A sparse semismooth Newton based proximal majorization-minimization algorithm for nonconvex square-root-loss regression problems

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Abstract

In this paper, we consider high-dimensional nonconvex square-root-loss regression problems and introduce a proximal majorization-minimization (PMM) algorithm for these problems. Our key idea for making the proposed PMM to be efficient is to develop a sparse semismooth Newton method to solve the corresponding subproblems. By using the Kurdyka-Lojasiewicz property exhibited in the underlying problems, we prove that the PMM algorithm converges to a $d$-stationary point. We also analyze the oracle property of the initial subproblem used in our algorithm. Extensive numerical experiments are presented to demonstrate the high efficiency of the proposed PMM algorithm.

Keywords: nonconvex square-root regression problems, proximal majorization-minimization, semismooth Newton method

1 Introduction

Sparsity estimation is one of the most important problems in statistics, machine learning and signal processing. One typical example on this aspect is to estimate a vector $\hat{\beta}$ from a high-dimensional linear regression model

$$b = X\hat{\beta} + \varepsilon,$$

where $b$ is the observed data, $X$ is the design matrix, and $\varepsilon$ is the error term.
where \( X \in \mathbb{R}^{m \times n} \) is the design matrix, \( b \in \mathbb{R}^m \) is the response vector, and \( \varepsilon \in \mathbb{R}^m \) is the noise vector for which each of its component \( \varepsilon_i \) has zero-mean and unknown variance \( \varsigma^2 \). In many applications, the number of attributes \( n \) is much larger than the sample size \( m \) and \( \beta \) is known to be sparse a priori. Under the assumption of sparsity, a regularizer which controls the overfitting and/or variable selection is often added to the model. One of the most commonly used regularizers in practice is the \( \ell_1 \) norm and the resulting model, first proposed in [42], is usually referred to as the Lasso model, which is given by

\[
\min_{\beta \in \mathbb{R}^n} \left\{ \frac{1}{2} \| X\beta - b \|^2 + \lambda \| \beta \|_1 \right\},
\]

(1)

where \( \| \cdot \| \) is the Euclidean norm in \( \mathbb{R}^m \). The Lasso estimator produced from (1) is computationally attractive because it minimizes a structured convex function. Moreover, when the error vector \( \varepsilon \) follows a normal distribution and suitable design conditions hold, this estimator achieves a near-oracle performance. Despite having these attractive features, the Lasso recovery of \( \beta \) relies on knowing the standard deviation of the noise. However, it is non-trivial to estimate the deviation when the feature dimension is large, particularly when it is much larger than the sample dimension. To overcome the aforementioned defect, the authors in [8] proposed a new estimator that solves the square-root Lasso (srLasso) model

\[
\min_{\beta \in \mathbb{R}^n} \left\{ \| X\beta - b \| + \lambda \| \beta \|_1 \right\},
\]

(2)

which eliminates the need to know or to pre-estimate the deviation. It has been shown in [8] that the srLasso estimator achieves the near-oracle rate of convergence under some suitable conditions, even though the noise level \( \varsigma \) is unknown. It is worth mentioning that the scaled Lasso proposed by Sun and Zhang [41] is essentially equivalent to the srLasso model (2). The solution approach proposed by the authors in [41] is to iteratively solve a sequence of Lasso problems, which can be expensive numerically. Moreover, Xu, Caramanis and Mannor [45] proved that the srLasso model (2) is equivalent to a robust linear regression problem subject to an uncertainty set that bounds the norm of the disturbance to each feature, which itself is an ideal approach of reducing sensitivity of linear regression.

The Lasso problem and the srLasso problem are both convex and computationally attractive. A number of algorithms, such as the accelerated proximal gradient (APG) method [3], interior-point method (IPM) [27], and least angle regression (LARS) [17] have been proposed to solve the Lasso problem (1). In a very recent work [31], Li, Sun and Toh proposed a highly efficient semismooth Newton augmented Lagrangian method to solve the Lasso problem (1). In contrast to the Lasso problem (1), there are currently no efficient algorithms for solving the more challenging srLasso problem (2) due to the fact that the square-root loss function in the objective is nonsmooth. Notably the alternating direction method of multipliers (ADMM) was applied to solve the srLasso problem (2) by the authors in [32]. However, as can be seen from the numerical experiments conducted later in this paper, the ADMM approach is not very efficient in solving many large-scale problems.

Going beyond the \( \ell_1 \) norm regularizer, other regularization functions for variables selection are often used to avoid overfitting in the area of support vector machines and other statistical applications. It has also been shown that, instead of a convex relaxation with the \( \ell_1 \) norm, a
proper nonconvex regularization can achieve a sparse estimation with fewer measurements, and is more robust against noises [11, 14]. After the pioneering work of Fan and Li [20], various nonconvex sparsity functions have been proposed as surrogates of the \( \ell_0 \) function in the last decade and they have been used as regularizers to avoid model overfitting (see e.g. [25]) in high-dimensional statistical learning. It has been proven that each of these nonconvex surrogate sparsity functions can be expressed as the difference of two convex functions [1, 29]. Given the d.c. (difference of convex functions) property of these nonconvex regularizers, it is natural for one to design a majorization-minimization algorithm to solve the nonconvex problem. Such an exploitation of the d.c. property of the regularization function had been considered in the majorized penalty approach proposed by Gao and Sun [23] for solving a rank constrained correlation matrix estimation problem.

In this paper, we aim to develop an efficient and robust algorithm for solving the following square-root regression problem

\[
\begin{align*}
\min_{\beta \in \mathbb{R}^n} \left\{ g(\beta) := \|X\beta - b\| + \lambda p(\beta) - q(\beta) \right\},
\end{align*}
\]  

(3)

where the first part of the regularization function \( p : \mathbb{R}^n \to \mathbb{R}_+ \) is a norm function whose proximal mapping is strongly semismooth and the second part \( q : \mathbb{R}^n \to \mathbb{R} \) is a convex smooth function (the dependence of \( q \) on \( \lambda \) has been dropped here). We should note that the assumption made on \( p \) is rather mild as the proximal mappings of many interesting functions, such as the \( l_1 \)-norm function, are strongly semismooth [33]. For the case when \( q \equiv 0 \), the oracle property of the model has been established in [40] when \( p \) is a weakly decomposable norm. For the need of efficient computations, here we shall extend the analysis to the same model but with the proximal terms \( \frac{\sigma}{2}\|\beta\|^2 + \frac{\tau}{2}\|X\beta - b\|^2 \) added, where \( \sigma \geq 0 \) and \( \tau \geq 0 \) are given parameters.

Based on the d.c. structure of the nonconvex regularizer in (3), we design a two stage proximal majorization-minimization (PMM) algorithm to solve the problem (3). In both stages of the PMM algorithm, the key step in each iteration is to solve a convex subproblem whose objective contains the sum of two nonsmooth functions, namely, the square-root-loss regression function and \( p(\cdot) \). One of the main contributions of this paper is in proposing a novel proximal majorization approach to solve the said subproblem via its dual by a highly efficient semismooth Newton method. We also analyze the convergence properties of our algorithm. By using the Kurdyka-Lojasiewicz (KL) property exhibited in the underlying problem, we prove that the PMM algorithm converges to a d-stationary point. In the last part of the paper, we present comprehensive numerical results to demonstrate the efficiency of our semismooth Newton based PMM algorithm. Based on the performance of our algorithm against two natural first-order methods, namely the primal based and dual based ADMM (for the convex case), we can safely conclude that our algorithmic framework is far superior for solving the square-root regression problem (3).

2 Preliminary

Let \( f : \mathbb{R}^n \to (-\infty, +\infty] \) be a proper closed convex function. The Fenchel conjugate of \( f \) is defined by \( f^*(x) := \sup_{y \in \mathbb{R}^n} \{ \langle y, x \rangle - f(y) \} \), the proximal mapping and the Moreau envelope.
function of $f$ with parameter $t > 0$ are defined, respectively, as
\[
\text{Prox}_f(x) := \arg\min_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2}\|y - x\|^2 \right\}, \quad \forall x \in \mathbb{R}^n,
\]
\[
\Phi_{tf}(x) := \min_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2t}\|y - x\|^2 \right\}, \quad \forall x \in \mathbb{R}^n.
\]

Let $t > 0$ be a given parameter. Then by Moreau’s identity theorem (see, e.g., Theorem 31.5 of [38]), we have that
\[
\text{Prox}_{tf}(x) + t\text{Prox}_{f^*}(x/t) = x, \quad \forall x \in \mathbb{R}^n.
\]

We also know from Proposition 13.37 of [39] that $\Phi_{tf}$ is continuously differentiable with
\[
\nabla \Phi_{tf}(x) = t^{-1}(x - \text{Prox}_{tf}(x)), \quad \forall x \in \mathbb{R}^n.
\]

Given a set $C \subseteq \mathbb{R}^n$ and an arbitrary collection of functions $\{f_i \ | i \in I\}$ on $\mathbb{R}^n$, we denote $\delta_C(\cdot)$ as the indicator function of $C$, namely $\delta_C(\beta) = 0$ if $\beta \in C$ and $\delta_C(\beta) = +\infty$ if $\beta \notin C$, $\text{conv}(C)$ as the convex hull of $C$ and $\text{conv}\{f_i \ | i \in I\}$ as the convex hull of the pointwise infimum of the collection. This means that $\text{conv}\{f_i \ | i \in I\}$ is the greatest convex function $f$ on $\mathbb{R}^n$ such that $f(\beta) \leq f_i(\beta)$ for any $\beta \in \mathbb{R}^n$ and $i \in I$.

Next we present some results in variational analysis from [39]. Let $\Psi : \mathcal{O} \to \mathbb{R}^m$ be a locally Lipschitz continuous vector-valued function defined on an open set $\mathcal{O} \subseteq \mathbb{R}^n$. It follows from Theorem 9.60 of [39] that $\Psi$ is F(réchét)-differentiable almost everywhere on $\mathcal{O}$. Let $D_\Psi$ be the set of all points where $\Psi$ is F-differentiable and $J\Psi(x) \in \mathbb{R}^{m \times n}$ be the Jacobian of $\Psi$ at $x \in D_\Psi$. For any $x \in \mathcal{O}$, the B-subdifferential of $\Psi$ at $x$ is defined by
\[
\partial_B \Psi(x) := \left\{ V \in \mathbb{R}^{m \times n} \mid \exists \{x^k\} \subseteq D_\Psi \text{ such that } \lim_{k \to \infty} x^k = x \text{ and } \lim_{k \to \infty} J\Psi(x^k) = V \right\}.
\]

The Clarke subdifferential of $\Psi$ at $x$ is defined as the convex hull of the B-subdifferential of $\Psi$ at $x$, that is $\partial C \Psi(x) := \text{conv}(\partial_B \Psi(x))$.

Let $\phi$ be defined from $\mathbb{R}^n$ to $\mathbb{R}$. The Clarke subdifferential of $\phi$ at $x \in \mathbb{R}^n$ is defined by
\[
\partial C \phi(x) := \left\{ h \in \mathbb{R}^n \mid \limsup_{x' \to x, t \downarrow 0} \frac{\phi(x' + ty) - \phi(x') - th^Ty}{t} \geq 0, \quad \forall y \in \mathbb{R}^n \right\}.
\]

The regular subdifferential of $\phi$ at $x \in \mathbb{R}^n$ is defined as
\[
\hat{\partial} \phi(x) := \left\{ h \in \mathbb{R}^n \mid \liminf_{x \neq y \to x} \frac{\phi(y) - \phi(x) - h^T(y - x)}{\|y - x\|} \geq 0 \right\}
\]
and the limiting subdifferential of $\phi$ at $x$ is defined as
\[
\partial \phi(x) := \left\{ h \in \mathbb{R}^n \mid \exists \{x^k\} \to x \text{ and } \{h^k\} \to h \text{ satisfying } h^k \in \hat{\partial} \phi(x^k), \quad \forall k \right\}.
\]

If $\phi$ is a convex function, then the Clarke subdifferential, the regular subdifferential and the limiting subdifferential of $\phi$ at $x$ coincide with the set of (transposed) subgradients of $\phi$ at $x$ in the sense of convex analysis.
We know from Theorem 10.1 of [32] that $0 \in \partial \phi(\bar{x})$ is a necessary condition for $\bar{x} \in \mathbb{R}^n$ to be a local minimizer of $\phi$. If the function $\phi$ is locally Lipschitz continuous near $\bar{x}$ and directionally differentiable at $\bar{x}$, then $0 \in \partial \phi(\bar{x})$ is equivalent to the directional-stationarity (d-stationarity) of $\bar{x}$, that is

$$\phi'(\bar{x}; h) := \lim_{\lambda \downarrow 0} \frac{\phi(\bar{x} + \lambda h) - \phi(\bar{x})}{\lambda} \geq 0, \quad \forall h \in \mathbb{R}^n.$$ 

In this paper, we will prove that the sequence generated by our algorithm converges to a d-stationary point of the problem.

For further discussions, we recall the concept of semismoothness originated from [34, 37] and other two definitions used in [44].

**Definition 2.1.** Let $F : \mathcal{O} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a locally Lipschitz continuous function and $\mathcal{K} : \mathcal{O} \rightrightarrows \mathbb{R}^{m \times n}$ be a nonempty and compact valued, upper-semicontinuous set-valued mapping on the open set $\mathcal{O}$. $F$ is said to be semismooth at $v \in \mathcal{O}$ with respect to the set-valued mapping $\mathcal{K}$ if $F$ is directionally differentiable at $v$ and for any $\Gamma \in \mathcal{K}(v + \Delta v)$ with $\Delta v \rightarrow 0$,

$$F(v + \Delta v) - F(v) - \Gamma \Delta v = o(\|\Delta v\|).$$

$F$ is said to be $\gamma$-order ($\gamma > 0$) (strongly, if $\gamma = 1$) semismooth at $v$ with respect to $\mathcal{K}$ if $F$ is semismooth at $v$ and for any $\Gamma \in \mathcal{K}(v + \Delta v)$,

$$F(v + \Delta v) - F(v) - \Gamma \Delta v = O(\|\Delta v\|^{1+\gamma}).$$

$F$ is called a semismooth ($\gamma$-order semismooth, strongly semismooth) function on $\mathcal{O}$ with respect to $\mathcal{K}$ if it is semismooth ($\gamma$-order semismooth, strongly semismooth) at every $v \in \mathcal{O}$ with respect to $\mathcal{K}$.

**Definition 2.2.** The norm function $p$ defined in $\mathbb{R}^n$ is said to be weakly decomposable for an index set $S \subset \{1, 2, \ldots, n\}$ if there exists a norm $p^S$ defined on $\mathbb{R}^{|S|}$ such that

$$p(\beta) \geq p(\beta_S) + p^S(\beta^S), \quad \forall \beta \in \mathbb{R}^n,$$

where $\hat{S} = \{1, 2, \ldots, n\}\setminus S$, $\beta_S = \beta \circ 1_S$ and $\beta^S := (\beta_j)_{j \in \hat{S}} \in \mathbb{R}^{|\hat{S}|}$. Here $1_S \in \mathbb{R}^n$ denotes the indicator vector of $S$ and “$\circ$” denotes the elementwise product.

The weakly decomposable property of a norm is a relaxation of the decomposability property of the $\ell_1$ norm. It has been proved in [10] that many interesting regularizers such as the sparse group Lasso and SLOPE are weakly decomposable. A set $S$ is said to be an allowed set if $p$ is a weakly decomposable norm for this set. We say that a vector $\beta \in \mathbb{R}^n$ satisfies the $(L, S)$-cone condition for a norm $p$ if $p^S(\beta^S) \leq L p(\beta_S)$ with $L > 0$ and $S$ being an allowed set.

**Definition 2.3.** Let $S$ be an allowed set of a weakly decomposable norm $p$ and $L > 0$ be a constant. Then the $p$-eigenvalue is defined as

$$\delta_p(L, S) := \min \left\{ \|X \beta_S - X \beta^S\| \mid p(\beta_S) = 1, \; p^S(\beta^S) \leq L \right\}.$$

The $p$-effective sparsity is defined as

$$\Gamma_p(L, S) := \frac{1}{\delta_p(L, S)}.$$

Note that the $p$-eigenvalue defined above is a generalization of the compatibility constant defined in [43].
3 The oracle property of the square-root regression problem with a generalized elastic-net regularization

We first consider the following convex problem without \( q \) in (3), that is

\[
\min_{\beta \in \mathbb{R}^n} \left\{ \|X\beta - b\| + \lambda p(\beta) \right\}.
\]

(6)

By adding proximal terms, we shall analyze the oracle property of the square-root regression problem with a generalized elastic-net regularization. For given \( \sigma \geq 0 \) and \( \tau \geq 0 \), it takes the following form

\[
\min_{\beta \in \mathbb{R}^n} \left\{ \|X\beta - b\| + \lambda p(\beta) + \frac{\sigma}{2}\|\beta\|^2 + \frac{\tau}{2}\|X\beta - b\|^2 \right\},
\]

whose optimal solution set is denoted as \( \Omega(\sigma, \tau) \). The purpose of this section is to study the oracle property of an estimator \( \hat{\beta} \in \Omega(\sigma, \tau) \) (whose residual is given by \( \hat{\epsilon} := b - X\hat{\beta} \)) to evaluate how good the estimator is in estimating the true vector \( \beta \).

For the given norm \( p \), the dual norm of \( p \) is given by

\[
p_*(\beta) := \max_{z \in \mathbb{R}^n} \left\{ \langle z, \beta \rangle \mid p(z) \leq 1 \right\}, \quad \forall \beta \in \mathbb{R}^n.
\]

For a weakly decomposable norm \( p \) with the allowed set \( S \), we let

\[
n_p = \frac{\lambda p(\beta)}{\|\epsilon\|}, \quad \lambda_0 = \frac{p_*(\epsilon^T X)}{\|\epsilon\|}, \quad \lambda_m = \max \left\{ \frac{p_*(\epsilon^T X)}{\|\epsilon\|}, \frac{p_*(\epsilon^T X)}{\|\epsilon\|}, p^S_*(\tilde{\beta}^S), p_*(\tilde{\beta}^S) \right\},
\]

(7)

\[
t_1 = 1 + \frac{\tau \|\epsilon\| + \frac{\sigma p_*(\beta)p(\beta)}{2\|\epsilon\|}}{\|\beta\|}, \quad t_2 = 2 + \frac{\tau + \frac{\sigma p_*(\beta)p(\beta)}{\|\epsilon\|}}{\|\beta\|},
\]

(8)

\[
c_u = t_1 + n_p, \quad a = \left( \lambda_0 + \frac{\sigma p_*(\beta)c_u}{\|\epsilon\|} \right) \frac{1}{\lambda},
\]

(9)

where \( p^S_\ast \) denotes the dual norm of \( p^S \).

Next we state two basic assumptions which are similar to those in [40]. The first assumption is about non-overfitting in the sense that if the optimal solution \( \hat{\beta} \) of (3) satisfies \( \|\hat{\epsilon}\| = 0 \), then it overfits. Furthermore, it has been proved in [36] that there exists a scalar \( \lambda^\ast \) such that the solution of the problem satisfies \( X\beta - b \neq 0 \) if \( \lambda > \lambda^\ast \). In other words, one can find the parameter \( \lambda \) to avoid overfitting.

**Assumption 3.1.** We assume that \( \|\hat{\epsilon}\| \neq 0 \) and \( a + \frac{2\lambda_0 n_p}{\lambda} < 1 \), where the constant \( a \) is defined in (4).

**Assumption 3.2.** The function \( p \) is a norm function and weakly decomposable in \( \mathbb{R}^n \) for an index set \( S \subset \{1, \ldots, n\} \), i.e., there exists a norm \( p^S \) defined on \( \mathbb{R}^{|S|} \) such that

\[
p(\beta) \geq p(\beta_S) + p^S(\beta^S), \quad \forall \beta \in \mathbb{R}^n.
\]

The above Assumption 3.2 goes back to [6] and was also used in [44] to give an assumption on the structure of the sparsity inducing norm. We also need some lemmas in order to prove the main theorem. First we introduce the following basic relationship between \( \lambda_0 \) and \( \lambda_m \).\(^1\)

\(^1\)Actually in [40] Stucky and van de Geer once employed this relationship without a proof. For the sake of clarity, we present this fact in the form of a lemma here and its proof in the appendix.
**Lemma 3.1.** Let $S$ be an allowed set of a weakly decomposable norm $p$. For the parameters $\lambda_0$ and $\lambda_m$ defined by (7), we have $\lambda_0 \leq \lambda_m$ and $p_*(\tilde{\beta}) \leq \lambda_m$.

The following lemma, which is from [44], shows that $p(\beta_S)$ is bounded by $\|X\beta\|$.

**Lemma 3.2.** Let $S$ be an allowed set of a weakly decomposable norm $p$ and $L > 0$ be a constant. Then the $p$-eigenvalue can be expressed in the following form:

$$\delta_p(L, S) = \min \left\{ \frac{\|X\beta\|}{p(\beta_S)} \left| \beta \text{ satisfies the } (L, S)\text{-cone condition and } \beta_S \neq 0 \right. \right\}.$$  

That is $p(\beta_S) \leq \Gamma_p(L, S)\|X\beta\|$.

An upper bound of $\hat{\epsilon}^T X (\hat{\beta} - \tilde{\beta})$, a lower bound and an upper bound of $\|\hat{\epsilon}\|$ are also important. They are presented in the following two lemmas.

**Lemma 3.3.** Suppose that Assumption [3, 4] holds. For the estimator $\tilde{\beta}$ of the generalized elastic-net square-root regression problem, we have

$$\hat{\epsilon}^T X (\hat{\beta} - \tilde{\beta}) \leq \left( \tau + \frac{1}{\|\hat{\epsilon}\|} \right)^{-1} \left( \lambda p(\tilde{\beta}) + \sigma \sigma_p(\tilde{\beta}) p(\tilde{\beta}) \right).$$

**Lemma 3.4.** Suppose that Assumption [3, 4] holds. We have

$$c_u := \frac{1 - a - \frac{2\lambda_0 n_p}{\lambda}}{2 + \left(1 + \frac{\sigma_p(\tilde{\beta})}{\lambda}\right)n_p} \leq \frac{\|\hat{\epsilon}\|}{\|\hat{\epsilon}\|} \leq c_u,$$

where the constants $c_u$ and $a$ are defined in (9).

Based on the above lemmas, we can present the following sharp oracle inequality on the prediction error.

**Theorem 3.1.** Let $\delta \in [0, 1)$. Under Assumptions [3, 4] and [3, 2] assume that $s_2 - \sqrt{s_2 - 4s_1 s_3} \leq \lambda < s_2 + \sqrt{s_2 - 4s_1 s_3}$ with $s_1 = \sigma \lambda_m p(\hat{\beta})$, $s_2 = 1 - \frac{\lambda m(\lambda + 2\sigma_1 + \sigma_1) p(\hat{\beta})}{\|\hat{\epsilon}\|} > 0$ and $s_3 = \lambda m(t_1 + t_2 + \sigma_1 t_2 + \sigma_1 t_2)$. For any $\hat{\beta} \in \Omega(\sigma, \tau)$, and any $\beta \in \mathbb{R}^n$ such that supp($\beta$) is a subset of $S$, we have that

$$\|X(\hat{\beta} - \tilde{\beta})\|^2 + 2\delta \left( (\hat{\lambda} - \tilde{\lambda}_m)p(\hat{\beta} \tilde{\beta}) + (\hat{\lambda} + \tilde{\lambda}_m)p(\hat{\beta}_S - \beta) \right) \|\hat{\epsilon}\|$$

$$\leq \|X(\beta - \tilde{\beta})\|^2 + \left( 1 + \delta \right)(\hat{\lambda} + \tilde{\lambda}_m)\Gamma_p(L_S, S)\|\hat{\epsilon}\|^2 + 2\sigma c_u \|\hat{\beta} - \tilde{\beta}\|\|\tilde{\beta} - \beta\|\|\hat{\epsilon}\|,$$

where

$$\hat{\lambda} := \frac{\lambda c_l}{1 + \tau c_l}, \quad \tilde{\lambda}_m := \lambda m(1 + \sigma c_u), \quad \hat{\lambda} := \lambda c_u, \quad L_S := \frac{\hat{\lambda} + \tilde{\lambda}_m}{\lambda - \lambda_m} \cdot \frac{1 + \delta}{1 - \delta}.\]$$

An important special case of Theorem 3.1 is to choose $\beta = \tilde{\beta}$ with supp($\tilde{\beta}$) $\subseteq$ $S$ allowed. Then only the $p$-effective sparsity term $\Gamma_p(L_S, S)$ appears in the upper bound.
Remark 3.1. Since \( \lim_{\sigma \downarrow 0} (s_2^2 - 4s_1 s_3) = \left(1 - \frac{3\lambda m p(\beta)}{\|\epsilon\|}\right)^2 > 0 \), we can find some \( \hat{\sigma} > 0 \) such that \( s_2^2 - 4s_1 s_3 > 0 \) holds if \( \sigma < \hat{\sigma} \). Theorem 3.1 is nearly the same as that in \([40]\) due to \( \frac{s_2}{\|s_1\|} = \frac{3\lambda m \|\epsilon\|}{\|\epsilon\| - 3\lambda m p(\beta)} \) and \( \lim_{\sigma \downarrow 0, \tau \downarrow 0} \frac{s_2 - s_2^2 - 4s_1 s_3}{2s_1} = +\infty \) with a different definition of \( \lambda_m \).

Remark 3.2. From Theorem 3.1 we can see that the upper bound is related to some random parts \( \lambda_m \) and \( \|\epsilon\| \). If we have Gaussian errors \( \epsilon \sim N(0, \sigma^2 I) \), then we know from Proposition 11 of \([40]\) that there exists an upper bound \( \lambda_m^u \) of \( \lambda_m \) such that \( \lambda_m \leq \lambda_m^u \) is valid with probability \( 1 - \alpha \) for a given constant \( \alpha \). Furthermore, it follows from \([28]\) that we can find an upper bound \( c_{1\sigma} \) and a lower bound \( c_{2\sigma} \) for \( \|\epsilon\| \) with a high probability. That is, if \( \lambda_m \) is replaced by \( \lambda_m^u \) and \( \|\epsilon\| \) is replaced by \( c_{1\sigma} \) or \( c_{2\sigma} \), then the sharp oracle bound with the Gaussian errors holds with a high probability.

4 The proximal majorization-minimization algorithm

To deal with the nonconvexity of the regularization function in the square-root regression problem \([3\text{]}\), we design a two stage proximal majorization-minimization (PMM) algorithm and solve a series of convex subproblems. In stage I, we first solve a problem by removing \( q \) from the original problem and adding appropriate proximal terms, to obtain an initial point to warm-start the algorithm in the second stage. In stage II, a series of majorized subproblems are solved to obtain a solution point.

The basic idea of the PMM algorithm is to linearize the concave term \(-q(\beta)\) in the objective function of \([3\text{]}\) at each iteration with respect to the current iterate, say \( \beta \). By doing so, the subproblem in each iteration is a convex minimization problem, which must be solved efficiently in order for the overall algorithm to be efficient. However, the objective function of the resulting subproblem contains the sum of two nonsmooth terms \( \|X\beta - b\| \) and \( p(\beta) \), and it is not obvious how such a problem can be solved efficiently. One important step we take in this paper is to add the proximal term \( \frac{\tau}{2}\|X\beta - \bar{X}\beta\|^2 \) to the objective function of the subproblem. Through this novel proximal term, the dual of the majorized subproblem can then be written explicitly as an unconstrained optimization problem. Moreover, its structure is highly conducive for one to apply the semismooth Newton (SSN) method to compute an approximate solution via solving a nonlinear system of equations.

4.1 A semismooth Newton method for the subproblems

For the purpose of our algorithm developments, given \( \sigma > 0, \tau > 0, \tilde{\beta} \in \mathbb{R}^n, \tilde{v} \in \mathbb{R}^n \), and \( \tilde{b} \in \mathbb{R}^m \), we consider the following minimization problem:

\[
\min_{\beta \in \mathbb{R}^n} \left\{ h(\beta; \sigma, \tau, \tilde{\beta}, \tilde{v}, \tilde{b}) := \|X\beta - b\| + \lambda p(\beta) - q(\tilde{\beta}) - \langle \tilde{v}, \beta - \tilde{\beta} \rangle + \frac{\sigma}{2}\|\beta - \tilde{\beta}\|^2 + \frac{\tau}{2}\|X\beta - \bar{X}\beta\|^2 \right\}. \tag{10}
\]

The optimization problem \([10]\) is equivalent to

\[
\min_{\beta \in \mathbb{R}^n, y \in \mathbb{R}^m} \left\{ \|y\| + \lambda p(\beta) - \langle \tilde{v}, \beta - \tilde{\beta} \rangle + \frac{\sigma}{2}\|\beta - \tilde{\beta}\|^2 + \frac{\tau}{2}\|y + b - \bar{b}\|^2 \bigm\| X\beta - y = b \right\}. \tag{11}
\]
The dual of problem (11) admits the following equivalent minimization form:

\[
\min_{u \in \mathbb{R}^m} \left\{ \varphi(u) := \langle u, b \rangle + \frac{1}{2\tau}\|\tau^{-1}u + \tilde{b} - b\|^2 - \|\text{Prox}_{\tau^{-1}1\|\|}(\tau^{-1}u + \tilde{b} - b)\| \right. \\
- \frac{1}{2\tau}\|\text{Prox}_{\tau\delta_B}(u + \tau(\tilde{b} - b))\|^2 + \frac{\sigma}{2}\|\tilde{\beta} + \sigma^{-1}(\tilde{v} - X^Tu)\|^2 \\
- \lambda_p \left( \text{Prox}_{\sigma^{-1}\lambda_p}(\tilde{\beta} + \sigma^{-1}(\tilde{v} - X^Tu)) \right) - \frac{1}{2\tau}\|\text{Prox}_{\sigma(\lambda_p)\|}(\sigma\tilde{\beta} + \tilde{v} - X^Tu)\|^2 \right\},
\]

where \( B = \{ x \mid \|x\| \leq 1 \} \). Let \( \bar{u} := \arg\min_{u \in \mathbb{R}^m} \varphi(u) \). Then the optimal solutions \( \bar{y}, \bar{\beta} \) to the primal problem (11) can be computed by

\[
\bar{y} = \text{Prox}_{\tau^{-1}\|\|}(\tau^{-1}\bar{u} + \bar{b} - b), \quad \bar{\beta} = \text{Prox}_{\sigma^{-1}\lambda_p}(\tilde{\beta} + \sigma^{-1}(\tilde{v} - X^T\bar{u})).
\]

Here we should emphasize the novelty in adding the proximal term \( \frac{1}{2\tau}\|X\beta - \tilde{b}\|^2 \) in (10). Without this term, the objective function in the dual problem (12) does not admit an analytical expression. As the reader may observe later, without the analytical expression given in (12), the algorithmic development in the rest of this subsection would break down. As a result, designing the PMM algorithm in the next subsection for solving (3) would also become impossible.

By Moreau’s identity (4) and the differentiability of the Moreau envelope functions of \( \| \cdot \| \) and \( \lambda_p \), we know that the function \( \varphi \) is convex and continuously differentiable and

\[
\nabla \varphi(u) = \text{Prox}_{\tau^{-1}\|\|}(\tau^{-1}u + \tilde{b} - b) - X\text{Prox}_{\sigma^{-1}\lambda_p}(\tilde{\beta} + \sigma^{-1}(\tilde{v} - X^Tu)) + b.
\]

Thus \( \bar{u} \) can be obtained via solving the following nonlinear system of equations:

\[
\nabla \varphi(u) = 0.
\]

In the rest of this subsection, we will discuss how we can apply the SSN method to compute an approximate solution of (13) efficiently. Since the mappings \( \text{Prox}_{\tau^{-1}\|\|}(\cdot) \) and \( \text{Prox}_{\tau^{-1}\lambda_p}(\cdot) \) are Lipschitz continuous, the following multifunction is well defined:

\[
\hat{\partial}^2 \varphi(u) := \sigma^{-1}X\partial\text{Prox}_{\sigma^{-1}\lambda_p}(\tilde{\beta} + \sigma^{-1}(\tilde{v} - X^Tu))X^T + \tau^{-1}\partial\text{Prox}_{\tau^{-1}\|\|}(\tau^{-1}u + \tilde{b} - b).
\]

Let \( U \in \partial\text{Prox}_{\sigma^{-1}\lambda_p}(\tilde{\beta} + \sigma^{-1}(\tilde{v} - X^Tu)) \) and \( V \in \partial\text{Prox}_{\tau^{-1}\|\|}(\tau^{-1}u + \tilde{b} - b) \). Then we have \( H := \sigma^{-1}XUX^T + \tau^{-1}V \in \hat{\partial}^2 \varphi(u) \). The following proposition demonstrates that \( H \) is nonsingular at the solution point that does not over fit, which, under Assumption 3.1, holds automatically when it is close to \( \tilde{\beta} \). This property is crucial to ensure the fast convergence of the SSN method for computing an approximate solution of (12).

**Proposition 4.1.** Suppose that the unique optimal solution \( \tilde{\beta} \) to the problem (10) satisfies \( \|X\tilde{\beta} - b\| \neq 0 \). Then all the elements of \( \hat{\partial}^2 \varphi(\bar{u}) \) are positive definite.

**Proof.** By the assumption, \( \bar{y} = X\tilde{\beta} - b \neq 0 \). Furthermore, \( \bar{y} = \text{Prox}_{\tau^{-1}\|\|}(\bar{u}) = \bar{u} - \tau^{-1}\Pi_B(\tau\bar{u}) \), where \( \bar{u} = \tau^{-1}\bar{u} + \tilde{b} - b \). Since \( \bar{y} \neq 0 \), it follows that \( \|\bar{u}\| \geq \frac{1}{\tau} \) and \( \text{Prox}_{\tau^{-1}\|\|}(\bar{u}) \) is differentiable with

\[
V := \nabla \text{Prox}_{\tau^{-1}\|\|}(\bar{u}) = \left(1 - \frac{1}{\tau\|\bar{u}\|}\right)I_m + \frac{\bar{u}\bar{u}^T}{\tau\|\bar{u}\|^3}.
\]
Hence for any $U \in \partial \text{Prox}_{\sigma^{-1}\lambda_p}(\tilde{\beta} + \sigma^{-1}(\tilde{v} - X^T\tilde{u}))$, $H = \sigma^{-1}XUX^T + \tau^{-1}V \in \tilde{\partial}^2 \varphi(\tilde{u})$. Since $V$ is positive definite and $XUX^T$ is positive semidefinite, $H$ is positive definite. This completes the proof.

Now we discuss how to apply the SSN method to solve the nonsmooth equation (13) to obtain an approximate solution efficiently. We first prove that $\nabla \varphi$ is strongly semismooth.

**Proposition 4.2.** The function $\nabla \varphi$ is strongly semismooth.

**Proof.** Firstly, we have assumed that the proximal operator $\text{Prox}_\rho(\cdot)$ is strongly semismooth. Secondly, by Proposition 4.3 of [12], it is known that the projection operator onto the second order cone is strongly semismooth. The strongly semismoothness of the proximal operator $\text{Prox}_{\|\cdot\|}(\cdot)$ then follows from Theorem 4 of [33], which states that if the projection onto the second order cone is strongly semismooth, then so is the proximal mapping of $\|\cdot\|$. From here, it is easy to prove the required result and we omit the details.

Now we can apply the SSN method to solve (13) as follows.

**Algorithm SSN**(σ,τ,\tilde{\beta},\tilde{v},\tilde{b}) with input $\sigma > 0$, $\tau > 0$, $\tilde{\beta}, \tilde{v} \in \mathbb{R}^n$, $\tilde{b} \in \mathbb{R}^m$. Choose $\mu \in (0,\frac{1}{2})$, $\eta \in (0,1)$, $\varrho \in (0,1)$, $\delta \in (0,1)$, and $u^0 \in \mathbb{R}^m$. For $j = 0,1,\ldots$, iterate the following steps:

**Step 1.** Choose $U^j \in \partial \text{Prox}_{\sigma^{-1}\lambda_p}(\tilde{\beta} + \sigma^{-1}(\tilde{v} - X^T u^j))$ and $V^j \in \partial \text{Prox}_{\tau^{-1}\|\cdot\|}(\tau^{-1}u^j + \tilde{b} - \bar{b})$. Let $H^j = \sigma^{-1}XU^jX^T + \tau^{-1}V^j$. Compute an approximate solution $\Delta u^j$ to the linear system

$$H^j \Delta u = -\nabla \varphi(u^j)$$

such that

$$\|H^j \Delta u^j + \nabla \varphi(u^j)\| \leq \min(\eta, \|\nabla \varphi(u^j)\|^{1+\varrho}). \quad (14)$$

**Step 2.** Set $\alpha_j = \delta^{t_j}$, where $t_j$ is the first nonnegative integer $t$ such that

$$\varphi(u^j + \delta^t u^j) \leq \varphi(u^j) + \mu \delta^t \nabla \varphi(u^j), \Delta u^j).$$

**Step 3.** Set $u^{j+1} = u^j + \alpha_j \Delta u^j$.

With Propositions 4.1 and 4.2, the SSN method can be proven to be globally convergent and locally superlinearly convergent. One may see Theorem 3.6 of [31] for the details. The local convergence rate for Algorithm SSN is stated in the next theorem without proof.

**Theorem 4.1.** Suppose that $\|X\tilde{\beta} - b\| \neq 0$ holds. Then the sequence $\{u^j\}$ generated by Algorithm SSN converges to the unique optimal solution $\bar{u}$ of the problem (12) and

$$\|u^{j+1} - \bar{u}\| = O(\|u^j - \bar{u}\|^{1+\varrho}).$$
4.2 The SSN based proximal majorization-minimization algorithm

In this subsection, we describe the details of the PMM algorithm for solving (3) wherein each subproblem is solved by the SSN method. We briefly present the PMM algorithm as follows.

**Algorithm PMM.** Let \( \sigma^{2,0} > 0, \tau^{2,0} > 0 \) be given parameters.

**Step 1.** Find \( \sigma^1 > 0, \tau^1 > 0 \) and compute
\[
\beta^0 \approx \arg\min_{\beta \in \mathbb{R}^n} \{ h(\beta; \sigma^1, \tau^1, 0, 0, b) \} \tag{15}
\]
via solving its dual problem such that the KKT residual of the problem \( (16) \) satisfies a prescribed stopping criterion. That is, given \( (\sigma, \tau, \tilde{\beta}, \tilde{\nu}, \tilde{b}) = (\sigma^1, \tau^1, 0, 0, b) \), apply the SSN method to find an approximate solution \( \sigma^{1,0}, \tau^{1,0} \) to (13) and then set \( \beta^0 = \text{Prox}_{\lambda p/\sigma^1}(-X^T u^0/\sigma^1) \). Let \( k = 0 \) and go to Step 2.1.

**Step 2.1 Compute**
\[
\beta^{k+1} = \arg\min_{\beta \in \mathbb{R}^n} \left\{ h(\beta; \sigma^{2,k}, \tau^{2,k}, \beta^k, \nabla q(\beta^k), X \beta^k) + \langle \delta^k, \beta - \beta^k \rangle \right\}
\]
via solving its dual problem. That is, given \( (\sigma, \tau, \tilde{\beta}, \tilde{\nu}, \tilde{b}) = (\sigma^{2,k}, \tau^{2,k}, \beta^k, \nabla q(\beta^k), X \beta^k) \), apply the SSN method to find an approximate solution \( u^{k+1} \) of (13) such that the error vector \( \delta^k \) satisfies the following accuracy condition:
\[
\|\delta^k\| \leq \frac{\sigma^{2,k}}{4} \|\beta^{k+1} - \beta^k\| + \frac{\tau^{2,k} X \beta^k - \beta^k}{2 \| \beta^{k+1} - \beta^k \|}, \tag{16}
\]
where \( \beta^{k+1} = \text{Prox}_{\lambda p/\sigma^{2,k}}(\beta^k + (\nabla q(\beta^k) - X^T u^{k+1})/\sigma^{2,k}) \).

**Step 2.2.** If \( \beta^{k+1} \) satisfies a prescribed stopping criterion, terminate; otherwise update \( \sigma^{2,k+1} = \rho_k \sigma^{2,k}, \tau^{2,k+1} = \rho_k \tau^{2,k} \) with \( \rho_k \in (0, 1) \) and return to Step 2.1 with \( k = k + 1 \).

Since \( h(\beta; \sigma^1, \tau^1, 0, 0, b) \) is bounded below, it has been proven in Proposition 4.19 of [21] that the optimal objective value of the problem \( (13) \) will converge to the optimal objective value of the problem \( (0) \) with a difference of \( q(0) \) as \( \sigma^1 \to 0, \tau^1 \to 0 \). We simply describe the convergence result of the algorithm in our first stage as follows and give a similar proof to that of Proposition 4.19 in [21].

**Theorem 4.2.** Let \( \bar{h}(\sigma^1, \tau^1) := \min_{\beta \in \mathbb{R}^n} \{ h(\beta; \sigma^1, \tau^1, 0, 0, b) \} \). Then we have
\[
\lim_{\sigma^1, \tau^1 \to 0} \bar{h}(\sigma^1, \tau^1) = \min_{\beta \in \mathbb{R}^n} \left\{ \| X \beta - b \| + \lambda p(\beta) - q(0) \right\}.
\]

**Proof.** For any \( \sigma^1, \tau^1 > 0 \) and \( \beta \in \mathbb{R}^n \), we have that
\[
\bar{h}(\sigma^1, \tau^1) \leq \| X \beta - b \| + \lambda p(\beta) - q(0) + \frac{\sigma^1}{2} \| \beta \|^2 + \frac{\tau^1}{2} \| X \beta - b \|^2.
\]
Therefore, \( \lim_{\sigma^1, \tau^1 \to 0} \bar{h}(\sigma^1, \tau^1) \leq \|X\beta - b\| + \lambda p(\beta) - q(0) \). That is

\[
\lim_{\sigma^1, \tau^1 \to 0} \bar{h}(\sigma^1, \tau^1) \leq \min_{\beta \in \mathbb{R}^n} \left\{ \|X\beta - b\| + \lambda p(\beta) - q(0) \right\}.
\]

Furthermore, \( \bar{h}(\sigma^1, \tau^1) \geq \min_{\beta \in \mathbb{R}^n} \left\{ \|X\beta - b\| + \lambda p(\beta) - q(0) \right\} \). The desired result follows.

### 4.3 Convergence analysis of the PMM algorithm

In this subsection, we analyze the convergence of the PMM algorithm. First we recall the definition of the KL property of a function (see [2, 4, 5] for more details). Let \( \eta > 0 \) and \( \Phi_\eta \) be the set of all concave functions \( \psi : [0, \eta) \to \mathbb{R}_+ \) such that

1. \( \psi(0) = 0 \);
2. \( \psi \) is continuous at 0 and continuously differentiable on \((0, \eta)\);
3. \( \psi'(x) > 0 \), for any \( x \in (0, \eta) \).

**Definition 4.1.** Let \( f : \mathbb{R}^n \to (-\infty, \infty] \) be a proper lower semi-continuous function and \( \bar{x} \in \text{dom}(\partial f) := \{x \in \text{dom}(f) | \partial f(x) \neq \emptyset\} \). The function \( f \) is said to have the KL property at \( \bar{x} \) if there exist \( \eta > 0 \), a neighbourhood \( \mathcal{U} \) of \( \bar{x} \) and a concave function \( \psi \in \Phi_\eta \) such that

\[
\psi'(f(x) - f(\bar{x})) \text{dist}(0, \partial f(x)) \geq 1, \quad \forall x \in \mathcal{U} \text{ and } f(x) < f(\bar{x}) + \eta,
\]

where \( \text{dist}(x, C) := \min_{y \in C} \|y - x\| \) is the distance from a point \( x \) to a nonempty closed set \( C \).

Furthermore, a function \( f \) is called a KL function if it satisfies the KL property at all points in \( \text{dom}\partial f \).

Note that a function is said to have the KL property at \( \bar{x} \) with an exponent \( \alpha \) if the function \( \psi \) in the definition of the KL property takes the form of \( \psi(x) = \gamma x^{1-\alpha} \) with \( \gamma > 0 \) and \( \alpha \in [0, 1) \).

For the function \( f(x) = x \), it has the KL property at any point with the exponent 0.

Now we are ready to conduct the convergence analysis of the PMM algorithm. Denote \( h_k(\beta) := h(\beta; \sigma^{2,k}, \tau^{2,k}, \beta^k, \nabla q(\beta^k), X\beta^k) \). At the \( k \)-th iteration of stage II, we have that

\[
\beta^{k+1} = \arg\min_{\beta \in \mathbb{R}^n} \left\{ h_k(\beta) + \langle \delta^k, \beta - \beta^k \rangle \right\}
\]

such that condition (16) is satisfied. The following lemma shows the descent property of the function \( h_k \).

**Lemma 4.1.** Let \( \beta^{k+1} \) be an approximate solution of the subproblem in the \( k \)-th iteration such that (16) holds. Then we have

\[
h_k(\beta^k) \geq h_k(\beta^{k+1}) - \frac{\sigma^{2,k}}{4}\|\beta^{k+1} - \beta^k\|^2 - \frac{\tau^{2,k}}{2}\|X\beta^{k+1} - X\beta^k\|^2.
\]
Proof. Since $h_k$ is a convex function and $-\delta^k \in \partial h_k(\beta^{k+1})$, we obtain
\[
h_k(\beta^k) - h_k(\beta^{k+1}) \geq \langle \delta^k, \beta^{k+1} - \beta^k \rangle \geq -\frac{\sigma^2 k}{4} \|\beta^{k+1} - \beta^k\|^2 - \frac{\tau^2 k}{2} \|X \beta^{k+1} - X \beta^k\|^2.
\]
The last inequality is valid since the condition (10) holds. The desired result follows. \qed

Next we recall the following lemma which is similar to that in [15, 35].

**Lemma 4.2.** The vector $\bar{\beta} \in \mathbb{R}^n$ is a $d$-stationary point of (3) if and only if there exist $\sigma, \tau \geq 0$ such that
\[
\bar{\beta} \in \arg\min_{\beta \in \mathbb{R}^n} \left\{ h(\beta; \sigma, \tau, \bar{\beta}, \nabla q(\bar{\beta}), X\bar{\beta}) \right\}.
\]

**Proof.** Recall the objective function $g$ defined in (4). Since $g$ is directionally differentiable at $\bar{\beta}$, we can see that $\bar{\beta}$ being a $d$-stationary point of $g$ is equivalent to $0 \in \partial g(\bar{\beta})$. It is easy to show that $\partial g(\bar{\beta}) = \partial \beta h(\bar{\beta}; \sigma, \tau, \bar{\beta}, \nabla q(\bar{\beta}), X\bar{\beta})$. For given $\sigma, \tau$ and $\bar{\beta}$, the function $h(\cdot; \sigma, \tau, \bar{\beta}, \nabla q(\bar{\beta}), X\bar{\beta})$ is convex. Thus $0 \in \partial \beta h(\bar{\beta}; \sigma, \tau, \bar{\beta}, \nabla q(\bar{\beta}), X\bar{\beta})$ is equivalent to $\bar{\beta} \in \arg\min_{\beta \in \mathbb{R}^n} \left\{ h(\beta; \sigma, \tau, \bar{\beta}, \nabla q(\bar{\beta}), X\bar{\beta}) \right\}$. This completes the proof. \qed

It has been proven in [15] that the sequence generated by the PMM algorithm converges to a directional stationary solution if the exact solutions of the subproblems are obtained. The following theorem shows that the result is also true if the subproblems are solved approximately.

**Theorem 4.3.** Suppose that the function $g$ in (3) is bounded below and Assumption 3.1 holds. Assume that $\{\sigma^k\}$ and $\{\tau^k\}$ are convergent sequences. Let $\{\beta^k\}$ be the sequence generated by the PMM algorithm. Then every cluster point of the sequence $\{\beta^k\}$, if exists, is a $d$-stationary point of (3).

**Proof.** Combing Lemma 4.1 and the convexity of $q$, we have
\[
g(\beta^k) = h_k(\beta^k) \geq h_k(\beta^{k+1}) - \frac{\sigma^2 k}{4} \|\beta^{k+1} - \beta^k\|^2 - \frac{\tau^2 k}{2} \|X \beta^{k+1} - X \beta^k\|^2
\]
\[
\geq g(\beta^{k+1}) + \frac{\sigma^2 k}{4} \|\beta^{k+1} - \beta^k\|^2.
\]
Therefore the sequence $\{g(\beta^k)\}$ is non-increasing. Since $g(\beta)$ is bounded below, the sequence $\{g(\beta^k)\}$ converges and so is the sequence $\{\|\beta^{k+1} - \beta^k\|\}$ which converges to zero. Next, we prove that the limit of a convergent subsequence of $\{\beta^k\}$ is a $d$-stationary point of (3). Let $\beta^\infty$ be the limit of a convergent subsequence $\{\beta^k\}_{k \in K}$. We can easily prove that $\{\beta^{k+1}\}_{k \in K}$ also converges to $\beta^\infty$. It follows from the definition of $\beta^{k+1}$ that
\[
h_k(\beta) \geq h_k(\beta^{k+1}) + \langle \delta^k, \beta^{k+1} - \beta \rangle \geq h_k(\beta^{k+1}) - \|\delta^k\| \|\beta^{k+1} - \beta\|, \quad \forall \beta \in \mathbb{R}^m.
\]
Letting $k(\in K) \to \infty$, we obtain that
\[
\beta^\infty \in \arg\min_{\beta \in \mathbb{R}^n} \left\{ h(\beta; \sigma^{2,\infty}, \tau^{2,\infty}, \beta^\infty, \nabla q(\beta^\infty), X\beta^\infty) \right\},
\]
where $\sigma^{2,\infty} = \lim_{k \to \infty} \sigma^{2,k} \geq 0$ and $\tau^{2,\infty} = \lim_{k \to \infty} \tau^{2,k} \geq 0$. The desired result follows from Lemma 4.2. This completes the proof. \qed
We can also establish the local convergence rate of the sequence \( \{ \beta^k \} \) under either an isolation assumption of the accumulation point or the KL property assumption.

**Theorem 4.4.** Suppose that the function \( g \) is bounded below and Assumption \( 3.4 \) holds. Let \( \{ \beta^k \} \) be the sequence generated by the PMM algorithm. Let \( B^\infty \) be the set of cluster points of the sequence \( \{ \beta^k \} \). If either one of the following two conditions holds,

(a) \( B^\infty \) contains an isolated element;

(b) The sequence \( \{ \beta^k \} \) is bounded; for all \( \beta \in B^\infty \), \( \nabla q \) is locally Lipschitz continuous near \( \beta \) and the function \( g \) has the KL property at \( \beta \);

then the whole sequence \( \{ \beta^k \} \) converges to an element of \( B^\infty \). Moreover, if condition (b) is satisfied such that \( \{ \beta^k \} \) converges to \( \beta^\infty \in B^\infty \) and the function \( g \) has the KL property at \( \beta^\infty \) with an exponent \( \alpha \in [0,1) \), then we have the following results:

(i) If \( \alpha = 0 \), then the sequence \( \{ \beta^k \} \) converges in a finite number of steps;

(ii) If \( \alpha \in (0, \frac{1}{4}] \), then the sequence \( \{ \beta^k \} \) converges \( R \)-linearly, that is, for all \( k \geq 1 \), there exist \( \nu > 0 \) and \( \eta \in [0,1) \) such that \( \| \beta^k - \beta^\infty \| \leq \nu k^\eta \);

(iii) If \( \alpha \in (\frac{1}{2}, 1) \), then the sequence \( \{ \beta^k \} \) converges \( R \)-sublinearly, that is, for all \( k \geq 1 \), there exists \( \nu > 0 \) such that \( \| \beta^k - \beta^\infty \| \leq \nu k - \frac{1}{\alpha - 1} \).

**Proof.** We know from Theorem 4.3 that \( \lim_{k \to \infty} \| \beta^{k+1} - \beta^k \| = 0 \). Then it follows from Proposition 8.3.10 of [19] that the sequence \( \{ \beta^k \} \) converges to an isolated element of \( B^\infty \) under the condition (a). In order to derive the convergence rate of the sequence \( \{ \beta^k \} \) under the condition (b), we first establish some properties of the sequence \( \{ \beta^k \} \), i.e.,

(1) \( g(\beta^k) \geq g(\beta^{k+1}) + \frac{2}{4} \| \beta^{k+1} - \beta^k \|^2 \);

(2) there exists a subsequence \( \{ \beta^{k_j} \} \) of \( \{ \beta^k \} \) such that \( \beta^{k_j} \to \beta^\infty \) with \( g(\beta^{k_j}) \to g(\beta^\infty) \) as \( j \to \infty \);

(3) for \( k \) sufficiently large, there exist a constant \( K > 0 \) and \( \xi^{k+1} \in \partial g(\beta^{k+1}) \) such that \( \| \xi^{k+1} \| \leq K \| \beta^{k+1} - \beta^k \| \).

The properties (1) and (2) are already known from Theorem 4.3. To establish the property (3), we first note that \( B^\infty \) is a nonempty, compact and connected set by Proposition 8.3.9 of [19]. Furthermore, let \( \xi^{k+1} = \nabla q(\beta^{k+1}) - \nabla q(\beta^k) - \sigma^2 \cdot (\beta^{k+1} - \beta^k) - \tau^2 X'X(\beta^{k+1} - \beta^k) - \delta^k \). We have that \( \xi^{k+1} \in \partial g(\beta^{k+1}) \). Since \( \nabla q \) is locally Lipschitz continuous near all \( \beta \in B^\infty \) and \( \| \delta^k \| \leq \frac{2}{4} \| \beta^{k+1} - \beta^k \|^2 + \frac{2}{4} \| X'X(\beta^{k+1} - \beta^k) - \delta^k \|^2 \), the property (3) holds for some constant \( K > 0 \) with \( \| \xi^{k+1} \| \leq K \| \beta^{k+1} - \beta^k \| \) for \( k \) sufficiently large. With the properties (1)-(3), the convergence rate of the sequence \( \{ \beta^k \} \) can be established similarly to that of Proposition 4 of [4].
5 Numerical experiments

In this section, we perform some numerical experiments to demonstrate the efficiency of our PMM algorithm for the square-root regression problems. We implemented the algorithm in MATLAB R2017a. All runs were performed on a PC (Intel Core 2 Duo 2.6 GHz with 4 GB RAM). We tested our algorithm on two types of data sets. The first set consists of synthetic data generated randomly in the high-sample-low-dimension setting. That is,\[ b = X\tilde{\beta} + \zeta \varepsilon, \quad \varepsilon \sim N(0, I). \]

Each row of the input data \( X \in \mathbb{R}^{m \times n} \) is generated randomly from the multivariate normal distribution \( N(0, \Sigma) \) with \( \Sigma \) as the covariance matrix. Now we present four examples which are similar to that in [48]. For each instance, we generate 8000 observations for the training data set, 2000 observations for the validation data set and 2000 observations for the test data set.

(a) In example 1, the problem has 800 predictors. Let \( \beta = (3, 1.5, 0, 0, 2, 0, 0, 0) \) and \( \tilde{\beta} = (\beta, \ldots, \beta)^T \) with the vector \( \beta = (0, 1) \).

(b) In example 2, the setting is the same as that in example 1 except that \( \tilde{\beta} = (\beta, \ldots, \beta)^T \) with the vector \( \beta = (0, 1) \).

(c) In example 3, we set \( \tilde{\beta} = (\beta, \ldots, \beta)^T \) with the vector \( \beta = (0, 1), \zeta = 15 \) and \( \Sigma_{ij} = 0.8^{|i-j|} \).

(d) In example 4, the problem has 4000 predictors. We choose \( \tilde{\beta} = (3, \ldots, 3, 0, \ldots, 0) \) and \( \zeta = 15 \). Let \( X_i \) be the \( i \)-th predictor of \( X \). For \( i \leq 1500 \), \( X_i \) is generated as follows:
\begin{align*}
X_i &= Z_1 + \tilde{\varepsilon}_i, \quad Z_1 \sim N(0, I), \quad i = 1, \ldots, 500, \\
X_i &= Z_2 + \tilde{\varepsilon}_i, \quad Z_2 \sim N(0, I), \quad i = 600, \ldots, 1000, \\
X_i &= Z_3 + \tilde{\varepsilon}_i, \quad Z_3 \sim N(0, I), \quad i = 1001, \ldots, 1500,
\end{align*}

with \( \tilde{\varepsilon}_i \sim N(0, 0.01I), i = 1, \ldots, 1500 \). For \( i > 1500 \), the predictor \( X_i \) is just white noise, i.e., \( X_i \sim N(0, I) \).

We also evaluated our algorithm on some large scale LIBSVM data sets \((X, b)\) [10] which are obtained from the UCI data repository [30]. As in [31], we used the method in [26] to expand the features of these data sets by using polynomial basis functions. The last digit in the names of the data sets, abalone7, bodyfat7, housing7, mpg7 and space9, indicate the order of the polynomial used to expand the features. The number of nonzero elements of a vector is defined as the minimal \( k \) such that
\[ \sum_{i=1}^{k} |\tilde{\beta}_i| \geq 0.9999 \| \beta \|_1, \]

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where $\tilde{\beta}$ is obtained by sorting $\beta$ such that $|\tilde{\beta}_1| \geq |\tilde{\beta}_2| \geq \ldots \geq |\tilde{\beta}_n|$.

In all the experiments, the parameter $\lambda$ is set to $\lambda = \lambda_c \Lambda$, where $\Lambda = 1.1 \Phi^{-1}(1 - 0.05/(2n))$ with $\Phi$ being the cumulative normal distribution function and $\Lambda$ is the theoretical choice recommended in [S] to compute a specific coefficient estimate. In order to obtain a reasonably sparse optimal solution, we choose different $\lambda_c$.

5.1 Numerical experiments for the convex square-root regression problems

In this section, we compare the performances of the alternating direction method of multipliers (ADMM) and our stage I algorithm for solving the convex square-root regression problem (6). For comparison purpose, we adopt the widely used ADMM algorithms for both the primal and dual problems of (6). For convenience, we use pADMM to denote the ADMM applied to the primal problem, dADMM to denote the ADMM applied to the dual problem, and PMM to denote our stage I algorithm for solving the convex square-root regression problem (6).

5.1.1 The ADMM for the problem (6)

In this subsection, we describe the implementation details of the ADMM for the problem (6). The convex problem (6) can be written equivalently as

$$\min_{\beta, z \in \mathbb{R}^n, y \in \mathbb{R}^m} \left\{ \|y\| + \lambda p(z) \mid X\beta - y = b, \beta - z = 0 \right\}.$$  \hspace{1cm} (17)

The dual problem corresponding to (17) has the following form

$$\min_{u, w \in \mathbb{R}^m, v \in \mathbb{R}^n} \left\{ \delta_B(w) + (\lambda p)^*(v) + \langle u, b \rangle \mid X^T u + v = 0, -u + w = 0 \right\}.$$  \hspace{1cm} (18)

Given $\zeta > 0$, the augmented Lagrangian functions corresponding to (17) and (18) are given by

$$L_\zeta(\beta, y, z; u, v) := \|y\| + \lambda p(z) + \langle u, X\beta - y - b \rangle + \frac{\zeta}{2} \|X\beta - y - b\|^2$$

$$+ \langle v, \beta - z \rangle + \frac{\zeta}{2} \|\beta - z\|^2,$$

$$\tilde{L}_\zeta(u, v; w; \beta, y) := \delta_B(w) + (\lambda p)^*(v) + \langle u, b \rangle - \langle \beta, X^T u + v \rangle + \frac{\zeta}{2} \|X^T u + v\|^2$$

$$- \langle y, -u + w \rangle + \frac{\zeta}{2} \| -u + w\|^2,$$

respectively. Based on the above augmented Lagrangian functions, the ADMMs ([16] [22]) for solving (17) and (18) are given as follows.
Algorithm pADMM for the primal problem (17). Let $\rho \in (0, (1 + \sqrt{5})/2)$, $\zeta > 0$ be given parameters. Choose $(y^0, z^0, u^0, v^0) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$, set $k = 0$ and iterate.

**Step 1.** Compute

$$\beta^{k+1} = \arg\min_{\beta \in \mathbb{R}^n} \left\{ \mathcal{L}_\zeta(\beta, y^k, z^k; u^k, v^k) \right\}$$

$$= (I_n + X^T X)^{-1}(z^k - \zeta^{-1}v^k + X^T (y^k + b - \zeta^{-1}u^k)),$$

$$(y^{k+1}, z^{k+1}) = \arg\min_{y \in \mathbb{R}^m, z \in \mathbb{R}^n} \left\{ \mathcal{L}_\zeta(\beta^{k+1}, y, z; u^k, v^k) \right\}$$

$$= \left( \text{Prox}_{\zeta^{-1}\|\cdot\|}(X\beta^{k+1} - b + \zeta^{-1}u^k), \text{Prox}_{\zeta^{-1}\lambda p}(\beta^{k+1} + \zeta^{-1}v^k) \right).$$

**Step 2.** Update

$$u^{k+1} = u^k + \rho\zeta(X\beta^{k+1} - y^{k+1} - b), \quad v^{k+1} = v^k + \rho\zeta(\beta^{k+1} - z^{k+1}).$$

If the prescribed stopping criterion is satisfied, terminate; otherwise return to Step 1 with $k = k + 1$.

Algorithm dADMM for the dual problem (18). Let $\rho \in (0, (1 + \sqrt{5})/2)$, $\zeta > 0$ be given parameters. Choose $(v^0, w^0, \beta^0, y^0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m$, set $k = 0$ and iterate.

**Step 1.** Compute

$$u^{k+1} = \arg\min_{u \in \mathbb{R}^m} \left\{ \mathcal{L}_\zeta(u, v^k, \beta^k, y^k) \right\}$$

$$= (I_m + XX^T)^{-1}(w^k - \zeta^{-1}y^k + X(-v^k + \zeta^{-1}\beta^k) - b),$$

$$(v^{k+1}, w^{k+1}) = \arg\min_{v \in \mathbb{R}^n, w \in \mathbb{R}^m} \left\{ \mathcal{L}_\zeta(u^{k+1}, v, w; \beta^k, y^k) \right\}$$

$$= \left( \text{Prox}_{\zeta^{-1}\lambda p}^*(\zeta^{-1}\beta^k - X^T u^{k+1}), \text{Prox}_{\zeta^{-1}\delta B}^*(\zeta^{-1}y^k + u^{k+1}) \right).$$

**Step 2.** Update

$$\beta^{k+1} = \beta^k - \rho\zeta(X^T u^{k+1} + v^{k+1}), \quad y^{k+1} = y^k - \rho\zeta(-u^{k+1} + w^{k+1}).$$

If the prescribed stopping criterion is satisfied, terminate; otherwise return to Step 1 with $k = k + 1$.

In the pADMM and dADMM, we set the parameter $\rho = 1.618$ and solve the linear system in Step 1 of pADMM and dADMM by using the Sherman-Morrison-Woodbury formula [24] if
it is necessary, i.e.,

\[
(I_n + X^TX)^{-1} = I_m - X \left( I_m + XX^T \right)^{-1} X^T,
\]

\[
(I_m + XX^T)^{-1} = I_n - X^T \left( I_n + X^TX \right)^{-1} X.
\]

Depending on the dimension \(n, m\) of the problem, we either solve the linear system with coefficient matrix \(I_m + XX^T\) (or \(I_n + X^TX\)) by Cholesky factorization or by an iterative solver such as the preconditioned conjugate gradient (PCG) method. We should mention that when the latter approach is used, the linear system only needs to be solved to a sufficient level of accuracy that depend on the progress of the algorithm without sacrificing the convergence of the ADMMs. For the details, we refer the reader to [13].

### 5.1.2 Stopping criteria

In order to measure the accuracy of an approximate optimal solution \(\beta\), we use the relative duality gap defined by

\[
\eta_G := \frac{|\text{pobj} - \text{dobj}|}{1 + |\text{pobj}| + |\text{dobj}|},
\]

where \(\text{pobj} := \|X\beta - b\| + \lambda p(\beta)\), \(\text{dobj} := -\langle u, b \rangle\) are the primal and dual objective values, respectively. We also adopt the relative KKT residual

\[
\eta_{\text{kkt}} := \frac{\left\| \beta - \text{Prox}_{\lambda p} \left( \beta - \frac{X^T(X\beta - b)}{\|X\beta - b\|} \right) \right\|}{1 + \|\beta\| + \|X^T(X\beta - b)\|/\|X\beta - b\|}
\]

to measure the accuracy of an approximate optimal solution \(\beta\). For a given tolerance, our stage I algorithm will be terminated if

\[
\eta_{\text{kkt}} < \epsilon_{\text{kkt}} = 10^{-6},
\]

or the number of iterations reaches the maximum of 200 while the ADMMs will be terminated if (19) is satisfied or the number of iterations reaches the maximum of 10000.

### 5.1.3 Numerical results for the srLasso problem

Here we compare the performance of different methods for solving the convex problem (2). In [40], it adopted the R package Flare [32] to solve the srLasso problem (2). As the algorithm in Flare is in fact the pADMM with unit steplength, we first compare our own implementation of the pADMM with that in the Flare package. For a fair comparison, our pADMM is also written in R. Since the stopping criterion of the Flare package is not stated explicitly, we first run the Flare package to obtain a primal objective value and then run our pADMM, which is terminated as soon as our primal objective value is smaller than that obtained by Flare. We note that since (2) is an unconstrained optimization problem, it is meaningful to compare the objective function values obtained by Flare and our pADMM.
We report the numerical results in Tables 1 and 2. The symbol “–” in Table 2 means that the Flare package fails to solve the problem due to excessive memory requirement. From Tables 1 and 2, we can observe that our pADMM is clearly faster than the Flare package. A possible cause of this difference may lie in the different strategies for dynamically updating the parameter $\zeta$ in the practical implementations of the pADMM. As our implementation of the pADMM is much more efficient than that in the Flare package, in the subsequent experiments, we will not compare the performance of our PMM algorithm with the Flare package but with our own pADMM.

**Table 1**: The performance of the Flare package and our pADMM on synthetic datasets for the srLasso problem.

| probname | $\lambda$ | $c$ | $pobj$ | time | $m$; $n$ |
|----------|-----------|-----|--------|------|---------|
| exmp1    | 1.0       | 3.8876+3 | 3.5799+3 | 11:26 | 8000;800 |
|          | 0.5       | 3.0501+3 | 1.9174+3 | 21:09 | 8000;800 |
|          | 0.1       | 1.0487+3 | 5.8738+2 | 28:42 | 8000;800 |
| exmp2    | 1.0       | 2.2422+3 | 2.2419+3 | 14:09 | 8000;800 |
|          | 0.5       | 1.8050+3 | 1.2811+3 | 27:18 | 8000;800 |
|          | 0.1       | 5.6150+2 | 4.6013+2 | 27:37 | 8000;800 |
| exmp3    | 1.0       | 2.4758+3 | 2.4569+3 | 10:05 | 8000;400 |
|          | 0.5       | 1.9819+3 | 1.9421+3 | 7:26  | 8000;400 |
|          | 0.1       | 1.4888+3 | 1.4438+3 | 7:14  | 8000;400 |
| exmp4    | 1.0       | 1.1210+4 | 1.1205+4 | 29:11 | 8000;400 |
|          | 0.5       | 1.0165+4 | 1.0165+4 | 1:43:48 | 8000;400 |
|          | 0.1       | 7.6846+3 | 3.4069+3 | 3:11:27 | 8000;400 |

**Table 2**: The performance of the Flare package and our pADMM on UCI datasets for the srLasso problem.

| probname               | $\lambda$ | $c$ | $pobj$ | time | $m$; $n$ |
|------------------------|-----------|-----|--------|------|---------|
| abalone.scale.expanded7 | 1.0       | –   | 2.3852+2 | –   | 25:57   |
|                        | 0.5       | –   | 2.0312+2 | –   | 25:32   |
|                        | 0.1       | –   | 1.5586+2 | –   | 26:29   |
| mpg.scale.expanded7    | 1.0       | 2.3550+2 | 2.3544+2 | 1:00 | 04      |
|                        | 0.5       | 1.5856+2 | 1.5831+2 | 57   | 03      |
|                        | 0.1       | 7.8656+1 | 7.8616+1 | 1.06 | 03      |
| space.ga.scale.expanded9 | 1.0       | 1.3113+1 | 1.3113+1 | 12:59 | 5:19    |
|                        | 0.5       | 2.2419+1 | 2.1607+1 | 9:01  | 2:00    |
|                        | 0.1       | 1.2950+1 | 1.1999+1 | 6:13  | 3:00    |

Next we conduct numerical experiments to evaluate the performance of the pADMM, dADMM and PMM. For the numerical results, we report the problem name (probname), the number of samples ($m$) and features ($n$), $\lambda_c$, the number of nonzero elements of $\beta$ (nnz), the relative duality gap ($\eta_G$), the primal objective value ($pobj$) and the computation time (time). For each synthetic data set, the models were fitted on the training data sets and the validation data sets were used to select the best regularization parameter $\lambda_c$. For the reported nnz and test error (the mean-squared error) on the test data set (testerror), both were obtained from the PMM algorithm. For each UCI data set, we adopted the tenfold cross validation to find the best regularization parameter. The PMM algorithm was used to perform the cross validation. The parameter $\lambda_c$ and the corresponding nnz obtained by the PMM algorithm are listed in Tables 3.
and 4. We do not report the test errors for the UCI data sets due to the lack of test data sets. In the implementation of the pADMM and dADMM, we first compute the (sparse) Cholesky decomposition of \( I_n + X^T X \) or \( I_m + XX^T \) and then solve the linear system of equations in each iteration by using the pre-computed Cholesky factor.

Table 3: The performance of different algorithms on synthetic datasets for the srLasso problem. In the table, “a”=PMM, “b”=pADMM, “c”=dADMM.

| probname   | \( \lambda \) | nnz | \( \%_{\text{kkt}} \) | \( \%_{\text{G}} \) | \( \text{pobj} \) | time | testerror |
|------------|----------------|-----|-------------------|-----------------|-----------------|------|-----------|
| exmp1      | 0.127          | 499 | 9.4-7 | 9.9-7 | 9.9-7 | 1.9-8 | 3.1-7 | 1.7-8 | 6.6996+2 | 6.6996+2 | 6.6996+2 | 30 | 3.49 | 2.44 | 9.4085+0 |
| exmp2      | 0.081          | 627 | 9.8-7 | 9.9-7 | 9.9-7 | 2.6-9 | 1.8-7 | 7.2-9 | 4.1824+2 | 4.1824+2 | 4.1824+2 | 32 | 1.49 | 2.26 | 9.4807+0 |
| exmp3      | 0.124          | 298 | 7.4-7 | 9.1-7 | 9.9-7 | 4.8-9 | 1.1-7 | 3.7-9 | 1.4476+3 | 1.4476+3 | 1.4476+3 | 08 | 17 | 1.05 | 2.3420+2 |
| exmp4      | 0.117          | 2845 | 7.7-7 | 9.9-7 | 9.5-7 | 8.6-10 | 5.2-7 | 3.3-7 | 3.6799+3 | 3.6799+3 | 3.6799+3 | 4.41 | 5.41 | 16.44 | 3.6759+2 |

Table 4: The performance of different algorithms on UCI datasets for the srLasso problem. In the table, “a”=PMM, “b”=pADMM, “c”=dADMM.

| probname                   | \( \lambda \) | nnz | \( \%_{\text{kkt}} \) | \( \%_{\text{G}} \) | \( \text{pobj} \) | time | testerror |
|---------------------------|----------------|-----|-------------------|-----------------|-----------------|------|-----------|
| E2006.test                | 0.019          | 1   | 5.7-7 | 3.2-7 | 9.7-7 | 2.5-7 | 3.8-8 | 8.1-9 | 2.1998+1 | 2.1998+1 | 2.1998+1 | 05 | 07 | 05 |
| log1p.E2006.test          | 0.260          | 201 | 7.9-7 | 1.2-4 | 1.2-3 | 2.8-6 | 3.0-3 | 3.9-5 | 2.1642+1 | 2.1713+1 | 2.1642+1 | 1:49 | 2:17:37 | 1:22:12 |
| pyrim.scale.expanded5     | 0.109          | 70  | 5.9-7 | 2.0-5 | 3.7-3 | 6.9-7 | 4.7-3 | 3.8-4 | 6.8301-1 | 6.9094-1 | 6.8308-1 | 18 | 20:31 | 12:10 |
| abalone.scale.expanded7   | 0.004          | 82  | 9.6-7 | 9.9-7 | 8.8-7 | 1.0-9 | 3.0-7 | 6.4-9 | 1.3495+2 | 1.3495+2 | 1.3495+2 | 07 | 1.32 | 7.57 |
| bodyfat.scale.expanded7   | 0.012          | 51  | 7.4-7 | 1.1-6 | 9.9-7 | 2.2-8 | 6.3-5 | 3.5-9 | 8.5770-2 | 8.5834-2 | 8.5767-2 | 10 | 27:59 | 8:38 |
| housing.scale.expanded7   | 0.079          | 134 | 6.5-7 | 9.9-7 | 3.4-5 | 5.9-9 | 2.6-5 | 1.1-5 | 9.2745+1 | 9.2748+1 | 9.2745+1 | 14 | 8.00 | 24:00 |
| mpg.scale.expanded7       | 0.053          | 45  | 7.0-7 | 9.9-7 | 9.9-7 | 5.8-9 | 2.4-6 | 1.5-7 | 6.0757+1 | 6.0757+1 | 6.0757+1 | 01 | 11 | 52 |
| space.ga.scale.expanded9  | 0.012          | 27  | 8.3-7 | 9.9-7 | 8.6-7 | 1.6-8 | 2.0-6 | 9.2-8 | 6.6746+0 | 6.6746+0 | 6.6746+0 | 05 | 2:27 | 51 |

Tables 3 and 4 show the performance of the three algorithms. For the synthetic datasets, the pADMM is more efficient than the dADMM in almost all cases. Furthermore, we can see that our PMM algorithm can solve all the problems to the required accuracy. The PMM algorithm not only takes much less time than the pADMM or dADMM does but also obtains more accurate solutions (in terms of \( \eta_{\text{kkt}} \)) in almost all cases.
5.2 Numerical experiments for the square-root regression problems with non-convex regularizers

In this section, we compare the performance of the ADMM and our PMM algorithm for solving the nonconvex square-root regression problem \((3)\). The relative KKT residual

\[
\tilde{\eta}_{\text{kkt}} := \frac{\|\beta - \text{Prox}_{\lambda p-q} \left( \beta - \frac{X^T(X\beta - b)}{\|X\beta - b\|} \right)\|}{1 + \|\beta\| + \frac{\|X^T(X\beta - b)\|}{\|X\beta - b\|}}
\]

(20)

is adopted to measure the accuracy of an approximate optimal solution \(\beta\). In our PMM algorithm, stage I is implemented to generate an initial point for stage II and is stopped if \(\eta_{\text{kkt}} < 10^{-4}\). The tested algorithms will be terminated if \(\tilde{\eta}_{\text{kkt}} < \tilde{\epsilon}_{\text{kkt}} = 10^{-6}\). In addition, the algorithms will also be stopped when they reach the pre-set maximum number of iterations (200 for stage II of the PMM and 10000 for the ADMM).

5.2.1 The ADMM for the problem \((3)\)

To describe the ADMM implemented (which is not guaranteed to converge though due to the nonconvexity) for solving the nonconvex square-root regression problem \((3)\), we first reformulate it to the following constrained problem:

\[
\min_{\beta, z \in \mathbb{R}^n, y \in \mathbb{R}^m} \{ \|y\| + \lambda p(z) - q(z) \mid X\beta - y = b, \beta - z = 0 \}.
\]

(21)

For \(\zeta > 0\), the augmented Lagrangian function of \((21)\) can be written as

\[
L_{\zeta}(\beta, y, z; u, v) := \|y\| + \lambda p(z) - q(z) + \langle u, X\beta - y - b \rangle + \frac{\zeta}{2} \|X\beta - y - b\|^2 \\
+ \langle v, \beta - z \rangle + \frac{\zeta}{2} \|\beta - z\|^2.
\]

The template of the ADMM for solving the problem \((3)\) is given as the following form.
Algorithm ADMM for the problem (3). Let $\zeta > 0$ be a given parameter. Choose $(y_0, z_0, u_0, v_0) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$, set $k = 0$ and iterate.

**Step 1.** Compute

$$
\beta^{k+1} = \arg\min_{\beta \in \mathbb{R}^n} \left\{ L_\zeta(\beta, y^k, z^k; u^k, v^k) \right\} = (I_n + X^TX)^{-1}(z^k - \zeta^{-1}v^k + X^T(y^k + b - \zeta^{-1}u^k)),
$$

$$(y^{k+1}, z^{k+1}) = \arg\min_{y \in \mathbb{R}^m, z \in \mathbb{R}^n} \left\{ L_\zeta(\beta^{k+1}, y, z; u^k, v^k) \right\} = \left( \text{Prox}_{\zeta^{-1}||\cdot||}(X\beta^{k+1} - b + \zeta^{-1}u^k), \text{Prox}_{\zeta^{-1}(\lambda p - q)}(\beta^{k+1} + \zeta^{-1}v^k) \right).$$

**Step 2.** Update

$$u^{k+1} = u^k + \zeta(X\beta^{k+1} - y^{k+1} - b), \quad v^{k+1} = v^k + \zeta(\beta^{k+1} - z^{k+1}).$$

If the prescribed stopping criterion is satisfied, terminate; otherwise return to Step 1 with $k = k + 1$.

5.2.2 Numerical experiments for the square-root regression problems with SCAD regularizations

The SCAD regularization involves a concave function $p_\lambda$, proposed in [20], that has the following properties: $p_\lambda(0) = 0$ and for $|t| > 0$,

$$p'_\lambda(|t|) = \begin{cases} 
\lambda, & \text{if } |t| \leq \lambda, \\
\frac{(a_s\lambda - |t|)_+}{a_s-1}, & \text{otherwise},
\end{cases}$$

for some given parameter $a_s > 2$. In the above, $(a_s\lambda - |t|)_+$ denotes the positive part of $a_s\lambda - |t|$. We can reformulate the expression of the SCAD regularization function as $\lambda p(\beta) - q(\beta)$ with $p(\beta) = \|\beta\|_1$ and

$$q(\beta) = \sum_{i=1}^n q_{\text{scad}}(\beta_i; a_s, \lambda), \quad q_{\text{scad}}(t; a_s, \lambda) = \begin{cases} 
0, & \text{if } |t| \leq \lambda, \\
\frac{(t-\lambda)^2}{2(a_s-1)}, & \text{if } \lambda \leq |t| \leq a_s\lambda, \\
\lambda|t| - \frac{a_s+1}{2}\lambda^2, & \text{if } |t| > a_s\lambda.
\end{cases}$$

The function $q(\beta)$ is continuously differentiable with

$$\frac{\partial q(\beta)}{\partial \beta_i} = \begin{cases} 
0, & \text{if } |\beta_i| \leq \lambda, \\
\frac{\text{sign}(\beta_i)(|\beta_i| - \lambda)}{a_s-1}, & \text{if } \lambda < |\beta_i| \leq a_s\lambda, \\
\lambda|\text{sign}(\beta_i)|, & \text{if } |\beta_i| > a_s\lambda.
\end{cases}$$

We can see that the SCAD regularization function associated with $\beta_i$ is increasing and concave in $[0, +\infty)$. It has been shown in [20] that the SCAD regularization usually performs better.
than the classical $\ell_1$ regularization in selecting significant variables without creating excessive biases.

The performance of the PMM algorithm and ADMM for the SCAD regularization with $a_s = 3.7$ are listed in Tables 5 and 6. We can see that in most cases, the PMM algorithm is not only much more efficient than the ADMM, but it can also obtain better objective function values. Although the objective value of the ADMM is less than that of the PMM algorithm in the `housing.scale.expanded7` data set, the solution of the PMM algorithm is more sparse than that of the ADMM with $\text{nnz}$ being 62 versus 6877.

**Table 5:** The performance of the ADMM and PMM on synthetic datasets for the SCAD regularization. In the table, “a” = PMM, “b” = ADMM.

| probname m; n | $\lambda_c$ | mzn | $\eta_{kk}$ a | $\eta_{kk}$ b | $\text{pobj}_a$ a | $\text{pobj}_b$ b | time a | time b | testerror |
|--------------|-------------|------|----------------|----------------|----------------|----------------|--------|--------|----------|
| exmp1 8000;800 | 0.145       | 460  | 5.9-7 | 5.9-1 | 5.9368+2 | 5.9392+2 | 20 | 3:39   | 9.2406+0 |
| exmp2 8000;800 | 0.087       | 616  | 5.9-7 | 1.0-1 | 4.0760+2 | 4.0777+2 | 28 | 3:33   | 9.3745+0 |
| exmp3 8000;400 | 0.230       | 293  | 7.8-7 | 2.7-1 | 1.5480+3 | 1.5529+3 | 10 | 2:02   | 2.3629+2 |
| exmp4 8000;4000| 0.153       | 2554 | 5.4-7 | 6.8-1 | 3.1837+3 | 3.1940+3 | 4:21 | 16:41  | 3.4480+2 |

**Table 6:** The performance of the ADMM and PMM on UCI datasets for the SCAD regularization. In the table, “a” = PMM, “b” = ADMM.

| probname m; n | $\lambda_c$ | mzn | $\eta_{kk}$ a | $\eta_{kk}$ b | $\text{pobj}_a$ a | $\text{pobj}_b$ b | time a | time b |
|--------------|-------------|------|----------------|----------------|----------------|----------------|--------|--------|
| E2006.test 3308;150358 | 0.071 | 1 | 2.2-8 | 9.0-7 | 2.2165+1 | 2.2165+1 | 08 | 12:51 |
| log1p.E2006.test 3308;1771946 | 0.257 | 207 | 2.1-7 | 5.9-3 | 2.1613+1 | 2.1366+2 | 3:50 | 2:36:14 |
| pyrim.scale.expanded5 74;201376 | 0.109 | 70 | 1.4-7 | 4.3-3 | 6.8301-1 | 7.2608-1 | 13 | 21:26 |
| abalone.scale.expanded7 4177;6435 | 0.011 | 49 | 9.9-7 | 6.9-1 | 1.3292+2 | 1.3864+2 | 12 | 21:41 |
| bodyfat.scale.expanded7 252;116280 | 0.201 | 2 | 3.9-8 | 7.6-2 | 9.4125-1 | 9.5136-1 | 06 | 25:51 |
| housing.scale.expanded7 506;77520 | 0.070 | 62 | 2.1-7 | 5.0-1 | 6.1449+1 | 5.7203+1 | 25 | 30:37 |
| mpg.scale.expanded7 392;3432 | 0.107 | 27 | 3.1-9 | 4.9-1 | 5.5558+1 | 5.9918+1 | 01 | 1:38 |
| space.ga.scale.expanded9 3107;5005 | 0.043 | 16 | 1.9-7 | 4.4-1 | 6.9072+0 | 9.0447+0 | 03 | 12:50 |
5.2.3 Numerical experiments for the square-root regression problems with MCP regularizations

In this subsection, we consider the regularization by a minimax concave penalty (MCP) function \([47]\). For two positive parameters \(a_m > 2\) and \(\lambda\), the MCP regularization can be defined as

\[
\lambda p(\beta) - q(\beta) \quad \text{with} \quad p(\beta) = 2\|\beta\|_1 \quad \text{and}
\]

\[
q(\beta) = \sum_{i=1}^{n} q^{\text{mcp}}(\beta_i; a_m, \lambda),
q^{\text{mcp}}(t; a_m, \lambda) = \begin{cases} 
\frac{t^2}{a_m}, & \text{if } |t| \leq a_m\lambda, \\
2\lambda|t| - a_m\lambda^2, & \text{if } |t| > a_m\lambda.
\end{cases}
\]

The function \(q(\beta)\) is continuously differentiable with its derivative given by

\[
\frac{\partial q(\beta)}{\partial \beta_i} = \begin{cases} 
\frac{2\beta_i}{a_m}, & \text{if } |\beta_i| \leq a_m\lambda, \\
2\lambda \text{sign}(\beta_i), & \text{if } |\beta_i| > a_m\lambda.
\end{cases}
\]

We evaluate the performance of our PMM algorithm on the same set of problems as in the last subsection with the MCP regularization. The numerical results are presented in Tables 7 and 8. In this case, we set the parameter \(a_m = 3.7\).

**Table 7:** The performance of the ADMM and PMM on synthetic datasets for the MCP regularization. In the table, “a”=PMM, “b”=ADMM.

| probname | \(\lambda\) | m | n | \(\eta_{k+1}\) | \(\eta_{k+2}\) | testerror |
|----------|-------------|---|---|----------------|----------------|------------|
| exmp1    | 0.209       | 8000;800 | 380 | 5.2-8 | 1.9-2 | 5.695+2 | 5.6091+2 | 9.3483+0 |
| exmp2    | 0.151       | 8000;800 | 535 | 2.7-7 | 1.3-1 | 4.5225+2 | 4.5414+2 | 9.4916+0 |
| exmp3    | 0.281       | 8000;400 | 267 | 9.3-7 | 1.5-1 | 1.3590+3 | 1.3617+3 | 2.3613+2 |
| exmp4    | 0.293       | 8000;4000| 1821| 9.4-7 | 6.9-2 | 4.1362+3 | 4.2741+3 | 3.8471+2 |

From the numerical results, one can see the efficiency and power of our SSN method based PMM algorithm. Note that though for the abalone.scale.expanded7 data set the objective value obtained by the ADMM is less than that obtained the PMM algorithm, the solution obtained by the PMM algorithm is more sparse than that by the ADMM with mznz being 55 versus 1039. Overall, our PMM algorithm is clearly more efficient and accurate than the ADMM on the tested datasets.

We have mentioned in the introduction that the scaled Lasso problem is equivalent to the srLasso problem \([2]\). However, in order to solve the scaled Lasso problem, we have to call an algorithm several times to solve the standard Lasso subproblems. However, by handling the srLasso problem \([2]\) directly, our algorithm is as fast as the highly efficient algorithm, LassoNAL \([31]\), for solving a single standard Lasso problem.
Table 8: The performance of the ADMM and PMM on UCI datasets for the MCP regularization. In the table, “a”=PMM, “b”=ADMM.

| probname             | λ    | mnz | η(k+1)  | pobj | time |
|----------------------|------|-----|---------|------|------|
|                      | c    | a   | b       | a    | b    |
| E2006.test           | 0.090| 1   | 2.4-8   | 9.3-7| 2.2077+1 | 2.2077+1 | 07 | 07 |
| log1p.E2006.test     | 0.261| 187 | 8.2-7   | 2.2-3| 2.1455+1 | 3.6500+1 | 04:09 | 2:20:46 |
| pyrim.scale.expanded5| 0.221| 43  | 9.9-7   | 7.0-3| 1.1428+0 | 4.6112+0 | 18 | 19:36 |
| abalone.scale.expanded7| 0.012| 55  | 7.1-7   | 1.6-5| 1.3271+2 | 1.2693+2 | 09 | 21:32 |
| bodyfat.scale.expanded7| 0.183| 2   | 2.8-7   | 5.3-6| 5.7347-1 | 5.8278-1 | 06 | 25:11 |
| housing.scale.expanded7| 0.282| 22  | 6.9-7   | 4.1-2| 1.1022+2 | 7.2573+2 | 26 | 30:09 |
| mpg.scale.expanded7  | 0.102| 23  | 7.3-7   | 2.1-2| 5.0964+1 | 5.9492+1 | 01 | 1:36 |
| space.ga.scale.expanded9| 0.046| 15  | 5.9-7   | 3.3-2| 6.6399+0 | 7.8456+0 | 05 | 12:46 |

6 Conclusion

In this paper, we proposed a two stage PMM algorithm to solve the square-root regression problems with nonconvex regularizations. We are able to achieve impressive computational efficiency for our algorithm by designing an innovative proximal majorization framework for the convex subproblem arising in each PMM iteration so that it can be solved via its dual by the SSN method. We presented the oracle property of the problem in stage I and analyzed the convergence of the PMM algorithm with its subproblems solved inexactly. Extensive numerical experiments have demonstrated the efficiency of our PMM algorithm when compared to other natural alternative algorithms such as the ADMM based algorithms in solving the problem of interest.

From the superior performance of our algorithm, it is natural for us to consider applying a similar proximal majorization-minimization algorithmic framework to design efficient algorithms to solve other square-root regression problems with structured sparsity requirements such as group sparsity in the regression coefficients [9]. We leave such an investigation as a future research topic.

7 Appendix

In this appendix, we first provide the proofs for Lemma 3.1, 3.3 and 3.4. Based on these results, we then give the proof for Theorem 3.1.

Proof for Lemma 3.1.
Proof. For a given allowed set $S$ of a weakly decomposable norm $p$, denote

$$C_1 = \left\{ z \in \mathbb{R}^n \mid p(z)S \leq 1, \ zS = 0 \right\}, \quad C_2 = \left\{ z \in \mathbb{R}^n \mid p(\hat{z})S \leq 1, \ zS = 0 \right\},$$

$$C = \left\{ z \in \mathbb{R}^n \mid p(z) + p(\hat{z})S \leq 1 \right\}.$$

Then we have that

$$\delta^*_C(\beta) = \max_{z \in \mathbb{R}^n} \left\{ \langle z, \beta \rangle \mid p(z)S \leq 1, \ zS = 0 \right\} = \max_{z \in \mathbb{R}^n} \left\{ \langle z, \beta z \rangle \mid p(z) \leq 1, \ zS = 0 \right\} \leq p_*(\beta),$$

$$\delta^*_C(\beta) = \max_{z \in \mathbb{R}^n} \left\{ \langle z, \beta \rangle \mid p(z)S \leq 1, \ zS = 0 \right\} = \max_{z \in \mathbb{R}^n} \left\{ \langle z, \beta \rangle \mid p(\hat{z})S \leq 1 \right\} = p_*(\beta).$$

Furthermore, on one hand, for any $x \in C_1, y \in C_2$ and $0 \leq t \leq 1$, it is easy to prove that $tx + (1-t)y \in C$. That is $\text{conv}(C_1 \cup C_2) \subseteq C$. On the other hand, for any $z \in C$ with $zS = 0$ or $\hat{z}S = 0$, it is clear that $z \in \text{conv}(C_1 \cup C_2)$; and for any $z \in C$ with $zS \neq 0$ and $\hat{z}S \neq 0$, we can find $x = \frac{\hat{z}}{p(zS)} \in C_1$ and $y = \frac{z}{1-p(zS)} \in C_2$ such that $z = p(zS)x + (1-p(zS))y$. Therefore we have shown that $C = \text{conv}(C_1 \cup C_2)$.

Due to Theorem 5.6 of [35], we can prove the following fact easily.

$$\text{conv}(\delta_{C_1}, \delta_{C_2})(\beta) = \delta_{\text{conv}(C_1 \cup C_2)}(\beta), \quad \forall \beta \in \mathbb{R}^n,$$

where $\text{conv}(\delta_{C_1}, \delta_{C_2})$ denotes the greatest convex function that is less than or equal to $\delta_{C_1}$ and $\delta_{C_2}$ pointwise over the entire $\mathbb{R}^n$. Based on the above basic results (22)-(24), $C = \text{conv}(C_1 \cup C_2)$ and Theorem 16.5 of [35], we have that

$$p_*(\beta) = \max_{z \in \mathbb{R}^n} \left\{ \langle \beta, z \rangle \mid p(z) \leq 1 \right\} \leq \max_{z \in \mathbb{R}^n} \left\{ \langle \beta, z \rangle \mid p(zS) + p(\hat{z})S \leq 1 \right\} = \delta_{\text{conv}(C_1 \cup C_2)}(\beta) = \left( \text{conv}(\delta_{C_1}, \delta_{C_2}) \right)_*(\beta) = \max \left\{ \delta_{C_1}(\beta), \delta_{C_2}(\beta) \right\} \leq \max \left\{ p_*(\betaS), p_*(\hat{z}) \right\}.$$ 

The desired results follow by taking $\beta = \hat{\beta}$ and dividing both sides of the above inequality by $\|e\|$ with $\beta = \epsilon^T X$, respectively.

Proof for Lemma 3.3.

Proof. Since $\hat{\beta} \in \text{argmin} \left\{ h(\beta; \sigma, \tau, 0, 0, b) \right\}$ and $p$ is a convex function, we have

$$-\frac{X^T(X\hat{\beta} - b)}{\|X\beta - b\|} - \sigma \hat{\beta} - \tau X^T(X\hat{\beta} - b) \in \lambda \partial p(\hat{\beta}).$$

Hence

$$\lambda p(\beta) \geq \lambda p(\hat{\beta}) + \left\langle \frac{X^T\hat{\epsilon}}{\|\hat{\epsilon}\|} - \sigma \hat{\beta} + \tau X^T\hat{\epsilon}, \beta - \hat{\beta} \right\rangle.$$
Let $\beta = \tilde{\beta}$. Then the inequality (25) can be rearranged to

$$\left( \tau + \frac{1}{\|\varepsilon\|} \right) \varepsilon^T X (\tilde{\beta} - \hat{\beta}) \leq \lambda p(\tilde{\beta}) - \lambda p(\hat{\beta}) + \sigma \beta^T (\tilde{\beta} - \hat{\beta}) \leq \lambda p(\tilde{\beta}) + \sigma \tilde{\beta}^T \tilde{\beta}$$

$$\leq \lambda p(\tilde{\beta}) + \sigma p_\ast(\tilde{\beta}) p(\hat{\beta}).$$

Note that the last inequality is obtained by the definition of $p_\ast$. The desired result now follows readily.

**Proof for Lemma 3.4.**

Proof. Since $\hat{\beta} \in \text{argmin}_{\beta \in \mathbb{R}^n} \{h(\beta; \sigma, \tau, 0, 0, b)\}$, we have $h(\hat{\beta}; \sigma, \tau, 0, 0, b) \leq h(\beta; \sigma, \tau, 0, 0, b)$. Thus, by the definition of the dual norm, we get

$$\|\varepsilon\| \leq \|\varepsilon\| + \frac{\tau}{2} \|\varepsilon\|^2 + \frac{\sigma}{2} \|\hat{\beta}\|^2 + \lambda p(\hat{\beta}) \leq \|\varepsilon\| + \frac{\tau}{2} \|\varepsilon\|^2 + \left( \lambda + \frac{\sigma p_\ast(\hat{\beta})}{2} \right) p(\hat{\beta}),$$

$$\lambda p(\hat{\beta}) \leq \|\varepsilon\| + \frac{\tau}{2} \|\varepsilon\|^2 + \frac{\sigma}{2} \|\hat{\beta}\|^2 + \lambda p(\hat{\beta}) \leq \|\varepsilon\| + \frac{\tau}{2} \|\varepsilon\|^2 + \left( \lambda + \frac{\sigma p_\ast(\hat{\beta})}{2} \right) p(\hat{\beta}).$$

Dividing both sides of (26) by $\|\varepsilon\|$, we obtain

$$\frac{\|\varepsilon\|}{\|\varepsilon\|} \leq 1 + \frac{\tau}{2} \|\varepsilon\| + n_p + \frac{\sigma p_\ast(\tilde{\beta}) p(\hat{\beta})}{2\|\varepsilon\|} = c_u,$$

where $c_u$ is defined in (21). In order to obtain the lower bound of $\|\varepsilon\|_{\|\varepsilon\|}$, we first use the triangle inequality $\|\varepsilon\| = \|\varepsilon - X(\tilde{\beta} - \hat{\beta})\| \geq \|\varepsilon\| - \|X(\tilde{\beta} - \hat{\beta})\|$, and then the upper bound of $\|X(\tilde{\beta} - \hat{\beta})\|$. By Lemma 3.3 and the definition of the dual norm, we have

$$\|X(\tilde{\beta} - \hat{\beta})\|^2 = \varepsilon^T X(\tilde{\beta} - \hat{\beta}) + \varepsilon^T X(\hat{\beta} - \tilde{\beta})$$

$$\leq \varepsilon^T X(\tilde{\beta} - \hat{\beta}) + \kappa \left( \lambda p(\hat{\beta}) + \sigma p_\ast(\hat{\beta}) p(\hat{\beta}) \right)$$

$$\leq \lambda p(\tilde{\beta} - \hat{\beta}) \|\varepsilon\| + \kappa \left( \lambda p(\hat{\beta}) + \sigma p_\ast(\hat{\beta}) p(\hat{\beta}) \right)$$

$$\leq \lambda_0 \left( p(\tilde{\beta}) + p(\hat{\beta}) \right) \|\varepsilon\| + \kappa \left( \lambda p(\hat{\beta}) + \sigma p_\ast(\hat{\beta}) p(\hat{\beta}) \right)$$

$$= \left( \lambda_0 \|\varepsilon\| + \lambda \kappa \right) p(\tilde{\beta}) + \left( \lambda_0 \|\varepsilon\| + \sigma \kappa p_\ast(\hat{\beta}) \right) p(\hat{\beta}),$$

where $\kappa = \left( \tau + \frac{1}{\|\varepsilon\|} \right)^{-1}$. Substituting the inequality (27) into the above formula, we can obtain

$$\|X(\tilde{\beta} - \hat{\beta})\|^2 \leq \frac{\|\varepsilon\| + \frac{\tau}{2} \|\varepsilon\|^2}{\lambda} \left( \lambda_0 \|\varepsilon\| + \sigma \kappa p_\ast(\hat{\beta}) \right)$$

$$+ \left( 2 \lambda_0 \|\varepsilon\| + \left( \lambda \kappa + \sigma \kappa p_\ast(\hat{\beta}) \right) + \frac{\sigma p_\ast(\hat{\beta})}{2\lambda} \left( \lambda_0 \|\varepsilon\| + \sigma \kappa p_\ast(\hat{\beta}) \right) \right) p(\hat{\beta}).$$
Rearranging the above inequality, we have

\[ \|X(\hat{\beta} - \bar{\beta})\| \leq \|\varepsilon\| \sqrt{\hat{a} + \frac{2\lambda_0 p(x)}{\|\varepsilon\|} + \frac{\|\varepsilon\|}{\|\varepsilon\|} \left(\frac{\lambda + \sigma p_*(\hat{\beta})}{1 + \tau\|\varepsilon\|}\right) \frac{p(\hat{\beta})}{\|\varepsilon\|}} \]

\[ \leq \|\varepsilon\| \sqrt{\frac{\lambda + \sigma p_*(\hat{\beta})}{\lambda} \left(1 + \frac{\sigma p_*(\hat{\beta})}{\lambda}\right)} \quad n_p, \]

where

\[ \hat{a} = \left(\frac{1 + \frac{\sigma p_*(\hat{\beta})}{\lambda}}{\lambda} + \frac{\sigma p_*(\hat{\beta})}{2\lambda\|\varepsilon\|}\right) \left(\frac{1 + \frac{\lambda + \sigma p_*(\hat{\beta})}{\lambda}}{\lambda}\right) \]

\[ \leq \left(\frac{1 + \frac{\sigma p_*(\hat{\beta})}{\lambda}}{\lambda} + \frac{\sigma p_*(\hat{\beta})}{2\lambda\|\varepsilon\|}\right) \left(\frac{\lambda + \sigma p_*(\hat{\beta})}{\lambda}\right) = a. \quad (28) \]

Therefore, by noting that \( \|X(\hat{\beta} - \bar{\beta})\| = \|\varepsilon - \hat{\varepsilon}\| \) and triangle inequality, we have

\[ \frac{\|\hat{\varepsilon}\|}{\|\varepsilon\|} \geq 1 - \sqrt{\hat{a} + \frac{2\lambda_0 n_p}{\lambda} + \frac{\|\hat{\varepsilon}\|}{\|\varepsilon\|} \left(1 + \frac{\sigma p_*(\hat{\beta})}{\lambda}\right) n_p}. \]

By rearranging the above inequality, in the case when \( \frac{\|\hat{\varepsilon}\|}{\|\varepsilon\|} < 1 \), we further derive that

\[ a + \frac{2\lambda_0 n_p}{\lambda} + \frac{\|\hat{\varepsilon}\|}{\|\varepsilon\|} \left(1 + \frac{\sigma p_*(\hat{\beta})}{\lambda}\right) n_p \geq \left(1 - \frac{\|\hat{\varepsilon}\|}{\|\varepsilon\|}\right)^2 \geq 1 - \frac{2\|\hat{\varepsilon}\|}{\|\varepsilon\|}. \]

Then we can obtain that

\[ \frac{\|\hat{\varepsilon}\|}{\|\varepsilon\|} \geq \frac{1 - a - \frac{2\lambda_0 n_p}{\lambda}}{2 + \left(1 + \frac{\sigma p_*(\hat{\beta})}{\lambda}\right) n_p} := c_1 > 0. \]

In the other case, if \( \frac{\|\hat{\varepsilon}\|}{\|\varepsilon\|} \geq 1 \), we have already obtain a lower bound that is larger than \( c_1 \). \qed

**Proof for Theorem 3.1.**

*Proof.* First we note that if the following inequality holds

\[ \langle X(\hat{\beta} - \bar{\beta}), X(\hat{\beta} - \beta) \rangle \leq -\delta \left(\lambda + \bar{\lambda}_m\right) p(\hat{\beta}_S - \beta) + (\lambda - \bar{\lambda}_m) p(\hat{\beta}_S) \|\varepsilon\| + \sigma c_\mu \|\hat{\beta} - \hat{\beta}\| \|\beta - \hat{\beta}\| \|\varepsilon\|, \]

then we can verify that the theorem is valid by the following simple calculations:

\[ \|X(\hat{\beta} - \bar{\beta})\|^2 - \|X(\beta - \bar{\beta})\|^2 + 2\delta \left(\lambda + \bar{\lambda}_m\right) p(\hat{\beta}_S - \beta) + (\lambda - \bar{\lambda}_m) p(\hat{\beta}_S) \|\varepsilon\| \]

\[ = 2\langle X(\hat{\beta} - \bar{\beta}), X(\hat{\beta} - \beta) \rangle - \|X(\beta - \hat{\beta})\|^2 + 2\delta \left(\lambda + \bar{\lambda}_m\right) p(\hat{\beta}_S - \beta) + (\lambda - \bar{\lambda}_m) p(\hat{\beta}_S) \|\varepsilon\| \]

\[ \leq -\|X(\beta - \hat{\beta})\|^2 + 2\sigma c_\mu \|\hat{\beta} - \hat{\beta}\| \|\beta - \hat{\beta}\| \|\varepsilon\| \]

\[ \leq 2\sigma c_\mu \|\hat{\beta} - \hat{\beta}\| \|\beta - \hat{\beta}\| \|\varepsilon\|. \]
Thus it is sufficient to show that the result is true if
\[
\langle X(\hat{\beta} - \beta), X(\hat{\beta} - \beta) \rangle \geq -\delta \left( (\hat{\lambda} + \bar{\lambda}_m)p(\hat{\beta}_S - \beta) + (\hat{\lambda} - \bar{\lambda}_m)p^S(\hat{\beta}^S) \right) \|\varepsilon\|
+ \sigma c_u \|\hat{\beta} - \beta\| \|\beta - \bar{\beta}\| \|\varepsilon\|.
\]
(29)

By the inequality (25) and the fact that \( \hat{\varepsilon} = X(\hat{\beta} - \hat{\beta} + \varepsilon) \), we can get
\[
\langle X(\hat{\beta} - \beta), X(\hat{\beta} - \beta) \rangle \leq \langle \varepsilon, X(\hat{\beta} - \beta) \rangle + \sigma \kappa \langle \hat{\beta}, \beta - \hat{\beta} \rangle + \lambda \kappa p(\beta),
\]
(30)

where \( \kappa := \left( \tau + \frac{1}{\|\varepsilon\|} \right)^{-1} \). Since supp(\( \beta \)) \subseteq S, it follows from the definition of the dual norm and the generalized Cauchy-Schwartz inequality that
\[
\langle \varepsilon, X(\hat{\beta} - \beta) \rangle = \langle \varepsilon, X(\hat{\beta}_S - \beta) \rangle + \langle \varepsilon, X(\hat{\beta}_S - \beta) \rangle
\leq \left( p_\ast((\varepsilon^T X)_S)p(\hat{\beta}_S - \beta) + p_\ast((\varepsilon^T X)_S)p^S(\hat{\beta}^S) \right) \leq \lambda m \left( p(\hat{\beta}_S - \beta) + p^S(\hat{\beta}^S) \right) \|\varepsilon\|. \quad (31)
\]

By substituting (31) into (30), we obtain
\[
\langle X(\hat{\beta} - \beta), X(\hat{\beta} - \beta) \rangle \leq \lambda m \left( p(\hat{\beta}_S - \beta) + p^S(\hat{\beta}^S) \right) \|\varepsilon\| + \sigma \kappa \langle \hat{\beta}, \beta - \hat{\beta} \rangle + \lambda \kappa p(\beta).
\]
(32)

Furthermore, by using the weak decomposability and the triangle inequality in (32) we derive
\[
\langle X(\hat{\beta} - \beta), X(\hat{\beta} - \beta) \rangle \leq \lambda m \left( p(\hat{\beta}_S - \beta) + p^S(\hat{\beta}^S) \right) \|\varepsilon\| + \sigma \kappa \langle \hat{\beta}, \beta - \hat{\beta} \rangle + \lambda \kappa \left( p(\hat{\beta}_S) + p(\hat{\beta}_S - \beta) \right).
\]
(33)

Then by eliminating \( \lambda \kappa p^S(\hat{\beta}_S) \) on both sides of (33) and using the weak decomposability, we get
\[
\langle X(\hat{\beta} - \beta), X(\hat{\beta} - \beta) \rangle \leq \lambda m \left( p(\hat{\beta}_S - \beta) + p^S(\hat{\beta}^S) \right) \|\varepsilon\| + \sigma \kappa \langle \hat{\beta}, \beta - \hat{\beta} \rangle + \lambda \kappa p(\beta)
\leq \lambda m \left( p(\hat{\beta}_S - \beta) + p^S(\hat{\beta}^S) \right) \|\varepsilon\| + \sigma \kappa \langle \hat{\beta}, \beta - \hat{\beta} \rangle + \sigma \kappa \langle \hat{\beta} - \beta, \beta - \beta \rangle + \lambda \kappa p(\hat{\beta}_S - \beta)
\leq \lambda m \left( p(\hat{\beta}_S - \beta) + p^S(\hat{\beta}^S) \right) \|\varepsilon\| + \lambda \kappa p(\beta - \hat{\beta}) + \lambda m \sigma \kappa \left( p(\hat{\beta}_S - \beta) + p^S(\hat{\beta}^S) \right)
+ \sigma \kappa \|\hat{\beta} - \beta\| \|\beta - \beta\|.
\]
(34)

By using the result of Lemma 3.4, the inequality (34) becomes
\[
\langle X(\hat{\beta} - \beta), X(\hat{\beta} - \beta) \rangle \geq \left( \hat{\lambda} - \bar{\lambda}_m \right) p^S(\hat{\beta}^S) \|\varepsilon\|
\leq \left( \hat{\lambda} + \bar{\lambda}_m \right) \|\varepsilon\| + \sigma c_u \|\hat{\beta} - \beta\| \|\beta - \hat{\beta}\| \|\varepsilon\|.
\]
(35)

From the condition (29) in (35) and simple rearrangement, we have that
\[
\left( \hat{\lambda} - \bar{\lambda}_m \right) (1 - \delta) p^S(\hat{\beta}^S) \leq \left( \hat{\lambda} + \bar{\lambda}_m \right) (1 + \delta) p(\hat{\beta}_S - \beta).
\]
By Lemma 3.1 we have $\lambda_0 \leq \lambda_m$ and $p_*(\beta) \leq \lambda_m$. Since
\[
\frac{\lambda - \lambda_m t_1 (1 + \sigma t_1 + \sigma n_p) - 2\lambda_m n_p}{\lambda \left(2 + n_p + \frac{\sigma p_x(\beta)p(\beta)}{\|e\|}\right)} < c_1 \leq \frac{1}{2 + n_p + \frac{\sigma p_x(\beta)p(\beta)}{\|e\|}},
\]
we can see that
\[
\hat{\lambda} > \frac{\lambda - \lambda_m t_1 (1 + \sigma t_1 + \sigma n_p) - 2\lambda_m n_p}{t_2 + n_p}.
\]
Then it is easy to find that if
\[
\frac{s_2 - \sqrt{s_2^2 - 4s_1 s_3}}{2s_1} < \lambda < \frac{s_2 + \sqrt{s_2^2 - 4s_1 s_3}}{2s_1}, \tag{36}
\]
then we have $\hat{\lambda} = \lambda c_l / (1 + \tau c_l) > \tilde{\lambda}_m = \lambda_m (1 + \sigma c_u)$.

Furthermore,
\[
p^S(\hat{\beta}^S) \leq \left(\frac{\hat{\lambda} + \tilde{\lambda}_m}{\lambda - \lambda_m}\right) \cdot \frac{1 + \delta}{1 - \delta} \cdot p(\beta)S - \beta).
\]

From the definition of $L_S$ and Lemma 3.2 with the assumption $\text{supp}(\beta) \subseteq S,$ it follows that
\[
p^S(\hat{\beta}^S) \leq L_S p(\hat{\beta}S - \beta), \quad p(\beta)S - \beta) \leq \Gamma_p(L_S, S)\|X(\hat{\beta} - \beta)\|.
\]

By using the inequality (35), we can derive that
\[
\langle X(\hat{\beta} - \beta), X(\hat{\tilde{\beta}} - \beta)\rangle + \delta \|\epsilon\| (\hat{\lambda} - \tilde{\lambda}_m) p^S(\hat{\beta}^S)
\leq (\hat{\lambda} + \tilde{\lambda}_m) p(\hat{\beta}S - \beta) \|\epsilon\| + \sigma c_u \|\beta - \beta\|\|\beta - \tilde{\beta}\|\|\epsilon\|
\leq (1 + \delta) (\hat{\lambda} + \tilde{\lambda}_m) \Gamma_p(L_S, S) \|X(\hat{\beta} - \beta)\|\|\epsilon\| \leq (1 + \delta) (\hat{\lambda} + \tilde{\lambda}_m) \Gamma_p(L_S, S) \|\hat{\beta} - \beta\|\|\beta - \tilde{\beta}\|\|\epsilon\|
\]

Noticing that
\[
2 \langle X(\hat{\beta} - \beta), X(\hat{\tilde{\beta}} - \beta)\rangle = \|X(\hat{\beta} - \beta)\|^2 - \|X(\beta - \tilde{\beta})\|^2 + \|X(\hat{\beta} - \beta)\|^2,
2(1 + \delta) (\hat{\lambda} + \tilde{\lambda}_m) \Gamma_p(L_S, S) \|X(\hat{\beta} - \beta)\|\|\epsilon\| \leq (1 + \delta) (\hat{\lambda} + \tilde{\lambda}_m) \Gamma_p(L_S, S) \|\hat{\beta} - \beta\|\|\beta - \tilde{\beta}\|\|\epsilon\|
\]
we get
\[
\|X(\hat{\beta} - \beta)\|^2 + 2\delta \left(\|\hat{\lambda} - \tilde{\lambda}_m\| p^S(\hat{\beta}^S) + (\hat{\lambda} + \tilde{\lambda}_m) p(\hat{\beta}S - \beta)\right) \|\epsilon\|
\leq \|X(\beta - \tilde{\beta})\|^2 + (1 + \delta)^2 (\hat{\lambda} + \tilde{\lambda}_m)^2 \Gamma_p^2(L_S, S) \|\epsilon\|^2 + 2\sigma c_u \|\beta - \tilde{\beta}\|\|\beta - \tilde{\beta}\|\|\epsilon\|.
\]
Therefore the oracle inequality holds and this completes the proof. \hfill \square
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