Canonical Formulation of the Light-Front Gluodynamics
and Quantization of the Non-Abelian Plane Waves

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Abstract

Without a gauge fixing, canonical variables for the light-front SU(2) gluodynamics are determined. The Gauss law is written in terms of the canonical variables. The system is qualified as a generalized dynamical system with first class constraints. Abelization is a specific feature of the formulation (most of the canonical variables transform nontrivially only under the action of an Abelian subgroup of the gauge transformations). At finite volume, a discrete spectrum of the light-front Hamiltonian \( P_+ \) is obtained in the sector of vanishing \( P_- \). We obtain, therefore, a quantized form of the classical solutions previously known as non-Abelian plane waves. Then, considering the infinite volume limit, we find that the presence of the mass gap depends on the way the infinite volume limit is taken, which may suggest the presence of different “phases” of the infinite volume theory. We also check that the formulation obtained is in accord with the standard perturbation theory if the latter is taken in the covariant gauges.

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I. INTRODUCTION

The light-front formulation of relativistic dynamics has been widely discussed since the work of Dirac [1]. Its virtue is the presence of a kinematic semi-positive observable, the momentum component $P_{-}$. It generates shifts along the longitudinal direction $x^{-} = (x^0 - x^3)/\sqrt{2}$ at a fixed light-front time $x^{+} = (x^0 + x^3)/\sqrt{2}$. As such, it should be quadratic in the canonical variables even in the presence of an interaction (such variables are called kinematic). For a system with no tachyons ($P^2 = P_{-}P_{+} - P_{\perp}^2 \geq 0$, $P_0 \geq 0$), $P_{-}$ is semi-positive, and the subspace annihilated by $P_{-}$ contains the translation invariant vacuum state. In addition, for a system with no massless states (i.e. having a mass gap), the above subspace contains only the states of vanishing four-momentum. Therefore, in the light-front formulation, finding the vacuum state and the Fock space of such a system are kinematic problems.

The light-front formulation also induces a specific kind of intuition believed to be valuable in many physical situations (for a review, see Refs. [2]). For example, the light-front formulation turns out to be closely related to the infinite momentum frame limit [3], and to the notion of constituent quarks [4]. We also note that the light-front formulation in a finite volume appears in discussions of recent developments of M-theory [5].

Thus, the light-front formulation of gauge theories and, in particular, of gluodynamics, is a worthwhile goal. However, up to the moment, the light-front formulation of the gauge theories is less well developed, in our opinion, than the conventional equal-time formulation. The reason for the above assertion is that, to overcome technical difficulties, the light-front formulation has been confined to fixed gauges (almost always, to the light-cone gauge [6–8], or to other gauges [9,10]). On the other hand, a general formulation of a gauge theory (see, e.g., Ref. [11]) may better start with a determination of the canonical variables prior to any gauge fixing. In the case of the equal-time formulation of the gauge theories, the canonical variables are $E$ and $A$. With this accomplished, one is to find the constraints and calculate the algebra of the Poisson brackets involving the constraints and the Hamiltonian. After that, the system is qualified as a generalized dynamical system with first class constraints (the case of the equal-time formulation of the gauge theories), and a well-developed machinery to treat such a system, in particular, by fixing a gauge and introducing the Faddeev–Popov ghosts, becomes available.

In this paper, we determine the canonical variables of the light-front $SU(2)$ gluodynamics without a gauge fixing. The light-front version of the Gauss law is also determined, and the system is qualified as a generalized dynamical system with first class constraints. Thus, the light-front formulation of a gauge theory is established on a par with the conventional equal-time formulation.

A specific feature of the formulation obtained is that it has a form of an Abelian gauge theory, because most of the canonical variables transform nontrivially only under the action of the Abelian subgroup of the gauge transformations which leaves the component $A_{-}$ of the gauge field invariant. Thus, we obtain a version of the Abelian projection [12] without a gauge fixing.

Quantization of the dynamical system obtained involves the ambiguity of ordering. We fix it in the simplest way, and then check that our choice leads to the standard Feynman rules in the covariant gauges.
We also consider a gauge invariant reduction of the dynamics to the configurations of zero $P_-$. To this end, we diagonalize $P_-$, i.e., identify the excitations carrying the nonzero quanta of the longitudinal momentum. Then, the reduction is obtained by nullifying the canonical variables corresponding to these quanta. If there is a mass gap in gluodynamics, and the light-front formulation is “correct” (i.e., equivalent to the equal-time formulation), we expect the light-front Hamiltonian $P_+$ to be vanishing on the equations of motion of this reduced dynamics. While the reduced dynamics is indeed much simpler than the complete one, the vanishing of $P_+$ is not evident.

This brings up another important issue, namely, the dependence of the formulation on the infrared regularization. To determine the canonical variables, we need to introduce a gauge invariant infrared cutoff. In this paper, we use a compactification on a torus imposing periodic boundary conditions on the gauge fields along the $x^-$ direction. Then, it turns out that the reduced $P_+$ does vanish on field configurations decaying fast enough at the infinity of the transverse plane, and is nontrivial on the configurations of nonzero asymptotics at the transverse infinity. Not surprisingly, in the latter case, the above reduced dynamics turns out to be a dynamics of the zero modes \[13\], i.e., of the fields with an imposed dependence on the longitudinal and transverse space coordinates. The classical solutions of the gauge field equations obtained in the framework of the reduced dynamics coincide up to a gauge with the previously known “non-Abelian plane wave” solutions due to Coleman \[14\]. The invariance of the quantum theory with respect to certain “large” gauge transformations gives a quantization condition for these non-Abelian plane waves. Therefore, the spectrum of $P_+$ on the subspace of vanishing $P_-$ turns out in this case to be discrete, bounded from below, and the quantum of the spectrum is proportional to $(g^2L)/V_\perp$, where $V_\perp$ is the volume of the compactified transverse plane, $g$ is the gauge coupling, and $L$ is the length of the compactified direction $x^-$. Note that this scaling of the quantum of the light-front energy holds for any nonzero number of the transverse dimensions, and the presence of the coupling makes the dimension correct.

We conclude that there is no mass gap in the finite volume light-front gluodynamics. On the other hand, the spectrum of the light-front gluodynamics at infinite volume is qualitatively dependent on the way the infinite volume limit is taken. If $L$ is taken to infinity first, the mass gap can be generated. If $V_\perp$ is taken to infinity before $L$, the resulting theory, if it exists, has no mass gap. Therefore, this can indicate that the finite volume theory contains markers ($V_\perp$ and $L$) for different “phases” of the infinite volume theory.

The rest of the paper is organized as follows. In Section 2, we define the canonical variables for $SU(2)$ gluodynamics without a gauge fixing, using Faddeev-Jackiw approach to constrained systems \[15\]. Then, in Section 3, we present the light-front Gauss law, which, in analogy with the equal-time formulation, generate the local gauge transformations of the canonical variables. In Section 4, we write the equation for the zero modes which is necessary to determine the light-front Hamiltonian. In Section 5, we discuss the quantization ambiguity, and check that the simplest prescription for fixing the ambiguity leads to the standard Feynman rules in the covariant gauges. In Section 6, we show that the reduced dynamics at $P_- = 0$ leads to non-Abelian plane waves solutions, and we find their quantum spectrum. Section 6 contains discussion and conclusion.
II. CANONICAL VARIABLES

We start with the action of the $SU(2)$ gluodynamics:

$$S_{\text{glue}} = \int dx^+ dx^- dx^\perp \left[ \frac{1}{2} F_{+}^{a} F_{+}^{a} + F_{-}^{a} F_{-}^{a} - \frac{1}{2} F_{12}^{a} F_{12}^{a} \right],$$  \hspace{1cm} (1)

where $x^\pm = (x^0 \pm x^3)/\sqrt{2}$; $x^\perp = x^{1,2}$; $F_{\mu\nu}^{a} = \partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + g\epsilon^{abc}A_{\mu}^{b}A_{\nu}^{c}$ is a strength tensor of the gauge field $A_{\mu}^{a}$; $a, b, c, \ldots$ are color indices running from 1 to 3, and $k$ is a Lorentz transverse index running from 1 to 2.

Our aim is to give a canonical formulation of the system (1) with $x^+$ as time. To this end, we will make a chain of transformations of the field variables. Every transformation will be one-to-one, or will introduce new auxiliary variables expressible via the initial variables. At each step, we will keep track of the form that the terms of the action with the time derivatives assume. Ultimately we will obtain the canonical form, $\sum_{i} p_{i} \dot{q}_{i}$ (the overdot denotes the time derivative), for these terms, and will recognize $p_{i}$ and $q_{i}$ as canonically conjugated variables. This way of treatment is in the spirit of the Faddeev–Jackiw approach to constrained systems [15].

What complicates this program is the way the time derivatives of $A_{k}^{a}$ enter the action (1): $(D_{-}A_{k})^{a} \dot{A}_{k}^{a}$, where $(D_{-}\Phi)^{a} = \partial_{-}\Phi^{a} + g\epsilon^{abc}A_{\mu}^{b}\Phi^{c}$ is the covariant derivative in the $x^-$ direction. A simplification of this term is the reason to confine the formulation to the light-cone gauge [7,8]. This approach is not available for us, as we explained above. Instead, to simplify this term, we suggest a transformation to new variables for $A_{k}^{a}$. We will denote them $\dot{A}_{k}^{a}$. The correspondence between the initial variables $A_{k}^{a}$ and the new variables $\dot{A}_{k}^{a}$ taken at the same moment of time is one-to-one and depends on the configuration of $A_{-}^{a}$ at the same moment of time.

Another ingredient of treating the term with the time derivatives of $A_{k}^{a}$ is taken in concord with Refs. [7,8]. That is, we compactify the theory along the $x^-$ direction. Namely, all the fields are considered to be periodic in $x^-$: $A_{k}^{a}(x^- = -L/2) = A_{k}^{a}(x^- = L/2)$. In this case, the spectrum of $D_{-}$ is discrete, and it becomes evident that the components of $A_{k}^{a}$ nullified by $D_{-}$ are nondynamical. One may neglect this subtlety at the expense of appearance of infrared divergences in the formulation.

To begin our chain of variable transformations, we start with the less problematic terms with the time derivative of $A_{-}^{a}$. The first term in the square brackets of Eq. (1) contains a square of the time derivative of $A_{-}^{a}$. In the case of the equal-time formulation, the time derivatives of all space components of the gauge field enter the action in this way. In the case under consideration, we will treat the time derivatives of the component $A_{-}^{a}$ in analogy with the equal-time formulation [11]. Namely, to get an action linear in time derivatives, we substitute the action (1) by an equivalent action:

$$S_{\text{glue}} = \int dx^+ dx^- dx^\perp \left[ \mathcal{E}^{a} F_{+}^{a} + F_{-}^{a} F_{-}^{a} - \frac{1}{2} (\mathcal{E}^{a} \mathcal{E}^{a} + F_{12}^{a} F_{12}^{a}) \right],$$  \hspace{1cm} (2)

where $\mathcal{E}^{a}$ is considered as an independent variable. To see the equivalence, take the variation with respect to $\mathcal{E}^{a}$ and substitute back in (1) its extremal value $F_{+}^{a}$. Now Eq. (2) is linear in the time derivatives, and the content of the round bracket gives the light-front energy.
yielded by the Noether procedure: \( P_\pm = \int dx^-dx^+ (F^a_{\pm}F^a_{\pm} + F^a_{12}F^a_{12})/2 \). The \( E^a \) will enter the definition of the canonical variable \( E^a = E^a + ... \) conjugated to \( A^a \) (see below Eq. (15)). The terms of \( E^a \) denoted by the tilde over the tilded and untilded bases. These expansions are an important ingredient of our approach.

The connection between the coefficients of the expansions of \( A^a_\pm \) as functional of \( A^a_\pm \) independent of \( x^- \). Below we systematically use the tilde over a quantity to denote the quantity gauge transformed to the light-cone gauge. The crucial point is that the transformation from \( A^a_\pm \) to \( \tilde{A}^a_\pm \) is one-to-one at fixed \( A^a_\pm \). Obviously, the transformation from \( A^a_\pm \) to \( \tilde{A}^a_\pm \) is not one-to-one. The transformation from \( A^a_\pm \) to \( \tilde{A}^a_\pm \) is one-to-one, but it involves time derivatives of \( A^a_\pm \). Thus, we keep the initial configurations \( A^a_\pm \) as independent variables and consider the variables \( \tilde{A}^a_\pm \) as functionals of the independent variables. For an illuminating discussion of the gauge transformation to the light-cone gauge see, for example, Ref. [16], where explicit formulas for \( U \) can be found.

We now need to express the term \( F^a_{-k}F^a_{+k} \) of Eq. (4), which contains the time derivatives of \( A^a_k \), in terms of the new set of variables \( \tilde{A}^a_\pm, \tilde{A}^a_k \). By the gauge invariance, the form of \( F^a_{-k}F^a_{+k} \) in terms of \( \tilde{A}^a_\pm, \tilde{A}^a_k \) is known: it is \( \tilde{F}^a_{-k}\tilde{F}^a_{+k} \). Thus, we need to express \( \tilde{A}^a_\pm \) in terms of \( A^a_\pm \). The connection between \( A^a_\pm \) and \( \tilde{A}^a_\pm \) (recall that \( \tilde{A}^a_\pm = \delta^a_\pm A^a_\pm \)) is easy to find considering a gauge invariant quantity expressible in terms of the component \( A^a_\pm \) alone. It is the trace of the large Wilson loop embracing the whole span of the compactified direction \( x^- \). Thus, in what follows, we will treat \( A^a_\pm \) as a known functional of \( A^a_\pm \).

To express \( \tilde{A}^a_\pm \) in terms of \( A^a_\pm \), we need to introduce special bases in the space of field configurations. The connection will be found between the coefficients of the expansions of the tilded and untilded fields over the tilded and untilded bases. These expansions are an important ingredient of our approach.

From now on, we switch over from the components with color indices to matrices: \( A^a_\mu = A^a_\mu \sigma^a/2 \), \( \sigma^a \) are the Pauli matrices. We will consider two sets of bases in the space of field configurations, \( \chi_\mu \) and \( \tilde{\chi}_\mu \), where \( \mu \) is an index described below, and every \( \chi_\mu, \tilde{\chi}_\mu \) is a traceless matrix field depending on the space-time coordinates.

The bases are complete and orthonormal, \( \langle \chi_\mu | \chi_\mu' \rangle = \delta_{\mu \mu'}, \langle \tilde{\chi}_\mu | \tilde{\chi}_\mu' \rangle = \delta_{\mu \mu'}, \) with respect to the following scalar product in the space of matrix fields:

\[
\langle \Phi | \Psi \rangle = \int dx^- 2 \text{Tr} \Phi \dagger \Psi.
\]

Therefore, any field is expressible as a sum over the bases: \( A_\mu = \sum_\mu \chi_\mu |\chi_\mu \rangle |A_\mu \rangle, \tilde{A}_\mu = \sum_\mu \tilde{\chi}_\mu |\tilde{\chi}_\mu \rangle |\tilde{A}_\mu \rangle \). For the components of the fields with respect to the bases, we will use the notation \( A^a_\mu = \langle \chi_\mu | A^a_\mu \rangle \). The fields will be expanded only over the corresponding bases:
tilted over tilted basis, untilded over untilded. Note that the expansion coefficients are independent of \(x^−\).

The bases \(\chi_p\) and \(\tilde{\chi}_p\) are the bases of eigenfunctions of the operators \(\hat{p} \equiv D−/i\) and \(\tilde{\hat{p}} \equiv \tilde{D}−/i\), respectively:

\[
\hat{p}\chi_p = p\chi_p, \quad \tilde{\hat{p}}\tilde{\chi}_p = p\tilde{\chi}_p.
\] (5)

Note that both \(\hat{p}\) and \(\tilde{\hat{p}}\) are Hermitian with the scalar product (4). Intuitively, the eigenvalues \(p\) correspond to the values of the quanta of the longitudinal momentum \(P−\). It is easy to understand that they are gauge invariant. Thus, there is no tilde over the eigenvalue \(p\) in the second of the Eqs. (5). We also indiscriminately interchange the label on the eigenfunction and the eigenvalue.

Now we give an explicit form of \(\tilde{\chi}_p\). Recalling the definitions \(\tilde{A}_a \equiv \delta^{3a}\alpha\) and \(\tilde{D}_−\Phi = \partial_−\Phi - ig[\tilde{A}_-, \Phi]\), it is easy to see that the eigenvalues of \(\tilde{\hat{p}}\) are

\[
p(n, \sigma) = \frac{2\pi n}{L} + \sigma g\alpha, \quad \sigma = -1, 0, +1,
\] (6)

where \(n\) is a number of the Fourier mode, and \(\sigma\) is an index to label the splitting of the level due to the presence of the gauge field. And the corresponding eigenfunctions are

\[
\tilde{\chi}_{p(n, \sigma)} = \exp(i\frac{2\pi n}{L}x^-)\sigma^3, \quad \tilde{\chi}_{p(n, \pm)} = \exp(i\frac{2\pi n}{L}x^-)(\sigma^1 \pm i\sigma^2). \] (7)

Now we give a representation for \(\chi_p\). If \(U\) is the unitary matrix of the gauge transformation from \(\tilde{A}_\mu\) to \(A_\mu\), \(A_\mu = U(\tilde{A}_\mu - \partial_\mu/(ig))U^\dagger\), it is easy to check that

\[
\chi_p = U\tilde{\chi}_p U^\dagger. \] (8)

In words: the eigenfunctions are transformed uniformly under the gauge transformations. The crucial difference between the tilded and untilded bases is that the one corresponding to the light-cone gauge is independent of the time and transverse space coordinates. We will exploit this property of the tilded basis in what follows.

Now we are ready to express \(\tilde{A}_p^\pm\) in terms of \(A_p^\pm\). To this end, consider a gauge invariant object \(\langle \chi_p|F_−\rangle\) at nonzero \(p\) (it is gauge invariant, because both \(\chi_p\) and \(F_−\) are transformed uniformly and the scalar product involves the trace). Calculated in the light-cone gauge, it is \(ip\tilde{A}_p^+\) (to see this, note that \(\langle \tilde{\chi}_p|\tilde{A}_-\rangle\) is vanishing at nonzero \(p\)). While in terms of the initial variables, it is \(ipA_p^+ - \tilde{A}_0^p\). Thus, we have

\[
\tilde{A}_p^+ = \frac{1}{ip}F_p^+ = A_p^+ - \frac{1}{ip}\tilde{A}_0^p, \quad p \neq 0.
\] (9)

These expressions are summarized as follows: \(\tilde{A}_p^+\) is linear in \(A_p^+\) and \(\tilde{A}_0^+\). This observation helps to keep track of the terms of the action involving the time derivatives of \(A_−\).

To express the zero mode of \(\tilde{A}_+\), \(\langle \chi_0|\tilde{A}_+\rangle \equiv \tilde{A}_0^+\), in terms of \(A_\pm\), we consider another gauge invariant object, \(\langle \chi_p|D_+\chi_p\rangle\), for an eigenvector \(\chi_p\) with nonzero commutator \([\chi_p, \chi_p^\dagger]\) = \(\epsilon(p)\chi_0\). The function \(\epsilon(p)\) above is gauge invariant and easy to calculate using Eq. (7).
Calculated in the light-cone gauge, \( \langle \chi_p | D_+ \chi_p \rangle = -ig \epsilon(p) \tilde{A}_+^0 \), while in the initial variables it is \( \epsilon(p)(\langle \tilde{\chi}_0 | U^+ \partial_+ U \rangle - ig A_+^0) \). To obtain the last expression, Eq. (8) was used. Thus,

\[
\tilde{A}_+^0 = A_+^0 - \frac{1}{ig} \langle \tilde{\chi}_0 | U^+ \partial_+ U \rangle. \tag{10}
\]

Now we go back to the term \( \langle \tilde{F}_{-k} | \tilde{F}_{+k} \rangle \) of the action (2). Using Eq. (9), the representations \( \tilde{F}_{-k} = \tilde{D}_- \tilde{A}_k - \partial_k \tilde{A}_- \), \( \tilde{F}_{+k} = \tilde{A}_k - \tilde{D}_k \tilde{A}_+ \), and the completeness of the base \( \tilde{\chi}_p \), we derive

\[
\int dx^+ dx^\perp \langle \tilde{F}_{-k} | \tilde{F}_{+k} \rangle = \int dx^+ dx^\perp \left[ \langle \tilde{D}_- \tilde{A}_k | \tilde{A}_k \rangle - \langle \partial_k \tilde{A}_k | \tilde{A}_- \rangle \right] + \sum_{p \neq 0} \left( \frac{1}{D_-} \tilde{D}_k \tilde{F}_{-k} | \tilde{\chi}_p \rangle \langle \chi_p | \tilde{F}_{+k} \rangle - \langle \tilde{D}_k \tilde{F}_{-k} | 0 \tilde{A}_+^0 \rangle \right]. \tag{11}
\]

The second term in the rhs of Eq. (11) comes from the product \( \langle \partial_k \tilde{A}_k | \tilde{A}_- \rangle \), and is obtained via integration by parts. The rhs is linear in \( F_{+\ast} \). The coefficient by \( F_{+\ast} \) will give a contribution to \( E \), the canonical variable conjugated to \( A_- \). Another contribution to \( E \) will come from the term \( \langle \partial_k \tilde{A}_k | \tilde{A}_- \rangle \) of Eq. (12) because \( \tilde{A}_- \) is linear in \( \tilde{A}_- \). To express \( \tilde{A}_- \) in terms of \( \tilde{A}_- \), we use Eq. (6) and the definition \( \tilde{A}_- = \sqrt{L} \tilde{\chi}_0 \alpha \), and express the time derivative in terms of the gauge invariant \( p \):

\[
\dot{\tilde{A}}_- = \frac{\sqrt{L}}{g} \tilde{\chi}_0 \dot{\tilde{p}}(n, +). \tag{12}
\]

The time derivative of the gauge invariant eigenvalue can be calculated as a time derivative of an expectation value over the untilded eigenvector of the untilded operator \( \tilde{\chi} \):

\[
\dot{\tilde{p}}(n, +) = \langle \chi_{p(n, +)} | \dot{\tilde{p}} \chi_{p(n, +)} \rangle = \frac{g}{\sqrt{L}} \langle \chi_0 | \dot{\tilde{A}}_- \rangle, \tag{13}
\]

where we have taken into account that \( [\chi_{p(n, -)}, \chi_{p(n, +)}] = \chi_0 / \sqrt{L} \).

With Eqs. (12)–(13), action (2) can be transformed to

\[
S_{glue} = \int dx^+ dx^\perp \left[ \langle E | \tilde{A}_- \rangle + \langle \tilde{D}_- \tilde{A}_k | \tilde{A}_k \rangle - \frac{1}{2} \left( \langle \mathcal{E} | \mathcal{E} \rangle + \langle \tilde{F}_{12} | \tilde{F}_{12} \rangle \right) \right. \\
- \langle E | D_- A_+ \rangle + \left. \langle \tilde{D}_k \tilde{F}_{-k} | 0 \tilde{A}_+^0 \rangle \right], \tag{14}
\]

where

\[
E = \mathcal{E} - \chi_0 \partial_k \tilde{A}_k^0 + \sum_{p \neq 0} \chi_p \left( \frac{1}{D_-} \tilde{D}_k \tilde{F}_{-k} \right)^p. \tag{15}
\]

In the above Eq. (14), \( E \) and \( A_- \) are the canonically conjugated variables, the content of the round brackets is the Hamiltonian, and the last line contains the Lagrange multiplier \( A_+ \). Eq. (13) is to be used to express \( \mathcal{E} \) of the Hamiltonian in terms of the canonical variables.

The last step in the variable change is to reveal the canonical variables connected with \( \tilde{A}_k \). Using the completeness of the basis \( \tilde{\chi}_p \), we rewrite the relevant term:

\[
\langle \tilde{D}_- \tilde{A}_k | \tilde{A}_k \rangle = 
\]
\[ i \sum_{p \neq 0} p (\hat{A}^p_k)^\dagger \hat{A}^p_k, \] where the dagger means complex conjugation. Note that the independence on the tilded basis of the time was crucial: we substituted the projections of the time derivative \( \hat{A}_k \) on the basis vectors by the time derivatives of the projections. Notice now that \( A^{(-p)} = (A^p)^\dagger \) for any Hermitian field \( A \). This observation makes evident that

\[
\langle \hat{D} - \hat{A}_k | \hat{A}_k \rangle = i \frac{1}{2} \sum_{p > 0} [(a^p_k)^\dagger \hat{a}^p_k - (\hat{a}^p_k)^\dagger a^p_k],
\]

where

\[
a^p_k = \sqrt{2p} \hat{A}^{(-p)}, \quad p > 0.
\]

Up to the total time derivative, the rhs of Eq. (16) takes the canonical form \( \sum_{p > 0} P^p_k \hat{Q}^p_k \) after the substitution \( a^p_k = (Q^p_k + i P^p_k)/\sqrt{2} \). We conclude that \( a^p_k(x^\perp) \) are the canonical variables with the following Poisson bracket:

\[
\{ a^p_k(x^\perp), (a^q_l)^\dagger(y^\perp) \} = i \delta^{pq} \delta_{kl} \delta(x^\perp - y^\perp).
\]

Notice that the lhs of Eq. (16) expressed with Eq. (17) in terms of the canonical variables could contain the time derivatives of \( p \), but these terms are cancelled against each other.

An important explanation is in order: Eq. (16) contains the inequality \( p > 0 \). It may seem that the fulfillment of this inequality depends on the configuration of \( \hat{A}_- \). This though is not the case, because we can treat the \( g \alpha \) of Eq. (6) as the splitting of the levels of \( \hat{p} \). As such, it is constrained by the inequality

\[
0 \leq \alpha \leq \frac{\pi}{gL}.
\]

In fact, the whole construction can be reformulated without use of the transformation to the light-cone gauge. In this reformulation, \( g \alpha \) is defined as the minimal splitting between the levels of \( \hat{p} \), and \( \chi_p \) are defined as the eigenfunctions of \( \hat{p} \). After this explanation, we can label the canonical variables \( a^p_k \) of Eq. (17) by \( n > 0 \) and the discrete variable \( \sigma = -1, 0, +1 \) related to \( p \): \( p = 2\pi n/L + \sigma g \alpha \). There are also degrees of freedom \( a^0_k \) corresponding to \( n = 0 \), \( p = g \alpha \). We will stick in what follows to the more compact labeling by the eigenvalue \( p > 0 \).

It is important that the dynamical variables \( a^p_k \) are not in one-to-one correspondence with \( \hat{A}_k \). The latter contains more information. Namely, it contains a zero mode \( \hat{B}_k (\equiv \hat{A}^0_k) \):

\[
\hat{A}_k = B_k \hat{x}_0 + \sum_{p > 0} \left[ \frac{\hat{x}_p}{\sqrt{2p}} (a^p_k)^\dagger + \frac{\hat{x}^\dagger_p}{\sqrt{2p}} a^p_k \right].
\]

The crucial observation is that the zero mode \( B_k \) enters only the Hamiltonian in the round bracket of Eq. (14), and not the terms involving the Lagrange multiplier \( A_+ \). The variation with respect to the latter gives the light-front version of the Gauss law. It consists of two components. The first component comes from the variation of the first term of the second line of Eq. (14), while the second component comes from the second term. Note that these terms can be varied independently, since the first term contains only the nonzero modes of \( A_+ \), while the second term depends only on the zero mode \( \hat{A}^0_k \). The variation of the Hamiltonian with respect to \( B_k \) gives an equation linear in \( B_k \). Solving it, one determines \( B_k \) in terms of the canonical variables.
III. THE GAUSS LAW

Next, we write down the components of the Gauss law. The first component is

\[ D_\perp E = 0, \]  

(21)

and the second,

\[ \Delta_\perp \tilde{A}_\perp^0 + g \sum_{p>0} \epsilon(p) (a_k^p)\dagger a_k^p = 0, \]  

(22)

where \( \Delta_\perp \equiv \partial_1^2 + \partial_2^2 \), and \( \epsilon(p) \) was previously introduced by the relation \([\chi_p, \chi_\perp^1] = \epsilon(p) \chi_0\). Note that the first component holds at any space-time point, while the second is independent

of \( x^- \) and holds at any point of the transverse plane at any moment of the light-front time. Also note that \( \epsilon(p) = -\sigma/\sqrt{L} \), where \( p = 2\pi n/L + \sigma g \alpha \).

The natural next step is to calculate the Poisson brackets between the canonical variables and the components of the Gauss law smeared with some weights. The expectation is that the components of the Gauss law will generate the local gauge transformations of the canonical variables. A simple calculation supports this expectation. The first component of the Gauss law (21) generates the gauge transformation of the canonical variables with the local parameter \( \phi \) orthogonal to the zero mode, \( \langle \chi_0 | \phi \rangle = 0 \). As the form of Eq. (21) shows, this transformation leaves \( a_k^p \) invariant. That we already know from the definition (17). To see this, notice that \( \tilde{A}_\perp^0 \) transforms as a charged field under the Abelian gauge transformation, and the charge is determined by the sign \( \sigma \). This can also be understood independently from the definition (17). To see this, notice that the Abelian transformations leave \( \chi_\perp \) invariant. If we take the basis \( \chi_p \) to be completely determined by the configuration of \( A_\perp \), the basis \( \chi_\perp \) is also invariant under these transformations. It follows from this observation that \( F_\perp^k \) transforms as a charged field under the Abelian gauge transformations. We conclude that the components of the light-front Gauss law generate the gauge transformations, which is quite analogous to the situation in the conventional equal time formulation.

Note how the zero mode component of the Gauss law (22) acts on \( E \):

\[ \{ \psi \Delta_\perp \tilde{A}_\perp^0, E \} = -\Delta_\perp \psi \chi_0. \]  

This is again in accord with our expectations, because it corresponds to the invariance of the zero mode of \( E = F_\perp \) under the Abelian subgroup of the gauge transformations (see Eq. (13) for the connection between \( E \) and \( E \)). To understand it, assume that \( B_k \equiv \tilde{A}_k^0 \) transforms as it should under the gauge transformation generated by \( \psi \chi_0 \). Then, there is a cancellation between the transformation of \( E \) and the transformation of \( B_k \) in Eq. (13) which leaves the zero mode of \( E \) invariant as promised.

To finally check the assumption that the components of the Gauss law generate the gauge transformation of all the fields involved, we should find \( B_k \) in terms of the canonical variables and check that the Poisson brackets between them and the components of the Gauss law generate the gauge transformations of \( B_k \). We postpone this check to make the following observations. As the components of the Gauss law generate the gauge transformations of
the canonical variables, the Poisson brackets of these components form a closed algebra and commute with the Hamiltonian, because the latter is gauge invariant. Therefore, the light-front gluodynamics is a generalized dynamical system with first class constraints. This is the main result of the paper. Another observation is that the canonical light-front formulation leads to a natural Abelianization of the theory. All the non-Abelian transformations act nontrivially only on $A_-$, leaving the rest of the dynamical variables invariant. An Abelian subgroup of the gauge transformations which leaves $A_-$ invariant is naturally singled out.

The zero mode of the gauge field plays the role of the Abelian gauge field in an Abelian gauge theory with the space-time dimension decreased by one (the $x^-$ dimension is “eaten” by the projection on the zero mode). This Abelian gauge field turns out to be nondynamical and expressible in terms of the dynamical variables.

**IV. THE ZERO MODE EQUATION**

We now work out the equation for $B_k$. To this end, we take a variation of the round bracket of Eq. (14) over $\tilde{A}_k$, convolute it with $\tilde{\chi}_0$, and require the convolution to vanish. It gives

$$
(- \Delta_\perp + \frac{g^2}{L} \sum_{p \neq 0, l} |\tilde{A}_l|^2)B_k = \partial_k E^0 + i g \sum_{p \neq 0} \epsilon(p) \left( E - \frac{1}{D_-} \tilde{D}_l \tilde{F}_{-l} \right)^p (\tilde{A}_k)^p \epsilon_{kl} - \epsilon_{kl}(\tilde{D}_l \tilde{F}_{12})^0, \quad (23)
$$

where $B_k$ is set to zero in the rhs, and $\epsilon_{kl}$ is the antisymmetric tensor with $\epsilon_{12} = 1$. This is the equation to determine $B_k$. First we note that it gives $B_k$ which transforms in the right way under the gauge transformations of the rhs. Second, to solve for $B_k$, we need to invert the operator acting on $B_k$. It is a Schrödinger operator with a positive potential. Thus, it may have a zero eigenvalue only at $a_k^p = 0$. In this case, the potential vanishes, and the equation does not restrict the contribution to $B_k$ which does not depend on $x^\perp$. Though, in this latter case, $B_k$ enters the Hamiltonian only via its derivatives over the transverse coordinates. Thus, we conclude that Eq. (23) suffices to determine the Hamiltonian of the light-front gluodynamics.

We mention here that Eq. (23) is in agreement with the equation obtained for the zero modes in Ref. [17]. In Ref. [17], the equation for the zero modes was obtained after gauge fixing. We stress that there is no need to fix the gauge to derive Eq. (23). The operator acting on $B$ in the lhs of Eq. (23) is gauge invariant.

**V. QUANTIZATION AND PERTURBATION THEORY**

It is possible to write down the expression for the Hamiltonian with $B_k$ excluded. This completes the formulation of the classical theory. The quantization consists in replacing the canonical brackets with commutators, and in prescribing the ordering of the operators in the Hamiltonian. If we consider the classical Hamiltonian with the Gauss law implemented, there is no ambiguity in ordering the operators $E$ and $A_-$, because they do not join one another in the Hamiltonian. But there is certainly an ambiguity of ordering for the transverse gluons creation-annihilation operators. We choose the simplest prescription of normal ordering.
Making this choice, we may contradict the results of the equal-time quantization, where its own ordering is assumed. That is, equal-time normal ordering may not coincide with the light-front normal ordering we have chosen. Thus, we must confirm that our prescription does not lead to a contradiction.

Here we demonstrate that there is no contradiction between the normal ordering light-front quantization and the standard perturbation theory in covariant gauges.

In checking this, we benefit from the fact that the obtained formulation is standard, and we can use the methods of Ref. [11] to write down the functional integral representation for vacuum expectations of the time ordered products of gauge invariant local operators (for example, for the products of $F^2_{\mu\nu}$). The time ordering is understood with respect to the light-front time $x^+$. In accord with that standard methods, the functional integral will run over the canonical variables, and the additional integration over $A_+, \tilde{A}_0$ will impose the Gauss law on the states featuring the derivation of the functional integral representation. To simplify the treatment we do not impose any boundary conditions, and consider instead a finite span $2T$ of the light-front time $x^+$ for the fields in the functional integral. When, in the limit $T \to \infty (1 - i\epsilon)$, the vacuum expectation is reproduced (up to normalization) due to the presence of $i\epsilon$ in the time span.

Next, we derive the Hamiltonian in the reverse route: introduce $B_k$ in the functional integral and restore it back in the action, go over from the integration over $a^\dagger, a, B_k$ to the integration over $A_k$ (see Eq. (20)), shift $E$ to $\mathcal{E}$ (see Eq. (15)), and, finally, integrate $\mathcal{E}$ out. This gives us the standard representation in terms of the functional integral over $A_\mu$, with the standard action $S_{\text{glue}}$ in the exponential. However, there is a subtle point: on the way back we pick up some determinants. They will potentially cause a difference between our formulation and the standard equal-time formulation. We now show that this difference vanishes in the limit of infinite volume, if dimensional regularization is used to regularize the infinite volume theory.

We pick up the first determinant when the integration over $B_k$ is introduced. $B_k$ enters the Hamiltonian quadratically. So, integrating out $B_k$ will return the initial expression for the Hamiltonian with $B_k$ excluded via Eq. (23), and also will produce a power of the determinant of the operator featuring the lhs of Eq. (23), det($-\Delta_\perp + g^2 \sum_p |A^p|^2/L$). The dependence of this determinant on the fields is explicitly suppressed by the inverse $L$. Therefore, in the limit of infinite $L$, this determinant introduces only an (infinite) multiplication constant in the measure of integration.

The second determinant appears when we go over from the integration over $B_k, a^\dagger, a$ to the integration over $A_k$. It is related to the presence of the denominators $\sqrt{2p}$ in Eq. (23). Since $p$ are the eigenvalues of $D_-$, the new determinant is a power of the determinant of $D_-$. And it is known that det$D_-$ is a constant independent of $A_-$, if the dimensional regularization is used to regularize the infrared divergences (see, for example, Ref. [18]).

We conclude that no determinants appear in the limit of infinite volume when the dimensional regularization is used.

The next step is to fix the gauge in the functional integral. Here we again enjoy our independence of the gauge fixing: we are not forced to consider the light-cone gauge, which has extra singularities in the gluon propagator. Instead, we consider covariant gauges, introducing Faddeev-Popov ghosts in the standard way.

As a result, we obtain the standard representation for the Green's functions in terms
of the functional integral. The only trace of the light-front approach left is the restriction of the fields to the strip \(-T \leq x^+ \leq T\). As we do not restrict the boundary values at \(|x^+| = T\) of the fields under the functional integration in any way, the classical solution of the linearized equations of motion related to the determination of the propagators should vanish on the boundaries of the strip. It is easy to check that the imaginary shift in \(T\) allows the presence of the unique solution to the classical linearized field equations which boundary values vanish at \(T \to \infty(1 - i\epsilon)\). Then, it is easy to check that the propagators appearing have the Feynman \(i\epsilon\) in the denominators.

We conclude that the normal ordering light-front quantization reproduces the standard Feynman rules in covariant gauges. We stress that the above derivation is possible only because the definition of the canonical variables we found does not depend on any gauge fixing.

VI. THE NON-ABELIAN PLANE WAVES REDUCTION

Now we turn to the analysis of the reduced dynamics by requiring \(P_-\) to vanish. To see the restriction this requirement sets on the dynamical variables, consider \(P_-\) as it is yielded by the Noether procedure:

\[
P_- = \int dx^+ \langle F_{-k}|F_{-k}\rangle.
\]

In terms of the canonical variables, it is

\[
P_- = \int dx^+ \left[ \sum_{p>0} p (a^p_k|^\dagger a^p_k - \tilde{A}^0_\perp \Delta_\perp \tilde{A}^0_\perp \right].
\]

With the zero mode component of the Gauss law \((22)\) and the relation \(\tilde{A}_- = \sqrt{L} \alpha \tilde{\chi}_0\) taken into account, it is

\[
P_- = \int dx^+ \sum_{p>0} \frac{2\pi n}{L} (a^p_k|^\dagger a^p_k),
\]

where \(n\) is related to \(p\) by \(p = 2\pi n/L + \sigma g\alpha\). Thus, we see that vanishing of \(a^p_k\) with \(p > g\alpha\) is necessary and sufficient for vanishing of \(P_-\). We note in passing that \(P_-\) is independent of the canonical pair \(E, A_-\). \(A_-\) enters \(P_-\) only indirectly via the zero mode component of the Gauss law \((22)\). The latter restricts the possible configurations of \(a^p_k\). In particular, integrating Eq. \((22)\) over the transverse plane we see that the total sum of the Abelian charges over the transverse plane should vanish all the time. This, in fact, implies that the only \(a^p_k\) which is allowed by the vanishing of \(P_-\) \((p = g\alpha)\) is forbidden by the Gauss law, because it has definite Abelian charge, and nothing can compensate it. Thus, we conclude that the reduction to the configurations of vanishing \(P_-\) is equivalent to the reduction to vanishing \(a^p_k\), i.e., to the reduction from Eq. \((20)\) to \(A_k = B_k \tilde{\chi}_0\).

This reduction simplifies the action \((14)\) dramatically. The second term of the square bracket vanishes, the round bracket is simplified because \(E\) becomes just \(E + \chi_0 \partial_\perp B_k\) (see Eq. \((13)\) and recall that \(B_k \equiv \tilde{A}^0_k\)), \(\tilde{F}_{12}\) becomes \((\partial_1 B_2 - \partial_2 B_1) \tilde{\chi}_0\), and the last term of the
square bracket becomes \((\partial_k \tilde{A}_0 \partial_k \tilde{A}_0^0)\). At no compactification over the \(x^-\) direction, in the light-cone gauge \(\tilde{A}_- = 0\), and at boundary conditions on \(E\) at the \(x^-\) infinity suppressing the zero mode \(E^0\), the classical equations of the reduced dynamics implied by this action are \(\partial_+ \tilde{A}_0^0 = 0, \Delta_\perp \tilde{A}_+ = 0, \tilde{A}_k = 0\). This reduction of the classical gauge equations is known since the work of Coleman \[14\]. Thus, we may say that the reduced dynamics of the paper generalizes one of Coleman’s non-Abelian plane waves for the case of the compactified \(x^-\). Our next aim is to see the Hamiltonian formulation of the paper at work. We will find the quantum spectrum of the non-Abelian plane waves.

To this end, we will work in the gauge \(D_+ A_+ = 0, \tilde{A}_+^0 = 0\), which is the light-front analog of the Weyl gauge. In this gauge, the reduced system is a Hamiltonian system with the Hamiltonian

\[
H_{\text{red}}(B) = \int dx^\perp \frac{1}{2} \left[ (E^0 + \partial_k B_k)^2 + (\partial_1 B_2 - \partial_2 B_1)^2 \right].
\]

Minimization with respect to \(B_k\) gives

\[
H_{\text{red}} = \frac{1}{2} \tilde{E}^2 V_\perp,
\]

where

\[
E \equiv \int \frac{dx^\perp}{V_\perp} \langle \chi_0 | E \rangle,
\]

and \(V_\perp \equiv \int dx^\perp\) is the volume of the transverse plane. Apart from the Hamiltonian \(28\), the reduced system is defined by the Gauss law \(D_e E = 0, \Delta_\perp \tilde{A}_0^0 = 0\). Notice that the Hamiltonian vanishes in the limit of the infinite transverse volume on the configurations of \(E\) with finite \(\int dx^\perp E^0\), because \(\tilde{E} \sim 1/V_\perp\) in this case.

Let us demonstrate that the phase space of the reduced system is parameterized by two variables. One is \(\tilde{E}\), another is \(\tilde{A} \equiv \int dx^\perp \tilde{A}_0^0 / V_\perp\). This is so because of all \(E^p\) only \(E^0\) is nonzero by the component of the Gauss law \(D_e E = 0\), and, under a gauge transformation, \(E^0\) can be shifted by the transverse Laplacian of a gauge parameter field (recall that \(\{ \int \psi \Delta_\perp \tilde{A}_0^0, E \} = -\Delta_\perp \psi \chi_0\)). Thus, the only piece of \(E\) which is simultaneously gauge invariant and allowed by the Gauss law is \(\tilde{E}\). The same holds with respect to \(A_\perp\): the only gauge invariant component of \(A_\perp\) is \(\tilde{A}_0^0\), and it should be independent of the transverse coordinates by the zero mode of the Gauss law. The reduction to a single gauge invariant component is achievable because \(A_\perp\) can be transformed to \(\tilde{A}_0^0\) by a gauge transformation, and the latter is a gauge invariant expressible in terms of the gauge invariant trace of a Wilson loop.

Notice the specific duality between \(E\) and \(A_\perp\): The Gauss law forbids nonzero \(E^p\) at \(p \neq 0\), while the gauge invariance admits only constant \(E^0\); for \(A_\perp\), the Gauss law and the gauge invariance interchange their roles in the reduction, \(i.e.,\) the gauge invariance forbids nonzero \(\tilde{A}_0^p\) at nonzero \(p\), while the Gauss law admits only constant contributions to \(\tilde{A}_0^0\). In this reasoning, we assumed that the theory is compactified also in the transverse directions \(x^{1,2}\).

Thus, we introduce two variables \(Q \equiv \tilde{A}_-,\) and \(P \equiv V_\perp \tilde{E}\). They suffice to parameterize the phase space of the reduced system, and the normalization is chosen to make them
canonically conjugated: \( \{ P, Q \} = 1 \). The Hamiltonian (28) in terms of these variables is 
\[ H_{\text{red}} = \frac{P^2}{2 V_\perp}. \]
Notice that \( V_\perp \) plays the role of the mass of a free nonrelativistic particle. The next crucial point is that \( Q \), in fact, should be compactified. To see this, consider a “large” gauge transformation generated by the Hermitian traceless matrix \( \omega \equiv g x - (2\pi/g\sqrt{L}) \chi_0 \). The transformed gauge field \( A'_U = U (A_- - \partial_+/(ig)) U^\dagger \), \( U = \exp(i\omega) \), is periodic in \( x^- \), if \( A_- \) is. Thus, \( U \) is indeed a gauge transformation of the compactified theory. Notice that this is not the case for any \( \omega' = \lambda \omega \), where \( \lambda \) is noninteger. Because of this, the above transformation cannot be continuously transformed to the trivial transformation. The presence of the “large” gauge transformations in a compactified theory is known since the work [19]. In the context of the light-front formulation, “large” gauge transformations have also been considered in Ref. [8], and utilized in Ref. [20] for the Schwinger model. The particular “large” gauge transformation we are considering here is a light-front analog of the equal-time finite-volume “central conjugations” of Ref. [21]. It is easy to check that this transformation leaves \( P \) invariant, and shifts \( Q \): 
\[ Q \rightarrow Q + \frac{2\pi}{g\sqrt{L}}. \]
Thus, the theory should be invariant with respect to this shift, because it is a remnant gauge transformation. The invariance is achieved by the condition on the wavefunctions in the \( Q \)-representation:
\[ \Psi \left( Q + \frac{2\pi}{g\sqrt{L}} \right) = e^{i\theta} \Psi(Q), \] (30)
where \( \theta \) is an angle, \( 0 \leq \theta < 2\pi \), parameterizing the theory.

In fact, the allowed values for the \( \theta \) in Eq. (30) are 0 and \( \pi \). This is the case because the double action of the above large gauge transformation on the gauge fields leaves both \( P \) and \( Q \) invariant. To see this, notice that \( Q \) is expressible in terms of the trace of a large Wilson loop, and the latter changes its sign under the single action of \( U \). For more explanations, see Refs. [19,21]. In what follows, we introduce the label \( e \) for the superselection sectors of the theory: \( e = 0 \) for \( \theta = 0 \) and \( e = 1 \) for \( \theta = \pi \). The notation is in accord with the one of Refs. [19,21] and is to remind of the connection with the values of the electric flux.

Then, in the sector \( e = 0 \) the wave functions are periodic, and in the sector \( e = 1 \), antiperiodic:
\[ \Psi(0) = (-1)^e \Psi \left( \frac{2\pi}{g\sqrt{L}} \right). \] (31)

Condition (31) singles out a discrete spectrum of the admissible values for \( P \):
\[ P(n) = g\sqrt{L} \left( n + \frac{e}{2} \right), \] (32)
where \( n \) is an integer, and \( e \) is either zero or unit “electric flux”. Recalling that \( H_{\text{red}} = P^2/(2V_\perp) = P_+|_{P_-=0} \), we conclude that the spectrum of \( P_+ \) in the subspace \( P_- = 0 \) is
\[ P_+|_{P_-=0}(n) = \frac{g^2L}{2V_\perp} \left( n + \frac{e}{2} \right)^2. \] (33)

At \( n = 0 \) and \( e = 1 \), it coincides with the “free energy of an electric flux” of ’t Hooft, see Eq. (9.2) of Ref. [19].
VII. DISCUSSION AND CONCLUSION

Our central result, Eq. (33), shows that the presence of a mass gap in infinite volume theory depends on the ordering of the limiting procedure. If one takes first the limit $V_\perp \to \infty$ then there is no mass gap, and if one takes first $L \to \infty$ then there is a mass gap.

We note that the dependence of the thermodynamic state on the limiting procedure is also present in statistical mechanics, where a non-unique limit is generally associated with some sort of first-order phase transition and may indeed be considered as a possible definition of a phase transition [22]. On a physically motivated way to select the “right” state see Ref. [23].

Thus, we interpret the nonexistence of the infinite volume limit in our case as the indication of the presence of the first-order phase transition in gluodynamics. This is an acceptable feature in view of the expected presence of the deconfinement phase transition and quark-gluon plasma in QCD [24]. Obviously, significant work is needed to reconcile our approach with what is known about the phase structure of QCD.

Having arrived at such simple spectrum, Eq. (33), could explain how previous approximate treatments yielded results seemingly valid beyond limitations of underlying assumptions [25].

There is similarity of our results with recent works on Yang-Mills fields decomposition [26,27] which leads to the Abelian dominance [12]. However, those equal time approaches are linked with choice of a proper gauge [26] or with a nonlocal variable transformation [27], while we have obtained our results without a gauge fixing in finite volume light-front formulation.

The next steps for development are the generalization to the $SU(N)$ case and the inclusion of fermions. In general, the results for the theory with fermions can differ drastically from pure gauge theory such as the gluodynamics considered here.

Note that the finite volume light-front formulation may play an important role for string theory, where one has to quantize a compactified system with first class constraints. The formulation obtained reproduces the standard Feynman rules in the covariant gauges. The spectrum (33) of the light-front Hamiltonian $P_+$ was determined in the subspace of zero $P_-$ for the case of the theory compactified on a torus. An unexpected feature of the spectrum is that the distance between the levels of $P_+$ may vanish in the limit of infinite volume, depending on the way the limit is taken. This suggests the possibility of the presence of massless states in certain “phases” of the infinite volume theory obtained via the limiting procedures with the vanishing quantum of $P_+$. There are obvious possibilities to further develop the formalism: to generalize for $SU(N)$, to include fermions, to develop the perturbation theory in various gauges at finite and at infinite volume, etc. Some of these problems will be addressed in Ref. [28].
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