A VARIANT PROOF OF Con(b < a)

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Abstract. We present a variation of the proof in [2] of Con(b < a), which in particular removes some of the obstacles to generalising the argument to cardinals κ > ω.

§1. Introduction. The generalisations of cardinal characteristics of the continuum to cardinals κ greater than ω has generated significant interest recently. A particular result that has so far resisted attempts at generalisation is the statement that b < a is consistent. Blass, Hyttinen and Zhang [1, Section 5] briefly survey the different approaches known for proving Con(b < a), highlighting the difficulties each presents for a generalisation.

We present here a variation on the proof of Con(b < a) given in [2], which we hope will be more amenable to generalisation. In particular, the proof in [2] relies on a rank argument, which of course cannot be naively generalised to uncountable κ. We show here that it may be replaced by a suitable formulation in terms of games, which does generalise to higher κ. Indeed, with this observation, the question of forcing bκ > aκ for some suitable large cardinal κ seems to boil down to interesting questions about the existence of suitable filters on κ.

§2. Preliminaries. Let κ be an infinite cardinal. A family A ⊆ [κ]κ is called almost disjoint if |A ∩ B| < κ for any two distinct members A and B of A. A is a maximal almost disjoint family (mad family, for short) if A is almost disjoint and maximal with this property. This means that for every C ∈ [κ]κ there is A ∈ A such that |A ∩ C| = κ. The almost disjointness number aκ is the least

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size of a mad family on $\kappa$ of size at least $cf(\kappa)$ (equivalently, of size $> cf(\kappa)$). In case $\kappa = \omega$ write $a$ for $a_\omega$.

Now assume $\kappa$ is a regular cardinal. For functions $f, g \in \kappa^\kappa$, say that $g$ eventually dominates $f$ ($f \leq^* g$ in symbols) if $f(\alpha) \leq g(\alpha)$ holds for all $\alpha$ beyond some $\alpha_0 < \kappa$. The unbounding number $b_\kappa$ is the least size of an unbounded family $F$ in the order $\langle \kappa^\kappa, \leq^* \rangle$. That is, for all $g \in \kappa^\kappa$ there is $f \in F$ with $f(\alpha) > g(\alpha)$ for cofinally many $\alpha$'s. Again we write $b$ instead of $b_\omega$.

Let $F$ be a filter on $\omega$. Mathias forcing $\mathbb{M}(F)$ with $F$ consists of conditions $\langle s, F \rangle$ such that $s \in [\omega]^<\omega$, $F \in F$, and $\max(s) < \min(F)$. $\mathbb{M}(F)$ is ordered by $\langle t, G \rangle \leq \langle s, F \rangle$ if $s \subseteq t \subseteq s \cup F$ and $G \subseteq F$. It is well-known and easy to see that $\mathbb{M}(F)$ is a $\sigma$-centered forcing which introduces a pseudointersection $Z$ of the filter $F$. This means that $Z \subseteq^* F$ for all $F \in F$, where $\subseteq^*$ denotes almost inclusion: $A \subseteq^* B$ iff $A \setminus B$ is finite.

In [2], the notion of pseudocontinuity is used. This notion and the corresponding basic lemma can be nicely phrased in terms of continuity with respect to an appropriate topology.

**Definition 1.** The initial segment topology on $\omega$ is the topology which has the (von Neumann) ordinals as open sets. We denote $\omega$ endowed with this topology by $\omega_i$.

**Definition 2.** A function to $\omega$ or $\omega^\omega$ is pseudocontinuous if it is continuous as a function to $\omega_i$ or $\omega^\omega_i$ respectively.

Thus, a pseudocontinuous function $F : X \to \omega$ is one such that for every $n \in \omega$, the set of $x$ in $X$ with image at most $n$ is open.

**Lemma 3.** Compact sets in $\omega_i$ and $\omega_i^\omega$ are bounded. In particular, any pseudocontinuous image in $\omega$ or $\omega^\omega$ of a compact set must be bounded.

**Proof.** The Lemma is clear for $\omega_i$. Similarly, compact $K \subset \omega_i^\omega$ are in fact bounded in the strict (not just $\leq^*$) sense. Otherwise, there would be some $m$ in $\omega$ such that $f(m)$ is unbounded in $\omega$ for $f \in K$, and then the open sets $O_{m,n} = \{ f \in \omega_i^\omega \mid f(m) \leq n \}$ for $n < \omega$ would form an open cover of $K$ with no finite subcover. $\Box$

As usual we may identify $\mathcal{P}(\omega)$ with $2^\omega$ by way of the map taking sets to their characteristic functions, $\chi : X \mapsto \chi_X$. We give $\mathcal{P}(\omega)$ the corresponding topology, making $\chi$ a homeomorphism from $\mathcal{P}(\omega)$ to the Cantor space $2^\omega$. 


A V A R I A N T  P R O O F  O F  C o n ( b < a )

**Definition 4.** For any cardinal \( \lambda \), we call a filter \( \mathcal{G} \subseteq \mathcal{P}(\omega) \) a \( K_\lambda \)-filter if it is generated by the union of fewer than \( \lambda \) many compact subsets of \( \mathcal{P}(\omega) \). We write \( K_\sigma \) for \( K_{\aleph_1} \).

**Lemma 5.** If \( K_0, \ldots, K_{n-1} \) are (finitely many) compact subsets of \( \mathcal{P}(\omega) \), then the pointwise intersection

\[
\bigwedge_{i<n} K_i = \left\{ \bigcap_{i<n} G_i \mid (G_0, \ldots, G_{n-1}) \in \prod_{i<n} K_i \right\}
\]

and the pointwise union

\[
\bigvee_{i<n} K_i = \left\{ \bigcup_{i<n} G_i \mid (G_0, \ldots, G_{n-1}) \in \prod_{i<n} K_i \right\}
\]

are compact. Furthermore, for any compact set \( K \subseteq \mathcal{P}(\omega) \), the upward closure

\[
\bar{K} = \{ A \in \mathcal{P}(\kappa) \mid \exists B \in K (A \supseteq B) \}
\]

is also compact.

**Proof.** The product \( \prod_{i<n} K_i \) is compact by the Tychonoff theorem, and the functions \( \mathcal{P}(\omega)^n \to \mathcal{P}(\omega) \) given by \( (G_0, \ldots, G_{n-1}) \mapsto \bigcap_{i<n} G_i \) and \( (G_0, \ldots, G_{n-1}) \mapsto \bigcup_{i<n} G_i \) are clearly continuous, so \( \bigwedge_{i<n} K_i \) and \( \bigvee_{i<n} K_i \) are compact. Finally, for compact \( K \subseteq \mathcal{P}(\omega) \), \( \bar{K} \) is just \( K \cup \mathcal{P}(\omega) \).

\( \Box \)

§3. **The proof.** We work in a model \( V \) of ZFC in which \( \lambda = \mathfrak{c}^V \) is a regular cardinal satisfying \( 2^\lambda = \lambda^+ \), and there is an unbounded, \(<^*\)-well-ordered sequence \( \langle f_\alpha : \alpha < \lambda \rangle \) of strictly increasing functions from \( \omega \) to \( \omega \). For example, any model of GCH will suffice as a ground model, and these properties will be preserved in intermediate stages of our forcing iteration.

Let \( \mathcal{A} \) be an infinite maximal almost disjoint family in \( V \) of subsets of \( \omega \).

**Theorem 6.** There is a ccc forcing \( \mathbb{P}(\mathcal{A}) \) such that

\[ \models_{\mathbb{P}(\mathcal{A})} \mathcal{A} \text{ is not mad and } \langle f_\alpha : \alpha < \lambda \rangle \text{ is still unbounded.} \]

**Proof.** Let \( \mathcal{F} = \mathcal{F}(\mathcal{A}) \) be the dual filter of \( \mathcal{A} \), that is, the filter generated by the sets whose complements are finite or in \( \mathcal{A} \). Note that this filter is proper: if for some \( k < \omega \) there were \( \{ A_i \mid i < k \} \subseteq \mathcal{A} \) such that \( |\bigcap_{i<k} \omega \setminus A_i| < \omega \), any other element of \( \mathcal{A} \) would have infinite intersection with one of the \( A_i \), violating almost disjointness. Note that the generic subset of \( \omega \) introduced by Mathias forcing with
\( \mathcal{F} \), or any filter extending \( \mathcal{F} \), will end the madness of \( A \), as it will be almost contained in \( \omega \setminus A \) for every \( A \in \mathcal{A} \).

First we add \( \lambda \) many Cohen reals. It is well-known that the unboundedness of \( \langle f_\alpha : \alpha < \lambda \rangle \) is preserved in this intermediate extension. In case \( \mathcal{A} \) is not mad anymore in this extension we are done. Also, if \( \mathcal{F} \) is contained in a \( K_\lambda \) filter \( \mathcal{G} \) in the intermediate extension, we may simply force with \( \mathbb{M}(\mathcal{G}) \) for it is well-known, and easy to see [2, 3.2], that Mathias forcing with a \( K_\lambda \)-filter does not destroy the unboundedness of \( \langle f_\alpha : \alpha < \lambda \rangle \). So assume that \( \mathcal{F} \) is not contained in any \( K_\lambda \)-filter.

We shall recursively construct a filter \( \mathcal{G} \supseteq \mathcal{F} \) such that furthermore (*): \( \mathbb{M}(\mathcal{G}) \langle f_\alpha : \alpha < \lambda \rangle \) is unbounded.

Along the construction we shall take care of every potential \( \mathbb{M}(\mathcal{G}) \)-name for a function in \( \omega^\omega \), either “killing it” or “sealing it off”.

To be precise: let us refer to partial functions \( \tau : [\omega]^<\omega \times \omega \rightarrow \omega \) as preterms, and let \( \mathcal{T} = \{ \tau_\beta : \beta < \lambda \} \) be an enumeration of the set of all preterms. Note in particular that if \( \mathcal{G} \supseteq \mathcal{F} \) is a filter and \( \dot{g} \) is an \( \mathbb{M}(\mathcal{G}) \)-name for a function in \( \omega^\omega \), then \( \tau = \tau_\beta \) given by

\[
\tau(s, m) = n \text{ iff } \exists G \in \mathcal{G} ( (s, G) \models \dot{g}(m) = n )
\]

is a preterm, the preterm associated with \( \dot{g} \). We shall constrain attention to names \( \dot{g} \) such that \( 1 \models \mathbb{M}(\mathcal{H}) \dot{g} \in \omega^\omega \), since every function from \( \omega \) to \( \omega \) in the generic extension has such a name; we call such names total names.

We construct filters \( \mathcal{G}_\beta \) for \( 0 \leq \beta \leq \lambda \), starting from \( \mathcal{G}_0 = \mathcal{F} \), such that

- for each \( \beta < \lambda \), \( \mathcal{G}_{\beta+1} \) is generated by \( \mathcal{G}_\beta \) and a \( K_\sigma \) filter \( \mathcal{H}_\beta \),
- \( \mathcal{G}_\delta = \bigcup_{\beta < \delta} \mathcal{G}_\beta \) for each limit ordinal \( \delta \leq \lambda \),

and either

(KILL): for all filters \( \mathcal{H} \supseteq \mathcal{G}_{\beta+1} \), \( \tau_\beta \) is not associated with any total \( \mathbb{M}(\mathcal{H}) \)-name, or

(SEAL): there is an \( \alpha < \lambda \) such that for all filters \( \mathcal{H} \supseteq \mathcal{G}_{\beta+1} \) and all \( \mathbb{M}(\mathcal{H}) \)-names \( \dot{g} \), if \( \tau_\beta = \tau_\beta \) then \( 1 \models \mathbb{M}(\mathcal{H}) \dot{g} \not\subseteq^* \dot{f}_\alpha \).

Clearly any filter \( \mathcal{G} \supseteq \mathcal{G}_\lambda \) will then satisfy (*).

So suppose \( \mathcal{G}_\beta \) has been defined for some \( \beta < \lambda \); we wish to find an appropriate \( K_\sigma \) filter \( \mathcal{H}_\beta \). Note that \( \mathcal{G}_\beta \) is generated by \( \mathcal{F} \) and a \( K_\lambda \) filter \( \mathcal{G}'_\beta \); without loss of generality we may assume that \( \mathcal{F} \) contains all cofinite subsets of \( \omega \). Let \( \mathcal{K}_\beta \) be a family of fewer than \( \lambda \) many compact subsets of \( 2^\omega \) generating \( \mathcal{G}'_\beta \). By Lemma 5 we may assume...
that $\mathcal{K}_\beta$ is closed under finite pointwise intersections, and that for all $K \in \mathcal{K}_\beta$, $K$ is upwards-closed under $\subseteq$, so that $G'_\beta = \bigcup \mathcal{K}_\beta$.

Everything that has come so far can actually be considered to have occurred in a partial extension model, between the original model and the full extension with $\lambda$-many Cohens. More explicitly, all (codes of) elements of $\mathcal{K}_\beta$ belong to this intermediate model.

Let $\subset \subseteq$ denote the strict end-extension relation on $[\omega]^{<\omega}$: that is, $s \subset s'$ if and only if $s \subset s'$ and $\max(s) < \min(s' \setminus s)$; define $\subseteq$, $\supseteq$ and $\supseteq$ accordingly.

In [2], a rank function was used. For our generalisation, we take a different approach using games, but use these games to much the same end as the rank function is used in [2]. It should be noted that our games are very closely related to the games independently introduced by Guzmán, Hrušák, and Martínez [3], also in the context of a proof of Con($b < a$).

Let $\tau = \tau_\beta$.

**Definition 7.** Given $\tau \in \mathcal{T}$, the $\tau$ nominalisation exercise is the following game. There are two players, Sensei and Student. On turn 0, Sensei chooses an $m \in \omega$ and $t_0 \in [\omega]^{<\omega}$. At odd stages $2d + 1$, Student plays a filter set $F(d) \in \mathcal{F}$ and a compact set $K(d) \in \mathcal{K}_\beta$. At even stages $2d + 2$, Sensei plays an element $t_{d+1}$ of $[\omega]^{<\omega}$ such that

- $t_{d+1}$ end-extends $t_d$
- $t_{d+1} \setminus t_d \subseteq F(d)$
- $t_{d+1} \setminus t_d$ meets every member of $K(d)$.

If there is $s \subseteq t_{d+1}$ end extending $t_0$ such that $(s, m) \in \text{dom}(\tau)$, Sensei declares Student to have passed and the game ends. If the game continues for infinitely many stages, then (clearly) Student has failed.

Note that, since $G_\beta$ is a filter, and by compactness of $K(d)$, a $t_{d+1}$ satisfying the requirements always exists. Also notice that if Student wins, he wins after finitely many steps. Hence the game is open and, by the classical Gale-Stewart Theorem, determined.

As in [2], we now distinguish two cases (in [2] they are Subcases), corresponding to options (KILL) and (SEAL) above.

**3.1. Case a.** There are $m \in \omega$ and $t_0 \in [\omega]^{<\omega}$ such that Sensei has a winning strategy in the $\tau$ nominalisation exercise with 0th move $(m, t_0)$: play will continue for infinitely many steps. In this case we
shall choose $\mathcal{H}_\beta$ in such a way that (KILL) holds: $\tau$ will not correspond to a name for a function $\omega \to \omega$ in the generic extension. The reader may wish to remember which case is which by the mnemonic "the $\tau$ that can be named is not the eternal $\tau$.

We shall actually work in the extension of such the intermediate model by one further Cohen function $c : \omega \to \omega$.

Consider the tree $T$ of all possible sequences of plays $(t_0, t_1, t_2, \ldots)$ for Sensei according to his strategy, corresponding to all possible plays of Student. Note that $T$ is infinitely branching since $\mathcal{F}$ extends the Frechet filter. Use the Cohen function $c$ to choose a branch through $T$, and denote the union of the $t_i$ of this branch by $G$.

There is no $(s, m)$ with $m$ from Sensei’s first move and $t_0 \subseteq s \subseteq G$ such that $(s, m) \in \text{dom}(\tau_\beta)$. Indeed otherwise, the $\tau_\beta$ nominalisation exercise would have ended once Sensei played $t_d$ sufficiently long to cover $s$. Thus, for any filter $\mathcal{H} \ni G$, $\tau \neq \tau_\beta$ for any total $M(\mathcal{H})$ name $\dot{g}$. We may therefore simply take $\mathcal{H}_\beta = \{G\}$ in order to satisfy (KILL). To check that $\{G\} \cup G_\beta$ generates a filter, consider any $F \in \mathcal{F}$ and $G' \in G'_\beta$, say $G'$ is in the compact set $K \in K_\beta$. For every $t_d \in T$, there is a successor node $t_{d+1}$ in the tree $T$ that is Sensei’s response, according to his strategy, to Student playing $F$ and $K$, and so in particular this $t_{d+1}$ meets the intersection of $F$ and every member of $K$. Thus, by Cohen genericity we have that $|G \cap F \cap G'| = \omega$, completing Case a. (Note that $G'$ may not belong to the intermediate model; this, however, is irrelevant for it is sufficient that $K$ does. By genericity the Cohen real $c$ will produce infinitely many $d$ such that $t_{d+1} \setminus t_d$ is contained in $F$ and meets every $G'' \in K$, and this is clearly absolute and thus also holds for $G'$.)

3.2. **Case b.** The negation of Case a: for every 0th move $(m, t_0)$ by Sensei, Student has a winning strategy in the $\tau_\beta$ nominalisation exercise. In this case we wish to choose $\mathcal{H}_\beta$ in such a way that (SEAL) holds.

Since Sensei chooses his moves from a countable set, there are clearly only countable many filter sets $F_\ell \in \mathcal{F}$, $\ell \in \omega$, which appear as $F(d)$ in some $2d + 1$st move of Student playing according to his strategy.

Suppose that for all but less than $\lambda$ many members $A$ of $\mathcal{A}$, there is $G \in G'_\beta$ such that $A \cap G$ is finite. Then, adding less than $\lambda$ many sets of the form $\omega \setminus A$, $A \in \mathcal{A}$, to $G'_\beta$ results in a $K_\lambda$ filter containing $\mathcal{F}$. This contradicts our initial assumption. Hence, for $\lambda$ many $A \in \mathcal{A}$, $A \cap G$ is infinite for all $G \in G'_\beta$. Let $A_j$, $j \in \omega$, be countably many
such $A$'s such that for each $j$ and $\ell$, $A_j$ is almost contained in $F_\ell$: this is possible because $F$ is the dual filter of the mad family $A$.

For each $G' \in \mathcal{G}_\beta'$, $k \in \omega$, $j \in \omega$, and finite subset $T$ of $[\omega]^{<\omega}$, we define a function $f_{G',k,j,T} : \omega \to \omega$ as follows.

$$f_{G',k,j,T}(m) = \min\{n \mid \text{for any partition } A_j = \bigcup_{i<k} B_i \text{ there is } i < k \text{ s.t. }$$

$$\forall t \in T \exists s \supseteq t (s \cap B_i \cap G' \land \tau_\beta(s,m) \leq n)\}.$$ 

**Lemma 8.** For every $G' \in \mathcal{G}_\beta'$, $k,j \in \omega$, and $T \in [[\omega]^{<\omega}]^\omega$, $f_{G',k,j,T}$ is well-defined.

**Proof.** Fix $m \in \omega$. Given a partition $\{B_i \mid i < k\}$ of $A_j$, let "$n$ suffices for $\{B_i \mid i < k\}$" mean the natural thing in the context of the definition of $f_{G',k,j,T}$, namely, that there is $i < k$ such that for every $t \in T$ there is $s \supseteq t$ with $s \cap B_i \subseteq B_i \cap G'$ and $\tau_\beta(s,m) \leq n$. So now fix a partition $\{B_i \mid i < k\}$ of $A_j$; we shall show that there is a $n \in \omega$ that suffices for it. Let $i < k$ be such that $|B_i \cap G' \cap G| = \omega$ for every $G \in \mathcal{G}_\beta'$: such an $i$ must exist, since $A_j$ has infinite intersection with every member of the filter $\mathcal{G}_\beta'$. Finally, fix $t \in T$.

Consider a play of the $\tau_\beta$ naming exercise in which Student follows his strategy, Sensei's 0th move is $(m,t_0)$ with $t_0 = t$, and his later moves always satisfy the additional requirement $t_{d+1} \cap t_d \subseteq B_i \cap G'$. Since $B_i$ is almost contained in all $F(d)$ played by Student according to his strategy and since $B_i$ has infinite intersection with all $G \in \mathcal{G}_\beta'$, Sensei always has a valid such move.

So we have that eventually Sensei plays a $t_d$ such that

$$\exists n_t \in \omega \exists s \subseteq t_d (s \supseteq t \land \tau_\beta(s,m) = n_t).$$

Of course, by the construction of the game, $s \cap t_0 \subseteq B_i \cap G'$. Taking such an $n_t$ for each $t \in T$ and setting $n = \max_{t \in T}(n_t)$, we have that $n$ suffices for $\{B_i \mid i < k\}$.

Now, with $k$ still fixed but allowing the partition $\{B_i \mid i < k\}$ to vary, let us denote by $n(\{B_i \mid i < k\})$ the least $n$ that suffices for $\{B_i \mid i < k\}$. The space of partitions of $A_j$ into $k$ pieces can be identified with $k^{A_j}$ and thus when endowed with the product topology is a compact topological space. Moreover, with this topology on the space of partitions, the function $n$ sending $\{B_i \mid i < k\}$ to $n(\{B_i \mid i < k\})$ is clearly pseudocontinuous, since $n$ being sufficient for $\{B_i \mid i < k\}$ is witnessed by finitely many finite tuples $s \cap t$ from $B_i$, which of course define an open set in $k^{A_j}$. Thus by Lemma 3 the image of
the function \( n \) is bounded below \( \omega \). The least such upper bound will be \( f_{G',k,j,T}(m) \), and it follows that \( f_{G',k,j,T} \) is well-defined. \( \dashv \)

**Lemma 9.** There exists an \( \alpha < \lambda \) such that for all \( G' \in \mathcal{G}'_\beta \), \( k, j \in \omega \) and \( T \in [[\omega]^{<\omega}]^{<\omega} \), \( f_\alpha \not\leq^* f_{G',k,j,T} \).

**Proof.** We first note that, given \( k, j \), \( T \), and compact \( K \in \mathcal{K}_\beta \), the function \( f_{k,j,T} \) sending \( G' \) to \( f_{G',k,j,T}(m) \) is pseudocontinuous from \( K \) to \( \omega^\omega \), by much the same argument as in the proof of Lemma 8. Indeed, fixing \( m \) and \( n \), \( \{G' \mid f_{G',k,j,T}(m) \leq n\} \) is open in \( K \).

We thus have from Lemma 3 that for each \( K \in \mathcal{K}_\beta \), \( f_{k,j,T} \) is bounded in \( \omega^\omega \), say by \( h_K \). Since \( \mathcal{K}_\beta \) has fewer than \( \lambda \) many elements, there is an \( \alpha < \lambda \) such that \( f_\alpha \) is not eventually dominated by any of the \( h_K \), and hence not by any \( f_{G',k,j,T} \). \( \dashv \)

We now show that \( \alpha \) as given by Lemma 9 will make (SEAL) hold for an appropriate choice of \( \mathcal{H}_\beta \). Given \( t \in [[\omega]^{<\omega}]^{<\omega} \), \( G \in \mathcal{P}(\omega) \), and \( m \in \omega \), let

\[
g^\beta_{t,G}(m) = \min\{n \mid \exists s \supseteq t(s \setminus t \subseteq G \land \tau_\beta(s, m) = n)\}
\]

if the set on the right hand side is non-empty, and otherwise put \( g^\beta_{t,G}(m) = \omega \). Thus, \( g^\beta_{t,G} \) is a function in \((\omega + 1)^\omega\). Let \( \alpha < \lambda \) be such that \( f_\alpha \) is not dominated by any \( f_{G',k,j,T} \), as given by Lemma 9, and define

\[
\mathcal{H}_\beta = \{H \subseteq \omega \mid \exists t \in [[\omega]^{<\omega}]^{<\omega} g^\beta_{t,\omega \setminus H} \geq^* f_\alpha\}.
\]

Note that given \( t \in [[\omega]^{<\omega}]^{<\omega} \) and \( m_0 \in \omega \), the set

\[
\{H \subseteq \omega \mid \forall m \geq m_0 (g^\beta_{t,\omega \setminus H}(m) \geq f_\alpha(m))\}
\]

is closed in \( \mathcal{P}(\omega) \), and hence compact. Therefore, \( \mathcal{H}_\beta \) is a \( K_\sigma \) set.

To see that this set is an appropriate choice of \( \mathcal{H}_\beta \) as called for above, we check the following.

**Claim 10.** Any filter \( \mathcal{H} \supseteq \mathcal{H}_\beta \) satisfies (SEAL).

**Proof.** Let \( \mathcal{H} \supseteq \mathcal{H}_\beta \) be a filter, and assume \( \tau_\beta = \dot{\tau}_g \) for some \( \dot{M}(\mathcal{H}) \)-name \( \dot{g} \) for a function in \( \omega^\omega \). Suppose there were \( (t, G) \in \dot{M}(\mathcal{H}) \) and \( m_0 \in \omega \) such that

\[
(t, G) \models_{\dot{M}(\mathcal{H})} \forall m \geq m_0 \left( \dot{g}(m) \geq \dot{f}_\alpha(m) \right).
\]

By the definition of \( g^\beta_{t,G} \), we must then also have \( g^\beta_{t,G}(m) \geq f_\alpha(m) \) for all \( m \geq m_0 \). So \( \omega \setminus G \in \mathcal{H}_\beta \subseteq \mathcal{H} \), contradicting the fact that \( \mathcal{H} \) is a filter. \( \dashv \)
CLAIM 11. \( H_\beta \cup G_\beta \) generates a filter.

PROOF. We take \( F \in \mathcal{F}, G' \in G'_\beta \), and for some \( k < \omega, H_i \in H_\beta \) for \( i < k \), and argue that \( F \cap G' \cap \bigcap_{i<k} H_i \) has cardinality \( \omega \). Assume for the sake of contradiction that \( F \cap G' \subseteq \bigcup_{i<k} \omega \setminus H_i \). For each \( i < k \), fix \( t_i \in [\omega]^{< \omega} \) such that \( g_{t_i, \omega \setminus H_i}^\beta \geq^* f_\alpha \). Also fix \( j \) such that \( A_j \subseteq^* F \). Without loss of generality, we may take \( a < \omega \) such that \( A_j \setminus a \subseteq F, F \cap G' \setminus a \subseteq \bigcup_{i<k} \omega \setminus H_i \) and \( \max(t_i) \geq a \) for every \( i < k \) (if necessary by extending each \( t_i \) with a sufficiently large element of \( \omega \setminus H_i \); this can only increase the values of \( g_{t_i, \omega \setminus H_i}^\beta \)). Fix \( m_0 \in \omega \) such that \( g_{t_i, \omega \setminus H_i}^\beta (m) \geq f_\alpha (m) \) for all \( m \geq m_0 \) and \( i < k \).

Let \( T = \{ t_i \mid i < k \} \) and let \( \{ B_i \mid i < k \} \) be a partition of \( A_j \) such that \( B_i \cap G' \setminus a \subseteq \omega \setminus H_i \) for all \( i < k \). By the definition of \( f_\alpha \), there is some \( m > m_0 \) such that \( f_\alpha (m) > f_{G', k, j, T} (m) \); take such a \( m \), and denote \( f_{G', k, j, T} (m) \) by \( n \). By the definition of \( f_{G', k, j, T} \), there is an \( i \) such that for all \( t \in T \), there is \( s \supseteq t \) such that \( \tau (s, m) \leq n \) and \( s \setminus t \) is a subset of the intersection of \( G' \) and \( B_i \). In particular, \( \min (s \setminus t_i) > \max (t_i) \geq a \), \( s \setminus t_i \subseteq B_i \cap G' \), and \( \tau (s, m) \leq n \). Thus \( s \setminus t_i \subseteq \omega \setminus H_i \), from which we have \( g_{t_i, \omega \setminus H_i}^\beta (m) \leq n < f_\alpha (m) \), contradicting the choice of \( m_0 \).

This completes the construction of \( G_{\beta+1} \) from \( G_{\beta} \), and hence the proof of Theorem 6.

We are now ready for the consistency of \( b < a \). Recall from the beginning of this section that our ground model \( V \) satisfies \( c = \lambda \) is regular, \( 2^\lambda = \lambda^+ \), and \( \langle f_\alpha : \alpha < \lambda \rangle \) is unbounded \( \langle \cdot \rangle^{\omega} \)-well-ordered.

THEOREM 12. There is a ccc forcing \( \mathbb{P} \) such that

\[ \models_{\mathbb{P}} b = \lambda^+ \text{ and } \langle f_\alpha : \alpha < \lambda \rangle \text{ is still unbounded.} \]

In particular, \( b \leq \lambda < \lambda^+ = a \) is consistent.

PROOF. Perform a finite support iteration of orderings of type \( \mathbb{P}(\mathcal{A}) \) of length \( \lambda^+ \), going through all (names for) mad families along the way by a bookkeeping argument (this is possible by the assumption \( 2^\lambda = \lambda^+ \)). The unboundedness of \( \langle f_\alpha : \alpha < \lambda \rangle \) is preserved in the successor step of the iteration by Theorem 6 and in the limit step, by standard preservation results.
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