I. INTRODUCTION

How does one reverse engineer a model for structure formation from observed phenomena? How unique is such an inversion? How robust are the phenomenological distinctions between broad classes of models? With the wealth of high-precision cosmological data expected in the near future from the cosmic microwave background (CMB), galaxy surveys, and the high redshift universe, the simple \(\Lambda\)CDM model of structure formation currently considered may be ruled out, forcing us to confront these difficult issues. In this paper, we take the first steps toward answering this question by examining a general standpoint what makes a model for structure formation behave as it does in linear perturbation theory.

A model for structure formation is completely specified by its initial conditions and the full temporal and spatial behavior of the stresses in its dark sector. The dark sector contains the elements in the model that do not interact with the photons at any observable redshift. It can include, but is not limited to, cold dark matter (CDM), neutrinos, and cosmological defects.

Unfortunately, the stress history of the dark sector is by definition not directly observable. Its effects come filtered through gravity as mediated by metric fluctuations. The translation of metric fluctuations into observables in the CMB and evolution of structure is well understood. Therefore, the main hurdle in the task of reconstructing a model from observations is to understand how stress histories translate into metric fluctuations and vice versa.

Our general philosophy here is to start from elements of the cosmological model that will likely survive the onslaught of data: general relativity and a universe whose deviations from homogeneity and isotropy are initially small. We proceed down the theory pipeline to the existing models of structure formation, making explicit the places where assumptions are made and hence could be altered. Where possible, we provide analytic solutions and approximations that highlight certain generic behavior and phenomena. These solutions are useful for describing the behavior of the existing models and are in most cases new or substantially more general than those found in the literature. In particular, we derive master solutions for models which contain multiple components with arbitrary equations of state and smooth, entropic, anisotropic, and sonic stresses.

We study the general structure of models for structure formation, with applications to the reverse engineering of the model from observations. Through a careful accounting of the degrees of freedom in covariant gravitational instability theory, we show that the evolution of structure is completely specified by the stress history of the dark sector. The study of smooth, entropic, sonic, scalar anisotropic, vector anisotropic, and tensor anisotropic stresses reveals the origin, robustness, and uniqueness of specific model phenomenology. We construct useful and illustrative analytic solutions that cover cases with multiple species of differing equations of state relevant to the current generation of models, especially those with effectively smooth components. We present a simple case study of models with phenomenologies similar to that of a \(\Lambda\)CDM model to highlight reverse-engineering issues. A critical-density universe dominated by a single type of dark matter with the appropriate stress history can mimic a \(\Lambda\)CDM model exactly.
II. CLASSIFICATION OF THEORIES

A. Initial Conditions

Perhaps the most fundamental difference between models for structure formation lies with their initial conditions. Currently, inflation is the only known means of laying down large-scale density or curvature perturbations in the early universe. Indeed, inflation in the more general sense of a period of superluminal expansion is required for the causal generation of large-scale power. It provides a means of connecting parts of the universe that are currently space-like separated, i.e. outside the current particle horizon. Models with initial curvature perturbations are usually called “adiabatic” models.

All other causal models begin with no density or curvature fluctuations on large scales and are hence called “isocurvature” models. In these models, stress gradients causally move matter around inside the horizon to form large-scale structure.

The generation mechanism is also responsible for determining the spectrum and statistics of the fluctuations. The simplest inflationary models predict a nearly scale-invariant and gaussian distribution of fluctuations [4] but higher order effects can break scale-invariance and generate non-gaussianity [5,6]. Defect perturbations are intrinsically non-gaussian but are typically also scale-invariant in the generalized sense of “scaling” [7]. We are primarily concerned here with the evolution of fluctuations from their initial state through the linear regime and do not consider these issues further. Note that changes in the spectrum of perturbations are simple to include in linear theory as evolutionary effects can be factored out into so-called “transfer functions”.

B. Perturbation Type

The perturbation type for the metric and matter fluctuations is the next most important distinction. A general linear fluctuation can be decomposed into scalar, vector and tensor components. These manifest themselves as density, vorticity, and gravitational wave perturbations respectively and do not interact in linear theory. The scalar modes are the only ones that grow through gravitational instability. Vector modes, on the other hand, always decay with the expansion. They can only be actively generated by shearing (or anisotropic) stress in the manner. Tensor modes are intermediate. Left to themselves, they propagate as gravity waves, but they generate and can be generated by transverse-traceless (quadrupolar) stresses in the matter.

The simplest inflationary models possess only scalar (“S”) fluctuations, and tensor fluctuations are generally cosmologically negligible in models with energy scales substantially below the Planck scale [8]. However, models whose initial conditions contain both scalar and tensor (“ST”) fluctuations are possible. Models with only S or ST generally have stresses that may be defined as functions of the metric, density and velocity perturbations and hence may be viewed as “passive” responses through equations of state.

“Active” models have stresses that are a consequence of complex internal dynamics in the dark sector that cannot be simply specified as responses to gravitational perturbations. Although this definition is not precise for scalar and tensor modes, the very presence of vector modes indicates an active source because these must be continuously generated to have an observable effect. Nevertheless, such models generally have all three types of perturbations (“SVT”).

C. Clustering Properties of Dark Matter

Finally, the nature of the dark components affects the evolution of perturbations. We define as “dark” any component that interacts with the CMB photons only gravitationally. Thus, even massless neutrinos are classified as dark matter in this scheme.

Stresses in the dark components change the evolution of the mean density with time and the response of the matter to gravitational compression. We will loosely type models whose expansion rate is driven by a compressible type of matter (on scales relevant to cosmological structures) as “clustered” models and those which possess matter that is incompressible as “smooth” models. We shall see that this distinction is in fact rather inexact as it is not time invariant: essentially all models pass through phases when they would be considered smooth or clustered on the relevant scale.

D. Phenomenology

The key to understanding the phenomenology of a given model for structure formation is the evolution of metric fluctuations, in particular the Newtonian gravitational potential. Its qualitative behavior is determined by the initial conditions, perturbation type, and dark matter content of the model. We illustrate this taxonomy scheme in Fig. 1.

The behavior of the gravitational potential is directly related to the evolution of density perturbations through the Poisson equation. Once its evolution is determined as a function of scale, not only is the present large-scale structure of the universe determined but also the whole time history of structure formation. The latter is important for predicting the properties and abundances of high-redshift objects.
The gravitational potential also generates CMB anisotropies through gravitational redshifts and is the ultimate source of all anisotropies from scalar perturbations. CMB phenomenology can be essentially read off of the time evolution of the gravitational potential, although this involves understanding the backreaction from density perturbations in the CMB itself.

Similar but simpler considerations apply for vector and tensor metric perturbations. They also generate anisotropies via gravitational redshifts but do not have unstable modes and hence do not affect large-scale structure formation in linear theory.

The difference between adiabatic and isocurvature models play a direct role in metric evolution because initial curvature (or gravitational potential) perturbations are present in one and absent in the other. The perturbation type changes the ratio of CMB anisotropies to large-scale structure. Finally, the dark matter properties affect the evolution of the gravitational potentials. Smooth components by definition do not contribute to the gravitational potential but do contribute to the expansion rate. They slow down the growth of structure and cause the gravitational potential to decay. Hence they decrease the amount of structure and increase the large-angle anisotropies of the CMB.

In summary, the observable properties of structure formation models are encapsulated in the time evolution of the metric fluctuations. This in turn is governed by the stress properties of the matter both through its initial conditions and intrinsic properties.

### E. Current Model Zoo

The archetypal model for structure formation is the standard cold dark matter model (sCDM), which is an adiabatic, passive, and clustered model. Here, an initial scale-invariant spectrum of adiabatic scalar ("S") perturbations collapses via the gravitational instability of pressureless cold dark matter. Although this model is no longer viable from an observational standpoint, it predicts phenomena sufficiently similar to the observations to act as a good starting point for model building. One of its failings is that it predicts too much small-scale power for the level of CMB anisotropies demanded by the COBE detection.

A simple variation of the sCDM model that attempts to address this problem involves tilting the initial spectrum of scalar perturbations (tCDM) to reduce small-scale power relative to large. Under certain inflationary scenarios, this brings about the addition of tensor perturbations that further reduce small-scale density perturbations relative to the COBE detection. Such a model would be a "ST" variant of sCDM.

The second class of variations involves changing the matter content so as to suppress the clustering of matter, yielding a “smooth” variant. The prototypical example is the ΛCDM model, where an additional component of matter that does not cluster replaces most of the CDM. Another examples is the OCDM model where spatial curvature plays the role of the smooth component. Those two represent examples where the additional component is smooth on all scales and for all time by definition. Variants where the matter is only smooth on small scales include the HCDM model (e.g., also called C+HDM and MDM) with a component of hot dark matter, the φCDM model with a scalar field component φ that tracks the background behavior of the matter, and QCDM with a general scalar field ("quintessence"). In a string-dominated universe (strCDM), the string network plays the role of a smooth component with the same equation of state as spatial curvature. Of course, one can have multiple smooth species as well, e.g. OACDM. The GDM class of models phenomenologically parameterizes all such models.

Replacing the CDM with GDM of a different equation of state but no stress perturbations is a phenomenological possibility (GDM) suggested by [17]. This is an adiabatic, passive, and clustered variant of CDM. We will use the designation “CDMv” to represent all such variants of the CDM model.

Isocurvature models have been proposed as alternatives to the CDMv class of models. The simplest examples are those in which the initial stress fluctuations are established by balancing the density perturbations of two different types of matter. Examples include the axion isocurvature (AXI) model and the primordial isocurvature baryon (PIB) model, where radiation density fluctuations are balanced by axions and baryons, respectively. The simplest versions involve only scalar fluctuations and hence are passive ("S") models. Models with and without smooth cosmological constant or spatial curvature components have been proposed. Versions with gaussian power-law initial conditions are observationally challenged but more complicated variations exist. Based on our work, one of us has constructed an isocurvature decaying dark matter (iDDM) model that defies conventional wisdom on isocurvature models and solves these observational problems.

Finally, topological defect models such as strings and textures fall into the isocurvature class but have fluctuations that are active ("STV"). The simplest versions obey scaling and have only clustering matter but fail to generate enough large-scale structure for the observed CMB anisotropies. Models with a smooth Λ component have been proposed to alleviate these problems.

Hybrid models can also be constructed. If defects form after the inflationary epoch, one has a model with adiabatic initial conditions and active perturbations. A string model with inflation and cold dark matter (SIC) is a concrete example. One can also add in smooth compo-
is $\gamma_{00} = -1$ and $\gamma_{0i} = \gamma_{i0} = 0$. Unless otherwise specified, overdots represent derivatives with respect to conformal time and primes derivatives with respect to $\ln a$. $c = 1$ throughout. The background curvature is given by $K = -H_0^2(1 - \Omega_{\text{tot}})$, where the Hubble constant is $H_0 = 100h\ \text{km s}^{-1}\ \text{Mpc}^{-1}$.

The ten degrees of freedom for the perturbations in the symmetric metric tensor $g_{\mu\nu}$ can be parameterized as

$$g^{00} = -a^2 (1 - 2A),$$
$$g^{0i} = -a^2 B^i,$$
$$g^{ij} = a^2 (\gamma^{ij} - 2H L \gamma^{ij} - 2H^2 ij),$$

(2)

We refer to the lapse $A$ as the potential, the three components of $B_i$ as the metric shift, $H_L$ as the curvature perturbation, and the five components of $H_T^{ij}$ as the metric shear following the conventions of [1,2].

Likewise, the symmetric stress-energy tensor can be parameterized by ten components

$$T^0_0 = -\rho - \delta\rho,$$
$$T^0_i = (\rho + p)(v_i - B_i),$$
$$T^0_0 = -(\rho + p)v^i,$$
$$T^{ij} = (\rho + p)\delta^{ij} + p\Pi^{ij},$$

(3)

i.e., the energy density and its perturbation $(\rho + \delta\rho)$, the isotropic stress $(\rho + p)$ and its perturbation $(\rho + p)$, the three components of the momentum density $(\rho + p)v_i$, and the five components of the anisotropic stress tensor $\Pi_{ij}$. Note that the metric shift $B_i$ enters in $T^0_0$ but not $T^{0i}$. Correspondingly, we shall see that $B_i$ enters into the momentum but not the energy conservation equation.

By writing the metric and stress energy tensor in this form, we have maintained general covariance. As a result, the equations of motion that result below take the same form for any coordinate system where linear perturbation theory holds. We reserve the term gauge invariant refers for objects that have the same value in each frame.

### III. COVARIANT PERTURBATION THEORY

#### A. General Definitions

We assume that the background is described by an FRW metric $\bar{g}_{\mu\nu} = a^2 \gamma_{\mu\nu}$ with scale factor $a(t)$ normalized to unity today and constant comoving curvature in the spatial metric $\gamma_{ij}$. Here greek indices run from 0 to 3 while latin indices run over the spatial part of the metric: $i, j = 1, 2, 3$. The component corresponding to conformal time

$$x^0 \equiv \eta = \int \frac{dt}{a(t)}$$

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(1)
F. 2. Scalar, vector, tensor decomposition. At the top of the tree of possibilities for structure formation models is the assumption that general relativity holds in the cosmological context and universe is homogeneous and isotropic in the mean with linear perturbations initially. Without further assumptions, the linear fluctuations may be expanded in scalar, vector, and tensor modes that do not interact while the fluctuations remain linear.

Fluctuations can be decomposed into the normal modes of the Laplacian operator

\[ \nabla^2 Q^{(0)} = -k^2 Q^{(0)}, \quad \nabla^2 Q_i^{(\pm 1)} = -k^2 Q_i^{(\pm 1)}, \quad \nabla^2 Q_{ij}^{(\pm 2)} = -k^2 Q_{ij}^{(\pm 2)}, \]

where vector and tensor modes satisfy a divergenceless and transverse-traceless condition respectively

\[ \nabla^i Q_i^{(\pm 1)} = 0, \quad \gamma^{ij} Q_{ij}^{(\pm 2)} = \nabla^i Q^{(\pm 2)} = 0. \]

In flat space, these correspond to plane waves times a local angular basis for the vectors and tensors.

Vector and tensor objects can of course be built out of scalar and vector modes through covariant differentation and the metric tensor

\[ Q_i^{(0)} = -k^{-1} \nabla_i Q^{(0)}, \quad Q_{ij}^{(0)} = (k^{-2} \nabla_i \nabla_j - \frac{1}{3} \gamma_{ij}) Q^{(0)}, \]

\[ Q_{ij}^{(\pm 1)} = -\frac{1}{2k} [\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}], \]

The perturbations in the kth eigenmode can now be written as

\[ A = \hat{A} Q_i^{(0)}, \quad H_L = \hat{H}_L Q_i^{(0)}, \quad \delta \rho = \hat{\delta} \rho Q_i^{(0)}, \]

which possess only scalar components,

\[ B_i = \sum_{m=-1}^{1} \hat{B}^{(m)} Q_i^{(m)}, \quad v_i = \sum_{m=-1}^{1} \hat{v}^{(m)} Q_i^{(m)}, \]

which possess scalar and vector components, and

\[ H_{Tij} = \sum_{m=-2}^{2} \hat{H}^{(m)} Q_{ij}^{(m)}, \quad \Pi_{ij} = \sum_{m=-2}^{2} \hat{\Pi}^{(m)} Q_{ij}^{(m)}, \]

which possess all three types. Here, scalar perturbations are denoted with superscript (0), which is elsewhere omitted. We will hereafter also omit the overhat in the normal mode amplitudes since real-space objects will no longer appear. Thus \( \hat{u}^{(0)} \equiv u \).

C. Gauge Covariant Equations

The equations of motion for the matter follow from the Einstein equations \( G_{\mu \nu} = 8\pi G T_{\mu \nu} \). Furthermore, the Bianchi identities guarantee \( T_{\mu \nu} \nu = 0 \) which represents covariant energy and momentum conservation.

For the background, energy conservation implies

\[ \dot{\rho} = -3 \frac{\dot{a}}{a} (1 + w) \rho, \]

where \( w = p/\rho \) determines the background equation of state. Due to the isotropy of the background, momentum conservation yields no additional constraint.

The Einstein equation determine the evolution of the scale factor through

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} a^2 \rho_c + \frac{8\pi G}{3} a^2 (\rho + \rho_S). \]

Here we have divided contributions to the expansion rate into the ordinary density and an effective density component that does not participate in gravitational collapse and is hence labeled “S” for smooth. The curvature provides the only component that is smooth by fiat.
\[
\rho_S = -\frac{3}{8\pi Ga^2}K, \tag{12}
\]
with \(w_S = -1/3\). Even the cosmological constant is kept smooth simply by dynamics. However, this notation is convenient for considering components that are approximately smooth. By keeping a general \(\rho_S\) and \(\rho\) here, we avoid lengthy rederivation of the equations of motion for such cases. Note that \(\rho + \rho_S = \rho_{\text{cr}}\), the so-called critical density, and we can define a critical equation of state \(w_{\text{cr}}\) by
\[
\rho_{\text{cr}} = -\frac{\dot{a}}{a}(1 + w_{\text{cr}})\rho_{\text{cr}}. \tag{13}
\]
Scalar matter perturbations obey the continuity and Euler equations
\[
\left[\frac{d}{d\eta} + \frac{1}{a}\frac{\dot{a}}{a}\right] \delta \rho = -(\rho + p)(kv + 3H_L), \tag{14}
\]
\[
\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}\right] (\rho + p)\frac{(v - B)}{k} = \delta p - \frac{2}{3}(1 - \frac{K}{k^2})p\Pi + (\rho + p)A, \tag{15}
\]
and place 2 constraints on the 4 scalar matter-variables.

The metric and matter are related by the Einstein equations
\[
(k^2 - 3K)[H_L + \frac{1}{3}H_T + \frac{\dot{a}}{a}\frac{\dot{a}}{a}k^2(kB - \dot{H}_T)] = 4\pi Ga^2 \left[\delta \rho + 3\frac{\dot{a}}{a}(\rho + p)(v - B)/k\right], \tag{16}
\]
\[
k^2(A + H_L + \frac{1}{3}H_T) + \left[\frac{d}{d\eta} + \frac{2\dot{a}}{a}\right] (kB - \dot{H}_T)
= 8\pi Ga^2 p\Pi, \tag{17}
\]
\[
\frac{\dot{a}}{a}A - \dot{H}_L - \frac{1}{3}H_T - \frac{K}{k^2}(kB - \dot{H}_T)
= 4\pi Ga^2(\rho + p)(v - B)/k, \tag{18}
\]
\[
\left[\frac{2\ddot{a}}{a} - \frac{2}{3}\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{d}{d\eta} + \frac{k^2}{3}\right] A - \left[\frac{d}{d\eta} + \frac{\dot{a}}{a}\right] (\dot{H}_L + \frac{1}{3}kB)
= 4\pi Ga^2(\delta p + \frac{1}{3}\dot{\rho}). \tag{19}
\]
Only two of these equations are functionally independent. The combination of these equations that corresponds to the conservation equation \(G_{\mu\nu}{}^{\nu} = 0\) is automatically satisfied by any choice of the 4 metric variables due to the Bianchi identities. The remaining degrees of freedom are related to gauge freedom as we shall see.

Momentum conservation for vector perturbations gives the Euler equation
\[
\left[\frac{d}{d\eta} + \frac{1}{a}\frac{\dot{a}}{a}\right] ([\rho + p](v^{(\pm 1)} - B^{(\pm 1)})/k]
= -\frac{1}{2}(1 - 2K/k^2)p\Pi^{(\pm 1)}, \tag{20}
\]
and the Einstein equations give
\[
(1 - 2K/k^2)(kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) = 16\pi Ga^2(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k, \tag{21}
\]
\[
\left[\frac{d}{d\eta} + \frac{2\dot{a}}{a}\right] (kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)})
= -8\pi Ga^2 p\Pi^{(\pm 1)}. \tag{22}
\]
Again the conservation equations nor the Bianchi identity say anything about tensor perturbations.

Although these relations are exact, they do not provide a closed system. There are in general 10 equations for 20 variables in the background and perturbations separately. For the background, homogeneity and isotropy brings this to 2 equations for the 3 variables \((a), (\rho, p), \) where the grouping distinguishes metric and matter categories. For the perturbations, the general relations are broken up into 4 equations for the 8 variables \((A, B, H_L, H_T), (\delta \rho, \delta p, (\rho + p)v, p\Pi)\) for the scalar perturbations, 2 equations for 4 variables \((B^{(\pm 1)}, H_T^{(\pm 1)}), (\rho^{(\pm 1)}, \Pi^{(\pm 1)})\) for each set of vector perturbations and 1 equation for 2 variables \((H_T^{(\pm 2)}), (\Pi^{(\pm 2)})\) for each set of tensor perturbations. We can express the remaining 1 + 10 degrees of freedom as the ability to choose the equation of state for the background \(w = p/\rho\), the 6 stress fluctuations \((\delta \rho, \delta \Pi, p\Pi^{(\pm 1)}, p\Pi^{(\pm 2)})\) for the perturbations, and the gauge (the 4 quantities \((\delta \eta, \delta x_i)\) for an arbitrary coordinate shift).

D. Multicomponent Generalization

The conservation equations \((14), (15)\) and \((20)\) are valid for each species whose stress-energy tensor is independently covariantly conserved. For example, they apply to the photon-baryon system and the dark sector which only interact through gravity. The Einstein equations \((16), (17), (18)\) and \((19)\) of course still hold with the appropriate summation over components, e.g. \(\rho = \sum_I \rho_I\). Note that we do not include the smooth component \(\rho_S\) in the multicomponent sum.

IV. COORDINATE CHOICE
A. Gauge Transformations

The additional four component freedom in the Einstein equations is fixed by a choice of coordinates that relate the perturbations to the underlying smooth background. The most general coordinate transformation associated with the $k$th normal mode is

$$
\tau = \tilde{\tau} + TQ^{(0)} ,
$$

$$
x_i = \tilde{x}_i + LQ_i^{(0)} + L(1)Q_i^{(1)} + L(-1)Q_i^{(-1)} ,
$$

(24)

$T$ corresponds to a choice in time slicing and $(L, L^{(1)}, L^{(-1)})$ a choice of spatial coordinates. Under the condition that metric distances be invariant, they transform the metric as

$$
A = \tilde{A} - \dot{T} - \frac{\dot{a}}{a}T ,
$$

$$
B = \tilde{B} + L + kT ,
$$

$$
H_L = \tilde{H}_L - \frac{k}{3}L - \frac{\dot{a}}{a}T ,
$$

$$
H_T = \tilde{H}_T + kL ,
$$

(25)

for the scalar perturbations and

$$
B^{(\pm 1)} = \tilde{B}^{(\pm 1)} + \dot{L}^{(\pm 1)} ,
$$

$$
H_T^{(\pm 1)} = \tilde{H}_T^{(\pm 1)} + kL^{(\pm 1)} ,
$$

(26)

for the vector perturbations.

Similarly, they transform the components of the stress-energy tensor as

$$
\delta \rho_J = \tilde{\delta} \rho_J - \dot{\rho}_J T ,
$$

$$
\delta \rho_J = \tilde{\delta} \rho_J - \dot{\rho}_J T ,
$$

$$
v_J = \tilde{v}_J + \dot{L} ,
$$

(27)

for the scalar perturbations and

$$
v_J^{(\pm 1)} = \tilde{v}_J^{(\pm 1)} + \dot{L}^{(\pm 1)} ,
$$

(28)

for the vector perturbations. All other quantities in the metric and matter are gauge invariant. In particular, the tensor modes do not exhibit gauge freedom since the transverse-traceless condition on $Q^{(\pm 2)}$ is sufficient to remove the gauge ambiguity. The gauge is thus fixed by conditions on the metric which fully specify the transformation $(T, L, L^{(\pm 1)})$ from an arbitrary frame.

It is important to bear in mind that both the metric fluctuations $(A, B, H_L, H_T)$ and the matter fluctuations $(\delta \rho, \delta \rho, [\rho + p]v)$ take on different numerical values in different frames even in this covariant notation. For example, if $\rho$ evolves in time, a density perturbation $\delta \rho$ arises simply from the warping of the time hypersurface on which the perturbation is defined. Thus, a density perturbation differs negligibly only between frames separated by (see Eq. [(10)])

$$
T \ll \left[ (1 + w) \frac{\dot{a}}{a} \right]^{-1} \frac{\delta \rho_J}{\rho_J} .
$$

(29)

The common gauge choices of the next section all agree on the density perturbation in the clustered component well inside the horizon.

B. Gauge Choice

Gauge freedom can be used to simplify the equations of motion. Most commonly, it is employed to convert certain Einstein equations to algebraic relations and/or eliminate relativistic effects from the conservation equations.

1. Vector Gauges

Let us first dispose of the vector degrees of freedom. There are two natural choices (1): $H_T^{(\pm 1)} = 0$ which fixes the gauge completely and $B^{(\pm 1)} = 0$ which leaves an arbitrary constant offset in $H_T^{(\pm 1)}$. The latter does not produce a dynamical effect and can always be eliminated by specifying an initial condition for $H_T^{(\pm 1)}$.

2. Comoving (Scalar) Gauge

It is useful to consider a scalar gauge where the metric and matter fluctuations are simply related (1). Inspection of the Euler and Einstein equations shows us that the coordinate choice $B = v$ simplifies the equations of motion greatly. This fixes the time slicing through $T = (\dot{v} - \dot{B})/k$. The additional condition $H_T = 0$ specifies that $L = \tilde{H}_T/k$ and fixes the gauge completely. We call this the comoving gauge since here the momentum density vanishes. The remaining metric variables are labeled $A = \xi$ and $H_L = \zeta$. This choice reduces the Euler equation to the algebraic relation for the potential

$$
(\rho + p)\xi = -\delta p + \frac{2}{3} (1 - 3K/k^2) \rho \Pi ,
$$

(30)

which is also simply related to the curvature through the Einstein equation (18)

$$
\dot{\zeta} = \frac{\dot{a}}{a} \xi + 4\pi G a^2 (\rho_S + p_S) v/k .
$$

(31)

Here we have again rewritten the curvature component as a smooth density contribution as in equation (18) for easy generalization to approximately smooth cases.

The simple relation between the metric and stress perturbation of equations (18) and (12) is what makes this gauge useful. A smooth component complicates these relations because of the difference between a frame that
is comoving with $v$ versus the total-momentum-weighted velocity $v\dot{\rho}/(\dot{\rho} + \rho S)$. With a constant comoving curvature, the continuity equation (14) reduces to an ordinary conservation equation since metric changes to the fiducial volume are absent.

3. Newtonian (Scalar) Gauge

Finally, the Newtonian gauge is defined by $B = H_T = 0$ and labels $A = \Psi$ and $H_L = \Phi$. The gauge is completely specified through $T = -\dot{B}/k + \ddot{H}_T/k^2$ and $L = -\dddot{H}_T/k$. The Einstein equations are reduced to algebraic relations that generalize the Poisson equation of Newtonian gravity

\[
(k^2 - 3K)\Phi = 4\pi G a^2 \left[ \delta \rho + 3 \frac{a'}{a} (\rho + p) v / k \right],
\]

\[
k^2 (\Phi + \Psi) = -8\pi G a^2 p \Pi.
\]

These algebraic relations and the fact that CMB anisotropies are simply related to $\Phi$ and $\Psi$ make this gauge useful.

4. Gauge-Covariant Variables

It is often useful to speak of the variables of say the comoving gauge while in a Newtonian representation. Bardeen [1] introduced a so-called “gauge-invariant” language that achieves this. We denote such techniques as gauge covariant since they amount to introducing covariant expressions for objects that take on the desired meaning only in a specific frame. The only objects that cannot be made gauge covariant are those that are ill-defined due to coordinate ambiguities.

To avoid confusion, we only use gauge-covariant variables to describe metric fluctuations ($\zeta$, $\xi$, $\Phi$, $\Psi$). Matter perturbations will always be represented in the comoving gauge unless otherwise specified. Note that $v$ is the same in comoving and Newtonian gauges.

The comoving curvature and density can be usefully expressed in Newtonian variables

\[
\zeta = \Phi + 2 (\Psi - \Phi') \frac{\rho S}{\rho},
\]

\[
4\pi G a^2 \delta \rho = (k^2 - 3K)\Phi,
\]

obtained through equations (23) and (13). Likewise equation (21) can be rewritten as

\[
\zeta' - \xi = (\Psi - \Phi') \frac{\rho S}{\rho}.
\]

Recall that primes represent derivatives with respect to $\ln a$.

As we shall see, employing both comoving and Newtonian metric variables in the covariant language allows us to exploit the simple relations to comoving stresses in the former and comoving density perturbations in the latter.

V. STRESS PHENOMENOLOGY

A. Stress Representation

We have seen that the stresses of the matter components completely determine the evolution of perturbations (see Figs. 3, 4, and 5). The background stress is completely determined by the equation of state $w$. The scalar stress fluctuations are determined by functional relations between the pressure or isotropic stress perturbations $\delta \rho$, anisotropic stress perturbation $p \Pi$ and the density perturbation $\delta \rho$. These relations may also involve hidden internal degrees of freedom. A model may also possess background stress without stress perturbations and vice versa. We call the former a smooth stress and the latter a seed stress. Vector and tensor perturbations likewise depend on the components $p \Pi(\pm 1)$ and $\rho \Pi(\pm 2)$ of the anisotropic stress tensor.

Scalar stress perturbations control the basic elements of the structure formation history, and so we pay particular attention to categorizing their properties. As discussed in §IV B, the isotropic scalar stress poses a special problem in that its value depends on the choice of coordinate or gauge. The comoving gauge (where the momentum density vanishes) provides a useful choice of gauge because of the simple relation between the metric and stress fluctuations. We will use these coordinates to define the total scalar stress

\[
S = -\xi = \frac{\delta \rho}{\rho + p} - \frac{2}{3} (1 - 3K/k^2) \frac{p}{\rho + p} \Pi.
\]

It is useful to isolate gauge invariant aspects of the stress perturbation. The anisotropic scalar stress

\[
S_{\Pi} = -8\pi G a^2 p \Pi / k^2,
\]

is gauge invariant by definition; this form of the anisotropic stress also enters into the Einstein equations separately from $S$.

An adiabatic stress perturbation obeys

\[
\delta p_A = (\rho + p) S_A = \frac{p'}{\rho'} \delta \rho.
\]

Although $\delta p_A$ is not gauge invariant, the adiabatic sound speed is

\[
c_s^2 = \frac{\delta p_A}{\delta \rho} = \frac{p'}{\rho'}.
\]

This gauge invariance implies that there are no coordinate ambiguities when discussing the pressure support of adiabatic fluctuations.
The remaining gauge-invariant pressure perturbation is
\[ S_T = \frac{1}{\rho + p} \left( \delta p - \frac{p'}{\rho} \delta \rho \right). \]  
(40)

Unlike adiabatic stresses, these may be present even when the comoving density perturbation \( \delta \rho \) is negligible (see Eq. [34]). Entropic stresses are the primary means of structure formation in most isocurvature models.

The comoving-gauge analogue of \( S_A \) and \( S_T \) are also useful. As long as \( \delta p / \delta \rho \) is less than unity, superhorizon stresses are negligible compared with curvature fluctuations. We call the part of \( \delta p / \delta \rho \) that is separable in time and space the comoving sound speed \( c_s^2 = f(k)g(\eta) \) and the accompanying stress sonic
\[ S_S = c_s^2 \frac{\delta \rho}{\rho + p}. \]  
(41)

Separability is not a gauge-invariant property, but this is not in itself a problem because the comoving frame is dynamically special.

We call the remaining isotropic stress the entropic stress
\[ S_E = \frac{1}{\rho + p} \left( \delta p - c_s^2 \delta \rho \right), \]  
(42)
such that the total stress is
\[ S = S_S + S_E + \frac{2}{3} \frac{k^2 - 3K}{8\pi G a^2 (\rho + p)} S_{\Pi}. \]  
(43)

Note that if the comoving sound speed equals the adiabatic sound speed \( c_S^2 = c_s^2 = p'/\rho' \) then \( S_S = S_A \) and \( S_E = S_T \).

**B. Seed Stress**

Seed stresses provide a special case with unique properties. The effect of seed perturbations are in fact simpler to understand than those of the fluid type because the problem decouples completely. If the seeds do not interact directly with other types of matter, the conservation equations [44] and [45] imply that the metric perturbations only affect the seed perturbations at second order since bare \( \rho_s \) and \( p_s \) terms may be dropped. They become
\[ \left[ \frac{d}{d\eta} + 3 \frac{\dot{a}}{a} \right] \delta \rho_s = -k(\rho_s + p_s) v_s - 3 \frac{\dot{a}}{a} \delta \rho_s, \]  
(44)
\[ \left[ \frac{d}{d\eta} + 4 \frac{\dot{a}}{a} \right] (\rho_s + p_s) \frac{v_s}{k} = \delta p_s - \frac{2}{3} (1 - 3K/k^2) p_s \Pi_s. \]  
(45)

The basic principles of how stress fluctuations affect the gravitational potential still hold but here there is a stress contribution that is truly external to the system of metric fluctuations. It is again possible to have large-scale entropic stress perturbation in the absence of initial curvature or density perturbation.

The formal solution to equations (44) and (45) are
\[ (\rho_s + p_s) v_s / k = a^{-4} \int d\eta a^4 [\delta \rho_s - \frac{2}{3} (1 - 3K/k^2) p_s \Pi_s], \]  
\[ \delta \rho_s = -a^{-3} \int d\eta a^3 [k(\rho_s + p_s) v_s + 3 \frac{\dot{a}}{a} \delta \rho_s]. \]  
(46)

The task of understanding a seed model like defects reduces to understanding its stresses, but this is a formidable task in realistic seed models such as cosmological defects (e.g. [23]).

**C. From Stresses to Curvature**

These stresses are the fundamental sources and sinks of the metric perturbations. We therefore seek to express the comoving and Newtonian curvatures in terms of the stresses. Combining equations [43], [45], and [46] with
\[ \Psi = -\Phi + S_{\Pi}, \]  
(47)
yields
\[ \zeta' = -S + [\Phi' + \Phi - S_{\Pi} ] \left( \frac{\rho_S}{\rho'} - 1 \right), \]  
(48)
and
\[ \frac{\sqrt{\rho}}{a} \left( \frac{a}{\sqrt{\rho}} \right)' = -\frac{1}{2} \frac{\rho'}{\rho} \zeta + S_{\Pi}. \]  
(49)

In a non-critical (non-flat, \( \rho \neq \rho_{cr} \)) universe, we combine these to eliminate \( \zeta \) from the evolution equation for the Newtonian curvature, giving
In a critical-density universe \((\rho = \rho_{\text{cr}})\), equations \((48)\) and \((49)\) can be formally solved as integrals over the stress fluctuations to yield

\[
\zeta(a) = \zeta(0) - \int \frac{da}{a} S ,
\]

and

\[
\Phi = \zeta - \frac{\sqrt{\rho}}{a} \int \frac{da}{\sqrt{\rho}} [\zeta - S - S_{\|}] + C \frac{\sqrt{\rho}}{a} .
\]

The last term is the decaying mode of \(\Phi\) where \(C = \text{const}\).

There are three general conclusions that we can draw from equations \((51)\) and \((52)\). The first is that in the absence of an initial comoving curvature perturbation \((\zeta(0) = 0)\), a stress fluctuation will generate one of order \(\zeta \sim -S\). The same goes for the Newtonian curvature \(\Phi \sim -S\). The reason for this behavior is that a stress gradient \(k \delta \rho\) generates a potential flow with \((\rho + p)^v \sim (k \eta) \delta \rho\) which generates a density perturbation of \(\delta \rho \sim -(k \eta)^2 \delta \rho\) and hence a curvature perturbation of \(\Phi \sim -\delta \rho/\rho_{\text{cr}}\). Note this intuitive argument fails for other gauge choices.

Second, starting with a curvature perturbation and assuming sonic stresses \(\delta \rho = c_{\text{s}}^2 \delta \rho\), it is clear that the same mechanism of generating flows will generate an opposing curvature fluctuation \(\Delta \Phi \sim -c_{\text{s}}^2 \delta \rho/\rho \sim -(c_{\text{s}}^2 k \eta)^2 \Phi\) that will destroy the initial curvature fluctuation when \(c_{\text{s}}^2 k \eta \sim 1\). In physical terms this occurs because pressure support prevents perturbations from collapsing and hence causes the curvature perturbation to redshift away. Finally, the anisotropic stress contributes to the total stress \(S\) and thus can both create and destroy comoving curvature fluctuations. Furthermore, it enters separately into the Newtonian curvature through equation \((52)\). This is because the Newtonian frame, unlike the comoving frame, is defined to be globally shear free \((B = H_T = 0)\). The coordinate transformation that maps the comoving frame to the shear free frame depends on the anisotropic stress and hence aliases background evolution into contributions to the Newtonian curvature.

Unfortunately, despite their general appearance, equations \((51)\) and \((52)\) are only formal solutions since the time evolution of the stress sources generally depend on the curvature fluctuation itself. We will use the special properties of smooth, anisotropic, entropic, and sonic stresses to address this problem in \(\text{VIII}\) and \(\text{VIII}\).

D. From Curvature to Observables

As discussed in \(\text{VIII}\), the Newtonian curvature is directly related to observables in the CMB and large scale structure. We are now in a position to quantify these relations. The Newtonian curvature \(\Phi\) and potential \(\Psi\) encapsulate all observable properties of scalar fluctuations.

FIG. 4. Vector perturbations. Vector perturbations simply decay from their initial value in the stress-free limit. The integral solution in the presence of vector stress is given in \(\text{VIII}\) and applies to defect models.

FIG. 5. Tensor perturbations. Tensor perturbations propagate as free gravity waves in the stress-free limit as is the case of a matter-dominated expansion. The integral solution in the presence of stresses is given in \(\text{VIII}\) and \(\text{VIII}\) and may be applied to propagation during radiation domination as well as defect sources.
The contribution of a given $k$-mode to the amplitude of $\ell$th multipole moment of the CMB anisotropy is given by

\[
\Theta_\ell \approx \int_0^{\eta_0} d\eta e^{-\tau} \left\{ \left[ \Psi - \Phi + \frac{1}{2} \tau (\Theta_\ell + \Psi) \right] \times j_\ell k (\eta_0 - \eta) + \tau v_0 j'_\ell k (\eta_0 - \eta) \right\},
\]

where we have dropped the small correction due to the polarization of the CMB, $\tau$ is the optical depth to Compton scattering between the present ($\eta_0$) and the epoch in question ($\eta$); $\Theta_\ell$ is the photon temperature perturbation in Newtonian gauge, and $v_0$ is the baryon velocity in Newtonian or comoving gauge. For an open universe, the spherical Bessel function $j_\ell$ is replaced by the hyperspherical bessel function. Note that the prime here and here only refers to a derivative with respect to the argument of the Bessel function. With the random phase assumption for the $k$-modes, the scalar contribution to power spectrum of the anisotropies is

\[
C_\ell = \frac{2}{\pi} \int \frac{dk}{k} k^3 (\Theta^2 + \Theta_\ell^2). \tag{54}
\]

In equation (5), the $\dot{\Psi} - \dot{\Phi}$ term leads to the so-called integrated Sachs-Wolfe (ISW) effect and contributes once the optical depth to scattering becomes small, i.e. after last scattering. The other terms, $\Theta_\ell + \Psi$ and $v_0$, are localized to the last scattering surface itself and represent the effective temperature of the distribution and the Doppler effect respectively. They are responsible for the so-called acoustic peaks in the CMB spectrum, the morphology of which can be directly read off of the metric driving terms [10].

The influence of time variability in $\Psi$ and $\Phi$ depends on how the variation rate $1/\Delta \tau$ compares with the perturbation crossing time for sound (before last scattering) and light (after last scattering). For variations on a much shorter time scale ($k \Delta \tau \ll 1$ or $k c_s \Delta \eta \ll 1$), only the total change $\Delta (\Psi - \Phi)$ is observable since all photons suffer a uniform gravitational redshift. For variations on a much longer timescale the effects mainly cancel out as the photons either traverse many wavelengths of the fluctuation during the variation with redshifts and blueshifts cancelling or equivalently undergo many acoustic oscillations. Changes whose duration are synchronized with the oscillation period are the most effective. These general considerations apply to metric variations of the vector and tensor type as well. Two commonly encountered examples are the cancellation cut off at high $\ell$ for a uniform potential decay and the driving of acoustic oscillations from synchronized potential decay in the radiation dominated epoch.

A constant potential also leads to observable effects through the effective temperature term $\Theta_0 + \Psi$. Since $\Theta_0$ is the temperature perturbation in Newtonian gauge, it can be obtained by a gauge transformation from the comoving gauge by noting that in that gauge both the density perturbation and the potential $\xi$ are negligible because stress perturbations must be negligible for the potential to remain constant. Equation (24) then imply

\[
\Theta_0 = \frac{\delta \rho_\gamma}{4 \rho_\gamma} - \frac{\dot{a}}{a} v/k, \quad \Psi = \frac{\dot{v}}{k} + \frac{\dot{a}}{a} v/k. \tag{55}
\]

For adiabatic fluctuations, $\delta \rho_\gamma/\rho_\gamma \propto \delta \rho/\rho$ and is hence negligible outside the horizon by virtue of the Poisson equation (33). Since $\Psi$ is a constant by assumption, these equations can be integrated to give

\[
\Theta_0 = -\frac{2}{3(1 + w)} \Psi, \tag{56}
\]

and hence $\Theta_0 + \Psi = \Psi(1 + 3w)/(3w + 3)$. This relation ultimately comes from the fact that $\Psi$ represents a time shift and $a \propto t^{2/3(1+w)}$ [30]. For $w = 0$, this reduces to the well-known result that the effective temperature is $\Psi/3$ in the adiabatic sCDM model [1]. Compared with this model, those that are dominated by the ISW term $\Delta (\Psi - \Phi) \approx 2 \Psi - S_\Omega$ like traditional isocurvature and smooth models potentially have up to 6 times the anisotropy for a given potential fluctuation.

The behavior of the density perturbations that underlies large-scale structure are even more directly related to the Newtonian curvature perturbation. Inside the horizon, all reasonable choices of gauge agree on the density perturbation. In particular, the comoving gauge density perturbation is algebraically related to $\Phi$ by equation (34) and hence allows a simple translation of results for one to the other. Consequently, the relation between temperature and potential fluctuations discussed in the last paragraph translates directly into a relation between density perturbations and CMB anisotropies.

With this relation, the so-called transfer function of the density perturbations below the current horizon can be simply read off of the time history in the potential without the usual gauge ambiguity in defining the initial density perturbation,

\[
T(k) = \frac{\Phi(\eta_0, k)}{\Phi(0, k)} \frac{\Phi(0, 0)}{\Phi(\eta_0, 0)} \tag{57}
\]

The power spectrum of density perturbations today is then proportional to $T(k)^2 P_{\text{initial}}$. Any process that makes the potential decay relative to the $k = 0$ mode will produce a downturn in $T(k)$ and a reduction of small-scale power.
VI. STRESS-FREE PERTURBATIONS

Adiabatic models for structure formation, those whose initial conditions contain true comoving curvature perturbations, all go through a period in which the scalar stress perturbation may be neglected. This is a direct consequence of causality. Stress gradients affect structure formation simply by the causal motion of matter. On scales larger than the horizon, the change in the comoving curvature perturbations due to causal motion always can be always be neglected.

We begin in \[\text{VI}A\] with adiabatic models that have no smooth components. Although the comoving curvature remains at the value set by the initial condition, the Newtonian curvature depends on the equation of state. We consider then in \[\text{VI}B\] the effect of a smooth component on the Newtonian curvature (and hence the density perturbation). For each case, we begin with a general description of the resultant phenomenology. We then illustrate the phenomenology with full solutions of the perturbation equations and discuss applications within the current generation of structure formation models of \[\text{II}I\]. We will follow this pattern throughout the paper. An overview of results is given in Fig. 6.

A. Clustered Case

We begin with the case in which there is no smooth component \((\rho_S = 0)\) and stress fluctuations are negligible compared with metric fluctuations \((S \ll \zeta)\). Here, equation (58) simply implies

\[
\zeta = \text{const.} \quad (58)
\]

Equation (52) then gives

\[
\Phi = \zeta \left(1 - \frac{\sqrt{\rho}}{a} \int \frac{da}{\sqrt{\rho}} + \frac{\sqrt{\rho}}{a} \int \frac{da}{\sqrt{\rho}} S_{\Pi} + C \sqrt{\rho} \right). \quad (59)
\]

We will focus only on the behavior of the first term. The second term may be neglected if \(S_{\Pi} \ll \zeta\). However, unlike \(S \ll \zeta\), \(S_{\Pi} \ll \zeta\) is not a consequence of causality and is mildly violated in the case of free radiation [11] and can be strongly violated in defect models [31]. We return to consider its effects in \[\text{VI}B\]. The last term is a decaying mode that carries no comoving curvature \((\zeta = 0)\). For a constant equation of state \(w\), it scales as

\[
\Phi \propto a^{-3(1+w)/2-1}. \quad (60)
\]

It is apparent from the form of the first term that since \(\rho\) cannot grow with time \((w \geq -1)\), the first integral in equation (54) goes to a constant between 0 and 1 with the two extremes representing \(w \rightarrow \infty\) and \(w = -1\) respectively. The integral is dominated by the most recent epoch; only the equation of state at the epoch in question matters. It is simple to show that during periods where \(w\) is approximately constant, \(\Phi\),

\[
\Phi \frac{1}{\zeta} \rightarrow \frac{3 + 3w}{5 + 3w}. \quad (61)
\]

At each epoch where \(w\) decreases from \(w_1\) to \(w_2\), the Newtonian curvature decreases by

\[
\Phi(w_1) - \Phi(w_2) = \frac{2}{(1 + w_1)(5 + 3w_2)}. \quad (62)
\]

These results are easy to understand in the nonrelativistic limit. In the absence of stresses, the gradients in the gravitational potential set up flows as \(v \sim (k\eta)\Phi\). The divergence of this flow generates density perturbations \(\delta \sim (1 + w)(k\eta)^2\Phi\) for constant \(w\). By the Poisson equation \(\Phi \sim (k\eta)^{-2}\delta\), this is exactly the rate of growth

![Stress Free](image_url)

\[
\begin{array}{|c|c|}
\hline
\text{Component} & \text{Example} \\
\hline
w_1, w_2 & \text{All Adiabatic Models except OCDM} \\
\hline
w_C = w_1, w_S = w_1, w_2 & \text{ACDM, GDM, OCDM, QCDM, HCDM, \Lambda HCDM, strCDM, OHCDM, QCDM, QHCDM, OCDM} \\
\hline
w_C = 0, w_S = -1, -1/3 & \text{(integral)} \, \text{OCDM} \\
\hline
\end{array}
\]

FIG. 6. Stress-free scalar perturbations. If all of the matter which drives the expansion participates in gravitational instability, the matter is said to be “clustered” otherwise there exists a “smooth” component. In either case, the perturbations depend only on the background (“backgrd.”) equation of state (middle row), which lead to exact solutions (bottom row) that describe behavior in a wide range of models.
needed to keep the potential constant. Here we have used the fact that \( 4\pi G a^2 \rho = 3(\dot{a}/a)^2/2 \sim \eta^{-2} \) if there are no smooth components. When \( w \) decreases from \( w_1 \) to \( w_2 \), the change affects the proportionality constants but not the scaling between \( \delta \) and \( \Phi \). This leads to a decrease in the Newtonian potential to a new constant.

**Full Solutions.** — If the matter is predominantly composed of two components with different but constant equations of state \( w_1 \) and \( w_2 \), we can solve equation (63) exactly provided \( S_H \ll \zeta \). The result is

\[
\frac{\Phi}{\zeta} = 1 - \frac{2}{5 + 3w_1} F \left[ \frac{1}{2}; 1 + n; \frac{y}{1 + y} \right]
\]

\[
= 1 - \frac{1}{3} \frac{1}{w_1 - w_2} (1 + y)^{1/2} y^{-n} B_{y/(1+y)} \left( n, \frac{1}{2} - n \right),
\]

where \( n \equiv (5 + 3w_1)/(6w_1 - 6w_2) \) and

\[
y = \rho_2/\rho_1 \propto a^{3(w_1 - w_2)}. \tag{64}
\]

Here, \( B_{x}(p, q) \) is the incomplete beta function and \( F(\alpha; \beta; \alpha; x) \) is Gauss’s hypergeometric function (i.e. \( _2F_1 \)). The combination \( y/(1+y) \) often enters into such solutions as it is just \( \rho_2/\rho_1 \), the fractional density perturbation supplied by component 2 as a function of time.

Any case with \( 2n \) equal to an integer can be expressed in elementary form. In particular, if \( n = N \) or \( N + 1/2 \) for an integer \( N \geq 0 \), then

\[
F \left( \frac{1}{2}; n + 1; x \right)
\]

\[
= 1 + \frac{1}{x} \sum_{k=0}^{N-1} \frac{\Gamma(n+1)\Gamma(n-k-1/2)}{\Gamma(n-k)\Gamma(n+1/2)} \left( \frac{x-1}{x} \right)^k + \left( \frac{x-1}{x} \right)^N \frac{\Gamma(n+1)\Gamma(n-N+1/2)}{\Gamma(n-N+1)\Gamma(n+1/2)} f_n,
\]

where

\[
f_n = \begin{cases} 
1/\sqrt{1-x}, & n = N; \\
\frac{1}{2}\frac{\ln{(1+\sqrt{1-x})}}{\sqrt{1-x}}, & n = N + \frac{1}{2}. 
\end{cases} \tag{66}
\]

**Applications.** — These solutions apply to any adiabatic model with a matter-dominated epoch and may be used to explore the behavior of perturbations entering and exiting the matter-dominated epoch. For example, the matter-radiation case reduces to

\[
\frac{\Phi}{\zeta} = \frac{3}{5} + \frac{2}{15y} - \frac{8}{15y^2} - \frac{16}{15y^3} + \frac{16\sqrt{1+y}}{15y^3}. \tag{67}
\]

This solution only holds in the absence of anisotropic stress perturbations and does not strictly apply to the usual radiation components. The neutrinos carry anisotropic stress [1] as do the photons after recombination. We consider how such effects can be taken into account in §VIIA.

---

**FIG. 7.** Stress-free clustered case. In the case where all contributions to the expansion rate also cluster, the comoving curvature is constant but the Newtonian curvature changes with the equation of state \( w \). For the case of the transition between \( w_1 = 0 \) and \( w_2 \) between \(-1 \) and \(-1/6 \), we show here the analytic solution of equation (63) is compared with a full numerical Boltzmann solution of a QCDM model (including radiation) for a mode that is outside the horizon at the given epoch. Notice that only for \( w_2 = -1 \) (cosmological constant) does the potential decay to zero as \( \rho_2/\rho_1 = y/(1+y) \rightarrow 1 \).

For adiabatic models where there exists a component with \( w_2 < 0 \), these solutions describe the exit from the matter-dominated epoch. A comparison of the analytic solution with full numerical solutions in those cosmologies is given in Fig. 7. A special case is \( w_1 = 0 \) and \( w_2 = -1 \), the matter to cosmological constant transition, where

\[
\frac{\Phi}{\zeta} = \frac{3}{5} \left[ 1 - \frac{1}{3}(1+y)^{1/2} y^{-5/6} B_{y/(1+y)} (5/6, -1/3) \right]. \tag{68}
\]

As is evident from equation (68), the cosmological constant is the only case where the Newtonian curvature decays to zero. Note in particular that \( w_2 = -1/3 \) does not correspond to the behavior of spatial curvature in spite of the fact that the effect on the expansion rate is the same. The presence of fluctuations in the \( w_2 \) component prevent the gravitational potential from decaying to zero. Across the \( w_1 = 0 \) (matter-dominated) to \( w_2 = -1/3 \) transition, the Newtonian curvature goes from \( 3\zeta/5 \) to \( \zeta/2 \). Thus, the string-dominated (strCDM) model which has \( w_{str} = -1/3 \) does not behave like an open model on the large scales relevant for the CMB (c.f. [14]).

**B. Smooth Components**

We next consider how smooth components affect the growth of structure. Although curvature is the only component that is smooth by definition and a cosmological
constant the only one that is smooth by dynamics, under certain circumstances other components can be approximately smooth.

We define a component \((S)\) to be smooth if its density fluctuations are small in comparison to those of the clustering components \((C)\), i.e. \(\delta \rho_S \ll \delta \rho_C\) regardless of whether \(\rho_S < \rho_C\). To be maintained dynamically, the respective energy fluxes must also satisfy \((\rho_S + p_S)v_S \ll (\rho_C + p_C)v_C\). In this section, we add the subscript \((C)\) on the remaining matter to remind the reader that part of the total matter density has been designated as effectively smooth in the division of equation \(14\). Our analysis also applies to cases where \(\delta \rho_S \ll \delta \rho_C\) where the time-average is over the dynamical time of the \(C\) component (see \(\text{VIII} C\)).

One cannot demand that \(\delta \rho_S\) be identically zero since the continuity equation \(14\) generates a density fluctuation as the metric curvature changes unless \(\varepsilon_{\rho} = \rho_S / \rho_C = -1\) (a cosmological constant). The fractional density fluctuation is generically at least of order the curvature fluctuation \(\Phi\). Since this term exceeds the energy flux term outside the horizon due to causality, energy conservation forbids smooth components (with \(\rho_S \neq 1\)) on these scales.

Components that are smooth within the horizon are possible. For example, the \(C\) component may be driven to collapse by potential gradients while the \(S\) component is supported against collapse by stress gradients (see \(\text{VIII} C\)). Here \(\delta \rho_S / \rho_S \sim \Phi \ll \delta \rho_C / \rho_C\). Since these stress gradients are set up exactly so as to keep the component smooth, one can replace such stress effects in equation \(50\) with an additional smooth density component \(\tilde{\rho}\). The remaining perturbations can then be approximated as stress-free and generate curvature fluctuations as

\[
\Phi'' + \left(1 - \frac{\rho_C''}{\rho_C} + \frac{1}{2} \frac{\rho_C'}{\rho_C} \right) \Phi' + \left( \frac{\rho_C'}{2} - \frac{\rho_C'}{\rho_C} \right) \Phi = 0.
\]

Now even for constant \(w_C\), \(\Phi' = 0\) no longer solves the equation of motion. Mathematically, \(\rho_S\) adds to the expansion drag (\(\Phi'\)) terms but not the gravitational (\(\Phi\)) terms. Physically, potential flows still create density perturbations as \(\rho_C \sim (\rho_C + p_C)(\kappa n)^2\Phi\) but the Poisson equation leads to a smaller potential \(\Phi = (\kappa n)^{-2} \rho_C / (\rho_C + \rho_S)\). As this process continues, \(\Phi\) decays away.

Equation \(53\) has simple solutions in the limit that \(\rho_C' \gg \rho_C\), as is usually the case when the smooth component dominates the expansion. In this case, the general solution to the equation is

\[
\Phi = C_1 a^{-1} + C_2 a^{-1} \int d \ln a \frac{a \rho_C'}{\rho_C^{1/2}}.
\]

Both terms represent decaying modes as long as \(w_C < 1/2\).

**Full Solutions.** — The full solution to equation \(53\) can be obtained analytically for a clustering component with a constant equation of state and present-day fractional density contribution pair \((w_1, \Omega_{S1})\), a smooth component with \((w_2, \Omega_{S2})\), and/or a second smooth component with \((w_2, \Omega_{S2})\):

\[
\Phi_j = y^{j/2 \Delta w} \left( \frac{y}{1 + y} \right)^{\alpha_j} \times F \left( \alpha_j, \alpha_j + 1, \frac{3 w_1 + 1}{6 \Delta w}, \frac{y}{1 + y} \right),
\]

where \(\Delta w = w_2 - w_1\) with

\[
\alpha_j = \frac{1 + 3 w_1}{12 \Delta w} \left[ 1 + \sqrt{1 + 24 \frac{\Omega_{C1}}{\Omega_{C1} + \Omega_{S1}} \frac{1 + w_1}{(1 + 3 w_1)^2}} \right],
\]

\[
y = \rho_2 / \rho_1 = \frac{\Omega_{S2}}{\Omega_{S1} + \Omega_{C1}} a^{-3 \Delta w},
\]

where \(j = 1, 2\) for the growing and decaying modes corresponding to \(-, +\) in the \(\alpha_j\) equation. Note that negative contributions to the critical density, e.g. positive spatial curvature \(K > 0\), are also covered by these solutions.

The hypergeometric solution allows one to identify elementary solutions more easily. In particular, cases where \(\alpha_j = -[N_1/2]/(3 \Delta w + 1)/6 \Delta w = N_1\), or \(2 \alpha_j - (3 \Delta w + 1)/6 \Delta w + 1/2 = N_1\) can be expressed in terms of elementary functions. Cases of the form \(F(a, b; a + N_1; x)\), \(F(a, b; a + N_1; x)\), or \(F(N_1/2, N_2/2; N_3/2; x)\) can be expressed in terms of elementary functions, incomplete beta functions, and/or complete elliptic integrals. Here, the \(N_i\) are integers and \(a \) and \(b \) are real numbers. For example, the case of \(w_1 = 0 \) and \(w_2 = -1/3 \) can be expressed in closed form for any \(\Omega_{C1}/(\Omega_{C1} + \Omega_{S1})\):

\[
\Phi_j \propto y^{\frac{2 \alpha_j}{a}} \frac{1 + \sqrt{1 + y}}{a^2 (1 + y^{2 \alpha_j})}.
\]

For \(w_1 = 0\), any case in which \(2 \alpha_j + 1/6 \) \(w_2 + 1/2\) is an integer can be simplified. One can also simplify the growing mode of cases with \(w_1 = \Omega_{S1} = 0 \) and \(w_2^{-1} \) equal to an integer.

Finally for completeness, there is a well-known special case of equation \(53\) that is not completely covered by equation \(71\) but that does have an integral solution. This involves a clustering component \(w = 0\) (CDM) and a smooth component composed of an arbitrary admixture of \(w = -1/3\) (curvature) and \(w = -1\) (Λ) pieces:

\[
\Phi \propto a^{-1} \rho_{ci}^{1/2} \int \frac{d \ln a}{a^2 \rho_{ci}^{1/2}}, \quad \Phi \propto a^{-1} \rho_{ci}^{1/2},
\]

for the growing and decaying modes respectively. Of course, if either the curvature or Λ component is negligible, the solutions have an analytic form described by
equation (71). Unfortunately, this solution cannot be
generalized to arbitrary combinations of smooth com-
ponents since it relies on the fact that \( w = -1/3 \) does not
accelerate the expansion and that \( w = -1 \) gives a con-
stant density contribution.

Applications. — Smooth components are widely found
in adiabatic models and their description in terms of
equation (71) are given in Table I.

In fact, all CDM variants (CDMv) go through a phase in
the radiation-dominated epoch when the perturbations
in the radiation are pressure supported leading to an
essentially smooth component \( \Omega_{\text{rad}} = \Omega_{S2} \) with \( w_2 = 1/3 \)
and also a smooth baryonic component \( \Omega_b = \Omega_{S1} \) with
\( w_1 = 0 \) that is held against collapse by Compton coupling
to the photons (see \( \ref{eq:2} \) and \( \ref{eq:3} \)). In this case, the
clustered matter is the CDM (\( \Omega_{\text{CDM}} = \Omega_{C1} \)), and the
solution (71) describes the evolution of the CDM density
perturbations via the Poisson equation. The behavior
deep in the radiation domain is given by equation (70),
which says that density perturbations actually grow log-
arithmically. The full solution (71) maps this logarith-
mic mode into a power-law growth across the matter-
radiation transition and is useful for determining the am-
plitude of small-scale fluctuations as a function of the
baryon content [35].

Smooth components that dominate at late times are
also described by equation (71). Many such models have
been proposed to reduce the amount of small-scale power
in the standard CDM model. The prototypical exam-
ple is the ΛCDM model where \( \Omega_{S2} = \Omega_{\Lambda} = 1 - \Omega_m \)
(\( w_2 = -1 \)) and the open model OCDM \( \Omega_{S2} = \Omega_K =
1 - \Omega_m \) (\( w_2 = -1/3 \)). The former case is special in that
it may alternately be considered a clustered component
(see \( \ref{eq:1} \)). The latter case must be considered as a
smooth component, and the growing mode reduces to
\[
\Phi \propto \frac{1}{y} \left( 1 + \frac{3}{y} - \frac{3}{y} \tan^{-1} \frac{y}{1+y} \right),
\]
where \( y \propto a \).

The obvious generalization of such models involves a
smooth component with an equation of state that dif-
fers from the curvature or cosmological constant exam-
ple obtained for the radiation dominated era for this flat \( \Omega_m = 0.35 \)
model.

**FIG. 8.** Single smooth component. The same as Fig. exception that \( w_2 \) is taken to be a smooth component. Here
equation (71) is compared to a QCDM model but for a mode
that is well inside the horizon at \( y = 1 \). Discrepancies at
early times are due to radiation contributions in the QCDM
models and are particularly pronounced for \( w_2 \to 0 \) since then
the epoch when the Q and matter components are equal is
driven into the radiation dominated era for this flat \( \Omega_m = 0.35 \)
model.

| \( w_1 \) | \( w_2 \) | \( \Omega_C \) | \( \Omega_S1 \) | \( \Omega_S2 \) | Model          |
|---------|---------|-------------|-------------|-------------|---------------|
| 0       | 1/3     | \( \Omega_{\text{cdm}} \) | \( \Omega_b \) | \( \Omega_{\text{rad}} \) | CDMv          |
| 0       | -       | \( \Omega_{\text{cdm+b}} \) | \( \Omega_r \) | -           | HCDM          |
| 0       | -1      | \( \Omega_{\text{cdm+b}} \) | 0           | \( \Omega_{\Lambda} \) | ΛCDM          |
| 0       | -1      | \( \Omega_{\text{cdm+b}} \) | \( \Omega_{\Lambda} \) | \( \Omega_{\Lambda} \) | ΛHCDM         |
| 0       | -1/3    | \( \Omega_{\text{cdm+b}} \) | 0           | \( \Omega_{K} \) | OMDM          |
| 0       | -1/3    | \( \Omega_{\text{cdm+b}} \) | \( \Omega_{K} \) | \( \Omega_{K} \) | OMDM          |
| 0       | -1/3    | \( \Omega_{\text{cdm+b}} \) | 0           | \( \Omega_{\text{str}} \) | strCDM        |
| 0       | -       | \( \Omega_{\text{cdm+b}} \) | \( \Omega_{\phi} \) | -           | φCDM          |
| \( w_Q \)| \( \Omega_{\text{cdm+b}} \) | 0           | \( \Omega_{Q} \) | \( \Omega_{Q} \) | QCDM, GDM     |
| \( w_Q \)| \( \Omega_{\text{cdm+b}} \) | \( \Omega_{r} \) | \( \Omega_{Q} \) | \( \Omega_{Q} \) | QHCDM, GDM    |
| \( w_Q \)| \( \Omega_{\text{cdm+b}} \) | \( \Omega_{\phi} \) | \( \Omega_{Q} \) | \( \Omega_{Q} \) | φQCDM, GDM    |

TABLE I. Correspondence between the analytic solution of
equation (71) and models with smooth components. \( \Omega_K \)
is the fractional effective density supplied by the curvature component.
C. Vector Perturbations

The behavior of vector modes is far simpler than that of scalar modes. Vector anisotropic stress can be neglected if \( H_T^{(±1)} \gg \rho \Pi^{(±1)} / \rho_c \). In the absence of stress perturbations, vector modes simply decay with the expansion (see equation [20]) is solved by

\[
(\rho + p)(\nu^{(±1)} - B^{(±1)}) \propto a^{-4},
\]

\[
k B^{(±1)} - H_T^{(±1)} \propto a^{-2},
\]

The metric source is related algebraically to this quantity by equation [21].

D. Tensor Perturbations

The tensor anisotropic stress is negligible if \( H_T^{(±2)} \gg \rho \Pi^{(±2)} / \rho_c \). In this limit, tensor metric fluctuations remain constant outside the horizon regardless of the expansion rate and propagate as free gravity waves inside of it. They are described by equation [23] with the sources set to zero.

Full Solutions.— If the expansion is dominated by a component with constant equation of state \( w_{cr} > -1/3 \), the fundamental modes of gravity waves are

\[
H_1 = \frac{2^{m+1} \Gamma(m + 3/2)}{\sqrt{\pi}} x^{-m} j_m(x),
\]

\[
H_2 = \frac{2^{m+1} \Gamma(m + 3/2)}{\sqrt{\pi}} x^{-m} n_m(x),
\]

with \( x = k \eta, m = (1 - 3w) / (1 + 3w) \) and \( \dot{k} = \sqrt{k^2 - 2 \dot{H}} \). Note we have normalized the modes so that \( H_1(0) = 1 \); this mode drops from unity into damped oscillations when the wavelength reaches some fraction of the horizon that decreases with \( m \) and hence increases with \( w \). In Fig. 10 we plot the derivative of these modes, as that determines its effect on the radiation through gravitational redshifts.

For \( w < -1/3 \), the universe accelerates and the horizon stops growing with the scale factor. This implies that gravity waves will also freeze out at some finite value related to their amplitude when the universe began accelerating. The solutions for the gravity wave behavior relative to the epoch of freeze-out follow the form of equation [77] with \( x = k \int_\eta^\infty d\eta \). The \( H_1 \) mode then has damped oscillations as \( x \) increases from negative values but freezes in to finite value as \( x \to 0 \) in the infinite future.

Applications.— The solutions above help us understand the phenomenology of tensor anisotropies in the CMB. The tensor analogue to equation [23] is

\[
\frac{\Theta_k}{2\ell + 1} \approx \sqrt{\frac{3}{8}} \frac{\ell!}{(\ell - 2)!} \int_{\eta_0}^{\infty} d\eta e^{-\tau} H_T^{(±2)} j_\ell(k \Delta \eta) (k \Delta \eta)^2,\]

where \( \Delta \eta = \eta_0 - \eta \). The power spectrum is again given by equation [54]. The appearance of the \( e^{-\tau} \) damping reflects the fact that anisotropic stress cannot be supported in the optically thick limit; we take \( \tau \to 0 \) in the examples here to consistently neglect all anisotropic stress effects. This in fact is a good approximation for tensor anisotropies in the neutrino background radiation.
VII. PURE STRESSES

We have shown in the previous section that in certain regimes stress perturbations can be ignored. However, to provide a complete history of structure formation, one must track the perturbations across all scales and time. We will first consider the pure stress cases in which the dominant stress contribution is anisotropic, entropic, or sonic, as defined in §IV. These prototypical cases have analytic solutions and are the starting point for the general cases discussed in §VIII. They also have direct application in many models. The anisotropic and sonic solutions are applicable to all adiabatic variants of the CDM model (CDMv). The entropic solutions show how prototypical isocurvature models such as the baryon (PIB) or axion (AXI) isocurvature models form structure. We outline these results in Fig. 12.

A. Anisotropic Stress

Anisotropic stresses play a special role because they enter directly into the Newtonian metric through $S_{\Pi}$, as opposed to other stresses which only contribute through the causal motion of the matter. This is despite the fact that the comoving curvature $\zeta$ depends only on $S$. In an isocurvature model, only the comoving curvature need vanish initially, not the Newtonian curvature.

Similarly, even though $\zeta$ remains constant above the horizon in adiabatic models, the Newtonian curvature $\Phi$ evolves under anisotropic stresses (see Eq. (72)). Recall that

$$S_{\Pi} = -8\pi Ga^2 p\Pi/k^2 = -3\left(\frac{\dot{a}}{a}\right)^2 \frac{p}{\rho_c} \Pi/k^2,$$

such that its effect is enhanced by $(p/\rho_c)(k\eta)^{-2}$ relative to $S$. Note, however, that once the universe enters a period when $(p/\rho_c)\Pi \ll \langle k\eta\rangle^2\zeta$, all traces of the anisotropic stress from any previous period vanish in the relation between $\Phi$ and $\zeta$. Its effect does not vanish from the CMB, however, since anisotropies record a time-integrated history of the gravitational potentials.

Full Solutions. — To close this system of equations, we need a relation between $S_{\Pi}$ and $\zeta$. We will consider two limiting cases: when $S_{\Pi} < \zeta$ as in the case of stresses from radiation backgrounds, and when $S_{\Pi} \gg \zeta$ as is possible with models involving active sources such as defects.

In the former case, anisotropic stress is generally created as a by-product of gravitational instability. Its anisotropic nature suggests that it can be created from shear in the velocity and metric, and its coordinate transformation properties demands that its source be gauge invariant. The linear combination of these sources that satisfies these requirements is $(k\nu - H_T)$. The anisotropic
FIG. 12. Scalar stress perturbations. Scalar stress perturbations are divided into three pure classes: anisotropic, entropic, and sonic. If one of these is dominant, then general techniques (third row) can be applied to analyze the resultant behavior of fluctuations. Many models (fourth row) do go through phases where the stresses are dominated by one of the pure stresses. The designations “all models” and “all adiabatic models” assume a neutrino background and fluctuations that were present before last scattering. Several cases with analytic solutions illustrate the range of behaviors.

stress can also have a dissipation timescale \( \eta_\Pi \). Together these considerations imply an evolution of the form

\[
\dot{\Pi} + \eta_\Pi^{-1} \Pi = 4(kv - H_T)\alpha .
\]

In this section, we are interested in large-scale effects and hence want the longest timescale for the dissipation; this is set by the expansion time. We will consider cases where the timescale is much smaller than the expansion time in \( \S VIII D \).

If we take \( \eta_\Pi^{-1} = 3\dot{a}/a \), we recover the phenomenological parameterization of \([7]\)

\[
\dot{\Pi} + \frac{3}{a} \dot{a} \Pi = 4(kv - H_T)\alpha ,
\]

which has the formal solution

\[
\Pi = 4a^{-3} \int d\eta a^3(kv - H_T)\alpha .
\]

\[\alpha = c^2_{vis}/w_g \] in the notation of \([7]\).

From equation \( (80) \), we find

\[
\Pi = -\frac{8}{3}a^{-3} \int d\eta a^3 \alpha \frac{k^2}{1 + w \rho \rho_c} \left( \frac{\dot{a}}{a} \right)^{-2} \left( \dot{\Phi} - \frac{\dot{a}}{a} \Psi \right) ,
\]

for modes well under the curvature scale. Employing equation \( (61) \) for the zeroth-order potentials and assuming constant \( \alpha \) and \( \rho = \rho_c \) yields

\[
\Pi = -2\alpha(k\eta)^2 \frac{1 + 3w}{(4 + 3w)(5 + 3w)} ,
\]

for constant \( w \). With this source in equation \( (52) \), the Newtonian potential becomes

\[
\frac{\Phi}{\zeta} = \frac{3 + 3w}{5 + 3w}(1 + \beta) ,
\]

with

\[
\beta = 16 \frac{w}{1 + w} \frac{1}{(4 + 3w)(5 + 3w)} .
\]
Notice that the integrals remain finite as long as $S$ one can build the third order relation for the curvature order form for the Newtonian potential the desired accuracy. For example, employing the second rate. If $\rho S/\zeta < \zeta$, solutions for $\Phi$ can be constructed by iteration. Here we show an example where the anisotropic stress is produced by massless neutrinos and compare numerical results from a Boltzmann code with the iterative solution. To first order, $S_S/\zeta = 4N_\nu/15(4.4 + N_\nu)$, where $N_\nu$ is the number of massless neutrino species.

if the curvature contributes negligibly to the expansion rate. If $\beta < 1$, this process may be repeated to obtain the desired accuracy. For example, employing the second order form for the Newtonian potential

$$\frac{\Psi}{\zeta} = -\frac{3 + 3w}{5 + 3w} \left\{ 1 - \frac{3}{2} \beta \left[ 1 - \frac{3}{2} (1 + w) \beta \right] (1 + w) \right\}, \quad (87)$$

one can build the third order relation for the curvature

$$\frac{\Phi}{\zeta} = \frac{3 + 3w}{5 + 3w} \left( 1 + \beta \left[ 1 - \frac{3}{2} \beta \right] \right) \times \left[ 1 - \frac{3}{2} (1 + w) \beta \right] (1 + w) \right\}. \quad (88)$$

In the opposite limit that the anisotropic stress contribution is large compared with the other perturbations ($S_S \gg \zeta, S_\Pi \gg S$), the integral solutions for the Newtonian metric reduce to

$$\Phi = \frac{\sqrt{\rho}}{a} \int \frac{da}{\sqrt{\rho}} S_S, \quad \Psi = -\Phi + S_\Pi. \quad (89)$$

Notice that the integrals remain finite as long as $S_S$ diverges at zero no faster than $a^{-5/2 - 3w/2}$ and the prefactor $\sqrt{\rho}/a$ is simply the decaying mode of the Newtonian curvature (see Eq. [59]).

Applications. — Collisionless radiation provides an application for these results. The anisotropic stress of radiation is related to the quadrupole moment as defined in equation (53) by

$$\Pi_\ell = \frac{12}{5} \Theta_2 \quad (90)$$

on scales much smaller than the curvature radius. Equation (53) also implicitly gives the equation of motion of the $\ell$th multipole as

$$\dot{\Theta}_\ell = k \left[ \frac{\ell}{2\ell - 1} \Theta_{\ell-1} - \frac{\ell + 1}{2\ell + 3} \Theta_{\ell+1} \right]. \quad (91)$$

This is an infinite set of coupled differential equations representing the fact that radiative stress depends on internal degrees of freedom of the distribution. However, [18] introduced an approximation based on the solution of these equations in the absence of sources

$$\frac{\Theta_\ell}{2\ell + 1} = C j_\ell(k\eta), \quad (92)$$

which allows us to express

$$\Theta_{\ell+1} = (2\ell + 3)(k\eta)^{-1} \Theta_{\ell} - \frac{2\ell + 3}{2\ell - 1} \Theta_{\ell}. \quad (93)$$

Applying this closure relation to the quadrupole, noting $(\dot{a}/a) = 1/\eta$ in the radiation-dominated era, and rewriting the dipole in covariant form $\Theta_1 = \nu - H_T$ shows that the anisotropic stress of free radiation in the radiation-dominated era obeys equation (81) with

$$\alpha = \rho_\gamma / p = \rho_\gamma / \rho_{\text{rad}}.$$

The photons actually do not behave in this manner before last scattering since their coupling to the baryons destroys any quadrupole moment in the distribution (see VIII D). However, the same analysis applies to the massless neutrinos, whose anisotropic stress can be approximately parameterized by $\alpha = \rho_\nu / \rho_{\text{rad}}$. Equation (81) shows that the change in $\Phi$ is enhanced by a factor $1 + (2/15)(\rho_\nu / \rho_{\text{rad}})$. With equation (54), we find the effective temperature of the CMB to be

$$\Theta_0 + \Psi = -\frac{1}{2} \frac{\Phi}{\eta} = -\frac{1}{3} \zeta(0) \left( 1 - \frac{4}{15} \frac{\rho_\nu}{\rho_\gamma} \right), \quad (94)$$

and the ISW combination to be

$$\Psi - \Phi = -\frac{4}{3} \zeta(0) \left( 1 - \frac{1}{15} \frac{\rho_\nu}{\rho_\gamma} \right), \quad (95)$$

to first order. These results are equivalent to but more physically transparent than those of [10]. In addition, equation (81) introduces second and third order corrections that are important for models with additional neutrino species or higher neutrino temperatures $T_\nu$.

$$\frac{\rho_\nu}{\rho_\gamma} = 0.681 \frac{N_\nu}{3} \left( \frac{1.401 T_\nu}{T_\gamma} \right), \quad (96)$$

as shown in Fig. 13.

Equations (54) and (92) imply that the neutrinos have significant but opposite effects on the effective temperature and ISW terms in the CMB anisotropy equation (53). The degeneracy is broken as the fluctuation
enters the horizon, as we shall see in §VII D, leading to
detectable effects in the acoustic peaks [43].
Finally, seed defect sources generally have large
anisotropic stress contributions outside the horizon.
Causality dictates that $p_s \Pi_s$ behaves as white noise above
the horizon which together with the so-called scaling
ansatz leads to an $\eta^{-1/2}$ temporal behavior. This
implies that $S_H \sim \eta^{-5/2} \sim \alpha^{-5(1+3w)/4}$. Since this diverges
slower than $\alpha^{-5/2-3w/2}$, it does not imply a divergence
of any observable [31] and contributes mainly to the de-
caying mode of the curvature [1]. Note that these con-
siderations assume vanishing spatial curvature.

B. Entropic Stress

We next consider the case in which entropic stress domi-
ninates the other stress components $S_E \gg S_S, S_H$. In
this limit $S_E$ does not depend on $\zeta$ and equations (52) and (53) are more than simply a formal solution: they
are the integral solutions for an arbitrary source in the
absence of smooth components. For instance, if the en-
trropic stress has a power-law behavior $S_E \propto \alpha^n$ then the comoving curvature will have the same behavior
$\zeta = -\int d\ln a S_E \propto \alpha^n$ assuming $\zeta(0) = 0$.

It is important to note that once $S_E$ turns off, $\zeta$ will
remain constant. Thus entropic stresses that act for only
a fixed amount of time generate curvature fluctuations
out of isocurvature initial conditions that then behave in
the same manner as initial curvature fluctuations.

A natural way of establishing an entropic stress is to
have two components whose sound speeds differ at some
point in the evolution. It is useful to introduce the “en-
tropy”

$$\sigma = \frac{\delta \rho_2}{\rho_2 + \rho_2} - \frac{\delta \rho_1}{\rho_1 + \rho_1},$$

(97)
since its equation of motion is simply

$$\dot{\sigma} = -k(v_2 - v_1)$$

(98)
by virtue of equation (14), assuming no direct energy ex-
change between species. The equivalence principle guar-
antees that under purely gravitational evolution the vel-
ocity differences vanish so that a constant $\sigma$ is a good
approximation outside the horizon. Tight coupling be-
tween the components also implies $\dot{\sigma} = 0$.

Let us assume that the density perturbations are ac-
companied by sonic stresses in the rest frame of each
component,

$$\Delta \rho_j = \delta \rho_j - \dot{\rho}_j (v_j - v) / k,$$
$$\Delta p_j = \delta p_j - \dot{\rho}_j (v_j - v) / k,$$

(99)
with $c_J^2 = \Delta p_j / \Delta \rho_j$. Under the constant entropy as-
sumption $v_j = v$ and with the definition of the combined
sonic stress

$$c_E^2 = \frac{\rho_1 c_1^2 + \rho_2 c_2^2}{\rho},$$

(100)
the entropic stress becomes

$$\frac{S_E}{\sigma} = \left( \frac{\rho_1 \rho_2'}{\rho_2} \right) (c_E^2 - c_2^2).$$

(101)

Consequently, entropic stresses are generated when the
sound speeds of the two components differ. Furthermore
they can be much larger than the sonic stresses if $\delta \rho / \rho \ll
\sigma$ as is the case for isocurvature models.

Full Solutions. — If the sonic stresses are also adi-
abatic ($c_J^2 = p_J'/\rho_J$), then $S_E = S_T$ and $S_T / \sigma =
(\rho_1 / \rho') / 3$. Equation (53) then yields

$$\zeta(a) = \frac{\sigma \rho_2'}{3 \rho} \left| \frac{a}{0} + \zeta(0) \right. \right.$$

(102)

The Newtonian potential follows from equation (52).

For example, in the constant $w_1$ and $w_2$ case,

$$\frac{S_T}{\sigma} = \frac{(1 + w_1)(1 + w_2) \Delta w}{[1 + w_1 + (1 + w_2)y] y},$$

(103)
$$\frac{\zeta}{\sigma} = \frac{1}{3} \frac{(1 + w_2)y}{1 + w_1 + (1 + w_2)y},$$

(104)
where recall $\Delta w = w_2 - w_1$ and $y = \rho_2 / \rho_1$ and we have
assumed isocurvature initial conditions ($\zeta(0) = 0$). Once
$\rho_2$ dominates the energy density, the entropic stress has
been converted into a constant comoving curvature $\zeta =
\sigma / 3$. This solution may then be substituted into equa-
tion (52) to obtain the integral solution

$$\frac{\Phi}{\sigma} = \frac{1}{3} \frac{1}{r + y} + \frac{1}{\sqrt{1 + y}} \int_0^y \frac{dy}{y^{n-1}}$$
$$\times \left[ 1 - \frac{x^2 + ry(3\Delta w + 1)}{(r + y)^2} \right],$$

(105)
where $r = (1 + w_1)/(1 + w_2)$ and $n = -(5 + 3w_1)/6 \Delta w$.

The Newtonian curvature has the limits

$$\frac{\Phi}{\sigma} = \frac{1}{5 + 9w_1 - 6w_2} y, \quad (y \ll 1),$$
$$\frac{1 + w_2}{1 + w_2}, \quad (y \gg 1).$$

(106)
The relation between the Newtonian and comoving cur-
vature in the $y \gg 1$ limit is exactly the same as the
stress-free clustered case of §VI A.

Another interesting case is when the two components
are initially both radiation $w_1(0) = w_2(0) = 1/3$ but the
second component $w_2$ becomes non-relativistic. Equa-
tion (102) implies that by the time $\rho_2 \gg \rho_1$, a curvature
fluctuation

$$\zeta = \frac{\sigma \rho_1}{3 \rho} \left| \frac{a}{0} \right. \right.$$

(107)
is generated from isocurvature conditions.

The more general solution \( (10) \) covers multicomponent models with non-adiabatic sonic stresses in one or both of the components and for example may be applied to scalar field models.

Applications. — Entropy perturbations between the matter and the radiation are the basis of the prototypical isocurvature models, i.e. the baryon isocurvature model (PIB) and the axion isocurvature model. In this case, the integral in equation \((105)\) can be explicitly solved to obtain \[33\]

\[
\frac{\Phi}{\sigma} = \frac{16 + 8y - 2y^2 + y^3 - 16\sqrt{1 + y}}{5y^3},
\]

(108)

where \( y \approx a \). Thus, the curvature grows as \( a \) in the radiation-dominated era only to freeze out at the amplitude reached at matter-radiation equality.

Motivated by our study, one of us constructed an isocurvature decaying dark matter (iDDM) model that utilizes the mechanism described above where density fluctuations in two radiation species are initially balanced \[22\]. The resulting constant superhorizon curvature fluctuation \( \zeta \) is a property generally associated with adiabatic models. Another interesting consequence is that the ratio of large-scale CMB temperature anisotropies to the Newtonian potential is neither \( 1 \) nor \( 2 \). This is because the photons possess initial perturbations. These can be chosen to make the ratio greater than or less that \( 2 \); in particular it can be arranged to be close to the adiabatic \( 1/3 \) relation.

C. Sonic Stress

Sonic stresses provide the final pure case. Here

\[
S = S_S = c_S^2 \frac{\delta \rho}{\rho + p} = c_S^2 \frac{\delta \rho}{\rho + p};
\]

(109)

recall that adiabatic stresses are a special case of a sonic stress where \( c_S^2 = c^2 \equiv \rho' / \rho' \). Because the stress fluctuation is related to the comoving density fluctuation by equation \((109)\) and in turn to the Newtonian curvature via the hybrid Poisson equation \((34)\), the evolution equation \((54)\) for the Newtonian curvature becomes a homogeneous second order differential equation.

\[
\frac{\rho'}{a \rho^2} \left( \frac{a \rho_{ci}^{1/2}}{\rho'} \Phi' \right)' + \left( \frac{1}{2} \frac{\rho_{ci}'}{\rho_{ci}} - \frac{\rho''}{\rho'} + k^2 s^2 \right) \Phi = 0,
\]

(110)

where the sound horizon is defined as

\[
s = \int d\eta c_C .
\]

Here, we have assumed that the wavelength is much smaller than the curvature scale, but all results are applicable to the general case with the replacement \( k \rightarrow \sqrt{k^2 - 3K} \).

For \( ks' \gg 1 \), we can approximate this as an oscillator equation with an effective mass

\[
(m_{\text{eff}} \Phi')' + k^2 s^2 m_{\text{eff}} \Phi = 0,
\]

(112)

with

\[
m_{\text{eff}} = \frac{a \rho_{ci}^{1/2}}{-\rho'}.
\]

(113)

Under the WKB assumption that the effective mass is slowly-varying compared with the frequency of oscillation, the fundamental solutions to this equation are

\[
\Phi_1 = \left( \frac{-\rho'}{cc} \right)^{1/2} \cos(ks),
\]

\[
\Phi_2 = \left( \frac{-\rho'}{cc} \right)^{1/2} \sin(ks).
\]

(114)

This equation says that once the fluctuation passes inside the sound horizon, the Newtonian potential oscillates and decays reflecting analogous behavior in the density perturbation through the Poisson equation.

Full Solutions. — Equation \((11)\) suggests that a change of variables to

\[
Q = \left( \frac{cc}{-\rho'} \right)^{1/2} \Phi,
\]

(115)

should simplify the equations of motion. Indeed, equation \((11)\) becomes

\[
\frac{d^2 Q}{d(k s)^2} + \left( 1 - \frac{F}{k^2 s^2} \right) Q = 0,
\]

(116)

where

\[
F = -\frac{s^2}{2s^2} \left\{ \frac{\rho_{ci}'}{\rho_{ci}} \rho' + \frac{\rho''}{\rho'} + \frac{3\rho'''}{2\rho^2} \right\}. \quad (117)
\]

If \( F \) is constant, then equation \((11)\) is a variant of the Bessel equation, and the solutions \( Q \) will approach \( \sin(ks) \) (up to a phase) at large \( ks \). Furthermore, when \( ks \gg |F|^{1/2} \), the term with \( F \) may simply be neglected, and again the solutions will be \( \sin(ks) \) with arbitrary phase. Because the latter solutions will match trivially onto the former, we reach the conclusion that if, for a given mode, \( F \) is constant until \( s \gg |F|^{1/2}/k \), then the appropriate Bessel function solution will hold for all times, regardless of how \( F \) varies once \( ks \gg |F|^{1/2} \). If \( cc \) and \( \rho \) vary on the Hubble time scale, then \( F \) is order unity, and the solution describes modes that are well inside the horizon before \( F \) begins to vary.
The simplest way to arrange $F$ to be constant is to have a constant equation of state $w_1$ and constant sound speed $c_C$. For constant $w_1$, $s$ is only defined for $w_1 > -1/3$, and we therefore set $\rho = \rho_{cr}$. Then

$$F = 6 \frac{1 + w_1}{(1 + 3w_1)^2},$$

independent of the sound speed. The growing and decaying solutions for $\Phi$ are then

$$\Phi_1 \propto \left( \frac{-\rho' k s}{c_C} \right)^{1/2} J_\nu(ks),$$

$$\Phi_2 \propto \left( \frac{-\rho' k s}{c_C} \right)^{1/2} N_\nu(ks),$$

where $\nu = (5 + 3w_1)/(2 + 6w_1)$. Again, once a given mode is well inside the sound horizon, the condition that $w_1$ and $c_C$ be constant can be relaxed.

**Applications.** — Adiabatic stress perturbations in a baryon-photon universe provide the prototypical example. In this case, the tight coupling between the photons and baryons through Compton scattering prevents the generation of entropy through the motion of matter (see VIIIC). The growing mode of equation (119) becomes

$$\frac{\Phi}{\zeta(0)} = \frac{3}{y^2} \left( k_y \right)^2 \left( 1 + \frac{3}{4} y^3 \right)^{3/4} \left[ \frac{\sin(ks)}{ks} - \cos(ks) \right],$$

with $k_y = (\dot{a}/a)\gamma = 1$ and

$$s = 4 \sqrt{2} \frac{1}{3} k_y \ln \frac{\sqrt{4 + 3y} + \sqrt{3y + 3}}{2 + \sqrt{3}}.$$  

We have chosen the normalization to match the stress-free solution $\Phi = 2\zeta/3$ initially. In this case, the acoustic oscillations of the photons are directly related to behavior of the potential via the Poisson equation

$$\Theta_0 = \frac{1}{3} \left( k_y \right)^2 \frac{y^2}{1 + 3y/4} \Phi$$

$$= \frac{1}{(1 + 3y/4)^{1/4}} \zeta(0) \cos(ks) \quad (ks \gg 1).$$

Since the acoustic oscillations are responsible for the acoustic peaks in the CMB from equation (33), this determines the morphology of the acoustic peaks in this simple adiabatic photon-baryon universe. The important result is that the acoustic oscillations follow a $\cos(ks)$ pattern in phase with an amplitude that is enhanced by the decay of the initial curvature perturbation. These results are in fact generic to adiabatic models due to similar evolution in the driving potentials [11].

A second example is provided by the QCDM and GDM models where the equation of state can go negative ($w < 0$) while the comoving sound speed remains real ($c_s^2 > 0$). In QCDM, $c_s^2 = 1$ by virtue of the scalar-field equations of motion. In GDM, it is allowed to have any positive value. The solutions are strictly valid only after the exotic component dominates the expansion rate and fluctuations. To test this solution, we show an extreme example where the dark matter is all in a “Q” component (no CDM) with $w_Q = 0$ (see Fig. 14). The small departures at early and late times are due to the radiation and baryonic contributions.

**VIII. MIXED STRESSES**

Although the purely anisotropic, entropic and sonic stress cases of the last section illustrate many aspects of stress phenomenology, when all three are present they can interact and create a diverse range of behavior. In this section, we study a few typical cases: sonic stress generation from entropic stress, entropic stress generation from sonic stress, and anisotropic stress dissipation of sonic stress. The first process is responsible for generating acoustic phenomena in all isocurvature models. The second process is responsible for the heat conduction in fluids and the generation of smooth components. The last process is responsible for viscous dissipation of acoustic phenomena and also provides an alternate means of generating a smooth component. We briefly discuss seed (defect) stresses, which carry not only all three types of scalar stress but also vector and tensor stresses as well. Finally, we consider the effect of tensor anisotropic stress
on gravitational waves and CMB anisotropies in passive models.

### A. Formal Solution

If the equations of motion for a sonic stress perturbation are known to be solved by $\Phi_1$ and $\Phi_2$, the full solution can be written in integral form as,

$$
\Phi = A\Phi_1(a) + B\Phi_2(a) + \int_0^a \frac{d\tilde{a}}{\tilde{a}} \left[ \Phi_1(\tilde{a})\Phi_2^\prime(\tilde{a}) - \Phi_1^\prime(\tilde{a})\Phi_2(\tilde{a}) \right] 
\times \left[ \frac{1}{2} \frac{\rho_s}{\rho_{cr}} S_E + S_\Pi^\prime + \left( \frac{1}{2} \frac{\rho_s^\prime}{\rho_{cr}} - \frac{\rho_s}{\rho_s^\prime} \right) S_\Pi \right],
$$

where $A$ and $B$ are arbitrary constants. This solution is merely formal unless the remaining stress sources can be specified independently of $\Phi$. We now consider some special cases.

### B. Entropic/Sonic Stress

A model that begins with isocurvature initial conditions will generate adiabatic density fluctuations through the action of entropic stresses, as discussed in VII B. These density perturbations carry with them sonic perturbations through equation (39) that stop the further growth of density perturbations through the same pressure support mechanism discussed in VII C. Therefore, entropic growth of curvature fluctuations generally ceases once the fluctuations cross the sound horizon of the dominant species.

**Full Solutions.** — The solutions for $\Phi$ in equation (119) along with the entropic stress from equation (101) in equation (123) allow us to construct the full solution in the presence of a constant entropy and $\sigma = \frac{\delta_b}{\rho_{cr}}$.

$$
\frac{\Phi}{\sigma} = \frac{3}{4y} \left( \frac{k_y}{k} \right)^2 \left[ 1 - \frac{\sqrt{3} k_y (4 + 3y)^{3/4}}{2 k y} \sin(k s) \right],
$$

which is again directly related to the temperature oscillations by equation (122). The first term in brackets is due to density perturbations in the baryons remaining from the constant entropy condition $\sigma = \delta_b - \frac{1}{2} \delta_s \approx \delta_b$. The second term represents decaying acoustic oscillations from the adiabatic pressure. The extra factor of $k_y/k$ reflects the fact that the curvature grows as $a$ until sound horizon crossing at $a_H \propto (k_y/k)^{1/4}$.

An interesting result is that the acoustic oscillation follows a $\sin(k s)$ relation, implying the opposite phase in the acoustic peaks compared with the $\cos(k s)$ adiabatic case. This result is rather generic to isocurvature models again due to the similar behavior of the driving potentials. For example, axionic isocurvature models where the entropy is between the radiation and the CDM also follows this pattern.

For this reason, isocurvature seed pressure also tends to generate this type of acoustic pattern [1]. These stresses are found in topological defect models; indeed the dominant scalar mode of strings, monopoles, textures, and non-topological textures do behave in this manner [2]. However, only in the latter two are the other modes sufficiently small at the first few peaks to yield clean acoustic peaks even for the scalar perturbations alone. Defect models generically have vector and tensor stresses that generate comparable levels of anisotropy and further obscure acoustic phenomena.

It is possible to construct isocurvature models with an “adiabatic” pattern of acoustic peaks. The simplest way to arrange this is to create constant comoving curvature perturbations $\zeta$ through entropic stresses that turn off well before the perturbation crosses the horizon. A concrete example of this kind of mechanism is given in II. As is clear from VII B and originally pointed out by [1], for an isocurvature model to generate scale-invariant curvature perturbations requires that the entropic stress have superhorizon scale correlations which cannot be generated causally without an inflationary epoch.

### C. Sonic/Entropic Stress

Likewise, an adiabatic or sonic fluctuation will not generally remain so as it crosses the horizon. Inside the horizon, the fact that components with different equations of state have different pressure responses to gravitational compression will cause the species to move independently. The generation of entropic stresses is in fact a primary mechanism for creating the smooth components of matter discussed in VII B.

**Full Solutions.** — The case of two components with constant $w_1 = 1/3$ and $w_2 = 0$ is of special interest because it corresponds to baryon-photon entropy perturbations, as in the PIB model before last scattering. The result is that on small scales where $k s \gg 1$ and $k/k_y \gg 1$, the curvature behaves as [33].
constructed out of combining the continuity equations \[ \rho_i \] of the two species and has the formal solution

\[ \sigma = \sigma(0) - k \int d\eta (v_2 - v_1). \] (127)

Recall the entropy is related to the entropic stress by equation \[ 101 \]. Note that entropy leads to entropic stress only if the two components differ in their equation of states \( \Delta \omega \neq 0 \).

Since we are interested in the generation of entropy, we will assume that the initial entropy perturbation \( \sigma(0) \) vanishes. Two interesting cases are when the entropy generation timescale \( \eta \) is small compared to the expansion time and when it is comparable to the expansion timescale.

In the former case, the entropy will lead to dissipative behavior if

\[ \sigma \approx -k \eta v, \] (128)

where \( \eta \) is some characteristic timescale for entropy generation. While this form is not gauge invariant even though \( \sigma \) is, the ambiguity vanishes inside the horizon. Hence, equation \[ 128 \] is a good approximation in the desired case \( \eta \ll \eta \). Substituting equation \[ 128 \] into \[ 101 \] and then \[ 13 \] in Newtonian gauge and assuming a solution of the form \( v \propto \exp(i \int \omega d\eta) \), we obtain the dispersion relation

\[ \omega = \pm kc_C - \frac{1}{2} \frac{\rho_1 \rho_2}{\rho_2^2} (c_s^2 - c_t^2) k^2 \eta, \] (129)

where recall \( c_C \) was defined in equation \[ 101 \]. Note that we have assumed \( \eta / \eta \ll 1 \) in order to replace rest-frame sound speeds with comoving sound speeds.

This describes an exponential damping of sound waves as the wavelength passes the “diffusion” scale \( k = (\eta / \eta)^{-1} \).

In the opposite regime, where the entropy is generated on the horizon scale, the sonic nature of the total system breaks down before acoustic oscillations even start. In this case, components can decouple completely and form subsystems where sonic, anisotropic, or smooth effects can occur separately. Here it is simpler to describe the behavior of individual subsystems through variables that are comoving with respect to the individual species \( J \).

The sound speed \( c_s \) and the contribution to the Newtonian potential are useful (see Eq. \[ 99 \])

\[ \Phi_J = 4\pi G \Delta \rho J / (k^2 - 3K), \] (130)

which implies \( \Phi = \sum_J \Phi_J \). If the density fluctuations in the other species are still below their own sound horizon, then the oscillating components can become effectively smooth in comparison. From that time forward, we can treat the system as having a smooth component, and the

\[ \eta \approx \frac{1}{6} k^2 \frac{\dot{\eta}}{R} \approx \frac{1}{6} k^2 \frac{\dot{\eta}}{R^2}. \] (132)

These entropic pressure techniques provide a simpler and more transparent derivation of this well-known result than exists in the literature. Note that heat-conduction damping is suppressed by \( R \) in the photon-dominated epoch, and we shall see that in that case viscous damping from the anisotropic stress is more important.

The opposite limit of an entropic timescale on order the expansion time \( \eta \approx \eta \) is applicable to all models with CDM. The CDM never participates in the acoustic oscillations of the baryon-photon system even during the radiation-dominated era. In Fig. \[ 13 \], we show an example of a perturbation deep in the radiation-dominated era in a CDM model with the usual neutrino content. Before
Horizon crossing $k \eta \ll 1$, the perturbations are adiabatic and the total potential $\Phi$ is constant with contributions from the photons $\Phi_\gamma$ and the neutrinos $\Phi_\nu$. The sound horizon for the radiation is $s = \eta/\sqrt{3}$ and once $k \eta \gg \sqrt{3}$, the potential contribution of the photons starts to decay as in equation (121). The neutrino contributions decay even faster due to their anisotropic stress, as we will discuss in the next section. The CDM contribution $\Phi_{\text{CDM}}$ then turns over and behaves as $\Phi_{\text{CDM}} \propto \ln(Ca)/a$ where $C$ is some constant [38]. This is exactly the behavior predicted by equation (121) assuming the radiation component is smooth. In actuality, the photon component is not smooth compared with the CDM component but oscillates sufficiently rapidly that its time-average is negligible.

These considerations also apply to models in which an additional component with $w_{\text{GDM}} < 0$ comes to dominate at late times. Quintessence and GDM models (see Fig. 16) are examples thereof. In §VII C, we considered the case where the GDM completely dominated the expansion. Here we consider the effect of adding a component of CDM.

The critical parameter here is the comoving sound speed of the GDM $c_{\text{GDM}}$. Slowly-rolling scalar fields found in the QCDM subcategory have $c_{\text{GDM}} = 1$, but components with lower sound speeds are possible in principle. The sound speed tells us how long after horizon crossing the GDM component stabilizes due to pressure support. If the perturbation is already within the sound horizon by the time the GDM comes to dominate the expansion, the solution returns to the clustered case of equation (59). We have seen in §VII A how anisotropic stress can generate Newtonian curvature perturbations but, like adiabatic stress, it can also destroy them. In this context, anisotropic stress represents the "frictional force" set up in response to the non-uniform bulk flow of matter and shear in the metric. As we shall see, this sort of behavior is not confined to fluids. It represents another mechanism for generating a smooth component.

Full Solutions. — The phenomenological parameterization of anisotropic stress in equation (81) yields two interesting limits. The short timescale limit $(k \eta_\text{I} \ll 1, \eta_\text{I}/\eta \ll 1)$ leads to viscous or collisional damping. Here, the anisotropic stress is algebraically related to the velocity in shear-free frames.

**D. Sonic/Anisotropic**

We have seen in §VII A how anisotropic stress can generate Newtonian curvature perturbations but, like adiabatic stress, it can also destroy them. In this context, anisotropic stress represents the "frictional force" set up in response to the non-uniform bulk flow of matter and shear in the metric. As we shall see, this sort of behavior is not confined to fluids. It represents another mechanism for generating a smooth component.

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which is the case for the baryon content implied by big bang nucleosynthesis (e.g. [43]).

Collisionless damping occurs in free radiation. Free radiation behaves in this “frictional” manner because gradients in the potential flow are dissipated as radiation streams from one part of the flow to the next. In terms of the multipole moments, power in the dipole gets transferred to the quadrupole and so on through the hierarchy equations (11). As such, the anisotropic stress acts as the gateway for anisotropy generation in the CMB. It also leads to a damping of density and velocity perturbations inside the horizon for all species of free radiation and is the reason that in Fig. 13 the neutrino contributions damp more rapidly than the photon contributions. More generally, it is responsible for smoothing out components when entropic stresses cannot be generated, e.g. when $c_s^2 - c_v^2 = 0$.

These dissipational terms may also be important in stabilizing other forms of matter. One way to generate a smooth density component is to introduce an effective viscous rather than sonic stress [17]: a mechanism of this type is thought to be involved in stabilizing the strCDM model [16]. Turok [50] suggested an extreme example of this sort, in which the comoving sound speed of the seeds is imaginary but the anisotropic stress is perfectly balanced to counter the otherwise exponential growth of density fluctuations.

E. Sonic/Entropic/Anisotropic Seed Stress

The seed stresses of topological defect models provide an example where all types of stresses coexist. While the consequences of a given seed stress for structure formation are straightforward to work out, the behavior of the seed stress itself is more difficult and requires simulations to work out in detail.

Full Solutions.— The two-point statistics of complicated defect models can be accurately modeled as the incoherent (quadrature) sum of a relatively small number ($\sim 10-20$) of simple seed stress histories [51,52]. Each individual source may then be determined by the techniques above (see Eq. 11 and 13).

Applications.— The simplest defect models typically have two other properties that have important phenomenological consequences. Defect models are causal in the classical sense. The stress perturbations $\delta p_\|$ and $p_\| \Pi_\|$ must fall off at least as white noise ($k^0$) outside the horizon and the initial curvature must vanish $(\zeta(0)) = 0)$. The traditional string and texture models also obey a scaling relation that states that the stress histories depend on wavelength only through the combination $k \eta$. This ensures self-similarity of the structure at horizon crossing and leads to nearly scale-invariant CMB anisotropies. Because the simplest versions of these models run into difficulties when CMB anisotropies are compared with large-scale structure, phenomenological models that do not obey the scaling relation have recently received some attention [25].

F. Vector Stress

We next consider the effect of vector anisotropic stresses. Recall that in the absence of vector stress, the vector perturbation decays. In order to generate an observable effect, vector perturbations must be continuously generated by vector anisotropic stresses. This implies that vector modes are unique to the active stress models, of which defects are an example.

Full Solutions.— The solutions in the presence of stress sources can be constructed via Green’s function techniques from the stress-free solutions of (7). For the vector modes, the solution becomes

$$v^{(\pm)} - B^{(\pm)} = a^{-4}(\rho + p)^{-1} \left[ C - \frac{1}{2} (1 - 2K/k^2) \right] \times k \int_0^n \tilde{\eta} \eta a^4 \rho \Pi^{(\pm)} ,$$

(136)

where $C =$-const. represents the decaying mode. The remaining metric perturbation $kB^{(\pm)} - \tilde{H}^{(\pm)}$ is related algebraically to equation (150) through equation (21).

Applications.— The discussion of scalar seed stress in defect models in VIII F also apply to vector stresses with a few additional considerations. Defect models generally have comparable scalar, vector, and tensor anisotropic stress sources above the horizon [51]. Since CMB anisotropies are primarily generated at horizon crossing, these sources tend to yield comparable anisotropy contributions for modes that cross after last scattering. Vector modes that cross before last scattering do not contribute due to suppression of tensor anisotropies in the CMB from scattering. Thus, vector modes can obscure the first few scalar acoustic features in defect models. On the other hand, vector modes have a special signature in the CMB polarization that may assist in their isolation [24,25].

G. Tensor Stresses

Tensor anisotropic stresses provide sources and sinks for gravity waves. It is well known that quadrupolar stresses generate gravity waves. Furthermore, a passing gravity wave will also impart some energy to the radiation backgrounds via differential gravitational redshifts and thereby decay.

Full Solutions.— An integral solution may be constructed out of the homogeneous solutions of (17) as
\[ H_T^{(\pm 2)}(\eta) = C_1 H_1(\eta) + C_2 H_2(\eta) \]
\[ + \int_0^{\eta} d\tilde{\eta} \frac{H_1(\tilde{\eta}) H_2(\eta) - H_1(\eta) H_2(\tilde{\eta})}{H_1(\tilde{\eta}) H_2(\tilde{\eta}) - H_1(\tilde{\eta}) H_2(\eta)} 8\pi G a^2 p \Pi^{(\pm 2)}, \]

where \( C_1 \) and \( C_2 \) are constants associated with the initial conditions.

**Applications.**—We show the damping effect of anisotropic stress in the radiation backgrounds in Fig. [17]. It is generally not included in standard Boltzmann codes that solve CMB anisotropies [54]. The reason is that it is negligible at large angles since the corresponding modes entered the horizon well into matter domination when the anisotropic stresses are negligible. Since this feedback effect is typically small, equation (137) may in principle be used to iterate to a solution from the free-gravity wave case of equation (78). However, the evolution of the anisotropic stress has a more important effect. Just as scalar anisotropic stresses in the photons are destroyed by scattering before last scattering (see §VIII D), tensor anisotropic stresses are destroyed as \( e^{-\tau} \). This cuts off the tensor contributions to the anisotropies before last scattering as indicated in equation (137) and shown in Fig. [7]. The feedback effect still exists, but the level of the anisotropy itself makes it too low to be observable for reasonable tensor to scalar ratios.

The Green’s function solution of equation (137) is more directly applicable to the seed stress case where \( p \Pi^{(\pm 2)} = p_s \Pi_s \). As in the vector stress case, the phenomenological result is that defect models tend to have significant tensor contributions above the angle the horizon subtends at last scattering. As in the passive models, their contribution is cut off below this scale due to scattering.

**IX. A DESIGNER APPLICATION**

With this general study in hand, we are now in a position to discuss prospects for reverse engineering the model for structure formation. Obviously, the specific route the inversion takes will depend on the results of ongoing experiments. Currently, the data favor a model with phenomenology like the \( \Lambda \)CDM model, e.g. the shape of the large-scale structure power spectrum [55–56], relative high to low redshift cluster abundances [57, 58], supernova luminosity distances to redshift of \( z \sim 0.5 \) [59, 60], and degree scale CMB anisotropies (e.g. [61]). If agreement between the \( \Lambda \)CDM model and future precision tests is good, can we use the methods developed here to modify the stress history, restore agreement, and in the process learn new information about the dark sector?

Let us address the first question. We know that the evolution of structure is completely defined by the stress history of the matter. Since the equations of state of the ordinary matter are known, the remaining element is the dark sector. To test the uniqueness of the \( \Lambda \)CDM model, we should look for alternate means of reproducing its stress history. The background stress history of the dark sector in a \( \Lambda \) model is given by

\[ w_{\Lambda \text{CDM}} = \frac{-a^3}{a^3 + \Omega_{\text{CDM}}/\Omega_{\Lambda}}, \]  

and its stress perturbations vanish. This suggests that a generalized dark matter (GDM) component of the type introduced by [67] and parameters

\[ w_{\text{GDM}} = w_{\Lambda \text{CDM}}, \quad c_{\text{GDM}}^2 = 0, \]  

should reproduce the phenomenology of the \( \Lambda \)CDM model. Recall that \( c_{\text{GDM}} \) is the sound speed in the frame comoving with the GDM (see §VIII C). To the extent that the comoving and GDM-comoving frames coincide, this form of dark matter exactly reproduces any mixture of \( \Lambda \) and CDM. We show an example in Figure [18]. Note that all classical cosmological and linear theory tests will return the same answer for the two models despite the fact that the GDM model is a single dark matter component model in a critical density universe! Non-linear effects are also identical if the same stress history is maintained throughout.

There are two lessons here. The first is that on purely phenomenological grounds we can do no more than measure the global properties of the dark sector. A multi-component model and a single component model with the same stress history are formally identical.

The second is that reproducing the phenomenology of a \( \Lambda \)CDM model is rather simple: it requires an equation of
state that varies from $w = 0$ at high redshift to $w \approx -2/3$ today and a form of matter that is free of large-scale stress gradients in its comoving frame. Thus, there exists a wide class of such single component models that fit the current data as well as the ΛCDM model.

On the other hand, variations on the conditions in equation (139) have observable consequences for future measurements. Relaxing the stress-free perturbation condition by raising the sound speed reduces the small-scale power in the model and delays the formation of high redshift objects. The remaining freedom in the stress history is associated with the equation of state $w$. We can quantify this by recalling that the gravitational potential depends only on the quantity (see equation 63)

$$\Delta \Phi \propto \frac{\sqrt{\rho}}{a} \int_0^a \frac{da}{\sqrt{\rho}}. \quad (140)$$

Large-scale structure constrains the value of this integral at the present ($a = 1$). CMB anisotropies from the ISW effect are sensitive to variations in this function that occur on the order of the light-travel time across a wavelength (see §IV). Since CMB anisotropies potentially probe nearly three orders of magnitude from the current horizon, variations at a fraction of a percent of the current expansion time are potentially visible in the CMB.

Similarly, distance measures such as the angular diameter distance to the last scattering surface ($a \sim 10^{-3}$) and the luminosity distance to high redshift objects ($a \sim 2/3$) probe the combination

$$\Delta \eta(a) \propto \int_a^1 \frac{da}{a^2 \rho^{1/2}}. \quad (141)$$

Current measures of $\Delta \Phi(a)$ and $\Delta \eta(a)$ are crude at best as they only probe their values at discrete epochs or averaged over long timescales. A sharp test of the Λ+CDM hypothesis that should be possible with future measures involves reconstructing the time evolution of the equation of state $w_{cr}$ through measures of $\Delta \Phi(a)$ and/or $\Delta \eta(a)$. Any combination of a cosmological constant and CDM will obey

$$w'_{cr} = 3w_{cr}(1 + w_{cr}). \quad (142)$$

The physical implication of this relation is that the background pressure is constant in time ($p' = 0$) so that the adiabatic sound speed vanishes ($c^2_s = p/\rho = 0$). If this condition is violated, we will have proven the existence of a new form of matter. If $w'_{cr} > 3w_{cr}(1 + w_{cr})$, then it can be supported by adiabatic stresses [64]; if not, then a form of matter with non-adiabatic stresses is required. Non-adiabatic stresses imply that the relation between the pressure and density perturbations does not follow that of their spatial averages.

The simplicity of the requirements of equation (139) raises the possibility of a unified description of the dark sector. A concrete but somewhat trivial example is a scalar field that rapidly oscillates around a non-zero potential minimum. The rapid oscillations average away all large-scale pressure effects save from the vacuum pressure of the potential minimum [53]. Unfortunately, the relationship between the constant and quadratic pieces of the scalar potential is left unexplained and so is no better than an explicit Λ+CDM model. Nonetheless, there may be more complicated examples, perhaps involving multiple fields, in which the mimic conditions (139) are approximately satisfied and that do unify the two behaviors in a true sense.

This discussion shows that a reverse-engineered model for structure formation will in general not be unique. On the other hand, the observables can be translated into constraints on the stress histories and phenomenological models of the dark sector. These in turn can assist in the search for compelling physical candidates to compose the dark sector.

X. DISCUSSION

Without any assumptions besides the validity of general relativity and nearly homogeneous and isotropic initial conditions, the evolution of structure is completely determined by the stress history of matter. We have studied the means by which stresses, both in the background...
and the fluctuations, can alter the observable properties of the model.

We have examined the effects of smooth, anisotropic, sonic, and entropic stresses in structure formation, including their interactions and ability to generate effectively smooth density components. We have illustrated these behaviors with analytic solutions for systems with multiple components of differing background equations of state, which can themselves be time dependent in several important cases. These solutions have applications to nearly all of the current models for structure formation and are substantially more general than those existing in the literature.

Although this study is not exhaustive, we have made explicit all of the assumption required to arrive at specific models and their accompanying phenomenology. In the process, we have exposed the limitations of traditional categorization schemes like that in Fig. 1. These distinctions can be blurred in cases where the usual assumptions do not apply. We summarize several notable cases here:

Initial Conditions:

Isocurvature initial conditions imply a growing comoving curvature outside the horizon on scales relevant to large-scale structure and degree-scale anisotropies. The comoving curvature grows outside the horizon only by the action of stress perturbations. Once stress perturbations are turned off, the curvature remains constant until horizon crossing or curvature domination. These considerations provide a means for mimicking the phenomenology of adiabatic models and are important for interpreting the implications of CMB acoustic peak phenomenology; however, they do not invalidate the conclusions of [11].

The Newtonian curvature $\Phi$ is simply proportional to the comoving curvature if the background equation of state is constant. The Newtonian curvature admits a decaying mode whereas the comoving curvature does not. The decaying mode can be stimulated by anisotropic stress perturbations outside the horizon but has observable consequences only through the contribution remaining at horizon crossing [1].

If the Newtonian potential $\Psi$ is constant from last scattering to the present, the observed temperature perturbation depends only on the equation of state and $\Psi$. The assumption here is that the comoving temperature perturbation is negligible and is only true if stress perturbations are also negligible compared with the comoving curvature for all time. The axion isocurvature model provides a simple counterexample. On the other hand, no assumptions about the anisotropic stress are necessary even when $\Phi = -\Psi$ no longer holds.

Isocurvature initial conditions predict an observed temperature perturbation of $2\Psi$ on scales larger than the horizon at last scattering. The assumption here is that the initial temperature perturbation in Newtonian gauge vanishes. This is not the case for models where the isocurvature conditions are established by balancing the photon density perturbations off another species of radiation. Furthermore, changes in the potential that are slowly varying compared with the light travel time across a perturbation do not affect the observed temperature.

Clustering Properties:

The smoothness of a component is gauge-dependent and hence has no physical meaning. The gauge dependence of a smooth component is not a problem per se as certain frames, e.g. the frame where the momentum of the component vanishes, are dynamically special. A smooth contribution to the density with $w_S \neq -1$ does violate covariant energy conservation in any coordinate system where the spatial curvature changes. Since the very presence of a smooth density component requires that the comoving curvature perturbation decays, there can be no identically smooth contributions in those coordinates except in the trivial zero curvature perturbation case. A component can be smooth relative to another species inside the horizon where the relativistic effects of curvature variation are negligible [12].

The behavior of a smooth component depends only on its equation of state $w_S$. Since all components except $\Lambda$ and curvature are clustered outside the horizon, the manner in which a component becomes smooth is observable. The presence of sonic (supportive) and anisotropic (dissipative) stresses are two possibilities, but others exist in principle.

A “clustered” model like standard CDM has no dynamically important smooth stresses or density contributions. The radiation backgrounds and baryons are effectively smooth well inside the horizon prior to recombination. Smooth contributions are generic to structure formation models.

Smooth component behavior implies small density perturbations $\delta \rho_S < \delta \rho_C$. A component can be effectively smooth even while possessing large density fluctuations. The crucial assumption is that their time-average density over the dynamical time of the clustered species is smooth. Density perturbations in a component that vary rapidly with the expansion time generally lead to no effect on the growth of structure. The radiation backgrounds mentioned above provide a familiar example.

The missing mass (clustered dark matter) and missing energy (smooth $w_S < -1/3$ dark matter) are separate problems. The stress history of the dark sector completely defines its properties for classical cosmology and structure formation. Any combination of components that produces the same stress history will produce the same phenomenology. As an example, we have constructed a toy model that that exactly reproduces the $\Lambda$CDM phenomenology but employs a single component.
of dark matter in a critical density universe. Variations in
the stress history produce models that satisfy the current
constraints equally well but are potentially distinguish-
able from ΛCDM.

Perturbation Type:

Scale-invariant gravity waves preferentially enhance
the low-order multipoles of CMB anisotropies. Enhancement
only occurs if the gravity wave amplitude decays close
to horizon crossing and is eliminated as the equa-
tion of state of the background drops.

There is a sharp distinction between active and passive
models for structure formation. Models that have been
labeled “passive” in the literature are those in which the
stress perturbations are simply related to density, veloc-
ity, and metric fluctuations by equations of state, sound
speeds, viscosity parameters, etc. Models in which there
is a component whose stresses have no fixed relation to
density and metric perturbations have been labeled “ac-
tive”. The issue is the number of internal degrees of
freedom that act to specify the stresses. For example,
scaling defect stress histories are typically approximated
by tens of parameters and those of particle dark matter
by three or fewer. In principle, there is a spectrum of pos-
sibilities between the two and a corresponding spectrum
of phenomenological consequences.

Our study is useful even if the current evidence sup-
porting ΛCDM-type phenomenology holds up. Even the
ΛCDM model itself does not have an entirely trivial stress
history as its dark sector includes the neutrino back-
ground radiation. The observability of the neutrino stress
history has been addressed numerically in [43]; we have
examined its physical origin here. Furthermore, the dark
sector could contain exotic features that produce more or
less ordinary phenomenology, and one needs to construct
sharp tests against alternatives. For example, a combi-
nation of cold dark matter and a cosmological constant
must obey $w'_{cr} = 3w_{cr}(1 + w_{cr})$. This relation also acts
as the dividing line between models with exotic and ordi-
nary matter. For exotic matter, the stress and density
perturbations obey a relation that opposes that between
the background stress and density; scalar fields are one
example of exotic matter.

In summary, a purely phenomenological reverse-
engineering of the model for structure formation will re-
quire the reconstruction of the time-averaged stress his-
tory of the dark sector. This inversion is generally not
unique. Nonetheless, if the observed phenomenology re-
ains close to that of our simplest models, our study of
stress phenomenology should provide the means for
constructing viable models. If the observations require
more radical departures, our study should be useful in
identifying the assumptions that are incorrect and assist
in the search for the correct phenomenological model for
structure formation.

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