DARBOUX TRANSFORMATION AND EXACT SOLITONIC SOLUTIONS OF INTEGRABLE COUPLED NONLINEAR WAVE EQUATION

IRFAN MAHMOOD AND HIRA SOHAIL

Abstract. In this article, we construct the Darboux solutions of integrable coupled nonlinear wave equation associated with Hirota Satsuma system in Darboux framework with their N-th generalization in terms of Wronskians through its Lax pair. We also derive the exact solitonic solutions for the coupled field variables of that system with the help of one and twofold Darboux transformations in the background of zero seed solution. This work also encloses the derivation of zero curvature representation for the integrable coupled nonlinear waves equation possessing traceless matrices through its existed Lax pair, which may be assumed to t in AKNS scheme as it usually involves the order 2 traceless matrices.

1. Introduction

The subsequent coupled nonlinear wave equation

\[
\begin{align*}
\left\{
\begin{array}{l}
u_t &= u_{xxx} + 6u u_x + 2 v v_x \\
v_t &= 2(u v)_x 
\end{array}
\right.
\] (1)

arises in the study of shallow-water waves while investigating the dynamical properties of long internal waves in the background of interactions [1] that coupled wave equation has also been found as lower-order reduction of integrable Ito system [2]. Moreover, the coupled nonlinear wave equation has earned much importance in integrable systems because it entails the following coupled KdV equation.

\[
\begin{align*}
\left\{
\begin{array}{l}
u_t &= a(u_{xxx} + 6u u_x) + 2 b v v_x \\
v_t &= -v_{xxx} - 3 u v_x 
\end{array}
\right.
\] (2)

and reduces to the actual KdV equation by setting the coupling variable \( v = 0 \). The CNW equation (1) is found integrable because it possesses the Lax representation [3] and also owns the hierarchy of higher-order equations with associated conserved quantities besides the Hamiltonian structure [2]. One of the fruitful additions to the above properties is its bilinear form with exact multisoliton solutions [4] through the Hirota bilinear approach, which may enhance its significance in the theory of integrable systems. Recently the number of distinct properties that systems have been explored [6] in framework
of complete discrimination system for polynomial method (CDSPM) \[7,8\] such as its explicit solutions with topological structure together with varieties of parametric solutions. In this paper, we make use of the Darboux transformation method \[DT\] \[9\] to explore the various integrable aspects of the CNW equation \[1\]. The Darboux transformation (DT) \[10\] has been acknowledged as one of the efficient tools in theory of integrable systems to explore the various algebraic and geometrical properties of nonlinear field equations with their solitonic solutions. The numerous successful implementations of DT in various physics and applied mathematics domains ensure its importance from the application point of view. Among the number of remarkable applications of DT few of them can be found in the analysis of electrodynamical features \[12\] in quantum cavity problems and in investigating geometrical properties of graphene \[13\] with exact solutions. Moreover, that method has also been applied fruitfully to construct the quasideterminant solutions of the Painlevé II equation \[14\] with its related Toda system \[15\] for its non-commutative analog and to construct the exact solutions of the generalized coupled dispersionless integrable system \[16\]. Here we derive the Darboux solution of CNW equation \[1\] in the Darboux framework and further imply calculating its two-fold Darboux expression. Moreover, by applying the successive iteration, we express the \(N\)-fold Darboux transformation to the determinantal form as the ratio of Wronskians. Subsequently, we construct the exact solitonic solutions, the one soliton, and two solitons, with interaction pictures in the background of zero seed solutions for both coupled field variables. This work also encloses the derivation of zero curvature representation for the integrable coupled nonlinear waves equation \[1\] possessing traceless matrices through its existed Lax pair, which may be assumed to fit in the AKNS scheme \[17\], as it usually involves the order 2 traceless matrices.

2. The Darboux solutions

In this section we construct the one and two fold Darboux transformations for field variable \(u\) associated to CNW equation \[1\] through its Lax pair. For that purpose let us pursue with subsequent Lax pair \[18\], systems of linear differential equations

\[
\psi_{xx} = (\lambda - u - \frac{\psi^2}{4\lambda})\psi 
\]

(3)

\[
\psi_t = (4\lambda + 2u)\psi_x - u_x\psi 
\]

(4)

further this can be shown that the compatibility condition of above system yields (NCW) equation \[6\] and here \(\lambda\) is spectral parameter.

Let consider equation \[3\] of above Lax pair which can be re-expressed under the Darboux Transformation \[9\] on arbitrary function \(\psi\)
\[
\psi[1] = \left( \frac{d}{dx} - \sigma_1 \right) \psi = \psi_x - \frac{\psi_{1x}}{\psi_1} \psi \tag{5}
\]
in following form

\[
\psi_{xx}[1] = (\lambda - u[1] - \frac{\psi^2[1]}{4\lambda}) \psi[1] \tag{6}
\]

where \(\psi_1\) is the particular solution of Lax system at \(\lambda = \lambda_1\). After substituting the the values for \(\psi[1]\) and \(\psi_{xx}[1]\) into (6) from transformation (5) then making some simplification by using the lax pair (3) and (4), we obtain the Darboux transformation for \(u\) in subsequent form

\[
u[1] = u + 2\sigma_{1x}. \tag{7}\]

The above result represents one fold Darboux transformation on field variable \(u\), here \(u[1]\) is the new solution presented in terms of seed solution \(\sigma_{1x} = \frac{d}{dx} \psi_1 x \psi_1\).

The second iteration on one fold DT (7) yields two fold Darboux solution of NCW equation, let the equation under next iteration

\[
u[2] = u[1] + 2\frac{d^2}{dx^2} \log \psi_2[1] \tag{8}\]

and the two step transformation on \(\psi\) can be defined from (5) as below

\[
\psi[2] = \left( \frac{d}{dx} - \frac{\psi_{2x}[1]}{\psi_2[1]} \right) \psi[1] \tag{9}
\]

\[
\psi[2] = \left( \frac{d}{dx} - \frac{\psi_{2x}[1]}{\psi_2[1]} \right) \left( \frac{d}{dx} - \frac{\psi_{1x}}{\psi_1} \right) \psi \tag{10}\]

again with the help of one fold transformation (5) on \(\psi\) one can construct the two fold transformation on \(\psi\) explicitly as below involving ratio of Wronskians

\[
\psi[2] = \left( \frac{d}{dx} - \sigma_1 \right) \psi_2 = \psi_{2x} - \frac{\psi_{1x}}{\psi_1} \psi_2 = \frac{W(\psi_1, \psi_2, \psi)}{W(\psi_1, \psi_2)} \tag{11}\]

At the moment this seems easy that we can calculate two fold solution \(u[2]\) in terms of old variables in following form

\[
u[2] = u + 2\frac{d^2}{dx^2} \log W(\psi_1, \psi_2). \tag{12}\]

Similarly, in the same way we may continue and N-fold Darboux transformations can be evaluated for \(\psi\) and for \(u\) in following forms respectively in terms of Wronskians

\[
\psi[N] = \frac{W(\psi_1, \psi_2, ..., \psi_N, \psi)}{W(\psi_1, \psi_2, ..., \psi_N)} \tag{13}\]

and
\[ u[N] = u + 2 \frac{d^2}{dx^2} \log W(\psi_1, \psi_2, \ldots, \psi_N) \quad (14) \]

where

\[
W(\psi_1, \psi_2, \ldots, \psi_N) = \begin{vmatrix}
\psi_1 & \psi_2 & \cdots & \psi_N \\
\psi_1^{(1)} & \psi_2^{(1)} & \cdots & \psi_N^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_1^{(n-1)} & \psi_2^{(n-1)} & \cdots & \psi_N^{(n-1)}
\end{vmatrix} \quad (15)
\]

in above expression \( \psi_j^{(i)} \) stands for \( i \)-th derivative of \( \psi_j \) with respect to \( x \) as \( \psi_j^{(i)} = \frac{d^i \psi_j}{dx^i} \).

### 3. Exact solitonic solutions

In this section, we work out exact solitonic solutions to CNW equation (1) through its one fold and two fold Darboux transformation in background of zero seed solution.

#### 3.1. One soliton solutions

Now the one fold Darboux transformation (7) with trivial solution \( u = 0 \) will take the following form

\[
u[1] = 2 \frac{d^2}{dx^2} \ln \psi_1 \quad (16)
\]

where as we can compute the value for \( \psi_1 \) from linear system (3) at \( \lambda = \lambda_1 \) which is given by

\[
\psi_1(x, t) = 2 \cosh(k_1 x + 4k_1 \lambda_1 t) \quad (17)
\]

Now after substituting the above value into second last expression (16) we get

\[
u[1] = 2 \frac{d^2}{dx^2} \ln [2 \cosh(k_1 x + 4k_1 \lambda_1 t)] \quad (18)
\]

and then doing some simplification we end with following exact one soliton solution

\[
u[1] = 2k_1^2 \text{sech}^2(k_1 x + 4k_1 \lambda_1 t) \quad (19)
\]

Finally the wave profile of one soliton solutions in one dimension as well as in case of three dimension can be shown as respectively in following figure.
Figure 1. (a) represents the one dimensional dynamics of one soliton solution where as in (b) its three dimensional profile has been shown.

Figure 2. represents the contour plot of one soliton solution

3.2. Two soliton solutions. Now the two fold Darboux transformation \([12]\) with trivial solution \(u=0\) will take the following form

\[
u[2] = 2 \frac{d^2}{dx^2} \ln W(\psi_1, \psi_2) \tag{20}\]

where

\[
W(\psi_1, \psi_2) = \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1^1 & \psi_2^1 \end{vmatrix} \tag{21}\]

we can compute the value for \(\psi_1\) and \(\psi_2\) from the linear system \([3]\) at \(\lambda = \lambda_1\) and \(\lambda = \lambda_2\) respectively which is given by

\[
\psi_1(x, t) = 2 \cosh (k_1 x + 4 k_1 \lambda_1 t) \tag{22}\]
\[ \psi_2(x, t) = 2 \sinh (k_2 x + 4k_2 \lambda_2 t) \] (23)

Now after substituting the above values into equation (20) we get

\[ u[2] = 2 \frac{d^2}{dx^2} \ln \left| \frac{\psi_1}{d\psi_1/dx} \frac{\psi_2}{d\psi_2/dx} \right| \] (24)

\[ u[2] = \frac{d^2}{dx^2} \ln \begin{vmatrix} 2 \cosh(k_1 x + 4k_1 \lambda_1 t) & 2 \sinh(k_2 x + 4k_2 \lambda_2 t) \\ 2k_1 \sinh(k_1 x + 4k_1 \lambda_1 t) & 2k_2 \cosh(k_2 x + 4k_2 \lambda_2 t) \end{vmatrix} \] (25)

and it can be seen immediately that the transformation of \( u \) is equal to

\[ u[2] = 4(k^2_2 - k^2_1) \left[ \frac{k^2_2 \cosh(2\gamma_1) + k^2_1 \cosh(2\gamma_2) + k^2_2 - k^2_1}{((k_2 - k_1) \cosh(\gamma_1 + \gamma_2) + (k_2 + k_1) \cosh(\gamma_1 - \gamma_2))^2} \right] \] (26)

where

\[ \gamma_1 = k_1 x + 4k_1 \lambda_1 t \] (27)

\[ \gamma_2 = k_2 x + 4k_2 \lambda_2 t \] (28)

**Figure 3.** (a) represents the one dimensional dynamics of two soliton solution before interaction where as in (b) one dimensional dynamics at the time of interaction has been shown.
Figure 4. represents the one dimensional dynamics of two soliton solution after interaction.
Figure 5. (a) represents the three dimensional dynamics of two soliton solution where as in (b) contour plot of two soliton solution has been shown.

4. Exact Solutions to coupled field variable $v$

In this section we work out the exact solitonic solutions for the coupled field variable $v$ converting on of coupled equation $v_t = 2(uv)_x$ to the ordinary from through the transformation converting

$$v(\eta) = v(x - \alpha t) \quad (29)$$

where $\alpha$ is a parameter, plays the role of velocity of travelling wave. Now one may calculate the following expression

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} = -\alpha \frac{\partial}{\partial \eta} \quad (30)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \eta} \quad (31)$$
to substitute into equation (1) and resulting equation yields its ordinary analogue

\[- \alpha \frac{dv}{d\eta} = 2 \frac{d}{d\eta}(uv). \quad (32)\]

Now integrating above equation with respect to \( \eta \) we obtain expression for \( v \) in terms of \( u \) in subsequent explicit form

\[ v = \frac{-A}{\alpha + 2u} \quad (33) \]

where \( A \) is constant of integration taking equal to unity and we may generate the all exact solutions for \( v \) with the help of existing soliton solutions of \( u \) as shown in below sections.

### 4.1. One soliton solution.

Let \( u \) with one soliton solution \( u[1] \) to construct an exact solution for \( v \) through the equation \( (19) \), now after substitution value of \( u[1] \) and taking \( \alpha = 4k_1^2 \), the above relation \( (33) \) yields first exact solution for \( v \) respectively with its contour form

\[ v[1] = \frac{1}{4k_1^2 + 4k_1^2 \text{sech}^2(k_1x + 4k_1\lambda_1t)} \quad (34) \]

with wave profile in \( 1 - D \) as well as in \( 3 - D \) shown below in following figures

Figure 6. (a) represents the one dimensional dynamics of one soliton solution where as in (b) its three dimension profile has been shown.
Figure 7. represents the contour plot of one soliton solution of $v$.

4.2. Two soliton solution. Similarly the second iteration $v[2] = -\frac{A}{\alpha + 2u[2]}$ on (33) produces the second exact solution with $\alpha = 8(k_2^2 - k_1^2)$ after simplification in succeeding form

$$v[2] = \frac{1}{8(k_2^2 - k_1^2)} \left[ 1 + \frac{k_2^2 \cosh(2\gamma_1) + k_1^2 \cosh(2\gamma_2) + k_2^2 - k_1^2}{[(k_2 - k_1) \cosh(\gamma_1 + \gamma_2) + (k_2 + k_1) \cosh(\gamma_1 - \gamma_2)]^2} \right]$$ (35)

The interaction pictures of above two solitons in $1-D$ and $3-D$ along with their contour analogue shown below respectively

- Figure 8. (a) represents the one dimensional dynamics of two soliton solution before interaction where as in (b) one dimensional dynamics at the time of interaction has been shown.
5. Reduction of Lax pair to Zero Curvature form

In this section, we construct zero curvature representation of CNW equation (1) for that purpose let commence our computation with subsequent system differential equations

\[
\begin{align*}
\frac{\partial}{\partial x} F &= UF \\
\frac{\partial}{\partial t} F &= VF
\end{align*}
\]  

(36)

whose compatibility condition yields zero curvature form

\[
\frac{\partial}{\partial t} U - \frac{\partial}{\partial x} V + [U, V] = 0
\]

(37)
here $F$ is arbitrary column vector whereas $U$ and $V$ are the traceless matrices of order 2 with $[U, V] = UV - VU$. Now we investigate the explicit expression for $U$ and $V$ which reproduce the CNW equation (1), let introduce a column vector as

$$F = (f_0, f_1)^T = (\psi, \psi_x)^T$$

(38)

$\psi$ in above setting satisfies Lax pair (3) and (4), now the system (36) can be written as

$$\partial_x \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = U \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}$$

(39)

and

$$\partial_t \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = V \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}$$

(40)

Now after doing some computations on incorporating the Lax pair (3) and (4) in matrix form, one may derive the explicit expressions for $U$ and $V$ as below Next we consider

$$U = \begin{bmatrix} 0 & 1 \\ \lambda - u - \frac{v^2}{4\lambda} & 0 \end{bmatrix}$$

(41)

$$V = \begin{bmatrix} -u_x \\ 4\lambda^2 - 2\lambda u - v^2 - 2u^2 - \frac{uv^2}{2\lambda} \\ u_x \\ (4\lambda + 2u) \end{bmatrix}$$

(42)

This is straightforward to show the the compatibility condition of system (36) yields the coupled nonlinear wave equation (1) and moreover the matrices $U$ and $V$ seem to be considered as in AKNS scheme.

6. Conclusion

In this article, we have derived the Darboux transformations for coupled field variables of coupled nonlinear waves equation associated to Hirota Satsuma system and then we generalized their Darboux solutions to $N$-th form in terms of Wronskians. We have also calculated the exact solitonic solutions through its one and two fold Darboux transformations in background of zero seed solutions. Finally we have introduced zero curvature representation for that coupled nonlinear waves equation with traceless matrices through its existed Lax pair. That zero curvature representation may be used to investigate the symmetries along with conserved quantities associated to that integrable coupled nonlinear wave equation and also fruitful to find its discrete analogue through the method [19] proposed by Ablowitz and Ladik.

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REFERENCES

[1] B. A. Kupershmidt, A coupled Korteweg-de Vries equation with dispersion, Journal of Physics A: Mathematical and General, vol. 18, 10, 571, (1985)
[2] M. Ito, Symmetries and conservation laws of a coupled nonlinear wave equation, Physics Letters A, 91, 7, 335–338 (1982)
[3] L. Debnath, Nonlinear partial differential equations for scientists and engineers, Springer, (2005)
[4] H. Zhao, Soliton solution of a multi-component higher-order Ito equation, Applied Mathematics Letters, vol. 26 7 681–686 (2013)
[5] C. Guha-Roy, Solutions of coupled KdV-type equations, International journal of theoretical physics, Vol.26, 8, 863–866, (1990)
[6] Y. Kai, J. Ji and Z. Yin, Exact solutions and dynamic properties of Ito-Type coupled nonlinear wave equations, Physics Letters A, 127780, (2021)
[7] L. Cheng-Shi, All single traveling wave solutions to (3+ 1)-dimensional Nizhnik–Novikov–Veselov equation, Communications in theoretical physics, vol.45, 6, 991–992, (2006)
[8] L. Cheng-Shi, Classification of all single travelling wave solutions to Calogero–Degasperis–Focas equation, Communications in Theoretical Physics, vol.48, 4,601, (2007)
[9] V. B. Matveev, VB, Darboux transformations and solitons, Springer, (1991)
[10] C. Gu, Chaohao, H. Hu, Hesheng, A. Hu and Z. Zhou, Darboux transformations in integrable systems: theory and their applications to geometry, Springer, (2004)
[11] A. Trisetyarso, Application of Darboux Transformation to solve Multisoliton Solution on Non-linear Schrödinger Equation, arXiv preprint arXiv:0910.0901, (2009)
[12] A. Trisetyarso, Erratum: “Correlation of Dirac potentials and atomic inversion in cavity quantum electrodynamics” J. Math. Phys. 51, 072103 (2010)
[13] A. Trisetyarso, Quantum Information and Computation 12, 989 (2012).
[14] I. Mahmood, Lax pair representation and Darboux transformation of noncommutative Painlevé’s second equation, Journal of Geometry and Physics, vol.62, 7, 1575–1582, (2012)
[15] I. Mahmood, Quasideterminant solutions of NC Painlevé II equation with the Toda solution at n= 1 as a seed solution in its Darboux transformation, Journal of Geometry and Physics, vol.95,127–136 (2015)
[16] M. Hassan, Darboux transformation of the generalized coupled dispersionless integrable system, Journal of Physics A: Mathematical and Theoretical, vol.42, 6,065203, (2009)
[17] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, “The inverse scattering transform - Fourier analysis for non-linear problems,” Stud. Appl. Math. 53, 249–315 (1974).
[18] A. B. Shabat, V. E. Adler, V.G. Marikhin, A.A. Mikhailov and V.V. Sokolov, Encyclopedia of integrable systems, Landau Institute for Theoretical Physics, vol.43, (2010)
[19] M. J. Ablowitz and J. F. Ladik, “Nonlinear differential– difference equations,” J. Math. Phys. 16, 508–603 (1975).