A 3D-SCHRÖDINGER OPERATOR UNDER MAGNETIC STEPS

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Abstract. We define a Schrödinger operator on the half-space with a discontinuous magnetic field having a piecewise-constant strength and a uniform direction. Motivated by applications in the theory of superconductivity, we study the infimum of the spectrum of the operator. We further give sufficient conditions on the strength and the direction of the magnetic field such that the aforementioned infimum is an eigenvalue of a reduced model operator on the half-plane.

1. Introduction

We consider a Schrödinger operator defined on the half-space and having a magnetic field with a piecewise-constant strength and a uniform direction. Such operator is interesting to be considered in new situations in the theory of superconductivity as we will describe later. We set the half-space to be $\mathbb{R}^3_+ := \{ x \in \mathbb{R}^3 | x = (x_1, x_2, x_3), \ x_2 > 0 \}$ and we split it in two regions in which the strength of the magnetic field is different as follows. Let $\alpha \in (0, \pi)$, using spherical coordinates, we define the domains $D^1_\alpha$ and $D^2_\alpha$ of $\mathbb{R}^3_+$:

$$D^1_\alpha = \{ x \in \mathbb{R}^3 | x = \rho (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi), \ \rho \in (0, \infty), \ 0 < \theta < \alpha, \ \phi \in (0, \pi) \}.$$ (1.1)

$$D^2_\alpha = \{ x \in \mathbb{R}^3 | x = \rho (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi), \ \rho \in (0, \infty), \ \alpha < \theta < \pi, \ \phi \in (0, \pi) \}.$$ (1.2)

Let $a \in [-1, 1) \setminus \{0\}$ and $1/2 \gamma \in [0, \pi/2]$, we introduce the following magnetic field in $\mathbb{R}^3_+$

$$B_{\alpha, \gamma, a} = (\cos \alpha \sin \gamma, \sin \alpha \sin \gamma, \cos \gamma) (\mathbb{1}_{D^1_\alpha} + a \mathbb{1}_{D^2_\alpha})$$ (1.3)

Here (and in the sequel) $\mathbb{1}_\sharp$ denotes the characteristic function corresponding to the set $\sharp$ (in this case $\sharp = D^1_\alpha, D^2_\alpha$). The function $s_{\alpha, a}$ represents the strength of the magnetic field (see Figure 1). The choice of the values $a$ in $[-1, 1) \setminus \{0\}$ will be discussed later (see Remark 1.2).

We consider the magnetic Neumann realization of the following self-adjoint operator on $\mathbb{R}^3_+$

$$\mathcal{L}_{\alpha, \gamma, a} = - (\nabla - i A_{\alpha, \gamma, a})^2,$$ (1.4)

where $A_{\alpha, \gamma, a} \in H^1_{\text{loc}}(\mathbb{R}^3_+, \mathbb{R}^3)$ is a magnetic potential such that $\text{curl} A_{\alpha, \gamma, a} = B_{\alpha, \gamma, a}$.

The domain of the operator $\mathcal{L}_{\alpha, \gamma, a}$ is

$$\mathcal{D}(\mathcal{L}_{\alpha, \gamma, a}) = \{ u \in L^2(\mathbb{R}^3_+) : (\nabla - i A_{\alpha, \gamma, a})^n u \in L^2(\mathbb{R}^3_+), \ \text{for } n \in \{1, 2\}, (\nabla - i A_{\alpha, \gamma, a})u \cdot (0, 1, 0) |_{\partial \mathbb{R}^3_+} = 0 \}.$$ (1.5)

The goal of the present paper is to study the bottom of the spectrum of $\mathcal{L}_{\alpha, \gamma, a}$.

\footnote{By symmetry considerations, we restrict the study to the case where $\gamma \in [0, \pi/2]$.}
1.1. **Motivation.** In the theory of superconductivity and in generic situations, a superconductor submitted to a sufficiently strong magnetic field loses permanently its superconducting properties when the intensity of the magnetic field exceeds a certain (unique) critical value—the so-called third critical field denoted by \( H_{C_3} \). We say that the material passes to the normal state (see [18, 40]). The Ginzburg–Landau (GL) model is used to study this phase transition from superconducting to normal states. This is naturally a three-dimensional (3D) model, but it is usually reduced to a two-dimensional (2D) one supposing that the superconductor is a long-cylindrical wire and that the direction of the magnetic field is perpendicular to the cross section of the wire (see e.g. [41]).

Within this context, 3D models were studied in the mathematical literature for more general (bounded or unbounded) domains, not necessarily cylinders, subjected to constant or smooth variable magnetic fields (see e.g. [24, 30, 34, 35, 38]). In particular, this literature considered a Schrödinger operator, \(-\overline{\hbar \nabla - iA}^2\), defined on an open and bounded set \( \Omega \subset \mathbb{R}^3 \), with smooth boundary or having edges, where \( A \in H^1_{\text{loc}}(\mathbb{R}^3) \) is a magnetic vector potential and \( \text{curl} A = B \) is the external magnetic field having a constant or a smooth variable strength. As the semiclassical parameter \( \hbar \) goes to 0, the third critical field \( H_{C_3} \) is estimated using the asymptotics of the first eigenvalue, \( \lambda(B; \Omega, \hbar) \), of this operator (see e.g. [17, Proposition 1.9], [2, 18, 20, 30]). Such asymptotics of \( \lambda(B; \Omega, \hbar) \) are usually obtained by using a variational argument where local energies are studied in different zones of the superconductor (like the interior, the boundary, or near the edges). The local study involves effective Schrödinger operators of the form \(-\overline{\nabla - iA}^2\), with magnetic fields having a constant strength, defined on unbounded domains like \( \mathbb{R}^3 \), \( \mathbb{R}^3_+ \) or infinite wedges (see [8, 30]). While the effective operator on \( \mathbb{R}^3 \) is related to the study in the interior of \( \Omega \), that on \( \mathbb{R}^3_+ \) is related to the study at the smooth boundary of \( \Omega \) and depends on the angle between the magnetic field and the boundary \( \mathbb{R}^3_+ \). Moreover, the effective operators

\footnote{Due to gauge invariance [18, Section 1.1], it is standard that the magnetic potential \( A \) contributes to the spectrum of \(-\overline{\hbar \nabla - iA}^2\) only through its associated magnetic field \( B \), which justifies the notation \( \lambda(B; \Omega, \hbar) \).}
on infinite wedges are considered ([32, 34, 35]) for the study near the edges of $\Omega$ (when exist), and depend on both the direction of the magnetic field and the opening angle of the wedge. Studying the effective models permit to determine the eventual localization of superconductivity in $\Omega$, before its breakdown. We refer the reader to the introduction in [35] for a brief explanation about the link between the original model on $\Omega$ and the various effective models (see also [8] for a more detailed explanation).

Back to the operator $\mathcal{L}_{\alpha,\gamma,a}$ defined in the present contribution, such a 3D operator with a discontinuous magnetic field was not considered yet in the literature. We expect $\mathcal{L}_{\alpha,\gamma,a}$ to be an effective operator which plays an essential role in the study of semi-classical problems similar to the aforementioned ones in $\Omega$, but in new situations where the magnetic field is piecewise-constant (see [19, Section 8]). A simple instance of such problems is given in what follows.

1.1.1. A simple application. Let $\mathbf{B} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a magnetic field such that

$$\mathbf{B}(x) = \mathbf{s}(x)(0,0,1), \text{ for } x = (x_1,x_2,x_3) \in \mathbb{R}^3,$$

where $\mathbf{s}$ is a step function in $\mathbb{R}^3$ representing the strength of the magnetic field and defined by

$$\mathbf{s}(x) = \mathbbm{1}_{x_2>0} + a \mathbbm{1}_{x_2<0},$$

with $a \in [-1,1] \setminus \{0\}$.

Let $\Omega$ be a set with a smooth boundary that intersects the plane $(x_1,x_3)$ transversally. We refer to this intersection by the discontinuity surface. In addition, we refer to the intersection between the discontinuity surface and the boundary of $\Omega$ by the discontinuity curve.

Considering the aforementioned operator $-(h \nabla - i \mathbf{A})^2$ in this instance of $\Omega$ and curl $\mathbf{A} = \mathbf{B}$, and using an approach similar to the one in [30], our operator $\mathcal{L}_{\alpha,\gamma,a}$ can be involved in the analysis of the semiclassical problem near a fixed point of the discontinuity curve. In this situation, $\alpha$ would represent the angle between the discontinuity surface and the boundary of $\Omega$ at the foregoing point, while $\gamma$ (modulo $-\pi$) would represent the angle between the magnetic field $\mathbf{B}$ and the discontinuity curve. With our results stated in Theorem 1.1 below, one expects that superconductivity eventually localizes near (some points of) the discontinuity curve. We plan to investigate such situations in a future work.

1.2. Main results. We recall the operator $\mathcal{L}_{\alpha,\gamma,a}$ introduced in (1.4)

$$\mathcal{L}_{\alpha,\gamma,a} = -(\nabla - i \mathbf{A}_{\alpha,\gamma,a})^2, \text{ in } \mathbb{R}^3_+,$$  

(1.6)

with the domain $\mathcal{D}(\mathcal{L}_{\alpha,\gamma,a})$ defined in (1.5). We consider the bottom of the spectrum of this operator

$$\lambda_{\alpha,\gamma,a} := \inf \mathrm{sp}(\mathcal{L}_{\alpha,\gamma,a}).$$  

(1.7)

Using a Fourier transform, the operator $\mathcal{L}_{\alpha,\gamma,a}$ can be decomposed into a family of 2D operators on $\mathbb{R}^2_+$, $\mathcal{L}_{\mathbf{A}_{\alpha,\gamma,a},\tau} + \mathcal{V}_{\mathbf{B}_{\alpha,\gamma,a},\tau}$, parametrized by $\tau \in \mathbb{R}$. These 2D operators are defined in Section 3. The bottom of the spectrum of these operators depends on $\alpha, \gamma, a$, and $\tau$, and is denoted by $\sigma(\alpha, \gamma, a, \tau)$. Having (see (3.17))

$$\lambda_{\alpha,\gamma,a} = \inf_{\tau \in \mathbb{R}} \sigma(\alpha, \gamma, a, \tau),$$

the examination of $\lambda_{\alpha,\gamma,a}$ reduces to that of the function $\tau \mapsto \sigma(\alpha, \gamma, a, \tau)$. This examination leads to an important comparison between $\lambda_{\alpha,\gamma,a}$ and other well-known spectral values, $\beta_a$ and $\zeta_{a_0}$, where $a \in [-1,1] \setminus \{0\}$ is the same parameter appearing in the definition of $\mathcal{L}_{\alpha,\gamma,a}$ and $\nu_0 := \arcsin(\sin \alpha \sin \gamma)$. The value $\beta_a$ is the bottom of the spectrum of a Schrödinger operator defined on $\mathbb{R}^3$ in (2.5), with a piecewise-constant magnetic field (splitting $\mathbb{R}^3$ in two half-spaces, the strength of the field takes the values 1 and $a$ in these half-spaces, respectively). The value $\zeta_{a_0}$ is the bottom of the spectrum of a magnetic
Neumann Schrödinger operator defined on $\mathbb{R}^d_+$ in (2.7), with a constant magnetic field making an angle $\nu_0$ with the $(x_1, x_3)$ plane.

**Theorem 1.1.** Let $a \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$, $\gamma \in [0, \pi/2]$, and $\nu_0 = \arcsin(\sin \alpha \sin \gamma)$. Let $\lambda_{\alpha, \gamma, a}$ be the bottom of the spectrum of the operator $L_{\alpha, \gamma, a}$ defined in (1.4). It holds

$$
\lambda_{\alpha, \gamma, a} \leq \min (\beta_a, |a|\zeta_{\nu_0}),
$$

where $\beta_a$ and $\zeta_{\nu_0}$ are respectively the bottom of the spectrum of the operators defined in (2.5) and (2.7).

Furthermore, if

$$
\lambda_{\alpha, \gamma, a} < \min (\beta_a, |a|\zeta_{\nu_0}),
$$

then there exists $\tau_* \in \mathbb{R}$ such that

$$
\lambda_{\alpha, \gamma, a} = \mathcal{G}(\alpha, \gamma, a, \tau_*)
$$

and $\mathcal{G}(\alpha, \gamma, a, \tau_*)$ is an eigenvalue of the operator $L_{\mathcal{A}_{\alpha, \gamma, a}} + V_{\mathcal{B}_{\alpha, \gamma, a}, \tau_*}$ defined in (3.7).

**Remark 1.2** (The choice of $a \in [-1, 1] \setminus \{0\}$). One can choose any two distinct real values $b_1$ and $b_2$ for the strength of the magnetic field $B_{\alpha, \gamma, a}$ respectively in $D_1^\ast$ and $D_2^\ast$. However, by a simple scaling argument, one can reduce the study to the case $b_1 = 1$ and $b_2 = a$, where $a$ is a value in $[-1, 1]$.

In the case $a = 0$, the energy $\beta_a$, appearing in Theorem 1.1, is equal to zero (see [25]). Hence, the comparison between the three energies $\lambda_{\alpha, \gamma, a}$, $\beta_a$, and $|a|\zeta_{\nu_0}$ is trivial:

$$
\lambda_{\alpha, \gamma, a} \geq \min (\beta_a, |a|\zeta_{\nu_0}) = 0.
$$

Moreover, our proof technically relies on the assumption $a \neq 0$ in many places, for instance when using translations to link our problem to the toy models in Section 2, which have well-explored spectra. We exclude the case $a = 0$ from our study.

**Remark 1.3** (About $\lambda(B; \Omega, h)$). Let $\Omega \subset \mathbb{R}^3$ and $B : \mathbb{R}^3 \to \mathbb{R}^3$ be respectively the domain and the magnetic field defined in Section 1.1.1. Also, recall $\lambda(B; \Omega, h)$, the lowest eigenvalue of the operator $-(h\nabla - iA)^2$ defined on $\Omega$ (see Section 1.1). When the bottom of the spectrum, $\lambda_{\alpha, \gamma, a}$, of the operator $L_{\alpha, \gamma, a}$ is an eigenvalue of a certain $L_{\mathcal{A}_{\alpha, \gamma, a}} + V_{\mathcal{B}_{\alpha, \gamma, a}, \tau_*}$ (see Theorem 1.1), one may use its corresponding eigenfunction to construct a trial function in $\Omega$, supported near the point(s) of the discontinuity curve corresponding to $(\alpha, \gamma, a)$, which yields an upper bound in the asymptotic estimates of $\lambda(B; \Omega, h)$ as $h$ goes to 0. See [2,18] for similar situations in domains of $\mathbb{R}^2$ or $\mathbb{R}^3$.

In Theorem 1.1, we gave sufficient conditions for $\lambda_{\alpha, \gamma, a}$ to be an eigenvalue of the operator $L_{\mathcal{A}_{\alpha, \gamma, a}} + V_{\mathcal{B}_{\alpha, \gamma, a}, \tau_*}$ in (3.7), for a certain $\tau_* \in \mathbb{R}$. Our next result provides a condition on $(\alpha, \gamma, a)$ such that (1.9) is realized.

**Proposition 1.4.** Let $a \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$, $\gamma \in [0, \pi/2]$, and $\nu_0 = \arcsin(\sin \alpha \sin \gamma)$. Consider the function $P[\alpha, \gamma, a] : (0, +\infty) \to \mathbb{R}$ defined by

$$
P[\alpha, \gamma, a](x) = A[\alpha, \gamma, a]x^2 - \frac{\pi}{2} \Lambda[\alpha, \gamma, a]x + \frac{\pi}{2},
$$

with

$$
A[\alpha, \gamma, a] := \frac{1}{128}(-1 + \coth \pi)\left\{ \pi \cos^2 \gamma \left[4(a-1)((a-e^\pi)e^{\pi-a}+(ae^\pi-1)e^a)\right] - (a-1)^2(e^{2\pi-2a}+e^{2a}) - 2e\pi(-4a+(3-2a+3a^2)\coth \pi) \right\} + 4(e^{2\pi}-1)\left[ - (a^2(\pi-a)+a)(-3+\cos(2\gamma)) + 2(a^2-1)\sin^2 \gamma \sin(2a) \right]
$$

and

$$
\Lambda[\alpha, \gamma, a] := \min (\beta_a, |a|\zeta_{\nu_0}).
$$
For the triplets $(\alpha, \gamma, a)$ in the colored region, $\lambda_{\alpha, \gamma, a}$ is an eigenvalue of the operator $L_{\alpha, \gamma, a} + V_{\beta, \alpha, \gamma, a}$, where $\beta(a)$ and $\zeta(\nu)$ are respectively the bottom of the spectrum of the operators defined in (2.5) and (2.7). If there exists $x = x(\alpha, \gamma, a) > 0$ such that $P(\alpha, \gamma, a)(x) < 0$, then
\[
\inf_{\tau} \sigma(\alpha, \gamma, a, \tau) \text{ is attained in } \mathbb{R}, \text{ i.e. there exists } \tau_0 \in \mathbb{R} \text{ satisfying}
\]
\[
\inf_{\tau} \sigma(\alpha, \gamma, a, \tau) = \sigma(\alpha, \gamma, a, \tau_0).
\]
Moreover, $\sigma(\alpha, \gamma, a, \tau_0)$ is an eigenvalue of the operator $L_{\alpha, \gamma, a} + V_{\beta, \alpha, \gamma, a}$ defined in (3.7).

Remark 1.5 (Admissible triplets $(\alpha, \gamma, a)$). In Section 2, we provide the following lower bound for the value $\Lambda[\alpha, \gamma, a]$ in (1.12)
\[
\Lambda[\alpha, \gamma, a] \geq |a| \Theta_0,
\]
where $\Theta_0$ is the de Gennes constant defined in (2.4). Moreover, [7] gives an explicit lower bound $\Theta_{\text{low}}$ of $\Theta_0$ equal to $0.590106125 \times 10^{-9}$. Hence, if one defines
\[
P_{\Theta_{\text{low}}}[\alpha, \gamma, a](x) := A[\alpha, \gamma, a]x^2 - \frac{\pi}{2}|a| \Theta_{\text{low}} x + \frac{\pi}{2},
\]
for $x > 0$ and $A[\alpha, \gamma, a]$ as in (1.11), one observes that $P[\alpha, \gamma, a](x) \leq P_{\Theta_{\text{low}}}[\alpha, \gamma, a](x)$. By computation, we get that for all $a \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$ and $\gamma \in [0, \pi/2]$, $P_{\Theta_{\text{low}}}[\alpha, \gamma, a](x)$ admits a minimum $\bar{x} > 0$. Using Mathematica, we plot the region of triplets $(\alpha, \gamma, a)$ satisfying
\[
\min_{x > 0} P_{\Theta_{\text{low}}}[\alpha, \gamma, a](x) = P_{\Theta_{\text{low}}}[\alpha, \gamma, a](\bar{x}) < 0.
\]
These triplets are represented by the colored region in Figure 2. Consequently, the corresponding $\lambda_{\alpha, \gamma, a} = \inf_{\tau} \sigma(\alpha, \gamma, a, \tau)$ is equal to $\sigma(\alpha, \gamma, a, \tau_0)$, for a certain $\tau_0 = \tau_0(\alpha, \gamma, a) \in \mathbb{R}$. Furthermore, $\sigma(\alpha, \gamma, a, \tau_0)$ is an eigenvalue of the corresponding operator $L_{\alpha, \gamma, a} + V_{\beta, \alpha, \gamma, a}$ defined in (3.7).

1.3. Paper organization. The rest of the paper is organized as follows. In Section 2, we recall some known operators in the plane and the half-space which are useful for our analysis. In Section 3, we decompose our operator into 2D reduced operators. For these reduced operators, we derive some properties of the bottom of essential spectrum and the bottom of the spectrum. The proof of the main results is then established in Section 4.

2. Known effective operators

In this section, we introduce useful linear Schrödinger operators on the plane and the half-space that were explored earlier in the literature.
2.1. An operator with a discontinuous magnetic field on the plane. Let \( a \in [-1, 1) \setminus \{0\} \). We consider a magnetic potential \( A_a \in H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) with the following associated piecewise-constant magnetic field

\[
\text{curl} \, A_a(x) = 1_{\{x_2 > 0\}}(x) + a 1_{\{x_2 < 0\}}(x), \quad (x_1, x_2) \in \mathbb{R}^2.
\]

We introduce the self-adjoint operator on \( \mathbb{R}^2 \)

\[
\mathcal{L}_a = -(\nabla - iA_a)^2,
\]

with domain

\[
D(\mathcal{L}_a) := \{ u \in L^2(\mathbb{R}^2) : (\nabla - iA_a)^n u \in L^2(\mathbb{R}^2), \text{ for } n \in \{1, 2\} \}.
\]

We denote the bottom of the spectrum by \( \beta_a \). The operator \( \mathcal{L}_a \) has been studied in [5, 6, 25, 26]; using a Fourier transform, \( \mathcal{L}_a \) was reduced to a family of Schrödinger operators on \( L^2(\mathbb{R}), h_a[\xi] \), parametrized by \( \xi \in \mathbb{R} \). For each fixed \( \xi \in \mathbb{R} \), the operator \( h_a[\xi] \) is defined by

\[
h_a[\xi] = \begin{cases} 
-\frac{d^2}{dx^2} + (at - \xi)^2, & t < 0, \\
-\frac{d^2}{dx^2} + (t - \xi)^2, & t > 0.
\end{cases}
\]  

(2.1)

We have (see [6])

\[
\beta_a = \inf_{\xi \in \mathbb{R}} \mu_a(\xi),
\]

(2.2)

where \( \mu_a(\xi) \) is the bottom of the spectrum of the operator \( h_a[\xi] \), i.e.,

\[
\mu_a(\xi) = \inf \text{sp}(h_a[\xi]).
\]

(2.3)

We collect the following useful properties of \( \beta_a \):

- For \( 0 < a < 1 \), \( \beta_a = a \) and \( \beta_a \) is not attained by \( \mu_a(\xi) \), for all \( \xi \in \mathbb{R} \).
- For \( -1 < a < 0 \), \( |a|\Theta_0 \leq \beta_a < |a| \) and \( \beta_a = \mu_a(\xi_a) \), for a certain (unique) \( \xi_a \in \mathbb{R} \).

Here, \( \Theta_0 \) is the de Gennes constant defined as the bottom of the spectrum of the magnetic Neumann realization of the Schrödinger operator \(- (\nabla - iA)^2\), with a unit magnetic field (\( \text{curl} A = 1 \)), on the half-plane (see e.g. [18])

\[
\Theta_0 = \inf \text{sp}[- (\nabla - iA)^2] \approx 0.59.
\]

(2.4)

Remark 2.1 (The value \( \beta_a \) as the bottom of spectrum of a Schrödinger operator on \( \mathbb{R}^3 \)). Consider the following Schrödinger operator on \( \mathbb{R}^3 \)

\[
\mathcal{L}_a := -(\nabla - iA_a)^2,
\]

(2.5)

with domain

\[
D(\mathcal{L}_a) := \{ u \in L^2(\mathbb{R}^3) : (\nabla - iA_a)_j u \in L^2(\mathbb{R}^3), \text{ for } j \in \{1, 2\} \},
\]

(2.6)

where \( A_a \in H^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3) \) is a magnetic potential such that the corresponding magnetic field has a piecewise-constant strength equal to \( 1_{\{x_2 > 0\}} + a 1_{\{x_2 < 0\}} \). One can easily show that \( \beta_a \) is equal to the bottom of the spectrum of the operator \( \mathcal{L}_a \).

2.2. An operator with a constant field on the half-space. Let \( \nu \in [0, \pi/2] \). We introduce the following magnetic field with a unit strength on \( \mathbb{R}^3_+ \)

\[
B_\nu = (0, \sin \nu, \cos \nu),
\]

and an associated magnetic potential \( A_\nu \in H^1_{\text{loc}}(\mathbb{R}^3_+, \mathbb{R}^3) \), (\( \text{curl} A_\nu = B_\nu \)). Note that \( B_\nu \) makes an angle \( \nu \) with the \((x_1, x_3)\) plane.

Now, we consider the magnetic Neumann realization of the following self-adjoint operator on the half space

\[
H_\nu = -(\nabla - iA_\nu)^2 \text{ in } L^2(\mathbb{R}^3_+),
\]

(2.7)

We denote by \( \zeta_\nu \) the bottom of the spectrum of \( H_\nu \),

\[
\zeta_\nu = \inf \text{sp}(H_\nu).
\]

(2.8)
We present the following useful properties of $\zeta$ (see e.g. [29–31]):

$$
\zeta_0 = \Theta_0, \quad \zeta_{\pi/2} = 1, \quad \zeta_\nu \in (\Theta_0, 1) \text{ for } \nu \in (0, \pi/2),
$$

(2.9)

where $\Theta_0$ is the de Gennes constant defined in (2.4).

3. The operator with magnetic steps on the half space

Let $a \in [-1, 1) \setminus \{0\}, \alpha \in (0, \pi)$ and $\gamma \in [0, \pi/2]$. We recall the operator $L_{\alpha, \gamma, a}$ with a discontinuous field on $\mathbb{R}^3_+$ introduced in (1.4)

$$
L_{\alpha, \gamma, a} = -\left(\nabla - iA_{\alpha, \gamma, a}\right)^2,
$$

(3.1)

with the domain defined in (1.5) as

$$
D(L_{\alpha, \gamma, a}) = \{u \in L^2(\mathbb{R}^3_+) : \left(\nabla - iA_{\alpha, \gamma, a}\right)^n u \in L^2(\mathbb{R}^3_+),
\quad \text{for } n \in \{1, 2\}, \left(\nabla - iA_{\alpha, \gamma, a}\right)u \cdot (0, 1, 0)|_{\partial \mathbb{R}^3_+} = 0\}.
$$

(3.2)

Using the min-max principle, we write the bottom of the spectrum of $L_{\alpha, \gamma, a}$ as

$$
\lambda_{\alpha, \gamma, a} = \inf_{u \in \text{Dom}Q_{\alpha, \gamma, a}, u \neq 0} \frac{Q_{\alpha, \gamma, a}(u)}{\|u\|^2_{L^2(\mathbb{R}^3_+)}},
$$

(3.3)

where $Q_{\alpha, \gamma, a}$ is the quadratic form associated to the operator $L_{\alpha, \gamma, a}$, defined by

$$
Q_{\alpha, \gamma, a}(u) = \|(\nabla - iA_{\alpha, \gamma, a})u\|^2_{L^2(\mathbb{R}^3_+)}
$$

on the domain

$$
D(Q_{\alpha, \gamma, a}) := \{u \in L^2(\mathbb{R}^3_+) : \left(\nabla - iA_{\alpha, \gamma, a}\right)u \in L^2(\mathbb{R}^3_+)\}.
$$

We also recall the magnetic field introduced in (1.3), and we denote by $b_j$, $j = 1, 2, 3$ its components:

$$
B_{\alpha, \gamma, a} = (\cos \alpha \sin \gamma, \sin \alpha \sin \gamma, \cos \gamma)s_{\alpha, a} =: (b_1, b_2, b_3),
$$

(3.4)

where $s_{\alpha, a} = \mathbb{1}_{\mathcal{D}_1^\alpha} + a \mathbb{1}_{\mathcal{D}_2^\alpha}$ (see Figure 1). Now, we fix the choice of the magnetic potential $A_{\alpha, \gamma, a}$. Let

$$
A_{\alpha, \gamma, a} = (A_1, A_2, A_3)
$$

(3.5)

such that

$$
A_1 = 0
$$

$$
A_2 = \begin{cases} 
\cos \gamma x_1 - (1 - a) \cot \alpha x_2 & \text{for } x \in \mathcal{D}_1^\alpha \\
\alpha \cos \gamma x_1 & \text{for } x \in \mathcal{D}_2^\alpha 
\end{cases}
$$

$$
A_3 = \begin{cases} 
x_2 \cos \alpha \sin \gamma - x_1 \sin \alpha \sin \gamma & \text{for } x \in \mathcal{D}_1^\alpha \\
\alpha(x_2 \cos \alpha \sin \gamma - x_1 \sin \alpha \sin \gamma) & \text{for } x \in \mathcal{D}_2^\alpha 
\end{cases}
$$

This choice of the vector potential guarantees the continuity of $A_{\alpha, \gamma, a}$ at the discontinuity plane $P_\alpha$ (see Figure 1), and consequently that $A_{\alpha, \gamma, a} \in H^1_{\text{loc}}(\mathbb{R}^3_+, \mathbb{R}^3)$. Moreover, with this vector potential, the operator $L_{\alpha, \gamma, a}$ is translation invariant in the $x_3$ variable. Hence, its spectrum is absolutely continuous, and a reduction of the study to a family of 2D operators is allowed as we see below.
3.1. \textbf{A family of reduced 2D operators}. Let $a \in (-1,1) \setminus \{0\}$, $\alpha \in (0, \pi)$, $\gamma \in [0, \pi/2]$. A partial Fourier transform in the $x_3$ variable yields the following decomposition of the operator $\mathcal{L}_{\alpha,\gamma,\tau}$ (see [39])

$$\mathcal{L}_{\alpha,\gamma,\tau} = \int_{\tau \in \mathbb{R}} (\mathcal{L}_{\alpha,\gamma,a} + V_{\mathcal{B}_{\alpha,\gamma,a}}) \, d\tau,$$

(3.6)

where

$$\mathcal{L}_{\alpha,\gamma,a} + V_{\mathcal{B}_{\alpha,\gamma,a}} \tau = -(\nabla - iA_{\alpha,\gamma,a})^2 + V_{\mathcal{B}_{\alpha,\gamma,a}} \tau$$

(3.7)

is a Schrödinger operator on $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$, parametrized by $\tau \in \mathbb{R}$ and such that we have the following.

- The magnetic potential $A_{\alpha,\gamma,a} := (A_1, A_2)$ represents the projection of the vector potential $A_{\alpha,\gamma,a}$ defined in (3.5) on $\mathbb{R}^2_+$, i.e.,

$$A_1 := 0, \quad A_2 := \begin{cases} \cos \gamma x_1 - (1-a) \cos \coth \alpha x_2 & \text{for } (x_1, x_2) \in D_1^a, \\ a \cos \gamma x_1 & \text{for } (x_1, x_2) \in D_2^a, \end{cases}$$

(3.8)

where $D_1^a$ and $D_2^a$ represent respectively the orthogonal projection of the regions $D_1^a$ and $D_2^a$ over the plane $(x_1, x_2)$:

$$D_1^a = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2) = \rho(\cos \theta, \sin \theta), \rho \in (0, \infty), 0 < \theta < \alpha\},$$

(3.9)

$$D_2^a = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2) = \rho(\cos \theta, \sin \theta), \rho \in (0, \infty), \alpha < \theta < \pi\}.$$

(3.10)

Note that $A_{\alpha,\gamma,a}$ satisfies

$$B_{\alpha} := \nabla \mathcal{A}_{\alpha,\gamma,a} = \mathcal{S}_{\alpha,a} \cos \gamma,$$

(3.11)

where $\mathcal{S}_{\alpha,a}$ is the step function defined in $\mathbb{R}^2_+$ by

$$\mathcal{S}_{\alpha,a} = 1_{D_1^a} + a 1_{D_2^a}.$$

(3.12)

- The field $B_{\alpha,\gamma,a}$ is a magnetic field that projects $B_{\alpha,\gamma,a}$ on $\mathbb{R}^2_+$ and it is defined as follows

$$B_{\alpha,\gamma,a} = (b_1, b_2) = (\cos \alpha \sin \gamma, \sin \alpha \sin \gamma) \mathcal{S}_{\alpha,a}.$$

(3.13)

Note that $B_{\alpha,\gamma,a}$ is discontinuous along the line $l_\alpha := P_\alpha \cap \mathbb{R}^2_+$ (see Figure 1), in the following we refer to $l_\alpha$ as the discontinuity line.

- The electric potential $V_{\mathcal{B}_{\alpha,\gamma,a}}$ is defined as

$$V_{\mathcal{B}_{\alpha,\gamma,a}} \tau = (x_1 b_2 - x_2 b_1 - \tau)^2,$$

(3.14)

$$= \begin{cases} (x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma - \tau)^2 & \text{for } (x_1, x_2) \in D_1^a, \\ a(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma - \tau)^2 & \text{for } (x_1, x_2) \in D_2^a. \end{cases}$$

We highlight the dependence of the electric potential on the magnetic field $B_{\alpha,\gamma,a}$.

We introduce the quadratic form associated to $\mathcal{L}_{\alpha,\gamma,a} + V_{\mathcal{B}_{\alpha,\gamma,a}} \tau$:

$$Q_{\alpha,\gamma,a}(u) = \int_{\mathbb{R}^2_+} ((\nabla - iA_{\alpha,\gamma,a})u)^2 + V_{\mathcal{B}_{\alpha,\gamma,a}} \tau |u|^2 \, dx_1 dx_2.$$

(3.15)

The form domain is

$$\mathcal{D}(Q_{\alpha,\gamma,a}) = \{u \in L^2(\mathbb{R}^2_+) : (\nabla - iA_{\alpha,\gamma,a})u \in L^2(\mathbb{R}^2_+), |x_1 b_2 - x_2 b_1| u \in L^2(\mathbb{R}^2_+)\}.$$
We denote by $\sigma(\alpha, \gamma, a, \tau)$ the bottom of the spectrum of the operator $\mathcal{L}_A^{\alpha,\gamma,a} + V_{B^{\gamma,a}}^{0,a,\tau}$.

We have

$$\sigma(\alpha, \gamma, a, \tau) = \inf \text{sp}(\mathcal{L}_A^{\alpha,\gamma,a} + V_{B^{\gamma,a}}^{0,a,\tau}) = \inf_{u \in \mathcal{D}(Q_{\gamma,a}^{\alpha}) \setminus \{0\}} \frac{Q_{\gamma,a}^{\alpha}(u)}{\|u\|^2_{L^2(\mathbb{R}^3_+)}}.$$  \hspace{1cm} (3.16)

Since the form domain is independent of $\tau$, the perturbation theory [27] ensures that the function $\tau \mapsto \sigma(\alpha, \gamma, a, \tau)$ is $C^\infty$. By (3.6), we have

$$\lambda_{\alpha,\gamma,a} = \inf_{\tau} \sigma(\alpha, \gamma, a, \tau).$$  \hspace{1cm} (3.17)

Hence, the study of $\lambda_{\alpha,\gamma,a}$ transforms to that of the associated band function $\tau \mapsto \sigma(\alpha, \gamma, a, \tau)$. This study will be the subject of the next subsections.

### 3.2. Case of a magnetic field parallel to the $x_3$-axis

We first treat the simple case when the magnetic field $B^{\gamma,a}_0 = (0,0,1)_{s,a}$ (i.e. when $\gamma = 0$). In this case, the field is parallel to the $x_3$-axis, thus $B^{0,0,a}_0 = 0$. The operator $\mathcal{L}_A^{\alpha,0,a} + V_{B^{0,0,a}}^{0,0,a,\tau}$ reduces to a simpler operator

$$\mathcal{L}_A^{\alpha,0,a} + V_{B^{0,0,a}}^{0,0,a,\tau} = -(\nabla - i A^{\alpha,0,a})^2 + \tau^2.$$  

For each $\tau \in \mathbb{R}$, the bottom of the spectrum of $\mathcal{L}_A^{\alpha,0,a} + V_{B^{0,0,a}}^{0,0,a,\tau}$ equals

$$\sigma(\alpha, 0, a, \tau) = \mu(\alpha, a) + \tau^2,$$

where $\mu(\alpha, a)$ is the bottom of the spectrum of the operator $\mathcal{L}_A^{\alpha,0,a} = -(\nabla + i A^{\alpha,0,a})^2$. It immediately follows that

$$\lambda_{\alpha,0,a,\mathbb{R}^3_+} = \inf_{\tau} \sigma(\alpha, 0, a, \tau) = \sigma(\alpha, 0, a, 0) = \mu(\alpha, a).$$  \hspace{1cm} (3.18)

We present some properties of the operator $\mathcal{L}_A^{\alpha,0,a}$, that is of $\mathcal{L}_A^{\alpha,0,a} + V_{B^{0,0,a}}^{0,0,a}$, obtained in [2, Section 3]. We denote by $\inf \text{sp}_{\text{ess}}$ the infimum of the essential spectrum. From [2, Theorem 3.1], we know that

$$\inf \text{sp}_{\text{ess}} \mathcal{L}_A^{\alpha,0,a} = \inf \text{sp}_{\text{ess}}(\mathcal{L}_A^{\alpha,0,a} + V_{B^{0,0,a}}^{0,0,a,\tau=0}) = |a| \Theta_0.$$  \hspace{1cm} (3.19)

As a consequence, if $\mu(\alpha, a) < |a| \Theta_0$ then $\mu(\alpha, a)$ is an eigenvalue of $\mathcal{L}_A^{\alpha,0,a} + V_{B^{0,0,a}}^{0,0,a}$. The foregoing properties will be used in the proof of Theorem 1.1, in the case $\gamma = 0$ (see Section 4).

### 3.3. Case of a magnetic field non-parallel to the $x_3$-axis

Now, we treat the case where the magnetic field $B^{\alpha,\gamma,a}$ is not parallel to the $x_3$-axis, that is the case when $\gamma \neq 0$ (see Figure 1). In this case, two auxiliary operators will be involved in the analysis. These operators are denoted by $H^{0,0,a}_{\text{bd}}$ and $H^{0,0,a}_{\text{up}}$ and are respectively defined on $\mathbb{R}^2_+$ and $\mathbb{R}^2$ with a constant (resp. piecewise constant) magnetic field. We refer to $H^{0,0,a}_{\text{bd}}$ as the ‘boundary operator’ since it will be an effective operator used in the proof of Proposition 3.7 while studying the operator $\mathcal{L}_A^{\alpha,\gamma,a} + V_{B^{\gamma,a}}^{\gamma,a,\tau}$ near the boundary of $\mathbb{R}^2_+$ away from the discontinuity line $l_a = P_a \cap \mathbb{R}^2_+$ (see Figure 1). Similarly, we refer to $H^{0,0,a}_{\text{up}}$ as the ‘step operator’ since it will be an effective operator in the study near the discontinuity line away from the boundary (see the proof of Proposition 3.3). We introduce these operators in what follows.
3.3.1. **The boundary operator.** Let $\tau \in \mathbb{R}$. We define $H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]$ as the magnetic Neumann realization of the following self-adjoint operator on $\mathbb{R}^2_+$

$$H_{\alpha,\gamma,a}^{\text{bnd}}[\tau] = - (\nabla - iA_{\alpha,\gamma,a}^{\text{bnd}})^2 + a(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma - \tau)^2,$$

(3.20)

where $A_{\alpha,\gamma,a}^{\text{bnd}} \in H^1_{\text{loc}}(\mathbb{R}^2_+)$ is a magnetic potential with an associated constant magnetic field $\text{curl} A_{\alpha,\gamma,a}^{\text{bnd}} = \cos \gamma$. This operator was studied in [34] in the case $a = 1$. Using translation, it was proven that the infimum of the spectrum of $H_{\alpha,\gamma,1}^{\text{bnd}}[\tau]$ is independent of $\tau$. More precisely, in [34, Lemma 2.3] it is shown that

$$\inf \text{sp}(H_{\alpha,\gamma,1}^{\text{bnd}}[\tau]) = \zeta_0, \quad \forall \tau \in \mathbb{R},$$

(3.21)

where $\zeta_0$ is the value defined in (2.8) for $\nu_0 = \arcsin(\sin \alpha \sin \gamma)$.

**Lemma 3.1** (Bottom of the spectrum of the boundary operator). Let $\alpha \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$ and let $\gamma \in (0, \pi/2]$. Let $\tau \in \mathbb{R}$. It holds

$$\inf \text{sp}(H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]) = |a| \inf \text{sp}(H_{\alpha,\gamma,1}^{\text{bnd}}[\tau]).$$

**Proof.** By a simple scaling argument, one can prove that $\inf \text{sp}(H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]) = |a| \inf \text{sp}(H_{\alpha,\gamma,1}^{\text{bnd}}[\tau])$. Combining this with (3.21) completes the proof. \qed

3.3.2. **The step operator.** Let $\tau \in \mathbb{R}$. We define $H_{\alpha,\gamma,a}^{\text{step}}[\tau]$ as the following self-adjoint operator on $\mathbb{R}^2$

$$H_{\alpha,\gamma,a}^{\text{step}}[\tau] = - (\nabla - iA_{\alpha,\gamma,a}^{\text{step}})^2 + |(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma) s_{\alpha,a}^{\text{step}} - \tau|^2,$$

(3.22)

where $A_{\alpha,\gamma,a}^{\text{step}} \in H^1_{\text{loc}}(\mathbb{R}^2)$ is such that $\text{curl} A_{\alpha,\gamma,a}^{\text{step}} = s_{\alpha,a}^{\text{step}} \cos \gamma$, and $s_{\alpha,a}^{\text{step}}$ is the following step function on $\mathbb{R}^2$

$$s_{\alpha,a}^{\text{step}} := \mathbbm{1}_{P_a^+} + a \mathbbm{1}_{P_a^-},$$

with

$$P_a^+ := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \sin \alpha - x_2 \cos \alpha > 0\},$$

$$P_a^- := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \sin \alpha - x_2 \cos \alpha < 0\}.$$

On can see the magnetic field $\text{curl} A_{\alpha,\gamma,a}^{\text{step}}$ in $\mathbb{R}^2$ as the analogous of the magnetic field $\text{curl} A_{\alpha,\gamma,a}$ in $\mathbb{R}^2_+$, defined in (3.11), with the sets $P_a^+$ and $P_a^-$ as the analogous of the sets $D_{\alpha}^1$ (in (3.9)) and $D_{\alpha}^2$ (in (3.10)) respectively.

The next lemma determines the infimum of the spectrum of $H_{\alpha,\gamma,a}^{\text{step}}[\tau]$.

**Lemma 3.2** (Bottom of the spectrum of the step operator). Let $\alpha \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$ and let $\gamma \in (0, \pi/2]$. Let $\tau \in \mathbb{R}$. It holds

$$\inf \text{sp}(H_{\alpha,\gamma,a}^{\text{step}}[\tau]) = \inf_{\xi \in \mathbb{R}} [\mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2],$$

where $\mu_a(\cdot)$ is the value defined in (2.3).

**Proof.** For simplicity, we denote $H_{\alpha,\gamma,a}^{\text{step}}[\tau]$, $A_{\alpha,\gamma,a}^{\text{step}}$ and $s_{\alpha,a}^{\text{step}}$ by $H^{\text{step}}$, $A^{\text{step}}$ and $s^{\text{step}}$ respectively. To estimate the bottom of the spectrum of $H^{\text{step}}$, we perform a rotation of angle $\alpha$ and get that the operator $H^{\text{step}}$ is unitarily equivalent to the following operator

$$H^{\text{step}} := - (\nabla - iA^{\text{step}})^2 + (x_2 \sin \gamma s^{\text{step}} + \tau)^2$$

defined on $\mathbb{R}^2$, with $\text{curl} \hat{A}^{\text{step}} = s^{\text{step}} \cos \gamma$ and $s^{\text{step}} := \mathbbm{1}_{\{x_2 < 0\}} + a \mathbbm{1}_{\{x_2 > 0\}}$. Thus, we get

$$\inf \text{sp}(H^{\text{step}}) = \inf \text{sp}(H^{\text{step}}).$$

(3.23)

\footnote{We refer to [34, Sec.1] for rotation invariance principles.}
Performing a suitable change of gauge, we choose \( \hat{A}^{\text{st}} = -(x_2 \cos \gamma \varepsilon^{\text{st}}) \). Then, we write the expression of \( \hat{H}^{\text{st}} \) explicitly as

\[
\hat{H}^{\text{st}} = -\left( \partial_{x_1} + ix_2 \cos \gamma \varepsilon^{\text{st}} \right)^2 - \partial_{x_2}^2 + (x_2 \sin \gamma \varepsilon^{\text{st}} + \tau)^2.
\]

By a Fourier transform in the \( x_1 \) variable, we get

\[
\hat{H}^{\text{st}} = \int_{\xi \in \mathbb{R}} \left( -\partial_{\xi}^2 + (\xi + x_2 \cos \gamma \varepsilon^{\text{st}})^2 + (x_2 \sin \gamma \varepsilon^{\text{st}} + \tau)^2 \right) d\xi, \tag{3.24}
\]

where \( -\partial_{\xi}^2 + (\xi + x_2 \cos \gamma \varepsilon^{\text{st}})^2 + (x_2 \sin \gamma \varepsilon^{\text{st}} + \tau)^2 \) is a self-adjoint fiber operator on \( \mathbb{R} \). Hence

\[
\inf \text{sp}(\hat{H}^{\text{st}}) = \inf \left[ \inf \text{sp}( -\partial_{\xi}^2 + (\xi + x_2 \cos \gamma \varepsilon^{\text{st}})^2 + (x_2 \sin \gamma \varepsilon^{\text{st}} + \tau)^2) \right].
\]

We can now rewrite

\[
(\xi + s_x^{\text{rot}} x_2 \cos \gamma)^2 + (s_x^{\text{rot}} x_2 \sin \gamma + \tau)^2 = (s_x^{\text{rot}} x_2 + \tau \sin \gamma + \xi \cos \gamma)^2 + (\xi \sin \gamma - \tau \cos \gamma)^2.
\]

Then using that \( \varepsilon^{\text{st}} = \mathbb{1}_{\{x_2 < 0\}} + a \mathbb{1}_{\{x_2 > 0\}} \), the fiber operator in (3.24) is unitary equivalent to the operator given by

\[
h_a[\tau \sin \gamma + \xi \cos \gamma] + (\xi \sin \gamma - \tau \cos \gamma)^2,
\]

where \( h_a[\cdot] \) is the operator defined in (2.1). Thus,

\[
\inf \text{sp}(\hat{H}^{\text{st}}) = \inf \text{sp}(h_a[\tau \sin \gamma + \xi \cos \gamma] + (\xi \sin \gamma - \tau \cos \gamma)^2). \tag{3.25}
\]

Moreover, we have

\[
\inf \text{sp}(h_a[\tau \sin \gamma + \xi \cos \gamma] + (\xi \sin \gamma - \tau \cos \gamma)^2) = \inf \text{sp}(h_a[\tau \sin \gamma + \xi \cos \gamma]) + (\xi \sin \gamma - \tau \cos \gamma)^2 = \mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2, \tag{3.26}
\]

where \( \mu_a(\cdot) \) is the bottom of the spectrum of \( h_a[\tau \sin \gamma + \xi \cos \gamma] \) (see (2.3)). Gathering (3.23), (3.25) and (3.26) completes the proof.

3.3.3. Bottom of the essential spectrum of the 2D reduced operator. In this section, we determine the infimum of the essential spectrum of the 2D operators \( L_{\mathcal{A}_{\alpha,\gamma,a},\tau} + V_{\mathcal{B}_{\alpha,\gamma,a},\tau} \) introduced in Section 3.1. For each \( a \in [-1,1] \setminus \{0\}, \gamma \in (0,\pi/2), \alpha \in (0,\pi) \) and \( \tau \in \mathbb{R} \), let

\[
\Sigma_{\text{ess}}(\alpha, \gamma, a, \tau) := \inf \text{sp}_{\text{ess}}(L_{\mathcal{A}_{\alpha,\gamma,a},\tau} + V_{\mathcal{B}_{\alpha,\gamma,a},\tau}). \tag{3.27}
\]

Knowing this infimum will be useful in determining values of \( (\alpha, \gamma, a, \tau) \) where the bottom of the spectrum \( \Sigma(\alpha, \gamma, a, \tau) \) of these operators is an eigenvalue. This will be used in establishing Theorem 1.1 later. The next proposition is the main result of this section.

**Proposition 3.3** (Characterization of \( \Sigma_{\text{ess}}(\alpha, \gamma, a, \tau) \)). Let \( a \in [-1,1] \setminus \{0\}, \alpha \in (0,\pi), \gamma \in (0,\pi/2) \) and \( \tau \in \mathbb{R} \). Let \( \Sigma_{\text{ess}}(\alpha, \gamma, a, \tau) \) be as in (3.27), we have

\[
\Sigma_{\text{ess}}(\alpha, \gamma, a, \tau) = \inf_{\xi \in \mathbb{R}} \mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2
\]

where \( \mu_a(\cdot) \) is the value defined in (2.3).

For the proof of Proposition 3.3 we need the following lemma.

**Lemma 3.4.** Let \( a \in [-1,1] \setminus \{0\}, \alpha \in (0,\pi), \gamma \in (0,\pi/2) \) and \( \tau \in \mathbb{R} \). Let \( \Sigma_{\text{ess}}(\alpha, \gamma, a, \tau) \) be as in (3.27). It holds

\[
\Sigma_{\text{ess}}(\alpha, \gamma, a, \tau) = \lim_{R \to +\infty} \Sigma(L_{\mathcal{A}_{\alpha,\gamma,a},\tau} + V_{\mathcal{B}_{\alpha,\gamma,a},\tau}, R),
\]
with
\[ \Sigma(L_{\alpha,\gamma,a} + V_{B_{\alpha,\gamma,a,\tau}}, \mathcal{B}_R) := \inf_{u \in C^\infty_0(\mathbb{R}^2_+) \cap \mathcal{B}_R} \frac{\sqrt{r}_{\alpha,\gamma,a}(u)}{\|u\|_{L^2(\mathbb{R}^2_+)}}, \]
where \( \mathcal{B}_R \) is a ball of radius \( R \) centered at the origin, \( \mathcal{B}_R^c \) is its complement in \( \mathbb{R}^2 \), and \( \sqrt{r}_{\alpha,\gamma,a} \) is the quadratic form defined in (3.15).

Lemma 3.4 is a well-known Persson-type result, useful to characterize the bottom of essential spectra. We refer the reader to [1,33,34] for this type of results, and [2, Appendix A] for a detailed proof in similar situations. Moreover, in the proof of Proposition 3.3, we shall see the importance of determining where the electric potential \( V_{B_{\alpha,\gamma,a,\tau}} \) attains its infimum and where it is big. To that end, we define the set
\[ \mathcal{Y}_{\alpha,\gamma,a,\tau} = \left\{ x \in \mathbb{R}^2_+ : V_{B_{\alpha,\gamma,a,\tau}}(x) = \inf_{y \in \mathbb{R}^2_+} V_{B_{\alpha,\gamma,a,\tau}}(y) \right\}. \] (3.28)
We note that \( \mathcal{Y}_{\alpha,\gamma,a,\tau} \) is not necessary \( V_{B_{\alpha,\gamma,a,\tau}}^{-1}(\{0\}) \); determining this set depends on the values of \( a \in [-1, 1) \setminus \{0\} \) and \( \tau \in \mathbb{R} \), as shown in what follows. We recall that \( V_{B_{\alpha,\gamma,a,\tau}} \) is defined for \( x = (x_1, x_2) \in \mathbb{R}^2_+ \) as
\[ V_{B_{\alpha,\gamma,a,\tau}}(x) = (x_1 b_2 - x_2 b_1 - \tau)^2, \]
where \( D^1_\alpha \) and \( D^2_\alpha \) are as in (3.9) and (3.10). We now define, for \( x = (x_1, x_2) \in \mathbb{R}^2 \),
\[ V_{B_{\alpha,\gamma,a,\tau}}^{(1)}(x) = (x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma - \tau)^2 \]
and the following subsets of \( \mathbb{R}^2 \)
\[ \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(1)} = \left( V_{B_{\alpha,\gamma,a,\tau}}^{(1)} \right)^{-1}(\{0\}) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \sin \alpha - x_2 \cos \alpha = \frac{\tau}{\sin \gamma} \right\}, \]
\[ \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(2)} = \left( V_{B_{\alpha,\gamma,a,\tau}}^{(2)} \right)^{-1}(\{0\}) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \sin \alpha - x_2 \cos \alpha = \frac{\tau}{\cos \gamma} \right\}. \] (3.29)
Note that \( \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(1)} \) and \( \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(2)} \) are two lines parallel to the discontinuity line \( l_\alpha \) of equation \( x_1 \sin \alpha - x_2 \cos \alpha = 0 \). Moreover, for \( x \in \mathbb{R}^2 \)
\[ V_{B_{\alpha,\gamma,a,\tau}}^{(1)}(x) = \sin^2 \gamma \text{ dist}^2(x, \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(1)}), \]
(3.30)
We keep denoting by \( \mathcal{Y}_{\alpha,\gamma,a,\tau} \) (resp. \( \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(2)} \)) the intersection between \( \mathbb{R}^2_+ \) and \( \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(1)} \) (resp. \( \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(2)} \)).

**Lemma 3.5** (The set \( \mathcal{Y}_{\alpha,\gamma,a,\tau} \)). Let \( a \in [-1, 1) \setminus \{0\} \), \( \alpha \in (0, \pi) \), and \( \gamma \in (0, \pi/2) \). Let \( \mathcal{Y}_{\alpha,\gamma,a,\tau} \subset \mathbb{R}^2 \) be the set defined in (3.28). It holds
\[ \mathcal{Y}_{\alpha,\gamma,a,\tau} = \begin{cases} l_\alpha & \text{if } a \in [-1, 0), \tau < 0 \\ \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(1)} \cup \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(2)} & \text{if } a \in [-1, 0), \tau \geq 0 \\ \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(1)} & \text{if } a \in (0, 1), \tau \geq 0 \\ \mathcal{Y}_{\alpha,\gamma,a,\tau}^{(2)} & \text{if } a \in (0, 1), \tau < 0. \end{cases} \]
Indeed (see Figures (3) and (4)).
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\[ \tau < 0 \]

\[ \tau \geq 0 \]

Figure 3. For \( \alpha \in (0, \pi) \), \( \gamma \in (0, \pi/2] \) and \( a \in [-1,0] \), the set \( \Upsilon_{\alpha,\gamma,a,\tau} \) is drawn in blue. For \( \tau \geq 0 \) (at right), \( \Upsilon_{\alpha,\gamma,a,\tau} = \Upsilon_{\alpha,\gamma,\tau}^{(1)} \cup \Upsilon_{\alpha,\gamma,a,\tau}^{(2)} \). For \( \tau < 0 \) (at left), \( \Upsilon_{\alpha,\gamma,a,\tau} = l_\alpha \).

Figure 4. For \( \alpha \in (0, \pi) \), \( \gamma \in (0, \pi/2] \) and \( a \in (0,1) \), the set \( \Upsilon_{\alpha,\gamma,a,\tau} \) is drawn in blue. For \( \tau \geq 0 \) (at right), \( \Upsilon_{\alpha,\gamma,a,\tau} = \Upsilon_{\alpha,\gamma,\tau}^{(1)} \cup \Upsilon_{\alpha,\gamma,a,\tau}^{(2)} \). For \( \tau < 0 \) (at left), \( \Upsilon_{\alpha,\gamma,a,\tau} = \Upsilon_{\alpha,\gamma,a,\tau}^{(2)} \).

- **Case** \( a \in [-1,0) \) and \( \tau < 0 \). One observes that \( V^{-1}_{B_{\alpha,\gamma,a,\tau}}(\{0\}) = \emptyset \). In this case,
  \[ \Upsilon_{\alpha,\gamma,a,\tau} = l_\alpha \quad \text{and} \quad \inf V_{B_{\alpha,\gamma,a,\tau}} = \tau^2. \]  \hfill (3.31)

- **Case** \( a \in [-1,0) \) and \( \tau \geq 0 \). Here,
  \[ \Upsilon_{\alpha,\gamma,a,\tau} = V^{-1}_{B_{\alpha,\gamma,a,\tau}}(\{0\}) = \Upsilon_{\alpha,\gamma,\tau}^{(1)} \cup \Upsilon_{\alpha,\gamma,a,\tau}^{(2)} \].
  Note that this set is \( l_\alpha \) for \( \tau = 0 \).

- **Case** \( a \in (0,1) \) and \( \tau < 0 \). In this case,
  \[ \Upsilon_{\alpha,\gamma,a,\tau} = V^{-1}_{B_{\alpha,\gamma,a,\tau}}(\{0\}) = \Upsilon_{\alpha,\gamma,a,\tau}^{(2)}. \]

- **Case** \( a \in (0,1) \) and \( \tau \geq 0 \). We have
  \[ \Upsilon_{\alpha,\gamma,a,\tau} = V^{-1}_{B_{\alpha,\gamma,a,\tau}}(\{0\}) = \Upsilon_{\alpha,\gamma,\tau}^{(1)}. \]

Again, this set is \( l_\alpha \) for \( \tau = 0 \).

Now, we prove Proposition 3.3.

**Proof of Proposition 3.3.** The idea of the proof is similar to that in [34, Proposition 3.2] (see also [2, Lemma 3.7]). However, one has to take into consideration the particular
We establish separately an upper bound and a lower bound for the limit above. Moreover, there exists a non-zero normalized function in $\mathcal{A}_{\alpha,\gamma,a}$ used frequently in the proof. Throughout the proof, we simplify their notation and denote them respectively by $H^{\text{stp}}$ and $Q$. 

In light of Lemma 3.4, it suffices to prove that

\[
\lim_{R \to +\infty} \Sigma(\mathcal{L}_{\alpha,\gamma,a} + \mathcal{V}_{\alpha,\gamma,a}, R) = \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + \left( \xi \sin \gamma - \tau \cos \gamma \right)^2 \right). \quad (3.32)
\]

We establish separately an upper bound and a lower bound for the limit above.

**Upper bound.** Let $\epsilon > 0$ and $R > 0$. Considering the operator $H^{\text{stp}}$, the min-max principle ensures the existence of a normalized function $u_{\epsilon}$, where

\[
\langle H^{\text{stp}} u_{\epsilon}, u_{\epsilon} \rangle < \inf_{\xi \in \mathbb{R}} \text{sp}(H^{\text{stp}}) + \epsilon \tag{3.33}
\]

where the last equality follows from Lemma 3.2. Let the function $u_{\epsilon,r}$ be the translation of $u_{\epsilon}$ by a vector $r$, i.e., $u_{\epsilon,r}(x) = u_{\epsilon}(x-r)$ for $x \in \mathbb{R}^2$, where $r$ is an upward direction vector of the discontinuity line $l_{\alpha}$. We have $u_{\epsilon,r} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. Moreover, there exists $r_0 > 0$ such that the function $u_{\epsilon,r}$ is supported in $\mathbb{R}^2_+ \cap \mathcal{B}_R$, for $|r| > r_0$. Using that $H^{\text{stp}}$ is invariant by translation in the $l_{\alpha}$ direction (see (3.32)), we get

\[
\langle H^{\text{stp}} u_{\epsilon,r}, u_{\epsilon,r} \rangle = \langle H^{\text{stp}} u_{\epsilon}, u_{\epsilon} \rangle. \tag{3.34}
\]

Combining (3.33) and (3.34) gives

\[
\langle H^{\text{stp}} u_{\epsilon,r}, u_{\epsilon,r} \rangle < \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + \left( \xi \sin \gamma - \tau \cos \gamma \right)^2 \right) + \epsilon.
\]

Now, using the support properties of $u_{\epsilon,r}$, a direct calculation shows that $\langle H^{\text{stp}}(u_{\epsilon,r}), u_{\epsilon,r} \rangle = \tilde{Q}(u_{\epsilon,r})$. Then,

\[
\tilde{Q}(u_{\epsilon,r}) < \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + \left( \xi \sin \gamma - \tau \cos \gamma \right)^2 \right) + \epsilon.
\]

Having $u_{\epsilon,r}$ a non-zero normalized function in $C_0^\infty(\mathbb{R}^2_+ \cap \mathcal{B}_R)$, we have

\[
\Sigma(\mathcal{L}_{\alpha,\gamma,a} + \mathcal{V}_{\alpha,\gamma,a}, R) = \inf_{u \in C_0^\infty(\mathbb{R}^2_+ \cap \mathcal{B}_R)} \frac{Q(u)}{\|u\|_{L^2(\mathbb{R}^2_+)}^2} < \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + \left( \xi \sin \gamma - \tau \cos \gamma \right)^2 \right) + \epsilon.
\]

Taking first $\epsilon \to 0$ and then $R \to +\infty$, we get the upper bound in (3.32).

**Lower bound.** Let $(\rho, \theta)$ be the polar coordinates in $\mathbb{R}^2$. We consider a partition of unity $(\chi_j^{\text{pol}})_{j=1,2,3} \subset C^\infty(\mathbb{R}_+ \times [0, \pi])$ such that: for $j \in \{1, 2, 3\}$, $0 \leq \chi_j^{\text{pol}} \leq 1$ and $\forall (\rho, \theta) \in \mathbb{R}_+ \times (0, \pi)$, $\chi_j^{\text{pol}}(\rho, \theta) = \chi_j^{\text{pol}}(1, \theta)$ and

\[
\chi_1^{\text{pol}}(\rho, \theta) = \begin{cases} 1 & \text{for } \theta \in \left(0, \frac{1}{8} \alpha\right], \\ 0 & \text{otherwise} \end{cases},
\]

\[
\chi_2^{\text{pol}}(\rho, \theta) = \begin{cases} 1 & \text{for } \theta \in \left[\frac{1}{4} \alpha, \frac{1}{4} \alpha + \frac{3\pi}{4}\right], \\ 0 & \text{otherwise} \end{cases},
\]

\[
\chi_3^{\text{pol}}(\rho, \theta) = \begin{cases} 1 & \text{for } \theta \in \left[\frac{1}{8} \alpha + \frac{7\pi}{8}, \pi\right], \\ 0 & \text{otherwise} \end{cases}.
\]

Moreover, $\sum_{j=1}^3 |\chi_j^{\text{pol}}|^2 = 1$ and $\sum_{j=1}^3 |\chi_j^{\text{pol}}|^2 \leq C$, where $C$ is a constant dependent on $\alpha$ but independent of $a$. Let $(\chi_j)_{j=1,\ldots,3}$ be the associated functions in Cartesian coordinates

\[
\chi_j(x_1, x_2) = \chi_j^{\text{pol}}(\rho, \theta), \quad (x_1, x_2) \in \mathbb{R}^2_+.
\]
For $R > 0$ and $u \in C^\infty_0(\mathbb{R}^2_+ \cap \overline{B}_R)$, we use the IMS formula to write (see [10, Theorem 3.2])

$$Q(u) = \sum_{j=1}^3 Q(\chi_j u) - \sum_{j=1}^3 \|u|\nabla \chi_j\|^2_{L^2(\mathbb{R}^2_+)}.$$  

(3.35)

We start by bounding the error term $\sum_{j=1}^3 \|u|\nabla \chi_j\|^2_{L^2(\mathbb{R}^2_+)}$. For $x = (x_1, x_2) \in \mathbb{R}^2_+$, we have

$$|\nabla_x \chi_j(x_1, x_2)|^2 = |\partial_r \chi_j^{pol}(\rho, \theta)|^2 + \frac{1}{r^2} |\partial_\theta \chi_j^{pol}(\rho, \theta)|^2 = \frac{1}{r^2} |\partial_\theta \chi_j^{pol}(\rho, \theta)|^2,$$

where the last equality follows from the fact that $\chi_j^{pol}$ is constant in the radial coordinate. Thus, using $\sum_{j=1}^3 |\chi_j^{pol}|^2 \leq C$ and that $u$ is supported outside $B_R$, we get

$$\sum_{j=1}^3 \|u|\nabla \chi_j\|^2_{L^2(\mathbb{R}^2_+)} \leq \frac{C}{R^2} \|u\|^2_{L^2(\mathbb{R}^2_+)}.$$

Next, we consider the main term, $\sum_{j=1}^3 Q(\chi_j u)$, in (3.35). We start by bounding $Q(\chi_2 u)$. Extending $\chi_2 u$ by zero over $\mathbb{R}^2$, we get that $\chi_2 u$ is in the domain of the operator $H^{\text{stp}}$. By Lemma 3.2, we notice that

$$Q(\chi_2 u) = \langle H^{\text{stp}}(\chi_2 u), \chi_2 u \rangle \geq \inf_{\xi \in \mathbb{R}} (\mu_\alpha (\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2) \|\chi_2 u\|^2.$$  

(3.36)

Now, we bound $Q(\chi_j u)$ for $j = 1, 3$. Here, we recall the sets $\Upsilon_{\alpha, \gamma, \tau}^{(1)}$ and $\Upsilon_{\alpha, \gamma, a, \tau}^{(2)}$ defined in (3.29). We choose a large $R_0 > 0$ and assume w.l.o.g that $\alpha \in (0, \pi/2)$, then an elementary computation yields for $R > R_0$ (see Figure 5):

$$\text{dist}(\text{supp}\chi_1 u, \Upsilon^{(1)}_{\alpha, \gamma, \tau}) \geq \frac{1}{|\sin \gamma|} |R \sin \left(\frac{3\alpha}{4}\right) - \tau|$$

and

$$\text{dist}(\text{supp}\chi_3 u, \Upsilon^{(2)}_{\alpha, \gamma, a, \tau}) \geq \frac{1}{|a \sin \gamma|} |aR \sin \alpha - \sin \gamma + \tau|.$$  

Hence, using the support properties of $\chi_1$, the definition of $V_{B_{\alpha, \gamma, \tau}}$ in (3.42), and (3.30)
we get for all \( x \in \text{supp} \chi_1 u \)

\[
V_{B_{a,\gamma,a,\tau}}(x) = V_{B_{a,\gamma,a,\tau}}^{(1)}(x) = \sin^2 \gamma \text{dist}^2(x, \mathcal{V}_{a,\gamma,a,\tau}^{(1)}) \geq R \sin \left( \frac{3\alpha}{4} \right) \sin \gamma - \tau^2.
\]

Similarly, using (3.30), we have for all \( x \in \text{supp} \chi_3 u \)

\[
V_{B_{a,\gamma,a,\tau}}(x) = V_{B_{a,\gamma,a,\tau}}^{(2)}(x) = a^2 \sin^2 \gamma \text{dist}^2(x, \mathcal{V}_{a,\gamma,a,\tau}^{(2)}) \geq |aR \sin \alpha \sin \gamma + \tau|^2.
\]

Thus, we can write

\[
Q(\chi_1 u) \geq R \sin \left( \frac{3\alpha}{4} \right) \sin \gamma - \tau ||\chi_1 u||^2 \quad Q(\chi_3 u) \geq |aR \sin \alpha \sin \gamma + \tau|^2 ||\chi_3 u||^2.
\]

Consequently for all \( R > R_0 \), (3.35), (3.36) and (3.37) imply

\[
\Sigma(\mathcal{L}_{A_{a,\gamma,a}} + V_{B_{a,\gamma,a,\tau}}, R) \geq \inf_{\xi \in \mathbb{R}} (\mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2) - \frac{C}{R^2}.
\]

Now, we state an immediate consequence of Proposition 3.3.

**Corollary 3.6.** For \( a \in [-1, 1) \setminus \{0\}, \alpha \in (0, \pi), \) and \( \gamma \in (0, \pi/2) \). Let \( \sigma_{\text{ess}}(\alpha, \gamma, a, \tau) \) be as in (3.27), we have

\[
\inf_{\tau \in \mathbb{R}} \sigma_{\text{ess}}(\alpha, \gamma, a, \tau) \geq \beta_a,
\]

where \( \beta_a \) is the value defined in (2.2).

**Proof.** By the definition of \( \beta_a \), we have

\[
\sigma_{\text{ess}}(\alpha, \gamma, a, \tau) = \inf_{\xi} (\mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2) \geq \inf_{\xi} \mu_a(\tau \sin \gamma + \xi \cos \gamma) + \inf_{\xi} (\xi \sin \gamma - \tau \cos \gamma)^2 \geq \beta_a.
\]

\[
\square
\]

3.3.4. **Bottom of the spectrum of the 2D reduced operator at infinity.** Now, we consider the bottom of the spectrum, \( \underline{\sigma}(\alpha, \gamma, a, \tau) \), of the operator \( \mathcal{L}_{A_{a,\gamma,a}} + V_{B_{a,\gamma,a,\tau}} \) as a function of \( \tau \).

In Proposition 3.7 below, we study the behavior of \( \underline{\sigma}(\alpha, \gamma, a, \tau) \) as \( |\tau| \) goes to infinity. We will use this proposition to provide a condition on \( (\alpha, \gamma, a) \) such that \( \inf_{\tau} \underline{\sigma}(\alpha, \gamma, a, \tau) \)—that is \( \lambda_{a,\gamma,a} \) (see (3.17))—is attained by some \( \tau \in \mathbb{R} \). This, together with the upper bound of the essential spectrum in Corollary 3.6, will be used to get the result in Theorem 1.1, when the strict inequality in (1.9) is satisfied.

**Proposition 3.7** (Characterization of \( \underline{\sigma}(\alpha, \gamma, a, \tau) \)). Let \( \alpha \in (0, \pi) \) and \( \gamma \in (0, \pi/2] \). For \( a \in [-1, 0) \), we have

\[
\lim_{\tau \to -\infty} \underline{\sigma}(\alpha, \gamma, a, \tau) = +\infty, \quad \lim_{\tau \to +\infty} \underline{\sigma}(\alpha, \gamma, a, \tau) = |a| \zeta_{\nu_0}.
\]

For \( a \in (0, 1) \), we have

\[
\lim_{\tau \to -\infty} \underline{\sigma}(\alpha, \gamma, a, \tau) = a \zeta_{\nu_0}, \quad \lim_{\tau \to +\infty} \underline{\sigma}(\alpha, \gamma, a, \tau) = \zeta_{\nu_0}.
\]

Here, \( \zeta_{\nu_0} \) is defined in (2.8) for \( \nu_0 = \arcsin(\sin \alpha \sin \gamma) \).

**Proof.** In this proof, we simplify the notation and write \( H_{a,\gamma,a}^{\text{bd}} \) for the ‘boundary operator’ \( H_{a,\gamma,a}^{\text{bd}}[\tau] \) in (3.21) and \( \overline{Q} \) for the quadratic form \( Q_{a,\gamma,a}^{\tau} \) in (3.15) associated to the operator \( \mathcal{L}_{A_{a,\gamma,a}} + V_{B_{a,\gamma,a,\tau}} \).
Case $a \in [-1,0]$. Establishing the limit when $\tau \to -\infty$ is straightforward. Indeed, considering the electric potential in (3.14), by (3.31) we have $\inf V_{B_0,\gamma,a,\tau} = \tau^2$, for any $\tau < 0$. Then, $\lim_{\tau \to -\infty} \inf V_{B_0,\gamma,a,\tau} = +\infty$. Hence,

$$\lim_{\tau \to -\infty} \mathcal{g}(\alpha, \gamma, a, \tau) = +\infty.$$ 

Now, we treat the case $\tau \to +\infty$. Here, the foregoing operator $H^{\text{bnd}}$ in (3.20) will be involved. By the min-max principle and Lemma 3.1, for any $\epsilon > 0$, there exists a normalized function $u_\epsilon \in C^\infty_0 (\mathbb{R}^2) \backslash \{0\}$ such that

$$\langle H^{\text{bnd}} u_\epsilon, u_\epsilon \rangle < |a| \zeta_0 + \epsilon. \quad (3.38)$$

We define the function $u_{\epsilon,\tau}$ as follows

$$u_{\epsilon,\tau}(x) = u_\epsilon \left( x_1 - \frac{\tau}{a \sin \alpha \sin \gamma} x_2 \right), \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$ 

For a sufficiently large $\tau$, we have $\operatorname{supp} u_{\epsilon,\tau} \subset D^2_\alpha$, where $D^2_\alpha$ is the set in (3.10). Performing a suitable change of gauge, in which we associate the function $\tilde{u}_{\epsilon,\tau}$ to the function $u_{\epsilon,\tau}$, we get

$$Q(\tilde{u}_{\epsilon,\tau}) = \langle (\mathcal{L}_{\Delta_{\alpha,\gamma,a}} + V_{B_0,\gamma,a,\tau}) \tilde{u}_{\epsilon,\tau}, \tilde{u}_{\epsilon,\tau} \rangle = \langle H^{\text{bnd}} u_{\epsilon,\tau}, u_{\epsilon,\tau} \rangle = \langle H^{\text{bnd}} u_\epsilon, u_\epsilon \rangle \leq |a| \zeta_0 + \epsilon,$$

where in the last inequality we used (3.38). Taking $\tau$ to $+\infty$, we get

$$\lim_{\tau \to +\infty} \sup_{\tau} \mathcal{g}(\alpha, \gamma, a, \tau) \leq |a| \zeta_0.$$ 

Next, we establish the lower bound for $\lim_{\tau \to +\infty} \mathcal{g}(\alpha, \gamma, a, \tau)$. We consider a partition of unity $(\tilde{\chi}_j)_{j \in \{1,2,3\}} \subset C^\infty(\mathbb{R})$ satisfying

$$\operatorname{supp} \tilde{\chi}_1 \subset \left( \frac{1}{4 \sin \gamma}, +\infty \right), \quad \operatorname{supp} \tilde{\chi}_2 \subset \left( \frac{1}{2 \alpha \sin \gamma}, \frac{1}{2 \sin \gamma} \right), \quad \operatorname{supp} \tilde{\chi}_3 \subset \left( -\infty, \frac{1}{4 \alpha \sin \gamma} \right)$$

such that

$$\sum_j |\tilde{\chi}_j|^2 = 1, \quad \sum_j |\tilde{\chi}'_j|^2 \leq C,$$

for a certain $C > 0$ independent of $\tau$. Let $(\chi_j)_{j \in \{1,2,3\}} \subset C^\infty(\mathbb{R}^2)$ be the partition of unity of $\mathbb{R}^2$ induced from $(\tilde{\chi}_j)_{j \in \{1,2,3\}}$ as follows

$$\chi_j(x_1, x_2) = \tilde{\chi}_j \left( \frac{x_1 \sin \alpha - x_2 \cos \alpha}{\tau} \right).$$

Consequently, we have for $j \in \{1,2,3\}$

$$\operatorname{supp} \chi_j \subset R_j, \quad \sum_j |\chi_j|^2 = 1, \quad \text{and} \quad \sum_j |\nabla \chi_j|^2 \leq \frac{C}{\tau^2},$$

where

$$R_1 := \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \sin \alpha - x_2 \cos \alpha > \frac{\tau}{4 \sin \gamma} \},$$

$$R_2 := \{ (x_1, x_2) \in \mathbb{R}^2 : \frac{\tau}{2 \alpha \sin \gamma} < x_1 \sin \alpha - x_2 \cos \alpha < \frac{\tau}{2 \sin \gamma} \},$$

$$R_3 := \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \sin \alpha - x_2 \cos \alpha < \frac{\tau}{4 \alpha \sin \gamma} \}.$$
Thus, for any $u \in D(Q)$ (see (3.1)), the IMS formula gives
\[ Q(u) = \sum_{j=1}^{3} Q(\chi_j u) - \sum_{j=1}^{3} \|u|\nabla\chi_j\|_{L^2(\mathbb{R}^2)}^2 \geq \sum_{j=1}^{3} Q(\chi_j u) - \frac{C}{\tau^2}. \tag{3.39} \]

We perform a suitable change of gauge and use (3.21), together with the support properties of $\chi_1u$, to get
\[ Q(\chi_1 u) = \int_{\mathbb{R}^2} \left( |(\nabla - iA_{\alpha,\gamma,a})(\chi_1 u)|^2 + V_{B_{\alpha,\gamma,a,\tau}}|\chi_1 u|^2 \right) \, dx_1 dx_2 \geq \zeta_0 \|\chi_1 u\|^2. \tag{3.40} \]

Similarly, considering the support of $\chi_3u$, doing a change of gauge and using Lemma 3.1, we find
\[ Q(\chi_3 u) \geq |a|\zeta_0 \|\chi_3 u\|^2. \tag{3.41} \]

Finally, considering the support of $\chi_2u$, a simple computation using the definition of the electric potential in (3.14) and Lemma 3.5 (see also Figure 3) gives
\[ V_{B_{\alpha,\gamma,a,\tau}} \geq \frac{\tau^2}{4}, \quad \text{for } x \in \text{supp } \chi_2 u. \tag{3.42} \]

Hence, there exists $\tau_0 > 0$ and $M > |a|\zeta_0$ such that for $\tau > \tau_0$
\[ Q(\chi_2 u) \geq M \|\chi_2 u\|^2. \tag{3.43} \]

Implementing (3.40), (3.41) and (3.43) in (3.39), we get for $a \in [-1, 0)$
\[ \lim_{\tau \to +\infty} \inf_{\tau} \sigma(\alpha, \gamma, a, \tau) \geq |a|\zeta_0. \]

Case $a \in (0, 1)$. Adopting a similar approach as above, using Lemma 3.5 for positive values of $a$, one can establish the results of the proposition in this case. We omit further computation details. □

4. PROOF OF THE MAIN RESULTS

**Proof of Theorem 1.1.** The proof in the case $\gamma = 0$, is a direct consequence of the results in Section 3.2. Indeed, from (3.18) and (3.19) it follows that
\[ \lambda_{\alpha,0,a} \leq |a|\Theta_0. \]

Now, from (2.9), we know that $\zeta_0 = \Theta_0$ and having $\beta_a \geq |a|\Theta_0$ (see Section 2.1), we get that
\[ \lambda_{\alpha,0,a} \leq \min(\beta_a, |a|\Theta_0) = \min(\beta_a, |a|\zeta_0). \]

Moreover, it follows from Section 3.2 that if $\lambda_{\alpha,0,a} < \min(\beta_a, |a|\zeta_0)$, then $\lambda_{\alpha,0,a}$ is an eigenvalue of the operator $L_{A_{\alpha,\gamma,a}} + V_{B_{\alpha,\gamma,a}}$, with the particular choice $\tau^* = 0$.

Next, we treat the case $\gamma \neq 0$. We first establish the upper bound of $\lambda_{\alpha,\gamma,a}$ in (1.8). The result is a consequence of Proposition 3.3 and Proposition 3.7, as it is shown below. We have (see (3.17))
\[ \lambda_{\alpha,\gamma,a} = \inf_{\tau} \sigma(\alpha, \gamma, a, \tau), \tag{4.1} \]
where $\sigma(\alpha, \gamma, a, \tau)$ is as in (3.16). We consider the following two cases.

**Case $a \in [-1, 0)$.** From Proposition 3.3, we have
\[ \sigma_{\text{ess}}(\alpha, \gamma, a, \tau) = \inf_{\mu_0} (\mu_0(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2), \]
where $\mu_0(\cdot)$ is introduced in (2.3). Let $\xi_0$ be the unique minimum of $\mu_0(\cdot)$ (see Section 2.1). For $\tau = \xi_0 \sin \gamma$, one can see that $\sigma_{\text{ess}}(\alpha, \gamma, a, \tau)$ is attained by $\xi = \xi_0 \cos \gamma$ and satisfies
\[ \sigma_{\text{ess}}(\alpha, \gamma, a, \xi_0 \sin \gamma) = \mu_0(\xi_0) = \beta_a. \]
This implies that
\[ \sigma(\alpha, \gamma, a, \tau) \leq \beta_a. \]  
(4.2)

Moreover, by Proposition 3.7, we have
\[ \sigma(\alpha, \gamma, a, \tau) \leq |a|\zeta_{\tau_0}. \]  
(4.3)

Combining (4.1)–(4.3) yields (1.8).

**Case** \( a \in (0, 1) \). By Proposition 3.7, we have
\[ \sigma(\alpha, \gamma, a, \tau) \leq a\zeta_{\tau_0}. \]
Moreover, \( \beta_a = a \) for \( a \in (0, 1) \) (see Section 2.1), and \( \zeta_{\tau_0} < 1 \) (see Section 2.2). This yields \( \lambda_{\alpha, \gamma, a} \leq a\zeta_{\tau_0} = \min(\beta_a, a\zeta_{\tau_0}) \).

Now, we consider the case when the strict inequality in (1.9) is satisfied. From Proposition 3.7, we have
\[ \inf_{\tau} \sigma(\alpha, \gamma, a, \tau) = \lambda_{\alpha, \gamma, a} < |a|\zeta_{\tau_0} = \min\left(\lim_{\tau \to -\infty} \sigma(\alpha, \gamma, a, \tau), \lim_{\tau \to +\infty} \sigma(\alpha, \gamma, a, \tau)\right). \]

Hence, \( \inf_{\tau} \sigma(\alpha, \gamma, a, \tau) \) is attained by some \( \tau_s \in \mathbb{R} \). Moreover, by Corollary 3.6 we know that
\[ \lambda_{\alpha, \gamma, a} = \sigma(\alpha, \gamma, a, \tau_s) < \beta_a \leq \sigma_{\text{ess}}(\alpha, \gamma, a, \tau_s). \]

We then deduce that \( \lambda_{\alpha, \gamma, a} \) is an eigenvalue of \( \mathcal{L}_{\mathbf{A}_{\alpha, \gamma, a}} + V_{\mathbf{B}_{\alpha, \gamma, a, \tau_s}} \). \( \square \)

**Proof of Proposition 1.4.** The proof is inspired by the construction done in [2, Proof of Proposition 3.9] while studying 2D smooth domains under discontinuous magnetic fields, and by [16, Proof of Theorem 1.1] while studying 2D corner domains under constant magnetic fields.

We fix \( \alpha \in [-1, 1] \setminus \{0\}, \alpha \in (0, \pi), \) and \( \gamma \in [0, \pi/2] \). Let \( \tau = 0 \). We define the function \( \varphi_{\alpha, \gamma, a} \in H^1_{\text{loc}}(\mathbb{R}^2_+) \) by
\[ \varphi_{\alpha, \gamma, a}(x_1, x_2) = \begin{cases} \left( \frac{1}{2}x_1x_2 + \frac{a-1}{2}x_2^2 \cot \alpha \right) \cos \gamma & \text{if } (x_1, x_2) \in D_1^1, \\ \frac{a}{2}x_1x_2 \cos \gamma & \text{if } (x_1, x_2) \in D_2^1. \end{cases} \]

This function satisfies \( \mathbf{A}_{\alpha, \gamma, a} = \tilde{\mathbf{A}}_{\alpha, \gamma, a} + \nabla \varphi_{\alpha, \gamma, a} \), where \( \mathbf{A}_{\alpha, \gamma, a} \) is the potential in (3.8), and \( \tilde{\mathbf{A}}_{\alpha, \gamma, a} = 1/2(-x_2, x_1)\mathbf{1}_{D_0^1} \) being the step function in (3.12) (see [28, Lemma 1.1] for the existence of such gauge functions in more general situations).

We define the quadratic form \( \tilde{Q}_{\alpha, \gamma, a} \) as follows
\[ \tilde{Q}_{\alpha, \gamma, a}(v) = \int_{\mathbb{R}^2_+} \left( |(\nabla - i\tilde{\mathbf{A}}_{\alpha, \gamma, a})v|^2 + V_{\mathbf{B}_{\alpha, \gamma, a, 0}}|v|^2 \right) dx_1 dx_2 \]
in the domain
\[ \mathcal{D}(\tilde{Q}_{\alpha, \gamma, a}) = \left\{ v \in L^2(\mathbb{R}^2_+) : (\nabla - i\tilde{\mathbf{A}}_{\alpha, \gamma, a})v \in L^2(\mathbb{R}^2_+), |x_1 \sin \alpha - x_2 \cos \alpha|v \in L^2(\mathbb{R}^2_+) \right\}, \]
where \( V_{\mathbf{B}_{\alpha, \gamma, a, 0}} = \mathbf{1}_{D_0^1}(x_1 \sin \gamma \sin \alpha - x_2 \sin \gamma \cos \alpha)^2 \) is the electric potential defined in (3.14) for \( \tau = 0 \). We explicitly express \( \tilde{Q}_{\alpha, \gamma, a}(v) \) by
\[ \int_{\mathbb{R}^2_+} \left( |(\partial_{x_1} + \frac{1}{2}i\mathbf{1}_{D_0^1}x_2 \cos \gamma)v|^2 + |(\partial_{x_2} - \frac{1}{2}i\mathbf{1}_{D_0^1}x_1 \cos \gamma)v|^2 + \mathbf{1}_{D_0^1}(x_1 \sin \gamma \sin \alpha - x_2 \sin \gamma \cos \alpha)^2 |v|^2 \right) dx_1 dx_2. \]

For any \( v \in \mathcal{D}(\tilde{Q}_{\alpha, \gamma, a}) \), we have
\[ \tilde{Q}_{\alpha, \gamma, a}(v) = \tilde{Q}_{\alpha, \gamma, a}^{\tau=0}(e^{i\varphi_{\alpha, \gamma, a}}v), \]
where \( Q_{\alpha,\gamma,\alpha} \) is the quadratic form in (3.15). In the rest of the proof, we write \( \bar{Q} \) for \( \bar{Q}_{\alpha,\gamma,\alpha} \) and \( \bar{s} \) for \( \bar{s}_{\alpha,\gamma,\alpha} \). We now express \( \bar{Q} \) in the polar coordinates \((\rho, \theta) \in (0, +\infty) \times (0, \pi) =: \tilde{D}_{\text{pol}} \) as follows

\[
\bar{Q}_{\text{pol}}(v) = \int_0^\pi \int_0^{+\infty} \left( |\partial_\rho v|^2 + \frac{1}{\rho^2} \left( \partial_\rho - i \bar{s}_{\text{pol}} \frac{\rho^2}{2} \cos \gamma \right) v \right)^2 + \bar{s}_{\text{pol}}^2 \rho^2 \sin^2 \gamma \sin^2 (\alpha - \theta) |v|^2 \rho \, d\rho \, d\theta,
\]

where \( \bar{s}_{\text{pol}}(\rho, \theta) = \bar{s}(x_1, x_2) \) and

\[
D(\bar{Q}_{\text{pol}}) = \left\{ v \in L^2_\rho(\tilde{D}_{\text{pol}}) : \partial_\rho v \in L^2_\rho(\tilde{D}_{\text{pol}}), \frac{1}{\rho} \left( \partial_\theta - i \bar{s}_{\text{pol}} \frac{\rho^2}{2} \cos \gamma \right) v \in L^2_\rho(\tilde{D}_{\text{pol}}), \rho v \in L^2_\rho(\tilde{D}_{\text{pol}}) \right\}.
\]

For any \( D \subset \mathbb{R}^2 \), we denote by \( L^2_\rho(D) \) the weighted space of weight \( \rho \). Consider further the quadratic form \( \tilde{Q}_{\text{pol}} \), defined on \( \tilde{D}_{\text{pol}} := (0, +\infty) \times (-\pi + \alpha, \alpha) \) by

\[
\tilde{Q}_{\text{pol}}(u) = \int_{-\pi + \alpha}^{\alpha} \int_{0}^{+\infty} \left( |\partial_\rho u|^2 + \frac{1}{\rho^2} \left( \partial_\rho + i \bar{s}_{\text{pol}} \frac{\rho^2}{2} \cos \gamma \right) u \right)^2 + \bar{s}_{\text{pol}}^2 \rho^2 \sin^2 \gamma \sin^2 \theta |u|^2 \rho \, d\rho \, d\theta,
\]

where

\[
D(\tilde{Q}_{\text{pol}}) = \left\{ u \in L^2_\rho(\tilde{D}_{\text{pol}}) : \partial_\rho u \in L^2_\rho(\tilde{D}_{\text{pol}}), \frac{1}{\rho} \left( \partial_\theta + i \bar{s}_{\text{pol}} \frac{\rho^2}{2} \cos \gamma \right) u \in L^2_\rho(\tilde{D}_{\text{pol}}), \rho u \in L^2_\rho(\tilde{D}_{\text{pol}}) \right\},
\]

and

\[
\bar{s}_{\text{pol}}(\rho, \theta) = \begin{cases} 
\alpha & \text{if } (\rho, \theta) \in (0, +\infty) \times (-\pi + \alpha, 0), \\
1 & \text{if } (\rho, \theta) \in (0, +\infty) \times (0, \alpha).
\end{cases}
\]

For any \( u \in \text{Dom} \tilde{Q}_{\text{pol}} \), we have \( \tilde{Q}_{\text{pol}}(u) = \tilde{Q}_{\text{pol}}(v) \), where \( v(\rho, \theta) = u(\rho, -\theta + \alpha) \).

In light of the computation above and from Theorem 1.1, a sufficient condition for \( \inf_{\tau} \bar{g}(\alpha, \gamma, \alpha, \tau) \) to be attained by some \( \tau_\ast \in \mathbb{R} \) and to be an eigenvalue of the operator \( \mathcal{L}_{\text{pol}} + \mathcal{V}_{\text{pol}} \) is to find a trial function \( u_\ast \in \text{Dom} \tilde{Q}_{\text{pol}} \) satisfying

\[
\tilde{Q}_{\text{pol}}(u_\ast) < \Lambda \| u_\ast \|^2_{L^2(\mathbb{R}^2)},
\]

where \( \Lambda = \Lambda[\alpha, \gamma, \alpha] \) is the minimum between \( \beta_\ast \) and \( |\alpha| \zeta_\alpha \). Towards this, we consider the function

\[
u_\ast(\rho, \theta) = e^{-\nu_\ast^2 \frac{\rho^2}{2}} e^{-i \rho \varphi(\theta)},
\]

where \( g : (-\pi + \alpha, \alpha) \rightarrow \mathbb{R} \) is a piecewise-differentiable function and \( \nu > 0 \). In what follows, we will suitably choose \( g \) and \( \nu \). We define the functional \( \mathcal{J} \) on \( \text{Dom} \tilde{Q}_{\text{pol}} \) by

\[
\mathcal{J}[u] = \tilde{Q}_{\text{pol}}(u) - \Lambda \| u \|^2_{L^2(\tilde{D}_{\text{pol}})}.
\]

The condition in (4.4) is now equivalent to

\[
\mathcal{J}[u_\ast] < 0.
\]

We compute \( \mathcal{J}[u_\ast] \) and get

\[
\mathcal{J}[u_\ast] = \int_0^{+\infty} \rho e^{-\nu_\ast^2} d\rho \int_{-\pi + \alpha}^{\alpha} \left( g^2(\theta) + g'^2(\theta) - \Lambda \right) d\theta
\]

\[- \int_0^{+\infty} \rho^2 e^{-\nu_\ast^2} d\rho \int_{-\pi + \alpha}^{\alpha} \bar{s}_{\text{pol}} g'(\theta) \cos \gamma \, d\theta
\]

\[+ \int_0^{+\infty} \rho^3 e^{-\nu_\ast^2} d\rho \int_{-\pi + \alpha}^{\alpha} \left( \nu^2 + \bar{s}_{\text{pol}}^2 \sin^2 \gamma \sin^2 \theta + \frac{1}{4} \bar{s}_{\text{pol}}^2 \cos^2 \gamma \right) d\theta.
\]
We use the following properties of $E_n = \int_0^{\infty} \rho^n e^{-r^2/\nu} \, d\nu$, for $n \geq 0$: $E_1 = 1/(2\nu)$, $E_2 = \sqrt{\pi}/(4\nu^{3/2})$, and $E_3 = 1/(2\nu^2)$ (see [21, Equations 3.461]). Hence, (4.6) becomes

\[
\mathcal{J}[u_0] = \frac{1}{2\nu} \int_{-\pi+\alpha}^{\alpha} \left( g^2(\theta) + g'^2(\theta) - \Lambda \right) \, d\theta - \frac{\sqrt{\pi}}{4\nu^{3/2}} \int_{-\pi+\alpha}^{\alpha} \tilde{s}_{\text{pol}} g'(\theta) \cos \gamma \, d\theta + \frac{1}{2\nu^2} \int_{-\pi+\alpha}^{\alpha} \left( \nu^2 + \tilde{s}_{\text{pol}}^2 \sin^2 \gamma \sin^2 \theta + \frac{1}{4} \tilde{s}_{\text{pol}}^2 \cos^2 \gamma \right) \, d\theta. \tag{4.6}
\]

Now, we choose

\[
g(\theta) = \begin{cases} c_1 e^\theta + c_2 e^{-\theta} & \text{if } -\pi + \alpha < \theta < 0, \\ c_3 e^\theta + c_4 e^{-\theta} & \text{if } 0 < \theta < \alpha,
\end{cases}
\]

where $c_i$, $i = 1, \ldots, 4$, are real coefficients satisfying the condition $c_1 + c_2 = c_3 + c_4$ which makes the function $g$ continuous on $(-\pi + \alpha, \alpha)$. Implementing this choice in (4.6) yields

\[
\mathcal{J}[u_0] = \left( \frac{2 - e^{-2\alpha} - e^{-2\pi+2\alpha} \nu}{2\nu} \right) c_1^2 + \left( \frac{-(e^{-2\alpha} + e^{2\pi-2\alpha})}{2\nu} \right) c_2^2 + \left( \frac{-(e^{-2\alpha} - e^{2\alpha})}{2\nu} \right) c_3^2 + \frac{(1 - e^{-2\alpha})}{\nu} c_1 c_2 + \frac{(-1 + e^{-2\alpha})}{\nu} c_1 c_3 + \frac{(-1 - e^{-2\alpha})}{\nu} c_2 c_3 + \frac{(1 - a - e^{-\alpha} + e^{-\pi+\alpha})\sqrt{\pi} \cos \gamma}{4\nu^2} c_1 + \frac{(1 - a - e^{-\alpha} + e^{-\pi+\alpha})\sqrt{\pi} \cos \gamma}{4\nu^2} c_2 + \frac{(e^{-\alpha} - e^{-\alpha})\sqrt{\pi} \cos \gamma}{4\nu^2} c_3 + \frac{4\nu^2 - 4\pi \nu \Lambda + (a^2(\pi - \alpha) + \alpha) \cos^2 \gamma}{8\nu^2} + \frac{2(2a^2(\pi - \alpha) + \alpha + (a^2 - 1) \cos \alpha \sin \alpha) \sin^2 \gamma}{8\nu^2}.
\]

Notice that $\mathcal{J}[u_0]$ is quadratic in $c_1$, $c_2$ and $c_3$. Minimizing $\mathcal{J}[u_0]$ with respect to these coefficients gives a unique solution $(c_1, c_2, c_3)$, which is

\[
c_1 = \frac{e^{\pi - 2\alpha}(-1 + a)e^\pi + (-1 + a)e^{\pi + 2\alpha} + 2e^\alpha(-a + e^\pi)\sqrt{\pi} \cos \gamma(-1 + \coth \pi)}{16\sqrt{\nu}},
\]

\[
c_2 = \frac{(-1 + a + (-1 + a)e^{2\alpha} - 2(-1 + a)e^\alpha)\sqrt{\pi} \cos \gamma(-1 + \coth \pi)}{16\sqrt{\nu}},
\]

\[
c_3 = \frac{e^{-\alpha}(-a + e^\pi + (-1 + a) \cosh(\pi - \alpha))\sqrt{\pi} \cos \gamma \csc \pi}{8\sqrt{\nu}}.
\]

We compute $\mathcal{J}[u_0]$ corresponding to the coefficients above, and get $\mathcal{J}[u_0] = P[\alpha, \gamma, a](x)$ with $x = \frac{1}{\nu} > 0$, where $P[\alpha, \gamma, a]$ is as in (1.10). This, together with the condition in (4.5), complete the proof. \(\square\)

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