Enhanced Binding in Quantum Field Theory

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May 5, 2014
Abstract

Enhanced binding in quantum field theory and related topics are reviewed, which suggests new possibility, beyond toy models, in the study of the stability of quantum field models. This Lecture Note reviews papers below:

1. F. Hiroshima and H. Spohn, Enhanced binding through coupling to quantum field, Ann. Henri Poincaré 2 (2001), 1159–1187.

2. F. Hiroshima and I. Sasaki, Enhanced binding of an $N$ particle system interacting with a scalar field I, Math. Z. 259 (2008), 657–680.

3. F. Hiroshima, H. Spohn and A. Suzuki, The no-binding regime of the Pauli-Fierz model, J. Math. Phys. 52 (2011), 062104.

4. C. Gérard, F. Hiroshima, A. Panati, and A. Suzuki, Absence of ground state of the Nelson model with variable coefficients, J. Funct. Anal. 262 (2012), 273–299.

This lecture note consists of three parts. Fundamental facts on Boson Fock space are introduced in Part I. Ref. 1 and 3 are reviewed in Part II and, Ref. 2 and 4 in Part III.

In Part I a symplectic structure of a Boson Fock space is studied and a projective unitary representation of an infinite dimensional symplectic group through Bogoliubov transformations is constructed.

In Part II the so-called Pauli-Fierz model (PF model) with the dipole approximation in non-relativistic quantum electrodynamics is investigated. This model describes a minimal interaction between a massless quantized radiation field and a quantum mechanical particle (electron) governed by Schrödinger operator. By applying the Bogoliubov transformation introduced in Part I we investigate the spectrum of the PF model. First the translation invariant case is considered and the dressed electron state with a fixed momentum is studied. Secondly the absence of ground state is proven by extending the Birman-Schwinger principle. Finally the enhanced binding of a ground state is discussed and the transition from unbinding to binding is shown.

In Part III the so-called $N$-body Nelson model is studied. This model describes a linear interaction between a scalar field and $N$-body quantum mechanical particles. First the enhanced binding is shown by checking the so-called stability condition. Secondly the Nelson model with variable coefficients is discussed, which model can be derived when the Minkowskian space-time is replaced by a static Riemannian manifold, and the absence of ground state is proven, if the variable mass decays to zero sufficiently fast. The strategy is based on a path measure argument.
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Part I

Boson Fock space and symplectic structures

1 Boson Fock space

1.1 Second quantization

Let $\mathcal{H}$ be a separable Hilbert space over the complex field $\mathbb{C}$ with the scalar product $\langle \cdot, \cdot \rangle_\mathcal{H}$. Here the scalar product is linear in the second component and antilinear in the first one. We omit subscript $\mathcal{H}$ unless confusion may arise. Consider the operation $\otimes^n_s$ of $n$-fold symmetric tensor product defined through the symmetrization operator

$$S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}, \quad n \geq 1,$$

(1.1)

where $f_1, \ldots, f_n \in \mathcal{H}$ and $\mathcal{P}_n$ denotes the permutation group of order $n$. Define $\otimes^n_s \mathcal{H} = S_n(\otimes^n \mathcal{H})$ with $\otimes^0_s \mathcal{H} = \mathbb{C}$. The space

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \otimes^n_s \mathcal{H},$$

(1.2)

is called the boson Fock space over $\mathcal{H}$. We simply denote $\mathcal{F}(\mathcal{H})$ by $\mathcal{F}$. The boson Fock space $\mathcal{F}$ can be identified with the space of $\ell_2$-sequences $(\Psi^{(n)})_{n \geq 0}$ such that $\Psi^{(n)} \in \otimes^n_s \mathcal{H}$ and $\|\Psi\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \|\Psi^{(n)}\|_{\mathcal{F}}^2 < \infty$. The boson Fock space $\mathcal{F}$ is a Hilbert space endowed with the scalar product

$$\langle \Psi, \Phi \rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} \langle \Psi^{(n)}, \Phi^{(n)} \rangle_{\otimes^n_s \mathcal{H}}.$$

(1.3)

The element $\Omega = (1, 0, 0, \ldots) \in \mathcal{F}$ is called Fock vacuum. In the description of the free quantum field the following operators acting in $\mathcal{F}$ are used. There are two fundamental operators, the creation operator denoted by $a^*(f)$, $f \in \mathcal{H}$, and the annihilation operator by $a(f)$, both acting on $\mathcal{F}$, defined by

$$(a^*(f)\Psi)^{(n)} = \begin{cases} \sqrt{n}S_n(f \otimes \Psi^{(n-1)}), & n \geq 1, \\ 0, & n = 0 \end{cases}$$

(1.4)
with domain $D(a^*(f)) = \{(\Psi^{(n)})_{n \geq 0} \in \mathcal{F} \mid \sum_{n=1}^{\infty} n\|S_n(f \otimes \Psi^{(n-1)})\|^2_\mathcal{F} < \infty\}$, and $a(f) = (a^*(f))^*$. As the terminology suggests, the action of $a^*(f)$ increases the number of bosons by one, while $a(f)$ decreases it by one. Since one is the adjoint operator of the other, the relation $(\Phi, a(f)\Psi)_{\mathcal{F}} = (a^*(f)\Phi, \Psi)_{\mathcal{F}}$ holds. Furthermore, since both operators are closable by the dense definition of their adjoints, we will use and denote their closed extensions by the same symbols. Let $D \subset \mathfrak{h}$ be a dense subset. It is known that

$$\mathcal{F} = \text{L.H.}\{a^*(f_1) \cdots a^*(f_n)\Omega, \Omega \mid f_j \in D, j = 1, \ldots, n, n \geq 1\}$$

(1.5)

where L.H. is a shorthand for the linear hull, and $\{\cdots\}$ denotes the closure in $\mathcal{F}$. The space

$$\mathcal{F}_{\text{fin}} = \{ (\Psi^{(n)})_{n \geq 0} \in \mathcal{F} \mid \Psi^{(m)} = 0 \text{ for all } m \geq M \text{ with some } M \}$$

(1.6)

is called finite particle subspace. The field operators $a, a^*$ leave $\mathcal{F}_{\text{fin}}$ invariant and satisfy the canonical commutation relations

$$[a(f), a^*(g)] = (\bar{f}, g)_\mathfrak{h} \mathbb{1}, \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0$$

(1.7)

on $\mathcal{F}_{\text{fin}}$. Given a bounded operator $T$ on $\mathfrak{h}$, the second quantization of $T$ is the operator $\Gamma(T)$ on $\mathcal{F}$ defined by

$$\Gamma(T) = \bigoplus_{n=0}^{\infty} \otimes^n T.$$  

(1.8)

Here it is understood that $\otimes^0 T = \mathbb{1}$. In most cases $\Gamma(T)$ is an unbounded operator resulting from the fact that it is given by a countable direct sum. However, for a contraction operator $T$, the second quantization $\Gamma(T)$ is also a contraction on $\mathcal{F}$. The map $\Gamma$ satisfies

$$\Gamma(S)\Gamma(T) = \Gamma(ST), \quad \Gamma(S)^* = \Gamma(S^*), \quad \Gamma(\mathbb{1}_\mathfrak{h}) = \mathbb{1}_\mathcal{F}.$$  

(1.9)

For a self-adjoint operator $h$ on $\mathfrak{h}$ the structure relations (1.9) imply in particular that $\{\Gamma(e^{ith})\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\mathcal{F}$. Then by the Stone theorem there exists a unique self-adjoint operator $d\Gamma(h)$ on $\mathcal{F}$ such that

$$\Gamma(e^{ith}) = e^{itd\Gamma(h)}, \quad t \in \mathbb{R}.$$  

(1.10)
The operator \( d\Gamma(h) \) is called the differential second quantization of \( h \) or simply second quantization of \( h \). Since \( d\Gamma(h) = -i \frac{d}{dt} \Gamma(e^{iht}) |_{t=0} \), we have

\[
d\Gamma(h) = 0 \oplus \bigoplus_{n=1}^{\infty} \left( \bigotimes_{j=1}^{n} 1_l \otimes \cdots \otimes h \otimes \cdots \otimes 1_l \right),
\]

where the overline denotes closure, and \( j \) on top of \( h \) indicates its position in the product. Thus the action of \( d\Gamma(h) \) is given by

\[
d\Gamma(h)\Omega = 0,
\]

\[
d\Gamma(h)a^*(f_1) \cdots a^*(f_n)\Omega = \sum_{j=1}^{n} a^*(f_1) \cdots a^*(hf_j) \cdots a^*(f_n)\Omega.
\]

It can also be seen by (1.11) that

\[
\sigma(d\Gamma(h)) = \left\{ \sum_{j=1}^{n} a_j \mid a_j \in \sigma(h), j = 1, \ldots, n, n \geq 1 \right\} \cup \{0\},
\]

\[
\sigma_p(d\Gamma(h)) = \left\{ \sum_{j=1}^{n} a_j \mid a_j \in \sigma_p(h), j = 1, \ldots, n, n \geq 1 \right\} \cup \{0\}.
\]

If \( 0 \not\in \sigma_p(h) \), the multiplicity of 0 in \( \sigma_p(d\Gamma(h)) \) is one. A crucial operator in quantum field theory is the boson number operator defined by the second quantization of the identity operator on \( h \): \( N = d\Gamma(1_l) \). Since \( N\Omega = 0 \) and \( Na^*(f_1) \cdots a^*(f_n)\Omega = na^*(f_1) \cdots a^*(f_n)\Omega \), it follows that \( \sigma(N) = \mathbb{N} \cup \{0\} \). We will use the following facts below.

**Proposition 1.1** Let \( h \) be a nonnegative self-adjoint operator, and \( f \in D(h^{-1/2}) \), \( \Psi \in D(d\Gamma(h)^{1/2}) \). Then \( \Psi \in D(a^2(f)) \) and

\[
\|a(f)\Psi\| \leq \|h^{-1/2}f\| \|d\Gamma(h)^{1/2}\Psi\|,
\]

\[
\|a^*(f)\Psi\| \leq \|h^{-1/2}f\| \|d\Gamma(h)^{1/2}\Psi\| + \|f\|\|\Psi\|.
\]

In particular, \( D(d\Gamma(h)^{1/2}) \subset D(a^2(f)) \), whenever \( f \in D(h^{-1/2}) \).
To obtain the commutation relations between $a^\sharp(f)$ and $d\Gamma(h)$, suppose that $f \in D(h^{-1/2}) \cap D(h)$. Then

$$[d\Gamma(h), a^\sharp(f)]\Psi = a^\sharp(hf)\Psi, \quad [d\Gamma(h), a(f)]\Psi = -a(hf)\Psi,$$ \hspace{1cm} (1.16)

for $\Psi \in D(d\Gamma(h)^{3/2}) \cap \mathcal{D}_{\text{fin}}$. By a limiting argument \ref{1.16} can be extended to $\Psi \in D(d\Gamma(h)^{3/2})$, and it is seen that $a^\sharp(f)$ maps $D(d\Gamma(h)^{3/2})$ into $D(d\Gamma(h))$. In general we can see that $a^\sharp(f)$ maps $D(d\Gamma(h)^{n+1/2})$ into $D(d\Gamma(h)^n)$ for all $n \geq 1$, when $f \in \bigcap_{n=1}^{\infty} D(h^{n/2})$. In particular, $a^\sharp(f)$ maps $\bigcap_{n=1}^{\infty} D(d\Gamma(h)^n)$ into itself. The creation and annihilation operators are realized as

$$(a(f)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n + 1} \int_{\mathbb{R}^d} f(k)\Psi^{(n+1)}(k, k_1, \ldots, k_n)dk, \quad n \geq 0, \hspace{1cm} (1.18)$$

$$(a^\ast(f)\Psi)^{(n)}(k_1, \ldots, k_n) = \begin{cases} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(k_j)\Psi^{(n-1)}(k_1, \ldots, k_j, \ldots, k_n), & n \geq 1, \\ 0, & n = 0. \end{cases} \hspace{1cm} (1.19)$$

Here $\Psi \in \mathcal{F}$ is denoted as a pointwise defined function for convenience, however, all of these expressions are to be understood in $L^2$-sense. Let $\omega : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the multiplication operator called dispersion relation given by

$$\omega(k) = \sqrt{|k|^2 + \nu^2}, \quad k \in \mathbb{R}^d, \hspace{1cm} (1.20)$$

with $\nu \geq 0$. Here $\nu$ describes the boson mass. The second quantization of the dispersion relation is

$$(d\Gamma(\omega)\Psi)^{(n)}(k_1, \ldots, k_n) = \left( \sum_{j=1}^{n} \omega(k_j) \right)\Psi^{(n)}(k_1, \ldots, k_n). \hspace{1cm} (1.21)$$

The self-adjoint operator $d\Gamma(\omega)$ is called the free field Hamiltonian on $\mathcal{F}(L^2(\mathbb{R}^d))$ and we use the notation

$$H_f = d\Gamma(\omega). \hspace{1cm} (1.22)$$
The spectrum of the free field Hamiltonian is \( \sigma(H_f) = [0, \infty) \), with component \( \sigma_p(H_f) = \{0\} \), which is of single multiplicity with \( H_f \Omega = 0 \). Then formally we may write the free field Hamiltonian as

\[
H_f = \int \omega(k) a^*(k) a(k) dk. \tag{1.23}
\]

Physically, this describes the total energy of the free field since \( a^*(k) a(k) \) gives the number of bosons carrying momentum \( k \), multiplied with the energy \( \omega(k) \) of a single boson, and integrated over all momenta. The commutation relations are

\[
[H_f, a(f)] = -a(\omega f), \quad [H_f, a^*(f)] = a^*(\omega f). \tag{1.24}
\]

The relative bound of \( a^2(f) \) with respect to the free field Hamiltonian \( H_f \) can be seen from (1.25) and (1.26). If \( f/\sqrt{\omega} \in L^2(\mathbb{R}^d) \), then

\[
\|a(f)\Psi\| \leq \|f/\sqrt{\omega}\| \|H_f^{1/2}\Psi\|, \tag{1.25}
\]

\[
\|a^*(f)\Psi\| \leq \|f/\sqrt{\omega}\| \|H_f^{1/2}\Psi\| + \|f\| \|\Psi\|. \tag{1.26}
\]

hold.

### 1.2 Segal fields

The creation and annihilation operators are not symmetric and do not commute. Roughly speaking, a creation operator corresponds to \( \frac{1}{\sqrt{2}}(x - \frac{d}{dx}) \) and an annihilation operator to \( \frac{1}{\sqrt{2}}(x + \frac{d}{dx}) \) in \( L^2(\mathbb{R}) \). We can, however, construct symmetric and commutative operators by combining the two field operators and this leads to Segal fields. The Segal field \( \Phi(f) \) on the boson Fock space \( \mathcal{F}(h) \) is defined by

\[
\Phi(f) = \frac{1}{\sqrt{2}}(a^*(\bar{f}) + a(f)), \quad f \in h, \tag{1.27}
\]

and its conjugate momentum by

\[
\Pi(f) = \frac{i}{\sqrt{2}}(a^*(\bar{f}) - a(f)), \quad f \in h. \tag{1.28}
\]

Here \( \bar{f} \) denotes the complex conjugate of \( f \). By the above definition both \( \Phi(f) \) and \( \Pi(g) \) are symmetric, however, not linear in \( f \) and \( g \) over \( \mathbb{C} \). Note that, in contrast,
they are linear operators over $\mathbb{R}$. It is straightforward to check that $[\Phi(f), \Pi(g)] = i\text{Re}(f,g) \mathbb{1}_h$, $[\Phi(f), \Phi(g)] = i\text{Im}(f,g) \mathbb{1}_h$ and $[\Pi(f), \Pi(g)] = i\text{Im}(f,g) \mathbb{1}_h$. In particular, for real-valued $f$ and $g$ the canonical commutation relations become

$$[\Phi(f), \Pi(g)] = i(f,g) \mathbb{1}_h, \quad [\Phi(f), \Phi(g)] = i\text{Im}(f,g) \mathbb{1}_h$$

and $[\Pi(f), \Pi(g)] = i\text{Im}(f,g) \mathbb{1}_h$. In particular, for real-valued $f$ and $g$ the canonical commutation relations become

$$[\Phi(f), \Pi(g)] = i(f,g) \mathbb{1}_h, \quad [\Phi(f), \Phi(g)] = i\text{Im}(f,g) \mathbb{1}_h, \quad [\Pi(f), \Pi(g)] = i\text{Im}(f,g) \mathbb{1}_h$$

Applying the inequalities \((1.25)\) and \((1.26)\) to $h = \mathbb{1}$, we see that $F_{\text{fin}}$ is the set of analytic vectors of $\Phi(f)$, i.e., $\lim_{m \to \infty} \sum_{n=0}^{m} \|\Phi(f)^n \Psi\|t^n/n! < \infty$ for $\Psi \in F_{\text{fin}}$ and $t \geq 0$. The following is a general result.

**Proposition 1.2 (Nelson’s analytic vector theorem)** Let $K$ be a symmetric operator on a Hilbert space. Assume that there exists a dense subspace $D \subset D(K)$ such that $\lim_{m \to \infty} \sum_{n=0}^{m} \|K^n f\|t^n/n! < \infty$, for $f \in D$ and some $t > 0$. Then $K$ is essentially self-adjoint on $D$, and $e^{-tK}\Phi = \text{s-lim}_{m \to \infty} \sum_{n=0}^{m} t^n K^n f/n!$ follows for $f \in D$.

By Nelson’s analytic vector theorem both $\Phi(f)$ and $\Pi(g)$ are essentially self-adjoint on $F_{\text{fin}}$. We keep denoting the closures of $\Phi(f)\big|_{F_{\text{fin}}}$ and $\Pi(g)\big|_{F_{\text{fin}}}$ by the same symbols.

### 1.3 Wick product

Loosely speaking, the so-called Wick product $:a^{\sharp}(f_1) \cdots a^{\sharp}(f_n):$ is defined in a product of creation and annihilation operators by moving the creation operators to the left and the annihilation operators to the right without taking the commutation relations into account. The Wick product $:\prod_{i=1}^{n} \Phi(g_i):$ is recursively defined by the equalities

$$:\Phi(f): = \Phi(f), \quad :\Phi(f) \prod_{i=1}^{n} \Phi(f_i): = \Phi(f) \prod_{i=1}^{n} \Phi(f_i) - \frac{1}{2} \sum_{j=1}^{n} (f, f_j) :\prod_{i \neq j} \Phi(f_i):$$

By the above definition we have

$$:\Phi(f)^n: = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} \Phi(f)^{n-2k} \left( -\frac{1}{4} \|f\|^2 \right)^k.$$  \hfill (1.30)
Note that \( \Phi(f_1) \cdots \Phi(f_n): \Omega = 2^{-n/2} a^*(f_1) \cdots a^*(f_n) \Omega \). From this
\[
\left( \prod_{i=1}^{n} \Phi(g_i): \Omega, \prod_{i=1}^{m} \Phi(f_i): \Omega \right) = \delta_{nm} 2^{-n/2} \sum_{\pi \in \mathcal{P}_n} \prod_{i=1}^{n} (g_i, f_{\pi(i)}) \tag{1.31}
\]
follows. The Wick product of the exponential can be computed directly to yield
\[
: e^{\alpha \Phi(f)}: \Omega = e^{-\left(\frac{1}{4}\alpha^2\|f\|^2\right)} e^{\alpha \Phi(f)} \Omega. \tag{1.32}
\]
Hence for real-valued \( f \) and \( g \),
\[
(\Omega, e^{\alpha \Phi(f)} \Omega_b) = e^{\left(\frac{1}{4}\alpha^2\|f\|^2\right)}, \quad \alpha \in \mathbb{C}. \tag{1.33}
\]
For example \( (\Omega, e^{i \Phi(f)} \Omega_b) = e^{-\left(\frac{1}{4}\|f\|^2\right)} \) and \( (\Omega, e^{\Phi(f)} \Omega_b) = e^{\left(\frac{1}{4}\|f\|^2\right)} \).
2 Symplectic structure

2.1 Infinite dimensional symplectic group

In this section we investigate an infinite dimensional symplectic group and its projective unitary representation on a Fock space. Symplectic transformations leave canonical commutation relations invariant. By symplectic transform of the annihilation operators and the creation operators \( \{a^\#\} \) we can construct operators \( \{b^\#\} \) satisfying the same canonical commutation relations. However it is not necessarily unitarily equivalent with each others if the dimension of the configuration space is infinity. We will see it in Proposition 2.1. We will also give an application of symplectic group in Section 4 to study the spectrum of some quadratic self-adjoint operator. The general reference of this section is [Ara91, Ber66, HI04, Rui77, Rui78, Seg70, Sha62].

Let \( C \) be a conjugation on \( \mathfrak{h} \), i.e., \( C \) is an antilinear isometry on \( \mathfrak{h} \) with \( C^2 = \mathbb{1} \). For \( f \in \mathfrak{h} \) and \( T \in B(\mathfrak{h}) \) \( (B = B(\mathfrak{h}) \) is the set of bounded linear operators on \( \mathfrak{h} \)), we define \( \tilde{f} \in \mathfrak{h} \) and \( \tilde{T} \in B(\mathfrak{h}) \) by \( \tilde{f} = Cf \) and \( \tilde{T} = CTC \). Let \( I_2 = I_2(\mathfrak{h}) \) denote the set of Hilbert-Schmidt operators on \( \mathfrak{h} \). We denote the norm (resp. Hilbert-Schmidt norm) of a bounded operator \( X \) on \( \mathfrak{h} \) by \( \|X\| \) (resp. \( \|X\|_2 \)). For \( S, T \in B \) we define

\[
A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}
\] (2.1)

by

\[
A(\phi \oplus \psi) = (S\phi + T\psi) \oplus (T\phi + S\psi).
\] (2.2)

Let

\[
J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.
\] (2.3)

Then \( J(\phi \oplus \psi) = \phi \oplus (-\psi) \). We define a symplectic group \( \mathfrak{Sp} \) by

\[
\mathfrak{Sp} = \left\{ A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \mid AJA^* = A^*JA = J \right\}
\] (2.4)

and the subgroup \( \mathfrak{Sp}_2 \subset \mathfrak{Sp} \) by

\[
\mathfrak{Sp}_2 = \left\{ A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{Sp} \mid T \in I_2 \right\}.
\] (2.5)

Here \( A^* \) is the adjoint of \( A \), i.e.,

\[
A^* = \begin{pmatrix} S^* & T^* \\ T^* & S^* \end{pmatrix}.
\]
Note also that the inverse of $A \in \mathfrak{sp}$ is given by

$$A^{-1} = J A^* J = \begin{pmatrix} S^* & -T^* \\ -T & S \end{pmatrix}.$$

We can see that $A$ induces the following maps:

$$
\begin{pmatrix} a(f) \\ a^*(f) \end{pmatrix} \mapsto \begin{pmatrix} b_A(f) \\ b_A^*(f) \end{pmatrix} = \begin{pmatrix} a^*(T f) + a(S f) \\ a^*(\bar{S} f) + a(\bar{T} f) \end{pmatrix}.
$$

(2.6)

Crucial fact is that the map leaves both canonical commutation relations

$$[b_A(f), b_A^*(g)] = (\bar{f}, g) \mathbb{1}, \quad [b_A(f), b_A(g)] = 0 = [b_A^*(f), b_A^*(g)],$$

(2.7)

and adjoint relation $(\Psi, b_A^*(f) \Phi) = (b_A(\bar{f}), \Psi, \Phi)$. Furthermore $a^\sharp$ can be represented in terms of $b_A^\sharp$:

$$a(f) = b_A(S^* f) - b_A^*(\bar{T^*} f),$$

(2.8)

$$a^*(f) = -b_A(T^* f) + b_A^*(S^* f).$$

(2.9)

In particular the Segal field and its conjugate are represented as

$$
\phi(f) = \frac{1}{\sqrt{2}} (b_A^*(S^* f - T^* \bar{f}) + b_A(S^* f - T^* \bar{f})),
$$

(2.10)

$$
\pi(f) = \frac{i}{\sqrt{2}} (b_A^*(S^* f + T^* \bar{f}) - b_A(S^* f + T^* \bar{f})).
$$

(2.11)

We will see below that $b_A^\sharp(f)$ and $a^\sharp(f)$ are unitarily equivalent if and only if $T$ is Hilbert Schmidt operator and will construct the unitary operator implementing this unitary equivalence.

**Proposition 2.1 (Necessary and sufficient condition to the unitary equivalence)** Let $A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{sp}_2$ and define $b_A^\sharp(f)$ by (2.6). Then there exists a unitary operator $U$ such that $U^{-1} b_A^\sharp(f) = a^\sharp(f)$ if and only if $T$ is the Hilbert Schmidt operator.

**Proof:** We give the proof of the necessary part only for the case of $\mathfrak{h} = L^2(\mathbb{R}^d)$. The proof of sufficient part is given in Proposition 2.5.
Set $\Omega_A = U\Omega$. Then $b_A(f)\Omega_A = 0$ for all $f \in L^2(\mathbb{R}^d)$. Hence $(a(Sf) + a^*(Tf))\Omega_A = 0$ for all $f \in L^2(\mathbb{R}^d)$. Let $P_n$ be the projection from $\mathcal{F}$ to the $n$-particle subspace. Then we have

$$a(Sf)P_{n+2}\Omega_A = P_{n+1}a(Sf)\Omega_A = -P_{n+1}a^*(Tf)\Omega_A = -a^*(Tf)P_n\Omega_A.$$ 

When $P_m\Omega_A = 0$, $a(Sf)P_{m+2}\Omega_A = 0$ for all $f \in L^2(\mathbb{R}^d)$, and then $a(f)P_{m+2}\Omega_A = 0$, since $S^{-1}$ exists. Hence $P_m\Omega_A = 0$ implies that $P_{m+2}\Omega_A = 0$. Since $b_A(f)\Omega_A = 0$, $a(Sf)P_1\Omega_A = 0$ and then $P_n\Omega_A = 0$ for all odd number $n$. If furthermore $P_0\Omega_A = 0$, $P_m\Omega_A = 0$ for all even number $m$, and it implies that $\Omega_A = 0$. Since $\Omega_A \neq 0$, $\kappa = P_0\Omega_A \neq 0$ follows. Notice that $\Phi = P_2\Omega_A$ is a function belonging to $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$a(Sf)P_2\Omega_A = -a^*(Tf)P_0\Omega_A. \quad (2.12)$$

We see that $a(Sf)P_2\Omega_A = \sqrt{2} \int (Sf)(k')\Phi(k', k)dk'$ and $-a^*(Tf)P_0\Omega_A(k) = \kappa Tf(k)$. Let $K_\Phi$ be the Hilbert Schmidt operator defined by $K_\Phi f(k) = \int f(k')\Phi(k', k)dk$. We then conclude that

$$(Tf)(k) = \frac{\sqrt{2}}{\kappa} \int (Sf)(k')\Phi(k', k)dk' = K_\Phi Sf(k).$$

Since $S$ is bounded and $K_\Phi$ is Hilbert-Schmidt, $T$ is Hilbert-Schmidt operator. \qed

### 2.2 Quadratic operators

Let $K \in I_2$ and $S \in B$. Then there exist two orthonormal systems $\{\psi_n\}$ and $\{\phi_n\}$ in $\mathfrak{h}$ and a positive sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ such that $Kf = \sum_{n=0}^\infty \lambda_n (\psi_n, f)\phi_n$ with $\sum_{n=0}^\infty \lambda_n^2 = \|K\|_2^2$, where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm.

**Lemma 2.2** [Rui77, Rui78, Ara90] Let $\{e_n\}$ be an arbitrary complete orthonormal system of $\mathfrak{h}$. Then for $\Psi \in \mathcal{F}_{\text{fin}}$, sequences

$$\left\{ \sum_{n=1}^M \lambda_n a^*(\bar{\psi}_n) a^*(\phi_n)\Psi \right\}, \left\{ \sum_{n=1}^M \lambda_n a(\bar{\psi}_n)a(\phi_n)\Psi \right\}, \left\{ \sum_{n=1}^M a^*(e_n)a(S^*e_n)\Psi \right\} \quad (2.13)$$

strongly converge as $M \to \infty$. 

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Proof: We check only the convergence of \( \sum_{n=1}^{M} \lambda_n a^*(\psi_n)a^*(\phi_n) \Psi \). The others are similar or rather simpler. We have

\[
\left\| \sum_{n=1}^{M} \lambda_n a^*(\psi_n)a^*(\phi_n) \Psi \right\|^2 = \sum_{m,n} \lambda_n \lambda_m (\Psi, a(\phi_n)a(\psi_m))\Psi
\]

and

\[
a(\psi_n)a(\psi_m)a(\phi_m) = \delta_{nm} a(\phi_n)a(\phi_m) + \delta_{nm} a(\phi_m)a(\phi_n) + (\psi_n, \phi_m)a(\phi_n)a(\phi_m) + (\phi_n, \psi_m)a(\phi_m)a(\phi_n) + a^*(\psi_m)a(\phi_m)a(\phi_n).
\]

We will estimate \( \sum_{m,n} \lambda_m \lambda_n (\Phi, (\cdot) \Phi) \) for (\cdot) = (2.14), (2.15), (2.16) separately. For (2.14) we have

\[
\sum_{m,n} \lambda_m \lambda_n (\Phi, (2.14) \Phi) = \lambda_n^2 \| a(\phi_n) \Phi \|^2 + \sum_n \lambda_n^2 \| a(\psi_n) \Phi \|^2 \leq 2 \sum_n \lambda_n^2 \| (N + 1) \Phi \|^2 = 2 \| K \|_2^2 \| (N + 1) \Phi \|^2.
\]

For the first term of \( \sum_{m,n} \lambda_m \lambda_n (\Phi, (2.15) \Phi) \) we have

\[
\sum_{m,n} \lambda_m \lambda_n (\psi_n, \phi_m)(\phi_n, \psi_m) \| \Phi \|^2 + \sum_{m,n} \lambda_m \lambda_n (\phi_n, \psi_m)(a(\psi_m) \Phi, a(\phi_n) \Phi).
\]

We see that

\[
\sum_{m,n} \lambda_m \lambda_n (\psi_n, \phi_m)(\phi_n, \psi_m) \| \Phi \|^2 = \sum_{m,n} \lambda_m \lambda_n ((\phi_m, \psi_n)\phi_n, \psi_m) \| \Phi \|^2
\]

\[
= \sum_m \lambda_m (K \phi_m, \psi_m) \| \Phi \|^2 \leq \left( \sum_n \lambda_n^2 \right)^{1/2} \left( \sum_m \| K \phi_m \|^2 \| \Phi \|^2 \right)^{1/2} = \| K \|_2 \| \Phi \|^2.
\]
For the second term let $\Phi = a^*(f_1) \cdots a^*(f_L)\Omega$. We have

\[
\sum_{m,n}^M \lambda_m \lambda_n (\overline{\psi}_n, \phi_m) a^*(\overline{\psi}_n) a(\phi_m) a^*(f_1) \cdots a^*(f_L)\Omega
\]

\[
= \sum_j \sum_{m,n}^M \lambda_m \lambda_n ((\phi_n, f_j) \overline{\psi}_m) a^*(f_1) \cdots a^*(f_j) \cdots a^*(f_L)\Omega
\]

\[
= \sum_j \sum_{m,n}^M \lambda_m a^*((\phi_n, f_j) (\overline{\psi}_n, \phi_m)) a^*(f_1) \cdots a^*(f_j) \cdots a^*(f_L)\Omega
\]

\[
= \sum_j \sum_m^M \lambda_m a^*(\lambda_m (K\overline{\phi}_m, f_j \overline{\psi}_m)) a^*(f_1) \cdots a^*(f_j) \cdots a^*(f_L)\Omega
\]

\[
= \sum_j a^*(\overline{K^*f_j}) a^*(f_1) \cdots a^*(f_j) \cdots a^*(f_L)\Omega.
\]

Then the second term converges. The second term of of $\sum_{m,n}^M \lambda_m \lambda_n (\Phi, (2.15)\Phi)$ is similarly estimated. Finally for $\sum_{m,n}^M \lambda_m \lambda_n (\Phi, (2.16)\Phi)$, we have

\[
\sum_{m,n}^M \lambda_m \lambda_n (\Phi, (2.16)\Phi) = \sum_n^M \lambda_n a(\psi_n) a(\overline{\phi}_n) a^*(f_1) \cdots a^*(f_L)\Omega
\]

\[
= \sum_j a((\phi_n, f_j) \psi_n) a^*(f_1) \cdots a^*(f_j) \cdots a^*(f_L)\Omega
\]

\[
= \sum_j a(K^*f_j) a^*(f_1) \cdots a^*(f_j) \cdots a^*(f_L)\Omega.
\]

Then $\sum_{m,n}^M \lambda_m \lambda_n (\Phi, (2.16)\Phi)$ also converges. $\square$

We can define for $\Psi \in \mathcal{F}_{\text{fin}}$

\[
\Delta_K^* \Psi = s - \lim_{M \to \infty} \sum_{n=1}^M \lambda_n a^*(\overline{\psi}_n) a^*(\phi_n) \Psi,
\]

(2.17)

\[
\Delta_K \Psi = s - \lim_{M \to \infty} \sum_{n=1}^M \lambda_n a(\overline{\psi}_n) a(\phi_n) \Psi,
\]

(2.18)

\[
N_S \Psi = s - \lim_{M \to \infty} \sum_{n=1}^M a^*(e_n) a(\overline{S^*e_n}) \Psi.
\]

(2.19)
Let \( \Psi = a^*(f_1) \cdots a^*(f_n) \Omega \). Then it is seen that
\[
\Delta_K \Psi = \sum_{i \neq j} (\bar{f}_i, (K + K^*) f_j) a^*(f_1) \cdots \hat{a}^*(f_i) \cdots \hat{a}^*(f_j) \cdots a^*(f_n) \Omega,
\tag{2.20}
\]
\[
N_S \Psi = \sum_{j=1}^n a^*(f_1) \cdots a^*(S f_j) \cdots a^*(f_n) \Omega.
\tag{2.21}
\]

We note that \((\Delta_K)^* = \Delta_K^*\) and \((N_S)^* = N_S^*\). Set \(K_T = K^*\). It can be also checked that on \(F_{\text{fin}}\),
\[
[\Delta_K^*, a(f)] = -a^*((K + K^T) f),
\tag{2.22}
\]
\[
[N_S, a(f)] = -a(S^T f),
\tag{2.24}
\]

From (2.22) and (2.23) it follows that
\[
\|\Delta_K^* \Omega\|^2 = (\Delta_K^* \Omega, \Delta_K^* \Omega) = \sum_{n} \lambda_n(\Omega, \Delta_K^* a^*(\bar{\psi}_n) a^*(\phi_n) \Omega)
\]
\[
= \sum_{n} \lambda_n(\Omega, a((K + K^T) \bar{\psi}_n) a^*(\phi_n) \Omega) = \sum_{n} \lambda_n((K + K^T) \bar{\psi}_n, \phi_n).
\]

Since \(\text{tr}(K^T) = \sum_n \lambda_n(\psi, T \phi_n)\), we have \(\|\Delta_K^* \Omega\|^2 = \text{tr}(K(K + K^T))\). Moreover
\[
\left\| \sum_{n=0}^N \frac{1}{n!} \left( -\frac{1}{2} \Delta_K^* \right)^n \Omega \right\|^2 = \sum_{n=0}^N a_n,
\]
where \(a_n = (2^n n!)^{-2} \|\Delta_K^* \|^2\). We set \(D_\infty = \bigcap_{k=1}^\infty D(N^k)\). We introduce a subset \(\bar{I}_2(\h) \subset I_2(\h)\) by
\[
\bar{I}_2(\h) = \{ K \in I_2(\h) | K = K^T, \|K\| < 1 \}.
\tag{2.26}
\]

**Proposition 2.3** Let \(K \in \bar{I}_2(\h)\). Then (1) and (2) hold:

(1) For all \(|z| \leq \|K\|^{-2}\) and \(k \geq 0\), the limit \(\lim_{N \to \infty} \sum_{n=0}^N n^k a_n z^n\) exists. In particular
\[
\sum_{n=0}^\infty a_n z^n = \det(\mathbb{1} - z K^* K)^{-\frac{1}{2}}.
\]
(2) For all $\Phi \in \mathcal{F}_{\text{fin}}$, the strong limit
\[ \exp \left( -\frac{1}{2} \Delta_K^* \right) \Phi = s - \lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!} \left( -\frac{1}{2} \Delta_K^* \right)^n \Phi, \] (2.27)
exists and belongs to $\mathcal{D}_\infty$.

Proof: Let $a_{n,N} = \left\| \frac{1}{n!} \left( -\frac{1}{2} \sum_{m=1}^{N} \lambda_n a^* (\phi_m) a (\psi_m) \right)^n \Omega \right\|$. Then we can see that
\[ \sum_{n=0}^{\infty} a_{n,N} \alpha^n = \frac{1}{\sqrt{\prod_{j=1}^{\infty} (1 - \alpha \lambda_j^2)}} \]
for $|\alpha| < 1$. By the limiting argument we have
\[ \sum_{n=0}^{\infty} a_n \alpha^n = \frac{1}{\sqrt{\prod_{j=1}^{\infty} (1 - \alpha \lambda_j^2)}} = [\det(\mathbb{I} - \alpha K^* K)]^{-1/2}. \]
In particular $\sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n \alpha^{n-k} < \infty$. Thus
\[ \left\| \sum_{n=k}^{\infty} \sqrt{n(n-1) \cdots (n-k+1)} \frac{1}{n!} (-\frac{1}{2} \Delta_K)^n \Omega \right\| < \infty \]
and
\[ \sum_{n=0}^{\infty} \frac{1}{n!} (-\frac{1}{2} \Delta_K)^n \Omega \in \mathcal{D}_\infty. \]
Furthermore $\sum_{n=0}^{\infty} \frac{1}{n!} (-\frac{1}{2} \Delta_K)^n \Phi$ converges for $\Phi = a^* (f_1) \cdots a^* (f_L) \Omega$. \hfill \Box

Suppose that $S \in B$ and $K \in I_2$. Then for $\Psi \in \mathcal{F}_{\text{fin}}$, we can also see (rather easier than (2.27)) that
\[ e^{-N_S} \Psi = s - \lim_{M \to \infty} \sum_{n=0}^{M} \frac{1}{n!} (-N_S)^n \Psi \] (2.28)
and
\[ e^{-\frac{1}{2} \Delta_K} \Psi = s - \lim_{M \to \infty} \sum_{n=0}^{M} \frac{1}{n!} \left( -\frac{1}{2} \Delta_K \right)^n \Psi \] (2.29)
exist, and $e^{-N_s}\Psi, e^{-\frac{1}{2}\Delta \kappa}\Psi \in \mathcal{F}_{\text{fin}}$. By (2.22)-(2.25) we can check the following commutation relations on $\mathcal{F}_{\text{fin}}$:

\[
[e^{-\frac{1}{2}\Delta \kappa}, a(f)] = \frac{1}{2}a^*((K + K^T)f)e^{-\frac{1}{2}\Delta \kappa}, \tag{2.30}
\]

\[
[e^{-\frac{1}{2}\Delta \kappa}, a^*(f)] = -\frac{1}{2}a((K + K^T)f)e^{-\frac{1}{2}\Delta \kappa}, \tag{2.31}
\]

\[
[x^{-N_s};, a(f)] = a(S^T f) x^{-N_s};, \tag{2.32}
\]

\[
[x^{-N_s};, a^*(f)] = -a^*(S f) x^{-N_s}.. \tag{2.33}
\]

**Corollary 2.4** Let $K_1 \in I_2, K_2$ and $K_2^{-1}$ be in $B$, and $K_1 K_2^{-1} \in \overline{I}_2$. Then

\[
\{a^*(K_1 f) + a(K_2 f)\} \Omega(K_1 K_2^{-1}) = 0, \quad f \in \mathfrak{h}, \tag{2.34}
\]

where $\Omega(K_1 K_2^{-1}) = \exp \left( -\frac{1}{2} \Delta^*_{K_1 K_2^{-1}} \right) \Omega$.

**Proof:** By (2.30), we can see that

\[
a(K_2 f) \exp \left( -\frac{1}{2} \Delta^*_{K_1 K_2^{-1}} \right) \Omega = -a^*(K_1 f) \exp \left( -\frac{1}{2} \Delta^*_{K_1 K_2^{-1}} \right) \Omega.
\]

The desired result follows. \qed

### 2.3 Bogoliubov transformations

In this section we construct a unitary operator implementing the unitary equivalence between $a^\sharp$ and $b_{\Lambda}^\sharp$ when $A \in \mathfrak{Sp}_2$.

#### 2.3.1 Homogeneous case

Let $A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{Sp}$. Then $A$ induces the following maps:

\[
\begin{pmatrix} a(f) \\ a^*(f) \end{pmatrix} \mapsto \begin{pmatrix} b_{\Lambda}(f) \\ b_{\Lambda}^*(f) \end{pmatrix} = \begin{pmatrix} a^*(T f) + a(S f) \\ a^*(\tilde{S} f) + a(T f) \end{pmatrix}. \tag{2.35}
\]

Formally we may write as $(b_{\Lambda}(f) b_{\Lambda}^*(f)) = (a(f) a^*(f)) A$. Since $b_{\Lambda}(f) [\mathcal{F}_{\text{fin}} \text{ (resp. } b_{\Lambda}^*(f))]$ is closable, we denote its closed extension by the same symbol $b_{\Lambda}(f)$ (resp $b_{\Lambda}^*(f)$).

It is seen in Proposition 2.1 that there exists a unitary operator $\mathcal{U}_{\Lambda}$ on $\mathcal{F}$ such that
\(\mathcal{U}_A^{-1} b_A^\#(f) \mathcal{U}_A = a^\#(f)\) if \(A \in \mathfrak{Sp}_2\). The condition \(\begin{pmatrix} S & T \\ T & S \end{pmatrix}\) is equivalent to the following algebraic relations:

\[
\begin{align*}
S^*S - T^*T &= \mathbb{I}, & (2.36) \\
S^*T - T^*S &= 0, & (2.37) \\
SS^* - TT^* &= \mathbb{I}, & (2.38) \\
TS^* - ST^* &= 0. & (2.39)
\end{align*}
\]

Using these algebraic relations we can prove that \(S^{-1} \in B, \|TS^{-1}\| < 1, (TS^{-1})^T = TS^{-1}\), and \((S^{-1}T)^T = S^{-1}T\). We set \(K_1 = TS^{-1}, \quad K_2 = \mathbb{I} - (S^{-1})^*, \quad K_3 = -S^{-1}T\).

Let \(\begin{pmatrix} S & T \\ T & S \end{pmatrix}\) \(\in \mathfrak{Sp}_2\). Since \(K_1 \in I_2, \quad K_1^T = K_1\) and \(\|K_1\| < 1\), i.e., \(K_1 \in \mathbb{I}_2\), we can see that by Proposition 2.3, \(U_A = \det(\mathbb{I} - K_1^*K_1)\mathcal{U}_A = \mathcal{U}_A\mathcal{U}^{-1}b_A^\#(f)\mathcal{U}_A = a^\#(f)\) (2.42) holds for all \(f \in \mathfrak{h}\).

**Proposition 2.5 (Homogeneous case)** Let \(A \in \mathfrak{Sp}_2\). Then \(\mathcal{U}_A\) can be uniquely extended to the unitary operator on \(\mathcal{F}\) and

\[
\mathcal{U}_A^{-1} b_A^\#(f) \mathcal{U}_A = a^\#(f)
\]

holds for all \(f \in \mathfrak{h}\).

**Proof:** Let \(\mathcal{U}_1 = e^{-\frac{\mathbb{I}}{2} \Delta_{K_1}}, \quad \mathcal{U}_2 = e^{-N_{K_2}}\), and \(\mathcal{U}_3 = e^{-\frac{\mathbb{I}}{2} \Delta_{K_3}}\). By commutation relations (2.30)-(2.33) we see that

\[
\mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 a^*(f) = \mathcal{U}_1 \mathcal{U}_2 a^*(f) \mathcal{U}_3 + \mathcal{U}_1 \mathcal{U}_2 a(-K_3 f) \mathcal{U}_3 = a^*((\mathbb{I} - K_2) f) \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 + \mathcal{U}_1 \mathcal{U}_2 a(-K_3 f) \mathcal{U}_3.
\]

Using \(\mathbb{I} - K_2 = (S^{-1})^* = \bar{S} - TS^{-1}\bar{T}\) and \(a^*((\mathbb{I} - K_2) f) = b_A^*(f) + a(-\bar{T} f) + a^*(-T S^{-1} \bar{T} f)\), we have

\[
a^*((\mathbb{I} - K_2) f) \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 = b_A^*(f) \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 + \mathcal{U}_1 a^*(-T S^{-1} \bar{T} f) \mathcal{U}_2 \mathcal{U}_3 + \mathcal{U}_1 \{a^*(K_1 \bar{T} f) + a(-\bar{T} f)\} \mathcal{U}_2 \mathcal{U}_3.
\]
Hence the right hand side above is identical with

\[ a^*((1 - K_2)f) \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 = b_\lambda^*(f) \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 + \mathcal{U}_1 \mathcal{U}_2 a(-T f) + a(K_2^T T f) \mathcal{U}_3. \]  

(2.44)

Combining (2.43) and (2.44), we obtain

\[ \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 a^*(f) = b_\lambda^*(f) \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 + \mathcal{U}_1 \mathcal{U}_2 a(-T f + K_2^T T f - K_3 f) \mathcal{U}_3. \]

Since \(-T + K_2^T T - K_3 = 0\), we get \( \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 a^*(f) \Phi = b_\lambda^*(f) \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 \Phi \) for all \( \Phi \in \mathcal{D}_\infty \) and \( f \in \mathfrak{h} \). I.e.,

\[ \mathcal{U}_\lambda a^2(f) \Phi = b_{\lambda}^*(f) \mathcal{U}_\lambda \Phi, \quad \Phi \in \mathcal{D}_\infty. \]  

(2.45)

From this, and the canonical commutation relations it follows that

\[ \| \mathcal{U}_\lambda a^*(f_1) \cdots a^*(f_n) \Omega \|^2 = \| b_{\lambda}^*(f_1) \cdots b_{\lambda}^*(f_n) \mathcal{U}_\lambda \Omega \|^2 = \| a^*(f_1) \cdots a^*(f_n) \Omega \|^2, \]

where we used that \( \| e^{-\frac{1}{2} \Delta_\lambda} \Omega \|^2 = \det(1 - K_1^* K_1)^{-1/2} \) and \( b_\lambda(f) e^{-\frac{1}{2} \Delta_\lambda} \Omega = 0 \). Then \( \mathcal{U}_\lambda \) is an isometry from \( \mathcal{F}_{\text{fin}} \) onto the dense subspace:

\[ \mathcal{E} = L.H.\{b_{\lambda}^*(f_1) \cdots b_{\lambda}^*(f_n) \mathcal{U}_\lambda \Omega, \mathcal{U}_\lambda \Omega | f_j \in \mathfrak{h}, j = 1, \ldots, n, n \geq 1 \}. \]

We notice that \( b_{\lambda}^*(f) \mathcal{E} \subset \mathcal{E} \) and \( a^2 \) can be represented in terms of a linear combination of \( b_{\lambda}^* \). See (2.8) and (2.9). By this we see that \( a^2(f) \) also leaves \( \mathcal{E} \) invariant: \( a^2(f) \mathcal{E} \subset \mathcal{E} \) for all \( f \in \mathfrak{h} \). Let \( \Psi \in \mathcal{E} \) and \( \Psi_N = \{\Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(N)}, 0, 0, \ldots\} \). Since \( \Psi_N \in \mathcal{F}_{\text{fin}} \), we see that \( \Psi_N \) is an analytic vector of \( \phi(f) = \frac{1}{\sqrt{2}} (a^*(f) + a(f)) \), which implies, together with \( a^2(f) \mathcal{E} \subset \mathcal{E} \), that \( e^{i\phi(f)} \Psi_N \in \overline{\mathcal{E}} \), and by a limiting argument \( e^{i\phi(f)} \mathcal{E} \subset \overline{\mathcal{E}} \). Thus \( e^{i\phi(f)} \mathcal{E} \subset \overline{\mathcal{E}} \) follows. By a limiting argument we have \( e^{i\phi(f)} \overline{\mathcal{E}} \subset \overline{\mathcal{E}} \). Thus \( \overline{\mathcal{E}} = \mathcal{F} \) by the irreducibility of \( \{\phi(f)\} \). Hence we conclude that \( \mathcal{U}_\lambda \) can be uniquely extended to a unitary operator on \( \mathcal{F} \). Then the proposition follows. \( \square \)

### 2.3.2 Inhomogeneous case

Let \( A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{sp} \) and \( L \in \mathfrak{h} \). Then it induces the map

\[ \begin{pmatrix} a(f) \\ a^*(f) \end{pmatrix} \mapsto \begin{pmatrix} b_{\lambda,L}(f) \\ b_{\lambda,L}^*(f) \end{pmatrix} = \begin{pmatrix} a^*(T f) + a(S f) + (L, f) \\ a^*(T f) + a(S f) + (L, f) \end{pmatrix}. \]  

(2.46)
It is clear that $b_{A,L}^\#$ satisfies canonical commutation relations: $[b_{A,L}(f), b_{A,L}^*(g)] = (\bar{f}, g)$ and $[b_{A,L}(f), b_{A,L}(g)] = 0 = [b_{A,L}^*(f), b_{A,L}^*(g)]$. Moreover $a^\#$ can be represented in terms of a linear sum of $b_{A,L}^\#$:

\[
a(f) = b_{A,L}(S^*f) - b_{A,L}^*(T^*f) + (-SL + TL, f),
\]

\[
a^*(f) = -b_{A,L}(T^*f) + b_{A,L}^*(S^*f) + (-\bar{S}L + TL, f).
\]

We define the operator $\pi(L)$ in $\mathcal{F}$ by $\pi(L) = i\{b_{A,L}(L) - b_{A,L}^*(\bar{L})\}$, which can be represented in terms of $a^\#$ by

\[
\pi(L) = i\{a^*(TL - \bar{S}L) - a(\bar{T}L - SL)\}.
\]

Let $S_{A,L} = \exp(-i\pi(L)) = \exp(b_{A,L}(L) - b_{A,L}^*(\bar{L}))$ and we define $\mathcal{U}_{A,L}$ by

\[
\mathcal{U}_{A,L} = S_{A,L} U_A.
\]

Here $S_{A,L}$ is called the displacement operator and $\mathcal{U}_{A,L}$ Bogoliubov transform.

**Proposition 2.6 (Inhomogeneous case)** Let $A \in \mathfrak{sp}_2$ and $L \in \mathfrak{h}$. Then we have

\[
\mathcal{U}_{A,L}^{-1} b_{A,L}^\#(f) \mathcal{U}_{A,L} = a^\#(f).
\]

**Proof:** Notice that $S_{A,L} b_{A,L}^\#(f) S_{A,L}^{-1} = b_{A,L}^\#(f)$. Then the proposition follows from Proposition 2.5. \qed

Suppose that $A \in \mathfrak{sp}_2$ and $L \in \mathfrak{h}$. Let $\Phi = \mathcal{U}_{A,L} \Omega$. For later use we compute $(a^*(f) \Omega, \Phi)$ and $(a^*(f) a^*(g) \Omega, \Phi)$.

**Lemma 2.7** Suppose that $A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{sp}_2$ and $L \in \mathfrak{h}$. Set $\xi = TL - \bar{S}L$ and $K = TS^{-1}$. Then it follows that

\[
\frac{(a^*(f) \Omega, \Phi)}{(\Omega, \Phi)} = (f, \xi) + (\bar{K}f, \bar{\xi}),
\]

\[
\frac{(a^*(f) a^*(g) \Omega, \Phi)}{(\Omega, \Phi)}
\]

\[
= (f, \xi)(g, \xi) + (g, \xi)(\bar{K}f, \bar{\xi}) + (f, \xi)(\bar{K}g, \bar{\xi}) + (\bar{K}g, \bar{\xi})(\bar{K}f, \bar{\xi}) - (f, \bar{K}\bar{g}).
\]
Proof: We set \( S = S_A L = \exp(a^*(\xi) - a(\bar{\xi})), \mathcal{U} = \mathcal{U}_A \) and \( J = (\Omega, \Phi) \). Notice that \( S^{-1} a(f)S = a(f) + (\bar{f}, \xi) \) and \( Sa(f)S^{-1} = a(f) - (\bar{f}, \xi) \). Then directly we have

\[
(a^*(f)\Omega, \Phi) = (\Omega, a(\bar{f})\mathcal{U}\Omega)
= (f, \xi)J + (\Omega, Sa(\bar{f})\mathcal{U}\Omega)
= (f, \xi)J + (S^{-1}\Omega, -a^*(K\bar{f})\mathcal{U}\Omega)
= (f, \xi)J + (-a(Kf)S^{-1}\Omega, \mathcal{U}\Omega)
= (f, \xi)J + (Kf, \bar{\xi})J.
\]

Then (2.52) follows. Next we see that

\[
(a^*(f)a^*(g)\Omega, \Phi)
= (\Omega, a(\bar{g})a(\bar{f})\Phi)
= (S^{-1}\Omega, (a(\bar{g}) + (g, \xi))(a(\bar{f}) + (f, \xi))\mathcal{U}\Omega)
= (S^{-1}\Omega, a(\bar{g})a(\bar{f})\mathcal{U}\Omega) + (g, \xi)(S^{-1}\Omega, a(\bar{f})\mathcal{U}\Omega)
\]

\[
+ (f, \xi)(S^{-1}\Omega, a(\bar{g})\mathcal{U}\Omega) + (g, \xi)(f, \xi)J
= (g, \xi)(f, \xi)J + (g, \xi)(Kf, \bar{\xi})J + (f, \xi)(Kg, \bar{\xi})J + (S^{-1}\Omega, a(\bar{g})a(\bar{f})\mathcal{U}\Omega)J.
\]

Moreover we have

\[
(S^{-1}\Omega, a(\bar{g})a(\bar{f})\mathcal{U}\Omega)
= (S^{-1}\Omega, a(\bar{g})(-a^*(K\bar{f}))\mathcal{U}\Omega)
= (S^{-1}\Omega, -a^*(K\bar{f})a(\bar{g})\mathcal{U}\Omega) - (f, K\bar{g})J
= (S^{-1}\Omega, a^*(K\bar{f})a^*(K\bar{g})\mathcal{U}\Omega) - (f, K\bar{g})J
= (a(K\bar{f})a(K\bar{g})S^{-1}\Omega, \mathcal{U}\Omega) - (f, K\bar{g})J
= (a(\bar{gf}) - (K\bar{f}, \xi)(a(\bar{g}g) - (K\bar{g}, \xi))\mathcal{U}\Omega) - (f, K\bar{g})J
= (K\bar{f}, \bar{\xi})J - (f, K\bar{g})J.
\]

Then (2.53) follows. \( \square \)

Lemma 2.8 Suppose that \( A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{sp}_2 \) and \( L \in \mathfrak{h} \). Set \( \xi = TL - \overline{SL} \) and \( K = TS^{-1} \). We assume that \( \bar{\xi} = \xi \) and \( \bar{f} = f \). Then

\[
((2a^*(f) + a^*(f)a^*(f))\Omega, \Phi) = (2\gamma + \gamma^2 - (f, Kf))(\Omega, \Phi), \quad (2.54)
\]

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where \( \gamma = (\xi, (\mathbb{1} + K)f) \). In particular for all \( p \in \mathbb{R} \), it follows that
\[
\frac{(p + a(\bar{f}) + a^*(f))^2\Omega, \Phi}{(\Omega, \Phi)} = (p + \gamma)^2 + (f, (\mathbb{1} - K)f).
\] (2.55)

**Proof:** By the assumptions we have
\[
(a^*(f)\Omega, \Phi) = \gamma,
\]
\[
(a^*(f)a^*(f)\Omega, \Phi) = (f, \xi)^2 + 2(\xi, f)(\xi, Kf) + (\xi, Kf)^2 - (f, Kf) = \gamma^2 - (f, Kf).
\]
Then (2.54) follows. Notice that
\[
(p + a(\bar{f}) + a^*(f))^2\Omega = (p^2 + 2pa^*(f) + a^*(f)a^*(f))\Omega + \|f\|^2\Omega.
\]
Then (2.55) follows from (2.54). \( \square \)

### 2.4 One parameter symplectic groups and 2-cocycles

In this section we review the pseudo unitary representation of symplectic groups \( \mathfrak{sp}_2 \). Let \( \mathcal{U}(\mathcal{F}) \) be the set of unitary operators on \( \mathcal{F} \). Then we can define the map
\[
\mathcal{U} : \mathfrak{sp}_2 \rightarrow U(\mathcal{F}).
\] (2.56)

Since
\[
\mathcal{U}_{AB}^{-1}\mathcal{U}_A\mathcal{U}_B a^\sharp(f) = a^\sharp(f)\mathcal{U}_{AB}^{-1}\mathcal{U}_A\mathcal{U}_B
\]
for all \( f \in \mathfrak{h} \), it follows that \( \mathcal{U}_{AB}^{-1}\mathcal{U}_A\mathcal{U}_B = \omega(A, B)\mathbb{1} \) with \( \omega(A, B) \in U(1) \). Thus \( \mathcal{U} \) gives a projective unitary representation of \( \mathfrak{sp}_2 \), i.e.,
\[
\mathcal{U}_A\mathcal{U}_B = \omega(A, B)\mathcal{U}_{AB}
\] (2.57)

where \( \omega(A, B) \in U(1) \) is 2-cocycle. Let
\[
\mathfrak{sp} = \left\{ A = \begin{pmatrix} S & T \\ T & \bar{S} \end{pmatrix} \bigg| AJ + JA^* = A^*J + JA = 0 \right\}
\] (2.58)
\[
= \left\{ A = \begin{pmatrix} S & T \\ T & \bar{S} \end{pmatrix} \bigg| S^* = -S, T^* = T \right\}.
\] (2.59)
If \( A \in \mathfrak{sp} \), then it can be shown that
\[
e^{tA} \in \mathfrak{sp}, \quad t \in \mathbb{R}.
\]
(2.60) is called the one-parameter symplectic group. Set
\[
\mathfrak{sp}_2 = \left\{ A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{sp} \mid T \in I_2 \right\}.
\]
(2.61)
If \( A \in \mathfrak{sp}_2 \), then we can also show that
\[
e^{tA} \in \mathfrak{sp}_2, \quad t \in \mathbb{R}.
\]
(2.62)
Let \( \mathcal{U}_t = \mathcal{U}_{e^{tA}} \) be the intertwining operator associated with \( A \in \mathfrak{sp}_2 \). Thus \( \mathcal{U}_t \) satisfies that
\[
\mathcal{U}_t \mathcal{U}_s = e^{i\rho(t,s)} \mathcal{U}_{t+s}
\]
(2.63)
with the so called local exponent \( \rho(t, s) \in \mathbb{R} \). The local exponent \( \rho(t, s) \) also satisfies relation:
\[
\rho(t, s) + \rho(t + s, r) = \rho(s, r) + \rho(t, s + r).
\]
(2.64)
We shall show the explicit form of the local exponent. Let \( A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{sp}_2 \).
Then \( e^{tA} = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{sp}_2 \) induces the map
\[
\begin{pmatrix} a(f) \\ a^*(f) \end{pmatrix} \mapsto \begin{pmatrix} b_t(f) \\ b_t^*(f) \end{pmatrix} = \begin{pmatrix} a^*(T_t f) + a(S_t f) \\ a^*(S_t f) + a(T_t f) \end{pmatrix}.
\]
(2.65)
The intertwining operator \( \mathcal{U}_t = \mathcal{U}_{e^{tA}} \) implements the map (2.65), i.e.,
\[
\mathcal{U}_t^{-1} b_t^* (f) \mathcal{U}_t = a^* (f)
\]
(2.66)
for all \( f \in \mathfrak{h} \). Furthermore we see that
\[
\frac{d}{dt} b_t (f) = a(S f) + a^*(T f)
\]
(2.67)
\[
\frac{d}{dt} b_t^* (f) = a(S f) + a^*(T f).
\]
(2.68)
Let $A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{sp}_2$. We define

$$\Delta(A) = \frac{i}{2}(\Delta_T^* - \Delta_T) - iN_S.$$  \hfill (2.69)

Operator $\Delta(A)$ is essentially self-adjoint on $\mathcal{F}_{\text{fin}}$ and we can also directly see commutation relations:

$$[i\Delta(A), a(f)] = a(Sf) + a^*(Tf),$$  \hfill (2.70)

$$[i\Delta(A), a^*(f)] = a(\bar{T}f) + a^*(\bar{S}f).$$  \hfill (2.71)

By using commutation relations we have

$$e^{it\Delta(A)}a(t)e^{-it\Delta(A)} = b(t)$$  \hfill (2.72)

and hence the equation $\mathcal{U}_te^{it\Delta(A)}a(t) = a(t)\mathcal{U}_te^{it\Delta(A)}$ is derived, i.e.,

$$[\mathcal{U}_te^{it\Delta(A)}, a(t)] = 0.$$

Thus for $A \in \mathfrak{sp}_2$ there exists $\theta_A(t) \in \mathbb{R}$ such that

$$\mathcal{U}_t = e^{i\theta_A(t)}e^{-it\Delta(A)}.$$  \hfill (2.73)

It is immediate from (2.73) that $[\mathcal{U}_t, \Delta(A)] = 0$.

**Lemma 2.9** Let $A \in \mathfrak{sp}_2$. Then the function $\theta_A(\cdot)$ is $C^1(\mathbb{R})$.

**Proof:** By the definition of $\theta(t)$ we have

$$e^{i\theta_A(t)} = \frac{(\Omega, \mathcal{U}_t\Omega)}{(\Omega, e^{it\Delta(A)}\Omega)}.$$

We can check that $(\Omega, \mathcal{U}_t\Omega)$ and $(\Omega, e^{it\Delta(A)}\Omega)$ are differentiable in $t$. Then the lemma follows. \hfill \Box

**Proposition 2.10 (Local exponent)** Let $A = \begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \mathfrak{sp}_2$ and we set $e^{tA} = \begin{pmatrix} S_t & T_t \\ T_t & S_t \end{pmatrix}$. Then
(1) $\theta(t) = \int_0^t \tau_r dr$ and $\tau_r = \frac{1}{2} \text{Im} \text{tr}(T^* T_r S_r^{-1})$.

(2) $\rho(t, s) = \int_0^t \tau_r dr + \int_s^t \tau_r dr - \int_0^{s+t} \tau_r dr$.

Proof: Let $K_t = T_t S_t^{-1}$. We notice that $\mathbb{R} \ni \det(1 - K_t^* K_t)^{1/4} = (\Omega, \mathcal{U}_t \Omega) = (\Omega, e^{itA(t)} e^{-it\Delta(A)} \Omega)$. Thus we see that

$$\frac{d}{dt} (\Omega, e^{itA(t)} e^{-it\Delta(A)} \Omega) = i \theta'_A(t)(\Omega, \mathcal{U}_t \Omega) + (\Omega, i\Delta(A) \mathcal{U}_t \Omega) \in \mathbb{R}$$

which implies that the imaginary part of the right hand side disappear and then

$$\theta'_A(t)(\Omega, \mathcal{U}_t \Omega) = -\text{Im}(\Omega, i\Delta(A) \mathcal{U}_t \Omega) = -\text{Im}\left(\frac{1}{2} \Delta^*_T \Omega, \Delta^*_K \Omega\right).$$

From this we have

$$\theta'_A(t) = -\text{Im}\frac{1}{2} \left(\Delta^*_T \Omega, \mathcal{U}_t \Omega\right) = -\frac{1}{2} \text{Im}(\Delta^*_T \Omega, e^{-\frac{i}{2} \Delta_K} \Omega) = \frac{1}{4} \text{Im}(\Delta^*_T \Omega, \Delta^*_K \Omega).$$

Since $(\Delta^*_T \Omega, \Delta^*_K \Omega) = (\Omega, [\Delta_{T^*}, \Delta^*_K] \Omega) = 2 \text{tr}(T^* K_t)$, we complete (1). The statement (2) follows from (1) immediately. \qed

From this proposition 2-cocycle $e^{i\rho(t,s)}$ vanishes if $\text{Im} \text{tr}(T^* T_r S_r^{-1}) = 0$. We give a sufficient condition to vanish the 2-cocycle.

**Corollary 2.11** Suppose $A = \begin{pmatrix} S & T \\ -T^* & S \end{pmatrix} \in \mathfrak{sp}_2$ and $\bar{S} = S = -S^*$ and $\bar{T} = T = T^*$. Then $\rho(t, s) = 0$, i.e., $\mathcal{U}_t = e^{it\Delta(A)}$. In particular $\mathcal{U}_t, t \in \mathbb{R}$, is the one-parameter unitary group.

**Example 2.12** Let $\mathfrak{h} = L^2(\mathbb{R})$. Define the Hilbert-Schmidt operator $T$ by $Tf(x) = \int K(x, y) f(y) dy$ with a real-valued function $K \in L^2(\mathbb{R} \times \mathbb{R})$. Let $h$ be a real-valued function such that $h \in L^\infty(\mathbb{R})$ and $h(-x) = -h(x)$. Define the operator $S = h(d/dx)$. Then $S$ and $T$ satisfy the condition in Corollary 2.11.
Part II
The Pauli-Fierz model

3 The Pauli-Fierz Hamiltonian

3.1 Introduction

It was well known that vacuum polarization divergences, self-energy divergences and another infinity plagued QED in 1930’s. When one attempted to compute calculate the contribution of radiative effects to the scattering of electrons by the Coulomb field of a nucleus, infrared divergences were encountered. In 1937 Bloch and Nordsieck [BN37] showed that this infrared divergence arose from the illegitimate neglect of processes involving the simultaneous emission of many photons, i.e., an emission of photons of very low frequencies yields divergences of an electromagnetic mass, a scattering cross section, etc. In 1938 according to a certain model describing an interaction between an electron and a quantized radiation field Pauli and Fierz [PF38] recognized that the quantized radiation field reacts back on the electron to produce an electromagnetic mass. This model is today the so-called Pauli-Fierz model, which is the main object in this paper\[1\]. The concept of mass renormalization in QED has its origin in these researches of Pauli and Fierz.

Here we look at a typical example of successes of the Pauli-Fierz model. Let $\alpha_\mu$, $\mu = 1, 2, 3$, and $\beta$ are $4 \times 4$ Hermitian matrices obeying the anticommutation relations $\{\alpha_\mu, \alpha_\nu\} = 2\delta_{\mu\nu}I$, $\{\alpha_\mu, \beta\} = 0$ and $\beta^2 = 1$. The Dirac Hamiltonian of a hydrogen-like atom is given by

$$D = \sum_{\mu=1}^{3} \alpha_\mu (-i \nabla_\mu) + \beta m - \frac{Ze^2}{|x|}, \quad (3.1)$$

where $Z$ is an atomic number, $e$ the charge of an electron, and $m$ the mass of an electron. Then $D$ has eigenvalues

$$E_{nj} = \frac{m}{\sqrt{1 + Z^2e^4\left(n - (j + 1/2) + \sqrt{(j + 1/2)^2 - Z^2e^4}\right)^2}}, \quad (3.2)$$

\[1\]In this note we take the dipole approximation of the standard Pauli-Fierz Hamiltonian.
Figure 1: Electron fluctuated by radiation in hydrogen atom

where \( n = 1, 2, ..., \) denotes the principal quantum number and \( j = l \pm 1/2 \) the total angular-momentum with the angular-momentum \( l = 0, ..., n - 1 \). Eigenvalue \( 2S_{1/2} \) corresponds to \( n = 2, j = 1/2, l = 0 \), and \( 2P_{1/2} \) to \( n = 2, j = 1/2, l = 1 \). Then the Dirac theory concludes that two levels \( 2S_{1/2} \) and \( 2P_{1/2} \) in a hydrogen-like atom sit at the same energy level, i.e.,

\[
2S_{1/2} = 2P_{1/2}.
\]

In 1947, by Lamb and Retherford \([LR47]\), it was experimentally observed, however, that

\[
2S_{1/2} > 2P_{1/2}.
\]

This discrepancy is called the Lamb-shift. Bethe \([Bet47]\) regarded the Lamb-shift as an evidence of a radiation reaction, and tentatively made a nonrelativistic calculation of the difference of the two levels. The resulting value was in remarkable agreement with the observation. In 1948, using the Pauli-Fierz model, Welton \([Wel48]\) gave an intuitive derivation to the Lamb-shift. He argued that a position-fluctuation of an electron through the radiation field will effectively modify the external potential \( V \) (Figure 1). The fluctuation was thought of as a Gaussian random variable \( \Delta x \), then an effective potential is formally given by a mean value of \( V(x + \Delta x) \);

\[
V_{\text{eff}}(x) = \langle V(x + \Delta x) \rangle_{\text{AVE}} = (2\pi C)^{-3/2} \int_{\mathbb{R}^3} V(y) e^{-|x-y|^2/(2C)} dy, \tag{3.3}
\]

with a certain positive constant \( C \). Then an electron Hamiltonian effectively turns out to be governed by a Hamiltonian with the external potential \( V_{\text{eff}} \) instead of
V. Welton gave an interpretation of the Lamb shift as the difference between the spectrum of the original Hamiltonian and an effective one.

In Part II we study the Pauli-Fierz Hamiltonian. The basic assumptions are as follows.

(1) We take the dipole approximation.

(2) We neglect spin.

The spectrum of this model is studied by the series of papers by A. Arai [Ara81-a, Ara81-b, Ara81-c, Ara83-a, Ara83-b]. Since we take the dipole approximation, the Pauli-Fierz Hamiltonian \( H \) is reduced to simple. Although it is not translation invariant, i.e., it does not commute with the total momentum, it commutes with particle momentum. Then \( H \) without external potential can be decomposable with respect to the momentum of the particle \( H = \int_{\mathbb{R}^d} H_p dp \) and we can diagonalize \( H_p \) for each fiber \( p \in \mathbb{R}^d \) by applying Bogoliubov transform studied in Section 2. This is a key observation in this section.

3.2 The Pauli-Fierz Hamiltonian with the dipole approximation

Let us assume that the electron moves in dimension \( d \geq 3 \). The physically reasonable dimension is \( d = 3 \). We denote by \( \mathcal{F} \) the boson Fock space over the one particle space \( L^2(\mathbb{R}^d \times \{1, \ldots, d-1\}) \). Here a photon is regarded as a transversal wave in \( d-1 \) directions. The Hilbert space \( \mathcal{H} \) of the coupled system is then given by

\[
\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}.
\]  

(3.4)

The annihilation operator \( a(f, j) \) and the creation operator \( a^*(g, j) \) satisfies canonical commutation relation:

\[
[a(f, j), a^*(g, j')] = \delta_{jj'}(f, g)_{L^2(\mathbb{R}^d)}, \quad [a(f, j), a(g, j')] = 0 = [a^*(f, j), a^*(g, j')]
\]

(3.5)

on the finite particle subspace \( \mathcal{F}_{\text{fin}} \) for \( f, g \in L^2(\mathbb{R}^d) \) and \( 1 \leq j, j' \leq d-1 \). The \( d \)-dimensional polarization vectors are written as

\[
c^j(k) = (c^j_1(k), \ldots, c^j_d(k)), \quad j = 1, \ldots, d-1,
\]

(3.6)
which satisfy \( e^i(k) \cdot e^j(k) = \delta_{ij} \) and \( e^i(k) \cdot k = 0 \) almost everywhere on \( \mathbb{R}^d \). Let

\[
H_f = d\Gamma(\omega)
\]

be the free field Hamiltonian with the dispersion relation

\[
\omega(k) = |k|.
\]

Let\(^2\)

\[
A_\mu(\eta) = \frac{1}{\sqrt{2}} \int \frac{e^j_\mu(k)}{\sqrt{\omega(k)}} (\hat{\eta}(-k)a^*(k,j) + \hat{\eta}(k)a(k,j)) \, dk;
\]

\[
\Pi_\mu(\eta) = i \frac{1}{\sqrt{2}} \int \sqrt{\omega(k)} e^j_\mu(k) (\hat{\eta}(-k)a^*(k,j) - \hat{\eta}(k)a(k,j)) \, dk.
\]

When \( \eta \) is real, then \( \hat{\eta}(k) = \hat{\eta}(-k) \) and \( A_\mu(\eta) \) and \( \pi_\mu(\eta) \) are symmetric. We have the following commutation relations on \( \mathcal{F}_{\text{fin}} \)

\[
[A_\mu(\eta), \Pi_\nu(\rho)] = i \int d_{\mu\nu}(k) \hat{\eta}(-k)\hat{\rho}(k) \, dk = i(d_{\mu\nu}\hat{\eta}, \hat{\rho}),
\]

\[
[A_\mu(\eta), A_\nu(\rho)] = 0,
\]

\[
[\Pi_\mu(\eta), \Pi_\nu(\rho)] = 0,
\]

where

\[
d_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}
\]

denotes the transversal delta function, and

\[
[H_f, A_\mu(\eta)] = -i\Pi_\mu(\eta),
\]

\[
[H_f, \Pi_\mu(\eta)] = iA_\mu(-\Delta\eta).
\]

The quantized radiation field \( A_\mu \) with a cutoff function \( \hat{\phi} \) is defined by

\[
A_\mu = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\omega(k)}} e^j_\mu(k) \left\{ \hat{\phi}(k)a^*(k,j) + \hat{\phi}(k)a(k,j) \right\} \, dk,
\]

\(^2\)Throughout Part II in this lecture note the summation over repeated indices is understood. The Greek letters \( \mu, \nu, \ldots \) and \( a, b \) run from 1 to \( d \), and \( i, j, k \) from 1 to \( d - 1 \).
and the quantized electric field, as its canonically conjugate, by

$$\Pi_\mu = \frac{1}{\sqrt{2}} \int \sqrt{\omega(k)} e^j_\mu(k) \left\{ \tilde{\phi}(k) a^*(k,j) - \phi(k)a(k,j) \right\} dk.$$  \hfill (3.16)

Here $\tilde{f}(k) = f(-k)$. They satisfy that

$$[A_\mu, \Pi_\nu] = i \int d_{\mu\nu}(k) \phi(-k) \phi(k) dk,$$  \hfill (3.17)

$$[A_\mu, A_\nu] = 0,$$  \hfill (3.18)

$$[\Pi_\mu, \Pi_\nu] = 0.$$  \hfill (3.19)

We define the Pauli-Fierz Hamiltonian.

**Definition 3.1 (Pauli-Fierz Hamiltonian!dipole approximation)** The Pauli-Fierz Hamiltonian $H$ with the dipole approximation is defined by

$$H = \frac{1}{2m} \left( -i \nabla \otimes 1 - \alpha 1 \otimes A \right)^2 + V \otimes 1 + 1 \otimes H_f,$$  \hfill (3.20)

where $\alpha \in \mathbb{R}$ denotes the coupling constant and $V : \mathbb{R}^d \to \mathbb{R}$ is an external potential.

In what follows we omit the tensor notation $\otimes$ for notational convenience. Thus $H$ is simply written as

$$\frac{1}{2m} \left( -i \nabla - \alpha A \right)^2 + V + H_f.$$  

We first of all state the self-adjointness of $H$.

**Proposition 3.2 (Self-adjointness)** \cite{Ara81-a} Suppose that $\tilde{\phi}/\omega, \sqrt{\omega} \phi \in L^2(\mathbb{R}^d)$ and $\phi(-k) = \overline{\phi}(k)$, and that $V$ is relatively bounded with respect to $-\frac{1}{2m} \Delta$ with a relative bound strictly smaller than one. Then $H$ is self-adjoint on $D(-\Delta) \cap D(H_f)$ and bounded below for arbitrary $\alpha \in \mathbb{R}$.

**Proof:** Let $V = 0$. Let $L = -\Delta + H_f + 1$. It can be seen that

$$||(HF, LG) - (LF, HG)|| \leq C\|L^{1/2}F\|\|L^{1/2}G\|$$

\cite{3} $D(V) \subset D(-\Delta)$ and $\|Vf\| \leq a\| - \frac{1}{2m} \Delta f\| + b\| f\|$ with some $a < 1$.  

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with some constant $C$. Then $H$ is essentially self-adjoint on $D(-\Delta) \cap D(H_f)$ by the Nelson commutator theorem. Furthermore by the inequality $\|HF\| \leq C\|LF\|$, the closedness of $H|_{D(-\Delta) \cap D(H_f)}$ follows. For nonzero $V$, by the diamagnetic inequality,

$$\left\| \left( \frac{1}{2m}(-i\nabla - \alpha A)^2 + H_f - z \right)^{-1} F \right\| \leq \left\| \left( \frac{1}{2m} \Delta + H_f - z \right)^{-1} F \right\|,$$

we can see that $V$ is also relatively bounded with respect to $\frac{1}{2m}(-i\nabla - \alpha A)^2 + H_f$ with a relative bound strictly smaller than one. Then the proposition follows by the Kato-Rellich theorem.

Let

$$A_\mu(x) = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\omega(k)}} e_{\mu}^j(k) \left( \hat{\phi}(-k)e^{-ikx}a^*(k,j) + \hat{\phi}(k)e^{ikx}a(k,j) \right) dk$$

for each $x \in \mathbb{R}^d$. Under the identification $\mathcal{H} = \int_{\mathbb{R}^d} \mathcal{F} dx$, we define

$$A_\mu = \int_{\mathbb{R}^d} A_\mu(x) dx.$$

Thus the Pauli-Fierz Hamiltonian without the dipole approximation is defined by

$$\frac{1}{2m}(-i\nabla \otimes \mathbb{1} - \alpha A)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes H_f. \quad (3.21)$$

The Pauli-Fierz Hamiltonian $H$ under consideration in this lecture note is defined by $(3.21)$ with $A_\mu$ replaced by $\mathbb{1} \otimes A_\mu(0)$.

### 3.3 Translation invariant Hamiltonian

Suppose that $V = 0$. Let us define the operator $K$ in $\mathcal{H}$ by

$$K = \frac{1}{2m}(-i\nabla - \alpha A)^2 + H_f. \quad (3.22)$$

Since $K$ commutes with $-i\nabla_\mu$, the Hilbert space $\mathcal{H}$ and the operator $K$ are decomposable with respect to the joint spectrum of $-i\nabla_\mu$, i.e.

$$\mathcal{H} = \int_{\mathbb{R}^d} \mathcal{F} dp,$$

$$K = \int_{\mathbb{R}^d} H_\mu dp$$

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where
\[ H_p = \frac{1}{2m} (p - \alpha A)^2 + H, \quad p \in \mathbb{R}^d. \] (3.23)

**Proposition 3.3 (Self-adjointness)** [Ara83-a] Suppose that \( \hat{\varphi}/\omega, \sqrt{\omega} \hat{\varphi} \in L^2(\mathbb{R}^d) \) and \( \hat{\varphi}(-k) = \overline{\hat{\varphi}(k)} \). Then \( H_p \) is self-adjoint on \( D(H) \) and bounded below for arbitrary \( \alpha \in \mathbb{R} \).

**Proof:** Let \( L = H + \mathbb{1} \). It can be seen that
\[ |(H_p F, L G) - (LF, H_p G)| \leq C \| L^{1/2} F \| \| L^{1/2} G \| \]
with some constant \( C \). Then \( H_p \) is essentially self-adjoint on \( D(H) \) by the Nelson commutator theorem. Furthermore by the inequality \( \| H_p F \| \leq C \| LF \| \), the closedness of \( H_p \) follows. Then the proposition follows. \( \square \)

The quadruple
\[ \mathcal{D} = \{ a^*, a, H, \Omega \} \]
satisfies the algebraic relations: 
\[ [a(f, j), a^*(g, j')] = \delta_{jj'}(\hat{f}, g), \quad [H, a(g)] = -a(\omega g), \]
\[ [H, a^*(g)] = a^*(\omega g) \text{ and } H\Omega = a(f)\Omega = 0. \]
In this section we construct operators \( B_p \) and \( B_p^* \) and a vector \( \Omega_p \) such that the quadruple
\[ \mathcal{D}_p = \{ B_p, B_p^*, H_p - E_p, \Omega_p \} \]
satisfies the same algebraic relations as those of \( \mathcal{D} \) for each \( p \in \mathbb{R}^d \), where \( E_p \) denotes the ground state energy of \( H_p \), i.e., \( E_p = \inf \sigma(H_p) \). We need in addition some technical assumptions on \( \hat{\varphi} \).

**Assumption 3.4** We suppose (1), (2), (3) or (1), (2'), (3):

1. \( \sqrt{\omega} \hat{\varphi}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^d), \quad \overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) \) and \( \hat{\varphi} \) is rotation invariant, i.e. \( \hat{\varphi}(k) = \hat{\varphi}(|k|) \).
2. \( \hat{\varphi}(k) \neq 0 \) for \( k \neq 0 \), and \( \rho(s) = |\hat{\varphi}(\sqrt{s})|^2 s^{d+2} \in L^1([0, \infty), ds) \) for some \( 1 < \epsilon \), and there exists \( 0 < C < 1 \) such that \( |\rho(s + h) - \rho(s)| \leq K|h|^C \) for all \( s \) and \( 0 \leq h \leq 1 \),
3. \( \hat{\varphi}(k) \neq 0 \) for \( \lambda \leq |k| \leq \Lambda \), and \( \hat{\varphi}(k) = 0 \) for \( |k| > \Lambda \) and \( |k| < \lambda \) with some \( \Lambda > 0 \) and \( \lambda > 0 \),

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(3) \( \| \hat{\phi} \omega^{(d-3)/2} \|_{\infty} < \infty \) and \( \| \hat{\phi} \omega^{(d-1)/2} \|_{\infty} < \infty \).

Assumption (1) is used for self-adjointness of \( H \), (2) or (2’) for the definition of a function \( D_+ \) and \( Q \) in Section 3.4, and (3) for an operator \( T_{\mu \nu} \) in Lemma 3.7. The main theorem in Section 3 is as follows.

**Theorem 3.5 (Diagonalization of \( H_p \))** Suppose Assumption 3.4.

(1) Let \( p = 0 \). Then there exists a unitary operator \( \mathcal{U}_0 : D(H_f) \to D(H_f) \) such that

\[
\mathcal{U}_0^{-1} H_0 \mathcal{U}_0 = H_f + g. \tag{3.24}
\]

(2) Suppose, in addition, \( \int \frac{\hat{\phi}^2}{\omega^3} dk < \infty \). Then for all \( p \in \mathbb{R}^d \), there exists a unitary operator \( \mathcal{U}_p : D(H_f) \to D(H_f) \) such that

\[
\mathcal{U}_p^{-1} H_p \mathcal{U}_p = \frac{1}{2m_{\text{eff}}} p^2 + H_f + g. \tag{3.25}
\]

Here the effective mass \( m_{\text{eff}} \) is given by

\[
m_{\text{eff}} = m + \alpha^2 \left( \frac{d-1}{d} \right) \| \hat{\phi} / \omega \|_2^2, \tag{3.26}
\]

and the additional constant \( g \) by

\[
g = \frac{d}{2\pi} \int_{-\infty}^{\infty} \frac{\alpha^2 \left( \frac{d-1}{d} \right) \| \frac{\hat{\phi}}{\sqrt{\pi \omega^2}} \|^2}{m + \alpha^2 \left( \frac{d-1}{d} \right) \| \frac{\hat{\phi}}{\sqrt{\pi \omega^2}} \|^2} dt. \tag{3.27}
\]

The condition \( \int \frac{\hat{\phi}^2}{\omega^3} dk < \infty \) is called the infrared regular condition, on the other hand \( \int \frac{\hat{\phi}^2}{\omega^3} dk = \infty \) infrared singular condition. The term \( \alpha^2 \left( \frac{d-1}{d} \right) \| \hat{\phi} / \omega \|_2^2 \) in \( m_{\text{eff}} \) is called self-energy. The self energy \( \alpha^2 \left( \frac{d-1}{d} \right) \int \frac{\hat{\phi}(k)^2}{|k|^2} dk \) has no singularity at the origin \( k = 0 \), since \( d \geq 3 \).

We furthermore define the unitary operator \( \mathcal{U} \) on \( \mathcal{H} = \int_{\mathbb{R}^d} \mathcal{F} \, dp \) by

\[
\mathcal{U} = \int_{\mathbb{R}^d} \mathcal{U}_p e^{i \frac{\pi}{2} N} dp, \tag{3.28}
\]

where \( N \) denotes the number operator in \( \mathcal{F} \).
Theorem 3.6 (Diagonalization of $H$) Suppose Assumption [3.4] and that $V$ is relatively bounded with respect to $-\frac{1}{2m}\Delta$ with a relative bound strictly smaller than one. Assume furthermore $\int \frac{\dot{\phi}^2}{\omega^3} dk < \infty$. Then, for each $\alpha \in \mathbb{R}$, $\mathcal{U}$ maps $D(-\Delta) \cap D(H_f)$ onto itself and

$$\mathcal{U}^{-1}H \mathcal{U} = H_{\text{eff}} + H_f + \delta V + g,$$

where $H_{\text{eff}}$ denotes the effective Hamiltonian given by

$$H_{\text{eff}} = -\frac{1}{2m_{\text{eff}}} \Delta + V,$$

and $\delta V$ is the perturbation given by

$$\delta V = T^{-1}VT - V,$$

with

$$T = \exp (-i(-i\nabla) \cdot K),$$

$$K_\mu = \frac{1}{\sqrt{2}} \int \frac{e^J_\mu(k)}{\sqrt{\omega(k)}} \left( \frac{\alpha \dot{\phi}(k)}{m_{\text{eff}}(k)\omega(k)} a^*(k,j) + \frac{\alpha \dot{\phi}(k)}{m_{\text{eff}}(k)\omega(k)} a(k,j) \right) dk. \quad (3.32)$$

Here the function $m_{\text{eff}}(k)$ is given by (3.39) below.

We shall give proofs of Theorems 3.6 and 3.5 in Section 3.5.

Formally

$$T^{-1}VT(x) = e^{\nabla \cdot K}Ve^{-\nabla \cdot K}(x) = V(x + K).$$

Thus

$$V(x + K) = \sum_{n=0}^{\infty} \frac{1}{n!}(\nabla \cdot K)^nV(x).$$

Thus the discrepancy between $V$ and $V_{\text{eff}}$ is given by $\sum_{n=1}^{\infty} \frac{1}{n!}(\nabla \cdot K)^nV(x)$. In particular

$$(\Omega, \sum_{n=1}^{\infty} \frac{1}{n!}(\nabla \cdot K)^nV(x)\Omega) = (\Omega, \sum_{n=2}^{\infty} \frac{1}{n!}(\nabla \cdot K)^nV(x)\Omega)$$

and $$(\Omega, \sum_{n=2}^{\infty} \frac{1}{n!}(\nabla \cdot K)^nV(x)\Omega) \sim \frac{1}{2}(\Omega, (\nabla \cdot K)^2V(x)\Omega).$$

Approximately the radiative effect changes $V$ to $V + \frac{1}{2}(\Omega, (\nabla \cdot K)^2V(x)\Omega)$. This gives an interpretation of the Lamb shift. See [Bet47, Wel48].

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3.4 Bogoliubov transformation

3.4.1 Algebraic relations

In order to prove Theorems 3.6 and 3.5 we prepare several lemmas.

Let us consider the time evolution of $A_\mu(f)$ by the Hamiltonian $H_p$. Let

$$A_\mu(f, t) = e^{itH_p}A_\mu(f)e^{-itH_p}$$

for $p \in \mathbb{R}^d$, and set $A_\mu(f, t) = \int A_\mu(x, t)f(x)dx$. Formally we have

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)A_\mu(x, t) = \frac{\alpha}{m}(p_\nu - \alpha A_\nu(x, t))\rho_{\nu\mu}(x), \quad (3.33)$$

where $\rho_{\nu\mu}(x) = (2\pi)^{-d/2} \int d\nu \hat{\phi}_\nu(k)\hat{\phi}_\mu(k)e^{ikx}dk$. We shall operator-theoretically solve (3.33) in what follows. Let us define

$$D(z) = m - \alpha^2 \left(\frac{d - 1}{d}\right) \int_{\mathbb{R}^d} \frac{\hat{\phi}(k)^2}{z - \omega(k)^2}dk, \quad z \in \mathbb{C}\setminus[0, \infty). \quad (3.34)$$

Lemma 3.7 Suppose Assumption 3.4.

(1) The function $D(z)$ is analytic and has no zero points in $\mathbb{C}\setminus[0, \infty)$.

(2) The function $D_\pm(s) = \lim_{\epsilon \downarrow 0} D(s \pm i\epsilon)$ exists for all $s \in [0, \infty)$, $D_\pm(0) = m_{\text{eff}}$ and $\lim_{s \to \infty} D_\pm(s) = m$.

(3) It follows that

$$D_+(s) - D_-(s) = \pi i \alpha^2 \left(\frac{d - 1}{d}\right) |S_{d-1}| \hat{\phi}(\sqrt{s})^2 s^{\frac{d-2}{2}},$$

where $|S_{d-1}| = 2\pi^{d/2}/\Gamma(\frac{d+1}{2})$ the volume of the $d - 1$-dimensional unit sphere $S_{d-1}$.

Proof: (1) is fundamental. We directly see that

$$D_\pm(s) = m - \alpha^2 \left(\frac{d - 1}{d}\right) |S_{d-1}| (H\rho(s) \mp \pi i \rho(s)), \quad (3.35)$$

where $\rho(s) = |\hat{\phi}(\sqrt{s})|^2 s^{\frac{d-2}{2}}$ and $H\rho$ denotes the Hilbert transform of $\rho$, i.e.,

$$H\rho(s) = \lim_{\epsilon \downarrow 0} \int_{|s - x| > \epsilon} \frac{\rho(x)}{s - x}dx.$$
Namely

\[ D_{\pm}(s) = m - \frac{\alpha^2}{2} \left( \frac{d-1}{d} \right) |S_{d-1}| \left( \lim_{\epsilon \downarrow 0} \int_{|s-x| > \epsilon} \frac{\hat{\phi}^2(x)|x|^\frac{d-2}{2}}{s-x} dx \mp \pi i |\hat{\phi}(\sqrt{s})|^2 s^\frac{d-2}{2} \right) . \]  

(3.36)

Then (2) and (3) follow from this. \( \square \)

**Lemma 3.8**

(1) Suppose (1),(2) and (3) of Assumption 3.4. Then there exists \( \epsilon > 0 \) such that \( |D_{\pm}(s)| > \epsilon \) for \( s \in [0, \infty) \).

(2) Suppose (1),(2') and (3) of Assumption 3.4. Then there exists \( \epsilon > 0 \) such that \( |D_{\pm}(s)| > \epsilon \) for \( s \in [\lambda^2, \Lambda^2] \) and \( D_{\pm}(s) \) has at most one zero point on open interval \( (\Lambda^2, \infty) \).

*Proof:* By Assumption 3.4 (2), the imaginary part of \( D_{\pm}(s) \) does not vanish, and Assumption 3.4 (2) implies that the real part of \( D_{\pm} \) is also Lipshitz continuous with the same order \( C \) as that of \( \rho(s) = |\hat{\phi}(\sqrt{s})|^2 s^\frac{d-2}{2} \), since the real part is the Hilbert transformation of \( \rho \). Then the real part of \( D_{\pm}(s) \) goes to \( m > 0 \) as \( s \to \infty \) [Tit37, p.145,5.15]. In particular, there exists \( \epsilon > 0 \) such that \( \sup_{s \in [0, \infty)} |D_{\pm}(s)| > \epsilon \).

Next Assumption 3.4 (2') implies that the imaginary part of \( D_{\pm}(s) \) \( \neq 0 \) for \( s \in [\lambda^2, \Lambda^2] \). The real part of \( D_{\pm}(s) \) is \( m - \frac{\alpha^2}{2} \left( \frac{d-1}{d} \right) |S_{d-1}| \int_{\Lambda^2}^\lambda \frac{|\hat{\phi}(\sqrt{s})|^2 s^\frac{d-2}{2}}{s-x} dx \). Then it is monotonously increasing on \( [0, \lambda^2) \cup (\Lambda^2, \infty) \) with \( D_{\pm}(0) = m + \alpha^2 \left( \frac{d-1}{d} \right) \|\hat{\phi}/\omega\|^2 \) and \( \lim_{s \to \infty} D_{\pm}(s) = m \). Then the lemma follows. \( \square \)

Define

\[ G_f(k) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \frac{f(k')}{(\omega(k)^2 - \omega(k')^2 + i\epsilon)(\omega(k)\omega(k'))^\frac{d-2}{2}} dk'. \]  

(3.37)

It is seen that

\[ G_f(k) = \frac{1}{2|k|^\frac{d-2}{2}} \left( HF(|k|^2) - \pi i [f(|k|)|k|]^{\frac{d-2}{2}} \right) , \]  

(3.38)

\[ \footnote{[HS01, (4.1)] is incorrect. ±2\pi i |\hat{\phi}(\sqrt{s})|^2 s^{(d-2)/2} is changed to ±\pi i |\hat{\phi}(\sqrt{s})|^2 s^{(d-2)/2}.} \]
where $HF$ denotes the Hilbert transform of $F(x) = [f](\sqrt{x})x^{\frac{d-2}{2}}$ and $[f](r) = \int_{S_{d-1}} f(r,v)dv$ and $v$ is the volume element on $S_{d-1}$. Define the running effective mass by

$$m_{\text{eff}}(k) = D_+(\omega(k)^2)$$

and we set

$$T_{\mu\nu}f = \delta_{\mu\nu}f + \alpha Q\omega^{\frac{d-2}{2}}G\omega^{\frac{d-2}{2}}d_{\mu\nu}\hat{\phi}f,$$  

(3.40)

where

$$Q(k) = \frac{\alpha\hat{\phi}(k)}{m_{\text{eff}}(k)}.$$  

(3.41)

The operator $T_{\mu\nu}$ is then given by

$$T_{\mu\nu}f(k) = \delta_{\mu\nu}f(k) + \alpha^2 \int \frac{d_{\mu\nu}(k')\hat{\phi}(k')\hat{\phi}(k)f(k')}{D_+(|k|^2)(|k|^2 - |k'|^2 + i0)}dk'.$$

Remark 3.9 We give a comment on the definition of $Q$. Under (2) of Assumption 3.4, $m_{\text{eff}}(k) \neq 0$ for all $k \in \mathbb{R}^d$, then $Q$ is well defined. On the other hand under (2') of Assumption 3.4, $m_{\text{eff}}(k)$ probably has one zero point in $(\Lambda^2, \infty)$, then $Q$ is well defined, since the support of the numerator is $\text{supp}\hat{\phi} = [\lambda, \Lambda]$. Notice also that $m_{\text{eff}}(k) \neq 0$ at least for $\lambda - \delta \leq |k| \leq \Lambda + \delta$ with some $\delta > 0$.

Example 3.10 (Running effective mass for sharp cutoff) Let us compute an effective mass $m_{\text{eff}}(k)$ with sharp cutoff. Let

$$\hat{\phi}(k) = \mathbb{I}_{[\lambda, \Lambda]}(|k|)$$  

(3.42)

be the indicator function on $\lambda \leq |k| \leq \Lambda$. Suppose $d = 3$. Then

$$H\rho(s) = \lim_{\epsilon \downarrow 0} \int_{|s - x| > \epsilon} \frac{\mathbb{I}_{[\lambda, \Lambda]}(\sqrt{x})\sqrt{x}}{s - x}dx = \lim_{\epsilon \downarrow 0} \int_{|s - x| > \epsilon} \frac{\mathbb{I}_{[\lambda^2, \Lambda^2]}(x)\sqrt{x}}{s - x}dx.$$  

Let $s \in [0, \lambda)$. Then we see that

$$H\rho(s) = \int_{\lambda^2}^{\Lambda^2} \frac{\sqrt{x}}{s - x}dx = \int_{\lambda}^{\Lambda} \frac{2t^2}{s - t^2}dt = -2(\Lambda - \lambda) - 2s \int_{\lambda}^{\Lambda} \frac{1}{t^2 - s}dt$$

$$= -2(\Lambda - \lambda) - \sqrt{s} \int_{\lambda}^{\Lambda} \frac{1}{t - \sqrt{s}} - \frac{1}{t + \sqrt{s}}dt$$

$$= -2(\Lambda - \lambda) + \sqrt{s} \log \left( \frac{(\lambda - \sqrt{s})(\sqrt{s} + \Lambda)}{(\sqrt{s} + \lambda)(\Lambda - \sqrt{s})} \right).$$
Let \( s \in (\Lambda, \infty) \). Then in a similar computation we have
\[
H_\rho(s) = -2(\Lambda - \lambda) + \sqrt{s} \log \left( \frac{(\sqrt{s} - \lambda)(\sqrt{s} + \Lambda)}{(\sqrt{s} + \lambda)(\sqrt{s} - \Lambda)} \right).
\]
Finally let \( s \in [\lambda, \Lambda] \).

\[
\int_{|s-x| > \epsilon} \frac{\mathbb{1}_{[\lambda, \Lambda]}(\sqrt{x})}{s-x} dx = \left( \int_{\lambda^2}^{s-\epsilon} + \int_{s+\epsilon}^{\Lambda^2} \right) \frac{\sqrt{x}}{s-x} dx
\]
\[
= \left( \int_{\lambda}^{\sqrt{s-\epsilon}} + \int_{\sqrt{s+\epsilon}}^{\Lambda} \right) \frac{2t^2}{s-t^2} dt
\]
\[
= 2 \left( \int_{\lambda}^{\sqrt{s-\epsilon}} + \int_{\sqrt{s+\epsilon}}^{\Lambda} \right) (-1 + \frac{s}{s-t^2}) dt
\]
\[
\rightarrow -2(\Lambda - \lambda) + \lim_{\epsilon \downarrow 0} 2s \left( \int_{\lambda}^{\sqrt{s-\epsilon}} + \int_{\sqrt{s+\epsilon}}^{\Lambda} \right) \frac{1}{s-t^2} ds
\]
as \( \epsilon \to 0 \). Directly we have
\[
2s \left( \int_{\lambda}^{\sqrt{s-\epsilon}} + \int_{\sqrt{s+\epsilon}}^{\Lambda} \right) \frac{1}{s-t^2} ds
\]
\[
= \sqrt{s} \left( \log \left( \frac{(\sqrt{s} + \sqrt{s-\epsilon})(\sqrt{s} - \lambda)}{(\sqrt{s} - \sqrt{s-\epsilon})(\sqrt{s} + \lambda)} \right) - \log \left( \frac{(\sqrt{s} + \sqrt{s+\epsilon})(\Lambda - \sqrt{s})}{(\sqrt{s} + \sqrt{s+\epsilon})(\sqrt{s} + \Lambda)} \right) \right)
\]
\[
= \sqrt{s} \log \left( \frac{(\sqrt{s} - \lambda)(\sqrt{s} + \Lambda) \sqrt{s} + \sqrt{s-\epsilon})(\sqrt{s} + \epsilon - \sqrt{s})}{(\sqrt{s} + \lambda)(\Lambda - \sqrt{s}) \sqrt{s} - \sqrt{s-\epsilon})(\sqrt{s} + \epsilon + \sqrt{s})} \right)
\]
\[
\rightarrow \sqrt{s} \log \left( \frac{(\sqrt{s} - \lambda)(\sqrt{s} + \Lambda)}{(\sqrt{s} + \lambda)(\Lambda - \sqrt{s})} \right)
\]
as \( \epsilon \to 0 \). Thus we obtain that
\[
H_\rho(s) = -2(\Lambda - \lambda) + \sqrt{s} \log \left( \frac{(\sqrt{s} + \Lambda)(\sqrt{s} - \lambda)}{(\sqrt{s} + \lambda)(\sqrt{s} - \Lambda)} \right).
\]
(3.43)
Since \( (d-1)|S_{d-1}| = 8\pi/3 \) for \( d = 3 \), the running effective mass with sharp cutoff (3.42) is given by
\[
m_{\text{eff}}(k) = m + \frac{8\pi\alpha^2}{3} (\Lambda - \lambda) - \frac{4\pi\alpha^2}{3} \left( |k| \log \left| \frac{(|k| + \Lambda)(|k| - \lambda)}{|(|k| + \Lambda)(|k| - \Lambda)} \right| - i\pi \mathbb{1}_{[\lambda, \Lambda]}(|k|) \sqrt{|k|} \right).
\]
(3.44)
Lemma 3.11 The operator $G$ is a bounded and antisymmetric operator on $L^2(\mathbb{R}^d)$, i.e., $G^* = -G$.

Proof: Let $f \in L^2(\mathbb{R}^d)$. The imaginary part of $Gf$ is $-\frac{\pi}{2} [f](|k|)$ and $||f|| \leq |S_{d-1}||f||$. The real part of $Gf$ is given by the Hilbert transform; $HF$ with $F(x) = [f](\sqrt{x})x^{\frac{d-2}{2}}$. The Hilbert transformation is bounded operator on $L^2(\mathbb{R}^d)$ with $||HF|| = \pi ||f||$ and then we have

$$
||HF||^2 = \frac{1}{4} \int \frac{|HF(|k|)|^2}{|k|^{d-2}} dk = \frac{1}{2} \int_0^\infty |HF(s)|^2 ds |S_{d-1}| = \frac{1}{2} |S_{d-1}|^2 \pi^2 ||F||^2
$$

$$
= \frac{1}{2} |S_{d-1}|^2 \pi^2 \int [f](\sqrt{s})^2 s^{\frac{d-2}{2}} ds = |S_{d-1}|^2 \pi^2 \int [f](r)^2 r^{d-2} dr
$$

$$
\leq |S_{d-1}|^2 \pi^2 \int f(r,v)^2 r^{d-2} dr dv = |S_{d-1}|^2 \pi^2 ||f||^2.
$$

Then the lemma follows. □

The operator $T_{\mu\nu}$, functions $Q$ and $\hat{\phi}$ satisfy some algebraic relations. We list them in the lemma below, where we assume that $m$ is not only positive but also negative. Precisely we assume that

$$m > -\left(\frac{d-1}{d}\right) \alpha^2 ||\hat{\phi}/\omega||^2, \quad m \neq 0 \quad (3.45)$$

for mathematical generality. When $0 > m > -\left(\frac{d-1}{d}\right) \alpha^2 ||\hat{\phi}/\omega||^2$, we see that $D(s) \in \mathbb{R}$ for $s \leq 0$, $\lim_{s \to -\infty} D(s) = m < 0$ and $D(0) = m + \left(\frac{d-1}{d}\right) \alpha^2 ||\hat{\phi}/\omega||^2 > 0$. Then $D(z)$ has a unique zero point of order one in $(-\infty, 0)$. We denote the zero point by $-E^2$ ($E > 0$), and $\gamma$ is defined only in the case of $m < 0$ by

$$\gamma = D'(-E^2)^{-1/2}, \quad (3.46)$$

where we can directly see that

$$D'(E^2) = \alpha^2 \left(\frac{d-1}{d}\right) \int \frac{||\hat{\phi}(k)||^2}{E^2 + \omega(k)^2} dk.$$

Let $T_{\mu\nu}^* = (T_{\mu\nu})^*$, i.e.,

$$T_{\mu\nu}^* f = \delta_{\mu\nu} f - \alpha d_{\mu\nu} \hat{\phi} \omega \frac{d-2}{2} G \omega^{\frac{d-2}{2}} Q f.$$

Note that $G^* = -G$. 

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Lemma 3.12 (Algebraic relations) Suppose Assumption 3.4 and (3.45). Let \( \theta(m) = \begin{cases} 1, & m < 0 \\ 0, & m > 0. \end{cases} \) Set

\[
\rho_{\mu\nu}(k) = \delta_{\mu\nu} - d_{\mu\nu}(k) = k_\mu k_\nu/|k|^2, \quad \eta = \frac{1}{(d-1)|S_{d-1}|}, \quad \gamma_\pm = D_+/D_-,
\]

\[
F_{\mu\nu} = d_{\mu\nu} \frac{\alpha \hat{\varphi}}{E^2 + \omega^2}, \quad [f](|k|) = \int_{S_{d-1}} f(|k|, v)dv.
\]

Then the operator \( T_{\mu\nu} \) has the following properties:

1. \( \|\omega^{n/2} T_{\mu\nu} f\| \leq C \|\omega^{n/2} f\| \) for \( n = -1, 0, 1 \).
2. \( \overline{\overline{T_{\mu\nu} f}} = T_{\mu\nu} f, \ T_{\mu\nu} = T_{\nu\mu} \) and if \( f \) is rotation invariant, then so is \( T_{\mu\nu} f \).
3. \( T_{\mu\nu}^* d_{\nu\alpha} T_{\alpha\beta} f = d_{\mu\beta} f - \theta(m) \gamma^2 (F_{ab}, f) F_{\alpha\mu} \). In particular

\[
e^r_{\mu} T_{\mu\nu}^* d_{\nu\alpha} T_{\alpha\beta} e^s_{\beta} f = \delta_{rs} f - \theta(m) \gamma^2 (e^s_{\beta} F_{ab}, f) e^r_{\mu} F_{\alpha\mu}.
\] (3.47)

4. \( T_{\mu\nu} d_{\nu\alpha} T_{\alpha\beta}^* f = d_{\mu\beta} f + \alpha(\rho_{\mu\beta} \hat{\varphi} \overline{\overline{Q}} - Q \hat{\varphi} \rho_{\mu\beta}) f \). In particular

\[
e^r_{\mu} T_{\mu\nu} d_{\nu\alpha} T_{\alpha\beta}^* e^s_{\beta} f = \delta_{rs} f.
\] (3.48)

5. \( T_{\mu\nu} f = \gamma_\pm T_{\mu\nu} f + (1 - \gamma_\pm) (\delta_{\mu\nu} f - \eta[d_{\mu\nu} f]) \).
6. \( \overline{\overline{T_{\mu\nu} d_{\nu\alpha} T_{\alpha\beta}^* f}} = d_{\mu\beta} f - (1 - \gamma_\pm) \eta[d_{\mu\beta} f] + \alpha(\rho_{\mu\beta} \hat{\varphi} \overline{\overline{Q}} - Q \hat{\varphi} \rho_{\mu\beta}) f \). In particular

\[
e^r_{\mu} \overline{\overline{T_{\mu\nu} d_{\nu\alpha} T_{\alpha\beta}^* e^s_{\beta}}} = e^r_{\mu} d_{\mu\beta} \hat{\varphi} e^s_{\beta} - (1 - \gamma_\pm) \eta e^r_{\mu} [d_{\mu\beta} \hat{\varphi} e^s_{\beta} f].
\] (3.49)

7. \( T_{\mu\nu}^* d_{\nu\alpha} h T_{\alpha\beta} = \overline{T_{\mu\nu}^* d_{\nu\alpha} h T_{\alpha\beta}} \) for rotation invariant function \( h \).
8. \([\omega^2, T_{\mu\nu}] f = \alpha(d_{\mu\beta} \hat{\varphi}, f) Q \) and \([\omega^2, T_{\mu\nu}^*] f = -\alpha(Q, f) d_{\mu\beta} \hat{\varphi} \).
9. \( \alpha T_{\mu\nu} \hat{\varphi} = \delta_{\mu\beta} mQ \). In particular \( \alpha e^r_{\mu} T_{\mu\nu} d_{\nu\alpha} \hat{\varphi} = \delta_{\mu\beta} mQ \).
10. \( e^r_{\mu} T_{\mu\nu} T_{\nu\alpha} e^s_{\alpha} f = \delta_{rs} f \).
11. \( \left( d_{\nu\alpha} \frac{Q}{\omega}, \frac{1}{\omega} T_{\mu\nu} f \right) = \frac{\alpha}{m_{\text{eff}}} \left( d_{\mu\beta} \frac{\hat{\varphi}}{\omega}, \frac{f}{\omega} \right) - \theta(m) \gamma^2 \frac{E^2}{\omega^2} (F_{\mu\alpha}, f) \).
12. \( \left( d_{\nu a} \frac{Q}{\sqrt{\omega^n}}, h \frac{1}{\omega} T_{\mu \nu} f \right) = \left( d_{\nu a} \frac{Q}{\sqrt{\omega^n}}, h \frac{1}{\omega} T_{\mu \nu} f \right) \) for rotation invariant function \( h \) with \( n = 0, 1, 2. \)

13. \( e^\mu T_{\mu \nu} F_{\nu a} = 0 \) \( (m < 0). \)

14. \( \alpha(\hat{\varphi}, F_{\mu \nu}) = -m \delta_{\mu \nu} \) \( (m < 0). \)

15. \( T^*_{\mu \nu} d_{\nu a} Q = \theta(m) \gamma^2 F_{\mu a}. \)

16. \( (F_{\mu a}, F_{a \nu}) = \delta_{\mu \nu} \frac{1}{\gamma^2} \) \( (m < 0). \)

Statements (3)-(7) are used in Lemma 3.14 to show some symplectic structure, (8) and (9) in Lemma 3.21 to show some commutation relations, (10) in the proof of Theorem 3.6, (11) and (12) in Lemma 3.17 and the proof of Theorem 3.6, and (13)-(16) in Section 3.10.

**Proof of Lemma 3.12**

Note that for rotation invariant functions \( f \) and \( g, \)
\[
(d_{\mu \nu}, f, g) = \delta_{\mu \nu} \left( \frac{d - 1}{d} \right) (f, g)
\]
and the identity
\[
\frac{\alpha^2}{2} \left( \frac{d - 1}{d} \right) |S_{d-1}| \hat{\varphi}(\sqrt{s})^2 s^{\frac{d-2}{2}} = \frac{1}{2\pi i} \left( \frac{1}{D_-(s)} - \frac{1}{D_+(s)} \right) \tag{3.50}
\]
holds by \( D_+(s) - D_-(s) = \pi i \alpha^2 \left( \frac{d-1}{d} \right) |S_{d-1}| \hat{\varphi}(\sqrt{s})^2 s^{\frac{d-2}{2}}. \) We set \( \hat{G} = \omega^{\frac{d-2}{2}} G \omega^{\frac{d-2}{2}} \) for the notational convenience.

1. By (3) of Assumption 3.4 we can see that
\[
\| \sqrt{\omega} T_{\mu \nu} f \| \leq \| \sqrt{\omega} \delta_{\mu \nu} f \| + |\alpha| \| \frac{1}{D_+\left(\omega^2(s)\right)} \| \omega^{\frac{d-3}{2}} \hat{\varphi} \|_{\infty} \| \omega^{\frac{d-3}{2}} \hat{\varphi} \|_{\infty} \| G \|_2 \| \sqrt{\omega} f \|,
\]
\[
\| \frac{1}{\sqrt{\omega}} T_{\mu \nu} f \| \leq \| \frac{1}{\sqrt{\omega}} \delta_{\mu \nu} f \| + |\alpha| \| \frac{1}{D_+\left(\omega^2(s)\right)} \| \omega^{\frac{d-3}{2}} \hat{\varphi} \|_{\infty} \| \omega^{\frac{d-3}{2}} \hat{\varphi} \|_{\infty} \| G \|_2 \| \frac{1}{\sqrt{\omega}} f \|,
\]
\[
\| T_{\mu \nu} f \| \leq \| \delta_{\mu \nu} f \| + |\alpha| \| \frac{1}{D_+\left(\omega^2(s)\right)} \| \omega^{\frac{d-3}{2}} \hat{\varphi} \|_{\infty} \| \omega^{\frac{d-3}{2}} \hat{\varphi} \|_{\infty} \| G \|_2 \| f \|.
\]

5When \( \hat{\varphi}(k) = 0 \) for \( |k| > \Lambda \) or \( |k| < \lambda \) (2') of Assumption 3.4, (3.50) is valid for \( s \in [\lambda^2, \Lambda^2]. \)
Then 1. follows. \[ \text{2. This follows from the definition of } T_{\mu\nu}. \]

3. We have

\[
(T_{\mu\nu}d_{\nu\alpha}T_{\alpha\beta}f, g) = (d_{\nu\alpha}T_{\alpha\beta}f, T_{\mu\nu}g)
\]

\[
= (d_{ab}f, g) + \alpha(f, d_{ab}Q\hat{G}d_{\mu\nu}\hat{\varphi}g) + \alpha(d_{ab}Q\hat{G}d_{ab}\hat{\varphi}f, g)
\]

\[
+ \alpha^2 \left( \frac{d - 1}{d} \right) (Q\hat{G}d_{ab}\hat{\varphi}f, Q\hat{G}d_{ab}\hat{\varphi}g)
\]

\[
= I + II + III + IV.
\]

We compute IV as

\[
IV = \lim_{\ell \to 0} \alpha^2 \int \frac{(d - 1) |Q(k)|^2 F(k', k'')}{(|k|^2 - |k'|^2 - it)(|k|^2 - |k''|^2 + it)} dkd'kd''
\]

\[
= \lim_{\ell \to 0} \frac{\alpha^2}{2} \int \left( \int_0^\infty \frac{(d - 1) \hat{\varphi}^2(\sqrt{s})s^{d-2}}{(s - |k'|^2 - it)(s - |k''|^2 + it)|D_+(s)|^2}ds \right) dk'd''
\]

\[
= \lim_{\ell \to 0} \frac{1}{2\pi i} \left( \int_0^\infty \frac{F(k', k'')}{(s - |k'|^2 - it)(s - |k''|^2 + it)} \frac{1}{D_-(s)} ds \right) dk'd''
\]

where \( F(k', k'') = d_{ab}(k')d_{\mu\nu}(k'')\hat{\varphi}(k')\hat{\varphi}(k'')\hat{f}(k')g(k'') \). By a contour integral on the cut plane \( C_{\epsilon, \delta, R} \) (Figure 2), we have

\[
IV = \lim_{\ell \to 0} \lim_{R \to \infty} \frac{1}{2\pi i} \int \left( \int_{C_{\epsilon, \delta, R}} \frac{-F(k', k'')}{(z - |k'|^2 - it)(z - |k''|^2 + it)D(z)} dz \right) dk'd''
\]

\[
= \lim_{\ell \to 0} \frac{-\alpha^2 F(k', k'')}{D(|k''|^2 - it)(|k''|^2 - |k''|^2 - 2it)} dk'd''
\]

\[
+ \lim_{\ell \to 0} \frac{-\alpha^2 F(k', k'')}{D(|k'|^2 + it)(|k'|^2 - |k''|^2 + 2it)} dk'd''
\]

\[
+ \lim_{\ell \to 0} \frac{-\alpha^2 F(k', k'')}{(E^2 - |k'|^2 + it)(E^2 + |k''|^2 - it)} dk'd''
\]

\[
= -\alpha^2 \left( f, d_{ab} \hat{\varphi} \hat{G}d_{\mu\nu}\hat{\varphi}g \right) - \alpha^2 \left( d_{ab} \hat{\varphi} \hat{G}d_{ab}\hat{\varphi}f, g \right) - \theta(m)\gamma^2(f, F_{ab})(F_{\mu\nu}, g).
\]

---

\[ \text{When } \hat{\varphi}(k) = 0 \text{ for } |k| > \Lambda \text{ or } |k| < \lambda (2') \text{ of Assumption 3.4, it is understood that} \]

\[ \| \frac{1}{D_\pm(|\omega|^2)} \| = \sup_{\lambda \leq |k| \leq \Lambda \| \frac{1}{D_\pm(|\omega|^2(k))} \|}. \]
Figure 2: Cut plane $C_{\epsilon, \delta, R}$

Then $IV = -II - III - \theta(m)\gamma^2(f, F_{ab})(F_{ab}, g)$. Hence the desired result is obtained.

4. We see that

\[
(T_{\mu\nu}d_{\nu a}T_{ab}^* f, g) = (d_{\mu b}f, g) - \alpha(d_{\mu b}\hat{\phi}Gf, g) - \alpha(f, d_{\mu b}\hat{\phi}GQg) + \alpha^2(d_{\mu b}\hat{\phi}GQf, \hat{\phi}GQg) = I + II + III + IV.
\]

We have

\[
IV = \lim_{t\to0} \alpha^2\delta_{\mu b} \int \frac{(d-1)\hat{\phi}(k)^2H(k', k'')}{(|k|^2 - |k'|^2 - it)(|k|^2 - |k''|^2 + it)}dkdk'dk'',
\]

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Then the desired result is obtained.

\[ \text{IV} = \lim_{t \to 0} \frac{\alpha^2}{2\delta_{\mu \nu}} \int \left( \int_0^{\infty} \frac{(d-1)\hat{\phi}(\sqrt{s})^2 s^{d-2}}{(s-|k''|^2-it)(s-|k'|^2+it)} ds \right) dk' dk'' \]

\[ = \lim_{t \to 0} \frac{\alpha^2}{2\delta_{\mu \nu}} \int \left( \int_0^{\infty} \frac{|D_+(s)|^2(\frac{d-1}{d})\hat{\phi}(\sqrt{s})^2 s^{d-2}}{(s-|k''|^2-it)(s-|k'|^2+it)|D_+(s)|^2} ds \right) dk' dk'' \]

\[ = \lim_{t \to 0} \frac{1}{2\pi i} \int \left( \int_0^{\infty} \frac{|D_+(s)|^2}{(s-|k''|^2-it)(s-|k'|^2+it)} ds \right) \left( \frac{1}{D_-(s)} - \frac{1}{D_+(s)} \right) dk' dk'' \]

It can be computed by a contour integral on the cut plane \( C_{\epsilon, \delta, R} \) as

\[ \text{IV} = \lim_{t \to 0} \lim_{R \to \infty} \frac{1}{2\pi i} \int_{C_{\epsilon, \delta, R}} \frac{D(z)H(k', k'')}{(z-|k''|^2-it)(z-|k'|^2+it)} dz dk' dk'' \]

\[ = \lim_{t \to 0} \delta_{\mu \nu} \int \frac{D(|k'|^2+it)}{|k'|^2+2it} - \frac{D(|k''|^2-it)}{|k''|^2-2it} H(k', k'') dk' dk'' \]

\[ = \alpha \delta_{\mu \nu}(\hat{\phi}GQf, g) + \alpha \delta_{\mu \nu}(\hat{\phi}GQf, g). \]

Then the desired result is obtained.

5. We have \( T_{\mu \nu}f(k) = \delta_{\mu \nu}f(k) + \lim_{t \to 0} \alpha Q(k) \int \frac{d_{\mu \nu}(k')\hat{\phi}(k')f(k')}{|k|^{2}-|k'|^{2}-it} dk' \), and

\[ \int \frac{d_{\mu \nu}(k')\hat{\phi}(k')f(k')}{|k|^{2}-|k'|^{2}-it} dk' \]

\[ = \int \frac{d_{\mu \nu}(k')\hat{\phi}(k')f(k')}{|k|^2 - |k'|^2 + it} dk' + 2i \int \frac{t d_{\mu \nu}(k')\hat{\phi}(k')f(k')}{(|k|^2 - |k'|^2)^2 + t^2} dk'. \]

Since \( \pi^{-1} \int_{x^2+x^2} f(x) dx \to f(0) \) as \( t \to 0 \), we see that

\[ 2i \int \frac{td_{\mu \nu}(k')\hat{\phi}(k')f(k')}{(|k|^2 - |k'|^2)^2 + t^2} dk' = i \int_{0}^{\infty} \frac{t[d_{\mu \nu}f]^{\sqrt{s}}\hat{\phi}(\sqrt{s})s^{\frac{d-2}{2}}}{(|k|^2 - s^2)^2 + t^2} ds \]

\[ \to \pi i[d_{\mu \nu}f](\omega)\hat{\phi}(\omega)\omega^{d-2}. \]
Then
\[ T_{\mu \nu} f = \delta_{\mu \nu} f + \alpha Q \left( \hat{G} d_{\mu \nu} \hat{\phi} f + \pi i \omega^{d-2} [d_{\mu \nu} f] (\omega) \hat{\phi} \right) \]

Notice that \( Q = \gamma \pm Q \). Hence
\[ T_{\mu \nu} f = \delta_{\mu \nu} f + \alpha \gamma \pm Q \left( \hat{G} d_{\mu \nu} \hat{\phi} f + \pi i \omega^{d-2} \hat{\phi} [d_{\mu \nu} f] (\omega) \right) = \gamma \pm T_{\mu \nu} f + \delta_{\mu \nu} (1 - \gamma) f + \pi i \alpha \gamma \pm Q [d_{\mu \nu} f] (\omega) \omega^{d-2}. \]

Since \( D_+(s) - D_-(s) = \pi i \alpha^2 \left( \frac{d-1}{d} \right) \hat{\phi}^2 (\sqrt{s}) s^{\frac{d-2}{2}} |S_{d-1}| \), we see that
\[ \pi i \alpha \omega^{d-2} \gamma \pm Q \hat{\phi} = \pi i \alpha \omega^{d-2} \alpha \frac{\hat{\phi}^2}{D_-(\omega^2)} = \eta \frac{D_+(\omega^2) - D_-(\omega^2)}{D_-(\omega^2)} = \eta (\gamma - 1). \]

Thus the desired result is obtained.

6. Note that \([d_{\mu \nu}] = \delta_{\mu \nu} / \eta\).
\[ T_{\mu \nu} d_{\nu a} T^*_{ab} f = \gamma \pm T_{\mu \nu} d_{\nu a} T^*_{ab} f + (1 - \gamma) (\delta_{\mu \nu} d_{\nu a} T^*_{ab} f - \eta [d_{\mu \nu} d_{\nu a} T^*_{ab} f]) \]

Notice that
\[ \delta_{\mu \nu} d_{\nu a} T^*_{ab} f - \eta [d_{\mu \nu} d_{\nu a} T^*_{ab} f] = d_{\mu b} f - \eta [d_{\mu b} f] + \alpha \rho_{\mu b} \hat{\phi} \hat{G} \hat{Q} f. \]

Thus
\[ T_{\mu \nu} d_{\nu a} T^*_{ab} f = d_{\mu b} f - (1 - \gamma) \eta [d_{\mu b} f] + \alpha (1 - \gamma) \rho_{\mu b} \hat{\phi} \hat{G} \hat{Q} f + \alpha \gamma \rho_{\mu b} \hat{\phi} \hat{Q} [d_{\mu b} f] - \eta \rho_{\mu b} \hat{\phi} \hat{G} \hat{Q} [d_{\mu b} f]. \]

Since \( e^*_{\mu \rho} \rho_{\mu b} = 0 \), we in particular obtain \([3.49]\).

7. We see that
\[ (T_{\mu \nu} d_{\nu a} h T^*_{ab} f, g) = (d_{\nu a} h T^*_{ab} f, T_{\mu \nu} g) = (\gamma \pm d_{\nu a} h T^*_{ab} f, \gamma \pm T_{\mu \nu} g) + (d_{\nu a} h (1 - \gamma) (\delta_{ab} f - \eta [d_{ab} f]), \gamma \pm T_{\mu \nu} g) + (d_{\nu a} h \gamma \pm T^*_{ab} f, (1 - \gamma) (\delta_{\mu \nu} g - \eta [d_{\mu \nu} g])) + (d_{\nu a} h (1 - \gamma) (\delta_{ab} f - \eta [d_{ab} f]), (1 - \gamma) (\delta_{\mu \nu} g - \eta [d_{\mu \nu} g])) = I + II + III + IV. \]
We have $I = (d_{\nu a} h_{a b} T_{\mu \nu} f, T_{\mu \nu} g)$. We will show that $II + III + IV = 0$. Since $h$ and $\gamma_{\pm}$ are rotation invariant, we have

$$II = \int \bar{h}(1 - \gamma_{\pm}) (d_{\mu b} \bar{f} g - \eta [d_{\mu a} g] [d_{a b} f]) dk$$

$$III = \int \bar{h}(1 - \gamma_{\pm}) (d_{\mu b} \bar{f} g - \eta [d_{\mu a} g] [d_{a b} f]) dk$$

$$IV = \int \bar{h}|1 - \gamma_{\pm}|^2 \times (d_{\mu b} \bar{f} g - \eta [d_{\mu a} g] [d_{a b} f] - \eta [d_{\mu a} g] [d_{a b} f] + \eta^2 [d_{\nu a} [d_{\mu a} g] [d_{a b} f]]) dk.$$

Notice that

$$(1 - \gamma_{\pm}) \gamma_{\pm} = \gamma_{\pm} - 1,$$

$$(1 - \gamma_{\pm}) \gamma_{\pm} = \gamma_{\pm} - 1,$$

$$|1 - \gamma_{\pm}|^2 + \gamma_{\pm} - 1 + \gamma_{\pm} - 1 = 0,$$

$$- \eta [d_{\mu a} g] [d_{\nu b} f] + \eta^2 [d_{\nu a} [d_{a b} f]] = - \eta [d_{\mu a} g] [d_{\nu b} f] + \eta \delta_{\nu a} [d_{\mu a} g] [d_{a b} f] = 0.$$

Then $II + III + IV = 0$ follows.

8. We see that

$$[\omega^2, T_{\mu \nu}] f = \lim_{t \to 0} \int \frac{(|k|^2 - |k'|^2) Q(k) d_{\mu b} (k') \hat{\varphi}(k') f(k')}{|k|^2 - |k'|^2 + it} dk' = \alpha (d_{\mu \nu} \hat{\varphi}, f) Q.$$

9. We see that

$$\alpha T_{\mu \nu} \hat{\varphi} = \alpha \delta_{\mu \nu} \hat{\varphi} + \alpha^2 \frac{\hat{\varphi}}{D_+} \hat{\varphi} d_{\mu \nu} \hat{\varphi}^2 = \alpha \delta_{\mu \nu} \hat{\varphi} \left(1 + \frac{\alpha^2 (\frac{d-1}{d}) \hat{\varphi}^2}{m - \alpha^2 (\frac{d-1}{d}) \hat{\varphi}^2} \right) = m \delta_{\mu \nu} Q.$$

Here we used that $D_+ = m - \alpha^2 (\frac{d-1}{d}) \hat{\varphi}^2$. In particular it follows that

$$\alpha e_{\mu}^* T_{\mu \nu} d_{\nu a} \hat{\varphi} = \alpha e_{\mu}^* T_{\mu \nu} \hat{\varphi} - e_{\mu}^* \rho_{\nu a} \hat{\varphi} = \alpha e_{\mu}^* T_{\mu \nu} \hat{\varphi} = m e_a^* Q.$$

10. This is shown in the same way as 4.

11. We have

$$\left( d_{\nu a} Q, \frac{1}{\omega} T_{\mu \nu} f \right) = \left( d_{\mu a} Q, \frac{f}{\omega} \right) + \left( d_{\nu a} Q, \frac{\alpha Q}{\omega} \bar{G} d_{\mu \nu} \hat{\varphi} f \right) = I + II.$$
By (3.50) we have

\[
\begin{align*}
\text{II} &= \lim_{\varepsilon \to 0} \alpha \int \frac{|Q(k)|^2 d_{\mu\nu}(k')d_{\nu\alpha}(k') \hat{\phi}(k') f(k')}{(|k|^2 - |k'|^2 + it)|k|^2} \, dk dk' \\
&= \lim_{\varepsilon \to 0} \frac{\alpha^2}{2} \int \left( \int_0^\infty \frac{\alpha^2 (\sqrt{s}) s^{d-2} d_{\mu a}(k') \hat{\phi}(k') f(k') S_{d-1}}{(s - |k'|^2 + it) s D_+(s)} \, ds \right) \, dk' \\
&= \lim_{\varepsilon \to 0} \frac{\alpha}{2\pi i} \int \left( \int_0^\infty \frac{\alpha}{(s - |k'|^2 + it) s} \left( \frac{1}{D_-(s)} - \frac{1}{D_+(s)} \right) F(k') \, ds \right) \, dk',
\end{align*}
\]

where \( F(k') = d_{\nu a}(k') \hat{\phi}(k') f(k') \). By a contour integral on the cut plane \( C_{\epsilon,\delta,R} \), we have

\[
\frac{1}{2\pi i} \int \frac{1}{(s - |k'|^2 + it) s} \left( \frac{1}{D_-(s)} - \frac{1}{D_+(s)} \right) \, ds
\]

\[
= -\frac{1}{2\pi i} \lim_{R \to \infty} \int_{C_{\epsilon,\delta,R}} \frac{1}{(z - |k'|^2 + it) z D(z)} \, dz
\]

\[
= -\frac{1}{(|k'|^2 - it) D(|k'|^2 - it)} - \frac{\theta(m)\gamma^2}{(E^2 + |k'|^2 + it)E^2} + \frac{1}{m_{\text{eff}}(|k'|^2 - it)}.
\]

Then

\[
\text{II}
\]

\[
= \lim_{\varepsilon \to 0} \int \left\{ \frac{\alpha F(k')}{m_{\text{eff}}(|k'|^2 - it)} - \frac{\alpha F(k')}{(|k'|^2 - it) D(|k'|^2 - it)} - \frac{\alpha \theta(m)\gamma^2 F(k')}{(E^2 + |k'|^2 + it)E^2} \right\} \, dk'
\]

\[
= \frac{\alpha}{m_{\text{eff}}} \left( d_{\mu a} \frac{\hat{\phi}}{\omega}, \frac{f}{\omega} \right) - \left( d_{\mu a} \frac{Q}{\omega}, \frac{f}{\omega} \right) - \theta(m) \frac{\gamma^2}{E^2} (F_{\mu a}, f).
\]

Hence we have

\[
1 + \text{II} = \frac{\alpha}{m_{\text{eff}}} \left( d_{\mu a} \frac{\hat{\phi}}{\omega}, \frac{f}{\omega} \right) - \theta(m) \frac{\gamma^2}{E^2} (F_{\mu a}, f).
\]

12. Note that \( \int h(k) d_{\mu\nu}(k) f(k) \, dk = \eta \int h(k) d_{\mu a}(k) [d_{a\nu} f] (|k|) \, dk \) for rotation invariant function \( h \). From 5. it follows that

\[
\left( d_{\nu a} \frac{Q}{\sqrt{\omega^2}}, h \frac{1}{\omega} T_{\mu \nu} f \right) = \left( \gamma_{\pm} d_{\nu a} \frac{Q}{\sqrt{\omega^2}}, \gamma_{\pm} \frac{1}{\omega} h T_{\mu \nu} f \right)
\]

\[
+ \left( \gamma_{\pm} d_{\nu a} \frac{Q}{\sqrt{\omega^2}}, h \frac{1}{\omega} (1 - \gamma_{\pm}) (\delta_{\mu \nu} f - \eta [d_{\mu \nu} f]) \right).
\]
Since $\gamma_{\pm}d_{\nu a}\frac{Q}{\sqrt{\omega}}h_{\nu a}^{\pm}(1-\gamma_{\pm})$ is rotation invariant, the second term vanishes. Then the claim is proven.

13. We have

$$(e_\mu^r T_{\mu\nu} F_{\nu a}, g) = (F_{\mu a}, e_\mu^r g) - (F_{\nu a}, \alpha d_{\mu\nu} \hat{\varphi} \hat{G} Q e_\mu^r g) = I - II.$$  

Then

$$II = \alpha^2 \int \frac{(d-1)}{d} \hat{\varphi}(k)^2 \hat{Q}(k') e^a_a(k') g(k') \frac{dk'}{(|k|^2 + E^2)(|k|^2 - |k'|^2 + it)}.$$  

We see that

$$\int \frac{\hat{\varphi}(k)^2}{(|k|^2 + E^2)(|k|^2 - |k'|^2 + it)} dk = \int \left( \frac{\hat{\varphi}(k)^2}{|k|^2 - |k'|^2 + it} - \frac{\hat{\varphi}(k)^2}{|k|^2 + E^2} \right) \frac{1}{E^2 + |k'|^2 - it} dk.$$  

By the definitions of $D_-$ and $-E^2$, we have

$$\lim_{t \to 0} \int \frac{\alpha^2 (d-1)}{d} \hat{\varphi}(k)^2 \frac{dk}{|k|^2 - |k'|^2 + it} = D_- (|k'|^2) - m,$$  

$$\int \frac{\alpha^2 (d-1)}{d} \hat{\varphi}(k)^2 \frac{dk}{|k|^2 + E^2} = D_- (-E^2) - m = -m.$$  

Thus we get

$$II = \int \frac{\hat{\varphi}(k') e^a_a(k') g(k')}{E^2 + |k'|^2} dk' = (F_{ba}, e^r_b g)$$

and $I - II = 0$ follows.

14. We see that $\alpha(\hat{\varphi}, F_{\mu\nu}) = \alpha^2 \delta_{\mu\nu} \left( \frac{d-1}{d} \right) \int \frac{\hat{\varphi}(k)^2}{E^2 + |k|^2} dk = -m \delta_{\mu\nu}$ by (3.52).

15. We see that

$$(T_{\mu\nu}^* d_{\nu a}Q, f) = (d_{\nu a}Q, \delta_{\mu\nu} f) + (d_{\nu a}Q, \alpha Q \hat{G} d_{\mu\nu} \hat{\varphi} f) = I + II.$$
We have

\[
II = \lim_{t \downarrow 0} \alpha \int \frac{|Q(k)|^2 d\mu_\alpha(k') d\nu_\alpha(k) \hat{\varphi}(k') f(k')}{|k|^2 - |k'|^2 + it} dk dk' = \lim_{t \downarrow 0} \alpha^3 \frac{1}{2} \int \left( \int_0^\infty \left( \frac{d-1}{d} \hat{\varphi}(\sqrt{s})^2 s^{d-2} d\mu_\alpha(k') \hat{\varphi}(k') f(k') |S_{d-1}| \right) (s - |k'|^2 + it)|D_+(s)|^2 ds \right) dk',
\]

where \( F(k') = d\mu_\alpha(k') \hat{\varphi}(k') f(k') \). By a contour integration on the cut plane \( C_{\epsilon, \delta, R} \), we compute as

\[
\frac{1}{2\pi i} \int_0^\infty \left( -\frac{1}{D(|k'|^2 - it)} + \frac{\gamma^2}{E^2 + |k'|^2 + it} \right) F(k') dk' = - (d\mu_\alpha Q, f) + \gamma^2 (F_{\mu\alpha}, f).
\]

Then we have

\[
II = \lim_{t \downarrow 0} \alpha \int \left( -\frac{1}{D(|k'|^2 - it)} + \frac{\gamma^2}{E^2 + |k'|^2 + it} \right) F(k') dk' = -(d_{\mu\alpha} Q, f) + \gamma^2 (F_{\mu\alpha}, f).
\]

Hence \( I + II = \gamma^2 (F_{\mu\alpha}, f) \) follows.

16. We have

\[
(F_{\mu\alpha}, F_{\nu\nu}) = \delta_{\mu\nu} \alpha^2 \left( \frac{d-1}{d} \right) \int \frac{\hat{\varphi}(k)^2}{(|k|^2 + E^2)^2} dk = \delta_{\mu\nu} \alpha^2 \left( \frac{d-1}{d} \right) \frac{\alpha^2}{2} \int_0^\infty \frac{\hat{\varphi}(\sqrt{s})^2 s^{d-2}}{(s + E^2)^2} ds = \frac{1}{2\pi i} \delta_{\mu\nu} \int_0^\infty \frac{D_+(s) - D_-(s)}{(s + E^2)^2} ds.
\]

Hence by a contour integral on the cut plane \( C_{\epsilon, \delta, R} \), we have

\[
(F_{\mu\alpha}, F_{\nu\nu}) = \lim_{R \to \infty} \frac{1}{2\pi i} \delta_{\mu\nu} \int_{C_{\epsilon, \delta, R}} \frac{D(z)}{(z + E^2)^2} dz = \delta_{\mu\nu} D'(-E^2).
\]

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Then the proof is complete. \qed

### 3.4.2 Intertwining operator

Now we introduce the class of functions. Let

\[ M_n = \{ f | \omega^n \hat{f} \in L^2(\mathbb{R}^d) \}. \tag{3.53} \]

Let \( \hat{A}_\mu(f) = A_\mu(\hat{f}) \) and \( \hat{\Pi}_\mu(g) = \Pi_\mu(\hat{g}) \), i.e.,

\[
\hat{A}_\mu(f) = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\omega(k)}} e^j_\mu(k) (a^*(k,j) f(k) + a(k,j) f(-k)) \, dk,
\]

\[
\hat{\Pi}_\mu(f) = \frac{i}{\sqrt{2}} \int \sqrt{\omega(k)} e^j_\mu(k) (a^*(k,j) f(k) - a(k,j) f(-k)) \, dk.
\]

Then \([\hat{A}_\mu(f), \hat{\Pi}_\nu(g)] = i(d_{\mu\nu}, \hat{g})\) holds, and

\[
[A_\mu, \hat{\Pi}_\nu(g)] = i(d_{\mu\nu}, \hat{\varphi}, \hat{g}) = i(d_{\mu\nu}, \tilde{\varphi}, g) = i(d_{\mu\nu}, \check{\varphi}, g) = i(d_{\mu\nu}, \hat{\varphi}, g),
\]

where we used \( \check{\varphi} = \hat{\varphi} \). Then we define

\[
B_p(f, j) = \frac{1}{\sqrt{2}} \left\{ \hat{A}_\mu \left( T^{*}_{\mu
u} \sqrt{\omega} e^j_\nu f \right) + i\hat{\Pi}_\mu \left( T^{*}_{\mu
u} \frac{1}{\sqrt{\omega}} e^j_\nu f \right) - \left( p \cdot e^j \frac{Q}{\omega}, \frac{f}{\sqrt{\omega}} \right) \right\}, \tag{3.54}
\]

\[
B^*_p(f, j) = \frac{1}{\sqrt{2}} \left\{ \hat{A}_\mu \left( \overline{T}^{*}_{\mu
u} \sqrt{\omega} e^j_\nu f \right) - i\hat{\Pi}_\mu \left( \overline{T}^{*}_{\mu\nu} \frac{1}{\sqrt{\omega}} e^j_\nu f \right) - \left( p \cdot e^j \frac{Q}{\omega}, \frac{f}{\sqrt{\omega}} \right) \right\} \tag{3.55}
\]

for \( f \in M_0 \cap M_{-1/2} \), where \( \mathcal{T} f(k) = \overline{f}(k) = f(-k) \).

**Remark 3.13** Note that condition \( f \in M_{-1/2} \) is not needed for the definition of \( B^*_p \) for \( p = 0 \).

We have

\[
B_p(f, j) = a(W_{+ij} f, i) + a^*(W_{-ij} f, i) - \left( p \cdot e^j \frac{Q}{\sqrt{2}\omega}, \frac{f}{\sqrt{\omega}} \right), \tag{3.56}
\]

\[
B^*_p(f, j) = a(W_{-ij} f, i) + a^*(W_{+ij} f, i) - \left( p \cdot e^j \frac{Q}{\sqrt{2}\omega}, \frac{f}{\sqrt{\omega}} \right). \tag{3.57}
\]
for \( f \in M_0 \cap M_{-1/2} \), where

\[
W_{+ij} = \frac{1}{2} e^i_\mu \left( \frac{1}{\sqrt{\omega}} T^*_{\mu
u} \sqrt{\omega} + \sqrt{\omega} T^*_{\mu
u} \frac{1}{\sqrt{\omega}} \right) \mathcal{F} e^j_\nu, \quad (3.58)
\]

\[
W_{-ij} = \frac{1}{2} e^i_\mu \left( \frac{1}{\sqrt{\omega}} T^*_{\mu
u} \sqrt{\omega} - \sqrt{\omega} T^*_{\mu
u} \frac{1}{\sqrt{\omega}} \right) e^j_\nu. \quad (3.59)
\]

Let \( W_\pm = (W_{\pm ij})_{1 \leq i,j \leq d-1} : \oplus^{d-1} L^2(\mathbb{R}^d) \to \oplus^{d-1} L^2(\mathbb{R}^d) \) and

\[
W = \begin{pmatrix} W_+ & W_- \\ W_- & W_+ \end{pmatrix} : \bigoplus^2 (\oplus^{d-1} L^2(\mathbb{R}^d)) \to \bigoplus^2 (\oplus^{d-1} L^2(\mathbb{R}^d)).
\]

We set

\[
b_W(f,j) = a(W_{+ij} f, i) + a^*(W_{-ij} f, i), \quad (3.60)
\]

\[
b_W^*(f,j) = a(W_{-ij}^* f, i) + a^*(W_{+ij}^* f, i). \quad (3.61)
\]

for \( f \in M_0 \).

**Lemma 3.14 (Symplectic structure)** Suppose Assumption \[3.4\] and that \( m > -\left(\frac{d-1}{d}\right) \alpha^2 \|\hat{\phi}/\omega\|^2 \), \( m \neq 0 \). Then

\[
W_+ W_+^* - W_- W_-^* = 1, \quad (3.62)
\]

\[
W_+^* W_- - W_-^* W_+ = 0, \quad (3.63)
\]

\[
W_+ W_+^* - W_- W_-^* = 1 + \theta(m) Z_+, \quad (3.64)
\]

\[
W_- W_+^* - W_+ W_-^* = \theta(m) Z_- \quad (3.65)
\]

where

\[
\theta(m) = \begin{cases} 1 & m < 0, \\ 0 & m > 0, \end{cases}
\]

\[
Z_{\pm, ij} f = \frac{1}{2} \gamma^2 \left( \sqrt{\omega} F^i_\mu \left( \frac{F^j_\mu}{\sqrt{\omega}} f \right) \pm \frac{1}{\sqrt{\omega}} F^i_\mu (\sqrt{\omega} F^j_\mu, f) \right),
\]

\[
F^i_\mu = \frac{\alpha \hat{\phi}^*}{E^2 + \omega^2 \epsilon^i_\mu}
\]

\[
\gamma = D'(-E^2)^{-1/2} = \left( \frac{d-1}{d} \right) \int \frac{\alpha^2 \hat{\phi}(k)^2}{E^2 + \omega(k)^2} dk \right)^{-1/2}
\]

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In particular in the case of $m > 0$, holds, i.e., $W \in \mathfrak{Sp}$.

Proof: These relations are proven by making use of algebraic relations stated in Lemma 3.12. By (3.48) in Lemma 3.12 we see that

$$W_{+}^{*}W_{+} - W_{-}^{*}W_{-} = 1,$$  
(3.66)

$$W_{+}^{*}W_{-} - W_{+}^{*}W_{-} = 0,$$  
(3.67)

$$W_{+}^{*}W_{+} - W_{-}^{*}W_{-} = 1,$$  
(3.68)

$$W_{-}^{*}W_{+} - W_{+}^{*}W_{-} = 0,$$  
(3.69)

Then (3.62) follows. By (3.49) of Lemma 3.12 we see that

$$e_{\mu}^{v} \sqrt{\omega T_{\mu \nu} d_{\nu a} T_{ab}^{*} e_{b}^{j'}} = e_{\mu}^{v} \frac{1}{\sqrt{\omega}} T_{\mu \nu} d_{\nu a} T_{ab}^{*} \sqrt{\omega} e_{b}^{j'},$$

Then

$$W_{+}^{*} - W_{-}^{*} = \delta_{jj'} \mathbb{1}.$$
Then from (3.47) it follows that
\[
W_{-jk}W^*_{-kj'} - W^*_{+jk}W_{-kj'} = \theta(m)Z_{-ij}.
\]
Then (3.62) follows.

Lemma 3.15 Suppose Assumption 3.4 and \( m > 0 \). Then \( W \in \mathfrak{sp}_2 \), i.e., \( W_\pm \in I_2 \).

Proof: It is enough to show that \( W_{-ij} \) is a Hilbert-Schmidt operator on \( L^2(\mathbb{R}^d) \) for each \( i, j \). By the definition of \( W_- \), we can see that \( W_{-ij} \) is the integral operator with the integral kernel:
\[
W_{-ij}(k, k') = \frac{\alpha^2}{2} \frac{\hat{\varphi}(k)\hat{\varphi}(k')e^i_{\mu}(k)e^{\dagger}_{\mu}(k')}{\sqrt{|k|}\sqrt{|k'|}((|k| + |k'|)D_-(|k'|^2))}.
\]
Since
\[
|W_{-ij}(k, k')| \leq C(\alpha^2/2)(\hat{\varphi}(k)/|k|)(\hat{\varphi}(k')/|k'|)
\]
with some constant \( C \) by the assumption \( \hat{\varphi}/\omega \in L^2(\mathbb{R}^d) \), \( W_{-ij}(\cdot, \cdot) \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \). Then \( W_{-ij} \) is a Hilbert-Schmidt operator. \( \square \)

Let \( m > 0 \). By the general result obtained in Section 2.1 we can see that canonical commutation relations hold:
\[
[B_p(f, j), B^*_p(g, j')] = \delta_{jj'}(\bar{f}, g),
[B^*_p(f, j), B^*_p(g, j')] = 0,
[B_p(f, j), B_p(g, j')] = 0,
\]
and the adjoint relation \( (F, B_p(f, j)G) = (B^*_p(\bar{f}, j)F, G) \) is satisfied. Furthermore for \( W = \begin{pmatrix} W_+ & W_- \\ W_- & W_+ \end{pmatrix} \) we have
\[
a(f, j) = b_W(W^*_{+ij}f, i) - b^*_W(W^*_{-ij}f, i), \quad (3.70)
\]
\[
a^*(f, j) = -b_W(W^*_{-ij}f, i) + b^*_W(W^*_{+ij}f, i) \quad (3.71)
\]
for \( f \in L^2(\mathbb{R}^d) \).
Lemma 3.16 (Intertwining operator) Let $m > 0$. Then the intertwining operator $\mathcal{U}_W$ associated with $W = \begin{pmatrix} W_+ & W_- \\ W_- & W_+ \end{pmatrix} \in \mathfrak{sp}_2$ is given by
\[
\mathcal{U}_W = C \exp \left( -\frac{1}{2} \Delta^*_{W_-W_+} \right) \exp \left( -N_{1-(W_+^{-1})^*} \right) \exp \left( -\frac{1}{2} \Delta_{-W_+W_-} \right)
\]
(3.72)
and the normalizing constant $C$ by $C = \det(1 - (W_+ W_-)^*)^{1/4}$.

Proof: This follows from Proposition 2.5.

3.4.3 Displacement operator

We will construct the displacement operator associated with $W = \begin{pmatrix} W_+ & W_- \\ W_- & W_+ \end{pmatrix}$ and the vector
\[
L = -p_\mu \begin{pmatrix} e^1_\mu \frac{Q}{\sqrt{\omega^3}} \\ \vdots \\ e^{d-1}_\mu \frac{Q}{\sqrt{\omega^3}} \end{pmatrix} \in \oplus^{d-1} L^2(\mathbb{R}^d).
\]
(3.73)

Lemma 3.17 (Displacement operator) Suppose that $\int \frac{\hat{\varphi}^2}{\omega^3} dk < \infty$. Then the displacement operator associated with $W = \begin{pmatrix} W_+ & W_- \\ W_- & W_+ \end{pmatrix}$ and $L$ is given by
\[
S_p = \exp(-i\Pi_p)
\]
(3.74)
with the generator
\[
\Pi_p = \frac{i}{\sqrt{2}} \frac{\alpha}{m_{\text{eff}}} \left\{ a^*(\frac{p \cdot e^j \hat{\varphi}}{\omega^{3/2}}, j) - a(\frac{p \cdot e^j \hat{\varphi}}{\omega^{3/2}}, j) \right\}.
\]
(3.75)

Proof: The generator of the displacement operator is given by
\[
\Pi_p = -\frac{i}{\sqrt{2}} (b_W(\frac{p \cdot e^j Q}{\omega^{3/2}}, j) - b^*_W(\frac{p \cdot e^j Q}{\omega^{3/2}}, j)).
\]
We compute the right hand side above. Then \( \Pi_p = \frac{i}{\sqrt{2}} (a^* (\bar{\xi}_j, j) - a (\xi_j, j)) \) with

\[
\xi_i = W_{+ij} \frac{p \cdot e_j Q}{\omega^{3/2}} - W_{-ij} \frac{p \cdot e_j \overline{Q}}{\omega^{3/2}}.
\]

By (7), (9) and (12) of Lemma 3.12, we can see that \( T_{\mu\nu}^* d_{\nu\alpha} \overline{Q} = T_{\mu\nu}^* d_{\nu\alpha} \overline{Q} \) and \( T_{\mu\nu}^* d_{\nu\alpha} \overline{Q} = T_{\mu\nu}^* d_{\nu\alpha} \overline{Q} \), and we have

\[
\xi_j = e_j^1 \sqrt{\omega} T_{\mu\nu}^* d_{\nu\alpha} p_a Q \omega^{3/2} = \frac{\alpha}{m_{\text{eff}}} \cdot e_j^1 \frac{\phi}{\omega^{3/2}}
\]

by (11) of Lemma 3.12 under the condition \( m > 0 \). \( \square \)

**Definition 3.18 (Bogoliubov transformation)** Let \( W = \left( \begin{array}{c} W_+ \\ W_- \end{array} \right) \) and \( p \in \mathbb{R}^d \). Suppose \( \int \frac{\hat{\phi}^2}{\omega^3} \, dk < \infty \). Then we define the unitary operator \( \mathcal{U}_p \) by

\[
\mathcal{U}_p = S_p \mathcal{U}_W.
\]  

(3.76)

**Remark 3.19** In the case of \( p = 0 \), we do not need to assume that \( \int \frac{\hat{\phi}^2}{\omega^3} \, dk < \infty \) in the definition of \( \mathcal{U}_p \) in (3.76).

**Lemma 3.20** Suppose Assumption 3.4. (1) Let \( p = 0 \). Then \( \mathcal{U}_0 \) maps \( D(H_1) \) onto itself and

\[
\mathcal{U}_0^{-1} B_0^* (f, j) \mathcal{U}_0 = a^* (f, j).
\]  

(3.77)

(2) In addition to Assumption 3.4, suppose that \( \int \frac{\hat{\phi}^2}{\omega^3} \, dk < \infty \). Then for all \( p \in \mathbb{R}^d \), \( \mathcal{U}_p \) maps \( D(H_1) \) onto itself and

\[
\mathcal{U}_p^{-1} B_p^* (f, j) \mathcal{U}_p = a^* (f, j).
\]  

(3.78)

**Proof:** This follows from the general results of Theorem 2.6. \( \square \)
3.5 Diagonalization and time evolution of radiation fields

3.5.1 Diagonalization

In this section we diagonalize $H_p$ by a unitary operator.

**Lemma 3.21** Suppose Assumption 3.4. Let $f \in M_1 \cap M_{1/2} \cap M_0 \cap M_{-1/2}$. Then for all $p \in \mathbb{R}^d$,

\[
\begin{align*}
[H_p, B_p(f, j)] &= -B_p(\omega f, j), \\
[H_p, B^*_p(f, j)] &= B^*_p(\omega f, j).
\end{align*}
\]

(3.79)  
(3.80)

*Proof:* By the algebraic relations in Lemma 3.12 we will check commutation relations, $[A_\mu, B^\sharp(f, j)]$ and $[H_f, B^\sharp(f, j)]$. By (9) of Lemma 3.12 we see that

\[
[A_\mu, B_p(f, j)] = -\frac{1}{\sqrt{2}} \left( d_{\mu a} \hat{\phi}, T^*_{ab} \frac{1}{\sqrt{\omega}} e^j_b f \right) = -\frac{1}{\sqrt{2}} \left( e^j_b T_{ab} d_{\mu a} \hat{\phi}, \frac{1}{\sqrt{\omega}} f \right) = -\frac{1}{\sqrt{2}} \alpha A_\mu.
\]

By taking the adjoint we also obtain

\[
[A_\mu, B^*_p(f, j)] = \frac{1}{\sqrt{2}} \left( e^j_b T_{ab} d_{\mu a} \hat{\phi}, \alpha A_\mu \right).
\]

Next we see that by the definition of $B_p(f, j)$,

\[
\begin{align*}
[H_f, B_p(f, j)] &= \frac{1}{\sqrt{2}} \left( H_f, \hat{\mu}(T^*_{\mu\nu} \sqrt{\omega} e^j_b f) + i \hat{\Pi}_\mu(T^*_{\mu\nu} \frac{1}{\sqrt{\omega}} e^j_b f) \right) \\
&= \frac{1}{\sqrt{2}} \left\{ -i \hat{\Pi}_\mu(T^*_{\mu\nu} \sqrt{\omega} e^j_b f) - \hat{\mu}(\omega^2 T^*_{\mu\nu} \frac{1}{\sqrt{\omega}} e^j_b f) \right\}.
\end{align*}
\]

By (8) of Lemma 3.12 we have

\[
[H_f, B_p(f, j)] = \frac{1}{\sqrt{2}} \left\{ i \hat{\Pi}_\mu(T^*_{\mu\nu} \frac{1}{\sqrt{\omega}} e^j_b \omega f) + \hat{\mu}(T^*_{\mu\nu} \sqrt{\omega} e^j_b \omega f) \right\} + \frac{\alpha}{\sqrt{2}} (e^j_b Q, \frac{1}{\sqrt{\omega}} f) A_\mu.
\]

Together with them we can see that

\[
\begin{align*}
[H_p, B_p(f, j)] &= \frac{1}{m} (p_\mu - \alpha A_\mu)(-\alpha) [A_\mu, B_p(f, j)] + [H_f, B_p(f, j)] \\
&= -\frac{1}{\sqrt{2}} \left\{ i \hat{\Pi}_\mu(T^*_{\mu\nu} \frac{1}{\sqrt{\omega}} e^j_b \omega f) + \hat{\mu}(T^*_{\mu\nu} \sqrt{\omega} e^j_b \omega f) \right\} + \frac{1}{\sqrt{2}} \left( \frac{p \cdot e^j_b}{\omega}, \frac{\omega f}{\sqrt{\omega}} \right) \\
&= -B_p(\omega f, j).
\end{align*}
\]

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Then (3.79) follows. (3.80) is similarly proven.

**Lemma 3.22** Suppose Assumption 3.4. Let $f \in M_0 \cap M_{-1/2}$. Then for all $p \in \mathbb{R}^d$, it follows that

$$e^{itH_p}B_p^*(f,j)e^{-itH_p} = B_p^*(e^{it\omega f,j}),$$  \hspace{1cm} (3.81)

$$e^{it\omega f,j}e^{-itH_p} = B_p(e^{-it\omega f,j}).$$  \hspace{1cm} (3.82)

**Proof:** Fix $j$ and $f$. Let $A = B_p^*(\bar{f},j) + B_p(f,j)$ and $\Pi = i(B_p^*(\bar{f},j) - B_p(f,j))$. Then $A$ and $\Pi$ are essentially self-adjoint. We denote the self-adjoint extensions by the same symbols. Let $A_t = B_p^*(e^{it\omega f,j}) + B_p(e^{-it\omega f,j})$ and $\bar{A}_t = e^{itH_p}Ae^{-itH_p}$. Then for $\Phi \in \mathcal{F}_\text{fin}$ we can see that $\frac{d}{dt}A_t\Phi = [H_p,A]\Phi$ by Lemma 3.21, and $\frac{d}{dt}\bar{A}_t\Phi = [H_p,A]\Phi$. Thus the function $F(t) = (\Phi, (A_t - \bar{A}_t)\Psi)$, $\Phi, \Psi \in \mathcal{F}_\text{fin}$, satisfies that $\frac{d}{dt}F(t) = 0$, and hence $F(t) = F(0) = 0$ for all $t$. Thus $A_t = \bar{A}_t$ on $\mathcal{F}_\text{fin}$. By a limiting argument $A_t \to A_t$ follows. Similarly we can see that $e^{itH_p}\Pi e^{-itH_p}(B_p^*(e^{it\omega f,j}) - B_p(e^{it\omega f,j}))$. Thus the lemma follows. $\Box$

Set $\Omega_p = \mathcal{U}_p\Omega$.

**Lemma 3.23** Suppose Assumption 3.4. Then (1) and (2) follow.

(1) $\Omega_0 \in \sigma_p(H_0)$.

(2) Suppose that $\int_{\omega^2}^{\omega_3} dk < \infty$. Then $\Omega_p \in \sigma_p(H_p)$ for all $p \in \mathbb{R}^d$.

**Proof:** We prove (2). Statement (1) is similarly proven. Since

$$B_p(f,j)\Phi = \mathcal{U}_p\alpha(f,j)\mathcal{U}_p^{-1}\Phi,$$

$B_p(f,j)\Phi = 0$ for all $f \in M_0 \cap M_{-1/2}$ implies that $\Phi = a\Omega_p$ with some $a \in \mathbb{C}$. By Lemma 3.22 we see that

$$B_p(f,j)e^{-itH_p}\Omega_p = e^{-itH_p}B_p(e^{-it\omega f,j})\Omega_p = 0$$

for all $f \in M_0 \cap M_{-1/2}$. Thus $e^{-itH_p}\Omega_p = a_t(p)\Omega_p$ with some $a_t \in \mathbb{C}$. By the unitary properties of $e^{itH_p}$ we see that $a_t(p)$ can be represented $a_t(p) = e^{itE_p}$ with some $E_p \in \mathbb{R}$. Thus $H_p\Omega_p = E_p\Omega_p$ follows. $\Box$
Proof of Theorem 3.5

Proof: Let $\mathcal{M} = L.H\{\prod_{i=1}^{n} B^*_p(f_i, j_i)\Omega_p | f_i \in M_0 \cap M_{-1/2}, 1 \leq j_i \leq d-1, n \geq 0\}$. Since $a^\sharp$ leaves invariant, $\mathcal{M}$ is dense in $\mathcal{F}$. Let $\Phi \in \mathcal{M}$. Then we have

$$e^{iH_p} \prod_{i=1}^{n} B^*_p(f_i, j_i)\Omega_p = \prod_{i=1}^{n} B^*_p(e^{i\omega f_i, j_i})e^{itE_p}\Omega_p = \mathcal{U}_p \prod_{i=1}^{n} a^*(e^{i\omega f_i, j_i})e^{itE_p}\Omega$$

$$= \mathcal{U}_p e^{it(H_f+E_p)} \prod_{i=1}^{n} a^*(f_i, j_i)\Omega = \mathcal{U}_p e^{it(H_f+E_p)} \mathcal{U}_p^{-1}\Phi.$$ 

Hence the theorem follows on $\mathcal{M}$. By a limiting argument the theorem is proven. □

We define the unitary operator on $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{F} dx$ by

$$\mathcal{U} = \int_{\mathbb{R}^d}^{\oplus} \mathcal{U}_p e^{i\pi N} dp.$$

(3.83)

Proof of Theorem 3.6

Proof: Let $V \in L^\infty(\mathbb{R}^d)$. By Theorem 3.5, we have

$$\mathcal{U}^{-1} H \mathcal{U} = H_{\text{eff}} + H_t + \mathcal{U}^{-1} V \mathcal{U} - V + g$$

(3.84)

on a core of the right-hand side above, e.g., $C_0^\infty(\mathbb{R}^d) \otimes_{\text{alg}} [\mathcal{F}_{\text{fin}} \cap D(H_t)]$. Since $H$ is self-adjoint on $D(-\Delta) \cap D(H_t)$, a limiting argument tells us that $\mathcal{U}$ maps $D(-\Delta) \cap D(H_t)$ onto itself and (3.84) is valid on $D(-\Delta) \cap D(H_t)$. We see that

$$\mathcal{U}^{-1} e^{-ik_x\cdot x} \mathcal{U} = e^{-ik_x} e^{-i\pi N} \mathcal{U}_W^{-1} \exp \left(-i\frac{\alpha}{m_{\text{eff}}} k \cdot \Pi\right) \mathcal{U}_W e^{i\pi N},$$

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where \( \Pi_\mu = \frac{i}{\sqrt{2}} \left\{ a^* \left( \frac{c^j_\mu \hat{\phi}}{\omega^{3/2}}, j \right) - a \left( \frac{c^j_\mu \hat{\phi}}{\omega^{3/2}}, j \right) \right\} \). We have

\[
\mathscr{U}_W^{-1} \Pi_\mu \mathscr{U}_W
\]

\[
= \frac{i}{\sqrt{2}} \mathscr{U}_W^{-1} \left\{ a^* \left( \frac{c^j_\mu \hat{\phi}}{\omega^{3/2}}, j \right) - a \left( \frac{c^j_\mu \hat{\phi}}{\omega^{3/2}}, j \right) \right\} \mathscr{U}_W
\]

\[
= \frac{i}{\sqrt{2}} \mathscr{U}_W^{-1}
\]

\[
\left\{ b_W \left( W^*_{+ij} \frac{c^j_\mu \hat{\phi}}{\omega^{3/2}} + W^*_{-ij} \frac{c^j_\mu \hat{\phi}}{\omega^{3/2}}, i \right) - b_W \left( W^*_{-ij} \frac{c^j_\mu \hat{\phi}}{\omega^{3/2}} + W^*_{+ij} \frac{c^j_\mu \hat{\phi}}{\omega^{3/2}}, i \right) \right\} \mathscr{U}_W
\]

By (10) and (11) of Lemma 3.12 we see that

\[
\omega^{1/2} c^i_a T_{ab} d_{b\mu} \frac{\hat{\phi}}{\omega^2} = \frac{m_{\text{eff}}}{\alpha} \omega^{1/2} c^i_a T_{ab} d_{\mu} \frac{Q}{\omega^2} = \frac{m_{\text{eff}}}{\alpha} \omega^{1/2} c^i_a T_{ab} T^*_{b\nu} d_{\nu} \frac{Q}{\omega^2} = \frac{m_{\text{eff}}}{\alpha} c^i_a \frac{Q}{\omega^{3/2}}.
\]

Since

\[
e^{-\frac{i}{2} \hat{\omega} N} i \left\{ a^* (\bar{g}, j) - a (g, j) \right\} e^{\frac{i}{2} \hat{\omega} N} = a^* (\bar{g}, j) + a (g, j),
\tag{3.85}
\]

we have

\[
e^{-\frac{i}{2} \hat{\omega} N} \mathscr{U}_W^{-1} \Pi_\mu \mathscr{U}_W e^{\frac{i}{2} \hat{\omega} N} = \frac{m_{\text{eff}}}{\alpha} \frac{1}{\sqrt{2}} \left\{ a^* \left( \frac{c^j_\mu \bar{Q}}{\omega^{3/2}}, j \right) + a \left( \frac{c^j_\mu Q}{\omega^{3/2}}, j \right) \right\} = \frac{m_{\text{eff}}}{\alpha} K_\mu.
\]

Hence

\[
\mathscr{U}^{-1} e^{-ikx} \mathscr{U} = e^{-ikx} e^{-ik \cdot K} = T^{-1} e^{-ikx} T.
\]

Let \( \rho \in C_0^\infty (\mathbb{R}^d) \) be such that \( \rho (x) \geq 0 \), \( \text{supp} \rho \subset \{ x \in \mathbb{R}^d | |x| \leq 1 \} \) and \( \int_{\mathbb{R}^d} \rho (x) dx = 1 \). Define \( \rho_\epsilon (x) = \rho (x/\epsilon) / \epsilon^d \), \( \epsilon > 0 \), and \( V_\epsilon = \rho_\epsilon \ast V \). We see that

\[
\mathscr{U}^{-1} \rho \mathscr{U} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \check{\rho} (k) \mathscr{U}^{-1} e^{-ikx} \mathscr{U} dk = \int_{\mathbb{R}^d} \check{\rho} (k) T^{-1} e^{-ikx} T dk = T^{-1} \rho T.
\]

Thus \( \mathscr{U}^{-1} V_\epsilon \mathscr{U} = T^{-1} V_\epsilon T \mathscr{U} \). By a limiting argument, we obtain (3.29) on \( D(-\Delta) \cap D(H_t) \). \( \Box \)
3.5.2 Time evolution of quantized radiation field

Now we can construct the solution to formal equation:

\[
\left( \frac{\partial^2}{\partial t^2} - \Delta \right) A_\mu(x, t) = \frac{\alpha}{m} (p_\nu - A_\nu(x, t)) p_{\nu\mu}(x).
\]

**Theorem 3.24 (Time evolution of A)** Suppose Assumption 3.4 and that \( f \) is real-valued and \( f \in M_0 \cap M_{-1/2} \). Then for all \( p \in \mathbb{R}^d \),

\[
e^{itH_p} A_\mu(f) e^{-itH_p} = \frac{1}{\sqrt{2}} \left\{ B_\nu^*(e^{it\omega} \epsilon^j \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{f}, j) + B_\nu(e^{-it\omega} \epsilon^j \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{f}, j) \right\} + \frac{\alpha}{m_{\text{eff}}} p_\nu (d_{\nu\mu} \hat{\phi}, \hat{f}).
\]

**Proof:** By the symplectic structure, \( W \in S_p, \) \( a^j \) can be represented in terms of \( B_\nu^j \):

\[
a(f, j) = -B_\nu^*(W_{+ij}^* f, i) + B_\nu(W_{+ij} f, i) + \frac{\alpha}{m_{\text{eff}}} \left( \frac{p \cdot e^j \hat{\phi}}{\sqrt{2\omega}}, \frac{f}{\sqrt{\omega}} \right), \tag{3.86}
\]

\[
a^*(f, j) = B_\nu^*(W_{-ij}^* f, i) - B_\nu(W_{-ij} f, i) + \frac{\alpha}{m_{\text{eff}}} \left( \frac{p \cdot e^j \hat{\phi}}{\sqrt{2\omega}}, \frac{f}{\omega} \right). \tag{3.87}
\]

Here we used (2.47) and (2.48) under \( S = W_+ , T = W_- \) and \( L = -p \cdot e^j Q/\omega^{3/2} \) and that

\[-W_{+ij}(-p \cdot e^j \frac{Q}{\omega^{3/2}}) + W_{-ij}(-p \cdot e^j \frac{\bar{Q}}{\omega^{3/2}}) = e^j \sqrt{\omega} T_{\mu\nu}^* \frac{1}{\sqrt{\omega}} e^j p \cdot e^j \frac{Q}{\omega^{3/2}} = \frac{\alpha}{m_{\text{eff}}} p \cdot e^j \frac{\hat{\phi}}{\omega^{3/2}}.
\]

The quantized radiation field \( A_\mu \) can be represented in terms of \( B_\nu^j \). Inserting (3.86) and (3.87) into \( A_\mu \) we can see that

\[
A_\mu(f) = \frac{1}{\sqrt{2}} \left\{ B_\nu^*(W_{+ij}^* \bar{R}_{\mu}^j - W_{-ij}^* \bar{R}_{\mu}^j, i) + B_\nu(W_{+ij} \bar{R}_{\mu}^j - W_{-ij} \bar{R}_{\mu}^j, i) \right\} + \frac{\alpha}{m_{\text{eff}}} p_\nu (d_{\nu\mu} \hat{\phi}, \hat{f}),
\]

\[\text{[HS01] (4.7) and (4.8)} \] is incorrect. \(- \frac{\alpha}{m_{\text{eff}}} (\frac{p \cdot e^j \hat{\phi}}{\sqrt{3\omega^{3/2}}}, \hat{f}) \) is changed to \(+ \frac{\alpha}{m_{\text{eff}}} (\frac{p \cdot e^j \hat{\phi}}{\sqrt{3\omega^{3/2}}}, \hat{f}) \) and \(- \frac{\alpha}{m_{\text{eff}}} (\frac{e^j \hat{\phi}}{\sqrt{3\omega^{3/2}}}, \hat{f}) \) to \(+ \frac{\alpha}{m_{\text{eff}}} (\frac{e^j \hat{\phi}}{\sqrt{3\omega^{3/2}}}, \hat{f}) \).

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where $R^i_{\mu} = e^i_{\mu} \hat{f} / \sqrt{\omega}$. Explicitly we can compute as

$$W^*_{+ij} R^i_{\mu} - W^*_{-ij} \tilde{R}^i_{\mu} = e^i_{\nu} \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{f}, \quad W^*_{+ij} \tilde{R}^i_{\mu} - W^*_{-ij} R^i_{\mu} = e^i_{\nu} \frac{1}{\sqrt{\omega}} T_{\nu\mu} \tilde{\hat{f}}. $$

Then

$$A_{\mu} = \frac{1}{\sqrt{2}} \left\{ B^*_{p}(e^j_{\nu} \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{f}, j) + B_{p}(e^j_{\nu} \frac{1}{\sqrt{\omega}} T_{\nu\mu} \tilde{\hat{f}}, j) \right\} + \frac{\alpha}{m_{\text{eff}}}(d_{\mu\nu} \tilde{\hat{\phi}}, \tilde{\hat{f}}).$$

follows and the theorem is obtained from Lemma 3.22.

From Theorem 3.24 we immediately see the corollary below.

Corollary 3.25 (Time evolution of $A$) Let $V = 0$. Suppose Assumption 3.4 and that $f$ is real-valued and $f \in M_0 \cap M_{-1/2}$. Then

$$e^{itH} A_{\mu}(f)e^{-itH}$$

$$= \frac{1}{\sqrt{2}} \left\{ B^*_{p}(e^{it\omega} e^j_{\nu} \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{f}, j) + B_{p}(e^{-it\omega} e^j_{\nu} \frac{1}{\sqrt{\omega}} T_{\nu\mu} \tilde{\hat{f}}, j) \right\} + \frac{\alpha}{m_{\text{eff}}}(-i \nabla_{\mu})(d_{\mu\nu} \tilde{\hat{\phi}}, \tilde{\hat{f}}).$$

Proof: Since $e^{itH} = \int_{\mathbb{R}^d} e^{iH_{\nu}} dp$,

$$e^{itH} A_{\mu}(f)e^{-itH} = \int_{\mathbb{R}^d} e^{iH_{\nu}} A_{\mu}(f)e^{-itH} dp.$$

Then the corollary follows from Theorem 3.24.

3.6 Dressed electron states

Let us study the relationship between the infrared regular/singular condition and the ground state of $H_p$.

Definition 3.26 (Dressed electron state) The ground state of $H_p$ is called the dressed electron state (DES).

Lemma 3.27 Suppose Assumption 3.4. Let $\Phi$ be an eigenvector of $H_p$. Then $B_p(f, j)\Phi = 0$, $j = 1, \ldots, d - 1$, for all $f \in M_0 \cap M_{-1/2}$.
Proof: Let $H_p \Phi = E\Phi$. Then

$$B_p(f,j)\Phi = e^{itH_p}e^{-itH_p}B_p(f,j)e^{itE}\Phi = e^{itH_p}B_p(e^{it\omega}f,j)e^{-itE}\Phi.$$ 

Let $\mathcal{M} = LH\{\prod_{i=1}^{n} b_W^*(f_i,j_i)\mathcal{W}_W|f_i \in M_0 \cap M_{-1/2}, 1 \leq i \leq n, n \geq 0\}$, where $\mathcal{W}_W$ denotes the intertwining operator associated with $W = \begin{pmatrix} W_+ & W_- \\ W_- & W_+ \end{pmatrix}$. Since $a^\sharp$ can be represented in terms of $B^\sharp_p$ and $B^\star_p(f,j)\mathcal{W}_W = -(p \cdot e^j Q \omega, f \sqrt{\omega})\mathcal{W}_W$, operator $a^\sharp$ leaves $\mathcal{M}$ invariant, and $\mathcal{M}$ is dense in $\mathcal{F}$. Let $\Psi = \prod_{i=1}^{n} B_p(f_i,j_i)\mathcal{W}_W$. We see that

$$B_p(e^{it\omega}f,j)\Psi = \sum_{i=1}^{n} (e^{-it\omega \bar{f}_i}B_p^*(f_i,j_i) \cdots B_p^*(f_i,j_i) \cdots B_p^*(f_n,j_n)\mathcal{W}_W$$

$$- (p \cdot e^{j Q \omega}, e^{-it\omega} f \sqrt{\omega})\Psi \to 0$$

as $t \to \infty$ by the Riemann-Lebesgue lemma. By a limiting argument we see that $e^{itH_p}B_p(e^{it\omega}f,j)e^{-itE}\Phi \to 0$ as $t \to \infty$. Hence we conclude that $B_p(f,j)\Phi = 0$. \qed

Theorem 3.28 (Existence and absence of DES) \cite{Ara83-a, Ara83-b} Suppose Assumption 3.4.

(1) Let $p = 0$. Then $H_0$ has a dressed electron state and it is unique.

(2) Suppose in addition that $\int \frac{\hat{\varphi}^2}{\omega^3}dk < \infty$. Then $H_p$ has a dressed electron state for all $p \in \mathbb{R}$, and it is unique.

(3) Suppose $\int \frac{\hat{\varphi}^2}{\omega^3}dk = \infty$ and $p \neq 0$. Then $\sigma_p(H_p) = \emptyset$. In particular $H_p$ has no ground state.

Proof: In the case of (1) and (2) we have the unitary equivalence $\mathcal{W}_p^{-1}H_p\mathcal{W}_p = H_f + E_p$. Since the Fock vacuum $\Omega$ is the ground state of $H_f + E_p$, $\mathcal{W}_p\Omega$ is the
dressed electron state of \( H_p \). Next we shall show (3). Let \( \Phi \) be any bound state of \( H_p \). Then \( B_p (f, j) \Phi = 0 \) for all \( f \in M_0 \cap M_{-1/2} \) by Lemma 3.27. Then

\[
0 = (F, B_p (f, j) \Phi) = (F, b_W (f, j) \Phi) - (p \cdot e^{jQ/\omega}, f/\sqrt{\omega}) (F, \Phi)
\]

for \( F \in \mathcal{F} \) such that \( F \in D(N) \) and \( (F, \Phi) \neq 0 \). Hence

\[
|(p \cdot e^{jQ/\omega}, f/\sqrt{\omega})| \leq C \| f \|
\]

holds with some constant \( C \), since \( \| a^\sharp (f) F \| \leq \| f \| \|(N + \mathbb{1})^{1/2} F \| \). Hence the functional \( f \mapsto (p \cdot e^{jQ/\omega}, f/\sqrt{\omega}) \) can be extended on \( M_0 (= L^2 (\mathbb{R}^d)) \). Note that

\[
\| p \cdot e^{jQ/\omega^3} \|^2 = p^2 \int |\hat{\varphi}(k)|^2 \frac{1}{\omega(k)^3} \frac{1}{D_+ (\omega(k)^2)} dk = \infty.
\]

The Riesz lemma yields that there exists \( g \in L^2 (\mathbb{R}^d) \) such that \( (g, f) = (p \cdot e^{jQ/\omega}, f/\sqrt{\omega}) \). It is however contradiction, since \( p \cdot e^{jQ/\omega^3} \not\in L^2 (\mathbb{R}^d) \).

See Figure 3

3.7 Ground state energy

3.7.1 Holomorphic property

The ground state energy \( E_p \) can be represented by \( W_\pm \) and \( \hat{\varphi} \).
Lemma 3.29 Suppose Assumption 3.4 and \( \int \frac{\hat{\phi}^2}{\omega^3} dk < \infty \). Then

\[
E_p = \frac{1}{2m}(p + \gamma(p))^2 + g,
\]

where

\[
\gamma(p) = -\frac{\alpha^2}{2m_{\text{eff}}} p \left( \frac{e^i \hat{\phi}}{\sqrt{\omega^3}}, (I + W_{-1})_{ij} \frac{e^i \hat{\phi}}{\sqrt{\omega}} \right),
\]

\[
g = \frac{1}{4m} \left( \frac{e^i \hat{\phi}}{\sqrt{\omega}}, (I - W_{-1})_{ij} \frac{e^i \hat{\phi}}{\sqrt{\omega}} \right).
\]

Proof: We notice that \( E_p = (H_p \Omega, \mathcal{V}_p \Omega) / (\Omega, \mathcal{V}_p \Omega) \),

\[
H_p \Omega = \frac{1}{2m} \sum_{\mu=1}^{d} ((p_\mu + a^*(f_{j,\mu}, j) + a(f_{j,\mu}, j))^2 \Omega, \mathcal{V}_p \Omega),
\]

and \( \mathcal{V}_p = \exp(a^*(\xi_j, j) - a(\xi_j, j)) \mathcal{V}_W \), where \( \xi_j = \frac{1}{\sqrt{2 m_{\text{eff}}}} p \cdot e^i \hat{\phi} / \sqrt{\omega} \) and \( f_{j,\mu} = -\frac{\alpha}{\sqrt{2}} e^i \hat{\phi} \). Then the lemma follows from Lemma 2.8. \( \square \)

We will show that \( E_p \) is holomorphic function of \( \alpha \) on some neighborhood of the real line. In what follows in this section we suppose (1),(2) and (3) of Assumption 3.4. Under (1), (2') and (3) of Assumption 3.4 a similar procedure is also shown. Set \( \mathcal{G} = \omega^{\frac{d-2}{2}} G \omega^{\frac{d-2}{2}} \). Let

\[
H(s) = \lim_{\epsilon \to 0} \left( -\frac{d-1}{d} \right) \int \frac{\hat{\phi}(k)^2}{s + i\epsilon - \omega(k)^2} dk.
\]

Then \( D_+(s) = m - \alpha^2 H(s) \) and

\[
T_{\mu\nu} f = \delta_{\mu\nu} f \left( \frac{1}{m_\alpha^2 - H(\omega^2)} \right) \hat{\phi} \hat{\phi} d_{\mu\nu} f.
\]

For \( \zeta \in \mathbb{C} \) we define \( T_{\mu\nu}(\zeta) \) by \( T_{\mu\nu} \) in (3.90) with \( m/\alpha^2 \) replaced by \( \zeta \), and \( W_{\pm ij}(\zeta) \) by \( W_{\pm ij} \) with \( T_{\mu\nu} \) replaced by \( T_{\mu\nu}(\zeta) \). We see that

\[^8 \text{Hir93 Lemma 5.12] is incorrect. It should be changed to (3.88).} \]
(1) $\text{Im}H(s) \neq 0$ for $s \neq 0$,

(2) $H(0) = -\left(\frac{d-1}{d}\right)\|\hat{\phi}/\omega\|^2 < 0$,

(3) $\lim_{s \to \infty} H(s) = 0$.

Let the image of $H(s)$, $s \geq 0$, be denoted by $H$. From (1)-(3) we can see that for each given $\epsilon > 0$, there exists $\delta > 0$ such that $\mathcal{O}_{\epsilon,\delta} = \{x + iy \in \mathbb{C} | x > \epsilon, |y| < \delta\}$ satisfies $\text{dist}(\mathcal{O}_{\epsilon,\delta}, H) > 0$. See Figure 4. We have

$$W_{\pm ij}(\zeta) = \delta_{ij} \mathbb{I} + e^i_\mu Y_{\mu \nu}^\pm(\zeta)e^j_\nu,$$

where $Y_{\mu \nu}^\pm(\zeta) = Y^\pm(\zeta)d_{\mu \nu}$ and

$$Y^+(\zeta) = \frac{1}{\zeta - H(\omega^2)}\hat{\phi} \left( \frac{1}{\sqrt{\omega}} \hat{G} \sqrt{\omega} + \sqrt{\omega}\hat{G} \frac{1}{\sqrt{\omega}} \right) \hat{\phi} \mathbb{T},$$

$$Y^-(\zeta) = \frac{1}{\zeta - H(\omega^2)}\hat{\phi} \left( \frac{1}{\sqrt{\omega}} \hat{G} \sqrt{\omega} - \sqrt{\omega}\hat{G} \frac{1}{\sqrt{\omega}} \right) \hat{\phi}. $$

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Let $\zeta_0 \in \mathcal{O}_\epsilon$. Then we expand $Y^{\pm}(\zeta)$ around $\zeta_0$ as
\[
Y^{+}(\zeta) = \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n}{(\zeta_0 - H(\omega^2))^{n+1}} \hat{\varphi} \left( \frac{1}{\sqrt{\omega}} \hat{G} \sqrt{\omega} + \sqrt{\omega} \hat{G} \frac{1}{\sqrt{\omega}} \right) \right\} (\zeta - \zeta_0)^n,
\]
\[
Y^{-}(\zeta) = \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n}{(\zeta_0 - H(\omega^2))^{n+1}} \hat{\varphi} \left( \frac{1}{\sqrt{\omega}} \hat{G} \sqrt{\omega} - \sqrt{\omega} \hat{G} \frac{1}{\sqrt{\omega}} \right) \right\} (\zeta - \zeta_0)^n.
\]
Thus $Y^{\pm}(\zeta)$ is analytic on $\mathcal{O}_{\epsilon, \delta}$, and then so is $W^{\pm}(\zeta)$. Furthermore $W^{\pm}(\zeta) \in \mathfrak{Sp}$ for $\zeta \in \mathbb{R}$. Thus we see that
\[
W^+_*(\zeta)W_+(\zeta) - W^+_*(\zeta)W_-(\zeta) = \mathbb{I} + \Delta_1(\zeta),
\]
\[
W^-_*(\zeta)W-(\zeta) - W^-_*(\zeta)W_+(\zeta) = \Delta_2(\zeta),
\]
where $\Delta_j(\zeta), j = 1, 2$, are bounded self-adjoint operators with
\[
\|\Delta_j(\zeta)\| < 1
\]
for $\zeta$ with sufficiently small imaginary part. Thus for each $R > 0$ we can define the open set $\mathcal{O}_{\epsilon, \delta', R} = \{x + iy \in \mathbb{C} | \epsilon < x < R, |y| < \delta'\}$ such that $\mathcal{O}_{\epsilon, \delta', R} \subset \mathcal{O}_{\epsilon, \delta}$ and for all $\zeta \in \mathcal{O}_{\epsilon, \delta', R}$, (3.91), (3.92), and (3.93) hold. Then $W^{-1}_+(\zeta)$ exists and is also holomorphic on $\mathcal{O}_{\epsilon, \delta', R}$. Define
\[
E_p(\zeta) = \frac{1}{2m} (p + \gamma(p, \zeta))^2 + g(\zeta),
\]
where $\gamma(p, \zeta)$ and $g(\zeta)$ are defined with $W_-$ replaced by $W_{\pm}(\zeta)$ for $\zeta \in \mathcal{O}_{\epsilon, \delta', R}$.

**Lemma 3.30** Suppose Assumption 3.4 and $\int \frac{\hat{\varphi}^2}{\omega^3} dk < \infty$. Then for each $\epsilon > 0$ and $R > 0$, there exits $\delta'$ such that $E_p(\zeta)$ is holomorphic on $\mathcal{O}_{\epsilon, \delta', R}$.

**Proof:** Since
\[
\gamma(p, \zeta)_\mu = -\frac{\alpha^2}{2m_{\text{eff}}} p_\nu \left( \frac{e_i^\nu \hat{\varphi}}{\sqrt{\omega^4}}, (\mathbb{I} + W_-(\zeta)W_+(\zeta)^{-1})_{ij} \frac{e_i^\nu \hat{\varphi}}{\sqrt{\omega^4}} \right)_{ij},
\]
\[
g(\zeta) = \frac{1}{4m} \left( \frac{e_i^\nu \hat{\varphi}}{\sqrt{\omega}}, (\mathbb{I} - W_-(\zeta)W_+(\zeta)^{-1})_{ij} \frac{e_i^\nu \hat{\varphi}}{\sqrt{\omega}} \right)_{ij},
\]
the theorem follows from the holomorphic properties of $W_-(\zeta)$ and $W_+(\zeta)^{-1}$. \qed
Lemma 3.31 Suppose Assumption 3.4 and \(\int \frac{\dot{\phi}^2}{\omega^3} dk < \infty\). Then there exists \(m_* > 0\) such that \(E_p = \frac{1}{2m} p^2 + g\).

Proof: Notice that

\[
\gamma(p) = -\frac{\alpha^2}{2m_{\text{eff}}} p_\mu (e^i_{\mu} \dot{\phi} \sqrt{\omega} (\mathbb{I} + W^- W^+)_{ij} e^i_{\mu} \dot{\phi})
= -\frac{\alpha^2}{m_{\text{eff}}} p_\mu \left( \frac{d-1}{d} \right) \|\phi/\omega\|^2 - \frac{\alpha^2}{m_{\text{eff}}} p_\nu \left( \frac{e^i_{\mu} \dot{\phi}}{\sqrt{\omega}^3}, e^i_a Y^- e^i_b (W^-)^{kj} e^i_{\mu} \dot{\phi} \right).
\]

Let \(\alpha^2\) be sufficiently small. We notice that \(W_+ = \mathbb{I} + Y\), where \(Y_{ij} = e^i_{\mu} Y^+ e^j_{\nu}\). The second term is computed as

\[
\left( \frac{e^i_{\mu} \dot{\phi}}{\sqrt{\omega}^3}, e^i_a Y^- e^i_b (W^-)^{kj} e^i_{\mu} \dot{\phi} \right) = \left( \frac{e^i_{\mu} \dot{\phi}}{\sqrt{\omega}^3}, e^i_a Y^- e^i_b ((\mathbb{I} + Y)^{-1})_{kj} e^i_{\mu} \dot{\phi} \right)
= \sum_{n=0}^{\infty} (-1)^n \left( \frac{e^i_{\mu} \dot{\phi}}{\sqrt{\omega}^3}, e^i_a Y^- e^i_b (Y^n)_{kj} e^i_{\mu} \dot{\phi} \right).
\]

Thus we directly see that

\[
\left( \frac{e^i_{\mu} \dot{\phi}}{\sqrt{\omega}^3}, e^i_a Y^- e^i_b (W^-)^{kj} e^i_{\mu} \dot{\phi} \right)
= \sum_{n=0}^{\infty} (-1)^n \left( \frac{e^i_{\mu} \dot{\phi}}{\sqrt{\omega}^3}, e^i_a Y^- e^i_b e^k_{\mu_1} e^k_{\nu_1} e^k_{\mu_2} e^k_{\nu_2} \cdots e^k_{\mu_n} e^k_{\nu_n} \dot{\phi} \right)
= \sum_{n=0}^{\infty} (-1)^n \left( \frac{\dot{\phi}}{\sqrt{\omega}^3}, d_{ab} Y^- d_{ab} d_{ab} d_{ab} Y^+ d_{ab} Y^+ d_{ab} Y^+ \cdots d_{ab} Y^+ \dot{\phi} \right)
= \delta_{\mu\nu} \sum_{n=0}^{\infty} (-1)^n \left( \frac{\dot{\phi}}{\sqrt{\omega}^3}, Y^{-n} \dot{\phi} \right).
\]

We set the right hand side by \(\delta_{\mu\nu} M\). Then we have

\[
E_p = \frac{1}{2m} (p + \gamma(p))^2 + g = \frac{p^2}{2m} \left( 1 - \frac{\alpha^2}{m_{\text{eff}}} \left( \frac{d-1}{d} \right) \|\phi/\omega\|^2 \right) + \frac{\alpha^2}{2m_{\text{eff}}} M + g.
\]

Hence the corollary follows for sufficiently small \(\alpha^2\). Since \(E_p\) is holomorphic on \(\mathcal{O}_{\epsilon, \delta, R}\) for arbitrary \(\epsilon > 0\) and \(R > 0\), the corollary follows for all \(\alpha \in \mathbb{R}\). \(\square\)
In Lemma 3.31 we suppose \( \int \frac{\hat{\varphi}^2}{\omega^3} \omega^3 dk < \infty \). This condition is, however, removed in the next section.

### 3.7.2 Explicit form of effective mass and ground state energy

In the present section we show that \( E_p \) is of the form \( \frac{1}{2m^*} p^2 + g \). The main theorem in this section is as follows.

**Theorem 3.32 (Explicit form of \( E_p \))** Suppose Assumption 3.4. Then \( E_p = \frac{1}{2m^*} p^2 + g \) for all \( \alpha \in \mathbb{R} \), where \( g \) is given by

\[
g = \frac{d}{2\pi} \int_{-\infty}^{\infty} \alpha^2 \left( \frac{d-1}{d} \right) \left\| \frac{\hat{\varphi}^2}{p+\omega^2} \right\|^2 dt.
\]

Note that we do not assume \( \int \frac{\hat{\varphi}^2}{\omega^3} \omega^3 dk < \infty \) in Theorem 3.32. Throughout this section we assume that \( \alpha^2 \) is sufficiently small unless otherwise stated.

Since a momentum lattice approximated \( H_p \) can be identified with a harmonic oscillator in \( L^2(\mathbb{R}^D) \) for some \( D \), \( E_p \) can be obtained through calculating the ground state energy of the harmonic oscillator.

First \( \omega \) is replaced by \( \omega_\epsilon(k) = \omega(k) + \epsilon \) for \( \epsilon > 0 \). For \( l = (l^1, \ldots, l^d) \in \mathbb{R}^d \), let \( |l| = \max_j |l^j| \). For the time being we suppose \( l \in (2\pi \mathbb{Z}/a)^d \), \( |l| \leq 2\pi L \) with some \( a \) and \( L \); \( l \) is a lattice point with the width \( 2\pi/a \) of the \( d \)-dimensional rectangle centered at the origin with the width \( 4\pi L \). The lattice points are named \( l_1, l_2, \ldots, l_\ell \), where \( \ell = (2[aL] + 1)^d \) denotes the number of lattice points and \( [z] \) denotes the integer part of \( z \in \mathbb{R} \). For \( l \in (2\pi \mathbb{Z}/a)^d \), we define the rectangle: \( \Gamma(l) = \left[ l^1, l^1 + \frac{2\pi}{a} \right] \times \cdots \times \left[ l^d, l^d + \frac{2\pi}{a} \right] \). Let

\[
Q_{jl} = \frac{1}{\sqrt{2}} \sqrt{\omega_\epsilon(l)} \left( \frac{a}{2\pi} \right)^{d/2} \left\{ a^*(\chi_{\Gamma(l)}(j)) + a(\chi_{\Gamma(l)}(j)) \right\},
\]

\[
P_{jl} = \frac{i}{\sqrt{2}} \sqrt{\omega_\epsilon(l)} \left( \frac{a}{2\pi} \right)^{d/2} \left\{ a^*(\chi_{\Gamma(l)}(j)) - a(\chi_{\Gamma(l)}(j)) \right\}.
\]

Then the Weyl relations hold,

\[
\exp(itP_{jl}) \exp(isQ_{j,l'}) = \exp\left(its\delta_{l,l_1} \delta_{l_2,l_1'} \cdots \delta_{l_{\ell},l_{\ell}'_1} \delta_{l_{\ell}',j} \right) \exp(isQ_{j,l'}) \exp(itP_{jl}), \quad t, s \in \mathbb{R}.
\]

(3.96)
Let $D = (d - 1)\ell$. We define the $D \times D$-diagonal matrix by

$$ A_0 = \begin{pmatrix} \omega_\epsilon(l_1)^2 \mathbb{I} & & \\ & \omega_\epsilon(l_2)^2 \mathbb{I} & \\ & & \ddots \\ & & & \omega_\epsilon(l_\ell)^2 \mathbb{I} \end{pmatrix}, $$

where $\mathbb{I}$ denotes the $(d - 1) \times (d - 1)$-identity matrix. Since $\epsilon > 0$, $A_0$ is a strictly positive matrix. We denote by $(f, g)_D$ the $D$-dimensional scalar product. Let $v_{\mu}^l = \hat{\phi}(l)e_{\mu}^j(l)$, and

$$ \vec{v}_{\mu} = \begin{pmatrix} v_{\mu 1}^l \\ \vdots \\ v_{\mu d-1}^l \\ \vdots \\ v_{\mu 1}^{\ell} \\ \vdots \\ v_{\mu d-1}^{\ell} \end{pmatrix} \in \mathbb{R}^D, \quad \mu = 1, \ldots, d. $$

For linear operator $T$, let $\langle T \rangle_D = (\vec{v}_{\mu}, T\vec{v}_{\mu})_D$. Suppose that $T : \mathbb{R}^d \to \mathbb{R}$ is a rotation invariant function. Let $T_{\text{diag}}$ be the $D \times D$-diagonal matrix with diagonal elements $T(l)$:

$$ T_{\text{diag}} = \begin{pmatrix} T(l_1) \mathbb{I} & & \\ & T(l_2) \mathbb{I} & \\ & & \ddots \\ & & & T(l_\ell) \mathbb{I} \end{pmatrix}. $$

Then $(\vec{v}_{\mu}, T_{\text{diag}}\vec{v}_{\nu})_D = \delta_{\mu\nu}(d-1)\sum_{|l|\leq 2\pi L} T(l)|\hat{\phi}(l)|^2$. Let $P = (P_{ji})_{1 \leq j \leq d-1, |l| \leq 2\pi L}$ and $Q = (Q_{ji})_{1 \leq j \leq d-1, |l| \leq 2\pi L}$. Then the momentum lattice approximated $H_p$ is written as

$$ H_{L,\alpha}^p(p) = \frac{1}{2m} (p - \alpha(\vec{v}, Q)_D)^2 + \frac{1}{2} (\langle P, P \rangle_D + \langle Q, A_0 Q \rangle_D) - \text{tr} \sqrt{A_0}, $$

where $p \in \mathbb{R}^d$ and $(\vec{v}, Q)_D = ((\vec{v}_1, Q)_D, \ldots, (\vec{v}_d, Q)_D)$.

---

9 $A_0$ given in [HS01, p.1176 in Appendix] is incorrect. Matrix elements $\omega_\epsilon(l_j)$ are changed to $\omega_\epsilon(l_j)^2$. 
Lemma 3.33 Suppose that $\epsilon > 0$. Let $\vec{p}$ and $\vec{q}$ be the momentum operator and its canonical position operator in $L^2(\mathbb{R}^D)$, respectively. Then there exists $\bar{M} \leq \infty$, a $D \times D$ nonnegative symmetric matrix $A$ and $\vec{f} \in \mathbb{R}^D$ such that

$$H_{L,a}^\epsilon(p) \cong \bigoplus_{j=1}^{\bar{M}} \left( \frac{1}{2}(\vec{p},\vec{p})_D + \frac{1}{2}(\vec{q},A\vec{q})_D + \frac{1}{2m}p^2 - \frac{1}{2}(\vec{f},A\vec{f})_D - \frac{1}{2}\text{tr}\sqrt{A_0} \right). \quad (3.97)$$

Proof: Define the $D \times D$-matrix by $P = \sum_{\mu=1}^{d}|\vec{v}_\mu\rangle\langle\vec{v}_\mu|$. Set $A = A_0 + \frac{\alpha^2}{m}P$. Note that $A$ is a strictly positive symmetric matrix, since $A_0$ is strictly positive and $P$ is nonnegative. In particular, $(A+a)^{-1}$ exists for $a \geq 0$. Let $\vec{f} = \vec{f}(p) = \frac{\alpha}{m}A^{-1}p_\mu\vec{v}_\mu \in \mathbb{R}^D$. Then we have

$$H_{L,a}^\epsilon(p) = \frac{1}{2}(P,P)_D + \frac{1}{2}((Q - \vec{f}),A(Q - \vec{f}))_D + \frac{1}{2m}p^2 - \frac{1}{2}(\vec{f},A\vec{f})_D - \frac{1}{2}\text{tr}\sqrt{A_0}.$$

By (3.96) and the von Neumann uniqueness theorem, there exists $\bar{M} \leq \infty$ and a unitary operator $U : \mathcal{F} \to \bigoplus_{j=1}^{\bar{M}} L^2(\mathbb{R}^D)$ implementing that

$$UP_{j}U^{-1} = \bigoplus_{j=1}^{\bar{M}}(-i\nabla_{x,j}),$$
$$UQ_{j}U^{-1} = \bigoplus_{j=1}^{\bar{M}}x_{x,j}.$$

Then $H_{L,a}^\epsilon(p)$ is unitarily equivalent with the direct sum of the harmonic oscillator:

$$\bigoplus_{j=1}^{\bar{M}} \left( \frac{1}{2}(\vec{p},\vec{p})_D + \frac{1}{2}((\vec{q} - \vec{f}),A(\vec{q} - \vec{f}))_D + \frac{1}{2m}p^2 - \frac{1}{2}(\vec{f},A\vec{f})_D - \frac{1}{2}\text{tr}\sqrt{A_0} \right)$$

in $\bigoplus_{j=1}^{\bar{M}} L^2(\mathbb{R}^D)$. By the shift $\vec{q} \to \vec{q} + \vec{f}$ implemented by a unitary operator, we obtain (3.97). \qed

Lemma 3.34 Suppose $\epsilon > 0$. Then

$$\inf \sigma(H_{L,a}^\epsilon(p)) = \frac{1}{2m}p^2 - \frac{1}{2}(\vec{f},A\vec{f})_D + \frac{1}{2}\text{tr}(\sqrt{A} - \sqrt{A_0}). \quad (3.98)$$

\footnote{Possibly $M = \infty$.}
Proof: Generally for the harmonic oscillator \( H_T = \frac{1}{2} (\vec{p}, \vec{p})_D + \frac{1}{2} (\vec{q}, T \vec{q})_D \) with a symmetric nonnegative matrix \( T \), inf \( \sigma (H_T) = \frac{1}{2} \text{tr} \sqrt{T} \). Hence

\[
\inf \sigma \left( \frac{1}{2} (\vec{p}, \vec{p})_D + \frac{1}{2} (\vec{q}, T \vec{q})_D \right) = \frac{1}{2} \text{tr} \sqrt{A}.
\]

Thus the ground state energy of \( H_{L,a}^\epsilon(p) \) is given by (3.98) by Lemma 3.33. \( \square \)

We calculate \((\vec{f}, A \vec{f})_D \) and \( \text{tr}(\sqrt{\tilde{A}} - \sqrt{A_0}) \) as follows. We note that

\[
(\vec{v}_\mu, A_0^{-1} \vec{v}_\nu) = \delta_{\mu\nu} \left( \frac{d-1}{d} \right) \sum_{|l| \leq 2\pi L} \frac{|\hat{\phi}(l)|^2}{\omega_{\epsilon}(l)^2}, \tag{3.99}
\]

\[
(\vec{v}_\mu, (s^2 + A_0)^{-1} \vec{v}_\nu) = \delta_{\mu\nu} \left( \frac{d-1}{d} \right) \sum_{|l| \leq 2\pi L} \frac{|\hat{\phi}(l)|^2}{s^2 + \omega_{\epsilon}(l)^2}, \tag{3.100}
\]

\[
(\vec{v}_\mu, (s^2 + A_0)^{-1} A_0 \vec{v}_\nu) = \delta_{\mu\nu} \left( \frac{d-1}{d} \right) \sum_{|l| \leq 2\pi L} \frac{\omega_{\epsilon}(l)^2 |\hat{\phi}(l)|^2}{s^2 + \omega_{\epsilon}(l)^2}. \tag{3.101}
\]

Furthermore \( A^{-1} = s - \lim_{N \to \infty} \sum_{n=1}^{N} \left(-\frac{\alpha^2}{m} A_0^{-1} P\right)^n A_0^{-1} \).

Lemma 3.35 Suppose \( \epsilon > 0 \). Then

\[
\frac{1}{2m} \frac{p^2}{2m} - \frac{1}{2} (\vec{f}, A \vec{f})_D = \frac{p^2}{2m} \frac{1}{1 + \frac{\alpha^2}{m} \theta}, \tag{3.102}
\]

where \( \theta = \left( \frac{d-1}{d} \right) \sum_{|l| \leq 2\pi L} \frac{|\hat{\phi}(l)|^2}{\omega_{\epsilon}(l)^2} \).
Proof: By (3.99) we have

\[(\vec{f}, A^{\dagger} \vec{f})_D = \frac{1}{m} m \alpha^2 \rho_{\mu \nu} (\vec{v}_\mu, A^{-1} \vec{v}_\nu)_D\]

\[= \frac{1}{m} m \alpha^2 \rho_{\mu \nu} \sum_{n=1}^{\infty} (-\frac{\alpha^2}{m})^{n-1} (\vec{v}_\mu, (A_0^{-1} P)^{n-1} A_0^{-1} \vec{v}_\nu)_D\]

\[= \frac{1}{m} m \alpha^2 \rho_{\mu \nu} \sum_{n=1}^{\infty} (-\frac{\alpha^2}{m})^{n-1} (\vec{v}_\mu, A_0^{-1} \vec{v}_{\mu_1})_D (\vec{v}_{\mu_1}, A_0^{-1} \vec{v}_{\mu_2})_D \cdots (\vec{v}_{\mu_{n-1}}, A_0^{-1} \vec{v}_\nu)_D\]

\[= \frac{1}{m} m \alpha^2 \rho_{\mu \nu} \sum_{n=1}^{\infty} (-\frac{\alpha^2}{m})^{n-1} \delta_{\mu \mu_1} \delta_{\mu_1 \mu_2} \cdots \delta_{\mu_{n-1} \nu} (-\frac{\alpha^2}{m})^{n-1} \theta^n\]

\[= \frac{1}{m} m \alpha^2 \rho_{\mu \nu} \sum_{n=1}^{\infty} (-\frac{\alpha^2}{m})^{n-1} \theta^n\]

\[= \frac{\alpha^2}{m} \rho_{\mu \nu} \sum_{n=1}^{\infty} (-\frac{\alpha^2}{m})^{n-1} \theta^n\]

\[= \frac{\alpha^2}{m} \rho_{\mu \nu} \frac{p^2}{1 + \frac{\alpha^2}{m} \theta m}.\]

Hence (3.102) follows. \[\square\]

Lemma 3.36 Suppose \( \epsilon > 0 \). Then

\[\frac{1}{2} \text{tr} (\sqrt{A} - \sqrt{A_0}) = \frac{d-1}{2\pi} \int_{-\infty}^{\infty} \frac{\alpha^2}{m} \sum_{|l| \leq 2\pi L} \frac{|\hat{\varphi}(l)|^2}{s^2 + \omega_{\epsilon}(l)^2} ds, \quad (3.103)\]

where \( \xi = \left(\frac{d-1}{d}\right) \sum_{|l| \leq 2\pi L} \frac{|\hat{\varphi}(l)|^2}{s^2 + \omega_{\epsilon}(l)^2}. \)

Proof: We see that

\[\text{tr} \sqrt{A} - \text{tr} \sqrt{A_0} = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{tr} \{A(s^2 + A)^{-1} - A_0(s^2 + A_0)^{-1}\} ds.\]

Let \( A_\infty = \sum_{n=1}^{\infty} \left\{ -\frac{\alpha^2}{m} P(s^2 + A_0)^{-1}\right\}^n \). We have

\[A(s^2 + A)^{-1} - A_0(s^2 + A_0)^{-1} = \frac{\alpha^2}{m} P(s^2 + A_0)^{-1} + A(s^2 + A_0)^{-1} A_\infty.\]
It follows that

\[
\text{tr} \frac{\alpha^2}{m} P (s^2 + A_0)^{-1} = \frac{\alpha^2}{m} \sum_{\mu=1}^{d} \sum_{\phi:\text{CONS}} (\phi, \vec{v}_\mu) D(\vec{v}_\mu, (s^2 + A_0)^{-1} \phi)_D,
\]

where \( \sum_{\phi:\text{CONS}} \) means to sum up all the vectors \( \phi_n \) in a complete orthonormal system (CONS). Take a CONS such that \( \left\{ \phi_1 = \frac{\vec{v}_\mu}{\|\vec{v}_\mu\|}, \phi_2, \phi_3, \cdots \right\} \). Then we have by (3.100)

\[
\text{tr} \frac{\alpha^2}{m} P (s^2 + A_0)^{-1} = \frac{\alpha^2}{m} \epsilon (s^2 + A_0)^{-1} = d \frac{\alpha^2}{m} \xi. \tag{3.104}
\]

We see that

\[
A (s^2 + A_0)^{-1} A_\infty = A_0 (s^2 + A_0)^{-1} A_\infty + \frac{\alpha^2}{m} P (s^2 + A_0)^{-1} A_\infty.
\]

It follows that

\[
\text{tr} A_0 (s^2 + A_0)^{-1} A_\infty
= \sum_{n=1}^{\infty} \left( -\frac{\alpha^2}{m} \right)^n \sum_{\phi:\text{CONS}} \left( (s^2 + A_0)^{-1} A_0 \phi, (P(s^2 + A_0)^{-1})^n \phi \right)_D
\]

\[
= \sum_{n=1}^{\infty} \left( -\frac{\alpha^2}{m} \right)^n \sum_{\phi:\text{CONS}} \left( (s^2 + A_0)^{-1} A_0 \phi, \vec{v}_{\mu_1} \right)_D \times
\]

\[
\times \left( (s^2 + A_0)^{-1} \vec{v}_{\mu_1}, \vec{v}_{\mu_2} \right)_D \cdots \left( (s^2 + A_0)^{-1} \vec{v}_{\mu_n}, \phi \right)_D.
\]
Take a CONS such that \( \phi_1 = \frac{(s^2 + A_0)^{-1} A_0 \bar{v}_{\mu_n}}{\|s^2 + A_0\|^{-1} A_0 \bar{v}_{\mu_n}} \), \( \phi_2, \phi_3, \cdots \). Since \(|\alpha|\) is sufficiently small, from (3.101) it follows that

\[
\text{tr}A_0(s^2 + A_0)^{-1}A_\infty = \sum_{n=1}^{\infty} \left(-\frac{\alpha^2}{m}\right)^n (s^2 + A_0)^{-1} \bar{v}_{\mu_n}, (s^2 + A_0)^{-1} A_0 \bar{v}_{\mu_1})_D \times
\]
\[
\times ((s^2 + A_0)^{-1} \bar{v}_{\mu_1}, \bar{v}_{\mu_2})_D \cdots ((s^2 + A_0)^{-1} \bar{v}_{\mu_{n-1}}, \bar{v}_{\mu_n})_D
\]
\[
= \sum_{n=1}^{\infty} \left(-\frac{\alpha^2}{m}\right)^n \delta_{\mu_1 \mu_2} \cdots \delta_{\mu_{n-1} \mu_n} \xi^{n-1} \epsilon (s^2 + A_0)^{-2} A_0
\]
\[
= -\frac{\alpha^2 \epsilon (s^2 + A_0)^{-2} A_0}{m} \left[\frac{\omega_\epsilon (l)^2 |\hat{\phi}(l)|^2}{(s^2 + \omega_\epsilon (l)^2)^2}\right]
\]

(3.105)

and

\[
\text{tr} \frac{\alpha^2}{m} P(s^2 + A_0)^{-1} A_\infty
\]
\[
= \sum_{n=1}^{\infty} \left(-\frac{\alpha^2}{m}\right)^n \frac{\alpha^2}{m} \sum_{\phi: \text{CONS}} (\phi, P(s^2 + A_0)^{-1} (P(s^2 + A_0)^{-1})^n \phi)_D
\]
\[
= \sum_{n=1}^{\infty} \left(-\frac{\alpha^2}{m}\right)^n \frac{\alpha^2}{m} \sum_{\phi: \text{CONS}} (\phi, \bar{v}_{\mu_1})_D (\bar{v}_{\mu_1}, (s^2 + A_0)^{-1} \bar{v}_{\mu_2})_D \cdots (\bar{v}_{\mu_{n+1}}, (s^2 + A_0)^{-1} \phi)_D.
\]

Take a CONS such that \( \{ \phi_1 = \frac{\bar{v}_{\mu_1}}{\|\bar{v}_{\mu_1}\|}, \phi_2, \phi_3, \cdots \} \). Then we see that

\[
\text{tr} \frac{\alpha^2}{m} P(s^2 + A_0)^{-1} A_\infty = -\sum_{n=1}^{\infty} \left(-\frac{\alpha^2}{m}\right)^{n+1} \delta_{\mu_1 \mu_2} \cdots \delta_{\mu_{n+1} \mu_1} \xi^{n+1} = \frac{\alpha^2}{m} \xi^2 \frac{\omega_\epsilon (l)^2 |\hat{\phi}(l)|^2}{(s^2 + \omega_\epsilon (l)^2)^2}.
\]

(3.106)

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Hence we have
\[
\text{tr} \left\{ A(s^2 + A)^{-1} - A_0(s^2 + A_0)^{-1} \right\} = \epsilon(s^2 + A_0)^{-2}A_0 \left( -\frac{\alpha^2}{m} \right) - d \left( \frac{\alpha^2}{m} \right)^2 + d \frac{\alpha^2}{m} \xi
\]
\[
= \frac{\alpha^2 d(d-1)}{m \left( 1 + \frac{\alpha^2}{m} \xi \right)} \sum_{|l| \leq 2\pi L} \left\{ \frac{|\hat{\varphi}(l)|^2}{s^2 + \omega_l(l)^2} - \frac{\omega_l(l)^2|\hat{\varphi}(l)|^2}{(s^2 + \omega_l(l)^2)^2} \right\}
\]
\[
= \frac{(d-1)\alpha^2 s^2}{1 + \frac{\alpha^2}{m} \xi} \sum_{|l| \leq 2\pi L} \frac{|\hat{\varphi}(l)|^2}{(s^2 + \omega_l(l)^2)^2}.
\]
Thus the lemma follows. \[\square\]

**Lemma 3.37** Suppose the same assumptions as in Lemma 3.33. Then
\[
\inf \sigma(H_{L,a}^\epsilon(p)) = \frac{p^2}{2(m + \alpha^2 \theta)} + d - 1 \int_{-\infty}^{\infty} \frac{\alpha^2 s^2}{m + \alpha^2 \xi} \sum_{|l| \leq 2\pi L} \frac{|\hat{\varphi}(l)|^2}{(s^2 + \omega_l(l)^2)^2} ds.
\]

**Proof:** It follows from Lemmas 3.35 and 3.36. \[\square\]

**The proof of Theorem 3.32**

**Proof:** Suppose that \( \int \frac{\hat{\varphi}^2}{\omega^3} dk < \infty \) and \( \alpha^2 \) is sufficiently small. We set
\[
m_{\text{eff}}(a, L, \epsilon) = m + \alpha^2 \theta,
\]
\[
g(a, L, \epsilon) = \frac{d - 1}{2\pi} \int_{-\infty}^{\infty} \frac{\alpha^2 s^2}{m + \alpha^2 \xi} \sum_{|l| \leq 2\pi L} \frac{|\hat{\varphi}(l)|^2}{(s^2 + \omega_l(l)^2)^2} ds.
\]

Note that \( m_{\text{eff}}(a, L, \epsilon) \to m_{\text{eff}} \) and \( g(a, L, \epsilon) \to g \) as \( a \to \infty, L \to \infty, \epsilon \to 0 \). Taking \( a \to \infty \) and then \( L \to \infty \), we see that \( H_{L,a}^\epsilon(p) \to H_p + \epsilon N \) uniformly in the resolvent sense, which yields that \( \inf \sigma(H_{L,a}^\epsilon(p)) \to \inf \sigma(H_p + \epsilon N) \). Hence
\[
\inf \sigma(H_p + \epsilon N) = \lim_{L \to \infty} \lim_{a \to \infty} \left( \frac{p^2}{2m_{\text{eff}}(a, L, \epsilon)} + g(a, L, \epsilon) \right) = \frac{p^2}{2m_{\text{eff}}(\epsilon)} + g(\epsilon),
\]
where $m_{\text{eff}}(\epsilon)$ and $g(\epsilon)$ are defined by $m_{\text{eff}}$ and $g$ with $\omega$ replaced by $\omega_\epsilon$. Since $H_p + \epsilon N \rightarrow H_p$ strongly on $D(H_p)$ as $\epsilon \rightarrow 0$, $H_p + \epsilon N \rightarrow H_p$ holds in the strong resolvent sense. Then it follows

$$\limsup_{\epsilon \rightarrow 0} \inf \sigma(H_p + \epsilon N) \leq \inf \sigma(H_p). \quad (3.107)$$

Furthermore, since $N \geq 0$, we have

$$\liminf_{\epsilon \rightarrow 0} \inf \sigma(H_p + \epsilon N) \geq \inf \sigma(H_p). \quad (3.108)$$

Combining (3.107) and (3.108) we have

$$\inf \sigma(H_p + \epsilon N) \rightarrow \inf \sigma(H_p) = E_p \quad \text{as} \quad \epsilon \rightarrow 0. \quad (3.109)$$

Hence the theorem follows for sufficiently small $\alpha^2$. The theorem is valid however for all $\alpha$ since $E_p$ is holomorphic on $\mathcal{O}_{\epsilon, \delta, R}$ for arbitrary $\epsilon > 0$ and $R > 0$ by Lemma 3.31.

Next we do not assume $\int \frac{\hat{\phi}^2}{\omega^3} dk < \infty$. Let $\hat{\phi}_n$ be a sequence such that $\hat{\phi}_n \omega^l \rightarrow \hat{\phi}/\omega^l$ for $l = 0, -1/2, -1$ and $\hat{\phi}_n/\omega^{3/2} \in L^2(\mathbb{R}^d)$. Then the ground state energy of $H_p$ with cutoff function $\hat{\phi}_n$, $H_p(n)$, is given by

$$E_p = \lim_{\epsilon \rightarrow 0} \left( \frac{p^2}{2m_{\text{eff}}(\epsilon)} + g(\epsilon) \right) = \frac{p^2}{2m_{\text{eff}}} + g.$$

Hence the theorem follows for sufficiently small $\alpha^2$. The theorem is valid however for all $\alpha$ since $E_p$ is holomorphic on $\mathcal{O}_{\epsilon, \delta, R}$ for arbitrary $\epsilon > 0$ and $R > 0$ by Lemma 3.31.

Next we do not assume $\int \frac{\hat{\phi}^2}{\omega^3} dk < \infty$. Let $\hat{\phi}_n$ be a sequence such that $\hat{\phi}_n \omega^l \rightarrow \hat{\phi}/\omega^l$ for $l = 0, -1/2, -1$ and $\hat{\phi}_n/\omega^{3/2} \in L^2(\mathbb{R}^d)$. Then the ground state energy of $H_p$ with cutoff function $\hat{\phi}_n$, $H_p(n)$, is given by

$$\frac{1}{2(m + \alpha^2(\frac{d-1}{d}) \|\hat{\phi}_n/\omega\|^2)} p^2 + \frac{d}{2 \pi} \int_{-\infty}^{\infty} \frac{\alpha^2(\frac{d-1}{d}) \|\hat{\phi}_n/\omega\|^2}{m + \alpha^2(\frac{d-1}{d}) \|\hat{\phi}_n/\omega\|^2} dt. \quad (3.109)$$

We can see that $H_p(n) \rightarrow H_p$ as $n \rightarrow \infty$ in the uniform resolvent sense. Then $\inf \sigma(H_p(n)) \rightarrow \inf \sigma(H_p)$ and (3.109) converges to $E_p$ as $n \rightarrow \infty$. Hence the theorem follows.

### 3.7.3 Ultraviolet cutoffs

In this subsection we assume that $d = 3$, $\alpha = 1$, the ultraviolet cutoff is given by the sharp cutoff

$$\hat{\phi}(k) = (2\pi)^{-3/2} \mathbb{1}_{[\lambda, \Lambda]}(k). \quad (3.110)$$

See Example 3.10. Set $g = g(\Lambda)$ for the emphasis of the dependence of the ultraviolet cutoff parameter $\Lambda$. We estimate the asymptotic behavior of $g(\Lambda)$ as $\Lambda \rightarrow \infty$. In
the case of $V = 0$, from Theorem 3.6 it follows that $g(\Lambda) = \inf \sigma(H)$. It is seen that $g(\Lambda)$ is monotonously increasing in $\Lambda$. Since

$$
\|\hat{\phi}/\sqrt{t^2 + \omega^2}\|^2 = 4\pi \left\{ (\Lambda - \lambda) + t \left( \arctan \frac{\lambda}{t} - \arctan \frac{\Lambda}{t} \right) \right\},
$$

and

$$
\|\hat{\phi}/(t^2 + \omega^2)\|^2 = 4\pi \left\{ \frac{1}{2t} \left( \arctan \frac{\Lambda}{t} - \arctan \frac{\lambda}{t} \right) + \frac{1}{2} \left( \frac{\lambda}{t^2 + \lambda^2} - \frac{\Lambda}{t^2 + \Lambda^2} \right) \right\},
$$

changing variable $t$ to $r = \Lambda/t$, we have the explicit form of ground state energy of $H$ with ultraviolet cutoff (3.110) and $V = 0$.

**Proposition 3.38 (Ground state energy)** Suppose that $V = 0$, $d = 3$, (3.110), and $\alpha = 1$. Then

$$
g(\Lambda) = 4\Lambda^2 \int_0^\infty \frac{(\arctan r - \frac{r}{1+r^2}) - \left( \arctan \frac{\lambda}{\Lambda} - \frac{r(\lambda)}{1+r^2(\frac{\lambda}{\Lambda})^2} \right)}{mr + \frac{8\pi}{3} \Lambda \left\{ (r - \arctan r) - (\arctan \frac{\lambda}{\Lambda}) \right\}} \frac{dr}{r^2}. \quad (3.111)
$$

**Theorem 3.39 (Asymptotic behavior of $g(\Lambda)$)** Assume that $m > \frac{8\pi \lambda}{3}$. Then

$$
\frac{8}{3} \left( \frac{3}{8\pi \ m} \right)^{1/2} \pi \frac{1}{2} \leq \lim_{\Lambda \to \infty} \frac{g(\Lambda)}{\Lambda^{3/2}} \leq \frac{8}{3} \left( \frac{9}{8\pi \ m} \right)^{1/2} \pi \frac{1}{2}.
$$

**Proof:** We decompose $\frac{g(\Lambda)}{4\Lambda}$ as $\frac{g(\Lambda)}{4\Lambda} = \int_0^{1/\Lambda^{1/4}} + \int_{1/\Lambda^{1/4}}^\infty = I_1(\Lambda) + I_2(\Lambda)$. It is enough to show that

$$
\frac{2}{3} \left( \frac{3}{8\pi \ m} \right)^{1/2} \pi \frac{1}{2} \leq \lim_{\Lambda \to \infty} \frac{I_1(\Lambda)}{\sqrt{\Lambda}} \leq \frac{2}{3} \left( \frac{9}{8\pi \ m} \right)^{1/2} \pi \frac{1}{2}, \quad (3.112)
$$

and that $\lim_{\Lambda \to \infty} \frac{I_2(\Lambda)}{\sqrt{\Lambda}} = 0$. Note that $\arctan x = \frac{x}{1+x^2} + \frac{2}{3} \frac{x^3}{(1+x^2)^2} + \frac{5}{3} \frac{x^5}{(1+x^2)^3} + \cdots$. We define functions $f$ and $h$ by

$$
\arctan r - \frac{r}{1+r^2} = \frac{2}{3} \frac{r^3}{(1+r^2)^2} + f(r),
$$

$$
r - \arctan r = \frac{r^3}{1+r^2} - h(r).
$$
It is satisfied that \( f(r) \geq 0, h(r) \geq 0, \lim_{r \to 0} \frac{f(r)}{r^3} = 0, \lim_{r \to 0} \frac{h(r)}{r^3} = \frac{2}{3} \) and

\[
h(r) = \frac{2}{3} \frac{r^3}{(1 + r^2)^2} + f(r) = \arctan r - \frac{r}{1 + r^2}.
\]

Let us set

\[
f_\Lambda(r) = \arctan \left( \frac{\lambda}{\Lambda} \right) - \frac{r \left( \frac{\lambda}{\Lambda} \right)}{1 + r^2 \left( \frac{\lambda}{\Lambda} \right)^2} > 0,
\]

\[
h_\Lambda(r) = r \left( \frac{\lambda}{\Lambda} \right) - \arctan \left( \frac{\lambda}{\Lambda} \right) > 0.
\]

Then \( I_1(\Lambda) \) is written as

\[
I_1(\Lambda) = \int_0^{1/\Lambda^{1/4}} \frac{2}{3} + \frac{(1 + r^2)^2}{r^2} (f(r) - f_\Lambda(r)) \frac{dr}{\frac{m}{\Lambda} + \frac{8\pi}{3} r^2 + \frac{8\pi}{3} \frac{1+r^2}{r} (h(r) + h_\Lambda(r)) (1 + r^2)^2}.
\]

It follows that for \( 0 \leq r \leq 1/\Lambda^{1/4} \),

\[
\frac{8\pi}{3} \frac{1+r^2}{r} (h(r) + h_\Lambda(r)) = \frac{8\pi}{3} r^2 (1 + r^2) \frac{h(r) + h_\Lambda(r)}{r^3} \leq \frac{8\pi}{3} \left( 1 + \frac{1}{\sqrt{\Lambda}} \right) r^2 \theta(\Lambda),
\]

where \( \theta(\Lambda) = \sup_{0 \leq r \leq 1/\Lambda^{1/4}} \frac{h(r)+h_\Lambda(r)}{r^3} \). We set the right hand side of (3.113) by \( r^2 \delta(\Lambda) \).

Note that

\[
\lim_{r \to 0} \frac{h(r) + h_\Lambda(r)}{r^3} = \frac{2}{3} + \frac{1}{3} \left( \frac{\lambda}{\Lambda} \right)^3, \quad \lim_{\Lambda \to \infty} \delta(\Lambda) = \frac{8\pi}{3} \frac{2}{3}.
\]

Moreover we have

\[
- \sup_{0 \leq r \leq (1/\Lambda^{1/4})} \frac{(1 + r^2)^2}{r^3} f_\Lambda(r) \leq \sup_{0 \leq r \leq (1/\Lambda^{1/4})} \frac{(1 + r^2)^2}{r^3} (f(r) - f_\Lambda(r)) \leq \sup_{0 \leq r \leq (1/\Lambda^{1/4})} \frac{(1 + r^2)^2}{r^3} f(r).
\]

Set \( \epsilon(\Lambda) = \max \left\{ \sup_{0 \leq r \leq (1/\Lambda^{1/4})} \frac{(1 + r^2)^2}{r^3} f(r), \sup_{0 \leq r \leq (1/\Lambda^{1/4})} \frac{(1 + r^2)^2}{r^3} f_\Lambda(r) \right\} \). It is trivial to see that \( \lim_{\Lambda \to \infty} \epsilon(\Lambda) = 0 \). Hence we have

\[
\frac{2}{3} - \epsilon(\Lambda) \int_0^{1/\Lambda^{1/4}} \frac{dr}{\frac{m}{\Lambda} + \left( \frac{m}{\Lambda} + \frac{8\pi}{3} \right) r^2} \leq I_1(\Lambda) \leq \frac{2}{3} + \left( \frac{\epsilon(\Lambda)}{1 - 1/\sqrt{\Lambda}} \right) \int_0^{1/\Lambda^{1/4}} \frac{dr}{\frac{m}{\Lambda} + \left( \frac{m}{\Lambda} + \frac{8\pi}{3} - \delta(\Lambda) \right) r^2}.
\]
Then a direct calculation yields that
\[
\lim_{\Lambda \to \infty} \frac{1}{\sqrt{\Lambda}} \int_0^{1/\Lambda^{1/4}} \frac{1}{m + \left(\frac{m}{\Lambda} + \frac{8\pi}{3} - \delta(\Lambda)\right)^2} dr = \lim_{\Lambda \to \infty} \frac{1}{\sqrt{m \left(\frac{m}{\Lambda} + \frac{8\pi}{3} - \delta(\Lambda)\right)}} \arctan \sqrt{\frac{m}{\Lambda}} \frac{m + \frac{8\pi}{3} - \delta(\Lambda)}{m/\sqrt{\Lambda}} = \left(\frac{9}{8\pi m}\right)^{1/2} \frac{\pi}{2}.
\]

Similarly we have
\[
\lim_{\Lambda \to \infty} \frac{1}{\sqrt{\Lambda}} \int_0^{1/\Lambda^{1/4}} \frac{1}{m + \left(\frac{m}{\Lambda} + \frac{8\pi}{3} - \delta(\Lambda)\right)^2} dr = \left(\frac{3}{8\pi m}\right)^{1/2} \frac{\pi}{2}.
\]
Thus
\[
\frac{2}{3} \left(\frac{3}{8\pi m}\right)^{1/2} \frac{\pi}{2} \leq \lim_{\Lambda \to \infty} \frac{1}{\sqrt{\Lambda}} I_1(\Lambda) \leq \frac{2}{3} \left(\frac{9}{8\pi m}\right)^{1/2} \frac{\pi}{2}.
\]
Hence (3.112) follows. Next we show that \( \lim_{\Lambda \to \infty} \frac{I_2(\Lambda)}{\sqrt{\Lambda}} = 0 \). Since
\[
(\arctan r - \frac{r}{1+r^2}) - \left(\arctan \left(\frac{\lambda}{\Lambda}\right) - \frac{r \left(\frac{\lambda}{\Lambda}\right)}{1 + r^2 \left(\frac{\lambda}{\Lambda}\right)^2}\right) \leq \frac{2}{3} \frac{r^3}{(1+r^2)^2} + \frac{53}{42} \frac{r^5}{(1+r^2)^2}
\]
and by the assumption \( m > 8\pi \lambda/3 \),
\[
\frac{m}{\Lambda} r + \frac{8\pi}{3} \left\{ (r - \arctan r) - \left(\arctan \frac{\lambda}{\Lambda} - \arctan \frac{r \left(\frac{\lambda}{\Lambda}\right)}{1 + r^2 \left(\frac{\lambda}{\Lambda}\right)^2}\right) \right\} > \frac{m}{\Lambda} r - \frac{\lambda}{\Lambda} \frac{8\pi}{3} r + \frac{8\pi}{3} \left\{ \frac{r^3}{1+r^2} - \frac{2}{3} \frac{r^3}{1+r^2} + \arctan (r \left(\frac{\lambda}{\Lambda}\right)) \right\} > \frac{8\pi}{9} \frac{r^3}{1+r^2}.
\]
Then
\[
\lim_{\Lambda \to \infty} \frac{1}{\sqrt{\Lambda}} I_2(\Lambda) \leq \lim_{\Lambda \to \infty} \frac{1}{\sqrt{\Lambda}} \int_{1/\Lambda^{1/4}}^{\infty} \frac{2}{3} \frac{r^3}{(1+r^2)^2} + \frac{53}{42} \frac{r^5}{(1+r^2)^2} dr = \lim_{\Lambda \to \infty} \frac{9}{8\pi} \frac{1}{\sqrt{\Lambda}} \int_{1/\Lambda^{1/4}}^{\infty} \frac{1}{1+r^2} \left(\frac{2}{3} + \frac{15}{8} r^2\right) dr = 0.
\]
Thus \( \lim_{\Lambda \to \infty} \frac{I_2(\Lambda)}{\sqrt{\Lambda}} = 0 \) follows, and then the theorem is proven. \( \square \)
3.7.4 Many particle system

We next consider an \( N \) particle system. We assume simply that each particle has mass \( m \) and there is no external potential. The Hamiltonian, \( H \), is defined as a self-adjoint operator acting on \( L^2(\mathbb{R}^3)^{\otimes N} \), and is given by

\[
H_N = \sum_{j=1}^{N} \frac{1}{2m} (-i \nabla_j - \alpha A_j)^2 + H_\text{f}, \tag{3.115}
\]

where

\[
A_{j,\mu} = \frac{1}{\sqrt{2}} \int \frac{e^{i\mu} }{\sqrt{\omega(k)}} \{ \hat{\phi}_j(k) a^*(k, j') + \hat{\phi}_j(k) a(k, j') \} \, dk.
\]

Let \( \inf \sigma(H_N) = g(\Lambda, N) \). We consider the two cases:

1. \( \hat{\phi}_j = \hat{\phi}, \ j = 1, \ldots, N \),
2. \( \text{supp} \hat{\phi}_j \cap \text{supp} \hat{\phi}_i \cap \{0\} = \emptyset, \ i \neq j \).

We will see below that the asymptotic behavior of \( g(\Lambda, N) \) as \( N \to \infty \) depends on ultraviolet cutoffs. In the case of (2) we intuitively expect that \( g(\Lambda, N) \approx N \), since \( N \) particles may have no interaction through quantized radiation fields.

**Proposition 3.40 (Ground state energy for many particle system)** In the case of (1),

\[
g(\Lambda, N) = \frac{N}{\pi} \int_{-\infty}^{\infty} \frac{\|t \hat{\phi}/(t^2 + \omega^2)\|^2}{m + \frac{2}{3} N \|\hat{\phi}/\sqrt{t^2 + \omega^2}\|^2} \, dt,
\]

in the case of (2),

\[
g(\Lambda, N) = \sum_{j=1}^{N} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\|t \hat{\phi}_j/(t^2 + \omega^2)\|^2}{m + \frac{2}{3} \|\hat{\phi}_j/\sqrt{t^2 + \omega^2}\|^2} \, dt.
\]

**Proof:** This is proven in a similar manner to Theorem 3.32.

In the case of (1) the following theorem holds.

**Theorem 3.41 (Asymptotic behavior of \( g(\Lambda) \))** We assume case (1) and \( m > 8\pi \lambda/3 \). Then

\[
\frac{8}{3} \left( \frac{3}{8\pi} \frac{1}{m} \right)^{1/2} \frac{\pi}{2} \leq \lim_{\Lambda, N \to \infty} \frac{g(\Lambda, N)}{\sqrt{N} \Lambda^{3/2}} \leq \frac{8}{3} \left( \frac{9}{8\pi} \frac{1}{m} \right)^{1/2} \frac{\pi}{2}.
\]
Proof: By Proposition 3.40 we have

\[
g(\Lambda, N) = \int_0^\infty \left( \arctan r - \frac{r}{1 + r^2} \right) \left( \arctan r \left( \frac{\Lambda}{\Lambda} \right) - \frac{r \left( \frac{\Lambda}{\Lambda} \right)}{1 + r^2 \left( \frac{\Lambda}{\Lambda} \right)^2} \right) dr.
\]

Then in the similar way as the proof of Theorem 3.41 we decompose it such as

\[
g(\Lambda, N) = \int_0^{1/(\Lambda N)^{1/4}} + \int_{1/(\Lambda N)^{1/4}}^\infty = I_1(\Lambda, N) + I_2(\Lambda, N),
\]

and it can be seen that

\[
\frac{2}{3} \left( \frac{3}{8\pi m} \right)^{1/2} \frac{\pi}{2} \leq \lim_{\Lambda, N \to \infty} \frac{I_1(\Lambda, N)}{\sqrt{N\Lambda}} \leq \frac{2}{3} \left( \frac{9}{8\pi m} \right)^{1/2} \frac{\pi}{2},
\]

and that \( \lim_{\Lambda, N \to \infty} \frac{I_2(\Lambda, N)}{\sqrt{N\Lambda}} = 0. \) Then the theorem follows.

\[\square\]

3.8 Self-energy term

3.8.1 Diagonalization and DES

In this section we investigate the \( A^2 \)-dependence of the ground state energy and DES. Let us define

\[
H_\epsilon = H_p - \frac{\alpha}{m} (-i\nabla) \cdot A + \epsilon \frac{\alpha^2}{2m} A^2 + H_t + V, \tag{3.116}
\]

where \( 0 \leq \epsilon \leq 1 \) denotes a parameter. In the case of \( \epsilon = 1 \), \( H_\epsilon \) describes \( H \) and in the case of \( \epsilon = 0 \), \( H \) without self-energy term \( A^2 \). So the parameter \( \epsilon \) interpolates between them. Neglecting external potential \( V \), we first study the translation invariant Hamiltonian defined by

\[
H_{\epsilon,p} = \frac{1}{2m} p^2 - \frac{\alpha}{m} p \cdot A + \epsilon \frac{\alpha^2}{2m} A^2 + H_t, \quad p \in \mathbb{R}^d. \tag{3.117}
\]

We can also diagonalize \( H_{\epsilon,p} \) in a similar manner to \( H_p \). Let

\[
D^\epsilon(z) = m - \epsilon \frac{\alpha^2}{2} \left( \frac{d - 1}{d} \right) \int \frac{\hat{\varphi}(k)^2}{z - \omega(k)^2} dk. \tag{3.118}
\]
Then we define \( D_+^\epsilon (s) \) and \( Q_\epsilon \) by \( D_+ (s) \) and \( Q \), respectively, with \( D(z) \) replaced by \( D^\epsilon (z) \). We also define \( T^\mu_\nu f = \delta^\mu_\nu f + \epsilon \alpha Q_\epsilon \omega^{d-2} G \omega^{d-2} d^\mu \hat{\varphi} f \). Furthermore the ground state energy \( E_{\epsilon, p} \) of \( H_{\epsilon, p} \) can be explicitly computed in the same manner as that of \( H \). The net result is as follows:

\[
E_{\epsilon, p} = \frac{1}{2m_\epsilon} p^2 + g_\epsilon,
\]

(3.119)

where

\[
\frac{1}{m_\epsilon} = \frac{1}{m} - \frac{\alpha^2 (d-1)}{m + \epsilon \alpha^2 (d-1)} \frac{\| \hat{\varphi} / \omega \|^2}{m},
\]

\[
g_\epsilon = \frac{d}{2\pi} \int_{-\infty}^{\infty} \frac{\epsilon \alpha^2 (d-1)}{m + \epsilon \alpha^2 (d-1)} \frac{\| \hat{\varphi} / \sqrt{t^2 + \omega^2} \|^2}{dt}.
\]

By replacing \( T^\mu_\nu \) with \( T^\epsilon_\mu_\nu \), we define \( W^\epsilon = \left( \begin{array}{cc} W^\epsilon_+ & W^\epsilon_- \\ W^\epsilon_- & W^\epsilon_+ \end{array} \right) \in \mathfrak{Sp}_2 \) and the intertwining operator \( \mathcal{U}(W^\epsilon) \). On the other hand the displacement operator is given by \( e^{-i\Pi_\epsilon} \) under the assumption \( \int \hat{\varphi}^2 / \omega^3 dk < \infty \), where

\[
\Pi_\epsilon = \frac{i}{\sqrt{2}} \frac{\alpha}{m_\epsilon^{\text{eff}}} \left\{ a^* \left( \frac{p \cdot e^j \hat{\varphi}}{\omega^{3/2}} , j \right) - a \left( \frac{p \cdot e^j \hat{\varphi}}{\omega^{3/2}} , j \right) \right\}, \quad p \in \mathbb{R}^d,
\]

(3.120)

with \( m_\epsilon^{\text{eff}} = m + \epsilon \alpha^2 (d-1) \| \hat{\varphi} / \omega \|^2 \). Then we define \( \mathcal{U}_{\epsilon, p} \) by \( \mathcal{U}_{\epsilon, p} = e^{-i\Pi_\epsilon} \mathcal{U}(W^\epsilon) \) for \( p \in \mathbb{R}^d \).

**Theorem 3.42 (Diagonalization of \( H_{\epsilon, p} \))** Suppose Assumption 3.4.

1. Let \( p = 0 \). Then

\[
\mathcal{U}_{\epsilon, 0}^{-1} \left( \epsilon \frac{\alpha^2}{2m} A^2 + H_f \right) \mathcal{U}_{\epsilon, 0} = g_\epsilon + H_f.
\]

(3.121)

2. Suppose \( \int \hat{\varphi}^2 / \omega^3 dk < \infty \). Then

\[
\mathcal{U}_{\epsilon, p}^{-1} H_{\epsilon, p} \mathcal{U}_{\epsilon, p} = \frac{1}{2m_\epsilon} p^2 + g_\epsilon + H_f.
\]

(3.122)

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Proof: The proof is similar to Theorem 3.5.

We define the unitary operator $\mathcal{U}_\epsilon$ on $\mathcal{H}$ by

$$\mathcal{U}_\epsilon = \int_{\mathbb{R}^d} \mathcal{U}_{\epsilon,p} e^{i \frac{p^2}{2}N} dp.$$  \hspace{1cm} (3.123)

**Theorem 3.43 (Diagonalization of $H_\epsilon$)** Suppose Assumption 3.4 and that $V$ is infinitesimally small with respect to $-\Delta$. Assume furthermore that

$$\int \hat{\varphi}_2 \omega^3 dk < \infty.$$  

Then $H_\epsilon$ is self-adjoint on $D(-\Delta) \cap D(H_f)$, and for each $\alpha \in \mathbb{R}$, $\mathcal{U}_\epsilon$ maps $D(-\Delta) \cap D(H_f)$ onto itself and

$$\mathcal{U}_\epsilon^{-1} H \mathcal{U}_\epsilon = -\frac{1}{2m_{\text{eff}}} \Delta + H_f + T_\epsilon^{-1} VT_\epsilon + g_\epsilon, \hspace{1cm} (3.124)$$

where

$$T_\epsilon = \exp \left( -i (-i \nabla) \cdot K^\epsilon \right), \hspace{1cm} (3.125)$$

$$K^\epsilon_{\mu} = \frac{1}{\sqrt{2}} \int \frac{e^{j}_\mu(k)}{\sqrt{\omega(k)}} \left( \frac{Q_\epsilon(k)}{\omega(k)} a^*(k,j) + \frac{Q_\epsilon(k)}{\omega(k)} a(k,j) \right) dk. \hspace{1cm} (3.126)$$

Proof: We set $H_\epsilon$ with $V = 0$ by $H_\epsilon(0)$. In a similar way to Proposition 3.2 we can show that $H_\epsilon(0)$ is self-adjoint on $D(-\Delta) \cap D(H_f)$. By the closed graph theorem there exists $C > 0$ such that $\|(-\Delta + H_f)F\| \leq C(\|H_\epsilon(0)\Phi\| + \|\Phi\|)$. Hence $V$ is infinitesimally small with respect to $H_\epsilon(0)$ and $H_\epsilon$ is self-adjoint on $D(-\Delta) \cap D(H_f)$ by the Kato-Rellich theorem. Statement (3.124) is proven in a similar manner to Theorem 3.6. \hfill \Box

**Corollary 3.44 (Existence and absence of DES of $H_{\epsilon,p}$)**

Suppose Assumption 3.4. Then (1)-(3) follow:

1. The Hamiltonian $H_{\epsilon,p}$ for $p = 0$ has a dressed electron state and it is unique.

2. Suppose $\int \frac{\hat{\varphi}^2}{\omega^3} dk < \infty$. Then $H_{\epsilon,p}$ has a dressed electron state for all $p \in \mathbb{R}^d$ and it is unique.

3. Suppose $\int \frac{\hat{\varphi}^2}{\omega^3} dk = \infty$. Then $H_{\epsilon,p}$ with $p \neq 0$ has no bound state.
Proof: The proof is similar to that of Theorem 3.6.

Corollary 3.45 Let $0 \leq \epsilon < 1$. Suppose Assumption 3.4 and that $V$ is nonnegative and infinitesimally small with respect to $-\Delta$. Assume also that $\int \frac{\hat{\phi}^2}{\omega^3} dk < \infty$. Let

$$\alpha_* = \frac{m}{1 - \epsilon \left( \frac{d-1}{d} \right) \| \hat{\phi} / \omega \|^2}.$$ 

Then $H_\epsilon$ is bounded from below for $\alpha^2 < \alpha_*$ and unbounded from below for $\alpha^2 > \alpha_*$. In particular $H_\epsilon$ has no ground state for $\alpha^2 > \alpha_*$. 

Proof: Let $H_\epsilon(0)$ be $H_\epsilon$ with $V = 0$. When $\alpha^2 < \alpha_*$ (resp. $\alpha^2 > \alpha_*$), $\frac{1}{m_\epsilon} > 0$ (resp. $\frac{1}{m_\epsilon} < 0$) follows. Then $H_\epsilon(0)$ is bounded from below (resp. unbounded from below). Since $V \leq 0$, $H_\epsilon$ is also unbounded from below for $\alpha^2 > \alpha_*$. Contrary to this, $H_\epsilon$ is bounded from below for $\alpha^2 < \alpha_*$, since $T^{-1}VT$ is infinitesimally small with respect to $-\frac{1}{2m_\epsilon} \Delta + H_f$. 

3.8.2 No self-energy term

We consider the special case: $\epsilon = 0$. Thus Hamiltonians are

$$H_0 = -\frac{1}{2m} \Delta - \frac{\alpha}{m} (-i \nabla) \cdot A + H_f + V, \quad (3.127)$$
$$H_{0,p} = \frac{1}{2m} p^2 - \frac{\alpha}{m} p \cdot A + H_f. \quad (3.128)$$

Then $D^\epsilon(z) = m$, $Q_\epsilon = \alpha \hat{\phi} / m$, $T^\epsilon_{\mu\nu} = \delta_{\mu\nu}$, $g = 0$, $W^\epsilon_+ = 1$ and $W^\epsilon_- = 0$. Thus the intertwining operator is the identity and $\mathcal{V}_{0,p} = e^{-i\Pi}$, where

$$\Pi = \frac{i}{\sqrt{2}} \frac{\alpha}{m} \left\{ a^*(\frac{p \cdot e_j \hat{\phi}}{\omega^{3/2}}, j) - a(\frac{p \cdot e_j \hat{\phi}}{\omega^{3/2}}, j) \right\}. \quad (3.129)$$

Suppose $\int \frac{\hat{\phi}^2}{\omega^3} dk < \infty$. Then

$$\mathcal{V}^{-1}_{0,p} H_{0,p} \mathcal{V}_{0,p} = \frac{1}{2m_0} p^2 + H_f. \quad (3.130)$$

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Let \( \mathcal{U}_0 = \int \mathcal{U}_0 \rho e^{i \pi N} dp \). Then

\[
\mathcal{U}_0^{-1} \left( -\frac{1}{2m} \Delta - \frac{\alpha}{m} (-i \nabla) \cdot A + H_f + V \right) \mathcal{U}_0 = -\frac{1}{2m_0} \Delta + H_f + T^{-1}VT, \tag{3.131}
\]

where

\[
\frac{1}{m_0} = \frac{1}{m} - \frac{\alpha^2(d-1)\|\hat{\varphi}/\omega\|^2}{m}, \tag{3.132}
\]

and \( T = \exp(-i(-i \nabla) \cdot K) \) with

\[
K_\mu = \frac{1}{\sqrt{2m}} \int \frac{\epsilon_\mu(k)}{\omega(k)^3} (\hat{\varphi}(k) a^*(k,j) + \hat{\varphi}(k) a(k,j)) dk.
\]

Thus formally we have \( T^{-1}VT = V(x + \alpha Z) \), where

\[
Z_\mu = \frac{1}{\sqrt{2}} \int \frac{\epsilon_\mu(k)}{\omega(k)^3} (\hat{\varphi}(k) a^*(k,j) + \hat{\varphi}(k) a(k,j)) dk.
\]

Let \( V \) be infinitesimally small with respect to \(-\Delta\). By (3.132) we see that \( H_0 \) is unbounded from below for \( \alpha^2 > \left( (d-1)\|\hat{\varphi}/\omega\|^2 \right)^{-1} \) and bounded from below for \( \alpha^2 < \left( (d-1)\|\hat{\varphi}/\omega\|^2 \right)^{-1} \).

3.9 Scaling limits

In this section we investigate scaling limits of Hamiltonian \( H \) and derive effective Hamiltonians. The general references in this section are [Ara90, Dav77, Dav79, Hir93, Hir97, Hir98, Hir99, Hir02].

3.9.1 Weak coupling limit

It is shown that one of the useful tool to derive effective objects is scaling limits. We introduce the scaling by \( a^2 \to \kappa a^2 \). Then the scaled Hamiltonian is of the form

\[
H(\kappa) = \frac{1}{2m} (-i \nabla - \alpha \kappa A)^2 + V + \kappa^2 H_f. \tag{3.133}
\]

We consider the asymptotic behavior of \( H(\kappa) \) as \( \kappa \to \infty \). The scaling (3.133) is equivalent to the substitution

\[
\omega \to \kappa^2 \omega, \quad \hat{\varphi} \to \kappa^2 \hat{\varphi}. \tag{3.134}
\]

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The operator $T_{\mu\nu}$ leaves invariant under this scaling, and hence $W_\pm$ and the intertwining operator $\mathcal{U}_W$ also leave invariant under (3.134). On the other hand the displacement operator $S_p$ is scaled as

$$S_p \to \exp \left( \frac{1}{\kappa} \sqrt{\frac{2}{m_{\text{eff}}}} \alpha \left( a^* \left( \frac{p \cdot e^j}{\omega^{3/2}} \right) - a \left( \frac{p \cdot e^j}{\omega^{3/2}} \right) \right) \right)$$

and then we have

$$\text{s-lim}_{\kappa \to \infty} S_p = \mathbb{1}.$$  

(3.136)

Let $\mathcal{U} = \int \mathbb{1} S_p \mathcal{U}_W e^{i \frac{2}{3} N} dp$. We recall that $g = \frac{d}{2\pi} \int_{-\infty}^{\infty} \frac{\alpha^2 (\frac{d-1}{d})}{m + \alpha^2 (\frac{d-1}{d})} \left( \frac{\phi^2}{\sqrt{t^2 + \omega^2}} \right)^2 dt$. Then $g$ is scaled as $g \to \kappa^2 g$. If we make the corresponding substitution in $\mathcal{U}$ and $\delta V$, we denote them by $\mathcal{U}_\kappa$ and $\delta V_\kappa$, respectively.

**Lemma 3.46** It follows that $\text{s-lim}_{\kappa \to \infty} \mathcal{U}_\kappa = \mathcal{U}_W$.

**Proof:** The lemma follows from (3.136) and the invariance of $\mathcal{U}_W$ under (3.134). $\square$

In order to consider the asymptotic behavior of $H(\kappa)$ as $\kappa \to \infty$, we introduce the energy renormalization $\kappa^2 g$.

**Theorem 3.47** (Weak coupling limit) (Hi02) Let $V$ be relatively bounded with respect to $-\frac{1}{2m} \Delta$ with a relative bound strictly smaller than one. Suppose Assumption 3.4 and $\int \frac{\phi^2}{\omega^3} dk < \infty$. Then for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\text{s-lim}_{\kappa \to \infty} (H(\kappa) - \kappa^2 g - z)^{-1} = (H_{\text{eff}} - z)^{-1} \otimes P_W,$$

(3.137)

where $P_W$ denotes the projection to the one dimensional subspace $\{ a \mathcal{U}_W \Omega | a \in \mathbb{C} \}$.

**Proof:** By the unitary transformation $\mathcal{U}_\kappa$ we have

$$(H(\kappa) - \kappa^2 g - z)^{-1} = \mathcal{U}_\kappa (H_{\text{eff}} + \kappa^2 H_\Gamma + \delta V_\kappa - z)^{-1} \mathcal{U}_\kappa^{-1}.$$  

(3.138)

We have already seen that $\text{s-lim}_{\kappa \to \infty} \mathcal{U}_\kappa = \mathcal{U}_W$. We can directly see that

1. $D(\delta V_\kappa) \supset D(H_{\text{eff}})$ and $\delta V_\kappa (H_{\text{eff}} + \lambda)^{-1}$ is bounded in $\mathcal{H}$ for large $\lambda > 0$ with $\lim_{\lambda \to \infty} \| \delta V_\kappa (H_{\text{eff}} + \lambda)^{-1} \| = 0$, 89
(2) \( \delta V_\kappa(H_{\text{eff}} + \lambda)^{-1} \) is strongly continuous in \( \kappa \),

(3) \( \text{s-lim}_{\kappa \to \infty} \delta V_\kappa(H_{\text{eff}} + \lambda)^{-1} = 0. \)

In the abstract formula in [Ara90], it has been established that (1)-(3) above imply that

\[
\text{s-lim}_{\kappa \to \infty} (H_{\text{eff}} + \kappa^2 H_f + \delta V_\kappa - z)^{-1} = (H_{\text{eff}} - z)^{-1} \otimes P_\Omega,
\]

(3.139)

where \( P_\Omega \) denotes the projection to \( \{ a\Omega | a \in \mathbb{C} \} \). Hence we can see that

\[
\text{s-lim}_{\kappa \to \infty} (H(\kappa) - \kappa^2 g - z)^{-1} = \text{s-lim}_{\kappa \to \infty} \mathcal{U}_\kappa (H_{\text{eff}} + \kappa^2 H_f + \delta V_\kappa - z)^{-1} \mathcal{U}_\kappa^{-1}
\]

\[
= \mathcal{U}_W ((H_{\text{eff}} - z)^{-1} \otimes P_\Omega) \mathcal{U}_W^{-1}
\]

\[
= (H_{\text{eff}} - z)^{-1} \otimes (\mathcal{U}_W P_\Omega \mathcal{U}_W^{-1})
\]

\[
= (H_{\text{eff}} - z)^{-1} \otimes P_W.
\]

Then the theorem is proven. \( \square \)

We notice that the projection \( P_W \) denotes the projection to the one dimensional subspace spanned by the unique ground state of operator \( H_{\rho=0} = \frac{1}{2m} A^2 + H_f \).

3.9.2 Strong coupling limit

In the previous section we study the weak coupling limit which is given by the asymptotic behavior of the scaled Hamiltonian:

\[
-\frac{1}{2m}\Delta - \kappa \frac{\alpha}{m} (-i \nabla) \cdot A + \kappa^2 (\frac{\alpha^2}{2m} A^2 + H_f) + V.
\]

We introduce another scaling. Let

\[
H_\epsilon(\kappa) = -\frac{1}{2m}\Delta - \kappa^2 \frac{\alpha}{m} (-i \nabla) \cdot A + \kappa^2 (\epsilon \frac{\alpha^2}{2m} A^2 + H_f) + V.
\]

(3.140)

The scaling (3.140) may reflect the interaction \((-i \nabla) \cdot A\) between the particle and the field rather than the weak coupling limit. As will be seen below under (3.134) the displacement operator does not disappear, and an effective potential appears instead of effective mass. The scaling in (3.140) corresponds to the substitution:

\[
\omega \to \kappa^2 \omega, \quad \hat{\phi} \to \kappa^3 \hat{\phi}, \quad \epsilon \to \epsilon / \kappa^2.
\]

(3.141)
We investigate the scaling limit of $H_\epsilon(\kappa)$ as $\kappa \to \infty$. Instead of energy renormalizations, in this scaling limit we need a mass renormalization. Under the scaling (3.141), $g$ and $\mathcal{U}$ are invariant. Let us define $m_{\text{ren}}$ by

$$\frac{1}{m_{\text{ren}}} = \frac{1}{m} + \frac{\alpha^2 \left( \frac{d-1}{d} \right) \| \hat{\varphi} / \omega \|^2}{m + \epsilon \alpha^2 \left( \frac{d-1}{d} \right) \| \hat{\varphi} / \omega \|^2} \frac{1}{m}. \quad (3.142)$$

Under the scaling (3.141), $m_{\text{ren}}$ is scaled as

$$\frac{1}{m_{\text{ren}}} \to \frac{1}{m_{\text{ren}}(\kappa)} = \frac{1}{m} + \frac{\kappa^2 \left( \frac{d-1}{d} \right) \| \hat{\varphi} / \omega \|^2}{m + \epsilon \alpha^2 \left( \frac{d-1}{d} \right) \| \hat{\varphi} / \omega \|^2} \frac{1}{m}. \quad (3.143)$$

We define the renormalized Hamiltonian by

$$H_{\text{ren}} = -\frac{1}{2m_{\text{ren}}(\kappa)} \Delta - \frac{\kappa^2 \alpha}{m} (-i \nabla) \cdot A + \kappa^2 \left( \frac{\alpha^2}{2m} A^2 + H_\ell \right) + V. \quad (3.144)$$

**Theorem 3.48** (Strong coupling limit) [Ara90, Hir93, Hir97, Hir02] Let $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ be relatively bounded with respect to $-\frac{1}{2m} \Delta$ with a relative bound strictly smaller than one. Suppose Assumption 3.4 and $\int \frac{\hat{\varphi}^2}{\omega^3} dk < \infty$. Then for $z \in \mathbb{C} \setminus \mathbb{R},$

$$\text{s-lim}_{\kappa \to \infty} (H_{\text{ren}} - z)^{-1} = e^{-i(-i \nabla) \cdot \Pi} \left( \left( -\frac{1}{2m} \Delta + V_{\text{eff}} + g - z \right)^{-1} \otimes P_W \right) e^{i(-i \nabla) \cdot \Pi}, \quad (3.145)$$

where

$$\Pi_{\mu} = \frac{i}{\sqrt{2} m} \left\{ a^* \left( \frac{e^{i \mu} \hat{\varphi}}{\omega^{3/2}} \right) - a \left( \frac{e^{i \mu} \hat{\varphi}}{\omega^{3/2}} \right) \right\}, \quad (3.146)$$

$$P_W = \mathcal{U}_W P_{\Omega} \mathcal{U}^{-1}_W, \quad (3.147)$$

$$V_{\text{eff}}(x) = V * P_C(x), \quad (3.148)$$

$$P_C(x) = (2\pi C)^{-d/2} e^{-|x|^2/(2C)}, \quad (3.149)$$

$$C = \frac{1}{2} \left( \frac{d-1}{d} \right) \| Q_e / \omega^{3/2} \|^2. \quad (3.150)$$

**Proof:** By the unitary transformation $\mathcal{U}$ we have

$$(H_{\text{ren}}(\kappa) - z)^{-1} = \mathcal{U} \left( -\frac{1}{2m} \Delta + \kappa^2 H_\ell + T^{-1} V T + g - z \right)^{-1} \mathcal{U}^{-1}. \quad (3.151)$$
### Table: Scaling and renormalization

|       | Scaling and renormalization | Effective Hamiltonian |
|-------|-----------------------------|-----------------------|
| WCL   | $\frac{1}{2m}(-i\nabla-\alpha\kappa A)^2 + V + \kappa^2 H_f - \kappa^2 g$ | $-\frac{1}{2m}\Delta + V$ |
| SCL   | $-\frac{1}{2m_{\text{ren}}(\kappa)}\Delta - \kappa^2 \frac{\alpha}{m}(-i\nabla)A + \kappa^2(\frac{\alpha^2}{2m}A^2 + H_f) + V$ | $-\frac{1}{2m}\Delta + V * P_C + g$ |

Figure 5: Scaling limits

By the abstract formula [Ara90] again, it has been established that

$$s\text{-lim}_{\kappa \to \infty} \left( \frac{1}{2m}\Delta + \kappa^2 H_f + T^{-1}VT + g - z \right)^{-1} = \left( \frac{1}{2m}\Delta + (\Omega, T^{-1}VT\Omega) + g - z \right)^{-1} \otimes P_\Omega.$$ 

Hence we can see that

$$s\text{-lim}_{\kappa \to \infty} (H_{\text{ren}}(\kappa) - z)^{-1}$$

$$= \mathcal{U} \left( \left( \frac{1}{2m}\Delta + (\Omega, T^{-1}VT\Omega) + g - z \right)^{-1} \otimes P_\Omega \right) \mathcal{U}^{-1}$$

$$= e^{-i(-i\nabla)\Pi} \left( \left( \frac{1}{2m}\Delta + V_{\text{eff}} + g - z \right)^{-1} \otimes P_W \right) \mathcal{U}^{-1}.$$

Then the theorem is proven. □

Potential $V_{\text{eff}}$ is called the effective potential. In the case of $\epsilon = 1$ the effective potential $V_{\text{eff}}$ and $C$ are given by

$$V_{\text{eff}}(x) = V * P_C,$$  \hspace{1cm} (3.151)

$$C = \alpha^2 \frac{1}{2} \left( \frac{d-1}{d} \right) \int \frac{|\hat{\varphi}(k)|^2}{m_{\text{eff}}(k)\omega(k)^2} dk,$$  \hspace{1cm} (3.152)

and in the case of $\epsilon = 0$,

$$V_{\text{eff}}(x) = V * P_C,$$  \hspace{1cm} (3.153)

$$C = \alpha^2 \frac{1}{2} \left( \frac{d-1}{d} \right) \int \frac{|\hat{\varphi}(k)|^2}{m^2\omega(k)^2} dk.$$  \hspace{1cm} (3.154)

The strong coupling limit gives a mathematical interpretation of the Lamb shift derived by (3.3). Namely the difference of the spectrum of $-\frac{1}{2m}\Delta + V$ and $-\frac{1}{2m}\Delta + V_{\text{eff}}$ approximately gives an interpretation of the Lamb shift. This was done in [Wel48] and see historical review [Sch94, p.306,(7.4.20)]. See also [Ara90, Ara11].

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3.10 Negative mass

We are interested in investigating the Hamiltonian with negative mass \( m < 0 \) from mathematical point of view. This is of course an unphysical assumption. In this section we suppose that
\[
- \sum_{\mu=1}^{3} \alpha^2 \| \lambda/\omega \|^2 < m < 0.
\]
In this case \( W = \begin{pmatrix} W_+ & W_- \\ W_- & W_+ \end{pmatrix} \not\in \mathfrak{sp} \). Then we define a new operators. Let us define \( X_\pm \) by
\begin{equation}
X_{\pm ij} f = W_{\pm ij} f + \frac{1}{2} \frac{1}{\sqrt{\omega}} e^i_{\mu} F_{\mu \nu} (\sqrt{\omega} e^j_{\nu} Q, f).
\end{equation}

Lemma 3.49 (Symplectic structure) Suppose Assumption 3.4. Then it follows that

\begin{align}
X_+^* X_+ - X_-^* X_- &= 1, \\
X_+^* X_- - X_-^* X_+ &= 0, \\
X_+ X_+^* - X_- X_-^* &= 1, \\
X_- X_+^* - X_+ X_-^* &= 0.
\end{align}

I.e., \( X = \begin{pmatrix} X_+ & X_-^* \\ X_- & X_+^* \end{pmatrix} \in \mathfrak{sp} \).

Proof: Let \( \xi = (\xi_{ij})_{1 \leq i, j \leq d-1} \) and \( \xi_{ij} = \frac{1}{2} \frac{1}{\sqrt{\omega}} e^i_{\mu} F_{\mu \nu} (\sqrt{\omega} e^j_{\nu} Q, \cdot) \). Then we have \( (\xi^*)_{ij} = \frac{1}{2} \sqrt{\omega} e^i_{\mu} Q (\frac{1}{\sqrt{\omega}} e^j_{\nu} F_{\mu \nu}, \cdot) \). Thus \( X_\pm = W_\pm + \xi \). We have
\begin{equation}
\text{LHS } (3.156) = 1 + \xi^* (W_+ - W_-) + (W_+^* - W_-^*) \xi
\end{equation}

By (13) of Lemma 3.12 we see that
\begin{align}
\xi^* (W_+ - W_-) f &= \frac{1}{2} \sqrt{\omega} e^i_{\mu} Q (\frac{1}{\sqrt{\omega}} e^k_{\nu} F_{\mu \nu}, (W_+ - W_-)_{kij} f) \\
&= \frac{1}{2} \sqrt{\omega} e^i_{\mu} Q (\frac{1}{\sqrt{\omega}} e^k_{\nu} F_{\mu \nu}, e^k_{a} \sqrt{\omega} T_{ab} \frac{1}{\sqrt{\omega}} e^l_{f} f) \\
&= \frac{1}{2} \sqrt{\omega} e^i_{\mu} Q (e^k_{b} T_{ab} F_{a \mu}, \frac{1}{\sqrt{\omega}} f) = 0.
\end{align}

We also see that
\begin{equation}
(W_+^* - W_-^*) \xi f = \frac{1}{2} (\sqrt{\omega} e^i_{\nu} Q, f) (W_+^* - W_-^*)_{ik} \frac{1}{\sqrt{\omega}} e^k_{\mu} F_{\mu \nu} = 0.
\end{equation}
Then (3.156) follows. Identity (3.157) is similarly proven. We have
\[ \text{LHS(3.158)} = 1 + Z_+ + (W_+ \xi^* - \bar{W}_- \bar{\xi}^*) + (\xi W_+^* - \bar{\xi} \bar{W}_+^*) + (\xi \xi^* - \bar{\xi} \bar{\xi}^*). \]
We see that \( \xi \xi^* - \bar{\xi} \bar{\xi}^* = 0 \). By (12) and (15) of Lemma 3.12 we have
\[ \bar{W}_- f = \frac{1}{4} e\mu \left( \frac{1}{\sqrt{\omega}} T_{\mu \nu} \omega d_{\nu a} Q(\frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f) - \frac{1}{4} e\mu \sqrt{\omega} \gamma^2 F_{\mu a} (\frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f) \right). \]
On the other hand
\[ W_+ \xi^* f = \frac{1}{4} e\mu \left( \frac{1}{\sqrt{\omega}} T_{\mu \nu} \omega d_{\nu a} Q(\frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f) + \frac{1}{4} e\mu \sqrt{\omega} \gamma^2 F_{\mu a} (\frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f) \right). \]
Hence
\[ W_+ \xi^* - \bar{W}_- \bar{\xi}^* = \frac{1}{2} e\mu \sqrt{\omega} \gamma^2 F_{\mu a} (\frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f) \]
follows. In a similar manner we have
\[ \bar{\xi} \bar{W}_+^* f = -\frac{1}{4} e\mu \sqrt{\omega} \gamma^2 F_{\mu a} (\frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f) \]
\[ \xi W_+^* f = \frac{1}{4} e\mu \sqrt{\omega} \gamma^2 F_{\mu a} (\frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f) \]
Then
\[ \xi W_+^* - \bar{\xi} \bar{W}_+^* = \frac{1}{2} e\mu \sqrt{\omega} \gamma^2 F_{\mu a} (\frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f). \]
Together with them we have
\[ (W_+ \xi^* - \bar{W}_- \bar{\xi}^*) + (\xi W_+^* - \bar{\xi} \bar{W}_+^*) + (\xi \xi^* - \bar{\xi} \bar{\xi}^*) = -Z_. \]
Then (3.158) follows. Finally we prove (3.159). We have
\[ \text{LHS(3.159)} = Z_+ + (W_+ \xi^* - \bar{W}_+ \bar{\xi}^*) + (\xi W_+^* - \bar{\xi} \bar{W}_+^*) + (\xi \xi^* - \bar{\xi} \bar{\xi}^*). \]
By (12) and (15) of Lemma 3.12 we have
\[
W_+\xi^* f = \frac{1}{4} e^i_\mu \left( \frac{1}{\sqrt{\omega}} T^{*\mu\nu} d_{\nu a} Q \left( \frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f \right) + \frac{1}{4} e^i_\mu \sqrt{\omega} T^{*\mu\nu} d_{\nu a} Q \left( \frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f \right) \right)
\]
\[
= \frac{1}{4} e^i_\mu \left( \frac{1}{\sqrt{\omega}} T^{*\mu\nu} d_{\nu a} Q \left( \frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f \right) + \frac{1}{4} e^i_\mu \sqrt{\omega} T^{*\mu\nu} d_{\nu a} Q \left( \frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f \right) \right)
\]
\[
= \frac{1}{4} e^i_\mu \left( \frac{1}{\sqrt{\omega}} T^{*\mu\nu} d_{\nu a} Q \left( \frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f \right) + \frac{1}{4} e^i_\mu \sqrt{\omega} \gamma^2 F_{\mu a} \left( \frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f \right) \right)
\]
\[
W_-\xi^* f = \frac{1}{4} e^i_\mu \left( \frac{1}{\sqrt{\omega}} T^{*\mu\nu} d_{\nu a} Q \left( \frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f \right) - \frac{1}{4} e^i_\mu \sqrt{\omega} \gamma^2 F_{\mu a} \left( \frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f \right) \right).
\]
Hence
\[
W_-\xi^* - \bar{W}_+\xi^* = -\frac{1}{2} \gamma^2 e^i_\mu \sqrt{\omega} F_{\mu a} \left( \frac{1}{\sqrt{\omega}} e^j_b F_{ab}, f \right).
\]
Similarly
\[
\bar{\xi} W^*_+ f = -\frac{1}{4} e^i_\mu F_{\mu a} (\sqrt{\omega} e^j_b Q, e^k_\omega \left( \frac{1}{\sqrt{\omega}} T_{ab} \sqrt{\omega} - \sqrt{\omega} T_{ab} \frac{1}{\sqrt{\omega}} \right) e^j_f)
\]
\[
= -\frac{1}{4} \frac{1}{\sqrt{\omega}} e^i_\mu F_{\mu a} (T_{ab} d_{\nu a} Q, \sqrt{\omega} e^j_f) + \frac{1}{4} e^i_\mu F_{\mu a} \left( d_{\nu a} Q, \omega T_{ab} \frac{1}{\sqrt{\omega}} e^j_f \right)
\]
\[
= -\frac{1}{4} \frac{1}{\sqrt{\omega}} e^i_\mu F_{\mu a} (T_{ab} d_{\nu a} Q, \sqrt{\omega} e^j_f) + \frac{1}{4} e^i_\mu F_{\mu a} \left( d_{\nu a} Q, \omega T_{ab} \frac{1}{\sqrt{\omega}} e^j_f \right)
\]
\[
= -\frac{1}{4} \frac{1}{\sqrt{\omega}} e^i_\mu F_{\mu a} (\gamma^2 F_{bv}, \sqrt{\omega} e^j_f) + \frac{1}{4} e^i_\mu F_{\mu a} \left( d_{\nu a} Q, \omega T_{ab} \frac{1}{\sqrt{\omega}} e^j_f \right)
\]
\[
\xi W_+ f = \frac{1}{4} \frac{1}{\sqrt{\omega}} e^i_\mu F_{\mu a} (\gamma^2 F_{bv}, \sqrt{\omega} e^j_f) + \frac{1}{4} e^i_\mu F_{\mu a} \left( d_{\nu a} Q, \omega T_{ab} \frac{1}{\sqrt{\omega}} e^j_f \right).
\]
Hence
\[
\xi W_+ - \bar{\xi} W^*_+ = \frac{1}{2} \frac{1}{\sqrt{\omega}} e^i_\mu \gamma^2 F_{\mu a} (\sqrt{\omega} e^j_b F_{bv}, )
\]
and then
\[
(W_-\xi^* - \bar{W}_+\xi^*) + (\xi W_+ + \bar{\xi} W^*_+) + (\xi \xi^* - \bar{\xi} \xi^*) = -Z_-
\]
Hence (3.159) follows. \qed

Although \( \begin{pmatrix} W_+ & W_- \\ W_- & W_+ \end{pmatrix} \notin \mathfrak{sp} \), it is shown that \( B^2_{p}(f, j) \) still satisfies canonical commutation relations and adjoint relation. We notice that \( a^2 \) can not be realized.
however in terms of $B^a_p$. For $p \in \mathbb{R}^d$, we define

$$C_\mu(p) = -\frac{1}{E} p_\mu + E \tilde{A}_a(F_{a\mu}) - \tilde{\Pi}_a(F_{a\mu}),$$

$$D_\mu(p) = -\frac{1}{E} p_\mu + E \tilde{A}_a(F_{a\mu}) + \tilde{\Pi}_a(F_{a\mu}).$$

Both $C_\mu(p)$ and $D_\mu(p)$ are essentially self-adjoint.

**Lemma 3.50** Suppose Assumption 3.4. Then it follows that

$$[C_\mu(p), D_\nu(p)] = 2i E \frac{1}{\gamma^2} \delta_{\mu\nu},$$

(3.160)

$$[C_\mu(p), C_\nu(p)] = 0,$$

(3.161)

$$[D_\mu(p), D_\nu(p)] = 0,$$

(3.162)

$$[B_p(f, j), D_\nu(p)] = [B_p(f, j), C_\nu(p)] = 0,$$

(3.163)

$$[B^*_p(f, j), D_\nu(p)] = [B^*_p(f, j), C_\nu(p)] = 0.$$  

(3.164)

**Proof:** We can directly see that

$$[C_\mu(p), D_\nu(p)] = 2E [\tilde{\Pi}_a(F_{a\mu}), \tilde{A}_b(F_{b\nu})] = 2E i (d_{ab} F_{a\mu}, \tilde{F}_{b\nu})$$

$$= 2E i (F_{b\mu}, F_{b\nu}) = 2i E \frac{1}{\gamma^2} \delta_{\mu\nu}$$

by (16) of Lemma 3.12. Both (3.161) and (3.162) also follow from (16) of Lemma 3.12. Both (3.163) and (3.164) follow from (13) of Lemma 3.12.

**Lemma 3.51** Suppose Assumption 3.4. Then it follows that

$$[H_0, C_\mu(p)] = iEC_\mu(p),$$

(3.165)

$$[H_0, D_\mu(p)] = -iED_\mu(p).$$

(3.166)

**Proof:** We have $[H_f, C_\mu(p)] = -i E \tilde{\Pi}_a(F_{a\mu}) - i \dot{A}_a(\omega^2 F_{a\mu})$. Note that

$$\omega^2 F_{a\mu} = \alpha \dot{\varphi} d_{a\mu} - E^2 F_{a\mu}.$$  

Then

$$[H_f, C_\mu(p)] = -i E \tilde{\Pi}_a(F_{a\mu}) - i \alpha A_\mu + i E^2 \dot{A}_a(F_{a\mu}).$$
On the other hand we can see that
\[ [A_\nu, C_\mu(p)] = -i(\bar{\phi}, F_{\mu\nu}) = i \frac{m}{\alpha} \delta_{\mu\nu} \]
by (14) of Lemma 3.12. Together with them we have
\[ [H_0, C_\mu(p)] = \frac{1}{m} (p_\nu - \alpha A_\nu)(-\alpha) \frac{mi}{\alpha} \delta_{\mu\nu} - iE \tilde{\Pi}_a(F_{a\mu}) - i\alpha A_\mu + iE^2 \hat{A}_a(F_{a\mu}) \]
\[ = -ip_\mu - iE \tilde{\Pi}_a(F_{a\mu}) + iE^2 \hat{A}_a(F_{a\mu}) = iEC_\mu(p). \]
Then (3.165) follows. (3.166) is similarly proven.

**Lemma 3.52** Suppose Assumption 3.4. Then it follows that
\[ e^{itH_0}C_\mu(p)e^{-itH_0} = e^{-te^2}C_\mu(p), \]
\[ e^{itH_0}D_\mu(p)e^{-itH_0} = e^{te^2}D_\mu(p). \]

**Proof:** Set \( C_t = e^{te^2}C_\mu(p) \) and \( \bar{C}_t = e^{itH_0}C_\mu(p)e^{-itH_0}. \) Then
\[ \frac{d}{dt}C_t = \frac{d}{dt} \bar{C}_t = i[H_0, C_\mu(p)] \]
and \( C_0 = \bar{C}_0. \) Then \( C_t = C_\mu(p) \) follows. The equality \( e^{itH_0}D_\mu(p)e^{-itH_0} = e^{te^2}D_\mu(p) \) is similarly proven. \( \square \)

**Theorem 3.53 (Time evolution of A)** Suppose Assumption 3.4. Then for all \( p \in \mathbb{R}^d \) and real-valued \( f \) such that \( f \in M_0 \cap M_{1/2} \cap M_{-1}, \)
\[ e^{itH_0}A_\mu e^{-itH_0} = \frac{1}{\sqrt{2}} \left\{ B_p^*(e^{it\omega}e_p^1 \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{\phi}) + B_p(e^{-it\omega}e_p^1 \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{\phi}) \right\} \]
\[ + \frac{\alpha}{m_{\text{eff}}} p_\nu (d_\mu \frac{\hat{\phi}}{\omega}, \frac{\hat{\theta}}{\omega}) + \frac{\gamma^2}{2E} (F_{\mu\nu}, f)(e^{te^2}C_\mu(p) + e^{te^2}D_\nu(p)). \] (3.167)

**Proof:** We shall show that
\[ A_\mu = \frac{1}{\sqrt{2}} \left\{ B_p^*(e_p^1 \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{\phi}) + B_p(e_p^1 \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{\phi}) \right\} \]
\[ + \frac{\alpha}{m_{\text{eff}}} p_\nu (d_\mu \frac{\hat{\phi}}{\omega}, \frac{\hat{\theta}}{\omega}) + \frac{\gamma^2}{2E} (F_{\mu\nu}, f)(C_\mu(p) + D_\nu(p)) \]
\[ = \frac{1}{\sqrt{2}} \left\{ B_p^*(e_p^1 \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{\phi}) + B_p(e_p^1 \frac{1}{\sqrt{\omega}} T_{\nu\mu} \hat{\phi}) \right\} \]
\[ + \frac{\alpha}{m_{\text{eff}}} p_\nu (d_\mu \frac{\hat{\phi}}{\omega}, \frac{\hat{\theta}}{\omega}) + \left( -\frac{\gamma^2}{E^2} p_\nu + \gamma^2 \hat{A}_a(F_{a\nu}) \right) (F_{\mu\nu}, \hat{\phi}). \] (3.168)
We see that

\[
B_p^* \left( e_i^j \frac{1}{\sqrt{\omega}} T_{\mu \nu} \hat{f} \right) = a(W_{-ij} e_i^j \frac{1}{\sqrt{\omega}} T_{\mu \nu} \hat{f}, i) + a^*(W_{+ij} e_i^j \frac{1}{\sqrt{\omega}} T_{\mu \nu} \hat{f}, i) - (p \cdot e_i^j \frac{\bar{Q}}{\sqrt{2\omega}}, \frac{1}{\omega} T_{\mu \nu} \hat{f}),
\]

By using (3) and (7) of Lemma 3.12 we compute the sum of test function of the creation operators as

\[
W_{+ij} e_i^j \frac{1}{\sqrt{\omega}} T_{\mu \nu} \hat{f} + W_{-ij} e_i^j \frac{1}{\sqrt{\omega}} T_{\mu \nu} \hat{f} = \frac{1}{\sqrt{\omega}} e_i^j \hat{f} - \gamma^2 (F_{\mu \nu}, \hat{f}) \frac{e_i^j F_{\nu a}}{\sqrt{\omega}} \tag{3.169}
\]

and that of the annihilation operators as

\[
W_{-ij} e_i^j \frac{1}{\sqrt{\omega}} T_{\mu \nu} \hat{f} + W_{+ij} e_i^j \frac{1}{\sqrt{\omega}} T_{\mu \nu} \hat{f} = \frac{1}{\sqrt{\omega}} e_i^j \hat{f} - \gamma^2 (F_{\mu \nu}, \hat{f}) \frac{e_i^j F_{\nu a}}{\sqrt{\omega}} \tag{3.170}
\]

By (11) and (12) of Lemma 3.12 we have

\[
- (p \cdot e_i^j \frac{\bar{Q}}{\sqrt{2\omega}}, \frac{1}{\omega} T_{\mu \nu} \hat{f}) - (p \cdot e_i^j \frac{\bar{Q}}{\sqrt{2\omega}}, \frac{1}{\omega} T_{\mu \nu} \hat{f}) = - \sqrt{2} p_a \left\{ \frac{\alpha}{m_{\text{eff}}} (d_{a \nu} \hat{\phi}, \frac{\hat{f}}{\omega}) - \frac{\gamma^2}{E^2} (F_{\mu \nu}, \hat{f}) \right\} \tag{3.171}
\]

From (3.169)-(3.171), (3.168) follows. (3.167) is derived from Lemmas 3.22 and 3.52.

\[\square\]
4 Binding and non-binding

4.1 Enhanced binding

Non-perturbative analysis of perturbation of eigenvalues embedded in the continuous spectrum has been developed in the last decade and has been applied to the mathematical rigorous analysis of Hamiltonians in quantum field theory. Among other things, stability and instability of quantum mechanical particle coupled to quantum fields have been investigated from mathematical point of view.

This section is the review of [HS01, Hir03, HSS11] and we also revise small errors found in [HS01, Hir03, HSS11].

Atoms consist of charged particles and they are necessarily coupled to the quantized radiation field. In the lowest approximation, this interaction can be ignored and one is led to a Schrödinger operator of the form

\[ H_p(m) = -\frac{1}{2m} \Delta + V \]  

for the particles only. Under suitable conditions on \( V \) the Schrödinger operator has a state of the lowest energy, the ground state of the atom. There has been renewed interest within mathematical physics to understand whether this ground state persists when the coupling to the radiation field is included. We will investigate here a related, but distinct problem.

In the non-relativistic approximation, the coupling to the radiation field is described by the Pauli-Fierz Hamiltonian discussed in the previous section:

\[ H = \frac{1}{2m} (\alpha A + A^\dagger)^2 + V + H_f \]  

acting on the Hilbert space \( \mathcal{H} \). In essence, \( V \) is short ranged and sufficiently shallow. The problem of the existence of the ground state for \( H \) is usually regarded as a stability property. One assumes that \( H \) has a ground state for \( \alpha = 0 \), which amounts to the existence of a ground state for \( H_p(m) \) and proves that \( H \) has also a ground state for \( \alpha \neq 0 \). It is then necessarily unique, since \( e^{-tH} \) has a positivity improving kernel in a suitable function space.

In contrast we assume that \( H \) has no ground state for \( \alpha = 0 \). In fact, this will be the case for a sufficiently shallow \( V \) in the space dimension three. We expect...
the interaction with the quantized radiation field to enhance binding. The non-binding potential should become binding at a sufficiently strong coupling strength. The enhanced binding is studied in e.g., [AK03, BLV05, BV04, CEH04, CVV03, HVV03, HS08, HS12, HS01, HSS11].

The physical reasoning behind such a result is simple. As the particle binds photons it acquires an effective mass $m_{\text{eff}} = m + \alpha^2 \left| \frac{\phi}{\omega} \right|^2$ which is increasing in $|\alpha|$ (Figure 6). Roughly speaking, $H$ may be replaced by

$$H_{\text{eff}} = -\frac{1}{2m_{\text{eff}}} \Delta + V,$$  \hspace{1cm} (4.3)

which binds for sufficiently strong $\alpha$. Indeed we can see that $H_{\text{eff}}$ can be derived through the weak coupling limit of $H$ in Section 3.9.1.

Next let us consider a transition from unbinding to binding as the mass $m$ is increased (Figure 7). More precisely, there is some critical mass, $m_c$, such that $H_p(m)$ has no ground state for $0 < m < m_c$ and a unique ground state for $m_c < m$. In fact, the critical mass is given by

$$\frac{1}{2m_c} = \left\| |V|^{1/2} (-\Delta)^{-1} |V|^{1/2} \right\|.$$  \hspace{1cm} (4.4)

On a heuristic level, through the dressing by photons the particle becomes effectively more heavy, which means that there is critical mass $m_c(\alpha)$ for the existence of a ground state. $m_c(\alpha)$ is expected to be decreasing as a function of $\alpha$ with $m_c(0) = m_c$. In particular, for fixed $m < m_c$, there should be an unbinding-binding transition as the coupling $\alpha$ is increased. In case $m > m_c$ more general techniques are available and the existence of a unique ground state for the full Hamiltonian $H$ is proven in
4.2 Absence of ground state

The unbinding for the Schrödinger operator $H_p(m) = -\frac{1}{2m_{\text{eff}}} \Delta + V$ is proven by the Birman-Schwinger principle. Formally one has

$$H_p(m) = \frac{1}{2m}(\Delta)^{1/2}(\mathbb{1} + 2m(\Delta)^{-1/2}V(\Delta)^{-1/2})(\Delta)^{1/2}.$$  

If $m$ is sufficiently small, then $2m(\Delta)^{-1/2}V(\Delta)^{-1/2}$ is a strict contraction. Hence the operator $\mathbb{1} + 2m(\Delta)^{-1/2}V(\Delta)^{-1/2}$ has a bounded inverse and $H_p(m)$ has no eigenvalue in $(-\infty, 0]$. More precisely the Birman-Schwinger principle states that

$$\dim \mathbb{1}_{(\frac{1}{2m}, \infty)}(V^{1/2}(\Delta)^{-1}V^{1/2}) \geq \dim \mathbb{1}_{(-\infty, 0]}(H_p(m)). \tag{4.5}$$

For small $m$ the left hand side equals 0 and thus $H_p(m)$ has no eigenvalues in $(-\infty, 0]$. Our approach will be to generalize (4.5) to the Pauli-Fierz model with the dipole approximation. We already see that $\hat{H}$ can be transformed by $\mathcal{U}$ and one arrives at $\mathcal{U}^{-1} \hat{H} \mathcal{U} = H_0(\alpha) + W + g$ as the sum of the free Hamiltonian

$$H_0(\alpha) = -\frac{1}{2m_{\text{eff}}} \Delta + H_f, \tag{4.6}$$
involving the effective mass of the dressed particle, the transformed interaction \( W = T^{-1}VT \), and the global energy shift \( g \). Effective mass \( m_{\text{eff}} \) is an increasing function of \( \alpha \).

Let \( h_0 = -\frac{1}{2} \Delta \). We assume that \( V \in L_{\text{loc}}^1(\mathbb{R}^d) \) and \( V \) is relatively form-bounded with respect to \( h_0 \) with relative bound \( a < 1 \), i.e., \( D(|V|^{1/2}) \supset D(h_0^{1/2}) \) and

\[
||V|^{1/2} \varphi||^2 \leq a||h_0^{1/2} \varphi||^2 + b||\varphi||^2, \quad \varphi \in D(h_0^{1/2}),
\]

(4.7)

with some \( b > 0 \). Under (4.7) the operators \( R_0 = \left( h_0 - E \right)^{-1/2} |V|^{1/2} \) for \( E < 0 \) are densely defined. From (4.7) it follows that \( R_E^* = |V|^{1/2} \left( h_0 - E \right)^{-1/2} \) is bounded and thus \( R_E \) is closable. We denote its closure by the same symbol. Let

\[
K_E = R_E^* R_E.
\]

(4.8)

Then \( K_E \ (E < 0) \) is a bounded, positive self-adjoint operator and it holds

\[
K_E f = |V|^{1/2} \left( h_0 - E \right)^{-1} |V|^{1/2} f, \quad f \in C_0^\infty(\mathbb{R}^d).
\]

Now let us consider the case \( E = 0 \). Let \( R_0 = h_0^{-1/2} |V|^{1/2} \). The self-adjoint operator \( h_0^{-1/2} \) has the integral kernel \( h_0^{-1/2}(x,y) = \frac{a_d}{|x-y|^{d-1}} \) for \( d \geq 3 \), where \( a_d = \sqrt{2\pi}^{(d-1)/2}/\Gamma(d/2) \). It holds that

\[
\left| (h_0^{-1/2} g, |V|^{1/2} f) \right| \leq a_d\|g\|_2\|V|^{1/2} f\|_{2d/(d+2)}
\]

for \( f, g \in C_0^\infty(\mathbb{R}^3) \) by the Hardy-Littlewood-Sobolev inequality. Since \( f \in C_0^\infty(\mathbb{R}^3) \) and \( V \in L_{\text{loc}}^1(\mathbb{R}^3) \), one concludes \( \|V|^{1/2} f\|_{2d/(d+2)} < \infty \). Thus \( |V|^{1/2} f \in D(h_0^{-1/2}) \) and \( R_0 \) is densely defined. Since \( V \) is relatively form-bounded with respect to \( h_0 \), \( R_0^* \) is also densely defined, and \( R_0 \) is closable. We denote the closure by the same symbol. We define

\[
K_0 = R_0^* R_0.
\]

(4.9)

Next let us introduce assumptions on the external potential \( V \).

**Assumption 4.1** \( V \) satisfies that (1) \( V \leq 0 \) and (2) \( R_0 \) is compact.

**Lemma 4.2** Suppose Assumption 4.1. Then

(1) \( R_E, R_E^* \) and \( K_E \ (E \leq 0) \) are compact.

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(2) \(\|K_E\|\) is continuous and monotonously increasing in \(E \leq 0\) and it holds that \(\lim_{E \to -\infty} \|K_E\| = 0\) and \(\lim_{E \to 0} \|K_E\| = \|K_0\|\).

**Proof:** Under (2) of Assumption 4.1, \(\|K_E\|\) is continuous and monotonously increasing in \(E \leq 0\) and it holds that \(\lim_{E \to -\infty} \|K_E\| = 0\) and \(\lim_{E \to 0} \|K_E\| = \|K_0\|\). Since \(\|K_E\|\) is continuous in \(E < 0\), we have to prove the left continuity at \(E = 0\). Since \(\|K_E\|\) is monotonously increasing in \(E\). Since \(R_0^*\) is bounded, \(\|K_E\|\) holds on \(L^2(\mathbb{R}^d)\) and

\[
K_E = R_0^* (h_0 - E)^{-1} h_0 \quad R_0
\]

for \(E \leq 0\). From this one concludes that \(\|K_E - K_{E'}\| \leq \|K_0\| \frac{|E - E'|}{|E'|} \) for \(E, E' < 0\). Hence \(\|K_E\|\) is continuous in \(E < 0\). We have to prove the left continuity at \(E = 0\). Since \(\|K_E\|\) is monotonously increasing in \(E\), one has \(\lim_{E \to 0} \|K_E\| \leq \|K_0\|\). By (4.11) we see that \(K_0 = s\)-limit of \(K_E\) and

\[
\|K_0 f\| = \lim_{E \to 0} \|K_E f\| \leq \left( \lim_{E \to 0} \inf \|K_E\| \right) \|f\|, \quad f \in L^2(\mathbb{R}^d).
\]

Hence we have \(\|K_0\| \leq \lim \inf_{E \to 0} \|K_E\|\) and \(\lim_{E \to 0} \|K_E\| = \|K_0\|\). It remains to prove that \(\lim_{E \to -\infty} \|K_E\| = 0\). Since \(R_0^*\) is compact, for any \(\epsilon > 0\), there exists a finite rank operator \(T_\epsilon = \sum_{k=1}^n (\varphi_k, \cdot) \psi_k\) such that \(n = n(\epsilon) < \infty\), \(\varphi_k, \psi_k \in L^2(\mathbb{R}^d)\) and \(\|R_0^* - T_\epsilon\| < \epsilon\). Then it holds that \(\|K_E\| \leq (\epsilon + \|T_\epsilon h_0 (h_0 - E)^{-1}\|) \|R_0\|\). For any \(f \in L^2(\mathbb{R}^d)\), we have

\[
\|T_\epsilon h_0 (h_0 - E)^{-1} f\| \leq \left( \sum_{k=1}^n \|h_0 (h_0 - E)^{-1} \varphi_k\| \|\psi_k\| \right) \|f\|
\]

and \(\lim_{E \to -\infty} \|T_\epsilon h_0 (h_0 - E)^{-1}\| = 0\), which completes (2).

Let \(H_P(m)\) be in (4.1). By (2) of Lemma 4.2, we have

\[
\lim_{E \to -\infty} \|V|^{1/2}(h_0 - E)^{-1/2}\| = 0.
\]

Therefore \(V\) is infinitesimally form bounded with respect to \(h_0\) and \(H_P(m)\) is the self-adjoint operator associated with the quadratic form

\[
f, g \mapsto \frac{1}{m} (h_0^{1/2} f, h_0^{1/2} g) + (|V|^{1/2} f, |V|^{1/2} g)
\]

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for \( f, g \in D(h_0^{1/2}) \). Note that the domain \( D(H_p(m)) \) is independent of \( m \).

Under (2) of Assumption 4.1, the essential spectrum of \( H_p(m) \) coincides with that of \( -\frac{1}{2m} \Delta \), hence \( \sigma_{ess}(H_p(m)) = [0, \infty) \). Next we will estimate the spectrum of \( H_p(m) \) contained in \((-\infty, 0]\). Let \( \mathbb{1}_O(T), O \subset \mathbb{R}, \) be the spectral resolution of self-adjoint operator \( T \) and set \( N_O(T) = \dim \text{Ran} \mathbb{1}_O(T) \). The Birman-Schwinger principle states that

\[
\begin{align*}
(E < 0) \quad & N_{(-\infty, \frac{E}{m}}(H_p(m)) = N_{\frac{1}{m}, \infty}(K_E), \\
(E = 0) \quad & N_{(-\infty, 0]}(H_p(m)) \leq N_{\frac{1}{m}, \infty}(K_0).
\end{align*}
\]

Now let us define the constant \( m_c \) by the inverse of the operator norm of \( K_0 \),

\[
m_c = \| K_0 \|^{-1}.
\]

**Lemma 4.3** Suppose Assumption 4.1.

(1) If \( m < m_c \), then \( N_{(-\infty, 0]}(H_p(m)) = 0 \).

(2) If \( m > m_c \), then \( N_{(-\infty, 0]}(H_p(m)) \geq 1 \).

**Proof:** It is immediate to see (1) by the Birman-Schwinger principle (4.12). Suppose \( m > m_c \). Then, using the continuity and monotonicity of \( E \to \| K_E \| \), see Lemma 4.2, there exists \( \epsilon > 0 \) such that \( m_c < \| K_{-\epsilon} \|^{-1} \leq m \). Since \( K_{-\epsilon} \) is positive and compact, \( \| K_{-\epsilon} \| \in \sigma_p(K_{-\epsilon}) \) follows and hence \( N_{\frac{1}{m}, \infty}(K_{-\epsilon}) \geq 1 \). Therefore (2) follows again from the Birman-Schwinger principle.

By Lemma 4.3, the critical mass at zero coupling is \( m_c(0) = m_c \). In the case \( m > m_c \), by the proof of Lemma 4.3 one concludes that the bottom of the spectrum of \( H_p(m) \) is strictly negative. For \( \epsilon > 0 \) we set \( m_\epsilon = \| K_{-\epsilon} \|^{-1} \).

**Corollary 4.4** Suppose Assumption 4.1 and \( m > m_\epsilon \). Then

\[
\inf \sigma(H_p(m)) \leq \frac{-\epsilon}{m}.
\]

**Proof:** The Birman-Schwinger principle states that \( 1 \leq N_{(-\infty, -\frac{\epsilon}{m}}(H_p(m)), \) since \( 1/m < \| K_{-\epsilon} \| \), which implies the corollary.

We extend the Birman-Schwinger type estimate to the Pauli-Fierz Hamiltonian.
Lemma 4.5 Suppose Assumption 4.1. If \( m < m_c \), then the zero coupling Hamiltonian \( H_p(m) + H_f \) has no ground state.

Proof: Since the Fock vacuum \( \Omega \) is the ground state of \( H_f \), \( H_p(m) + H_f \) has a ground state if and only if \( H_p(m) \) has a ground state. But \( H_p(m) \) has no ground state by Lemma 4.3. Therefore \( H_p(m) + H_f \) has no ground state.

From now on we discuss \( \mathcal{U}^{-1} H \mathcal{U} \) with \( \alpha \neq 0 \). We have

\[
\mathcal{U}^{-1} H \mathcal{U} = H_0(\alpha) + W + g,
\]

where \( H_0(\alpha) = -\frac{1}{2m_{\text{eff}}} \Delta + H_f \) and \( W = T^{-1} V T \) and \( T \) is given in (4.31).

Theorem 4.6 (Absence of ground state) [HSS11] Suppose Assumptions 3.4 and 4.1. If \( m_{\text{eff}} < m_c \), then \( H \) has no ground state.

Proof: Since \( g \) is a constant, we prove the absence of ground state of \( H_0(\alpha) + W \). Since \( V \) is negative, so is \( W \). Hence \( \inf \sigma(H_0(\alpha) + W) \leq \inf \sigma(H_0(\alpha)) = 0 \). Then it suffices to show that \( H_0(\alpha) + W \) has no eigenvalues in \( (-\infty, 0] \). Let \( E \in (-\infty, 0] \) and set

\[
K_E = |W|^{1/2}(H_0(\alpha) - E)^{-1}|W|^{1/2},
\]

where \( |W|^{1/2} \) is defined by the functional calculus. We shall prove now that if \( H_0(\alpha) + W \) has eigenvalue \( E \in (-\infty, 0] \), then \( K_E \) has eigenvalue 1. Suppose that \( (H_0(\alpha) + W - E)\varphi = 0 \) and \( \varphi \neq 0 \), then \( K_E|W|^{1/2}\varphi = |W|^{1/2}\varphi \) holds. Moreover if \( |W|^{1/2}\varphi = 0 \), then \( W\varphi = 0 \) and hence \( (H_0(\alpha) - E)\varphi = 0 \), but \( H_0(\alpha) \) has no eigenvalue by Lemma 4.5. Then \( |W|^{1/2}\varphi \neq 0 \) is concluded and \( K_E \) has eigenvalue 1. Then it is sufficient to see \( \|K_E\| < 1 \) to show that \( H_0(\alpha) + W \) has no eigenvalues in \( (-\infty, 0] \). Notice that \( -\frac{1}{2m_{\text{eff}}} \Delta \) and \( T \) commute, and

\[
\left\|(-\Delta)^{1/2}(H_0(\alpha) - E)^{-1}(-\Delta)^{1/2}\right\| \leq 2m_{\text{eff}}.
\]

Then we have

\[
\|K_E\| \leq \left\|V\right\|^2 \left(-\frac{1}{2m_{\text{eff}}} \Delta\right)^{-1/2} = m_{\text{eff}}\|K_0\| = \frac{m_{\text{eff}}}{m_c} < 1
\]

and the proof is complete. \( \square \)
Now we give examples of potentials $V$ satisfying Assumption 4.1. The self-adjoint operator $h_0^{-1}$ has the integral kernel

$$h_0^{-1}(x, y) = \frac{b_d}{|x - y|^{d-2}}, \quad d \geq 3,$$

with $b_d = 2\Gamma(\frac{d}{2} - 1)/\pi^{\frac{d}{2} - 2}$. It holds that

$$(f, K_0f) = \int dx \int dy f(x)K_0(x, y)f(y), \quad (4.17)$$

where

$$K_0(x, y) = b_d \frac{|V(x)|^{1/2}|V(y)|^{1/2}}{|x - y|^{d-2}}, \quad d \geq 3, \quad (4.18)$$

is the integral kernel of operator $K_0$. We recall that the Rollnik class $\mathcal{R}$ of potentials is defined by

$$\mathcal{R} = \left\{ V \left| \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \frac{|V(x)V(y)|}{|x - y|^2} < \infty \right. \right\}. \quad (4.19)$$

Let $d = 3$. By the Hardy-Littlewood-Sobolev inequality, $\mathcal{R} \supset L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ with $1/p + 1/r = 4/3$. In particular, $L^{3/2}(\mathbb{R}^3) \subset \mathcal{R}$.

**Example 4.7 (d = 3 and Rollnik class)** Let $d = 3$. Suppose that $V$ is negative and $V \in \mathcal{R}$. Then $K_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Hence $K_0$ is Hilbert-Schmidt and Assumption 4.1 is satisfied.

The example can be extended to dimensions $d \geq 3$.

**Example 4.8 (d \geq 3 and $V \in L^{d/2}(\mathbb{R}^d)$)** Let $L^p_w(\mathbb{R}^d)$ be the set of Lebesgue measurable function $u$ such that $\sup_{\beta > 0} \beta \left| \left\{ x \in \mathbb{R}^d \left| u(x) > \beta \right. \right\} \right|^{1/p}_L < \infty$, where $|E|_L$ denotes the Lebesgue measure of $E \subset \mathbb{R}^d$. Let $g \in L^p(\mathbb{R}^d)$ and $u \in L^p_w(\mathbb{R}^d)$ for $2 < p < \infty$. Define the operator $B_{u,g}$ by

$$B_{u,g}h = (2\pi)^{-d/2} \int e^{ikx}u(k)g(x)h(x)dx.$$

It is shown in [Cwi77, Theorem, p.97] that $B_{u,g}$ is a compact operator on $L^2(\mathbb{R}^d)$. It is known that $u(k) = 2|k|^{-1} \in L^p_w(\mathbb{R}^d)$ for $d \geq 3$. Let $F$ denote Fourier transform on $L^2(\mathbb{R}^d)$, and suppose that $V \in L^{d/2}(\mathbb{R}^d)$. Then $B_{u,V^{1/2}}$ is compact on $L^2(\mathbb{R}^d)$ and then $R_0^* = FB_{u,V^{1/2}}F^{-1}$ is compact. Thus $R_0$ is also compact.
Assume that $V \in L^{d/2}(\mathbb{R}^d)$. Let us now see the critical mass of zero coupling $m_c = m_0$. By the Hardy-Littlewood-Sobolev inequality, we have

$$|(f, K_0 f)| \leq D_V \|f\|_2^2,$$

where $D_V = \sqrt{2\pi} \frac{\Gamma(d/2 - 1)}{\Gamma(d/2 + 1)} \left( \frac{\Gamma(d)}{\Gamma(d/2 + 1)} \right)^{2/d} \|V\|_{d/2}^2$. (4.19) is proved by Lieb [Lie83]. Then $\|K_0\| \leq D_V$. By this bound we have $m_c \geq D_V^{-1}$. In particular in the case of $d = 3$,

$$m_c \geq \frac{3}{\sqrt{2\pi^{3/2}} \Gamma(3/2)} \|V\|_{3/2}^2.$$

(4.20)

4.3 Existence of ground state

In this section we investigate the existence of ground state of $H$ for sufficiently large $|\alpha|$. Let us define the Pauli-Fierz Hamiltonian with scaled external potential $V_\kappa(x) = V(x/\kappa)/\kappa^2$ by

$$\frac{1}{2m} (-i\nabla - \alpha A)^2 + V_\kappa + H_f.$$  (4.21)

We also define $H(\kappa)$ by $H$ with $a^z$ replaced by $\kappa a^z$. Then

$$H(\kappa) = \frac{1}{2m} (-i\nabla - \kappa A)^2 + V + \kappa^2 H_f.$$  (4.22)

We can see the unitary equivalence:

$$\kappa^{-2} H(\kappa) \approx \frac{1}{2m} (-i\nabla - \alpha A)^2 + V_\kappa + H_f.$$

Then $H(\kappa)$ has a ground state if and only if (4.21) has a ground state. We furthermore introduce assumptions on the external potential $V$ and ultraviolet cutoff $\hat{\varphi}$. Recall that $Q(k) = \alpha \hat{\varphi}(k)/m_{\text{eff}}(k)$.

Assumption 4.9 The external potential $V$ and the ultraviolet cutoff $\hat{\varphi}$ satisfy:

1. $V \in C^1(\mathbb{R}^d)$ and $\nabla V \in L^\infty(\mathbb{R}^d)$,

2. $\hat{\varphi}/\omega^{5/2} \in L^2(\mathbb{R}^d)$,
(3) $\sup_\alpha \|Q/\omega^{n/2}\| < \infty$, $n = 3, 4, 5$\footnote{HS01 Theorem 4.14 is incorrect. The effective mass $m_{\text{eff}}$ in [HS01] Theorem 4.14 should be changed to $m_{\text{eff}}(k)$, and we need assumption (3) to show the enhanced binding.}

**Example 4.10** We give an example of ultraviolet cutoff satisfying both of Assumption 3.4 and Assumption 4.9 (2) and (3). Suppose that $d = 3$ and $\hat{\varphi}(k) = 1_{[\lambda, \Lambda]}(|k|)$ is the sharp cutoff function. See Example 3.10. Then

$$|m_{\text{eff}}(k)| \geq \alpha^2 \frac{4\pi^2}{3} 1_{[\lambda, \Lambda]}(\omega(k)) \sqrt{\omega(k)}.$$  

We have

$$\|Q/\omega^{n/2}\|^2 \leq \frac{1}{\alpha^2} \frac{3}{4\pi^2} \int_{\lambda \leq |k| \leq \Lambda} \frac{1}{\omega(k)^{n+1}} dk.$$  

In particular it follows that $\lim_{\alpha \to \infty} \|Q/\omega^{n/2}\| = 0$.

We drop $g$ for instance. We reset

$$H(\kappa) = H_{\text{eff}} + \kappa^2 H_\ell + \delta V_\kappa. \quad (4.23)$$  

In Theorem 3.47 we show that $H(\kappa)$ converges to $H_{\text{eff}}$ as $\kappa \to \infty$ in some sense. It suggests that $H(\kappa)$ with sufficiently large $\kappa$ has a ground state if $H_{\text{eff}}$ does. Let $m < m_c$ and $\epsilon > 0$. We define

$$\alpha_\epsilon = \left( \left( \frac{d-1}{d} \right) \|\hat{\varphi}/\omega\|^2 \right)^{-1/2} \sqrt{m_\epsilon - m}, \quad \epsilon > 0, \quad (4.24)$$

$$\alpha_0 = \left( \left( \frac{d-1}{d} \right) \|\hat{\varphi}/\omega\|^2 \right)^{-1/2} \sqrt{m_c - m}, \quad (4.25)$$

where we recall that $m_\epsilon = \|K_\epsilon\|^{-1}$ for $\epsilon \geq 0$. Note that

1. $|\alpha| < \alpha_0$ if and only if $m_{\text{eff}} < m_c$;
2. $|\alpha| > \alpha_\epsilon$ if and only if $m_{\text{eff}} > m_\epsilon$.

Note that $\alpha_0 < \alpha_\epsilon$ because of $m_\epsilon > m_c$. Since $\lim_{\epsilon \downarrow 0} m_\epsilon = m_c$, it holds that $\lim_{\epsilon \downarrow 0} \alpha_\epsilon = \alpha_0$. We note that for $|\alpha| > \alpha_\epsilon$, $H_{\text{eff}}$ has a ground state with negative ground state energy.
4.3.1 Massive case

We introduce an artificial mass of photon, $\varepsilon > 0$, and define

$$H^\varepsilon(\kappa) = H_{\text{eff}} + \delta V_\kappa + \kappa^2 H^\varepsilon_f,$$

where

$$H^\varepsilon_f = H_{f} + \varepsilon N = \int (\omega(k) + \varepsilon) a^*(k, j) a(k, j) dk.$$

Using a momentum lattice approximation we will prove that $H^\varepsilon(\kappa)$ has a ground state. Let $\Gamma(l, a)$, $l = (l_1, \cdots, l_d) \in \mathbb{Z}^d$, $a > 0$, be the momentum lattice with spacing $1/a$, i.e., $\Gamma(l, a) = [l_1/a, (l_1+1)/a] \times \cdots \times [l_d/a, (l_d+1)/a]$. Then

$$\chi_{\Gamma(l,a)}(k) = \begin{cases} 0, & k \not\in \Gamma(l, a), \\ \frac{a_d}{2}, & k \in \Gamma(l, a). \end{cases}$$

For $L > 0$ we define the momentum-lattice-approximated Hamiltonian by

$$H^\varepsilon_{a,L}(\kappa) = H_{\text{eff}} + \kappa^2 H^\varepsilon'_{f} + \delta V'_\kappa,$$  (4.26)

where $H^\varepsilon_f$ and $\delta V'_\kappa$ are momentum-lattice-approximated operators given by

$$H^\varepsilon_f = H_{f, a,L} = \int \left( \sum_{|l| \leq L} \chi_{\Gamma(l,a)}(k) (\omega(l) + \varepsilon) \right) a^*(k, j) a(k, j) dk,$$

$$\delta V'_\kappa = \delta V_{a,L} = V(\cdot + K_{a,L}/\kappa) - V$$

and $K_{a,L} = (K_{a,L,1}, \cdots, K_{a,L,d})$ is the column of the field operator defined by

$$K_{a,L,\mu} = \frac{1}{\sqrt{2}} \int \sum_{|l| \leq L} \chi_{\Gamma(l,a)}(k) \left(g_\mu(l, j) a^*(k, j) + g_\mu(l, j) a(k, j)\right) dk.$$

Here we set $g_\mu(l, j) = e_j^\mu(k) Q(k)/\omega(k)^{3/2}$. We can show that

$$\|K_{a,L,\mu}\Psi\| \leq C \left( \left\| \sum_{|l| \leq L} \frac{\chi_{\Gamma(l,a)} Q(l)}{\sqrt{\omega(l) (l)^{3/2}}} \right\| + \left\| \sum_{|l| \leq L} \frac{\chi_{\Gamma(l,a)} Q(l)}{\omega(l) (l)^{3/2}} \right\| \right) \left( \|H^\varepsilon'_{f}\|^{1/2} \Psi\| + \|\Psi\| \right)$$

(4.27)

with some constant $C$. Here we used the bound

$$c_1 \|H^\varepsilon'_{f}\|^{1/2} \Psi\| \leq \|H^\varepsilon'_{f}\|^{1/2} \Psi\| \leq c_2 \|H^\varepsilon'_{f}\|^{1/2} \Psi\|$$

with some constants $c_1$ and $c_2$.  

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Lemma 4.11 It follows that \( \lim_{L \to \infty} \lim_{a \to \infty} H^\varepsilon_{a,L}(\kappa) = H^\varepsilon(\kappa) \) in the uniform resolvent sense.

Proof: It can be seen that there exists a constant \( c_{a,L} \) such that
\[
\| (H^\varepsilon_f - H^\varepsilon) \Psi \| \leq c_{a,L} \| H^\varepsilon_f \Psi \|
\]
and \( \lim_{L \to \infty} \lim_{a \to \infty} c_{a,L} = 0 \). Moreover
\[
\| (\delta V'_\kappa - \delta V_\kappa) \Psi \| = \| (V(\cdot + K_{a,L}/\kappa) - V(\cdot + K/\kappa)) \Psi \|
\leq \frac{1}{\kappa} \| \nabla \mu V \|_\infty \| (K_\mu - K_{a,L,\mu}) \Psi \|.
\]
Since \( \| (K_\mu - K_{a,L,\mu}) \Psi \| \leq c'_{a,L}(\| (H^\varepsilon_f)^{1/2} \Psi \| + \| \Psi \|) \), where
\[
c'_{a,L} = C \left( \left\| \frac{1}{\sqrt{\omega}} \left\{ \frac{Q}{\omega^{3/2}} - \sum_{|l| \leq L} \chi_{\Gamma(l,a)}(l) \frac{Q(l)}{\omega(l)^{3/2}} \right\} \right\| + \left\| \frac{Q}{\omega^{3/2}} - \sum_{|l| \leq L} \chi_{\Gamma(l,a)}(l) \frac{Q(l)}{\omega(l)^{3/2}} \right\| \right)
\]
with some constant \( C \), and \( c'_{a,L} \) satisfies that \( \lim_{L \to \infty} \lim_{a \to \infty} c'_{a,L} = 0 \), we have
\[
\| (H^\varepsilon(\kappa) - z)^{-1} \Psi - (H^\varepsilon_{a,L}(\kappa) - z)^{-1} \Psi \|
\leq \| (H^\varepsilon_{a,L}(\kappa) - z)^{-1} \| \| (H^\varepsilon_{a,L}(\kappa) - H^\varepsilon(\kappa))(H^\varepsilon(\kappa) - z)^{-1} \Psi \|
\leq \frac{\max_\mu \| \nabla \mu V \|_\infty}{|\text{Im} z|} c_{a,L}' \left( \| (H^\varepsilon_f)^{1/2} (H^\varepsilon(\kappa) - z)^{-1} \Psi \| + \| (H^\varepsilon(\kappa) - z)^{-1} \Psi \| \right)
\leq \frac{\max_\mu \| \nabla \mu V \|_\infty}{|\text{Im} z|} c_{a,L}' \| (H^\varepsilon_f)^{1/2} (H^\varepsilon(\kappa) - z)^{-1} \Psi \|.
\]
Since \( \| H^\varepsilon_f (H^\varepsilon(\kappa) - z)^{-1} \Psi \| \leq C' \| \Psi \| \) with some constant \( C' \), we have
\[
\| (H^\varepsilon(\kappa) - z)^{-1} \Psi - (H^\varepsilon_{a,L}(\kappa) - z)^{-1} \Psi \| \leq c''_{a,L} \| \Psi \|
\]
with \( c''_{a,L} \) such that \( \lim_{L \to \infty} \lim_{a \to \infty} c''_{a,L} = 0 \). Hence the lemma follows. \( \square \)

Let \( f \in L^2(\mathbb{R}^d) \). We identify
\[
\ell^2(\mathbb{Z}^d) \ni \{ f(l) \}_{l \in \mathbb{Z}^d} \cong a^{d/2} \sum_{l \in \mathbb{Z}^d} f(l) \chi_{\Gamma(l,a)}(\cdot) \in L^2(\mathbb{R}^d).
\]

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By this identification we regard $\ell^2 = \ell^2(\mathbb{Z}^d)$ as the subspace of $L^2(\mathbb{R}^d)$. Let
\[ H_a = L^2(\mathbb{R}^d) \otimes \mathcal{F}(\ell^2 \otimes \mathbb{C}^{d-1}), \quad (4.28) \]
\[ \mathcal{K} = \bigoplus_{n=1}^{\infty} \mathcal{F}(n)(\ell^{2_\perp} \otimes \mathbb{C}^{d-1}). \quad (4.29) \]

Then the following fundamental identification follows:
\[ \mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}(L^2(\mathbb{R}^d \times \{1, \ldots, d-1\})) \]
\[ \cong L^2(\mathbb{R}^d) \otimes \mathcal{F}(L^2(\mathbb{R}^d) \otimes \mathbb{C}^{d-1}) \]
\[ \cong L^2(\mathbb{R}^d) \otimes \mathcal{F}([\ell^2 \oplus \ell^{2_\perp}] \otimes \mathbb{C}^{d-1}) \]
\[ \cong L^2(\mathbb{R}^d) \otimes \mathcal{F}(\ell^{2_\perp} \otimes \mathbb{C}^{d-1}) \]
\[ = \mathcal{H}_a \otimes (\mathcal{H} \oplus \mathbb{C}) \]
\[ \cong (\mathcal{H}_a \otimes \mathcal{H}) \oplus \mathcal{H}_a. \]

We have
\[ \mathcal{H} \cong (\mathcal{H}_a \otimes \mathcal{H}) \oplus \mathcal{H}_a. \]

In particular we can see that
\[ \mathcal{H}_a^\perp \cong \mathcal{H}_a \otimes \mathcal{H} \quad (4.30) \]

and that $H^\varepsilon_{a,L}(\kappa)$ is reduced by $\mathcal{H}_a$. We set
\[ K = H^\varepsilon_{a,L}(\kappa)\bigg|_{\mathcal{H}_a}, \]
\[ K^\perp = H^\varepsilon_{a,L}(\kappa)\bigg|_{\mathcal{H}_a^\perp}. \]

Then
\[ H^\varepsilon_{a,L}(\kappa) = K^\perp \oplus K. \]

We can immediately see the lemma below:

**Lemma 4.12** Under the identification (4.30), we have
\[ K^\perp \cong K \otimes 1 + 1 \otimes \kappa^2 H^\varepsilon_1 \bigg|_{\mathcal{H}}. \]

In particular $\inf \sigma(K^\perp) \geq \inf \sigma(K) + \varepsilon$. 

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In what follows we estimate the spectrum of $K$.

**Lemma 4.13** Let $\Psi \in D(-\Delta) \cap D(H^1)$. Then

1. $\Psi \in D(\delta V_\kappa)$ and
   \[ \|\delta V_\kappa \Psi\| \leq \theta_\kappa (\|H^{1/2}_\kappa\| + \|\Psi\|), \tag{4.31} \]
   where $\theta_\kappa = \frac{1}{\kappa} C\|\nabla V\|_\infty (\|Q/\omega^2\| + \|Q/\omega^{3/2}\|)$ with some constant $C$,
2. $\Psi \in D(\delta V'_\kappa)$ and
   \[ \|\delta V'_\kappa \Psi\| \leq \theta'_\kappa (\|H^{1/2}_\kappa\| + \|\Psi\|), \tag{4.32} \]
   where
   \[ \theta'_\kappa = \theta_{\kappa,\alpha,L} = \frac{1}{\kappa} C'\|\nabla V\|_\infty \left( \left\| \frac{1}{\sqrt{\omega}} \sum_{|l| \leq L} \frac{\chi_{\Gamma(l)}}{\omega(l)^{3/2}} \right\| + \left\| \sum_{|l| \leq L} \frac{\chi_{\Gamma(l)}}{\omega(l)^{3/2}} \right\| \right) \]
   with some constant $C'$.

**Proof:** We have $\|\delta V_\kappa \Psi\| \leq \frac{1}{\kappa} \|\nabla V\|_\infty \|K_{\kappa,\alpha,L}\|$. Then (4.31) follows. (4.32) is similarly proven. \[ \square \]

**Lemma 4.14** It follows that

\[ \inf \sigma(H^\kappa) \leq \inf \sigma(H_{\text{eff}}) + \frac{3\theta_\kappa}{2}, \]
\[ \inf \sigma(H^\kappa_{\alpha,L}(\kappa)) \leq \inf \sigma(H_{\text{eff}}) + \frac{3\theta'_\kappa}{2}. \]

**Proof:** We write $A \leq B$, if $D(B) \subset D(A)$ and $(\psi, A\psi) \leq (\psi, B\psi)$ for $\psi \in D(B)$. We have

\[ |(\Psi, \delta V_\kappa \Psi)| \leq \theta_\kappa \left\{ \|\Psi\| (\|H^{1/2}_\kappa\| + \|\Psi\|) \right\} \leq (\Psi, \theta_\kappa (\frac{3}{2} + \frac{1}{2}H^\kappa)) \Psi. \]

Thus we can get the bound

\[ -\theta_\kappa \left( \frac{1}{2}H^\kappa + \frac{3}{2} \right) \leq \delta V_\kappa \leq \theta_\kappa \left( \frac{1}{2}H^\kappa + \frac{3}{2} \right). \tag{4.33} \]
Hence for $f \in C_0^\infty(\mathbb{R}^d)$,
\[
\inf \sigma(H^\varepsilon(\kappa)) \leq (f \otimes \Omega, H^\varepsilon(\kappa)f \otimes \Omega) \leq (f, (H_{\text{eff}} + \frac{3}{2}\theta^\kappa)f).
\]

In particular, since $C_0^\infty(\mathbb{R}^d)$ is a core of $H_{\text{eff}}$, we have
\[
\inf \sigma(H^\varepsilon(\kappa)) \leq \inf \sigma(H_{\text{eff}}) + \frac{3\theta^\kappa}{2}.
\]

Similarly we have
\[
-\theta^\kappa' \left(\frac{1}{2}H^\varepsilon' + \frac{3}{2}\right) \leq \delta V^\varepsilon' \leq \theta^\kappa' \left(\frac{1}{2}H^\varepsilon' + \frac{3}{2}\right) \quad (4.34)
\]
and hence
\[
\inf \sigma(H^\varepsilon_{a,L}(\kappa)) \leq \inf \sigma(H_{\text{eff}}) + \frac{3\theta^\kappa'}{2}.
\]

Then the lemma follows. \hfill \Box

We set $\Sigma = \inf \sigma(H_{\text{eff}})$ and $\overline{H_{\text{eff}} - \Sigma}$. Suppose $|\alpha| > \alpha_\epsilon$. Since $m_{\text{eff}} > m_\epsilon > m_\epsilon/2$,
\[
\Sigma \leq \inf \sigma(H_p(m_\epsilon)) \leq -\frac{\epsilon}{2m_\epsilon} \quad (4.35)
\]
by Corollary 4.4. In particular
\[
|\Sigma| > 0. \quad (4.36)
\]

For a self-adjoint operator $M$, the spectral projection of $M$ on a Borel set $\mathcal{B} \subset \mathbb{R}$ is denoted by $E^M_{\mathcal{B}}$.

**Lemma 4.15** Suppose $|\alpha| > \alpha_\epsilon$. Let $a$, $L$ and $\kappa$ be sufficiently large such that $\min\{|\Sigma|/3, 2\kappa^2\} > \theta^\kappa'$. Then for $\varepsilon$ such that $|\Sigma| > 3\theta^\kappa + \varepsilon$, we have
\[
K - \inf \sigma(K) - \varepsilon \geq E^\overline{H_{\text{eff}}}_{[0,|\Sigma|]} \otimes \left(\kappa^2 - \frac{\theta^\kappa'}{2}H^\varepsilon' - 3\theta^\kappa - \varepsilon\right).
\]
Proof: We directly see by Lemma 4.14 that

\[ K - \inf \sigma(K) - \varepsilon \]

\[ = H_{\text{eff}} + \delta V' + \kappa^2 H_i' - \inf \sigma(H) - \varepsilon \]

\[ \geq H_{\text{eff}} + \delta V' + \kappa^2 H_i' - \frac{3}{2} \theta'_\kappa - \Sigma - \varepsilon \]

\[ \geq H_{\text{eff}} + (\kappa^2 - \frac{\theta'}{2}) H_i' - \frac{3}{2} \theta'_\kappa - \frac{3}{2} \theta'_\kappa - \Sigma - \varepsilon \]

\[ = H_{\text{eff}} + (\kappa^2 - \frac{\theta'}{2}) H_i' - 3 \theta'_\kappa - \varepsilon \]

\[ \geq |\Sigma| E_{|\Sigma|,\infty}^{H_{\text{eff}}} \otimes 1 - \theta''_\kappa (E_{|\Sigma|,\infty}^{H_{\text{eff}}} + E_{|\Sigma|,\infty}^{\Sigma}) \otimes 1 + (\kappa^2 - \frac{\theta'}{2}) (E_{|\Sigma|,\infty}^{H_{\text{eff}}} + E_{|\Sigma|,\infty}^{\Sigma}) \otimes H_i', \]

where \( \theta''_\kappa = 3 \theta'_\kappa + \varepsilon \). Then

\[ K - \inf \sigma(K) - \varepsilon \]

\[ \geq (|\Sigma| - \theta''_\kappa) E_{|\Sigma|,\infty}^{H_{\text{eff}}} \otimes 1 + (\kappa^2 - \frac{\theta'}{2}) E_{|\Sigma|,\infty}^{H_{\text{eff}}} \otimes H_i' \]

\[ + E_{|\Sigma|,\infty}^{H_{\text{eff}}} \otimes \left( (\kappa^2 - \frac{\theta'}{2}) H_i' - \theta''_\kappa \right). \]

Since \(|\Sigma| - \theta''_\kappa = |\Sigma| - 3 \theta'_\kappa - \varepsilon > 0\) and \(\kappa^2 - \frac{\theta'}{2} > 0\) by the assumption, we have

\[ K - \inf \sigma(K) - \varepsilon \geq E_{|\Sigma|,\infty}^{H_{\text{eff}}} \otimes \left( (\kappa^2 - \frac{\theta'}{2}) H_i' - \theta''_\kappa \right). \]

Thus the lemma follows. \(\square\)

Set \(T = K - \inf \sigma(K) - \varepsilon\) as an operator in \(\mathcal{H}_a\). Define \(\mathcal{H}_a(+) = E_T^{T_{[0,\infty)}} \mathcal{H}_a\) and \(\mathcal{H}_a(-) = E_T^{T_{(-\infty,0)}} \mathcal{H}_a\).

Lemma 4.16 Suppose \(|\alpha| > \alpha_\varepsilon\) and that \(\min\{|\Sigma|/3, 2\kappa^2| > \theta'_\kappa\}. Then for \(\varepsilon\) such that \(|\Sigma| > 3 \theta'_\kappa + \varepsilon\), \(T|_{\mathcal{H}_a(-)}\) has a purely discrete spectrum, i.e.,

\[ \sigma(K) \cap [\inf \sigma(K), \inf \sigma(K) + \varepsilon] \subset \sigma_{\text{disc}}(K). \]

Proof: Let \(\{\phi_n\}_n\) be a complete orthonormal system of \(\mathcal{H}_a(-)\) and \(\{\psi_m\}_m\) that of \(\mathcal{H}_a(+)\). We see that by Lemma 4.15

\[ 0 \geq \text{tr} T|_{\mathcal{H}_a(-)} = \sum_n (\phi_n, T\phi_n) \geq \sum_n (\phi_n, T'\phi_n), \]

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where \( T' = E_{(0,|\Sigma|)}^{H_{\Sigma}} \otimes \left( (\kappa^2 - \frac{\theta'}{2}) H_{\epsilon}^{\epsilon'} - 3\theta' - \varepsilon \right) \). Set \( T'_\epsilon = T'E_{(-\infty,0)}^{T'} \). Then

\[
0 \geq \text{tr} T[\mathcal{H}_{\epsilon}(-)] \geq \sum_n (\phi_n, T'_{\epsilon} \phi_n) \geq \sum_n (\phi_n, T'_{\epsilon} \phi_n) + \sum_m (\psi_m, T'_{\epsilon} \psi_m) = \text{tr} T'_{\epsilon}.
\]

Hence we obtain that

\[
\left| \text{tr} T[\mathcal{H}_{\epsilon}(-)] \right| \leq \left| \text{tr} T'_{\epsilon} \right| = \text{tr} E_{(0,|\Sigma|)}^{H_{\Sigma}} \times \left| \text{tr} \left( (\kappa^2 - \frac{\theta'}{2}) H_{\epsilon}^{\epsilon'} \right) \mathcal{H}_{\epsilon} - 3\theta' - \varepsilon \right|,
\]

where \((\cdots)_-\) denotes the negative part of \((\cdots)\). Since \( \sigma\left( (\kappa^2 - \frac{\theta'}{2}) H_{\epsilon}^{\epsilon'} \right) = \sigma_{\text{disc}}\left( (\kappa^2 - \frac{\theta'}{2}) H_{\epsilon}^{\epsilon'} \right) \) and \( \left| \text{tr} E_{(0,|\Sigma|)}^{H_{\Sigma}} \right| < \infty \), it follows that \( \left| \text{tr} T[\mathcal{H}_{\epsilon}(-)] \right| < \infty \). Thus the lemma follows. \( \square \)

**Lemma 4.17** Suppose that \( \min\{|\Sigma|/3, 2\kappa^2\} > \theta'_{\kappa} \). Then for \( \varepsilon \) such that \( |\Sigma| > 3\theta'_{\kappa} + \varepsilon \), it follows that

\[
\sigma(H^{\epsilon}_{\kappa,L}(\kappa)) \cap [\inf \sigma(H^{\epsilon}_{\kappa,L}(\kappa)), \inf \sigma(H^{\epsilon}_{\kappa,L}(\kappa)) + \varepsilon) \subset \sigma_{\text{disc}}(H^{\epsilon}_{\kappa,L}(\kappa)).
\]

**Proof:** We have by Lemmas 4.12 and 4.16

\[
\sigma(H^{\epsilon}_{\kappa,L}(\kappa)) = \sigma(K^\perp) \cup \sigma(K),
\sigma(K^\perp) \subset [\inf \sigma(K) + \varepsilon, \infty),
\sigma(K) \cap [\inf \sigma(K), \inf \sigma(K) + \varepsilon) \subset \sigma_{\text{disc}}(K).
\]

Notice that \( \inf \sigma(K) = \inf \sigma(H^{\epsilon}_{\kappa,L}(\kappa)) \). Then the lemma follows. \( \square \)

Now we can show the existence of ground state of massive Hamiltonian \( H^{\epsilon}(\kappa) \).

**Lemma 4.18** Suppose \( |\alpha| > \alpha_\epsilon \) and that \( \min\{|\Sigma|/3, 2\kappa^2\} > \theta_{\kappa} \). Then for \( \varepsilon \) such that \( |\Sigma| > 3\theta_{\kappa} + \varepsilon \),

\[
\sigma(H^{\epsilon}(\kappa)) \cap [\inf \sigma(H^{\epsilon}(\kappa)), \inf \sigma(H^{\epsilon}(\kappa)) + \varepsilon) \subset \sigma_{\text{disc}}(H^{\epsilon}(\kappa)).
\]

In particular \( H^{\epsilon}(\kappa) \) has a ground state.

**Proof:** Note that \( \lim_{L \to \infty} \lim_{a \to \infty} \theta'_{\kappa} = \theta_{\kappa} \). Then by Lemmas 4.11 and 4.17, the lemma follows. \( \square \)

See Figure 8 for the spectrum of massive Pauli-Fierz Hamiltonian.
4.3.2 Massless case

A ground state of $H^\varepsilon$ is denoted by $\Psi^\varepsilon$.

**Lemma 4.19** Suppose $|\alpha| > \alpha_\varepsilon$, Assumptions 3.4 and 4.9, and that $\min\{|\Sigma|/3, 2\} > \theta$. Then for $\varepsilon$ such that $|\Sigma| > 3\theta_\kappa + \varepsilon$,

$$\frac{\|N^{1/2}\Psi^\varepsilon\|}{\|\Psi^\varepsilon\|} \leq \frac{1}{\kappa^2} C \|Q/\omega^{5/2}\| (\max_\mu \|\nabla_\mu V\|_\infty)$$

(4.37)

with some constant $C$.

**Proof:** We set $E = \inf \sigma(H^\varepsilon(\kappa))$. Since

$$[H^\varepsilon(\kappa), a(k, j)] = -(\omega(k) + \varepsilon)a(k, j) + [\delta V_\kappa, a(k, j)],$$

we have

$$H^\varepsilon(\kappa)a(k, j)\Psi^\varepsilon = -(\omega(k) + \varepsilon)a(k, j)\Psi^\varepsilon + E a(k, j)\Psi^\varepsilon + [\delta V_\kappa, a(k, j)]\Psi^\varepsilon.$$

Hence we derive that

$$(H^\varepsilon(\kappa) - E + \omega(k) + \varepsilon)a(k, j)\Psi^\varepsilon = [\delta V_\kappa, a(k, j)]\Psi^\varepsilon$$

(4.38)

and

$$[\delta V_\kappa, a(k, j)] = \left[ V(\cdot + \frac{1}{\kappa} K), a(k, j) \right] = T^{-1}_\kappa \left[ V, T_\kappa a(k, j)T^{-1}_\kappa \right] T_\kappa.$$

Since

$$T_\kappa a(k, j)T^{-1}_\kappa = a(k, j) - \frac{1}{\kappa} \frac{i}{\sqrt{2}} (-i \nabla_\nu) g^{\nu}(k, j),$$

it follows that

$$[\delta V_\kappa, a(k, j)] = T^{-1}_\kappa \left[ V, -\frac{i}{\sqrt{2}\kappa} (-i \nabla_\nu) g^{\nu}(k, j) \right] T_\kappa$$

$$= \frac{1}{\kappa} T^{-1}_\kappa \left( \frac{1}{\sqrt{2}} (\nabla_\nu V) g^{\nu}(k, j) \right) T_\kappa.$$

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Thus we obtain the pull-through formula:

$$a(k, j)\Psi_\epsilon = \frac{1}{\kappa}(H^\epsilon(\kappa) - E + \omega(k) + \epsilon)^{-1}T_{\kappa}^{-1}\left(\frac{1}{\sqrt{2}}(\nabla_\nu V)g''(k, j)\right)T_{\kappa}\Psi_\epsilon. \quad (4.39)$$

Using identity (4.39) we see that

$$\parallel N_{1/2}\Psi_\epsilon \parallel^2 = \sum_{j=1}^{d-1} \int \parallel a(k, j)\Psi_\epsilon \parallel^2 dk = \frac{1}{2k^2} \sum_{j=1}^{d-1} \int \parallel(H^\epsilon(\kappa) - E + \omega(k) + \epsilon)^{-1}T_{\kappa}^{-1}(\nabla_\nu V)g''(k, j)T_{\kappa}\Psi_\epsilon \parallel^2 dk$$

We then estimate as

$$\parallel N_{1/2}\Psi_\epsilon \parallel^2 \leq \frac{1}{2k^2} \sum_{j=1}^{d-1} \int \left(\frac{1}{\omega(k)}\parallel\nabla_\nu V\parallel_\infty\right)^2 |g''(k, j)|^2 dk \leq \frac{1}{k^2} C(\max_\mu \parallel\nabla_\mu V\parallel_\infty)^2 \parallel Q/\omega^{5/2} \parallel^2.$$

Hence the lemma follows.

**Lemma 4.20** Suppose $|\alpha| > \alpha_\epsilon$, Assumptions 3.4 and 4.9. Let $P_\Omega$ be the projection onto $\{\alpha\Omega \mid \alpha \in \mathbb{C}\}$ and $Q_\Omega = E^{H_{eff}(\kappa+\delta, \infty)}(P_\Omega \psi)$ with some $\delta > 0$ such that $\delta > \frac{3}{2}\theta_\kappa$. Suppose that $\min\{|\Sigma|/3, 2\kappa^2\} > \theta_\kappa$. Then for $\epsilon$ such that $|\Sigma| > 3\theta_\kappa + \epsilon$, it follows that

$$\parallel Q_\Omega\Psi_\epsilon \parallel \leq \sqrt[\frac{\theta_\kappa}{\delta - \frac{3}{2}\theta_\kappa}}. \quad (4.40)$$

**Proof:** Since $(\Psi_\epsilon, Q_\Omega(H^\epsilon(\kappa) - \inf\sigma(H^\epsilon(\kappa)))\Psi_\epsilon) = 0$, we have

$$(\Psi_\epsilon, Q_\Omega(H_{eff} - \inf\sigma(H^\epsilon(\kappa)))\Psi_\epsilon) = -(\Psi_\epsilon, Q_\Omega\delta V_{\kappa}\Psi_\epsilon).$$

The left-hand side above is estimated as

$$(\Psi_\epsilon, Q_\Omega(H_{eff} - \inf\sigma(H^\epsilon(\kappa)))\Psi_\epsilon) \geq (\Sigma + \delta - \inf\sigma(H^\epsilon(\kappa)))(\Psi_\epsilon, Q_\Omega\Psi_\epsilon).$$
Note that
\[ \Sigma + \delta - \inf \sigma(H^\epsilon(\kappa)) \geq \Sigma + \delta - \frac{3}{2} \theta_\kappa = \delta - \frac{3}{2} \theta_\kappa > 0. \]

Then
\[ (\Psi_\epsilon, Q_\Omega(H_{\text{eff}} - \inf \sigma(H^\epsilon(\kappa)) + g)\Psi_\epsilon) \geq (\delta - \frac{3}{2} \theta_\kappa)\|Q_\Omega\Psi_\epsilon\|^2 > 0. \]

Moreover
\[
|((\Psi_\epsilon, Q_\Omega \delta V_\kappa \Psi_\epsilon)| = |(\delta V_\kappa Q_\Omega \Psi_\epsilon, \Psi_\epsilon)| \leq \|\delta V_\kappa Q_\Omega \Psi_\epsilon\| \|\Psi_\epsilon\|
\leq \theta_\kappa \left( \|H^{1/2}_\epsilon Q_\Omega \Psi_\epsilon\| + \|Q_\Omega \Psi_\epsilon\| \right) \|\Psi_\epsilon\|
= \theta_\kappa \|Q_\Omega \Psi_\epsilon\| \|\Psi_\epsilon\| \leq \theta_\kappa \|\Psi_\epsilon\|^2.
\]

Hence we have
\[ 0 < (\delta - \frac{3}{2} \theta_\kappa)\|Q_\Omega\Psi_\epsilon\|^2 \leq \theta_\kappa \|\Psi_\epsilon\|^2. \]

The lemma follows. \qed

We normalize \( \Psi_\epsilon \), i.e., \( \|\Psi_\epsilon\| = 1 \). Take a subsequence \( \epsilon' \) such that \( \Psi_{\epsilon'} \) weakly converges to a vector \( \varphi_\kappa \) as \( \epsilon' \to \infty \).

**Proposition 4.21** [AH97, Lemma 4.9] Let \( S_n \) and \( S \) be self-adjoint operators on a Hilbert space \( h \), which have a common core \( D \) such that \( S_n \to S \) on \( D \) strongly as \( n \to \infty \). Let \( \psi_n \) be a normalized eigenvector of \( S_n \) such that \( S_n = E_n \psi_n \), \( E = \lim_{n \to \infty} E_n \) and the weak limit \( \psi = w - \lim_{n \to \infty} \psi_n \neq 0 \) exist. Then \( S\psi = E\psi \). In particular if \( E_n \) is the ground state energy, then \( E \) is the ground state energy of \( S \) and \( \psi \) is a ground state of \( S \).

**Proof:** Since \( S_n \) converges to \( S \) in the strong resolvent sense by the assumption, we can see that \( \lim_{n \to \infty}(\phi, (S_n - z)^{-1}\psi_n) = (\phi, (S_n - z)^{-1}\psi) \) for any \( \phi \in \mathfrak{h} \). This implies that \( (S_n - z)^{-1}\psi = (E - z)^{-1}\psi \) and then \( S\psi = E\psi \). \qed

Now we are in the position to state the main theorem in Section 4.

**Theorem 4.22** (Enhanced binding) [HS01] Suppose Assumptions 3.4 and 4.9. Then for any \( \epsilon > 0 \), there exists \( \kappa_\epsilon \) such that for all \( \kappa > \kappa_\epsilon \), \( H(\kappa) \) has a unique ground state for all \( \alpha \) such that \( |\alpha| > \alpha_\epsilon \).
Proof: Let $E_\varepsilon = \inf \sigma(H^\varepsilon(\kappa))$ and $E = \inf \sigma(H(\kappa))$. Since $H^\varepsilon(\kappa) \to H(\kappa)$ as $\varepsilon \to 0$ in the strong resolvent sense, $\limsup_{\varepsilon \to 0} E_\varepsilon \leq E$ follows. On the other hand we notice that $E_\varepsilon \geq E + \varepsilon \langle \Psi_\varepsilon, N\Psi_\varepsilon \rangle$. Since $\varepsilon \langle \Psi_\varepsilon, N\Psi_\varepsilon \rangle \to 0$ as $\varepsilon \to 0$, $\liminf_{\varepsilon \to 0} E_\varepsilon \geq E$ follows. Thus $\lim_{\varepsilon \to 0} E_\varepsilon = E$. By Proposition 4.21 it is enough to prove $\varphi_g \neq 0$.

Note that $1 \otimes N + (E^{H_{\text{eff}}}_{[\Sigma, \Sigma+\delta]} + E^{H_{\text{eff}}}_{[\Sigma+N, \Sigma+\delta]}) \otimes P_\Omega = 1 \otimes N + E^{H_{\text{eff}}}_{[\Sigma, \Sigma+\delta]} \otimes P_\Omega + Q_\Omega \geq 1$, and

$$E^{H_{\text{eff}}}_{[\Sigma, \Sigma+\delta]} \otimes P_\Omega \geq 1 - 1 \otimes N - Q_\Omega. \quad (4.41)$$

Suppose that $\min\{|\Sigma|/3, 2\kappa^2\} > \theta_\kappa$ and $\delta > \frac{3}{2}\theta_\kappa$. Then for $\varepsilon'$ such that $|\Sigma| > 3\theta_\kappa + \varepsilon'$, we have by (4.41), Lemmas 4.19 and 4.20,

$$(\Psi_{\varepsilon'}, E^{H_{\text{eff}}}_{[\Sigma, \Sigma+\delta]} \otimes P_\Omega \Psi_{\varepsilon'}) \geq 1 - (\Psi_{\varepsilon'}, N\Psi_{\varepsilon'}) - (\Psi_{\varepsilon'}, Q_\Omega \Psi_{\varepsilon'}) \geq 1 - \frac{1}{\kappa^2} C\|Q/\omega^{5/2}\|(\max_\mu \|\nabla_\mu V\|_\infty) - \frac{\theta_\kappa}{\delta} - \frac{3}{2}\theta_\kappa.$$

Note that $\sup_\alpha \|Q/\omega^{5/2}\| < \infty$ and $\lim_{\kappa \to \infty} \frac{\theta_\kappa}{\delta - \frac{3}{2}\theta_\kappa} = 0$ uniformly with respect to $\alpha$. Hence for sufficiently large $\kappa$, $(\Psi_{\varepsilon'}, E^{H_{\text{eff}}}_{[\Sigma, \Sigma+\delta]} \otimes P_\Omega \Psi_{\varepsilon'}) > \eta$ follows uniformly in $\varepsilon'$ and $\alpha$ with some $\eta > 0$. Take $\varepsilon' \to 0$ on both sides above. Since $E^{H_{\text{eff}}}_{[\Sigma, \Sigma+\delta]} \otimes P_\Omega$ is a finite rank operator, we see that $E^{H_{\text{eff}}}_{[\Sigma, \Sigma+\delta]} \otimes P_\Omega \Psi_{\varepsilon'} \to E^{H_{\text{eff}}}_{[\Sigma, \Sigma+\delta]} \otimes P_\Omega \varphi_g$ strongly and $(\varphi_g, E^{H_{\text{eff}}}_{[\Sigma, \Sigma+\delta]} \otimes P_\Omega \varphi_g) > \eta$. In particular $\varphi_g \neq 0$. Then $\varphi_g$ is a ground state of $H(\kappa)$.

We can also show the existence of ground state for the Pauli-Fierz Hamiltonian without scaling parameter.

**Theorem 4.23 (Enhanced binding, no scaling)** [HS01] Let $\kappa = 1$, i.e., the Hamiltonian is not scaled. Suppose Assumption 3.4, and (1) and (2) of Assumption 4.9, and that

$$\lim_{\alpha \to \infty} \|Q/\omega^{n/2}\| = 0, \quad n = 3, 4, 5. \quad (4.42)$$

Then there exists $\alpha_s > \alpha_\varepsilon$ such that for all $\alpha$ with $|\alpha| > \alpha_s$, $H$ has a ground state.
Proof: By (4.42) we can see that $\theta_\kappa \to 0$ and $\|Q/\omega^{5/2}\| \to 0$ as $\alpha \to \infty$. Then Lemma 4.18 holds for sufficiently large $\alpha$ with $\kappa = 1$. Then the massive ground state $\Psi_\varepsilon$ exists. Furthermore we have

$$\langle \Psi_\varepsilon', E^{H_{\text{eff}}}_{\Sigma,\Sigma^+} \otimes P_\Omega \Psi_\varepsilon' \rangle \geq 1 - C\|Q/\omega^{5/2}\|(\max_{\mu} \|\nabla_\mu V\|_\infty) - \frac{\theta_1}{\delta - \frac{3}{2}\theta_1},$$

where $\theta_1$ is $\theta_\kappa$ with $\kappa = 1$. Since $\lim_{|\alpha| \to \infty} \|Q/\omega^{5/2}\| = 0$ and $\lim_{|\alpha| \to \infty} \frac{\theta_1}{\delta - \frac{3}{2}\theta_1} = 0$, we can conclude that $\varphi_\varepsilon \neq 0$ for sufficiently large $|\alpha|$. Then the corollary follows. □

An example of (4.42) is given in Example 4.10. See Figure 9 for the spectrum of massless Pauli-Fierz Hamiltonian.

4.4 Transition from unbinding to binding

In the previous sections we show the absence and the existence of ground state. Combining these results we can construct examples of the Pauli-Fierz Hamiltonian having transition from unbinding to binding according to the value of coupling constant $\alpha$.

Lemma 4.24 Suppose Assumptions 3.4 and 4.1. Then $H(\kappa)$ has no ground state for all $\kappa > 0$ and all $\alpha$ such that $|\alpha| < \alpha_0$.

Proof: Define the unitary operator $u_\kappa$ by $(u_\kappa f)(x) = k^{d/2} f(x/\kappa)$. Then we infer

$$V_\kappa = \kappa^{-2} u_\kappa V u_\kappa^{-1}, -\Delta = \kappa^{-2} u_\kappa (-\Delta) u_\kappa^{-1}$$

and

$$\|V_\kappa|^{1/2}(-\Delta)^{-1}|V_\kappa|^{1/2}\| = \kappa^{-2} \|u_\kappa|V|^{1/2}u_\kappa^{-1}(-\Delta)^{-1}u_\kappa|V|^{1/2}u_\kappa^{-1}\| = \|K_0\|.$$  

Then the lemma follows from Theorem 4.6. □

Theorem 4.25 (Transition from unbinding to binding) Suppose Assumptions 3.4, 4.1 and 4.9. Let arbitrary $\delta > 0$ be given. Then there exists an external potential $\tilde{V}$ and constants $\alpha_+ > \alpha_-$ such that
(1) $0 < \alpha_+ - \alpha_- < \delta$;

(2) $H$ has a ground state for $|\alpha| > \alpha_+$ but no ground state for $|\alpha| < \alpha_-$. 

Proof: For $\delta > 0$ we take $\epsilon > 0$ such that $\alpha_+ - \alpha_0 < \delta$. Take a sufficiently large $\kappa$, and set $\tilde{V}(x) = V(x/\kappa)/\kappa^2$. Define $H$ by the Pauli-Fierz Hamiltonian with potential $\tilde{V}$. Then $H$ has a ground state for $|\alpha| > \alpha_\epsilon$ by Theorem 4.22 and $H$ has no ground state for $|\alpha| < \alpha_0$ by Lemma 4.24. Set $\alpha_\epsilon = \alpha_+$ and $\alpha_0 = \alpha_-$. Then the proof is completed. 

4.5 Enhanced binding by UV cutoff

We can also consider the enhanced binding by UV cutoff. In Example 3.10 we give the example of UV cutoff function:

$$\hat{\phi}(k) = 1_{[\lambda, \Lambda]}(k).$$

(4.43)

In this section we suppose that $\hat{\phi}$ is (4.43) and the dimension $d = 3$. Thus $m_{\text{eff}} = m + \frac{8}{3}\pi\alpha^2(\Lambda - \lambda)$ and we have the corollary below.

Corollary 4.26 (Absence of ground state) Suppose Assumptions 3.4 and 4.1 and

$$\Lambda < \frac{8}{3\pi}\alpha^{-2}(m_c - m) + \lambda.$$ 

(4.44)

Then $H$ has no ground state.

Proof: (4.44) implies that $m_{\text{eff}} < m_c$. Then the corollary follows from Theorem 4.6. 

We can also show the existence of ground state for sufficiently large $\Lambda$.

Corollary 4.27 (Enhanced binding) Suppose Assumption 3.4 and (1) and (2) of Assumption 4.9. Then there exists $\Lambda_\ast$ such that $H$ has a ground state for $\Lambda > \Lambda_\ast$.

Proof: We notice that

$$m_{\text{eff}}(k) = m + \alpha^2 \frac{8\pi}{3}(\Lambda - \lambda)$$

$$- \frac{\alpha^2 8\pi}{2} \frac{1}{3} \left( |k| \log \frac{|k| + \Lambda(|k| - \lambda)}{|k| + \lambda(|k| - \Lambda)} - i\pi 1_{[\lambda, \Lambda]}(|k|) \sqrt{|k|} \right).$$

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Then we have
\[ \|Q/\omega^{n/2}\|^2 = \alpha^2 \int_{\lambda \leq |k| \leq \Lambda} \frac{1}{m_{\text{eff}}(k)^2 \omega(k)^n} dk \]
and
\[ \mathbb{1}_{[\lambda, \Lambda]}(k) \frac{1}{m_{\text{eff}}(k)^2 \omega(k)^n} \leq \left( \frac{3}{4\pi^2 \alpha^2} \right)^2 \frac{1}{\omega(k)^{n+1}}. \]
Since the right and side above is integrable for \( n = 3, 4 \) and \( 5 \). The the Lebesgue dominated convergence theorem yields that \( \lim_{\Lambda \to \infty} \|Q/\omega^{n/2}\| = 0 \). Hence in a similar way to Theorem 4.23 we can prove the corollary. \( \square \)
Part III
The Nelson model

5 The Nelson Hamiltonian

5.1 The Nelson Hamiltonian

We begin with giving the definition of the Nelson Hamiltonian. Let \( \mathcal{F} \) be the Boson Fock space over \( L^2(\mathbb{R}^d) \). The creation operator and the annihilation operator satisfy canonical commutation relations on \( \mathcal{F}_{\text{fin}} \):

\[
[a(f), a^*(g)] = (f, g), \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)]. \tag{5.1}
\]

For \( h \in L^2(\mathbb{R}^d) \), the field operator is defined by

\[
\phi(h) = \frac{1}{\sqrt{2}} \int \left( a^*(k)\hat{h}(-k) + a(k)\hat{h}(k) \right) dk. \tag{5.2}
\]

Let \( H_f \) be the free field Hamiltonian with the dispersion relation \( \omega(k) = |k| \).

In order to study the binding of \( N \) particle system by the linear coupling with the scalar quantum field, \( N \) particles assumed to be independent of each other, and then there is no external potential linking two particles. Thus the \( N \) particle Hamiltonian \( H_p \) is defined by the self-adjoint operator on \( L^2(\mathbb{R}^{dN}) \) by

\[
H_p = \sum_{j=1}^{N} \left( -\frac{1}{2m_j} \Delta_j + V_j \right), \tag{5.3}
\]

where \( m_j \) is the mass of the \( j \)-th particle and \( V_j = V_j(x_j) \) external potential depending only on \( x_j \). Hamiltonian \( H_p \) does not necessarily have ground states. For example, with sufficiently shallow potential \( V_j \)'s, \( H_p \) has no ground states.

The state space of the \( N \) particle coupled to the scalar quantum field is

\[
\mathcal{H} = L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}. \tag{5.4}
\]

Let

\[
H_0 = H_p \otimes 1 + 1 \otimes H_f \tag{5.5}
\]

be the non-interacting Hamiltonian.
Definition 5.1 (Nelson Hamiltonian) The Nelson Hamiltonian $H$ on $\mathcal{H}$ is defined by

$$H = H_0 + H_1,$$

(5.6)

where $H_1$ is given by

$$H_1 = \sum_{j=1}^{N} \alpha_j \int_{\mathbb{R}^d} \phi_j(x_j) dx.$$ 

Here $\alpha_j$’s are real coupling constants, we identify $H$ as $H \sim \int_{\mathbb{R}^d} F dx$ and $\phi_j(x)$, $x \in \mathbb{R}^d$, is given by

$$\phi_j(x) = \frac{1}{\sqrt{2}} \int (a^*(k)\lambda_j(-k)e^{-ikx} + a(k)\lambda_j(k)e^{ikx}) dk.$$ 

We will give assumptions on $\lambda_j$ later. Since the semigroup $e^{-tH}$ is ergodic, the uniqueness of the ground state of $H$ follows.

The existence of ground states of the Nelson Hamiltonian has been investigated in the last decade. [BFS98-a, BFS98-b] proved the existence of ground states under some conditions. [Ger00, Spo99] remove the weak coupling condition, namely they show the existence of ground states of the Nelson Hamiltonian for arbitrary values of a coupling constant. [Sas05] shows the existence of a ground state with general external potentials including the Coulomb potential, and [HHS05] shows the existence of a ground state without cutoffs.

5.2 Enhanced binding

As is mentioned in the previous section, the existence of the ground state of the Nelson Hamiltonian has been proven under some general conditions. One of fundamental assumption among them is that $H_p$ has a ground state. In this note we remove this condition.

If there is no interaction between particles, the $j$-th particle is governed only by the potential $V_j$. In this case, if $V_j$’s are sufficiently shallow, external potential $\sum_{j=1}^{N} V_j$ can not trap these particles. But if these particles attractively interact with each other by an effective potential derived from the scalar quantum field, particles close up with each other and behave just like as one particle with heavy mass.
\[ \sum_{j=1}^{N} m_j. \] We will see that effective potential is of the form
\[
V_{\text{eff}}(x) = -\frac{1}{4} \sum_{i \neq j}^{N} \alpha_i \alpha_j \int_{\mathbb{R}^d} \frac{\lambda_i(-k) \lambda_j(k)}{\omega(k)} e^{-ik(x_i-x_j)} dk. \tag{5.7}
\]

Effective potential \( V_{\text{eff}} \) depends on the choice of cutoff function \( \lambda_j \)'s. A typical example of \( V_{\text{eff}} \) is a three dimensional \( N \)-body smeared Coulomb potential:
\[
V_{\text{eff}}(x_1, \ldots, x_N) = -\frac{1}{8\pi} \sum_{i \neq j}^{N} \frac{\alpha_i \alpha_j}{|x_i - x_j|} \varpi(|x_i - x_j|),
\]
where \( \varpi(|x|) > 0 \) holds for a sufficiently small \( |x| \). For this case it is determined by signs of \( \alpha_1, \ldots, \alpha_N \) whether \( V_{\text{eff}} \) is attractive or repulsive for sufficiently small \( |x_i - x_j| \). We can see from (5.15) that an identical sign of coupling constants and
\[ \text{supp} \lambda_i \cap \text{supp} \lambda_j \neq \emptyset, \quad i \neq j, \]
derive attractive effective potentials, and which enhances binding of the system. If \( N \) is large enough, this one particle has sufficiently heavy mass and is bounded by external potential \( \sum_{j=1}^{N} V_j \), and finally it is trapped. See Figure 10. Heuristically we see that
\[
H \sim -\frac{1}{2\sum_{j=1}^{N} m_j} \Delta + \sum_{j=1}^{N} V_j = \frac{1}{\sum_{j=1}^{N} m_j} \left( -\frac{1}{2} \Delta + (\sum_{j=1}^{N} m_j) \sum_{j=1}^{N} V_j \right). \tag{5.8}
\]
5.3 Weak coupling limit

In this section we see the relationship between an enhanced binding and a weak coupling limit, which has been seen in the case of the Pauli-Fierz model in Section 3.9.1. In our model under consideration, it is seen that the enhanced binding is derived from the effective potential \( V_{\text{eff}} \) which is the sum of potentials between two particles. Alternatively the effective potential can be derived from a weak coupling limit [Dav77, Dav79, Hir98, Hir99], which is one of a key ingredient of this paper. We outline a weak coupling limit by path measures. Let us introduce a scaling in the Nelson model as

\[
H(\kappa) = H_p + \kappa^2 H_f + \kappa H_I,
\]

(5.9)

where \( \kappa > 0 \) is a scaling parameter. Let \( \mathcal{X}^- = C([0, \infty); \mathbb{R}^{dN}) \) be the set of continuous paths valued in \( \mathbb{R}^{dN} \). Then \( e^{-tH(\kappa)} \) can be expressed by a path measure as

\[
(f \otimes \Omega, e^{-tH(\kappa)} g \otimes \Omega)_{\mathcal{H}} = \int_{\mathcal{X}^- \times \mathbb{R}^{dN}} f(X_0)g(X_t)e^{-\int_0^T V(X_s)ds}e^{W_\kappa}d\mathcal{W}^x dx,
\]

(5.10)

where \( X_t = (X^1_t, ..., X^N_t), X^j_t(w) = w^j(t) \in \mathbb{R}^d, w = (w^1, ..., w^N) \in \mathcal{X}^-, \) denotes the point evaluation of \( w \in \mathcal{X}^-, d\mathcal{W}^x \) the Wiener measure on \( \mathcal{X}^- \) starting from \( x \) at \( t = 0 \) and

\[
V(X_s) = \sum_{j=1}^N V_j(X^j_s),
\]

(5.11)

\[
W_\kappa = \frac{1}{4} \sum_{i,j=1}^N \alpha_i \alpha_j \int_0^T ds \int_0^T dt \int_{\mathbb{R}^d} \lambda_i(-k)\lambda_j(k)\kappa^2 e^{-\kappa^2|s-t|\omega(k)}e^{-ik\cdot(X^i_s-X^j_t)} dk.
\]

(5.12)

Informally taking \( \kappa \to \infty \) in (5.12), we see that only the diagonal part of \( \int_0^T ds \int_0^T dt \) survives and the off diagonal part is dumped by the factor

\[
\kappa^2 e^{-\kappa^2|s-t|\omega(k)} \to \delta(s-t)\frac{2}{\omega(k)}.
\]
as \( \kappa \to \infty \). Thus we have

\[
W_\kappa \to \frac{1}{2} \sum_{i,j=1}^{N} \alpha_i \alpha_j \int_0^{T} ds \int_{\mathbb{R}^d} \frac{\lambda_i(-k)\lambda_j(k)}{\omega(k)} e^{-ik(X_i^s - X_j^s)} dk
\]

as \( \kappa \to \infty \). Combining the right-hand side of \( (5.13) \) with \( \int_0^{T} V(X_s) ds \) in \( (5.10) \), we can derive the Feynman-Kac type formula:

\[
\lim_{\kappa \to \infty} (5.10) = \int_{\mathcal{X}_+ \times \mathbb{R}^dN} f(X_0)g(X_T)e^{-\int_0^{T} (V(X_s) + V_{\text{eff}}(X_s) + G) ds} d\mathcal{W}^x dx,
\]

where

\[
V_{\text{eff}}(x_1, \ldots, x_n) = -\frac{1}{4} \sum_{i \neq j}^{N} \alpha_i \alpha_j \int_{\mathbb{R}^d} \frac{\lambda_i(-k)\lambda_j(k)}{\omega(k)} e^{-ik(x_i - x_j)} dk
\]

and \( G \) is the constant derived from the diagonal part of \( (5.13) \), which is given by

\[
G = -\frac{1}{4} \sum_{j=1}^{N} \alpha_j^2 \int_{\mathbb{R}^d} \frac{\lambda_j(-k)\lambda_j(k)}{\omega(k)} dk.
\]

Note that when \( \text{supp}\lambda_i \cap \text{supp}\lambda_j = \emptyset, i \neq j \), the effective potential \( V_{\text{eff}} \) vanishes and only the constant \( G \) remains. Let

\[
H_{\text{eff}} = \sum_{j=1}^{N} \left( -\frac{1}{2m_j} \Delta_j + V_j \right) + V_{\text{eff}}.
\]

Actually \( (5.14) \) can be shown rigorously.

**Proposition 5.2 (Weak coupling limit)** [Dav77, Dav79, Hir98, Hir99] Let \( t > 0 \).

Then

\[
\lim_{\kappa \to \infty} e^{-tH(\kappa)} = e^{-t(H_{\text{eff}} + G)} \otimes P_{\Omega},
\]

where \( P_{\Omega} \) denotes the projection onto \( \{ z \Omega | z \in \mathbb{C} \} \subset \mathcal{F} \). In particular for \( f, g \in L^2(\mathbb{R}^dN) \),

\[
\lim_{\kappa \to \infty} (f \otimes \Omega, e^{-tH(\kappa)} g \otimes \Omega)_{\mathcal{F}} = (f, e^{-t(H_{\text{eff}} + G)} g)_{L^2(\mathbb{R}^dN)}.
\]
Proposition 5.2 is interesting in both the stochastic analysis and the operator theory. Probabilistically, through a weak coupling limit as is seen in Proposition 5.2 one can derive a Markov process from a non Markov process. We will see it below. The family of measures $\{\mu_\kappa^x\}_{\kappa>0}$ on the path space $\mathcal{X}_+$ is given by

$$\mu_\kappa^x(dX) = e^{-\int_0^1 V(X_s) ds} e^{W_x} dW_x. \tag{5.18}$$

The double integral $W_\kappa$ in (5.18) is independent of $x$ and breaks a Markov property of the stochastic process $(X_s)_{s>0}$, and

$$T_{\kappa,s} : f \mapsto \int_{\mathcal{X}_+} f(X_s) \mu_\kappa(dX)$$

does not define a semigroup on $L^2(\mathbb{R}^{dN})$. By Proposition 5.2 however, the Markov propertyrevives as $\kappa \to \infty$, and we have $T_{\infty,s} = e^{-s(H_{\text{eff}} + G)}$.

Furthermore Proposition 5.2 also suggests that $H(\kappa) \sim H_{\text{eff}} + G$ for a sufficiently large $\kappa$. Actually we can show that $H(\kappa)$ is isomorphic to a self-adjoint operator of the form

$$H_{\text{eff}} + \kappa^2 H_f + \frac{1}{\kappa} H_1 + \frac{1}{\kappa^2} H_2 + \text{constant} \tag{5.19}$$

with some operators $H_1$ and $H_2$. It is checked that under some condition $H_{\text{eff}}$ has a ground state for $\alpha_j$’s such that $0 < \alpha_c < |\alpha_j|$, $j = 1, \ldots, N$, for some $\alpha_c$, which suggests by (5.19) that for a sufficiently large $\kappa$, $H(\kappa)$ also has a ground state for $\alpha_j$ with $\alpha_c < |\alpha_j| < \alpha_c(\kappa)$, $j = 1, \ldots, N$, for some $\alpha_c(\kappa)$. This is actually proved by checking stability conditions for (5.19) under some assumptions. This is an idea to show the enhanced binding for the Nelson model. Note that we do not need to assume the existence of ground state of $H_p$, namely $H(\kappa)$ with $\alpha_1 = \cdots = \alpha_N = 0$ may have no ground state.
6 Binding

6.1 Existence of ground states

In order to show the enhanced binding we check the so-called stability condition. The stability condition implies that the lowest two cluster threshold of $H$ is strictly larger than the ground state energy of $H$. Then intuitively atom can not be ionized and thus the ground state is stable. We introduce assumptions:

Assumption 6.1 For all $j = 1, \ldots, N$, (i),(ii),(iii) and (iv) are fulfilled.

(i) $\lambda_j(-k) = \overline{\lambda_j(k)}$ and $\lambda_j/\sqrt{\omega} \in L^2(\mathbb{R}^d)$.

(ii) There exists an open set $S \subset \mathbb{R}^d$ such that $\bar{S} = \text{supp} \lambda_j$ and $\lambda_j \in C^1(S)$.

(iii) For all $R > 0$, $S_R = \{k \in S ||k| < R\}$ has a cone property.

(iv) For all $p \in [1, 2)$ and all $R > 0$, $|\nabla_k \lambda_j| \in L^p(S_R)$.

Condition (i) guarantees that $H_I$ is a symmetric operator. In order to show the existence of a ground state, we applied a method invented in [GLL01]. Precisely, we used the photon derivative bound and the Rellich-Kondrachov theorem. The conditions (ii)-(iv) are required to verify these procedures in the proof of Proposition 6.4 below. It is easily proven that $H$ is self-adjoint on $D(H) = D(H_p) \cap D(H_I)$ and bounded from below for an arbitrary $\alpha_j \in \mathbb{R}$.

Assumptions (V1) and (V2) are also introduced:

(V1) There exists $\alpha_c > 0$ such that $\inf \sigma(H_{\text{eff}}) \in \sigma_{\text{disc}}(H_{\text{eff}})$ for $\alpha_j$ with $|\alpha_j| > \alpha_c$, $j = 1, \ldots, N$.

(V2) $V_j(-\Delta + 1)^{-1}$, $j = 1, \ldots, N$, are compact.

The main theorem in Section 6 is stated below.

Theorem 6.2 (Enhanced binding) [HS08] Let $\lambda_j/\omega \in L^2(\mathbb{R}^d)$, $j = 1, \ldots, N$, and assume Assumption 6.1 (V1) and (V2). Fix a sufficiently large $\kappa > 0$. Then there exists $\alpha_c(\kappa)$ such that for $\alpha_j$ with $\alpha_c < |\alpha_j| < \alpha_c(\kappa)$, $j = 1, \ldots, N$, $H(\kappa)$ has a ground state, where $\alpha_c(\kappa)$ is possibly infinity.
Proof: We give a proof in Section 6.2.

The scaling parameter $\kappa$ in Theorem 6.2 can be regarded as a dummy and absorbed into $m_j$, $V_j$ and $\lambda_j$, $j = 1, \ldots, N$. Let $\kappa$ be sufficiently large. Define

$$\hat{H} = \sum_{j=1}^{N} \left( -\frac{1}{2\hat{m}_j} \Delta_j + \hat{V}_j \right) + \sum_{j=1}^{N} \alpha_j \hat{\phi}_j + H_f,$$

where $\hat{m}_j = m_j \kappa^2$, $\hat{V}_j = V_j / \kappa^2$ and $\hat{\phi}_j$ is defined by $\phi_j$ with $\lambda_j$ replaced by $\lambda_j / \kappa$.

Corollary 6.3 Let $\lambda_j / \omega \in L^2(\mathbb{R}^d)$, $j = 1, \ldots, N$, and assume Assumption 6.1, (V1) and (V2). Then $\hat{H}$ has a ground state for $\alpha_c < |\alpha_j| < \alpha_c(\kappa)$, $j = 1, \ldots, N$, where $\alpha_c(\kappa)$ is introduced in Theorem 6.2.

Proof: We have $\kappa^{-2}H(\kappa) = \hat{H}$. Then by Theorem 6.2 $\hat{H}$ has a ground state.

6.2 Stability conditions

Let $\lambda_j / \omega \in L^2(\mathbb{R}^d)$, $j = 1, \ldots, N$, and define the unitary operator $T$ on $\mathcal{H}$ by

$$T = \exp \left(-i \frac{1}{\kappa} \sum_{j=1}^{N} \alpha_j \pi_j \right),$$

where

$$\pi_j = \int_{\mathbb{R}^d N}^{\oplus} \pi_j(x_j) dx$$

with

$$\pi_j(x) = \frac{i}{\sqrt{2}} \int \left(a^*(k)e^{-ik \cdot x} \frac{\lambda_j(-k)}{\omega(k)} - a(k)e^{ik \cdot x} \frac{\lambda_j(k)}{\omega(k)} \right) dk.$$

Then we can show that $T$ maps $D(H)$ onto itself and

$$T^{-1}H(\kappa)T = \sum_{j=1}^{N} \left\{ \frac{1}{2m_j} \left( -i \nabla_j - \frac{\alpha_j}{\kappa} A_j \right)^2 + V_j - \frac{\alpha_j^2}{2} \| \lambda_j / \sqrt{\omega} \|^2 \right\} + \kappa^2 H_f + V_{eff} = H_{eff} + \kappa^2 H_f + H'(\kappa).$$
where \( A_j = \int_{\mathbb{R}^d} A_j(x_j) dx \) with
\[
A_j(x) = \frac{1}{\sqrt{2}} \int k \left( a^*(k)e^{-ikx} \frac{\lambda_j(-k)}{\omega(k)} + a(k)e^{ikx} \frac{\lambda_j(k)}{\omega(k)} \right) dk
\]
and
\[
H'(\kappa) = \sum_{j=1}^N \left\{ \frac{1}{\kappa} \frac{\alpha_j}{2m_j} ((-i \nabla_j) \cdot A_j + A_j (-i \nabla_j)) + \frac{1}{\kappa^2} \frac{\alpha_j}{2m_j} A_j^2 - \frac{\alpha_j}{2} \| \lambda_j / \sqrt{\omega} \|^2 \right\}.
\]
Let us set \( C_N = \{1, ..., N\} \). For \( \beta \subseteq C_N \), we define
\[
H_0(\beta) = H_0(\beta, \kappa) = \sum_{j \in \beta} \frac{1}{2m_j} \left( (-i \nabla_j - \frac{\alpha_j}{\kappa} A_j)^2 + \kappa^2 H_f + V_{\text{eff}}(\beta),
\]
\[
V_{\text{eff}}(\beta) = \begin{cases} 
-\frac{1}{4} \sum_{i,j \in \beta, i \neq j} \alpha_i \alpha_j \int_{\mathbb{R}^d} \frac{\lambda_i(-k)\lambda_j(k)}{\omega(k)} e^{-ik(x_i-x_j)} dk, & |\beta| \geq 2, \\
0, & |\beta| = 0, 1,
\end{cases}
\]
\[
H_V(\beta) = H_V(\beta, \kappa) = H_0(\beta) + \sum_{j \in \beta} V_j,
\]
where \(|\beta| = \# \beta\). Simply we set \( H_V = H_V(C_N) \).
\[
H_V = H(\kappa) - \frac{1}{4} \sum_{j=1}^N \alpha_j^2 \| \lambda_j \|^2
\]
has ground states if and only if \( H(\kappa) \) does, since \( \sum_{j=1}^N \alpha_j^2 \| \lambda_j \|^2 / 4 \) is a fixed number. The operators \( H_0(\beta) \) and \( H_V(\beta) \) are self-adjoint operators acting on \( L^2(\mathbb{R}^{|\beta|}) \otimes \mathcal{F} \).
We set
\[
E_V(\kappa) = \inf \sigma(H_V),
\]
\[
E_V(\kappa, \beta) = \inf \sigma(H_V(\beta)),
\]
\[
E_0(\kappa, \beta) = \inf \sigma(H_0(\beta)),
\]
\[
E_V(\kappa, \emptyset) = 0.
\]
The lowest two cluster threshold \( \Sigma_V(\kappa) \) is defined by
\[
\Sigma_V(\kappa) = \min \{ E_V(\kappa, \beta) + E_0(\kappa, \beta^c) | \beta \subseteq C_N \}.
\]
To establish the existence of ground state of \( H(\kappa) \), we use the next proposition:
Figure 11: Ionization $\mathcal{E}_V(\beta^c) + \mathcal{E}(\beta)$

**Proposition 6.4 [GLL01]** Let $\Sigma_V(\kappa) - E_V(\kappa) > 0$. Then $H(\kappa)$ has a ground state.

For $\beta \subset C_N$, we set the Schrödinger operators in $L^2(\mathbb{R}^{|\beta|})$ by

$$h_0(\beta) = -\sum_{j \in \beta} \frac{1}{2m_j} \Delta_j + V_{\text{eff}}(\beta),$$
$$h_V(\beta) = h_0(\beta) + \sum_{j \in \beta} V_j,$$
$$\mathcal{E}_0(\beta) = \inf \sigma(h_0(\beta)),$$
$$\mathcal{E}_V(\beta) = \inf \sigma(h_V(\beta)),$$

where $h_0(\emptyset) = 0$ and $h_V(\emptyset) = 0$. Furthermore we simply put

$$h_V = h_V(C_N) = H_{\text{eff}}, \quad \mathcal{E}_V = \inf \sigma(h_V). \quad (6.3)$$

We define the lowest two cluster threshold for $h_V$ by (Figure 11)

$$\Xi_V = \min\{\mathcal{E}_V(\beta) + \mathcal{E}_0(\beta^c)|\beta \subseteq C_N\} \quad (6.4)$$

and we set

$$V_{\text{eff}ij}(x) = -\frac{1}{4} \alpha_i \alpha_j \int_{\mathbb{R}^d} \frac{\lambda_i(-k)\lambda_j(k)}{\omega(k)} e^{-ik \cdot x} dk, \quad i \neq j.$$

**Lemma 6.5** Effective potentials $V_{\text{eff}ij}$, $i,j = 1,\ldots,N$, are relatively compact with respect to $-\Delta$.  

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Proof: Since $\lambda_i, \lambda_j/\omega \in L^1(\mathbb{R}^d)$, $i, j = 1, \ldots, N$, we can see that $V_{\text{eff}}_{ij}(x)$ is continuous in $x$ and $\lim_{|x| \to \infty} V_{\text{eff}}_{ij}(x) = 0$ by the Riemann-Lebesgue theorem. In particular $V_{\text{eff}}_{ij}$ is relatively compact with respect to the $d$-dimensional Laplacian.

We want to estimate $\inf \sigma_{\text{ess}}(H_{\text{eff}})$. For Hamiltonians with the center of mass motion removed, the bottom of the essential spectrum is estimated by HVZ theorem.

Lemma 6.6 Assume (V2). Then $\sigma_{\text{ess}}(H_{\text{eff}}) = [\Xi_{\nu}, \infty)$.

Proof: We may assume that $V_i, V_{\text{eff}}_{ij} \in C^\infty_0(\mathbb{R}^d)$ by Proposition 6.17 below. Then there exists a normalized sequence $\{g_n\}_n \subset C^\infty_0(\mathbb{R}^{dN})$ such that

$$\text{supp} g_n \subset \{ x \in \mathbb{R}^{dN} \mid V_i(x) = 0, V_{\text{eff}}_{ij}(x_i - x_j) = 0 \}$$

and $(g_n, h_V(\beta) g_n) = \sum_{j \in \beta} (g_n, -\frac{1}{2m_j} \Delta g_n) \to 0$ as $n \to \infty$. Then we have

$$\mathcal{E}_V(\beta) + \mathcal{E}_0(\beta^c) \leq 0. \quad (6.5)$$

Let $\tilde{j}_\beta \in C^\infty(\mathbb{R}^d)$, $\beta \in C_N$, be a Ruelle-Simon partition of unity, which satisfy (i)-(v) below:

(i) $\sum_{\beta \subset C_N} \tilde{j}_\beta(x)^2 = 1$,

(ii) $\tilde{j}_\beta(Cx) = \tilde{j}_\beta(x)$ for $|x| = 1$, $C \geq 1$ and $\beta \neq C_N$,

(iii) $\text{supp} \tilde{j}_\beta \subset \left\{ x \in \mathbb{R}^d \mid \min_{i \in \beta, j \in \beta^c} \{|x_i - x_j|, |x_j|\} \geq c|x| \right\}$ for some $c > 0$,

(iv) $\tilde{j}_\beta(x) = 0$ for $|x| < \frac{1}{2}$ and $\beta \neq C_N$,

(v) $\tilde{j}_{C_N}$ has a compact support.

For a constant $R > 0$ we put $j_\beta(x) = \tilde{j}_\beta(x/R)$. Note that for each $\beta \subset C_N$,

$$H_{\text{eff}} = h_V(\beta) \otimes \mathbb{1} + \mathbb{1} \otimes h_0(\beta^c) + I_\beta,$$

where

$$I_\beta = \sum_{i \in \beta^c} \mathbb{1} \otimes V_i(x_i) + \sum_{i \in \beta, j \in \beta^c} V_{\text{eff}}_{ij}(x_i - x_j).$$
Here we identify as $L^2(\mathbb{R}^d) \cong L^2(\mathbb{R}^{d,\beta}) \otimes L^2(\mathbb{R}^{d,\beta_c})$. By the IMS localization formula [CFKS87, Theorem 3.2 and p. 34], we have

$$H_{\text{eff}} = j_{C_N} H_{\text{eff}} j_{C_N} + \sum_{\beta \subseteq C_N} j_{\beta} \left( h_{V}(\beta) \otimes 1 + 1 \otimes h_{0}(\beta_c) \right) j_{\beta} + \sum_{\beta \subseteq C_N} j_{\beta}^2 I_{\beta} - \frac{1}{2} \sum_{\beta \subseteq C_N} |\nabla j_{\beta}|^2.$$

Since $j_{C_N} \left( \sum_{j=1}^{N} V_j + V_{\text{eff}} \right)$ and $\sum_{\beta \subseteq C_N} j_{\beta}^2 I_{\beta}$ are relatively compact with respect to the $dN$-dimensional Laplacian by the property (iii) and (v), it is seen that the essential spectrum of $H_{\text{eff}}$ coincides with that of

$$j_{C_N} \left( -\frac{1}{2} \sum_{j=1}^{N} \Delta_j \right) j_{C_N} + \sum_{\beta \subseteq C_N} j_{\beta} \left( h_{V}(\beta) \otimes 1 + 1 \otimes h_{0}(\beta_c) \right) j_{\beta} - \frac{1}{2} \sum_{\beta \subseteq C_N} |\nabla j_{\beta}|^2.$$

We have

$$\sum_{\beta \subseteq C_N} j_{\beta} \left( h_{V}(\beta) \otimes 1 + 1 \otimes h_{0}(\beta_c) \right) j_{\beta} \geq \sum_{\beta \subseteq C_N} (E_{V}(\beta) + E_{0}(\beta_c) ) j_{\beta}^2.$$

By (ii) and (v),

$$\left\| \frac{1}{2} \sum_{\beta \subseteq C_N} |\nabla j_{\beta}|^2 \right\| \leq \frac{C}{R^2}$$

with some constant $C$ independent of $R$. Hence we obtain that

$$\inf \sigma_{\text{ess}}(H_{\text{eff}}) \geq \min_{x \in \mathbb{R}^d} \sum_{\beta \subseteq C_N} (E_{V}(\beta) + E_{0}(\beta_c) ) j_{\beta}^2 - \frac{C}{R^2} \geq \Xi_V - \frac{C}{R^2}$$

for all $R > 0$. Here we used (i) and (6.5). Thus $\sigma_{\text{ess}}(H_{\text{eff}}) \subset [\Xi_V, \infty)$ follows.

Next we shall prove the reverse inclusion $\sigma_{\text{ess}}(H_{\text{eff}}) \supset [\Xi_V, \infty)$. Fix $\beta \subset C_N$. Let $\{\psi_n^V\}_{n=1}^{\infty} \subset C^\infty(\mathbb{R}^{d,\beta})$ be a minimizing sequence of $h_{V}(\beta)$ so that

$$\lim_{n \to \infty} \| (h_{V}(\beta) - E_{V}(\beta)) \psi^V_n \| = 0, \quad \|\psi^V_n\| = 1$$

and $\{\psi_n^0\}_{n=1}^{\infty} \subset C^\infty(\mathbb{R}^{d,\beta_c})$ a normalized sequence such that

$$\lim_{n \to \infty} \| (h_{0}(\beta_c) - E_{0}(\beta_c) - K) \psi^0_n \| = 0, \quad (6.6)$$

where $K \geq 0$ is a constant. Note that since $\sigma(h_{0}(\beta_c) = [E_{0}(\beta_c), \infty)$, $\psi_n^0$ such as (6.6) exists. By the translation invariance of $h_{0}(\beta_c)$, for any function $\tau : \mathbb{N} \to \mathbb{R}^d$, 134
the shifted sequence $\psi_n^0(x_{j_1} - \tau_n, \ldots, x_{j_{|\beta|}} - \tau_n)$ also satisfies (6.6). Let $R_n > 0$ be a constant satisfying

$$\operatorname{supp} \psi_n^V \subset \left\{ x = (x_{j_1}, \ldots, x_{j_{|\beta|}}) \in \mathbb{R}^{d|\beta|} ||x_{j_i}| < R_n, j_i \in \beta, i = 1, \ldots, |\beta| \right\}.$$ 

We take $\tau$ such that

$$\operatorname{supp} \psi_n^0(\cdot - \tau_n, \ldots, \cdot - \tau_n) \subset \left\{ x = (x_{k_1}, \ldots, x_{k_{|\beta|}}) \in \mathbb{R}^{d|\beta|} ||x_{k_i}| \geq R_n + n, k_i \in \beta^c, i = 1, \ldots, |\beta| \right\}.$$ 

We set $\Psi_n(x_1 \cdots x_N) = \psi_n^V(x_{j_1} \cdots x_{j_{|\beta|}}) \otimes \psi_n^0(x_{k_1} - \tau_n \cdots x_{k_{|\beta|}} - \tau_n) \in L^2(\mathbb{R}^{dN})$. Then, for all $i, j$ with $i \in \beta, j \in \beta^c$, we have

$$\|V_{\text{eff},ij}(x_i - x_j)\Psi_n\| \leq \sup_{x \in \mathbb{R}^d, |x| > n} |V_{\text{eff},ij}(x)| \to 0, \quad (n \to \infty),$$

$$\|V_j(x_j)\Psi_n\| \leq \sup_{x \in \mathbb{R}^d, |x| \geq R_n + n} |V_j(x)| \to 0, \quad (n \to \infty).$$

Hence, by a triangle inequality, we have that

$$\|(H_{\text{eff}} - \mathcal{E}_V(\beta) - \mathcal{E}_0(\beta^c) - K)\Psi_n\| \to 0, \quad (n \to \infty).$$

Therefore $[\mathcal{E}_V(\beta) + \mathcal{E}_0(\beta^c) + K, \infty) \subset \sigma(H_{\text{eff}})$. Since $\beta \subset \subset C_N$ and $K > 0$ are arbitrary, $[\Xi_V, \infty) \subset \sigma_{\text{ess}}(H_{\text{eff}})$ follows. Thus the proof is complete.

We define $\Delta_p(\alpha_1, \ldots, \alpha_N) = \Xi_V - \mathcal{E}_V$.

**Corollary 6.7** Assume (V1) and (V.2). Then $\Delta_p(\alpha_1, \ldots, \alpha_N) > 0$ follows for $\alpha_j$ with $|\alpha_j| > \alpha_c, j = 1, \ldots, N$.

**Proof:** Since $\inf \sigma_{\text{ess}}(H_{\text{eff}}) = \Xi_V$ by Lemma 6.6 and $\inf \sigma(H_{\text{eff}}) \in \sigma_{\text{disc}}(H_{\text{eff}})$ by (V1), the corollary follows from $\Delta_p(\alpha_1, \ldots, \alpha_N) = \inf \sigma_{\text{ess}}(H_{\text{eff}}) - \inf \sigma(H_{\text{eff}}) > 0$. \hfill $\square$

**Lemma 6.8** For an arbitrary $\kappa > 0$, it follows that $\Sigma_V(\kappa) \geq \Xi_V$.

**Proof:** It is well known that $H_V(\beta)$ can be realized as a self-adjoint operator on a Hilbert space $\mathcal{H}_Q = L^2(\mathbb{R}^{d|\beta|}) \otimes L^2(Q, d\mu)$ with some probability space $(Q, \mu)$, which is called a Schrödinger representation. It is established in e.g., [LHB11] that

$$(\Psi, e^{-iH_V(\beta)}\Phi)_{\mathcal{H}_Q} \leq (||\Psi||, e^{-t(h_V(\beta) + \kappa^2H_i)}||\Phi||)_{\mathcal{H}_Q}.$$
Hence for any $\beta \subset C_N$, it follows that

$$\inf \sigma (h_V(\beta) + \kappa^2 H_t) \leq \inf \sigma (H_V(\beta)).$$

Since $\inf \sigma (H_t) = 0$ and $\inf \sigma (h_V(\beta) \otimes 1 + \kappa^2 1 \otimes H_t) = \inf \sigma (h_V(\beta))$, we can obtain

$$\inf \sigma (h_V(\beta)) \leq \inf \sigma (H_V(\beta))$$

for arbitrary $\beta \in C_N$. Then the lemma follows from the definition of lowest two cluster thresholds. $\Box$

**Lemma 6.9** Assume (V1). Then

$$E(\kappa) \leq \mathcal{E}_V + \kappa^{-2} \sum_{j=1}^{N} \alpha_j^2 \|\lambda_j\|^2 / (4m_j)$$

for $\alpha_j$ with $|\alpha_j| > \alpha_c$, $j = 1, ..., N$.

**Proof:** By (V1), $H_{\text{eff}}$ has a normalized ground state $u$ for $\alpha_j$ with $|\alpha_j| > \alpha_c$, $j = 1, ..., N$. Set $\Psi = u \otimes \Omega$. Then

$$E(\kappa) \leq \langle \Psi, H(\kappa) \Psi \rangle \leq \langle u, H_{\text{eff}} u \rangle + \sum_{j=1}^{N} \frac{\alpha_j}{2m_j \kappa} 2 \Re(i \nabla_j \Psi, A_j \Psi) + \sum_{j=1}^{N} \frac{\alpha_j^2}{2m_j \kappa^2} \|A_j \Psi\|^2$$

$$= \mathcal{E}_V + \sum_{j=1}^{N} \frac{\alpha_j^2}{4m_j \kappa^2} \|\lambda_j\|^2.$$ 

Here we used that $\langle \nabla_j \Psi, A_j \Psi \rangle = \frac{1}{\sqrt{2}} \sum_{\mu=1}^{d} \langle \nabla_{x_{\mu}} u \otimes \Omega, u \otimes a^* (k_{\mu} e^{-ik \cdot x} / \omega) \Omega \rangle = 0$. Then the lemma follows. $\Box$

**Proof of Theorem 6.2**

By Lemmas 6.8 and 6.9 we have

$$\Sigma_V(\kappa) - E(\kappa) \geq \Xi_V - \mathcal{E}_V - \sum_{j=1}^{N} \frac{\alpha_j^2}{4m_j \kappa^2} \|\lambda_j\|^2 = \Delta_p(\alpha_1, ..., \alpha_N) - \sum_{j=1}^{N} \frac{\alpha_j^2}{4m_j \kappa^2} \|\lambda_j\|^2.$$ 

Note that $\Delta_p(\alpha_1, ..., \alpha_N) > 0$ is continuous in $\alpha_1, ..., \alpha_N$. Then for a sufficiently large $\kappa$, there exists $\alpha_c(\kappa) > \alpha_c$ such that for $\alpha_c < |\alpha_j| < \alpha_c(\kappa)$, $j = 1, ..., N$, $\Sigma_V(\kappa) - E(\kappa) > 0$. Thus $H(\kappa)$ has a ground state for such $\alpha_j$’s by Proposition 6.4. $\Box$
6.3 Examples

6.3.1 Example of effective potential

We show a typical example of cutoff function and effective potentials. We introduce the assumption below. Let \( \lambda_j = \rho_j / \sqrt{\omega} \), \( j = 1, \ldots, N \), with rotation invariant nonnegative functions \( \rho_j \). In this case, by \([5.15]\), effective potential \( V_{\text{eff}} \) is explicitly computed as

\[
V_{\text{eff}}(x_1, \cdots, x_N) = -\frac{1}{4} \sum_{i \neq j}^N \alpha_i \alpha_j \sqrt{\frac{(2\pi)^d}{|x_i - x_j|^{(d-1)/2}}} \int_0^\infty \frac{r^{(d-1)/2}}{r^2} \rho_i(r) \rho_j(r) \sqrt{|x_i - x_j|} J_{\frac{d-2}{2}}(r) dr.
\]

(6.7)

Here \( J_\nu \) is the Bessel function: \( J_\nu(x) = (\frac{x}{2})^\nu \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n} \). We can see that \( V_{\text{eff}} \) satisfies that

1. \( V_{\text{eff}} \) is continuous,

2. \( \lim_{|x| \to \infty} V_{ij}(x) = 0 \),

3. \( V_{\text{eff}}(0) < V_{\text{eff}}(x) \) for all \( x \in \mathbb{R}^d \) but \( x \neq 0 \).

In particular, when \( d = 3 \) and \( \rho_j \) is the indicator function such as

\[
\rho_j(k) = \begin{cases} 
0 & |k| < \kappa, \\
\frac{1}{\sqrt{(2\pi)^3}} & \kappa < |k| < \Lambda, \\
0 & |k| \geq \Lambda,
\end{cases}
\]

(6.8)

we see that

\[
V_{\text{eff}}(x_1, \cdots, x_N) = -\frac{1}{8\pi^2} \sum_{i \neq j}^N \frac{\alpha_i \alpha_j}{|x_i - x_j|} \int_{|x_i - x_j|}^{\Lambda |x_i - x_j|} \frac{\sin r}{r} dr.
\]

(6.9)

For sufficiently small \( |x_i - x_j| \), \( i, j = 1, \ldots, N \), and \( \alpha_j \) with an identical sign, the effective potential \([6.9]\) is attractive.
6.3.2 Example of external potential

We give an example of $V_1, \ldots, V_N$ satisfying assumption (V1). Assume simply that $V_1 = \cdots = V_N = V$, $\alpha_1 = \cdots = \alpha_N = \alpha$, $\lambda_1 = \cdots = \lambda_N = \lambda$ and $m_1 = \cdots = m_N = m$. Then

$$V_{\text{eff}}_{ij}(x) = W(x) = -\frac{\alpha^2}{4} \int_{\mathbb{R}^d} \frac{\lambda(k)}{\omega(k)} |\lambda(k)|^2 e^{-ik \cdot x} dk$$

for all $i \neq j$. Let

$$h_V(\alpha) = \sum_{j=1}^N \left( -\frac{1}{2m} \Delta_j + V(x_j) \right) + \alpha^2 \sum_{j \neq l} W(x_j - x_l),$$

which acts on $L^2(\mathbb{R}^{dN})$. We assume (W1)-(W3) below:

(W1) $V$ is relatively compact with respect to the $d$-dimensional Laplacian $\Delta$, and $\sigma(-\Delta/2m + V) = [0, \infty)$.

(W2) $W$ satisfies that $-\infty < W(0) = \text{ess. inf} \, W(x) < \text{ess. inf} \, W(x)$ for all $\epsilon > 0$.

(W3) $\inf \sigma\left(-\Delta/(2Nm) + NV\right) \in \sigma_{\text{disc}}\left(-\Delta/(2Nm) + NV\right)$.

Remark 6.10 Note that examples of $V_{\text{eff}}$ given in (6.7) satisfies (W2), and remember that $\lim_{|x| \to \infty} W(x) = 0$ and $W(x)$ is relatively compact with respect to the $d$-dimensional Laplacian. See Lemma 6.5. The condition (W1) means that the external potential $V$ is shallow and the non-interacting Hamiltonian $h^V(0)$ has no negative energy bound state.

Theorem 6.11 Assume (W1)-(W3). Then, there exists $\alpha_c > 0$ such that for all $\alpha$ with $|\alpha| > \alpha_c$, $\inf \sigma(h_V(\alpha)) \in \sigma_{\text{disc}}(h_V(\alpha))$. Namely $h_V(\alpha)$ for $|\alpha| > \alpha_c$ has a ground state.

To prove Theorem 6.11 we need several lemmas. For $\beta \subset C_N$, we define

$$h_0(\alpha, \beta) = -\frac{1}{2m} \sum_{j \in \beta} \Delta_j + \alpha^2 \sum_{j,l \in \beta, j \neq l} W(x_j - x_l),$$

$$h_V(\alpha, \beta) = h_0(\alpha, \beta) + \sum_{j \in \beta} V(x_j),$$

$$\mathcal{E}_0(\alpha, \beta) = \inf \sigma(h_0(\alpha, \beta)),$$

$$\mathcal{E}_V(\alpha, \beta) = \inf \sigma(h_V(\alpha, \beta)).$$
where $E_V(\alpha, \emptyset) = 0$ and $E_0(\alpha, \emptyset) = 0$. Simply we set $E_V(\alpha, C_N) = E_V(\alpha)$ and $E_0(\alpha, C_N) = E_0(\alpha)$. Let $\Xi_V(\alpha)$ denote the lowest two cluster threshold of $h_V(\alpha)$ defined by (6.4). Then by (W1) and Lemma 6.6, we have

$$\sigma_{ess}(h_V(\alpha)) = [\Xi_V(\alpha), \infty).$$

(6.10)

**Lemma 6.12** Let $\beta \subsetneq C_N$ but $\beta \neq \emptyset$. Then there exists $a_1 > 0$ such that, for all $\alpha$ with $|\alpha| > a_1$,

$$E_0(\alpha) < E_V(\alpha, \beta) + E_0(\alpha, \beta^c).$$

(6.11)

**Proof:** Since $h_0(\alpha, \beta)/\alpha^2$ and $h_V(\alpha, \beta)/\alpha^2$ converge to $\sum_{j \in \beta} W(x_j - x_i)$ in the uniform resolvent sense, by (W2), one can show that

$$\lim_{\alpha \to \infty} \frac{E_V(\alpha, \beta)}{\alpha^2} = \lim_{\alpha \to \infty} \frac{E_0(\alpha, \beta)}{\alpha^2} = |\beta|(|\beta| - 1)W(0).$$

Hence

$$\lim_{\alpha \to \infty} \frac{E_0(\alpha)}{\alpha^2} = N(N - 1)W(0)$$

and

$$\lim_{\alpha \to \infty} \frac{1}{\alpha^2} (E_V(\alpha, \beta) + E_0(\alpha, \beta^c)) = \left\{ |\beta|(|\beta| - 1) + |\beta^c|(|\beta^c| - 1) \right\} W(0)$$

$$= \left\{ N(N - 1) + 2|\beta|(|\beta| - N) \right\} W(0).$$

Since $|\beta|(|\beta| - N) \leq -1$ and $W(0) < 0$ by (W2), we see that there exists $a_1 > 0$ such that (6.11) holds for all $\alpha$ with $|\alpha| > a_1$. □

Let $X = (x_1, \ldots, x_N)^t \in \mathbb{R}^{dN}$ and $Y = (x_c, y_1, \ldots, y_{N-1})^t$ be its Jacobi coordinates:

$$x_c = \frac{1}{N} \sum_{j=1}^N x_j, \quad y_j = x_{j+1} - \frac{1}{j} \sum_{i=1}^j x_i, \quad j = 1, \ldots, N - 1.$$
Let $T \in \text{GL}(N, \mathbb{R})$ be such that $Y = TX$. Note that

$$T = \begin{bmatrix}
\frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \cdots & \cdots & \frac{1}{N} \\
-1 & 1 & 0 & \cdots & \cdots & 0 \\
-\frac{1}{2} & -\frac{1}{3} & 1 & 0 & 0 & 0 \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
-\frac{1}{N-1} & -\frac{1}{N-1} & -\frac{1}{N-1} & \cdots & -\frac{1}{N-1} & 1
\end{bmatrix}$$

and

$$T^{-1} = \begin{bmatrix}
1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{5} & \cdots & \frac{1}{N} \\
1 & 1 & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{5} & \cdots & -\frac{1}{N} \\
1 & 0 & 1 & -\frac{1}{4} & -\frac{1}{5} & \cdots & -\frac{1}{N} \\
1 & 0 & 0 & 1 & -\frac{1}{5} & \cdots & -\frac{1}{N} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & \cdots & \cdots & 0 & \frac{N-2}{N-1} \\
1 & 0 & \cdots & \cdots & \cdots & 0 & \frac{N-1}{N}
\end{bmatrix}.$$ 

Matrix $T$ induces the unitary operator $U : L^2(\mathbb{R}^d_N) \to L^2(\mathbb{R}^d_N)$ defined by

$$(U\psi)(Y) = \psi \circ T^{-1}(Y).$$

We have

$$U h_0(\alpha) U^{-1} = -\frac{1}{2Nm} \Delta x - \sum_{j=1}^{N} \frac{1}{2\mu_j} \Delta y_j + \alpha^2 \sum_{j \neq l}^{N} W(x_j(Y) - x_l(Y)),$$

$$U h_V(\alpha) U^{-1} = U h_0(\alpha) U^{-1} + \sum_{j=1}^{N} V(x_j(Y)),$$

where $\mu_j = jm/(j + 1)$ is a reduced mass and $x_j(Y) = (T^{-1}Y)_j$. Let $k(\alpha)$ be $h_0(\alpha)$ with the center of mass motion removed:

$$k(\alpha) = -\sum_{j=1}^{N} \frac{1}{2\mu_j} \Delta y_j + \alpha^2 \sum_{j \neq l}^{N} W(x_j(Y) - x_l(Y)).$$

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Set $\mathbb{R}^{dN} = \mathbb{R}^d_x \oplus \mathbb{R}^{d(N-1)}_y = \chi_c \oplus \chi^\perp_c$. Since $x_j(Y) - x_i(Y)$, $i, j = 1, \ldots, N - 1$, depend only on $y_1, \ldots, y_{N-1} \in \chi^\perp_c$, $k(\alpha)$ is a self-adjoint operator acting in $L^2(\chi^\perp_c)$.

**Lemma 6.13** There exists $a_2 > 0$ such that $\inf \sigma(k(\alpha)) \in \sigma_{\text{disc}}(k(\alpha))$ for all $\alpha$ with $|\alpha| > a_2$.

**Proof:** Note that $\lim_{|x| \to \infty} W(x) = 0$. Let $\chi, \bar{\chi} \in C^\infty(\mathbb{R})$ be such that $\chi(x)^2 + \bar{\chi}(x)^2 = 1$ with $\chi(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 2. \end{cases}$ For a parameter $R$, we set

$$\chi_R(y) = \chi(|y_1|/R), \quad \bar{\chi}_R(y_1) = \bar{\chi}(|y_1|/R), \quad y_1 \in \mathbb{R}^d,$$

$$\theta_R(Y_1) = \chi(|Y_1|/2R), \quad \bar{\theta}_R(Y_1) = \bar{\chi}(|Y_1|/2R), \quad Y_1 = (y_2, \ldots, y_{N-1}) \in \mathbb{R}^{d(N-2)}.$$

By the IMS localization formula, we have

$$k(\alpha) = \chi_R \theta_R k(\alpha) \theta_R \chi_R + \chi_R \bar{\theta}_R k(\alpha) \bar{\theta}_R \chi_R + \bar{\chi}_R k(\alpha) \bar{\chi}_R + B(R), \quad (6.12)$$

where

$$B(R) = -\frac{1}{2} \chi^2_R |\nabla \theta_R|^2 - \frac{1}{2} \chi^2_R |\nabla \bar{\theta}_R|^2 - \frac{1}{2} |\nabla \chi_R|^2 - \frac{1}{2} |\nabla \bar{\chi}_R|^2.$$

Here $B(R) : L^2(\chi^\perp_c) \to L^2(\chi^\perp_c)$ is a bounded operator with the bound

$$\|B(R)\| \leq \frac{C}{R^2},$$

where $C$ is a constant independent of $R$. Let us define $k'(\alpha)$ by $k(\alpha)$ with the first term $\chi_R \theta_R k(\alpha) \theta_R \chi_R$ in (6.12) replaced by $\chi_R \theta_R (-\sum_{j=1}^N (1/2 \mu_j) \Delta y_j) \theta_R \chi_R$:

$$k'(\alpha) = \chi_R \theta_R \left( -\sum_{j=1}^N \frac{1}{2 \mu_j} \Delta y_j \right) \theta_R \chi_R + \chi_R \bar{\theta}_R k(\alpha) \bar{\theta}_R \chi_R + \bar{\chi}_R k(\alpha) \bar{\chi}_R + B(R).$$

Since the difference between $k(\alpha)$ and $k'(\alpha)$ is $\chi^2_R \theta^2_R \alpha^2 \sum_{j \neq l} W(x_j(Y) - x_l(Y))$, and which is relatively compact with respect to the kinetic term $-\sum_{j=1}^N (2 \mu_j)^{-1} \Delta y_j$ by Remark 6.10, we have $\sigma_{\text{ess}}(k(\alpha)) = \sigma_{\text{ess}}(k'(\alpha))$. Moreover $k'(\alpha)$ can be estimated
from below as
\[
\begin{align*}
k'(\alpha) &\geq \chi R^2 \beta R E \left( k(\alpha) - \alpha^2 W(x_2(Y) - x_3(Y)) - \alpha^2 W(x_3(Y) - x_2(Y)) \right) \tag{6.13} \\
&\quad + \chi R^2 \beta R \alpha^2 \left( W(x_2(Y) - x_3(Y)) + W(x_3(Y) - x_2(Y)) \right) \tag{6.14} \\
&\quad + \chi R E \left( k(\alpha) - \alpha^2 W(x_1(Y) - x_2(Y)) - \alpha^2 W(x_2(Y) - x_1(Y)) \right) \tag{6.15} \\
&\quad + \chi R^2 \alpha^2 \left( W(x_1(Y) - x_2(Y)) + W(x_2(Y) - x_1(Y)) \right) \tag{6.16} \\
&\quad - C/R^2. \tag{6.17}
\end{align*}
\]

Note that \( y_1 = x_2(Y) - x_1(Y) \) and \( x_3(Y) - x_2(Y) = y_2 - y_1/2 \). We have
\[
|\text{(6.14)}| \leq 2 \sup_{|y_1|<2R,|y_2|>4R} \alpha^2 |W(y_2 - y_1/2)| \leq 2\alpha^2 \sup_{|y|>3R} |W(y)|, \tag{6.14}
\]
\[
|\text{(6.16)}| \leq 2 \sup_{|y_1|>2R} \alpha^2 |W(y_1)|. \tag{6.16}
\]

Since we assume that \( \lim_{|x|\to\infty} W(x) = 0 \), we obtain that \( \lim_{R\to\infty} \|\text{(6.14)}\| = 0 \) and \( \lim_{R\to\infty} \|\text{(6.16)}\| = 0 \). Thus, for all \( R > 0 \) we have
\[
\inf \sigma_{ess}(k(\alpha)) = \inf \sigma_{ess}(k'(\alpha))
\geq \inf_{Y \in R^{4(N-1)}} \left[ \text{(6.13)} + \text{(6.15)} \right] - \|\text{(6.14)}\| - \|\text{(6.16)}\| - C/R^2
\geq \min \{ E(k(\alpha) - \alpha^2 W(x_1 - x_2) - \alpha^2 W(x_2 - x_1)) ,
\quad E(k(\alpha) - \alpha^2 W(x_2 - x_3) - \alpha^2 W(x_3 - x_2)) \} + o(R), \tag{6.18}
\]

where \( \lim_{R\to\infty} o(R)/R = 0 \). It is seen that
\[
\lim_{\alpha\to\infty} \frac{1}{\alpha^2} E \left( k(\alpha) - \alpha^2 W(x_1 - x_2) - \alpha^2 W(x_2 - x_1) \right) = [N(N-1) - 2]W(0), \tag{6.19}
\]
\[
\lim_{\alpha\to\infty} \frac{1}{\alpha^2} E \left( k(\alpha) - \alpha^2 W(x_2 - x_3) - \alpha^2 W(x_3 - x_2) \right) = [N(N-1) - 2]W(0), \tag{6.20}
\]
\[
\lim_{\alpha\to\infty} \frac{E(k(\alpha))}{\alpha^2} = N(N-1)W(0). \tag{6.21}
\]

Therefore combining (6.18)-(6.21) we obtain that
\[
\lim_{\alpha\to\infty} \frac{1}{\alpha^2} \left( \inf \sigma_{ess}(k(\alpha)) - \inf \sigma(k(\alpha)) \right) \geq -2W(0). \tag{6.22}
\]
Since $W(0) < 0$ by (W2), there exists $a_2 > 0$ such that $\inf \sigma_{ess}(k(\alpha)) - \inf \sigma(k(\alpha)) > 0$ for $|\alpha| > a_2$. This implies the desired result.

\begin{lemma}
Let $u_{\alpha}$ be a normalized ground state of $k(\alpha)$, where $|\alpha| > a_2$. Then $|u_{\alpha}(y_1, \ldots, y_{N-1})|^2 \to \delta(y_1) \cdots \delta(y_{N-1})$ as $\alpha \to \infty$ in the sense of distributions.

\begin{proof}
It suffices to show that for all $\epsilon > 0$,
\[
\lim_{\alpha \to \infty} \int_{|Y_0| > \epsilon} |u_{\alpha}(Y_0)|^2 dY_0 = 0, \quad Y_0 = (y_1, \ldots, y_{N-1}). \quad (6.23)
\]
We prove (6.23) by a reductive absurdity. Assume that
\[
\lim \inf \ell \to \infty \int_{|Y_0| > \epsilon} |u_{\alpha_{\ell}}(Y_0)|^2 dY_0 > 0
\]
for some constant $\epsilon > 0$ and some sequence $\{\alpha_{\ell}\}_{\ell=1}^{\infty} \subset \mathbb{R}$ such that $\alpha_{\ell} \to \infty (\ell \to \infty)$.
We can take a subsequence $\{\hat{\alpha}_{\ell}\}_{\ell=1}^{\infty} \subset \{\alpha_{\ell}\}_{\ell=1}^{\infty}$ so that
\[
\gamma = \lim \ell \to \infty \int_{|Y_0| > \epsilon} |u_{\hat{\alpha}_{\ell}}(Y_0)|^2 dY_0 > 0.
\]
Since $k(\alpha)/\alpha^2 \geq N(N-1)W(0)$ and $\lim_{\alpha \to \infty} E(k(\alpha)/\alpha^2) = N(N-1)W(0)$, we have
\[
N(N-1)W(0) = \lim_{\ell \to \infty} \frac{1}{\hat{\alpha}_{\ell}^2} (u_{\hat{\alpha}_{\ell}}, k(\hat{\alpha}_{\ell})u_{\hat{\alpha}_{\ell}}) = \lim_{\ell \to \infty} \left( u_{\hat{\alpha}_{\ell}}, \sum_{j \neq l}^{N} W(x_j(Y_0) - x_l(Y_0))u_{\hat{\alpha}_{\ell}} \right)
\geq (1 - \gamma)N(N-1)W(0) + \gamma \inf_{|Y_0| > \epsilon} \sum_{j \neq l}^{N} W(x_j(Y_0) - x_l(Y_0))
\geq N(N-1)W(0).
\]
Thus we have
\[
\inf_{|Y_0| > \epsilon} \sum_{j \neq l}^{N} W(x_j(Y_0) - x_l(Y_0)) = N(N-1)W(0). \quad (6.24)
\]
By (W2) and (6.24) there exists a sequence $Z_n = (z_{1,n}, \ldots, z_{(N-1),n}) \in \mathbb{R}^{d(N-1)}$ such that $|Z_n| > \epsilon$ and $\lim_{n \to \infty} (x_j(Z_n) - x_l(Z_n)) \to 0$ for $j \neq l$. By the definition of
Let $u_\alpha$ be a ground state of $k(\alpha) = U h_0(\alpha) U^{-1}$. By Proposition 6.17, we may assume that $V \in C_0^\infty(\mathbb{R}^d)$. Let $|\alpha| > a_2$. Let $v \in C_0^\infty(\mathbb{R}^d)$ be a normalized vector such that
\[
\left(v, \left(-\frac{1}{2N} \Delta x_c + N V(x_c)\right)v\right) < 0.
\] (6.25)
Such a vector exists by (W3). We set $\Psi(Y) = \Psi(x_c, Y_0) = v(x_c) u_\alpha(y_1, \ldots, y_{N-1}) \in \mathbb{R}^{dN}$. Then
\[
(\Psi, U h_V(\alpha) U^{-1} \Psi) = -\frac{1}{2mN} (v, \Delta x_c v) + \mathcal{E}_0(\alpha) + \left(\Psi, \sum_{j=1}^{N} V(x_j(Y)) \Psi\right).
\] (6.26)
We define
\[
V_{j,*}^\alpha(x_c) = \int_{\mathbb{R}^{d(N-1)}} dy_1 \cdots dy_{N-1} V(x_j(Y)) |u_\alpha(y_1, \ldots, y_{N-1})|^2, \quad j = 1, \ldots, N.
\]
By Lemma 6.14, we have
\[
\lim_{\alpha \to \infty} \left(\Psi, \sum_{j=1}^{N} V(x_j(Y)) \Psi\right) = \lim_{\alpha \to \infty} \sum_{j=1}^{N} (v, V_{j,*}^\alpha v) = (v, NV(x_c)v).
\]
Therefore, by (6.25) and (6.26), $(\Psi, h_V(\alpha) \Psi) < \mathcal{E}_0(\alpha)$ for $|\alpha| > a_3$ with some $a_3 > 0$. By this inequality, Lemma 6.12 and (6.10), we conclude that for $\alpha$ with $|\alpha| > \alpha_c = \max\{a_1, a_3\}$,
\[
\Xi_V(\alpha) - \mathcal{E}_V(\alpha) \geq \mathcal{E}_0(\alpha) - \mathcal{E}_V(\alpha) > 0.
\]
Then the theorem follows. \qed

We give a general lemma.
Lemma 6.15 Let $K_\epsilon$, $\epsilon > 0$, and $K$ be self-adjoint operators on a Hilbert space $\mathcal{K}$ and $\sigma_{\text{ess}}(K_\epsilon) = [\xi, \infty)$. Suppose that $\lim_{\epsilon \to 0} K_\epsilon = K$ in the uniform resolvent sense, and $\lim_{\epsilon \to 0} \xi_\epsilon = \xi$. Then $\sigma_{\text{ess}}(K) = [\xi, \infty)$. In particular $\lim_{\epsilon \to 0} \inf \sigma_{\text{ess}}(K_\epsilon) = \inf \sigma_{\text{ess}}(K)$.

Proof: Let $a > \xi$. Then there exists $\epsilon_0$ such that for all $\epsilon$ with $\epsilon < \epsilon_0$, $\xi_\epsilon < a$, from which we have $a \in \sigma(K_\epsilon)$ for all $\epsilon < \epsilon_0$. Since $K_\epsilon$ uniformly converges to $K$ in the resolvent sense, $a \in \sigma(K)$ follows from [RS80, Theorem VIII.23 and p.291]. Since $a$ is arbitrary, $(\xi, \infty) \subset \sigma(K)$ follows and then

$$[\xi, \infty) \subset \sigma_{\text{ess}}(K).$$

It is enough to show $\inf \sigma_{\text{ess}}(K) = \xi$. Let $\lambda \in [\inf \sigma_{\text{ess}}(K), \xi)$ but $\lambda \notin \sigma(K)$. Note that for all sufficiently small $\epsilon$, $\lambda \notin \sigma(K_\epsilon)$ by [RS80, Theorem VIII.24]. Since $\mathbb{R} \setminus \sigma(K)$ is an open set, there exists $\delta > 0$ such that $(\lambda - \delta, \lambda + \delta) \notin \sigma(K)$. Let $P_A(T)$ denote the spectral projection of a self-adjoint operator $T$ on a Borel set $A \subset \mathbb{R}$. We have $\lim_{\epsilon \to 0} P_{[\inf \sigma_{\text{ess}}(K), \delta']}(K_\epsilon) = P_{[\inf \sigma_{\text{ess}}(K) - \delta', \lambda]}(K)$ uniformly by [RS80, Theorem VIII.23(b)]. In particular, for some $\delta' > 0$,

$$\|P_{[\inf \sigma_{\text{ess}}(K), \delta']}(K_\epsilon) - P_{[\inf \sigma_{\text{ess}}(K) - \delta', \lambda]}(K)\| < 1.$$

Then $P_{[\inf \sigma_{\text{ess}}(K), \delta']}(K_\epsilon)\mathcal{K}$ is isomorphic to $P_{[\inf \sigma_{\text{ess}}(K) - \delta', \lambda]}(K)\mathcal{K}$. Hence the dimension of $P_{[\inf \sigma_{\text{ess}}(K), \delta']}(K_\epsilon)\mathcal{K}$ is finite, since that of $P_{[\inf \sigma_{\text{ess}}(K) - \delta', \lambda]}(K)\mathcal{K}$ is finite. Thus $(\inf \sigma_{\text{ess}}(K) - \delta', \lambda) \cap \sigma(K) \subset \sigma_{\text{disc}}(K)$. This is a contradiction. Hence we have $[\inf \sigma_{\text{ess}}(K), \xi) \subset \sigma(K)$. Suppose that $\inf \sigma_{\text{ess}}(K) < \xi$. Let $\tau > 0$ be sufficiently small. Note that $(\inf \sigma_{\text{ess}}(K) - \tau, \inf \sigma_{\text{ess}}(K) + \tau) \subset \sigma_{\text{disc}}(K_\epsilon)$ for all sufficiently small $\epsilon$. Let $\theta \in C_0^\infty(\mathbb{R})$ satisfy that

$$\theta(z) = \begin{cases} 1, & |z - \inf \sigma_{\text{ess}}(K)| < \tau, \\ 0, & |z - \inf \sigma_{\text{ess}}(K)| > 2\tau. \end{cases}$$

Then we have $\lim_{\epsilon \to 0} \theta(K_\epsilon) = \theta(K)$ uniformly by [RS80, Theorem VIII.20]. Since $\theta(K_\epsilon)$ is a finite rank operator for all sufficiently small $\epsilon$, $\theta(K)$ has to be a compact operator. It contradicts with the fact, however, that the spectrum of $\theta(K)$ is continuous. Then we can conclude that $\inf \sigma_{\text{ess}}(K) = \xi$ and the proof is complete. \(\Box\)

Let $V : \mathbb{R}^d \to \mathbb{R}$ be a real-valued measurable function.
Lemma 6.16 Let $\Delta$ be the $d$-dimensional Laplacian. Assume that $V(-\Delta + 1)^{-1}$ is a compact operator. Then there exists a sequence $\{V^\epsilon\}_{\epsilon > 0}$ such that $V^\epsilon \in C_0^\infty(\mathbb{R}^d)$ and $\lim_{\epsilon \to 0} V^\epsilon(-\Delta + 1)^{-1} = V(-\Delta + 1)^{-1}$ uniformly.

Proof: Generally, let $A$ be a compact operator and $\{B_n\}_n$ bounded operators such that $s-lim_{n \to \infty} B_n = 0$, then $B_n A \to 0$ as $n \to \infty$ in the operator norm. Since $V(-\Delta + 1)^{-1}$ is a compact operator, we obtain that for a sufficiently large $R > 0$,

$$
\|(1 - \chi_R) V(-\Delta + 1)^{-1}\| < \epsilon/3, \quad (6.27)
$$

where $\chi_R$ characteristic function of $\{x \in \mathbb{R}^d | |x| < R\}$. Let $\chi^{(n)}$ denote the characteristic function of $\{x \in \mathbb{R}^d | V(x) < n\}$. Since $(1 - \chi^{(n)}) \to 0$ strongly as $n \to \infty$,

$$
\|(1 - \chi^{(n)}) \chi_R V(-\Delta + 1)^{-1}\| < \epsilon/3 \quad (6.28)
$$

for a sufficiently large $n$. Since $C_0^\infty((\mathbb{R}^d))$ is dense in $L^2(\mathbb{R}^d)$, there exists a sequence $\{V_m\}_m \subset C_0^\infty((\mathbb{R}^d))$

$$
\|V_m - \chi_R \chi^{(n)} V\|_{L^2(\mathbb{R}^d)} \to 0
$$

as $m \to \infty$. Since $\chi_R \chi^{(n)} V$ has a compact support and is bounded, we obtain that $s-lim_{m \to \infty} V_m = \chi_R \chi^{(n)} V$ as an operator. Thus for a sufficiently large $m$,

$$
\|(V_m - \chi_R \chi^{(n)} V)(-\Delta + 1)^{-1}\| < \epsilon/3. \quad (6.29)
$$

By $(6.27)$-$(6.29)$ we can obtain that for an arbitrary $\epsilon > 0$, $\|(V - V_m)(-\Delta + 1)^{-1}\| < \epsilon$ for a sufficiently large $m$. Thus the lemma follows by setting $V_m = V^\epsilon$. \qed

Let $\beta \subset C_N$. Set

$$
k_0(\beta) = -\sum_{j \in \beta} \frac{1}{2m_j} \Delta_j + \sum_{i,j \in \beta} V_{ij}, \quad k_V(\beta) = h_0(\beta) + \sum_{j \in \beta} V_j
$$

with $V_i \in L^2_{loc}(\mathbb{R}^d)$ and $V_{ij} \in L^2_{loc}(\mathbb{R}^d)$ such that $V_i(-\Delta + 1)^{-1}$ and $V_{ij}(-\Delta + 1)^{-1}$ are compact operators. We define $K = k_V(C_N)$. Let

$$
\Xi_V = \min_{\beta \subset C_N} \{\inf_{\beta \subset C_N} \sigma(k_0(\beta)) + \inf_{\beta \subset C_N} \sigma(k_V(\beta))\} \quad (6.30)
$$

be the lowest two cluster threshold of $K$. 146
Proposition 6.17 There exist sequences \( \{ V_i^\epsilon \} \), \( \{ V_{ij}^\epsilon \} \subset C_0^\infty (\mathbb{R}^d) \), \( i, j = 1, ..., N \), such that

\[
(1) \lim_{\epsilon \to 0} \Xi V(\epsilon) = \Xi V, \quad (2) \lim_{\epsilon \to 0} \inf \sigma_{\text{ess}}(K(\epsilon)) = \inf \sigma_{\text{ess}}(K),
\]

where \( \Xi V(\epsilon) \) (resp. \( K(\epsilon) \)) is \( \Xi V \) (resp. \( K \)) with \( V_i \) and \( V_{ij} \) replaced by \( V_i^\epsilon \) and \( V_{ij}^\epsilon \), respectively.

Proof: By Lemma 6.16 there exist sequences \( \{ V_i^\epsilon \}_{\epsilon > 0}, \{ V_{ij}^\epsilon \}_{\epsilon > 0} \subset C_0^\infty (\mathbb{R}^d) \), such that

\[
V_i^\epsilon (x_i)(-\Delta_i + 1)^{-1} \to V_i(x_i)(-\Delta_i + 1)^{-1}
\]

and

\[
V_{ij}^\epsilon (x_i - x_j)(-\Delta_i - \Delta_j + 1)^{-1} \to V_{ij}(x_i - x_j)(-\Delta_i - \Delta_j + 1)^{-1}
\]

uniformly as \( \epsilon \to 0 \) for \( i, j = 1, ..., N \). Hence \( \inf \sigma (k_V(\epsilon)) \) and \( \inf \sigma (k_0(\epsilon)) \) converge to \( \inf \sigma (k_V) \) and \( \inf \sigma (k_0) \) as \( \epsilon \to 0 \), respectively. Then (1) follows from the definition (6.30). By this and the uniform convergence of \( K(\epsilon) \) to \( K \) in the resolvent sense, Lemma 6.15 yields (2). \( \square \)
7 Absence of ground state

7.1 Introduction

7.1.1 Stability and the decay of variable mass

In this section we study a general version of the Nelson model, i.e., the Nelson model with a variable coefficients. This model is an extension of the standard Nelson model when the Minkowskian space-time is replaced by a static pseudo Riemannian manifold. It is studied in the series of papers [GHPS09, GHPS11, GHPS12-a, GHPS12-b].

In this section the absence of ground state of the Nelson model with variable coefficients is discussed under infrared singularity condition. Throughout this section we assume that

\[ d = 3. \]

The Hamiltonian of the Nelson Hamiltonian with a variable coefficients is defined formally by

\[
H = \frac{1}{2} \sum_{\mu, \nu=1}^{3} D_\mu A_{\mu \nu}(x) D_\nu + W(x) + \int \omega(D, x) a^*(x) a(x) dx \\
+ \frac{1}{\sqrt{2}} \int \omega^{-1/2}(D, x) \rho(x - x) (a^*(x) + a(x)) dx,
\]

(7.1)

where \( a(x) \) and \( a^*(x) \) the annihilation operator and the creation operator in the position representation, respectively, \( \rho \) a nonnegative cutoff function and \( \omega = h^{1/2} \) a dispersion relation with a position dependent variable mass \( m(x) \):

\[
h = h(D, x) = \sum_{\mu, \nu=1}^{3} c(x)^{-1} D_\mu a_{\mu \nu}(x) D_\nu c(x)^{-1} + m^2(x).
\]

(7.2)

We give examples of (7.2) in the next section. In [GHPS11] the existence of ground states of \( H \) is shown when

\[
m(x) \geq a \langle x \rangle^{-1},
\]

(7.3)

where \( \langle x \rangle = (1 + |x|^2)^{1/2} \). Then we study the case of

\[
m(x) \leq a \langle x \rangle^{-\beta/2}, \quad \beta < 2
\]

in this lecture note. See Figure [12].
The standard Nelson model is defined by $H$ with $A_{\mu\nu}(x)$ replaced by $\delta_{\mu\nu}$, $a_{\mu\nu}(x)$ by $\delta_{\mu\nu}$ and $m(x)$ by a constant $m \geq 0$. By the condition $\rho \geq 0$, $\hat{\rho}(0) > 0$ follows, and the integral $\int |\hat{\rho}(k)|^2/\omega(k)^3 dk$ is finite if and only if $m > 0$ since $d = 3$. Thus $m > 0$ corresponds to the infrared regular condition and $m = 0$ to the infrared singular condition.

### 7.1.2 Klein-Gordon equation on pseudo Riemannian manifold

In quantum field theory the dispersion relation $\omega = \sqrt{-\Delta + m^2}$ can be derived from the Klein-Gordon equation:

$$\frac{\partial^2 \phi}{\partial t^2} = -\omega^2 \phi. \quad (7.4)$$

On the other hand the dispersion relation with variable coefficients can be derived from the Klein-Gordon equation on a pseudo Riemannian manifold. We here give an example of a Klein-Gordon equation defined on a static pseudo Riemannian manifold $\mathcal{M}$ such that a short range potential $v(x) = O((x)^{-\beta-2})$ appears.

Let $\vec{x} = (t, x) = (x_0, x) \in \mathbb{R} \times \mathbb{R}^3$ and $\mathcal{M}$ the 4 dimensional pseudo Riemannian manifold equipped with the metric tensor:

$$g(\vec{x}) = g(x) = \begin{pmatrix}
    e^{-\theta(x)} & 0 & 0 & 0 \\
    0 & -e^{-\theta(x)} & 0 & 0 \\
    0 & 0 & -e^{-\theta(x)} & 0 \\
    0 & 0 & 0 & -e^{-\theta(x)}
\end{pmatrix}. \quad (7.5)$$

Note that $g$ depends on $x$ but independent of $t$. The line element associated with $g$ is given by

$$ds^2 = e^{-\theta(x)} dt \otimes dt - e^{-\theta(x)} \sum_{j=1}^3 dx^j \otimes dx^j. \quad (7.6)$$

![Table](image)

| Condition | Existence | Absence |
|-----------|-----------|---------|
| $m(x) \geq a(x)^{-1}$ | Exist | Not exist |
| $m(x) \leq a(x)^{-\beta}$, $\beta > 1$ | Not exist | Exist |

Figure 12: Existence and absence of ground state
The Klein-Gordon equation on $\mathcal{M}$ is

$$\Box_g \phi + m^2 \phi = 0, \quad (7.7)$$

where the d’Alembertian operator is defined by

$$\Box_g = e^{\theta(x)} \partial_t^2 - e^{2\theta(x)} \sum_j \partial_j e^{-\theta(x)} \partial_j. \quad (7.8)$$

Thus the Klein-Gordon equation (7.7) is reduced to the equation

$$\frac{\partial^2 \phi}{\partial t^2} = K_0 \phi, \quad (7.9)$$

where

$$K_0 = e^{\theta(x)} \sum_j \partial_j e^{-\theta(x)} \partial_j - e^{-\theta(x)} m^2. \quad (7.10)$$

The operator $K_0$ is symmetric on the weighted $L^2$ space $L^2(\mathbb{R}^d; e^{-\theta(x)} dx)$. Now we transform the operator $K_0$ to the one on $L^2(\mathbb{R}^3)$. This is done by the unitary map $U_0 : L^2(\mathbb{R}^d; e^{-\theta(x)} dx) \to L^2(\mathbb{R}^d), f \mapsto e^{-(1/2)\theta} f$.

**Lemma 7.1** There exist functions $\theta$ and $v$ such that $U_0 K_0 U_0^{-1} = \Delta - v$, $v(x) = O(\langle x \rangle^{-\beta - 2})$ for $\beta \geq 0$, and $-\Delta + v$ has no non-positive eigenvalues.

Hence the Klein-Gordon equation (7.9) is transformed to the equation

$$\frac{\partial^2 \phi}{\partial t^2} - \Delta \phi + v \phi = 0 \quad (7.11)$$

on $L^2(\mathbb{R}^3)$, and the dispersion relation is given by $\sqrt{-\Delta + v}$. Although the proof of Lemma 7.1 is straightforward, we shall show this statement through a more general scheme in what follows.

Suppose that $g = (g_{\mu\nu}), \mu, \nu = 0, 1, 2, 3$, is a metric tensor on $\mathbb{R}^4$ such that

1. $g_{\mu\nu}(x) = g_{\mu\nu}(x), \text{ i.e., it is independent of time } t$,
2. $g_{0j}(x) = g_{j0}(x) = 0, j = 1, 2, 3$,
3. $g_{ij}(x) = -\gamma_{ij}(x)$, where $\gamma = (\gamma_{ij})$ denotes a 3-dimensional Riemannian metric.
Namely
\[ g = \begin{bmatrix} g_{00} & 0 \\ 0 & -\gamma \end{bmatrix}. \tag{7.12} \]

Let \( \mathcal{M} \) be a pseudo Riemannian manifold equipped with the metric tensor \( g \) satisfying (1)-(3) above. Then the line element on \( \mathcal{M} \) is given by
\[ ds^2 = g_{00}(x)dt \otimes dt - \sum_{i,j=1}^{3} \gamma_{ij}(x)dx^i \otimes dx^j. \]

Let \( g^{-1} = (g^{\mu\nu}) \) denote the inverse of \( g \). In particular \( 1/g_{00} = g^{00} \). We also denote the inverse of \( \gamma \) by \( \gamma^{-1} = (\gamma^{ij}) \). The Klein-Gordon equation on the static pseudo Riemannian manifold \( \mathcal{M} \) is generally given by
\[ \Box_g \phi + (m^2 + \eta R) \phi = 0, \tag{7.13} \]
where \( \eta \) is a constant, \( R \) the scalar curvature of \( \mathcal{M} \), and \( \Box_g \) is the d’Alembertian operator on \( \mathcal{M} \), which is given by
\[ \Box_g = \sum_{\mu,\nu=0}^{3} \frac{1}{\sqrt{|\det g|}} \partial_{\mu} g^{\mu\nu} \sqrt{|\det g|} \partial_{\nu}. \tag{7.14} \]

Let us assume that \( g_{00}(x) > 0 \). Then (7.13) is rewritten as
\[ \frac{\partial^2 \phi}{\partial t^2} = K \phi, \tag{7.15} \]
where
\[ K = g_{00} \left( \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{3} \partial_j \sqrt{|\det g|} \gamma^{ji} \partial_i - m^2 - \eta R \right). \tag{7.16} \]

The operator \( K \) is symmetric on \( L^2(\mathbb{R}^3; \rho(x)dx) \), where
\[ \rho = \frac{\sqrt{|\det g|}}{g_{00}} = g_{00}^{-1/2} \sqrt{|\det \gamma|}. \tag{7.17} \]

Now let us transform the operator \( K \) on \( L^2(\mathbb{R}^3; \rho(x)dx) \) to the one on \( L^2(\mathbb{R}^3) \). Define the unitary operator \( U : L^2(\mathbb{R}^3; \rho(x)dx) \to L^2(\mathbb{R}^3) \) by
\[ Uf = \rho^{3/2} f. \tag{7.18} \]
Let \( \rho_i = \partial_i \rho \) and \( \partial_i \partial_j \rho = \rho_{ij} \) for notational simplicity. Furthermore we set \( \alpha^{ij} = g_{00} \gamma^{ij} \) and \( \partial_k \alpha^{ij} = \alpha^{ij}_k \). Since \( U^{-1} \partial_j U = \partial_j + \rho_j - \frac{\rho_j}{2 \rho} \), we see that as an operator identity

\[
U^{-1} \left( \sum_{i,j=1}^{3} \partial_i g_{00} \gamma^{ij} \partial_j \right) U = g_{00} \sum_{i,j=1}^{3} \gamma^{ij} \partial_i \partial_j + V_1 + V_2, \quad (7.19)
\]

where

\[
V_1 = \sum_{i,j=1}^{3} \left( \alpha^{ij}_i + \frac{n^{ij} \rho_i}{\rho} \right) \partial_j,
\]

\[
V_2 = \frac{1}{4} \sum_{i,j=1}^{3} \left( 2 \alpha^{ij} \frac{\rho_j}{\rho} + 2 \alpha^{ij} \frac{\rho_{ij} \rho_i}{\rho} - \alpha^{ij} \frac{\rho_{ij} \rho_i}{\rho} \right).
\]

Set \(|\det g| = G\) and \( \partial_i G = G_i \). Hence we have

\[
V_1 = g_{00} \sum_{i,j=1}^{3} \left( \gamma^{ij}_i + \frac{G_i}{2G} \right) \partial_j,
\]

where \( \gamma^{ij}_i = \partial_i \gamma^{ij} \), and directly we can see that

\[
g_{00} \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{3} \partial_i \sqrt{|\det g|} \gamma^{ij} \partial_j = V_1 + g_{00} \sum_{i,j=1}^{3} \gamma^{ij} \partial_i \partial_j. \quad (7.20)
\]

Comparing \((7.19)\) with \((7.20)\) we obtain that

\[
U^{-1} \left( \sum_{i,j=1}^{3} \partial_i g_{00} \gamma^{ij} \partial_j - V_2 \right) U = g_{00} \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{3} \partial_i \sqrt{|\det g|} \gamma^{ij} \partial_j. \quad (7.21)
\]

Then we proved the lemma below.

**Lemma 7.2** It follows that

\[
UKU^{-1} = \sum_{i,j=1}^{3} \partial_i g_{00} \gamma^{ij} \partial_j - v, \quad (7.22)
\]

where \( v = g_{00}(m^2 + \eta \mathcal{R}) + V_2 \).
By Lemma 7.2 (7.15) is transformed to the equation:

$$\frac{\partial^2 \phi}{\partial t^2} = \left( \sum_{i,j=1}^{3} \partial_i g_{00} \gamma^{ij} \partial_j - v \right) \phi$$

(7.23)
on $L^2(\mathbb{R}^d)$.

**Proof of Lemma 7.1** Now we come back to the proof of Lemma 7.1. Set

$$g_{\mu \nu}(x) = \begin{cases} e^{-\theta(x)}, & \mu = \nu = 0, \\ -e^{-\theta(x)}, & \mu = \nu = 1, 2, 3, \\ 0, & \mu \neq \nu. \end{cases}$$

Then

$$\rho = \frac{\sqrt{|\det g|}}{g_{00}} = e^{-\theta}, \quad \alpha^{ij} = g_{00} \gamma^{ij} = \delta_{ij},$$

(7.24)
and $UKU^{-1} = \Delta - v$ follows by (7.22), where, inserting (7.24) to $v$, we have

$$v = e^{-\theta}(m^2 + \eta R) - \frac{\Delta \theta}{2} + \frac{|
abla \theta|^2}{4}. \quad (7.25)$$

Taking $\eta = 0$, $m = 0$, and $\theta(x) = 2a \langle x \rangle^{-\beta}$, we obtain

$$v(x) = -a \langle x \rangle^{-\beta-4}(\beta(\beta - 1)|x|^2 - 3\beta) + a^2 \beta^2 \langle x \rangle^{-2\beta-4}|x|^2. \quad (7.26)$$

In the case of $0 \leq \beta \leq 1$ and $a > 0$, we see that $v \geq 0$ and $v = O(\langle x \rangle^{-\beta - 2})$. Furthermore $-\Delta + v$ has no non-positive eigenvalues. In the case of $\beta > 1$ and $a < 0$, we see that however $v \not\geq 0$. We can estimate the number of non-positive eigenvalues of $-\Delta + v$ by the Lieb-Thirring inequality [Lie76]:

$$\# \{\text{eigenvalues of } -\Delta + v \leq 0\} \leq a_3 \int |v_-(x)|^{3/2} dx, \quad (7.27)$$

where $v_-$ denotes the negative part of $v$ and $a_3$ is a constant independent of $v$. This yields that $-\Delta + v$ has no non-positive eigenvalues for sufficiently small $a$. Thus the lemma holds. 

$\square$
7.2 The Nelson model with a variable mass

7.2.1 Dirichlet forms and symmetric semigroups

Before going to study the Nelson Hamiltonian with variable coefficients we review fundamental properties of Dirichlet forms and symmetric semigroups, which will be used in the next sections. The general reference in this section is [Dav89].

We assume that the dimension of the configuration space is $d$. Let $(E,D)$ be a symmetric quadratic form on $L^2(\mathbb{R}^d)$ with a form domain $D$. $(E,D)$ is Markovian if and only if for arbitrary $\epsilon > 0$, there exists $\rho_\epsilon$ such that

1. $\rho_\epsilon(t) = t$ for $t \in [0, 1]$, $-\epsilon \leq \rho_\epsilon(t) \leq 1 + \epsilon$ for all $t \in \mathbb{R}$, $0 \leq \rho_\epsilon(t) - \rho_\epsilon(s) \leq t - s$ for $s < t$,

2. $\rho_\epsilon \circ f \in D$ and $E(\rho_\epsilon \circ f, \rho_\epsilon \circ f) \leq E(f, f)$ holds for $f \in D$.

A Markovian closed symmetric form $(E,D)$ is called the Dirichlet form. When $C_0^{\infty}(\mathbb{R}^d)$ is a form core of the Dirichlet form $(E,D)$, it is called a regular Dirichlet form. When $f, g \in D$ satisfies $\text{supp} f \cap \text{supp} g = \emptyset$, $E(f, g) = 0$. Then $(E,D)$ is called a local Dirichlet form.

Let $g_{\mu\nu} \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $(g_{\mu\nu}(x))_{1 \leq \mu, \nu \leq d} = g(x)$ satisfy

$$\lambda_1(x) \mathbb{I} \leq g(x) \leq \lambda_2(x) \mathbb{I}$$

with strictly positive continuous functions $\lambda_j$. Define

$$E_g(f, g) = \sum_{\mu, \nu = 1}^{d} \int_{\mathbb{R}^d} g_{\mu\nu}(x) \partial_{\nu} f(x) \partial_{\nu} g(x) dx$$

for $f, g \in C_0^{\infty}(\mathbb{R}^d)$.

**Proposition 7.3** $E_g$ is closable quadratic form on $C_0^{\infty}(\mathbb{R}^d)$.

**Proof:** See [Dav89, Theorem 1.2.6]. \(\square\)

We denote the closure by $\bar{E}_g$.

**Proposition 7.4** Let $L$ be the self-adjoint operator associated with $\bar{E}_g$. Then

1. $e^{-tL}$, $t \geq 0$, is contraction from $L^p(\mathbb{R}^d)$ to itself for all $1 \leq p \leq \infty$,
(ii) \( e^{-tL}, t \geq 0, \) is positivity preserving.

**Proof:** See [Dav89, Theorem 1.3.5]. □

**Proposition 7.5** Suppose that \( K > 0 \) be a self-adjoint operator such that \( e^{-tK} \) is positivity preserving and \( e^{-tK} \) is bounded on \( L^\infty(\mathbb{R}^d) \). Let \( \mathcal{E} \) denote the quadratic form associated with \( K \). Then

1. A bound of the form
   \[
   \|e^{-tK}f\|_\infty \leq C_1 t^{-\alpha/4} \|f\|_2
   \]
   with \( \alpha > 2 \) for all \( t \geq 0 \) and all \( f \in L^2(\mathbb{R}^d) \) is equivalent to
   \[
   \|f\|_{2(\alpha/(\alpha-2)}^2 \leq C_2 \mathcal{E}(f,f).
   \]

2. Suppose a bound \( \|e^{-tK}f\|_\infty \leq C_1 \|f\|_2 \) follows for all \( t \geq 0 \) and all \( f \in L^2(\mathbb{R}^d) \). Then \( e^{-tK}f \) has an integral kernel \( e^{-tK}(x,y) \) for all \( t \geq 0 \) which satisfies that
   \[0 \leq e^{-tK}(x,y) \leq C_2 t/2 \]
   almost everywhere.

**Proof:** See [Dav89, Theorem 2.4.2] for (1), and [Dav89, Lemma 2.1.2] for (2). □

**Remark 7.6** Let \( K > 0 \) be a self-adjoint operator in \( L^2(\mathbb{R}^d) \) such that \( e^{-tK} \) is positivity preserving and \( e^{-tK} \) is bounded on \( L^\infty(\mathbb{R}^d) \). Then \( e^{-tK} \) is bounded on \( L^p(\mathbb{R}^d) \) for \( 1 \leq p \leq \infty \).

Suppose that \( L \) is the self-adjoint operator associated with the quadratic form \( \mathcal{E}_g \) defined by (7.29) but \( \lambda_1(x) \) and \( \lambda_2(x) \) in (7.28) are replaced by positive constants \( \lambda_1 \) and \( \lambda_2 \), respectively. Then \( L \) is called a strictly elliptic operator.

**Proposition 7.7** Let \( L \) be a strictly elliptic operator. Then \( e^{-tL} \) has an integral kernel \( e^{-tL}(x,y) \) and has Gaussian bounds:

\[
C_1 e^{C_2 t \Delta(x,y)} \leq e^{-tL}(x,y) \leq C_3 e^{C_4 t \Delta(x,y)}.
\]

**Proof:** The upper and lower Gaussian bounds are proven in [Dav89, Corollary 3.2.8] and [Dav89, Theorem 3.3.4], respectively. □
7.2.2 Schrödinger operators with divergence form

We define the Schrödinger operator $K$ on $L^2(\mathbb{R}^3)$ with variable coefficients. Let $K_0$ be defined formally by

$$K_0 = \frac{1}{2} \sum_{\mu,\nu=1}^{3} D_{\mu} A_{\mu\nu}(x) D_{\nu},$$

(7.30)

where $D_{\mu} = -i \nabla_{\mu}$ with the domain $D(D_{\mu}) = H^1(\mathbb{R}^3)$ describes the momentum operator and $A = A(x) = (A_{\mu\nu}(x))_{1 \leq \mu,\nu \leq 3}$ is a $3 \times 3$ symmetric matrix for each $x \in \mathbb{R}^3$. We give the rigorous definition of $K_0$ through a quadratic form. We introduce an assumption on $A(x)$.

**Assumption 7.8 (Uniform elliptic condition)** Suppose that each $A_{\mu\nu}, 1 \leq \mu,\nu \leq 3$, is a measurable function, and $A$ is uniformly elliptic, i.e., there exist constants $C_0 > 0$ and $C_1 > 0$ such that

$$C_0 \|f\|^2 \leq \mathcal{E}(f,f) \leq C_1 \|f\|^2,$$  

(7.31)

Let $\mathcal{E}_1(f,g)$ and $\mathcal{E}_A(f,g)$ be the quadratic forms defined by

$$\mathcal{E}_A(f,g) = \frac{1}{2} \sum_{\mu,\nu=1}^{3} \int A_{\mu\nu}(x) \partial_{\mu} f(x) \cdot \partial_{\nu} g(x) dx$$

(7.32)

and

$$\mathcal{E}_1(f,g) = \frac{1}{2} \sum_{\mu=1}^{3} \int \partial_{\mu} f(x) \cdot \partial_{\mu} g(x) dx$$

(7.33)

with the form domain $H^1(\mathbb{R}^3)$. Under Assumption 7.8 we have

$$C_0 \mathcal{E}_1(f,g) \leq \mathcal{E}_A(f,f) \leq C_1 \mathcal{E}_1(f,f), \quad f \in H^1(\mathbb{R}^3).$$

(7.34)

From this inequality we can see that $(\mathcal{E}_A, H^1(\mathbb{R}^3))$ is a closed semibounded form. We define $K_0$ by the unique self-adjoint operator associated with $\mathcal{E}_A$: there exists a nonnegative self-adjoint operator $K_0$ such that

$$\mathcal{E}_A(f,g) = (K_0^{1/2} f, K_0^{1/2} g)$$

(7.35)

with $H^1(\mathbb{R}^3) = D(K_0^{1/2})$. In general it is not easy to specify the operator domain of $K_0$. We can however specify it under some regularity conditions on $A_{\mu\nu}(x)$. Let

$$W^{n,\infty} = \{ f \in L^\infty(\mathbb{R}^3) \mid \partial^z f \in L^\infty(\mathbb{R}^3) \text{ for } |z| \leq n \},$$

(157)
where \( \partial \) denotes the distributional differential operator on \( L^1_{\text{loc}}(\mathbb{R}^3) \). It is fundamental that for \( f \in W^{1,\infty}(\mathbb{R}^3) \) and \( u \in H^1(\mathbb{R}^3) \), we have \( fu \in H^1(\mathbb{R}^3) \) and \( \partial_\mu(fu) = (\partial_\mu f)u + f\partial_\mu u \) for \( \mu = 1, 2, 3 \).

**Lemma 7.9** Suppose that each \( A_{\mu\nu} \), \( 1 \leq \mu, \nu \leq 3 \), satisfies \( A_{\mu\nu} \in W^{1,\infty}(\mathbb{R}^3) \), and Assumptions 7.8. Then \( D(K_0) = H^2(\mathbb{R}^3) \) and

\[
K_0 f = \sum_{\mu,\nu=1}^3 D_\mu(A_{\mu\nu}(x) D_\nu f).
\]

**Proof:** Since \( H^2(\mathbb{R}^3) \subset D(K_0) \) is trivial, it is enough to see \( H^2(\mathbb{R}^3) \supset D(K_0) \). Let \( f \in K_0 \) and \( T_t^\mu = e^{itD_\mu} \). Note that \( T_t^\mu f(x) = f(x + te_\mu) \), where \( e_\mu \) is the unit vector in \( \mathbb{R}^3 \) to the \( \mu \)th direction, and \( D_\nu T_t^\mu f = T_t^\mu D_\nu f \) follows for \( f \in H^1(\mathbb{R}^3) \) with \( \mu, \nu = 1, \ldots, d \). It is a fundamental fact that \( f \in H^1(\mathbb{R}^3) \) if and only if

\[
\sup_{0 \in (0,1]} \left\| \frac{1}{t} (T_t^\mu - 1) f \right\|_{L^2} < \infty, \quad \mu = 1, \ldots, d.
\]  

(7.36)

Furthermore

\[
\sup_{0 \in (0,1]} \left\| \frac{1}{t} (T_t^\mu - 1) f \right\|_{H^1(\mathbb{R}^3)} \leq \|f\|_{H^1(\mathbb{R}^3)}, \quad \mu = 1, \ldots, d
\]  

(7.37)

holds for \( f \in H^1(\mathbb{R}^3) \). Then if \( f \in H^1(\mathbb{R}^3) \) satisfies that

\[
\sup_{0 \in (0,1]} \left\| \frac{1}{t} (T_t^\mu - 1) f \right\|_{H^1(\mathbb{R}^3)} < \infty, \quad \mu = 1, \ldots, d,
\]  

(7.38)

then \( f \in H^2(\mathbb{R}^3) \). Let \( \|f\|_{\mathcal{E}_A}^2 = \|f\|^2 + \mathcal{E}_A(f, f) \). We fix \( \alpha = 1, \ldots, d \), and set

\[
\Delta_t f(x) = \frac{1}{t} (T_t^\alpha - 1) f(x) = \frac{1}{t} (f(x + te_\alpha) - f(x)).
\]

Let \( f \in D(K_0)(\subset H^1(\mathbb{R}^3)) \) and set \( f_t = \Delta_t f \). We will show that

\[
\sup_{t \in [0,1]} \|f_t\|_{H^1(\mathbb{R}^3)} < \infty.
\]  

(7.39)

We have \( \|f_t\|_{\mathcal{E}_A}^2 = (\Delta_t f, f_t)_{\mathcal{E}_A} = P_t + Q_t \), where

\[
P_t = -(f, \Delta_{-t} f_t)_{L^2} - (K_0 f, \Delta_{-t} f_t)_{L^2},
\]

\[
Q_t = (f, \Delta_{-t} f_t)_{\mathcal{E}_A} + (\Delta_t f, f_t)_{\mathcal{E}_A}.
\]
We have
\[ |P_t| \leq \|f_t\|_{H^1(\mathbb{R}^3)}(\|f\| + \|K_0 f\|) \leq \|f_t\|_{E_A}(\|f\| + \|K_0 f\|), \]
while
\[ Q_t = (f, \Delta_{-t} f_t) + (\Delta_{-t} f, f_t) + \sum_{\mu, \nu = 1}^3 ((A_{\nu \mu} D_{\mu} f, D_{\nu} \Delta_{-t} f_t) + (A_{\nu \mu} D_{\mu} \Delta_{-t} f, D_{\nu} f_t)). \]

We have
\[ (A_{\nu \mu} D_{\mu} f, D_{\nu} \Delta_{-t} f_t) + (A_{\nu \mu} D_{\mu} \Delta_{-t} f, D_{\nu} f_t) = (A_{\nu \mu} D_{\mu} \Delta_{-t} f - \Delta_{-t} (A_{\nu \mu} D_{\nu} f), D_{\nu} f_t) \]
\[ = (-\Delta_{-t} A_{\nu \mu} \cdot T_{\nu}(D_{\mu} f), D_{\nu} f_t). \]

Then
\[ |(A_{\nu \mu} D_{\mu} f, D_{\nu} \Delta_{-t} f_t) + (A_{\nu \mu} D_{\mu} \Delta_{-t} f, D_{\nu} f_t)| \leq \|\Delta_{-t} A_{\nu \mu}\|_{\infty} \|f\|_{H^1(\mathbb{R}^3)}\|f_t\|_{H^1(\mathbb{R}^3)} \]
and
\[ |Q_t| \leq C\|f\|_{H^1(\mathbb{R}^3)}\|f_t\|_{E_A} \]
follows with some constant C independent of t. Then we see that
\[ \|f_t\|_{E_A}^2 \leq \|f_t\|_{H^1(\mathbb{R}^3)}(\|f\| + \|K_0 f\|) + C\|f\|_{H^1(\mathbb{R}^3)}\|f_t\|_{E_A} \]
and
\[ \sup_{t \in (0, 1]} \|f_t\|_{H^1(\mathbb{R}^3)} \leq \sup_{t \in (0, 1]} \|f_t\|_{E_A} \leq \|f\| + \|K_0 f\| + C\|f\|_{H^1(\mathbb{R}^3)} < \infty. \]

Then (7.39) follows and the lemma is proven. \(\square\)

We furthermore introduce the assumption on external potentials \(W\).

**Assumption 7.10 (Confining potential)** \(W \in L^1_{\text{loc}}(\mathbb{R}^3)\) and there exist \(\delta > 0\) and \(C > 0\) such that
\[ W(x) \geq C \langle x \rangle^{2\delta}. \] (7.40)

The Schrödinger operator \(K\) on \(L^2(\mathbb{R}^3)\) with kinetic term \(K_0\) and an external potential \(W\) satisfying Assumption 7.10 is defined by the quadratic form sum. Let
\[ \mathcal{E}(f, g) = \mathcal{E}_A(f, g) + (W^{1/2} f, W^{1/2} g) \] (7.41)
with the form domain $C_0^\infty(\mathbb{R}^3)$. The quadratic form $\mathcal{E}$ is semibounded and then closable. We denote the closure of $\mathcal{E}$ by the same symbol. We define $K$ as the unique self-adjoint operator associated with the quadratic form $\mathcal{E}$:

$$\mathcal{E}(f,g) = (K^{1/2}f, K^{1/2}g)$$

for $f, g$ in the quadratic form domain of $\mathcal{E}$. We describe it as

$$K = K_0 + W.$$  \hfill (7.43)

Lemma 7.11 (Compact resolvent) Suppose Assumptions 7.8 and 7.10. Then $K$ has a compact resolvent and in particular it has a ground state.

Proof: In general a nonnegative self-adjoint operator $T$ has a compact resolvent if and only if $D_T(b) = \{f \in D(T^{1/2}) \mid \|f\| < 1, \|T^{1/2}f\| \leq b\}$ is a compact set for all $b > 0$. See e.g., [RS78, Theorem XIII.64]. Let $L = -\frac{1}{2} \Delta + W$. Then $L$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ by Kato’s inequality, and since $D_L(b)$ is compact for all $b$, $L$ has a compact resolvent. By Assumption 7.8 we see that

$$\|L^{1/2}f\| \leq C_0^{-1}\|K^{1/2}f\|$$  \hfill (7.44)

for $f \in C_0^\infty(\mathbb{R}^3)$, where constant $C_0$ is given by (7.31). By a limiting argument (7.44) holds true for $f \in D(K^{1/2})$, and $D_K(b) \subset D_L(b/C_0)$ follows. Then $D_K(b)$ is compact for all $b > 0$, thus $K$ has a compact resolvent. \hfill $\Box$

In addition to Assumptions 7.8 and 7.10, suppose Assumption 7.18 (Lipschitz condition) mentioned later. It will be proven in Corollary 7.25 that the normalized ground state $\varphi_\nu$ of $K$ is strictly positive and unique. Define the probability measure on $\mathbb{R}^3$ by

$$d\mu_\nu = \varphi_\nu^2(x)dx$$  \hfill (7.45)

and we set

$$\mathcal{H}_\nu = L^2(\mathbb{R}^3; d\mu_\nu).$$  \hfill (7.46)

We transform $K$ by the ground state transformation for later use. Let

$$U_\nu : \mathcal{H}_\nu \rightarrow L^2(\mathbb{R}^3), \ f \mapsto \varphi_\nu f.$$
Let $L_p$ be the unitarily transformed operator of $K - \inf\sigma(K)$ defined by
\[ L_p = U_p^{-1}(K - \inf\sigma(K))U_p \] (7.47)
with the domain $D(L_p) = U_p^{-1}D(K)$. We note that
\[ (f, L_p g)_p = (\varphi_p f, K \varphi_p g)_{L^2} - \inf\sigma(K)(\varphi_p f, \varphi_p g)_{L^2}. \]

7.2.3 Scalar quantum fields

In the previous section we discuss the particle part. In the present section we introduce a scalar quantum field. Let us begin with defining a scalar field in the Schrödinger representation. We use the notation $E_P$ for the expectation with respect to a probability measure $P$, i.e.,
\[ \int \cdots dP = E_P[\cdots]. \]

Let $\mathcal{Q} = \mathcal{S}_R(\mathbb{R}^3)$ be the set of real-valued rapidly decreasing and infinite-times differentiable functions on $\mathbb{R}^3$. There exist a $\sigma$-field $\Sigma$, a probability measure $\mu$ on $(\mathcal{Q}, \Sigma)$ and a Gaussian random variable $\phi(f)$ indexed by $f \in L^2_{\mathbb{R}}(\mathbb{R}^3)$ such that
\[ E_{\mu}[\phi(f)] = 0 \] (7.48)
and the covariance given by
\[ E_{\mu}[\phi(f)\phi(g)] = \frac{1}{2}(f, g)_{L^2}, \] (7.49)
and henceforth
\[ E_{\mu}[e^{z\phi(f)}] = e^{(z^2/4)\|f\|^2}, \quad z \in \mathbb{C}. \] (7.50)
For general $f \in L^2(\mathbb{R}^3)$, $\phi(f)$ is defined by $\phi(f) = \phi(\Re f) + i\phi(\Im f)$. Thus $\phi(f)$ is linear in $f$ over $\mathbb{C}$. The boson Fock space is defined by $L^2(\mathcal{Q}, d\mu) = L^2(\mathcal{Q})$. The identity function $1 \in L^2(\mathcal{Q})$ is called the Fock vacuum. It is known that the linear hull of
\[ \{1\} \cup \{\phi(f_1)\cdots\phi(f_n) : |f_j \in L^2(\mathbb{R}^3), j = 1, \ldots, n, n \geq 1\} \] (7.51)
is dense in $L^2(\mathcal{D})$, where $:\phi(f_1) \cdots \phi(f_n) :$ denotes the Wick product inductively defined by
\[
\phi(f) := \phi(f), \\
\phi(f) \prod_{j=1}^{n} \phi(f_j) := \phi(f) \prod_{j=1}^{n} \phi(f_j) : - \frac{1}{2} \sum_{k=1}^{n} (f, f_k) : \prod_{j \neq k}^{n} \phi(f_j) : .
\]

For a contraction operator $T$ on $L^2(\mathbb{R}^3)$, define the second quantization $\Gamma(T) : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ by $\Gamma(T) \mathbb{1} = \mathbb{1}$ and
\[
\Gamma(T) : \phi(f_1) \cdots \phi(f_n) := \phi(Tf_1) \cdots \phi(Tf_n) : .
\]

Then $\Gamma(T)$ is also contraction on $L^2(\mathbb{R}^3)$ and can be uniquely extended to the contraction operator on the hole space $L^2(\mathcal{D})$, which is denoted by the same symbol $\Gamma(T)$. We can check that $\Gamma(T) \Gamma(S) = \Gamma(TS)$. Then $\{\Gamma(e^{-ith})\}_{t \in \mathbb{R}}$ for a self-adjoint operator $h$ defines the strongly continuous one-parameter unitary group on $L^2(\mathcal{D})$. The unique self-adjoint generator of $\{\Gamma(e^{-ith})\}_{t \in \mathbb{R}}$ is denoted by $d\Gamma(h)$, i.e.,
\[
\Gamma(e^{-ith}) = e^{-itd\Gamma(h)}, \quad t \in \mathbb{R}.
\]

### 7.2.4 The Nelson model with a variable mass

For the standard Nelson model the dispersion relation is given by $(-\Delta + m^2)^{1/2}$ with a constant $m \geq 0$. In this note $m$ is replaced by a positive function $m(x)$ and $-\Delta$ by the divergence form $\sum_{\mu, \nu=1}^{3} c(x)^{-1} D_{\mu} a_{\mu \nu}(x) D_{\nu} c(x)^{-1}$. Let
\[
h = h(D, x) = \sum_{\mu, \nu=1}^{3} c(x)^{-1} D_{\mu} a_{\mu \nu}(x) D_{\nu} c(x)^{-1} + m^2(x).
\]

In the same way as $K_0$ we define $h$ by the quadratic form, then the following assumption is introduced.

**Assumption 7.12 (Condition on $\omega$)** Let $a = a(x) = (a_{\mu \nu}(x))_{1 \leq \mu, \nu \leq 3}$.

1. **Uniform elliptic condition** $a_{\mu \nu} \in W^{1, \infty}$ and there exist constants $C_0 > 0$ and $C_1 > 0$ such that
\[
C_0 \mathbb{1} \leq a(x) \leq C_1 \mathbb{1}.
\]
(2) **(Uniform bound)** There exist $0 < C_0$ and $0 < C_1$ such that $C_0 \leq c(x) \leq C_1$ and $c \in W^{2,\infty}$.

(3) **(Decay of the variable mass)** There exists $\beta > 2$ such that

$$m(x) \leq \langle x \rangle^{-\beta/2}.$$  

(7.56)

Under Assumption 7.12 let us define the semibounded quadratic form

$$(f, g) \mapsto \mathcal{E}_a(f, g)$$

$$= \sum_{\mu, \nu=1}^{3} \int a_{\mu\nu}(x) \partial_{\mu} \left( \frac{1}{c(x)} f(x) \right) \cdot \partial_{\nu} \left( \frac{1}{c(x)} g(x) \right) \, dx + \int m^2(x) f(x) g(x) \, dx$$

for $f, g \in H^1(\mathbb{R}^3)$, which is closable. Notice that $c^{-1} f \in H^1(\mathbb{R}^3)$ if $f \in H^1(\mathbb{R}^3)$, since $c^{-1} \in W^{2,\infty}$, and $\partial_{\mu}(c^{-1} f) = \partial c^{-1} \cdot f + c^{-1} \cdot \partial_{\mu} f$.

**Definition 7.13 (Dispersion relation with a variable mass)** Operator $h$ is defined by the nonnegative self-adjoint operator associated with the closure of $\mathcal{E}_a$, and the self-adjoint operator $\omega$ on $L^2(\mathbb{R}^3)$ is defined by

$$\omega = h^{1/2}.$$  

(7.57)

**Lemma 7.14** Suppose Assumption 7.12. Then $h$ is self-adjoint on $H^2(\mathbb{R}^3)$, and $\inf \sigma(h) = 0$ but 0 is not an eigenvalue of $h$. In particular $\ker \omega = 0$.

**Proof:** Directly we can see that $c^{-1} f \in H^2(\mathbb{R}^3)$ if $f \in H^2(\mathbb{R}^3)$ and

$$h f = h_0 f + v f,$$  

(7.58)

where

$$h_0 f = \sum_{\mu, \nu=1}^{3} D_{\mu}(c^{-1} a_{\mu\nu} c^{-1} D_{\nu} f),$$  

(7.59)

and, by assumptions $c \in W^{2,\infty}$ and $a_{\mu\nu} \in W^{1,\infty}$, $v$ is the bounded multiplication operator given by

$$v = m^2 + \sum_{\mu, \nu=1}^{3} \left( c^{-1}(\partial_{\nu} a_{\mu\nu})(\partial_{\mu} c^{-1}) + c^{-1} a_{\mu\nu}(\partial_{\mu} \partial_{\nu} c^{-1}) \right).$$

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Since \( D(h_0) = H^2(\mathbb{R}^3) \) by \( c^{-1}a_\mu c^{-1} \in W^{1,\infty} \), \( h \) is self-adjoint on \( H^2(\mathbb{R}^3) \) by the Kato-Rellich theorem. By (7.58) we have

\[
\mathcal{E}_a(f,f) \leq D_1 \mathcal{E}_1(c^{-1}f,c^{-1}f) + (mf,mf), \tag{7.60}
\]

\[
D_2 \mathcal{E}_1(c^{-1}f,c^{-1}f) + (mf,mf) \leq \mathcal{E}_a(f,f) \tag{7.61}
\]

with some constants \( D_1 \) and \( D_2 \). Notice that \( -D_j \Delta + m^2 \) has no zero eigenvector and \( \sigma(-D_j \Delta + m^2) = [0, \infty) \), since \( m^2 \) is a compact perturbation of \( -C \Delta \). By (7.60), \( h \) has also no zero eigenvector and \( \inf \sigma(h) = 0. \) □

**Definition 7.15 (The Nelson Hamiltonian with a variable mass)**

The Nelson Hamiltonian with a variable mass \( m(x) \) and a cutoff function \( \rho \) is defined by

\[
H = L_p \otimes \mathbb{1} + \mathbb{1} \otimes H_f + \phi_{\rho} \tag{7.62}
\]

on the tensor product Hilbert space \( \mathcal{H} = \mathcal{H}_p \otimes L^2(\mathcal{Q}) \), where we set the coupling constant \( \alpha \) as \( \alpha = 1 \), \( L_p \) is defined by (7.47), the free field Hamiltonian \( H_f \) by \( H_f = d\Gamma(\omega) \) and the field operator \( \phi_{\rho} \) is given by

\[
\phi_{\rho} = \int_{\mathbb{R}^3} \phi_{\rho}(x)d\mu_p \tag{7.63}
\]

with \( \phi_{\rho}(x) = \phi(\omega^{-1/2}\rho(\cdot - x)) \). Here we used the identification \( \mathcal{H} \cong \int_{\mathbb{R}^3} L^2(\mathcal{Q})d\mu_p. \)

Thus the Nelson Hamiltonian is a linear operator defined on the \( L^2 \)-space over the probability space \( (\mathbb{R}^3 \times \mathcal{Q}, d\mu_p \otimes d\mu) \).

**Assumption 7.16 (Condition on \( \rho \))** The ultraviolet cutoff function \( \rho \) satisfies that

\[
(1) \ \rho \geq 0, \quad (2) \ \hat{\rho}/\sqrt{|k|} \in L^2(\mathbb{R}^3), \quad (3) \ \hat{\rho}/|k| \in L^2(\mathbb{R}^3). \tag{7.64}
\]

We will use (1) of Assumption 7.16 in the proof of Lemma 7.42

**Proposition 7.17** Suppose Assumptions 7.8, 7.10, 7.12 and 7.16. Then the Nelson Hamiltonian \( H \) is self-adjoint on \( D(L_p) \cap D(H_f) \), and bounded from below. Furthermore it is essentially self-adjoint on any core of \( L_p + H_f \).
Proof: By Assumption 7.12 it follows that
\[
\sup_x \| \omega^{-n/2} \rho(\cdot - x) \| \leq C \| \hat{\rho} / |k|^{n/2} \|
\] (7.65)
for \( n = 1, 2 \) with some \( C \). (7.65) is shown in Corollary 7.40 later. Then \( \phi_\rho(x) \) is infinitesimally small with respect to \( H_t \) for each \( x \in \mathbb{R}^3 \). Then \( \phi_\rho \) is infinitesimally small with respect to \( L_p + H_t \), and the proposition follows by the Kato-Rellich theorem.

\[7.3 \text{ Feynman-Kac formula and diffusions}\]

In this section we construct a functional integral representation of the one-parameter heat semigroup \( e^{-tH} \).

\[7.3.1 \text{ Super-exponential decay}\]

The following assumption ensures the existence and uniqueness of a stochastic differential equation. Let \( C^n_b(\mathbb{R}^3) = \{ f \in C^n(\mathbb{R}^3) | f^m \in L^\infty(\mathbb{R}^3), |m| \leq n \} \).

**Assumption 7.18 (Lipshitz condition)** Suppose that \( A_{\mu \nu} \in C^1_b(\mathbb{R}^3) \), \( \mu, \nu = 1, 2, 3 \), and \( b_\nu(x) = \frac{1}{2} \sum_{\mu=1}^3 \partial_\mu A_{\mu \nu}(x) \) and the \( 3 \times 3 \) matrix \( (\sigma_{\mu \nu}(x))_{1 \leq \mu, \nu \leq 3} = \sigma(x) = \sqrt{A(x)} \) satisfy the Lipshitz condition:
\[
|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y| \tag{7.66}
\]
for arbitrary \( x, y \in \mathbb{R}^3 \) with some constant \( D \) independent of \( x \) and \( y \), where \( |\sigma(x)| = \sqrt{\sum_{\mu, \nu=1}^3 |\sigma_{\mu \nu}(x)|^2} \).

**Lemma 7.19** Suppose Assumptions 7.8 and 7.18. Then \( D(K_0) = H^2(\mathbb{R}^3) \) and \( K_0 f = \sum_{\mu, \nu=1}^3 D_{\mu}(A_{\mu \nu}D_{\nu}f) \) for \( f \in H^2(\mathbb{R}^3) \).

**Proof:** We see that \( A_{\mu \nu} \in W^{1, \infty}(\mathbb{R}^3) \). Then the lemma immediately follows from Lemma 7.9.

Let us consider the stochastic differential equation:
\[
\begin{cases}
    dX^\nu_t = \sigma_\nu(X_t) \cdot dB_t + b_\nu(X_t) dt, \\
    X^\nu_0 = x^\nu,
\end{cases}
\] (7.67)

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on the probability space \((\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+), \mathcal{W})\), where we recall \(\mathcal{X}_+ = C([0, \infty); \mathbb{R}^3)\), \(\mathcal{B}(\mathcal{X}_+)\) is the \(\sigma\)-field generated by cylinder sets and \(\mathcal{W}\) the Wiener measure starting at 0. We denote \(\mathbb{E}_W\) by \(\mathbb{E}\) unless confusions may arise. \((B_t)_{t \geq 0}\) denotes the 3-dimensional Brownian motion on \((\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+), \mathcal{W})\). The drift term \(b_\nu\) and the diffusion term \(\sigma_\nu = (\sigma_{\nu 1}, \sigma_{\nu 2}, \sigma_{\nu 3})\) are defined in Assumption 7.18. Note that \(b_\nu\) and \(\sigma_{\mu\nu}\) are bounded; \(\|b_\nu\|_\infty < \infty\) and \(\|\sigma_{\mu\nu}\|_\infty < \infty\). Let \((F_t)_{t \geq 0}\) be the natural filtration of the Brownian motion: \(F_t = \sigma(B_s, 0 \leq s \leq t)\).

**Proposition 7.20** Suppose Assumption 7.18. Then \((7.67)\) has the unique solution \(X^x = (X^x_t)_{t \geq 0}\) which is a diffusion process with respect to the filtration \((F_t)_{t \geq 0}\). Namely \(X^x\) has continuous sample paths and Markov property:

\[
\mathbb{E}\left[f(X^x_{s+t})|F_s\right] = \mathbb{E}\left[f(X^x_t)\right], \tag{7.68}
\]

where \(\mathbb{E}\left[f(X^x_t)\right]\) is \(\mathbb{E}[f(X^y_t)]\) evaluated at \(y = X^x_s\).

From \((7.68)\) we can show that

\[
T_t f(x) = \mathbb{E}\left[f(X^x_t)\right] \tag{7.69}
\]

satisfies the semigroup property \(T_s T_t f = T_{s+t} f\) on \(L^\infty(\mathbb{R}^3)\). In the next proposition we show indeed that \(T_t f \in L^2(\mathbb{R}^3)\) for \(f \in L^2(\mathbb{R}^3)\). Namely \(T_t\) defines a semigroup not only on \(L^\infty(\mathbb{R}^3)\) but also on \(L^2(\mathbb{R}^3)\).

In order to show that \(T_t : L^\infty \to L^\infty\) can be extended to a semigroup on \(L^2(\mathbb{R}^3)\), we introduce a Dirichlet form. Suppose Assumption 7.8. We see that \((\mathcal{E}_A, H^1(\mathbb{R}^3))\) is a local and regular Dirichlet form. It is a fundamental fact that there exist a probability measure \(\nu^x\) on \((\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+))\) and a coordinate process \(Z = (Z_t)_{t \geq 0}\) such that (1) \(\nu^x(Z_0 = x) = 1\), (2) \(Z\) is a symmetric diffusion process with respect to the natural filtration \(\mathcal{M}_t = \sigma(Z_s, 0 \leq s \leq t)\), (3)

\[
S_t f(x) = \mathbb{E}_{\nu^x}[f(Z_t)] \tag{7.70}
\]

defines the semigroup, and (4) for each \(t \geq 0\),

\[
(e^{-tK_0} f)(x) = (S_t f)(x), \quad a.e. \quad x \in \mathbb{R}^3. \tag{7.71}
\]

See e.g., [Fuk80, Lemma 4.3.1].
Proposition 7.21 \textbf{\((L^2\) extension)} Let \(f \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\). Suppose Assumptions 7.8 and 7.18. Then
\[ T_t f = e^{-tK_0} f, \quad t \geq 0, \quad \text{a.e.} \]  
(7.72)
In particular \(T_t [L^2 \cap L^\infty] = e^{-tK_0}\), where \(\ldots\) denotes the closure in \(L^2(\mathbb{R}^3)\).

Proof: Let \(f \in C_0^\infty(\mathbb{R}^3)\). We set
\[ M_t = f(Z_t) - \int_0^t (K_0f)(Z_s)ds, \]
\[ N_t = f(X^x_t) - \int_0^t (K_0f)(X^x_s)ds. \]
By the Itô formula we have
\[ f(X^x_t) - f(x) = \int_0^t (K_0f)(X^x_s)ds + \sum_{\mu=1}^3 \int_0^t (\partial_\mu f)(X^x_s)\sigma_\mu(X^x_s) \cdot dB_s. \]
Hence
\[ N_t = f(x) + \sum_{\mu=1}^3 \int_0^t (\partial_\mu f)(X^x_s)\sigma_\mu(X^x_s) \cdot dB_s \]
and then \((N_t)_{t \geq 0}\) is martingale on \((\mathcal{D}_+, \mathcal{B}(\mathcal{D}_+), \mathcal{W})\) with respect to \((\mathcal{F}_t)_{t \geq 0}\), while we can see that
\[ \mathbb{E}_\nu[M_{t+s}|\mathcal{F}_s] = \mathbb{E}[f(Z_{t+s})|\mathcal{F}_s] - \int_s^t K_0f(Z_r)dr - \mathbb{E}\left[ \int_s^{t+s} K_0f(Z_r)dr | \mathcal{F}_s \right]. \]
Here \(\mathcal{F}_t = \sigma(Z_s, 0 \leq s \leq t)\). Let \(p(t, y, A)\) be the probability transition kernel of \(Z_t\) under \(\nu^x\). Then by the Markov property of \(Z\) we have
\[ \mathbb{E}[f(Z_{t+s})|\mathcal{F}_s] = \int f(y)p(t, Z_s, dy) = (e^{-tK_0}f)(Z_s) \]
and
\[ \mathbb{E}\left[ \int_s^{t+s} (K_0f)(Z_r)dr | \mathcal{F}_s \right] = \int_s^{t+s}dr \int (K_0f)(y)p(r-s, Z_s, dy) \]
\[ = \int_0^t dr \int (K_0f)(y)p(r, Z_s, dy) = \int_0^t (e^{-rK_0}K_0f)(Z_s)dr = (e^{-tK_0}f)(Z_s) - f(Z_s). \]
Then we see that
\[ E[M_{t+s} | \mathcal{F}_s] = f(Z_s) - \int_0^s K_0 f(Z_r) dr = M_s \]
and we conclude that \((M_t)_{t \geq 0}\) is also martingale with respect to \((\mathcal{F}_t)_{t \geq 0}\). By the uniqueness of martingale problem (e.g., [RW00, Theorem (24.1)], [SV06, Chapter 8], [KS91, Section 5.4.E]), it follows that \(\nu^x = W \circ (X^x)^{-1}\). In particular
\[ E[\nu^x[f(Z_t)]] = E[W[f(X^x_t)]], \]
which is equivalent to \(T_t f = S_t f\). Then the proposition follows from (7.71). \(\square\)

In order to see some properties of \(e^{-tK_0}\), we give a Gaussian bound of the integral kernel of \(e^{-tK_0}\). When \(\|e^{-tL} f\|_\infty \leq C_t \|f\|_2\) is satisfied for all \(t > 0\) and all \(f \in L^2(\mathbb{R}^3)\), \(e^{-tL}\) is called ultracontractivity.

**Proposition 7.22 (Kernels)** Suppose Assumption 7.8. Then \(e^{-tK_0}\) is ultracontractive, has an integral kernel, and the kernel satisfies that
\[ C_1 e^{tC_2 \Delta}(x, y) \leq e^{-tK_0}(x, y) \leq C_3 e^{tC_4 \Delta}(x, y) \] (7.73)
with some constants \(C_j, j = 1, 2, 3, 4\), where
\[ e^{t\Delta}(x, y) = (2\pi T)^{-3/2} \exp(-|x - y|^2/(2T)) \]
is the 3-dimensional heat kernel.

**Proof:** See Propositions 7.4, 7.5 and 7.7. \(\square\)

We prove the Feynman-Kac formula of \(e^{-t(K_0 + W)}\) for general \(W\). Let \(h_0 = (-1/2)\Delta\). Suppose that \(W\) is form bounded with respect to \(h_0\) with a relative bound \(b\), i.e.,
\[ \lim_{E \to \infty} \|W^{1/2}(h_0 + E)^{-1/2} f\|/\|f\| = b. \]
By Proposition 7.22 we notice that
\[ |(f, e^{-tK_0} g)| \leq (|f|, C e^{-tCh_0} |g|), \] (7.74)
where $C'$ and $C$ are nonnegative constants. Let $T$ be nonnegative self-adjoint operator. Then $(T + E)^{-1/2} = \pi^{-1/2} \int_0^\infty t^{-1/2} e^{-(T + E)dt}$ for $E > 0$. From this formula and (7.74) it follows that
\[
|(K_0 + E)^{-1/2} f(x)| \leq C'(C_{h_0} + E)^{-1/2} |f|(x). \tag{7.75}
\]
Hence
\[
\|W\|^{1/2} (K_0 + E)^{-1/2} \|f\| \leq C'\|W\|^{1/2} (C_{h_0} + E)^{-1/2} \|f\| \tag{7.76}
\]
and we have
\[
\lim_{E \to \infty} \|W\|^{1/2} (K_0 + E)^{-1/2} \|f\| = C'C^{-1/2}b.
\]
Then $W$ is also relatively form bounded with respect to $K_0$ with a relative bound $< C'C^{-1/2}b$. We introduce an assumption on $W$.

**Assumption 7.23** Let $W = W_+ - W_-$, where $W_\pm = \max\{\pm W, 0\}$. Suppose $W_+ \in L_{\text{loc}}^1(\mathbb{R}^d)$ and $W_-$ is relatively form bounded with respect to $h_0$ with a relative bound $b$ such that
\[
C'C^{-1/2}b < 1, \tag{7.77}
\]
where constants $C, C'$ are in (7.74).

Suppose Assumptions 7.8 and 7.23. Then by the KLMN theorem
\[
K = K_0 + W_+ - W_- \tag{7.78}
\]
can be defined as a self-adjoint operator. Here $\dot{\pm}$ denotes the quadratic form sum.

**Proposition 7.24 (Feynman-Kac formula)** Suppose Assumptions 7.8, 7.18 and 7.23. Let $K$ be given by (7.78). Then
\[
(g, e^{-tK} f) = \int d\mu_0 \mathbb{E} \left[ \overline{g(x)} f(X_t^x) e^{-\int_0^t W(X_s^x)ds} \right]. \tag{7.79}
\]
In particular
\[
(e^{-tK} f)(x) = \mathbb{E} \left[ f(X_t^x) e^{-\int_0^t W(X_s^x)ds} \right]. \tag{7.80}
\]
Proof: Suppose first that W is bounded and continuous. By the Trotter-Kato product formula we have

\[
(f, e^{-tK} g) = \lim_{n \to \infty} (f, (e^{-(t/n)K_0} e^{-(t/n)W})^n g).
\] (7.81)

By Proposition 7.21 we have for \(t_0 \leq t_1 \leq \cdots \leq t_n\),

\[
(e^{-(t_1-t_0)K_0} f_1 \cdots e^{-(t_n-t_{n-1})K_0} f_n)(x) = \mathbb{E} [f_1(X_{t_1-t_0}) e^{-(t_2-t_1)K_0} f_1 \cdots e^{-(t_n-t_{n-1})K_0} f_n] (X_{t_1-t_0}).
\]

By the Markov property (7.68) we also have

\[
\mathbb{E} [f_1(X_{t_1-t_0})] = \mathbb{E} [f_1(X_{t_2-t_1})] \cdots = \mathbb{E} [f_1(X_{t_n-t_{n-1}})]
\]

Inductively we obtain that

\[
(e^{-(t_1-t_0)K_0} f_1 \cdots e^{-(t_n-t_{n-1})K_0} f_n)(x) = \mathbb{E} \left[ \prod_{j=1}^{n} f_j(X_{t_j-t_{j-1}}) \right].
\] (7.82)

Combining the Trotter product formula (7.81) and (7.82) with \(t_j = tj/n\), we have

\[
(f, e^{-tK} g) = \lim_{n \to \infty} \int dx \bar{f}(x) \mathbb{E} \left[ e^{-\left\{ t/n \right\} \sum_{j=1}^{n} W(X_{t_j/n})} g(X_{t}) \right].
\] (7.83)

Since \(s \mapsto X_s^\omega(\omega)\) has continuous paths, \(W(X_s^\omega)\) is continuous in \(s \in [0, t]\) for each \(\omega\). Therefore \(\sum_{j=1}^{n} t_j/n W(X_{t_j/n}) \to \int_0^t W(X_s^\omega)ds\) as \(n \to \infty\) for each \(\omega\) and exists as a Riemann integral.

In order to extend \(W\) to more general class, we use a standard limiting argument. To do that, suppose that \(W \in L^\infty\) and \(W_n(x) = \phi(x/n)(W * j_n)(x)\), where \(j_n = n^3 \phi(xn)\) with \(\phi \in C_0^\infty(\mathbb{R}^3)\) such that \(0 \leq \phi \leq 1\), \(\int \phi(x)dx = 1\) and \(\phi(0) = 1\). Then \(W_n\) is bounded and continuous, moreover \(W_n(y) \to W(y)\) as \(n \to \infty\) for \(y \not\in \mathcal{N}\) with some null set \(\mathcal{N}\). Notice that \(\mathbb{E}[1_{X_s^\omega \in \mathcal{N}}] = \int 1_{y \in \mathcal{N}} e^{-tK_0(x,y)}dy = 0\) and thus \(\int_0^t ds \mathbb{E}[1_{X_s^\omega \in \mathcal{N}}] = \mathbb{E} \left[ \int_0^t ds 1_{X_s^\omega \in \mathcal{N}} \right] = 0\) by Fubini’s lemma. Thus for each
where the form domains are given by $m$ strongly as $q$ sense, which implies that $K_n$ as adjoint operators satisfy sequence of forms (see [Kat76, VIII. Theorem 3.11]) the associated positive self-

\[ \int d\mu_p E \left[ f(x) g(x) e^{-\int_0^t W_n(x^2) ds} \right] \rightarrow \int d\mu_p E \left[ f(x) g(x) e^{-\int_0^t W(x^2) ds} \right] \]

as $n \rightarrow \infty$. On the other hand, $e^{-t(K_0 + W_n)} \rightarrow e^{-t(K_0 + W)}$ strongly as $n \rightarrow \infty$, since $K_0 + W_n$ converges to $K_0 + W$ on the common domain $H^2(\mathbb{R}^3)$. Next define

\[ W_{+,n}(x) = \begin{cases} W_+(x), & W_+(x) < n, \\ n, & W_+(x) \geq n, \end{cases} \quad W_{-,m}(x) = \begin{cases} W_-(x), & W_-(x) < m, \\ m, & W_-(x) \geq m. \end{cases} \]

Note that $Q(K_0) = H^1(\mathbb{R}^3)$, where $Q(T)$ denotes the form domain of $T$, i.e., $Q(T) = D(|T|^{1/2})$. Define the closed quadratic forms

\[
q_{n,m}(f,f) = (K_0^{1/2} f, K_0^{1/2} f) + (W_{+,n}^{1/2} f, W_{+,n}^{1/2} f) - (W_{-,m}^{1/2} f, W_{-,m}^{1/2} f), \\
q_{n,\infty}(f,f) = (K_0^{1/2} f, K_0^{1/2} f) + (W_{+,n}^{1/2} f, W_{+,n}^{1/2} f) - (W_{-,\infty}^{1/2} f, W_{-,\infty}^{1/2} f), \\
q_{\infty,\infty}(f,f) = (K_0^{1/2} f, K_0^{1/2} f) + (W_{+,\infty}^{1/2} f, W_{+,\infty}^{1/2} f) - (W_{-,\infty}^{1/2} f, W_{-,\infty}^{1/2} f),
\]

where the form domains are given by

\[
Q(q_{n,m}) = H^1(\mathbb{R}^3), \quad Q(q_{n,\infty}) = H^1(\mathbb{R}^3), \quad Q(q_{\infty,\infty}) = H^1(\mathbb{R}^3) \cap Q(W_+).
\]

Note that

\[
q_{n,m} \geq q_{n,m+1} \geq q_{n,m+2} \geq \ldots \geq q_{n,\infty}
\]

and $q_{n,m} \rightarrow q_{n,\infty}$ in the sense of quadratic forms on $\cup_m Q(q_{n,m}) = H^1(\mathbb{R}^3)$. Since $q_{n,\infty}$ is closed on $H^1(\mathbb{R}^3)$, by the monotone convergence theorem for a non-increasing sequence of forms (see [Kat76 VIII. Theorem 3.11]) the associated positive self-adjoint operators satisfy $K_0 + W_{+,n} - W_{-,m} \rightarrow K_0 + W_{+,n} - W_{-}$ in strong resolvent sense, which implies that

\[
e^{-t(K_0 + W_{+,n} - W_{-,m})} \rightarrow e^{-t(K_0 + W_{+,n} - W_{-})} \quad (7.84)
\]

strongly as $m \rightarrow \infty$ for all $t \geq 0$. Similarly, we have

\[
q_{n,\infty} \leq q_{n+1,\infty} \leq q_{n+2,\infty} \leq \ldots \leq q_{\infty,\infty}
\]

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and \( q_{n,\infty} \to q_{\infty,\infty} \) in quadratic form sense on
\[
\{ f \in \cap_n Q(q_{n,\infty}) \mid \sup_n q_{n,\infty}(f, f) < \infty \} = H^1(\mathbb{R}^3) \cap Q(W_+).
\]

Hence by the monotone convergence theorem for a non-decreasing sequence of forms (see [Kat76] VIII. Theorem 3.13 and p.575) we obtain
\[
e^{-t(K_0 + W_{+,n} - W_-)} \to e^{-t(K_0 + W_+ - W_-)} ,
\]
strongly as \( n \to \infty \). On the other hand, we can see that
\[
\int dx \mathbb{E} \left[ e^{-\int_0^t (W_{+,n} - W_-)(X_s^x)ds} \right] \to \int dx \mathbb{E} \left[ e^{-\int_0^t (W_+ - W_-)(X_s^x)ds} \right]
\]
as \( m \to \infty \). Moreover,
\[
\int dx \mathbb{E} \left[ e^{-\int_0^t (W_{+,n} - W_-)(X_s^x)ds} \right] \to \int dx \mathbb{E} \left[ e^{-\int_0^t (W_+ - W_-)(X_s^x)ds} \right]
\]
as \( n \to \infty \), by (7.85) and the dominated convergence theorem. Thus the proof is complete.

\[\square\]

**Corollary 7.25 (Positivity improving)** Suppose Assumptions 7.8, 7.10 and 7.18. Then \( e^{-tK} \) is positivity improving. In particular the ground state of \( K \) is strictly positive and unique.

**Proof:** Let \( f \geq 0 \) and \( g \geq 0 \) but \( f \neq 0 \) and \( g \neq 0 \). It is enough to show that \( (f, e^{-tK} g) > 0 \). Let \( \text{supp} f = D_f \) and \( \text{supp} g = D_g \). We first show that for each \( x \in \mathbb{R}^3 \),
\[
W \left( \int_0^t W(X_s^x)ds = \infty \right) = 0.
\]
(7.86)

Let us recall that \( (B_t)_{t \geq 0} \) is the Brownian motion on \( (\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+), W) \). Let \( N \in \mathbb{N} \). Since \( W \in L^1_{\text{loc}}(\mathbb{R}^3) \), \( \mathbb{1}_N W \in L^1(\mathbb{R}^3) \) and then by Proposition 7.22
\[
\int dx \mathbb{E} \left[ \int_0^t \mathbb{1}_N W(X_s^x)ds \right] = \int dx \int_0^t \mathbb{E} \left[ \mathbb{1}_N W(X_s^x) \right] ds
\leq \int dx C_3 \int_0^t \mathbb{E} \left[ \mathbb{1}_N W(B_{2C_3t} + x) \right] ds \leq \| \mathbb{1}_N W \|_{L^1} C_3 t < \infty.
\]
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Thus $\mathcal{W}(\mathbb{I}_N W(X^s_t)ds < \infty) = 1$ and there exists $\mathcal{N}_N$ such that $\mathcal{W}(\mathcal{N}_N) = 0$ and $\int_0^t \mathbb{I}_N W(X^s_t(\omega))ds < \infty$ for arbitrary $\omega \in \mathcal{B}_+ \setminus \mathcal{N}_N$. Let $\mathcal{N} = \cup_{\mathcal{N}_N \in \mathcal{N}}$. Thus

$$\int_0^t \mathbb{I}_N W(X^s_t(\omega))ds < \infty$$

for arbitrary $N \in \mathbb{N}$ and $\omega \in \mathcal{B}_+ \setminus \mathcal{N}$. Since $X^s_t$ is continuous in $s$, for each $\omega \in \mathcal{B}_+ \setminus \mathcal{N}$, there exists $N = N(\omega)$ such that $\sup_{0 \leq s \leq t} X^s_t(\omega) < N$. Then $W(X^s_t(\omega)) = \mathbb{I}_N W(X^s_t(\omega))$ for $0 \leq s \leq t$, and

$$\int_0^t W(X^s_t(\omega))ds = \int_0^t \mathbb{I}_N W(X^s_t(\omega))ds < \infty.$$

This implies (7.86) and $e^{-\int_0^t W(X^s_t(\omega))ds} > 0$ for a.e. $\omega \in \mathcal{B}_+$. By the Feynman-Kac formula, it is sufficient to see that $\int dx \mathbb{E}[f(X^s_0)g(X^s_t)] > 0$. Let

$$D^s_g = \{\omega \in \mathcal{B}_+ | X^s_t(\omega) \in D_g\}.$$

Thus

$$\int_{D_f} dx \mathbb{E}[\mathbb{I}_{D_g}] = (\mathbb{I}_{D_f}, e^{-tK_0} \mathbb{I}_{D_g}) \geq C_1 (\mathbb{I}_{D_f}, e^{C_2t\Delta} \mathbb{I}_{D_g}) > 0.$$

Then the measure of $\cup_{x \in D_f} D^s_g (\subset \mathbb{R}^3 \times \mathcal{B}_+)$ is strictly positive with respect to $dx \otimes d\mathcal{W}$ and $f(X^s_0)g(X^s_t) > 0$ on $\cup_{x \in D_f} D^s_g$. Then

$$\int dx \mathbb{E}[f(X^s_0)g(X^s_t)] \geq \int_{D_f} dx \int_{D^s_g} f(X^s_0)g(X^s_t)d\mathcal{W} > 0$$

and the corollary follows. \hfill \Box

**Corollary 7.26 (Ultracontractivity)** Suppose Assumptions 7.8, 7.10 and 7.18. Then $e^{-tK}$ is ultracontractive.

**Proof:** Note that $e^{-\int_0^t W(X^s_t)ds} \leq 1$. By the Feynman-Kac formula, we have

$$| (e^{-tK}f)(x) | \leq (\mathbb{E}[|f(X^s_t)|^2])^{1/2}.$$

By Proposition 7.22 we have

$$\mathbb{E}[|f(X^s_t)|^2] = (e^{-tK_0}|f|^2)(x) \leq C_3 \left( e^{C_4t\Delta} |f|^2 \right)(x) \leq C t^{-3/2} \|f\|_{L^2}^2.$$

Then $\|e^{-tK}f\|_\infty \leq C t^{-3/4} \|f\|_{L^2}$ and the corollary follows. \hfill \Box

We can also prove the theorem below.
Theorem 7.27 (Super-exponential decay) Suppose Assumptions 7.8, 7.10 and 7.18. Then there exists a constant $\gamma > 0$ such that

$$e^{\gamma |x|^{d+1}} \varphi_p \in H^1(\mathbb{R}^3).$$ (7.87)

Proof: Let $F \in C^\infty(\mathbb{R}^3)$ be real, bounded with all derivatives. Then for $u \in D(K)$ we have the Agmon identity:

$$\int \frac{1}{2} \nabla(e^F \bar{u}) \cdot A \nabla(e^F u) dx + \int e^{2F}(W - \frac{1}{2} \nabla F \cdot A \nabla F) |u|^2 dx = \int e^{2F} \bar{u} K u dx + 2i \text{Im} \int e^{2F} \nabla \bar{u} \cdot A \nabla F dx.$$

Applying this identity to the ground state $\varphi_p$, we obtain that $e^{\gamma |x|^{d+1}} \varphi_p \in L^2(\mathbb{R}^3)$ and $\nabla(e^{\gamma |x|^{d+1}} \varphi_p) \in L^2(\mathbb{R}^3)$. \qed

7.3.2 Diffusion processes

We can also construct a Markov process $X = (X_t)_{t \in \mathbb{R}}$ on the whole real line $\mathbb{R}$ associated with the semigroup $e^{-tL_p}$ by a stochastic differential equation. Let

$$\mathcal{X} = C(\mathbb{R}; \mathbb{R}^3).$$

Proposition 7.28 (Diffusion process associated with $e^{-tL_p}$) Let $X_t(\omega) = \omega(t)$ be the coordinate process on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Suppose Assumptions 7.8, 7.10 and 7.18. Then there exists a probability measure $P^x$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ satisfying (1)-(4) below:

1. (Initial distribution) $P^x(X_0 = x) = 1$.

2. (Reflection symmetry) Two processes $(X_t)_{t \geq 0}$ and $(X_s)_{s \leq 0}$ are independent and $X_{-t} \overset{d}{=} X_t$.\footnote{\hspace{1cm}X \overset{d}{=} Y means that X and Y has the same distribution.}
Let $(\mathcal{F}_t^+)_{t \geq 0} = \sigma(X_s, 0 \leq s \leq t)$ and $(\mathcal{F}_t^-)_{t \leq 0} = \sigma(X_{-t}, -t \leq s \leq 0)$ be filtrations. Then $(X_t)_{t \geq 0}$ and $(X_s)_{s \leq 0}$ are diffusion processes with respect to $(\mathcal{F}_t^+)_{t \geq 0}$ and $(\mathcal{F}_t^-)_{t \leq 0}$, respectively, i.e.,

$$
\mathbb{E}_{P_X}[X_{t+s}|\mathcal{F}_t^+] = \mathbb{E}_{P_X}[X_{t+s}|\sigma(X_s)] = \mathbb{E}_{P_X}[X_t],
$$

$$
\mathbb{E}_{P_X}[X_{t-s}|\mathcal{F}_t^-] = \mathbb{E}_{P_X}[X_{t-s}|\sigma(X_{-s})] = \mathbb{E}_{P_X}[X_{-t}]
$$

for $s, t \geq 0$, and $X_t$ is continuous in $t \in \mathbb{R}$, where $\mathbb{E}_{P_X}$ means $\mathbb{E}_y$ evaluated at $y = X_s$.

(4) (Shift invariance) It follows that

$$
\int d\mu_p \mathbb{E}_{P_X} \left[ f_0(X_{t_0}) \cdots f_n(X_{t_n}) \right] = (f_0, e^{-(t_1-t_0)L_p} f_1 \cdots e^{-(t_n-t_{n-1})L_p} f_n)_{\mathcal{H}_p}
$$

(7.88)

for $f_j \in L^\infty(\mathbb{R}^3)$, $j = 1, \ldots, n$, and then the finite dimensional distribution of $X$ is shift-invariant, i.e.,

$$
\int d\mu_p \mathbb{E}_{P_X} \left[ \prod_{j=1}^n f_j(X_{t_j}) \right] = \int d\mu_p \mathbb{E}_{P_X} \left[ \prod_{j=1}^n f_j(X_{t_j+s}) \right], \quad s \in \mathbb{R},
$$

for any bounded Borel measurable functions $f_j$, $j = 1, \ldots, n$.

In order to prove Proposition 7.28 we need several steps. An outline of constructing a diffusion process $(X_t)_{t \in \mathbb{R}}$ is as follows.

For $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ let the set function $\nu_{t_0, \ldots, t_n} : \prod_{j=0}^n \mathcal{B}(\mathbb{R}^3) \to \mathbb{R}$ be given by

$$
\nu_{t_0, \ldots, t_n} \left( \prod_{i=0}^n A_i \right) = (\mathbb{1}_{A_0}, e^{-(t_1-t_0)L_p} \mathbb{1}_{A_1} \cdots e^{-(t_n-t_{n-1})L_p} \mathbb{1}_{A_n})_{\mathcal{H}_p},
$$

(7.89)

and for $0 \leq t$, $\nu_t : \mathcal{B}(\mathbb{R}^3) \to \mathbb{R}$ by

$$
\nu_t(A) = (\mathbb{1}, e^{-tL_p} \mathbb{1}_A)_{\mathcal{H}_p} = (\mathbb{1}, \mathbb{1}_A)_{\mathcal{H}_p},
$$

(7.90)

where $\mathcal{B}(\mathbb{R}^3)$ denotes the Borel $\sigma$-field of $\mathbb{R}^3$. We show an outline of steps of the proof.
(Step 1) By the Kolmogorov extension theorem we can construct a probability measure \( \nu_\infty \) on \((\mathbb{R}^3)^{[0,\infty)}\) from the family of probability measures given by (7.89) and (7.90), and we define a stochastic process \( Y = (Y_t)_{t \geq 0} \) on a probability space \(((\mathbb{R}^3)^{[0,\infty)}, \mathcal{B}((\mathbb{R}^3)^{[0,\infty)}), \nu_\infty)\) such that finite dimensional distributions of \( Y \) is given by the right-hand side of (7.89) and (7.90). We also show the existence of the continuous version \((\tilde{Y}_t)_{t \geq 0}\) on the same probability space.

(Step 2) Let \( Q = \nu_\infty \circ \tilde{Y}^{-1} \) be the image measure of \( \nu_\infty \) on \((\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+))\), where \( \mathcal{X}_+ = C([0, \infty); \mathbb{R}^3) \). Let \((\tilde{Y}_t)_{t \geq 0}\) be the coordinate process on the probability space \((\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+), Q)\), i.e., \( \tilde{Y}_t(\omega) = \omega(t) \) for \( \omega \in \mathcal{X}_+ \). Notice that \( \tilde{Y} \overset{d}{=} \tilde{Y} \).

(Step 3) Define a regular conditional probability measure by \( Q^x(\cdot) = Q(\cdot|\tilde{Y}_0 = x) \). Then the stochastic process \((\tilde{Y}_t)_{t \geq 0}\) on a probability space \((\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+), Q^x)\) satisfies

\[
(f_0, e^{-(t_1-t_0)L_p} f_1 \cdots e^{-(t_n-t_{n-1})L_p} f_n) \mathcal{E} = \int d\mu_p \mathbb{E}_{Q^x} \left[ \prod_{j=0}^n f_j(\tilde{Y}_{t_j}) \right]
\]

for \( 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n \) and we can show that \( \tilde{Y} \) is a diffusion process with respect to the natural filtration \( \sigma(\tilde{Y}_s, 0 \leq s \leq t) \).

(Step 4) We extend \( \tilde{Y} \) to a process of the whole real line. Define a stochastic process \( \tilde{X}_t(\omega) = \left\{ \begin{array}{ll} \tilde{Y}_t(\omega_1), & t \geq 0, \\ \tilde{Y}_{-t}(\omega_2), & t < 0 \end{array} \right. \) on the product probability space \((\mathcal{X}_+ \times \mathcal{X}_+, \mathcal{B}(\mathcal{X}_+) \times \mathcal{B}(\mathcal{X}_+), Q^x \times Q^x)\). This is a continuous process.

(Step 5) We will prove Proposition 7.28 in this step. Let \( P^x \) be the image measure given by \( P^x = Q_x \circ X^{-1} \) on \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\), where \( \mathcal{X} = C(\mathbb{R}; \mathbb{R}^3) \). Then the coordinate process \((X_t)_{t \in \mathbb{R}}\) on the probability space \( X = (\mathcal{X}, \mathcal{B}(\mathcal{X}), P^x) \) satisfies that

\[
\int d\mu_p \mathbb{E}_{P^x} [f_0(X_{t_0}) \cdots f_n(X_{t_n})] = (f_0, e^{-(t_1-t_0)L_p} f_1 \cdots e^{-(t_n-t_{n-1})L_p} f_n) \mathcal{E}
\]

for \(-\infty < t_0 \leq t_1 \leq \cdots \leq t_n\). By this we can see that \( X = (X_t)_{t \in \mathbb{R}} \) satisfies (1)-(4) of Proposition 7.28.
(Step 1) The family of set functions \(\{\nu_t\}_{t \in \mathbb{R}^+} \) given by (7.99) and (7.100) satisfies the consistency condition:

\[
\nu_{t_0 \ldots t_n + m} \left( \prod_{i=0}^{n} A_i \times \prod_{i=n+1}^{n+m} \mathbb{R}^3 \right) = \nu_{t_0 \ldots t_n} \left( \prod_{i=0}^{n} A_i \right)
\]

and by the Kolmogorov extension theorem [KS91, Theorem 2.2] there exists a probability measure \(\nu_\infty\) on \((\mathbb{R}^3)^{[0, \infty)}, \mathcal{B}((\mathbb{R}^3)^{[0, \infty)})\) such that

\[
\nu_t(A) = \mathbb{E}_{\nu_\infty}[\mathbb{I}_A(Y_t)], \quad \nu_{t_0 \ldots t_n} \left( \prod_{i=0}^{n} A_i \right) = \mathbb{E}_{\nu_\infty}[\prod_{j=0}^{n} \mathbb{I}_{A_j}(Y_t)], \quad n \geq 1,
\]

where \(Y_t(\omega) = \omega(t), \omega \in (\mathbb{R}^3)^{[0, \infty)}\), is the coordinate process. Then the process \(Y = (Y_t)_{t \geq 0}\) on the probability space \((\mathbb{R}^3)^{[0, \infty)}, \mathcal{B}((\mathbb{R}^3)^{[0, \infty)}), \nu_\infty)\) satisfies that

\[
(f_0, e^{-(t_1 - t_0)L_p} f_1 \ldots e^{-(t_n - t_{n-1})L_p} f_n)_{\mathcal{H}_p} = \mathbb{E}_{\nu_\infty} \left[ \prod_{j=0}^{n} f_j(Y_t) \right],
\]

\[
(1, f)_{\mathcal{H}_p} = (1, e^{-tL_p} f)_{\mathcal{H}_p} = \mathbb{E}_{\nu_\infty} [f(Y_t)] = \mathbb{E}_{\nu_\infty} [f(Y_0)]
\]

for \(f_j \in L^\infty(\mathbb{R}^3), j = 0, 1, \ldots, n\).

(Step 2) We now see that the process \(Y\) has a continuous version.

**Lemma 7.29** The process \(Y\) on \((\mathbb{R}^3)^{[0, \infty)}, \mathcal{B}((\mathbb{R}^3)^{[0, \infty)}), \nu_\infty)\) has a continuous version.

**Proof:** We note that by (7.95), (7.96) and Proposition 7.24 \(E_{\nu_\infty}[|Y_t - Y_s|^{2n}]\) can be expressed in terms of the diffusion process \(X_t = (X_t^s)_{t \geq 0}\). Since

\[
E_{\nu_\infty}[|Y_t - Y_s|^{2n}] = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \mathbb{E}_{\nu_\infty} [Y_t^{2n-k}Y_s^k],
\]

the left hand side above can be expressed in terms of \(e^{-tL_p}\) as

\[
E_{\nu_\infty}[|Y_t - Y_s|^{2n}] = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (x^{2n-k}, e^{-(t-s)L_p}x^k)_{\mathcal{H}_p}
\]

\[
= \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (x^{2n-k} \varphi_p, e^{-(t-s)K}x^k \varphi_p)_{L^2} e^{(t-s)\text{inf}(L_p)}.
\]

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Furthermore by Feynman-Kac formula, i.e., Proposition 7.24, the right-hand side above can be expressed in terms of $X^x = (X^x_t)_{t \geq 0}$ as

\[ E_{\nu_\infty} |Y_t - Y_s|^{2n} = \int d\mu_p \mathbb{E} \left[ |X^x_{t-s} - X^x_0|^{2n} \varphi_p(X^x_0) \varphi_p(X^x_{t-s}) e^{-\int_0^{t-s} W(X^x_r)dr} \right] e^{(t-s) \inf_\nu L_p}. \]

Since $W \geq 0$,

\[ E_{\nu_\infty} |Y_t - Y_s|^{2n} \leq \|\varphi_p\|_\infty^2 \int d\mu_p \mathbb{E} \left[ |X^x_{t-s} - X^x_0|^{2n} \right]. \]

We next estimate $E [ |X^x_t - X^x_s|^{2n} ]$. Since $X^x_t$ is the solution to the stochastic differential equation:

\[ X^x_{t+s} - X^x_s = \int_s^t b(X^x_r)dr + \int_s^t \sigma(X^x_r) \cdot dB_r, \]

we have

\[ E [ |X^x_{t+s} - X^x_s|^{2n} ] \leq 2^{n-1} \mathbb{E} \left[ \left| \frac{t-s}{2^n} \|b\|_\infty^{2n} + \sum_{\nu=1}^3 \int_s^t \sigma_{\mu\nu}(X^x_r) dB_r^\nu \right|^{2n} \right]. \]

By the Burkholder-Davies-Gundy inequality [KS91, Theorem 3.28], we have

\[ E \left[ \left| \int_s^t \sigma_{\mu\nu}(X^x_r) dB_r^\nu \right|^{2n} \right] \leq (n(2n-1))^n |t-s|^{n-1} \mathbb{E} \left[ \int_s^t |\sigma_{\mu\nu}(X^x_r)|^{2n} dr \right] \leq (n(2n-1))^n |t-s|^{n} \|\sigma_{\mu\nu}\|_\infty^{2n}. \]

Then $E [ |X^x_t - X^x_s|^{2n} ] \leq C|t-s|^n$ with some constant $C$ independent of $s$ and $t$, and

\[ E_{\nu_\infty} |Y_t - Y_s|^{2n} \leq C|t-s|^n \] (7.97)

follows. Thus $Y = (Y_t)_{t \geq 0}$ has a continuous version by Kolmogorov-Čentov continuity theorem [KS91, Theorem 2.8]. □

Let $\bar{Y} = (\bar{Y}_t)_{t \geq 0}$ be the continuous version of $Y$ on $((\mathbb{R}^3)^{0,\infty}, \mathcal{B}((\mathbb{R}^3)^{0,\infty}), \nu_\infty)$. The image measure of $\nu_\infty$ on $(\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+))$ with respect to $\bar{Y}$ is denoted by $Q$, i.e., $Q = \nu_\infty \circ \bar{Y}^{-1}$, and $\bar{Y}_t(\omega) = \omega(t)$ for $\omega \in \mathcal{X}_+$ is the coordinate process. Then we
constructed a stochastic process \( \tilde{Y} = (\tilde{Y}_t)_{t \geq 0} \) on \( (\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+), Q) \) such that \( \hat{Y} \overset{d}{=} \tilde{Y} \).

Then (7.95) and (7.96) can be expressed in terms of \( \tilde{Y} \) as

\[
(f_0, e^{-(t_1-t_0)L_p}f_1 \cdots e^{-(t_n-t_{n-1})L_p}f_n)_{\mathcal{F}_p} = \mathbb{E}_Q \left[ \prod_{j=0}^{n} f_j(\tilde{Y}_{t_j}) \right],
\]

\[
(1, f)_{\mathcal{F}_p} = (1, e^{-tL_p}f)_{\mathcal{F}_p} = \mathbb{E}_Q \left[ f(\tilde{Y}_t) \right] = \mathbb{E}_Q \left[ f(\tilde{Y}_0) \right]
\]

for \( 0 \leq t \) and \( 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n \).

(Step 3) Define the regular conditional probability measure on \( \mathcal{X}_+ \) by

\[
Q^x(\cdot) = Q(\cdot | \tilde{Y}_0 = x)
\]

for each \( x \in \mathbb{R}^3 \). It is well defined, since \( \mathcal{X}_+ \) is a Polish space (completely separable metrizable space). See e.g., [KS91, Theorems 3.18. and 3.19]. Since the distribution of \( \tilde{Y}_0 \) equals to \( d\mu_p \), note that \( Q(A) = \int d\mu_p \mathbb{E}_{Q^x}[1_A] \). Then the stochastic process \( \tilde{Y} = (\tilde{Y}_t)_{t \geq 0} \) on \( (\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+), Q^x) \) satisfies

\[
(f_0, e^{-(t_1-t_0)L_p}f_1 \cdots e^{-(t_n-t_{n-1})L_p}f_n)_{\mathcal{F}_p} = \int d\mu_p \mathbb{E}_{Q^x} \left[ \prod_{j=0}^{n} f_j(\tilde{Y}_{t_j}) \right],
\]

\[
(1, e^{-tL_p}f)_{\mathcal{F}_p} = (1, f)_{\mathcal{F}_p} = \int dx \varphi_p^2(x)\mathbb{E}_{Q^x} \left[ f(\tilde{Y}_0) \right] = \int dx \varphi_p^2(x)f(x).
\]

Lemma 7.30 \( \tilde{Y} \) is a Markov process on \( (\mathcal{X}_+, \mathcal{B}(\mathcal{X}_+), Q^x) \) with respect to the natural filtration \( (\mathcal{M}_r)_{r \geq 0} \), where \( \mathcal{M}_s = \sigma(\tilde{Y}_r, 0 \leq r \leq s) \).

Proof: Let

\[
p_t(x, A) = \left( e^{-tL_p}1_A \right)(x), \quad A \in \mathcal{B}(\mathbb{R}^3), \quad t \geq 0.
\]

Notice that \( p_t(x, A) = \mathbb{E}[1_A(X^x_t)] \). Then the finite dimensional distribution of \( \tilde{Y} \) is

\[
\mathbb{E}_{Q^x} \left[ \prod_{j=1}^{n} 1_{A_j}(\tilde{Y}_{t_j}) \right] = \int \prod_{j=1}^{n} 1_{A_j}(x_j) \prod_{j=1}^{n} p_{t_j-t_{j-1}}(x_{j-1}, dx_j)
\]

with \( t_0 = 0 \) and \( x_0 = x \) by (7.99). We show that \( p_t(x, A) \) is a probability transition kernel, i.e., (1) \( p_t(x, \cdot) \) is a probability measure on \( \mathcal{B}(\mathbb{R}^3) \), (2) \( p_t(x, A) \) is Borel measurable with respect to \( x \), (3) the Chapman-Kolmogorov equality

\[
\int p(s, y, A)p(t, x, dy) = p(s + t, x, A)
\]

(7.102)
is satisfied. Note that \( e^{-tL_p} \) is positivity improving. Then \( 0 \leq e^{-tL_p} f \leq \mathbb{1} \) for all function \( f \) such that \( 0 \leq f \leq \mathbb{1} \), and \( e^{-tL_p} \mathbb{1} = \mathbb{1} \) follows. Then \( p_t(x, \cdot) \) is the probability measure on \( \mathbb{R}^3 \) with \( p_t(x, \mathbb{R}^3) = 1 \), and (1) follows. (2) is trivial. From the semigroup property \( e^{-sL_p} e^{-tL_p} \mathbb{1}_A = e^{-(s+t)L_p} \mathbb{1}_A \), the Chapman-Kolmogorov equality \((7.102)\) follows. Hence \( p_t(x, A) \) is a probability transition kernel. We write \( \mathbb{E} \) for \( \mathbb{E}_{Q^x} \) for notational simplicity. From the identity

\[
\mathbb{E}[\mathbb{1}_A(\tilde{Y}_t)\mathbb{E}[f(\tilde{Y}_r)|\sigma(\tilde{Y}_t)]] = \mathbb{E}[\mathbb{1}_A(\tilde{Y}_t)f(\tilde{Y}_r)]
\]

for \( r > t \), it follows that

\[
\int \mathbb{1}_A(y)\mathbb{E}[f(\tilde{Y}_r)|\tilde{Y}_t = y]P_t(dy) = \int P_t(dy)\mathbb{1}_A(y)\int f(y')p(r - t, y, dy'),
\]

where \( P_t(dy) \) denotes the distribution of \( \tilde{Y}_t \) on \( \mathbb{R}^3 \). Thus

\[
\mathbb{E}[f(\tilde{Y}_r)|\tilde{Y}_t = y] = \int f(y')p(r - t, y, dy')
\]

follows almost everywhere \( y \) with respect to \( P_t(dy) \). Then

\[
\mathbb{E}[f(\tilde{Y}_r)|\sigma(\tilde{Y}_t)] = \int f(y)p(r - t, \tilde{Y}_t, dy)
\]

and

\[
\mathbb{E}[\mathbb{1}_A(\tilde{Y}_t)|\sigma(\tilde{Y}_t)] = p(r - t, \tilde{Y}_t, A)
\]

(7.103) follow. By using (7.103) and the Chapman-Kolmogorov equality we can show that

\[
\mathbb{E}\left[\mathbb{1}_A(\tilde{Y}_{t+s}) \prod_{j=0}^n \mathbb{1}_{A_j}(\tilde{Y}_{t_j})\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_A(\tilde{Y}_t)|\sigma(\tilde{Y}_s)\right]\prod_{j=0}^n \mathbb{1}_{A_j}(\tilde{Y}_{t_j})\right]
\]

for \( t_0 \leq \cdots \leq t_n \leq s \). This implies that \( \mathbb{E}[\mathbb{1}_A(\tilde{Y}_{t+s})|\mathcal{M}] = \mathbb{E}[\mathbb{1}_A(\tilde{Y}_t)|\sigma(\tilde{Y}_s)] \). Then \( \tilde{Y} \) is Markov with respect to the natural filtration under the measure \( Q^x \).

**Step 4** We extend \( \tilde{Y} = (\tilde{Y}_t)_{t \geq 0} \) to a process on the whole real line \( \mathbb{R} \). Set \( \tilde{\mathcal{X}}_+ = \mathcal{X}_+ \times \mathcal{X}_+ \), \( \tilde{\mathcal{M}} = \mathcal{B}(\mathcal{X}_+) \times \mathcal{B}(\mathcal{X}_+) \) and \( \tilde{Q}^x = Q^x \times Q^x \). Let \( (\tilde{X}_t)_{t \in \mathbb{R}} \) be the stochastic process on the product space \((\tilde{\mathcal{X}}_+, \tilde{\mathcal{M}}, \tilde{Q}^x)\), defined by for \( \omega = (\omega_1, \omega_2) \in \tilde{\mathcal{X}}_+ \),

\[
\tilde{X}_t(\omega) = \begin{cases} 
\tilde{Y}_t(\omega_1), & t \geq 0, \\
\tilde{Y}_{-t}(\omega_2), & t < 0.
\end{cases}
\]
Note that $X_0 = x$ almost surely with respect to $Q^x$ and $X_t$ is continuous in $t$ almost surely. It is trivial to see that $X_t$, $t \geq 0$, and $X_s$, $s \leq 0$, are independent, and $\dot{X}_t \overset{d}{=} \dot{X}_{-t}$.

**Step 5** Proof of Proposition 7.28

The image measure of $Q^x$ on $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$ with respect to $X$ is denoted by $P^x$, i.e., $P^x = Q^x \circ X^{-1}$. Let $X(\omega) = \omega(t), t \in \mathbb{R}, \omega \in \mathcal{F}$, be the coordinate process. Then we can see that

$$X_t \overset{d}{=} \tilde{Y}_t \quad (t \geq 0), \quad X_t \overset{d}{=} \tilde{Y}_{-t} \quad (t \leq 0). \quad (7.105)$$

Since by (Step 3), $(\tilde{Y}_t)_{t \geq 0}$ and $(\tilde{Y}_{-t})_{t \leq 0}$ are Markov processes with respect to the natural filtration $\sigma(\tilde{Y}_s, 0 \leq s \leq t)$ and $\sigma(\tilde{Y}_s, -t \leq s \leq 0)$, respectively, $(X_t)_{t \geq 0}$ and $(X_t)_{t \leq 0}$ are also Markov processes with respect to $(\mathcal{F}^+_t)_{t \geq 0}$ and $(\mathcal{F}^-_t)_{t \leq 0}$, respectively, where

$$\mathcal{F}^+_t = \sigma(X_s, 0 \leq s \leq t), \quad \mathcal{F}^-_t = \sigma(X_s, -t \leq s \leq 0).$$

Thus the diffusion property (3) follows. We also see that $(X_s)_{s \leq 0}$ and $(X_t)_{t \geq 0}$ are independent and $X_{-t} \overset{d}{=} X_t$ by (7.105) and (Step 4). Thus reflection symmetry (2) follows.

**Lemma 7.31** Let $-\infty < t_0 \leq t_1 \leq \cdots \leq t_n$. Then

$$\int d\mu_p \mathbb{E}_P [f_0(X_{t_0}) \cdots f_n(X_{t_n})] = (f_0, e^{-(t_1-t_0)L_p} f_1 \cdots e^{-(t_n-t_{n-1}L_p} f_n) \underline{x}_p. \quad (7.106)$$

**Proof:** Let $t_0 \leq \cdots \leq t_n \leq 0 \leq t_{n+1} \leq \cdots t_{n+m}$. Then we have by the independence of $(X_s)_{s \leq 0}$ and $(X_t)_{t \geq 0}$,

$$\int d\mu_p \mathbb{E}_P [f_0(X_{t_0}) \cdots f_{n+m}(X_{t_{n+m}})]$$

$$= \int d\mu_p \mathbb{E}_P [f_0(X_{t_0}) \cdots f_n(X_{t_n})] \mathbb{E}_P [f_{n+1}(X_{t_{n+1}}) \cdots f_{n+m}(X_{t_{n+m}})].$$

Since we have

$$\mathbb{E}_P [f_{n+1}(X_{t_{n+1}}) \cdots f_{n+m}(X_{t_{n+m}})]$$

$$= (e^{-t_{n+1}L_p} f_{n+1} e^{-(t_{n+2} - t_{n+1})L_p} f_{n+2} \cdots e^{-(t_{n+m} - t_{n+m-1})L_p} f_{n+m}) (x) \quad (7.107)$$

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and
\[
\mathbb{E}_{P_x} \left[ f_0(X_{t_0}) \cdots f_n(X_{t_n}) \right] = \mathbb{E}_{P_x} \left[ f_0(\tilde{Y}_{-t_0}) \cdots f_n(\tilde{Y}_{-t_n}) \right] \\
= \left( e^{+t_0 L_p} f_n e^{-((t_n-t_{n-1}) L_p f_{n-1}) \cdots e^{-(t_1-t_0) L_p f_1} \right). \tag{7.108}
\]

By (7.107) and (7.108) we have
\[
\int d\mu P_x \mathbb{E}_{P_x} \left[ f_0(X_{t_0}) \cdots f_{n+m}(X_{t_{n+m}}) \right] \\
= (e^{+t_n L_p} f_n \cdots e^{-(t_1-t_0) L_p f_1} e^{-t_{n+1} L_p f_{n+1}} \cdots e^{-(t_{n+m}-t_{n+m-1}) L_p f_{n+m})} \mathcal{H}_p. \\
= (f_1 e^{-(t_1-t_0) L_p} f_2 \cdots e^{-(t_{n+m}-t_{n+m-1}) L_p f_{n+m}}) \mathcal{H}_p.
\]

Hence (7.106) follows. \qed

From Lemma 7.31 it follows that for any \( s \in \mathbb{R} \),
\[
\int d\mu P_x \mathbb{E}_{P_x} \left[ \prod_{j=0}^n f_j(X_{t_j+s}) \right] = \int d\mu P_x \mathbb{E}_{P_x} \left[ \prod_{j=0}^n f_j(X_{t_j+s}) \right].
\]

Hence shift invariance (4) is obtained. \qed

We denote \( \mathbb{E}^x \) for \( \mathbb{E}_{P_x} \) in what follows.

### 7.4 Absence of ground state

#### 7.4.1 The Nelson model by path measures

Now we construct a Feynman-Kac formula for \( e^{-tH} \) by using the diffusion process \( X \). Let \( \phi_E(f) \) be the Euclidean scalar field on a probability space \((\mathcal{D}_E, \Sigma_E, \mu_E)\), which is the Gaussian random variable indexed by \( f \in L^2(\mathbb{R}^4) \) with \( \mathbb{E}_{\mu_E}[\phi_E(f)] = 0 \) and the covariance given by
\[
\mathbb{E}_{\mu_E}[\phi_E(f)\phi_E(g)] = \frac{1}{2} \langle \hat{f}, \hat{g} \rangle.
\]

Euclidean scalar field \( L^2(\mathcal{D}_E) \) and \( L^2(\mathcal{D}) \) are connected through some isometry \( j_t \). Let \( j_t: L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^4) \) be given by
\[
j_t f(x_0, x) = \frac{1}{2\pi} \int dk_0 e^{-i(t-x_0)k_0} \left( \omega^{1/2}(\omega^2 + |k_0|^2)^{-1/2} f \right)(x). \tag{7.109}
\]
Then we can have the formula \((j_s f, j_t g) = (f, e^{-|t-s|\omega} g)\). In particular
\[
j_s^* j_s = e^{-|t-s|\omega}.
\] (7.110)

Let \(J_t = \Gamma(j_t) : L^2(\mathcal{D}) \to L^2(\mathcal{D}_E)\) be the isometry defined by \(J_t 1 = 1\) and
\[
J_t \prod_{j=1}^n \phi(f_j) = \prod_{j=1}^n \phi_E(j_t f_j).
\]
From the definition of \(J_t\), the identity
\[
J_s^* J_s = e^{-|t-s|H_t}
\] (7.111)
follows. Thus the semigroup \(e^{-tH_t}\) can be factorized by \(J_t\) and it can be expressed as
\[
(\Phi, e^{-tH} \Psi)_{L^2(\mathcal{D})} = (J_0 \Phi, J_t \Psi)_{L^2(\mathcal{D}_E)}.
\] (7.112)

**Theorem 7.32 (Feynman-Kac formula)**

**Suppose Assumptions 7.8, 7.10, 7.12, 7.16 and 7.18.** Then we have
\[
(F, e^{-TH} G)_{\mathcal{D}} = \int d\mu_p \mathbb{E}_x \left[ (J_0 F(X_0), e^{\phi_p(K_T)} J_T G(X_T))_{L^2(\mathcal{D}_E)} \right],
\] (7.113)
where \(K_T = \int_0^T j_s \omega^{-1/2} \rho(\cdot - X_s) ds\) is an \(L^2(\mathbb{R}^4)\)-valued integral. In particular it follows that
\[
(1, e^{-TH} 1)_{\mathcal{D}} = \int d\mu_p \mathbb{E}_x \left[ e^{(1/2) \int_0^T dt \int_0^T ds \mathcal{W}(X_t, X_s, |t-s|)} \right],
\] (7.114)
where \(1 \in L^2(\mathbb{R}^3 \times \mathcal{D})\) and
\[
\mathcal{W}(X, Y, |t|) = \frac{1}{2} (\rho(\cdot - X), \omega^{-1} e^{-|t|\omega} \rho(\cdot - Y)).
\] (7.115)

**Proof:** By the Trotter-Kato product formula:
\[
e^{-tH} = s\lim_{n\to\infty} \left( e^{-(t/n)\phi_p} e^{-(t/n)H_t} \right)^n,
\]

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the factorization formula (7.111), and Markov property of $E_t = J_t J_t^*$, we have

$$(F, e^{-tH}G) = \lim_{n \to \infty} \int d\mu_p \mathbb{E}^x \left[ \left( J_0 F(X_0), e^{-\sum_{j=0}^{n} \frac{t\rho(jt)}{\pi} (-X_{jt/n})} J_t G(X_t) \right) \right].$$

Note that the map $\mathbb{R} \to L^2(\mathbb{R}^3)$, $s \mapsto \omega^{-1/2} \rho(-X_s)$, is strongly continuous almost surely. Hence the map $\mathbb{R} \to L^2(\mathcal{E}_E)$, $s \mapsto \phi_E(j_t \rho(-X_s))$, is also strongly continuous. By a simple limiting argument (7.113) follows. Let $F = G = 1$. Since $\phi_E$ is a Gaussian random variable, we have

$$(1, e^{-tH}1) = \int \mathbb{E}^x \left[ (1, e^{\phi_E(K_T)} 1) \right] = \int \mathbb{E}^x \left[ e^{(1/4)\|K_T\|^2_{L^2(\mathbb{R}^4)}} \right].$$

Hence $\|K_T\|^2_{L^2(\mathbb{R}^4)} = \int dt \int ds \mathcal{W}(X_t, X_s, |t-s|)$ and (7.114) follows.

7.4.2 Absence of ground states

From Theorem 7.32 we can obtain a useful lemma to show the absence of ground states.

**Theorem 7.33 (Positivity improving)** Suppose Assumptions 7.8, 7.10, 7.12, 7.16 and 7.18. Then $e^{-tH}$ is positivity improving for all $t > 0$.

**Proof:** Let $F, G \in L^2(\mathbb{R}^3 \times \mathcal{E})$ be such that $F \geq 0$ and $G \geq 0$ but $F \neq 0$ and $G \neq 0$. Define $\mathcal{D}_F = \{x \in \mathbb{R}^3 | F(x, \cdot) \neq 0\}$ and $\mathcal{D}_G = \{x \in \mathbb{R}^3 | G(x, \cdot) \neq 0\}$. Note that $\int_{\mathcal{D}_F} dx > 0$ and $\int_{\mathcal{D}_G} dx > 0$. Let $\mathcal{K}^x = \{\omega \in \mathcal{E}| X_0(\omega) = x, X_t(\omega) \in \mathcal{D}_F\}$. It follows that

$$\int_{\mathcal{D}_F} d\mu_p \int_{\mathcal{K}^x} dP^x = (1, e^{-tL_p} 1) \mathcal{K}^x = (\varphi_p 1_{\mathcal{D}_F}, e^{-tK} \varphi_p 1_{\mathcal{D}_F}) L^2 e^{t \inf_\sigma(K)} > 0$$

by Lemma 2.49. Thus $\mathbb{R}^3 \times \mathcal{E} \supset \cup_{x \in \mathcal{D}_F} \mathcal{K}^x$ has a positive measure with respect to $d\mu_p dP^x$ and

$$(F, e^{-tH}G) \geq \int_{\mathcal{D}_F} d\mu_p \int_{\mathcal{K}^x} dP^x (J_0 F, e^{\phi(K_\omega)} J_t G(X_t)) > 0,$$

since $(J_0 F(X_0(\omega)), e^{\phi(K_\omega)} J_t G(X_t(\omega))) > 0$ for $\omega \in \cup_{x \in \mathcal{D}_F} \mathcal{K}^x$. Then the theorem follows.

By Theorem 7.33 and Perron-Frobenius arguments, we immediately have the corollary.
Corollary 7.34 (Uniqueness of ground state) Suppose Assumptions 7.8, 7.10, 7.12, 7.16 and 7.18. Then the ground state of \( \varphi_g \) is unique and \( \varphi_g > 0 \) if it exists.

In particular \((\mathbb{1}, \varphi_g) > 0\).

From Corollary 7.34 we can see that \( e^{-TH} \mathbb{1}/\|e^{-TH} \mathbb{1}\| \) converges to the ground state as \( T \to \infty \) if the ground state exists. We then define \( \gamma(T) \) by

\[
\gamma(T) = \frac{(\mathbb{1}, e^{-TH} \mathbb{1})}{(\mathbb{1}, e^{-2TH} \mathbb{1})}, \quad T > 0.
\]  

(7.117)

Lemma 7.35 Suppose Assumptions 7.8, 7.10, 7.12, 7.16 and 7.18. Let \( P_\Delta, \Delta \subset \mathbb{R} \), denote the spectral projection of \( H \) associated with \( \Delta \cap \sigma(H) \). Let \( E = \inf \sigma(H) \).

Then it follows that \( \lim_{T \to \infty} \gamma(T) = \|P_{\{E\}} \mathbb{1}\|^2 \). In particular \( H \) has a ground state if and only if \( \lim_{T \to \infty} \gamma(T) \neq 0 \).

Proof: Assume that \( E = 0 \). Thus \( \lim_{T \to \infty} e^{-TH} = P_{\{E\}} \). If 0 is an eigenvalue, then by Corollary 7.34 and Perron-Frobenius arguments, \( P_{\{E\}} = (u, \cdot)u \) for some \( u > 0 \). It follows that \( \lim_{T \to \infty} \gamma(T) = (u, \mathbb{1})^2 > 0 \). Next we prove the sufficient part. Assume now that there exists a sequence \( T_n \to +\infty \) such that \( \delta(T_n) \geq \epsilon \). This implies that \( (\mathbb{1}, e^{-T_nH} \mathbb{1}) \geq \epsilon (\mathbb{1}, e^{-2T_nH} \mathbb{1})^{1/2} \). Letting \( n \to \infty \), we obtain that \( \|P_{\{E\}} \mathbb{1}\| \geq \epsilon \). Then \( H \) has a ground state.

The denominator of \( \gamma(T) \) is computed as

\[
\|e^{-TH} \mathbb{1}\|^2 = (\mathbb{1}, e^{-2TH} \mathbb{1}) = \int d\mu_{\mathbb{P}} e^{(1/2) \int_T -T ds f^T_t ds f^T_t dt \varphi(X_s, X_t, |s-t|)}
\]

by the shift invariance (Proposition 7.28) of \( X_t \). Then \( \gamma(T) \) can be expressed as

\[
\gamma(T) = \frac{\left(\int d\mu_{\mathbb{P}} e^{(1/2) \int_T -T ds f^T_t ds f^T_t dt \varphi(X_s, X_t, |s-t|)}\right)^2}{\int d\mu_{\mathbb{P}} e^{(1/2) \int_T -T ds f^T_t ds f^T_t dt \varphi(X_s, X_t, |s-t|)}}.
\]  

(7.118)

Let \( \mu_T \) be the probability measure on \((\mathbb{R}^3 \times \mathcal{X}, \mathcal{B}(\mathbb{R}^3) \times \mathcal{B}(\mathcal{X}))\) defined by for \( O \in \mathcal{B}(\mathbb{R}^3) \times \mathcal{B}(\mathcal{X}), \)

\[
\mu_T(O) = \frac{1}{Z_T} \int d\mu_{\mathbb{P}} e^{(1/2) \int_T -T ds f^T_t ds f^T_t dt \varphi(X_s, X_t, |s-t|)},
\]  

(7.119)

where \( Z_T \) denotes the normalizing constant such that \( \mu_T \) becomes a probability measure.

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Lemma 7.36 Suppose Assumptions 7.8, 7.10, 7.12, 7.16 and 7.18. Then it follows that
\[ \gamma(T) \leq \mathbb{E}_{\mu_T}[e^{-\int_0^T ds \int_0^T dt W(X_s, X_t, |s-t|)}]. \] (7.120)

Proof: The numerator of (7.118) can be estimated by the Schwartz inequality with respect to \(d\mu_p\) and the reflection symmetry of \(X\), and then
\[ \left( \int d\mu_p \mathbb{E}^x[e^{(1/2) \int_0^T ds \int_0^T dt W}] \right)^2 \leq \int d\mu_p \left( \mathbb{E}^x[e^{(1/2) \int_0^T ds \int_0^T dt W}] \right)^2. \]

Since \(X_t\) and \(X_s\) for \(s \leq 0 \leq t\) are independent, thus we have
\[ \left( \int d\mu_p \mathbb{E}^x[e^{(1/2) \int_0^T ds \int_0^T dt W}] \right)^2 \leq \int d\mu_p \mathbb{E}^x[e^{(1/2)(\int_0^T ds \int_0^T dt W + \int_0^T ds \int_0^T dt W)}]. \]

Moreover \(\int_{-T}^0 \int_{-T}^0 + \int_0^T \int_0^T = \int_{-T}^T \int_{-T}^T - 2 \int_{-T}^0 \int_0^T\) yields that (Figure 13)
\[ \left( \int d\mu_p \mathbb{E}^x[e^{(1/2) \int_0^T ds \int_0^T dt W}] \right)^2 \leq \int d\mu_p \mathbb{E}^x[e^{-\int_0^T ds \int_0^T dt W} + (1/2) \int_{-T}^T ds \int_{-T}^T dt W]. \]

Then the lemma follows. \(\square\)

In order to show that the right-hand side of (7.120) converges to zero as \(T \to \infty\), we estimate its upper bound. Let
\[ \mathcal{W}_\infty(X, Y, |t|) = \frac{1}{2} \int dk \hat{\rho}(k)^2 e^{-ik \cdot (X-Y)} e^{-|t||k|} \] (7.121)

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or it is expressed in the position representation as

\[
\Psi_{\infty}(X, Y, |t|) = \frac{1}{2}(\rho(\cdot - X), \omega_\infty^{-1} e^{-t|\omega_\infty|} \rho(\cdot - Y))
\]  

(7.122)

with

\[
\omega_\infty = \sqrt{-\Delta}.
\]  

(7.123)

The next proposition on the upper and lower Gaussian bound of the integral kernel \(e^{-t\omega^2}(x, y)\) is the key ingredient of the proof of the absence of ground states of \(H\).

**Proposition 7.37** Suppose Assumption 7.12. Then the semigroup \(e^{-t\omega^2}\) has an integral kernel \(e^{-t\omega^2}(x, y)\), and there exist constants \(C_1, \cdots, C_4\) such that

\[
C_1 e^{-C_2 t\omega^2}(x, y) \leq e^{-t\omega^2}(x, y) \leq C_3 e^{-C_4 t\omega^2}(x, y)
\]  

(7.124)

for \(t \geq 0\) and a.e. \(x, y \in \mathbb{R}^3\).

**Proof:** Conjugating by the unitary \(U : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3, e^2(x)dx)\), \(f \mapsto c^{-1} f\), we obtain

\[
\tilde{\omega}^2 = U\omega^2 U^{-1} = \tilde{h}_0 + m^2(x),
\]

where

\[
\tilde{h}_0 = \tilde{h}_0(D, x) = c^{-2}(x) \sum_{\mu=1}^3 D_\mu a_{\mu\nu}(x) D_\nu.
\]

Throughout \((f, g)_2\) denotes \(\int \tilde{f}(x) g(x)c^2(x)dx\) and \(\|f\|_p^p = \int |f(x)|^p c^2(x)dx\). Note that \(C_0 \int |f(x)|^p dx \leq \|f\|_p^p \leq C_1 \int |f(x)|^p dx\). Let \(f \in C_0^\infty(\mathbb{R}^3)\). Then

\[
C_0(f, \tilde{h}_0 f) \leq \mathcal{E}_0(f, f) \leq C_1(f, \tilde{h}_0 f).
\]

From this \(D(\tilde{h}_0^{1/2}) = H^1(\mathbb{R}^3)\) follows and

\[
C_0(\tilde{h}_0^{1/2} f, \tilde{h}_0^{1/2} f)_2 \leq \mathcal{E}_0(f, f) \leq C_1(\tilde{h}_0^{1/2} f, \tilde{h}_0^{1/2} f)_2
\]  

(7.125)

for \(f \in H^1(\mathbb{R}^3)\). Notice that \(e^{-t\tilde{\omega}}(x, y)\) denotes an integral kernel of \(e^{-t\omega^2}\) with respect to the measure \(c^2(x)dx\), while \(e^{-t\omega^2}(x, y)\) is that of \(e^{-t\omega^2}\) with respect to \(dx\). Since

\[
(f, e^{-t\omega^2} g)_{L^2(\mathbb{R}^3)} = \int dx \int dy f(x)e^{-t\omega^2}(x, y)g(y) = (U f, e^{-t\tilde{\omega}^2} U g)_2
\]

\[
= \int c^2(x)dx \int c^2(y)dy \frac{1}{c(x)} f(x)e^{-t\omega^2}(x, y) \frac{1}{c(y)} g(y),
\]

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we note that
\[ e^{-t\omega^2}(x, y) = c(x)e^{-t\tilde{\omega}^2}(x, y)c(y) \]
almost everywhere. So it suffices to prove proposition for \( e^{-t\tilde{\omega}^2} \). We know from [PE84, Theorems 3.4 and 3.6] that \( e^{-t\tilde{\omega}_0} \) has an integral kernel with
\[ C_1 e^{-C_2 t\omega} (x, y) \leq e^{-t\tilde{\omega}_0} (x, y) \leq C_3 e^{-C_4 t\omega} (x, y) \]  
(7.126)
for a.e. \( x, y \in \mathbb{R}^3 \) and all \( t > 0 \). Notice that \( e^{-t(\tilde{\omega}_0 + V)} \), \( \sup_x |V(x)| < \infty \), is positivity preserving and bounded on \( L^\infty(\mathbb{R}^3) \). This can be proven by the Trotter product formula. Then by (7.126) we see that \( \|e^{-t\tilde{\omega}_0} f\|_\infty \leq C t^{-3/4} \|f\|_2 \), which is equivalent to
\[ \|f\|_6^2 \leq C(\tilde{\omega}_0^{1/2} f, \tilde{\omega}_0^{1/2} f)_2 \]  
(7.127)
by Proposition 7.5. Since \( m^2(x) \geq 0 \), the upper bound follows from the Trotter product formula. Let us now prove the lower bound, following [Sem97, Theorem 6.1]. Since \( m^2(x) \leq \langle x \rangle - \beta \) with \( \beta > 2 \), we see that \( m^2 \in L^{3/2}(\mathbb{R}^3) \). Then we have \( (m^2 f, f) \leq \|m^2\|_{3/2} \|f\|_6^2 \). By the Sobolev inequality we see that \( \|f\|_6 \leq C_1 \mathcal{E}_4(f, f) \), and then together with (7.125),
\[ \gamma(m^2 f, f)_2 \leq (\tilde{\omega}_0^{1/2} f, \tilde{\omega}_0^{1/2} f)_2 \]
for some \( \gamma > 0 \). Set now \( w(x) = -\gamma m^2(x)/4 \). Then \( \tilde{\omega}_0 + 2w \geq \frac{1}{2} \tilde{\omega}_0 \) in the sense of form. We see, together with (7.127), that
\[ \|f\|_6^2 \leq C_2((\tilde{\omega}_0 + 2w)^{1/2} f, (\tilde{\omega}_0 + 2w)^{1/2} f)_2 \]  
(7.128)
By Proposition 7.5 and the fact that \( e^{-t(\tilde{\omega}_0 + w)} \) is positivity preserving and bounded on \( L^\infty(\mathbb{R}^3) \), (7.128) is equivalent to
\[ \|e^{-t(\tilde{\omega}_0 + 2w)} f\|_\infty \leq C_3 t^{-3/4} \|f\|_2 \]
and \( e^{-t(\tilde{\omega}_0 + 2w)} \) is ultracontractive. Then \( e^{-t(\tilde{\omega}_0 + 2w)} \) has an integral kernel, furthermore it can be estimated as
\[ e^{-t(\tilde{\omega}_0 + 2w)}(x, y) \leq C_4 t^{-3/2} \]
almost everywhere. We prove in Lemma 7.38 that \( \lambda \mapsto e^{-t(\tilde{\omega}_0 + \lambda w)}(x, y) \) is logarithmically convex for all \( t > 0 \) and a.e. \( x, y \in \mathbb{R}^3 \). Then we have
\[ e^{-t(\tilde{\omega}_0 + w)}(x, y) = e^{-t(\tilde{\omega}_0 + \frac{1}{2} 0 w + \frac{1}{2} 2w)}(x, y) \leq t^{-3/4} e^{-\tilde{\omega}_0}(x, y)^{1/2}. \]  
(7.129)
Applying again the log-convexity we get that

$$e^{-t\tilde{h}_0(x,y)} = e^{-t(\tilde{h}_0 + sm^2 + (1-s)w)(x,y)} \leq e^{-t(\tilde{h}_0 + m^2)(x,y)} e^{-t(\tilde{h}_0 + w)(x,y)^{1-s}}$$

with \( s = \gamma/(4 + \gamma) \). Hence using (7.129) we obtain

$$e^{-t\tilde{h}_0(x,y)} e^{-t(1+s)/2} \leq e^{-t(\tilde{h}_0 + m^2)(x,y)} e^{-t(\tilde{h}_0 + w)(x,y)^{1-s}}$$

which, together with (7.126), implies the proposition. \( \square \)

It remains to show the log-convexity of \( e^{-t(\tilde{h}_0 + w)}(x,y) \).

**Lemma 7.38** \( \mathbb{R} \ni \lambda \mapsto e^{-t(\tilde{h}_0 + \lambda w)}(x,y) \in \mathbb{R} \) is logarithmically convex for all \( t > 0 \) and a.e. \( x, y \in \mathbb{R}^3 \), i.e., for \( 0 \leq s \leq 1 \),

$$e^{-t(\tilde{h}_0 + s\lambda + (1-s)\lambda')w)(x,y)} \leq e^{-t(\tilde{h}_0 + \lambda w)}(x,y) e^{-t(\tilde{h}_0 + \lambda' w)(x,y)^{1-s}}.$$

**Proof:** Set \( t = 1 \). By the Trotter product formula we have

$$e^{-(\tilde{h}_0 + \lambda w)}(x,y) = \left( s - \lim_{n \to \infty} \left( e^{-\tilde{h}_0/n} e^{-\lambda w/n} \right)^n \right) (x,y). \quad (7.130)$$

Let \( A_\lambda(x,y) \) and \( B_\lambda(x,y) \) be the kernels of two operators \( A_\lambda \) and \( B_\lambda \) assumed to be log-convex in \( \lambda \). Then the kernel of \( A_\lambda B_\lambda \):

$$A_\lambda B_\lambda(x,y) = \int_{\mathbb{R}^3} A_\lambda(x,z) B_\lambda(z,y) dz$$

is also log-convex in \( \lambda \). Then the kernel of \( e^{-\tilde{h}_0/n} e^{-\lambda w/n}(x,y) = e^{-\tilde{h}_0/n}(x,y) e^{-\lambda w(y)/n} \) is log-convex in \( \lambda \). Then the lemma follows from the Trotter product formula (7.130). \( \square \)

**Corollary 7.39 (Positivity improving)** Suppose Assumption 7.12. Then \( e^{-t\omega^2} \) is positivity improving.

**Proof:** This immediately follows from the Gaussian bound (7.124). \( \square \)

**Corollary 7.40** Suppose Assumption 7.12. Then it follows that

$$\|\omega^{-n/2} f\| \leq C \|\omega_{\infty}^{-n/2} f\|.$$
Proof: Since $\omega^{-n/2} = C_n \int_0^\infty e^{-\omega^2 t(n+4)/4} dt$ with $C_n = (\int_0^\infty e^{-s(n+4)/4} ds)^{-1}$. Hence the corollary follows from Proposition 7.37. \qed

Lemma 7.41 Suppose Assumptions 7.12 and 7.16. Then

$$\mathcal{W}(X, Y, |t|) \geq 0, \quad \mathcal{W}_\infty(X, Y, |t|) \geq 0$$

and there exist constants $C_j > 0$, $j = 1, 2, 3, 4$, such that

$$C_1 \mathcal{W}_\infty(X, Y, C_2|t|) \leq \mathcal{W}(X, Y, |t|) \leq C_3 \mathcal{W}_\infty(X, Y, C_4|t|) \quad (7.131)$$

for all $X, Y \in \mathbb{R}^3$ and $t \in \mathbb{R}$. In particular it follows that

$$\gamma(T) \leq \mathbb{E}_{\mu_T}\left[ e^{-C_1 \int_{-t}^t ds \int_t^0 dt \mathcal{W}_\infty(X_s, X_t, C_2|s-t|)} \right]. \quad (7.132)$$

Proof: Set $\rho_X(x) = \rho(x - X)$. We note that the function $f(x) = e^{-\sqrt{x}}$ on $[0, \infty)$ is completely monotone, i.e., $(-1)^n df(x)/dx^n \geq 0$ and that $f(+0) = 0$. Then there exists a Borel probability measure $m$ on $[0, \infty)$ such that

$$e^{-\sqrt{x}} = \int_0^\infty e^{-sx} dm(s)$$

and it is indeed exactly given by

$$dm(s) = \frac{1}{2\sqrt{\pi}} \frac{e^{-1/(4s)}}{s^{3/2}} ds.$$

Hence

$$e^{-t\omega} = \int_0^\infty e^{-st^2\omega^2} dm(s) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{te^{-t^2/(4s)}}{s^{3/2}} e^{-s\omega^2} ds.$$

It follows that

$$\mathcal{W}(X, Y, |t|) = \frac{1}{2} \int_t^\infty dr \rho_X e^{-\omega^2 \rho_Y} = \frac{1}{4\sqrt{\pi}} \int_t^\infty dr \int_0^\infty \frac{re^{-r^2/(4p)}}{p^{3/2}} (\rho_X e^{-\rho_Y^2}) dp.$$ 

Hence $\mathcal{W}(X, Y, |t|) > 0$ follows, since $e^{-p\omega^2}$ is positivity improving for $p > 0$. $\mathcal{W}_\infty(X, Y, |t|) > 0$ also follows in the same way as above. Since $\rho_X$ and $\rho_Y$ are
nonnegative, by the Gaussian bound $c_1 e^{-c_2 t \omega_\infty(x, y)} \leq e^{-t \omega^2(x, y)} \leq c_3 e^{-c_4 t \omega_\infty(x, y)}$, we can see that by changing a variable,

$$c_1 c_2 \mathcal{W}_\infty(X, Y, \sqrt{c_2} t) \leq \mathcal{W}(X, Y, |t|) \leq c_3 c_4 \mathcal{W}_\infty(X, Y, \sqrt{c_4} |t|).$$

Then the lemma follows. \hfill \Box

Let us take $\lambda$ such that

$$\frac{1}{\delta + 1} < \lambda < 1,$$

where $\delta$ is the positive constant given in Assumption 7.10. Let

$$A_T = \mathbb{R}^3 \times \left\{ \sup_{|s| \leq T} |X_s| \leq T^\lambda \right\} \subset \mathbb{R}^3 \times \mathcal{X}.$$  \hfill (7.134)

We divide the right-hand side of (7.132) into $E_{\mu_T} [1 L_{A_T} \cdots] + E_{\mu_T} [1 L_{A_T^c} \cdots]$. Then in order to prove the absence of ground state it is enough to show that $\lim_{T \to \infty} E_{\mu_T} [1 L_{A_T} \cdots] = 0$ and $\lim_{T \to \infty} E_{\mu_T} [1 L_{A_T^c} \cdots] = 0$.

**Lemma 7.42** [LMS02] Suppose Assumptions 7.8, 7.10, 7.12, 7.16 and 7.18. Then it follows that

$$\lim_{T \to \infty} E_{\mu_T} \left[ \mathbb{I}_{A_T} e^{-C_1 \int_0^T ds f_0^T dt \mathcal{W}_\infty(X_s, X_t, C_2 |s-t|)} \right] = 0.$$  \hfill (7.135)

**Proof:** Since the integral kernel of $e^{-|t| \omega_\infty}$ is

$$e^{-|t| \omega_\infty}(x, y) = \frac{1}{\pi^2} \frac{|t|}{(|x-y|^2 + |t|^2)^2},$$

we have

$$\mathcal{W}_\infty(X, Y, |t|) = \frac{1}{4\pi^2} \int dx \int dy \frac{\rho(x) \rho(y)}{|(x-X) - (y-Y)|^2 + |t|^2}. \hfill (7.136)$$

On $A_T$ we know that $|(X_s - x) - (X_t - y)|^2 + |t-s|^2 \leq 8T^{2\lambda} + 2|x-y|^2 + |t-s|^2$. Let

$$\Delta_T = \{(s, t)| 0 \leq s \leq T, 0 \leq t \leq T, 0 \leq s + t \leq T/\sqrt{2}\},$$

$$\Delta_T^c = \{(s, t)| 0 \leq s \leq T/\sqrt{2}, -s \leq t \leq s\}.$$
Since
\[
\int_{-T}^{0} dt \int_{0}^{T} dt \frac{1}{a^2 + |t - s|^2} \geq \int \int_{\Delta_T} ds dt \frac{1}{a^2 + |s + t|^2} = \int \int_{\Delta_T} ds dt \frac{1}{a^2 + s^2} = \log \left( \frac{a^2 + T^2/2}{a^2} \right),
\]
we have
\[
\mathbb{1}_{A_T} \int_{-T}^{0} ds \int_{0}^{T} dt \mathcal{W}_\infty(X_s, X_t, C_2|s - t|) = \frac{1}{4\pi^2} \mathbb{1}_{A_T} \int_{-T}^{0} ds \int_{0}^{T} dt \int dx dy \frac{\rho(x)\rho(y)}{8T^{2\lambda} + 2|x - y|^2 + C_2|t - s|^2} \geq \frac{1}{4C_2\pi^2} \mathbb{1}_{A_T} \int dx dy \rho(x)\rho(y) \log \left( \frac{8T^{2\lambda} + 2|x - y|^2 + C_2T^2/2}{8T^{2\lambda} + 2|x - y|^2} \right).
\]
Note that \( \rho \geq 0 \) and \( \lambda < 1 \). Since the right-hand side above goes to \(+\infty\) as \( T \to \infty \), (7.135) follows.

**Lemma 7.43** Suppose Assumptions 7.8, 7.10, 7.12, 7.16, and 7.18. Then it follows that
\[
\lim_{T \to \infty} \mathbb{E}_{\mu_T} \left[ \mathbb{1}_{A_T} e^{-C_1 \int_{-T}^{0} ds \int_{0}^{T} dt \mathcal{W}_\infty(X_s, X_t, C_2|s - t|)} \right] = 0. \tag{7.137}
\]

**Proof:** Note that
\[
\int_{-T}^{0} ds \int_{0}^{T} dt \mathcal{W}_\infty(X_s, X_t, |s - t|) \leq \frac{T}{2} \|\omega^{-1}_\infty \rho\|^2 \tag{7.138}
\]
and
\[
\int_{-T}^{T} ds \int_{-T}^{T} dt \mathcal{W}_\infty(X_s, X_t, |s - t|) \leq 4T \|\omega^{-1}_\infty \rho\|^2. \tag{7.139}
\]
Then
\[
\mathbb{E}_{\mu_T} \left[ \mathbb{1}_{A_T^c} e^{-\int_{-T}^{0} ds \int_{0}^{T} dt \mathcal{W}_\infty} \right] \leq e^{(T/2)\|\omega^{-1}_\infty \rho\|^2} \mathbb{E}_{\mu_T} \left[ \mathbb{1}_{A_T^c} \right].
\]

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By the Schwartz inequality we have

$$e^{(T/2)||\omega^{-1}_{\rho}||^2}E_{\mu_T}[1_{A_T}] = e^{(T/2)||\omega^{-1}_{\rho}||^2}\frac{\int d\mu_p\mathbb{E}^{x}[1_{A_T}e^{(1/2)\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}}]}{\int d\mu_p\mathbb{E}^{x}[e^{(1/2)\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}}]}$$

$$\leq e^{(T/2)||\omega^{-1}_{\rho}||^2}\left(\frac{\int d\mu_p\mathbb{E}^{x}[e^{\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}}]}{\int d\mu_p\mathbb{E}^{x}[e^{(1/2)\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}}]}\right)^{1/2}\frac{\int d\mu_p\mathbb{E}^{x}[1_{A_T}]}{\int d\mu_p\mathbb{E}^{x}[e^{(1/2)\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}}]}.$$  (7.140)

By Lemma 7.41 bounds

$$C_1\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}_{\infty}(X_s, X_t, C_2|s-t|) \leq \int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}(X_s, X_t, |s-t|)$$

and

$$\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}(X_s, X_t, |s-t|) \leq C_3\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}_{\infty}(X_s, X_t, C_4|s-t|)$$

are derived. Then we obtain

$$\left(\frac{\int d\mu_p\mathbb{E}^{x}[e^{\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}}]}{\int d\mu_p\mathbb{E}^{x}[e^{(1/2)\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}}]}\right)^{1/2} \leq \left(\frac{\int d\mu_p\mathbb{E}^{x}[e^{C_3\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}_{\infty}(X_s, X_t, C_4|s-t|)}]}{\int d\mu_p\mathbb{E}^{x}[e^{(1/2)\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}_{\infty}(X_s, X_t, C_4|s-t|)}]}\right)^{1/2}$$

and by (7.139) there exists $\epsilon > 0$ such that

$$\left(\frac{\int d\mu_p\mathbb{E}^{x}[e^{\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}}]}{\int d\mu_p\mathbb{E}^{x}[e^{(1/2)\int_{-T}^{T}ds\int_{-T}^{T}dt\mathcal{W}}]}\right)^{1/2} \leq e^{T||\omega^{-1}_{\rho}||^2}.$$  (7.141)

It remains to estimate $\int d\mu_p\mathbb{E}^{x}[1_{A_T}]$ in (7.140). There exists an at most polynomially growth function $\xi(T)$ such that

$$\int d\mu_p\mathbb{E}^{x}[1_{A_T}] \leq \xi(T) \exp(-cT^{\lambda+1})$$  (7.142)

with some constant $c > 0$. This is proven in Lemma 7.46 below. By (7.140), (7.141) and (7.142) we have

$$\lim_{T\to\infty} \mathbb{E}_{\mu_T}[1_{A_T}] \leq \lim_{T\to\infty} \xi(T)e^{-cT^{\lambda+1}}e^{(c+1/2)T||\omega^{-1}_{\rho}||^2} = 0,$$  (7.143)
since $\frac{1}{\delta + 1} < \lambda < 1$. Then (7.137) follows. \qed

Let us now consider some path properties of $X$ to show (7.142).

**Proposition 7.44** Let $P(B) = \int 1_B d\mu_p dP^x$ be the probability measure on $\mathbb{R}^3 \times \mathcal{X}$ and $\Lambda > 0$. Suppose Assumptions 7.8, 7.10, and 7.18. Suppose that $f \in C(\mathbb{R}^3) \cap D(L_p^{1/2})$. Then it follows that

$$P\left(\sup_{0 \leq s \leq T} |f(X_s)| \geq \Lambda\right) \leq e^{\Lambda \left[ (f, f)_{\mathcal{H}_p} + T(L_p^{1/2} f, L_p^{1/2} f)_{\mathcal{H}_p}\right]^{1/2}}.$$  

(7.144)

**Proof:** The proof is a modification of that of [KV86, Lemma 1.4 and Theorem 1.12]. Set $T_j = Tj/2^n$, $j = 0, 1, ..., 2^n$ and we fix $T$ and $n$. Let $G = \{ x \in \mathbb{R}^3 ||f(x)| \geq \Lambda\}$, then the stopping time $\tau$ is defined by

$$\tau = \inf\{ T_j \geq 0 | X_{T_j} \in G \}.$$  

Then it follows that

$$P\left(\sup_{j=0,...,2^n} |f(X_{T_j})| \geq \Lambda\right) = P(\tau \leq T).$$  

We estimate the right-hand side above. Let $0 < \chi < 1$ be fixed and we choose a suitable $\chi$ later. We see that

$$P(\tau \leq T) = \int d\mu_p E^x[\mathbbm{1}_{\tau \leq T}] \leq \int d\mu_p E^x[\chi^{\tau - T}]$$

$$\leq \chi^{-T} \int d\mu_p E^x[\chi^{\tau}] \leq \chi^{-T} \left( \int d\mu_p (E^x[\chi^{\tau}])^2 \right)^{1/2}.  \quad (7.145)$$

Let $0 \leq \psi$ be any function such that $\psi(x) \geq 1$ on $G$. Then the Dirichlet principle

$$\int d\mu_p (E^x[\chi^{\tau}])^2 \leq (\psi, \psi)_{\mathcal{H}_p} + \frac{\chi^{T/2^n}}{1 - \chi^{-T/2^n}} (\psi, (\mathbbm{1} - e^{-(T/2^n)L_p})\psi)_{\mathcal{H}_p}  \quad (7.146)$$

follows. We prove this in the next lemma. Inserting

$$|f(x)|/\Lambda = \begin{cases} \geq 1, & x \in G, \\ |f(x)|/\Lambda, & x \in G^c, \end{cases}$$

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into $\psi$ in \num{7.146}, we have
\[
\int d\mu_p(\mathbb{E}[\mathcal{X}^\tau])^2 \leq \frac{1}{\Lambda^2} (f, f)_{\mathcal{H}_p} + \frac{\chi^{T/2n}}{1 - \chi^{T/2n}} \frac{1}{\Lambda^2} (|f|, (\mathbb{1} - e^{-(T/2n)\mathbb{L}_p})|f|)_{\mathcal{H}_p}.
\]  
(7.147)

Since $e^{-(T/2n)\mathbb{L}_p}$ is positivity improving and then
\[
(|f|, (\mathbb{1} - e^{-(T/2n)\mathbb{L}_p})|f|)_{\mathcal{H}_p} \leq (f, (\mathbb{1} - e^{-(T/2n)\mathbb{L}_p})f)_{\mathcal{H}_p},
\]
we have by \num{7.145},
\[
P\left(\sup_{j=0,\ldots,2^n} |f(X_{T_j})| \geq \Lambda \right) \leq \frac{\chi^{-T}}{\Lambda} \left[ (f, f)_{\mathcal{H}_p} + \frac{\chi^{T/2n}}{1 - \chi^{T/2n}} (f, (\mathbb{1} - e^{-(T/2n)\mathbb{L}_p})f)_{\mathcal{H}_p} \right]^{1/2}.
\]  
(7.148)

Set $\chi = e^{-1/T}$. Then by $\frac{\chi^{T/2n}}{1 - \chi^{T/2n}} \leq 2^n$, we have
\[
P\left(\sup_{j=0,\ldots,2^n} |f(X_{T_j})| \geq \Lambda \right) \leq \frac{e}{\Lambda} \left[ (f, f)_{\mathcal{H}_p} + 2^n (f, (\mathbb{1} - e^{-(T/2n)\mathbb{L}_p})f)_{\mathcal{H}_p} \right]^{1/2}.
\]  
(7.149)

Since $(f, (\mathbb{1} - e^{-(T/2n)\mathbb{L}_p})f)_{\mathcal{H}_p} \leq (T/2^n)(L_p^{1/2}f, L_p^{1/2}f)$, we obtain that
\[
P\left(\sup_{j=0,\ldots,2^n} |f(X_{T_j})| \geq \Lambda \right) \leq \frac{e}{\Lambda} \left[ (f, f)_{\mathcal{H}_p} + T(L_p^{1/2}f, L_p^{1/2}f)_{\mathcal{H}_p} \right]^{1/2}.
\]  
(7.149)

Take $n \to \infty$ on both sides of \num{7.149}. By the Lebesgue dominated convergence theorem,
\[
\lim_{n \to \infty} P\left(\sup_{j=0,\ldots,2^n} |f(X_{T_j})| \geq \Lambda \right) = P\left(\lim_{n \to \infty} \sup_{j=0,\ldots,2^n} |f(X_{T_j})| \geq \Lambda \right).
\]

Since $f(X_t)$ is continuous in $t$, $\lim_{n \to \infty} \sup_{j=0,\ldots,2^n} |f(X_{T_j})| = \sup_{0 \leq s \leq T} |f(X_s)|$ follows. Then we complete the proposition. \hfill \square

It remains to show the Dirichlet principle \num{7.146}.

**Lemma 7.45 (Dirichlet principle)** Suppose Assumptions 7.8, 7.10 and 7.18. Then it follows that
\[
\int d\mu_p(\mathbb{E}[\mathcal{X}^\tau])^2 \leq (\psi, \psi)_{\mathcal{H}_p} + \frac{\chi^{T/2n}}{1 - \chi^{T/2n}} (\psi, (\mathbb{1} - e^{-(T/2n)\mathbb{L}_p})\psi)_{\mathcal{H}_p}
\]  
(7.150)

for any function $\psi \geq 0$ such that $\psi(x) \geq 1$ on $G$. 

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Proof: Define the function \( \psi_\chi \) by \( \psi_\chi(x) = \mathbb{E}^x[\chi^\tau] \). By the definition of \( \tau \) we can see that

\[
\psi_\chi(x) = 1, \quad x \in G, \tag{7.151}
\]
since \( \tau = 0 \) when \( X_s \) stars from the inside of \( G \). Let \( \mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t) \) be the natural filtration of \( (X_t)_{t \geq 0} \). By the Markov property of \( X \), we can directly see that

\[
e^{-\left(T/2^n\right)L_p} \psi_\chi(x) = \mathbb{E}^x[\mathbb{E}^{X_{T/2^n}}[\chi^\tau]] = \mathbb{E}^x[\mathbb{E}^{\chi^\tau} | \mathcal{F}_{T/2^n}] = \mathbb{E}^x[\chi^\tau], \tag{7.152}
\]
where \( \theta_t \) is the shift on \( \mathcal{X}^c \), which is defined by \( (\theta_t \omega)(s) = \omega(s + t) \) for \( \omega \in \mathcal{X}^c \). Note that

\[
(\tau \circ \theta_{T/2^n})(\omega) = \tau(\omega) - T/2^n \geq 0, \tag{7.153}
\]
when \( x = X_0(\omega) \in G^c \). Hence by (7.152) and (7.153) we have the identity:

\[
\chi^{T/2^n} e^{-\left(T/2^n\right)L_p} \psi_\chi(x) = \psi_\chi(x), \quad x \in G^c. \tag{7.154}
\]

It is trivial to see that

\[
\int d\mu_p(\mathbb{E}^x[\chi])^2 = (\psi_\chi, \psi_\chi)_G \leq (\psi_\chi, \psi_\chi)_p + \frac{\chi^{T/2^n}}{1 - \chi^{T/2^n}} (\psi_\chi, (1 - e^{-\left(T/2^n\right)L_p}) \psi_\chi)_p.
\]

Let us define \( (f, g)_G = \int_G d\mu_p f(x)g(x) \). By the relation (7.154) we can compute the right-hand side above as

\[
(\psi_\chi, \psi_\chi)_G + \frac{\chi^{T/2^n}}{1 - \chi^{T/2^n}} (\psi_\chi, (1 - e^{-\left(T/2^n\right)L_p}) \psi_\chi)_G. \tag{7.155}
\]

Since

\[
(\psi_\chi, (1 - e^{-\left(T/2^n\right)L_p}) \psi_\chi)_G \\
= (\psi_\chi 1_G, (1 - e^{-\left(T/2^n\right)L_p}) \psi_\chi 1_G)_p + (\psi_\chi 1_G, (1 - e^{-\left(T/2^n\right)L_p}) \psi_\chi 1_{G^c})_p \\
= (\psi_\chi 1_G, (1 - e^{-\left(T/2^n\right)L_p}) \psi_\chi 1_G)_p - (\psi_\chi 1_G, e^{-\left(T/2^n\right)L_p} \psi_\chi 1_{G^c})_p \\
\leq (\psi_\chi 1_G, (1 - e^{-\left(T/2^n\right)L_p}) \psi_\chi 1_G)_p.
\]

Hence

\[
\int d\mu_p(\mathbb{E}^x[\chi])^2 \leq (\psi_\chi 1_G, \psi_\chi 1_G)_p + \frac{\chi^{T/2^n}}{1 - \chi^{T/2^n}} (\psi_\chi 1_G, (1 - e^{-\left(T/2^n\right)L_p}) \psi_\chi 1_G)_p. \tag{7.156}
\]
Note that $\psi_\chi \mathbb{1}_G(x) \leq \psi(x)$ for all $x \in \mathbb{R}^3$. Then
\[
(\psi_\chi \mathbb{1}_G; \psi_\chi \mathbb{1}_G)_{\mathcal{H}^p} + \frac{\chi^{T/2^n}}{1 - \chi^{T/2^n}}(\psi_\chi \mathbb{1}_G, (\mathbb{1} - e^{-(T/2^n)L_p})\psi_\chi \mathbb{1}_G)_{\mathcal{H}^p} \leq (\psi, \psi)_{\mathcal{H}^p} + \frac{\chi^{T/2^n}}{1 - \chi^{T/2^n}}(\psi, (\mathbb{1} - e^{-(T/2^n)L_p})\psi)_{\mathcal{H}^p}.
\] (7.157)

Combining (7.156) with (7.157), we prove the lemma.

**Lemma 7.46** (7.142) holds.

**Proof:** Suppose that $f \in C^\infty(\mathbb{R}^3)$, $f(-x) = f(x)$ and
\[
f(x) = \begin{cases} |x|, & |x| \geq T^\lambda, \\ \leq |x|, & T^\lambda - 1 < |x| < T^\lambda, \\ 0, & |x| \leq T^\lambda - 1. \end{cases}
\]

Then we see that
\[
\int d\mu \mathbb{E}^x[\mathbb{1}_{A_T^\lambda}] = \int d\mu \mathbb{E}^x[\mathbb{1}_{\sup|s|<T, |X_s|>T^\lambda}] = \int d\mu \mathbb{E}^x[\mathbb{1}_{\sup|s|<T, |f(X_s)|>T^\lambda}].
\] (7.158)

By the reflection symmetry of $(X_t)_{t \in \mathbb{R}}$ we have
\[
\int d\mu \mathbb{E}^x[\mathbb{1}_{\sup|s|<T, |f(X_s)|>T^\lambda}] = 2 \int d\mu \mathbb{E}^x[\mathbb{1}_{\sup|s|<T, |f(X_s)|>T^\lambda}]
\]
and by Proposition 7.44 we have
\[
\int d\mu \mathbb{E}^x[\sup_{|s|<T} |f(X_s)| > T^\lambda] \leq \frac{2e}{T^\lambda} \left( (f, f)_{\mathcal{H}^p} + T(1/2)_{\mathbb{L}^p} f, (1/2)_{\mathbb{L}^p} f)_{\mathcal{H}^p} \right)^{1/2}. \tag{7.159}
\]

We estimate the right-hand side of (7.159). First we show $f\varphi_\Lambda \in D(K)$. Let $C_0^\infty(\mathbb{R}^3) \ni f_R(x) = \mathcal{X}(x/R)f(x)$, where $\mathcal{X} \in C_0^\infty(\mathbb{R}^3)$ and
\[
\mathcal{X}(x) = \begin{cases} 1, & |x| < 1, \\ < 1, & 1 \leq |x| \leq 2, \\ 0, & |x| \geq 2. \end{cases}
\]
Since $A_{\mu\nu}$ satisfies the Lipshitz condition (Assumption 7.18), $A_{\mu\nu} \in W^{1,\infty}(\mathbb{R}^3)$. Then by Lemma 7.9, we see that $\varphi_p \in H^2(\mathbb{R}^3)$, $f_R\varphi_p \in D(K)$, $(K-E)\varphi_p = 0$, and

$$K f_R \varphi_p = \sum_{\mu,\nu=1}^3 (D_\mu A_{\mu\nu}(D_\nu f_R)\varphi_p + (D_\mu f) A_{\mu\nu} D_\nu \varphi_p) + E f_R \varphi_p.$$ 

We see that

$$f_R \varphi_p \to f \varphi_p,$$

$$(D_\mu f_R) A_{\mu\nu} D_\nu \varphi_p \to (D_\mu f) A_{\mu\nu} D_\nu \varphi_p,$$

$$D_\mu A_{\mu\nu}(D_\nu f_R) \varphi_p = D_\mu A_{\mu\nu} \cdot (D_\nu f_R) \varphi_p + A_{\mu\nu} \cdot D_\mu (D_\nu f_R) \cdot \varphi_p + A_{\mu\nu} (D_\nu f_R) \cdot D_\mu \varphi_p$$

$$\to ((D_\mu A_{\mu\nu})(D_\nu f) + A_{\mu\nu}(D_\mu D_\nu f)) \varphi_p$$

as $R \to \infty$ in $L^2(\mathbb{R}^3)$. Since $K$ is closed, $f \varphi_p \in D(K)$ follows. By the estimate above we also see that

$$(L^{1/2}_p, L^{1/2}_p) \varphi_p = (f \varphi_p, (D_\mu f) A_{\mu\nu} D_\nu \varphi_p + (D_\nu A_{\mu\nu})(D_\nu f) \varphi_p + A_{\nu\mu} f^{\mu\nu} \varphi_p).$$

By the spatial super-exponential decay $\varphi_p(x) \leq C e^{-\gamma|x|^{\delta+1}}$ derived in (7.87) we have

$$\|f \varphi_p\|^2 = \int f(x)^2 \varphi_p^2(x) dx \leq C^2 e^{-2\gamma T^{\lambda(\delta+1)}} \int |x|^2 e^{-2\gamma |x|^{\delta+1}} dx. \tag{7.160}$$

Note that $D_\mu f, D_\mu D_\nu f \in L^\infty(\mathbb{R}^3)$. Then

$$(L^{1/2}_p, L^{1/2}_p) \varphi_p \leq \|f \varphi_p\| \|A_{\mu\nu} D_\nu \varphi_p + A_{\mu\nu}^{\nu\mu} \varphi_p + A_{\mu\nu} \varphi_p\| \leq C' e^{-\gamma T^{\lambda(\delta+1)}} \|A_{\mu\nu} D_\nu \varphi_p + A_{\mu\nu}^{\nu\mu} \varphi_p + A_{\mu\nu} \varphi_p\|$$

follows. Similarly

$$(f, f) \varphi_p \leq C^2 e^{-2\gamma T^{\lambda(\delta+1)}} \int |x|^2 e^{-2\gamma |x|^{\delta+1}} dx$$

is also derived. Hence we have

$$E_{\mu_T} [1_{A_p}] \leq T^{-\lambda} \sqrt{a + T b e^{-(\gamma/2) T^{\lambda(\delta+1)}}}$$

with some constant $a$ and $b$. This completes the proof. \hfill \square

Now we are in the position to state the main theorem.
Theorem 7.47 (Absence of ground state) Suppose Assumptions 7.8, 7.10, 7.12, 7.16 and 7.18. Then there is no ground states of $H$.

Proof: Since $\gamma(T) \leq E_{\mu_T} \left[ e^{-C_1 \int_{-T}^0 ds \int_0^T dt \mathbb{W}_\infty(X_s, X_t, C_2|s-t|) \right]$ and

$$\lim_{T \to \infty} E_{\mu_T} \left[ e^{-C_1 \int_{-T}^0 ds \int_0^T dt \mathbb{W}_\infty(X_s, X_t, C_2|s-t|) \right] = 0$$

by Lemmas 7.42 and 7.43 we obtain $\lim_{T \to \infty} \gamma(T) = 0$. Then the theorem follows.

Acknowledgments

FH acknowledges support of Grant-in-Aid for Science Research (B) 20340032 from JSPS and Grant-in-Aid for Challenging Exploratory Research 22654018 from JSPS. HS is thankful to the hospitality of Mathematics-for-Industry of Kyushu University from October 22 of 2009 to January 7 of 2010, where part of this work has been done.

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