Darcy-Brinkman Ferro convection with temperature dependent viscosity

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Abstract. The method of small perturbation is used to examine the variable viscosity effect on Darcy-Brinkman ferroconvection. Taking the viscosity to be temperature dependent and assuming the fluid and solid matrix to be in local thermal equilibrium, the eigenvalues of the stationary instability are computed using the higher order Galerkin method. The study reveals that viscosity variation effect is to hasten the threshold of porous medium ferroconvection and its destabilizing effect is enhanced when the magnetic mechanism is effective. Further the convection cell size at the onset of ferroconvection is significantly affected by the variable viscosity effect. The impact of porous and magnetic parameters on the stability is also discussed.

1. Introduction

The Rayleigh-Bénard instability with variable viscosity has been paid ample attention thanks to its connection in engineering applications of heat transfer (Platten and Legros [1], Gebhart et al [2]). The variable viscosity effect on convection, making use of exponential forms and truncated series expansions, has been dealt with by several researchers. Palm [3] pointed out the formation of steady hexagonal cells at the convection onset when the variable viscosity effect was taken into account.

Torrance and Turcotte [4] analyzed the influence of large viscosity variations on convective instability of Rayleigh-Bénard type. Horne and Sullivan [5] examined the variable viscosity effect on natural convection of water with permeable formations. They perceived that the convective motion becomes unstable at values of Rayleigh number that are apparently moderate. White [6] studied convective instability with temperature-dependent viscosity and made clear that stable hexagonal and square patterns appear with an increase in viscosity ratio. Chakraborty and Borkakati [7] analyzed consequences of viscosity variation in an electrically conducting fluid. Based on energy inequalities and assuming a linear relation for the viscosity variation, Saravananand Brindha [8] derived sufficient conditions applicable to convective motion driven by applied pressure gradient and volumetric heat sources.

Ferrofluids are essentially colloidal suspensions comprising surfactant-coated submicron sized magnetic particles in a liquid medium. They exhibit a variety of unusual properties and can be controlled by magnetic field gradients. Motivated by the fact that the existence of magnetic forces change the critical values associated with the natural convection, several researchers investigated the problem of magnetic fluid convective instability of Rayleigh-Bénard type (Finlayson...
Ramanathan and Muchikel [15] analyzed the problem of porous medium ferroconvection with viscosity varying with temperature. It is clarified that oscillatory instability is ruled out and only stationary mode of instability is possible. They further showed that variable viscosity effect boosts the onset of porous medium ferroconvection. Assuming a viscosity varying exponentially with temperature, Nanjundappa et al [16] studied the Marangoni-Bénardferroconvective instability problem. They made it clear that the variable viscosity plays a larger role on the stability characteristics. Very recently Prakash [17] examined the effects of both MFD viscosity and non-uniformity in the basic temperature profiles on thermomagnetic convection in ferrofluids.

While the effect of viscosity variation with respect to temperature fluctuations on the threshold of convective instability has been studied by means of exponential and reduced Taylor series forms, we have not come across any investigation concerning ferromagnetic porous medium instability with an inverse linear relation for viscosity variation. In this paper we therefore study the problem of ferroconvective porous medium instability taking into account an inverse linear relation for the viscosity varying with temperature. Considering more realistic boundary conditions, the Galerkin technique is used to figure out the critical values of porous medium thermomagnetic instability.

**Figure 1.** Schematic of the problem.

2. **Mathematical Formulation**

A horizontal porous layer of ferromagnetic fluid of thickness $d$ with the lower and upper boundaries having uniform temperatures is considered. The temperatures of the lower and upper boundaries are $T_1$ and $T_o$ (with $T_1 > T_o$) respectively. A uniform magnetic field $\vec{H}_o$ is applied in the direction of the z-axis (see figure 1).

The governing equations describing flow in an incompressible, non-conducting magnetic fluid saturated porous layer are (Finlayson [9] and Maruthamanikandan [18])

$$\nabla \cdot \vec{q} = 0, \quad (1)$$

$$\rho_o \left[ \frac{1}{e} \frac{\partial \vec{q}}{\partial t} + \frac{1}{e^2} (\vec{q} \cdot \nabla) \vec{q} \right] = -\nabla p + \rho \vec{g} - \frac{\mu_t}{k} \vec{q} + \nabla \cdot (\vec{H} \vec{B}) + \nabla \cdot \left[ \vec{\mu} \left( \nabla \vec{q} + \nabla \vec{q}^T \right) \right], \quad (2)$$
\begin{equation}
\varepsilon \left[ \rho_o C_{v,H} - \mu_q \vec{H} \cdot \left( \frac{\partial \vec{M}}{\partial t} \right)_{v,H} \right] \left[ \frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T \right] \cdot \left[ \frac{\partial H}{\partial t} + (\vec{q} \cdot \nabla)\vec{H} \right] = k_1 \nabla^2 T, \tag{3}
\end{equation}

\begin{equation}
\rho = \rho_o \left[ 1 - \alpha (T - T_a) \right], \tag{4}
\end{equation}

\begin{equation}
\vec{M} = \frac{\vec{H}}{H} M(H,T), \tag{5}
\end{equation}

\begin{equation}
M = M_o + \chi (H - H_o) - K(T - T_a), \tag{6}
\end{equation}

where \( \vec{q} = (u,v,w) \) is the fluid velocity, \( \rho_o \) is a reference density, \( \varepsilon \) is the porosity, \( t \) is the time, \( p \) is the pressure, \( \ddot{g} \) is the acceleration due to gravity, \( \rho \) is the fluid density, \( \mu_t \) is the dynamic viscosity, \( \bar{\mu}_t \) is the effective viscosity, \( \kappa \) is the permeability of the porous medium, \( \vec{H} \) is the magnetic field, \( \vec{B} \) is the magnetic induction, \( T \) is the temperature, \( \mu_o \) is the magnetic permeability, \( \vec{M} \) is the magnetization, \( k_1 \) is the thermal conductivity, \( \alpha \) is the thermal expansion coefficient, \( T_a \) is the arithmetic mean of temperatures at the boundaries, \( \nabla \) is the vector differential operator, \( C_{v,H} \) is the specific heat at constant volume and magnetic field, \( M_o \) is the reference magnetization, \( \chi \) is the magnetic susceptibility, \( K \) is the pyromagnetic coefficient and \( Tr \) denotes the transpose.

The relevant Maxwell equations are

\begin{equation}
\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \vec{0}, \quad \vec{B} = \mu_o \left( \vec{H} + \vec{M} \right). \tag{7}
\end{equation}

The fluid viscosity is taken to be temperature-dependent in the following forms

\begin{equation}
\mu_t (T) = \frac{\mu_1}{1 + \delta(T - T_a)}, \quad \bar{\mu}_t (T) = \frac{\mu_2}{1 + \delta(T - T_a)}, \tag{8}
\end{equation}

where \( \mu_1 \) and \( \mu_2 \) are values of \( \mu_t \) and \( \bar{\mu}_t \) at \( T = T_a \) and \( 0 < \delta < 1 \). Equations characterising the basic state are introduced in the form (with subscript \( b \) representing the basic state quantities)

\begin{equation}
\vec{q} = \vec{q}_b = (0,0,0), \quad \rho = \rho_b(z), \quad T = T_b(z), \quad p = p_b(z), \quad \mu_f = \mu_{fb}(z), \quad \vec{H} = \vec{H}_b = (0,0,H_b(z)), \quad \vec{M} = \vec{M}_b = (0,0,M_b(z)). \tag{9}
\end{equation}

The solution pertaining to the basic state reads

\begin{equation}
\rho_b = \rho_o \left[ 1 - \alpha \beta z \right], \quad \vec{H}_b = \left[ H_o - \frac{K \beta z}{1 + \chi} \right] \hat{k}, \quad \vec{M}_b = \left[ M_o + \frac{K \beta z}{1 + \chi} \right] \hat{k}, \quad \vec{B}_b = \mu_o \left( \vec{H}_o + \vec{M}_o \right) \hat{k}, \quad \mu_{fb}(T) = \frac{\mu_1}{1 - \delta \beta z}, \quad \bar{\mu}_{fb}(T) = \frac{\mu_2}{1 - \delta \beta z}, \tag{10}
\end{equation}

where \( \beta = \frac{T_1 - T_0}{d} \). The perturbed state equations involving infinitesimally small perturbations are
\[ \bar{q} = \bar{q}_b + \bar{q}' = (u', v', w'), \quad p = p_b + p', \quad \rho = \rho_b + \rho', \quad T = T_b + T', \quad \mu_f = \mu_{f_b} + \mu_{f_b}', \quad \bar{m} = M + M', \quad \phi = \phi_b + \phi', \quad H = H_b + H'. \]  

(11)

where the perturbed quantities are indicated by the primes. The linearized equations governing small perturbations therefore take the form

\[ \rho_0 \frac{\partial}{\partial t} \left( \nabla^2 w' \right) = \alpha \rho_0 \sigma \nabla^2 T' - \frac{\mu_k}{k} \nabla^2 w' + \frac{\mu_k}{1 + \chi} \nabla^2 T' - \frac{\mu_k}{1 + \chi} \sigma \nabla^2 \phi' \]

\[ - \frac{\partial^2 \mu_k}{\partial \zeta^2} \left[ \nabla^2 w' - \frac{\partial^2 w'}{\partial \zeta^2} \right] + 2 \frac{\partial \mu_k}{\partial \zeta} \nabla^2 w' + \bar{m} \nabla^4 w', \]

(12)

\[ \left( \rho_0 C \right)_1 \frac{\partial T'}{\partial t} - \mu_k KT_0 \nabla^2 (\nabla^2 w') = k_1 \nabla^2 T' + \left[ \left( \rho_0 C \right)_2 - \frac{\mu_k^2 T_a}{1 + \chi} \right] \beta w', \]

(13)

\[ \left( 1 + \frac{M_0}{H_0} \right) \nabla^2 \phi' + (1 + \chi) \frac{\partial^2 \phi'}{\partial \zeta^2} - K \frac{\partial^2 \phi'}{\partial \zeta^2} = 0, \]

(14)

where

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \left( \rho_0 C \right)_1 = \epsilon \rho_0 C_{V,H} + \epsilon \mu_0 K H_0 + (1 - \epsilon)(\rho_0 C)_s, \quad \left( \rho_0 C \right)_2 = \epsilon \rho_0 C_{V,H} + \epsilon \mu_0 K H_0. \]

The normal mode solution is adopted and the same has the form

\[ \begin{bmatrix} w' \\ T' \\ \phi' \end{bmatrix} = \begin{bmatrix} \Theta(z) \\ \phi(z) \end{bmatrix} \exp \left[ i(lx + my) + \sigma t \right], \]

(15)

where \( l \) and \( m \) are respectively the wavenumbers in \( x \) and \( y \) directions and \( \sigma \) is the growth rate. Substitution of equation (15) into equations (12) – (14) leads to

\[ \rho_0 \sigma \left( D^2 - k_h^2 \right) W = - \alpha \rho_0 \sigma \nabla^2 \Theta - \frac{\mu_k}{k(1 - \delta \beta \zeta)} \left( D^2 - k_h^2 \right) W - \frac{\mu_k}{1 + \chi} \nabla^2 \Theta + \mu_0 K \beta k^2_D \Phi \]

\[ + \frac{2\mu_0 \delta^2 \beta^2}{(1 - \delta \beta \zeta)} \left( D^2 + k^2_h \right) W + \frac{2\mu_0 \delta \beta}{(1 - \delta \beta \zeta)} \left( D^2 - k^2_h \right) DW + \frac{\mu_2}{(1 - \delta \beta \zeta)} \left( D^2 - k^2_h \right) W, \]

(16)

\[ \left( \rho_0 C \right)_1 \sigma \Theta - \mu_0 KT_0 \sigma \Phi = k_h \left( D^2 - k_h^2 \right) \Theta + \left[ \left( \rho_0 C \right)_2 - \frac{\mu_k^2 T_a}{1 + \chi} \right] \beta W, \]

(17)

\[ \left( 1 + \chi \right) D^2 \Phi - \left( 1 + \frac{M_0}{H_0} \right) k^2_h \Phi - KD \Theta = 0, \]

(18)

where \( D = \frac{d}{dz} \) and \( k^2_h = l^2 + m^2 \) is the overall horizontal wavenumber. Non-dimensionalizing equations (16) – (18) using the scaling
\[
W^* = \frac{W_d}{\kappa}, \quad \Phi^* = \frac{\Phi(1+\chi)}{K\beta d^2}, \quad \sigma^* = \frac{\sigma d^2}{\kappa}, \quad \Theta^* = \frac{\Theta}{\beta d}, \quad z^* = \frac{z}{d}, \quad a = k_d d, \quad \{19\}
\]

we obtain
\[
\frac{\sigma}{Pr}(D^2 - a^2)W = -(R + N)a^2\Theta + Na^2D\Phi - Da^{-1}g(z)\left(D^2 - a^2\right)W + 2AV^2g^3(z)\left(D^2 + a^2\right)W
\]
\[
+ 2AVg^2(z)\left(D^2 - a^2\right)DW + Ag(z)\left(D^2 - a^2\right)^2 W,
\]
\[
\lambda \sigma \Theta = \left(D^2 - a^2\right)\Theta + W, \quad \{20\}
\]
\[
D^2\Phi - M_3a^2\Phi - D\Theta = 0, \quad \{21\}
\]
where \(Pr = \frac{\mu_\lambda}{\rho_\sigma \kappa}\) is Prandtl number, \(Da^{-1} = \frac{d^2}{k}\) is inverse Darcy number, \(A = \frac{\mu_\lambda}{\mu_\kappa}\) is Brinkman number, \(R = \frac{\alpha \rho_\sigma g \beta d^4}{\mu_\kappa}\) is the Rayleigh number, \(N = \frac{\mu_\sigma K^2 \beta^2 d^4}{\mu_\kappa (1+\chi) \kappa}\) is magnetic Rayleigh number, \(V = \delta \beta d\) is the variable viscosity parameter and \(g(z) = (1 - Vz)^{-1}\).

The appropriate boundary conditions are (Finlayson [9])
\[
W = DW = \Theta = 0 \quad \text{at} \quad z = \pm 1/2, \quad \{23\}
\]
\[
D\Phi + \frac{a\Phi}{1+\chi} = 0 \quad \text{at} \quad z = 1/2, \quad D\Phi - \frac{a\Phi}{1+\chi} = 0 \quad \text{at} \quad z = -1/2 . \quad \{23\}
\]

Since the occurrence of oscillatory instability is ruled out (Maruthamanikandan [18]), the stability equations for stationary instability (with \(\sigma = 0\)) are thus given by
\[
Ag(z)\left(D^2 - a^2\right)^2 W + 2AVg^2(z)\left(D^2 - a^2\right)DW + 2AV^2g^3(z)\left(D^2 + a^2\right)W
\]
\[
- Da^{-1}g(z)\left(D^2 - a^2\right)W - (R + N)a^2\Theta + Na^2D\Phi = 0, \quad \{24\}
\]
\[
\left(D^2 - a^2\right)\Theta + W = 0, \quad \{25\}
\]
\[
\left[D^2 - M_3a^2\right]\Phi - D\Theta = 0. \quad \{26\}
\]

### 3. Method of Solution
Since the system comprising equations (24) – (26) has space varying coefficients, an approximate solution of the eigenvalue problem can be obtained by resorting to the Galerkin method (Finlayson [19]). The trial functions \(W_i = \left(z^2 - \frac{1}{4}\right)^{i+1}\), \(\Theta_i = \left(z^2 - \frac{1}{4}\right)^i\) and \(\Phi_i = z^{2i-1}\) are employed in the computations of critical values.
4. Results and Discussion
The variable viscosity effect on Darcy-Brinkman ferroconvection is investigated. Realistic hydrodynamic boundary conditions and general magnetic boundary conditions are considered. An inverse linear relationship is considered for the viscosity variation with temperature. The local thermal equilibrium condition for the fluid and solid matrix is assumed. The critical values associated with stationary instability are computed by the higher order Galerkin method. The results of the study are indicated by means of figures 2 –6 and table 1. The thermal Rayleigh number $R$ turned out to be a function of both magnetic and non-magnetic parameters. The change in critical Rayleigh number $R_c$ with magnetic Rayleigh number $N$, Brinkman number $A$, inverse Darcy number $Da^{-1}$, variable viscosity parameter $V$, non-buoyancy-magnetization parameter $M_3$ and magnetic susceptibility $\chi$ is exhibited in figures 2 –6.

![Figure 2. $R_c$ variation with $N$ and $A$.](image1)

![Figure 3. $R_c$ variation with $N$ and $Da^{-1}$.](image2)

The influence of porous parameters $A$ and $Da^{-1}$ on the stability is portrayed in figures 2 and 3. As can be seen, an increase in both $Da^{-1}$ and $A$ results in postponement of porous medium ferroconvection. This is due to the reduction in the permeability of the porous media following the increase in $Da^{-1}$. Besides, increasing the porous parameter $A$ boosts the viscous effect, which is responsible for slowing down convective instability. It is remarkable to note that the system is destabilized more slowly by virtue of the magnetic mechanism when both the porous parameters $Da^{-1}$ and $A$ are large enough.
We see from figure 4 that the parameter $V$ designating variable viscosity effect is to advance the threshold of ferromagnetic convection. Further, the destabilizing effect of $V$ is heightened when the magnetic Rayleigh number $N$ is large. In figure 5, the deviation in $R_c$ with respect to $N$ and $M_3$ is exhibited. The parameter $M_3$ signifies the shift towards nonlinearity associated with the magnetic equation of state. The critical number $R_c$ decreases monotonically when $M_3$ is increased thereby favouring porous medium ferroconvective instability.

It is comprehended from figure 6 that an increase in the magnetic susceptibility $\chi$ gives rise to the same trend for $R_c$ implying the constraining nature of $\chi$ albeit its stabilizing influence is negligibly small.

| $N$ | $V = 0$ | $V = 0.5$ | $V = 1$ |
|-----|---------|-----------|---------|
| 0   | 3.145   | 3.116     | 3.019   |

Table 1. Dependence of $a_c$ with $N$ and $V$. 

Figure 4. $R_c$ variation with $N$ and $V$.

Figure 5. $R_c$ variation with $N$ and $M_3$.

Figure 6. $R_c$ variation with $N$ and $\chi$. 
Computations also reveal that the convection cell size is more sensitive to the porous parameters $A$ and $Da^{-1}$, and the opposite is true for the magnetic parameters $M_3$ and $\chi$. Further, it is evident from table 1 that the variable viscosity parameter $V$ tends to amplify the convection cell size at the threshold of ferroconvection. Furthermore, in the limit of $N = Da^{-1} = V = 0$ and $A = 1$, one is able to obtain the classical values of the critical numbers $a_c = 3.117$ and $R_c = 1707.76$ (Chandrasekhar [20]).

5. Conclusions
Ferromagnetic porous medium convection with variable viscosity effect when the viscosity varying with temperature and obeying an inverse linear relationship is investigated. The investigation has led to the following conclusions:

- The onset of ferroconvection is advanced due to the presence of variable viscosity.
- The destabilizing nature of variable viscosity becomes more distinct when the magnetic mechanism is more effective.
- The size of convection cell is more vulnerable to the constraints of porous medium and variable viscosity.

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