Generalized Kraus operators for the one-qubit depolarizing quantum channel

M. Arsenijević · J. Jeknić-Dugić · M. Dugić

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Abstract Microscopic Hamiltonian models of the composite system “open system + environment” typically do not provide the operator-sum Kraus form of the open system’s dynamical map. With the use of a recently developed method [17], we derive the Kraus operators starting from the microscopic Hamiltonian model, i.e. from the proper master equation, of the one-qubit depolarizing channel. Those Kraus operators generalize the standard counterparts, which are widely used in the literature. Comparison of the standard and the here obtained Kraus operators is performed via investigating dynamical change of the Bloch sphere volume, entropy production and the open system’s state trace distance. We compare the generalized with the standard Kraus operators for both single qubit as well as regarding the occurrence of the entanglement sudden death for a pair of initially correlated qubits. We find that the generalized Kraus operators describe the less deteriorating quantum channel than the standard ones.

Keywords Open quantum systems · Kraus operator-sum decomposition · Qubit operations

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1 Introduction

The differential master equations and the integral operator-sum Kraus form of the completely positive dynamical maps are the basic however mutually non-equivalent
methods in modern open quantum systems theory [1][2][3]. On the one hand, for the completely positive (CP) dynamical maps, solution of a master equation can always be expressed as a quantum operation [1] in the operator-sum representation [2][3][4]. On the other hand, an operator-sum representation cannot necessarily be presented in a master-equation form [2][3][4]. Hence the relative advantage of the operator-sum formalism.

Nevertheless, in the case of the Markovian (and particularly time-homogeneous semigroup) dynamics [2][3][4], when the methods are interchangeable, there is a physical advantage of the master-equations formalism that is worth emphasizing. Actually, the master equations formalism fully respects the microscopic details in the Hamiltonian form of the composite system “open system + environment”. These details regard the realistic physical situations and, in principle, open the door for the better control of the open system’s dynamics. However, such microscopic-model details are typically absent from the operator-sum representation of the open-system’s dynamics. That is, the Kraus operators are often constructed without the clear and unambiguous microscopic-model criteria [4]. Hence the natural, as yet virtually intact, question arises: whether or not the widely used Kraus operators [4] reliably represent the microscopically modelled quantum noisy channels?

In this paper we answer this question for the quantum depolarizing channel [4][9][10][11][12][13][14][15] while the analogous results for the (generalized) amplitude and phase damping processes can be found in [16]. We make use of a recently devised method for deriving the Kraus operators from a local-in-time master equation for the finite-dimensional open systems [17]. Formally, this procedure is equivalent to solving the master equation, whether Markovian, non-Markovian, of Lindblad form or not. The method can also be used in the opposite direction.

We first derive an interaction-picture master equation and hence derive the novel Kraus operators that generalize the standard ones. The new Kraus operators can be termed “generalized” as the proper choice of the model parameters reduce the new to the standard Kraus operators for the depolarizing (DP) channel. Dissent of the novel from the standard Kraus operators is investigated via temporal behavior of the Bloch sphere volume, the state trace-distance and entropy production. We find that, in short time intervals, the generalized DP channel is less deteriorating than the standard DP channel, while in the asymptotic limit the channels are indistinguishable. This finding is confirmed by investigating entanglement dynamics for a pair of qubits initially in a maximally entangled state.

Structure of this paper is as follows. In Section 2 we overview the basics of the procedure devised in Ref. [17]. Based on a master equation for the depolarizing channel that is derived in Appendix, in Section 3.1 we derive the generalized Kraus operators for the depolarizing channel. In Section 3.2 we perform comparison of the standard and the here obtained generalized depolarizing processes. Section 4 is discussion and we conclude in Section 5.

2 Overview of the method

In Ref. [17], the authors developed a general procedure for deriving a Kraus decomposition from a known master equation and vice versa, regarding the finite-
dimensional quantum systems. The only assumption is that the master equation is local in time.

The so-called Nakajima-Zwanzig projection method \[2,3\] gives the following master equation for the system’s density operator $\hat{\rho}_S(t)$, $(\hbar = 1)$:

$$\frac{d\hat{\rho}_S(t)}{dt} = -i[\hat{H}, \hat{\rho}_S(t)] + \int_0^t K_{t,s}[\hat{\rho}_S(s)]ds,$$

(1)

where $\hat{H}$ represents the system’s self-Hamiltonian (that includes the so-called Lamb-shift term) and $K_{t,s}$ is the memory kernel which accounts for the non-unitary effects due to the environment.

Certain processes can be written in a local-in-time form \[2,3,6,7\]:

$$\dot{\hat{\rho}}_S(t) = \Lambda_t(\hat{\rho}_S(t)),$$

(2)

where $\Lambda_t$ is a linear map which preserves hermiticity, positivity and unit trace of $\hat{\rho}_S(t)$ and has the property:

$$tr\Lambda_t(\hat{\rho}_S(t)) = 0.$$ (3)

Alternatively, dynamics can be presented in a non-differential, “integral” form \[2,3,17\]:

$$\hat{\rho}_S(t) = \phi_t(\hat{\rho}_S(0)),$$

(4)

where $\phi_t$ is a completely positive and trace preserving, linear map.

It can be shown \[17\], that the linear maps $\Lambda_t$ and $\phi_t$ are connected via the matrix differential equation:

$$\dot{F} = LF,$$

(5)

where the $F$ matrix represents the $\phi_t$ map while the matrix elements of $L$ are given by:

$$L_{kl} = tr[\hat{g}_k \Lambda(\hat{g}_l)].$$

(6)

In eq. (6), \{$\hat{g}_k$\} is an orthonormal basis of the Hermitian operators acting on the system’s Hilbert state space. For the time independent $\Lambda_t$, i.e. $L$, eq. (5) has the unique solution:

$$F = e^{Lt}.$$ (7)

Complete positivity of the map $\phi_t$ (and hence of the matrix $F$) is equivalent to the positivity of the, so called, Choi matrix, $S$ \[17,8\] whose elements are defined as \[17\]:

$$S_{nm} = \sum_{s,r} F_{sr} tr[\hat{g}_r \hat{g}_s \Lambda(\hat{g}_n)].$$

(8)

where $F_{sr}$s are entries of the $F$ matrix, eq.(5).

With the use of eq. (8), eq.(5) takes the form:

$$\phi(\hat{\rho}_S(0)) = \sum_{nm} S_{nm} \hat{g}_n \hat{\rho}_S(0) \hat{g}_m^\dagger,$$

(9)

which, after diagonalization of the $S$ matrix:

$$S = UDU^\dagger,$$

(10)

gives rise to a Kraus decomposition. The eigenvalues $d_i$ and the eigenvectors of the $S$ matrix constitute the diagonal matrix $D$ and the unitary matrix $U = (u_{ij})$,
respectively; columns of the unitary matrix $U$ are the normalized eigenvectors of the $S$ matrix. Then the Kraus operators:

$$E_i = \sum_j \sqrt{d_{ij}} u_{ji} G_j$$

yield the Kraus decomposition of the dynamical map $\phi_t$:

$$\phi_t(\hat{\rho}_S(0)) = \sum_k \hat{E}_k(t) \hat{\rho}_S(0) \hat{E}_k^\dagger(t).$$

Hence the chain of the construction is established: from a master equation to calculate $L$, then via relation (7) to obtain the matrix $F$ and, due to eq.(8) and eq.(10), to calculate the Kraus operators, eq.(11).

3 The depolarizing channel

While microscopic derivation for the phase and amplitude damping master equations can be found in the literature [3], to the best of our knowledge, this is not the case for the master equation concerning the depolarizing process, eq.(14). One possible form (due to the unitary freedom) of the standard depolarizing Kraus operators is given by [4,9,10,13]:

$$\hat{E}_0 = \sqrt{\frac{1}{4} - 3p(t)} \hat{I}, \quad \hat{E}_i = \sqrt{p(t)} \hat{\sigma}_i (i = x, y, z) \quad (13)$$

with the associated, phenomenological master equation in the interaction picture:

$$\frac{d\hat{\rho}_S(t)}{dt} = \gamma \sum_i (2\hat{\sigma}_i \hat{\rho}_S(t) \hat{\sigma}_i^\dagger - \hat{\sigma}_i^2 \hat{\rho}_S(t) - \hat{\rho}_S(t) \hat{\sigma}_i^2)$$

where the Pauli sigma-operators $\hat{\sigma}_i (i = x, y, z)$ pertain to the qubit’s degrees of freedom and $\gamma$ is so-called damping constant, while $p(t) = 1 - e^{-\gamma t}$.

In this section, for a microscopic (Hamiltonian) model we derive the Kraus operators starting from a Markovian (Lindblad-form) master equation, which generalizes eq.(14) and therefore leads to a generalization of the standard Kraus operators, eq.(13). Thereby we implicitly test derivability of eq.(13) from eq.(14) that here will not be made explicit. Analogous derivation regarding the GAD and PD channels can be found in [16].

The microscopic model for the depolarizing process we are interested in is given by the following Hamiltonian [3]:

$$\hat{H} = \hat{H}_S \otimes \hat{I}_E + \hat{I}_S \otimes \hat{H}_E + \hat{A}_x \otimes \hat{B}_x + \hat{A}_y \otimes \hat{B}_y + \hat{A}_z \otimes \hat{B}_z. \quad (15)$$

The terms in eq.(15) [in the physical units of $\hbar = 1$]: the self-Hamiltonians $\hat{H}_S = \frac{\omega_0}{2} \hat{\sigma}_z$ and $\hat{H}_E = \sum_i \int_{0}^{\omega_{\text{max}}} d\omega \hat{a}_{\omega i}^\dagger \hat{a}_{\omega i}$ for the $S$ system and the environment (thermal bath) $E$, respectively, while the interaction terms $\hat{A}_i \otimes \hat{B}_i = \hat{\sigma}_i \otimes \int_{0}^{\omega_{\text{max}}} d\omega h(\omega)(\hat{a}_{\omega i}^\dagger + \hat{a}_{\omega i})$, $i = x, y, z$, with the coupling functions $h(\omega)$ accounting for the interaction strength per frequency. From eq.(15) we can learn that
the environment $E$ effectively acts as a set of three independent bosonic environments, $E_i,i = x,y,z$, with the standard, Bose-operators, commutation relations $[\hat{a}_{\omega_i},\hat{a}_{\omega_j}^\dagger] = \delta(\omega - \omega')\delta_{ij}, i,j = x,y,z$.

Our approach is minimalist in the sense that we do not introduce assumptions that might force us to leave the standard physical models. Therefore we are concerned with the time-homogeneous $L$, cf. eq. (6), as well as with the Ohmic spectral density.

3.1 Kraus operators for the generalized depolarizing channel

In Appendix we derive the following, Markovian (Lindblad-form), master equation for the microscopic model eq. (15) of the depolarizing channel:

$$\frac{d\hat{\rho}_S(t)}{dt} = -i\Delta[\hat{\sigma}_z,\hat{\rho}_S(t)] + \gamma_{zz}(0,T)[(\hat{\sigma}_z\hat{\rho}_S(t)\hat{\sigma}_z - \hat{\rho}_S(t)) + \frac{\gamma_{xx}(\omega_0,T) + \gamma_{yy}(\omega_0,T)}{2}(\hat{\sigma}_x\hat{\rho}_S(t)\hat{\sigma}_x - \hat{\rho}_S(t))] + \frac{\gamma_{xx}(\omega_0,T) + \gamma_{yy}(\omega_0,T)}{2}(\hat{\sigma}_y\hat{\rho}_S(t)\hat{\sigma}_y - \hat{\rho}_S(t)).$$  

(16)

Definitions of the constants appearing in eq. (16) can be found in Appendix where we strongly emphasize that the microscopic model eq. (15) allows for the Markovian dynamics eq. (16) only if the spectral density is of the Ohmic type and $\omega_0/\omega_c \ll 1$, while the high temperature limit gives rise to the constraint $k_B T/\hbar\omega_0 \gg 1$ and $\omega_c$ is the cutoff frequency.

Eq. (16) straightforwardly reduces to the standard one, eq. (14), for $\Delta = 0$ (no Lamb shift term) and $2\gamma_{zz}(0,T) = \gamma_{xx}(\omega_0,T) + \gamma_{yy}(\omega_0,T)$. From eq. (15) it follows the same interaction-strength per frequency $\hbar\omega(t)$, which gives rise the same interaction strength, $\alpha$ for all environments—see Appendix.

To ease the calculation we introduce: $x = \Delta$, $y = \gamma_{zz}$ and $z = \frac{\gamma_{xx} + \gamma_{yy}}{2}$ with which the equation (16) reads:

$$\frac{d\hat{\rho}_S(t)}{dt} = -ix[\hat{\sigma}_z,\hat{\rho}_S(t)] + y(\hat{\sigma}_z\hat{\rho}_S(t)\hat{\sigma}_z - \hat{\rho}_S(t)) + z(\hat{\sigma}_x\hat{\rho}_S(t)\hat{\sigma}_x - \hat{\rho}_S(t)) + z(\hat{\sigma}_y\hat{\rho}_S(t)\hat{\sigma}_y - \hat{\rho}_S(t)).$$  

(17)

Then, the use of eq. (6) gives rise to:

$$L = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -2(y + z) & -2x & 0 \\
0 & 2x & -2(y + z) & 0 \\
0 & 0 & 0 & -4x
\end{pmatrix}.$$  

(18)

Multiplication of the matrix $L$ by $\frac{1}{2(y + z)}$ and introduction of the new parameters, $\theta = \frac{x}{y + z}$, $\Omega = -\frac{2x}{y + z}$, $\tau = 2(y + z)t$, give:

$$L' = \frac{L}{2(y + z)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 - \theta & 0 \\
0 & \theta & -1 & 0 \\
0 & 0 & 0 & \Omega
\end{pmatrix}.$$  

(19)
while $\Omega \in (-2, 0)$, $\tau \in (-\infty, \infty)$.

Now eq. (7) can be written as:

$$F = e^{L'\tau},$$  \hspace{1cm} (20)$$

that makes easy the calculation of the exponential matrix $F$:

$$F = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{-\tau} \cos(\theta \tau) & -e^{-\tau} \sin(\theta \tau) & 0 \\
0 & e^{-\tau} \sin(\theta \tau) & e^{-\tau} \cos(\theta \tau) & 0 \\
0 & 0 & 0 & e^{\tau \Omega}
\end{pmatrix},$$  \hspace{1cm} (21)$$

as well as of the Choi matrix, cf. eq. (8):

$$S = \begin{pmatrix}
e^{-\tau} \cos(\theta \tau) + \frac{\tau \Omega}{2} + \frac{i}{2} & 0 & 0 & ie^{-\tau} \sin(\theta \tau) \\
0 & \frac{1}{2} - \frac{\tau \Omega}{2} & 0 & 0 \\
- ie^{-\tau} \sin(\theta \tau) & 0 & 0 & -e^{-\tau} \cos(\theta \tau) + \frac{\tau \Omega}{2} + \frac{i}{2}
\end{pmatrix}.$$  \hspace{1cm} (22)$$

Diagonalization of the $S$ matrix, eq. (22), gives the eigenvalues:

$$\frac{1}{2} \left(1 - e^{\tau \Omega}\right),$$  \hspace{1cm} (23a)$$

$$\frac{1}{2} \left(1 - e^{\tau \Omega}\right),$$  \hspace{1cm} (23b)$$

$$\frac{1}{2} e^{-\tau} \left(-2 + e^{\tau} + e^{\tau + \Omega}\right),$$  \hspace{1cm} (23c)$$

$$\frac{1}{2} e^{-\tau} \left(2 + e^{\tau} + e^{\tau + \Omega}\right),$$  \hspace{1cm} (23d)$$

and the respective non-normalized eigenvectors:

$$\{0, 0, 1, 0\},$$  \hspace{1cm} (24a)$$

$$\{0, 1, 0, 0\},$$  \hspace{1cm} (24b)$$

$$\{i(\cot[\theta \tau] - \csc[\theta \tau]), 0, 0, 1\},$$  \hspace{1cm} (24c)$$

$$\{i(\cot[\theta \tau] + \csc[\theta \tau]), 0, 0, 1\}\}.$$  \hspace{1cm} (24d)$$

With the use of eq. (11), from eqs. (23)-(24) follow the desired Kraus matrices:

$$E_1 = \begin{pmatrix}0 \\
\frac{i}{2} \sqrt{1 - e^{\tau \Omega}} \\
0 \\
-\frac{i}{2} \sqrt{1 - e^{\tau \Omega}}
\end{pmatrix},$$  \hspace{1cm} (25)$$

$$E_2 = \begin{pmatrix}0 \\
\frac{i}{2} \sqrt{1 - e^{\tau \Omega}} \\
0 \\
\frac{i}{2} \sqrt{1 - e^{\tau \Omega}}
\end{pmatrix},$$  \hspace{1cm} (26)$$

$$E_3 = \begin{pmatrix}\frac{1}{2} \left(1 - i \tan \left(\frac{\theta \tau}{2}\right)\right) \sqrt{\frac{e^{\tau \Omega} - 2e^{-\tau} + 1}{\tan^2\left(\frac{\theta \tau}{2}\right) + 1}} & 0 \\
0 & \frac{1}{2} \left(1 - i \tan \left(\frac{\theta \tau}{2}\right)\right) \sqrt{\frac{e^{\tau \Omega} - 2e^{-\tau} + 1}{\tan^2\left(\frac{\theta \tau}{2}\right) + 1}}
\end{pmatrix}.$$  \hspace{1cm} (27)$$
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\begin{align*}
E_4 &= \left( \begin{array}{cc}
\frac{1}{2} \left( 1 + i \cot \left( \frac{\Omega}{2} \right) \right) \sqrt{\frac{e^{\tau \Omega} + 2 e^{-\tau} + 1}{\cot^2 \left( \frac{\Omega}{2} \right) + 1}} & 0 \\
0 & \frac{1}{2} i \left( \cot \left( \frac{\Omega}{2} \right) + i \right) \sqrt{\frac{e^{\tau \Omega} + 2 e^{-\tau} + 1}{\cot^2 \left( \frac{\Omega}{2} \right) + 1}}
\end{array} \right) \tag{28}
\end{align*}

It is easy to show that the Kraus matrices (25)-(28) satisfy the completeness relation \( \sum_k \hat{E}_k(t)^\dagger \hat{E}_k(t) = \hat{I} \) and also reduce to the standard ones in eq.(13) with the following choice of the parameters: \( \theta = 0 \) (equivalently \( x = 0 \), neglecting the Lamb shift) and \( \Omega = -1 \) (equivalently \( y = z \)). That is, for this choice of the parameters values, from eqs. (25)-(28) easily follow approximations of the Kraus operators eqs.(25)-(28), respectively:

\begin{align*}
E_1 &= \left( \begin{array}{cc}
0 & -\frac{1}{2} i \sqrt{1 - e^{-\tau}} \\
\frac{1}{2} \sqrt{1 - e^{-\tau}} & 0
\end{array} \right), \tag{29}
E_2 &= \left( \begin{array}{cc}
0 & -\frac{1}{2} i \sqrt{1 - e^{-\tau}} \\
\frac{1}{2} \sqrt{1 - e^{-\tau}} & 0
\end{array} \right), \tag{30}
E_3 &= \left( \begin{array}{cc}
\frac{1}{2} \sqrt{1 - e^{-\tau}} & 0 \\
0 & -\frac{1}{2} \sqrt{1 - e^{-\tau}}
\end{array} \right), \tag{31}
E_4 &= \left( \begin{array}{cc}
-\frac{1}{2} i e^{-\frac{\tau}{2}} \sqrt{3 + e^\tau} & 0 \\
0 & -\frac{1}{2} i e^{-\frac{\tau}{2}} \sqrt{3 + e^\tau}
\end{array} \right). \tag{32}
\end{align*}

The matrices (29)-(31) are the \( \hat{\sigma}_z = |0\rangle \langle 0| - |1\rangle \langle 1| \) representation of the standard Kraus operators in eq.(13) while the matrix \( E'_4 = 2 \hat{E}_4 \) exhibits a unitary freedom for the Kraus operator \( E_4 \) in eq.(13).

3.2 Comparison of the generalized and the standard depolarizing channels

The Kraus matrices eqs.(25)- (28) represent a generalization of the standard DP channel eq.(13) (i.e. (29)-(32)). In this section we compare these two kinds of the noisy channels by comparing their dynamical effects regarding the Bloch-sphere volume, the von Neumann entropy and the trace distance. In order to give precise meaning of the task, we first provide definitions of these standard quantum-information quantities.

The Bloch sphere is a geometrical representation of the single-qubit state space where every pure state can be written as:

\[ \hat{\rho} = \frac{1}{2} \left( \hat{I} + \mathbf{n} \cdot \hat{\sigma} \right) \tag{33} \]

where the vector \( \mathbf{n} \) of the unit length, \( |\mathbf{n}| = 1 \), corresponds to the points on the Bloch sphere determined by the spherical coordinates \( \mathbf{n} = (\sin v \cos u, \sin v \sin u, \cos v) \) and uniquely determine a pure quantum state of a single qubit; \( v \in [0, \pi] \), \( u \in [0, 2\pi] \). The points inside the sphere correspond one-to-one to the mixed states for every \( |\mathbf{n}| < 1 \). The noisy channel effects can be geometrically presented as the Bloch-sphere deformation [4]. Denoting the volume of the [deformed] Bloch sphere \( V(t) \) in an instant of time \( t \), the relative speed of the volume-change is defined as:
\[ \kappa(t) = \frac{1}{V_0} \frac{dV(t)}{dt} \]  

(34)

where \( V_0 \) is the volume at \( t = 0 \).

The von Neumann entropy of a quantum state \( \hat{\rho} \) in an instant \( t \) of time

\[ S(t) = -\text{tr}(\hat{\rho}(t) \ln \hat{\rho}(t)), \]  

(35)

increases for all kinds of the noisy channels [4]. Increase of the von Neumann entropy is an alternative (non-geometrical) description of the Bloch-sphere deformation that brings the information-theoretic description of the regarded processes [4] that cannot be uniquely deduced from eq. (34).

The so-called trace distance

\[ T(\hat{\rho}, \hat{\sigma}) := \frac{1}{2} ||\hat{\rho} - \hat{\sigma}||_1 = \frac{1}{2} \text{tr} \left[ \sqrt{(\hat{\rho} - \hat{\sigma})(\hat{\rho} - \hat{\sigma})^\dagger} \right] \]  

(36)

quantifies how much the quantum states \( \hat{\rho} \) and \( \hat{\sigma} \) differ from each other [4]. Hence comparison of the regarded channels is possible with the use of eq. (36) by setting \( \hat{\rho}(t) \) and \( \hat{\sigma}(t) \) as the states produced by the generalized and by the standard DP channel, respectively.

For the generalized DP channel, i.e. from eqs. (25-28) follows:

\[ \begin{align*}
\phi_\tau(I) &= \hat{I}, \\
\phi_\tau(\hat{\sigma}_x) &= e^{-\tau} (\hat{\sigma}_x \cos \theta \tau + \hat{\sigma}_y \sin \theta \tau), \\
\phi_\tau(\hat{\sigma}_y) &= e^{-\tau} (\hat{\sigma}_y \cos \theta \tau - \hat{\sigma}_x \sin \theta \tau), \\
\phi_\tau(\hat{\sigma}_z) &= e^{\tau \Omega} \hat{\sigma}_z,
\end{align*} \]  

(37a-37d)

that is

\[ \phi_\tau(\hat{\rho}) = \frac{1}{2} [\hat{I} + e^{-\tau} \sin v \cos(u + \theta \tau) \hat{\sigma}_x + e^{-\tau} \sin u \sin(v + \theta \tau) \hat{\sigma}_y + e^{\tau \Omega} \cos v \hat{\sigma}_z], \]  

(38)

and equivalently:

\[ \begin{align*}
\phi_\tau(\hat{\rho}) &= \frac{1}{2} [\hat{I} + e^{-2(\Omega + 2 \tau)} \sin v \cos(u + 2\tau) \hat{\sigma}_x + e^{-2\Omega \tau} \sin u \sin(v + 2\tau) \hat{\sigma}_y + e^{-4\Omega \tau} \cos v \hat{\sigma}_z].
\end{align*} \]  

(39)

Expressions (38) and (39) are solutions of the master equation eq. (16). It is well known that for Pauli channel Pauli matrices are eigenvectors of the channel, but, as can be seen from eqs. (37), it is not the case for generalized depolarizing (GDP) channel.

From eq. (39) we analytically obtain the time dependence

\[ V(\tau) = \frac{4\pi}{3} e^{\tau(\Omega - 2)}, \]  

(40)

and the relative change

\[ \kappa(t) = -4(2z + y)e^{-4(2z + y)t} \]  

(41)
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Fig. 1 (left) The depolarizing channel for: $T = 50$, $\alpha = 0.02$, $\omega_0 = 1$ and $\omega_c = 15$. The big sphere is for $t = 0$, the small sphere is obtained for the standard DP channel ($x = 0$ and $y = z$), while the ellipsoid concerns the generalized DP channel, both for $t = 0.05$. (Right) The relative change of the Bloch sphere volume. The dashed line is for the standard DP channel and the thick one for the generalized DP channel, respectively.

of the deformed-Bloch-sphere volume. The von Neumann entropy and the trace distance are numerically calculated for the initial state defined by $u = 0 = v$, eq.(33). Comparison with the standard DP channel follows from placing $x = 0$ and $y = z$ in eq.(39).

Fig.1(left) exhibits faster deformation of the Bloch sphere for the standard than for the generalized DP channel, while Fig.1(right) depicts the volume relative change, eq.(41).

From Fig.2(top left) we can see that, for long times, the trace distance tends to be equal for both channels.

Time dependence of the entropy and the rate of its change are depicted on Fig.3.

Expectably, operation of the GDP channel is sensitive to the values of the parameters $T$, $\alpha$ and $\omega_c$. Qualitative similarity of the GDP-channel operations for certain values of the parameters is depicted in Figs.4 and 5.

From Figs. 4 and 5 increase in temperature $T$ and the coupling constant $\alpha$ gives rise to the more efficient action of the channel. To this end, relatively the most relevant contribution is due to the strength of the interaction-which, of course, cannot be arbitrarily large for the weak-coupling Markovian processes such as the GDP channel. The contribution of the cutoff frequency comes essentially from the better satisfied formal condition for Markovianity, i.e. form the inequality $\omega_0/\omega_c \ll 1$, and hence is of the secondary importance.

4 Discussion

Microscopic derivation of the Kraus operators is poorly addressed in the literature. Nevertheless, microscopically derived Kraus operators provide the physically richer and mathematically the more accurate description of the realistic physical processes. Hence we can expect that the parameters control can in principle give rise to the open system’s partial control. Bearing in mind that the error-correction
protocols [4] are adapted to the concrete forms of the Kraus operators, the knowledge of the exact, generalized Kraus operators eqs. (25)-(28) can help in devising the more reliable and more accurate error-correction protocols [18,19,20,21,22]. To this end, the results will be presented elsewhere. Here we just emphasize that our results regard the short-time intervals of interest for the realistic quantum information/computation protocols/algorithms.

Practical implementation of the quantum information-theoretic protocols [4] assumes that the quantum “hardware” operates as it is theoretically described. This non-trivial assumption is here theoretically addressed in the context of the one-qubit quantum noise channels. The results regarding the amplitude damping and the phase damping quantum channels are presented in [16].

On the other hand, to the best of our knowledge, the generalized Kraus operators for the depolarizing channel, eqs. (25)-(28), are here derived for the first time exhibiting a subtle yet physically important dissent from the standard and widely
Fig. 4 The GDP process ellipsoids for different parameters values, starting from the largest one, respectively: $T = 50, \alpha = 0.005, \omega_c = 15$; $T = 100, \alpha = 0.005, \omega_c = 50$; $T = 50, \alpha = 0.02, \omega_c = 15$; $T = 50, \alpha = 0.02, \omega_c = 50$; $T = 100, \alpha = 0.02, \omega_c = 15$. The parameters $\omega_0 = 1$ and $t = 0.05$ are kept fixed. The ellipsoids pertaining to $(T = 50, \alpha = 0.02, \omega_c = 15)$ and $(T = 50, \alpha = 0.02, \omega_c = 50)$ are mutually almost indistinguishable.

Fig. 5 (Top left) The relative change of the Bloch sphere volume. (Top right) The trace-distance change. (Bottom) The entropy change. For all figures the meaning of the lines is the same: thinner dashed line $(T = 50, \alpha = 0.005, \omega_c = 15)$, dot-dashed line $(T = 100, \alpha = 0.005 i \omega_c = 50)$, thinner solid line $(T = 50, \alpha = 0.02 i \omega_c = 15)$, thicker dashed line $(T = 50, \alpha = 0.02 i \omega_c = 50)$, thicker solid line $(T = 100, \alpha = 0.02 i \omega_c = 15)$. The parameter $\omega_0 = 1$ is kept fixed.

used ones, eq. (13). On this basis we can recognize the following research lines of interest.

From Figs. 4 and 5 we can learn that higher temperature and strong interaction give rise to the more efficient and faster action of the generalized depolarizing channel that, at least in principle, can be experimentally tested. On the other hand, transition to the Schrödinger picture:

$$\hat{E}_i^S(t) = \hat{U}(t)\hat{E}_i(t),$$

(42)
where $\hat{U}(t) = e^{-\hat{H}q t}$ and $\hat{H}_q$ is the qubit’s self-Hamiltonian, provides a study of the external-fields influence on the single-qubit’s dynamics.

Temporal change of the Bloch sphere, Section 3.2, reveals that the standard channel is more detrimental than the here introduced generalized channel for the short time intervals.

However, for long times we observe the full match of the standard and the here derived generalized DP channel. This can be seen in the asymptotic limit ($\tau \to \infty$) as follows. On the one hand, the state in eq.(38) clearly satisfies $\lim_{\tau \to \infty} \phi_{\tau}(\hat{\rho}) = \hat{I}/2$, which is the defining feature of the standard DP channel, eq.(13). On the other hand, in the asymptotic limit, all the Kraus operators, eqs. (25)-(28), (29)-(32) and eq.(13), give the unique set of Kraus operators:

$$E_1 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$E_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$E_3 = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

$$E_4 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

that obviously satisfy the completeness relation.

With the use of eqs.(62)-(64), the condition $2\gamma_{zz}(0, T) = \gamma_{xx}(\omega_0, T) + \gamma_{yy}(\omega_0, T)$ (Section 3.1) for reducing the GDP to the DP channel, numerically gives the unique (physically relevant, i.e. non-negative) value $\omega_0 = 0$ for every temperature $T$. Then neglecting the Lamb-shift term makes the interaction-picture equation (14) valid also in the Schrödinger picture.

As an application, we consider entanglement dynamics for a pair of qubits regarding the following typical scenario in quantum information-processing. Alice and Bob share a pair of the initially entangled qubits (denoted 1 and 2) and locally and independently perform operations on their respective qubits. The qubits are sufficiently remote from each other and hence do not mutually interact while being subjected to the mutually independent, local GDP channels. The task is to investigate dynamics of entanglement for the pair 1 + 2.

Being spatially remote, the qubits are dynamically independent. Therefore the Kraus operators for the combined 1 + 2 system are the tensor-product Kraus operators for the individual qubits. Transition to the Schrödinger picture is fulfilled by the unitary operations, $U_i(t) = \exp(-it\hat{H}_i/\hbar)$ for the qubits $i = 1, 2$, independently of each other. Given the combined-system’s state (in the Schrödinger picture):

$$\rho(t) = \sum_{i,j} U_1(t) K_{1i}(t) \otimes U_2(t) K_{2j}(t) |\Psi(0)\rangle \langle \Psi(0)| K^*_1(t) U^*_1(t) \otimes K^*_2(t) U^*_2(t),$$

we calculate the so-called concurrence for a bipartite mixed state $\rho(t)$ that is defined as [23]:

$$C(\rho(t)) = \max \{0, A(t)\},$$

where
Fig. 6 Concurrence dynamics for the standard DP channel (solid line) and the GDP channel (dotted line). The parameters: $\omega_1 = 0.1$, $\omega_2 = 0.2$, $T = 10$, $\alpha = 0.02$.

$$A(t) = \sqrt{\lambda_1(t)} - \sqrt{\lambda_2(t)} - \sqrt{\lambda_3(t)} - \sqrt{\lambda_4(t)}, \quad (49)$$

with the eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ of

$$\rho(t)(\sigma_{1y} \otimes \sigma_{2y})\rho^*(t)(\sigma_{1y} \otimes \sigma_{2y}). \quad (50)$$

In (50), $\sigma_{iy}$ is the $y$-Pauli-matrix for the $i$th qubit, while $*$ denotes the standard complex conjugation. For the initial entangled state we choose $|\Psi\rangle = [|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2]/\sqrt{2}$, where we omit the symbol of the tensor-product and $|i\rangle, i = 0, 1$ represent the eigenbasis of the $\sigma_z$ Pauli matrix.

Bearing in mind the definition of the so-called entanglement-of-formation $[23]$:

$$\mathcal{E}(\rho) = H \left(1 + \sqrt{1 + C^2(t)}\right) \quad (51)$$

where $H(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$, from Fig. 6 we conclude about the occurrence of the “entanglement sudden death” $[24]$ also for the GDP channel. That is, for the concurrence $C(t) = 0$ for some $t$, the entanglement of formation $\mathcal{E} = 0$ in the same instant of time—the loss of the initial entanglement in the instant of time $t$. Fig. 6 justifies the above finding that the GDP channel is less deteriorating (slower occurrence of the entanglement sudden death) than the standard DP channel.

5 Conclusion

Microscopic derivation of the Kraus operators is a new emerging line of research of foundational as well as of interest for application in modern open quantum systems theory. The microscopically derived Kraus operators for the generalized depolarizing channel are here presented for the first time while carrying the clear signatures of the microscopic-models details. Implications for the possible open-system's control and for an elaboration of the known error-correction protocols will be presented elsewhere.
A Derivation of the generalized depolarizing master equation

Here we derive a Markovian master equation, which describes a completely positive and trace preserving, homogeneous Markovian process for the microscopic model eq. (15) in the body text. Such processes are known to be presented by a master equation of the general Lindblad form [3,12]:

\[ \dot{\rho}_S(t) = -i[H_{LS}, \rho_S(t)] + \sum_{k,\ell} \gamma_k(\omega) [A_k(\omega) \dot{\rho}_S(t) \hat{A}_k^\dagger(\omega) - \frac{1}{2} \{ \hat{A}_k(\omega) \hat{A}_k^\dagger(\omega), \rho_S(t) \}], \]  

(52)

where the Lamb shift:

\[ \dot{H}_{LS} = \sum_{k,\ell} S_{k\ell}(\omega) \hat{A}_k^\dagger(\omega) \hat{A}_k(\omega), \]  

(53)

and the correlation (damping) functions \( \gamma_k(\omega) \):

\[ \gamma_k(\omega) = 2\pi \text{tr} \left[ \hat{B}_k(\omega) \hat{B}_k \rho_{\text{Th}} \right], \]  

(54)

while \( \hat{\rho}_{\text{Th}} \) is the environment’s thermal equilibrium state, \( \hat{\rho}_{\text{Th}} = \exp(-\beta\hat{H}_E)/Z, \) on the inverse temperature \( \beta = 1/k_B T \).

With the definition [3],

\[ \hat{A}_k(\omega) = \sum_{\epsilon' = -\epsilon = \omega} \langle \psi_{\epsilon'} | \hat{A}_k | \psi_\epsilon \rangle | \psi_{\epsilon'} \rangle, \]  

(55)

where \( | \psi_\epsilon \rangle \) are eigenstates of the system’s self-Hamiltonian, \( \hat{H}_S \), and \( \hat{A}_k \) appearing in eq. (15) of the body text, and analogously for \( \hat{B}_k(\omega), k = x, y, z \), we obtain:

\[ \hat{A}_1(0) = \hat{\sigma}_z, \quad \hat{A}_1(\omega_0) = 0, \quad \hat{A}_1(-\omega_0) = 0, \]  

(56a)

\[ \hat{A}_2(0) = 0, \quad \hat{A}_2(\omega_0) = \hat{\sigma}_-, \quad \hat{A}_2(-\omega_0) = \hat{\sigma}_+, \]  

(56b)

\[ \hat{A}_3(0) = 0, \quad \hat{A}_3(\omega_0) = i\hat{\sigma}_-, \quad \hat{A}_3(-\omega_0) = -i\hat{\sigma}_+. \]  

(56c)

From the second term in eq. (52) and having in mind eq. (56), after some algebra we obtain the dissipator, \( D(\dot{\rho}_S) \):

\[
\mathcal{D}(\dot{\rho}_S) = \gamma_{xx}(0)[\hat{\sigma}_z \dot{\rho}_S \hat{\sigma}_z - \dot{\rho}_S] + \gamma_{yy}(\omega_0)[\hat{\sigma}_- \dot{\rho}_S \hat{\sigma}_+ - \frac{1}{2} \{ \hat{\sigma}_+ \hat{\sigma}_-, \dot{\rho}_S \}] + \\
+ \gamma_{yx}(\omega_0)[i\hat{\sigma}_- \dot{\rho}_S \hat{\sigma}_+ - \frac{i}{2} \{ \hat{\sigma}_+ \hat{\sigma}_-, \dot{\rho}_S \}] + \gamma_{xy}(\omega_0)[\hat{\sigma}_- \dot{\rho}_S \hat{\sigma}_+ - \frac{1}{2} \{ \hat{\sigma}_+ \hat{\sigma}_-, \dot{\rho}_S \}] + \\
+ \gamma_{yy}(-\omega_0)[-i\hat{\sigma}_+ \dot{\rho}_S \hat{\sigma}_- + \frac{i}{2} \{ \hat{\sigma}_+ \hat{\sigma}_-, \dot{\rho}_S \}] + \gamma_{yx}(-\omega_0)[-i\hat{\sigma}_+ \dot{\rho}_S \hat{\sigma}_- + \frac{i}{2} \{ \hat{\sigma}_+ \hat{\sigma}_-, \dot{\rho}_S \}]
\]  

(57)

where \( \hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y) \).

In accordance with eq. (56), the initial thermal state of the bath \( E \) can be chosen:

\[ \dot{\rho}_{\text{Ei}}(0) = \hat{\rho}_{\text{Ei}} \otimes \hat{\rho}_{\text{Ei}} \otimes \hat{\rho}_{\text{Ei}}, \]  

(58)

where the thermal states \( \hat{\rho}_{\text{Ei}} = \exp(-\beta\hat{H}_{E_i})/Z_{E_i}, i = x, y, z, \) and \( \sum_i \hat{H}_{E_i} = \hat{H}_E \), for the inverse temperature \( \beta \). That is, the qubit’s environment \( E \) can be regarded as formally consisting of three noninteracting subsystems (modes) so that these modes, represented as the independent environments \( E_i \), are independently coupled to the system’s \( \hat{\sigma}_z, \hat{\sigma}_y \) and \( \hat{\sigma}_x \) Pauli
Generalized Kraus operators for the one-qubit depolarizing quantum channel

operators. Then it easily follows that the cross terms in eq. (57), \( \gamma_{ij}, i \neq j = x, y, z \), fall-off as a direct consequence of eq. (54). Now, for noninteracting \( E_i \), environments, the dissipator eq. (57) reads:

\[
\mathcal{D}(\dot{\rho}_S) = \gamma_{zz}(0)[\sigma_x \dot{\rho}_S \sigma_x - \dot{\rho}_S] + \gamma_{yy}(\omega_0)[\sigma_y \dot{\rho}_S \sigma_y - \dot{\rho}_S] + \frac{1}{2} \{\sigma_x \sigma_x, \dot{\rho}_S\} + \\
+ \gamma_{xx}(\omega_0)[\sigma_x \dot{\rho}_S \sigma_x - \dot{\rho}_S] + \frac{1}{2} \{\sigma_x \sigma_x, \dot{\rho}_S\} + \\
+ \gamma_{yy}(\omega_0)[\sigma_y \dot{\rho}_S \sigma_y - \dot{\rho}_S] + \frac{1}{2} \{\sigma_y \sigma_y, \dot{\rho}_S\} + \\
+ \gamma_{xx}(\omega_0)[\sigma_x \dot{\rho}_S \sigma_x - \dot{\rho}_S] + \frac{1}{2} \{\sigma_x \sigma_x, \dot{\rho}_S\} + \\
+ \gamma_{yy}(\omega_0)[\sigma_y \dot{\rho}_S \sigma_y - \dot{\rho}_S] + \frac{1}{2} \{\sigma_y \sigma_y, \dot{\rho}_S\}.
\]

(59)

The model eq. (15) and eq. (58) limit the choice of the parameters in order to obtain Markovian dynamics eq. (52) as follows [3]. First, it is required that the so-called spectral density, \( J(\omega) \), be of the Ohmic kind. Second, the inequality \( \omega_0/\omega_c \ll 1 \) is required, where \( \omega_c \) is the cutoff frequency. Finally, the high temperature limit is required for the depolarizing process, formally defined by \( k_B T/\hbar \omega_0 \gg 1 \).

From eq. (54), the damping function \( \gamma_{zz}(0) \):

\[
\gamma_{zz}(0) = 2 \pi \text{tr}[\dot{B}_z(0) \dot{B}_z \rho_{E1H}(0)],
\]

(60)

where

\[
\dot{B}_z = \int_{-\omega_{\text{max}}}^{\omega_{\text{max}}} d\omega \dot{B}_z(\omega), \quad \text{with}\quad \dot{B}_z(\omega) = h(\omega) a_\omega, \quad \text{for} \quad \omega > 0.
\]

Since \( \dot{B}_z(\omega = 0) \) is not well-defined, we stick to [3]:

\[
\gamma_{zz}(0) = 2 \pi \lim_{\omega \to 0} \text{tr}[\dot{B}_z(0) \dot{B}_z \rho_{E1H}(0)],
\]

(61)

with the often used Ohmic spectral density \( J(\omega) = \alpha \omega e^{-\omega/\omega_c} \) thus obtaining:

\[
\gamma_{zz}(0) = 2 \pi \lim_{\omega \to 0} J(|\omega|)|n(|\omega|)|,
\]

(62)

while

\[
\gamma_{yy}(\omega_0) = \frac{\pi}{2} J(\omega_0)[|n(\omega_0)| + 1]
\]

(63)

and

\[
\gamma_{yy}(\omega_0) = \frac{\pi}{2} J(\omega_0)|n(\omega_0)|,
\]

(64)

and analogously for \( \gamma_{xx} \); the constant \( \alpha \) is the interaction strength, i.e. the weak-coupling constant.

It should be stressed here that, for high temperature, \( |n(\omega_0)| = [e^{(\omega_0/T) - 1}]^{-1} \gg 1 \) and thus:

\[
\gamma_{yy}(\omega_0) \approx \gamma_{yy}(\omega_0) = \frac{\pi}{2} J(\omega_0)|n(\omega_0)|,
\]

(65)

and analogously for \( \gamma_{xx} \). Hence the dissipator eq. (57) simplifies:

\[
\mathcal{D}(\dot{\rho}_S) = \gamma_{zz}(0)[\sigma_x \dot{\rho}_S \sigma_x - \dot{\rho}_S] + \\
+ \gamma_{yy}(\omega_0)[\sigma_y \dot{\rho}_S \sigma_y - \dot{\rho}_S] + \frac{1}{2} \{\sigma_x \sigma_x, \dot{\rho}_S\} - \\
- \frac{1}{2} \{\sigma_y \sigma_y, \dot{\rho}_S\} + \\
- \gamma_{xx}(\omega_0)[\sigma_x \dot{\rho}_S \sigma_x - \dot{\rho}_S] + \frac{1}{2} \{\sigma_x \sigma_x, \dot{\rho}_S\} - \\
- \gamma_{yy}(\omega_0)[\sigma_y \dot{\rho}_S \sigma_y - \dot{\rho}_S] + \frac{1}{2} \{\sigma_y \sigma_y, \dot{\rho}_S\}.
\]

(66)

With the use of the identities: \( \dot{\sigma}_x \rho \sigma_x + \dot{\sigma}_x \rho \sigma_x = \frac{1}{2} \{\sigma_x \rho \sigma_x + \sigma_x \rho \sigma_x\} \) and \( \{\sigma_x \sigma_x, \rho + \dot{\sigma}_x \rho \sigma_x + \dot{\sigma}_x \rho \sigma_x\} = 2 \rho \), eq. (60) reads:

\[
\mathcal{D}(\dot{\rho}_S) = \gamma_{zz}(0)[\sigma_x \dot{\rho}_S \sigma_x - \dot{\rho}_S] + \\
+ \gamma_{yy}(\omega_0) + \gamma_{xx}(\omega_0) \frac{1}{2} \{\sigma_x \rho \sigma_x - \dot{\rho}_S\} + \\
+ \gamma_{yy}(\omega_0) + \gamma_{xx}(\omega_0) \frac{1}{2} \{\sigma_y \rho \sigma_y - \dot{\rho}_S\}.
\]

(67)
The damping rates

\[ \hat{H}_{LS} = S_{11}(0)\hat{I} + S_{22}(\omega_0)\hat{\sigma}_+\hat{\sigma}_- + S_{33}(\omega_0)\hat{\sigma}_+\hat{\sigma}_- + 
+ S_{22}(-\omega_0)\hat{\sigma}_-\hat{\sigma}_+ + S_{33}(-\omega_0)\hat{\sigma}_-\hat{\sigma}_+. \]  

(68)

Calculation of \( S_{11}(\pm\omega_0) \) follows via the so called Sochozki’s formulae [3]:

\[ S_{22}(\omega_0) = \frac{1}{4} P.V. \int_0^{\omega_{\text{max}}} d\omega' J(\omega') \left\{ \frac{n(\omega')}{(\omega_0 - \omega')^2} + \frac{n(\omega')}{(\omega_0 + \omega')^2} \right\}, \]  

(69)

and

\[ S_{22}(-\omega_0) = -\frac{1}{4} P.V. \int_0^{\omega_{\text{max}}} d\omega' J(\omega') \left\{ \frac{n(\omega')}{(\omega_0 + \omega')^2} + \frac{n(\omega')}{(\omega_0 - \omega')^2} \right\}, \]  

(70)

and similarly for \( S_{33}(\pm\omega_0) \).

At this point, the following conditions facilitate the analysis: the spectral density \( J(\omega) \) is assumed to be the same for all interaction-terms in eq. (15), while, for high temperature, \( S_{22}(-\omega_0) = -S_{22}(\omega_0) \) and \( S_{33}(-\omega_0) = -S_{33}(\omega_0) \). Hence:

\[ \hat{H}_{LS} = S_{11}(0)\hat{I} + (S_{22}(\omega_0) + S_{33}(\omega_0))\hat{\sigma}_x. \]  

(71)

Equations (69) and (70), in the high temperature limit, take the form:

\[ S_{22}(\omega_0) = \frac{\omega_0}{2} P.V. \int_0^{\omega_{\text{max}}} d\omega' J(\omega') \frac{n(\omega')}{\omega_0^2 - \omega'^2} \]  

(72)

and

\[ S_{22}(-\omega_0) = -\frac{\omega_0}{2} P.V. \int_0^{\omega_{\text{max}}} d\omega' J(\omega') \frac{n(\omega')}{\omega_0^2 - \omega'^2} = -S_{22}(\omega_0), \]  

(73)

and analogously for \( S_{33}(\pm\omega_0) \).

From eqs. (71)–(73), the Lamb shift finally obtains the form:

\[ \hat{H}_{LS} = S_{11}(0)\hat{I} + (S_{22}(\omega_0) + S_{33}(\omega_0))\hat{\sigma}_x = S_{11}(0)\hat{I} + 
+ \frac{\omega_0}{2} \left( P.V. \int_0^{\omega_{\text{max}}} d\omega' J(\omega') \frac{n(\omega')}{\omega_0^2 - \omega'^2} + P.V. \int_0^{\omega_{\text{max}}} d\omega'' J(\omega'') \frac{n(\omega'')}{\omega_0^2 - \omega''^2} \right) \hat{\sigma}_x \equiv S_{11}(0)\hat{I} + \Delta \hat{\sigma}_x. \]  

(74)

Collecting all the above results follows the Markovian master equation:

\[ \frac{d\hat{\sigma}_x(t)}{dt} = -i\Delta [\hat{\sigma}_x, \hat{\rho}_S(t)] + \gamma_{xx}(0)(\hat{\sigma}_x \hat{\rho}_S(t) - \hat{\rho}_S(t) \hat{\sigma}_x) + 
+ \gamma_{xx}(\omega_0) + \frac{\gamma_{yy}(\omega_0)}{2} (\hat{\sigma}_y \hat{\rho}_S(t) - \hat{\rho}_S(t)), \]  

(75)

The damping rates \( \gamma_{xx} \) and \( \gamma_{xx} \) are of the same form. Eq. (77) is the master equation eq. (100) in the body text.

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