Symmetric Properties for Dirichlet-Type Multiple \((p, q)\)-L-Function

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Abstract: The main aim of this article is to investigate some interesting symmetric identities for the Dirichlet-type multiple \((p, q)\)-L function. We use this function to examine the symmetry of the generalized higher-order \((p, q)\)-Euler polynomials related to \(\chi\). First, the generalized higher-order \((p, q)\)-Euler numbers and polynomials related to \(\chi\) are defined. We also give a few new symmetric properties for the Dirichlet-type multiple \((p, q)\)-L-function and generalized higher-order \((p, q)\)-Euler polynomials related to \(\chi\).

Keywords: Dirichlet-type multiple \((p, q)\)-L function; generalized higher-order \((p, q)\)-Euler polynomials related to \(\chi\); symmetric identities; generalized alternating \((p, q)\)-power sums

1. Introduction

In the extensive literature of number theory we can find a systematic and extensive investigation of the ordinary Bernoulli, Euler, Genocchi and tangent polynomials and their corresponding numbers, as well as \(q\)-extensions, \((p, q)\)-extensions, and many generalizations. Many mathematicians have presented several interesting results of these general polynomial systems, including some explicit series representations in terms of the Hurwitz zeta function and generalized zeta function. Barnes [1] introduced the generalized multiple Hurwitz zeta function. Choi and Srivastava [2] defined and studied the multiple Hurwitz–Euler eta function. Hwang and Ryoo [3,4] introduced the multiple \((p, q)\)-Hurwitz–Euler eta function and the multiple twisted \((p, q)\)-L function. Recently, He [5] investigated some identities of symmetry for Carlitz-type \(q\)-Bernoulli numbers and polynomials in a complex field. Kim et al. [6] obtained several identities of symmetry for Carlitz-type \(q\)-Euler numbers and polynomials in a complex field.

By \(\mathbb{N}\) we denote the set of natural numbers and \(\mathbb{N}_0 = \{0\} \cup \mathbb{N}\). Set

\[ [w]_q = \frac{1 - q^w}{1 - q}, 0 < q < 1 \text{ and } [w]_{p,q} = \frac{p^w - q^w}{p - q}, p \neq q. \]

Note that if \(p = 1\), then \(\lim_{q \to 1} [w]_{p,q} = w\).

\[ \sum_{l_1=0}^{n} \cdots \sum_{l_r=0}^{n} = \sum_{l_1=0}^{n} \cdots \sum_{l_r=0}^{n} \]

The binomial formulae are known as

\[ (1 - c)^n = \sum_{l=0}^{n} \binom{n}{l} (-1)^l c^l, \text{ where } \binom{n}{l} = \frac{n(n-1)\cdots(n-l+1)}{l!}, \]
and
\[
\frac{1}{(1-c)^n} = (1-c)^{-n} = \sum_{l=0}^{\infty} \binom{-n}{l} (-1)^l c^l = \sum_{l=0}^{\infty} \binom{n+l-1}{l} c^l.
\]

Choi and Srivastava [2] constructed the multiple Hurwitz–Euler eta function \( \eta_r(s,a) \) defined by the following \( r \)-ple series:

\[
\eta_r(s,a) = \sum_{h_1,\ldots,h_r} \frac{(-1)^{h_1+\cdots+h_r}}{(h_1+\cdots+h_r+a)^s}, \quad (\text{Re}(s) > 0; a > 0; r \in \mathbb{N}).
\]

It is known that \( \eta_r(s,a) \) can be continued analytically to be a whole complex \( s \)-plane (see [2]).

The ordinary Euler numbers \( E_n \) and polynomials \( E_n(z) \), together with their familiar generalizations \( E_n^{(r)} \) and Euler polynomials \( E_n^{(r)}(z) \) of order \( r \), are usually defined by means of the following generating functions:

\[
\left( \frac{2}{e^t+1} \right)^r = \sum_{m=0}^{\infty} E_n^{(r)} t^m, \quad (|t| < \pi),
\]

and

\[
\left( \frac{2}{e^t+1} \right)^r e^{zt} = \sum_{m=0}^{\infty} E_n^{(r)}(z) t^m, \quad (|t| < \pi),
\]

so that the ordinary Euler numbers \( E_n \) and polynomials \( E_n(z) \) are given, respectively, by

\[
E_n := E_n^{(1)} \quad \text{and} \quad E_n(z) := E_n^{(1)}(z).
\]

Some interesting properties of the generalized \((p,q)\)-Euler numbers and polynomials were first investigated by Ryoo [7]. We begin by recalling here definitions of generalized \((p,q)\)-Euler numbers and polynomials as follows. Let \( r \) be a positive integer and let \( \chi \) be Dirichlet’s character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \).

**Definition 1.** For \( 0 < q < p \leq 1 \), the generalized \((p,q)\)-Euler numbers \( E_{n,\chi,p,q} \) and polynomials \( E_{n,\chi,p,q}(z) \) related to \( \chi \) are defined by means of the generating functions

\[
\sum_{n=0}^{\infty} E_{n,\chi,p,q} t^n = [2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) q^m e^{[z+m]_p t}.
\]

and

\[
\sum_{n=0}^{\infty} E_{n,\chi,p,q}(z) t^n = [2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) q^m e^{[z+m]_p t}.
\]

respectively.

The multiple \((p,q)\)-Hurwitz–Euler eta functions of the \((p,q)\)-extension of the multiple Hurwitz–Euler eta functions were constructed in [3]. The aim of this paper is that a new generalization of the multiple \((p,q)\)-Hurwitz–Euler eta function related to \( \chi \) is constructed and studied. The sections have the following contents. In Section 2, Carlitz-type generalized higher-order \((p,q)\)-Euler numbers and polynomials related to \( \chi \) are defined. We derive some of their relevant properties. In Section 3, by using the complex integral and the Dirichlet character with conductor, we can also define the Dirichlet-type multiple \((p,q)\)-L function. That is, Dirichlet-type multiple \((p,q)\)-L functions are the generalization of multiple \((p,q)\)-Hurwitz–Euler eta functions. In Section 4, first, the summation is calculated by applying a new calculation technique using the Dirichlet character with conductor, and the symmetry of the Dirichlet-type multiple \((p,q)\)-L function is derived using the properties of the Dirichlet character. Specifically, if \( \chi \) is a trivial character (equal to 1 for all \( n \)), then the results of [3] are the same as our results. Next, we use this symmetric
property for the Dirichlet-type multiple \((p, q)\)-L function to obtain a symmetric identity about Carlitz-type higher-order \((p, q)\)-Euler numbers and polynomials related to \(\chi\).

The following diagram shows the various types of Euler polynomials and generalized Euler polynomials. Those polynomials in the first and second rows of the diagram were studied by Hwang and Ryoo [3,4]; Ryoo, Kim and Jang [8]; Ryoo [7,9]; Abramowicz and Stegun [10]; and Yang and Qiao [11]; and Srivastava [12] and Y. Simsek [13], respectively. The study of these has produced beneficial results in number theory. The best known in number theory is the one-variable zeta function, specifically the Riemann zeta function and its generalizations. Research on the multi-zeta function began to develop more intensively from the beginning of the 21st century (see [1,2,14]). First, the Dirichlet-type multiple \((p, q)\)-L function is introduced as an important motive of this paper with the expansion of this research. We also establish symmetric identities for such functions and generalized higher-order \((p, q)\)-Euler polynomials related to a Dirichlet character \(\chi\) in a complex field. Next, the motivation of this paper is to obtain some explicit properties and symmetric identities for Carlitz-type generalized higher-order \((p, q)\)-Euler polynomials related to Dirichlet’s character \(\chi\) in the third row of the diagram.

\[
\sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!} = \left(\frac{e^{zt} - 1}{z}\right) e^{zt} \quad \text{(Euler polynomials)}
\]

\[
\sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!} = \left(\frac{2}{e^{zt} + 1}\right) e^{zt} \quad \text{(generalized Euler polynomials)}
\]

\[
\sum_{m=0}^{\infty} E_{n,p,q}(z) \frac{t^n}{n!} = \sum_{m=0}^{\infty} E_{n,p,q}(z) \frac{t^n}{n!} = \left[2\right]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{mz} e^{zt} \quad \text{(generalized \((p, q)\)-Euler polynomials)}
\]

\[
\sum_{m=0}^{\infty} E_{n,p,q}(z) \frac{t^n}{n!} = \sum_{m=0}^{\infty} E_{n,p,q}(z) \frac{t^n}{n!} = \left[2\right]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{mz} e^{zt} \quad \text{(generalized \((p, q)\)-Euler polynomials)}
\]

\[
\sum_{m=0}^{\infty} E_{n,p,q}(z) \frac{t^n}{n!} = \sum_{m=0}^{\infty} E_{n,p,q}(z) \frac{t^n}{n!} = \left[2\right]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{mz} e^{zt} \quad \text{(generalized \((p, q)\)-Euler polynomials)}
\]

\[
\sum_{m=0}^{\infty} E_{n,p,q}(z) \frac{t^n}{n!} = \sum_{m=0}^{\infty} E_{n,p,q}(z) \frac{t^n}{n!} = \left[2\right]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{mz} e^{zt} \quad \text{(generalized \((p, q)\)-Euler polynomials)}
\]

\[
2. \text{ Generalized Higher-Order \((p, q)\)-Euler Numbers and Polynomials Related to } \chi
\]

In this section, we construct generalized higher-order \((p, q)\)-Euler numbers and polynomials related to \(\chi\) and derive some of their relevant properties. Let \(\chi\) be Dirichlet’s character with conductor \(d \in \mathbb{N}\) with \(d \equiv 1 \pmod{2}\), and \(r\) be a positive integer.

**Definition 2.** For \(r \in \mathbb{N}\), the generalized higher-order \((p, q)\)-Euler polynomials \(E_{n,p,q}^{(r)}(z)\) related to \(\chi\) are defined by the following generating function:

\[
\sum_{n=0}^{\infty} E_{n,p,q}^{(r)}(z) \frac{t^n}{n!} = [2]_q \sum_{m_1, \ldots, m_r=0}^{\infty} \left(\frac{r}{\prod_{i=1}^{r} \chi(m_i)}\right) \times (-1)^{m_1+\cdots+m_r} q^{m_1+\cdots+m_r} e^{m_1+\cdots+m_r+z \eta q t}.
\]

where \(z = 0, E_{n,p,q}^{(r)}(0)\) are called the generalized higher-order \((p, q)\)-Euler numbers \(E_{n,p,q}^{(r)}(z)\) related to \(\chi\). Note that if \(r = 1\), then \(E_{n,p,q}^{(1)} = E_{n,p,q} \) and \(E_{n,p,q}^{(r)}(z) = E_{n,p,q}(z)\). Observe that if \(q \rightarrow 1, p = 1\) and \(\chi\) is a trivial character (equal to 1 for all \(n\)), then \(E_{n,p,q}^{(r)} \rightarrow E_n^{(r)}\) and \(E_{n,p,q}^{(r)}(z) \rightarrow E_n^{(r)}(z)\).
Definition 3. The generalized higher-order \((h, p, q)\)-Euler polynomials \(E_{n,h,p,q}(z)\) related to \(\chi\) are defined by the following generating function:

\[
\sum_{n=0}^{\infty} E_{n,h,p,q}(z) \frac{t^n}{n!} = [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{i=1}^{r} m_i} q^{\sum_{i=1}^{r} m_i} \left[ \sum_{i=1}^{r} m_i + z \right] \frac{t^n}{n!},
\]

(2)

where \(z = 0, E_{n,h,p,q} = E_{n,h,p,q}(0)\) are called the generalized higher-order \((h, p, q)\)-Euler numbers \(E_{n,h,p,q}\) related to \(\chi\). Remark that if \(h = 0\), then \(E_{n,h,p,q} = E_{n,p,q}\) and \(E_{n,h,p,q}(z) = E_{n,h,p,q}(z)\). Observe that if \(q \to 1, p = 1, \chi\) is a trivial character (equal to 1 for all \(n\)), then \(E_{n,h,p,q} \to E_{n}(z)\) and \(E_{n,h,p,q}(z) \to E_{n}(z)\).

Theorem 1. For \(n \in \mathbb{Z}_0\), we have

\[
E_{n,h,p,q}(z) = \left[ 2 \right]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{i=1}^{r} m_i} q^{\sum_{i=1}^{r} m_i} \left[ \sum_{i=1}^{r} m_i + z \right] \frac{t^n}{n!}.
\]

Proof. By using the above Definition (1), we get

\[
\sum_{n=0}^{\infty} E_{n,h,p,q}(z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \left[ 2 \right]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left( \prod_{i=1}^{r} \chi(m_i) \right) (-1)^{\sum_{i=1}^{r} m_i} q^{\sum_{i=1}^{r} m_i} \left[ \sum_{i=1}^{r} m_i + z \right] \frac{t^n}{n!} \right).
\]

Equating the coefficients of \(\frac{t^n}{n!}\), the statement has been proved.

If \(\chi\) is a trivial character (equal to 1 for all \(n\)) in the above theorem we obtain

\[
E_{n,h,p,q}(z) = \left[ 2 \right]_q^r \left( \frac{1}{p-q} \right) \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) (-1)^{l} p^l q^l \left[ \sum_{i=1}^{r} a_i \right] \frac{t^n}{n!}.
\]

From (1) and (2), we note that

\[
E_{n,h,p,q}(z + y) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) p^{(n-l)} y^l \frac{t^{n-l}}{n-l} E_{n,h,p,q}(z),
\]

\[
E_{n,h,p,q}(z) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) q^l \frac{t^{n-l}}{n-l} E_{n,h,p,q}(z).
\]

(4)
Theorem 2. For any positive integer $m$, we have

$$E_{r, \chi, p, q}^{(r)}(z) = 2^r q \sum_{m=0}^{\infty} \left( \frac{m + r - 1}{m} \right) (-1)^m \times \sum_{a_1, \ldots, a_r = 0}^{d-1} \prod_{i=1}^{r} \chi(a_i) \left( \sum_{i=1}^{-a_i + m} \right) \left[ \sum_{i=1}^{a_i + z + md} \right]_p^u.$$ 

Proof. By Taylor–Maclaurin series expansion of $(1 - b)^{-n}$, we have

$$\left( \frac{1}{1 + q^{d(l+1)}p^{d(n-l)}} \right)^r = \sum_{m=0}^{\infty} \left( \frac{m + r - 1}{m} \right) (-1)^m q^{dm(1+1)} p^{dm(n-l)}.$$

By the binomial expansion and Theorem 1, we also obtain the desired result immediately. \( \square \)

3. Dirichlet-Type Multiple $(p, q)$-L Function

In this section, we construct the Dirichlet-type multiple $(p, q)$-L-function. This function interpolates the generalized higher-order $(p, q)$-Euler polynomials related to $\chi$ at negative integers. The multiple $(p, q)$-L-function can be defined as follows (see [3]):

Definition 4. For $s, z \in \mathbb{C}$ with Re$(z) > 0$, the multiple $(p, q)$-L-function $L_{p, q}^{(r)}(s, z)$ is defined by

$$L_{p, q}^{(r)}(s, z) = 2^r q \sum_{m_1, \ldots, m_r = 0}^{\infty} \prod_{i=1}^{r} \chi(m_i) q^{m_1 + \cdots + m_r}.$$ 

Note that if $q \to 1, p = 1$, then $L_{p, q}^{(r)}(s, a) = 2^r \eta_r(s, a)$.

Definition 5. For $s, z \in \mathbb{C}$ with Re$(z) > 0$, the Dirichlet-type multiple $(p, q)$-L-function $L_{\chi, p, q}^{(r)}(s, z)$ is defined by

$$L_{\chi, p, q}^{(r)}(s, z) = 2^r q \sum_{m_1, \ldots, m_r = 0}^{\infty} \prod_{i=1}^{r} \chi(m_i) q^{m_1 + \cdots + m_r}.$$ 

Observe that if $q \to 1, p = 1$, and $\chi$ is a trivial character (equal to 1 for all $n$), then $L_{\chi, p, q}^{(r)}(s, z) = \eta_r(s, z)$. Note that if $\chi$ is a trivial character (equal to 1 for all $n$), $L_{\chi, p, q}^{(r)}(s, z) = L_{p, q}^{(r)}(s, z)$.

Let

$$\tilde{F}^{(r)}_{\chi, p, q}(z, t) = \sum_{n=0}^{\infty} E_{\chi, p, q}^{(r)}(z) \frac{t^n}{n!}$$

$$= 2^r q \sum_{m_1, \ldots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} \prod_{i=1}^{r} \chi(m_i) q^{m_1 + \cdots + m_r} |z|_{p, q}^t.$$ 

(5)

Theorem 3. For $r \in \mathbb{N}$, we have

$$\frac{1}{\Gamma(s)} \int_0^{\infty} \tilde{F}^{(r)}_{\chi, p, q}(z, -t) t^{s-1} dt = L_{\chi, p, q}^{(r)}(s, z),$$ 

(6)

where $\Gamma(s) = \int_0^{\infty} z^{s-1} e^{-z} dz.$
Proof. Applying the Mellin transformation to Equation (5), and Definition 5, we obtain

\[
\frac{1}{\Gamma(s)} \int_0^\infty \mathcal{F}_s^{(r)}(z, t) t^{-s-1} dt
\]

\[
= \frac{2^r}{\Gamma(s)} \sum_{m_1, \ldots, m_r=0}^\infty (-1)^{m_1+\cdots+m_r} \prod_{i=1}^r \chi(m_i) \frac{m_1+\cdots+m_r+z}{p,q} t^{m_1+\cdots+m_r} t^{-1} dt
\]

\[
= \frac{2^r}{\Gamma(s)} \sum_{m_1, \ldots, m_r=0}^\infty (-1)^{m_1+\cdots+m_r} \prod_{i=1}^r \chi(m_i) \frac{m_1+\cdots+m_r+z}{p,q} \int_0^\infty z^{s-1} e^{-z} dz
\]

\[
= \frac{2^r}{\Gamma(s)} \sum_{m_1, \ldots, m_r=0}^\infty (-1)^{m_1+\cdots+m_r} \prod_{i=1}^r \chi(m_i) \frac{m_1+\cdots+m_r+z}{p,q}
\]

\[
= \mathcal{L}_{r, p,q}(s, z).
\]

This completes the proof of Theorem 3. \( \square \)

The value of the Dirichlet-type multiple \((p, q)\)-L-function \(L_{r, p,q}(s, z)\) at negative integers is given explicitly by the following theorem:

Theorem 4. Let \( n \in \mathbb{N} \). Then we obtain

\[
L_{r, p,q}^{(r)}(-n, z) = E_{n, r, p,q}(z).
\]

Proof. Again, by using Equations (5) and (6), we have

\[
L_{r, p,q}^{(r)}(s, z) = \frac{1}{\Gamma(s)} \int_0^\infty \mathcal{F}_s^{(r)}(z, t) t^{-s-1} dt
\]

\[
= \frac{1}{\Gamma(s)} \sum_{m=0}^\infty E_{m, r, p,q}(z) \frac{(-1)^m}{m!} \int_0^\infty t^{m+s-1} dt.
\]

Observe that

\[
\Gamma(-n) = \int_0^\infty e^{-z} z^{-n-1} dz
\]

\[
= \lim_{z \to 0} \frac{2\pi i}{n!} \left( \frac{d}{dz} \right)^n (z^{n+1} e^{-z} z^{-n-1})
\]

\[
= 2\pi i \frac{(-1)^n}{n!}.
\]

For \( n \in \mathbb{N} \), let us take \( s = -n \) in Equation (7). Then, by Equations (7) and (8), and Cauchy residue theorem, we obtain

\[
L_{r, p,q}^{(r)}(-n, z) = \lim_{s \to -n} \frac{1}{\Gamma(s)} \sum_{m=0}^\infty E_{m, r, p,q}(z) \frac{(-1)^m}{m!} \int_0^\infty t^{m-n-1} dt
\]

\[
= 2\pi i \frac{1}{\Gamma(s)} \left( \frac{E_{r, p,q}(z)}{n!} \right) \frac{(-1)^n}{n!}
\]

\[
= 2\pi i \frac{1}{2\pi i \frac{(-1)^n}{n!}} \left( \frac{E_{r, p,q}(z)}{n!} \right) = E_{n, r, p,q}(z).
\]

Hence the proof is completed. \( \square \)

4. Some Symmetric Identities for the Dirichlet-Type Multiple \((p, q)\)-L Function

In this section, we obtain some symmetric identities for the Dirichlet-type multiple \((p, q)\)-L function. Finally, we get symmetric identities for generalized higher-order \((p, q)\)-Euler polynomials \(E_{r, p,q}(z)\) related to \( \chi \) using the symmetric properties for the Dirichlet-type multiple \((p, q)\)-L function \(L_{r, p,q}(s, z)\).
Let $a, b \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$, $b \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain certain symmetry identities for the multiple $(p, q)$-$L$ function.

**Theorem 5.** Let $a, b \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$, $b \equiv 1 \pmod{2}$. Then we obtain

$$
[b]_{p, q}^a [2]_{p, q}^b \sum_{j_1, \ldots, j_r=0}^{da-1} (-1)^{\sum_j} q^{\sum_j} \left( \prod_{j=1}^r \chi(j) \right)
\times L_{x_{p, q}}^{(r)} \left( s, bz + \frac{b}{a}(j_1 + \cdots + j_r) \right)
= [a]_{p, q}^a [2]_{p, q}^b \sum_{j_1, \ldots, j_r=0}^{db-1} (-1)^{\sum_j} q^{\sum_j} \left( \prod_{j=1}^r \chi(j) \right)
\times L_{x_{p, q}}^{(r)} \left( s, az + \frac{a}{b}(j_1 + \cdots + j_r) \right).
$$

**Proof.** Note that $[zy]_{p, q} = [z]_{p, q}^a [y]_{p, q}$ for any $z, y \in \mathbb{C}$. In Definition 5, by substituting $bz + \frac{b}{a}(j_1 + \cdots + j_r)$ for $z$ and replacing $q$ and $p$ with $q^a$ and $p^a$, respectively, we derive the next result

$$
\frac{1}{[2]_{p, q}^a} L_{x_{p, q}}^{(r)} \left( s, bz + \frac{b}{a}(j_1 + \cdots + j_r) \right)
= \sum_{m_1, \ldots, m_r=0}^\infty \frac{(-1)^{m_1 + \cdots + m_r} (\prod_{j=1}^r \chi(m_j)) q^{am_1 + \cdots + am_r}}{m_1 + \cdots + m_r + bz + \frac{b}{a}(j_1 + \cdots + j_r)} [p_{p, q}^a]
$$

$$
= \sum_{m_1, \ldots, m_k=0}^\infty \frac{(-1)^{m_1 + \cdots + m_r} (\prod_{j=1}^r \chi(m_j)) q^{am_1 + \cdots + am_r}}{a(m_1 + \cdots + m_k) + abz + b(j_1 + \cdots + j_k)} [a]_{p, q}^a
$$

$$
= \sum_{m_1, \ldots, m_k=0}^\infty \frac{(-1)^{m_1 + \cdots + m_r} (\prod_{j=1}^r \chi(m_j)) q^{am_1 + \cdots + am_r}}{a(m_1 + \cdots + m_k) + abz + b(j_1 + \cdots + j_k)} [a]_{p, q}^a
$$

$$
= \sum_{m_1, \ldots, m_k=0}^\infty \frac{(-1)^{m_1 + \cdots + m_r} (\prod_{j=1}^r \chi(m_j)) q^{am_1 + \cdots + am_r}}{a(m_1 + \cdots + m_k) + abz + b(j_1 + \cdots + j_k)} [a]_{p, q}^a
$$
By using the same method as in Equation (10), we have

\[
\frac{\sum_{m_1, \ldots, m_s=0}^{d-1} \left( -1 \right)^{\sum_{i=1}^s m_i} \chi(i) \prod_{i=1}^r \chi(i) \right)}{\sum_{m_1, \ldots, m_s=0}^{d-1} \left( -1 \right)^{\sum_{i=1}^s m_i} \chi(i) \prod_{i=1}^r \chi(i) \right)}
\]

Thus, from Equation (9), we can derive the following equation.

\[
\frac{\sum_{m_1, \ldots, m_s=0}^{d-1} \left( -1 \right)^{\sum_{i=1}^s m_i} \chi(i) \prod_{i=1}^r \chi(i) \right)}{\sum_{m_1, \ldots, m_s=0}^{d-1} \left( -1 \right)^{\sum_{i=1}^s m_i} \chi(i) \prod_{i=1}^r \chi(i) \right)}
\]

By using the same method as in Equation (10), we have

\[
\frac{\sum_{m_1, \ldots, m_s=0}^{d-1} \left( -1 \right)^{\sum_{i=1}^s m_i} \chi(i) \prod_{i=1}^r \chi(i) \right)}{\sum_{m_1, \ldots, m_s=0}^{d-1} \left( -1 \right)^{\sum_{i=1}^s m_i} \chi(i) \prod_{i=1}^r \chi(i) \right)}
\]

Therefore, the claimed result is obtained. □

Taking \( b = 1 \) and if \( \chi \) is a trivial character (equal to 1 for all \( n \)) in the above theorem, we have the following corollary.
Corollary 1. Let $a \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain
\[
L_{pq}^{(r)}(s, az) = \frac{[2]^r}{[2]^r[^2]_q[a]_{p,q}^a} \sum_{j_1, \ldots, j_r = 0}^{a-1} (-1)^{\sum_{i=1}^{r} h_i} q^{\sum_{i=1}^{r} h_i} L_{pq}^{(r)}(s, z + \frac{h_1 + \cdots + h_r}{a}).
\]

If $p = 1$ in Corollary 1, then we obtain the following corollary.

Corollary 2. Let $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain
\[
L_{q}^{(r)}(s, mz) = \frac{[2]^r}{[2]^r[^2]_q[m]_{q}^r} \sum_{j_1, \ldots, j_r = 0}^{m-1} (-1)^{\sum_{i=1}^{r} h_i} q^{\sum_{i=1}^{r} h_i} L_{q}^{(r)}(s, z + \frac{h_1 + \cdots + h_r}{m}).
\]

If $q \to 1$ and $p = 1$ in Corollary 2, then we obtain the following corollary.

Corollary 3. Let $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain
\[
\eta_r(s, z) = \frac{1}{m^n} \sum_{j_1, \ldots, j_r = 0}^{m-1} (-1)^{h_1 + \cdots + h_r} \eta_r(s, z + \frac{h_1 + \cdots + h_r}{m}).
\]

Let us take $s = -n$ in Theorem 5. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain certain symmetry identities for generalized higher-order $(p, q)$-Euler polynomials.

Theorem 6. Let $a, b \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$, $b \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain
\[
[a]^n_{p,q}[2]^r_{q} \sum_{j_1, \ldots, j_r = 0}^{a-1} (-1)^{\sum_{i=1}^{r} h_i} \left( \prod_{i=1}^{r} \chi(j_i) \right) p^{\sum_{i=1}^{r} h_i} \times E_{n, \chi, p, q^r} \left( bz + \frac{b}{a} (j_1 + \cdots + j_r) \right)
= [b]^n_{p,q}[2]^r_{q} \sum_{j_1, \ldots, j_r = 0}^{b-1} (-1)^{\sum_{i=1}^{r} h_i} \left( \prod_{i=1}^{r} \chi(j_i) \right) q^{\sum_{i=1}^{r} h_i} \times E_{n, \chi, p, q^r} \left( az + \frac{a}{b} (j_1 + \cdots + j_r) \right).
\]

Proof. By Theorems 4 and 5, we obtain the following theorem. \(\square\)

If $\chi$ is a trivial character (equal to 1 for all $n$) and taking $b = 1$ in Theorem 6, we have the following corollary.

Corollary 4. Let $a \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain
\[
E_{n, p, q}(az) = \frac{[2]^r}{[2]^r[^2]_q[a]_{p,q}^a} \sum_{j_1, \ldots, j_r = 0}^{a-1} (-1)^{\sum_{i=1}^{r} h_i} q^{\sum_{i=1}^{r} h_i} E_{n, \chi, p, q^r} \left( z + \frac{h_1 + \cdots + h_r}{a} \right).
\]

If $q \to 1$ and $p = 1$ in Equation (12), then we obtain the following corollary (see [10]).

Corollary 5. Let $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_0$, we obtain
\[
E_{n}^{(r)}(z) = m^n \sum_{j_1, \ldots, j_r = 0}^{m-1} (-1)^{h_1 + \cdots + h_r} E_{n}^{(r)} \left( z + \frac{h_1 + \cdots + h_r}{m} \right).
\]
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By Equation (3), we have
\[
\frac{d_a - 1}{j_1, \ldots, j_r = 0} (-1)^{j_r-1} \left( \prod_{j=1}^{r} \chi(j) \right) q^{b_j} E_{n, i X, p^s, a^r}^{(r)} (b z, \frac{b}{a} (j_1 + \cdots + j_k))
\]
\[
= \frac{d_a - 1}{j_1, \ldots, j_r = 0} (-1)^{j_r-1} \left( \prod_{j=1}^{r} \chi(j) \right) q^{b_j} E_{n, i X, p^s, a^r}^{(r)} (b z, \frac{b}{a} (j_1 + \cdots + j_k))
\]
\[
\times \sum_{i=0}^{n} \left( \prod_{i=0}^{n} \chi(i) \right) q^{b(n-i)(j_1 + \cdots + j_r)} E_{n-i X, p^s, a^r}^{(r)} (b z, \frac{b}{a} (j_1 + \cdots + j_k))
\]
\[
= \frac{d_a - 1}{j_1, \ldots, j_r = 0} (-1)^{j_r-1} \left( \prod_{j=1}^{r} \chi(j) \right) q^{b_j} E_{n, i X, p^s, a^r}^{(r)} (b z, \frac{b}{a} (j_1 + \cdots + j_k))
\]
\[
\times \sum_{i=0}^{a-1} \left( \prod_{i=0}^{n} \chi(i) \right) q^{b(n-i+1)(j_1 + \cdots + j_r)} [j_1 + \cdots + j_r]^{i, p^s, a^r}.
\]

Hence we have the following theorem.

**Theorem 7.** Let \( a, b \in \mathbb{N} \) with \( a \equiv 1 \ (\text{mod} \ 2) \), \( b \equiv 1 \ (\text{mod} \ 2) \). For \( r \in \mathbb{N} \) and \( n \in \mathbb{Z}_0 \), we obtain
\[
\frac{d_a - 1}{j_1, \ldots, j_r = 0} (-1)^{j_r-1} \left( \prod_{j=1}^{r} \chi(j) \right) q^{b_j} E_{n, i X, p^s, a^r}^{(r)} (b z, \frac{b}{a} (j_1 + \cdots + j_k))
\]
\[
= \sum_{i=0}^{n} \left( \prod_{i=0}^{n} \chi(i) \right) q^{b(n-i+1)(j_1 + \cdots + j_r)} [j_1 + \cdots + j_r]^{i, p^s, a^r}.
\]

For each integer \( n \geq 0 \), let
\[
A_{n, i X, p^s, a^r}(w) = \frac{d_a - 1}{j_1, \ldots, j_r = 0} (-1)^{j_r-1} \left( \prod_{j=1}^{r} \chi(j) \right) q^{b(j_1 + \cdots + j_k)} [j_1 + \cdots + j_k]^{i, p^s, a^r}.
\]

The above sum \( A_{n, i X, p^s, a^r}(w) \) are called the alternating generalized \((p, q)\)-power sums. By Theorem 7, we have
\[
[2]_{q, a}^{n, p^s, a^r} \sum_{j_1, \ldots, j_r = 0}^{d_a - 1} (-1)^{j_r-1} \left( \prod_{j=1}^{r} \chi(j) \right) q^{b_j} E_{n, i X, p^s, a^r}^{(r)} (b z, \frac{b}{a} (j_1 + \cdots + j_k))
\]
\[
= [2]_{q, a}^{n, p^s, a^r} \sum_{i=0}^{n} \left( \prod_{i=0}^{n} \chi(i) \right) q^{b(n-i+1)(j_1 + \cdots + j_r)} [j_1 + \cdots + j_r]^{i, p^s, a^r}
\]
\[
\times A_{n, i X, p^s, a^r}(w) (d a)
\]
By using the same method as in Equation (13), we have

\[
[2]_{p,q}^r [b]_p^n \sum_{i=0}^{dB-1} (-1)^{\Sigma_{i-1}^h} \left( \prod_{i=1}^r \chi(j_i) \right) \times E^{(r)}_{n_i,j_i,p^h,q^h}(az) \\
\times [2]_{p,q}^r \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [a]_p^i [b]_p^{n-i} p^{dbzi} E^{(r,i)}_{n-i,p^h,q^h}(az) \\
= [2]_{p,q}^r \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [a]_p^i [b]_p^{n-i} p^{dbzi} E^{(r,i)}_{n-i,p^h,q^h}(az) \\
\times A^{(r)}_{n,j,i,q^h,p^h}(db) \\
\times A^{(r)}_{n,j,i,q^h,p^h}(da).
\]

Therefore, by Equations (13) and (14) and Theorem 6, we have the following theorem.

**Theorem 8.** Let \(a, b \in \mathbb{N}\) with \(a \equiv 1 \mod 2\), \(b \equiv 1 \mod 2\). For \(r \in \mathbb{N}\) and \(n \in \mathbb{Z}_0\), we obtain

\[
[2]_{p,q}^r \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [a]_p^i [b]_p^{n-i} p^{dbzi} E^{(r,i)}_{n-i,p^h,q^h}(az) \\
\times A^{(r)}_{n,j,i,q^h,p^h}(db) \\
= [2]_{p,q}^r \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [b]_p^i [a]_p^{n-i} p^{dbzi} E^{(r,i)}_{n-i,p^h,q^h}(az) \\
\times A^{(r)}_{n,j,i,q^h,p^h}(da).
\]

By Theorem 8, we have the interesting symmetric identity for the generalized higher-order \((h, p, q)\)-Euler numbers \(E^{(r,i)}_{n_i,j_i,p^h,q^h}\) related to \(\chi\).

**Corollary 6.** Let \(a, b \in \mathbb{N}\) with \(a \equiv 1 \mod 2\), \(b \equiv 1 \mod 2\). For \(k \in \mathbb{N}\) and \(n \in \mathbb{Z}_0\), we obtain

\[
[2]_{p,q}^r \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [a]_p^i [b]_p^{n-i} p^{dbzi} E^{(r,i)}_{n-i,p^h,q^h}(az) \\
\times A^{(r)}_{n,j,i,q^h,p^h}(db) \\
= [2]_{p,q}^r \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [b]_p^i [a]_p^{n-i} p^{dbzi} E^{(r,i)}_{n-i,p^h,q^h}(az) \\
\times A^{(r)}_{n,j,i,q^h,p^h}(da).
\]

If \(p = 1\) in Corollary 6, then we obtain the following corollary.

**Corollary 7.** Let \(a, b \in \mathbb{N}\) with \(a \equiv 1 \mod 2\), \(b \equiv 1 \mod 2\). For \(k \in \mathbb{N}\) and \(n \in \mathbb{Z}_0\), we obtain

\[
[2]_{p,q}^r \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [a]_q^i [b]_q^{n-i} A^{(r)}_{n,j,i,q^h}(db) E^{(r,i)}_{n-i,p^h,q^h} \\
= [2]_{p,q}^r \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [b]_q^i [a]_q^{n-i} A^{(r)}_{n,j,i,q^h}(da) E^{(r,i)}_{n-i,p^h,q^h},
\]

where

\[
A^{(r)}_{n,j,i,q^h}(w) = \sum_{j_1,\ldots,j_r=0}^{w-1} (-1)^{\Sigma_{j=1}^r h} \left( \prod_{i=1}^r \chi(j_i) \right) q^{(n-i+1)(\Sigma_{j=1}^r h)}[j_1 \cdots + j_k]_q,
\]

If \(\chi\) is a trivial character (equal to 1 for all \(n\)), \(q \to 1, p = 1, r = 1\) in Theorem 6, then we have the following theorem for Euler polynomials, which are symmetric in \(a\) and \(b\) (see [11]).
Corollary 8. Let $a,b \in \mathbb{N}$ with $a \equiv 1 \pmod{2}$, $b \equiv 1 \pmod{2}$. Then we obtain

$$a^n \sum_{j=0}^{a-1} (-1)^j E_n \left( b z + \frac{b}{a} j \right) = b^n \sum_{j=0}^{b-1} (-1)^j E_n \left( a z + \frac{a}{b} j \right).$$

5. Conclusions and Future Directions

The motivation of this paper was to investigate some explicit identities for the generalized higher-order $(p,q)$-Euler polynomials $E_{n,p,q}^{(r)}(z)$ attached to $\chi$ in the third row of the diagram at page 3. First, we defined the generalized higher-order $(p,q)$-Euler polynomials $E_{n,p,q}^{(r)}(z)$ related to $\chi$ (see Definition 2). In order to obtain a symmetry identity for the generalized higher-order $(p,q)$-Euler polynomials $E_{n,p,q}^{(r)}(z)$ related to $\chi$, we constructed the Dirichlet-type multiple $(p,q)$-$L$ function (see Definition 5). In Theorem 5, we gave a symmetry identity for the Dirichlet-type multiple $(p,q)$-$L$ function $L_{\chi,p,q}^{(r)}(s,z)$. Next, using the Dirichlet-type multiple $(p,q)$-$L$-function $L_{\chi,p,q}^{(r)}(s,z)$, we also obtained some symmetry identities for the generalized higher-order $(p,q)$-Euler polynomials $E_{n,p,q}^{(r)}(z)$ related to $\chi$ (see Theorems 6 and 7). Finally, we also obtained the explicit identities related to the generalized higher-order $(p,q)$-Euler polynomials $E_{n,p,q}^{(r)}(z)$ related to $\chi$, the alternating generalized $(p,q)$-power sums (see Theorems 1, 2 and 8, and Corollaries 6 and 7). In particular, if we take $\chi = 1$ in all results of this article, then [3] is the special case of our results. The Dirichlet character is the most important character in the algebra. Many mathematicians conduct research related to the Dirichlet character. In our paper, we wanted to generalize our known polynomial using this Dirichlet character, which has many applications in the algebra.

We also want to further study the solution of equation $E_{n,p,q}^{(r)}(z) = 0$ as an application. In our next research, we will draw distribution pictures of solutions of this equation using computer graphics and also investigate the symmetry of distribution pictures of the solutions, which may give clues about the properties of the solutions. Our referees mentioned that multi-zeta values have significant connections to arithmetic geometry. They also asked about the relation between arithmetic geometry and Carlitz $q$-analogues. Although we do not know how arithmetic geometry is related to Carlitz $q$-analogues, this is very helpful question. We have included reference 14 because it suggests several applications of special functions.

Author Contributions: All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Dong-A university research fund.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees for their valuable comments, which improved the original manuscript in its present form.

Conflicts of Interest: The authors declare no conflicts of interest.

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