Generalized fixed-point algebras
of certain actions on crossed products.

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Abstract: Let $G$ and $H$ be two locally compact groups acting on a C*-algebra $A$ by commuting actions $\lambda$ and $\sigma$. We construct an action on $A \times_\lambda G$ out of $\sigma$ and a unitary 2-cocycle $u$. For $A$ commutative, and free and proper actions $\lambda$ and $\sigma$, we show that if the roles of $\lambda$ and $\sigma$ are reversed, and $u$ is replaced by $u^*$, then the corresponding generalized fixed-point algebras, in the sense of Rieffel, are strong-Morita equivalent. We apply this result to the computation of the K-theory of quantum Heisenberg manifolds.

Introduction. Given two commuting actions $\lambda$ and $\sigma$ of locally compact groups $G$ and $H$, respectively, on a C*-algebra $A$, we study the action $\gamma^{\sigma,u}$ of $H$ on $A \times_\lambda G$ defined by

$$(\gamma^{\sigma,u}_h \Phi)(x) = u(x,h)\sigma_h(\Phi(x)),$$

where $\Phi \in C_c(G,A)$, $h \in H$, $x \in G$, $u(x,h)$ is a unitary element of the center of the multiplier algebra of $A$, and $u$ satisfies the cocycle conditions

$$u(x_1x_2,h) = u(x_1,h)\lambda_{x_1}(u(x_2,h)) \quad \text{and} \quad u(x,h_1h_2) = u(x,h_1)\sigma_{h_1}(u(x,h_2)).$$

The study of this situation was originally motivated by the example of quantum Heisenberg manifolds ([Rf5]), which can be described as generalized fixed-point algebras ([Rf4]) of actions of this form, when $A = C_0(R \times T)$, and $G = H = Z$.

This work is organized as follows. In Section 1 we define the action $\gamma^{\sigma,u}$ and show that for $G$ and $H$ second countable, and $A$ separable, the crossed product $A \times_\lambda G \times_{\gamma^{\sigma,u}} H$ is isomorphic to a certain twisted crossed product of the algebra $A$ by the group $G \times H$.

In Section 2 we assume that the algebra $A$ is commutative and show that for free and proper actions $\lambda$ and $\sigma$, the generalized fixed-point algebra of $A \times_\lambda G$ under $\gamma^{\sigma,u}$ and that of $A \times_\sigma H$ under $\gamma^{\lambda,u^*}$ are strong-Morita equivalent.

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In Section 3 we apply these results to show that the K-groups of the quantum Heisenberg manifolds do not depend on the deformation constant. This enables us to compute them, by calculating them in the commutative case.

In what follows, for a C*-algebra \(A\), \(\mathcal{M}(A)\) denotes its multiplier algebra, \(\mathcal{Z}(A)\) its center and \(\mathcal{U}(A)\) the group of unitary elements in \(A\). All actions of locally compact groups on C*-algebras are assumed to be strongly continuous. All integrations on a group \(G\) are with respect to a fixed left Haar measure \(\mu_G\) with modular function \(\Delta_G\).

1 Actions on crossed products.

For locally compact groups \(G\) and \(H\) acting on a C*-algebra \(A\) by commuting actions \(\lambda\) and \(\sigma\), respectively, and a 2-cocycle on \(G \times H\), we define an action \(\gamma^{\sigma,v}\) of \(H\) on \(A \times_{\lambda} G\). We show in Proposition 1.3 that, when \(A\) is separable, and \(G\) and \(H\) are second-countable, the crossed product \(A \times_{\lambda} G \times_{\gamma^{\sigma,v}} H\) is a twisted crossed product of \(A\) by \(G \times H\).

**Proposition 1.1** Let \(G\) be a group acting on a C*-algebra \(A\) by an action \(\lambda\), and let \(v : G \rightarrow U \mathcal{Z} \mathcal{M}(A)\) verify the cocycle condition

\[v(xy) = v(x)\lambda_x(v(y)).\]

Let \(\sigma \in \text{Aut}(A)\) commute with \(\lambda\), and, for \(\Phi \in C_c(G, A)\), define

\[(\gamma^{\sigma,v}\Phi)(x) = v(x)\sigma(\Phi(x)).\]

Then \(\gamma^{\sigma,v}\) extends to an automorphism on \(A \times_{\lambda} G\).

**Proof:**

Let \((\Pi, V)\) be a covariant representation of the C*-dynamical system \(C^*(G, A, \lambda)\) on a Hilbert space \(\mathcal{H}\), and let \(\Pi \times U\) denote its integrated form. Let \(\Pi^\sigma\) denote the representation of \(A\) on \(\mathcal{H}\) defined by \(\Pi^\sigma(a) = \Pi(\sigma(a))\), and let \(\tilde{V}\) be the unitary representation of \(G\) on \(\mathcal{H}\) given by \(\tilde{V}_x = \Pi(v(x))V_x\), where \(\Pi\) also denotes its extension to \(\mathcal{M}\). Then \((\Pi^\sigma, \tilde{V})\) is a covariant representation of \(C^*(G, A, \lambda)\): for \(x \in G\), and \(a \in A\) we have

\[\tilde{V}_x \Pi^\sigma(a) \tilde{V}_{x^{-1}} = \Pi(v(x))V_x \Pi(\sigma(a))\Pi(v(x^{-1}))V_{x^{-1}} = \]
We now show that for \( \Phi \in C_c(G,A) \) we have that 
\[
(\Pi \times V)(\gamma_{\sigma,a}\Phi) = (\Pi^\sigma \times \tilde{V})(\Phi),
\]
which ends the proof: for any \( \xi \in \mathcal{H} \), we have 
\[
[(\Pi \times V)(\gamma_{\sigma,a}\Phi)](\xi) = \int_G \Pi[(\gamma^\sigma_{a}\Phi)(x)]V_x\xi dx = \int_G \Pi(v(x))\Pi[(\sigma\Phi)(x)]V_x\xi dx = \int_G \Pi^\sigma[\Phi(x)]\tilde{V}_x\xi dx = [(\Pi^\sigma \times \tilde{V})(\Phi)](\xi).
\]
Q.E.D.

**Proposition 1.2** Assume that \( G, \lambda, \) and \( A \) are as in Proposition 1.1 and that \( H \) is a locally compact group acting on \( A \) by an action \( \sigma \) commuting with \( \lambda \). Let \( u : G \times H \rightarrow \mathcal{UZM}(A) \) be continuous for the strict topology in \( \mathcal{M}(A) \), and satisfy
\[
u(xy,h) = u(x,h)\lambda_xu(y,h) \quad \text{and} \quad u(x, hg) = u(x, h)\sigma_hu(x, g),
\]
for \( x, y \in G \) and \( h, g \in H \). For \( h \in H \) and \( \Phi \in C_c(G, A) \), let
\[
(\gamma^\sigma_{u,h}\Phi)(x) = u(x, h)\sigma_h(\Phi(x)).
\]
Then \( h \mapsto \gamma^\sigma_{u,h} \) is a (strongly continuous) action of \( H \) on \( A \times_\lambda G \).

**Proof:**

By Proposition 1.1 we have that \( \gamma^\sigma_{h,1} \in Aut(A \times_\lambda G) \), for all \( h \in H \). Besides, the cocycle condition implies that \( \gamma^\sigma_{h_1,h_2}_{u,h} \Phi(x) = \gamma^\sigma_{h_1,h_2}_{u,h} \Phi(x) \).

Finally, \( h \mapsto \gamma^\sigma_{u,h} \Phi \) is continuous for any \( \Phi \in C_c(G, A) \):
\[
\| \gamma^\sigma_{u,h} \Phi - \gamma^\sigma_{u,h_0} \Phi \|_{A \times_\lambda G} \leq \| \gamma^\sigma_{u,h} \Phi - \gamma^\sigma_{u,h_0} \Phi \|_{L^1(G, A)} = \int_G \| u(x,h)\sigma_h(\Phi(x)) - u(x,h_0)\sigma_{h_0}(\Phi(x)) \|_A dx \leq \int_{\text{supp}(\Phi)} \| \sigma_h(\Phi(x)) - \sigma_{h_0}(\Phi(x)) \|_A + \| (u(x,h) - u(x,h_0))\sigma_{h_0}(\Phi(x)) \|_A dx,
\]
which converges to 0 when \( h \) goes to \( h_0 \), because \( u \) is continuous, and \( \sigma \) is strongly continuous.
Q.E.D.

Next Proposition shows that the double crossed product $A \times_\lambda G \times_{\gamma,\sigma,u} H$ is isomorphic to a twisted crossed product. Since twisted crossed products are defined for separable algebras and second-countable groups, we add these conditions.

**Proposition 1.3** Let $G$, $H$, $A$, $u$, $\lambda$, $\sigma$ and $\gamma_{\sigma,u}$ be as in Proposition 1.2. If $A$ is separable and $H$ and $G$ are second-countable, then $A \times_\lambda G \times_{\gamma,\sigma,u} H$ is isomorphic to the twisted crossed product $A \times_{(\lambda,\sigma),U} (G \times H)$, where

$$(\lambda,\sigma)_{(x,h)}(a) = \lambda_x \sigma_h(a) \text{ and } U((x_0,h_0),(x_1,h_1)) = \lambda_{x_0}(u(x_1,h_0)).$$

**Proof:**

First notice that $((\lambda,\sigma),U)$ is a twisted action of $G \times H$ on $A$: conditions a), b) and c) in [PR, Def. 2.1] are easily checked, and, for $(x_0,h_0), (x_1,h_1)$, and $(x_2,h_2)$ in $G \times H$, we have

$$(\lambda,\sigma)_{(x_0,h_0)}[U((x_1,h_1),(x_2,h_2))]U((x_0,h_0),(x_1,x_2,h_1h_2)) =$$

$$= \lambda_{x_0} \sigma_{h_0} \lambda_{x_1}(u(x_2,h_1)) \lambda_{x_0}(u(x_1x_2,h_0)) = \lambda_{x_0x_1}(u(x_2,h_0h_1)) \lambda_{x_0}(u(x_1,h_0)) =$$

$$= U((x_0x_1,h_0h_1),(x_2,h_2))U((x_0,h_0),(x_1,h_1)).$$

We now construct maps $i_A : A \to \mathcal{M}(A \times_\lambda G \times_{\gamma,\sigma,u} H)$ and $i_{G \times H} : G \times H \to \mathcal{U}\mathcal{M}(A \times_\lambda G \times_{\gamma,\sigma,u} H)$ satisfying

$$i_A((\lambda,\sigma)_{(x,h)}(a)) = i_{G \times H}(x,h)i_A(a)i_{G \times H}(x,h)^* \text{ and } i_{G \times H}(x_0,h_0)i_{G \times H}(x_1,h_1) = i_A(U((x_0,h_0),(x_1,h_1))i_{G \times H}(x_0x_1,h_0h_1),$$

for all $x_i \in G$, $h_i \in H$, and $a \in A$.

If $\alpha$ is an action of a group $K$ on a C*-algebra $B$, $b \in \mathcal{M}(B)$, and $\mu$ is a bounded complex Radon measure with compact support on $G$, let $M(b,\mu)$ denote the multiplier of $B \times_\alpha K$ defined by

$$(M(b,\mu)f)(t) = b \int_K \alpha_s(f(s^{-1}t))d\mu(s),$$

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fpr $f \in C_c(K, B)$. 
Now define
\[
i_A(a) = M(M(a, \delta_1), \delta_1, \delta_1) \quad \text{and} \quad i_{G \times H}(x, h) = M(M(1, \delta_x), \delta_h),
\]
where $\delta_i$ denotes the point mass at $t$.
For $f \in C_c(G \times H, A)$, explicit formulas are given by:
\[
(i_A(a)f)(x, h) = af(x, h), \quad \text{and}
\]
\[
(i_{G \times H}(x, h_0)f)(x, h) = u^*(x_0, h_0)u(x, h_0)\lambda_{x_0}\sigma_{h_0}((f(x_0^{-1}x, h_0^{-1}h))).
\]
It follows that
\[
(i_{G \times H}^*(x_0, h_0)f)(x, h) = u(x, h_0^{-1})\sigma_{h_0}^{-1}\lambda_{x_0}^{-1}(f(x_0x, h_0h)).
\]
The pair $(i_A, i_{G \times H})$ is covariant:
\[
(i_{G \times H}(x_0, h_0)i_A(a)i_{G \times H}(x_0, h_0)f)(x, h) =
\]
\[
= u^*(x_0, h_0)u(x, h_0)\lambda_{x_0}\sigma_{h_0}[au(x_0^{-1}x, h_0^{-1})\sigma_{h_0}^{-1}\lambda_{x_0}^{-1}(f(x, h))] =
\]
\[
= (i_A(\lambda_{x_0}\sigma_{h_0}(a))f)(x, h),
\]
and
\[
(i_{G \times H}(x_0, h_0)i_{G \times H}(x_1, h_1))(x, h) =
\]
\[
u^*(x_0, h_0)u(x, h_0)\lambda_{x_0}\sigma_{h_0}[u^*(x_1, h_1)u(x_0^{-1}x, h_1^{-1})\lambda_{x_1}\sigma_{h_1}(f(x_1^{-1}x_0^{-1}x, h_1^{-1}h_0^{-1}h))] =
\]
\[
= \lambda_{x_0}u(x_1, h_0)u^*(x_0x_1, h_0h_1)u(x, h_0h_1)\lambda_{x_0x_1}\sigma_{h_0h_1}(f(x_1^{-1}x_0^{-1}x, h_1^{-1}h_0^{-1}h)) =
\]
\[
= U((x_0, h_0), (x_1, h_1))i_{G \times H}((x_0x_1, h_0h_1)f)(x, y).
\]
We next show that for any covariant representation $(\Pi, V)$ of $(A, G \times H, (\lambda, \sigma), U)$ on a Hilbert space $H$ there is an integrated form $\Pi \times V$ on $A \times \lambda G \times \gamma, u \times H$.
Let $V_G$ and $V_H$ be the restrictions of $V$ to $G$ and $H$, respectively. Then $(\Pi, V_G)$ is a covariant representation of $(A, G, \lambda)$ and, if $\Pi \times V_G$ denotes its integrated form, then $(\Pi \times V_G, V_H)$ is a covariant representation of $(A \times \lambda G, G, \gamma, u)$.
So $\Pi \times V_G \times V_H$ is a non-degenerate representation of $A \times \lambda G \times \gamma, u \times H$ and
\[
\Pi = \Pi \times V_G \times V_H \circ i_A \quad \text{and} \quad V = \Pi \times V_G \times V_H \circ i_{G \times H}.
\]
Finally, the set $\{i_A \times i_{G \times H}(f) : f \in L^1(G \times H, A)\}$, where
\[
[i_A \times i_{G \times H}(f)](x, h) = \int_{G \times H} i_A[f(x, h)]i_{G \times H}(x, h)d(x, y)
\]
is a dense subspace of $A \times \lambda G \times \gamma, u \times H$, which ends the proof.

Q.E.D.
2 The generalized fixed-point algebras.

With the example of quantum Heisenberg manifolds in mind, we now discuss the situation described in Section 1 in the case of some particular actions λ and σ on a commutative C*-algebra \( C_0(M) \). We prove that if the action σ is proper, then so is \( \gamma^{\sigma,u} \) (in the sense of [Rf4]), and that if σ is also free then \( \gamma^{\sigma,u} \) is saturated ([Rf4]). Besides, for λ and σ free and proper, the generalized fixed-point algebras under \( \gamma^{\sigma,u} \) and \( \gamma^{\lambda,u} \) respectively are strong-Morita equivalent.

More specifically, we show that the space \( C_c(M) \) can be made into a dense submodule of an equivalence bimodule for the generalized fixed-point algebras. Part of this is done by adapting to our situation the techniques of [Rf3, Situation 10].

Assumptions and notation. Throughout this section \( M \) denotes a locally compact Hausdorff space, and \( \beta M \) its Stone-Cech compactification. The groups G and H act on \( M \) by commuting actions \( \lambda \) and \( \sigma \), respectively. In this context, if \( T \) denotes the unit circle, the cocycle \( u \) of Section 1 consists of continuous functions \( u(x,h) : M \to T \), for \( (x,h) \in G \times H \), such that, for any \( f \in C_0(M) \) the map \( (x,h) \to u(x,h)f \) is continuous for the supremum norm. As in Section 1 we require the cocycle conditions:

\[
\begin{align*}
\gamma^\sigma u(x_1x_2,h) &= \gamma^\sigma u(x_1,h)\lambda_{x_1}u(x_2,h) \\
\gamma^\sigma u(x,h_1h_2) &= \gamma^\sigma u(x,h_1)\sigma_{h_1}u(x,h_2),
\end{align*}
\]

for \( x,x_i \in G \) and \( h, h_i \in H \). Notice that if these conditions are satisfied for \( u \) they also hold for \( u^* \). We denote by \( \gamma^{\sigma,u} \) and \( \gamma^{\lambda,u} \) the actions of \( H \) and \( G \) on \( C_0(M) \times_\lambda G \) and \( C_0(M) \times_\sigma H \) respectively, as defined in Proposition 1.2.

**Proposition 2.1** In the notation above, if σ is proper, so is the action \( \gamma^{\sigma,u} \) of \( H \) on \( C_0(M) \times_\lambda G \). The generalized fixed-point algebra \( D^{\sigma,u} \) of \( C_0(M) \times_\lambda G \) under \( \gamma^{\sigma,u} \) consists of the closure in \( \mathcal{M}(C_0(M) \times_\lambda G) \) of the linear span of the set \( \{ P_{\sigma,u}(E^*F) : E,F \in C_c(M \times G) \} \), where \( P_{\sigma,u} \) denotes the linear map \( P_{\sigma,u} : C_c(M \times G) \to \mathcal{M}(C_0(M) \times_\lambda G) \) defined by

\[
(P_{\sigma,u}(F))(m,x) = \int_H (\gamma_h^{\sigma,u}(F))(m,x)dh,
\]

for \( F \in C_c(M \times G) \), and \( (m,x) \in M \times G \).

Furthermore, \( P_{\sigma,u} \) satisfies

i) \( P_{\sigma,u}(F^*) = P_{\sigma,u}(F)^* \)
ii) \( P_{\sigma,u}(F) \geq 0 \), for \( F \geq 0 \), where \( F \) and \( P_{\sigma,u}(F) \) are viewed as elements of \( M(C_0(M) \times \lambda G) \).

iii) \( P_{\sigma,u}(F*\Phi) = P_{\sigma,u}(F)*\Phi \) and \( P_{\sigma,u}(\Phi*F) = \Phi*P_{\sigma,u}(F) \), for any \( \Phi \in M(C_0(M) \times \lambda G) \) carrying \( C_c(M \times G) \) into itself and such that \( \gamma_{\sigma,u}(\Phi) = \Phi \) for any \( h \in H \).

Proof:

We check conditions 1) and 2) of \cite{RF1} Def. 1.2. Let \( B = C_c(M \times G) \).

Then \( B \) is a dense *-subalgebra of \( C_0(M) \times \lambda G \), and it is invariant under \( \gamma_{\sigma,u} \).

We now show that, for \( E,F \in B \), the map \( h \to E*\gamma_{\sigma,u}^h(F*) \) is in \( L^1(H,C_0(M) \times \lambda G) \). For \((m,x) \in M \times G \) we have

\[
[E*\gamma_{\sigma,u}^h(F*)](m,x) = \int_G E(m,y)|u(y^{-1}x,h)|(\lambda_{y^{-1}}m)F(\lambda_{y^{-1}}\sigma_{h^{-1}}m,x^{-1}y)\Delta_G(x^{-1}y)dy.
\]

Since \( \sigma \) is proper and \( \text{supp}(E) \) and \( \text{supp}(F) \) are compact, then the set

\[ \{h \in H : \sigma_{h^{-1}}\lambda_{x^{-1}}m \in \text{supp}_M(F) \text{ for } (m,x) \in \text{supp}_M(E)\times\text{supp}_G(E)\text{supp}_G(F)^{-1}\} \]

is compact. Therefore \( h \to E*\gamma_{\sigma,u}^h(F*) \) and \( h \to \Delta_H^{-1/2}(h)E*\gamma_{\sigma,u}^h(F*) \) are in \( C_c(H,B) \subseteq L^1(H,M(C_0(M) \times \lambda G)) \).

For \( F \in B \) and \( m_0 \in M \), let \( N \) be a neighborhood of \( m_0 \) with compact closure. Then there exists a compact set \( K \) in \( H \) such that

\[
P_{\sigma,u}(F)(m,x) = \int_K (\gamma_{\sigma,u}^h F)(m,x)dh,
\]

for all \((m,x) \in N \times G\), which shows that \( P_{\sigma,u}(F) \) is continuous. Since \( \text{supp}_G(P_{\sigma,u}(F)) \) is compact, then \( P_{\sigma,u}(F) \) is bounded on \( \text{supp}_M(F) \times G \).

Besides, for all \((m,x) \in M \times G \) and \( h \in H \), we have \(|P_{\sigma,u}F(m,x)| = |P_{\sigma,u}F(\sigma_h m,x)|\), and \( \text{supp}_M(P_{\sigma,u}(F)) \subset \sigma_H(\text{supp}_M(F)) \).

Therefore \( P_{\sigma,u}(F) \in C_c(\beta M \times G) \subseteq M(C_0(M) \times \lambda G) \), and, as a multiplier, \( P_{\sigma,u}(F) \) carries \( B \) into itself.

Notice now that the fact that \( h \to E*\gamma_{\sigma,u}^h(F) \) is in \( L^1(H,C_0(M) \times \lambda G) \) implies that the integral \( \int_H \gamma_{\sigma,u}^h(F)dh \) makes sense as an integral in the completion of \( M(C_0(M) \times \lambda G) \), viewed as a locally convex linear space, for the topology induced by the set of seminorms \( \{\|F : F \in B\} \), where

\[
\|\Phi\|_F = \|F*\Phi\|_{C_0(M) \times \lambda G} + \|\Phi*F\|_{C_0(M) \times \lambda G}
\]
for Φ ∈ ℳ(C₀(M) ×₀ G).

A straightforward application of Fubini’s theorem shows that
\[ \int_H (E * \gamma_\sigma^u)(F)(m, x)dh = (E * P_\sigma(F))(m, x), \]
for any E, F ∈ B, (m, x) ∈ M × G, and it follows that
\[ \int_H \gamma_\sigma^u(F)dh = P_\sigma(F), \]
in the sense mentioned above.

Also, since the positive cone is closed, and involution and the extension of \( \gamma_\sigma^u \) are continuous for the topology of \( ℳ(C₀(M) × G) \) defined above, \( P_\sigma \) satisfies i), ii), and iii) stated above.

Set now \( \langle E, F >_\sigma = P_\sigma(E^* F) \), for \( E, F \in B \). We have shown that \( \gamma_\sigma^u \) is proper. The generalized fixed-point algebra \( D_\sigma^u \) of \( C₀(M) ×₀ G \) under \( \gamma_\sigma^u \) consists of the closure in \( ℳ(C₀(M) ×₀ G) \) of the linear span of the set \( \{ \langle E, F >_\sigma : E, F \in B \} \).

Q.E.D.

**Lemma 2.2** Assume that σ is proper and let \( \{ \Phi_{N,\epsilon,K} \} \) be a net in
\( C_c(M × G × H) \), indexed by decreasing neighborhoods \( N \) of \( 1_G × H \), decreasing \( \epsilon > 0 \), and increasing compact subsets \( K \) of \( M \), satisfying

i) \( \text{supp}_{G × H}(\Phi_{N,\epsilon,K}) \subset N \)

ii) \( |\int_{G × H} \Phi_{N,\epsilon,K}(m, x, h)dxdh - 1| < \epsilon, \text{ for all } m \in K \)

iii) There exists a real number \( R \) such that \( \int_{G × H} |\Phi_{N,\epsilon,K}(m, x, h)|dxdh \leq R, \text{ for all } m \in K, \text{ and for all } K, \epsilon \text{ and } N. \)

Then \( \{ \Phi_{N,\epsilon,K} \} \) is an approximate identity for \( C_c(M × G × H) \subset C₀(M) ×₀ G × \gamma_\sigma^u H \) in the inductive limit topology.

**Proof:**

Let \( \psi \in C_c(M × G × H) \) and \( \delta > 0 \) be given. Then
\[ |(\Phi_{N,\epsilon,K} * \psi - \psi)(m, x, h)| \leq \]
\[ \leq |\int_{H × G} [u^*(y, k)(m)u(x, k)(m) - 1]\Phi_{N,\epsilon,K}(m, y, k)\psi(\sigma_{k^{-1}}y^{-1}m, y^{-1}x, k^{-1}h)dxdy| + \]

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\begin{align*}
+\left|\int_{H \times G} \Phi_{N,\epsilon,K}(m,y,k)dydk - 1\right|\Psi(m,x,h)| + \\
+\int_{H \times G} \Phi_{N,\epsilon,K}(m,y,k)[\Psi(\sigma^{-1}\lambda y^{-1}m, y^{-1}x, k^{-1}h) - \Psi(m,x,h)]dydk \leq \delta,
\end{align*}
for appropriate choices of \(\epsilon\) and \(N\).

Q.E.D.

**Proposition 2.3** If the action \(\sigma\) is free and proper, then \(\gamma^\sigma_u\) is saturated.

**Proof:**

Let \(J\) denote the ideal of \(C^*_r(H, C_0(M \times \lambda G))\) consisting of maps \(h \mapsto \Delta_H^{-1/2}(h)E \ast \gamma_h^\sigma u(F^*),\) for \(E, F \in C_c(M \times G)\). In order to show that \(J\) is dense in \(C^*_r(H, C_0(M \times \lambda G))\) we prove that \(J\) contains an approximate identity for \(C_c(M \times G \times H)\).

Let \(N, \epsilon,\) and \(K\) as in Lemma 2.2 be given. We assume without loss of generality that the closure of \(N\) is compact. Fix an open set \(U\) with compact closure such that \(K \subset U\). Choose neighborhoods \(N_G\) and \(N_H\) of \(1_G\) and \(1_H\), respectively, such that \(N_G \times N_H \subset \mathcal{N}_G\) and \(|u^*(y,h)(m)u(x,h)(m) - 1| < \epsilon_1\) for all \(h \in N_H, m \in U, x, y \in V, V\) being a fixed open set with compact closure containing \(N_G\), and for some \(\epsilon_1\) to be chosen later.

The action of \(G \times H\) on \(M \times G\) defined by \((x,h)(m,y) = (\lambda x \sigma_h m, xy)\) is free and proper, so for each \((m,y) \in K \times \mathcal{N}_G\) we can choose ([Rf3, Situation 10]) a neighborhood \(U_{(m,y)} \subset U \times V\) of \((m,y)\) such that

\[\{(x,h) : (x,h)(U_{(m,y)}) \cap U_{(m,y)} \neq \emptyset\} \subset \mathcal{N}_G \times N_H.\]

Take a finite subcover \(\{U_1, U_2, \ldots, U_n\}\) of \(\{U_{(m,y)}\}_{(m,y) \in K \times \mathcal{N}_G}\) and, for each \(i = 1, \ldots, n\), let \(F_i \in C^*_c(M \times G)\) be such that \(\text{supp}(F_i) \subset U_i\), and \(\int_G \sum_i F_i(m,x)dx = 1\) for all \(m \in K\).

Now we can find ([Rf3, Situation 10]) functions \(G_i \in C^*_c(M \times G)\) such that \(\text{supp}(G_i) \subset \text{supp}(F_i)\), and

\[|F_i(m,y) - G_i(m,y) \int_{G \times H} G_i(\lambda x^{-1} \sigma_h^{-1} m, x^{-1}y)dxdh| < \epsilon_3,\]

for all \((m,y) \in M \times G,\) and some \(\epsilon_3\) to be chosen later.
Now set
\[
\Phi_{N,\epsilon,K}(m,x,h) = \sum_i \Delta_{H}^{-1/2}(h) [G_i \ast \gamma_h^{\sigma,u}(G_i^*)](m,x).
\]

Then,
\[
\left| \int_{H \times G} \Phi_{N,\epsilon,K}(m,x,h) dx dh - 1 \right| =
\]
\[
= \sum_i \int_{G \times G \times H} \Delta_G(x^{-1}y)[u^*(y,h)u(x,h)](m)G_i(m,y)G(\lambda_{x^{-1}\sigma^{-1}h^{-1}m,x^{-1}y}) dx dy dh - \sum_i \int_G F_i(m,y) dy \leq
\]
\[
\leq \left| \sum_i \int_V ([u^*(y,h)(m)u(x,h)(m)\Delta_G(x^{-1}y)-1]G_i(m,y) \int_{G \times H} G_i(\lambda_{x^{-1}\sigma^{-1}h^{-1}m,x^{-1}y}) dx dh) dy \right| + \left| \sum_i \int_V G_i(m,y) \int_{G \times H} G_i((\lambda_{x^{-1}\sigma^{-1}h^{-1}m,x^{-1}y}) dx dh - F_i(m,y) dy \right| < \epsilon,
\]
für appropriatem Choices of \( \epsilon_1, \epsilon_2, \text{and} \ \epsilon_3. \)

Besides, \( \text{supp}(\Phi_{N,\epsilon,K}) \subset N_G \times N_H \subset N. \) Finally, a similar argument shows that from some \( N_0 \) and \( \epsilon_0 \) on we have
\[
\int_{H \times G} |\Phi_{N,\epsilon,K}(m,x,h)| dx dh \leq R,
\]
for some real number \( R, \text{and all} \ m \in K. \)

Q.E.D.

Assumptions. We next compare the generalized fixed-point algebras obtained when the roles of \( \sigma \) and \( \lambda \) are reversed. That is why we require symmetric conditions on these two actions. So, we assume from now on that both \( \lambda \) and \( \sigma \) are free and proper actions.

Notation. Let \( C^{\sigma,u} \) denote the subalgebra of \( M(C_0(M) \times \lambda G) \) consisting of functions \( \Phi \in C_c(\beta M \times G) \) such that the projection of \( \text{supp}_M(\Phi) \) on \( M/H \) is precompact and \( \gamma_h^{\sigma,u}\Phi = \Phi \) for all \( h \in H. \)

Remark 2.4 Notice that, for \( F \in C_c(M \times G), \) we have that \( \text{supp}_M(P_{\sigma,u}F) \subset \sigma_H(\text{supp}_M(F)) \), and therefore \( C^{\sigma,u} \) contains the image of \( P_{\sigma,u}. \)
Lemma 2.5 Let \( \{ \Phi_{N,\epsilon} \} \) be a net in \( C^{\sigma,u} \), indexed by decreasing neighborhoods \( N \) of \( 1_G \), increasing compact subsets \( K \) of \( M \), and decreasing \( \epsilon > 0 \), and such that

1) \( \text{supp}_G(\Phi_{N,\epsilon,K}) \subseteq N \)

2) \( |\int_G \Delta^{1/2}_G(x)(\Phi_{N,\epsilon}(m,x))dx - 1| < \epsilon \) for all \( m \in K \).

3) There is a real number \( R \) such that \( \int_G |\Phi_{N,\epsilon,K}(m,x)|dx \leq R \), for all \( m \in K \), and for all \( N \) and \( \epsilon \) from some \( N_0 \) and \( \epsilon_0 \) on.

Then \( \{ \Phi_{N,\epsilon,K} \} \) is an approximate identity for \( C^{\sigma,u} \).

Proof: Let \( \Psi \in C^{\sigma,u} \) and \( \delta > 0 \) be given. Fix a neighborhood \( N' \) of \( 1_G \) with compact closure, and let \( K' \subseteq M \) be a compact set such that \( \Pi_H(\text{supp}_M \Psi) \subseteq \Pi_H(K') \), where \( \Pi_H \) denotes the canonical projection on \( M/H \).

As in Lemma 2.2, we can find \( N_0 \subseteq N' \), \( \epsilon_0 \), and \( K_0 \) such that, from \( N_0 \), \( \epsilon_0 \), and \( K_0 \) on, we have

\[ |(\Phi_{N,\epsilon,K} \ast \Psi - \Psi)(m,x)| < \delta, \]

for all \( m \in \lambda_{N'}(K') \).

Therefore, if \( m \in \text{supp}(\Phi_{N,\epsilon,K} \ast \Psi - \Psi) \), then we have that \( \sigma_h m \in \lambda_{N'}(K') \), for some \( h \in H \). On the other hand we have that

\[ |(\Phi_{N,\epsilon,K} \ast \Psi - \Psi)(\sigma_h m, x)| = |(\Phi_{N,\epsilon,K} \ast \Psi - \Psi)(m, x)|, \]

for all \( h \in H \), \( m \in M \), and \( x \in G \), because \( \Phi_{N,\epsilon,K} \) and \( \Psi \in C^{\sigma,u} \). This shows that \( |(\Phi_{N,\epsilon,K} \ast \Psi - \Psi)(m,x)| < \delta \) for all \( m \in M \). Therefore \( \Phi_{N,\epsilon,K} \ast \Psi \) converge to \( \Psi \) in the multiplier algebra norm.

Q.E.D.

Remark 2.6 Notice that Lemma 2.3 above also holds, with a similar proof, if condition 2) is replaced by

\[ 2') |\int_G \Phi_{N,\epsilon,K}(m,x)dx - 1| < \epsilon \] for all \( m \in K \).

Proposition 2.7 The generalized fixed point algebra \( D^{\sigma,u} \) is the closure in \( \mathcal{M}(C_0(M) \times \chi_G) \) of \( C^{\sigma,u} \).
Proof:

In view of property iii) in Proposition 2.1, it suffices to show that the span of the set
\[ \{ P_{\sigma,u}(E^* F) : E, F \in C_c(M \times G) \} \]
contains an approximate identity for \( C^{\sigma,u} \).

For a given compact set \( K \subset M \), let us fix an open set \( U \) of compact closure containing \( K \). Then the set \( L = \{ h \in H : \sigma_h m \in \overline{U} \text{ for some } m \in K \} \) is compact.

Let \( N \) be a given neighborhood of \( 1_G \) and \( \epsilon > 0 \). As in [Ref], Sit. 10, first lemma, we can take an open cover \( \{ U_1, U_2, ..., U_n \} \) of \( K \), such that \( U_i \subseteq U \) and \( U_i \cap \lambda U_i \neq \emptyset \) only if \( x \in N \). For each \( i = 1, ..., n \), let \( H_i \in C_c^+(M \times G) \) be such that \( \text{supp}(H_i) \subset U_i \times N \), and \( \sum_i H_i \) is strictly positive on \( K \times 1_G \). Then \( \sum_i \int_{H \times G} H_i(\sigma_{h^{-1}m}, y) dhdy > 0 \) for all \( m \in K \). Therefore, we can find functions \( F_i \in C_c^+(M \times G) \) such that \( \text{supp}(F_i) \subset \text{supp}(H_i) \) and \( \int_{H \times G} F_i(\sigma_{h^{-1}m}, y) dhdy = 1 \) for all \( m \in K \). Now, the action of \( G \) on \( M \times G \) given by \( \alpha_x(m, y) = (\lambda x m, xy) \) is free and proper, so the second lemma in [Ref], Situation 10 applies and for each \( i = 1, ..., n \) we can find \( G_i \in C_c^+(M \times G) \) such that \( \text{supp}(G_i) \subseteq \text{supp}(F_i) \) and

\[ |F_i(m, y) - G_i(m, y) \int_G G_i(\lambda x^{-1}m, x^{-1}y) dx| < \delta/n, \]

for all \( m \in M, y \in G \), and some positive number \( \delta \) to be chosen later. Set now \( \Phi_{N,\epsilon,K} = \sum_{i=1}^n P_{\sigma,u}(G_i \ast J_i) \), where \( J_i(m, x) = G_i(\lambda x^{-1}m, x^{-1}) \). We have

\[ \Phi_{N,\epsilon,K}(m, x) = \sum_i \int_U u(x, h) \int_G G_i(\sigma_{h^{-1}m}, y) G_i(\sigma_{h^{-1}\lambda x^{-1}m}, x^{-1}y) dy, \]

so, since \( \text{supp}(G_i) \subseteq \text{supp}(F_i) \), it follows that \( \text{supp}_G(\Phi_{N,\epsilon,K}) \subseteq N \).

Besides, if \( m \in K \),

\[ | \int \Phi_{N,\epsilon,K}(m, x) dx - 1 | = \]

\[ = | \sum_i \int_U \int_G [u(x, h) G_i(\sigma_{h^{-1}m}, y) \int_G G_i(\sigma_{h^{-1}\lambda x^{-1}m}, x^{-1}y) dx - F_i(\sigma_{h^{-1}m}, y)] dy dh | < \epsilon, \]

for a suitable choice of \( \delta \), if \( N \) is chosen to have \( |u(x, h) - 1| \) small enough for all \( x \in N \) and \( h \in L \).

Finally, from some \( \epsilon_0 \) and \( N_0 \) on, \( \| \Phi_{N,\epsilon,K}(m, x) \| dx \leq R \), for some real number \( R \) and all \( m \in K \).

Then, by Remark 2.6, \( \{ \Phi_{N,\epsilon,K} \} \) is an approximate identity for \( C^{\sigma,u} \).
Remark 2.8 The span of the set
\[ \{ P_{\sigma,u}(F) : F(m,x) = \Delta^{-1/2}(x)e_i(m)\mathcal{F}(\lambda_{x^{-1}}m), e \in C_c(M) \} \]
contains an approximate identity for \( C_{\sigma,u} \).

Proof:
In the notation of Proposition 2.7, let \( \{ f_i \} \subset C^+_c(M) \) be such that \( \text{supp}(f_i) \subset U_i, \) and \( \int_H \sum f_i(\sigma_{y^{-1}}m) > 0, \) for all \( m \in K. \) Since the action \( \lambda \) is proper we can get \( g_i \in C^+_c(M) \) such that \( \text{supp}(g_i) \subseteq \text{supp}(f_i) \) and \( |f_i(m) - g_i(m)\int_G g_i(\lambda_{x^{-1}}m)dx| < \delta \) for all \( m \in M \) and a given positive number \( \delta. \) Then, if we let \( L_i(m,x) = \Delta^{-1/2}(x)g_i(m)g_i(\lambda_{x^{-1}}m) \) we have that, for an appropriate choice of \( \delta \) in terms of \( \epsilon, \) the function \( \Phi_{N,\epsilon,K} = \sum_i P_{\sigma,u}(L_i) \) can be shown (by an argument quite similar to that in Proposition 2.7) to satisfy the hypotheses of Lemma 2.5.

Q.E.D.

Notation. We denote by \( \lambda \prec, \succ \) and \( \prec, \succ \lambda \) the \( C_c(M \times G) \)-valued maps defined on \( C_c(M) \times C_c(M) \) by
\[
\lambda \prec f, g \succ (m,x) = \Delta^{-1/2}_G(x)f(m)\mathcal{F}(\lambda_{x^{-1}}m)
\]
and
\[
\prec f, g \succ \lambda (m,x) = \Delta^{-1/2}_G(x)\mathcal{F}(m)g(\lambda_{x^{-1}}m),
\]
where \( f, g \in C_c(M). \)

Remark 2.9 It is a well known result ([NR, Situation 2]) that \( C_c(M) \) is a left (resp. right) \( C_c(M \times G) \)-rigged module for \( \lambda \prec, \succ \) (resp. \( \prec, \succ \lambda \)) and the actions given by:
\[
(\Phi \cdot f)(m) = \int_G \Delta^{1/2}_G(y)\Phi(m,y)f(\lambda_{y^{-1}}m)dy
\]
and \((f \cdot \Phi)(m) = \int_G \Delta_G^{-1/2}(y)\Phi(\lambda^{-1}m, y^{-1})f(\lambda^{-1}m)dy\), for \(\Phi \in C_c(M \times G)\) It is easily checked that, by taking \(\Phi \in C_c(\beta M \times G)\) in the formulas above, one makes \(C_c(M)\) into a \(C_c(\beta M \times G)\)-module with inner product. Of course it is no longer a rigged space because the condition of density fails.

**Proposition 2.10** Let \(C^\sigma,u \subseteq C_c(\beta M \times G)\) act on \(C_c(M)\) on the left and on the right as in Remark 2.3. For \(f, g \in C_c(M)\) define

\[
< f, g >_{D^\sigma,u} = P_{\sigma,u}(< f, g >_\lambda) \quad \text{and} \quad D^\sigma,u < f, g > = P_{\sigma,u}(\lambda < f, g >).
\]

Then \(C_c(M)\) is a left (resp. right) \(C^\sigma,u\)-rigged space with respect to \(D^\sigma,u <, >\) (resp. \(<, >_{D^\sigma,u}\)).

**Proof:**

The density condition follows from Remark 2.8. All other properties follow immediately from the fact that \(\lambda <, >\) and \(<, >_\lambda\) are inner products and from Remark 2.4 and properties i), ii), and iii) of \(P_{\sigma,u}\) shown in Proposition 2.1.

Q.E.D.

We are now ready to show the main result of this section.

**Theorem 2.11** Let \(\lambda\) and \(\sigma\) be free and proper commuting actions of locally compact groups \(G\) and \(H\) respectively on a locally compact space \(M\). Let \(u\) be a cocycle as in Proposition 1.3. Then the generalized fixed-point algebras \(D^\sigma,u\) and \(D^{\lambda,u^*}\) of the actions \(\gamma^\sigma,u\) and \(\gamma^{\lambda,u^*}\) on \(C_0(M) \times_\lambda G\) and \(C_0(M) \times_\sigma H\), respectively, are strong-Morita equivalent.

**Proof:**

By Proposition 2.10, \(C_c(M)\) is a left \(C^\sigma,u\)-rigged space and a right \(C^{\lambda,u^*}\)-rigged space under

\[
(\Phi \cdot f)(m) = \int_G \Delta_G^{-1/2}(y)\Phi(m, y)f(\lambda^{-1}m)dy, \quad D^{\sigma,u} < f, g > = P_{\sigma,u}(\lambda < f, g >),
\]
\((f \cdot \Psi)(m) = \int_H \Delta_H^{-1/2}(h)\Psi(\sigma h^{-1}m, h^{-1})f(\sigma h^{-1}m)dh,\)

and \(<f, g > D_{\lambda,u^*} = P_{\lambda,u^*}(<f, g >_\sigma),\)

where \(f, g \in C_c(M), \Phi \in C^{\sigma,u} \) and \(\Psi \in C^{\lambda,u^*}.\)

Then \(C_c(M)\) is an \(C^{\sigma,u}-C^{\lambda,u^*}\) bimodule: for \(\Phi, \Psi\) and \(f\) as above we have

\[
[(\Phi \cdot f) \cdot \Psi](m) = \int_H \int_G \Delta_H^{-1/2}(h)\Delta_G^{1/2}(y)\Psi(\sigma h^{-1}m, y)\Phi(\sigma h^{-1}m, y^{-1})f(\lambda y^{-1}m)\eta(h, k)dhdy = [\Phi \cdot (f \cdot \Psi)](m).
\]

Besides, for \(e, f, g \in C_c(M)\), we have

\[
(D^{\sigma,u} < e, f > \cdot \eta)(m) = \int_G \int_H u(y, h)e(\sigma h^{-1}m)\overline{\Phi}(\sigma h^{-1}m)g(\lambda y^{-1}m)dhdy = (e < f, g > D_{\lambda,u^*})(m).
\]

We now prove the continuity of the module structures with respect to the inner products.

Fix a measure \(\mu\) of full support on \(M\). Then, by [Ph 6.1] and [Pd 7.7.5], we have faithful representations \(\Pi\) of \(C^{\sigma,u}\) on \(L^2(M \times G)\) and \(\Theta\) of \(Im(P_{\lambda,u^*})\) on \(L^2(M \times H)\) given by

\[
(\Pi_{\Phi} \xi)(m, x) = \int_G \Phi(\lambda x m, y)\xi(m, y^{-1}x)dx,
\]

and \((\Theta_{\Psi} \eta)(m, h) = \int_H \Psi(\sigma h m, k)\eta(m, k^{-1}h)dk,\)

where \(\Phi \in C^{\sigma,u}, \Psi \in C^{\lambda,u^*}, \xi \in L^2(M \times G)\) and \(\eta \in L^2(M \times H).\)

Now, for \(f \in C_c(M)\) and \(\eta \in L^2(M \times H)\)

\[
< \Theta < f, f >_{D_{\lambda,u^*}^*} \eta, \eta >_{L^2(M \times H)} = \int_{M \times G \times H \times H} \Delta_H^{-1/2}(k)\overline{\Phi}(\lambda y^{-1}m)\eta(\lambda y^{-1}m, k^{-1}h)dkdhdym = \|\xi(f, \eta)\|_{L^2(M \times G)}^2,
\]

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where $\xi(f, \eta) \in L^2(M \times G)$ is given by

$$(\xi(f, \eta))(m, x) = \int_H u^*(x, h^{-1}) \Delta_H^{-1/2}(h) f(\lambda_{x-1}^{-1} \sigma_h m) \eta(m, h) dh.$$ 

Then, if $\Phi \in C^{\sigma,u}$

$$\langle \xi(\Phi \cdot f, \eta) \rangle(m, x) =$$

$$= \int_G \int_H u^*(x, h^{-1}) \Delta_H^{-1/2}(h) \Delta_G^{1/2}(y) \Phi(\lambda_{x-1}^{-1} \sigma_h m, y) f(\lambda_{y-1}^{-1} \sigma_h m) \eta(m, h) dh dy =$$

$$= (U \Pi_\Phi U \xi)(f, \eta)(m, x),$$

where $U$ denotes the unitary operator on $L^2(M \times G)$ defined by $(U \xi)(m, x) = \Delta_H^{-1/2}(x) \xi(m, x^{-1})$.

Thus we have

$$< \Theta_{\Phi \cdot f, \Phi \cdot f}, \eta, \eta >_{L^2(M \times H)} = ||\xi(\Phi \cdot f, \eta)||^2 = ||U \Pi_\Phi U \xi(\Phi \cdot f, \eta)||^2 \leq$$

$$\leq ||\Phi||^2 ||\xi(f, \eta)||^2 < \Theta_{f, f}, \eta, \eta >_{L^2(M \times H)},$$

and it follows that

$$< \Phi \cdot f, \Phi \cdot f >_{D^{\lambda,u}*} \leq ||\Phi||^2 < f, f >_{D^{\lambda,u}*},$$

as elements of $D^{\lambda,u*}$. Analogously, one shows that, for $f \in C_c(M)$ and $\xi \in L^2(M \times G)$

$$< \Pi_{D^{\sigma,u}}, f >, \xi, \eta >_{L^2(M \times G)} ||\eta(f, \xi)||^2,$$

for some $\eta(f, \xi) \in L^2(M \times H)$, and that, for $\Psi \in C^{\lambda,u*}$ one has

$$\eta(f \cdot \Psi, \xi) = (V \Theta_{\Psi', V}) (\eta(f, \xi)),$$

where $V$ denotes the unitary operator in $L^2(M \times H)$ defined by

$$(V \eta)(m, h) = \Delta_H^{-1/2}(h) \eta(m, h^{-1}).$$

It follows that

$$D^{\sigma,u} < f \cdot \Psi, f \cdot \Psi > \leq ||\Psi||_{D^{\sigma,u}}^2 < f, f >,$$

as elements of $D^{\sigma,u}$.

Thus, we have proven that $C_c(M)$ is an $C^{\sigma,u} - C^{\lambda,u*}$ equivalence bimodule. Now, if we define on $C_c(M)$ the norms

$$||f||_{D^{\sigma,u}}^2 = ||D^{\sigma,u} f, f >|| \quad \text{and} \quad ||f||_{D^{\lambda,u*}}^2 = ||f, f >_{D^{\lambda,u*}}||,$$

it follows from [Rf1, 3.1] that $||f||_{D^{\sigma,u}} = ||f||_{D^{\lambda,u*}}$ and that the completion of $C_c(M)$ with respect to this norm gives, by continuity, an equivalence bimodule between $D^{\sigma,u}$ and $D^{\lambda,u*}$.
Corollary 2.12  With the assumptions of Theorem 2.11, the algebras $C^*_r(H, C_0(M) \times \chi G)$ and $C^*_r(G, C_0(M) \times \sigma H)$ are strong-Morita equivalent.

Proof:
The proof follows from Proposition 2.3, Theorem 2.11, and [Rf4, 1.7].

Q.E.D.

3 Applications to quantum Heisenberg manifolds.

In this section we apply the previous results to the computation of the K-groups of the quantum Heisenberg manifolds. We recall the basic results and definitions concerning those algebras. We refer the reader to [Rf2] for further details.

For each positive integer $c$, the Heisenberg manifold $M_c$ consists of the quotient $G/D_c$, where $G$ is the Heisenberg group

$$ G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} ; \text{ for real numbers } x, y, z \right\} $$

and $D_c$ is the discrete subgroup obtained when $x, y$ and $cz$ above are integers.

The set of non-zero Poisson brackets on $M_c$ that are invariant under the action of $G$ by left translation can be parametrized by two real numbers $\mu$ and $\nu$, with $\mu^2 + \nu^2 \neq 0$. A deformation quantization $\{D^c_{\mu \nu}\}_{h \in \mathbb{R}}$ of $M_c$ in the direction of a given invariant Poisson bracket $\Lambda_{\mu \nu}$ was constructed in [Rf5].

The algebra $D^c_{\mu \nu}$ can be described as a generalized fixed-point algebra as follows. Let $M = R \times T$ and $\lambda^h$ and $\sigma$ be the commuting actions of $Z$ on $M$ induced by the homeomorphisms

$$ \lambda^h(x, y) = (x + 2h\mu, y + 2h\nu) \quad \text{and} \quad \sigma(x, y) = (x - 1, y). $$

Consider the action $\rho$ of $Z$ on $C_0(R \times T) \times \lambda^h Z$ given by

$$ (\rho_k \Phi)(x, y, p) = e(ckp(y - hp\nu))\Phi(x + k, y, p), $$

$$ k \in \mathbb{R}, \quad \Phi \in C_0(R \times T) \times \lambda^h Z. $$
where \( e(x) = \exp(2\pi i x) \) for any real number \( x \). The action \( \rho \) defined above corresponds to the action \( \rho \) defined in [Rf5, p.539], after taking Fourier transform in the third variable to get the algebra denoted in that paper by \( A_h \), and viewing \( A_h \) as a dense *-subalgebra of \( C_0(R \times T) \times_{\lambda^h} \mathbb{Z} \) via the embedding \( J \) defined in [Rf5, p.547].

Notice that, for \( M = R \times T, \ G = H = \mathbb{Z}, \) and \( h \neq 0 \), the actions \( \lambda^h \) and \( \sigma \) satisfy the hypotheses of Section 3 and that the action \( \rho \) defined above corresponds, in that context, to the action we denoted by \( \gamma_{\sigma,u} \), where \( u : \mathbb{Z} \times \mathbb{Z} \to \mathcal{U}M(C_0(R \times T)) \) is the 2-cocycle defined by
\[
u \mapsto e(ckp(y - \bar{h}p\nu)),
\]
for \( p, k \in \mathbb{Z} \). Besides, [Rf5, Theorem 5.4] shows that the algebra \( D^c_{\mu \nu} \) consisting of functions \( \Phi \in C_c(\beta(R \times T) \times \mathbb{Z}) \) satisfying \( \rho_k(\Phi) = \Phi \) for all \( k \in \mathbb{Z} \) is a dense *-subalgebra of \( D^c_{\mu \nu} \). This follows from Remark 2.4, Proposition 2.7, and from the fact that \( R \times T/\sigma \) is compact.

**Remark 3.1** We will also use the fact that the algebra \( \hat{D}^c_{\mu \nu} \) consisting of functions \( \Phi \in C_c(\beta(R \times T) \times \mathbb{Z}) \) satisfying \( \rho_k(\Phi) = \Phi \) for all \( k \in \mathbb{Z} \) is a dense *-subalgebra of \( D^c_{\mu \nu} \). This follows from Remark 2.4, Proposition 2.7, and from the fact that \( R \times T/\sigma \) is compact.

**Theorem 3.2** For \( h \neq 0 \) the K-groups of \( D^c_{\mu \nu} \) do not depend on \( h \).

**Proof:**

It follows from Theorem 2.11 that, for \( h \neq 0 \), \( D^c_{\mu \nu} \) is strong-Morita equivalent to the generalized fixed-point algebra \( E^c_{\mu \nu} \) of \( C_0(R \times T) \times_{\sigma} \mathbb{Z} \) under the action \( \gamma^h \) of \( \mathbb{Z} \) defined by
\[
(\gamma^h_{\mu}(\Phi))(x, y, k) = e(ckp(y - \bar{h}p\nu))\Phi(x - 2p\mu, y - 2p\nu, k).
\]
Now, by Proposition 2.3, \( \gamma^h \) is saturated, so we have ([Rf4, Corollary 1.7]) that \( D^c_{\mu \nu} \) is strong-Morita equivalent to \( C_0(R \times T) \times_{\sigma} \mathbb{Z} \times_{\gamma^h} \mathbb{Z} \).

Besides, \( h \mapsto \lambda^h \) is a homotopy between the \( \lambda^h \)'s, which shows ([3, 10.5.2]) that the K-groups of \( C_0(R \times T) \times_{\sigma} \mathbb{Z} \times_{\gamma^h} \mathbb{Z} \) do not depend on \( h \).

On the other hand, since strong-Morita equivalent separable C*-algebras are stably isomorphic ([BGR]) and therefore have the same K-groups, we have proven that the K-groups of \( D^c_{\mu \nu} \), for \( h \neq 0 \), do not depend on \( h \).
Notation. Since the algebras $D_{c,h}^\mu \bar{\nu}$ and $D_{h,\mu,\nu}^{c,1}$ are isomorphic, we drop from now on the constant $\hbar$ from our notation and absorb it into the parameters $\mu, \nu.$

Remark 3.3 Notice that, since for any pair of integers $k$ and $l$ the algebras $D_{c,\mu}^\nu$ and $D_{c,\mu+k,\nu+l}$ are isomorphic ([A7]), the assumption $\hbar \neq 0$ in Theorem 3.2 can be dropped.

Theorem 3.4 $K_0(D_{c,\mu}^\nu) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_c$ and $K_1(D_{c,\mu}^\nu) \cong \mathbb{Z}^3.$

Proof: In view of Theorem 3.2 and Remark 3.3, it suffices to prove the theorem for the commutative case where $D_{c,\mu}^\nu = C(M_c).$

After reparametrizing the Heisenberg group we get that $M_c = G/H_c$ where

$$G = \left\{ \begin{pmatrix} 1 & y & z/c \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

and

$$H_c = \left\{ \begin{pmatrix} 1 & m & p/c \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : m, p, q \in \mathbb{Z} \right\}$$

We first use [Ro, Corollary 3] to reduce the proof to the computation of the K-theory of $C^*(H_c).$

The group $C^*$-algebra $C^*(H_c)$ is strong-Morita equivalent to $C(G/H_c) \times G,$ where $G$ acts by left translation [Rf2, Example 1]. Now, $G$ is nilpotent and simply connected so we have

$$G = R \times \mathbb{R} \times \mathbb{R}$$

as a semi-direct product.

Therefore

$$C(G/H_c) \times G \simeq C(G/H_c) \times R \times \mathbb{R},$$

and Connes’-Thom isomorphism ([B, 10.2.2]) gives

$$K_i(C^*(H_c)) = K_i(C(G/H_c) \times G) = K_{1-i}(C(G/H_c)) = K_{1-i}(C(M_c)).$$
So it suffices to compute $K_i(C^*(H_c))$. The computation was made in [AP, Prop. 1.4] for the case $c=1$, and the general case can be obtained with slight modifications to their proof. We first write $H_c$ as a semi-direct product, so its group $C^*$-algebra can be expressed as a crossed product algebra. Then, by using the Pimsner-Voiculescu exact sequence ([Bl, 10.2.1]), we get its $K$-groups.

Let

$$N = \left\{ \begin{pmatrix} 1 & m & p/c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : m, p, c \in Z \right\} \quad \text{and} \quad K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : q \in Z \right\}$$

Then $H_c = N \rtimes_{\alpha_c} K$, where $\alpha_c$ is conjugation. If we identify in the obvious way $N$ and $K$ with $Z^2$ and $Z$ respectively, we have that $H_c \cong Z^2 \rtimes_{\alpha_c} Z$, where $\alpha_c(q)(m, p) = (m, p - cmq)$. Then the Pimsner-Voiculescu exact sequence yields:

$$\begin{align*}
K_0(C(T^2)) & \xrightarrow{id - \alpha_c^*} K_0(C(T^2)) \quad \xrightarrow{i_*} \quad K_0(H_c) \\
\delta & \quad \uparrow \quad \delta \\
K_1(H_c) & \quad \xleftarrow{i_*} \quad K_1(C(T^2)) \quad \xrightarrow{id - \alpha_c^*} \quad K_1(C(T^2))
\end{align*}$$

It was shown on [AP, Prop. 1.4] that $id = \alpha_1$ on $K_0(C(T^2))$ and, since $\alpha_c = \alpha_1^c$, it follows that $id = \alpha_c$ on $K_0(C(T^2))$ for any $c$. Thus we get the following short exact sequences:

$$
\begin{align*}
0 & \rightarrow \ Z^2 \rightarrow K_0(H_c) \xrightarrow{\delta} \text{Ker}(id - \alpha_c) \rightarrow 0 \\
0 & \rightarrow K_1(C(T^2))/\text{Ker}(id - \alpha_c) \rightarrow K_1(H_c) \xrightarrow{\delta} \ Z^2 \rightarrow 0 
\end{align*}
$$

where $id - \alpha_c$ is the map on $K_1(C(T^2))$.

Let us now compute $id - \alpha_c$ on $K_1(C(T^2))$. We have identified $C(T^2)$ with $C^*(Z^2)$ via Fourier transform, so the automorphism $\alpha_c$ on $C(T^2)$ becomes $(\alpha_c f)(x, y) = f(x - cy, y)$. Now, $K_1(C(T^2)) = Z^2$ if we identify $[u_1]_{K_1}$ and $[u_2]_{K_2}$ with $(1, 0)$ and $(0, 1)$ in $Z^2$, respectively, where $u_1(x, y) = e(x)$, $u_2(x, y) = e(y)$ for all $(x, y) \in T^2$. Then, for $(a, b) \in Z^2$ we have

$$(id - \alpha_c)(a, b) = (a, b) - (a, b - ac) = (0, ac).$$
This shows that
\[ \text{Ker}(\text{id} - \alpha_{c_e}) = Z \oplus \{0\} \subset Z^2, \quad \text{Im}(\text{id} - \alpha_{c_e}) = \{0\} \oplus cZ \subset Z^2. \]

So the exact sequences above become:
\[
\begin{align*}
0 & \rightarrow Z^2 \rightarrow K_0(H_c) \rightarrow Z \rightarrow 0 \\
0 & \rightarrow Z + Z_c \rightarrow K_1(H_c) \rightarrow Z^2 \rightarrow 0.
\end{align*}
\]
Therefore
\[
K_1(D_{\mu\nu}^c) = K_0(H_c) = Z^3 \quad \text{and} \quad K_0(D_{\mu\nu}^c) = K_1(H_c) = Z^3 + Z_c.
\]

Q.E.D

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