Feedback Particle Filter on Matrix Lie Groups

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Abstract—This paper is concerned with the problem of continuous-time nonlinear filtering for stochastic processes on a compact and connected matrix Lie group without boundary, e.g. \( SO(n) \) and \( SE(n) \), in the presence of real-valued observations. This problem is important to numerous applications in attitude estimation, visual tracking and robotic localization. The main contribution of this paper is to derive the feedback particle filter (FPF) algorithm for this problem. In its general form, the FPF provides a coordinate-free description of the filter that furthermore satisfies the geometric constraints of the manifold. The particle dynamics are encapsulated in a Stratonovich stochastic differential equation that preserves the feedback structure of the original Euclidean FPF. Specific examples for \( SO(2) \) and \( SO(3) \) are provided to help illustrate the filter using the phase and the quaternion coordinates, respectively.

I. INTRODUCTION

There has been an increasing interest in the nonlinear filtering community to explore geometric approaches for handling constrained systems. In many cases, the constraints are described by smooth Riemannian manifolds, in particular the Lie groups. Engineering applications of filtering on Lie groups include: (i) attitude estimation of satellites or aircrafts [7], [4]; (ii) visual tracking of humans or objects [12], [20]; and (iii) localization of mobile robots [3], [37]. In these applications, the Lie groups of interest are primarily the matrix groups such as the special orthogonal group \( SO(2) \) or \( SO(3) \) and the special Euclidean group \( SE(3) \).

This paper considers the continuous-time nonlinear filtering problem for matrix Lie groups in the presence of real-valued observations. The objective is to obtain a generalization of the feedback particle filter (FPF) (see [41]) in this non-Euclidean setting. FPF is a continuous-time filtering algorithm that extends the feedback structure of the Kalman filter to general nonlinear non-Gaussian filtering problems. For application problems in the Euclidean space, evaluation and comparison of FPF against the conventional particle filter appears in [6], [33], [34].

The contributions of this paper are as follows:

- **Feedback particle filter for Lie groups.** The extension of the FPF for matrix Lie groups is derived. The particle dynamics, expressed in their Stratonovich form, respect the manifold constraints. Even in the manifold setting, the FPF is i) shown to admit an error correction feedback structure, and ii) proved to be an exact algorithm. Exactness means that, in the limit of large number of particles, the empirical distribution of the particles exactly matches the posterior distribution.

- **Poisson equation on Lie groups.** Numerical implementation of the FPF requires approximation of the solution of a linear Poisson equation. The equation is described for the Lie group in an intrinsic coordinate-free manner. For computational purposes, a Galerkin scheme is proposed to approximate the solution.

- **Algorithms.** Specific examples for \( SO(2) \) and \( SO(3) \) are worked out, including expressions for the filter and the Poisson equation using a canonical choice of coordinates—the phase coordinate for \( SO(2) \) and the quaternion coordinate for \( SO(3) \).

Filtering of stochastic processes in non-Euclidean spaces has a rich history; c.f., [28], [14]. In recent years, the focus has been on computational approaches to approximate the solution. Such approaches have been developed, e.g., by extending the classical extended Kalman filter (EKF) to Riemannian manifolds. EKF-based extensions have appeared for both discrete-time [4], [2] and for continuous time settings [7], [9]. Deterministic nonlinear observers have also been considered for \( SO(3) \) [26], [39], [5], \( SE(3) \) [17], as well as for systems with other types of symmetry and invariance properties [23], [8]. A closely related theme is the use of non-commutative harmonic analysis for characterizing error propagation for rigid bodies [30], [19], [25]. These algorithms have also been applied extensively, e.g., for attitude estimation [13], [42], [27]. Non-parametric approaches such as the particle filter (PF) have also been developed for Riemannian manifolds [11], with extensive applications to visual tracking and localization [21], [20], [12]. Typically, PF algorithms adopt discrete-time description of the dynamics and are based on importance sampling. Closely related approaches, such as the Rao-Blackwellized particle filter [22], [3] and the unscented Kalman filter (UKF) [16], have also been investigated for the Lie groups.

The remainder of this paper is organized as follows: After a brief overview in Sec. III of relevant preliminaries for matrix Lie groups, the filtering problem is formulated in Sec. IV. In Sec. V, the generalization of the FPF algorithm to matrix Lie groups is presented, including both theory and algorithms. In Sec. VI, examples for \( SO(2) \) and \( SO(3) \) are discussed. All the proofs appear in the appendix.

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II. PRELIMINARIES

This section includes a brief review of matrix Lie groups. The intent is to fix the notation used in subsequent sections.

The general linear group, denoted as $GL(n; \mathbb{R})$, is the group of $n \times n$ invertible matrices, where the group operation is matrix multiplication. The identity element is the identity matrix, denoted as $I$. A matrix Lie group, denoted as $G$, is a closed subgroup of $GL(n; \mathbb{R})$. The Lie algebra of $G$, denoted as $\mathfrak{g}$, is the set of matrices $V$ such that the matrix exponential, $\exp(V)$, is in $G$. $\mathfrak{g}$ is a vector space whose dimension, denoted as $d$, equals the dimension of the group. $\mathfrak{g}$ is equipped with an inner product, denoted as $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, and an orthonormal basis $\{E_1, \ldots, E_d\}$ with $\langle E_i, E_j \rangle_{\mathfrak{g}} = \delta_{ij}$. The space of smooth real-valued functions $f : G \to \mathbb{R}$ is denoted as $C^\infty(G)$.

Example: The special orthogonal group $SO(3)$ is the group of $3 \times 3$ matrices $R$ such that $RR^T = I$ and $\det(R) = 1$. The Lie algebra $so(3)$ is the 3-dimensional vector space of skew-symmetric matrices. An inner product is given by $\langle \Omega_1, \Omega_2 \rangle_{\mathfrak{g}} = (1/2) \text{Tr}(\Omega_1^T \Omega_2)$, for $\Omega_1, \Omega_2 \in \mathfrak{g}$, and an orthonormal basis $\{E_1, E_2, E_3\}$ of $so(3)$ is given by,

$$
E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

These matrices have the physical interpretation of generating rotations about the three canonical axes (denoted as $e_1, e_2, e_3$) in $\mathbb{R}^3$. $\det(\cdot)$ and $\text{Tr}(\cdot)$ denote the determinant and trace of a matrix.

The Lie algebra can be identified with the tangent space at the identity matrix $I$, and can furthermore be used to construct a basis $\{E_1, \ldots, E_d\}$ for the tangent space at $x \in G$, where $E_n^x = xe_n$ for $n = 1, \ldots, d$. Therefore, a smooth vector field, denoted as $\mathcal{V}$, is expressed as

$$
\mathcal{V}(x) = v_1(x)E_1 + \cdots + v_d(x)E_d,
$$

with $v_n(x) \in C^\infty(G)$ for $n = 1, \ldots, d$. We write

$$
\mathcal{V} = vX,
$$

where $V(x) := v_1(x)E_1 + \cdots + v_d(x)E_d$ is an element of the Lie algebra $\mathfrak{g}$ for each $x \in G$.

With a slight abuse of notation, the action of the vector field $\mathcal{V}$ on $\psi \in C^\infty(G)$ is denoted as

$$
V \cdot f(x) := \frac{d}{dt} \bigg|_{t=0} f(t\exp(Vx)).
$$

The coordinates of $V$ are denoted as $v(x) := [v_1(x), \ldots, v_d(x)]$. The inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ induces an inner product of two vector fields,

$$
\langle \mathcal{V}, \mathcal{W} \rangle(x) := \langle V, W \rangle_{\mathfrak{g}}(x) = \sum_{n=1}^{d} v_n(x)w_n(x).
$$

It is understood that $V : G \to \mathfrak{g}$, where the identification with a vector field is through $\mathfrak{g}$, and the action is defined in $\mathfrak{g}$. We will not use the $^\vee$ or $^\wedge$ notation to move between $V$ and its coordinates $v$, as is customary in the Lie groups [10].

[9]. This is because the $^\vee$ notation is reserved for expectation, consistent with its use in stochastic processes.

As an example, we define next the notation for the vector field $\text{grad}(\psi)$ for $\psi \in C^\infty(G)$,

$$
\text{grad}(\psi)(x) = xK(x),
$$

where $K(x) = E_1 \cdot \phi(x)E_1 + \cdots + E_d \cdot \phi(x)E_d \in \mathfrak{g}$, and according to (2), $(E_n \cdot \phi)(x) := \frac{d}{dt} \big|_{t=0} \phi(x\exp(tE_n))$ for $n = 1, \ldots, d$. The vector field acts on a function $f \in C^\infty(G)$ as,

$$
K \cdot f(x) = \sum_{n=1}^{d} E_n \cdot \phi(x)E_n \cdot f(x) = \langle \text{grad}(\psi), \text{grad}(f) \rangle(x).
$$

Apart from smooth functions, we will also need to consider other types of function spaces: For a probability measure $\pi$ on $G$, $L^2(G; \pi)$ denotes the Hilbert space of functions on $G$ that satisfy $\pi(|f|^2) < \infty$. $H^1(G; \pi)$ denotes the Hilbert space of functions $f$ such that $f$ and $E_n \cdot f$ (defined in the weak sense) are all in $L^2(G; \pi)$.

III. NONLINEAR FILTERING PROBLEM ON LIE GROUPS

A. Problem statement

We consider the following continuous-time system evolving on a Lie group $G$ with real-valued observations:

$$
dX_t = X_tV_0(X_t)\, dt + X_tV_1 \circ dB_t, \quad (5a)
$$

$$
dZ_t = h(X_t)\, dt + dW_t, \quad (5b)
$$

where $X_t \in G$ is the state at time $t$, $Z_t \in \mathbb{R}$ is the observation, $V_0 : G \to \mathfrak{g}$, $V_1 \in \mathfrak{g}$, $\{B_t\}$ and $\{W_t\}$ are mutually independent real-valued standard Wiener processes, which are also independent of the initial state $X_0$. The $\circ$ before $dB_t$ indicates that the stochastic differential equation (SDE) (5a) is defined in its Stratonovich form.

Remark 1: On a smooth manifold, SDEs are usually constructed in their Stratonovich form instead of their Itô form. The former respects the intrinsic geometry of the manifold [29], while the latter requires special geometric structures [15], and in general does not maintain the manifold constraint [10].

The objective of the filtering problem is to compute the conditional distribution of $X_t$ given the history of observations $\mathcal{F}_t = \sigma(Z_s: s \leq t)$. The conditional distribution, denoted as $\pi^*_t$, acts on a function $f \in C^\infty(G)$ according to,

$$
\pi^*_t(f) := \mathbb{E}(f(X_t)|\mathcal{F}_t).
$$

$\pi^*_t$ is referred to as the filtered estimate.

B. Filtering equation

The filtering equation describes the evolution of the conditional distribution $\pi^*_t$. For the system (5a) and (5b), the Kushner-Stratonovich (K-S) filtering equation is (see [1]),

$$
\pi^*_t(f) = \pi^*_0(f) + \int_0^t \pi^*_s(L^\vee f)\, ds + \int_0^t \left( \pi^*_s(fh) - \pi^*_s(h)\pi^*_s(f) \right)(dZ_s - \pi^*_s(h)\, ds),
$$

(6)
for any \( f \in C^\infty(G) \), where the operator \( \mathcal{L}^* \) is given by,
\[
\mathcal{L}^* f = V_0 \cdot f + \frac{1}{2} V_1 \cdot (V_1 \cdot f).
\] (7)

IV. FEEDBACK PARTICLE FILTER ON LIE GROUPS

This section extends the FPF algorithm originally proposed in [41] to matrix Lie groups, with necessary modifications to the original framework to account for the manifold structure.

A. Particle dynamics and control architecture

The feedback particle filter on a matrix Lie group \( G \) is a controlled system comprising of \( N \) stochastic processes \( \{X_i\}_{i=1}^N \) with \( X_i \in G \). The particles are modeled by the Stratonovich SDE,
\[
dX_i^t = X_i^t (V_0 (X_i^t) + u(X_i^t, t)) \, dt + X_i^t V_1 \, dB_i^t + X_i^t K(X_i^t, t) \, dZ_t,
\] (8)
where \( u(x, t) \), \( K(x, t) \) are called control and gain function, respectively. These functions need to be chosen. The coordinates of \( u \) and \( K \) are denoted as \([u_1, ..., u_l]\) and \([k_1, ..., k_l]\) respectively. Admissibility requirement is imposed on \( u \) and \( K \):

**Definition 1**: (Admissible Input): The functions \( u(x, t) \) and \( K(x, t) \) are admissible if, for each \( t \geq 0 \), they are \( \mathcal{F}_t \)-measurable and we have \( E[\sum_n |u_n(X_i^t, t)|] < \infty \), \( E[\sum_n |k_n(X_i^t, t)|^2] < \infty \).

The conditional distribution of the particle \( X_i^t \) given \( \mathcal{F}_t \) is denoted by \( \pi_t \), which acts on \( f \in C^\infty(G) \) according to,
\[
\pi_t(f) := E[f(X_i^t)|\mathcal{F}_t].
\]
The evolution PDE for \( \pi_t \) is given by the proposition below. The proof appears in Appendix A.

**Proposition 1**: Consider the particles \( X_i^t \) with dynamics described by (8). The forward evolution equation of the conditional distribution \( \pi_t \) is given by,
\[
\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(\mathcal{L} f) \, ds + \int_0^t \pi_s(K \cdot f) \, dZ_s,
\] (9)
for any \( f \in C^\infty(G) \), where the operator \( \mathcal{L} \) is,
\[
\mathcal{L} f = (V_0 + u) \cdot f + \frac{1}{2} V_1 \cdot (V_1 \cdot f) + \frac{1}{2} K \cdot (K \cdot f).
\] (10)

**Problem statement**: There are two types of conditional distributions:

- \( \pi_t^* \): The conditional dist. of \( X_t \) given \( \mathcal{F}_t \).
- \( \pi_t \): The conditional dist. of \( X_i^t \) given \( \mathcal{F}_t \).

The functions \( \{u(x, t), K(x, t)\} \) are said to be exact if \( \pi_t = \pi_t^* \) for all \( t \geq 0 \). Thus, the objective is to choose \( \{u, K\} \) such that, given \( \pi_0 = \pi_0^* \), the evolution of the two conditional distributions are identical (see (6) and (9)).

**Solution**: The FPF on Lie groups represents the following choice of the gain function \( K \) and the control function \( u \):

1. **Gain function**: The gain function is obtained by solving a Poisson equation. Specifically, at each time \( t \), Let \( \phi \in H^1(G; \pi) \) be the solution of:
\[
\pi_t(\langle \text{grad}(\phi) \rangle, \langle \text{grad}(\psi) \rangle) = \pi_t(\langle h - \hat{h} \rangle \psi),
\] (11)
for all \( \psi \in H^1(G; \pi) \), where \( \hat{h} = \pi_t(h) \). The gain function \( K \) is then given by, \( xK(x, t) = \text{grad}(\phi)(x) \). Noting that (see [3]),
\[
\text{grad}(\phi)(x) = E_1 \cdot \phi(x) E_1^t + \cdots + E_d \cdot \phi(x) E_d^t,
\]
where recall \( E_n^t = xE_n \), we have,
\[
K(x, t) = k_1(x, t) E_1 + \cdots + k_d(x, t) E_d,
\]
with coordinates,
\[
k_n(x, t) = E_n \cdot \phi(x), \text{ for } n = 1, ..., d. \ (12)
\]

2. **Control function**: The function \( u(x, t) \) is obtained as,
\[
u(x, t) = -\frac{1}{2} K(x, t) (h(x) + \hat{h}_t).
\] (13)

The consistency between \( \pi_t^* \) and \( \pi_t \) is asserted in the following theorem. The proof is contained in appendix [9].

**Theorem 1**: Let \( \pi_t^* \) and \( \pi_t \) satisfy the forward evolution equations (6) and (9), respectively. Suppose that the gain function \( K(x, t) \) obtained using (12), and the control function \( u(x, t) \) obtained using (13) are admissible. Then, assume \( \pi_0 = \pi_0^* \), we have,
\[
\pi_t(f) = \pi_t^*(f),
\]
for all \( t \geq 0 \) and all function \( f \in C^\infty(G) \).

**Remark 2**: The admissibility of the control input depends on the existence, uniqueness and regularity of the solution \( \phi \) of the Poisson equation (11). For the Euclidean space, this theory is developed in [24], [40] based on spectral estimates for the \( \pi_t^* \). Extensions of these estimates to the manifold settings is a subject of the continuing work.

B. Galerkin approximation

The Poisson equation (11) needs to be solved at each time step. A Galerkin method is presented below to obtain an approximate solution. Since the time \( t \) is fixed, the explicit dependence on \( t \) is suppressed in what follows. So, \( \pi_t \) is denoted as \( \pi \); \( \phi \) is denoted as \( \phi \) etc.

The function \( \phi(x) \) is approximated as,
\[
\phi(x) = \sum_{l=1}^L \kappa_l \psi_l(x),
\]
where \( \{ \psi_i \}_{i=1}^L \) are a given (assumed) set of basis functions on the manifold \( G \). Using (12), the coordinates of the gain function \( K \) are then given by,

\[
k_n(x) = \sum_{i=1}^L \kappa_i E_n \cdot \psi_i(x).
\]

The finite-dimensional approximation of the Poisson equation (11) is to choose coefficients \( \{ \kappa_i \}_{i=1}^L \) such that

\[
\sum_{i=1}^L \kappa_i \pi((\text{grad}(\psi_i), \text{grad}(\psi_n))) = \pi((h - \hat{h})\psi_n),
\]

for all \( \psi \in \text{span}\{\psi_1, \ldots, \psi_L\} \subset H^1(G; \pi) \). On taking \( \psi = \psi_1, \ldots, \psi_L \), (16) is compactly written as a linear matrix equation,

\[
A\kappa = b,
\]

where \( \kappa := [\kappa_1, \ldots, \kappa_L] \) is a \( L \times 1 \) column vector that needs to be computed. The \( L \times L \) matrix \( A \) and the \( L \times 1 \) vector \( b \) are defined and approximated as,

\[
[A]_{lm} = \pi((\text{grad}(\psi_l), \text{grad}(\psi_m))) \\
\approx \frac{1}{N} \sum_{i=1}^N (\text{grad}(\psi_l)(X_i), \text{grad}(\psi_m)(X_i)),
\]

\[
= \frac{1}{N} \sum_{i=1}^N \sum_{m=l}^d (E_n \cdot \psi_l)(X_i)(E_n \cdot \psi_m)(X_i),
\]

\[
b_l = \pi((h - \hat{h})\psi_l) \approx \frac{1}{N} \sum_{i=1}^N (h(X_i) - \hat{h})\psi_l(X_i),
\]

where \( \hat{h} \approx \frac{1}{N} \sum_{i=1}^N h(X_i) \).

**Remark 3:** The Galerkin method is completely adapted to the data. That is, no explicit computation of the distribution is ever required. Instead, one only needs to evaluate a given set of basis functions at the particles \( X_i \). The choice of basis functions \( \{ \psi_i \}_{i=1}^L \) depends upon the problem. The functions \( E_n \cdot \psi_i \) can typically be computed in an offline fashion. This is illustrated with examples in the next section.

**V. Examples**

This section contains two examples to illustrate the construction and implementation of the feedback particle filter.

**A. FPF on SO(2)**

\( SO(2) \) is a 1-dimensional Lie group of rotation matrices \( R \) such that \( RR^T = I \) and \( \det(R) = 1 \). An arbitrary element is expressed as,

\[
R = R(\theta) = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix},
\]

where \( \theta \in S^1 \) is defined as the phase coordinate.

The general form of the nonlinear filtering problem on \( SO(2) \) is:

\[
dR_t = R_t \omega(R_t)E \, dt + R_t E \circ dB_t,
\]

\[
dZ_t = h(R_t) \, dt + dW_t,
\]

where \( \omega(\cdot) \) and \( h(\cdot) \) are given real-valued functions on \( SO(2) \), \( \{B_t \}, \{W_t \} \) are independent standard Wiener processes in \( \mathbb{R} \), and \( E \) is a basis of the Lie algebra \( so(2) \),

\[
E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

The function \( \omega(\cdot) \) has physical interpretation of the (local) angular velocity.

The gain function \( K(R,t) = k(R,t)E \) is a matrix in \( so(2) \) with the coordinate \( k(R,t) \), a real-valued function on \( SO(2) \). By the identification of \( SO(2) \) and \( S^1 \) in terms of the phase coordinate \( \theta \), define \( k(R,t) = k(\theta,t) \). Similarly, define \( \omega(\theta) = \omega(R(\theta)), h(\theta) = h(R(\theta)) \), and \( \phi(\theta) = \phi(R(\theta)) \).

It is straightforward to see that the filter expressed in the phase coordinate is given by,

\[
d\theta_t^\dagger = (\omega(\theta_t^\dagger)) \, dt + dB_t^\dagger + k(\theta_t^\dagger, t) \circ (dZ_t - \frac{(h(\theta_t^\dagger) + \hat{h}) \, dt}{2}) \mod 2\pi.
\]

At each time \( t \), \( k(\theta_t \circ dt + dB_t^\dagger \mod 2\pi) \) is obtained by solving the boundary value problem with respect to the phase coordinate. We suppress dependence on \( t \), and write \( k(\theta) \) for \( k(\theta,t) \). With a slight abuse of notation, the action of \( E \) on a smooth function is (see (20)),

\[
E \cdot \phi(\theta) = \frac{\partial \phi}{\partial \theta}(\theta),
\]

and the Poisson equation (11) is expressed as,

\[
\pi((E \cdot \phi)(E \cdot \psi)) = \pi((h - \hat{h})\psi),
\]

and needs to hold for all \( \psi \in H^1(S^1; \pi) \). If \( \pi \) has a probability density function \( p \) on \( S^1 \), then one can write (23) as,

\[
\int_0^{2\pi} \frac{\partial \phi}{\partial \theta}(\theta) \frac{\partial \psi}{\partial \theta}(\theta) p(\theta) \, d\theta = \int_0^{2\pi} (h(\theta) - \hat{h})\psi(\theta) p(\theta) \, d\theta.
\]

The solution of the Poisson equation on \( S^1 \) is approximated using a Fourier series basis. In the simplest case, these are just the first Fourier modes, in which case,

\[
\phi(\theta) = \kappa_1 \sin(\theta) + \kappa_2 \cos(\theta),
\]

leading to the following formula in light of (17):

\[
\frac{1}{N} \left[ \sum_i \cos^2(\theta_i^\dagger) - \sum_i \cos(\theta_i^\dagger) \sin(\theta_i^\dagger) \right] [\kappa_1] \\
= \frac{1}{N} \left[ \sum_i (h(\theta_i^\dagger) - \hat{h}) \sin(\theta_i^\dagger) \right].
\]

Finally, the gain function is obtained as,

\[
k(\theta) = \kappa_1 \cos(\theta) - \kappa_2 \sin(\theta).
\]

The resulting algorithm appears in [36] where it is referred to as a coupled oscillator FPF. The filter is applicable to the problem of gait estimation in locomotion systems [35].
B. FPF on SO(3)

SO(3) is a 3-dimensional Lie group (see the example in Sec. [I] for notation). The nonlinear filtering problem is,

\[
dR_t = R_t \Omega dt + R_t V_1 \circ dB_t, \\
dZ_t = h(R_t) dt + dW_t,
\]

where \( \Omega, \ V_1 \in \text{so}(3) \), and we write \( \Omega = \omega_1 E_1 + \omega_2 E_2 + \omega_3 E_3 \). The coordinates \( \omega_k \) may in general depend on \( R_t \).

The feedback particle filter is given by,

\[
dR_t' = R_t' \Omega dt + R_t' V_1 \circ dB_t + \nabla R_t' K(R_t', t) \circ \left( dZ_t - \frac{h(R_t') + \tilde{h}_t}{2} dt \right),
\]

where the gain function \( K(R,t) = k_1(R,t)E_1 + k_2(R,t)E_2 + k_3(R,t)E_3 \) is an element of the Lie algebra so(3). The coordinates \( (k_1,k_2,k_3) \) are obtained by solving a Poisson equation ([I]). We propose the following as basis functions:

\[
\phi_1(R) = \frac{1}{2} e_2^T (R - R^T) e_3, \quad \phi_2(R) = \frac{1}{2} e_3^T (R - R^T) e_1, \\
\phi_3(R) = \frac{1}{2} e_1^T (R - R^T) e_2, \quad \phi_4(R) = \frac{1}{2} (\text{Tr}(R) - 1),
\]

where \( e_1, e_2, e_3 \) are the canonical basis of \( \mathbb{R}^3 \). The action of the basis \( E_1, E_2, E_3 \) is easily computed and given in Table-I.

### Table I

| \( E_1 \) | \( E_2 \) | \( E_3 \) |
|-----|-----|-----|
| \( \phi_1 \) | \( -(R_{22} + R_{33})/2 \) | \( R_{21}/2 \) |
| \( \phi_2 \) | \( R_{12}/2 \) | \( -(R_{11} + R_{33})/2 \) |
| \( \phi_3 \) | \( R_{13}/2 \) | \( R_{23}/2 \) |
| \( \phi_4 \) | \( (R_{23} - R_{12})/2 \) | \( (R_{31} - R_{13})/2 \) |

Using Table-I, the \( 4 \times 4 \) matrix \( A \) and the \( 4 \times 1 \) vector \( b \) are assembled according to ([I] and [I]), respectively. The solution of the linear equation ([I]) is a \( 4 \times 1 \) vector, denoted as \( \kappa = [\kappa_1, \kappa_2, \kappa_3, \kappa_4] \). Denoting

\[
\Upsilon = \begin{bmatrix} \kappa_1 & -\kappa_3 & \kappa_2 \\
\kappa_3 & \kappa_4 & -\kappa_1 \\
-\kappa_2 & \kappa_1 & \kappa_4 \end{bmatrix},
\]

the coordinate functions of the gain have a succinct representation,

\[
k_n(R) = \frac{1}{2} \text{Tr}(RE_n \Upsilon), \quad \text{for } n = 1, 2, 3.
\]

For computational reasons, quaternions is a preferred choice for simulating rotations in SO(3) ([I], [I]). A unit quaternion has a general form,

\[
q = \left( \cos \left( \frac{\theta}{2} \right), \sin \left( \frac{\theta}{2} \right) \omega_1, \sin \left( \frac{\theta}{2} \right) \omega_2, \sin \left( \frac{\theta}{2} \right) \omega_3 \right)^T,
\]

which represents rotation of angle \( \theta \) about the axis defined by the unit vector \( (\omega_1, \omega_2, \omega_3)^T \). A quaternion is also written as \( q = (q_0, q_1, q_2, q_3)^T \).

In the following, the FPF is described for the quaternion coordinates. In these coordinates, the four basis functions (counterparts of (29) are,

\[
\phi_1(q) = 2q_1q_0, \quad \phi_2(q) = 2q_1q_0, \\
\phi_3(q) = 2q_1q_0, \quad \phi_4(q) = 2q_0^2 - 1.
\]

In order to compute the matrix \( A \) and the vector \( b \), the formulae for the action of \( E_1, E_2, E_3 \) on these basis functions appear in Table-II.

### Table II

| \( E_1 \) | \( E_2 \) | \( E_3 \) |
|-----|-----|-----|
| \( \phi_1 \) | \( \bar{q}_1^2 - \bar{q}_0^2 \) | \( 2q_1q_0 - 2q_3q_0 \) |
| \( \phi_2 \) | \( 2q_1q_0 + \bar{q}_0 \bar{q}_1 \) | \( -2q_0 + \bar{q}_0 \bar{q}_2 \) |
| \( \phi_3 \) | \( q_1q_3 - q_2q_0 \) | \( q_2 + 2q_3 \) |
| \( \phi_4 \) | \( 2q_1q_0 \) | \( 2q_2q_0 \) |

As before, the solution of the linear matrix equation is denoted as \( \kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4) \), and the coordinates of the gain function are obtained as,

\[
k_n(q,t) = \frac{1}{2} \text{Tr}(R(q)E_n \Upsilon),
\]

where \( \Upsilon \) is defined in (30), and \( R(q) \) is obtained using the conversion rule between rotation matrices and quaternions (see [I]).

Finally, the filter in the quaternion coordinates has the following form,

\[
dq'_f = \frac{1}{2} \Lambda(V(q'_f)) q'_f + \frac{1}{2} \Lambda(K(q'_f, t)) q'_f \circ (dZ_t - \frac{h(q'_f) + \tilde{h}_t}{2} dt),
\]

where \( K(q'_f, t), V(q'_f) \in \text{so}(3) \), \( V(q'_f) = \Omega dt + V_1 \circ dB_t \) and the \( 4 \times 4 \) matrix \( \Lambda(K) \) is given by,

\[
\Lambda(K) := \begin{bmatrix} 0 & -k_1 & -k_2 & -k_3 \\
k_1 & 0 & k_3 & -k_2 \\
k_2 & -k_3 & 0 & k_1 \\
k_3 & k_2 & -k_1 & 0 \end{bmatrix},
\]

and similarly for \( \Lambda(V(q'_f)) \).

Remark 4: Consider the special case where the dynamics are restricted to the subgroup \( SO(2) \) of \( SO(3) \). In this case, the filter (33) for \( SO(3) \) reduces to the filter (22) for \( SO(2) \). To see this, note that with the axis of rotation \((\omega_1, \omega_2, \omega_3)\) fixed, the four basis functions are given by,

\[
\phi_1(q) = 2q_1q_0 = \sin(\theta)\omega_1, \quad \phi_2(q) = 2q_1q_0 = \sin(\theta)\omega_2, \\
\phi_3(q) = 2q_1q_0 = \sin(\theta)\omega_3, \quad \phi_4(q) = 2q_0^2 - 1 = \cos(\theta).
\]

These functions span a 2-dimensional space, same as the Fourier basis functions \{\sin(\theta), \cos(\theta)\} for the \( SO(2) \) problem.
VI. CONCLUSIONS

In this paper, the generalization of the feedback particle filter to the continuous-time filtering problem on matrix Lie groups was presented. The formulation was shown to respect the intrinsic geometry of the manifold and preserve the error correction-based feedback structure of the original FPF. Algorithms were described and illustrated with examples for SO(2) and SO(3).

The continuing research includes application and evaluation of the filter to attitude estimation and robot localization; comparison of the FPF with existing algorithms based on EKF and the particle filter; and extension of the FPF for filtering stochastic processes where the observation also evolves on manifold (c.f., [28], [31]).

APPENDIX
A. Proof of Proposition [7]

The solution \( X_t \) to the Stratonovich SDE (38) is a continuous semimartingale on the Lie group. Furthermore, for any smooth function \( f: G \to \mathbb{R} \), \( f(X_t) \) is also a continuous semimartingale [32], satisfying,

\[
df(X_t) = (V_0 + u) \cdot f(X_t) \, dt + V_1 \cdot f(X_t) \circ dB_t^f + (K \cdot f)(X_t) \, dZ_t.
\]

(34)

To avoid the technical difficulty in taking expectation of the Stratonovich stochastic integrals, we convert (34) to its Itô form using the formula given in [38]: For continuous semi-martingales \( X,Y,Z \),

\[
Y \circ dX = Y \, dX + \frac{1}{2} Y \, dX \, dY,
\]

(35)

\[
(X \circ dY) \, dZ = X \, (dY \, dZ).
\]

(36)

To convert the second term of the right hand side of (34), take \( Y \) in (35) to be \( (V_1 \cdot f)(X_t) \) and \( X \) to be \( B_t^f \), we have,

\[
(V_1 \cdot f)(X_t) \circ dB_t^f = (V_1 \cdot f)(X_t) \, dB_t^f + \frac{1}{2} d(V_1 \cdot f)(X_t) \, dB_t^f.
\]

(37)

Then replace \( f \) by \( V_1 \cdot f \) in (34) to obtain,

\[
d(V_1 \cdot f) = (V_0 + u) \cdot (V_1 \cdot f) \, dt + V_1 \cdot (V_1 \cdot f) \circ dB_t^f + K \cdot (V_1 \cdot f) \, dZ_t.
\]

Using (36) and Itô’s rule,

\[
d(V_1 \cdot f)(X_t) \, dB_t^f = V_1 \cdot (V_1 \cdot f)(X_t) \, dr.
\]

(38)

which when substituted in (37) yields,

\[
V_1 \cdot f(X_t) \circ dB_t^f = V_1 \cdot f(X_t) \, dB_t^f + \frac{1}{2} V_1 \cdot (V_1 \cdot f)(X_t) \, dr.
\]

The third term on the right hand side of (34) is similarly converted. The Itô form of (34) is then given by,

\[
df(X_t) = \mathcal{L} f(X_t) \, dt + V_1 \cdot f(X_t) \, dB_t^f + (K \cdot f)(X_t,t) \, dZ_t,
\]

where the operator \( \mathcal{L} \) is defined by,

\[
\mathcal{L} f := (V_0 + u) \cdot f + \frac{1}{2} V_1 \cdot (V_1 \cdot f) + \frac{1}{2} K \cdot (K \cdot f).
\]

(39)

The solution of \( f(X_t) \) is obtained as,

\[
f(X_t) = f(X_0) + \int_0^t \mathcal{L} f(X_s) \, ds + \int_0^t V_1 \cdot f(X_s) \, dB_s^f + \int_0^t (K \cdot f)(X_s) \, dZ_s.
\]

By taking conditional expectation on both sides and interchanging expectation and integration,

\[
\pi_s(f) = \pi_0(f) + \int_0^s \pi_s(\mathcal{L} f) \, ds + \int_s^t \pi_s(K \cdot f) \, dZ_t,
\]

which is the desired formula (30).

B. Proof of Theorem [7]

Using (3) and (9) and the expressions for the operators \( \mathcal{L}^* \) and \( \mathcal{L} \), it suffices to show that

\[
\pi_s(u \cdot f) \, ds + \frac{1}{2} \pi_s((K \cdot f) + \pi_s(K \cdot f) \, dZ_s
\]

\[
= \left( \pi_s(fh) - \pi_s(h) \pi_s(f) \right) \, (dZ_s - \pi_s(h) \, ds),
\]

(40)

for all \( 0 \leq s \leq t \), and all \( f \in C^\infty(G) \).

On taking \( \psi = f \) in (11) and using the formula (4) for the inner product,

\[
\pi_s(K \cdot f) = \pi_s((h - \pi_s(h))f).
\]

(41)

Using the expression (13) for the control function and noting \( \bar{h} = \pi_s(h) \),

\[
u \cdot f = -\frac{1}{2} \pi_s((h - \pi_s(h))K \cdot f - \pi_s(h)K \cdot f).
\]

Using (41) repeatedly then leads to,

\[
\pi_s(u \cdot f) = -\frac{1}{2} \pi_s((h - \pi_s(h))K \cdot f - \pi_s(h)K \cdot f)
\]

\[
= -\frac{1}{2} \pi_s((K \cdot (K \cdot f)) - \pi_s(h) \pi_s((h - \pi_s(h))f).
\]

(42)

The desired equality (40) is now verified by substituting in (41) and (42).

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