In this paper, a formula, describing a threshold of the regenerative multi-pass Beam Breakup (BBU) for a single dipole higher order mode with arbitrary polarization in a two-pass accelerator with a general-form, 4x4 recirculation matrix, is derived. Also a new two-dimensional BBU code is introduced. To illustrate specifics of the BBU in two dimensions, the formula is used to calculate the threshold in several cases including two-dimensional uncoupled optics, reflecting optics, and rotating optics. The analytical results are compared to results of simulation obtained with the new code. At the end of the paper, a mathematical relation between transfer matrices between cavities of the accelerating structure and recirculation matrices for each cavity, which must be satisfied in order to successfully suppress the BBU by reflection or rotation in several cavities, is presented.

1 Introduction

The regenerative multi-pass BBU arises from interaction of an intense recirculated beam with High Order Modes (HOM) of the accelerating structures. In a two-pass machine, the beam deflected off-axis by the HOM magnetic field on the first pass comes back to the cavity with a displacement on the second pass. The recirculated beam induces the HOM voltage, depending on a magnitude and direction of the beam displacement. Thus, the beam constitutes a feedback that can become unstable if the beam current reaches a threshold.

A formula describing the BBU threshold for a single HOM in one dimension was presented in [1],[2]. This formula is valid if recirculation optics is uncoupled and HOMs deflect only in the horizontal or vertical planes. In reality, however, HOMs are not necessarily bound to the $x$ or $y$ planes. Besides, the recirculator optics and the field of the accelerating structures
can introduce betatron coupling. An early consideration of the regenerative multi-pass BBU in two dimensions was given by R. Rand and T. Smith in [3]. In their paper, the authors investigated a possibility of suppressing the BBU by means of rotating or reflecting the vector of the beam displacement on the second pass. In this paper, a general formula for the BBU threshold for a single dipole HOM with arbitrary polarization in a two-pass machine with a general-form, 4x4 recirculation matrix is presented. Section 2 presents a derivation of the formula. It also contains formulas describing evolution of the HOM voltage above and below the threshold. The third section briefly describes a new two-dimensional tracking BBU code. The formula and the program were used to calculate the threshold in several simple cases including uncoupled recirculator optics, reflecting optics, and 90°-rotating optics. The analytical and simulation results are presented and compared to each other in Section 4. For effective suppression of the BBU by rotation or reflection in several cavities simultaneously, the recirculation matrix for each cavity has to be of a form that provides rotation or reflection. To preserve the form of the recirculation matrix from cavity to cavity, transfer matrices between the cavities have to satisfy a condition discussed in Section 5.

2 BBU threshold in two-dimensional systems and HOM voltage evolution above and below the threshold

Imperfections and asymmetrical features of cavities such as couplers violate two-dimensional symmetry of cavity geometry and split the frequencies of degenerate modes. A typical separation between two orthogonal polarizations of the same mode in a superconducting RF cavity is of the order of several hundred kHz to several MHz. Besides, imperfections cause the frequency of HOMs to vary from cavity to cavity. The variation of the HOM frequency from cavity to cavity is also of the order of several hundred kHz to several MHz.

Simulation results and experimental data presented elsewhere (see, for example, [4], [5]) suggest that dipole TM HOMs with a quality factor of $10^5 - 10^6$ will limit the maximum recirculated current to the order of tens to hundreds milliamps. Assuming a quality factor of $10^5 - 10^6$ and a typical HOM frequency of the order of 2 GHz, one concludes that a typical bandwidth of dangerous HOMs is substantially smaller than a typical separation between modes. This allows one to consider each dangerous HOM separately if the number of cavities in the accelerating structures is smaller than a ratio of the typical HOM frequency variation in the cavities and the typical bandwidth of dangerous modes.

2.1 Two-dimensional formula for the BBU threshold

A bunch traveling through a cavity interacts with cavity HOMs. It changes the voltage of the HOMs and the energy stored in the modes. In a one-dimensional case, when the bunch
displacement $x$ and the vector of the HOM polarization are collinear, a variation of the energy stored in a dipole TM HOM produced by a point-like bunch, $\Delta U$, is given by [6]

$$\Delta U = -qV_a \cos(\phi) \frac{x}{a} + \frac{q}{2} V_q \frac{x}{a}$$

(1)

where $V_a$ is the accelerating HOM voltage at the radius of the beam pipe $a$ induced by previous bunches, $\phi$ is the phase of the point-like bunch with respect to the peak of the HOM electric field, and $q$ is the bunch charge. $V_q$ in (1) is the accelerating voltage at the beam pipe radius $a$ induced by the bunch passing through the cavity:

$$V_q = q a^2 \frac{\omega}{2} \left( \frac{\omega}{c} \right)^2 \left( \frac{R}{Q} \right) \frac{x}{a}$$

(2)

where $\omega$ is the HOM angular frequency. In a case of instability, $V_q$ is much smaller than $V_a$. Thus, we can approximate (1) as

$$\Delta U = -qV_a \cos(\phi) \frac{x}{a}$$

(3)

In two dimensions, the projection of the beam displacement on the vector of mode polarization has to be substituted for the transverse displacement $x \vec{n} = \cos(\alpha)\hat{x} + \sin(\alpha)\hat{y}$. Therefore

$$x \rightarrow \vec{d} \cdot \vec{n} = x \cos(\alpha) + y \sin(\alpha)$$

(4)

where $\vec{d}$ is the displacement vector $(x, y)$ and $\alpha$ is the mode polarization angle in the $xy$ coordinate system. The energy deposited by the bunch in the cavity on the first and second passes can be written as

$$\Delta U_1 = -qV_a \cos(\phi)(x_1 \cos(\alpha) + y_1 \sin(\alpha))$$

(5)

$$\Delta U_2 = -qV_a \cos(\phi + \omega T_r)(x_2 \cos(\alpha) + y_2 \sin(\alpha))$$

(6)

where $T_r$ is the recirculation time. Subscripts 1 and 2 in the last equation denote the first and second passes respectively. It is assumed in (5-6) that the voltage amplitude $V_a$ does not change significantly during recirculation. As will be shown later, a limited range of beam current values where this assumption is valid always exists around the threshold current. Exactly at threshold, $V_a$ is constant. Coordinates of the recirculated bunch, $x_2$ and $y_2$, can be expressed via the bunch coordinates before deflection and the HOM accelerating voltage on the first pass as

$$x_2 = m_{11}x + m_{12}x' + m_{13}y_1 + m_{14}y_1' - \frac{qV_a}{\omega a p} \sin(\phi)(m_{12} \cos(\alpha) + m_{14} \sin(\alpha))$$

(7)

$$y_2 = m_{31}x + m_{32}x' + m_{33}y_1 + m_{34}y_1' - \frac{qV_a}{\omega a p} \sin(\phi)(m_{32} \cos(\alpha) + m_{34} \sin(\alpha))$$

(8)
where \( p \) is the beam momentum. In the last two equations, the transverse HOM voltage that deflects the beam was expressed via the accelerating HOM voltage and its phase as [6]

\[
V_\perp = -\frac{c}{\omega} \frac{V_a}{a} \sin(\phi)
\] (9)

The average power deposited by the beam in the HOM is equal to the average energy deposited by individual bunches times the bunch repetition frequency:

\[
\dot{U}_{\text{beam}} = \langle \Delta U_1 + \Delta U_2 \rangle \cdot f_b
\] (10)

The averaging is done respectively to the phase of the HOM, \( \phi \), taken at moments when bunches pass through the cavity on the first pass. Ohmic losses in the cavity can be expressed as [6]

\[
P_c = \frac{V_a^2}{(\omega/c)^2 a^2 (R/Q)}
\] (11)

The energy balance equation for the HOM stored energy is

\[
\dot{U} = \dot{U}_{\text{beam}} - P_c = \langle \Delta U_1 + \Delta U_2 \rangle \cdot f_b - P_c
\] (12)

Terms proportional to \( \cos(\phi) \), \( \sin(\phi) \), \( \cos(\phi + \omega T_r) \), and \( \sin(\phi + \omega T_r) \) yield zero after averaging if \( x_1, x_1' \) and \( y_2, y_2' \) in (7-8) are slow varying steering errors. If the HOM frequency is not equal to a harmonic of the bunch repetition rate, terms proportional to \( \cos(\phi) \cdot \sin(\phi) \) also yield zero and the average value of the \( \sin^2(\phi) \) is equal to 1/2. Thus, Equation (12) can be rewritten as

\[
\dot{U} = \dot{U}_{\text{beam}} - P_c = \langle \Delta U_1 + \Delta U_2 \rangle \cdot f_b - P_c
\] (13)

where \( m_{12}^* = m_{12} \cos^2(\alpha) + (m_{14} + m_{32}) \sin(\alpha) \cos(\alpha) + m_{34} \sin^2(\alpha) \). At threshold, \( \dot{U} \) is equal to zero. Thus, the threshold current is given by the following equation

\[
I_{th} \frac{m_{12}^* q}{2p} \omega \sin(\omega T_r) + \frac{1}{(\omega/c)^2 R} = 0
\] (14)

which yields the threshold current as

\[
I_{th} = -\frac{2pc}{q \frac{\omega}{c} (R/Q) Q m_{12}^* \sin(\omega T_r)}
\] (15)

\[
m_{12}^* = m_{12} \cos^2(\alpha) + (m_{14} + m_{32}) \sin(\alpha) \cos(\alpha) + m_{34} \sin^2(\alpha)
\]
Precise knowledge of the phase $\phi$ in (13) is required to find the threshold if the HOM frequency is equal to a harmonic of the bunch repetition rate. Because the presented method does not provide this information, in a case of the resonance, one has to calculate the phase, using other methods, or proceed, using computer simulations. Note that the resonance between potentially dangerous HOMs and beam harmonics should be avoided by a proper choice of cavity and beam parameters at the design stage.

For negative values of the product $m_{12}^* \sin(\omega T_r)$, the threshold given by (15) perfectly agrees with simulation results presented later in this paper. Equation (15) yields absolute beam stability if $m_{12}^* \sin(\omega T_r)$ is positive. The simulations results, however, show that the beam can be unstable at extremely high values of the beam current even if $m_{12}^* \sin(\omega T_r) > 0$. The ratio of the threshold in the "pseudo-stable" region to the threshold for $m_{12}^* \sin(\omega T_r) < 0$ is of the order of $10^3 - 10^4$, corresponding to strong, but not complete, suppression of the BBU. In one-dimensional cases, similar dependence of the threshold on $m_{12}^* \sin(\omega T_r)$ was predicted analytically and observed in simulations in early works by G.A. Krafft, J.J. Bisognano, and S. Laubach [1] and later by G. Hoffstaetter and I. Bazarov [7].

2.2 Evolution of the HOM voltage below and above the threshold

In the region where $m_{12}^* \sin(\omega T_r) < 0$, Equation (13) can be used to describe evolution of the HOM voltage below and above the threshold. Because the energy $U$ stored in the HOM can be expressed via the accelerating voltage $V_a/a$ as [6]

$$U = \frac{V_a^2}{a^2 \left( \frac{\omega}{c} \right)^2 \left( \frac{R}{Q} \right)}$$

(16)

equations (16) and (13) yield an equation for the HOM stored energy

$$\frac{dU}{U} = -dt \frac{\omega}{Q} \frac{I_{th} - I}{I_{th}}$$

(17)

The solution of the last equation is

$$U = U_0 \exp \left( -t \frac{\omega}{Q} \frac{I_{th} - I}{I_{th}} \right)$$

(18)

The HOM voltage depends on time as

$$V = V_0 \exp \left( -t \frac{\omega}{2Q} \frac{I_{th} - I}{I_{th}} \right)$$

(19)

Using equations (18) and (19), one can introduce the effective quality factor $Q_{eff}$ for the beam-HOM system, given by the equation

$$Q_{eff} = Q \frac{I_{th}}{I_{th} - I}$$

(20)
At zero current, $Q_{\text{eff}}$ is equal to the HOM quality factor $Q$. As the beam current increases, $Q_{\text{eff}}$ becomes larger and turns to infinity at the threshold. That is, oscillations the HOM voltage and beam coordinates do not decay. If the beam current exceeds the threshold current, the beam-HOM system becomes unstable. The amplitude of oscillations grows exponentially. Equation (20) can be also rewritten in terms of the decay time as

$$\tau_{\text{eff}} = \tau \frac{I_{\text{th}}}{I_{\text{th}} - I}$$

Equations (18) and (19) were derived within the assumption that the recirculation time is much smaller than the instability growth/decay time, $\tau_{\text{eff}}$. This assumption is valid if

$$\frac{|\delta I|}{I_{\text{th}}} \ll \frac{\tau}{T_r} \quad (22)$$

where $\delta I$ is the difference between the threshold current and the beam current, $I_{\text{th}} - I$. Assuming the HOM decay time, $\tau$, of the order of $10^{-5} - 10^{-4}$ sec, which corresponds to the quality factor $Q$ of the order of $10^5 - 10^6$, and a kilometer-long recirculation pass, one can easily calculate that the current range where (18) and (19) are valid is approximately equal to a doubled value of the threshold current.

### 3 New two-dimensional tracking BBU code

In the present configuration, the code simulates the beam dynamics in two-pass recirculating accelerators. In a two-pass machine, bunches that are accelerated through a linac for the first time are interspaced with recirculated bunches. Time intervals between bunches depend on the bunch repetition frequency and the recirculation time. The code assumes that the distance between bunches does not change in the linac. The code also assumes that bunches are point-like and cavities are infinitely short. The number of cavities and the number of HOMs per cavity are not limited in the code.

The tracking algorithm consists of four main steps:

- A freshly injected bunch propagates through the entire linac. During its passage through the linac, the bunch induces the voltage of HOMs and gets deflected by the HOM voltage excited by previous bunches. After its passage through the linac, the bunch is stored in an array. The array contains all bunches present in the linac on the first pass and in the recirculation pass.

- The code updates the voltage of all HOMs to the moment of arrival of the recirculated bunch that immediately follows the injected bunch.
The code extracts the first recirculated bunch from the array, multiplies the bunch coordinates by the recirculation matrix, and runs the bunch through the entire linac.

The code updates the voltage of all HOMs in the linac to the moment of arrival of the next injected bunch.

Steps 1-4 are repeated until the simulation time exceeds the run time specified in the input.

The voltage of each HOM has two components: real and imaginary. The real and imaginary parts of the voltage are proportional to the electric and magnetic HOM fields respectively. A bunch modifies the real part of the HOM voltage when it passes through a cavity according to

$$\delta V_r = \frac{q c}{2} \left( \frac{\omega}{c} \right)^2 \left( R \frac{Q}{x \cos(\alpha) + y \sin(\alpha)} \right)$$

where $\alpha$ is the mode polarization angle. The imaginary part of the voltage deflects bunches according to

$$\delta x' = \frac{V_i \cos(\alpha)}{V_b}, \quad \delta y' = \frac{V_i \sin(\alpha)}{V_b}$$

where $V_b$ is the beam voltage, $pc/q$. To calculate evolution of the HOM voltage between bunch passages through the cavity, the code uses the following transformation

$$\begin{bmatrix} V_r \\ V_i \end{bmatrix}_{t+dt} = \exp \left( -\frac{\omega dt}{2Q} \right) \begin{bmatrix} \cos(\omega dt) & -\sin(\omega dt) \\ \sin(\omega dt) & \cos(\omega dt) \end{bmatrix} \begin{bmatrix} V_r \\ V_i \end{bmatrix}_t$$

where $\omega$ and $Q$ are the angular frequency and the quality factor of the HOM respectively, and $dt$ corresponds to the time interval between bunches. The transformation given by (25) is equivalent to rotation of a complex vector with exponentially decaying amplitude. The code uses 4x4 transfer matrices to calculate transverse coordinates of bunches.

The first version of the code has been developed and validated. The code has been used to simulate beam breakup in several one-dimensional cases including the JLab FEL Upgrade. The results of simulations were within a 3% agreement with results simulated by TDBBU [8] and MATBBU [9]. For one-dimensional cases, the new code gave results identical to those produced by the code "bi" [10], developed by I. Bazarov of Cornell. Simulation results for several two-dimensional cases are presented later in this paper.

For all simulated cases, the new code was faster than TDBBU and MATBBU by an order of magnitude or more depending on the particular problem. The new code was also more than a factor of 3 faster than "bi". Both codes were run in the same conditions including a computer, an operational system, and a compiler. This gain in the execution speed comparatively to "bi" was due to differences in tracking algorithms used by the codes and the use of the C++ Standard Template Library by "bi".
4 Examples of the regenerative BBU in two-dimensional systems

To illustrate specifics of the regenerative BBU in two dimensions, Equation (15) and the newly developed code were used to calculate the BBU threshold in a two-pass machine consisting of a recirculator and a single cavity, containing a dipole TM HOM. HOM parameters were set to values typical for TM11n modes in the third cryomodule of the JLab FEL Upgrade [4]: the HOM frequency was 2113.9 MHz, $R/Q$ was 30 Ohm, and $Q$ was $5 \cdot 10^6$. The recirculation time and the bunch repetition frequency were 433.2 nsec and 74.85 MHz respectively, identical to those in the JLab FEL Upgrade. The beam energy was 70 MeV. Three cases of recirculation optics were examined: an uncoupled matrix, a matrix interchanging $x$ and $y$ motion (pseudo-reflection), and a matrix rotating the vector of the beam displacement by 90° with respect to the mode polarization. Although values of the matrix elements were randomly chosen, their magnitude roughly corresponded to that of recirculation matrix elements of the JLAB FEL Upgrade.

4.1 Uncoupled recirculation matrix

The recirculation matrix has the following form

$$M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 1.1 & 0 & 0 & 0 \\ 0.325 & 1.5 & 0 & 0 \\ 0 & 0 & 0.9 & m_{34} \\ 0 & 0 & -0.19/m_{34} & 0.9 \end{bmatrix}$$

(26)

Figure 1 shows the BBU threshold calculated from (15) and simulated by the new code vs. HOM rotation angle for $m_{34}$ equal to -4, -2, -1, and 4 meters. The threshold given by (15) is inversely proportional to

$$m_{12}^* = m_{12} \cos^2(\alpha) + m_{34} \sin^2(\alpha)$$

(27)

for positive values of $m_{12}^*$. The simulation results perfectly agree with the analytical results for $m_{12}^* > 0$.

For negative values of $m_{34}$, the threshold current given by (15) becomes infinite if the mode angle exceeds the critical angle given by

$$\alpha_{st} = \arctan\left(\sqrt{-\frac{m_{12}}{m_{34}}}\right)$$

(28)

However, the code finds the threshold even for $m_{12}^* < 0$. As follows from Figure 1, the threshold for $m_{12}^* < 0$ is of the order of 10-100 A. The code was not able to find any threshold for $\alpha$ equal to $\alpha_{st}$ given by (28), indicating an infinite threshold current at this point.
Figure 1: BBU threshold for the recirculation matrix given by (26), calculated from (15) and simulated by the new code, vs. HOM rotation angle. For $m_{12}^* = m_{12} \cos^2(\alpha) + m_{34} \sin^2(\alpha) > 0$, analytical and simulations results practically coincide with each other. For $m_{12}^* < 0$, the simulation results show strong BBU suppression instead of infinite threshold predicted by (15).

4.2 $x - y$-motion interchange (pseudo-reflection)

The transfer matrix

$$ M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \quad (29) $$

interchanges the vertical and horizontal betatron planes. If the 2x2 blocks $A$ and $B$ are the same, the matrix (29) reflects the beam respectively to the $x = y$ plane. According to (15), the threshold for the recirculation matrix (29) is inversely proportional to

$$ m_{12}^* = (m_{14} + m_{32}) \cdot \frac{\sin(2\alpha)}{2} \quad (30) $$

for $m_{12}^* > 0$ and becomes infinite for $m_{12}^* < 0$.

Figure 2 shows the BBU threshold for the recirculation matrix

$$ M = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} \quad (31) $$

where $A$ is the same as in (26), calculated from (15) and simulated by the new code. The simulated results are in better than 0.1% agreement with the analytical results for $0^\circ < \alpha <$
90°. For negative values of \(\sin(2\alpha)\), the threshold simulated by the code is on the order of 20-200 A for \(95° < \alpha < 175°\). The code was not able to find the threshold for \(\alpha\) equal to 90° and 180° degrees, that indicates an infinite threshold at these points.

![Figure 2: BBU threshold for the reflecting matrix (31), calculated from (15) and simulated by the new code, vs. HOM rotation angle.](image)

4.3 90° rotation

The matrix, providing 90° rotation, has the following form

\[
M = \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}
\]  

(32)

where \(A\) is the same as in (26). The expression

\[
m_{12}^* = m_{12} \cos^2(\alpha) + (m_{14} + m_{32}) \sin(\alpha) \cos(\alpha) + m_{34} \sin^2(\alpha)
\]

(33)

is zero for (32), yielding an infinite threshold current in the case of 90° rotation. Attempts to find the BBU threshold using the new code also failed. The simulations were stopped at a beam current of 10\(^6\) A.
5 Mathematical relations for transfer matrices between cavities

Let’s consider a two-pass accelerator with two accelerating cavities and denote recirculation matrices for the first and the second cavities as \( M_1 \) and \( M_2 \) respectively. Let’s also assume that both \( M_1 \) and \( M_2 \) are of a form that provides effective suppression of the BBU. The recirculation matrices are related to each other via transfer matrices between the cavities on the first and the second pass, \( T_1 \) and \( T_2 \) respectively, as

\[
M_2 = T_2 M_1 T_1^{-1} \tag{34}
\]

Both \( T_1 \) and \( T_2 \) have to satisfy (34) for the particular form of the recirculation matrices \( M_1 \) and \( M_2 \). Equation (34) can be transformed into a simpler form. The matrix \( T_1 \) is symplectic, as well as the other matrices, and satisfies the equation

\[
T_1^T S T_1 = S \tag{35}
\]

where \( S \) is the 4 by 4 matrix

\[
S = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix} \tag{36}
\]

Using equation (35), one can express the inverse matrix \( T_1^{-1} \) via \( T_1 \) as

\[
T_1^{-1} = -S T_1^T S \tag{37}
\]

Equations (34) and (37) yield the relation between \( T_1 \) and \( T_2 \) which does not require calculation of the inverse matrix:

\[
M_2 = -T_2 M_1 S T_1^T S \tag{38}
\]

The last equation also allows one to express the condition (34) as a set of equations for 2x2 block sub-matrices of \( M_{1,2} \), \( T_{1,2} \), and \( S \). To illustrate the method let’s consider two examples.

5.1 Uncoupled transfer matrices \( T_{1,2} \), recirculation matrices \( M_{1,2} \) providing 90°-rotation

Equation (34) can be written as

\[
M_2 = T_2 M_1 T_1^{-1} = \begin{bmatrix}
X_2 & 0 \\
0 & Y_2
\end{bmatrix} \begin{bmatrix}
0 & A \\
-A & 0
\end{bmatrix} \begin{bmatrix}
X_1^{-1} & 0 \\
0 & Y_1^{-1}
\end{bmatrix} = \begin{bmatrix}
0 & X_2A Y_1^{-1} \\
-Y_2AX_1^{-1} & 0
\end{bmatrix} \tag{39}
\]
A condition that $M_2$ has to have the same form as $M_1$ yields the relation between $2x2$ block sub-matrices of $T_1$ and $T_2$

$$X_2AY_1^{-1} = Y_2AX_1^{-1} \quad (40)$$

which can be rewritten as

$$X_2^{-1}Y_2 = AY_1^{-1}X_1A^{-1} \quad (41)$$

Designing optics with specific transfer matrices for the first and the second passes that satisfy equation (41) can be difficult. One can notice that equations (34) and (41) are automatically satisfied if the transfer matrices $T_1, T_2$ are axisymmetric with the diagonal blocks $X$ and $Y$ equal to each other. Thus, axisymmetric elements with no coupling preserve the rotational form of the recirculation matrix from cavity to cavity. For example, two identical nearby solenoids of opposite polarity constitute an optical element with an axisymmetric transfer matrix that does not change its form with energy.

### 5.2 Uncoupled transfer matrices $T_{1,2}$, $x – y$-exchanging recirculation matrices $M_{1,2}$

According to Equation (30) and Figure 2, pseudo-reflection, provided by recirculation matrices of the form (29), effectively suppresses the BBU if HOMs are bound to the horizontal and vertical planes. It is trivial to show that all uncoupled transfer matrices satisfy Equation (34) for recirculation matrices of the form (29)

$$M_2 = T_2M_1T_1^{-1} = \begin{bmatrix} X_2 & 0 \\ 0 & Y_2 \end{bmatrix} \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} X_1^{-1} & 0 \\ 0 & Y_1^{-1} \end{bmatrix} = \begin{bmatrix} 0 & X_2AY_1^{-1} \\ Y_2BX_1^{-1} & 0 \end{bmatrix} \quad (42)$$

Thus, an optical insertion interchanging the vertical and horizontal planes installed in the recirculation pass can simultaneously suppress HOMs in different cavities if the HOMs are bound to the horizontal and vertical planes. Such an insertion was installed in the recirculation pass of the JLab FEL Upgrade in Summer, 2004 [11]. The insertion consists of 5 skew-quadrupoles and reflects the beam respectively to the $x = y$ plane. Initial experience with the reflector showed an increase of the BBU threshold current from 5 mA to a level above 8.5 mA. The maximum current was limited by the injector performance at the time of the experiment. Farther studies of the reflector performance are planned in the Fall of 2004.

### 6 Conclusions

Equation (15), describing the BBU threshold for a single TM dipole HOM with arbitrary polarization and a general-form, $4x4$ recirculation matrix in a two-pass accelerator, has been derived. The equation can be used to estimate the BBU threshold in a case of many high-$Q$ HOMs if a typical separation between the HOMs exceeds the bandwidth of the modes.
Reflection or rotation produced by the recirculation optics can significantly suppress or completely eliminate the regenerative BBU. This result was first predicted by R. Rand and T. Smith in [3] and confirmed in this work analytically and by simulations. Additionally, it was discovered in this work that reflection provides BBU suppression by a factor of $10^3 - 10^4$ for a wide range of values of the mode polarization angle.

To provide BBU suppression by reflection or rotation simultaneously in several cavities of the accelerating structure, the transfer matrices between the cavities have to satisfy equation (34).

Suppression of BBU by rotation in several cavities simultaneously can be difficult if axi-asymmetric elements, such as quadrupoles and dipoles, are used between cavities.

If HOMs are bound to the horizontal and vertical planes, the BBU can be effectively suppressed by an optical transformation with a matrix of the form

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

where $A$ and $B$ are 2x2 block matrices. An optical insertion providing the $x-y$ motion exchange can consist of solenoids and/or skew quadrupoles and can be located in the recirculation pass of a two pass machine.

A new two-dimensional tracking BBU code has been developed at Jefferson Lab. The formula for the BBU threshold derived in this paper has been used to validate the code. The new code is faster than existing codes TDBBU and MATBBU by an order of magnitude or more depending on the particular problem.

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