Representations of noncommutative quantum mechanics and symmetries

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Abstract. We present a unified approach to representations of quantum mechanics on noncommutative spaces with general constant commutators of phase-space variables. We find two phases and duality relations among them in arbitrary dimensions. Conditions for physical equivalence of different representations of a given system are analysed. Symmetries and classification of phase spaces are discussed. Especially, the dynamical symmetry of a physical system is investigated. Finally, we apply our analyses to the two-dimensional harmonic oscillator and the Landau problem.

1 Introduction

The problem of quantum mechanics on noncommutative spaces can be understood in the framework of deformation quantization. It is a subject with a long history starting with works of Wigner, Weyl and von Neumann (see Refs.[1] for a recent review). More recently, the investigation of noncommutative quantum mechanics was inspired by the development that led to noncommutative field theory. Namely, it was realized that low-energy effective field theory of various D-brane configurations has a configuration space which is described in terms of noncommuting, matrix-valued coordinate fields [2]. Then, it was shown that, in a certain limit, the entire string dynamics can be described by minimally coupled gauge theory on noncommutative space [3]. Intensive studies of field theories on various noncommutative spaces [4] were also inspired by connection with M-theory compactifications [5] and more recently, by the matrix formulation of the quantum Hall effect [6]. In order to study phenomenological consequences of noncommutativity, a noncommutative deformation of Standard model have been constructed and analysed [7].

In the last two years a lot of work has been done in analysing and understanding quantum mechanics (QM) on noncommutative (NC) spaces [8–18] and also in applying it to different physical systems in order to test its relevance to the real world [19,20]. Still, there are many different views and approaches to noncommutative physics [21–23]. Some important questions, such as physical equivalence of different noncommutative systems, as well as their relation to ordinary quantum mechanics with canonical variables have not been completely resolved. The symmetries and the physical content in different phases have not been completely elucidated, even in the simplest case of harmonic oscillator on the noncommutative plane.

In this paper, we present a unified approach to representations of NCQM in arbitrary dimensions. Conditions for physical equivalence of different representations of a given system are analysed. We show that there exist two phases in parameter space. Phase I can be viewed as a smooth deformation of ordinary QM, where all physical quantities have a smooth limit to physical quantities in ordinary QM. Phase II is qualitatively different from phase I and cannot be continuously connected to ordinary QM. There is a discrete duality transformation connecting the two phases.

Furthermore, we investigate symmetry transformations preserving commutators, the Hamiltonian and also the dynamical symmetry of the physical system. We analyse the angular momentum generators, and give conditions for their existence.

We demonstrate our general results on the simple example of harmonic oscillator on a noncommutative plane. Especially, we describe dynamical symmetry structure and discuss uncertainty relations. Finally, we briefly comment on the NC Landau problem.

2 Noncommutative quantum mechanics and its representations

Let us start with the two-dimensional noncommutative coordinate plane $X_1, X_2$ and the corresponding momenta

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While in phase II the number of such states is finite. It may have unusual properties. Namely, it was shown in usual sense may not exist. Moreover, even when it exists, \( M \) is symmetric matrix \( M \) is parametrized by 6 real parameters

\[
M = \begin{pmatrix}
0 & h_1 & \theta & \phi_1 \\
-h_1 & 0 & \phi_2 & B \\
-\theta & -\phi_2 & 0 & h_2 \\
-\phi_1 & -B & -h_2 & 0
\end{pmatrix}
\]  

(2)

and the determinant \( \det M = (h_1 h_2 - \theta B + \phi_1 \phi_2)^2 \) is positive. The critical point \( \det M = 0 \) divides the space of the parameters into two phases: phase I for \( \kappa = h_1 h_2 - \theta B + \phi_1 \phi_2 > 0 \) and phase II for \( \kappa < 0 \). The ordinary, commutative space \( M_0 \)

\[
M_0 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]  

(3)

has \( \kappa = 1 \) and belongs to phase I. Therefore, we can view phase I as a continuous, smooth deformation of ordinary quantum mechanics. The critical point \( \kappa = 0 \) corresponds to reduction of dimensions in phase space and to infinite degeneracy of states and is related to the (noncommutative) Landau problem [9] (see also the "exotic" approach [21]).

If we define the angular momentum \( J \) as

\[
[J, X_a] = i \varepsilon_{ab} X_b, \quad [J, P_a] = i \varepsilon_{ab} P_b, \quad a, b = 1, 2
\]  

(4)

or

\[
[J, U_i] = i E_{ij} U_j, \quad i, j = 1, 2, 3, 4
\]  

(5)

then for a given regular matrix \( M \) we can construct the angular momentum only if \( [E, M] = 0 \). This condition is fulfilled when \( \phi_1 = \phi_2 = h_1 = h_2 \). Then

\[
J = -\frac{1}{2} (EM^{-1})_{ij} U_j U_j.
\]  

(6)

We see that for general \( M \) the angular momentum in the usual sense may not exist. Moreover, even when it exists, it may have unusual properties. Namely, it was shown in Ref.[12] that in phase I system could have an infinite number of states for a given value of the angular momentum, while in phase II the number of such states is finite.

Now, let us assume that the Hamiltonian of the system describes the motion of a single particle on a noncommutative plane:

\[
H = \frac{1}{2} \dot{p}^2 + V(X^2),
\]  

(7)

with a discrete spectrum \( E_{n_1,n_2} \), where \( n_1, n_2 \) are non-negative integers. The pair \( (H(U), M) \) defines a system with a given energy spectrum and the corresponding energy eigenfunctions. We wish to characterize all systems \( (H'(U'), M') \) with the same spectrum. The class of such systems is very large and can be defined by all real, nonlinear, regular transformations \( U_i' = U_i'(U_j), \quad U_i = U_i(U_j) \).

We restrict ourselves to linear transformations \( GL(4, \mathbb{R}) \) in order to keep the matrix elements \( M'_{ij} \) independent of phase-space variables. Among these, of special interest are \( O(4) \) orthogonal transformations changing commutation relations, and the group of transformations isomorphic to \( Sp(4) \) keeping \( M \) invariant. Systems with the same energy spectrum connected by transformations that keep commutation relations invariant are physically equivalent. In both cases, the Hamiltonian generally changes, but the energy spectrum is invariant.

Let us consider \( O(4) \) transformations. The important property [24,14] is that there exists an orthogonal transformation \( R \) such that

\[
\tilde{R}^T MR = \begin{pmatrix}
0 & |\omega_1| & 0 & 0 \\
|\omega_1| & 0 & 0 & 0 \\
0 & 0 & 0 & |\omega_2| \\
0 & 0 & -|\omega_2| & 0
\end{pmatrix},
\]  

(8)

where \( |\omega_1| \geq |\omega_2| \geq 0 \) and \( \det M = \omega_1^2 \omega_2^2 \geq 0 \). The matrix \( \tilde{R} \) is unique up to the transformations \( S \in SO(4) \):

\[
\tilde{R} = S \tilde{R}, \quad S^T MS = M.
\]  

(9)

The first (second) phase is characterized by \( \det \tilde{R} = +1 \) (\( \det \tilde{R} = -1 \)). For the two-dimensional case, the eigenvalues of the general matrix \( iM \), Eq.(2), are

\[
\omega_{1,2} = \pm \frac{1}{2} \sqrt{(\theta - B)^2 + \phi_1^2 + \phi_2^2 + (h_1 + h_2)^2} \pm \frac{1}{2} \sqrt{(\theta + B)^2 + \phi_1^2 + \phi_2^2 + (h_1 - h_2)^2}.
\]  

(10)

Notice that \( \omega_1 \) is always positive, while \( \omega_2 \) changes the sign at the critical point \( \det M = 0 \), i.e., when \( \theta B - \phi_1 \phi_2 = h_1 h_2 \).

The matrix \( \tilde{R} \) is universal, i.e., there exists the matrix \( R \in SO(4) \), with \( \det R = 1 \) such that

\[
R^T MR = \begin{pmatrix}
0 & \omega_1 & 0 & 0 \\
-\omega_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_2 \\
0 & 0 & -\omega_2 & 0
\end{pmatrix} = J_\omega,
\]  

(11)

regardless of \( \omega_2 \) being positive, zero, or negative. When \( \omega_2 < 0 \), we use \( \tilde{R} = RF \) to obtain Eq.(8), where the flip matrix \( F \in O(4) \) is given by

\[
F = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]  

(12)

At the critical point (\( \omega_2 = 0 \)) both \( R \) and \( RF \) satisfy Eq.(8).

The most general orthogonal matrix \( R \) depends on six continuous parameters. For a fixed values \( \{\omega_1, \omega_2\} \), the number of parameters of the matrix \( M \) is the same as
the number of parameters of $R$. As we have already mentioned, there exist orthogonal matrices that commute with $M$, Eq.(9), and these matrices form a group isomorphic to $U(1) \times U(1)$. We can use this symmetry to fix two parameters in the matrix $M$, and we choose $h_1 = h_2 = 1$ or $h_1 = -h_2 = 1$. This parametrization covers all pairs $\{\omega_1, \omega_2\}$ such that $\omega_1^2 + \omega_2^2 \geq 2$.

The eigenvalues $\omega_1, \omega_2$ have the meaning of the "Planck" constants for new variables:

$$U^0_i = R^T_{ik} U_k,$$
$$[X^0_a, P^0_b] = i\omega_a \delta_{ab}, \ [X^0_a, X^0_b] = [P^0_a, P^0_b] = 0. \quad (13)$$

We have transformed the noncommutative system $(H(U), M)$ into $(H(RU^0), J_U)$ keeping the energy spectrum of the system invariant. Note that for system $(H(RU^0), J_U)$ we cannot define the angular momentum. In order to connect a noncommutative system with a quantum mechanical system in ordinary space, we perform the following transformation:

$$U^0 = D u^0 = \begin{pmatrix} \sqrt{\omega_1} & 0 & 0 & 0 \\ 0 & \sqrt{\omega_1} & 0 & 0 \\ 0 & 0 & \sqrt{\omega_2} & 0 \\ 0 & 0 & 0 & \sqrt{\omega_2} \end{pmatrix} u^0, \quad (14)$$

where the variables $u^0 = \{u^0_1, u^0_2, u^0_3, u^0_4\}$ are canonical, i.e., $[x^0_a, p^0_b] = i\delta_{ab}$ and $[x^0_a, x^0_b] = [p^0_a, p^0_b] = 0$. Now we have obtained $H(U) = H(RD_0 U^0)$ with the standard canonical relations $D_0$, Eq.(3). Transformation $D$, Eq.(14) is valid in both phases, but at the critical point it becomes singular. Also note that the composition $L_0 = RD$ has a smooth limit when $M \to M_0$.

In order to make contact with other representations in the literature [9,15], we perform a symplectic transformation on the canonical variables $u^0_i$:

$$u_i = S_{ij} u^0_j = V u^0_i V^\dagger, \quad (15)$$

where $S$ commutes with $M_0$ and

$$V = \exp \left( i \sum v_{ij} u^0_i u^0_j \right), \quad VV^\dagger = 1,$$

is a unitary operator corresponding to the symplectic transformation $S$. This symplectic transformation generates a class of ordinary quantum mechanical systems which are physically (unitary) equivalent. Of course, the initial system $(H(U), M)$ is not physically equivalent to the canonical ones, but all corresponding physical quantities can be uniquely determined. In Fig. 1 we show a simple graphic description of connection between different representations of NC quantum mechanics.

There is a "mirror-symmetric" diagram for phase II, obtained using the flip matrix $F$ (12), where $U' = R'R^T U$, $M' = R'R^T M R R^T$, and $L' = R'R^T L F$. The matrix $R'$ is any special orthogonal matrix. The universality of matrix $R$ means that we can choose $R'$ and $R$ to have the same functional dependence on matrix elements $M_{ij}'$ and $M_{ij}$, respectively. We have the discrete $Z_2$ symmetry connecting two components of group $O(4)$, or more generally, $G(4, R)$.

Starting from the matrix $M$, we can construct the matrix $R$ by finding eigenvalues and eigenvectors of the matrix $iM$, i.e., $R = U_M U_0^\dagger$, where

$$U_M^\dagger (iM) U_M = U_j^\dagger (iJ_0) U_j = \text{diag}(\omega_1, -\omega_1, \omega_2, -\omega_2).$$

For example, for $\phi_1 = \phi_2 = 0$, we can write the matrix $R$ in the following form:

$$R = \begin{pmatrix} \cos \varphi & 0 & \sin \varphi & 0 \\ 0 & \sin \varphi & 0 & \cos \varphi \\ 0 & \cos \varphi & 0 & -\sin \varphi \\ -\sin \varphi & 0 & \cos \varphi & 0 \end{pmatrix} \quad (16)$$

where we choose $\varphi \in (0, \pi/2)$, $\theta \geq 0$, $\theta + B \geq 0$ and

$$\cos \varphi = \frac{1}{\sqrt{1 + (B + \omega_2)^2}} = \frac{\omega_2 + \theta}{\sqrt{1 + (\omega_2 + \theta)^2}} = \sqrt{\frac{\omega_1 - B}{\omega_1 + \omega_2}},$$

The basic relations are

$$\omega_1 \omega_2 = 1 - \theta B,$$
$$\omega_1 - \omega_2 = \theta + B,$$
$$\omega_1 + \omega_2 = \sqrt{(\theta - B)^2 + 4}.$$

An interesting example of matrix $R$ is obtained in the case $\theta = B$, which corresponds to $\varphi = \pi/4$ in Eq.(16). In that case matrix $R$ does not depend on noncommutativity parameters.

The $R$ matrix was discussed in Ref.[14] in the context of the $*$-eigenvalue problem, but only in phase I. The authors of Ref.[14] stated that the matrix $R$ became singular at the critical point. However, we wish to emphasize that...
the matrix $R$ is universal orthogonal matrix valid in both phases and even at the critical point. The transformations $L$, $S$ and $L_0$ were discussed in Refs.[9,15] for the case of two-dimensional harmonic oscillator and parametrization $\hbar = \hbar_2 = 1$ and $\phi_1 = \phi_2 = 0$, with the identification $u^\prime = \{Q,P\}$ and $u = \{\alpha, \beta\}$. The authors of Ref.[9] treated the two phases separately over-looking the universality of the transformation $L_0 = RD$, whereas in Ref.[15] phase II was not analysed. We point out that two systems $(H(U), M)$ and $(H(U'), M')$ with the same energy spectrum and $M \neq M'$ are physically not equivalent. The condition for physical equivalence is the same energy spectrum and the same commutation relations $M = M'$. Hence, even within the same phase two systems with the same energy spectrum can be quite different.

### 3 Two phases, duality and symmetries in arbitrary dimensions

Construction of different representations of quantum mechanics on a noncommutative plane can be easily generalized to arbitrary dimensions $D$. The regular, antisymmetric matrix $M$ is parametrized by $D(2D - 1)$ real parameters. We can classify noncommutative spaces according to $\{\omega_1, \omega_2, \ldots, \omega_D\}$, eigenvalues of the Hermitean matrix $iM$. The determinant of the matrix $M$ is positive, $\det M = \omega_1^2 \cdots \omega_D^2$. The critical point $\det M = 0$ divides the space of the parameters in two phases. In phase I, $\kappa = \omega_1 \cdots \omega_D > 0$, and in phase II, $\kappa < 0$. The critical point $\kappa = 0$ may have interesting physical applications, like the Landau problem in two dimensions.

In $D$-dimensions, angular momentum operators are generators of coordinate space rotations:

\[
\begin{align*}
[J_{ab}, X_c] &= i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})X_d, \\
[J_{ab}, P_c] &= i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})P_d,
\end{align*}
\]

and generally,

\[
[J_{ab}, U_i] = (E_{ab})_{ij}U_j, \quad i, j = 1, \ldots, 2D.
\]

For a regular matrix $M$ we can construct the angular momentum generators $J_{ab} = -\frac{1}{2}(E_{ab}M^{-1})_{ij}U_iU_j$ only if $[E_{ab}, M] = 0$, for all $a, b = 1, \ldots, D$.

There are two sets of important transformations in the $2D$ phase space. One is a group of linear transformations $U_i^\prime = S_iU_j$ preserving $M$, $S^T MS = M$. These transformations form a group $G(M)$ isomorphic to $Sp(2D)$. For every transformation $S$ there exists a unitary operator $V \sim \exp(i\sum v_iU_iU_j)$, and any two systems connected by such an $S$ transformation are physically (unitary) equivalent.

The other important set of transformations are orthogonal transformations $O(2D)$ preserving the spectrum of the matrix $(iM)$, i.e., preserving $\omega_1, \ldots, \omega_D$ up to the signs. Transformations $R \in SO(2D)$ with $\det R = 1$ keep the system in the same phase. There is a discrete $Z_2$ transformation that changes the sign of one eigenvalue, we choose $\omega_D$ for definiteness. We represent this transformation using the flip matrix $F$, $F_{ii} = 1, i = 1, \ldots, 2(D - 1)$, $F_{2D-1,2D} = F_{2D,2D-1} = 1$, and all other matrix elements zero. There is a simple example of this duality transformation that connects the two phases, obtained by choosing $R' = FRF$ (see Fig.1):

\[
\begin{align*}
\omega_D &= -\omega_D', \omega_i = \omega_i', i = 1, \ldots, D - 1, \\
FMF &= M', \prod i\omega_i = -\prod i\omega_i'.
\end{align*}
\]

In general, duality is characterized by $|\omega_i| = |\omega_i'|, \forall i$ and $\kappa = -\kappa'$.

The matrix $M$ can be brought to the $J_{ab}$ form by the orthogonal transformation $R$, see Eq.(11). This $R \in SO(2D)$ matrix is unique up to orthogonal transformations that preserve $M$. For a fixed values $\{\omega_1, \ldots, \omega_D\}$, the number of parameters of the matrix $M$ is the same as the number of parameters of $R$. The group of orthogonal transformations keeping $M$ invariant, $SO(2D) \cap G(M)$, is isomorphic to $(U(1))^D$ in the generic case. Using this freedom we can fix $M_{2D-1,2D} = 1, \forall i$ or we can put $M_{2D-1,2D} = -1$. So, using the symmetry we reduce the number of continuous phase-space parameters to $2D(D - 1)$.

For a special choice of phase-space parameters $M_{ij}$, we can enlarge the symmetry group $[U(1)]^D$. The symmetries are characterized by degeneracy of eigenvalues $|\omega_i|$. If $k_1, \ldots, \kappa_a$ are frequencies of appearance of $|\omega_1, \ldots, \omega_D|$ in the spectrum of the matrix $iM$, then the symmetry group $SO(2D) \cap G(M) \sim U(k_1) \times \cdots \times U(k_a)$, where $\sum k = D$. Obviously, the largest symmetry group is $U(D)$. The sign of the product of eigenvalues determines the phase and the degeneracy among $|\omega_i|$’s determines the complete symmetry structure of phase space. In this way, we classify noncommutative spaces according to $\{\omega_1, \omega_2, \ldots, \omega_D\}$.

Figure 1 is, of course, valid in any number of dimensions, and we can construct corresponding transformations in a way analogous to the two-dimensional case.

After defining the Hamiltonian we can also discuss the group of linear transformations $G(H) \subset GI(2D, \mathbb{R})$ that keep Hamiltonian invariant. For the noncommutative harmonic oscillator, this group is $O(2D)$. The degenerate energy levels for a given Hamiltonian are described by a set of orthogonal eigenstates transforming according to an irreducible representation of the dynamical symmetry group. The dynamical symmetry group $G(H, M)$ is a group of all transformations preserving both, $M$ and the Hamiltonian, i.e., $G(H, M) = G(H) \cap G(M)$. For the fixed Hamiltonian the dynamical symmetry depends on $M$, so, by changing the parameters of the matrix $M$ we can change $G(H, M)$ from $G_{\text{min}}(H, M)$ to $G_{\text{max}}(H, M)$. For the noncommutative harmonic oscillator, the minimal dynamical symmetry group is $[U(1)]^D$, and the maximal symmetry is $U(D)$. Note, however, that after fixing both the Hamiltonian and $M$, all systems connected to $(H, M)$ by linear transformations will have dynamical symmetry groups isomorphic to each other.

Hence, different choices of $M$ correspond to different dynamical symmetry. This can be viewed as a new mechanism of symmetry breaking with the origin in (phase)space...
structure. There are possible applications to bound states in atomic, nuclear and particle physics. From the symmetry-breaking effects in these systems one can, in principle, extract upper limits on the noncommutative parameters.

### 4 Harmonic oscillator - an example

In order to illustrate general claims from the preceding sections, we choose a simple harmonic oscillator in two dimensions as an example. The $O(4)$ invariant Hamiltonian in this case is

\[ H = \frac{1}{2} \sum_{i=1}^{4} U_i^2. \]

(18)

The constants $\hbar, m$ and $\omega$ are absorbed in phase-space variables. We parametrize the matrix $M$ by four parameters:

\[ M = \begin{pmatrix} 0 & 1 & \theta & \phi_1 \\ -1 & 0 & \phi_2 & B \\ -\theta & -\phi_2 & 0 & 1 \\ -\phi_1 & -B & -1 & 0 \end{pmatrix}. \]

(19)

The eigenvalues of the matrix $iM$ are

\[ \omega_{1,2} = \frac{1}{2} \sqrt{(\theta - B)^2 + (\phi_1 + \phi_2)^2} + \frac{1}{2} \sqrt{(\theta + B)^2 + (\phi_1 - \phi_2)^2}, \]

(20)

and the spectrum of the Hamiltonian (18) is $E = \omega_1(n_1 + 1/2) + \omega_2(n_2 + 1/2)$ [9], see Eq. (27) below. If the product of eigenvalues is positive, we are in phase I, and if negative in phase II. The frequency $\omega_1$ is always positive, and $\omega_2$ changes the sign in phase II. Duality relations between the two phases are obtained by demanding that physical systems in both phases have the same energy spectrum. 

In the simple case $\phi_1 = \phi_2 = 0$, we have one-to-one correspondence between $(\theta, B)$ and $(\theta', B')$

\[ \theta = \frac{1}{2} \left[ \sqrt{(\theta - B)^2 + 4 + \sqrt{(\theta + B)^2 - 4}} \right], \]

\[ B = \frac{1}{2} \left[ \sqrt{(\theta - B)^2 + 4 - \sqrt{(\theta + B)^2 - 4}} \right], \]

(21)

and

\[ \theta' = \frac{1}{2} \left[ \sqrt{(\theta + B)^2 + 4 + \sqrt{(\theta - B)^2 - 4}} \right], \]

\[ B' = \frac{1}{2} \left[ \sqrt{(\theta - B)^2 + 4 - \sqrt{(\theta + B)^2 - 4}} \right]. \]

(22)

A comment is in order. Notice that relations (21) and (22) are valid for $|\theta + B| > 2$ and $|\theta' + B'| > 2$, respectively. This is the sole consequence of the oversimplified parametrization $\phi_1 = \phi_2 = 0$. For every point in parameter space there exists a dual point, we just have to allow for the most general parametrization of $M$. Finally, from $\omega_1 \omega_2 = -\omega'_1 \omega'_2$, we obtain

\[ 1 - \theta B = \theta'B' - 1. \]

(23)

This condition is necessary but not sufficient in order to have energy spectra in two phases identical. A special case ($\theta = \theta'$) of this relation was obtained in Ref.[9], by considering the limit from the fuzzy sphere to the plane, for the Landau problem.

Although the systems depicted in Fig. 1 are physically distinct, the dynamical symmetry groups are all isomorphic to each other. At every point in Fig. 1 the generic symmetry $\omega_1 \neq |\omega_2|$ is $U(1) \times U(1)$. We have only one quadratic symmetry generator, in addition to the Hamiltonian

\[ \mathcal{G} = \sum_{i,j} C_{ij} U_i U_j, \quad [\mathcal{G}, H] = 0. \]

(24)

The matrix $C$ is symmetric, commutes with $M$, $[C, M] = 0$, and we can always choose $\text{Tr} C = 0$. Then, $C^2$ is proportional to the identity matrix. Namely, the $C^0$ matrix for the system $(U_0, J_\omega)$ is $C^0 \sim \text{diag}(1, 1, -1)$. Using the transformation $U = RT^0$ we obtain $C = RC^0\text{R}^T$ implying $C^2 \sim \mathbb{I}_{4 \times 4}$. For the matrix $M$ (19), the generator commuting with the Hamiltonian (18) is

\[ \mathcal{G}_0 = \frac{1}{2\sqrt{\omega_1|\omega_2|}} \left( X_1^2 + P_1^2 - X_2^2 + P_2^2 \right). \]

(25)

One is tempted to call this generator the angular momentum, but this requires caution, as we have already discussed. For example, in the system $(H(RT^0), J_\omega)$ we cannot construct the angular momentum because $E[J_\omega] \neq 0$.

However, the symmetry generator for this system is

\[ \mathcal{G}_0 = \frac{1}{2\sqrt{\omega_1|\omega_2|}} \left( X_1^2 + P_1^2 - X_2^2 + P_2^2 \right). \]

(26)

There are special points in parameter space of enhanced symmetry. In the special case $\omega_1 = \omega_2$ (in phase I), we have the $U(2)$ symmetry group. In this case $h_1 = h_2 = 1$, $B = -\theta$ and $\phi_1 = \phi_2 = \phi$ and we can construct three generators of dynamical symmetry satisfying the $SU(2)$ algebra $[L_i, L_j] = i\epsilon_{ijk} L_k$,

\[ L_1 = \frac{1}{1 + \theta^2 + \phi^2} \left[ X_1 P_2 - X_2 P_1 - \phi(X_1 P_1 + X_2 P_2) \right. \]

\[ - \left. \theta \left( X_1^2 + X_2^2 + P_1^2 - P_2^2 \right) \right], \]

\[ L_2 = \frac{1}{1 + \theta^2 + \phi^2} \left[ -P_1 P_2 - X_1 X_2 + \theta(X_1 P_1 - X_2 P_2) \right. \]

\[ + \left. \phi \left( X_1^2 - X_2^2 + P_1^2 - P_2^2 \right) \right], \]

\[ L_3 = \frac{1}{1 + \theta^2 + \phi^2} \left[ \frac{1}{2} \left( X_1^2 - X_2^2 + P_1^2 - P_2^2 \right) \right. \]

\[ + \left. \theta(X_1 P_2 + X_2 P_1) - \phi(X_1 X_2 - P_1 P_2) \right]. \]

(26)
The dual point with $\omega_1 = -\omega_2$, with $SU(2)$ symmetry in phase II, is obtained with $B = \theta, \phi_1 = -\phi_2 = \phi, h_1 = -h_2 = 1$. We wish to emphasize that $SU(2)$ symmetry exists only for a special choice of parameters, and is not a dynamical symmetry of the Hamiltonian in the generic case (in contrast to the claims in Ref.[15]).

The transformations $L, L_0, S, D, R$ connecting different representations (see Fig. 1) of the harmonic oscillator on the noncommutative plane were discussed in the preceding section, and, partly, in the literature [9,14,15]. Using the matrix $L_0 = RD$ we can transform the Hamiltonian (18) into an ordinary QM system:

$$H(U) = H(RDUU^0) = \frac{1}{2}L_{ij}^0 L_{ij}^0 u_i^0 u_j^0$$
$$= \frac{\omega_1}{2}(u_1^0 u_1^0 + u_2^0 u_2^0) + \frac{\omega_2}{2}(u_3^0 u_3^0 + u_4^0 u_4^0). \quad (27)$$

Next, we calculate matrix elements of observables starting from ordinary harmonic oscillator observables:

$$\langle U_1 \cdots U_k \rangle = L_{ij_1}^0 \cdots L_{ij_k}^0 (u_{ij_1}^0 \cdots u_{ij_k}^0).$$

For quadratic observables in the ground state we use $\langle u_i^0 u_j^0 \rangle = 1/2$, $\langle u_i^0 u_i^0 \rangle = \langle u_i^0 u_i^0 \rangle = i/2$, all others are zero. For a special case $\phi_1 = \phi_2 = 0$, we use the matrix $R$ Eq.(16) to obtain

$$\langle X_1^2 \rangle = \langle X_2^2 \rangle = \frac{1}{2} \omega_1 \cos^2 \varphi + |\omega_2| \sin^2 \varphi,$$
$$\langle P_1^2 \rangle = \langle P_2^2 \rangle = \frac{1}{2} \omega_1 \sin^2 \varphi + |\omega_2| \cos^2 \varphi. \quad (28)$$

These expressions are universal, i.e., they are valid in both phases and at the critical point.

Here, we would like to comment uncertainty relations following from commutation rules which define the theory. In the simple case $\phi_1 = \phi_2 = 0$, we have four nontrivial uncertainty relations $\Delta U_i \Delta U_j \geq |M_{ij}|/2$, i.e.,

$$\langle X_a^2 \rangle \langle P_a^2 \rangle \geq \frac{1}{4}, \quad a = 1, 2,$$
$$\langle X_1^2 \rangle \langle X_2^2 \rangle \geq \frac{\theta^2}{4} \langle P_1^2 \rangle \langle P_2^2 \rangle \geq \frac{B^2}{4}. \quad (30)$$

We calculate the left-hand-side of relations (29,30) in the ground state, using (28) and (17). In phase I we can saturate the first two relations (29) for $\theta = B$. In phase II we can saturate the other two relations (30) for any $B$ and $\theta$. At the critical point $\theta B = 1$ all four relations are saturated in the ground state. In the special case in phase I, $B = 0, \theta \neq 0$, one of the four uncertainty relations are saturated, in agreement with the theorem valid for quantum mechanics on the noncommutative plane with $B = 0$ [25]. This short analysis also indicates that physics in different phases is qualitatively different and depends crucially on $M$.

An especially interesting physical system is the Landau problem in the noncommutative plane, defined by $H = P^2/2$ and the matrix $M$

$$M = \begin{pmatrix} 0 & 1 & \theta & 0 \\ -1 & 0 & 0 & B \\ -\theta & 0 & 0 & 1 \\ 0 & -B & -1 & 0 \end{pmatrix}. \quad (31)$$

This problem can be treated as a noncommutative harmonic oscillator $H = P^2/2 + \omega X^2/2$, in the limit when $\omega \to 0$. We simply define $U_1 = \omega X_1, U_3 = \omega X_2$ to obtain a new matrix $M_\omega$

$$M_\omega = \begin{pmatrix} 0 & \omega & \omega^2 \theta & 0 \\ -\omega & 0 & 0 & B \\ -\omega^2 \theta & 0 & 0 & \omega \\ 0 & -B & -\omega & 0 \end{pmatrix}, \quad (32)$$

with the determinant $\det M_\omega = \omega^4 (1 - \theta B)^2$. We find the magnetic length (the minimum spatial extent of the wavefunction in the ground state) in a universal form, valid in both phases and at the critical point:

$$\langle X_1^2 + X_2^2 \rangle = \langle r^2 \rangle = \frac{1}{\sqrt{(\omega^2 \theta - B)^2 + 4 \omega^2}}. \quad (33)$$

In the limit $\omega \to 0$, eigenvalues of the matrix $M_\omega$ are $\omega_1 = |B|, \omega_2 = 0$ and magnetic length is

$$\langle r^2 \rangle = \begin{cases} \frac{\theta B}{\theta B'} & \text{if } \theta B < 1, \\ \frac{B}{B'} & \text{if } \theta B' > 1. \end{cases} \quad (34)$$

For $|B| = B'$ these two expressions are the same if the duality relation (23) holds.

The above representation of the noncommutative Landau problem as a case of noncommutative harmonic oscillator with $\omega \to 0$ can also be viewed as a noncommutative harmonic oscillator with $\omega \neq 0$, at the critical point $\theta B = 1$. The connection between parameters is $\omega^2 \theta + 1/\theta = B$. If we insist on having the same magnetic length in both pictures, we can fix $\theta$ and $\omega$.

However, these systems (the Landau problem with $\omega = 0$ and the harmonic oscillator with $\omega \neq 0$) are not physically equivalent. They have just the same energy spectrum and the same magnetic length if we choose so. A simple way to see this is to consider uncertainty relations in phases I and II for the Landau problem, and at the critical point for the harmonic oscillator. Here we wish to emphasize once more that only system having equal both the spectrum of the Hamiltonian and the matrix of commutators $M$ are physically equivalent.

### 5 Conclusion

We have presented a unified approach to NCQM in terms of noncommutative coordinates and momenta in arbitrary
dimensions and for arbitrary c-number commutation relations. We have considered all representations of NCQM connected by linear transformations from $Gl(2D, \mathbb{R})$ preserving the property that commutation relations remain independent of phase-space variables and keeping the energy spectrum of the system fixed. Among these only representations connected by transformations preserving the commutation relations are physically equivalent. We classify noncommutative spaces according to the eigenvalues of the matrix $iM$, $\{\omega_1, \omega_2, \ldots, \omega_D\}$. The sign of the product of eigenvalues determines the phase and the degeneracy among $|\omega_i|$'s determine the complete symmetry structure of phase space. Since orthogonal transformations keep the spectrum of the matrix $iM$ fixed, they have been analysed in detail. We have shown that for general $M$ the angular momentum operator in the usual sense might not exist, and we have given the condition for its existence. An important result is that two physically distinct phases exist in arbitrary dimensions and that they are connected by discrete duality transformations.

Besides the symmetry structure of space, we have also discussed the dynamical symmetry of a physical system and proposed a new mechanism for symmetry breaking, originating from phase-space structure.

In our approach to symmetries, there is no physical principle what $H$ and $M$ we have to choose in terms of noncommuting variables $U$. One way to test the idea of noncommutativity is to choose the Hamiltonian as in ordinary quantum mechanics, and to search for (tiny) symmetry breaking effects induced by the phase-space structure $M$. The opposite way [16] is to fix the dynamical symmetry structure as in ordinary quantum mechanics. In the latter case, the differences should appear in matrix elements of observables and energy eigenstates. Of course, one can choose a combination of both approaches. Regardless of the approach, noncommutativity offers a new explanation of symmetry breaking, or change in probability amplitudes as a consequence of phase-space (space-time) structure.

Our general approach enabled us to obtain new results even in the simplest case of two-dimensional harmonic oscillator. We expect that we shall also obtain physically interesting results in the $D = 3$ case, currently under investigation.

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