Covering morphisms of internal groupoids in the models of a semi-abelian theory

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Abstract
In this paper, for given an algebraic theory $T$ whose category $\text{Set}^T$ of models is semi-abelian, we consider the topological models of $T$ called topological $T$-algebras and obtain some results related to the fundamental groups of topological $T$-algebras. We also deal with the internal groupoid structure in the category of models providing that the fundamental groupoid deduces a functor from topological $T$-algebras to the internal groupoids in $\text{Set}^T$ and prove a criterion for the lifting of such an internal groupoid structure to the covering groupoids.

1 Introduction

Semi-abelian categories are Barr-exact, protomodular which means the short five lemma holds, have finite coproducts and have zero objects [8]. For example all abelian categories, the category of all groups, of rings without unit, of $\Omega$-groups, of Heyting semi-lattices, of locally boolean distributive lattices, of loops, of presheaves or sheaves of these are semi-abelian categories.

Let $T$ be an algebraic theory in the sense of Lawvere [10]. A model of $T$ in the category of sets is called $T$-algebra and a model of $T$ in the category of topological spaces is called topological $T$-algebra. An algebraic theory $T$ whose category $\text{Set}^T$ of the models is semi-abelian is called semi-abelian theory and a model of such a theory is called semi-abelian algebra. Such a theory is characterized in [5, Theorem 1.1]. The category of topological $T$-algebras and continuous $T$-homomorphisms between them is denoted by $\text{Top}^T$. For example when $T$ is the theory of groups, $\text{Top}^T$ becomes the category of topological groups.

In this paper for a semi-abelian theory $T$ we prove that the fundamental group of a topological $T$-algebra is a $T$-algebra and obtain topological $T$-algebras corresponding

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to the subalgebras. We also prove that the fundamental groupoid of a topological $\mathbb{T}$-algebra is an internal groupoid in $\text{Set}^{\mathbb{T}}$ and obtain some results on covering morphisms of internal groupoids.

On the one hand in 

some properties of topological groups such as being Hausdorff, compact, connected and etc. have been generalized to the topological $\mathbb{T}$-algebras for a semi-abelian algebra. Likewise similar results have been obtained in 

for more wider class of algebras called as topological protomodular algebras which have $n$-constants rather than unique constant.

On the other hand we know from 

Theorem 10.34] that for a connected topological space $X$ which has a universal cover, $x_0 \in X$ and a subgroup $G$ of the fundamental group $\pi_1(X,x_0)$ at $x_0$, there is a covering map $p: (\tilde{X}_G, \tilde{x}_0) \rightarrow (X,x_0)$ of pointed spaces, with characteristic group $G$ and by 

Theorem 10.42] whenever $X$ is a topological group, $\tilde{X}_G$ becomes a topological group such that $p$ is a group homomorphism. Using this method Mucuk and Şahan in 

recently have generalized some results on covering groups of topological groups to the topological groups with operations whose idea comes from Higgins 

Orzech 

and 

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For a given semi-abelian theory $\mathbb{T}$ and a topological $\mathbb{T}$-algebra $A$, we start our work by proving that the fundamental groupoid $\pi_1(A,e)$ at the constant $e \in A$ becomes a $\mathbb{T}$-algebra. Then assuming that $A'$ is a sub $\mathbb{T}$-algebra of $A$ and $B$ is the fundamental group $\pi_1(A',e)$, we prove that $\tilde{A}_B$ corresponding to $B$ as in 

Theorem 10.34] is a topological $\mathbb{T}$-algebra such that $p: (\tilde{A}_B,e) \rightarrow (A,e)$ is a topological $\mathbb{T}$-homomorphism.

Next we define internal category in the category $\text{Set}^{\mathbb{T}}$ of semi-abelian algebras and prove that the fundamental groupoid $\pi A$ of a topological $\mathbb{T}$-algebra $A$ is an internal groupoid in $\text{Set}^{\mathbb{T}}$. We continue proving an equivalence of the categories in Theorem 

and finally we obtain a criterion for the lifting of internal groupoid structure to the covering groupoids considering the internal groupoid structure in the category $\text{Set}^{\mathbb{T}}$.

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2 Preliminaries on covering groupoids and topological semi-abelian algebras

A groupoid is a small category in which each arrow is an isomorphism (see 

and 

for more discussion on groupoids). More precisely a groupoid $G$ has a set $G$ of arrows and a set $\text{Ob}(G)$ of objects together with source and target point maps $s, t: G \rightarrow \text{Ob}(G)$ and object inclusion map $\epsilon: \text{Ob}(G) \rightarrow G$ such that $s \epsilon = t \epsilon = 1_{\text{Ob}(G)}$. There exists a partial composition defined by 

$G_1 \times_s G \rightarrow G$, $(g, h) \mapsto g \circ h$, where $G_1 \times_s G$ is the pullback of $t$ and $s$. Here if $g, h \in G$ and $t(g) = s(h)$, then the composite $g \circ h$ exists such that $s(g \circ h) = s(g)$ and $t(g \circ h) = t(h)$. Further, this partial composition is associative, for $x \in \text{Ob}(G)$ the arrow $\epsilon(x)$ acts as the identity and it is denoted by $1_x$, and each arrow $g$ has an inverse $g^{-1}$ such that $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g \circ g^{-1} = \epsilon(s(g))$, $g^{-1} \circ g = \epsilon(t((g))$. The map $G \rightarrow G$, $g \mapsto g^{-1}$ is called the inversion. In a groupoid $G$, the source and target points, the object inclusion, the inversion maps and the partial
composition are called structural maps.

An example of a groupoid is the fundamental groupoid \(\pi(X)\) of a topological space \(X\), where the objects are points of \(X\) and arrows, say from \(x\) to \(y\) are the homotopy classes of the paths in \(A\), relative to the end points, with source point \(x\) and final point \(y\). The partial composition on the homotopy classes is defined by the concatenation of the paths. A group is also a groupoid with one object.

In a groupoid \(G\) for \(x, y \in \text{Ob}(G)\) we write \(G(x, y)\) for the set of all arrows with source points \(x\) and target points \(y\). According to [4] \(G\) is transitive if for all \(x, y \in \text{Ob}(G)\), the set \(G(x, y)\) is not empty; for \(x \in \text{Ob}(G)\) the star of \(x\) is defined as \(\{g \in G \mid s(g) = x\}\) and denoted by \(\text{St}_G x\); and the object group \(G(x, x)\) at \(x\) is denoted by \(G(x)\).

A functor \(p: H \to G\) of groupoids is called a morphism of groupoids. A groupoid morphism \(p: H \to G\) is said to be covering morphism and \(H\) covering groupoid of \(G\) if for each \(\tilde{x} \in \text{Ob}(H)\) the restriction \(\text{St}_H \tilde{x} \to \text{St}_G p(\tilde{x})\) is bijective. A covering morphism \(p: H \to G\) in which both \(H\) and \(G\) are transitive is called universal when \(H\) covers every cover of \(G\) in the sense that for every covering morphism \(q: K \to G\) there is a unique morphism of groupoids \(r: H \to K\) such that \(r \circ q = p\).

For a groupoid morphism \(p: H \to G\) and an object \(\tilde{x}\) of \(H\) we call the subgroup \(p(\text{Ob}(\tilde{x}))\) of \((p\tilde{x})\) as characteristic group of \(p\) at \(\tilde{x}\).

An action of a groupoid \(G\) on a set \(A\) is defined in [7, pp.373] as consisting of two functions \(\omega: A \to \text{Ob}(G)\) and \(\varphi: A_\omega \times_s G \to A, (a, g) \mapsto ag\), where \(A_\omega \times_s G\) is the pullback of \(\omega\) and \(s\), subject to the following conditions:

1. \(\omega(ag) = t(g)\) for \((a, g) \in A_\omega \times_s G\);
2. \(a(g \circ h) = (ag)h\) for \((g, h) \in G_t \times_s G\) and \((a, g) \in A_\omega \times_s G\);
3. \(a(\omega(a)) = a\) for \(a \in A\).

According to [7, 10.4.2] for given such an action of groupoid \(G\) on a set \(A\), the semi-direct product groupoid \(G \ltimes A\) with object set \(A\) is defined such that the arrows from \(a\) to \(b\) are the pairs \((g, a)\) with \(g \in G(\omega(a), \omega(b))\) and \(ag = b\). The partial composition is given by

\[(g, a) \circ (h, b) = (g \circ h, a)\]

when \(b = ag\). The projection map \(p: G \ltimes A \to G\) defined on objects by \(\omega\) and on arrows by \((g, a) \mapsto g\) is a covering morphism.

Since in the proof of Theorem 4.10 we need some details of the following result we remind a sketch proof from [7, 10.4.3].

**Theorem 2.1.** Let \(x\) be an object of a transitive groupoid \(G\), and let \(C\) be a subgroup of the object group \(G(x)\). Then there exists a covering morphism \(q: (H_C, \tilde{x}) \to (G, x)\) with characteristic group \(C\).

**Proof:** Let \(A_C\) be the set of cosets \(C \circ g = \{c \circ g \mid c \in C\}\) for \(g\) in \(\text{St}_G x\). Let \(\omega: A_C \to \text{Ob}(G)\) be a map, which maps \(C \circ g\) to the target point of \(g\). The function \(\omega\) is
well defined, because if \( C \circ g = C \circ h \) then \( t(g) = t(h) \). The groupoid \( G \) acts on \( A = A_C \) by
\[
\varphi: A_\omega \times_s G \to A, (C \circ g, h) \mapsto C \circ (g \circ h).
\]
The required groupoid \( H_C \) is taken to be the semi-direct product groupoid \( G \ltimes A_C \).
Then the projection \( q: H_C \to G \) given on objects by \( \omega: A_C \to \text{Ob}(G) \) and on arrows by \( (h, C \circ g) \mapsto h \), is a covering morphism of groupoids and has the characteristic group \( C \).
Here the partial composition on \( H_C \) is defined by
\[
(k, C \circ g) \circ (l, C \circ h) = (k \circ l, C \circ g)
\]
whenever \( C \circ h = C \circ g \circ k \). The required object \( \tilde{x} \in H_C \) is the coset \( C \). \( \square \)

An algebraic theory \( \mathbb{T} \) in the sense of Lawvere [10, pp.109] is a category with objects
\[
T^0, T, T^2, T^3, \ldots
\]
where \( T^n \) is \( n \)-copies of the distinguish object \( T \), and with \( m \) arrows
\[
\pi_i^{(m)}: T^m \to T, i = 0, 1, 2, \ldots, m - 1
\]
for each \( m \) such that for any \( m \) arrows
\[
\tau_i: T^m \to T, i = 0, 1, \ldots, m - 1
\]
in \( \mathbb{T} \) there is exactly one arrow
\[
(\tau_0, \tau_1, \ldots, \tau_{m-1}): T^m \to T^m
\]
so that
\[
(\tau_0, \tau_1, \ldots, \tau_{m-1}) \circ \pi_i^{(m)} = \tau_i \quad (i = 0, 1, \ldots, m - 1)
\]
The arrows \( \tau: T^m \to T \) of this category are called \( n \)-ary operations and in particular, the 0-ary operations \( T^0 \to T \) are called constants of the theory \( \mathbb{T} \).
Throughout the paper by an algebraic theory \( \mathbb{T} \) we mean the theory in the sense of Lawvere as stated (see also Borceux [2] for an equivalent set theoretical interpretation).
A product preserving functor \( F: \mathbb{T} \to \text{Set} \) is called a model of the theory or a \( \mathbb{T} \)-algebra and natural transformations between \( \mathbb{T} \)-algebras are called \( \mathbb{T} \)-homomorphisms. Hence a \( \mathbb{T} \)-homomorphism is a map between sets commuting with all operations of the theory. Let \( \text{Set}_T \) be the category of models of the algebraic theory \( \mathbb{T} \) whose objects are \( \mathbb{T} \)-algebras and arrows are \( \mathbb{T} \)-homomorphisms.

A semi-abelian theory is characterized in [5, Theorem 1.1] as follows. We write \( e \) rather than 0 for the constant of the theory \( \mathbb{T} \) to distinguish from 0 in a path \( \beta: [0, 1] \to A \) for a topological \( \mathbb{T} \)-algebra \( A \).

**Theorem 2.2.** An algebraic theory \( \mathbb{T} \) has a semi-abelian category \( \text{Set}_T \) of models precisely when, for some natural number \( n \), the theory \( \mathbb{T} \) contains

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1. a unique constant $e$

2. $n$ binary operations $\alpha_1(X,Y), \alpha_2(X,Y), \ldots, \alpha_n(X,Y)$ satisfying $\alpha_i(X,X) = e$

3. an $(n+1)$-ary operation $\theta(X_1, X_2, \ldots, X_{n+1})$ satisfying
   \[ \theta(\alpha_1(X,Y), \alpha_2(X,Y), \ldots, \alpha_n(X,Y), Y) = X. \]

Here we remark that, in general, $T$ admits many more operations than simply $\alpha_i$ and $\theta$; and the choice in $T$ of the operations $\alpha_i$ and $\theta$ as indicated is not unique. We mean such a theory by **semi-abelian theory** and the corresponding $T$-algebras by **semi-abelian algebras**.

For example; each algebraic theory $T$ which has a unique constant and a group operation ‘+’ is semi-abelian. This is in particular the case for groups, abelian groups, $\Omega$-groups, modules on a ring, rings or algebras without unit, Lie algebras, etc. In Theorem 2.2 one chooses $n = 1$ and $\alpha_1(X,Y) = X - Y$, $\theta(X,Y) = X + Y$.

Replacing $\text{Set}$ with $\text{Top}$ in the definition of $T$-algebra, we obtain the categorical notion of topological $T$-algebra as a functor $F: T \to \text{Top}$. An equivalent set theoretical definition of a topological $T$-algebra is given in [4, Definition 5] as follows.

**Definition 2.3.** Given an algebraic theory $T$, by a **topological $T$-algebra** we mean a topological space $A$ provided with the structure of a $T$-algebra, in such a way that every operation $\tau: T^n \to T$ of $T$ induces a continuous mapping $\tau_A: A^n \to A, (a_1, \ldots, a_n) \mapsto \tau(a_1, \ldots, a_n)$.

We write $\text{Top}^T$ for the category of topological $T$-algebras and continuous $T$-homomorphisms between them. If $T$ is a semi-abelian theory, the corresponding topological $T$-algebras will be also called **topological semi-abelian algebras**.

We recall from [4] that a theory $T$ is called **protomodular** if the category $\text{Set}^T$ of models of the theory is protomodular as defined by Bourn in [6] and a protomodular theory generalizing semi-abelian theory is characterized by Bourn and Janelidze in [5] as a theory with $n$-constants $e_1, \ldots, e_n$ satisfying the similar axioms of Theorem 2.2. The models of such a theory are called **protomodular algebras**. Protomodular categories include all Abelian categories, the category of all groups, loops or even semi-loops, rings with or without unit, associative algebras with or without unit, Lie algebras, Jordan algebras, Boolean algebras, Heyting algebras, Boolean rings, Heyting semi-lattices, and so on. If in Definition 2.3 $T$ is a protomodular theory, the corresponding topological $T$-algebras are called **topological protomodular algebras**. It was proved in [4] that some results about topological semi-abelian algebras studied in [3] can be extended to the topological protomodular algebras.

We also recall that in an algebraic theory $T$, the set of constants is the free algebra on the empty set of generators and is trivially the initial object in the category $\text{Set}^T$ of $T$-algebras. If $T$ has a unique constant $e$, the initial object is thus reduced to the singleton $\{e\}$ and therefore, becomes trivially also a final object, that is, a zero object in $\text{Set}^T$. A theory $T$ is equivalent to the dual of the category of finitely generated $T$-algebras. In this equivalence, the object $T^n$ of the theory corresponds by duality to the
free algebra $F(n)$ on $n$ generators. In particular the object $T^0$ corresponds to the free algebra on the empty set, that is, the zero algebra $\{e\}$. We can now prove the following Lemma which is used later in some proofs.

**Lemma 2.4.** Let $T$ be a semi-abelian theory and $A$ a $T$-algebra with constant $e \in A$. Then an $n$-ary mapping $\tau: A^n \to A$ maps $(e, \ldots, e)$ to $e$.

**Proof:** Let $\tau: A^n \to A$ be an $n$-ary mapping of the theory $T$. It corresponds by duality to a $T$-homomorphism $t: F(1) \to F(n)$ between the corresponding free algebras. By the definition of a free algebra, given a $T$-algebra $A$, we have a bijection between

$$\text{Set}(\{1, 2, \ldots, n\}, A) \to \text{Set}^T(F(n), A)$$

Choosing $n$-elements $a_1, \ldots, a_n$ in the $T$-algebra $A$ is the same as choosing a $T$-homomorphism $\beta: F(n) \to A$. The composite $\beta \circ t: F(1) \to F(n) \to A$ of $t$ and $\beta$ is a $T$-homomorphism $F(1) \to A$ and corresponds thus to the single element $\tau(a_1, \ldots, a_n)$ of $A$.

Suppose now that $a_1$ to $a_n$ are all $e$. This means that the $T$-homomorphism $\beta$ maps all generators of $F(n)$ to $e$, that is, $\beta$ factors through the zero object $\{e\}$

$$\beta: F(n) \to \{e\} \to A$$

But then the composite $T$-homomorphism $\beta \circ t$ factors through $\{e\}$ as well

$$\beta \circ t: F(1) \to F(n) \to \{e\} \to A$$

This composite is thus the zero $T$-homomorphism, which maps the generator of $F(1)$ on $e$ in $A$. Hence we have that $\tau(e, \ldots, e) = e$.

\(\square\)

We remind the following construction from [17] pp.295-302.

Let $X$ be a topological space with a base point $x_0$ and $G$ a subgroup of the fundamental group $\pi_1(X, x_0)$. Let $P(X, x_0)$ be the set of all paths $\beta$ in $X$ with source point $x_0$. Then the relation on $P(X, x_0)$ defined by $\beta \simeq \gamma$ if and only if $\beta(1) = \gamma(1)$ and $[\beta \circ \gamma^{-1}] \in G$, is an equivalence relation, where $\circ$ denotes the concatenation of the paths. Denote the equivalence class of $\beta$ by $\langle \beta \rangle_G$ and define $\tilde{X}_G$ as the set of all such equivalence classes of the paths in $X$ with source point $x_0$. Define a function $p: \tilde{X}_G \to X$ by $p(\langle \beta \rangle_G) = \beta(1)$. Let $\beta_0$ be the constant path at $x_0$ and $\tilde{x}_0 = \langle \beta_0 \rangle_G \in \tilde{X}_G$. If $\beta \in P(X, x_0)$ and $U$ is an open neighbourhood of $\langle 1 \rangle$, then a path of the form $\beta \circ \lambda$, where $\lambda$ is a path in $U$ with $\lambda(0) = \beta(1)$, is called a continuation of $\beta$. For a $\langle \beta \rangle_G \in \tilde{X}_G$ and an open neighbourhood $U$ of $\beta(1)$, let $\langle (\beta)_{G, U} \rangle = \{ \langle (\beta \circ \lambda) \rangle_G : \lambda(I) \subseteq U \}$. Then the subsets $\langle (\beta)_{G, U} \rangle$ form a basis for a topology on $\tilde{X}_G$ such that the map $p: (\tilde{X}_G, \tilde{x}_0) \to (X, x_0)$ is continuous [17] Lemma 10.31. We also know from [17] Theorem 10.34 that if $X$ is connected and has a universal cover, then $p: (\tilde{X}_G, \tilde{x}_0) \to (X, x_0)$ is a covering map with characteristic group $G$. 

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3 Topological $\mathcal{T}$-algebras

In this section for a semi-abelian theory $\mathcal{T}$, we apply the construction of [17, pp.295-302] stated above to the topological $\mathcal{T}$-algebras. We first need the following preparation:

Let $\mathcal{T}$ be a semi-abelian theory, $A$ a topological $\mathcal{T}$-algebra with the unique constant $e$ and $P(A, e)$ the set of all paths in $A$ with source points $e$. For every $n$-ary continuous mapping $\tau: A^n \to A$ and the paths $\beta_1, \ldots, \beta_n$ of $P(A, e)$ we have a continuous mapping $[0,1] \to A$ defined by

\[
\tau(\beta_1, \ldots, \beta_n)(t) = \tau(\beta_1(t), \ldots, \beta_n(t))
\]

for $t \in [0,1]$. Then by Lemma 2.4 we have that

\[
\tau(\beta_1, \ldots, \beta_n)(0) = \tau(\beta_1(0), \ldots, \beta_n(0)) = \tau(e, \ldots, e) = e
\]

and therefore $\tau(\beta_1, \ldots, \beta_n)$ is a path of $P(A, e)$.

We also have

\[
(\tau(\beta_1, \ldots, \beta_n))^{-1} = \tau(\beta_1^{-1}, \ldots, \beta_n^{-1})
\]

where, for a path, say $\beta$ the inverse $\beta^{-1}$ denotes the inverse path defined by $\beta^{-1}(t) = \beta(1 - t)$ for $t \in [0,1]$. Then by the evaluation of the concatenation of the paths in $A$ at $t \in [0,1]$ we have that the interchange rule

\[
\tau(\beta_1, \ldots, \beta_n) \circ \tau(\gamma_1, \ldots, \gamma_n) = \tau(\beta_1 \circ \gamma_1, \ldots, \beta_n \circ \gamma_n)
\]

holds whenever the concatenations of the paths $\beta_i$ and $\gamma_i$ are defined, where ‘$\circ$’ denotes the concatenation of the paths. More precisely evaluating the concatenations of these paths at $t \in [0,1]$ for the left side of Eq.[3] we have

\[
(\tau(\beta_1, \ldots, \beta_n) \circ \tau(\gamma_1, \ldots, \gamma_n))(t) = \begin{cases} 
\tau(\beta_1(2t), \ldots, \beta_n(2t)), & 0 \leq t \leq \frac{1}{2} \\
\tau(\gamma_1(2t - 1), \ldots, \gamma_n(2t - 1)), & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

and for the right side

\[
(\tau(\beta_1 \circ \gamma_1, \ldots, \beta_n \circ \gamma_n)(t)) = \tau((\beta_1 \circ \gamma_1)(t), \ldots, (\beta_n \circ \gamma_n)(t))
\]

= $\tau(\beta_1(2t), \ldots, \beta_n(2t))$

if $0 \leq t \leq \frac{1}{2}$ and

\[
\tau(\beta_1 \circ \gamma_1, \ldots, \beta_n \circ \gamma_n)(t) = \tau((\beta_1 \circ \gamma_1)(t), \ldots, (\beta_n \circ \gamma_n)(t))
\]

= $\tau(\gamma_1(2t - 1), \ldots, \gamma_n(2t - 1))$

if for $\frac{1}{2} \leq t \leq 1$. This proves that Eq[3] is satisfied.
Theorem 3.1. Let $\mathbb{T}$ be a semi-abelian theory. If $A$ is a topological $\mathbb{T}$-algebra with the unique constant $e$, then the fundamental group $\pi_1(A, e)$ becomes a $\mathbb{T}$-algebra.

Proof: Let $A$ be a topological $\mathbb{T}$-algebra with the constant element $e$. Hence categorically it represents a product preserving functor $F_A: \mathbb{T} \to \text{Top}$ in which $F_A(T) = A$, for the distinguish object $T$ of $\mathbb{T}$. Then for an $n$-ary operation $\tau: T^n \to T$ of the theory $\mathbb{T}$ we have a mapping

$$\pi_1(A, e)^n \to \pi_1(A, e)$$

defined by

$$([\beta_1], \ldots, [\beta_n]) \mapsto [\tau(\beta_1, \ldots, \beta_n)] \quad \text{(4)}$$

for $[\beta_i] \in \pi_1(A, e)$ ($1 \leq i \leq n$). Here $\tau(\beta_1, \ldots, \beta_n)$ is the path defined by Eq.1. The mapping defined on $\pi_1(A, e)^n$ is well defined by the continuity of the mapping $\tau: A^n \to A$. We now prove that according to these mappings, $\pi_1(A, e)$ becomes a $\mathbb{T}$-algebra with the constant element $\tilde{e}$, which is the homotopy class of the constant path at $e \in A$. An arrow

$$(\tau_0, \ldots, \tau_{m-1}): T^n \to T^m$$

of the theory $\mathbb{T}$ constitutes the mapping

$$\pi_1(A, e)^n \to \pi_1(A, e)^m$$

defined by

$$([\beta_1], \ldots, [\beta_n]) \mapsto ([\tau_0(\beta_1, \ldots, \beta_n)], \ldots, [\tau_{m-1}(\beta_1, \ldots, \beta_n)]$$

Hence by this evaluation we have a product preserving functor

$$\pi_1 F_A: \mathbb{T} \to \text{Set}$$

induced by the distinguished object $T$ of the theory $\mathbb{T}$

$$(\pi_1 F_A)(T) = \pi_1(F_A(T), e) = \pi_1(A, e)$$

The axioms of Theorem 2.2 can be checked as follows.

$$\alpha_i([\beta], [\beta]) = [\alpha_i(\beta, \beta)] = \tilde{e}$$

for any binary mapping $\alpha_i$ and $n + 1$-ary mapping $\theta$

$$\theta(\alpha_1([\beta], [\gamma]), \ldots, \alpha_n([\beta], [\gamma]), [\gamma]) = \theta([\alpha_1(\beta, \gamma)], \ldots, [\alpha_n(\beta, \gamma)], [\gamma]) \quad \text{(by Eq.4)}$$

$$= [\theta(\alpha_1(\beta, \gamma), \ldots, \alpha_n(\beta, \gamma), \gamma)] \quad \text{(by Eq.4)}$$

$$= [\beta] \quad \text{(by Theorem 2.2)}$$

for $[\beta], [\gamma] \in \pi_1(A, e)$. Therefore $\pi_1(A, e)$ becomes a $\mathbb{T}$-algebra for the same semi-abelian theory $\mathbb{T}$. \qed
Proposition 3.2. Let $\mathbb{T}$ be a semi-abelian theory. Then we have a functor $\pi_1: \text{Top}^\mathbb{T} \to \text{Set}^\mathbb{T}$ assigning each topological $\mathbb{T}$-algebra $A$ to the $\mathbb{T}$-algebra $\pi_1(A,e)$.

Proof: Let $f: A \to B$ be a continuous $\mathbb{T}$-homomorphism. Then by the following evaluation, the induced map $f_* = \pi_1 f: \pi_1(A,e) \to \pi_1(B,e)$ becomes a $\mathbb{T}$-homomorphism

$$f_*(\tau([\gamma_1], \ldots, [\gamma_n]) = f_*(\tau(\gamma_1, \ldots, \gamma_n))$$

The axioms for $\pi_1$ to be a functor are straightforward and therefore omitted. \qed

We recall from [3, Theorem A.2] that for a semi-abelian theory $\mathbb{T}$, a sub $\mathbb{T}$-algebra $B$ of a $\mathbb{T}$-algebra $A$ is normal if for every operation $\tau(X_1, \ldots, X_k, Y_1, \ldots, Y_l)$ of the theory such that $\tau(X_1, \ldots, X_k, e, \ldots, e) = e$ one has $\tau(a_1, \ldots, a_k, b_1, \ldots, b_l) \in B$ for $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_l \in B$.

Lemma 3.3. Let $A$ be a topological $\mathbb{T}$-algebra for a semi-abelian theory $\mathbb{T}$ with constant $e$. If $B$ is a normal sub $\mathbb{T}$-algebra of $A$, then $\pi_1(B,e)$ becomes a normal sub $\mathbb{T}$-algebra of $\pi_1(A,e)$.

Proof: Let $\tau(X_1, \ldots, X_k, Y_1, \ldots, Y_l)$ be an operation of the theory such that $\tau(X_1, \ldots, X_k, e, \ldots, e) = e$; and let $[\gamma_1], \ldots, [\gamma_k] \in \pi_1(A,e)$ and $[\beta_1], \ldots, [\beta_l] \in \pi_1(B,e)$. Since $B$ is a normal sub $\mathbb{T}$-algebra of $A$ we have

$$\tau(\gamma_1, \ldots, \gamma_k, \beta_1, \ldots, \beta_l)(t) = \tau(\gamma_1(t), \ldots, \gamma_k(t), \beta_1(t), \ldots, \beta_l(t)) \in B$$

and therefore

$$\tau([\gamma_1], \ldots, [\gamma_k], [\beta_1], \ldots, [\beta_l]) = [\tau(\gamma_1, \ldots, \gamma_k, \beta_1, \ldots, \beta_l)] \in \pi_1(B,e)$$

Hence $\pi_1(B,e)$ is a normal sub $\mathbb{T}$-algebra of $\pi_1(A,e)$. \qed

Let $\mathbb{T}$ be a semi-abelian theory with the constant $e$ and $A$ a topological $\mathbb{T}$-algebra. Assume that $A'$ is a sub $\mathbb{T}$-algebra of $A$ and $B$ is the fundamental group $\pi_1(A',e)$. Then $B$ is a sub $\mathbb{T}$-algebra of $\pi_1(A,e)$; and therefore by [17, Lemma 10.31] we have a topological space $\tilde{A}_B$ and a continuous map $p: (\tilde{A}_B, \tilde{e}) \to (A,e)$ between topological spaces, corresponding to the subgroup $B$ of $\pi_1(A,e)$. Hence we can prove the following theorem for topological $\mathbb{T}$-algebras.

Theorem 3.4. Let $\mathbb{T}$ be a semi-abelian theory with the constant $e$ and $A$ a topological $\mathbb{T}$-algebra. Let $A'$ be a sub $\mathbb{T}$-algebra of $A$ and $B$ the fundamental group $\pi_1(A',e)$. Then $\tilde{A}_B$ becomes a topological $\mathbb{T}$-algebra such that the map $p: (\tilde{A}_B, \tilde{e}) \to (A,e)$ is a continuous $\mathbb{T}$-homomorphism.

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\textbf{Proof:} Since $B$ is a subgroup of the fundamental group $\pi_1(A, e)$, by [17, Lemma 10.31] we have a continuous map $p: (\tilde{A}_B, \bar{c}) \to (A, e)$ of topological spaces corresponding to $B$. Hence $\tilde{A}_B$ is defined as the set of equivalence classes defined via $B$. Then each mapping defined by Eq. 4 reduces a mapping defined by

$$\tau((\beta_1), \cdots, (\beta_n)) = (\tau(\beta_1, \cdots, \beta_n)).$$

for $(\beta_1), \cdots, (\beta_n) \in \tilde{A}_B$. We now prove that this mapping is well defined: For the paths $\beta_1, \ldots, \beta_n$ and $\gamma_1, \ldots, \gamma_n$ in $P(A, e)$ with $\beta_i(1) = \gamma_i(1)$ one has that

$$[\tau(\beta_1, \ldots, \beta_n) \cdot (\tau(\gamma_1, \ldots, \gamma_n))^{-1}] = [\tau(\beta_1, \ldots, \beta_n) \circ (\gamma_1^{-1}, \ldots, \gamma_n^{-1})]$$

(by Eq.2)

$$= [(\beta_1 \circ \gamma_1^{-1}, \ldots, \beta_n \circ \gamma_n^{-1})]$$

(by Eq.3)

$$= \tau(\beta_1 \circ \gamma_1^{-1}, \ldots, \beta_n \circ \gamma_n^{-1}).$$

(by Eq.4)

Since $B$ is a sub $T$-algebra of $\pi_1(A, e)$ one has $\tau([\beta_1 \circ \gamma_1^{-1}], \ldots, [\beta_n \circ \gamma_n^{-1}]) \in B$ whenever $[\beta_i \circ \gamma_i^{-1}] \in B$ and therefore the mapping $\tau$ defined on $\tilde{A}_B$ is well defined. Hence we have a product preserving functor $F_B: T \to \text{Set}$ induced by $F_B(T) = \tilde{A}_B$, for the distinguished object of $T$. The axioms of Theorem 2.2 for the mappings defined in Eq. 5 are satisfied and hence $\tilde{A}_B$ becomes a semi-abelian algebra. The map $p$ is a $T$-homomorphism by the details

$$p(\tau((\beta_1), \cdots, (\beta_n))) = p(\tau(\beta_1, \cdots, \beta_n))$$

(by Eq.5)

$$= (\tau(\beta_1, \cdots, \beta_n))(1)$$

$$= \tau(\beta_1(1), \cdots, \beta_n(1))$$

(by Eq.1)

$$= \tau(p((\beta_1), \cdots, p((\beta_n))))$$

for $(\beta_1), \cdots, (\beta_n) \in \tilde{A}_B$.

To prove that $\tilde{A}_B$ is a topological $T$-algebra, we now prove that each $n$-ary mapping $\tau$ of $\tilde{A}_B$ defined by Eq.5 is continuous. Let $\beta = (\beta_1, \ldots, \beta_n)$ and let $(V, (\tau(\beta)))$ be a base open neighbourhood of $(\tau(\beta))$. Then $V$ is an open neighbourhood of

$$\tau(\beta(1)) = \tau(\beta_1(1), \ldots, \beta_n(1)).$$

and since the mapping $\tau$ on $A$ is continuous, there are respectively open neighbourhoods $U_1, \ldots, U_n$ of $\beta_1(1), \ldots, \beta_n(1)$ such that

$$\tau(U_1 \times \cdots \times U_n) \subseteq V.$$ 

Setting $U = U_1 \times \cdots \times U_n$, one obtains $\tau(U, (\beta)) \subseteq (V, (\tau(\beta)))$ which concludes that $n$-ary mapping $\tau'$ on $\tilde{A}_B$ is continuous (see the proof of [17, Lemma 10.31]). \qed
4 Covering morphisms of internal groupoids in semi-abelian categories

Let $\mathbb{T}$ be a semi-abelian theory. In this section we define internal groupoid in the semi-abelian category $\mathbf{Set}^\mathbb{T}$ of $\mathbb{T}$-algebras and extend some results of \cite{1} about the coverings of internal groupoids in the groups with operations to the internal groupoids in the category $\mathbf{Set}^\mathbb{T}$. To define an internal groupoid in the semi-abelian category $\mathbf{Set}^\mathbb{T}$ of $\mathbb{T}$-algebras we comply with the notations for groupoids given in Section 2.

Similar to the notion of an internal category in the category of groups with operations as defined in \cite{16}, the definition of internal groupoid in the category $\mathbf{Set}^\mathbb{T}$ of models for a semi-abelian theory $\mathbb{T}$ is given as follows.

Definition 4.1. Let $\mathbb{T}$ be a semi-abelian theory. An internal groupoid in $\mathbf{Set}^\mathbb{T}$ is a groupoid $G$ in which the set $\text{Ob}(G)$ of objects and the set $G$ of arrows are both $\mathbb{T}$-algebras; and the source and target point maps $s, t: G \rightarrow \text{Ob}(G)$, the object inclusion map $\epsilon: \text{Ob}(G) \rightarrow G$, the partial composite $\circ: G \times_s G \rightarrow G, (g, h) \mapsto g \circ h$ and the inversion $G \rightarrow G, g \mapsto g^{-1}$ are all $\mathbb{T}$-homomorphisms.

Note that the partial composite `$\circ$' is a $\mathbb{T}$-homomorphism if and only if for every $n$-ary mapping $\tau$ the interchange rule
\begin{equation}
\tau(g_1 \circ h_1, \ldots, g_n \circ h_n) = \tau(g_1, \ldots, g_n) \circ \tau(h_1, \ldots, h_n) \tag{6}
\end{equation}
is satisfied for $g_1, \ldots, g_n \in G$ and $h_1, \ldots, h_n \in G$ whenever one side composition is defined. For the category of internal groupoids in $\mathbf{Set}^\mathbb{T}$ we use the notation $\text{Gpd}(\mathbf{Set}^\mathbb{T})$.

In particular if $\mathbb{T}$ is the group theory, then $\mathbf{Set}^\mathbb{T}$ becomes the category of groups and hence an internal groupoid in $\mathbf{Set}^\mathbb{T}$ becomes a group-groupoid which is also called in literature as group objects or 2-group.

Remark 4.2. Let $G$ be an internal groupoid in $\mathbf{Set}^\mathbb{T}$ for a semi-abelian theory $\mathbb{T}$. Then we have the following:

1. Since the inversion $G \rightarrow G, g \mapsto g^{-1}$ is a $\mathbb{T}$-homomorphism for any $n$-ary mapping $\tau$ and $g_1, \ldots, g_n \in G$ we have
\begin{equation}
(\tau(g_1, \ldots, g_n))^{-1} = \tau(g_1^{-1}, \ldots, g_n^{-1}) \tag{7}
\end{equation}
2. Since the object inclusion map $\epsilon: \text{Ob}(G) \rightarrow G, x \mapsto 1_x$ is a $\mathbb{T}$-homomorphism, the identity arrow $1_e$ at the constant $e \in \text{Ob}(G)$ is the constant arrow of $G$ and

$1_{\tau(x_1, \ldots, x_n)} = \tau(1_{x_1}, \ldots, 1_{x_n})$.

Lemma 4.3. For a semi-abelian theory $\mathbb{T}$, if $G$ is an internal groupoid in $\mathbf{Set}^\mathbb{T}$ with unique constant $e \in \text{Ob}(G)$, then we have the following:

1. $\text{St}_{Ge}$ is a sub $\mathbb{T}$-algebra of $G$. 

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2. \( G(e) \), the object group at \( e \in \text{Ob}(G) \), is a sub \( \mathbb{T} \)-algebra of \( G \).

**Proof:**

1. Let \( G \) be an internal groupoid in \( \text{Set}^{\mathbb{T}} \). Since the source point map \( s: G \to \text{Ob}(G) \) is a \( \mathbb{T} \)-homomorphism, by Lemma 2.4 for \( g_1, \ldots, g_n \in \text{St}_{G}e \) we have

\[
\begin{align*}
s(\tau(g_1, \ldots, g_n)) &= \tau(s(g_1), \ldots, s(g_n)) \\
&= \tau(e, \ldots, e) \\
&= e
\end{align*}
\]

and therefore \( \tau(g_1, \ldots, g_n) \in \text{St}_{G}e \) when \( g_1, \ldots, g_n \in \text{St}_{G}e \). Hence \( \text{St}_{G}e \) becomes a sub \( \mathbb{T} \)-algebra of \( G \) with unique constant 1.

2. In addition to the proof of (1) we need to prove that similar axiom are satisfied for the target point map \( t: G \to \text{Ob}(G) \). Since \( t \) is a \( \mathbb{T} \)-homomorphism by Lemma 2.4 for \( g_1, \ldots, g_n \in G(e) \) we have

\[
\begin{align*}
t(\tau(g_1, \ldots, g_n)) &= \tau(t(g_1), \ldots, t(g_n)) \\
&= \tau(e, \ldots, e) \\
&= e
\end{align*}
\]

and therefore \( \tau(g_1, \ldots, g_n) \in G(e) \) when \( g_1, \ldots, g_n \in G(e) \). Hence \( G(e) \) becomes a sub \( \mathbb{T} \)-algebra of \( G \).

\( \square \)

**Example 4.4.** Let \( \mathbb{T} \) be a semi-abelian theory and let \( A \) be a \( \mathbb{T} \)-algebra. Then the groupoid \( G = A \times A \) with object set \( A \) such that a pair \((a, b)\) is an arrow from \( a \) to \( b \) with inverse arrow \((b, a)\) and the composition is defined by \((a, b) \circ (b, c) = (a, c)\), becomes an internal groupoid in \( \text{Set}^{\mathbb{T}} \).

Here an \( n \)-ary mapping \( \tau \) on \( G \) is defined by

\[
\tau(\{(a_1, b_1), \ldots, (a_n, b_n)\}) = (\tau(a_1, \ldots, a_n), \tau(b_1, \ldots, b_n))
\]

One can check that for \( g_i = (a_i, b_i) \), \( h_i = (b_i, a_i) \) (1 \( \leq \) \( i \leq \) \( n \)) the following interchange rule holds

\[
\tau(g_1 \circ h_1, \ldots, g_n \circ h_n) = \tau(g_1, \ldots, g_n) \circ \tau(h_1, \ldots, h_n)
\]

The following result enables us to produce more examples of the internal groupoids in \( \mathbb{T} \)-algebras.

**Theorem 4.5.** Let \( \mathbb{T} \) be a semi-abelian theory. If \( A \) is a topological \( \mathbb{T} \)-algebra, then the fundamental groupoid \( \pi A \) becomes an internal groupoid in the semi-abelian category \( \text{Set}^{\mathbb{T}} \) of \( \mathbb{T} \)-algebras.
Proof: Let $A$ be a topological $\mathbb{T}$-algebra for a semi-abelian theory $\mathbb{T}$. As similar to Eq\[\ref{eq:4.3}\] one can define $n$-ary mappings for the fundamental groupoid $\pi A$. Since $A$ can be represented by a product preserving functor $F_A: \mathbb{T} \to \text{Top}$ such that $F_A(T) = A$, for the distinguish object $T$ of the theory $\mathbb{T}$ we have a functor $\pi F_A: \mathbb{T} \to \text{Set}$ induced by $(\pi F_A)(T) = \pi F_A(T) = \pi A$. If $\tau: \mathbb{T}^n \to \mathbb{T}^m$ is an operation of the theory $\mathbb{T}$, then we have $(\pi F_A)(\tau): (\pi A)^n \to (\pi A)^m$, a mapping. Hence $\pi F_A: \mathbb{T} \to \text{Set}$ becomes a product preserving functor. We can prove that the axioms of Theorem \[\ref{thm:2.2}\] are satisfied as similar to the proof of Theorem \[\ref{thm:3.1}\]. Therefore $\pi A$, as the set of arrows is a $\mathbb{T}$-algebra. The interchange rule

$$\tau([\beta_1] \circ [\gamma_1], \ldots, [\beta_n] \circ [\gamma_n]) = \tau([\beta_1], \ldots, [\beta_n] \circ [\gamma_1], \ldots, [\gamma_n])$$

for $\pi A$ can be checked by the following evaluating the concatenation of the paths:

$$\tau([\beta_1] \circ [\gamma_1], \ldots, [\beta_n] \circ [\gamma_n]) = \tau([\beta_1 \circ [\gamma_1], \ldots, [\beta_n \circ [\gamma_n]])$$

(by Eq\[\ref{eq:4.3}\])

$$= \tau([\beta_1 \circ [\gamma_1], \ldots, [\beta_n \circ [\gamma_n]])$$

(by Eq\[\ref{eq:3}\])

$$= \tau([\beta_1, \ldots, [\beta_n] \circ [\gamma_1], \ldots, [\gamma_n])$$

(by Eq\[\ref{eq:4}\])

The other details to complete the proof are straightforward.

As a result of Theorem \[\ref{thm:4.5}\] for a semi-abelian theory $\mathbb{T}$, we have a functor $\pi: \text{Top}^\mathbb{T} \to \text{Gpd}(\text{Set}^\mathbb{T})$ assigning each topological $\mathbb{T}$-algebra $A$ to the internal groupoid $\pi A$ in $\text{Set}^\mathbb{T}$.

Lemma 4.6. Let $A$ be a topological $\mathbb{T}$-algebra for a semi-abelian theory $\mathbb{T}$. If $B$ is a normal sub $\mathbb{T}$-algebra of $A$, then $\pi(B)$ becomes a normal sub $\mathbb{T}$-algebra of $\pi(A)$.

Proof: The proof can be done as similar to the proof of Lemma \[\ref{lem:3.3}\].

Proposition 4.7. For a semi-abelian theory $\mathbb{T}$; if $A$ and $B$ are topological $\mathbb{T}$-algebras, then $\pi(A \times B)$ and $\pi A \times \pi B$ are isomorphic as internal groupoids in $\text{Set}^\mathbb{T}$.

Proof: By \[\cite{7}\, 6.4.4\] we know that the map $f: \pi(A \times B) \to \pi A \times \pi B$ defined by $f(\beta) = ([\beta_A], [\beta_B])$ for a homotopy class $\beta \in \pi(A \times B)$ is an isomorphism of the underlying groupoids, where $\beta_A$ and $\beta_B$ are the projections of the path $\beta$ on $A$ and $B$ respectively.

Replacing $A$ with $A \times B$ in Theorem \[\ref{thm:4.5}\] we have that $\pi(A \times B)$ is an internal groupoid, where $n$-ary mappings of $\pi(A \times B)$ are defined by

$$\tau([\beta_1], \ldots, [\beta_n]) = [\tau(\beta_1, \ldots, \beta_n)]$$

$$= [\tau((\beta_1A, \beta_1B), \ldots, (\beta_nA, \beta_nB)))]$$

$$= [(\tau(\beta_1A, \ldots, \beta_nA), \tau(\beta_1B, \ldots, \beta_nB))]$$

for $[\beta_1], \ldots, [\beta_n] \in \pi(A \times B)$. Further we now check that $f$ is a morphism of the internal groupoids in $\text{Set}^\mathbb{T}$.

$$f(\tau([\beta_1], \ldots, [\beta_n])) = ([\tau(\beta_1A, \ldots, \beta_nA)], [\tau(\beta_1B, \ldots, \beta_nB))]$$
On the other hand
\[
\tau(f[\beta_1], \ldots, f[\beta_n]) = \tau(\beta_1, \ldots, \beta_n) = (\tau(\beta_1, \ldots, \beta_n), \tau(\beta_1, \ldots, \beta_n))
\]
and therefore \( f \) is a morphism of the internal groupoids in \( \text{Set}^\mathbb{T} \).

Let \( G \) be an internal groupoid in a certain category of groups with operations and \( X \) a group with operations. The action of \( G \) on \( X \) is defined in [1, Definition 3.11]. This definition is generalized to the internal groupoids in the category \( \text{Set}^\mathbb{T} \) of semi-abelian \( \mathbb{T} \)-algebras as follows:

**Definition 4.8.** Let \( \mathbb{T} \) be a semi-abelian theory and \( G \) an internal groupoid in the category \( \text{Set}^\mathbb{T} \) of semi-abelian \( \mathbb{T} \)-algebras. Let \( A \) be a \( \mathbb{T} \)-algebra and \( \omega: A \to \text{Ob}(G) \) a \( \mathbb{T} \)-homomorphism. If the underlying groupoid \( G \) acts on the underlying set of \( A \) via \( \omega \) such that \( \varphi: A_\omega \times_s G \to A, (a, g) \mapsto ag \) is also a \( \mathbb{T} \)-homomorphism, then we say that the internal groupoid \( G \) acts on \( \mathbb{T} \)-algebra \( A \) via \( \omega \).

We write \((A, \omega, \varphi)\) for such an action. Here note that \( \varphi: A_\omega \times_s G \to A, (a, g) \mapsto ag \) is a \( \mathbb{T} \)-homomorphism in \( \text{Set}^\mathbb{T} \) if and only if for \( a_1, \ldots, a_n \in A \) and \( g_1, \ldots, g_n \in G \)

\[
\tau(a_1, \ldots, a_n) \tau(g_1, \ldots, g_n) = \tau(a_1g_1, \ldots, a_ng_n)
\]
whenever one side is defined.

**Example 4.9.** Let \( \mathbb{T} \) be a semi-abelian theory and \( p: H \to G \) a morphism of internal groupoids in the semi-abelian category \( \text{Set}^\mathbb{T} \) of \( \mathbb{T} \)-algebras such that \( p \) is a covering morphism on the underlying groupoids. Then the internal groupoid \( G \) acts on the \( \mathbb{T} \)-algebra \( A = \text{Ob}(H) \) via \( \text{Ob}(p): A \to \text{Ob}(G) \) assigning to \( a \in A \) and \( g \in \text{St}_{GP}(a) \) the target of the unique lifting \( \tilde{g} \) of \( g \) in \( H \) with source \( a \). Clearly the underlying groupoid of \( G \) acts on the underlying set and by evaluating the uniqueness of the lifting, the condition Eq[9] is satisfied for \( a_1, \ldots, a_n \in A \) and \( g_1, \ldots, g_n \in G \) whenever one side is defined.

The Characterization of Theorem 2.1 for semi-abelian theory is as follows:

**Theorem 4.10.** Let \( \mathbb{T} \) be a semi-abelian theory with unique constant \( e \) and \( G \) an internal groupoid in \( \text{Set}^\mathbb{T} \) such that the underlying groupoid is transitive. Let \( G(e) \) be the object group at \( e \in \text{Ob}(G) \) and \( C \) a subgroup and a sub \( \mathbb{T} \)-algebra of \( G(e) \). Then the set \( AC \) of cosets \( C \circ g = \{c \circ g \mid c \in C\} \) for \( g \) in \( \text{St}_{GC} \) becomes a \( \mathbb{T} \)-algebra and the internal groupoid \( G \) acts on \( AC \) by \((C \circ a \circ g) = C \circ a \circ g \).
Hence the axioms of Theorem 2.2 are satisfied. Therefore

\[ \tau(C \circ g_1, \ldots, C \circ g_n) = C \circ \tau(g_1, \ldots, g_n) \]  

(by Eq.10)

Here note that by Lemma 4.3 (1), \( \tau(g_1, \ldots, g_n) \in \text{St}_{G,e} \) whenever \( g_1, \ldots, g_n \in \text{St}_{G,e} \). We now prove that the \( n \)-ary mappings \( \tau \) are well defined. Let \( C \circ g_1 = C \circ h_1, \ldots, \) and \( C \circ g_n = C \circ h_n \). Since \( C \) is a subgroup of \( G(e) \) we have \( g_1 \circ h_1^{-1} \in C, \ldots, g_n \circ h_n^{-1} \in C \) and therefore

\[ \tau(g_1, \ldots, g_n) \circ (\tau(h_1, \ldots, h_n))^{-1} = \tau(g_1, \ldots, g_n) \circ (\tau(h_1^{-1}, \ldots, h_n^{-1}) \]  

(by Eq.4)

\[ = \tau(g_1 \circ h_1^{-1}, \ldots, g_n \circ h_n^{-1}) \]  

(by Eq.5)

Since \( C \) is a sub \( T \)-algebra of \( G(e) \) we have that \( \tau(g_1 \circ h_1^{-1}, \ldots, g_n \circ h_n^{-1}) \in C \) and so \( \tau(g_1, \ldots, g_n) \circ (\tau(h_1, \ldots, h_n))^{-1} \in C \). Hence \( C \circ \tau(g_1, \ldots, g_n) = C \circ \tau(h_1, \ldots, h_n) \) and the \( n \)-ary mappings \( \tau \) defined by Eq.10 are well defined. Hence we have a functor \( F_C : T \rightarrow \text{Set} \) defined by \( F_C(T) = A_C \). Moreover for 2-ary mappings \( \alpha_i \) by Eq.10 we have

\[ \alpha_i(C \circ g, C \circ g) = C \circ \alpha_i(g, g) = C \circ e = C \]

By the Eq.10 we have the following evaluation.

\[ \theta(\alpha_1(C \circ g, C \circ h), \ldots, \alpha_n(C \circ g, C \circ h), C \circ h) = \theta(C \circ \alpha_1(g, h), \ldots, C \circ \alpha_n(g, h), C \circ h) \]

\[ = C \circ \theta(\alpha_1(g, h), \ldots, \alpha_n(g, h), h) \]

\[ = C \circ g \]  

(by Theorem 2.2 (3))

Hence the axioms of Theorem 2.2 are satisfied. Therefore \( A_C \) becomes a semi-abelian algebra. Here the underlying groupoid of \( G \) acts on the set \( A = A_C \) by

\[ A \times_s G \rightarrow A_i(C \circ g, h) \mapsto (C \circ g)h = C \circ g \circ h \]  

(11)

via the map \( \omega : A_C \rightarrow \text{Ob}(G), C \circ g \mapsto t(g) \).

For \( a_i = C \circ g_i \in A \) \((i = 1, \ldots, n)\) the following evaluations prove that the interchange rule Eq.9 is satisfied:

\[ \tau(a_1, \ldots, a_n) \tau(h_1, \ldots, h_n) = \tau(C \circ g_1, \ldots, C \circ g_n) \tau(h_1, \ldots, h_n) \]

\[ = (C \circ \tau(g_1, \ldots, g_n)) \tau(h_1, \ldots, h_n) \]  

(by Eq.10)

\[ = C \circ \tau(g_1, \ldots, g_n) \circ \tau(h_1, \ldots, h_n) \]  

(by Eq.11)

\[ = C \circ \tau(g_1 \circ h_1, \ldots, g_n \circ h_n) \]  

(by Eq.6)

and

\[ \tau(a_1 h_1, \ldots, a_n h_n) = \tau((C \circ g_1) h_1, \ldots, (C \circ g_n) h_n) \]

\[ = \tau(C \circ g_1 h_1, \ldots, (C \circ g_n) h_n) \]  

(by Eq.11)

\[ = C \circ \tau(g_1 h_1, \ldots, g_n h_n) \]  

(by Eq.10)
Hence $\tau(a_1, \ldots, a_n)\tau(h_1, \ldots, h_n) = \tau(a_1h_1, \ldots, a_nh_n)$ and therefore the interchange rule in Eq.9 is satisfied. □

We now give a concrete example to Theorem 4.10.

Example 4.11. Let $T$ be a semi-abelian algebra with a constant $e$ and $A$ a topological $T$-algebra. Then by Theorem 4.10, $G = \pi A$ becomes an internal groupoid in $\text{Set}^T$ and for a sub-$T$-algebra $B$ of $A$, the fundamental group $C = \pi_1(B, e)$ becomes a sub-$T$-algebra of $G(e) = \pi_1(A, e)$. Hence by Theorem 4.10, the set $AC$ of all cosets $\{C o g \mid g \in St_{\pi A} e\}$ is a $T$-algebra and the internal groupoid $\pi A$ acts on $AC$.

**Theorem 4.12.** Let $T$ be a semi-abelian theory. Let $G$ be an internal groupoid in $\text{Set}^T$ and $A$ a $T$-algebra. Suppose that $G$ acts on the $T$-algebra $A$ via a $T$-homomorphism $\omega: A \to Ob(G)$. Then the semi-direct product groupoid $G \rtimes A$ becomes an internal groupoid in $\text{Set}^T$ such that the projection $p: G \rtimes A \to G$ defined on objects by $\omega$ and on arrows by $(g, a) \mapsto g$ is a morphism of internal groupoids which is a covering morphism on the underlying groupoids.

**Proof:** By [7, 10.42] we know that the projection map $p: G \rtimes A \to G$ is a covering morphism of groupoids. Then the semi-direct product groupoid $H = G \rtimes A$ becomes a $T$-algebra by the $n$-ary mappings defined by

$$\tau((g_1, a_1), \ldots, (g_n, a_n)) = (\tau(g_1, \ldots, g_n), \tau(a_1, \ldots, a_n)) \quad (12)$$

In addition the source and target point maps $s, t: H \to A$, the object inclusion map $\epsilon: A \to H$ and the partial composition $\circ: H \times_s H \to H, (h, k) \mapsto h \circ k$ are $T$-homomorphisms. Hence $G \rtimes A$ becomes an internal groupoid in $\text{Set}^T$. Moreover for given arrows $(g_1, a_1), \ldots, (g_n, a_n)$ of $G \rtimes A$ and any $n$-ary mapping $\tau$ by the following evaluation, $p$ is a $T$-homomorphism.

$$p(\tau((g_1, a_1), \ldots, (g_n, a_n))) = p(\tau(g_1, \ldots, g_n), \tau(a_1, \ldots, a_n))$$

$$= \tau(g_1, \ldots, g_n)$$

$$= \tau(p(g_1, a_1), \ldots, p(g_n, a_n))$$

$\square$

Let $G$ be an internal groupoid in $\text{Set}^T$ for a semi-abelian theory $T$. Then we have a category $\text{Act}_{Gpd(\text{Set}^T)}/G$ whose objects are actions $(A, \omega, \varphi)$ of the internal groupoid $G$ on $T$-algebras and morphisms, say from $(A, \omega, \varphi)$ to $(A', \omega', \varphi')$ are $T$-homomorphisms $f: A \to A'$ such that $\omega = \omega'f$ and $f(ag) = (fa)g$ whenever $ag$ is defined.

Let $\text{Cov}_{Gpd(\text{Set}^T)}/G$ be the category whose objects are the morphisms $p: H \to G$ of internal groupoids in $\text{Set}^T$ such that $p$ is a covering morphism of groupoids and arrows are commutative diagrams of morphisms of internal groupoids
where $p$ and $q$ are covering morphisms on the underlying groupoids. Similarly writing such a diagram as a triple $(f; p, q)$, the composition of the arrows in $\text{Cov}_{\text{Gpd}(\text{Set}^T)}/G$ is defined by $(f; p, q) \circ (g; q, r) = (fg; p, q)$.

We can now prove the equivalence of these categories as follows.

**Theorem 4.13.** Let $T$ be a semi-abelian theory and $G$ an internal groupoid in $\text{Set}^T$. Then the categories $\text{Act}_{\text{Gpd}(\text{Set}^T)}/G$ and $\text{Cov}_{\text{Gpd}(\text{Set}^T)}/G$ are equivalent.

**Proof:** If $(A, \omega, \phi)$ is an object of $\text{Act}_{\text{Gpd}(\text{Set}^T)}/G$, then by Theorem 4.12, we have a morphism $p: G \ltimes A \rightarrow G$ of internal groupoids in $\text{Set}^T$, which is a covering morphism on the underlying groupoids. This gives us a functor $\Gamma: \text{Act}_{\text{Gpd}(\text{Set}^T)}/G \rightarrow \text{Cov}_{\text{Gpd}(\text{Set}^T)}/G$.

Conversely if $p: H \rightarrow G$ is a morphism of internal groupoids in $\text{Set}^T$ which is a covering morphism on the underlying groupoids, then by Example 4.9 we have an action of the internal groupoid $G$ on the $T$-algebra $A = \text{Ob}(H)$ via $p: A \rightarrow \text{Ob}(H)$. In this way we define a functor $\Phi: \text{Cov}_{\text{Gpd}(\text{Set}^T)}/G \rightarrow \text{Act}_{\text{Gpd}(\text{Set}^T)}/G$.

The natural equivalences $\Gamma \Phi \simeq 1$ and $\Phi \Gamma \simeq 1$ follow. $\square$

**Definition 4.14.** Let $T$ be a semi-abelian theory, $H$ a groupoid and $G$ an internal groupoid in $\text{Set}^T$ with constant $e \in \text{Ob}(G)$. Suppose that $p: H \rightarrow G$ is a covering morphism of groupoids and $e' \in \text{Ob}(H)$. We say that $T$-algebraic structure of $G$ lifts to $H$ if $H$ becomes an internal groupoid in $\text{Set}^T$ such that $p$ is a morphism of internal groupoids.

Using Theorem 4.10 and Theorem 4.12 we now give a criterion for the $T$-algebraic structure of an internal groupoid $G$ in the semi-abelian category $\text{Set}^T$ lifts to a covering groupoid.

**Theorem 4.15.** Let $T$ be a semi-abelian theory with unique constant $e$, $H$ a groupoid and $G$ an internal groupoid in $\text{Set}^T$ whose underlying groupoid is transitive. Suppose that $p: H \rightarrow G$ is a covering morphism of underlying groupoids, $e' \in \text{Ob}(H)$ such that $p(e') = e$ and the characteristic group $C$ of $p$ at $e'$ is a sub $T$-algebra of $G(e)$. Then the $T$-algebraic structure of $G$ lifts to $H$.

**Proof:** Let $C$ be the characteristic group of the covering morphism $p: H \rightarrow G$ at $e' \in \text{Ob}(H)$. By Theorem 2.1 we have a covering morphism of groupoids $q: H_C \rightarrow G$ with the characteristic group $C$. Since the covering morphisms $p$ and $q$ are equivalent we can replace $p$ with $q$ and prove that the $T$-algebraic structure of $G$ lifts to $H_C = G \ltimes A_C$. By Theorem 4.10 $A_C$ becomes a $T$-algebra and $G$ acts on $A_C$; and by Theorem 4.12 $T$-algebraic structure of $G$ lifts to $H_C$ which completes the proof. $\square$
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