A spectral gap theorem in simple Lie groups

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Abstract

We establish the spectral gap property for dense subgroups generated by algebraic elements in any compact simple Lie group, generalizing earlier results of Bourgain and Gamburd for unitary groups.

1 Introduction

The purpose of the paper is to study the spectral gap property for measures on a compact simple Lie group $G$. If $\mu$ is a Borel probability measure on $G$, we say that $\mu$ has a spectral gap if the spectral radius of the corresponding operator on $L^2_0(G)$ – the space of mean-zero square integrable functions on $G$ – is strictly less than 1. We also say that $\mu$ is almost Diophantine if it satisfies, for some positive constants $C_1$ and $c_2$, for $n$ large enough and for any proper closed subgroup $H$,

$$\mu^\ast n(\{x \in G \mid d(x,H) \leq e^{-C_1 n}\}) \leq e^{-c_2 n}.$$  

Using the discretized Product Theorem proved in [14] and the techniques developed by Bourgain and Gamburd in [4] for the group $SU(2)$, we prove the following theorem.

**Theorem 1.1.** Let $G$ be a connected compact simple Lie group and $\mu$ be a Borel probability measure on $G$. Then $\mu$ has a spectral gap if and only if it is almost Diophantine.

A measure $\mu$ on the compact simple Lie group $G$ is called adapted if its support generates a dense subgroup of $G$. It is not known whether every adapted probability measure on the compact simple Lie group $G$ is almost Diophantine, but it is natural to conjecture a affirmative answer to this question. In this direction, Bourgain and Gamburd proved that if $\mu$ is an adapted probability measure on $SU(d)$ supported on elements with algebraic entries, then $\mu$ has a spectral gap. We generalize their result to an arbitrary simple group, and prove the following, using the theory of random matrix products over arbitrary local fields, as exposed in [3].

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Theorem 1.2. Let $G$ be a connected compact simple Lie group and $U$ a fixed basis for its Lie algebra. Let $\mu$ be an adapted probability measure on $G$ and assume that for any $g$ in the support of $\mu$, the matrix of $\text{Ad}_g$ in the basis $U$ has algebraic entries. Then $\mu$ is almost Diophantine, and therefore has a spectral gap.

In the case $G$ is the group $SO(n)$ of rotations of the Euclidean space of dimension $n$, Theorem 1.2 is used by Lindenstrauss and Varjú [11] to study absolute continuity of self-similar measures defined by isometries of the Euclidean space described by matrices with algebraic coefficients.

The plan of the paper is simple: in Section 2 we prove Theorem 1.1, in Section 3 we prove Theorem 1.2.

For us, a compact simple Lie group will be a compact real Lie group whose Lie algebra is simple. We will also make use of some classical notation:

- The Landau notation: $O(\epsilon)$ stands for a quantity bounded in absolute value by $C\epsilon$, for some constant $C$ (generally depending on the ambient group $G$).
- The Vinogradov notation: we write $x \ll y$ if, $x \leq Cy$ for some constant $C$ (again, possibly depending on the ambient group). We will also write $x \asymp y$ if $x \ll y$ and $x \gg y$, and similarly. For two real valued functions $\varphi$ and $\psi$ on $G$, we write $\varphi \ll \psi$ if there exists an absolute constant $C$ such that for all $x$ in $G$, $\varphi(x) \leq C \cdot \psi(x)$.

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2 The spectral gap property

Let $G$ be a connected compact simple Lie group. If $\mu$ is a Borel probability measure on $G$, we define an averaging operator $T_\mu$ on the space $L^2_0(G)$ of mean-zero square-integrable functions by the formula

$$T_\mu f(x) = \int_G f(xg) \, d\mu(g), \quad \forall f \in L^2_0(G).$$

Definition 2.1. We say that a probability measure $\mu$ on $G$ has a spectral gap if the spectral radius of the averaging operator $T_\mu$ on the space $L^2_0(G)$ is strictly less than one.

The purpose of this section is to relate the spectral gap property to the following Diophantine property of measures.
**Definition 2.2.** We say that a probability measure $\mu$ on $G$ is *almost Diophantine* if there exist positive constants $C_1$ and $c_2$ such that for $n$ large enough, for any proper closed connected subgroup $H$,

$$\mu^\ast n(H^{(e^{-C_1n})}) \leq e^{-c_2n}, \quad (1)$$

where $H^{(\rho)}$ denotes the neighborhood of size $\rho$ of the closed subgroup $H$: $H^{(\rho)} = \{ x \in G \mid d(x, H) \leq \rho \}$.

With this definition, we have the following theorem.

**Theorem 2.3** (Spectral gap for almost Diophantine measures). Let $G$ be a connected compact simple Lie group. A Borel probability measure $\mu$ on $G$ has a spectral gap if and only if it is almost Diophantine.

**Remark 1.** The spectral radius of the averaging operator $T_\mu$ on $L^2_0(G)$ is less than one if and only if the spectral radius of $T_\mu T_\bar{\mu} = T_{\mu \ast \bar{\mu}}$ is less than one. This shows that it will be enough to prove the Theorem 2.3 in the case $\mu$ is symmetric.

We start by proving the trivial implication: if $\mu$ has a spectral gap, then it must be almost Diophantine.

*Spectral gap $\implies$ Almost Diophantine.* Suppose $\mu$ has a spectral gap, and let $c > 0$ such that the spectral radius of $T_\mu$ satisfies $RS(T_\mu) \leq e^{-c}$. Let $d$ be the dimension of $G$ and let $H$ be a maximal proper closed subgroup of $G$ of dimension $p$. For $\delta > 0$, we can bound the $L^2$-norm of the indicator function of the $2\delta$-neighborhood of $H$:

$$\| 1_{H^{(2\delta)}} \|_2 \ll \delta^{\frac{d-p}{2}}.$$ 

Therefore, for $n$ larger than $\frac{d-p}{2c} \log \frac{1}{\delta}$, we have

$$\| T^n_\mu 1_{H^{(2\delta)}} \|_2 \ll \delta^{d-p}.$$ 

Making the left-hand side explicit, we find

$$\sqrt{\int_G \mu^\ast n(xH^{(2\delta)})^2 \, dx} \ll \delta^{d-p}$$

and this implies,

$$\mu^\ast n(H^{(\delta)}) \ll \delta^{\frac{d-p}{2}}.$$

Choosing $C_1 \leq \frac{2c}{d-p}$ and $c_2 = c$, and letting $\delta = e^{-C_1n}$, this shows that $\mu$ is almost Diophantine.

To prove the converse implication in Theorem 2.3 we use the strategy developed by Bourgain and Gamburd. If $A$ is a subset of a metric space, for $\delta > 0$, we denote by $N(A, \delta)$ the minimal cardinality of a covering of $A$ by balls of radius $\delta$. We have the following Product Theorem [14, Theorem 3.9].
Theorem 2.4. Let $G$ be a simple Lie group of dimension $d$. There exists a neighborhood $U$ of the identity in $G$ such that the following holds. Given $\alpha \in (0, d)$ and $\kappa > 0$, there exists $\epsilon_0 = \epsilon_0(\alpha, \kappa) > 0$ such that, for $\delta > 0$ sufficiently small, if $A \subset U$ is a set satisfying

1. $N(A, \delta) \leq \delta^{-d+\alpha-\epsilon_0}$, 
2. for all $\rho \geq \delta$, $N(A, \rho) \geq \rho^{-\kappa} \delta^{\epsilon_0}$, 
3. $N(AAA, \delta) \leq \delta^{-\epsilon_0} N(A, \delta)$,

then $A$ is included in a neighborhood of size $\delta^\tau$ of a proper closed connected subgroup of $G$.

We will use Theorem 2.4 to derive a flattening statement for measures. For $\delta > 0$, we let

$$P_\delta = \frac{\mathbf{1}_{B(1, \delta)}}{|B(1, \delta)|}$$

(where $| \cdot |$ is the volume associated to the Haar probability measure on $G$) and if $\mu$ is a probability measure on $G$, we denote by $\mu_\delta$ the function approximating $\mu$ at scale $\delta$:

$$\mu_\delta = \mu * P_\delta.$$

Lemma 2.5 ($L^2$-flattening). Let $G$ be a connected compact simple Lie group. Given $\alpha, \kappa > 0$, there exists $\epsilon > 0$ such that the following holds for any $\delta > 0$ small enough.

Suppose $\mu$ is a symmetric Borel probability measure on $G$ such that one has

1. $\|\mu_\delta\|_2^2 \geq \delta^{-\alpha}$,
2. for any $\rho \geq \delta$ and any closed connected subgroup $H$, $\mu_* \mu (H^{(\rho)}) \leq \delta^{-\tau} \rho^\kappa$.

Then,

$$\|\mu_\delta * \mu_\delta\|_2 \leq \delta^\epsilon \|\mu_\delta\|_2.$$

The proof goes by approximating the measure $\mu_\delta$ by dyadic level sets. We say that a collection of sets $\{X_i\}_{i \in I}$ is essentially disjoint if for some constant $C$ depending only on the ambient group $G$, any intersection of more than $C$ distinct sets $X_i$ is empty. We will use the following lemma.

Lemma 2.6. Let $G$ be a compact Lie group, $\mu$ a Borel probability measure on $G$ and $\delta > 0$. There exist subsets $A_i$, $0 \leq i < \log \frac{1}{\delta}$ such that

1. $\mu_\delta \ll \sum_i 2^i \mathbf{1}_{A_i} \ll \mu_{4\delta}$
2. Each $A_i$ is an essentially disjoint union of balls of radius $\delta$.

Proof. A proof in the case $G = SU(2)$ is given in [10] and also applies in this more general setting, mutatis mutandis. 

To derive Lemma 2.5, we will also use the non-commutative Balog-Szemerédi-Gowers Lemma, due to Tao. If $A$ and $B$ are two subsets of a metric group $G$, we define the multiplicative energy of $A$ and $B$ at scale $\delta$ by

$$E_\delta(A, B) = N(\{(a, b, a', b') \in A \times B \times A \times B \mid d(ab, a'b') \leq \delta\}, \delta).$$

(See [15] for elementary properties.) We have the following important theorem (see Tao [15, Theorem 6.10]).
**Theorem 2.7** (Non-commutative Balog-Szemerédi-Gowers Lemma). Let $G$ be a compact Lie group with a Riemannian metric. There exists a constant $C > 0$ depending only on $G$ such that the following holds for any $\delta > 0$ and any $K \geq 2$.

Suppose that $A$ and $B$ are non-empty subsets of $G$ such that

$$E_\delta(A, B) \geq \frac{1}{K} N(A, \delta)^{1/2} N(B, \delta)^{1/2}.$$ 

Then there exists a $K^C$-approximate subgroup $H$ and elements $x, y$ in $G$ such that

- $N(H, \delta) \leq K^C \cdot N(A, \delta)^{1/2} N(B, \delta)^{1/2}$
- $N(A \cap xH, \delta) \geq K^{-C} \cdot N(A, \delta)$
- $N(B \cap yH, \delta) \geq K^{-C} \cdot N(B, \delta)$.

Recall that a subset $H$ of $G$ is called a $K$-approximate subgroup if it is symmetric and there exists a finite symmetric set $X \subset H$ of cardinality at most $K$ such that $HH \subset XH$.

We are now ready to prove Lemma 2.5.

**Proof of Lemma 2.5.** Write $\mu_\delta \ll \sum_i 2^i 1_{A_i} \ll \mu_{4\delta}$ as in Lemma 2.6. Note that for all $i$, one has

$$2^i |A_i|^{1/2} = \|2^i 1_{A_i}\|_2 \ll \|\mu_{4\delta}\|_2 \simeq \|\mu_\delta\|_2,$$

and

$$2^i |A_i| \simeq 2^i \delta^d N(A_i, \delta) \ll 1.$$

Assume for a contradiction that for some $\epsilon > 0$,

$$\|\mu_\delta * \mu_\delta\|_2 \geq \delta^\epsilon \|\mu_\delta\|_2,$$

with $\delta > 0$ arbitrarily small. This gives,

$$\delta^\epsilon \|\mu_\delta\|_2 \ll \| \sum_{i,j} 2^i 1_{A_i} * 2^j 1_{A_j} \|_2$$

$$\leq \sum_{i,j} \| 2^i 1_{A_i} * 2^j 1_{A_j} \|_2,$$

and as the sum on the right-hand side contains at most $O((\log \delta)^2)$ terms, we must have, for some $i$ and $j$,

$$\| 2^i 1_{A_i} * 2^j 1_{A_j} \|_2 \gg \frac{\delta^\epsilon}{(\log \delta)^2} \|\mu_\delta\|_2 \geq \delta^{O(\epsilon)} \|\mu_\delta\|_2.$$

Therefore,

$$\delta^{O(\epsilon)} \|\mu_\delta\|_2 \leq \| 2^i 1_{A_i} * 2^j 1_{A_j} \|_2 \leq \| 2^i 1_{A_i} \|_2 \| 2^j 1_{A_j} \|_2 \ll 2^i |A_i| \|\mu_\delta\|_2. \quad (2)$$
This implies,
\[ 2^i |A_i| = \delta^{O(\epsilon)} \text{ and similarly } 2^j |A_j| = \delta^{O(\epsilon)}. \] (3)

So we have the following lower bound on the multiplicative energy of \( A_i \) and \( A_j \):
\[
E_\delta(A_i, A_j) \gg \delta^{-3d} \| A_i \ast A_j \|_2^2
\geq \delta^{-3d+O(\epsilon)} 2^{-2i-2j} \| \mu_\delta \|_2^2
\geq \delta^{-3d+O(\epsilon)} 2^{-i-j} |A_i|^{\frac{1}{2}} |A_j|^{\frac{1}{2}} = \delta^{O(\epsilon)} N(A_i, \delta)^{\frac{1}{2}} N(A_j, \delta)^{\frac{1}{2}}.
\]

By Theorem [2.7], there exists a \( \delta^{-O(\epsilon)} \)-approximate subgroup \( \tilde{H} \) and elements \( x, y \) in \( G \) such that
\[
N(\tilde{H}, \delta) \leq \delta^{-O(\epsilon)} N(A_i, \delta)^{\frac{1}{2}} N(A_j, \delta)^{\frac{1}{2}}, \tag{4}
\]
\[
N(x\tilde{H} \cap A_i, \delta) \geq \delta^{O(\epsilon)} N(A_i, \delta) \text{ and } N(\tilde{H}y \cap A_j, \delta) \geq \delta^{O(\epsilon)} N(A_j, \delta). \tag{5}
\]

We may replace \( \tilde{H} \) by its \( \delta \)-neighborhood, and then, \( \mu_\delta(x\tilde{H}) \geq \delta^{O(\epsilon)} \). Let \( U \) be a neighborhood of the identity in \( G \) as in Theorem [2.3] let \( r > 0 \) be such that \( B(1, 2r) \subset U \), and cover \( x\tilde{H} \) by \( O(1) \) balls of radius \( r \). One of these balls \( B \) must satisfy \( \mu_\delta(x\tilde{H} \cap B) \geq \delta^{O(\epsilon)} \) and thus,
\[
\mu_\delta \ast \mu_\delta(\tilde{H}^2 \cap U) \geq \mu_\delta(\tilde{H}^{-1} \cap B^{-1}) \mu_\delta(x\tilde{H} \cap B) \geq \delta^{O(\epsilon)}.
\]

On the other hand, by (2) and (3),
\[
\delta^{O(\epsilon)} \| \mu_\delta \|_2 \leq \| 2^i \mathbb{1}_{A_i} \|_2 \| 2^j \mathbb{1}_{A_j} \|_2 \leq \| 2^j \mathbb{1}_{A_j} \|_2 \leq \delta^{O(\epsilon)} 2^{j/2},
\]
so that \( 2^j \geq \delta^{^{-\alpha+O(\epsilon)}} \) and similarly \( 2^i \geq \delta^{^{-\alpha+O(\epsilon)}} \). This implies
\[
N(A_j, \delta) \leq \delta^{-d+\alpha-O(\epsilon)} \text{ and similarly } N(A_i, \delta) \leq \delta^{-d+\alpha-O(\epsilon)}.
\]

The set \( \tilde{H} \) is a \( \delta^{-O(\epsilon)} \)-approximate subgroup, so \( N(\tilde{H}^2, \delta) \leq \delta^{-O(\epsilon)} N(\tilde{H}, \delta) \). Recalling Inequality (3), we find
\[
N(\tilde{H}^2 \cap U, \delta) \leq N(\tilde{H}^2, \delta) \leq \delta^{-d+\alpha-O(\epsilon)}.
\]

On the other hand, \( \mu_\delta \ast \mu_\delta(\tilde{H}^2 \cap U) \geq \delta^{O(\epsilon)} \) so the second assumption on \( \mu_\delta \) forces, for any \( \rho \geq \delta \) (note that any ball of radius \( \rho \) is included in the \( \rho \)-neighborhood of some proper closed connected subgroup),
\[
N(\tilde{H}^2 \cap U, \rho) \geq \rho^{-\kappa \delta^{O(\epsilon)}}.
\]

Thus, provided we have chosen \( \epsilon > 0 \) small enough, the set \( \tilde{H}^2 \cap U \) satisfies the assumptions of Theorem [2.4] and so must be included in the \( \delta^{-\tau} \)-neighborhood of a proper closed connected subgroup \( H \) of \( G \), contradicting the assumption \( \mu \ast \mu(\tilde{H}^{(\delta^\tau)}) \leq \delta^{-\epsilon \delta^{\kappa \tau}} \). \( \square \)
The idea is now to apply repeatedly that Flattening Lemma to obtain:

**Lemma 2.8.** Let $\mu$ be a symmetric almost Diophantine measure on a connected compact simple Lie group $G$. There exists a constant $C_0 = C_0(\mu)$ such that for any $\delta = e^{-C_0 n} > 0$ small enough,

$$\|(\mu^* \log \frac{1}{\delta})\|_2 \leq \delta^{-\frac{1}{4}}.$$

**Remark 2.** The constant $\frac{1}{4}$ could be replaced in this lemma by any fixed positive constant $\alpha$. Of course, $C_0$ would then depend on $\alpha$.

**Proof.** We first check that a suitable power $\nu = \mu^{c \log \frac{1}{\delta}}$ satisfies the second condition of Lemma 2.5. Since $\mu$ is almost Diophantine, taking $n = \frac{1}{C_1} \log \frac{1}{\delta}$ in Equation (1) shows that when $\delta < \delta_0$, for any proper closed connected subgroup $H$,

$$\mu^{\frac{1}{4}c \log \frac{1}{\delta}}(H(\delta)) \leq \delta^{\frac{1}{4}}.$$

If $xH$ is a left coset of a closed subgroup $H$ and $m$ any symmetric measure, we have

$$m(xH(\delta))^2 \leq m * m(H(2\delta)).$$

Therefore, denoting $c = \frac{1}{4C_1}$ and $\kappa = \frac{1}{C_1^{\frac{1}{4}}}$, we have, for all $\delta < \delta_0$, for any left coset $xH$ of a proper closed connected subgroup,

$$\mu^{2c \log \frac{1}{\delta}}(xH(\delta)) \leq \delta^\kappa.$$

Now, if $H$ is a closed subgroup and $m$ and $m'$ are any two probability measures on $G$, we have

$$m * m'(H(\delta)) \leq \sup_{x \in G} m'(xH(\delta)).$$

Therefore, if $\delta < \rho < \delta_0$, we have, for any proper closed connected subgroup $H$,

$$\mu^{2c \log \frac{1}{\delta}}(H(\rho)) \leq \max_x \mu^{2c \log \frac{1}{\rho}}(xH(\rho)) \leq \rho^\kappa.$$

In other terms, for $\delta > 0$ small enough, the measure $\nu := \mu^{c \log \frac{1}{\delta}}$ satisfies the second condition of Lemma 2.5.

We now apply Lemma 2.5 repeatedly, starting with the measure $\nu$. If $\|\nu_\delta\|_2 \leq \delta^{-\frac{1}{4}}$, then we have what we want. Otherwise, Lemma 2.5 applied to $\nu_\delta$ with $\alpha = \frac{1}{2}$ shows that

$$\|(\nu * \nu)_\delta\|_2 \ll \|\nu_\delta * \nu_\delta\|_2 \leq \delta^\epsilon \|\nu_\delta\|_2.$$

We then repeat the same procedure, replacing $\nu$ by $\nu * \nu$, and so on (note that the computations made above for $\nu$ also show that all the convolution powers of $\nu$ will satisfy the second condition of Lemma 2.5). After at most $\frac{d}{\epsilon}$ iterations, the procedure must stop, i.e. we must have,

$$\|(\mu^{C_0 \log \frac{1}{\delta}})_\delta\|_2 = \|(\nu^{2^d})_\delta\|_2 \leq \delta^{-\frac{1}{4}}.$$
The end of the proof of Theorem 2.3 relies on the high-multiplicity of irreducible representations in the regular representation $L^2(G)$. Recall that the irreducible representations of $G$ are in bijection with dominant analytically integral weights (see e.g. [9]). We denote by $\pi_\lambda$ the irreducible representation of $G$ with highest weight $\lambda$. If $\mu$ is a finite Borel measure on $G$, the Fourier coefficient of $\mu$ at $\lambda$ is

$$\hat{\mu}(\lambda) = \int_G \pi_\lambda(g) \, d\mu(g).$$

By Lemma 2.8, all we need to show is the following.

**Lemma 2.9.** Let $\mu$ be a Borel probability measure on a compact semisimple Lie group $G$ such that for some constant $C$, for all $\delta = e^{-Cn} > 0$ small enough ($n$ a positive integer),

$$\| (\mu^{*C \log \frac{1}{\delta}})_\delta \|_2 \leq \delta^{-\frac{1}{4}}.$$

Then $\mu$ has a spectral gap in $L^2(G)$.

**Proof.** Since the representation $V_\lambda$ occurs in $L^2(G)$ with multiplicity $\dim V_\lambda$, the Parseval Formula for $(\mu^{*C \log \frac{1}{\delta}})_\delta$ gives

$$\| (\mu^{*C \log \frac{1}{\delta}})_\delta \|_2^2 = \sum_\lambda (\dim V_\lambda) \| \hat{\mu}(\lambda)^{C \log \frac{1}{\delta}} \hat{P}_\delta(\lambda) \|_{HS}^2,$$

where $\| \cdot \|_{HS}$ is the Hilbert-Schmidt norm. Moreover, it is easily seen that we may bound the distance (in operator norm) from $\hat{P}_\delta(\lambda)$ to the identity (see for instance [13, Lemme 3.1]): for some constant $c > 0$ depending only on $G$, we have, whenever $\| \lambda \| \leq c\delta^{-1}$,

$$\| \hat{P}_\delta(\lambda) - Id_{V_\lambda} \|_{op} \leq \frac{1}{2}.$$ 

Therefore for any $\lambda$ such that $\| \lambda \| \leq c\delta^{-1}$, using (6) and the assumption of the lemma,

$$\delta^{-\frac{1}{2}} \geq \frac{1}{4} (\dim V_\lambda) \| \hat{\mu}(\lambda)^{C \log \frac{1}{\delta}} \|_{op}^2.$$

Now, as a consequence of the Weyl dimension Formula, we have, for some constant $c$ depending only on $G$, for any representation $V_\lambda$ with highest weight $\lambda$ [13, Lemme 3.2],

$$\dim V_\lambda \geq c\| \lambda \|.$$

Taking $\lambda$ with $e^{-C}c\delta^{-1} \leq \| \lambda \| \leq c\delta^{-1}$ in the above equation (7), we find

$$\| \hat{\mu}(\lambda)^{C \log \frac{1}{\delta}} \|_{op}^2 \ll \delta^{\frac{1}{2}}.$$

However, the spectral radius of an operator $T$ satisfies, for any integer,

$$RS(T) \leq \| T^n \|_{op}^{\frac{1}{n}},$$

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so that for some absolute constant $K$, we have

$$RS(\hat{\mu}(\lambda)) \leq (K\delta^{1/4})^{\log \frac{1}{\delta}}$$

$$= e^{-\frac{1}{14} K^{1/4} \log \frac{1}{\delta}}$$

which is bounded away from 1 as long as $\delta$ is sufficiently small, i.e. as long as $\lambda$ is sufficiently large. As the spectral radius of $T_\mu$ in $L^2_0(G)$ is equal to the supremum of all $RS(\hat{\mu}(\lambda))$ for $\lambda \neq 0$, this finishes the proof. \qed

3 Measures supported on algebraic elements

In this section, we fix a basis for the Lie algebra $\mathfrak{g}$. We say that an element $g \in G$ is algebraic if the entries of the matrix of $\text{Ad} g$ in that fixed basis are algebraic numbers. Recall that a probability measure on $G$ is called adapted if its support generates a dense subgroup of $G$. We want to prove the following.

**Theorem 3.1.** Let $G$ be a connected compact simple Lie group. If $\mu$ is an adapted probability measure on $G$ whose support consists of algebraic elements, then $\mu$ has a spectral gap.

**Remark 3.** We have already explained in Remark 1 that it is enough to prove such a theorem for a symmetric measure $\mu$. Moreover, if $\mu$ is symmetric, under the assumptions of the theorem, we may always find a symmetric finitely supported adapted measure $\nu$ that is absolutely continuous with respect to $\mu$. It is readily seen that if $\nu$ has a spectral gap, then so has $\mu$, so we may assume in the proof of Theorem 3.1 that $\mu$ is finitely supported.

The proof has two parts. First, we show that, given a proper closed subgroup $H$, the probability $\mu^\ast n(H)$ decays exponentially, with a rate that does not depend on $H$. This part is based on the theory of product of random matrices, as developed by Furstenberg, Guivarc’h and others; the central input is Theorem 3.4 below. The difficult point in the proof is to reduce to the case where the subgroup generated by the support of $\mu$ acts proximally. While writing this paper, we learnt from Emmanuel Breuillard that an alternative approach was to derived an improved version of Theorem 3.4 that applies also to some non-proximal representations [7]. Some partial results on this issue were also obtained previously by Aoun [11].

Then, we show that when the support of $\mu$ consists of algebraic elements, the measure $\mu$ is almost Diophantine. This second part is based on an application of the effective arithmetic Nullstellensatz, and relies crucially on the algebraic assumption on the elements of the support of $\mu$.

3.1 Transience of closed subgroups

We want to prove the following.
Proposition 3.2. Let $\mu$ be an adapted finitely supported symmetric probability measure on a connected compact simple Lie group $G$. Then, there exists a constant $\kappa = \kappa(\mu)$ such that for $n \geq n_0$, for any proper closed subgroup $H < G$, 
\[
    \mu_n^*(H) \leq e^{-\kappa n}.
\]

The proposition is based on the following lemma.

Lemma 3.3. Let $\Gamma = \langle S \rangle$ be a finitely generated dense subgroup in $G$. There exists a finite collection of vector spaces $S_i$, $1 \leq i \leq s$, over local fields $K_i$, such that the following holds:

- for each $i \in \{1, \ldots, s\}$, the group $\Gamma$ acts proximally and strongly irreducibly on $S_i$;
- for any proper closed subgroup $H < G$ such that $\Gamma \cap H$ is infinite, there exists an $i \in \{1, \ldots, s\}$ for which $\Gamma \cap H$ stabilizes a proper linear subspace of $S_i$.

Let us explain how this lemma implies Proposition 3.2 when combined with the following important result of random matrix products theory [3, Proposition 12.3] (see also [6, Theorem 4.4]).

Theorem 3.4. Let $K$ be a local field and $S$ be a finite dimensional vector space over $K$. Suppose $\mu$ is a measure on $GL(S)$ such that the semigroup $\Gamma$ generated by the support of $\mu$ acts proximally on $S$. Then, there exists a constant $\kappa = \kappa(\mu)$ such that for any integer $n$ large enough, for any vector $v \in S$ and any hyperplane $V < S$,
\[
    \mu_n^*(\{g \in GL(S) \mid g \cdot v \in V\}) \leq e^{-\kappa n}.
\]

Proof of Proposition 3.2. Let $\Gamma$ be the group generated by the support of $\mu$. Given a proper closed connected subgroup $H$ of $G$, we distinguish two cases. First case; $\Gamma \cap H$ is finite.

By Selberg’s Lemma, $\Gamma$ contains a torsion free subgroup of finite index $N_0$. Hence the cardinality of $\Gamma \cap H$ is bounded by $N_0$ and the uniform exponential decay of $\mu_n^*(H) = \mu^*(\Gamma \cap H)$ is a direct consequence of Kesten’s Theorem [8, Corollary 3] since $\Gamma$ is not amenable.

Second case; $\Gamma \cap H$ is infinite.

Let $S_i$, $1 \leq i \leq s$, be the vector spaces given by Lemma 3.3. For each $i$, the measure $\mu$ may be viewed as a measure on $GL(S_i)$. Choose $\kappa > 0$ such that the conclusion of Theorem 3.4 holds for each $S_i$. Choose $i$ such that $\Gamma \cap H$ stabilizes a proper subspace $L$ of $S_i$. We then have, for $n$ large enough,
\[
    \mu_n^*(\{g \in \Gamma \mid g \cdot L = L\}) \leq e^{-\kappa n},
\]
so that
\[
    \mu_n^*(H) = \mu_n^*(H \cap \Gamma) \leq e^{-\kappa n}.
\]

Before turning to the proof of Lemma 3.3, let us recall the setting. The group $\Gamma$ is a dense finitely generated free subgroup of the connected compact simple group $G$, and $k$ is the field generated by the coefficients of the elements $\text{Ad} \, g$, for $g$ in $\Gamma$. As $\Gamma$ is dense in $G$, we may view $G$ as the group of real points of an algebraic group $G$ defined over $k$. Whenever $K$ is a field containing $k$, we will denote by $G(K)$ the group of $K$-points of $G$. Similarly, if $V$ is a linear representation of $G$ defined over $K$, we will write $V(K)$ for the associated $K$-vector space, on which $G(K)$ acts.

In the case when $\Gamma$ acts proximally on the adjoint representation $g(K)$, for some local field $K$ containing $k$, the proof of Lemma 3.3 is substantially simpler. This is the content of the next lemma.

**Lemma 3.5.** Assume that $\Gamma$ acts proximally on $g(K)$, for some local field $K$ containing $k$. Then,
- the group $\Gamma$ acts proximally and strongly irreducibly on $g(K)$;
- for any proper closed subgroup $H < G$ such that $\Gamma \cap H$ is infinite, $\Gamma \cap H$ stabilizes a proper linear subspace of $g(K)$.

**Proof.** By assumption, $\Gamma$ acts proximally on $g(K)$. As $\Gamma$ is dense in $G$, it is Zariski dense in $G(K)$, and therefore $\Gamma$ acts strongly irreducibly on $g(K)$.

Now if $H$ is a proper closed infinite subgroup of $G$ such that $\Gamma \cap H$ is infinite, then $\Gamma \cap H$ stabilizes the (complex) Lie algebra of the Zariski closure of $\Gamma \cap H$. This is a proper subspace $L < g_C$ defined over $k$ (and hence, over $K$), so that $\Gamma \cap H$ stabilizes a proper subspace of $g(K)$. \hfill $\square$

Let $\Delta \subset E$ ($E$ a Euclidean space of dimension $\text{rk} \, G$) be the root system of $G$, choose a basis $\Pi$ for $\Delta$, and let $C$ be the associated Weyl chamber. If $\omega$ is a dominant weight, with associated irreducible representation $V_\omega$, we denote by $\omega^*$ the dominant weight of the dual irreducible representation $(V_\omega)^*$. We observe the following:

**Lemma 3.6.** Let $\tilde{\alpha}$ be the largest root of $\Delta$. Either $\tilde{\alpha} = \omega$ is a fundamental weight, or $\tilde{\alpha} = \omega + \omega^*$ is the sum of a fundamental weight and its dual (those two might coincide).

**Proof.** Let $\rho$ be the sum of all fundamental weights of $\Delta$. Choose a fundamental weight $\omega$ minimizing $\langle \omega, \rho \rangle$. The adjoint representation can be viewed as a subrepresentation of $\text{End} \, V_\omega \simeq V_\omega \otimes (V_\omega)^*$. Comparing the highest weights, we find that $\tilde{\alpha}$ can be written

$$\tilde{\alpha} = \omega + \omega^* - \sum_i n_i \alpha_i, \quad n_i \in \mathbb{N}, \quad \alpha_i \text{ simple roots}.$$

Taking the inner product with $\rho$, we find that $\langle \tilde{\alpha}, \rho \rangle \leq 2 \langle \omega, \rho \rangle$ and in case of equality, we must have all $n_i$ equal to zero i.e. $\tilde{\alpha} = \omega + \omega^*$. On the other hand, if the inequality is strict, by minimality of $\langle \omega, \rho \rangle$, the dominant weight $\tilde{\alpha}$ must be fundamental (not necessarily $\omega$, though). This proves the lemma. \hfill $\square$

Finally, we recall the following fact.
Lemma 3.7. Assume $\Gamma$ acts proximally on $V^\omega(K)$, for some local field $K$ containing $k$. Then, $\Gamma$ acts proximally on $V^{\omega + \omega^*}(K)$.

Proof. This is an immediate consequence of the fact that if $\Gamma$ acts proximally on a vector space $V$, then we may find an element $\gamma$ in $\Gamma$ such that both $\gamma$ and $\gamma^{-1}$ act proximally on $V$, see [2, Lemme 3.9].

According to Lemma 3.6, write $\tilde{\alpha} = \omega$ or $\tilde{\alpha} = \omega + \omega^*$. Putting together Lemma 3.5 and Lemma 3.7, we find that Lemma 3.3 holds whenever $\Gamma$ acts proximally on $V^\omega(K)$ (or $V^{\omega^*}(K)$) for some local field $K$. Therefore, for the rest of the proof of Lemma 3.3, we assume (writing the largest root $\tilde{\alpha} = \omega + \omega^*$ or $\tilde{\alpha} = \omega$, for some fundamental weight $\omega$):

There is no local field $K$ such that $\Gamma$ acts proximally on $V^\omega(K)$. (8)

To prove Lemma 3.3, we start by defining a certain family of irreducible complex representations of $G$. For any nonzero vector $X$ in the Weyl chamber $C$ of $\Delta$, we let

$$E_X = \{ \alpha \in \Delta \mid \langle \alpha, X \rangle \text{ is maximal} \}$$

and

$$m_X = \text{card } E_X.$$

Note that the largest root $\tilde{\alpha}$ of $\Delta$ always belongs to $E_X$ so that $E_X = \{ \alpha \in \Delta \mid \langle \tilde{\alpha} - \alpha, X \rangle = 0 \}$.

Finally, we define a dominant weight $\omega_X$ by

$$\omega_X = \sum_{\alpha \in E_X} \alpha,$$

and denote by $S_X$ the irreducible representation of $G$ with highest weight $\omega_X$. A simple way to check that $\omega_X$ is indeed a dominant weight is to construct $S_X$ explicitly as follows. Write the decomposition of $\mathfrak{g}_C$ into root spaces for some maximal torus $T$:

$$\mathfrak{g}_C = \mathfrak{t}_C \oplus \left( \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right).$$

Each $\mathfrak{g}_\alpha$ is one-dimensional, so write $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$. The representation $S_X$ is the subrepresentation of $\bigwedge^{m_X} \mathfrak{g}_C$ generated by the vector

$$\xi_X = \bigwedge_{\alpha \in E_X} E_\alpha \in \bigwedge^{m_X} \mathfrak{g}_C.$$

The spaces $S_i$ of Lemma 3.3 will be constructed as representations $S_X(K)$, where the local field $K$ will be suitably chosen as to arrange that the action of $\Gamma$ is proximal. The difficult point will be to prove the existence of a proper stable subspace under $\Gamma \cap H$, when $H$ is a closed subgroup. For that, one crucial observation is the following fact about faces of root systems.

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Lemma 3.8. Let $\Delta$ be an irreducible root system with a given basis $\Pi$. Denote by $\tilde{\alpha}$ the largest root of $\Delta$, and let $X$ be a nonzero vector in the Weyl chamber $C$. In the case $\tilde{\alpha} = \omega + \omega^*$ and $\omega \neq \omega^*$, assume $X$ not collinear to $\omega$ nor to $\omega^*$.

We define the face of $\Delta$ associated to $X$ by
\[
E_X = \{ \alpha \in \Delta \mid \langle \tilde{\alpha} - \alpha, X \rangle = 0 \},
\]
and denote by $W_{\tilde{\alpha}}$ the stabilizer of $\tilde{\alpha}$ in the Weyl group $W$ of $\Delta$. Then,
\[
\bigcap_{w \in W_{\tilde{\alpha}}} w \cdot \mathcal{E}_X = \{ \tilde{\alpha} \}.
\]

Proof. Letting $E'_X = \tilde{\alpha} - E_X$, we want to check that
\[
\bigcap_{w \in W_{\tilde{\alpha}}} w \cdot E'_X = \{ 0 \}.
\]

For sake of clarity, we deal first with the case when $\tilde{\alpha}$ is proportional to some fundamental weight $\omega = \omega_{i_0}$. Any element $u$ in $E'_X$ can be written $u = \tilde{\alpha} - \alpha$, so that
\[
\langle u, \tilde{\alpha} \rangle = \| \tilde{\alpha} \|^2 - \langle \alpha, \tilde{\alpha} \rangle,
\]
and, as $\tilde{\alpha}$ has maximal norm among the roots, this shows,
\[
\forall u \in E'_X \setminus \{ 0 \}, \quad \langle u, \tilde{\alpha} \rangle > 0. \tag{9}
\]

On the other hand, since the largest root $\tilde{\alpha}$ is proportional to a fundamental weight, the elements of $E$ invariant under $W_{\tilde{\alpha}}$ are proportional to $\tilde{\alpha}$. This implies that the element $\frac{1}{|W_{\tilde{\alpha}}|} \sum_{w \in W_{\tilde{\alpha}}} w \in \text{End} E$ is just the orthogonal projection to $\mathbb{R}\tilde{\alpha}$, so that
\[
\frac{1}{|W_{\tilde{\alpha}}|} \sum_{w \in W_{\tilde{\alpha}}} w \cdot X = \langle X, \frac{\tilde{\alpha}}{\| \tilde{\alpha} \|^2} \rangle \tilde{\alpha},
\]
is a nonzero multiple of $\tilde{\alpha}$. This implies in particular that
\[
\bigcap_{w \in W_{\tilde{\alpha}}} w \cdot X^\perp \subset \tilde{\alpha}^\perp.
\]

Recalling (9), we indeed find
\[
\bigcap_{w \in W_{\tilde{\alpha}}} w \cdot \mathcal{E}'_X \subset \mathcal{E}_X \cap \bigcap_{w \in W_{\tilde{\alpha}}} w \cdot X^\perp \subset \mathcal{E}'_X \cap \tilde{\alpha}^\perp = \{ 0 \}.
\]

We deal now with the case $\tilde{\alpha} = \omega + \omega^*$, with $\omega \neq \omega^*$. This means that the group $G$ is of type $A_{\ell}$, i.e. locally isomorphic to $SU(\ell + 1)$. Note that this is exactly the case studied by Bourgain and Gamburd in [5]. We may modify the above argument in the following way. The element $\frac{1}{|W_{\tilde{\alpha}}|} \sum_{w \in W_{\tilde{\alpha}}} w$ is the orthogonal projection on the subspace $\mathbb{R}\omega \oplus \mathbb{R}\omega^*$. As $X$ is not collinear to $\omega$ nor to $\omega^*$, we have
\[
\frac{1}{|W_{\tilde{\alpha}}|} \sum_{w \in W_{\tilde{\alpha}}} w \cdot X = a\omega + b\omega^*, \quad \text{for some } a, b > 0.
\]
so that
\[ \bigcap_{w \in W_\tilde{\alpha}} w \cdot X^\perp \subset (a\omega + b\omega^*)^\perp. \]

Then we observe that any element \( u \) in \( E'_X \) is a sum of simple roots:
\[ u = \sum_{\alpha \in \Pi} n_\alpha \alpha \]
and as \( \tilde{\alpha} = \omega + \omega^* \) has maximal norm among the roots, we must have \( n_\alpha \geq 1 \) for \( \alpha \) the simple root corresponding to \( \omega \) or \( \omega^* \). This implies in particular
\[ \forall u \in E'_X \setminus \{0\}, \langle u, a\omega + b\omega^* \rangle > 0. \]

As before, this yields
\[ \bigcap_{w \in W_\tilde{\alpha}} w \cdot E'_X \subset E'_X \cap \bigcap_{w \in W_\tilde{\alpha}} w \cdot X^\perp \subset E'_X \cap (a\omega + b\omega^*)^\perp = \{0\}. \]

This property of root systems implies the following result about non-irreducibility of the representations \( S_X \) under proper subgroups of \( G \).

**Lemma 3.9.** Let \( G \) be a connected compact simple Lie group with root system \( \Delta \), let \( X \) be a nonzero vector in the Weyl chamber \( C \). In the case \( \tilde{\alpha} = \omega + \omega^* \) and \( \omega \neq \omega^* \), assume \( X \) is not collinear to \( \omega \) nor to \( \omega^* \). If \( H \) is a proper closed positive dimensional subgroup of \( G \) such that for some \( \gamma \) in \( H \), the vector \( \xi_X \) above is an eigenvector of \( \gamma \) whose associated eigenvalue has multiplicity one.

Then, the representation \( S_X \) is not irreducible under the action of \( H \).

**Proof.** Denote by \( L \) the complexification of the Lie algebra of \( H \), by \( L^\perp \) its orthogonal for the Killing form, and write
\[ \bigwedge^{m_X} \mathfrak{g}_C = \bigoplus_{j=0}^{m_X} \bigwedge^j L \wedge \bigwedge^{m_X-j} L^\perp. \]

All the subspaces on the right-hand side of the formula are stable under the action of \( \gamma \) (in fact, of \( H \)), so that the eigenvector \( \xi_X \), whose associated eigenvalue has multiplicity one, must belong to one of them, say
\[ \xi_X \in \bigwedge^j L \wedge \bigwedge^{m_X-j} L^\perp. \]  

The subspace \( S_X \cap \bigwedge^j L \wedge \bigwedge^{m_X-j} L^\perp \) is a nonzero subspace of \( S_X \) that is invariant under \( H \). Suppose for a contradiction that it is equal to the whole of \( S_X \), i.e. that
\[ S_X \subset \bigwedge^j L \wedge \bigwedge^{m_X-j} L^\perp. \]

Let \( F \) be the subspace of \( \mathfrak{g}_C \) generated by the \( E_\alpha \), for \( \alpha \) in \( \mathcal{E}_X \). By (10), we have
\[ F = F \cap L \oplus F \cap L^\perp. \]
As the largest root $\tilde{\alpha}$ is always in $E_X$, the vector $E_{\tilde{\alpha}}$ is in $F$, and therefore,

$$p_L(E_{\tilde{\alpha}}) \in F,$$

where $p_L$ denotes the orthogonal projections from $g_C$ to $L$. Now, let $w$ be an element of the Weyl group of $\Delta$ fixing $\tilde{\alpha}$. By (11) and the fact that $S_X$ is stable under $G$, we have

$$w \cdot \xi_X \in \bigwedge^j L \wedge \bigwedge^{mX - j} L^\perp.$$

Reasoning as before, this yields, since $\tilde{\alpha}$ is invariant under $w$,

$$p_L(E_{\tilde{\alpha}}) \in w \cdot F.$$

Therefore, letting $w$ describe the stabilizer $W_{\tilde{\alpha}}$ of the largest root, we obtain

$$p_L(E_{\tilde{\alpha}}) \in \bigcap_{w \in W_{\tilde{\alpha}}} w \cdot F.$$

However, by Lemma 3.8, the intersection on the right reduces to $C_{E_{\tilde{\alpha}}}$. If $p_L(E_{\tilde{\alpha}}) \neq 0$, we find $E_{\tilde{\alpha}} \in L$. Otherwise, $E_{\tilde{\alpha}} \in L^\perp$. To conclude, we observe that by (11) and the fact that $S_X$ is stable under $G$, we have, for any $g$ in $G$,

$$g \cdot \xi_X \in \bigwedge^j L \wedge \bigwedge^{mX - j} L^\perp,$$

so that we can reason exactly as before, just conjugating the maximal torus $T$, the root-spaces and the space $F$ by the element $g$. This yields

$$g \cdot E_{\tilde{\alpha}} \in L \quad \text{or} \quad g \cdot E_{\tilde{\alpha}} \in L^\perp.$$

Exchanging if necessary $L$ and $L^\perp$, we may assume without loss of generality that for a set $A \subset G$ of positive Haar measure in $G$, we have

$$\forall g \in A, \quad g \cdot E_{\tilde{\alpha}} \in L,$$

which is easily seen to imply $L = g_C$ contradicting the assumption that $H$ is a proper closed connected subgroup of $G$.

Thus, we have shown that $S_X \cap \bigwedge^j L \wedge \bigwedge^{mX - j} L^\perp$ is a proper subspace of $S_X$ that is invariant under $H$. In particular, $S_X$ is not irreducible under $H$. \[\square\]

**Remark 4.** Note that the fact that $S_X$ is not irreducible under $H$ also implies that it is not irreducible under any conjugate $aHa^{-1}$ of $H$.

We are now ready to conclude the proof of Proposition 3.2 by deriving Lemma 3.3.

**Proof of Lemma 3.3.** Clearly, it suffices to deal with maximal proper closed subgroups $H$. There are only finitely many such maximal subgroups, up to conjugation by elements of $G$. Denote by $T$ a finite set of representatives modulo conjugation of all maximal closed subgroups $H$ that admit a conjugate $H_0$ such that $H_0 \cap \Gamma$ is infinite. We may require that for each $H_0$ in $T$, the intersection $\Gamma \cap H_0$ is infinite. For each such $H_0$, we will construct a vector space $S$ over a local field $K$ and a representation of $\Gamma$ in $S$ such that:
the group $\Gamma$ acts proximally and strongly irreducibly on $S$.

- if $H$ is any conjugate of $H_0$, then $H \cap \Gamma$ stabilizes a proper subspace of $S$.

As $\Gamma \cap H_0$ is infinite, it contains a non-torsion element $\gamma$. Then, $\text{Ad} \gamma$ has an eigenvalue $\lambda$ that is not a root of unity. If $k$ is the field generated by the coefficients of all $\text{Ad} g$, $g \in \Gamma$, by \cite[Lemma 4.1]{15}, we may choose an embedding of $k(\lambda)$ into a local field $K_\nu$ such that $|\lambda|_\nu > 1$.

Denote by $\Delta$ the root system of $G$ and by $E$ the Euclidean space containing it. For some $X_0 \in E$, the eigenvalues of $\text{Ad} \gamma$ are: 1 (with multiplicity $\text{rk} G$) and the $e^{i\langle \alpha, X_0 \rangle}$, $\alpha \in \Delta$.

As $|\cdot|_\nu$ is multiplicative, there exists a unique $X \in E$ such that

$$\forall \alpha \in \Delta, \quad \log |e^{i\langle \alpha, X_0 \rangle}|_\nu = \langle \alpha, X \rangle.$$

We choose a basis for $\Delta$ such that $X$ lies in the Weyl chamber $C$ and consider the associated complex irreducible representation of $G$ introduced earlier as $S_X$. We choose a finite extension $K$ of $K_\nu$ containing all extensions of $k$ of degree at most $\dim S_X$ and such that $G$ is split over $K$. The representation $S_X$ is then defined over $K$, and we set $S = S_X(K)$. As $\Gamma$ is a Zariski dense subgroup of $G(K)$, $S$ is a strongly irreducible and proximal representation of $\Gamma$.

On the other hand, writing the largest root $\tilde{\alpha} = \omega$ or $\tilde{\alpha} = \omega + \omega^*$, Assumption (8) implies that the element $X$ is not collinear to $\omega$ nor to $\omega^*$. Moreover, the vector $\xi_X$ is the eigenvector of $\gamma$ associated to the unique eigenvalue of maximal modulus in $K_\nu$, so that Lemma 3.9 shows that $S_X$ is not irreducible under $H_0$.

As we already observed, this implies that whenever $H$ is conjugate to $H_0$, $S_X$ is not irreducible under $H$.

Thus, if $H$ is any conjugate of $H_0$, applying Lemma 3.10 below to the set of $\text{Ad} g$, for $g \in \Gamma \cap H$, we obtain an extension $K' > K$ of degree at most $\dim S_X$ and a proper subspace of $S_X$ defined over $K'$ that is stable under $\Gamma \cap H$. This yields a proper subspace of $S$ stable under $\Gamma \cap H$ and finishes the proof.

For convenience of the reader, we recall the following easy linear algebra lemma, which we just used in the above proof.

**Lemma 3.10.** Let $A$ be a subset of $SU(d)$ whose elements have coefficients in a field $k < \mathbb{C}$, and suppose $A$ stabilizes a proper subspace $V$ of $\mathbb{C}^d$. Then there exists an extension $k' > k$ of degree at most $d$ and a proper subspace $V'$ defined over $k'$ and stable under $A$.

**Proof.** The set of solutions $x \in \text{End}(\mathbb{C}^d)$ to

$$\forall a \in A, \ ax = xa,$$  \hspace{1cm} (12)

is a vector space defined over $k$, it contains both the identity and the orthogonal projection on the proper stable subspace, so it has dimension at least two. Therefore, we may find a solution $x$ that has coefficients in $k$ and is not a homothety. Then, pick an eigenvalue $\lambda$ of $x$, let $k' = k(\lambda)$ and $V' = \ker(x - \lambda I)$; this solves the problem.
3.2 From a closed subgroup to a small neighborhood

Let $S$ be a finite set of algebraic elements in $G$, and let $\Gamma = \langle S \rangle$ be the subgroup generated by $S$. We endow $\Gamma$ with the word metric associated to the generating system $S$, and denote by $B_\Gamma(n)$ the ball of radius $n$ centered at the identity, for that metric. If $L$ is a proper subspace of the Lie algebra $\mathfrak{g}$ of $G$, we let

$$H_L = \{ g \in G \mid (\text{Ad} \ g)L = L \}.$$ 

The key proposition is the following.

**Proposition 3.11.** Let $G$ be a connected compact simple group and $\Gamma$ a dense subgroup generated by a finite set $S$ of algebraic elements of $G$. There exist a constant $C_1 = C_1(S)$ and an integer $n_0$ such that for any integer $n \geq n_0$, for any proper subspace $L_0 < \mathfrak{g}$, there exists a proper closed subgroup $H_1 < G$ such that

$$B_\Gamma(n) \cap H_{L_0}^{(e^{-C_1 n})} \subset B_\Gamma(n) \cap H_1.$$ 

With this proposition, let us prove Theorem 3.1.

**Proof of Theorem 3.1.** By Theorem 2.3, it suffices to check that $\mu$ is almost Diophantine. Let $C_1$ be the constant given by Proposition 3.11. For $H$ a proper closed subgroup of $G$ we want to bound $\mu^*(H^{(e^{-C_1 n})})$. If $H$ is finite we conclude as in the proof of Lemma 3.3 using Selberg’s Lemma and Kesten’s Theorem, so we may as well assume that $H$ is positive dimensional. Denote by $L_0$ its Lie algebra. By Proposition 3.11

$$B_\Gamma(n) \cap H_{L_0}^{(e^{-C_1 n})} \subset B_\Gamma(n) \cap H_1,$$

and therefore, by Proposition 3.2 (taking $c_2 = \kappa > 0$),

$$\mu^*(H^{(e^{-C_1 n})}) \leq \mu^*(H_1) \leq e^{-c_2 n},$$

and $\mu$ is almost Diophantine. \(\square\)

To prove Proposition 3.11 we want to use an effective version of Hilbert’s Nullstellensatz. For that, we need to set up some notation.

Let $e_i$, $1 \leq i \leq d$, be a basis for $\mathfrak{g}_C$, and define, for $I \subset \{1, \ldots, d\}$,

$$e_I = \bigwedge_{i \in I} e_i.$$ 

The family $(e_I)_{|I|=1}$ is a basis for $\bigwedge^I \mathfrak{g}_C$. Denote $W_\ell \subset \bigwedge^\ell \mathfrak{g}_C$ the set of pure tensors, i.e., the set of elements in $\bigwedge^\ell \mathfrak{g}_C$ that can be written $v_1 \wedge v_2 \wedge \cdots \wedge v_\ell$ for some $v_i$’s in $\mathfrak{g}_C$. It is easy to check that $W_\ell$ is an algebraic subvariety of $\bigwedge^\ell \mathfrak{g}_C$ defined over the rationals and therefore, we may choose a finite collection...
of polynomials \((R_j)_{1 \leq j \leq C}\) with integer coefficients in \(\binom{d}{\ell}\) variables such that for any \(v = \sum v_\ell e_\ell\) in \(\Lambda^\ell \mathfrak{g}_C\),

\[
v \in W_\ell \iff \forall j, \, R_j((v_\ell)_{|I|=\ell}) = 0.
\]

We also define a family of polynomial maps \(P_{I_0,g} : \mathbb{C}^{\binom{d}{\ell}-1} \rightarrow \Lambda^\ell \mathfrak{g}_C\) for \(I_0 \subset \{1, \ldots, d\}\) with \(|I_0| = \ell\) and \(g \in G\), in the following way. The polynomial \(P_{I_0,g}\) has \(\binom{d}{\ell}-1\) variables \(v_\ell, i\), indexed by all subsets \(I\) of \(\{1, \ldots, d\}\) of cardinality \(\ell\) except \(I_0\), and is defined by

\[
P_{I_0,g}(v_I) = g \cdot v - v,
\]

where \(v = e_{I_0} + \sum_{I \neq I_0} v_\ell e_\ell\).

**Definition 3.12.** If \(P\) is a polynomial map \(\mathbb{C}^n \rightarrow \mathbb{C}^b\) with coefficients in a number field \(k\) (in the canonical bases), we define the size of \(P\) by

\[
\|P\| = \max\{|\sigma(c)| ; c \text{ coefficient of } P, \, \sigma \in \text{Hom}_Q(k, \mathbb{C})\}.
\]

Let \(k\) be the number field generated by the coefficients of all \(\text{Ad} \ g\), for \(g \in \Gamma\), and denote by \(O_k\) its ring of integers. We have the following obvious lemma.

**Lemma 3.13.** There exists a positive integer \(q = q(S)\) such that if \(g \in B_\Gamma(n)\), then \(q^n P_{I_0,g}\) has coefficients in \(O_k\) and

\[
\|q^n P_{I_0,g}\| \leq q^{2n}.
\]

We are now ready to derive Proposition 3.11. The letter \(C\) denotes any constant that depends only on \(G\); this constant will change along the proof.

**Proof of Proposition 3.11.** Let \(L_0\) be an \(\ell\)-dimensional subspace of \(\mathfrak{g}\) with orthonormal basis \((u_i)_{1 \leq i \leq \ell}\). Write \(u = u_1 \wedge \cdots \wedge u_\ell = \sum u_\ell e_\ell\). As \(L_0\) is defined over the reals, \(H_{L_0} \cdot u = \pm u\). We assume for simplicity that \(H_{L_0} \cdot u = u\). For some \(I_0\), we have \(|u_{I_0}| \geq \frac{1}{\ell}\) for some constant \(C\) depending only on \(\dim G\). We let \(u' = \frac{1}{|u_{I_0}|} u_{I_0}\), so that \(\|u'\| \leq C\). We claim that if we choose \(C_1\) large enough, then, for \(n \geq n_0\) \((C_1, n_0\) independent of \(L_0\)), the family of polynomials \(\mathcal{P} = \{R_i\} \cup \{P_{I_0,g}\}_{g \in H_{L_0}^{-n-C_1} \cap B_\Gamma(n)}\) must have a common zero in \(\mathbb{C}^{\binom{d}{\ell}-1}\).

Suppose for a contradiction that this is not the case. By the above lemma, there is a positive integer \(q\) depending only on \(S\) such that for all \(P \in \mathcal{P}\), \(q^n P\) has coefficients in \(O_k\) and for all \(P \in \mathcal{P}\),

\[
\|q^n P\| \leq q^{2n}.
\]

As the \(P_{I_0,g}\) have bounded degree (in fact, degree 1) we may extract from the family \(q^n \mathcal{P}\) polynomials \(P_j, 1 \leq j \leq C\) generating the same ideal as \(\mathcal{P}\). By the effective Nullstellensatz \(\text{[12]}\) Theorem IV, if the family of polynomials \(\mathcal{P}\) has

\[\text{[1]}\]Otherwise, one should use polynomials \(P_{I_0,g}(v)\) defining the subvariety \(\{v \mid g \cdot v \pm v = 0\}\).
no common zero, then there exist an element $a \in O_k$ and polynomials $Q_j$ with coefficients in $O_k$, such that

$$a = \sum Q_j P_j \quad (13)$$

and

$$\forall j, \quad \|Q_j\| \leq q^{Cn} \quad \deg Q_j \leq C \quad \text{and} \quad \|a\| \leq q^{Cn}. \quad (14)$$

Now, we want to evaluate (13) at $u'$ to get a contradiction.

First, we observe that for any $P$ in $q^n P$ (in particular, for any $P_j$),

$$|P(u')| \leq Cq^ne^{-C_1 n}.$$  

Indeed, if $P$ is one of the $R_i$'s, we have $P(u') = 0$ because $u'$ is a pure tensor; and if $P = P_{I_0,g}$, using that $g \in H_{L_0}^{\varepsilon (e^{-Cn})}$ and that $H_{L_0}$ fixes $u'$, we also find the desired estimate.

Second, by (14) and the fact that $\|u'\| \leq C$, we have, for each $j$,

$$|Q_j(u')| \leq C q^{Cn}.$$  

Finally, as $a$ is a nonzero element of $O_k$ of size at most $q^{Cn}$, we have a lower bound on its complex absolute value (for a constant $M$ depending only on $O_k$):

$$q^{-Mn} \leq |a|.$$  

Thus,

$$q^{-Mn} \leq |a| \leq \sum |Q_j(u')||P_j(u')| \leq C q^{Cn} e^{-C_1 n},$$

which yields a contradiction provided we have chosen $C_1$ large enough (in terms of $C$, $q$ and $M$).

Now let $(v_I)_{I \neq I_0}$ be a common zero for the family $P$. As, for each $i$, $R_i((v_I)) = 0$, the vector $v = e_{I_0} + \sum_{I \neq I_0} v_I e_I$ is a pure tensor: $v = v_1 \wedge \cdots \wedge v_l$. Moreover, for all $g \in B_1(n) \cap H_{L_0}^{\varepsilon (e^{-Cn})}$, $g \cdot v = v$, so that the subspace $L_1 = \text{Span} v_i$ is stable under $g$. In other terms, $g \in H_{L_1}$, which is what we wanted to show.  

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