Records in Fractal Stochastic Processes

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The records statistics in stationary and non-stationary fractal time series is studied extensively. By calculating various concepts in record dynamics, we find some interesting results. In stationary fractional Gaussian noises, we observe a universal behavior for the whole range of Hurst exponents. However, for non-stationary fractional Brownian motions the record dynamics is crucially dependent on the memory, which plays the role of a non-stationarity index, here. Indeed, the deviation from the results of the stationary case increases by increasing the Hurst exponent in fractional Brownian motions. We demonstrate that the memory governs the dynamics of the records as long as it causes non-stationarity in fractal stochastic processes, otherwise, it has no impact on the records statistics.

I. INTRODUCTION

Emergent extreme events are a key characteristic of complex dynamical systems, and the investigation of such events is a pivotal problem in understanding and predicting various complex systems [1–3]. Recently, the extreme event studies have been developed using various approaches, such as the density of states [3, 4], first-passage and return-time statistics [5, 6], persistence [7], interoccurrence time statistics [8, 9], and record statistics [10–12]. Among them, there has been considerable interest in investigating the record statistics, which has found many fruitful applications in diverse complex systems, such as spin glasses [13, 14], adaptive processes [15], domain wall dynamics [16], avalanche dynamics [17], stock prices [18, 19], global warming [20, 21], growing network [22], high-temperature superconductors [23], the ant movements dynamics [24], flood dynamics [25], sport statistics [26, 27], earthquakes [28, 29], and evolutionary biology [30–32].

A record is an entry in a series that is larger (or smaller) than all previous entries. From another point of view, records are extreme rare events related to an increasing (or decreasing) threshold. The investigation of the record statistics was started by the pioneering work of Chandler [33], which was based on independent and identically distributed (i.i.d.) stochastic time series. One of the most important findings of the record statistics in such random processes is their universal characteristics in the sense that they are completely independent of the underlying distribution [34]. To include more complicated systems, many research studies have been performed. For example, the record statistics of independent random variables from non-identical distributions [35], broadening distributions [36, 37], and time-dependent distributions (linear drift) [38, 39] are also well understood. However, many time series in real situations are correlated. In this respect, records in symmetric random walks has been well studied, recently [40], which triggers a series of new research studies, e.g., record statistics of biased random walks and Levy flights [21, 22], multiple independent random walks [21], and continuous-time random walks [40]. The main aim of such studies is to understand and model the statistics of records in observational data by comparing with various kinds of stochastic processes. Despite the striking significance and numerous applications of fractal processes in various areas of science, much less is known about their record statistics. In this regard, it is important to improve our understanding of the record dynamics in such a broader class of stochastic processes.

In order to model turbulent flows, fractional Brownian motion (fBm), a generalization of the more well-known Brownian motion, was introduced many decades ago [41, 42], and has become one of the most studied stochastic processes widely used in a variety of fields, including physics, probability, statistics, hydrology, economy, biology, and many others [43–49]. A fBm is a self-similar Gaussian process with stationary increments (called fractional Gaussian noise-fGn) and possesses long-range linear correlation which depends on a parameter, called the Hurst exponent, $H$ [50], where $0 < H < 1$. The case $H = 1/2$ corresponds to the ordinary Brownian motion in which successive increments are statistically independent of one another. For $H > 1/2$, the increments of the process are positively correlated (persistent), and for $H < 1/2$, consecutive increments are more likely to have opposite signs (anti-persistent).

The rest of this paper is organized as follows. In section II, we review and discuss some basic definitions and general findings in the record theory. In particular, we present important findings from two most studied time series: an i.i.d. process and a symmetric random walk. Next, by using extensive numerical analysis, we study the record statistics of non-stationary fractional Brownian motions as well as stationary fractional Gaussian noises in section III. We show that for stationary fractal series a robust universal behavior in the record dynamics is observed. We also demonstrate that the non-stationarity in fractal processes destroys such a universality observed in the stationary case. Finally, we conclude in section IV.
This means that the average record number
and defined as the sum over the record indicator
is another quantity of particular interest (see Fig. 1),

In practice, \( \bar{n} \) at time step \( n \) series. The number of records
after collecting and averaging many samples of the time

Hence, the ensemble averaging of the record indicator \( \bar{n} \)
to introduce a quantity called the record indicator, as

\[ X(t) \]

FIG. 1. A realization of a symmetric random walk \( X(t) \) with
n steps. Here the number of records is \( R_n = 5 \). The \( \tau_i \)'s
are the ages of the records and the \( \tau_i \)'s are the records
values.

II. GENERAL CONCEPTS IN RECORDS

In a series of random variables \( X_1, X_2, ..., X_n \), the data
point \( X_n \) is a record if \( X_n > \max\{X_0, X_1, ..., X_{n-1}\} \).

Fig. 1 represents the records evolution in a symmetric
random walk, in which filled red dots show the records.

One of the fundamental quantities in the records theory
is the record rate, \( P_n \), defined as the probability that
\( X_n \) is a record. To simplify the analysis, it is helpful
to introduce a quantity called the record indicator, as
follows:

\[ I_n = \begin{cases} 1, & \text{if } X_n \text{ is a record,} \\ 0, & \text{otherwise} \end{cases} \quad (1) \]

Then, in terms of the record indicator, we can write

\[ P_n = \text{Prob}[I_n = 1] \quad (2) \]

Hence, the ensemble averaging of the record indicator \( \bar{I}_n \)
at time step \( n \), reads,

\[ \bar{I}_n = P_n \times 1 + (1 - P_n) \times 0 = P_n \quad (3) \]

In practice, \( \bar{I}_n \) represents the value that one would obtain
after collecting and averaging many samples of the time
series. The number of records \( R_n \) that occur up to time
\( n \) is another quantity of particular interest (see Fig. 1),
and defined as the sum over the record indicator

\[ R_n = \sum_{i=1}^{n} I_i \quad (4) \]

This means that the average record number \( \bar{R}_n \) is just
the sum over the record rate \( P_n \),

\[ \bar{R}_n = \sum_{i=1}^{n} P_i \quad (5) \]

For i.i.d. random time series, due to the equal probability of the record occurrence at each time step, one finds that \( P_n = 1/n \), which is completely independent of the underlying distribution of the series [14, 15]. By substituting this into Eq. 5 one obtains for the average record number \( \bar{R}_n \) estimates \( R_n \) and \( \gamma \), where \( \gamma \approx 0.577215 \ldots \) is
the Euler-Mascheroni constant [16]. The mean number
of records is an increasing (but extremely slow) function
of time, due to the fact that the probability of the record
occurrence decays with time. Also, the distribution of
the record number, \( P(R_n) \), over a large number of re-
alizations of i.i.d. series approaches a Gaussian function,
with mean and variance of \( \ln n \) [17], as follows:

\[ P(R_n) = \frac{1}{\sqrt{2\pi \ln n}} e^{-\frac{(R_n - 1)^2}{2 \ln n}} \quad (6) \]

Similar universal properties have been found for the
record statistics of symmetric random walks, by Majum-
dar and Ziff [18]. They showed that for a symmetric
random walk \( R_n \) is approximately \( \sqrt{4n/\pi} \) and \( P_n \) is approximately \( 1/\sqrt{\pi n} \) as \( n \to \infty \).

As it can be seen, the record rate \( P_n \) of symmetric ran-
dom walks decays much slower than that of i.i.d. random
series. They also computed the record number distribution,
and showed that it approaches a half-Gaussian form of

\[ P(R_n) = \frac{1}{\sqrt{\pi n}} e^{-\frac{R_n^2}{\pi n}} \quad (7) \]

where the most probable value of \( R_n \) is zero. We note
that the standard deviation of \( R_n \) is large \( (\sigma R_n \sim \sqrt{n}) \)
for large \( n \), in contrast to the i.i.d. case which is small
\( (\sigma R_n \sim \sqrt{\ln n}) \) compared to the mean. This indicates
that the record number distribution of a (correlated) symmetric random walk is significantly broader than that
of an (uncorrelated) i.i.d. time series.

In order to search for the time characteristics of the
record statistics, one may take into account the record
ages. Let \( \tau = [\tau_1, \tau_2, ..., \tau_N] \) denote the time intervals
between successive records. On the other word, \( \tau_i \) is the
age of the \( i \)th record; i.e., it denotes the time up to which
the \( i \)th record survives (see Fig. 1). Note that the last
record, i.e., the \( R_n \)th record, still stays a record at the
\( n \)th step since there are no more record breaking events
after. The typical age of a record can be estimated as
\( \tau_{typ} = n/R_n \), which behaves as \( n/\ln n \) and \( \sqrt{\ln n}/4 \) for
i.i.d. random walk time series, respectively [18].

However, there are rare records with completely differ-
ent age behavior. For example, one may consider the
longest age \( \tau_{max} = \max[\tau_1, \tau_2, ..., \tau_N] \) of the records. In
particular, the asymptotic large \( n \) behavior of the average
of \( \tau_{max} \) can be expressed explicitly as \( \bar{\tau}_{max} \approx cn \),
with \( c \approx 0.624330 \) and 0.626508 for i.i.d. and symmetric
random walk series, respectively [18]. This shows that the
longest age of the records is much larger than the
typical one.

Further, one may ask wether the record occurrences
are correlated. To quantify such a correlation, we can

\[ P(R_n) = \frac{1}{\sqrt{2\pi \ln n}} e^{-\frac{(R_n - 1)^2}{2 \ln n}} \quad (6) \]

\[ P(R_n) = \frac{1}{\sqrt{\pi n}} e^{-\frac{R_n^2}{\pi n}} \quad (7) \]
take into account the joint probability \( P_{n,m} = \text{Prob}[I_n = 1 \text{ and } I_m = 1] \) of the occurrence of two records at time steps \( n \) and \( m \). It was shown [14] that for i.i.d. time series, the individual record events are uncorrelated, and our knowledge about the present record tells us nothing about the future one. This means that the record events of \( n \) and \( m \) are completely independent if \( m \neq n \), and the joint probability becomes \( P_{n,m} = P_n P_m \). In the language of probability theory, for the conditional probability we have \( P_{n|m} = P_{n,m}/P_m \), which defines the occurrence probability of a record at time step \( n \), given that a record has occurred in time step \( m \). Thus, in the case of i.i.d. processes, \( P_{n|m} \) is simply equal to \( P_n \). For a symmetric random walk, records are not independent, and the conditional probability of two successive records is given by \( P_{n+1|n} = 1/2 \) for large \( n \) [37]. This shows that the record events in symmetric random walks effectively attract each other, i.e., there is an increased probability for the occurrence of a record if another record has occurred in the previous time step [37].

### III. Records in Fractals

In this section, we present an extensive numerical analysis of the record statistics for fractal time series with various Hurst exponents. We discuss the effect of correlation, as well as stationarity/non-stationarity on various aspects of the record dynamics. In order to investigate the record statistics of fractional processes, we construct such time series through a generic \( 1/f \) noise with Fourier filtering method [52]. Then, we search for various record features of fractional time series of size up to \( N = 10^8 \). Also, all calculations have been performed by averaging over \( 10^6 \) different realizations.

#### A. Record rate

At first, we construct series of record events, corresponding to fractional Brownian motions and fractional Gaussian noises with different Hurst exponents, and search for various statistics. In Fig. 2(a), we plotted in logarithmic scale, the record rates \( P_n \) for fBm series with different Hurst exponents of 0.1, 0.3, 0.5, 0.7, and 0.9. We fit power-law functions (dashed lines) of the form of \( P_n \sim n^{-\delta} \) to the record rates, and find that \( \delta = 1 - H \). Majumdar and Ziff showed [39] that the probability \( P_n \) for a record in the \( n \)th event of a symmetric random walk is the same as the corresponding persistence probability \( Q_n = \text{Prob}[X_1, X_2, ..., X_n < 0] \), which is the probability that a random walker starting from the origin stays below the origin without crossing it for the next \( n \) steps, and given by \( Q_n = P_n = 1/\sqrt{2n} \) as \( n \to \infty \). On the other hand, it was shown [52] that the persistence probability for a fBm series is given by \( Q_n \sim n^{H-1} \). Here, we find numerically such a power-law behavior for the record rate of fractional Brownian motions.

![FIG. 2. The record rates \( P_n \) for (a) fBm, and (b) fGn series with five different Hurst exponents. The dashed lines represent the fitted power-law functions, discussed in section III A.](image)

We also search for the record rates corresponding to fractional Gaussian noises. Fig. 2(b), shows log-log plots of the record rates, \( P_n \), for different Hurst exponents. Interestingly, all the curves collapse into a universal power-law function of the form of \( P_n \sim 1/n \) (dashed line) for the whole range of Hurst exponents. This result is the same as the one obtained for an i.i.d. random series. Indeed, we find that the record rate in stationary fractional Gaussian noises is independent of the underlying memory in the time series. We will discuss this finding in more details in section III D by searching for correlations between the records.

#### B. Record number

As we mentioned above, the record number is an important quantity in the record theory. Fig. 3(a) demonstrates the average record number \( R_n \) versus time for fBm series with five different Hurst exponents. We find a power-law behavior (dashed lines) of the form of \( R_n \sim n^{H} \). However, a completely different behavior is observed for fGn series. Fig. 3(b) shows semi-logarithmic
and with three different Hurst exponents of 0.0, 0.5, and 10. The inset in (b) shows the fitting parameter $c$ in $\ln n \sim \ln n + c$ for different Hurst exponents. Also, the dashed line in (b) shows a $\ln n$ function, for comparison.

To find out more about the nature of the record number fluctuations, it is helpful to take into account the record number distributions $P(R_n)$. In Fig. 5(a), we plotted the normalized probability distributions of the record number for fBm series of sizes $n = 10^3$, $10^4$, and $10^5$, and with three different Hurst exponents of 0.3, 0.5, and 0.7. Due to better visibility, the curves in Fig. 5(a) are shifted vertically by a factor of 0.01 and 100 for $H = 0.3$ and $H = 0.7$, respectively. The dashed lines show fitted functions of the form

$$y \sim x^\alpha e^{-\frac{x^2}{2}}$$

where $x = R_n/\sigma_{R_n}$, $y = \sigma_{R_n} P(R_n)$, and $\alpha$ is positive/zero/negative for Hurst exponents smaller than/equal to/larger than 0.5. We observe that $P(R_n)$ approaches an asymmetric Gaussian form, as $H \rightarrow 0$, and tends to a power-law functional form, as $H \rightarrow 1$. Therefore, in non-stationary fractal processes, increasing $H$ broadens the record number distribution. Note that for $H = 0.5$, we find the half-Gaussian distribution of Eq. 7 as expected.

Fig. 5(b) shows the record number distributions for stationary fGn series, with different sizes and Hurst exponents. The curves are normalized to zero mean and unit standard deviation. As it can be seen, all curves
collapse into a universal Gaussian function of the form of

\[ y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \]  

(9)

where \( x = (R_n - \bar{R}_n)/\sigma_{R_n} \) and \( y = \sigma_{R_n} P(R_n) \). This indicates that the record number distribution of stationary fractal time series is the same as that of an i.i.d random series of Eq. 8 and is completely independent of the memory aspects of the original process. We note here that for non-stationary fractal series, the width of the record number distribution increases as the Hurst exponent increases, in contrast with the case of stationary fGn series. This indicates that for high values of the Hurst exponent in an fBm series, the record parameters can largely fluctuate about their mean values, and thus we should be careful when we talk about such average values.

We are also interested in questions like “how long should one wait for the occurrence of the next record?”. In this respect, we calculate the age distribution \( P(\tau) \) of the records for both fGn and fBm series with various Hurst exponents. Fig. 6(a) represents such distributions for fBm series with three Hurst exponents of 0.2, 0.5, and 0.8. The dashed lines in both figures represent the corresponding fitted power-law functions, discussed in section III C.

\[ P(\tau) \sim \tau^{-\beta} \]  

(10)

with \( \beta = 1 + H \). In the case of fGn series, we find again a universal power-law behavior of the form of \( P(\tau) \sim \tau^{-1} \) (dashed line in Fig. 6(b)) for different Hurst exponents. It is interesting to note here that any questions about the average of the record age does not make sense, since due to the power-law behavior of the age distribution with exponents between \(-2\) and \(-1\), \( \tau = \int \tau P(\tau)d\tau \) diverges as \( \tau \to \infty \).

C. Record ages

FIG. 5. The normalized record number distributions, \( P(R_n) \), for (a) fBm, and (b) fGn series with three different Hurst exponents of 0.3, 0.5, and 0.7 and three different sizes. In (a), the plots for \( H = 0.3 \) and \( H = 0.7 \) are shifted vertically by a factor of 0.01 and 100 for better visibility. In (b) all curves with different Hurst exponents collapse into a universal Gaussian function. The dashed lines in both figures represent the corresponding fitted functions, discussed in section III B.

FIG. 6. The age distributions \( P(\tau) \) of the records for (a) fBm, and (b) fGn series of size \( n = 10^5 \) with three different Hurst exponents of 0.2, 0.5, and 0.8. The dashed lines in both figures represent the corresponding fitted power-law functions, discussed in section III C.
D. Correlation in records

At the end, we are interested in correlations between the individual record events. To quantify the correlation properties between two records at time steps \( n \) and \( m \), we introduce a correlation index, \( C_{n,m} \), as follows:

\[
C_{n,m} = \frac{\bar{T}_n \bar{T}_m - \bar{I}_n \bar{I}_m}{\sigma_{\bar{T}} \sigma_{\bar{I}}} \tag{11}
\]

where \( \sigma_{\bar{T}}^2 = \bar{T}^2 - \bar{T} \). Due to the binary property of the record indicator, \( \bar{T}^2 = \bar{I} \), and one simply gets \( \sigma_{\bar{T}}^2 = \bar{T} \). On the other hand, the joint probability \( P_{n,m} \) of the occurrence of two record events can be written, in terms of the record indicator \( I_n \), as

\[
P_{n,m} = \text{Prob}\{I_n = 1 \text{ and } I_m = 1\} = \bar{I}_n \bar{I}_m \tag{12}
\]

Then, by using Eq. (3), we have

\[
C_{n,m} = \frac{P_{n,m} - \bar{P}_n \bar{P}_m}{\sqrt{P_n - \bar{P}_n^2} \sqrt{P_m - \bar{P}_m^2}} \tag{13}
\]

Hence, for an independent sequence with \( P_{n,m} = P_n P_m \), we obtain \( C_{n,m} = 0 \). From now on, we concentrate our attention only on the correlation between two successive points, i.e., \( C_{n+1,n} \). According to the power-law form of the record rate, we get \( P_{n+1}/P_n = (1 + 1/n)^{-\delta} \). Thus, one can assume that \( P_{n+1} \approx P_n \) for \( n \gg 1 \). By using the relationship \( P_{n+1,n} = P_{n+1}P_n \), we have

\[
C_{n+1,n} \approx \frac{P_{n+1}P_n - P_n^2}{P_n - P_n^2} \tag{14}
\]

For \( n \gg 1 \), we can also neglect \( P_n^2 \) in comparison with \( P_n \), and then finally we get \( C_{n+1,n} \approx P_{n+1|n} \). We know that for a fBm process, the conditional probability \( P_{n+1|n} \) equals to the probability of the occurrence of a positive increment, and equals to \( H \). Thus, we find \( C_{n+1,n} \approx H \) which is independent of \( n \). To check this, we plotted in Fig. 7(a) the time behavior of the correlation index, \( C_{n+1,n} \), for fractal time series with different Hurst exponents. We observe that for fBm series, \( C_{n+1,n} \approx H \), and is approximately independent of time for \( n \gg 1 \), as expected. Also, for stationary fGn series, we find \( C_{n+1,n} \approx 0 \), independent of the Hurst exponent. This indicates that the record events in stationary fractal time series are uncorrelated for large \( n \). For better understanding, we also plotted in Fig. 7(b), the correlation index \( C_{n+1,n} \) at a fixed time step \( n = 5000 \), for various Hurst exponents. As it can be seen, we find a linear behavior of \( C_{n+1,n} \) versus \( H \) for fBm series, in perfect agreement with our analytical result. Interestingly, for fGn series with various Hurst exponents, the record dynamics is uncorrelated and independent of the Hurst exponent. We find an important result: the record statistics of stationary fractal time series are uncorrelated, even with the presence of long-range linear correlations in the original process. In fact, this feature influences all aspects of the record dynamics, causing the observed universal behavior in the records statistics, mentioned above.

IV. CONCLUSION

In conclusion, we investigated the record statistics of stationary and non-stationary fractal time series, i.e., fractional Gaussian noises and fractional Brownian motions, respectively. By doing extensive numerical calculations, we found a power-law functional form for many record features such as the record rates and the age distributions. We have also investigated other dynamical characteristics of the records, such as correlation between the record events, and found that the record dynamics are uncorrelated in stationary fractal time series, in spite of the existence of long-range linear correlation in the original series. Further, we have shown that for stationary fractal time series, a universal behavior is observed in all aspects of the record dynamics, independent of the memory inherited in the process. On the other hand, for non-stationary fractal processes, the record dynamics...
strongly depends on the Hurst exponent. This deviation from the results of the stationary case is increased by increasing the Hurst exponent, which can be considered as the degree of non-stationarity in such fractal series. Finally, we have demonstrated that the memory in fractal time series can affect the record dynamics, only if the existing correlation can produce non-stationarity in such fractal processes.

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