Preschool Mathematics Intervention Can Significantly Improve Student Learning Trajectories Through Elementary School

Denis Dumas  
University of Denver

Daniel McNeish  
Arizona State University

Julie Sarama  
Douglas Clements  
University of Denver

In today’s increasingly technology- and information-based society, mathematical skills are critical to all students’ long-term economic and social success (Jang, 2016). Consequently, the mathematical development of young children is currently seen as a key indicator of our society’s readiness to meet the challenges of the future, a viewpoint that has been highlighted in scholarly research (Clements & Sarama, 2011), policy reports (Ginsburg, Lee, & Boyd, 2008), and even the popular press (National Public Radio, 2017). Unfortunately, current evidence suggests that U.S. students—especially students from traditionally underrepresented and underresourced groups—do not exhibit similar mathematical achievement as their same-age peers in other industrialized countries around the world (Mullis, Martin, Foy, & Arora, 2012). In light of this wide-reaching concern, educational researchers have worked to develop instructional interventions to improve the mathematical ability of preschool aged students, some of which (e.g., Building Blocks, Clements & Sarama, 2007; Clements, Sarama, Wolfe, & Spitler, 2013; Sarama, Clements, Wolfe, & Spitler, 2012) have demonstrated the capability to significantly improve mathematical outcomes, with particularly promising findings for groups of students that have been historically underrepresented in mathematically oriented professions (e.g., African Americans; Clements, Sarama, Spitler, Lange, & Wolfe, 2011; Schenke, Nguyen, Watts, Sarama, & Clements, 2017).

Given the demonstrated efficacy of these interventions, a further question arises: If an instructional intervention has improved a young child’s mathematical achievement in preschool, how does this impact their long-term outcomes through elementary school? In this article, we use a relatively novel statistical framework, Dynamic Measurement Modeling, that takes both intra- and interindividual student differences across time into account, to demonstrate that while students who receive a short-term intervention in preschool may not differ from a control group in terms of their long-term mathematics outcomes at the end of elementary school, they do exhibit significantly steeper growth curves as they approach their eventual skill level. In addition, this significant improvement of learning rate in elementary school benefited minority (i.e., Black or Latinx) students most, highlighting the critical societal need for research-based mathematics curricula in preschool.
preschool, what does that imply for the trajectory of their mathematical development throughout elementary school? Do those students who began elementary school ahead of their peers in math retain their advantage over the subsequent years? How does the long-term effect of an instructional intervention interact with other salient student background characteristics (e.g., sex, race/ethnicity)? The answers to these questions have wide-reaching policy ramifications, but contradictory or unclear findings within the mathematical development literature currently preclude the formulation of meaningful policy recommendations related to early childhood mathematics education (Bailey, Duncan, Watts, Clements, & Sarama, 2018; Cobb & Jackson, 2008).

That is, some studies indicate that benefits from preschool instructional intervention—in comparison to a control group that did not receive the intervention—do not persist; that is, that effect sizes “fade” (Administration for Children and Families, 2010; Leak et al., 2012; Natriello, McDill, & Pallas, 1990; Preschool Curriculum Evaluation Research Consortium, 2008; Turner, Ritter, Robertson, & Featherston, 2006). Such reports reify the treatment effect of an intervention (measured via an effect size statistic) as an entity that would ideally persist perpetually throughout student academic development. Such a perspective conceptualizes students’ intervention-related gains in comparison to their control-group peers as a static object carried by the students who would ideally continue to lift the intervention-group students’ achievement above the control group forever. However, intervention effects are, by the very definition of an intervention, exceptions to the normal developmental course for these students in their schools. Alternatively, interventions may provide students with new concepts, skills, and dispositions that temporarily change the trajectory of the students’ educational course. Because the new trajectories are exceptions, multiple processes may vitiating their positive effects over time, such as institutionalization of programs that assume low levels of mathematical knowledge and focus on lower level skills and cultures of low expectations for certain groups (e.g., kindergarten and first-grade instruction often covers material children already know even without pre-K experience; Carpenter & Moser, 1984; Engel, Claessens, & Finch, 2013; Van den Heuvel-Panhuizen, 1996). Left without continual, progressive support, children’s nascent learning trajectories may revert to their original course. In contrast, major benefits from a preschool intervention may also be detectable via a close examination of student growth trajectories: A hypothesis that is currently untested in the relevant literature.

To address this research and policy issue, student learning trajectories in mathematics must be conceptualized as an ongoing phenomenon: Students were improving on their mathematical skills at a particular rate before the intervention, the intervention occurred, and then students continued to learn math for years following the intervention. In the ensuing years, students may learn new material at an improved (e.g., more rapid) rate as a result of the earlier intervention, but also possibly not (Campbell, Pungello, Miller-Johnson, Burchinal, & Ramey, 2001; Grimm, Ram, & Hamagami, 2011). In this way, the shape of student growth trajectories in mathematics over the course of elementary school may be conceptualized as meaningful evidence of the efficacy of a preschool intervention to affect the future learning of students.

In this study, the effects of a mathematics instructional intervention, administered during preschool, on the student-specific nonlinear growth trajectories of mathematical ability through elementary school will be systematically examined. Specifically, data are drawn from a large-scale randomized control trial that was developed as part of an evaluation of a model of scale up that included the Building Blocks curriculum in preschool, with a highly diverse and majority low–socioeconomic status sample of students also being assessed on their mathematical ability in kindergarten, first, third, fourth, and fifth grade (Clements et al., 2013; Sarama & Clements, 2013). A recently developed methodological paradigm, Dynamic Measurement Modeling (DMM; Dumas & McNeish, 2017; McNeish & Dumas, 2017) is applied to these data to estimate nonlinear growth trajectories for every individual student in that data set. Then, student-specific parameter estimates associated with those learning trajectories are utilized to inform inferences about mathematical development in elementary school, and the effects of early instructional intervention on the course of that development.

Nonlinear Learning Trajectories in Educational Research

From the earliest days of psychological research on learning (Ebbinghaus, 1885), through present-day investigations in cognitive science (Donner & Hardy, 2015; Resing, Bakker, Pronk, & Elliott, 2017), student improvement on a particular skill has been commonly observed to follow a recognizable and consistent pattern: initial learning gains tend to occur rapidly, but growth decelerates over time, with the student’s ability to perform that particular skill eventually leveling off. For example, in one widely cited sequence of meta-analytic studies from a decade ago (i.e., Bloom, Hill, Black, & Lipsey, 2008; Hill, Bloom, Black, & Lipsey, 2008), effect sizes associated with learning gains across multiple domains of learning in schools were shown to decrease as students age, indicating that student learning growth, on average across many included studies, was decelerating across developmental time. Today, such nonlinear learning curves are familiar to most educational practitioners and researchers, and the term “learning curve” is commonly utilized in popular parlance to describe the process by which a particular skill can be developed.
However, despite their ubiquity in popular discourse, limitations in data availability, computational power, or statistical methodology have meant that student-specific nonlinear growth trajectories (i.e., learning curves) are almost never modeled in large-scale educational research (see Cameron, Grimm, Steele, Castro-Schilo, & Grissmer, 2015, and Campbell et al., 2001, for notable exceptions). Today, the vast majority of educational researchers have utilized outcome data collected at one particular time point (e.g., Kim & Petscher, 2016), or, when student data are collected at multiple time points, linear change among measured outcomes across those time points (e.g., Dumas, McNeish, Schreiber-Gregory, Durning, & Torre, 2019; Jitendra et al., 2013; Nesbitt, Farran, & Fuhs, 2015), as evidence of learning occurring within students. Such research practice, however typical in our field, does not fully capture changes in the shape of student nonlinear growth trajectories that may occur in response to instruction, and as such represents a major current limitation in the educational research literature. In addition, even when nonlinear growth models are applied to educational data, those growth models are typically “marginal” in nature, meaning they model average growth within groups of students (e.g., Morgan, Farkas, Hillemeyer, & Maczuga, 2016; Shanley, 2016). As such, these marginal growth models cannot generate growth parameters for individual students, limiting the substantive inferences that can be made about learning and instruction.

In addition to the scientific limitations of these current methods, the continued use of single time-point student scores, or linear change among those scores, may exacerbate existing equity and social justice issues in educational research and measurement. This is because, by modeling average student scores linearly across time, such methods implicitly include an assumption of student rank-order preservation, which relegates all naturally occurring nonlinear growth—and the concomitant shifts in the relative standing of students in terms of their skill level over time—to a residual error term (McNeish & Matta, 2018). By ignoring the nonlinearity and student-specificity of learning growth trajectories, well-meaning researchers may inadvertently mis-specify their models to the advantage of students who enter a linear and marginal (i.e., equality of slopes) conceptualization of student growth. For instance, mathematics education researchers perennially observe an achievement gap, in which underresourced students systematically underperform their more privileged peers in math (Bohrnstedt, Kitmitto, Ogut, Sherman, & Chan, 2015; Lee, 2002; Reardon & Galindo, 2009). Such a gap is specifically well-documented in longitudinal research, with the additional finding that early achievement gaps in mathematics tend to widen throughout schooling (Burchinal et al., 2011; Cameron et al., 2015; Klibanoff, Levine, Huttenlocher, Vasilyeva, & Hedges, 2006). Although such a widening gap phenomenon could theoretically be compatible with linear growth, it could never be compatible with a marginal model in which all students’ slopes are equal. This is because, if all developmental slopes are equal, students’ mathematical ability may grow, but the differences among the students’ scores (i.e., the gaps) must remain the same magnitude. In contrast to a marginal model, a nonlinear growth model in which all parameters are estimated for each student may specifically capture a widening achievement gap if the steep initial section of a “learning curve”—shaped trajectory is steeper on average for socially dominant students or remains reasonably steep for a longer period of time on average for socially dominant students.

In addition, longitudinal research on mathematical ability has shown that interyear correlations (i.e., correlations between achievement scores from one year to another) among student mathematical outcomes tend to increase over developmental time, implying that these skills are stabilizing over the course of schooling (Bailey et al., 2018; Baumert, Nagy, & Lehmann, 2012). Such an observation of increasing interyear stability is highly suggestive of nonlinear “learning curve” shaped growth, in which student growth rates on measured mathematical skills may be decelerating across time. Moreover, the effects of instructional interventions that significantly improve the mathematical ability of young children appear to weaken over the course of elementary school (Bailey et al., 2016; Smith, Cobb, Farran, Cordray, & Munter, 2013). This finding indicates that students who receive an intervention early on (e.g., in preschool) may outperform a comparison group initially, but, assuming the intervention does not
continue, the advantage conferred to students by the intervention disappears after a few years. As with the observation of increasing interyear stability, the diminishing returns or “fadeout” of early math intervention reported in the literature (e.g., Kang, Duncan, Clements, Sarama, & Bailey, 2018) are not compatible with a conceptualization of linear growth. This is because, if such a linear growth trajectory truly occurred, the students who were advantaged early on in their development (e.g., by an effective instructional intervention) would necessarily remain advantaged over time, because the constant linearity of their learning trajectory would keep them ahead of their peers. Indeed, recent work that utilized a nonlinear decelerating growth model found that achievement gaps among high-resourced and lower resourced students did not increase over time in mathematics (Helbling, Tomasik, & Moser, 2019; Mok, McNerney, Zhu, & Or, 2015). These findings allow for the possibility that, by specifying a growth model as linear and therefore assuming that student learning rates are constant across developmental time, researchers may be overestimating the learning capacity of socially dominant children who enter schooling with higher levels of mathematics knowledge on average, or who grow faster on average earlier on. In contrast, a model that allows the rate of learning to vary across time for every individual student may reveal that lower-resourced students can be predicted to catch up to their higher resourced peers (e.g., Dumas & McNeish, 2017), but linear growth models are unable to account for this possibility.

For these reasons, there is ample evidence to hypothesize that mathematical development over the course of elementary school follows a nonlinear “learning curve” trajectory, although of course the trajectories of individual students may exhibit subtle differences in shape (i.e., there is a need to model student-specific growth trajectories). Therefore, in this investigation, a methodological approach that specifically models individual nonlinear growth curves for every student in a given data set will be applied.

**Applying a Novel Method: Dynamic Measurement Modeling**

Within educational and psychological research that focuses on understanding differences in learning trajectories and capacity among students (e.g., Calero, Belen, & Robles, 2011; Resing et al., 2017), single time-point assessment scores cannot meaningfully form the basis of psychological inferences about learning. Instead, in a method termed Dynamic Assessment (DA; Tzuriel, 2001), students are systematically measured on a particular skill multiple times, with standardized learning opportunities interspersed between those assessment occasions. DA methods have historically provided richer information about research participants than is possible with single time point (i.e., static) testing practices (Elliott, Resing, & Beckmann, 2018; Grigorenko & Sternberg, 1998), but DA is resource intensive and therefore has only historically been applied to small samples or within clinical contexts.

In response to this methodological challenge, a statistical modeling framework capable of estimating quantities associated with DA, but with much larger samples, called Dynamic Measurement Modeling was recently introduced (McNeish & Dumas, 2017). DMM also draws meaningfully on a growth modeling framework that was originally formulated in biochemistry (i.e., the Michaelis-Menten model, Michaelis & Menten, 1913; English et al., 2006), but reparametrizes that growth function as a nonlinear mixed-effects model (Cudeck & Harring, 2007) to individually model student-specific trajectories. Specifically, DMM describes the learning trajectory of every individual student in a longitudinal data set in terms of three parameters: an intercept that represents the initial skill level of the student at the first time point in the model, an upper asymptote that represents the predicted final skill level of the student, and a midpoint parameter that represents the time point at which a student’s skill level is halfway between their intercept and asymptote.

Using terminology more nested within the educational research discipline, the DMM asymptotes have been previously termed “learning capacities” because they are meant to describe students’ predicted level of future skill attainment, and the midpoint parameters have been described as “learning rates” because they provide information about the rapidity with which students approach their predicted asymptotic level over developmental time (Dumas & McNeish, 2017; McNeish & Dumas, 2018). See Figure 1 for a visual depiction of the relations among these DMM parameters. In addition, DMM is theoretically akin to existing latent measurement models such as item-response models: a theoretical similarity that means the conditional reliability of DMM capacity scores is calculable across the full distribution of students (McNeish & Dumas, 2018; Nicewander, 2018).

These methodological details mean that DMM can be used to reliably model student-specific nonlinear growth trajectories in large data sets. For example, prior studies (e.g., Dumas & McNeish, 2017, 2018) using DMM have focused on the Early-Child Longitudinal Study–Kindergarten (ECLS-K) 1999 data set, revealing a clear decelerating growth trajectory in mathematics and reading ability scores (Cameron et al., 2015). DMM has also been shown to improve the consequential validity of measurement in both mathematics (Dumas & McNeish, 2017) and reading (Dumas & McNeish, 2018), by demonstrating that no substantial differences in student learning capacity scores exist based on socioeconomic status, race/ethnicity, and gender in the ECLS-K data set. This empirical finding would have been hidden with traditional methods. However, DMM has
never before been applied to a data set in which a particular instructional intervention was administered to a subset of students to determine if that intervention affected the shape of student learning trajectories, such an application of DMM is the focus of this investigation.

**Data Source: The TRIAD Project**

After validating the efficacy of the *Building Blocks* curriculum (Clements & Sarama, 2007; Clements et al., 2013) to improve mathematical outcomes of preschool students (Clements & Sarama, 2007, 2008), the next challenge was taking it to scale. To do so, the Technology-enhanced, Research-based, Instruction, Assessment, and professional Development (TRIAD) scale-up model, which included 10 research-based guidelines (Sarama & Clements, 2013; Sarama, Clements, Starkey, Klein, & Wakeley, 2008) was created. One critical feature of the TRIAD scale-up model was a variety of professional development opportunities for teachers aimed at promoting their knowledge of the intervention and its purposes, high-quality student-teacher interactions, and equity in classroom instruction (see Clements, Sarama, Wolfe, & Spitsler, 2015, for a focused fidelity study of the implementation of this intervention). The data for the present study were taken from the evaluation of an implementation of the TRIAD model using the Building Blocks curriculum.

**Design of the TRIAD Evaluations**

The implementation of the TRIAD model was evaluated in two related, large-scale studies. The first and main study evaluated its implementation in preschool, with a follow-up into kindergarten and first grade. The second study extended the evaluation of the TRIAD model from the original preschool to first grades to include the fourth and fifth grades, with no additional interventions. The first study was a cluster randomized trial in which the TRIAD model was implemented in 42 schools in two city districts serving low-resource communities, randomly assigned to three conditions, with a total participation of 1,305 students in 106 classrooms. By the end of first grade, 1,172 students from 347 classrooms in 172 schools completed all assessments. All 42 schools were represented, with the three treatment groups maintaining their original percentages. In preschool, the two experimental interventions were identical, but one (TRIAD-Follow Through or TRIAD-FT) included follow through in the kindergarten and first-grade years, whereas the other experimental condition (TRIAD-NFT) did not. The TRIAD-FT kindergarten and first-grade teachers received information about what at least some of their entering students had learned in their preschool year and how to build on it. TRIAD coaches provided support through monthly classroom visits, always including use of formative assessments to support decisions about differentiating instruction (Clements et al., 2013; Sarama et al., 2012).

The second study measured the persistence of the TRIAD intervention effects into the project’s seventh year; that is, up to 4 years following the end of the treatment for the Follow Through (TRIAD-FT) group and 6 years following the end of the treatment for the TRIAD-NFT group (Clements et al., 2019). By the end of fifth grade, 781 students from 338 classrooms in 153 schools completed all assessments. Between first grade and fifth grade, the

![Depiction of a hypothetical student-specific dynamic measurement modeling (DMM) curve. The relations among the intercept, midpoint (learning rate), and asymptote (learning capacity) parameters are visualized here.](image)

**FIGURE 1.** Depiction of a hypothetical student-specific dynamic measurement modeling (DMM) curve. The relations among the intercept, midpoint (learning rate), and asymptote (learning capacity) parameters are visualized here.
overall attrition was 36% and the three groups did not experience substantially different attrition rates (34% for TRIAD-FT, 40% for TRIAD-NFT, and 33% for control). At all grade levels, none of the baseline differences (free/reduced price lunch, gender, disability status) were greater than 0.25 standard deviations (in absolute value), which aligns with the reasonable threshold employed by What Works Clearinghouse (Clements et al., 2019).

**Instruments and Scoring**

Students’ mathematical ability was assessed at seven time points: pre- and postintervention assessments in preschool, and then additional measurement occasions in kindergarten, first, third, fourth, and fifth grade (i.e., seven total time points). Mathematical ability for pre-K to Grade 1 was measured using the Research-based Early Math Assessment (REMA; Clements, Sarama, & Liu, 2008). The REMA assesses young students’ conceptual and procedural knowledge of mathematics, as well as problem-solving and strategic competencies. Abilities are assessed according to theoretically and empirically based developmental progressions (National Research Council, 2009; Sarama & Clements, 2009). Topics represented on the REMA included verbal counting, object counting, subitizing, number comparison, number sequencing, connection of numerals to quantities, number composition and decomposition, adding and subtracting, and place value. In addition, shape recognition, shape composition and decomposition, congruence, construction of shapes, and spatial imagery; and additional topics include measurement, patterning, and reasoning are also present on the REMA. The developmental progression of items as well the fit of individual items to their scoring model has been reported in earlier research (Clements et al., 2008).

The REMA measures mathematical competence as a latent trait using an item response theory (IRT) scoring model, yielding a score that locates students on a common ability scale with a consistent, justifiable metric that allows for accurate comparisons across ages and meaningful comparison of change scores, even when initial scores differ (B. D. Wright & Stone, 1979). The 225 items on the REMA are presented to students in order of item difficulty, and students stop after four consecutive errors on each of the number and geometry section items. Based on the expected growth in mathematical competency from preschool to first grade, administration at the first grade time point began with Item 30 of the number section and Item 6 in the geometry section. All assessment sessions were videotaped, and each item coded by a trained coder for correctness and for solution strategy; 10% of the assessments were double-coded. Both assessors and coders were blind to the group membership of the students. Continuous coder calibration by an expert coder (one tape per coder per week) was performed to mitigate coder drift. Calibration feedback was sent to coders, alerting them to any undue variance from coding protocols. Previous analysis of the assessment data showed that the reliability of the scores was strong (Clements et al., 2008); on the sample used in the present investigation, the reliability was .92. In addition, REMAs scores had a correlation of .86 with a different measure of preschool mathematics achievement (Clements et al., 2008), the Child Math Assessment: Preschool Battery (Klein, Starkey, & Wakeley, 2000), and a correlation of .74 with the Woodcock-Johnson Applied Problems subscale for a pre-K specific subset of 19 items (Weiland et al., 2012). Because the REMA is a research instrument used primarily for studies within the educational sciences and not heavily utilized within large-scale educational testing practice, the REMA does not yet feature national norming information for comparison of the TRIAD sample to a nationally normative distribution.

At the beginning and end of fourth grade and at the end of fifth grade, students’ mathematical knowledge was measured using the Tools for Early Assessment in Mathematics 3–5 (Clements, Sarama, Khasanova, & Van Dine, 2012). The TEAM 3–5 is a paper-and-pencil assessment that can be administered in a group setting. It is aligned with the same developmental progressions as the REMA and TEAM Pre-K–2 although some topics that are relevant in the youngest students (e.g., simple counting, subitizing, shape recognition) are “retired,” while others are introduced or receive greater emphasis (e.g., multiplication and division, fractions and decimals, measurement of area and volume, coordinate systems, and more sophisticated analysis of geometric shapes). In the current sample, the TEAM 3-5 was found to have good internal reliability (Cronbach’s α = 0.91). Furthermore, correlations between the assessment and state Grade 5 achievement tests in New York, r(351) = 0.82, p < .001, and Massachusetts, r(110) = 0.76, p < .001, were strong for the subset of students for which state tests were available (approximately 40% of the full sample). As with the REMA, the TEAM 3-5 was also converted to a standardized Rasch–IRT score, and this IRT approach allowed for the vertical scaling of the scores across time, to allow for an objective investigation of student learning trajectories using longitudinal methodology. In addition to these mathematics assessment scores, the TRIAD data set also includes a number of salient background characteristics about students and their families that are relevant to this investigation (e.g., student sex, race/ethnicity).

**Fitting the Dynamic Measurement Model: Linear or Nonlinear?**

To determine whether linear growth, or a nonlinear learning curve, shaped and decelerating trajectory best modeled the TRIAD data, we compared the fit of a longitudinal mixed-effects model that modeled development in...
mathematics as linear, quadratic, or as following the Michaelis-Menten nonlinear trajectory. All models featured a heterogeneous error structure (i.e., no assumption of homoscedasticity, or equal error variances over time, was made; Grimm & Widaman, 2010); see the supplemental materials included with this article for full information on model estimation strategies. Figure 2 shows the mean REMA scores within the entire TRIAD sample at each time point, with the Michaelis-Menten, quadratic, and linear growth trajectories plotted over them. Visually, it can be seen that the nonlinear “learning curve” shaped Michaelis-Menten model much better describes the average growth trajectory in this data set than does the linear model. This visually apparent pattern is shown to be statistically supported through the inspection of relevant fit statistics, which show that the Michaelis-Menten model fits the TRIAD data much better than the linear model (Michaelis-Menten, BIC = 12,235; Quadratic, BIC = 13,954; Linear, BIC = 16,706). Therefore, the nonlinear Michaelis-Menten model was retained for further analysis.

Beyond improvement in fit, it is important to differentiate between models that are nonlinear in the variables versus models that are nonlinear in the parameters. The commonly used quadratic model is nonlinear in the variables (e.g., time is squared), but remains linear in the parameters because the effect of linear time is simply added to the effect of quadratic time. Accounting for nonlinearity in this way is known to lead to curves that may provide a decent local approximation of the phenomenon being modeled but possess parameters that are often of little interpretable utility (Cudeck & du Toit, 2002; Grimm, Ram, & Estabrook, 2016). Models that are nonlinear in the parameters (e.g., where parameters can be included in exponential, fractions, or products with other parameters) are generally more capable of estimating meaningful quantities that characterize nonlinear growth, especially of mental attributes relevant to educational research (Cudeck, 1996).

It should also be noted here that a DMM conceptual framework is compatible with a number of functions that are nonlinear in their parameters including Richards, Gompertz, von Bertalanffy, or Schnute curves. Each of these curves feature monotonically increasing growth that eventually culminates in an asymptote, but differ from the Michaelis-Menten function in terms of their general nonlinear shape as well as the number of parameters used to estimate the curve and the particular scale and formulation of those parameters (see McNeish & Dumas, 2017, or McNeish, Dumas, & Grimm, 2019, for a full and detailed discussion of the various nonlinear functions that are compatible with DMM). In this study, the marginal (i.e., without the random effects) fit of each of these functions, including the quadratic growth function, to the TRIAD data was assessed, and the Michaelis-Menten model demonstrated the most advantageous fit, including the smallest mean-squared error. Therefore, both the empiricism and conceptual discussion (including Figure 1) in this article focus on a DMM that utilizes the three-parameter Michaelis-Menten function, which has also been the best-fitting functional form in previous DMM research using other educational data sets (e.g., Dumas & McNeish, 2017; McNeish & Dumas, 2018).

**DMM Results and Interpretation**

After retaining the Michaelis-Menten based DMM for further analysis, intercept, midpoint, and asymptote scores for every student in the data set were computed via Empirical Bayes predictions. Taken together, these three parameters describe the shape of student-specific learning trajectories. See Figure 3 for student-specific DMM trajectory plots for 50 students from the TRIAD data set, drawn without regard to treatment group membership. Recently, researchers (McNeish
Asymptote (Capacity Scores) 0.59 0.77 0.35
Midpoint (Learning Rates) 0.50 2.02
Intercept 0.46

Asymptotic levels of elementary mathematics skill, the development of which tends to level off as students progress through elementary school. In the case of the TRIAD data set, the DMM model predicts that the students will eventually reach relatively similar amounts of elementary mathematics skill before their development asymptotically levels off. Of course, individual differences will still exist across students on other abilities, but the ability that the REMA is designed to measure (i.e., elementary mathematics) is predicted not to be highly variable asymptotically.

Therefore, despite the lower variability in their starting and predicted end points, students develop their math skills at highly variable rates. In addition, all three of the DMM parameters were positively correlated in this sample, implying that those students who entered preschool ahead of their peers in mathematics also had higher predicted capacities on average. In addition, students with higher capacities, on average, also took longer to reach the half-way point on their learning trajectory (i.e., their midpoints were higher). However, this pattern was far from hard and fast (i.e., the correlations are only moderate in strength), and the correlation between the midpoints and the capacities ($r = .77$) was stronger than the correlation between the intercept and the capacities ($r = .59$), or the correlation between the intercept and the midpoints ($r = .50$). One reason why the examination of these correlations is relevant to this investigation is because they provide quantitative insight into the intraindividual patterns in student learning trajectories found in this data set, and contextualize the purpose of the intervention in terms of DMM. In this case, this correlational pattern highlights the importance of early learning to later achievement, with those students entering preschool ahead of their peers being predicted to have higher asymptotic ability later in life. In many ways, one purpose of the TRIAD intervention being examined here is to decouple these parameters by steepening the learning trajectories of students in the intervention group (i.e., lowering the midpoints of their growth curves), without decreasing student future capacity.
TABLE 2
Dynamic Measurement Scores Across and Within Intervention Conditions

| Parameter                  | Full Sample     | Treatment Group | Control Group  |
|----------------------------|-----------------|-----------------|---------------|
| Initial Intercept          | −3.22 (0.03)    | −3.18 (0.04)    | −3.27 (0.05)  |
| Midpoint (learning rates)  | 1.93 (0.02)     | 1.89 (0.02)     | 2.00 (0.02)   |
| Asymptote (capacity scores)| 2.94 (0.06)     | 2.91 (0.07)     | 3.02 (0.09)   |

Note. Standard errors for Dynamic Measurement Modeling scores are in parentheses. Significant differences exist between treatment and control group only for the midpoint parameters. Intercept and asymptote scores are on the same scale (i.e., the scale of mathematics ability), while the midpoint scores are on the scale of years since the start of preschool.

**Intervention Effect**

Having observed these general intraindividual differences, we turned to the question of the efficacy of the TRIAD intervention in influencing the learning trajectories of the treatment group of students. It should be noted here that, originally, the TRIAD intervention featured two treatment groups (TRIAD-FT and Triad-NFT); in both of the TRIAD treatment groups, students received the same and the same amount of the Building Blocks instructional intervention. They differed only in a follow-through teacher professional development component in kindergarten and first grade. Furthermore, our DMM analysis showed that no statistically significant or substantively meaningful differences existed in student learning trajectories across the two treatment groups (see online Supplemental Material for details). Therefore, for our main analysis, we combined those treatment groups in this study and analyze these data with only two groups: A treatment group that received the Building Blocks intervention, and a control group that did not, resulting in unequal analytic sample sizes in the control group (N = 378, 29%) and combined treatment group (N = 927, 71%).

Table 2 depicts the mean scores for each of the three DMM parameters across the treatment and control group. Because random assignment for the TRIAD intervention was done at the school level, the equality of these means was tested through a multivariate linear mixed-effects model that accounted for the school-based nestedness of these data in the calculation of the significance tests. Because the TRIAD sample was composed of 42 schools, which is near the border for minimum sample size requirements at the second level of this model (Chang, 2015), small sample size corrections to the estimation of this multilevel model were performed (see details in online Supplemental Material). This multilevel modeling approach revealed that both the intercepts and the asymptotes were similar across groups, leading to nonsignificant differences in both the intercept, $t(47.5) = 1.50, p = .14, d = .12$, and capacity scores, $t(65) = −0.98, p = .33, d = .08$. It should be noted here that, although the effect is not significant, the descriptive differences in the capacity scores actually favored the control group (see Table 2). The similar intercept scores across the groups is to-be-expected because of the random assignment for the intervention (i.e., the intercept corresponds to a time that precedes the intervention). However, the equality of the capacities could not definitively have been expected. This finding implies that the TRIAD scale up of the Building Blocks intervention in preschool did not shift students’ predicted asymptotic level of elementary mathematical skill compared with students in the control condition.

Although the TRIAD data set does not allow for the direct comparison of DMM capacity estimates to actual adulthood levels of elementary mathematical skill, some recent work with DMM models (i.e., McNeish et al., 2019) does support the criterion validity of these scores to much later in life. In McNeish et al.’s (2019) recent study, a life span verbal ability data set that followed participants from ages 3 to 72 years (i.e., the Berkeley Growth Study; Bayley, 1949) was utilized to demonstrate that DMM capacity estimates generated from childhood and adolescent ability scores explained nearly three times as much variance in ability at age 72 years than did extrapolating directly from IRT scores. Given this finding, there is reasonable evidence to consider DMM capacity scores viable estimates of eventual skill level, and therefore use them as outcome scores when testing the effect of the TRIAD intervention on the course of students’ mathematical learning trajectories.

However, in an analysis of the midpoints, it was found that the students in the treatment condition did indeed learn at a significantly faster rate than those in the control condition through elementary school (i.e., their midpoint parameters were significantly lower; $t(76.5) = −3.86, p < .01, d = .30$), while maintaining statistically equal asymptotic capacity estimates. This effect implies that the TRIAD intervention was successful at improving students’ rate of learning through elementary school such that the students who received the intervention developed half of their eventual predicted mathematics skill-level significantly sooner than their peers in the control group, even though that asymptotic capacity estimate was statistically equal across groups. So, although the treatment and control group were similar in the amount of mathematical skill they had on entering preschool, as well as their predicted asymptotic future level of elementary mathematical skill, the treatment group students approached that upper asymptote more rapidly than did the
control group. To help visualize this finding, Figure 4 shows the histogram of the student-specific random effects for each of the DMM parameters, separated by treatment group status. For intercepts and the asymptotes in Figure 4, there is very little separation between the treatment and control groups which shows that students are largely the same at baseline and in predicted capacity. For the midpoints, there is notable divergence between the treatment and control groups, demonstrating that although the treatment is not lifting the asymptotes, it does steepen the learning curves. In other words, students in both the treatment and control conditions are going to the same eventual skill level, on average; however, the students in the treatment group are arriving there faster.

Such an augmented rate-of-learning in mathematics may be helpful to the academic development of young students, because it can benefit them in a number of ways including increased motivation and self-concept for mathematics learning (Becker & Neumann, 2018), possible skill transfer to other unmeasured academic areas (e.g., science) early in elementary school (Marcus, Haden, & Uttal, 2018), and a decrease in negative emotions (e.g., frustration) associated with early mathematics learning (Ahmed, van der Werf, Kuyper, & Minnaert, 2013). It should be noted here, however, that hypotheses concerning the potential transfer of elementary mathematical knowledge to other domains of learning (e.g., science) or to more advanced areas of mathematics (e.g., algebra) are not testable using the TRIAD data set, and therefore must remain reasoned literature-based conjectures at this point.

Interaction With Demographic Background

Given the significance of the treatment effect on the student’s learning rates (i.e., midpoint parameters), an interaction term between the treatment and student sex on the midpoints, as well as between the treatment and student race/ethnicity on the midpoints was tested in the multivariate

FIGURE 4. Comparison of histograms of student-specific random effects for the intercepts (upper left), asymptotes (upper right), and midpoints (bottom) for treatment (gray) and control (black) groups.
linear mixed-effects model (which also accounted for school-based nestedness). This interaction term was non-significant for student sex, \( r(1258) = 1.19, p = .23, d = .08 \), implying that the TRIAD intervention was equally effective at steepening mathematics learning trajectories of male and female students. Given perennially observed sex differences in mathematics learning outcomes in the United States (Geary et al., 2019; Reilly, Neumann, & Andrews, 2015), the capability of Building Blocks to equally steepen mathematics learning trajectories for both male and female students is educationally important.

The interaction term between the treatment and student race/ethnicity on the learning rate parameters was found to be significant, \( F(6,108) = 6.21, p < .01 \), with students who were Black or Latinx benefiting more (i.e., steepening their trajectory by reducing their midpoint parameter) from the treatment than did their White peers (\( d \) effect sizes for the treatment on the learning rates were Black = .34, Latinx = .25, and White = .12). It should be noted here that the sample size of other ethnicities (e.g., Asian/Pacific Islander) was not sufficient in the TRIAD data set to test treatment-interaction terms for those groups. Importantly, all race/ethnicity groups of students (including White students) displayed steeper learning trajectories (i.e., reduced midpoint parameters) from the Building Blocks intervention, but traditionally underrepresented race/ethnicity groups in mathematics (i.e., Black and Latinx students) benefitted most from the preschool instructional intervention. Given the entrenched nature of the U.S. achievement gap between White students and their peers of color (Bohrnstedt et al., 2015; Burchinal et al., 2011), such a finding may illustrate the equity-related importance of preschool mathematics intervention in general and the efficacy of the Building Blocks curriculum in particular.

Implications for Early Mathematics Intervention

This study has been the first empirical investigation to apply the DMM nonlinear student-specific growth methodology (Dumas & McNeish, 2018) to answer research questions related to the efficacy of an instructional intervention. As such, the findings from this study differ in critical ways from past examinations of the same learning phenomena (i.e., early mathematics development) that have utilized more constrained methods (e.g., Schenke et al., 2017). Here, we posit and briefly present three principal findings that can be drawn from the present investigation that could not have been similarly observed in previously existing work.

Growth Trajectories Are Highly Variable Across Students

In this study, both the initial amount of elementary mathematical skill with which students entered preschool and their predicted eventual asymptotic level of elementary mathematical skill showed no statistical differences between the treatment and control groups. In addition, neither of these two parameters (intercept and asymptote) was particularly variable, and students varied slightly more on their intercepts than they did on their predicted asymptotes. This pattern suggests that students in the TRIAD data set both began preschool relatively similarly in terms of their elementary mathematical skill and were predicted to reach relatively similar levels of elementary mathematical skill eventually. However, the rate with which students developed this skill was much more variable (approximately 5 times as variable) as the intercepts and asymptotes. Given this variability, the DMM learning rates of individual students, rather than their eventual achievement, may be a much more fertile ground for investigating the efficacy of instructional interventions. This finding is also interesting in regards to previous work that has suggested that preschool intervention effects fade over time (e.g., Bailey et al., 2016): The results of the present study suggest that students do not vary substantially on their eventual asymptotic level of elementary mathematical skill, implying that the detection of distal intervention effects may be complicated by that low variability. Related to the finding of midpoint variability, one interesting future direction for this line of research with the TRIAD data set or other related data sets would be to include a growth mixture modeling framework (Harring, 2012; Muthén & Shedden, 1999), which would allow for nuanced differences in learning trajectory across students to be used to identify latent classes within the TRIAD students. Such a future study may identify that subsets of students who received certain aspects of the intervention or who have certain background characteristics exhibit systematic differences in the shape of their mathematical learning curves, although of course such a study remains a future direction at this point.

Preschool Intervention Can Significantly Steepen Learning Trajectories

Despite the statistical equivalence of the intercept and asymptotic parameters across the treatment and control group in this study, the Building Blocks intervention was successful in significantly shortening the time it took for students to develop half of their eventual predicted elementary mathematical skill. This finding means that the learning trajectories of students in the treatment group were significantly steeper, indicating their growth early in elementary school was more rapidly approaching their asymptotic capacity. Furthermore, it may be hypothesized from this pattern that, should the intervention have lasted longer into elementary school, treatment group students may have experienced accelerated learning for longer, leading either to an even stronger effect on their learning rates, or potentially a positive effect on their capacity scores as well,
although of course this hypothesis must remain a future
direction until longer term intervention-based data sets
become available. In this way, although the Building Blocks
intervention in preschool did not significantly alter the
capacity of the students in the treatment group to develop
mathematical skill (an observation that is in accordance
with previous fadeout findings), it did improve the rate at
which students approached their asymptotic capacity.
Therefore, by shifting the focus of investigation from event-
tual skill level (i.e., capacity scores) and to the shape of
the learning trajectory over time (i.e., by comparing the learn-
ing rates), the positive effect of the preschool intervention
through elementary school is detectable.

Early Intervention Benefits Minority Students Most

In the United States, White, or European American stu-
dents, on average, tend to outperform their Black or Latinx
peers in mathematics achievement (Reardon & Galindo,
2009). However, in this study, we showed that students of
color benefitted more from the Building Blocks interven-
tion than did their White peers, meaning that the learning
rates for Black and Latinx students were most steepened
by the intervention. This finding implies that early inter-
vention may be a key component for achieving a main
goal of educational research and practice: equal opportu-
nity for students, regardless of background, to learn and
develop mathematical skills. It is important to note here
that the intervention effect on the DMM capacity scores
was statistically equal across race/ethnicity groups, while
the intervention effect on the DMM learning rate scores
significantly interacted with both African American and
Hispanic status in the substantively positive direction (i.e.,
midpoints were lowered): implying that the intervention
did not support the long-term mathematical development
of any ethnic or cultural group more than the others, but it
did improve the learning rate of minoritized students more
than European American students. In addition, follow-
through research-based curricula is necessary if children
attend poor quality schools (Brooks-Gunn, 2003), which,
in the United States, is especially more likely for students
of color (Currie & Thomas, 2000). There is a cumulative
positive effect of students experiencing consecutive years
of high-quality teaching, and a cumulative negative effect
of low-quality teaching (Ballou, Sanders, & Wright, 2004;
Jordan, Mendro, & Weerasinghe, 1997; Sanders & Horn,
1998; Sanders & Rivers, 1996; S. P. Wright, Horn, &
Sanders, 1997). Unfortunately, the latter is more probable
for children from historically disadvantaged social groups
(Akiba, LeTendre, & Scribner, 2007; Darling-Hammond,
2006). Therefore, the continued application of research-
based curricula throughout elementary school (as opposed
to only in preschool as in this investigation) may be neces-
sary to achieve equitable educational outcomes. Of course,
previous research on preschool intervention effects using
other longitudinal methodologies (e.g., Burchinal et al.,
2011) has also highlighted the need for research-based
curricula or quality teaching throughout elementary school
as a requirement for equitable educational outcomes: This
study underscores that key societal need with support from
a newer and potentially more detailed methodology.

Acknowledgments

This research was supported by the Institute of Education
Sciences (IES), U.S. Department of Education, through grants
R305K05157 and R305A110188 and also by the National
Science Foundation (NSF), through grants ESI-9730804 and
REC-0228440. The opinions expressed are those of the authors
and do not represent views of the IES or NSF. Although the
research is concerned with the scale-up model, not particular cur-
ricula, a minor component of the intervention used in this
research has been published by the authors, who thus could have
a vested interest in the results. An external auditor oversaw the
research design, data collection, and analysis and other research-
ers independently confirmed findings and procedures. The
authors wish to express appreciation to the school districts, teach-
ers, and students who participated in this research.

Note

1. We focus on the midpoints in text, but these analyses also
included the intercepts and asymptotes as well, which did not
reveal any meaningful differences related to the intervention. Full
results for the intercepts and slopes are included in the supplemen-
tal material for interested readers.

References

Administration for Children and Families. (2010). Head Start
Impact Study: Final report. Washington, DC: Author.
Ahmed, W., van der Werf, G., Kuypers, H., & Minnaert, A. (2013).
Emotions, self-regulated learning, and achievement in math-
ematics: A growth curve analysis. Journal of Educational
Psychology, 105, 150–161. doi:10.1037/a0030160
Akiba, M., LeTendre, G. K., & Scribner, J. P. (2007). Teacher
quality, opportunity gap, and national achievement in 46 coun-
tries. Educational Researcher, 36, 369–387. doi:10.3102/0013
189X07308739
Bailey, D. H., Duncan, G. J., Watts, T., Clements, D. H., & Sarama,
J. (2018). Risky business: Correlation and causation in longitu-
dinal studies of skill development. American Psychologist, 73,
81–94.
Bailey, D. H., Nguyen, T., Jenkins, J. M., Domina, T., Clements,
D. H., & Sarama, J. S. (2016). Fadeout in an early mathematics
intervention: Constraining content or preexisting differences?
Developmental Psychology, 52, 1457–1469.
Ballou, D., Sanders, W. L., & Wright, P. (2004). Controlling for
student background in value-added assessment of teachers.
Journal of Educational and Behavioral Statistics, 29, 37–65.
doi:10.3102/10769986029001037
Baumert, J., Nuy, G., & Lehmann, R. (2012). Cumulative
advantages and the emergence of social and ethnic inequality:
Matthew effects in reading and mathematics development
within elementary schools? Child Development, 83, 1347–1367. doi:10.1111/j.1467-8624.2012.01779.x
Bayley, N. (1949). Consistency and variability in the growth of intelligence from birth to eighteen years. Journal of Genetic Psychology, 75, 165–196.
Becker, M., & Neumann, M. (2018). Longitudinal big-fish-little-pond effects on academic self-concept development during the transition from elementary to secondary schooling. Journal of Educational Psychology, 110, 882–897. doi:10.1037/edu0000233
Bloom, H. S., Hill, C. J., Black, A. R., & Lipsey, M. W. (2008). Performance trajectories and performance gaps as achievement effect-size benchmarks for educational interventions. Journal of Research on Educational Effectiveness, 1, 289–328. doi:10.1080/19345740802400072
Bohnstedt, G., Kitmitto, S., Ogut, B., Sherman, D., & Chan, D. (2015). School composition and the black-white achievement gap (NCES 2015-018). Washington, DC: National Center for Education Statistics. Retrieved from https://eric.ed.gov/?id=ED560723
Brooks-Gunn, J. (2003). Do you believe in magic? What we can expect from early childhood intervention programs. Social Policy Report, 17(1), 3–14.
Burchinal, M., McCartney, K., Steinberg, L., Crosnoe, R., Friedman, S. L., McLoyd, V., . . . NICHD Early Child Care Research Network. (2011). Examining the Black–White achievement gap among low-income children using the NICHD Study of Early Child Care and Youth Development. Child Development, 82, 1404–1420. doi:10.1111/j.1467-8624.2011.01620.x
Calero, M. D., Belen, G.-M. M., & Robles, M. A. (2011). Learning potential in high IQ children: The contribution of dynamic assessment to the identification of gifted children. Learning and Individual Differences, 21, 176–181. doi:10.1016/j.lindif.2010.11.025
Cameron, C. E., Grimm, K. J., Steele, J. S., Castro-Schilo, L., & Grissmer, D. W. (2015). Nonlinear Gompertz curve models of achievement gaps in mathematics and reading. Journal of Educational Psychology, 107, 789–804. doi:10.1037/edu0000009
Campbell, F. A., Pungello, E. P., Miller-Johnson, S., Burchinal, M., & Ramey, C. T. (2001). The development of cognitive and academic abilities: Growth curves from an early childhood educational experiment. Developmental Psychology, 37, 231–242.
Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. Journal for Research in Mathematics Education, 15, 179–202. doi:10.2307/488348
Chang, W. (2015). Sufficient sample sizes for the multivariate multilevel regression model (Unpublished doctoral dissertation). University of Texas at Austin.
Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the building blocks project. Journal for Research in Mathematics Education, 38, 136–163. doi:10.2307/30034954
Clements, D. H., & Sarama, J. (2008). Experimental evaluation of the effects of a research-based preschool mathematics curriculum. American Educational Research Journal, 45, 443–494.
Clements, D. H., & Sarama, J. (2011). Early childhood mathematics intervention. Science, 333, 968–970.
Ebbinghaus, H. (1885). Memory: A contribution to experimental psychology. New York, NY: Dover.

Elliott, J. G., Resing, W. C. M., & Beckmann, J. F. (2018). Dynamic assessment: A case of unfulfilled potential? Educational Review, 70, 7–17. doi:10.1080/0013189X.2018.1396806

Engel, M., Claessens, A., & Finch, M. A. (2013). Teaching students what they already know? The (mis)alignment between mathematics instructional content and student knowledge in kindergarten. Educational Evaluation and Policy Analysis, 35, 157–178. doi:10.3102/0162373112461850

English, B. P., Min, W., van Oijen, A. M., Lee, K. T., Luo, G., Sun, H., . . . Xie, X. S. (2006). Ever-fluctuating single enzyme molecules: Michaelis-Menten equation revisited. Nature Chemical Biology, 2, 87–94.

Geary, D. C., Hoard, M. K., Nugent, L., Chu, F., Scofield, J. E., & Ferguson Hibbard, D. (2019). Sex differences in mathematics anxiety and attitudes: Concurrent and longitudinal relations to mathematical competence. Journal of Educational Psychology. Advance online publication. doi:10.1037/edu000355

Ginsburg, H. P., Lee, J. S., & Boyd, J. S. (2008). Mathematics education for young children: What it is and how to promote it (Society for Research in Child Development). Retrieved from https://files.eric.ed.gov/fulltext/ED521700.pdf

Grigorenko, E. L., & Sternberg, R. J. (1998). Dynamic test-taking. Psychological Bulletin, 124, 75–111. doi:10.1037/0033-2909.124.1.75

Grimm, K. J., Ram, N., & Estabrook, R. (2016). Growth modeling: Structural equation and multilevel modeling approaches. New York, NY: Guilford Press.

Grimm, K. J., Ram, N., & Hamagami, F. (2011). Nonlinear growth curves in developmental research. Child Development, 82, 1357–1371. doi:10.1111/j.1467-8624.2011.01630.x

Grimm, K. J., & Widaman, K. F. (2010). Residual structures in latent growth curve modeling. Structural Equation Modeling, 17, 424–442.

Harrington, J. R. (2012). Finite mixtures of nonlinear mixed-effects models. In J. R. Harrington & G. R. Hancock (Eds.), Advances in longitudinal methods in the social and behavioral sciences (pp. 159–191). Charlotte, NC: Information Age. (2012-31006-007)

Helbing, L. A., Tomasik, M. J., & Moser, U. (2019). Long-term trajectories of academic performance in the context of social disparities: Longitudinal findings from Switzerland. Journal of Educational Psychology. Advance online publication. doi:10.1037/edu000341

Hill, C. J., Bloom, H. S., Black, A. R., & Lipsey, M. W. (2008). Empirical benchmarks for interpreting effect sizes in research. Child Development Perspectives, 2, 172–177. doi:10.1111/j.1750-8606.2008.00061.x

Jang, H. (2016). Identifying 21st century STEM competencies using workplace data. Journal of Science Education and Technology, 25, 284–301. doi:10.1007/s10956-015-9593-1

Jitendra, A. K., Dupuis, D. N., Rodriguez, M. C., Zaslowsky, A. F., Slater, S., Cozine-Corroy, K., & Church, C. (2013). A randomized controlled trial of the impact of schema-based instruction on mathematical outcomes for third-grade students with mathematics difficulties. Elementary School Journal, 114, 252–276.

Jordan, H., Mendoza, R., & Weersing, D. (1997, July). Teacher effects on longitudinal student achievement. Paper presented at the National Evaluation Institute, Indianapolis, IN.

Kang, C. Y., Duncan, G. J., Clements, D. H., Sarama, J., & Bailey, D. H. (2018). The roles of transfer of learning and forgetting in the persistence and fadeout of early childhood mathematics interventions. Journal of Educational Psychology, 111, 590–603. doi:10.1037/edu0000297

Kim, Y. S. G., & Petscher, Y. (2016). Prosodic sensitivity and reading: An investigation of pathways of relations using a latent variable approach. Journal of Educational Psychology, 108, 630–645.

Klein, A., Starkey, P., & Wakeley, A. (2000). Child Math Assessment: Preschool battery (CMA). Berkeley: University of California.

Klibanoff, R. S., Levine, S. C., Huttenlocher, J., Vasilyeva, M., & Hedges, L. V. (2006). Preschool children’s mathematical knowledge: The effect of teacher math talk. Developmental Psychology, 42, 59–69.

Leak, J., Duncan, G. J., Li, W., Magnuson, K., Schindler, H., & Yoshikawa, H. (2012). Is timing everything? How early childhood education program cognitive and achievement impacts vary by starting age, program duration and time since the end of the program. Irvine: University of California.

Lee, J. (2002). Racial and ethnic achievement gap trends: Reversing the progress toward equity? Educational Researcher, 31, 3–12.

Marcus, M., Haden, C. A., & Uttal, D. H. (2018). Promoting children’s learning and transfer across informal science, technology, engineering, and mathematics learning experiences. Journal of Experimental Child Psychology, 175, 80–95. doi:10.1016/j.jecp.2018.06.003

McNeish, D., & Dumas, D. (2017). Non-linear growth models as psychometric models: A second-order growth curve model for measuring potential. Multivariate Behavioral Research, 52, 61–85.

McNeish, D., & Dumas, D. (2018). Calculating conditional reliability for dynamic measurement model capacity estimates. Journal of Educational Measurement, 55, 614–634. doi:10.1111/jedm.12195

McNeish, D., Dumas, D., & Grimm, K. (2019). Estimating new quantities from longitudinal test scores to improve forecasts of future performance [Preprint]. Retrieved from https://doi.org/10.31234/osf.io/stp5f

McNeish, D., & Matta, T. (2018). Differentiating between mixed-effects and latent-curve approaches to growth modeling. Behavior Research Methods, 50, 1398–1414.

Michaelis, L., & Menten, M. L. (1913). The kinetics of the inversion effect. BioChem Z, 49, 333–369.

Mok, M. M. C., McInerney, D. M., Zhu, J., & Or, A. (2015). Growth trajectories of mathematics achievement: Longitudinal tracking of student academic progress. British Journal of Educational Psychology, 85, 154–171. doi:10.1111/bjep.12060

Morgan, P. L., Farkas, G., Hillemeier, M. M., & Maczuga, S. (2016). Science achievement gaps begin very early, persist, and are largely explained by modifiable factors. Educational Researcher, 45, 18–35. doi:10.3102/0013189X16633182

Mullis, I. V. S., Martin, M. O., Foy, P., & Arora, A. (2012). TIMSS 2011 international results in mathematics (International Association for the Evaluation of Educational Achievement). Retrieved from https://eric.ed.gov/?id=ED544554

Muthén, B., & Shedden, K. (1999). Finite mixture modeling with mixture outcomes using the EM algorithm. Biometrics, 55, 463–469.
National Public Radio. (2017). *We’re all born with mathematical abilities*. Retrieved from https://www.npr.org/sections/ed/2017/08/01/530053714/guess-what-were-all-born-with-mathematical-abilities

Natriello, G., McDill, E. L., & Pallas, A. M. (1990). *Schooling disadvantaged children: Racing against catastrophe*. New York, NY: Teachers College Press.

Nesbitt, K. T., Farran, D. C., & Fuhs, M. W. (2015). Executive function skills and academic achievement gains in prekindergarten: Contributions of learning-related behaviors. *Developmental Psychology, 51*, 865–878. doi:10.1037/dev0000021

Nicewander, W. A. (2018). Conditional reliability coefficients for test scores. *Psychological Methods, 23*, 351–362.

Preschool Curriculum Evaluation Research Consortium. (2008). *Effects of preschool curriculum programs on school readiness* (NCER 2008-2009). Retrieved from https://ies.ed.gov/ncer/pubs/20082009/

Reardon, S. F., & Galindo, C. (2009). The Hispanic-White achievement gap in math and reading in the elementary grades. *American Educational Research Journal, 46*, 853–891. doi:10.3102/0028312009333184

Reilly, D., Neumann, D. L., & Andrews, G. (2015). Sex differences in mathematics and science achievement: A meta-analysis of National Assessment of Educational Progress assessments. *Journal of Educational Psychology, 107*, 645–662. doi:10.1037/edu0000012

Resing, W. C. M., Bakker, M., Pronk, C. M. E., & Elliott, J. G. (2017). Progression paths in children’s problem solving: The influence of dynamic testing, initial variability, and working memory. *Journal of Experimental Child Psychology, 153*, 83–109.

Sanders, W. L., & Horn, S. P. (1998). Research findings from the Tennessee Value-Added Assessment System (TVAAS) database: Implications for educational evaluation and research. *Journal of Personnel Evaluation in Education, 12*, 247–256.

Sanders, W. L., & Rivers, J. C. (1996). *Cumulative and residual effects of teachers on future student academic achievement* (Research Progress Report). Knoxville: University of Tennessee Value-Added Research and Assessment Center.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Routledge.

Sarama, J., & Clements, D. H. (2013). Lessons learned in the implementation of the TRIAD scale-up model: Teaching early mathematics with trajectories and technologies. In T. Halle, A. Metz, & I. Martinez-Beck (Eds.), *Applying implementation science in early childhood programs and systems* (pp. 173–191). Baltimore, MD: Paul H Brookes.

Sarama, J., Clements, D. H., Starkey, P., Klein, A., & Wakeley, A. (2008). Scaling up the implementation of a pre-kindergarten mathematics curriculum: Teaching for understanding with trajectories and technologies. *Journal of Research on Educational Effectiveness, 1*, 89–119.

Sarama, J., Clements, D. H., Wolfe, C. B., & Spitler, M. E. (2012). Longitudinal evaluation of a scale-up model for teaching mathematics with trajectories and technologies. *Journal of Research on Educational Effectiveness, 5*, 105–135. doi:10.1080/11934574.2011.627980

Schenke, K., Nguyen, T., Watts, T. W., Sarama, J., & Clements, D. H. (2017). Differential effects of the classroom on African American and non-African American’s mathematics achievement. *Journal of Educational Psychology, 109*, 794–811. doi:10.1037/edu0000165

Shanley, L. (2016). Evaluating longitudinal mathematics achievement growth: Modeling and measurement considerations for assessing academic progress. *Educational Researcher, 45*, 347–357.

Smith, T. M., Cobb, P., Farran, D. C., Cordray, D. S., & Munter, C. (2013). Evaluating math recovery assessing the causal impact of a diagnostic tutoring program on student achievement. *American Educational Research Journal, 50*, 397–428. doi:10.3102/0002831212469045

Turner, R. C., Ritter, G. W., Robertson, A. H., & Featherston, L. (2006, April). *Does the impact of preschool child care on cognition and behavior persist throughout the elementary years?* Paper presented at the American Educational Research Association, San Francisco, CA.

Tzuriel, D. (2001). *Dynamic assessment of young children*. New York, NY: Kluwer Academic.

Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education*. Utrecht, Netherlands: Freudenthal Institute, Utrecht University.

Weiland, C., Wolfe, C. B., Hurwitz, M. D., Clements, D. H., Sarama, J. H., & Yoshikawa, H. (2012). Early mathematics assessment: Validation of the short form of a prekindergarten and kindergarten mathematics measure. *Educational Psychology, 32*, 311–333.

Wright, B. D., & Stone, M. H. (1979). *Best test design: Rasch measurement*. Chicago, IL: Mesa Press.

Wright, S. P., Horn, S. P., & Sanders, W. L. (1997). Teacher and classroom context effects on student achievement: Implications for teacher evaluation. *Journal of Personnel Evaluation in Education, 11*, 57–67.

**Authors**

DENIS DUMAS is an assistant professor of research methods and statistics in the Morgridge College of Education at the University of Denver. His research interests are in understanding student learning and cognition through the application and refinement of quantitative research methods.

DANIEL MCNEISH is an assistant professor of quantitative psychology at Arizona State University. His research interest is the development and testing of statistical methods for psychological and educational data with challenging data structures.

JULIE SARAMA is Kennedy Endowed Professor in innovative learning technologies at the Morgridge College of Education, University of Denver. Her research interests are in the development and evaluation of research-based educational technology, especially in early mathematics.

DOUGLAS CLEMENTS is Kennedy Endowed Professor in early childhood learning at the Morgridge College of Education, University of Denver. His research interests are in the learning and teaching of early mathematics, with an emphasis on the development of research-based curricula.