Lower bounds for Galois orbits of special points on Shimura varieties: a point-counting approach

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Abstract
Let $S$ be a Shimura variety. We conjecture that the heights of special points in $S(\mathbb{Q})$ are discriminant negligible with respect to some Weil height function $h : S(\mathbb{Q}) \to \mathbb{R}$. Assuming this conjecture to be true, we prove that the sizes of the Galois orbits of special points grow as a fixed power of their discriminant (an invariant we will define in the text). In particular, we give a new proof of a theorem of Tsimerman on lower bounds for Galois degrees of special points in Shimura varieties of abelian type. This gives a new proof of the André–Oort conjecture for such varieties that avoids the use of Masser–Wüstholz isogeny estimates, replacing them by a point-counting argument.

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Dedicated to the memory of Bas Edixhoven

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1 Introduction

For terminology, facts and notations concerning Shimura varieties, we refer to [11] and references therein. Let \((G, X)\) be a Shimura datum. We assume that \(G\) is semisimple of adjoint type. This assumption does not cause any loss of generality with regards to the problem of bounding Galois degrees and applications to André–Oort type questions: one can always reduce to this situation (see for example Proposition 2.2 of [6]).

Let \(K\) be a compact open subgroup of \(G(\mathbb{A}_f)\) and \(X\) a connected component of \(X\). We let \(G(\mathbb{Q})^+\) be the group of \(\mathbb{Q}\)-points of \(G\) contained in the neutral component of \(G(\mathbb{R})\) and we let \(\Gamma = K \cap G(\mathbb{Q})^+\).

This data defines the Shimura variety

\[ Sh_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \]

and its distinguished connected component \(S := \Gamma \backslash X\). By standard abuse of terminology, we will refer to \(S\) as a ‘Shimura variety’ (even if technically it is only a connected component of a Shimura variety).

The variety \(S\) is a quasi-projective algebraic variety admitting a canonical model over an explicitly described number field \(F\) of degree bounded in terms of the data \((G, X)\) and \(K\).

Let \(\pi : X \to S\) denote the quotient map by \(\Gamma\), and choose \(\Lambda \subset X\) a semialgebraic fundamental set for the action of \(\Gamma\) on \(X\) as in Theorem 3.1 of [10].

Recall that special points on \(S\) are algebraic points defined over abelian extensions of \(F\). Estimating the degrees of these extensions is a difficult problem. To a special point one attaches a quantity which we call its ‘discriminant’ and it is conjectured that degrees of special points (over \(F\)) grow as a power of their discriminant. At present, this conjecture is only known under the assumption of the Generalised Riemann Hypothesis (see [16]). It is also known for all Shimura varieties of abelian type (see [14] and [15]). This result relies on two major ingredients: one—the averaged Colmez formula for Faltings heights of CM abelian varieties and two—the Masser–Wüstholz isogeny theorem.

The Masser–Wüstholz theorem does not seem to be easily generalisable to the case of general Shimura varieties. In this paper we replace its usage with a counting theorem due to the first author which holds for all Shimura varieties. The question of bounding the height remains a major obstacle. We formulate a conjecture on height bounds, assuming which, we are able to obtain the required estimate for the degrees of special points.

To formulate our conjecture we need to introduce some technical notations. Let \(p\) be a special point of \(S\) and write \(p = \pi(x)\) with \(x \in \Lambda\). Let \(T\) be the Mumford-Tate group of \(x\). For a definition of the Mumford-Tate group we refer to [11]. By definition of a special point, \(T\) is an algebraic torus. Let \(K^m_T\) be the maximal compact open subgroup of \(T(\mathbb{A}_f)\) (\(\mathbb{A}_f\) denotes the finite adeles of \(\mathbb{Q}\)) and \(K_T\) the compact open subgroup \(K \cap T(\mathbb{A}_f)\) of \(T(\mathbb{A}_f)\). Let \(L\) be the splitting field of \(T\) i.e. the smallest extension of \(\mathbb{Q}\) such that \(T_L\) is a split torus. Under our assumption (\(G\) is of adjoint type), \(L\) is a Galois CM field. We let \(d_L\) be the absolute value of the discriminant of \(L\).
Definition 1 With the notations introduced above, we define the discriminant of $p$ as
\[ \text{disc}(p) = [K^m_T : K_T]d_L. \]

We now formulate our conjecture on heights. In what follows we write $a = O_b(c)$ (respectively, $a = \text{poly}_b(c)$) to indicate that $a \leq \gamma(b) \cdot c$ (respectively, $a \leq (c + 1)^{\gamma(b)}$) where $\gamma(\cdot)$ is some universally fixed function (which may be different for each occurrence of this notation in the text). Here $a$ and $c$ denote natural numbers, and $b$ can involve one or several arguments of any type. We also allow several arguments in $\text{poly}_b(c_1, \ldots, c_n)$ which we interpret as $\text{poly}_b(c_1 + \cdots + c_n)$. We will also sometimes use notation $\gg_S$ or $\ll_S$ to mean bigger than (resp. smaller than) up to a constant depending on $S$ only.

Conjecture 2 (Conjecture on heights of special points on $S$) There exists a Weil height function $h: S(\overline{\mathbb{Q}}) \to \mathbb{R}$ such that the following holds.

Let $p \in S$ be a special point, then for any $\varepsilon > 0$ we have
\[ h(p) = O_{S,\varepsilon}(\text{disc}(p)^{\varepsilon}). \]

We then say that the heights of special points are discriminant-negligible (some authors use the terminology ‘sub-polynomial in the discriminant’).

Remark 3 The Shimura variety $S$ admits a Baily–Borel compactification $\overline{S}$ defined over the same field as that of $S$, and thus there is a a natural Weil height function $h: S(\overline{\mathbb{Q}}) \to \mathbb{R}$. This is a natural candidate to study. The other important compactification is the toroidal one, also yielding a Weil height function. From well-known properties of Weil heights (see for example [9, B.3]) follows that if conjecture 2 holds for one choice of Weil height on $S$ it does hold for every choice.

Our main goal in the present paper is to prove the following.

Theorem 1 Assume Conjecture 2 for a Shimura variety $S$. There are $C > 0$ and $\varepsilon > 0$, depending only on $S$ and $F$, such that the following holds. Let $p$ be a special point of $S$, then
\[ [F(p) : F] > C \text{disc}(p)^{\varepsilon}. \]

The conclusion of Theorem 1 for an arbitrary Shimura variety $S$ is the only missing ingredient in a proof of the André–Oort conjecture in full generality using the Pila–Zannier strategy (see [14] and [8]).

We have:

Theorem 2 The following hold:

1. Assume conjecture 2 holds for the Shimura variety $S$. Then the André–Oort conjecture holds for $S$ and any mixed Shimura variety whose pure part is $S$.
2. Assume that $S$ is of abelian type. Then the André–Oort conjecture holds for $S$ and any mixed Shimura variety whose pure part is $S$. 
The conclusions of the theorem follow from Theorem 1 using the Pila–Zannier strategy. For details we refer to [8] and [14].

We will deduce Theorem 1 from Theorem 3 in the next section. The idea originates from a paper of the second author [13], and applies more generally to deduce a lower bound for the degrees of special points from the corresponding height upper bounds in a variety of contexts (for instance, for torsion points on abelian varieties). The same proof in fact gives a slightly more refined statement, which we state below. For

\[ p \in S \]

a special point, let \( S(p) \) denote the smallest (zero-dimensional) special subvariety of \( S \) that contains \( p \). It consists of the points of the image of the Shimura morphism \( Sh_{KT}(T, \{ x \}) \) in \( Sh_K(G, X) \) induced by the inclusion of Shimura datum \( (T, \{ x \}) \) in \( (G, X) \), contained in the component \( S \). Since the number of components of \( Sh_K(G, X) \) is independent of \( p \), in all our statements and arguments, we do not differentiate between the cardinality of \( S(p) \) and the image of \( Sh_{KT}(T, \{ x \}) \) in \( Sh_K(G, X) \). We actually prove the following.

**Proposition 4** Let \( p \in S \) and let \( h \) denote the maximum of \( h(q) \) for \( q \in S(p) \). Then

\[
\text{disc}(p) < C ([F(p) : F] + h)^\kappa
\]

(2)

for some positive constants \( C, \kappa \) depending only on \( S \).

We describe the counting theorem which replaces the Masser–Wüstholz isogeny estimates in Tsimerman’s strategy. Recall (see [17] Section 3.3 and references therein), that \( X \) is a subset of a projective variety \( \tilde{X} \) (its compact dual), naturally defined over \( \overline{Q} \) and that this \( \overline{Q} \) structure is \( G(\overline{Q}) \)-invariant. We can thus talk of the set of algebraic points \( X(\overline{Q}) \) of \( X \). The embedding \( X \hookrightarrow \tilde{X} \) also provides us with a height function on \( X(\overline{Q}) \) and thus for any Weil height function on \( S(\overline{Q}) \), we obtain a Weil height function \( h \) on \((X \times S)(\overline{Q})\).

Our main technical tool is the following point-counting result. Let

\[
Z_S \subset X \times S, \quad Z_S := \{(x, s) : x \in \Lambda, s = \pi(x)\}
\]

(3)

denote the graph of \( \pi \) restricted to the fundamental domain \( \Lambda \), and denote

\[
Z_S(f, h) := \{(x, s) \in Z_S : [F(x, s) : F] \leq f, h(x, s) \leq h\}
\]

(4)

**Theorem 3** We have an upper bound \( \#Z_S(f, h) = \text{poly}_S(f, h) \).

Our proof of Theorem 3 is based on a polylogarithmic counting theorem [2] by the first author, sharpening Pila–Wilkie’s theorem for sets defined using leaves of foliations over number fields.

We apply this to a canonical foliation associated to the variety \( S \) to deduce Theorem 3. Note however that the results of [2] only directly apply to counting in compact domains. We overcome this by analyzing the degeneration of the counting constants as a function of the distance to the boundary of the Shimura variety. The crucial input is provided by the fact that the connection giving rise to the canonical foliation above admits regular singularities.
Tsimerman, in [14] (Corollary 3.2), proved that for a principally polarised abelian variety $A$ of dimension $g$ which is simple and has CM by the ring of integers $O_E$, the Faltings height $h_F(A)$ is $|\text{disc}(E)|$-negligible. This is a consequence of the averaged Colmez formula (see [1] and [18]).

Faltings’ comparison between $h_F$ and a Weil height (see [7]) shows that our height conjecture 2 holds for all special points on $A_g$ (for all $g > 0$) corresponding to simple abelian varieties with CM by a ring of integers of a CM field of degree $2g$ (with uniform constants). From Proposition 4 one then deduces that the corresponding Galois lower bounds hold for these special points, thus giving a new proof of Tsimerman’s bound [14, Theorem 1.1]. This in turn implies lower bounds for the Galois degrees of all special points of $A_g$ for all $g > 0$ (see [14, Theorem 5.1] and its proof) and the André–Oort conjecture for all Shimura varieties of abelian type.

We have allowed a slight sloppiness. The Shimura variety $A_g$ is not defined by a group of adjoint type: the centre of $GSp_{2g}$ is $G_m$. However, passing to the adjoint Shimura variety $A_g^{ad}$ does not change the discriminant of $p$ and the morphism $A_g \to A_g^{ad}$ is finite, hence the height bound remains true.

The main issue is the deduction of degree (lower) bounds from the height bounds. Tsimerman’s method relies heavily on the use of Masser–Wüstholz isogeny estimates, which are only known for abelian varieties. There is no known (even conjectural) analogue of the Masser–Wüstholz theorem for general Shimura varieties. Our method instead uses a point counting result by the first author adapted to the context of Shimura varieties.

2 Proof of Theorem 1

In this section we deduce Theorem 1 from Theorem 3 (the point-counting result). We will require a few standard properties of special points summarised below.

In what follows we make the assumption that $K$ is neat and that $K$ is a product $K = \prod_p K_p$ where $K_p$ is a compact open subgroup of $G(\mathbb{Q}_p)$. This assumption does not alter any of our bounds since replacing an arbitrary $K$ by such a subgroup only changes constants by a bounded amount and will not affect our estimates.

We consider a special point $p$ of $S$. Let $x$ be a point of $\Lambda$ such that $p = \pi(x)$. We let $T$ be the Mumford–Tate group of $x$ and let $L$ be the splitting fields the splitting field of $T$. We let $d$ be the discriminant of $p$.

We let $S(p)$ be the smallest zero dimensional special subvariety of $S$ containing $p$. It is a zero dimensional Shimura variety embedded in $Sh_K(G, X)$ via the inclusion of Shimura data

$$(T, x) \subset (G, X)$$

All points in $S(p)$ have discriminant $d$. The number of points in this zero dimensional Shimura variety $S(p)$ is $#(T(\mathbb{Q}))\backslash T(\mathbb{A}_f)/K_T$.

Remark 5 As already mentioned in the introduction, we have allowed a slight inaccuracy. Strictly speaking, the special subvariety defined by the inclusion $(T, x) \subset (G, X)$
may be spread among several components of $Sh_K(G, \mathbf{X})$. Nevertheless, since we are interested in the estimates up to constants depending only on $G, \mathbf{X}$ and $K$, this does not change the estimates.

**Proposition 6** The number of points in $S(p)$ is at least $d^c$, for some $c = c(S) > 0$.

Furthermore, all points of $S(p)$ have the same degree over $F$.

**Proof** The second claim follows from the definition of canonical models of zero dimensional Shimura varieties (see [11] or [6]). Indeed, let $r_x : \text{Res}_{L/Q} G_{m, L} \to T$ be the reciprocity map attached to the data $(T, x)$ (see [6] for example). Let $U$ be $r_x(L \otimes \mathbb{A}_f) \subset T(\mathbb{A}_f)$. The fact that $T$ is commutative immediately shows that for any $t$ the size of the image of $U \cdot t$ in $T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/K_T$ is the size of the image of $U$, proving the second claim.

As for the first claim, first note that

$$\#(T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/K_T) \gg_S [K^m_T : K_T] \times \#(T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/K^m_T).$$

Since we have assumed $G$ to be adjoint, $T(\mathbb{R})$ is compact and therefore $K^m_T \cap T(\mathbb{Q})$ is finite. Its size is bounded in terms of $S$ only (actually, only in terms of dim$(G)$) thus justifying $\gg_S$.

Again, since $T(\mathbb{R})$ is compact, we can apply Theorem 2.3 of [16] (note that the same result has been obtained independently, at the same time and by the same method by Tsimerman—see [15]). By that Theorem, we have

$$\#(T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/K^m_T) \gg_S d^a_L,$$

where $a$ depends on $S$ only. The result follows with $c = \min(1, a)$. \hfill $\Box$

**Proposition 7** With $p$ and $x$ as above, we have

$$H(x) = \text{poly}_S(d).$$

**Proof** This is a consequence of Theorem 1.4 (d) of [5] by Daw and Orr. They prove that

$$H(x) \ll_S c_1^{i(T)} [K^m_T : K_T]^{c_2} d_3^{c_3},$$

where $c_1, c_2, c_3$ are constants depending only on $S$. The function $i(T)$ is the number of primes such that $K^m_{T, p} \neq K_{T, p}$ (under our assumption that $K = \prod_p K_p$, we have $K_T = \prod_p K_{T, p}$). We may assume that $c_1 > 1$ (otherwise the factor $c_1^{i(T)}$ disappears).

Using $2^{i(T)} \leq [K^m_T : K_T]$, we see that

$$c_1^{\log([K^m_T : K_T])} \geq c_1^{i(T)/2}$$

and thus

$$c_1^{i(T)} \leq [K^m_T : K_T]^{2 \log(c_1)}.$$
We conclude that \( H(x) = \text{poly}_S(d) \).

By Proposition 6, there are at least \( d^c \) special points of degree \( f \) over \( F \). By Proposition 7 and Conjecture 2 each point \( q \) in \( S(p) \) gives rise to a pair \( (x_q, q) \in Z_S \) with

\[
h(x_q, q) = \log(\text{poly}_S(d)) + O_S,\varepsilon(d^\varepsilon) \quad \text{for every } \varepsilon > 0. \tag{5}\]

Comparing this with Theorem 3 we have

\[
d^c < \#Z_S(f, O_{S,\varepsilon}(d^\varepsilon)) = \text{poly}_S(f, O_{S,\varepsilon}(d^\varepsilon)). \tag{6}\]

Choosing now \( \varepsilon \) to be sufficiently small, we conclude that \( f > \text{const}(S, \varepsilon')d^{c/N-\varepsilon'} \) where \( N \) is the degree on the polynomial on the right hand side and \( \varepsilon' \) is any positive number. This concludes the proof of Theorem 1. Note that the above reasoning proves Proposition 4.

### 3 Point counting with foliations: proof of Theorem 3

In this section we recall a result from [2] that will be needed in the sequel. We state only the result that we require in the present text, allowing us to slightly simplify the presentation. We refer the reader to [2] for some more general forms of the counting theorem.

We use the following notation. If \( D \subset \mathbb{C} \) (resp. \( A \subset \mathbb{C} \)) is a disc of radius \( r \) (resp. annulus of inner radius \( r_1 \) and outer radius \( r_2 \)) and \( \delta \in (0, 1) \), we denote by \( D^\delta \) (resp \( A^\delta \)) the disc (resp. annulus) with the same center and radius \( \delta^{-1}r \) (resp \( \delta r_1, \delta^{-1}r_2 \)). We extend this notation coordinatewise to polydiscs or polyannuli in \( \mathbb{C}^n \). Similarly if \( B \subset \mathbb{C}^n \) is a ball we denote by \( B^\delta \) the ball with the same center and radius \( \delta^{-1}r \).

#### 3.1 The variety

Let \( \mathbb{M} \subset \mathbb{A}^N_K \) be an irreducible affine variety defined over a number field \( K \). We equip \( \mathbb{M} \) with the standard Euclidean metric from \( \mathbb{A}^N \), denoted dist, and denote by \( B_R \subset \mathbb{M} \) the intersection of \( \mathbb{M} \) with the ball of radius \( R \) around the origin in \( \mathbb{A}^N \).

**Remark 8** In our setting, the ambient variety will not necessarily be affine. It is implicitly understood that in applying Theorem 4 below, we first reduce to an affine cover. Note that the notion of metric dist inherited from the ambient affine space is not canonical, and depends on this choice of affine charts.

#### 3.2 The foliation

Let \( \xi := (\xi_1, \ldots, \xi_n) \) denote \( n \) commuting, pointwise linearly independent rational vector fields on \( \mathbb{M} \) defined over \( K \). We denote by \( \mathcal{F} \) the foliation of \( \mathbb{M} \) generated by \( \xi \).
For every \( p \in \mathbb{M} \) denote by \( \mathcal{L}_p \) the germ of the leaf of \( \mathcal{F} \) through \( p \). We have a germ of a holomorphic map \( \varphi_p : (\mathbb{C}^n, 0) \to \mathcal{L}_p \) satisfying \( \partial \varphi_p / \partial x_i = \xi_i \) for \( i = 1, \ldots, n \). We refer to this coordinate chart as the \( \xi \)-coordinates on \( \mathcal{L}_p \). If \( \varphi_p \) continues holomorphically to a ball \( B \subset \mathbb{C}^n \) around the origin then we call \( B := \varphi_p(B) \) a \( \xi \)-ball. If \( \varphi_p \) extends to \( B^\delta \) we denote \( B^\delta := \varphi_p(B^\delta) \).

### 3.3 Counting algebraic points

Finally we are ready to state the point counting result. We fix: \( \ell \in \mathbb{N}; \) a map \( \Phi \in \mathcal{O}(\mathbb{M})^\ell \) defined over \( K \); and a \( \xi \)-ball \( B \subset \mathbb{B}_R \) of radius at most \( R \). Set

\[
A = A_{\Phi, B} := \Phi(B^2) \subset \mathbb{C}^\ell.
\] (7)

We denote by \( h : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0} \) the absolute logarithmic Weil height and set

\[
A(g, h) := \{ p \in A : [\mathbb{Q}(p) : \mathbb{Q}] \leq g \text{ and } h(p) \leq h \}.
\] (8)

We denote by \( \delta_\xi \) (resp. \( \delta_\Phi \)) the maximum of the degree and the log-height of \( \xi \) (resp. \( \Phi \)). The reader may see [2] for the precise definition, or simply consider the degrees and the heights of the coordinates of \( \xi, \Phi \) thought of as regular function on the affine variety \( \mathbb{M} \), i.e. as polynomials. The following is a direct consequence of [2, Corollary 6], applied with \( V = \mathbb{M} \).

**Theorem 4** Suppose that for every \( p \in \mathbb{M} \) the germ \( \Phi|_{\mathcal{L}_p} \) is a finite map, and \( \Phi(\mathcal{L}_p) \) contains no germs of algebraic curves. Then

\[
\#A(g, h) = \text{poly}_{\mathbb{M}, \ell}(\delta_\xi, \delta_\Phi, \log R, g, h).
\] (9)

### 4 Counting special points

In this section, we prove Theorem 3 by applying the counting results of [2] to an appropriate foliation.

Here again, we make the assumption that \( K \) is neat and therefore \( \Gamma = G(\mathbb{Q}^+) \cap K \) acts without fixed points. This does not change the estimates.

The variety \( S \) is equipped with a standard principal \( G(\mathbb{C}) \)-bundle over \( S \) given by

\[
P = \Gamma \backslash (G(\mathbb{C}) \times X).
\] (10)

We reiterate that the reference for this and the following facts is [12, Chapter III, p.58] (note that our definition (10) agrees with the Definition after Lemma 3.1 in loc. cit. by our assumption that \( G \) is adjoint).

Let us briefly recall (see [11] and [12]) that each point \( x \in X \) defines a Hodge character

\[
\mu_x : \mathbb{G}_{m, \mathbb{C}} \to \mathbb{G}_{\mathbb{C}}.
\]
Fixing a faithful representation of $G_\mathbb{Q}$ we obtain a variation of rational Hodge structures over $X$ (and by passing to a quotient on $S$) and $\mu_x$ gives rise to a Hodge filtration $F_{\mu_x}$ on the corresponding fibre of the variation of the Hodge structure. Then $\tilde{X}$ can be identified with a $G(\mathbb{C})$-conjugacy class of such filtrations and the Borel embedding is $x \mapsto F_{\mu_x}$. Recall that the projective variety $\tilde{X}$ is defined over the field $F$ (this is explicitly explained in [17], section 3.3).

Reverting to the $G(\mathbb{C})$-bundle $P$, the corresponding flat $G(\mathbb{C})$-structure corresponds to a flat, regular-singular $G(\mathbb{C})$-connection on $P$, which is also defined over $F$ (this follows from Theorem 5.1 of [12]). There is a $G(\mathbb{C})$-equivariant map

$$\beta : P \to \tilde{X}, \quad \beta([g, x]) = g^{-1}F_{\mu_x}. \quad (11)$$

Denote by $\pi_P : P \to S$ the projection map. Then

$$\pi_P([1, x]) = \pi \circ \beta([1, x]). \quad (12)$$

There exists a finite collection of affine opens $\{U_i \subset S\}$ and trivialising charts $P|_{U_i} \simeq U_i \times G$. In this trivialisation the canonical connection $\nabla$ takes the form

$$\nabla(\sigma) = d\sigma - \Omega_i \cdot \sigma, \quad \Omega_i \in g(\Omega^1(U_i)) \quad (13)$$

where $\sigma$ is a section of $P|_{U_i}$, $g$ is the complex Lie algebra associated to $G$, and $\Omega^1(U_i)$ denotes the space of algebraic one-forms on $U_i$. Recall that $\nabla$, and hence $\Omega_i$, are defined over $F$.

By Hironaka’s desingularisation theorem we may fix a projective variety $\bar{U}_i$, also defined over $F$, such that $U_i \subset \bar{U}_i$ is dense and $\bar{U}_i \setminus U_i$ consists of normal crossings divisors. That is, around every point $s \in \bar{U}_i$ there exists a system of parameters $(x_1, \ldots, x_n)$ such that $\bar{U}_i \setminus U_i$ is given by the zero locus of some monomial in these parameters, which we assume for simplicity to be

$$\bar{U}_i \setminus U_i = \{x_{i,1} \cdots x_{i,k_i} = 0\}, \quad k_i \in \{0, \ldots, n\}. \quad (14)$$

By compactness of each $\bar{U}_i$ in the complex topology, we may cover $\bar{S}$ (see Remark 3) by a finite set of complex polydiscs $B_\alpha$ with a system of parameters $(x_{\alpha,1}, \ldots, x_{\alpha,n})$, defined over $F$, and

$$B_\alpha^0 := S \cap B_\alpha = B_\alpha \cap \{x_{\alpha,1} \cdots x_{\alpha,k_\alpha} \neq 0\}. \quad (15)$$

We may assume that $(B_\alpha^0)^{1/4} \subset U_i$. Moreover we may after rescaling assume that each $B_\alpha$ is the unit polydisc. We denote the number of polydiscs by $N$. We denote by

$$S(f, h) = \{s \in S(\mathbb{C}) \mid h(s) \leq h, [F(s) : F] \leq f\}.$$

**Lemma 9** A fraction of at least $1/(2N)$ of the points in $S(f, h)$ lie in the polyannulus

$$A_\varepsilon := \{\varepsilon < |x| < 1\}^{\times k_\alpha} \times \{|x| < 1\}^{\times n-k_\alpha}, \quad \varepsilon = e^{-O_S(Nh)} \quad (16)$$
inside one of the polydiscs $B_\alpha$ above.

We note that the constant in $O_S(Nh)$ above also depends on the affine cover and the choice of polydiscs but we may consider it fixed.

**Proof** Let $s \in S(f, h)$, so that $h(s) \leq h$ and $[F(s) : F] \leq f$. Note that $S(f, h)$ is invariant under the action of the Galois group $\text{Gal}(\bar{F}/F)$. Denote by $O_s \subset S(f, h)$ the Galois orbit of $s$. Since the coordinates $x_{\alpha, i}$ are defined over $F$, it holds for the naive Weil-height $h(x_{\alpha, i})$ that

$$h(x_{\alpha, i}(s)) = O_S(h), \quad \alpha = 1, \ldots, N, i = 1, \ldots, n.$$ 

A fraction of $1/N$ of the conjugates of $s$ belong to a single $B_\alpha$—denote these by $O_{s, \alpha}$. Without loss of generality we may assume that $s \in B_\alpha$ (otherwise replace $s$ by a conjugate). As $h(x) = h(1/x)$ for $x \neq 0$ it follows from the definition of the Weil-height as a sum of local heights [3, 1.5.7] that

$$\sum_{s_\alpha \in O_{s, \alpha}} -\log |x_{\alpha, i}(s_\alpha)| = O_S(\#O_s h), \quad i = 1, \ldots, k_\alpha.$$ 

At this point we have used that $B_\alpha$ is a polydisc of radius 1. As $\#O_{s, \alpha} \geq \#O_s / N$ we also have $O_S(\#O_s h) = O_S(N\#O_{s, \alpha} h)$. In particular at least $1 - 1/(2k_\alpha)$ of the points in $O_{s, \alpha}$ satisfy $-\log |x_{\alpha, i}(s_\alpha)| = O_S(2k_\alpha Nh) = O_S(Nh)$. Repeating this for $i = 1, \ldots, k_\alpha$, we see that half the points in $O_{s, \alpha}$ satisfy this estimate for all these coordinates concurrently. This proves the claim. \hfill $\square$

According to Lemma 9 it will suffice to count the $(f, h)$-points in the polyannulus $A_\varepsilon$ inside each polydisc $B_\alpha$. So below we fix one $U = U_i$ and one $B = B_\alpha \subset U$, with the corresponding coordinate system $(x_1, \ldots, x_n)$ and $k = k_\alpha$. We also write $\Omega = \Omega_i$.

We consider the space $\mathbb{M} = P|_U = U \times G(\mathbb{C})$ with its foliation $\mathcal{F}$ by horizontal sections of $\nabla$. Explicitly, this is the foliation generated by the vector fields $\xi_1, \ldots, \xi_n$ where

$$\xi_j = \frac{\partial}{\partial x_j} + \Omega \left( \frac{\partial}{\partial x_j} \right) \cdot g \quad \text{ (17)}$$

for $(x, g) \in U \times G(\mathbb{C})$. The leaves of $\mathcal{F}$ are given by $\{L_g\}_{g \in G(\mathbb{C})}$ where

$$L_g = \{ [x, g] : x \in X \} \subset P. \quad \text{ (18)}$$

In particular set $L := L_1$, and recall that $L_g = g \cdot L_1$.

Consider the map

$$\Phi : P \to \tilde{X} \times S, \quad \Phi = (\beta, \pi_p), \quad \text{ (19)}$$

and note that $\Phi(L_1)$ is the graph of $\pi$, and in particular $\Phi(L_1) \cap (\Lambda \times S) = Z_S$ (where we identify $\Lambda$ with its image in $\tilde{X}$ under the map $X \to \tilde{X}$).
Lemma 10 The fibers of $\Phi$ are zero-dimensional on every leaf $L_g$, and $\Phi(L_g)$ contains no germs of algebraic curves.

Proof Since $L_g = gL_1$ and $\beta$ is $G(\mathbb{C})$-equivariant, it is enough to prove the claim for $L_1$. The first claim is obvious, since both $\beta$ and $\pi_1\rho$ each form a local system of coordinates at every point of $L_1$. Since $\Phi(L_1)$ is the graph of $\pi$, the second claim follows from general functional transcendence statements for Shimura varieties. However, we give a completely elementary argument below.

Suppose that the graph $G_\pi$ of $\pi : X \to S$ contains the germ of an irreducible algebraic curve $C \subset \hat{X} \times S$ at a point $p \in \hat{X} \times S$. We will show that in this case $C \subset X \times S$. Since $X$ is a bounded symmetric domain, this implies that $C$ projects to a point in $\hat{X}$, contradicting the fact that the germ of $C$ at $p$ is a subset of $G_\pi$.

To prove the claim let $q = (q_X, q_S) \in C$ and we will prove $q_X \in X$. Recall that $C$, as an irreducible curve, is pathwise connected, and let $\gamma = (\gamma_X, \gamma_S) : [0, 1] \to C$ be a path with $\gamma(0) = p$ and $\gamma(1) = q$. Let $\gamma_X' : [0, 1] \to X$ denote the unique lifting of the path $\gamma_X$ along $\pi$ with $\gamma'(0) = p_X$. We will show that $\gamma_X \equiv \gamma_X'$, and in particular $q_X = \gamma_X(1) \in X$ as claimed. To see this, let $t_0$ denote the maximal $t$ such that for every $t \leq t_0$ we have $\gamma_X(t) = \gamma_X'(t)$, and suppose toward contradiction that $t_0 < 1$. The path $\gamma|_{[0,t_0]}$ belongs $C \cap G_\pi$, and by analytically continuing along this path we see that the germ of $C$ at $\gamma(t_0)$ belongs to $G_\pi$ as well. In particular $\pi(\gamma_X(t)) = \gamma_S(t)$ for every $t$ in some small neighborhood of $t_0$. Thus $\gamma_X$ remains the (unique) lifting of $\gamma_S$, and agrees with $\gamma_X'$, in this neighborhood of $t_0$—contradicting the maximality of $t_0$. □

We will obtain an upper bound for $\#Z_\xi(f, h)$ by applying Theorem 4 to the foliation $\mathcal{F}$, the leaf $L$, and the map $\Phi$ constructed above. However, since Theorem 4 only directly applies to $\xi$-balls, a further covering argument will be needed. We will require the following growth estimate for the leaf $L$. Let $\hat{B}^0 \subset B$ denote the domain

$$\hat{B}^0 = B \setminus \bigcup_{i=1}^{k} \{x_i \in (-\infty, 0]\}. \quad (20)$$

(Note that $\hat{B}^0 \subset B^0$. Fix some basepoint $(s_0, g_0) \in L$ with $s_0 \in \hat{B}^0$, and let $g : \hat{B}^0 \to G$ be the multivalued function given by analytically continuing from $g(s_0) = g_0$ as a flat section of $\nabla$. This corresponds to a subset of $L$ that we denote $L_{\hat{B}^0, s_0, g_0}$.

Lemma 11 Consider $g$ constructed above and $s \in A_{\xi}^{1/2} \cap \hat{B}^0$. Then

$$\log \|g(s)\| = \text{poly}_S(\|\log \xi\|) = \text{poly}_S(h). \quad (21)$$

Proof This essentially follows by the regularity of the canonical connection $\nabla$. More explicitly, since $\pi_1(B^0)$ is commutative, it follows that the monodromy operators $M_i$ along $x_i = 0$, for $i = 1, \ldots, k$, commute. Let $L_i$ be such that

$$\exp(2\pi i L_i) = M_i^{-1}$$
and such that $L_1, \ldots, L_n$ (pairwise) commute (that this choice is possible follows from [4, Lemma IV.4.5]). Then
\[
\hat{g} = x_1^{L_1} \cdots x_k^{L_k} g
\] (22)
is a univalued matrix function. As a flat section of the regular connection $\nabla$, $g$ admits regular growth along any analytic curve in $B$. The same is then obviously true for $\hat{g}$, and we conclude that $\hat{g}$ is meromorphic in $B$, with poles along $x_i = 0$ for $i = 1, \ldots, k$.

The estimate for $\hat{g}(s)$ follows by standard theory of meromorphic functions. A similar estimate for $x_i^{L_i}$ follows immediately by direct computation.

Now we return to the proof of Theorem 3. Recall that the map $\pi : X \to S$ restricted to $\Lambda$ is definable in the o-minimal structure $\mathbb{R}_{\text{an, exp}}$ (see [10], Theorem 4.1). It follows that $\Lambda \cap \pi^{-1}(\hat{B}^\circ)$ has finitely many connected components (with their number depending on $S$). The part of $Z_S$ lying over $\hat{B}^\circ$ is therefore contained in the union of finitely many sets of the form $\Phi(L_{B^\circ, s_0, g_0})$. It will suffice to count the algebraic points in each of these sets separately. Denote one such set by $Z'$. Below.

To finish the counting, set $\varepsilon = e^{-O_S(h)}$ (with $O_S(h) = O_S(Nh)$ but we have fixed $N$) as in Lemma 9. Up to the constant factor $1/(2N)$, it will suffice to count the points lying over $A_\varepsilon$. Cover $A_\varepsilon$ by polydiscs $B_j$, with $B_j^{1/2} \subset A_\varepsilon^{1/2}$ and
\[
\#\{B_j\} = \text{poly}_S(|\log \varepsilon|) = \text{poly}_S(h). \tag{23}
\]

This can be achieved by a simple logarithmic subdivision process. Namely, for each $i = 1, \ldots, k$ we use $O(1)$ discs to cover $\{1/2 < |x_j| < 1\}$, then $O(1)$-discs to cover $\{1/4 < |x_j| < 1/2\}$ and so on. Taking direct products of the collections obtained for each coordinate (and the unit disc for $x_{i+1}, \ldots, x_n$) gives the required collection.

Let $B_j$ be the $\xi$-ball given in the $x$-coordinates by $B_j$, and in the $g$-coordinates by analytically continuing from $s_0$ to the $B_j$ along a path inside $\hat{B}^\circ$. In light of Lemma 11 each $B_j$ is contained in a ball of radius $R$ in $M$, with $\log R = \text{poly}_S(h)$. By construction the union of the images of images of $\Phi(B_j)$ covers $Z'$. Finally, applying Theorem 4 to each $B_j$ finishes the proof.

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