two directions.

One development was started by Pimsner and Popa in [7] as the relative entropy $H(A|B)$ for arbitrary von Neumann subalgebras $A$ and $B$ of $M$, (see [4] for many interesting results in this direction).

The other was a generalization due to Connes in [1] by changing the trace $\tau$ to a state $\phi$ on $M$. He defined the relative entropy $H_\phi(A|B)$ for two subalgebras $A$ and $B$ of $M$ with respect to $\phi$.

We modify the Connes-Størmer relative entropy $H(A|B)$ and the Connes relative entropy $H_\phi(A|B)$ with respect to a state $\phi$, and we introduce the corresponding two kinds of the constant $h(A|B)$ and

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In the case where $A = M$, $h(M \mid B)$ is nothing else but the Connes-Størmer relative entropy $H(A \mid B)$. In general, $0 \leq h_\phi(A\mid B) \leq H_\phi(A\mid B)$, and if $M$ is an abelian von Neumann algebra, then $h_\phi(A \mid B) = H_\phi(A \mid B)$ so that it coincides with the conditional entropy in the ergodic theory.

In this paper, we restrict our subjects to the maximal abelian subalgebras (abbreviated as MASA’s) of the $n \times n$ complex matrices $M_n(\mathbb{C})$.

If $A$ and $B$ are two MASA’s of $M_n(\mathbb{C})$, then there exists a unitary matrix $u$ with $B = uAu^*$, which we denote by $u(A, B)$. Each unitary matrix $u$ induces a bistochastic matrix $b(u)$, which is a typical example of a bistochastic matrix.

For a bistochastic matrix $b$, it is introduced in [9] the notion of the entropy $H(b)$ and the weighted entropy $H_\lambda(b)$ with respect to $\lambda$.

Here we show a relation between $h_\phi(A \mid B)$ for a pair $\{A, B\}$ of MASA’s and the $H(b(u(A, B)))$. This is a consequence of our computation on $h_\phi(D \mid uDu^*)$, where $\phi$ is a state of $M_n(\mathbb{C})$ and $D$ is the set of $n \times n$ diagonal matrices corresponding to the eigenvectors of $\phi$. The $h_\phi(D \mid uDu^*)$ satisfies the following relation:

$$h_\phi(D \mid uDu^*) = H_\lambda(b(u)^*) + S(\phi|_D) - S(\phi|_{uDu^*}),$$

where $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ is the eigenvalues for $\phi$ and $S(\psi)$ means the entropy of a positive linear functional $\psi$.

As the special case where $\phi$ is the normalized trace, we have

$$h(A \mid B) = H(b(u(A, B))).$$

In the case of the Connes-Størmer relative entropy, for a given two MASA’s $A$ and $B$ of $M_n(\mathbb{C})$, $H(A \mid B)$ is not equal to $H(b(u(A, B)))$ in general. For an example, see [6, Appendix].

The above results show that $h_\phi(A\mid B)$ for MASA’s of $M_n(\mathbb{C})$ are determined by the entropy for the related unistochastic matrices in the sense of [9]. Also, the value $h(D \mid uDu^*)$ is depending on the inner automorphism $Ad_u$ defined by the unitary $u$, and we can consider these values as a kind of conditional entropy for $Ad_u$ conditioned by $D$.

Two MASA’s $A$ and $B$ of $M_n(\mathbb{C})$ are orthogonal in the sense of Popa [8] if

$$A \cap B = \mathbb{C}1 \quad \text{and} \quad E_A E_B = E_B E_A = E_{A\cap B} = E_{\mathbb{C}1}.$$
where $E_A$ is the conditional expectation of $M_n(\mathbb{C})$ onto $A$ such that $\tau \circ E_A = \tau$ for the normalized trace $\tau$ of $M_n(\mathbb{C})$. This means that

\[
B \subset M_n(\mathbb{C}) \\
\cup \cup \\
\mathbb{C}1 \subset A
\]

is a commuting square in the sense of [3].

We have that a pair $\{A, B\}$ of MASA's is orthogonal if and only if $h(A|B)$ takes the maximal value, and then the value is log $n$.

2. Preliminaries

In this section, we summarize notations, terminologies and basic facts which we need later.

Let $M$ be a finite von Neumann algebra, and let $\tau$ be a fixed faithful normal tracial state. By a von Neumann subalgebra $A$ of $M$, we mean that $A$ has the same identity with $M$. Let $E_A$ be the $\tau$-conditional expectations on $A$.

2.1. Relative entropy.

2.1.1. Relative entropy of Connes-Størmer. First, we review about the formulae of the relative entropy by Connes and Størmer in [2] (cf. [4]).

Let $S$ be the set of all finite families $(x_i)$ of positive elements in $M$ with $1 = \sum_i x_i$. Let $A$ and $B$ be two von Neumann subalgebras of $M$.

The relative entropy $H(A|B)$ is

\[
H(A \mid B) = \sup_{(x_i) \in S} \sum_i (\tau\eta E_B(x_i) - \tau \eta E_A(x_i)).
\]

Here $\eta(t) = -t \log t, \ (0 < t \leq 1)$ and $\eta(0) = 0$.

Let $\phi$ be a normal state on $M$. Let $\Phi$ be the set of all finite families $(\phi_i)$ of positive linear functionals on $M$ with $\phi = \sum_i \phi_i$.

The relative entropy $H_{\phi}(A|B)$ of $A$ and $B$ with respect to $\phi$ is

\[
H_{\phi}(A \mid B) = \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i \mid_A, \phi \mid_A) - S(\phi_i \mid_B, \phi \mid_B))
\]

where $S(\psi|\varphi)$ is the relative entropy for two positive linear functionals $\psi$ and $\varphi$, and $H_\tau(A|B) = H(A \mid B)$. 
2.1.2. Relative entropy of positive linear functionals. After, we need the precise form $S(\psi|\varphi)$ in the case of finite dimensional algebras $C$, mainly the full matrix algebra $M_n(\mathbb{C})$ (the set of the $n \times n$ matrix $x = (x(i,j))_{ij}$ with $x(i,j) \in \mathbb{C}$).

We denote by $\text{Tr}$ the canonical trace on $C$, that is, $\text{Tr}(p) = 1$ for every minimal projection $p \in C$.

Let $\psi$ be a positive linear functional on $C$. We denote by $Q_\psi$ the density operator of $\psi$, that is, $Q_\psi \in C$ is a unique positive operator with

$$\psi(x) = \text{Tr}(Q_\psi x), \quad (x \in C),$$

and the von Neumann entropy of $\psi$ is given by

$$S(\psi) = \text{Tr}(\eta(Q_\psi)),$$

Let $\psi$ and $\varphi$ be two positive linear functionals on $C$. If $\psi \leq \lambda \varphi$ for some $\lambda > 0$, then the relative entropy of $\psi$ and $\varphi$ is given as

$$S(\psi, \varphi) = \text{Tr}(Q_\psi (\log Q_\psi - \log Q_\varphi)),$$

(cf. [4], [5]).

2.1.3. Conditional relative entropy. Let $A$ and $B$ be von Neumann sub-algebras of $M$. We modify $H(A|B)$ and $H_\varphi(A|B)$ and as a replacement of $H(A|B)$ (resp. $H_\varphi(A|B)$) we define the following constant $h(A|B)$ (resp. $h_\varphi(A|B)$).

The conditional relative entropy $h(A | B)$ of $A$ and $B$ conditioned by $A$ is

$$h(A | B) = \sup_{(x_i) \in S} \sum_i (\tau \eta E_B(E_A(x_i)) - \tau \eta E_A(x_i)).$$

Let $S(A) \subset S$ be the set of all finite families $(x_i)$ of positive elements in $A$ with $1 = \sum_i x_i$. Then it is clear that

$$h(A | B) = \sup_{(x_i) \in S(A)} \sum_i (\tau \eta E_B(x_i) - \tau \eta(x_i)).$$

Let $S'(A) \subset S(A)$ be the set of all finite families $(x_i)$ with each $x_i$ a scalar multiple of a projection in $A$. Then by the same proof with in [7],

$$h(A | B) = \sup_{(x_i) \in S'(A)} \sum_i (\tau \eta E_B(x_i) - \tau \eta(x_i)).$$

Hence, if $A$ is finite dimensional, we only need to consider the families consisting of scalar multiples of orthogonal minimal projections.

Let $\varphi$ be a normal state of $M$. 

The conditional relative entropy of $A$ and $B$ with respect to $\phi$ conditioned by $A$ is

$$h_{\phi}(A|B) = \sup_{(\phi_i) \in \Phi} \sum_i (S(\phi_i|A, \phi|A) - S((\phi_i \circ E_A)|B, (\phi \circ E_A)|B)).$$

Assume that $M$ is finite dimensional and that the density matrix of $\phi$ is contained in $A$. Let $\Phi(A) \subset \Phi$ be the set of all finite families $(\phi_i)$ whose density operators $(Q_i)_i$ are contained in $A$. Since the density matrix of $\phi_i \circ E_A$ is $E_A(Q_i)$, we have

$$h_{\phi}(A|B) = \sup_{(\phi_i) \in \Phi(A)} \sum_i (S(\phi_i|A, \phi|A) - S(\phi_i|B, \phi|B)).$$

2.1.4. Remark. We will study about $h_{\phi}(A|B)$ and $h(A|B)$ for von Neumann subalgebras $A$ and $B$ of a general finite von Neumann algebra $M$ elsewhere. Here we just remark the following facts.

1. If $\phi$ is the normalized trace $\tau$, then by [4, Theorem 2.3.1(x)]

$$h_{\tau}(A|B) = h(A|B).$$

2. It is clear that $h(M|B)$ is nothing else but the Connes-Stømer relative entropy $H(M|B)$, and we have the relation with Index by Pimsner-Popa [7].

3. In general, $0 \leq h_{\phi}(A|B) \leq H_{\phi}(A|B)$. If $M$ is an abelian von Neumann algebra, then $h_{\phi}(A|B)$ coincides with the conditional entropy in the ergodic theory by a similar proof as in [4, p. 158].

2.2. Unistochastic matrices and the entropy. When a matrix $x \in M_n(\mathbb{C})$ is given, we denote the $(i,j)$-component of $x$ by $x(i,j)$. A matrix $b \in M_n(\mathbb{C})$ is called bistochastic if $b(i,j) \geq 0$ for all $i,j = 1, \cdots, n$, $\sum_{i=1}^n b(i,j) = 1$ for all $j = 1, \cdots, n$ and $\sum_{j=1}^n b(i,j) = 1$ for all $i = 1, \cdots, n$. Let $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ be a probability vector.

The weighted entropy $H_{\lambda}(b)$ of a bistochastic matrix $b$ with respect to $\lambda$ and the entropy $H(b)$ for a bistochastic matrix $b$ are given in [9] as the following forms, respectively:

$$H_{\lambda}(b) = \sum_{k=1}^n \lambda_k \sum_{j=1}^n \eta(b(j,k)),$$

and

$$H(b) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \eta(b(i,j)).$$
Let $u$ be a $n \times n$ unitary matrix. The unistochastic matrix $b$ defined by $u$ is the bistochastic matrix given as

$$b(i, j) = |u(i, j)|^2, \ (i, j = 1, 2, \ldots, n).$$

3. Results

Lemma 1. Let $A$ be a maximal abelian subalgebra of $M_n(\mathbb{C})$, and let $\{p_1, \ldots, p_n\}$ be the minimal projections of $A$.

(1) If $\psi$ is a positive linear functional of $M_n(\mathbb{C})$, then

$$S(\psi |_A) = \sum_{j=1}^{n} \eta(\psi(p_j)).$$

(2) If $\phi$ is a state of $M_n(\mathbb{C})$ and if $\phi = \sum_i \phi_i$ is a finite decomposition of $\phi$ into a sum of positive linear functionals, then

$$\sum_i S(\phi_i |_A, \phi |_A) = - \sum_i S(\phi_i |_A) + S(\phi |_A).$$

Proof. (1) Since the density operator of $\psi |_A$ is written as $\sum_j \psi(p_j)p_j$, we have

$$S(\psi |_A) = \text{Tr}_A(\sum_j \eta(\psi(p_j))p_j) = \sum_{j=1}^{n} \eta(\psi(p_j)).$$

(2) We denote the density operator of $\phi_i |_A$ by $Q_i$ and the density operator of $\phi |_A$ by $Q$, then

$$\sum_i S(\phi_i |_A, \phi |_A) = \sum_i \text{Tr}(Q_i(\log Q_i - \log Q))$$

$$= - \sum_i S(\phi_i |_A) - \text{Tr}(\sum_i Q_i \log Q)$$

$$= - \sum_i S(\phi_i |_A) + S(\phi |_A).$$

Let $\phi$ be a state of $M_n(\mathbb{C})$. We number the set of the eigenvalues of $Q_\phi$ as $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$. Let us decompose $Q_\phi$ into the form $Q_\phi = \sum_{i=1}^{n} \lambda_i e_i$, where $\{e_1, \cdots, e_n\}$ is a family of mutually orthogonal minimal projections in $M_n(\mathbb{C})$, which we fix. Let $D$ be the MASA generated by the projections $\{e_1, \cdots, e_n\}$, which we denote by $D(\phi)$ when we need.

Let $\{e_{kl}\}_{k,l=1,\ldots,n}$ be a system of matrix units of $M_n(\mathbb{C})$ such that $e_{ii} = e_i$ for all $i = 1, \cdots, n$. We give the matrix representation for
each $x \in M_n(\mathbb{C})$ depending on this matrix units $\{e_{kl}\}_{k,l=1,\ldots,n}$ so that $D = D(\phi)$ is the diagonal algebra. Let $u \in M_n(\mathbb{C})$ be a unitary, and let $b(u)$ be the unistochastic matrix defined by $u$. Under these situations, we have the following theorem.

**Theorem 2.** Let $\phi$ be a state of $M_n(\mathbb{C})$, and let $D$ be the diagonal algebra of $M_n(\mathbb{C})$. Let $u \in M_n(\mathbb{C})$ be a unitary, then

$$h_\phi(D | uDu^*) = H_\lambda(b(u)^*) + S(\phi|_D) - S(\phi|_{uDu^*}).$$

**Proof.** First we remark that $Q_\phi$ is contained in $D$ and $S(\phi) = S(\phi|_D)$. Using the matrix representation of the unitary $u$, the matrix representation of each $ue_ju^*$ is given by

$$ue_ju^* = \sum_{k,l} u(k,j)\overline{u(l,j)}e_{k,l} = (u(k,j)\overline{u(l,j)})_{k,l}.$$ 

Let $(\phi_i)_{i=1,\ldots,n}$ be the positive linear functionals of $M_n(\mathbb{C})$ such that $Q_{\phi_i} = \lambda_ie_i$ for all $i$. Then $\phi = \sum_i \phi_i$ gives a finite decomposition of $\phi$, and

$$\sum_j \eta(\phi_i(ue_ju^*)) = \sum_j \eta(\lambda_i | u(i,j) |^2)$$

$$= \sum_j (\eta(\lambda_i)| u(i,j) |^2 + \lambda_i \eta(| u(i,j) |^2))$$

$$= \eta(\lambda_i) + \lambda_i \sum_j \eta(| u(i,j) |^2).$$

Hence by Lemma 1,

$$\sum_{i=1}^n (S(\phi_i | D, \phi | D) - S(\phi_i |_{uDu^*}, \phi |_{uDu^*}))$$

$$= -\sum_i \sum_j \eta(\phi_i(e_{ij})) + \sum_j \eta(\lambda_j)$$

$$+ \sum_i \sum_j \eta(\phi_i(ue_ju^*)) - \sum_j \eta(\phi(ue_ju^*))$$

$$= \sum_i \sum_j \eta(\phi_i(ue_ju^*)) - \sum_j \eta(\phi(ue_ju^*))$$

$$= S(\phi|_D) + H_\lambda(b(u)^*) - S(\phi|_{uDu^*}).$$

Thus

$$H_\phi(D | uDu^*) \geq H_\lambda(b(u)^*) + S(\phi|_D) - S(\phi|_{uDu^*}).$$
To prove the opposite inequality, assume that \((\phi)_{i \in I}\) be a given finite family which is contained in \(\Phi(D)\). We denote \(Q_\phi\), simply by \(Q_i\).

Then
\[
\lambda_l = \sum_{i \in I} Q_i(l, l), \quad \text{and} \quad Q_i(l, k) = 0 \quad \text{if} \quad l \neq k,
\]
for all \(l, k = 1, \cdots, n\) and \(i \in I\), and
\[
\phi_i(ue_ju^*) = \sum_k Q_i(k, k)|u(k, j)|^2 \quad \text{and} \quad \phi_i(e_j) = Q_i(j, j),
\]
for all \(i \in I\) and \(j = 1, \cdots, n\).

Since \(\eta(st) = \eta(s)t + s\eta(t)\) and \(\eta(s + t) \leq \eta(s) + \eta(t)\) for all positive numbers \(s\) and \(t\), by using that \(\sum_j |u(k, j)|^2 = 1\) for all \(k\), we have that
\[
\sum_i S(\phi_i|uDu^*) - \sum_i S(\phi_i|D)
= \sum_i \sum_j \eta(\sum_k Q_i(k, k)|u(k, j)|^2) - \sum_i \sum_j \eta(Q_i(j, j))
\leq \sum_i \sum_j \left\{ \sum_k \eta(Q_i(k, k))|u(k, j)|^2 + \sum_k Q_i(k, k)\eta(|u(k, j)|^2) \right\}
- \sum_i \sum_j \eta(Q_i(j, j))
= \sum_i \sum_k \eta(Q_i(k, k))\left( \sum_j |u(k, j)|^2 \right) + \sum_j \sum_k \left\{ \left( \sum_i Q_i(k, k) \right)\eta(|u(k, j)|^2) \right\}
- \sum_i \sum_j \eta(Q_i(j, j))
= \sum_i \sum_k \eta(Q_i(k, k)) + \sum_j \sum_k \lambda_k \eta(|u(k, j)|^2) - \sum_i \sum_j \eta(Q_i(j, j))
= H_\lambda(b(u^*))
\]
These imply that by Lemma 1,
\[
\sum_{i \in I} \{ S(\phi_i|D, \phi|D) - S(\phi_i|uDu^*, \phi|uDu^*) \}
= -\sum_i S(\phi_i|D) + S(\phi|D) + \sum_i S(\phi_i|uDu^*) - S(\phi|uDu^*)
\leq H_\lambda(b(u^*)) + S(\phi|D) - S(\phi|uDu^*)
\]
Hence we have always that
\[
h_\phi(D|uDu^*) \leq H_\lambda(b(u^*)) + S(\phi|D) - S(\phi|uDu^*),
\]
and

\[ h_\phi(D|uDu^*) = H_\lambda(b(u)^*) + S(\phi|_D) - S(\phi|_{uD^*u^*}). \]

□

Let \( D \) be the algebra of diagonal \( n \times n \) matrices. Let \( \phi \) be a state of \( M_n(\mathbb{C}) \), and let \( \psi \) be a unitary such that \( vDv^* = D(\psi) \). Let \( \psi_v \) be the state given by the inner perturbation by \( \psi : \psi_v(x) = vxx^* \). Then we have the following relation between \( h(\psi_v) \) and the set of \( \{ h_\phi(\psi_v) \} \).

**Corollary 3.** Let \( D \) be the algebra of diagonal \( n \times n \) matrices, and let \( \psi \) be a unitary. Then

\[ h(\psi_v) = H_\lambda(b(\psi)^*) + S(\phi_v|_D) - S(\psi_v|_{uD^*u^*}). \]

Proof. The first equality is clear because \( h(\psi_v) = h(\psi_v) \) and \( H_\lambda(b(\psi)^*) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \eta(| u(i, j) |^2) \) where the eigenvalues for \( \tau \) is \( \lambda = \{ 1/n, \cdots, 1/n \} \). We denote the minimal projections of \( D \) by \( \{ e_1, \cdots, e_n \} \). Let \( \phi \) be a state, and let \( \{ \lambda_1, \lambda_2, \cdots, \lambda_n \} \) be eigenvalues of \( Q_\phi \). We decompose \( Q_\phi = \sum_{i=1}^{n} \lambda_i p_i \) by mutually orthogonal minimal projections \( \{ p_1, p_2, \cdots, p_n \} \). Let \( \psi_v \) be a unitary with \( \psi_i e_i = p_i \) for all \( i \). Then

\[ Q_{\psi_v} = \sum_i \lambda_i e_i = \psi_v Q_{\psi_v}. \]

That is, \( D(\psi_v) = D \) and \( S(\psi_v) = S((\psi_v|_D) \). Since

\[ S(\psi_v) \leq S((\psi_v|_{uD^*u^*}) \]

(cf. [4, Theorem 2.2.2 (vii)]), we have by Theorem 2 that, for the unistochastic matrix \( b(\psi) \) defined by \( u \),

\[
\begin{align*}
    h_{\psi_v}(D|uDu^*) &= H_\lambda(b(\psi)^*) + S((\psi_v|_D) - S((\psi_v|_{uD^*u^*}) \\
    &\leq H_\lambda(b(\psi)^*) \\
    &\leq H(b(\psi)^*) = H(b(\psi)) = h(D|uDu^*).
\end{align*}
\]

□

As another easy consequence of Theorem 2, by the property of the normalized trace \( \tau \), we have the following statement for general MASA’s \( A \) and \( B \) :
Corollary 4. Let $A$ and $B$ be maximal abelian subalgebras of $M_n(\mathbb{C})$. Then
\[ h(A \mid B) = H(b(u)) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \eta(\|u(i, j)\|^2) \]
where the $(u(i, j))_{ij}$ for $u = u(A, B)$ is given by the matrix units whose minimal projections generates $A$.

Corollary 5. Let $\{A_0, B_0\}$ be a pair of maximal abelian subalgebras of $M_n(\mathbb{C})$. Then $\{A_0, B_0\}$ is an orthogonal pair if and only if
\[ h(A_0 \mid B_0) = \log n = \max h(A \mid B), \]
where the maximum is taken over the set of a pair $\{A, B\}$ of maximal abelian subalgebras of $M_n(\mathbb{C})$.

Proof. Let $u \in M_n(\mathbb{C})$ be a unitary with $uAu^* = B$. By a characterization of Popa([8]), a pair $\{A, B\}$ of MASA’s is orthogonal if and only if $\tau(aubu^*) = 0$ for all $a, b \in A$ with $\tau(a) = \tau(b) = 0$. This implies that $\{A, B\}$ is orthogonal if and only if
\[ |u(j, k)| = 1/\sqrt{n} \quad \text{for all} \quad j, k. \]
Hence we have the conclusion. \qed

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