Decoherence by wave packet separation and collective neutrino oscillations

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In collaboration with J. Kopp and M. Lindner
(arXiv:1405.7275)
ν oscillations in dense neutrino backgrounds

(Early Universe, supernovae) – interesting collective oscillation effects, absent in the case of the usual MSW oscillations.

- Synchronized neutrino oscillations
- Bi-polar oscillations
- Spectral splits & swaps, multiple spectral splits
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- Taking into account previously not included features often changes the results qualitatively
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This work: study of effects of decoherence by wave packet separation on collective ν oscillations
Coherent neutrino - neutrino scattering

2-flavour evolution equation:

\[
\diamond \quad i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} -\frac{\Delta m^2_{4E}}{4E} \cos 2\theta_0 + V_e(t) + V_{ee} \\ \frac{\Delta m^2_{4E}}{4E} \sin 2\theta_0 + V_{\mu e} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}
\]

For isotropic backgrounds:

\[
V_e(t) = \sqrt{2} G_F n_e(t) \quad \nu_e + e \rightarrow \nu_e + e \quad \text{(Wolfenstein, 1978)}
\]

\[
V_{ee} = \sqrt{2} G_F (2n_{\nu_e} + n_{\nu_\mu}) \quad \nu_e + \nu_\mu \rightarrow \nu_e + \nu_\mu \quad \text{(Fuller et al., 1987; Nötzold & Raffelt, 1988)}
\]

\[
V_{\mu\mu} = \sqrt{2} G_F (n_{\nu_e} + 2n_{\nu_\mu})
\]

\[
V_{\alpha\beta} = \sqrt{2} G_F \int \frac{d^3q}{(2\pi)^3} \left[ \rho_{\alpha\beta}(\vec{q}) - \bar{\rho}_{\alpha\beta}(\vec{q}) \right] \quad \nu_\alpha(\vec{p}) + \nu_\beta(\vec{q}) \rightarrow \nu_\alpha(\vec{q}) + \nu_\beta(\vec{p})
\quad \text{(Pantaleone, 1992; Sigl & Raffelt, 1993)}
\]

In general backgrs.:

\[
V_{\alpha\beta} = \sqrt{2} G_F \int \frac{d^3q}{(2\pi)^3} \left[ \rho_{\alpha\beta}(\vec{q}) - \bar{\rho}_{\alpha\beta}(\vec{q}) \right] (1 - \vec{v}_p \cdot \vec{v}_q)
\]
The Schrödinger evolution equation

\[ i \frac{d}{dt} \nu = H \nu \]

Even in the case of pure neutrino states it is convenient to introduce neutrino density matrix in the flavour space:

\[ \rho_{\alpha \beta} = \langle \nu_\alpha \nu_\beta^* \rangle \]  

(Dolgov 1981; Sigl & Raffelt 1993, ...)  

\langle \ldots \rangle: \text{ summation over all neutrinos in the system and averaging over } \text{“microscopically large but macroscopically small” spatial volumes} \]

Equation of motion for the density matrix:

\[ \dot{\rho} = -i[H, \rho] \]
A convenient method of analysis and visualization of flavour transitions (especially useful for oscillations in dense neutrino backgrounds!)

For 2-flavour oscillations:

\[ \rho = \frac{n_\nu}{2} \left( P_\omega^0 + \vec{P}_\omega \cdot \vec{\sigma} \right), \quad H = \frac{1}{2} \left( H_0 + \vec{H}_\omega \cdot \vec{\sigma} \right), \quad \omega \equiv \frac{\Delta m^2}{2p}. \]
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From \( \dot{\rho} = -i[H, \rho] \Rightarrow \dot{P}_\omega^0 = 0 \) \( (n_\nu P_\omega^0 = n_\nu \omega = \text{const}) \),

\[ \vec{P}_\omega = \vec{H}_\omega \times \vec{P}_\omega \]

Precession of flavour “spin” around the “magnetic field” \( \vec{H}_\omega \).
Flavour spin

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$$\diamondsuit \quad \dot{\vec{P}}_\omega = \vec{H}_\omega \times \vec{P}_\omega$$

Precession of flavour “spin” around the “magnetic field” $\vec{H}_\omega$.

$$\vec{H}_\omega = \omega \vec{B} + \lambda \vec{L} + \mu \vec{D} ,$$
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\[
\vec{H}_\omega = \omega \vec{B} + \lambda \vec{L} + \mu \vec{D},
\]

\[
\vec{B} = (\sin 2\theta_0, 0, -\cos 2\theta_0), \quad \vec{L} = (0, 0, 1),
\]

\[
\lambda = \sqrt{2} G_F n_e, \quad \mu = \sqrt{2} G_F n_\nu, \quad \vec{D} = \int_{-\infty}^{\infty} \vec{P}_\omega d\omega \tag{4}
\]
Flavour spin

The convention for antineutrinos: $\bar{\rho}_{\alpha\beta} = \langle \bar{\nu}_\beta \bar{\nu}_\alpha^* \rangle$ and

$$\bar{\rho} = \frac{1}{2} n_\nu (P^0_{-\omega} - \vec{P}_{-\omega} \vec{\sigma}).$$

$\vec{P}_{-\omega}$ describes the antineutrino flavour. $P^0_{-\omega}$: $n_\nu P^0_{-\omega} = n_{\bar{\nu}_\omega}.$
The convention for antineutrinos: $\bar{\rho}_{\alpha\beta} = \langle \bar{\nu}_\beta \bar{\nu}_\alpha^* \rangle$ and

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For $\nu_e \leftrightarrow \nu_x$ oscillations ($\nu_x = \nu_\mu, \nu_\tau$):

$$\rho_{ee} = \frac{n_\nu}{2} [P_{\omega}^{0} + P_{\omega 3}],$$

$$\rho_{xx} = \frac{n_\nu}{2} [P_{\omega}^{0} - P_{\omega 3}].$$
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(Figs. from Duan, Fuller & Qian, arXiv:1001.2799)
Rotating away ordinary matter effects

If the density of ordinary matter is constant or nearly constant \( \Rightarrow \) effects of ordinary matter (\( \lambda \vec{L} \) term) can be removed by going into a frame rotating around \( \vec{L} \) and replacing \( \theta_0 \rightarrow \theta \) (\( \theta \) small).

\[
\vec{H}_\omega = \omega \vec{B} + \lambda \vec{L} + \mu \vec{D} \quad \Rightarrow \quad \vec{H}_\omega = \omega \vec{B} + \mu \vec{D}
\]

with

\[
\vec{B} = (\sin 2\theta, 0, -\cos 2\theta),
\]

\( \theta \ll 1 \). Used in many papers (also by us).
Collective oscill. – many interesting effects

Synchronized oscillations:

Thermal spectrum, \( E_0 = 2.2T \)

\[
\kappa \equiv \frac{2\sqrt{2}G_F n_\nu E_0}{\Delta m^2}, \quad \sin 2\theta_0 = 0.8, \quad \tau \equiv (\Delta m^2/2p_0)t
\]

(From Pastor, Raffelt & Semikoz, hep-ph/0109035)
Collective oscill. – many interesting effects

Spectral splits (Duan, Fuller, Carlson & Qian 2006; Raffelt & Smirnov, 2007):

\[ E_\nu = \pm \frac{\delta m^2}{2\omega} \]

Multiple spectral splits:

(Dasgupta, Dighe, Raffelt & Smirnov, 2009)
Review: Duan, Fuller & Qian, 1001.2799.
In quantum theory moving particles are described by wave packets!
Decoherence by wave packet separation

W. packets of different propagation eigenstates propagate with different group velocities.
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Coherence time: $\Delta v_g \cdot t_{coh} \simeq \sigma_x$. 
Decoherence by wave packet separation

W. packets of different propagation eigenstates propagate with different group velocities

Coherence time: $\Delta v_g \cdot t_{coh} \approx \sigma_x$. Coherence length: $L_{coh} = v_g \cdot t_{coh} \approx \frac{v_g}{\Delta v_g} \sigma_x$. 
Decoherence by wave packet separation

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Coherence time: $\Delta v_g \cdot t_{coh} \simeq \sigma_x$.

Coherence length: $L_{coh} = v_g \cdot t_{coh} \simeq \frac{v_g}{\Delta v_g} \sigma_x$.

In vacuum:

$$\frac{\Delta v_g}{v_g} \simeq \frac{\Delta m^2}{2p^2} \Rightarrow L_{coh} \simeq \frac{2p^2}{\Delta m^2} \sigma_x$$
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\]

In ordinary matter:

\[
\frac{\Delta v_g}{v_g} \simeq \frac{\Delta m^2}{2p^2} \cdot \frac{\Delta m^2}{2p} - V_e c_2 \sqrt{\left( \frac{\Delta m^2}{2p} c_2 - V_e \right)^2 + \left( \frac{\Delta m^2}{2p} s_2 \right)^2}
\]

Typically of the same order as in vacuum (exception: close to the MSW res.).
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Coherence time: $\Delta v_g \cdot t_{coh} \simeq \sigma_x$. Coherence length: $L_{coh} = v_g \cdot t_{coh} \simeq \frac{v_g}{\Delta v_g} \sigma_x$.

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Typically of the same order as in vacuum (exception: close to the MSW res.).

In dense neutrino backgrounds: $\Delta v_g = (\partial/\partial p) \Delta E$ depends on the solution of the evol. equation.
Decoherence of astrophys. and cosmol. \( \nu S \)

In core-collapse SN and in the early Universe neutrinos are produced at very high densities \( \Rightarrow \) neutrino wave packets are very short!

For SN neutrinos:

\[
\sigma_{xP} \sim 10^{-11} \text{ cm} \quad (\text{Kersten 2012})
\]

(our estimates give similar results).

- WPs of different neutrinos do not overlap if \( n_\nu < \sigma_{xP}^{-3} \sim 10^{33} \text{ cm}^{-3} \).
- For SN neutrinos with \( E \sim 10 \text{ MeV} \) and \( \Delta m^2 = 2.5 \times 10^{-3} \text{ eV}^2 \):
  \[ L_{\text{coh}} \sim 10 \text{ km} \]

By the time SN neutrinos reach \( r \sim 100 \text{ km} \) a large fraction of them should have already lost their coherence \( \Rightarrow \)

Decoherence may affect significantly collective neutrino oscillations!
Decoherence by wave packet separation

WP separation: In the propagation eigenstate basis off-diagonal elements of the density matrix $\rho_{ik} = \langle \nu_i \nu_k^* \rangle$ get suppressed.
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Assuming this suppression to be exponential with time:

\[
\dot{\rho} = -i[H, \rho] - \frac{1}{L_{coh}} (\rho - T[\rho])
\]

$T[\rho]$: in the propagation eigenstate basis selects the diagonal part of $\rho$. $(\rho - T[\rho])$ contains only off-diagonal elements of $\rho$ ⇒ leads to their exponential suppression with $t$. N.B.: $\text{tr}(\rho)$ is conserved!
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Evolution equation for individual flavour spins:

$$\dot{\vec{P}}^a_\omega = \vec{H}_\omega \times \vec{P}^a_\omega - \frac{1}{L_{\text{coh}}} \left( \vec{P}^a_\omega - \frac{(\vec{P}^a_\omega \cdot \vec{H}_\omega)}{\vec{H}_\omega^2} \vec{H}_\omega \right).$$
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\]

Define \( \vec{P}_\omega \equiv \sum_a \vec{P}^a_\omega \); summing over different neutrinos \( \implies \)
Decoherence by wave packet separation

Equation of motion for the flavour spin vector $\vec{P}_\omega$:

$$\dot{\vec{P}}_\omega = \vec{H}_\omega \times \vec{P}_\omega - \frac{1}{L_{\text{coh}}} \left( \vec{P}_\omega - \frac{(\vec{P}_\omega \cdot \vec{H}_\omega)}{\vec{H}_\omega^2} \vec{H}_\omega \right)$$

$$(...) = \vec{P}_\omega^\perp$ – component of $\vec{P}_\omega$ orthogonal to $\vec{H}_\omega$. Leads to exponential suppression of $\vec{P}_\omega^\perp$ with time (tends to align or anti-align $\vec{P}_\omega$ with $\vec{H}_\omega$).

Once the alignment is complete – no more evolution.

EoM for $\vec{P}_\omega$ is similar to equation describing dynamical decoherence through interaction with the environment:

$$\dot{\vec{P}} = \vec{H} \times \vec{P} - D \vec{P}_T$$  \hspace{1cm} (Stodolsky1987)

$\vec{P}_T$ is orthogonal to $\vec{V}$ (part of $\vec{H}$ responsible for inter. w/ the environment).

N.B.: $|\vec{P}_\omega|$ is not conserved! (Unlike in the no-decoherence case).
Decoherence by wave packet separation

E.g. for 2-flavor $\nu_e \rightarrow \nu_x$ oscillations in vacuum:

$$P_{ee} = c^4 + s^4 + 2c^2s^2 e^{-t/L_{coh}} \cos \phi,$$

$$P_{ex} = 2c^2s^2(1 - e^{-t/L_{coh}} \cos \phi).$$

$$\phi \equiv (\Delta m^2 / 2E)t.$$

Simplified neutrino systems (toy models):

- **Monochromatic** (or nearly monochromatic) uniform and isotropic gas of neutrinos (of a given flavour at $t = 0$) – synchronized oscillations w/o WP separation;

- **Uniform and isotropic gas** consisting initially of neutrinos and antineutrinos of a given flavour and energy ("flavour pendulum")
“Monochromatic” single-flavour $\nu$ ensemble

“Monochromatic”: $|\Delta \omega| \sim |\omega|(\sigma_E/E) \ll \mu$.

EoM:

$$\dot{\vec{P}}_\omega = \vec{H}_\omega \times \vec{P}_\omega - \frac{1}{L_{\text{coh}}} \left( \vec{P}_\omega - \frac{(\vec{P}_\omega \cdot \vec{H}_\omega)}{H_\omega^2} \vec{H}_\omega \right)$$

$$\vec{H}_\omega = \omega \vec{B} + \mu \vec{P}_\omega, \quad \vec{P}_\omega(0) = P_\omega^0 \hat{e}_3.$$

In the absence of decoherence: $\vec{P}_\omega$ and $\vec{H}_\omega$ precess around $\vec{B}$.

For a non-monochromatic neutrino ensemble with $|\Delta \omega| \ll \mu$:

~ a synchronized precession of all $\vec{P}_\omega$ around $\vec{B}$ with a common angular velocity $\omega_{\text{sync}}$.

With damping: a combination of precession of $\vec{P}_\omega$ and $\vec{H}_\omega$ around $\vec{B}$ and change of the angles between them + shrinkage of $\vec{P}_\omega$. 
“Monochromatic” single-flavour $\nu$ ensemble

$$\vec{H}_\omega = \omega \vec{B} + \mu \vec{P}_\omega \quad \Rightarrow$$

$\vec{P}_\omega$, $\vec{H}_\omega$ and $\vec{B}$ are always in the same plane rotating around $\vec{B}$ with angular velocity $\omega$ $\Rightarrow$ go to rotating frame

$\vec{P}_\omega$ shrinks but the length of $\vec{H}_\omega$ is preserved by the evolution! ($\dot{\vec{H}}_\omega = \mu \dot{\vec{P}}_\omega$)

$\Rightarrow$ The evolution of $\vec{H}_\omega$ in the rotating frame is a simple rotation w.r.t. $\vec{B}$. 
\[ \overrightarrow{H_\omega} = \omega \overrightarrow{B} + \mu \overrightarrow{P_\omega} \]
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$\vec{H}_\omega = \omega \vec{B} + \mu \vec{P}_\omega$
\[ \mathbf{H}_\omega = \omega \mathbf{B} + \mu \mathbf{P}_\omega \]
For $\omega > 0$:

$\vec{P}_\omega$ asymptotically aligns with $\vec{B}$ when $|\vec{H}_\omega| > |\omega \vec{B}| = \omega$, i.e.

$$P_\omega(0) > 2(\omega/\mu) \cos 2\theta$$

In the opposite case: $\vec{P}_\omega$ anti-aligns with $\omega \vec{B}$. 
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For $|\vec{H}_\omega| = \omega$, i.e.

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$- \vec{P}_\omega$ shrinks to zero (complete flavour depolarization).
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Alignment of $\vec{P}_\omega$ with $\omega \vec{B}$: an (almost) complete flavour transition possible.

Anti-alignment: little evolution.
Analytic solution for arbitrary $t$

In the rotating frame:

$$\dot{\vec{H}}_\omega = -\frac{\omega}{L_{coh}} \left[ \frac{(\vec{B} \cdot \vec{H}_\omega) \vec{H}_\omega}{\vec{H}_\omega^2} - \vec{B} \right].$$

$H_\parallel(t)$ and $H_\perp(t)$ — components of $\vec{H}_\omega$ parallel and orthogonal to $\vec{B}$ in the corotating frame. Solutions:

$$H_\parallel(t) = H_0 \frac{\eta(t) - 1}{\eta(t) + 1}, \quad H_\perp^2(t) = H_0^2 \frac{4\eta(t)}{[\eta(t) + 1]^2}$$

with $H_0 \equiv |\vec{H}_\omega(0)|$ and

$$\eta(t) \equiv \left( \frac{H_0 + H_\parallel(0)}{H_0 - H_\parallel(0)} \right) \exp \left[ \frac{2\omega}{L_{coh}H_0} t \right].$$
The components of $\vec{P}_\omega(t)$:

$$P_{\parallel}(t) = (H_{\parallel}(t) - \omega)/\mu, \quad P_{\perp}(t) = H_{\perp}(t)/\mu.$$ 

Asymptotic values of $\vec{H}_\omega$ and $\vec{P}_\omega$ do not depend on $L_{\text{coh}}$.

$L_{\text{coh}}$: determines the time scale $t_{as}$ over which asymptotics is reached.

From expression for $\eta(t)$:

$$t_{as} \sim L_{\text{coh}} \quad \text{for} \quad |\omega| \gtrsim \mu P_\omega(0);$$

$$t_{as} \sim L_{\text{coh}} \frac{\mu P_\omega(0)}{|\omega|} \gg L_{\text{coh}} \quad \text{for} |\omega| \ll \mu P_\omega(0).$$
Bipolar systems

Toy model: isotropic and homogeneous gas consisting initially of monochromatic $\nu_e$ and $\bar{\nu}_e$ with

$$\frac{n_{\nu}}{n_{\bar{\nu}}} = \frac{1 + \epsilon/2}{1 - \epsilon/2}, \quad |\epsilon| < 2.$$ 

Evolution equations:

$$\dot{\mathbf{P}}_\omega = \mathbf{H}_\omega \times \mathbf{P}_\omega - \frac{1}{L_{\text{coh}}} \left( \mathbf{P}_\omega - \frac{\mathbf{P}_\omega \cdot \mathbf{H}_\omega}{H^2_\omega} \mathbf{H}_\omega \right),$$

$$\dot{\mathbf{P}}_{-\omega} = \mathbf{H}_{-\omega} \times \mathbf{P}_{-\omega} - \frac{1}{L_{\text{coh}}} \left( \mathbf{P}_{-\omega} - \frac{\mathbf{P}_{-\omega} \cdot \mathbf{H}_{-\omega}}{H^2_{-\omega}} \mathbf{H}_{-\omega} \right)$$

with

$$\mathbf{H}_{\pm \omega} = \pm \omega \mathbf{B} + \mu \mathbf{D}.$$ 

Initial conditions:

$$\mathbf{P}_\omega(0) = (1 + \epsilon/2)\mathbf{e}_3, \quad \mathbf{P}_{-\omega}(0) = -(1 - \epsilon/2)\mathbf{e}_3.$$
New variables:

\[ \vec{D} \equiv \vec{P}_\omega + \vec{P}_{-\omega}, \quad \vec{Q} \equiv \vec{P}_\omega - \vec{P}_{-\omega} - (\omega/\mu)\vec{B}, \quad \vec{S} \equiv \vec{P}_\omega - \vec{P}_{-\omega}. \]

Satisfy EoMs

\[
\begin{align*}
\dot{\vec{D}} &= \omega \vec{B} \times \vec{Q} - \frac{1}{L_{\text{coh}}} \left[ - K_1 \mu \vec{D} - K_2 \omega \vec{B} \right], \\
\dot{\vec{Q}} &= \mu \vec{D} \times \vec{Q} - \frac{1}{L_{\text{coh}}} \left[ \vec{Q} - K_1 \omega \vec{B} - K_2 \mu \vec{D} \right],
\end{align*}
\]

with

\[
\begin{align*}
K_1 &\equiv \frac{\vec{P}_\omega \cdot \vec{H}_\omega}{\vec{H}_\omega^2} + \frac{\vec{P}_{-\omega} \cdot \vec{H}_{-\omega}}{\vec{H}_{-\omega}^2} - \frac{1}{\mu}, \\
K_2 &\equiv \frac{\vec{P}_\omega \cdot \vec{H}_\omega}{\vec{H}_\omega^2} - \frac{\vec{P}_{-\omega} \cdot \vec{H}_{-\omega}}{\vec{H}_{-\omega}^2}.
\end{align*}
\]
From evolution eqs.:

\[
\mu \vec{D} \cdot \dot{\vec{D}} + \omega \vec{B} \cdot \dot{\vec{Q}} = 0,
\]

\[
\omega \vec{B} \cdot \dot{\vec{D}} + \mu \vec{D} \cdot \dot{\vec{Q}} = 0.
\]

First relation can be integrated:

\[
E_{\text{tot}} \equiv \frac{\mu \vec{D}^2}{2} + \omega \vec{B} \cdot \vec{Q} = \text{const}.
\]

\(E_{\text{tot}}\): conserved total energy of a system of interacting flavour spins in an external “magnetic field” \(\vec{B}\).

Damping due to decoherence thus does not destroy conservation of \(E_{\text{tot}}\)!

Case considered: \(\omega < 0\) (inverted hierarchy), \(\epsilon \geq 0\).
Asymptotically:

\[ \vec{P}_\omega = a \frac{\vec{H}_\omega}{\mu}, \quad \vec{P}_{-\omega} = b \frac{\vec{H}_{-\omega}}{\mu} \]

with constant \(a\) and \(b\).

\(\Rightarrow\) Two possibilities:
Asymptotically:

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(i) \(a = b = 1/2\) \(\Rightarrow\) \(\vec{Q} = 0\), \(\vec{D}\) undefined. \(\vec{S} = \vec{P}_\omega - \vec{P}_{-\omega}\) asymptotically aligned with \(\omega \vec{B}\). But: \(\vec{P}_\omega\) and \(\vec{P}_{-\omega}\) not individually aligned with \(\pm \vec{B}\) \(\Rightarrow\) partial alignment regime.
Bipolar systems – contd.

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(ii) \( a, b \neq 1/2 \),

\[ \vec{D} = \frac{a - b}{1 - (a + b)} \frac{\omega}{\mu} \vec{B}, \]

\[ \vec{Q} = -\frac{(1 - 2a)(1 - 2b)}{1 - (a + b)} \frac{\omega}{\mu} \vec{B}. \]

\( \vec{P}_\omega, \vec{P}_{-\omega} \) (and so \( \vec{D}, \vec{S} \) and \( \vec{Q} \)) are all asymptotically aligned with \( \pm \vec{B} \) (full alignment regime).
Bipolar systems – contd.

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Which of the two regimes is actually realized depends on the value of \( \epsilon \).
Bipolar systems – contd.

Partial alignment regime: $\epsilon < \epsilon_1$.
Full alignment regime: $\epsilon \geq \epsilon_1$.

Partial alignment regime: $\vec{D}$ is not defined by asympt. conditions, but $|\vec{D}|$ can be found from conservation of $E_{\text{tot}}$ (since $\vec{Q} = 0$).
Also: the projection of $\vec{D}$ on $\vec{B}$ can be approximately found. In the partial alignment regime

$$| \int \vec{Q} \cdot \dot{\vec{D}} dt | \ll | \int \vec{D} \cdot \dot{\vec{Q}} dt |$$

$\Rightarrow$ the second relation between derivatives can be approx. integrated.
⇒ Asymptotically:

\[ \vec{B} \cdot \vec{D} \simeq -2 \mu \epsilon /|\omega|. \]

Matching \( \vec{B} \cdot \vec{D} / D \) at the border between the two regimes:

\[ \epsilon_1 \simeq \left[ \frac{2x^3(2 \cos 2\theta - x)}{4 - x^2} \right]^{1/2}, \quad x \equiv \frac{|\omega|}{\mu}. \]

For \( x > 2 \cos 2\theta \simeq 2 \) full alignment regime is realized for all \( \epsilon \).
Bipolar systems – contd.

Full alignment regime: \( \epsilon \geq \epsilon_1 \).
\( \vec{D} \) and \( \vec{S} \) anti-aligned with \( \vec{B} \).

3 regions:

1. \( \epsilon_1 < \epsilon < \epsilon_2 \)
2. \( \epsilon_2 < \epsilon < \epsilon_3 \)
3. \( \epsilon_3 < \epsilon \leq 2 \).

\[ \frac{\omega}{\mu} = -0.5 \]

\( \nu - \nu \) asymmetry parameter \( \epsilon \)
Bipolar systems – contd.

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3 regions:

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Critical values of \( \epsilon \):

\[
\epsilon_2 \simeq \frac{|\omega|}{\mu}, \quad \epsilon_3 \simeq \frac{[4x(x^2 - 4x \cos 2\theta + 4)^{3/2}]^{1/2}}{2 - x \cos 2\theta}.
\]
Bipolar systems – contd.

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Region (2): \( \vec{D} \sim \vec{S} \), i.e. \( \vec{P}_{-\omega} \sim 0 \) (\( \sim \) complete flavour depolarization of antineutrinos). \( \vec{P}_\omega \) can be found from \( E_{\text{tot}} \).
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Critical values of $\epsilon$:

$$\epsilon_2 \simeq \frac{|\omega|}{\mu}, \quad \epsilon_3 \simeq \left[ \frac{4x(x^2 - 4x \cos 2\theta + 4)^{3/2}}{2 - x \cos 2\theta} \right]^{1/2}.$$

Region (2): $\vec{D} \sim \vec{S}$, i.e. $\vec{P}_{-\omega} \sim 0$ ($\sim$ complete flavour depolarization of antineutrinos). $\vec{P}_\omega$ can be found from $E_{\text{tot}}$.

Region 3: $\vec{D} \sim -\epsilon \vec{B}$, $\vec{S} \sim -2\vec{B}$, i.e.

$$\vec{P}_\omega \sim -(1 + \epsilon/2)\vec{B}, \quad \vec{P}_{-\omega} \sim -(1 - \epsilon/2)\vec{B}.$$

Final configuration of flavour spins nearly coincides with the initial one.
Conclusions

- In core-collapse supernovae and in the early Universe neutrino production processes are well localized – neutrino wave packets are very short.

- Since WPs of propagation eigenstates move with different group velocities, they separate very quickly.

- This leads to decoherence effects which may strongly affect collective neutrino oscillations.

- Two simplified neutrino systems considered, qualitatively new effects found (e.g. almost complete neutrino flavour conversion in the case of constant $\mu$).

- Effects of decoherence by WP separation in realistic settings still remain to be studied.
Good news and bad news
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Bad news: We are rather far from having the full understanding of collective effects in neutrino flavour transitions in SNe.
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Good news: We are rather far from having the full understanding of collective effects in neutrino flavour transitions in SNe.
The Schrödinger evolution equation

\[ i \frac{d}{dt} \nu^a = H \nu^a \]

– single particle equation, though \( H = H[\rho] \) depends on the states of all neutrinos in the system:

\[ \rho_{\alpha\beta} = \sum_a \rho_{\alpha\beta}^a, \quad \rho_{\alpha\beta}^a = \langle \nu_{\alpha}^a \nu_{\beta}^{a*} \rangle \]

Can one use single-particle evolution equations in a system of \( N \) neutrinos?

– Yes! (Friedland & Lunardini, 2003)

Flavour spin vector:

\[ \vec{P}_\omega = \sum_a \vec{P}_\omega^a \]
Density matrix in flavour space: $\rho_{\alpha\beta} = \langle \nu_\alpha \nu^*_\beta \rangle$ (Dolgov 1981; Sigl & Raffelt 1993, ...)
Density matrix formalism

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\( \langle \ldots \rangle \): summation over all neutrinos in the system and averaging over “microscopically large but macroscopically small” spatial volumes
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\[ \rho = \rho(t, \vec{x}, \vec{p}) \] is classical in all quantum numbers except discrete species numbers (flavour, mass, matter) that lead to its matrix structure.
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From quantum-theoretic perspective: \( \vec{x}, \vec{p} \) are the expectation values over the corresponding wave functions (wave packets).
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Equation of motion for the density matrix:

$$ \dot{\rho} = -i[H, \rho] $$
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Liouville equation:

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial t} + \vec{x} \frac{\partial}{\partial \vec{x}} + \vec{p} \frac{\partial}{\partial \vec{p}} \rightarrow \frac{\partial}{\partial t} + \vec{x} \frac{\partial}{\partial \vec{x}} \rightarrow \frac{\partial}{\partial t}$$

(only forward scatt.) (uniform $\nu$ bckgrs.)
In quantum theory moving particles are described by wave packets!
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Density matrix: Averaging over volume elements \( d^3x = \bar{r}^3, \quad \bar{r} \gg \sigma_x P \)
\( (d^3x \text{ large compared to the volume of the WP; may contain many neutrinos}). \)
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For non-overlapping wave packets of different neutrinos (no Pauli blocking effects): for each individual neutrino still have

$$\vec{x}^j \approx \vec{v}^j t$$

$\Rightarrow$ $x$-dependence can be eliminated in favour of $t$-dependence.
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$$\bar{x}^j \approx \bar{v}^j t$$

$⇒ x$-dependence can be eliminated in favour of $t$-dependence.

Formalism is very similar to the standard one, except in one important respect: Possibility of separation of wave packets of different mass (or matter) eigenstates composing each neutrino flavour state must be taken into account.
In quantum theory moving particles are described by wave packets!
In quantum theory moving particles are described by wave packets!

For neutrino oscillations in vacuum:

\[
|\nu_{\alpha}(\vec{x}, t)\rangle = \sum_i U_{\alpha i}^* |\nu_i^{\text{mass}}(\vec{x}, t)\rangle = \sum_i U_{\alpha i}^* \Psi_i(\vec{x}, t)|\nu_i^{\text{mass}}\rangle
\]
In quantum theory moving particles are described by wave packets!

For neutrino oscillations in vacuum:

\[ |\nu_\alpha^\text{fl}(x, t)\rangle = \sum_i U_{\alpha i}^* |\nu_i^\text{mass}(x, t)\rangle = \sum_i U_{\alpha i}^* \Psi_i(x, t)|\nu_i^\text{mass}\rangle \]

The coordinate-space wave function of the \(i\)th mass eigenstate (w. packet):

\[
\Psi_i(x, t) = \int \frac{d^3p}{(2\pi)^3} f_i(p) e^{i\vec{p}\vec{x} - iE_i(p)t}
\]

Mom. distribution function \(f_i(p)\): sharp maximum at \(\vec{p} = \vec{P}\) \((\sigma_{pP} \ll P)\).
QM wave packet approach

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For neutrino oscillations in vacuum:

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Mom. distribution function \(f_i(\vec{p})\): sharp maximum at \(\vec{p} = \vec{P}\) \((\sigma_{pP} \ll P)\).

\[
E_i(p) = E_i(P) + \left. \frac{\partial E_i(p)}{\partial \vec{p}} \right|_P (\vec{p} - \vec{P}) + \frac{1}{2} \left. \frac{\partial^2 E_i(p)}{\partial \vec{p}^2} \right|_{\vec{p}_0} (\vec{p} - \vec{P})^2 + \ldots
\]

\[
\vec{v}_i = \frac{\partial E_i(p)}{\partial \vec{p}} = \frac{\vec{p}}{E_i}, \quad \alpha = \frac{\partial^2 E_i(p)}{\partial \vec{p}^2} = \frac{m_i^2}{E_i^2}
\]
\[ \Psi_i(\vec{x}, t) \approx e^{-iE_i(P)t + i\vec{P}\vec{x}} g_i(\vec{x} - \vec{v}_i t) \quad (\alpha \to 0) \]

\[ g_i(\vec{x} - \vec{v}_i t) \equiv \int \frac{d^3p_1}{(2\pi)^3} f_i(\vec{p}_1 + \vec{P}) e^{i\vec{p}_1(\vec{x} - \vec{v}_i t)} \]
QM wave packet approach

\[ \Psi_i(\vec{x}, t) \simeq e^{-iE_i(P)t+\vec{P}\vec{x}} g_i(\vec{x} - \vec{v}_i t), \quad (\alpha \to 0) \]

\[ g_i(\vec{x} - \vec{v}_i t) \equiv \int \frac{d^3p_1}{(2\pi)^3} f_i(\vec{p}_1 + \vec{P}) e^{i\vec{p}_1(\vec{x} - \vec{v}_i t)} \]

Peak of the wave packet: \(\vec{x} - \vec{v}_i t = 0\) – propagates along the classical trajectory. Away from the peak:

\[ \vec{x} \neq \vec{v}_i t, \quad \text{but} \quad |\vec{x} - \vec{v}_i t| \lesssim \sigma_{xP}. \]
QM wave packet approach

\[ \Psi_i(\vec{x}, t) \sim e^{-iE_i(P)t+i\vec{p}\cdot\vec{x}} g_i(\vec{x} - \vec{v}_i t) \quad (\alpha \to 0) \]

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Spatial length of the WP.:  \( \sigma_{xP} \sim 1/\sigma_{pP} . \) E.g. for Gaussian WP

\[ g_i(\vec{x} - \vec{v}_i t) \propto \exp\left[-\frac{(\vec{x} - \vec{v}_i t)^2}{4\sigma_{xP}^2}\right] . \]
What is $\sigma_x$ in the expression for $L_{coh}$?

$$\sigma_x \approx \max\{\sigma_{xP}, \sigma_{xD}\} \quad \text{(Kiers, Nussinov & Weiss, 1996)}$$

E.g. for Gaussian wave packets

$$\sigma_x = \sqrt{\sigma_{xP}^2 + \sigma_{xD}^2}.$$ 

Over the distance $x$ WPs separate by $d = \Delta v_g(x/v_g) = (\Delta m^2/2E^2)x$.

If $x > (2E^2/\Delta m^2)\sigma_{xP} \Rightarrow d > \sigma_{xP}, \text{ WPs no longer overlap.}$

But: Detection process may restore coherence!

If the detection process has high energy resolution (small $\delta E_D$), its duration (time scale) $\delta t_D$ is large.
Possible coherence restoration at detection

If \( \delta t_D > d/v_g \) the “slow” WP will arrive at the detector before the detection is over \( \Rightarrow \) amplitudes of different WPs interfere in the detector.

\[
\delta t_D \sim \sigma_{xD}/v_g; \quad \delta t_D > d/v_g \Rightarrow \sigma_{xD} > \frac{\Delta v_g}{v_g}x
\]

For \( x < \frac{v_g}{\Delta v_g} \sigma_{xD} \) coherence is restored even if \( x > \frac{v_g}{\Delta v_g} \sigma_{xP} \).

The coherence length is

\[
\diamond \quad L_{coh} \approx \frac{v_g}{\Delta v_g} \max\{\sigma_{xP}, \sigma_{xD}\}
\]

For SN \( \nu \) detection at the Earth \( \Rightarrow \) no propagation coherence restoration!

E.g for \( x \sim 10 \text{ kpc} \), \( \Delta m^2 \sim 2.5 \times 10^{-3} \text{ eV}^2 \) and \( E \sim 20 \text{ MeV} \) coherence restoration would require \( \sigma_{xD} > 0.96 \text{ km} \)!