Fixed-Time Convergent Distributed Observer Design of Linear Systems: A Kernel-Based Approach

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Abstract—The robust distributed state estimation for a class of continuous-time linear time-invariant systems is achieved by a novel kernel-based distributed observer, which, for the first time, ensures fixed-time convergence properties. The communication network between the agents is prescribed by a directed graph in which each node involves a fixed-time convergent estimator. The local observer estimates and broadcasts the observable states among neighbors so that the full-state vector can be recovered at each node and the estimation error reaches zero after a predefined fixed time in the absence of perturbation. This represents a new distributed estimation framework that enables faster convergence speed and further reduced information exchange compared to a conventional Luenberger-like approach. The ubiquitous time-varying communication delay across the network is suitably compensated by a prediction scheme. Moreover, the robustness of the algorithm in the presence of bounded measurement and process noise is characterized. Numerical simulations and comparisons demonstrate the effectiveness of the observer and its advantages over the existing methods.

Index Terms—Communication network, distributed observer, fixed-time convergence, Volterra operator.

I. INTRODUCTION

Large-scale systems are encountered more frequently in real-world applications, such as power networks, intelligent transportation systems, and other cyber-physical systems. Such systems have an increasing demand for flexibility and scalability. The continuous growth of communication technology has enabled the development of decentralized and distributed solutions, which can perform collaborative tasks by using multiagent communications. This has posed new challenges in control theory, including distributed consensus control, distribution estimation, and so on [1], [2].

State estimation represents one of the most important problems in control. Motivated by previous developments in the centralized observer, this article focuses on the distributed observer, where the outputs of a large-scale system are measured by a sensor network and only a small portion of the system output is available at each sensor node. Therefore, the main challenge is that the state of the system is not fully observable at any sensor node. The goal is to design a distributed observer, such that the full state of the system can be collaboratively reconstructed by each agent using local measurement and proper neighboring communication [3], [4], [5], [6].

A variety of distributed linear time-invariant (LTI) state estimation approaches have been reported in the literature under different formulations inherit from the centralized approaches, including the Kalman filter and Luenberger observer. A comprehensive overview of existing distributed observers can be found in [7]. It is noteworthy that the design of a distributed observer is highly influenced by the communication graph. In [8], a distributed Kalman filtering algorithm was proposed for an undirected and connected communication graph. With the same assumption on the communication graph, a distributed Luenberger-type observer was presented in [9]. More recently, research efforts are paid to more general directed graphs [10], [11], [12], [13]. Necessary and sufficient observability conditions for designing a distributed observer are stated in [11], [14], [15]. The study [16], on top of the Luenberger observer-based scheme, introduces a multihop staircase decomposition mechanism, which makes it possible to lower information exchange and to relax the common assumptions of strongly connected graphs compared to the majority of distributed observers in the literature. Most of the existing methods are based on the Luenberger observer, which permits a single-agent-based design and implementation and ensures that the local state estimate of each agent asymptotically converges to the system state. On the other hand, Slim et al. [17], [18] propose alternative solutions to distributed state estimation by using the homogeneous technique [19]. As such, the state estimation error decays within a small finite time.

An important challenge in distributed estimation that has not been extensively addressed is the communication delay throughout the network. In the majority of existing works, the effect of delays is omitted, while its presence may drastically influence the estimation performance. Very recently, Slim et al. [20] propose a time-delay distributed observer, which guarantees exponential stability in the presence of time-varying but conservatively known (upper bound is available) communication delays, and the convergence rate can be designed up to a maximum total delay.

In this article, we study the distributed observer problem of a continuous-time LTI system, where the communication between agents may involve time-varying delays, as assumed in [20]. The main contribution of this article lies in a novel fixed-time convergent distributed observer based on a cross-agent information sharing mechanism. The method provides an example of how distributed estimation systems can benefit from fixed-time convergence properties. The key to the fixed-time observer is the Volterra integral operators with specialized kernel functions, as inspired by the centralized counterpart [21].

Manuscript received 17 July 2022; accepted 17 September 2022. Date of publication 4 October 2022; date of current version 28 July 2023. This work was supported in part by EPSRC under Grant EP/W028662/1 and in part by The Royal Society under Grant RGS/R1/21125. The work of Peng Li was supported by the Guangdong Basic and Applied Basic Research Foundation under Grant 2021A151510262 and Grant 2022A151511274. Recommended by Associate Editor D. Efimov. (Corresponding author: Fei Teng.)

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2022.3212005.

Digital Object Identifier 10.1109/TAC.2022.3212005
contrast to the majority of methods in the literature that require the full-dimensional state estimates to be shared among neighboring nodes, the proposed scheme enables a reduction of the transmitted data over the communication links by invoking a rank condition and the effect of delays in communication networks is compensated. Finally, the robustness of the proposed method against measurement noise and perturbations is characterized.

The rest of this article is organized as follows. The state estimation problem formulation and mathematical preliminaries are given in Section II. Section III introduces the main algorithm, and its robustness against disturbances and measurement noise is analyzed in Section IV. Simulation examples are presented in Section V. Finally, Section VI concludes this article.

II. PROBLEM STATEMENT AND PRELIMINARY

A. Problem Setting

Notation: Let $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq0}$ denote the real, the nonnegative real sets, and the strict positive real sets of numbers, respectively. Given a vector $x \in \mathbb{R}^n$, we denote $|x|$ as the Euclidean norm of $x$. Given a time-varying vector $x(t) \in \mathbb{R}^m, t \in \mathbb{R}_{\geq0}$, we will denote $|x|_\infty = \sup_{t \geq 0} |x(t)|$, as the quantity $|x|_\infty = \sup_{t \geq 0} |x(t)|$. Assuming that $x(t)$ is kth order differentiable, the kth order derivative of $x(t)$ is denoted by $x^{(k)}(t)$.

In this article, a directed graph is denoted by $G = (\mathcal{N}, \mathcal{E}, A)$, where $\mathcal{N} = \{1, 2, \ldots, N\}$ is a set of nodes, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is a set of edges, and $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ denotes the adjacency matrix. The element $a_{ij}$ is the weight of the edge $(i, j)$, and $a_{ii} = 1$ if and only if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Specifically, $(i, j) \in \mathcal{E}$ means that the ith node can send information to the jth node. The set of neighbors of node $i$ is denoted by $N_i = \{i : (i, j) \in \mathcal{E}\}$. A graph $G$ is strongly connected if there exists a directed path between $\forall i, j \in \mathcal{N}, i \neq j$.

Consider the following continuous LTI system:

$$\dot{x} = Ax, \quad y = Cx$$

(1)

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^m$ is the output, $A \in \mathbb{R}^{n \times n}$, and $C \in \mathbb{R}^{m \times n}$. The system (1) is sensed by N distributed agents $y_i = C_i x$ with $y = \{y_1, y_2, \ldots, y_N\}$, where $y_i \in \mathbb{R}^m, \sum_{i=1}^{N} m_i = m$ and $C = \{C_1, C_2, \ldots, C_N\}$. For each node/subsystem $i \in \mathcal{N}$, $y_i$ is the only output that is available for node $i$. Neighboring relations between distinct pairs of agents are characterized by a directed graph $G$. We assume throughout that $C_i \neq 0 \forall i \in \mathcal{N}$ and $C_i \neq C_j, i \neq j, i, j \in \mathcal{N}$. For the sake of further analysis, let $\mathcal{O} \triangleq \text{obs}(A, C)$ and $\mathcal{O}_I \triangleq \text{obs}(A_i, C_i)$ be the observability matrices of the pairs $(A, C)$ and $(A_i, C_i)$, respectively. The date transmission between agents may be impact by time-varying delays.

Assumption 1: The pair $(A, C)$ is observable, but the pair $(A_i, C_i)$ is not fully observable.

The problem investigated in this article is defined as follows.

Problem 1: Given the system (1) subject to a communication topology $G$, how to design a distributed observer with the estimated state $\hat{x}_i, \forall i \in \mathcal{N}$, such that the estimation error goes to 0 within a fixed time,

$$|\hat{x}_i(t) - x(t)| = 0 \quad \forall t \geq \tau$$

(2)

where $\tau \in \mathbb{R}_{>0}$ is a known finite time.

B. Volterra Operator and Bivariate Feedthrough Nonasymptotic Kernel (BF-NK)

Volterra operator and nonasymptotic kernel functions are the key tools to the observer design in this article. To introduce later a distributed fixed-time observer, here we briefly recall the basic concepts [21].

Given a function belongs to the Hilbert space of locally integrable function with domain $\mathbb{R}_{>0}$ and range $\mathbb{R}$, i.e., $w \in \mathcal{L}_{}^{2}(\mathbb{R}_{>0})$, its image by the Volterra operator $V_k$ induced by a Hilbert–Schmidt ($\mathcal{H}_S$) kernel function $K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is denoted by $[V_k w]$ of the form

$$[V_k w](t) = \int_{0}^{t} K(t,\tau)w(\tau)d\tau, \quad t \in \mathbb{R}_{>0}.$$

Definition 1 (BF-NK [21]): If a kernel $K \in \mathcal{H}_S$, which is at least $(i-1)$th order differentiable with respect to the second argument, verifies the conditions $K^{(j)}(t,0) = 0 \forall t \in \mathbb{R}_{>0}$ and $K^{(j)}(t,\tau) \neq 0 \forall \tau \neq 0$ for all $j \in \{0, 1, \ldots, i-1\}$, it is called an $r$th order BF-NK.

Lemma 1 (22]): For a given $i \geq 0$, consider a signal defined as a function of time $w \in \mathcal{L}_{}^{2}(\mathbb{R}_{>0})$ that admits the $r$th derivative in $\mathbb{R}_{>0}$ and a kernel function $K \in \mathcal{H}_S$, having the $r$th derivative with respect to the second argument, denoted as $K$. After successive integral by parts, it holds that

$$[V_k w](t) = \sum_{j=0}^{i-1} (-1)^{i-j-1} w^{(j)}(t)K^{(i-j-1)}(t, t) + \sum_{j=0}^{i-1} (-1)^{i-j} w^{(j)}(0)K^{(i-j-1)}(0, t) + (-1)^i [V_k w](0),$$

(3)

which shows that the function $[V_k w](t)$ is nonanticipative with respect to the lower order derivatives $w, w^{(1)}, \ldots, w^{(i-1)}$.

Owing to the definition of the BF-NK, induced by a BF-NK $K_h$, the Volterra image (3) reduces to

$$[V_k w](t) = \sum_{j=0}^{i-1} (-1)^{i-j-1} w^{(j)}(t)K^{(i-j-1)}(t, t) + (-1)^i [V_k w](0),$$

(4)

A typical class of $\delta$th order BF-NKs that we will use in this article have the form

$$K_h(t, \tau) = e^{-\omega_h(t-\tau)}(1 - e^{-\overline{\omega}_h\delta}),$$

which is parameterized by $\omega_h \in \mathbb{R}_{>0}$ and $\overline{\omega}_h \in \mathbb{R}_{>0}$. As it can be seen, all the nonasymptoticity conditions up to the $\delta$th order are met, thanks to the factor $(1 - e^{-\overline{\omega}_h\delta})$, regardless of the choice of $\omega_h$ and $\overline{\omega}_h$. The Volterra image signal $[V_k w](t)$ is bounded, and $\overline{\omega}_h$ strictly positive, it holds that the Volterra operators...
induced by the proposed kernels is bounded-input–bounded-output (BIBO) stable with respect to $w$.

### III. FIXED-TIME CONVERGENT DISTRIBUTED OBSERVER

In this section, the solution method to Problem 1 is presented. In the first instance, data transmission and communication delays within the sensor network are omitted. Under such a condition, a new distributed observer framework is designed. Then, the algorithm is modified to accommodate various network delays.

#### A. Delay-Free Case

From Assumption 1, the state vector $x$ is not fully observable from a single sensor node. Nevertheless, by resorting to the commonly used observability decomposition technique of each subsystem [9], [10], [18], it is possible to partially estimate $x$. Let $n_i$ denote the rank of the observability matrix of $(A, C_i)$, which is $n_i = \text{rank}(O_i) < n$. There exists an orthogonal matrix $T_i \in \mathbb{R}^{n \times n}$ that enables the state transformation $\tilde{x}_i = T_i x$, and $\tilde{x}_i$ admits the following decomposition $\tilde{x}_i = [\tilde{x}_{io} \ \tilde{x}_{ia}]^T = [T_{io} \ T_{ia}]^T x$, where $\tilde{x}_{io}$ represents the observable part and $\tilde{x}_{ia}$ stands for the unobservable part. The dynamics of $\tilde{x}_i$ follows $\dot{\tilde{x}}_i = \tilde{A}_i \tilde{x}_i, \ y_i = C_i \tilde{x}_i$, where

$$\tilde{A}_i = T_i A T_i^T = \begin{bmatrix} A_{io} & 0 \\ A_{ia} & A_{ia} \end{bmatrix}, \quad \tilde{C}_i = C_i T_i^T = [C_{io} \ 0]$$

with $A_{io} \in \mathbb{R}^{n_i \times n_i}, \ A_{ia} \in \mathbb{R}^{(n-n_i) \times (n-n_i)}, \ A_{ia} \in \mathbb{R}^{(n-n_i) \times n_i}, \ C_{io} \in \mathbb{R}^{m \times n_i}$, $T_{io} \in \mathbb{R}^{n \times n_i}$, and $T_{ia} \in \mathbb{R}^{n \times (n-n_i) \times n}$. Furthermore, $(A_{io}, C_{io})$ is observable, and the dynamics of the observer part is governed by

$$\dot{\tilde{x}}_{io} = A_{io} \tilde{x}_{io}, \ y_i = C_i \tilde{x}_{io}. \quad (7)$$

Next, a finite and fixed time convergent observer [21] is applied to estimate the observable part $\tilde{x}_{io} \in \mathbb{R}^{n_i}$, which will then be used to recover the full-state vector through communication.

Thanks to the observability of $(A_{io}, C_{io})$, there exists a linear coordinate transformation $z_i = T_{io} \tilde{x}_{io}$ with $T_{io} \in \mathbb{R}^{n \times n_i}$ such that the system (7) can be rewritten in the observer canonical form with respect to $z_i$

$$\dot{z}_i = A_{i,z} z_i, \ y_i = C_{i,z} z_i \quad (8)$$

where

$$A_{i,z} = T_{io} A_{io} T_{io}^{-1} = \begin{bmatrix} a_{n_i-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_1 & 0 \\ 0 & \cdots & 0 & a_0 \end{bmatrix}.$$
From Lemma 2 and the fact that $T_{iα}$ is full rank, we have

$$\begin{align*}
\text{rank}[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})] &+ \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})] = \sum_{i \in \mathcal{N}_i} n_i \\
\leq \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})] &+ \text{rank}[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})] = \sum_{i \in \mathcal{N}_i} n_i \\
\leq \min\{\text{rank}[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})], \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})]\} &+ \min\{\text{rank}[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})], \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})]\} \\
\downarrow & (a)\quad \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})] \leq \min\{\text{rank}[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})], \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})]\} \\
\text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})] &+ \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})] = \text{rank}[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})] = \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})]
\end{align*}$$

where $(a)$ comes from the fact that

$$\max_{i \in \mathcal{N}_i} n_i \leq \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})] \leq \text{rank}[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})] = \sum_{i \in \mathcal{N}_i} n_i.$$ 

This completes the proof.

Proposition 1 bridges the gap between traditional observability conditions based on the system matrices and the invertibility of the transformation matrix $T_{iα}$, which is instrumental for the following analysis.

**Lemma 3 (27):** For a given directed graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$, $\{c_{i1}\} \in \mathbb{R}^{N \times N}$ is full rank, we have

$$\text{rank}\[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})\] + \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})] = \sum_{i \in \mathcal{N}_i} n_i \leq \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})] + \text{rank}[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})] \leq \min\{\text{rank}[\text{diag}_{i \in \mathcal{N}_i}(T_{iα})], \text{rank}[\text{col}_{i \in \mathcal{N}_i}(T_{iα})]\}.$$

**Algorithm 1:** Offline Optimisation of the Data Acquistion Scheme.

**Input:** system matrices $A$ and $C$; graph adjacency matrix $A$; node number $N$

**Output:** optimized CN sets $\mathcal{CN}_{\text{opt}}$

1. **Initialisation:** iteration index $k = 1, l = 1$;
2. **while** $\mathcal{CN}_{\text{opt}}$ is not obtained **do**
3. calculate $D_l$;
4. for $k \leftarrow 1$ to $N$ **do**
5. if $\mathcal{CN}_{\text{opt}}^{\text{opt}}$ is not obtained then
6. calculate $N_k^\text{opt}$ based on $D_l$
7. optimise $\mathcal{CN}_{\text{opt}}^{\text{opt}}$ using (16) and identify $T_{jα}$
8. end
9. end
10. $l = l + 1$;
11. end

Consider $h^L_i$ the $L$-step neighboring set of node $i$ required for exchanging data (possibly cross-agent) with agent $i$, $\min|\mathcal{CN}_{\text{opt}}|$ is found according to

$$\{h^1_i, h^2_i, \ldots, h^L_i\} = \arg\min_{h^L_i} \{\mathcal{CN}_{\text{opt}}\},$$

such that

$$\text{rank}\left[\begin{bmatrix}
T_{iα} \\
\text{col}_{j \in \mathcal{CN}_{\text{opt}}}(T_{jα})
\end{bmatrix}\right] = n \quad (16)$$

provided $P < |\mathcal{N}|$ the minimum step value to ensure the rank condition, such that

$$\text{rank}\left[\begin{bmatrix}
T_{iα} \\
\text{col}_{j \in \mathcal{CN}_{\text{opt}}}(T_{jα})
\end{bmatrix}\right] = n > \text{rank}\left[\begin{bmatrix}
T_{iα} \\
\text{col}_{j \in \mathcal{CN}_{\text{opt}}}(T_{jα})
\end{bmatrix}\right]$$

It is worth noting that the optimization problem (16) does not necessarily lead to the minimum $|\mathcal{CN}_{\text{opt}}|$ in a global sense as a smaller $|\mathcal{CN}_{\text{opt}}|$ may be obtained by searching up to a step value greater than $P$. However, by constraining the outreach step value at $P$, it is beneficial for mitigating the impact of cross-agent communication delay, as will be discussed later on in Section III-B. By (16), we provide the offline optimization algorithm for the selection of a CN set as summarized in Algorithm III-A. It is noteworthy that under Assumption 3, the solution to the optimization problem (16) may not be unique unless additional constraints are imposed. Moreover, in case that $\sum_{j} n_j > n - n_i, j \in \mathcal{CN}_{\text{opt}}$, the matrix $\begin{bmatrix}
T_{iα} \\
\text{col}_{j \in \mathcal{CN}_{\text{opt}}}(T_{jα})
\end{bmatrix}$ has more than $n$ rows, which implies information redundancy. In this context, Algorithm III-A also extracts $n - n_i$ rows from $\begin{bmatrix}
T_{iα} \\
\text{col}_{j \in \mathcal{CN}_{\text{opt}}}(T_{jα})
\end{bmatrix}$ without further constraints, for any agent $i$, its CN set may not be unique and redundant information might be exchanged. In the following, we show how to find a class of optimized CN sets $\mathcal{CN}_{\text{opt}} = \{\mathcal{CN}_{\text{opt}}^1, \ldots, \mathcal{CN}_{\text{opt}}^N\}$ in terms of the communication cost for the proposed distributed observer, and how to avoid redundant data exchange.

**Assumption 3:** The graph $\mathcal{G}$ modelling the communication network of the distributed system (1) has an equal communication cost across all edges.
discussion of this subject is beyond the scope of this article, but it is envisaged to be done in future work.

As \( T_{ij} \), \( \forall j \in \mathcal{N}_i^{opt} \) is determined offline, it is known to each node \( i \) when the communication network is initialized. Furthermore, the datasets required by each node \( i \) in real time for global state observation is defined as

\[
\mathcal{T}_{ij} = \{ \hat{z}_j^i \} \quad \forall j \in \mathcal{N}_i^{opt}
\]

where \( \hat{z}_j^i \) is the local estimate of the \( z_j^i \) that fulfills \( z_j^i = T_{ij} x \).

In view of the linear relation (14), each agent \( i \) can obtain the full-state vector by

\[
\hat{x}_i(t) = e^{A^T} \begin{bmatrix} T_{io}^\tau \left( T_{ji}^\tau \right)^{-1} \hat{z}_i(t - \tau) \\ \text{col}_{j \in \mathcal{N}_i^{opt}} (\hat{z}_j^i(t - \tau)) \end{bmatrix} \quad \forall t > 0
\]

provided the datasets \( \mathcal{T}_{ij} \), \( \forall j \in \mathcal{N}_i^{opt} \) via communication. Hence, the fixed-time convergent condition (2) can be achieved. However, in practice, due to the various delays except in the network, (18) will not work without further provisions, which will be provided in the next section.

**B. Delayed Case**

We now have all the ingredients to propose our main algorithm for the practical case, where network delays exist. For the sake of further analysis, let \( \tau_{ij} \) be the time-varying delay present in gathering the information set \( \mathcal{T}_{ij} \) from \( j \).

**Assumption 4:** For any \( i \in \mathcal{N}, \) the accumulated delays is bounded, such that \( \sum_{j} \tau_{ij} \leq \tau \quad \forall j \in \mathcal{N}_i^{opt}, \) with \( \tau \) a known positive constant.

Under Assumption 4, we assume that all the sensor nodes and observers have synchronized clocks and include time-stamps in the data transmission [28]. As such, each node \( i \) can identify at \( t \geq \tau \) a set of \( \mathcal{T}_{ij}(t - \tau) \) \( \forall j \in \mathcal{N}_i^{opt} \) with synchronized delay. Combined with \( \hat{z}_i(t - \tau) \) the distributed observer can be designed, as shown in the following theorem.

**Theorem 1 (Distributed Fixed-Time Observer):** Under Assumptions 1, 2, and 4, given the distributed system (1), the fixed-time estimation scheme (13) and the intercommunication mechanism determined by Algorithm 1, for each node \( i \in \mathcal{N}, \) the local state estimate \( \hat{x}_i(t) \) \( \forall t \geq \tau + t_d \) obtained by

\[
\hat{x}_i(t) = e^{A^T} \begin{bmatrix} T_{io}^\tau \left( T_{ji}^\tau \right)^{-1} \hat{z}_i(t - \tau) \\ \text{col}_{j \in \mathcal{N}_i^{opt}} (\hat{z}_j^i(t - \tau)) \end{bmatrix}
\]

for all \( t \geq \tau + t_d \) is equal to \( x(t) \), such that the condition (2) is fulfilled.

**Proof:** Thanks to the finite-time local observers (13) that are activated at \( t = t_d \) and the information sets \( \mathcal{T}_{ij} \) received from the CN set, node \( i \) is able to reconstruct delayed estimates

\[
\begin{bmatrix} \hat{z}_i(t - \tau) \\ \text{col}_{j \in \mathcal{N}_i^{opt}} (\hat{z}_j^i(t - \tau)) \end{bmatrix} = \begin{bmatrix} z_i(t - \tau) \\ \text{col}_{j \in \mathcal{N}_i^{opt}} (z_j^i(t - \tau)) \end{bmatrix}
\]

for all \( t \geq \tau + t_d \) is equal to \( x(t) \). Hence, from (18) and (1), it is immediate to show the following relationship by using (19):

\[
\hat{x}_i(t) = e^{A^T} x(t - \tau) = x(t) \quad \forall t \geq \tau + t_d
\]

which completes the proof. \( \square \)

**Remark 2:** In contrast to the existing distributed observers (e.g., Luenberger-like observers) where agents only communicate with their neighbors (i.e., \( \mathcal{N}_i \)), the proposed method relies on a cross-agent communication strategy that enables an agent \( i \) to communicate with \( j \notin \mathcal{N}_i \). This feature enables the optimization Algorithm III-A, and the resulting data flow may turn out to be efficient and useful in practice to reduce communication load. More specifically, in the proposed distributed observer, the accumulated data flow into a node is of dimension \( (n - n_i) \); thus, the dimension of the data flow through a communication channel (i.e., an edge, in one direction) is below \( (n - n_i) \). However, in Luenberger-like distributed observers [10], the data transmitted along any edge are of dimension \( n \), and each node has to manage to collect \( n \mid \mathcal{N}_i \mid \)-dimensional data. Despite the delay introduced by the cross-agent communication, the influence of a bounded delay can be compensated using an open-loop prediction scheme.

**Remark 3:** With the proposed cross-agent communication strategy, the proposed estimation scheme remains valid if the outputs \( y_i \) are shared instead of the local state estimates \( z_i \) (see Proposition 1 that builds the connection between conventional observability and the invertibility of the coordinate transformation from \( x \) to \( z \)). Nevertheless, sharing the outputs directly may sacrifice privacy-preserving properties of the method. Particularly, when one or more sensors/communication links are attacked, it could expose more sensor nodes to the attacker. For this reason, the state estimate sharing strategy is adopted in the proposed framework. Moreover, the cross-agent communication strategy may be applied to either Luenberger-type asymptotic [10] or finite-time [17] observers, which can also leads to reduced communication, as discussed in Remark 2.

**IV. ROBUSTNESS ANALYSIS OF THE OBSERVER**

This section analyzes the robustness of the proposed observer against measurement and process disturbances. Assuming the presence of the bounded model uncertainty and sensor disturbance, \( \| d_x \|_{\infty} \leq \bar{d}_x, \| d_y \|_{\infty} \leq \bar{d}_y \) in (1), such that

\[
\dot{x}_d = Ax_d + dz_x, \quad y_d = Cx_d + dy
\]

where \( x_d \) denotes state variable under the effect of \( d_x(t) \). In this context, for ith subsystem, it holds that \( \dot{\hat{x}}_{i,d} = \dot{A}_i \hat{x}_{i,d} + T_{io} \dot{z}_i + T_{io} dz_i, y_{d,i} = C_i \hat{x}_{i,d} + d_{y,i} \), where \( \hat{x}_{i,d} = [\hat{x}_{i,0,d}, \ldots, \hat{x}_{i,n_i-1,d}]^T \). By analogy to (8), the observable part follows

\[
\dot{\hat{z}}_{i,d} = A_i \hat{z}_{i,d} + dz_{i,d}, y_{d,i} = C_i \hat{z}_{i,d} + d_{y,i}
\]

where \( dz_{i,d} = [dz_{i,0,d}, \ldots, dz_{i,n_i-1,d}]^T \) with

\[
\gamma_{i,h} \hat{z}_{i,d} = \lambda_{i,h,d}
\]

In the noisy environment, the state estimator (13) gives

\[
\hat{x}_{i,d} = \Gamma_i^{-1} \hat{A}_i \hat{x}_i \quad \forall t \geq t_d
\]

and

\[
\hat{\lambda}_{i,h,d} = (-1)^{n_i-1} \Gamma_i^{-1} V_{K_i^{(p)}} C_i \hat{z}_{i,d} + \sum_{p=0}^{n_i-1} \lambda_{i,h,d} (-1)^p V_{K_i^{(p)}} d_{y,i}
\]

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\]

Comparing (23) and (24), the estimation error of \( z_{i,d} \) takes on the form

\[
e_{z,i} \triangleq z_{i,h,d} - \hat{z}_{i,h,d} = \Gamma_i^{-1} e_{x,i}\]
where \( \epsilon_{A,i} = [\epsilon_{i,0}, \epsilon_{i,1}, \ldots, \epsilon_{i,n_i-1}] \), and

\[
\epsilon_{i,h} \triangleq \lambda_{i,h} - \hat{\lambda}_{i,h} = \sum_{p=0}^{n_i-1} (-1)^p [V_{K(p)} d_{i,n_i-1-p} - \sum_{p=0}^{n_i-1} a_p (-1)^p [V_{K(p)} d_{i,p}]].
\]

The effects of both measurement noise \( d_{y,i,p} \) and model uncertainty \( d_{e} \) are embedded in \( \epsilon_{A,i} \) in the form of Volterra images, i.e., \( [V_{K(p)} d_{y,i}] \triangleq \epsilon_{y,i,p,h} \) and \( [V_{K(p)} d_{x,i}] \triangleq \epsilon_{d,i,p,h} \). Consequently, \( \epsilon_{y,i,p,h} \) are not affected by model uncertainty and disturbances. Therefore, the distributed observation of error defined in (25) is bounded by

\[
|\epsilon_{i,h}| \leq \tau_{d_{y,i}},n_i + \sum_{p=0}^{n_i-1} |a_p| \tau_{e_{y,i}},p,h + \sum_{p=0}^{n_i-1} \tau_{d_{x,i}},p,h \triangleq \tau_{e_{i,h}}.
\]

As such, by stacking \( \tau_{e_{i}},h \) induced by different kernels, one can obtain the vector bound as \( \tau_{\epsilon_{A,i}} \triangleq [\tau_{e_{i,0}}, \ldots, \tau_{e_{i,n_i-1}}] \). Consequently, the observation error defined in (25) is bounded by \( \tau_{\epsilon_{A,i}} \leq \|\Gamma_i^{-1}\| \|\tau_{A,i}\| \).

Taking the communication delay into account, the compensation in (19) is written as

\[
\dot{\hat{x}}_{i,d}(t) = e^{A_\tau} \left[ \begin{array}{c} T_{i_\alpha} \\ \text{col}_{j \in \mathcal{CN}_i^{\text{opt}}} (T_{i_\alpha}) \end{array} \right]^{-1} \left[ \begin{array}{c} \hat{x}_{i,d}(t - \tau) \\ \text{col}_{j \in \mathcal{CN}_i^{\text{opt}}} (\hat{x}_{j,d}(t - \tau)) \end{array} \right].
\]

However, recalling (21), during the delay \( \tau \), \( d_{x} \) introduce extra effects that can be expressed as

\[
\epsilon_{d,x,\tau} = \int_{t-\tau}^t e^{A(t-\tau)} d_x(t) d\tau.
\]

Being \( A \) Hurwitz, it is straightforward to conclude that \( \epsilon_{d,x,\tau} \) is bounded with an upper bound \( \tau_{\epsilon_{d,x,\tau}} \geq \|f_\tau - e^{A(t-\tau)} d_x(t) d\tau\| \).

Notably, for any \( i \in \mathcal{N} \) in (18), \( \Gamma_i \) and \( T_{i_\alpha} \) are not affected by model uncertainty and disturbances. Therefore, the distributed observation of \( x_{i,d} \) remains bounded as long as \( d_x \) and \( d_{e} \) are bounded, i.e., for all \( t \geq \tau + t_\tau \),

\[
|\epsilon_{i,d}| \leq \left[ \begin{array}{c} T_{i_\alpha} \\ \text{col}_{j \in \mathcal{CN}_i^{\text{opt}}} (T_{i_\alpha}) \end{array} \right]^{-1} \left[ \begin{array}{c} \tau_{x_i} \\ \text{col}_{j \in \mathcal{CN}_i^{\text{opt}}} (\tau_{x_j}) \end{array} \right] + \tau_{d_{x}},\tau.
\]

### V. NUMERICAL EXAMPLES

In this section, the effectiveness of the proposed distributed observer is examined by a few numerical examples. Consider a linear system [10] of order \( n = 6 \) with four local sensors, i.e., \( N = 4 \) and \( [n_1, n_2, n_3, n_4] = [2515] \). System parameters are given as follows:

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & -1 & 1 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 \\
-8 & 1 & -1 & -1 & -2 & 0 \\
4 & -0.5 & 0.5 & 0 & -0 & 4
\end{bmatrix}
\]
The state estimates of the proposed method in the delayed and noisy scenarios are shown in Fig. 4. Under the delayed network, the asymptotic method [10] shows the nonconvergent performance, while the error of the finite-time observer [17] stays bounded. However, the proposed method demonstrates its advantage in terms of dealing with network delays by showing the most accurate state estimation.

Finally, a noisy scenario is simulated where the outputs are corrupted by a uniformly distributed random noise within $[-0.2, 0.2]$ and the system dynamics are perturbed by a sinusoidal uncertainty $0.1 \sin(50\tau)$. The effects of both disturbances and the same delay considered in the previous example, the estimation error of all three methods are compared in Fig. 3, where the proposed method outperforms the other two in terms of steady-state accuracy. From the state estimates shown in Fig. 4, the proposed method converges within a fixed time $t_3 + \tau = 1.27$ s. This arises from that once the proposed observer is activated at $t_3 = 1$ s, it requires at most $\tau$ to transmit neighboring information ensuring the fully observable in each subsystem.

VI. Conclusion

A fixed-time convergent observer is proposed for distributed state estimation of a large-scale system with directed communication topologies. The fast convergence properties enable the data transmission delay to be compensated a posteriori. As such, cross-agent communication is utilized and it yields a more effective data exchange mechanism with an optimized (minimized) data flow. The boundedness of the estimation error has been confirmed, subject to bounded measurement and process disturbances. Numerical examples and comparisons with the existing method have been shown to verify the effectiveness of the proposed method. Future research efforts will be devoted to including a time-varying communication graph, event-triggered communication, as well as to consider more general nonlinear systems and cyber security issues.

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