A Class of Reducible Cyclic Codes and Their Weight Distribution

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Abstract

In this paper, a family of reducible cyclic codes over $F_p$ whose duals have four zeros is presented, where $p$ is an odd prime. Furthermore, the weight distribution of these cyclic codes is determined.

**Key words and phrases:** cyclic code, quadratic form, weight distribution.

**MSC:** 94B15, 11T71.

1 INTRODUCTION

Throughout this paper, let $m \geq 5$ be an odd integer and $k$ be any positive integer such that $\gcd(m, k) = 1$. Let $p$ be an odd prime and $\pi$ be a primitive element of the finite field $F_p^n$.

Recall that an $[n, l, d]$ linear code $C$ over $F_p$ is a linear subspace of $F_p^n$ with dimension $l$ and minimum Hamming distance $d$. Let $A_i$ denote the number of codewords in $C$ with Hamming weight $i$. The sequence $(A_0, A_1, A_2, \ldots, A_n)$ is called the weight distribution of the code $C$. $C$ is called cyclic if for any $(c_0, c_1, \ldots, c_{n-1}) \in C$, then $(c_{n-1}, c_0, \ldots, c_{n-2}) \in C$. A linear code $C$ in $F_p^n$ is cyclic if and only if $C$ is an ideal of the polynomial residue class ring $F_p[x]/(x^n - 1)$. Since $F_p[x]/(x^n - 1)$ is a principal ideal ring, every cyclic code corresponds to a principal ideal $(g(x))$ of the multiples of a polynomial $g(x)$ which is the monic polynomial of lowest degree in the ideal. This polynomial $g(x)$ is called the generator polynomial, and $h(x) = (x^n - 1)/g(x)$ is called the parity-check polynomial of the code $C$. We also recall that a cyclic code is called irreducible if its parity-check polynomial is irreducible over $F_p$, and reducible, otherwise.

Determining the weight distribution of a linear code is an important research object in coding theory. For cyclic codes, the error-correcting capability may not be as good as with some other linear codes in general. However, because of their good algebraic structure, the weight distribution of some cyclic codes can be determined by algebraic

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techniques, exponential sums for example. Besides, cyclic codes have wide applications in storage and communication systems because they have efficient encoding and decoding algorithms. Therefore, the weight distributions of cyclic codes is not only a problem of theoretical interest, but also of practical importance. For information on the weight distribution of cyclic codes, the reader is referred to \[1,10,13,21\].

Let \(h_0(x), h_1(x), h_2(x)\) and \(h_3(x)\) be the minimal polynomials of \(\pi^{-1}, \pi^{-2}, \pi^{-p^k+1}\) and \(\pi^{-p^k+1}\) over \(\mathbb{F}_p\), respectively. It is easy to check that \(h_0(x), h_1(x), h_2(x)\) and \(h_3(x)\) are polynomials of degree \(m\) and are pairwise distinct when \(m \geq 5\). Define \(h(x) = h_0(x)h_1(x)h_2(x)h_3(x)\). Then \(h(x)\) has degree \(4m\) and is a factor of \(xp^{m-1} - 1\).

Let \(C_{(p,m,k)}\) be the cyclic code with parity-check polynomial \(h(x)\). Then \(C_{(p,m,k)}\) has length \(p^m - 1\) and dimension \(4m\). Moreover, it can be expressed as

\[
C_{(p,m,k)} = \{ c(\alpha,\beta,\gamma,\delta) : \alpha, \beta, \gamma, \delta \in \mathbb{F}_{p^m} \},
\]

where

\[
c(\alpha,\beta,\gamma,\delta) = (Tr(\alpha \pi^{(p^k+1)t} + \beta \pi^{(p^k+1)t} + \gamma \pi^{2t} + \delta \pi^t))_{t=0}^{p^m-2}
\]

and \(Tr\) is the trace map from \(\mathbb{F}_{p^m}\) to \(\mathbb{F}_p\). Let \(h'(x) = h_1(x)h_2(x)h_3(x)\) and \(C'\) be the cyclic code with parity-check polynomial \(h'(x)\). Then \(C'\) is a subcode of \(C_{(p,m,k)}\) with dimension \(3m\). From \[19\] and \[21\], we can obtain the weight distribution of \(C_{(p,m,k)}\) when \(m\) is odd. In this paper, we will determine the weight distribution of \(C_{(p,m,k)}\). For doing this, we need to determine the value distribution of the multi-sets

\[
\{ T(\alpha, \beta, \gamma) = \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}(\alpha x^{p^k+1} + \beta x^k + \gamma x)} : \alpha, \beta, \gamma \in \mathbb{F}_p \}
\]

and

\[
\{ S(\alpha, \beta, \gamma, \delta) = \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}(\alpha x^{p^k+1} + \beta x^k + \gamma x^2 + \delta x)} : \alpha, \beta, \gamma, \delta \in \mathbb{F}_p \},
\]

where \(\zeta_p = e^{2\pi i/p}\). The rest of this paper is organized as follows. Some preliminaries will be introduced in Section 2. The weight distribution of the cyclic code \(C_{(p,m,k)}\) will be given in Section 3.

## 2 Mathematical Foundations

In this section, we first give a brief introduction to the theory of quadratic forms over finite fields. By fixing a basis \(v_1, v_2, \ldots, v_m\) of \(\mathbb{F}_{p^m}\) over \(\mathbb{F}_p\), each \(x \in \mathbb{F}_{p^m}\) can be uniquely expressed as

\[
x = x_1 v_1 + x_2 v_2 + \cdots + x_m v_m,
\]

where \(x_i \in \mathbb{F}_p\) for \(1 \leq i \leq m\). Then \(\mathbb{F}_{p^m}\) is isomorphic to the \(m\)-dimensional linear space \(\mathbb{F}_p^m\). In other words, we have the following \(\mathbb{F}_p\)-linear isomorphism:

\[
x = x_1 v_1 + x_2 v_2 + \cdots + x_m v_m \mapsto X = (x_1, x_2, \ldots, x_m).
\]

For a quadratic form \(F\), there exists a symmetric matrix \(H\) of order \(m\) over \(\mathbb{F}_p\) such that \(F(X) = XH'X\), where \(X = (x_1, x_2, \ldots, x_m) \in \mathbb{F}_p^m\) and \(X'\) denotes the transpose.
of $X$. The rank of the quadratic form $F$ is defined as the codimension of the $\mathbb{F}_p$-vector space $V = \{ x \in \mathbb{F}_{p^m} : f(x+z) - f(x) = 0 \text{ for all } z \in \mathbb{F}_{p^m} \}$, i.e., the rank of $H$. Then there exists a nonsingular matrix $M$ of order $m$ over $\mathbb{F}_p$ such that $MHM'$ is a diagonal matrix \( (\mathbb{1}_{m}) \). Under the nonsingular linear substitution $X = ZM$ with $Z = (z_1, z_2, \ldots, z_m) \in \mathbb{F}_p^m$, then $F(X) = ZMHM'Z' = \sum_{i=1}^{r} d_i z_i^2$, where $r$ is the rank of $F(X)$ and $d_i \in \mathbb{F}_p$. We can recall that the Legendre symbol \( (\frac{a}{p}) \) has the value 1 if $a$ is a quadratic residue mod $p$, -1 if $a$ is a quadratic nonresidue mod $p$, and zero if $p|a$. Let $\triangle = d_1d_2\cdots d_r$ (we assume $\triangle = 0$ when $r = 0$). Then the \( (\frac{\triangle}{p}) \) is an invariant of $H$ under the action of $M \in GL_m(\mathbb{F}_p)$.

For any fixed $(\alpha, \beta, \gamma) \in \mathbb{F}_p^3$, let $Q_{\alpha,\beta,\gamma}(X) = Tr(\alpha x^{p^{2k+1}} + \beta x^{p^k+1} + \gamma x^2)$. Then its induced quadratic form is

$$F_{\alpha,\beta,\gamma}(X) = Tr(\alpha \sum_{i=1}^{m} x_i v_i)^{p^{2k+1}} + \beta (\sum_{i=1}^{m} x_i v_i)^{p^k+1} + \gamma (\sum_{i=1}^{m} x_i v_i)^2$$

$$= \sum_{i,j=1}^{m} Tr(\alpha x_i^{p^{2k+1}} v_j + \beta x_i^{p^k+1} v_j + \gamma v_i v_j) x_i x_j$$

$$= XH_{\alpha,\beta,\gamma}X',$$

where $H_{\alpha,\beta,\gamma} = (h_{i,j})$ and

$$h_{i,j} = \frac{1}{2} Tr(\alpha(v_i^{p^{2k+1}} v_j + v_i v_j^{p^{2k+1}}) + \beta(v_i^{p^k+1} v_j + v_i v_j^{p^k+1}) + \gamma v_i v_j), \ 1 \leq i, j \leq m.$$

Then we have the following:

$$T(\alpha, \beta, \gamma) = \sum_{x \in \mathbb{F}_p^m} \zeta_p^{Tr(\alpha x^{p^{2k+1}} + \beta x^{p^k+1} + \gamma x^2)} = \sum_{X \in \mathbb{F}_p^m} \zeta_p^{XH_{\alpha,\beta,\gamma}X'}$$

and

$$S(\alpha, \beta, \gamma, \delta) = \sum_{x \in \mathbb{F}_p^m} \zeta_p^{Tr(\alpha x^{p^{2k+1}} + \beta x^{p^k+1} + \gamma x^2 + \delta x)} = \sum_{X \in \mathbb{F}_p^m} \zeta_p^{XH_{\alpha,\beta,\gamma}X' + A_{\delta} X'},$$

where $A_{\delta} = (Tr(\delta v_1), Tr(\delta v_2), \ldots, Tr(\delta v_m))$. Hence, in order to determine the value distribution of the two multi-sets

$$\{ T(\alpha, \beta, \gamma) \}_{\alpha, \beta, \gamma \in \mathbb{F}_p^m}$$

and

$$\{ S(\alpha, \beta, \gamma, \delta) \}_{\alpha, \beta, \gamma, \delta \in \mathbb{F}_p^m},$$

we need the following lemmas.

**Lemma 2.1 (\cite{1})** Let $F(X) = XHX'$ be a quadratic form in $m$ variables of rank $r$ over $\mathbb{F}_p$, then

1. $\sum_{X \in \mathbb{F}_p^m} \zeta_p^{F(X)} = (\frac{\triangle}{p})(p^*)^{\frac{1}{2}} p^{m-r}$, where $p^* = (-1)^{\frac{p-1}{2}} p^{\ell}$;
2. for \( A = (a_1, a_2, \ldots, a_m) \in \mathbb{F}_p^m \), if \( 2YH + A = 0 \) has a solution \( Y = B \in \mathbb{F}_p^m \), then 
\[
\sum_{X \in \mathbb{F}_p^m} \zeta_{\mathbb{F}_p}^{F(X)+AX'} = \zeta_{\mathbb{F}_p}^{t} \sum_{X \in \mathbb{F}_p^m} \zeta_{\mathbb{F}_p}^{F(X)}, \text{ where } t = \frac{1}{2}AB' \in \mathbb{F}_p, \text{ otherwise }
\]
\[
\sum_{X \in \mathbb{F}_p^m} \zeta_{\mathbb{F}_p}^{F(X)+AX'} = 0.
\]

Lemma 2.2 ([19, 21])
1. For any \((\alpha, \beta, \gamma) \in \mathbb{F}_p^3 \setminus \{(0, 0, 0)\}, \text{ rank}(H_{\alpha, \beta, \gamma}) \text{ is at least } m - 4.
2. Let \( n_i \) be the number of \((\alpha, \beta, \gamma)\) such that \( r_{\alpha, \beta, \gamma} = m - i \) for \( 0 \leq i \leq 4 \). Then
\[
n_1 = \frac{p^{m+1}(p^{2m-2} - p^{2m-3} + p^m - p^m - 1)}{p^2 - 1}
\]
and
\[
n_3 = \frac{p^{m-3}(p^{m-1} - 1)(p^m - 1)}{p^2 - 1}.
\]

The following result is important to determine the multiplicity of each value of \( T(\alpha, \beta, \gamma) \).

Lemma 2.3
For the exponential sum \( T(\alpha, \beta, \gamma) \), we have
\[
\sum_{\alpha, \beta, \gamma \in \mathbb{F}_p^m} T^2(\alpha, \beta, \gamma) = \begin{cases} p^{3m} (2p^m - 1), & q \equiv 1 \pmod{4}, \\ p^{3m}, & q \equiv 3 \pmod{4}, \end{cases}
\]
and
\[
\sum_{\alpha, \beta, \gamma \in \mathbb{F}_p^m} T^4(\alpha, \beta, \gamma) = \begin{cases} p^{3m}((2p^m - 1)^2 + (p - 1)(p^m - 1)(2p^m - p - 1)), & q \equiv 1 \pmod{4}, \\ p^{3m}(1 + (p + 1)(p^m - 1)(2p^m - p + 1)), & q \equiv 3 \pmod{4}. \end{cases}
\]

We prove this lemma only for the case that \( p \equiv 3 \pmod{4} \). The proof for the case that \( p \equiv 1 \pmod{4} \) is similar and omitted. To prove this, the following two lemmas are necessary.

Lemma 2.4
Let \( p \equiv 3 \pmod{4} \) and let \( N_2 \) denote the number of solutions \((x_1, x_2) \in \mathbb{F}_p^2\) of the following system of equations
\[
\begin{align*}
x_1^2 + x_2^2 &= 0 \\
x_1^{p^k+1} + x_2^{p^k+1} &= 0 \\
x_1^{p^{2k}+1} + x_2^{p^{2k}+1} &= 0.
\end{align*}
\]
Then \( N_2 = 1 \).

Proof. This system of equations have only one solution \((0, 0)\), since \(-1\) is a non-square when \( p \equiv 3 \pmod{4} \). \( \blacksquare \)
Lemma 2.5  Let \( p \equiv 3 \pmod{4} \) and let \( N_4 \) denote the number of solutions \((x_1, x_2, x_3, x_4) \in \mathbb{F}_{p^m}^4\) of the following system of equations

\[
\begin{align*}
&x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \\
&x_1^{p^k+1} + x_2^{p^k+1} + x_3^{p^k+1} + x_4^{p^k+1} = 0 \\
&x_1^{2k+1} + x_2^{2k+1} + x_3^{2k+1} + x_4^{2k+1} = 0.
\end{align*}
\]

Then \( N_4 = (p + 1)(p^m - 1)(2p^m - p + 1) + 1\).

Proof. See Appendix. \( \blacksquare \)

Now we are ready to prove Lemma 2.3 in the case of \( p \equiv 3 \pmod{4} \).

Proof of Lemma 2.3  From Eq. (1), we have

\[
\sum_{(\alpha, \beta, \gamma) \in \mathbb{F}_{p^m}^3} T^2(\alpha, \beta, \gamma) = \sum_{(\alpha, \beta, \gamma) \in \mathbb{F}_{p^m}^3} \zeta_p T(\alpha x_1^{2k+1} + \beta x_2^{2k+1} + \gamma x_3^{2k+1}) \sum_{x_2 \in \mathbb{F}_{p^m}} \zeta_p T(\alpha x_2^{2k+1} + \beta x_2^{2k+1} + \gamma x_2^2) \sum_{x_1 \in \mathbb{F}_{p^m}} \zeta_p T(\alpha x_1^{2k+1} + \beta x_2^{2k+1} + \gamma x_2^2)
\]

\[
= \sum_{(x_1, x_2) \in \mathbb{F}_{p^m}^2} \sum_{\alpha \in \mathbb{F}_{p^m}} \zeta_p T(\alpha x_1^{2k+1} + x_2^{2k+1}) \sum_{\beta \in \mathbb{F}_{p^m}} \zeta_p T(\beta x_1^{2k+1} + x_2^{2k+1}) \sum_{\gamma \in \mathbb{F}_{p^m}} \zeta_p T(\gamma x_1^2 + x_2^2)
\]

\[
= p^{3m} \# W,
\]

where

\[
W = \{(x_1, x_2) \in \mathbb{F}_{p^m}^2 : x_1^2 + x_2^2 = 0, x_1^{p^k+1} + x_2^{p^k+1} = 0, x_1^{2k+1} + x_2^{2k+1} = 0 \}.
\]

Then by Lemma 2.4, we have

\[
\sum_{(\alpha, \beta, \gamma) \in \mathbb{F}_{p^m}^3} T^2(\alpha, \beta, \gamma) = p^{3m}.
\]

Similarly, by Lemmas 2.3 and 2.5 we have

\[
\sum_{(\alpha, \beta, \gamma) \in \mathbb{F}_{p^m}^3} T^4(\alpha, \beta, \gamma) = p^{3m} (1 + (p + 1)(p^m - 1)(2p^m - p + 1)).
\]

3  RESULTS ON EXPONENTIAL SUMS AND THE WEIGHT DISTRIBUTION OF THE CYCLIC CODE

We follow the notation and conditions fixed in Section 1 and 2.

Theorem 3.1  Let \( m \geq 5 \) be an odd integer, \( k \) be any positive integer such that \( \gcd(m, k) = 1 \). Then the value distribution of the multi-set \( \{T(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in \mathbb{F}_{p^m}\} \) is given by Table 1.
Proof. By Lemma 2.1 for $\varepsilon = \pm 1$ and $0 \leq i \leq 4$, we define

$$n_{\varepsilon,i} = \begin{cases} \#\{(\alpha, \beta, \gamma) \in F^3_{p_m} \setminus \{(0,0,0)\} \mid T(\alpha, \beta, \gamma) = \varepsilon^{-2i} \} & m - i \text{ even}, \\ \#\{(\alpha, \beta, \gamma) \in F^3_{p_m} \setminus \{(0,0,0)\} \mid T(\alpha, \beta, \gamma) = \varepsilon^{-2i} \} & m - i \text{ odd}. \end{cases}$$

- $i = 1, 3$. In this case, $m - i$ is even. According to the results of [19] and [21], we can obtain

$$n_{1,1} = \frac{(p^{m+1} + p^{(m+3)/2})(p^{2m} - p^{2m-2} - p^{2m-3} + p^{m-2} - p^{m-3} - 1)}{2(p^2 - 1)},$$

$$n_{-1,1} = \frac{(p^{m-1} - p^{(m+3)/2})(p^{2m} - p^{2m-2} - p^{2m-3} + p^{m-2} - p^{m-3} - 1)}{2(p^2 - 1)},$$

$$n_{1,3} = \frac{(p^{m-3} + p^{(m-3)/2})(p^{m-1} - 1)(p^m - 1)}{2(p^2 - 1)},$$

$$n_{-1,3} = \frac{(p^{m-3} - p^{(m-3)/2})(p^{m-1} - 1)(p^m - 1)}{2(p^2 - 1)}.$$

- $i = 0, 2, 4$. In this case, $m - i$ is odd. By the same method in [9], we also have

$$n_{1,i} = n_{-1,i} = \frac{1}{2}n_i.$$

Moreover, we have

$$\sum_{(\alpha, \beta, \gamma) \in F^3_{p_m}} T^2(\alpha, \beta, \gamma) = p^{2m} - (n_{1,0} + n_{-1,0})p^m + (n_{1,1} + n_{-1,1})p^{m+1} + (n_{1,2} + n_{-1,2})p^{m+2} + (n_{1,3} + n_{-1,3})p^{m+3} + (n_{1,4} + n_{-1,4})p^{m+4}$$

and similarly,

$$\sum_{(\alpha, \beta, \gamma) \in F^3_{p_m}} T^4(\alpha, \beta, \gamma) = p^{4m} + 2n_{1,0}p^{2m} + n_{1,1}p^{2m+1} + 2n_{1,2}p^{2m+2} + n_{1,3}p^{2m+3} + 2n_{1,4}p^{2m+4}.$$
The proof is completed.

### Table 1: Value Distribution of $T(\alpha, \beta, \gamma)$

| Value                              | Frequency |
|------------------------------------|-----------|
| $\sqrt{p^{n/2-1}}, -\sqrt{p^{n/2-1}}$ | $n_{1,0}(n_{-1,0})$ |
| $p^{m+1}/2$                        | $n_{1,1}$ |
| $-p^{m+1}/2$                       | $n_{-1,1}$ |
| $\sqrt{p^{n/2-1}}, -\sqrt{p^{n/2-1}}$ | $n_{1,2}(n_{-1,2})$ |
| $p^{m+1}/2$                        | $n_{1,3}$ |
| $-p^{m+1}/2$                       | $n_{-1,3}$ |
| $\sqrt{p^{n/2-1}}, -\sqrt{p^{n/2-1}}$ | $n_{1,4}(n_{-1,4})$ |
| $p^{m}$                            | 1         |

Until now, we have determined the value distribution of the multi-set $\{T(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in \mathbb{F}_{p^m}\}$. The value distribution of the multi-set $\{S(\alpha, \beta, \gamma, \delta) \mid \alpha, \beta, \gamma, \delta \in \mathbb{F}_{p^m}\}$ can be determined by the following theorem.

**Theorem 3.2** Let $m \geq 5$ be an odd integer, $k$ be any positive integer such that $\gcd(m, k) = 1$. Then the value distribution of the multi-set $\{S(\alpha, \beta, \gamma, \delta) \mid \alpha, \beta, \gamma, \delta \in \mathbb{F}_{p^m}\}$ is given by Table 2.

**Proof.** By Lemma 2.1, $S(\alpha, \beta, \gamma, \delta)$ takes values from the set

$$\{0, \pm \zeta_p^{m/2}, \pm \zeta_p^{m/2} \sqrt{p^{m/2-1}} : j \in \mathbb{F}_p, 0 \leq i \leq 4\}.$$

Then for $\varepsilon = \pm 1$, we define

$$n_{\varepsilon, i, j} = \begin{cases} 
\#\{(\alpha, \beta, \gamma, \delta) \in \mathbb{F}_{p^m}^4 \setminus \{(0, 0, 0, 0)\} \mid S(\alpha, \beta, \gamma, \delta) = \varepsilon \zeta_p^{m/2} \} & m - i \text{ even,} \\
\#\{(\alpha, \beta, \gamma, \delta) \in \mathbb{F}_{p^m}^4 \setminus \{(0, 0, 0, 0)\} \mid S(\alpha, \beta, \gamma, \delta) = \varepsilon \zeta_p^{m/2} \sqrt{p^{m/2-1}} \} & m - i \text{ odd.} 
\end{cases}$$

and

$$\omega = \#\{(\alpha, \beta, \gamma, \delta) \in \mathbb{F}_{p^m}^4 \setminus \{(0, 0, 0, 0)\} \mid S(\alpha, \beta, \gamma, \delta) = 0\}.$$

By the same method in [9], we have the following results.

- **i = 1, 3.** In this case, $m - i$ is even. For $\varepsilon = \pm 1$ and $j \in \mathbb{F}_p^*$, we have
  $$n_{\varepsilon, i, 0} = (p^{m-i-1} + \varepsilon(p-1)p^{m-i-2})n_{\varepsilon, i},$$
  $$n_{\varepsilon, i, j} = (p^{m-i-1} - \varepsilon p^{m-i-2})n_{\varepsilon, i}.$$  

- **i = 0, 2, 4.** In this case, $m - i$ is odd. For $\varepsilon = \pm 1$ and $j \in \mathbb{F}_p^*$, we also have
  $$n_{\varepsilon, i, 0} = p^{m-i-1}n_{\varepsilon, i},$$
  $$n_{\varepsilon, i, j} = (p^{m-i-1} + \varepsilon (-j/p)p^{m-i-2})n_{\varepsilon, i}.$$
Furthermore, we have
\[ \omega = p^m - 1 + (p^m - p^{m-1})n_1 + (p^m - p^{m-2})n_2 + (p^m - p^{m-3})n_3 + (p^m - p^{m-4})n_4. \]

Summarizing the discussion above completes the proof of this theorem. ■

### Table 2: Value Distribution of \( S(\alpha, \beta, \gamma, \delta) \)

| Value                                      | Frequency          |
|--------------------------------------------|--------------------|
| \( \sqrt{p^m} p^\frac{m-1}{2}, -\sqrt{p^m} p^\frac{m-1}{2} \) | \( n_1,0,0(n-1,0,0) \) |
| \( \zeta_j^{\frac{m+1}{2}}, j \in \mathbb{F}_p^* \) | \( n_1,0,j \) |
| \( -\zeta_j^{\frac{m+1}{2}}, j \in \mathbb{F}_p^* \) | \( n_1,0,j \) |
| \( p^\frac{m+1}{2} \)                          | \( n_{1,1,0} \)   |
| \( n_{1,1,0} \)                              |                    |
| \( \zeta_j^{\frac{m+1}{2}}, j \in \mathbb{F}_p^* \) | \( n_{1,1,j} \) |
| \( -\zeta_j^{\frac{m+1}{2}}, j \in \mathbb{F}_p^* \) | \( n_{1,1,j} \) |
| \( \sqrt{p^m} p^\frac{m+3}{2}, -\sqrt{p^m} p^\frac{m+3}{2} \) | \( n_{1,2,0}(n-1,2,0) \) |
| \( \zeta_j^{\frac{m+3}{2}}, j \in \mathbb{F}_p^* \) | \( n_{1,2,j} \) |
| \( -\zeta_j^{\frac{m+3}{2}}, j \in \mathbb{F}_p^* \) | \( n_{1,2,j} \) |
| \( p^\frac{m+3}{2} \)                          | \( n_{1,3,0} \)   |
| \( -p^\frac{m+3}{2} \)                        | \( n_{1,3,0} \)   |
| \( \zeta_j^{\frac{m+3}{2}}, j \in \mathbb{F}_p^* \) | \( n_{1,3,j} \) |
| \( -\zeta_j^{\frac{m+3}{2}}, j \in \mathbb{F}_p^* \) | \( n_{1,3,j} \) |
| \( \sqrt{p^m} p^\frac{m+3}{2}, -\sqrt{p^m} p^\frac{m+3}{2} \) | \( n_{1,4,0}(n-1,4,0) \) |
| \( \zeta_j^{\frac{m+3}{2}}, j \in \mathbb{F}_p^* \) | \( n_{1,4,j} \) |
| \( -\zeta_j^{\frac{m+3}{2}}, j \in \mathbb{F}_p^* \) | \( n_{1,4,j} \) |
| 0                                          | \( \omega \)       |
| \( p^m \)                                   | 1                  |

**Theorem 3.3** Let \( m \geq 5 \) be an odd integer and \( k \) be any positive integer such that \( \gcd(m, k) = 1 \). \( C_{(p,m,k)} \) is a cyclic code over \( \mathbb{F}_p \) with parameters \([p^m - 1, 4m, (p - 1)(p^{m-1} - p^{\frac{m-3}{2}})]\). Moreover, the weight distribution of \( C_{(p,m,k)} \) is given in Table 3.

**Proof.** According to the discussion in section 1, the length and dimension of \( C_{(p,m,k)} \) are clearly. In terms of exponential sums, the weight of the codeword \( e_{(\alpha,\beta,\gamma,\delta)} = \)
\((c_0, c_1, \ldots, c_{p^m-2})\) in \(C_{(p,m,k)}\) is given by
\[
W(c_{(\alpha,\beta,\gamma,\delta)}) = \# \{0 \leq t \leq p^m - 2 : c_t \neq 0\} = p^m - 1 - \frac{1}{p} \sum_{t=0}^{p^m-2} \sum_{y \in \mathbb{F}_p} \zeta_p y^{w(t)}
\]
\[
= p^m - 1 - \frac{1}{p} \sum_{t=0}^{p^m-2} \sum_{y \in \mathbb{F}_p} \zeta_p y^{T r(\alpha \pi^{2k}+1+\beta \pi^{k+1}+\gamma \pi^{2t}+\delta x)}
\]
\[
= p^m - 1 - \frac{1}{p} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p y^{T r(\alpha x^{2k+1+\beta x^{k+1}+\gamma x^2+\delta x})}
\]
\[
= p^{m-1}(p-1) - \frac{1}{p} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p y^{T r(\alpha x^{2k+1+\beta x^{k+1}+\gamma x^2+\delta x})}
\]
\[
= p^{m-1}(p-1) - \frac{1}{p} R(\alpha, \beta, \gamma, \delta),
\]
where
\[
R(\alpha, \beta, \gamma, \delta) = \sum_{y \in \mathbb{F}_p^*} S(\alpha y, \beta y, \gamma y, \delta).
\]

Again by the same method in [9], we have the following results.

- \(i = 1, 3\). In this case, \(m - i\) is even. For \(\varepsilon = \pm 1\) and \(j \in \mathbb{F}_p^*\), we have
  - if \(S(\alpha, \beta, \gamma, \delta) = \varepsilon p^{m+i}\), then
    \[R(\alpha, \beta, \gamma, \delta) = \varepsilon (p-1)p^{m+i};\]
  - if \(S(\alpha, \beta, \gamma, \delta) = \varepsilon \zeta_p^{m+i}\), then
    \[R(\alpha, \beta, \gamma, \delta) = -\varepsilon p^{m+i} .\]

- \(i = 0, 2, 4\). In this case, \(m - i\) is odd. For \(\varepsilon = \pm 1\) and \(j \in \mathbb{F}_p^*\), we also have
  - if \(S(\alpha, \beta, \gamma, \delta) = \varepsilon \sqrt{p} p^{m+j}\), then
    \[R(\alpha, \beta, \gamma, \delta) = 0;\]
  - if \(S(\alpha, \beta, \gamma, \delta) = \varepsilon \zeta_p^{m+j}\sqrt{p} p^{m+i+1}\), then
    \[R(\alpha, \beta, \gamma, \delta) = \varepsilon \left(\frac{p}{p}\right)^{m+i+1}.\]

Summarizing the discussion above, together with Theorem 3.2, the proof is completed.

The following is an example of these codes.

**Example 3.4** Let \(p = 3\), \(m = 5\) and \(k = 1\). The the code \(C_{(3,5,1)}\) is a \([242, 20, 81]\) cyclic code over \(\mathbb{F}_3\) with weight enumerator
\[
1 + 484z^{81} + 72600z^{108} + 6853440z^{135} + 84092580z^{144} + 947952720z^{153} + 1618713316z^{162} + 782825472z^{171} + 42810768z^{180} + 3455760z^{189} + 7260z^{216}.
\]
which is completely in agreement with the results presented in Table 3.
Table 3: Weight Distribution of $C(p,m,k)$

| Weight | Frequency |
|--------|-----------|
| 0      | 1         |
| $(p - 1)p^{m-1}$ | $\omega + 2n_{1,0,0} + 2n_{1,2,0} + 2n_{1,4,0}$ |
| $(p - 1)p^{m-1} - p^{m-1}/2$ | $(p - 1)n_{-\left(\frac{1}{p}\right),0,1} + (p - 1)n_{-1,1,1}$ |
| $(p - 1)p^{m-1} + p^{m-1}/2$ | $(p - 1)n_{-\left(\frac{1}{p}\right),0,1} + (p - 1)n_{-1,1,1}$ |
| $(p - 1)p^{m-1} - (p - 1)p^{m-1}/2$ | $n_{1,1,0}$ |
| $(p - 1)p^{m-1} + (p - 1)p^{m-1}/2$ | $n_{-1,1,0}$ |
| $(p - 1)p^{m-1} - p^{m-1}/2$ | $(p - 1)n_{\left(\frac{1}{p}\right),2,1} + (p - 1)n_{-1,3,1}$ |
| $(p - 1)p^{m-1} + p^{m-1}/2$ | $(p - 1)n_{\left(\frac{1}{p}\right),2,1} + (p - 1)n_{-1,3,1}$ |
| $(p - 1)p^{m-1} - (p - 1)p^{m-1}/2$ | $n_{1,3,0}$ |
| $(p - 1)p^{m-1} + (p - 1)p^{m-1}/2$ | $n_{-1,3,0}$ |
| $(p - 1)p^{m-1} - p^{m-1}/2$ | $(p - 1)n_{\left(\frac{1}{p}\right),4,1}$ |
| $(p - 1)p^{m-1} + p^{m-1}/2$ | $(p - 1)n_{\left(\frac{1}{p}\right),4,1}$ |

APPENDIX

In the following discussion, let $d_1(d_2, \text{resp.})$ denote $p^k + 1(p^{2k} + 1, \text{resp.})$.

**Proof of Lemma 2.5** For any $(\overline{\alpha}, \overline{b}, \overline{c}) \in \mathbb{F}_p^m$, let $N_{1(\overline{\alpha},\overline{b},\overline{c})}$ and $N_{2(\overline{\alpha},\overline{b},\overline{c})}$ denote the number of solutions of the following two system of equations

\[
\begin{align*}
\left\{ 
\begin{array}{l}
 x_1^2 + x_2^2 = \overline{\alpha} \\
 x_1^d_1 + x_2^d_2 = \overline{b} \\
 x_1^d_1 + x_2^d_2 = \overline{c}
\end{array}
\right.
\end{align*}
\]

(5)

\[
\begin{align*}
\left\{ 
\begin{array}{l}
 x_3^2 + x_4^2 = -\overline{\alpha} \\
 x_3^d_3 + x_4^d_4 = -\overline{b} \\
 x_3^d_3 + x_4^d_4 = -\overline{c}
\end{array}
\right.
\end{align*}
\]

(6)

Then we have

\[
N_4 = \sum_{(\overline{\alpha},\overline{b},\overline{c}) \in \mathbb{F}_p^3} N_{1(\overline{\alpha},\overline{b},\overline{c})} N_{2(\overline{\alpha},\overline{b},\overline{c})}.
\]

Case 1, when $\overline{\alpha} = 0$. In this case, (5) and (6) have solutions if and only if $\overline{b} = \overline{c} = 0$ since $-1$ is a non-square. Moreover, $N_{1(0,0,0)} = N_{2(0,0,0)} = 1$.

Case 2, when $\overline{\alpha} \neq 0$. In this case, if $\overline{b} = 0$ or $\overline{c} = 0$, neither (5) nor (6) has solutions. So in the following, we consider the problem only when $\overline{b} \neq 0$ and $\overline{c} \neq 0$.

- $\overline{\alpha}$ is a nonzero square, $\overline{b} \neq 0$ and $\overline{c} \neq 0$. In this case, for any fixed $\overline{\alpha}$, (5) has the
same number of solutions as
\[
\begin{align*}
x_1^2 + x_2^2 &= 1 \\
x_1^{d_1} + x_2^{d_2} &= b \\
x_1^{d_2} + x_2^{d_2} &= c
\end{align*}
\] (7)
and (6) has the same number of solutions as
\[
\begin{align*}
x_3^2 + x_4^2 &= -1 \\
x_3^{d_1} + x_4^{d_1} &= -b \\
x_3^{d_2} + x_4^{d_2} &= -c
\end{align*}
\] (8)
where \( b = \frac{d_1}{\pi^{d_1}} \) and \( c = \frac{d_2}{\pi^{d_2}} \). Clearly, \((b, c)\) runs through \( \mathbb{F}_p^\times \) as \((d_1, d_2)\) does.

According to the proof of Lemma 3.5 and 3.6, it can be easy to see that for any fixed \((b, c)\) such that (7) have \(2(p+1)\) solutions, then for \((-b, -c), (8)\) also have \(2(p+1)\) solutions. Therefore, in this case we have
\[
\sum_{(\pi, \sigma, \tau) \in \mathbb{F}_p^2} N_1(\pi, \tau, \sigma)N_2(\pi, \sigma, \tau)
= (p^m - 1) \left\{(p+1)^2 + (2p+1)^2 \frac{p^m - p}{2(p+1)} \right\}
= (p+1)(p^m - 1)(2p^m - p + 1).
\]

• \( \pi \) is a non-square. In this case, for any fixed \( \pi \), (5) has the same number of solutions as
\[
\begin{align*}
x_1^2 + x_2^2 &= -1 \\
x_1^{d_1} + x_2^{d_1} &= -b \\
x_1^{d_2} + x_2^{d_2} &= -c
\end{align*}
\]
and equation system (6) has the same number of solutions as
\[
\begin{align*}
x_3^2 + x_4^2 &= 1 \\
x_3^{d_1} + x_4^{d_1} &= b \\
x_3^{d_2} + x_4^{d_2} &= c
\end{align*}
\]
It can be easily seen that this case is equivalent to the case when \( \pi \) is a nonzero square. So when \( \pi \) is a non-square, we also have
\[
\sum_{(\pi, \sigma, \tau) \in \mathbb{F}_p^2} N_1(\pi, \sigma, \tau)N_2(\pi, \tau, \sigma)
= (p+1)(p^m - 1)(2p^m - p + 1).
\]
Summarizing the two cases above, we have
\[
N_4 = (p+1)(p^m - 1)(2p^m - p + 1) + 1.
\]

**Lemma 3.5** Let \( N_1(b, c) \) denote the number of solutions \((x_1, x_2) \in \mathbb{F}_p^2\) of (7), where \((b, c) \in \mathbb{F}_p^2\). Then we have the following conclusions.
1. \( N_{1(1,1)} = p + 1; \)

2. When \((b, c)\) runs through \( \mathbb{F}_{p^m}^2 \setminus \{(1, 1)\}, \)

\[
N_{1(b,c)} = \begin{cases} 
2(p + 1), & \text{for } \frac{p^m - b}{p + 1} \text{ times}, \\
0, & \text{for the rest}. 
\end{cases}
\]

**Proof.** We first compute the number \( N_{1(b)} \) of solutions \((x_1, x_2) \in \mathbb{F}_{p^m}^2 \) of the following system of equations

\[
\begin{align*}
  x_1^2 + x_2^2 &= 1 \\
  x_1^{d_1} + x_2^{d_1} &= b.
\end{align*}
\]

(9)

When \( p \equiv 3 \pmod{4}, -1 \) is a non-square in \( \mathbb{F}_{p^m}. \) However, we can choose \( t \in \mathbb{F}_{p^m} \) such that \( t^2 = -1. \) From the first equation of (9), by setting \( \theta = x_1 - tx_2 \in \mathbb{F}_{p^m}, \) we can have

\[
x_1 = \frac{\theta + \theta^{-1}}{2}, x_2 = \frac{t(\theta - \theta^{-1})}{2}.
\]

(10)

Substituting (10) into the second equation of (9), we obtain

\[
\theta^{p^k + 1} + \theta^{-p^k - 1} = 2b.
\]

(11)

Denote \( \theta^{p^k + 1} \) by \( w, \) Eq. (11) is equivalent to

\[
w^2 - 2bw + 1 = 0.
\]

(12)

If Eq. (12) has no solution, i.e., \( b^2 - 1 \) is a non-square of \( \mathbb{F}_{p^m}, \) then \( N_b = 0. \) Otherwise, let \( w_1 \) and \( w_2 = w_1^{-1} \) be two solutions of (12). Since \( x_1 \in \mathbb{F}_{p^m}, \) then the following holds:

\[
\frac{\theta + \theta^{-1}}{2} = \frac{(\theta + \theta^{-1})^p}{2} = \frac{\theta^{p^m} + \theta^{-p^m}}{2},
\]

which implies \( \theta^{p^m + 1} = 1 \) or \( \theta^{-p^m - 1} = 1. \)

- If \( \theta^{p^m + 1} = 1, \) then \( x_2^{p^m} = \frac{t(\theta - \theta^{-1})}{2}p^m = \frac{t(\theta - \theta^{-1})}{2} \) is a solution since \( t^p = -t. \) And then \( x_2 \in \mathbb{F}_{p^m}. \) According to the discussion above, we have

\[
\theta^{p^k + 1} = w_1, \theta^{p^m + 1} = 1,
\]

(13)

or

\[
\theta^{p^k + 1} = w_1^{-1}, \theta^{p^m + 1} = 1.
\]

(14)

If \( \theta_1 \) and \( \theta_2 \) are two solutions of (13), then \( (\theta_1/\theta_2)^{p^k + 1} = 1 = (\theta_1/\theta_2)^{p^m + 1}. \) Observe that \( \gcd(p^k + 1, p^m + 1) = p + 1, \) then \( (\theta_1/\theta_2)^{p^k + 1} = 1. \) So if (13) has solutions, then it has exactly \( p + 1 \) solutions. If \( w_1 = w_1^{-1}, \) then (14) is the same as (13). In this case we have \( w_1 = \pm 1 \) and then from Eq. (12), \( b = \pm 1. \) But when \( b = -1, \theta^{p^k + 1} = w_1 = -1. \) By \( \theta^{p^m + 1} = 1 \) and \( \gcd(p^m + 1, 2(p^k + 1)) = p + 1, \) we have \( \theta^{p^k + 1} = 1. \) And then \( \theta^{p^k + 1} = 1, \) which is a contradiction. So in the following we only consider \( b = 1. \) In this case, \( w_1 = 1 \) and (14) have \( p + 1 \) solutions of \( \theta. \)
Moreover, we have \( p + 1 \) solutions of (9). If \( w_1 \neq w_1^{-1} \), then (14) has the same number of solutions as (13). Moreover, their solutions are distinct since \( w_1 \neq \pm 1 \). Therefore, (13) and (14) both have \( p + 1 \) solutions or no solutions in \( \mathbb{F}_{p^2} \). By (10), \((x_1, x_2)\) is uniquely determined by \( \theta \). Then (9) has \( 2(p + 1) \) solutions or no solutions in \( \mathbb{F}_{p^m}^2 \).

- If \( \theta^{p^m+1} \neq 1 \), then \( \theta^{p^m-1} = 1 \). In this case, \( \theta \in \mathbb{F}_{p^m}^* \). Since \( t \in \mathbb{F}_{p^m}^* \), so \( x_2 = \frac{t(\theta - \theta^{-1})}{2} \in \mathbb{F}_{p^m}^* \) if and only if \( \theta = \theta^{-1} \). Then \( \theta^{p^m+1} = \theta^{p^m-1} \cdot \theta^2 = 1 \) which is a contradiction.

Until now, we have \( N_{1(1)} = p + 1 \) and \( N_{1(b)} = 0 \) or \( 2(p + 1) \) for \( b \neq 1 \). And as in Lemma 5.4 in [21], we define

\[
T = \#\{b \in \mathbb{F}_{p^m}^* : N_{1(b)} = 2(p + 1)\}.
\]

Then we have

\[
T = \frac{p^m - p}{2(p + 1)}.
\]

Similarly, we also have \( c \) is uniquely determined by \( b \) and \( c = 1 \) if and only if \( b = 1 \). The proof is finished.

**Lemma 3.6** Let \( N_{2(b,c)} \) denote the number of solutions \((x_1, x_2) \in \mathbb{F}_{p^m}^2 \) of (8), where \((b, c) \in \mathbb{F}_{p^2}^* \). Then we have the following conclusions.

1. \( N_{2(1,1)} = p + 1 \).
2. When \((b, c)\) runs through \( \mathbb{F}_{p^2}^* \setminus \{(1, 1)\} \),

\[
N_{2(b,c)} = \begin{cases} 
2(p + 1), & \text{for } \frac{p^m - p}{2(p + 1)} \text{ times}, \\
0, & \text{for the rest}.
\end{cases}
\]

**Proof.** The proof is similar to the proof of the lemma above, so we omit the details.

**References**

[1] L.D. Baumert, R.J. McEliece, Weights of irreducible cyclic codes, *Inf. Contr.*, 20, no. 2 (1972), 158-175.

[2] L.D. Baumert, J. Mykkeltveit, Weight distribution of some irreducible cyclic codes, *DSN Progr. Rep.*, 16 (1973), 128-131.

[3] A.R. Calderbank, J.M. Goethals, Three-weight codes and association schemes, *Philips J. Res.*, 39 (1984), 143-152.

[4] C. Carlet, C. Ding, J. Yuan, Linear codes from highly nonlinear functions and their secret sharing schemes, *IEEE Trans. Inf. Theory*, 51, no. 6 (2005), 2089-2102.

[5] C. Ding, The weight distribution of some irreducible cyclic codes, *IEEE Trans. Inf. Theory*, 55, no. 3 (2009), 955-960.
[6] C. Ding, J. Yang, Hamming weights in irreducible cyclic codes, *Discrete Mathematics*, **313**, no. 4 (2013), 434-446.

[7] C. Ding, Y. Liu, C. Ma, L. Zeng, The weight distributions of the duals of cyclic codes with two zeros, *IEEE Trans. Inf. Theory*, **57**, no. 12 (2011), 8000-8006.

[8] K. Feng, J. Luo, Value distributions of exponential sums from perfect nonlinear functions and their applications, *IEEE Trans. Inf. Theory*, **53**, no. 7 (2007), 3035-3041.

[9] K. Feng, J. Luo, Weight distribution of some reducible cyclic codes, *Finite Fields Appl.*, **14**, no. 2 (2008), 390-409.

[10] T. Feng, On cyclic codes of length $2^r - 1$ with two zeros whose dual codes have three weights, *Des. Codes Cryptogr.*, **62** (2012), 253-258.

[11] T. Feng, KaHin Leung, Qing Xiang, Binary cyclic codes with two primitive nonzeros, *Sci. China Math.*, **56**, no. 7 (2012), 1403-1412.

[12] R. Lidl, H. Niederreiter, Finite fields, *Addison-Wesley Publishing Inc.*, (1983).

[13] Y. Liu, H. Yan, C. Liu, A Class of Six-weight Cyclic Codes and Their Weight Distribution, *arXiv: 1311.3391*, (2013).

[14] J. Luo, K. Feng, On the weight distribution of two classes of cyclic codes, *IEEE Trans. Inf. Theory*, **54**, no. 12 (2008), 5332-5344.

[15] C. Ma, L. Zeng, Y. Liu, D. Feng, C. Ding, The weight enumerator of a class of cyclic codes, *IEEE Trans. Inf. Theory*, **57**, no. 1 (2011), 397-402.

[16] J. Yuan, C. Carlet, C. Ding, The weight distribution of a class of linear codes from perfect nonlinear functions, *IEEE Trans. Inf. Theory*, **52**, no. 2 (2006), 712-717.

[17] B. Wang, C. Tang, Y. Qi, Y. Yang, M. Xu, The weight distributions of cyclic codes and elliptic curves, *IEEE Trans. Inf. Theory*, **58**, no. 12 (2012), 7253-7259.

[18] M. Xiong, The weight distributions of a class of cyclic codes, *Finite Fields Appl.*, **18**, no. 5 (2012), 933-945.

[19] D. Zheng, X. Wang, X. Zeng, L. Hu, The weight distribution of a family of p-ary cyclic codes, *Des. Codes Cryptogr.*, (2013).

[20] Z. Zhou, C. Ding, A class of three-weight cyclic codes, *Finite Fields Appl.*, **25** (2014), 79-93.

[21] Z. Zhou, C. Ding, J. Luo, A. Zhang, A family of five-weight cyclic codes and their weight enumerators, *IEEE Trans. Inf. Theory*, **59**, no. 10 (2013), 6674-6682.