LONG-TIME ASYMPTOTIC OF STABLE DAWSON-WATANABE PROCESSES IN SUPERCRITICAL REGIMES

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Abstract. Let \( W = (W_t)_{t \geq 0} \) be a supercritical \( \alpha \)-stable Dawson-Watanabe process (with \( \alpha \in (0, 2) \)) and \( f \) be a test function in the domain of \( -(-\Delta)^{\frac{\alpha}{2}} \) satisfying some integrability condition. Assuming the initial measure \( W_0 \) has a finite positive moment, we determine the long-time asymptotic of all orders of \( W_t(f) \). In particular, it is shown that the local behavior of \( W_t \) in long-time is completely determined by the asymptotic of the total mass \( W_t(1) \), a global characteristic.

1. Introduction

Let \( W = (W_t, t \geq 0) \) be a Dawson-Watanabe process starting from a finite measure \( m \) with motion generator \( -(-\Delta)^{\frac{\alpha}{2}} \) on \( \mathbb{R}^d \) (\( \alpha \in (0, 2) \)) and linear growth \( \beta > 0 \). More precisely, \( W \) is a measure-valued Markov process such that the process

\[
t \to M_t(f) := W_t(f) - m(f) - \int_0^t W_s \left( -(-\Delta)^{\frac{\alpha}{2}} + \beta \right) f \, ds
\]

(1.1)

is a martingale with quadratic variation \( \langle M^W(f) \rangle_t = \int_0^t W_s \left( f^2 \right) ds \) for all \( f \in C^2_b(\mathbb{R}^d) \). The law of \( W \) is denoted by \( \mathbb{P}^m \). Throughout the paper, we assume that the initial measure \( m \) has a finite positive moment, that is

\[
\int_{\mathbb{R}^d} |x|^a m(dx) < \infty \text{ for some } a > 0.
\]

(1.2)

The case when \( \beta > 0 \) is known as supercritical branching regime. The cases \( \beta < 0 \) and \( \beta = 0 \) are known respectively as subcritical and critical branching regimes which, however, are not considered in the current article. For a fixed test function \( f \) with sufficient regularity and integrability, we investigate the long-time asymptotic of \( W_t(f) \) in supercritical branching regimes. To state the main result precisely, we

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prepare some notation. For each multi-index
$k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, we denote

$$|k| = k_1 + k_2 + \cdots + k_d, \quad k! = k_1!k_2! \cdots k_d!, \quad x^k = x_1^{k_1}x_2^{k_2}\cdots x_d^{k_d},$$

and define the constant $\theta^k_{d,\alpha}$ and the $\sigma$-finite signed measure $\lambda^k_d$ on $\mathbb{R}^d$ respectively

$$\theta^k_{d,\alpha} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} \theta^k d\theta \quad \text{and} \quad \lambda^k_d(dy) = \frac{1}{k!} y^k dy. \quad (1.3)$$

Obviously $\lambda^0_d$ is the Lebesgue measure $\lambda_d$. In supercritical regimes ($\beta > 0$), it is well-known that the limit $\lim_{t \to \infty} e^{-\beta t} W_t(1)$ exists almost surely and is a well-defined random variable. We denote $\tilde{W}_\infty(1) = \lim_{t \to \infty} e^{-\beta t} W_t(1)$.

**Theorem 1.1.** Let $N$ be a non-negative integer and $f$ be a function in $D(-(-\Delta)^{\frac{\alpha}{2}})$ satisfying

$$\int_{\mathbb{R}^d} |f(x)||x|^N dx < \infty. \quad (1.4)$$

Then, with $\mathbb{P}_m$-probability one,

$$\lim_{t \to \infty} t^{\frac{N\alpha}{2}} \left| W_t(f) - \tilde{W}_\infty(1) \sum_{k \in \mathbb{N}^d: |k| \leq N \atop |k| \text{ is even}} (-1)^{|k|} t^{-\frac{|k|}{\alpha}} \theta^k_{d,\alpha} \lambda^k_d(f) \right| = 0. \quad (1.5)$$

Written another way, we have

$$t^{\frac{d}{\alpha}} W_t(f) = \tilde{W}_\infty(1) \sum_{k \in \mathbb{N}^d: |k| \leq N \atop |k| \text{ is even}} (-1)^{|k|} t^{-\frac{|k|}{\alpha}} \theta^k_{d,\alpha} \lambda^k_d(f) + o(t^{-\frac{N}{\alpha}}), \quad (1.6)$$

where $\lim_{t \to \infty} t^{\frac{N}{\alpha}} o(t^{-\frac{N}{\alpha}}) = 0$ almost surely.

Herein, $D(-(-\Delta)^{\frac{\alpha}{2}})$ denotes the domain of the weak generator of the $\alpha$-stable process (see the following section for a precise definition). Theorem 1.1 extends results of Kouritzin and Ren [KR14] in which the first order asymptotic ($N = 0$) was identified. For a heuristic explanation of long-time limits of supercritical superprocesses and their connection with strong laws of large numbers, we refer to [KLS16, Section 2] and [KL17, Subsection 2.2].

The higher order asymptotic expansions (1.5) and (1.6) are obtained by combining the method initiated by Asmussen and Hering [AH76] and an asymptotic expansion of the $\alpha$-stable semigroup (see Proposition 4.1 below). When $-(-\Delta)^{\frac{\alpha}{2}}$ is replaced by the generator of an Ornstein-Uhlenbeck process, similar results have
been obtained by Adamczak and Milòś [AM15]. In such case, because of the exponential rates in the expansion of the Ornstein-Uhlenbeck semigroup, convergences in distribution are expected in the asymptotic of high orders (which are called central limit theorems). On the other hand, the rates in the asymptotic expansion of the $\alpha$-stable semigroup are those of polynomials (see (4.1) below) and are negligible under the exponential growing expected total mass $W_t(1)$, which leads to almost sure limits in the asymptotic of all high orders. In view of Theorem 1.1, it is interesting to observe that the local behavior of $W_t$ in long time is completely determined by the asymptotic of the total mass $W_t(1)$, which is a global characteristic.

We conclude the introduction with an outline of the article. Section 2 reviews the martingale formulations of Dawson-Watanabe processes. In Section 3, we investigate the long-time asymptotic of $W_t$ against some special test functions. The proof of Theorem 1.1 is presented in Section 4.

2. Martingale formulations

We use $\nu(f)$ and $\langle f, \nu \rangle$ to denote $\int_{\mathbb{R}^d} f \, d\nu$ for a measure $\nu$ and an integrable function $f$. Let $T_t$ be the semigroup corresponding to a symmetric $\alpha$-stable process acting on $b\mathcal{E}(\mathbb{R}^d)$, the space of bounded Borel measurable functions on $\mathbb{R}^d$. In particular, for every $f \in b\mathcal{E}(\mathbb{R}^d)$,

$$T_t f(x) = \int_{\mathbb{R}^d} p_t(x - y) f(y) dy,$$

where $p_t$ is the probability transition kernel

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i x \cdot \theta} e^{-|\theta|^\alpha} d\theta. \quad (2.2)$$

Let $\hat{f}$ denote the Fourier transform of $f$ with the normalization $\hat{f}(\theta) = \int_{\mathbb{R}^d} e^{-i \theta \cdot x} f(x) dx$. Using Fourier transform, $P_t f$ takes a simpler form

$$T_t f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i x \cdot \theta - |\theta|^\alpha} \hat{f}(\theta) d\theta. \quad (2.3)$$

The weak domain of $-(-\Delta)^{\frac{\alpha}{2}}$, denoted by $\mathcal{D}(-(-\Delta)^{\frac{\alpha}{2}})$, is the collection of all functions $f$ in $b\mathcal{E}(\mathbb{R}^d)$ such that the limit

$$\lim_{t \downarrow 0} \frac{1}{t} (T_t f - f)$$

exists pointwise and is a bounded measurable function on $\mathbb{R}^d$. If $f$ belongs to $\mathcal{D}(-(-\Delta)^{\frac{\alpha}{2}})$, we denote the above limit by $-(-\Delta)^{\frac{\alpha}{2}} f$. 

We define \( \tilde{W}_t = e^{-\beta t} W_t \) and \( \tilde{M}_t = \int_0^t e^{-\beta s} dM_s \). It follows from (1.1) and Itô formula that for every \( f \in \mathcal{D}(-(-\Delta)^{\frac{\alpha}{2}}) \)
\[
\tilde{W}_t(f) = m(f) + \int_0^t \tilde{W}_s(-(-\Delta)^{\frac{\alpha}{2}} f) ds + \tilde{M}_t(f) .
\] (2.4)

Note that \( t \to \tilde{M}_t(f) \) is a martingale with quadratic variation
\[
\langle \tilde{M}(f) \rangle_t = \int_0^t e^{-2\beta s} W_s(f^2) ds = \int_0^t e^{-\beta s} \tilde{W}_s(f^2) ds .
\] (2.5)

The measure-valued process \( (\tilde{M}_t)_{t \geq 0} \) can be considered as a worthy martingale measure (cf. [Wal86]) with dominating measure \( K(dx, dy, ds) = \tilde{W}_s(dx \times dy) ds \). In this sense, for every deterministic function \( f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) and \( t > 0 \) satisfying
\[
\mathbb{E} \int_0^t \int_{\mathbb{R}^d} e^{-2\beta s} W_s(f^2) ds < \infty ,
\]
one can define the stochastic integration \( \tilde{M}_t(f) := \int_0^t \int_{\mathbb{R}^d} f(s, y) d\tilde{M}(s, y) \) such that \( (\tilde{M}_t(f))_{t \geq 0} \) is a martingale with quadratic variation
\[
\langle \tilde{M}(f) \rangle_t = \int_0^t e^{-\beta s} \tilde{W}_s(f^2) ds .
\] (2.5)

It follows that (see [Per02, pg. 167]) for every \( f \in b\mathcal{E}(\mathbb{R}^d) \)
\[
\tilde{W}_t(f) = m(P_t f) + \int_0^t \int_{\mathbb{R}^d} T_{t-s} f(y) dM(r, y) ,
\] (2.6)
which is called the Green function representation. From (2.6), one derives the following two important identities
\[
\tilde{W}_t(f) - \tilde{W}_s(T_{t-s} f) = \int_s^t \int_{\mathbb{R}^d} T_{t-r} f(y) d\tilde{M}(r, y)
\] (2.7)

and
\[
\mathbb{P}_m \tilde{W}_t(f) = m(T_t f) ,
\] (2.8)
which are valid for all \( 0 \leq s \leq t \) and \( f \in b\mathcal{E}(\mathbb{R}^d) \). The following estimate is intrinsic to supercritical regimes and plays a central role in our approach.

**Lemma 2.1.** For every \( f \in b\mathcal{E}(\mathbb{R}^d) \) and \( t \geq s \geq 0 \), we have
\[
\mathbb{P}_m \left[ \left( \tilde{W}_t(f) - \tilde{W}_s(T_{t-s} f) \right)^2 \right] \lesssim m(1) \| f \|_\infty^2 e^{-\beta s} .
\]
Proof. From (2.7), (2.5) and (2.8)
\[
\mathbb{P}_m \left[ \left( \tilde{W}_t(f) - \tilde{W}_s(T_{t-s}f) \right)^2 \right] = \mathbb{P}_m \int_s^t \tilde{W}_r((T_{t-r}f)^2)e^{-\beta r} dr
\]
\[
= \int_s^t \langle T_r(T_{t-r}f)^2, m \rangle e^{-\beta r} dr.
\]
We observe that \( T_r(T_{t-r}f)^2 \leq T_rT_{t-r}(f^2) = T_t(f^2) \) by Jensen’s inequality and \( \|T_t f^2\|_\infty \leq \|f\|_\infty^2 \). Hence, \( \langle T_r(T_{t-r}f)^2, m \rangle \leq \|T_t(f^2)\|_\infty m(1) \leq m(1)\|f\|_\infty^2 \). It follows that
\[
\mathbb{P}_m \left[ \left( \tilde{W}_t(f) - \tilde{W}_s(T_{t-s}f) \right)^2 \right] \leq m(1)\|f\|_\infty^2 \int_s^t e^{-\beta r} dr,
\]
which yields the result. \( \Box \)

3. Characteristic martingales

For every \( x, \theta \in \mathbb{R}^d \), we denote \( e_\theta(x) = e^{i\theta \cdot x} \), \( \cos_\theta(x) = \cos(\theta \cdot x) \) and \( \sin_\theta(x) = \sin(\theta \cdot x) \) and recall the assumption (1.2) on \( m \) and the definition of \( \vartheta_{d,\alpha}^k \) in (1.3). We investigate the long-time asymptotic of \( \tilde{M}_t(e_\theta) \).

Lemma 3.1. For every \( \theta \in \mathbb{R}^d \),
\[
\mathbb{P}_m \left[ \sup_{t \geq 0} |\tilde{M}_t(e_\theta) - \tilde{M}_t(1)|^2 \right] \lesssim |\theta|^{2\wedge \alpha} + |\theta|^\alpha.
\]

Proof. For each \( \theta \in \mathbb{R}^d \), \((\tilde{M}_t(e_\theta))_{t \geq 0}\) is a complex valued martingale whose real and imaginary parts have quadratic variations satisfying
\[
\langle \Re \tilde{M}(e_\theta) \rangle_t = \int_0^t \tilde{W}_r (\cos_\theta^2) e^{-\beta r} dr \quad \text{and} \quad \langle \Im \tilde{M}(e_\theta) \rangle_t = \int_0^t \tilde{W}_r (\sin_\theta^2) e^{-\beta r} dr.
\]
Together with martingale maximal inequality, we see that
\[
\mathbb{P}_m \left[ \sup_{t \geq 0} |\tilde{M}_t(e_\theta) - \tilde{M}_t(1)|^2 \right] \lesssim \mathbb{P}_m \left[ \int_0^\infty \tilde{W}_r ( (\cos_\theta - 1)^2 + \sin_\theta^2 ) e^{-\beta r} dr \right].
\]
Hence, using the elementary identity \( (\cos_\theta - 1)^2 + \sin_\theta^2 = 4 \sin_\theta^2/2 \) and (2.8), we obtain
\[
\mathbb{P}_m \left[ \sup_{t \geq 0} |\tilde{M}_t(e_\theta) - \tilde{M}_t(1)|^2 \right] \leq \int_0^\infty \langle T_r (\sin_{\theta/2}^2), m \rangle e^{-\beta r} dr.
\]
Note that for every \( x \in \mathbb{R}^d \)
\[
2T_r \sin_{\theta/2}^2(x) = 1 - \cos_\theta(x)e^{-r|\theta/2|^\alpha} = (1 - \cos_\theta(x))e^{-r|\theta/2|^\alpha} + 1 - e^{-r|\theta/2|^\alpha}
\]
\[
\lesssim (1 \wedge |\theta||x|^2) + r|\theta|^\alpha.
\]
These estimates and (1.2) implies the result.

**Lemma 3.2.** \( \tilde{M}_t(e_\theta) \) converges almost surely and in the mean-square sense to limit \( \tilde{M}_\infty(e_\theta) \) for each \( \theta \in \mathbb{R}^d \). In addition, the following relation holds

\[
\tilde{W}_\infty(1) = m(1) + \tilde{M}_\infty(1).
\] (3.2)

**Proof.** Using (3.1) we have

\[
\mathbb{P}_m \left| \tilde{M}_t(e_\theta) \right|^2 = \int_0^t \mathbb{P}_m \tilde{W}_s(1) e^{-\beta r} dr,
\]

which together with (2.8) implies \( \sup_{t \geq 0} \mathbb{P}_m \left| \tilde{M}_t(e_\theta) \right|^2 < \infty \). Hence, by the martingale convergence theorem, \( \lim_{t \to \infty} \tilde{M}_t(e_\theta) \) exists almost surely and in mean-square sense for each \( \theta \in \mathbb{R}^d \). The relation (3.2) follows from here (by setting \( \theta = 0 \)) and the relation (2.6) with \( f \equiv 1 \). \( \square \)

**Proposition 3.3.** Let \( \rho(t) = t^\kappa \) for some \( \kappa \in (0, 1) \). With \( \mathbb{P}_m \)-probability one, we have for every \( k \in \mathbb{N}^d \) that

\[
\lim_{t \to \infty} t^{d+\frac{|k|}{\alpha}} \tilde{W}_{\rho(t)}(\partial^k p_{t-\rho(t)}) = \begin{cases} 
0 & \text{if } |k| \text{ is odd} \\
(-1)^{|k|} \partial^k \tilde{W}_{\infty}(1) & \text{if } |k| \text{ is even}.
\end{cases} \] (3.3)

**Proof.** It suffices to restrict on the event \( \tilde{W}_\infty(1) \neq 0 \). We note that for every function \( f \in L^1(\mathbb{R}^d) \), by Fubini’s theorem,

\[
\tilde{W}_t(f) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \tilde{W}_t(e_\theta) \hat{f}(\theta) d\theta.
\] (3.4)

Hence,

\[
\tilde{W}_{\rho(t)}(\partial^k p_{t-\rho(t)}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t-\rho(t)|\theta|^\alpha} \tilde{W}_{\rho(t)}(e_\theta)(i\theta)^k d\theta.
\]

In addition, from (2.4), we obtain

\[
\tilde{W}_{\rho(t)}(e_\theta) = m(e_\theta) - |\theta|^\alpha \int_0^{\rho(t)} \tilde{W}_s(e_\theta) ds + \tilde{M}_{\rho(t)}(e_\theta).
\]

It follows that

\[
\tilde{W}_{\rho(t)}(\partial^k p_{t-\rho(t)}) = I_1 + I_2 + I_3 + I_4,
\] (3.5)
where

\[
I_1 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} m(e_\theta)(i\theta)^k d\theta ,
\]

\[
I_2 = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} \int_0^{\rho(t)} \bar{W}_s(e_\theta) ds |\theta|^\alpha (i\theta)^k d\theta ,
\]

\[
I_3 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} \left[ \bar{M}_{\rho(t)}(e_\theta) - \bar{M}_{\rho(t)}(1) \right] (i\theta)^k d\theta ,
\]

\[
I_4 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} (i\theta)^k d\theta \bar{M}_{\rho(t)}(1) .
\]

Applying Lemma 3.2, we see that

\[
\lim_{t \to \infty} t^{\frac{d+|k|}{\alpha}} I_4 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} (i\theta)^k d\theta \bar{M}_\infty(1) . \tag{3.6}
\]

We will show that

\[
\lim_{t \to \infty} t^{\frac{d+|k|}{\alpha}} I_1 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} (i\theta)^k d\theta m(1) , \tag{3.7}
\]

\[
\lim_{t \to \infty} t^{\frac{d+|k|}{\alpha}} I_2 = 0 \quad \text{and} \quad \lim_{t \to \infty} t^{\frac{d+|k|}{\alpha}} I_3 = 0 . \tag{3.8}
\]

By a change of variable, we have

\[
I_1 = t^{\frac{d+|k|}{\alpha}} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(1-\frac{\rho(t)}{t})|\theta|^\alpha} m(e_{t^{-1/\alpha}\theta})(i\theta)^k d\theta .
\]

This, together with dominated convergence theorem yields (3.7). For $I_2$, we observe that

\[
|I_2| \lesssim \bar{W}_\infty(1) \rho(t) \int_{\mathbb{R}^d} e^{-(t-\rho(t))|\theta|^\alpha} |\theta| |k|+\alpha d\theta
\]

\[
\lesssim \bar{W}_\infty(1) \frac{\rho(t)}{t} t^{\frac{d+|k|}{\alpha}} \int_{\mathbb{R}^d} e^{-(1-\frac{a(t)}{t})|\theta|^\alpha} |\theta| |k|+\alpha d\theta ,
\]

which due to sublinearity of $\rho$ immediately implies the first assertion in (3.8). For $I_3$, putting $a_n = e^n$ and utilizing the Borel-Cantelli lemma, we merely need to show

\[
\sum_{n \geq 1} \mathbb{P}_m \left( \sup_{a_n t \leq a_{n+1}} t^{\frac{d+|k|}{\alpha}} |I_3| \right)^2 < \infty . \tag{3.9}
\]
Applying Lemma 3.1, we have
\[
\mathbb{P}_m \left( \sup_{a_n \leq t \leq a_{n+1}} t^{-\frac{d+[k]}{n}} |I_3| \right)^2 \lesssim \left( \frac{a_{n+1}}{a_n} \right)^{d+[k]} \int_{\mathbb{R}^d} e^{-\frac{(a_n-a_{n+1})}{|\theta|^\alpha}} |\theta|^{2\lambda d} \left( a_{n+1}^{-2\lambda d/\alpha} + a_{n+1}^{-1} |\theta|^\alpha \right) |\theta|^{[k]} d\theta.
\]
By Jensen’s inequality, we have
\[
\mathbb{P}_m \left( \sup_{a_n \leq t \leq a_{n+1}} t^{-\frac{d+[k]}{n}} |I_3| \right)^2 \lesssim \left( \frac{a_{n+1}}{a_n} \right)^{d+[k]} \int_{\mathbb{R}^d} e^{-\frac{(a_n-a_{n+1})}{|\theta|^\alpha}} |\theta|^\alpha |\theta|^{k} d\theta.
\]
Applying Proposition 3.2, we see that
\[
\mathbb{P}_m \left( \sup_{a_n \leq t \leq a_{n+1}} t^{-\frac{d+[k]}{n}} |I_3| \right)^2 \lesssim \left( \frac{a_{n+1}}{a_n} \right)^{d+[k]} \int_{\mathbb{R}^d} e^{-\frac{(a_n-a_{n+1})}{|\theta|^\alpha}} (a_{n+1}^{-2\lambda d/\alpha} + a_{n+1}^{-1} |\theta|^\alpha) |\theta|^{k} d\theta.
\]
Observing that \( \frac{a_{n+1}}{a_n} = e, \lim_n (\frac{a_{n+1}}{a_n} - \frac{a_{n+1}}{a_{n+1}}) = e^{-1} \) and \( \sum_n a_n^{-\epsilon} < \infty \) for any \( \epsilon > 0 \), the above estimate implies (3.9).

Finally, combining (3.5), (3.6), (3.7) and (3.8) yields
\[
\lim_{t \to \infty} t^{-\frac{d+[k]}{a}} W_{\rho(t)}(\partial^k p_{t-\rho(t)}) = \left( \frac{i^{[k]}}{(2\pi)^d} \right) \int_{\mathbb{R}^d} e^{-|\theta|^\alpha} \theta^k d\theta \left( m(1) + \tilde{M}_\infty(1) \right). \]
The equality (3.3) follows from the above relation and (3.2), after observing that \( X_{\rho(t)}(\partial^k p_{t-\rho(t)}) \) is a real number.

\[\square\]

4. Proof of the main result

We begin with an asymptotic expansion of \( T_t \) as \( t \to \infty \). If \( k = (k_1, \ldots, k_d) \in \mathbb{N}^d \) is a multi-index and \( f \) is a sufficiently smooth test function, we define \( \partial^k f = \partial^1_{k_1} \partial^2_{k_2} \cdots \partial^d_{k_d} f \). The following semigroup expansion is proved in [KL17, Proposition 3.2].

**Proposition 4.1** (Semigroup expansion). Let \( f \) be a measurable function on \( \mathbb{R}^d \) and \( N \) be a non-negative integer such that (1.4) holds. Then, we have
\[
\lim_{t \to \infty} \sum_{x \in \mathbb{R}^d} \left| T_t f(x) - \sum_{k \in \mathbb{N}^d, |k| \leq N} \frac{(-1)^{|k|}}{k!} \int_{\mathbb{R}^d} f(y) y^k dy \partial^k p_t(x) \right| = 0. \tag{4.1}
\]
Remark 4.2. For semigroups with discrete spectra such as the Ornstein-Uhlenbeck semigroup, similar asymptotic expansions can be obtained via spectral decompositions. Although the \(\alpha\)-stable semigroup does not belong to this class, such expansion can be obtained using Taylor’s expansion. We refer to [KL17] for a proof of the above result.

Set \(t_n = n^\delta\) for some \(\delta \in (0, 1)\) sufficiently small so that
\[
\delta \frac{N + d}{\alpha} + \delta < 1. \tag{4.2}
\]

We first show that the sequence \(\{t_n\}_{n} \) determines the long-time asymptotic of \(\tilde{W}_t\).

Lemma 4.3. For every \(f \in \mathcal{D}(-(-\Delta)^{\frac{\alpha}{2}})\), we have
\[
\lim_{n} \sup_{t \in [t_n, t_{n+1}]} |t \frac{d N}{\alpha} \tilde{W}_t(f) - t_n \frac{d N}{\alpha} \tilde{W}_{t_n}(f)| = 0 \quad \mathbb{P}_m\text{-a.s.} \tag{4.3}
\]

Proof. We observe that
\[
\sup_{t \in [t_n, t_{n+1}]} |t \frac{d N}{\alpha} \tilde{W}_t(f) - t_n \frac{d N}{\alpha} \tilde{W}_{t_n}(f)| \leq J_1 + J_2 + J_3,
\]
where
\[
J_1 = \sup_{t \in [t_n, t_{n+1}]} t \frac{N + d}{\alpha} |\tilde{W}_t(f) - \tilde{W}_t(T_{t_{n+1}-t} f)|,
\]
\[
J_2 = \sup_{t \in [t_n, t_{n+1}]} t \frac{N + d}{\alpha} |\tilde{W}_t(T_{t_{n+1}-t} f) - \tilde{W}_{t_n}(T_{t_{n+1}-t_n} f)|,
\]
\[
J_3 = \sup_{t \in [t_n, t_{n+1}]} t \frac{N + d}{\alpha} |\tilde{W}_{t_n}(T_{t_{n+1}-t_n} f) - t_n \frac{N + d}{\alpha} \tilde{W}_{t_n}(f)|.
\]

Hence, it suffices to show \(\lim_n J_1 = \lim_n J_2 = \lim_n J_3 = 0\) almost surely. Indeed, we have
\[
J_1 \leq \sup_{t \in [t_n, t_{n+1}]} t \frac{N + d}{\alpha} \|T_{t_{n+1}-t} f - f\|_\infty \sup_{t \in [t_n, t_{n+1}]} \tilde{W}_t(1).
\]

Since \(f \in \mathcal{D}(-(-\Delta)^{\frac{\alpha}{2}})\), we see that
\[
\sup_{t \in [t_n, t_{n+1}]} t \frac{N + d}{\alpha} \|T_{t_{n+1}-t} f - f\|_\infty \lesssim t \frac{N + d}{\alpha} |t_{n+1} - t_n|, \tag{4.4}
\]
which converges to 0 because of the range of \(\delta\) in (4.2). In addition,
\[
\lim_{n} \sup_{t \in [t_n, t_{n+1}]} \tilde{W}_t(1) \leq \lim_{t \to \infty} \tilde{W}_t(1) = \tilde{W}_\infty(1). \tag{4.5}
\]
Hence, \( \lim_n J_1 = 0 \) almost surely. For \( J_2 \), we observe from (2.7) that for every \( t \in [t_n, t_{n+1}] \),
\[
\tilde{W}_t(T_{t_{n+1}-t} f) - \tilde{W}_{t_n}(T_{t_{n+1}-t_n} f) = \int_{t_n}^t \int_{\mathbb{R}^d} T_{t_{n+1}-r} f(y) d\tilde{M}(r, y).
\]

Fixing \( \varepsilon > 0 \) and applying martingale maximal inequality as well as (2.5) and (2.8), we have
\[
\mathbb{P}_m(J_2 > \varepsilon) \leq \mathbb{P}_m \left(\sup_{t \in [t_n, t_{n+1}]} \left| \int_{t_n}^t \int_{\mathbb{R}^d} T_{t_{n+1}-r} f(y) d\tilde{M}(r, y) \right| > \varepsilon \right)
\leq e^{-2} t_{n+1}^{\frac{N+2d}{n}} \mathbb{P}_m \left( \int_{t_n}^t \int_{\mathbb{R}^d} T_{t_{n+1}-r} f(y) d\tilde{M}(r, y) \right)^2
\leq e^{-2} t_{n+1}^{\frac{N+2d}{n}} \int_{t_n}^t \langle T_r ((T_{t_{n+1}-r} f)^2), m \rangle e^{-\beta_r} dr.
\]

As in the proof of Lemma 2.1, an application of Jensen’s inequality gives
\[
\langle T_r ((T_{t_{n+1}-r} f)^2), m \rangle \leq \langle T_r (f^2), m \rangle \leq m(1) \| f \|_\infty^2.
\]

It follows that \( \mathbb{P}_m(J_2 > \varepsilon) \lesssim e^{-2} t_{n+1}^{\frac{N+2d}{n}} e^{-\beta_t} \) and, hence \( \sum_n \mathbb{P}_m(J_2 > \varepsilon) < \infty \). Applying Borel-Cantelli lemma, we find that \( \lim_n J_2 = 0 \) almost surely. \( J_3 \) can be treated analogously as \( J_1 \). Indeed, we have
\[
J_3 \leq \sup_{t \in [t_n, t_{n+1}]} \| t \frac{N+2d}{n} T_{t_{n+1}-t_n} f - t \frac{N+2d}{n} f \|_\infty \sup_{t \in [t_n, t_{n+1}]} \tilde{W}_t(1).
\]

By triangle inequality, we see that \( \sup_{t \in [t_n, t_{n+1}]} \| t \frac{N+2d}{n} T_{t_{n+1}-t_n} f - t \frac{N+2d}{n} f \|_\infty \) is at most
\[
\sup_{t \in [t_n, t_{n+1}]} \frac{N+2d}{n} \| T_{t_{n+1}-t_n} f - f \|_\infty + \left( \frac{N+2d}{n} - \frac{N+2d}{n+1} \right) \| f \|_\infty,
\]
which converges to 0 by (4.4) and (4.2). In conjunction with (4.5), these estimates imply that \( \lim_n J_3 = 0 \) almost surely. \( \square \)

**Proof of Theorem 1.1.** We are going to obtain the limit (1.5) along the sequence \( \{t_n\}_n \).

We put \( \rho(t) = \sqrt{t} \) and
\[
L_t f = \sum_{k \in \mathbb{N}^d, |k| \leq N} (-1)^{|k|} a_d^k(f) \partial^k p_t \quad \forall t \geq 0.
\]

We will show that
\[
\lim_{n} t_n^{-\frac{N+2d}{n}} |\tilde{W}_{t_n}(f) - \tilde{W}_{\rho(t_n)}(L_{t_n-\rho(t_n)} f)| = 0 \quad \mathbb{P}_m\text{-a.s.}
\]
From Lemma 2.1, we see that
\[
\mathbb{P}_m \left[ \left( \tilde{W}_{t_n}(f) - \tilde{W}_{\rho(t_n)}(T_{t_n-\rho(t_n)} f) \right)^2 \right] \lesssim e^{-\beta \rho(t_n)}.
\]
This implies that
\[
\sum_n \mathbb{P}_m \left[ 2^{N+d} \left( \tilde{W}_{t_n}(f) - \tilde{W}_{\rho(t_n)}(T_{t_n-\rho(t_n)} f) \right)^2 \right] \lesssim \sum_n 2^{N+d} e^{-\beta \rho(t_n)} < \infty.
\]
An application of Borel-Cantelli lemma yields
\[
\lim_n t_n^{N+d} \left| \tilde{W}_{t_n}(f) - \tilde{W}_{\rho(t_n)}(T_{t_n-\rho(t_n)} f) \right| = 0 \quad \mathbb{P}_m\text{-a.s.} \tag{4.8}
\]
In addition,
\[
\frac{t_n^{N+d}}{\rho(t_n)} \left| \tilde{W}_{\rho(t_n)}(T_{t_n-\rho(t_n)} f) - \tilde{W}_{\rho(t_n)}(L_{t_n-\rho(t_n)} f) \right| \leq t_n^{N+d} \left| T_{t_n-\rho(t_n)} f - L_{t_n-\rho(t_n)} f \right| \mathbb{P}(\tilde{W}_{\rho(t_n)}(1)).
\]
We note that \( \lim_n \tilde{W}_{\rho(t_n)}(1) = \tilde{W}_\infty(1) \) and \( \lim_n \frac{t_n^{N+d}}{\rho(t_n)} \left| T_{t_n-\rho(t_n)} f - L_{t_n-\rho(t_n)} f \right| \mathbb{P}(\tilde{W}_{\rho(t_n)}(1)) = 0 \) (by (4.1)). It follows that
\[
\lim_n t_n^{N+d} \left| \tilde{W}_{\rho(t_n)}(T_{t_n-\rho(t_n)} f) - \tilde{W}_{\rho(t_n)}(L_{t_n-\rho(t_n)} f) \right| = 0 \quad \mathbb{P}_m\text{-a.s.},
\]
which together with (4.8) implies (4.7). More precisely, we have shown
\[
\lim_n t_n^{N+d} \left| \tilde{W}_{t_n}(f) - \sum_{k \in \mathbb{N}^d, |k| \leq N} (-1)^{|k|} \lambda_d^k (f) \tilde{W}_{\rho(t_n)}(\partial^k p_{t_n-\rho(t_n)}) \right| = 0 \quad \mathbb{P}_m\text{-a.s.}
\]
Applying Proposition 3.3, we see that for every \( k \in \mathbb{N}^d \),
\[
\lim_n t_n^{\frac{d+|k|}{a}} \tilde{W}_{\rho(t_n)}(\partial^k p_{t_n-\rho(t_n)}) = \begin{cases} 
0 & \text{if } |k| \text{ is odd} \\
(-1)^{|k|/2} \partial_d^k \tilde{W}_{\infty}(1) & \text{if } |k| \text{ is even}.
\end{cases}
\]
Combining these limits together yields
\[
\lim_n t_n^{N+d} \left| \tilde{W}_{t_n}(f) - \tilde{W}_\infty(1) \sum_{k \in \mathbb{N}^d, |k| \leq N} (-1)^{|k|/2} t_n^{\frac{d+|k|}{a}} \partial_d^k \lambda_d^k (f) \right| = 0.
\]
Applying Lemma 4.3, we find that the above limit implies (1.5). \qed
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