A note on special duality triads and their operator valued counterparts

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Abstract

We shall work with the so called duality triads following Kwaśniewski. In particular in this note we propose some extensions of them - hence we choose such special class of triads that admit - all at once - a unified combinatorial interpretation in a way Konvalina does. The proposed extensions contain also definition of operator valued arrays of Konvalina-like $C_{n,k}$ - for the sake of future investigation and applications.

1 Preliminaries.

At first let us quote - following [1, 2] the basic definitions introducing the so called - duality triads.

As underlined in [1, 2] these duality triads arise as a natural and sine qua non object for example in dynamical data basis' models. For this and other motivations and other "triad sources" of inspiration - we refer to [1, 2] and references therein. We shall consider now the infinite array of numbers $c_{n,k}$ satisfying the following 3-term recurrence equation (for $n, k$ being nonnegative integers and $i_k, q_k, d_k$ being integer numbers):

\[
\begin{align*}
c_{n+1,k} &= i_{k-1} c_{n,k-1} + q_k c_{n,k} + d_{k+1} c_{n,k+1} \\
c_{0,0} &= 1 \\
c_{0,k} &= 0 \text{ for } k > 0
\end{align*}
\] (1)
Then the dual recurrence with respect to (1) is of the form:

\[
\begin{cases}
x \phi_k(x) = d_k \phi_{k-1}(x) + q_k \phi_k(x) + i_k \phi_{k+1}(x) \\
\phi_0(x) = 1 \\
\phi_{-1}(x) = 0
\end{cases}
\]

where \( \{\phi_k(x)\}_{k \geq 0} \) is a polynomial sequence such that \( \deg \phi_k(x) = k \) and

\[
x^n = \sum_{k=0}^{n} c_{n,k} \phi_k(x)
\]

Given the recurrence equations (1) and (2) one can derive the identity (3) (for proof see [1]). From (3) one can observe that the sequence of numbers \( c_{n,k} \) is the sequence of expansion coefficients of polynomials \( x^n \) in the basis of polynomials \( \{\phi_k(x)\}_{k \geq 0} \). In combinatorics \( c_{n,k} \) are called connection constants [3, 4].

The equations (1), (2), (3) compose the duality triads and the coefficients \( c_{n,k} \) are called the triad coefficients, while the polynomials \( \{\phi_k(x)\}_{k \geq 0} \) are to be refer to as triad polynomials. The first important and illustrative examples of duality triad polynomials were delivered in [2] from where all our quotations come.

## 2 Konvalina triads.

In [5, 6] Konvalina presents a unified simultaneous combinatorial interpretation for both binomial coefficients of the first and second kind and Gaussian coefficients of the first and second kind and Stirling numbers of the first and second kind.

Let us present it in brief. Consider as in [6] \( n \) different boxes containing different balls - the \( i \)-th box contains \( w_i \) balls \( (w_i \geq 1) \). The number \( w_i \) is called the weight of box \( i \) and vector \( \vec{w} = (w_1, w_2, \ldots, w_n) \) is the weight vector.

**Definition 2.1.** The generalized Konvalina binomial coefficient of the first kind with vector weight \( \vec{w} \), denoted \( \mathcal{C}_k^n(\vec{w}) \), is the number of ways to select \( k \) balls from \( n \) different boxes \( (i_1 < i_2 < \ldots < i_k) \) and then taking one ball from each of the \( k \) selected boxes (there are \( w_{i_1} w_{i_2} \cdots w_{i_k} \) possibilities):

\[
\mathcal{C}_k^n(\vec{w}) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} w_{i_1} w_{i_2} \cdots w_{i_k}
\]

(4)
Definition 2.2. The generalized Konvalina binomial coefficient of the second kind with vector weight $-\overrightarrow{w}$, denoted $S^n_k(-\overrightarrow{w})$, is the number of ways to select $k$ balls from $n$ not necessarily different boxes ($i_1 \leq i_2 \leq \ldots \leq i_k$) and then taking one ball from each of the $k$ selected boxes (there are $w_{i_1}w_{i_2}\ldots w_{i_k}$ possibilities):

$$S^n_k(-\overrightarrow{w}) = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n} w_{i_1}w_{i_2}\ldots w_{i_k} \quad (5)$$

Counting the ways of selections $k$ balls from $n$ different or not necessarily different boxes, respectively, with dependence either the last box was chosen or not one can derive the following recurrences:

$$C^n_k(-\overrightarrow{w}) = C^n_{k-1}(-\overrightarrow{w}) + w_n C^n_{k-1}(-\overrightarrow{w}) \quad (6)$$

$$S^n_k(-\overrightarrow{w}) = S^n_{k-1}(-\overrightarrow{w}) + w_n S^n_{k-1}(-\overrightarrow{w}) \quad (7)$$

Setting $S^n_k(-\overrightarrow{w}) = S^n_{n-k}(-\overrightarrow{w})$ one can get the equivalent to (7) recurrence:

$$S^n_k(-\overrightarrow{w}) = w_k S^n_{k-1}(-\overrightarrow{w}) + S^n_{k-1}(-\overrightarrow{w}) \quad (8)$$

Observation 2.1. Let $\overrightarrow{w}$ be a weight vector. Then from Definitions 2.1, 2.2 one can observe:

(a) $C^n_0(-\overrightarrow{w}) = 1$ for $n \geq 0$ and $C^n_k(-\overrightarrow{w}) = 0$ for $k > 0$

(b) $S^n_0(-\overrightarrow{w}) = 1$ for $n \geq 0$ and $S^n_k(-\overrightarrow{w}) = 0$ for $k > 0$

The Konvalina coefficients $C^n_k(-\overrightarrow{w})$, $S^n_k(-\overrightarrow{w})$ for the vector weight $\overrightarrow{w}$ along with conditions given in the observation above constitute two classes of triad coefficients with $i_{k-1} = w_n$, $q_k = 1$, $d_k = 0$ and $i_k = 1$, $q_k = w_k$, $d_k = 0$, respectively. Then let us define two classes of duality triads:

Definition 2.3. Konvalina triads of the first kind are triads satisfying the following two 2-terms dual recurrence equations and the third coming next equation interrelating the first two ones:

$$\begin{cases} C^{n+1}_k(-\overrightarrow{w}) = w_n C^n_{k-1}(-\overrightarrow{w}) + C^n_k(-\overrightarrow{w}) \\ C^n_0(-\overrightarrow{w}) = 1 \\ C^n_k(-\overrightarrow{w}) = 0 \text{ for } k > 0 \end{cases} \quad (9)$$

$$\begin{cases} x\phi_k(x) = \phi_k(x) + i_k\phi_{k+1}(x) \text{ where } i_{k-1} = w_n \\ \phi_0(x) = 1 \\ \phi_{-1}(x) = 0 \end{cases} \quad (10)$$
where \( \{ \phi_k(x) \}_{k \geq 0} \) is a polynomial sequence such that \( \deg \phi_k(x) = k \) and

\[
x^n = \sum_{k=0}^{n} C_k^n(\overrightarrow{w}) \phi_k(x)
\]  

(11)

**Definition 2.4.** Konvalina triads of the second kind are triads satisfying the following two 2-terms dual recurrence equations and the third coming next equation interrelating the first two ones:

\[
\begin{align*}
S_{k+1}^n(\overrightarrow{w}) &= S_k^n(\overrightarrow{w}) + w_k S_{k-1}^n(\overrightarrow{w}) \\
S_0^n(\overrightarrow{w}) &= 1 \\
S_{k-1}^n(\overrightarrow{w}) &= 0 \text{ for } k > 0
\end{align*}
\]  

(12)

\[
\begin{align*}
x \phi_k(x) &= w_k \phi_k(x) + \phi_{k+1}(x) \\
\phi_0(x) &= 1 \\
\phi_{-1}(x) &= 0
\end{align*}
\]  

(13)

where \( \{ \phi_k(x) \}_{k \geq 0} \) is a polynomial sequence such that \( \deg \phi_k(x) = k \) and

\[
x^n = \sum_{k=0}^{n} S_k^n(\overrightarrow{w}) \phi_k(x)
\]  

(14)

Let us now quote from [1, 2] few simplest examples of triads which are to be here now Konvalina triads of the first and second kind.

**Example 2.1.** Pascal triad:

\[
\begin{align*}
\binom{n+1}{k} &= \binom{n}{k-1} + \binom{n}{k} \\
\binom{n}{0} &= 1 \\
\binom{n}{k} &= 0 \text{ for } k > 0
\end{align*}
\]  

\[
\begin{align*}
x \phi_k(x) &= \phi_k(x) + \phi_{k+1}(x) \\
\phi_0(x) &= 1 \\
\phi_{-1}(x) &= 0
\end{align*}
\]  

where

\[
x^n = \sum_{k=0}^{n} \binom{n}{k} \phi_k(x).
\]

Hence

\[\phi_k(x) = (x - 1)^k.\]
The coefficients $c_{n,k} = \binom{n}{k}$ are Konvalina triads coefficients of the first kind with vector weight $\vec{w} = (1, 1, \ldots, 1)$:

$$\binom{n}{k} = C^n_k(\vec{w}).$$

**Example 2.2.** Stirling triad:

\[
\begin{cases}
\binom{n}{k} = k \binom{n-1}{k} + \binom{n-1}{k-1} \\
\binom{0}{0} = 1 \\
\binom{0}{k} = 0 \text{ for } k > 0
\end{cases}
\]

where

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \phi_k(x).$$

Hence

$$\phi_k(x) = x^k = x(x-1)(x-2)\cdots(x-k+1).$$

The coefficients $c_{n,k} = \binom{n}{k}$ are Konvalina triads coefficients of the second kind with vector weight $\vec{w} = (1, 2, 3, \ldots, n)$:

$$\binom{n}{k} = S^n_{n-k}(\vec{w}).$$

**Example 2.3.** $q$-Gaussian triad:

\[
\begin{cases}
\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q \\
\binom{0}{0}_q = 1 \\
\binom{k}{0}_q = 0 \text{ for } k > 0
\end{cases}
\]

where

$$x\phi_k(x) = q^k \phi_k(x) + \phi_{k+1}(x)$$

$$\phi_0(x) = 1$$

$$\phi_{-1}(x) = 0$$

5
where
\[ x^n = \sum_{k=0}^{n} \binom{n}{k}_q \phi_k(x). \]

Hence
\[ \phi_k(x) = \prod_{i=0}^{k-1} (x - q^i). \]

The coefficients \( c_{n,k} = \binom{n}{k}_q \) are Konvalina triads coefficients of the second kind with vector weight \( \overrightarrow{w} = (1, q, q^2, \ldots, q^{n-1}) \):
\[ \binom{n}{k}_q = S^{n-k+1}(\overrightarrow{w}). \]

Though the Stirling numbers of the first kind \( \left[ \begin{array}{c} n \\ k \end{array} \right] \)-satisfying (6) -are the Konvalina generalized binomial coefficients \( \left[ \begin{array}{c} n \\ k \end{array} \right] = C^{n-1}_{n-k}(\overrightarrow{w}) \) where \( \overrightarrow{w} = (1, 2, \cdots, n) \) (see [5, 6]) due the well known recurrence equation:
\[
\begin{cases}
\left[ \begin{array}{c} n \\ k \end{array} \right] = (n - 1) \left[ \begin{array}{c} n - 1 \\ k \end{array} \right] + \left[ \begin{array}{c} n - 1 \\ k - 1 \end{array} \right] \\
\left[ \begin{array}{c} 0 \\ k \end{array} \right] = 1 \\
\left[ \begin{array}{c} 0 \\ k \end{array} \right] = 0 \text{ for } k > 0
\end{cases}
\]

(15)

they don’t constitute the Konvalina triads coefficients \( \left[ \begin{array}{c} n \\ k \end{array} \right] \). The non-triad polynomials that anyhow might be eventually associated with the Stirling numbers of the second kind - constitute - as seen from what follows - a one \( n \) - parameter family of polynomial sequences:
\[
\begin{cases}
x\phi_k(x) = (n - 1)\phi_k(x) + \phi_{k+1}(x) \\
\phi_0(x) = 1 \\
\phi_{-1}(x) = 0
\end{cases}
\]

The other important and illustrative examples of duality triad polynomials were delivered in [1, 2] from where all our quotations come.
3 Properties of Konvalina triads coefficients.

The Konvalina triads coefficients deliver array of combinatorics interpreted. These Konvalina triad coefficients that share the following elementary properties.

**Proposition 3.1.** Let $\mathbf{w}$ be a vector weight of $n$ different boxes. Then:

1. $C^n_k(\mathbf{w}) = \sum_{i=0}^{n} w_i C^{i-1}_{k-1}(\mathbf{w})$

2. $S^n_k(\mathbf{w}) = \sum_{i=0}^{n} w_i S^{i}_{k-1}(\mathbf{w})$

**Proof.** The proof runs counting ways of selections $k$ balls from $n$ different boxes with vector weight $\mathbf{w}$ choosing at first $k$ different (not necessarily different) boxes, respectively, and next taking one ball per box from selected boxes with respect to the last box that has been selected. $A_i$ ($1 \leq i \leq n$) denotes the class of $k$-selections that the box $i$ is the last box has been selected. $|A_i| = w_i C^{i-1}_{k-1}$ as one ball is taken from the box $i$ and $k-1$ balls from boxes $\{1, 2, 3, \ldots, i-1\}$ (box repetitions not allowed). $|A_i| = w_i S^{i}_{k-1}$ if box repetitions allowed (one ball is taken from the box $i$ and $k-1$ balls from boxes $\{1, 2, 3, \ldots, i\}$). Hence

\[
C^n_k(\mathbf{w}) = \sum_{i=1}^{n} w_i C^{i-1}_{k-1}(\mathbf{w}).
\]

\[
S^n_k(\mathbf{w}) = \sum_{i=1}^{n} w_i S^{i}_{k-1}(\mathbf{w}).
\]

**Proposition 3.2.** Let $\mathbf{w}$ be a vector weight of $n$ different boxes. Then:

1. $C^{n+m}_k(\mathbf{w}) = \sum_{i \geq 0} C^n_i(\mathbf{w}) C^m_{k-i}(\mathbf{v})$ where $\mathbf{v} = (w_{n+1}, w_{n+2}, \ldots, w_{n+m})$.

2. $S^{n+m}_k(\mathbf{w}) = \sum_{i \geq 0} S^n_i(\mathbf{w}) S^m_{k-i}(\mathbf{v})$ where $\mathbf{v} = (w_{n+1}, w_{n+2}, \ldots, w_{n+m})$.

**Proof.** Selecting $k$ balls from $n$ different boxes with vector weight $\mathbf{w}$ choosing at first $k$ different (not necessarily different) boxes, respectively, and next taking one ball per box, $i$ balls are taken from $i$ boxes selected from $\{1, 2, \ldots, n\}$ and $k-i$ balls from $k-i$ boxes selected from $\{n+1, n+2, \ldots, n+m\}$ ($0 \leq i \leq k$). Hence thesis.

From Proposition 3.1 one can get the following well-known identities:
Corollary 3.1. Let us denote vectors: \( \vec{1} = (1, 1, \ldots, 1) \), \( \vec{i} = (1, 2, 3, \ldots, n) \), \( \vec{q} = (1, q, q^2, \ldots, q^{n-1}) \).

1. \( C_{k+1}^{n+k+1}(\vec{1}) = \sum_{i=0}^{n+k+1} C_i^i(\vec{1}) = \sum_{i=0}^n C_i^{k+1}(\vec{1}) \) hence
   \[
   \binom{n+k+1}{k+1} = \sum_{i=0}^n \binom{k+i}{k}.
   \]

2. \( C_{k+1}^{n+1}(\vec{i}) = \sum_{i=0}^{n+1} C_i^{i-1}(\vec{i}) = \sum_{i=0}^n C_i^{i}(\vec{i}) \) hence
   \[
   \binom{n+1}{k+1} = \sum_{i=0}^n \binom{i}{k}.
   \]

3. \( S_{k+1}^n(\vec{i}) = \sum_{i=0}^n iS_i^i(\vec{i}) = \sum_{i=0}^n iS_i^i(\vec{i}) \) hence
   \[
   \left\{ \binom{n+k+1}{n} = \sum_{i=0}^n \left\{ \binom{k+i}{i} \right\} \right. \]

4. \( C_{k+1}^{n+k}(\vec{i}) = \sum_{i=0}^{n+k} iC_i^{i-1}(\vec{i}) = \sum_{i=k+1}^{n+k} iC_i^{i-1}(\vec{i}) = \sum_{i=1}^n (k+i)C_i^{k+i-1}(\vec{i}) \) hence
   \[
   \binom{n+k+1}{n} = \sum_{i=0}^n (k+i) \binom{k+i}{i}.
   \]

5. \( S_{k+1}^{n+1}(\vec{q}) = \sum_{i=0}^{n+1} q^{i-1}S_i^i(\vec{q}) = \sum_{i=0}^n q^iS_i^{i+1}(\vec{q}) \) hence
   \[
   \binom{n+k+1}{k+1} = \sum_{i=0}^n \binom{k+i}{k} q^i.
   \]

Proposition (3.2) in particular is the Cauchy identity:

\[
\binom{n+m}{k} = C_{k+m}^n(\vec{1}) = \sum_{i \geq 0} C_i^n(\vec{i})C_{k-i}^m(\vec{1}) = \sum_{i \geq 0} \binom{n}{i}\binom{m}{k-i}
\]

4 \( \hat{q}_\psi \) Konvalina-like operators

In this section we introduce as suggested by Kwasniewski to the present author the operator valued binomial array matrix elements \([7, 8]\) and then consequently operator valued Gaussian arrays and operator valued Stirling
arrays. Apart from the already existing applications in [7, 8] a similar in character operator valued infinite arrays appear also in physics (see for more Katriel and Kibler [9]). The idea of considering operator valued arrays (this time) in the setting of Konvalina generalized binomial coefficients is presented in what follows. All operators are supposed to act on the algebra of formal series including the subalgebra $P = \mathbb{F}[x]$ of polynomials of a single variable $x$ over the field $\mathbb{F}$ of characteristic 0. We thus ensure the sufficient background for the constructions to follow (see for more in [12]).

We shall use the upside-down notation introduced in [11, 10] and applied for example in [7, 8] which are of the source importance for what follows. At first then let us re-introduce this $\psi$-notation.

Consider $\mathfrak{I}$ - the family of functions' sequences such that:

$$\mathfrak{I} = \{ \psi; R \supset [a, b] ; q \in [a, b] ; \psi (q) : Z \to \mathbb{F} ; \psi_0 (q) = 1 ; \psi_n (q) \neq 0; \psi_{-n} (q) = 0; n \in \mathbb{N}\}.$$ 

We shall call $\psi = \{ \psi_n (q) \}_{n \geq 0} ; \psi_n (q) \neq 0; n \geq 0$ and $\psi_0 (q) = 1$ an admissible sequence. Consequently the symbol $n_\psi$ denotes

$$n_\psi = \frac{\psi_{n-1} (q)}{\psi_n (q)}, \quad n \geq 0.$$ 

Then $\psi$-factorial and lower $\psi$-factorial are given accordingly by

$$n_\psi! \equiv \psi_{n-1}^{-1} (q) \equiv n_\psi (n-1)_\psi (n-2)_\psi (n-3)_\psi \ldots 2_\psi 1_\psi; \quad 0_\psi! = 1$$

$$n^k_\psi = n_\psi (n-1)_\psi \cdots (n-k+1)_\psi.$$ 

**Example 4.1.** Taking an admissible sequence $\psi_n (q) = (n_q!)^{-1}$ where $n_q = 1 + q + q^2 + \cdots + q^{n-1}$ is well known $q$-deformation of a natural number $n$ we obtain $n_\psi = n_q$. While $q = 1$, $n_q = n$.

For the definition of $q$-mutator operator to follow see: Definition (3.2) in [11] and (5.2) in [10] for the most general case.

**Definition 4.1.** [11, 11] Let $\psi$ be an admissible sequence. The $\hat{q}_\psi$ operator is a linear operator acting on the algebra $P$ defined in the basis $\{ x^n \}_{n \geq 0}$ as follows:

$$\hat{q}_\psi x^n = \frac{(n-1)_\psi - 1}{n_\psi} x^n, \quad n \geq 0.$$ 

We shall call it after Kwaśniewski the $\hat{q}_\psi$-mutator operator.
Definition 4.2. Let $\psi$ be an admissible sequence. The $n\hat{q}_\psi$ operator is a linear operator acting on the algebra $P$ defined as follows:

$$n\hat{q}_\psi = 1 + \hat{q}_\psi + \hat{q}_\psi^2 + \cdots + \hat{q}_\psi^{n-1}, \quad n > 0$$

Observation 4.1. With the choice $\psi_n(q) = (nq)^{-1}$ one has of course:

1. $\hat{q}_\psi x^n = qx^n$ for $n \geq 0$.
2. $\hat{q}_\psi^n x^n = q^n x^n$ for $n \geq 0$.
3. $n\hat{q}_\psi x^n = nq x^n$ and for $q = 1$ $n\hat{q}_\psi x^n = nx^n$ is just the operator of multiplication by $n$.

Let us now come over to the announced extensions of $\binom{n}{k}$, $\left[\begin{array}{c}n \\ k\end{array}\right]$ and $\left\{\begin{array}{c}n \\ k\end{array}\right\}$ as to become operator valued arrays and from now here on denoted correspondingly as: $\binom{n}{k}_{\hat{q}_\psi}$, $\left[\begin{array}{c}n \\ k\end{array}\right]_{\hat{q}_\psi}$ and $\left\{\begin{array}{c}n \\ k\end{array}\right\}_{\hat{q}_\psi}$.

For that to do consider first the Konvalina binomial coefficients. Let $w(\hat{q}_\psi) = (w_1(\hat{q}_\psi), w_2(\hat{q}_\psi), \ldots, w_n(\hat{q}_\psi))$ denotes the vector of linear operators such that $w_i(\hat{q}_\psi) = \sum_{k=0}^{i-1} a_{i,k} \hat{q}_\psi^k$ where $a_{i,k} \in F$ for $0 \leq k \leq i - 1$.

Definition 4.3. The $\hat{q}_\psi$ KonKwa operator of the first kind, denoted $C^n_k(w(\hat{q}_\psi))$ - is a linear operator defined on $P$ as follows:

$$C^n_k(w(\hat{q}_\psi)) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} w_{i_1}(\hat{q}_\psi) w_{i_2}(\hat{q}_\psi) \cdots w_{i_k}(\hat{q}_\psi).$$

Definition 4.4. The $\hat{q}_\psi$ KonKwa operator of the second kind - denoted $S^n_k(w(\hat{q}_\psi))$ - is a linear operator defined on $P$ as follows:

$$S^n_k(w(\hat{q}_\psi)) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} w_{i_1}(\hat{q}_\psi) w_{i_2}(\hat{q}_\psi) \cdots w_{i_k}(\hat{q}_\psi).$$

While setting $w(\hat{q}_\psi) = (1, \hat{q}_\psi, \hat{q}_\psi^2, \ldots, \hat{q}_\psi^{n-1})$ (1 is identity operator) we obtain an extension of binomial coefficient:

$$\left(\begin{array}{c}n \\ k\end{array}\right)_{\hat{q}_\psi} = S^n_{k+1}(w(\hat{q}_\psi))$$

(16)
and choosing the \( \psi \) admissible sequence \( \psi_n(q) = \frac{1}{n^q!} \binom{n}{k} \) and putting \( q = 1 \) one ends up with the ordinary binomial coefficient (treated as linear operator of multiplication the polynomial \( x^n \) by \( \binom{n}{k} \), respectively).

Setting now \( \hat{\psi}_n(w) = (1, 2, 3, \ldots, n) \) we obtain an extension of Stirling numbers:

\[
\left\{ \begin{array}{c}
\binom{n}{k} \\
\end{array} \right\}_{\hat{\psi}} = S_{n-k}(\hat{\psi}_n(w))
\]

(17)

\[
\left[ \begin{array}{c}
\binom{n}{k} \\
\end{array} \right]_{\hat{\psi}} = C_{n-k}(\hat{\psi}_n(w))
\]

(18)

and again choosing the \( \psi \) admissible sequence \( \psi_n(q) = \frac{1}{n^q!} \binom{n}{k} \) and putting \( q = 1 \) one ends up with the ordinary Stirling numbers (treated as linear operator of multiplication by \( \binom{n}{k} \), \( \binom{n}{k} \), \( \left[ \begin{array}{c}
\binom{n}{k} \\
\end{array} \right]_{\hat{\psi}} \) respectively).

The clue observation and hope for further investigation and possible applications of the objects introduced above is that the \( \hat{\psi}_n \) KonKwa operators satisfy the same shape identities and recurrence equations as the Konvalina generalized binomial coefficients. These are all obtained via replacing \( \hat{\psi}_n(w) \) by \( w(\hat{\psi}_n) \).

5 \( \psi \)-extensions of Konvalina binomial coefficients

Another kind of extensions of Konvalina binomial coefficients one can obtain by extending the vector weight of boxes. Let \( \hat{w}_n = (w_1, w_2, \ldots, w_n) \) be such extension of the vector weight \( \hat{w} \) that \( w_i \in \{n_i : n \geq 0\} \). Then

\[
C_n^k(\hat{w}_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} w_{i_1} w_{i_2} \cdots w_{i_k}
\]

(19)

is called the \( \psi \)-extended generalized Konvalina binomial coefficients of the first kind and

\[
S_n^k(\hat{w}_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} w_{i_1} w_{i_2} \cdots w_{i_k}
\]

(20)
is called the $\psi$-extended generalized Konvalina binomial coefficients of the second kind. The initial conditions are following:

$$S^n_0 = C^n_0(\overrightarrow{w_\psi}) = 1 \quad \text{for} \quad n \geq 0 \quad \text{and} \quad S^n_k = C^n_k(\overrightarrow{w_\psi}) = 0 \quad \text{for} \quad k > 0.$$  

Expectedly, these $\psi$-extended generalized Konvalina binomial coefficients have combinatorial interpretation for $n \geq 0$ and $n_\psi$ being nonnegative integers - for example Fibonacci numbers.

Leave however this not easy question apart and let us do what we can right now. Let us take a look at the vector weight $\overrightarrow{w_\psi} = (1_\psi, 2_\psi, \ldots , n_\psi)$. We obtain $\psi$-Stirling numbers:

$$\left[ \begin{array}{c} n \\ k \end{array} \right] \psi = C^{n-1}_{n-k}(\overrightarrow{w_\psi}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} (i_1)_{\psi} (i_2)_{\psi} \cdots (i_{n-k})_{\psi} \quad (21)$$

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} \psi = S^k_{n-k}(\overrightarrow{w_\psi}) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-k} \leq k} (i_1)_{\psi} (i_2)_{\psi} \cdots (i_{n-k})_{\psi} \quad (22)$$

Choosing the $\psi$ admissible sequence $\psi_n(q) = \frac{1}{n_q}$ $\psi$-Stirling numbers become $q$-Stirling numbers $\left\{ \begin{array}{c} n \\ k \end{array} \right\} \psi = \left\{ \begin{array}{c} n \\ k \end{array} \right\} q$, $\left[ \begin{array}{c} n \\ k \end{array} \right] \psi = \left[ \begin{array}{c} n \\ k \end{array} \right] q$ and for $q = 1$ they are ordinary Stirling numbers.

One can easy observe that

$$\sum_{n \geq 0} \left\{ \begin{array}{c} n \\ k \end{array} \right\} \psi x^n = \frac{x^k}{(1 - 1_\psi x) (1 - 2_\psi x) \cdots (1 - k_\psi x)} \quad (23)$$

and

$$\sum_{k \geq 0} \left[ \begin{array}{c} n \\ k \end{array} \right] \psi x^k = \Psi^n(x), \quad \Psi^n(x) = x (x + 1_\psi) (x + 2_\psi) \cdots (x + (n - 1)_\psi) \quad (24)$$

In [13] Kwaśniewski among others introduces following Wagner [14] the so-called Comtet $\psi$-Stirling numbers of the second kind, denoted there $\left\{ \begin{array}{c} n \\ k \end{array} \right\} \sim_\psi$ and defined equivalently among others also by the identity (22) as well as the Comtet $\left[ \begin{array}{c} n \\ k \end{array} \right] \sim_\psi$ Stirling numbers of the first kind different from our $\left[ \begin{array}{c} n \\ k \end{array} \right] \psi$ Stirling numbers of the first kind.

Apart from extension of Stirling numbers of the second kind to Comtet numbers case there are also vastly considered other $q$-extended numbers...
$S_q(n, k) = q^{(k)} \tilde{S}_q(n, k)$ where $\tilde{S}_q(n, k) = \left\{ \begin{array}{c} n \\ k \end{array} \right\}_q$ (see [13] for representative references and see also comments therein). May be then for example these $S_q(n, k)$ be also treated as another $q$-extended generalized Konvalina binomial coefficients? And what about their eventual operator valued arrays counterparts?

These and other questions we leave for further examination and investigation.

Extensions of generalized Konvalina binomial coefficients introduced in this paper constitute only two classes of extensions: $\hat{q}_\psi$-operator and $\psi$-extension Comtet one in terminology of [13]. The investigation is at preliminary stage - as seen from the all above. We expect to say soon more on other extensions which can be generated while choosing different vector weights $\overrightarrow{w}$.

References

[1] A. K. Kwaśniewski: On duality triads, Bull. Soc. Sci. Letters Łódź 53 Rech. sur déf. 42 (2003), 11.

[2] A. K. Kwaśniewski: On Fibonomial and other triangles versus duality triads, Bull. Soc. Sci. Letters Łódź 53 Rech. sur déf. 42 (2003), 27.

[3] S. M. Roman: The umbral calculus, Academic Press, New York 1984.

[4] E. Damiani, O. D’Antona, G. Naldi: On the connection Constants, Studies in Appl. Math. 85(4) (1991), 157.

[5] J. Konvalina: Generalized binomial coefficients and subset-space problem, Adv. in Appl. Math. 21 (1998), 228.

[6] J. Konvalina: A Unified Interpretation of the Binomial Coefficients, the Stirling Numbers and the Gaussian Coefficients, The Am. Math. Month. 107 (2000), 901.

[7] A. K. Kwaśniewski: Cauchy $\hat{q}_\psi$-identity and $\hat{q}_\psi$-Fermat matrix via $\hat{q}_\psi$-muting variables of $\hat{q}_\psi$-Extended Finite Operator Calculus

arXiv:math.CO/0403107 v1 5 March 2004
[8] A. K. Kwański: Pascal like matrices - an accessible factory of one source identities and resulting applications, Proc. Jangjeon Math. 7(2) -in press, ArXiv:math.CO/0403123 v1 7 March 2004

[9] J.Katriel, M. Kibler: Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers J. Phys. A: Math. Gen. 25 (1992), 2683.

[10] A. K. Kwański: Towards $\psi$-extension of finite operator calculus of Rota, Rep. Math. Phys. 47(4) (2001), 305.

[11] A. K. Kwański: On extended finite operator calculus of Rota and quantum groups, Integral Transforms and Special Functions 4(2) (2001), 333.

[12] A. K. Kwański: Main theorems of extended finite operator calculus Integral Transforms and Special Functions, 14 No 6 (2003): 499-516

[13] A. K. Kwański: On umbral extensions of Stirling numbers and Dobinski-like formulas ArXiv:math.CO/0411002 30 October 2004

[14] C. G. Wagner: Partition Statistics and q-Bell Numbers ($q = -1$) Journal of Integer Sequences 7 (2004). http://www.emis.de/journals/JIS/VOL7/Wagner/wagner3.html