Hypothesis testing for eigenspaces of covariance matrix

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Abstract

Eigenspaces of covariance matrices play an important role in statistical machine learning, arising in variety of modern algorithms. Quantitatively, it is convenient to describe the eigenspaces in terms of spectral projectors. This work focuses on hypothesis testing for the spectral projectors, both in one- and two-sample scenario. We present new tests, based on a specific matrix norm developed in order to utilize the structure of the spectral projectors. A new resampling technique of independent interest is introduced and analyzed: it serves as an alternative to the well-known multiplier bootstrap, significantly reducing computational complexity of bootstrap-based methods. We provide theoretical guarantees for the type-I error of our procedures, which remarkably improve the previously obtained results in the field. Moreover, we analyze power of our tests. Numerical experiments illustrate good performance of the proposed methods compared to previously developed ones.

1 Introduction

1.1 Background

We consider a traditional statistical scenario, where we observe n i.i.d. zero-mean random vectors $X_1, \ldots, X_n$ in dimension $d$. Let $X$ be a generic random vector with the same distribution. The geometric structure of the data is described by the covariance matrix

$$\Sigma = E \left[ XX^\top \right].$$

The simplest estimator of $\Sigma$ is the sample covariance matrix

$$\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n X_iX_i^\top.$$

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The covariance matrix estimation is one of the fundamental problems in statistics: it extends far beyond the sample covariance matrix and has been very well studied under various structural assumptions and different robustification techniques. Some representative works over the past decade include Bickel and Levina (2008a,b); Lam and Fan (2009); Cai and Liu (2011); Cai and Zhou (2012); Koltchinskii and Lounici (2017a); Avella-Medina et al. (2018); Mendelson and Zhivotovsky (2019), among many others. Problem of hypothesis testing for covariance matrix was considered in Cai and Ma (2013).

However, in order to develop successful methods for modern machine learning problems, one has to go further then just covariance matrices. In particular, eigenstructure of the covariance matrix contains a lot of meaningful information:

- In dimension reduction, Principal Component Analysis (PCA) (Pearson (1901)) projects given high-dimensional observations onto low-dimensional subspace spanned by some number of the leading eigenvectors.
- Factor Models (e.g. Fan et al. (2008, 2011, 2016); Li et al. (2018)), surprisingly closely related to PCA (see Fan et al. (2013)), also make use of the eigenstructure of the covariance matrix to estimate underlying factors and loadings.
- Spectral methods in clustering and community detection (von Luxburg (2007)) rely on the eigenvectors of specifically constructed Laplacian matrix (which in some cases can be modelled as covariance matrix).

(See Fan et al. (2018) for the exposition of problems that can be approached with Spectral/PCA-based techniques.) To that end, a careful statistical analysis is required for the eigenvectors, or, more generally, for the spectral projector of the covariance matrix $\Sigma$:

$$P_J = \sum_{k \in I_J} u_k u_k^\top,$$

where $\{u_k\}_{k=1}^d$ is an orthonormal basis of ordered eigenvectors of $\Sigma$, $J$ specifies the set of eigenspaces of interest and the set $I_J$ consists of the indices of the respective eigenvectors. Its empirical version $\hat{P}_J$ is computed from $\hat{\Sigma}$. The reason why we focus on the spectral projectors rather than working directly with the eigenvectors is that there is always an ambiguity in eigenvectors, while spectral projectors are in one-to-one correspondence with the subspace spanned by the eigenvectors, which is really what plays a role. Together with the mentioned progress on the covariance matrix estimation, the prominent Davis-Kahan inequality (Davis and Kahan (1970)) makes the question of statistical estimation of the true spectral projector relatively easy. In contrast, statistical inference (uncertainty quantification, hypothesis testing and confidence sets) for eigenspaces, or in particular for principal components, is significantly less studied but longstanding problem.

Anderson (1963) was the first paper to study asymptotic distribution of an eigenvector of the sample covariance matrix $\hat{\Sigma}$. It proposes the following asymptotically $\chi^2_{d-1}$-distributed
statistic to test whether the $k$-th eigenvector $u_k$ of $\Sigma$ is equal (up to a sign) to some specified unit vector $u^\circ$:

$$n \left( \lambda_k(\hat{\Sigma}) u^\circ \top \hat{\Sigma}^{-1} u^\circ + u^\circ \top \hat{\Sigma} u^\circ / \lambda_k(\hat{\Sigma}) - 2 \right),$$

where $\lambda_k(\hat{\Sigma})$ denotes the $k$-th eigenvalue of $\hat{\Sigma}$. Le Cam’s asymptotic theory was utilized in Hallin et al. (2010) to derive a test for the same problem in case of elliptical distributions, while Paindaveine et al. (2018) studied the test from Hallin et al. (2010) even further in the regime where the spectral gap vanishes. Some other asymptotic results for subspaces spanned by eigenvectors are derived in Tyler (1981, 1983).

The two-sample problem also has a long history dating several decades back. A descriptive technique for comparison of principal components of two or more groups was discussed in Krzanowski (1979). Accompanying empirical results were presented in Krzanowski (1982). A more theoretically justified approach was suggested by Schott (1988), which considers the test statistic

$$m \sum_{k=1}^{m} \left[ \lambda_k(\hat{\Sigma}_a) + \lambda_k(\hat{\Sigma}_b) - \lambda_k(\hat{\Sigma}_a + \hat{\Sigma}_b) \right],$$

where $\hat{\Sigma}_a$ and $\hat{\Sigma}_b$ are the sample covariance matrices of two samples $X_1^a, \ldots, X_{n_a}^a$ and $X_1^b, \ldots, X_{n_b}^b$, respectively. It is proven that the limiting distribution of this test statistic under null is generalized $\chi^2$. A more sophisticated, again asymptotically $\chi^2$, test statistic was developed in Schott (1991). Furthermore, Fujioka (1993) proposed a method, based on the trace of the specific matrix:

$$\text{Tr} \left[ U_2^{(a)} \top U_1^{(b)} U_1^{(b)} \top U_2^{(a)} \right],$$

where $U_1^{(b)} = [u_1^{(b)}, \ldots, u_m^{(b)}]$ consists of the leading $m$ eigenvectors of $\hat{\Sigma}_b$ and $U_2^{(a)} = [u_{m+1}^{(a)}, \ldots, u_d^{(a)}]$ consists of the last $(d - m)$ eigenvectors of $\hat{\Sigma}_a$.

The above methods are asymptotic and are valid only for a fixed dimension $d$ and a sample size $n$ growing to infinity. A new line of research in this area was initiated by Koltchinskii and Lounici (2017b), which obtained the normal approximation for the squared Frobenius distance $\left\| \hat{\mathcal{P}}_J - \mathcal{P}_J \right\|_F^2$, providing finite sample error bounds for Kolmogorov distance. However, this result could not be used directly for the statistical inference as the mean and the variance of the normal distribution approximating $\left\| \hat{\mathcal{P}}_J - \mathcal{P}_J \right\|_F^2$ depend on the true unknown $\Sigma$; the idea of splitting the sample into three parts to estimate mean and variance was just mentioned. A follow-up paper of Koltchinskii and Lounici (2017c) formalized this sample splitting idea, and derived completely data-driven test statistic with known approximating distribution. Another approximation was proposed in Naumov et al. (2019). The focus of that paper is on constructing confidence sets for the true spectral projector, so the multiplier bootstrap was employed to deal with unknown parameters of the limiting distribution. Silin and Spokoiny (2018) proposed to use Bayesian inference instead of bootstrap, at the same time extending the results from Naumov et al. (2019) to non-Gaussian data. Even though these works did not pose the hypothesis testing problem, it is straightforward to develop one-sample tests based on their results.
1.2 Contributions

The aim of this work is to develop statistical procedures for testing one- and two-sample hypotheses about underlying eigenspaces of covariance matrices.

We try to address the following challenges:

- High-dimensionality. Rates obtained in the previous works require $d^3 \ll n$ (except the cases of Gaussian data with small effective rank), which significantly restricts the applicability of the proposed methods.

- Heavy-tailed data. While most of the discussed literature focuses only on the Gaussian data, the ability to move beyond Gaussian or sub-Gaussian distributions is crucial for modern applications, especially in finance.

- Computational complexity. Over the past decades, the multiplier bootstrap (also called wild bootstrap) has been one of the main tools for statistical inference. However, to generate one bootstrap sample, a statistician needs to perform $O(n)$ operations (to generate $n$ bootstrap weights), which can lead to intractable running time of a bootstrap-based procedure.

Our main contributions can be summarized as follows:

- We develop new statistical procedures for one- and two-sample hypothesis testing for eigenspaces of covariance matrix. The tests are based on newly developed matrix norm, which is designed by taking the structure of spectral projectors into account. In fact, we develop a family of tests that provides flexibility in controlling the trade-off between the closeness to the desired type-I error and power.

- We propose a new resampling technique of independent interest, which can be considered as an alternative to the multiplier bootstrap. While possessing the same statistical properties, this technique reduces computational complexity by $n$ times compared to the bootstrap.

- Our theoretical results for the presented procedures include both validity guarantees (type-I error close to the desired level) as well as power analysis (probability of rejection of null hypothesis goes to one under alternative). The results do not rely on the Gaussianity or sub-Gaussianity of the data. Moreover, we demonstrate a significant progress in obtaining dimension-free bounds for this problem. In some setups (e.g. Factor Models) the dependence on $d$ is remarkably improved compared to the previous works.

- The numerical study confirms good properties of our algorithms. The proposed procedures outperform the variety of previously developed methods in a wide diversity of settings.
1.3 Structure of the paper

The paper is organized as follows. We conclude the introduction with defining necessary notations in Subsection 1.4. The general framework and problem formulation are presented in Section 2. The proposed testing procedures are described in Section 3. Their theoretical properties are analyzed in Section 4. In Section 5 we apply the developed methods to Factor Models. Section 6 provides some numerical simulations. The comparison with other works is presented in Section 7. Section 8 is devoted to the main proofs. Finally, Appendix A and Appendix B gather auxiliary results and proofs, respectively.

1.4 Notations

The following notations are used throughout the work. For positive integers $k$ and $l$, we write $[k]$ as shorthand for the set \{1, 2, \ldots, k\} and $[k : l]$ for \{k, k + 1, \ldots, l\}. The space of $k$-dimensional real-valued vectors is denoted by $\mathbb{R}^k$. The space of real-valued matrices of size $k \times l$ is denoted by $\mathbb{R}^{k \times l}$. We use $0_k$ for the zero vector in $\mathbb{R}^k$, $O_{k \times l}$ for $k \times l$ matrix of zeros and $I_k$ for the identity matrix of size $k \times k$. For a matrix $A$, we denote by $A_{\{i:j\},\{k:l\}}$ its submatrix formed by intersection of rows \{i, i + 1, \ldots, j\} and columns \{k, k + 1, \ldots, l\}. Let $\text{supp}[x]$ denote the support of a vector $x$.

For a vector $x \in \mathbb{R}^k$, $\|x\|$ denotes its Euclidean norm. By $S^k$ we denote unit sphere in $\mathbb{R}^k$.

For a matrix $A \in \mathbb{R}^{k \times l}$, notations $\|A\|$, $\|A\|_F$ and $\|A\|_\ast$ mean spectral norm (largest singular value), Frobenius norm (square root of sum of squared singular values) and nuclear norm (sum of singular values), respectively, while $\|A\|_{\max}$ denotes maximal absolute elementwise norm. $\text{Tr}[\cdot]$ and $\text{rank}[\cdot]$ stand for trace and rank.

For two real numbers $a$ and $b$, by $a \lor b$ and $a \land b$ we mean their maximum and minimum, respectively. The relation $a \lesssim b$ means that there exists an absolute constant $C$, different from place to place, such that $a \leq Cb$, while $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$. When this constant has a subscript or argument, i.e. $C_\gamma$ or $C(\gamma)$, it specifies that this constant may be different for different values of variable $\gamma$, but does not depend on anything else.

2 Setup and statistical problem

2.1 Setup

Let $X_1, \ldots, X_n$ be i.i.d. mean zero random vectors in $\mathbb{R}^d$ and $X$ be a generic random vector from the same distribution. We store the observed data in a matrix

$$
\mathbf{X} = [X_1, \ldots, X_n] \in \mathbb{R}^{d \times n}.
$$

The covariance matrix of the data is

$$
\Sigma = \text{Cov}[X] = \mathbb{E} [XX^\top] \in \mathbb{R}^{d \times d}.
$$
Typically, $\Sigma$ is unknown, and one estimates it using its sample version $\hat{\Sigma}$:

$$\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} X_i X_i^\top \in \mathbb{R}^{d \times d}.$$ 

Let us introduce some notations. Let $\sigma_1 \geq \ldots \geq \sigma_d$ be the ordered eigenvalues of $\Sigma$ (assume all eigenvalues are strictly positive). Introduce groups of indices $\mathcal{I}_r = \{j \in [d] : \mu_r = \sigma_j\}$ and denote by $m_r$ the multiplicity factor $|\mathcal{I}_r|$ for all $r \in [q]$. The corresponding eigenvectors are denoted as $u_1, \ldots, u_d$.

Define projector on $r$-th eigenspace as $P_r = \sum_{k \in \mathcal{I}_r} u_k u_k^\top$ for $r \in [q]$. Similarly, suppose that $\hat{\Sigma}$ has $d$ eigenvalues $\hat{\sigma}_1 > \ldots > \hat{\sigma}_d$ (distinct with probability one). The corresponding eigenvectors are $\hat{u}_1, \ldots, \hat{u}_d$.

Suppose we are interested in the sum of some of the $q$ eigenspaces of $\Sigma$. In particular, let $J = \{r_1, r_1 + 1, \ldots, r_2\}$ be a set of consecutive indices of eigenspaces of interest. Define also

$$\mathcal{I}_J = \bigcup_{r \in J} \mathcal{I}_r.$$ 

Quantitatively, sum of the $J$ eigenspaces of $\Sigma$ is described by the projector onto this subspace, defined as

$$P_J \overset{\text{def}}{=} \sum_{r \in J} P_r = \sum_{r \in J} \sum_{k \in \mathcal{I}_r} u_k u_k^\top \in \mathbb{R}^{d \times d}.$$ 

Its empirical counterpart is given by

$$\hat{P}_J \overset{\text{def}}{=} \sum_{r \in J} \hat{P}_r = \sum_{r \in J} \sum_{k \in \mathcal{I}_r} \hat{u}_k \hat{u}_k^\top \in \mathbb{R}^{d \times d}.$$ 

The rank of these projectors is $m \overset{\text{def}}{=} |\mathcal{I}_J| = \sum_{r \in J} m_r$. As an example, when $\mathcal{I}_J = \{1, \ldots, m\}$, then $P_J$ consists of the projector onto the eigenspace spanned by the eigenvectors of the top $m$ distinguished eigenvalues, while $\hat{P}_J$ is its empirical counterpart. For brevity, we occasionally will be using the notation $P_{J^c} \overset{\text{def}}{=} I_d - P_J$.

### 2.2 Statistical problem

One may be interested in testing hypothesis about $P_J$. The hypothesis testing problem

$$H_0^{(1)} : P_J = P^o \quad \text{vs} \quad H_1^{(1)} : P_J \neq P^o$$

for a given projector $P^o$ of rank $m$ is the main focus of our work.

Two-sample problem is also of great interest. Suppose we have two i.i.d. samples: $X_1^a, \ldots, X_{2n_a}^a$ and $X_1^b, \ldots, X_{2n_b}^b$ (it will be clear later why we denote the sizes of the samples as $2n_a$ and $2n_b$; assume for simplicity they are even numbers). As previously, we store them as $\mathbf{X}^a$ and $\mathbf{X}^b$. 


Let the true covariance matrix of the first sample be $\Sigma_a$ and the true covariance matrix of the second sample be $\Sigma_b$. Let $\mathcal{J}_a$ be a set of consecutive indices of eigenspaces of $\Sigma_a$ and $\mathcal{J}_b$ be a set of consecutive indices of eigenspaces of $\Sigma_b$. Sets $\mathcal{I}_{\mathcal{J}_a}$ and $\mathcal{I}_{\mathcal{J}_b}$ contain the indices of the associated ordered eigenvectors; it makes sense to require $|\mathcal{I}_{\mathcal{J}_a}| = |\mathcal{I}_{\mathcal{J}_b}| = m$ (so in both one- and two-sample problems $m$ denotes the dimension of the subspace being tested). We denote by $P_a$ and $P_b$ the corresponding projectors of rank $m$:

$$P_a = \sum_{k \in \mathcal{I}_{\mathcal{J}_a}} u^a_k u^a_k^\top,$$
$$P_b = \sum_{k \in \mathcal{I}_{\mathcal{J}_b}} u^b_k u^b_k^\top,$$

where $\{u^a_k\}_{k=1}^d$ and $\{u^b_k\}_{k=1}^d$ are the sets of ordered (w.r.t. the associated eigenvalues) eigenvectors of $\Sigma_a$ and $\Sigma_b$. Here, in order to avoid excessive sub- and superscripts, we slightly abuse the notation: $P_a$ and $P_b$ should not be confused with $P_r$ for $r \in [q]$ from one-sample case. In addition to the one-sample problem stated above, we will propose a method for the following hypothesis testing problem

$$H_0^{(2)} : P_a = P_b \; \text{ vs } \; H_1^{(2)} : P_a \neq P_b.$$

In both of these problems, a statistician is often given a desired level of the test $\alpha$. However, in most of the situations (including our setting), creating a reasonable test with type-I error exactly $\alpha$ is difficult or impossible. Our goal is to develop tests, whose type-I errors will be close to the level $\alpha$, and provide finite sample guarantees for the discrepancy between them.

## 3 Testing procedure

Previous works of Koltchinskii and Lounici (2017b,c); Naumov et al. (2019); Silin and Spokoiny (2018) considered the Frobenius norm $\sqrt{n} \| \hat{P}_\mathcal{J} - P_\mathcal{J} \|_F$ and would suggest this object as a basis for one-sample testing procedure. Another interesting random quantity to analyze would be the spectral norm

$$\tilde{Q}^{(1)} \overset{\text{def}}{=} \sqrt{n} \| \hat{P}_\mathcal{J} - P_\mathcal{J} \|.$$

(Here and further the superscript specifies whether we are in context of one-sample or two-sample problem.) However, as we will see, current techniques doesn’t allow us to obtain an approximation to the distribution of $\tilde{Q}^{(1)}$ that is accurate in high dimensions. This prevents us from developing a test based on this random quantity, and forces us to construct a new, less conventional and more problem-specific, matrix norm that will have better theoretical properties.

The matrix norm, which our test statistic will be based on, is introduced in the following definition.
Definition 3.1. Let $P \in \mathbb{R}^{d \times d}$ be a projector of rank $m$. Fix $\Gamma = [\Gamma_1 \ \Gamma_2] \in \mathbb{R}^{d \times d}$ with $\Gamma_1 \in \mathbb{R}^{d \times m}, \Gamma_2 \in \mathbb{R}^{d \times (d - m)}$ satisfying

\begin{align*}
\Gamma_1 \Gamma_1^\top &= P, \quad \Gamma_1^\top \Gamma_1 = I_m, \\
\Gamma_2 \Gamma_2^\top &= I_d - P, \quad \Gamma_2^\top \Gamma_2 = I_{d - m}.
\end{align*}

Let also $s_1 \in [m]$ and $s_2 \in [d - m]$. Then, for any symmetric matrix $A \in \mathbb{R}^{d \times d}$ define

$$
\|A\|(P, \Gamma, s_1, s_2) \overset{\text{def}}{=} \frac{1}{2} \|\Gamma_1^\top A \Gamma_1\| + \frac{1}{2} \|\Gamma_2^\top A \Gamma_2\| + \max_{k \in [m - s_1 + 1]} \|\Gamma_1^\top A \Gamma_2\|_{k:(k+s_1-1),\ldots:k:(l+s_2-1)}.
$$

Let us briefly describe the role of $\Gamma_1, \Gamma_2$ and $s_1, s_2$ in the above definition. As can be seen from (3.1), the columns of $\Gamma_1$ form an orthonormal basis in the subspace associated with $P$, while the columns of $\Gamma_2$ form an orthonormal basis in the orthogonal complement. Thus, $\Gamma_1$ can be found as the set of eigenvectors of $P$ corresponding to the eigenvalue 1 of multiplicity $m$, and $\Gamma_2$ is the set of eigenvectors of $P$ corresponding to the eigenvalue 0 of multiplicity $(d - m)$, i.e. the eigendecomposition of $P$ looks like

$$
P = [\Gamma_1 \ \Gamma_2] \begin{bmatrix} I_m & O_{m \times (d - m)} \\
O_{(d - m) \times m} & O_{(d - m) \times (d - m)} \end{bmatrix} \begin{bmatrix} \Gamma_1^\top \\
\Gamma_2^\top \end{bmatrix}.
$$

This rotation is necessary for our future theoretical analysis. Clearly, $\Gamma_1$ and $\Gamma_2$ satisfying (3.1) are not unique, but a specific choice will not play any role in the sequel. The first two terms will be negligible under null hypothesis while allowing us to improve the power of the test (“power enhancement”); the third term is the main term that will give us the desired approximation. The integers $s_1$ and $s_2$ parametrize the family of norms and give flexibility in the test that we will develop: as we will see, the test based on the norm with $s_1 = s_2 = 1$ will have better guarantees under null hypothesis and weaker power (less omnibus), while taking largest possible values $s_1 = m, s_2 = d - m$ yields the test with potentially unstable behaviour under $H_0$ but omnibus. Figure 1 further explains Definition 3.1.

We state some useful properties of this operator in the next proposition.

Proposition 3.1 (Properties of $\|\cdot\|(P, \Gamma, s_1, s_2)$). Fix arbitrary $P, \Gamma = [\Gamma_1 \ \Gamma_2], s_1, s_2$ as in Definition 3.1. Then, the following holds:

(i) $\|\cdot\|(P, \Gamma, s_1, s_2)$ is indeed a norm on the space of symmetric matrices.

(ii) $\|\cdot\|(P, \Gamma, s_1, s_2)$ is equivalent to the spectral norm: for any symmetric $A \in \mathbb{R}^{d \times d}$

$$
\frac{1}{2} \sqrt{\frac{s_1}{m} \cdot \frac{s_2}{d - m}} \cdot \|A\| \leq \|A\|(P, \Gamma, s_1, s_2) \leq 2 \|A\|.
$$
Figure 1: Graphical illustration of how $\|A\|_{(P, \Gamma, s_1, s_2)}$ is computed. In this example, we take $d = 8$, $m = 3$, $s_1 = 2$, $s_2 = 3.$ Consider the rotated matrix $\tilde{A} = \Gamma^\top A \Gamma$ and split it into four blocks: $m \times m$ top left block (blue), $(d - m) \times (d - m)$ bottom right block (pink), bottom left $(d - m) \times m$ block (white) and top right $m \times (d - m)$ block. Then $\|A\|_{(P, \Gamma, s_1, s_2)}$ is computed as half of the sum of spectral norms of blue and pink blocks, plus the largest spectral norm of gray submatrices, for which we have $(m - s_1 + 1) \cdot (d - m - s_2 + 1) = 6$ options.

3.1 One-sample test

Our one-sample test is based on the following random quantity

$$Q^{(1)} \overset{\text{def}}{=} \sqrt{n} \|\hat{P}_J - P^0\|_{(P^0, \Gamma^0, s_1, s_2)},$$

where $\Gamma^0$ satisfying properties (3.1) for $P^0$ (as in Definition 3.1) is chosen arbitrarily.

**Remark 3.1** (Link between $Q^{(1)}$ and $\tilde{Q}^{(1)}$). Under $H^{(1)}_0$ it holds $P^0 = P_J$, and the random quantity of interest becomes $Q^{(1)} = \sqrt{n} \|\hat{P}_J - P_J\|_{(P_J, \Gamma^0, s_1, s_2)}$. Take $s_1 = m$, $s_2 = d - m$. Note that even in this case,

$$\|\hat{P}_J - P_J\|_{(P_J, \Gamma^0, m, d-m)} \neq \|\hat{P}_J - P_J\|,$$

though we have bounds as in Proposition 3.1. However, as will be seen in the proofs, due to a specific structure of spectral projectors, it holds

$$\|\hat{P}_J - P_J\|_{(P_J, \Gamma^0, m, d-m)} \approx \|\hat{P}_J - P_J\|$$

up to higher-order terms with high probability, and, moreover, $\tilde{Q}^{(1)} = \sqrt{n} \|\hat{P}_J - P^0\|$ can also be used as the test statistics with the same theoretical guarantees under null hypothesis as for $\sqrt{n} \|\hat{P}_J - P^0\|_{(P^0, \Gamma^0, m, d-m)}$.

If we knew the quantiles $q^{(1)}(\alpha)$ of the distribution of $Q^{(1)}$ under $H^{(1)}_0$, we would use the following test

$$\phi_{\alpha}(X) = 1 \left\{ \sqrt{n} \|\hat{P}_J - P^0\|_{(P^0, \Gamma^0, s_1, s_2)} \geq q^{(1)}(\alpha) \right\}.$$
which has type-I error exactly $\alpha$. However, in practice the distribution of $Q^{(1)}$ is unavailable to us, since even if we could obtain closed-from approximation to it, it would depend heavily on the underlying unknown covariance $\Sigma$. Hence, we suggest two approaches to approximate $q^{(1)}(\alpha)$.

**Approach 1: Bootstrap-based test**

Let us apply the idea of multiplier bootstrap to approximate the unknown distribution of $Q(1)$ under $H_0^{(1)}$. Consider $\eta_1, \ldots, \eta_n \overset{i.i.d.}{\sim} \mathcal{N}(1, 1)$. Define $\Sigma^B \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n \eta_i X_i X_i^\top$ and the corresponding projector $P^B_J$ from $\Sigma^B$. Consider the random quantity

$$Q_B^{(1)} \overset{\text{def}}{=} \sqrt{n} \|P^B_J - \hat{P}_J\|_{(P^*, \Gamma^*, s_1, s_2)}.$$

The hope is that $(Q_B^{(1)} | X) \overset{d}{\approx} Q^{(1)}$ under $H_0^{(1)}$ with high probability. At the same time, the distribution of $(Q_B^{(1)} | X)$ is available to us and can be sampled to find its $\alpha$-quantile $q_B^{(1)}(\alpha)$.

**Approach 2: Frequentist-Bayes related test**

We also propose another resampling technique to approximate the unknown distribution of $Q(1)$ under $H_0^{(1)}$. Consider $Z_1, \ldots, Z_n \overset{i.i.d.}{\sim} \mathcal{N}(0, \hat{\Sigma})$. Define $\Sigma^F \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n Z_i Z_i^\top$ and the corresponding projector $P^F_J$ from $\Sigma^F$. Consider the random quantity

$$Q_F^{(1)} \overset{\text{def}}{=} \sqrt{n} \|P^F_J - \hat{P}_J\|_{(P^*, \Gamma^*, s_1, s_2)}.$$

Similarly to Approach 1, we expect that $(Q_F^{(1)} | X) \overset{d}{\approx} Q^{(1)}$ under $H_0^{(1)}$ with high probability. Again, the distribution of $(Q_F^{(1)} | X)$ is available to us and can be sampled in order to find its $\alpha$-quantile $q_F^{(1)}(\alpha)$. Note that instead of sampling $Z_1, \ldots, Z_n \overset{i.i.d.}{\sim} \mathcal{N}(0, \hat{\Sigma})$, we can directly generate $\Sigma^F \sim \frac{1}{n} \cdot \text{Wishart}(n, \hat{\Sigma})$, which is more computationally efficient.

**Remark 3.2 (Relation to Frequentist Bayes).** One may be curious why we call Approach 2 “Frequentist-Bayes related”. It turns out, that this resampling method somehow arises from the Bayesian inference conducted in Silin and Spokoiny (2018). Due to space limitations, we do not elaborate on this connection in our work.

Based on one of the presented resampling strategies, we summarize our test method as in Algorithm 1.

### 3.2 Two-sample test

In one-sample problem we have a null hypothesis projector $P^o$ given to us, and can use it in our test statistic. Specifically, we use $\| \cdot \|_{(P^*, \Gamma^*, s_1, s_2)}$-norm. In contrast, in two-sample problem we have only two samples, while no $P^o$ is provided, so the one-sample procedure cannot be straightforwardly extended, as it is not clear what norm to use.
Algorithm 1: One-sample testing procedure

**Input:** Data $\mathbf{X} = [X_1, \ldots, X_n]$, set $\mathcal{I} \mathcal{J}$, null hypothesis projector $\mathbf{P}^o$ of rank $|\mathcal{I} \mathcal{J}|$, desired level $\alpha$.

**Hyperparameters:** $s_1, s_2$, number of resampling iterations $N$.

Set $m := |\mathcal{I} \mathcal{J}|$;

Compute $\hat{\Sigma} := \frac{1}{n} \sum_{i=1}^{n} X_i X_i^\top$;

Compute the corresponding projector $\hat{\mathbf{P}}_J$ from $\hat{\Sigma}$;

Fix $\Gamma_1^o \in \mathbb{R}^{d \times m}$ such that $\Gamma_1^o \Gamma_1^{o\top} = \mathbf{I}_m$ and $\Gamma_1^o \Gamma_1^{o\top} = \mathbf{P}^o$;

Fix $\Gamma_2^o \in \mathbb{R}^{d \times (d - m)}$ such that $\Gamma_2^o \Gamma_2^{o\top} = \mathbf{I}_{d - m}$ and $\Gamma_2^o \Gamma_2^{o\top} = \mathbf{I}_d - \mathbf{P}^o$;

Apply Bootstrap-based resampling:

for $k = 1, \ldots, N$ do

Sample $\eta_1, \ldots, \eta_n \overset{i.i.d.}{\sim} \mathcal{N}(1, 1)$;

Compute $\Sigma^B := \frac{1}{n} \sum_{i=1}^{n} \eta_i X_i X_i^\top$;

Compute the corresponding projector $\mathbf{P}^B_J$ from $\Sigma^B$;

Compute $k$-th realization $Q_R^{(1)}(k) := \sqrt{n} \| \mathbf{P}^B_J - \hat{\mathbf{P}}_J \|_{(\mathbf{P}^o, \Gamma^o, s_1, s_2)}$;

end

or apply Frequentist-Bayes related resampling:

for $k = 1, \ldots, N$ do

Sample $\Sigma^F := \frac{1}{n} \cdot \text{Wishart}(n, \hat{\Sigma})$;

Compute the corresponding projector $\mathbf{P}^F_J$ from $\Sigma^F$;

Compute $k$-th realization $Q_R^{(1)}(k) := \sqrt{n} \| \mathbf{P}^F_J - \hat{\mathbf{P}}_J \|_{(\mathbf{P}^o, \Gamma^o, s_1, s_2)}$;

end

Compute $q_R^{(1)}(\alpha) := \alpha$-quantile of $\{Q_R^{(1)}(k)\}_{k=1}^N$;

**Result:**

$\phi_R^\alpha(\mathbf{X}) := 1 \{ \sqrt{n} \| \hat{\mathbf{P}}_J - \mathbf{P}^o \|_{(\mathbf{P}^o, \Gamma^o, s_1, s_2)} \geq q_R^{(1)}(\alpha) \}$,

$p$-value($\mathbf{X}$) := $\frac{1}{N} \sum_{k=1}^{N} 1 \{ \sqrt{n} \| \hat{\mathbf{P}}_J - \mathbf{P}^o \|_{(\mathbf{P}^o, \Gamma^o, s_1, s_2)} \geq Q_R^{(1)}(k) \}$.
To overcome this difficulty, we split each of the samples $X_1^a, \ldots, X_{2n_a}^a$ and $X_1^b, \ldots, X_{2n_b}^b$ into two equal parts. The second part will be used to learn $P^o$ and $\Gamma^o$ and the first part will be used to construct a test. More specifically, define

$$
\hat{\Sigma}_a \equiv \frac{1}{n_a} \sum_{i=1}^{n_a} X_i^a X_i^a^\top, \quad \hat{\Sigma}_b \equiv \frac{1}{n_b} \sum_{i=1}^{n_b} X_i^b X_i^b^\top, \\
\hat{\Sigma}_a \equiv \frac{1}{n_a} \sum_{i=n_a+1}^{2n_a} X_i^a X_i^a^\top, \quad \hat{\Sigma}_b \equiv \frac{1}{n_b} \sum_{i=n_b+1}^{2n_b} X_i^b X_i^b^\top.
$$

Denote by $\hat{P}_a, \hat{P}_a$ the corresponding projectors of $\hat{\Sigma}_a, \Sigma_a$ associated with $\mathcal{I}_{\mathcal{J}_a}$, and by $\hat{P}_b, \hat{P}_b$ the corresponding projectors of $\hat{\Sigma}_b, \Sigma_b$ associated with $\mathcal{I}_{\mathcal{J}_b}$. Introduce

$$
P \equiv \arg \min_{\text{projector}, \rank(P) = m} \left\{ \|P - \hat{P}_a\|_F^2 + \|P - \hat{P}_b\|_F^2 \right\}.
$$

One can show that it can be easily computed: $P = \Psi \Psi^\top$, where $\Psi \in \mathbb{R}^{d \times m}$ consists of the eigenvectors of $(\hat{P}_a + \hat{P}_b)$ associated with $m$ largest eigenvalues. Fix $\Gamma$ satisfying properties (3.1) for $P$ in an arbitrary way. Define the symmetric (w.r.t. change of sample $a$ and sample $b$) test statistic

$$
Q^{(2)} \equiv \sqrt{\frac{n_a n_b}{n_a + n_b}} \|\hat{P}_a - \hat{P}_b\|_{(P, \Gamma, s_1, s_2)}.
$$

To estimate its distribution, we again employ one of the presented approaches: Bootstrap-based or Frequentist-Bayes related. Both of them are straightforwardly extended from one-sample case and lead to the following random quantities:

$$
Q^{(2)}_B = \sqrt{\frac{n_a n_b}{n_a + n_b}} \|\hat{P}_a - \hat{P}_b\|_{(P, \Gamma, s_1, s_2)},
$$

and

$$
Q^{(2)}_F = \sqrt{\frac{n_a n_b}{n_a + n_b}} \|\hat{P}_a - \hat{P}_b\|_{(P, \Gamma, s_1, s_2)}.
$$

Note that here $\hat{P}_a^B, \hat{P}_b^B, \hat{P}_a^F, \hat{P}_b^F$ correspond only to the first halves of the samples and has nothing to do with the second halves. Now the hope is that $(Q^{(2)}_B | X^a, X^b) \overset{d}{\approx} (Q^{(2)} | \Gamma)$ and $(Q^{(2)}_F | X^a, X^b) \overset{d}{\approx} (Q^{(2)} | \Gamma)$ with high probability. This brings us directly to Algorithm 2.

4 Theoretical properties

Before stating our assumptions and theoretical results, we introduce some important characteristics of the true covariance $\Sigma$ that will appear in the error bounds. In particular, the relative rank of $\Sigma$ (see Jirak and Wahl (2018)) is

$$
r_r(\Sigma) \equiv \sum_{s \neq r} \frac{m_s \mu_s}{\mu_r - \mu_s} + \frac{m_r \mu_r}{\min(\mu_{r-1} - \mu_r, \mu_r - \mu_{r+1})} \quad \text{for all } r \in [q].
$$
Algorithm 2: Two-sample testing procedure

**Input:** Data $X^a = [X^a_1, \ldots, X^a_{2na}]$ and $X^b = [X^b_1, \ldots, X^b_{2nb}]$;
sets $\mathcal{I}^a$ and $\mathcal{I}^b$ of the same size, desired level $\alpha$.

**Hyperparameters:** $s_1, s_2$, number of resampling iterations $N$.
Set $m := |\mathcal{I}^a| = |\mathcal{I}^b|$;
Compute $\Sigma_a := \frac{1}{na} \sum_{i=1}^{na} X^a_i X^a_i^T$ and $\Sigma_a := \frac{1}{na} \sum_{i=na+1}^{2na} X^a_i X^a_i^T$;
Compute $\Sigma_b := \frac{1}{nb} \sum_{i=1}^{nb} X^b_i X^b_i^T$ and $\Sigma_b := \frac{1}{nb} \sum_{i=nb+1}^{2nb} X^b_i X^b_i^T$;
Compute the corresponding projectors $\hat{P}_a$, $\hat{P}_b$;
Compute $\overline{P} := \arg \min_{P: \text{projector, rank}(P) = m} \left\{ \|P - \overline{P}_a\|_F + \|P - \overline{P}_b\|_F^2 \right\}$ using eigendecomposition;
Fix $\Gamma_1 \in \mathbb{R}^{d \times m}$ such that $\Gamma_1^\top \Gamma_1 = I_m$ and $\Gamma_1 \Gamma_1^\top = P_1$;
Fix $\Gamma_2 \in \mathbb{R}^{d \times (d-m)}$ such that $\Gamma_2^\top \Gamma_2 = I_{d-m}$ and $\Gamma_2 \Gamma_2^\top = I_d - P_1$;
Apply Bootstrap-based resampling:
for $k = 1, \ldots, N$ do
    Sample $\eta^a_1, \ldots, \eta^a_{na} \overset{i.i.d.}{\sim} \mathcal{N}(1,1)$ and $\eta^b_1, \ldots, \eta^b_{nb} \overset{i.i.d.}{\sim} \mathcal{N}(1,1)$;
    Compute $\Sigma^B_a := \frac{1}{na} \sum_{i=1}^{na} \eta^a_i X^a_i X^a_i^T$ and $\Sigma^B_a := \frac{1}{na} \sum_{i=na+1}^{2na} \eta^a_i X^a_i X^a_i^T$;
    Compute the corresponding projectors $\hat{P}^B_a$ from $\Sigma^B_a$ and $\hat{P}^B_b$ from $\Sigma^B_b$;
    Compute $k$-th realization $Q_R^{(2)}(k) := \sqrt{\frac{na \cdot nb}{na + nb}} \| (\hat{P}^B_a - \hat{P}_a) - (\hat{P}^B_b - \hat{P}_b) \|_{(\overline{P}, \Gamma^{s_1}, s_2)}$;
end
or apply Frequentist-Bayes related resampling:
for $k = 1, \ldots, N$ do
    Sample $\Sigma^F_a := \frac{1}{na} \cdot \text{Wishart}(na, \hat{\Sigma}_a)$ and $\Sigma^F_b := \frac{1}{nb} \cdot \text{Wishart}(nb, \hat{\Sigma}_b)$;
    Compute the corresponding projectors $\hat{P}^F_a$ from $\Sigma^F_a$ and $\hat{P}^F_b$ from $\Sigma^F_b$;
    Compute $k$-th realization $Q_R^{(2)}(k) := \sqrt{\frac{na \cdot nb}{na + nb}} \| (\hat{P}^F_a - \hat{P}_a) - (\hat{P}^F_b - \hat{P}_b) \|_{(\overline{P}, \Gamma^{s_1}, s_2)}$;
end
Compute $q_R^{(2)}(\alpha) := \alpha$-quantile of $\{Q_R^{(2)}(k)\}_{k=1}^N$;
**Result:**
\[
\phi^R(X^a, X^b) := \frac{1}{N} \sum_{k=1}^N 1 \left\{ \sqrt{\frac{na \cdot nb}{na + nb}} \| \hat{P}_a - \hat{P}_b \|_{(\overline{P}, \Gamma^{s_1}, s_2)} \geq q_R^{(2)}(\alpha) \right\},
\]
\[
p_{\text{value}}(X^a, X^b) := \frac{1}{N} \sum_{k=1}^N 1 \left\{ \sqrt{\frac{na \cdot nb}{na + nb}} \| \hat{P}_a - \hat{P}_b \|_{(\overline{P}, \Gamma^{s_1}, s_2)} \geq Q_R^{(2)}(k) \right\}.
\]
It turns out that the following quantity will play role of effective dimension:

\[
d_J(\Sigma) \triangleq \left( \sum_{r \in J} \left( r_r(\Sigma) \sqrt{\sum_{s \neq r} m_r \mu_r m_s \mu_s \left( \mu_r - \mu_s \right)^2} \right) \right)^{2/3}.
\]

Other important quantities appearing in the theorems are

\[
\kappa_J(\Sigma) \triangleq \min_{r \in J, s \notin J} \sqrt{\mu_r \mu_s |\mu_r - \mu_s|}, \quad \kappa_J(\Sigma) \triangleq \max_{r \in J, s \notin J} \sqrt{\mu_r \mu_s |\mu_r - \mu_s|}, \quad \kappa_J(\Sigma) \triangleq \frac{\kappa_J(\Sigma)}{\kappa_J(\Sigma)}.
\]

The last quantity \( \kappa_J(\Sigma) \) can be interpreted as a kind of condition number, but with respect to splitting the eigenvalues into two groups associated with \( J \) and \( J^c \). Throughout the paper, when it does not cause ambiguity (in context of one-sample problem) we write \( d, \kappa, \kappa \) instead of \( d_J(\Sigma), \kappa_J(\Sigma), \kappa_J(\Sigma), \kappa_J(\Sigma) \), respectively, to keep the notation light.

**Remark 4.1.** Later in Section 5 we will focus on factor models. Under the standard assumptions imposed in that field, we will see that that above quantities in this case are of the following order:

\[
d \asymp m^{5/3}, \quad \kappa \asymp \frac{1}{\sqrt{d}}, \quad \kappa \asymp 1,
\]

where \( m \) plays role of the number of common factors in the model. The fact that the effective dimension \( d \) in this situation does not depend on the full dimension \( d \) (can even be finite) will help us to significantly weaken the relation between \( d \) and \( n \) required for the validity of the tests, compared the the previous works.

### 4.1 Assumptions

We start by specifying the assumptions which will be required in our theorems.

**Assumption 4.1 (Uncorrelatedness).** \( v^T P_J XX^T P_J \tilde{v} \) and \( w^T P_{J^c} XX^T P_{J^c} \tilde{w} \) are uncorrelated for all \( v, \tilde{v}, w, \tilde{w} \in \mathbb{R}^d \).

**Remark 4.2.** Any of the following conditions is sufficient for Assumption 4.1:

(i) \( P_J X \) and \( P_{J^c} X \) are independent (these random vectors are always orthogonal, and consequently uncorrelated; this condition is somewhat stronger);

(ii) The components of \( \Sigma^{-1/2} X \) are independent;

(iii) \( X \) is Gaussian random vector.

Additionally note, that (iii) implies (ii), (ii) implies (i).

**Assumption 4.2 (Tail bound).** \( \Sigma^{-1/2} X \) is jointly sub-Weibull random vector with parameter \( 0 < \beta \leq 2 \) (see Kuchibhotla and Chakrabortty (2018)). That is, there exists a constant \( c > 0 \) such that

\[
\| \Sigma^{-1/2} X \|_{J, \psi_\beta} \triangleq \sup_{u \in S^{d-1}} \| u^T \Sigma^{-1/2} X \|_{\psi_\beta} \leq c < \infty,
\]

\[14\]
where $\| \cdot \|_{\psi_\beta}$ is the Orlicz norm for $\psi_\beta = e^{x\beta} - 1$. The following tail bound takes place:

$$P \left[ |u^\top \Sigma^{-1/2} X| \geq t \right] \leq 2 \exp \left( - (t/c)^\beta \right),$$

for all $u \in S^{d-1}$ and $t > 0$.

**Remark 4.3.** Case $\beta = 2$ corresponds to sub-Gaussian distribution of $X$. We restrict ourselves to the case $\beta \leq 2$, since it is unreasonable to expect tails lighter than Gaussian in applications. Nevertheless, our results extend easily to $\beta > 2$ by replacing $\beta$ with $(\beta \wedge 2)$ in all of the further error bounds.

Let us introduce some auxiliary quantities and rates, which will appear in our bounds:

$$p = p_{d,n,s_1,s_2} \overset{\text{def}}{=} \exp \left( (s_1 + s_2) \log(3n) + 2 \log(d) \right),$$

$$\psi_n \overset{\text{def}}{=} C_\beta c^2 \left( \sqrt{\frac{\log(n) + \log(d)}{n}} + \frac{(\log(n) + \log(d))^{2/\beta}}{n} \right),$$

$$\tilde{\psi}_n \overset{\text{def}}{=} C c^2 \left( \frac{\log(n) + \log(2d^2)}{\sqrt{n}} \right)^{\frac{2}{\beta} + \frac{1}{2}},$$

$$\zeta[\delta] \overset{\text{def}}{=} \delta \left( \log \left( \frac{ep}{\delta} \right) \right)^{1/2} \text{ for all } \delta > 0,$$

$$\vartheta[\delta] \overset{\text{def}}{=} \delta^{1/3} \left( \log \left( \frac{ep}{\delta} \right) \right)^{2/3} \text{ for all } \delta > 0.$$

The constants $C_\beta$ and $C$ are properly chosen and come from the proofs of the theorems in the sequel. The functions $\zeta[\cdot]$ and $\vartheta[\cdot]$ are introduced just for convenience to avoid long expressions with logarithmic factors. Now we state an additional assumption.

**Assumption 4.3.** The following holds:

(i) $\psi_n \max_{r \in J} r(\Sigma) \leq 1/12.$

(ii) $\tilde{\psi}_n \max_{r \in J} r(\Sigma) \leq 1/12.$

### 4.2 Validity

#### 4.2.1 One-sample test

In this subsection we work under $H_0^{(1)}$, so that $P_{\mathcal{J}} = P^*$ and

$$Q^{(1)} = \sqrt{n} \| \tilde{P}_{\mathcal{J}} - P_{\mathcal{J}} \| (P_{\mathcal{J}}, P^*, s_1, s_2).$$

Our first result provides approximation for the distribution of $Q^{(1)}$ under $H_0^{(1)}$.

**Theorem 4.1** (One-sample test; test statistics approximation). Let the data $X$ satisfy Assumptions 4.1, 4.2, 4.3(i). Then there exists a Gaussian vector $Y \in \mathbb{R}^p$, with specific covariance structure (presented in the proof) that depends on $\Sigma$, such that under $H_0^{(1)}$ holds

$$\sup_{z \in \mathbb{R}} \left| P \left[ Q^{(1)} \leq z \right] - P \left[ \max_{j \in [p]} Y_j \leq z \right] \right| \leq \diamondsuit^{(1)},$$
where

\[
\hat{\Phi}^{(1)} = C_\kappa \left\{ \hat{\Phi}^{GA} + \zeta \left[ \sqrt{n} \psi_n^2 d^{3/2} / E \right] \right\},
\]

\[
\hat{\Phi}^{GA} = 8^{1/(2\beta)} \left( \frac{(\log(pm))^7}{n} + c^2 \left( \frac{(\log(2pn^2))^{3+4/\beta}}{n} \right) \right)^{1/2}.
\]

(4.1)

Moreover, the same result holds for spectral norm test statistics \( \tilde{Q}^{(1)} = \sqrt{n}\|\hat{P}_J - P^o\| \), if we take \( s_1 = m, s_2 = d - m \).

This theorem gives understanding of how to prove the next two validity results. For the validity of Approach 1 we need to define an additional quantity and an assumption on it.

**Assumption 4.4.** Define

\[
\Delta_B \overset{\text{def}}{=} C_\beta c^4 \kappa^2 \left( \sqrt{\log(pm) n} + (\log(n))^{2/\beta} (\log(pm))^{4/\beta} \right).
\]

Here again \( C_\beta \) comes from the corresponding proof. Suppose \( \Delta_B \leq 1/2 \).

**Theorem 4.2** (One-sample test; validity of Approach 1). Let the data \( \mathbf{X} \) satisfy Assumptions 4.1, 4.2, 4.3. Also suppose Assumption 4.4 is fulfilled. Then under \( H_0^{(1)} \) with probability \( 1 - 1/n \)

\[
\sup_{z \in \mathbb{R}} \left| P \left[ Q^{(1)} \leq z \right] - P \left[ Q_B^{(1)} \leq z \mid \mathbf{X} \right] \right| \leq \hat{\Phi}^{(1)}_B,
\]

where

\[
\hat{\Phi}^{(1)}_B \overset{\text{def}}{=} C_\kappa \left\{ \hat{\Phi}^{GA} + \zeta \left[ \sqrt{n} (\tilde{\psi}_n + \psi_n)^2 d^{3/2} / E \right] + \psi[\Delta_B] \right\}
\]

with \( \hat{\Phi}^{GA} \) from (4.1).

Moreover, the same result holds for spectral norm test statistics \( \tilde{Q}^{(1)} = \sqrt{n}\|\hat{P}_J - P^o\| \) and \( \tilde{Q}_B^{(1)} = \sqrt{n}\|P_{J}^F - \hat{P}_J\| \), if we take \( s_1 = m, s_2 = d - m \).

Similarly, the validity of Approach 2 requires the following assumption.

**Assumption 4.5.** Define

\[
\Delta_F \overset{\text{def}}{=} |J| C_\beta c^2 \kappa^2 \left( \sqrt{\log(pm) n} + (\log(n))^{1/\beta} (\log(pm))^{2/\beta} \right),
\]

As above, \( C_\beta \) comes from the corresponding proof. Suppose \( \Delta_F \leq 1/2 \).

**Theorem 4.3** (One-sample test; validity of Approach 2). Let the data \( \mathbf{X} \) satisfy Assumptions 4.1, 4.2, 4.3. Also suppose Assumption 4.5 is fulfilled. Then under \( H_0^{(1)} \) with probability \( 1 - 1/n \)

\[
\sup_{z \in \mathbb{R}} \left| P \left[ Q^{(1)} \leq z \right] - P \left[ Q_F^{(1)} \leq z \mid \mathbf{X} \right] \right| \leq \hat{\Phi}^{(1)}_F,
\]

where

\[
\hat{\Phi}^{(1)}_F \overset{\text{def}}{=} C_\kappa \left\{ \hat{\Phi}^{GA} + \zeta \left[ \sqrt{n} (\tilde{\psi}_n + \psi_n)^2 d^{3/2} / E \right] + \psi[\Delta_F] \right\},
\]

with \( \hat{\Phi}^{GA} \) from (4.1).

Moreover, the same result holds for spectral norm test statistics \( \tilde{Q}^{(1)} = \sqrt{n}\|\hat{P}_J - P^o\| \) and \( \tilde{Q}_F^{(1)} = \sqrt{n}\|P_{J}^F - \hat{P}_J\| \), if we take \( s_1 = m, s_2 = d - m \).
The previous two results imply that both Approach 1 and Approach 2 have type-I error close to the desired level \( \alpha \). This is formalized in the following Corollary.

**Corollary 4.4** (One-sample test; type-I error). (i) Assume the conditions of Theorem 4.2 are fulfilled. Define
\[
q_{B}^{(1)}(\alpha) \overset{\text{def}}{=} \inf \{ \gamma > 0 : P \left[ Q_{B}^{(1)} > \gamma \mid X \right] \leq \alpha \}.
\]
Then
\[
\sup_{\alpha \in (0;1)} \left| P \left[ Q^{(1)} > q_{B}^{(1)}(\alpha) \right] - \alpha \right| \leq \diamondsuit_{B}^{(1)} + \frac{1}{n},
\]
where \( \diamondsuit_{B}^{(1)} \) is the total error term from Theorem 4.2.

(ii) Assume the conditions of Theorem 4.3 are fulfilled. Define
\[
q_{F}^{(1)}(\alpha) \overset{\text{def}}{=} \inf \{ \gamma > 0 : P \left[ Q_{F}^{(1)} > \gamma \mid X \right] \leq \alpha \}.
\]
Then
\[
\sup_{\alpha \in (0;1)} \left| P \left[ Q^{(1)} > q_{F}^{(1)}(\alpha) \right] - \alpha \right| \leq \diamondsuit_{F}^{(1)} + \frac{1}{n},
\]
where \( \diamondsuit_{F}^{(1)} \) is the total error term from Theorem 4.3.

**Remark 4.4.** For the sake of illustration, let us treat \( \beta \) as fixed and omit the logarithmic terms. Then the error bounds on the Kolmogorov distance in the previous theorems become more transparent and can be bounded by:
\[
C_{\kappa} \left\{ \left( \frac{(s_{1} + s_{2})^{7}}{n} \right)^{1/8} + \left( \frac{(s_{1} + s_{2})^{4/\beta + 2}}{n} \right)^{1/3} + \frac{1}{\kappa} \left( \frac{(s_{1} + s_{2}) d^{3}}{n} \right)^{1/2} \right\},
\]
which in case of the spectral norm reduces to
\[
C_{\kappa} \left\{ \left( \frac{d^{7}}{n} \right)^{1/8} + \left( \frac{d^{4/\beta + 2}}{n} \right)^{1/3} \right\}.
\]

Note additionally, that with slightly different technique used in previous works of Koltchinskii and Lounici (2017b); Naumov et al. (2019); Silin and Spokoiny (2018), \( d^{3} \) can be replaced by \( d^{2} \). This will improve our bound in case when \( d \asymp d \), however will be worse if \( d \ll d \). Since the main motivation behind our work is Factor Models, where \( d \asymp m^{5/3} \ll d \), we choose to present the result with \( d^{3} \). We preview the bound that will be obtained in case of Factor Models (take \( s_{1} = s_{2} = 1 \) for simplicity):
\[
C \left\{ \frac{1}{n^{1/8}} + m^{5/2} \sqrt{\frac{d}{n}} \right\}.
\]

**4.2.2 Two-sample test**

Similar theoretical properties are obtained for the two-sample problem. Before we state them, we introduce one more version of effective dimension that will show up:
\[
\bar{d}_{j}(\Sigma) \overset{\text{def}}{=} \sum_{r \in j} \sum_{s \notin j} \frac{m_{r} m_{s} \mu_{r} \mu_{s}}{(\mu_{r} - \mu_{s})^{2}}.
\]
Also, we define "total effective dimensions" for two samples as
\[d_{a,b} \overset{\text{def}}{=} (d_{Ja}(\Sigma_a)^{3/2} + d_{Ja}(\Sigma_b)^{3/2})^{2/3},\]
\[\overline{d}_{a,b} \overset{\text{def}}{=} (\overline{d}_{Ja}(\Sigma_a)^{1/2} + \overline{d}_{Ja}(\Sigma_b)^{1/2})^2,\]
and
\[\overline{\kappa}_{a,b} \overset{\text{def}}{=} \overline{\kappa}_{Ja}(\Sigma_a) \vee \overline{\kappa}_{Ja}(\Sigma_b), \quad \kappa_{a,b} \overset{\text{def}}{=} \kappa_{Ja}(\Sigma_a) \wedge \kappa_{Ja}(\Sigma_b), \quad \kappa_{a,b} \overset{\text{def}}{=} \overline{\kappa}_{a,b} / \overline{\kappa}_{a,b}.
\]

Define \(p_{a,b}\) in a similar fashion as \(p\), but with \(n\) replaced by \(n_a + n_b\).

**Theorem 4.5** (Two-sample test; test statistic approximation). Let the data \(X^a\) and \(X^b\) satisfy Assumptions 4.1, 4.2, 4.3(i) (with \(n\) replaced by \(n_a \wedge n_b\)). Additionally, assume \(d_{a,b}^{3/2} \overline{d}_{a,b}^{1/2} \psi_{n_a \wedge n_b} \leq d_{a,b}^{3/2} + \overline{d}_{a,b}\). Then there exists a Gaussian vector \(Y^{a,b} \in \mathbb{R}^{p_{a,b}}\), with specific covariance structure (presented in the proof) that depends on \(\Sigma_a\) and \(\Sigma_b\), such that under \(H_0^{(2)}\) holds
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ Q^{(2)} \leq z \mid \Gamma \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_j^{a,b} \leq z \mid \Gamma \right] \right| \leq \Diamond^{(2)},
\]
with probability \(1 - 1/n_a - 1/n_b\), where
\[\Diamond^{(2)} \overset{\text{def}}{=} C_{\kappa_{a,b}} \left\{ \phi^{GA} + \zeta \left\{ \sqrt{\frac{n_a n_b}{n_a + n_b}} \psi_{n_a \wedge n_b}^2 (d_{a,b}^{3/2} + \overline{d}_{a,b}) / \overline{\kappa}_{a,b} \right\} \right\},\]
\[\phi^{GA} \overset{\text{def}}{=} 8^{3/(2\beta)} \left( \frac{(\log(p_{a,b}(n_a + n_b)))^3}{n_a + n_b} \right)^{1/8} + c^2 \left( \frac{(\log(2p_{a,b}(n_a + n_b)))^{3+4/\beta}}{n_a + n_b} \right)^{1/2} + \frac{1}{n_a} + \frac{1}{n_b}.
\]

To state validity of Approach 1 and Approach 2 in two-sample problem, let \(\Delta_B^{a,b}\) be defined as \(\Delta_B\) and \(\Delta_F^{a,b}\) be defined as \(\Delta_F\) with \(n, \kappa, |J|\) replaced with \(n_a \wedge n_b, \kappa_{a,b}, |J_a| \vee |J_b|\), respectively. Then we have the following theorems.

**Theorem 4.6** (Two-sample test; validity of Approach 1). Let the data \(X^a\) and \(X^b\) satisfy Assumptions 4.1, 4.2, 4.3 (with \(n\) replaced by \(n_a \wedge n_b\)). Additionally, assume \(\Delta_B^{a,b} \leq 1/2\) and \(d_{a,b}^{3/2} \overline{d}_{a,b} \psi_{n_a \wedge n_b} \leq d_{a,b}^{3/2} + \overline{d}_{a,b}\). Then under \(H_0^{(2)}\) with probability \(1 - 1/n_a - 1/n_b\)
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ Q^{(2)} \leq z \mid \Gamma \right] - \mathbb{P} \left[ Q_B^{(2)} \leq z \mid X^a, X^b \right] \right| \leq \Diamond_B^{(2)},
\]
where
\[\Diamond_B^{(2)} \overset{\text{def}}{=} C_{\kappa_{a,b}} \left\{ \phi^{GA} + \zeta \left\{ \sqrt{\frac{n_a n_b}{n_a + n_b}} (\psi_{n_a \wedge n_b} + \overline{\psi}_{n_a \wedge n_b})^2 (d_{a,b}^{3/2} + \overline{d}_{a,b}) / \overline{\kappa}_{a,b} \right\} + \partial \left[ \Delta_B^{a,b} \right] \right\},\]
with \(\phi^{GA}\) from (4.2).

**Theorem 4.7** (Two-sample test; validity of Approach 2). Let the data \(X^a\) and \(X^b\) satisfy Assumptions 4.1, 4.2, 4.3 (with \(n\) replaced by \(n_a \wedge n_b\)). Additionally, assume \(\Delta_F^{a,b} \leq 1/2\) and \(d_{a,b}^{3/2} \overline{d}_{a,b} \overline{\psi}_{n_a \wedge n_b} \leq d_{a,b}^{3/2} + \overline{d}_{a,b}\). Then under \(H_0^{(2)}\) with probability \(1 - 1/n_a - 1/n_b\)
\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ Q^{(2)} \leq z \mid \Gamma \right] - \mathbb{P} \left[ Q_F^{(2)} \leq z \mid X^a, X^b \right] \right| \leq \Diamond_F^{(2)},
\]
where

$$
\hat{\Diamond}^{(2)}_F \overset{\text{def}}{=} C_{n_a, n_b} \left\{ \sqrt{\frac{n_a n_b}{n_a + n_b}} \left[ \psi_{n_a \wedge n_b} + \tilde{\psi}_{n_a \wedge n_b} \right]^2 \left( \hat{d}_{a,b}^{3/2} + \hat{\Delta}_{a,b} \right) + \vartheta \left[ \Delta_{F,a,b} \right] \right\},
$$

with $\hat{\Diamond}^{GA}$ from (4.2).

**Remark 4.5.** The condition $\hat{d}_{a,b}^{3/2} \hat{\Delta}_{a,b} \leq d_{a,b}^{3/2} + \hat{d}_{a,b}$ in the theorems above is technical and is imposed just to slightly simplify the bounds.

Similarly to one-sample case, we have the following guarantees for type-I error.

**Corollary 4.8** (Two-sample test; type-I error). (i) Assume the conditions of Theorem 4.6 are fulfilled. Define

$$
q^{(2)}_B(\alpha) \overset{\text{def}}{=} \inf \left\{ \gamma > 0 : \Pr \left[ \mathcal{Q}^{(2)}_B > \gamma \mid \mathcal{X} \right] \leq \alpha \right\}.
$$

Then

$$
\sup_{\alpha \in (0; 1)} \left| \Pr \left[ \mathcal{Q}^{(2)} > q^{(2)}_B(\alpha) \right] - \alpha \right| \leq \hat{\Diamond}^{(2)}_B + \frac{1}{n_a} + \frac{1}{n_b},
$$

where $\hat{\Diamond}^{(2)}_B$ is the complete error term from Theorem 4.6.

(ii) Assume the conditions of Theorem 4.7 are fulfilled. Define

$$
q^{(2)}_F(\alpha) \overset{\text{def}}{=} \inf \left\{ \gamma > 0 : \Pr \left[ \mathcal{Q}^{(1)}_F > \gamma \mid \mathcal{X} \right] \leq \alpha \right\}.
$$

Then

$$
\sup_{\alpha \in (0; 1)} \left| \Pr \left[ \mathcal{Q}^{(2)} > q^{(2)}_F(\alpha) \right] - \alpha \right| \leq \hat{\Diamond}^{(2)}_F + \frac{1}{n_a} + \frac{1}{n_b},
$$

where $\hat{\Diamond}^{(2)}_F$ is the complete error term from Theorem 4.7.

As in one-sample case, more transparent expression for the error bound can be seen in Remark 4.4, with $n$ replaced by $n_a \wedge n_b$ and $d$ replaced by $d_{a,b}^{3/2} + \hat{d}_{a,b}^{3/2}$.

**Remark 4.6.** One can also consider two-sample tests based on the spectral norm. In this case, there is no need to split the sample and condition on $\mathcal{F}$, as there are no unknown rotations involved. Similar results holds true for spectral norm test statistics

$$
\tilde{\mathcal{Q}}^{(2)}_B = \sqrt{\frac{n_a n_b}{n_a + n_b}} \| \hat{\mathcal{P}}_a - \hat{\mathcal{P}}_b \|,
$$

$$
\tilde{\mathcal{Q}}^{(2)}_F = \sqrt{\frac{n_a n_b}{n_a + n_b}} \| (\mathcal{P}^B_a - \hat{\mathcal{P}}_a) - (\mathcal{P}^B_b - \hat{\mathcal{P}}_b) \|,
$$

$$
\tilde{\mathcal{Q}}^{(2)}_F = \sqrt{\frac{n_a n_b}{n_a + n_b}} \| (\mathcal{P}^F_a - \hat{\mathcal{P}}_a) - (\mathcal{P}^F_b - \hat{\mathcal{P}}_b) \|,
$$

if we put $s_1 = m$, $s_2 = d - m$ in the error bounds (here $n_a$ and $n_b$ are the sizes of the whole samples $a$ and $b$).
4.3 Power analysis

After we understood the behavior of our procedures under the null hypothesis, we are also interested in the behavior under the alternatives. In particular, the question is whether the power of our procedure goes to 1 and under which conditions. We first answer this question for the one-sample test.

**Theorem 4.9** (One-sample test; power). Under $H_1^{(1)}$ assume $\|P_J - P^0\|_{(P^0, \Gamma^0, s_1, s_2)} \geq \lambda_n/\sqrt{n}$, where $\lambda_n$ satisfies

$$\liminf_{n \to \infty} \frac{\lambda_n}{\sqrt{n} \left( \psi_n + \tilde{\psi}_n \right) d_1^{1/2} + (\psi_n + \tilde{\psi}_n)^2 d_3^{3/2} / 2} \geq C$$

for some absolute constant $C > 0$. Then

(i) the power of Approach 1

$$P \left[ Q^{(1)} > \gamma_B^{(1)}(\alpha) \right] \to 1 \text{ as } n \to \infty;$$

(ii) the power of Approach 2

$$P \left[ Q^{(1)} > \gamma_F^{(1)}(\alpha) \right] \to 1 \text{ as } n \to \infty.$$

An important question is how restrictive the assumption $\|P_J - P^0\|_{(P^0, \Gamma^0, s_1, s_2)} \geq \lambda_n/\sqrt{n}$ is. It would be more natural to assume $\|P_J - P^0\| \geq \lambda_n/\sqrt{n}$, which is significantly weaker condition in the worst case, when the bound $\|P_J - P^0\| \leq 2 \sqrt{m(d-m) \psi_n^2 + \tilde{\psi}_n^2}$ is close to being tight. However, due to the first two “power enhancement” terms in the definition of $\|\cdot\|_{(P^0, \Gamma^0, s_1, s_2)}$, the bound is tight only in very specific cases, while if $P_J - P^0$ is random (not adversarially chosen), we can expect $\|P_J - P^0\|_{(P^0, \Gamma^0, s_1, s_2)} - \|P_J - P^0\|_{(P^0, \Gamma^0, s_1, s_2)}$ (from our numerical experiences). This makes the assumptions of Theorem 4.9 reasonable.

Now we provide guarantees for power of the two-sample tests.

**Theorem 4.10** (Two-sample test; power). Under $H_1^{(2)}$ assume

$$\|P_a - P_b\| \geq \lambda_{n_a, n_b} \cdot 2 \sqrt{m(d-m) / s_1 s_2} \sqrt{n_a + n_b / n_a n_b},$$

where $\lambda_{n_a, n_b}$ satisfies

$$\liminf_{n_a, n_b \to \infty} \frac{\lambda_{n_a, n_b}}{\sqrt{n_a n_b / (n_a + n_b)} \left( \psi_{n_a \wedge n_b} + \tilde{\psi}_{n_a \wedge n_b} \right) d_1^{1/2} + (\psi_{n_a \wedge n_b} + \tilde{\psi}_{n_a \wedge n_b})^2 d_3^{3/2}} \geq C$$

for some absolute constant $C > 0$. Then

(i) the power of Approach 1

$$P \left[ Q^{(2)} > \gamma_B^{(2)}(\alpha) \right] \to 1 \text{ as } n_a, n_b \to \infty;$$
(ii) the power of Approach 2

$$\mathbb{P}\left[ Q^{(2)} > \gamma^{(2)}_F (\alpha) \right] \to 1 \quad \text{as } n_a, n_b \to \infty.$$ 

One can notice that here, unlike the one-sample case, we make the assumption for the spectral norm

$$\|P_a - P_b\| \geq \lambda_{n_a,n_b} \cdot 2 \sqrt{\frac{m(d-m)}{s_1s_2}} \sqrt{\frac{n_a+n_b}{n_a n_b}},$$

because it is more convenient for the proof. However, we pay the factor $2 \sqrt{m(d-m)/(s_1s_2)}$ for avoiding spectral norm as our test statistic. Again, this factor corresponds to worst-case scenario for very specific $P_a - P_b$, while in most cases this condition can be much weaker, i.e.

$$\|P_a - P_b\| \gtrsim \lambda_{n_a,n_b} \sqrt{\frac{n_a+n_b}{n_a n_b}}$$

for the most of non-adversarial choices of pairs of $P_a$ and $P_b$.

Due to the space limitations, the optimality analysis of the presented tests is left for the future work.

5 Application to Factor Models

Factor model (FM) specifies the data generating process to be

$$X_i = B f_i + \xi_i \quad \text{for } i \in [n],$$

where

- $B \in \mathbb{R}^{d \times m}$ is deterministic loading matrix,
- $f_i \in \mathbb{R}^m$ is a random vector of $m$ common factors,
- $\xi_i \in \mathbb{R}^d$ is a random idiosyncratic component.

If we put $F \overset{\text{def}}{=} [f_1, \ldots, f_n]^\top \in \mathbb{R}^{n \times m}$ and $\Xi \overset{\text{def}}{=} [\xi_1, \ldots, \xi_n] \in \mathbb{R}^{d \times n}$, the factor model can be rewritten in matrix form

$$X = BF^\top + \Xi.$$

It is natural to assume that $\{\xi_i\}_{i=1}^n$ are uncorrelated with $\{f_i\}_{i=1}^n$. In addition, for simplicity, we assume $\{f_i\}_{i=1}^n$ are i.i.d. and so are $\{\xi_i\}_{i=1}^n$. In literature this assumption is often relaxed to strong mixing condition, allowing weak dependence between pairs of consecutive factors and idiosyncratic components; we believe our general results can be extended to weakly dependent $X_1, \ldots, X_n$ as well, however we stick to original i.i.d. framework to avoid technical details.

As in the literature, it is important to mention, that FM is not identifiable. In particular, for any invertible $H \in \mathbb{R}^{m \times m}$ it holds $BF^\top = (BH)(H^{-1}F^\top)$, so that the loading matrix $BH$ and the factors $F(H^{-1})^\top$ are as good in explaining $X$ as $B$ and $F$. However, $\overline{\text{span}}[B]$ and $\overline{\text{span}}[F]$ are identifiable; indeed, for any $H$ as above holds $\overline{\text{span}}[B] = \overline{\text{span}}[BH]$ and $\overline{\text{span}}[F] =$
span \([F(H^{-1})^\top]\). Our work exploits terminology of spectral projectors rather than subspaces, so we remind that there is one-to-one correspondence between subspace \(\text{span}[A]\) and the projector \(A(A^\top A)^{-1}A^\top\) for any matrix \(A\) with linearly independent columns.

**Remark 5.1.** We also note that usually it is assumed that \(\text{Cov}[f_i] = I_m\) and \(B^\top B\) is diagonal, in order to bring some concreteness in derivations. This reduces ambiguity of parametrization but does not solve completely the identifiability issue. For our purposes this assumption does not play any role.

The covariance matrix under this model looks like

\[
\Sigma = B \text{Cov}[f_i] B^\top + \text{Cov}[\xi_1].
\]

(5.2)

We will be interested in \(m\) principal eigenvectors of \(\Sigma\). Define \(J\) so that \(I_J = \{1, \ldots, m\}\), implying that \(P_J\) is the projector onto subspace spanned by \(m\) principal eigenvectors of \(\Sigma\).

Before formulating specific hypothesis and applying our general scheme, let us recall standard assumptions from FM literature and their implications on the rate in our general framework.

**Assumption 5.1.** The eigenvalues of \(\Sigma\) are distinct and there exist absolute constants \(L_1, L_2, L_3 > 0\) such that

- \(L_1d \geq \mu_1 > \ldots > \mu_m \geq L_2d\).
- \(L_2 \geq \mu_{m+1} > \ldots > \mu_d \geq L_3\).

This assumption readily implies the following:

\[
d \approx m^{5/3}, \quad \kappa \approx \kappa \approx \frac{1}{\sqrt{d}}, \quad \kappa \approx 1,
\]

so if we were to apply out testing procedure for the \(m\)-dimensional principal eigenspace, the error rate would be

\[
C \left\{ \left( \frac{(s_1 + s_2)^7}{n} \right)^{1/8} + \left( \frac{(s_1 + s_2)^{4/3+2}}{n} \right)^{1/3} + m^{5/2} \left( \frac{(s_1 + s_2)d}{n} \right)^{1/2} \right\},
\]

or, if \(s_1 = s_2 = 1\), simply

\[
C \left\{ \frac{1}{n^{1/8}} + m^{5/2} \sqrt{\frac{d}{n}} \right\}.
\]

Now we demonstrate how our general framework reduces to testing loading matrices. From (5.2), it is clear that \(B\) is closely related to the space spanned by the eigenvectors of top \(m\) eigenvalues; see Proposition 5.1 below. At the same time, by multiplying \(B^\top\) on both sides of equation (5.1), assuming the that noise is smoothed out, we have \(f_i \approx (B^\top B)^{-1}B^\top X_i\). Since the matrix \(B^\top B\) plays only the normalization role, \(B^\top X_i\) is really the estimate of the latent factors. Thus, testing \(B\) lying in a specific space amounts to testing whether the latent factors
are the known factors such as the famous Fama-French 3-factor or 5-factor models, see Fama and French (1993, 2015).

Suppose we have some guess \( B^\circ \) for the unknown underlying loading matrix \( B \). Define \( P^\circ = B^\circ (B^\circ \top B^\circ)^{-1} B^\circ \top \), i.e. \( P^\circ \) is projector onto \( \text{span}[B^\circ] \). Another scenario could be that instead of \( B^\circ \) we are given projector \( P^\circ \) from the very beginning. The corresponding testing problem writes as

\[
H_0 : \text{span}[B] = \text{span}[B^\circ] \quad \text{vs} \quad H_1 : \text{span}[B] \neq \text{span}[B^\circ]
\]

or equivalently

\[
H_0 : P^\circ = B(B^\top B)^{-1} B^\top \quad \text{vs} \quad H_1 : P^\circ \neq B(B^\top B)^{-1} B^\top
\]

The following proposition attempts to bridge the gap between this testing problem and our framework.

**Proposition 5.1.** Under Assumption 5.1, it holds

\[
\|P_J - B(B^\top B)^{-1} B^\top\| = O\left(\frac{1}{d}\right).
\]

We notice that the true projector \( P_J \) of \( \Sigma \) onto the \( m \) principal directions is not exactly corresponds to \( \text{span}[B] \). This is not satisfactory for us, because if we push this additional error term through the proof, we will get another \( \sqrt{n/d} \) term in the final bound, while we already have \( \sqrt{d/n} \) term. This will make our results meaningless. However, we can artificially remove the contribution of idiosyncratic components to our underlying eigenspaces by assuming additional conditions on the interaction between factors and idiosyncratic components.

**Proposition 5.2.** Under condition \( \text{Cov}[\xi_1] B = O_{d \times m} \), it holds

\[
P_J = B(B^\top B)^{-1} B^\top.
\]

This allows to rewrite the hypothesis in familiar form:

\[
H_0 : P_J = P^\circ \quad \text{vs} \quad H_1 : P_J \neq P^\circ
\]

Now we can directly apply Algorithm 1. Our procedure uses \( \hat{P}_J \), which naturally arises in FM context, since in POET (see Fan et al. (2013)) \( B \) is estimated by

\[
\hat{B} \overset{\text{def}}{=} [\hat{\sigma}_1 \hat{u}_1, \ldots, \hat{\sigma}_m \hat{u}_m],
\]

leading exactly to \( \hat{B}(\hat{B}^\top \hat{B})^{-1} \hat{B}^\top = \hat{P}_J \). As before, one can use the test statistic \( Q^{(1)} = \sqrt{n}\|\hat{P}_J - P^\circ\|_{(P^\circ, \Gamma_{n_1, n_2})} \), whose distribution can be approximated by one of two approaches.

Likewise, the two-sample problem can be solved. As explained above, this amounts to test whether the latent factors are the same between two groups (e.g. treatment vs. control, pre-financial crisis vs. post financial crisis). If we have two samples \( X_1^a, \ldots, X_{2n_a}^a \) and \( X_1^b, \ldots, X_{2n_b}^b \).
generated from two FMs, e.g. with $B^a, F^a$ and $B^b, F^b$, respectively, and we want to understand whether their loading matrices span the same subspace, this is equivalent to testing

$$H_0 : \ P_a = P_b \quad \text{vs} \quad H_1 : \ P_a \neq P_b$$

where $P_a$ and $P_b$ are projectors onto $m$ principal eigenspaces of underlying covariance matrices of the samples $a$ and $b$, respectively. Algorithm 2 can be employed to deal with such a problem.

6 Simulation studies

6.1 Construction of the covariance matrix for null and alternative hypothesis

In order to clearly describe the setup of our experiments, we start with a toy example on a plane, i.e. $d=2$. Suppose we are interested in testing whether the first principal direction of our 2-dimensional data is aligned with the first axis of our coordinate system, i.e. $J = \{1\}$, $m = 1$ and the hypothetical leading eigenvector is $u^o = [1,0]^\top$. In this case, the spectral projector is 

$$P^o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$ 

To empirically study the power of our method, we generate the data in such a way that its leading eigenvector is obtained by rotating $u^o$ by angle $\varphi$, i.e. $u_\varphi = [\cos \varphi, \sin \varphi]^\top$, while the orthogonal direction is given by $v_\varphi = [-\sin \varphi, \cos \varphi]^\top$. The associated true projectors are 

$$P_1 = u_\varphi u_\varphi^\top = \begin{bmatrix} \cos^2 \varphi & \cos \varphi \sin \varphi \\ \cos \varphi \sin \varphi & \sin^2 \varphi \end{bmatrix}, \quad P_2 = v_\varphi v_\varphi^\top = \begin{bmatrix} \sin^2 \varphi & -\cos \varphi \sin \varphi \\ -\cos \varphi \sin \varphi & \cos^2 \varphi \end{bmatrix}.$$ 

If the corresponding variances along these directions (eigenvalues of the covariance matrix) are $\mu_1$ and $\mu_2$ ($\mu_1 > \mu_2$), then the true covariance matrix of the data is formed as 

$$\Sigma^{(\varphi)} = \mu_1 P_1 + \mu_2 P_2 = \begin{bmatrix} \mu_1 \cos^2 \varphi + \mu_2 \sin^2 \varphi & (\mu_1 - \mu_2) \cos \varphi \sin \varphi \\ (\mu_1 - \mu_2) \cos \varphi \sin \varphi & \mu_1 \sin^2 \varphi + \mu_2 \cos^2 \varphi \end{bmatrix}.$$ 

Thereby, $\varphi = 0$ corresponds to null hypothesis $P^o = P_1$, while under the alternative the larger deviations of angle $\varphi$ from 0 (until one point) mean that $P_1$ is further from $P^o$. So, the suitable data for our experiment can be generated from some distribution with the covariance matrix $\Sigma^{(\varphi)}$ with varying $\varphi$. Our goal is to verify that with growing $\varphi$ our methods reject the null hypothesis more often and eventually this probability approaches 1 and check the size of the test when $\varphi = 0$. See Figure 2 for visualization of the construction.

Now we extend this setting to higher dimensions. In dimension $d$ we are interested in the subspace spanned by $m$ leading eigenvectors, i.e. $J = \{1, \ldots, m\}$ (for simplicity we explain the procedure assuming all eigenvalues are distinct; it will be clear how to extend the construction to the case of multiplicities within the first $m$ eigenvalues and within the last $(d-m)$ eigenvalues).
Figure 2: Construction of alternative hypothesis data. Here in yellow we depict sub-level sets of Gaussian density with mean zero and covariance $\Sigma^{(0)}$ and $\Sigma^{(\varphi)}$ in order to understand how the clouds of data look like in each case.

Without loss of generality, we assume that under null hypothesis the eigenvectors are aligned with the axes of the coordinate system, so that the hypothetical projector is

$$P^o = \begin{bmatrix} I_m & \mathbf{0}_{m \times (d-m)} \\ \mathbf{0}_{(d-m) \times m} & \mathbf{0}_{(d-m) \times (d-m)} \end{bmatrix},$$

and the default covariance matrix is diagonal with descending entries and characterized only by eigenvalues $\mu_1 > \ldots > \mu_m > \mu_{m+1} > \ldots > \mu_d$:

$$\Sigma^{(0)} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_m \\ \mu_{m+1} \\ \vdots \\ \mu_d \end{bmatrix},$$

To generate the data under alternative, we rotate the plane containing the first and $(m+1)$-th axes by the angle $\varphi$, i.e. the leading eigenvector becomes

$$u_\varphi = \underbrace{[\cos \varphi, 0, \ldots, 0, \sin \varphi, 0, \ldots, 0]}_m,$$

while $(m+1)$-th eigenvector turns into

$$v_\varphi = \underbrace{[-\sin \varphi, 0, \ldots, 0, \cos \varphi, 0, \ldots, 0]}_m.$$

The covariance matrix is then

$$\Sigma^{(\varphi)} = \mu_1 u_\varphi u_\varphi^\top + \sum_{r=2}^m \mu_r P_r + \mu_{m+1} v_\varphi v_\varphi^\top + \sum_{r=m+2}^d \mu_r P_r.$$
or explicitly

\[
\Sigma^{(\varphi)} = \begin{bmatrix}
\mu_1 \cos^2 \varphi + \mu_{m+1} \sin^2 \varphi & 0 & \ldots & 0 & (\mu_1 - \mu_{m+1}) \cos \varphi \sin \varphi & 0 & \ldots & 0 \\
0 & \mu_2 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \mu_m & 0 & \ldots & \ldots & \ldots & 0 \\
(\mu_1 - \mu_{m+1}) \cos \varphi \sin \varphi & 0 & \ldots & 0 & \mu_1 \sin^2 \varphi + \mu_{m+1} \cos^2 \varphi & 0 & \ldots & 0 \\
0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \mu_d
\end{bmatrix}.
\]

### 6.2 Description of regimes and data distributions

In our experiments we focus on three regimes:

- **Factor Model regime**: we take \( m = 8 \), and \( \mu_1 = 5d \), \( \mu_2 = 4d \), \( \mu_3 = 3.5d \), \( \mu_4 = 3d \), \( \mu_5 = 2.5d \), \( \mu_6 = 2d \), \( \mu_7 = 1.5d \), \( \mu_8 = d \) and \( \mu_9, \ldots, \mu_d \) uniformly distributed in \([0.5; 1.5]\) and sorted.

- **Spiked regime**: we take \( m = 1 \) and \( \mu_1 = 10 \), \( \mu_2 = 6 \), \( \mu_3 = 3 \), \( \mu_4 = 1 \) (of multiplicity \( d - 3 \)).

- **Decay regime**: we take \( m = 5 \) and \( \mu_1 = 10 \), \( \mu_2 = 9 \), \( \mu_3 = 8 \), \( \mu_4 = 7 \), \( \mu_5 = 6 \), \( \mu_k = 2^{-(k-6)} \) for \( k = 6, \ldots, d \).

We consider two types of data distributions:

- **Gaussian distribution** with mean zero and proper covariance \( \Sigma \).

- **Laplace distribution**: we generate components of vector \( \tilde{X} \) as independent Laplace r.v.’s with scale parameter \( 1/\sqrt{2} \) (so that each component has unit variance), and then put our observation \( X = \Sigma^{1/2} \tilde{X} \) (so that \( X \) has covariance matrix \( \Sigma \)).

Once we fix a regime, a data distribution and methods that we want to compare, we conduct the simulations for the sample size in range \( n \in \{500, 1500, 5000\} \) and the dimension in range \( d \in \{50, 150\} \). Significance level is fixed to be \( \alpha = 0.05 \). In one-sample problem, for each \( n, d \) we perform the following:

- We try a number of angles \( \varphi \) (including \( \varphi = 0 \)) — they are chosen differently in different settings in order to illustrate the transition of the power from \( \alpha \) to 1.

- For each nonzero angle \( \varphi \) we generate 100 samples \( \tilde{X} \) of size \( n \) in dimension \( d \) with the covariance matrix \( \Sigma^{(\varphi)} \) specified by the formula above and regime. For angle \( \varphi = 0 \) we generate 1000 samples, since it is important to estimate type-I error accurately.

- For each sample we apply each method to test hypothesis \( \mathbf{P}_\mathcal{J} = \mathbf{P}^0 \) vs \( \mathbf{P}_\mathcal{J} \neq \mathbf{P}^0 \). Since some of the methods are resampling-based, we fix the number of resampling \( N = 2000 \).
We estimate the power (for non-zero angles) and type-I error (for $\varphi = 0$) of the test simply as the fraction of samples, for which the null hypothesis was rejected.

The steps for two-sample problem are similar, but $X^a$ generated from distribution with covariance matrix $\Sigma^{(\varphi)}$ and $X^b$ generated from distribution with covariance matrix $\Sigma^{(-\varphi)}$. The testing problem is changed accordingly.

### 6.3 Experimental results

Now we describe three scenarios. In each scenario we compare our methods with the methods from previous literature, suitable for each particular situation.

**Scenario 1**

In this scenario we consider one-sample problem in Factor Model regime with Laplace distribution. We compare the following methods already discussed above:

- **“Fr-Bootstrap”**: Frobenius norm test statistic + Bootstrap (Naumov et al. (2019)).
- **“Fr-Bayes”**: Frobenius norm test statistic + Frequentist Bayes (Silin and Spokoiny (2018)).
- **“Spectral-1”**: Spectral norm test statistic $\tilde{Q}(1)$ + Approach 1.
- **“Spectral-2”**: Spectral norm test statistic $\tilde{Q}(1)$ + Approach 2.
- **“New-1”**: $\| \cdot \|_{(P,\Gamma,s_1,s_2)}$-norm test statistic $Q^{(1)}$ (with $s_1 = s_2 = 1$) + Approach 1.
- **“New-2”**: $\| \cdot \|_{(P,\Gamma,s_1,s_2)}$-norm test statistic $Q^{(1)}$ (with $s_1 = s_2 = 1$) + Approach 2.

In some of the settings (with relatively large $n$ and $d$) we are not able to run bootstrap based methods (“Fr-Bootstrap”, “Spectral-1”, “New-1”), since their computational time is too intensive.

The results are presented in Figure 3. We observe that the bootstrap-based methods are extremely conservative, almost never rejecting null hypothesis. This also implies very weak power of such methods. Our procedures are also slightly conservative, but the power of “New-1” and “New-2” significantly outperforms the methods based on Spectral and Frobenius norms. In further scenarios, we exclude bootstrap-based approaches.

**Scenario 2**

The next scenario considers one-sample problem in spiked regime with Gaussian distribution. We compare the following methods:

- **“HPV-LeCam”**: Le Cam optimal test (Hallin et al. (2010))
- **“Fr-DataDriven”**: Frobenius norm test statistic + Sample splitting strategy (Koltchinskii and Lounici (2017c))
- **“Spectral-2”**: same as in the previous scenario.
“New-2”: same as in the previous scenario.

The results are presented in Figure 4. Main conclusion here is that “HPV-LeCam” dramatically fails when the sample size is not significantly larger than the dimension, and even in the opposite case its power is quite weak. Quite unexpectedly, “Fr-DataDriven” behaves well under null (even though this method requires the dimension going to infinity). However, its power is inferior to “Spectral-2” and “New-2”, which perform very similar in this setting, though again slightly conservative under null.

Scenario 3
In the last scenario we focus on two-sample problem in decay regime with Laplace distribution. We compare the following methods:

• “Schott”: the procedure proposed in Schott (1988)
• “Fujioka”: the procedure proposed in Fujioka (1993).
• “Spectral-2”: Spectral norm test statistic $\tilde{Q}^{(2)}$ + Approach 2.
• “New-2”: $\| \cdot \|_{(P_J, \Gamma, s_1, s_2)}$-norm test statistic $Q^{(2)}$ (with $s_1 = s_2 = 1$) + Approach 2.

The results are presented in Figure 5. The quality of the four compared methods in this scenario is approximately similar, with “New-2” being slightly weaker then the others. The explanation is that “New-2” requires to split the sample into two halves, and effectively only half of the data is used to measure the discrepancy. Another important observation is that in this scenario with decay regime, the behaviour under null is very stable, and doesn’t really seem to depend on the dimension. This promises that the type-I error bounds can be made dependant on some notion of effective rank, rather then the full dimension.

7 Discussion

7.1 Building confidence sets for $P_J$

Even though our work focuses on hypothesis testing, the idea can be used for constructing the confidence sets for the true spectral projector $P_J$ from the data $[X_1, \ldots, X_{2n}] = X$. Split the sample into two equal parts, compute the sample covariance matrices $\hat{\Sigma}$ and $\Sigma$ based on the first and second halves of the sample, respectively. Let $\hat{P}_J$ and $\bar{P}_J$ be the corresponding projectors. Fix $\Gamma$ satisfying (3.1) for $P_J$. For a given confidence level $(1 - \alpha)$, consider sets

$$C_{B}^{1-\alpha}(X) \overset{\text{def}}{=} \left\{ P \in \mathbb{R}^{d \times d} \text{ projector of rank } m : \sqrt{n}\|P - \hat{P}_J\|_{(P_J, \Gamma, s_1, s_2)} \leq q^B_{\alpha} \right\},$$
$$C_{F}^{1-\alpha}(X) \overset{\text{def}}{=} \left\{ P \in \mathbb{R}^{d \times d} \text{ projector of rank } m : \sqrt{n}\|P - \bar{P}_J\|_{(P_J, \Gamma, s_1, s_2)} \leq q^F_{\alpha} \right\},$$

where $q_{\alpha}^{B}$ and $q_{\alpha}^{F}$ are the $\alpha$-quantiles of $(\sqrt{n}\|P_J^B - \hat{P}_J\|_{(P_J, \Gamma, s_1, s_2)} \mid X)$ and $(\sqrt{n}\|P_J^F - \bar{P}_J\|_{(P_J, \Gamma, s_1, s_2)} \mid X)$, respectively ($P_J^B$ and $P_J^F$ depend on the first half of the sample.
Figure 3: Experiments for Scenario 1: One-sample problem, FM regime with $m = 8$, Laplace distribution.
Figure 4: Experiments for Scenario 2: One-sample problem, spiked regime with $m = 1$, Laplace distribution.
Figure 5: Experiments for Scenario 3: Two-sample problem, decay regime with $m = 5$, Laplace distribution.
only). One can show that the theoretical properties of the coverage probabilities of these sets are similar to the theoretical properties of type-I error in one-sample testing problem.

7.2 Why do we use this test statistic?

The reasons behind very non-trivial construction of our test statistic are partially similar to Han et al. (2016), which addresses similar testing problem, but for covariance matrices, rather than spectral projectors. Specifically, Han et al. (2016) try to approximate the distribution of $\|\hat{\Sigma} - \Sigma\|$ and apply the bootstrap inference for it. So, their original idea is to work with spectral norm. However, to approximate the distribution of $\|\hat{\Sigma} - \Sigma\|$ they require $d^9 \ll n$, and to approximate the distribution of $\|\hat{\Pi_J} - \Pi_J\|$ we need $d^7 \ll n$. To avoid this restrictive assumption, Han et al. (2016) introduce a parameter $s$, which helps to connect $\|\hat{\Sigma} - \Sigma\|$ with $\|\hat{\Sigma} - \Sigma\|_{\text{max}}$ in a “smooth” way: in particular, they consider the $s$-sparse largest eigenvalue

$$
\sup_{v \in \mathcal{V}(s,d)} v^T (\hat{\Sigma} - \Sigma) v,
$$

where $\mathcal{V}(s,d) \overset{\text{def}}{=} \{v \in S^{d-1} : |\text{supp}(v)| \leq s\}$. The quality of approximation of distribution of this quantity is expressed in terms of $s^9/n$ (omitting logarithmic factors), as we take supremum over smaller set. We could try to follow the same logic and work with the test statistic (here we focus on one one-sample framework for simplicity)

$$
\sup_{v \in \mathcal{V}(s,d)} v^T (\hat{\Pi_J} - \Pi_J) v,
$$

however, in this case it follows from our proof that under $H_0^{(1)}$

$$\text{Var}[v^T (\hat{\Pi_J} - \Pi_J) v] \asymp C_\Sigma \cdot v^T \Pi_J v \cdot v^T \Pi_J v,$$

which prevents us from applying the main tool in our analysis, Gaussian approximation (see Chernozhukov et al. (2013), Theorem 2.2), as we cannot guarantee the lower bound

$$\text{Var}[v^T (\hat{\Pi_J} - \Pi_J) v] \geq c_1 > 0$$

uniformly over $v$. In the case of covariance matrices from Han et al. (2016) this problem is solved either by assuming $\lambda_{\text{min}}(\Sigma) \geq c_1 > 0$, or by considering the normalized version

$$\sup_{v \in \mathcal{V}(s,d)} \frac{v^T (\hat{\Sigma} - \Sigma) v}{v^T \Sigma v}.$$

In our situation this normalization would lead to

$$\sup_{v \in \mathcal{V}(s,d)} \frac{v^T (\hat{\Pi_J} - \Pi_J) v}{v^T \Pi_J v \cdot v^T (I_d - \Pi_J) v},$$

which is a reasonable object in theory. However, from practical perspective such a normalization leads to computational issues (in addition to intractability of combinatorial optimization over
\(V(s, d)\). It turns out that the rotation \(\Gamma\) that we introduce in Definition 3.1 (here \(\Gamma\) corresponds to \(P_J\)) plays role of the normalization and can be used instead: specifically, we could consider

\[
\sup_{v \in \mathbb{V}(s_1, m), w \in \mathbb{V}(s_2, d-m)} [v^\top w^\top] \Gamma^\top (\hat{P}_J - P_J) \Gamma \begin{bmatrix} v \\ w \end{bmatrix}.
\]

One may check that in this case under \(H_0^{(1)}\)

\[
\text{Var} \left[ [v^\top w^\top] \Gamma^\top (\hat{P}_J - P_J) \Gamma \begin{bmatrix} v \\ w \end{bmatrix} \right] \cong C_\Sigma,
\]

which makes Gaussian approximation applicable. Moreover, to avoid computation intractability caused by optimization over \(\mathbb{V}(s, d)\), or more specifically in our case \(\mathbb{V}(s_1, m)\) and \(\mathbb{V}(s_2, d-m)\), we replace them by sets \(D_{s_1}^m\) and \(D_{s_2}^{d-m}\), where \(D^d\) consist of unit vectors in \(\mathbb{R}^d\), whose support consists of \(s\) consecutive coordinates. So, it is another way to provide smooth connection between extreme cases using the parameters \(s_1\) and \(s_2\), and it would lead to

\[
\sup_{v \in D_{s_1}^m, w \in D_{s_2}^{d-m}} [v^\top w^\top] \Gamma^\top (\hat{P}_J - P_J) \Gamma \begin{bmatrix} v \\ w \end{bmatrix},
\]

which can be written also as

\[
\sup_{v \in D_{s_1}^m, w \in D_{s_2}^{d-m}} \left( v^\top \Gamma_1^\top (\hat{P}_J - P_J) \Gamma_1 v + w^\top \Gamma_2^\top (\hat{P}_J - P_J) \Gamma_2 w + 2v^\top \Gamma_1^\top (\hat{P}_J - P_J) \Gamma_2 w \right).
\]

We go even further, and the last step of explaining the reasons behind specific construction of our test statistic is the observation that under \(H_0^{(1)}\), due to specific structure of spectral projectors, the first two terms in the above display become negligible, and in fact we can replace them with \(\|\Gamma_1^\top (\hat{P}_J - P_J) \Gamma_1\|\) and \(\|\Gamma_2^\top (\hat{P}_J - P_J) \Gamma_2\|\), which still will be negligible. The reason behind this change is that while the properties under null hypothesis are not spoiled, the discrimination power under alternative hypothesis of the sum of these spectral norms is better rather that of

\[
\sup_{v \in D_{s_1}^m, w \in D_{s_2}^{d-m}} \left( v^\top \Gamma_1^\top (\hat{P}_J - P_J) \Gamma_1 v + w^\top \Gamma_2^\top (\hat{P}_J - P_J) \Gamma_2 w \right).\]

In other words, this allows to gain in power for free (“power enhancement”). This leads to the final version of our test statistic

\[
\|\Gamma_1^\top (\hat{P}_J - P_J) \Gamma_1\| + \|\Gamma_2^\top (\hat{P}_J - P_J) \Gamma_2\| + 2 \sup_{v \in D_{s_1}^m, w \in D_{s_2}^{d-m}} v^\top \Gamma_1^\top (\hat{P}_J - P_J) \Gamma_2 w,
\]

which gives an equivalent (up to a factor 2) definition of our norm.

### 7.3 Comparison with covariance matrix testing in Han et al. (2016)

As discussed above, Han et al. (2016) focuses on a problem on bootstrap inference for \(s\)-sparse largest eigenvalue of \((\hat{\Sigma} - \Sigma)\), and, consequently, applies the results to hypotheses testing setting
for covariance matrices. Now we want to compare different aspects of our work and Han et al. (2016).

While we deal with a different problem of hypothesis testing for spectral projectors, the results rely on similar idea of Gaussian approximation for maxima of sums random vectors after $\varepsilon$-net argument for supremum. Also, both works try to replace spectral norm to get better rates: Han et al. (2016) works with $s$-sparse largest eigenvalue, and we consider $\| \cdot \|_{(\mathbf{P}, \Gamma, s_1, s_2)}$-norm. Here we highlight what differs our work from Han et al. (2016), apart from the fact that the objects of interest for us are spectral projectors, rather than covariance matrices.

- As can be seen from the previous subsection, generalization of $s$-sparse largest eigenvalue norm is not straightforward.

- New norm brings computational tractability for the test statistic. Han et al. (2016) claims that they compute $s$-sparse largest eigenvalue using truncated power method, but in fact this method doesn’t apply to their framework, see Yuan et al. (2013).

- Proofs of Han et al. (2016) and ours are based on different results. While proof of Han et al. (2016) uses coupling inequality for maxima of sums of random vectors (see Corollary 4.1 of Chernozhukov et al. (2014)), we employ Gaussian approximation technique (see Theorem 2.2 of Chernozhukov et al. (2013)). Though these results are closely tied and derived by the same authors, it turns out that the latter allows to obtain slightly better rate: for instance, if we consider test statistic based on spectral norm, the results of Han et al. (2016) require (omitting logarithmic terms) $d^9/n \ll 1$, while we assume a bit weaker $d^7/n \ll 1$.

- Bootstrap inference has been very popular for this kind of statistical problems where the limiting distribution of the test statistic depends on unknown parameters of the model and/or, in addition, is very unfriendly to work with, as in our case. However, as we already mentioned, multiplier bootstrap suffers from one computational issue, since to generate one bootstrap sample, one needs to generate $n$ random weights $\eta_1, \ldots, \eta_n$. Hence, in our work, along with the standard bootstrap method (Approach 1) we suggested the method, linked to Frequentist Bayes. The computational complexity of Approach 2 (specifically, of its “resampling” stage) does not depend on $n$, hence, it is significantly more efficient than Approach 1.

- Continuing the previous point, we note that the implementation of the bootstrap procedure in Han et al. (2016) does not allow to build confidence sets, since their bootstrap test statistic $\tilde{B}_{max}$ (see equation (2.3) from Han et al. (2016)) depends on $\Sigma$, thus is known only under $H_0$. In general, testing hypotheses and constructing confidence sets are known to be dual problems; however, constructing confidence sets is more difficult in a sense that we never know $\Sigma$ in this case, while for testing hypotheses we have a hypothetical covariance $\Sigma^\circ$, which can be used in test statistic and coincides with the true one under $H_0$. 34
In contrast, we provide the procedure for building confidence sets.

- Finally, the results of Han et al. (2016) assume sub-Gaussian data distribution, even though they mention that it can be relaxed. In modern applications the data are often heavy-tailed, and the extension beyond sub-Gaussianity becomes crucial. We make use of results from recent paper of Kuchibhotla and Chakrabortty (2018) that provides user-friendly framework for dealing with sub-Weibull distributions considered in our work.

7.4 Comparison with previous works on inference for projectors

Previous works on the topic are Koltchinskii and Lounici (2017b,c); Naumov et al. (2019); Silin and Spokoiny (2018). Here we discuss how our work differs from these papers.

- All of the mentioned works do not state the problem as hypothesis testing, and hence, they do not analyze power of the test that can be proposed based on their results. Furthermore, two-sample case was not considered as well.

- Koltchinskii and Lounici (2017b,c); Naumov et al. (2019) rely on Gaussianity of the data distributions, which is, undoubtedly, extremely restrictive. Silin and Spokoiny (2018) work under significantly weaker “covariance concentration condition” (cov. conc.) of the form $\|\hat{\Sigma} - \Sigma\| \leq \delta_{n,d} \|\Sigma\|$ with high probability, and the rate $\delta_{n,d}$ appears in their bounds. In our work, no parametric assumption is imposed as well. Moreover, as was already mentioned, not only our results apply to sub-Gaussian case, but extend to distributions with heavier tails.

- The quantity of interest in all of the mentioned works is squared Frobenius distance between projectors $\|\hat{P}_J - P_J\|_F^2$, and the limiting distributions in Koltchinskii and Lounici (2017b); Naumov et al. (2019); Silin and Spokoiny (2018) depends on the unknown true covariance $\Sigma$, hence, statistical inference requires some special treatment. Koltchinskii and Lounici (2017b) suggest splitting the sample into three parts to make statistical inference. In Koltchinskii and Lounici (2017c) the developed limiting distribution doesn’t depend on $\Sigma$, which makes statistical inference straightforward. Naumov et al. (2019) uses multiplier bootstrap, which is computationally intensive, as we pointed out previously. Silin and Spokoiny (2018) suggests Bayesian inference, that actually serves as a basis for our Approach 2.

Unlike these works, we consider completely different test statistic, and though the limiting distribution does depend on the underlying true covariance as well, we present both the multiplier approach and the approach emerging from Bayesian perspective to make a valid calibration for our test.

- All of the mentioned works use linear approximation for spectral projectors and bound the remainder term as in Lemma 2 of Koltchinskii and Lounici (2016). While for Gaussian data distribution this result is sufficient to state dimension-free bounds (thanks to the
sample covariance concentration in terms of effective rank, see Koltchinskii and Lounici (2017a), Corollary 2), without Gaussianity the appearance of the term $\sqrt{d^2/n}$ in the error bounds seems to be inevitable. In contrast, to bound the remainder term in linear approximation for projectors, we use new tight results from Jirak and Wahl (2018), which allow to state the bounds in terms of relative rank. As a result, the dependence on the dimension is much better for example in Factor Model setting.

We summarize the comparison of the methods in the following table. The column “Complexity” specifies how many times we need to compute the corresponding norm in the procedures. In the last two columns we compare the required relations between the dimension $d$ and the sample size $n$ in two important regimes: Factor Model (FM) regime and Spiked ($\text{Tr}[\Sigma] \approx d$) regime.

| Method | Idea | Data  | Complexity | FM rate | Spiked rate |
|--------|------|-------|------------|---------|-------------|
| Koltchinskii, Lounici (2017b,c) | Data-driven | Gaussian | $O(d^2)$ | not appl. | $1 \ll d \ll n$ |
| Naumov et al. (2019) | Bootstrap | Gaussian | $O(Nnd^2)$ | $d^2 \ll n$ | $d^6 \ll n$ |
| Silin, Spokoiny (2018) | Bayes | Cov. conc. | $O(Nd^2)$ | $d^{3.5} \ll n$ | $d^{3.5} \ll n$ |
| Our Approach 1 | Bootstrap | Sub-Weibull | $O(Nnd^2)$ | $d \ll n$ | $d^3 \ll n$ |
| Our Approach 2 | $\approx$ Bayes | Sub-Weibull | $O(Nd^2)$ | $d \ll n$ | $d^3 \ll n$ |

We again mention that in the Spiked regime the condition $d^3 \ll n$ can be improved to $d^2 \ll n$, but we do not pursue this goal in current work but rather focus on FM regime.
8 Main proofs

8.1 Key ingredients and outline of the proof

We start by presenting the key ingredients that our main theorems relies on. All the lemmas stated in this subsection are either borrowed from the literature or proved below in the end of the paper.

8.1.1 Concentration of sample covariance for sub-Weibull distributions

The first lemma describes how the sample covariance concentrates under Assumption 4.2. Our concentration for covariance is written in somewhat specific form; the reason for that will be justified in the next subsection.

Lemma 8.1. Let the data satisfy Assumption 4.2. Then the following bound holds with probability $1 - 1/n$:

$$\max_{s,t \in [q]} \frac{\|P_s(\hat{\Sigma} - \Sigma)P_t\|_F}{\sqrt{m_s \mu_s m_t \mu_t}} \leq \psi_n,$$

where

$$\psi_n \overset{\text{def}}{=} C_\beta c^2 \left( \sqrt{\frac{\log(n) + \log(d)}{n}} + \frac{(\log(n))^{1/\beta}(\log(n) + \log(d))^{2/\beta}}{n} \right).$$

8.1.2 Linear approximation of projectors

The projector of a covariance matrix is a complicated nonlinear operator. We use machinery from Jirak and Wahl (2018), unlike previous works (e.g. Naumov et al. (2019); Silin and Spokoiny (2018)) which were based on Koltchinskii and Lounici (2016), Lemma 2. Novel result from Jirak and Wahl (2018) allows to obtain linear approximation for spectral projectors with remainder term bounded by dimension-free rate even for non-Gaussian distributions. We slightly modify it to prepare it for our framework.

Lemma 8.2. Let $\tilde{\Sigma}$ be perturbed covariance matrix. Take

$$x = \max_{s,t \in [q]} \frac{\|P_s(\tilde{\Sigma} - \Sigma)P_t\|_F}{\sqrt{m_s \mu_s m_t \mu_t}}$$

and assume

$$x \max_{r \in J} r_r(\Sigma) \leq \frac{1}{6}. \quad (8.1)$$

Then following decomposition holds:

$$\tilde{P}_J - P_J = L_J(\tilde{\Sigma} - \Sigma) + R_J(\tilde{\Sigma} - \Sigma), \quad (8.2)$$

where the linear part is

$$L_J(\tilde{\Sigma} - \Sigma) \overset{\text{def}}{=} \sum_{r \in J} \sum_{s \in J^c} \frac{P_r(\tilde{\Sigma} - \Sigma)P_s + P_s(\tilde{\Sigma} - \Sigma)P_r}{\mu_r - \mu_s}.$$
and the remainder term satisfies

\[
\|R_J(\tilde{\Sigma} - \Sigma)\|_F \leq Cx^2 \sum_{r \in J} r_r(\Sigma) \sqrt{\sum_{s \neq r} \frac{M_r m_r m_s \mu_s}{(\mu_r - \mu_s)^2}} = Cx^2 d_J(\Sigma)^{3/2}.
\] (8.3)

The following lemma deals with the remainder term using the previous lemma.

**Lemma 8.3.** Let \(\tilde{\Sigma}\) be perturbed covariance matrix (potentially depending on the data and additional source of randomness; e.g. \(\Sigma^B\) or \(\Sigma^F\)) and \(\tilde{P}_J\) is the corresponding projector. Assume for some rate \(\tilde{\psi}_n\) holds

\[
P\left[ \max_{s, t \in [q]} \|P_s(\tilde{\Sigma} - \hat{\Sigma})P_t\|_F \leq \tilde{\psi}_n \right] \geq 1 - \frac{1}{n},
\] (8.4)

with probability \(1 - 1/n\), and

\[
(\psi_n \lor \tilde{\psi}_n) \max_{r \in J} r_r(\Sigma) \leq \frac{1}{12}.
\]

Then the following approximation holds with probability \(1 - 2/n\)

\[
P\left[ \left| \|\tilde{P}_J - \hat{P}_J\|_{(P_J, \Gamma^o, s_1, s_2)} - \|L_J(\tilde{\Sigma} - \hat{\Sigma})\|_{(P_J, \Gamma^o, s_1, s_2)} \right| \leq C(\psi_n + \tilde{\psi}_n)^2 d_J(\Sigma)^{3/2} \right] \geq 1 - \frac{1}{n}.
\] (8.5)

As one can see, the setting of the lemma is pretty general, and we are going to apply it in the sequel with different \(\tilde{\Sigma}\).

### 8.1.3 Alternative representation of \(\| \cdot \|_{(P, \Gamma, s_1, s_2)}\)

For the clarity of presentation, in Section 3 we introduced user-friendly definition of \(\| \cdot \|_{(P, \Gamma, s_1, s_2)}\) that doesn’t require any extra definitions. However, in our proofs it will be more convenient to work with \(\| \cdot \|_{(P, \Gamma, s_1, s_2)}\) expressed in a slightly different way.

**Definition 8.1.** Let \(k\) be an integer and \(s \in [k]\). Define

\[
D^k_s \overset{\text{def}}{=} \left\{ x \in S^{k-1} \mid \max_{j \neq 0} x_j \leq \min_{j \neq 0} x_j \leq s - 1 \right\} = \bigcup_{l=0}^{k-s} \left\{ [0^T, y^T, 0_{k-l-s}^T] \mid y \in S^{s-1} \right\}.
\]

**Example 8.1.** Consider, first, extreme cases:

- \(s = 1\): we have \(D^k_1 = \{ \pm e_j \}_{j=1}^k\), where \(e_j \in \mathbb{R}^k\) is \(j\)-th standard basis vector.
- \(s = k\): we have \(D^k_k = S^{k-1}\).

To illustrate Definition 8.1 in a less trivial case, take \(k = 7\) and \(s = 3\). Then \(D^7_3\) consists of \(k\)-dimensional unit vectors with support contained in the shadow area of one of the following \(k - s + 1 = 5\) variants, depicted on Figure 6.
Lemma 8.4 (Additional properties of $\| \cdot \|_{(P, \Gamma, s_1, s_2)}$). Fix arbitrary $P$ of rank $m$, $\Gamma = [\Gamma_1 \Gamma_2]$, $s_1$, $s_2$ as in Definition 3.1. Then, the following holds:

(i) $\| \cdot \|_{(P, \Gamma, s_1, s_2)}$ can be alternatively represented as

$$
\|A\|_{(P, \Gamma, s_1, s_2)} = \frac{1}{2} \| \Gamma_1^T A \Gamma_1 \| + \frac{1}{2} \| \Gamma_2^T A \Gamma_2 \| + \sup_{v \in D^m_{s_1}} v^T \Gamma_1^T A \Gamma_2 w.
$$

(ii) For a symmetric $A \in \mathbb{R}^{d \times d}$ of the form $A = PA(I_d - P) + (I_d - P)AP$ we have

$$
\|A\|_{(P, \Gamma, s_1, s_2)} = \sup_{v \in D^m_{s_1}} v^T \Gamma_1^T A \Gamma_2 w.
$$

If, additionally, $s_1 = m$ and $s_2 = d - m$, then

$$
\|A\|_{(P, \Gamma, m, d-m)} = \|A\|,
$$

while in case $s_1 = 1$ and $s_2 = 1$ we get

$$
\|A\|_{(P, \Gamma, 1, 1)} = \|\Gamma_1^T A \Gamma_2\|_{\text{max}}.
$$

8.1.4 $\varepsilon$-net argument

Let $N_\varepsilon(D^m_{s_1})$ and $N_\varepsilon(D^{d-m}_{s_2})$ be proper $\varepsilon$-nets of $D^m_{s_1}$ and $D^{d-m}_{s_2}$, respectively, w.r.t. Euclidean distance. Let us explicitly demonstrate how we construct them; namely, we construct $N_\varepsilon(D^m_{s_1})$, while $N_\varepsilon(D^{d-m}_{s_2})$ can be constructed similarly. Consider a proper $\epsilon$-net of $S^{s_1-1}$ w.r.t. Euclidean distance and denote it as $N_\epsilon(S^{s_1-1})$. Take

$$
N_\epsilon(D^m_{s_1}) = \left\{ [0^T_k, v^T, 0^{T}_{m-k-s_1}]^T \mid v \in N_\epsilon(S^{s_1-1}), k \in \{0, \ldots, m-s_1\} \right\}.
$$

By Definition 8.1 it is trivial to see that $N_\epsilon(D^m_{s_1})$ is indeed an $\varepsilon$-net of $D^m_{s_1}$. 
Consider all possible pairs \((\Gamma_1 v, \Gamma_2 w)\) such that \(v \in N_\varepsilon(D_{s_1}^m), w \in N_\varepsilon(D_{s_2}^{d-m})\). Enumerate them \(\{(v_j, w_j)\}_{j=1}^p\) with \(p = p(\varepsilon, d, m, s_1, s_2) = |N_\varepsilon(D_{s_1}^m)| \cdot |N_\varepsilon(D_{s_2}^{d-m})|\). Note that the constructed \(\varepsilon\)-net is different for different \(P\) and \(\Gamma\).

The following lemma provides standard approximation of infinite supremum by finite maximum over the \(\varepsilon\)-net.

**Lemma 8.5 (Discretization).** Let \(P, \Gamma, s_1, s_2\) be as in Definition 3.1. For any symmetric matrix \(A \in \mathbb{R}^{d \times d}\) satisfying

\[
A = PA(I_d - P) + (I_d - P)AP
\]

the following bounds hold:

\[
\max_{j \in [p]} v_j^\top Aw_j \leq \|A\|_{(P, \Gamma, s_1, s_2)} \leq \frac{1}{1 - 2\varepsilon} \max_{j \in [p]} v_j^\top Aw_j.
\]

The size of the \(\varepsilon\)-net can be bounded according to the following lemma.

**Lemma 8.6 (Covering number).** The following bound holds:

\[
\log(p(\varepsilon, d, m, s_1, s_2)) \leq (s_1 + s_2) \log\left(\frac{3}{\varepsilon}\right) + 2 \log(d).
\]

In our proofs, we will use \(\varepsilon = 1/n\). This fixes \(p\) to be

\[
p \leq \exp((s_1 + s_2) \log(3n) + 2 \log(d)),
\]

8.1.5 Gaussian approximation, anti-concentration and comparison for maxima

We will be using Gaussian approximation, anti-concentration and comparison results for maximum of a random vector. Before we state the specific results from Chernozhukov et al. (2013, 2015), let us introduce the framework from these papers. Suppose we have a collection of \(n\) independent zero-mean random vectors in \(\mathbb{R}^p\):

\[x_i = \{x_{ij}\}_{j=1}^p, \quad i \in [n].\]

Let \(y_i \sim \mathcal{N}_p(0, \text{Cov}(x_i)), i \in [n]\) be a collection of Gaussian vectors in \(\mathbb{R}^p\) with the same covariances as these of \(x_i\)’s. Denote

\[
\bar{E}[\cdot] \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n E[\cdot], \quad \text{e.g.} \quad \bar{E}[x_{ij}^2] = \frac{1}{n} \sum_{i=1}^n E[x_{ij}^2].
\]

Based on that, introduce

\[
M_k \overset{\text{def}}{=} \max_{j \in [p]} \left(\bar{E}[|x_{ij}|^k]\right)^{1/k}.
\]

Finally, define \(u_x(\gamma)\) as the smallest \(u\) such that

\[
\mathbb{P}\left[|x_{ij}| \leq u \cdot \left(\bar{E}[|x_{ij}|^2]\right)^{1/2} \quad \forall i \in [n], \forall j \in [p]\right] \geq 1 - \gamma,
\]

and define \(u_y(\gamma)\) similarly for the Gaussian counterpart \(y_{ij}\), and denote \(u(\gamma) = u_x(\gamma) \vee u_y(\gamma)\). Now we are ready to state the results. The first one is Gaussian approximation for maxima of sum of random vectors.
Lemma 8.7 (Chernozhukov et al. (2013), Theorem 2.2: Main result 1, Gaussian approximation). Suppose that there are some constants \( 0 < c_1 < C_1 \) such that \( c_1 \leq \bar{E} \left[ x_{ij}^2 \right] \leq C_1 \) for all \( j \in [p] \). Then for every \( \gamma \in (0, 1) \),

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \max_{j \in [p]} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_{ij} \leq z \right] - \mathbb{P} \left[ \max_{j \in [p]} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} y_{ij} \leq z \right] \right| \leq C \left\{ n^{-1/8} \left( M_3^{3/4} \lor M_4^{1/2} \right) \left( \log(n \gamma) \right)^{7/8} + n^{-1/2} \left( \log(p \gamma) \right)^{3/2} u(\gamma) + \gamma \right\},
\]

where \( C > 0 \) is a constant that depends on \( c_1 \) and \( C_1 \) only.

The following is the anti-concentration result from Chernozhukov et al. (2015).

Lemma 8.8 (Chernozhukov et al. (2015), Corollary 1: Anti-concentration). Let \( (Z_1, \ldots, Z_p)^\top \) be a centered Gaussian random vector in \( \mathbb{R}^p \) with \( \sigma_j^2 \overset{\text{def}}{=} \mathbb{E} \left[ Z_j^2 \right] > 0 \) for all \( j \in [p] \). Let \( \underline{\sigma} \overset{\text{def}}{=} \min_{j \in [p]} \sigma_j \) and \( \overline{\sigma} \overset{\text{def}}{=} \max_{j \in [p]} \sigma_j \). Then for every \( \epsilon > 0 \),

\[
\sup_{z \in \mathbb{R}} \mathbb{P} \left[ \max_{j \in [p]} Z_j - z \right] \leq \epsilon \leq C \epsilon \sqrt{1 \lor \log(p/\epsilon)},
\]

where \( C > 0 \) depends only on \( \underline{\sigma} \) and \( \overline{\sigma} \).

The following is the comparison result from Chernozhukov et al. (2015).

Lemma 8.9 (Chernozhukov et al. (2015), Theorem 2: Comparison of distributions). Let \( Z = (Z_1, \ldots, Z_p)^\top \) and \( Y = (Y_1, \ldots, Y_p)^\top \) be centered Gaussian random vectors in \( \mathbb{R}^p \) with covariances \( \{\sigma_{jk}^Z\}_{j,k=1}^p \) and \( \{\sigma_{jk}^Y\}_{j,k=1}^p \), respectively. Define

\[
\Delta \overset{\text{def}}{=} \max_{j,k \in [p]} |\sigma_{jk}^Z - \sigma_{jk}^Y| \text{ and } a_p \overset{\text{def}}{=} \mathbb{E} \left[ \max_{j \in [p]} (Y_j / \sigma_{jj}^Y) \right].
\]

Suppose that \( p \geq 2 \) and \( \sigma_{jj}^Y > 0 \) for all \( j \in [p] \). Then

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \max_{j \in [p]} Z_j \leq z \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_j \leq z \right] \right| \leq C \Delta^{1/3} \left\{ 1 \lor a_p^2 \lor \log(1/\Delta) \right\}^{1/3} (\log p)^{1/3},
\]

where \( C > 0 \) depends only on \( \min_{j \in [p]} \sigma_{jj}^Y \) and \( \max_{j \in [p]} \sigma_{jj}^Y \). Moreover, in the worst case, \( a_p \leq \sqrt{2 \log p} \), so that

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \max_{j \in [p]} Z_j \leq z \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_j \leq z \right] \right| \leq C' \Delta^{1/3} \left\{ 1 \lor \log(p/\Delta) \right\}^{2/3},
\]

where as before \( C' > 0 \) depends only on \( \min_{j \in [p]} \sigma_{jj}^Y \) and \( \max_{j \in [p]} \sigma_{jj}^Y \).

Now we are equipped to proceed with the proof of the main results.
8.2 Proof of Theorem 4.1

8.2.1 Approximation by finite maximum

Throughout the proof we work with $\| \cdot \|_{(x_i, \Gamma^\circ, s_1, s_2)}$-norm, and apply all lemmas from previous subsection with $P = P_J$ and $\Gamma = \Gamma^\circ$.

In accordance with Lemma 8.3 (applied with $\hat{\Sigma} = \Sigma, \hat{P}_J = P_J$), we start by working with the linear part $\sqrt{n} L_J(\hat{\Sigma} - \Sigma)$ of $\sqrt{n}(\hat{P}_J - P_J)$. Let $\{(v_j, w_j)\}_{j=1}^p$ be the $\varepsilon$-net constructed in subsection 8.1.4 for $P_J$ and $\Gamma^\circ$ with $\varepsilon = 1/n$. Observe that $L_J(\hat{\Sigma} - \Sigma)$ satisfies the condition of Lemma 8.5 with $P = P_J$, hence, for

$$L_{disc} \equiv \max_{j \in [p]} v_j^\top \left( \sum_{r \in J \setminus s, \hat{J}} \frac{P_r(\hat{\Sigma} - \Sigma)P_s + P_s(\hat{\Sigma} - \Sigma)P_r}{\mu_r - \mu_s} \right) w_j$$

$$= \max_{j \in [p]} \sum_{r \in J \setminus s, \hat{J}} v_j^\top \frac{P_r(\hat{\Sigma} - \Sigma)P_s w_j}{\mu_r - \mu_s}$$

we have

$$L_{disc} \leq \| L_J(\hat{\Sigma} - \Sigma) \|_{(x_i, \Gamma^\circ, s_1, s_2)} \leq \frac{1}{1 - 2\varepsilon} L_{disc}. \quad (8.7)$$

Now let us represent $L_{disc}$ in a different way. Introduce for $i \in [n], j \in [p]$

$$x_{ij} \equiv \sum_{r \in J \setminus s, \hat{J}} v_j^\top \frac{P_r(X_i \top X_i - \Sigma)P_s w_j}{\mu_r - \mu_s} = \sum_{r \in J \setminus s, \hat{J}} v_j^\top \frac{P_r X_i \top X_i P_s w_j}{\mu_r - \mu_s}.$$

Therefore,

$$\sqrt{n} L_{disc} = \max_{j \in [p]} \frac{1}{\sqrt{n}} \sum_{i=1}^n x_{ij}.$$

We can arrange this random variables as i.i.d. centered random vectors in $\mathbb{R}^p$:

$$x_i \equiv \{x_{ij}\}_{j=1}^p.$$

Lemma 8.7 suggests that the distribution of $\max_{j \in [p]} \frac{1}{\sqrt{n}} \sum_{i=1}^n x_{ij}$ can be approximated by the distribution of its Gaussian analogue $\max_{j \in [p]} \frac{1}{\sqrt{n}} \sum_{i=1}^n y_{ij}$, where $y_i \equiv \{y_{ij}\}_{j=1}^p$ are i.i.d. centered Gaussian random vectors with the same covariance structure as $x_i$’s. In other terms, introducing

$$Y \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n y_i \sim \mathcal{N}_p \left( 0, \frac{1}{n} \sum_{i=1}^n \text{Cov}(x_i) \right) \sim \mathcal{N}_p \left( 0, \text{Cov}(x_1) \right),$$

we would like to use the distribution of $\max_{j \in [p]} Y_j$ as the approximation for the distribution of $\sqrt{n} L_{disc}$, consequently for the distribution of $\sqrt{n} \| L_J(\hat{\Sigma} - \Sigma) \|_{(P_J, \Gamma^\circ, s_1, s_2)}$, and eventually for the distribution of $\sqrt{n} \| \hat{P}_J - P_J \|_{(P_J, \Gamma^\circ, s_1, s_2)}$.  

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8.2.2 Verifying the conditions

In order to apply Lemma 8.7 to our situation, we need to verify the conditions and compute some quantities.

Computing the covariance: Let us start with computing the covariance of \( x_i \)'s (and hence, the covariance of the Gaussian \( y_i \)'s and \( Y \)). First, using Assumption 4.1, we compute for \( i \in [n], j \in [p], r, r' \in J, s, s' \notin J \)

\[
E \left[ v_j^\top P_r X_i X_i^\top P_s w_j \cdot v_k^\top P_{r'} X_i X_i^\top P_{s'} w_k \right] \\
= E \left[ v_j^\top P_r (P_{\mathcal{J}} X_i) \cdot (P_{\mathcal{J}^c} X_i)^\top P_s w_j \cdot v_k^\top P_{r'} (P_{\mathcal{J}} X_i) \cdot (P_{\mathcal{J}^c} X_i)^\top P_{s'} w_k \right] \\
= E \left[ v_j^\top P_r X_i X_i^\top P_r' v_k \right] \cdot E \left[ w_j^\top P_s X_i X_i^\top P_{s'} w_k \right] \\
= \delta_{r,r'} \delta_{s,s'} \mu_r \mu_s \cdot (v_j^\top P_r v_k) \cdot (w_j^\top P_s w_k),
\]

where \( \delta_{.,.} \) is the Kronecker delta. Further,

\[
E[x_{ij} x_{ik}] = \sum_{r \in J} \sum_{s \notin J} \frac{\mu_r \mu_s}{(\mu_r - \mu_s)^2} (v_j^\top P_r v_k) \cdot (w_j^\top P_s w_k).
\]

So,

\[
\text{Cov}(x_i) = \{ \sigma_{jk}^\Sigma \}_{j,k = 1}^p, \quad \text{where}
\]

\[
\sigma_{jk}^\Sigma = \sum_{r \in J} \sum_{s \notin J} \frac{\mu_r \mu_s}{(\mu_r - \mu_s)^2} (v_j^\top P_r v_k) \cdot (w_j^\top P_s w_k).
\]

Then, let us show the existence of \( c_1 \) and \( C_1 \) required in Lemma 8.7, bound \( M_3 \) and \( M_4 \) and estimate \( u(\gamma) \).

Showing the existence of \( c_1 \) and \( C_1 \): to do that, write

\[
\mathbb{E} \left[ x_{ij}^2 \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ x_{ij}^2 \right] \overset{\text{i.i.d.}}{=} \mathbb{E} \left[ x_{ij}^2 \right] = \sigma_{jj}^\Sigma = \sum_{r \in J} \sum_{s \notin J} \frac{\mu_r \mu_s}{(\mu_r - \mu_s)^2} (v_j^\top P_r v_j) \cdot (w_j^\top P_s w_j).
\]

Note that

\[
\sigma_{jj}^\Sigma \leq \max_{r \in J, s \notin J} \frac{\mu_r \mu_s}{(\mu_r - \mu_s)^2} \sum_{r \in J} \sum_{s \notin J} (v_j^\top P_r v_j) \cdot (w_j^\top P_s w_j) = \kappa^2 \cdot (v_j^\top P_{\mathcal{J}} v_j) \cdot (w_j^\top P_{\mathcal{J}^c} w_j) = \kappa^2,
\]

\[
\sigma_{jj}^\Sigma \geq \min_{r \in J, s \notin J} \frac{\mu_r \mu_s}{(\mu_r - \mu_s)^2} \sum_{r \in J} \sum_{s \notin J} (v_j^\top P_r v_j) \cdot (w_j^\top P_s w_j) = \kappa^2 \cdot (v_j^\top P_{\mathcal{J}} v_j) \cdot (w_j^\top P_{\mathcal{J}^c} w_j) = \kappa^2.
\]

This implies that there exist \( c_1 = \kappa^2 > 0 \) and \( C_1 = \kappa^2 > 0 \) satisfying the condition

\[
c_1 \leq \mathbb{E} \left[ x_{ij}^2 \right] \leq C_1
\]

for all \( j \in [p] \).
Upperbounding $M_3$ and $M_4$: we have
\[
|x_{ij}| = \left| \sum_{r \in J} \sum_{s \notin J} \frac{v_j^\top P_r X_i X_i^\top P_s w_j}{\mu_r - \mu_s} \right| \leq \max_{r \in J, s \notin J} \frac{\sqrt{|\mu_r - \mu_s|}}{\mu_r - \mu_s} \sum_{r \in J} |v_j^\top P_r \Sigma^{-1/2} X_i| \cdot \sum_{s \notin J} |w_j^\top P_s \Sigma^{-1/2} X_i|.
\]
Let us deal with $\sum_{r \in J} |v_j^\top P_r \Sigma^{-1/2} X_i|$. Represent
\[
|v_j^\top P_r \Sigma^{-1/2} X_i| = |\bar{v}_{j,r}^\top \Sigma^{-1/2} X_i|,
\]
where $\bar{v}_{j,r}$ is either $P_r v_j$ or $(-P_r v_j)$ depending on the sign of $v_j^\top P_r \Sigma^{-1/2} X_i$. Note that $\{\bar{v}_{j,r}\}_{r \in J}$ are orthogonal. Define $\bar{v}_j \equiv \sum_{r \in J} \bar{v}_{j,r}$ with the squared norm
\[
||\bar{v}_j||^2 = \sum_{r \in J} ||\bar{v}_{j,r}||^2 = \sum_{r \in J} ||P_r v_j||^2 = v_j^\top P_j v_j = 1,
\]
where the first equality is due to orthogonality. Hence,
\[
\sum_{r \in J} |v_j^\top P_r \Sigma^{-1/2} X_i| = \sum_{r \in J} \bar{v}_{j,r}^\top \Sigma^{-1/2} X_i = \bar{v}_j^\top \Sigma^{-1/2} X_i.
\]
Similarly,
\[
\sum_{s \notin J} |w_j^\top P_s \Sigma^{-1/2} X_i| = \sum_{s \notin J} \bar{w}_{j,s}^\top \Sigma^{-1/2} X_i = \bar{w}_j^\top \Sigma^{-1/2} X_i
\]
with some $||\bar{w}_j|| = 1$. Thus,
\[
x_{ij} \leq \bar{\kappa} \cdot \bar{v}_j^\top \Sigma^{-1/2} X_i \cdot \bar{w}_j^\top \Sigma^{-1/2} X_i \tag{8.8}
\]
with $||\bar{v}_j|| = 1$, $||\bar{w}_j|| = 1$ and $\bar{v}_j, \bar{w}_j$ are orthogonal. Therefore,
\[
(\bar{\mathbb{E}} [x_{ij}^4])^{1/4} = \left( \frac{1}{n} \sum_{i=1}^n \mathbb{E} [x_{ij}^4] \right)^{1/4} \overset{i.i.d.}{=} \mathbb{E} [x_{ij}^4]^{1/4} \leq \bar{\kappa} \cdot \mathbb{E} [(\bar{v}_j^\top \Sigma^{-1/2} X_i)^4 (\bar{w}_j^\top \Sigma^{-1/2} X_i)^4]^{1/4}
\]
\[
= \bar{\kappa} \cdot \mathbb{E} [(\bar{v}_j^\top \Sigma^{-1/2} X_i)^8]^{1/8} \mathbb{E} [(\bar{w}_j^\top \Sigma^{-1/2} X_i)^8]^{1/8} \lesssim 8^{2/\beta} \bar{\kappa} \text{ for all } j \in [p],
\]
where we used Cauchy-Schwarz inequality and the moment bound for sub-Weibull distributions, see Kuchibhotla and Chakrabortty (2018), p.7, together with Assumption 4.2. So, $M_4 \lesssim 8^{2/\beta} \bar{\kappa}$. By Jensen’s inequality, $M_3 \leq M_4$.

Estimating $u_x(\gamma), u_y(\gamma)$ and $u(\gamma)$: using $|x_{ij}| \leq \bar{\kappa} \cdot |\bar{v}_j^\top \Sigma^{-1/2} X_i| \cdot |\bar{w}_j^\top \Sigma^{-1/2} X_i|$, for arbitrary $u > 0$ we write
\[
P \left[ x_{ij} > u \cdot (\mathbb{E} [x_{ij}^2])^{1/2} \forall i \in [n] \forall j \in [p] \right]
\]
\[
\leq P \left[ \bar{\kappa} \cdot |\bar{v}_j^\top \Sigma^{-1/2} X_i| \cdot |\bar{w}_j^\top \Sigma^{-1/2} X_i| > u \cdot \bar{\kappa} \forall i \in [n] \forall j \in [p] \right]
\]
\[
= P \left[ |\bar{v}_j^\top \Sigma^{-1/2} X_i| \cdot |\bar{w}_j^\top \Sigma^{-1/2} X_i| > u / \bar{\kappa} \forall i \in [n] \forall j \in [p] \right]
\]
\[
\overset{\text{union bound}}{\leq} \sum_{i=1}^n \sum_{j=1}^p P \left[ |\bar{v}_j^\top \Sigma^{-1/2} X_i| \cdot |\bar{w}_j^\top \Sigma^{-1/2} X_i| > u / \bar{\kappa} \right] \leq n p \cdot 2 \exp \left( - \left( \frac{u}{\kappa c^2} \right)^{3/2} \right),
\]
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with the last inequality is due to Assumption 4.2 and Lemma A.1. This, by definition of \( u_x(\gamma) \), implies

\[
u_x(\gamma) \leq \kappa c^2 \left( \log(2pn) + \log(1/\gamma) \right)^{2/\beta}.
\]

At the same time, \( y_{ij} \) is centered Gaussian random variable with the variance \( \mathbb{E} [y_{ij}^2] = \bar{E} [y_{ij}^2] \), so \( y_{ij}/(\bar{E} [y_{ij}^2])^{1/2} \sim \mathcal{N}(0, 1) \). Thus,

\[
P \left[ |y_{ij}| > u \cdot (\bar{E} [y_{ij}^2])^{1/2} \; \forall i \in [n] \; \forall j \in [p] \right] = P \left[ |y_{ij}/(\bar{E} [y_{ij}^2])^{1/2}| > u \; \forall i \in [n] \; \forall j \in [p] \right] \leq \sum_{i=1}^{n} \sum_{j=1}^{p} n \cdot P \left[ |y_{ij}/(\bar{E} [y_{ij}^2])^{1/2}| > u \right] = np \cdot P [|\mathcal{N}(0, 1)| > u] \leq np \cdot 2 \exp \left( -u^2/2 \right),
\]

which yields by definition of \( u_y(\gamma) \)

\[
u_y(\gamma) \leq \sqrt{2(\log(2pn) + \log(1/\gamma))}.
\]

For our purposes we can take \( \gamma = 1/n \). Therefore,

\[
u(\gamma) = u_x(\gamma) \vee u_y(\gamma) \lesssim \kappa c^2 \left( \log(2pn^2) \right)^{2/\beta}.
\]

### 8.2.3 Applying Gaussian approximation and anti-concentration

To catch the dependence on \( \kappa \) and \( \pi \) more carefully, we apply Lemma 8.7 not to \( x_{ij} \) and \( y_{ij} \), but rather to \( x'_{ij} := x_{ij}/\kappa \) and \( y'_{ij} := y_{ij}/\kappa \). Then, the conditions verified above translate into

\[
1 \leq \bar{E} \left[ x'^2_{ij} \right] \leq \frac{\kappa^2}{\kappa^2} = \kappa^2,
\]

\[
M'_k = \frac{M_k}{\kappa} \lesssim 8^{2/\beta} \kappa \quad \text{for } k = 3, 4,
\]

\[
u'(\gamma) = u(\gamma) \lesssim \kappa c^2 \left( \log(2pn^2) \right)^{2/\beta}.
\]

Obviously, passing from \( x_{ij}, y_{ij} \) to \( x'_{ij}, y'_{ij} \) does not change Kolmogorov distance, so we proved

\[
\sup_{z \in \mathbb{R}} \left| P \left[ \sqrt{n}L_{\text{disc}} \leq z \right] - P \left[ \max_{j \in [p]} Y_j \leq z \right] \right| = \sup_{z \in \mathbb{R}} \left| P \left[ \sqrt{n}L_{\text{disc}}/\kappa \leq z \right] - P \left[ \max_{j \in [p]} Y_j/\kappa \leq z \right] \right| \leq C(1, \kappa^2) \left\{ 8^{3/(2\beta)} \kappa^{3/4} \left( \frac{\log(2pn^2)^{7}}{n} \right)^{1/8} + \kappa c^2 \left( \frac{\log(2pn^2)^{3+4/\beta}}{n} \right)^{1/2} + \frac{1}{n} \right\}
\]

\[
\leq C(\kappa) \left\{ 8^{3/(2\beta)} \left( \frac{\log(2pn^2)^{7}}{n} \right)^{1/8} + c^2 \left( \frac{\log(2pn^2)^{3+4/\beta}}{n} \right)^{1/2} \right\}.
\]

Crucial observation here is that the obtained bound depends on \( \kappa, \pi \) only through \( \kappa = \pi/\kappa \).

Getting back from finite maximum to supremum of infinite-state process: now we want to derive the same result, but for \( \sqrt{n} \|L_x(\Sigma - \Sigma_{\beta})\|_{(P, \Gamma^0, s_1, s_2)} \) rather than \( \sqrt{n}L_{\text{disc}} \). To do that, we
Thus, taking

\[ \Diamond \overset{\text{def}}{=} \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \max_{j \in [p]} Y_j \leq z \right] - \mathbb{P} \left[ \frac{1}{1 - 2\varepsilon} \max_{j \in [p]} Y_j \leq z \right] \right| \]

\[ = \sup_{z \in \mathbb{R}} \mathbb{P} \left[ \max_{j \in [p]} Y_j \in [(1 - 2\varepsilon)z, z] \right] = \mathbb{P} \left[ \max_{j \in [p]} \left( Y_j / \kappa \right) \in [(1 - 2\varepsilon)z, z] \right], \]

where we again pass from \( \max_{j \in [p]} Y_j \) to \( \max_{j \in [p]} (Y_j / \kappa) \). First, notice that since each \( (Y_j / \kappa) \) is Gaussian with variance at most \( \pi^2 / \kappa^2 \), then all \( (Y_j / \kappa) \) are sub-Gaussian with parameter \( \kappa^2 \). Then, e.g. by Lemma 5.2 (Maximal tail inequality for sub-Gaussian random variables) from van Handel (2018), we have for all \( \delta \in (0, 1) \)

\[ \mathbb{P} \left[ \max_{j \in [p]} (Y_j / \kappa) \leq \kappa \sqrt{\log(p) + \log(1/\delta)} \right] \geq 1 - \delta. \]

Thus, taking \( \delta = 1/n \) and assuming \( \varepsilon \leq 1/4 \), for \( z \geq 2\kappa \sqrt{\log(pn)} \geq \frac{\kappa \sqrt{\log(p) + \log(1/\delta)}}{1 - 2\varepsilon} \) we have

\[ \mathbb{P} \left[ \max_{j \in [p]} (Y_j / \kappa) \leq (1 - 2\varepsilon)z \right] \geq 1 - \frac{1}{n}, \]

which implies \( \Diamond \leq 1/n \). On the other hand, for \( z \leq 2\kappa \sqrt{\log(pn)} \) it is better to apply the anti-concentration for Gaussian random vector. Applied to our setting, Lemma 8.8 implies

\[ \mathbb{P} \left[ \max_{j \in [p]} (Y_j / \kappa) \in [(1 - 2\varepsilon)z, z] \right] \leq C \varepsilon z \sqrt{1 + \log(p/\varepsilon z)} \leq C \varepsilon z \sqrt{\log(ep/\varepsilon z)} \]

\[ \leq 2C \varepsilon \sqrt{\log(pn) \cdot \log \left( \frac{ep}{2\kappa \varepsilon \sqrt{\log(pn)}} \right)} \leq 2C \varepsilon \sqrt{\log(pn) \cdot \log \left( \frac{p}{\kappa \varepsilon} \right)}, \]

where \( C \) depends only on \( \min(\sigma_{jj}^2/\kappa^2) \) and \( \max(\sigma_{jj}^2/\kappa^2) \), but effectively on \( \kappa^2 / \kappa^2 = 1 \) and \( \pi^2 / \kappa^2 = \kappa^2 \). Here we used also that \( \varepsilon z \sqrt{\log(ep/\varepsilon z)} \) is increasing in \( z \) together with assumption \( 2\kappa \varepsilon \sqrt{\log(pn)} \leq ep \) (which anyway should be fulfilled, otherwise our results makes no sense). Combining the bounds on \( \Diamond \) for two different regimes of \( z \) and recalling \( \varepsilon = 1/n \), we get

\[ \Diamond \leq 2C \kappa \varepsilon \sqrt{\log(pn) \cdot \log \left( \frac{p}{\kappa \varepsilon} \right)} \vee \frac{1}{n} \leq 2C \kappa \log(pn) \frac{n}{n} + \frac{1}{n} \leq \frac{(2C + 1) \kappa \log(pn)}{n}. \]

This bound, together with (8.9) and bounds (8.7), yields

\[ \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \sqrt{n} \| L_{\mathcal{F}}(\hat{\Sigma} - \Sigma) \|_{(p, \Gamma, \Gamma, s_1, s_2)} \leq z \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_j \leq z \right] \right| \leq \]

\[ \leq C \kappa \left\{ 8^{3/(2\beta)} \left( \frac{\log(pn^2)}{n} \right)^{1/8} + \right\} + \Diamond. \]

Adjusting the dependence on \( \kappa \) in \( C \kappa \) makes \( \Diamond \) negligible compared to the current error term. We obtained

\[ \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \sqrt{n} \| L_{\mathcal{F}}(\hat{\Sigma} - \Sigma) \|_{(p, \Gamma, \Gamma, s_1, s_2)} \leq z \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_j \leq z \right] \right| \leq \]

\[ \leq C \kappa \left\{ 8^{3/(2\beta)} \left( \frac{\log(pn^2)}{n} \right)^{1/8} + \right\}. \quad (8.10) \]
8.2.4 From linear part to projectors

We have for all $z \in \mathbb{R}$ with $\delta = \sqrt{n}C\psi_n^2d_\mathcal{J}(\Sigma)^{3/2}$ (C from (8.5) of Lemma 8.3) the following:

$$
P\left[\sqrt{n}\|\hat{P}_\mathcal{J} - P_\mathcal{J}\|_{(P_\mathcal{J},\Gamma^g,s_1,s_2)} \geq z\right] - P\left[\max_{j \in [p]} Y_j \geq z\right]
\leq P\left[\sqrt{n}\|\hat{P}_\mathcal{J} - P_\mathcal{J}\|_{(P_\mathcal{J},\Gamma^g,s_1,s_2)} - \sqrt{n}\|L_\mathcal{J}(\hat{\Sigma} - \Sigma)\|_{(P_\mathcal{J},\Gamma^g,s_1,s_2)} \geq \delta\right] +
+ \left\{P\left[\sqrt{n}\|L_\mathcal{J}(\hat{\Sigma} - \Sigma)\|_{(P_\mathcal{J},\Gamma^g,s_1,s_2)} \geq z - \delta\right] - P\left[\max_{j \in [p]} Y_j \geq z - \delta\right]\right\} +
+ \left\{P\left[\max_{j \in [p]} Y_j \geq z - \delta\right] - P\left[\max_{j \in [p]} Y_j \geq z\right]\right\}.
$$

The first term is bounded by $2/n$ due to Lemma 8.3 (applied with $\hat{\Sigma} = \Sigma, \hat{P}_\mathcal{J} = P_\mathcal{J}$) combined with Lemma 8.1 and Assumption 4.3 (i). Next, the second term is bounded according to (8.10). For the third term we apply Lemma 8.8 again and obtain

$$
P\left[\max_{j \in [p]} Y_j \geq z - \delta\right] - P\left[\max_{j \in [p]} Y_j \geq z\right] = P\left[\max_{j \in [p]} (Y_j/\kappa) \geq z/\kappa - \delta/\kappa\right] - P\left[\max_{j \in [p]} (Y_j/\kappa) \geq z/\kappa\right]
\leq C(1, \kappa^2)\frac{\delta}{\kappa} \cdot \sqrt{1 \vee \log\left(\frac{p}{\delta/\kappa}\right)} \leq C_\kappa \frac{\delta}{\kappa^2} \sqrt{\log\left(\frac{ep\kappa}{\delta}\right)}.
$$

The opposite inequality for

$$
P\left[\max_{j \in [p]} Y_j \geq z\right] - P\left[\sqrt{n}\|\hat{P}_\mathcal{J} - P_\mathcal{J}\|_{(P_\mathcal{J},\Gamma^g,s_1,s_2)} \geq z\right]
$$

can be obtained in a similar way. Putting all the bounds together, we obtain the desired approximation.

As to the spectral norm test statistic $\tilde{Q}^{(1)}$, since by Lemma 8.4 (ii)

$$
\|L_\mathcal{J}(\hat{\Sigma} - \Sigma)\| = \|L_\mathcal{J}(\hat{\Sigma} - \Sigma)\|_{(P_\mathcal{J},\Gamma^g,m,d_m)},
$$

and trivially

$$
\|\hat{P}_\mathcal{J} - P_\mathcal{J}\| - \|L_\mathcal{J}(\hat{\Sigma} - \Sigma)\| \leq \|R_\mathcal{J}(\hat{\Sigma} - \Sigma)\|,
$$

the same proof applies and yields the desired bound with $s_1 = m$ and $s_2 = d - m$.

8.3 Proof of Theorem 4.2

Proof is quite similar to the proof of Theorem 4.1, so we skip the technical details and focus only on the key parts that are different.

As in the proof of Theorem 4.1, we start with concentration written in specific form, but this time we are interested in concentration of $\Sigma^B$ around $\hat{\Sigma}$ conditionally on $X$.

**Lemma 8.10.** With probability $1 - 1/n$ it holds

$$
P\left(\max_{s,t \in [q]} \frac{\|P_s(\Sigma^B - \hat{\Sigma})P_t\|_F}{\sqrt{m_s\mu_s m_t\mu_t}} \geq \psi_n:\ X \right) \leq \frac{1}{n}.$$
\[
\tilde{\psi}_n \overset{\text{def}}{=} C \frac{e^{\frac{1}{2} \left( \log(n) + \log(2d^2) \right) + \frac{1}{2} \beta_j}}{\sqrt{n}}.
\]

Due to Lemma 8.3 (this time applied with \( \tilde{\Sigma} = \Sigma^B, \tilde{P}_J = P^B_J \)) combined with Lemma 8.10 and Assumption 4.3,

\[
P \left[ \sqrt{n} \left\| P^B_J - \hat{P}_J \right\| \left( P^B_J, \Gamma^{s_1, s_2}, \tilde{\Sigma} \right) \right) \geq \sqrt{n} \left\| L_J \left( \Sigma^B - \hat{\Sigma} \right) \right\| \left( P^B_J, \Gamma^{s_1, s_2} \right) \geq \frac{\sqrt{n} C (\psi_n + \tilde{\psi}_n)^2 d_J (\Sigma)}{\sqrt{n}} \right] \leq 1 \cdot \frac{1}{n}
\]

with probability \( 1 - 1/n \). Therefore, it again makes sense to work with the linear part \( \sqrt{n} L_J (\Sigma^B - \hat{\Sigma}) \) of \( \sqrt{n} (P^B_J - \hat{P}_J) \). Introduce

\[
L^B_{\text{disc}} \overset{\text{def}}{=} \max_{j \in [p]} v_j^T \left( \sum_{r \in J, s \notin J} \frac{P_r (\Sigma^B - \hat{\Sigma}) P_s + P_s (\Sigma^B - \hat{\Sigma}) P_r}{\mu_r - \mu_s} \right) w_j = \max_{j \in [p]} \left( \sum_{r \in J, s \notin J} \frac{v_j^T P_r (\Sigma^B - \hat{\Sigma}) P_s w_j}{\mu_r - \mu_s} \right).
\]

We again apply discretization step: by Lemma 8.5,

\[
L^B_{\text{disc}} \leq \left\| L_J (\Sigma^B - \hat{\Sigma}) \right\| (P_J, \Gamma^{s_1, s_2}) \leq \frac{1}{1 - 2\varepsilon} L^B_{\text{disc}}.
\]

Now let us represent \( \sqrt{n} L^B_{\text{disc}} \) in a different way. Introduce for \( i \in [n], j \in [p] \)

\[
x^B_{ij} \overset{\text{def}}{=} \sum_{r \in J, s \notin J} \frac{v_j^T P_r (\eta_i X_i X_i^\top - X_i X_i^\top P_s w_j)}{\mu_r - \mu_s} = (\eta_i - 1) \cdot x_{ij},
\]

where the random variables \( x_{ij}, i \in [n], j \in [p] \) are from the proof of Theorem 4.1. Therefore,

\[
\sqrt{n} L^B_{\text{disc}} = \max_{j \in [p]} \frac{1}{\sqrt{n}} \sum_{i=1}^n x^B_{ij}.
\]

Similarly to \( x_i = \{x_{ij}\}_{j=1}^p, i \in [n], \) we can arrange these random variables as i.i.d. \( p \)-dimensional centered random vectors

\[
x^B_i \overset{\text{def}}{=} \{x^B_{ij}\}_{j=1}^p.
\]

Note that conditionally on the data \( \mathbf{X} \) these vectors are automatically Gaussian with the covariance

\[
\text{Cov} (x^B_i | \mathbf{X}) = x_i x_i^\top, \quad i = 1, \ldots, n.
\]

Hence, conditionally on \( \mathbf{X} \), the random vector

\[
Y^B \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^n x^B_i
\]

is Gaussian with covariance

\[
\text{Cov} (Y^B | \mathbf{X}) = \frac{1}{n} \sum_{i=1}^n x_i x_i^\top.
\]
So, the Gaussian approximation step is not needed, unlike in the proof of Theorem 4.1. To proceed, we need to show that \( \text{Cov}(Y^B \mid X) \) is close to \( \text{Cov}(Y) \) with \( Y \) from the proof of Theorem 4.1. Recall that

\[
\text{Cov}(Y) = \text{Cov}(x_1) = E \left[ \text{Cov}(Y^B \mid X) \right].
\]

The next lemma takes place.

**Lemma 8.11.** With probability \( 1 - 1/n \) it holds

\[
\| \text{Cov}(Y^B \mid X) - \text{Cov}(Y) \|_{\max} \leq \kappa^2 \Delta_B,
\]

where

\[
\Delta_B \overset{\text{def}}{=} C_\beta c^4 \kappa^2 \left( \sqrt{\frac{\log(pn)}{n}} + \frac{\log(n)^{2/\beta} (\log(pn))^{4/\beta}}{n} \right).
\]

Moreover, by Assumption 4.4, \( \Delta_B \leq 1/2 \).

Define \( \Omega \) to be the event from Lemma 8.11, \( P[\Omega] \geq 1 - 1/n \). On \( \Omega \) for all \( j \in [p] \)

\[
| \text{Var}(Y^B_j) - \text{Var}(Y_j) | = \left| \text{Cov}(Y^B \mid X) - \text{Cov}(Y) \right|_{j,j} \leq \kappa^2 \Delta_B \leq \frac{\kappa^2}{2}.
\]

So, since \( \sqrt{n}L^B_{\text{disc}} = \max_{j \in [p]} Y^B_j \) and \( \text{Var}(Y^B_j) \leq \text{Var}(Y_j) + \frac{\kappa^2}{2} \leq \bar{\kappa}^2 + \frac{\kappa^2}{2} \leq 2\bar{\kappa}^2 \),

\[
\text{Var}(Y^B_j) \geq \text{Var}(Y_j) - \frac{\kappa^2}{2} \geq \kappa^2 - \frac{\kappa^2}{2} = \frac{\kappa^2}{2},
\]

the same approach as in subsection 8.2.3, applied conditionally on \( X \), together with bounds (8.12) implies

\[
\sup_{z \in \mathbb{R}} \left| P \left[ \sqrt{n} \| L_{\mathcal{J}}(\Sigma^B - \hat{\Sigma}) \|_{(P_{\mathcal{J}},\Gamma^{s_1,s_2})} \leq z \mid X \right] - P \left[ \max_{j \in [p]} Y^B_j \leq z \right] \right| \leq
\]

\[
\frac{C_\kappa \log(pn)}{n}
\]

on \( \Omega \).

Next, we deal with nonlinearity similarly to subsection 8.2.4, this time taking \( \delta = \sqrt{n}C(\psi_n + \tilde{\psi}_n)^2 d_{\mathcal{J}}(\Sigma)^{3/2} \) and applying Lemma 8.3 with \( \tilde{\Sigma} = \Sigma^B, \tilde{\psi}_n = \psi_n \). Omitting the details, we obtain

\[
\sup_{z \in \mathbb{R}} \left| P \left[ \sqrt{n} \left\| \hat{P}_{\mathcal{J}} - \hat{\Sigma} \right\|_{(P_{\mathcal{J}},\Gamma^{s_1,s_2})} \leq z \mid X \right] - P \left[ \max_{j \in [p]} Y^B_j \leq z \mid X \right] \right| \leq
\]

\[
C_\kappa \left( \frac{\log(pn)}{n} + \frac{\delta}{\kappa} \sqrt{\log \left( \frac{ep}{\delta/\kappa} \right)} + \frac{1}{n} \right)
\]

on \( \Omega \).
The only thing left is to compare the distributions of
\[
\left( \max_{j \in [p]} Y^B_j \right) \text{ and } \max_{j \in [p]} Y_j.
\]

On Ω we showed
\[
\max_{j,k \in [p]} |\text{Cov}(Y^B_j/\kappa | X) - \text{Cov}(Y_j/\kappa)|_{jk} \leq \Delta_B.
\]

Applying Lemma 8.9 results in
\[
\sup_{z \in R} \left| P \left[ \max_{j \in [p]} Y^B_j \leq z \right] - P \left[ \max_{j \in [p]} Y_j \leq z \right] \right| \leq \Delta B
\]
on Ω. Putting all the bounds together with Theorem 4.1, we obtain the desired.

### 8.4 Proof of Theorem 4.3

The proof is quite similar to Theorem 4.2, so we follow the same steps. First we formulate the concentration result, this time for \( \Sigma^F \).

**Lemma 8.12.** With probability \( 1 - 1/n \) it holds
\[
P \left( \max_{s,t \in [g]} \left\| P_s (\Sigma^F - \hat{\Sigma}) P_t \right\|_F \geq \tilde{\psi}_n \left| X \right| \right) \leq \frac{1}{n},
\]
where
\[
\tilde{\psi}_n \overset{\text{def}}{=} C C_2 \left( \log(n) + \log(2d^2) \right)^{3/2} \sqrt{n}.
\]

Due to Lemma 8.3 (applied with \( \tilde{\Sigma} = \Sigma^F \), \( \tilde{P}_J = P_J^F \)) combined with Lemma 8.12 and Assumption 4.3,
\[
P \left[ \sqrt{n} \left\| P_J^F - \hat{P}_J \right\|_{(P_J^F, \Gamma^0, s_1, s_2)} - \sqrt{n} \left\| L_J (\Sigma^F - \hat{\Sigma}) \right\|_{(P_J^F, \Gamma^0, s_1, s_2)} \geq \tilde{\psi}_n \right) \leq \frac{1}{n},
\]
with probability \( 1 - 1/n \). We again elaborate on the linear term \( L_J (\Sigma^F - \Sigma) \). Define its discretized version
\[
L^F_{\text{disc}} \overset{\text{def}}{=} \max_{j \in [p]} \sum_{r \in J, s \notin J} \sum_{r \in J, s \notin J} v_j^{\top} \left( \sum_{r \in J, s \notin J} \sum_{r \in J, s \notin J} P_r (\Sigma^F - \hat{\Sigma}) P_s \frac{\mu_r - \mu_s}{\mu_r - \mu_s} \right) w_j
\]

\[
= \max_{j \in [p]} \sum_{r \in J, s \notin J} v_j^{\top} \frac{P_r (\Sigma^F - \hat{\Sigma}) P_s w_j}{\mu_r - \mu_s}.
\]
and by Lemma 8.5 obtain the bounds

\[ L^F_{\text{disc}} \leq \|L_J(\Sigma^F - \hat{\Sigma})\| (p, \Gamma^{\circ}, s_1, s_2) \leq \frac{1}{1 - 2\varepsilon} L^F_{\text{disc}}. \]

Introducing for \( i \in [n], j \in [p] \)

\[ x^F_{ij} \overset{\text{def}}{=} \sum_{r \in \mathcal{J}} \sum_{s \in \mathcal{J}} v^T_j P_r (Z_i Z^T - \hat{\Sigma}) P_s w_j, \]

we represent

\[ \sqrt{n} L^F_{\text{disc}} = \max_{j \in [p]} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x^F_{ij}. \]

The \( p \)-dimensional centered random vectors

\[ x^F_i \overset{\text{def}}{=} \{ x^F_{ij} \}_{j=1}^p \]

are i.i.d., and we need to compute their covariance conditionally on \( X \). For any fixed indices \( i \in [n], j, k \in [p], r, r' \in \mathcal{J}, s, s' \notin \mathcal{J} \), similarly to Subsection 8.2.2, “Computing the covariance” part, we have

\[
\begin{align*}
E \left[ v_j^T P_r Z_i Z^T_j P_s w_j \cdot v_k^T P_{r'} Z_i Z^T_j P_{s'} w_k | X \right] &= \left( v_j^T P_r \hat{\Sigma} P_{r'} v_k \right) \cdot \left( w_j^T P_s \hat{\Sigma} P_{s'} w_k \right) - \left( v_j^T P_r \hat{\Sigma} P_{s' w_k} \right) \cdot \left( w_j^T P_s \hat{\Sigma} P_{r'} w_k \right),
\end{align*}
\]

where we take into account that for Gaussian \( Z \) it holds that \( P_J Z \) and \( P_{J'} Z \) are independent (which implies Assumption 4.1 for \( X \)). Thus,

\[
\begin{align*}
E \left[ v_j^T P_r (Z_i Z^T_j - \hat{\Sigma}) P_s w_j \cdot v_k^T P_{r'} (Z_i Z^T_j - \hat{\Sigma}) P_{s'} w_k | X \right] &= \left( v_j^T P_r \hat{\Sigma} P_{r'} v_k \right) \cdot \left( w_j^T P_s \hat{\Sigma} P_{s'} w_k \right) - \left( v_j^T P_r \hat{\Sigma} P_{s'} w_k \right) \cdot \left( w_j^T P_s \hat{\Sigma} P_{r'} w_k \right),
\end{align*}
\]

and

\[
\begin{align*}
\text{Cov}(x^F_i | X)_{j,k} &= E \left[ x^F_{ij} x^F_{jk} | X \right] = \sum_{r \in \mathcal{J}} \sum_{s \in \mathcal{J}} \sum_{r' \notin \mathcal{J}} \sum_{s' \notin \mathcal{J}} \frac{(v_j^T P_r \hat{\Sigma} P_{r'} v_k) \cdot (w_j^T P_s \hat{\Sigma} P_{s'} w_k) - (v_j^T P_r \hat{\Sigma} P_{s'} w_k) \cdot (w_j^T P_s \hat{\Sigma} P_{r'} w_k)}{(\mu_r - \mu_s)(\mu_{r'} - \mu_{s'})}.
\end{align*}
\]

We again have to use Lemma 8.7 to pass from \( x^F_i \)'s to their Gaussian counterparts \( y^F_i \)'s with the same covariance. Introduce

\[ Y^F \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} y^F_i \sim N_p \left( 0, \text{Cov}(x^F_1 | X) \right) \]

to approximate the distribution of interest by \( \max_{j \in [p]} Y^F_j \). Recall that in the proof of Theorem 4.1 we had for \( Y \) and \( x_i \)'s

\[
\text{Cov}(Y)_{j,k} = \text{Cov}(x_1)_{j,k} = \sum_{r \in \mathcal{J}} \sum_{s \notin \mathcal{J}} \frac{\mu_r \mu_s}{(\mu_r - \mu_s)^2} (v_j^T P_r v_k) \cdot (w_j^T P_s w_k),
\]

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which alternatively can be expressed as
\[
(Cov(Y))_{j,k} = [Cov(x_1)]_{j,k} = \\
\sum_{r \in \mathcal{J}} \sum_{s \in \mathcal{J}} (v_j^\top P_r \Sigma P_r v_k) \cdot (w_j^\top P_s \Sigma P_s w_k) - (v_j^\top P_r \Sigma P_s w_j) \cdot (v_k^\top P_r \Sigma P_s w_k).
\]

Observe that the difference between this expression and the expression for
\[
\delta = \delta \cdot \sum_{r \in \mathcal{J}} \sum_{s \in \mathcal{J}} \cdot (v_j^\top P_r \Sigma P_r v_k) \cdot (w_j^\top P_s \Sigma P_s w_k) - (v_j^\top P_r \Sigma P_s w_j) \cdot (v_k^\top P_r \Sigma P_s w_k).
\]

above is that the true covariance \(\Sigma\) replaces the empirical one \(\hat{\Sigma}\). In the next lemma we bound the maximal element-wise absolute difference between this two covariances.

**Lemma 8.13.** With probability \(1 - 1/n\) it holds
\[
\|Cov(Y^F | X) - Cov(Y)\|_{\max} \leq L^2 \Delta_F,
\]
where
\[
\Delta_F \overset{\text{def}}{=} |\mathcal{J}| C_B \epsilon^2 \kappa^2 \left( \sqrt{\log(np) + \log(|\mathcal{J}|)} + \frac{(\log(n))^{1/3} (\log(np) + \log(|\mathcal{J}|))^{2/3}}{n} \right).
\] (8.14)

Moreover, by Assumption 4.5, \(\Delta_F \leq 1/2\).

Let \(\Omega\) be the event from Lemma 8.13, \(P[\Omega] \geq 1 - 1/n\). On this event
\[
\text{Var}(Y^F_j) \leq \text{Var}(Y_j) + \frac{\kappa^2}{2} \leq \pi^2 + \frac{\kappa^2}{2} \leq 2\pi^2,
\]
\[
\text{Var}(Y^F_j) \geq \text{Var}(Y_j) - \frac{\kappa^2}{2} \geq \kappa^2 - \frac{\kappa^2}{2} = \frac{\kappa^2}{2},
\]
and all the arguments from the proof of Theorem 4.1 work with slightly shifted variances and \(\psi_n^2\) replaced by \((\psi_n + \tilde{\psi}_n)^2\). Nonlinearity is treated similarly to subsection 8.2.4, this time taking \(\delta = \sqrt{n}C(\psi_n + \tilde{\psi}_n)^2 d_\mathcal{F}(\Sigma)^{3/2}\) and applying Lemma 8.3 with \(\tilde{\Sigma} = \Sigma^F, \tilde{\psi}_n = \tilde{\psi}_n\). So,
\[
\sup_{x \in \mathbb{R}} |P \left[ \sqrt{n} \| P_{\mathcal{F}} - \hat{P}_{\mathcal{F}} \|_{(p,\Gamma_n,s_1,s_2)} \leq z \mid X \right] - P \left[ \max_{j \in \mathcal{J}} Y^F_j \leq z \mid X \right] | \leq \\
\leq C_\kappa \left\{ \hat{\Delta} + \zeta \left[ \sqrt{n}(\psi_n + \tilde{\psi}_n)^2 \| \mathcal{D}_\mathcal{F}(\Sigma)^{3/2} \| \right] \right\}
\]
on \(\Omega\), in addition to already established result
\[
\sup_{x \in \mathbb{R}} |P \left[ \sqrt{n} \| \hat{P}_{\mathcal{F}} - P_{\mathcal{F}} \|_{(p,\Gamma_n,s_1,s_2)} \leq z \right] - P \left[ \max_{j \in \mathcal{J}} Y_j \leq z \right] | \leq \\
\leq C_\kappa \left\{ \hat{\Delta} + \zeta \left[ \sqrt{n}(\psi_n)^2 \| \mathcal{D}_\mathcal{F}(\Sigma)^{3/2} \| \right] \right\}.
\]
The only thing left is to apply Gaussian comparison Lemma 8.9 to
\[
\max_{j \in \mathcal{J}} (Y^F_j) \quad \text{and} \quad (\max_{j \in \mathcal{J}} (Y^F_j) \mid X)
\]
with \(\Delta = \Delta_F\) from Lemma 8.13.
8.5 Proof of Theorem 4.5

We start with the following lemma, which is modification of Lemma 8.3. Not only does it allow to get rid of the remainder term, but at the same time replaces $\| \cdot \|_{(P,R,s_1,s_2)}$ by some $\| \cdot \|_{(P^*,\Gamma^*,s_1,s_2)}$ with deterministic $P^*$. We provide a simplified version to be used in this proof; a more general one, suitable for proofs of Theorem 4.6 and Theorem 4.7, can be established similarly to Lemma 8.3.

**Lemma 8.14.** Let under $H_0^{(2)}$ be $P^* \overset{\text{def}}{=} P_a = P_b$. For any $\Gamma^*$, there exists $\Gamma^* = [\Gamma^*_1 \Gamma^*_2] \in \mathbb{R}^{d \times d}$ with $\Gamma^*_1 \in \mathbb{R}^{d \times m}, \Gamma^*_2 \in \mathbb{R}^{d \times (d-m)}$ satisfying

$$\begin{align*}
\Gamma^*_1 \Gamma^*_1^\top &= P^*, \quad \Gamma^*_1^\top \Gamma^*_1 = I_m, \\
\Gamma^*_2 \Gamma^*_2^\top &= I_d - P^*, \quad \Gamma^*_2^\top \Gamma^*_2 = I_{d-m},
\end{align*}$$

such that the following holds:

$$\begin{align*}
P \left[ \| \tilde{P}_a - \tilde{P}_b \|_{(P^*,\Gamma^*,s_1,s_2)} - \| L_a(\tilde{\Sigma}_a - \Sigma_a) - L_b(\tilde{\Sigma}_b - \Sigma_b) \|_{(P^*,\Gamma^*,s_1,s_2)} \right] &\leq C \psi^2_{n_a \wedge n_b} \left( d_{a,b} + d_{a,b}^{3/2} \right) |\Gamma^*| \\
&\geq 1 - \frac{1}{n_a} - \frac{1}{n_b}
\end{align*}$$

with probability $1 - 1/n_a - 1/n_b$.

Now we start elaborating on $\| L_a(\tilde{\Sigma}_a - \Sigma_a) - L_b(\tilde{\Sigma}_b - \Sigma_b) \|_{(P^*,\Gamma^*,s_1,s_2)}$ in a similar fashion as the proof of Theorem 4.1. The only difference is that in the rest of the proof, all the probabilities, expectations and variances are conditional on $\Gamma$. Let this time $(\Gamma^*_1v, \Gamma^*_2w)$ for $v \in \mathcal{N}(\mathcal{D}^m_{s_1}),$ $w \in \mathcal{N}(\mathcal{D}^{d-m}_{s_2})$. We again take $\varepsilon = 1/n$, which fixes $p$ to be

$$p \leq \exp \left( (s_1 + s_2) \log(3n) + \log(2d) \right).$$

Note that both $L_a(\tilde{\Sigma}_a - \Sigma_a)$ and $L_b(\tilde{\Sigma}_b - \Sigma_b)$ satisfy

$$\begin{align*}
L_a(\tilde{\Sigma}_a - \Sigma_a) &= P^* L_a(\tilde{\Sigma}_a - \Sigma_a)(I_d - P^*) + (I_d - P^*) L_a(\tilde{\Sigma}_a - \Sigma_a) P^*, \\
L_b(\tilde{\Sigma}_b - \Sigma_b) &= P^* L_b(\tilde{\Sigma}_b - \Sigma_b)(I_d - P^*) + (I_d - P^*) L_b(\tilde{\Sigma}_b - \Sigma_b) P^*,
\end{align*}$$

so Lemma 8.5 yield

$$L_{\text{disc}}^{a,b} \leq \| L_a(\tilde{\Sigma}_a - \Sigma_a) - L_b(\tilde{\Sigma}_b - \Sigma_b) \|_{(P^*,\Gamma^*,s_1,s_2)} \leq \frac{1}{1 - 2\varepsilon} L_{\text{disc}}^{a,b}, \quad (8.15)$$

where $L_{\text{disc}}^{a,b} = \max_{j \in [p]} \sum_{i \in [Q]} (\Gamma^* \tilde{\Sigma}_j - \Sigma_j)(I_d - P^*) \tilde{\Sigma}_j (I_d - P^*)$. The same quantity can be expressed as a sum. For all $j \in [p]$ introduce $x_{ij}^a$ for $i \in [n_a]$ and $x_{ij}^b$ for $i \in [n_b]$, which are analogs of $x_{ij}$.

Then

$$L_{\text{disc}}^{a,b} = \max_{j \in [p]} \left( \frac{1}{n_a} \sum_{i=1}^{n_a} x_{ij}^a - \frac{1}{n_b} \sum_{i=1}^{n_b} x_{ij}^b \right).$$

Gaussian counterpart of $\sqrt{\frac{n_a n_b}{n_a + n_b}} L_{\text{disc}}^{a,b}$ is given by $\max_{j \in [p]} Y_{j}^{a,b}$, where

$$Y_{j}^{a,b} = \sqrt{\frac{n_a n_b}{n_a + n_b}} \left( \frac{1}{n_a} \sum_{i=1}^{n_a} y_{ij}^a - \frac{1}{n_b} \sum_{i=1}^{n_b} y_{ij}^b \right).$$

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with \( y_i^a | \Gamma \sim \mathcal{N} \left( 0, \text{Cov}(x_i^a | \Gamma) \right) \) for all \( i \in [n_a] \) and \( y_i^b | \Gamma \sim \mathcal{N} \left( 0, \text{Cov}(x_i^b | \Gamma) \right) \) for all \( i \in [n_b] \).

The conditions of Lemma 8.7, verified in Subsection 8.2.2, can be treated here likewise. Note that similar to \( \text{Var}(x_{ij}) \), we can lower and upper bound

\[
\kappa_{\mathcal{J}_a}(\Sigma_a)^2 \leq \text{Var}(x_{ij}^a | \Gamma) \leq \kappa_{\mathcal{J}_b}(\Sigma_a)^2,
\]

\[
\kappa_{\mathcal{J}_b}(\Sigma_b)^2 \leq \text{Var}(x_{ij}^b | \Gamma) \leq \kappa_{\mathcal{J}_b}(\Sigma_b)^2.
\]

Furthermore, direct computation shows

\[
\text{Var}(Y_{ij}^{a,b} | \Gamma) = \frac{n_b \text{Var}(x_{ij}^a | \Gamma) + n_a \text{Var}(x_{ij}^b | \Gamma)}{n_a + n_b},
\]

implying

\[
\kappa_{\mathcal{J}_a}(\Sigma_a)^2 \land \kappa_{\mathcal{J}_b}(\Sigma_b)^2 \leq \text{Var}(Y_{ij}^{a,b} | \Gamma) \leq \kappa_{\mathcal{J}_a}(\Sigma_a)^2 \lor \kappa_{\mathcal{J}_b}(\Sigma_b)^2.
\]

So, the existence of \( c_1, C_1 > 0 \) lower- and upperbounding the variance is established. Upper bound on \( M_3, M_4 \) can be obtained as well, and it will be

\[
8^{2/\beta} \cdot (\kappa_{\mathcal{J}_a}(\Sigma_a)^2 \lor \kappa_{\mathcal{J}_b}(\Sigma_b)^2).
\]

Upper bound on \( u(\gamma) \) again follows likewise and becomes (for \( \gamma = 1/n_a + 1/n_b \))

\[
u(\gamma) \lesssim \kappa_{a,b} c^2 \left( \log \left( 2p(n_a + n_b)^2 \right) \right)^{2/\beta}.
\]

Lemma 8.7 then yields almost surely

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \sqrt{\frac{n_a n_b}{n_a + n_b}} L_{\text{disc}}^{a,b} \leq z | \Gamma \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_{ij}^{a,b} \leq z | \Gamma \right] \right| \leq C_{\kappa_{a,b}} \left\{ 8^{3/2(\beta)} \left( \frac{\left( \log(p(n_a + n_b)^2) \right)^{7}}{n_a + n_b} \right)^{1/8} + c^2 \left( \frac{\left( \log(2p(n_a + n_b)^2) \right)^{3+4/\beta}}{n_a + n_b} \right)^{1/2} \right. \\
\left. + \frac{1}{n_a + n_b} \right\},
\]

and by similar to Subsection 8.2.3 reasoning, bounds (8.15) allow to pass from discretized version to infinite-state supremum, omitting the negligible additional error term:

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left[ \sqrt{\frac{n_a n_b}{n_a + n_b}} \| L_a(\Sigma_a - \Sigma_a) - L_a(\Sigma_b - \Sigma_b) \|_{(p^*, p^*)} \leq z | \Gamma \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_{ij}^{a,b} \leq z | \Gamma \right] \right| \leq C_{\kappa_{a,b}} \left\{ 8^{3/2(\beta)} \left( \frac{\left( \log(p(n_a + n_b)^2) \right)^{7}}{n_a + n_b} \right)^{1/8} + c^2 \left( \frac{\left( \log(2p(n_a + n_b)^2) \right)^{3+4/\beta}}{n_a + n_b} \right)^{1/2} \right. \\
\left. + \frac{1}{n_a + n_b} \right\},
\]

The last step is to use Lemma 8.14 to finalize the result for projectors. As in Subsection 8.2.4,
we have with \( \delta = \sqrt{\frac{n_a n_b}{n_a + n_b}} C_{\psi}^{1/2} (\tilde{d}_{a,b} + d_{a,b}^{3/2}) \)

\[
\mathbb{P} \left[ \sqrt{\frac{n_a n_b}{n_a + n_b}} \| \hat{P}_a - \hat{P}_b \| (\mathbf{P}, \Gamma, s_1, s_2) \geq z \left| \Gamma \right] \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_{j}^{a,b} \geq z \left| \Gamma \right] \right] \\
\leq \mathbb{P} \left[ \sqrt{\frac{n_a n_b}{n_a + n_b}} \| L_a(\hat{\Sigma}_a - \Sigma_a) - L_b(\hat{\Sigma}_b - \Sigma_b) \| (\mathbf{P}, \Gamma, s_1, s_2) \geq \delta \left| \Gamma \right] \right] + \\
\{ \mathbb{P} \left[ \sqrt{\frac{n_a n_b}{n_a + n_b}} \| L_a(\hat{\Sigma}_a - \Sigma_a) - L_b(\hat{\Sigma}_b - \Sigma_b) \| (\mathbf{P}, \Gamma, s_1, s_2) \geq z - \delta \left| \Gamma \right] \right] - \\
\mathbb{P} \left[ \max_{j \in [p]} Y_{j}^{a,b} \geq z - \delta \left| \Gamma \right] \right] \} + \\
\{ \mathbb{P} \left[ \max_{j \in [p]} Y_{j}^{a,b} \geq z - \delta \left| \Gamma \right] \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_{j}^{a,b} \geq z \left| \Gamma \right] \right] \} .
\]

By Lemma 8.14, the first term is at most \( 1/n_a + 1/n_b \) with probability \( 1 - 1/n_a - 1/n_b \). The second term is bounded by \( (8.16) \). For the third term we apply Lemma 8.8 and get

\[
\mathbb{P} \left[ \max_{j \in [p]} Y_{j}^{a,b} \geq z - \delta \left| \Gamma \right] \right] - \mathbb{P} \left[ \max_{j \in [p]} Y_{j}^{a,b} \geq z \left| \Gamma \right] \right] \leq C_{\kappa,a,b} \delta \kappa J_a(\Sigma_b) \vee \kappa J_b(\Sigma_a) (\log (e\delta) )^{1/2} .
\]

The opposite inequality can be obtained similarly. This concludes the proof.

### 8.6 Proofs of Theorem 4.6 and Theorem 4.7

The proofs repeat proofs of Theorem 4.2 and Theorem 4.3, respectively, with Theorem 4.5 used in place of Theorem 4.1.

### 8.7 Proofs of Corollary 4.4 and Corollary 4.8

To prove Corollary 4.4, it is enough to repeat the proof of Corollary 2.3 of Silin and Spokoiny (2018), applied with our Theorem 4.2 and Theorem 4.3. Corollary 4.8 is slightly trickier, since we condition on \( \Gamma \). However, still the proof of Corollary 2.3 of Silin and Spokoiny (2018), applied with, for example, Theorem 4.6, yields

\[
\sup_{\alpha \in (0;1)} \left| \mathbb{P} \left[ Q^{(2)} > \gamma_B^{[S]} \left| \Gamma \right] \right] - \alpha \right| \leq \frac{1}{n_a} + \frac{1}{n_b}
\]

with probability \( 1 - 1/n_a - 1/n_b \). Integrating \( \Gamma \) out, we obtain the desired.

### 8.8 Proof of Theorem 4.9 and 4.10

Let us demonstrate the proof of Theorem 4.9 first. The proofs for (i) and (ii) are identical, so let us only focus on (i).

Recall \( \gamma_B^{(1)}(\alpha) \) from Corollary 4.4. By triangle inequality and the assumption of the theorem,
\[
P \left[ \sqrt{n} \| \hat{P}_J - P^o \|_{(P^o, \Gamma^o, s_1, s_2)} \geq \gamma_B^{(1)}(\alpha) \right] \geq \\
\geq P \left[ \sqrt{n} \| P_J - P^o \|_{(P^o, \Gamma^o, s_1, s_2)} - \sqrt{n} \| \hat{P}_J - P_J \|_{(P^o, \Gamma^o, s_1, s_2)} \geq \gamma_B^{(1)}(\alpha) \right] \\
= P \left[ \sqrt{n} \| \hat{P}_J - P_J \|_{(P^o, \Gamma^o, s_1, s_2)} \leq \lambda_n - \gamma_B^{(1)}(\alpha) \right].
\]

Proposition 3.1 (ii) implies
\[
P \left[ \sqrt{n} \| \hat{P}_J - P^o \|_{(P^o, \Gamma^o, s_1, s_2)} \geq \gamma_B^{(1)}(\alpha) \right] \geq \\
\geq P \left[ \sqrt{n} \| \hat{P}_J - P^o \|_{(P^o, \Gamma^o, s_1, s_2)} \geq \gamma_B^{(1)}(\alpha) \right] \\
\geq P \left[ \sqrt{n} \| \hat{P}_J - P_J \| \leq \lambda_n - \gamma_B^{(1)}(\alpha) \right],
\]
where \(\gamma_B^{(1)}(\alpha)\) is \(\alpha\)-quantile of \(\sqrt{n} \| P_B^o - \hat{P}_J \|\). In the proof of Lemma 8.14 we will show that with probability \(1 - 1/n\) we have the bound
\[
\sqrt{n} \| \hat{P}_J - P_J \| \leq \sqrt{n} \Phi_n \left( \psi_n d^{1/2} + \tilde{\psi}_n d^{3/2} \right),
\]
and similarly
\[
\sqrt{n} \| P_B^o - \hat{P}_J \| \leq \sqrt{n} \Phi_n \left( \tilde{\psi}_n d^{1/2} + \tilde{\psi}_n d^{3/2} \right),
\]
which means that \(\gamma_B^{(1)}(\alpha)\) is at most of the same order treating \(\alpha\) as constant. Denoting for shortness \(\Phi_n = \sqrt{n} \Phi_n \left( (\psi_n + \tilde{\psi}_n) d^{1/2} + (\psi_n + \tilde{\psi}_n)^2 d^{3/2} \right)\), if \(C \geq 2C\), this ensures that
\[
P \left[ \sqrt{n} \| \hat{P}_J - P_J \| \leq \lambda_n - \gamma_B^{(1)}(\alpha) \right] = P \left[ \sqrt{n} \| \hat{P}_J - P_J \| \leq \Phi_n \left( \frac{\lambda_n}{\Phi_n} - \frac{\gamma_B^{(1)}(\alpha)}{\Phi_n} \right) \right] \to 1
\]
as \(n \to \infty\), since \(\liminf_{n \to \infty} \frac{\lambda_n}{\Phi_n} \geq C / C \geq 2\) by condition (4.3) and \(\gamma_B^{(1)}(\alpha) / \Phi_n \leq 1\). This concludes the proof.

The proof of Theorem 4.10 repeats the proof above, with the only difference that we will also need to apply the inequality
\[
\| P_a - P_b \|_{(P, \Gamma, s_1, s_2)} \geq \frac{1}{2} \sqrt{m(d-m)} \| P_a - P_b \| \geq \lambda_{n_a, n_b} \sqrt{\frac{n_a + n_b}{n_a n_b}}.
\]
A Auxiliary results from literature

A.1 Results from Kuchibhotla and Chakrabortty (2018)

Proposition A.1 (Kuchibhotla and Chakrabortty (2018), Proposition S.3.2). If $W_i$, $i \in [k]$ are (possibly dependent) random variables satisfying $\|W_i\|_{\psi_{\alpha_i}} < \infty$ for some $\alpha_i > 0$, then

$$\left\| \prod_{i=1}^{k} W_i \right\|_{\psi_{\beta}} \leq \prod_{i=1}^{k} \|W_i\|_{\psi_{\alpha_i}},$$

where $1/\beta = \sum_{i=1}^{k} 1/\alpha_i$.

Theorem A.2 (Kuchibhotla and Chakrabortty (2018), Theorem 4.1). Let $X_1, \ldots, X_n$ be independent random vectors in $\mathbb{R}^p$ satisfying

$$\max_{i \in [n]} \|X_i(j)\|_{\psi_{\beta}} \leq K_{n,p} < \infty \text{ for some } 0 < \beta \leq 2.$$

Fix $n, p \geq 1$. Then for any $t \geq 0$, with probability at least $1 - 3e^{-z}$,

$$\max_{j,k \in [p]} \left| \frac{1}{n} \sum_{i=1}^{n} X_i(j)X_i(k) - E[X_i(j)X_i(k)] \right| \leq 7A_{n,p} \sqrt{\frac{z + 2\log(p)}{n}} + \frac{C_{\beta}K_{n,p}^2(\log(2n))^{1/\beta}(z + 2\log(p))^{2/\beta}}{n},$$

where $C_{\beta} > 0$ is a constant depending only on $\beta$, and $A_{n,p}^2$ is given by

$$A_{n,p} \overset{\text{def}}{=} \max_{j,k \in [p]} \frac{1}{n} \sum_{i=1}^{n} \text{Var}(X_i(j)X_i(k)).$$

Remark A.1. Remark 4.1 from Kuchibhotla and Chakrabortty (2018) claims $A_{n,p} \leq C_{\beta}K_{n,p}^2$, so in every application of Theorem A.2 we use this small fact without further notice.

A.2 Results from Jirak and Wahl (2018)

Recent paper Jirak and Wahl (2018) considers infinite-dimensional Hilbert space $\mathcal{H}$ and two covariance operators $\Sigma, \hat{\Sigma}$ with perturbation $E \overset{\text{def}}{=} \hat{\Sigma} - \Sigma$. The notations for eigenvalues (and distinct eigenvalues), eigenvectors and projectors are similar to ours. Furthermore, the following intuitive notations are employed:

$$\text{Tr}_{\geq r_0}(\Sigma) \overset{\text{def}}{=} \sum_{r \geq r_0} m_r \mu_r$$

and

$$P_{\geq r_0} \overset{\text{def}}{=} \sum_{r \geq r_0} P_r.$$

They also define the resolvent

$$R_r = \sum_{s \neq r} \frac{1}{\mu_s - \mu_r} P_s.$$

Now we are ready to state relative perturbation bounds for eigenvalues and projectors.
Theorem A.3 (Jirak and Wahl (2018), Theorem 3). Let $r \geq 1$. Consider $r_0 \geq 1$ such that $\mu_{r_0} \leq \mu_r/2$. Let $x > 0$ be such that for all $s, t < r_0$,

$$\|P_sEP_t\|_F, \|P_sEP_{r_0}\|_F, \|P_{r_0}EP_{r_0}\|_F \leq x. \tag{A.1}$$

Suppose that

$$r_r(\Sigma) \leq 1/(6x). \tag{A.2}$$

Then we have

$$\frac{1}{m_r \mu_r} \sum_{k=1}^{m_r} \lambda_k(\hat{P}_r(\hat{\Sigma} - \mu_r \Gamma \hat{\Sigma})) - \lambda_k(P_rEP_r) \leq Cx^2 r_r(\Sigma), \tag{A.3}$$

where $\lambda_k(\cdot)$ denotes the $k$-th largest eigenvalue. In particular, if $j$ is the smallest integer such that $j \in I_r$, then

$$\frac{1}{m_r \mu_r} \lambda_j - \mu_r - \lambda_1(P_rEP_r) \leq Cx^2 r_r(\Sigma).$$

Theorem A.4 (Jirak and Wahl (2018), Theorem 4). Let $r \geq 1$. Consider $r_0 \geq 1$ such that $\mu_{r_0} \leq \mu_r/2$. Let $x$ be such that (A.1) holds. Moreover, suppose that Condition (A.2) holds. Then we have

$$\|\hat{P}_r - P_r - R_rEP_r - P_rER_r\|_F \leq Cx^2 r_r(\Sigma) \sqrt{\sum_{s \neq r} \frac{m_r \mu_r m_s \mu_s}{(\mu_r - \mu_s^2)}} \tag{A.4}$$

and

$$\left|\|\hat{P}_r - P_r\|_F^2 - 2\|R_rEP_r\|_F^2\right| \leq Cx^3 r_r(\Sigma) \sum_{s \neq r} \frac{m_r \mu_r m_s \mu_s}{(\mu_r - \mu_s^2)}. \tag{A.5}$$

B Auxiliary proofs

Proof of Proposition 3.1.

(i) Homogeneity is trivial. Triangle inequality follows directly from triangle inequality for spectral norm combined with triangle inequality for maximum. The only property to check is that $\|A\|_{(p, 1, s_1, s_2)} = 0$ implies $A = 0$. Indeed, as we will see in (ii), $\|A\|_{(p, 1, s_1, s_2)} = 0$ implies $\|A\| = 0$, and thus $A = 0$, since spectral norm is a norm.

(ii) To prove the desired bounds, it is more convenient to use the representation from Lemma 8.4 (i) (which is proved independently slightly later). The upper bound is trivially implied by the inequalities

$$\|\Gamma_1^TA\Gamma_1\| \leq \|A\|, \quad \|\Gamma_2^TA\Gamma_2\| \leq \|A\|,$$

$$\sup_{w \in D_{\Sigma}^{d-m}} v^T \Gamma_1^TA\Gamma_2 w \leq \sup_{w \in S^{d-m-1}} v^T \Gamma_1^TA\Gamma_2 w \leq \sup_{w \in S^{d-1}} v^T A w = \|A\|.$$ 

Let us prove the lower bound. Let $\tilde{u}$ be the eigenvector corresponding to largest absolute eigenvalue of $A$. Define $\tilde{v} = \Gamma_1^T\tilde{u} \in \mathbb{R}^m$ and $\tilde{w} = \Gamma_2^T\tilde{u} \in \mathbb{R}^{d-m}$, so that $\tilde{u} = \Gamma_1\tilde{v} + \Gamma_2\tilde{w}$. Note
since \( \|v\| \leq 1 \) and \( \|\tilde{w}\| \leq 1 \). Then

\[
\|A\| = \|\tilde{u}^T A \tilde{u}\| = \left| (\Gamma_1 \tilde{v} + \Gamma_2 \tilde{w})^T A (\Gamma_1 \tilde{v} + \Gamma_2 \tilde{w}) \right|
\leq \|\tilde{v}^T \Gamma_1^T \Gamma_1 \tilde{v}\| + \|\tilde{w}^T \Gamma_2^T \Gamma_2 \tilde{w}\| + 2\|\tilde{v}^T \Gamma_1^T \Gamma_2 \tilde{w}\|
\leq \|\Gamma_1^T \Gamma_1\| + \|\Gamma_2^T \Gamma_2\| + 2\|\tilde{v}^T \Gamma_1^T \Gamma_2 \tilde{w}\|.
\]

To bound \( \|\tilde{v}^T \Gamma_1^T \Gamma_2 \tilde{w}\| \) in terms of \( \sup_{w \in D_{d-m}^2} v^T \Gamma_1^T \Gamma_2 w \), let us decompose

\[
\tilde{v} = \sum_{k=1}^{\lfloor m/s_1 \rfloor} \tilde{v}^{(k)},
\tilde{w} = \sum_{l=1}^{\lfloor (d-m)/s_2 \rfloor} \tilde{w}^{(l)},
\]

with

\[
\text{supp}(\tilde{v}^{(k)}) \subseteq \{(k-1)s_1 + 1, \ldots, ks_1\} \quad \text{for all} \; k \in [m/s_1],
\text{supp}(\tilde{w}^{(l)}) \subseteq \{(l-1)s_2 + 1, \ldots, ls_2\} \quad \text{for all} \; l \in [(d-m)/s_2],
\]

where \( \text{supp}(\cdot) \) denotes support of a vector. Therefore,

\[
\|\tilde{v}^T \Gamma_1^T \Gamma_2 \tilde{w}\| = \left| \left( \sum_{k=1}^{\lfloor m/s_1 \rfloor} \tilde{v}^{(k)} \right)^T \Gamma_1^T \Gamma_2 \left( \sum_{l=1}^{\lfloor (d-m)/s_2 \rfloor} \tilde{w}^{(l)} \right) \right|
\leq \sum_{k=1}^{\lfloor m/s_1 \rfloor} \sum_{l=1}^{\lfloor (d-m)/s_2 \rfloor} \left| (\tilde{v}^{(k)})^T \Gamma_1^T \Gamma_2 \tilde{w}^{(l)} \right|
= \sum_{k=1}^{\lfloor m/s_1 \rfloor} \sum_{l=1}^{\lfloor (d-m)/s_2 \rfloor} \left| (\tilde{v}^{(k)})^T \Gamma_1^T \Gamma_2 \tilde{w}^{(l)} \right| \cdot \|\tilde{v}^{(k)}\| \cdot \|\tilde{w}^{(l)}\|
\leq \sup_{w \in D_{d-m}^2} v^T \Gamma_1^T \Gamma_2 w \cdot \sum_{k=1}^{\lfloor m/s_1 \rfloor} \|\tilde{v}^{(k)}\| \cdot \sum_{l=1}^{\lfloor (d-m)/s_2 \rfloor} \|\tilde{w}^{(l)}\|
\leq \sqrt{\frac{m}{s_1}} \cdot \sqrt{\frac{d-m}{s_2}} \cdot \sup_{w \in D_{d-m}^2} v^T \Gamma_1^T \Gamma_2 w.
\]

Here we used

\[
\sum_{k=1}^{\lfloor m/s_1 \rfloor} \|\tilde{v}^{(k)}\| \leq \sqrt{\frac{m}{s_1}} \sum_{k=1}^{\lfloor m/s_1 \rfloor} \|\tilde{v}^{(k)}\|^2 = \sqrt{\frac{m^2}{s_1}} \|\tilde{v}\|^2 \leq \sqrt{\frac{m}{s_1}},
\]
since \( \{\tilde{v}^{(k)}\}_{k=1}^{\lfloor m/s_1 \rfloor} \) are orthogonal, and similarly \( \sum_{l=1}^{\lfloor (d-m)/s_2 \rfloor} \|\tilde{w}^{(l)}\| \leq \sqrt{\frac{(d-m)}{s_2}} \).
Hence,

\[ \|A\| \leq \|\Gamma_1^T A \Gamma_1\| + \|\Gamma_2^T A \Gamma_2\| + 2|\tilde{v}^T \Gamma_1^T A \Gamma_2 \tilde{w}| \]
\[ \leq \|\Gamma_1^T A \Gamma_1\| + \|\Gamma_2^T A \Gamma_2\| + 2 \sqrt{\frac{m}{s_1} \cdot \frac{d - m}{s_2}} \cdot \sup_{u \in D_{c_1}^n} v^T \Gamma_1^T A \Gamma_2 w \]
\[ \leq 2\sqrt{\frac{m}{s_1} \cdot \frac{d - m}{s_2}} \cdot \|A\|_{(P, F_{s_1, s_2})}. \]

**Proof of Lemma 8.1.** Fix \( s, t \in [q] \). Expanding squared Frobenius norm over the basis of eigenvectors \( \{u_j\}_{j=1}^d \), we have

\[ \|P_s(\hat{\Sigma} - \Sigma)P_t\|_F^2 = \sum_{j,k=1}^d \left( u_j^T P_s(\hat{\Sigma} - \Sigma)P_t u_k \right)^2 = \sum_{j \in I_s} \sum_{k \in I_t} \left( u_j^T (\hat{\Sigma} - \Sigma) u_k \right)^2 \]
\[ \leq m_s m_t \max_{j \in I_s} \max_{k \in I_t} \left( u_j^T (\Sigma - \hat{\Sigma}) u_k \right)^2 = m_s m_t \max_{j \in I_s} \max_{k \in I_t} \left( u_j^T (\Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} - I_d) u_k \right)^2. \]

Hence,

\[ \max_{s, t \in [q]} \frac{\|P_s(\hat{\Sigma} - \Sigma)P_t\|_F}{\sqrt{m_s m_t m_u m_t}} \leq \max_{s, t \in [q]} \max_{j \in I_s} \max_{k \in I_t} \left| u_j^T (\Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} - I_d) u_k \right| \]
\[ = \max_{j, k \in [d]} \left| u_j^T (\Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} - I_d) u_k \right| = \max_{j, k \in [d]} \left| U^T \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} U - I_d \right|_{j,k} \]
\[ = \|U^T \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} U - I_d\|_{\text{max}}, \]

which is maximum absolute elementwise norm of the difference between sample and true covariance matrices of random vectors \( \{U^T \Sigma^{-1/2} X_i\}_{i=1}^n \), where columns of \( U \) are eigenvectors \( \{u_j\}_{j=1}^d \). This fits the framework of Theorem A.2. The joint Orlicz norm of these vectors is

\[ \|U^T \Sigma^{-1/2} X_i\|_{J, \phi_\beta} = \|\Sigma^{-1/2} X_i\|_{J, \phi_\beta} \leq c < \infty, \quad i \in [n], \]

due to Assumption 4.2. Therefore, Theorem A.2 applied with \( U^T \Sigma^{-1/2} X_i \) instead of \( X_i \), \( d \) instead of \( p \), \( K_{n,p} = c \) and \( z = \log(3n) \) implies

\[ \|U^T \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2} U - I_d\|_{\text{max}} \leq C_{\beta} c^2 \left( \sqrt{\frac{\log(3n) + 2 \log(d)}{n}} + \frac{(\log(2n))^{1/\beta} (\log(3n) + 2 \log(d))^{2/\beta}}{n} \right) \]

with probability \( 1 - 1/n \).

\[ \square \]
Proof of Lemma 8.2. Theorem A.4 is stated for infinite-dimensional Hilbert space $\mathcal{H}$, so we can take $\mathcal{H}$ to be some space, in which $\mathbb{R}^d$ is embedded. Consider covariance operator $\Sigma_{\mathcal{H}}$ that acts on an element of $\mathcal{H}$ is the same way as $\Sigma$ acts on the first $d$ components of this element. Similarly, $\Sigma_{\mathcal{H}}$ is a counterpart of $\Sigma$. Operator $\Sigma_{\mathcal{H}}$ has $q+1$ distinct eigenvalues: the first $q$ are all the eigenvalues of $\Sigma$, specifically $\mu_1, \ldots, \mu_q$, and the last one is $\mu_{q+1} = 0$. The corresponding projectors for the first $q$ eigenvalues coincide with $P_1, \ldots, P_q$ and the last projector is $P_{q+1} = I - \sum_{r \in [q]} P_r$ (here $I$ is identity operator in $\mathcal{H}$).

Now we apply Theorem A.4 for every $r \in J$ with $r_0 = q + 1$. Let us verify the conditions. Note that $\mu_{r_0} = 0 \leq \mu_r/2$. The first inequality of Condition (A.1) is satisfied automatically by the specific choice of $x$. A bit tricky things are happening to the second and third inequalities of Condition (A.1). Observe that

$$\|P_s(\Sigma_{\mathcal{H}} - \Sigma_{\mathcal{H}})P_{\geq r_0}\|_{\mathcal{F}} = 0, \quad \|P_{\geq r_0}(\Sigma_{\mathcal{H}} - \Sigma_{\mathcal{H}})P_{\geq r_0}\|_{\mathcal{F}} = 0, \quad \text{Tr}_{\geq r_0}(\Sigma_{\mathcal{H}}) = 0,$$

so the second and third inequalities of Condition (A.1) become $0/0 \leq x$, which doesn’t allow us to apply this result rigorously. However, from the analysis of the proof of Theorem 4 of Jirak and Wahl (2018) it is clear that these inequalities can be replaced by

$$\|P_s(\Sigma_{\mathcal{H}} - \Sigma_{\mathcal{H}})P_{\geq r_0}\|_{\mathcal{F}} \leq x \cdot \sqrt{m_s \mu_s \text{Tr}_{\geq r_0}(\Sigma_{\mathcal{H}})},$$
$$\|P_{\geq r_0}(\Sigma_{\mathcal{H}} - \Sigma_{\mathcal{H}})P_{\geq r_0}\|_{\mathcal{F}} \leq x \cdot \text{Tr}_{\geq r_0}(\Sigma_{\mathcal{H}}),$$

and all the derivation stays true (division by $\text{Tr}_{\geq r_0}(\Sigma_{\mathcal{H}})$ actually never appears in the proof). In our situation, these inequalities reduce to $0 \leq x \cdot 0$, which holds true. Finally, Condition (A.2) is fulfilled due to Condition (8.1).

Thus, we obtain the following: for all $r \in J$

$$\|\tilde{P}_r - P_r - R_r(\tilde{\Sigma} - \Sigma)P_r - P_r(\tilde{\Sigma} - \Sigma)R_r\|_{\mathcal{F}} \leq C x^2 r_r(\Sigma) \sqrt{\sum_{s \neq r} m_r \mu_r m_s \mu_s \frac{r_r(\Sigma)}{(\mu_r - \mu_s)^2}},$$

which, by triangle inequality leads to

$$\left\|\sum_{r \in J} \tilde{P}_r - \sum_{r \in J} P_r - \sum_{r \in J} R_r(\tilde{\Sigma} - \Sigma)P_r + P_r(\tilde{\Sigma} - \Sigma)R_r\right\|_{\mathcal{F}} \leq$$
$$\leq C x^2 \sum_{r \in J} \left(r_r(\Sigma) \sqrt{\sum_{s \neq r} m_r \mu_r m_s \mu_s \frac{r_r(\Sigma)}{(\mu_r - \mu_s)^2}}\right).$$

The only thing left to note is

$$\sum_{r \in J} \left(R_r(\tilde{\Sigma} - \Sigma)P_r + P_r(\tilde{\Sigma} - \Sigma)R_r\right) = \sum_{r \in J} \sum_{s \notin J} \frac{P_r(\tilde{\Sigma} - \Sigma)P_s + P_s(\tilde{\Sigma} - \Sigma)P_r}{\mu_r - \mu_s},$$

which can be seen from inserting the resolvents and observing that the terms of the type $P_r(\tilde{\Sigma} - \Sigma)P_{r'}$, $r, r' \in J$, $r \neq r'$ cancel out since they appear exactly twice in the sum with different signs. \qed
Proof of Lemma 8.3. Denote
\[
x = \max_{s,t \in [i]} \frac{\| P_s(\hat{\Sigma} - \Sigma) P_t \|_F}{\sqrt{m_s \mu_s m_t \mu_t}}, \quad \tilde{x} = \max_{s,t \in [i]} \frac{\| P_s(\hat{\Sigma} - \Sigma) P_t \|_F}{\sqrt{m_s \mu_s m_t \mu_t}}.
\]
Let \( \Omega \) be the event on which \( x \leq \psi_n \) and \( \tilde{\Omega} = \tilde{\Omega}(X) \) be the event on which \( (\tilde{x} \leq \tilde{\psi}_n | X) \). By Lemma 8.1, \( P[\Omega] \geq 1 - 1/n \). By Condition (8.4) \( P[\tilde{\Omega} | X] \geq 1 - 1/n \) on some event \( \Omega' \) with \( P[\Omega'] \geq 1 - 1/n \). By union bound, \( P[\Omega \cap \Omega'] \geq 1 - 2/n \).

As in Lemma 8.2, we decompose
\[
\tilde{P}_J - \tilde{P}_J = (\tilde{P}_J - P_J) - (\tilde{P}_J - P_J)
\]
\[
= L_J(\hat{\Sigma} - \Sigma) + R_J(\hat{\Sigma} - \Sigma) - L_J(\hat{\Sigma} - \Sigma) - R_J(\hat{\Sigma} - \Sigma)
\]
\[
= L_J(\hat{\Sigma} - \Sigma) + R_J(\hat{\Sigma} - \Sigma) - R_J(\hat{\Sigma} - \Sigma).
\]
Then, by Proposition 3.1 (i), (ii)
\[
\left\| \tilde{P}_J - P_J \right\|_{(P_J, \Gamma^s, \psi_1, \psi_2)} - \left\| L_J(\hat{\Sigma} - \Sigma) \right\|_{(P_J, \Gamma^s, \psi_1, \psi_2)} \leq \left\| R_J(\hat{\Sigma} - \Sigma) \right\|_{(P_J, \Gamma^s, \psi_1, \psi_2)} + \left\| R_J(\hat{\Sigma} - \Sigma) \right\|_{(P_J, \Gamma^s, \psi_1, \psi_2)} \leq 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| + 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\|.
\]
Further, on \( \Omega \cap \Omega' \) with \( \delta = 4C(\psi_n + \tilde{\psi}_n)^2 d_J(\Sigma)^{3/2} \) (with \( C \) from Lemma 8.2) we have
\[
P \left[ 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| + 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| > \delta \bigg| X \right] =
\]
\[
= P \left[ 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| + 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| > \delta \bigg| X; \tilde{x} > \tilde{\psi}_n \right] \cdot P \left[ \tilde{x} > \tilde{\psi}_n \bigg| X \right] +
\]
\[
P \left[ 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| + 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| > \delta \bigg| X; \tilde{x} \leq \tilde{\psi}_n \right] \cdot P \left[ \tilde{x} \leq \tilde{\psi}_n \bigg| X \right]
\]
\[
\leq 1 \cdot \frac{1}{n} + P \left[ 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| + 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| > \delta \bigg| X; \tilde{x} \leq \tilde{\psi}_n \right] \cdot 1.
\]
So far we have used only that we are on \( \Omega' \). Since we are also on \( \Omega \), \( x \leq \psi_n \) implies \( x \max_{r \in J} r_r(\Sigma) \leq 1/12 \), yielding that Condition (8.1) is fulfilled. Thus, by Lemma 8.2
\[
\left\| R_J(\hat{\Sigma} - \Sigma) \right\| \leq Cx^2 d_J(\Sigma)^{3/2}.
\]
Similarly, when \( \tilde{x} \leq \tilde{\psi}_n \), Condition (8.1) is satisfied for \( (x + \tilde{x}) \), and Lemma 8.2 claims
\[
\left\| R_J(\hat{\Sigma} - \Sigma) \right\| \leq C(x + \tilde{x})^2 d_J(\Sigma)^{3/2}.
\]
Therefore, on \( \Omega \)
\[
P \left[ 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| + 2 \left\| R_J(\hat{\Sigma} - \Sigma) \right\| > \delta \bigg| X; \tilde{x} \leq \tilde{\psi}_n \right] = 0,
\]
yielding the desired. \( \square \)
Proof of Lemma 8.4.

(i) The first two terms coincide with the original definition, so we have to make sure that the third one coincides as well. From the definitions of $D_{s1}^m$ and $D_{s2}^{d-m}$, we have

$$\sup_{v \in D_{s1}^m} \sup_{w \in D_{s2}^{d-m}} v^\top \Gamma_1^\top A \Gamma_2 w = \max_{k \in \{0, \ldots, m-s_1\}} \sup_{l \in \{0, \ldots, d-m-s_2\}} \left[ 0^\top_k, v^\top, 0^\top_{m-k-s_1} \right] \Gamma_1^\top A \Gamma_2 \left[ 0_l \ w \right]_{0_{d-m-l-s_2}}$$

$$= \max_{k \in \{0, \ldots, m-s_1\}} \sup_{v \in S_{s1}^m-1} \sup_{w \in S_{s2}^{d-m-1}} v^\top \left[ \Gamma_1^\top A \Gamma_2 \right]_{[k+1:k+s_1],[l+1:l+s_2]} w$$

$$= \max_{k \in \{0, \ldots, m-s_1\}} \max_{l \in \{0, \ldots, d-m-s_2\}} \left\| \left[ \Gamma_1^\top A \Gamma_2 \right]_{[(k+1):(k+s_1)],[(l+1):(l+s_2)]} \right\|,$$

as desired.

(ii) Note that $P \Gamma_1 = \Gamma_1$, $P \Gamma_2 = O_{d \times (d-m)}$, $(I_d - P) \Gamma_1 = O_{d \times m}$, $(I_d - P) \Gamma_2 = \Gamma_2$. Hence, plugging $A = PA(I_d - P) + (I_d - P)AP$ into the definition, we notice that the first two terms $\|\Gamma_1^\top A \Gamma_1\|/2$ and $\|\Gamma_2^\top A \Gamma_2\|/2$ disappear and $\|A\|_{(P,\Gamma_1,\Gamma_2)}$ is expressed by the third term only.

Let us take $s_1 = m$ and $s_2 = d - m$. We can represent spectral norm as

$$\|A\| = \sup_{u \in S^{d-1}} |u^\top A u| = \sup_{u \in S^{d-1}} |u^\top [PA(I_d - P) + (I_d - P)AP] u|$$

$$= 2 \sup_{u \in S^{d-1}} |u^\top PA(I_d - P) u|.$$  \hspace{1cm} (B.1)

Indeed, for any $u \in S^{d-1}$ we can take

$$v = \pm \Gamma_1^\top P u / \|Pu\| \quad \text{and} \quad w = \pm \Gamma_2^\top (I_d - P) u / \|(I_d - P) u\|$$

(it is straightforward to check $v \in S^{m-1}$ and $w \in S^{d-m-1}$) and obtain

$$2 |u^\top PA(I_d - P) u| = 2 |v^\top \Gamma_1^\top A \Gamma_2 w| \cdot \|Pu\| \cdot \|(I_d - P) u\| \leq |v^\top \Gamma_1^\top A \Gamma_2 w|.$$  

Conversely, for any $v \in S^{m-1}$ and $w \in S^{d-m-1}$ we can take $u = (\Gamma_1^\top v + \Gamma_2^\top w) / \sqrt{2}$ (again, easy to see that $u \in S^{d-1}$) and obtain

$$2 u^\top PA(I_d - P) u = v^\top \Gamma_1^\top A \Gamma_2 w.$$  

This proves (B.1), and, consequently,

$$\sup_{v \in D_m^m} \sup_{w \in D_{d-m}^{d-m}} v^\top \Gamma_1^\top A \Gamma_2 w = \|A\|.$$  

If we take $s_1 = 1$ and $s_2 = 1$, then

$$\sup_{v \in D_m^m} \sup_{w \in D_{d-m}^{d-m}} v^\top \Gamma_1^\top A \Gamma_2 w = \max_{j,k \in [d]} |e_j^\top \Gamma_1^\top A \Gamma_2 e_k| = \max_{j,k \in [d]} |[\Gamma_1^\top A \Gamma_2]_{j,k}| = \|\Gamma_1^\top A \Gamma_2\|_{\max},$$

where $\{e_j\}_{j=1}^d$ are standard basis vectors in $\mathbb{R}^d$. \qed
Proof of Lemma 8.5. For any $v \in D^m_{s_1}$ denote the closest to $v$ vector of $N_\varepsilon(D^m_{s_1})$ as $\pi(v)$, that is, $\|v - \pi(v)\| \leq \varepsilon$. Similarly, for any $w \in D^d_{s_2}$ denote the closest to $w$ vector of $N_\varepsilon(D^d_{s_2})$ as $\rho(w)$, that is, $\|w - \rho(w)\| \leq \varepsilon$. The construction (8.6) allows without loss of generality assume $(v - \pi(v)) \in D^m_{s_1}$ and $(w - \rho(w)) \in D^d_{s_2}$.

By Lemma 8.4 (ii),
\[
\|A\|_{(p, \Gamma, s_1, s_2)} = \sup_{v \in D^m_{s_1}} \sup_{w \in D^d_{s_2}} v^\top \Gamma_1^\top A \Gamma_2 w.
\]

We have the following standard chain of equalities and inequalities:
\[
\begin{align*}
\|A\|_{(p, \Gamma, s_1, s_2)} &= \sup_{v \in D^m_{s_1}} \sup_{w \in D^d_{s_2}} v^\top \Gamma_1^\top A \Gamma_2 w \\
&= \sup_{v \in D^m_{s_1}} \sup_{w \in D^d_{s_2}} \left[ v^\top \Gamma_1^\top A \Gamma_2 w - \pi(v)^\top \Gamma_1^\top A \Gamma_2 \rho(w) + \pi(v)^\top \Gamma_1^\top A \Gamma_2 \rho(w) \right] \\
&\leq \sup_{v \in D^m_{s_1}} \sup_{w \in D^d_{s_2}} \left[ v^\top \Gamma_1^\top A \Gamma_2 w - \pi(v)^\top \Gamma_1^\top A \Gamma_2 \rho(w) \right] + \sup_{v \in D^m_{s_1}} \sup_{w \in D^d_{s_2}} \pi(v)^\top \Gamma_1^\top A \Gamma_2 \rho(w) \\
&= \sup_{v \in D^m_{s_1}} \sup_{w \in D^d_{s_2}} \left\{ (v - \pi(v))^\top \Gamma_1^\top A \Gamma_2 w + \pi(v)^\top \Gamma_1^\top A \Gamma_2 (w - \rho(w)) \right\} + \\
&\quad + \max_{(v, w) \in N_\varepsilon(D^m_{s_1}) \times N_\varepsilon(D^d_{s_2})} v^\top \Gamma_1^\top A \Gamma_2 w \\
&\leq \varepsilon \cdot \sup_{v \in D^m_{s_1}} \sup_{w \in D^d_{s_2}} \left\{ (v - \pi(v))^\top \|v - \pi(v)\|_2 \Gamma_1^\top A \Gamma_2 w + \pi(v)^\top \Gamma_1^\top A \Gamma_2 \frac{(w - \rho(w))}{\|w - \rho(w)\|_2} \right\} + \max_{j \in [p]} v^j_\top A w_j \\
&\leq 2\varepsilon \cdot \sup_{v \in D^m_{s_1}} \sup_{w \in D^d_{s_2}} v^\top \Gamma_1^\top A \Gamma_2 w + \max_{j \in [p]} v^j_\top A w_j = 2\varepsilon \cdot \|A\|_{(p, \Gamma, s_1, s_2)} + \max_{j \in [p]} v^j_\top A w_j.
\end{align*}
\]

Therefore, we obtain
\[
\max_{j \in [p]} v^j_\top A w_j \leq \|A\|_{(p, \Gamma, s_1, s_2)} \leq \frac{1}{1 - 2\varepsilon} \max_{j \in [p]} v^j_\top A w_j.
\]

\[\square\]

Proof of Lemma 8.6. It is a well-known fact that
\[
N_\varepsilon(S^{s_1 - 1}) \leq \left(\frac{3}{\varepsilon}\right)^{s_1}, \quad N_\varepsilon(S^{s_2 - 1}) \leq \left(\frac{3}{\varepsilon}\right)^{s_2},
\]
e.g. see Lemma 5.13 of van Handel (2018). From the construction (8.6) follows
\[
N_\varepsilon(D^m_{s_1}) \leq (m - s_1 + 1) \cdot N_\varepsilon(S^{s_1 - 1}) \leq (m - s_1 + 1) \cdot \left(\frac{3}{\varepsilon}\right)^{s_1},
\]
\[
N_\varepsilon(D^d_{s_2}) \leq (d - m - s_2 + 1) \cdot N_\varepsilon(S^{s_2 - 1}) \leq (d - m - s_2 + 1) \cdot \left(\frac{3}{\varepsilon}\right)^{s_2}.
\]
Taking logarithm of $p(\varepsilon, d, m, s_1, s_2) = |N_\varepsilon(D^m_{s_1})| \cdot |N_\varepsilon(D^d_{s_2})|$, we get the desired bound.

\[\square\]

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Proof of Lemma 8.10. We start with following the proof of Lemma 8.1. Fix \( s, t \in [q] \). Expanding squared Frobenius norm over the basis of eigenvectors \( \{u_j\}_{j=1}^d \) and using the definition of \( \Sigma^B \), we have

\[
\|P_s(\Sigma^B - \hat{\Sigma})P_t\|_F^2 = \sum_{j,k=1}^d \left( u_j^T P_s(\Sigma^B - \hat{\Sigma})P_t u_k \right)^2 = \sum_{j \in I_s} \sum_{k \in I_t} \left( u_j^T (\Sigma^B - \hat{\Sigma})u_k \right)^2 \\
\leq m_s m_t \max_{j \in I_s} \left( u_j^T (\Sigma^B - \hat{\Sigma})u_k \right)^2 = m_s m_t \max_{j \in I_s} \left( u_j^T \frac{1}{n} \sum_{i=1}^n (\eta_i - 1)X_iX_i^T u_k \right)^2 \\
= m_s m_t m \max_{j \in I_s} \left( u_j^T \frac{1}{n} \sum_{i=1}^n (\eta_i - 1)(\Sigma^{-1/2}X_i)(\Sigma^{-1/2}X_i)^T u_k \right)^2.
\]

Hence,

\[
\max_{s,t \in [q]} \frac{\|P_s(\Sigma^B - \hat{\Sigma})P_t\|_F^2}{\sqrt{m_s m_t m}} \leq \max_{s,t \in [q]} \left| u_j^T \frac{1}{n} \sum_{i=1}^n (\eta_i - 1)(\Sigma^{-1/2}X_i)(\Sigma^{-1/2}X_i)^T u_k \right| \\
= \max_{j,k \in [d]} \left| \frac{1}{n} \sum_{i=1}^n (\eta_i - 1)(U^T \Sigma^{-1/2}X_i)(U^T \Sigma^{-1/2}X_i)^T \right|_{j,k} \\
= \max_{j,k \in [d]} \left| \frac{1}{n} \sum_{i=1}^n (\eta_i - 1)(U^T \Sigma^{-1/2}X_i)_j(U^T \Sigma^{-1/2}X_i)_k \right|.
\]

For arbitrary \( j, k \in [d] \), since \( \eta_i \overset{i.i.d.}{\sim} \mathcal{N}(1, 1) \), conditionally on \( \mathbf{X} \) we have

\[
\frac{1}{n} \sum_{i=1}^n (\eta_i - 1)(U^T \Sigma^{-1/2}X_i)_j(U^T \Sigma^{-1/2}X_i)_k \sim \mathcal{N} \left( 0, \frac{1}{n^2} \sum_{i=1}^n (U^T \Sigma^{-1/2}X_i)_j^2(U^T \Sigma^{-1/2}X_i)_k^2 \right).
\]

Consider the event \( \Omega \) defined as

\[
\left\{ \max_{j,k \in [d]} \frac{1}{n} \sum_{i=1}^n (U^T \Sigma^{-1/2}X_i)_j^2(U^T \Sigma^{-1/2}X_i)_k^2 \leq \sigma^2 \right\}
\]

with \( \sigma \overset{\text{def}}{=} \sqrt{\left( \log(n) + \log(2d^2) \right)^{2/3}} \).

Let us verify that \( P(\Omega) \geq 1 - 1/n \). By Proposition A.1 and Assumption 4.2,

\[
\left\| \frac{1}{n} \sum_{i=1}^n (U^T \Sigma^{-1/2}X_i)_j^2(U^T \Sigma^{-1/2}X_i)_k^2 \right\|_{\psi/4} \leq \left\| (U^T \Sigma^{-1/2}X_1)_j^2(U^T \Sigma^{-1/2}X_1)_k^2 \right\|_{\psi/4} \\
\leq \left\| (U^T \Sigma^{-1/2}X_1)_j \right\|_{\psi/2} \left\| (U^T \Sigma^{-1/2}X_1)_k \right\|_{\psi/2} \leq C^4 < \infty,
\]

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we have

\[ \frac{1}{n} \sum_{i=1}^{n} (U^\top \Sigma^{-1/2} X_i)^2 (U^\top \Sigma^{-1/2} X_i) \geq \sigma^2 \]

yielding

\[ \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} (U^\top \Sigma^{-1/2} X_i)^2 (U^\top \Sigma^{-1/2} X_i) \geq \sigma^2 \right) \leq 2 \exp \left( -\left( \sigma / c^2 \right)^{3/2} \right), \]

By union bound,

\[ \mathbb{P} \left( \max_{j,k \in [d]} \frac{1}{n} \sum_{i=1}^{n} (U^\top \Sigma^{-1/2} X_i)^2 (U^\top \Sigma^{-1/2} X_i) \geq \sigma^2 \right) \leq 2d^2 \exp \left( -\left( \sigma / c^2 \right)^{3/2} \right), \]

which, plugging \( \sigma \) in and using definition of \( \Omega \) can be rewritten as \( \mathbb{P}(\Omega^c) \leq 1/n. \)

Further, on \( \Omega \) we have for all \( j, k \in [d] \) Gaussian tail inequality

\[ \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} (\eta_i - 1)(U^\top \Sigma^{-1/2} X_i)^2 (U^\top \Sigma^{-1/2} X_i) \right| \geq z \right) \leq 2e^{-nz^2/\sigma^2}, \]

thus, by union bound

\[ \mathbb{P} \left( \max_{j,k \in [d]} \left| \frac{1}{n} \sum_{i=1}^{n} (\eta_i - 1)(U^\top \Sigma^{-1/2} X_i)^2 (U^\top \Sigma^{-1/2} X_i) \right| \geq z \right) \leq 2d^2 e^{-nz^2/\sigma^2}. \]

We conclude the proof by taking \( z = \sqrt{\frac{\sigma^2 \log(n) + \log(2d^2)}{n}}. \)

**Proof of Lemma 8.11.** Fix arbitrary \( i \in [n], j \in [p] \). Let us bound \( \|x_{ij}\|_{\psi_{1/2}} \). From (8.8)

\[ |x_{ij}| \leq \kappa \cdot \overline{v}_j \Sigma^{-1/2} X_i \cdot \overline{w}_j \Sigma^{-1/2} X_i, \]

where \( \overline{v}_j, \overline{w}_j \in S^{d-1} \). Hence,

\[ \|x_{ij}\|_{\psi_{1/2}} \leq \kappa \|\overline{v}_j \Sigma^{-1/2} X_i\|_{\psi_{1/2}} \|\overline{w}_j \Sigma^{-1/2} X_i\|_{\psi_{1/2}} \leq \kappa c^2, \]

where we used Proposition A.1 and Assumption 4.2. Now the claim follows from Theorem A.2 with \( X_i = x_i, \beta/2 \) taken as \( \beta, K_{n,p} = \kappa c^2 \) and \( z = \log(3n) \).

**Proof of Lemma 8.12.** The idea is similar to the proof of Lemma 8.10. Fix \( s, t \in [q] \). Expanding squared Frobenius norm over the basis of eigenvectors \( \{u_j\}_{j=1}^d \) and using the definition of \( \Sigma^F \), we have

\[ \|P_s(\Sigma^F - \hat{\Sigma})P_t\|_F^2 = \sum_{j,k=1}^{d} \left( u_j^\top P_s(\Sigma^F - \hat{\Sigma})P_t u_k \right)^2 = \sum_{j \in I_s, k \in I_t} \left( u_j^\top (\Sigma^F - \hat{\Sigma}) u_k \right)^2 \]

\[ \leq m_s m_t \max_{j \in I_s, k \in I_t} \left( u_j^\top (\Sigma^F - \hat{\Sigma}) u_k \right)^2 = m_s m_t \max_{j \in I_s} \left( u_j^\top \Sigma^{-1/2} (\Sigma^F - \hat{\Sigma}) \Sigma^{-1/2} u_k \right)^2. \]

Hence,

\[ \max_{s,t \in [q]} \|P_s(\Sigma^F - \hat{\Sigma})P_t\|_F \leq \max_{s,t \in [q]} \max_{j \in I_s, k \in I_t} \left| u_j^\top \Sigma^{-1/2} (\Sigma^F - \hat{\Sigma}) \Sigma^{-1/2} u_k \right| \]

\[ = \max_{j \in [d]} \left| u_j^\top \Sigma^{-1/2} (\Sigma^F - \hat{\Sigma}) \Sigma^{-1/2} u_k \right|. \]
For arbitrary \( j, k \in [d] \), by the definition of \( \Sigma^F \), we have
\[
\left. u_j^\top \Sigma^{-1/2}(\Sigma^F - \hat{\Sigma})\Sigma^{-1/2}u_k = \right. \\
= \frac{1}{n} \sum_{i=1}^{n} \left. \left\{ (u_j^\top \Sigma^{-1/2}Z_i)(u_k^\top \Sigma^{-1/2}Z_i) - E \left[ (u_j^\top \Sigma^{-1/2}Z_i)(u_k^\top \Sigma^{-1/2}Z_i) \right] \right\} \right,
\]
where, conditionally on \( \mathbf{X} \),
\[
u_j^\top \Sigma^{-1/2}Z_i \sim \mathcal{N}(0, u_j^\top \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2}u_j), \\
u_k^\top \Sigma^{-1/2}Z_i \sim \mathcal{N}(0, u_k^\top \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2}u_k),
\]
since \( Z_i \overset{i.i.d.}{\sim} \mathcal{N}_d(0, \hat{\Sigma}) \) given \( \mathbf{X} \). Consider the event \( \Omega \) defined as
\[
\left\{ \max_{j \in [d]} u_j^\top \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2}u_j \leq \sigma^2 \right\}
\]
with \( \sigma \overset{\text{def}}{=} c \left((\log(n) + \log(2d))\right)^{1/\beta} \).

Let us verify that \( P(\Omega) \geq 1 - 1/n \). Fix \( j \in [d] \). By Proposition A.1 and Assumption 4.2,
\[
\left\| u_j^\top \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2}u_j \right\|_{\psi_{\beta/2}} = \left\| \frac{1}{n} \sum_{i=1}^{n} (u_j^\top \Sigma^{-1/2}X_i)^2 \right\|_{\psi_{\beta/2}} \leq \left\| (u_j^\top \Sigma^{-1/2}X_i)^2 \right\|_{\psi_{\beta/2}} \leq \left\| u_j^\top \Sigma^{-1/2}X_i \right\|_{\psi_{\beta}}^{2} \leq c^2 < \infty,
\]
yielding
\[
P \left( u_j^\top \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2}u_j \geq \sigma^2 \right) \leq 2 \exp \left( -\frac{(\sigma/c)^2}{\beta} \right).
\]
By union bound,
\[
P \left( \max_{j \in [d]} u_j^\top \Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2}u_j \geq \sigma^2 \right) \leq 2d \exp \left( -\frac{(\sigma/c)^2}{\beta} \right),
\]
which, plugging in \( \sigma \) and using definition of \( \Omega \), can be rewritten as \( P(\Omega^c) \leq 1/n \).

Further, on \( \Omega \) for arbitrary \( j \in [d] \), \( (u_j^\top \Sigma^{-1/2}Z_i) \) is \( \sigma^2 \)-subgaussian, and \( \left\| u_j^\top \Sigma^{-1/2}Z_i \right\|_{\psi_{2}} \leq C\sigma \). Hence, for arbitrary \( j, k \in [d] \)
\[
\left\| (u_j^\top \Sigma^{-1/2}Z_i)(u_k^\top \Sigma^{-1/2}Z_i) \right\|_{\psi_{1}} \leq \left\| u_j^\top \Sigma^{-1/2}Z_i \right\|_{\psi_{2}} \left\| u_k^\top \Sigma^{-1/2}Z_i \right\|_{\psi_{2}} \leq C\sigma^2
\]
due to Proposition A.1. So, \( (u_j^\top \Sigma^{-1/2}Z_i)(u_k^\top \Sigma^{-1/2}Z_i) \) is subexponential and by, for instance, Exercise 2.7.10 of Vershynin (2018) the centered version is also subexponential (but with different multiplicative constant factor in the Orlicz norm):
\[
\left\| (u_j^\top \Sigma^{-1/2}Z_i)(u_k^\top \Sigma^{-1/2}Z_i) - E \left[ (u_j^\top \Sigma^{-1/2}Z_i)(u_k^\top \Sigma^{-1/2}Z_i) \right] \right\|_{\psi_{1}} \leq C\sigma^2.
\]
Further, by Bernstein inequality (e.g. Corollary 2.8.3 of Vershynin (2018)) on \( \Omega \)
\[
P \left( \left\| \frac{1}{n} \sum_{i=1}^{n} \left\{ (u_j^\top \Sigma^{-1/2}Z_i)(u_k^\top \Sigma^{-1/2}Z_i) - E \left[ (u_j^\top \Sigma^{-1/2}Z_i)(u_k^\top \Sigma^{-1/2}Z_i) \right] \right\} \right| \geq z \right| \mathbf{X} \right) \leq \\
\leq 2 \exp \left( -Cn \left\{ \frac{z^2}{\sigma^4} \lor \frac{z}{\sigma^2} \right\} \right).
\]
and by union bound
\[
P \left( \max_{j,k \in [d]} \left| u_j^T \Sigma^{-1/2} (\Sigma F - \hat{\Sigma}) \Sigma^{-1/2} u_k \right| \geq z \right) \leq 2d^2 \exp \left( -Cn \left\{ \frac{z^2}{\sigma^4} \wedge \frac{z}{\sigma^2} \right\} \right).
\]

We conclude the proof by taking \( z = C\sigma^2 \sqrt{\frac{\log(n) + \log(2d^2)}{n}} \) (assuming \( \log(d)/n < 1 \)). \( \square \)

**Proof of Lemma 8.13.** Fix \( j, k \in [p] \). We have to bound
\[
\left[ \text{Cov}(Y^F | \mathbf{X}) - \text{Cov}(Y) \right]_{j,k} =
\sum_{r \in J, s \notin J} \sum_{r' \in J, s' \notin J} \left( v_j^T P_r \hat{\Sigma} P_{s'} v_k \right) \cdot \left( w_j^T P_s \hat{\Sigma} P_{s'} w_k \right) - \left( v_j^T P_r \hat{\Sigma} P_{s'} w_k \right) \cdot \left( w_j^T P_s \hat{\Sigma} P_{s'} w_k \right)
\]
\[
- \sum_{r \in J, s \notin J} \sum_{r' \in J, s' \notin J} \left( v_j^T P_r \Sigma P_{s'} v_k \right) \cdot \left( w_j^T P_s \Sigma P_{s'} w_k \right) - \left( v_j^T P_r \Sigma P_{s'} w_k \right) \cdot \left( w_j^T P_s \Sigma P_{s'} w_k \right)
\]

To simplify the expression, define auxiliary matrices
\[
B_j = \sum_{r \in J, s \notin J} P_r v_j w_j^T P_s \mu_r - \mu_s \in \mathbb{R}^{d \times d},
\]
\[
B_k = \sum_{r \in J, s \notin J} P_r v_k w_k^T P_s \mu_r - \mu_s \in \mathbb{R}^{d \times d}.
\]

Using cyclic property of the trace, we obtain
\[
\sum_{r \in J, s \notin J} \sum_{r' \in J, s' \notin J} \left( v_j^T P_r \hat{\Sigma} P_{s'} v_k \right) \cdot \left( w_j^T P_s \hat{\Sigma} P_{s'} w_k \right) \frac{1}{(\mu_r - \mu_s)(\mu_r' - \mu_s')}
\]
\[
= \text{Tr} \left[ \sum_{r \in J, s \notin J} \sum_{r' \in J, s' \notin J} \left( v_j^T P_r \hat{\Sigma} P_{s'} v_k \right) \cdot \left( w_j^T P_s \hat{\Sigma} P_{s'} w_k \right) \frac{1}{(\mu_r - \mu_s)(\mu_r' - \mu_s')} \right]
\]
\[
= \text{Tr} \left[ \hat{\Sigma} \left( \sum_{r' \in J, s' \notin J} P_{s'} w_k^T P_{s'} \mu_r' - \mu_s' \right) \hat{\Sigma} \left( \sum_{r' \in J, s' \notin J} P_{s'} w_j^T P_{s'} \mu_r - \mu_s \right) \right]
\]
\[
= \text{Tr} \left[ \hat{\Sigma} B_k \hat{\Sigma} B_j^T \right].
\]

Similarly,
\[
\sum_{r \in J, s \notin J} \sum_{r' \in J, s' \notin J} \left( v_j^T P_r \Sigma P_{s'} v_k \right) \cdot \left( w_j^T P_s \Sigma P_{s'} w_k \right) \frac{1}{(\mu_r - \mu_s)(\mu_r' - \mu_s')}
\]
\[
= \text{Tr} \left[ \Sigma B_k \Sigma B_j^T \right].
\]
In a slightly different fashion, we derive
\[
\sum_{r \in J} \sum_{s \in J} \frac{(v^T_j P_r \hat{\Sigma} P_s w_j) \cdot (v^T_k P_r \hat{\Sigma} P_s w_k)}{(\mu_r - \mu_s)(\mu_{r'} - \mu_{s'})} = \sum_{r \in J} \sum_{s \not\in J} \frac{v^T_j P_r \hat{\Sigma} P_s w_j}{\mu_r - \mu_s} \cdot \sum_{r' \in J} \sum_{s' \not\in J} \frac{v^T_k P_r \hat{\Sigma} P_s w_k}{\mu_{r'} - \mu_{s'}}
\]
\[
= \text{Tr} \left[ \sum_{r \in J} \sum_{s \in J} \frac{v^T_j P_r \hat{\Sigma} P_s w_j}{\mu_r - \mu_s} \cdot \text{Tr} \left[ \sum_{r' \in J} \sum_{s' \not\in J} \frac{v^T_k P_r \hat{\Sigma} P_s w_k}{\mu_{r'} - \mu_{s'}} \right] \right]
\]
\[
= \text{Tr} \left[ \hat{\Sigma} \left( \sum_{r \in J} \sum_{s \in J} \frac{P_s w_j v^T_j P_r}{\mu_r - \mu_s} \right) \right] \cdot \text{Tr} \left[ \hat{\Sigma} \left( \sum_{r' \in J} \sum_{s' \not\in J} \frac{P_{s'} w_k v^T_k P_{r'}}{\mu_{r'} - \mu_{s'}} \right) \right]
\]
\[
= \text{Tr} \left[ \hat{\Sigma} B^T_j \right] \cdot \text{Tr} \left[ \hat{\Sigma} B^T_k \right].
\]
Similarly,
\[
\sum_{r \in J} \sum_{s \not\in J} \frac{(v^T_j P_r \Sigma P_s w_j) \cdot (v^T_k P_{r'} \Sigma P_{s'} w_k)}{(\mu_r - \mu_s)(\mu_{r'} - \mu_{s'})} = \text{Tr} \left[ \Sigma B^T_j \right] \cdot \text{Tr} \left[ \Sigma B^T_k \right].
\]
Hence, our expression reduces to
\[
[Cov(Y^F | X) - Cov(Y)]_{j,k} = \text{Tr} \left[ \hat{\Sigma} B_k \hat{\Sigma} B_j^T \right] - \text{Tr} \left[ \Sigma B_k \Sigma B_j^T \right] - \text{Tr} \left[ \hat{\Sigma} B_j^T \right] \cdot \text{Tr} \left[ \hat{\Sigma} B_k^T \right] + \text{Tr} \left[ \Sigma B_j^T \right] \cdot \text{Tr} \left[ \Sigma B_k^T \right].
\]
Note that actually \( \text{Tr} [\Sigma B_j^T] = \text{Tr} [\Sigma B_k^T] = 0 \), so
\[
[Cov(Y^F | X) - Cov(Y)]_{j,k} = \text{Tr} \left[ \hat{\Sigma} B_k \hat{\Sigma} B_j^T \right] - \text{Tr} \left[ \Sigma B_k \Sigma B_j^T \right] - \text{Tr} \left[ (\hat{\Sigma} - \Sigma) B_j^T \right] \cdot \text{Tr} \left[ (\hat{\Sigma} - \Sigma) B_k^T \right].
\]
Adding and subtracting \( \text{Tr} \left[ \Sigma B_k \Sigma B_j^T \right] \), we get
\[
[Cov(Y^F | X) - Cov(Y)]_{j,k} = \text{Tr} \left[ (\hat{\Sigma} - \Sigma) B_k \hat{\Sigma} B_j^T \right] + \text{Tr} \left[ \Sigma B_k (\hat{\Sigma} - \Sigma) B_j^T \right] - \text{Tr} \left[ (\hat{\Sigma} - \Sigma) B_j^T \right] \cdot \text{Tr} \left[ (\hat{\Sigma} - \Sigma) B_k^T \right] - \text{Tr} \left[ (\hat{\Sigma} - \Sigma) B_j^T \right] \cdot \text{Tr} \left[ (\hat{\Sigma} - \Sigma) B_k^T \right].
\]
It is easy to see, that if we define
\[
\widetilde{B}_j = \sum_{r \in J} \sum_{s \not\in J} \frac{\sqrt{\mu_r \mu_s}}{\mu_r - \mu_s} P_r v_j w_j^T P_s \in \mathbb{R}^{d \times d},
\]
\[
\widetilde{B}_k = \sum_{r \in J} \sum_{s \not\in J} \frac{\sqrt{\mu_r \mu_s}}{\mu_r - \mu_s} P_r v_k w_k^T P_s \in \mathbb{R}^{d \times d},
\]

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then we can rewrite

\[
\left[ \text{Cov}(Y^F \mid X) - \text{Cov}(Y) \right]_{j,k} = \]

\[
= \text{Tr} \left[ (\Sigma^{-1/2} \Sigma^{-1/2} - I_d) \tilde{B}_k (\Sigma^{-1/2} \Sigma^{-1/2} - I_d) \tilde{B}_j^T \right] + \]

\[
+ \text{Tr} \left[ (\Sigma^{-1/2} \Sigma^{-1/2} - I_d) \tilde{B}_k \tilde{B}_j^T + \tilde{B}_j^T \tilde{B}_k \right] - \]

\[
- \text{Tr} \left[ (\Sigma^{-1/2} \Sigma^{-1/2} - I_d) \tilde{B}_j^T \right] \cdot \text{Tr} \left[ (\Sigma^{-1/2} \Sigma^{-1/2} - I_d) \tilde{B}_j^T \right] \]

\[
=: T_1 + T_2 - T_3.
\]

It suffices to bound each $|T_1|, |T_2|, |T_3|$ with probability $1 - 1/(3m^2)$ to get the bound on

\[
\left[ \text{Cov}(Y^F \mid X) - \text{Cov}(Y) \right]_{j,k}
\]

with probability $1 - 1/(np^2)$. Before we do so, let us bound some important quantities. To slightly simplify some expressions, introduce for all $r \in J$

\[
\overline{w}_{j,r} \overset{\text{def}}{=} \sum_{s \in J} \sqrt{\mu_r - \mu_s} P_s w_j,
\]

\[
\overline{w}_{k,r} \overset{\text{def}}{=} \sum_{s \in J} \sqrt{\mu_r - \mu_s} P_s w_k;
\]

so that

\[
\tilde{B}_j = \sum_{r \in J} P_r v_j \overline{w}_{j,r}^T,
\]

\[
\tilde{B}_k = \sum_{r \in J} P_r v_k \overline{w}_{k,r}^T.
\]

Note that $\|\overline{w}_{j,r}\| \leq \overline{\kappa}$. We have

\[
\|B_j\|_* = \|B_j^T\|_* = \left\| \sum_{r \in J} P_r v_j \overline{w}_{j,r}^T \right\|_* \leq \sum_{r \in J} \|P_r v_j \overline{w}_{j,r}^T\|_* = \sum_{r \in J} \|P_r v_j\| \|\overline{w}_{j,r}\|
\]

\[
\leq \overline{\kappa} \sum_{r \in J} \|P_r v_j\| \leq \sqrt{|J| \overline{\kappa}} \sqrt{\sum_{r \in J} \|P_r v_j\|^2} = \sqrt{|J| \overline{\kappa}}.
\]

Similarly, $\|B_k\|_* = \|B_k^T\|_* \leq \sqrt{|J| \overline{\kappa}}$. Moreover,

\[
\|\tilde{B}_k \tilde{B}_j^T + \tilde{B}_j^T \tilde{B}_k\|_* \leq 2 \|B_k\|_* \|B_j^T\|_* \leq 2 |J| \overline{\kappa}^2.
\]

Note also that $\text{rank}(\tilde{B}_h) \leq |J|, \text{rank}(\tilde{B}_j) \leq |J|$ and $\text{rank}(\tilde{B}_h \tilde{B}_j^T + \tilde{B}_j^T \tilde{B}_h) \leq 2 |J|$. Now we want to show, that for any matrix $D \in \mathbb{R}^{d \times d}$ of rank $h \in [d]$ it holds

\[
\left| \text{Tr} \left[ (\Sigma^{-1/2} \Sigma^{-1/2} - I_d) D \right] \right| \leq \]

\[
\leq \|D\|_* : C_{\beta} c^2 \left( \frac{\log(n) + \log(h)}{n} + \frac{(\log(n))^{1/\beta}(\log(n) + \log(h))^{2/\beta}}{n} \right) \quad (B.2)
\]

with probability $1 - 1/(3m^2)$. Indeed, let $D = \sum_{i=1}^h \sigma_i(D) a_i b_i^T$ be SVD of $D$ with singular values
\[ \sigma_l(D) \text{ and left and right singular vectors } a_l, b_l \in S^{d-1}, \ l \in [h]. \] Then
\[
\left| \text{Tr} \left[ (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d)D \right] \right| = \left| \text{Tr} \left[ (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) \sum_{l=1}^{h} \sigma_l(D) a_l b_l^T \right] \right|
\]
\[
= \left| \sum_{l=1}^{h} \sigma_l(D) b_l^T (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) a_l \right| \leq \max_{l \in [h]} \left| b_l^T (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) a_l \right| \sum_{l=1}^{h} \sigma_l(D)
\]
\[
= \|D\|_* \cdot \max_{l \in [h]} \left| b_l^T (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) a_l \right| .
\]

Now (B.2) follows from Theorem A.2 applied with 2h instead of p, \( K_{n,p} = c^{-1/\beta}, \ X_i(2l - 1) = a_l^T \Sigma^{-1/2} X_i, \ X_i(2l) = b_l^T \Sigma^{-1/2} X_i, \ l \in [h], i \in [n] \) and \( z = \log(9np^2) \).

Applying (B.2) to \( |T_2| \) gives with probability \( 1 - 1/(3np^2) \)
\[
|T_2| \leq \bar{\sigma}^2 \nu_n
\]
where
\[
\nu_n \overset{\text{def}}{=} C_\beta c^2 \left( \frac{\sqrt{\log(np) + \log(|J|)}}{n} + \frac{(\log(n))^{1/\beta}(\log(np) + \log(|J|))^{2/\beta}}{n} \right).
\]

Similarly, with probability \( 1 - 1/(3np^2) \)
\[
|T_3| \leq \bar{\sigma}^2 \nu_n^2.
\]

Finally, \( |T_1| \) can be bounded in the same way. Let
\[
\tilde{B}_k = \sum_{l=1}^{\lfloor J \rfloor} \sigma_l(B_k) a_l^{(k)} b_l^{(k)}^T,
\]
\[
\tilde{B}_j^T = \sum_{l=1}^{\lfloor J \rfloor} \sigma_l(B_j^T) a_l^{(j)} b_l^{(j)}^T
\]
be SVD of \( \tilde{B}_k \) and \( \tilde{B}_j^T \). Then
\[
|T_1| = \left| \text{Tr} \left[ (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) \tilde{B}_k (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) \tilde{B}_j^T \right] \right|
\]
\[
= \left| \text{Tr} \left[ (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) \left( \sum_{l=1}^{\lfloor J \rfloor} \sigma_l(\tilde{B}_k) a_l^{(k)} b_l^{(k)}^T \right) \times \right.
\]
\[
\left. \times (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) \left( \sum_{l=1}^{\lfloor J \rfloor} \sigma_l(\tilde{B}_j^T) a_l^{(j)} b_l^{(j)}^T \right) \right]\right|
\]
\[
\leq \sum_{l_1=1}^{\lfloor J \rfloor} \sum_{l_2=1}^{\lfloor J \rfloor} \sigma_{l_1}(\tilde{B}_k) \sigma_{l_2}(\tilde{B}_j^T) \cdot \left| b_{l_2}^{(j)} (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) a_{l_2}^{(k)} \right| \times
\]
\[
\times \left| b_{l_1}^{(k)} (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) a_{l_1}^{(j)} \right| 
\]
\[
\leq \| \tilde{B}_k \|_* \| \tilde{B}_j^T \|_* \max_{l_1, l_2 \in [\lfloor J \rfloor]} \left| b_{l_2}^{(j)} (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) a_{l_2}^{(k)} \right| \times
\]
\[
\times \left| b_{l_1}^{(k)} (\Sigma^{-1/2} \Sigma \Sigma^{-1/2} - I_d) a_{l_1}^{(j)} \right| .
\]
By yet another application of Theorem A.2
\[ |T_1| \leq |J| \nu_n^2 \]
with probability $1 - 1/(3np^3)$. Putting all the bounds together, we derive
\[ \left| \left[ \text{Cov}(Y^F \mid X) - \text{Cov}(Y) \right]_{j,k} \right| \leq |J| \nu_n^2 (\nu_n + \nu_n^2) \]
with probability $1 - 1/(np^2)$. Union bound concludes the proof:
\[ \max_{j,k \in [p]} \left| \left[ \text{Cov}(Y^F \mid X) - \text{Cov}(Y) \right]_{j,k} \right| \leq |J| \nu_n^2 (\nu_n + \nu_n^2) \]
with probability $1 - 1/n$.

**Proof of Lemma 8.14.** We first construct proper $\Gamma^*$. Recall that $\Gamma_1$ and $\Gamma_2$ which satisfy $\Gamma_1^T \Gamma_1 = \mathbf{P}$, $\Gamma_2^T \Gamma_1 = \mathbf{I}_m$, $\Gamma_2^T \Gamma_2 = \mathbf{I}_d - \mathbf{P}$ and $\Gamma_2^T \Gamma_2 = \mathbf{I}_{d-m}$ are fixed at the beginning of our procedure. At the same time, by Davis-Kahan theorem (e.g. Theorem 2 from Yu et al. (2015)), there exist $\Gamma_1^* \text{ and } \Gamma_2^*$ such that $\Gamma_1^T \Gamma_1^* = \mathbf{P}^*$, $\Gamma_1^T \Gamma_1 = \mathbf{I}_m$, $\Gamma_2^T \Gamma_2^* = \mathbf{I}_d - \mathbf{P}^*$ and $\Gamma_2^T \Gamma_2 = \mathbf{I}_{d-m}$, and
\[ \|\Gamma_1^* - \Gamma_1\|_F \leq 2^{3/2} \|\mathbf{P} - \mathbf{P}^*\|_F, \quad \|\Gamma_2^* - \Gamma_2\|_F \leq 2^{3/2} \|\mathbf{P} - \mathbf{P}^*\|_F. \]

Then, as in Lemma 8.2, denoting the linear parts of $\hat{\mathbf{P}}_a - \mathbf{P}_a$ and $\hat{\mathbf{P}}_b - \mathbf{P}_b$ as $L_a(\hat{\Sigma}_a - \Sigma_a)$ and $L_b(\hat{\Sigma}_b - \Sigma_b)$ and the remainder terms as $R_a(\hat{\Sigma}_a - \Sigma_a)$ and $R_b(\hat{\Sigma}_b - \Sigma_b)$, we decompose
\[ \hat{\mathbf{P}}_a - \mathbf{P}_a = (\mathbf{P}_a - \mathbf{P}^*) - (\hat{\mathbf{P}}_b - \mathbf{P}^*) = (\hat{\mathbf{P}}_a - \mathbf{P}_a) - (\hat{\mathbf{P}}_b - \mathbf{P}_b) \]
\[ = L_a(\hat{\Sigma}_a - \Sigma_a) + R_a(\hat{\Sigma}_a - \Sigma_a) - L_b(\hat{\Sigma}_b - \Sigma_b) - R_b(\hat{\Sigma}_b - \Sigma_b). \]

We first state some auxiliary bounds.

Bounds on the linear parts: Let $\hat{x}_a$, $\hat{x}_b$, $\hat{\tau}_a$ and $\hat{\tau}_b$ be the same quantities as $x$ in Lemma 8.2, but now for $(\hat{\Sigma}_a - \Sigma_a)$, $(\hat{\Sigma}_b - \Sigma_b)$, $(\hat{\Sigma}_a - \Sigma_a)$ and $(\hat{\Sigma}_b - \Sigma_b)$, respectively. Since we can bound
\[ \|L_{\mathcal{J}}(\hat{\Sigma} - \Sigma)\|_F^2 = \] 
\[ = \sum_{r \in \mathcal{J}} \sum_{s \notin \mathcal{J}} \frac{\mathbf{P}_r(\hat{\Sigma} - \Sigma)\mathbf{P}_s + \mathbf{P}_s(\hat{\Sigma} - \Sigma)\mathbf{P}_r}{\mu_r - \mu_s} \|\mathbf{P}_r(\hat{\Sigma} - \Sigma)\mathbf{P}_s\|_F^2 = 2 \sum_{r \in \mathcal{J}} \sum_{s \notin \mathcal{J}} \frac{\mathbf{P}_r(\hat{\Sigma} - \Sigma)\mathbf{P}_s}{(\mu_r - \mu_s)^2} \]
\[ \leq 2 \left( \sum_{r \in \mathcal{J}} \sum_{s \notin \mathcal{J}} \frac{m_r m_s}{(\mu_r - \mu_s)^2} \right) \left( \max_{r \in \mathcal{J}, s \notin \mathcal{J}} \frac{\|\mathbf{P}_r(\hat{\Sigma} - \Sigma)\mathbf{P}_s\|_F}{\sqrt{m_r m_s}} \right)^2 \leq 2 \mathcal{D}_{\mathcal{J}}(\Sigma)^2 \]

similar bounds apply to $L_a(\hat{\Sigma}_a - \Sigma_a)$, $L_b(\hat{\Sigma}_b - \Sigma_b)$, $L_a(\Sigma_a - \Sigma_a)$ and $L_b(\Sigma_b - \Sigma_b)$ and it holds
\[ \|L_a(\hat{\Sigma}_a - \Sigma_a)\|_F \leq \sqrt{2} \mathcal{D}_{\mathcal{J}_a}(\Sigma_a)^{1/2} \hat{x}_a, \]
\[ \|L_b(\hat{\Sigma}_b - \Sigma_b)\|_F \leq \sqrt{2} \mathcal{D}_{\mathcal{J}_b}(\Sigma_b)^{1/2} \hat{x}_b, \]
\[ \|L_a(\Sigma_a - \Sigma_a)\|_F \leq \sqrt{2} \mathcal{D}_{\mathcal{J}_a}(\Sigma_a)^{1/2} \hat{x}_a, \]
\[ \|L_b(\Sigma_b - \Sigma_b)\|_F \leq \sqrt{2} \mathcal{D}_{\mathcal{J}_b}(\Sigma_b)^{1/2} \hat{x}_b. \]
Main part: Denote for shortness for the rest of the proof

\[ A \overset{\text{def}}{=} L_a(\hat{\Sigma}_a - \Sigma_a) - L_b(\hat{\Sigma}_b - \Sigma_b). \]

Let us bound \[ \|\hat{P}_a - \hat{P}_b\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} - \|A\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} \]. First, if \( \hat{x}^a \leq \psi_{n_a} \) and \( \hat{x}^b \leq \psi_{n_b} \), then

\[
\begin{align*}
\|\hat{P}_a - \hat{P}_b\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} - \|A\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} &\leq R_a(\hat{\Sigma}_a - \Sigma_a)\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} + R_b(\hat{\Sigma}_b - \Sigma_b) \|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} \\
&\leq 2\|R_a(\hat{\Sigma}_a - \Sigma_a)\| + 2\|R_b(\hat{\Sigma}_b - \Sigma_b)\| \\
&\leq 2C\tilde{x}^a_a)^2 d_{\mathcal{J}_a}(\Sigma_a)^{3/2} + 2C(\tilde{x}^b)^2 d_{\mathcal{J}_b}(\Sigma_b)^{3/2} \\
&\leq 2C\psi_{n_a}^{2/3} \psi_{n_b}^{2/3} \frac{d_{a,b}^{3/2}}{a,b}.
\end{align*}
\]

where we used Proposition 3.1 (i), (ii) and Lemma 8.2 (Condition 8.1 is fulfilled by Assumption 4.3 (i)). Next, we bound \[ \|A\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} - \|A\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} \]. By Definition 3.1, it is clear that

\[
\begin{align*}
\|A\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} - \|A\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} &\leq \|\Gamma_1^\top A\Gamma_1 - \|\Gamma_1^\top A\Gamma_1^*\| + \|\Gamma_2^\top A\Gamma_2 - \|\Gamma_2^\top A\Gamma_2^*\| + \\
&\quad + \sup_{v \in \mathcal{D}_{s_1}^m} v^\top \Gamma_1^\top A\Gamma_2 w - \sup_{w \in \mathcal{D}_{s_2}^{d-m}} w^\top \Gamma_1^\top A\Gamma_2^* w.
\end{align*}
\]

Each of the three terms can be bounded similarly, so let us bound just the first one. Adding and subtracting the mixed term \( \|\Gamma_1^\top A\Gamma_1^*\| \), we obtain

\[
\begin{align*}
\|\Gamma_1^\top A\Gamma_1 - \|\Gamma_1^\top A\Gamma_1^*\| &\leq \|\Gamma_1^\top A(\Gamma_1 - \Gamma_1^*)\| + \|\Gamma_1^\top A\Gamma_1^* - \|\Gamma_1^\top A\Gamma_1^*\| \\
&\leq \|\Gamma_1^\top A(\Gamma_1 - \Gamma_1^*)\| + \|\Gamma_1^\top A\Gamma_1^* - \|\Gamma_1^\top A\Gamma_1^*\| \\
&\leq 2\|A\|\|\Gamma_1 - \Gamma_1^*\| \leq 2^{5/2}\|A\|\|\bar{P} - P^*\|_F.
\end{align*}
\]

So, if \( \hat{x}^a \leq \psi_{n_a} \) and \( \hat{x}^b \leq \psi_{n_b} \), then

\[
\begin{align*}
\|\hat{P}_a - \hat{P}_b\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} - \|A\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} &\leq 3 \cdot 2^{5/2}\|A\|\|\bar{P} - P^*\|_F \\
&\leq 3 \cdot 2^{5/2}\left(\|L_a(\hat{\Sigma}_a - \Sigma_a)\| + \|L_b(\hat{\Sigma}_b - \Sigma_b)\|\right)\|\bar{P} - P^*\|_F \\
&\leq 3 \cdot 2^{5/2}\left(\tilde{x}^a_a d_{\mathcal{J}_a}(\Sigma_a)^{1/2} + \tilde{x}^b_b d_{\mathcal{J}_b}(\Sigma_b)^{1/2}\right)\|\bar{P} - P^*\|_F \\
&\leq 3 \cdot 2^{5/2}\psi_{n_a}^{1/2} \psi_{n_b}^{1/2} \frac{d_{a,b}^{1/2}}{a,b} \|\bar{P} - P^*\|_F.
\end{align*}
\]

Hence, if \( \hat{x}^a \leq \psi_{n_a} \) and \( \hat{x}^b \leq \psi_{n_b} \), then

\[
\begin{align*}
\|\hat{P}_a - \hat{P}_b\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} - \|A\|_{(\mathbf{P}, \mathbf{G}, s_1, s_2)} &\leq 2C\psi_{n_a}^{2/3} \psi_{n_b}^{2/3} \frac{d_{a,b}^{3/2}}{a,b} + 3 \cdot 2^{5/2}\psi_{n_a}^{1/2} \psi_{n_b}^{1/2} \|\bar{P} - P^*\|_F,
\end{align*}
\]

which implies, introducing event \( \Omega = \{\hat{x}^a \leq \psi_{n_a}, \hat{x}^b \leq \psi_{n_b}\} \) with \( \mathbb{P}[\Omega] \geq 1 - 1/n_a - 1/n_b \) (by
Lemma 8.1 and union bound,

\[
\begin{align*}
\mathbb{P} \left[ \| \hat{\mathbf{P}}_a - \hat{\mathbf{P}}_b \|_{(\mathbf{P},t_{s_1,s_2})} - \| \mathbf{P}^* \|_{(\mathbf{P}^*,t_{s_1,s_2})} \right] &> C \left( \psi_{n_a \wedge n_b}^2 d_{a,b}^3 + \psi_{n_a \wedge n_b} \bar{d}_{a,b}^{1/2} \cdot \| \mathbf{P} - \mathbf{P}^* \|_F \right) \mid \Gamma \right] = \\
&= \mathbb{P} \left[ \| \hat{\mathbf{P}}_a - \hat{\mathbf{P}}_b \|_{(\mathbf{P},t_{s_1,s_2})} - \| \mathbf{P}^* \|_{(\mathbf{P}^*,t_{s_1,s_2})} \right] > C \left( \psi_{n_a \wedge n_b}^2 d_{a,b}^3 + \psi_{n_a \wedge n_b} \bar{d}_{a,b}^{1/2} \cdot \| \mathbf{P} - \mathbf{P}^* \|_F \right) \mid \Gamma; \Omega \right] \times \\
&\quad \times \mathbb{P}[\Omega] + \\
&\quad + \mathbb{P} \left[ \| \hat{\mathbf{P}}_a - \hat{\mathbf{P}}_b \|_{(\mathbf{P},t_{s_1,s_2})} - \| \mathbf{P}^* \|_{(\mathbf{P}^*,t_{s_1,s_2})} \right] > C \left( \psi_{n_a \wedge n_b}^2 d_{a,b}^3 + \psi_{n_a \wedge n_b} \bar{d}_{a,b}^{1/2} \cdot \| \mathbf{P} - \mathbf{P}^* \|_F \right) \mid \Gamma; \Omega^c \right] \times \\
&\quad \times \mathbb{P}[\Omega^c] \\
&\leq 0 \cdot 1 + 1 \cdot \left( \frac{1}{n_a} + \frac{1}{n_b} \right) = \frac{1}{n_a} + \frac{1}{n_b}.
\end{align*}
\]

Now it is left to bound \( \| \mathbf{P} - \mathbf{P}^* \|_F \) with high probability.

By definition of \( \mathbf{P} \) given by (3.2)

\[
\| \mathbf{P} - \mathbf{P}_a \|_F^2 + \| \mathbf{P} - \mathbf{P}_b \|_F^2 \leq \| \mathbf{P}^* - \mathbf{P}_a \|_F^2 + \| \mathbf{P}^* - \mathbf{P}_b \|_F^2 = \| \mathbf{P}_a - \mathbf{P}_a \|_F^2 + \| \mathbf{P}_b - \mathbf{P}_b \|_F^2.
\]

Therefore,

\[
\| \mathbf{P} - \mathbf{P}^* \|_F^2 = \frac{1}{2} \left( \| \mathbf{P} - \mathbf{P}_a \|_F^2 + \| \mathbf{P} - \mathbf{P}_b \|_F^2 \right) \\
\leq \| \mathbf{P} - \mathbf{P}_a \|_F^2 + \| \mathbf{P} - \mathbf{P}_b \|_F^2 + \| \mathbf{P}_a - \mathbf{P}_a \|_F^2 + \| \mathbf{P}_b - \mathbf{P}_b \|_F^2 \\
\leq 2 \left( \| \mathbf{P}_a - \mathbf{P}_a \|_F^2 + \| \mathbf{P}_b - \mathbf{P}_b \|_F^2 \right).
\]

Hence, if \( \varpi_a \leq \psi_{n_a} \) and \( \varpi_b \leq \psi_{n_b} \), then

\[
\| \mathbf{P} - \mathbf{P}^* \|_F \leq \sqrt{2} \left( \| \mathbf{P}_a - \mathbf{P}_a \|_F + \| \mathbf{P}_b - \mathbf{P}_b \|_F \right) \\
\leq \sqrt{2} \left( \| L_a(\Sigma - \Sigma) \|_F + \| R_a(\Sigma - \Sigma) \|_F + \| L_b(\Sigma - \Sigma) \|_F + \| R_b(\Sigma - \Sigma) \|_F \right) \\
\leq C \left( \psi_{n_a \wedge n_b} \left[ d_{a}^{1/2} + d_{b}^{1/2} \right] + \psi_{n_a \wedge n_b} \left[ d_{a}^{3/2} + d_{b}^{3/2} \right] \right) \\
= C \left( \psi_{n_a \wedge n_b} \bar{d}_{a,b}^{1/2} + \psi_{n_a \wedge n_b} \bar{d}_{a,b}^{3/2} \right),
\]

where we used bounds (B.3) and Lemma 8.2 (again, Condition 8.1 is fulfilled by Assumption 4.3 (i)). Since, probability of the event \( \varpi_a \leq \psi_{n_a} \) and \( \varpi_b \leq \psi_{n_b} \) is at least \( 1 - 1/n_a - 1/n_b \) (again by Lemma 8.1 and union bound), we conclude (adjusting the constants and using technical assumption to simplify the bound):

\[
\mathbb{P} \left[ \| \hat{\mathbf{P}}_a - \hat{\mathbf{P}}_b \|_{(\mathbf{P},t_{s_1,s_2})} - \| \mathbf{P}^* \|_{(\mathbf{P}^*,t_{s_1,s_2})} \right] > C \psi_{n_a \wedge n_b}^2 \left( d_{a,b}^3 + \bar{d}_{a,b} \right) \mid \Gamma \right] \leq \frac{1}{n_a} + \frac{1}{n_b}
\]

with probability \( 1 - 1/n_a - 1/n_b \).

\[\square\]

**Proof of Proposition 5.1.** Let us first prove that the spectral projector onto the sum of \( m \) eigenspaces corresponding to non-zero eigenvalues of \( \mathbf{B} \operatorname{Cov}[\mathbf{f}_i] \mathbf{B}^T \) is given by \( \mathbf{B} (\mathbf{B}^T \mathbf{B})^{-1} \mathbf{B}^T \).

Consider the eigendecomposition of \( (\mathbf{B}^T \mathbf{B})^{1/2} \operatorname{Cov}[\mathbf{f}_i] (\mathbf{B}^T \mathbf{B})^{1/2} \):

\[
(\mathbf{B}^T \mathbf{B})^{1/2} \operatorname{Cov}[\mathbf{f}_i] (\mathbf{B}^T \mathbf{B})^{1/2} = QDQ^T,
\]

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where \( Q \in \mathbb{R}^{m \times m} \) is orthogonal and \( D \in \mathbb{R}^{m \times m} \) is diagonal. Take \( H = (B^\top B)^{-1/2}Q \). Then
\[
(BH)^\top BH = H^\top B^\top BH = Q^\top (B^\top B)^{-1/2}B^\top B(B^\top B)^{-1/2}Q = Q^\top Q = I_m,
\]
i.e. columns of \( BH \) are orthogonal and have unit length. Also,
\[
H^{-1} \text{Cov}[f_i](H^{-1})^\top = Q^\top (B^\top B)^{1/2} \text{Cov}[f_i](B^\top B)^{1/2}Q = Q^\top QDQ^\top Q = D,
\]
i.e. diagonal. Therefore,
\[
(BH)[H^{-1} \text{Cov}[f_i](H^{-1})^\top](BH)^\top
\]
is a valid eigendecomposition of \( B \text{Cov}[f_i]B^\top \). The spectral projector of interest is then exactly
\[
(BH)(BH)^\top = BHH^\top B^\top = B(B^\top B)^{-1/2}QQ^\top (B^\top B)^{-1/2}B^\top = B(B^\top B)^{-1}B^\top.
\]

Now we just apply Davis-Kahan theorem to \( \Sigma \) and \( B \text{Cov}[f_i]B^\top \) to get
\[
\|P_J - B(B^\top B)^{-1}B^\top\| \lesssim \frac{\|\text{Cov}[\xi]\|}{\mu_m - \mu_{m+1}} = O\left(\frac{1}{d}\right),
\]
where we used Assumption 5.1.

\( \square \)

**Proof of Proposition 5.2.** The condition \( \text{Cov}[\xi]B = 0_{d \times m} \) implies that any projector of \( \text{Cov}[\xi] \) corresponding to non-zero eigenvalue is orthogonal to \( B(B^\top B)^{-1}B^\top \). This means that the projectors of \( \Sigma \) and \( B \text{Cov}[f_i]B^\top \) onto the first \( m \) eigenspaces coincide. \( \square \)

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