THE DYNAMICS OF CONJUNCTIVE AND DISJUNCTIVE BOOLEAN NETWORKS

ABDUL SALAM JARRAH, REINHARD LAUBENBACHER, ALAN VELIZ-CUBA

Abstract. The relationship between the properties of a dynamical system and the structure of its defining equations has long been studied in many contexts. Here we study this problem for the class of conjunctive (resp. disjunctive) Boolean networks, that is, Boolean networks in which all Boolean functions are constructed with the AND (resp. OR) operator only. The main results of this paper describe network dynamics in terms of the structure of the network dependency graph (topology). For a given such network, all possible limit cycle lengths are computed and lower and upper bounds for the number of cycles of each length are given. In particular, the exact number of fixed points is obtained. The bounds are in terms of structural features of the dependency graph and its partially ordered set of strongly connected components. For networks with strongly connected dependency graph, the exact cycle structure is computed.

1. Introduction

The understanding of the relationship between structural features of dynamical systems and the resulting dynamics is an important problem that has been studied extensively in the dynamical systems literature. For example, work by Golubitsky and Stewart [1] about coupled cell dynamical systems given by coupled systems of ODEs attempts to obtain information about system dynamics from the topology of the graph of couplings. Albert and Othmer [2] used a Boolean network model to show that the expression patterns of the segment polarity in Drosophila are determined by the topology of its gene regulatory network. Thomas et al [3] conjectured that negative feedback loops are necessary for periodic dynamics whereas positive feedback loops are necessary for multistationarity. These conjectures have been the subject of many published articles [4; 5; 6]. In [7] we demonstrated that networks with a large number of independent negative feedback loops tend to have few limit cycles and these cycles are usually long.

In this paper we study the effect of the network topology on the dynamics of a family of Boolean networks. Boolean networks in general, and cellular automata in particular, have long been used to model and simulate a wide range of phenomena, from logic circuits in engineering and gene regulatory networks in molecular biology [2; 5; 6; 10; 11; 12] to population dynamics and the spread of epidemics [13; 14]. Especially for large networks, e.g., many agent-based simulations, it becomes infeasible to simulate the system extensively in order to obtain information about its dynamic properties, even if such simulation is possible. In such instances it...
becomes important from a practical point of view to be able to derive information about network dynamics from structural information. But the problem is of interest in its own right, in particular in the more general context of time-discrete dynamical systems over general finite fields. These have been studied extensively, see, for example, [12, 16, 17, 18, 19]. They have a wide range of applications in engineering [12, 20, 21, 22, 23, 24, 25] and recently in computational biology [26, 27].

Let $\mathbb{F}_2 := \{0, 1\}$ be the Galois field with two elements. We view a Boolean network on $n$ variables as a dynamical system $f = (f_1, \ldots, f_n) : \mathbb{F}_2^n \to \mathbb{F}_2^n$. Here, each of the coordinate functions $f_i : \mathbb{F}_2^n \to \mathbb{F}_2$ is a Boolean function which can be written uniquely as a polynomial where the exponent of each variable in each term is at most one [28]. In particular $\text{AND}(a, b) = ab$ and $\text{OR}(a, b) = a + b + ab$. We use polynomial forms of Boolean functions throughout this paper.

Two directed graphs are usually assigned to each such system: The dependency graph which encodes the static relationships among the nodes of the network and the phase space which describe the dynamic behavior of the network. In this paper we focus on the question of deriving information about the phase space of a Boolean network from the structure of its dependency graph. Next we define these graphs.

The dependency graph $\mathcal{D}(f)$ of $f$ has $n$ vertices corresponding to the Boolean variables $x_1, \ldots, x_n$ of $f$. There is a directed edge $i \to j$ if $x_i$ appears in the function $f_j$. That is, $\mathcal{D}(f)$ encodes the variable dependencies in $f$. It is similar to the coupling graph of Stewart and Golubitsky.

Example 1.1. The dependency graph of $f = (x_2x_3, x_1, x_2, x_3x_4, x_1x_6, x_3x_4x_5) : \mathbb{F}_2^6 \to \mathbb{F}_2^6$ is the directed graph in Figure 1.

![Figure 1. The dependency graph of f from Example 1.1](image)

The dynamics of $f$ is encoded by its phase space, denoted by $\mathcal{S}(f)$. It is the directed graph with vertex set $\mathbb{F}_2^n$ and a directed edge from $u$ to $v$ if $f(u) = v$. For each $u \in \mathbb{F}_2^n$, the sequence $\{u, f(u), f^2(u), \ldots\}$ is called the orbit of $u$. If $u = f^t(u)$ and $t$ is the smallest such number, the sequence $\{u, f(u), f^2(u), \ldots, f^{t-1}(u)\}$ is called a limit cycle of length $t$ denoted by $C_t$, and $u$ is called a periodic point of period $t$. The point $u$ is called a fixed point if $f(u) = u$. If every limit cycle is of length 1, the system $f$ is called a fixed-point system. Since $\mathbb{F}_2^n$ is finite, every orbit must include a limit cycle. We denote the cycle structure of $f$ in the form of the generating function

$$C(f) = \sum_{i=1}^{\infty} C(f)_i C_i,$$
where $C(f)_i$ denotes the number of cycles $C_i$ of length $i$ in the phase space of $f$. Since the phase space of $f$ is finite, $C(f)_i = 0$ for almost all $i$.

The height of $f$, denoted by $h(f)$, is the least positive integer $s$ such that $f^s(u)$ is a periodic point for all $u \in \mathbb{F}_2^n$. A component of the phase space $\mathcal{S}(f)$ consists of a limit cycle and all orbits of $f$ that contain it. Hence, the phase space is a disjoint union of components. We define the period of $f$, denoted by $p(f)$, to be the least common multiple of the lengths of all limit cycles in the phase space of $f$.

Example 1.2. Let $f : \mathbb{F}_2^3 \rightarrow \mathbb{F}_2^3$ be given by $f(x_1, x_2, x_3) = (x_2 x_3, x_1 + x_3, x_1 x_2)$. The phase space of $f$ has two components, containing one fixed point and one limit cycle of length two, that is $C(f) = C_1 + C_2$, see Figure 2. It is clear from the phase space that $h(f) = p(f) = 2$. The phase space in Figure 2 was generated using DVD [29].

![Figure 2. The phase space $\mathcal{S}(f)$ of the system $f$ from Example 1.2.](image)

Without exhaustive iteration and just by analyzing $\mathcal{D}(f)$ what can we say about $\mathcal{S}(f)$? Namely, what is the period of $f$, the height of $f$, or the generating function $C(f)$? This question is NP-hard in general, so it is important to limit the class of Boolean networks considered. Next we list some of the main known results.

- When all coordinate Boolean functions are the XOR function (that is, the functions are linear polynomials), the above questions have been answered completely. In fact the questions have been answered for linear systems over any Galois field [12; 20; 30]. We have developed and implemented algorithms that answer the questions above, see [31].
- For Boolean networks where all coordinate functions are symmetric threshold functions, it has been shown that all cycles in the phase space are either fixed points or of length two [32].
- For Boolean cellular automata with the majority rule, the number of fixed points was determined in [33], and
- In [34], the authors studied AND-OR networks (Boolean network where each local function is either an OR or AND function) with directed dependency graphs. Formulae for the maximum number of fixed points are obtained, and
- The main result in [35] is an upper bound for the number of fixed points in Boolean regulatory networks.

In this paper we focus on the class of conjunctive and disjunctive Boolean networks, that is, Boolean networks where all of their coordinate functions are either
the AND function or the OR function. The following represent main previous attempts to mathematically analyze this class of Boolean networks.

- A family of this class of networks have been analyzed in [36]. The authors studied conjunctive Boolean networks (which they called OR-nets) and disjunctive Boolean networks (AND-nets) on undirected dependency graph. That is, on graphs where each edge is bidirectional and hence the dependency graph consists of cycles of length two. In particular, their result that OR-nets have only fixed points and possibly a limit cycle of length two [36, Lemma 1] follows directly from our results as we explain in Remark 2.8.

- In [37], the authors study a smaller family where each edge is undirected and each node in the network has a self loop. In particular, they showed that disjunctive Boolean networks (which they called OR-PDS) have only fixed points as limit cycles [37, Theorem 3.3]; this follows from our results as we show in Remark 2.8.

- Another family consists of the conjunctive Boolean cellular automata and was analyzed in [16]. In particular, upper bounds for the number of cycles are given. Here we study the whole class of conjunctive Boolean networks and provide lower and upper bounds for the number of their limit cycles. In particular, we present formulas for the exact number of fixed points of any conjunctive or disjunctive boolean network, see Equation (7.4).

In this paper we focus only on conjunctive Boolean networks, since for any disjunctive Boolean network there exists a conjunctive Boolean network that has exactly the same dynamics after relabeling the 0 and 1 and hence we have the following theorem.

**Theorem 1.3.** Let \( f = (f_1, \ldots, f_n) : \mathbb{F}_2^n \to \mathbb{F}_2^n \) where \( f_i = x_{i_1} \lor \cdots \lor x_{i_j} \) be any disjunctive Boolean network. Consider the conjunctive Boolean network \( g = (g_1, \ldots, g_n) : \mathbb{F}_2^n \to \mathbb{F}_2^n \), where \( g_i = x_{i_1} \land \cdots \land x_{i_j} \). Then the two phase spaces \( S(f) \) and \( S(g) \) are isomorphic as directed graphs.

**Proof.** Let \( \neg : \mathbb{F}_2^n \to \mathbb{F}_2^n \) be defined by \( \neg(x_1, \ldots, x_n) = (1 + x_1, \ldots, 1 + x_n) \). Then it is easy to see that \( f(x_1, \ldots, x_n) = (\neg \circ g \circ \neg)(x_1, \ldots, x_n) \). Thus \( S(f) \) and \( S(g) \) are isomorphic directed graphs. \( \square \)

**Remark 1.4.** Let \( G \) be a graph on \( n \) vertices such that the in-degree for each vertex is non-zero (\( G \) has no source vertex). Then there is one and only one conjunctive network \( f \) on \( n \) nodes such that \( D(f) = G \). Thus there is a one-to-one correspondence between the set of all conjunctive Boolean networks on \( n \) nodes where none of the local functions is constant and the set of directed graphs on \( n \) vertices where none of the vertices is a source.

This correspondence was used in [18] to find the period of a given Boolean monomial system (conjunctive Boolean network) and to decide when that system is a fixed point system as we will recall in the next section. In this paper we present upper and lower bounds on the number of cycles of any length in the phase space of any conjunctive Boolean network. Furthermore, we give upper bounds for the height.

In the next section, we recall some results from graph theory as well as results about powers of positive matrices that we will use to obtain upper bounds for the lengths of transients.
2. The Relationship Between Dependency Graph and Dynamics

Let \( f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n \) be a conjunctive Boolean network, \( G = \mathcal{D}(f) \) and \( A \) the adjacency matrix of \( G \). We will assume here and in the remainder of the paper that none of the Boolean coordinate functions of \( f \) are constant, that is, all vertices of \( G \) have positive in-degree.

2.1. The Adjacency Matrix. Define the following relation on the vertices of \( G \): \( a \sim b \) if and only if there is a directed path from \( a \) to \( b \) and a directed path from \( b \) to \( a \). It is easy to check that \( \sim \) is an equivalence relation. Suppose there are \( t \) equivalence classes \( V_1, \ldots, V_t \). For each equivalence class \( V_i \), the subgraph \( G_i \) with the vertex set \( V_i \) is called a strongly connected component of \( G \). The graph \( G \) is called strongly connected if \( G \) has a unique strongly connected component.

There exists a permutation matrix \( P \) that permutes the rows and columns of \( A \) such that

\[
PA P^{-1} = \begin{bmatrix}
A_1 & A_{12} & \cdots & A_{1t} \\
0 & A_2 & \cdots & A_{2t} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_t
\end{bmatrix}
\]

where \( A_i \) is the adjacency matrix of the component \( G_i \), and \( A_{ij} \) represents the edges from the component \( G_i \) to \( G_j \), see [38, Theorem 3.2.4]. The form in (2.1) is called the Frobenius Normal Form of \( A \).

Remark 2.1. The effect of the matrix \( P \) on the dependency graph is the relabeling of the vertices of \( G \) such that the diagonal blocks correspond to strongly connected components of \( G \). In Example 1.1 above, the adjacency matrix of the dependency graph is in the normal form.

Example 1.1 (Cont.). The dependency graph \( G \) in Figure 1 has 3 strongly connected components and their vertex sets are: \( V_1 = \{1, 2, 3\} \), \( V_2 = \{4\} \), and \( V_3 = \{5, 6\} \), see Figure 3 (left).

For any non-empty strongly connected component \( G_i \), let \( h_i \) be the conjunctive Boolean network with dependency graph \( \mathcal{D}(h_i) = G_i \). Let \( h : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n \) be the conjunctive Boolean network defined by \( h = (h_1, \ldots, h_t) \). That is, the dependency graph of \( h \) is the disjoint union of the strongly connected graphs \( G_1, \ldots, G_t \).

Example 1.1 (Cont.). The conjunctive Boolean network \( h \) corresponding to the disjoint union is \( h : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n \) and given by \( h(x_1, \ldots, x_6) = (h_1(x_1, x_2, x_3), h_2(x_4), h_3(x_5, x_6)) \) where

\[
h_1(x_1, x_2, x_3) = (x_2x_3, x_1, x_2), \quad h_2(x_4) = x_4, \quad h_3(x_5, x_6) = (x_5, x_6).
\]

Now define the following relation among the strongly connected components \( G_1, \ldots, G_t \) of the dependency graph \( \mathcal{D}(f) \) of the network \( f \).

(2.2) \( G_i \preceq G_j \) if there is at least one edge from a vertex in \( G_i \) to a vertex in \( G_j \).

Since \( G_1, \ldots, G_t \) are the strongly connected components of \( \mathcal{D}(f) \), the set of strongly components with the relation \( \preceq \) is a partially ordered set \( \mathcal{P} \). In this paper, we relate the dynamics of \( f \) to the dynamics of its strongly connected components and their poset \( \mathcal{P} \).

Example 1.1 (Cont.). The poset of the strongly connected components of \( f \) is in Figure 3 (right).
For any non-negative matrix $A$, the sequence $\{A, A^2, \ldots\}$ has been studied extensively, see, for example, [38, 39]. Next we use the Boolean operators AND and OR to examine the sequence of powers of $A$ and infer results about the conjunctive Boolean network that corresponds to an adjacency matrix $A$.

2.2. Powers of Boolean Matrices. Let $A, B$ be $n \times n$ Boolean matrices (all entries are either 0 or 1). Define $A \otimes B$ such that $(A \otimes B)_{ij} = \bigvee_{k=1}^{n} (A_{ik} \land B_{kj})$, where $\lor$ (resp. $\land$) is the Boolean OR (resp. AND) operator.

Proposition 2.2. Let $A, B$ be as above and let $f, g : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ be the two conjunctive Boolean networks that correspond to the adjacency matrices $A$ and $B$, respectively. That is, $f_i = x_1^{a_{i1}}x_2^{a_{i2}} \ldots x_n^{a_{in}}$ and $g_i = x_1^{b_{i1}}x_2^{b_{i2}} \ldots x_n^{b_{in}}$ for all $i$. Then the adjacency matrix corresponding to $f \circ g$ is $A \otimes B$.

Proof. It is easy to see that, for all $i$,

$$f_i(g_1, \ldots, g_n) = g_1^{a_{i1}} \ldots g_n^{a_{in}}$$
$$= (x_1^{b_{11}}x_2^{b_{12}} \ldots x_n^{b_{1n}})^{a_{i1}} \ldots (x_1^{b_{n1}}x_2^{b_{n2}} \ldots x_n^{b_{nn}})^{a_{in}}$$
$$= x_1^{a_{i1}b_{11} + \ldots + a_{in}b_{1n}} \ldots x_n^{a_{i1}b_{nn} + \ldots + a_{in}b_{nn}}$$
$$= \bigvee_{j=1}^{n} a_{ij}b_{1j} \ldots \bigvee_{j=1}^{n} a_{in}b_{nj}$$

Since we are working over $\mathbb{F}_2$, for all $1 \leq k \leq n$, we have $x_k^k = x_k$ for all positive integers $q$. Thus

$$x_k \text{ divides } f_i(g_1, \ldots, g_n) \iff \sum_{j=1}^{n} a_{ij}b_{jk} \geq 1$$
$$\iff a_{ij0} b_{j0k} = 1 \text{ for some } j_0$$
$$\iff a_{ij0} \land b_{j0k} = 1 \text{ for some } j_0$$
$$\iff \bigvee_{j=1}^{n} a_{ij} \land b_{jk} = 1$$
$$\iff (A \otimes B)_{ik} = 1.$$

Thus the matrix $(A \otimes B)$ is the adjacency matrix of $f \circ g$. 

Throughout this paper, we use $A^s$ to denote $A \otimes \cdots \otimes A$.

**Figure 3.** The strongly connected components of $f$ (left) and their poset (right).
Corollary 2.3. Let $f$ be a conjunctive Boolean network and let $A$ be the adjacency matrix of its dependency graph $\mathcal{D}(f)$. Then $A^*$ is the adjacency matrix of $f^*$.

2.3. The Loop Number. An invariant of a strongly connected graph, called the loop number, was defined in [18]. We generalize this definition to any directed graph.

Definition 2.4. The loop number of a strongly connected graph is the greatest common divisor of the lengths of its simple (no repeated vertices) directed cycles. We define the loop number of a trivial strongly connected graph to be 0. The loop number of any directed graph $G$ is the least common multiple of the loop numbers of its non-trivial strongly connected components.

Remark 2.5. Let $G$ be a directed graph and $A$ its adjacency matrix.

1. The loop number of $G$ is the same as the index of cyclicity of $G$ as in [40] and the index of imprimitivity of $A$ as in [38, 41].
2. The loop number can be computed in polynomial time, see [18] for an algorithm.
3. If the loop number of $G$ is 1, the adjacency matrix $A$ of $G$ is called primitive, see [38].

Example 1.1 (Cont.). The loop numbers of $V_1, V_2, V_3$ are 1, 1, 2, respectively. See Figure 3(left). In particular, the loop number of $G$ is 2.

Definition 2.6. The exponent of an irreducible matrix $A$ with loop number $c$ is the least positive integer $k$ such that $A^k + c = A^k$. The following lemma follows from Proposition 2.2 and the definition above.

Lemma 2.7. Let $f$ be a conjunctive Boolean network and let $A$ be the adjacency matrix of its dependency graph $\mathcal{D}(f)$. Suppose the loop number of $A$ is $c$ and its exponent is $k$. Then $h(f) = k$ and $p(f)$ divides $c$. In particular, $C(f) = 0$ for every $i \mid c$ and hence Equation (1.1) becomes

$$C(f) = \sum_{i \mid c} C(f)C(i).$$

Example 1.1 (Cont.). The phase space $S(f)$ has $2^6$ vertices, its cycle structure is $C(f) = 4C_1 + 1C_2$. In particular, the period of $f$ is 2 which is the same as the loop number of its dependency graph $G$.

Remark 2.8. It is clear that if $x_i$ appears in $f_i$ for all $i$, then the loop number of $\mathcal{D}(f)$ is one and hence $S(f)$ has only fixed points as limit cycles, this was shown in [37, Theorem 3.3]. Also if each edge in the dependency graph is undirected, then the dependency graph is made up of simple cycles of length 2 and hence the loop number of $\mathcal{D}(f)$ is either two or one. Thus $S(f)$ has only fixed points and possibly cycles of length two, which was shown in [36, Lemma 1].

For the case when $\mathcal{D}(f)$ is strongly connected $p(f) = c$ as was shown in [18, Theorem 4.13]; in particular, $S(f)$ has a simple cycle of length $l$ if and only if $l$ divides $c$. In general, however, this is not the case.

Example 2.9. Consider the Boolean network $f = (x_2, x_1, x_2x_5, x_3, x_4) : \mathbb{F}_2^5 \rightarrow \mathbb{F}_2^5$. The graph $\mathcal{D}(f)$ has two strongly connected components with loop numbers 2 and
3 respectively, and hence the loop number of $D(f)$ is 6. However, it is easy to see, using DVD [29], that $C(f) = 3C_1 + IC_2 + 2C_3$, and hence $f$ has no cycle of length 6.

The exponent has been studied extensively and upper bounds are known. [40, Theorem 3.11] presents an upper bound for the exponent of any irreducible matrix. Using the lemma above, we rewrite this upper bound for the height of any Boolean monomial system with a strongly connected dependency graph.

**Theorem 2.10.** Let $f$ be a conjunctive Boolean network with strongly connected dependency graph $D(f)$, and suppose the loop number of $D(f)$ is $c$. Then

$$h(f) \leq \begin{cases} (n-1)^2 + 1, & \text{if } c = 1; \\ \max\{n-1, n^2 - \frac{1}{2} + \frac{n^2}{c} - 3n + 2c\}, & \text{if } c > 1. \end{cases}$$

**Proof.** The proof follows from Proposition 2.2 above and [40, Theorem 3.11]. □

Furthermore, [40, Theorem 3.20] also implies an upper bound for the height of any conjunctive Boolean network.

**Theorem 2.11.** Let $f$ be any conjunctive Boolean network on $n$ nodes. Then $h(f) \leq 2n^2 - 3n + 2$.

**Proof.** The proof follows from Proposition 2.2 above and [40, Theorem 3.20]. □

We close this section with a classical theorem about positive powers of Boolean matrices. This theorem has the remarkable corollary that almost all conjunctive Boolean networks have a strongly connected dependency graph and, furthermore, have only fixed points as limit cycles.

**Theorem 2.12.** [38, Theorem 3.5.11] Let $N(n)$ be the number of conjunctive Boolean networks on $n$ nodes with strongly connected dependency graph and loop number 1. Then

$$\lim_{n \to \infty} \frac{N(n)}{2n^2} = 1.$$

In particular, since there are $2n^2$ different conjunctive Boolean networks on $n$ nodes, as $n \to \infty$, almost all conjunctive Boolean networks on $n$ nodes have a strongly connected dependency graph and have only fixed points as limit cycles.

Although there is a wealth of information about powers of non-negative matrices such as the transient length or possible cycle length, very little seems to be known about the number of cycles of a given length in the phase space and that is the main goal of this paper. In the next section we give a complete answer to this problem for conjunctive Boolean networks with strongly connected dependency graph. For this class of systems, we find the number of cycles of any possible length in the phase space.

### 3. Networks with Strongly Connected Dependency Graph

In this section we give an exact formula for the cycle structure of conjunctive Boolean networks with strongly connected dependency graphs. Since "almost all" conjunctive Boolean networks have this property by Theorem 2.12, one may consider this result as giving a complete answer for conjunctive Boolean networks in the limit. However, in the next section we will also consider networks with general dependency graphs and give upper and lower bounds for the cycle structure.
Before deriving the desired state space results we prove some needed facts about general finite dynamical systems.

**Lemma 3.1.** Let $f : X \to X$ be a finite dynamical system, with $X$ a finite set, and let $u \in X$ be a periodic point of period $t$.

1. If $f^s(u) = u$, then $t$ divides $s$.
2. The period of $f^j(u)$ is $t$, for any $j \geq 1$.
3. If $f^s(u) = f^j(u)$, then $t$ divides $s - j$.

**Proof.** Since $t$ is the period of $u$ and $f^s(u) = u$, by definition $s \geq t$. Thus $s = qt + r$, where $0 \leq r < t$. Now $u = f^s(u) = f^r(f^{qt}(u)) = f^r(u)$. Since $r < t$ and $t$ is the period of $u$, $r = 0$ and hence $t$ divides $s$. That proves (1). The proof of (2) is straightforward, since $f^t(f^j(u)) = f^j(f^t(u)) = f^j(u)$. Now we prove (3). Suppose $s \geq j$. Since $f^s(u) = f^j(u)$, we have $f^{s-j}(f^j(u)) = f^s(u) = f^j(u)$. Thus, by (1), the period of $f^j(u)$ divides $s - j$. But the period of $f^j(u)$ is $t$, by (2).

**Lemma 3.2.** Let $f : X \to X$ be a finite dynamical system. Then, $p(f) = c$ and $h(f) = d$ if and only if $c$ and $d$ are the least positive integers such that $f^{m+c}(u) = f^m(u)$ for all $m \geq d$ and $u \in X$.

**Proof.** Suppose that $p(f) = c$ and $h(f) = d$. Then for all $u \in X$ and $m \geq d$, $f^m(u)$ is a periodic point and hence its period divides $c$. Thus $f^{m+c}(u) = f^c(f^m(u)) = f^m(u)$.

Now suppose $c$ and $d$ are the least positive integers such that $f^{m+c}(u) = f^m(u)$ for all $m \geq d$ and $u \in X$. We want to show that $p(f) = c$ and $h(f) = d$. It is clear that $f^d(u)$ is periodic for all $u \in X$ and $d$ is the least such positive number. Thus the height of $f$ is $d$. Also, the period $f$ is $c$, since $c$ is the least positive integer such that $f^c(u) = u$ for any periodic point $u \in X$.

**Theorem 3.3.** Let $f : X \to X$ and $g : Y \to Y$ be two finite dynamical systems. Define the system $h : X \times Y \to X \times Y$ by $h(u, u') = (f(u), g(u'))$. Then $\mathcal{S}(h) = \mathcal{S}(f) \times \mathcal{S}(g)$.

**Proof.** This follows from the fact that $h(u, u') = (v, v')$ if and only if $f(u) = v$ and $g(u') = v'$ for all $u \in X$ and $u' \in Y$.

**Corollary 3.4.** Let $f$, $g$ and $h$ be as in Theorem 3.3. Then the period of $h$ is $p(h) = \text{lcm}\{p(f), p(g)\}$ and its height is $h(h) = \max\{h(f), h(g)\}$.

**Proof.** It is easy to see that a set $A \subset X \times Y$ is a cycle in $\mathcal{S}(h)$ if and only if $A_X$ (resp. $A_Y$) is a cycle in $\mathcal{S}(f)$ (resp. $\mathcal{S}(g)$), where $A_X := \{u \in X : (u, v) \in A$ for some $v \in Y\}$. Furthermore, it is clear that the length of the cycle $A$ is the least common multiple of the lengths of $A_X$ and $A_Y$. Thus $p(h) = \text{lcm}\{p(f), p(g)\}$. The proof of $h(h) = \max\{h(f), h(g)\}$ follows from the definition of height.
Corollary 3.5. Let \( f, g \) and \( h \) be as above. Then, the cycle structure of \( h \) is \( C(h) = C(f)C(g) := \sum_{m \mid p(h)} C(h)_m C_m, \) where

\[
C(h)_m = \sum_{\substack{s \mid p(f) \\ t \mid p(g) \\ \text{lcm}(s,t) = m}} \gcd(s,t) C(f)_s C(g)_t.
\]

That is, the generating function for the cycle structure of \( h \) is the product of the generating functions of \( f \) and \( g \), where \( C_s \cdot C_t = \gcd(s,t)C_{\text{lcm}(s,t)} \).

Let \( f : \mathbb{F}_2^n \to \mathbb{F}_2^n \) be a conjunctive Boolean network. Assume that the dependency graph \( \mathcal{D}(f) \) of \( f \) is strongly connected with loop number \( c \). For any divisor \( k \) of \( c \), it is well-known that the set of vertices of \( \mathcal{D}(f) \) can be partitioned into \( c \) non-empty sets \( W_1, \ldots, W_k \) such that each edge of \( \mathcal{D}(f) \) is an edge from a vertex in \( W_i \) to a vertex in \( W_{i+1} \) for some \( i \) with \( 1 \leq i \leq k \) and \( W_{k+1} = W_1 \). For a proof of this fact see [18, Lemma 3.4.1(iii)] or [18, Lemma 4.7].

By definition the length of any cycle in the phase space \( \mathcal{S}(f) \) of \( f \) divides \( c \), the period of \( f \). For any positive integers \( p, k \) that divide \( c \), let \( A(p) \) be the set of periodic points of period \( p \) and let \( D(k) := \bigcup_{p \mid k} A(p) \).

Lemma 3.6. The cardinality of the set \( D(k) \) is \( |D(k)| = 2^k \).

Proof. Let \( \Phi : \mathbb{F}_2^k \to D(k) \) be defined by

\[
\Phi(x_1, \ldots, x_k) = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_k, \ldots, x_k),
\]

where, without loss of generality, \( W_i = \{ v_{i,1}, \ldots, v_{i,s_i} \} \) for all \( 1 \leq i \leq k \). We will show that \( \Phi \) is a bijection. First we show that \( \Phi \) is well-defined. Let \( z = \Phi(x_1, \ldots, x_k) \), by [18, Theorem 4.10], there exists a positive integer \( m \) such that

\[
f^{mk+j}(z) = (x_{j+1}, \ldots, x_{j+1}, x_{j+2}, \ldots, x_{j+2}, \ldots, x_{j}, \ldots, x_{j}).
\]

In particular, \( f^k(z) = f^{mk+k}(z) = z \). Thus \( z = \Phi(x_1, \ldots, x_k) \in D(k) \). Since it is clear that \( \Phi \) is one-to one, it is left to show that \( \Phi \) is onto.

Let \( z \in D(k) \) which means that \( f^k(z) = z \). It is enough to show that \( z_{i,h} = z_{i,g} \) for all \( 1 \leq h, g \leq s_i \) and \( 1 \leq i \leq k \). Suppose not. Without loss of generality, let \( z_{i,h} = 1 \) and \( z_{i,g} = 0 \). Since \( k \) divides \( c \), \( f^c(z) = z \). But there is a path from \( v_{i,h} \) and \( v_{i,g} \) of length divides \( c \), contradiction. Hence \( \Phi \) is a bijection and \( |D(k)| = 2^k \). \( \square \)

Corollary 3.7. If \( p \) is a prime number and \( p^k \) divides \( c \) for some \( k \geq 1 \), then

\[
|A(p^k)| = 2^{p^k} - 2^{p^{k-1}}.
\]

Proof. It is clear that \( |D(1)| = 2 \). Namely, \( (0, \ldots, 0) \) and \( (1, \ldots, 1) \) are the only two fixed points. Now if \( p \) is prime and \( k \geq 1 \), then the proof follows from the fact that \( D(p^k) = D(p^{k-1}) \cup A(p^k) \), where \( \cup \) is the disjoint union. \( \square \)

The following theorem gives the exact number of periodic points of any possible length.
Theorem 3.8. Let $f$ be a conjunctive Boolean network whose dependency graph is strongly connected and has loop number $c$. If $c = 1$, then $f$ has the two fixed points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ and no other limit cycles of any length. If $c > 1$ and $m$ is a divisor of $c$, then

$$|A(m)| = \sum_{i_1=0}^{1} \cdots \sum_{i_r=0}^{1} (-1)^{i_1+i_2+\cdots+i_r} 2^{p_1^{k_1}r_1} p_2^{k_2} r_2 \cdots p_r^{k_r} r_r,$$

where $m = \prod_{i=1}^{r} p_i^{k_i}$ is the prime factorization of $m$, that is $p_1, \ldots, p_r$ are distinct primes and $k_i \geq 1$ for all $i$.

Proof. The statement for $c = 1$ is part of the previous corollary. Now suppose that $c > 1$. For $1 \leq j \leq r$, let $m_j = p_j^{k_j-1} \prod_{i=1, i \neq j}^{r} p_i^{k_i}$. Then $D(m) = A(m) \bigcup (\bigcup_{j=1}^{r} D(m_j))$, where $\bigcup$ is a disjoint union, in particular,

$$A(m) = D(m) \setminus \bigcup_{j=1}^{r} D(m_j).$$

The formula follows from the inclusion-exclusion principle and the disjoint union above. \qed

Corollary 3.9. If $m$ divides $c$, then the number of cycles of length $m$ in the phase space of $f$ is $C(f)_m = \left| \frac{A(m)}{m} \right|$. Hence the cycle structure of $f$ is

$$C(f) = \sum_{m|c} \left| \frac{A(m)}{m} \right| C_m.$$

Remark 3.10. Notice that the cycle structure of $f$ depends on its loop number only. In particular, a conjunctive Boolean network with loop number 1 on a strongly connected dependency graph only has as limit cycles the two fixed points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$, regardless of how many vertices its dependency graph has.

4. Networks with general dependency graph

Let $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a conjunctive Boolean network with dependency graph $\mathcal{D}(f)$. Without loss of generality, the adjacency matrix of $\mathcal{D}(f)$ is in the form \(\mathcal{A}\). Let $G_1, \ldots, G_t$ be the strongly connected components of $\mathcal{D}(f)$ corresponding to the matrices $A_1, \ldots, A_t$, respectively. Furthermore, suppose that none of the $G_i$ is trivial (i.e., $A_i$ is the zero matrix). For $1 \leq i \leq t$, let $h_i$ be the conjunctive Boolean network that has $G_i$ as its dependency graph and suppose that the loop number of $h_i$ is $l_i$. In particular, the loop number of $f$ is $l := \text{lcm}(l_1, \ldots, l_t)$.

In the remainder of the paper, we present lower and upper bounds for the number of cycles of a given length. We use the strongly connected components of the dependency graph and their poset to infer the cycle structure of $f$.

Let $G_1$ and $G_2$ be two strongly connected components in $\mathcal{D}(f)$ and suppose $G_1 \preceq G_2$. Furthermore, assume the vertex set of $G_1$ (resp. $G_2$) is $\{x_{i_1}, \ldots, x_{i_s}\}$ (resp. $\{x_{j_1}, \ldots, x_{j_t}\}$). Without loss of generality, let $x_{i_1} \to x_{j_1}$ be a directed edge in $\mathcal{D}(f)$ and let $\mathcal{D}'$ be the graph $\mathcal{D}(f)$ after deleting the edge $(x_{i_1}, x_{j_1})$. Let $g$ be the conjunctive Boolean network such that $\mathcal{D}(g) = \mathcal{D}'$.

Theorem 4.1. Any cycle in the phase space of $f$ is a cycle in the phase space of $g$. In particular $C(f) \leq C(g)$ componentwise.
Proof. Let \( C := \{ u, f(u), \ldots, f^{m-1}(u) \} \) be a cycle of length \( m \) in \( \mathcal{S}(f) \). To show that \( C \) is a cycle in \( \mathcal{S}(g) \), it is enough to show that, whenever \( x_i \)-value in \( u \) is 0, there exists \( x_{j_\omega} \in G_2 \), such that there is an edge from \( x_{j_\omega} \) to \( x_j \), and the \( x_{j_\omega} \)-value in \( u \) is 0. Thus, the value of \( x_{j_\omega} \) is determined already by the value of \( x_{j_\omega} \) and the edge \((x_{i-1}, x_j)\) does not make a difference here and hence \( C \) is a cycle in \( \mathcal{S}(g) \).

Suppose the loop number of \( G_1 \) (resp. \( G_2 \)) is a (resp. \( b \)) large enough, see [13 Corollary 4.4]. Thus there is a path from \( x_i \) to \( x_{j_\omega} \), of length \( qa + 1 \) for any \( q \geq T \). Also, there is a directed path from \( x_{j_\omega} \) to \( x_{j_\omega} \) of length \( qb - 1 \) for any \( q \geq T \). This implies the existence of an edge from \( x_i \) to \( x_{j_\omega} \) of length \( q(a + b) \) for all \( q \geq T \).

Now \( u = f^m(u) = f^{mk}(u) \), for all \( k \geq 1 \). Choose \( q, k \) large enough such that \( q(a + b) = km \). Then the value of \( x_{j_\omega} \) in \( f^{q(a+b)}(u) \) is equal to zero, since there is a path from \( x_i \) to \( x_{j_\omega} \) of length \( q(a + b) \) and the value of \( x_i \) is zero. Therefore, the value of \( x_{j_\omega} \) in \( u \) is zero, since \( u = \sum f^{mk}(u) = f^{q(a+b)}(u) \).

**Corollary 4.2.** Let \( f \) and \( \mathcal{D}(f) \) be as above. For any two strongly connected components in \( \mathcal{D}(f) \) that are connected, drop all but one of the edges. Let \( \mathcal{D}' \) be the new graph and let \( g \) be the conjunctive Boolean network such that \( \mathcal{D}(g) = \mathcal{D}' \). Then any cycle in \( \mathcal{S}(f) \) is a cycle in \( \mathcal{S}(g) \). In particular \( \mathcal{C}(f) \leq \mathcal{C}(g) \) componentwise.

Notice that there are many different possible \( g \) one can get and each one of them has the same poset as \( f \) and provides an upper bound for the cycle structure of \( f \). However, we do not have a polynomial formula for the cycle structure of \( g \). When we delete all edges between any two strongly connected components we get an easy formula for the cycle structure for the corresponding system as we describe below. In Section 7 we will present an improved upper bound for the cycle structure of \( f \).

Let \( h : F^n_2 \to F^n_2 \) be the conjunctive Boolean network with the disjoint union of \( G_1, \ldots, G_t \) as its dependency graph. That is, \( h = (h_1, \ldots, h_t) \). For \( h \), there are no edges between any two strongly connected components of the dependency graph of the network. Its cycle structure can be completely determined from the cycles structures of the \( h_i \) alone.

**Theorem 4.3.** Let \( \mathcal{C}(h_i) = \sum_{j_i \mid l_i} a_{i,j} \mathcal{C}_j \) be the cycle structure of \( h_i \). Then the cycle structure of \( h \) is \( \mathcal{C}(h) = \prod_{i=1}^t \mathcal{C}(h_i) \) and the number of cycles of length \( m \) (where \( m \mid l \) in the phase space of \( h \)) is

\[
\sum_{j_i \mid l_i} \frac{j_1 \cdots j_t}{\text{lcm}(j_1, \ldots, j_t)} \prod_{i=1}^t a_{i,j_i}.
\]

**Proof.** This follows directly from Corollary 3.3 \( \square \)

**Corollary 4.4.** Let \( f \) and \( h \) be as above. The number of cycles of any length in the phase space of \( f \) is less than or equal to the number of cycles of that length in the phase space of \( h \). That is \( \mathcal{C}(f) \leq \mathcal{C}(h) \) componentwise.

**Example 2.9 (Cont.)** Here \( h = (h_1, h_2) \), where \( h_1(x_1, x_2) = (x_2, x_1) \) and \( h_2(x_3, x_4, x_5) = (x_5, x_3, x_4) \). It is easy to check that \( \mathcal{C}(h_1) = 2\mathcal{C}_1 + 1\mathcal{C}_2 \), and \( \mathcal{C}(h_2) = 2\mathcal{C}_1 + 2\mathcal{C}_2 \). By Theorem 4.3 \( \mathcal{C}(h) = 4\mathcal{C}_1 + 2\mathcal{C}_2 + 4\mathcal{C}_3 + 2\mathcal{C}_6 \). In particular, \( \mathcal{C}(f) \leq \mathcal{C}(h) \).
5. Bounds on the cycle structure

Let \( f, h, G_1, \ldots, G_t \) be as above and let \( l_i \) be the loop number of \( G_i \). Furthermore, let \( P \) be the poset of the strongly connected components. Let \( \Omega \) be the set of all maximal antichains in \( P \). For \( J \subseteq [t] = \{1, \ldots, t\} \), let \( x_J := \prod_{j \in J} x_j \) and let \( \bar{J} := \{1, \ldots, t\} \setminus J \).

**Definition 5.1.** Let \( A \) be a subset of the set of limit cycles of \( h \) and let \( s_i \) be the number of limit cycles of length \( i \) in \( A \). We define and denote \( \|A\| \) by

\[
\|A\| := \sum_i s_i C_i.
\]

**Remark 5.2.** Using the inclusion-exclusion principle, if \( A, A_1, \ldots, A_s \) are subsets of the set of limit cycles of \( h \) and \( A = \bigcup_{1 \leq i \leq s} A_i \), then,

\[
\|A\| = \sum_{J \subseteq [s]} (-1)^{|J|+1} \|\bigcap_{j \in J} A_j\|.
\]

**Definition 5.3.** For any subset \( J \subseteq [t] \), let

- \( J^\leq := \{k : G_j \preceq G_k \text{ for some } j \in J\} \),
- \( J^\geq := \{k : G_j \succeq G_k \text{ for some } j \in J\} \),
- \( J^< := \{k : G_j < G_k \text{ for some } j \in J\} \), and \( J^> := \{k : G_j > G_k \text{ for some } j \in J\} \).

A limit cycle \( C \) in the phase space of \( f \) is \( J^0 \) (resp. \( J^1 \)) if the \( G_j \) component of \( C \) is 0 (resp. 1) for all \( j \in J \).

Notice that the sets defined above are closely related to upper and lower order ideals in posets, see [42, p. 100].

**Example 1.1 (Cont.).** Let \( J = \{2\} \). Then \( J^\leq = \{2, 3\} \), \( J^\geq = \{1, 2\} \), \( J^< = \{3\} \), and \( J^> = \{1\} \).

**Definition 5.4.** A limit cycle \( C \) in \( S(h) \) is \( J \)-regular if \( C \) is both \( (J^<)_0 \) and \( (J^>)_1 \), for some maximal antichain \( J \) in \( P \). A limit cycle is called regular if it is \( J \)-regular for some maximal antichain \( J \). That is, an element of a \( J \)-regular limit cycle has 0 in all entries corresponding to strongly connected components lying above components in \( J \) and 1 in all entries corresponding to components lying below components in \( J \).

**Lemma 5.5.** Let \( J \subseteq \Omega \) be a set of maximal antichains. For \( J \in J \), let \( A_J \) be the set of all \( J \)-regular limit cycles in the phase space of \( f \). Then,

\[
\| \bigcap_{J \in J} A_J \| = \prod_{j \in \bigcap_{J \in J} J} C(h_j).
\]

**Proof.** It is easy to see that a regular limit cycle \( C \in \bigcap_{J \in J} A_J \) if and only if the only non trivial components of \( C \) are in \( j \in \bigcap_{J \in J} J \). \( \Box \)

6. A Lower Bound

It is easy to see that a regular cycle is actually a limit cycle in the phase space of \( f \), and hence the numbers of regular cycles of different length provide lower bounds for the number of limit cycles of different lengths in the phase space of \( f \). Next we express these numbers in terms of the cycle structure \( C(h_i) \) of the strongly connected components.
Lemma 6.1. The cycle structure of the regular limit cycles in $S(h)$ is

$$\sum_{J \subseteq \Omega} (-1)^{|J|+1} \prod_{j \in \bigcap_{J \in J} J} C(h_j).$$

Proof. By definition, the set of regular limit cycles is $\bigcup_{J \in \Omega} A_J$. Now, using the inclusion-exclusion principle and Lemma 5.5, it follows that

$$\bigcup_{J \in \Omega} A_J = \sum_{J \subseteq \Omega} (-1)^{|J|+1} \bigcap_{J \in J} A_J = \sum_{J \subseteq \Omega} (-1)^{|J|+1} \prod_{j \in \bigcap_{J \in J} J} C(h_j).$$

Theorem 6.2. Consider the function

$$L(x_1, \ldots, x_t) := \sum_{J \subseteq \Omega} (-1)^{|J|+1} \prod_{j \in \bigcap_{J \in J} J} x_i.$$

Then, for any conjunctive Boolean network $f$ with subnetworks $h_1, \ldots, h_t$ and $\Omega$ it’s set of maximal antichains in the poset of $f$, we have

$$L(C(h_1), \ldots, C(h_t)) \leq C(f).$$

Here, the function $L$ is evaluated using the “multiplication” described in Corollary 3.5. This inequality provides a sharp lower bound on the number of limit cycles of $f$ of a given length.

Proof. The inequality follows from the previous lemma. Now to show that this lower bound is sharp, it is enough to present a Boolean monomial dynamical system $f$ such that $C(f) = L(C(h_1), \ldots, C(h_t))$.

Let $f$ be such that whenever there is a directed edge from $G_i$ to $G_j$ in the poset $p$, there is a directed edge from every vertex in $G_i$ to every vertex in $G_j$ in the dependency graph of $f$. It is enough to check that any cycle in the phase space of $f$ is regular. Let $C$ be a cycle in the phase space of $f$. Let $C'$ be the projection of $C$ on the strongly connected component $G_j$. It is clear that $C'$ is cycle in the $C(h_j)$. Suppose that $C'$ is non-trivial. Now if $G_i \preceq G_j$, then all entries corresponding to $G_i$ must be one. On the other hand, if $G_i \succeq G_j$, then all entries corresponding to $G_i$ must be zero. Therefore, $C$ is regular.

Note that the left side of the inequality (6.2) is a polynomial function in the variables $C(h_i)$, with coefficients that are functions of the cycle numbers of various lengths of the $h_i$ and the anti-chains of the poset of strongly connected components. That is, the sharp lower bound is a polynomial function depending exclusively on measures of the network topology.

7. An Upper Bound

Next we present a polynomial whose coefficients provide upper bounds for the number of possible limit cycles in the phase space of $f$. Similar to the lower bound polynomial, this upper bound polynomial is in terms of the strongly connected components of $f$ and their poset. However, this upper bound is not sharp. In fact, we will show that it is not possible to give a sharp upper bound that is in
a polynomial form with constant coefficients. First we prove the following
set-theoretic equality which we need in this section.

**Lemma 7.1.** Let \( \{ A_i : i \in \Delta \} \) be a finite collection of finite sets of limit cycles in \( S(h) \), \( A = \bigcap A_i \), and \( B_i \subseteq A_i \). Then, for \( K \subseteq \Delta \),

\[
\| \bigcap_{k \in K} (A_k \setminus B_k) \| = \sum_{L \subseteq K} (-1)^{|L|} \| A \cap \bigcap_{k \in L} B_k \|. 
\]

**Proof.** To begin with, observe that

\[
\bigcap_{k \in K} (A_k \setminus B_k) = \bigcap_{k \in K} (A \setminus B_k) 
= A \setminus \bigcup_{k \in K} (A \cap B_k). 
\]

Then, using the inclusion-exclusion principle we obtain:

\[
\| \bigcap_{k \in K} (A_k \setminus B_k) \| = \| A \setminus \bigcup_{k \in K} (A \cap B_k) \|
= \sum_{L \subseteq K} (-1)^{|L|} \| A \cap \bigcap_{k \in L} B_k \|. 
\]

\[\square\]

**Definition 7.2.** A limit cycle \( C \) in \( S(h) \) is called *admissible* if, for any \( j \in [t] \),

1. If the coordinates of states in \( C \) corresponding to component \( G_j \) are 0, then \( C \) is \( (j \leq) 0 \); and
2. If the coordinates of states in \( C \) corresponding to component \( G_j \) are 1, then \( C \) is \( (j \geq) 1 \).

Notice that any limit cycle in the phase space of \( f \) is admissible. In particular, we have the following lemma.

**Lemma 7.3.** The number of admissible limit cycles in \( S(h) \) is an upper bound for the number of limit cycles in \( S(f) \).

For \( J \subseteq [t] \), let \( Z_J \) be the set of all \( J_0 \) limit cycles and let \( O_J \) be the set of all \( J_1 \) limit cycles. Let \( K, L \subseteq [t] \), then it is easy to see that \( Z_K \cap Z_L = Z_{K \cup L} \), \( O_K \cap O_L = O_{K \cup L} \), and the cycle structure of \( Z_K \cap O_L \) is

\[
\| Z_K \cap O_L \| = \langle K, L \rangle \prod_{j \in K \cup L} C(h_j), 
\]

where

\[
\langle K, L \rangle = \begin{cases} 0, & \text{if } K \cap L \neq \emptyset, \\ 1, & \text{if } K \cap L = \emptyset. 
\end{cases}
\]

**Theorem 7.4.** The cycle structure of the admissible limit cycles in \( S(h) \) is equal to

\[
\sum_{I \subseteq N \subseteq [t]} (\prod_{j \in I \cap N} (-1)^{|I|+|M|+|I|+|J|} \langle I_N, J_M \rangle \prod_{k \in I_N \cup J_M} C(h_k)),
\]

where \( I_N = I^> \cup N, J_M = J^< \cup M \).
Proof. By definition, it follows that the set of admissible limit cycles is

$$\text{Adm} := \{\text{limit cycles in } S(h)\} \setminus \bigcup_{i,j \in [t]} (Z_{(i)} \setminus Z_{(i)}^+) \cup (O_{(j)} \setminus O_{(j)}^+).$$

Using the inclusion-exclusion principle, it follows that

\begin{equation}
\|\text{Adm}\| = \prod_{k \in [t]} C(h_k) + \sum_{\substack{N \subseteq [t] \ N \cup M \neq \emptyset \ M \subseteq [t] \ N \cap M = \emptyset}} (-1)^{|N|+|M|} \| \bigcap_{i \in N} (Z_{(i)} \setminus Z_{(i)}^+) \cap \bigcap_{j \in M} (O_{(j)} \setminus O_{(j)}^+) \|. \tag{7.1}
\end{equation}

On the other hand, since $\bigcap_{i \in N} Z_{(i)} = Z_N$, $\bigcap_{j \in M} O_{(j)} = O_M$, $Z_i \subseteq Z_{(i)}^+$ and $O_j \subseteq O_{(j)}$ it follows from Lemma \ref{lemma} that

$$\| \bigcap_{i \in N} (Z_{(i)} \setminus Z_{(i)}^+) \cap \bigcap_{j \in M} O_{(j)} \setminus O_{(j)}^+ \|$$

\begin{align*}
= & \sum_{\substack{I \subseteq N \ J \subseteq M \ N \cup J \neq \emptyset \ J \subseteq M}} (-1)^{|I|+|J|} \| (Z_N \cap O_M) \cap (\bigcap_{i \in N} Z_{(i)}^+ \cap \bigcap_{j \in M} O_{(j)}^+) \|
= & \sum_{\substack{I \subseteq N \ J \subseteq M \ N \cup J \neq \emptyset \ J \subseteq M}} (-1)^{|I|+|J|} \| (Z_N \cap O_M) \cap (\bigcup_{i \in N} Z_{(i)}^+ \cap O_{\bigcup_{j \in M} (j)}) \|
= & \sum_{\substack{I \subseteq N \ J \subseteq M \ N \cup J \neq \emptyset \ J \subseteq M}} (-1)^{|I|+|J|} \| Z_N \cap O_M \cap Z_{I}^+ \cap O_{J}^+ \|
= & \sum_{\substack{I \subseteq N \ J \subseteq M \ N \cup J \neq \emptyset \ J \subseteq M}} (-1)^{|I|+|J|} \| Z_{I} \cap O_{J}^+ \|
= & \sum_{\substack{I \subseteq N \ J \subseteq M \ N \cup J \neq \emptyset \ J \subseteq M}} (-1)^{|I|+|J|} \langle I_N, J_M^+ \rangle \prod_{k \in I_N \cup J_M^+} C(h_k).
\end{align*}

Equation (7.1) then becomes

$$\|\text{Adm}\| = \prod_{k \in [t]} C(h_k) + \sum_{\substack{N \subseteq [t] \ M \subseteq [t] \ N \cap M = \emptyset \ N \cup M \neq \emptyset}} (-1)^{|N|+|M|} \sum_{\substack{I \subseteq N \ J \subseteq M \ N \cup J \neq \emptyset \ J \subseteq M}} (-1)^{|I|+|J|} \langle I_N, J_M^+ \rangle \prod_{k \in I_N \cup J_M^+} C(h_k)$$

\begin{align*}
= & \prod_{k \in [t]} C(h_k) + \sum_{\substack{I \subseteq N \subseteq [t] \ J \subseteq M \subseteq [t] \ N \cup M \neq \emptyset \ N \cap M = \emptyset}} (-1)^{|N|+|M|+|I|+|J|} \langle I_N, J_M^+ \rangle \prod_{k \in I_N \cup J_M^+} C(h_k)
= & \sum_{\substack{I \subseteq N \subseteq [t] \ J \subseteq M \subseteq [t] \ N \cup M \neq \emptyset \ N \cap M = \emptyset}} (-1)^{|N|+|M|+|I|+|J|} \langle I_N, J_M^+ \rangle \prod_{k \in I_N \cup J_M^+} C(h_k).
\end{align*}

\[\square\]
Corollary 7.5. For $K, L \subseteq [t]$, let
\[
\phi_{(K,L)}(x_1, \ldots, x_t) = \begin{cases} 
0, & \text{if } K \cap L \neq \emptyset, \\
\sum_{I \subseteq K, J \subseteq L} a_{IJ}x_I, & \text{if } K \cap L = \emptyset.
\end{cases}
\]

Consider the function
\[
U(x_1, \ldots, x_t) = \sum_{I \subseteq N, J \subseteq M} (-1)^{|N|+|M|+|I|+|J|} \phi_{(I, J)}(x_1, \ldots, x_t)
\]
then
\[
C(f) \leq U(C(h_1), \ldots, C(h_l)),
\]
which provides therefore an upper bound for the cycle structure of $f$.

This upper bound is clearly not sharp, since it is easy to find admissible limit cycles in the phase space of $h$ that are not cycles in the phase space of $f$. The advantage of this upper bound, however, is that it is a polynomial in terms of the strongly connected components and their poset. Furthermore, we will show next that this bound gives the exact number of fixed points.

Theorem 7.6. Let $f$ be a conjunctive Boolean network. Then any admissible fixed point is regular and hence the number of fixed points $C(f)_0$ in the phase space of $f$ is
\[
C(f)_0 = \sum_{J \subseteq \Omega} (-1)^{|J|+1} 2^{|\bigcap_{J \in \mathcal{J}} J|}.
\]

Proof. It follows from the definitions that any admissible fixed point is regular, and hence $U(2C_1, \ldots, 2C_1) = \mathcal{L}(2C_1, \ldots, 2C_1)$. In particular, the number of fixed points $C(f)_0$ is the coefficient of $C_1$ in $\mathcal{L}(2C_1, \ldots, 2C_1)$. By Equation (6.1), we get
\[
\mathcal{L}(2C_1, \ldots, 2C_1) = \sum_{J \subseteq \Omega} (-1)^{|J|+1} \prod_{j \in \bigcap_{J \in \mathcal{J}} J} 2C_1 = \sum_{J \subseteq \Omega} (-1)^{|J|+1} 2^{|\bigcap_{J \in \mathcal{J}} J|} C_1.
\]

\[\square\]

7.1. An example. Each system $g$ from Corollary 4.2 has the same poset as $f$ and the cycle structure of each $g$ is an upper bound for the cycle structure of $f$. However, using the poset alone, or using the cycle structure of the strongly connected components alone, one can not expect to find a polynomial form with constant coefficients that would provide a sharp upper bound, as we now show.

Let $p$ be the poset on two nodes $G_1$ and $G_2$ where $G_1 \leq G_2$. Let $f$ be a conjunctive Boolean network with a dependency graph that has only two strongly connected components which are connected by one edge. Let $F$ be the set of all such systems. We will show that there is no polynomial $W(x_1, x_2) = \sum_{i,j} a_{ij}x_i^1x_j^2$ such that, for all $f \in F$, $W(C(h_1), C(h_2)) = C(f)$, where $h_1$ and $h_2$ are the conjunctive Boolean networks corresponding to the two strongly connected components of $f$.

Suppose the loop number of $h_1$ is 1 and that of $h_2$ is $q$ where $q$ is prime. Then $C(h_1) = 2C_1$ and $C(h_2) = 2C_1 + \frac{2^q - 2}{q} C_q$. It is easy to check that $C(f) = 3C_1 + \frac{2^q - 2}{q} C_q$. Since $C(f) = W(C(h_1), C(h_2))$, we have

\[\]
\[
\frac{2^q - 2}{q} C_q + 3C_1 = W(2C_1, 2C_1 + \frac{2^q - 2}{q} C_q)
\]
\[
= \sum_{i,j} a_{ij} (2C_1)^i (2C_1 + \frac{2^q - 2}{q} C_q)^j
\]
\[
= \sum_{i,j} 2^i a_{ij} (2C_1 + \frac{2^q - 2}{q} C_q)^j
\]
\[
= \sum_{i,j} 2^i a_{ij} [2^j C_1 + [(2 + \frac{2^q - 2}{q})^j - 2^j] C_q]
\]
\[
= \sum_{i,j} 2^{i+j} a_{ij} C_1 + \sum_{i,j} 2^i a_{ij} [(2 + \frac{2^q - 2}{q})^j - 2^j] C_q.
\]

Equating coefficients, for any \( q \), we have \( \frac{2^q - 2}{q} = \sum_{i,j} 2^i a_{ij} [(2 + \frac{2^q - 2}{q})^j - 2^j] \).
Therefore, \( a_{ij} = 0 \) for all \( i, j \geq 1 \) and hence \( W(x_1, x_2) \) must be of the form \( W(x_1, x_2) = a_{11} x_1 x_2 + a_{10} x_1 + a_{01} x_2 - a_{00} \).

Now suppose the loop number of \( h_1 \) is \( p \) and that of \( h_2 \) is \( q \). Then any cycle in the phase space of \( f \) is regular and hence it is easy to check that \( C(f) = C(h_1) + C(h_2) - C_1 \). In particular, \( a_{11} = 0 \) and hence \( W(x_1, x_2) = a_{10} x_1 + a_{01} x_2 - a_{00} \).

However, for the case when the loop number of \( h_1 \) is 4 and that of \( h_2 \) is 6, we have \( C(f) = 3C_1 + 3C_2 + 2C_3 + 4C_4 + 11C_6 + 2C_{12}, \ C(h_1) = 2C_1 + C_2 + 3C_4 \) and \( C(h_2) = 2C_1 + C_2 + 2C_3 + 9C_6 \). In particular, \( C(f) \neq W(C(h_1), C(h_2)) \).

8. Discussion

In this paper we have focused on the class of conjunctive and disjunctive Boolean networks and have treated the problem of predicting network dynamics from the network topology. Such networks are entirely determined by the topology of their dependency graphs, so it should be possible in principle to extract all information about the dynamics from this topology. This problem has been the subject of many research articles and been solved for the class of XOR Boolean networks, that is, networks where all nodes use the XOR Boolean operator. Determining the dynamics of a network from its topology has practical importance, for instance in the study of dynamical processes on social networks, such as the spread of an infectious disease. Public health interventions, such as quarantine of selected individuals or closure of certain institutions or means of travel, often attempt to alter network dynamics by altering network topology. The current study can be seen as a theoretical study that begins to elucidate the role of certain features of the network topology in supporting certain types of dynamics. Another example is discussed in [2], where it is shown that the dynamics of a certain gene regulatory network in the fruit fly \textit{Drosophila melanogaster} is determined by the topology of the network.

We have shown that if the dependency graph of the network is strongly connected, then the key topological determinant of dynamics is the loop number, or index of imprimitivity, of the graph, and one can describe the cycle structure exactly. The approach here is to look at powers of the incidence matrix of the graph in the max-plus algebra. In light of Theorem 2.12 one could argue that this is the prevalent case for large networks. What is more, this theorem shows that large
networks tend to have loop number one, so that their dynamics is extraordinarily simple, consisting only of two fixed points.

For general dependency graphs, we give lower and upper bounds on the cycle structure in the form of a polynomial function determined by structural measures of the graph. The lower bound is sharp.

In some sense, the main contribution of this paper lies in the identification of a class of networks that provide an excellent case study of the relationship between network topology and dynamics, which can provide hypotheses for more general classes of networks.

References

[1] I. Stewart, M. Golubitsky, M. Pivato, Symmetry groupoids and patterns of synchrony in coupled cell networks, SIAM J. Appl. Dyn. 2 (2003) 609–646.
[2] R. Albert, H. Othmer, The topology of the regulatory interactions predicts the expression pattern of the segment polarity genes in *Drosophila melanogaster*, J. Theor. Biol. 223 (2003) 1–18.
[3] R. Thomas, M. Kaufman, Multistationarity, the basis of cell differentiation and memory. II. Structural conditions of multistationarity and other nontrivial behavior, Chaos 11 (1) (2001) 170–179.
[4] C. Soule, Graphic requirements for multistationarity, ComPlexUs 1 (2003) 123–133.
[5] E. Plahte, T. Mestl, S. Omholt, Feedback loops, stability and multistationarity in dynamical systems, Journal of Biological Systems 3 (2) (1995) 409–413.
[6] J. Gauzé, Positive and negative circuits in dynamical systems, Journal of Biological Systems 6 (1) (1998) 11–15.
[7] E. Sontag, A. Veliz-Cuba, R. laubenbacher, A. Jarrah, The effect of negative feedback loops on the dynamics of Boolean networks, Biophysical Journal. URL [Online:DOI:10.1529/biophysj.107.125021]
[8] S. Kauffman, Homeostasis and differentiation in random genetic control networks, Nature 224 (1969) 177–178.
[9] S. Kauffman, Metabolic stability and epigenesis in randomly constructed genetic nets, J. Theor. Biol. 22 (1969) 437–467.
[10] Y.-K. Kwon, K.-H. Cho, Boolean dynamics of biological networks with multiple coupled feedback loops, Biophysical Journal 92 (2007) 2975–2981.
[11] M. I. Davidich, S. Bornholdt, Boolean network model predicts cell cycle sequence of fission yeast, PLoS ONE 3 (2) (2008) e1672.
[12] B. Elspas, The theory of autonomous linear sequential networks, IRE Transaction on Circuit Theory (1959) 45–60.
[13] B. Youssef, Cellular Automata, Lecture Notes in Computer Science, Springer Berlin, 2004, Ch. Simulation of Cell Population Dynamics Using 3-D Cellular Automata, pp. 561–570.
[14] G. C. Sirakoulis, I. Karafyllidis, A. Thanailakis, A cellular automaton model for the effects of population movement and vaccination on epidemic propagation, Ecological Modelling 133 (3) (2000) 209–223.
[15] F. Vivaldi, Cellular automata and finite fields, Physica D Nonlinear Phenomena 79 (1994) 115–131.
[16] R. Bartlett, M. Garzon, Monomial cellular automata, Complex Systems 7 (5) (1993) 367–388.
[17] G. Manzini, L. Margara, Invertible linear cellular automata over $\mathbb{Z}_m$: Algorithmic and dynamical aspects, Journal of Computer and System Sciences 56 (1998) 60–67.
[18] O. Colón-Reyes, R. Laubenbacher, B. Pareigis, Boolean monomial dynamical systems, Annals of Combinatorics 8 (2004) 425–439.
[19] O. Colón-Reyes, A. S. Jarrah, R. Laubenbacher, B. Sturmfels, Monomial dynamical systems over finite fields, Complex Systems 16 (4) (2006) 333–342.
[20] P. Cull, Linear analysis of switching nets, Kybernetik 8 (1971) 31–39.
[21] M. LeBorgne, A. Benveniste, P. LeGuernic, Polynomial dynamical systems over finite fields, in: G. Jacob, F. Lamnabhi-Lagarrigue (Eds.), Algebraic Computing in Control, Vol. 165 of Lecture Notes in Control and Information Sciences, Springer, New York, 1991, pp. 212–222.
[22] H. Marchand, M. Le Borgne, On the optimal control of polynomial dynamical systems over $\mathbb{Z}/p\mathbb{Z}$, in: 4th International Workshop on Discrete Event Systems, Cagliari, Italy, 1998, pp. 385–390.
[23] H. Marchand, M. Le Borgne, Partial order control of discrete event systems modelled as polynomial dynamical systems, Control Applications, 1998. Proceedings of the 1998 IEEE International Conference on 2 (1998) 817–821.
[24] D. Milligan, M. Wilson, The behavior of affine boolean sequential networks, Connection Science 5 (2) (1993) 153–167.
[25] M. Wilson, D. Milligan, Cyclic behavior of autonomous synchronous boolean networks: Some theorems and conjectures, Connection Science 4 (2) (1992) 143–154.
[26] R. Laubenbacher, B. Stigler, A computational algebra approach to the reverse-engineering of gene regulatory networks, Journal of Theoretical Biology 229 (2004) 523–537.
[27] A. Jarrah, R. Laubenbacher, B. Stigler, M. Stillman, Reverse-engineering of polynomial dynamical systems, Adv. in Appl. Math. 39 (4) (2007) 477–489.
[28] R. Lidl, H. Niederreiter, Finite fields, 2nd Edition, Vol. 20 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1997.
[29] A. Jarrah, R. Laubenbacher, H. Vastani, DVD: Discrete visual dynamics, World Wide Web, http://dvd.vbi.vt.edu.
[30] A. Hernández-Toledo, Linear finite dynamical systems, Communications in Algebra 33 (9) (2005) 2977–2989.
[31] A. Jarrah, R. Laubenbacher, M. S. P. Vera-Licona, An efficient algorithm for the phase space structure of linear dynamical systems over finite fields, submitted (2007).
[32] E. Goles, J. Olivos, Periodic behaviour of generalized threshold functions, Discrete mathematics 30 (1980) 187–189.
[33] Z. Agur, A. S. Fraenkel, S. T. Klein, The number of fixed points of the majority rule, Discrete Math. 70 (3) (1988) 295–302.
[34] J. Aracena, J. Demongeot, E. Goles, Fixed points and maximal independent sets in AND-OR networks, Discrete Appl. Math. 138 (3) (2004) 277–288.
[35] J. Aracena, Maximum number of fixed points in regulatory boolean networks, Bulletin of Mathematical Biology.
[36] E. Goles, G. Hernández, Dynamical behavior of Kauffman networks with and-or gates, Journal of Biological Systems 8 (2) (2000) 151–175.

[37] C. L. Barrett, W. Y. C. Chen, M. J. Zheng, Discrete dynamical systems on graphs and boolean functions, Math. Comput. Simul. 66 (6) (2004) 487–497.

[38] R. A. Brualdi, H. J. Ryser, Combinatorial matrix theory, Vol. 39 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1991.

[39] V. N. Sachkov, V. E. Tarakanov, Combinatorics of nonnegative matrices, Vol. 213 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 2002, translated from the 2000 Russian original by Valentin F. Kolchin.

[40] B. D. Schutter, B. D. Moor, On the sequence of consecutive powers of a matrix in a Boolean algebra, SIAM Journal on Matrix Analysis and Applications 21 (1) (2000) 328–354. URL [citeseer.ist.psu.edu/deschutter99sequence.html](http://citeseer.ist.psu.edu/deschutter99sequence.html)

[41] A. Berman, R. J. Plemmons, Nonnegative matrices in the mathematical sciences, Vol. 9 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994, revised reprint of the 1979 original.

[42] R. P. Stanley, Enumerative Combinatorics I, Translations of Mathematical Monographs, Wadsworth Inc., Belmont, CA, 1986.

(Abdul Salam Jarrah, Reinhard Laubenbacher, Alan Veliz-Cuba) Virginia Bioinformatics Institute, Virginia Tech, Blacksburg, VA 24061-0477, USA

E-mail address: {ajarrah, reinhard, alanavc}@vbi.vt.edu