POWER SPECTRUM FROM WEAK-SHEAR DATA

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ABSTRACT

We demonstrate that the aperture mass as a measure for cosmic shear closely approximates (to better than $\approx 5\%$) the scaled and shifted power spectrum of the projected mass density. This cosmological weak-lensing information can thus be used to directly infer the projected matter power spectrum with high accuracy. As an application, we show that aperture-mass observations can be used to constrain the cosmic density parameter and the power-spectrum amplitude. We show that, for a particular example, it should be possible to constrain $\Omega_0$ to within $\approx \pm 2\%$, and $\sigma_8$ to within $\approx \pm 8\%$ using weak-shear data on a square-shaped field of $8^\circ$ side length.

1. INTRODUCTION

A new measure for cosmic shear, the aperture mass $\langle M_{ap} \rangle$, was recently proposed by Schneider et al. (1998; hereafter S98). It was shown there that $\langle M_{ap}^2 \rangle$ is related to the power spectrum $P_k$ of the projected density fluctuations filtered with a narrow function. Here, we demonstrate in Sect. 2 that an accurate approximation to $\langle M_{ap}^2 \rangle$ can be constructed which directly yields $P_k$. In essence, this makes the projected matter power spectrum a directly observable quantity, so that no deconvolution algorithms need to be invoked. We then use our approximation in Sect. 3 to infer the cosmic density parameter $\Omega_0$ and the power-spectrum normalisation $\sigma_8$ from simulated data. We present our conclusions in Sect. 4.

2. APERTURE MASS AND APPROXIMATIONS

2.1. Effective convergence

The two-point statistics of the gravitational lens properties of the large-scale structure can be described, to high accuracy, in terms of the power spectrum of an equivalent single lens plane matter distribution (see, e.g., Kaiser 1998; S98; and references therein), which is given by

$$P_k(l) = \frac{9 H_0^2 \Omega_0^2}{4 c^2} \int_0^{\infty} dw \frac{W^2(w)}{a^2(w)} P_0 \left( \frac{l}{f_k(w)} \right) w, \quad (1)$$

where $\vec{\theta}$ is the Fourier conjugate to the angle $\theta$, $w$ and $f_k(w)$ are the comoving radial and angular-diameter distance, respectively, $a = (1 + z)^{-1}$ is the scale factor, $P_0(k, w)$ is the density-perturbation power spectrum, and $w$ is the weight function

$$W(w) = \int_0^{\infty} dw' G(w') \frac{f_k(w' - w)}{f_k(w')}, \quad (2)$$

which depends on the probability distribution $G(w)$ of source distances. The upper integration limit $w_1$ is the comoving horizon distance, here defined as the comoving distance corresponding to redshift infinity. Quite intuitively, eq. (1) relates the lensing power on angular scales $\theta = 2 \pi l c^{-1}$ to the power in density fluctuations at a comoving scale $2 \pi k c^{-1} = f_k(w) \theta$.

We parameterise the source-distance distribution $G(w)$ as a function of redshift, $G_z(z)$, specified by

$$G_z(z) = \frac{\beta}{z_0 \Gamma(3/\beta)} z^2 \exp \left[ - \left( \frac{z}{z_0} \right)^\beta \right], \quad (3)$$

It is normalised to $0 \leq z < \infty$ and provides a good fit to the observed redshift distribution (e.g. Smail et al. 1995). The mean redshift $\langle z \rangle$ is proportional to $z_0$. For $\beta = 1.5$ which we assume throughout, $\langle z \rangle \approx 1.505 z_0$.

2.2. Aperture mass

The aperture mass $M_{ap}(\theta)$ as a function of smoothing scale $\theta$ is defined in terms of a weighted average within a circle of radius $\theta$ of the surface mass density of the equivalent single lens plane. It can readily be obtained from the observed image ellipticities of faint background galaxies which provide an unbiased estimate of the shear $\gamma$:

$$M_{ap}(\theta) = \int d^2 \theta Q(|\vec{\theta}|) \gamma(\vec{\theta}), \quad (4)$$

by replacing the integral over the shear by a sum over galaxy ellipticities. Here, $\gamma$ is the tangential component of the shear relative to the aperture centre, and $Q$ is an appropriately chosen weight function which is non-zero only for $0 \leq \theta < \theta$. The mean-squared aperture mass is related to the effective-convergence power spectrum through

$$\langle M_{ap}^2 \rangle(\theta) = 2\pi \int_0^{\infty} d\theta P_k(l) J^2(\theta), \quad (5)$$

where $\langle \rangle$ is related to the filter function $Q$ (see S98). For $Q(\theta) = \frac{6}{\pi} \frac{\theta^2 (\theta^2 - \psi^2)}{\psi^6}$, it is

$$J(\eta) = \frac{12}{\pi \eta^2} J_4(\eta), \quad (6)$$

and $J_4(\eta)$ is the fourth-order Bessel function of the first kind. $J^2(\eta)$ peaks at $\eta_0 \approx 4.11$. Examples for the rms aperture mass $\langle M_{ap}^2 \rangle^{1/2}$ are plotted in Fig. 1 as a function of aperture radius $\theta$.

It is crucial in eqs. (1) and (5) to take the non-linear evolution of the density power spectrum into account. For aperture radii of order a few arc minutes, $\theta = 3 \times 10^{-4}$ rad, the peak $\eta_0$ of $J^2(\eta)$ translates to $l \approx 1.4 \times 10^4$, which corresponds to $2 \pi k c^{-1} \approx 1 h^{-1}$ Mpc for sources around redshift unity, i.e. the physical scales of density perturbations to which the aperture mass is most sensitive to are well in the non-linear regime of evolution. We assume a CDM power spectrum and describe its non-linear evolution as given by Peacock & Dodds (1996). We normalise the spectrum to the local abundance of rich clusters by choosing $\sigma_8$ as derived by Viana & Liddle (1996) and Eke, Cole & Frenk (1996).
2.3. Signal-to-noise ratio

Although an $rms$ aperture-mass amplitude appears around one per cent, the signal-to-noise ratio of the aperture mass can be quite high. There are three sources of noise in a measurement of $\langle M_{\text{ap}}^2 \rangle$. Since it will be inferred from distortions of galaxy ellipticities, the intrinsic non-vanishing ellipticity of the galaxies provides one source of noise, and the random positions of the galaxies provides another. The third source of noise is due to cosmic variance. Assuming a large number of galaxies $N$ per aperture, and neglecting the kurtosis of the aperture mass, the dispersion of $M_{\text{ap}}$ in a single aperture is (see S98)

$$\sigma^2(\langle M_{\text{ap}}^2 \rangle) \approx \left( \frac{6\sigma^2}{5\sqrt{2N}} + \sqrt{2}\langle M_{\text{ap}}^2 \rangle \right)^2,$$

where $\sigma_k \approx 0.2$ is the intrinsic dispersion of the galaxy ellipticities. On angular scales of a few arc minutes and smaller, the galaxies dominate the noise, while the cosmic variance dominates on larger scales. Of course, $M_{\text{ap}}$ will be measured in a large number of apertures $N_{\text{ap}}$ rather than a single one. If the apertures are independent, the dispersion (7) is reduced by a factor of $N_{\text{ap}}^{1/2}$, and the ensemble variance becomes

$$\bar{\sigma} = \frac{\sigma(\langle M_{\text{ap}}^2 \rangle)}{N_{\text{ap}}^{1/2}}.$$

Typical signal-to-noise ratios reach values of $\gtrsim 5$ for aperture radii of a few arc minutes and data fields of a single degree in size.

2.4. Approximate $rms$ aperture mass

The strong peak of $J^2(\eta) –$ see Fig. 2 in S98 – motivates the approximation of $J^2$ by a delta function,

$$J^2(\eta) \approx A \delta_0(\eta - \eta_0),$$

with $A = 512/(1155\pi^3) \approx 1.43 \times 10^{-2}$, $\eta_0 = 693\pi/512 \approx 4.25$, as determined from the norm and mean of $J^2$. Then the expression for the aperture mass (5) becomes particularly simple,

$$\langle M_{\text{ap}}^2 \rangle(\theta) \approx \langle M_{\text{ap}}^2 \rangle = 6(5\pi)^{-1} \eta_0^{-2} P_k(\eta_0/\theta).$$

If this provides a good approximation, the observable $rms$ aperture mass at angular scale $\theta$ would directly yield the effective-convergence power spectrum $P_k(l)$ at wave number $\eta_0/\theta$, and thus a most direct and straightforward measure for dark-matter fluctuations. We therefore have to investigate whether replacing the filter function $J^2(\eta)$ by a delta function can be justified.

Let us first assume that $P_k(l)$ can be approximated locally as a power law in $l$ over a range in which $J^2$ differs significantly from zero,

$$P_k(l) \approx B l^{n_{\text{eff}}}.$$

Then, the true aperture mass (5) becomes

$$\langle M_{\text{ap}}^2 \rangle(\theta) = \frac{144B}{\pi^{3/2}} \Gamma\left(\frac{3-n_{\text{eff}}}{2}\right) \Gamma\left(\frac{3+n_{\text{eff}}}{2}\right) \Gamma\left(\frac{6-n_{\text{eff}}}{2}\right) \eta_0^{-2} \theta^{-2-n_{\text{eff}}},$$

where the Gamma functions arise from integrating the power-law spectrum times the squared fourth-order Bessel function. The approximate aperture mass $\langle M_{\text{ap}}^2 \rangle$ defined in eq. (10) becomes

$$\langle M_{\text{ap}}^2 \rangle(\theta) = 6(5\pi)^{-1} B \eta_0^{n_{\text{eff}}} \theta^{-(2+n_{\text{eff}})}.$$

The relative deviation of the approximation from the true aperture mass is a function of the effective-power-law exponent $n_{\text{eff}}$ only. It is plotted in Fig. 1. Clearly, the deviation of $\langle M_{\text{ap}}^2 \rangle$ from $\langle M_{\text{ap}}^2 \rangle$ is very small, less than five per cent, for effective power-spectrum slopes in the range $-1.5 \lesssim n_{\text{eff}} \lesssim 0.5$, and this is exactly the range of slopes in the $l$ interval contributing most of the power to the aperture mass for aperture radii of $\gtrsim 1'$. It is therefore fair to say that, especially in the presence of measurement errors, the aperture mass and its approximation (10) can be considered equivalent. Figure 2 shows $\langle M_{\text{ap}}^2 \rangle$ and $\langle M_{\text{ap}}^2 \rangle$ for three cluster-normalised CDM models as examples for realistic non-power-law spectra, and emphasises the conclusions from Fig. 1.

The approximation (10) to the aperture mass is excellent. This immediately implies that measurements of the aperture mass directly measure the effective-convergence power spectrum, and therefore the latter effectively becomes an observable quantity,

$$P_k(l) \approx (5\pi/6) \left(\eta_0/l\right)^2 \langle M_{\text{ap}}^2 \rangle(\eta_0/l).$$

This provides the most straightforward access to a quantity of paramount importance for cosmology, i.e. the dark-matter power spectrum.

![Fig. 1.](image-url) — Relative deviation between the true aperture mass $\langle M_{\text{ap}}^2 \rangle$ and the approximate aperture mass $\langle M_{\text{ap}}^2 \rangle$ as defined in eq. (10), for an assumed effective-convergence power spectrum that is a power law in $l$. For the most relevant range of effective power-law exponents, $-1.5 \lesssim n_{\text{eff}} \lesssim 0.5$, the relative deviation is less than five per cent.

3. Applications

It is now interesting to investigate a specific application of our results. Let us assume we are given a large, quadratic data field of angular size $L$, into which we place apertures of radius $\theta$. We can place apertures of a fixed size densely on the field since the correlation of the aperture mass in neighbouring apertures is very small due to the narrowness of the filter function $J^2(\eta)$ (see Fig. 8 in S98). However, we want to sample $\langle M_{\text{ap}}^2 \rangle$ at a variety of aperture sizes rather than a single one. Starting from a
Therefore, we assume that the number decorrelate quickly with differing radius and centre separation – for different aperture sizes are not completely uncorrelated, but encompass on average twice as many galaxies. Clearly, the data It can safely be assumed that the redshift distribution as to minimise (15). The sensitivity of the aperture mass to perturbation power spectrum, we can then vary parameters such as to normalise the power spectrum for simulating the input data, and choose either \( \Omega_\Lambda = 0 \) or \( \Omega_\Lambda = 1 - \Omega_0 \). \( \chi^2 \) contours for the former case are shown in Fig. 4.

The solid contours in the left panel of Fig. 4 show that at the 1-\( \sigma \) significance level, \( \Omega_0 \) and \( \sigma_8 \) can be constrained to relative accuracies of \( \approx \pm 27\% \) and \( \approx \pm 8\% \), respectively. Table 2 gives 1-\( \sigma \) limits for fields of 4° and 8° side length. Similar experiments using \( \Omega_\Lambda = 1 - \Omega_0 \) instead of \( \Omega_\Lambda = 0 \) yield a minimum \( \chi^2 \gtrsim 10 \), so that spatially flat cosmologies can be rejected at very high significance using the data simulated with \( \Omega_\Lambda = 0 \).

Of course, the quality of the results depends on how accurately we guess the shape of the true power spectrum. For the contours in Fig. 4, we vary two parameters and keep all others fixed. In Fig. 5 for example, we keep \( h = 0.7 \) constant, so that the shape parameter \( \Gamma = \Omega_0 h \) of the power spectrum is \( \Gamma = f \Gamma' \) with \( f \neq 1 \), the best-fitting parameter combination \( (\Omega_0, \sigma_8) \) would shift towards \( f \Omega_0 \) along the “valley” indicated by the \( \chi^2 \) contours without substantially degrading the quality of the fit. This degeneracy reflects the degeneracy in the cluster abundance used to normalise the power spectrum for simulating the input data, because \( \langle M_{ap}^2 \rangle^{1/2} \) essentially measures the halo abundance at intermediate redshifts, \( z \sim 0.3 \). To illustrate this point, we indicate the present-day cluster-abundance constraint as the long-dashed curve in Fig. 4. It is seen to follow well the “valley” in the \( \chi^2 \) contours for low \( \Omega_0 \), but it departs increasingly for increasing \( \Omega_0 \) because structure grows more rapidly within \( z \sim 0.3 \) and today for higher \( \Omega_0 \). The parameter degeneracy with the cosmological constant can be broken using the skewness of \( M_{ap} \), for which the degeneracy between \( \Omega_0 \) and \( \Omega_\Lambda \) is different (Van Waerbeke, Bernardeau & Mellier 1999).

Although our assumptions concerning the number of independent apertures that can be placed inside a given data field and the neglect of the kurtosis in the approximation for \( \sigma(M_{ap}) \) may be slightly optimistic, we believe that these effects do not
In contrast to other methods for constraining $P_{\kappa}(l)$ through cosmic-shear data (Kaiser 1998; Seljak 1998; Van Waerbeke et al. 1999), the aperture-mass method allows for a simple combined analysis of independent data fields scattered across the sky. Furthermore, since $(M_{ap}^2)^{1/2}$ is a scalar quantity directly obtained from observed galaxy-image ellipticities, its full probability-distribution function can be derived and used for parameter extraction.

The approximate aperture mass can then straightforwardly be applied to constrain cosmological parameters from observed aperture masses. We simulated observations of $\langle M_{ap}^2 \rangle$ and their expected errors and showed that $\Omega_0$ and $\sigma_8$ can be recovered with relative accuracies of $\approx \pm 27\%$ and $\approx \pm 8\%$, respectively, using weak-shear data on a square field with 8° side length. For that, we have assumed that the parameters of the source redshift distribution are sufficiently well known. We believe that this is not a serious limitation because of the high reliability and accuracy of photometric redshift determinations (e.g., Benitez 1998, and references therein). In fact, the source-redshift dependence of $(M_{ap}^2)$ can be used as a consistency check: With increasing source redshift, $(M_{ap}^2)$ increases, while the skewness of $M_{ap}$ decreases.

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TABLE 1.—1-σ limits to $\Omega_0$ and $\sigma_8$ obtained with fields of 4° and 8° side length.

| $L$   | input $\Omega_0$ | fit $\Omega_0$ | input $\sigma_8$ | fit $\sigma_8$ |
|-------|-----------------|----------------|-----------------|---------------|
| 4°    | 0.3 ± 0.12      | 0.25 ± 0.12    | 0.85 ± 0.12     | 0.93 ± 0.12   |
| 8°    | 0.3 ± 0.08      | 0.26 ± 0.08    | 0.85 ± 0.08     | 0.93 ± 0.08   |

4. CONCLUSIONS

We have shown here that a recently proposed measure for cosmic shear, the dispersion of the aperture mass $\langle M_{ap}^2 \rangle$, can be approximated by the local value of the power spectrum $P_{\kappa}$ of the projected matter fluctuations at wave number $k_0/\theta$ to a relative accuracy better than 5%. This is possible because the aperture mass is a convolution of $P_{\kappa}$ with a narrow filter function which for this purpose can safely be approximated by a Dirac delta function.