**Q-spectral and L-spectral radius of subgroup graphs of dihedral group**

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**Abstract.** Research on Q-spectral and L-spectral radius of graph has been attracted many attentions. In other hand, several graphs associated with group have been introduced. Based on the absence of research on Q-spectral and L-spectral radius of subgroup graph of dihedral group, we do this research. We compute Q-spectral and L-spectral radius of subgroup graph of dihedral group and their complement, for several normal subgroups. Q-spectrum and L-spectrum of these graphs are also observed and we conclude that all graphs we discussed in this paper are Q-integral dan L-integral.

1. Introduction

For finite simple graph $G$ of order $p$, its signless Laplacian matrix is defined by $Q(G) = \Delta(G) + A(G)$ and its Laplacian matrix is defined by $L(G) = \Delta(G) - A(G)$, where $\Delta(G)$ is the vertex degree of $G$ and $A(G)$ is adjacency matrix of $G$. The $Q$-polynomial of $Q(G)$ is $p_Q(q) = \det(\Delta(G) - qI)$ and the $L$-polynomial of $L(G)$ is $p_L(\lambda) = \det(L(G) - \lambda I)$, where $I$ is identity matrix of dimension $p$. The largest eigenvalue of $Q(G)$ and $L(G)$ are named $Q$-spectral and $L$-spectral radius of $G$, respectively. The set of all distinct $Q$-eigenvalues with their multiplicities is called $Q$-spectrum and the set of all distinct $L$-eigenvalues with their multiplicities is called $L$-spectrum.

$Q$-spectral and $L$-spectral radius have received a great deal of attention and several researches have been reported. Some researches on $Q$-spectral radius and its sharp bound for various graphs can be seen in [1-4]. Sharp bound of $L$-spectral radius of graphs has also been studied, such as in [5-12].

Graphs associated with a finite group have been introduced, for example commuting graph [13], non-commuting graph [14], conjugate graph [15] and inverse graph [16], and seem to be an interesting area of research. Researches on signless Laplacian and Laplacian spectra of graphs associated with group have been conducted, such as [17-19]. In [20], Anderson et al. introduced the concept of subgroup graph of given subgroup $H$ of a group $G$ as a directed graph and denoted by $\Gamma_H(G)$. When the subgroup $H$ is normal in $G$, then $\Gamma_H(G)$ is an undirected simple graph [21].

We are interested in doing research on $Q$-spectral and $L$-spectral radius of graph associated with group. This paper is aimed to determine $Q$-spectral and $L$-spectral radius of subgroup graphs of dihedral group and their complements. The $Q$-spectral and $L$-spectral of these subgroup graphs are also observed.
2. Literature Review

A graph $G$ contained a finite non-empty set $V(G)$ of vertices together with a possibly empty set $E(G)$ of edges. The cardinality of $V(G)$ is called the order of $G$, while the cardinality of $E(G)$ is called the size of $G$. An empty graph is a graph of size 0. Two vertices $u$ and $v$ in $G$ are adjacent if $uv \in E(G)$. The degree of vertex $u$ in $G$ is defined as the number of vertices that adjacent with $u$ and denoted by $\deg(u)$.

Let $K_n$ denote a complete graph with $n$ vertices and $K_{m,n}$ denote a complete bipartite graph with partition sets $V_1$ and $V_2$ where $|V_1| = m$ and $|V_2| = n$. Then, $K_{m,n}$ has order $m + n$ and size $mn$ [22].

For more general, a complete multipartite graph with $k$ partition sets $V_1, V_2, \ldots, V_k$ ($k > 1$) where $|V_i| = n_i$ for $1 \leq i \leq k$ is denoted by $K_{n_1, n_2, \ldots, n_k}$. Graph $K_{n_1, n_2, \ldots, n_k}$ has order $n = \sum_{i=1}^{k} n_i$. The union $G = G_1 \cup G_2$ of two graphs $G_1$ and $G_2$ with $V(G_1) \cup V(G_2) = \emptyset$ is a graph that $V(G_1 \cup V(G_2))$ and $E(G) = E(G_1) \cup E(G_2)$ [23]. The graph $K_{n}^c$ is the empty graph of order $n$ [24]. The graph $K_{m,n}^c$ is $K_m \cup K_n$. Since $G = G_1$ [22] then $K_m \cup K_n = K_{m,n}^c$.

Let $G$ be a graph of order $p$. Let the adjacency matrix of $G$ is $A(G)$ and the degree matrix of $G$ is $D(G)$. Then the matrix $Q(G) = D(G) + A(G)$ is named the signless Laplacian matrix of $G$ [25,26] and $L(G) = D(G) - A(G)$ is named the Laplacian matrix of $G$ [27]. The $Q$-polynomial of $Q(G)$ is $p_Q(q) = \det(Q(G) - qI)$ [28] and the $L$-polynomial of $L(G)$ is $p_L(\lambda) = \det(L(G) - \lambda I)$, where $I$ is identity matrix of dimension $p$ [2]. The roots of characteristics equation associated with a matrix are called eigenvalues [29]. The eigenvalues of $Q(G)$ are called $Q$-eigenvalues of $G$ and the eigenvalues of $L(G)$ are called $L$-eigenvalues of $G$. Since $Q(G)$ and $L(G)$ are real and symmetric matrices then their eigenvalues are real and nonnegative [10,30] and can be arranged as $q_p \geq q_{p-1} \geq \cdots \geq q_2 \geq q_1$ and $\lambda_p \geq \lambda_{p-1} \geq \cdots \geq \lambda_2 \geq \lambda_1$, respectively. The largest eigenvalue $q_p$ of $Q(G)$ is called $Q$-spectral radius of $G$ [31] and the largest eigenvalue $\lambda_p$ of $L(G)$ is called $L$-spectral radius of $G$ [5].

Let $q_t > q_{t-1} > \cdots > q_2 > q_1$ are $t$ distinct $Q$-eigenvalues with the corresponding multiplicities $m_t, m_{t-1}, \ldots, m_2, m_1$. Then, $Q$-spectrum of $G$ is defined by $\text{spec}_Q(G) = \begin{bmatrix} q_t & q_{t-1} & \cdots & q_2 & q_1 \\ m_t & m_{t-1} & \cdots & m_2 & m_1 \end{bmatrix}$.

If every $Q$-eigenvalues of $G$ are integer then $G$ is called $Q$-integral [28]. $L$-spectrum of $G$ is defined in similar manner, and if every $L$-eigenvalues of $G$ are integer then $G$ is called $L$-integral [32].

The following are the results of previous research that will be used in this paper.

**Result 1** [2]. $Q$-polynomial of complete multipartite graph $K_{n_1, n_2, \ldots, n_k}$ of order $n$ is

$$p_Q(q) = (-1)^n \left( \sum_{i=1}^{k} \frac{n_i}{n - 2n_i - q} + 1 \right) \prod_{i=1}^{k} (n - 2n_i - q)(n - n_i - q)^{n_i - 1}.$$ 

$Q$-polynomial in Result 1 can be expressed as

$$p_Q(q) = \prod_{i=1}^{k} (q - n + n_i)^{n_i - 1} \prod_{i=1}^{k} (q - n + 2n_i) \left( 1 - \sum_{i=1}^{k} \frac{n_i}{q - n + 2n_i} \right) [28,33]$$

**Result 2** [34]. $Q$-eigenvalues of $K_n$ are $2(n - 1)$ and $n - 2$ with their multiplicities are $1$ and $n - 1$, respectively.

**Result 3** [35]. $Q$-polynomial of bipartite graphs is equal to $L$-polynomial.

**Result 4** [36]. $Q$-eigenvalues of complete graph $K_n$ are $n$ and $0$ with multiplicities $n - 1$ and $1$, respectively.

**Result 5** [37]. Let $C = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ is a block symmetric matrix of order 2. The eigenvalues of $C$ are those of $A + B$ together with those of $A - B$.

3. Main Results

Based on Anderson et al. [20] and Kakeri and Erfanian [21], if $G$ is a group and $H$ is its normal subgroup then the subgroup graph $\Gamma_H(G)$ of $G$ and its complement $\overline{\Gamma_H(G)}$ are undirected simple graphs. So, we focus on the normal subgroup of dihedral group along this paper.
The dihedral group $D_{2n}$ ($n \geq 3$) has $2n$ elements that consist of $n$ rotations $1, r, r^2, \ldots, r^{n-1}$ and $n$ reflection $s, sr, sr^2, \ldots, sr^{n-1}$. The order of $r$ is $n$ ($|r| = n$) and the order of $sr^i$ is $2$ ($|sr^i| = 2$) for $i = 1, 2, \ldots, n$. By using its generator, we can write $D_{2n} = \langle r, s \rangle = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$. It is well known that $sr \neq rs$ and $sr^i = r^{-i}s$. Hence, composition of two reflections is a rotation. For odd $n$, all normal subgroups of $D_{2n}$ are $\langle 1 \rangle$, $\langle r^d \rangle$ for all $d$ dividing $n$ and $D_{2n}$ itself. For even $n$, all normal subgroups of $D_{2n}$ are $\langle 1 \rangle, \langle r^d \rangle$ for all $d$ dividing $n$, $\langle r^2, s \rangle, \langle r^2, rs \rangle$ and $D_{2n}$ itself.

By definition of subgroup graph, we have $\Gamma_{D_{2n}}(D_{2n})$ is complete graph of order $2n$, for $n \geq 3$. So, $\Gamma_{D_{2n}}(D_{2n})$ is empty graph of order $2n$. The fact leads us to our first result.

**Theorem 1.**

(a) $Q$-spectral radius of $\Gamma_{D_{2n}}(D_{2n})$ is $4n - 2$ and $L$-spectral radius of $\Gamma_{D_{2n}}(D_{2n})$ is $2n$.

(b) $\text{Spec}_Q\left(\Gamma_{D_{2n}}(D_{2n})\right) = \left\{\frac{2n-2}{1}, \frac{2n-1}{2n-1}\right\}$ and $\text{Spec}_L\left(\Gamma_{D_{2n}}(D_{2n})\right) = \left\{\frac{2n}{2n-1}, 0\right\}$.

(c) $Q$-spectral and $L$-spectral radius of $\Gamma_{D_{2n}}(D_{2n})$ are $0$.

**Proof.** It is straightforward from Result 2 and then Result 4. ♦

The normal subgroup $\langle 1 \rangle$ has only identity element of $D_{2n}$. Therefore, $xy \in \langle 1 \rangle$ if and only if $y = x^{-1}$ in $D_{2n}$. We know that $(r^i)^{-1} = r^{n-i}$ and $(sr^i)^{-1} = sr^{-i}$ for odd and even $n$, and in addition $(r^{n/2})^{-1} = r^{n/2}$ for even $n$. Because graph in this paper is simple graph, then $sr^i$ and $r^{n/2}$ are not adjacent to themselves in $\Gamma_{\langle 1 \rangle}(D_{2n})$. Hence, only $r^i$ and $r^{n-i}$ are adjacent in $\Gamma_{\langle 1 \rangle}(D_{2n})$ for $i \neq n/2$.

Now, we have the following results on subgroup graph $\Gamma_{\langle 1 \rangle}(D_{2n})$, for $n \geq 3$.

**Theorem 2.**

(a) $Q$-spectral and $L$-spectral radius of $\Gamma_{\langle 1 \rangle}(D_{2n})$ are $2$.

(b) $\text{Spec}_Q\left(\Gamma_{\langle 1 \rangle}(D_{2n})\right) = \text{Spec}_L\left(\Gamma_{\langle 1 \rangle}(D_{2n})\right) = \left\{\frac{2}{(n-1)/2}, \frac{0}{(3n+1)/2}\right\}$ for odd $n$ and

\[\text{Spec}_Q\left(\Gamma_{\langle 1 \rangle}(D_{2n})\right) = \text{Spec}_L\left(\Gamma_{\langle 1 \rangle}(D_{2n})\right) = \left\{\frac{2}{(n-2)/2}, \frac{0}{(3n+2)/2}\right\}
\]

for even $n$.

(c) $L$-spectral radius of $\Gamma_{\langle 1 \rangle}(D_{2n})$ are $2n$.

(d) $\text{Spec}_L\left(\Gamma_{\langle 1 \rangle}(D_{2n})\right) = \left\{\frac{2n}{3n-1/2}, \frac{0}{(n-1)/2}\right\}$ for odd $n$

and $\text{Spec}_L\left(\Gamma_{\langle 1 \rangle}(D_{2n})\right) = \left\{\frac{2n}{3n/2}, \frac{0}{(n-2)/2}\right\}$ for even $n$.

The next results are for subgroup graph $\Gamma_{\langle r \rangle}(D_{2n})$ of dihedral group $D_{2n}$, where $n \geq 3$.

**Theorem 3.**

(a) $Q$-spectral radius of $\Gamma_{\langle r \rangle}(D_{2n})$ is $2(n-1)$ and $L$-spectral radius of $\Gamma_{\langle r \rangle}(D_{2n})$ is $n$.

(b) $\text{Spec}_Q\left(\Gamma_{\langle r \rangle}(D_{2n})\right) = \text{Spec}_L\left(\Gamma_{\langle r \rangle}(D_{2n})\right) = \left\{\frac{2(n-1)}{2}, \frac{n-2}{2(n-1)}\right\}$

(c) $Q$-spectral and $L$-spectral radius of $\Gamma_{\langle r \rangle}(D_{2n})$ are $2n$.

(d) $\text{Spec}_Q\left(\Gamma_{\langle r \rangle}(D_{2n})\right) = \text{Spec}_L\left(\Gamma_{\langle r \rangle}(D_{2n})\right) = \left\{\frac{2n}{1}, \frac{n}{2(n-1)}\right\}$

**Proof.**

(a) Subgroup graph $\Gamma_{\langle r \rangle}(D_{2n})$ is disconnected with two components and each component is a complete graph of order $n$. Hence, $\text{deg}(v) = n - 1$, for all $v \in \Gamma_{\langle r \rangle}(D_{2n})$. Therefore, $Q\left(\Gamma_{\langle r \rangle}(D_{2n})\right) = \begin{bmatrix} A & O \\ O & A \end{bmatrix}$, where $A = [a_{ij}]$ is matrix of order $n$ with $a_{ij} = 1$ otherwise and $O$ is zero matrix of order $n$. Using Result 5 on $\begin{bmatrix} A & Q \\ O & A \end{bmatrix}$ and then Result 2 on $A + O$ and $O - A$, we have the $Q$-eigenvalues are $2(n-1)$ and $n - 2$ with their multiplicities are $2$ and $2(n-1)$, respectively. In other hand, $L\left(\Gamma_{\langle r \rangle}(D_{2n})\right) = \begin{bmatrix} B & O \\ O & B \end{bmatrix}$, where $B = [b_{ij}]$ is matrix of
order $n$ with $b_{ij} = n - 1$ for $i = j$ and $b_{ij} = -1$ otherwise and $O$ is zero matrix of order $n$. With similar fashion, we have the $L$-eigenvalues are $n$ and $0$ with their multiplicities are $2(n - 1)$ and $2$, respectively. It completes the proof.

(b) From the proof of (a), $Q$-polynomial and $L$-polynomial of $\Gamma_r(D_{2n})$ are $p_Q(q) = (q - (2n - 2))^2(q - (n - 2))^{2n-2}$ and $p_L(\lambda) = (\lambda - n)^2\lambda^{2n-2}$. So, we have the desired proof.

(c) Since $\Gamma_r(D_{2n}) = K_n \cup K_n$, then $\overline{\Gamma_r(D_{2n})} = K_n \cup K_n$. By Result 1, $p_Q(q) = (q - (2n - n))^{2n-2} q$. Because $\overline{\Gamma_r(D_{2n})}$ is complete bipartite graph, by Result 3 we have $p_L(\lambda) = (\lambda - (n - 2)^2 \lambda). So, 2n is the largest eigenvalue and the proof is complete.

(d) It is clear from (c).

Normal subgroup $\langle r^2 \rangle$ of dihedral group $D_{2n}$, where $n \geq 4$ and $n$ is even, is $\langle r^2 \rangle = \{1, r^2, r^4, ..., r^{n-2}\}$ and $r^i r^j, s r^i r^j \in \langle r^2 \rangle$ if and only if $i$ and $j$ both even or both odd, for $1 \leq i, j \leq n - 2$. Therefore, subgroup graph $\Gamma_{r^2}(D_{2n})$ has four components and each component is complete graph $K_n/2$. So, we have the following results.

**Theorem 4.**

(a) $Q$-spectral radius of $\Gamma_{r^2}(D_{2n})$ is $n - 2$ and $L$-spectral radius of $\Gamma_{r^2}(D_{2n})$ is $n/2$, for even $n$ and $n \geq 4$.

(b) $\text{spec}_Q(\Gamma_{r^2}(D_{2n})) = \left[ \frac{n - 2}{4}, \frac{n - 4}{2(n - 2)} \right]$ and $\text{spec}_L(\Gamma_{r^2}(D_{2n})) = \left[ \frac{n}{2}, \frac{0}{4} \right]$.

(c) $Q$-spectral radius of $\Gamma_{r^2}(D_{2n})$ is $3n$ and $L$-spectral radius of $\Gamma_{r^2}(D_{2n})$ is $2n$, where $n$ is even and $n \geq 4$.

(d) $\text{spec}_Q(\Gamma_{r^2}(D_{2n})) = \left[ \frac{3n}{3}, \frac{3n}{2}, \frac{n}{3}, \frac{n}{2}, \frac{n}{3} \right]$ and $\text{spec}_L(\Gamma_{r^2}(D_{2n})) = \left[ \frac{2n}{3}, \frac{3n}{2}, \frac{0}{1} \right]$.

**Proof.**

(a) The $Q$-polynomial of $\Gamma_{r^2}(D_{2n})$ is $p_Q(q) = (-1)^{\frac{n}{4}}(q - (n - 2))^4(q - (n - 4))^2(n - 2)$.

and $L$-polynomial of $\Gamma_{r^2}(D_{2n})$ is $p_L(\lambda) = (-1)^{\frac{n}{2}}(\lambda - \frac{n}{2})^{2(n - 2)} \lambda^4$.

(b) It is clear from (a).

(c) Complement of subgroup graph $\overline{\Gamma_{r^2}(D_{2n})}$ is complete multipartite $K_n/2, n/2, n/2, n/2$ of order $2n$. By using Result 1, then $Q$-polynomial of $\overline{\Gamma_{r^2}(D_{2n})}$ is $p_Q(\lambda) = (\lambda - 3n)^4(\lambda - \frac{3n}{2})^{2(n - 2)}(\lambda - n)^3$.

And we have $L$-polynomial of $\overline{\Gamma_{r^2}(D_{2n})}$ is $p(\lambda) = (\lambda - 2n)^3(\lambda - \frac{3n}{2})^{2(n - 2)} \lambda$.

(d) It is clear from (c).

The normal subgroup $\langle r^2, s \rangle$ of $D_{2n}$ for even $n$ and $n \geq 4$ is $\langle r^2, s \rangle = \{1, r^2, r^4, ..., r^{n-2}, s, r^2, r^4, ..., s r^{n-2}\}$ and $(s^k r^i)(s^k r^j) \in \langle r^2, s \rangle$ if and only if $i$ and $j$ both even or both odd, for $1 \leq i, j \leq n - 2$ and $k = 0, 1$. Therefore, subgroup graph $\Gamma_{r^2}(D_{2n})$ has two components and each component is complete graph $K_n$ of order $n$. Then, subgroup graph $\Gamma_{r^2}(D_{2n})$ is isomorphic to $\Gamma_r(D_{2n})$. The following results are obvious.

**Theorem 5.**

(a) $Q$-spectral radius of $\Gamma_{r^2}(D_{2n})$ is $2(n - 1)$ and $L$-spectral radius of $\Gamma_{r^2}(D_{2n})$ is $n$. 

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(b) $\text{spec}_Q(\Gamma_{(r^2,s)}(D_{2n})) = \left[\frac{2(n-1)}{2} \right]^{n-2} \frac{n-2}{2(n-1)}$ and $\text{spec}_L(\Gamma_{(r^2,s)}(D_{2n})) = \left[\frac{n}{2(n-1)}\right]^0$.

(c) $Q$-spectral and $L$-spectral radius of $\Gamma_{(r^2,s)}(D_{2n})$ are 2n.

(d) $\text{spec}_Q(\Gamma_{(r^2,s)}(D_{2n})) = \text{spec}_L(\Gamma_{(r^2,s)}(D_{2n})) = \left[\frac{2n}{1} \frac{n}{2(n-1)}\right]^0$.

For even $n$ and $n \geq 4$, we also can observe that subgroup graph $\Gamma_{(r^2,r^2)}(D_{2n})$ is isomorphic to $\Gamma_{(r^2,s)}(D_{2n})$ and the following result is obvious.

**Theorem 6.**

(a) $Q$-spectral radius of $\Gamma_{(r^2,r^2)}(D_{2n})$ is $2(n-1)$ and $L$-spectral radius of $\Gamma_{(r^2,r^2)}(D_{2n})$ is $n$.

(b) $\text{spec}_Q(\Gamma_{(r^2,r^2)}(D_{2n})) = \left[\frac{2(n-1)}{2} \frac{n-2}{2(n-1)}\right]$ and $\text{spec}_L(\Gamma_{(r^2,r^2)}(D_{2n})) = \left[\frac{n}{2(n-1)}\right]^0$.

(c) $Q$-spectral and $L$-spectral radius of $\Gamma_{(r^2,r^2)}(D_{2n})$ are 2n.

(d) $\text{spec}_Q(\Gamma_{(r^2,r^2)}(D_{2n})) = \text{spec}_L(\Gamma_{(r^2,r^2)}(D_{2n})) = \left[\frac{2n}{1} \frac{n-2}{2(n-1)}\right]^0$.

4. Conclusion

We have computed $Q$-spectral and $L$-spectral radius of subgroup graphs of dihedral group $D_{2n}$ and their complement. According to our results, we can conclude that $\Gamma_{D_{2n}}(D_{2n})$ and $\Gamma_{(r^2,s)}(D_{2n})$ and their complement are $Q$-integral and $L$-integral, for all $n$ and $n \geq 3$. For even $n$ and $n \geq 4$, the subgroup graphs $\Gamma_{(r^2,s)}(D_{2n})$, $\Gamma_{(r^2,s)}(D_{2n})$, $\Gamma_{(r^2,r^2)}(D_{2n})$ and their complement also $Q$-integral and $L$-integral.

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