LOCAL EQUILIBRIUM IN PLANAR NON INTERACTING PARTICLE SYSTEMS

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Abstract. Particles are injected to a large planar rectangle through the boundary. Assuming that the particles move independently from one another and the boundary is also absorbing, we identify a set of abstract conditions which imply the local equilibrium of the particle density in diffusive scaling limit. We verify that our abstract conditions hold in two examples: iid random walks and the periodic Lorentz process.

1. Introduction

A major open problem in mathematical statistical mechanics is to rigorously derive macroscopic laws of physics, such as Fourier’s law of heat conduction, from underlying microscopic principles [2]. A realistic microscopic model should consist of a macroscopic domain inside which the microscopic particles are subject to some bulk dynamics and interact with a heat bath on the boundary. If the temperature of the heat bath varies along the boundary, then one would like to study the emergence of local equilibrium (i.e. the existence of a well defined temperature at microscopic or mesoscopic locations inside the domain).

There are two separate classes of models for the particle dynamics. The first class is stochastic, namely Markov processes. Because of the Markov property, the future of the system can equally be described no matter what happened in the past and so Markov processes provide an excellent opportunity to derive beautiful mathematical results. Indeed, the results oftentimes go much beyond the derivation of the heat equations (such as second order fluctuations or other PDEs). We don’t attempt to review the literature of such Markov models here so we refer the reader to the classical surveys [22, 17].

The second class of models are realistic (Hamiltonian) deterministic dynamical systems. Proving that the bulk dynamics obey the heat equation becomes considerably harder for such deterministic systems. However, a notable realistic Hamiltonian system for which rigorous results are available is the Sinai billiard [21]. In Sinai billiards, point particles fly freely among fixed convex bodies and elastically collide on their boundaries. In this case, a rigorous study of a variant of the problem prescribed in the first paragraph is possible when ’temperature’ is replaced by ’particle
density’ and ‘the heat bath’ is replaced by ‘varying chemical potential’. Indeed, the point particles do not interact with one another and so there is no exchange of energies. Furthermore, the trajectory of each particle satisfies the central limit theorem \[3, 4, 5\] leading to the heat equation in the bulk. However, even in this case of a ‘non-interacting particle system’, a better understanding of the boundary phenomena is desirable.

We now describe the problem to be studied here. Let \(D \subset \mathbb{R}^2\) be a bounded domain with piece-wise smooth boundary and let particles be injected to the large domain \(LD\) for \(L \gg 1\) through its boundary. The particles will then perform some independent motion \(Z\) on a lattice inside \(LD\). The boundary is also absorbing so most particles are killed (i.e. absorbed) shortly after injection. However, some will survive for a long time and find their way deep into the interior of \(LD\). The problem now is to show that the limiting density profile of particles is governed by the heat equation when time is rescaled by \(L^2\) and by the Laplace equation when time is infinite, where, in both cases, the boundary conditions are given by the injection rate. We will refer to the first case as hydrodynamic and the second one as hydrostatic limit. This terminology is somewhat unusual since there is no energy exchange here, but we find it natural since we are studying the scaling limits of particle systems. Specifically, we look at the problem of proving local equilibrium of the particle density profile in systems forced out of equilibrium when the particle injection rate varies along the boundary of the domain.

In this paper, we identify an abstract framework for which we can solve the problem presented in the previous paragraph, that is, we can prove the local equilibrium in both the hydrodynamic and the hydrostatic limit in the case \(D\) is a rectangle. This framework is general enough to include two basic examples: (1) when \(Z\) is an iid random walk, and (2) when \(Z\) is given by the spatially periodic extension of the Sinai billiard (called periodic Lorentz process). The abstract framework is given by some hypotheses (H1)-(H3) (see Section 2.1). The main hypothesis is (H2), which is a conditional local invariance principle conditioned on survival of the particle. In case of the periodic Lorentz gas, our result provides a natural extension of \[12\] from one dimensional domains (i.e. line segments) to two dimensional rectangles. See Figure 1 for the case of periodic Lorentz gas: particles, indicated by blue dots, are injected from the left ("West") side of a large rectangle while the entire boundary of the rectangle is absorbing.

Complementary to our approach, there is a classical proof based on the idea of duality. If the particle motion in the bulk has a nice dual process and the injection on the boundary is chosen very carefully, then the problem can be reformulated in terms of the hitting of \(\partial D\) by the dual process starting from the bulk. The approach by duality thus gives similar results with two major differences: it is more general in the sense that \(D\) can be any domain with piece-wise smooth boundary, but it is
more restrictive in the sense that requires both the existence of a nice dual process and a very specific way of injection on the boundary.

In case of our two basic examples, the dual process (in fact a certain reverted process) is essentially the same as the original process. We present the proof by duality in Section 4. In case of Markov processes, the proof by duality is very well known even for some interacting particle systems, see e.g. [18] (we do not review the literature and do not study any of these stochastic interacting particle systems here). The proof by duality is not surprising for the Lorentz gas either, but it was not observed in [12] and so our Proposition 4.3 (with trivial changes to include a 1 dimensional macroscopic domain) gives a simple new proof of the main results of [12] in case of a very special injection mechanism, which is essentially given by the Lebesgue measure. The utility of this special injection mechanism is limited, however, since no reasonable heat bath is likely to preserve the invariant measure of the bulk dynamics (see e.g. [1]).

Because the proof by duality requires a very rigid structure of both the bulk dynamics and the injection, it is essential to develop other tools which do not require such a rigid structure. Such tools are exemplified by our main results in Sections 2 and 3. Indeed, our injection procedure (2.3) is quite general: besides the dependence on the macroscopic position we allow the injection rate to depend on the microscopic geometry through some function $A$ and on time through another function $B$. In case of deterministic systems, the only source of randomness is the choice of the initial
condition according to an initial probability measure. Once the initial condition is fixed, \( Z \) is deterministic. We allow a lot of initial measures. For example any ”standard pair” \[9\] in case of Sinai billiards. In our context a useful way of thinking about standard pairs is that they are conditional measures corresponding to a given past symbolic trajectory of the particle. Except for the special choice of \( A \) and \( B \) as in Proposition 4.1, we believe that our results are new even in case of random walks. Finally, we believe that some ideas presented here could be of use for studying deterministic interacting particle systems as well.

The rest of this paper is organized as follows. In Section 2 we provide the basic definitions and the main result Theorem 2.1 in our abstract framework. In Section 3 we present our two basic examples – namely, the random walk and the Lorentz gas. In Section 4 we discuss the approach by duality. In Section 5 we prove Theorem 2.1. The most technical part of this work is the verification of the conditional local invariance principle (H2) for the Lorentz gas, which is presented in Section 7. This section is heavily built upon the standard pair technique of Chernov and Dolgopyat \[9\] and tools from \[12\], such as the mixing local limit theorem. The necessary background is summarized in Section 6.

2. Abstract setup

2.1. Non-interacting particle systems. Let \( \mathcal{L} \subseteq \mathbb{R}^2 \) be a lattice of dimension 2. We consider the graph \( G \) with vertices \( \mathcal{L} \) and edges joining \( l \) with \( l + w_j \) for all \( l \in \mathcal{L} \) and \( j = 1, ..., J \) for a fixed set \( \{w_1, ..., w_J\} \subseteq \mathcal{L} \). For \( z \in \mathbb{R}^2 \), let \( \langle z \rangle \) be the closest \( l \in \mathcal{L} \) to \( z \) with the property that \( l_1 \geq z_1 \) (if there is more than one such lattice points, then choose the smallest in lexicographic order).

Let \((S, \mathbb{P})\) be a probability space and \( Z_t (t \geq 0) \) be an \( \mathcal{L} \) valued stochastic process. That is, \( Z_t : S \rightarrow \mathcal{L} \) for every \( t \geq 0 \). We assume that \( Z \) is continuous from the right and has left limits. In other words, \( Z \) is a càdlàg function (i.e. for almost every \( s \in S \) fixed, \( Z \) jumps at random times \( t \) from a lattice point \( Z_{t^-} \) to another lattice point \( Z_t \)). We do not assume that \( Z \) is Markovian.

Now let \( D = [0, A] \times [0, 1] \) for a fixed positive real \( A \). Fix a non-negative continuous functions \( f : [0, 1] \rightarrow \mathbb{R} \) and write \( F : \partial D \rightarrow \mathbb{R} \),

\[
F(z) = \begin{cases} 
  f(y) & \text{if } z = (0, y) \\
  0 & \text{otherwise.}
\end{cases}
\]

We will consider the following Dirichlet problems

\[
\Delta u = 0, \quad u|_{\partial D} = \varsigma F,
\]

(2.1)

\[
v_t = \frac{1}{2} [v_{xx} + v_{yy}], \quad v(t, x, y)|_{(x,y) \in \partial D} = \varsigma F, \quad v(0, x, y) = 0.
\]

(2.2)
We are actually interested in the Dirichlet problem where $F$ is permitted to be nonzero for all boundary points (as in (2.10)), but this case follows from linearity. By classical theory, there is a unique solution to both the Laplace equation (2.1) and the heat equation (2.2) and furthermore $\lim_{t \to \infty} v(t, x, y) = u(x, y)$. Of course this is true for much more general domains $D$, e.g. when $\partial D$ is piecewise-smooth with no cusps.

For $L \gg 1$, let $D_L = (LD) \cap L$, 

$$\partial D_L = \{ l \in D_L : l \text{ is connected to a point outside of } D_L \},$$

and 

$$\partial_W D_L = \{ l \in D_L : l \text{ is connected to a point } l' \text{ with } l'_0 < 0 \}.$$ 

Here $\partial_W$ stands for West boundary as points in $\partial_W D_L$ are close to the ”West” side of the rectangle $D_L$. Given $l \in \partial D_L$, let 

$$J(l) = \{ j = 1, ..., J : l + w_j \notin D_L \}$$

We consider the following process for $L \gg 1$. First, for some $t \in \mathbb{R}_+ \cup \{ \infty \}$, let $\Theta_t$ be a Poisson point process on $(-t, 0] \times \partial_W D_L$ with intensity measure 

$$(2.3) \quad A(J(l))B(s)f(l_2/L)d\text{Leb}(s)d\text{counting}(l),$$

where $l \in \partial D_L$ and $A : 2^{\{1, ..., J\}} \to \mathbb{R}_+$ and $B : \mathbb{R} \to \mathbb{R}_+$ are fixed functions. We assume that $B$ is continuous, periodic with period 1, and $\int_0^1 B = 1$. One example is $A(J) = |J|$ and $B = 1$. However, we want to allow more general functions to accommodate for more general behavior of the heat bath.

For each point $(T, l) \in \Theta$, we start an iid copy of $Z$ at time $T$ from position $l$ and we kill it at 

$$(2.4) \quad \tau^* = \inf\{ t > T : Z_T \notin D_L \},$$

the first exit from $D_L$. In the case $Z$ is not Markovian, the initial condition $Z_T = l$ may not define the distribution of $Z_{T+t}$ for $t > 0$ uniquely. In this case, we allow multiple choices of this distribution but we require that $Z_{T+t} = l$ only depends on $l$ through $J(l)$. That is, if $l, l' \in \partial D_L$, with $J(l) = J(l')$ and $(T, l), (T', l') \in \Theta$, then we require that for all $t \geq 0$, and for all $\tilde{l} \in \mathcal{L}$,

$$\mathbb{P}(Z_{T+t} = \tilde{l} | Z_T = l) = \mathbb{P}(Z_{T'+t} = l' + \tilde{l} | Z_{T'} = l').$$

This procedure is to be interpreted as injecting a particle to the domain $D_L$ at time $T$ through an edge $(l_-, l)$ of the graph $G$, where $l_- \notin D_L$, $l \in D_L$ and letting particles evolve independently from one another until coming back to the absorbing boundary. The specific mechanism of injection through $(l_-, l)$ only depends on $j = 1, ..., J$, where $l - l_- = w_j$. Let $\Lambda_t(l)$ be the number of particles at site $l$ at time $T = 0$. We start with the following abstract result.
Theorem 2.1. Assume that (H1) - (H3) are satisfied. Then for any $z$ in the interior of $D$
\begin{equation}
\lim_{L \to \infty} \mathbb{E}(\Lambda_\infty(\langle zL \rangle)) = u(z)
\end{equation}
and
\begin{equation}
\lim_{L \to \infty} \mathbb{E}(\Lambda_{tL^2}(\langle zL \rangle)) = v(t, z)
\end{equation}
where $u$ and $v$ are defined by (2.1) and (2.2) with some $\varsigma$.

To define our hypotheses (H1) - (H3), we need some definitions.
Let $W_t$ be a standard Brownian motion. Let
\[ \phi(\eta, \gamma, \xi) = \lim_{dt \to 0} \frac{1}{dt} \mathbb{P}(W_1 \in [\gamma, \gamma + dt], \min_{t \in [0,1]} W_t > 0, \max_{t \in [0,1]} W_t < \xi | W_0 = \eta). \]
It is known (see e.g. [15]) that for any $0 < \gamma, \eta < \xi$, the following formula holds
\begin{equation}
\phi(\eta, \gamma, \xi) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left( \exp \left( -\frac{(\gamma - \eta - 2n\xi)^2}{2} \right) - \exp \left( -\frac{(\gamma + \eta + 2n\xi)^2}{2} \right) \right).
\end{equation}

Recall that the Brownian meander is a stochastic process on $[0,1]$ obtained by conditioning a standard Brownian motion to stay positive on $[0,1]$ (which has zero probability but the definition still makes sense by conditioning on staying above $-\varepsilon$, letting $\varepsilon \to 0$ and taking weak limit, see e.g. [14]). Let $\mathcal{X}(t)$ be a Brownian meander and $\mathcal{M}(t) = \max_{0 \leq s \leq t} \mathcal{X}(s)$ its maximum. Then it is proven in [11, Theorem 5] that the function
\[ \psi(\alpha, \beta) = \lim_{dt \to 0} \frac{1}{dt} \mathbb{P}(\mathcal{X}(1) \in [\alpha, \alpha + dt], \mathcal{M}(1) < \beta) \]
for any $0 < \alpha < \beta$ satisfies
\begin{equation}
\psi(\alpha, \beta) = \sum_{k=-\infty}^{\infty} (2k\beta + \alpha) \exp \left( -\frac{(2k\beta + \alpha)^2}{2} \right).
\end{equation}

Note that the formulas (2.7) and (2.8) are closely related as the Brownian meander is closely related to the Brownian motion. Indeed, by the definition of Brownian meander, $\psi(\alpha, \beta) = \lim_{\eta \to 0} \phi(\eta, \alpha, \beta) / \int_{0}^{\beta} \phi(\eta, \alpha', \beta) d\alpha'$. We refer to [11] for more details.

Let us write $\mathcal{Z}_t = (\mathcal{X}_t, \mathcal{Y}_t)$. Denote
\[ \tau^X_x = \begin{cases} \min\{t > 0 : \mathcal{X}_t > x\} & \text{if } x > 0 \\ \min\{t > 0 : \mathcal{X}_t < x\} & \text{if } x \leq 0. \end{cases} \]
We define $\tau^Y_y$ analogously.

Now we make the following assumptions:
(H1) **Vertical rational dependence** There is some \( l \in L \), \( l \neq 0 \) so that \( l_1 = 0 \).

Let \( (0, 0) = l^{(0)}, l^{(1)}, l^{(2)} \ldots \) be the enumeration of points \( l \in \partial D_L \) which are connected to lattice points with negative first coordinate in increasing order of second coordinate (that is \( l^{(j)}_2 \leq l^{(j+1)}_2 \)). If there are points \( l^{(j)}, l^{(j+1)} \) with the same second coordinate, then we order them in increasing order of the first coordinate. Let \( K \) be the smallest positive integer so that

\[
(2.9) \quad l^{(K)}_1 = 0.
\]

By condition (H1), \( K \) exists. Now we say that the lattice point \( l \in \partial D_L \) is of type \( k \) with \( k = 1, \ldots, K \) if there exists an integer \( m \) so that \( l = l^{(mK+k)} \).

(H2) **Conditional local invariance principle**

There are constants \( c_1, \ldots, c_K \) so that for any \( 0 < \alpha < \beta \) and for any \( 0 < \eta, \gamma < \xi \) the following holds. If \( l \in \partial D_L \) is of type \( k \), and \( l_2 = \eta \sqrt{T} \), then

\[
\lim_{T \to \infty} T^{3/2} P \left( \tau_{\alpha, \beta}^T > T \mid Z_0 = l \right) = c_k \psi(\alpha, \beta) \phi(\eta, \gamma, \xi).
\]

Furthermore, for any \( \varepsilon > 0 \), the convergence is uniform for \( \varepsilon < \alpha < \alpha + \varepsilon < \beta < 1/\varepsilon \) and \( \varepsilon < \eta < \eta + \varepsilon < \xi < 1/\varepsilon \), \( \varepsilon < \gamma < \gamma + \varepsilon < \xi \).

(H3) **Moderate deviation bounds**

For any \( x \in (0, 1) \) and \( y \in (-1, 1) \), and for any \( l = l^{(0)}, \ldots, l^{(K-1)} \)

\[
\lim_{\delta \to 0} \lim_{L \to \infty} \int_{[0, \delta L^2] \cup [L^2/\delta, \infty]} L \mathbb{P} (Z_t = \langle (xL, yL) \rangle, \min \{\tau_0^X, \tau_L^X\} > t \mid Z_0 = l) \, dt = 0
\]

2.2. **Local equilibrium.** Consider now the Dirichlet problems

\[
(2.10) \quad \Delta \tilde{u} = 0, \tilde{u}|_{\partial D} = \tilde{F},
\]

\[
(2.11) \quad \tilde{v}_t = \frac{1}{2} \left[ \tilde{v}_{xx} + \tilde{v}_{yy} \right], \quad \tilde{v}(t, x, y)|_{(x, y) \in \partial D} = \tilde{F}, \tilde{v}(0, x, y) = 0.
\]

Here \( \tilde{F} \) is defined by \( \tilde{F} : \partial D \to \mathbb{R} \),

\[
\tilde{F}(z) = \begin{cases} 
\varsigma_W f_W(y) & \text{if } z = (0, y) \\
\varsigma_S f_S(x) & \text{if } z = (x, 0) \\
\varsigma_E f_E(y) & \text{if } z = (A, y) \\
\varsigma_N f_N(y) & \text{if } z = (x, 1), 
\end{cases}
\]

where \( f_E, f_W : [0, 1] \to \mathbb{R} \), \( f_N, f_S : [0, A] \to \mathbb{R} \) are given non-negative continuous functions and \( \varsigma_{W/S/E/N} \) are non-negative real numbers (\( W, S, E, N \) stand for West, South, North and East). We perform the same procedure of injecting particles and
absorbing them on the boundary as before, but now we inject from all 4 sides of the rectangle. Let \( \tilde{\Lambda}_t \) denote the resulting measure defined as \( \Lambda_t \).

We say that \( \mathcal{Z} \) satisfies that **local equilibrium** (LE) if for any \( t \in \mathbb{R}_+ \cup \{\infty\} \), for any \( k \in \mathbb{Z}_+ \), for any \( z_1, \ldots, z_k \) distinct points in the interior \( D \) and for any distinct lattice points \( l_1, \ldots, l_k \in \mathcal{L} \), the joint distribution of

\[
W_{t,i,j,L} := \tilde{\Lambda}_{tL^2}(\langle z_i L \rangle + l_j), \quad i, j = 1, \ldots, k
\]

converge weakly as \( L \to \infty \) to independent Poisson random variables \( W_{t,i,j,\infty} \) with expectation \( \tilde{v}(t, z_i) \) (or \( \tilde{u}(z_i) \) in case \( t = \infty \)), where \( \tilde{v} \) is defined by (2.11) (and \( \tilde{u} \) is defined by (2.10)) with some constants \( \varsigma_{W/S/E/N} \). The points \( \langle z_i L \rangle + l_j \), \( j = 1, \ldots, k \) can be thought of as lying in a microscopic region near \( \langle z_i L \rangle \). In particular, each point \( \langle z_i L \rangle + l_j \) is a finite distance from \( \langle z_i L \rangle \) so that it is in a "local" region of \( z_i \) as \( L \) becomes large. Indeed, the term **local equilibrium** refers to the fact that the limiting distribution does not depend on \( j \). We call the case \( t \in \mathbb{R}_+ \) **local equilibrium in the hydrodynamic limit** and the case \( t = \infty \) **local equilibrium in the hydrostatic limit**. Since in our case both hold at the same time, we simply refer to these properties as **local equilibrium**.

Finally, we say that a lattice \( \mathcal{L} \) is **rational** if there are non-zero lattice points \( l^{(K_1),1}, l^{(K_2),2} \) in \( \mathcal{L} \) so that \( l_1^{(K_1),1} = l_2^{(K_2),2} = 0 \). Without loss of generality, we assume that \( l_1^{(K_1),1} > 0 \) and \( l_2^{(K_1),1} \) is the smallest among such vectors with respect to the ordering introduced right after (H1) (and likewise for \( l^{(K_2),2} \), except that in the ordering, the role of the first and second coordinates are swapped). Clearly, if \( \mathcal{L} \) is rational, then (H1) holds with \( K = K_1 \) (and likewise, a variant of (H1), where the two coordinates are swapped, holds with \( K = K_2 \)).

Next, we show some examples, where we can verify conditions (H1) - (H3) and also prove (LE).

### 3. Basic examples

#### 3.1. Random walks

Let \( \tilde{\mathcal{L}} \subset \mathbb{R}^2 \) be a 2 dimensional lattice. Let \( \tilde{\mathcal{P}} \) be a finitely supported probability measure on \( \tilde{\mathcal{L}} \) with zero expectation. We assume that there are finitely many lattice points \( \tilde{w}_1, \ldots, \tilde{w}_J \) so that \( \tilde{\mathcal{P}}(\tilde{w}_j) > 0 \) and \( \sum \tilde{\mathcal{P}}(\tilde{w}_j) = 1 \). To avoid degeneracy, we assume that the group generated by \( \tilde{w}_j \)’s is \( \tilde{\mathcal{L}} \).

Let \( \tilde{\mathcal{Z}} \) be a homogeneous Markov process: at exponential distributed times, \( \tilde{\mathcal{Z}} \) jumps with a jump distribution given by \( \tilde{\mathcal{P}} \). That is, the generator \( \tilde{G} \) of \( \tilde{\mathcal{Z}} \) is defined by

\[
(\tilde{G}f)(l) = \sum_{j=1}^J \tilde{\mathcal{P}}(\tilde{w}_j)[f(\tilde{w}_j + l) - f(l)]
\]
for test functions $f : \tilde{\mathcal{C}} \to \mathbb{R}$. By the central limit theorem, $\tilde{Z}_t/\sqrt{t}$ converges weakly to a Gaussian distribution with mean zero and some covariance matrix $\Sigma$. Furthermore, the non-degeneracy assumption ensures that $\Sigma$ is positive definite. Now we define $L = \Sigma^{-1/2} \tilde{\mathcal{C}}$, $w_j = \Sigma^{-1/2} \tilde{w}_j$, $P(w_j) = \tilde{P}(\tilde{w}_j)$, $Z = \Sigma^{-1/2} \tilde{Z}$.

**Proposition 3.1.** If $L$ is a rational lattice, then in the above model (H1) - (H3) hold.

We do not give a proof of Proposition 3.1 as it follows from a much simplified version of our proof of Theorem 3.2. In fact, the one dimensional version of (H2) and (H3) is known for random walks, see [6, 7]. We find it likely that the two dimensional version is also known but we could not find a reference.

### 3.2. Lorentz gas.

#### 3.2.1. Definitions.

We start with the definition of Sinai billiards [21]. Consider a finite collection of strictly convex disjoint subsets $B_1, ..., B_t$ of the 2-torus with $C^3$ boundary. The complement of these sets is denoted by $D_0 = T^2 \setminus \bigcup_{i=1}^t B_i$ and is called the configuration space. A point particle flies with constant speed inside $D_0$ and undergoes specular reflection upon reaching $\partial D_0$ (i.e. angle of incidence equals the angle of reflection). Since the speed is conserved, we obtain a continuous time dynamical system $\Phi_t$, $t \in \mathbb{R}$ on the phase space $\Omega_0 = D_0 \times S^1$. The Sinai billiard flow $\Phi_0$ preserves the Lebesgue measure on $\Omega_0$ (denoted by $\mu_0$). We assume the finite horizon condition, i.e. that the sets $B_i$ are chosen in such a way that the free flight time is bounded. Similarly, we define the periodic Lorentz gas when the phase space is lifted to the universal cover. That is, the configuration space is $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{(m,n) \in \mathbb{Z}^2} \bigcup_{i=1}^t (B_i + (m,n))$, where we identify $D_0$ with $\mathcal{D} \cap [-1/2, 1/2)^2$. We choose this identification in such a way that $(-1/2, -1/2) \notin \mathcal{D}$. The phase space is $\Omega = \mathcal{D} \times S^1$ and the billiard flow is denoted by $\Phi^f$. It preserves the $\sigma$-finite measure $\mu$, which is $\mu_0$ times the counting measure on $\mathbb{Z}^2$.

Now we construct the stochastic process which is the projection of the billiard flow, $\Phi^f$ onto $\mathbb{Z}^2$. Given $(q, v) \in \Omega$, let $\Pi_{\mathbb{Z}^2}(q, v) = (k, l) \in \mathbb{Z}^2$ if $q \in (k, l) + [-1/2, 1/2)^2$ and let $\Pi_{\mathcal{D}_0}(q, v) = q_0$, $\Pi_{\Omega_0}(q, v) = (q_0, v)$ if $q = q_0 + \Pi_{\mathbb{Z}^2}(q, v)$. We also put $\tilde{Z}_t(q, v) = \Pi_{\mathbb{Z}^2}(\Phi^f_t(q, v))$. Thus any probability measure on $D_0$ induces a stochastic process $\tilde{Z}_t$. It is important to note that here the randomness only appears in the initial condition. Once $(q, v)$ is fixed, then $\tilde{Z}_t$ is uniquely defined for every $t$.

We will also need the billiard map $F_0$, which is defined as the Poincaré section corresponding to the collisions, that is $F_0 : \mathcal{M}_0 \to \mathcal{M}_0$, where

$$\mathcal{M}_0 = \{(q, v) \in \partial D_0 \times S^1 : \langle v, n \rangle \geq 0\},$$

where $n$ is normal to $\partial D_0$ at $q$ pointing inside $D_0$. The phase space of the billiard map, $\mathcal{M}_0$, thus corresponds to collisions where by convention we use the post-collisional
velocity \( v \). \( F_0 \) preserves the probability measure \( \nu_0 \) defined by 
\[ d\nu_0 = c \cos \phi dr d\phi, \]
where \((r, \phi)\) are coordinates on \( M_0 \); \( r \) is arclength parameter and \( \phi \in [-\pi/2, \pi/2] \) is the angle between \( v \) and \( n \). The definitions of \( M, F, \nu \) are analogous.

Fix a measure given by an arbitrary proper standard family (The exact definition will be given in Section 6. One example is the invariant measure \( \nu \)). This measure induces a stochastic process \( \tilde{Z}_t \). Furthermore, \( \tilde{Z}_t \) satisfies the central limit theorem with a covariance matrix which is independent of the standard family. That is, there exists a positive definite \( 2 \times 2 \) matrix \( \Sigma \) so that \( \tilde{Z}_t T / \sqrt{T} \) converges weakly as \( T \to \infty \) to the Gaussian distribution with mean zero and covariance matrix \( \Sigma \) (see e.g. [5]).

Now let \( L = \Sigma^{-1/2} \mathbb{Z}^2 \), \( Z_t = \Sigma^{-1/2} \tilde{Z}_t \). The invariance principle holds as well. That is, \( Z_t T / \sqrt{T}, t \in [0,1] \) converges weakly to a standard Brownian motion (see e.g. [8]).

Recalling that \((-1/2, 1/2) \notin D\), the graph \( G \) on \( L \) induced by \( Z \) satisfies that \((l, l')\) is an edge in \( G \) if and only if \( \Sigma^{1/2} l \) and \( \Sigma^{1/2} l' \) are nearest neighbors in \( \mathbb{Z}^2 \). We will assume this in the sequel.

**Theorem 3.2.** In the setting described above, assume that \( L \) is a rational lattice. Then (H1)-(H3) hold.

Theorem 3.2 does not claim that \( \varsigma > 0 \). In fact, there are standard families for which \( \varsigma = 0 \). This is not surprising since in \( Z_t \) can be deterministic for a bounded time. In particular, we can choose a standard family so that \( Z_t < 0 \) almost surely for a fixed \( t \) and so all particles will be absorbed within bounded amount of time. However, there are standard families for which \( \varsigma > 0 \) (e.g., for the invariant measure \( \nu \)). In general, we cannot compute \( \varsigma \) even if it is positive.

Note that we assumed that \( L \) is a rational lattice, which immediately gives (H1) and the variant of (H1) when the vertical and the horizontal coordinates are swapped. This is a highly non trivial assumption and we expect this not to hold for a typical billiard table. However, we have some examples when it does hold due to some extra symmetry. We discuss these examples in Section 3.2.2. The proof of (H2) and (H3) will be given in Section 7.

In case of deterministic systems like the Lorentz gas, a natural extension of (LE) is a finer counting problem: that is to only count particles in a given nice subset of \( \Omega_0 \) (for example, those that are close to a given scatterer). Let us fix and open set 

\[ A \subset \Omega_0 \text{ with } \mu_0(\partial A) = 0 \]

and update the definition of \( \tilde{\Lambda}_t \) so as we only count particles at phase \((q, v)\) that satisfies \( \Pi_{\Omega_0}(q, v) \in A \). Let the resulting measure be \( \tilde{\Lambda}_A^t \) and let us say that detailed local equilibrium (DLE) holds if there is some \( \varsigma \) so that for every \( A \) as in (3.2), the definition of (LE) with \( \tilde{\Lambda} \) replaced by \( \tilde{\Lambda}_A \) holds with the constant \( \varsigma \mu_0(A) \).

**Theorem 3.3.** Under the assumptions of Theorem 3.2 (LE) and (DLE) hold.
Proof of Theorem 3.3 assuming Theorem 3.2. As observed in [12], the derivation of (LE) from (2.5) and (2.6) is straightforward. Let \( M = (tL^2, 0) \times \partial D_L \times \Omega_0 \). Let \( G : M \to D_L \times \Omega_0 \cup \{ \infty \} \), where \( G(s, t, (q, v)) = \Phi^s(q + \Sigma^{1/2}I, v) \) if the particle has not been absorbed by time \( s \) and \( G(s, t, (q, v)) = \infty \) otherwise. Since the initial conditions of particles is given by a Poisson point process (PPP) on \( M \), the mapping and restriction theorems for PPP (see e.g. sections 2.2 and 2.3 in [16]) give that \( \{G(s_i, t_i, x_i)\}_{G(s_i, t_i, x_i) \neq \infty} \) forms a PPP on \( D_L \times \Omega_0 \). Letting \( L \to \infty \), the intensity measure of this PPP converges by Theorem 2.1 and Theorem 3.2 (for particles injected on the West or on the East this is immediate. For particles injected on the North or the South, this follows from the variant of Theorem 2.1 when the role of \( x \) and \( y \) are swapped. Since \( L \) is assumed to be rational, (H1) holds even in this case). Thus in the limit \( L \to \infty \), we obtain a PPP with intensity measure as on the right hand side of (2.5) and (2.6). This implies (LE). The proof of (DLE) is analogous, except that when we verify (H2), we only need to take into account particles at phase \( (q, v) \) that satisfies \( \Pi_{\Omega_0}(q, v) \in A \). This requires a very minor change in the proof (see the remark after Theorem 6.4). □

3.2.2. Symmetry Conditions.

Example 3.4. Assume that \( D_0 \) is invariant under a 90 degree rotation or a vertical or horizontal reflection of the unit square. Then \( \mathcal{L} \) is rational.

Proof. Let us assume that \( D_0 \) is invariant under a rotation by 90 degrees. Then the probability density function (pdf) of the limiting distribution of \( \sqrt{t} \mathbf{Z}_t \) also needs to be invariant under the rotation by 90 degrees. Since this is a normal distribution, the isocontours of the pdf are ellipses. The only ellipses invariant under the rotation by 90 degrees are circles. This means that there is a positive real number \( \sigma \) so that \( \Sigma = \sigma^2 I_2 \). Similarly, if \( D_0 \) is invariant under reflection to vertical or horizontal axis, then the isocontours of limiting normal distribution are ellipses with semi axes parallel to the coordinate axes and so \( \Sigma \) is diagonal. □

In the above examples, \( \mathcal{L} \) is generated by \( \sigma_1^{-1}[1,0]^T \) and \( \sigma_2^{-1}[0,1]^T \). Consequently, \( K_1 = K_2 = 1 \). In this sense, these examples are the simplest possible ones (Figure 1 shows a configuration which symmetric with respect to the vertical axis, and is repeated over a \( 10 \times 6 \) rectangle). Our next example is less trivial as \( K_2 = 2 \).

Example 3.5. Consider a scatterer configuration on the regular hexagon that is invariant under the rotation by 120 degrees and satisfy all other assumptions (that is, the scatterers are smooth, disjoint, strictly convex and the configuration has finite horizon). One such example is only one scatterer which is a disc, centered at the center of the hexagon and with radius large enough to ensure that \( D \) is of finite horizon. By tiling the plane with regular hexagons, we obtain the Lorentz
gas as before. As in the previous example, the isocontours of the limiting normal distribution are invariant under the rotation by 120 degrees; hence they are circles and $\Sigma = \sigma^2 I_2$. In this case, $\tilde{Z}_t$ for any $t$ takes values in the set of tiles of the hexagonal tiling. Let $\mathcal{L}$ be the lattice generated by the vectors $\sigma^{-1}[0,1]^T$ and $\sigma^{-1}[^{\sqrt{3}}/2,1/2]^T$ and $\mathcal{G}$ be the graph with vertices $\mathcal{L}$ and edges between points at distance $\sigma^{-1}$. That is, $\mathcal{G}$ forms the triangular grid, dual to the hexagonal tiling (see Figure 2, the edges of $\mathcal{G}$ are denoted by dotted lines). In this example, $K_1 = 1$ and $K_2 = 2$. Indeed, on the horizontal boundary, we see an alternating sequence of two kinds of hexagons (ignoring the very first and the very last one): one of them has 5 neighbors in $D_L$ and the other one only has 3. A particle injected to a uniform random location on the first type hexagon has higher chance of staying in $D_L$ than in case of the second type hexagon. Thus we expect that $c_1 \neq c_2$ in the variant of (H2), when the vertical and horizontal coordinates are swapped.

4. Duality

4.1. Random walks. The definitions given in Section 2.1 easily extend to more general domains $D$ with piece-wise smooth boundary. One minor difference is that in (2.3) instead of $f(t_2/L)$ we need to choose a slightly different argument of $f$ as $t_2/L$ may not be on $\partial D$ and $f$ may not be defined (e.g. one can choose the closest point on $\partial D$ to $t/L$). Since $f$ is continuous, the exact choice is irrelevant as long as it is a bounded distance from $t/L$. To keep notations simple, we will write $f(t/L)$, where $f$ is a continuous function defined on $\partial D$ (there is no need to introduce $F$).

**Proposition 4.1.** Consider a random walk as in Section 3.1 and let

\[
A(J) = \sum_{j \in J} P(w_j)
\]
and $B = 1$. Then the conclusion of Theorem 2.11 and (LE) hold with $\varsigma = 1$ without assuming the rationality of $\mathcal{L}$ and for general bounded domains $D$ with piece-wise smooth boundary and no cusps.

**Proof.** We are only going to prove (2.6). A proof of (2.5) can be obtained by replacing $t$ by $\infty$ in the proof below and (LE) can be proved as in Theorem 3.3.

The key idea of the proof is duality. Specifically, we use the fact that the reversed Markov process is also Markovian. Let $\hat{Z}$ be the discretized version of $Z$. That is, $\hat{Z}_0 = Z_0, \hat{Z}_n = Z_{t_n}$ where $t_n$ is the time of the $n$th jump of $Z$. The reverted random walk $\hat{Z}'$ is defined by the generator

$$(G'f)(l) = \sum_{j=1}^{J} \mathcal{P}(w_j)[f(-w_j + l) - f(l)]$$

and $\hat{Z}'$ is the discretized version of $Z'$ (defined analogously to $\hat{Z}$).

Note that for any $N$, $\hat{Z}$ induces a measure $\mathbb{P}_{\hat{Z}}$ on $\mathcal{L}^N$ by

$$\mathbb{P}_{\hat{Z}}(l_0, ..., l_{N-1}) = \mathbb{P}(\hat{Z}_1 = l_1, ..., \hat{Z}_{N-1} = l_{N-1}|\hat{Z}_0 = l_0).$$

Let us define $\mathbb{P}_{\hat{Z}'}$, analogously. Then by definition of $\hat{Z}'$, for any sequence $l_0, ..., l_M \in \mathcal{L}$,

$$(4.2) \quad \mathbb{P}_{\hat{Z}}(l_0, ..., l_M) = \mathbb{P}_{\hat{Z}'}(l_M, ..., l_0).$$

For fixed $L, z \in D, t \in \mathbb{R}^+, \ell \in \partial D_L$ and $M$, let $\mathcal{A} = \mathcal{A}_{L,z,t,\ell,M}$ be the set of length $M$ trajectories from $\ell$ to $\langle zL \rangle$ staying inside $D_L$, i.e.

$$\mathcal{A} = \{(l_0, ..., l_M) : l_0 = \ell, \forall i = 0, ..., M-1 : \exists j = 1, ..., J : l_{i+1} - l_i = w_j, l_i \in D_L, l_M = \langle zL \rangle\}.$$ 

For a subset $\mathcal{B} \subset \mathcal{L}^{M+1}$ and a lattice point $\hat{l}$, let

$$\mathcal{B}' = \{(l_M, ..., l_0) : (l_0, ..., l_M) \in \mathcal{B}\},$$

and

$$\hat{\mathcal{B}} = \{(\hat{l}, l_0, ..., l_M) : (l_0, ..., l_M) \in \mathcal{B}\}, \quad \hat{\mathcal{B}'} = \{(l_0, ..., l_M, \hat{l}) : (l_0, ..., l_M) \in \mathcal{B}\}.$$ 

Then by (4.2), we have

$$\mathbb{P}_{\hat{Z}}(\mathcal{A}_{L,z,t,\ell,M}) = \mathbb{P}_{\hat{Z}'}(\mathcal{A}'_{L,z,t,\ell,M}).$$

Furthermore, for any $L_{-1} \notin D_L$, which is connected to $\ell$ in $\mathcal{G}$,

$$(4.3) \quad \mathbb{P}_{\hat{Z}}(L_{-1}A_{L,z,t,\ell,M}) = \mathbb{P}_{\hat{Z}'}(A'_{L,z,t,\ell,M}L_{-1}).$$

Let $T$ be the first hitting time of $\mathcal{L} \setminus D_L$ by $\hat{Z}'$. Then (4.3) is equal to

$$\mathbb{P}(T = M + 1, \hat{Z}'_{M+1} = L_{-1}, \hat{Z}'_M = \ell | \hat{Z}'_0 = \langle zL \rangle).$$
To turn to continuous time, let $\tau^{*} \ast$ be the first time $Z'$ leaves $D_L$. Then we have
\[ P(\tau^{*} \ast < t L^2, Z^{\prime*} = l_{-1}, Z^{\prime*}_{\tau^{*} \ast} = 0 | Z_0 = \langle zL \rangle) = \sum_{M=0}^{\infty} F_{M+1}(t L^2) P_{Z}(l_{-1} A_{L,z,t,l,M}), \]
where $F_N(.)$ is the cumulative distribution function of the Gamma distribution with shape parameter $N$ and scale parameter 1 (that is, it is the sum of $N$ iid exponential random variables, each with expectation 1). Indeed, (4.4) holds since the time of jumps of the Markov process $Z'$ are independent of the location of the jump. On the other hand, we have
\[ \sum_{M=0}^{\infty} F_{M+1}(t L^2) P_{Z}(l_{-1} A_{L,z,t,l,M}) = P(l_{-1} \tau^{*} \ast = l_{-1}, Z^{\prime*}_{\tau^{*} \ast} = l_{-1} | Z_0 = \langle zL \rangle) = \int_{0}^{t L^2} P(Z_s = \langle zL \rangle, \forall s' \in [0, s], Z_{s'} \in D_L | Z_0 = l_{-1}) ds. \]
(4.5)
Since $B = 1$, we have
\[ \Lambda_{t L^2}(\langle zL \rangle) = \sum_{l_{-1} \in \partial D_L} A_{L} f(l/L) \int_{0}^{t L^2} P(Z_s = \langle zL \rangle, \forall s' \in [0, s], Z_{s'} \in D_L | Z_0 = l_{-1}) ds. \]
(4.6)
Thus by (4.1) and (4.5), we have
\[ \Lambda_{t L^2}(\langle zL \rangle) = \sum_{l_{-1} \in \partial D_L} \sum_{l_{-1} < l \in L \setminus D_L} f(l/L) \sum_{M=0}^{\infty} F_{M+1}(t L^2) P_{Z}(l_{-1} A_{L,z,t,l,M}) \]
and so by (4.4),
\[ \Lambda_{t L^2}(\langle zL \rangle) = \mathbb{E} \left( f \left( \frac{Z^{\prime*}_{\tau^{*} \ast}}{L} \right) 1_{\tau^{*} \ast < t L^2} | Z_0 = \langle zL \rangle \right). \]
(4.7)
Now the right hand side of (4.7) converges, as $L \to \infty$ to
\[ \mathbb{E} \left( f \left( W_{T^*} \right) 1_{T^* < t} | W_0 = z \right), \]
where $W_t$ is a standard planar Brownian motion and $T^*$ is the hitting time of $\mathbb{R}^2 \setminus D$ by $W$. (This follows from Donsker’s theorem and the continuous mapping theorem. A more detailed proof of (4.8) for the case $t = \infty$ can be found in e.g. [20, Proposition 3].) Let $W$ be a diffusion process whose first coordinate is deterministic with constant 1 drift and whose second and third coordinates are independent standard Brownian motions. Applying Dynkin’s formula for $W$ with $W_0 = (-t, z)$, the stopping time $\tau$ as the first hitting time of $\mathbb{R}^3 \setminus ([-t, 0] \times D)$, and with the test function $v(-s, \tilde{z})$, where $v$ is defined by (2.2), we conclude that (4.8) satisfies (2.6) with $\varsigma = 1$. 
We record a remark for later reference:

**Remark 4.2.** Note that the proof of Proposition 4.1 does not use Theorem 2.1. Thus we already have an example (random walks), where both the assumptions and the conclusion of Theorem 2.1 are verified (by Proposition 3.1 and Proposition 4.1 respectively).

### 4.2. Lorentz gas

Let $D$ be a bounded domain with piece-wise smooth boundary and no cusps. In the setup of Section 3.2, given $D$, $L$, and $l \in \partial D_L$, we consider the following initial measure. For any $l' \in L \setminus D$, $\tilde{l} := \Sigma l'$ is a nearest neighbor of $l := \Sigma l$ in $\mathbb{Z}^2$ (that is, $\tilde{l} - l' \in \{w_1 = (0, -1), w_2 = (0, 1), w_3 = (-1, 0), w_4 = (1, 0)\}$) by our assumption in Section 3.2. Let $E = E_{l,l'} \subset \mathbb{R}^2$ be the line segment on the boundary of $\tilde{l} + [-1/2, 1/2]^2$ and $\tilde{l} + [-1/2, 1/2]^2$. Define

\[ N = N_{l,l'} = \{(q, v) \in \Omega : q \in E, \langle v, \tilde{l} - l' \rangle > 0\}. \]

Let $\text{type}(l, l') = j$ if $\tilde{l} - l' = w_j$ and $\zeta_j : N_{0, \Sigma^{-1/2}w_j} \to \mathbb{R}_+$ be the first return to $N_{0, \Sigma^{-1/2}w_j}$ in the compact Sinai billiard, that is

\[ \zeta_j = \min \{s : \Phi^s_0(q, v) \in N_{0, \Sigma^{-1/2}w_j}\}. \]

Let us also write

\[ \tilde{\zeta}_j = \int_{N_{0, \Sigma^{-1/2}w_j}} \zeta_j d\varrho_{0, \Sigma^{-1/2}w_j}. \]

Next, we define the finite measure $\varrho = \varrho_{l,l'}$ on $N$ by

\[ d\varrho = \frac{1}{2\zeta_j} \cos(\langle v, \tilde{l} - l' \rangle) dq dv, \]

where $\text{type}(l, l') = j$. Note that $\varrho(N) = |E_{l, -w_j}|/\tilde{\zeta}_j$ and so it may not be a probability measure. Now the initial condition $\mathcal{G}$ is given by the normalized sum of these measures for all neighbors $l'$. That is,

\[ \nu_{\mathcal{G}} = \frac{1}{\sum_{j \in \mathcal{J}(l)} |E_{l, -w_j}| \zeta_j} \sum_{j \in \mathcal{J}(l)} \varrho_{l, -w_j}. \]

By definition, $\nu_{\mathcal{G}}$ is a probability measure. Next, we choose

\[ \mathcal{A}(\mathcal{J}(l)) = \sum_{j \in \mathcal{J}(l)} \frac{|E_{l, -w_j}|}{\zeta_j} \]

(which clearly depends on $l$ only through $\mathcal{J}(l)$) and $\mathcal{B} = 1$. This choice guarantees that particles are being continuously injected through the entire boundary of $D_L$ with a measure which is simply the projection of the invariant measure $\mu$ to the Poincaré
section on the boundary of $D_L$. Because of this very special choice of $\nu_G, A, B$, we have

**Proposition 4.3.** With the above choice, the conclusion of Theorem 1.1 (LE) and (DLE) hold with $\zeta = 1$ without assuming the rationality of $\mathcal{L}$ and for general bounded domains $D$ with piece-wise smooth boundary and no cusps.

**Proof.** The proof is similar to that of Proposition 4.1. We use duality and it is sufficient to verify (2.6).

We claim that there is some $s^* > 0$ so that for any $(q, v) \in \mathcal{N}_{l,v}$ and any $s \in [0, s^*], \mathcal{Z}_s(q, v) \in \{I, V\}$. Furthermore, if there is some $s \in [0, s^*]$ with $\mathcal{Z}_s(q, v) = l'$, then $\mathcal{Z}_{s^*}(q, v) = l'$. Indeed, the first statement follows from the assumption that $(-1/2, 1/2) \notin D$ and the second follows from the fact that visiting $l$, then $l'$ and then $l$ again requires at least 2 collisions and so we choose $s^*$ shorter than the minimal free flight.

Next, for any $(l, l')$ as above, by the definition of $\varrho$ and by the fact that $s^* < \min \zeta$, we have for measurable sets $B \subset \bigcup_{s \in [0, s^*]} \Phi^s(\mathcal{N}_{l,v})$

\[ (4.9) \quad \int B d\mu = \int_0^{s^*} \left( \int B d\Phi^s(\varrho_{l,v}) \right) ds. \]

By the definition of $\nu_G, A$ and $B$, we have

\[ \Lambda_{IL^2}(\langle zL \rangle) = \sum_{t \in \partial D_L} \sum_{v \in \mathcal{L} \setminus D_L:(l', l) \in G} f(l/L) \]

\[ \int_0^{tL^2} \int_{\mathcal{N}_{l,v}} \{ (q, v) : \forall s' \in [0, s], \mathcal{Z}_{s'}(q, v) \in D_L, \mathcal{Z}_s(q, v) = \langle zL \rangle \} d\varrho_{l,v}(q, v) ds. \]

For fixed $t$ and $L$, let $K \in \mathbb{Z}_+$ so that $Ks^* \leq tL^2 < (K+1)s^*$. To simplify formulas, let us assume that $Ks^* = tL^2$ holds (it is easy to check that the contribution of $s \in [Ks^*, tL^2]$ is negligible). Now for $k = 1, \ldots, K$ we apply (4.9) with $B_{l,v,k} = \{ (q, v) \in \bigcup_{s \in [0, s^*]} \Phi^s(\mathcal{N}_{l,v}) : \forall s' \in [0, (k-1)s^*], \mathcal{Z}_{s'}(q, v) \in D_L, \mathcal{Z}_{(k-1)s^*}(q, v) = \langle zL \rangle \}$ and the definition of $s^*$ to conclude

\[ \int_{(k-1)s^*}^{ks^*} \int_{\mathcal{N}_{l,v}} \{ (q, v) : \forall s' \in [0, s], \mathcal{Z}_{s'}(q, v) \in D_L, \mathcal{Z}_s(q, v) = \langle zL \rangle \} d\varrho_{l,v}(q, v) ds = \int B_{l,v,k} d\mu \]

and so

\[ (4.10) \quad \Lambda_{IL^2}(\langle zL \rangle) = \sum_{t \in \partial D_L} \sum_{v \in \mathcal{L} \setminus D_L:(l', l) \in G} f(l/L) \sum_{k=1}^K \int (B_k) d\mu. \]
Now we recall the involution (also known as time reversibility) property of billiards. For \((q, v) \in \Omega\), let \(\mathcal{I}(q, v) = (q, -v)\). Then \(\mathcal{I}\) preserves \(\mu\) and anticommutes with the flow. That is,
\[
\Phi^{-s} \circ \mathcal{I} = \mathcal{I} \circ \Phi^s.
\]
(see e.g. [10, Section 2.14]). Thus

\[
\int B_{l', k} d\mu = \int B'_{l', k} d\mu,
\]
where

\[
B'_{l', k} = \{(q, v) \in \Omega : \mathcal{Z}_0(q, v) = \langle zL \rangle
\}
\]
and

\[
\exists s \in [(k - 1)s^*, ks^*] : \forall s' \in [0, s] : \mathcal{Z}_{s'} \in D_L, \Pi_{D} \Phi^s(q, v) \in E_{l, v}\}.
\]

Using the notation (2.4) and combining (4.10), (4.11) and (4.12), we conclude

\[
\Lambda_{tL^2}(\langle zL \rangle) = \int \int f \left( \frac{Z_{r^*}}{L} \right) 1_{r^* < tL^2} d\mu
\]

By the invariance principle, the right hand side of (4.13) converges as \(L \to \infty\) to (4.8). As in Proposition 4.1, (2.6) follows.

\[
\square
\]

5. PROOF OF THEOREM 2.1

The keep the notations simpler, we assume that \(a = 1\) (the proof extends to any \(a > 0\) with no new ideas). We will prove (2.5) first. Let \(z = (x, y)\) be a point in the interior of \(D\). By definition, we have

\[
\mathbb{E}(\Lambda(\langle zL \rangle)) = \int_0^\infty \sum_{l \in \partial W L} A(J(l)) B(t) f \left( \frac{l^2}{L} \right) P \left( \mathcal{Z}_t = \langle (x, y)L \rangle, \min\{\tau^y, \tau^y_L, \tau^x, \tau^x_L\} > t | \mathcal{Z}_0 = l \right) dt
\]

with \(I_j = I_j(L, x, y, \delta)\) for \(j = 1, 2, 3\). Noting that

\[
\lim_{\delta \to 0} \lim_{L \to \infty} I_2 + I_3 = 0.
\]

by (H3), it remains to prove

\[
\lim_{\delta \to 0} \lim_{L \to \infty} I_1 = u(z).
\]
Let $\Psi_{\delta'} : [0, 1] \to [0, 1]$ be defined by

$$
\Psi_{\delta'}(y) = \begin{cases} 
0 & \text{if } y < \delta' \\
\frac{1}{\delta'} y - 1 & \text{if } \delta' \leq y < 2\delta' \\
1 & \text{if } 2\delta' \leq y < 1 - 2\delta' \\
\frac{1}{\delta'} y - 1 + \frac{1}{\delta'} & \text{if } 1 - 2\delta' \leq y < 1 - \delta' \\
0 & \text{if } y > 1 - \delta' 
\end{cases}
$$

and write $f_{\delta'}(y) = f(y)\Psi_{\delta'}(y)$.

To prove (5.3), we first write $I_1 = I_{11} + I_{12}$ with $I_{1,k} = I_{1,k}(L, x, y, \delta, \delta')$ for $k = 1, 2$, where $I_{11}$ and $I_{12}$ are obtained from $I_1$ by replacing $f$ by $f_{\delta'}$ and $f - f_{\delta'}$, respectively. To verify (5.3), it is sufficient to prove

$$(5.4) \quad \lim_{\delta' \to 0} \lim_{\delta \to 0} \lim_{L \to \infty} I_{11} = u(z)$$

and

$$(5.5) \quad \lim_{\delta' \to 0} \lim_{\delta \to 0} \lim_{L \to \infty} I_{12} = 0$$

To simplify notations, we will write $I_{11}^\infty = \lim_{L \to \infty} I_{11}$ and $I_{11,0}^\infty := \lim_{\delta \to 0} I_{11}^\infty$.

Let us consider the following truncated version of (2.1)

$$(5.6) \quad \Delta \hat{u} = 0, \quad \hat{u}|_{\partial \Omega} = \varsigma F_{\delta'},$$

where $F_{\delta'}$ is defined as $F$ except that $f$ is replaced by $f_{\delta'}$.

**Proposition 5.1.** For any $\delta' \in (0, 1/4)$, $I_{11,0}^\infty$ is the solution of (5.6).

**Proof.** The proof consists of two steps. First, we prove that $I_{11}^{\infty,0}$ exists; then we show that it satisfies (5.6).

**Step 1: $I_{11}^{\infty,0}$ exists**

Let us define $B = l_{(K)}^{(K)}$, where $K$ is defined by (2.9). To simplify formulas, let us write $\bar{\tau} = \min\{\tau_0, \tau_L^Y, \tau_0^X, \tau_L^X\}$. Also observe that by transitivity of $G$, there are constants $A_1, \ldots, A_K$ so that for any $m \in \mathbb{N}$, for any $k = 1, \ldots, K$, $A(j(l^{(mK+k)})) = \ldots$
Now, we compute

$$I_{11} = \sum_{l \in \partial W_2} A(\mathcal{J}(l)) \int_{\delta L^2}^{t_{L/\delta}} B(t) f_{\delta'} \left( \frac{l_2}{L} \right) \mathbb{P}(Z_t = \langle (x, y) L \rangle, \bar{\tau} > t | Z_0 = l) dt$$

$$= \sum_{m=\delta L/B}^{(1-\delta)L/B} \sum_{k=1}^K A_k \int_{\delta L^2}^{t_{L/\delta}} B(t) f_{\delta'} \left( \frac{l_2^{(mK+k)}}{L} \right) \mathbb{P}(Z_t = \langle (x, y) L \rangle, \bar{\tau} > t | Z_0 = l^{(mK+k)}) dt$$

$$= \sum_{m=\delta L/B}^{(1-\delta)L/B} \sum_{k=1}^K A_k \int_{\delta}^{1/\delta} B(sL^2) f_{\delta'} \left( \frac{l_2^{(mK+k)}}{L} \right) \mathbb{P}(Z_{sL^2} = \langle (x, y) L \rangle, \bar{\tau} > sL^2 | Z_0 = l^{(mK+k)} \rangle L^2 ds$$

Now using (H2) with $T = sL^2$, $\alpha = x/\sqrt{s}$, $\beta = 1/\sqrt{s}$, $\eta = \frac{l_2^{(mK+k)}}{(L\sqrt{s})}$, $\gamma = y/\sqrt{s}$, $\xi = 1/\sqrt{s}$, we obtain

$$I_{11} \sim \sum_{m=\delta L/B}^{(1-\delta)L/B} \sum_{k=1}^K A_k c_k$$

$$\int_{\delta}^{1/\delta} B(sL^2) f_{\delta'} \left( \frac{l_2^{(mK+k)}}{L} \right) s^{-3/2} L^{-1} \psi \left( \frac{x}{\sqrt{s}}, \frac{1}{\sqrt{s}} \right) \phi \left( \frac{l_2^{(mK)}}{L\sqrt{s}}, \frac{y}{\sqrt{s}}, \frac{1}{\sqrt{s}} \right) ds$$

by uniform convergence, where $a_L \sim b_L$ means that $\lim_{L \to \infty} a_L/b_L = 1$. Let us write

$$\bar{c} = \frac{1}{K} \sum_{k=1}^K A_k c_k.$$

Then

$$I_{11} \sim \frac{\bar{c} K}{B} \int_{\delta}^{1/\delta} B(sL^2) s^{-3/2} \psi \left( \frac{x}{\sqrt{s}}, \frac{1}{\sqrt{s}} \right) \left[ \sum_{m=\delta L/B}^{(1-\delta)L/B} \frac{B}{L} f_{\delta'} \left( \frac{l_2^{(mK)}}{L} \right) \phi \left( \frac{l_2^{(mK)}}{L\sqrt{s}}, \frac{y}{\sqrt{s}}, \frac{1}{\sqrt{s}} \right) \right] ds.$$

Replacing the Riemann sum with the corresponding Riemann integral, we obtain

$$I_{11} \sim \frac{\bar{c} K}{B} \int_{\delta}^{1/\delta} B(sL^2) s^{-3/2} \psi \left( \frac{x}{\sqrt{s}}, \frac{1}{\sqrt{s}} \right) \left[ \int_{\delta}^{1-\delta} f_{\delta'}(\sigma) \phi \left( \frac{\sigma}{\sqrt{s}}, \frac{y}{\sqrt{s}}, \frac{1}{\sqrt{s}} \right) d\sigma \right] ds$$

(we are permitted to do this because of uniform convergence of the bracketed expression in $s$). Since the integrand in the last formula is uniformly continuous in
s and since \( B \) is periodic with period 1 and \( f_0^1 B = 1 \), we can take the limit \( L \to \infty \) to conclude that \( I_{11}^\infty \) exists and is equal to

\[
\frac{cK}{B} \int_\delta^{1/\delta} s^{-3/2} \psi \left( \frac{x}{\sqrt{s}}, \frac{1}{\sqrt{s}} \right) \left[ \int_{\delta'}^{1-s'} f_{\delta'}(\sigma) \phi \left( \frac{\sigma}{\sqrt{s}}, \frac{y}{\sqrt{s}}, \frac{1}{\sqrt{s}} \right) d\sigma \right] ds.
\]

Now we substitute (2.7) and (2.8) to the above to conclude

\[
I_{11}^\infty = \frac{cK}{B} \int_\delta^{1-\delta'} \int_{\delta'}^1 \sum_{k=-\infty}^\infty \sum_{n=-\infty}^\infty \left( \frac{1}{s^2} (2k + x) \exp \left( - \frac{(2k + x)^2}{2s} \right) \frac{1}{\sqrt{2\pi}} \right) f_{\delta'}(\sigma) \left[ \exp \left( - \frac{(y - \sigma - 2n)^2}{2s} \right) - \exp \left( - \frac{(y + \sigma + 2n)^2}{2s} \right) \right] ds d\sigma.
\]

Clearly, the sum is absolutely and uniformly convergent and so we can write the sums in front of the integrals. Thus

\[
I_{11}^\infty = \frac{cK}{B} \sum_{k=-\infty}^\infty \sum_{n=-\infty}^\infty \int_{\delta'}^{1-\delta'} R(k, n, \delta, \sigma, s, x, y) d\sigma,
\]

where

\[
R(k, n, \delta, \sigma, s, x, y) = \frac{x + 2k}{\sqrt{2\pi}} f_{\delta'}(\sigma)
\]

\[
\ast \int_{\delta'}^{1/\delta} \frac{1}{s^2} \left[ \exp \left( - \frac{(2k + x)^2 + (y - \sigma - 2n)^2}{2s} \right) - \exp \left( - \frac{(2k + x)^2 + (y + \sigma + 2n)^2}{2s} \right) \right] ds.
\]

Making the substitution \( \omega = (2s)^{-\frac{1}{2}} \) (and so \( 4\omega d\omega = -ds/s^2 \)) and letting \( P_1 = (2k + x)^2 + (y - \sigma - 2n)^2 \) and \( P_2 = (2k + x)^2 + (y + \sigma + 2n)^2 \), we get:

\[
R(k, n, \delta, \sigma, x, y) = \frac{4(x + 2k)}{\sqrt{2\pi}} f_{\delta'}(\sigma) \int_{\sqrt{2\delta}}^{\sqrt{2s}} \omega \left[ \exp(-P_1 \omega^2) - \exp(-P_2 \omega^2) \right] d\omega
\]

\[
= -\frac{2(x + 2k)}{\sqrt{2\pi}} f_{\delta'}(\sigma) \left[ \frac{1}{P_1} \exp \left( -\frac{P_1}{2\delta} \right) - \frac{1}{P_2} \exp \left( -\frac{P_2}{2\delta} \right) \right]
\]

\[
+ \frac{2(x + 2k)}{\sqrt{2\pi}} f_{\delta'}(\sigma) \left[ \frac{1}{P_1} \exp \left( -\frac{P_1 \delta}{2} \right) - \frac{1}{P_2} \exp \left( -\frac{P_2 \delta}{2} \right) \right]
\]

\[
=: R_1 + R_2.
\]

Clearly, we have

\[
\lim_{\delta \to 0} \sum_n \sum_k R_1 = 0
\]
and as Lemma 5.2 shows,
\[
\lim_{\delta \to 0} \sum_n \sum_k R_2 = \sum_n \sum_k \lim_{\delta \to 0} R_2.
\]
So we get
\[
\lim_{\delta \to 0} R(k, n, \delta, \sigma, x, y) = R(k, n, \sigma, x, y) = \frac{2(x + 2k)}{\sqrt{2\pi}} f_\psi(\sigma) \left[ \frac{1}{P_1} - \frac{1}{P_2} \right].
\]
and hence
\[
(5.7) \quad I_{11}^{\infty,0} = \frac{\bar{c}K}{B} \int_{\delta}^{1-\delta'} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(k, n, \sigma, x, y) d\sigma.
\]
To complete Step 1, it remains to verify

**Lemma 5.2.** Let \( u(z) = \exp(-z)/z \). And let \( P_1 \) and \( P_2 \) be as defined above. Then for \( \delta \in \mathbb{R}, x \in [0,1], \sigma \in [0,1] \), and \( k, n \) not both 0, the following sum converges uniformly in \( \delta, x, \sigma \), and the convergence is uniform in \( \delta \), \( x \), and \( \sigma \) as \( M \to \infty \).

\[
\sum_{k=-M}^{M} \sum_{n=-M}^{M} (2k + x) \delta [u(P_1 \delta) - u(P_2 \delta)].
\]

**Proof.** Let us write
\[
P_3 = (2k + x)^2 + (y - \sigma + 2n)^2, \quad P_4 = (2k + x)^2 + (y + \sigma - 2n)^2.
\]
We will show
\[
(5.8) \quad \lim_{M \to \infty} \left\{ \sum_{k:|k|> M} \sum_{n=1}^{\infty} + \sum_{k=-M}^{M} \sum_{n=-M}^{\infty} \right\} |S| = 0,
\]
where
\[
S = S(k, n, \delta, \sigma, x, y) = (2k + x) \delta [u(P_1 \delta) - u(P_2 \delta) + u(P_3 \delta) - u(P_4 \delta)].
\]
and the convergence is uniform in \( \delta, x, \sigma \). First, observe that
\[
P_1 - P_2 = -4(\sigma + 2n)y, \quad P_3 - P_4 = 4(2n - \sigma)y.
\]
By the mean value theorem,
\[
u(P_1 \delta) - u(P_2 \delta) = u'(P_1' \delta)(P_1 - P_2) \delta, \quad u(P_3 \delta) - u(P_4 \delta) = u'(P_3' \delta)(P_3 - P_4) \delta
\]
for some \( P_1' \in (P_1, P_2) \) and \( P_3' \in (P_4, P_3) \). Using the mean value theorem again, we conclude
\[
u(P_1 \delta) - u(P_2 \delta) + u(P_3 \delta) - u(P_4 \delta) = -4\sigma y \delta [u'(P_1' \delta) + u'(P_3' \delta)] - 8ny\delta^2 (P_1' - P_3') u''(P_1' \delta)
\]
for some \( P_1'' \in (P_4, P_2) \). In the sequel, \( C \) denotes a universal constant (independent of \( k, n, x, y, \delta, \sigma, L \) or any other parameters), whose value is unimportant and can even
change from line to line. Now using the estimates $|u'(z)| < C/z^2$, $|u''(z)| < C/z^3$ for any real number $z$, we have

$$|S| \leq C \left( \frac{|k|}{(k^2 + n^2)^2} + \frac{|k|n^2}{(k^2 + n^2)^3} \right)$$

Thus we conclude

$$\sum_{k=M}^{\infty} \sum_{n=1}^{k} |S| \leq C \sum_{k=M}^{\infty} \sum_{n=1}^{k} \frac{1}{k^3} \leq C/M$$

and likewise

$$\sum_{n=M}^{\infty} \sum_{k=0}^{n-1} |S| \leq C \sum_{n=M}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n^3} \leq C/M$$

We have verified (5.8). The lemma follows. □

**Step 2**: $I_{11}^{\infty,0}$ satisfies (5.6)

We give two independent proofs for Step 2. The first proof is shorter and easily generalizes to the case of finite $t$. The second proof shows that the formulas derived above are tractable (at least in case $t = \infty$).

**Proof 1**: Step 1 shows that for any stochastic process $Z_t$ satisfying (H1)-(H3), the limit (5.7) is the same. Recalling Remark 4.2, we already have examples when (H1)-(H3) as well as the conclusion of the theorem holds. Thus $I_{11}^{\infty,0}$ has to satisfy (5.6). To finish the first proof, we identify the constant $\varsigma$.

Let us consider the simplest possible random walk, called the simple symmetric random walk. That is, $w_1 = (0, -\sqrt{2})^T$, $w_2 = (0, \sqrt{2})^T$, $w_3 = (-\sqrt{2}, 0)^T$, $w_4 = (\sqrt{2}, 0)^T$ and

$$P(w_i) = \frac{1}{4} \text{ for } i = 1, ..., 4.$$ 

In this case, $L = (\sqrt{2}Z)^2$ and by the central limit theorem, $Z_t/\sqrt{t}$ converges to a 2 dimensional standard normal random variable (we chose the normalization $\sqrt{2}$ so that the limiting covariance matrix is identity and so $Z$ fits into the framework of Proposition 4.1). In this case, we clearly have $K = 1$, $B = \sqrt{2}$, $A_1 = 1/4$ (and $B = 1$). Thus $\bar{c} = c_1/4$. Next we claim that now $c_1 = 4/\sqrt{\pi}$. To prove the claim, first note that

$$\lim_{T \to \infty} \sqrt{T}P(\tau_0^X > T | A_0 = 0) = \frac{2}{\sqrt{\pi}}$$

(this follows from e.g., [19, Proposition 5.1.2]). The proof of (H2) is based on the fact that under the assumption $\tau_0^X > T$, $Z_{[T]}/\sqrt{T}$, $0 \leq t \leq 1$ converges to a stochastic process whose first coordinate is a Brownian meander and the second coordinate is a Brownian motion. Furthermore, the local limit theorem also holds under the
assumption $\tau_0 X > T$ which gives (H2) (see the details in Section 7). This local limit theorem combined with (5.9) gives $c_1 = \frac{2}{\sqrt{\pi}} \text{covol}(L) = \frac{4}{\sqrt{\pi}}$ which proves the claim.

Thus in case $K = 1$, $B = \sqrt{2}$, $\bar{c} = 1/\sqrt{\pi}$, (5.7) satisfies (5.6) with $\varsigma = 1$. Since (5.7) is linear in $\bar{c} K/B$, we conclude that in case of general $K, B$ and $\bar{c}$, (5.7) satisfies (5.6) with

$$c_1 = \frac{2}{\sqrt{\pi}} \text{covol}(L) = \frac{4}{\sqrt{\pi}}$$

(5.10)

Proof 2:

**Step 2’a: $I_{11}^{\infty,0}$ is harmonic**

An elementary computation shows that $R(k,n,\delta,\sigma,x,y)$, as a function of $x, y \in (0, 1)^2$ is harmonic for any $k$ and $n$. Since the derivatives of $R(k,n,\sigma,x,y)$ with respect to $x$ and $y$ converge uniformly in a neighborhood of $x, y$, the Laplacian can be taken inside the sum in (5.7). It follows that $I_{11}^{\infty,0}$ is harmonic.

**Step 2’b: $I_{11}^{\infty,0}$ satisfies the boundary conditions of (5.6)**

Recall (5.7) from Step 1. Let us first consider the case when $|n| + |k| > 0$. In this case, there is uniform convergence in $x, y$ and $\sigma$ so we can write the limit inside the sum and the integral:

$$\bar{c} K \frac{B}{B} \sum_{k,n \in \mathbb{Z}} \int_{\delta'}^{1-\delta'} \lim_{(x,y) \to (0,y_0)} R(k,n,\sigma,x,y) d\sigma.$$ 

We can directly compute this limit as

$$\int_{\delta'}^{1-\delta'} \lim_{(x,y) \to (0,y_0)} R(k,n,\sigma,x,y) d\sigma = \int_{\delta'}^{1-\delta'} R(k,n,\sigma,0,y_0) d\sigma$$

$$= \int_{\delta'}^{1-\delta'} \frac{16ky_0(\sigma + 2n)}{[2k^2 + (y_0 - \sigma - 2n)^2][2k^2 + (y_0 + \sigma + 2n)^2]} d\sigma.$$ 

We see that for each $n$, these terms are antisymmetric in $k$, so that summing over $k$ and $n$, with $|n| + |k| > 0$, all of the terms cancel. Now we consider the case $n = k = 0$. This term gives:

$$\lim_{(x,y) \to (0,y_0)} I_{11}^{\infty,0} = \frac{\bar{c} K}{B} \frac{8}{\sqrt{2\pi}} \lim_{(x,y) \to (0,y_0)} \int_{\delta'}^{1-\delta'} \frac{16ky_0(\sigma + 2n)}{[2k^2 + (y_0 - \sigma - 2n)^2][2k^2 + (y_0 + \sigma + 2n)^2]} d\sigma.$$ 

To compute the above integral assume first that $\delta' < y_0 < 1 - \delta'$, and decompose it as

$$\int_{\delta'}^{1-\delta'} \ldots d\sigma = \int_{y_0 - Ax}^{y_0 - Ax} \ldots d\sigma + \int_{y_0 - Ax}^{y_0 - Ax} \ldots d\sigma =: I_{111} + I_{112}$$

for some large constant $A$. 

First, we compute $I_{111}$. For $y_0$ and $A$ fixed, and for $x$ and $|y-y_0|$ small, $y f_{y'}(\sigma)/[x^2+(y+\sigma)^2]$ is close to $f_{y'}(y_0)/(4y_0)$ uniformly in $\sigma$ as in $I_{111}$. Indeed, this follows from the continuity of $f_{y'}$. Thus we can write this term in front of the integral. Now it remains to compute \[
abla_{y_0} x \sigma /[x^2+(y_0-\sigma)^2] d\sigma.\]
Let us apply the substitution $\rho = (\sigma - y_0)/x$. Then the previous integral becomes \[
abla_{y_0} x \rho/(1 + \rho^2) d\rho + \nabla_{y_0} y_0/(1 + \rho^2) d\rho.
\]
The first integral here is zero as the integrand is an odd function. The second integral is $\pi y_0(1 + o_A(1))$. We conclude \[\lim_{(x,y) \to (0,y_0)} I_{111} = \frac{\pi}{4} f_{y'}(y_0)(1 + o_A(1)).\]

Next, we claim \[\lim_{(x,y) \to (0,y_0)} I_{112} = o_A(1).\]
To prove (5.12), we compute \[
\nabla_{y_0} x \sigma_0 / [x^2+(y_0-\sigma)^2] = \int_{y_0}^{y_0+A} x \sigma_0 / [x^2+(y_0-\sigma)^2] d\sigma.
\]
This estimate, combined with a similar computation for the domain $[\delta', y_0-A\delta']$, verifies (5.12). Next, if $y_0 < \delta'$ or $y_0 > 1 - \delta'$, then clearly $I_{111} = 0$ and $I_{112} = o_A(1)$. Now combining (5.11) and (5.12), we obtain the boundary conditions of (5.6) on the "West side" (that is when $x = 0$) with the constant \[\varsigma = \frac{\sqrt{2\pi\bar{c}K}}{B}.
\]
which coincides with (5.10).

Checking the boundary conditions on the other three sides is easier. First, recall that
\[ R(n, k, \sigma, x, y) = \frac{2(x + 2k)}{\sqrt{2\pi}} \frac{(y + \sigma + 2n)^2 - (y - \sigma - 2n)^2}{[(2k + x)^2 + (y - \sigma - 2n)^2][(2k + x)^2 + (y + \sigma + 2n)^2]} \]

Thus for every \( k = 0, 1, 2, \ldots \), we have \( R(n, k, \sigma, 1, y) = -R(n, -k - 1, \sigma, 1, y) \) and so \( \sum_{k \in \mathbb{Z}} R(n, k, \sigma, 1, y) = 0 \) for every \( n \). It follows that \( \lim_{y \to 0} I_{11}^{\infty, 0} = 0 \). Clearly, \( R(n, k, \sigma, 0) = 0 \) for every \( n \) and \( k \) and so \( \lim_{y \to 0} I_{11}^{\infty, 0} = 0 \). Finally, to prove \( \lim_{y \to 1} I_{11}^{\infty, 0} = 0 \), let us write
\[ \lim_{y \to 1} I_{11}^{\infty, 0} = \sum_{k \in \mathbb{Z}} \frac{2(x + 2k)}{\sqrt{2\pi}} \sum_{n} \frac{1}{P_1(n)} - \frac{1}{P_2(n)}, \]
where \( P_1(n) = (2k + x)^2 + (1 - \sigma - 2n)^2 \) and \( P_2(n) = (2k + x)^2 + (1 + \sigma + 2n)^2 \). Now observe that \( P_2(n) = P_1(n + 1) \). Thus the sum over \( n \) is telescopic and so by absolute convergence, \( \lim_{y \to 1} I_{11}^{\infty, 0} = 0 \). We have finished the proof of Step 2'b. □

Now we finish the proof of (2.5). First note that Proposition 5.1 implies (5.4). Thus it remains to verify (5.5). Consider the following Dirichlet problem:
\[
(5.13) \quad \begin{cases} \Delta U = 0 \text{ in } (0, 1) \times (-1, 2), \\ U(0, y) = \mathcal{S}(f(y) - f_{\delta'}(y)), U(1, y) = U(x, -1) = U(x, 2) = 0, \end{cases}
\]
where \( f \) and \( f_{\delta'} \) are identically zero on \([-1, 0] \cup [1, 2]\]. Now the proof of Proposition 5.1 applied on the domain \((0, 1) \times (-1, 2)\) with boundary condition given by \( f - f_{\delta'} \) implies that for any \( \delta', x, y \) fixed,
\[ \lim_{\delta' \to 0} \lim_{L \to \infty} I_{12} \leq U(x, y). \]

Indeed, on the one hand if the particles are only killed upon leaving \((0, L) \times (-L, 2L)\), then we obtain an upper bound on the number of surviving particles in case when particles are killed upon leaving \((0, L) \times (0, L)\). On the other hand, the proof of Proposition 5.1 is applicable on the larger domain since the boundary condition is identically zero in a neighborhood of the corners.

Now since the function \( f - f_{\delta'} \) is supported on the union of two intervals with total length \( 4\delta' \) and is bounded uniformly in \( \delta' \), we have \( \lim_{\delta' \to 0} U(x, y) = 0 \) for all \( x, y \) fixed. Thus (5.5) follows and the proof of (2.5) is complete.

The proof of (2.6) is similar, so we only explain the differences. First, the decomposition (5.1) now reads
\[
\int_{-\delta L}^{\delta L} \ldots dt + \int_{0}^{\delta L} \ldots dt =: I_1 + I_2.
\]
In particular, $I_3$ is missing and $I_2$ is negligible as before. We decompose $I_1 = I_{11} + I_{12}$ as before. Proceeding as in Step 1 of the proof of Proposition 5.1, we obtain

$$
\lim_{\delta \to 0} \lim_{L \to \infty} I_{11} = \frac{cK}{B} \int_{\delta'}^{1-\delta'} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(t, k, n, \sigma, x, y) d\sigma,
$$

where

$$
R(t, k, n, \sigma, x, y) = \frac{2(x + 2k)}{\sqrt{2\pi}} f_\sigma(\sigma) \left[ \frac{1}{P_1} \exp \left( -\frac{P_1}{2t} \right) - \frac{1}{P_2} \exp \left( -\frac{P_2}{2t} \right) \right].
$$

The first proof of Step 2 in Proposition 5.1 is the same as before. We refer not to give a second proof of Step 2 as in the time dependent case, the formulas in Step 2’a become substantially longer. Finally, the proof of (5.5) is again analogous to the previous case with $U$ as in (5.13) replaced by the unique solution $V(t, x, y) : \mathbb{R}_{\geq 0} \times (0, 1) \times (-1, 2) \to \mathbb{R}$ of

$$
\begin{cases}
V_t = \frac{1}{2} [V_{xx} + V_{yy}], \\
V(t, 0, y) = \varsigma(f(y) - f_\sigma'(y)), V(t, 1, y) = V(t, x, -1) = V(t, x, 2) = 0, V(0, x, y) = 0.
\end{cases}
$$

We have finished the proof of Theorem 2.1.

6. Background on Lorentz gas

6.1. Preliminaries. Here, we review some results for the Lorentz gas that are necessary to the proof of Theorem 3.2. We refer the reader to [10] for an in depth discussion. Let us use the notation of Section 3.2.

The map $F_0$ is hyperbolic in the sense that there are stable and unstable cone fields $C^u/s_x \subset T_x M_0$ so that $D_x F_0(C^u_x) \subset C^u_{F_0(x)}$ and $D_x F_0^{-1}(C^u_x) \subset C^2_{F_0^{-1}(x)}$ and for all $v \in C^u_x$, $\|D_x F_0(v)\| \geq \Lambda \|v\|$ (and likewise for all $v \in C^s_x$, $\|D_x F_0^{-1}(v)\| \geq \Lambda \|v\|$). Furthermore, stable and unstable manifolds exist through almost every point, but not through every point because of singularities due to grazing collisions. In fact, the presence of these singularities makes the study of billiards particularly peculiar.

Let us use the coordinates $(r, \varphi)$ on $M_0$ where $r$ is the arc length parameter of $\partial D_0$ and $\varphi \in [-\pi/2, \pi/2]$ is the angle between the postcollisonal velocity and the normal vector to $D$. A curve $W \subset M_0$ is called unstable if for every $x \in W$, $T_x W$ is in the unstable cone. Furthermore, an unstable curve $W$ is called weakly homogeneous if it does not intersect any singularity and there exists $k = 0, k_0, k_0 + 1, \ldots$ so that for all $x = (r, \varphi) \in [(k + 1)^{-2}, k^{-2}]$ if $|k| > k_0$ or $|\varphi| < k_0^{-2}$. In other words, weakly homogeneous unstable curves are required to be disjoint from the real singularities of $F_0$ as well as secondary singularities $\varphi = \pm k^{-2}$ for $|k| \geq k_0$. A weakly homogeneous unstable curve is called homogeneous if it satisfies certain extra regularity properties.
whose exact form are not needed for us (see the distortion and curvature bounds in \[9\] Section 4.3).

A pair $\ell = (W, \rho)$ is called a \textit{standard pair} if $W$ is a homogeneous unstable curve and $\rho$ is a probability measure on $W$ so that

$$\left| \log \frac{d\rho}{d\text{Leb}}(x) - \log \frac{d\rho}{d\text{Leb}}(y) \right| \leq C_0 \frac{|W(x, y)|}{|W|^{2/3}}$$

where $C_0$ is a universal constant and $|.|$ stands for arc length. Here and in the sequel log stands for logarithm with base $e$. We will also use the notation $\log_2$ for the logarithm with base 2.

Given $\ell$, we denote by $\nu_\ell$ the probability measure generated by $\rho$ and $\text{length}(\ell) = \text{length}(W)$. Due to the singularities, an image of a homogeneous unstable curve will be a collection of unstable curves. Furthermore, the regularity of $\rho$ is chosen in a way that is preserved by $F_0$. Thus the image of a standard pair under $F_0$ is the weighted average of standard pairs. Thus it is convenient to introduce the notion of a \textit{standard family}: a weighted average of standard pairs. Specifically, let us say that $\mathcal{G} = \{\ell_a = (W_a, \rho_a)\}_{a \in \mathcal{A}}$, $\lambda$ is a standard family if $\ell_a$ are standard pairs, $W_a$'s are disjoint and $\lambda$ is a probability measure on the index set $\mathcal{A}$. The standard family $\mathcal{G}$ induces a measure $\nu_\mathcal{G}$ on $M_0$ by

$$\nu_\mathcal{G}(B) = \int_\mathcal{A} \nu_\ell(B \cap W_a) d\lambda(a)$$

for Borel sets $B \subset M_0$. For a given homogeneous unstable curve $W$, and for $x \in W$, we denote by $r(x)$ the distance from $x$ to the closest endpoint of $W$, measured along $W$. We denote by $r_n(x)$ the distance from $F_0^n(x)$ to the closest endpoint of $W'$, where $W'$ is the maximal homogeneous curve in the image $F^n(W)$ containing $F_0^n(x)$. We define the $Z$ function of a standard family by

$$Z_\mathcal{G} = \sup_{\varepsilon > 0} \frac{\nu_\mathcal{G}(r < \varepsilon)}{\varepsilon}.$$

Note that we assumed that the curves in a standard family are disjoint and so the function $r$ is well defined. Now we are ready to state the last missing technical piece of Theorem 3.2. $\mathcal{G}$ is any standard family with a finite $Z$ function. Examples include any standard pair or the invariant measure $\nu_0$.

A fundamental property of Sinai billiards is that the expansion wins over fragmentation. That is, most of the weight carried by the image of a standard pair is concentrated on long curves. The precise statement, called Growth lemma is the following (see \[9\] Prop. 4.9, 4.10):

\textbf{Lemma 6.1.} For any standard pair $\ell = (W, \rho)$ and any $n \in \mathbb{Z}_+$,

$$\nu_\ell(A \circ F_0^n) = \sum_i c_{n,i} \nu_{\ell_{n,i}}(A),$$

(6.1)
where \( c_{n,i} > 0, \sum_i c_{n,i} = 1 \) and \( \ell_{n,i} = (W_{n,i}, \rho_{n,i}) \) are standard pairs so that \( \cup_i W_{n,i} = F^n_0(W) \) and \( \rho_{n,i} \) is a constant times the push-forward of \( \rho \) by \( F^n_0 \). Furthermore, there are universal constants \( \kappa, C \) so that for any \( n > \kappa \log \text{length}(\ell) \) and for any \( \varepsilon > 0 \)

\[
\sum_{i : \text{length}(\ell_{n,i}) < \varepsilon} c_{n,i} < C \varepsilon.
\]

We will refer to (6.1) as Markov decomposition. A simple consequence of the Growth lemma is the following lemma, which is proven in e.g. [10, Proposition 7.17]

**Lemma 6.2.** There are constants \( c_1, c_2 \) and \( \theta < 1 \) depending only on \( D_0 \) so that for any standard family \( G \) with finite \( Z \) function and for any \( n \),

\[
Z_{F^n_0(G)} \leq c_1 \theta^n Z_G + c_2.
\]

Let \( \kappa : M_0 \to \mathbb{R}^2 \) be the free flight vector and \( \check{\kappa} : M_0 \to \mathbb{R}^2 \) be the discrete free flight vector, that is \( \check{\kappa}(q,v) = \Pi_{Z^2}(F_0(q,v)) - \Pi_{Z^2}(q,v) \). Let us also write \( \bar{\kappa} = \int |\kappa| d\nu_0 \in \mathbb{R}_+ \).

Let

\[
\check{Z}_n(q,v) = \sum_{j=0}^{n-1} \check{\kappa}(F^j_0(q,v)).
\]

Similarly to the flow, we write \( \check{Z}_n = (\check{X}_n, \check{Y}_n) \). Put

\[
\tau_0^{\check{X}} = \min\{n > 0 : \check{X}_n < 0\}
\]

and for \( x \neq 0 \), put

\[
\tau_x^{\check{X}} = \min\{n > 0 : \check{X}_n = x\}
\]

(and likewise with \( \check{X} \) replaced by \( \check{Y} \)).

The next result is the extension of the central limit theorem to a functional variant in both discrete and continuous times (see e.g. [8]).

**Theorem 6.3** (Invariance principle). Fix a standard pair \( \ell \) and consider the stochastic processes \( \check{Z}_t, \check{Z}_n \) induced by the initial condition \( \ell \). Then

(a) \( \check{Z}_t/\sqrt{T}, t \in [0, 1] \) converges weakly as \( T \to \infty \) to a planar Brownian motion with zero mean and covariance matrix \( \Sigma \) (introduced in Section 3.2) uniformly for \( \ell \) satisfying \( |\log \text{length}(\ell)| > T^{1/4} \).

(b) With the notation \( \check{\Sigma} = \bar{\kappa} \Sigma \) we have \( \check{Z}_{[tN]}/\sqrt{N}, t \in [0, 1] \) converges weakly as \( N \to \infty \) to a planar Brownian motion with zero mean and covariance matrix \( \check{\Sigma} \) uniformly for \( \ell \) satisfying \( |\log \text{length}(\ell)| > N^{1/4} \).

Another extension of the central limit theorem is the so-called mixing local limit theorem, which we discuss next.
6.2. Mixing local limit theorem. Recall (6.2). Let us also define
\[ F_n(q, v) = \sum_{j=0}^{n-1} |\kappa(F_0^j(q, v))|. \]
Given \( x \in \mathbb{R}^2 \), \( y \in \mathbb{R} \) and a standard pair \( \ell \) let us denote by \( \vartheta_n \) the push-forward of \( \nu_\ell \) by the map
\[ (q, v) \mapsto (\hat{Z}_n(q, v) - \langle x \sqrt{n} \rangle, F_n(q, v) - n\bar{\kappa} - y\sqrt{n}, F_0^n(q, v)). \]
That is, \( \vartheta_n = \vartheta_n(\ell, x, y) \) is a measure on \( \mathbb{Z}^2 \times \mathbb{R} \times \mathcal{M}_0 \). Fix an open set \( A \subset \Omega_0 \) as in (3.2) and define \( A \subset \mathbb{Z}^2 \times \mathbb{R} \times \mathcal{M}_0 \) so that \( ((k, l), -t, (q, v)) \in A \) if and only if \( \Pi_{\mathbb{Z}^2}(q, v) = (k, l), \Pi_{\mathbb{Z}^2}(\Phi^t(q, v)) = 0, \Phi^t(q, v) \in A \) and \( |\kappa(q, v)| > t \). That is, \( A \) contains phase points \( (q + (k, l), v) \) and corresponding flight times \( t \) so that a flight of length \( t \) from \( (q + (k, l), v) \) is free and arrives in the set \( A \). By the finite horizon assumption, \( A \) is bounded.

Let \( g_\Sigma \) denote the Gaussian density with zero mean and covariance matrix \( \Sigma \).

The version of the mixing local limit theorem (MLLT) that we consider here is the following

**Theorem 6.4.** There is a positive definite \( 3 \times 3 \) matrix \( \widehat{\Sigma} \) whose top left \( 2 \times 2 \) minor is \( \widehat{\Sigma} \) and constants \( C, C_1, C_2 \) so that for any standard pair \( \ell \) with \( |\log \text{length}(\ell)| < n^{1/4} \) the following hold

(a) for any \( (x, y) \in \mathbb{R}^3 \) and for any \( A \) as in (3.2),
\[ \lim_{n \to \infty} n^{3/2} \vartheta_n(A) = \mu_0(A)\bar{\kappa}g_\Sigma(x, y) \]
uniformly for \( x, y \) in compact subsets of \( \mathbb{R}^3 \).

(b) for any \( (x, y) \in \mathbb{R}^3 \) and for any \( A \) as in (3.2) and for any positive integer \( n \),
\[ n^{3/2} \vartheta_n(A) < C_1g_{C\Sigma}(x, y) + C_2^{-1/2}. \]

A variant of Theorem 6.4 was proved in [12, Lemma 2.8]. Specifically, [12, Lemma 2.8] covers the case when \( \hat{Z} \) is replaced by \( \hat{X} \) and \( A = \Omega_0 \) in the definition of \( \vartheta_n \) (we included the more general case of \( A \) to accommodate for (DLE) as in Theorem 3.3). Since the proof directly applies here as well (except for one minor adjustment), we only discuss this minor adjustment and don’t repeat the entire proof.

**Proof.** First, we need some definitions. For a bounded Hölder function \( f : \mathcal{M}_0 \to \mathbb{R}^d \), we define \( S(f) \) as the smallest closed additive subgroup of \( \mathbb{R}^d \) that supports the values of \( f - r \) for some \( r \in \mathbb{R}^d \). Let us write \( f \sim g \) if \( f \) and \( g \) are cohomologous. That is, \( f(x) = g(x) + h(x) - h(F_0(x)) \) for a measurable \( h \) and for all \( x \in \mathcal{M}_0 \). We say that \( f \) is minimal if \( M(f) = S(f) \), where
\[ M(f) = \cap_{g \sim f} S(g). \]
The only minor adjustment that is needed in the proof of [12, Lemma 2.8] is that we need to show that
\[ f := (\tilde{\kappa}, |\kappa| - \bar{\kappa}) : \mathcal{M}_0 \to \mathbb{R}^3 \]
is minimal. That is, \( M(f) = \mathbb{Z}^2 \times \mathbb{R} \). (Heuristically, there is a clear obstruction to the MLLT in its present form if \( M(f) \) is a proper subgroup of \( \mathbb{Z}^2 \times \mathbb{R} \). It turns out that, similarly to the case of IID random variables, this is the only possible obstruction.) This generalizes [12, Lemma A.3], which shows that \( \tilde{\tilde{f}} := (\tilde{\kappa}_1, |\kappa| - \bar{\kappa}) \)
is minimal, that is
\[
(6.3) \quad M(\tilde{\tilde{f}}) = \mathbb{Z} \times \mathbb{R}.
\]
To establish the minimality of \( f \), it is enough to prove the following. If \( M(f) \) is a proper subgroup of \( \mathbb{Z}^2 \times \mathbb{R} \), then there are real numbers \( \alpha, r \) and two measurable functions \( h : \mathcal{M}_0 \to \mathbb{R}, g : \mathcal{M}_0 \to \mathbb{Z} \) so that
\[
(6.4) \quad |\kappa(q,v)| = h(q,v) - h(F_0(q,v)) + r + \alpha g(q,v).
\]
Indeed, a contradiction follows from (6.4) as in [12]. To prove (6.4), we first recall that by [23, Theorem 5.1], \( \tilde{\kappa} \) is minimal. Thus the projection of \( M(f) \) to the first two coordinates needs to be \( \mathbb{Z}^2 \). In particular, there exist \( e_1 = (0, 0, \alpha)^T, e_2 = (1, 0, \beta)^T, e_3 = (0, 1, \gamma)^T \) in \( M(f) \). If \( M(f) \) is a proper subgroup of \( \mathbb{Z}^2 \times \mathbb{R} \), then there exists a minimal \( \alpha > 0 \) with the property that \( e_1 \in M(f) \). Now we claim that \( e_1, e_2, e_3 \) generate \( M(f) \). Indeed, by the choice of \( \alpha \), \( e_1 \) generates \( M(f) \cap \{(0, 0, z), z \in \mathbb{R}\} \) and so \( e_1, e_2, e_3 \) generate
\[
M(f) \cap \{(x, y, z) : (x, y) \in \{(1, 0), (0, 1)\}\}.
\]
Since the projection of \( M(f) \) to the first two coordinates in \( \mathbb{Z}^2 \), the claim follows.

Thus there are constants \( r_1, r_2, r_3 \) so that for every \((q, v) \in \mathcal{M}_0 \) there are integers \( m, n, k \) (depending on \((q, v)\)) so that
\[
(6.5) \quad (\tilde{\kappa}_1(q,v), \tilde{\kappa}_2(q,v), |\kappa(q,v)|)^T = (r_1, r_2, r_3)^T - me_1 + ne_2 + ke_3 + (h_1, h_2, h_3)^T(q,v) - (h_1, h_2, h_3)^T(F_0(q,v))
\]
From the first coordinate of (6.5) we have
\[
n = \tilde{\kappa}_1 - r_1 - h_1 \circ F_0
\]
and likewise from the second coordinate we have
\[
k = \tilde{\kappa}_2 - r_2 - h_2 \circ F_0
\]
Substituting these to the third coordinate of the equation (6.5), we find
\[
(6.6) \quad |\kappa(q,v)| - \tilde{\kappa}_1 \tilde{\kappa}_2(q,v) - \gamma \tilde{\kappa}_2(q,v) = m\alpha + \tilde{h}(q,v) - h(F_0(q,v)),
\]
where \( \tilde{r} = r_3 - r_1 \beta - r_2 \gamma \) and \( \tilde{h} = h_3 - \beta h_1 - \gamma h_2 \). Fix now \((q, v)\) and write \( F_0(q, v) = (q_1, v_1) \). Note that by reverting the free flight, we have \( F_0(q_1, -v_1) = (q, -v) \). Applying (6.6) to \((q_1, -v_1)\), we obtain

\[
(6.7) \quad |\kappa(q, v)| - \tilde{r} + \beta \tilde{K}_1(q, v) + \gamma \tilde{K}_2(q, v) = m' \alpha + \tilde{h}(q_1, -v_1) - \tilde{h}(F_0(q_1, -v_1)).
\]

Finally, adding (6.6) to (6.7), we obtain (6.4) with \( r = \tilde{r}, h(q, v) = \frac{1}{2} \tilde{h}(q, v) + \tilde{h}(q_1 - v_1) \) and \( g(q, v) = \frac{m + m'}{2} \). This completes the proof of (6.4). \( \square \)

7. Proof of Theorem 3.2

7.1. Change of coordinates. Since \( \mathcal{L} \) is rational, we have \( \mathcal{M} := \Sigma^{1/2}(K_1) \in \mathbb{Z}^2 \) and \( \mathcal{N} := \Sigma^{1/2}(K_2) \in \mathbb{Z}^2 \). Furthermore, \( \mathcal{M} \) and \( \mathcal{N} \) are primitive lattice vectors (i.e. their coordinates are coprime due to the definition of \((K_1), (K_2)\)). Now we introduce an enlarged fundamental domain for the Lorentz gas. Let \( \mathbb{Z}' \) be the subset of \( \mathbb{Z}^2 \) containing the origin and those points of \( \mathbb{Z}^2 \) that are in the interior of the parallelogram with vertices \( 0, \mathcal{M}, \mathcal{N}, \mathcal{M} + \mathcal{N} \). Let \( T' = \bigcup_{z \in \mathbb{Z}'} [z - 1/2, z + 1/2]^2 / \sim \), where \( P \sim Q \) if \( P - Q \) is in the lattice generated by \( \mathcal{M}, \mathcal{N} \). That is, \( T' \) is a union of unit squares and \( \sim \) is a pairing of all parallel sides on the boundary of \( T' \). In particular, \( T' \) is a flat torus. Now we put \( \mathcal{D}'_0 = T' \setminus \bigcup_{z \in \mathbb{Z}'} \bigcup_{i=1}^k (B_i + z) \). See Figure 3 for the special case \( \mathcal{M} = (1, 3) \) and \( \mathcal{N} = (2, 1) \). \( T' \) is the polygon with bold boundary (modulo the identification).

We are going to study the Sinai billiard in \( \mathcal{D}'_0 \) and so we define \( \Phi''_0, \Omega'_0, \mu'_0, \mathcal{M}', \mathcal{F}', \nu'_0 \) exactly as before using the larger configuration space \( \mathcal{D}'_0 \). Note that \( \Phi''_0 \) is a factor of \( \Phi'_0 \) by the map \( \iota : \Omega'_0 \to \Omega_0, \iota : (q, v) \mapsto (\bar{q}, v) \), where \( q \in \mathcal{D}'_0, \bar{q} \in \mathcal{D}_0 \) and \( \bar{q} = q \pmod{\mathbb{Z}^2} \). Also note that \( \Phi'' \) is an extension of both \( \Phi'_0 \) and \( \Phi''_0 \).
Given \((q, v) \in \Omega\), we write \(\Pi'_\mathbb{Z}_t(q, v) = (m, n)\) if \(q \in (m \mathbb{M}, n \mathbb{M}) + T'\) and \(\Pi'_{\mathbb{Z}_t}(q, v) = q_0\) if \(q = q_0 + \Pi'_{\mathbb{Z}_t}(q, v) * (\mathbb{M}, \mathbb{M})\), where * means multiplication coordinate-wise. Let us write \(\mathcal{Z}_t'(q, v) = \Pi'_\mathbb{Z}_t(\Phi^t(q, v))\).

Note that for any \((k, l) \in \mathbb{Z}^2\) we can find a unique \((k_0, l_0) \in \mathbb{Z}'\) with \((k, l) \sim (k_0, l_0)\) and a unique \((m, n)\) so that \((k, l) = (m \mathbb{M}, n \mathbb{M}) + (k_0, l_0)\). Let us write \([(k, l)] = (k_0, l_0)\) and \([(\mathbb{Z}_t)] = (m, n)\). Note that

\[
[[\tilde{z}_t(q, v)]] = \Sigma^{-1/2}(\tilde{z}_t(q, v) - [\tilde{z}_t(q, v)]) = \mathcal{Z}_t'(q, v).
\]

Given \((q, v) \in \Omega\), we write \(\Pi'_{\mathbb{Z}_t}(q, v) = \Pi'_\mathbb{Z}_t(\Phi^t(q, v)) = [\tilde{z}_t(q, v)] \mathcal{E}\) stands for extension). We will also write \([[\ell']] = [[\Pi_{\mathbb{Z}_t}(q, v)]]\) for any \((q, v)\) in the support of \(\nu'_\ell\) (we assume that the standard pairs are supported in one cell) and likewise \([[\mathcal{G}']]\) for standard families. All definitions and results in Section 6 extend to \(\Phi'_0\). We will use those notations and results with a prime in the superscript.

### 7.2. Proof of (H2).

We claim that (H2) follows from

(H2') For any proper standard family \(\mathcal{G}'\) there is some \(C_{\mathcal{G}'}\) so that for any \(0 < \alpha < \beta\) and for any \(0 < \eta, \gamma < \xi\) and for any \(z' \in \mathbb{Z}'\), if \([[\mathcal{G}']] = (0, [\eta \sqrt{T}])\), then

\[
\lim_{T \to \infty} T^{3/2} \nu'_{\mathcal{G}'} \left( \mathcal{Z}'_T = \langle \alpha, \gamma \rangle \sqrt{T}, \mathcal{E}_T = z', \min\{\tau_{0}^{\omega}, \tau_{\xi \sqrt{T}}^{\omega}, \tau_{0}^{\omega}, \tau_{\beta \sqrt{T}}^{\omega}\} > T \right) = C_{\mathcal{G}'}(\alpha, \beta) \phi(\eta, \gamma, \xi)
\]

Furthermore, for any \(\varepsilon > 0\), the convergence is uniform for \(0 < \alpha < \alpha + \varepsilon < \beta < 1/\varepsilon, \varepsilon < \eta < \gamma + \varepsilon < \xi < 1/\varepsilon, \varepsilon < \gamma < \gamma + \varepsilon < \xi\).

To prove the claim, first recall that by (7.1), \(\mathcal{Z}_t = \mathcal{Z}_t + \Sigma^{-1/2} \mathcal{E}_t\). To compare the initial conditions in (H2) and (H2'), note that given any standard family \(\mathcal{G}\) on \(\mathcal{M}_0\), there are exactly \(Z := |Z'|\) corresponding standard families \(\mathcal{G}'_1, \ldots, \mathcal{G}'_Z\) on \(\mathcal{M}'_0\) that project to \(\mathcal{G}\) along \(t\). Indeed, for any point \((q, v) \in \Omega_0, t^{-1}((q, v)) = \{(q + z', v), z' \in Z'\}\). Recall that the free flight is bounded by 1 and so the initial condition in (H2), i.e. \(Z_0 = 1\) and \(\mathbb{P}\) being induced by a standard family \(\mathcal{G}\) corresponds to an initial condition given by \(\mathcal{G}'_1\), for some \(z' = 1, \ldots, Z\) in (H2'). Indeed, the type of \(I\) uniquely defines \(z'\). Thus \(\mathcal{G}\) and the type of \(I\) in (H2') is replaced by \(\mathcal{G}'\) in (H2). Since \(\mathcal{E}_t\) is bounded, the claim follows.

Note that for a given standard family \(\mathcal{G}\) and two lift ups \(\mathcal{G}'_{z_1}, \mathcal{G}'_{z_2}, z'_1 \neq z'_2 \in Z'\), the constants \(C_{\mathcal{G}'_{z_1}}, C_{\mathcal{G}'_{z_2}}\) can be different. As we will see later,

\[
C_{\mathcal{G}'_{z_2}} = \lim_{T \to \infty} \nu_{\mathcal{G}'_{z_2}}(\tau_{0}^{\omega} > T) / \sqrt{T}.
\]

Thus e.g. in Figure 3 \(C_{\mathcal{G}'_{(1,1)}} \geq C_{\mathcal{G}'_{(1,2)}}\) for all standard families \(\mathcal{G}\). This inequality is strict in case of some standard pairs. To prove this, note that in case of Figure 3 \(\tau_{0}^{\omega}(q, v) > T\) is equivalent to \((\tilde{z}_t)_2 \leq \sqrt{3(\tilde{z}_t)}_1\) for all \(t \leq T\). Now observe that
\( \tau_0^V(q_0 + (1, 2), v) > T \) implies \( \tau_0^V(q_0 + (1, 1), v) > T \), but the converse implication does not hold.

We will prove \((H2')\). The proof is built upon the results of \([13, 12]\). In particular, \([12, \text{Proposition 3.8}]\) gives that under the assumptions of \((H2')\),

\[
\lim_{T \to \infty} T \nu_{T^*} \left( \mathcal{X}_T' = [\alpha \sqrt{T}], \mathcal{E}_T = z', \min\{\tau_{0, \mathcal{X}_T'}, \tau_{\beta \sqrt{T}}\} > T \right) = C_{\mathcal{G}'} \psi(\alpha, \beta)
\]

with \(C_{\mathcal{G}'}\) defined by \((7.2)\). Furthermore, \([12, \text{Proposition 3.9}]\) gives that under the assumptions of \((H2')\),

\[
\lim_{T \to \infty} \sqrt{T} \nu_{T^*} \left( \mathcal{Y}_T' = [\gamma \sqrt{T}], \mathcal{E}_T = z', \min\{\tau_{0, \mathcal{Y}_T'}, \tau_{\xi \sqrt{T}}\} > T \right) = \phi(\eta, \gamma, \xi)
\]

We interpret \((7.3)\) as the one dimensional version of \((H2')\). If the events on the left hand sides of \((7.3)\) and \((7.4)\) were independent, then \((H2')\) would follow immediately. By the invariance principle, \(\mathcal{X}_T'\) and \(\mathcal{Y}_T'\) are asymptotically independent (since by the change of coordinates, the covariance matrix is identity) but this yet is not enough to conclude \((H2')\) as the events considered here have small probabilities. Thus we cannot derive \((H2')\) directly from \((7.3)\) and \((7.4)\); we instead have to revisit their proofs. Since we only need to make minor changes to their proofs, we give details only at places where changes are needed and otherwise refer to \([12]\) (and sometimes give a sketch).

First we need some lemmas. Recall the notations introduced for the billiard ball map in Section 6. To simplify some notations a little, we will write

\[
\tau_{a}^{\mathcal{X}} = \min\{\tau_{a}^{\mathcal{X}'}, \tau_{-a}^{\mathcal{X}'}\},
\]

and likewise for \(\tilde{\mathcal{X}}'\) replaced by \(\tilde{\mathcal{Y}}'\).

**Lemma 7.1.** There are constants \(C_3, C_4\) depending only on \(D\) so that for every standard pair \(\ell'\) with \([[[\ell']]] = (0, 0)\), for every \(m > C_3 \log \text{length}(\ell)\) and for every \(L\),

\[
\nu_{T^*} \left( \tau_{Lm}^{\mathcal{Y}} < \tau_{m}^{\mathcal{X}} \right) < 0.51^L + \frac{C_4 L}{m^{500}}.
\]

**Proof.** Let us fix a positive constant \(\eta\) so that the probability that a standard planar Brownian motion \(W_t\) leaves the the box \([-1, 1]^2\) through the North or South side (and not through the East or West side) is at most 0.505 whenever the \(y\)-coordinate of \(W_0\), denoted by \((W_0)_2\), satisfies \(|(W_0)_2| < \eta\). We are going to use the invariance principle and the above estimate inductively \(L\) times to derive the lemma. Each time the North or South side is reached, we apply a Markov decomposition and discard too short curves (hence the second term on the right hand side of \((7.5)\)). Key to this argument is the fact that the limiting Brownian motion has a diagonal covariance matrix, which is guaranteed by the change of coordinates from Section 7.1. Now we give the details of the proof.
Choosing $C_3$ large and using Lemma 6.2, we can guarantee that the standard family $\mathcal{G} := F_0^m(\ell')$ has a bounded $Z$ function (e.g. $Z_\mathcal{G} < 2c_2$, where $c_2$ is defined in Lemma 6.2). Such standard families are sometimes called proper). Recall that we assumed that the free flight is bounded by $1$. Thus for any standard pair $\ell'' = (W'', \rho'')$ in $\mathcal{G}$, $||[[\ell'']]|| \leq \eta m$. If $\text{length}(\ell'') < m^{-500}$, then we estimate $\nu_{\ell''}(\mathcal{C}) \leq 1$, where $\mathcal{C} = \{ \tau_{Lm}^{[\mathcal{Y}]} < \tau_{m}^{[\mathcal{X}']} \}$. By the Growth lemma, the measure carried by such standard pairs in $\mathcal{G}$ is bounded by $C_4m^{-500}$. Let us now assume that $\text{length}(\ell'') > m^{-500}$. Then by the choice of $\eta$ and by invariance principle (assuming as we can that $m$ is large enough),

$$\nu_{\ell''}(\tau_m^{[\mathcal{Y}]} < \tau_m^{[\mathcal{X}']}) \leq 0.51.$$ 

Now let $\ell'' = (W'', \rho'')$ be a standard pair in the standard family $\mathcal{G}_1 := F_0^{m}(\ell'')$.

Note that there exists a constant $T_{\ell''}$ so that for any $x \in W''$ with $F_0^{m}(\ell'') \in W''$, $\tau_m^{[\mathcal{Y}]} = T_{\ell''}$. Indeed, this follows from the definition of homogeneous unstable curves. Now we distinguish two cases. Let us say that $\ell''$ is of type 1 if $T_{\ell''} > m$ or length($\ell''$) < $m^{-750}$. For type 1 standard pairs $\ell''$, we use the trivial bound $\nu_{\ell''}(\mathcal{C}) \leq 1$. By [12, Lemma 5.1], the measure carried by standard pairs $\ell''$ with $T_{\ell''} > m^3$ is bounded by $Cm^{-999}$. Thus by the growth lemma, the measure carried by standard pairs $\ell''$ with $T_{\ell''} \leq m^3$ and length($\ell''$) < $m^{-750}$ is bounded by $Cm^{-947}$. Thus the total contribution of type 1 standard pairs is bounded by $C_4m^{-500}$. Let us say that $\ell''$ is of type 2 if not of type 1. By the invariance principle and by the definition of $\eta$, for every type 2 standard pair $\ell''$, we have

$$\nu_{\ell''}(\tau_m^{[\mathcal{Y}]} < \tau_m^{[\mathcal{X}']}) \leq 0.51.$$ 

Thus we have derived

$$\nu_{\ell''}(\tau_m^{[\mathcal{Y}]} < \tau_m^{[\mathcal{X}']}) \leq 0.51^2 + \frac{2C_4}{m^{500}}.$$ 

Following the above procedure inductively, we obtain the lemma.

\[\square\]

**Lemma 7.2.** For every $\delta > 0$ and for every $\xi > 0$ there exists $M_0$ and $\bar{L}$ so that for every standard pair $\ell$ with $||[\ell''']|| = (0,0)$ and length($\ell''$) > $\delta$, and for every $M > M_0$,

$$\nu_{\ell} \left( \tau_{Lm}^{[\mathcal{X}']} < \tau_{M}^{[\mathcal{X}']} \big| \tau_{M}^{[\mathcal{X}']} < \tau_{0}^{[\mathcal{X}']} \right) < \xi$$

**Proof.** [13, Lemma 11.1(a)] says that

\[ (7.6) \quad \bar{c} = \bar{c}(\ell') = \lim_{M \to \infty} M \nu_{\ell'}(\tau_M^{[\mathcal{X}']} < \tau_{0}^{[\mathcal{X}']} \big| \tau_{Lm}^{[\mathcal{X}']} < \tau_{M}^{[\mathcal{X}']} \) \]

is finite. We will use the proof of that lemma to prove our lemma. Let us recall the main steps of the proof.
Let $t_k = \tau_2^{\nu'}$ and

$$s_k = \min\{n > t_k : \tilde{X}'_n < 0 \text{ or } \tilde{X}'_n = 2^{k+1}\}.$$  

Let now $\ell''$ be a standard pair with

$$[\ell''] = 2^k \text{ and length}(\ell'') > 2^{-100k}$$

(we will consider $\ell''$ in the image of $\ell'$ under the map $(\mathcal{F}')^{t_k}$). The proof of [13 Lemma 11.1(a)] is based on the following identity (see [13, Lemma 11.2]):

$$\nu(\ell'') \left( t_{k+1} < \tau_0^{\nu'} \text{ and } r_{t_{k+1}}' > 2^{-100(k+1)} \right) = \frac{1}{2} + O(2^{-k\zeta})$$

with a universal positive constant $\zeta$. Fixing an arbitrary $\varepsilon > 0$, one can choose $k_0$ large enough so that an induction on $k = k_0, ..., \log_2 M$ using (7.8) gives that

$$|M \nu''(s_k = t_{k+1}, r'_{t_k} > 2^{-100(k+1)})| \leq \varepsilon,$$

which implies (7.6) (by the Growth lemma, the measure of the points where $r_{s_k} < 2^{-100(k+1)}$ for some $k < \log_2 M$ can be neglected). We refer the reader to [13] for more details.

Now we turn to the proof of our lemma. Let us put $m_k = 2^k$, $\tilde{k} = (\log_2 M) - k$ and

$$L_k = \begin{cases} 2^k & \text{if } k_0 \leq k < \frac{1}{2} \log_2 M \\ K1.5^k & \text{if } \frac{1}{2} \log_2 M \leq k < \log_2 M \end{cases}$$

with some $K = K(\xi)$ to be specified later. Assuming that $k_0$ is bigger than a universal constant (as we can), we have $m_k > 100C_1 \log(1/m_k)$. Thus Lemma [13] imply that for all standard pairs satisfying (7.7):

$$\nu(\ell'') \left( \min\{\tau_{[\ell'']}^{\nu'} - L_k m_k, \tau_{[\ell'']}^{\nu'} + L_k m_k \} < \min\{\tau_0^{\nu'}, \tau_2 m_k\} \right) < 0.51 L_k + \frac{C_1 L_k}{m_{k100}},$$

which combined with (7.8) gives

$$\nu''(t_{k+1} < \min\{\tau_0^{\nu'}, \tau_{[\ell'']}^{\nu'} - L_k 2^k, \tau_{[\ell'']}^{\nu'} + L_k 2^k \} \text{ and } r'_{t_{k+1}} > 2^{-100(k+1)}) = \frac{1}{2} + E_{k, \ell''},$$

where

$$- C'2^{-k\zeta} - 0.51 L_k - \frac{C_1 L_k}{m_{k100}} < E_{k, \ell''} \leq C''2^{-k\zeta},$$

with a universal constant $C'$. Now we revisit the inductive proof of (7.9). Let us write

$$\mathcal{P} = \nu''(s_k = t_{k+1}, r'_{s_k} > 2^{-100(k+1)}, \frac{\tau_{[\ell'']}^{\nu'}}{M+\sum_{j=k_0} L_j 2^j} > s_k \text{ for } k = k_0, ..., \log_2 M - 1).$$
Using (7.10) inductively, we find
\[ P = \nu_{\ell'}(\tau_{2k_0} < \min\{\tau_0^{\tilde{c}'}, \tau_M^{\tilde{c}'}\}) \prod_{k=k_0}^{\log_2 M-1} \frac{1}{2}(1 + E_k), \]
where \( E_k \) satisfies the same inequalities (7.11) as \( E_{k',\tilde{l}'} \). As before, choosing \( k_0 \) and \( M \) large, we can guarantee
\[ (7.13) \quad P > \frac{\tilde{c} - \xi'/10}{M} \prod_{k=k_0}^{\log_2 M-1} (1 + E_k), \]
where \( \xi' = \xi \tilde{c}/2 \). Let us write
\[ (7.14) \quad \prod_{k=k_0}^{\log_2 M-1} (1 + |E_k|) = \exp \left( \sum_{k=k_0}^{\log_2 M-1} \log(1 + |E_k|) \right) \leq \exp \left( \sum_{k=k_0}^{\log_2 M-1} |E_k| \right). \]
Later we will show that
\[ (7.15) \quad \sum_{k=k_0}^{\log_2 M-1} \left( C' 2^{-k\xi'} + 0.51L_k \frac{C_4L_k}{m_{500}} \right) < \frac{\xi'}{10\tilde{c}} = \frac{\xi}{20}. \]

Before proving (7.15), let us show how it implies the lemma. Combining (7.13), (7.14) and (7.15), we find
\[ (7.16) \quad P > \frac{\tilde{c} - \xi'}{M}. \]
Next observe that the event in (7.12) implies that
\[ \tau_L^{\tilde{c}'} > \tau_{\tilde{c}'}^M, \]
where \( \tilde{L} = 1 + \frac{1}{M} \sum_{k=k_0}^{\log_2 M} L_k 2^k \). The next computation shows that \( \tilde{L} \) is bounded by a constant \( \tilde{L} = \tilde{L}(\xi) \) uniformly in \( M \):
\[
1 + \frac{1}{M} \sum_{k=k_0}^{\log_2 M} L_k 2^k = 1 + \frac{1}{M} \sum_{k=k_0}^{\log_2 M} 4^k + \frac{K}{M} \sum_{k=\frac{1}{2}}^{\log_2 M} 1.5^k 2^k \leq 5 + \frac{K}{M} \sum_{k=0}^{\log_2 M-1} 1.5^k 2^{\log_2 M-k} \leq 5 + K \sum_{k=0}^{\infty} \left( \frac{3}{4} \right)^{\tilde{k}} = 5 + 4K =: \tilde{L}.
\]
Thus we find
\[
\nu_{\ell'} \left( \tau_L^{\tilde{c}'} < \tau_M^{\tilde{c}'} \right) < 1 - \frac{P}{\nu_{\ell'}(\tau_M^{\tilde{c}'})} \leq 1 - \frac{\tilde{c} - \xi'}{\tilde{c} + \xi'} \leq \xi,
\]
where the penultimate inequality uses \((7.16)\) and the last one uses the definition of \(\xi'\). This proves the lemma. It remains to verify \((7.15)\).

To prove \((7.15)\), first choose \(K = K(\xi)\) large, so that
\[
\sum_{k=\frac{1}{2} \log_2 M}^{\log_2 M} 0.51^L_k < \sum_{k=0}^{\infty} 0.51^2^k 1.5^k < \frac{\xi}{100}.
\]
Then we compute
\[
\sum_{k=k_0}^{\log_2 M} C' 2^{-k} \xi < \frac{\xi}{100},
\]
\[
\sum_{k=k_0}^{\frac{1}{2} \log_2 M} 0.51^L_k < \sum_{k=k_0}^{\infty} 0.51^2^k < \frac{\xi}{100}
\]
and
\[
\sum_{k=k_0}^{\frac{1}{2} \log_2 M} \frac{L_k}{m_{500}^k} < \sum_{k=k_0}^{\infty} 2^{-499} k^2 \xi < \frac{\xi}{100}.
\]
(Note that we can ensure the last inequality in all of the three displayed formulas above by increasing \(k_0 = k_0(\xi)\) if necessary.) Finally, we have
\[
\sum_{k=\frac{1}{4} \log_2 M}^{\log_2 M} \frac{L_k}{m_{500}^k} < \log_2 M \frac{1.5^{\log_2 M}}{2250} \log_2 M = o(M^{-249}) < \frac{\xi}{100},
\]
which completes the proof of \((7.15)\).

**Lemma 7.3.** For every \(\eta_1, \eta_2 > 0\) there exists \(\varepsilon_0\) so that for every \(\varepsilon < \varepsilon_0\) and for every \(\delta > 0\) there is some \(N_0\) so that for all \(N > N_0\) and for all standard pair \(\ell'\), with \([[\ell']] = (0, 0)\), length(\(\ell\)) > \(\delta\), we have
\[
\nu_{\ell'} \left( \tau_{\varepsilon \sqrt{N}}^{\hat{X}_\ell'} < \min \{ \tau_{\eta_1 \sqrt{N}}, \tau_{\eta_1 \sqrt{N}}, \varepsilon N \} \bigg| \tau_0^{\hat{X}_0} > N \right) > 1 - \eta_2.
\]

**Proof.** [12] Lemma 5.2] implies that
\[
\nu_{\ell'} \left( \tau_{\varepsilon \sqrt{N}}^{\hat{X}_\ell'} < \varepsilon N, \bigg| \tau_0^{\hat{X}_0} > N \right) > 1 - \frac{\eta_2}{2},
\]
and [13] Theorem 8] implies that
\[
\lim_{T \to \infty} \nu_{\ell'} (\tau_0^{\hat{X}_0} > N) / \sqrt{N} =: \tilde{C}_{\ell'}
\]
is finite for all standard pairs and non-zero for some. Thus it suffices to prove
\[
\nu_{\ell'} (ABC) < \frac{\eta_2 \tilde{C}_{\ell'}}{4 \sqrt{N}},
\]
where
\[ A = \{ \tau_{\varepsilon \sqrt{N}} > \min\{ \tau_{\eta_1 \sqrt{N}}, \tau_{-\eta_1 \sqrt{N}} \} \}, \quad B = \{ \tau_{\varepsilon \sqrt{N}} < \varepsilon N \}, \quad C = \{ \tau_{0} > N \}. \]

To prove (7.18), let us write
\[ D = \{ \tau_{\varepsilon \sqrt{N}} < \tau_{\varepsilon \sqrt{N}} \} \]
and
\[ \nu_{\ell}(ABC) = \nu_{\ell}(ABCD) \leq \nu_{\ell}(AD)\nu_{\ell}(C|ABD) =: I \ast II. \]

To estimate II, we use Markov decomposition at time \( \tau_{\varepsilon \sqrt{N}} \). By the invariance principle, II is asymptotic (as \( N \to \infty \)) to the probability that the maximum of the standard Brownian motion before time 1 is less than \( \hat{c} \varepsilon \) which is bounded from above by \( \bar{c} \varepsilon \). Let \( \bar{c} = \bar{c}(\ell) \) as in (7.6) and let \( \xi = \frac{\eta_2 \varepsilon}{\sqrt{N}} \). Lemma 7.2 gives \( \bar{L} = \bar{L}(\xi) \). Then we choose \( \varepsilon_0 < \eta_1 / \bar{L} \). Now Lemma 7.2 implies that
\[ I = \nu_{\ell}(A|D)\nu_{\ell}(D) \leq \xi \frac{\bar{c}}{\varepsilon \sqrt{N}} \]
and so (7.18) follows.

Next, we have the following extension of [12, Theorem 3.5] to two dimensions.

**Proposition 7.4.** The process \( \hat{Z}_{0N} \) induced by the measure \( \nu_{G'}(\tau_{0} > N) \) converges weakly as \( N \to \infty \) to the planar stochastic process with independent coordinates, whose first coordinate is a Brownian meander and the second coordinate is a standard Brownian motion.

The proof of Proposition 7.4 is the same as that of [12, Theorem 3.5] except that [12, Lemma 5.2] is replaced by our Lemma 7.3. The sketch of the proof is as follows. Under the assumption \( \tau_{0} > N \), with high probability, we have \( \tau_{\varepsilon \sqrt{N}} < \min\{ \tau_{\eta_1 \sqrt{N}}, \tau_{-\eta_1 \sqrt{N}} \}, \varepsilon N \}. \) Then we use the invariance principle starting at time \( \tau_{\varepsilon \sqrt{N}} \).

The invariance principle is applicable since \( \nu_{\ell}(\tau_{0} > N) \) is bounded from below for \( \ell'' \) with \( \ell'' > \delta_0 \) and \( \| \ell'' \| = \varepsilon \sqrt{N} \) for fixed \( \varepsilon \). Thus we obtain a planar Brownian motion with identity covariance matrix, whose first coordinate starts from \( \varepsilon \) and does not reach 0 before time 1 and whose second coordinate starts from a position with absolute value less than \( \eta_1 \). Choosing \( \eta_1 \) small (and consequently \( \varepsilon \) small), the distribution of this process is close to the one described in the lemma.

(H2') is a local version of Proposition 7.4 in continuous time. The proof of (H2') is again analogous to the one dimensional case given in [12, Proposition 3.8]. Although the proof is quite lengthy, let us give a short sketch. Let \( N = T / \bar{K}, N_1 = (1 - \delta_t)N \) with a small \( \delta_t \) and partition the rectangle \( R_T := [0, \beta \sqrt{T}] \times [0, \xi \sqrt{T}] \) into boxes \( B_k \) with side length \( \delta_s \sqrt{T} \) with some fixed \( \delta_s \) small. Proposition 7.4 gives the asymptotic
probability (for $T$ large, other parameters fixed) of arriving in a box $B_k$ after discrete time $N_1$. Then for any given box $B_k$ and any given a standard pair $\ell'$ in this box as initial condition (with $\text{length}(\ell') > \delta_0$ for some fixed $\delta_0$), we need to find the probability that in the remaining continuous time before $T$ but after the first $N_1$ collisions, the particle arrives in the cell $\langle \alpha\sqrt{T}, \gamma\sqrt{T} \rangle$. To give an upper bound, we use the MLLT by simply ignoring the requirement that, in the remaining $\approx \delta_t T$ time, the particle has to stay inside $R_T$. Switching from discrete to continuous time is a non-trivial step. For ”typical” number of collisions, Theorem 6.4(a) is used, whereas the contribution of non-typical number of collisions is negligible by Theorem 6.4(b). This gives the upper bound in (H2'). To prove the lower bound, one needs to verify that the error made by ignoring the requirement that the particle has to stay inside $R_T$ for the last $\approx \delta_t T$ time is negligible. If a particle leaves $R_T$ and returns to $\langle \alpha\sqrt{T}, \gamma\sqrt{T} \rangle$, then in particular it has to travel a distance $\min\{\alpha, 1-\alpha, \gamma, 1-\gamma\}\sqrt{T}$ during time $\delta_t T$. This has small probability which gives the lower bound in (H2') (in [12] $\delta_t$ is chosen small given $\alpha \in (0, 1)$, now we need to choose it small given $\alpha, \gamma \in (0, 1)$). No other substantial change is required.

7.3. Proof of (H3). As in case of (H2), we use the change of coordinates to reformulate (H3) as

(H3') For any $x \in (0, 1)$ and $y \in (-1, 1)$, and for any proper standard family $G'$ with $\{G'\} = (0, 0)$

$$\lim_{\delta \to 0} \lim_{L \to \infty} \int_{[0,\delta L^2] \cup [L^2/\delta, \infty]} L\nu_{G'}(Z'_t = \langle (xL, yL) \rangle, \min\{\tau^x_{0}, \tau^y_{0}\} > t)dt = 0.$$  

The fact that (H3') implies (H3) follows the same way as we proved that (H2') implies (H2). In fact, this case is easier as contrary to the case of (H2). We only need an upper bound here and so we can ignore the requirement $E_t = z'$ at the cost of losing a constant multiplier.

As in the upper bound of (H2'), we can derive that for given $(x, y) \in (0, 1)^2$ and $\varepsilon > 0$ there exists $\delta$ so that for large enough $L$ and for any $t < \delta L^2$,

$$\nu_{G'}(Z'_t = \langle (xL, yL) \rangle, |\tau^y_{xL/2} < \tau^x_{0}\rangle < \frac{\varepsilon}{L^2}.$$  

Using this estimate, the proof of (H3') follows as in [12] Lemma 7.2].

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References

[1] Bonetto, F., Chernov, N., Korepanov, A., Lebowitz, J. Spatial Structure of Stationary Nonequilibrium States in the Thermostatted Periodic Lorentz Gas, *Journal of Statistical Physics* **146**, 1221–1243, (2012).

[2] Bonetto, F.; Lebowitz, J. L.; Rey-Bellet, L. Fourier’s law: a challenge to theorists, Proceedings ICMP-2000, Imp. Coll. Press, London, pp. 128–150.

[3] Bunimovich, L. A., Sinai Ya. G., Statistical properties of Lorentz gas with periodic configuration of scatterers, *Comm. Math. Phys.* **78** 479–497, (1981).

[4] Bunimovich, L.A., Sinai, Y.G., Chernov, N.I., Markov partitions for two dimensional hyperbolic billiard, Uspekhi Mat. Nauk **45** 3, 97—134, (1990).

[5] Bunimovich L. A., Sinai Ya. G., Chernov N. I., Statistical properties of two-dimensional hyperbolic billiards, *Russian Math. Surveys*, **46** (1991) 47–106.

[6] Caravenna, F., A local limit theorem for random walks conditioned to stay positive, *Probability theory and related fields* **133** 4, 508–530 (2005).

[7] Caravenna, F., Chaumont, L., Invariance principles for random walks conditioned to stay positive *Annales de l’IHP Probabilités et statistiques* **44** 1, 170–190 (2008).

[8] Chernov, N., Advanced statistical properties of dispersing billiards. J. Stat. Phys. 122 (2006), no. 6, 1061–1094

[9] Chernov, N., Dolgopyat, D., Brownian Brownian motion. I. *Mem. Amer. Math. Soc.* **198** (2009), no. 927, viii+193 pp.

[10] Chernov, N., Markarian, R., Chaotic billiards. Mathematical Surveys and Monographs, 127. American Mathematical Society, Providence, R.I., 2006.

[11] Chung, K. L., Excursions in Brownian motion, *Aktiv fur Matematik* 155–177, (1976).

[12] Dolgopyat, D., Nándori, P., Non equilibrium density profiles in Lorentz tubes with thermostated boundaries *Communications on Pure and Applied Mathematics* **69** 4: 649–692 (2016).

[13] Dolgopyat D., Szász D., Varjú T. Recurrence properties of planar Lorentz process, *Duke Math. J.* **142** (2008) 241–281.

[14] Durrett, R. T., Iglehart, D. L., Miller, D. R.: Weak convergence to Brownian meander and Brownian excursion, *Annals of Probability* **5** 1 117–129 (1977).

[15] Freedman, D., Brownian motion and diffusion, Holden-Day, 1971.

[16] Kingman J. F. C. Poisson processes, *Oxford Studies in Probability* **3** (1993) viii+104 pp, Oxford University Press, New York.

[17] Kipnis, C., Landim, C., Scaling limits of interacting particle systems, Springer-Verlag Berlin Heidelberg, Grundlehren der mathematischen Wissenschaften 320 (1999).

[18] Kipnis, C., Marchioro, C., Presutti, E., Heat flow in an exactly solvable model, *Journal of Statistical Physics*, **27**, 65–74 (1982).

[19] Lawler, G., Limic, V., Random Walk: A modern introduction (Cambridge Studies in Advanced Mathematics). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511750854 (2010).

[20] Li, Y., Nándori, P., Young, L.-S., Local thermal equilibrium for certain stochastic models of heat transport, *Journal of Statistical Physics*, **163**, 1: 61–91, (2016).

[21] Sinai, Ya. G.: Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards, *Russian Math. Surve.* **25** 137–189 (1970).

[22] Spohn, H., Large Scale Dynamics of Interacting Particles, Springer-Verlag Berlin Heidelberg (1991).
[23] Szász, D. Varjú, T., Local limit theorem for the Lorentz process and its recurrence in the plane. 
Ergodic Theory Dynam. Systems \textbf{24}, 1, 257–278 (2004).

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