DISTRIBUTIVE QUOTIENTS

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Abstract. We note that each lattice \( L \) has a unique largest distributive quotient, of which every distributive quotient of \( L \) is itself a quotient.

Let \( L \) be a lattice, with meet and join denoted by \( \land \) and \( \lor \) respectively. A congruence on \( L \) is an equivalence relation \( \theta \) on \( L \) that is compatible with meet and join in the sense

\[
(a_1, a_2) \in \theta \text{ and } (b_1, b_2) \in \theta \Rightarrow (a_1 \land b_1, a_2 \land b_2) \in \theta \text{ and } (a_1 \lor b_1, a_2 \lor b_2) \in \theta
\]

whenever \( a_1, b_1, a_2, b_2 \in L \); the corresponding quotient lattice is the set \( L/\theta \) of blocks (or equivalence classes) with meet and join well-defined by

\[
[a]_\theta \land [b]_\theta = [a \land b]_\theta, \quad [a]_\theta \lor [b]_\theta = [a \lor b]_\theta
\]

whenever \( a, b \in L \). A lattice \( L \) is distributive (notation: \( L \in \mathbb{D} \)) exactly when it satisfies either (hence each) of the equivalent conditions

\[
\forall a, b, c \in L \quad a \land (b \lor c) = (a \land b) \lor (a \land c),
\]

\[
\forall a, b, c \in L \quad a \lor (b \land c) = (a \lor b) \land (a \lor c).
\]

The purpose of this brief note is to record certain elementary facts regarding the distributive quotients of an arbitrary lattice: in particular, the fact that each lattice \( L \) has a unique largest distributive quotient, of which every distributive quotient of \( L \) is itself a quotient.

The congruences on \( L \) themselves constitute a lattice \( \Theta(L) = \text{Con}(L) \) in which meet is intersection and the join of two congruences is the transitive closure of their union. Within \( \Theta(L) \) we single out those congruences of \( L \) relative to which the quotient is distributive:

\[
\Theta_D(L) = \{ \theta \in \Theta(L) : L/\theta \in \mathbb{D} \}.
\]

Note that \( \Theta_D(L) \) is a filter in \( \Theta(L) \): that is, an up-set closed under finite intersections. In fact, more is true.

**Theorem 1.** \( \Theta_D(L) \subseteq \Theta(L) \) is an up-set that is closed under arbitrary intersections.

**Proof.** \( \Theta_D(L) \) is an up-set: if \( \Theta_D(L) \ni \theta_0 \subseteq \theta \in \Theta(L) \) then the inclusion \( \theta_0 \subseteq \theta \) induces a surjective homomorphism \( L/\theta_0 \to L/\theta \) realizing \( L/\theta \) as a quotient of the distributive lattice \( L/\theta_0 \); thus \( L/\theta \in \mathbb{D} \) and so \( \theta \in \Theta_D(L) \). \( \Theta_D(L) \) is closed under arbitrary intersections: if \( \theta_\lambda \in \Theta_D(L) \) for each \( \lambda \in \Lambda \) then the canonical map

\[
L \to \prod_\lambda (L/\theta_\lambda) : a \mapsto ([a]_{\theta_\lambda})_\lambda
\]

factors through an injective homomorphism

\[
L/(\cap_\lambda \theta_\lambda) \to \prod_\lambda (L/\theta_\lambda) \in \mathbb{D};
\]

thus \( L/(\cap_\lambda \theta_\lambda) \in \mathbb{D} \) and so \( \cap_\lambda \theta_\lambda \in \Theta_D(L) \). \( \square \)
Thus, \( \Theta_D(L) \) has as least element its infimum
\[
\delta_L = \bigwedge \Theta_D(L) = \bigcap \{ \theta : \theta \in \Theta_D(L) \} \in \Theta_D(L)
\]
and so \( \Theta_D(L) \) is principal with \( \delta_L \) as generator:
\[
\Theta_D(L) = \uparrow \delta_L = \{ \theta \in \Theta(L) : \theta \supseteq \delta_L \}.
\]
Observe that \( L/\delta_L \) is the largest distributive quotient of \( L \): in fact, if \( L/\theta \) is any distributive quotient of \( L \) then \( \theta \in \Theta_D(L) = \uparrow \delta_L \); thus, \( \theta \) contains \( \delta_L \) and so there is a canonical surjective homomorphism \( L/\delta_L \to L/\theta \).

We may identify the generator \( \delta \) for an arbitrary quotient as follows.

**Theorem 2.** If \( \theta \in \Theta(L) \) then \( \delta(\theta/\theta) = \delta_L \lor \theta/\theta \).

*Proof.** For convenience, write \( \delta = \delta_L \). On the one hand, the isomorphism
\[
(L/\theta)/(\delta \lor \theta/\theta) \equiv L/(\delta \lor \theta) \in \mathbb{D}
\]
places the congruence \( \delta \lor \theta/\theta \) in \( \Theta_D(L/\theta) \). On the other hand, let \( \phi \) be a congruence of \( L \) containing \( \theta \): if \( \phi/\theta \in \Theta_D(L/\theta) \) then the isomorphism
\[
L/\phi \equiv (L/\theta)/(\phi/\theta) \in \mathbb{D}
\]
forces \( \phi \in \Theta_D(L) \) so \( \phi \) also contains \( \delta \) and \( \phi \in \uparrow (\delta \lor \theta) \). Conclusion: \( \Theta_D(L/\theta) = \uparrow (\delta \lor \theta/\theta) \). \( \square \)

We may identify the generator \( \delta \) for a finite product as follows.

**Theorem 3.** \( \delta(L_1 \times L_2) = \delta_{L_1} \times \delta_{L_2} \).

*Proof.** For convenience, write \( \delta_1 = \delta_{L_1} \) and \( \delta_2 = \delta_{L_2} \). On the one hand, the isomorphism
\[
(L_1 \times L_2)/(\delta_1 \times \delta_2) \equiv (L_1/\delta_1) \times (L_2/\delta_2) \in \mathbb{D}
\]
places \( \delta_1 \times \delta_2 \) in \( \Theta_D(L_1 \times L_2) \). On the other hand, each \( \theta \in \Theta(L_1 \times L_2) \) has the form \( \theta_1 \times \theta_2 \) for \( \theta_1 \in \Theta(L_1) \) and \( \theta_2 \in \Theta(L_2) \); now, if \( \theta \in \Theta_D(L_1 \times L_2) \) then
\[
(L_1/\theta_1) \times (L_2/\theta_2) \equiv (L_1 \times L_2)/\theta \in \mathbb{D}
\]
forces \( (L_1/\theta_1) \in \mathbb{D} \) and \( (L_2/\theta_2) \in \mathbb{D} \) so that \( \theta_1 \in \Theta_D(L_1) = \uparrow \delta_1 \) and \( \theta_2 \in \Theta_D(L_2) = \uparrow \delta_2 \) whence \( \theta = \theta_1 \times \theta_2 \in \uparrow (\delta_1 \times \delta_2) \). Conclusion: \( \Theta_D(L_1 \times L_2) = \uparrow (\delta_1 \times \delta_2) \). \( \square \)

Let us identify the generator \( \delta_L \) of \( \Theta_D(L) \) for a lattice \( L \) in some basic examples.

**Example 0.** If \( L \) is a distributive lattice then, as each quotient of \( L \) is distributive, \( \Theta_D(L) = \Theta(L) \) and \( \delta_L = \theta \) is the equality (or diagonal) relation on \( L \).

**Example 1.** The non-distributive ‘diamond’ \( M_3 \) is simple; it follows at once that \( \Theta_D(M_3) = \{1\} \), so that \( \delta_{M_3} = \mathbb{1} = M_3 \times M_3 \) is the trivial congruence that fully collapses \( M_3 \).

**Example 2.** The non-modular ‘pentagon’ \( N_5 = \{0,a,b,c,1\} \) with \( a > b \) yields a distributive quotient as soon as \( a \) and \( b \) are identified. Accordingly, \( \delta_{N_5} \) is the principal congruence \( \theta(a,b) \): that is, the smallest congruence containing the pair \( (a,b) \); its only nontrivial block is the doubleton \( \{a,b\} \).

**Example 3.** The free modular lattice \( F_{\mathbb{M}}(3) \) on three generators \( x,y,z \) admits a unique homomorphism to the free distributive lattice \( F_D(3) \) on \( x,y,z \) respecting the generators; the kernel of this homomorphism is the principal congruence \( \theta(u,v) \) that identifies \( u = (y \lor z) \land (z \lor x) \land (x \lor y) \) and \( v = (y \lor z) \lor (z \lor x) \lor (x \lor y) \). As no smaller congruence can yield a distributive quotient, \( \delta_{F_{\mathbb{M}}(3)} = \theta(u,v) \). The nontrivial blocks of this congruence are six doubletons and the diamond with top \( u \) and bottom \( v \).

**Example 4.** The case of the free lattice \( F(n) \) on \( n \) generators is similar: \( \delta_{F(n)} \) is the kernel of the unique homomorphism \( F(n) \to F_D(n) \) that respects all \( n \) generators.
**Remark 1.** We have considered only the class $\mathbb{D}$ of distributive lattices, but entirely similar considerations apply to the class $\mathbb{M}$ of modular lattices. Indeed, they apply to any equational class $\mathbb{K}$ of lattices: as $\mathbb{K}$ is closed under the formation of quotients, products and sublattices, if $L$ is any lattice then the filter
\[
\Theta_{\mathbb{K}}(L) = \{\theta \in \Theta(L) : L/\theta \in \mathbb{K}\}
\]
is principal with generator
\[
\kappa_L = \bigwedge \Theta_{\mathbb{K}}(L) = \bigcap \{\theta : \theta \in \Theta_{\mathbb{K}}(L)\} \in \Theta_{\mathbb{K}}(L).
\]
In particular, $L$ has a largest quotient $L/\kappa_L$ in any equational class $\mathbb{K}$ and each quotient of $L$ in $\mathbb{K}$ is actually a quotient of $L/\kappa_L$.

**Remark 2.** This offers a perspective on the free lattice $F_{\mathbb{K}}(P)$ over the class $\mathbb{K}$ generated by the poset $P$ as discussed in section 5 of [1]: thus, let $F(P)$ be the free lattice generated by $P$ as in [1] Corollary 5.7; the lattice $F_{\mathbb{K}}(P)$ arises as the quotient of $F(P)$ modulo the congruence $\kappa_{F(P)}$ when the elements of $P \subseteq F(P)$ lie in different blocks of $\kappa_{F(P)}$.

**References**

[1] G. Grätzer, *Lattice Theory: First Concepts and Distributive Lattices*, W.H. Freeman and Company (1971); Dover Publications (2009).

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