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Dynamical Inference for Transitions in Stochastic Systems with $\alpha$–stable Lévy Noise

Ting Gao$^1$, Jinqiao Duan$^1$, Xingye Kan$^2$ & Zhuan Cheng$^1$

1. Department of Applied Mathematics, Illinois Institute of Technology
   Chicago, IL 60616, USA
   E-mail: tinggao0716@gmail.com, duan@iit.edu

2. School of Mathematics, University of Minnesota
   Minneapolis, MN 55414, USA
   E-mail: xkan@umn.edu

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Abstract

A goal of data assimilation is to infer stochastic dynamical behaviors with available observations. We consider transition phenomena between metastable states for a stochastic system with (non-Gaussian) $\alpha$–stable Lévy noise. With either discrete time or continuous time observations, we infer such transitions between metastable states by computing the corresponding nonlocal Zakai equation (and its discrete time counterpart) and examining the most probable orbits for the state system. Examples are presented to demonstrate this approach.

Short Title: Transitions in Non-Gaussian Stochastic Systems

Key Words: Nonlocal Zakai equation; nonlocal Laplace operator; infer mean exit time; transitions between metastable states; most probable orbits

PACS (2010): 05.40.Ca, 02.50.Fz, 05.40.Fb, 05.40.Jc

1 Introduction

Random fluctuations in nonlinear systems in engineering and science are often non-Gaussian [32]. For instance, it has been argued that diffusion by geophysical turbulence [28] corresponds to a series of “pauses”, when the particle is trapped by a coherent structure, and “flights” or “jumps” or other extreme events, when the particle moves in the jet flow. Paleoclimatic data

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[9, 10] also indicate such irregular processes. There are also experimental demonstrations of Lévy flights in foraging theory and rapid geographical spread of emergent infectious disease. Humphries et. al. [15] used GPS to track the wandering black bowed albatrosses around an Island in Southern Indian Ocean to study the movement patterns of searching food. They found that by fitting the data of the movement steps, the movement patterns obeys the power-law property with power parameter $\alpha = 1.25$. To get the data set of human mobility that covers all length scales, Brockmann [5] collected data by online bill trackers, which give successive spatial-temporal trajectories with a very high resolution. When fitting the data of probability of bill traveling at certain distances within a short period of time (less than one week), he found power-law distribution property with power parameter $\alpha = 1.6$, and observed that $\alpha$–stable Lévy motions are strikingly similar to practical data of human influenza.

Lévy motions are thought to be appropriate models for a class of important non-Gaussian processes with jumps [26, 4, 25]. Recall that a Lévy motion $L(t)$, or $L_t$, is a stochastic process with stationary and independent increments. That is, for any $s, t$ with $0 \leq s < t$, the distribution of $L_t - L_s$ only depends on $t - s$, and for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $L_{t_i} - L_{t_{i-1}}$, $i = 1, \cdots, n$, are independent. A Lévy motion has a version whose sample paths are almost surely right continuous with left limits.

Stochastic differential equations (SDEs) with non-Gaussian Lévy noises have attracted much attention recently [11, 2, 27]. To be specific, let us consider the following $n$-dimensional stochastic state system:

$$dX_t = f(X_t, t)dt + dL_t^\alpha, \quad X_0 = x_0,$$

where $f$ is a vector field (also called a drift), and $L_t^\alpha$ is a symmetric $\alpha$–stable Lévy motion ($0 < \alpha < 2$), defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that we have either

(i) a discrete time $m$-dimensional stochastic observation system:

$$y_k = h(x_k, t_k) + \sqrt{R_k}v_k, \quad k = 0, 1, \cdots,$$

where $v_k$ is a white sequence of Gaussian random variables, i.e. $v_k$’s are mutually independent standard normal random variables, and $R_k$ is a sequence of nonnegative numbers;

or

(ii) a continuous time $m$-dimensional stochastic observation system:

$$dY_t = h(X_t, t)dt + \sqrt{\varepsilon} dW_t, \quad Y_0 = y_0,$$

where $h$ is a given vector field (sometimes called a sensor), $\varepsilon$ is the noise intensity and $W_t$ is a Brownian motion.

In the present paper, we estimate system states, with help of observations, and in particular, we try to capture transitions between metastable
states by examining most probable paths for system states. We also infer
mean exit time from metastable region, using available observations. Our
data assimilation approach allows the state system (1) to be a stochastic
system with non-Gaussian Lévy noise $L_\alpha^\alpha$, while most existing works involve
with (Gaussian) Brownian noise \[3\]. Moreover, the observation system (3)
can also contain non-Gaussian Lévy noise, because this will only introduce a
random term in the Zakai equation (16) which does not involve a numerical
difficulty. For recent works on data assimilation and stochastic filtering for
state systems with non-Gaussian noise, see \[24, 29, 14, 23\]. A limitation of
our approach is that the state systems need to be low dimensional, because
we use a finite difference scheme \[13, 31\] to simulate the Zakai equation (16)
which is a stochastic partial differential equation.

This paper is organized as follows. We consider state estimates with
discrete time and continuous time observations in Sections 2 and 3, re-
information about the system state $X_t$, and hence will be useful in our investigation of state estimation. Also note that the non-Gaussianity of the Lévy noise manifests as nonlocality (an integral term) in the generator. The adjoint operator for the generator $A$ is

$$A^*p = -\partial_x(f(x,t)p(x,t)) + \sigma \int_{\mathbb{R}^1 \setminus \{0\}} [p(x+y,t) - p(x,t) - I_{\{|y|<1\}} y \partial_x p(x,t)] \nu_\alpha(dy).$$

(7)

The Fokker-Planck equation for the SDE (4) is ([2, 11]):

$$p_t = A^*p.$$  

(8)

Let $Y_{t_k}$ be the sigma-field generated by \{y_0, y_1, \ldots, y_k\}. Similarly as in [17], we have the following theorem which determines the time evolution of the conditional probability density function $p(x, t \mid Y_{t_k})$. For convenience, we often write $p(x, t)$ for $p(x, t \mid Y_{t_k})$.

**Theorem 1. (Conditional Density for Continuous-discrete Problems).** Let system (4) satisfy the hypotheses that $f$ is Lipschitz in space and the initial state $X_0$, with the property $E(|X_0|^2) < \infty$, is independent of \{L_t^\alpha, t \in [t_0, T]\}. Suppose that the prior density $p(x, t)$ for (4) exists and is once continuously differentiable with respect to $t$ and twice with respect to $x$. Let $h$ be continuous in both arguments and bounded for each $t_k$ with probability 1.

Then, between observations, the conditional density $p(x, t \mid Y_t)$ satisfies the Fokker-Planck equation

$$dp(x, t \mid Y_{t_k}) = A^*p dt, \quad t_k \leq t < t_{k+1}, \quad p(x, t_0 \mid Y_{t_0}) = p(x_{t_0}).$$

(9)

where $A^*$ is the operator in (7). At an observation (at $t_k$), the conditional density satisfies the following difference equation

$$p(x, t_k \mid Y_{t_k}) = \frac{p(y_k \mid x)p(x, t_k \mid Y_{t_k})}{\int_{\mathbb{R}^1} p(y_k \mid \xi)p(\xi, t_k \mid Y_{t_k})d\xi},$$

(10)

where $p(y_k \mid x)$ is

$$p(y_k \mid x) = (1/(2\pi)^{1/2}|R_k|^{1/2})exp\{-\frac{1}{2}[y_k - h(x, t_k)]^TR_k^{-1}[y_k - h(x, t_k)]\}.$$  

(11)

**Proof.** The conditional density in the absence of observation, satisfies the Fokker-Planck equation. Therefore, between observations, conditional density $p(x, t \mid Y_{t_k})$ satisfies the Fokker-Planck equation (9).

Thus, it remains to determine the relationship between $p(x, t_k \mid Y_{t_k})$ and

$$p(x, t_k \mid Y_{t_k}) \equiv p(x, t_k \mid Y_{t_{k-1}}).$$
Since \( p(x, t_k|Y_{t_k}) = p(x, t_k|y_k, Y_{t_k-1}) \), we have by Bayes’ rule
\[
p(x, t_k|Y_{t_k}) = \frac{p(y_k|x, t_k, Y_{t_k-1})P(x, t_k|Y_{t_k-1})}{p(y_k|Y_{t_k-1})} = \frac{p(y_k|x, Y_{t_k-1})P(x, t_k|Y_{t_k-1})}{p(y_k|Y_{t_k-1})}.
\]

Now, since the noise \( \{v_k\} \) is white,
\[
p(y_k|x, t_k, Y_{t_k-1}) = p(y_k|x_k).
\]

Similarly, we compute
\[
p(y_k|Y_{t_k-1}) = \int p(y_k|x)p(x, t_k|Y_{t_k-1})dx.
\]

Therefore,
\[
p(x, t_k|Y_{t_k}) = \frac{p(y_k|x)p(x, t_k|Y_{t_k-1})}{\int_{R^1} p(y_k|\xi)p(\xi, t_k|Y_{t_k-1})d\xi}.
\]

This completes the proof. \( \square \)

This theorem provides the foundation for computing conditional density for system state of SDE (4), under discrete time observations.

Define \( x_m(t) \triangleq \text{maximizer for } \max_{x \in R^1} p(x, t) \).

This provides the most probable orbit ([11, 8]) starting at \( x_0 \). These most probable orbits are the maximal likely orbits for a dynamical system under noisy fluctuations.

Let us consider an example.

**Example 1.** Let us consider a scalar system with state equation
\[
dX_t = 4(X_t - X_t^3)dt + dL_t^\alpha, \quad X_0 = x_0.
\]

The discrete-time scalar observation is:
\[
y_k = h(x_k, t) + \sqrt{R_k}v_k,
\]

with \( h(x, t) = x \), and \( R_k \equiv 0.1 \). We take the magnitude \( \sigma = \sqrt{0.24} \) for the jump measure in \( L_t^\alpha \).

In the absence of Lévy noise, this system has two stable states: \(-1\) and \(+1\). When the noise kicks in, these two states are no longer fixed. The random system evolution near these two states, together with possible transitions between them, is sometimes called a metastable phenomenon [22]. For convenience, we call \(-1\) and \(+1\) (and random motions nearby) metastable states.

The corresponding nonlocal Fokker-Planck equation is computed on \((-2.5, 2.5)\) with a finite difference scheme ([12, 13]). The nonlocal Laplacian operator (see (6)) in this equation involves an integral on the whole space (including the
Figure 1: Example 1 – Conditional probability and state estimation when $\alpha = 1.5$. The initial density for $x_0$ is a Gaussian distribution centered at $-1$ with variance $\frac{1}{20}$. Online version: The red curve in the right panel is the estimated state orbit (i.e., most probable orbit).

part outside the computational domain); the value of $p$ in this external domain is specified via an extrapolation ([13]). Space stepsize $\Delta x = 0.05$ and time stepsize $\Delta t = 0.001$. The initial probability density is taken either as a Gaussian distribution or a uniform distribution.

In Figures 1 and 2, we show the conditional density $p(x, t)$, together with the corresponding most probable orbit (taken as the state estimation for $X_t$), together with observations $y_k$ and a state path $X_t$. The sample paths for $L^\alpha_t$ is generated by a direct method in [16, Ch. 3], and the state path $X_t$ is generated by a Euler scheme as reviewed in [11, Ch. 7]. The initial density for $x_0$ is Gaussian in Figure 1 and uniform in Figure 2. In the numerical simulations, we take $y_0$ as defined in (13) via $x_0$. Notice that the estimated state captures multiple transitions from $-1$ to $+1$, during the time period $0 < t < 50$.

3 Inferring transitions with continuous time observations

We consider the following scalar state system with a symmetric $\alpha$—stable Lévy motion

$$dX_t = f(X_t)dt + dL^\alpha_t, \quad X_0 = x_0, \quad (14)$$

together with a continuous time scalar observation system:

$$dY_t = h(X_t)dt + \sqrt{\varepsilon} \, dW_t, \quad Y_0 = y_0, \quad (15)$$

where $h$ is a given vector field (sometimes called a sensor function), $\varepsilon$ is the noise intensity and $W_t$ is a Brownian motion. We assume the drift $f$ and
Figure 2: Example 1 – Conditional probability and state estimation when $\alpha = 1.5$. The initial density for $x_0$ is a uniform distribution on $(-1.5, -0.5)$. Online version: The red curve in the right panel is the estimated state orbit (i.e., most probable orbit).

the sensor function $h$ are autonomous (i.e., do not depend on time explicitly) for simplicity. When the drift depends on time explicitly, see [6] about tools quantifying stochastic dynamics (without observation).

Let $\mathcal{Y}_t$ be the sigma-field generated by $Y_s$, for $0 \leq s \leq t$. The unnormalized conditional probability density $p(x, t | \mathcal{Y}_t)$ satisfies a nonlocal Zakai equation ([24, 14, 23]):

$$dp(x, t | \mathcal{Y}_t) = A^* p(x, t | \mathcal{Y}_t) \, dt + h(x) p(x, t | \mathcal{Y}_t) \, dY_t,$$  \hspace{1cm} (16)

where $A^*$ is the adjoint operator (7) of the generator $A$ in (6), and $p(x, 0 | Y_0)$ is the initial density of $X_t$ (say a uniform distribution near the metastable state $-1$, or a Gaussian distribution centered at $-1$). The Zakai equation (16) may be numerically solved with a finite difference method based on [12, 13] together with a discretization of the noisy term at the current space-time point and $dY_t \approx Y_{t+\Delta t} - Y_t$. The initial probability density is taken either as a Gaussian distribution or a uniform distribution. For other numerical methods, see, for example, [19, 3, 18, 7, 34].

Remark 1. The normalized conditional probability density $p(x, t | \mathcal{Y}_t)$ satisfies the nonlinear Kushner’s equation [3, 18],

$$dp(x, t | \mathcal{Y}_t) = A^* p(x, t | \mathcal{Y}_t) \, dt + (h(x) - \hat{h}(x))(dZ_t - \hat{h}(x)dt)p(x, t | \mathcal{Y}_t)$$

$$= A^* p(x, t | \mathcal{Y}_t) \, dt + h(x)p(x, t | \mathcal{Y}_t)dZ_t - \hat{h}pdt - \hat{h}pdZ_t + \hat{h}pdZ_t,$$

where $\hat{h}(x)$ is the mathematical expectation of $h(X_t)$, with respect to $p$.

The conditional density $p(x, t | \mathcal{Y}_t)$ provides information for the system evolution. With the observation, we can infer possible transitions from the
metastable state $-1$ to the metastable state $+1$, within a time range $(0, T)$. If the system starts with a probability distribution $p_0(x)$ near the metastable state $x = -1$, then the conditional density $p(x, t|\mathcal{Y}_t)$ helps us to infer whether the system will get near the other metastable state $+1$, and vice versa. This may be achieved by examining the most probable orbits for the system, under the observation. Define $x_m(t) \triangleq \text{maximizer for } \max_{x \in \mathbb{R}} p(x, t|\mathcal{Y}_t)$. This provides the most probable orbit ([11, 8]) starting at $x_0$. The most probable orbit depends on observational samples, as it is computed from conditional probability density $p(x, t|\mathcal{Y}_t)$. We take this as our state estimation for $X_t$, as in [20].

We have also tried to infer the mean exit time for orbit $X_t$, starting at $x_0$, from a domain $D$. Define the first exit time $\tau(x_0, \omega)$ of $X_t$ from a bounded domain $D$ as

$$\tau(x_0, \omega) \triangleq \inf\{t \geq 0, X_t(x_0, \omega) \notin D\}.$$ 

Similarly, the first exit time of the most probable orbit $x_m(t, x_0, \omega)$ (starting at $x_0$), with extra information provided by observation $Y_t$, is then

$$\tilde{\tau}(x_0, \omega) \triangleq \inf\{t \geq 0, x_m(t, x_0, \omega) \notin D\}.$$

The mean exit time, without or with the observation $Y_t$, is then respectively denoted by

$$u(x_0) = \mathbb{E}[\tau(x_0, \omega)],$$

$$v(x_0) = \mathbb{E}[\tau(x_0, \omega)|\mathcal{Y}_t],$$

denoted by $u(x_0)$ and $v(x_0)$ for starting point $x_0 \in D$. These are computed via ensemble averaging on first exit time, with simulated sample-wise orbits $X_t$ and $x_m(t)$. However, we have noted that our ‘filter’, the most probable orbit $x_m(t)$, appears not to reproduce the mean exit time accurately. See Example 2 below.

The conditional probability density $p(x, t|\mathcal{Y}_t)$ can also be used to infer other dynamical information, e.g., the likelihood that the system will get to the neighborhood $(0.75, 1.25)$ of the metastable state $x = +1$, at time $T$. This is can be computed via (after normalizing $p(x, t|\mathcal{Y}_t)$)

$$\mathbb{P}(X_T \in (0.75, 1.25)) = \int_{0.75}^{1.25} p(x, T|\mathcal{Y}_t) dx.$$ 

Let us illustrate our approach by an example.

**Example 2.** Let us consider the following scalar SDE state equation with a symmetric $\alpha$–stable Lévy motion:

$$dX_t = 4(X_t - X_t^3) dt + dL^\alpha, \quad X_0 = x_0.$$ 

The scalar observation equation is given by

$$dY_t = X_t dt + \sqrt{\varepsilon} dW_t,$$
Figure 3: Example 2 – Conditional probability (top), state estimation (middle) and rms error (bottom) for $\alpha = 1.5$, $T = 50$ and $\varepsilon = 0.1$. The initial density is Gaussian centered at $-1$ with variance $\frac{1}{20}$. 
Figure 4: Example 2 – Conditional probability (top), state estimation (middle), and rms error (bottom) for $\alpha = 1.5$, $T = 50$ and $\varepsilon = 0.05$. The initial density is uniform on $(-1.6, -0.4)$. 
Figure 5: Example 2 – Conditional probability (top), state estimation (middle) and rms error (bottom) for $\alpha = 0.5$, $T = 10$ and $\varepsilon = 0.1$. The initial density is Gaussian centered at $-1$ with variance $\frac{1}{20}$. 
with $\varepsilon$ the noise intensity. When noise is absent, the state system has two 
stable states: $-1$ and $+1$. In this example, the vector field $f(x) = 4(x - x^3)$ 
and the sensor is $h(x) = x$. We take the magnitude $\sigma = \sqrt{0.24}$ for the jump 
measure in $L_\alpha^t$.

The corresponding nonlocal Zakai equation is solved by a finite difference 
scheme with space stepsize, satisfying the time-space stepsize relation in [13, 
Proposition 1], such as space stepsize $\Delta x = 0.2$ and time stepsize $\Delta t = 
0.0025$. We have conducted numerous numerical experiments with various 
parameters $\alpha, \varepsilon$, and time interval $0 < t < T$.

In Figures 3-5, we show the conditional density $p(x, t)$, the corresponding 
most probable orbit $x_m(t)$ (taken as the state estimation for the state orbit 
$X_t$), and the root-mean-square (rms) error, for various parameters $\alpha, \varepsilon$ and 
time interval length $T$. The sample paths for $L^x_\alpha$ is generated by a direct 
method in [16, Ch. 3], and the state orbit $X_t$ is generated by a Euler scheme 
as reviewed in [11, Ch. 7]. The initial density for $x_0$ is either Gaussian or 
uniform. In these simulations, we take $y_0$ to be the same as $x_0$.

In these figures, the state orbit $X_t$ and the most probable orbit $x_m(t)$ 
are shown with a single sample. The rms error at time $t$, between a state 
orbit (‘a true orbit’) $X_t$ and the most probable orbits $x_m(t)$ is defined by 
$\sqrt{E[(X_t - x_m(t))^2]}$. It is generated by ensemble averaging multiple sample 
paths (typically 30 sample paths in our simulations). Note that only the 
most probable orbits (which correspond to sample-wise solutions of the Zakai 
stochastic partial differential equation) are varied.

Notice that the estimated state orbit captures the transitions between 
metastable states $-1$ and $+1$, during the time period $0 < t < T$. The state 
orbit $X_t$ has jumps and our estimated state orbit does not always reproduce 
large jumps (which are finite in number); see Figures 3 (middle) and Figure 
4 (middle). This is more so for $\alpha < 1$ when there are more frequent large 
jumps; see Figure 5 (middle). This also shows up in the error estimation, 
where larger discrepancies are contributed by larger jumps in state orbit $X_t$. 
This is one limitation of our method, as we use most probable orbit as our 
state estimation.

We have further tested whether our ‘filter’ , the most probable orbit 
$x_m(t)$, can reproduce the mean exit time, and found that it appears not to 
reproduce it accurately. For example, taking $\alpha = 1.5$ with the starting point 
$x_0 = -1$ (a metastable state) and its neighborhood $D = (-1.15, -0.85)$, the 
original state system’s mean exit time is $u(-1) = 7.0812$, while the mean 
exit time for the most probable orbit is $v(-1) = 7.4221$ (with 30 samples in 
simulating the mean). When changing to different domain $D$, together with 
other starting points $x_0$ with various number of samples, we have noted simi-
lar discrepancies. This is likely due to the following reason: The original 
state orbit $X_t$ have countable jumps (which facilitate ‘exits’ from $D$) of size 
smaller than the distance between the metastable states ($-1$ and $+1$), while 
the most probable orbit $x_m(t)$ have much fewer small jumps (see the middle
plot in Figure 3) and thus have different (perhaps longer) exit time. A related evidence for this comment is from our recent work [8]: When a scalar state system only involves Brownian motion, the most probable orbit $x_m(t)$ is solution of an ordinary differential equation and thus is indeed one order more smooth than the state orbit $X_t$ (Here $x_m(t)$ has derivative in time, while $X_t$ is only continuous in time and does not have time derivative).

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