RELLICH-CHRISTIANSON TYPE IDENTITIES FOR THE NEUMANN DATA MASS OF DIRICHLET EIGENFUNCTIONS ON POLYTOPES

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Abstract. We consider the Dirichlet eigenvalue problem on a simple polytope. We use the Rellich identity to obtain an explicit formula expressing the Dirichlet eigenvalue in terms of the Neumann data on the faces of the polytope of the corresponding eigenfunction. The formula is particular simple for polytopes admitting an inscribed ball tangent to all the faces. Our result could be viewed as a generalization of similar identities for simplices recently found by Christianson [1,2].

1. Introduction and main results

Let $P$ be a simple (i.e. non self-intersecting) $n$-dimensional polytope in $\mathbb{R}^n$ with faces $F_1, \ldots, F_k$. Let $f$ be a solution of the Dirichlet problem with eigenvalue $\lambda$:

\[
\begin{cases}
(\Delta + \lambda)f = 0 & \text{in } P \\
f = 0 & \text{on } \partial P
\end{cases}
\]

where $\Delta$ is the Laplacian and $f$ is assumed to be normalized $\|f\|_{L^2} = 1$. We are interested in the Neumann data mass on each face of the polytope, defined on the face $F_i$ by

\[
\int_{F_i} |\partial_\nu f|^2 dV
\]

where $\partial_\nu$ is the outward pointing normal derivative on $F_i$.

In the case of a compact manifold in $\mathbb{R}^n$, general upper and lower bounds for the Neumann data on the whole boundary are known [3], while in the specific case of $n$-dimensional simplices, Christianson showed the equidistribution of the Neumann data on the faces [2]. Inspired by these results, we investigate, using the Rellich identity, similar equalities for the Neumann data mass of a Dirichlet eigenfunction on the boundary of a polytope.

As Christianson showed with the example of a square [1], it is not true in general that the Neumann data mass is equidistributed between each faces. So we will instead consider the sum over all the faces.

We first define some notations to express our result in simple terms. Given a point $p$, we write $C_i(p)$ for the pyramid with base the face $F_i$ and apex $p$. We define $\text{Vol}_n(C_i(p))$ to be the signed volume of $C_i(p)$, with the sign given
by the sign of \(-v \cdot \nu\) where \(v\) is any vector from the base to the apex of the pyramid and \(\nu\) is the outward normal vector to \(F_i\). This is defined such that if the polytope is convex and \(p\) lies inside it, all the \(\text{Vol}^*_n(C_i(p))\) are positive, while some are negative when \(p\) is outside of the polytope (see figure 1 for an example). We also let \(\text{dist}^*(p,F_i)\) be the signed distance between \(p\) and \(F_i\) with the same sign as \(\text{Vol}^*_n(C_i(p))\). With this notation our result is

**Theorem 1.** Let \(P\) a simple \(n\)-polytope with faces \(F_1, \ldots, F_k\) and let \(p \in \mathbb{R}^n\) be an arbitrary point. Then

\[
\sum_{i=1}^{k} \text{dist}^*(p, F_i) \int_{F_i} |\partial_{\nu} f|^2 dV = 2\lambda
\]

or equivalently

\[
\sum_{i=1}^{k} \frac{\text{Vol}^*_n(C_i(p))}{\text{Vol}_{n-1}(F_i)} \int_{F_i} |\partial_{\nu} f|^2 dV = \frac{2\lambda}{n}
\]

**Remark 1.** If there exists a point \(p\) making all the pyramids \(C_1(p), \ldots, C_k(p)\) have the same signed volume, the second equality can be rewritten as

\[
\sum_{i=1}^{k} \frac{\int_{F_i} |\partial_{\nu} f|^2 dV}{\text{Vol}_{n-1}(F_i)} = \frac{2k\lambda}{n \text{Vol}_n(P)}.
\]

Such a point does not exist for all polytopes. For example, on a polygon \(P\), this point would be at the intersection of lines \(L_1, \ldots, L_k\) where \(L_i\) is parallel to \(F_i\) with a distance of \(\frac{2\text{Vol}_2(P)}{\text{Vol}_1(F_i)}\) between the two. It is clear when considering a trapezoid that those lines need not intersect at a common point.

**Remark 2.** Christianson’s result [2] on simplices can be obtained by putting \(p\) on one of the vertex of the simplex to obtain the equality for the face
opposite to it. Indeed in this case, all the pyramid $C_i(p)$ have volume 0 except the one with base the opposite face. In fact the proof of the theorem was inspired by his proof in [1] and simplified using the Rellich identity.

The equality [1] implies a simpler result when considering polytopes with special geometric properties,

**Corollary 1.** If $P$ is a tangential polytope, i.e. $P$ has an inscribed ball tangent to all its faces, then

\[
\int_{\partial P} |\partial_\nu f|^2 dV = \frac{\text{Vol}_{n-1}(\partial P)}{\text{Vol}_n(P)} \frac{2\lambda}{n}.
\]

Restricting ourselves to polygons, one can also follow up on remark[1] and try to find geometric properties of polygon with inner triangles of same area:

**Corollary 2.** If $Q$ is a quadrilateral with one diagonal intersecting the other diagonal in its middle then there exists a point $p$ such that the $\text{Vol}^*_{n}(C_i(p))$ are all equal and

\[
\sum_{i=1}^{4} \int_{F_i} |\partial_\nu f|^2 dV \frac{\text{Vol}_1(F_i)}{\text{Vol}_2(Q)} = \frac{4\lambda}{\text{Vol}_2(Q)}.
\]

**Remark 3.** This corollary applies to any parallelogram.

2. Proofs

**Proof of Theorem** [1] Without loss of generality, after a translation in $\mathbb{R}^n$, we can assume $p$ is at the origin. Let $p_i$ be the orthogonal projection of $p$ on $F_i$. In the following, we consider points in $\mathbb{R}^n$ as vectors.

We consider Rellich’s identity for Dirichlet eigenvalues [1]:

\[
2\lambda = \int_{\partial P} (\nu, q)|\partial_\nu f|^2 dV(q).
\]

This integral is split over the faces $F_i$ of the polytope, writing $\nu_i$ for the normal vector to $F_i$. One then remark that $F_i \subset p_t + p_i^\perp$, where $p_t^\perp$ is the orthogonal hyperplane to $p_t$, and by construction, $p_t$ and $\nu_t$ are parallel so $(\nu_t, q) = (\nu_t, p_t)$ for all $q \in F_i$. Thus

\[
\int_{F_i} (\nu_i, q)|\partial_\nu f|^2 dV(q) = (\nu_i, p_i) \int_{F_i} |\partial_\nu f|^2 dV.
\]

Defining $I_i = \int_{F_i} |\partial_\nu f|^2 dV$ and summing over all the sides of the polytope, we obtain the first equality of the theorem

\[
2\lambda = \int_{\partial P} (\nu, q)|\partial_\nu f|^2 dS(q) = \sum_{i=1}^{k} (\nu_i, p_i) I_i
= \sum_{i=1}^{k} |p_i|(\nu_i, p_i / |p_i|) I_i
= \sum_{i=1}^{k} \text{dist}^*(p, F_i) I_i
\]
Here \( \text{dist}^*(p, F_i) = |p_i| (\nu_i, p_i/|p_i|) = \text{dist}(p, F_i)(\nu, p_i/|p_i|) \) is the signed distance between \( p \) and \( F_i \). Finally we use the formula for the volume of a pyramid, \( \text{Vol}_n(C_i(p)) = \frac{1}{n} \text{dist}(p, F_i) \text{Vol}_{n-1}(F_i) \), to obtain

\[
\frac{2\lambda}{n} = \sum_{i=1}^{k} \frac{\text{Vol}^*_n(C_i(p))}{\text{Vol}_{n-1}(F_i)} I_i
\]

\( \square \)

**Proof of Corollary 1.** In the case of a tangential polytope \( P \) the point \( p \), apex of the pyramids, is set to be the center of the inscribed ball. Then all the distances \( \text{dist}^*(p, F_i) \) are equal to some \( h \) by definition of tangential polytope. Then by the first equality of theorem 1,

\[
\sum_{i=1}^{k} \int_{F_i} |\partial f|^2 dV = \frac{2\lambda}{h}
\]

We also have for the volume of the boundary \( \partial P \),

\[
\text{Vol}_{n-1}(\partial P) = \sum_{i=1}^{k} \text{Vol}_{n-1}(F_i) = \sum_{i=1}^{k} \frac{n \text{Vol}_n(C_i(p))}{h} = \frac{n}{h} \text{Vol}_n(P)
\]

thus

\[
\sum_{i=1}^{k} \int_{F_i} |\partial f|^2 dV = \frac{\text{Vol}_{n-1}(\partial P) \cdot 2\lambda}{\text{Vol}_n(P) n}
\]

\( \square \)

**Proof of Corollary 2.** The vertices of the quadrilateral \( Q \) are labeled \( A, B, C, D \) and the diagonal \( D_1, D_2 \) between \( A, C \) and \( B, D \) respectively. We suppose without loss of generality that \( D_1 \) passes through the middle of \( D_2 \) and name \( E \) their intersection point (see figure 2).
We will show that by taking $p$ in the middle of $D_1$, the triangles $ABp$, $BCp$, $CDp$ and $DAp$ have the same area thus allowing us to use equation (3).

Let $B'$ and $D'$ be the orthogonal projection on $AC$ of $B$ and of $D$ respectively. The angles $\hat{B}EB$ and $\hat{D}ED'$ are equal and so are $\hat{BB'}E$ and $\hat{DD'}E$. Thus the triangles $B'BE$ and $DD'E$ are similar. But since $E$ is the middle of $BD$, the edges $BE$ and $ED$ have the same length which implies that the triangles $B'BE$ and $DD'E$ are in fact congruent. Thus the lengths of $BB'$ and $DD'$ are equal.

By taking $p$ in the middle of $AC$, we obtain that $Ap$ and $Cp$ have the same length. Then since the area of a triangle is given by the length of its base times its height, we conclude that the triangles $ABp$, $BCp$, $CDp$ and $DAp$ have the same area. Equation (3) then gives the desired result.

□

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References

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