A NOTE ON A STABILITY RESULT FOR THE FANO PLANE

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ABSTRACT. In this note, we adapt the proof of the (Turán) Stability Theorem for the Fano plane [9, Theorem 1.2] to find an explicit dependency between the parameters $\varepsilon$ and $\delta$. This is useful in the solution of a multicolored version for hypergraphs of an extremal problem about edge-colorings, known as the Erdős-Rothschild problem, which may be considered for the Fano plane.

1. Introduction

This note deals with stability in uniform hypergraphs. As usual, for an integer $r \geq 2$, an $r$-uniform hypergraph $H = (V, E)$ is a pair consisting of a vertex set $V$ and of a set $E \subseteq \binom{V}{r}$ of hyperedges, where $\binom{V}{r}$ is the set of all subsets of $V$ with cardinality $r$. For a fixed $r$-uniform hypergraph $F$, we say that a hypergraph $H = (V, E)$ is $F$-free if it does not contain a copy of $F$ as a subhypergraph. Let $\text{Forb}_F(n)$ denote the family of all labeled $F$-free $r$-uniform hypergraphs on $n$ vertices.

Given an $r$-uniform hypergraph $F$, the hypergraph Turán problem for $F$ consists of determining the Turán number $\text{ex}(n, F) = \max\{|E(H)| : H \in \text{Forb}_F(n)\}$ and of characterizing those hypergraphs $H \in \text{Forb}_F(n)$ such that $|E(H)| = \text{ex}(n, F)$, which are said to be $F$-extremal hypergraphs.

It is often the case that the extremal hypergraph is unique up to isomorphism, and that any $F$-free configuration $H = (V, E)$ such that $|E|$ is close to the maximum size must be structurally similar to the extremal configuration for $F$. This is known as stability, a concept that was observed in the graph-theoretical setting by Erdős and Simonovits [12] and that may be naturally extended to the hypergraph Turán problem. We should also mention that, compared with the graph case, much less is known about the Turán problem for hypergraphs. For more information about this problem and stability results for hypergraphs, we refer the reader to surveys by Keevash [8] and by Mubayi and Verstraëte [10].

The Fano plane, hereafter denoted by Fano, is the single linear 3-uniform hypergraph on seven vertices and seven hyperedges. (A hypergraph $H = (V, E)$ is linear if $|e \cap f| \leq 1$ for all hyperedges $e, f \in E, e \neq f$.) It is the projective plane over the field with two elements. Sós [13] conjectured that the unique extremal hypergraph for the Fano plane is the balanced, complete, bipartite 3-uniform hypergraph $B_n$ on $n$ vertices. This conjecture was verified for sufficiently large $n$ by Keevash and Sudakov [9] and, independently, by Füredi and Simonovits [7]. An earlier paper of De Caen and Füredi [2]...
already established that
\[ \lim_{n \to \infty} \frac{\text{ex}(n, \text{Fano})}{\binom{n}{3}} = \frac{3}{4}. \]
Recently, Bellmann and Reiher [1] proved that the conjecture holds for every \( n \geq 8 \),
which is best possible. This means that, for \( n \geq 8 \),
\[ \text{ex}(n, \text{Fano}) = \left( \floor{\frac{n}{2}} \right) \cdot \left( \Ceil{\frac{n}{2}} \right) + \left( \Ceil{\frac{n}{2}} \right) \cdot \left( \floor{\frac{n}{2}} \right). \]
In the papers of Keevash and Sudakov and of Füredi and Simonovits, the extremality
of \( B_n \) was obtained using stability. The precise result proved in [9, Theorem 1.2] is the
following. For a hypergraph \( H = (V, E) \) and a subset \( A \subseteq V \), we write \( e_H(A) \) for the
number of hyperedges \( e_H \) of \( H \) such that \( e \subseteq A \).

**Theorem 1.1.** For all \( \varepsilon > 0 \), there are \( \delta > 0 \) and \( n_0 \) with the following property.
If \( H = (V, E) \) is a Fano-free 3-uniform hypergraph on \( n \geq n_0 \) vertices with at least
\( (1 - \delta) \frac{n^3}{8} \) hyperedges, then there is a partition \( V(H) = A \cup B \) with the property that
\( e_H(A) + e_H(B) < \varepsilon n^3 \).

In this note, we wish to compute the dependency between \( \delta \) and \( \varepsilon \) explicitly. Our
motivation was to compute explicit bounds for an Erdős-Rothschild-type problem in-
volving the Fano plane [4], where we apply a stability method.
With a careful analysis of the proof of [9, Theorem 1.2] and performing some modi-
fications to tighten the bounds, we obtain the following result.

**Theorem 1.2.** For any fixed \( 0 < \delta < (1/36)^8 \), there exists \( n_0 \) such that the following holds for all \( n \geq n_0 \). If \( H = (V, E) \) is a Fano-free 3-uniform hypergraph on \( n \geq n_0 \)
vertices with \( \text{ex}(n, \text{Fano}) - \delta n^3/8 \) hyperedges, then there is a partition \( V(H) = A \cup B \) so that
\( e_H(A) + e_H(B) \leq (9/16)(3^2 \cdot 2^4 \cdot 139)^{1/8} n^3 < 1.94 \delta^{1/64} n^3 \).

## 2. Preliminaries

In this section, we state some results that will be useful in our proof of Theorem 1.2.
Several of the lemmas come from the paper of Keevash and Sudakov [9].

Let \( K_4^{(3)} \) be the complete 3-uniform hypergraph on four vertices, which is sometimes
called tetrahedron. The Turán number \( \text{ex}(n, K_4^{(3)}) \) is still not known, and we shall use
the following upper bound.

**Lemma 2.1 (Chung and Lu [3]).** Let \( H \) be a 3-uniform hypergraph on \( n \geq n_0 \) vertices
with at least \( (3 + \sqrt{17})/12 + o(1)) \binom{n}{3} \approx 0.594 \binom{n}{3} \) hyperedges. Then \( H \) contains a copy
of \( K_4^{(3)} \).

Numerical computations associated with a semidefinite program (see Razborov [11]
and [5]) lead to a better upper bound, but Chung and Lu’s bound suffices for our
purposes.

We shall also consider \( K^{(3)}(2, 2, 2) \), the 3-uniform hypergraph with six vertices whose
vertex set may be partitioned as \( V_1 \cup V_2 \cup V_3 \), where \( |V_i| = 2 \) for every \( i \in [3] \), and
whose edge set is given by all triples \( e \) such that \( |e \cap V_i| = 1 \) for every \( i \in [3] \). Note
that \( K^{(3)}(2, 2, 2) \) contains eight edges. This hypergraph is sometimes called octahedron.
Moreover, let \( K_\ell \) be the complete graph on \( \ell \) vertices.
For a 3-uniform hypergraph \( H = (V, E) \) and a vertex \( x \in V \), whose neighborhood is denoted by \( N_H(x) \), the link graph \( L(x) = (V', E') \) has vertex set \( V' = N_H(x) \) and edge set \( E' = \{ e-x : x \in e \text{ and } e \in E \} \). Of course, the number of edges of \( L(x) \) is the degree of vertex \( x \) in \( H \). It is convenient to consider several link graphs simultaneously, regarding them as a multigraph. This is a loopless graph in which each edge has some non-negative integral multiplicity. If \( S \subseteq V \) is a set of vertices of \( H \), then the link multigraph \( L(S) \) of \( S \) is the multigraph given by the union of the link graphs of each vertex in \( S \). Let \( G(S) \) be the subgraph of the link multigraph \( L(S) \) obtained by removing \( S \).

**Lemma 2.2.** Let \( H = (V, E) \) be a 3-uniform hypergraph.

(i) Let \( \{x_1, x_2, x_3\} \in E \) be a hyperedge in \( H \) and let \( L(x_i) \) be the link graph of vertex \( x_i \). If \( G(\{x_1, x_2, x_3\}) \) contains four vertices spanning a complete graph \( K_4 \) whose edge set can be partitioned into three matchings \( M_1, M_2, M_3 \) with \( M_i \subseteq L(x_i) \) for all \( i = 1, 2, 3 \), then \( H \) contains a Fano plane.

(ii) If there is a vertex \( x \) whose link graph \( L(x) \) contains three pairwise vertex-disjoint edges \( e_1, e_2, e_3 \) such that all triples \( \{x_1, x_2, x_3\} \), with \( x_i \in e_i \) for \( i \in \{1, 2, 3\} \), are hyperedges in \( H \), then \( H \) contains a Fano plane.

**Lemma 2.3 (Füredi and Kündgen [6]).** Every loopless multigraph on \( n \) vertices, where every four vertices span at most 20 edges, has at most \( 3\binom{n}{2} + n - 2 \) edges.

**Lemma 2.4 (De Caen and Füredi [2]).** Let \( H \) be a 3-uniform hypergraph which contains a copy of \( K_4^{(3)} \) with vertex set \( S \). Let \( L(S) \) be the link multigraph of \( S \) and let \( G(S) \) be the multigraph obtained from \( L(S) \) by deleting \( S \). If the subgraph \( G(S) \) contains a set of four vertices spanning at least 21 edges, then \( H \) contains a Fano plane.

Parts of the following lemma are from [9].

**Lemma 2.5.** Let \( H = (V, E) \) be a 3-uniform hypergraph which contains a copy of \( K_4^{(3)} \) with vertex set \( S \). Let \( E' \) be the set of edges in \( G(S) \) of multiplicity at least 3.

(i) If \( E' \) contains a copy of \( K_4 \) that has at least one edge of multiplicity 4, then \( H \) contains a Fano plane.

(ii) If \( E' \) contains a copy of \( K_5 \), then \( H \) contains a Fano plane.

(iii) If \( G(S) \) contains a copy of \( K_4 \) with vertex set \( \{y_1, y_2, y_3, y_4\} \), where the edges \( \{y_1, y_2\} \) and \( \{y_1, y_3\} \) both have multiplicity 4, and edge \( \{y_2, y_3\} \) has multiplicity at least 3, and the other three edges have in some order multiplicities at least 4, 3, 2, then \( H \) contains a Fano plane.

(iv) If \( G(S) \) contains a copy of \( K_4 \) with vertex set \( \{y_1, y_2, y_3, y_4\} \), where edge \( \{y_1, y_2\} \) has multiplicity 4 and edge \( \{y_3, y_4\} \) has multiplicity at least 2, and edge \( \{y_1, y_3\} \) has multiplicity at least 2 and edge \( \{y_2, y_3\} \) has multiplicity 4, and the remaining edges have multiplicities at least 3, then \( H \) contains a Fano plane.

(v) If \( G(S) \) contains a copy of \( K_4 \) with vertex set \( \{y_1, y_2, y_3, y_4\} \), where edges \( \{y_1, y_2\} \) and \( \{y_1, y_3\} \) have multiplicities 4 and edge \( \{y_2, y_3\} \) has multiplicity at least 3, and edge \( \{y_1, y_4\} \) has multiplicity 4 and edge \( \{y_2, y_4\} \) has multiplicity at least 2 and edge \( \{y_3, y_4\} \) has multiplicity at least 1, then \( H \) contains a Fano plane.

**Proof.** We prove parts (i), (iv) and (v), the others are similar. For part (i), partition the edge set of the copy of \( K_4 \) into three matchings \( M_1, M_2, M_3 \) of size two and assume
that $M_3$ contains an edge with multiplicity 4. Since the edges of $M_1$ and $M_2$ have multiplicity at least three, we may find distinct vertices $x_1, x_2 \in S$ such that the edges of $M_1$ lie in $L(x_1)$ and the edges of $M_2$ lie in $L(x_2)$. The edge with least multiplicity in $M_3$ must contain a vertex $x_3 \notin \{x_1, x_2\}$. The other edge has multiplicity 4 and hence contains $x_3$. We apply Lemma 2.2 (i) to get a copy of a Fano plane.

For part (iv), let $S = \{u_1, u_2, u_3, u_4\}$ and let $L(u_i)$ be the link graph for $u_i$, $i = 1, 2, 3, 4$. Partition the edge set of the copy of $K_4$ with vertex set $\{y_1, y_2, y_3, y_4\}$ into three edge-disjoint matchings $M_1, M_2, M_3$. Let $J_i$ be the indices of the link graphs that contain both edges of $M_i$. Then, in some order, we have the following lower bounds on the sizes of the $J_i$: $4 + 2 - 4 = 2$, and $2 + 3 - 4 = 1$, and $4 + 3 - 4 = 3$. The result follows by Lemma 2.2 (i) and Hall’s theorem.

The proof of part (v) is similar to that of part (iv). Namely, let $S = \{u_1, u_2, u_3, u_4\}$ and let $L(u_i)$ be the link graph for $u_i$, $i = 1, 2, 3, 4$. Partition the edges of the copy of $K_4$ on $\{y_1, y_2, y_3, y_4\}$ into three edge-disjoint matchings $M_1, M_2, M_3$. Let $J_i$ be the indices of the link graphs that contain both edges of $M_i$. Then, in some order, we have the following lower bounds on the sizes of the $J_i$: $4 + 1 - 4 = 1$, and $2 + 4 - 4 = 2$, and $4 + 3 - 4 = 3$. The result follows by Lemma 2.2 (i) and Hall’s theorem.

Lemma 2.6. For every fixed $0 < \alpha < 1/6$, there is a constant $n_0$ such that the following holds for every integer $n \geq n_0$. Let $H = (V, E)$ be a 3-uniform hypergraph with $|V| = n$ and $|E| \geq \alpha n^3$. Then, the number of copies of $K^{(3)}(2, 2, 2)$ in $H$ is at least

$$c' \cdot \alpha^8 n^6 = \frac{37}{211} \cdot \alpha^8 n^6.$$

Proof. Let $\alpha < 1/6$ and $H$ as in the statement of the lemma, fix $H_0 = H$ and consider the following deletion process starting with $j = 0$. While there is a vertex $v_i \in V(H_j)$ whose degree satisfies $d_{H_j}(v_i) \leq n^{3/2}$, define $H_{j+1} = H_j - v_i$. Let $H' = (V', E')$ be the hypergraph obtained at the end of this process, where $V' = \{v_1, \ldots, v_{n'}\}$. Assume that $xn$ vertices have been deleted upon reaching termination, where $0 \leq x \leq 1$. Then we must have

$$xn^\frac{5}{2} + \binom{(1 - x)n}{3} \geq \alpha n^3,$$

which implies, for $n \geq 4/\alpha^2$,

$$\frac{(1 - x)n^3}{6} \geq \frac{\alpha n^3}{2},$$

thus $x \leq 1 - (3\alpha)^{1/3}$. So $H' = (V', E')$ has $|V'| = n' \geq (3\alpha)^{1/3}n$ vertices and at least

$$\alpha n^3 - (1 - (3\alpha)^{1/3})n^\frac{5}{2} \geq \frac{\alpha n^3}{2}$$

hyperedges, for $n \geq 4/\alpha^2$. Let $\alpha' = \alpha/2$, hence $|E'| \geq \alpha'n^3$.

For any real number $y > 1$, set $\binom{y}{2} = \frac{y(y-1)}{2}$. For $y \geq 2a$, we will use the inequality

$$a \cdot \binom{y}{2} \geq \frac{y^2}{4a}.$$

We shall give a lower bound on the number of 4-cycles in the link graphs $L(v)$ for all $v \in V'$. Consider the link graph $L(v_i)$ of vertex $v_i \in V'$ in $H'$, so that $L(v_i)$ contains
Let $d_i = \deg_{H'}(v_i)$ edges, $i = 1, \ldots, n'$. Let $d_i(v_j)$ be the degree of vertex $v_j \in V'$ in the link graph $L(v_i)$, $j = 1, \ldots, n'$, where, for convenience, we assume that $v_i$ lies in $L(v_i)$ with $d_i(v_i) = 0$. The number of (unordered) pairs of edges incident with vertex $v_j$ in $L(v_i)$, that is the number of pairs of triples $\{v_i, v_j, v\} \in E'$, is $\left(\frac{d_i(v_j)}{2}\right)$. Summing over all $j$ and using the convexity of the function $f(x) = \left(\frac{x}{2}\right)$, i.e., $\sum_{i=1}^\ell \left(\frac{x_i}{2}\right) \geq \ell \cdot \left(\frac{\sum_{i=1}^\ell x_i}{\ell}\right)$, the total number of pairs of edges in $L(v_i)$ incident with vertex in $V(L(v_i))$ is

$$\sum_{j=1}^{n'} \left(\frac{d_i(v_j)}{2}\right) \geq n' \cdot \left(\frac{\sum_{j=1}^{n'} d_i(v_j)}{n'}\right) = n' \cdot \left(\frac{2d_i}{n}\right) \geq n \cdot \left(\frac{2d_i}{n}\right) \geq \frac{d_i^2}{n}. \quad (2)$$

To count the number $m_i(C_4)$ of 4-cycles in the link graph $L(v_i)$, consider the expression

$$2 \cdot m_i(C_4) = \sum_{\{v_j, v_k\} \in [V(L(v_i))]^2} \left|N_{L(v_i)}(v_j) \cap N_{L(v_i)}(v_k)\right|.$$

By an averaging argument, we obtain

$$2 \cdot m_i(C_4) \geq \left(\begin{array}{c} n' \\ 2 \end{array}\right) \cdot \left(\frac{\sum_{\{v_j, v_k\} \in [V(L(v_i))]^2} |N_{L(v_i)}(v_j) \cap N_{L(v_i)}(v_k)|}{\left(\begin{array}{c} n' \\ 2 \end{array}\right)}\right)$$

$$= \left(\begin{array}{c} n' \\ 2 \end{array}\right) \cdot \left(\frac{\sum_{i=1}^{n'} \frac{d_i(v_i)}{2}}{\left(\begin{array}{c} n' \\ 2 \end{array}\right)}\right)$$

$$\geq \left(\begin{array}{c} n' \\ 2 \end{array}\right) \cdot \left(\frac{d_i^2/n}{\left(\begin{array}{c} n' \\ 2 \end{array}\right)}\right) \geq \frac{d_i^4}{2n^4}. \quad (3)$$

Inequality (3), together with $\sum_{i=1}^{n'} d_i \geq 3\alpha' n^3$ and the convexity of the function $g(x) = x^4$, leads to

$$\sum_{i=1}^{n'} m_i(C_4) \geq \sum_{i=1}^{n'} \frac{d_i^4}{4n^4} \geq \frac{n'}{4n^4} \cdot \left(\frac{\sum_{i=1}^{n'} d_i}{n'}\right)^4 \geq \frac{1}{4n^4} \cdot \frac{(3\alpha' n^3)^4}{n^3} \geq \frac{3^4}{4} \cdot \alpha' \cdot n^5. \quad (4)$$

Notice that the same cycle $C_4$ in the link graphs $L(v_i)$ and $L(v_j)$, $i \neq j$, gives a copy of $K^{(3)}(2, 2, 2)$ in $H'$. We may now use an averaging argument to find a lower bound on the number of copies of $K^{(3)}(2, 2, 2)$ in $H'$ (hence in $H$). If both link graphs $L(v_i)$ and $L(v_j)$ have a copy of a $C_4$ on the vertex set $w_i, w_j, w_k, w_\ell$, the copies must be the same in order to produce a copy of $K^{(3)}(2, 2, 2)$ in $H$. For fixed vertices $w_i, w_j, w_k, w_\ell$, let $C_4(\{w_i, w_j; w_k, w_\ell\})$ be the number of link graphs in $L(v_1), \ldots, L(v_n')$ that contain a copy of $C_4$ with vertex set $\{w_i, w_j, w_k, w_\ell\}$ such that the bipartition is $\{w_i, w_j\} \cup \{w_k, w_\ell\}$. As we sum over all possible copies of $C_4$ in the sum below, since every copy
of $K^{(3)}(2, 2, 2)$ is counted three times, we have
\[
\frac{1}{3} \sum_{(i,j,k,\ell) \in [n']^4} \left( \binom{C_4(\{w_i, w_j; w_k, w_\ell\})}{2} + \binom{C_4(\{w_i, w_k; w_j, w_\ell\})}{2} + \binom{C_4(\{w_i, w_\ell; w_j, w_k\})}{2} \right)
\geq \frac{1}{3} \sum_{(i,j,k,\ell) \in [n']^4} 3 \left( \binom{\sum_{m(C_4)}}{2} \right)
\geq \frac{1}{3} \cdot 3 \left( \frac{n'}{4} \right) \cdot \left( \frac{\sum_{m(C_4)}}{2} \right)
\geq \frac{3^7}{23} \cdot \alpha^8 \cdot n^6
\]
\[(\alpha' = \alpha/2) \geq \frac{3^7 \cdot \alpha^8 \cdot n^6}{2^{11}}, \]
as claimed. \qed

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Our argument follows the steps of [9, Theorem 1.2], but we adjust the constants in a way that gives us a better dependency between $\delta$ and $\epsilon$. No attempt was made to find good bounds on $n_0$.

Proof of Theorem 1.2 Throughout the proof we assume that $n$ is sufficiently large and $\delta > 0$ is fixed and sufficiently small. With foresight, we fix $\delta < (1/36)^8$ and, using $c' = 3^7/2^{11}$ from Lemma 2.6, we also fix
\[
\delta_1 = (5/3)\delta^{1/2},
\delta_2 = \delta^{1/2},
\delta_3 = \delta_2^{1/2},
\delta_4 = \delta_1^{1/2},
\delta_5 = 6\delta_3 = 6\delta_1^{1/2},
\delta_6 = 18\delta_2^{1/2},
\delta_7 = \sqrt{2006}\delta_3^{1/8},
\delta_8 = 90\delta_4^{1/4},
\delta_9 = 417\delta_5^{1/8},
\delta_{10} = 9\delta_3 = 9\delta_1^{1/2},
\delta_{11} = (27/64)(834/c')^{1/8}\delta_1^{1/64} < 1.94\delta_1^{1/64}.
\]
Let $H$ be a 3-uniform Fano-free hypergraph on $n \geq n_0$ vertices whose number of hyperedges is at least $\text{ex}(n, \text{Fano}) - \delta n^3/8$. If $H$ contains a vertex of degree smaller than $(1 - \delta_1)3n^2/8 + 3n$, then we delete this vertex and continue this process as long as there are vertices of degree less than $(1 - \delta_1)3n^2/8 + 3n$. If we had deleted $\delta_2 n - 4$ vertices, then we would have arrived at a hypergraph on $(1 - \delta_2)n + 4$ vertices with at
least

\[
\text{ex}(n, \text{Fano}) - \delta n^3/8 - (\delta_2 n - 4)((1 - \delta_1)3n^2/8 + 3n) \\
\geq (1 - 2\delta - 3\delta_2(1 - \delta_1)) \text{ex}(n, \text{Fano}) + 12(1 - \delta_1)n^2/8 - 3\delta_2 n^2 + 12n
\]

hyperedges, for \( n \) sufficiently large. We claim that, for \( n \) sufficiently large, for the choice

\[ \delta_1 = (5/3)\delta^{1/2} \text{ and } \delta_2 = \delta^{1/2} \] this is larger than

\[
\text{ex}((1 - \delta_2)n + 4, \text{Fano}),
\]

which would imply that there is a Fano plane in \( H \), a contradiction. We will show that

\[
1 - 2\delta - 3\delta_2(1 - \delta_1) > (1 - \delta_2)^3,
\]

which will confirm the claim for \( n \) sufficiently large. Now (5) is equivalent to

\[
-2\delta + 3\delta_1\delta_2 - 3\delta_2^2 + \delta_3^2 > 0.
\]

By our choice of \( \delta_1, \delta_2 \) by (3) this is equivalent to

\[
\delta_3^2 > 0,
\]

as claimed. Let \( V_0 \) be the subset of \( V \) with all vertices that have not been deleted in this process. Hence all degrees in the subhypergraph \( H[V_0] \) induced by \( V_0 \) are at least

\[
(1 - \delta_1)3n^2/8 + 3n.
\]

Our discussion implies that \( n_0 = |V_0| \geq n - \delta^{1/2}n + 4 \). Note that the number of hyperedges of \( H \) that are not in \( H[V_0] \) is at most \( \delta_2 n(1 - \delta_3)3n^2/8 + 3\delta_2 n^2 \leq 3\delta^{1/2}n^3/8 \), for sufficiently large \( n \). Since \( \delta < 1/400 \), the subhypergraph \( H[V_0] \) contains at least

\[
\text{ex}(n, \text{Fano}) - \delta n^3/8 - 3\delta^{1/2}n^2/8 \geq 0.6 \cdot \left( \frac{n_0}{3} \right)
\]

hyperedges for sufficiently large \( n \), and by Lemma 2.1 it contains a copy of the complete hypergraph \( K_4^{(3)} \) with vertex set \( S = \{a, b, c, d\} \). Let \( L(a), L(b), L(c) \) and \( L(d) \) be the link graphs of the vertices in \( S \). The number of edges in \( L(S) \) that are incident with at least one vertex in \( S \) is at most \( 12n \), as any vertex \( x \in V \) is incident with at most three vertices of \( S \) in each of the four link graphs. Let \( V_1 = V_0 \setminus S \) and \( n_1 = |V_1| \geq (1 - \delta_2)n \).

The link multigraph \( G = G(S) \) contains at least \( e(L(a)) + e(L(b)) + e(L(c)) + e(L(d)) - 12n \geq 4[(1 - \delta_1)3n^2/8 + 3n] - 12n \geq (1 - \delta_1)3n^2/2 \) edges. Theorem 1.2 is a consequence of the following structural result for the multigraph \( G \).

Lemma 3.1. There is a partition of \( S \) into two sets of size two, say \( \{a, b\} \) and \( \{c, d\} \), for which the following holds. There exists a partition \( V_1 = A \cup B \) of the vertex set \( V_1 \) such that all but at most \( \delta_0 n_1^2 \) pairs \( \{v, w\} \subset V_1 \) satisfy the following:

(i) Every pair contained in \( A \) is an edge in both link graphs \( L(a) \) and \( L(b) \) but not an edge in \( L(c) \) or \( L(d) \).

(ii) Every pair contained in \( B \) is an edge in both link graphs \( L(c) \) and \( L(d) \) but not in \( L(a) \) or \( L(b) \).

(iii) Every crossing pair between \( A \) and \( B \) is an edge in all four link graphs \( L(a), L(b), L(c) \) and \( L(d) \).

Moreover, the partition satisfies

\[ |A|, |B| \geq \frac{n_1}{4}. \]
Before proving Lemma 3.1, we show that it implies the desired result Theorem 1.2.

Let \( V_1 = A' \cup B' \) denote the partition given by Lemma 3.1. Define a partition \( A \cup B \) of \( V \) by adding each vertex of \( V \setminus V_1 \) to \( A' \) or \( B' \) arbitrarily. We choose \( \delta_{11} \) satisfying \( \delta_{11} = (27/64) (2 \delta_9/c')^{1/8} \), where \( c' \) is given in Lemma 2.6.

To obtain the required bound \( e(A) + e(B) < 2 \delta_{11} n_1^3 \), we suppose, for a contradiction, that one of the classes, say \( A \), contains at least \( \delta_{11} n_1^3 \) hyperedges of \( H \). By the definition of \( V_1 \), at least

\[
\delta_{11} n_1^3 - 3 \delta_2 n^3 / 8 \geq \frac{\delta_{11} n_1^3}{2^{\delta_2}} \geq \frac{\delta_{11} n_1^3}{2^{\delta_2}} \cdot |A'|^3 \geq \frac{64 \delta_{11} n_1^3}{27 \cdot 2^{\delta_2}} \cdot |A'|^3
\]

of these hyperedges lie in \( A' \). This expression cannot be greater than \( \binom{|A'|}{3} \), so that Lemma 2.6 applies and by (14), \( H[A'] \) contains at least

\[
\frac{c' (27/64)^8 \cdot 2^{\delta_2} : n_1^{24}}{2 |A'|^{24}} \cdot |A'|^6 \binom{|A'|}{2} \geq \frac{\delta_9 n_1^2 |A'|^4}{16}
\]

copies of \( K^{(3)}(2, 2, 2) \). If we find three vertex-disjoint edges \( e_1, e_2, e_3 \) in \( L(a) \) such that \( e_1, e_2, e_3 \) form three classes of the partition of some copy of \( K^{(3)}(2, 2, 2) \), then \( H \) contains a Fano plane by Lemma 2.2 (ii), a contradiction. On the other hand, by Lemma 3.1 (i) only at most \( \delta_9 n_1^2 \) pairs of vertices in \( A' \) are not edges of \( L(a) \). Clearly, these pairs can be contained in the vertex sets of at most \( \delta_9 n_1^2 \binom{|A'|}{4} \leq \delta_9 n_1^2 |A'|^4 / 24 \) copies of \( K^{(3)}(2, 2, 2) \), thus there is a copy of \( K^{(3)}(2, 2, 2) \) in \( H[A'] \) such that all three pairs in the partition of its vertex set belong to \( L(a) \), yielding a copy of a Fano plane by Lemma 2.2 (ii), which is the desired contradiction.

**Proof of Lemma 3.1.** We shall construct a sequence \( V_1 \supseteq V_2 \supseteq \cdots \supseteq V_5 \) of subsets of the vertex set \( V_1 \) of \( H[V_1] \), where \( |V_j| = n_j, j = 1, \ldots, 5 \), and \( V_1 \) is the set containing \( n_1 \geq n - \delta_2 n \) vertices with degree at least \((1 - \delta_1)3n^2 / 8\) that has been defined in the first part of the proof.

For \( S = \{ a, b, c, d \} \), consider the link multigraph \( G(S) \) on \( n_1 \geq (1 - \delta_2)n \) vertices with at least \((1 - \delta_1)3n_1^2 / 2\) edges. If \( G \) contains a vertex of degree less than \((1 - \delta_3)n_1 \), then we delete this vertex. As before, we continue this deletion process until no vertex satisfies this property. If we had deleted \( \delta_3 n_1 \) vertices, then we would have arrived at a multigraph with at least

\[
(1 - \delta_1 - 2 \delta_4 (1 - \delta_3)) 3n_1^2 / 2
\]

edges. We claim that for \( \delta_3 = \delta_4 = \delta_1^{1/2} \) this is at least

\[
3 [(1 - \delta_4)n_1]^2 / 2.
\]

Indeed, the inequality

\[
1 - \delta_1 - 2 \delta_4 (1 - \delta_3) \geq (1 - \delta_4)^2
\]

is equivalent to

\[
- \delta_1 + 2 \delta_3 \delta_4 - \delta_4^2 \geq 0,
\]

which holds by our choice of \( \delta_3 = \delta_4 = \delta_1^{1/2} \). By Lemma 2.3, we find four vertices spanning at least 21 edges, hence by Lemma 2.4, we have a Fano plane, a contradiction.
Thus, we deleted at most \( \delta_4 n_1 = \sqrt{(5/3)\delta_1^{1/4}} n_1 \) vertices. This produces a subset \( V_2 \subseteq V_1 \) with \( n_2 = |V_2| \geq n_1 - \sqrt{(5/3)\delta_1^{1/4}} n_1 \). We deleted at most \( \delta_4 n_1 (1 - \delta_3) 3n_1 \leq 3\delta_1^{1/2} n_1^2 \) edges from \( G \). By construction, all degrees in the subgraph \( G[V_2] \) are at least \( (1 - \delta_3) 3n_1 \geq (1 - \delta_3) 3n_2 \). We shall refer to this inequality as the degree condition.

Next, we distinguish two cases according to whether the subgraph \( G[V_2] \) contains three vertices spanning at most ten edges or not. As it turns out, the second case will lead to a contradiction.

**Case 1:** Suppose that every three vertices in \( G[V_2] \) span at most ten edges. In \( G[V_2] \) there must be some edge of multiplicity 4; otherwise, denoting the number of edges of multiplicity 3 in \( G[V_2] \) by \( e_3 \), the degree condition leads to

\[
2 \left( \frac{n_2}{2} - e_3 \right) + 3e_3 \geq (1 - \delta_3) 3n_2^2/2,
\]

which implies \( e_3 \geq (1 - 3\delta_3) n_2^2/2 \). Then, by Turán’s theorem for \( \delta_3 < 1/12 \), the subgraph \( G[V_2] \) contains a complete graph \( K_5 \) whose edges have multiplicity at least 3. By Lemma 25(ii) the hypergraph \( H \) must contain a Fano plane, a contradiction.

Let \( \{p, q\} \) be an edge of multiplicity 4. As we are in case 1, for each vertex \( r \) in \( G[V_2] \) there are at most six edges between \( r \) and \( \{p, q\} \). Furthermore, by the degree condition, there are at least \( (1 - \delta_3) 6n_2 - 8 \) edges between \( \{p, q\} \) and \( V_2 - \{p, q\} \) in \( G[V_2] \).

We partition \( V_2 - \{p, q\} \) into four sets \( A, B, C, D \) such that for each vertex \( x \) in \( A \) the edge \( \{x, p\} \) has multiplicity 4 and edge \( \{x, q\} \) has multiplicity 2, for each vertex \( x \) in \( B \) the edge \( \{x, p\} \) has multiplicity 2 and edge \( \{x, q\} \) has multiplicity 4 and for each \( x \) in \( C \) both edges \( \{x, p\} \) and \( \{x, q\} \) have multiplicity 3. For each vertex \( x \) in \( D \) the sum of the multiplicities of the edges \( \{p, x\} \) and \( \{q, x\} \) is at most 5.

In the following we will show that the sizes of both sets \( C \) and \( D \) are small.

Let \( a_i \) be the number of vertices in \( V_2 - \{p, q\} \) that are connected to \( \{p, q\} \) via \( i \) edges. Note that, because we are in case 1, \( i \in \{0, \ldots, 6\} \). We have \( \sum_{i=0}^{6} a_i = n_2 - 2 \), and

\[
5(n_2 - 2 - a_6) + 6a_6 \geq \sum_{i=0}^{6} i \cdot a_i \geq (1 - \delta_3) 6n_2 - 8,
\]

which is equivalent to

\[
a_6 \geq (1 - 6\delta_3)n_2 + 2.
\]

Hence, \( \sum_{i=0}^{5} a_i \leq 6\delta_3 n_2 = 6\sqrt{(5/3)\delta_1^{1/4}} n_2 \), and we set \( \delta_5 = 6\delta_3 = 6\delta_1^{1/2} = 6\sqrt{(5/3)\delta_1^{1/4}} \). This implies that the set \( D \) in the partition contains at most \( \delta_5 n_2 \) vertices. Let \( V_3 = V_2 - D \). Note that \( n_3 = |V_3| \geq (1 - \delta_5)n_2 \) and that \( e(G[V_3]) \geq e(G[V_2]) - 4\delta_5 n_2^2 \).

Now we consider the set \( C \). If the subgraph \( G[C] \) contains an edge \( \{x, y\} \) of multiplicity at least 3, then \( p, q, x, y \) form a \( K_4 \) in \( G[V_3] \) satisfying the hypothesis of Lemma 25(i), so \( H \) must contain a Fano plane, a contradiction. Thus subgraph \( G[C] \) contains only edges of multiplicity at most 2. Moreover, if there is an edge \( \{x, y\} \) with multiplicity at least 2 between \( A \cup B \) and \( C \), then \( p, q, x, y \) form a \( K_4 \) as in Lemma 25(iv), and \( H \) must contain a Fano plane, a contradiction. Therefore, all edges between \( C \) and \( A \cup B \) have multiplicity at most 1. Since we are in case 1, every edge inside \( A \) or \( B \) has multiplicity at most 2. Thus, the maximum possible number of edges in \( A \cup B \)
is achieved when every edge inside $A$ or $B$ has multiplicity 2 and every edge with one vertex in $A$ and one vertex in $B$ has multiplicity 4. Therefore,

$$\frac{3}{2}n_1^2 - 29.5\delta_1^{1/2}n_2^2$$

$$\leq (1 - \delta_1)\frac{3}{2}n_1^2 - 3\delta_4n_1^2 - 4\delta_5n_2^2$$

$$\leq e(G[V_3])$$

$$\leq 2\left(\frac{|A|}{2}\right) + 2\left(\frac{|B|}{2}\right) + 4|A||B| + 2\left(\frac{|C|}{2}\right) + |C||A \cup B|.$$  \hspace{1cm} (9)

For fixed size $|A| + |B|$, by taking the derivative, \(9\) is maximum for $|A| = |B| = m$. Then, with $|C| = n_3 - 2m$, the term \(9\) is at most

$$6m^2 - 2m|C| + |C|^2 - |C| = 6m^2 - 2mn_3 + n_3^2 - n_3 \leq 6m^2 - 2mn_1 + n_1^2,$$

and therefore with \(9\)

$$\frac{3n_1^2}{2} - 29.5\delta_1^{1/2}n_2^2 \leq 6m^2 - 2mn_1 + n_1^2,$$

thus, for $n$ sufficiently large,

$$m \geq \frac{n_1}{6} + \frac{n_1}{3} \cdot \sqrt{1 - 45\delta_1^{1/2} + 1}.$$

Using $\sqrt{1 - x} \geq 1 - 3x/5$ for $0 \leq x \leq 5/9$, with $\delta_1^{1/2} \leq 1/81$, we infer

$$m \geq \frac{n_1}{2}(1 - 18\delta_1^{1/2}) + 1,$$

thus $|C| \leq 18\delta_1^{1/2}n_1$. Let $\delta_6 = 18\delta_1^{1/2}$. We produce $V_4$ by deleting all vertices in $C$, which leads to the deletion of at most $4\delta_5n_2^2$ edges incident with them. For the sake of simplicity, we also delete $p$ and $q$ from $G[V_3]$. Therefore $G[V_4]$ has $n_4 \geq n_3 - 18\delta_1^{1/2}n_1$ vertices.

Now we show that $A$ and $B$ satisfy the conditions of the lemma (we note that, near the end of the proof, each vertex in $C \cup D \cup (V \setminus V_1)$ will be added arbitrarily to $A$ or $B$). Note that to obtain $G[V_4]$, at most

$$(3\delta_1^{1/2} + 4\delta_5 + 4\delta_6)n_1^2 \leq 128\delta_1^{1/4}n_1^2$$  \hspace{1cm} (10)

edges have been removed from $G = G[V_1]$.

Let $E'$ be the set of all pairs of vertices within $A$ or $B$ of multiplicity at most 1 and pairs of vertices between $A$ and $B$ of multiplicity at most 3. Since $|A| + |B| = n_4$, we deduce

$$(1 - \delta_1)\frac{n_1^2}{2} \leq e(G[V_4]) \leq 2\left(\frac{|A|}{2}\right) + 2\left(\frac{|B|}{2}\right) + 4|A||B| + 128\delta_1^{1/4}n_1^2 - |E'|$$

$$\leq 3\left(\frac{|A| + |B|}{2}\right) - \frac{(|A| - |B|)^2}{2} + 128\delta_1^{1/4}n_1^2 - |E'|$$

$$\leq 3\left(\frac{n_4}{2}\right) - \frac{(|A| - |B|)^2}{2} + 128\delta_1^{1/4}n_1^2 - |E'|$$

$$\leq 3\left(\frac{n_1}{2}\right) - \frac{(|A| - |B|)^2}{2} + 128\delta_1^{1/4}n_1^2 - |E'|.$$
or, for $\delta < 4^{4/5}$,
\[
\frac{(|A| - |B|)^2}{2} + |E'| \leq \left(\frac{3}{2} \delta_1 + 128 \delta^{1/4}\right) n_1^2 \leq 130 \delta^{1/4} n_1^2.
\]
Therefore, we infer $|E'| \leq 130 \delta^{1/4} n_1^2$ and with $\delta_7 = \sqrt{260} \delta^{1/8}$ also
\[
||A| - |B|| \leq \delta_7 n_1,
\]
which, combined with $|A| + |B| = n_4$, implies
\[
\frac{n_4}{2} - \frac{\delta_7 n_1}{2} \leq |A|, |B| \leq \frac{n_4}{2} + \frac{\delta_7 n_1}{2}.
\]
(11)
To find a lower bound on $|A|$ in terms of $n_1$, we first note that
\[
n_4 \geq n_3 - 18 \delta_1^{1/2} n_1 \\
\geq (1 - \delta_5)n_2 - 18 \delta_1^{1/2} n_1 \\
\geq (1 - \delta_5)[n_1 - \delta_1^{1/2} n_1] - 18 \delta_1^{1/2} n_1 \\
\geq n_1 - 25 \delta_1^{1/2} n_1.
\]
With (11), this leads to
\[
|A| \geq \frac{n_4}{2} - \frac{\sqrt{65}}{2} \delta^{1/8} n_1 \\
\geq \frac{n_1}{2} - \frac{25}{2} \delta^{1/2} n_1 - \sqrt{260} \delta^{1/8} n_1/2 \\
= \frac{n_1}{2} - \left(\frac{25}{2} \sqrt{\frac{5}{3}} \delta^{1/8} + \sqrt{65}\right) \delta^{1/8} n_1 \\
\geq \frac{n_1}{2} - 9 \delta^{1/8} n_1.
\]
(12)
We used that $\delta < (1/36)^{8}$, which also leads to
\[
|A| \geq \frac{n_1}{2} - 9 \delta^{1/8} n_1 \geq \frac{n_1}{4}.
\]
(13)
The same applies to $|B|$. Then, since $|A| + |B| \leq n_1$, inequality (13) implies
\[
\frac{n_1}{4} \leq |A|, |B| \leq \frac{3n_1}{4}.
\]
(14)
By (10), for all but at most
\[
|E'| + 128 \delta^{1/4} n_1^2 \leq 258 \delta^{1/4} n_1^2
\]
pairs $\{x, y\}$ of vertices $x, y \in V_1$, the multiplicity of $\{x, y\}$ in $G[V_4]$ is equal to 2, if $\{x, y\} \subset A$ or $\{x, y\} \subset B$, or equal to 4, if $x \in A$ and $y \in B$. It remains to show that most pairs $\{x, y\} \subset A$ are edges in the same two link graphs and most pairs $\{x, y\} \subset B$ are edges in the two other link graphs.
Let $d(x)$ be the degree of vertex $x$ in $G[V_4]$. For a vertex $x \in A$, let $B(x) \subseteq B$ be the set of vertices in $B$ joined to $x$ by an edge of multiplicity 4. For a vertex $u \in B$, let
A(u) ⊆ A be defined correspondingly. Since \(|A|\) and \(|B|\) are each at most \(n_4/2 + \delta \tau n_1\) and at least \(n_4/2 - \delta \tau n_1\), and their sum is \(n_4\), we have

\[
(1 - \delta_3)3n_2 - 4(\delta_5 + \delta_6)n_1 < d(x)
\]

\[
< 2|A| + 4|B| - (|B| - |B(x)|)
\]

\[
\leq 2(n_4/2 - \delta \tau n_1/2) + 4(n_4/2 + \delta \tau n_1/2) - (|B| - |B(x)|)
\]

\[
= 3n_4 + \delta \tau n_1 - (|B| - |B(x)|),
\]

so

\[
|B| - |B(x)| \leq (3\delta_3 + 4\delta_5 + 4\delta_6 + \delta \tau)n_1
\]

\[
\leq (3\delta_1^{1/2} + 24\delta_1^{1/2} + 72\delta_1^{1/4} + 17\delta_1^{1/4})n_1
\]

\[
\leq (27\delta_1^{1/4} + 89)\delta_1^{1/4}n_1 \leq 90\delta_1^{1/4}n_1,
\]

and hence \(|B| - |B(x)| < \delta_8 n_1\), where \(\delta_8 = 90\delta_1^{1/4}\). This means that every vertex \(x \in V_4\) is incident with at most \(\delta_8 n_1\) vertices of the other side with edges that do not have multiplicity 4.

Now fix an edge \(\{x, y\}\) of multiplicity 2 inside \(A\). Without loss of generality assume that it lies in the edge set of both \(L_A(a)\) and \(L_A(b)\), where set subscripts indicate subgraphs of the link graph induced by the corresponding set. At this point, we would know that most almost all pairs of vertices in \(A\) are edges of multiplicity 2 and that the same holds for \(B\). Since \(|B| - |B(x) \cap B(y)| < 2\delta_8 n_1\), we can delete all the vertices in \(B - (B(x) \cap B(y))\) and assume that \(B = B(x) \cap B(y)\). Now no edge \(\{w, z\}\) of \(B\) can be in \(L_B(a)\), as then partitioning the edges of a copy of the complete graph \(K_4\) on the vertex set \(\{w, x, y, z\}\) as \(M_a = \{\{x, y\}, \{w, z\}\}, M_b = \{\{w, x\}, \{y, z\}\}, M_c = \{\{w, y\}, \{x, z\}\}\) gives a Fano plane by Lemma \(2.2\) (i). Similarly no edge in \(B\) can be in \(L_B(b)\).

Now let \(\{u, v\}\) be an edge of multiplicity 2 in \(B\). Then, as we just proved, it lies in the edge set of both \(L_B(c)\) and \(L_B(d)\). So arguing as above, and deleting the at most \(2\delta_8 n_1\) vertices in \(A - (A(u) \cap A(v))\), we can assume that all vertices in \(A\) are adjacent to both \(u\) and \(v\) by edges of multiplicity 4, and therefore no edge in \(A\) can be in \(L(c)\) or \(L(d)\). More formally, the above vertex-deletions produce a graph \(G_5\) on \(n_5 \geq n_4 - 4\delta_8 n_1\) vertices. Now we distribute all the vertices of \(V_1\) that have been deleted up to this point to make \(A \cup B\) a partition of \(V_1\). At this point we find the upper bound \(\delta_8 n_1^2\) on the number of pairs of vertices in \(V_1\) that do not satisfy the conditions of Lemma \(3.1\) by summing the upper bounds (15) and \(4\delta_8 n_1^2\) on the number of pairs deleted from \(G\), that is,

\[
258\delta_1^{1/4}n_1^2 + 4\delta_8 n_1^2 \leq 258\delta_1^{1/4}n_1^2 + 4 \cdot 90\delta_1^{1/4}n_1^2
\]

\[
\leq (258\delta_1^{1/8} + 360(5/3)^{1/4}) \delta_1^{1/8}n_1^2
\]

\[
\leq 417\delta_1^{1/8}n_1^2,
\]

and we set \(\delta_0 = 417\delta_1^{1/8}\). This shows that \(A \cup B\) is the required partition, and completes the analysis of Case 1.

Case 2: Suppose next that there exist three vertices \(p, q, r\) in \(G[V_2]\) that span at least 11 edges. Without loss of generality we can assume that \(\{p, q\}\) and \(\{p, r\}\) have multiplicity 4 and \(\{q, r\}\) has multiplicity at least 3. By the degree condition, there are at least \((1 - \delta_3)9n_2 - 24\) edges between \(\{p, q, r\}\) and \(V_2 - \{p, q, r\}\).
By Lemma 2.4 for each vertex $s$ the sum of the multiplicities of all edges from $s \in V_2 - \{p, q, r\}$ to $\{p, q, r\}$ is at most 9. Let $a_i$ be the number of vertices in $V_2 - \{p, q, r\}$ that are connected to $\{p, q, r\}$ via $i$ edges, $i = 0, \ldots, 9$. We have $\sum_{i=0}^{9} a_i = n_2 - 3$, and

$$8(n_2 - 3 - a_9) + 9a_9 \geq \sum_{i=0}^{9} i \cdot a_i \geq (1 - \delta_3)9n_2 - 24,$$

which implies $a_9 \geq (1 - 9\delta_3)n_2$, and $\sum_{i=0}^{8} a_i \leq 9\delta_3n_2 = 9\delta_1^{1/2}n_2$. We set $\delta_1 = 9\delta_1^{1/2}$ and we produce $V_3$ by deleting at most $\delta_1 n_2$ vertices from $V_2 - \{p, q, r\}$ that are each connected to $\{p, q, r\}$ by less than nine edges. These vertices altogether are incident with at most $4\delta_1 n_2^2$ edges. Therefore, $G[V_3]$ has $n_3 \geq n_2 - \delta_1 n_2$ vertices.

By Lemma 2.5 (i) and (iii) no vertex $s \in V_3 - \{p, q, r\}$ can be adjacent to vertices in $\{p, q, r\}$ by edges with multiplicities 3, 3, 3 or 4, 3, 2. Since $V_3$ only contains vertices counted by $a_9$, the degree pattern in $\{p, q, r\}$ of all vertices $s \in V_3 - \{p, q, r\}$ is 4, 4, 1, in particular, the multiplicities of the edges $\{p, s\}$, $\{q, s\}$ and $\{r, s\}$ are 1, 4, 4 in this order, as otherwise, by Lemma 2.5 (v), $H$ would contain a copy of the Fano plane. Suppose that there is some edge $\{s, t\}$ in $G[V_3 - \{p, q, r\}]$ of multiplicity at least 2. Then the vertices $q, r, s, t$ span at least 21 edges, and by Lemma 2.4 $H$ contains a copy of the Fano plane. Thus, all edges in $G[V_3 - \{p, q, r\}]$ have multiplicity at most 1. On the other hand, by the degree condition, the subgraph $G[V_3 - \{p, q, r\}]$ contains at least

$$(1 - \delta_3)3n_2^2/2 - 4\delta_1 n_2^2 - 12n_2 \geq 3n_2^2/2 - 38\delta_1 n_2^2 \geq (3/2 - 38\delta_1^{1/2}) n_2^3$$

edges. For $\delta_1 < (1/38)^2$, this is not possible. Therefore, case 2 always leads to a contradiction. This completes the proof of Lemma 3.1. $\square$

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