MAX DEHN, AXEL THUE, AND THE UNDECIDABLE

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INTRODUCTION

Dehn was not the only mathematician to develop the question that came to be known as the word problem (see Chapters 6 and 7). In addition to Dehn’s approach through geometric group theory, the word problem was formulated independently by Axel Thue for general tree structures in 1910 and for semigroups in 1914. In his book with Bruce Chandler, Dehn’s student Wilhelm Magnus remarked that Dehn and Thue knew each other and mentioned the amazing parallel between their discoveries:

What appears to be incidental or, if one prefers, miraculous, is the fact that independent of Dehn and independent of topology, a contemporary mathematician had begun to ask questions of the type of the word problem in combinatorial group theory, but in an even more general and highly abstract setting. We are referring to the work of Thue, who may be considered as the founder of a general theory of semigroups. With one widely quoted exception, this work of his is largely forgotten nowadays. We do not know whether Dehn was influenced by Thue, and we have reasons to doubt it. We know that Dehn knew Thue personally, but only very superficially. Dehn mentioned Thue’s work on occasion, observing that Thue’s papers dealt with combinatorial problems. But he never used them, and indeed there is no known direct application of Thue’s work to Dehn’s group-theoretic problems.\(^1\)

A similar statement occurs in Magnus’s Footnote 5]. There Magnus also remarked that Dehn’s wife Toni did not recall that Max Dehn had ever mentioned Axel Thue in her presence although Dehn had visited Norway quite a few times and was skilled in the Norwegian language. Moreover, there are no known personal relations among the students of Dehn and Thue’s only student, Thoralf Skolem. While Dehn’s work had spread quickly, we do not know when mathematicians became aware of Thue’s work on semigroups and what we now call Thue systems. We only know that a paper of Emil Post from 1947 mentions that Thue’s paper from 1914 was pointed out to Post by Alonzo Church. Around 1935–1955, the word problem became an attractive challenge for people working in the theory of computation (alias recursion theory) because it was, besides the Entscheidungsproblem, historically one of the first genuine mathematical problems which appeared to be potentially undecidable\(^2\). Our investigations indicate that Alonzo Church, Emil Post and other people at Princeton were a major driving force in bringing the word

\(^1\) Date: Final version, **.

\(^2\)
problem and the theory of computation together, thus placing the heritage of Dehn and Thue in the right historical context.

Many problems in mathematics are accessible through computation and algorithms. The example of the Euclidean algorithm quite prominently shows how effective mathematical thinking can be in inventing algorithms. Gottfried Wilhelm Leibniz was the first scientist who expressed in a precise way the role of a device (which he called calculus ratiocinator) being able to decide about the truth of all reasonable statements, not necessarily restricted to mathematics, by a sort of logical computation. Although Leibniz’s thoughts remained in an abstract realm, he worked on the realization of an arithmetic calculating machine during his whole life. The Analytic Engine conceived by Charles Babbage and Ada Lovelace was another attempt – albeit unsuccessful – towards a programmable computer. Finally, from about 1940 on, Konrad Zuse at Berlin and John von Neumann at Princeton started to construct the first fully Turing complete computers Z3 and ENIAC. This was of course the beginning of a success story of incredible impact.

In the early 20th century, the notion of algorithmic computability still had no underlying mathematically sound theory. Nevertheless people had a pragmatic idea what computability was supposed to mean, i.e., to reach a result in a finite number of computational steps. An example for this is the formulation of Hilbert’s 10th problem:

Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.³

In another direction, Hilbert started his program in proof theory (Hilbert’s program) after 1917 to obtain a solid foundation of mathematics with a method he called finitistic.⁴ Hilbert showed a lot of optimism⁵ that all metamathematical questions could be settled within a mathematical proof theory. In 1928, together with Ackermann, he posed his famous Entscheidungsproblem (decision problem). It asks for an algorithm in the spirit of Leibniz, which decides the provability of statements in first order axiomatic theories. By Gödel’s completeness theorem for first order logic, which he proved in his dissertation from 1929, this is equivalent to asking for satisfiability in all possible set-theoretic models. In 1931, Gödel’s two famous incompleteness theorems were discovered.⁶ The first theorem shows that first-order Dedekind–Peano arithmetic is incomplete in the sense that there are statements which are neither provable nor disprovable but true in the standard model. The second theorem states that the consistency of a theory at least as rich as Dedekind–Peano arithmetic cannot be proved as a syntactic formula within the theory. Hilbert’s proof theoretic program had to be modified in the sequel. Gödel’s proof used primitive recursive functions and the technique of Gödel numberings of arithmetic statements and proofs. Tarski had independently shown that arithmetic truth predicates exist only outside the realm of the theory, which implies Gödel’s incompleteness theorem.⁷ From his correspondence with John von Neumann we know that Gödel was aware of this result. However, he was not able to solve the Entscheidungsproblem at that time since the theory of computable functions was only developed in full depth after 1936. That a given mathematical problem like Hilbert’s 10th problem or the word problem might be undecidable was probably considered unlikely by most people before 1931.
But after Gödel’s achievements this became a more realistic possibility. We will see, however, that Dehn and Thue already realized the difficulty of the word problem very clearly around 1910.

A full-fledged theory of computation emerged around 1936 through the work of Church, Gödel, Herbrand, Kleene, Markov, Post and Turing. Using this, the undecidability of the Entscheidungsproblem and of the related Halteproblem was shown. See [Davis1965] for the precise history of these developments. In addition, Church showed the undecidability of the word problem for finitely generated semigroups in 1937. It still took many more years before Post and Markov gave the first proof of the undecidability of the word problem for finitely presented semigroups in 1947. Another five years passed until the word problem for finitely presented groups was shown to be undecidable by William Boone and Pyotr Novikov in 1952. Two decades later, the undecidability of Hilbert’s 10th problem was shown by Yuri Matiyasevich in 1970, building on work of Martin Davis, Hilary Putnam and Julia Robinson, see [Matiyasevich1970].

In the following text, we describe the impact that both Dehn and Thue had on the community of recursion theory, i.e., the theory of computation, and we shed some light on the period between 1936 and 1955 during which many people worked on proving the (un)solvability of the word problem. I would like to thank Steve Batterson, Martin Davis, Catherine Goldstein, Jemma Lorenat, John McCleary, Carl–Fredrik Nyberg–Brodda, Edmund Robertson, David Rowe, Marjorie Senechal, Reinhard Siegmund–Schultze, Jörn Steuding and Marcia Tucker for helpful remarks and assistance.

1. Max Dehn and the Word Problem for Groups

In his paper [Dehn1911] on the word problem, Dehn used the presentation of a (finitely presented) group \(G\) by generators and relations. As Dehn remarked, this concept had been studied before in detail by Walther von Dyck [vonDyck1882]. It came up even earlier in the work of William Rowan Hamilton.

In this paper, Dehn phrased the word problem, the conjugacy problem and the isomorphism problem for groups. The word problem asked to decide whether a given word \(w\) in \(G\) is equal to 1. The conjugacy problem extended this question to determine whether two words \(w, w'\) in \(G\) are conjugate and if they are, find \(u\) such that \(w' = uwu^{-1}\). The conjugacy problem implies the word problem, since a word \(w\) is equal to 1 if and only if it is conjugate to 1. Finally, the isomorphism problem aimed to determine whether two given groups \(G\) and \(G'\) are isomorphic.

In his own words, Dehn formulated the word problem, which he called Identitätsproblem, as follows:

Let an arbitrary element of a group be given by its composition out of generators. One shall provide a method which decides in a finite number of steps whether this element is equal to the identity or not.\(^{10}\)

Dehn was aware that the word problem might turn out to be difficult for a general group. He wrote:

Here we have three fundamental problems whose solution is very important and probably not possible without a thorough study of the subject.\(^{11}\)
Hence, Dehn was aware of the difficulty of the word problem, perhaps even of its potential unsolvability. In his considerations, Dehn used what he called the Gruppenbild (Cayley graph). In figure 1, this is illustrated for the dihedral group $D_5$. In [Dehn1912], Dehn solved the word problem for infinite surface groups, i.e., fundamental groups of orientable closed 2-manifolds.

![Cayley graph of $D_5$](image)

Figure 1. (Undirected) Cayley graph of $D_5$ with two generators $\sigma = (1 \, 2 \, 3 \, 4 \, 5)$ (rotation) and $\tau = (2 \, 5)(3 \, 4)$ (reflection, dashed). All figures in this text were prepared by the author using tikz.

The idea of this proof is described by Dehn in [Dehn1912]. In modern language, he wrote that one needs to prove that any non-trivial closed loop in the Cayley graph of a surface group $G$ of genus $g \geq 2$ contains more than half of the defining relations, or can be freely reduced. In this way, Dehn provided an algorithm (called Dehn’s algorithm today) to solve the word problem for $G$ and some other groups. It can be enumerated in a quite general form:

1. Let any freely reduced word $w = w_0$ be given. We construct a finite sequence $w_0, w_1, \ldots, w_n$ of freely reduced words by recursion such that $w = w_0$ and the lengths decrease $|w_0| > |w_1| > \cdots > |w_i| > \cdots$.
2. If $w_i$ is already constructed and empty, i.e., $w_i = 1$, then terminate.
3. If $w_i$ contains a subword $a$ such that for some relation $r = ab$ and $|a| > |r|/2$, then replace $a$ by $b^{-1}$ in $w_i$ and obtain $w_{i+1}$.
4. If not, terminate at step $i$.

There is the notion of a Dehn presentation for groups which is sufficient for Dehn’s algorithm to work, see [Miller2014, p. 345]. An example where Dehn’s algorithm does not apply is the genus one case, i.e., the fundamental group of the torus, and—more generally—the free abelian group $\mathbb{Z}^n$ for $n \geq 2$ [Miller2014, p. 345]. Note that the word problem is nevertheless easy to solve for free abelian groups of finite rank.

There are large classes of groups beyond surface groups for genus $g \geq 2$ to which Dehn’s algorithm can be extended. One direction where this was successful is the field called small cancellation theory. It deals with (finitely presented) groups...
where the relations have small overlap. We refrain from presenting any definitions and refer to the books [LyndonSchupp1977] and [Sims1994] for an account of this theory. Historically, small cancellation theory was mainly developed in [Tartakovskii1949], [Greendlinger1960], [Lyndon1966] and [Schupp1968]. For example, in [Greendlinger1960] it is proved that a group satisfying a small cancellation property denoted by $C'(1/6)$ has solvable word problem.

Small cancellation is not a geometric concept. A geometric class of finitely presented groups where Dehn’s algorithm works are word–hyperbolic groups which satisfy certain metric conditions on the Cayley graph, see [Gromov1987]. Small cancellation groups satisfying the $C'(1/6)$–condition are examples of word–hyperbolic groups. It is a theorem due to Gromov and Olshanskii that for a general group $G$ – in the sense that $G$ is in some way chosen randomly – the Dehn algorithm solves the word problem for $G$, see [Gromov1987] [Olshanskii1992].

For other algorithms related to the word problem see [KnuthBendix1970] and [ToddCoxeter1936]. The Knuth–Bendix algorithm [19] for completing term rewriting systems can be used to solve the word problem for the large class of automatic groups [Epstein et al.1992] which contains word–hyperbolic groups and braid groups. The Todd–Coxeter algorithm, which is primarily a coset enumeration method for finite index subgroups, can also be applied to the word problem. The historical survey of John Stillwell [Stillwell1982] on the word problem contains many examples of finitely presented groups with a solvable word problem. In the following table we list some of them:

| Type of group                  | Reference               |
|-------------------------------|-------------------------|
| Surface groups                | Dehn1912                |
| Trefoil knot group            | Dehn1914                |
| Subgroups of free groups (abelian or not) | Nielsen1921 |
| Braid groups                  | Artin1925               |
| One–relator groups            | Magnus1932              |
| Residually finite groups      | McKinsey1943            |
| Hypo–abelian groups           | Engel1949               |
| Linear groups                 | Rabin1960               |
| Knot groups                   | Waldhausen1968          |
| Hyperbolic groups             | Alonso et al.1991       |
| Automatic groups              | Epstein et al.1992      |

Among these people, Engel and Magnus were students of Dehn (see chapter 6). [21] Magnus proved his Freiheitssatz [22] in 1930 to treat the one–relator case. Amazingly, the word problem for one–relator semigroups is still open, see [Nyberg–Brodda2021]. Other finitely presented groups for which the word problem has been solved are finite groups, polycyclic groups, Coxeter groups and finitely presented simple groups. We refer to the textbooks [LyndonSchupp1977], [Sims1994] for these and other cases.

2. Axel Thue and the Word Problem for Semigroups

Axel Thue was a number theorist with broad interests and he was well–known for his work in arithmetic far beyond Norway. He held a chair position in applied mathematics at Oslo from 1903 on. Some of Thue’s most important work in number theory is concerned with diophantine equations. For example, he looked at
integer solutions of equations $f(x, y) = c$ for a homogenous polynomial $f$ with integer coefficients and showed that the number of those solutions is finite, provided certain conditions on $f$ are valid, in particular the degree of $f$ needs to be at least three. Such results were later extended by Carl Ludwig Siegel and are the basis of finiteness conjectures in modern arithmetic geometry (on Siegel, see chapter 5). In the same paper [Thue, p. 232], published in Crelle’s Journal in 1909, Thue looked at generalizations of Liouville’s result which bounds the approximation of irrational algebraic numbers by rational numbers from below. Thue’s results later were strengthened by Siegel in his 1929 dissertation under Edmund Landau and in 1955 by Klaus Friedrich Roth who obtained an optimal estimate. Today the final result is known as the Thue–Siegel–Roth theorem. As a consequence, Roth received a fields medal during the ICM at Edinburgh in 1958.

Thue claimed that it happened often that he discovered results which were previously obtained by others. For example, he wrote in a letter to Elling Holst from 1902 [Thue, p. xxii] that he had discovered the transcendence of $e$ and $\pi$ independently of Hermite and Lindemann during his time as a teacher at the technical college in Trondheim, i.e., between 1894 and 1902.

Among Thue’s many papers are also four quite abstract papers about trees, words, semigroups and term rewriting which were written in German and belong to mathematical logic:

- "Über unendliche Zeichenreihen" [Thue, p. 139–158] from 1906.
- "Die Lösung eines Spezialfalles eines generellen logischen Problems" [Thue, p. 273–310] from 1910.
- "Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen" [Thue, p. 413–477] from 1912.
- "Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln" [Thue, p. 493–524] from 1914.

The papers from 1906 and 1912 present the general theory of trees and words. For example, in the 1906 paper Thue proves theorems which assert that there are infinitely long sequences consisting of three or four letters which are square–free, i.e., no finite length word $B$ occurs twice as $BB$ in the sequence. The 1906 paper continues by showing that there is an infinite sequence

$$011010011011001011001001011001101001 \cdots$$

in two letters which is cube–free, i.e., no finite word $B$ occurs as $BBB$. Thue’s 1912 paper elaborates on the case of two and three symbols even more and classifies irreducible sequences on two letters. See [Hedlund1967] for all this.

It turned out that such a sequence had already been discovered before Thue by Eugène Prouhet in 1851 (solving the Tarry–Escot problem) and later, independently, by others. The sequence shows that an infinite chess game is possible without violating certain chess regulations [Morse1938, MorseHedlund1944].

Thue’s paper from 1910 introduced a very general philosophical (or logical) problem which he phrased in a metamathematical language. In modern language, he considered term rewriting systems for tree–like structures. The 1914 paper is concerned with words (Zeichenreihen) instead of binary trees. The underlying algorithmic problems in the case of words are known as (Semi–)Thue systems or as term rewriting systems [Buchi1989, p. 181].
Let us describe some more details of Thue’s work. We look at finite, binary, rooted trees as in figure 2 (a copy of figure 3 from Thue, p. 275). The outer leaves correspond to variables $A$–$F$ of a certain type (either of type $p$ or $q$ in figure 2). Thue explains that for him there is a theory of a certain logical kind behind all this (called Begriffe and Begriffskategorien by Thue). In the inner nodes going to the root, each time two values (of type $p$ resp. $q$ in figure 2) are combined by a binary operation into a new value of the indicated new type. Hence, going all the way to the root corresponds to the computation of a tree automaton which computes a value of type $p$ from the given values of the entry variables $A$–$F$.

These trees may be viewed as objects representing certain algebraic or logical terms, as in the following example of the associativity of addition:

$$(A + B) + C = A + (B + C).$$

The two trees corresponding to both sides of the equation are displayed in figure 3. Vice versa, a binary tree corresponds to a term. In summary, we see that Thue had already imagined the famous correspondence between trees and terms. Generalizations of this occur in Post’s work on canonical systems [Post1921]. Now, term rewriting means that such trees are transformed into other trees in single steps by replacing (i.e., rewriting) parts according to certain rules. Thue thought of this term rewriting problem as an algorithmic problem about the relation between two given trees $A$ and $B$ in his 1910 paper:
... so we ask in other words, whether one can find trees $C_1C_2 \ldots C_h$, such that $A \sim C_1 \sim C_2 \sim \ldots \sim C_h \sim B$.\textsuperscript{27}

Thue even claimed its possible undecidability by continuing:

> A solution of this problem in the most general case may perhaps be connected with unsurmountable difficulties.\textsuperscript{28}

Thue’s 1910 paper essentially contains the word problem without any relations [Büchi1989, p. 235]. In the 1914 paper, Thue reduced this problem from binary to unary trees, i.e., to words or strings of letters (Zeichenreihen) and he introduced (Semi-)Thue systems consisting of a finite set of words $a_1, \ldots, a_n$ over a given countable alphabet together with a finite set of operations (called productions) given by pairs of words $(g, h)$. Any word of the form $xgy$ with possibly empty words $x, y$ may be replaced by the word $xhy$ for any given production $(g, h)$. A (Semi-)Thue system is called Thue system, if for each production $(g, h)$ also the inverse production $(h, g)$ is contained in $G$. By composing words via concatenation, Thue systems can be viewed as semigroups with a finite presentation. This paper resembles a remarkable point in history where the idea of a semigroup was born. With this setup, the term rewriting problem becomes the word problem for (finitely generated) semigroups in [Thue, p. 494]. Notice that one replaces the question $w = 1$ in the word problem for groups by $w_1 = w_2$ for two words $w_1, w_2$ in the case of semigroups. Thue describes the problem as follows: assuming an arbitrary choice of given words $A$ and $B$, to find a method through which one can always decide after a computable number of operations whether any two given words are equivalent with respect to $A$ and $B$.\textsuperscript{29}

The 1914 paper is cited very frequently in the literature, for example by Post in 1947 [Post1947], whereas the three other papers are mostly unknown. Richard Büchi speculates in [Büchi1989, p. 235] that Post might have known Thue’s papers already in 1921, when he wrote his paper [Post1921] on canonical systems which may be seen as a continuation (and extension) of Thue’s ideas. But this is not reflected in the first sentence of [Post1947]:

> Alonzo Church suggested to the writer that a certain problem of Thue might be proved unsolvable.\textsuperscript{30}

3. DEHN, THUE, AND THE PRINCETON COMMUNITY

Dehn’s student Wilhelm Magnus was a faculty member in Frankfurt from 1933 to 1938. He rejected the Nazi government in public and was suspended from office for this reason. During the second world war he had to work in a private company. In 1947 he was appointed to Göttingen but moved to the United States one year later and finally became member of the Courant Institute in 1950. Already in the academic year 1934/35, Magnus visited Princeton. This fact alone implies that in the mid 30’s the Princeton community was fully aware of the word problem. This applies in particular to the prominent topologists Solomon Lefschetz, James Alexander, Ralph Fox and Marston Morse (who arrived in 1935). The books by Lefschetz [Lefschetz1930] (1930) and Kurt Reidemeister [Reidemeister1932] (1932) refer to Dehn. Alexander and Fox were experts in knot theory at Princeton. We do not know much about the dissemination of the work of Axel Thue. Although we suspect that his work on number theory, in particular the paper from 1909, had been well-known to many people, his four articles on logic were probably not.
the other hand, with the help of Princeton librarians we found out that the journal in which Thue had published his papers was on the shelves in Princeton university between 1894 to 1960. Princeton University and the Institute for Advanced Studies (IAS) play a major role in the development of the theory of computability and in the history of the word problem. While the IAS was officially independent from Princeton, there was significant overlap among the early mathematics faculty, including Oswald Veblen, John von Neumann and James Alexander (see [Dyson2012]).

Veblen was of Norwegian descent although born in the United States in 1880. As professor at Princeton, Veblen spent the fall of 1913 visiting Oslo, Göttingen and Berlin [Batterson2007]. Veblen and Thue were both participants at the 1913 Scandinavian congress of mathematics, but we do not know whether they met at this occasion or at any time. After 1932, Veblen became a leading figure in the newly founded IAS at Princeton.

In both of his academic positions, Veblen supported the hiring of people in seemingly remote areas like mathematical logic. For example, the Polish immigrant Emil Post spent the year 1920–1921 at Princeton as a postdoc fellow. During this time he wrote his famous article on canonical systems [Post1921]. Later he spent time at Columbia, Cornell and New York, often interrupted by periods in which he suffered from manic attacks.

Without doubt, Veblen had an important impact via his student Alonzo Church who began studying at Princeton in 1924 and finished his dissertation under Veblen in 1927. Church was then a postdoc in Göttingen and Amsterdam between 1927 and 1928. He joined the Princeton faculty in 1929 and stayed until his retirement in 1967. After that he continued to teach at UCLA until 1992. Church had an impressive roster of students. His list of students include Boone, Collins, Davis, Henkin, Kleene, Rabin, Rogers, Rosser, Scott, Smullyan and Turing who all contributed to the theory of computation, the word problem or related areas of logic in some essential way.

4. THE RISE OF THE UNDECIDABLE

As we already mentioned, the year 1936 was the annus mirabilis for the theory of computation, also called recursion theory. It saw the birth of four notions of computability: the λ-calculus of Church [Church1936], the concept of a Turing machine [Turing1936], another machine concept by Post [Post1936], and the notion of partial recursive function (alias μ–recursive functions) by Kleene [Kleene1936], the latter building up on the work of Dedekind (between 1872 and 1888), Peano (1889), Skolem (1923), Gödel and Herbrand (1930–1934). Surprisingly, these four definitions are equivalent. It is conjectured that there is no other feasible notion of computability beyond them. This statement is often called Church’s thesis [32].

Equipped with a notion of a (partially defined) computable function \( f: \mathbb{N} \to \mathbb{N} \), one can define recursively enumerable sets \( S \subset \mathbb{N} \) as domains, or equivalently, as images of such maps. A set \( S \subset \mathbb{N} \) is called decidable, if \( S \) and its complement are both recursively enumerable, i.e., the characteristic function of \( S \) is computable. In this way, the algorithmic (un)solvability of a logical or mathematical problem, i.e., the computation of the characteristic function of the set \( S \) of Gödel numbers associated to the instances of the problem, is related to the (un)decidability of \( S \).
The existence of undecidable sets is the central paradigm in this theory. First examples in this direction were given by sets of natural numbers related to undecidable problems like the Entscheidungsproblem and the Halteproblem.\footnote{After it had been shown that these problems were undecidable (i.e., algorithmically unsolvable), people were looking for more traditional math problems for which undecidability could be shown. It turned out that the word problem for groups and semigroups was a suitable candidate. Other undecidable sets later occurred in the negative solution of Hilbert’s tenth problem.\footnote{In 1937, Church announced that he could prove the undecidability of the word problem for a particular finitely generated semigroup which is not finitely presented:}}

By a semigroup is meant a set in which the product of any two elements is a unique element of the set, the multiplication being associative but not necessarily obeying a law of cancellation. Consider the system of combinators, in the sense of Rosser (Duke Mathematical Journal, vol. 1 (1935), p. 336), allowing as equivalence operations $r$–conversions, $p$–conversions, and also the operations (allowed by Curry) of replacing $BI$ by $I$ and inversely. This system is a semigroup, with identity element $I$, if we take as multiplication the operation (introduced by Curry) which is denoted by Rosser as $\times$. From the relations $ab = T b \times T a \times B \times T$ and $T(ab) = T b \times T a \times B$ it follows that every element is expressible as a product formed out of the four particular elements $TI, TJ, B, T$. The semigroup thus has a finite set of generators, although the set of generating relations must apparently be infinite. There is, however, an effective process of writing out the series of generating relations to as many terms as desired; also an effective means of distinguishing generating relations from others. From the results of the author (American Journal of Mathematics, vol. 58 (1936), pp. 345–363), it follows that the word problem of this semigroup is unsolvable. (Received April 14, 1937.)\footnote{Higman type embedding theorems, which were available only much later, can be used to show that Church’s construction can be embedded into a finitely presented semigroup. Post proved the undecidability of the word problem for finitely presented semigroups (without cancellation) in 1947 \cite{Post1947}. He mentioned that Church pointed out the 1914 paper of Thue to him. The same result was also proved in the same year (but independently) in \cite{Markov1947} by A. A. Markov jr., the son of the famous mathematician who invented Markov processes. We refer to the survey article of Miller \cite{Miller2014} and Rotman’s book \cite{Rotman1995} for more details on these and the following results.}

The method Post used was to associate a (Semi–)Thue system $G_T$ to any Turing machine $T$ \cite{Post1947}. In a different way, the undecidability of the Halteproblem can be used to show the undecidability of the word problem for some (Semi–)Thue system $G_T$, see \cite{Oberschelp1993, §33} and \cite{Büchi1989, p. 181}. Turing proved the undecidability of the word problem for semigroups admitting cancellation in 1950 \cite{Turing1950} in an attempt to obtain the full result for groups. However, this result did not imply the corresponding result for groups, since the semigroups used in the proof at least a priori cannot be embedded into groups.
The word problem for groups was successfully attacked during the following years (see chapters 6 and 7 for further details). Max Dehn died in 1952 shortly before Novikov announced his proof of the undecidability of the word problem for groups in [Novikov1952]. Novikov’s published proof of this result in [Novikov1955] uses Turing’s result from the 1950 paper although it employs a different method. Church’s student William Boone independently proved the undecidability of the word problem for (finitely presented) groups during his thesis. His final results were published in [Boone1959] after a long series of six papers [Boone1954–57]. Boone used Post’s semigroup approach [Post1946] for his proof. It is known that Fox and Gödel had many conversations with Boone during this work. John Britton independently gave a proof in 1958 and later developed Boone’s and other methods further [Britton1963]. There is a fascinating set of technical results, called Britton’s lemma and Novikov’s principal lemma, in Boone’s, Britton’s and Novikov’s proofs which turned out to be related to each other, see [Miller2014, p. 355] and [Rotman1995, Ch. 12].

In the sequel, much simpler proofs were discovered in parallel with developments in group theory. One of the shortest proofs uses Higman’s embedding theorem from [Higman1961] and we describe it in the following section. The same method also implies that there exists a finitely presented group $G$ containing isomorphic copies of all finitely generated groups having solvable word problem. This group $G$ then does not have a solvable word problem. In other words, there is no uniform algorithm for all finitely presented groups that have a solvable word problem.\(^{36}\)

We remark that there are many other properties of groups which cannot be recognized algorithmically, e.g., the properties of being trivial, finite, abelian, nilpotent, solvable, free, torsion-free, residually finite, simple or automatic.\(^{37}\) It is not difficult to prove the related undecidability of the homeomorphism problem\(^{38}\) for manifolds from this.

5. Explicit Unsolvable Examples

As of today, many construction principles are known that yield finitely presented groups for which the word problem is unsolvable (i.e., undecidable). One particular method is quite simple and goes back to work of Higman and others [Higman1961]. To obtain such an example, take the finitely generated (and recursively presented) group

$$G = \langle a, b, c, d \mid a^{-e}ba^e = c^{-e}de^e \forall e \in E \rangle$$

where $E \subset \mathbb{N}$ is a recursively enumerable, but non-recursive, set, i.e., an undecidable set. Then use Higman’s embedding theorem [Higman1961] to embed $G$ into a finitely presented group $G'$ with unsolvable word problem. Explicit examples are given in [Borislav1969] and [Collins1986].

An simple example of Gregory S. Tseytin [Tseytin1957, Collins1986] for a semigroup with unsolvable word problem – even in the stronger sense that on a fixed word $w$ the decision problem $w' = w$ for any other word $w'$ is undecidable – is given by

$$G = \langle a, b, c, d, e \mid ac = ca, ad = da, bc = cb, bd = db, ce = eca, de = edb, cdca = cdcac, caaa = aaaa, daaa = aaaa \rangle$$

The word in question is $w = aaaa$. There is an example with 2 generators and 3 relations in [Matiyasevich1967].
A yes/no decision problem for an infinite set of mathematical objects is called decidable (alias algorithmically solvable, or recursive) if the set $S$ of Gödel numbers of the involved mathematical objects is a decidable subset of $\mathbb{N}$. This means that there is an algorithm which has output 1 if $n \in S$ and output 0 else.

The finitistic approach somehow rejects the use of infinitely many steps. This is related but not the same as the intuitionistic and constructivist approach of Brouwer, Kronecker, and Weyl which Hilbert disliked. Gödel’s system $T$, or equivalently Gentzen’s proof for the consistency of arithmetic, may also be considered as finitistic in some sense.

Gödel only published the first theorem, see [Gödel2003, Vol. I] and [Plato2020] for the full story. Here, completeness has a different meaning than in Gödel’s dissertation. This means that the set of Gödel numbers of true statements in the standard model of the natural numbers is not an arithmetic set, hence not even recursively enumerable.

Church’s example was not finitely presented, as it had infinitely many relations. This is usually denoted by $G = \langle s_1, \ldots, s_n \mid r_1, \ldots, r_m \rangle$. A relation $r$ is given by a word $r$, and the notation amounts to identifying every occurrence of $r$ with the trivial word, i.e., setting $r = 1$. Here, a word $w$ (of length $\ell$) is a finite combination of generators, possibly with repetition: $w = g_1^{\pm 1} \cdots g_k^{\pm 1}$. The length of a word $w$ is denoted by $|w|$. The inverse $w^{-1}$ of a word $w$ is obtained by inverting all $g_i$ involved and reversing the order, e.g., $(g_1g_2)^{-1} = g_2^{-1}g_1^{-1}$.

German original in [Dehn1911]: 1. Das Identitätsproblem: Jede festwerte Element der Gruppe ist durch seine Zusammensetzung aus den Erzeugenden gegeben. Man soll eine Methode angeben, um mit einer endlichen Anzahl von Schritten zu entscheiden, ob dies Element der Identität gleich ist oder nicht.

German original in [Dehn1911]: Hier sind es vor allem drei fundamentale Probleme, deren Lösung sehr wichtig und wohl nicht ohne eindringliches Studium der Materie möglich ist.

Magnitude in [ChandlerMagnus1982] p. 55] cites Dehn as follows: Solving the word problem for groups may be as impossible as solving all mathematical problems.

Given a finitely presented group $G = \langle S \mid R \rangle$ with set of generators $S = \{s_1, \ldots, s_n\}$, the vertices are all elements of $G$, and the (directed) edges connect $g$ and $gs$ for every $g \in G$ and $s \in S \cup S^{-1}$. The edges are usually colored.

These are of the form $G = \langle a_1, b_1, \ldots, a_p, b_p \mid \prod_{i=1}^{q} a_i b_i a_i^{-1} b_i^{-1} \rangle$ with one relation.

German original: Zum Beweise haben wir bloß zu zeigen, daß jedes geschlossene Streckenzug in dem Gruppenbild, also in dem 4p-Eckennetz, mit einem Netzpolygon mehr als 2p Seiten gemein hat oder zweimal in entgegengesetztem Sinne und nacheinander durchlaufene Strecken besitzt.

Freely reduced words have no substrings of the form $x^{-1}x$ or $xx^{-1}$.

Hyperbolic groups are defined as follows. Consider the Cayley graph of $G$ and endow it with its graph metric. Then $G$ is word-hyperbolic, if the resulting topological space is hyperbolic in the sense of [Gromov1987], i.e., there is a constant $\delta > 0$ such that any triangle is $\delta$-thin.

Donald Knuth was a great-grandson of Thue via Thue–Skolem–Ore–Hall–Knuth.

As later observed in [Huber–Dyson1964] as well as in [Mostowski1966].

A list of students of Dehn is contained in [MagnusMoufang1954].

The Freiheitssatz asserts that leaving away at least one generator appearing in the relation induces a free subgroup in any one-relator group $G$.

The equation defines a plane curve in the projective plane of the same degree with equation $f(x, y) = c \cdot x^\deg(f)$. The other conditions on $f$ which we did not mention take care that this curve is not rational, i.e., the image of a projective line.

The theorem asserts that for every algebraic number $\alpha$ and every $\varepsilon > 0$ the inequality $|\alpha - \frac{p}{q}| < q^{-\varepsilon}$ has only finitely many solutions in coprime integers $p, q$. 

Notes

1See [ChandlerMagnus1982, p. 54].

2A yes/no decision problem for an infinite set of mathematical objects is called decidable (alias algorithmically solvable, or recursive) if the set $S$ of Gödel numbers of the involved mathematical objects is a decidable subset of $\mathbb{N}$. This means that there is an algorithm which has output 1 if $n \in S$ and output 0 else.

3German original in [Hilbert1900]: Eine diophantische Gleichung mit irgendwelchen Unbekannten und rationalen ganzen Zahlenkoeffizienten sei vorgelegt: Man soll ein Verfahren angeben, nach welchem sich mittels einer endlichen Zahl von Operationen entscheiden läßt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist.

4The finitistic approach somehow rejects the use of infinitely many steps. This is related but not the same as the intuitionistic and constructivist approach of Brouwer, Kronecker, and Weyl which Hilbert disliked. Gödel’s system $T$, or equivalently Gentzen’s proof for the consistency of arithmetic, may also be considered as finitistic in some sense.

5See Hilbert’s famous words: “Wir müssen wissen, wir werden wissen”.

6Gödel only published the first theorem, see [Gödel2003, Vol. I] and [Plato2020] for the full story. Here, completeness has a different meaning than in Gödel’s dissertation.

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9This is usually denoted by $G = \langle s_1, \ldots, s_n \mid r_1, \ldots, r_m \rangle$. A relation $r$ is given by a word $r$, and the notation amounts to identifying every occurrence of $r$ with the trivial word, i.e., setting $r = 1$. Here, a word $w$ (of length $\ell$) is a finite combination of generators, possibly with repetition: $w = g_1^{\pm 1} \cdots g_k^{\pm 1}$. The length of a word $w$ is denoted by $|w|$. The inverse $w^{-1}$ of a word $w$ is obtained by inverting all $g_i$ involved and reversing the order, e.g., $(g_1g_2)^{-1} = g_2^{-1}g_1^{-1}$.

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24The theorem asserts that for every algebraic number $\alpha$ and every $\varepsilon > 0$ the inequality $|\alpha - \frac{p}{q}| < q^{-\varepsilon}$ has only finitely many solutions in coprime integers $p, q$. 

Notes
A sequence is irreducible if it is square–free, i.e., no consecutive blocks $BB$ appear.

Notably, Marston Morse (1921), see Morse1921, Kurt Mahler (1929), and the chess player Max Euwe (1929).

German original in Thue, p. 280]: ... so fragen wir mit anderen Worten, ob man solche Bäume $C_1C_2\ldots C_h$ finden kann, sodass $A \sim C_1 \sim C_2 \sim \ldots \sim C_h \sim B$.

See SteinbyThomas2000. German original in Thue, p. 280]: Eine Lösung dieser Aufgabe im allgemeinsten Falle dürfte vielleicht mit unüberwindlichen Schwierigkeiten verbunden sein.

German original in Thue, p. 494]: Bei beliebiger Wahl der gegebenen Zeichenreihen $A$ und $B$ eine Methode zu finden, durch welche man nach einer berechenbaren Anzahl von Operationen immer entscheiden kann, ob zwei beliebige gegebene Zeichenreihen in Bezug auf die Reihen $A$ und $B$ äquivalent sind oder nicht.

See Post1947.

The Kristiana Videnskabs Selskabets Skrifter, Mathematis–Naturvidenskabelig Klasse I, superseded after 1924 by the journal of the academy Skrifte r utgitt av det Norske Videnskaps–Akademi i Oslo I, Matematiske–Naturvidenskabelig Klasse.

Historically more correct it should be called Church–Markov–Post–Turing thesis. The relevant literature in this field is reprinted in Davis1965.

The Halteproblem has the following simple interpretation. If we look at computable partial functions $f: \mathbb{N} \rightarrow \mathbb{N}$, then it is possible to define a sequence of Turing machines $T_n$ labeled by $n \in \mathbb{N}$ such that each computable partial function $f$ can be computed by at least one $T_n$. Then the set $S$ of all $n$ such that $T_n$ halts on input $n$ is undecidable. See Davis1965.

This Hilbertian problem asks for an algorithm to decide whether a polynomial system of equations over the integers has a non–trivial integer solution. This problem turned out to be undecidable by showing that every recursively enumerable set is diophantine, i.e., the projection of the zero set of a system of integer polynomial equations. By applying this to an arbitrary undecidable set $S$, one shows that the family $X_s$ of zero sets over every $s \in S$ has the property that one cannot decide whether $X_s$ is empty or not.

See Church1937.

This fact is called the Boone–Rogers theorem.

This is a theorem of Adian and Rabin, see Miller2014, p. 366 for a short proof.

This problem asks for deciding whether two given $n$–manifolds are homeomorphic (for $n \geq 4$). This was treated by A. A. Markov in 1958.

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