Almost paracontact almost paracomplex Riemannian manifolds as extensions of 2-dimensional space-forms

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Abstract. Almost paracontact Riemannian manifolds of the lowest dimension are studied, whose paracontact distributions are equipped with an almost paracomplex structure. These manifolds are constructed as a product of a real line and a 2-dimensional Riemannian space-form. Their metric is obtained in two ways: as a cone metric and as a hyperbolic extension of the metric of the underlying paracomplex 2-manifold. The resulting manifolds are studied and characterised in terms of the classification used and their curvature properties.

Introduction

In 1976, I. Sato [9] introduced the concept of an almost paracontact structure compatible with a Riemannian metric as an analogue of an almost contact Riemannian manifold. The study of the differential geometry of these manifolds began with [1], [10], [11] by I. Sato, T. Adati and T. Miyazawa. After that, in [8], S. Sasaki defined the notion of an almost paracontact Riemannian manifold of type \((p, q)\), where \(p\) and \(q\) are the multiples of the eigenvalues \(+1\) and \(−1\) of the paracontact endomorphism, respectively. It also has a simple eigenvalue of 0.

In [4], M. Manev and M. Staikova gave a classification of almost paracontact Riemannian manifolds \(\mathcal{M}, \phi, \xi, \eta, g\) of type \((n, n)\). The dimension of \(\mathcal{M}\) is \(2n + 1\) and the induced almost product structure \(P\) of \(\phi\) on the paracontact distribution \(\ker(\eta)\) is traceless, i.e. \(P\) is an almost paracomplex structure. Because of this, the present authors called them almost paracontact almost paracomplex Riemannian manifolds in [5] and continued their study together with S. Ivanov and H. Manev (e.g. [3, 5, 6]).

In the present paper, we study the geometry of almost paracontact almost paracomplex Riemannian manifolds of the lowest dimension 3. In Section 1, we recall some necessary facts about the studied manifolds. In Section 2 and Section 3, we use two different approaches to construct a manifold of the studied type as a product of a real line and a 2-dimensional manifold. In the first case, \(g\) is the cone metric, and in a second one, \(g\) is the so-called hyperbolic extension, introduced in [3], where the underlying manifold is paraholomorphic paracomplex Riemannian. Our goal in the present work is to study the basic curvature properties of the resulting manifolds.
1. Preliminaries

1.1. Almost paracontact almost paracomplex Riemannian manifolds. Let \((\mathcal{M}, \phi, \xi, \eta, g)\) be an almost paracontact almost paracomplex Riemannian manifold (abbr. apapR manifold). This means that \(\mathcal{M}\) is a \((2n+1)\)-dimensional differentiable manifold, \(\phi\) is a paracontact endomorphism of the tangent bundle \(T\mathcal{M}\), \(\xi\) is a characteristic vector field and \(\eta\) is its dual 1-form, and \(g\) is a compatible Riemannian metric, such that:

\[
\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi \xi = 0, \quad \text{tr}\phi = 0,
\]

\[
g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y),
\]

where \(I\) denotes the identity on \(T\mathcal{M}\) \cite{9, 4}.

Here and further \(x, y, z, w\) will stand for arbitrary elements of the Lie algebra \(\mathcal{X}(\mathcal{M})\) of tangent vector fields on \(\mathcal{M}\) or vectors in the tangent space \(T_p\mathcal{M}\) at \(p \in \mathcal{M}\).

The associated metric \(\tilde{g}\) of \(g\) on \((\mathcal{M}, \phi, \xi, \eta, g)\) is determined by \(\tilde{g}(x, y) = g(x, \phi y) + \eta(x)\eta(y)\). In \cite{4}, it is shown that \(\tilde{g}\) is a compatible metric with \((\mathcal{M}, \phi, \xi, \eta, g)\) as \(g\), but \(\tilde{g}\) is a pseudo-Riemannian metric of signature \((n+1, n)\).

The fundamental tensor \(F\) of type \((0, 3)\) on \((\mathcal{M}, \phi, \xi, \eta, g)\) is defined by

\[
F(x, y, z) = g((\nabla_x \phi)y, z),
\]

where \(\nabla\) is the Levi-Civita connection of \(g\). The basic properties of \(F\) with respect to the structure are the following:

\[
F(x, y, z) = F(x, z, y) = -F(x, \phi y, \phi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).
\]

The Lee forms of \((\mathcal{M}, \phi, \xi, \eta, g)\) are the following 1-forms associated with \(F\):

\[
\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \phi e_j, z), \quad \omega(z) = F(\xi, \xi, z),
\]

where \(g^{ij}\) are the components of the inverse matrix of \(g\) with respect to a basis \(\{\xi, e_i\}\) \((i = 1, 2, \ldots, 2n)\) of \(T_p\mathcal{M}\) at an arbitrary point \(p \in \mathcal{M}\).

A classification of apapR manifolds is made in \cite{4}. It consists of 11 basic classes \(\mathcal{F}_1\), \(\mathcal{F}_2\), \ldots, \(\mathcal{F}_{11}\) and it is made with respect to \(F\). The components \(F_i\) of \(F\) corresponding to \(\mathcal{F}_i\) are determined in \cite{5}. Namely, \((\mathcal{M}, \phi, \xi, \eta, g)\) belongs to \(\mathcal{F}_i\) \((i \in \{1, 2, \ldots, 11\}\) if and only if the equality \(F = F_i\) is valid. Moreover, \((\mathcal{M}, \phi, \xi, \eta, g) \in \mathcal{F}_i \oplus \mathcal{F}_j \oplus \cdots\) if and only if the following condition is satisfied \(F = F_i + F_j + \cdots\).

In the present work, we consider the case of the lowest dimension of \((\mathcal{M}, \phi, \xi, \eta, g)\), i.e. \(\dim \mathcal{M} = 3\). Let \(\{e_0, e_1, e_2\}\) be a \(\phi\)-basis of \(T_p\mathcal{M}\), which satisfies the following conditions:

\[
\phi e_0 = 0, \quad \phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \xi = e_0, \quad \eta(e_0) = 1, \quad \eta(e_1) = \eta(e_2) = 0, \quad g(e_i, e_j) = \delta_{ij}, \quad i, j \in \{0, 1, 2\}.
\]

The components of \(F\), \(\theta\), \(\theta^*\), \(\omega\) with respect to \(\{e_0, e_1, e_2\}\) are denoted by \(F_{ijk} = F(e_i, e_j, e_k)\), \(\theta_k = \theta(e_k)\), \(\theta^*_k = \theta^*(e_k)\) and \(\omega_k = \omega(e_k)\), respectively. According to \cite{5}, we have the following:

\[
\theta_0 = F_{110} + F_{220}, \quad \theta_1 = F_{111} = -F_{122} = -\theta^*_2, \quad \theta_0 = F_{120} + F_{210}, \quad \theta_2 = F_{222} = -F_{211} = -\theta^*_1, \quad \omega_0 = 0, \quad \omega_1 = F_{001}, \quad \omega_2 = F_{002}.
\]
Then, if $F^s$, $s \in \{1, 2, \ldots, 11\}$, are the components of $F$ in the corresponding basic classes $F_s$ and $x = x^i e_i$, $y = y^i e_i$, $z = z^i e_i$ are arbitrary vectors in $T_p \mathcal{M}$, we have the following: 

\begin{align}
F^1(x, y, z) &= (x^1 \theta_1 - x^2 \theta_2) (y^1 z^1 - y^2 z^2); \\
F^2(x, y, z) &= F^3(x, y, z) = 0; \\
F^4(x, y, z) &= \frac{1}{2} \theta_0 \left\{ x^1 \left( y^0 z^1 + y^1 z^0 \right) + x^2 \left( y^0 z^2 + y^2 z^0 \right) \right\}; \\
F^5(x, y, z) &= \frac{1}{2} \theta_0 \left\{ x^1 \left( y^0 z^2 + y^2 z^0 \right) + x^2 \left( y^0 z^1 + y^1 z^0 \right) \right\}; \\
F^6(x, y, z) &= F^7(x, y, z) = 0; \\
F^8(x, y, z) &= \lambda \{ x^1 \left( y^0 z^1 + y^1 z^0 \right) - x^2 \left( y^0 z^2 + y^2 z^0 \right) \}, \\
\lambda &= F_{110} = -F_{220}; \\
F^9(x, y, z) &= \mu \{ x^1 \left( y^0 z^2 + y^2 z^0 \right) - x^2 \left( y^0 z^1 + y^1 z^0 \right) \}, \\
\mu &= F_{120} = -F_{210}; \\
F^{10}(x, y, z) &= \nu x^0 \left( y^1 z^1 - y^2 z^2 \right), \\
\nu &= F_{011} = -F_{022}; \\
F^{11}(x, y, z) &= x^0 \{ \omega_1 \left( y^0 z^1 + y^1 z^0 \right) + \omega_2 \left( y^0 z^2 + y^2 z^0 \right) \}.
\end{align}

By virtue of the latter equations, the 3-dimensional manifolds of the considered type can belong only to the basic classes $F_1, F_4, F_5, F_8, F_9, F_{10}, F_{11}$ and their direct sums [3].

The curvature $(1, 3)$-tensor $R$ of $\nabla$ is defined as usually by $R = [\nabla, \nabla] - \nabla[\ ]$. The corresponding $(0,4)$-tensor is denoted by the same letter and it is determined by $R(x, y, z, w) = g(R(x, y)z, w)$.

The Ricci tensor $\rho$ and the scalar curvature $\tau$ for $R$ as well as their associated quantities are defined respectively by:

\begin{align}
\rho(y, z) &= g^{ij} R(e_i, y, z, e_j), \\
\rho^*(y, z) &= g^{ij} R(e_i, y, z, \phi e_j), \\
\tau &= g^{ij} \rho(e_i, e_j), \\
\tau^* &= g^{ij} \rho^*(e_i, e_j).
\end{align}

The following tensors are essential curvature-like tensors of type $(1,3)$:

\begin{align}
\pi_1(x, y,z) &= g(y, z)x - g(x, z)y, \\
\pi_2(x, y,z) &= g(y, \phi z)\phi x - g(x, \phi z)\phi y.
\end{align}

Their corresponding curvature-like $(0,4)$-tensors are determined by $g$ as usually, $\pi_i(x, y, z, w) = g(\pi_i(x, y, z, w), i = 1, 2$.

Let $\alpha$ be a non-degenerate 2-plane in $T_p \mathcal{M}$, $p \in \mathcal{M}$, having a basis $\{x, y\}$. The sectional curvature $k(\alpha; p)$ is determined by

\begin{align}
k(\alpha; p) &= \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}.
\end{align}

1.2. Almost paracomplex Riemannian manifold. Consider a differentiable manifold $\mathcal{N}$ of arbitrary dimension. Let us recall that an almost product structure $P$ on $\mathcal{N}$ is an endomorphism in the tangent bundle $T\mathcal{N}$ of $\mathcal{N}$ such that $P^2$ is the identity $I$ in $T\mathcal{N}$, but $P$ does not coincide with $I$. Then, such a manifold $(\mathcal{N}, P)$ is called an almost product manifold. In the particular case when the eigenvalues +1 and -1 of $P$ have the same multiplicity $n$, the structure $P$ is called an almost paracomplex structure and $(\mathcal{N}, P)$ is known as an almost paracomplex manifold of dimension $2n$ [2]. In this case $P$ is traceless, i.e. $\text{tr}P = 0$. This kind of manifolds are also known as almost product Riemannian manifolds with $\text{tr}P = 0$ in [12].
As it is known, the $2n$-dimensional paracontact distribution $\mathcal{H} = \ker(\eta)$ of $(\mathcal{M}, \phi, \xi, \eta, g)$ can be considered as an almost paracomplex manifold $\mathcal{N}$ equipped with an almost paracomplex structure $P = \phi|_\mathcal{H}$ and a metric $h = g|_\mathcal{H}$, where $\phi|_\mathcal{H}$ and $g|_\mathcal{H}$ are the restrictions of $\phi$ and $g$ on $\mathcal{H}$, respectively.

Let $x', y', z', w'$ denote arbitrary vector fields or vectors on $\mathcal{H}$ of $\mathcal{M}$.

Since $g$ is a Riemannian metric of $(\mathcal{M}, \phi, \xi, \eta)$, then $h$ is the corresponding Riemannian metric on $\mathcal{H}$ and due to (1.1) it is compatible with $P$ as follows

$$h(Px', Py') = h(x', y').$$

The associated Riemannian metric $\tilde{h}$ of $h$ is determined by

$$\tilde{h}(x', y') = h(x', Py')$$

and it is a pseudo-Riemannian of signature $(n, n)$.

Let us note that an $2n$-dimensional manifold $\mathcal{N}$, which is equipped with an almost paracomplex structure $P$ and a Riemannian metric $h$ satisfying $|1.9|$, is known as an almost paracomplex Riemannian manifold $(\mathcal{N}, P, h)$.

In [7], A. M. Naveira gave a classification of almost product Riemannian manifolds $(\mathcal{N}, P, h)$ with respect to the covariant derivative $\nabla'P$ for the Levi-Civita connection $\nabla'$ of $h$.

Furthermore, using this classification, M. Staikova and K. Gribachev present a classification of almost paracomplex Riemannian manifolds $(\mathcal{N}, P, h)$ in [12]. The basic classes of the Staikova-Gribachev classification are three, $W_1$, $W_2$ and $W_3$. Their intersection is the class $W_0$ determined by the condition $\nabla'P = 0$. The manifolds of the latter class are known as locally product Riemannian manifolds [7], Riemannian $P$-manifolds [12] or paraholomorphic paracomplex Riemannian manifolds [3]. In the present paper, we call such manifolds briefly $W_0$-manifolds.

Let us remark that $W_1$ contains the manifolds which are locally conformal equivalent to $W_0$-manifolds.

In the present paper we deal with the case of almost paracomplex Riemannian manifolds $(\mathcal{N}, P, h)$ of the lowest dimension, i.e. $\dim\mathcal{N} = 2$. As it is known such a manifold is a space-form, i.e. the manifold has a curvature tensor of the form $R' = k' \pi'_1$, where $k'$ is its pointwise constant sectional curvature, and $\pi'_1$ is the essential curvature-like tensor, such as $\pi_1$ in [14], but with respect to $h$. Moreover, each $(\mathcal{N}, P, h)$ is a $W_1$-manifold determined by

$$F'(x', y', z') = \frac{1}{2}\{h(x', y')\theta'(z') + h(x', z')\theta'(y')$$

$$+ h(x', Py')\theta^*(z') + h(x', Pz')\theta^*(y')\},$$

where $F'$ is the fundamental tensor of $(\mathcal{N}, P, h)$ defined by the following equality $F'(x', y', z') = h((\nabla'_x P)y', z')$, $\theta'$ is the Lee form determined analogously as in (1.2) and $\theta^* = -\theta' \circ P$ is valid [12].

2. CONE OVER A 2-DIMENSIONAL PARACOMPLEX RIEMANNIAN SPACE-FORM

Let us consider $C(\mathcal{N}) = \mathbb{R}^+ \times \mathcal{N}$, the cone over a 2-dimensional paracomplex space-form $(\mathcal{N}, P, h)$, where $\mathbb{R}^+$ is the set of positive reals. We introduce a Riemannian metric $g$ on $C(\mathcal{N})$ defined by

$$g((x', a\frac{d}{dt}), (y', b\frac{d}{dt})) = t^2 h(x', y') + ab,$$

where $t$ is the coordinate on $\mathbb{R}^+$ and $a, b$ are differentiable functions on $C(\mathcal{N})$. 

Proposition 2.1. The manifold $(\mathcal{C}(\mathcal{N}), \phi, \xi, \eta, g)$ is a 3-dimensional para-$\mathcal{R}$ manifold.

According to \eqref{2.1} and \eqref{2.2} and the well-known Koszul equality
\begin{equation}
2g(\nabla_xy, z) = xg(y, z) + yg(z, x) - zg(x, y) + g([x, y], z) + g([z, x], y) + g([z, y], x),
\end{equation}
we get the following equalities for the Levi-Civita connection $\nabla$ of $g$:
\begin{align*}
g(\nabla_{x'}y', z') &= t^2 h(\nabla_x y, z), \quad g(\nabla_{x'}y', z') = t^2 h(x', y'), \quad g(\nabla_{x}y, x) = t h(x', y'),
g(\nabla_{x'}y, x) = -t h(x, y'), \quad g(\nabla_{x}y, x) = t h(x', y').
\end{align*}

Bearing in mind the latter equalities, we obtain the covariant derivatives with respect to $\nabla$ as follows
\begin{equation}
\nabla_{x'}y = \nabla_{x}y' - \frac{1}{t}g(x', y') \xi, \quad \nabla_{x}y' = \frac{1}{t}y' \xi, \quad \nabla_{x}x' = \frac{1}{t}x'.
\end{equation}

By virtue of \eqref{2.4}, we get
\begin{align*}
R(x', y')z' &= \frac{1}{t^2}(k' - 1)\pi_1(x', y')z', \\
R(x', y')z &= R(x', \xi)y' = R(x', y') = R(x', \xi) = R(x', \xi) = R(x', y') = R(x', \xi) = 0.
\end{align*}

Thus, by direct computations, we establish the following
\begin{equation}
R(x', y', z', w') = \frac{1}{t^2}(k' - 1)\pi_1(x', y', z', w'),
\end{equation}
\begin{align*}
R(x, y, z, x') &= R(x, \xi,y', z') = R(x', y, \xi, z') = R(x', y', \xi, x) = 0.
\end{align*}

Using \eqref{1.3} and \eqref{2.1}, we determine the components $h_{ij} = h(e_i, e_j)$ and $g_{ij} = g(e_i, e_j)$ with respect to the basis. The non-zero ones of them are the following:
\begin{equation}
h_{11} = h_{22} = \frac{1}{t^2}, \quad g_{00} = g_{11} = g_{22} = 1.
\end{equation}

Taking into account \eqref{2.3}, \eqref{2.4} and \eqref{2.0}, we determine the components of the covariant derivatives of $e_i$ with respect to $\nabla$:
\begin{align*}
\nabla_{e_i}e_1 &= \nabla_{e_i}e_1 - \frac{1}{t}e_0, \quad \nabla_{e_i}e_2 = \nabla_{e_i}e_2, \quad \nabla_{e_i}e_0 = \frac{1}{t}e_1, \\
\nabla_{e_0}e_1 &= \frac{1}{t}e_1, \quad \nabla_{e_0}e_2 = \frac{1}{t}e_2, \quad \nabla_{e_0}e_0 = 0.
\end{align*}

Bearing in mind \eqref{2.2}, \eqref{2.3} and \eqref{2.7}, we obtain the components $F_{ijk}$ of $F$ with respect to the basis $\{e_0, e_1, e_2\}$. The non-zero ones of them are:
\begin{align*}
F_{111} &= -F_{122} = \theta_1', \quad F_{222} = -F_{211} = \theta_2', \\
F_{120} &= F_{102} = F_{210} = F_{201} = -\frac{1}{t},
\end{align*}
where $\theta_i' = \theta'(e_i)$ for $i = 1, 2$. 

We equip $\mathcal{C}(\mathcal{N})$ with an almost paracontact almost paracomplex structure $(\phi, \xi, \eta)$ by the following way
\begin{equation}
\phi|_\mathcal{N} = P, \quad \xi = \frac{dt}{t}, \quad \eta = dt, \quad \phi\xi = 0, \quad \eta \circ \phi = 0.
\end{equation}

Obviously, we establish the truthfulness of the following
According to (1.4), we compute the components of the Lee forms of \((C(N), \phi, \xi, \eta, g)\) and the non-zero ones of them are:

\[
\begin{align*}
\theta_1 &= \theta'_1, \\
\theta_2 &= \theta'_2, \\
\theta^*_1 &= -\theta'_2, \\
\theta^*_2 &= -\theta'_1, \\
\theta^*_0 &= -\frac{2}{t}.
\end{align*}
\]

By virtue of (1.5) and (2.8), we establish the following form of the tensor \(F\):

\[
F(x, y, z) = (F^1 + F^5)(x, y, z).
\]

According to (1.5) and (1.10), the non-zero components of \(F^1\) and \(F^5\) are the following:

\[
\begin{align*}
F^1_{111} &= -F^1_{122} = \theta_1, \\
F^1_{222} &= -F^1_{211} = \theta_2, \\
F^5_{120} &= F^5_{102} = F^5_{210} = F^5_{201} = \frac{1}{2}\theta^*_0.
\end{align*}
\]

Thus, we obtain the following

**Theorem 2.2.** The 3-dimensional apapR manifold \((C(N), \phi, \xi, \eta, g)\):

1. belongs to \(F_1 \oplus F_5\),
2. belongs to \(F_5\) if and only if \((N, P, h)\) is a \(W_0\)-manifold,
3. cannot belong to \(F_1\).

**Proof.** We compare (2.8), (2.9) and (2.10) to accomplish the proof. \(\square\)

Next, bearing in mind (2.5), (2.6) and (2.7), we determine the basic components \(R_{ijkl} = R(e_i, e_j, e_k, e_l)\) of \(R\). The non-zero ones of them are obtained by the basic symmetries of \(R\) and the following

\[
R_{1212} = -\frac{1}{t^2}(k' - 1).
\]

Thus, it is valid the following

**Theorem 2.3.** The 3-dimensional apapR manifold \((C(N), \phi, \xi, \eta, g)\) is flat if and only if \(k' = 1\).

Moreover, using (1.8), (2.6) and (2.11), we compute the basic sectional curvatures \(k_{ij} = k(e_i, e_j)\) determined by the basis \(\{e_i, e_j\}\) of the corresponding 2-plane as follows

\[
k_{12} = \frac{1}{t^2}(k' - 1), \quad k_{01} = k_{02} = 0.
\]

By virtue of (1.6), (2.6) and (2.11), we get the basic components \(\rho_{jk} = \rho(e_j, e_k)\) and \(\rho^*_{jk} = \rho^*(e_j, e_k)\) of \(\rho\) and \(\rho^*\), respectively, as well as the values of \(\tau\) and \(\tau^*\).

The non-zero ones of them are the following:

\[
\rho_{11} = \rho_{22} = -\rho^*_{12} = -\rho^*_{21} = \frac{1}{t^2}(k' - 1).
\]

Taking into account (2.12) and (2.13), we get the following

**Theorem 2.4.** The following properties of \((C(N), \phi, \xi, \eta, g)\) are valid:

1. its sectional curvatures of \(\xi\)-sections vanish;
2. it is \(*\)-scalar flat, i.e. \(\tau^* = 0\);
3. \(\tau < 0\) if and only if \(k' < 1\);
4. \(\tau > 0\) if and only if \(k' > 1\).
3. Hyperbolic extension of a 2-dimensional paracomplex Riemannian space-form

In this section, we construct a special type of a 3-dimensional warped product manifold $\mathcal{S}(\mathcal{N})$ of $\mathbb{R}^+$ and a paracomplex space-form $(\mathcal{N}, P, h)$ from the class $\mathcal{W}_1$. Then, $(\mathcal{N}, P, h)$ is defined by (1.10) for some 1-form $\theta$.

Let $dt$ be the coordinate 1-form on $\mathbb{R}^+$ and let us introduce an almost parcontact almost paracomplex structure and a Riemannian metric on $\mathcal{S}(\mathcal{N})$ as follows

\begin{equation}
\phi|_\mathcal{H} = P, \quad \xi = \frac{d}{dt}, \quad \eta = dt, \quad g = dt^2 + \cosh 2t h + \sinh 2t \tilde{h}.
\end{equation}

Then, it is easy to check the following

**Proposition 3.1.** The manifold $(\mathcal{S}(\mathcal{N}), \phi, \xi, \eta, g)$ is a 3-dimensional apapR manifold.

In the partial case, when $(\mathcal{N}, P, h)$ is paraholomorphic, i.e. $\nabla' P = 0$, then the Riemannian manifold $(\mathcal{S}(\mathcal{N}), \phi, \xi, \eta, g)$ is para-Sasaki-like [3].

Taking into account (1.10), (2.3) and (3.1), we get the following formulæ for the Levi-Civita connections $\nabla$ and $\nabla'$ of $g$ and $h$, respectively:

\begin{equation}
\nabla_x y' = \nabla'_x y' + \frac{1}{2} \sinh 2t \left\{ g(x', y') \theta'^t - g(x', P y') P \theta'^z \right\} - g(x', P y') \xi,
\end{equation}

\[
\nabla_\xi y' = P y', \quad \nabla'_x \xi = P x', \quad \nabla'_\xi \xi = 0.
\]

In the latter equalities and further, $\theta'^t$ denotes the dual vector of $\theta'$ with respect to $h$ on $(\mathcal{N}, P, h)$. Analogously, $\theta'^z$ stands for the dual vector of $\theta$ with respect to $g$ on $(\mathcal{S}(\mathcal{N}), \phi, \xi, \eta, g)$. Bearing in mind (3.1), we have the following relations

\[
\theta'|_\mathcal{H} = \cosh 2t \theta'^z - \sinh 2t P \theta'^z, \quad \theta(z') = \theta'(z').
\]

Taking into account (3.2), we compute the following

\[
R(x', y') z' = k' \left\{ \cosh 2t \pi_1(x', y') z' - \sinh 2t P \pi_2(x', y') z' \right\} - \pi_2(x', y') z'
\]

\[
+ \frac{1}{2} \sinh 2t \left\{ g(y', z') \nabla_x \theta'^t - g(x', z') \nabla_y \theta'^z \right\}
\]

\[
- g(y', P z') \nabla_x P \theta'^z + g(x', P z') \nabla_y P \theta'^z \right\} - g(x', P y') \nabla_x \xi
\]

\[
\frac{1}{2} \left\{ \pi_1 - \pi_2 \right\} (x', y', z', \theta'^z) \right\} \{ \sinh 2t P \theta|^H + 2 \cosh 2t \xi \}
\]

\[
R(x', y') \xi = \frac{1}{2} \left\{ \pi_1 - \pi_2 \right\} (x', y') \theta'|_\mathcal{H},
\]

\[
R(\xi, y') z' = \frac{1}{2} \left\{ \pi_1 - \pi_2 \right\} (\theta'|_\mathcal{H}, y') z' - g(y', z') \xi,
\]

\[
R(x', \xi) z' = \frac{1}{2} \left\{ \pi_1 - \pi_2 \right\} (x', \theta'|_\mathcal{H}) z' + g(x', z') \xi.
\]

Bearing in mind (3.1) and (3.3), we obtain the following
\begin{equation}
R(x', y', z', w') = k' \{ \cosh 2t \pi_1 (x', y', z', w') - \sinh 2t \pi_2 (x', y', z', Pw') \} \\
- \pi_2 (x', y', z', w') + \frac{1}{2} \sinh 2t \{ g(y', z') g (\nabla_x \theta^{\pi}, w') \\
- g(x', z') g (\nabla_y \theta^{\pi}, w') \\
- g(x', Pz') g (\nabla_x P\theta^{\pi}, w') \\
+ g(x', Pz') g (\nabla_y P\theta^{\pi}, w') \} \\
+ \frac{1}{2} \sinh 2t \{ \pi_1 - \pi_2 \} (x', y', z', \theta^{\pi})(Pw'),
\end{equation}

(3.4)

According to (1.3) and (3.1), we determine the components $g_{ij}$ and $h_{ij}$ as follows:
\begin{equation}
g_{00} = g_{11} = g_{22} = 1, \quad g_{12} = g_{21} = 0, \quad h_{11} = h_{22} = \cosh 2t, \quad h_{12} = h_{21} = -\sinh 2t.
\end{equation}

(3.5)

Bearing in mind (2.3), (3.2) and (3.3), we obtain:
\begin{equation}
\begin{align*}
\nabla_{e_1} e_1 &= \nabla'_{e_1} e_1 + \frac{1}{2} \sinh 2t \theta^{\pi}, \\
\nabla_{e_2} e_2 &= \nabla'_{e_2} e_2 + \frac{1}{2} \sinh 2t \theta^{\pi}, \\
\nabla_{e_0} e_1 &= \nabla'_{e_0} e_1 = e_2, \\
\nabla_{e_0} e_2 &= \nabla'_{e_0} e_2 = e_1, \\
\nabla_{e_0} e_0 &= 0.
\end{align*}
\end{equation}

(3.6)

Then, applying (3.1), (3.5) and (3.6), we determine the components $F_{ijk}$ of $F$.

The non-zero of them are the following:
\begin{equation}
\begin{align*}
F_{111} &= -F_{122} = \theta'_1, \\
F_{222} &= -F_{211} = \theta'_2, \\
F_{101} &= F_{110} = F_{202} = F_{220} = -1.
\end{align*}
\end{equation}

(3.7)

According to (1.4), we get the components of the Lee forms of $(S(N), \phi, \xi, \eta, g)$ with respect to the basis. The non-zero of them are:
\begin{equation}
\theta_0 = -2, \quad \theta_1 = -\theta'_2 = \theta'_1, \quad \theta_2 = -\theta'_1 = \theta'_2.
\end{equation}

(3.8)

By virtue of (1.3) and (3.7), we establish the truthfulness of the following equality
\begin{equation}
F(x, y, z) = (F^1 + F^4)(x, y, z).
\end{equation}

By virtue of (3.9) and (1.10), we have the following non-zero components of $F^1$ and $F^4$ with respect to the basis (1.3):
\begin{equation}
\begin{align*}
F^1_{111} &= -F^1_{122} = \theta'_1, \\
F^1_{222} &= -F^1_{211} = \theta'_2, \\
F^4_{101} &= F^4_{110} = F^4_{202} = F^4_{220} = \frac{1}{2} \theta_0.
\end{align*}
\end{equation}

(3.9)

Thus, we get the following

**Theorem 3.2.** The 3-dimensional apopR manifold $(S(N), \phi, \xi, \eta, g)$:

(1) belongs to $F_1 \oplus F_4$,
(2) belongs to $F_1$ if and only if $(N, P, h)$ is a $W_0$-manifold,
(3) cannot belong to $F_4$. 

Proof. It follows from the above by comparing (3.7), (3.8) and (3.9).

Taking into account (3.4), (3.5) and (3.6), we obtain the components $R_{ijkl}$ of $R$ with respect to the basis. The non-zero of them are determined by the basic symmetries of $R$ and the following equalities

\begin{equation}
R_{1221} = k' \cosh 2t + 1
\end{equation}

\begin{equation}
R_{1210} = -\theta'_1, \quad R_{1220} = \theta'_1, \quad R_{0110} = R_{0220} = -1.
\end{equation}

Bu virtue of (1.8), (3.5) and (3.10), we calculate the basic sectional curvatures $k_{ij}$ as follows

\begin{equation}
k_{12} = R_{1221}, \quad k_{01} = k_{02} = -1.
\end{equation}

Furthermore, from (1.6), (3.5) and (3.10), we obtain the basic components $\rho_{jk}$ and $\rho^*_jk$ as well as the values $\tau$ and $\tau^*$ as follows:

\begin{equation}
\rho_{11} = \rho_{22} = R_{1221} - 1, \quad \rho_{00} = -2, \quad \rho_{01} = \rho_{11} = \rho_{22} = 0,
\end{equation}

\begin{equation}
\rho_{12} = \rho_{21} = 0, \quad \rho_{01} = \rho_{01} = \theta'_1, \quad \rho_{02} = \rho_{20} = \theta'_2
\end{equation}

Using (3.11) and (3.12), we conclude the following

**Proposition 3.3.** The manifold $(S(N), \phi, \xi, \eta, g)$ has the following properties:

(1) It has constant negative $\xi$-sectional curvatures;
(2) It is $\star$-scalar flat.

Bearing in mind (3.12) and (3.5) for $g_{ij}$, we obtain the following

**Theorem 3.4.** The following properties of $(S(N), \phi, \xi, \eta, g)$ are equivalent:

(1) $(N, P, h)$ is a $\mathcal{W}_0$-manifold;
(2) $\rho = k' \cosh 2t g - (2 + k' \cosh 2t) \eta \otimes \eta$;
(3) $\rho^* = -(1 + k' \cosh 2t) (\tilde{g} - \eta \otimes \eta)$.

Let us remark that (1) of the latter theorem is equivalent to the fact that $(S(N), \phi, \xi, \eta, g)$ is an $F_2$-manifold, according to (2) of Theorem 3.2.

The expression of the Ricci tensor in (2) of Theorem 3.4 means that, in this special case, $(S(N), \phi, \xi, \eta, g)$ is an para-$\eta$-Einstein manifold following [6].

Then, we obtain immediately the following

**Corollary 3.5.** If $(N, P, h)$ is a $\mathcal{W}_0$-manifold, then $(S(N), \phi, \xi, \eta, g)$ has the properties:

(1) $\tau = 2(k' \cosh 2t - 1)$;
(2) $k' < 0$ if and only if $\tau \leq 2(k' - 1) < -2$;
(3) $k' = 0$ if and only if $\tau = -2$;
(4) $k' > 0$ if and only if $-2 < 2(k' - 1) \leq \tau$.

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