LIGHT PROPAGATION IN
GENERALLY COVARIANT ELECTRODYNAMICS
AND THE FRESNEL EQUATION∗

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Within the framework of generally covariant (pre-metric) electrodynamics, we specify
a local vacuum spacetime relation between the excitation \(H = (D, \mathcal{H})\) and the field
strength \(F = (E, B)\). We study the propagation of electromagnetic waves in such a
spacetime by Hadamard’s method and arrive, with the constitutive tensor density \(\kappa \sim \partial H/\partial F\), at a Fresnel equation which is algebraic of 4th order in the wave covector. We
determine how the different pieces of \(\kappa\), in particular the axion and the skewon pieces,
affect the propagation of light.

Keywords: electrodynamics; metric; skewon; axion.

1. Introduction

Electromagnetic wave propagation is a very important physical phenomenon in
classical field theory. In general, the geometrical structure of spacetime as well
as the intrinsic properties and the motion of material media can affect the light
propagation. In the generally covariant pre-metric approach to electrodynamics,1
the axioms of electric charge and of magnetic flux conservation manifest themselves
in the Maxwell equations for the excitation \(H = (D, \mathcal{H})\) and the field strength \(F = (E, B)\):

\[dH = J, \quad dF = 0.\] (1)

These equations should be supplemented by a constitutive law \(H = H(F)\). The
latter relation contains the crucial information about the underlying physical continuum (i.e., spacetime and/or material medium). Mathematically, this constitutive law arises either from a suitable phenomenological theory of a medium or from the
electromagnetic field Lagrangian. It can be a nonlinear or even nonlocal relation
between the electromagnetic excitation and the field strength.

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Earlier, we have investigated the propagation of waves in the most general linear theory. Here we present some results which hold true for all electrodynamic models with an arbitrary local spacetime relation. This is how we call the constitutive law if it applies to spacetime ("the vacuum") itself.

If local coordinates $x^i$ are given, with $i,j,... = 0,1,2,3$, we can decompose the excitation and field strength 2-forms into their components according to

$$H = \frac{1}{2} H_{ij} \, dx^i \wedge dx^j, \quad F = \frac{1}{2} F_{ij} \, dx^i \wedge dx^j.$$  \hfill (2)

2. Wave propagation: Fresnel equation

We will study the propagation of a discontinuity of the electromagnetic field following the lines of Ref. \cite{4}, see also Refs. \cite{5,6}. The surface of discontinuity $S$ is defined locally by a function $\Phi$ such that $\Phi = \text{const}$ on $S$. Across $S$, the geometric Hadamard conditions are satisfied:

$$\left[F_{ij}\right] = 0, \quad \left[H_{ij}\right] = 0,$$

$$\left[\partial_i F_{jk}\right] = q_i f_{jk}, \quad \left[\partial_i H_{jk}\right] = q_i h_{jk}.$$  \hfill (3)

$$\left[H_{ij}\right] = 0, \quad \left[H_{ij}\right] = 0,$$

$$\left[\partial_i H_{jk}\right] = q_i h_{jk}.$$  \hfill (4)

Here $\left[F\right](x)$ denotes the discontinuity of a function $F$ across $S$, and $q_i := \partial_i \Phi$ is the wave covector. Since the spacetime relation $H(F)$ determines the excitation in terms of the field strength, the corresponding tensors $f_{ij}$ and $h_{ij}$, describing the jumps of the derivatives of field strength and excitation, are related by

$$h_{ij} = \frac{1}{2} \kappa_{ijkl} f_{kl}, \quad \text{with} \quad \kappa_{ijkl} := \frac{\partial H_{ij}}{\partial F_{kl}}.$$  \hfill (5)

In linear electrodynamics, the components of the constitutive tensor $\kappa_{ijkl}$ are constant (or, at least, independent of the electromagnetic field). But in general $\kappa_{ijkl}$ is a function of the electromagnetic field, the velocity of matter, the temperature, and other physical and geometrical variables. Quite remarkably, however, all the earlier results obtained for linear electrodynamics remain also valid in the general case because whatever nonlinear spacetime relation $H(F)$ may exist, the relation between the jumps of the field derivatives, according to (3), is always linear.

If we use Maxwell’s equations (1), then (3) and (4) yield

$$\epsilon^{ijkl} q_j h_{kl} = 0, \quad \epsilon^{ijkl} q_j f_{kl} = 0.$$  \hfill (6)

Introducing the conventional constitutive matrix

$$\chi^{ijkl} = \frac{1}{2} \epsilon^{ijmn} \kappa_{mnkl}$$  \hfill (7)

and making use of (5), we rewrite the above system as

$$\chi^{ijkl} q_j f_{kl} = 0, \quad \epsilon^{ijkl} q_j f_{kl} = 0.$$  \hfill (8)

Solving the last equation by $f_{ij} = q_i a_j - q_j a_i$, we finally reduce (6) to

$$\chi^{ijkl} q_j q_k a_l = 0.$$  \hfill (9)
This algebraic system has a nontrivial solution for $a_i$ only when the determinant of the matrix on the l.h.s. vanishes. The latter gives rise to our covariant Fresnel equation

$$G^{ijkl}(\chi) q_i q_j q_k q_l = 0,$$

with the fourth order Fresnel tensor density of weight +1 defined by

$$G^{ijkl}(\chi) := \frac{1}{4!} \epsilon_{mnop} \epsilon_{rstu} \chi^{mnr}(i) \chi^{psk} j) q^{tu}.$$

It is totally symmetric, $G^{ijkl}(\chi) = G^{ijkl}(\chi)$, and thus has 35 independent components.

3. Spacetime relation: The emergence of the axion and the skewon fields

The quantity $\kappa^{ijkl}$ or, equivalently, $\chi^{ijkl}(x)$, characterizes the electromagnetic properties of the vacuum and is as such of universal importance. The untwisted tensor density $\chi^{ijkl}(x)$ of weight +1 has 36 independent components. If we decompose it into irreducible pieces with respect to the 6-dimensional linear group, then we find

$$\chi^{ijkl} = (1)\chi^{ijkl} + (2)\chi^{ijkl} + (3)\chi^{ijkl}, \quad \text{with} \quad 36 = 20 \oplus 15 \oplus 1$$

independent components, respectively. The irreducible pieces of $\chi$ are defined as follows:

$$(2)\chi^{ijkl} := \frac{1}{2}(\chi^{ijkl} - \chi^{klij}) = -(2)\chi^{klij}, \quad (3)\chi^{ijkl} := \chi^{ijkl},$$

$$(1)\chi^{ijkl} := \chi^{ijkl} - (2)\chi^{ijkl} - (3)\chi^{ijkl} = (1)\chi^{klij}.\quad (13)$$

Since no metric is available, we cannot form traces. The Abelian axion piece $^{(3)}\chi^{ijkl} := \alpha(x) \epsilon^{ijkl}$ has been introduced by Nieves and Pal. It yields P- and CP-violating terms in the field equations. Thus all constitutive functions in (12) can claim respectability from a physical point of view in the framework of linear response theory.

Alternatively, by means of mere contractions, we can decompose

$$\kappa_{ijkl} = (1)\kappa_{ijkl} + (2)\kappa_{ijkl} + (3)\kappa_{ijkl}$$

$$= (1)\kappa_{ijkl} + 2\delta_{[i}^{[k} \delta_{j]}^{j]} + \frac{1}{6} \kappa \delta_{[i}^{[k} \delta_{j]}^{j]}.$$ 

Here $\kappa_i^k := \kappa_{il}^{kl}$, $\kappa := \kappa_{kl}^k = \kappa_{kl}^{kl}$, and the traceless piece is $\delta_{[i}^{[k} := \kappa_{i}^{k} - \frac{1}{4} \kappa \delta_{i}^{k}$.

By introducing the pseudotensorial skewon field and the pseudoscalar axion field by

$$S_{i}^{j} = - \frac{1}{2} \kappa_{i}^{k} = S_{i}^{j}, \quad \alpha = \frac{1}{12} \kappa,$$

$$\kappa_{ijkl} = \frac{1}{4!} \epsilon_{mnop} \epsilon_{rstu} \chi^{mnr}(i) \chi^{psk} j) q^{tu}.$$
we ultimately find the basic relations between the corresponding irreducible pieces:

\begin{align}
(1) \chi_{ijkl} &= \frac{1}{2} \epsilon_{ijmn} (1) \kappa_{mn}^{~~kl}, \\
(2) \chi_{ijkl} &= \frac{1}{2} \epsilon_{ijmn} (2) \kappa_{mn}^{~~kl} = \epsilon_{ijm}^{[k} S_{ml]} - \epsilon_{klm}^{[i} S_{mj]}, \\
(3) \chi_{ijkl} &= \frac{1}{2} \epsilon_{ijmn} (3) \kappa_{mn}^{~~kl} = \alpha \epsilon_{ijkl}.
\end{align}

4. Properties of the Fresnel tensor density

The various irreducible pieces of the constitutive tensor affect the wave propagation in different ways. Technically, this can be determined by studying how a certain irreducible piece \((a) \chi\) contributes to the Fresnel tensor density \((11)\) and, thereby, to the Fresnel equation \((10)\) which governs the wave covectors.

We will demonstrate here the general properties of the Fresnel tensor density. As a preparatory step, let us take \(\chi = \phi + \psi\). Then, using a compact notation by omitting the indices, we have quite generally

\[ G(\chi) = G(\phi) + G(\psi) + \frac{1}{4!} (O_1 + O_2 + O_3 + T_1 + T_2 + T_3). \]

Here the mixed terms \(O_a\) contain one \(\psi\)-factor and the \(T_a\)'s two \(\psi\)-factors. Postponing the symmetrization over \(i, j, k, l\) to the very last moment, these terms read explicitly as follows:

\begin{align}
O_1(\phi, \psi, \phi) &= \epsilon_{mnpq} \epsilon_{rstu} \phi^{mnri} \psi^{jpsk} \phi^{lqtu}, \\
O_2(\psi, \phi, \phi) &= \epsilon_{mnpq} \epsilon_{rstu} \psi^{mnri} \phi^{jpsk} \phi^{lqtu}, \\
O_3(\phi, \phi, \psi) &= \epsilon_{mnpq} \epsilon_{rstu} \phi^{mnri} \phi^{jpsk} \psi^{lqtu}, \\
T_1(\psi, \phi, \psi) &= \epsilon_{mnpq} \epsilon_{rstu} \psi^{mnri} \psi^{jpsk} \psi^{lqtu}, \\
T_2(\psi, \psi, \phi) &= \epsilon_{mnpq} \epsilon_{rstu} \psi^{mnri} \psi^{jpsk} \phi^{lqtu}, \\
T_3(\phi, \psi, \psi) &= \epsilon_{mnpq} \epsilon_{rstu} \phi^{mnri} \psi^{jpsk} \psi^{lqtu}.
\end{align}

Now we are in position to prove the following properties of the Fresnel tensor:

1) \(G^{(3)}(\chi) = 0\). \textbf{Proof}: By direct substitution of \((3)\chi_{ijkl} = \alpha \epsilon_{ijkl}\), we have

\[ \epsilon_{mnpq} \epsilon_{rstu} \phi^{mnri} \psi^{jpsk} \phi^{lqtu} = 4\alpha^3 \epsilon_{ijkl}. \]

This vanishes upon symmetrization over the indices \(i, j, k, l\).

2) \(G^{(2)}(\chi) = 0\). \textbf{Proof}: Using the symmetry properties \((13)\) and especially the skew symmetry \((2)\chi_{ijkl} = -(2)\chi_{klji}\), we find

\[ \epsilon_{mnpq} \epsilon_{rstu} \phi^{mnri} \psi^{jpsk} \phi^{lqtu} = -\epsilon_{mnpq} \epsilon_{rstu} \phi^{rinn} \psi^{skjp} \phi^{ltauq} = -\epsilon_{mnpq} \epsilon_{rstu} \phi^{tunq} \psi^{kspj} \phi^{irmn} = -\epsilon_{mnpq} \epsilon_{rstu} \phi^{rmrl} \psi^{kpsj} \phi^{lqtu}. \]
Upon the symmetrization over the indices $i, j, k, l$, we then get $G^{(2)}(\chi) = - G^{(2)}(\chi)$ or $G^{(2)}(\chi) = 0$.

3) $G^{(1)}(\chi) + (2)^{\chi} + (3)^{\chi} = G^{(1)}(\chi) + (2)^{\chi}$. **Proof:** Let us put $\phi = (1)^{\chi} + (2)^{\chi}$ and $\psi = (3)^{\chi}$ in the formula (27). Then we can verify that all the mixed terms (21)-(26) vanish. Indeed, using $(3)^{\chi}_{ijkl} = \alpha \varepsilon_{ijkl}$, we find:

\[
O_1 = 2\alpha \epsilon_{mnpq} \phi_{ijkl} \phi_{ijkl} (\delta^t_t \delta^a_a + \delta^a_t \delta^t_a + \delta^t_t \delta^a_a ) = 2\alpha \epsilon_{mnpq} (\phi_{mnpq} + \phi_{mnpq} + \phi_{mnpq} ).
\]

This is zero when we impose the symmetrization over $i, j, k, l$. Similarly, we find

\[
O_2 = -2\alpha \epsilon_{rstu} (\delta^t_t \delta^q_q - \delta^t_t \delta^q_q ) \phi^{ipsk} \phi_{ijkl} = 2\alpha \epsilon_{rstu} (-\phi^{irsk} \phi_{ijkl} + \phi^{irsk} \phi_{ijkl} )
\]

and

\[
O_3 = -2\alpha \epsilon_{mnpq} (\delta^t_t \delta^t_t - \delta^a_a \delta^a_a ) \phi^{ipsk} = 2\alpha \epsilon_{mnpq} (-\phi^{mnpq} \phi^{ipsk} + \phi^{mnpq} \phi^{ipsk} ).
\]

Both expressions vanish when we impose symmetrization over $i, j, k, l$. The proof that all $T$'s are equal to zero reduces to the above formulas in which we simply need to replace one of the $\phi$ factors by the Levi-Civita $\epsilon$. Consequently, (20) yields $G(\phi + \psi) = G(\phi) + G(\psi) = G(\phi)$, since $G(\psi) = G^{(3)}(\chi) = 0$.

4) $G^{(2)}(\chi) + (3)^{\chi} = 0$. **Proof:** Take $\phi = (2)^{\chi}$ and $\psi = (3)^{\chi}$. Then the proof reduces to the above case. Formally, we can simply consider a special case of the formula $G^{(1)}(\chi) + (2)^{\chi} + (3)^{\chi} = G^{(1)}(\chi) + (2)^{\chi}$ by putting $(1)^{\chi} = 0$ and using the property $G^{(2)}(\chi) = 0$.

5. Discussion and conclusion

The properties of the Fresnel tensor simplify greatly the final analysis of the light propagation in the general electrodynamical theory.

As we saw, neither the axion piece $(2)^{\chi}$ nor the skewon piece $(3)^{\chi}$ (alone or together) can provide electromagnetic wave propagation.

The presence of the principal part $(1)^{\chi}$ of the constitutive tensor is indispensable for the existence of nontrivial electromagnetic waves and ultimately for the existence of the light cone structure on spacetime. In any case, the Fresnel tensor density reads

\[
G^{ijkl}(\chi) = G^{ijkl}(1)^{\chi} + (2)^{\chi}.
\]

Furthermore, in general

\[
G^{ijkl}(1)^{\chi} + (2)^{\chi} \neq G^{ijkl}(1)^{\chi},
\]

which means that the skewon field does influence the Fresnel equation and thus, eventually, the light cone structure. An example of this general result can be found in the asymmetric constitutive tensor studied by Nieves and Pal.
Actually, we can use (20) to find more exactly the contribution of the skewon to the Fresnel tensor. One can straightforwardly see that
\[
O_1((1)\chi, (2)\chi, (1)\chi) = 0, \quad O_2 ((2)\chi, (1)\chi, (1)\chi) + O_3 ((1)\chi, (1)\chi, (2)\chi) = 0, \quad (29)
\]
\[
T_2((2)\chi, (2)\chi, (1)\chi) = T_3 (1)\chi, (2)\chi, (2)\chi). \quad (30)
\]
The proof is analogous to the demonstration of the property \( G^{(2)}(\chi) = 0 \) and is based directly on the skew symmetry \( (2)\chi^{ijkl} = -(2)\chi^{klij} \). As a result, (20) yields
\[
G^{(1)}(\chi) + (2)\chi = G^{(1)}(\chi) + \frac{1}{4} \left[ T_1((2)\chi, (1)\chi, (2)\chi) + 2T_2((2)\chi, (2)\chi, (1)\chi) \right].
\]
Substituting here the explicit form of the skewon part (18), we finally find
\[
G^{ijkl}(\chi) = G^{ijkl}((1)\chi + (2)\chi) = G^{ijkl}((1)\chi) + (1)\chi^{m(ij)kS_mS_n}. \quad (31)
\]
Our basic results on the light propagation are then that \((1)\chi \neq 0\), otherwise there is no orderly wave propagation, the axion piece \( \alpha \) is left arbitrary, it doesn’t influence light propagation locally, whereas the skewon piece \( \mathcal{S}_{ij} \) makes itself felt by means of equation (31). The detailed study of how exactly these 15 functions \( \mathcal{S}_{ij} \) affect the light cone is left for future work.

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