ABOUT BLOW UP OF SOLUTIONS WITH ARBITRARY POSITIVE INITIAL ENERGY TO NONLINEAR WAVE EQUATIONS

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Abstract. We show that blow up of solutions with arbitrary positive initial energy of the Cauchy problem for the abstract wave equation of the form $P u_{tt} + A u = F(u)$ (*) in a Hilbert space, where $P, A$ are positive linear operators and $F(\cdot)$ is a continuously differentiable gradient operator can be obtained from the result of H.A. Levine on the growth of solutions of the Cauchy problem for (*). This result is applied to the study of initial boundary value problems for nonlinear Klein-Gordon equations, generalized Boussinesq equations and nonlinear plate equations. A result on blow up of solutions with positive initial energy of the initial boundary value problem for wave equation under nonlinear boundary condition is also obtained.

1. Introduction

We consider the following problem

$$Pu_{tt} + A u = F(u), \quad (1.1)$$

$$u(0) = u_0, \quad u_t(0) = u_1 \quad (1.2)$$

in a Hilbert space $H$ with the inner product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$. We denote by $u$ a vector-function with domain $[0, T)$ and range $D$, where $D$ is a dense linear subspace of $H$. Suppose that $P, A$ are symmetric, positive definite, linear operators defined on $D$, $F(\cdot) : D \to H$ is a nonlinear gradient operator defined on $D$ with the potential $G(u) : D \to \mathbb{R}$. We assume also that

$$(F(v), v) \geq 2(1 + 2\alpha)G(v) - 2R_0, \quad \forall v \in D \quad (1.3)$$

for some $\alpha > 0, R_0 \geq 0$.

For the sake of simplicity it is assumed that $u(t)$ is a strong solution of (1.1), i.e. a solution $u$ for which all terms in (1.1) are elements of $L^2(0, T; H)$ and $u(\cdot), u_t(\cdot) \in C(0, T; H)$.

The idea of the concavity method of H. A. Levine introduced in [11] is based on a construction of some positive functional $\Psi(t) = \psi(u(t))$, which is defined in terms of the local solution of the problem (the local solvability of the problem is therefore required) and proving that the function $\Psi(t)$ satisfies the inequality (1.4) given in the following statement:

Lemma 1.1. (see [11]) Let $\Psi(t)$ be a positive, twice differentiable function, which satisfies the inequality

$$\Psi''(t)\Psi(t) - (1 + \alpha) [\Psi'(t)]^2 \geq 0, \quad t \geq t_0 \quad (1.4)$$

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with some $\alpha > 0$. If $\Psi(t_0) > 0$ and $\Psi'(t_0) > 0$, then there exists a time $t_1 \leq t_0 + \frac{\Psi(0)}{\alpha \Psi'(0)}$ such that $\Psi(t) \to +\infty$ as $t \to t_1$.

The concavity method and its modifications was used in the study of various nonlinear partial differential equations (see e.g. [1], [4], [6], [12], [13], [23], [20], [21]).

There is a number of papers devoted to the question of blow up of solutions to the Cauchy problem and initial boundary value problems for nonlinear wave equations with arbitrary large initial energy.

One of the first results of this type is the result of H. Levine and G. Todorova [15]. The concavity method and its modifications is employed to find sufficient conditions of blow up of solutions to the Cauchy problem and initial boundary value problems for nonlinear Klein-Gordon equation, damped Kirchhoff-type equation, generalized Boussinesq equation, quasilinear strongly damped wave equations and some other equations (see, e.g.[2], [5]–[10], [16], [17], [19], [22], [24] and references therein).

Our aim is to show that blow up of solutions with arbitrary positive initial energy of the problem (1.1) actually can be established by using the Lemma 1.1 and the following theorem on growth of solutions of the problem obtained in [11].

**Theorem 1.2.** (11) Suppose that the $P, A : D \to H$ are positive symmetric operators, $F(\cdot) : D \to H$ satisfies the condition (1.3) and $u$ is a solution of the problem (1.1), (1.2). Suppose that the initial data satisfy the conditions

$$ (u_0, Pu_1)/(u_0, Pu_0) > 0, \quad (1.5) $$

$$ \frac{1}{2}(u_0, Au_0) + \frac{1}{2}(Pu_1, u_1) - G(u_0) + \frac{R_0}{(1 + 2\alpha)} < \frac{1}{2}(u_0, Pu_1)^2/(u_0, Pu_0). \quad (1.6) $$

Then

$$ \lim_{t \to +\infty} (u(t), Pu(t)) = +\infty, $$

if $u(\cdot)$ exists on $(0, +\infty)$.

2. Blow Up of Solutions to Abstract Wave Equations.

In this section we find sufficient conditions for finite-time blow up of solutions to the problem (1.1), (1.2) when the initial energy may take arbitrary positive values.

**Theorem 2.1.** Suppose that the operators $P, A$ and $F$ satisfy all the conditions of Theorem 1.2 and suppose that there exists $a_0 > 0$ such that

$$(Av, v) \geq a_0(Pv, v), \quad \forall v \in D. \quad (2.1)$$

Then there exists $t_1 > 0$ such that

$$ \lim_{t \to t_1^-} (Pu(t), u(t)) = +\infty. \quad (2.2) $$

**Proof.** Assume that all solutions of the problem (1.1), (1.2) are global solutions, i.e. they are defined for all $t \in (0, +\infty)$. Thanks to the Theorem 1.2 if $u$ is a solution of the problem (1.1), (1.2), then

$$ \Psi(t) := (Pu(t), u(t)) \to +\infty \quad as \quad t \to \infty. \quad (2.3) $$
On the other hand
\[ \Psi'(t) = 2(Pu_t(t), u(t)), \]
\[ \Psi''(t) = 2(Pu_t(t), u_t(t)) + 2(Pu_{tt}(t), u(t)). \]

Employing the equation (1.1) and the condition (1.3) we get
\[ \Psi''(t) = 2(Pu_t(t), u_t(t)) - 2(Au(t), u(t)) + 2(F(u(t), u(t))) \geq 2(Pu_t(t), u_t(t)) - 2(Au(t), u(t)) + 4(1 + 2\alpha)G(u(t) - 4R_0. \]

As usual, we define the energy as
\[ E(t) := \frac{1}{2}(Pu_t(t), u(t)) + \frac{1}{2}(Au(t), u(t)) - G(u(t)), \]
and find that
\[ E(t) = E(0) = \frac{1}{2}(u_0, Au_0) + \frac{1}{2}(Pu_1, u_1) - G(u_0), \quad t > 0. \]

By using the energy equality (2.5) we obtain from the last inequality that
\[ \Psi''(t) \geq 4(1 + 2\alpha) \left[ -\frac{1}{2}(Pu_t(t), u_t(t)) - \frac{1}{2}(Au(t), u(t)) + G(u(t)) \right] + 4(\alpha + 1)(Pu_t(t), u_t(t)) + 4\alpha(Au(t), u(t)) - 4R_0 \]
\[ \geq -4(1 + 2\alpha)E(0) - 4R_0 + 4(\alpha + 1)(Pu_t(t), u_t(t)) + 4\alpha(Au(t), u(t)). \quad (2.6) \]

Thus, by using the Cauchy - Schwarz inequality and the condition (2.1) we obtain
\[ \Psi''(t)\Psi(t) - (1 + \alpha)[\Psi'(t)]^2 \geq -4(1 + 2\alpha)E(0) - 4R_0 \Psi(t) + 4\alpha a_0 \Psi^2(t) \]
\[ + 4(\alpha + 1) [(Pu_t(t), u_t(t))(Pu(t), u(t)) - (Pu_t(t), u(t))^2] \geq [4\alpha a_0 \Psi(t) - 4(1 + 2\alpha)E(0) - 4R_0 \Psi(t). \quad (2.7) \]

Thanks to the Theorem 1.2 the function \( \Psi(t) \) tends to \(+\infty\) as \( t \to +\infty \). Therefore, there exists \( t_* > 0 \) such that
\[ \alpha a_0 \Psi(t) - 4(1 + 2\alpha)E(0) - 4R_0 \geq \delta > 0, \quad \forall t \geq t_* \]

Hence, (2.7) implies that
\[ \Psi''(t)\Psi(t) - (1 + \alpha)[\Psi'(t)]^2 \geq 0, \quad \forall t \geq t_* \]

Moreover, in view of the assumption on \( t_* \), by using (2.1) we easily deduce from (2.6) that
\[ \Psi''(t) \geq \delta > 0 \quad \forall t \geq t_* \]

Consequently, there exists some \( t_0 \geq t_* \) such that \( \Psi'(t_0) > 0 \). Now we can apply the Lemma 1.1 and deduce that
\[ (Pu(t), u(t)) \to +\infty, \quad ast \to \ast. \]
3. Examples of Nonlinear Wave equations

1. Nonlinear Klein-Gordon Equation

Let \( u \) be a local strong solution to the Cauchy problem

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u + m^2 u &= |u|^2 u, \quad x \in \mathbb{R}^3, \; t > 0, \\
u(x, 0) &= u_0(x), \; \partial_t u(x, 0) = u_1(x),
\end{align*}
\]

where \( m > 0 \) is a given number, \( u_0 \in H^1(\mathbb{R}^n) \), \( u_1 \in L^2(\mathbb{R}^n) \) are given compactly supported functions.

The equation can be written in the form \((1.1)\) with \( P = I \), \( A = -\Delta + m^2 I \) and \( F(u) = |u|^2 u \). It follows from Theorem 2.1 that if \( (u_0, u_1) > \left[ \|u_1\|^2 + \|\nabla u_0\|^2 + m^2 \|u_0\|^2 - \frac{1}{2} \int_\mathbb{R} |u_0(x)|^4 dx \right]^{\frac{1}{2}} \|u_0\| \)

then the solution of the problem \((3.1)\) blows up in a finite time. If \( u_0 \) is a smooth nonnegative, nontrivial compactly supported function then for \( u_1 = \frac{1}{\sqrt{2}} u_0^2 \) the initial energy is

\[
E(0) = \frac{1}{2} \|\nabla u_0\|^2 + \frac{m^2}{2} \|u_0\|^2
\]

and the condition \((3.2)\) takes the form

\[
\int_\mathbb{R} |u_0(x)|^3 > \sqrt{2} \left[ \|\nabla u_0\|^2 + m^2 \|u_0\|^2 \right]^{\frac{3}{2}} \|u_0\|.
\]

It is clear that there is a wide class of functions \( u_0 \) for which the energy takes any large value and the condition \((3.3)\) holds true.

Remark 3.1. The Theorem \((2.1)\) holds true also for solutions of the initial boundary value problem for the nonlinear wave equation under the homogeneous Dirichlet boundary condition:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u + m^2 u &= |u|^p u, \quad x \in \Omega, \; t > 0, \\
u(x, 0) &= u_0(x), \; \partial_t u(x, 0) = u_1(x), \quad x \in \Omega, \\
u(x, t) &= 0, \quad x \in \partial \Omega, \; t > 0,
\end{align*}
\]

where \( p \) is an arbitrary positive number if \( n = 1, 2 \) and \( p \in (0, \frac{2}{n-2}) \) if \( n \geq 3 \).

Let us note that this result easily follows from the results of T. Cazenave obtained in \[3\] for solutions of the problem \((3.4)\) and the Theorem \([1,2]\) of H. A. Levine.

Indeed, T. Cazenave proved that each solution of the problem \((3.4)\) either blows up in a finite time or is uniformly bounded.

Thus, if the functions \( u_0, u_1 \) satisfy the conditions of Theorem \([1,2]\) that is

\[
(u_0, u_1) > \left[ \|u_1\|^2 + \|\nabla u_0\|^2 + m^2 \|u_0\|^2 - \frac{2}{p + 2} \int_\mathbb{R} |u_0(x)|^{p+2} dx \right]^{\frac{1}{2}} \|u_0\|
\]

then the corresponding local solution of the problem \((3.4)\) can not be continued on the whole interval \([0, \infty)\), i.e. it must blow up in a finite time.
Example 3.2. Generalized Boussinesq Equation

Similarly we can find sufficient conditions for blow up of solutions with arbitrary positive initial energy for the generalized Boussinesq equation

$$\partial^2_t u - a \Delta u_{tt} - \Delta u + \nu \Delta^2 u + \Delta f(u) = 0, \ x \in \Omega, t > 0$$

(3.5)

under the homogeneous Dirichlet boundary conditions

$$u = \Delta u = 0, \ x \in \partial \Omega, t > 0,$$

where $f(u) = |u|^m u + P_{m-1}(u)$, $m \geq 1$ is a given integer, $a \geq 0$, $\nu > 0$ are given numbers, $\Omega \in \mathbb{R}^n$ is a bounded domain and $P_{m-1}(u)$ is a polynomial of order $\leq m - 1$. Applying $(-\Delta)^{-1}$ to (3.5), where $-\Delta$ is the Laplace operator under Dirichlet boundary conditions, we obtain an equation of the form (1.1) with $P = (-\Delta)^{-1} + aI$, $A = I - \nu \Delta$ and $F$ replaced by $f$. It is easy to see that there is $R_0 \geq 0$ such that $f$ satisfies (1.3) with $G(u) = \int_\Omega \int_0^u f(s) ds dx$ and $\alpha = \frac{m}{4}$. We consider initial data $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$. Since $\Omega$ is bounded, the Poincare inequality assures that the assumption (2.1) is verified. Hence, the conclusion of Theorem 2.1 holds provided that the assumptions (1.2) are fullfilled, that is $u_0, u_1$ satisfy

$$(Pu_0, u_1) > 0,$$

$$\frac{1}{2} (Pu_0, u_1)^2 > E(u_0, u_1) + \frac{R_0}{1 + \frac{m}{2}},$$

where

$$E(u_0, u_1) = \frac{1}{2} (Pu_1, u_1) + \frac{1}{2} (Au_0, u_0) - \int_\Omega \int_0^{u_0} f(s) ds dx.$$

Let us prove that for any given number $K^2 > 0$ and any pair of functions $[\hat{u}_0, \hat{u}_1] \in H^1_0(\Omega) \times L^2(\Omega)$ with

$$(Pu_0, \hat{u}_1) = \theta > 0, (Pu_0, \hat{u}_0) = (Pu_1, \hat{u}_1) = 1,$$

there are uncountably infinitely many data of the form $u_i = c_i \hat{u}_i$, $c_i > 0$, $i = 0, 1$, such that $E(u_0, u_1) = K^2$ and the above conditions are satisfied. Here, note that necessarily $0 < \theta \leq 1$. Observe that in this case it is enough to verify only the latter of the two conditions above. Rewriting this condition for the initial data of the form described above we find

$$\frac{1}{2} c_1^2 \theta^2 > E(c_0 \hat{u}_0, c_1 \hat{u}_1) + \frac{2R_0}{m + 2}.$$

Thus, the question is, given $K^2 > 0$, can we find $c_0, c_1 > 0$ so that this inequality is satisfied together with the equality $E(c_0 \hat{u}_0, c_1 \hat{u}_1) = K^2$. So, let $K^2 > 0$ be given and fix any $c_1 > \theta^{-1} \left[ 2K^2 + \frac{4R_0}{m + 2} \right]^\frac{1}{2}$. Note that, if for this fixed value of $c_1$ we can find $c_0 > 0$ such that $E(c_0 \hat{u}_0, c_1 \hat{u}_1) = K^2$, then the inequality condition above is automatically satisfied, and we are done. It is easy to see that it is possible for any such $c_1$. Indeed, since $\theta \leq 1$, we have

$$\frac{1}{2} c_1^2 > K^2,$$
and consequently the continuous function
\[ H(c_0) := E(c_0 \hat{u}_0, c_1 \hat{u}_1) - K^2 = \frac{1}{2} c_1^2 - K^2 + \frac{1}{2} c_0^2(A \hat{u}_0, \hat{u}_0) - \int_0^{c_0 \hat{u}_0} \int_{\Omega} f(s) ds dx \]
has the property
\[ \lim_{c_0 \to 0^+} H(c_0) = \frac{1}{2} c_1^2 - K^2 > 0. \]
Moreover, by the structure of \( f \) we have
\[ \lim_{c_0 \to \infty} H(c_0) = -\infty. \]
Due to the intermediate value theorem we deduce that there is \( c_0 > 0 \) such that \( H(c_0) = 0 \), and this finishes the proof.

**Example 3.3. Nonlinear Plate Equations** It is clear that we can apply Theorem 2.1 to find sufficient conditions of blow up of solutions to initial boundary value problems for the nonlinear plate equations of the form
\[ u_{tt} + \Delta^2 u + \left( a_1 + b_1 \int_\Omega u_{x_1}^2 dx \right) u_{x_1 x_1} + \left( a_2 + b_2 \int_\Omega u_{x_2}^2 dx \right) u_{x_2 x_2} = 0, \ x \in \Omega, t > 0, \]
and
\[ u_{tt} + \Delta^2 u = f(u), \ x \in \Omega, t > 0, \]
under the boundary conditions
\[ u = \Delta u = 0, \ x \in \partial \Omega, \]
where \( f(\cdot) : \mathbb{R} \to \mathbb{R} \) is a continuous function which satisfies the condition
\[ f(s) s - 2(1 + 2\alpha) F(s) \geq -r_0, \ \forall s \in \mathbb{R}, \quad (3.6) \]
\( r_0 \geq 0, a_1, a_2 \in \mathbb{R}, b_1 > 0, b_2 > 0 \) are given numbers, \( F(s) := \int_0^s f(\tau) d\tau \) and \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with sufficiently smooth boundary \( \partial \Omega \).

**Remark 3.4.** Applying Theorem 2.1 we can obtain similar results on blow up of solutions to
- initial boundary value problem for improved Boussinesq equation
  \[ u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u + \Delta^2 u_{tt} + \Delta(f(u)) = 0, \]
  where \( f(\cdot) \) is a smooth function which satisfies (3.6),
- Cauchy problem and initial boundary value for system of nonlinear Klein-Gordon equation
  \[
  \begin{cases}
  u_{tt} - \Delta u + m^2 u = uv^2 + h_1(x), \\
  v_{tt} - \Delta v + \mu^2 v = vu^2 + h_2(x),
  \end{cases}
  \]
where \( h_1, h_2 \in L^2(\mathbb{R}^3) \) are given functions.
4. Second Order Wave Equation Under Nonlinear Boundary Conditions

In this section we consider the following problem

\[ u_{tt} - \Delta u + bu = 0, \quad x \in \Omega, t > 0, \quad (4.1) \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (4.2) \]
\[ \frac{\partial u}{\partial n} = f(u), \quad x \in \partial \Omega, t > 0, \quad (4.3) \]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with sufficiently smooth boundary, \( b > 0 \) is a given number, \( \vec{n} \) denotes the outward directed normal to \( \partial \Omega \) and \( f(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) is nonlinear term that satisfies the condition (3.6).

The energy equality in this case has the form

\[ E(t) := \frac{1}{2} \|u_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2} \|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{b}{2} \|u(t)\|^2_{L^2(\Omega)} - \int_{\partial \Omega} F(u(x, t))d\sigma = E_0, \quad (4.4) \]

where

\[ E_0 = \frac{1}{2} \|u_1\|^2_{L^2(\Omega)} + \frac{1}{2} \|\nabla u_0\|^2_{L^2(\Omega)} + \frac{b}{2} \|u_0\|^2_{L^2(\Omega)} - \int_{\partial \Omega} F(u_0(x))d\sigma. \]

Set

\[ \Psi(t) = \|u(t)\|^2_{L^2(\Omega)}, \quad t \geq 0, \]

where \( u(t) \) is a solution of the problem (4.1) - (4.3). Then employing the equation (4.1), the boundary condition (4.3) and the condition (3.6) we obtain

\[ \Psi''(t) \geq 2\|u_t(t)\|^2_{L^2(\Omega)} - 2\|\nabla u(t)\|^2_{L^2(\Omega)} - 2b\|u(t)\|^2_{L^2(\Omega)} + 4(1 + 2\alpha) \int_{\partial \Omega} F(u)d\sigma - 2R_0, \]

where \( R_0 = |\partial \Omega|r_0 \). Utilizing the energy equality (4.4) from the last inequality we obtain that

\[ \Psi''(t) \geq -4(1 + 2\alpha)E_0 - 2R_0 + 4(1 + \alpha)\|u_t(t)\|^2_{L^2(\Omega)} + 4\alpha\|\nabla u(t)\|^2_{L^2(\Omega)} + 4\alpha b\|u(t)\|^2_{L^2(\Omega)}. \quad (4.5) \]

Employing (4.5), similar to the proof of the Theorem 1.2 we can show that if

\[ \frac{\langle u_0, u_1 \rangle_{L^2(\Omega)}}{\|u_0\|^2_{L^2(\Omega)}} > 2E_0 + \frac{R_0}{1 + 2\alpha} > 0, \quad (4.6) \]

then

\[ \Psi(t) = \|u(t)\|^2_{L^2(\Omega)} \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty. \quad (4.7) \]

Finally arguing as in the proof of the Theorem 2.1 we get the inequality

\[ \Psi''(t)\Psi(t) - (1 + \alpha)[\Psi'(t)]^2 \geq (4\alpha b\Psi(t) - 4(1 + 2\alpha)E_0 - 2R_0)\Psi(t). \]

Thanks to the last inequality and the Lemma 1.1 we proved the following

**Theorem 4.1.** If the conditions (4.6) are satisfied, then the interval of existence \([0, T)\) of solution to the problem (4.1)-(4.3) is finite. Moreover

\[ \|u(t)\|^2_{L^2(\Omega)} \rightarrow +\infty \quad \text{as} \quad t \rightarrow T^-. \]
Remark 4.2. Theorem 4.1 holds true also for the equation
\[ u_{tt} - \Delta u = 0, \quad x \in \Omega, \ t > 0 \]
when a nonlinear boundary condition of the form
\[ u(x,t) = 0, \quad x \in \Gamma_1, \ \frac{\partial u}{\partial n} = f(u), \quad x \in \Gamma_2, \ t > 0, \]
where \( \Gamma_1 \cup \Gamma_2 = \partial \Omega, \ mes(\Gamma_1) \neq 0 \).

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