ON COFINITELY WEAK RAD-SUPPLEMENTED MODULES

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Abstract. In this paper, necessary and sufficient conditions for a quotient module are found to be a cofinitely weak Rad-supplemented module under which circumstances. Nevertheless, some relations are investigated between cofinitely Rad-supplemented modules and cofinitely weak Rad-supplemented modules. Lastly, we show that an arbitrary ring $R$ is a left Noetherian $V$–ring if and only if every weak Rad-supplemented $R$–module is injective.

1. Introduction

Throughout the paper, $R$ will be an associative ring with identity, $M$ will be an $R$–module and all modules will be unital left $R$–modules unless otherwise specified. By $N \leq M$, we mean that $N$ is a submodule of $M$. Recall that a submodule $L$ of $M$ is small in $M$ and denoted by $L \ll M$, if $M \neq L + K$ for every proper submodule $K$ of $M$. A submodule $S$ of $M$ is said to be essential in $M$ and denoted by $S \leq M$, if $S \cap N \neq 0$ for every nonzero submodule $N \leq M$. We write $\text{Rad}(M)$ for the Jacobson radical of a module $M$. An $R$–module $M$ is called supplemented, if every submodule $N$ of $M$ has a supplement in $M$, i.e. a submodule $K$ is minimal with respect to $M = N + K$. $K$ is supplement of $N$ in $M$ if and only if $M = N + K$ and $N \cap K \ll K$ [16].

If $M = N + K$ and $N \cap K \ll M$, then $K$ and $N$ are called weak supplements of each other. Also $M$ is called a weakly supplemented module if every submodule of $M$ has a weak supplement in $M$ [13, 18]. By using this definition, Büyükaşk and Lomp showed in [6] that a ring $R$ is left perfect if and only if every left $R$–module is weakly supplemented if and only if $R$ is semilocal and the radical of the countably infinite free left $R$–module has a weak supplement. Furthermore Alizade and Büyükaşk showed that a ring $R$ is semilocal if and only if every direct product of simple modules is weakly supplemented [3].

In [17], Xue introduced Rad-supplemented modules. Let $M$ be an $R$–module. Let $M$ be an $R$–module, $N$ and $K$ be any submodules of $M$ with $M = N + K$. If $N \cap K \leq \text{Rad}(K)$
(N ∩ K \leq \text{Rad}(M))$, then $K$ is called a \textit{(weak) Rad-supplement} of $N$ in $M$. Besides, $M$ is called \textit{(weakly) Rad-supplemented} module provided that each submodule has a (weak) Rad-supplement in $M$. For characterizations of Rad-supplemented and weak Rad-supplemented modules, we refer to [15] and [17]. Since the Jacobson radical of a module is the sum of all small submodules, every supplement is a Rad-supplement.

Certain modules whose maximal submodules have supplements are studied in [1]. Also in the same paper, cofinitely supplemented modules are introduced. A submodule $N$ of $M$ is said to be \textit{cofinite} if $\frac{M}{N}$ is finitely generated. $M$ is called \textit{cofinitely (weak) supplemented} if every cofinite submodule has a (weak) supplement in $M$ [1, 2]. Nevertheless, it is known by [1, Theorem 2.8] and [2, Theorem 2.11] that an $R$–module $M$ is cofinitely (weak) supplemented if and only if every maximal submodule of $M$ has a (weak) supplement in $M$. Clearly, supplemented modules are cofinitely supplemented and weakly supplemented modules are cofinitely weak supplemented ones.

$M$ is called \textit{cofinitely Rad-supplemented} if every cofinite submodule of $M$ has a Rad-supplement [5]. Since every submodule of a finitely generated module is cofinite, a finitely generated module is Rad-supplemented if and only if it is cofinitely Rad-supplemented. According to [12], if every cofinite submodule of $M$ has a Rad-supplement that is a direct summand of $M$, then $M$ is called a \textit{cofinitely \textit{Rad-supplemented}} module.

In a present paper [10], a module is called \textit{cofinitely weak Rad-supplemented} if every cofinite submodule has a weak Rad-supplement and \textit{totally cofinitely weak Rad-supplemented} if every submodule is \textit{cofinitely weak Rad-supplemented}. Also it is proved in [10] that any arbitrary sum of cofinitely weak Rad-supplemented modules is a cofinitely weak Rad-supplemented module. Clearly this implies that any finite direct sum of cofinitely weak Rad-supplemented modules is also cofinitely weak Rad-supplemented. We will show that an infinite direct sum of totally cofinitely weak Rad-supplemented modules is totally cofinitely weak Rad-supplemented under certain conditions. Also we will prove that every torsion module over a Dedekind domain is a cofinitely weak Rad-supplemented module and find some conditions to show when any module over a Dedekind domain is cofinitely weak Rad-supplemented.

2. Main Results

Following [5], a module $M$ is called \textit{w–local} if it has a unique maximal submodule.

\textbf{Theorem 1.} Every \textit{w–local} module is cofinitely weak Rad-supplemented.

\textit{Proof.} Let $M$ be a module and $U$ be a cofinite submodule of $M$. Since $\frac{M}{U}$ is finitely generated, it has a maximal submodule such as $P^\ell$. Therefore $P$ is a maximal
submodule of $M$. Then we have $U + M = M$ and $U \cap M = U \subseteq P = \text{Rad}(M)$. Hence $M$ is cofinitely weak Rad-supplemented.

Recall that a module $M$ is called refinable (or suitable), if for any submodules $U, V \leq M$ with $U + V = M$, there exists a direct summand $U_1$ of $M$ with $U_1 \leq U$ and $U_1 + V = M$.

**Theorem 2.** Let $M$ be a refinable $R$-module. Then the following are equivalent:

(i) $M$ is $\oplus$-cofinitely Rad-supplemented,
(ii) $M$ is cofinitely Rad-supplemented,
(iii) $M$ is cofinitely weak Rad-supplemented.

**Proof.** The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (i)$ Let $M$ be a cofinitely weak Rad-supplemented module and $N$ be a cofinite submodule of $M$. Then, we have $M = N + K$ and $N \cap K \leq \text{Rad}(M)$ where $K$ is a submodule of $M$. Since $M$ is a refinable module, it has a direct summand $L$ such that $L \leq K$ and $M = L + N$. Following this, $N \cap L \leq N \cap K \leq \text{Rad}(M)$ implies that $L$ is weak Rad-supplement of $N$. By using [14, Proposition 4], we get that $L$ is Rad-supplement of $N$. Therefore, $M$ is $\oplus$-cofinitely Rad-supplemented.

A ring $R$ is called a left $V$-ring if every simple left $R$-module is injective.

**Theorem 3.** For an arbitrary ring $R$, the following are equivalent:

(i) Every weakly Rad-supplemented $R$-module is injective,
(ii) $R$ is a left Noetherian $V$-ring.

**Proof.** $(i) \Rightarrow (ii)$ Assume that $M$ is a $\oplus$-supplemented $R$-module. Since $M$ is weak Rad-supplemented, it is an injective module. By Proposition 5.3 in [11] we get that $R$ is a left Noetherian $V$-ring.

$(ii) \Rightarrow (i)$ Let $M$ be a weakly Rad-supplemented module. Since $R$ is a left Noetherian $V$-ring, we get $\text{Rad}(M) = 0$ by Villamayor theorem in [7]. Then, $M$ is semisimple and so $\oplus$-supplemented. Again using Proposition 5.3 in [11], we obtain $M$ is an injective module.

**Corollary 1.** Let $R$ be a commutative ring. Then, every weakly Rad-supplemented $R$-module is injective if and only if $R$ is semisimple.

**Proof.** Suppose that every weakly Rad-supplemented module is injective. By using the preceding theorem, we can say that $R$ is a left Noetherian $V$-ring. Thus, $R$ is semisimple by Proposition 1 and first corollary of [7]. The other side of the proof is obvious by [16, 20.3].

**Theorem 4.** Let $M$ be a module and $N$ be a submodule of $M$. If every cofinite submodule containing $N$ of $M$ has a weak Rad-supplement in $M$, then $\frac{M}{N}$ is cofinitely weak Rad-supplemented.
Proof. Let \( \frac{U}{N} \) be a cofinite submodule of \( \frac{M}{N} \). Since \( \frac{(U)}{(N)} \cong \frac{M}{U} \), we get that \( U \) is a cofinite submodule of \( M \) containing \( N \). Hence, we can find a submodule \( V \) of \( M \) such that \( M = U + V \) and \( U \cap V \leq \text{Rad}(M) \). By using Proposition 3.2 of [15], we can deduce that \( \frac{(U+N)}{N} \) is a weak Rad-supplement of \( \frac{U}{N} \) in \( \frac{M}{N} \). Therefore, \( \frac{M}{N} \) is a cofinitely weak Rad-supplemented module.

Remark. While a quotient module of a module is a cofinitely weak Rad-supplemented module, it may not be a cofinitely weak Rad-supplemented module. For example, \( \mathbb{Z}/2\mathbb{Z} \) isn’t cofinitely weak Rad-supplemented but \( \mathbb{Z}/p\mathbb{Z} \) is cofinitely weak Rad-supplemented for any prime number \( p \).

Proposition 1. Let \( M \) be a cofinitely weak Rad-supplemented \( R \)-module. Then every Rad-supplement in \( M \) is cofinitely weak Rad-supplemented.

Proof. Let \( V \) be a Rad-supplement of \( U \) in \( M \). That means \( M = U + V \) and \( U \cap V \leq \text{Rad}(V) \). Since \( \frac{M}{V} = \frac{(U+V)}{V} \cong \frac{V}{V} \), we get that \( \frac{V}{V} \) is a cofinitely weak Rad-supplement of \( \frac{U}{V} \) in \( \frac{M}{V} \). Therefore, \( \frac{M}{V} \) is cofinitely weak Rad-supplemented by [10, Proposition 6]. Theorem 4 in the same paper implies that \( V \) is cofinitely weak Rad-supplemented.

Theorem 5. Let \( R \) be a Dedekind domain and \( M \) be a torsion \( R \)-module. Then \( M \) is cofinitely weak Rad-supplemented.

Proof. By [3, Corollary 2.7], we have \( \frac{M}{\text{Rad}(M)} \) is semisimple and so cofinitely weak Rad-supplemented.

Theorem 6. Let \( R \) be a Dedekind domain, \( \frac{M}{\text{Rad}(M)} \) be finitely generated and \( \text{Rad}(M) \leq M \). If \( \text{Rad}(M) \) is cofinitely weak Rad-supplemented, then \( M \) is cofinitely weak Rad-supplemented.

Proof. Suppose that \( \frac{M}{\text{Rad}(M)} \) is generated by \( m_1 + \text{Rad}(M), m_2 + \text{Rad}(M), \ldots, m_n + \text{Rad}(M) \). Then, for finitely generated submodule \( K = Rm_1 + Rm_2 + \ldots + Rm_n \), we have \( M = \text{Rad}(M) + K \) and \( K \cap \text{Rad}(M) \) is finitely generated as \( K \) is finitely generated. So \( K \cap \text{Rad}(M) \ll M \) by Lemma 2.3 in [3]. That is to say, \( K \) is a weak supplement of \( \text{Rad}(M) \) of \( M \). Since \( \text{Rad}(M) \leq M \), we get \( \frac{M}{\text{Rad}(M)} \) is torsion. Besides this, Proposition 9.15 of [4] implies that \( \text{Rad} \left( \frac{M}{\text{Rad}(M)} \right) = 0 \). Hence \( \frac{M}{\text{Rad}(M)} \) is semisimple by Corollary 2.7 in [3]. If we consider \( 0 \to \text{Rad}(M) \to M \to \frac{M}{\text{Rad}(M)} \to 0 \), then \( M \) is cofinitely weak Rad-supplemented by Theorem 7 in [10].

Proposition 2. Let \( R \) be a non-semilocal commutative domain. If \( M \) is totally cofinitely weak Rad-supplemented, then \( M \) is torsion.

Proof. Suppose that \( \text{Ann} (m) = 0_R \) for some \( m \in M \). Then we have \( Rm \cong R_R \). Since \( Rm \) is cofinitely weak Rad-supplemented, \( R_R \) is also (cofinitely) weak Rad-supplemented. Then by 17.2 of [8], \( R \) is a semilocal ring which gives a contradiction. Thus, \( M \) is a torsion module.
Theorem 7. Let $R$ be an arbitrary ring and $M = \bigoplus_{i \in I} M_i$ such that $M_i$ is totally cofinitely weak Rad-supplemented for all $i \in I$. If $U = \bigoplus_{i \in I} (U \cap M_i)$ for every submodule $U$ of $M$, then $M$ is totally cofinitely weak Rad-supplemented.

Proof. Assume that $U$ is a submodule of $M$ and $V$ is a cofinite submodule of $U$ where $U = \bigoplus_{i \in I} (U \cap M_i)$. Since $V = \bigoplus_{i \in I} (V \cap M_i)$ and $\frac{U}{V} \cong \bigoplus_{i \in I} \left( \frac{U \cap M_i}{V \cap M_i} \right)$, we get that $V \cap M_i$ is a cofinite submodule of $U \cap M_i$ for all $i \in I$. We know that $U \cap M_i$ is cofinitely weak Rad-supplemented. Therefore $V \cap M_i$ has a weak Rad-supplement $K_i$ in $U \cap M_i$ for all $i \in I$. Let $K = \bigoplus_{i \in I} K_i$. Then we obtain $U = V + K$ and $V \cap K \leq \text{Rad}(U)$. As a result, $U$ is cofinitely weak Rad-supplemented and so $M$ is totally cofinitely weak Rad-supplemented. \hfill \square

Let $R$ be a Dedekind domain and $M$ be an $R$–module. By $\Omega$, we denote the set of all maximal ideals of $R$. The submodule $T_P(M) = \{ m \in M | P^n m = 0 \text{ for some } n \geq 1 \}$ is called the $P$–primary part of $M$.

Theorem 8. Let $R$ be a non-semilocal Dedekind domain. Then, $M$ is a totally cofinitely weak Rad–supplemented module if and only if $M$ is torsion and $T_P(M)$ is totally cofinitely weak Rad-supplemented for every $P \in \Omega$.

Proof. Assume that $M$ is a totally cofinitely weak Rad-supplemented module. Then $M$ is torsion by Proposition 2. On the other hand $T_P(M)$ is totally cofinitely weak Rad-supplemented for every $P \in \Omega$. Because every submodule of a totally cofinitely weak Rad-supplemented module is a totally cofinitely weak Rad-supplemented module.

Conversely, we can write $M = \bigoplus_{P \in \Omega} T_P(M)$ by Proposition 6.9 in [9]. Let $N$ be a submodule of $M$. Since $M$ is torsion, $N$ is also a torsion module. By using the same proposition, we can write that $N = \bigoplus_{P \in \Omega} T_P(N)$. Therefore, $\bigoplus_{P \in \Omega} T_P(N) = \bigoplus_{P \in \Omega} (N \cap T_P(M))$ and $T_P(M)$ is totally cofinitely weak Rad-supplemented for every $P \in \Omega$. As a result, $M$ is totally cofinitely weak Rad-supplemented by the preceding theorem. \hfill \square

Theorem 9. Any torsion module over a Dedekind domain is totally cofinitely weak Rad–supplemented.

Proof. Let $R$ be a Dedekind domain, $M$ be a torsion $R$–module and $N$ be a submodule of $M$. Due to Corollary 2.7 of [3], $\frac{N}{\text{Rad}(N)}$ is semisimple and so it is cofinitely weak Rad–supplemented. Therefore $N$ is cofinitely weak Rad–supplemented by Theorem 4 of [10]. \hfill \square

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