Classification of Lie bialgebras over current algebras

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Abstract

In this paper we give a classification of Lie bialgebra structures on Lie algebras of type $g[[x]]$ and $g[x]$, where $g$ is a simple complex finite dimensional Lie algebra.

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1 Introduction

In what follows $F$ is an algebraically closed field of characteristic zero. By a quantum group we mean a Hopf algebra $A$ over the series $F[[h]]$, where $h$ is a formal parameter such that:

1. $A/hA \cong U(L)$ as a Hopf algebra. Here $U(L)$ is the universal enveloping algebra of some Lie algebra $L$ viewed as a Hopf algebra with comultiplication $\Delta_0 = a \otimes 1 + 1 \otimes a$, $a \in L$.

2. $A \cong V[[h]]$ as a topological $F[[h]]$-module for some vector space $V$ over $F$.

The important example of quantum groups are the quantum universal enveloping algebra $U_h(g)$, the quantum affine Kac-Moody algebra $U_h(\hat{g})$ and the Yangian $Y(g)$, here $g$ in a finite-dimensional simple complex Lie algebra.

If $A$ is a quantum group and $A/hA \cong U(L)$ then $L$ is equipped with a Lie bialgebra structure $\delta : L \rightarrow L \otimes L$, $\delta(b) = h^{-1}(\Delta(a) - \Delta^{op}(a)) \mod (h)$, $b \in L$, here $a$ is a preimage of $b$ in $A$. The Lie bialgebra $(L, \delta)$ is called the classical limit of the quantum group $A$.

P. Etingof and D. Kazhdan proved that an arbitrary Lie bialgebra is a classical limit of some quantum group (see [6] [7]). Moreover, they proved that for any Lie algebra $L$ there exists a one-to-one correspondence between Lie bialgebra structures on $L[[h]]$ and those quantum groups which have $L$ with some $\delta$ as the classical limit.

The aim of this paper is classification of Lie bialgebras on the current algebras $g[[x]]$ and $g[x]$, which is an important step towards classification of quantum groups such that $A/hA \cong U(g[[x]])$ or $A/hA \cong U(g[x])$.

We proceed in the following way.
I. Any Lie bialgebra structure \( \delta \) on a Lie algebra \( L \) gives rise to an embedding \( L \subset D(L, \delta) \) into the Drinfeld double algebra \( D(L, \delta) \) equipped with a nondegenerate skew-symmetric bilinear form (see [5]).

We prove in Theorem 2.10 that there are four types of possible Drinfeld doubles of bialgebras on \( g \). They correspond to the classical \( r \)-matrices

\[
\begin{align*}
 r_1(x, y) &= 0, \quad r_2(x, y) = \frac{\Omega}{x - y}, \\
 r_3(x, y) &= \frac{x \Omega}{x - y} + r_{DJ}, \\
 r_4(x, y) &= \frac{xy \Omega}{x - y}
\end{align*}
\]

where \( \Omega \) is a Casimir (invariant) element of the \( g \)-module \( g \otimes g \) and \( r_{DJ} \) is the Drinfeld-Jimbo classical \( r \)-matrix which will be defined below.

Throughout this paper we use the following root space decomposition (or Cartan decomposition) \( g = h \oplus (\oplus \alpha g_\alpha) \). Here \( h \) is a Cartan subalgebra and \( \{\alpha\} \) is the set of roots with respect to \( h \). We also denote by \( \alpha_i, i = 1, 2, \ldots, \text{rank}(g) \), the set of simple roots, and \( \alpha_0 = -\alpha_{\text{max}} \). We define the positive integers \( k_i \) using the following relation:

\[
\sum_{i=0}^{\text{rank}(g)} k_i \alpha_i = 0, \text{ where } k_0 = 1.
\]

The Drinfeld-Jimbo classical \( r \)-matrix is defined by the following formula:

\[
r_{DJ} = \sum_{\alpha > 0} e_\alpha \otimes e_{-\alpha} + \frac{1}{2} \sum_{i=1}^{\text{rank}(g)} h_{\alpha_i} \otimes h'_{\alpha_i}.
\]

Here \( e_\alpha \in g_\alpha \) are chosen such that the Killing form \( K(e_\alpha, e_{-\alpha}) = 1, h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}] \) and \( \{h'_{\alpha_i}\} \) is the basis of \( h \) dual to \( \{h_{\alpha_i}\} \) with respect to the Killing form.

We also consider the multivariable case \( g[[x_1, x_2, \ldots, x_n]] \). We prove in Theorem 3.1 that there exists only the trivial double over \( g[[x_1, x_2, \ldots, x_n]] \) for \( n \geq 2 \).

II. Using the classification of the Drinfeld doubles on \( g[[x]] \), in Theorems 4.11, 4.16, 4.19 we obtain a classification of the doubles on \( g[[x]] \). The classification table contains seven separate cases and a family of algebras, which can be parametrized by \( \mathbb{CP}^1/\mathbb{Z}_2 \).

III. If Lie bialgebras \( (L, \delta_1) \) and \( (L, \delta_2) \) lead to the same Drinfeld double, then \( \delta_1 - \delta_2 \) is called the classical twist. Using the theory of the maximal orders we classify classical twists in all the cases for \( g[[x]] \) (see Theorems 5.3, 5.5, 5.6, 5.7).

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2 Classification over series

In what follows \( L = g[[x]] \) is the algebra of infinite series over a finite dimensional simple Lie algebra \( g \). Let \( M \) be the maximal ideal of \( F[[x]] \) which consists of series
with zero constant term. A functional \( f : L \to F \) (resp. \( F[[x]] \to F \)) is called a distribution if there exists \( m \geq 1 \), such that \( f(g \otimes M^m) = 0 \) (resp. \( f(M^m) = 0 \)).

Let \( L^* \) denote the space of all distributions on \( L \). Let \( L \otimes L \) be the (topological) tensor square of \( L \). A comultiplication \( \delta : L \to L \otimes L \) defines an algebra structure on \( L^* \) via

\[
\delta(a) = \sum a_{(1)} \otimes a_{(2)}; f, g \in L^*, (fg)(a) = \sum f(a_{(1)})g(a_{(2)}).
\]

Following V. Drinfeld (see [5]) we call \((L, \delta)\) a Lie bialgebra if

- the induced algebra structure on \( L^* \) is a Lie algebra,
- \( \delta \) is a derivation of \( L \) into the \( L \)-module \( L \otimes L \).

The vector space \( D = L \oplus L^* \) is equipped with a natural nondegenerate symmetric bilinear form

\[
(L|L) = (L^*|L^*) = (0), \quad (a|f) = f(a)
\]

for \( a \in L, f \in L^* \). Moreover, there exists a unique Lie algebra structure on \( D \), which extends the multiplications on \( L \) and \( L^* \) respectively and makes the form defined above invariant. We call \( D \) the Drinfeld double of the bialgebra \((L, \delta)\) and denote it \( D(L, \delta) \).

**Lemma 2.1.** Let \( L = \mathfrak{g}[[x]] \). As a \( \mathfrak{g} \)-module \( D(L, \delta) \) is a direct sum of regular (adjoint) \( \mathfrak{g} \)-modules.

**Proof.** Let \( U(\mathfrak{g}) \) be the universal enveloping algebra of the Lie algebra \( \mathfrak{g} \). For a \( \mathfrak{g} \)-module \( V \) let \( \text{Ann}(V) \) denote the ideal \( \text{Ann}(V) = \{ a \in U : aV = 0 \} \). We will start with the following observations: if \( V, W \) are finite dimensional irreducible \( \mathfrak{g} \)-modules and \( \text{Ann}(W)V = 0 \) then \( V \cong W \). Indeed, let \( w_1, \ldots, w_n \) be a basis of \( W \), \( \text{Ann}(w_i) = \{ a \in U : aw_i = 0 \}, 1 \leq i \leq n \), are the corresponding maximal left ideals of \( U(\mathfrak{g}) \),

\[
\text{Ann}(W) = \text{Ann}(w_1) \cap \cdots \cap \text{Ann}(w_n).
\]

If \( v \) is a nonzero element of \( V \) then by our assumption \( \text{Ann}(W) \subset \text{Ann}(V) \). Hence \( V \cong U(\mathfrak{g})/\text{Ann}(v) \) is a homomorphic image of the module \( U(\mathfrak{g})/\text{Ann}(W) \), which is a submodule of \( U(\mathfrak{g})/\text{Ann}(w_1) \oplus \cdots U(\mathfrak{g})/\text{Ann}(w_n) \cong W \oplus \cdots W \). This implies that \( V \cong W \).

Let \( P = \text{Ann}(\mathfrak{g}) \) be the annihilator of the regular \( \mathfrak{g} \)-module. Clearly \( PL = 0 \).

The algebra \( U(\mathfrak{g}) \) has an involution \( * : U(\mathfrak{g}) \to U(\mathfrak{g}) \), which sends an arbitrary element \( a \in \mathfrak{g} \) to \(-a\). Since the algebra \( \mathfrak{g} \) is equipped with a symmetric nondegenerate bilinear form it follows that \( P^* = P \).

For an arbitrary \( x \in U(\mathfrak{g}) \), arbitrary element \( d_1, d_2 \in D \) we have \( (xd_1|d_2) = (d_1|x^*d_2) \). Hence

\[
(PD|L) = (D|P^*L) = (D|PL) = (0).
\]
Hence, $PD \subset L^\perp = L$. Therefore $PPD = (0)$. Since $P$ is the kernel of the homomorphism $U(\mathfrak{g}) \to \text{End}_F(\mathfrak{g})$, it has finite codimension in $U(\mathfrak{g})$. Let

$$U(\mathfrak{g}) = P + \sum_{i=1}^k F a_i, a_i \in U(\mathfrak{g}).$$

Since the algebra $U(\mathfrak{g})$ is Noetherian the ideal $P$ is finitely generated as a left ideal, $P = \sum_{i=1}^l U(\mathfrak{g}) p_i$. Now, for an arbitrary element $d \in D$ we have

$$U(\mathfrak{g})d = \sum_{i=1}^k F a_i d + Pd \leq \sum_{i=1}^k F a_i d + \sum_{i,j} F a_i p_j d + P^2 d = \sum_{i} F a_i d + \sum_{i,j} F a_i p_j d.$$ 

We proved that $\dim_F U(\mathfrak{g})d < \infty$ and therefore $D$ is a direct sum of irreducible finite dimensional $\mathfrak{g}$-modules. From the observation above and from $P^2 D = (0)$ it follows that all these irreducible modules are isomorphic to the regular module. The lemma is proved.

**Corollary 2.2.** There exists an associative commutative $F$-algebra $A$ with 1 such that $D \cong \mathfrak{g} \otimes_F A$.

**Proof.** The corollary immediately follows from the lemma above and from Proposition 2.2 in [2].

**Lemma 2.3.** Let $A$ be an associate commutative $F$-algebra with 1 and let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $F$. Suppose that $( \cdot | \cdot )$ is a symmetric invariant nondegenerate bilinear form on $\mathfrak{g} \otimes_F A$. Then there exists a linear functional $t : A \to F$ such that for arbitrary elements $x, y \in \mathfrak{g}$ and $a, b \in A$ we have

$$(x \otimes a | y \otimes b) = K(x, y) \cdot t(ab),$$

where $K(x, y)$ is the Killing form on $\mathfrak{g}$.

**Proof.** Consider the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha} \mathfrak{g}_\alpha)$. Fix $\alpha \in \Delta$ and choose elements $e_\alpha \in \mathfrak{g}_\alpha, e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $K(e_\alpha, e_{-\alpha}) = 1$. Define the functional $t : A \to F$ via $t(a) = (e_\alpha \otimes 1 | e_{-\alpha} \otimes a)$.

Choose a basis $e_1, \ldots, e_n$ in $\mathfrak{g}$ and the dual basis $e^1, \ldots, e^n$ with respect to the Killing form. Suppose that the Casimir operator $C = \sum_{i=1}^n \text{ad}(e_i)\text{ad}(e^i)$ acts on $\mathfrak{g}$...
as multiplication by $\gamma \in F, \gamma \neq 0$. Choose $a, b \in A$ and let $C(a) : g \otimes A \to g \otimes A$ be defined by

$$C(a) = \sum_{i=1}^{n} \text{ad}(e_i \otimes 1)\text{ad}(e_i \otimes a).$$

Then $e_\alpha \otimes ab = \frac{1}{\gamma} C(a)(e_\alpha \otimes b)$. Since the form $( \mid )$ is invariant, we have

$$\begin{align*}
(e_\alpha \otimes a | e_\alpha \otimes b) &= \frac{1}{\gamma} (C(a)(e_\alpha \otimes 1)| e_\alpha \otimes b) \\
 &= \frac{1}{\gamma} (e_\alpha \otimes 1| C(a)(e_\alpha \otimes b)) \\
 &= (e_\alpha \otimes 1| e_\alpha \otimes ab) \\
 &= t(ab)
\end{align*}$$

Now let $\beta$ be an arbitrary root. Choose $e_\beta \in g_\beta, e_{-\beta} \in g_{-\beta}$. We have

$$([e_\alpha, e_\beta] \otimes a \mid [e_\alpha, e_{-\beta}] \otimes b) = (e_\alpha \otimes 1| [e_\beta, [e_\alpha, e_{-\beta}]] \otimes ab).$$

Let $[e_\beta, [e_\alpha, e_{-\beta}]] = \xi e_\alpha, \xi \in F$. Then

$$(e_\alpha \otimes 1| [e_\beta, [e_\alpha, e_{-\beta}]] \otimes ab) = \xi (e_\alpha \otimes 1| e_\alpha \otimes ab) = \xi t(ab).$$

On the other hand,

$$K([e_\alpha, e_\beta], [e_\alpha, e_{-\beta}]) = K([e_\alpha, e_\beta], [e_\alpha, e_{-\beta}]) = \xi.$$

Hence,

$$([e_\alpha, e_\beta] \otimes a \mid [e_\alpha, e_{-\beta}] \otimes b) = K([e_\alpha, e_\beta], [e_\alpha, e_{-\beta}])t(ab).$$

This implies the lemma. \qed

Let $A$ be an associative commutative unital $F$-algebra with a functional $t : A \to F$.

**Definition 2.4.** We call $(A, t : A \to F)$ a trace extension of $F[[x]]$ if

(i) the bilinear form $(a|b) = t(ab)$ is nondegenerate,

(ii) $F[[x]]^\perp = \{a \in A : (F[[x]]|a) = (0)\} = F[[x]]$,

(iii) for an arbitrary distribution $f : F[[x]] \to F$ there exists an element $a \in A$ such that $f(b) = (b|a)$ for an arbitrary $b \in F[[x]]$.

**Remark 2.5.** If $F[[x]] \subset A' \subset A''$, both $A'$ and $A''$ are trace extensions of $F[[x]]$ and the trace of $A''$ is an extension of the trace of $A'$, then $A' = A''$. 

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Example 2.6. Let $F((x_e))$ be the algebra of Laurent series in $x_e$ with the identity $e$. Let
\[ Ff + Fxf + \cdots + Fx_f^{n-1} = F[x_f|x_f^n = 0] \]
be the algebra of truncated polynomials in $x_f$ with the identity $f: n \geq 1$. The algebra $F[[x]]$ embeds into the direct sum \( \left( \sum_{i=0}^{n-1} Fx_f^i \right) \oplus F((x_e)) \) via $x \rightarrow x_f + x_e$.

Choose a sequence $\alpha = (\alpha_i \in F, -\infty < i \leq n - 2)$ and define a trace
\[
t: \left( \sum_{i=0}^{n-1} Fx_f^i \right) \oplus F((x_e)) \rightarrow F
\]
via $t(x_e^i) = 0$ for $i \geq n$, $t(x_e^{n-1}) = 1$, $t(x_f^i) = \alpha_i$ for $i \leq n - 2$ and $t(x_f^{n-1}) = -1$, $t(x_f^i) = -\alpha_i$ for $0 \leq i \leq n - 2$ if $n \geq 2$.

The algebra
\[
A(n, \alpha) = \left( \sum_{i=0}^{n-1} Fx_f^i \right) \oplus F((x_e))
\]
with the trace $t$ is a trace extension of $F[[x]]$. If $n = 0$ then we let $A(n, \alpha) = F((x)), t(x^{-1}) = 1, t(x^i) = \alpha_i, i \leq -2$.

Example 2.7. (trivial extension) Let $A(\infty) = \sum_{i=0}^{\infty} F a_i + F[[x]]$ with the multiplication $a_i a_j = 0, a_i x^j = a_{i-j}$ if $i \geq j$, otherwise $a_i x^j = 0; t(a_0) = 1, t(a_i) = 0$ for $i \geq 1$.

Definition 2.8. Two trace extensions $(A, t)$ and $(A', t')$ are isomorphic if there exists an algebra isomorphism $f: A \rightarrow A'$ which is identical on $F[[x]]$ and a nonzero scalar $\xi \in F$ such that $t'(f(a)) = \xi t(a)$ for an arbitrary $a \in A$.

Let $(A, t)$ be a trace extension of $F[[x]]$ and let $\phi$ be an automorphism of the algebra $A$, such that $F[[x]]^\phi = F[[x]]$. Define a new trace $t^\phi : A \rightarrow F$ via $t^\phi(a) = t(\phi(a))$. Then $A^\phi = (A, t^\phi)$ is a trace extension of $F[[x]]$ as well.

Consider the group $G = \text{Aut}_F F[[x]]$ of infinite series $x + \gamma_2 x^2 + \gamma_3 x^3 + \ldots, \gamma_i \in F$, with respect to substitution. For $\phi \in G$ the mapping $x_e \rightarrow \phi(x_e), x_f \rightarrow \phi(x_f)$ induces an automorphism of the algebra $A(n, \alpha), F[[x]]^\phi = F[[x]]$. Clearly there exists a sequence $\beta = (\beta_i, i \leq n - 2)$ such that $A(n, \alpha)^{\phi} = A(n, \beta)$.

Proposition 2.9. Let $(A, t)$ be a trace extension of $F[[x]]$. Then either $(A, t) \cong A(\infty)$ or $(A, t) \cong A(n, \alpha)$ for some $n \geq 0, \alpha = (\alpha_i \in F, -\infty < i \leq n - 2)$.

Proof. Consider the descending chain of ideals $A \geq x A \geq x^2 A \geq \ldots$

Case 1. All inclusions $x^n A > x^{n+1} A, n \geq 0$ are strict. We claim that in this case $x^n A \cap F[[x]] = x^n F[[x]]$ for an arbitrary $n \geq 0$. For $n = 0$ the equality is obvious. Choose a minimal $n \geq 1$ such that $x^n F[[x]] \nsubseteq x^n A \cap F[[x]]$. Then $x^n F[[x]] \nsubseteq x^n A \cap F[[x]] < x^{n-1} A \cap F[[x]] = x^{n-1} F[[x]]$. The codimension of $x^n F[[x]]$ in $x^{n-1} F[[x]]$ is 1. Hence $x^n A \cap F[[x]] = x^{n-1} A \cap F[[x]]$. This implies
that $x^{n-1} \in x^n A$ and therefore $x^{n-1} A \subset x^n A$, which contradicts our assumption. Since $F[[x]] \subset A$ is a trace extension there exists an element $a_0' \in A$ such that $t(a_0') = 1$, $t(Ma_0') = 0$. Then $xa_0' \in F[[x]]^{\perp} = F[[x]]$. By the above $xa_0' \in xF[[x]]$. Hence there exists an element $b_0 \in F[[x]]$ such that $xa_0' = xb_0$. Let $a_0 = a'_0 - b_0$. Then $a_0x = 0$, $t(a_0) = 1$, $t(a_0 M) = (0)$. Suppose that we have found elements $a_0, \ldots, a_n \in A$ such that

$$x^i a_j = \begin{cases} a_{j-i} & \text{for } j \geq i \\ 0 & \text{otherwise.} \end{cases}$$

As above there exists an element $a_{n+1}' \in A$ such that $t(x^i a_{n+1}') = \delta_{i,n+1}$. Then $t(x^i(xa_{n+1}')) = \delta_{i,n}$, which implies that $xa_{n+1}' = a_n + b_n, b_n \in F[[x]]$. Now $x^{n+1} a_{n+1}' = x^{n+1} a_n + x^{n+1} b_n$. Hence $x^{n+1} b_n \in x^{n+2} A \cap F[[x]] = x^{n+2} F[[x]]$. Hence $b_n = xc_n, c_n \in F[[x]]$. Let $a_{n+1} = a_{n+1}' - c_n$. We have $x a_{n+1} = x a_{n+1}' - b_n = a_n$. Thus all assumptions about the element $a_{n+1}'$ are satisfied.

For arbitrary $i, j \geq 0$ we have $a_j = x^{i+1} a_{i+j+1}$, which implies $a_i a_j = (x^{i+1} a_j)a_{i+j+1} = 0$. Now $A$ contains a subalgebra $A(\infty) = \text{Span}(a_{i,j}, i \geq 0)+F[[x]]$ which is a trace extension of $F[[x]]$. In view of the Remark 2.5 $A = A(\infty)$.

**Case 2.** There exists $n \geq 0$ such that $x^n \in x^{n+1} A$. Let $n$ be minimal with this property. Let $x^n = x^{n+1} a, a \in A$. Consider the element $e = x^n a^n$. We have $e^2 = (x^{2n}a^n)a^n = x^n a^n = e$.

If $n = 0$ then $e = 1$. Let $f = 1 - e$ and denote $x_e = xe, x_f = xf$. The element $x_e$ is invertible in

$$eA : x_e^{-1} = \begin{cases} x_e^{-1} a^n & \text{if } n \geq 1 \\ a & \text{if } n = 0. \end{cases}$$

From $ax^{n+1} = x^n$ it follows that $ex^n = a^n x^{2k} = x^n$. Hence $x_f^n = 0$. By the minimality of $n$, $a x^n \neq x^{n-1}$, which implies that $e x^{n-1} \neq x^{n-1}$ and therefore $x_f^{n-1} \neq 0$.

The mapping $e F[[x]] \rightarrow F((x_e))$ is an isomorphism, we will identify $eF[[x]]$ with $F((x_e))$. Now $A$ contains the subalgebra $\left( \sum_{i=0}^{n-1} F x_f^i \right) \oplus F((x_e))$. For an arbitrary $i \geq 0$ we have $t(x^i) = t(x_f^i) + t(x_e^i) = 0$. If $t(x_e^{n-1}) = 0$ then $x_e^{n-1} \in F[[x]]^{\perp} = F[[x]]$, which contradicts $x_e^{n-1} \neq 0$. Hence for $n \geq 1$, $t(x_e^{n-1}) \neq 0$. It is clear that for $n = 0$ the element $x_e^{-1}$ does not lie in $F[[x]]$ as well, and therefore $t(x_e^{-1}) \neq 0$. Scaling the trace we can assume that $t(x_e^{n-1}) = 1, t(x_f^{n-1}) = -1$. For $-\infty < i \leq n-2$ let $a_i = t(x_e^i), \alpha = (a_i, -\infty < i \leq n-2)$. Then

$$\left( \sum_{i=0}^{n-1} F x_f^i \right) \oplus F((x_e)) \cong A(n, \alpha).$$

By Remark 2.5

$$\left( \sum_{i=0}^{n-1} F x_f^i \right) \oplus F((x_e)) = A.$$
Let $\mathfrak{g}$ be, as above, a simple finite dimensional Lie algebra, $L = \mathfrak{g} \otimes F A(n, \alpha)$, the nondegenerate symmetric bilinear form on $L$ is defined via $(x \otimes u|y \otimes v) = K(x, y)t(xy)$, where $x, y \in \mathfrak{g}$ and $u, v \in A(n, \alpha)$, $K(x, y)$ is the Killing form.

Our aim is to prove the following theorem.

**Theorem 2.10.**

(i) $D$ contains a subalgebra $W$ such that $(W|W) = (0)$ and $D = \mathfrak{g} \otimes F[[x]] + W$ is a direct sum if and only if $0 \leq n \leq 2$.

(ii) For $0 \leq n \leq 2$ there exists a unique $\phi \in \text{Auto} F[[x]]$ such that $A(n, \alpha) \cong A(n, 0)(\phi)$.

This theorem immediately implies that the Drinfeld double of a Lie bialgebra $(\mathfrak{g} \otimes F[[x]], \delta)$ is isomorphic to to $\mathfrak{g} \otimes A(\infty)$ or to $\mathfrak{g} \otimes A(n, 0)(\phi)$, $0 \leq n \leq 2$.

In what follows we will often write $af$ instead of $a \otimes f$ for $a \in \mathfrak{g}, f \in A(n, \alpha)$.

Suppose that $D$ contains a subalgebra $W$ such that $(W|W) = (0)$ and $D = \mathfrak{g} \otimes F[[x]] + W$ is a direct sum of vector spaces. Suppose that $n \geq 3$. For an arbitrary element $a \in \mathfrak{g}$ fix elements $p(a) = \sum_{i \geq 0} p_i(a)x^i, q(a) = \sum_{i \geq 0} q_i(a)x^i; p_i(a), q_i(a) \in \mathfrak{g}$, such that $ax_e^{n-1} + p(a), ax_e^{n-2} + q(a) \in W$.

Since $W$ is isotropic, for arbitrary elements $a, b \in \mathfrak{g}$, we have

$$(ax_e^{n-1} + p(a))|bx_e^{n-1} + p(b)) = (ax_e^{n-1}|p(b)) + (p(a)|bx_e^{n-1})$$

$$= K(a, p_0(b)) + K(p_0(a), b) = 0$$

and similarly

$$(ax_e^{n-1} + p(a))|bx_e^{n-2} + q(b)) = K(a, q_0(b)) + K(p_1(a), b) = 0$$

Define $p_0^T : \mathfrak{g} \rightarrow \mathfrak{g}$ by $K(p_0(a), b) = K(a, p_0^T(b))$. Then we see that the linear transformation $p_0 = -p_0^T$ is skew-symmetric with respect to $K$ and $p_1^T + q_0 = 0$.

Let $\text{im } p_0, \text{im } q_0$ denote the images of $p_0, q_0$ respectively. Then

$$\ker p_0 = (\text{im } p_0)^\perp, \quad \ker p_1 = (\text{im } q_0)^\perp.$$

Consider the subspace

$$S = \ker p_0 \cap \ker p_1 = (\text{im } p_0 + \text{im } q_0)^\perp.$$ 

In what follows $O(k)$ will stand for an element lying in $\sum_{i \geq k} \mathfrak{g} \otimes x_i^+ + \sum_{i \geq k} \mathfrak{g} \otimes x_i^-.$

**Lemma 2.11.** If $ax_e^k + bx_f^k + O(k + 1) \in W$, $a, b \in \mathfrak{g}$, $k > 1$, then $a = 0$.

**Proof.** If $ax_e^k + bx_f^k + O(k + 1) \in W$, then $k \leq n - 1$. Indeed, if $k \geq n$ then $x_e^k = 0$, $x_e^k = x^k$ and the element lies in $\mathfrak{g} \otimes F[[x]]$. Now choose a counterexample with a
maximal \( k \). Since \( a \neq 0 \) there exists an element \( c \in g \) such that \([c, a], a \neq 0\). The subalgebra \( W \) contains an element \( cx^{-1} + O(0), O(0) \in g \otimes F[[x]] \). Now,

\[
[[cx^{-1} + O(0), ax^k + bx^k + O(k + 1)], ax^k + bx^k + O(k + 1)] = [[c, a], a]x^{2k-1} + O(2k) \in W
\]

But if \( k > 1 \) then \( 2k - 1 > k \). The lemma is proved. \( \Box \)

**Lemma 2.12.** For an arbitrary element \( a \in S \) the subalgebra \( W \) contains an element of the form \( ax^{-1} + O(n) \).

**Proof.** Let \( a \in S \). Then \( p_0(a) = p_1(a) = 0 \) and therefore \( ax^{-1} + p_2(a)x^2 + p_3(a)x^3 \ldots \in W \). Choose a minimal \( k \) such that \( p_k(a) \neq 0 \). If \( k \leq n - 1 \), then \( p_k(a)x^k + O(k + 1) \in W \), which contradicts Lemma 2.11.

If \( k = n - 1 \), then \( (a + p_{n-1}(a))x^{-1} + p_{n-1}(a)x^{-1} + O(n) \in W \). Again by Lemma 2.3 \( a + p_{n-1}(a) = 0 \), hence \( p_{n-1}(a)x^{-1} + O(n) = -ax^{-1} + O(n) \in W \). If \( k \geq n \), then \( ax^{-1} + O(n) \in W \), which again contradicts Lemma 2.11.

Finally, if for any \( k \geq 2 \) we have \( p_k(a) = 0 \), then \( ax^{-1} \in W \), which contradicts Lemma 2.11. The lemma is proved. \( \Box \)

**Lemma 2.13.** \( S^\perp = \{a \in g : \text{there exists } b \in g \text{ such that } af + be + O(1) \in W\} = \{a \in g : a + O(1) \in W\} \).

**Proof.** Let \( X_1 = \{a \in g : \text{there exists } b \in g \text{ such that } af + be + O(1) \in W\}, X_2 = \{a \in g : a + O(1) \in W\} \). Obviously, \( X_2 \subseteq X_1 \). Since \( S^\perp = \text{im } p_0 + \text{im } q_0 \) it follows that \( S^\perp \subseteq X_2 \). Now, let \( a \in X_1, af + be + O(1) \in W \). Choose an arbitrary element \( c \in S \). By Lemma 2.12 there exists an element \( cX^{-1} + O(n) \in W \). Hence \( (af + be + O(1)0x^{-1} + O(n)) = K(a, c) = 0, a \in S^\perp \). The lemma is proved. \( \Box \)

**Lemma 2.14.** For an arbitrary element \( a \in g \) there exists an element of the form \( ae + O(1) \) in \( W \).

**Proof.** For an arbitrary elements \( a \in g \) there exist elements \( h_i(a) \in g, i \geq 0 \) such that \( ae + \sum_{i \geq 0} h_i(a)x^i \in W \). Then \( h_0(a)f + (a + h_0(a))e + O(1) \in W \). By Lemma 2.12 there exists an element of the type \( h_0(a) + O(1) \) in \( W \). Now, \( (ae + \sum_{i \geq 0} h_i(a)x^i) - (h_0(a) + O(1)) = ae + O(1) \in W \). The lemma is proved. \( \Box \)

**Lemma 2.15.** If \( ax_e + bx_f + O(2) \in W, a, b \in g \), then \( a = 0 \).

**Proof.** Let \( I = \{a \in g : \text{there exists } b \in g \text{ such that } ax_e + bx_f + O(2) \in W\}, a \in I \).

By Lemma 2.14 for an arbitrary element \( c \in g \) there exists and element \( ec + O(1) \in W \). Now \( [ax_e + bx_f + O(2), ec + O(1)] = [a, c]x_e + O(2) \in W \). Hence \( I \) is an ideal in \( g \), hence \( I = (0) \) or \( I = g \). Choose another pair of elements \( a', c' \in I, c' \in g \). By the above there above an element of the type \( [a', c'] + O(2) \in W \). Hence

\[
[[a, c]x_e + O(2), [a', c']x_e + O(2)] = [[a, c], [a', c']][x_e^2 + O(3)] \in W.
\]

By Lemma 2.3 \([a, c], [a', c'] = 0 \) and thus \([I, g], [I, g] = 0 \) which implies that \( I = (0) \). The lemma is proved. \( \Box \)
Lemma 2.16. For an arbitrary element \( a \in \mathfrak{g} \) there exists an element of the form \( ax_f + O(2) \) in \( W \).

Proof. If \( a \in \ker p_0 \), then \( p_1(a)x + ax_n^{n-1} + p_2(a)x^2 + \cdots = p_1(a)x + O(2) \in W \). Hence, by Lemma 2.15 \( p_1(a) = 0 \). We proved that \( \ker p_0 \subseteq \ker p_1 \), which implies that \( \text{im} p_0 = (\ker p_0)^\perp \supseteq (\ker p_1)^\perp = \text{im} q_0 \).

Hence for an arbitrary element \( a \in \mathfrak{g} \) there exists \( b \in \mathfrak{g} \) such that \( p_0(b) = q_0(a) \). We have also

\[
q_0(a) + (q_1(a) + a)x_e + q_1(a)x_f + O(2) \in W, \quad p_0(b) + p_1(b)x + O(2) \in W.
\]

Subtracting these two inclusions we get

\[
(q_1(a) + a - p_1(b))x_e + (q_1(a) - p_1(b))x_f + O(2) \in W.
\]

By Lemma 2.15 \( q_1(a) - p_1(b) = -a \), hence \(-ax_f + O(2) \in W \), which proves the lemma.

Proposition 2.17. If \( \mathfrak{g} \otimes A(n, \alpha) \) has a Lagrangian subalgebra \( W \) such that \( \mathfrak{g} \otimes A(n, \alpha) = \mathfrak{g} \otimes F[[x]] + W \) is a direct sum, then \( n \leq 2 \).

Proof. Assume that \( n > 2 \). Since \( \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \), commuting elements of the type \( ax_f + O(2) \in W \) we conclude that for an arbitrary element \( a \in \mathfrak{g} \) there exists an element of the type \( ax_f^{n-2} + O(n - 1) \in W \). Let \( b \in \mathfrak{g} \), \( bx_f + O(2) \in W \). Then

\[
(ax_f^{n-2} + O(n - 1)) (bx_f + O(2)) = K(a, b) = 0.
\]

Hence \( K(\mathfrak{g}, \mathfrak{g}) = (0) \), a contradiction. The proposition is proved.

Now our aim is to prove the following.

Proposition 2.18. If \( A(2, \alpha) \) is a trace extension of \( F[[x]] \) and \( \mathfrak{g} \otimes A(2, \alpha) \) has a Lagrangian subalgebra \( W \), \( \mathfrak{g} \otimes A(2, \alpha) = \mathfrak{g} \otimes F[[x]] + W \), then \( \alpha_0 = 0 \).

Recall that \( A(2, \alpha) = F((x_e)) + Ff + Fx_f \). Hence for an arbitrary element \( a \in \mathfrak{g} \) there exist elements \( p_i(a), q_i(a) \in \mathfrak{g}, i \geq 0 \) such that

\[
ax_e + p_0(a) + p_1(a)x + p_2(a)x^2 + \cdots \in W
\]

\[
ae + q_0(a) + q_1(a)x + \cdots \in W.
\]

Denote \( S = \ker p_0 \cap \ker q_0 \). Just as in the proof of Proposition 2.17

\[
\left( ax_e + \sum_{i \geq 0} p_i(a)x^i \big| bx_e + \sum_{i \geq 0} p_i(b)x^i \right) = 0
\]

implies \( K(p_0(a), b) + K(a, p_0(b)) = 0 \) and

\[
\left( ax_e + \sum_{i \geq 0} p_i(a)x^i \big| be + \sum_{i \geq 0} q_i(b)x^i \right) = 0
\]
implies that
\[ K(a, b) + K(a, q_0(b)) + \alpha_0 K(p_0(a), b) + K(p_1(a), b) = K(a, (1 + q_0)(b)) + K((\alpha_0 p_0 + p_1)(a), b) = 0. \]

Hence,
\[ p_0^T = -p_0, \quad (\alpha_0 p_0 + p_1)^T = -(1 + q_0). \]
This implies \( S^\perp = \text{im } p_0 + \text{im } (1 + q_0). \)

**Lemma 2.19.** \( S = \{ a \in g : ax_e + O(2) \in W \}. \)

*Proof.* The inclusion of \( S \) in the right hand side is obvious. Now let \( z = ax_e + O(2) \in W. \) We have
\[ g \otimes A(2, \alpha) = \sum_{k=1}^{\infty} g x_e^{-k} + ge + gx_e + g \otimes F[[x]]. \]
Comparing degrees, we see that \( z \in g e + gx_e + g \otimes F[[x]]. \) Since \( z \in W \) it follows that
\[ z = (a_0 e + q_0(a_0) + q_1(a_0)x + \ldots) + (a_1 x_e + p_0(a_1) + p_1(a_1)x + \ldots), \]
where \( a_0, a_1 \in g. \) Since \( z \in x_e + O(2) \) it follows that \( a_0 e + q_0(a_0) + p_0(a_1) = 0, \)
\( q_1(a_0)x + a_1 x_e + p_1(a_1)x \in g x_e, \) which implies \( a_0 = 0, \) \( p_0(a_1) = 0, \) \( p_1(a_1) = 0. \)
Therefore \( a \in S. \) The lemma is proved.

**Lemma 2.20.** \( S^\perp = \{ a \in g : \text{there exists } b \in g, \text{ such that } ae + bf + O(1) \in W \}. \)

*Proof.* It is clear that \( S^\perp = \text{im } p_0 + \text{im } (1 + q_0) \) lies in the right hand side. Now suppose that \( a \in g, b \in g, \) and \( ae + bf + O(1) \in W. \) Let \( c \in S. \) Then by Lemma 2.19 we have \( cx_e + O(2) \in W. \) Hence,
\[ (ae + bf + O(1) | cx_e + O(2)) = K(a, c) = 0, \]
which implies the other inclusion. The lemma is proved.

**Lemma 2.21.** \( [S,S] = (0). \)

*Proof.* Let \( a, b \in S. \) Then there exist elements \( ax_e + O(2), bx_e + O(2) \) lying in \( W. \) Hence,
\[ ax_e + O(2), bx_e + O(2) = [a,b]x^2 + O(3) \in W \cap g \otimes F[[x]] = (0), \]
which implies \([a,b] = 0. \) The lemma is proved.

**Lemma 2.22.** \( [g,S] \subset \{ a \in g : ae + O(1) \in W \} \subset S^\perp. \)
Proof. Choose an element \( a \in S \), \( ax_e + O(2) \in W \). For an arbitrary element \( b \in \mathfrak{g} \) there exists an element of the type \( bx_e^{-1} + O(0) \) in \( W \). Hence,

\[
[ax_e + O(2), bx_e^{-1} + O(0)] = [a, b]e + O(1) \in W.
\]

By Lemma 2.20 we see that \([a, b] \in S^\perp \). The lemma is proved.\( \square \)

Lemma 2.23. \([S^\perp, S] \subseteq S \).

Proof. Let \( a \in S \), \( ax_e + O(2) \in W \) and \( b \in S^\perp \), \( be + cf + O(1) \in W \) for some element \( c \in \mathfrak{g} \). Then

\[
[ax_e + O(2), be + cf + O(1)] = [a, b]x_e + O(2) \in W
\]

which implies the claim. The lemma is proved.\( \square \)

Recall that an abelian subalgebra \( I \) of a Lie algebra \( L \) is called an inner ideal if \([[L, I], I] \subseteq I \). Lemmas 2.20, 2.21, and 2.22 imply that \( S \) is an inner ideal of the Lie algebra \( \mathfrak{g} \).

Lemma 2.24. If \( S \neq 0 \) then \( \alpha_0 = 0 \).

Proof. If \( S \neq 0 \) then \( S \) contains a minimal nonzero inner ideal of \( \mathfrak{g} \). In [3] it is proved that minimal inner ideals in a semisimple finite dimensional Lie algebra over an algebraically closed field of zero characteristic are one dimensional. Hence \( S \ni a \neq 0, [a, [a, \mathfrak{g}]] = Fa \). By the Morozov Lemma (see [10]) there exists an element \( b \in \mathfrak{g} \) such that \( a, b = [a, b] \), \( b \) is an \( \mathfrak{sl}_2 \)-triple, that is \( [h, a] = 2a, [h, b] = -2b \). Since \( h \in [a, \mathfrak{g}] \in [S, \mathfrak{g}] \) by Lemma 2.22 it follows that there exists an element \( he + O(1) \) in \( W \). Choose elements \( c, d \in \mathfrak{g} \) such that \( he + cx_e + dx_f + O(2) \in W \).

Since \( a \in S \) there exists an element \( ax_e + O(2) \) in \( W \). Choose an element \( v \in \mathfrak{g} \) such that \( ax_e + vx^2 + O(3) \in W \). We have

\[
[he + cx_e + dx_f + O(2), ax_e + vx^2 + O(3)] = 2ax_e + ([h, y] + [c, a])x^2 + O(3) \in W.
\]

Now

\[
(2ax_e + ([h, y] + [c, a])x^2 + O(3)) - 2(ax_e + vx^2 + O(3)) \in W \cap \mathfrak{g} \otimes F[[x]] = (0).
\]

Hence \([h, y] + [c, a] = 2v\). This implies \( K(b, [h, y]) + K(b, [c, a]) = 2K(b, y) \). But \( K(b, [h, y]) = K(b, h), [h, y] = 2K(b, y) \). Hence \( K(b, [c, a]) = K(h, c) = 0 \).

On the other hand since the element \( he + cx_e + dx_f + O(2) \) lies in \( W \), we have \((he + cx_e + dx_f + O(2))he + cx_e + dx_f + O(2) = \alpha_0 K(h, h) + 2K(h, c) = 0 \).

Since \( K(h, h) \neq 0 \) it follows that \( \alpha_0 = 0 \). The lemma is proved.\( \square \)

Lemma 2.25. \( \ker p_0 \neq (0) \).
Proof. If \( \ker p_0 = (0) \), then \( p_0 \) is invertible. Denote \( f = p_0^{-1} \). Since \( W \) is a subalgebra we have
\[
\left[ ax_c + \sum_i p_1(a) x_i^i, bx_c + \sum_i p_1(b) x_i^i \right] = ([a, p_0(b)] + [p_0(a), b]) x_c + [p_0(a), p_0(b)] + \sum_{i \geq 1} (\ldots) x_i
\]
Therefore \( p_0([p_0(a), b] + [a, p_0(b)]) = [p_0(a), p_0(b)] \) or, equivalently, \( f([a, b]) = [f(a), b] + [a, f(b)] \). Hence, \( f \) is a derivation, which contradicts it being invertible. The lemma is proved. \( \Box \)

Now our aim is to prove that \( \alpha_0 = 0 \). We will assume therefore the contrary. In particular \( S = (0) \) by Lemma 2.24

**Lemma 2.26.** For an arbitrary element \( a \in \ker p_0 \) there exists an element of the form \( ax_f + O(2) \) in \( W \).

**Proof.** Let \( a \in \ker p_0 \). Then
\[
ax_c + p_1(a) + O(2) = (a + p_1(a)) x_c + p_1(a) x_f + O(2) \in W.
\]
For an arbitrary element \( b \in \mathfrak{g} \) choose \( bx_c^{-1} + O(0) \in W \). Then
\[
[[bx_c^{-1} + O(0), (a + p_1(a)) x_c + p_1(a) x_f + O(2)], (a + p_1(a)) x_c + p_1(a) x_f + O(2)] = [[b, a + p_1(a)], a + p_1(a)] x_c + O(2) \in W.
\]
Hence, \( [[\mathfrak{g}, a + p_1(a)], a + p_1(a)] \subseteq S = (0) \), which implies that \( a + p_1(a) = 0 \). Now,
\[
(a + p_1(a)) x_c + p_1(a) x_f + O(2) = -ax_f + O(2) \in W.
\]
The lemma is proved. \( \Box \)

**Lemma 2.27.** For an arbitrary element \( a \in \mathfrak{g} \) there exists a unique element \( b \in \mathfrak{g} \) such that \( ae + bx_c + O(2) \in W \).

**Proof.** Let \( T = \{ a \in \mathfrak{g} : ae + O(1) \in W \} \). Since \( S = (0) \), \( S^\perp = \mathfrak{g} \). By Lemma 2.20 it follows that for an arbitrary element \( c \in \mathfrak{g} \) there exists \( d \in \mathfrak{g} \) such that \( ce + df + O(1) \in W \). Let \( a \in T \). Then
\[
[ae + O(1), ce + df + O(1)] = [a,c]e + O(1) \in W.
\]
Hence \( [T, \mathfrak{g}] \subseteq T \). Hence \( T = \mathfrak{g} \) or \( T = (0) \). Suppose that \( T = (0) \). For an arbitrary element \( a \in \mathfrak{g} \) we have \( ae + q_0(a) + O(1) \in W \). Hence \( \ker q_0 \subseteq T = (0) \), so \( q_0 \) is invertible. This implies that for an arbitrary element \( b \in \mathfrak{g} = \text{im } q_0 \) there exists an element \( c \in \mathfrak{g} \) such that \( bf + ce + O(1) \in W \).
Then \(Ax + T^2.25\). We have shown that \(T = g\). Now it is easy to see that for an arbitrary element \(a \in [T, T] = g\) there exists an element \(b \in g\), such that \(ae + bx_e + O(2) \in W\).

Now, if \(ae + bx_e + O(2)\) and \(ae + b'x_e + O(2)\) both belong to \(W\), then \((b - b')x_e + O(2) \in W\), which implies \(b - b' \in S = (0)\). The lemma is proved. \(\square\)

Now we are ready to finish the proof of Proposition 2.18

**Proof.** We will show that the assumption \(\alpha_0 \neq 0\) leads to a contradiction. By Lemma 2.27 there exists a map \(h : \mathfrak{g} \to \mathfrak{g}\) such that for an arbitrary \(a \in \mathfrak{g}\) the element \(ae + bx_e + O(2)\) lies in \(W\) if and only if \(b = h(a)\). Commuting two such elements, we get

\[
[ae + h(a)x_e + O(2), be + h(b)x_e + O(2)] = [a, b]e + ([h(a), b] + [a, h(b)])x_e + O(2) \in W.
\]

Hence, \(h([a, b]) = [h(a), b] + [a, h(b)]\), i.e. \(h\) is a derivation of \(\mathfrak{g}\). On the other hand, because of the isotropy of \(W\) we have

\[
(ae + h(a)x_e + O(2))|e + h(b)x_e + O(2)) = \alpha_0 K(a, b) + K(h(a), b) + K(a, h(b)) = 0.
\]

Since \(h\) is a derivation it follows that \(K(h(a), b) + K(a, h(b)) = 0\). Thus if \(\alpha_0 \neq 0\) then \(K(a, b) = 0\) for arbitrary \(a, b \in \mathfrak{g}\), which is a contradiction. Proposition 2.18 is proved. \(\square\)

**End of the proof of Theorem 2.10**

**Proof.** If \(A = A(2, \alpha)\), \(\alpha_0 = 0\) then it is not difficult to find coefficients \(\xi_i \in F, i \geq 1\) such that for \(y_e = x_e + \sum_{i \geq 1} \xi_i x_e^{i+1}\) we have \(t(y_e^{-k}) = 0\) for \(k = 1, 2, \ldots\). Let

\[
x_e = y_e + \sum_{i \geq 1} \eta_i y_e^{i+1} \quad \text{and} \quad \phi(x) = x + \sum_{i \geq 1} \eta_i x^{i+1}.
\]

Then \(A(2, \alpha) \cong A(2, 0)\). If \(n = 1\) then

\[
A(1, \alpha) = F((x_e)) \oplus Ff, \quad t(e) = 1, \quad t(f) = -1.
\]

Again there exists a series

\[
y_e = x_e + \sum_{i \geq 1} \xi_i x_e^{i+1}, \quad x_e = y + e + \sum_{i \geq 1} \eta_i y_e^{i+1}
\]
such that \(t(y_e^{-k}) = 0\) for \(k = 1, 2, \ldots\). Then \(A(1, \alpha) \cong A(1, 0)\) for \(\phi(x) = x + \sum_{i \geq 1} \eta_i x^{i+1}\).
In the case of $n = 0$ we have $A(0,\alpha) = F((x))$, $t(x^{-1}) = 1$ and there exist series $y = x + \sum_{i \geq 1} \xi_i x^{i+1}$, $x = y + \sum_{i \geq 1} \eta_i y^{i+1}$ such that $t(y^k) = 0$ for $k = 2, 3, \ldots$. Then

$$A(0,\alpha) \cong A(0,0)^{(\phi)}, \quad \phi(x) = x + \sum_{i \geq 1} \eta_i x^{i+1}.$$ 

This finishes the proof of Theorem 2.10.

**Remark 2.28.** Let us consider the group of $F[[x]]$-linear automorphisms of $g[[x]]$, which we will denote by $Aut_{F[[x]]}(g[[x]])$. Each automorphism $U \in Aut_{F[[x]]}(g[[x]])$ can be extended to an automorphism $\tilde{U}$ of $A(n,\alpha)$. The automorphism $\tilde{U}$ preserves the sequence $\{\alpha_i\}$.

3 Multivariable case

Let us now consider the algebra of the series $F[[X]] = F[[x_1, \ldots, x_n]]$ of $n \geq 2$ variables. Let $F[[X]] \subseteq A$, $t : A \to F$, be a trace extension, that is, the bilinear form $(a|b) = t(ab)$ on $A$ is nondegenerate,

$$F[[X]]^\perp = \{a \in A : (F[[X]]|a) = (0)\} = F[[X]]$$

and for an arbitrary distribution $f : F[[X]] \to F$ there exists an element $a \in A$ such that $f(b) = (b|a)$ for an arbitrary $b \in F[[X]]$.

**Theorem 3.1.** The trace extension $(A, t)$ is isomorphic to a trivial extension.

**Proof.** For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_n$ denote $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Choose elements $b_\alpha \in \alpha \in \mathbb{Z}_n$ such that $t(b_\alpha x_\beta) = \delta_{\alpha,\beta}$, the Kronecker symbol. Our aim is to construct elements $a_\alpha \in b_\alpha + A, \alpha \in \mathbb{Z}_n$ such that

$$a_\alpha x^\beta = \begin{cases} a_{\alpha-\beta} & \text{if } \alpha \geq \beta \\ 0 & \text{otherwise.} \end{cases}$$

We will proceed by induction on $|\alpha| = \alpha_1 + \cdots + \alpha_n$. To start we assume $a_\alpha = 0$ for all $\alpha \in \mathbb{Z}_n \setminus \mathbb{Z}_n$. Let $\epsilon(i) = (0,1,0,\ldots,0)$ (with 1 on the $i$th place).

Choose $\alpha \in \mathbb{Z}_n$ and suppose that the elements $a_{\alpha-\epsilon(i)}$ have been chosen. For an arbitrary $i, 1 \leq i \leq n$ we have $(b_\alpha x_i - a_{\alpha-\epsilon(i)}|F[[X]]) = (0)$. This implies that $b_\alpha x_i-a_{\alpha-\epsilon(i)} = g_{\alpha,i} \in F[[X]]$. Furthermore, $b_\alpha x_i x_j - a_{\alpha-\epsilon(i) - \epsilon(j)} = g_{\alpha,i} x_j = g_{\alpha,j} x_i$ for $i \neq j$. Hence there exists an element $p_\alpha \in F[[X]]$ such that $g_{\alpha,i} = p_\alpha x_i$. We then set $a_\alpha = \{b_\alpha - p_\alpha\}$. It is easy to see that the elements $a_\alpha, \alpha \in \mathbb{Z}_n$ satisfy the above assumptions. Choose $\alpha, \beta \in \mathbb{Z}_n$ and choose $\gamma \in \mathbb{Z}_n$ such that $\gamma > \alpha$. Then $a_\beta = a_{\beta+\gamma} x^\gamma$ and $a_\alpha a_\beta = a_\alpha x^\gamma a_{\beta+\gamma} = 0$. It is easy to see that $A = F[[X]] + \sum F a_\alpha$ is the trivial trace extension of $F[[X]]$. Theorem 3.1 is proved.

\qed
4 Classification over polynomials

The aim of this part is to use Theorem 4.2.10 to classify classical doubles over polynomials. In this section we assume that $F = \mathbb{C}$.

Lemma 4.1. Let $\delta : \mathfrak{g}[x] \to \mathfrak{g}[x] \otimes \mathfrak{g}[y] = (\mathfrak{g} \otimes \mathfrak{g})[x, y]$ be a Lie bialgebra structure on $\mathfrak{g}[x]$. Then it can be extended to $\delta : \mathfrak{g}[[x]] \to (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$.

Proof. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, we can write $ax^2 = [bx, cx]$ where $a, b, z \in \mathfrak{g}$. Hence
\[
\delta(ax^2) = \delta([bx, cx]) = [\delta(bx), cx \otimes 1 + 1 \otimes cy] + [bx \otimes 1 + 1 \otimes by, d(cx)] \in (x, y) \cdot (\mathfrak{g} \otimes \mathfrak{g})[[x, y]].
\]
A simple induction shows that
\[
\delta(ax^n) \in (x, y)^{n-1}(\mathfrak{g} \otimes \mathfrak{g})[[x, y]].
\]
Therefore $\delta(\sum_{n=0}^{\infty} a_n x^n)$ can be defined as $\sum_{n=0}^{\infty} \delta(a_n x^n)$.

It follows from Theorem 2.10 that the classical double $D(\mathfrak{g}[[x]], \delta) \cong \mathfrak{g} \otimes A(n, \alpha)$ where $n = 0, 1, 2$. Our plan is to use the condition $\delta(\mathfrak{g}[x]) \subseteq (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ to get a description of all possible sequences $\alpha$. Since the proofs in all three cases are similar, we will provide detailed proofs in case $n = 0$.

In this case $\mathfrak{g} \otimes A(0, \alpha) \cong \mathfrak{g}(x)$ and the trace $t : \mathbb{C}(\langle x \rangle) \to \mathbb{C}$ is given by the formula $t(x^n) = 0$, $n \geq 0$ and $t(x^{-1}) = 1$, $t(x^{-k}) = a_{k-1}$ for $k \geq 2$. Denote $1 + \sum_{i \geq 1} a_i x^i$ by $a(x)$. Then, evidently the canonical form in the double $A(0, \alpha)$ is given by the formula $(f_1(x)/f_2(x)) = \operatorname{Res}_{x=0}(K'(f_1, f_2) \cdot a(x))$, where $K'$ is the Killing form of the Lie algebra $\mathfrak{g}(x)$ over $\mathbb{C}(\langle x \rangle)$.

Lemma 4.2. Let $W \subset \mathfrak{g} \otimes A(0, \alpha)$ be a subspace. Let $W^\perp$ be the orthogonal complement of $W$ in $\mathfrak{g} \otimes A(0, \alpha) = \mathfrak{g}(x)$ with respect to $(f_1/f_2)_0 = \operatorname{Res}_{x=0}K'(f_1, f_2)$. Then the orthogonal complement $W^\perp$ of $W$ with respect to the form $(f_1/f_2) = \operatorname{Res}_{x=0}(K'(f_1, f_2) \cdot a(x))$ is $1/\alpha(x)W^\perp$.

Theorem 4.3. Let $\mathfrak{g} \otimes A(0, \alpha) = \mathfrak{g}[[x]] \oplus W$, the $W$ is a Lagrangian subalgebra of $A(0, \alpha)$ corresponding to $\delta$. Then $W$ is bounded, i.e., there exists $N$ such that $W \subseteq x^N\mathfrak{g}[x^{-1}]$.

Proof. Let $\{E_k\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form. Since $\mathfrak{g} \otimes A(0, \alpha) = \mathfrak{g}[[x]] \oplus x^{-1}\mathfrak{g}[x^{-1}]$, there exists a basis of $W$, which consists of the following elements:
\[
E_{k, n} = E_k x^{-n} + p_{k, n}(x), n = -1, -2, \ldots,
\]
where $p_{k, n} \in \mathfrak{g}[[x]]$. By Lemma 4.3 the dual basis in $\mathfrak{g}[[x]]$ with respect to the form $(f_1, f_2) = \operatorname{Res}_{x=0}(K'(f_1, f_2) \cdot a(x))$ consists of the elements $1/\alpha(x)E_k x^{n-1} \in \mathfrak{g}[[x]]$. Then
\[
\delta(f(x)) = [f(x) \otimes 1 + 1 \otimes f(y), r(x, y)] = \frac{1}{\alpha(x)} \left( \frac{\Omega}{y-x} + R(x, y) \right),
\]
where \( \Omega = \sum E_k \otimes E_k \in g \otimes g \) is invariant and \( R(x, y) \in (g \otimes g)[[x, y]] \). Let

\[
\frac{1}{a(x)} = 1 + b_1 x + \cdots + b_n x^n \ldots
\]

Then

\[
\frac{1}{a(x)} R(x, y) = \sum_{k,n} E_k x^{n-1} \left( 1 + b_1 x + \cdots + b_k x^k \ldots \right) \otimes p_{k,n}(y)
\]

is well defined because

\[
\frac{1}{a(x)} R(x, y) = \sum_{n,k} E_k x^n \odot \left\{ \text{finite sum of } p_{\alpha,k}(y) \right\}
\]

Let us rewrite \( \frac{1}{a(x)} R(x, y) = \sum_{i,j} c_{ij} x^i y^j \in (g \otimes g)[[x, y]] \) and let us compute \( \tilde{\mathcal{O}}(E) \) for \( E \in g \). We get \( \tilde{\mathcal{O}}(E) = \sum_{i,j} c_{ij} E \otimes 1 + 1 \otimes E x^i y^j \). Since \( \tilde{\mathcal{O}}(E) \in (g \otimes g)[x, y] \) we deduce that there exists a number \( N \) such that \( [c_{ij}, E \otimes 1 + 1 \otimes E] = 0 \) for all \( i, j \geq N \). It follows that

\[
\frac{1}{a(x)} R(x, y) = P(x, y) + p(x, y) \Omega, \text{ where } P(x, y) = (g \otimes g)[x, y] \text{ and } p(x, y) \in C[[x, y]].
\]

Now, we can rewrite

\[
r(x, y) = \frac{\Omega}{a(x)(y-x)} + \frac{(y-x)p(x,y)\Omega}{y-x} + P(x, y) = \frac{A(x,y)\Omega}{y-x} + P(x, y).
\]

Here \( A(x, y) \in C[[x, y]] \).

Now, let us compute for \( E \in g \),

\[
\tilde{\mathcal{O}}(E x) = A(x, y)[\Omega, E \otimes 1] + [P(x, y), E x \otimes 1 + 1 \otimes E y].
\]

Since the second summand is polynomial and \( \tilde{\mathcal{O}}(E x) \) is polynomial, we have proved that \( A(x, y) \) is polynomial.

Let us notice that

\[
\frac{x^i y^j}{y-x} = \sum_{k \geq 0} x^{k+i} y^{k-1+j}
\]

and therefore we can rewrite \( r(x, y) = \sum_{k,n} E_k x^n \otimes s_{k,n}(y) \), where \( s_{k,n}(y) \in y^N g[y^{-1}] \) with \( N \) being maximal of the degrees of \( A(x, y) \) and \( P(x, y) \) in \( y \). Let \( \{ E_k x^n \} \) and \( E_{k,n}(y) \) be the dual bases of \( g[[x]] \) and \( W \) with respect to \( \langle f_1, f_2 \rangle = \text{Res}_{x=0}(K'(f_1, f_2) \cdot a(x)) \). Then

\[
r(x, y) = \sum E_k x^n \otimes E_{k,n}(y)
\]

and we see that \( s_{k,n}(y) = E_{k,n}(y) \) and \( W \subset y^N g[y^{-1}] \). The theorem is proved.

Maximal orders in loop algebras

In what follows we need the so-called orders in \( g((x^{-1})) \). Proofs of the results below can be found in [15, 16].

**Definition 4.4.** Let \( W \subseteq g((x^{-1})) \) be a Lie subalgebra. We say that \( W \) is an order if there exists \( n, k \) such that \( x^{-n} g[[x^{-1}]] \subseteq W \subseteq x^k g[[x^{-1}]] \).
Example 4.5. $W_0 = g[[x^{-1}]]$.

Let us consider the group $\text{Aut}_{\mathbb{C}[x]}(g[x])$. Clearly, there exists a natural embedding $i : \text{Aut}_{\mathbb{C}[x]}(g[x]) \hookrightarrow \text{Aut}_{\mathbb{C}((x^{-1}))}(g((x^{-1})))$. If $\sigma(x) \in \text{Aut}_{\mathbb{C}[x]}(g[x])$, then abusing notations we denote $i(\sigma(x))$ by $\sigma(x)$.

**Definition 4.6.** We say that two orders $W_1$ and $W_2$ are gauge equivalent if there exists $\sigma(x) \in \text{Aut}_{\mathbb{C}[x]}(g[x])$ such that $\sigma(x)W_1 = W_2$.

Now, let us introduce some orders in $g((x^{-1}))$, which will be denoted by $\mathcal{O}_h$.

Fix a Cartan subalgebra $h \subset g$. Let $R$ be the corresponding set of roots and $\Gamma$ the set of simple roots. Denote by $g_{\alpha}$ the root space corresponding to $\alpha \in R$. Consider the valuation on $\mathbb{C}((x^{-1}))$ defined by $v(\sum_{k \geq n} a_kx^{-k}) = n$, $a_n \neq 0$. For every root $\alpha \in R$ and every $h \in \mathfrak{h}(\mathbb{R})$, set

$$M_\alpha(h) := \{ f \in \mathbb{C}((x^{-1})) : v(f) \geq \alpha(h) \}.$$  

Consider $\mathcal{O}_h := h[[x^{-1}]] \oplus (\bigoplus_{\alpha \in R} M_\alpha(h)g_{\alpha})$. It is not difficult to see that $\mathcal{O}_h$ is a Lie subalgebra and, moreover, an order in $g((x^{-1}))$.

Let us consider the following standard simplex

$$\{ h \in \eta(\mathbb{R}) : \alpha(h) \geq 0 \; \forall \alpha \in \Gamma, \; \alpha_{\text{max}}(h) \leq 1 \}.$$  

The vertices of this simplex are $0$ and $h_i$, where $h_i$ are uniquely defined by the condition $\alpha_j(h_i) = \delta_{ij}/k_i$. Here $\alpha_{\text{max}} = \sum k_j\alpha_j$, $\alpha_j \in \Gamma$. Clearly, there exists a one-to-one correspondence between the vertices of the standard simplex and the vertices of the extended Dynkin diagram of $g := \alpha_0 := \alpha_{\text{max}} \leftrightarrow 0$, $\alpha_i \leftrightarrow h_i$. In what follows we will denote $\mathcal{O}_{h_i}$ by $\mathcal{O}_{\alpha_i}$.

**Theorem 4.7.** Let $W \subset g((x^{-1}))$ be an order such that $W + g[x] = g((x^{-1}))$. Then there exists $\sigma(x) \in \text{Aut}_{\mathbb{C}[x]}(g[x])$ such that $\sigma(x)W \subseteq \mathcal{O}_{\alpha_i}$ for some $\alpha_i \in \Gamma$.

**Remark 4.8.** Generally speaking, $\alpha_i$ is not unique, it may happen that there exists $\sigma_1(x) \in \text{Aut}_{\mathbb{C}[x]}(g[x])$ such that $\sigma_1(x)W \subseteq \mathcal{O}_{\alpha_i}$.

**Application of maximal orders to classification of Lie bialgebra structures on $g[x]$**

We continue with the case $D(g[x], \delta) \cong g \otimes A(0, \alpha)$, where $\delta$ is a natural extension of $\delta : g[x] \to (g \otimes g)[x, y]$. Let $g \otimes A(0, \alpha) = g[[x]] \oplus W$ be the Manin triple corresponding to $\delta$. Then according to Theorem 4.3, $W \subseteq x^Ng[x^{-1}]$ for some $N$. Since $W = W^\perp$, we see that $(x^Ng[x^{-1}])^\perp \subset W \subset x^Ng[x^{-1}]$.

**Lemma 4.9.**

$$(x^Ng[x^{-1}])^\perp = \frac{1}{a(x)}x^{-N-2}g[x^{-1}]$$
Theorem 4.11. Assume that $W \supseteq 3$. Therefore, $\sigma \neq H$. Hence it is not hard to compute ($x^{-N}g[x^{-1}]\perp \alpha = x^{-N-2}g[x^{-1}]$).

**Corollary 4.10.** $\frac{a(x)}{x}$ is a polynomial of the form $1 + b_1x + \ldots b_kx^k$, $k \leq N + 2$.

**Proof.** We have
\[
x^{-N-2}g[x^{-1}] \subseteq W \subset x^N g[x^{-1}],
\]
which proves the corollary.

In order to formulate the main result of this section let us make some remarks.

1. $W \subset g[x, x^{-1}] \subset g((x^{-1}))$.
2. $W \cdot \mathbb{C}[[x^{-1}]]$ is an order in $g((x^{-1}))$ because $W \cdot \mathbb{C}[[x^{-1}]] \subseteq x^N g[[x^{-1}]]$ and
\[
W \cdot \mathbb{C}[[x^{-1}]] \supseteq x^{-N-2}(1 + b_1x + \ldots + b_kx^k) \cdot g[[x^{-1}]]
\]
\[
\supseteq x^{-N-k-2}(b_k + b_{k-1}x + \ldots + x^{-k}) \cdot g[[x^{-1}]]
\]
\[
= x^{-N-k-2}g[[x^{-1}]]
\]
since $b_k + b_{k-1}x + \ldots + x^{-k}$ is a unit in $\mathbb{C}[[x^{-1}]]$.
3. Therefore, $W \cdot \mathbb{C}[[x^{-1}]]$ is an order in $g((x^{-1}))$.
4. There exists $\sigma(x) \in \text{Aut}_{\mathbb{C}[x]}(g[x])$ such that $\sigma(x)(W \cdot \mathbb{C}[[x^{-1}]])) \subseteq \mathbb{O}_{\alpha_i}$.

**Theorem 4.11.** Assume that $\sigma(x)(W \cdot \mathbb{C}[[x^{-1}]])) \subseteq \mathbb{O}_{\alpha_i}$. Then

- $\frac{a(x)}{x}$ is a polynomial of degree at most 2 if $k_i = 1$,
- $\frac{a(x)}{x}$ is a polynomial of degree at most 1 if $k_i > 1$.

**Proof.** Clearly $\sigma(x)W = W_1$ also defines a Manin triple $g \otimes A(0, \alpha) = g[[x]] \oplus W_1$ and $W_1 \subseteq g[x, x^{-1}]$. It follows that $W_1 \subset g[x, x^{-1}] \cap \mathbb{O}_{\alpha_i}$. Let us describe $g[x, x^{-1}] \cap \mathbb{O}_{\alpha_i}$. For each $r$, $-k_i \leq r \leq k_i$, let $R_r$ denote the set of all roots, which contain $\alpha_i$ with coefficient $r$. Let $g_0 = h \oplus \sum_{\beta \in R_0} g_\beta$ and $g_r = \sum_{\beta \in R_r} g_\beta$. Then
\[
\mathbb{O}_{\alpha_i} = \sum_{r=1}^{k_i} x^{-1}\mathbb{C}[[x^{-1}]]g_r + \sum_{r=-k_i}^{0} \mathbb{C}[[x^{-1}]]g_r + x\mathbb{C}[[x^{-1}]]g_{-k_i}.
\]

Hence
\[
\mathbb{O}_{\alpha_i} \cap g[x, x^{-1}] = \sum_{r=1-k_i}^{0} \mathbb{C}[[x^{-1}]]g_r + \sum_{r=-k_i}^{0} \mathbb{C}[x^{-1}]g_r + x\mathbb{C}[x^{-1}]g_{-k_i}.
\]

It is not hard to compute $(\mathbb{O}_{\alpha_i} \cap g[x, x^{-1}])\perp \alpha$: we will get
\[
x^{-3}\mathbb{C}[x^{-1}]g_{k_i} + \sum_{r=0}^{k_i-1} x^{-2} + \sum_{r=-1}^{-k_i} \mathbb{C}[x^{-1}]g_r.
\]
By Lemma 4.2 we have

\[
\frac{1}{a(x)} \left( x^{-3}\mathcal{C}[x^{-1}]\mathfrak{g}_{k_1} + \sum_{r=0}^{k_1-1} x^{-2}\mathcal{C}[x^{-1}]\mathfrak{g}_r + \sum_{r=-k_1}^{-1} x^{-1}\mathcal{C}[x^{-1}]\mathfrak{g}_r \right) \subset W_1 \subset \left( \sum_{r=1}^{k_1} x^{-1}\mathcal{C}[x^{-1}]\mathfrak{g}_r + \sum_{r=1-k_1}^{0} \mathcal{C}[x^{-1}]\mathfrak{g}_r + x\mathcal{C}[x^{-1}]\mathfrak{g}_{-k_1} \right).
\]

We see that if \( k_1 = 1 \), then

\[
(O_{\alpha_i} \cap \mathfrak{g}[x, x^{-1}])^{i=0} = x^{-2}(O_{\alpha_i} \cap \mathfrak{g}[x, x^{-1}])
\]

and

\[
x^{-2} \frac{1}{a(x)} (O_{\alpha_i} \cap \mathfrak{g}[x, x^{-1}]) \subseteq O_{\alpha_i} \cap \mathfrak{g}[x, x^{-1}]
\]

Then \( \frac{1}{a(x)} \) is a polynomial of degree at most 2.

If \( k_i \geq 1 \), we can consider the \( \mathfrak{g}_{k_i-1} \)-component. It is easy to see that

\[
\frac{1}{a(x)} x^{-2}\mathcal{C}[x^{-1}]\mathfrak{g}_{k_i-1} \subseteq x^{-1}\mathcal{C}[x^{-1}]\mathfrak{g}_{k_i-1}.
\]

Therefore, \( \deg \left( \frac{1}{a(x)} \right) \leq 1 \). The Theorem is proved.

\[
\square
\]

**Corollary 4.12.** If \( D(\mathfrak{g}[x]), \delta = \mathfrak{g} \otimes A(0, \alpha) \), then \( D(\mathfrak{g}[x], \delta) \cong \mathfrak{g}[x, x^{-1}] \) as Lie algebras. The canonical form on \( \mathfrak{g}[x, x^{-1}] \) is given by the formula \((f_1(x), f_2(x)) = Res_{x=0}(K(f_1, f_2)\alpha(x))\). Up to automorphism \( \gamma : \mathbb{C}[x, x^{-1}] \rightarrow \mathbb{C}[x, x^{-1}] \) given by \( \gamma(x) = cx, c \in \mathbb{C} \), we have the following possibilities for \( a(x) \):

\[
a(x) = 1 \quad (\text{case } A1)
\]

\[
a(x) = \frac{1}{1-x} \quad (\text{case } A2)
\]

\[
a(x) = \frac{1}{(1-x)^2} \quad (\text{case } A3)
\]

\[
a(x) = \frac{1}{(1-m_1x)(1-m_2x)} \quad (\text{case } A4)
\]

In the last case \( m_1 \neq m_2, m_1m_2 \neq 0 \), and the corresponding canonical forms are parameterized by \( \frac{m_1}{m_2} \in (\mathbb{CP}^1 \setminus \{0, 1, \infty \})/\mathbb{Z}_2 \).

The case A1 is well known and \( W \) can be chosen as \( W_1 = x^{-1}\mathfrak{g}[x^{-1}] \). In the case A2, one can easily check that \( W \) can be chosen as

\[
W_2 = \text{Span}\{(x^{-1} - \frac{1}{2})h_i, (x^{-1} - 1)e_{-\alpha}, x^{-1}e_{\alpha}, (1 - x^{-1})x^{-1}\mathfrak{g}[x^{-1}]\}.
\]

In the case A3, one can check that the complementary \( W \) can be chosen as

\[
W_3 = \text{Span}\{(x^{-1} - 1)e_{\pm\alpha}, (x^{-1} - 1)h_i, (1 - x)^2x^{-2}\mathfrak{g}[x^{-1}]\}.
\]
Finally, for $A_4$ one can check that

$$W_{4,m_1,m_2} = \text{Span}\{(x^{-1} - m_1)e_\alpha, (x^{-1} - m_2)\alpha, (x^{-1} - \frac{m_1 + m_2}{2})h_i, (1 - m_1)x(1 - m_2)x^{-2}g[x^{-1}]\}$$

is a Lagrangian subalgebra complementary to $g[x]$.

Let $\sigma$ be the Cartan involution of $g[x, x^{-1}]$, $\sigma(e_\alpha) = e_{-\alpha}$ for simple roots $\alpha_i$. Then $\sigma(g[x]) = g[x]$ and $\sigma(W_{4,m_1,m_2}) = W_{4,m_2,m_1}$, that implies that the forms corresponding to $m_1/m_2$ and $m_2/m_1$ provide isomorphic Lie bialgebras on $g[x]$.

Therefore we conclude that the variety of Lie bialgebras of type $A_4$ is isomorphic to $(\mathbb{CP}^1 \setminus \{0, 1, \infty\})/\mathbb{Z}_2$.

**Corollary 4.13.** The corresponding $r$-matrices are:

$$r_{A1}(x, y) = \frac{\Omega}{y - x}$$

$$r_{A2}(x, y) = \frac{1 - x}{y - x} \Omega - r_{DJ}$$

$$r_{A3}(x, y) = \frac{(x - 1)(y - 1)}{y - x} \Omega$$

$$r_{A4,m_1,m_2}(x, y) = \frac{1 - (m_1 + m_2)u + m_1m_2xy}{y - x} \Omega - r_{m_1,m_2}$$

Here $r_{m_1,m_2} = \sum_{\alpha > 0} m_1 e_{-\alpha} \otimes e_\alpha + m_2 e_\alpha \otimes e_{-\alpha} + \frac{m_1 + m_2}{4} h_i \otimes h_i$ and $r_{DJ} = \frac{1}{2}(r_{-1,1} + \Omega)$, which is the Drinfeld-Jimbo $r$-matrix.

The dual bases are given below:

**Case A1:**

$$\left\{e_{-\alpha} y^{k-1}, e_\alpha y^{k-1}, \frac{1}{2} h_i y^{k-1}\right\}, \quad k = 0, 1, \ldots$$

**Case A2:**

$$\left\{(y^{-1} - 1)e_{-\alpha}, y^{-1}e_\alpha, \frac{1}{2}(y^{-1} - \frac{1}{2})h_i, (y^{-1} - 1)y^{-k}e_{-\alpha}, (y^{-1} - 1)y^{-k}e_\alpha, \frac{1}{2}(y^{-1} - 1)y^{-k}h_i\right\}, \quad k = 1, 2, \ldots$$

**Case A3:**

$$\left\{(y^{-1} - 1)e_{-\alpha}, (y^{-1} - 1)e_\alpha, \frac{1}{2}(y^{-1} - 1)h_i, (1 - y^2)y^{-k-1}e_{-\alpha}, (1 - y^2)y^{-k-1}e_\alpha, \frac{1}{2}(1 - y^2)y^{-k-1}h_i\right\}, \quad k = 1, 2, \ldots$$

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Case A4:

\[(y^{-1} - m_2)e_{-\alpha}, \ (y^{-1} - m_1)e_{\alpha}, \ \frac{1}{2}(y^{-1} - \frac{m_1 + m_2}{2})h_i, \ y^{-k}(y^{-1} - m_1)(y^{-1} - m_2)e_{-\alpha}\]  
\[y^{-k}(y^{-1} - m_1)(y^{-1} - m_2)e_{-\alpha}, \ \frac{1}{2}(y^{-1} - m_1)(y^{-1} - m_2)h_i, \]  
\[k = 1, 2, \ldots \]

**Remark 4.14.** Complementary Lagrangian subalgebras \(W\) are not unique. The corresponding \(r\)-matrices have the form \(r(x, y) = r_A(x, y) + p(x, y)\), where

\[p(x, y) \in \langle g \otimes g \rangle[x, y] \quad \text{and} \quad p(x, y) + p_{21}(y, x) = 0.\]

Later we will present combinatorial data describing all \(W\).

**Remark 4.15.** For the case \(g = sl_2\), the \(r\)-matrix

\[r_{A4, m_1, m_2}(x, y) = \frac{1 - (m_1 + m_2)u + m_1 m_2 uv}{y - x} \Omega - r_{m_1, m_2}\]

is the classical limit of the quantum \(R\)-matrix considered in [8] in connection with exactly solvable stochastic processes.

Now we discuss the two remaining cases

- \(D(\mathfrak{g}[[x]], \delta) = \mathfrak{g}((x)) \oplus \mathfrak{g}\) (case B),
- \(D(\mathfrak{g}[[x]], \delta) = \mathfrak{g}((x)) \oplus \mathfrak{g}[\varepsilon]\), where \(\varepsilon^2 = 0\) (case C).

We remind the reader that the canonical form in case B is given by the following formula:

\[(f_1(x) + g_1) f_2(x) + g_2 = t(K(f_1, f_2)) - K(g_1, g_2),\]

where the trace \(t : \mathbb{C}((x)) \to \mathbb{C}\) is defined by the formulas: \(t(x^k) = 0 \) for \(k \geq 1\), \(t(1) = 1\) and \(t(x^{-k}) = b_k\).

Let us denote the series \(1 + b_1 x + b_2 x^2 + \ldots\) by \(b(x)\). The following results can be proved similarly to that of the case A.

**Theorem 4.16.** Assume that

\[\sigma(x)(W(\mathbb{C}[x^{-1}] \oplus \mathbb{C})) \subseteq \mathfrak{g}_\alpha \oplus \mathfrak{g}.\]

Then \(\frac{1}{b(x)}\) is a polynomial of degree at most 1 if \(k_i = 1\) and \(b(x) = 1\) if \(k_i > 1\).

**Corollary 4.17.** If \(D(\mathfrak{g}[[x]], \delta) = \mathfrak{g}((x)) \oplus \mathfrak{g}\), then \(D(\mathfrak{g}[[x]], \delta) = \mathfrak{g}[x, x^{-1}] \oplus \mathfrak{g}\) as Lie algebras. The canonical form on \(D(\mathfrak{g}[[x]], \delta)\) in this case is given by the formula

\[(f_1(x) + g_1 f_2(x) + g_2) = \text{Res}_{x=0}(K'(f_1, f_2)x^{-1}b(x)) - K(g_1, g_2).\]

Up to automorphism \(\gamma : \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}]\) given by \(\gamma(x) = cx, c \in \mathbb{C}\), we have two possibilities \(b_1(x) = 1\) (case B1) and \(b_2(x) = \frac{1}{1-\alpha}\) (case B2).
Remark 4.18. At this point we note that we need to present $W_1$ and $W_2$, which are Lagrangian subalgebras of $\mathfrak{g}[x, x^{-1}] \oplus \mathfrak{g}$ transversal to $\mathfrak{g}[x] \subset \mathfrak{g}[x, x^{-1}] \oplus \mathfrak{g}$ with respect to the canonical forms determined by $b_1(x)$ and $b_2(x)$.

In the case $B_1$ we can choose

$$W_1 = \text{Span}\{x^{-1}\mathfrak{g}[x^{-1}], (e_\alpha, 0), (0, e_{-\alpha}), (h_i, -h_i)\}.$$ 

In the case $B_2$, it is easy to verify that

$$W_2 = \text{Span}\{(1 - x^{-1})\mathfrak{g}[x^{-1}], (e_\alpha, 0), (0, e_{-\alpha}), (h_i, -h_i)\}$$

satisfies all the conditions we need. The corresponding $r$-matrices are:

$$r_{B_1}(x, y) = \frac{x}{y - x} \Omega + r_{DJ}, \quad r_{B_2}(x, y) = \frac{x(1 - y)}{y - x} \Omega + r_{DJ}.$$ 

Now let us treat the last case $C$. We have:

$$D(\mathfrak{g}[x], \tilde{\delta}) = \mathfrak{g}((x)) \oplus \mathfrak{g}[\varepsilon], \quad \varepsilon^2 = 0.$$

The canonical form in the case $C$ is given by the following formula

$$(f_1(x) + g_1 \varepsilon + h_1, f_2(x) + g_2 \varepsilon + h_2) = t(K'(f_1, f_2) - K(g_1, h_2) - K(h_1, g_2) - c_1 K(h_1, h_2)),$$

where $t : \mathbb{C}((x)) \to \mathbb{C}$ is defined in the following way: $t(x^k) = 0$ for $k \geq 2$, $t(x) = 1$, $t(1) = c_1$, $t(x^{-k}) = c_{k+1}$.

As before, let us denote series $1 + c_1 x + c_2 x^2 + \ldots$ by $c(x)$. Then we have the following results.

**Theorem 4.19.** Assume that

$$\sigma(x)(W(\mathbb{C}[[x^{-1}]] \oplus \mathbb{C})) \subset \mathbb{O}_\alpha \oplus \mathfrak{g}[\varepsilon].$$

Then

- $c(x) = 1$ if $k_i = 1$.
- $\mathbb{O}_\alpha \oplus \mathfrak{g}[\varepsilon]$ does not contain Lagrangian subalgebras if $k_i > 1$.

This theorem implies that the case $C$ coincides with the so-called 4th structure considered in [17]. In particular, $W = \mathfrak{g}[x^{-1}] + \varepsilon \mathfrak{g}$ is a Lagrangian subalgebra transversal to $\mathfrak{g}[x]$ embedded into $\mathfrak{g}[x, x^{-1}] \oplus \mathfrak{g}[\varepsilon]$ as follows:

$$a_0 + a_1 x + \ldots + a_n x^n \longrightarrow (a_0 + a_1 x + \ldots + a_n x^n) \oplus (a_0 + a_1 \varepsilon).$$

The corresponding $r$-matrix is

$$r_{C}(x, y) = \frac{xy}{y - x} \Omega.$$
5 Description of all Lie bialgebra structures on current polynomial Lie algebras

The aim of this section is to describe all Lagrangian subalgebras $W$ of $D(g[x], \delta)$ such that $D(g[x], \delta) = g[x] \oplus W$, direct sum of vector spaces.

It will be done along with classification of solutions of the classical Yang–Baxter equation of certain types. Throughout this section we assume that $W \oplus g[x] = D(g[x], \delta)$.

First of all, we have proved that $W \subset xN_{Og[x]}$. Furthermore, we know that applying some $\sigma(x) \in \text{Aut}_{C[x]}(g[x])$, we can achieve that

- $\sigma(x)W \subset O_{\alpha_i}$ in the case A,
- $\sigma(x)W \subset O_{\alpha_i} \oplus g$ in the case B,
- $\sigma(x)W \subset O_{\alpha_i} \oplus g[\varepsilon]$ in the case C.

Let $p^-_{\alpha_i} \subset g$ be the parabolic subalgebra corresponding to $\alpha_i$, i.e. is generated by all $e_{-\alpha}$ and those $e_{\alpha}$, which do not contain $\alpha_i$ in their simple root decomposition.

Definition 5.1. The data $F(\alpha_i, k_i, L, B)$ consists of a subalgebra $L \subset g$ such that $L + p^-_{\alpha_i} = g$ and a 2-cocycle $B$ on $L$ such that $B$ is nondegenerate on $L \cap p^-_{\alpha_i}$.

It was proved in [14], [15] that in the case A1 there exists a one-to-one correspondence between the sets $\{W \subset O_{\alpha_i}, k_i = 1\}$ and $\{F(\alpha_i, 1, L, B)\}$. In the sequel we will use the following notation $\{W \subset O_{\alpha_i}, k_i = 1\} \leftrightarrow \{F(\alpha_i, 1, L, B)\}$.

Remark 5.2. In the case A1 the set $\{W \subset O_{\alpha_i}, k_i = 3\}$ has a similar but more complicated description and not much is known for other $k_i$.

It turns out that in the cases A3 and C the corresponding Lagrangian subalgebras transversal to $g[x]$ can be completely described by the data $\{F(\alpha_i, 1, L, B)\}$.

Theorem 5.3. 1. Case A3:
(a) $\{W \subset O_{\alpha_i}, k_i = 1\} \leftrightarrow \{F(\alpha_i, 1, L, B)\}$,
(b) the set $\{W \subset O_{\alpha_i}, k_i \geq 2\}$ is empty.

2. Case C:
(a) $\{W \subset O_{\alpha_i} \oplus g[\varepsilon], k_i = 1\} \leftrightarrow \{F(\alpha_i, 1, L, B)\}$,
(b) the set $\{W \subset O_{\alpha_i} \oplus g[\varepsilon], k_i \geq 2\}$ is empty.

Proof. It will be explained later why Cases A3 and C have one and the same description. Then Case C(a) was considered in [13], Case A3(b) follows from Theorem 4.11.

For a description of the Lagrangian subalgebras transversal to $g[x]$ in the remaining cases we will define new data $BD(\alpha_i, k_i, \Gamma_1, \Gamma_2, \tau, s)$.

Definition 5.4. $BD(\alpha_i, k_i, \Gamma_1, \Gamma_2, \tau, s)$ consists of:
1. $\alpha_i \in \hat{\Gamma}$, which is the extended Dynkin diagram of $g$.
2. $\Gamma_1, \Gamma_2 \subset \hat{\Gamma}$ which are such that $\alpha_i \notin \Gamma_1$ and $\alpha_0 \notin \Gamma_2$.
3. $\tau : \Gamma_1 \rightarrow \Gamma_2$ is admissible in sense of Belavin-Drinfeld, i.e. $\langle \tau(\alpha), \tau(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Gamma_1$ and $\tau^n$ is not defined for large $n$.
4. $s \in \Lambda^2 V$, where $V \subset \mathfrak{h}$ is defined by the linear system $(\alpha - \tau(\alpha))(h) = 0$, $\alpha \in \Gamma_1$.

The following theorem can be deduced from [12]

**Theorem 5.5.** In the case $B1$, $\{W \subset \mathcal{O}_{\alpha_i} \oplus g\} \xrightarrow{1-1} \{BD(\alpha_i, k_i, \Gamma_1, \Gamma_2, \tau, s)\}$.

Later we will explain why this result implies

**Theorem 5.6.** The set $\{W \subset \mathcal{O}_{\alpha_i}\} \xrightarrow{1-1} \{BD(\alpha_i, k_i, \Gamma_1, \Gamma_2, \tau, s)\}$ in the case $A2$.

In the remaining cases $A4$ and $B2$, the results are analogous to the theorems above.

**Theorem 5.7.** Both sets $\{W \subset \mathcal{O}_{\alpha_i}, k_i = 1\}$ (case $A4$) and $\{W \subset \mathcal{O}_{\alpha_i} \oplus g, k_i = 1\}$ (case $B2$) are in a one-to-one correspondence with the set $\{BD(\alpha_i, 1, \Gamma_1, \Gamma_2, \tau, s)\}$. The sets $\{W \subset \mathcal{O}_{\alpha_i}\}$ (case $A4$) and $\{W \subset \mathcal{O}_{\alpha_i} \oplus g\}$ (Case $B2$) are empty if $k_i > 1$.

6 **Connection with solutions of the classical Yang-Baxter equation**

We already know that the classical $r$-matrices $r_{X_i}(x, y)$ define Lie bialgebras on $\mathfrak{g}[x]$. Clearly, if $r(x, y) = r_{X_i}(x, y) + p(x, y)$ satisfies the classical Yang-Baxter equation, then $r(x, y)$ defines a Lie bialgebra on $\mathfrak{g}[x]$. Here $X = A, B, C$ and $p(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]$ is such that $p(x, y) = -p^{21}(y, x)$.

It is well known that the classical doubles constructed from $r_{X_i}(x, y)$ and $r(x, y) = r_{X_i}(x, y) + p(x, y)$ are isomorphic as Lie algebras with canonical forms (for proofs see [13] and [11]).

Therefore, the following result is true:

**Theorem 6.1.** Solutions of the classical Yang-Baxter equation of the form

$$r(x, y) = r_{X_i}(x, y) + p(x, y)$$

are in a 1-1 correspondence with the Lagrangian subalgebras $W$ of the respective double satisfying conditions

1. $W \oplus \mathfrak{g}[x] = D(\mathfrak{g}[x], \delta)$ (cases $A, B, C$).
2. $W \subset x^N \mathfrak{g}[x^{-1}]$ (case $A$), $W \subset x^N \mathfrak{g}[x^{-1}] \oplus \mathfrak{g}$ (case $B$) and $W \subset x^N \mathfrak{g}[x^{-1}] \oplus \mathfrak{g}[\varepsilon]$ (case $C$).
Corollary 6.2. In the following pairs of cases, there is a one-to-one correspondence between the corresponding sets of Lagrangian subalgebras of $D(g[x])$ transversal to $g[x]$: $A2$ and $B1$, $A3$ and $C$, $A4$ and $B2$.

Proof. We start the proof with the following observation: In each case there exists automorphisms of $g[x]$ of the form $\tau_i : x \to px + q$, $p, q, s_i \in \mathbb{C}$ such that

$$(\tau_1 \otimes \tau_1)(r_{B1}) = s_1 r_{A2}, \quad (\tau_2 \otimes \tau_2)(r_{C}) = s_2 r_{A3}, \quad (\tau_1 \otimes \tau_1)(r_{B2}) = s_3 r_{A4}$$

(it immediately follows from Corollary 4.13 and Remark 4.18). Therefore, the correspondence between the three cases exists.

Let us explain how the corresponding Lagrangian subalgebras relate to each other. Consider for instance the cases $A3$ and $C$. Assume that $\alpha_i$ has coefficient 1 in decomposition $\alpha_{\text{max}} = \sum k_i \alpha_i$. We can also assume that the Lagrangian subalgebra $W_C$ is contained in $O_{\alpha_i} \oplus g[\varepsilon]$. Then $W_C$ contains the orthogonal complement to $O_{\alpha_i} \oplus g[\varepsilon]$. The latter is $O_{\alpha_i}$. Therefore, $W_C$ is uniquely defined by its image $\hat{W}_C$ under the canonical projection

$$O_{\alpha_i} \oplus g[\varepsilon] \rightarrow g[\varepsilon]$$

Since $W_C$ is transversal to $g[x]$, it is easy to see that $\hat{W}_C$ should be transversal to the image of $g[x] \cap (O_{\alpha_i} \oplus g[\varepsilon])$ under the same projection. This image is isomorphic to $p_{\alpha_i}^{-1}(g[\varepsilon])$, where $(p_{\alpha_i}^{-1})$ is the orthogonal complement to $p_{\alpha_i}$ in $g$ with respect to the Killing form.

Let us turn to the case $A3$. We are looking for a Lagrangian subalgebra $W_{A3} \subset O_{\alpha_i} \cap g[x, x^{-1}]$ transversal to $g[x]$. Then $W_{A3}$ contains the orthogonal complement to $O_{\alpha_i} \cap g[x, x^{-1}]$, which is $x^{-2}(1 - x^2)(O_{\alpha_i} \cap g[x, x^{-1}])$ because $\alpha_i$ has coefficient 1 in the decomposition of the maximal root, and it was proved in [15] that

$$O_{\alpha_i} \cap g[x, x^{-1}] = \text{Ad}(X(x))(O_{\alpha_i} \cap g[x, x^{-1}]),$$

where $X(x) \in \text{Ad}(g[x, x^{-1}])$. It follows that

$$O_{\alpha_i} \cap g[x, x^{-1}] \approx g[\gamma],$$

where $\gamma = x^{-1} - 1$ and $\gamma^2 = 0$. Again we see that $W_{A3}$ is uniquely defined by its image $W_{A3}$ in $g[\gamma]$ and $W_{A3}$ should be transversal to the image of $g[x] \cap O_{\alpha_i}$ in $g[\gamma]$. The latter is again $p_{\alpha_i}^{-1} + p_{\alpha_i}^{-1}$, what completes the proof of the theorem for the cases $A3$ and $C$. Two other cases can be considered in a similar way. □

7 Remarks about quantization

It was proved in [9] that if $(L_1, \delta_1)$ and $(L_2, \delta_2)$ are such that there exists an isometry $\varphi : D(L_1, \delta_1) \rightarrow D(L_1, \delta_2)$ (with respect to the canonical forms) satisfying
\( \varphi(L_1) = L_2 \), then we can find quantizations of \((L_1, \delta_1)\) and \((L_2, \delta_2)\) denoted by \((H_1, \Delta_1)\) and \((H_2, \Delta_2)\) such that \(H_1\) and \(H_2\) are isomorphic as algebras. Moreover, if \(\varphi: H_1 \to H_2\) is this isomorphism, then

\[
(\varphi^{-1} \otimes \varphi^{-1}) \Delta_2(\varphi(a)) = F \Delta_1(a) F^{-1},
\]

where \(F\) satisfies the so called cocycle equation

\[
F_{12}(\Delta_1 \otimes id)(F) = F_{23}(id \otimes \Delta_1)(F).
\]

We call \(F\) a quantum twist.

Hence, we see that all Lie bialgebras related to one and the same case \(X_i\) \((X = A, B, C)\) can be quantized in such a way that they will be isomorphic as algebras and their co-algebra structures will differ by a quantum twist \(F\).

- Quantization of \(A_1\) is called Yangian \([5]\).
- Quantization of \(B_1\) is \(U_q(g[x])\) \([12]\).
- One can show that quantization of \(C_1\) is \(U(g) \otimes RDY(g)\), where \(RDY(g)\) is the restricted dual Hopf algebra to the Yangian \(Y(g)\).
- Quantization of \(A_2\) is the so-called Drinfeldian, \(Drin(g)\) (see \([18]\)).
- Quantizations of the cases \(A_3, B_2, A_{4,m_1,m_2}\) are not known yet.

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