FOLIATIONS BY STATIONARY DISKS
OF ALMOST COMPLEX DOMAINS

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Abstract. We study the problem of existence of stationary disks for
domains in almost complex manifolds. As a consequence of our results,
we prove that any almost complex domains which is a small deforma-
tions of a strictly linearly convex domain $D \subset \mathbb{C}^n$ with standard complex
structure admits a singular foliation by stationary disks passing through
any given internal point. Similar results are given for foliation by sta-
tionary disks through a given boundary point.

Introduction

Let $(M, J)$ be an almost complex manifold and $D \subset M$ a strongly pseu-
doconvex domain with smooth boundary. Given a point $x_0 \in D$, let us
call foliation by stationary disks of $(D, x_0)$ any collection of stationary disks
centered at $x_0$ and smoothly parameterized by the points of a unit sphere
$S = \{ v \in T_{x_0}D : \|v\| = 1 \}$ for some Euclidean norm $\| \cdot \|$ on $T_{x_0}D$. By
“stationary disk” we mean any $J$-holomorphic embedding $f : \Delta \to D$ of the
unit disk $\Delta \subset \mathbb{C}$ that satisfies the definition of Coupet, Gaussier and Sukhov
in [4], which naturally generalizes the usual notion of Lempert’s stationary
disks for bounded domains in $\mathbb{C}^n$.

In case $(M, J) = (\mathbb{C}^n, J_{st})$, natural examples of foliations by stationary
disks are given by the straight disks through the origin of the pseudoconvex,
smoothly bounded complete circular domains $D$ in $\mathbb{C}^n$. Other interesting
examples are provided by the celebrated results by Lempert on Kobayashi
extremal disks in strictly linearly convex domains ([13, 14, 15]). In fact, an immediate consequence of those results is that for any smoothly bounded, strictly linearly convex domain $D \subset \mathbb{C}^n$ and any $x_o \in D$, the Kobayashi extremal disks of $D$ through $x_o$ give a foliation by stationary disks of $(D, x_o)$. The existence of a foliation by stationary disks is also one of the main properties of the smoothly bounded domains of circular type, a class of domains in $\mathbb{C}^n$ with an exhaustion of a special kind, which naturally include all complete strictly pseudoconvex bounded circular domains, all bounded strictly linearly convex domains and, more generally, all strictly pseudoconvex domains with (singular) foliations given Kobayashi extremal disks satisfying some special regularity conditions ([18, 19]).

In all these cases, the foliation by stationary disks $\mathcal{F}^{(x_o)}$ can be used to construct a so-called generalized Riemann map, i.e. a homeomorphism $\varphi : \overline{B^n} \to \overline{D}$, which is smooth on $B^n \setminus \{0\}$ and maps the straight complex lines in $B^n$ through 0 into corresponding disks of $\mathcal{F}^{(x_o)}$. This generalized Riemann map have been often used in at least two important research areas: a) generalizations of Fefferman’s theorem on boundary regularity of biholomorphisms between pseudoconvex domains; b) Green functions with logarithmic pole for Monge-Ampère equations and plurisubharmonic exhaustions of pseudoconvex domains (see e.g. [13, 14, 11, 24, 7]).

At the best of our knowledge, the first use of foliations by stationary disks in the contest of almost complex manifolds can be found in [4]. There, the authors generalize Lempert’s notion of stationary disks in the almost complex setting and show the existence of a foliation by stationary disks of the unit ball $B^n \subset \mathbb{C}^n$, endowed with an almost complex structure $J$ which is a sufficiently small deformation of standard complex structure $J_{st}$. The corresponding generalized Riemann map has been used to prove $C^\infty$-regularity of biholomorphisms between two almost complex domains $(B^n, J)$ and $(B^n, J')$ of this kind, which admit $C^1$-extensions up to the boundary (see also [20]). Later, Gaussier and Sukhov proved in [9] showed that the hypothesis of $C^1$-extendibility can be removed and that the result holds true for any pair of smoothly bounded, strictly pseudoconvex almost complex domains, proving Fefferman’s theorem in almost complex setting in full generality (see also [5]).

Motivated by these results and possible applications on plurisubharmonic exhaustions, in this paper we determine more general situations in which the existence of foliations by stationary disks (and hence of generalized Riemann maps) is granted. Basically, we follow the approach of [4]. We first consider the differential problem that characterizes the stationary disks of an almost complex domain $(D, J_o)$ and we explicitly determine the associated linearized operator $\mathcal{R}$ at a given stationary disk $f_o : \overline{D} \to \overline{D}$. When $\mathcal{R}$ is invertible, we say that $\partial D$ is good relatively to the pair $(f_o, J_o)$. A direct application of the Implicit Function Theorem implies that if $\partial D$ is good, then there exists stationary disks in a neighborhood of $f_o$ also when $J_o$ is
replaced by a sufficiently close almost complex structure $J$. On the base of this observation, one has that if an almost complex domain $(D, J_0)$ has a foliation $F(x_0)$ of stationary disks through $x_0$, and if the boundary is good for $(f_0, J_0)$ for any $f_0 \in F(x_0)$, then there exists a foliation for $(D, J)$ of stationary disks passing through $x_0$, also when $J_0$ is replaced by a sufficiently close almost complex structure $J \neq J_0$.

Secondly, by a line of arguments that goes back to Lempert and Pang ([13, 17]; see also [23, 22, 4, 20]), we are able to prove that any smoothly bounded, strictly linearly convex domain $D \subset \mathbb{C}^n$ has a boundary which is “good” for any of its stationary disks. This fact and previous observation bring directly to our result, which generalizes the quoted Coupet, Gaussier and Sukhov’s theorem on the unit ball: if a smoothly bounded, strongly pseudoconvex domain $D$ in an almost complex manifold $(M, J)$ is biholomorphic to a strictly linearly convex domain $\tilde{D} \subset (\mathbb{C}^n, J')$, endowed with small deformation $J'$ of $J_{st}$, then there exists a foliation by stationary disks of $(D, x_0)$ for any $x_0 \in D$ (Theorem 4.1).

This shows that the class of almost complex domains, admitting a foliation by stationary disks, is indeed much larger than the class considered in [4]. In fact, via a diffeomorphism $\varphi : U \rightarrow V \subset \mathbb{C}^n$ mapping $\overline{D}$ onto $\overline{V}$, one obtains the existence of foliations by stationary disks on $(B^n, J')$ also when $J' = \varphi_*\hat{J}$ is not a small deformation of $J_{st}$.

We also prove that, for any almost complex domain $(D, J)$ as above and with $J'$ sufficiently close to $J_{st}$, there exists a generalized Riemann map $\varphi : \overline{B^n} \rightarrow \overline{D}$ for any $x_0 \in D$ and the function $u = (\varphi^{-1})^*u_o : D \rightarrow ]-\infty, 0[$ of $u_o(z) \overset{\text{def}}{=} \log(|z|)$ is a plurisubharmonic exhaustion for $D$. When $J$ is integrable, $u$ is a solution of the Monge-Ampère equation $(\partial \overline{\partial} u)^n = 0$ with boundary data $u|_{\partial D} = 0$ and logarithmic singularity at $x_0$. It would be interesting to know if this and other related properties have counterparts in almost complex setting.

Finally, we consider the families $G^{(x_0,a)}$, formed by all stationary disks in a given almost complex domain $(D, J)$, passing through a given boundary point $x_0 \in \partial D$ and with tangent vector $v$ at $x_0$, with inner product $<v, \nu>$ with the unit normal $\nu_{x_0}$ larger than a value $a \geq 0$. In case $D$ is a strictly convex domain in $\mathbb{C}^n$, the disks in $G^{(x_0,a)}$ give a (regular) foliation of a certain subdomain $D^{(x_0,a)} \subset D$ that coincide with $D$ in case $a = 0$ ([6]). We prove that if $a > 0$, this is true also when the standard complex structure $J_{st}$ is replaced by an almost complex structure $J$ sufficiently close to $J_{st}$ and we therefore have an analogue of the previous results also for what concerns foliations of conical subdomains $D^{(x_0,a)}$, $a > 0$, of almost complex domains. A proof for the case $a = 0$ seems to be at the moment out of reach, because the family of stationary disks $G^{(x_0,0)}$ is not parameterized by a compact set, in contrast with all other considered situations.

As final remark, notice that when $J$ is integrable, the regular foliations $G^{(x_0,a)}$ determine analogues of the Riemann map and have been used in
Finally, for any $\alpha, \epsilon > 0$, a Banach space $H$ of the functions $f$ and the definition of stationary disks in almost complex domains. In §3, we consider the so-called foliations of circular type, prove their stability under small deformations of $J$ in case of a “good boundary”. In §4, general conditions for a boundary “to be good” are given and are used to show that any strictly linearly convex domain has a “good” boundary. This and the results of §5 give our main Theorem 4.1 as immediate consequence. Section §5 is devoted to the quoted results on foliations of conical subdomains.

2. Preliminaries

2.1. Notations. Given a real manifold $M$ and a system of coordinates $\xi = (z^i) : \mathcal{U} \subset M \rightarrow \mathbb{C}^n$, we call associated coordinates on $T^*M$ the coordinates $\xi^i = (x^i, p_i)$, where for any $\alpha \in T^*_x M$ the “$p_i$” are the components of $\alpha = p_i dx^i$ in the basis $(dx^i)$. If $(M, J)$ is an almost complex manifold of real dimension $2n$, we call system of complex coordinates any local diffeomorphism $\xi = (z^i) : \mathcal{U} \subset M \rightarrow \mathbb{C}^n$. We call them holomorphic whenever $J$ is integrable and $\xi = (z^i)$ is a chart of the corresponding complex manifold structure of $(M, J)$. We also call associated complex coordinates on $T^*M$ the complex coordinates $\xi = (\bar{z}^i, w_j) : \pi^{-1}(\mathcal{U}) \subset T^*M \rightarrow \mathbb{C}^{2n}$, where the $w_i$’s are defined for any 1-form $\alpha$ by the expression $\alpha = w_idz^i + \bar{w}_i\bar{dz}^i$.

For any Banach space $X$ and $\mathcal{U} \subset \mathbb{R}^M, \alpha \in [0,1[$, we denote by $C^\alpha(\mathcal{U}, X)$ the Banach space of the functions $f : \mathcal{U} \rightarrow X$ such that

$$\left\| f \right\|_{C^\alpha} \overset{\text{def}}{=} \sup_{\zeta \in \mathcal{U}} \left\| f(\zeta) \right\| + \sup_{\theta, \eta \in \mathcal{U}, \theta \neq \eta} \frac{\left\| f(\theta) - f(\eta) \right\|}{\left| \theta - \eta \right|^\alpha} < \infty.$$

If $\alpha = m + \beta$, for some $m \in \mathbb{N}$ and $\beta \in [0,1[$, we denote by $C^\alpha(\mathcal{U}, X)$ the Banach space $C^\alpha(\mathcal{U}, X) = \{ r \in C^m(\mathcal{U}, X) : D^\nu r \in C^\beta(\mathcal{U}, X), |\nu| \leq m \}$. Finally, for any $\alpha, \epsilon > 0$, we set $C^{\alpha, \epsilon}(\Delta, \mathbb{C}^n) = C^\epsilon(\Delta, \mathbb{C}^n) \cap C^{\alpha}(\Delta, \mathbb{C}^n)$ and $H(\Delta, \mathbb{C}^n) = C^\epsilon(\Delta, \mathbb{C}^n) \cap Hol(\Delta, \mathbb{C}^n)$.

2.2. Lifts of $J$-holomorphic disks. We recall that a $\mathcal{C}^\alpha$-map $f : M \rightarrow M'$, $1 \leq \alpha$, between two almost complex manifolds $(M, J)$, $(M', J')$ is called $(J, J')$-holomorphic if and only if $\partial_{J, J'} f(v) = 0$ for any $v \in TM$, where $\partial_{J, J'} f$ is the operator

$$\partial_{J, J'} f : TM \rightarrow TM', \quad \partial_{J, J'} f(v) \overset{\text{def}}{=} f_*(J(v)) - J'(f_*(v)). \quad (2.1)$$

When $(M, J) = (\mathbb{C}^n, J_{st})$, we will shortly write $\partial_{J'}$ for $\partial_{J_{st}, J'}$. A $J_{st}$-holomorphic disk of $(M, J)$ is a $(J_{st}, J)$-holomorphic map $f : \Delta \rightarrow M$ from the unit disk $\Delta \subset \mathbb{C}$ into $(M, J)$. Recall that $\partial_{J} f = 0$ if and only if $\partial_{J} f \left( \frac{\partial}{\partial x} |_{x+iy} \right) = 0$ at any $x + iy \in \Delta$ (see e.g. [11]).
If \((M, J)\) is a complex manifold, the cotangent bundle \(T^*M\) is naturally endowed with an \textit{integrable} complex structure \(\mathcal{J}\), determined by the identifications of open subsets \(U \subset M\) with open subsets of \(\mathbb{C}^n\) and by the identifications of the sets \(T^*M|_U\) with open subsets of \(\mathbb{C}^{2n} = T^*\mathbb{C}^n\). When \(J\) is not integrable, these identifications are no longer valid, but there still exists a natural almost complex structure \(\mathcal{J}\) on \(T^*M\), which reduces to the usual one if \(J\) is integrable ([10]). The main properties of \(\mathcal{J}\) are summarized in the next proposition. Here, \(J^j_i(x)\) are the components of \(J = J^i_j \frac{\partial}{\partial x^i} \otimes dx^j\) in a system of real coordinates \(\xi = (x^i)\).

**Proposition 2.1.** [10] For any almost complex manifold \((M, J)\), there exists a unique almost complex structure \(\mathcal{J}\) on \(T^*M\) with the following properties:

i) the projection \(\pi : T^*M \to M\) is \((\mathcal{J}, J)\)-holomorphic;
ii) for any \((J, J')\)-biholomorphism \(f : M \to N\) between two almost complex manifolds \((M, J)\) and \((N, J')\), the induced map \(\hat{f} : T^*N \to T^*M\) is \((\mathcal{J}', \mathcal{J})\)-holomorphic;
iii) if \(J\) is integrable, then also \(\mathcal{J}\) is integrable and coincides with the natural complex structure of \(T^*M\);
iv) in a system of coordinates

\[
\tilde{\xi} = (x^1, \ldots, x^{2n}, p_1, \ldots, p_{2n}) : \pi^{-1}(U) \subset T^*M \to \mathbb{R}^{4n},
\]

(2.2) associated with \(\xi = (x^i)\), the tensor \(\mathcal{J}\) is of the form

\[
\mathcal{J} = J^a_i \frac{\partial}{\partial x^a} \otimes dx^i + J^a_i \frac{\partial}{\partial p_i} \otimes dp_a + \\
\frac{1}{2} p_a \left( -J^a_{i,j} + J^a_i J^j_m - J^f_{i,m} J^m_j - J^f_{j,m} J^m_i \right) \frac{\partial}{\partial p_j} \otimes dx^i.
\]

(2.3)

The almost complex structure \(\mathcal{J}\) is called \textit{canonical lift} of \(J\) on \(T^*M\).

**Lemma 2.2.** Let \(\mathcal{J}\) be the canonical lift on \(T^*M\) of an almost complex structure \(J\). For any \(0 \neq t \in \mathbb{R}\), the map \(\varphi_t : T^*M \to T^*M\) defined by \(\varphi_t(\alpha) = t \cdot \alpha\) is a \(\mathcal{J}\)-biholomorphic diffeomorphisms, i.e. \(\varphi_t \circ \mathcal{J} = \mathcal{J} \circ \varphi_{t*}\).

**Proof.** Writing \(\varphi_t\) in a system of coordinates \(2.2\), one has that \(\varphi_t(x^i, p_j) = (x^i, tp_j)\). Using \(2.3\), the claim is then immediately checked. \(\square\)

Given a \(J\)-holomorphic disk \(f : \Delta \to (M, J)\), we call \textit{lift} of \(f\) any \(\mathcal{J}\)-holomorphic disk \(\hat{f} : \Delta \to (T^*M, \mathcal{J})\) so that \(f = \pi \circ \hat{f}\).

2.3. \textbf{Stationary disks.} Let \(\Gamma \subset M\) be a smooth hypersurface of an almost complex manifold \((M, J)\). The \textit{conormal bundle} of \(\Gamma\) is defined as

\[
\mathcal{N} \overset{\text{def}}{=} \{ \alpha \in T_x^*M, \ x \in \Gamma : \alpha|_{\Gamma} \equiv 0 \} \subset T^*M|_{\Gamma}.
\]

(2.4)

In the following, we denote by \(\mathcal{N}_* = \mathcal{N} \setminus \{\text{zero section}\}\) and \textit{when we mention “the conormal bundle” we will always mean} \(\mathcal{N}_*\).
The CR structure of $\Gamma$ is defined as the pair $(\mathcal{D}, J)$ given by the distribution

$$\mathcal{D} = \bigcup_{x \in \Gamma} \mathcal{D}_x \subset T\Gamma, \quad \mathcal{D}_x \overset{\text{def}}{=} \{ v \in T_x \Gamma : J(v) \in T_x \Gamma \} \quad (2.5)$$

endowed with the family $J = \{ J_x \}$ of complex structures $J_x \overset{\text{def}}{=} J|_{\mathcal{D}_x}$. A defining 1-form for $\mathcal{D}$ is a 1-form on $\Gamma$ so that $\ker \vartheta|_x = \mathcal{D}_x$ for any $x \in \Gamma$. The Levi form at $x$ is the quadratic form $L_x : \mathcal{D}_x \to \mathbb{R}$ defined by $L_x(v) \overset{\text{def}}{=} -d\vartheta_x(v, Jv)$ for any $v \in \mathcal{D}_x$ and (up a scalar factor) it is independent on the choice of $\vartheta$. This last property follows immediately from the fact that for any vector field $X^{(v)} \in \mathcal{D}$ so that $X^{(v)}_x = v$ one has

$$L_x(v) = -d\vartheta_x(X^{(v)}, JX^{(v)}) = \vartheta_x([X^{(v)}, JX^{(v)}]). \quad (2.6)$$

An oriented hypersurface $\Gamma \subset M$ is called strongly pseudoconvex if $L_x$ is positive definite at every $x \in \Gamma$ when determined by a defining 1-form $\vartheta$ with $\vartheta_x(Jn) > 0$ for any $n$ pointing in the “outwards” direction. If $D \subset M$ is a bounded domain with smooth boundary $\partial D$, we say that $D$ is strongly pseudoconvex when $\partial D$, oriented so that the ”outwards” directions are pointing outside $D$, is strongly pseudoconvex.

The following notion of “stationary disk” for domains in almost complex manifolds was considered for the first time by Coupet, Gaussier and Sukhov in [13]. It generalizes the notion of stationary disks of bounded domains in $\mathbb{C}^n$ ([13] [24]).

**Definition 2.3.** Let $D \subset M$ be a domain with smooth boundary and $\mathcal{N}_x$ the conormal bundle of $\partial D$. Given $\alpha \geq 1, \varepsilon > 0$, a map $f : \Delta \to M$ is called $C^{\alpha,\varepsilon}$-stationary disk of $D$ if

i) $f|_{\Delta}$ is a $J$-holomorphic embedding and $f(\partial \Delta) \subset \partial D$;

ii) there exists a lift $\hat{f} : \hat{\Delta} \to T^*M$ of $f$ so that

$$\zeta^{-1} \cdot \hat{f}(\zeta) \in \mathcal{N}_x \quad \text{for any } \zeta \in \partial \Delta \quad (2.7)$$

and $\hat{\zeta} \circ \hat{f} \in C^{\alpha,\varepsilon}(\hat{\Delta}, \mathbb{C}^{2n})$ for some complex coordinates $\hat{\zeta} = (z^i, w_j)$ around $\hat{f}(\hat{\Delta})$. Here “·” denotes the usual $\mathbb{C}$-action on $T^*M$, i.e.

$$\zeta \cdot \alpha \overset{\text{def}}{=} \text{Re}(\zeta)\alpha - \text{Im}(\zeta)J^*\alpha \quad \text{for any } \alpha \in T^*M, \ z \in \mathbb{C}. \quad (2.8)$$

In the following, the values of $\alpha$ and $\varepsilon$ are considered as fixed and by “stationary” we always mean “$C^{\alpha,\varepsilon}$-stationary”. Moreover, given a stationary disk $f$, the maps $\hat{f}$ satisfying (ii) are called stationary lifts of $f$.

**Lemma 2.4.** i) If $D \subset M$ is a smoothly bounded, strongly pseudoconvex domain and $f : \Delta \to M$ is a non-constant stationary disk of $D$, then $f(\Delta) \subset \overline{D}$ and $f(\zeta) \in \partial D$ if and only if $\zeta \in \partial \Delta$.

ii) For any $t \in \mathbb{R}_+$ and any stationary lift $\hat{f}$ of a stationary disk $f : \Delta \to \overline{D}$, also the map $\hat{f}_t(\zeta) \overset{\text{def}}{=} (\varphi_t \circ \hat{f})(\zeta) = t \cdot \hat{f}(\zeta)$ is a stationary lift of $f$. 

Proof. (i) If $D$ is strongly pseudoconvex, it is known that there exists a defining function $\rho : U \subset M \to \mathbb{R}$ for $D$ which is $J$-plurisubharmonic, i.e. so that $\rho \circ f : \Delta \to \mathbb{R}$ is strictly subharmonic for any $J$-holomorphic disk $f : \Delta \to U$ (see e.g. [5], p.14). Since $\rho \circ f|_{\partial \Delta} = 0$, the claim follows from the maximum principle.

(ii) It follows from the fact that $\tilde{f}_t$ satisfies (2.7) and that the diffeomorphism $\varphi_t$ is a $J$-biholomorphism by Lemma 2.2.

We conclude recalling the following theorem that generalizes a well-known result by Webster to the almost complex setting ([26]).

**Theorem 2.5.** [21] Let $\Gamma$ be a strongly pseudoconvex hypersurface in an almost complex manifold $(M, J)$ and $N_\ast \subset T^*M$ its conormal bundle with the zero section excluded. Then $N_\ast$ is a totally real submanifold of $(T^*M, J)$.

### 3. Foliations by stationary disks and deformations of almost complex structures

#### 3.1. The Riemann-Hilbert problem for stationary disks.

In this and the next sections, $D$ is a strongly pseudoconvex domain in an almost complex manifold $(M, J)$ with smooth boundary $\partial D$ with conormal bundle $N \subset T^*M|_{\partial D}$. We also assume that $\overline{D} \subset M$ is contained in a globally coordinatizable open subset $U \subset M$ or, equivalently, that $D$ is a domain of $M = \mathbb{R}^{2n} \simeq \mathbb{C}^n$ equipped with a non-standard complex structure $J$. We also assume that $D$ has a smooth defining function $\rho : U \subset M \to \mathbb{R}$ on $U$, so that

$$D = \{ x \in M : \rho(x) < 0 \} \quad \text{and} \quad d\rho_x \neq 0 \quad \text{for any} \ x \in \Gamma = \partial D.$$  

We want to study the differential problem that characterizes the lifts $\hat{f} : \Delta \to T^*M$ of stationary disks of $D$. First of all, consider the map

$$\tilde{\rho} : \mathbb{R}_+ \times T^*M|_{U} \longrightarrow \mathbb{R} \times T^*M|_{U}, \quad \tilde{\rho}(t, \alpha) \overset{\text{def}}{=} (\rho(\pi(\alpha)), \alpha - t \cdot d\rho_{\pi(\alpha)}) . \quad (3.1)$$

Notice that the bundle $N_\ast = N \setminus \{ \text{zero section} \}$, which is a $2n$-dimensional submanifold of $T^*M$, can be identified with the level set

$$\{(t, \alpha) : t \neq 0, \tilde{\rho}(t, \alpha) = (0, 0, t \cdot d\rho_{\pi(\alpha)}) \} \subset \mathbb{R}_+ \times T^*M|_{U},$$

which is a $2n$-dimensional submanifold of $\mathbb{R}_+ \times T^*M$. Therefore, using a system of coordinates $\hat{\xi} = (x^i, p_j)$ on $T^*M|_U$, associated with coordinates $\xi = (x^i)$, we may identify $\mathbb{R}_+ \times T^*M|_{U}$ with an open subset $\mathcal{V} \subset \mathbb{R}^{4n+1}$ and $N_\ast$ with the level set in $\mathcal{V}$ defined by

$$N_\ast \simeq \{ (t, \alpha) \in \mathcal{V} : \tilde{\rho}(t, \alpha) = 0, \quad 1 \leq i \leq 2n + 1 \} .$$

By a direct check of the rank of the Jacobian, one can check that the map $\tilde{\rho} = (\tilde{\rho}^1, \ldots, \tilde{\rho}^{2n+1})$ is a smooth defining function for $N_\ast$.  

We now consider the map \( r : \mathbb{C} \times \mathcal{V} \subset \mathbb{C} \times \mathbb{R}^{4n+1} \rightarrow \mathbb{R}^{2n+1} \), defined by
\[
r(\zeta, t, \alpha) \overset{\text{def}}{=} \left( \bar{\rho}^1(t, \zeta^{-1} \cdot \alpha), \ldots, \bar{\rho}^n(t, \zeta^{-1} \cdot \alpha) \right).
\] (3.2)
Here, the product \( \zeta^{-1} \cdot \alpha \) is as in (2.8). By definition, a disk \( f : \overline{\Delta} \rightarrow \overline{D} \subset \mathbb{R}^{2n} \) is stationary if and only if there exists \( \hat{f} \in (C^{\alpha,\epsilon}(\Delta); \mathbb{C}^{2n}) \) and \( \lambda \in C^{(\partial \Delta; \mathbb{R})} \) so that
\[
\begin{align*}
\bar{J} f(\zeta) & = 0, \quad \zeta \in \Delta \\
r(\zeta, \lambda(\zeta), \hat{f}(\zeta)) & = 0, \quad \zeta \in \partial \Delta
\end{align*}
\] (3.3)
where \( \bar{J} \) is as in (2.1). \( \bar{J} \) is stationary if and only if there exists \( \hat{f} \in (C^{\alpha,\epsilon}(\Delta); \mathbb{C}^{2n}) \) and \( \lambda \in C^{(\partial \Delta; \mathbb{R})} \) so that
\[
\begin{align*}
\bar{J} f(\zeta) & = 0, \quad \zeta \in \Delta \\
r(\zeta, \lambda(\zeta), \hat{f}(\zeta)) & = 0, \quad \zeta \in \partial \Delta
\end{align*}
\] (3.3)

The differential problem (3.3) belongs to a class often called of \textit{generalized Riemann-Hilbert problems} (see f.i. \cite{16}, Ch. VII).

3.2. 
\textbf{Stability under small deformations of the data.} Consider a fixed almost complex structure \( J = J_o \), a point \( x_0 \in D(\subset \mathbb{R}^{2n}) \) and a vector \( v_o \in T_{x_0}D \simeq \mathbb{R}^{2n} \) and denote by \( \mathcal{R}(J_o, x_0, v_o) = (\mathcal{R}_1, \ldots, \mathcal{R}_5) \) the operator from \( C^{\alpha,\epsilon}(\Delta; \mathbb{C}^{2n}) \times C^{(\partial \Delta; \mathbb{R})} \times \mathbb{R}^{n} \) into \( C^{\alpha-1,\epsilon}(\Delta; \mathbb{C}^{2n}) \times C^{(\partial \Delta; \mathbb{R})} \times \mathbb{C}^{n} \times \mathbb{C}^n \times \mathbb{R} \) with components \( \mathcal{R}_i \) defined by
\[
\begin{align*}
\mathcal{R}_1(\hat{f}, \lambda, \mu) & \overset{\text{def}}{=} \overline{\mathcal{J}_a} \hat{f}, \quad \mathcal{R}_2(\hat{f}, \lambda, \mu) \overset{\text{def}}{=} r(\zeta, \lambda(\zeta), \hat{f}(\zeta)) \\
\mathcal{R}_3(\hat{f}, \lambda, \mu) & \overset{\text{def}}{=} |(\hat{f})|_{\zeta=0} - x_0, \quad \mathcal{R}_4(\hat{f}, \lambda, \mu) \overset{\text{def}}{=} \pi(\hat{f}) \left( \left. \frac{\partial}{\partial x} \right|_{\zeta=0} \right) - \mu v_o \\
\mathcal{R}_5(\hat{f}, \lambda, \mu) & \overset{\text{def}}{=} \hat{f} \left( \pi(\hat{f}) \left( \left. \frac{\partial}{\partial x} \right|_1 \right) \right) - 1
\end{align*}
\] (3.4)

Notice also that, by Hopf’s Lemma and Lemma 2.4 (ii), for any stationary disk, there exists a stationary lift satisfying \( \hat{f} \left( \pi(\hat{f}) \left( \left. \frac{\partial}{\partial x} \right|_1 \right) \right) = 1 \). So, by the previous section, the existence of a stationary disk \( f : \Delta \rightarrow D \) with \( f(0) = x_o \) and \( f \left( \left. \frac{\partial}{\partial x} \right|_0 \right) \in \mathbb{R}^{2n} \) is equivalent to the existence of a solution to
\[
\mathcal{R}(J_o, x_0, v_o)(\hat{f}, \lambda, \mu) = 0.
\] (3.5)

Let \((\hat{f}_o, \lambda_o, \mu_o)\) be solution of (3.5) and \( \mathcal{R}(J_o, x_0, v_o)(\hat{f}_o, \lambda_o, \mu_o) \overset{\text{def}}{=} \mathcal{R}(J_o, x_0, v_o)(\hat{f}_o, \lambda_o, \mu_o) \) the linearized operator at \((\hat{f}_o, \lambda_o, \mu_o)\) determined by \( \mathcal{R}(J_o, x_0, v_o) \). Now, by the Implicit Function Theorem (see e.g. \cite{12}), when \( \mathcal{R} \) is invertible, there exists a solution to the problem \( \mathcal{R}(J_t, x_t, v_t)(\hat{f}, \lambda, \mu) = 0 \) for any smooth deformation \((J_t, x_t, v_t)\) of \((J_o, x_o, v_o)\) for \( t \) sufficiently small \( t \) and \( \dim \ker \mathcal{R}(J_o, x_0, v_o)(\hat{f}_o, \lambda_o, \mu_o) \) is equal to the dimension of the solutions space. This motivates the following:

\textbf{Definition 3.1.} Let \( f_o : \overline{\Delta} \rightarrow \overline{D} \) be a stationary disk of \((D, J_o)\) with \( x_o = f(0) \) and \( v_o = (\hat{f}) \left( \left. \frac{\partial}{\partial x} \right|_{\zeta=0} \right) \). We call \( \partial D \) a \textit{good boundary for} \((J_o, f_o)\)
if there is a lift $\tilde{f}_o$ of $f_o$ and a function $\lambda_o$ so that $(\tilde{f}_o, \lambda_o, 1)$ is a solution to (3.5) and the linearized operator $\mathcal{R} = \mathcal{R}_{(J_o,x_o,v_o;\tilde{f}_o,\lambda_o,1)}$ is invertible.

The Implicit Function Theorem and previous remarks brings immediately to the next proposition. In the statement, we denote by $g$ a fixed Riemannian metric $g = g_{ij} dx^i \otimes dx^j$ on a neighborhood of $\overline{D}$ and by $g^\ast = g_{ij} dx^i \otimes dx^j + g^3 dp_1 \otimes dp_j$ the corresponding Riemannian metric on $T^*M$. We also set

$$||J - J'||_D^{(1)} = \sup_{x \in \overline{D}, v \in T(T_M^* \cdot M)} \frac{\|\mathcal{R}(v) - \mathcal{R}'(v)\|}{\|v\|_g^\ast},$$

where $\|\cdot\|_g^\ast$ is the norm function determined by $g^\ast$. The topology determined by the norm $|| \cdot ||_D^{(1)}$ is clearly independent on the choice of $g$.

**Proposition 3.2.** Let $f_o : \overline{D} \rightarrow \overline{D}$ be a stationary disk of $D \subset (M, J_o)$ with $x_o = f_o(0)$ and $v_o = f_o^\ast \left( \frac{\partial}{\partial z} |_{z=0} \right)$. If $\partial D$ is a good boundary for $(J_o, f_o)$, there exists a neighborhood $V \subset D$ of $x_o$, a neighborhood $W \subset T D$ of $v_o$, with $\pi(W) = V \subset D$ and a real number $\varepsilon > 0$ so that, for any $x \in V$, $v \in W$ and $||J - J_o||_D^{(1)} < \varepsilon$, there exists a unique stationary disk $f$ of $(D, J)$ so that

$$f(0) = x, \quad f^\ast \left( \frac{\partial}{\partial x} |_{z=0} \right) = \mu v \quad \text{for some} \quad \mu \neq 0. \quad (3.7)$$

The disk $f$ depends differentially on $x$, $v$ and $J$ and, given $m_o > 0$, one can choose $\varepsilon$, $W$ and $V = \pi(W)$ so that $\sup_{\xi \in \Sigma} \text{dist}_g(f(\xi), f_o(\xi)) < m_o$.

### 3.3. Foliations of circular type and their stability.

#### 3.3.1. Blow-up of an almost complex domain at one point.

Let $x_o$ be a point of the almost complex manifold $(M, J)$ and $\xi = (z^i) : U \rightarrow \mathbb{C}^n$ a system of complex coordinates with

$$\xi(x_o) = 0, \quad \xi^s(J|_{x_o}) = J_{\text{st}}|_{x_o}. \quad (3.8)$$

Consider the blow up $\pi : \tilde{U} \rightarrow \xi(U) \subset \mathbb{C}^n$ of $\xi(U)$ at $0$, i.e. the submanifold of $\mathbb{C}^n \times \mathbb{C}^{P^m-1}$ defined by $\tilde{U} = \{ (z, [w]) : z \in [w], z \in U \} \subset \mathbb{C}^n \times \mathbb{C}^{P^m-1}$. The standard projection $\pi(z, [w]) = z$ composed with $\xi^{-1}$ determines a diffeomorphism between $\tilde{U \setminus \pi^{-1}(0)}$ and $U \setminus \{0\}$ that we use to glue $\tilde{U}$ with $M \setminus \{x_o\}$ and obtain a manifold $\tilde{M}$ that we call blow up of $(M, J)$ at $x_o$.

At a first glance, this construction seems to depend on the choice of the complex coordinates $\xi = (z^i)$. But indeed the real manifold structure of $\tilde{M}$ depends only on the linear map $J_{x_o} : T_{x_o}M \rightarrow T_{x_o}M$. This fact is a direct consequence of the following simple lemma.

**Lemma 3.3.** Consider two sets of complex coordinates $\xi = (z^i)$ and $\xi' = (z'^i)$ on $U$ satisfying $\xi(x_o) = \xi'(x_o) = 0$ and $\xi^s(J_{x_o}) = \xi'^s(J_{x_o}) = J_{\text{st}}|_{x_o}$. Then the diffeomorphism $\tilde{\phi} = \pi^{-1} \circ \xi' \circ \xi^{-1} \circ \pi$ of $\tilde{U \setminus \pi^{-1}(0)}$ into itself admits a unique smooth extension on $\tilde{U}$. It follows that the blow up $\tilde{M}$, defined using
the chart $\xi = (z^i)$, is naturally diffeomorphic to the one constructed using
the chart $\xi' = (z'^i)$.

**Proof.** By construction, the map $\varphi = \xi' \circ \xi^{-1}$ is so that $\varphi_{*0} \circ J_{st} = J_{st} \circ \varphi_{*0}$
and hence it is of the form

$$\varphi(z) = \psi(z) + g(z)$$

where $\psi$ is the $\mathbb{C}$-linear map $\psi = \varphi_{*0} : \mathbb{C}^n \to \mathbb{C}^n$ and $g : U \to U$ is an
infinitesimal of the second order in $|z|$. Since

$$\bar{\varphi}(z, [z]) = (\pi^{-1} \circ \varphi \circ \pi)(z, [z]) = (\psi(z) + g(z), [\psi(z) + g(z)]) ,$$
an explicit computation in coordinates shows that $\bar{\varphi}$ extends smoothly on
$\pi^{-1}(0) \subset \tilde{U}$ by setting $\bar{\varphi}(0, [v]) \overset{\text{def}}{=} (0, [\psi(v)])$ for any $[v] \in \mathbb{CP}^{n-1}$.

3.3.2. **Foliations of circular type.** Let $D$ be a smoothly bounded, strongly
pseudoconvex domain in $(M, J)$ and denote also by $\overline{D} \subset \tilde{M}$ the blow up of
$D$ at a point $x_0$ as defined in the previous section. For any stationary disk
$f : \overline{\Delta} \to \overline{D}$ with $f(0) = x_0$ and $f_*(\frac{\partial}{\partial x_0}) = v$ there exists a unique map
$\bar{f} : \overline{\Delta} \to \overline{D}$ so that $\pi \circ f(\zeta) = f(\zeta)$ for any $\zeta \neq 0$. In fact, if we identify $\overline{D}$
with a domain in $\tilde{U} \subset \mathbb{C}^n \times \mathbb{CP}^{n-1}$ by means of a chart like in (3.8), the
lifted map $\bar{f}$ is of the form

$$\bar{f}(\zeta) = \begin{cases} (f(\zeta), [f(\zeta)]) & \text{when } \zeta \neq 0 , \\ (0, [v]) & \text{when } \zeta = 0 . \end{cases} \tag{3.10}$$

Since $f$ is $J$-holomorphic (and hence $f_*(J_{st}|0) = J_{st}|0$), we may write

$$f(\zeta) = h(\zeta) + g(\zeta)$$

for some holomorphic disk $h : \overline{\Delta} \to \tilde{U} \subset \mathbb{C}^n$ and a smooth map $g : \overline{\Delta} \to \tilde{U}$
which is infinitesimal of second order in $|\zeta|$. Using this, one can check that
$\bar{f}$ is smooth also at $0$. We call $\bar{f}$ the smooth lift of $f$ at $\overline{D}$.

**Definition 3.4.** Let $x_0 \in D$ and $\overline{D}$ as above and denote by $\mathcal{F}(x_0)$ the family
of all stationary disks of $D$ with $f(0) = x_0$. We call $\mathcal{F}(x_0)$ foliation of circular
type of the pointed domain $(D, x_0)$ if the following conditions are satisfied:

i) for any $v \in T_{x_0}D$, there exists a unique disk $f^{(v)} \in \mathcal{F}(x_0)$ such that

$$f^{(v)}_*(\frac{\partial}{\partial x_0}|0) = \mu \cdot v$$

for some $0 \neq \mu \in \mathbb{R}$;

ii) previous an identification $(T_{x_0}D, J_{st}) \simeq (\mathbb{C}^n, J_{st})$, the map

$$\exp : \overline{B}^n \subset \overline{\mathbb{C}}^n \to \overline{D} , \quad \exp(v, [v]) \overset{\text{def}}{=} \overline{\varphi}(\zeta(\bar{v})) , \tag{3.12}$$

between the blow up at $0$ of $\mathbb{B}^n \subset \mathbb{C}^n$ and the blow up of $D$ at $x_0$ is smooth
with a smooth extension up to the boundary, which induces a
diffeomorphism between the boundaries $\exp|_{\partial \mathbb{B}^n} : \partial \mathbb{B}^n \to \partial D$.

If $\mathcal{F}(x_0)$ is a foliation of circular type, we call $x_0$ center of the foliation and
$D$ a domain of circular type w.r.t. to $J$. 

3.3.3. Stability under small deformations of foliations of circular type.

**Proposition 3.5.** Let $D$ be of circular type w.r.t. to $x_o$, and with center $x_o$. If $\partial D$ is a good boundary for $(o, f_o)$ for any stationary disk $f_o \in \mathcal{F}(x_o)$, then there exists $\varepsilon > 0$ and an open neighborhood $U \subset D$ of $x_o$ so that for any $J$ with $\|J - J_o\|_D^{(1)} < \varepsilon$ and any $x \in U$, the point $x$ is a center of a foliation of circular type of $D$ w.r.t. the almost complex structure $J$.

**Proof.** Using a system of coordinates $\xi = (x^i)$ on a neighborhood $W$ of $x_o$, let us identify $W$ with an open subset of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and its tangent space with $TW \cong \mathbb{R} \times \mathbb{R}^{2n} \subset \mathbb{R}^{4n}$. Pick also the same Euclidean inner product $\langle , \rangle$ on all tangent spaces in $TW \cong \mathbb{R} \times \mathbb{R}^{2n}$. By definitions, for any $v_o \in S^{2n-1} = \{ v \in T_{x_o}M : \langle v, v \rangle = 1 \}$, there is a unique stationary disk $f_o \in \mathcal{F}(x_o)$ with $f_o (\frac{\partial}{\partial \xi_0}) = \mu \cdot v_o$ for some $\mu \neq 0$.

By Proposition 3.2, there exists a neighborhood $U(v_o)$ of $x_o$, a neighborhood $V(v_o) \subset S^{2n-1}$ and $\varepsilon(v_o) > 0$, so that, for any $y \in U(v_o)$, $v \in V(v_o) \subset T_yM \simeq T_{x_o}M \simeq \mathbb{R}^{2n}$ and $J$ with $\|J - J_o\|_D^{(1)} < \varepsilon(v_o)$, there exists a unique disk $\widetilde{f}$, which is stationary for $D$ w.r.t. $J$, passing through $y$ and with $\widetilde{f}_o (\frac{\partial}{\partial \xi_0})$ parallel to $v$. By compactness of $S^{2n-1}$, there exists a finite number of vectors $v_1, \ldots, v_N \in S^{2n-1}$ so that the corresponding open sets $V(v_i) \subset S^{2n-1}$ give an open covering of $S^{2n-1}$. We conclude that, for any point $y \in \tilde{U} = \bigcap_{i=1}^N U(v_i)$, $\|J - J_o\|_D^{(1)} < \min_i \varepsilon(v_i)$ and $v \in T_yM$, there exists a unique disk passing through $y$, which is stationary w.r.t. $J$ and with $\widetilde{f}_o (\frac{\partial}{\partial \xi_0})$ parallel to $\frac{v}{\|v\|}$ in $S^{2n-1}$. In particular, the disks in $\mathcal{F}(y)$, $y \in \tilde{U}$, satisfy Definition 3.4(i).

Consider now the map $\exp : \overline{B^n} \to \overline{D}$ in (3.12). By Proposition 3.2 it is smooth and depends smoothly on $y$ and $J$. Moreover, if $J = J_o$ and $y = x_o$, it is a diffeomorphism between manifolds with boundaries. Hence, there exists $\tilde{U} \subset \tilde{U}$ and $\varepsilon < \min_i \varepsilon(v_i)$ so that $\exp_o$ is invertible at all points of $\overline{B^n}$ whenever $y \in \tilde{U}$ and $\|J - J_o\|_D^{(1)} < \varepsilon$. In these cases, $\exp$ is a local homeomorphism from the compact set $\overline{B^n}$ to $\overline{D}$ and hence is a covering map of $\overline{D}$. Being $\overline{B^n}$ simply connected, it is a diffeomorphism, i.e. also (ii) of Definition 3.4 holds true. \(\square\)

4. Conditions that force a boundary to be “good”

In this section we are going to prove a result (Theorem 4.6), which provides a condition for the existence of foliations by stationary disks of a pointed domain $(D, x_o)$ endowed with a small deformation of the standard complex structure. An immediate consequence of this and of the contents of §3 is represented by the following theorem.
Theorem 4.1. Let $D \subset M$ be a smoothly bounded, strongly pseudoconvex domain in an almost complex manifold $(M, J)$. If there is a local diffeomorphism $\varphi : U \subset M \to \mathbb{C}^n$, so that $\hat{D} = \varphi(D)$ is a strictly linearly convex domain $\hat{D} \subset \mathbb{C}^n$ and $\varphi_* J$ is sufficiently close to $J_{st}$ in $C^1$-norm, then $D$ is a domain of circular type w.r.t. $J$ and any point is a center.

Roughly speaking, this shows that if one defines a suitable topology on the set of almost complex domains admitting foliations of circular type, such space contains a whole open neighborhood of the class of strictly linearly convex domains of $\mathbb{C}^n$.

4.1. The linearized operator $\mathcal{R} = \mathcal{R}_{(f_o, \lambda_o, \mu_o)}$. First of all, we want to determine an explicit expression for the tangent map $\mathcal{R} = \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$ at $(f_o, \lambda_o, \mu_o)$ of the operator (3.4). For this, recall that, being $\hat{f}_o : \Sigma \to T^*M$ a $J$-holomorphic disk, one can always find a system of complex coordinates $(z^i)$ on a neighborhood $W$ of $\hat{f}_o(\Sigma)$, in such a way that, identifying $W$ with an open subset of $\mathbb{C}^{2n}$, one has $|J_z| = J_{st}|_z$ at any $z \in f(\Sigma)$. Moreover, by Hopf lemma and being the defining function $\rho$ strongly plurisubharmonic, we have that $d\rho(f_\ast(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})) = d\rho(\text{Re} (z^1 \frac{\partial}{\partial z^1})) \neq 0$ at all points of $f(\partial \Sigma)$. In these coordinates, the tangent map of $\mathcal{R}_1 = \overline{\partial}_{f_o}$ at $\hat{f}_o$ is

$$\mathcal{R}_1(\hat{h}) = \frac{\partial \hat{h}}{\partial \zeta} + \frac{1}{2i} D(\mathbb{J} - J_{st})_{f_o} \cdot \hat{h},$$

(4.1)

where $D(\mathbb{J} - J_{st})_{f_o}$ is the real differential of the matrix valued function $\zeta \mapsto (\mathbb{J} - J_{st})_{f_o(\zeta)}$. In matrix notation, $D(\mathbb{J} - J_{st})_{f_o} \cdot \hat{h}$ can be written as

$$\left(D(\mathbb{J} - J_{st})_{f_o} \cdot \hat{h}\right)_\zeta = A(\zeta) \cdot \hat{h}(\zeta) + B(\zeta) \cdot \overline{\hat{h}}(\zeta),$$

for some $A, B : \Sigma \to M_{n \times n}(\mathbb{C})$ and $\mathcal{R}_1$ assumes the form

$$\mathcal{R}_1(\hat{h}) = \frac{\partial \hat{h}}{\partial \zeta} + A \cdot \hat{h} + B \cdot \overline{\hat{h}}.$$ 

(4.2)

Consider now the tangent map $\mathcal{R}_2$. By previous remarks, the defining function $\tilde{\rho} = (\tilde{\rho}^1, \ldots, \tilde{\rho}^{2n+1})$ in (3.1) is locally equivalent to

$$\tilde{\rho}(t, \alpha) = \left(\rho(\alpha), t - \frac{\alpha \left(\text{Re} (z^1 \frac{\partial}{\partial z^1})\right)}{d\rho \left(\text{Re} (z^1 \frac{\partial}{\partial z^1})\right)|_{\tilde{\rho}(\alpha)}}\right).$$
where \( g : \mathcal{W} \subset T^* M_{|U} \to \mathbb{R}^{2n} \) is the defining function for \( \mathcal{N}_* \) obtained by replacing \( t = \frac{\alpha (\text{Re}(z^1 \partial_{\bar z^1}))}{\partial \rho (\text{Re}(z^1 \partial_{\bar z^1}))}_{|\pi(a)} \) in all places of \( \tilde{\rho} \). If we set

\[
\tilde{r}(\zeta, t, \alpha) \overset{\text{def}}{=} \left( g(\zeta, \alpha), t - \frac{\alpha (\text{Re}(z^1 \partial_{\bar z^1}))}{\partial \rho (\text{Re}(z^1 \partial_{\bar z^1}))}_{|\pi(a)} \right), \quad \text{with } g(\zeta, \alpha) \overset{\text{def}}{=} g(\zeta^{-1} \cdot \alpha) \quad (4.3)
\]

we see that \( \mathcal{R}_2 \) is equivalent to the tangent map of the operator

\[
\mathcal{R}_2(\hat{f} \big|_{\partial \Delta}, \lambda) = \left( g(\cdot, \hat{f}(\cdot)), \lambda - \frac{\hat{f}(\cdot) \left( \text{Re}(z^1 \partial_{\bar z^1}) \right)}{\partial \rho (\text{Re}(z^1 \partial_{\bar z^1}))}_{|\pi(\hat{f}(\cdot))} \right)
\]

and hence of the form

\[
\mathcal{R}_2(\hat{h}, \tau) = \left( 2 \text{Re}(G \cdot \hat{h}|_{\partial \Delta}), \tau - \text{g}(\hat{h}) \right) \quad (4.4)
\]

where \( g \) is obtained by linearization of the map \( \hat{f} \mapsto \frac{\hat{f}(\cdot) \left( \text{Re}(z^1 \partial_{\bar z^1}) \right)}{\partial \rho (\text{Re}(z^1 \partial_{\bar z^1}))}_{|\pi(\hat{f}(\cdot))} \) and \( G \) is the matrix valued map on \( \partial \Delta \) defined by

\[
G(\zeta) = \begin{pmatrix}
\frac{\partial \rho^1}{\partial z^1}(\zeta, \hat{f}(\zeta)) & \ldots & \frac{\partial \rho^1}{\partial z^n}(\zeta, \hat{f}(\zeta)) \\
\vdots & \ddots & \vdots \\
\frac{\partial \rho^n}{\partial z^1}(\zeta, \hat{f}(\zeta)) & \ldots & \frac{\partial \rho^n}{\partial z^n}(\zeta, \hat{f}(\zeta))
\end{pmatrix}, \quad \zeta \in \partial \Delta \quad (4.5)
\]

By Theorem 2.5, \( \mathcal{N}_* \) is totally real w.r.t. \( \mathcal{J} \) and hence, by our choice of the coordinates, it is totally real also w.r.t. \( J_{st} \) on a neighborhood of \( \hat{f}(\partial \Delta) \). This implies that

\[
\det (G(\zeta)) \neq 0, \quad \text{for any } \zeta \in \partial \Delta . \quad (4.6)
\]

Finally, the maps \( \mathcal{R}_3, \mathcal{R}_4 \) and \( \mathcal{R}_5 \) are easily seen to be (here \( h \overset{\text{def}}{=} \pi \circ \hat{h} \))

\[
\mathcal{R}_3(\hat{h}) = h(0), \quad \mathcal{R}_4(\hat{h}, \sigma) = \left. \frac{\partial h}{\partial x} \right|_{\zeta = 0} - \sigma v_o, \quad \mathcal{R}_5(\hat{h}) = \left. \hat{f}_o \left( \frac{\partial h}{\partial x} \right|_1 \right) + \hat{h} \left( \frac{\partial f_o}{\partial x} \right|_1) .
\]

4.2. **The operator** \( R_{A,B,G} = (\mathcal{R}_1, \mathcal{R}_2) \). Consider the operator

\[
R_{A,B,G} = (\mathcal{R}_1, \mathcal{R}_2) = \left( \frac{\partial \hat{h}}{\partial \zeta} + A \cdot \hat{h} + B \cdot \overline{\hat{h}}, 2 \text{Re}(G \cdot \hat{h}) \right),
\]

which is a well-known Fredholm operator related with the generalized Riemann-Hilbert problems. In the next theorem, we recall some information that will be used in the sequel (see e.g. Thm. 3.2.5, Thm. 3.3.1 in [27]).

**Theorem 4.2.** If \( G \) satisfies (4.6), the operator \( R_{A,B,G} \) is Fredholm with index \( \nu = 2n - \frac{1}{i \pi} \int_{\partial \Delta} d \arg(\det(G)) \) and hence is surjective if and only if

\[
\dim \ker R_{A,B,G} = 2n - \frac{1}{i \pi} \int_{\partial \Delta} d \arg(\det(G)) . \quad (4.7)
\]
Next, we need to recall a lemma due to Globevnik and some of its direct consequences, which give a way to establish the surjectivity of $R_{A,B,G}$ in case of integrable complex structures. But in order to state them, we first need to recall the definition of “canonical system” (see e.g. [8]). In what follows, for any holomorphic function $g : U \subset \mathbb{C} \to \mathbb{C}^N$ on a neighborhood of $\infty$ and with at most one pole at $\infty$, we call order (zero of) $g$ the integer $k$ such that $g = \frac{1}{z^k} g_0$ for some $g_0$ which is holomorphic at $\infty$ and with $g_0(\infty) \neq 0$.

**Definition 4.3.** Given $A \in \mathcal{C}^\epsilon(\partial \Delta, \text{GL}(N, \mathbb{C}))$, with $\epsilon \in ]0,1[$, consider the problem consisting of finding a continuous map $\Psi^+ : \overline{\Delta} \to \mathbb{C}^N$, holomorphic on $\Delta$, and a continuous map $\Psi^- : \mathbb{C} \setminus \Delta \to \mathbb{C}^N$, holomorphic on $\overline{\mathbb{C} \setminus \Delta}$ and with at most one pole at $\infty$, so that

$$\Psi^+(\zeta) = A(\zeta) \cdot \Psi^-(\zeta) \quad \zeta \in \partial \Delta .$$

A canonical system of $A$ is any collection of solutions $\Phi_j = (\Phi^+_j, \Phi^-_j)$, $1 \leq j \leq N$, of the problem (4.8) so that

i) $\Phi^+_j(\zeta) = [\Phi^+_1(\zeta), \ldots, \Phi^+_N(\zeta)]$ is in $\text{GL}(N, \mathbb{C})$ for any $\zeta \in \overline{\Delta}$;

ii) $\Phi^-_j(\zeta) = [\Phi^-_1(\zeta), \ldots, \Phi^-_N(\zeta)]$ is in $\text{GL}(N, \mathbb{C})$ for any $\zeta \in \mathbb{C} \setminus \Delta$;

iii) the order $k$ of $\det \Phi^-_j$ at $\infty$ is equal to the sum of the orders $k_j$ of the columns $\Phi^-_j$.

If $\{\Phi_j = (\Phi^+_j, \Phi^-_j)\}$ is a canonical system of $A$, the orders $k_j$ of the $\Phi^-_j$’s are called partial indices of $A$. The sum $k = \sum k_j$ is called total index of $A$.

An important fact is that, up to reordering, the partial indices and the total index depend only on $A$ and not on the considered canonical system. We may now recall the following lemma by Globevnik, which can be considered as a corollary of N. P. Vekua’s factorization theorem ([25]).

**Lemma 4.4.** ([8], Lemma 5.1) Let $L \in \mathcal{C}^\epsilon(\partial \Delta, \text{GL}(N, \mathbb{C}))$, with $\epsilon \in ]0,1[$. Then there is a map $\Theta : \overline{\Delta} \to \text{GL}(N, \mathbb{C})$ in $\mathcal{H}^\epsilon(\overline{\Delta}, \mathbb{C}^{N^2})$, such that

$$L(\zeta) \cdot \overline{L(\zeta)}^{-1} = \Theta(\zeta) \cdot \Lambda(\zeta) \cdot \overline{\Theta^{-1}(\zeta)}$$

with $\Lambda(\zeta) = \begin{pmatrix} \zeta^{k_1} & 0 & \ldots & 0 \\ 0 & \zeta^{k_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \zeta^{k_N} \end{pmatrix}$ (4.9)

for any $\zeta \in \partial \Delta$, where $k_1, \ldots, k_N$ are the partial indices of $A(\cdot) \overset{\text{def}}{=} L(\cdot) \cdot \overline{L(\cdot)}^{-1}|_{\partial \Delta}$.

The integers $k_i$ of the previous lemma and the sum $k = \sum_{i=1}^N k_i$ are the same for all maps $L' = M|_{\partial \Delta} \cdot L$ with $M : \overline{\Delta} \to \text{GL}(N, \mathbb{C})$ in $\mathcal{H}^\epsilon(\overline{\Delta}, \mathbb{C}^{N^2})$. They are called partial indices and total index of $L$, respectively.

Consider now the map $G(\zeta)$ in (4.5) and let $\Theta^G$ be a map that gives a decomposition (4.9) for $L(\zeta) = G^{-1}(\zeta)$. We set

$$A^G \overset{\text{def}}{=} (\Theta^G)^{-1} \cdot A \cdot \Theta^G , \quad B^G \overset{\text{def}}{=} (\Theta^G)^{-1} \cdot B \cdot \Theta^G . \quad (4.10)$$
It is immediate to realize that the linear map
\[ \tilde{h} \mapsto -\tilde{h} = (\Theta G)^{-1} \cdot \tilde{h} \]
is an isomorphism between \( \ker R_{A,B,G} \) the space of solutions of the problem
\[
\begin{cases}
\bar{\partial} \tilde{h} + A G \cdot \tilde{h} + B G \cdot \tilde{h} = 0 , & \zeta \in \Delta \\
\tilde{h}^i(\zeta) = \zeta^k \tilde{h}^i(\zeta) , & 1 \leq i \leq 2n, & \zeta \in \partial \Delta
\end{cases}
\]
where the \( k_i \) are the partial indices of \( L = G^{-1} \).

**Lemma 4.5.** The operator \( R_{A,B,G} \) is surjective if and only if \( \dim \ker R_{A,B,G} = 2n + k \), with \( k = \sum_{i=1}^{2n} k_i \). Moreover, when \( A = B = 0 \), \( R_{0,0,G} \) is surjective if and only if \( k_i \geq -1 \) for any \( 1 \leq i \leq 2n \).

**Proof.** The first claim follows from Theorem 4.2 and from
\[
\dim \ker R_{A,B,G} = 2n - \frac{1}{i \pi} \int_{\partial \Delta} d \arg(\det(G))
\]
\[
= 2n + \frac{1}{2 \pi i} \int_{\partial \Delta} d \arg \left( \det(G^{-1} \cdot \bar{G}) \right)
\]
\[
= 2n + \frac{1}{\pi i} \int_{\partial \Delta} d \arg(\det(\Theta G)) + \sum_{i=1}^{2n} \frac{1}{2 \pi i} \int_{\partial \Delta} d \arg(\zeta^k)
\]
\[
= 2n + k
\]
where we used the fact that \( \det(\Theta G) \) is holomorphic and never zero in \( \Delta \).

Assume now that \( A = B = 0 \) and recall that the elements of \( \ker R_{0,0,G} \) are in natural correspondence with the elements \( \tilde{h} = (\tilde{h}^1, \ldots, \tilde{h}^{2n}) \in H^\varepsilon(\Delta, \mathbb{C}^{2n}) \) that solve (4.11) and hence of the form \( \tilde{h}^i(\zeta) = \sum_{\ell \geq 0} a_{\ell}^i \zeta^\ell \) with coefficients \( a_{\ell}^i \in \mathbb{C} \) so that the boundary conditions are satisfied, i.e.
\[
\begin{cases}
    a_{\ell}^i = 0 & \text{when } \ell \geq \max\{k_i + 1, 0\} \\
    a_{\ell}^i = a_{\ell-k_i}^i & \text{when } k_i \geq 0 \text{ and } 0 \leq \ell \leq k_i
\end{cases}
\]
From this, a simple check shows that \( \dim \ker R_{0,0,G} = \sum_{k_i \geq 0} (k_i + 1) \). Since
\[
2n + k = 2n - \sum_{k_i \leq -1} (|k_i| - 1) - (\# \{k_i \leq -1\}) + \sum_{k_i \geq 0} k_i =
\]
\[
= (\# \{k_i \geq 0\}) + \sum_{k_i \geq 0} k_i - \sum_{k_i \leq -1} (|k_i| - 1) = \sum_{k_i \geq 0} (k_i + 1) - \sum_{k_i \leq -1} (|k_i| - 1)
\]
it follows that \( \dim \ker R_{0,0,G} = 2n + k \) if and only if \( \sum_{k_i \leq -1} (|k_i| - 1) = 0 \), i.e. \( k_i \geq -1 \) for any \( 1 \leq i \leq 2n \). \( \square \)
4.3. The operator $\mathcal{R} = (R_{0,0,G}, R_3, R_4, R_5)$ for convex domains in $\mathbb{C}^n$.

**Theorem 4.6.** Let $D$ be a domain in $(\mathbb{C}^n, J_{st})$, with smooth boundary and let $f_0 : \overline{\Delta} \to \overline{D}$ a stationary disk $D$. If there is a neighborhood $\mathcal{U}$ of $f_0(\overline{D})$ where $\mathcal{U} \cap \overline{D}$ is strictly linearly convex, then $\partial D$ is good for $(J_{st}, f_0)$.

**Proof.** We first need the following:

**Lemma 4.7** ([17], Prop. 2.36, Thm. 2.45). Let $f_0 : \Delta \to \overline{D}$ as above. Then there exists a system of complex coordinates $(z^i)$ and a defining function $\rho$ for $\partial D$ on a neighborhood $\mathcal{V}$ of $f(\Delta)$, such that $f_0(\zeta) = (\zeta, 0, \ldots, 0)$ and

$$\rho = -1 + |z|^2 + \sum_{\alpha, \beta=2}^n \delta_{\alpha \beta} z^\alpha \overline{z}^\beta + \text{Re} \left( \sum_{\alpha, \beta=1}^n B_{\alpha \beta} z^\alpha \overline{z}^\beta \right) + r(z^1, \ldots, z^n) \quad (4.13)$$

with $r$ smooth function so that $|r(z)| \leq c|z|^3$ for some $c > 0$ for all $z \in \mathcal{V}$.

Secondly, we need the following lemma, from which the theorem will follows almost immediately. There, we denote by $(z^i)$ the coordinates in previous lemma and by $(z^i, w_i)$ the associated complex coordinates for $T^*\mathbb{C}^n$ (see [21]).

**Lemma 4.8.** Let $\mathcal{R} = (R_{0,0,G}, R_3, R_4, R_5)$ be the linear operator defined in [4.7] using the coordinates $(z^i, w_j)$. Then:

i) The partial indices of $G^{-1}$ are $k_1 = 2$, $k_2 = 0$ and $k_j = 1$ for all $j \geq 2$. In particular, $R_{0,0,G}$ is surjective and $\dim \ker R_{0,0,G} = 4n+1$.

ii) The restrictions of $R_3, R_4$ on $\ker R_{0,0,G}$ are surjective.

**Proof.** (i) If $\rho$ is the defining function $(4.13)$, the components of the function $\varrho(\zeta, \alpha)$, defined in (4.3), are (up to multiplication by a nowhere vanishing smooth function)

$$\varrho^1 = -1 + |z|^2 + \sum_{\alpha, \beta=2}^n \delta_{\alpha \beta} z^\alpha \overline{z}^\beta + \text{Re} \left( \sum_{\alpha, \beta=1}^n B_{\alpha \beta} z^\alpha \overline{z}^\beta \right) + O(|z|^3)$$

$$\varrho^2 = i \left\{ 2|z|^2 (\zeta^{-1} w_1 - \overline{\zeta^{-1} w_1}) - (z^1 \zeta^{-1} w_1 + z^1 \overline{\zeta^{-1} w_1})(\overline{z^1} - z^1) \right\} + O(|z|^2)$$

$$\varrho^{2n-1} = 2|z|^2 (\zeta^{-1} w_\alpha + \overline{\zeta^{-1} w_\alpha}) - (z^1 \zeta^{-1} w_1 + \overline{z^1 \zeta^{-1} w_1}) \left\{ (\delta_{\alpha \beta} z^\beta + B_{\alpha \beta} z^\beta) + (\overline{\delta_{\alpha \beta} z^\beta + B_{\alpha \beta} z^\beta}) \right\} + O(|z|^2)$$

$$\varrho^{2n} = i \left\{ 2|z|^2 (\zeta^{-1} w_\alpha - \overline{\zeta^{-1} w_\alpha}) - (z^1 \zeta^{-1} w_1 + \overline{z^1 \zeta^{-1} w_1}) \left\{ (\delta_{\alpha \beta} z^\beta + B_{\alpha \beta} z^\beta) - (\delta_{\alpha \beta} z^\beta + B_{\alpha \beta} z^\beta) \right\} \right\} + O(|z|^2)$$

with $2 \leq \alpha \leq n$. Hence, the matrix $(4.5)$ is (up to reordering of columns)

$$G(\zeta) = \begin{pmatrix} G_1(\zeta) & 0 \\ 0 & G_2(\zeta) \end{pmatrix}$$
In order to conclude, we only need to check that \( R \) or each of them, there is a unique stationary lift \( \hat{\xi} \). By Thm. 4.8 in [17], there exists a neighborhood \( \mathcal{W} \subset D \) of \( x_0 \) and a neighborhood \( \mathcal{W}' \subset \partial I_{x_0} \) of \( v_0 \), so that for any \( x \in \mathcal{W}, v \in \mathcal{W}' \) there exists exactly two stationary disks \( f^{(x,v_0)}, f^{(x,v)} : \Delta \to \overline{D} \) satisfying

\[
f^{(x,v_0)}(0) = x, \quad f^{(x,v)} \left( \frac{\partial}{\partial x} \bigg|_0 \right) = v_0, \quad f^{(x,v)}(0) = x_0, \quad f^{(x,v)} \left( \frac{\partial}{\partial x} \bigg|_0 \right) = v
\]

For each of them, there is a unique stationary lift \( \hat{f}^{(x,v_0)} \) and \( \hat{f}^{(x,v)} \) satisfying certain normalization conditions (i.e. so that \( \zeta \cdot \hat{f}^{(x,v_0)}(\zeta) \) and \( \zeta \cdot \hat{f}^{(x,v)}(\zeta) \) are the so-called dual maps - see [17], Def. 2.10). These lifts depend smoothly on the coordinates of the point \( x \) and the vector \( v \) and for any curves \( \gamma_t \in D \) and \( \gamma'_t \in T_{x_t} D \) with \( \gamma_0 = x_0 \) and \( \gamma'_0 = v_0 \), the 1-parameter families of stationary lifts \( \hat{f}_t \) and \( \hat{f}'_t \) are so that \( \hat{h}(\zeta) = \frac{d\hat{f}(\zeta)}{dt} \bigg|_{t=0} \) are in \( \ker R_{0,0,G} \). Moreover, by construction,

\[
R_3(\hat{h}) = \pi(\hat{h}(0)) = i_{v_0} \in \mathbb{C}^n, \quad R_4(\hat{h}, \sigma) = \frac{\partial(\pi \circ \hat{h})}{\partial x} \bigg|_0 - \sigma v_0 = i'_0 - \sigma v_0 \in \mathbb{C}^n.
\]

Since \( v_0 \) is transversal to \( \partial I_{x_0} \), by the arbitrariness of \( \gamma_t \) and \( \gamma'_t \) it follows that \( R_3|_{\ker R_{0,0,G}} \) and \( R_4|_{\ker R_{0,0,G}} \) are both surjective. \( \square \)

By the previous lemma, \( \dim \ker R_{0,0,G} \cap \ker R_3 \cap \ker R_4 = 1 \). So, in order to conclude, we only need to check that \( R_3|_{\ker R_{0,0,G} \cap \ker R_3 \cap \ker R_4} \) is surjective onto \( \mathbb{R} \) or, equivalently, that there is \( 0 \neq \hat{h} \in \ker R_{0,0,G} \cap \ker R_3 \cap \ker R_4 \) so that \( R_3(\hat{h}) = \hat{h}(1,0,\ldots,0) \neq 0 \). But an element of this kind is
given by 
\[ \hat{h}(\zeta) = \frac{d\varphi_t(f_0(\zeta))}{dt} \bigg|_0 = (\zeta, 1, 0, \ldots, 0), \]
where we denote by \( \varphi_t \) the diffeomorphism considered in Lemma 2.2 and the proof is concluded. □

**Remark 4.9.** Lemma 4.8 (i) corrects and generalizes a computation in [4], where, by a minor mistake, the partial indices of \( G^{-1} \) in case \( D = B^n \) are claimed to be all equal to 1.

5. Other non-singular foliations by stationary disks

5.1. **Foliations of horospherical type.** As before, \((M, J)\) is an almost complex manifold of dimension \(2n\). Let \( x_o \in M \) and consider a Riemannian metric \( <,> \) on a neighborhood \( \mathcal{U} \) so that \( <,> |_{x_o} \) is \( J \)-Hermitian. For instance, if \( \mathcal{U} \) is identified with an open subset of \( \mathbb{C}^n \) so that \( J|_{x_o} = J_{st}|_{x_o} \), we may assume that \( <,> \) is the standard Hermitian metric of \( \mathbb{C}^n \). Denote also by \( \nabla \) the Levi-Civita connection of \( <,> \).

**Definition 5.1.** Let \( f : \Delta \to M \) be a \( J \)-holomorphic disk, which is \( C^1 \) up to the boundary and with \( v_o = f^*(\frac{\partial}{\partial x}|_1) \neq 0 \). We call **parameter of tangency** at \( x_o = f(1) \) the real number
\[ p(f; x_o) \overset{\text{def}}{=} \left( \nabla v_o \left( f^* \left( \frac{\partial}{\partial x} \right) \right), Jv_o \right). \tag{5.14} \]

This number depends on the first order jet of \( <,> \) at \( x_o \), but if two \( J \)-holomorphic disks \( f, h \) are so that
\[ x_o = f(1) = h(1), \quad v_o = f^* \left( \frac{\partial}{\partial x}|_1 \right) = h^* \left( \frac{\partial}{\partial x}|_1 \right), \]
then their parameters of tangency are the same for a choice of \( <,> \) if and only if they are the same for any other choice of the metric. In fact, if we consider a new metric \( <,>', \) with Levi-Civita connection \( \nabla' \), then \( S = \nabla' - \nabla \) is a tensor field of type \((1,2)\) so that
\[ \left( \nabla_{v_o} \left( f^* \left( \frac{\partial}{\partial x} \right) \right) - \nabla_{v_o} \left( h^* \left( \frac{\partial}{\partial x} \right) \right) \right)|_1 = S(v_o, v_o - v_o) = 0 \]

Moreover, a simple computation shows that any disk \( h = f \circ \varphi \) where \( \varphi \in \text{Aut}(\Delta) \) with \( \varphi(1) = 1, \varphi'(1) = 1 \), satisfies
\[ \nabla_{v_o} \left( h^* \left( \frac{\partial}{\partial x} \right) \right) = \lambda J f^* \left( \frac{\partial}{\partial x} \right) = \lambda J v_o \quad \text{for some} \ \lambda \in \mathbb{R}. \]

Therefore, for any given \( \bar{\lambda} \), one can choose \( \varphi \) so that \( p(f \circ \varphi; x_o) = \bar{\lambda}. \) Moreover, \( p(f \circ \varphi; x_o) = p(f; x_o) \) if and only if \( \varphi = Id_\Delta \) and \( f = h. \)

Consider now a bounded, strictly convex domain in \((\mathbb{C}^n, J_{st})\) with smooth boundary and let \( x_o \in \partial D \) and \( \nu \) the outward unit normal to \( \partial D \) in \( x_o. \) By
For any \( v_o \in T_{x_o}M \) so that \( < \nu, v_o >> 0 \) and for any \( \lambda \in \mathbb{R} \), there exists a unique stationary disk \( f(v_o, \lambda) : \overline{x} \rightarrow \overline{D} \) so that
\[
f(v_o, \lambda)(1) = x_o, \quad f_*(\frac{\partial}{\partial x} |_1) = v_o, \quad p(f; x_o) = \lambda.
\]
(5.15)

If we denote by \( H_{x_o} = \{ v \in T_{x_o}C^n : < \nu, v >> 0 \} \subset T_{x_o}C^n \), we have that the exponential map \( \Phi^{(D, x_o)} : H_{x_o} \times (\Delta \setminus \{1\}) \rightarrow \overline{D} \setminus \{x_o\} \) defined by \( \Phi^{(D, x_o)}(v; \xi) = f(v, 0)(\xi) \) is a diffeomorphism.

We now consider the following definition. As before, \( D \) is a smoothly bounded, strictly pseudoconvex domain in the almost complex manifold \((M, J)\) and, for any given \( x_o \in \partial D \), we denote by \( \nu \) the outward unit normal to \( \partial D \) in \( x_o \) w.r.t. some Riemannian metric \(<,>\), which is \( J \) Hermitian at \( x_o \). Finally, for any real number \( a > 0 \), we denote by \( C(a) \) the open cone
\[
C(a) = \{ v \in T_{x_o}M : < v, \nu >> a \} \subset T_{x_o}M.
\]

**Definition 5.2.** For any \( x_o \in \partial D \) and \( a > 0 \), let \( G^{(x_o)} \) be the family of stationary disks \( f : \overline{x} \rightarrow \overline{D} \) with \( f(1) = x_o \) and by \( G^{(x_o, a)} \subset G^{(x_o)} \) the subfamily of disks with \( f_*(\frac{\partial}{\partial x} |_1) \in C(a) \). Denote also by \( D^{(x_o, a)} \subset D \) the union of all images of the disks in \( G^{(x_o, a)} \).

We say that \( G^{(x_o)} \) is a foliation of horospherical type for \( D \) (resp. \( G^{(x_o, a)} \) is a good foliation for \( D^{(x_o, a)} \)) if the following conditions are satisfied:

i) for any \( v \in T_{x_o}M \) so that \( < v, \nu >> 0 \) (resp. for any \( v \in C(a) \)) and for any \( \lambda \in \mathbb{R} \) there exists a unique \( f(v, \lambda) \in G^{(x_o)} \) so that
\[
f_*(v, \lambda) \left(\frac{\partial}{\partial x} |_1\right) = v, \quad p(f; x_o) = \lambda
\]
(5.16)

ii) the map \( \exp : \overline{B}^n \setminus \{y_0\} \rightarrow \overline{D} \setminus \{x_o\}, y_0 \overset{def}{=} (1, 0, \ldots, 0) \), defined by
\[
\exp(\Phi^{(B^n, y_0)}(v, \xi)) \overset{def}{=} f(v, 0)(\xi)
\]
(5.17)
is a diffeomorphism on \( B^n \) (resp. on \( B^n(y_0, a) \)), extends smoothly at all points of the closure, different from \( y_0 \), and induces an homeomorphism between the closures of the two domains.

If \( G^{(x_o, a)} \) with \( a > 0 \) is a good foliation for \( D^{(x_o, a)} \), we say that \( D^{(x_o, a)} \subset D \) is a good conical subdomain with vertex in \( x_o \). If \( G^{(x_o)} \) is a foliation of horospherical type, we say that \( x_o \) is a center at infinity for \( D \) and \( D \) is of horospherical type.

By the results in [6], any strictly convex domain \( D \) in \((C^n, J_{st})\) is a domain of horospherical type with center at infinity at any point of the boundary.

5.1.1. *Stability of foliations of horospherical type.* In analogy with [8, 2], let us consider the nonlinear operator \( \mathcal{R}'(J_o, x_o, v_o, \nu) = (\mathcal{R}'_1, \ldots, \mathcal{R}'_6) \) from \( C^{\alpha, \xi}(\overline{x}, C^{2n}) \times C^\epsilon(\partial \Delta; \mathbb{R}) \) into \( C^{\alpha-1, \xi}(\overline{x}, C^{2n}) \times C^\epsilon(\partial \Delta; \mathbb{R}^{2n+1}) \times \partial D \times C^n \times \mathbb{R} \times \mathbb{R} \), where
\[
\mathcal{R}'_1(\tilde{f}, \lambda) = \overline{J}_o \tilde{f}, \quad \mathcal{R}'_2(\tilde{f}, \lambda) = r(\zeta, \lambda(\zeta), \tilde{f}(\zeta))
\]
$R'_3(f, \lambda) = \pi(f)|_{\zeta=1} - x_o$, $R'_4(f, \lambda) = \pi(f)_* \left( \frac{\partial}{\partial x}_{\zeta=1} \right) - \nu_o$,

$R'_5(f, \lambda) = p(\pi(f); x_o) - \nu$,

$R'_6(f, \lambda) = \tilde{f} \left( \pi(\tilde{f})_* \left( \frac{\partial}{\partial x}_{\zeta=1} \right) \right) - 1$.

Given a stationary disk $f_o : \overline{\Delta} \to \overline{D}$ with $x_o = f(1)$, $\nu_o = f_{o*} \left( \frac{\partial}{\partial x}_{\zeta=0} \right)$ and $p(f_o; x_o) = \nu_o$, we say that $\partial D$ is a horospherically good boundary for $(J_o, f_o)$ if $f_o$ admits a lift $\tilde{f}_o$ so that $(\tilde{f}_o, \lambda_o)$ is a solution of $R'_5(\tilde{f}_o, \lambda) = 0$ and the tangent operator $\mathcal{N}'$ of $R'_5(\tilde{f}_o, \lambda)$ at $(\tilde{f}_o, \lambda)$ is invertible.

Again, by the Implicit Function Theorem, if $\partial D$ is a horospherically good boundary for $(J_o, f_o)$, there is a neighborhood $\mathcal{V} \subset \partial D$ of $x_o$, a neighborhood $\mathcal{W} \subset TD$ of $\nu_o$, with $\pi(\mathcal{W}) = \mathcal{V}$ and a real number $\varepsilon > 0$ so that, for any $x \in \mathcal{V}$, $v \in \mathcal{W}$, $|\nu - \nu_o| < \varepsilon$ and $\|J - J_o\|_{\mathcal{D}} < \varepsilon$, there exists a unique disk $f$ in $D$ with

$$f(1) = x, \quad f_{*} \left( \frac{\partial}{\partial x}_{\zeta=1} \right) = v \quad \text{and} \quad p(f; x) = \nu, \quad (5.18)$$

which is stationary for $D$ w.r.t. the almost complex structure $J$. The dependence of $f$ on $x, v, \nu$ and $J$ is differentiable and, given $m_o > 0$ and a metric $g$, one can choose $\varepsilon, \mathcal{W}$ and $\mathcal{V} = \pi(\mathcal{W})$, so that $\sup_{\zeta \in \Delta} \text{dist}_g(f(\zeta), f_o(\zeta)) < m_o$.

So, in analogy with Proposition 3.2, we have:

**Proposition 5.3.** Let $D^{(x_o, a)} \subset D$, $a > 0$, be a good conical subdomain w.r.t. to $J_o$ with vertex in $x_o \in \partial D$. If $\partial D$ is a good boundary for $(J_o, f_o)$ for any stationary disk $f_o \in \mathcal{C}^{(x_o, a)}$, there exists $\varepsilon > 0$ and an open neighborhood $\mathcal{U} \subset \partial D$ of $x_o$ so that for any $J$ with $\|J - J_o\|_{\mathcal{D}} < \varepsilon$ and any $x \in \mathcal{U}$ and $|a' - a| < \varepsilon$, the point $x$ is vertex for a good foliation for $D^{(x, a')}$ relatively to the almost complex structure $J$.

**Proof.** The proof can be obtained following the same steps of the proof of Prop. 6 in [4] and we give here only a sketch of it. First of all, using the Implicit Function Theorem and the compactness of $\overline{C}^{(a)} \cap S^{2n-1} \subset T_{x_o}M$, one can determine $\mathcal{U}$ and $\varepsilon$ so that $\mathcal{G}^{(x, a')}$ satisfies (i) for Definition 5.2 for any almost complex structure such that $\|J - J_o\|_{\mathcal{D}} < \varepsilon$ and for any $x \in \mathcal{U}$, $|a' - a| < \varepsilon$. Using the Implicit Function Theorem once again, one can also assume that for all these $J$, $x$ and $a$, the map “exp”, defined in (ii) of that definition, is a local diffeomorphism at all points. It remains to be checked that $\mathcal{U}$ and $\varepsilon$ can be chosen so that “exp” is also injective. From this and a possible further restriction of $\mathcal{U}$ and $\varepsilon$, we obtain that “exp” is a diffeomorphism and satisfies all other requirements of (ii). To prove injectivity, one may argue by contradiction as in Step 2 of the proof of Prop. 6 in [4]. In fact, if one assumes that “exp” is never injective for
any choice of $U$ and $\varepsilon$, one can construct sequences of complex structures $J_j$, of vertices $x_j$ and of pairs $y_j \neq y'_j \in B^n$, so that $J_j \to J_0$, $x_j \to x_0$ and corresponding exponential maps $\exp^{(j)}$ are so that $\exp^{(j)}(y_j) = \exp^{(j)}(y'_j)$ for all $j$. Using compactness and Implicit Function Theorem, one can select a subsequence $z_{jm} = \exp^{(j)}(y_{jm})$, with $y_{jm} \to y_0$, $y'_j \to y'_0$ with $y_0 \neq y'_0$ and $z_{jm} \to z_0 = \exp(y_0) = \exp(y'_0) \in \overline{D(x_0,0)}$, contradicting the hypothesis of bijectivity of “$\exp$” on $\overline{D(x_0,0)}$.

We remark that the tangent operator $\mathcal{R}' = (\mathcal{R}'_1, \ldots, \mathcal{R}'_0)$ of $\mathcal{R}'_{(J_0, x_0, \nu_0, \nu_0)}$ at $(\mathcal{f}_0, \lambda)$ is so that $(\mathcal{R}'_1, \mathcal{R}'_2) = R_{A,B,G}$ (see 4.2 for definition) and hence it coincides with operator $R_{0,0,G}$ when $D \subset \mathbb{C}^n$. If $D$ is a strictly linearly convex domain in $\mathbb{C}^n$, by Lemma 4.8 (1), the dimension of $\ker(\mathcal{R}'_1, \mathcal{R}'_2) = 4n+1$. From this, the results in [6] and a line of argument which is essentially the same of the proofs of Lemma 4.8 (1) and Theorem 4.6 one gets that $\mathcal{R}' = (\mathcal{R}'_1, \ldots, \mathcal{R}'_0)$ is invertible also in this case. By Proposition 5.3 the following result is obtained.

**Theorem 5.4.** Let $D \subset M$ be a smoothly bounded, strongly pseudoconvex domain in an almost complex manifold $(M, J_0)$ and $a > 0$ be any fixed positive real number. If there is a local diffeomorphism $\varphi : U \subset M \to \mathbb{C}^n$, so that $\hat{D} = \varphi(D)$ is a strictly linearly convex domain $\hat{D} \subset \mathbb{C}^n$ and $\varphi(J_0)$ is sufficiently close to $J_{st}$ in a $C^1$-norm, then, for any $x_0 \in \partial D$, the subset $D(x_0,0) \subset D$ is a good conical subdomain.

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