Indecomposable representations of the Euclidean algebra $\mathfrak{e}(3)$ from irreducible representations of the symplectic algebra $\mathfrak{sp}(4, \mathbb{C})$.

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Abstract. The Euclidean group $E(3)$ is the noncompact, semidirect product group $E(3) \cong \text{SO}(3) \rtimes \mathbb{R}^3$. It is the Lie group of orientation-preserving isometries of 3-dimensional Euclidean space. The Euclidean algebra $\mathfrak{e}(3)$ is the complexification of the Lie algebra of $E(3)$. We embed $\mathfrak{e}(3)$ into the 10-dimensional symplectic algebra $\mathfrak{sp}(4, \mathbb{C})$, the simple Lie algebra of type $C_2$. We show that, up to conjugation by an element of $\text{Sp}(4, \mathbb{C})$, there is only one embedding of $\mathfrak{e}(3)$ into $\mathfrak{sp}(4, \mathbb{C})$, and then prove that the irreducible representations of $\mathfrak{sp}(4, \mathbb{C})$ remain indecomposable upon restriction to $\mathfrak{e}(3)$, thus creating a new class of indecomposable $\mathfrak{e}(3)$-representations.

1. Introduction
The Euclidean group $E(3)$ is the noncompact, semidirect product group $E(3) \cong \text{SO}(3) \rtimes \mathbb{R}^3$. It is the Lie group of orientation-preserving isometries of 3-dimensional Euclidean space. The Euclidean algebra $\mathfrak{e}(3)$ is the complexification of the Lie algebra of $E(3)$. Both $E(3)$ and $\mathfrak{e}(3)$ have found significant applications in physics, examples of which are described in [4].

The finite-dimensional irreducible representations of $\mathfrak{e}(3)$ are not very interesting, but classifying its indecomposable representations remains a significant challenge. We remind the reader that a representation is irreducible if it has no proper subrepresentations. It is indecomposable if it is not isomorphic to a direct sum of two nonzero subrepresentations. A representation may be indecomposable without being irreducible.

Although a full classification of $\mathfrak{e}(3)$-indecomposable representations remains elusive, constructing large classes of indecomposable representations that may be classified is a viable option. One interesting possibility is to embed $\mathfrak{e}(3)$ into an algebra for which the irreps are known, and then consider their restrictions to $\mathfrak{e}(3)$, regarded as a subalgebra. In some cases, it may be possible to show that the resulting representations of $\mathfrak{e}(3)$ are indecomposable.

This direction of research has been pursued, for instance, by Douglas and Premat [3], who show that irreducible $\mathfrak{sl}(3, \mathbb{C})$-modules remain $\mathfrak{e}(2)$-indecomposable, and later in a tour de force by Casati et al. [2] who established that irreducible $\mathfrak{sl}(3, \mathbb{C})$ and $\mathfrak{sp}(4, \mathbb{C})$-modules remain indecomposable modules of the Diamond Lie algebra under appropriate embeddings. The Diamond Lie algebra is a central extension of the Poincaré Lie algebra in two dimensions.
In the current article we begin by embedding the Euclidean algebra \( \mathfrak{e}(3) \) into the 10-dimensional symplectic algebra \( \mathfrak{sp}(4, \mathbb{C}) \), the simple Lie algebra of type \( C_2 \). It is the smallest simple Lie algebra into which \( \mathfrak{e}(3) \) embeds. In fact, we are able to show that, up to conjugation by an element of \( \mathfrak{sp}(4, \mathbb{C}) \), there is only one such embedding. We then show that the finite-dimensional irreducible representations of \( \mathfrak{sp}(4, \mathbb{C}) \) remain indecomposable upon restriction to \( \mathfrak{e}(3) \) under this embedding, thus creating a new class of indecomposable \( \mathfrak{e}(3) \)-representations.

The article is organized as follows. In section 2 we describe the basis and commutation relations of \( \mathfrak{e}(3) \). Section 3 records information about the simple Lie algebra \( \mathfrak{sp}(4, \mathbb{C}) \) and its irreducible representations that will be employed in the following section. In section 4 we prove that, up to conjugation by an element in \( \mathfrak{sp}(4, \mathbb{C}) \), there is only one embedding of \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \). In section 5 we prove that irreducible representations of \( \mathfrak{sp}(4, \mathbb{C}) \) remain \( \mathfrak{e}(3) \)-indecomposable under the (essentially) unique embedding of \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \).

2. The Euclidean algebra \( \mathfrak{e}(3) \)

The Euclidean algebra \( \mathfrak{e}(3) \) is the complexification of the Lie algebra of the Euclidean Lie group \( E(3) \). For a more detailed discussion of \( E(3) \) and the calculation of its Lie algebra we refer the reader to Hall [6]. The Euclidean algebra \( \mathfrak{e}(3) \) has basis \( E, H, F, P_0, P_\pm \), and nonzero commutation relations

\[
\begin{align*}
[H, E] &= 2E, & [H, F] &= -2F, & [E, F] &= H, \\
[H, P_\pm] &= \pm 2P_\pm, & [E, P_0] &= -P_+, & [F, P_0] &= -P_-, \\
[F, P_+] &= -2P_0, & [E, P_-] &= -2P_0.
\end{align*}
\]

One can easily see that \( \langle E, H, F \rangle \cong \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3) \), and that \( \{P_0, P_\pm\} \) is an abelian ideal of \( \mathfrak{e}(3) \).

Moreover, under the adjoint action, the action of the \( \mathfrak{so}(3) \) subalgebra \( \langle E, H, F \rangle \) decomposes as the direct sum of two three-dimensional irreps, \( \langle E, H, F \rangle \) and \( \{P_0, P_\pm\} \).

3. The symplectic algebra \( \mathfrak{sp}(4, \mathbb{C}) \) and its irreducible representations

The symplectic algebra \( \mathfrak{sp}(4, \mathbb{C}) \) is the Lie algebra of \( 4 \times 4 \) complex matrices \( X \) satisfying \( JX^T J = X \), where \( J \) is the \( 4 \times 4 \) matrix

\[
J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]

It is the 10-dimensional simple Lie algebra of type \( C_2 \) (equivalent to the simple Lie algebra of type \( B_2 \)). Let

\[
\{x_i, y_i, h_j : 1 \leq i \leq 4, 1 \leq j \leq 2\}
\]

be the Chevalley basis of \( \mathfrak{sp}(4, \mathbb{C}) \) with nontrivial commutation relations

\[
\begin{align*}
[h_i, h_j] &= 0, & [h_i, x_j] &= C_{ij} x_j, & [h_i, y_j] &= -C_{ij} y_j, & [x_i, y_j] &= \delta_{ij} h_i, \\
[h_1, x_2] &= 2x_4, & [h_1, y_2] &= -2y_4, & [h_2, x_3] &= x_3, & [h_2, y_3] &= -y_3, \\
[x_1, x_2] &= -x_3, & [x_1, x_3] &= -2x_4, & [x_1, y_3] &= 2y_2, & [x_1, y_4] &= y_3, \\
[x_2, y_3] &= y_1, & [x_3, y_1] &= 2x_2, & [x_3, y_2] &= -x_1, & [x_3, y_4] &= h_1 + 2h_2, \\
[x_3, y_4] &= -y_1, & [x_4, y_1] &= x_3, & [x_4, y_3] &= -x_1, & [x_4, y_4] &= h_1 + h_2, \\
y_1 y_2 &= y_3, & [y_1, y_3] &= 2y_4,
\end{align*}
\]

for \( 1 \leq i, j \leq 2 \) and where \( C \) is the Cartan matrix of type \( C_2 \)

\[
C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.
\]
For $i = 1$ or 2, define $\Lambda_i \in \mathfrak{h}^*$ by $\Lambda_i(h_i) = \delta_{ij}$. For each $\lambda = m_1\Lambda_1 + m_2\Lambda_2 \in \mathfrak{h}^*$ with nonnegative integers $m_1, m_2$, there exists a finite dimensional irreducible $\mathfrak{sp}(4, \mathbb{C})$-module $V(m_1, m_2)$ which can be realized as the quotient of the universal enveloping algebra $U(\mathfrak{sp}(4, \mathbb{C}))$ by the left ideal $J_\lambda$ generated by $x_i, h_i - \lambda(h_i), y_i^{1+\lambda(h_i)}, 1 \leq i \leq 2$ (here the action of $U(\mathfrak{sp}(4, \mathbb{C}))$ on itself and on $V(m_1, m_2)$ is given by left multiplication). We will denote the element $1 + J_\lambda$ of $V(m_1, m_2)$ by $v_\lambda$. The dimension of $V(m_1, m_2)$ follows in a straightforward manner from Weyl’s character formula [7]:

$$\dim(V(m_1, m_2)) = \frac{1}{3!} (m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)(m_1 + 2m_2 + 3).$$

Bliem [1] showed that $V(m_1, m_2)$ has basis

$$\mathcal{B} = \{y_1^{a_1}y_2^{a_2}y_3^{a_3}y_4^{a_4}v_\lambda\},$$

subject to the relations

$$0 \leq a_4 \leq m_2, \quad 2a_4 \leq a_3 \leq m_1 + 2a_4, \quad \frac{1}{2} \leq a_3 \leq a_2 + a_3 - 2a_4, \quad 0 \leq a_1 \leq m_1 + 2a_2 - 2a_3 + 2a_4.$$ (8)

Bliem’s basis, however, is not convenient for the present purposes. We use the spanning set of $V(m_1, m_2)$ described in the following Lemma.

**Lemma 3.1** The $\mathfrak{sp}(4, \mathbb{C})$-module $V(m_1, m_2)$ is spanned by the (not necessarily linearly independent) set

$$\mathcal{S}_\lambda = \{y_1^a y_2^b y_3^c y_4^d v_\lambda\}$$

subject to the relations

$$b \leq m_2, \quad c \leq m_1 + 2m_2, \quad d \leq m_1 + m_2, \quad a + c + 2d \leq 2m_1 + 2m_2.$$ (10)

**Proof:** The set $\{y_1^a y_2^b y_3^c y_4^d v_\lambda\}$ with nonnegative integers $a, b, c, d$ spans $V(m_1, m_2)$ [7]. Since $y_2, y_3$ and $y_4$ commute, a defining relation of $V(m_1, m_2)$ implies $b \leq m_2$.

Note that $(h_1 + 2h_2) \cdot y_1^ay_2^by_3^cy_4^dv_\lambda = (m_1 + 2m_2 - 2(b + c + d))v_\lambda$ and $(h_1 + h_2) \cdot y_1^ay_2^by_3^cy_4^dv_\lambda = (m_1 + m_2 - a + c + 2d)v_\lambda$. Hence, since $(x_3, h_1 + 2h_2, y_3)$ and $(x_4, h_1 + h_2, y_4)$ are each isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ implies the remaining conditions given in (10).

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4. Classification of Embeddings of $\mathfrak{e}(3)$ into $\mathfrak{sp}(4, \mathbb{C})$

4.1. Restriction to $\mathfrak{so}(3)_{\mathbb{C}}$

Suppose $T : \mathfrak{e}(3) \hookrightarrow \mathfrak{sp}(4, \mathbb{C})$ is an embedding of Lie algebras. The restriction of $T$ to the $\mathfrak{so}(3)_{\mathbb{C}}$ subalgebra is a four-dimensional representation of $\mathfrak{so}(3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}}$, so it must be one of the following: $4V^0$, $V^1 \perp 2V^0$, $V^1 \perp V^0$, $2V^{1/2}$, or $V^{3/2}$, where $V^j$ is the $(2j + 1)$-dimensional representation of $\mathfrak{so}(3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}}$. Of these, the first is not an embedding.

From (1), we note that, when $H$ acts on $V^1$, its highest eigenvalue is 2. Accordingly, its highest eigenvalue on $V^j$ is $2j$. If $T|_{\mathfrak{so}(3)_{\mathbb{C}}} \cong V^{3/2}$, then $T(H)$ will have eigenvalues 3, 1, −1, −3. Under its action on $\mathfrak{sl}(4, \mathbb{C})$ under the $\mathfrak{sl}(4, \mathbb{C})$-adjoint action, $T(H)$ will have highest eigenvalue $3 - (-3) = 6$, which means that the action of $T(\mathfrak{so}(3)_{\mathbb{C}})$ on $\mathfrak{sl}(4, \mathbb{C})$ will include a copy of $V^3$. Since $V^3$ has dimension 7, it cannot be complementary to $\mathfrak{sp}(4, \mathbb{C})$, which has dimension 10 inside the 15-dimensional space $\mathfrak{sl}(4, \mathbb{C})$. This implies that the action of $T(\mathfrak{so}(3)_{\mathbb{C}})$ on $\mathfrak{sp}(4, \mathbb{C})$ contains a copy of $V^3$. But this is impossible, since we know that the $\mathfrak{so}(3)_{\mathbb{C}}$ action on $\mathfrak{e}(3)$ and
hence the $T(\mathfrak{so}(3)_{\mathbb{C}})$ action on $T(\mathfrak{e}(3)) \subset \mathfrak{sp}(4, \mathbb{C})$ decomposes as $V^1 \oplus V^1$, which has dimension 6.

If $T|_{\mathfrak{so}(3)_{\mathbb{C}}} \cong V^{1/2} \oplus 2V^0$, then $T(H)$ has eigenvalues $1, -1, 0, 0$. Under the $\mathfrak{sl}(4, \mathbb{C})$-adjoint action, $T(H)$ will have highest eigenvalue $1 - (-1) = 2$, but only with multiplicity 1. This means that the action of $T(\mathfrak{so}(3)_{\mathbb{C}})$ on $T(\mathfrak{e}(3)) \subset \mathfrak{sl}(4, \mathbb{C})$ cannot contain $V^1 \oplus V^1$, a contradiction.

If $T|_{\mathfrak{so}(3)_{\mathbb{C}}} \cong V^1 \oplus V^0$, then $T(H)$ has eigenvalues $2, -2, 0, 0$. It must be conjugate by an element of $Sp(4, \mathbb{C})$ to the diagonal matrix $\text{diag}(2, 0, -2, 0)$. Under the $\mathfrak{sp}(4, \mathbb{C})$-adjoint action, this matrix has highest eigenvalue $2 - (-2) = 4$, implying that the action of $T(\mathfrak{so}(3)_{\mathbb{C}})$ on $\mathfrak{sp}(4, \mathbb{C})$ contains a copy of $V^2$. Since $\dim(V^2) = 5$, $\mathfrak{sp}(4, \mathbb{C})$ cannot also contain $V^1 \oplus V^1$, so this case is also impossible.

The only remaining possibility is that $T|_{\mathfrak{so}(3)_{\mathbb{C}}}$ must decompose as $V^{1/2} \oplus V^{1/2}$.

### 4.2. Embeddings of $\mathfrak{so}(3)_{\mathbb{C}}$ into $\mathfrak{sp}(4, \mathbb{C})$

Without loss of generality, we can assume that $T$ takes the element $H \in \mathfrak{so}(3)_{\mathbb{C}} \subset \mathfrak{e}(3)$ into the diagonal Cartan subalgebra of $\mathfrak{sp}(4, \mathbb{C})$. The result of the preceding subsection shows that we can assume that $T(H) = \text{diag}(1, -1, -1, 1)$. The element $T(H)$ has eigenspaces under the adjoint action in $\mathfrak{sp}(4, \mathbb{C})$ as follows:

The 0-eigenspace is spanned by

$$E_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad E' = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad (11)$$

The $-2$-eigenspace is spanned by

$$F_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad (12)$$

The 0-eigenspace is spanned by $Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ and

$$H_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (13)$$

One obvious embedding of $\mathfrak{so}(3)_{\mathbb{C}}$ in $\mathfrak{sp}(4, \mathbb{C})$ is spanned by $H_0, E_0, F_0$. In fact, up to conjugacy by elements of $Sp(4, \mathbb{C})$, it is the only one that can arise as the restriction of an embedding of $\mathfrak{e}(3)$.

**Proposition 4.1** *Any embedding $T$ of $\mathfrak{so}(3)_{\mathbb{C}}$ into $\mathfrak{sp}(4, \mathbb{C})$ which decomposes as $V^{1/2} \oplus V^{1/2}$ is conjugate by an element of $Sp(4, \mathbb{C})$ to the embedding given by $T_0(H) = H_0, T_0(E) = E_0, T_0(F) = F_0$.*
Proof: As above, we can assume that \( T(H) = H_0 \). Then \( T(E) \), an element of the 2-eigenspace for \( T(H) \), must be a linear combination of \( E_0, E', E'' \), say \( T(E) = aE_0 + bE' + cE'' \), with \( a, b, c \in \mathbb{C} \).

Likewise, \( T(F) = rF_0 + sF' + tF'' \), where \( r, s, t \in \mathbb{C} \).

We calculate that

\[
[T(E), T(F)] = \begin{pmatrix}
    ar + bt & 0 & 0 & as + br \\
    0 & -cs - ar & as + br & 0 \\
    0 & cr + at & -ar - bt & 0 \\
    cr + at & 0 & 0 & cs + ar
\end{pmatrix}.
\]

(14)

For this to equal \( T(H) = H_0 \), we must have \( as + br = 0 \), \( cr + at = 0 \), \( ar + bt = 1 \), and \( cs + ar = 1 \).

The last two conditions imply \( bt = cs \). From this we find that the vector \( (r, s, t) \) is a complex scalar multiple of \((a, -b, -c)\). If the constant is \( \lambda \), it all comes down to \( \lambda(a^2 - bc) = 1 \).

In particular, this cannot happen if \( a^2 - bc = 0 \), and otherwise it determines the value of \( \lambda \), namely \( \lambda = \frac{1}{a^2 - bc} \). Note that \( a^2 - bc \neq 0 \) is equivalent to requiring that \( T(E) \) have rank 2, so it is a necessary condition for a representation of the form \( V^{1/2} \oplus V^{1/2} \). We conclude that \((r, s, t) = \left(\frac{a}{a^2 - bc}, \frac{-b}{a^2 - bc}, \frac{-c}{a^2 - bc}\right)\).

The question arises whether there is a \( g \in Sp(4, \mathbb{C}) \) such that \( T(X) = gT_0(X)g^{-1} \), for all \( X \in \mathfrak{so}(3)_\mathbb{C} \). In particular, such a \( g \) would have to satisfy \( gH_0g^{-1} = H_0 \). Combined with the condition \( g \in Sp(4, \mathbb{C}) \), this forces \( g \) to be of the form

\[
g = \begin{pmatrix}
    \alpha & 0 & 0 & \beta \\
    0 & u\alpha & u\beta & 0 \\
    0 & \gamma & \delta & 0 \\
    \frac{2}{u} & 0 & 0 & \frac{\delta}{u}
\end{pmatrix},
\]

(15)

where \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{C}) \) and \( u \in \mathbb{C}^\times \). A simple calculation then shows that

\[
gE_0g^{-1} = \begin{pmatrix}
    0 & \frac{a\beta + b\gamma}{u} & -2\alpha\beta & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    \frac{2\gamma\delta}{u^2} & -\frac{a\delta + b\gamma}{u} & 0 & 0
\end{pmatrix}.
\]

(16)

The question is whether this can be made equal to

\[
T(E) = aE_0 + bE' + cE'' = \begin{pmatrix}
    0 & a & -b & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & c & -a & 0
\end{pmatrix},
\]

(17)

subject to the condition \( a^2 - bc \neq 0 \).

Assuming that \( b \neq 0 \), we can solve and find that if \( \alpha \) is an arbitrary nonzero complex number, \( u \) is a square root of \( \frac{1}{u^2 - bc} \), \( \beta = \frac{b}{2u} \), \( \gamma = \frac{au + \alpha}{u^2 + \alpha^2 - bc} \), and \( \delta = \frac{1 + \alpha u}{2u} \), then the displayed matrices in (16) and (17) are equal. If \( b = 0 \), then it works with \( \alpha = 1, \beta = 0, u = 1, \gamma = \frac{c}{2u} \), and \( \delta = a \).

In any event, provided \( a^2 - bc \neq 0 \), it is possible to find \( g \in Sp(4, \mathbb{C}) \) so that \( gT_0(X)g^{-1} = T(X) \), for \( X = H, E \). Another straightforward calculation shows that the same relation then also holds for \( X = F \), and hence for all \( X \in \mathfrak{so}(3)_\mathbb{C} \). \( \square \)
4.3. Embeddings of \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \)

We may embed \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \) as follows:

\[
\psi(H) = H_0, \quad \psi(E) = E_0, \quad \psi(F) = F_0 \\
\psi(P_0) = H', \quad \psi(P_+) = 2E', \quad \psi(P_-) = -2F'.
\] (18)

In the following theorem we establish that \( \psi \) is (essentially) the unique embedding of \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \).

**Theorem 4.2** Up to conjugacy by elements of \( \text{Sp}(4, \mathbb{C}) \), \( \psi \) is the only embedding of \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \).

**Proof:** Let \( T: \mathfrak{e}(3) \hookrightarrow \mathfrak{sp}(4, \mathbb{C}) \) be an embedding of \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \). Then, \( T \) restricts to an embedding of \( \mathfrak{so}(3)_\mathbb{C} \) which decomposes as \( V^{1/2} \oplus V^{-1/2} \). One copy of \( V^{1/2} \) is the adjoint action of \( \mathfrak{so}(3)_\mathbb{C} \) on its image \( T(\mathfrak{so}(3)_\mathbb{C}) \). The other is the action of \( \mathfrak{so}(3)_\mathbb{C} \) on the image \( T(\mathfrak{p}) \) of the abelian ideal \( \mathfrak{p} = \langle P_0, P_+, P_- \rangle \).

As has been shown in Proposition 4.1, we can assume that \( T(\mathfrak{so}(3)_\mathbb{C}) \) is the span of \( H_0, E_0, F_0 \). The element \( T(P_+) \) must be a linear combination of the elements \( E_0, E', E'' \) in the 2-eigenspace, say \( T(P_+) = x E_0 + y E' + z E'' \), for some \( x, y, z \in \mathbb{C} \).

In this case, we must have

\[
T(P_0) = -\frac{1}{2} \left[ F_0, T(P_+) \right] = -\frac{1}{2} \left( x[F_0, E_0] + y[F_0, E'] + z[F_0, E''] \right) = \frac{x}{2} H_0 + \frac{y}{2} H' + \frac{z}{2} H'', \\
T(P_-) = -\left[ F_0, T(P_0) \right] = -\frac{1}{2} \left( x[F_0, H_0] + y[F_0, H'] + z[F_0, H''] \right) = -x F_0 - y F' - z F''.
\]

Since \( \mathfrak{p} \) is an abelian ideal, its image under \( T \) must be abelian. We find that

\[
\left[ T(P_+), T(P_-) \right] = \begin{pmatrix}
-x^2 - yz & 0 & 0 & -2xy \\
0 & x^2 + yz & -2xy & 0 \\
0 & -2xz & x^2 + yz & 0 \\
-2xz & 0 & 0 & -x^2 - yz
\end{pmatrix},
\] (19)

so the requirement is

\[
x^2 + yz = 0, \quad xy = 0, \quad xz = 0.
\] (20) (21) (22)

It is easy to check that these conditions imply that \( T(\mathfrak{p}) \) is abelian, i.e., that \( T(P_0), T(P_+), \) and \( T(P_-) \) all commute.

If \( x \neq 0 \), then (21) and (22) force \( y = z = 0 \), in which case (20) fails. So we must have \( x = 0 \). From (20), it then follows that \( y = 0 \) or \( z = 0 \).

So any embedding of \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \) is \( \text{Sp}(4, \mathbb{C}) \)-conjugate either to the embedding \( T \) (which is precisely \( \psi \)) or to \( T' \):

\[
T(H) = H_0 \quad T'(H) = H_0 \\
T(E) = E_0 \quad T'(E) = E_0 \\
T(F) = F_0 \quad T'(F) = F_0 \\
T(F_0) = H' \quad T'(F_0) = H'' \\
T(P_+) = 2E' \quad T'(P_+) = 2E'' \\
T(P_-) = -2F' \quad T'(P_-) = -2F''
\]
But it is easy to check that these two embeddings are conjugate by the following element of $Sp(4, \mathbb{C})$:

$$w = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (23)

i.e., for all $X \in e(3)$, $wT(X)w^{-1} = T'(X)$. We have just established the result. \hfill \Box

5. Representations of $e(3)$ from representations of $sp(4, \mathbb{C})$

With respect to the Chevalley basis of $sp(4, \mathbb{C})$ given in Eq. (4), we may explicitly embed $e(3)$ into $sp(4, \mathbb{C})$ as follows:

$$\phi : e(3) \hookrightarrow sp(4, \mathbb{C}), \quad E \rightarrow x_3, \quad H \rightarrow h_1 + 2h_2, \quad F \rightarrow y_3, \quad P_+ \rightarrow -2x_2, \quad P_0 \rightarrow y_1, \quad P_- \rightarrow 2y_4.$$  \hspace{1cm} (24)

Hence, any embedding of $e(3)$ into $sp(4, \mathbb{C})$ is equivalent to $\phi$ under conjugation by an element in $Sp(4, \mathbb{C})$. In particular, $\phi$ is equivalent to $\psi$ defined in Eq. (18). The following Lemma will be used to prove the indecomposability of $V(m_1, m_2)$ in Theorem 5.2 below. Its proof is omitted since it may be established in a straightforward manner by direct computation.

**Lemma 5.1** Let $y_1^a y_2^b y_3^c y_4^d v_\lambda$ be a nonzero element of $S_\lambda$ ($S_\lambda$ defined in Lemma 3.1). Then

$$H \cdot y_1^a y_2^b y_3^c y_4^d v_\lambda = (m_1 + 2m_2 - 2(b + c + d))y_1^a y_2^b y_3^c y_4^d v_\lambda,$$  \hspace{1cm} (25)

$$P_+ \cdot y_1^a y_2^b y_3^c y_4^d v_\lambda = -2b(m_2 - b + 1)y_1^a y_2^{b-1} y_3^{c+1} y_4 v_\lambda - 2c(c - 1)y_1^a y_2^b y_3^{c-2} y_4^{d+1} v_\lambda + 2cy_1^a y_2 y_3^{c-1} y_4^{d-1} v_\lambda.$$  \hspace{1cm} (26)

The above equation is interpreted in the sense that if $b = 0$, the first term on the right side is zero, if $c = 0$, the second and third terms on the right are zero, and if $c = 1$, the second term on the right is zero.

Let $b + c + d = m_2$, then

$$(P_+)^{m_2} \cdot y_1^a y_2^b y_3^c y_4^d v_\lambda = \begin{cases} (-2)^{m_2}(\Pi_{i=1}^{m_2} i(m_2 - i + 1)) y_1^a v_\lambda & \text{for } a \leq m_1, c = d = 0, \\ 0 & \text{for } a + c > m_1, d = 0 \text{ or } d > 0, \\ \alpha y_1^{a+c} v_\lambda & \text{for } d = 0, a + c \leq m_1, \end{cases} \hspace{1cm} (27)$$

where $\alpha$ is a (possibly zero) scalar.

**Theorem 5.2** The irreducible $sp(4, \mathbb{C})$-module $V(m_1, m_2)$ is $\mathfrak{e}(3)$-indecomposable.

**Proof:** Suppose $V(m_1, m_2) \cong M \oplus M'$. Upon restriction to the subalgebra $\langle E, H, F \rangle \cong sl(2, \mathbb{C})$, the representations $V(m_1, m_2)$, $M$ and $M'$ decompose into $H$-weight spaces. One of $M$ or $M'$ must contain an element

$$y_2^{m_2} v_\lambda + \sum_{a, b, c, d} \alpha(a, b, c, d) y_1^a y_2^b y_3^c y_4^d v_\lambda,$$  \hspace{1cm} (28)
for $y_1^a y_2^b y_3^c y_4^d v_\lambda \in S_\lambda$, scalars $a(b,c,d)$, and the sum over indices $(a,b,c,d)$ such that $b+c+d = m_2$, $(a,b,c,d) \neq (0,m_2,0,0)$. This element has $H$-weight equal to $m_1$. Without loss of generality, let this element belong to $M$. Considering Eqs. (27) and (28),

$$
(P_+^{m_2}) \cdot (y_2^{m_2} v_\lambda + \sum_{a,b,c,d} \alpha(a,b,c,d) y_1^a y_2^b y_3^c y_4^d v_\lambda) = (-2)^{m_2} (\Pi_{i=1}^{m_2} (m_2 - i + 1)) v_\lambda + \sum_{a,c} \beta(a,c) y_1^a y_2^c v_\lambda \in M,
$$

(29)

where $1 \leq a + c \leq m_1$ and $\beta(a,c)$ are scalars. One can easily show that $v_\lambda, y_1 v_\lambda, ..., y_1^{m_1} v_\lambda$ is a linearly independent set. For $m_2 > 0$, $(-2)^{m_2} (\Pi_{i=1}^{m_2} (m_2 - i + 1)) \neq 0$, so that the coefficient of $v_\lambda$ after the application of $(P_+)^{m_2}$ to Eq. (28) is nonzero. Recalling that $\phi(P_0) = y_1$, a straightforward application of $P_0$ to the right hand side of Eq. (29) then yields

$$
y_1^a v_\lambda \in M, \text{ for } 0 \leq a \leq m_1.
$$

(30)

We will now proceed by induction on the $H$-weight of the $H$-weight spaces of $V(m_1, m_2)$ to show that each $H$-weight space belongs to $M$.

The $H$-weight space $V(m_1, m_2)_{m_1 + 2m_2}$ has basis $v_\lambda, y_1 v_\lambda, y_1^2 v_\lambda, ..., y_1^{m_1} v_\lambda$. Eq. (35) implies that $V(m_1, m_2)_{m_1 + 2m_2}$ is the $H$-weight space of highest $H$-weight. Eq. (30) implies that $V(m_1, m_2)_{m_1 + 2m_2} \subseteq M$.

Let $w$, such that $-(m_1 + 2m_2) \leq w < m_1 + 2m_2$, be a fixed $H$-weight of $V(m_1, m_2)$, and suppose $V(m_1, m_2)_w \subseteq M$ for all $i > 0$ such that $w + 2i \leq m_1 + 2m_2$. Let the nonzero element $y_1^a y_2^b y_3^c y_4^d v_\lambda \in S_\lambda$ be such that $m_1 + 2m_2 - (b + c + d) = w$. We proceed in cases to show that $y_1^a y_2^b y_3^c y_4^d v_\lambda \in M$:

Case 1. $d > 0$: The $H$-weight of $y_1^a y_2^b y_3^c y_4^{d-1} v_\lambda$ is $w + 2$ and the vector is thus contained in $M$. Noting that $\phi(P_+) = 2y_4$ and $[y_4, y_i] = 0$, for $i = 1, 2, 3$, we have $P_+ \cdot y_1^a y_2^b y_3^c y_4^{d-1} v_\lambda = 2 y_1^a y_2^b y_3^c y_4^d v_\lambda \in M$.

Case 2. $d = 0$ and $c > 0$: The $H$-weight of $y_1^a y_2^b y_3^{c-1} v_\lambda$ is $w + 2$ and hence the vector is contained in $M$. Then, noting that $\phi(F) = y_3, [y_3, y_2] = 0$ and $\phi(P_0) = y_1, (P_0^b F) \cdot y_1^a y_2^{b} y_3^{c-1} v_\lambda = y_1^a y_2^b y_3^c v_\lambda \in M$.

Case 3. $d = 0$, $c = 0$ and $b > 0$: In this case $w = m_1 + 2m_2 - 2b$. As mentioned after Eq. (28), we have

$$
y_2^{m_2} v_\lambda + \sum_{a', b', c', d'} \alpha'(a', b', c', d') y_1^{a'} y_2^{b'} y_3^{c'} y_4^{d'} v_\lambda \in M,
$$

(31)

where the sum is over $a', b', c', d'$ such that $b' + c' + d' = m_2$ and $(a', b', c', d') \neq (0, m_2, 0, 0)$. From Eq. (26) we then have

$$
(P_+)^{m_2-b} \cdot (y_2^{m_2} v_\lambda + \sum_{a', b', c', d'} \alpha'(a', b', c', d') y_1^{a'} y_2^{b'} y_3^{c'} y_4^{d'} v_\lambda) = (-2)^{m_2-b}(\Pi_{i=1}^{m_2-b} (m_2 - i + 1)) y_2^b v_\lambda
$$

$$
+ \sum_{a'', b'', c'', d''} \alpha''(a'', b'', c'', d'') y_1^{a''} y_2^{b''} y_3^{c''} y_4^{d''} v_\lambda \in M,
$$

(32)

for scalars $\alpha''(a'', b'', c'', d'')$ where $b'' + c'' + d'' = b$ and $(a'', b'', c'', d'') \neq (0, b, 0, 0)$. For $m_2-b > 0$, $(-2)^{m_2-b}(\Pi_{i=1}^{m_2-b} (m_2 - i + 1)) \neq 0$, hence the coefficient of $y_2^b v_\lambda$ after the application of $(P_+)^{m_2-b}$
to Eq. (31) is nonzero. Each nonzero element \( y_1^a y_2^b y_3^c y_4^d v_\lambda \) in the sum above is contained in \( M \) by the above two cases, except for terms where \((a'', b'', c'', d'') = (a'', b, 0, 0), a'' > 0\). Hence,

\[
y_2^b v_\lambda + \sum_{a'' > 0} \beta(a'') y_1^{a''} y_2^b v_\lambda \in M,
\]

for scalars \( \beta(a'') \). Since \( V(m_1, m_2) \) is finite-dimensional and \( y_2^b v_\lambda \neq 0 \), there exists \( N \geq 0 \) such that \( y_1^N y_2^b v_\lambda \neq 0 \), but \( y_1^{N+1} y_2^b v_\lambda = 0 \). It is easily shown that the set \( \{y_2^b v_\lambda, y_1 y_2^b v_\lambda, ..., y_1^N y_2^b v_\lambda\} \) is linearly independent.

Recalling that \( \phi(P_0) = y_1 \), a straightforward application of \( P_0 \) to Eq. (33) yields

\[
y_2^b v_\lambda \in M.
\]

Finally we have

\[
P_0^a \cdot y_2^b v_\lambda = y_1^a y_2^b v_\lambda \in M.
\]

Hence, each \( H \)-weight space of \( V(m_1, m_2) \) is contained in \( M \) and thus \( V(m_1, m_2) \subseteq M \). Hence \( V(m_1, m_2) = M \) and \( M' = 0 \). Thus \( V(m_1, m_2) \) is indecomposable. \( \Box \)

An interesting consequence of the proof of Theorem 5.2 is that \( V(m_1, m_2) \) is generated by the single element \( y_2^{m_2} v_\lambda \) as an \( \mathfrak{e}(3) \)-module.

6. Conclusions

We have shown that the irreducible \( \mathfrak{sp}(4, \mathbb{C}) \)-modules \( V(m_1, m_2) \) remain indecomposable as \( \mathfrak{e}(3) \)-modules under the embedding described in Eq. (24), which we have shown is essentially the only embedding of \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \). Thus, we have created a new class of indecomposable \( \mathfrak{e}(3) \)-modules. An important and interesting consequence of this construction is that we can decompose the tensor product of representations within this class: The tensor product decomposition of indecomposable \( \mathfrak{e}(3) \)-modules in this class follows from the well-known tensor product decomposition of irreducible \( \mathfrak{sp}(4, \mathbb{C}) \)-modules. One illustrative example is the following 400-dimensional tensor product

\[
V(2, 0) \otimes V(1, 2) \cong \mathfrak{e}(3) \otimes V(3, 2) \oplus V(1, 3) \oplus V(3, 1) \oplus 2V(1, 2) \oplus V(3, 0) \oplus V(1, 1).
\]

The above decomposition was calculated with the assistance of the computer algebra system GAP [5].

The embedding of \( \mathfrak{e}(3) \) into \( \mathfrak{sp}(4, \mathbb{C}) \) does not correspond to an embedding of the group \( E(3) \) into \( Sp(4, \mathbb{C}) \). In fact, it corresponds to an embedding into \( Sp(4, \mathbb{C}) \) of the two-fold cover \( SU(2) \times \mathbb{R}^3 \) of \( E(3) = SO(3) \times \mathbb{R}^3 \). In the terminology of Douglas and Repka [4], it is a “spinor” embedding.

It has recently been pointed out to the authors that the techniques of this paper can be improved and generalized, an idea that will be pursued in future work.

Acknowledgments

The work of A.D. is partially supported by the Professional Staff Congress/City University of New York (PSC/CUNY). The work of J.R. is partially supported by the Natural Sciences and Engineering Research Council (NSERC)
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