Moduli of Prym curves

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Abstract

Here we focus on $\overline{R}_g$, the compactification of the moduli space of curves of genus $g$ together with an unramified double cover, constructed by Arnaud Beauville in order to compactify the Prym mapping. We present an alternative description of $\overline{R}_g$, inspired by the moduli space of spin curves of Maurizio Cornalba, and we discuss in detail its main features, both from a geometrical and a combinatorial point of view.

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Introduction

Let $X$ be a smooth curve of genus $g$. As it is well known (see for instance [Bea96], p. 104, or [ACGH85], Appendix B, § 2, 13.), a square root of $\mathcal{O}_X$ corresponds to an unramified double cover of $X$.

A compactification $\overline{R}_g$ of the moduli space of curves of genus $g$ together with an unramified double cover was constructed by Arnaud Beauville ([Bea77], Section 6; see also [DS81], Theorem 1.1) by means of admissible double covers of stable curves. This moduli space was introduced as a tool to compactify the mapping which associates to a curve plus a 2-sheeted cover the corresponding Prym variety; however, we believe that it is interesting also in its own and worthy of a closer inspection.

Here we explore some of the geometrical and combinatorial properties of $\overline{R}_g$. In order to do that, we present a description of this scheme which is different from the original one and is inspired by the construction performed by Maurizio Cornalba in [Cor89] of the moduli space of spin curves $\overline{S}_g$. This

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is a natural compactification over \( \overline{M}_g \) of the space of pairs \((X, \zeta)\), where \( \zeta \in \text{Pic } X \) is a square root of the canonical bundle \( K_X \).

In Section 1 we define a Prym curve to be just the analogue of a spin curve; Cornalba’s approach can be easily adapted to our context and allows us to put a structure of projective variety on the set \( \overline{Pr}_g \) of isomorphism classes of Prym curves of genus \( g \). This variety has two irreducible components \( \overline{Pr}_g^- \) and \( \overline{Pr}_g^+ \), where \( \overline{Pr}_g^- \simeq \overline{M}_g \) contains “trivial” Prym curves; moreover, by comparing Prym curves and admissible double covers, we give an explicit isomorphism between \( \overline{Pr}_g^+ \) and \( R_g \) over \( \overline{M}_g \).

Next, in Section 2 we reproduce the arguments in [Fon02] in order to show that \( \overline{Pr}_g \) is endowed with a natural injective morphism into the compactification of the universal Picard variety constructed by Lucia Caporaso in [Cap94], just like \( S_g \).

Finally, in Section 3 we turn to the combinatorics of \( \overline{Pr}_g \). Applying the same approach used in [CC03] for spin curves, we study the ramification of the morphism \( \overline{Pr}_g \to \overline{M}_g \) over the boundary. We describe the numerical properties of the scheme-theoretical fiber \( Pr_Z \) over a point \([Z] \in \overline{M}_g\), which turn out to depend only on the dual graph \( \Gamma_Z \) of \( Z \). From this combinatorial description, it follows that the morphisms \( \overline{Pr}_g \to \overline{M}_g \) and \( \overline{S}_g \to \overline{M}_g \) ramify in a different way.

The moduli space \( \overline{R}_g \) of admissible double covers has been studied also by Mira Bernstein in [Ber99], where \( \overline{R}_g \) is shown to be of general type for \( g = 17, 19, 21, 23 \) ([Ber99], Corollary 3.1.7) (for \( g \geq 24 \) it is obvious, since \( \overline{M}_g \) is).

We work over the field \( \mathbb{C} \) of complex numbers.

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# 1 Prym curves and admissible double covers

**1.1. Defining the objects.** Let \( X \) be a Deligne-Mumford semistable curve and \( E \) an irreducible component of \( X \). One says that \( E \) is exceptional if it is smooth, rational, and meets the other components in exactly two points. Moreover, one says that \( X \) is quasistable if any two distinct exceptional components of \( X \) are disjoint. The stable model of \( X \) is the stable curve \( Z \) obtained from \( X \) by contracting each exceptional component to a point. In the sequel, \( \widetilde{X} \) will denote the subcurve \( \widetilde{X} \subset \bigcup_i E_i \) obtained from \( X \) by removing all exceptional components.

We fix an integer \( g \geq 2 \).
Definition 1. A Prym curve of genus $g$ is the datum of $(X, \eta, \beta)$ where $X$ is a quasistable curve of genus $g$, $\eta \in \text{Pic}X$, and $\beta: \eta \otimes^2 \rightarrow \mathcal{O}_X$ is a homomorphism of invertible sheaves satisfying the following conditions:

(i) $\eta$ has total degree 0 on $X$ and degree 1 on every exceptional component of $X$;

(ii) $\beta$ is non zero at a general point of every non-exceptional component of $X$.

We say that $X$ is the support of the Prym curve $(X, \eta, \beta)$.

An isomorphism between two Prym curves $(X, \eta, \beta)$ and $(X', \eta', \beta')$ is an isomorphism $\sigma: X \rightarrow X'$ such that there exists an isomorphism $\tau: \sigma^*(\eta') \rightarrow \eta$ which makes the following diagram commute:

$$
\begin{array}{ccc}
\sigma^*(\eta') \otimes^2 & \xrightarrow{\tau \otimes^2} & \eta \otimes^2 \\
\downarrow \sigma^* & & \downarrow \beta \\
\sigma^*(\mathcal{O}_{X'}) & \sim & \mathcal{O}_X.
\end{array}
$$

Let $(X, \eta, \beta)$ be a Prym curve and let $E_1, \ldots, E_r$ be the exceptional components of $X$. From the definition it follows that $\beta$ vanishes identically on all $E_i$ and induces an isomorphism

$$
\eta \otimes^2|_{\tilde{X}} \sim \mathcal{O}_X(-q_1^1 - q_1^2 - \cdots - q_r^1 - q_r^2),
$$

where $\{q_i^1, q_i^2\} = \tilde{X} \cap E_i$ for $i = 1, \ldots, r$. In particular, when $X$ is smooth, $\eta$ is just a point of order two in the Picard group of $X$. The number of such points, as it is well-known, is exactly $2^{2g}$.

We denote by $\text{Aut}(X, \eta, \beta)$ the group of automorphisms of the Prym curve $(X, \eta, \beta)$. As in [Cor89], p. 565, one can show that $\text{Aut}(X, \eta, \beta)$ is finite.

We say that an isomorphism between two Prym curves $(X, \eta, \beta)$ and $(X, \eta', \beta')$ having the same support is inessential if it induces the identity on the stable model of $X$. We denote by $\text{Aut}_0(X, \eta, \beta) \subseteq \text{Aut}(X, \eta, \beta)$ the subgroup of inessential automorphisms. We have the following

Lemma 2 ([Cor89], Lemma 2.1). There exists an inessential isomorphism between two Prym curves $(X, \eta, \beta)$ and $(X, \eta', \beta')$ if and only if

$$
\eta|_{\tilde{X}} \simeq \eta'|_{\tilde{X}}.
$$

---

1Observe that we are adopting the convention that the datum of $\tau$ is not included in the definition of isomorphism, as in [Cor89]. This is different from the convention in [Cor91]; see [Cor91], end of section 2, for a discussion about this.
So the set of isomorphism classes of Prym curves supported on $X$ is in bijection with the set of square roots of $O_{\tilde{X}}(-q_1^1 - q_1^2 - \cdots - q_r^1 - q_r^2)$ in $\text{Pic}\, \tilde{X}$, modulo the action of the group of automorphisms of $\tilde{X}$ fixing $q_1^1, q_1^2, \ldots, q_r^1, q_r^2$.

A family of Prym curves is a flat family of quasistable curves $f: \mathcal{X} \to S$ with an invertible sheaf $\eta$ on $\mathcal{X}$ and a homomorphism

$$\beta: \eta^{\otimes 2} \to \mathcal{O}_{\mathcal{X}}$$

such that the restriction of these data to any fiber of $f$ gives rise to a Prym curve. An isomorphism between two families of Prym curves $(\mathcal{X} \to S, \eta, \beta)$ and $(\mathcal{X}' \to S, \eta', \beta')$ over $S$ is an isomorphism $\sigma: \mathcal{X} \to \mathcal{X}'$ over $S$ such that there exists an isomorphism $\tau: \sigma^*(\eta') \to \eta$ compatible with the canonical isomorphism between $\sigma^*(\mathcal{O}_{\mathcal{X}'})$ and $\mathcal{O}_{\mathcal{X}}$.

We define the moduli functor associated to Prym curves in the obvious way: $\mathcal{P}_{rg}$ is the contravariant functor from the category of schemes to the one of sets, which to every scheme $S$ associates the set $\mathcal{P}_{rg}(S)$ of isomorphism classes of families of Prym curves of genus $g$ over $S$.

1.2. The universal deformation. Fix a Prym curve $(X, \eta, \beta)$, call $Z$ the stable model of $X$ and denote by $E_1, \ldots, E_r$ the exceptional components of $X$. Let $Z' \to B'$ be the universal deformation of $Z$, where $B'$ is the unit policylinder in $\mathbb{C}^{3g-3}$ with coordinates $t_1, \ldots, t_{3g-3}$ such that $\{t_i = 0\} \subset B'$ is the locus where the node corresponding to $E_i$ persists for $i = 1, \ldots, r$. Let $B$ be another unit policylinder in $\mathbb{C}^{3g-3}$ with coordinates $\tau_1, \ldots, \tau_{3g-3}$, and consider the map $B \to B'$ given by $t_i = \tau_{2i}$ for $1 \leq i \leq r$ and $t_i = \tau_i$ for $i > r$. Call $Z$ the pull-back of $Z'$ to $B$. For $i \in \{1, \ldots, r\}$ the family $\mathcal{Z}_{\{\tau_i = 0\}} \to \{\tau_i = 0\} \subset B$ has a section $V_i$, corresponding to the locus of the $i$th node. Let $\mathcal{X} \to Z$ be the blow-up of $V_1, \ldots, V_r$ and call $\mathcal{E}_1, \ldots, \mathcal{E}_r$ the exceptional divisors.

$$\begin{array}{ccc}
\mathcal{X} & \to & Z \\
\downarrow & & \downarrow \\
B & \to & B'
\end{array}$$

The variety $\mathcal{X}$ is smooth and $\mathcal{X} \to B$ is a family of quasistable curves, with $X$ as central fiber. Up to an inessential automorphism, we can assume that $\eta^{\otimes 2} \simeq \mathcal{O}_X(-\sum_i \mathcal{E}_i)|_X$ and that this isomorphism is induced by $\beta$. By shrinking $B$ if necessary, we can extend $\eta$ to $\eta \in \text{Pic}\, \mathcal{X}$ such that $\eta^{\otimes 2} \simeq \mathcal{O}_X(-\sum_i \mathcal{E}_i)$. Denote by $\beta$ the composition of this isomorphism with the natural inclusion $\mathcal{O}_X(-\sum_i \mathcal{E}_i) \hookrightarrow \mathcal{O}_X$. Then $(\mathcal{X} \to B, \eta, \beta)$ is a family of Prym curves, and there is a morphism $\psi: X \to \mathcal{X}$ which induces an isomorphism of Prym curves between $(X, \eta, \beta)$ and the fiber of the family
over \( b_0 = (0, \ldots, 0) \in B \). This family provides a \textit{universal deformation} of 
\((X, \eta, \beta)\):

**Theorem 3.** Let \((X' \to T, \eta', \beta')\) be a family of Prym curves and let \( \varphi: X \to X' \) be a morphism which induces an isomorphism of Prym curves between 
\((X, \eta, \beta)\) and the fiber of the family over \( t_0 \in T \).

Then, possibly after shrinking \( T \), there exists a unique morphism \( \gamma: T \to B \) satisfying the following conditions:

\begin{enumerate}
\item \( \gamma(t_0) = b_0 \);
\item there is a cartesian diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{\delta} & X \\
\downarrow \quad & & \downarrow \\
T & \xrightarrow{\gamma} & B \\
\end{array}
\]
\item \( \eta' \simeq \delta^*(\eta) \) and \( \beta' = \delta^*(\beta) \);
\item \( \delta \circ \varphi = \psi \).
\end{enumerate}

Since the proof of [Cor89], Proposition 4.6 applies verbatim to our case, we
omit the proof of Theorem 3.

1.3. The moduli scheme. Let \( \overline{Pr}_g \) be the set of isomorphism classes of
Prym curves of genus \( g \). We define a natural structure of analytic variety on 
\( \overline{Pr}_g \) following [Cor89], § 5.

Consider a Prym curve \((X, \eta, \beta)\) and its universal deformation \((X \to B, \eta, \beta)\) constructed in 1.2. By the universality, the group \( \text{Aut}(X, \eta, \beta) \) acts
on \( B \) and on \( X \). This action has the following crucial property:

**Lemma 4 ([Cor89], Lemma 5.1).** Let \( b_1, b_2 \in B \) and let \((X_{b_1}, \eta_{b_1}, \beta_{b_1})\) and 
\((X_{b_2}, \eta_{b_2}, \beta_{b_2})\) be the fibers of the universal family over \( b_1 \) and \( b_2 \) respectively.
Then there exists \( \sigma \in \text{Aut}(X, \eta, \beta) \) such that \( \sigma(b_1) = b_2 \) if and only if the 
Prym curves \((X_{b_1}, \eta_{b_1}, \beta_{b_1})\) and \((X_{b_2}, \eta_{b_2}, \beta_{b_2})\) are isomorphic.

Lemma 4 implies that the natural (set-theoretical) map \( B \to \overline{Pr}_g \), associating
to \( b \in B \) the isomorphism class of the fiber over \( b \), descends to a well-defined, 
injective map
\[
J: B/\text{Aut}(X, \eta, \beta) \to \overline{Pr}_g.
\]

This allows to define a complex structure on the subset \( \text{Im} \ J \subseteq \overline{Pr}_g \). Since 
\( \overline{Pr}_g \) is covered by these subsets, in order to get a complex structure on \( \overline{Pr}_g \)
\footnote{Where we still denote by \( \sigma \) the automorphism of \( B \) induced by \( \sigma \).}
we just have to check that the complex structures are compatible on the overlaps.

This compatibility will follow from the following remark, which is an immediate consequence of the construction of the universal family in 1.2:

- the family of Prym curves \((X \to B, \eta, \beta)\) is a universal deformation for any of its fibers.

In fact, assume that there are two Prym curves \((X_1, \eta_1, \beta_1)\) and \((X_2, \eta_2, \beta_2)\) such that the images of the associated maps \(J_1, J_2\) intersect. Choose a Prym curve \((X_3, \eta_3, \beta_3)\) corresponding to a point in \(\text{Im} J_1 \cap \text{Im} J_2\). Let \(B_i (i = 1, 2, 3)\) be the basis of the universal deformation of \((X_i, \eta_i, \beta_i)\). Then by the remark above, for \(i = 1, 2\) there are natural open immersions \(h_i: B_3 \to B_i\), equivariant with respect to the actions of the automorphism groups. Hence \(h_i\) induces an open immersion \(h_i: B_3/\text{Aut}(X_3, \eta_3, \beta_3) \to B_i/\text{Aut}(X_i, \eta_i, \beta_i)\), and \(J_3 = J_i \circ h_i\).

Observe now that the morphisms

\[
B/\text{Aut}(X, \eta, \beta) \longrightarrow B'/\text{Aut}(Z)
\]

glue together and yield a morphism \(p: \overline{Pr}_g \to \overline{M}_g\). Clearly \(p\) is finite, as a morphism between analytic varieties (see [Ray71]). Hence \(\overline{Pr}_g\) is projective, because \(\overline{M}_g\) is. The variety \(\overline{Pr}_g\) has finite quotient singularities; in particular, it is normal.

The degree of \(p\) is \(2^{2g}\). The fiber over a smooth curve \(Z\) is just the set of points of order two in its Picard group, modulo the action of \(\text{Aut}(Z)\) if non trivial. When \(Z\) is a stable curve, the set-theoretical fiber over \([Z]\) consists of isomorphism classes of Prym curves \((X, \eta, \beta)\) such that the stable model of \(X\) is \(Z\). In section 3 we will describe precisely the scheme-theoretical fiber over \([Z]\), following [Cor89] and [CC03]. We will show that \(p\) is étale over \(\overline{M}_g^0 \setminus D_{\text{irr}}\), where \(\overline{M}_g^0\) is the locus of stable curves with trivial automorphism group and \(D_{\text{irr}}\) is the boundary component whose general member is an irreducible curve with one node.

Finally, \(\overline{Pr}_g\) is a coarse moduli space for the functor \(\overline{Pr}_g\). For any family of Prym curves over a scheme \(T\), the associated moduli morphism \(T \to \overline{Pr}_g\) is locally defined by Theorem 3.

Let \(\overline{Pr}_g^-\) be the closed subvariety of \(\overline{Pr}_g\) consisting of classes of Prym curves \((X, \eta, \beta)\) where \(\eta \simeq \mathcal{O}_X\). Observe that when \(\eta\) is trivial, the curve \(X\) is stable. So \(\overline{Pr}_g^-\) is the image of the obvious section of \(p: \overline{Pr}_g \to \overline{M}_g\), and it is an irreducible (and connected) component of \(\overline{Pr}_g\), isomorphic to \(\overline{M}_g\).

Let \(\overline{Pr}_g^+\) be the complement of \(\overline{Pr}_g^-\) in \(\overline{Pr}_g\), and denote by \(Pr^+_g\) its open subset consisting of classes of Prym curves supported on smooth curves. Then
Pr$^+_g$ parametrizes connected unramified double covers of smooth curves of genus $g$; it is well-known that this moduli space is irreducible, being a finite quotient of the moduli space of smooth curves of genus $g$ with a level 2 structure, which is irreducible by [DM69]. So $\overline{Pr}_g^+$ is an irreducible component of $\overline{Pr}_g$.

1.4. Admissible double covers. Consider a pair $(C, i)$ where $C$ is a stable curve of genus $2g - 1$ and $i$ is an involution of $C$ such that:

- the set $I$ of fixed points of $i$ is contained in $\text{Sing } C$;
- for any fixed node, $i$ does not exchange the two branches of the curve.

Then the quotient $Z := C/i$ is a stable curve of genus $g$, and $\pi: C \to Z$ is a finite morphism of degree 2, étale over $Z \setminus \pi(I)$. This is called an admissible double cover. Remark that $\pi$ is not a cover in the usual sense, since it is not flat at $I$.

The moduli space $\overline{R}_g$ of admissible double covers of stable curves of genus $g$ is constructed [Bea77], Section 6 (see also [DS81, ABH01]), as the moduli space for pairs $(C, i)$ as above.

An isomorphism of two admissible covers $\pi_1: C_1 \to Z_1$ and $\pi_2: C_2 \to Z_2$ is an isomorphism $\varphi: Z_1 \to Z_2$ such that there exists$^3$ an isomorphism $\widetilde{\varphi}: C_1 \sim C_2$ with $\pi_2 \circ \widetilde{\varphi} = \varphi \circ \pi_1$.

We denote by $\text{Aut}(C \to Z)$ the automorphism group of the admissible cover $C \to Z$, so $\text{Aut}(C \to Z) \subseteq \text{Aut}(Z)$. All elements of $\text{Aut}(C \to Z)$ are induced by automorphisms of $C$, different from $i$, and that commute with $i$.

Let $(C, i)$ be as above; we describe its universal deformation. Let $C' \to W'$ be a universal deformation of $C$. By the universality, there are compatible involutions $i'$ of $W'$ and $\tilde{i}^\prime$ of $C'$, extending the action of $i$ on the central fiber. Let $W \subset W'$ be the locus fixed by $i'$, $C \to W$ the induced family and $i$ the restriction of $i'$ to $C$. Then $(C, i) \to W$ is a universal deformation of $(C, i)$ and the corresponding family of admissible double covers is $C \to Q := C/i \to W$.

We are going to show that $\overline{R}_g$ is isomorphic over $\overline{M}_g$ to the irreducible component $\overline{Pr}_g^+$ of $\overline{Pr}_g$.

First of all we define a map $\Phi$ from the set of non trivial Prym curves of genus $g$ to the set of admissible double covers of stable curves of genus $g$.

Let $\xi = (X, \eta, \beta)$ be a Prym curve with $\eta \not\cong O_X$; then $\Phi(\xi)$ will be an admissible double cover of the stable model $Z$ of $X$, constructed as follows. The homomorphism $\beta$ induces an isomorphism

$$\eta|_X^{\otimes(-2)} \cong O_X(q_1+q_2^2+\cdots+q_r^1+q_r^2).$$

$^3$Given $\varphi$, there are exactly two choices for $\widetilde{\varphi}$; if $\widetilde{\varphi}$ is one, the other is $\widetilde{\varphi} \circ i_1$. 

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This determines a double cover \( \tilde{\pi}: \tilde{C} \to \tilde{X} \), ramified over \( q_1^i, q_1^1, \ldots, q_r^i, q_r^2 \), which are smooth points of \( \tilde{X} \). Now call \( C_\xi \) the stable curve obtained identifying \( \tilde{\pi}^{-1}(q_1^i) \) with \( \tilde{\pi}^{-1}(q_i^2) \) for all \( i = 1, \ldots, r \). Then the induced map \( C_\xi \to Z \) is the admissible double cover \( \Phi(\xi) \).

Now consider two Prym curves \( \xi_1 \) and \( \xi_2 \) supported respectively on \( X_1 \) and \( X_2 \). Suppose that \( \sigma: X_1 \to X_2 \) induces an isomorphism between \( \xi_1 \) and \( \xi_2 \). Let \( \sigma: Z_1 \to Z_2 \) be the induced isomorphism between the stable models. Then it is easy to see that \( \sigma \) is an isomorphism between the admissible covers \( \Phi(\xi_1) \) and \( \Phi(\xi_2) \). Moreover, any isomorphism between \( \Phi(\xi_1) \) and \( \Phi(\xi_2) \) is obtained in this way. Hence we have an exact sequence of automorphism groups:

\[
1 \to \text{Aut}_0(\xi) \to \text{Aut}(\xi) \to \text{Aut}(C_\xi \to Z) \to 1. \tag{1}
\]

We show that \( \Phi \) is surjective. Let \( C \to Z \) be an admissible double cover, \( I \subset C \) the set of fixed points of the involution and \( J \subset Z \) their images. Let \( \tilde{C} \to C \) and \( \nu: \tilde{X} \to Z \) be the normalizations of \( C \) at \( I \) and of \( Z \) at \( J \) respectively. Then \( \nu \) extends to an involution on \( \tilde{C} \), whose quotient is \( \tilde{X} \), namely: \( \tilde{C} \to \tilde{X} \) is a double cover, ramified over \( q_1^i, q_1^1, \ldots, q_r^i, q_r^2 \), where \( r = |J| \) and \( \nu(q_1^i) = \nu(q_i^2) \in J \) for \( i = 1, \ldots, r \). Let \( L \in \text{Pic}\tilde{X} \) be the associated line bundle, satisfying \( L^{\otimes 2} \cong O_{\tilde{X}}(q_1^1 + q_1^2 + \cdots + q_r^1 + q_r^2) \). Finally let \( X \) be the quasi-stable curve obtained by attaching to \( \tilde{X} \) \( r \) rational components \( E_1, \ldots, E_r \) such that \( E_i \cap \tilde{X} = \{ q_1^i, q_2^i \} \). Choose \( \eta \in \text{Pic}X \) having degree 1 on all \( E_i \) and such that \( \eta|_{\tilde{X}} = L^{\otimes (1)} \). Let \( \beta: \eta^{\otimes 2} \to O_X \) be a homomorphism which agrees with \( \eta|_{\tilde{X}} \cong O_{\tilde{X}}(-q_1^1 - q_1^2 - \cdots - q_r^1 - q_r^2) \to O_X \) on \( \tilde{X} \). Then \( \xi = (X, \eta, \beta) \) is a Prym curve with \( \xi \neq O_X \), and \( C \to Z \) is \( \Phi(\xi) \). For different choices of \( \eta \), the corresponding Prym curves differ by an inessential isomorphism.

**Proposition 5.** The map \( \Phi \) just defined induces an isomorphism

\[
\hat{\Phi}: \overline{P^+_{r,g}} \longrightarrow \overline{R_g}
\]

over \( \overline{M_g} \).

**Proof.** By what precedes, \( \Phi \) induces a bijection \( \hat{\Phi}: \overline{P^+_{r,g}} \to \overline{R_g} \). The statement will follow if we prove that \( \hat{\Phi} \) is a local isomorphism at every point of \( \overline{P^+_{r,g}} \).

Fix a point \( \xi = (X, \eta, \beta) \in \overline{P^+_{r,g}} \) and consider its universal deformation \( (\mathcal{X} \to B, \eta, \beta) \) constructed in 1.2. Keeping the notations of 1.2, the line bundle \( \eta^{\otimes (1)} \) determines a double cover \( \overline{\mathcal{P}} \to \mathcal{X} \), ramified over \( \mathcal{E}_1, \ldots, \mathcal{E}_r \). The divisor \( \mathcal{E}_i \) is a \( \mathbb{P}^1 \)-bundle over \( V_i \subset B \), and the restriction of its normal
bundle to a non trivial fiber $F$ is $(\mathcal{N}_{\mathcal{E}_i/\mathcal{X}})|_F \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$. The inverse image $\mathcal{E}_i$ of $\mathcal{E}_i$ in $\mathcal{P}$ is again a $\mathbb{P}^1$-bundle over $V_i \subset B$, but now the restriction of its normal bundle to a non trivial fiber $\mathcal{F}$ is $(\mathcal{N}_{\mathcal{E}_i/\mathcal{P}})|_\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$. Let $\mathcal{P} \to \mathcal{P}$ be the blow-down of $\mathcal{E}_1, \ldots, \mathcal{E}_r$. We get a diagram

$$
\begin{array}{c}
\mathcal{P} \downarrow \downarrow \downarrow \rightarrow \mathcal{P} \\
\mathcal{X} \downarrow \downarrow \downarrow \rightarrow \mathcal{Z} \rightarrow B
\end{array}
$$

where $\mathcal{P} \to \mathcal{Z} \to B$ is a family of admissible double covers whose central fiber is $C_\xi \to Z$. Therefore, up to shrinking $B$, there exists a morphism $B \to W$ such that $\mathcal{P} \to \mathcal{Z} \to B$ is obtained by pull-back from the universal deformation $\mathcal{C} \to Q \to W$ of $C_\xi \to Z$. Now notice that $Q \to W$ is a family of stable curves of genus $g$, with $Z$ as central fiber: so (again up to shrinking) it must be a pull-back of the universal deformation $Z' \to B'$. In the end we get a diagram:

$$
\begin{array}{c}
\mathcal{P} \downarrow \downarrow \downarrow \rightarrow \mathcal{P} \rightarrow \mathcal{C} \\
\mathcal{X} \downarrow \downarrow \downarrow \rightarrow \mathcal{Z} \rightarrow Q \rightarrow Z' \\
\mathcal{B} \downarrow \downarrow \downarrow \rightarrow \mathcal{W} \rightarrow \mathcal{B}'
\end{array}
$$

We can assume that $\varphi$ and $\psi$ are surjective. Observe that both maps are equivariant with respect to the actions of the automorphism groups indicated in the diagram.

Clearly $\varphi$ is just the restriction of $\Phi$ to the set of Prym curves parametrized by $B$.

Now by (1), $\varphi(b_1) = \varphi(b_2)$ if and only if there exists an inessential isomorphism between $(X_{b_1}, \eta_{b_1}, \beta_{b_1})$ and $(X_{b_2}, \eta_{b_2}, \beta_{b_2})$. Hence $\varphi$ induces an equivariant isomorphism $\hat{\varphi}$:

$$
\begin{array}{c}
B/\text{Aut}_0(\xi) \hat{\varphi} \downarrow \downarrow \downarrow \rightarrow \mathcal{W} \rightarrow \mathcal{B}' \\
\text{Aut}(\xi)/\text{Aut}_0(\xi) \downarrow \downarrow \downarrow \rightarrow \text{Aut}(C_\xi \to Z) \downarrow \downarrow \downarrow \rightarrow \text{Aut}(Z)
\end{array}
$$
and finally if we mod out by all the automorphism groups, we get

\[
\begin{array}{c}
B/\text{Aut}(\xi) \xrightarrow{\sim} \text{W/Aut}(C \to Z) \longrightarrow B'/\text{Aut}Z \\
\downarrow \Phi \downarrow \downarrow \Phi \downarrow \downarrow \downarrow \\
\overline{Pr}_g \longrightarrow \overline{R}_g \longrightarrow \overline{M}_g.
\end{array}
\]

This shows that \( \Phi \) is a local isomorphism in \( \xi \).

\[\text{\(\square\)}\]

2 Embedding \( \overline{Pr}_g \) in the compactified Picard variety

Let \( g \geq 3 \). For every integer \( d \), there is a universal Picard variety

\[ P_{d,g} \longrightarrow \mathcal{M}_g^0 \]

whose fiber \( J^d(X) \) over a point \( X \) of \( \mathcal{M}_g^0 \) parametrizes line bundles on \( X \) of degree \( d \), modulo isomorphism. Denote by \( P_{r,0} \) the inverse image of \( \mathcal{M}_g^0 \) under the finite morphism \( \overline{Pr}_g \to \overline{M}_g \); then we have a commutative diagram

\[ \begin{array}{c}
P_{r,0}^0 \hookrightarrow P_{0,g} \\
\downarrow \downarrow \downarrow \\
\mathcal{M}_g^0
\end{array} \]

Assume \( d \geq 20(g - 1) \); this is not a real restriction, since for all \( t \in \mathbb{Z}_{\geq 0} \) there is a natural isomorphism \( P_{d,g} \cong P_{d+t(2g-2),g} \). Then \( P_{d,g} \) has a natural compactification \( \overline{P}_{d,g} \), endowed with a natural morphism \( \phi_d: \overline{P}_{d,g} \to \overline{M}_g \), such that \( \phi_d^{-1}(\mathcal{M}_g^0) = P_{d,g} \). It was constructed in [Cap94] as a GIT quotient

\[ \pi_d: H_d \longrightarrow H_d // G = \overline{P}_{d,g}, \]

where \( G = \text{SL}(d - g + 1) \) and

\[ H_d := \{ h \in \text{Hilb}_{d-g}^{d-x-g+1} \mid h \text{ is G-semistable and the corresponding curve is connected} \} \]

(\( \text{the action of } G \text{ is linearized by a suitable embedding of } \text{Hilb}_{d-g}^{d-x-g+1} \text{ in a Grassmannian} \)).
Fix now and in the sequel an integer $t \geq 10$ and define

$$K_{2t(g-1)} := \{ h \in \text{Hilb}_{2t(g-1)-g} | \text{there is a Prym curve } (X, \eta, \beta) \text{ and an embedding } h_t : X \to \mathbb{P}^{2t(g-1)-g} \text{ induced by } \eta \otimes \omega_X^{\otimes t},$$

such that $h$ is the Hilbert point of $h_t(X)\}.$

Our result is the following.

**Theorem 6.** The set $K_{2t(g-1)}$ is contained in $H_{2t(g-1)}$; consider its projection

$$\Pi_t := \pi_{2t(g-1)}(K_{2t(g-1)}) \subset \mathcal{P}_{2t(g-1),g}.$$

There is a natural injective morphism

$$f_t : \mathcal{P}_{t,g} \longrightarrow \mathcal{P}_{2t(g-1),g}$$

whose image is $\Pi_t.$

In particular, the Theorem implies that $\Pi_t$ is a closed subvariety of $\mathcal{P}_{2t(g-1),g}$.

The proof of Theorem 6 will be achieved in several steps and will take the rest of this section. The argument is the one used in [Fon02] to show the existence of an injective morphism $\overline{S_g} \to \overline{\mathcal{P}_{2t+1}(g-1),g}$ of the moduli space of spin curves in the corresponding compactified Picard variety.

One can define (see [Cap94], §8.1) the contravariant functor $\mathcal{P}_{d,g}$ from the category of schemes to the one of sets, which to every scheme $S$ associates the set $\mathcal{P}_{d,g}(S)$ of equivalence classes of polarized families of quasistable curves of genus $g$

$$f : (X, \mathcal{L}) \longrightarrow S$$

such that $\mathcal{L}$ is a relatively very ample line bundle of degree $d$ whose multidegree satisfies the following Basic Inequality on each fiber.

**Definition 7.** Let $X = \bigcup_{i=1}^n X_i$ be a projective, nodal, connected curve of arithmetic genus $g$, where the $X_i$’s are the irreducible components of $X$. We say that the multidegree $(d_1, \ldots, d_n)$ satisfies the Basic Inequality if for every complete subcurve $Y$ of $X$ of arithmetic genus $g_Y$ we have

$$m_Y \leq d_Y \leq m_Y + k_Y$$

where

$$d_Y = \sum_{X_i \subseteq Y} d_i, \quad k_Y = |Y \cap X \setminus Y| \quad \text{and} \quad m_Y = \frac{d}{g-1} \left( g_Y - 1 + \frac{k_Y}{2} \right) - \frac{k_Y}{2}$$

(see [Cap94] p. 611 and p. 614).
Two families over $S$, $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ are equivalent if there exists an $S$-isomorphism $\sigma: \mathcal{X} \to \mathcal{X}'$ and a line bundle $M$ on $S$ such that $\sigma^*\mathcal{L}' \cong \mathcal{L} \otimes f^*M$.

By [Cap94], Proposition 8.1, there is a morphism of functors:

$$\overline{P}_{d,g} \rightarrow \text{Hom}(\cdot, \overline{P}_{d,g}) \quad (2)$$

and $\overline{P}_{d,g}$ coarsely represents $\overline{P}_{d,g}$ if and only if

$$(d - g + 1, 2g - 2) = 1. \quad (3)$$

**Proposition 8.** For every integer $t \geq 10$ there is a natural morphism:

$$f_t: \overline{P}_{r,g} \rightarrow \overline{P}_{2t(g-1),g}. \quad (4)$$

**Proof.** First of all, notice that in this case (3) does not hold, so the points of $\overline{P}_{2t(g-1),g}$ are not in one-to-one correspondence with equivalence classes of very ample line bundles of degree $2t(g-1)$ on quasistable curves, satisfying the Basic Inequality (see [Cap94], p. 654). However, we claim that the thesis can be deduced from the existence of a morphism of functors:

$$F_t: \overline{P}_{r,g} \rightarrow \overline{P}_{2t(g-1),g}. \quad (4)$$

Indeed, since $\overline{P}_{r,g}$ coarsely represents $\overline{P}_{r,g}$, any morphism of functors $\overline{P}_{r,g} \rightarrow \text{Hom}(\cdot, T)$ induces a morphism of schemes $\overline{P}_{r,g} \rightarrow T$, so the claim follows from (2). Now, a morphism of functors as (4) is the datum for any scheme $S$ of a set-theoretical map

$$F_t(S): \overline{P}_{r,g}(S) \rightarrow \overline{P}_{2t(g-1),g}(S),$$

satisfying obvious compatibility conditions. Let us define $F_t(S)$ in the following way:

$$(f: \mathcal{X} \to S, \eta, \beta) \mapsto (f: (\mathcal{X}, \eta \otimes \omega_f^\otimes) \to S).$$

In order to prove that $F_t(S)$ is well-defined, the only non-trivial matter is to check that the multidegree of $\eta \otimes \omega_f^\otimes$ satisfies the Basic Inequality on each fiber, so the thesis follows from the next Lemma. 

**Lemma 9.** Let $(X, \eta, \beta)$ be a Prym curve. If $Y$ is a complete subcurve of $X$ and $d_Y$ is the degree of $(\eta \otimes \omega_X^\otimes)|_Y$, then $m_Y \leq d_Y \leq m_Y + k_Y$ in the notation of the Basic Inequality. Moreover, if $d_Y = m_Y$ then $k_Y := |\tilde{Y} \cap \tilde{X} \setminus \tilde{Y}| = 0$. 

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Proof. In the present case, the Basic Inequality simplifies as follows:

\[-\frac{k_Y}{2} \leq e_Y \leq \frac{k_Y}{2},\]

where \(e_Y := \deg \eta_Y\). By the definition of a Prym curve, the degree \(e_Y\) depends only on the exceptional components of \(X\) intersecting \(Y\).

For any exceptional component \(E\) of \(X\) with \(E \subseteq X \setminus Y\), let \(m := |E \cap Y|\). The contribution of \(E\) to \(k_Y\) is \(m\), while its contribution to \(e_Y\) is \(-\frac{m}{2}\).

Next, for any exceptional component \(E\) of \(X\) with \(E \subseteq Y\), let \(l := |E \cap X \setminus Y|\). The contribution of \(E\) to \(k_Y\) is \(l\), while its contribution to \(e_Y\) is \(1 - \frac{2-l}{2} > \frac{l}{2}\).

Summing up, we see that the Basic Inequality holds. Finally, if \(\tilde{k}_Y \neq 0\), then there exists a non-exceptional component of \(X\) intersecting \(Y\). Such a component contributes at least 1 to \(k_Y\), but it does not affect \(e_Y\); hence \(-\frac{k_Y}{2} < e_Y\) and the proof is over.

By applying [Cap94], Proposition 6.1, from the first part of Lemma 9 we deduce

\[K_{2t(g-1)} \subset H_{2t(g-1)}\).

Moreover, the second part of the same Lemma provides a crucial information on Hilbert points corresponding to Prym curves.

**Lemma 10.** If \(h \in K_{2t(g-1)}\), then the orbit of \(h\) is closed in the semistable locus.

**Proof.** Let \((X, \eta, \beta)\) be a Prym curve such that \(h\) is the Hilbert point of an embedding \(h_t : X \to \mathbb{P}^{2t(g-1)-g}\) induced by \(\eta \otimes \omega_X^{\otimes t}\). Just recall the first part of [Cap94], Lemma 6.1, which says that the orbit of \(h\) is closed in the semistable locus if and only if \(\tilde{k}_Y = 0\) for every subcurve \(Y\) of \(X\) such that \(d_Y = m_Y\), so the thesis is a direct consequence of Lemma 9.

**Proof of Theorem 6.** It is easy to check that \(f_t(\overline{Pr}_g) = \Pi_t\). Indeed, if \((X, \eta, \beta) \in \overline{Pr}_g\), then any choice of a base for \(H^0(X, \eta \otimes \omega_X^{\otimes t})\) gives an embedding \(h_t : X \to \mathbb{P}^{2t(g-1)-g}\) and \(f_t(X, \eta, \beta) = \pi_{2t(g-1)}(h)\), where \(h \in K_{2t(g-1)}\) is the Hilbert point of \(h_t(X)\). Conversely, if \(\pi_{2t(g-1)}(h) \in \Pi_t\), then there is a Prym curve \((X, \eta, \beta)\) and an embedding \(h_t : X \to \mathbb{P}^{2t(g-1)-g}\) such that \(h\) is the Hilbert point of \(h_t(X)\) and \(f_t(X, \eta, \beta) = \pi_{2t(g-1)}(h)\).

Next we claim that \(f_t\) is injective. Indeed, let \((X, \eta, \beta)\) and \((X', \eta', \beta')\) be two Prym curves and assume that \(f_t(X, \eta, \beta) = f_t(X', \eta', \beta')\). Choose bases for \(H^0(X, \eta \otimes \omega_X^{\otimes t})\) and \(H^0(X', \eta' \otimes \omega_X^{\otimes t})\) and embed \(X\) and \(X'\) in \(\mathbb{P}^{2t(g-1)-g}\). If \(h\) and \(h'\) are the corresponding Hilbert points, then \(\pi_{2t(g-1)}(h) = \pi_{2t(g-1)}(h') = 0\) and so \(f_t(X, \eta, \beta) = f_t(X', \eta', \beta')\).
\[ \pi_{2g-1}(h') \text{ and the Fundamental Theorem of GIT implies that } O_G(h) \text{ and } O_G(h') \text{ intersect in the semistable locus. It follows from Lemma 10 that } O_G(h) \cap O_G(h') \neq \emptyset, \text{ so } O_G(h) = O_G(h') \text{ and there is an isomorphism } \sigma: (X, \eta, \beta) \rightarrow (X', \eta', \beta'). \]

Observe that Theorem 6 and Lemma 10 imply that \( K_{2g-1} \) is a constructible set in \( H_{2g-1} \).

3 Fiberwise description

Let \( Z \) be a stable curve of genus \( g \). We recall that the dual graph \( \Gamma_Z \) of \( Z \) is the graph whose vertices are the irreducible components of \( Z \) and whose edges are the nodes of \( Z \). The first Betti number of \( \Gamma_Z \) is \( b_1(\Gamma_Z) = \delta - \gamma + 1 = g - g'' \), where \( \delta \) is the number of nodes of \( Z \), \( \gamma \) the number of its irreducible components and \( g'' \) the genus of its normalization.

We denote by \( Pr_Z \) the scheme parametrizing Prym curves \( (X, \eta, \beta) \) such that the stable model of \( X \) is \( Z \), modulo inessential isomorphisms, and by \( S_Z \) the analogue for spin curves. Since by Lemma 2 the homomorphism \( \beta \) is not relevant in determining the inessential isomorphism class of \( (X, \eta, \beta) \), in this section we will omit it and just write \( (X, \eta) \).

When \( Aut(Z) = \{ \text{Id}_Z \} \), \( Pr_Z \) is the scheme-theoretical fiber over \( [Z] \) of the morphism \( p: Pr_g \rightarrow \mathcal{M}_g \). Recall that \( p \) is finite of degree \( 2^{2g} \), and \( \text{é}tale \) over \( \mathcal{M}_g^0 \).

For any 0-dimensional scheme \( P \) we denote by \( L(P) \) the set of integers occurring as multiplicities of components of \( P \).

In this section we describe the numerical properties of \( Pr_Z \), namely the number of irreducible components and their multiplicities, showing that they depend only on the dual graph \( \Gamma_Z \) of \( Z \). Using this, we give some properties of \( L(Pr_Z) \), and show that in some cases the set of multiplicities \( L(Pr_Z) \) gives informations on \( Z \). In particular, we show that the morphism \( Pr_g \rightarrow \mathcal{M}_g \) is \( \text{étale} \) over \( \mathcal{M}_g^0 \setminus D_{irr} \).

We use the techniques and results of [CC03], where the same questions about the numerics of \( S_Z \) are studied (see also [CS03], § 3). Quite surprisingly, the schemes \( P_Z \) and \( S_Z \) are not isomorphic in general.

Finally we will show with an example that, differently from the case of spin curves, the set of multiplicities \( L(Pr_Z) \) appearing in \( Pr_Z \) does not always identify curves having two smooth components.

Let \( X \) be a quasistable curve having \( Z \) as stable model and consider the set
\[ \Delta_X := \{ z \in \text{Sing } Z \mid z \text{ is not the image of an exceptional component of } X \}. \]
Given $Z$, the quasistable curve $X$ is determined by $\Delta_X$, or equivalently by $\Delta_c^\varepsilon : = \text{Sing } Z \setminus \Delta_X = \{\text{images in } Z \text{ of the exceptional components of } X\}$.

Remark that any subset of $\text{Sing } Z$ can be seen as a subgraph of the dual graph $\Gamma_Z$ of $Z$.

We recall that the \textit{valency} of a vertex of a graph is the number of edges ending in that vertex and a graph $\Gamma$ is \textit{eulerian} if it has all even valencies. Thus $\Gamma_Z$ is eulerian if and only if for any irreducible component $C$ of $Z$, $|C \cap Z \setminus C|$ is even. The set $\mathcal{C}_\Gamma$ of all eulerian subgraphs of $\Gamma$ is called the cycle space of $\Gamma$. There is a natural identification of $\mathcal{C}_\Gamma$ with $H_1(\Gamma, \mathbb{Z}_2)$, so $|\mathcal{C}_\Gamma| = 2^{b_1(\Gamma)}$ (see [CC03]). Reasoning as in [CC03], Section 1.3, we can show the following:

\textbf{Proposition 11.} \textit{Let $X$ be a quasistable curve having $Z$ as stable model.}

\textit{The curve $X$ is the support of a Prym curve if and only if $\Delta_c^\varepsilon$ is eulerian.}

\textit{If so, there are $2^{2g^\nu + b_1(\Delta_X)}$ different choices for $\eta \in \text{Pic } X$ such that $(X, \eta) \in \text{Pr}_Z$.}

\textit{For each such $\eta$, the point $(X, \eta)$ has multiplicity $2^{b_1(\Gamma_Z) - b_1(\Delta_X)}$ in $\text{Pr}_Z$.}

Hence the number of irreducible components of $\text{Pr}_Z$ is

$$2^{2g^\nu} \cdot \sum_{\Sigma \in \mathcal{C}_{\Gamma_Z}} 2^{b_1(\Sigma^c)},$$

and its set of multiplicities is given by

$$L(\text{Pr}_Z) = \{2^{b_1(\Gamma_Z) - b_1(\Delta)} | \Delta^c \in \mathcal{C}_{\Gamma_Z}\}.$$ 

Remark that since $|\mathcal{C}_{\Gamma_Z}| = 2^{b_1(\Gamma_Z)}$, we can check immediately from the proposition that the length of $\text{Pr}_Z$ is

$$\sum_{\Sigma \in \mathcal{C}_{\Gamma_Z}} (2^{2g^\nu + b_1(\Sigma^c)} \cdot 2^{b_1(\Gamma_Z) - b_1(\Sigma^c)}) = 2^{b_1(\Gamma_Z)} \cdot 2^{2g^\nu + b_1(\Gamma_Z)} = 2^{2g}.$$ 

As a consequence of Proposition 11, we see that

\begin{itemize}
  \item a point $(X, \eta)$ in $\text{Pr}_Z$ is non reduced if and only if $X$ is non stable.
\end{itemize}

\textit{Example (curves having two smooth components).} Let $Z = C_1 \cup C_2$, $C_i$ smooth irreducible, $|C_1 \cap C_2| = \delta \geq 2$. 

\begin{center}
  \includegraphics[width=0.1\textwidth]{Gamma_Z}
\end{center}
Let $X$ be a quasistable curve having $Z$ as stable model and let $\Delta_X$ be the corresponding subset of $\text{Sing} \ Z$. The subgraph $\Delta_X$ is eulerian if and only if $|\Delta_X^e|$ is even. Therefore $X$ is support of a Prym curve if and only if it has an even number $2r$ of exceptional components. If so, for each choice of $\eta \in \text{Pic} \ X$ such that $(X, \eta) \in Pr_Z$, this point will have multiplicity $2^{b_1(\Gamma_Z) - b_1(\Delta_X)}$. We have $b_1(\Gamma_Z) = \delta - 1$ and $|\Delta_X| = \delta - 2r$, so

$$b_1(\Delta_X) = \begin{cases} 
\delta - 2r - 1 & \text{if } 2r \leq \delta - 2, \\
0 & \text{if } \delta - 1 \leq 2r \leq \delta 
\end{cases}$$

and we get

$$L(Pr_Z) = \{2^{2r} \mid 0 \leq r \leq \frac{1}{2}\delta - 1\} \cup \{2^{\delta-1}\}.$$  

**Proposition 12 (combinatorial properties of $L(Pr_Z)$).** The following properties hold:

1. $1 \in L(Pr_Z)$;
2. $\max L(Pr_Z) = 2^{b_1(\Gamma_Z)}$;
3. $2^g \in L(Pr_Z)$ if and only if $Z$ has only rational components;
4. $Pr_Z$ is reduced if and only if $Z$ is of compact type;
5. if $\Gamma_Z$ is an eulerian graph, then $L(Pr_Z) = L(S_Z)$.

**Proof.** (1) Choosing $\Delta_X = \Gamma_Z$, we get $X = Z$; since the empty set is trivially in $C_{\Gamma_Z}$, there always exists $\eta \in \text{Pic} \ Z$ such that $(Z, \eta) \in Pr_Z$. This $\eta$ is a square root of $O_Z$; there are $2^{2g'' + b_1(\Gamma_Z)}$ choices for it, and it will appear with multiplicity 1 in $Pr_Z$. So $1 \in L(Pr_Z)$.

(2) From Proposition 11 we get $\max L(Pr_Z) \leq 2^{b_1(\Gamma_Z)}$. Set $M = \max\{b_1(\Sigma) \mid \Sigma \in C_{\Gamma_Z}\}$ and let $\Sigma_0 \in C_{\Gamma_Z}$ be such that $b_1(\Sigma_0) = M$. By Proposition 11, we know that $2^{b_1(\Gamma_Z) - b_1(\Sigma_0)} \in L(Pr_Z)$. We claim that $b_1(\Sigma_0^c) = 0$. Indeed, if not, $\Sigma_0^c$ contains a subgraph $\sigma$ with $b_1(\sigma) = 1$ and having all valencies equal to 2. Then $\Sigma_0 \cup \sigma \in C_{\Gamma_Z}$ and $b_1(\Sigma_0 \cup \sigma) > M$, a contradiction. Hence we have points of multiplicity $2^{b_1(\Gamma_Z)}$ in $Pr_Z$, so $\max L(Pr_Z) = 2^{b_1(\Gamma_Z)}$.

Property (3) is immediate from (2), since $b_1(\Gamma_Z) = g$ if and only $g'' = 0$.

Also property (4) is immediate from (2), because $L(Pr_Z) = \{1\}$ if and only if $b_1(\Gamma_Z) = 0$.

(5) Assume that $\Gamma_Z$ is eulerian. Then $\Delta_X^e \in C_{\Gamma_Z}$ if and only if $\Delta_X \in C_{\Gamma_Z}$, so we have

$$L(Pr_Z) = L(S_Z) = \{2^{b_1(\Gamma_Z) - b_1(\Delta_X)} \mid \Delta_X \in C_{\Gamma_Z}\}$$

(see [CC03] for the description of $L(S_Z)$).
Property (4) implies the following

**Corollary 13.** The morphism \( p: \overline{\text{Pr}}_g \to \overline{\mathcal{M}}_g \) is étale over \( \overline{\mathcal{M}}_g^0 \setminus D_{\text{irr}} \).

Consider now property (1) of Proposition 12. It shows, in particular, that in general \( \text{Pr}_Z \) and \( S_Z \) are not isomorphic and do not have the same set of multiplicities: indeed, for spin curves, it can very well happen that \( 1 \not\in L(S_Z) \) (see example after Corollary 14).

The following shows that in some cases, \( L(\text{Pr}_Z) \) gives informations on \( Z \).

**Corollary 14.** Let \( Z \) be a stable curve and \( \nu: Z^\nu \to Z \) its normalization. Assume that for every irreducible component \( C \) of \( Z \), the number \( |\nu^{-1}(C \cap \text{Sing} Z)| \) is even and at least 4.

(i) If \( 2b_1(\Gamma_Z)^{-2} \not\in L(\text{Pr}_Z) \), then \( Z = C_1 \cup C_2 \), with \( C_1 \) and \( C_2 \) smooth and irreducible.

(ii) If \( 2b_1(\Gamma_Z)^{-3} \not\in L(\text{Pr}_Z) \), then either \( Z \) is irreducible with two nodes, or \( Z = C_1 \cup C_2 \cup C_3 \), with \( C_i \) smooth irreducible and \( |C_i \cap C_j| = 2 \) for \( 1 \leq i < j \leq 3 \).

**Proof.** By hypothesis \( \Gamma_Z \) is eulerian, so property (5) says that \( L(\text{Pr}_Z) = L(S_Z) \). Then (i) follow immediately from [CC03], Theorem 11. Let us show (ii). If \( b_1(\Gamma_Z) \geq 4 \), by property (2) we can apply [CC03], Theorem 13; then \( Z \) has three smooth components meeting each other in two points. Assume \( b_1(\Gamma_Z) \leq 3 \) and let \( \delta, \gamma \) be the number of nodes and of irreducible components of \( Z \). Since all vertices of \( \Gamma_Z \) have valency at least 4, we have \( \delta \geq 2\gamma \), so \( \gamma \leq b_1(\Gamma_Z) - 1 \leq 2 \). Then by an easy check we see that the only possibility which satisfies all the hypotheses is \( \gamma = 1 \) and \( \delta = 2 \).

In [CC03] it is shown (Theorem 11) that \( L(S_Z) \) allows to recover curves having two smooth components. Instead, when the number of nodes is odd, it is no more true that these curves are characterized by \( L(\text{Pr}_Z) \). For instance, consider the graphs:
It is easy to see that if $Z_1, Z_2$ are stable curves with $\Gamma_{Z_i} = \Gamma_i$ for $i = 1, 2$, we have $L(P_{Z_1}) = L(P_{Z_2}) = \{1, 4, 16\}$, while $L(S_{Z_1}) = \{4, 8, 16\}$ and $L(S_{Z_2}) = \{2, 8, 16\}$.

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