Gauge invariance of elementary particle processes taking place in presence of a background magnetic field

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Abstract

Elementary particle scatterings and decays in presence of a background magnetic field are very common in physics, specially after the observation that the core of the neutron stars can sustain a magnetic field of the order of $10^{13}$ G. The important point about these calculations is that they are done in a background of a gauge field and as a result the calculations are prone to gauge arbitrariness. In this work we will investigate how this gauge arbitrariness is eradicated in processes where the initial and final particles taking part in the interactions are electrically neutral. Some comments on those processes where the initial or final state consists of electrically charged particles is presented at the end of the article.

1 Introduction

Calculations of elementary particle decays and scattering cross-sections in presence of a background magnetic field are commonly found in literature [1, 2, 3, 4]. These calculations became more important after it was understood that the neutron star cores can sustain magnetic fields of the order of $10^{13}$ G or more. In presence of such strong magnetic fields all the particles which have a magnetic moment are bound to get affected and consequently their properties like self-energy, decay rates or scattering cross-sections are modified. The interesting feature of a background magnetic field is that it not only modifies properties of particles with magnetic moments but can also affect the properties of particles which do not have any magnetic moment. The obvious question is, how is it possible? To give a satisfactory answer to this question we take the example of standard model neutrinos and their self-energy. In the standard model, neutrinos have no electric charge hence they do not have any direct coupling to photons in any renormalizable quantum field theory. The standard Dirac contribution to the magnetic moment, which comes from the vector coupling of a fermion to the photon, is therefore absent for the neutrino. In the standard model of electroweak interactions, the neutrinos cannot have any anomalous magnetic moment either. The reason is: anomalous magnetic moment comes from chirality-flipping interactions $\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}$, and neutrinos cannot have such interactions because there are no right-chiral neutrinos in the standard model. Consequently the standard model neutrinos cannot interact with the background magnetic field. If one looks at the Feynman diagrams contributing to the neutrino self-energy in Fig. 1, one will see that the self-energy diagrams contain charged lepton, $\ell$, propagators and $W^-$ propagator. Although the neutrinos in this case have no coupling with the background fields the virtual particles in the loops can have couplings with the background field and so the result of the self-energy calculations are bound to be affected by the background fields.

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As a second example we take the case of the vacuum polarization of the photon. To one-loop the Feynman diagram is shown in Fig. 2. In this case also the virtual particles propagating in the loops must be charged particles and as a result the magnetic field will affect the expression of the vacuum polarization result. There can be various other cases, as an example the four photon interactions in QED, in which the magnetic field can affect properties of electrically neutral particles.

The calculations of the quantities like the neutrino self-energy and the vacuum polarization of the photons in presence of a background magnetic field is similar to the calculations of them in absence of the same, except that now one has to use the modified two-point functions of the charged particles. We will discuss about the charged fermion two-point functions in presence of a magnetic field in section 3.

Before going into further details about how to calculate various elementary particle processes in presence of a background uniform magnetic field, we must be careful about the fact that we are calculating physical quantities in presence of a gauge field. The background field must be specified by some suitable choice of gauge. Then the question is whether the quantities which we calculate are in the end gauge invariant? If the answer is no than there is no utility of doing all those calculations because they will be arbitrary to the gauge which one uses. In this article we will try to understand how the gauge arbitrariness of the various calculations mentioned above can be tackled. We will mainly focus on all those elementary particle processes whose Feynman diagrams do not have any charged particle external legs. Processes like Compton scattering, pair annihilation and so on will be dealt briefly in the penultimate section.

The article is organized in the following way. In section 2 we will set the preliminaries required to follow the following sections and doing so we will also set the mathematical notations which we will follow. In section 3 we will discuss about the form of the Schwinger phase accompanying the charged fermion two-point function in presence of a magnetic field and try to understand its properties. The next section will deal with the effects of the phase factor on loop calculations. It will be discussed in which circumstances we can neglect the phase factor for an actual calculation and when this cannot be done. In section 4 we show the Dirac equation solutions in a magnetic field background and a pure gauge background and comment on their gauge transformation properties and the difficulties we come across to prove background gauge invariance of those processes which involve charged particles in the external lines of the Feynman diagrams.
2 Conventions and notations

In presence of a background magnetic field we can decompose the photon field as follows:

\[ A^\mu(x) = A_D^\mu(x) + A_B^\mu(x), \]  

(1)

where \( A_D^\mu(x) \) is the dynamical photon field and \( A_B^\mu(x) \) is the classical background field which gives rise to the magnetic field. If the uniform background classical magnetic field is called \( B \) then we must have:

\[ B = \nabla \times A_B(x), \]  

(2)

where \( A_B^\mu(x) = (0, A_B(x)) \). In presence of the background magnetic field we can also write the field strength tensor as:

\[ F^{\mu\nu}(x) = F_D^{\mu\nu}(x) + F_B^{\mu\nu}, \]  

(3)

where \( F_D^{\mu\nu}(x) = \partial^\mu A_D^\nu(x) - \partial^\nu A_D^\mu(x) \) and \( F_B^{ij} = \partial^i A_B^j(x) - \partial^j A_B^i(x) \) is a constant as given in Eq. (2).

The QED Lagrangian can be written as:

\[ L = \overline{\psi}(i \gamma^\mu D^\mu - m) \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \]  

(4)

where \( D^\mu = \partial^\mu + ieq A^\mu \) is the covariant derivative of the fermion fields. Here \( e \) is the magnitude of the electronic charge and \( q \) designates the sign, for electrons \( q = -1 \) and for positrons \( q = +1 \). In this article we will only be talking about those cases where \(|q| = 1\). The gauge invariant Lagrangian can also be written as:

\[ L = \overline{\psi} [i \gamma^\mu \Pi^\mu - m] \psi - eq \overline{\psi} \gamma^\mu \psi A_D^\mu - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \]  

(5)

where \( \Pi^\mu = i \partial^\mu - eq A_D^\mu \) is the kinetic momentum of the fermions in presence of the background field. The commutation relations of interest are:

\[ [x^\mu, \Pi^\nu] = -ig^{\mu\nu}, \quad \text{and} \quad [\Pi^\mu, \Pi^\nu] = -ieq F_\nu^{\mu\nu}, \]  

(6)

where,

\[ g^{\mu\nu} = \text{diag}(1, -1, -1, -1). \]  

(7)

Now we can talk about the gauge invariance of the Lagrangian given in Eq. (4). We have to be particularly careful since there are two kind of gauge invariances here. The first kind is about the gauge invariance of the Lagrangian under a gauge transformation of the dynamical photon field \( A_D^\mu \rightarrow A_D^\mu + \partial^\mu \omega \) where \( \omega \) is some well behaved function of space and time. Under this kind of a gauge transformation the fermion fields will transform like \( \psi \rightarrow \psi e^{-ieq\omega} \). But this does not exhaust all the gauge transformation possibilities of the Lagrangian given in Eq. (5), we can still have a gauge transformation of the background gauge field \( A_B^\mu \) as \( A_B^\mu \rightarrow A_B^\mu + \partial^\mu \lambda \) where \( \lambda \) is a well behaved function of space-time coordinates, which will leave the Lagrangian in Eq. (5) invariant provided the fields transforms like \( \psi \rightarrow \psi e^{-ieq\lambda} \).

The Euler-Lagrange equation of \( \psi \) is:

\[ (\Pi - m)\psi(x) = 0, \]  

(8)

where \( \Pi = \gamma_\mu \Pi^\mu \). Generally the inverse of the kinetic part of the Lagrangian (for Eq. (5) the quantity sandwiched between the fields in the first term in the right hand side of the equation) gives the two-point Greens function of the theory, which are the building blocks of subsequent developments. In the present circumstance, the two-point functions are also defined as:

\[ (\Pi - m)S_B(x, x') = \delta^4(x - x'), \]  

(9)
which can also be written in a matrix form:

\[ S_B = \frac{1}{\Pi - m}, \]  

where \( S_B \) is a matrix, in fact the inverse matrix of \( \Pi - m \), and the two-point Greens function is \( \langle x'|S_B|x \rangle = S_B(x, x') \). So from the definition of the two-point function given in Eq. (9) we see that the fermion two-point functions are defined not in a gauge invariant way but has explicit dependence on the background gauge field. We will talk about this gauge dependence of the two-point function in the next section.

In the Feynman diagrams appearing in Fig. 1 and Fig. 2 the photons represent the dynamical photons \( A_\mu^D \) appearing in the Lagrangian given in Eq. (5). The background field \( A_\mu^B \) is tacitly taken into care by the fermion propagators or other charged particle propagators appearing inside the loops. Another interesting point which must be noted is that although the Lagrangian as written down in Eq. (5) is Lorentz invariant but it contains \( A_\mu^B \) and \( A_\mu^D \) satisfies Eq. (2) with the condition that \( A_\mu^B(x) = 0 \). The condition as given in Eq. (2) is not Lorentz invariant i.e., a magnetic field is not a Lorentz invariant quantity. So if we demand that we will be working in presence of a uniform classical background magnetic field then we are restricting the Lorentz covariance of the theory.

Lastly we fix the notations and conventions which we will be following in this article. In future the magnitude of the uniform background magnetic field will always be denoted by the symbol \( B \) and we will take it to be pointed along the \( z \)-direction of the coordinate axis. The magnetic field vector pointing in the \( z \)-direction will be given by

\[ B = \hat{z}B, \]  

where \( \hat{z} \) is the unit vector along the \( z \)-axis.

As Lorentz covariance is restricted and the magnetic field chooses a particular direction in space, the 4-vector structure breaks down into a perpendicular part and a parallel part. If \( a^\mu \) is a 4-vector, then

\[
\begin{align*}
a_\parallel^\mu &= (a^0, 0, 0, a^3), \\
a_\perp^\mu &= (0, a^1, a^2, 0),
\end{align*}
\]

such that

\[ a^\mu = a_\parallel^\mu + a_\perp^\mu. \]  

Also in our convention

\[ g_{\mu\nu} = g_{\parallel \mu \nu} + g_{\perp \mu \nu}, \]  

where

\[
\begin{align*}
g_{\parallel \mu \nu} &= (1, 0, 0, -1), \\
g_{\perp \mu \nu} &= (0, -1, -1, 0).
\end{align*}
\]

Terms as \( a_\parallel^2 \) and \( a_\perp^2 \) stands for:

\[
\begin{align*}
a_\parallel^2 &= g_{\parallel \mu \nu} a^\mu a^\nu, \\
&= (a_1^2 - (a_3)^2), \\
a_\perp^2 &= -g_{\perp \mu \nu} a^\mu a^\nu, \\
&= (a_1^2 + (a_3)^2),
\end{align*}
\]

such that

\[ a^2 = a_\parallel^2 - a_\perp^2. \]
Also in our case,
\[ F_B^{12} = -F_B^{21} = B. \] (23)

With this amount of preliminary understanding about the fermion two-point functions in a background magnetic field and the notations which we will follow we move to the next section which deals with the explicit form of the two-point fermionic Greens functions.

3 The phase factor appearing in the two-point Greens function of a charged fermion

In a much celebrated paper Schwinger derived what will be the form of the fermionic two-point function in presence of constant electromagnetic fields [5, 6, 7]. In the present article we will only study the behaviour of the two-point function in presence of a pure magnetic field. Later on it was possible to find a curious momentum space description of the Schwinger’s two-point function in presence of a magnetic field. It can be noted that the above two-point function is not a free field propagator, it contains interactions with the background magnetic field. This fact makes the two-point function to be translationally non-invariant. The reason why it becomes translationally non-invariant is closely linked to its gauge transformation properties which we will discuss shortly afterwards in this section. As translational invariance is lost we cannot ideally Fourier transform \( S_B(x, x') \) into something analogous to \( S_B(p) \), but the actual two-point function contains a multiplicative factor which is translation invariant and has all the good properties that a two-point function should have.

Schwinger’s calculation of the fermion two-point function relies on solving the operator formula given in Eq. (10) and then finding out the matrix element \( \langle x' | S_B | x \rangle = S_B(x, x') \). The two-point function can be expressed as [7, 8, 9, 10]:
\[ iS_B(x, x') = \kappa(x, x') \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-x')} iS_B(p), \] (24)

here \( x \) stands for the coordinate 4-vector as usual. \( S_B(p) \) is expressed as an integral over a variable usually (though confusingly) called the ‘proper time’:
\[ iS_B(p) = \int_0^\infty ds \, e^{\Phi(p,s)} G(p, s). \] (25)

The quantities \( \Phi(p, s) \) and \( G(p, s) \) can be written in the following way:
\[ \Phi(p, s) = is \left( p^2 - \frac{\tan(eqBs)}{eqBs} p^2 - m^2 \right) - \epsilon s, \] (26)
\[ G(p, s) = \frac{e^{-ieqBs\Sigma}}{\cos(eqBs)} \left( \hat{p}^\parallel + \frac{e^{-ieqBs\Sigma}}{\cos(eqBs)} \hat{p}^\perp + m \right) = (1 + i\Sigma_z \tan(eqBs))(\hat{p}^\parallel + m) + \sec^2(eqBs)\hat{p}^\perp. \] (27)

In the above expressions \( \Sigma_z = i\gamma^1\gamma^2 \), where the gamma matrices are taken to be in the standard Dirac-Pauli representation. \( \hat{p}^\parallel = g^{\mu\nu}\gamma^\mu p^\nu \) and \( \hat{p}^\perp = g^{\mu\nu}\gamma^\mu p^\nu \) while the symbols \( p^\parallel \) and \( p^\perp \) are explained in the last section. \( \epsilon \) is an infinitesimal positive quantity introduced for the convergence of the integrals. For convenience henceforth we will call the expression of the two-point function as given in Eq. (24) as the Schwinger two-point function.

In a typical loop diagram, one therefore will have to perform not only integrations over the loop momenta, but also over the proper time variables. From the above equations we see that \( iS_B(p) \) is manifestly translation invariant, and it has another interesting property. As \( B \to 0 \), \( \Phi(p, s) \to is(p^\parallel -
\[ p_\perp^2 - m^2 - \epsilon s = is(p^2 - m^2) - \epsilon s \quad \text{and} \quad G(p, s) \rightarrow (\not p + m) + \not p_\perp = \not p + m. \]

Therefore when \( B \rightarrow 0 \) we have:

\[
iS_B(p) \rightarrow \int_0^\infty ds \, e^{i(s(p^2 - m^2) - \epsilon s(\not p + m))},
\]

\[
= \frac{i(\not p + m)}{p^2 - m^2 + i\epsilon},
\]

which is the normal fermion propagator in absence of any background magnetic field. Moreover it is seen that \( iS_B(p) \) consists of \( Bs \) and not the gauge fields, so \( iS_B(p) \) is not only translation invariant but gauge invariant also. This leads us to the conclusion that the gauge dependence of the two-point function \( iS_B(x, x') \) must present in the function \( \kappa(x, x') \).

Not going into any detailed description of \( \kappa(x, x') \) we can simply understand its necessity in the two-point function. In presence of a background gauge field the gauge transformation property of the fields of the charged fermions are different at two different space-time points. Unless there is some factor in the two-point function which can connect these two fields at different space-time points with different gauge transformation properties, the calculations involving charged fermion two-point functions will not be manifestly gauge covariant. The gauge transformed fields comes with phase factors where the phase depends upon the space-time point where the gauge transformation is made. The fermionic fields at two different space-time points will therefore have two different phase factors. To make a connection between them \( \kappa(x, x') \) must also be some form of a phase. Conventionally it is named the phase-factor. The phase factor, as calculated by Schwinger \[5\], is given as:

\[
\kappa(x, x') = \exp \{ieq I(x, x')\}
\]

where

\[
I(x, x') = \int_{x'}^x d\xi_\mu \left[ A^\mu_{\lambda}(\xi) + \frac{1}{2} F^{\mu\nu}(\xi - x')\nu \right].
\]

From Eq. (31) we notice that the phase factor breaks the translation invariance of the two-point function.

For a constant background field we can always write the gauge field as

\[
A^\mu_{\lambda}(\xi) = -\frac{1}{2} F^{\mu\nu} \xi_\nu + \partial^\mu \lambda(\xi),
\]

where \( \lambda(\xi) \) is an arbitrary well behaved function and depends upon our choice of gauge. Using the above relation in conjunction with Eq. (31) we can simplify the integration appearing in the phase factor as

\[
I(x, x') = \int_{x'}^x d\xi_\mu \left[ -\frac{1}{2} F^{\mu\nu} x'_\nu + \partial^\mu \lambda(\xi) \right].
\]

Using the constancy of the field strength tensor the above expression can be written as

\[
I(x, x') = \frac{1}{2} x_\mu F^{\mu\nu} x'_\nu + \lambda(x) - \lambda(x').
\]

From Eq. (34) we can immediately see if we set \( x = x' \), in other words if we integrate over any closed contour in space-time \( I(x, x') \) vanishes. Thus \( I(x, x') \) connecting two points in space-time is independent of the path joining them, and as a result the phase factor of the Schwinger two-point function joining the points \( x' \) and \( x \) in Eq. (30) is also path independent.

Utilizing the path independence of the phase factor of the two-point function the general convention is to choose a straight line path connecting the two points \( x' \) and \( x \). Points on this path are represented by

\[
\xi^\mu = (1 - \zeta)x'^\mu + \zeta x^\mu,
\]

\[
\int_{x'}^x d\xi_\mu \left[ -\frac{1}{2} F^{\mu\nu} x'_\nu + \partial^\mu \lambda(\xi) \right].
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\int_{x'}^x d\xi_\mu \left[ -\frac{1}{2} F^{\mu\nu} x'_\nu + \partial^\mu \lambda(\xi) \right].
\]
where the parameter \( \zeta \) ranges from 0 to 1. Using Eq. (31) and the straight line path given in Eq. (35) one gets

\[
I(x, x') = \int_{x'}^{x} d\xi A_B^\mu(\xi) + \frac{\zeta}{2} \int_{0}^{1} d\zeta (x_\mu - x'_\mu) F_B^{\mu\nu}(x_\nu - x'_\nu),
\]

\[
= \int_{x'}^{x} d\xi A_B^\mu(\xi). \tag{36}
\]

Using Eq. (32) for the gauge field we can retrieve Eq. (34).

The form of Eq. (30) when \( I(x, x') \) is as given in Eq. (36) is very similar to a Wilson line but it is not a general Wilson line. A general Wilson line in presence of a U(1) field is defined as:

\[
U(x, x') = \exp \left[ i e q \int_P d\xi A_B^\mu \right], \tag{37}
\]

where \( P \) specifies a particular path joining the points \( x \) and \( x' \) in space-time. From Eq. (36) we see that in this case \( I(x, x') \) is obtained by integrating \( A_B^\mu(\xi) \) only on a straight line path joining \( x' \) and \( x \) as given in Eq. (35). For a general Wilson line as defined in Eq. (37) the gauge invariant Wilson loop will be:

\[
U(x, x) = \exp \left[ i e q \frac{e}{2} \oint_S d\sigma F_B^{\mu\nu} \right],
\]

where \( S \) is a surface that spans the closed integration loop and \( d\sigma^{\mu\nu} \) is an area element on this surface. As such \( U(x, x') \) will not be unity in general but as the integration defining \( I(x, x') \) in Eq. (36) is only on a straight line path so in this case \( \kappa(x, x) \) will always be unity.

From Eq. (34) it is clear that the phase factor is dependent on the form of the function \( \lambda(\xi) \), that is to say the fermion two-point function is dependent on the gauge in which the constant background magnetic field is specified. Suppose we are working in such a gauge that \( \lambda(\xi) = 0 \), and then we make a gauge transformation of the background field as

\[
A_B^\mu \rightarrow A_B^\mu + \partial^\mu \lambda(\xi) \tag{39}
\]

then from Eq. (34) it follows that the fermion two-point function will transform as

\[
iS_B(x, x') \rightarrow \exp(ie\lambda(x))iS_B(x, x') \exp(-ie\lambda(x')) . \tag{40}
\]

under the gauge transformation. As because we are working in presence of a background gauge field, the fields of the charged particles and their two-point functions both become background gauge dependent. This is the reason why the phase factor arises in the expression of the two-point function.

In the \( B \rightarrow 0 \) limit \( iS_B(p) \) goes to the normal fermionic propagators in absence of any magnetic field, but what is the fate of \( \kappa(x, x') \) when \( B \rightarrow 0 \)? The answer is closely related to the way one choose \( A_B^\mu \). There are many equivalent ways of writing \( A_B^\mu \) as:

\[
A_B^0 = A_B^y = A_B^z = 0 , \quad A_B^x = -yB + b . \tag{41}
\]

or

\[
A_B^0 = A_B^z = A_B^y = 0 , \quad A_B^x = yB + c . \tag{42}
\]

or

\[
A_B^0 = A_B^z = 0 , \quad A_B^y = \frac{1}{2}xB + d , \quad A_B^x = -\frac{1}{2}yB + g , \tag{43}
\]

where \( b, c, d \) and \( g \) are constants. In the above equations \( x, y \) are just coordinates and not 4-vectors. All of these above choices gives a magnetic field along the \( z \)-axis of the proper magnitude and the point to note about the above choices of the gauge fields is that all of them goes to zero as \( B \rightarrow 0 \) if \( b = c = d = g = 0 \).
When \( b = c = d = g = 0 \) and \( B \to 0 \) it is noted that \( iS_B(p) \) approaches the free propagator form as it should be in absence of the gauge fields and \( \kappa(x, x') \) also approaches unity, as is expected. But \( b = c = d = g = 0 \) is not the most general gauge choice which produces a magnetic field along the \( z \)-direction. With the general choice of the gauges as is specified in Eq. (11), Eq. (12) or Eq. (13) it is clear that \( \kappa(x, x') \) does not approach unity as \( B \to 0 \). If we take the choice of the background gauge field as given in Eq. (11) then \( \kappa(x, x') \to e^{ieqb(x-x')} \) as \( B \to 0 \). So when \( B \to 0 \) in this case we will have,

\[
iS_B(x, x') \to e^{ieqb(x-x')} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \frac{i(p + m)}{p^2 - m^2 + ie}, \tag{44}
\]

which is translation invariant but not the form which we expect when there is no background magnetic field present.

A somewhat similar situation arises when we deal with pure gauge configurations. The Schwinger two-point function as given in Eq. (24) is specific for the case of those gauges which gives a background magnetic field. In the general derivation by Schwinger \cite{5} the form of the fermionic two-point function was derived for general constant \( F^\mu_\nu \). If we choose a pure gauge field which does not give rise to any electric or magnetic field as

\[
A_B^\alpha(\xi) = k\delta^\alpha_\xi, \tag{45}
\]

where \( k \) is a constant, then also we can find the Schwinger two-point function and it differs from the vacuum propagator only by the phase factor. Using Eq. (40) in this case we will have,

\[
iS_B(x, x') = e^{ieqb(x^\alpha x_\alpha - x'^\alpha x'_\alpha)} \int \frac{d^4p}{2\pi^2} e^{-ip\cdot(x-x')} \frac{i(p + m)}{p^2 - m^2 + ie}. \tag{46}
\]

This form of the propagator is not translation invariant and does not match our expected form. The exponential factor multiplying the propagator of the free fermions is annoying. In the next section we will explicitly show how to tackle these problems.

A brief summary of the properties about the Schwinger two-point function is presented below. The Schwinger two-point function, in general, is a function of \( A_B^\alpha \) and is defined for all gauge configurations including pure gauges. In the present section only its form in presence of a uniform magnetic field was presented. As a consequence of this fact when one takes the \( B \to 0 \) limit of it, the free-fermion propagator is reproduced up to multiplicative phase, which designates a trivial pure background gauge configuration. In other words the \( B \to 0 \) limit of the two-point function means a transition (not a gauge transformation) of \( iS_B(x, x') \) where initially the gauge fields followed Eq. (2) with \( A_B^\mu(0) = 0 \) and finally \( A_B^\alpha(x) = \partial \rho(x) \) for some well behaved function \( \rho(x) \). The gauges which produces a magnetic field along the \( z \)-axis may contain constant terms as \( b, c, d, g \) and when \( B = 0 \) all the gauges in Eq. (11), Eq. (12) and Eq. (13) becomes trivial pure gauge configurations. So in general the Schwinger two-point function as given in Eq. (24) can be continuously transformed from the case where it is a function of the gauge fields which produces a magnetic field to the case where it is a function of pure gauge fields. This fact has an interesting outcome. It is known that in presence of an external magnetic field the transverse momenta \( p_\perp \) do not represent gauge invariant degrees of freedom. But still the expression of the Schwinger two-point function contains \( p_\perp^2 \) explicitly. The presence of \( p_\perp^2 \) is there only because the two-point function is a function of \( A_B^\alpha(x) \) and the domain of \( A_B^\alpha(x) \) which can give rise to a magnetic field can be continuously transformed into trivial pure gauge configurations.

4 Effects of \( \kappa(x, x') \) on calculations

As we have discussed previously in section \( \mathbf{3} \) the phase factor in the Schwinger two-point function appears because the fermion two-point function attaches two points with different gauge transformation properties. Till now we have only talked about fermion two-point functions but one can also find out the Greens functions of charged scalars and vector particles in presence of a uniform background magnetic
field. It is seen that all of these two-point functions can be written as a product of a translationally invariant part and a background gauge dependent part. In fact all of the gauge dependent parts of the various two-point functions are same and are equal to $\kappa(x, x')$ [11]. The charged gauge boson two-point functions also depend upon other gauge parameters, like Feynman gauge parameter or Landau gauge parameter, but those are related to the dynamical gauge invariance. As we are not interested in an actual loop calculation we will not explicitly write down the forms of all the two-point functions here but for further discussions we will only utilize the fact that the translationally non-invariant part of all the two-point functions of charged particles are functionally equivalent to $\kappa(x, x')$. So all the points we will use to prove that a two-point fermion loop is independent of the choice of the background gauge will also apply for a loop like the one which is to the left in Fig. 1.

In this section we will study various cases and try to understand what will be the effect of $\kappa(x, x')$ on loop calculations. To understand its importance we take an example of the one loop photon vacuum polarization in QED. Let $P$ and $Q$ be the two space-time coordinates where the photon line interacts with the virtual charged fermions. If we are interested in finding out the overall phase factor accompanying the vacuum polarization tensor then we will have to use the Schwinger two-point function for the charged fermions. The contribution from the phase factors $\kappa(Q, P)$ and $\kappa(P, Q)$ to the loop integral in coordinate space, which we denote as $\Phi(P, Q)$ will be according to Eq. (30) and Eq. (34)

$$\Phi(P, Q) = \kappa(Q, P)\kappa(P, Q) = \exp \left\{ i e g \frac{1}{2} \left[ P_\mu F_B^{\mu\nu} Q_\nu + Q_\mu F_B^{\mu\nu} P_\nu \right] \right\} ,$$

(47)

which reduces to unity because of the antisymmetry of $F_B^{\mu\nu}$. From Eq. (47) it is seen that the phase factor's contribution in the one loop calculation is trivial and obviously gauge invariant. The same analysis also holds for the diagram in the left of Fig. 1.

Next we take a loop with three charged particle two-point functions connecting three space-time points $P, Q, R$. The overall phase contribution to the loop integral in the coordinate space can then be calculated using Eq. (30) and Eq. (34) and is given by

$$\Phi(P, Q, R) = \kappa(Q, P)\kappa(R, Q)\kappa(P, R) = \exp \left\{ i e g \frac{1}{2} \left[ P_\mu F_B^{\mu\nu} Q_\nu + Q_\mu F_B^{\mu\nu} R_\nu + R_\mu F_B^{\mu\nu} P_\nu \right] \right\} .$$

(48)

As all the phase factors are of the same form the first point to notice is that the contribution from the function $\lambda(\xi)$ cancel out in the overall factor, showing that the contribution is explicitly gauge invariant.

The next point which requires to be discussed is about the path independence of the phase factor. From section 3 we know that $\kappa(Q, P)$ is independent of the path which joins them, but here the path independence of $\kappa(Q, P)$ does not imply

$$\Phi(P, Q, R) = \kappa(Q, P)\kappa(R, Q)\kappa(P, R) = 1 ,$$

(49)

or from Eq. (51)

$$I(P, P) = I(Q, P) + I(R, Q) + I(P, R) = 0 .$$

(50)

Instead of the above expectation we get a finite contribution from Eq. (48). As $I(Q, P)$ consists of the product of the two end points instead of their difference, the different phase factors from the different paths connecting two intermediate points of the loop when multiplied does not reduce to unity. In general if,

$$I(Q, P) = f(Q) - f(P) ,$$

(51)

where $f(P)$ is some well-behaved function of space and time, then Eq. (50) will hold and consequently Eq. (49) will be true. In this regard we can say that the factor which multiplied the the normal fermion propagator in absence of any magnetic field in Eq. (44) is of the form as given in Eq. (51) and consequently they will not pose any problems for actual loop calculations.
4.1 Similarity of $\Phi$ with the Wilson loop

In the previous discussions it was shown that the phase factor contribution to the one loop calculations of various cases are explicitly gauge invariant. In the case of the photon vacuum polarization it was shown that it contributes nothing for the phase factor. The contribution from the phase factors for cases where one has three or more than three vertices can be understood in another way. If we have a loop with three or more vertices then the overall phase will be $\oint_L d\xi^\mu A^\mu_B(\xi)$ where $L$ designates the path which joins all the vertices in straight lines. Then from generalized Stokes theorem we can write:

$$\oint_L d\xi^\mu A^\mu_B(\xi) = \frac{1}{2} \int_S d\sigma_{\mu\nu} F_{\mu\nu}^B(\xi),$$  \hspace{1cm} (52)

where $S$ is the area of the loop enclosed by the straight lines and $d\sigma_{\mu\nu}$ is the infinitesimal surface area in the $\xi_\mu - \xi_\nu$ plane and $F_{\mu\nu}^B$ is the field strength tensor. In this case we notice that $\kappa(x,x')$ can be interpreted as the Wilson loop which is a gauge invariant quantity.

For pure gauge configurations we have seen in Eq. (46) that the fermion two-point function in presence of a pure gauge field comes with an unwanted phase. In the level of two-point functions this problem remains but if we look at loop integrals then this problem of the unwanted phase disappears. For the specific gauge choice in Eq. (45) we see that $I(P,Q)$ is of the form as given in Eq. (51) and consequently the overall phase factor will be trivial. But the result is not dependent on the particular form of the pure gauge chosen. From Eq. (52) we can see in all those cases where one calculates some process which contains three or more than three vertices in presence of a background pure gauge field $\Phi = 1$ because for a pure gauge configuration $F_{\mu\nu}^B(\xi) = 0$.

From Eq. (52) we can generalize that the overall phase depends on the flux of the magnetic field attached to the area of the loop. Given an arbitrary diagram we can initially calculate this flux to find out the overall phase. The typical nature of the overall phase function $\Phi(P,Q)$ is that $\Phi(P,P)$ is always unity. Similarly for all those graphs containing only two charged particle two-point functions, meeting each other at vertices $P$ and $Q$, the overall phase function $\Phi(P,Q) = 1$. But if the number of vertices in the loop, containing solely charged particles, is three or more then $\Phi$ is non trivial and it behaves like the Wilson loop as defined in Eq. (38).

5 Processes with charged particles as external legs in their Feynman diagrams

Except the processes which we have discussed in the previous sections there are many other processes where there are charged particles in the external legs of the Feynman diagrams, as Compton scattering, electron electron-neutrino scattering, pair creation and annihilation and so on. In addition to the charged external particles all these processes may require virtual charged particles for their happening. In this cases the standard procedure is to solve the Dirac equation in presence of the background magnetic field. But in these cases the calculations loose manifest gauge invariance because we have to solve Eq. (8) choosing some form of the background gauge and that will fix the gauge in this case. To make the point clear we actually solve the Dirac equation in presence of a background magnetic field in this section. For stationary states, we can write the solution of the Dirac equation as:

$$\psi = e^{-iEt} \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$  \hspace{1cm} (53)

where $\phi$ and $\chi$ are 2-component objects. We use the Pauli-Dirac representation of the Dirac matrices. In this notation, we can write

$$\begin{align*}
(E-m)\phi &= \sigma \cdot (-i\nabla - eqA_B)\chi, \\
(E+m)\chi &= \sigma \cdot (-i\nabla - eqA_B)\phi,
\end{align*} \hspace{1cm} (54) \hspace{1cm} (55)$$
where \( \sigma \) designates the Pauli matrices. Eliminating \( \chi \), we obtain
\[
(E^2 - m^2) \phi = \left[ \sigma \cdot (\pi - eqA_B) \right]^2 \phi.
\]
Now we choose the background gauge field configuration as given in Eq. (41) with \( b = 0 \) and with this choice Eq. (56) reduces to the form
\[
(E^2 - m^2) \phi = \left[ -\nabla^2 + (eqB)^2 y^2 - eqB(2iy\frac{\partial}{\partial x} + \sigma_z) \right]\phi.
\]
Here \( \sigma_z \) is the diagonal Pauli matrix. With this choice of our background gauge field the positive energy solutions of the Dirac equation for an electron is given as:
\[
e^{-i\nu \cdot X_n} U_+(y, n, p_y),
\]
here \( X^\mu \) denotes the space-time coordinate and \( \nu \cdot X = E_n t - \nu \cdot X = E_n t - p_x x + p_z z \). Here we have introduced the notation \( X \) for the vector coordinate \( x \), which is one of the components of \( X \), and \( X_n \) for the vector \( X \) with its \( y \)-component set equal to zero. In other words, \( \nu \cdot X_n = p_x x + p_z z \), where \( p_x \) and \( p_z \) denote the eigenvalues of momentum in the \( x \) and \( z \) directions.\(^1\) The energy \( E_n \) is given as:
\[
E_n^2 = m^2 + p_x^2 + 2neB,
\]
giving the relativistic form of Landau energy levels where \( n \) is a natural number. The form of \( U_s \) for \( s = +1, -1 \) are given by:\(^2\)
\[
U_+(y, n, p_y) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z}{E_n + m} I_{n-1}(\xi) \\ -\frac{\sqrt{2neB}}{E_n + m} I_{n-1}(\xi) \end{pmatrix}, \quad U_-(y, n, p_y) = \begin{pmatrix} 0 \\ I_n(\xi) \\ -\frac{\sqrt{2neB}}{E_n + m} I_n(\xi) \\ -\frac{p_z}{E_n + m} I_n(\xi) \end{pmatrix}.
\]
In the above expressions
\[
\xi = \sqrt{eB} \left( y + \frac{p_z}{qB} \right),
\]
and
\[
I_\nu(\xi) = \left( \frac{\sqrt{eB}}{\nu! 2\nu \sqrt{\pi}} \right)^{1/2} e^{-\xi^2/(2\nu)} I_\nu(\xi),
\]
where \( \nu \) are also natural numbers, and \( H_\nu(\xi) \) are Hermite polynomial functions of order \( \nu \).

The negative energy solutions are:
\[
e^{i\nu \cdot X_n} V_+(y, n, p_y),
\]
where
\[
V_+(y, n, p_y) = \begin{pmatrix} \frac{p_z}{E_n + m} I_{n-1}(\tilde{\xi}) \\ -\frac{\sqrt{2neB}}{E_n + m} I_{n-1}(\tilde{\xi}) \\ \frac{p_z}{E_n + m} I_n(\tilde{\xi}) \\ I_n(\tilde{\xi}) \end{pmatrix}, \quad V_-(y, n, p_y) = \begin{pmatrix} I_{n-1}(\tilde{\xi}) \\ 0 \\ -\frac{\sqrt{2neB}}{E_n + m} I_{n-1}(\tilde{\xi}) \\ I_n(\tilde{\xi}) \end{pmatrix}.
\]
\(^1\)It is to be understood that whenever we write the spatial component of any vector with a lettered subscript, it would imply the corresponding contravariant component of the relevant 4-vector.
where $\tilde{\xi}$ is obtained from $\xi$ by changing the sign of the $p_x$-term. The calculation of the above solutions are given in the appendix at the end.

The above solutions are obtained by solving the Dirac equation for a particular background gauge field configuration as specified in Eq. (41) with $b = 0$. If we designate the Dirac solutions in a background magnetic field by $\psi_B(x)$ then the above solution is one of them but we can easily gauge transform this solution to obtain other solutions. All the gauge configurations we have specified in Eq. (41), Eq. (42) and Eq. (43) can be connected by smooth gauge transformations. Specifically if we take $b = 0$ and $c = 0$ then we can get the gauge fields in Eq. (42) from those of Eq. (41) by the transformation $A_\mu^B \rightarrow A_\mu^B + \partial^\mu \lambda$ where $\lambda(x, y) = xyB$. In this case $\psi_B(x) = \psi_B(x)e^{-ieqyB}$ where $\psi_B(x)$ is the solution for the gauge configuration as given in Eq. (41) and $\psi_B'(x)$ is the solution of the Dirac equation for the gauge configuration as given in Eq. (42) for $b = c = 0$. In the above solutions the transverse momenta, $p_x$ and $p_y$ are just spurious degrees of freedom, they depend on the choice of the gauge. In the above solutions we will not find $p_y$ as the gauge we chose to work with contained $y$, but had we started with the gauge fields as given in Eq. (42) then $p_x$ should have been absent and if we had worked with the gauge fields as specified in Eq. (43) none of $p_x$ or $p_y$ should have appeared in our calculations. The arbitrariness of $p_x$ and $p_y$ reflects the fact that in presence of electromagnetic gauge fields $p_x$ and $p_y$ are not the proper quantities to work with. But in what ever background gauge we choose to work with the expression of the energy as given in Eq. (59), which is a physically measurable gauge invariant quantity, remains the same.

It can be a curious exercise to see if a pure gauge configuration like $A^\mu_A = A^\mu_B = A^\mu_1 = 0$ and $A^\mu_2 = xk$, where $k$ is a constant, can produce any effect on the Dirac solutions. We saw in our previous discussion with the Schwinger two-point functions that a pure gauge configuration modifies its form but cannot affect loop calculations involving them. In the present case also it can be shown that pure gauge configurations although modifies the shape of the Dirac solutions but the free-particle solutions can be easily distinguished from the complete solutions. With the gauge which we have chosen we can proceed similarly to the case where we were solving the Dirac equation in a magnetic field, the equation corresponding to Eq. (59) in this case will be:

$$ (E^2 - m^2)\phi = \left[ \left(-i \frac{\partial}{\partial x} - eqxk \right)^2 - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right] \phi. \hspace{1cm} (65) $$

Looking at the above equation we can propose a possible solution as:

$$ \phi = e^{ip\cdot X}$ f(x), \hspace{1cm} (66) $$

where $f(x)$ is a 2-component matrix which depends only on the $x$-coordinate. In the present case $p \cdot X \equiv p_y y + p_z z$. Putting the above solution into Eq. (65) we get the differential equation for $f(x)$ which looks like:

$$ \left(-i \frac{d}{dx} - eqxk \right)^2 f(x) = (E^2 - m^2 - p_y^2 - p_z^2)f(x). \hspace{1cm} (67) $$

Now we can always assume a solution of the above equation of the form:

$$ f(x) = e^{ix(a + \frac{1}{2}eqx)} \eta_\pm, \hspace{1cm} (68) $$

where $a$ is a constant and $\eta_\pm$ are the standard two-component spinors of the form

$$ \eta_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hspace{1cm} \eta_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \hspace{1cm} (69) $$

Now if we put this solution into Eq. (67) to get

$$ (E^2 - m^2 - p_y^2 - p_z^2 - a^2)f(x) = 0, \hspace{1cm} (70) $$

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which predicts $E^2 - m^2 - p_y^2 - p_z^2 - a^2 = 0$ and is the free-particle energy-momentum relation if we assume the constant $a$ to be the momentum canonically conjugate to $x$, i.e. if $a = p_x$. So Eq. (66) can now be written as:

$$\phi = e^{i\frac{px}{2}e^{\frac{i\pi k x^2}{2}}\eta_{\pm}}$$

(71)

which is similar to the free-particle solution up to the phase $e^{i\frac{\pi k x^2}{2}}$. Using the expression of $\phi$ we can find $\chi$ from Eq. (55). It can be seen easily that the complete solution of the Dirac equation in the pure gauge which we chose is the free-particle solution times a phase. The above solution can also be interpreted in another way, all the solutions of the Dirac equation in presence of a pure gauge field are related to the free-particle solutions (solutions with $A_{B}^{\mu} = 0$) by a gauge rotation. In the present case the rotation phase is simply $e^{\frac{i\pi k x^2}{2}}$. In presence of a magnetic field the Dirac solutions show that motion of the electrons transverse to the magnetic field directions gets quantized yielding discrete eigenvalues as $n$. This fact is crucial in making the $B \to 0$ limit non trivial in the present circumstances.

The solutions as given in Eq. (60) and Eq. (64) can be used to write the Dirac field in the usual sense and quantize the system in presence of a magnetic field [13, 14]. Then using those fields one can define a two-point function of the fermions. But as the Dirac solutions which we obtained are solved for a particular gauge field configuration the calculations with them will not be manifestly gauge invariant but reflect the choice of the gauge. This situation is unlike the one which we faced when we were talking about the Schwinger two-point functions where gauge invariance of the loops can be manifestly proved.

It may be tempting to write:

$$iS_{B}(x, x') = \langle \Omega | T \{ \Psi_{B}(x) \bar{\Psi}_{B}(x') \} | \Omega \rangle ,$$

(72)

where $\Psi_{B}(x)$ is the Dirac field in presence of a background magnetic field and $| \Omega \rangle$ is the appropriate vacuum state here, it is not the same as free-field vacuum as now the vacuum contains an infinite number of photons. $iS_{B}(x, x')$ is the Schwinger two-point function as defined in Eq. (24). If we analyze the left and the right hand sides of the above equation we will see that such a relation cannot be correct. From the discussion presented in the end of section 3 it was observed that the Schwinger two-point function in the right is a function of $A_{B}^{\mu}$ and is defined for all gauge configurations including pure gauges and consequently it had dependence on the transverse momenta components $p_{\perp}$ and had a smooth $B \to 0$ limit. On the other hand $\Psi_{B}(x)$ which appears on the right hand side of Eq. (72) is made up of Dirac solutions in a particular gauge which only follows Eq. (2) and as a result of this the $p_{\perp}$ components are spurious. There is no transformation which transforms Eq. (60) or Eq. (64) to the pure gauge solutions as discussed in the previous paragraph. Precisely, the right hand side of Eq. (72) is not a function of a general $A_{B}^{\mu}(x)$ and this fact makes it different from the left hand side. As a consequence it is impossible to obtain the Schwinger two-point function by manipulating on the Dirac solutions which we obtain by solving Dirac equation in a magnetic field.

From the discussions in this section we see that it is impossible to prove generally the gauge invariance of all those elementary particle processes which includes charged particles as external lines. The way of proving it must be particular and will depend upon the process under consideration.

6 Conclusion

In this article we tried to analyze the background gauge invariance of elementary particle processes in presence of a uniform background magnetic field. It was shown that the background gauge invariance of the calculations of certain class of processes requires a term in the fermion two-point functions, in presence of a magnetic field, which must have proper covariance properties. In loops solely composed of charged particles the Schwinger phase will not contribute until the loop has three or more than three vertices. It was shown that pure gauge configurations will not alter any properties of the calculations and $B \to 0$ limit of the calculations yield the expected results. All the results stated are to one-loop but it is expected that the result is more general and can be proved for higher loop cases also in a way similar
to the one presented in this article. Lastly we discuss all those particular processes which has one or more charged particles as external lines in their Feynman diagrams. In this cases it becomes impossible to furnish a general proof which will show that all these processes are background gauge invariant, as the calculations heavily rely on the effect of the particular gauge chosen to solve the Dirac equation in presence of a background magnetic field. The only way left to prove these processes to be background gauge invariant is to calculate a process in multiple gauges and show that the result is the same.

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Appendix

A Solution of the Dirac equation in a constant background magnetic field

In the standard Dirac-Pauli representation:

\[
\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]  \tag{73}

where each block represents a 2 \times 2 matrix. Noticing that the coordinates-ordinates \( x \) and \( z \) do not appear in Eq. (57) except through the derivatives a possible solution of it can be:

\[
\phi = e^{ip \cdot x} f(y),
\]  \tag{74}

where \( f(y) \) is a 2-component matrix which depends only on the \( y \)-coordinate, and possibly some momentum components, as we will see shortly. There will be two independent solutions for \( f(y) \), which can be taken, without any loss of generality, to be the eigenstates of \( \sigma_z \) with eigenvalues \( s = \pm 1 \). This means that we choose the two independent solutions in the form

\[
\begin{align*}
  f_+(y) &= \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix}, \\
  f_-(y) &= \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix}.
\end{align*}
\]  \tag{75}

Since \( \sigma_z f_s = s f_s \), the differential equations satisfied by \( F_s \) is

\[
\frac{d^2 F_s}{dy^2} - (eqBy + px)^2 F_s + (E^2 - m^2 - p_z^2 + eqBs)F_s = 0,
\]  \tag{76}

which is obtained from Eq. (57). The solution is obtained by using the dimensionless variable

\[
\xi = \sqrt{e|q|B} \left( y + \frac{px}{eqB} \right),
\]  \tag{77}

which transforms Eq. (76) to the form

\[
\left[ \frac{d^2}{d\xi^2} - \xi^2 + a_s \right] F_s = 0,
\]  \tag{78}

where

\[
a_s = \frac{E^2 - m^2 - p_z^2 + eqBs}{e|q|B}.
\]  \tag{79}

This is a special form of Hermite’s equation, and the solutions exist provided \( a_s = 2\nu + 1 \) for \( \nu = 0, 1, 2, \cdots \). This provides the energy eigenvalues

\[
E^2 = m^2 + p_z^2 + (2\nu + 1)e|q|B - eqBs,
\]  \tag{80}
and the solutions for $F_s$ are

$$N_\nu e^{-\xi^2/2}H_\nu(\xi) \equiv I_\nu(\xi),$$  \hspace{1cm} (81)

where $H_\nu$ are Hermite polynomials of order $\nu$, and $N_\nu$ are normalizations which we take to be

$$N_\nu = \left(\frac{\sqrt{e|q|B}}{\nu! 2^\nu \sqrt{\pi}}\right)^{1/2}. \hspace{1cm} (82)$$

With our choice of normalization, the functions $I_\nu$ satisfy the completeness relation

$$\sum_\nu I_\nu(\xi)I_\nu(\xi) = \sqrt{|q|B} \delta(\xi - \xi_*) = \delta(y - y_*), \hspace{1cm} (83)$$

where $\xi_*$ is obtained by replacing $y$ by $y_*$ in Eq. (77).

So far, $q$ was arbitrary. We now specialize to the case of electrons, for which $q = -1$. The solutions are then conveniently classified by the energy eigenvalues

$$E_n^2 = m^2 + p_z^2 + 2neB, \hspace{1cm} (84)$$

which is the relativistic form of Landau energy levels. The solutions are two fold degenerate in general: for $s = 1$, $\nu = n - 1$ and for $s = -1$, $\nu = n$. In the case of $n = 0$, only the second solution is available since $\nu$ cannot be negative. The solutions can have positive or negative energies. We will denote the positive square root of the right side by $E_n$. Representing the solution corresponding to this $n$-th Landau level by a superscript $n$, we can then write for the positive energy solutions,

$$f_+^{(n)}(y) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \end{pmatrix}, \quad f_-^{(n)}(y) = \begin{pmatrix} 0 \\ I_n(\xi) \end{pmatrix}. \hspace{1cm} (85)$$

For $n = 0$, the solution $f_+$ does not exist. We will consistently incorporate this fact by defining

$$I_{-1}(y) = 0, \hspace{1cm} (86)$$

in addition to the definition of $I_n$ in Eq. (81) for non-negative integers $n$.

The solutions in Eq. (85) determine the upper components of the spinors through Eq. (74). The lower components, denoted by $\chi$ earlier, can be solved using Eq. (55).

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