A new formulation for the 3-D Euler equations with an application to subsonic flows in a cylinder

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Abstract

In this paper, a new formulation for the three dimensional Euler equations is derived. Since the Euler system is hyperbolic-elliptic coupled in a subsonic region, so an effective decoupling of the hyperbolic and elliptic modes is essential for any development of the theory. The key idea in our formulation is to use the Bernoulli’s law to reduce the dimension of the velocity field by defining new variables \( (1, \beta_2 = \frac{u_2}{u_1}, \beta_3 = \frac{u_3}{u_1}) \) and replacing \( u_1 \) by the Bernoulli’s function \( B \) through \( u_1^2 = \frac{2(B - h(\rho))}{1 + \beta_2^2 + \beta_3^2} \). We find a conserved quantity for flows with a constant Bernoulli’s function, which behaves like the scaled vorticity in the 2-D case. More surprisingly, a system of new conservation laws can be derived, which is new even in the two dimensional case. We use this new formulation to construct a smooth subsonic Euler flow in a rectangular cylinder, which is also required to be adjacent to some special subsonic states. The same idea can be applied to obtain similar information for the 3-D incompressible Euler equations, the self-similar Euler equations, the steady Euler equations with damping, the steady Euler-Poisson equations and the steady Euler-Maxwell equations.

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1 Introduction

The three dimensional steady isentropic compressible Euler Equations expressing the
conservations of mass and momentum, take the following form:

\[
\begin{align*}
(\rho u_1)_{x_1} + (\rho u_2)_{x_2} + (\rho u_3)_{x_3} &= 0, \\
(\rho u_1^1)_{x_1} + (\rho u_1 u_2)_{x_2} + (\rho u_1 u_3)_{x_3} + p_{x_1} &= 0, \\
(\rho u_1 u_2)_{x_1} + (\rho u_2^2)_{x_2} + (\rho u_2 u_3)_{x_3} + p_{x_2} &= 0, \\
(\rho u_1 u_3)_{x_1} + (\rho u_2 u_3)_{x_2} + (\rho u_3^3)_{x_3} + p_{x_3} &= 0
\end{align*}
\]

(1.1)

where \( \mathbf{u} = (u_1, u_2, u_3) \) is the velocity field, \( p \) is the pressure and \( \rho \) is the density. We consider the ideal polytropic gas, which means that \( p(\rho) = A\rho^\gamma \), where \( A \) is a positive constant depending on the specific entropy and \( \gamma \in (1, 3) \) is the adiabatic exponent. For simplicity, we take \( A = \frac{1}{2} \). It follows from the momentum equations that the following important Bernoulli’s law holds:

\[
\mathbf{u} \cdot \nabla B = 0.
\]

(1.2)

Here the Bernoulli’s function is defined to be

\[
B = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) + h(\rho),
\]

where \( h(\rho) \) is the enthalpy satisfying \( h'(\rho) = \frac{p'(\rho)}{\rho} = \frac{c^2(\rho)}{\rho} \).

The original motivation to start such a research is the following: we try to construct a smooth subsonic flow in a three dimensional finitely long nozzle by formulating suitable or physically acceptable boundary conditions on upstream and downstream. Since the Euler system is hyperbolic-elliptic coupled in a subsonic region, the main task is to understand the coupling properties of the hyperbolic and elliptic modes in subsonic flows by finding a decomposition of the Euler system to effectively decouple the different modes. We consider this as a first step to get a close look at the difficult 3-D transonic shock problems in a divergent nozzle and hope that this may shed light on that problem. For the study of transonic shock in an infinitely long nozzle, one may refer to [8, 7]. For the study of transonic shock problem in a duct or a De Laval nozzle, one may refer to [3, 9, 10, 11, 13, 22, 23, 24] and [36, 37, 38, 39] for more details. This is a natural continuation of previous results in the two dimensional case discussed in [16, 30, 31].

There have been many relevant studies on the 2-D steady compressible Euler equations, especially for subsonic flows around a given profile and subsonic flows in an infinitely long nozzle. Profound understanding has been achieved both physically and mathematically for subsonic flows around a given profile by Frankl and Keldysh [20], L. Bers [4, 5], M. Shiffman [28] and many other authors [14, 18, 19, 26, 27]. By a variational method, Shiffman [28] proved that there is an optimal range \([0, M_\infty)\) for Mach number \( M_0 \) at infinity that ensures the existence and uniqueness of a subsonic potential flow around a given profile with finite energy. Bers [4] presented a new proof of the existence and improved the uniqueness of a steady two-dimensional subsonic potential flow of a perfect gas around a given profile by removing the restriction of finite energy. In the recent works by Xie and Xin [33, 34] for subsonic potential flows in an infinitely long nozzle, they showed that there exists a critical value such that a global uniform subsonic flow exists uniquely in a general nozzle as long as the incoming mass flux is less than the critical value. They also obtained a class of subsonic-sonic flows by investigating the properties of these uniform subsonic flows and employing
the compensated compactness method. Such a strategy can not be applied to the case in
a multi-dimensional infinitely long nozzle \((n \geq 3)\), but a different formulation involving
the potential function has been employed by Du, Xin and Yan [17] to obtain the existence and
uniqueness results for subsonic potential flows in a multi-dimensional infinitely long nozzle
\((n \geq 2)\). The new ingredients of their analysis are methods of calculus of variations, the
Moser iteration techniques for the potential equation and a blow-up argument for infinitely
long nozzles. Concerning subsonic Euler flows in a 2-D infinitely long nozzle, Xie and Xin
[35] explored the special structure of the 2-D Euler system and reduced the Euler system
to a single second order quasi-linear elliptic equation, which yields the existence of subsonic
flows when the variation of the Bernoulli’s function is sufficiently small and the mass flux is
in a suitable regime with an upper critical value. For subsonic Euler flow in an axisymmetric
nozzle, one may refer to [15].

There are also some results doing this direction for both the 3-D compressible and incom-
pressible Euler equations [10, 11, 23, 38, 39, 11, 29]. In particular, Xin and Yin [39]
developed a decomposition of the Euler equations for the uniqueness proof of the symmetric transonic
shock solution and showed that the pressure satisfies a second order elliptic equation. They
also studied the properties of the shock front. In [23], they extended their techniques to
obtain the uniqueness results in the case of variable end pressure. In [10], the author studied
the transonic shock problems in 3-D nozzle with a general section, where he gave a decom-
position of Euler equations in which the elliptic part and hyperbolic part is separated at the
level of principle. To deal with the loss of derivative in the velocity field, the author first
obtained the vorticity by solving transport equations, then the velocity field was obtained
by solving an elliptic system with complex boundary conditions. However, it seems difficult
to apply these techniques to construct a subsonic Euler flow in a 3-D bounded nozzle. In
[1], the author formulated an inflow-outflow problem for 3-D incompressible Euler system by
prescribing the Bernoulli’s function, the normal component of the vorticity field at the inlet
of the flow region and the normal component of the velocity field on the whole boundary of
the flow region. The velocity field is obtained by solving a first order elliptic system. Tang
and Xin [29] have identified a class of additional boundary conditions for the vorticities and
established the existence and uniqueness of solutions with non-vanishing vorticity for the
three dimensional stationary incompressible Euler equations on simply connected bounded
three dimensional domains with smooth boundary.

In [16, 30, 31], we have characterized a set of physically acceptable boundary conditions
that ensure the existence and uniqueness of a subsonic irrotational flow in a finitely long
flat nozzle by prescribing the incoming flow angle and the Bernoulli’s function at the inlet
and the end pressure at the exit. In [16], we show that if the incoming flow is horizontal at
the inlet and an appropriate pressure is prescribed at the exit, then there exist two positive
constants \(m_0\) and \(m_1\) with \(m_0 < m_1\), such that a global smooth subsonic irrotational flow
exists uniquely in the nozzle, provided that the incoming mass flux \(m \in [m_0, m_1]\). The
boundary conditions we have imposed in the 2-D case have a natural extension in the 3-D
case. One may also prescribe the incoming flow angles \((\frac{u_2}{u_1}, \frac{u_3}{u_1})\) and the Bernoulli’s function \(B\n
at the inlet and the end pressure at the exit of the nozzle. We conjecture that such a problem
should be well-posed for the 3-D Euler equations. These important clues help us a lot to find
an effective decomposition for the 3-D Euler equations. It is natural to reformulate the Euler
system in terms of new variables \(s = \ln \rho, \beta_2 = \frac{u_2}{u_1}, \beta_3 = \frac{u_3}{u_1}, B\). Here one should note that
due to the Bernoulli’s law (1.2), the Bernoulli’s function $B$ will possess the same regularity as the boundary data at the inlet, which is quite different from the velocity field. So the key idea in our new formulation in terms of $(s, \beta_2, \beta_3, B)$ is to use the Bernoulli’s law to reduce the dimension of the velocity field by employing \( (1, \beta_2 = \frac{u_2}{u_1}, \beta_3 = \frac{u_3}{u_1}) \) and replace $u_1$ by $B$ through the simple algebraic formulae \( u_1^2 = 2\left(\frac{B - h(\rho)}{1 + \beta_2^2 + \beta_3^2}\right) \). In this way, we can explore the role of the Bernoulli’s law in greater depth and hope that may simplify the Euler equations a little bit. In the case of no vacuum and stagnation points, the original Euler system is equivalent to (2.2). It is shown that $s$ and $\partial_2\beta_2 + \partial_3\beta_3$ satisfy an elliptic system. At a first sight, one may regard $(\beta_2, \beta_3)$ as hyperbolic modes since they satisfy transport equations (4.10), which will induce a loss of one derivative estimates when integrating along the particle path. Our first observation is that \( W = \partial_2\beta_3 - \partial_3\beta_2 + \beta_2\partial_1\beta_2 - \beta_2\partial_1\beta_3 \) is conserved along each particle path, at least for the case of flows with a constant Bernoulli’s function. Indeed, $W$ satisfies the equation (3.6), which behaves like the scaled vorticity in the 2-D case. And a physical interpretation of $W$ is also given, see section 3 for more details. This, together with the first three equations in (2.2) may help to recover the loss of one derivative in the velocity field, so that the velocity field will possess the same regularity as the pressure.

Our second observation is a system of new conservation laws, which is somehow surprising. The original motivation comes from our study on the 2-D Euler system in a curved nozzle (See [30]), where we use the flow angle $\Theta = \arctan \left(\frac{u_2}{u_1}\right)$ instead of the angular velocity $w = \frac{u_2}{u_1}$, to get a clear insight of the role played by the curvatures of the nozzle walls. We try to find a proper substitute in the 3-D case and choose the “sine function” of the flow angles. Finally, a system of new conservation laws arises, whose physical interpretation is still unclear so far. Even in the 2-D case, these conservation laws are new as far as we know. Yet, it seems that these information are still not enough to deal with the subsonic-sonic limit for a sequence of uniformly bounded 2-D Euler flows. How to use these new information to obtain some deeper understanding on the compressible Euler system is still under consideration.

Back to our original problem, we can construct a smooth subsonic flow in a rectangular cylinder satisfying the prescribed incoming flow angle and the Bernoulli’s function at the inlet and the given end pressure at the exit, which is also required to be adjacent to some special subsonic states. We emphasize that the background can have large vorticity. We develop an iteration scheme to obtain the solution as a fixed point of a contractive operator. The velocity field $\beta_2$ and $\beta_3$ will be solved by integrating along the particle path. The corner singularities are avoided by the reflection technique. The delicate nonlinear coupling between the hyperbolic modes and elliptic modes $(\beta_2, \beta_3)$ makes it extremely difficult to have an effective iteration scheme in a 3-D nozzle with general section. How to develop an effective iteration scheme to obtain a smooth subsonic flow in a general 3-D nozzle is still under investigation.

It is worthy noting that our idea can be applied to the 3-D steady incompressible Euler equations without any essential changes. That is to say, we can employ the Bernoulli’s law to get a new decomposition for the 3-D incompressible Euler system. A new conserved quantity and a system of new conservation laws can be derived as for the compressible case. As we will see, the steady incompressible Euler system is always a hyperbolic-elliptic coupled system. We can construct a smooth incompressible Euler flow adjacent to some special states in a rectangular cylinder, which will satisfy the prescribed incoming flow angles, the Bernoulli’s
function at the inlet and the end pressure at the exit. Our strategy can also be applied to the self-similar Euler equations, the steady Euler equations with damping, the steady Euler-Poisson equations and the steady Euler-Maxwell equations.

The rest of the paper will be organized as follows. In the next section, we will present a new formulation for the compressible Euler equations and carry out a characteristic analysis to get an insight into the structure of the Euler system. The new conserved quantity and a system of new conservation laws will be derived in section 3. A smooth subsonic flow in a rectangular cylinder is constructed in section 4 by employing the previous new formulation. In section 5, we apply the same idea to give a new formulation of the incompressible Euler system. Some detailed calculations will be given in the appendix (section 6).

2 A new formulation of the 3-D Isentropic Euler equations

In this section, we will give a new formulation of the Euler equations and carry out a characteristic analysis for the new system to get some insights on the coupling structure of hyperbolic and elliptic modes in the compressible Euler equations.

In the whole paper, it is assumed that the flow does not contain vacuum and any stagnation points, which means that $\rho > 0$ and $(u_1, u_2, u_3) \neq (0, 0, 0)$ respectively. Without loss of generality, it is assumed that $u_1 > 0$.

Define the following three new variables: $\beta_2 = \frac{u_2}{u_1}, \beta_3 = \frac{u_3}{u_1}, s = \ln \rho$. Multiplying the first equation in (1.1) by $-\frac{u_i - 1}{\rho u_1^2}$, dividing the $i$-th equation in (1.1) by $\rho u_1^2$ and adding them together for $i = 2, 3, 4$, we obtain the following new system:

$$\begin{cases}
\partial_1 \beta_2 + \beta_3 \partial_3 \beta_2 - \beta_2 \partial_3 \beta_3 - \beta_2 \partial_1 s - \beta_2^2 \partial_2 s + \frac{c^2(\rho)}{u_1^2} \partial_2 s - \beta_2 \beta_3 \partial_3 s = 0, \\
\partial_1 \beta_3 + \beta_2 \partial_2 \beta_3 - \beta_3 \partial_2 \beta_2 - \beta_3 \partial_1 s - \beta_2 \beta_3 \partial_2 s + \frac{c^2(\rho)}{u_1^2} \partial_3 s - \beta_3^2 \partial_3 s = 0, \\
-\partial_2 \beta_2 - \partial_3 \beta_3 + \frac{c^2(\rho)}{u_1^2} \partial_1 s - \partial_1 s - \beta_2 \partial_2 s - \beta_3 \partial_3 s = 0, \\
\partial_1 B + \beta_2 \partial_2 B + \beta_3 \partial_3 B = 0.
\end{cases}$$

(2.1)

Using the third equation, we rewrite the above system as follows:

$$\begin{cases}
\partial_1 s + \beta_2 \partial_2 s + \beta_3 \partial_3 s - \frac{c^2(\rho)}{u_1^2} \partial_1 s + \partial_2 \beta_2 + \partial_3 \beta_3 = 0, \\
\partial_1 \beta_2 + \beta_2 \partial_2 \beta_2 + \beta_3 \partial_3 \beta_2 - \frac{c^2(\rho)}{u_1^2} \beta_2 \partial_1 s + \frac{c^2(\rho)}{u_1^2} \partial_2 s = 0, \\
\partial_1 \beta_3 + \beta_2 \partial_2 \beta_3 + \beta_3 \partial_3 \beta_3 - \frac{c^2(\rho)}{u_1^2} \beta_3 \partial_1 s + \frac{c^2(\rho)}{u_1^2} \partial_3 s = 0, \\
\partial_1 B + \beta_2 \partial_2 B + \beta_3 \partial_3 B = 0.
\end{cases}$$

(2.2)
It is easy to verify that the above system (2.2) is equivalent to (1.1), provided that the flow is smooth and does not contain vacuum and stagnation point.

We focus mainly on subsonic flows in which the Euler equations become hyperbolic-elliptic coupled equations. Now we carry out a characteristic analysis to get an insight of the structure of the Euler equations.

We rewrite (2.2) in a matrix form as

\[ M_1 \partial_t U + M_2 \partial_2 U + M_3 \partial_3 U = 0. \]  

Here \( U = (s, \beta_2, \beta_3, B)^T \) and

\[
M_1 = \begin{bmatrix}
1 - \frac{c^2(\rho)}{u_1^2} & 0 & 0 & 0 \\
-\frac{c^2(\rho)}{u_1^2} \beta_2 & 1 & 0 & 0 \\
-\frac{c^2(\rho)}{u_1^2} \beta_3 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, M_2 = \begin{bmatrix}
\beta_2 & 1 & 0 & 0 \\
\frac{c^2(\rho)}{u_1^2} \beta_2 & \beta_2 & 0 & 0 \\
0 & 0 & \beta_2 & 0 \\
0 & 0 & 0 & \beta_2 \\
\end{bmatrix}, \quad M_3 = \begin{bmatrix}
\beta_3 & 0 & 1 & 0 \\
0 & \beta_3 & 0 & 0 \\
0 & 0 & 0 & \beta_3 \\
\end{bmatrix}.
\]

Denote by \( \lambda \) the root of \( \det(\lambda M_1 - \xi_2 M_2 - \xi_3 M_3) \). A simple calculation shows that
\[
\det(\lambda M_1 - \xi_2 M_2 - \xi_3 M_3) = (\lambda - \beta \cdot \xi)^2[(1 - \frac{c^2(\rho)}{u_1^2})\lambda^2 - 2\lambda(\beta \cdot \xi) + (\beta \cdot \xi)^2 - \frac{c^2(\rho)}{u_1^2}(\xi_2^2 + \xi_3^2)],
\]
and it has one real root \( \lambda_r = \beta \cdot \xi \) with multiplicity 2 and two conjugate complex roots \( \lambda_+^\pm = \lambda_R \pm i\lambda_I \), where \( \lambda_R = \frac{a_2}{a_1}, \lambda_I = \sqrt{a_1 a_3 - a_2^2} \) and \( a_1 = 1 - \frac{c^2(\rho)}{u_1^2}, a_2 = \beta \cdot \xi = \beta_2 \xi_2 + \beta_3 \xi_3, a_3 = (\beta \cdot \xi)^2 - \frac{c^2(\rho)}{u_1^2}(\xi_2^2 + \xi_3^2) \). And \( a_1 a_3 - a_2^2 = \frac{c^2(\rho)}{u_1^2}[(\frac{c^2(\rho)}{u_1^2} - 1)(\xi_2^2 + \xi_3^2) - (\beta \cdot \xi)^2] > 0 \) if the flow is subsonic, i.e. \( c^2(\rho) > u_1^2 + u_2^2 + u_3^2 \). The corresponding left eigenvectors to \( \lambda_r \) are the following

\[
\mathbf{l}_1 = (0, \xi_3 + \beta_3(\beta \cdot \xi), -\xi_2 - \beta_2(\beta \cdot \xi), 0), \quad \mathbf{l}_2 = (0, 0, 0, 1).
\]

The corresponding left eigenvectors to \( \lambda_+^\pm \) are

\[
\mathbf{l}_1^\pm = \mathbf{l}_R \pm i\mathbf{l}_I = (\frac{a_2}{a_1} - a_2, \xi_2, \xi_3, 0) \pm i(\frac{\sqrt{a_1 a_3 - a_2^2}}{a_1}, 0, 0, 0).
\]

The differential operators corresponding to \( \mathbf{l}_R \) and \( \mathbf{l}_I \) are \( \mathbf{D}_1 = (\frac{c^2(\rho)}{u_1^2 - c^2(\rho)}(\beta_2 \partial_2 + \beta_3 \partial_3), \partial_2, \partial_3, 0) \) and \( \mathbf{D}_2 = (1, 0, 0, 0) \). The action of these two differential operations on (2.2) yields an elliptic system for \( s \) and \( \omega = \partial_2 \beta_2 + \partial_3 \beta_3 \).
\[
\begin{cases}
\partial_1 s + \beta_2 \partial_2 s + \beta_3 \partial_3 s - \frac{c^2(\rho)}{u_1^2} \partial_1 s + \varpi = 0,
\partial_1 \varpi + \frac{u_1^2}{u_1^2 - c^2(\rho)} (\beta_2 \partial_2 + \beta_3 \partial_3) \varpi + \frac{c^2(\rho)}{u_1^2 - c^2(\rho)} \sum_{i=2}^3 \beta_i (\beta_2 \partial_2 + \beta_3 \partial_3) \partial_i s + \\
\sum_{i=2}^3 \partial_i (\frac{c^2(\rho)}{u_1^2 - c^2(\rho)} \partial_i s) + \frac{c^2(\rho)}{u_1^2 - c^2(\rho)} \partial_1 s (\beta_2 \partial_2 + \beta_3 \partial_3) (1 - \frac{c^2(\rho)}{u_1^2}) - \sum_{i=2}^3 \partial_i (\frac{c^2(\rho)}{u_1^2} \beta_i) \partial_i s \\
+ \frac{c^2(\rho)}{u_1^2 - c^2(\rho)} \sum_{i=2}^3 (\beta_2 \partial_2 + \beta_3 \partial_3) \beta_i \partial_i s + (\partial_2 \beta_2)^2 + 2 \partial_2 \beta_3 \partial_3 \beta_2 + (\partial_3 \beta_3)^2 = 0
\end{cases}
\] (2.4)

Remark 2.1. The differential operator corresponding to \(I^2_r\) is \(D_4 = (0, 0, 0, 1)\). The action of \(D_4\) on (2.2) just yields the fourth equation in (2.2), indicating that the Bernoulli’s function is conserved along the particle path. One may expect that the action of the differential operator corresponding to \(I^1_r\) would show us another conserved quantity for hyperbolic modes. Indeed, the differential operator corresponding to \(I^1_r\) is \(D_3 = (0, \partial_3 + \beta_3 (\beta_2 \partial_2 + \beta_3 \partial_3), -\partial_2 - \beta_2 (\beta_2 \partial_2 + \beta_3 \partial_3), 0) = (0, \partial_3 - \beta_3 \partial_1 + \beta_3 D, -\partial_2 + \beta_2 \partial_1 - \beta_2 D, 0)\), where \(D = \partial_1 + \beta_2 \partial_2 + \beta_3 \partial_3\). Applying \(D_3\) to (2.2) and tracing the principal term, one can find that \(DW\) is the principal term involved, where \(W = (\beta_3 \partial_1 - \beta_3) \beta_2 - (\beta_2 \partial_1 - \beta_2) \beta_3\). However, we do not perform such a complex calculation here. It turns out that one just need to take the action of \((0, \partial_3 - \beta_3 \partial_1, -\partial_2 + \beta_2 \partial_1, 0)\) on (2.3). See the next section for more details.

Remark 2.2. Here \(s\) and \(\varpi = \partial_2 \beta_2 + \partial_3 \beta_3\) satisfy an elliptic system, which is similar to the observation made in [10, 11]. Indeed, the authors in [10, 11] carried out a characteristic analysis directly to the non-isentropic Euler equations and obtained a decomposition for the Euler equations. They defined a new variable \(v = \partial_2 u_2 + \partial_3 u_3 - \frac{u_2}{u_1} \partial_2 u_1 - \frac{u_3}{u_1} \partial_3 u_1\) and showed that \(v\) and the pressure \(p\) satisfy the following elliptic system:

\[
\begin{cases}
\partial_1 v + \lambda_1 (\partial_2^2 + \partial_3^2) p + f_1(u_1, u_2, u_3, p, S) = 0,
\partial_1 v - \lambda_2 \partial_1 p = f_2(u_1, u_2, u_3, p, S).
\end{cases}
\] (2.5)

Here \(f_i(u_1, u_2, u_3, p, S)\) for \(i = 1, 2\) are functions of \((u_1, u_2, u_3, p, S)\) and the first and second order derivatives of \((u_1, u_2, u_3, p, S)\). Simple calculations show that \(\varpi = \frac{v}{u_1}\). Our formulation is different from [10, 11] in the sense that we use the Bernoulli’s law to reduce the dimension of the velocity field by employing \((1, \beta_2 = \frac{u_2}{u_1}, \beta_3 = \frac{u_3}{u_1})\). Due to the Bernoulli’s law (1.2), the Bernoulli’s function \(B\) will not lose regularity when integrating along the particle path, which is different from \(u_1\). We replace \(u_1\) by \(B\) through the simple algebraic formulae \(u_1^2 = \frac{2(B - h(\rho))}{1 + \beta_2^2 + \beta_3^2}\).

3 The 3-D Euler equations with a constant Bernoulli’s function
3.1 A conserved quantity

Before going into the details, we state the following simple but important fact. It is known that a two dimensional Euler flow is irrotational is equivalent to that the Bernoulli’s function is a constant. However, this fact is not true in the three dimensional case. Indeed, by employing the following identity in vector calculus

\[ \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times \text{curl} \mathbf{u}, \]  

(3.1)

and the conservation of momentum, we have

\[ \nabla \left( \frac{1}{2} |\mathbf{u}|^2 + h(\rho) \right) - \mathbf{u} \times \text{curl} \mathbf{u} = 0. \]  

(3.2)

So if \( \mathbf{w} = \text{curl} \mathbf{u} = (w_1, w_2, w_3) \equiv 0 \), then \( \nabla B \equiv 0 \) and \( B \) is a constant. However, the constant Bernoulli’s function only implies the vorticity field is parallel to the velocity field. One needs an additional condition such as \( \partial_2 u_3 - \partial_3 u_2 \equiv 0 \) to guarantee that \( \mathbf{w} = 0 \). In a word, the constant Bernoulli’s function and \( \partial_2 u_3 - \partial_3 u_2 \equiv 0 \) will imply that the flow must be irrotational. However, \( \partial_2 u_3 - \partial_3 u_2 \) is not invariant under Galilean transformation. Also the three component of \( \text{curl} \mathbf{u} \) are of the same status, there is no reasons to choose one of them specially. The following interesting calculation is due to the motivation that we try to find a better substitute of \( \frac{\partial_2 \beta_3 - \partial_3 \beta_2 + \beta_3 \partial_1 \beta_2 - \beta_2 \partial_1 \beta_3}{\partial_3 \beta_3} \) may serve as a good candidate. That is to say, \( \beta_2 \) and \( \beta_3 \) as follows:

\[
\begin{aligned}
G^{-1}(\partial_1 \beta_2 + \beta_2 \partial_2 \beta_2 + \beta_3 \partial_3 \beta_2) - \beta_2 \partial_1 K(s) + \partial_2 K(s) & = 0, \\
G^{-1}(\beta_3 \partial_3 - \beta_2 \partial_2 \beta_3 + \beta_3 \partial_3 \beta_3) - \beta_3 \partial_1 K(s) + \partial_3 K(s) & = 0, \\
\end{aligned}
\]

(3.3)

Apply \( \gamma \partial_1 - \partial_3 \) and \( -\beta_2 \partial_1 + \partial_2 \) to the above two equations respectively and add them together, one can show that \( W \) satisfies the following equation.

\[ \mathbf{D} W + (\partial_2 \beta_2 + \partial_3 \beta_3) W - G \partial_1 K(s) W - G^{-1} W D G = 0. \]  

(3.4)

It follows from the first equation in (2.2) that \( W \) satisfies the following equation:

\[ \mathbf{D} W - W D s - G^{-1} W D G = 0. \]  

(3.5)
That is
\[ D \left( \frac{W}{\rho G} \right) = 0. \] (3.6)

One may refer to the appendix for the detailed computations.

Simple calculations show that \( W = \beta \cdot (\nabla \times \beta) = \frac{u_1 w_1}{u_1 u_1} + \frac{u_2 w_2}{u_1 u_1} + \frac{u_3 w_3}{u_1 u_1} = \frac{1}{u_1^2} (u \cdot w) \),
here \( \beta = (1, \beta_2, \beta_3) \), which is the component of “scaled vorticity” along the “scaled velocity”.

Then (3.6) becomes
\[ D \left( \frac{u \cdot w}{\rho ||u||^2} \right) = 0. \] (3.7)

The three components of \( \text{curl} u \) are functions dependent, so the “scaled helicity” \( \frac{W}{\rho G} \) as a linear combination of \( w_1, w_2, w_3 \), which is also invariant under Galilean transformation, seems to be a better candidate for the hyperbolic mode.

Since the vorticity field is parallel to the velocity field, that is \( w = \text{curl} u = \mu(x) u \), then we have
\[ W = \frac{u_1 w_1}{u_1 u_1} + \frac{u_2 w_2}{u_1 u_1} + \frac{u_3 w_3}{u_1 u_1} = \mu(x)(1 + \beta_2^2 + \beta_3^2) = \mu(x)G. \]

Then (3.6) becomes
\[ D \left( \frac{\mu(x)}{\rho} \right) = 0. \] (3.8)

Hence \( \frac{\mu(x)}{\rho} \) is conserved along the stream line, and plays a similar role as the “scaled vorticity” in the two dimensional case. Recall that in 2-D case, we have
\( (u_1 \partial_1 + u_2 \partial_2) \left( \frac{\partial_1 u_2 - \partial_2 u_1}{\rho} \right) = 0. \)

Historically, the stationary solution of Euler equations with the vorticity field paralleling to the velocity field was called Beltrami flow and have been investigated for over a century. Arnol’d [2] had identified a class of flows with presumably chaotic streamline, that is the Arnol’d-Beltrami-Childress (ABC) flows with \( u = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x) \). Turbulent flows can be understood as a superposition of Beltrami flows has been proposed in the work of Constantin and Majda [12]. There are also some literatures on compressible Beltrami flows, for example [25]. Indeed, (3.8) can be derived in the following simple way: using the divergent free condition for the vorticity field, \( \text{curl} u = \mu(x) u \) will imply that \( \text{div}(\mu(x)u) = 0 \), then by employing the density equation, we directly obtain (3.8).

Now we explain the reason why we spend so much time and energy on the above complicated calculations for (3.8). The point is our calculations also work for the isentropic Euler equations. For the isentropic Euler system, similar calculations as above show that \( W \) satisfies the following equation:
\[ \text{D} \left( \frac{W}{\rho G} \right) + c^2(\rho) \left[ \frac{1}{2(B-h(\rho))} \right] \left[ (\partial_2 B \partial_3 s - \partial_3 B \partial_2 s) \right] = 0. \] (3.9)

That is
\[ \text{D} \left( \frac{W}{\rho G} \right) + \frac{c^2(\rho)}{2\rho^2(B-h(\rho))^2} \left[ (1, \beta_2, \beta_3) \cdot (\nabla \rho \times \nabla B) \right] = 0. \] (3.10)
Remark 3.1. Our original goal is to construct a smooth subsonic flow satisfying the prescribed incoming flow angles and the Bernoulli’s function at the inlet and the given end pressure at the exit. Assume that the boundary data are in $C^{2,\alpha}$. Since the Bernoulli’s function is conserved and will not lose regularity when integrating along the particle path, then $B \in C^{2,\alpha}$. It follows from (2.2) that $s$ satisfies the second order elliptic equation

$$\partial_1\left(\frac{c_2^2(\rho)}{u_1^2} - 1\right)\partial_1 s - \beta_2 \partial_2 s - \beta_3 \partial_3 s + \partial_2\left(-\beta_2 \partial_1 s + \left(\frac{c_2^2(\rho)}{u_1^2} - \beta_2^2\right)\partial_2 s - \beta_2 \beta_3 \partial_3 s\right)$$

$$+ \partial_3\left(-\beta_3 \partial_1 s - \beta_2 \beta_3 \partial_2 s + \left(\frac{c_2^2(\rho)}{u_1^2} - \beta_3^2\right)\partial_3 s\right) - \left(\frac{c_2^2(\rho)}{u_1^2} - 1\right)\partial_1 s - \beta_2 \partial_2 s - \beta_3 \partial_3 s$$

$$= -\left((\partial_2 \beta_2)^2 + (\partial_3 \beta_3)^2 + 2\partial_2 \partial_3 \beta_2\right).$$

(3.11)

This shows that $s$ also possesses $C^{2,\alpha}$ regularity in general. However, $\beta_2$ and $\beta_3$ will lose one order derivative in the process of integrating the transport equations. Roughly speaking, the quantity $W$ may help to raise the regularity of $\beta_2$ and $\beta_3$. Since $W$ satisfies (3.10), $W$ should have $C^{1,\alpha}$ regularity in general. Combining this with the first three equations in (2.2), one can obtain a divergent-curl system for $\beta_2$ and $\beta_3$, which may help to raise one more regularity for $\beta_2$ and $\beta_3$. So the velocity field would possess the same regularity as that of the pressure. The above informal argument indicates that whether the Bernoulli’s function is constant or not is so essential for enhancing the regularity of the velocity field.

Remark 3.2. Indeed, one can also derive the equations (3.6) and (3.7) directly from the original Euler equations (1.1). It follows from the original Euler equations that the vorticity $\mathbf{w}$ satisfies the following equations:

$$u_1 \partial_1 \mathbf{w} + u_2 \partial_2 \mathbf{w} + u_3 \partial_3 \mathbf{w} + \text{div}\mathbf{u} \cdot \mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{u} = 0. \quad (3.12)$$

Combine these with the density equation, we have

$$(u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3) \left(\frac{\mathbf{w}}{\rho}\right) - \left(\frac{\mathbf{w} \cdot \nabla}{\rho}\right) \mathbf{u} = 0. \quad (3.13)$$

It follows from (3.12) that $\mathbf{u} \cdot \mathbf{w}$ satisfies

$$(\mathbf{u} \cdot \nabla)(\mathbf{u} \cdot \mathbf{w}) + \text{div}\mathbf{u} \cdot (\mathbf{u} \cdot \mathbf{w}) - (\mathbf{w} \cdot \nabla)||\mathbf{u}||^2 = 0. \quad (3.14)$$

Employing the Bernoulli’s law and the density equation, one can derive the following equation:

$$||\mathbf{u}||^2(\mathbf{u} \cdot \nabla)\left(\frac{\mathbf{u} \cdot \mathbf{w}}{\rho}\right) - \frac{\mathbf{u} \cdot \mathbf{w}}{\rho}(\mathbf{u} \cdot \nabla)(||\mathbf{u}||^2) + \frac{2c_2^2(\rho)}{\rho^2}\left(||\mathbf{u}||^2(\mathbf{w} \cdot \nabla)\rho - (\mathbf{u} \cdot \mathbf{w})(\mathbf{u} \cdot \nabla)\rho\right) = 0. \quad (3.15)$$

That is

$$(\mathbf{u} \cdot \nabla)\left(\frac{\mathbf{u} \cdot \mathbf{w}}{\rho||\mathbf{u}||^2}\right) + \frac{2c_2^2(\rho)}{\rho^2||\mathbf{u}||^4}\left(||\mathbf{u}||^2(\mathbf{w} \cdot \nabla)\rho - (\mathbf{u} \cdot \mathbf{w})(\mathbf{u} \cdot \nabla)\rho\right) = 0. \quad (3.16)$$

If $\nabla B \equiv 0$, that is $\mathbf{u} \parallel \mathbf{w}$, then the last term drops out. We obtain (3.7) again. It should be emphasized that (3.15) holds also for flows with stagnation points. Hence if $\mathbf{u} \cdot \mathbf{w} \equiv 0$ at the inlet, it holds also in the whole nozzle.
Remark 3.3. We also remark here that (3.2) implies that the Bernoulli’s function is conserved not only along the stream line, but also the vortex line. The last conclusion seems not getting enough attention before. Indeed, since the vorticity field is divergent free, we obtain the following new conservation law:

$$\text{div}(B\text{curl}\mathbf{u}) = 0.$$  \hspace{1cm} (3.17)

3.2 A system of new conservation laws

In the following, we assume that the Bernoulli’s function is a constant and the speed $|\mathbf{u}|$ has a positive lower bound $\delta$. It is easy to see that $G$ satisfies the following Riccati-type equation:

$$DG - \frac{c^2(\rho)}{(B - h(\rho))} \partial_1 s G^2 + \frac{c^2(\rho)}{(B - h(\rho))} G \partial_1 s = 0.$$  \hspace{1cm} (3.18)

Define three new variables: $K_1 = G^{-\frac{1}{2}}$, $K_i = \frac{\beta_i}{G^2}$, $i = 2, 3$. Here $K_2$ and $K_3$ behave like the “sine function” of “the flow angles”. In the following, we derive the equations satisfied by $K_2$ and $K_3$. To simplify the notation, we define two new functions $I(\rho) = \int_0^\rho \frac{c^2(t)}{2(B - h(t))} I(t)^{-2} dt$ and $Q(\rho) = \int_0^\rho \frac{c^2(t)}{2(B - h(t))} I(t)^{-2} dt$. Since the speed $|\mathbf{u}|$ has a lower bound $\delta$, $I(\rho)$ and $Q(\rho)$ are well-defined and $Q(\rho) > 0$ for $\rho > 0$. One should note that $I(\rho)$ satisfies $I'(\rho) = \frac{c^2(\rho)}{2\rho(B - h(\rho))}$.

It follows from (3.18) that $K_1$ satisfies:

$$DK_1 - \frac{c^2(\rho)}{2\rho(B - h(\rho))} K_1 D\rho + \frac{c^2(\rho)}{2\rho(B - h(\rho))} K_1^{-1} \partial_1 \rho = 0.$$  \hspace{1cm} (3.19)

While the first equation in (2.2) yields

$$\frac{K_1}{\rho} D\rho - \frac{c^2(\rho)}{2\rho(B - h(\rho))} K_1^{-1} \partial_1 \rho + K_1 (\partial_2 \beta_2 + \partial_3 \beta_3) = 0.$$  \hspace{1cm} (3.20)

Combing these two equations, we obtain

$$\frac{K_1}{\rho} D\rho + DK_1 - \frac{c^2(\rho)}{2\rho(B - h(\rho))} K_1 D\rho + K_1 (\partial_2 \beta_2 + \partial_3 \beta_3) = 0.$$  \hspace{1cm} (3.21)

That is

$$\frac{1}{\rho} \mathbf{K} \cdot \nabla \rho + \text{div} \mathbf{K} - \frac{c^2(\rho)}{2\rho(B - h(\rho))} \mathbf{K} \cdot \nabla \rho = 0.$$  \hspace{1cm} (3.22)

Using the definition of $I(\rho)$, we have the following new conservation law, which is similar to the conservation of mass:

$$\text{div}(\rho \mathbf{A}) = 0.$$  \hspace{1cm} (3.23)

where $\mathbf{A} = \frac{1}{I(\rho)} \mathbf{K} = \frac{1}{I(\rho)} (K_1, K_2, K_3)$. 

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Similarly, one can also derive the equations satisfied by $K_i, i = 1, 2, 3$, as

\[
\begin{align*}
DK_1 - \frac{c^2(\rho)}{2\rho(B - h(\rho))}K_1D\rho + \frac{c^2(\rho)}{2\rho(B - h(\rho))}K_1^{-1}\partial_1\rho &= 0, \\
DK_2 - \frac{c^2(\rho)}{2\rho(B - h(\rho))}K_2D\rho + \frac{c^2(\rho)}{2\rho(B - h(\rho))}K_1^{-1}\partial_2\rho &= 0, \\
DK_3 - \frac{c^2(\rho)}{2\rho(B - h(\rho))}K_3D\rho + \frac{c^2(\rho)}{2\rho(B - h(\rho))}K_1^{-1}\partial_3\rho &= 0.
\end{align*}
\]  

That is:

\[
\begin{align*}
\rho K \cdot \nabla K_1 &= K_1 \frac{\rho K \cdot \nabla I(\rho)}{I(\rho)} + \frac{c^2(\rho)}{2(B - h(\rho))}\partial_1\rho = 0, \\
\rho K \cdot \nabla K_2 &= K_2 \frac{\rho K \cdot \nabla I(\rho)}{I(\rho)} + \frac{c^2(\rho)}{2(B - h(\rho))}\partial_2\rho = 0, \\
\rho K \cdot \nabla K_3 &= K_3 \frac{\rho K \cdot \nabla I(\rho)}{I(\rho)} + \frac{c^2(\rho)}{2(B - h(\rho))}\partial_3\rho = 0.
\end{align*}
\]

With the help of (3.23), we obtain the following conservation laws, which are similar to the conservation of momentum:

\[
\begin{align*}
\partial_1(\rho A_1^2) + \partial_2(\rho A_1 A_2) + \partial_3(\rho A_1 A_3) + \partial_1 Q(\rho) &= 0, \\
\partial_1(\rho A_1 A_2) + \partial_2(\rho A_2^2) + \partial_3(\rho A_2 A_3) + \partial_2 Q(\rho) &= 0, \\
\partial_1(\rho A_1 A_3) + \partial_2(\rho A_2 A_3) + \partial_3(\rho A_3^2) + \partial_3 Q(\rho) &= 0.
\end{align*}
\]  

For the isentropic Euler equations, since $(\partial_1 + \beta_2\partial_2 + \beta_3\partial_3)B = 0$, we still have

\[
\frac{1}{\rho}K \cdot \nabla \rho + div K - \frac{1}{I(\rho, B)}K \cdot \nabla I(\rho, B) = 0,
\]  

where $I(\rho, B) = e^{\int_0^\rho \frac{c^2(\tau)}{2(B - h(\tau))}d\tau}$. Hence one can still obtain the following conservation law:

\[
\partial_1(\rho)\frac{K_1}{I(\rho, B)} + \partial_2(\rho)\frac{K_2}{I(\rho, B)} + \partial_3(\rho)\frac{K_3}{I(\rho, B)} = 0.
\]  

However, similar conservation laws as (3.26) are not true in general.

For the non-isentropic Euler equations, one can perform the same operations to get similar information. Actually, the non-isentropic Euler equations can be rewritten as

\[
\begin{align*}
\frac{1}{\rho}(\partial_1\rho + \beta_2\partial_2\rho + \beta_3\partial_3\rho) - \frac{1}{\rho u_1^2}\partial_1 P + \partial_2\beta_2 + \partial_3\beta_3 &= 0, \\
\partial_1\beta_2 + \beta_2\partial_2\beta_2 + \beta_3\partial_3\beta_2 - \frac{1}{\rho u_1^2}\beta_2\partial_1 P + \frac{1}{\rho u_1^2}\partial_2 P &= 0, \\
\partial_1\beta_3 + \beta_2\partial_2\beta_3 + \beta_3\partial_3\beta_3 - \frac{1}{\rho u_1^2}\beta_3\partial_1 P + \frac{1}{\rho u_1^2}\partial_3 P &= 0, \\
\partial_1 B + \beta_2\partial_2 B + \beta_3\partial_3 B &= 0, \\
\partial_1 S + \beta_2\partial_2 S + \beta_3\partial_3 S &= 0.
\end{align*}
\]
In this case, \( B = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) + h(\rho, S) = \frac{1}{2}u_1^2(1 + \beta_2^2 + \beta_3^2) + h(\rho, S) \), where \( h(\rho, S) = \int_0^\rho \frac{\partial P(\tau, S)}{\tau} d\tau \). Hence \( u_2^2 = \frac{2(B - h(\rho, S))}{1 + \beta_2^2 + \beta_3^2} \).

Similar calculations as previous sections show that

\[
W = \partial_2 \beta_3 - \partial_3 \beta_2 + \beta_3 \partial_1 \beta_2 - \beta_2 \partial_1 \beta_3
\]

satisfies the following equation:

\[
D\left(\frac{W}{\rho G}\right) + \frac{1}{2\rho^3(P, S)(B - h(P, S))^2} \left\{ (B - h(P, S))\partial S \rho(P, S) \beta \cdot (\nabla P \times \nabla S) + \rho(P, S)[\beta \cdot (\nabla P \times \nabla B) - \partial S h(P, S) \beta \cdot (\nabla P \times \nabla S)] \right\} = 0, \tag{3.30}
\]

where we have rewritten \( \rho \) and \( h \) as smooth functions of \((P, S)\).

Also a similar conservation law holds:

\[
\text{div}\left(\rho \frac{K}{I(\rho, B, S)}\right) = 0, \tag{3.31}
\]

where \( I(\rho, B, S) = e^{\int_0^\rho \frac{\partial P(\tau, S)}{\tau} d\tau} \).

**Remark 3.4.** Since \( G \) satisfies a Riccati-type equation (3.18), one may expect some blow-up results. However, the ambiguous sign of \( \partial_1 s \) in the coefficient of \( G^2 \) in (3.18) makes the whole argument nontrivial. One should note that \( G \) blows up means that the fluid will turn around in the flow region.

**Remark 3.5.** The physical interpretations of (3.23) and (3.26) are not clear at this moment. How to use these new information to obtain some useful results is an interesting problem and will be investigated further.

**Remark 3.6.** One may perform the same operation as we have done in Remark 3.3 to (3.23) and (3.26) to obtain a new conservation law involving \( \text{curl} A \). However, one should note that the Bernoulli’s function for (3.26) is just a constant, so no new information can be obtained in this case.

**Remark 3.7.** One may perform similar operations as we have done for the original Euler equations, to the new conservation laws (3.23) and (3.26), it turns out that one will get (3.23) and (3.26) again. That is, (3.23) and (3.26) is invariant under our operations.

**Remark 3.8.** These calculations certainly work for the two dimensional Euler system. Indeed, \( W \) will always be zero. We can obtain a system of new conservation laws as (3.23) and (3.26) for potential flows and a new conservation law as (3.28) or (3.31) for the 2-D Euler equations. Even in this case, all these information are completely new as far as we know. However, it seems that these new information are still not enough to deal with the subsonic-sonic limit for the 2-D Euler flow.

**Remark 3.9.** We may apply the same idea to the self-similar compressible Euler equations,
which take the following form:

\[
\begin{aligned}
\{(\rho U_1)_{\xi_1} + (\rho U_2)_{\xi_2} + (\rho U_3)_{\xi_3} + 3\rho &= 0, \\
\rho U_1 \partial_{\xi_1} U_1 + \rho U_2 \partial_{\xi_2} U_1 + \rho U_3 \partial_{\xi_3} U_1 + \partial_{\xi_1} p + \rho U_1 &= 0, \\
\rho U_1 \partial_{\xi_1} U_2 + \rho U_2 \partial_{\xi_2} U_2 + \rho U_3 \partial_{\xi_3} U_2 + \partial_{\xi_2} p + \rho U_2 &= 0, \\
\rho U_1 \partial_{\xi_1} U_3 + \rho U_2 \partial_{\xi_2} U_3 + \rho U_3 \partial_{\xi_3} U_3 + \partial_{\xi_3} p + \rho U_3 &= 0.
\end{aligned}
\] (3.32)

Here \(\xi = (\xi_1, \xi_2, \xi_3) = (\frac{x_1}{l}, \frac{x_2}{l}, \frac{x_3}{l})\) and \((U_1, U_2, U_3)(\xi) = (u_1, u_2, u_3)(\xi) - (\xi_1, \xi_2, \xi_3)\). The main difference in (3.32) is that the Bernoulli’s law does not hold any more. One can still define \(s = \ln \rho, \beta_2 = \frac{U_2}{U_1}, \beta_3 = \frac{U_3}{U_1}, B = \frac{1}{2}(U_1^2 + U_2^2 + U_3^2) + h(\rho)\) and obtain a new formulation for (3.32):

\[
\begin{aligned}
(\partial_{\xi_1} s + \beta_2 \partial_{\xi_2} s + \beta_3 \partial_{\xi_3} s) - \frac{c^2(\rho)}{U_1^2} \partial_{\xi_1} s + \partial_2 \beta_2 + \partial_3 \beta_3 + \frac{2}{U_1} &= 0, \\
\partial_{\xi_1} \beta_2 + \beta_2 \partial_{\xi_2} \beta_2 + \beta_3 \partial_{\xi_3} \beta_2 - \frac{c^2(\rho)}{U_1^2} \beta_2 \partial_{\xi_1} s + \frac{c^2(\rho)}{U_1^2} \partial_{\xi_2} s &= 0, \\
\partial_{\xi_1} \beta_3 + \beta_2 \partial_{\xi_2} \beta_3 + \beta_3 \partial_{\xi_3} \beta_3 - \frac{c^2(\rho)}{U_1^2} \beta_3 \partial_{\xi_1} s + \frac{c^2(\rho)}{U_1^2} \partial_{\xi_3} s &= 0, \\
\partial_{\xi_1} B + \beta_2 \partial_{\xi_2} B + \beta_3 \partial_{\xi_3} B + \frac{(U_1^2 + U_2^2 + U_3^2)}{U_1} &= 0.
\end{aligned}
\] (3.33)

Although the Bernoulli’s law does not hold in this case, the Bernoulli’s function \(B\) still has the same regularity as that of the pressure, and \(W = \partial_2 \beta_2 - \partial_3 \beta_3 - \beta_2 \partial_1 \beta_2 - \beta_3 \partial_1 \beta_3\) may also help to raise the regularity of \(\beta_2\) and \(\beta_3\). Hence the reformulation (3.33) may provide some new insights on the self-similar Euler equations. Same arguments work for the steady Euler equations with damping.

Remark 3.10. In [32], we will employ the same idea to investigate the structural stability of some steady subsonic solutions for 1-D Euler-Poisson system under multi-dimensional perturbations of suitable boundary conditions and background charge.

4 Subsonic Euler flows in a rectangular cylinder

In this section, we try to construct a subsonic Euler flow in a rectangular cylinder by imposing suitable boundary conditions at the inlet and exit. The cylinder will be \(\Omega = [0, 1] \times [0, 1] \times [0, 1]\). Actually the flow we construct below will be adjacent to some special subsonic states \((\rho_0, u_0(x_2, x_3), 0, 0)\). Here \(\rho_0\) is a constant, \(u_0(x_2, x_3) \in C^{3,\alpha}([0, 1] \times [0, 1])\) satisfying \(0 < u_0(x_2, x_3) < c(\rho_0)\). We also set \(B_0(x_2, x_3) = \frac{1}{2} u_0^2(x_2, x_3) + h(\rho_0), s_0 = \ln \rho_0\).
At the inlet of the nozzle \( x_1 = 0 \), we impose the flow angles and the Bernoulli’s function:

\[
\begin{align*}
\beta_i(0, x_2, x_3) &= \epsilon \beta_i^{in}(x_2, x_3), \quad i = 2, 3, \\
B(0, x_2, x_3) &= B_0(x_2, x_3) + \epsilon B^{in}(x_2, x_3). 
\end{align*}
\] (4.1)

Here the compatibility conditions should be satisfied:

\[
\begin{align*}
\partial^j_2 \beta_2^{in}(0, x_3) &= \partial^j_2 \beta_2^{in}(1, x_3) = 0, \\
\partial^j_3 \beta_3^{in}(x_2, 0) &= \partial^j_3 \beta_3^{in}(x_2, 1) = 0, \quad j = 0, 2, \\
\partial^k_2 B^{in}(0, x_3) &= \partial^k_2 B^{in}(1, x_3) = \partial^k_3 B^{in}(x_2, 0) = \partial^k_3 B^{in}(x_2, 1) = 0, \quad k = 1, 3.
\end{align*}
\] (4.2)

At the exit of the nozzle \( x_1 = 1 \), we prescribe the end pressure:

\[
p(1, x_2, x_3) = \frac{1}{\gamma} e^{\gamma e (s_0 + \epsilon s_e(x_2, x_3))}.
\] (4.3)

Here we also require that \( s_e \) satisfies the following compatibility conditions:

\[
\begin{align*}
\partial^j_2 s_e(0, x_3) &= \partial^j_2 s_e(1, x_3) = 0, \\
\partial^j_3 s_e(x_2, 0) &= \partial^j_3 s_e(x_2, 1) = 0, \quad j = 1, 3.
\end{align*}
\] (4.4)

While on the nozzle walls, the usual slip boundary condition is imposed:

\[
\begin{align*}
u_2(x_1, 0, x_3) &= u_2(x_1, 1, x_3) = 0, \\
u_3(x_1, x_2, 0) &= u_3(x_1, x_2, 1) = 0.
\end{align*}
\] (4.5)

Mathematically, we are going to prove that \((1.1)\) with boundary conditions \((4.1),(4.3)\) and \((4.5)\) satisfying compatibility conditions \((4.2)\) and \((4.4)\), has a unique subsonic solution.

### 4.1 Extension to the domain \( \Omega_e = [0, 1] \times T^2 \)

Suppose that the flow \((\rho, u_1, u_2, u_3) \in C^{3, \alpha}(\bar{\Omega}) \times C^{2, \alpha}(\bar{\Omega})^3\) to be constructed has the following properties:

\[
\begin{align*}
\partial^j_2 (\rho, u_1)(x_1, 0, x_3) &= \partial^j_2 (\rho, u_1)(x_1, 1, x_3) = 0, \\
\partial^j_2 (\rho, u_1)(x_1, x_2, 0) &= \partial^j_2 (\rho, u_1)(x_1, x_2, 1) = 0, \quad j = 1, 3, \\
\partial^k_2 u_2(x_1, 0, x_3) &= \partial^k_2 u_2(x_1, 1, x_3) = 0, \\
\partial^k_3 u_3(x_1, x_2, 0) &= \partial^k_3 u_3(x_1, x_2, 1) = 0, \quad k = 0, 2.
\end{align*}
\] (4.6)
Then we may extend \((\rho, u_1, u_2, u_3)\) in the following way, to \((\hat{\rho}, \hat{u}_1, \hat{u}_2, \hat{u}_3)\) ∈ \(C^{3,\alpha}([0,1] \times \mathbb{R}^2) \times C^{2,\alpha}([0,1] \times \mathbb{R}^2)^3\):

For \((x_1, x_2, x_3)\) ∈ \([0,1] \times [-1,1] \times [-1,1]\), we define \((\hat{\rho}, \hat{u}_1, \hat{u}_2, \hat{u}_3)\) as follows

\[
(\hat{\rho}, \hat{u}_1, \hat{u}_2, \hat{u}_3)(x) = \begin{cases}
(\rho, u_1, u_2, u_3)(x_1, x_2, x_3), & \text{if } (x_2, x_3) \in [0,1] \times [0,1], \\
(\rho, u_1, -u_2, u_3)(x_1, -x_2, x_3), & \text{if } (x_2, x_3) \in [-1,0] \times [0,1], \\
(\rho, u_1, u_2, -u_3)(x_1, x_2, -x_3), & \text{if } (x_2, x_3) \in [0,1] \times [-1,0], \\
(\rho, u_1, -u_2, -u_3)(x_1, -x_2, -x_3), & \text{if } (x_2, x_3) \in [-1,0] \times [-1,0].
\end{cases}
\]

Then we extend \((\hat{\rho}, \hat{u}_1, \hat{u}_2, \hat{u}_3)\) periodically to \([0,1] \times \mathbb{R}^2\) with period 2. It is easy to verify that \((\hat{\rho}, \hat{u}_1, \hat{u}_2, \hat{u}_3)\) belong to \(C^{3,\alpha}([0,1] \times \mathbb{R}^2) \times C^{2,\alpha}([0,1] \times \mathbb{R}^2)^3\). Moreover, \((\hat{\rho}, \hat{u}_1, \hat{u}_2, \hat{u}_3)\) satisfy the Compressible Euler equations. Due to these reasons, one may directly work on the domain \([0,1] \times \mathbb{T}^2\) (Here \(\mathbb{T}^2\) is a 2-torus), so that the difficulty caused by corner singularities and slip boundary conditions can be avoided. We may extend \((u_0, B_0, \beta_2^{in}, \beta_3^{in}, B^{in}, s_e)\) to \(\mathbb{T}^2\), which will still be denoted by \((u_0, B_0, \beta_2^{in}, \beta_3^{in}, B^{in}, s_e)\).

### 4.2 Main results

The main result in this section is the following existence and uniqueness theorem.

**Theorem 4.1.** Given \((\beta_2^{in}, \beta_3^{in}, B^{in}, s_e) \in C^{3,\alpha}(\mathbb{T}^2)\) satisfying the compatibility conditions \((4.2)\) and \((4.1)\), there exists a positive small number \(\epsilon_0\), which depends on the background subsonic state \((\rho_0, u_0(x_2, x_3), 0, 0)\) and \((\beta_2^{in}, \beta_3^{in}, B^{in}, s_e)\), such that if \(0 < \epsilon < \epsilon_0\), then there exists a unique smooth subsonic flow \((\rho, u_1, u_2, u_3)\in C^{3,\alpha}(\Omega_e) \times C^{2,\alpha}(\Omega_e)^3\) to \((1.1)\) satisfying boundary conditions \((4.1), (4.3)\) and \((4.5)\). Moreover, the following estimate holds:

\[
\|(u_1, u_2, u_3) - (u_0(x_2, x_3), 0, 0)\|_{C^{2,\alpha}(\Omega_e)} + \|\rho - \rho_0\|_{C^{3,\alpha}(\Omega_e)} \leq C\epsilon. \tag{4.7}
\]

Here \(C\) is a constant depending on \((\rho_0, u_0(x_2, x_3), 0, 0)\) and \((\beta_2^{in}, \beta_3^{in}, B^{in}, s_e)\).

Since we are looking for a smooth subsonic Euler flow in the whole nozzle, which is also close to some special subsonic states, no vacuum and stagnation points will appear in the flow region, we may use the new formulation developed in the section \(^2\) and section \(^3\). Hence to prove the above theorem, it is equivalent to show that the following system has a unique
smooth subsonic solution:

\[
\begin{aligned}
&\partial_1 s + \beta_2 \partial_2 s + \beta_3 \partial_3 s - \frac{c^2(\rho)}{u_1^2} \partial_1 s + \partial_2 \beta_2 + \partial_3 \beta_3 = 0, \\
&\partial_1 \beta_2 + \beta_2 \partial_2 \beta_2 + \beta_3 \partial_3 \beta_2 - \frac{c^2(\rho)}{u_1^2} \beta_2 \partial_1 s + \frac{c^2(\rho)}{u_1^2} \partial_2 s = 0, \\
&\partial_1 \beta_3 + \beta_2 \partial_2 \beta_3 + \beta_3 \partial_3 \beta_3 - \frac{c^2(\rho)}{u_1^2} \beta_3 \partial_1 s + \frac{c^2(\rho)}{u_1^2} \partial_3 s = 0, \\
&\partial_1 B + \beta_2 \partial_2 B + \beta_3 \partial_3 B = 0, \\
&\beta_i(0, x_2, x_3) = \epsilon \beta_i^{in}(x_2, x_3), \quad \text{for } i = 2, 3, \\
&B(0, x_2, x_3) = B_0 + \epsilon B^{in}(x_2, x_3), \quad \text{and } u_1(0, x_2, x_3) > 0, \\
&s(1, x_2, x_3) = s_0 + \epsilon s_e(x_2, x_3).
\end{aligned}
\]

We can derive that \( \bar{s} = s - s_0, \) \( \beta_2, \beta_3 \) and \( \bar{B} = B - B_0 \) satisfy the following equations, respectively:

\[
\begin{aligned}
&\partial_1 \left( \frac{c^2(\rho)}{u_1^2} - 1 \right) \partial_1 \bar{s} - \beta_2 \partial_2 \bar{s} - \beta_3 \partial_3 \bar{s} \right) + \partial_2 \left( - \beta_2 \partial_1 \bar{s} + \left( \frac{c^2(\rho)}{u_1^2} - \beta_2^2 \right) \partial_2 \bar{s} - \beta_2 \beta_3 \partial_3 \bar{s} \right) \\
&\quad + \partial_3 \left( - \beta_3 \partial_1 \bar{s} - \beta_2 \beta_3 \partial_2 \bar{s} + \left( \frac{c^2(\rho)}{u_1^2} - \beta_3^2 \right) \partial_3 \bar{s} \right) - \left( \frac{c^2(\rho)}{u_1^2} - 1 \right) \partial_1 \bar{s} - \beta_2 \partial_2 \bar{s} - \beta_3 \partial_3 \bar{s} \\
&\quad + (\partial_2 \beta_2)^2 + (\partial_3 \beta_3)^2 + 2 \partial_2 \beta_2 \partial_3 \beta_3 = 0, \\
&\quad \left( \frac{c^2(\rho)}{u_1^2} - 1 \right) \partial_1 \bar{s} - \beta_2 \partial_2 \bar{s} - \beta_3 \partial_3 \bar{s} \right) (0, x_2, x_3) = \epsilon (\partial_2 \beta_2^{in} + \partial_3 \beta_3^{in}), \\
&\bar{s}(1, x_2, x_3) = \epsilon s_e(x_2, x_3).
\end{aligned}
\]

\[
\begin{aligned}
&\partial_1 \beta_2 + \beta_2 \partial_2 \beta_2 + \beta_3 \partial_3 \beta_2 - \frac{c^2(\rho)}{u_1^2} \beta_2 \partial_1 \bar{s} + \frac{c^2(\rho)}{u_1^2} \partial_2 \bar{s} = 0, \\
&\partial_1 \beta_3 + \beta_2 \partial_2 \beta_3 + \beta_3 \partial_3 \beta_3 - \frac{c^2(\rho)}{u_1^2} \beta_3 \partial_1 \bar{s} + \frac{c^2(\rho)}{u_1^2} \partial_3 \bar{s} = 0, \\
&\beta_2(0, x_2, x_3) = \epsilon \beta_2^{in}(x_2, x_3), \\
&\beta_3(0, x_2, x_3) = \epsilon \beta_3^{in}(x_2, x_3), \\
&\quad \left( \frac{c^2(\rho)}{u_1^2} - 1 \right) \partial_1 \bar{s} - \beta_2 \partial_2 \bar{s} - \beta_3 \partial_3 \bar{s} \right) (0, x_2, x_3) = \epsilon (\partial_2 \beta_2^{in} + \partial_3 \beta_3^{in}), \\
&B(0, x_2, x_3) = \epsilon B^{in}(x_2, x_3).
\end{aligned}
\]
4.3 Proof of Theorem 4.1

The main idea is simple: we construct an operator $\Lambda$ on a suitable space, which will be bounded in a high order norm and contraction in a low order norm.

The solution class will be given by

$$\Xi = \{ (\tilde{s}, \beta_2, \beta_3, \tilde{B}) : \| \tilde{s} \|_{C^{3,\alpha} (\Omega_\epsilon)} \leq \delta, \| \beta_2, \beta_3, \tilde{B} \|_{C^{2,\alpha} (\Omega_\epsilon)} \leq \delta \}$$

Here $\delta$ will be determined later. For any given $(\tilde{s}, \beta_2, \beta_3, \tilde{B}) \in \Xi$, we define an operator $\Lambda : (\tilde{s}, \beta_2, \beta_3, \tilde{B}) \mapsto (\tilde{s}, \beta_2, \beta_3, \tilde{B})$ mapping from $\Xi$ to itself, through the following iteration scheme.

Step 1. To obtain $\tilde{s}$ by solving the following linearized elliptic system:

$$\begin{align*}
\partial_1 \left( \left( \frac{\partial^2 (\tilde{\rho})}{\partial y^2} - 1 \right) \partial_1 \tilde{s} - \beta_2 \partial_2 \tilde{s} - \beta_3 \partial_3 \tilde{s} \right) + \partial_2 \left( - \beta_2 \partial_1 \tilde{s} + \left( \frac{\partial^2 (\tilde{\rho})}{\partial y^2} - \beta_3 \right) \partial_2 \tilde{s} - \beta_2 \beta_3 \partial_3 \tilde{s} \right) + \partial_3 \left( - \beta_3 \partial_1 \tilde{s} - \beta_2 \beta_3 \partial_2 \tilde{s} + \left( \frac{\partial^2 (\tilde{\rho})}{\partial y^2} - \beta_3 \right) \partial_3 \tilde{s} \right) &= F_1,
\left( \frac{\partial^2 (\tilde{\rho})}{\partial y^2} - 1 \right) \partial_1 \tilde{s} - \beta_2 \partial_2 \tilde{s} - \beta_3 \partial_3 \tilde{s} \right) (0, x_2, x_3) &= \epsilon (\partial_2 \beta_2^{in} + \partial_3 \beta_3^{in}),
\end{align*}$$

$$\tilde{s}(1, x_2, x_3) = \epsilon \delta_\epsilon(x_2, x_3).$$

(4.12)

Here $F_1 = \left( \frac{\partial^2 (\tilde{\rho})}{\partial y^2} - 1 \right) \partial_1 \tilde{s} - \beta_2 \partial_2 \tilde{s} - \beta_3 \partial_3 \tilde{s} \right)^2 - \left( (\partial_2 \tilde{\rho})^2 + (\partial_3 \tilde{\rho})^2 + 2 \partial_2 \beta_2 \partial_3 \beta_3 \right)$, $\tilde{\rho} = \tilde{s} + s_0$

and $\tilde{u}_1^2 = \frac{2 (B_0 + \tilde{B} - h(\tilde{\rho}))}{1 + \beta_2^2 + \beta_3^2}$.

By the standard elliptic estimates, we can show that (4.12) has a unique solution $\tilde{s} \in C^{3,\alpha}(\tilde{\Omega})$ with the following estimate:

$$\| \tilde{s} \|_{C^{3,\alpha} (\Omega_\epsilon)} \leq C_1 \left( \| F_1 \|_{C^{1,\alpha} (\Omega_\epsilon)} + \epsilon \| \partial_2 \beta_2^{in} + \partial_3 \beta_3^{in} \|_{C^{2,\alpha} (T^2)} + \epsilon \| s_\epsilon \|_{C^{3,\alpha} (T^2)} \right) \leq C_2 (\delta^2 + \epsilon).$$

(4.13)

Here $C_1$ depends only on the background solution and $\Omega_\epsilon$, and $C_2$ depends also on $(\beta_2^{in}, \beta_3^{in}, B^{in}, s_\epsilon)$.

Step 2. To obtain $\beta_2, \beta_3$ by solving the following hyperbolic equations:

$$\begin{align*}
\partial_1 \beta_2 + \tilde{\beta}_2 \partial_2 \beta_2 + \tilde{\beta}_3 \partial_3 \beta_2 - \frac{\partial^2 (\tilde{\rho})}{\partial y^2} \tilde{\beta}_2 \partial_1 \tilde{s} + \frac{\partial^2 (\tilde{\rho})}{\partial y^2} \tilde{\beta}_2 \partial_2 \tilde{s} &= 0,
\partial_1 \beta_3 + \tilde{\beta}_2 \partial_2 \beta_3 + \tilde{\beta}_3 \partial_3 \beta_3 - \frac{\partial^2 (\tilde{\rho})}{\partial y^2} \tilde{\beta}_3 \partial_1 \tilde{s} + \frac{\partial^2 (\tilde{\rho})}{\partial y^2} \tilde{\beta}_3 \partial_2 \tilde{s} &= 0,
\beta_2 (0, x_2, x_3) = \epsilon \beta_2^{in}(x_2, x_3),
\beta_3 (0, x_2, x_3) = \epsilon \beta_3^{in}(x_2, x_3).
\end{align*}$$

(4.14)
The particle path \((\tau, \tilde{x}_2(\tau; x), \tilde{x}_3(\tau; x))\) through \((x_1, x_2, x_3)\), is defined by the following ordinary differential equations:

\[
\begin{align*}
\frac{d\tilde{x}_2(\tau; x)}{d\tau} &= \tilde{\beta}_2(\tau, \tilde{x}_2(\tau; x), \tilde{x}_3(\tau; x)), \\
\frac{d\tilde{x}_3(\tau; x)}{d\tau} &= \tilde{\beta}_3(\tau, \tilde{x}_2(\tau; x), \tilde{x}_3(\tau; x)), \\
\tilde{x}_2(x_1; x) &= x_2, \\
\tilde{x}_2(x_1; x) &= x_3.
\end{align*}
\]  

(4.15)

Since \((\tilde{\beta}_2, \tilde{\beta}_3) \in C^{2,\alpha}(\Omega_e)\), it is easy to prove that \((\tilde{x}_2(0; x), \tilde{x}_3(0; x))\) belong to \(C^{2,\alpha}(\Omega_e)\) with respect to \(x\). Indeed, we have the following estimates:

\[
\|(\tilde{x}_2(0; x), \tilde{x}_3(0; x))\|_{C^{2,\alpha}(\Omega_e)} \leq C_3. \tag{4.16}
\]

Then by the characteristic method, it holds that:

\[
\begin{align*}
\beta_2(x) &= \epsilon \beta_2^{in}(\tilde{x}_2(0; x), \tilde{x}_3(0; x)) + \int_0^{x_1} \left( \frac{\partial^2 \rho}{a_1^2} (\tilde{\beta}_2 \partial_1 \tilde{s} - \partial_2 \tilde{s}) \right) (\tau, \tilde{x}_2(\tau; x), \tilde{x}_3(\tau; x)) d\tau, \\
\beta_3(x) &= \epsilon \beta_3^{in}(\tilde{x}_2(0; x), \tilde{x}_3(0; x)) + \int_0^{x_1} \left( \frac{\partial^2 \rho}{a_1^2} (\tilde{\beta}_3 \partial_1 \tilde{s} - \partial_3 \tilde{s}) \right) (\tau, \tilde{x}_2(\tau; x), \tilde{x}_3(\tau; x)) d\tau.
\end{align*}
\]  

(4.17)

These enable one to obtain the following estimate:

\[
\| (\beta_2(x), \beta_3(x)) \|_{C^{2,\alpha}(\Omega_e)} \leq C_4 [\epsilon \| (\beta_2^{in}, \beta_3^{in}) \|_{C^{2,\alpha}(T^2)} + \| \tilde{s} \|_{C^{3,\alpha}(\Omega)}] \leq C_5 (\delta^2 + \epsilon). \tag{4.18}
\]

Step 3. To resolve \(\bar{B}\). Solving

\[
\begin{align*}
\partial_1 \bar{B} + \tilde{\beta}_2 \partial_2 \bar{B} + \tilde{\beta}_3 \partial_3 \bar{B} &= -\beta_2 \partial_2 B_0 - \beta_3 \partial_3 B_0, \\
\bar{B}(0, x_2, x_3) &= \epsilon B^{in}(x_2, x_3).
\end{align*}
\]  

(4.19)

As above, \(\bar{B}\) has the following formulae

\[
\bar{B} = \epsilon B^{in}(\tilde{x}_2(0; x), \tilde{x}_3(0; x)) + \int_0^{x_1} \left( -\beta_2 \partial_2 B_0 - \beta_3 \partial_3 B_0 \right) (\tau, \tilde{x}_2(\tau; x), \tilde{x}_3(\tau; x)) d\tau.
\]

Furthermore, the following estimate holds:

\[
\| \bar{B} \|_{C^{2,\alpha}(\Omega_e)} \leq \epsilon \| B^{in} \|_{C^{2,\alpha}(T^2)} + \tilde{C}_5 \| (\beta_2(x), \beta_3(x)) \|_{C^{2,\alpha}(\Omega_e)} \leq C_6 (\delta^2 + \epsilon). \tag{4.20}
\]

Collecting all the estimates (4.13), (4.18) and (4.20) gives

\[
\| \tilde{s} \|_{C^{3,\alpha}(\Omega_e)} \leq C_7 (\delta^2 + \epsilon), \quad \| \beta_2, \beta_3, \bar{B} \|_{C^{2,\alpha}(\Omega_e)} \leq C_7 (\delta^2 + \epsilon). \tag{4.21}
\]

Here \(C_7 = max\{C_2, C_5, C_6\}\), which depends only on the background solution, \(\Omega_e\) and \((\beta_2^{in}, \beta_3^{in}, B^{in}, s_e)\).
Choose $\epsilon_1$ small enough, such that if $\epsilon \leq \epsilon_1$, then $C_7^2 \epsilon < \frac{1}{4}$. Set $\delta = 2C_7\epsilon$, then $C_7\delta < \frac{1}{2}$ and $C_7(\delta^2 + \epsilon) < C_7\epsilon + \frac{1}{2}\delta = \delta$. This implies that $\Lambda$ maps $\Xi$ to itself.

It remains to show that the mapping $\Lambda : \Xi \rightarrow \Xi$ is a contraction operator. Suppose $\Lambda : (\tilde{s}^k, \tilde{\beta}_2^k, \tilde{\beta}_3^k, \tilde{B}^k) \mapsto (s^k, \beta_2^k, \beta_3^k, B^k)$ for $k = 1, 2$. Define the difference $(Y_1, Y_2, Y_3, Y_4) = (s^1 - \tilde{s}^2, \tilde{\beta}_2^1 - \beta_2^2, \tilde{\beta}_3^1 - \beta_3^2, \tilde{B}^1 - B^2)$ and $(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4) = (\tilde{s}^1 - \tilde{s}^2, \tilde{\beta}_2^1 - \beta_2^2, \tilde{\beta}_3^1 - \beta_3^2, \tilde{B}^1 - \tilde{B}^2)$.

Step 1. Estimate of $Y_1$.

We rewrite the equation satisfied by $\tilde{s}$ as follows:

$$
\sum_{i,j=1}^{3} \partial_i \left( a_{ij}(\tilde{s}, \tilde{\beta}_2, \tilde{\beta}_3, B) \partial_j \tilde{s} \right) = \left( \sum_{i,j=1}^{3} a_{ij}(\tilde{s}, \tilde{\beta}_2, \tilde{\beta}_3, B) \partial_j \tilde{s} \right)^2 - \left( (\partial_2 \tilde{\beta}_2)^2 + (\partial_3 \tilde{\beta}_3)^2 + 2\partial_2 \tilde{\beta}_3 \partial_3 \tilde{\beta}_2 \right).
$$

Then $Y_1$ satisfies the following elliptic system:

$$
\begin{align*}
\sum_{i,j=1}^{3} \partial_i \left( a_{ij}(s^1, \beta_2^1, \beta_3^1, \tilde{B}^1) \partial_j Y_1 \right) &= -\sum_{i=1}^{3} \partial_i G_i + H_1 - H_2, \\
- \sum_{j=1}^{3} a_{1j}(\tilde{s}^1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{B}^1) \partial_j Y_1(0, x_2, x_3) &= G_1, \\
Y_1(1, x_2, x_3) &= 0.
\end{align*}
$$

Here

$$
\begin{align*}
G_i &= \sum_{j=1}^{3} \left( a_{ij}(\tilde{s}^1, \beta_2^1, \beta_3^1, \tilde{B}^1) - a_{ij}(s^2, \beta_2^2, \beta_3^2, B^2) \right) \partial_j \tilde{s}^2, \\
H_1 &= \left( \sum_{i=1}^{3} a_{1j}(s^1, \beta_2, \beta_3, \tilde{B}^1) \partial_j \tilde{s} \right)^2 - \left( \sum_{i=1}^{3} a_{1j}(s^2, \beta_2, \beta_3, B^2) \partial_j \tilde{s} \right)^2, \\
H_2 &= \left( (\partial_2 \tilde{\beta}_2)^2 + (\partial_3 \tilde{\beta}_3)^2 + 2\partial_2 \tilde{\beta}_3 \partial_3 \tilde{\beta}_2 \right) - \left( (\partial_2 \beta_2^2)^2 + (\partial_3 \beta_3^2)^2 + 2\partial_2 \beta_3^2 \partial_3 \beta_2^2 \right).
\end{align*}
$$

Thus the following estimate holds:

$$
\|Y_1\|_{C^{2,\alpha}(\Omega_\epsilon)} \leq C_8 \left( \sum_{i=1}^{3} \|G_i\|_{C^{1,\alpha}(\Omega_\epsilon)} + \|(H_1, H_2)\|_{C^{\alpha}(\Omega_\epsilon)} + \|G_i\|_{C^{1,\alpha}(T^2)} \right)
$$

$$
\leq C_9\delta \left( \|\tilde{Y}_1\|_{C^{2,\alpha}(\Omega_\epsilon)} + \|\tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4\|_{C^{1,\alpha}(\Omega_\epsilon)} \right).
$$

Step 2. Estimate of $Y_2, Y_3$. 
It follows from (4.14), $Y_2$ and $Y_3$ satisfy the following system:

\[
\begin{align*}
\partial_1 Y_2 + \beta_2^1 \partial_2 Y_2 + \beta_3^1 \partial_3 Y_2 - \frac{c^2(\beta^1)}{(\bar{u}_1)^2} (\beta_2^1 \partial_1 - \partial_2) Y_1 &= K_1, \\
\partial_1 Y_3 + \beta_2^1 \partial_2 Y_3 + \beta_3^1 \partial_3 Y_3 - \frac{c^2(\beta^1)}{(\bar{u}_1)^2} (\beta_3^1 \partial_1 - \partial_3) Y_1 &= K_2, \\
Y_2(0, x_2, x_3) &= 0, \\
Y_3(0, x_2, x_3) &= 0.
\end{align*}
\]  

(4.26)

Here

\[
\begin{align*}
K_1 &= - (\bar{Y}_2 \partial_2 \beta_2^2 + \bar{Y}_3 \partial_3 \beta_3^2) + \left( \frac{c^2(\beta^1)}{(\bar{u}_1)^2} \beta_2^2 + \frac{c^2(\beta^2)}{(\bar{u}_1)^2} \beta_3^2 \right) \partial_1 s^2 - \left( \frac{c^2(\beta^1)}{(\bar{u}_1)^2} - \frac{c^2(\beta^2)}{(\bar{u}_1)^2} \right) \partial_2 s^2, \\
K_2 &= - (\bar{Y}_2 \partial_2 \beta_2^3 + \bar{Y}_3 \partial_3 \beta_3^3) + \left( \frac{c^2(\beta^1)}{(\bar{u}_1)^2} \beta_3^3 - \frac{c^2(\beta^2)}{(\bar{u}_1)^2} \beta_3^3 \right) \partial_1 s^2 - \left( \frac{c^2(\beta^1)}{(\bar{u}_1)^2} - \frac{c^2(\beta^2)}{(\bar{u}_1)^2} \right) \partial_3 s^2.
\end{align*}
\]  

(4.27)

Then it follows directly that:

\[
\|(Y_2, Y_3)\|_{C^{1, \alpha}(\Omega_\epsilon)} \leq C_{10} \left( \delta \|(\bar{Y}_1, \bar{Y}_2, \bar{Y}_3, \bar{Y}_4)\|_{C^{1, \alpha}(\Omega_\epsilon)} + \|\nabla Y_1\|_{C^{1, \alpha}(\Omega_\epsilon)} \right) \\
\leq C_{11} \delta \left( \|\bar{Y}_1\|_{C^{2, \alpha}(\Omega_\epsilon)} + |\bar{Y}_2, \bar{Y}_3, \bar{Y}_4|_{C^{1, \alpha}(\Omega_\epsilon)} \right).
\]  

(4.28)

Step 3. Estimate of $Y_4$.

Indeed, $Y_4$ satisfies the following equation:

\[
\begin{align*}
\partial_1 \bar{Y}_4 + \beta_2^1 \partial_2 Y_4 + \beta_3^1 \partial_3 Y_4 &= - \bar{Y}_2 \partial_2 \bar{B}^2 - \bar{Y}_3 \partial_3 \bar{B}^2 - Y_2 \partial_2 B_0 - Y_3 \partial_3 B_0, \\
\bar{Y}_4(0, x_2, x_3) &= 0.
\end{align*}
\]  

(4.29)

Then we have the following estimate:

\[
\|Y_4\|_{C^{1, \alpha}(\Omega_\epsilon)} \leq C_{12} \delta \|(\bar{Y}_2, \bar{Y}_3)\|_{C^{1, \alpha}(\Omega_\epsilon)} + C_{12} \|(Y_2, Y_3)\|_{C^{1, \alpha}(\Omega_\epsilon)} \\
\leq C_{13} \delta \left( \|\bar{Y}_1\|_{C^{2, \alpha}(\Omega_\epsilon)} + |\bar{Y}_2, \bar{Y}_3, \bar{Y}_4|_{C^{1, \alpha}(\Omega_\epsilon)} \right).
\]  

(4.30)

Set $C_{14} = \max\{C_9, C_{11}, C_{13}\}$, then take $\epsilon_2$ small enough such that $C_{14} \epsilon_2 < 1$. Now we set $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$. Then if $0 < \epsilon < \epsilon_0$, $\Lambda$ maps $\Xi$ to itself and is contraction in low order norm. Hence $\Lambda$ has a unique fixed point, which will be the solution to (4.8). We have finished our proof.

There are a few remarks in order.

Remark 4.2. We remark that the vorticity of the background can be large.
Remark 4.3. The above iteration scheme can not be applied to a 3-D nozzle with general sections. In general, the velocity field $\beta_2$ and $\beta_3$ will lose one order derivative when integrating along the particle path. The delicate nonlinear coupling between the hyperbolic modes and elliptic modes ($\beta_2, \beta_3$) makes it extremely difficult to develop an effective iteration scheme in a general 3-D nozzle. How to effectively explore the good property of the quantity $W = \partial_2\beta_3 - \partial_3\beta_2 + \beta_3\partial_1\beta_2 - \beta_2\partial_1\beta_3$ will be investigated in the forthcoming paper.

5 A new formulation of the three dimensional incompressible Euler equations

5.1 A new formulation for the incompressible Euler equations

We apply the same idea to get a new formulation for the incompressible Euler equations. The original incompressible Euler equations take the following form:

\[
\begin{align*}
\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 &= 0, \\
u_1 \partial_1 u_1 + u_2 \partial_2 u_1 + u_3 \partial_3 u_1 + \partial_1 p &= 0, \\
u_1 \partial_1 u_2 + u_2 \partial_2 u_2 + u_3 \partial_3 u_2 + \partial_2 p &= 0, \\
u_1 \partial_1 u_3 + u_2 \partial_2 u_3 + u_3 \partial_3 u_2 + \partial_3 p &= 0.
\end{align*}
\]

(5.1)

Define $\beta_i = \frac{u_i}{u_1}, i = 2, 3$ and the Bernoulli’s function $B = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) + p$. The inner product of $u$ and the last three equations yields the Bernoulli’s law:

\[
u_1 \partial_1 B + u_2 \partial_2 B + u_3 \partial_3 B = 0.
\]

(5.2)

Multiplying the first equation in (5.1) by $-\frac{u_i - 1}{u_1^2}$, dividing the i-th equation in (5.1) by $u_1^2$ and adding them together for $i = 2, 3, 4$, we obtain the following new system:

\[
\begin{align*}
\partial_1 \beta_2 + \beta_3 \partial_3 \beta_2 - \beta_2 \partial_3 \beta_3 + \frac{1}{u_1^2} \partial_2 p &= 0, \\
\partial_1 \beta_3 + \beta_2 \partial_2 \beta_3 - \beta_3 \partial_2 \beta_2 + \frac{1}{u_1^2} \partial_3 p &= 0, \\
\partial_2 \beta_2 + \partial_3 \beta_3 - \frac{1}{u_1^2} \partial_1 p &= 0, \\
\partial_1 B + \beta_2 \partial_2 B + \beta_3 \partial_3 B &= 0.
\end{align*}
\]

(5.3)
Using the third equation, we rewrite the above system as follows:

\[
\begin{align*}
\partial_2\beta_2 + \partial_3\beta_3 - \frac{1}{u_1^2}\partial_1 p &= 0, \\
\partial_1 \beta_2 + \beta_2 \partial_2 \beta_2 + \beta_3 \partial_3 \beta_2 - \frac{\beta_2}{u_1^2}\partial_1 p + \frac{1}{u_1^2}\partial_2 p &= 0, \\
\partial_1 \beta_3 + \beta_2 \partial_2 \beta_3 + \beta_3 \partial_3 \beta_3 - \frac{\beta_3}{u_1^2}\partial_1 p + \frac{1}{u_1^2}\partial_3 p &= 0, \\
\partial_1 B + \beta_2 \partial_2 B + \beta_3 \partial_3 B &= 0.
\end{align*}
\] (5.4)

We also carry out a characteristic analysis to get an insight of the structure of incompressible Euler equations.

We rewrite (5.4) in a matrix form like the following

\[ M_1 \partial_1 U + M_2 \partial_2 U + M_3 \partial_3 U = 0. \] (5.5)

Here \( U = (p, \beta_2, \beta_3, B)^T \) and

\[
M_1 = \begin{bmatrix}
\frac{1}{u_1^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{u_1^2} & 1 \\
\frac{1}{u_1} & 0 & -1 & 0
\end{bmatrix},
M_2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & \frac{1}{u_1^2} & 0 \\
0 & 0 & \beta_2 & 0 \\
0 & 0 & 0 & \beta_2
\end{bmatrix},
M_3 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & \beta_3 & 0 & 0 \\
0 & 0 & 0 & \beta_3 \\
0 & 0 & 0 & \beta_3
\end{bmatrix}.
\]

Denote by \( \lambda \) the root of \( \det(\lambda M_1 - \xi_2 M_2 - \xi_3 M_3) \). A simple calculation shows that

\[ \det(\lambda M_1 - \xi_2 M_2 - \xi_3 M_3) = -\frac{1}{u_1}(\lambda - \beta \cdot \xi)^2[\lambda^2 + (\xi_2^2 + \xi_3^2)] \]

then we find that it has one real root \( \lambda_r = \beta \cdot \xi \) with multiplicity 2 and two conjugate complex roots \( \lambda_c^\pm = \pm i \lambda_I \), where \( \lambda_I = \sqrt{\xi_2^2 + \xi_3^2} \). The corresponding left eigenvectors to \( \lambda_r \) are the following

\[ L_1^I = (0, \xi_3 + \beta_3(\beta \cdot \xi), -\xi_2 - \beta_2(\beta \cdot \xi), 0), L_2^I = (0, 0, 0, 1). \]

The corresponding left eigenvectors to \( \lambda_c^\pm \) are

\[ L_c^\pm = L_R \pm i L_I = (-\beta \cdot \xi, \xi_2, \xi_3, 0) \pm i(\sqrt{\xi_2^2 + \xi_3^2}, 0, 0, 0). \]

The differential operators corresponding to \( L_R \) and \( L_I \) are \((-\beta_2 \partial_2 + \beta_3 \partial_3, \partial_2, \partial_3, 0)\) and \((1, 0, 0, 0)\). The action of these two differential operations will give us an elliptic system. We denote by \( \varpi = \partial_2 \beta_2 + \partial_3 \beta_3 \) and find that \( \varpi \) and \( p \) satisfy the following elliptic system:

\[ \begin{align*}
-\frac{1}{u_1^2}\partial_1 p + \varpi &= 0, \\
\partial_1 \varpi + \partial_2(\frac{1}{u_1^2}\partial_2 p) + \partial_3(\frac{1}{u_1^2}\partial_3 p) - \partial_2(\frac{\beta_2}{u_1^2}\partial_1 p) - \partial_3(\frac{\beta_3}{u_1^2}\partial_1 p) \\
&\quad + (\beta_2 \partial_2 + \beta_3 \partial_3)(\frac{1}{u_1^2}\partial_1 p) + (\partial_2 \beta_2)^2 + (\partial_3 \beta_3)^2 + 2\partial_2 \beta_2 \partial_3 \beta_2 &= 0.
\end{align*} \] (5.6)
Remark 5.1. The steady 3-D incompressible Euler system is always an hyperbolic-elliptic coupled system. Here our point of view is different from previous works [1] and [29]. Indeed, in the works of [1] and [29], the authors regarded the pressure \( p \) as the Lagrange multiplier for the divergent free condition.

5.2 The incompressible Euler equations with a constant Bernoulli’s function

In the following, we assume that the Bernoulli’s function is a constant and the speed \(|u|\) has a positive lower bound \( \delta \), then we have \( u_1^2 = \frac{2(B - p)}{1 + \beta_2^2 + \beta_3^2} \). As in the compressible case, the vorticity field will be parallel to the velocity field and \( W = \partial_2 \beta_3 - \partial_3 \beta_2 + \beta_3 \partial_1 \beta_2 - \beta_2 \partial_1 \beta_3 \) will take the place of \( \partial_2 u_3 - \partial_3 u_2 \). We define \( G = 1 + \beta_2^2 + \beta_3^2 \) and a new function \( K(p) \) of \( p \) satisfying \( K'(p) = \frac{1}{2(B - p)} \). and then rewrite the equations satisfied by \( \beta_2 \) and \( \beta_3 \) as follows:

\[
\begin{align*}
G^{-1} (\partial_1 \beta_2 + \beta_2 \partial_2 \beta_2 + \beta_3 \partial_3 \beta_2) - \beta_2 \partial_1 K(p) + \partial_2 K(p) &= 0, \\
G^{-1} (\partial_1 \beta_3 + \beta_2 \partial_2 \beta_3 + \beta_3 \partial_3 \beta_3) - \beta_3 \partial_1 K(p) + \partial_3 K(p) &= 0,
\end{align*}
\]

(5.7)

Applying \( \beta_3 \partial_1 - \partial_2 \) and \( -\beta_2 \partial_1 + \partial_2 \) to the above two equations and adding them together, then one can obtain the equation for \( W \):

\[
DW + (\partial_2 \beta_2 + \partial_3 \beta_3) W - G \partial_1 K(p) W - G^{-1} WDG = 0.
\]

(5.8)

Due to the first equation in (5.3), \( W \) satisfies the following equation:

\[
DW - G^{-1} WDG = 0.
\]

(5.9)

That is

\[
D\left(\frac{W}{G}\right) = 0.
\]

(5.10)

It is easy to find \( G \) satisfies the following Riccati-type equation:

\[
DG = \frac{1}{(B - p)} \partial_1 p G^2 + \frac{1}{(B - p)} G \partial_1 p = 0.
\]

(5.11)

Define three new variables \( K_1 = G^{-\frac{1}{2}}, K_i = \frac{\beta_i}{G^2}, i = 2, 3 \). Here \( K_2 \) and \( K_3 \) behave as the sine function of “the flow angles”. Define also two new functions \( I(p) = e^{\int_p^0 \frac{1}{2(B - t)} dt} \) and \( Q(p) = \int_p^0 \frac{1}{2(B - t)} I(t)^{-2} dt \), where \( p \) is a given positive reference pressure. Since the speed \(|u|\) has a lower bound \( \delta \), it is easy to verify that the above two integrals are convergent, so \( I(p) \) and \( Q(p) \) are well-defined. One should note that \( I(p) \) satisfies \( \frac{I'(p)}{I(p)} = \frac{1}{2(B - p)} \).

It follows from (5.11) that \( K_1 \) satisfies:

\[
DK_1 - \frac{1}{2(B - p)} K_1 \partial_1 p + \frac{1}{2(B - p)} K_1^{-1} \partial_1 p = 0.
\]

(5.12)
This equation, together with \([5.4]\) yields:

\[
\begin{cases}
- \frac{1}{2(B - p)} K_1 \mathbf{D} p + \partial_1 K_1 + \partial_2 K_2 + \partial_3 K_3 = 0, \\
\mathbf{D} K_1 - \frac{1}{2(B - p)} K_1 \mathbf{D} p + \frac{1}{2(B - p)} K_1^{-1} \partial_1 p = 0, \\
\mathbf{D} K_2 - \frac{1}{2(B - p)} K_2 \mathbf{D} p + \frac{1}{2(B - p)} K_1^{-1} \partial_2 p = 0, \\
\mathbf{D} K_3 - \frac{1}{2(B - p)} K_3 \mathbf{D} p + \frac{1}{2(B - p)} K_1^{-1} \partial_3 p = 0.
\end{cases}
\] (5.13)

It follows from the definition of \(I(p)\) and \(Q(p)\) that

\[
\begin{cases}
- \frac{1}{I(p)} K_1 \mathbf{D} I(p) + \partial_1 K_1 + \partial_2 K_2 + \partial_3 K_3 = 0, \\
\mathbf{D} K_1 - \frac{1}{I(p)} K_1 \mathbf{D} I(p) + \frac{1}{I(p)} K_1^{-1} \partial_1 I(p) = 0, \\
\mathbf{D} K_2 - \frac{1}{I(p)} K_2 \mathbf{D} I(p) + \frac{1}{I(p)} K_1^{-1} \partial_2 I(p) = 0, \\
\mathbf{D} K_3 - \frac{1}{I(p)} K_3 \mathbf{D} I(p) + \frac{1}{I(p)} K_1^{-1} \partial_3 I(p) = 0.
\end{cases}
\] (5.14)

Setting \(A_i = \frac{K_i}{I(p)} \), \(i = 1, 2, 3\), we obtain the following conservation laws:

\[
\begin{cases}
\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 = 0, \\
\partial_1 (A_1^2) + \partial_2 (A_1 A_2) + \partial_3 (A_1 A_3) + \partial_1 Q(p) = 0, \\
\partial_1 (A_1 A_2) + \partial_2 (A_2^2) + \partial_3 (A_2 A_3) + \partial_2 Q(p) = 0, \\
\partial_1 (A_1 A_3) + \partial_2 (A_2 A_3) + \partial_3 (A_3^2) + \partial_3 Q(p) = 0.
\end{cases}
\] (5.15)

For the incompressible Euler equations, \(W\) satisfies the following equation:

\[
(\partial_1 + \beta_2 \partial_2 + \beta_3 \partial_3) \left( \frac{W}{G} \right) + \frac{1}{2(B - p)^2} [(1, \beta_2, \beta_3) \cdot (\nabla p \times \nabla B)] = 0.
\] (5.16)

And the following conservation law also holds:

\[
\partial_1 \left( \frac{K_1}{I(p, B)} \right) + \partial_2 \left( \frac{K_2}{I(p, B)} \right) + \partial_3 \left( \frac{K_3}{I(p, B)} \right) = 0.
\] (5.17)

**Remark 5.2.** As an application, one can apply the techniques developed in section 4 to construct a smooth incompressible Euler flow in a rectangular cylinder, which satisfies the given incoming flow angles and the Bernoulli’s function at the inlet and the end pressure at the exit. Indeed, our background solution will be denoted by \((p_0, u_0(x_2, x_3), 0, 0)\). The corresponding
Bernoulli’s function will be \( B_0(x_2, x_3) = \frac{1}{2} u_0^2(x_2, x_3) + p_0 \). At the exit of the nozzle, we will prescribe the end pressure:

\[
p(1, x_2, x_3) = p_0 + e p_e(x_2, x_3).
\]

(5.18)

Here it is required that \( p_e(x_2, x_3) \) should satisfy the same compatibility conditions as (4.4) for \( s_e(x_2, x_3) \) in the previous section. For the incoming flow angles and the Bernoulli’s function and the slip boundary conditions, we should impose the same compatibility conditions as the compressible case. Then we can obtain the following existence and uniqueness results:

**Theorem 5.3.** Given \((\beta_2^m, \beta_3^m, B^m, p_e) \in C^{3,\alpha}(T^2)\) satisfying the same compatibility conditions as (4.2) and (4.4) for the compressible case, there exists a positive small number \( \epsilon_0 \), which depends on the background subsonic state \((p_0, u_0(x_2, x_3), 0, 0) \) and \((\beta_2^m, \beta_3^m, B^m, p_e) \), such that if \( 0 < \epsilon < \epsilon_0 \), then there exists a unique smooth subsonic flow \((p, u_1, u_2, u_3) \in C^{3,\alpha}(\Omega_e) \times (C^{2,\alpha}(\Omega_e))^3 \) to (5.1) satisfying boundary conditions (6.1), (5.18) and (4.5). Moreover, the following estimate holds:

\[
\|(u_1, u_2, u_3) - (u_0(x_2, x_3), 0, 0)\|_{C^{2,\alpha}(\Omega_e)} + \|p - p_0\|_{C^{3,\alpha}(\Omega_e)} \leq C \epsilon.
\]

(5.19)

Here \( C \) is a constant depending on \((p_0, u_0(x_2, x_3), 0, 0) \) and \((\beta_2^m, \beta_3^m, B^m, p_e) \).

6 Appendix

In this appendix, we give the detailed calculations for (3.4), (3.6). Applying \(-\beta_3 \partial_1 + \partial_3 \) and \( \beta_2 \partial_1 - \partial_2 \) to (3.3) and adding them together, we obtain

\[
0 = (\beta_3 \partial_1 - \partial_3)(G^{-1} \mathbf{D} \beta_2) - (\beta_2 \partial_1 - \partial_2)(G^{-1} \mathbf{D} \beta_3)
- \left[(\beta_3 \partial_1 - \partial_3)(\beta_2 \partial_1 - \partial_2)K(s) - (\beta_2 \partial_1 - \partial_2)(\beta_3 \partial_1 - \partial_3)K(s)\right]
= (\beta_3 \partial_1 - \partial_3)(G^{-1} \mathbf{D} \beta_2) - (\beta_2 \partial_1 - \partial_2)(G^{-1} \mathbf{D} \beta_3) - W \partial_1 K(s)
= J - W \partial_1 K(s).
\]

While

\[
J = G^{-1} \mathbf{D}((\beta_3 \partial_1 - \partial_3) \beta_2 - (\beta_2 \partial_1 - \partial_2) \beta_3)
+ G^{-1} \sum_{j=1}^{3} ((\beta_3 \partial_1 - \partial_3) \beta_j \partial_j \beta_2 - (\beta_2 \partial_1 - \partial_2) \beta_j \partial_j \beta_3)
+ G^{-1} \sum_{j=1}^{3} \beta_j \left\{(\beta_3 \partial_1 - \partial_3) \partial_j - \partial_j (\beta_3 \partial_1 - \partial_3)\right\} \beta_2
- \left\{((\beta_2 \partial_1 - \partial_2) \partial_j - \partial_j (\beta_2 \partial_1 - \partial_2)) \beta_3\right\}
+ (\beta_3 \partial_1 - \partial_3) G^{-1} \mathbf{D} \beta_2 - (\beta_2 \partial_1 - \partial_2) G^{-1} \mathbf{D} \beta_3
= G^{-1} \mathbf{D} W + J_1 + J_2 + J_3.
\]

Now we compute \( J_i, i = 1, 2, 3 \) respectively.
\[ J_1 = G^{-1} \left[ W(\partial_2 \beta_2 + \partial_3 \beta_3) - (\beta_3 \partial_1 - \partial_3) \beta_2 \partial_3 \beta_3 + (\beta_3 \partial_1 - \partial_3) \beta_3 \partial_2 \beta_2 - (\beta_2 \partial_1 - \partial_2) \beta_2 \partial_3 \beta_3 \right] \]

\[ = G^{-1} \left[ W(\partial_2 \beta_2 + \partial_3 \beta_3) + Z \right]. \]

Here \( Z = \beta_3 \partial_1 \beta_2 \partial_2 \beta_3 - \beta_3 \partial_1 \beta_2 \partial_3 \beta_3 + \beta_2 \partial_1 \beta_3 \partial_2 \beta_2 - \beta_2 \partial_1 \beta_2 \partial_2 \beta_3. \)

\[ J_2 = G^{-1} \sum_{j=1}^{3} \beta_j \left\{ [\beta_3(\partial_1 - \partial_3)] \partial_j - \partial_j (\beta_3 \partial_1 - \partial_3) \right\} \beta_2 \]

\[ = G^{-1} Z. \]

\[ J_3 = -G^{-2} \left[ (\beta_3 \partial_1 - \partial_3) GD \beta_2 - (\beta_2 \partial_1 - \partial_2) GD \beta_3 \right] \]

\[ = -2G^{-2} \sum_{j=1}^{3} \beta_j (\beta_3 \partial_1 - \partial_3) \beta_j D \beta_2 - \beta_j (\beta_2 \partial_1 - \partial_2) \beta_j D \beta_3 \]

\[ = -2G^{-2} [W(\beta_2 D \beta_2 + \beta_3 D \beta_3) + J_{31}]. \]

Here

\[ J_{31} = \beta_2 (\beta_2 \partial_1 - \partial_2) \beta_3 D \beta_2 + \beta_3 (\beta_3 \partial_1 - \partial_3) \beta_2 D \beta_2 \]

\[ - \beta_3 (\beta_3 \partial_1 - \partial_3) \beta_2 D \beta_3 - \beta_2 (\beta_2 \partial_1 - \partial_2) \beta_3 D \beta_3 \]

\[ = \beta_2 \left[ (\beta_2 \partial_1 - \partial_2) \beta_3 D \beta_2 - (\beta_2 \partial_1 - \partial_2) \beta_2 D \beta_3 \right] \]

\[ + \beta_3 \left[ (\beta_3 \partial_1 - \partial_3) \beta_3 D \beta_2 - (\beta_2 \partial_1 - \partial_2) \beta_2 D \beta_3 \right] \]

\[ = \beta_2 \left[ (\beta_2 \partial_1 - \partial_2) \beta_3 \partial_1 \beta_2 - (\beta_2 \partial_1 - \partial_2) \beta_2 \partial_1 \beta_3 \right] \]

\[ + \beta_3 \left[ (\beta_3 \partial_1 - \partial_3) \beta_3 \partial_1 \beta_2 - (\beta_3 \partial_1 - \partial_3) \beta_2 \partial_1 \beta_3 \right] \]

\[ + \left[ (\beta_2 \partial_1 - \partial_2) \beta_3 (\beta_2 \partial_2 \beta_2 + \beta_2 \beta_3 \partial_2 \beta_2) - (\beta_2 \partial_1 - \partial_2) \beta_2 (\beta_2 \partial_2 \beta_3 + \beta_2 \beta_3 \partial_2 \beta_3) \right] \]

\[ + \left[ (\beta_3 \partial_1 - \partial_3) \beta_3 (\beta_2 \partial_3 \beta_2 + \beta_2 \beta_3 \partial_3 \beta_2) - (\beta_3 \partial_1 - \partial_3) \beta_2 (\beta_2 \partial_3 \beta_3 + \beta_2 \beta_3 \partial_3 \beta_3) \right] \]

\[ = GZ. \]

Hence we have \( J_3 = -2G^{-2} [W(\beta_2 D \beta_2 + \beta_3 D \beta_3)] - 2G^{-1} Z. \) Substitute these calculations into the above formula to get:
\[ 0 = J - W \partial_1 K(s) = J_1 + J_2 + J_3 + G^{-1}DW - W \partial_1 K(s) \]
\[ = G^{-1}[W(\partial_2\beta_2 + \partial_3\beta_3) + Z] + G^{-1}Z - 2G^{-2}[W(\beta_2D\beta_2 + \beta_3D\beta_3)] \]
\[ - 2G^{-1}Z + G^{-1}DW - W \partial_1 K(s) \]
\[ = G^{-1}DW + G^{-1}(\partial_2\beta_2 + \partial_3\beta_3 - G\partial_1 K(s))W - G^{-2}W DG \]
\[ = G^{-1}DW - G^{-1}WD\bar{s} - G^{-2}W DG = \rho D\left(\frac{W}{\rho G}\right). \]

This implies that \((\partial_1 + \beta_2\partial_2 + \beta_3\partial_3)(\frac{W}{\rho G}) = 0.\)

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