A decidability result for the halting of cellular automata in the pentagrid

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Abstract
In this paper, we investigate the halting problem for deterministic cellular automaton in the pentagrid. We prove that the problem is decidable when the cellular automaton starts its computation from a finite configuration and when it has at two states, one of them being a quiescent state.

Keywords: tilings, hyperbolic geometry, cellular automata, halting problem, decidability

1 Introduction

Let us start with generalities about cellular automata. A cellular automaton is defined by two basic objects: the space of its cells and the finite automaton, a copy of which lies in each cell. The space of cells is assumed to be homogeneous enough in order to ensure that each cell has the same number of neighbours. This condition is naturally satisfied if the space of cells is associated to a tiling which is a tessellation based on a single regular tile. Then, each cell is associated to a tile which is called the support of the cell. Each cell has a state belonging to some finite set \( \mathcal{L} \), called the set of states. As \( \mathcal{L} \) is finite, it can be seen as the alphabet used by the finite automaton which equips the cells. The cellular automaton evolves in a discrete time provided by a clock. At time \( t \), each cell updates its state according to the current value of its state at time \( t \) and the values at the same time of the states of its neighbours. These current states constitute the neighbourhood to which the finite automaton associates a new state which will be the current state of the cell at time \( t+1 \). The way with which such an association is performed is called a rule of the cellular automaton. There are finitely many rules constituting the program of the cellular automaton.
A quiescent state is a state ξ such that the cell remains in state ξ if all its neighbours are also in state ξ. The corresponding rule is called the quiescent rule. Usually, we call that state white and it is denoted by W. A configuration at time t is the set of cells which are in a non-quiescent state together with the position of their supports in the tiling. Traditionally, the initial configuration of a cellular automaton is finite. This means that at time 0, the time which marks the beginning of the computation, the set of cells which are in the non-quiescent state is finite. Define the distance of a cell c to a cell d by the smallest number of cells needed to link c to d by a sequence of cells where two consecutive ones are neighbours. Then, define the disc D(c, n) of center c and radius n as the set of cells d whose distance from c is at most n. If we fix a cell c as origin of the space, there is a smallest number N₀ such that the initial configuration is contained in D(c, N₀). This means that all cells outside D(c, N₀) are in the quiescent state. Call such an N₀ the initial border number. The reason of the index 0 will be clear later. Let C(c, n) be the set of cells whose distance from c is exactly n. The definition of N₀ also entails that C(c, N₀) contains at least one non-quiescent state. In this setting, the halting of a cellular automaton is reached by two identical consecutive configurations. Accordingly, there is a number k and a time t such that the configurations at time t and t+1 are both contained in D(c, k) and they are equal.

From now on, when we say cellular automaton, we need to understand deterministic cellular automaton with a quiescent state. The term deterministic means that a unique new state is associated to the current state of a cell and the current states of its neighbours.

From various papers of the author, we know the following on cellular automata in hyperbolic spaces: in the tessellations {5, 4}, {7, 3} and {5, 3, 4}, namely the pentagrid, the heptagrid and the dodecagrid respectively, it is possible to construct weakly universal cellular automata with two states only. In the case of the dodecagrid, the constructed automaton is rotation invariant, we remind the definition in Section 3. In the case of the pentagrid and of the heptagrid, the rules are not rotation invariant. Moreover, in the case of the pentagrid, we assume the Moore neighbourhood, i.e. we assume that the neighbours of the cell are the cells which share at least a vertex with it. It is known that with rotation invariant rules and von Neumann neighbourhood, which means that the neighbours of a cell share a side with it, there is a strongly universal cellular automaton in the pentagrid with ten states, see [4]. This means that the cellular automaton which is universal starts its computation from a finite configuration. If we relax the rotation invariance, there is a weakly universal cellular automaton in the pentagrid with five states. And so, results concerning rotation invariance are also interesting.

Very little is known if we change something in the above assumptions.

The present paper is devoted to the proof of our main result:

**Theorem 1** For any deterministic cellular automaton in the pentagrid, if its initial configuration is finite and if it has at most two states with one of them being quiescent, then its halting problem is decidable.
The proof is split into two propositions dealing first with rotation invariance in Section 3, then when that condition is relaxed, see Section 4. In Section 5 we study what happens in an infinite motion of the cellular automaton when such a motion occurs. In Section 6 we present to the reader a minimal introduction of the pentagrid and of the implementation of cellular automata in that context. Section 8 brings in a few reflections on the topic.

We now turn to hyperbolic geometry and the tiling we consider in which the cellular automata later considered evolves.

2 The pentagrid

In this paper, we use the model of the hyperbolic geometry which is known as the Poincaré’s disc. A disc is fixed, call it the unit disc. Let $D$ be the open unit disc. The model $\mathcal{M}$ of the hyperbolic plane we consider is defined in $D$ which we call the support of $\mathcal{M}$. The points in $\mathcal{M}$ are the points of the open disc. The lines in $\mathcal{M}$ are the traces in $D$ of circles which are orthogonal to $\partial D$, the border of $D$ and the traces in $D$ of straight lines which pass through the centre of $D$.

Figure 1 represents a line $\ell$ and a point $A$ out of $\ell$. The figure also shows us four lines which pass through $A$. The line $s$ cuts $\ell$ and is therefore called a secant with $\ell$. The lines $p$ and $q$ touch $\ell$ on $\partial D$. The points $P$ and $Q$ where, respectively, $p$ and $q$ touch $\ell$ are called points at infinity of the hyperbolic plane but do not belong to that plane. The lines $p$ and $q$ are called parallel to $\ell$. At last, but not the least, the line $m$ does not cut $\ell$ and it also does not touch it neither in $D$ nor on its border, nor outside $D$. The line $m$ is called non-secant with $\ell$. It is proved that two lines of the hyperbolic plane are non-secant if and only if they have a unique common perpendicular.

![Figure 1](image.png)

**Figure 1** Poincaré’s disc model of the hyperbolic plane. Here, the various relations between a line, a point out of the line with other lines passing through the point.
A theorem by Poincaré tells us that there are infinitely many tessellations in the hyperbolic plane whose basic tile is a triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$ provided that the positive numbers $p$, $q$, and $r$ satisfy

$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < 1,$$

which simply means that the triangle with these angles lives in the hyperbolic plane. As a consequence, if we consider $P$ the regular convex polygon with $p$ sides and with interior angle $\frac{2\pi}{q}$, $P$ tiles the plane by recursive reflections in its sides and in the sides of its images if and only if

$$\frac{\pi}{p} + \frac{\pi}{q} < \frac{1}{2}.$$

When this is the case, the corresponding tessellation is denoted by $\{p, q\}$.

We call the tessellation $\{5, 4\}$ which is illustrated by Figure 2 the pentagrid. In [1, 2], it is proved that the pentagrid is spanned by a tree. The left-hand side of Figure 3 shows us a quarter of the pentagrid spanned by the tree illustrated on the right-hand side of the same figure. That tree is called the Fibonacci tree. The reason of this name comes from the properties of the tree. The tree is a finitely branched tree generated by two rules:

$B \rightarrow BW$ and $W \rightarrow BW$. \hspace{1cm} (RF)

Indeed, we split the nodes into two kinds: black nodes and white nodes. Black nodes have two sons, as suggested by the above rules, a black son, the left-hand side son, and a white son, the right-hand side one. White nodes have three sons, a black son, the leftmost one, and two white sons, the others. The root of the tree is a white node. It is not difficult to prove, see [1, 2] from the
above rules that there are exactly \( f_{2n+1} \) nodes lying on the \( n^{th} \) level of the tree, where \( f_k \) is the Fibonacci sequence where \( f_0 = f_1 = 1 \).

There is another, more striking property. Number the nodes of the tree, starting from the root, which receives 1, and level after level and, on each level from left to right. Then, represent these numbers in the Fibonacci sequence, choosing the one whose number of digits is the biggest. Then, if \([n]\) is that Fibonacci representation of \( n \), the black son of the node \( n \) has the number represented by \([n00]\) if \( n \) is attached to a black node and the middle son of the node \( n \) has the number with the same representation if \( n \) is attached to a white node. This property is called the preferred son property which can be checked on the right-hand side picture of Figure 3.

Figure 3 To left: A sector of the pentagrid generated by the Fibonacci tree illustrated to right. In the right-hand side picture: under each node, vertically, we represented the Fibonacci representation of the number attached to the node. We can check the preferred son property.

In the sequel, we shall also call Fibonacci tree a tree whose root is a black node and to which the rules \((RF)\) are applied to its sons and, recursively, to the sons of its sons. In such a Fibonacci tree, the number of nodes on the \( n^{th} \) level of the tree is \( f_{2n} \).

Note that in Figure 2 a tile seems to play a different role than the others. It is the tile which contains the centre of the support of \( D \). As can be seen in Figure 2 not much can be seen from the tiling. We can very well see the central tile, also well its neighbours, but going further from the central tile, we can see the tiles less and less. In fact, as the hyperbolic plane has no centre, the pentagrid too has no tile playing a central role. We can view the support of our model as a window over the hyperbolic plane. We can imagine that we fly over that plane, that the window is a screen on the board of our space craft. The centre of that window is simply the point of the hyperbolic plane over which our space craft is flying. Indeed, we fly with instruments only, those which we just defined. This window property of the Poincaré’s disc stresses that so little can be represented of this space in its Euclidean models. It is the reason why
we choose the disc model.

We fix a tile \( \tau_0 \) which we call from now on the **central tile** and we shall consider that the central tile is the tile in which the centre of \( D \) lies in the figures. As illustrated by the left-hand side of Figure 4 around the central tile, we can assemble five quarters as those defined in the left-hand side picture of Figure 3 in order to construct the whole pentagrid. We call these quarters **sectors**. In each sector, the tiling is spanned by the Fibonacci tree. It is not difficult to prove that the tiles which lie on the level \( k \) of a Fibonacci tree of a sector are at the distance \( k \) from the central tile. We call Fibonacci **circle** of level \( n \) the set of tiles \( C(\tau_0, n) \) denoted by \( \mathcal{F}_n \). Similarly, we call Fibonacci **discs** the sets \( D(\tau_0, n) \) which we denote by \( \mathcal{D}_n \). Note that \( \mathcal{D}_n \) is the union of the Fibonacci circles \( \mathcal{F}_k \) with \( 0 \leq k \leq n \). In Figure 4 we illustrate the notion of Fibonacci circles and discs by marking in blue, green and gray the tiles which belong to \( \mathcal{F}_3 \) and by marking in pink those which belong to \( \mathcal{D}_2 \).

We call the Fibonacci representation we attached to the number given to a node \( \nu \) of a Fibonacci tree the **coordinate** of \( \nu \) denoted by \( [\nu] \). We identify the node with its number \( \nu \) and sometimes by \( [\nu] \). We locate the tiles of the pentagrid with 0 for the central tile and for the other tiles with two numbers: the number of the sector in which the tile lies and the number of the node in the Fibonacci tree which spans the sector, as clear from Figure 4. We extend the coordinate of a tile by appending the number of its sector. We shall also say that the central tile is the support of the **central cell**. Again, the central cell is the cell on which we focus our attention at the given moment of our argumentation.

![Figure 4](image)

*Figure 4* To left, how sectors are assembled around the central cell in order to get the pentagrid. To right, the Fibonacci circle of level 3. Together with the tiles of the Fibonacci circle, the tiles in pink, i.e. the central cell and the tiles of levels 0 and 1 constitute the Fibonacci disc of level 2.

These considerations allow us to implement cellular automata in the pentagrid as performed in [1, 3]. As mentioned in the introduction, to each tile we
associate a cell of the cellular automaton. We shall also identify the cell by the coordinates of its support, or its number depending on the context. If \( \eta \) is the state of the cell attached to the tile \( \nu \), we say that \( \nu \) is also an \( \eta \)-cell. In order to note the rules of a deterministic cellular automaton in the pentagrid, we introduce a numbering of the sides of each tile. The numbering starts from 1 and it is increased by 1 for each side while counterclockwise turning around the tile. For the central cell, side 1 is fixed once and for all and for the other tiles, side 1 is the side of the tile shared with its father, the central cell being the father of the root of the tree. Neighbour \( i \) of a cell \( \nu \) shares with \( \nu \) the side \( i \) of \( \nu \). The precision is required because the side shared by two tiles do not receive the same number in both tiles.

If \( \eta_0 \) is the current state of the cell, if \( \eta_1^0 \) is its new state and if \( \eta_i \) is the state of its neighbour \( i \) at the current time, then the rule giving \( \eta_1^0 \) from \( \eta_0 \) and the \( \eta_i \)’s is written as a word in \( \{\mathcal{L}\}^* \), where \( \mathcal{L} \) is the set of states of the cellular automaton: \( \eta_0\eta_1...\eta_5\eta_1^0 \). The underscore is put under \( \eta_0 \) and \( \eta_1^0 \) in order to facilitate the reading. In a rule \( \eta_0\eta_1...\eta_5\eta_1^0 \), we say that \( \eta_0\eta_1...\eta_5 \) is the context of the rule and we say that the word \( \eta_1...\eta_5 \) is the state neighbourhood of the cell.

### 3 Rotation invariant cellular automata in the pentagrid with two states

By definition, the rules of a cellular automaton \( A \) in the pentagrid are said to be invariant by rotation, in short rotation invariant and \( A \) is said to be a rotation invariant cellular automaton, if for each rule present in the program of \( A \), namely, \( \eta_0\eta_1...\eta_5\eta_1^0 \), the rules \( \eta_0\eta_\pi(1)...\eta_\pi(5)\eta_1^0 \) are also present, where \( \pi \) runs over the circular permutations on \([1..5]\). When the cellular automaton is rotation invariant, we usually indicate the rule where, after the current state, we have the state of neighbour 1.

The goal of this section is to prove

**Proposition 1** *For any deterministic cellular automaton in the pentagrid, if its initial configuration is finite, if it has two states with one of them being quiescent, and if its rules are rotation invariant, then its halting problem is decidable.*

Our proof is based on the following considerations. If the halting of the computation of a cellular automaton halts, it means that the computation remains in some \( D_N \) for ever. Note that the computation may remain within some \( D_N \) and not halt. But in that case, after a certain time, the computation becomes periodic. And this can be detected: it is enough to find two identical configurations during the computation: this generalizes the situation of the halting. What is not that easy to detect is the case when the configuration extends to infinity in that sense that for each circle \( F_k \), there is a time when that circle contains a non quiescent cell.
Let us closer look at such a case. Let \( N_0 \) be the initial border number. We know that there is at least one tile \( \nu \) of \( \mathcal{F}_{N_0} \) which is a B-cell, at time 0. Call \( \mathcal{F}_{N_0} \) the front at time 0. The front at time \( t \) is \( \mathcal{F}_{N_t} \), where \( N_t \) is the smallest \( k \) such that \( \mathcal{F}_k \) contains all configurations at time \( \tau \), with \( \tau \leq t \), and such that all cells outside \( D_k \) are quiescent. This is the reason why the initial border number is denoted by \( N_0 \): \( \mathcal{F}_{N_0} \) is the front at time 0.

Our proof of Proposition \( \Box \) lies on the analysis of how a B-cell on the front at time \( t \) can propagate to the front at time \( t+1 \). If we can prove that \( N_t \) is a non-decreasing function of \( t \) which tends to infinity, we then prove that the computation of the cellular automaton does not halt. The main property which will allow us to detect such a situation is that a cell on \( \mathcal{F}_{n+1} \), has at most two neighbours on \( \mathcal{F}_n \) and the others on \( \mathcal{F}_{n+2} \). So that if \( \nu \) is node of the front which is a B-cell, the state neighbourhood of its sons is either \( B\omega^4 \) or \( B\omega^3 \). That situation occurs if and only if the node \( \nu-1 \) of the front is also a B-cell. We say that a B-cell is isolated on \( \mathcal{F}_n \) if \( \nu \) being its support, \( \nu-1 \) and \( \nu+1 \) are both \( \omega \)-cells. These considerations significantly reduce the number of rules to consider and, consequently, the number of cases to scrutinize. More precisely, we have the following lemma.

**Lemma 1** Let \( A \) be a deterministic cellular automaton on the pentagrid with two states, one being quiescent, and whose rules are rotation invariant. If the rule \( B\omega^4 \) occurs in the program of \( A \), the front at time \( t+k \) is the same as the front at time \( t+1 \) for \( k \geq 2 \), i.e. \( N_{t+k} = N_{t+1} \) for the same values of \( k \). If it is not the case, i.e. if the rule \( B\omega^3 \) occurs in the program of \( A \), then if the front at time \( t \) contains a B-cell, the front at time \( t+1 \) also contains a B-cell, i.e. we have \( N_{t+1} = N_{t+1} \).

Proof of the lemma. Let \( \nu \) be the tile of \( \mathcal{F}_{N_0} \), which is a B-cell. Assume that \( \nu \) is an isolated B-cell of the front at time \( t \). Let \( \sigma \) be a son of \( \nu \). Whether \( \sigma \) is a black node or a white one, \( \sigma \) is a \( \omega \)-cell as well as its sons. Accordingly, its state neighbourhood is \( B\omega^4 \) so that the rule \( B\omega^4 \) applies. Consequently, \( \sigma \) remains a \( \omega \)-cell at time \( t \).

If \( \nu+1 \) is also a B-cell at time \( t \). Let \( \sigma \) be the black son of \( \nu+1 \). Then, the state neighbourhood of \( \nu \) is \( B\omega^3 \). If the program of \( A \) contains the rule \( B\omega^3 \), then \( \sigma \) remains a \( \omega \)-cell at time \( t+1 \) as well as the other sons of \( \nu \). If the program contains the rule \( B\omega^3 \), then \( \sigma \) becomes a B-cell at time \( t+1 \) but the cells \( \sigma-1 \) and \( \sigma+1 \) are white nodes, so that whatever the state of their father, they remain \( \omega \)-cells at time \( t+1 \) as either the quiescent rule or the rule \( B\omega^4 \) applies to them. Accordingly, in that case, the cell \( \sigma \) is an isolated B-cell of the front at time \( t+1 \). Now, from what we proved in the previous paragraph shows us that the sons of \( \sigma \) remain \( \omega \)-cells at the time \( t+1 \) so that the front at time \( t+2 \) is the same as at time \( t+1 \) and it remains the same afterwards. This proves the part of the lemma concerning the rule \( B\omega^4 \).

Now, assume that the rule \( B\omega^3 \) occurs in the program of \( A \). From our previous study on the sons of \( \nu \), at least one of them is a white node which means that its neighbourhood is \( B\omega^4 \). Accordingly, if \( \nu \) is a B-cell, that white son becomes a B-cell at the next time, so that \( \mathcal{F}_{N_{t+1}} = \mathcal{F}_{N_{t+1}} \). \( \Box \)
We are now in position to prove Proposition 1. If the initial configuration is empty, \( i.e. \) if all tiles are \( W \)-cells at time 0, there is nothing to prove: the configuration remains empty for ever. Accordingly, if the initial configuration is not empty, \( N \) is definite, so that \( F_N \) contains at least one \( B \)-cell. From Lemma 1, if the rule \( W_BW^4B \) occurs in the program of \( A \), the front moves by one step forward at each time, so that the computation of the cellular automaton does not halt.

If that rule does not occur then, necessarily, the rule \( W_BW^4W \) is present in the program of \( A \). From Lemma 1, we know that at most, we have \( F_{t+1} = F_1 \) but that necessarily, \( F_t = F_1 \) for \( k \geq 1 \).

4 When the rules are not rotation invariant

Here again, we deal with a deterministic cellular automaton with a quiescent state which starts its computation from a finite configuration. But in this section, we relax the assumption of rotation invariance. The convention we fixed in Section 2 for the numbering of the sides of a tile have their full meaning in this section. And so, a rule \( \eta_0 \eta_1 \ldots \eta_5 \eta_1 \) may be different from a rule \( \eta_0 \eta_{\pi(1)} \ldots \eta_{\pi(5)} \eta_1 \) where \( \pi \) is a permutation over \([1..5]\). Note that this time, the order of the letters in the state neighbourhood associated to the rule is meaningful.

Consider a cell \( \nu \in F_{n+1} \). In all cases, its neighbour 1 is its father which by construction belongs to \( F_n \). If \( \nu \) is a black node, as already noticed in previous sections, \( \nu \) has two neighbours exactly which belong to \( F_n \): neighbour 1, as it is the father and also neighbour 2. Consider \( N_0 \) the initial border number. From what we just noticed, a rule can make a state \( B \) move from \( F_{N_0} \) to \( F_{N_0+1} \) if its state neighbourhood starts with \( BW \), \( WB \) or \( B2 \): the last two cases may happen if the considered cell of \( F_{N_0+1} \) is a black node. As an example, the state neighbourhood of the tile \( \nu \) of \( F_{N_0+1} \) cannot be \( WWBWW \); if a rule whose state neighbourhood is \( WWBWW \) is applied to a cell of \( F_n \), its neighbour which is a \( B \)-cell belongs to \( F_{N_0+1} \).

Lemma 2 Let \( A \) be a deterministic cellular automaton on the pentagrid with two states where one of them is a quiescent state. If the program of \( A \) contains the rule \( WBW^4B \) and the rule \( WBBW^3W \), then \( F_{N_1+k} = F_{N_1} \) for all positive integer \( k \) with \( k \geq 2 \).

Proof of the lemma. The proof comes from the fact that the state neighbourhood of a the son of a node \( \nu \) which is an isolated \( B \)-cell of the front is \( BW^4 \). If \( \nu \) is a \( B \)-cell and if \( \nu+1 \) is a \( W \)-cell, then the state neighbourhood of the leftmost son of \( \nu+1 \) is \( WBW^3 \). Now, the sons of an isolated \( B \)-cell of the front at time \( t \) remain quiescent at time \( t+1 \). If the program of \( A \) contains the rule \( WB^2W^3B \), then if the state pattern \( BB \) is present on the front at time \( t \), say at the nodes \( \nu \) and \( \nu+1 \), then the just mentioned rule applies to the leftmost son \( \sigma \) of \( \nu+1 \), but the white sons of \( \nu \) and those of \( \nu+1 \) remains \( W \)-cells at time \( N_t+2 \). Accordingly, \( \sigma \) is an isolated \( B \)-node of the level \( N_t+1 \) so that, from the just previous study, all cells on \( F_{N_t+2} \) remain quiescent, so that \( F_{N_t+k} = F_{N_t+2} \) for all positive integer \( k \).
Clearly, the same conclusion holds if the program of \( A \) contains the rule \( \text{WB}^2\text{W}^3\text{W}^3 \).
\[
\square
\]

Let \( \nu_1, \ldots, \nu+k \) a sequence of consecutive nodes on the circle \( F_n \). Then, the word \( \eta_{1}\eta_{k} \) with \( \eta_i \in \{ \text{B},\text{W} \}, i \in \{1..k\} \), is called a state pattern.

| Rule | State Pattern |
|------|---------------|
| \text{BW} | \text{WB}^2\text{B}\text{W}^3\text{W}^3 | \text{WB}^4\text{W}^3 | \text{WB}^4\text{W}^3 | \text{WB}^4\text{W}^3 |
| \text{WB} | \text{WB}^2\text{W}^3\text{B} | \text{WB}^2\text{W}^3\text{W} | \text{WB}^2\text{W}^3\text{W} | \text{WB}^2\text{W}^3\text{W} |
| \text{BB} | \text{WB}^2\text{W}^3\text{B} | \text{WB}^2\text{W}^3\text{W} | \text{WB}^2\text{W}^3\text{W} | \text{WB}^2\text{W}^3\text{W} |

Let us now prove Theorem 1. From Lemma 2, the computation remains within \( D_{N_1+2} \) if the program of \( A \) contains both rules \( \text{WB} \) and \( \text{WB} \). Accordingly, we may assume that it contains either the rule \( \text{WB} \) or the rule \( \text{WB} \).

Consider the case when the rule \( \text{WB} \) belongs to the program of \( A \). If a node \( \nu \) of the front at time \( t \) is a B-cell, from the proof of Lemma 2 we know that there is also a B-cell on the front at time \( t+1 \) and that we have \( N_{t+1} = N_t+1 \) as any white son of \( \nu \) is a B-cell on \( F_{N_t+1} \) at time \( N_t+1 \).

Consider the case when the rules \( \text{WB} \) and \( \text{WB} \) belong to the program of \( A \). If a node \( \nu \) of the front at time \( t \) is a B-cell we have to look at the case when the state pattern \( \text{WB} \) is present on the front or not. As by definition the front contains at least one B-cell, if the state pattern \( \text{WB} \) is not present, this means that all tiles of the front are B-cells. In that case, all black nodes of \( F_{N_t+1} \) have the state neighbour \( \text{B}^2\text{W}^3 \). Accordingly, as the white nodes of \( F_{N_t+1} \) remain quiescent at time \( t+1 \), the evolution depends on the rule whose context is \( \text{WB}^2\text{W}^3 \).

If the rule is \( \text{WB} \), then the black nodes of \( F_{N_t+1} \) remain quiescent at time \( t+1 \), which entails that all nodes of \( F_{N_t+1} \) remain quiescent at time \( t+1 \). Now, if at time \( t+1 \) at least one node of \( F_{N_t} \) at time \( t \) becomes a \( \text{W} \)-cell at time \( t+1 \) and at least one remains a B-cell, then the pattern \( \text{WB} \) occurs, say on the nodes \( \nu \) and \( \nu+1 \). Then, if \( \sigma \) is the black node of \( \nu+1 \), its state neighbourhood at time \( t+1 \) is \( \text{WB}^2\text{W}^3 \), so that the rule \( \text{WB} \) applies and \( \sigma \) becomes a B-cell at time \( t+2 \). From the rule \( \text{WB} \), we know that the white sons of the B-cells on \( F_{N_t} \) remain \( \text{W} \)-cells and so the black nodes on \( F_{N_{t+1}} \) are isolated B-cells on the level \( N_t+1 \). Accordingly, the rule \( \text{WB} \) applies to their black sons on \( F_{N_{t+2}} \) at time \( t+2 \). The argument applies again to those nodes which are also isolated B-cells on the new front. So that \( F_{N_{t+k}} = F_{N_t+k} \) for all positive integer \( k \).

We remain with the situation when all the nodes of the front at time \( t \) are B-cells and all of them become \( \text{W} \)-cells at time \( t+1 \). We can repeat the above analysis to time \( t+2 \). If at that time all nodes are again B-cells, say that this situation is an alternation of B and \( \text{W} \). If such a situation is repeated long enough, as in that case the front does not go beyond \( D_{N_t+1} \), the computation remains for ever within that disc and so the computation is periodic. We know that such an evolution can be detected: it is enough to observe two identical configurations. If this is not the case, we find a situation where the front contains the state

Table 1

Rules of a deterministic cellular automaton on the pentagrid with two states, \( \text{W} \) being the quiescent state, which apply to the sons of a node of the front.
pattern BW so that the rule WB applies endlessly as already seen.

If the program of A contains the rule BB, as it also contains the rule BW, the state pattern BW occurs on \( F_{N_t+1} \) at time \( t+1 \) as soon as the pattern BB occurs on \( F_{N_t} \) at time \( t \). If the pattern BB does not occur, clearly, the pattern BW occurs on \( F_{N_t} \) at time \( t \), so that we have the same conclusion as we had with the rule BB when the pattern BW occurs on the front: a non-halting computation which is detected by the occurrence of that pattern.

We can summarize the discussion as follows:

**Table 2** Table of the evolutions of the computation of A depending on its rules of Table 1 and on the patterns which can be seen on the front.

| rules | front  | evolution                        |
|-------|--------|----------------------------------|
| BW    | a B-cell | \( N_{t+1} = N_t+1 \) from some \( t_0 \) |
| BW, WB | any     | within \( D_{N_0} \) from some \( t_0 \) |
| BW,WB | a BW    | \( N_{t+1} = N_t+1 \) from some \( t_0 \) |
| WB,WB | never BW | within \( D_{N_0} \) from some \( t_0 \) |

Accordingly, as we have analyzed all possible cases each of one can be detected, we conclude that the proof of Theorem 1 is completed.

\( \square \)

5 Propagation of the front

Although we solved the question about the halting problem for such cellular automata, it can be interesting to examine their behaviour in the case when the computation does not halt with an unbounded occurrence of non quiescent cells. We shall focus on the front. Up to now, we have seen that a motion to infinity exactly means that the front is increasing starting from some time \( t_0 \). This happens in different settings as shown by Table 2. It could be interesting to have more information about such a motion. However, as the situation may be intricate in some cases as can be seen in the proof of Theorem 1 when the rules are BW, WB and BB, we shall restrict our attention to what happens on the front. We shall see that with two states only the study of this restricted aspect is not that trivial.

From Table 2 we know that we basically have to consider two cases: the case when the program of A contains the rule BW and the case when it contains the rule BW together with the rule WB in the case that the pattern BW appears at some time on the front.

Consider that latter case. From the proof of Theorem 1 we know that if the pattern BW appears on the front at time \( t \), it will also appear on the front at time \( t+1 \). But the proof has given us a more exact information. If \( \nu \) is the node of the B-cell of some BW pattern of the front at time \( t \), the application of the rule WB to the black son \( \sigma \) of \( \nu+1 \) produces a BW patterns on the nodes \( \sigma \) and \( \sigma+1 \) as \( \sigma+1 \) remains a quiescent cell due to the fact that \( \nu+1 \) is a W-cell and that \( \sigma+1 \) is a white node. And so, \( \sigma \) and \( \sigma+1 \) define a BW pattern on the
front at time $t+1$. Remark that the B-cell of the pattern BW on $F_{N_{t+1}}$ is isolated. The same arguments can be repeated to the black son of $\sigma+1$. Consequently, a pattern BW where the B-cell is isolated on the front at time $t$ generates a sequence of such patterns on each front at time $t+k$, with $k$ being a positive integer, the B-cell of such a pattern being isolated and being the black son of the W-cell of the same pattern on the previous front. We can call this sequence a line of patterns BW. Accordingly, if there are $k$ patterns BW on the front at time $t$, each of them generates a line of patterns BW on the successive fronts after time $t$.

From now on, consider the case when the program of $A$ contains the rule BW.

If a B-cell occurs on the front at time $t$ on the node $\nu$, the white sons of $\nu$ become B-cells at time $t+1$, as seen in the proof of Theorem 1. Accordingly, not only the front at time $t+1$ contains a B-cell, it also contains the pattern BB. If $\nu+1$ is a W-cell, its black son is a B-cell if and only if the program of $A$ contains the rule WB. In that case the front at time $t+1$ contains the pattern BBB. The occurrence of the pattern BB on the front raises the question of which rule BB or WB belongs to the program of $A$.

The easiest case to analyze is the case when together with the rule BW we also have the rules WB and BB.

**Figure 5** The program contains the rules BW, WB and BB. From left to right, times 0, 1, 2, 3 and 4. It is assumed that once a node is a B-cell, it remains in this situation. The light pink cells represent the circles which are behind the front at time $t$.

In the pictures of Figure 5, the result of applying the rules BW, WB and BB, respectively, yields the cells in blue, purple and green, respectively. Clearly, the white neighbour of the purple neighbour of the central cell is its father, see the pictures for times 1, 2, 3 and 4. It is assumed that the state B is permanent: once a cell gets that state, it remains unchanged. The figure also assumes that we start from a single B-cell on the front at time 0. That cell is placed as the central cell of the Figures 5, 6 and 7 in order to focus the attention on the evolution of the computation from that cell.

In Figure A, contrarily to Figure 5, it is assumed that a B-cell at time $t$ becomes quiescent and remains in that state later on. Note that this representation allows us to better see the propagation of the front in the case of the motions ruled by the occurrence of the rule BW in the program of $A$. As we assume that the rules WB and BB also belong to the program of $A$, we can easily see that the sons of a B-cell in node $\nu$ at time $t$ are B-cells at time $t+1$ whatever the states at time $t$ of the nodes $\nu-1$ and $\nu+1$. Accordingly, on the front at time $t+k$ the B-cells occupy at least the whole level $k$ of the Fibonacci tree rooted at $\nu$, whether $\nu$ is a black node or a white one.
The program contains the rules $BW$, $WB$ and $BB$. From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a $B$-cell, it becomes a quiescent cell at the next time.

Still assume that we have the rules $BW$ and $WB$, but that we have the rule $BB$.

The program contains the rules $BW$, $WB$ and $BB$. From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a $B$-cell, it becomes a quiescent cell at the next time.

Figure 7 illustrates the propagation of the front in that case, starting with a single $B$-cell on the front at the initial time. The picture at time 3 in that figure indicates that the pattern $BBB$ appears on the front of that time and the picture at time 4 seems to indicate the same property and that no new pattern appears. Let us prove this property.

**Lemma 3** Let $A$ be a deterministic cellular automaton on the pentagrid with two states, one of them being quiescent. Assume that the program of $A$ contains the rule $BW$. The states of the cells attached to the white sons of a white node $\nu$ at time $t+1$ are the state of the cell of $\nu$ at time $t$.

Proof of the lemma. Indeed, let $\nu$ be a cell of the front at time $t$ supported by a white node. Its white sons belong to $F_{Nt+1}$ and, as white nodes, they have one neighbour on $F_{Nt}$ and four of them on $F_{Nt+2}$. At time $t$, those four neighbours are $W$-cells by definition of the front at time $t$, so that the quiescent rule applies if $\nu$ is a $W$-cell and the rule $BW$ applies if $\nu$ is a $B$-cell. In both cases, we get the conclusion of the lemma.

This lemma shows that among the sons of a white node on the front, two of them always have the same state at the next time. We can now state:

**Lemma 4** Let $A$ be a deterministic cellular automaton on the pentagrid with two states, one of them being quiescent. Assume that the program of $A$ contains the rules $BW$, $WB$ and $BB$. Then, the front at time $t$, with $t \geq 3$ does not contain neither the pattern $WBW$ nor the pattern $BBBB$.  

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Proof of the lemma. Assume that the pattern $WBW$ occurs at time $t+2$. Let $\nu$, $\nu+1$ and $\nu+2$ be the nodes supporting that pattern. From the rules and from Lemma 3, the nodes $\nu$, $\nu+1$ and $\nu+2$ cannot have the same father which should be a white node. Accordingly, the fathers of $\nu+1$ and $\nu+2$ are different, say $\varphi+1$ and $\varphi+2$. Assume that $\varphi+2$ is the father of $\nu+1$ and $\nu+2$, so that $\nu+1$ is a black node and $\nu+2$ is a white one. Also, $\nu$ must be a white son of $\varphi+1$. By Lemma 3 both $\varphi+1$ and $\varphi+2$ should be $W$-cells, so that by the quiescent rule, $\nu+1$ should be a $W$-cell, a contradiction with our assumption. And so, we have that $\varphi+1$ is the father of $\nu$ and $\nu+1$ and that $\varphi+2$ is the father of $\nu+2$.

Then, $\varphi+1$, which we consider on the front at time $t+1$ cannot be a white node as its white sons would bear different states at time $t+2$. So $\varphi+1$ is a black node and $\nu$ is its black son and $\nu+1$ is its white one. Accordingly, $\varphi+1$ is a $B$-cell at time $t+1$. As $\nu+2$ is a $W$-cell at time $t+2$, $\varphi+2$ must also be a $B$-cell at time $t+1$. Let us look at what happens at time $t+3$. Let $\sigma$ be the white son of $\nu$ which is a black node. Then, the rule $W$ apply to $\sigma$, $\sigma+1$, $\sigma+2$ and $\sigma+3$ producing the pattern $WBBB$. Now, the rule $WB$ applies to $\sigma+4$ as that node is a black one, and by Lemma 3 $\sigma+5$ is quiescent, so that starting from $\sigma$, the sons of $\nu$, $\nu+1$ and $\nu+2$ produce the state pattern $WBBBW$ at time $t+3$. Applying the rules in a similar way, at time $t+4$, starting from the rightmost son of $\sigma$, we obtain the pattern $WBBBBWBBBBW$ where the rightmost $W$ is the first white son of $\sigma+5$.

Now, consider the case of a pattern $WBBBB$ on the front at time $t+1$, and let $\nu$ be the node which gives the leftmost $W$ at that time. By Lemma 3 $\nu$, $\nu+1$ and $\nu+2$ cannot be the sons of a white node $\nu$. A similar contradiction would occur if we assume that $\nu$ and $\nu+1$ are sons of a black node. We conclude that $\nu$ is the rightmost son of a node $\varphi$. If we assume that $\nu+2$ and $\nu+3$ are the sons of $\varphi+1$ which should accordingly be a black node supporting a $B$-cell, we have a contradiction between the state of $\varphi+2$ at time $t$, which would be a white node and that of $\nu+3$ and $\nu+4$ at time $t+1$. Accordingly, $\varphi+1$ must be a white node and $\nu+4$ and $\nu+5$ are sons of $\varphi+2$, so that we find the situation associated with the pattern $WBW$.

We have seen that the pattern produced by the sons of the nodes supporting $WBBBB$ does not contain neither $WBW$ nor $BBBB$. It contains four occurrences of $BB$, two of them being separated by a single $W$.

Now, let us look at the pattern $WBBW$ which we assume to be on the front at time $t+1$. Let $\nu$ be the node which supports the left-hand side $W$ of the pattern. The nodes $\nu$, $\nu+1$ and $\nu+2$ can be the sons of a white node $\varphi+1$ which is necessarily a $B$-cell at time $t$. As $\nu$ is a $W$-cell at time $t+1$, $\varphi$ must be a $B$-cell at time $t$. This indicates which kind of nodes are $\nu$, $\nu+1$ and $\nu+2$ and, clearly, $\nu+3$ is a black node. Applying the rules to the sons of these nodes, we get that, from $\sigma$ is the rightmost son of $\nu$ until the leftmost white son of $\nu+3$, the nodes $\nu+i$ produce the pattern $WBBBBWBBW$ at time $t+2$.

However, $\nu+1$, $\nu+2$ and $\nu+3$ cannot be the sons of a white node as $\nu+2$ and $\nu+3$ have different states. Another disposition for the fathers of the node we have seen is that the father of $\nu$, say $\varphi$, is a black node, so that $\varphi+1$ is a white one. Necessarily, $\varphi$ is a $B$-cell at time $t$ and $\varphi+1$ is a white one. Looking at the
sons of the nodes \( \nu, \nu+1, \nu+2 \) and \( \nu+3 \), starting from the rightmost son \( \sigma \) of \( \nu \) until the leftmost white son of \( \nu+3 \) we find this time the pattern \text{BBBWBBW}.

Let us now consider the pattern \text{WBBBW} on the front at time \( t+2 \).

Again, let \( \nu \) be the node which supports the leftmost \( \text{W} \)-cell of this pattern. We can see that the nodes \( \nu, \nu+1 \) and \( \nu+2 \) can be the sons of a node \( \varphi \) which must be a \( \text{B} \)-cell at time \( t \) while the node \( \varphi+1 \) must be a \( \text{W} \)-cell at the same time. It is not difficult to see that under that assumption on \( \varphi \) and \( \varphi+1 \) with respect to the nodes \( \nu+i \), if \( \sigma \) is the rightmost son of \( \nu \), we get the pattern \text{BBBWBBWBBW} on the front at time \( t+2 \) until the leftmost white son of \( \nu+4 \).

If \( \nu \) and \( \nu+1 \) would be the sons of a black node \( \varphi \) while the other nodes \( \nu+i \) would be the sons of the necessarily white node \( \varphi+1 \). The nodes \( \nu+3 \) and \( \nu+4 \) have different colours at time \( t+1 \), a contradiction with Lemma 3.

And so, another configuration which this time is possible, is that the nodes \( \nu+i \) we consider have three fathers: \( \varphi \) is the father of \( \nu \) only, \( \varphi+1 \) is a white node or a black one, respectively, it does not matter, and \( \varphi+2 \) is the father of \( \nu+4 \), or of \( \nu+3 \) and \( \nu+4 \), respectively. In both cases, \( \nu+1 \) is a black node, it is the important point. It can be checked that in both cases, if \( \sigma \) is the rightmost son of \( \nu \), the pattern on the front at time \( t+2 \) starting from \( \sigma \) and ending on the first white son of \( \nu+4 \) is \text{BBBWBBWBBW}.

With this analysis, the proof of Lemma 4 is completed. \( \square \)

We have analyzed the situation when the program of \( A \) contains the rules \text{BW} and \text{WB}. With programs containing the rule \text{BB}, we remain to consider the case of the rule \text{BB}. Figure 8 illustrates the propagation of the front whatever the rule \text{BB} or \text{BB}.

**Figure 8** The program contains the rules \text{BW} and \text{WB}. In green and lighter dark blue the \( \text{B} \)-cells produced when using the rule \text{BB} too. When the program contains the rule \text{BB}, the \( \text{B} \)-cells are restricted to the dark blue cells. From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a \( \text{B} \)-cell, it becomes a quiescent cell at the next time.

In fact, the figure illustrates both cases: as mentioned in the caption of the figure, a different coloration is applied to the cells produced directly by the application of the rule \text{BB} or to further applications of all rules in the tree rooted at the node where a first application of the rule \text{BB} was performed. When the rules \text{BW}, \text{WB} and \text{BB} are applied starting from an isolated \( \text{B} \)-cell supported by a node \( \nu \) on the front at time \( t \), the evolution of the computation concerns the Fibonacci tree rooted at \( \nu \) and on the front at time \( t+k \), the trace of that computation is the whole level \( k \) of that tree. Say that a node \( \nu \) is **hereditary white** if there is a sequence of \( k \) white nodes \( \nu_i \), with \( i \in \{1..k\} \) such that \( \nu_{i+1} \) is a white son of \( \nu_i \) with \( i \in \{1..k-1\} \) and \( \nu = \nu_k \). When the rule \text{BB} is used in
place of the rule BB, the trace is restricted to hereditary white nodes only.

We can summarize our analysis by appending the Table 3 to Table 2. The table assumes that we start from a B-cell supported by a white node of the front at time $t$. In order to better analyze the patterns, we remind the reader that the number of nodes on the level $k$ of a Fibonacci tree rooted at a white node, a black node, respectively, is $f_{2k+1}$, $f_{2k}$, respectively.

**Table 3** Patterns on the front at time $t+k$ when the program of $A$ contains the rule $BW$ starting from a B-cell in a white node of the front at time $t$.

| rules  | patterns at $t+k$                                                                 |
|--------|----------------------------------------------------------------------------------|
| BW, WB, BB | WB$^{f_{2k+1}+f_{2k-2}}$                                                          |
| BW, WB, BB | WBBW, WBBBW in a range wider than $f_{2k+1} + f_{2k-2}$ nodes                  |
| BW, WB, BB | B on hereditary white nodes in a range wider than $f_{2k+1}$ nodes               |

**Figure 9** The program contains the rotation invariant rules $BW$ and $BB$. In green the B-cells produced when using the rule BB too. From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a B-cell, it becomes a quiescent cell at the next time.

We remain to append an information regarding the case when the program of $A$ is rotation invariant. The first remark is that in such a situation, there is no difference between the rules $BW$ and $WB$ as well as between the rules $BW$ and $WB$. As we assume the rule $BW$, there is no consideration of a rule $WB$. This also means that in a situation where we applied the rule $WB$ when rotation invariance is relaxed, in the case of rotation invariance we apply the rule $BW$. However, note that the rules $BW$ and $WB$ are contradictory under rotation invariance as in that case, $WB$ is the same rule as $BW$ which, by definition, is opposite to $BW$. And so, we are concerned with the first two rows of Table 3. However, there is a special phenomenon which occurs here and may not occur in the situation where we deal when the rotation invariance does not take place. It is illustrated by Figures 9 and 10.

In Figure 9 we assume that besides the rule $BW$, the rule $BB$ too belongs to the program of $A$. In this case too, the rules $BW$ and $WB$ are the same up to a circular permutation on the neighbours.

Comparing Figure 6 with Figure 9 on one hand and Figure 7 with Figure 10 on the other hand we can see in both cases that the B-cells are at the same places during the propagation.
The program contains the rotation invariant rules $BW$ and $BB$. From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a $B$-cell, it becomes a quiescent cell at the next time.

6 Conclusion

Of course, the first question is what can be said for three states? That issue is more difficult. We already have seen a rather difficult situation in the proof of Theorem 1 when it could happen that the front enters a blinking between all cells in the state $B$ and all of them in the state $W$ at the next time and conversely. As then the front remains at the same place during a certain time, the discussion was how long such a blinking might last. A worse situation occurs with three states for something which we could ignore with two states: the point is what happens behind the front? In fact, in case a node changes its state from $W$ to $B$ behind the front, the worse thing it might happen is that another line of $B$-cells might propagate but, in that case, another line already occurred so that we are in the situation of a constant advance of the front. Accordingly, it does not change the situation for what is the halting of the computation.

The things are different with three states. Let the three states being $W$, $B$ and $R$, where $W$ is the quiescent state which is associated to the quiescent rule possessed by the program of our cellular automaton. As a third state enters the play, we may have the following rules:

- $BWB$: $W$ $BW$ $4$ $B$,
- $BWR$: $W$ $BW$ $4$ $R$,
- $RWB$: $W$ $RW$ $4$ $B$,
- $RWR$: $W$ $RW$ $4$ $R$,
- $RWW$: $W$ $RW$ $4$ $W$.

Clearly, if we have the rule $BWB$, or the rule $RWR$, we have a constant progression of the front once a non-quiescent cell occurs on the front. A similar conclusion occurs if we have both rules $BWR$ and $RWB$: they call each other in some sense, again once a non-quiescent cell occurs on the front. What happens if, instead of both rules $BWR$ and $RWB$ we have, for instance, both rules $BWR$ and $RWW$? In that case, assume that the rule $BWR$ applies to the node $\varphi$ of the front at time $t$. Let $\nu$ be the first white son of $\varphi$ and let $\sigma$ be the first white son of $\nu$. Then, at time $t+1$ $\nu$ becomes an $R$-cell and, due to $RWW$, at time $t+2$, $\sigma$ becomes remains a quiescent cell. Now, it may happen that at time $t+2$, $\nu$ becomes a $B$-cell. In that case, $\sigma$ becomes an $R$-cell at time $t+3$. However, even if the transformation of $\nu$ from an $R$-cell to a $B$-cell happens at time $t+2$, we are not guaranteed that the same transformation will happen for $\sigma$ at time $t+4$. The reason is that in those cases, the transformation depends on what happened behind the front. Note that in our discussions with a single non-quiescent state, it was enough to look at the rules which apply to a quiescent cell and not to look at those
which apply to a cell in a non-quiescent state although in the figures, in order to obtain nice pictures, we made implicit assumptions on rules applied to a $B$-cell or to a $W$-cell behind the front whose neighbourhood may be different from $BW^4$, $WBW^3$ or $B^2W^3$. If we ignore the complex discussion involving a huge number of rules, we might expect an argument on how long we have to wait for a new transformation of $\nu$ from $R$ to $B$. Even if we have an argument on the number of possible configurations within $D_N$, to repeat the same argument to $\sigma$ requires to consider the number of possible configurations within $D_{N+1}$ which is much bigger. Accordingly, this leads to no conclusion, so that the case with three states is open, even with rotation invariance.

Other questions may be considered. We know that strong universality is possible for deterministic cellular automata on the pentagrid with a quiescent state with ten states, see [4]. That cellular automaton is rotation invariant. What can be performed if we relax rotation invariance? The answer is not straightforward as the cellular automaton of [4] is based on a cellular automaton on the line which is strongly universal with eleven states and six states of that automaton could be could be absorbed by the cellular automaton of the pentagrid which implements the cellular automaton on the line. And so, for that direction two, a new approach is needed.

Accordingly, as the gap between two states and ten states seems to be not a small one, there is still a huge amount of work ahead.

References

[1] M. Margenstern, New Tools for Cellular Automata of the Hyperbolic Plane, *Journal of Universal Computer Science*, **6**(12), (2000), 1226–1252.

[2] M. Margenstern, *Cellular Automata in Hyperbolic Spaces*, vol. 1, *Theory*, Old City Publishing, Philadelphia, (2007), 422p.

[3] M. Margenstern, *Cellular Automata in Hyperbolic Spaces*, vol. 2, *Implementation and computations*, Old City Publishing, Philadelphia, (2008), 360p.

[4] M. Margenstern, About Strongly Universal Cellular Automata, *Electronic Proceedings in Theoretical Computer Science*, **128**, (2013), Machines, Computations and Universality (MCU 2013), T. Neary and M. Cook (Eds.), 93-125.