MATRIX UNITS IN THE SYMMETRIC GROUP ALGEBRA, AND
UNITARY INTEGRATION

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Abstract. In this paper, we establish an explicit isomorphism between the
symmetric group algebra \( \mathbb{C}[S_d] \) and the path algebra of the Young graph \( \mathcal{Y}_d \), by ex-
pressing the matrix elements \( E^\lambda_{T,S} \) as a linear combination of group elements.

We then investigate applications of this result. As a main application, we obtain
new formulas, alternative to Weingarten calculus, to integrate polynomials with
respect to the Haar measure on the unitary group. In particular, we obtain a
closed formula for the law of moments of the first \( k \) rows of the unitary group \( U_n \),
uniform in \( n \geq k \).

1. Introduction

A fundamental result from representation theory is that the group algebra of a
finite group \( G \) is isomorphic to a matrix algebra (see [CST10]). In particular, a
group algebra is spanned by a family of matrix units \( E^\lambda_{T,S} \) where \( \lambda \in \hat{G} \)
and for which the multiplication follows the normal rules:

\[
E^\lambda_{T,S} = \delta_{S,R} \delta_{\lambda,\beta} E^\lambda_{T,M}.
\]

It is a difficult problem in general to compute these matrix elements in terms of the
abstract elements of the group algebra. In this paper, we provide such a formula for
the symmetric group algebra, and discuss some applications of these formula.

Our first main result can be stated as follows: the elements \( E^\lambda_{T,S} := E^\lambda_{T,S} \sigma_{S,T} E^\lambda_{T,S} \)
form, up to a non-zero multiplicative factor, a set of matrix units for the group
algebra \( \mathbb{C}[S_d] \), where \( T \) and \( S \) denote standard fillings of the Young diagram \( \lambda \), \( E^\lambda_T \)
is a shorthand notation for \( E^\lambda_{TT} \), and \( \sigma_{S,T} \) is the unique permutation transforming
\( S \) into \( T \). Using available formulas for \( E^\lambda_T \) ([CST10], Theorem 3.4.11) we obtain
explicit formulas for the matrix units as linear combinations of permutations. Similar
formulas have been obtained in [RW92] in a more general setup, in a recursive way,
rather than directly as a combination of group elements.

We will discuss two applications of these formulas. The first result concerns formulas
for inclusion of matrix units \( E^\lambda_{T,S} \in \mathbb{C}[S_d] \subset \mathbb{C}[S_{d+1}] \), and conversely formulas for
the projection of matrix units in \( \mathbb{C}[S_{d+1}] \) onto \( \mathbb{C}[S_d] \). Not surprisingly, these formulas
depend of the structure of the Young graph, which encodes complete information on
the irreducible representations of the symmetric groups of all order.

A second and more important byproduct is an application to the theory of inte-
gration over the unitary group with respect to the Haar measure.

An explicit formula for the integral of a polynomial against the Haar measure
has remained elusive until quite recently (see [Co03, CS06]). We propose here a
completely new method for performing this task. Our main result can be stated as
follows:

Theorem 1. For all \( d \)-tuples of indices \( I = (i_1, \ldots, i_d), J = (j_1, \ldots, j_d), K =
(k_1, \ldots, k_d), L = (l_1, \ldots, l_d) \), the following formula holds true:

\[
\int_{U_n} u_{i_1,j_1} \cdots u_{i_d,j_d} \bar{u}_{k_1,l_1} \cdots \bar{u}_{k_d,l_d} d\mu(U) = \sum_{\lambda \vdash d; S,T \in \text{Stab}(\lambda), l(\lambda) \leq n} \frac{\langle e_{J,L}, E_{S,T}^\lambda \rangle \langle E_{S,T}^\lambda, e_{I,K} \rangle}{||E_{S,T}^\lambda||^2},
\]

where \( E_{S,T}^\lambda \) are multiples of matrix units in \( \mathbb{C}_n[S_d] \subset M_n^{\otimes d} \), as defined in Theorem 6,
and \( e_{I,J} = e_{i_1,j_1} \otimes \cdots \otimes e_{i_d,j_d} \) are the standard matrix units in \( M_n^{\otimes d} \).

The paper is organized as follows: in section 2 we begin with a review of the
necessary background concerning Young diagrams, the Young graph \( \Upsilon \), and the rep-
resentation theory of the symmetric groups \( S_d, d \geq 1 \).

In section 3 we derive explicit formulas for the matrix units \( E_{T,S}^\lambda \in \mathbb{C}[S_d] \), and
describe the behaviour of these matrix units with respect to projection and inclusion
between symmetric groups of different orders.

Finally, in section 4 we show how these matrix formulas can be used in the
calculation of polynomial integrals over the unitary group \( U_n \subset GL_n(\mathbb{C}) \) with respect
to the Haar measure \( \mu = \mu_n \).

2. Reminder: Young Diagrams and Irreducible \( S_d \) Modules

In order to construct matrix elements for \( \mathbb{C}[S_d] \), we first need to recall the con-
struction of irreducible \( S_d \) modules. We follow the lines of Vershik and Okounkov in
[VO05], which contains all the details of this section. The representation theory
of the symmetric groups relies primarily on the combinatorial structure of Young
diagrams.

A Young diagram \( \lambda \) is defined as a finite non-increasing sequence of integers \( \lambda_1 \geq
\ldots \geq \lambda_k > 0 \). The size of the diagram \( \lambda \) is given by \( |\lambda| = \sum_i \lambda_i \), and we use the
notation \( \lambda \vdash d \) to mean \( \lambda \) is a Young diagram of size \( d \). The number \( k \) is known
as the length of \( \lambda \) and is denoted \( l(\lambda) \). Young diagrams have a useful graphical
representation as a collection of boxes, with three conventional representations shown
below (English, French, and Russian). For example, the graphical representation of
the diagram \( \lambda = (4, 3, 3, 2, 1) \) is shown below.
In what follows, we adopt the English convention for drawing Young diagrams. The Young graph \( \mathbb{Y} \) (the first 4 levels of which is shown below in figure 2) is the infinite directed graph whose vertex set is the set of all Young diagrams, and where a diagram of size \( d \) is connected to one of size \( d+1 \) if the two differ by exactly one box. The truncated Young graph \( \mathbb{Y}_d \) consists only of the first \( d \) levels of \( \mathbb{Y} \). Given any diagram \( \lambda \), we can consider the set \( \text{Stab}(\lambda) \) of all paths in the Young graph starting at the unique block of size 1 and ending at \( \lambda \). Equivalently, such a path in the Young graph corresponds in a natural way to a filling of the boxes of \( \lambda \) with the numbers \( 1, \ldots, d \) so that the numbers are increasing along every row and column of the diagram. Such a filling is called standard, which explains the usage of \( \text{Stab}(\lambda) \), denoting the set of standard tableaux. To each standard filling \( T \) of the diagram \( \lambda \) of size \( d \), we can associate the content vector \( c(T) = (a_1(T), \ldots, a_d(T)) \), where \( a_i(T) \) is the difference between the \( x \) and \( y \) coordinates of the \( i^{th} \) box added according to the filling \( T \). This vector encodes important information about the representations of \( S_d \) which we will discuss later.

Using the Young graph we can construct all irreducible \( S_d \) modules as follows: let \( V_\lambda \) be the free complex vector space with orthonormal basis consisting of elements \( w_T \) where \( T \in \text{Stab}(\lambda) \). We define the action of \( S_d \) on this space by reducing to the case of Coxeter transpositions \( s_i = (i,i+1) \), which generate the symmetric group \( S_d \). If \( s_i \) is a Coxeter transposition, and \( T \in \text{Stab}(\lambda) \), we consider the following two cases:

1. If the filling \( s_i T \) (which is obtained by replacing \( k \) with \( s_i(k) \) in the filled diagram \( \lambda \)) is not standard, define \( s_i(w_T) \) to be \( w_T \) if \( i \) and \( i+1 \) lie in the same row of \( T \), and \( -w_T \) if they lie in the same column of \( T \).
2. If the filling \( S = s_i T \) is also standard, define \( s_i(w_T) = \frac{1}{r}w_T + \sqrt{1 - \frac{1}{r^2}}w_S \) and \( s_i(w_S) = \sqrt{1 - \frac{1}{r^2}}w_T - \frac{1}{r}w_S \), where \( r \) is the axial distance defined by \( r = r_i(T) = a_{i+1}(T) - a_i(T) \).
The following result can be found in [VO05].

**Theorem 2.** Indexing over all partitions $\lambda$ of $d$ boxes, the set of $V_\lambda$ defined above constitute a full set of pairwise non isomorphic irreducible representations of $S_d$. Hence, we have a $\ast$-algebra isomorphism $\mathbb{C}[S_d] = \bigoplus_{\lambda \vdash d} \text{End}(V_\lambda)$.

The combinatorial structure of Young diagrams is equivalent to the algebraic structure of the chain $\mathbb{C}[S_1] \subset \mathbb{C}[S_2] \subset \mathbb{C}[S_3] \subset \ldots$ of symmetric group algebras. This algebraic structure is encoded in the so called *Bratteli diagram* of the chain of algebras. This is the directed graph whose vertices consist of all isomorphism classes of irreducible $S_d$ modules, for all integers $d \geq 1$. We connect two isomorphism classes $\lambda$ of $S_d$ and $\beta$ of $S_{d+1}$ by exactly $k$ directed edges (from $\lambda$ to $\beta$) when the multiplicity of $\lambda$ in the restriction of $\beta$ to $S_d$ is $k$. The following is a summary of the important properties of the Bratteli diagram of the symmetric groups, and can be found in [VO05]:

**Theorem 3 ([VO05]).** The Bratteli diagram of the symmetric groups is (graph theoretically) isomorphic to the Young graph, whose vertices are Young diagrams of size $d$ (for all natural numbers $d \geq 1$), and whose edges are determined by the inclusion of Young diagrams of size $d$ into Young diagrams of size $d + 1$.

We obtain a canonical basis, called the *Young Basis* of an irreducible $S_d$ module $V_\lambda$ as follows: for each path $T$ in the Young graph (Bratteli diagram) from the
unique $S_1$ module $\lambda_1$ to $\lambda$, say $T = \lambda_1 \to \lambda_2 \to \ldots \to \lambda$, choose a vector $v_T$ in $V_\lambda$ such that $v_T \in V_{\lambda_i}$ for all $i$. Doing so for each path in the Young graph gives a basis indexed by paths in the Young graph from $\lambda_1$ to $\lambda$, or equivalently, indexed by standard fillings of the diagram $\lambda$. The simplicity of the Bratteli diagram ensures that this basis is uniquely determined up to scalar multiplication. One of the most important properties of this basis is its connection to the Gelfand Tsetlin algebra, the subalgebra $\text{GZ}(d) \subset \mathbb{C}[S_d]$ generated by the centers $Z(\mathbb{C}[S_1]), \ldots, Z(\mathbb{C}[S_d])$. Because of the simplicity of the Bratteli diagram, this subalgebra contains all projections from irreducible $S_k + 1$ modules to irreducible $S_k$ modules. Hence this subalgebra contains the maximally abelian subalgebra of operators diagonal on the Young basis. Being abelian itself, this implies that the GZ algebra consists entirely of operators diagonal on the Young basis.

In order to study the structure of the GZ-algebra, we consider the elements $X_i \in \mathbb{C}[S_d]$ defined as $X_1 = 0, X_2 = (1, 2), X_3 = (1, 3) + (2, 3), \ldots$. These are called the Young-Jucys Murphy elements of the group algebra $\mathbb{C}[S_d]$, and their spectrum on the Young basis is the key to the analysis of the Bratteli diagram described above. The following facts summarize the relationship between $\text{GZ}(d)$, the Young basis, and the Jucys Murphy elements, and can be found in [VO05]:

1. $\text{GZ}(d)$ is the maximally abelian subalgebra of $\mathbb{C}[S_d]$ consisting of those elements whose Fourier transform is a diagonal operator with respect to the Young basis.
2. $\text{GZ}(d)$ is generated by the elements $X_1, X_2, \ldots, X_d$.
3. The Young basis of a representation $V_\lambda$ is completely determined by the eigenvalues of $X_1, \ldots, X_d$ acting on it.

For any Young vector $v_T \in V_\lambda$, denote the eigenvalues of $X_1, X_2, \ldots, X_d$ on $v_T$ by the vector $c(T) = (a_1(T), \ldots, a_d(T))$. The set of all such vectors is called Spec$(d)$ in [VO05]. It can be shown that for a path $T$ in the Young graph, $c(T)$ is in fact the content vector of the standard filling associated to $T$. This is a highly important and nontrivial fact, and a complete proof can be found in [VO05] or [CST10].

Because the Young basis is unique up to scalar multiplication, it is possible (see [VO05]) to choose normalized coefficients such that the Coxeter generators act on the normalized basis $\{w_T | T \in \text{Stab}(\lambda)\}$ according the rules (1) and (2) from the beginning of this section.

Given the decomposition $\mathbb{C}[S_d] \cong \bigoplus_{\lambda \vdash d} \text{End}(V_\lambda)$, we obtain a set of matrix units $E^\lambda_{T,S}$ for $\mathbb{C}[S_d]$. In particular, $E^\lambda_{T,S}$ is the operator defined by $E^\lambda_{T,S}w_R = \delta_{R,S}w_T$ where $R$ is another standard filling of $\lambda$. We will denote the minimal projection $E^\lambda_{T,T}$ simply by $E^\lambda_T$. By our choice of basis, these elements lie in $\text{GZ}(d)$, and so they can be written as polynomials in the YJM elements. In the next section we will discuss
the polynomials $p^\lambda_T$ such that $E^\lambda_T = p^\lambda_T(X_1, ..., X_d)$, and we will use these to obtain explicit formulas for $E^\lambda_{T,S}$ as elements of $\mathbb{C}[S_d]$.

3. Matrix Units in $\mathbb{C}[S_d]$

3.1. Minimal Projections. Before giving formulas for the general matrix unit, $E^\lambda_{T,S}$, we will first recall the formulas for the minimal projections $E^\lambda_T \in \mathbb{C}[S_d]$ as polynomials in the Young Jucys Murphy elements.

If $T \in \text{Stab}(\lambda)$ is a standard filling of $\lambda$, denote by $\overline{T}$ the standard tableau of $d - 1$ boxes obtained by removing the box containing $d$ from $T$, and denote by $\overline{\lambda}$ the shape of $\overline{T}$.

**Proposition 4.** Let $T$ and $\overline{T}$ be as above. Then as a polynomial in the YJM elements, we have

$$E^\lambda_T = E^\overline{\lambda}_\overline{T} \left( \prod_{S \neq T, S \in T} \frac{(a_d(S) - X_d)}{(a_d(S) - a_d(T))} \right).$$

This result can be found in [CST10], and follows from the fact that the right hand side behaves as a minimal projection should, in that it maps $v_T$ to $v_T$, and maps all other basis vectors to zero. This formula is similar in spirit to the formula for the spectral projections of a self adjoint operator $A \in M_n(\mathbb{C})$ with a simple spectrum $\lambda_i \neq \lambda_j$, $i \neq j$ (i.e. no repeated eigenvalues):

$$E_{\lambda_i} = \prod_{i \neq j} \frac{(A - \lambda_j \text{Id})}{(\lambda_i - \lambda_j)}.$$

As a simple example, consider the path $T$ shown in figure 2. Using proposition 4, we have $E^\lambda_T = E^\overline{\lambda}_{\overline{T}} \left( \frac{1}{4} (2 - X_4)(2 + X_4) \right)$. Repeating for $E^\overline{\lambda}_{\overline{T}}$ we have $E^\overline{\lambda}_{\overline{T}} = E^\overline{\lambda}_{\overline{T}} \frac{1}{3} (2 + X_3)$. Finally $E^\overline{\lambda}_{\overline{T}} = \frac{1}{2} (1 - X_2)$. This gives $E^\lambda_T = \frac{1}{6} (2 + X_4)(2 - X_4)(2 + X_3)(1 - X_2)$.

From a computational point of view, specifically for calculating unitary integrals, we are free to use multiples of minimal projections:

$$\tilde{E}_{\lambda_i} = \prod_{i \neq j} (A - \lambda_j \text{Id}),$$

which are computationally less expensive. However, for all integration formulas we present in this paper, we will assume that the minimal projections have been normalized as in proposition 4.
3.2. Main result. In this section we present explicit formulas for the matrix units $E^\lambda_{T,S}$. Since the formulas rely on the pointwise action of the symmetric group on fillings of a given tableau $\lambda$, we recall that if $T$ is a filling of $\lambda$ and $\sigma \in S_d$, then $\sigma T$ is the filling of $\lambda$ obtained by replacing $i$ in $T$ by $\sigma(i)$. This defines a transitive action of $S_d$ on the set of fillings of a given diagram $\lambda$.

We start with the following:

**Proposition 5.** Let $T$ and $S$ be standard fillings of the diagram $\lambda$, and suppose $T = s_i S$. Then $E^\lambda_{T,S} = \sqrt{\frac{r^2}{r^2 - 1}}(E^\lambda_{T,s_i}E^\lambda_S)$, where $r$ is the $i$th axial distance of the filling $T$.

*Proof.* Using formula (2) from page 3, we see that the right hand side of the equality sends $w_S$ to $w_T$ and sends all other basic elements to zero. Thus it must be $E^\lambda_{T,S}$. □

In order to generalize the previous result to arbitrary standard fillings $T$ and $S$ of a diagram $\lambda$, we need the notion of an *admissible transposition* for the diagram $T$.

A Coxeter generator $s_i = (i, i+1)$ is called *admissible* for $T$ if the filling $S = s_i T$ is also standard. Any two standard fillings of the same diagram can be transformed into one another by a sequence of admissible Coxeter generators. The minimal number of admissible Coxeter transpositions required is called the *Coxeter distance between $T$ and $S$*, and will be denoted by $d(T, S)$. Given a diagram $\lambda$, the Coxeter distance $d$ is a well defined metric on the set of paths $\text{Stab}(\lambda)$.

**Theorem 6.** Let $T, S \in \text{Stab}(\lambda)$. Let $\sigma$ be the permutation sending $T$ to $S$. Then, there exists $c \neq 0$ such that $E^\lambda_{T,S} = cE^\lambda_T\sigma^{-1}E^\lambda_S$.

*Proof.* The case when $T$ and $S$ differ by a single Coxeter generator was proven above, so we shall prove the result by induction on the Coxeter distance $d(T, S)$. If said distance is $k + 1$ we can send $T$ to $S$ via $R \in \text{Stab}(\lambda)$, where $d(T, R) = k$ and $d(R, S) = 1$. Suppose that $T$ is sent to $R$ with $\sigma$ and $R$ is sent to $S$ with $s_i$. The following calculation gives the desired result:

$$E^\lambda_T\sigma^{-1}s_iE^\lambda_S = E^\lambda_T\sigma^{-1}\text{Id}s_iE^\lambda_S = E^\lambda_T\sigma^{-1}E^\lambda_Rs_iE^\lambda_S + \sum_{L \neq R} E^\lambda_T\sigma^{-1}E^\lambda_Ls_iE^\lambda_S$$

$$= E^\lambda_T\sigma^{-1}E^\lambda_Rs_iE^\lambda_S + E^\lambda_T\sigma^{-1}E^\lambda_Ss_iE^\lambda_S$$

Both summands are proportional to $E^\lambda_{T,S}$, but it is possible the whole sum is zero. We must prove that $E^\lambda_T(\sigma^{-1}w_S) = 0$. However, since $\sigma^{-1}$ can, by assumption, be written as the product of $k$ Coxeter generators, we have that $\sigma^{-1}w_S$ lies in the span...
of those $w_Q$ with $d(S,Q) \leq k$. Since, in particular, $d(S,T) = k + 1$ we have that $E_T^\lambda \sigma^{-1} w_S = 0$ and so the theorem holds. □

The non-zero scalar $c$ actually has a closed form which is a clear generalization of proposition 5. Given the permutation $\sigma$ above, decompose it into the minimal number of Coxeter generators $\sigma = s_{i_k} \ldots s_{i_1}$, so that $T \to T_1 \to \cdots \to S$ is transformed into $S$ via the generators $s_{i_1}, \ldots, s_{i_k}$. Let $r_m = a_{i_m+1}(T_m) - a_{i_m}(T_m)$.

**Proposition 7.** The non-zero scalar $c$ appearing in Theorem 6 is given by $\sqrt{\prod_i (\frac{r_i^2}{r_i^2 - 1})}$.

**Remark**
While it is possible to have two different minimal decompositions of a given permutation into admissible transpositions, the constant in the previous proposition does not depend on the particular decomposition. We are unaware of a direct combinatorial proof of this fact.

For the integration formulas in section 4, we will ignore the nonzero scalar $c$ and simply use the orthogonal (but non-normalized) basis $E_T^\lambda \sigma S T E_S^\lambda$ of $\mathbb{C}[S_d]$.

### 3.3. Comparison with known Formulas.
In [RW92], the authors construct a family of matrix units for the type $A$ Hecke algebras $H_d(q)$, which gives the symmetric group algebra $\mathbb{C}[S_d]$ when $q = 1$. In particular, these algebras have generators $g_1, \ldots, g_{d-1}$ satisfying the relations

1. $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for all $1 \leq i \leq d - 2$
2. $g_i g_j = g_j g_i$ for all $|i - j| \geq 2$
3. $g_i^2 = (q - 1) g_i + q$ for $1 \leq i \leq d - 1$.

For almost all $q$ this algebra is isomorphic to $\mathbb{C}[S_d]$, replacing $g_i$ with $(i, i + 1)$. The construction of the matrix units given in [RW92] are more general than our construction, as they apply to $H_d(q)$ for all $q, d$. However, they are computationally far more expensive than the formulas we present in this section. Indeed, in order to construct $E_T^\lambda S \sigma T E_S^\lambda$ we need only construct $E_T, E_S$ and then multiply by the permutation $\sigma$ presented in this section. The presentation in [RW92] cannot rely on any such element $\sigma_{ST}$.

Actually, the formulas in [RW92] have a normalizing constant in front, but for our purposes (in particular for calculating unitary integrals in the next section) the normalizing constants are not needed, only the orthogonality of the matrix units. The formulas in this section and in [RW92] of course give the same results, but as we are concerned with unitary integrals (and hence with the symmetric group algebra) and with computation, our formulas are an improvement over those in [RW92], as they require fewer applications of induction on the size of the diagram $\lambda$. 
3.4. **Inclusion Rules for** $\mathbb{C}[S_d] \subset \mathbb{C}[S_{d+1}]$. Having formulas for all $E_{T, S}^\lambda$ in $\mathbb{C}[S_d]$ leads to a natural question involving the inclusion of $\mathbb{C}[S_d] \subset \mathbb{C}[S_{d+1}]$. Recall that $S_d \subset S_{d+1}$, where a permutation of $d$ elements is viewed as a permutation of $d + 1$ elements fixing the $(d + 1)^{st}$. This extends to an inclusion of $\ast$-algebras $\mathbb{C}[S_d] \subset \mathbb{C}[S_{d+1}]$.

Given a Young diagram $\lambda \vdash d$, and given $T, S \in \text{Stab}(\lambda)$, we have an element $E_{T, S}^\lambda$ in $\mathbb{C}[S_d]$. As an element of $\mathbb{C}[S_{d+1}]$, there is a unique decomposition

$$E_{T, S}^\lambda = \sum_{\beta \vdash d+1} \sum_{R, M \in \text{Stab}(\beta)} \alpha_{T, S}^\beta(R, M) E_{R, M}^\beta,$$

where $\alpha_{T, S}^\beta(R, M)$ are complex numbers depending on $\beta, T, S, R,$ and $M$. We would like a characterization of these coefficients.

**Theorem 8.** Let $\lambda \vdash d$ and $T \in \text{Stab}(\lambda)$. Then as an element of $\mathbb{C}[S_{d+1}]$, $E_{T}^\lambda$ is projection onto the subspace $\bigoplus S \mathbb{C}w_S$ spanned by all $w_S$ such that $S = T$. Hence,

$$E_{T}^\lambda = \sum S E_{S}^\beta,$$

where we sum over all $S$ such that $S = T$ and $\bar{\beta} = \lambda$.

**Proof.** We need to verify that if $S = T$, then $E_{T}^\lambda(w_S) = w_S$, otherwise $E_{T}^\lambda(w_S) = 0$. Write $T = \lambda_1 \to \lambda_2 \to \ldots \to \lambda_d = \lambda$, and let $T_i$ be the path $\lambda_1 \to \ldots \to \lambda_i$. Then we can write

$$E_{T}^\lambda = \prod_{i=1}^{d} \prod_{S, T_S \neq T_i} \frac{(a_i(S) - X_i)}{(a_i(S) - a_i(T_i))}$$

If $S = T$ then $a_i(T) = a_i(S)$ for each $i = 1, ..., d$, and hence the polynomial above will send $w_S$ to $w_S$ as required. If $S \neq T$ then the polynomial above will send $w_S$ to 0 by construction. 

**Lemma 9.** Suppose $T, S \in \text{Stab}(\lambda)$, where $\lambda \vdash d$. Then, as an element of $\mathbb{C}[S_{d+1}]$,

$$E_{T, S}^\lambda = \sum_{R, M} E_{R, M}^\beta,$$

where we sum over all $R$ and $M$ of the same shape $\beta$ such that $R = T$ and $M = S$.

**Proof.** We begin with the case when $T$ and $S$ differ by a Coxeter transposition $s_i$. Whenever $M = S$, the right hand side of the equation above sends $w_M$ to $w_R$ for the unique $R$ of the same shape as $M$ with $R = T$. Otherwise, $w_M$ is sent to 0. Hence we need to verify that the left hand side does the same. We can right $E_{T, S}^\lambda$ as $c E_{T}^\lambda s_i E_{S}^\lambda$ where $c$ is the nonzero constant from proposition 5. Since $s_i S = T$ implies $s_i M = R$,
we have that $cE_\lambda^T s_i E_\lambda^S(w_M) = w_R$ when we are in the first situation above, otherwise $cE_\lambda^T s_i E_\lambda^S(w_M) = 0$, as required.

For the general case, we proceed by induction on the Coxeter distance between $T$ and $S$. The case when $d(T, S) = 1$ was proven above. Now suppose $d(T, S) = k + 1$. Choose $R$ such that $d(T, R) = k$ and $d(R, S) = 1$. We have $E_\lambda^T s_i E_\lambda^S(w_M) = w_R$ when we are in the first situation above, otherwise $cE_\lambda^T s_i E_\lambda^S(w_M) = 0$, as required.

For the general case, we proceed by induction on the Coxeter distance between $T$ and $S$. The case when $d(T, S) = 1$ was proven above. Now suppose $d(T, S) = k + 1$. Choose $R$ such that $d(T, R) = k$ and $d(R, S) = 1$. We have $E_\lambda^T s_i E_\lambda^S(w_M) = w_R$ when we are in the first situation above, otherwise $cE_\lambda^T s_i E_\lambda^S(w_M) = 0$, as required.

□

The dual question to that of the inclusion of matrix units $E_\lambda^T s_i E_\lambda^S \in \mathbb{C}[S_d] \subset \mathbb{C}[S_{d+1}]$ is the following: how do the matrix units $E_\lambda^T s_i E_\lambda^S$ decompose upon restriction from $\mathbb{C}[S_{d+1}]$ to $\mathbb{C}[S_d]$. To be more precise, we define the map $E : \mathbb{C}[S_{d+1}] \to \mathbb{C}[S_d]$ by stipulating that for $\sigma \in S_{d+1}$, $E(\sigma) = \sigma$ if $\sigma \in S_d$ and 0 otherwise. This map is just projection from $\mathbb{C}[S_{d+1}]$ to $\mathbb{C}[S_d]$, which we call the conditional expectation. Our question then is the following: for $E_\lambda^T s_i E_\lambda^S \in \mathbb{C}[S_{d+1}]$, what are the coefficients of $E(\sigma) = \sum_{\beta \vdash d} \sum_{R,M \in \text{Stab}(\beta)} \alpha_\beta T,S(R, M) E_\beta^{R,M}$ as an element of $\mathbb{C}[S_d]$?

**Theorem 10.** If $T$ and $S$ have the same shape $\beta$, then $E(\sigma) = \frac{\dim(V_\beta)}{\dim(V_\lambda)} E_\beta^{T,S}$. Otherwise, $E(\sigma) = 0$.

The proof of this, and a more general result can be found in [RW92], where the author looks at the setting of a chain $A_1 \subset A_2 \subset \ldots$ of finite dimensional, semi-simple complex algebras whose Bratteli diagram has simple branching rules. As the symmetric group algebras form such a chain, the formulas in [RW92] apply. However, they are computationally far more complex, and the proofs don’t take advantage of the combinatorial structure of the Bratteli diagram that the results in [VO05] afford.

4. Application to Unitary matrix integrals

4.1. Algebraic preliminaries. In this section, we are interested in the following problem. Let $\mu$ be the normalized Haar measure on the unitary group $U_n$ and $u_{ij} : U_n \to \mathbb{C}$ the $ij$ coordinate map. We are interested in computing all moments of $\mu$, or equivalently, all integrals

$$\int_{U_n} u_{i_1j_1} \cdots u_{i_dj_d} \pi_{k_1l_1} \cdots \pi_{k_dl_d} d\mu(U)$$

Let us start with the following tensor reformulation:

**Lemma 11.**

$$\int_{U_n} u_{i_1j_1} \cdots u_{i_dj_d} \pi_{k_1l_1} \cdots \pi_{k_dl_d} d\mu(U) = \int_{U_n} tr(e_{K,L} U^\otimes d e_{J,L} U^* \otimes d) d\mu(U)$$
In order to calculate these integrals explicitly, we use an algebraic result connecting the representation theory of the symmetric groups with those of the unitary groups. Recall that the vector space \((\mathbb{C}^n)^{\otimes d}\) is both a \(S_d\) module and a \(U_n\) module. The action of \(S_d\) is given by the homomorphism \(p_n^d : \mathbb{C}[S_d] \to \text{End}((\mathbb{C}^n)^{\otimes d})\), with \(p_n^d(\sigma)(v_1 \otimes \cdots \otimes v_d) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}\). The action of \(U_n\) is given by the homomorphism \(\rho : U_n \to \text{GL}((\mathbb{C}^n)^{\otimes d})\), with \(\rho(U)(v_1 \otimes \cdots \otimes v_d) = (Uv_1) \otimes \cdots \otimes (Uv_d)\). It is clear that these actions commute with one another, so that \((\mathbb{C}^n)^{\otimes d}\) is in fact a \(\mathbb{C}[S_d] \times U_n\) module with the following structure:

**Theorem 12** (Schur-Weyl Duality, [CST10], Theorem 8.2.10). The multiplicity free decomposition of \((\mathbb{C}^n)^{\otimes d}\) into irreducible \(S_d \times U_n\) modules is given by \(\bigoplus_{l(\lambda) \leq n} V_\lambda \otimes U^\lambda\), where \(U^\lambda = \text{End}_{S_d}(V_\lambda, (\mathbb{C}^n)^{\otimes d})\), and \(V_\lambda\) is the irreducible representation of \(S_d\) corresponding to the partition \(\lambda\) of \(d\).

Note that restricting to the subgroups \(1 \times U_n \cong U_n\) and \(S_d \times 1 \cong S_d\) gives the decomposition of \((\mathbb{C}^n)^{\otimes d}\) as an \(U_n\) and \(S_d\) module. In particular, the multiplicity of \(V_\lambda\) in \((\mathbb{C}^n)^{\otimes d}\) is zero for \(l(\lambda) > n\). This fact will be used later in calculating integrals over sub-rectangles in \(U_n\).

4.2. A reminder of Weingarten calculus. Before supplying a new integration formula, let us recall the existing integration technique, known as Weingarten calculus. The idea goes back to [Wei78].

**Definition**

Consider the set \(P(d)^U\) of pair partitions of \(\{1, \ldots, 2d\}\) linking an element \(< d\) with an element \(\geq d\).

1. We plug multi-indices \(i = (i_1, \ldots, i_{2d})\) into partitions \(p \in P(d)^U\), and we set \(\delta_{pi} = 1\) if all blocks of \(p\) contain equal indices of \(i\), and \(\delta_{pi} = 0\) if not.

2. The Gram matrix of partitions (of index \(n \geq 4\)) is given by \(G_{n,d}(p, q) = n|^{n|^{\vee q}|}\), where \(\vee\) is the set-theoretic sup, and \(|.|\) is the number of blocks.

3. The Weingarten matrix \(W_{n,d}\) is the inverse of \(G_{n,d}\).

We can view elements of \(P(d)^U\) as permutations in \(S_d\) acting on \(\text{End}((\mathbb{C}^n)^{\otimes d})\). The Gram matrix of this basis with respect to the scalar product induced by the non-normalized canonical trace is nothing but \(G_{k,n}\), as shown by the following computation:

\[
\langle p, q \rangle = \sum_i \delta_{pi} \delta_{qi} = \sum_i \delta_{p\vee q, i} = n|^{n|^{\vee q}|}
\]

With these notations, we have the following result.

**Theorem 13.** [Col03, CS06] The Haar functional is given by

\[
\int u_{i_1,j_1} \cdots u_{i_k,j_k} = \sum_{pq} \delta_{pi} \delta_{qj} W_{n,k}(p, q)
\]
where the sum is over all pairs of diagrams $p, q \in NC(k)$.

The original proof can be found in [Col03, C’S06], and simplified proofs are available in the more general quantum group setup [BC07a, BC07b, BC08]. See also [CM09].

### 4.3. A new integration formula.

Theorem 12 implies that the map $p^d_n$ is injective on the subalgebra $\mathbb{C}_n[S_d] \cong \bigoplus_{l(\lambda) \leq n} \text{End}(V_\lambda)$, and hence we will view $\mathbb{C}_n[S_d]$ as being contained in $\text{End}((\mathbb{C}^n)^{\otimes d})$ via the map $p^d_n$. Further, the subalgebra $\mathbb{C}_n[S_d] \cong \bigoplus_{l(\lambda) \leq n} \text{End}(V_\lambda)$ is contained in the centralizer $\text{End}_{U_n}((\mathbb{C}^n)^{\otimes d})$. Theorem 7 implies that we actually have equality, giving the following proposition:

**Proposition 14.** For any $A \in M_n^{\otimes d}$, we have

$$\int_{U_n} U^{\otimes d} A(U^{-1})^{\otimes d} d\mu(U) = \mathbb{E}(A)$$

where $\mathbb{E}$ is the orthogonal projection with respect to the Hilbert Schmidt norm of $M_n^{\otimes d}$ onto the sub algebra $\mathbb{C}_n[S_d] \subset \text{End}((\mathbb{C}^n)^{\otimes d})$.

Strictly speaking the projection above is orthogonal with respect to the $U_n$ invariant inner product on $M_n^{\otimes d}$, which in our case happens to be the Hilbert Schmidt inner product.

Proposition 14, together with the following elementary fact, will allow us to obtain specific formulas for polynomial integrals. The orthogonal projection above is normally referred to as conditional expectation.

**Proposition 15.**

$$\mathbb{E}(A) = \sum_{\lambda \vdash d; S, T \in \text{Stab}(\lambda), l(\lambda) \leq n} \langle A, E_{S, T} \rangle E_{S, T} / ||E_{S, T}||^2$$

Before proving the main result of this section, we recall the following fact concerning integrating a function $f : G \to V$, where $G$ is a compact topological group and $V$ is a Hilbert space:

$$\langle \int_G f(g) d\mu(g), w \rangle = \int_G \langle f(g), w \rangle d\mu(g)$$

**Theorem 16.** The following integration formula holds true

$$\int_{U_n} u_{i_1j_1} \cdots u_{i_dj_d} \bar{u}_{k_1l_1} \cdots \bar{u}_{k_dl_d} d\mu(U) = \sum_{\lambda \vdash d; S, T \in \text{Stab}(\lambda), l(\lambda) \leq n} \frac{\langle e_{I, K}, E^\lambda_{S, T} \rangle \langle E^\lambda_{S, T}, e_{I, L} \rangle}{||E^\lambda_{S, T}||^2}.$$

**Proof.** We begin with noting that

$$\int_U e_{K, I} U^{\otimes d} e_{I, L} U^{* \otimes d} d\mu(U) = \sum_M \langle \mathbb{E}(e_{I, L}), e_{I, M} \rangle e_{K, M}.$$
Indeed, \[ \int_U E_{K,I} U^\otimes d E_{I,L} U^* \otimes d \mu(U) = \sum_{R,M} \langle \int_U E_{K,I} U^\otimes d E_{I,L} U^* \otimes d \mu(U), E_{R,M} \rangle E_{R,M}, \]
and one notes that
\[
\langle \int_U e_{K,I} U^\otimes d e_{I,L} U^* \otimes d \mu(U), e_{R,M} \rangle = \langle \mathbb{E}(e_{I,L}), e_{I,M} \rangle \delta_{K,R}.
\]

Taking the trace gives the first equality. The second comes from decomposing \( \mathbb{E}(e_{I,L}) \) in terms of the basis \( E_{T,S}^\lambda \) from the previous section. \( \square \)

4.4. **Examples.** Now, we investigate a few examples.

First we start with \( \int |u_{1,1}|^2 d\mu(U) \). One has \( \int |u_{1,1}|^2 d\mu(U) = \langle \mathbb{E}(e_{1,1}), e_{1,1} \rangle = \langle \frac{1}{n} \text{Id}, e_{1,1} \rangle = \frac{1}{n}. \)

Next, we look at \( \int |u_{1,1}|^4 d\mu(U) \). Here, we have two matrix units in \( \mathbb{C}[S_2] \), namely \( E_T = \frac{1}{2} (1 + (1, 2)) \) and \( E_S = \frac{1}{2} (1 - (1, 2)) \).

In \( M_n \otimes M_n \), these elements are
\[
\frac{1}{2} (\text{Id} \otimes \text{Id} + \sum_{i,j} e_{i,j} \otimes e_{j,i}), \frac{1}{2} (\text{Id} \otimes \text{Id} - \sum_{i,j} e_{i,j} \otimes e_{j,i}).
\]

Therefore,
\[
\int |u_{1,1}|^4 d\mu(U) = \langle \mathbb{E}(e_{1,1} \otimes e_{1,1}), e_{1,1} \otimes e_{1,1} \rangle
\]
\[
= ||E_T||^{-2} \langle e_{1,1} \otimes e_{1,1}, E_T \rangle + ||E_S||^{-2} \langle e_{1,1} \otimes e_{1,1}, E_S \rangle
\]
\[
= ||E_T||^{-2} \langle e_{1,1} \otimes e_{1,1}, \frac{1}{2} (\text{Id} \otimes \text{Id} + \sum_{i,j} e_{i,j} \otimes e_{j,i}) \rangle +
\]
\[
||E_S||^{-2} \langle e_{1,1} \otimes e_{1,1}, \frac{1}{2} (\text{Id} \otimes \text{Id} - \sum_{i,j} e_{i,j} \otimes e_{j,i}) \rangle.
\]

Next we calculate the quantities \( ||E_T||^2 \) and \( ||E_S||^2 \):
\[ ||E_T||^2 = \frac{1}{4} \langle \text{Id} \otimes \text{Id} + \sum_{i,j} e_{i,j} \otimes e_{j,i}, \text{Id} \otimes \text{Id} + \sum_{i,j} e_{i,j} \otimes e_{j,i} \rangle \]
\[ = \frac{1}{4} \left( \langle \text{Id} \otimes \text{Id}, \text{Id} \otimes \text{Id} \rangle + 2 \langle \text{Id} \otimes \text{Id}, \sum_{i,j} e_{i,j} \otimes e_{j,i} \rangle + \right. \]
\[ \left. \sum_{i,j,k,l} \langle e_{i,j} \otimes e_{j,i}, e_{k,l} \otimes e_{l,k} \rangle \right) \]
\[ = \frac{1}{4} \left( n^2 + 2n + n^2 \right) = \frac{n(n+1)}{2} \]

\[ ||E_S||^2 = \frac{1}{4} \langle \text{Id} \otimes \text{Id} - \sum_{i,j} e_{i,j} \otimes e_{j,i}, \text{Id} \otimes \text{Id} - \sum_{i,j} e_{i,j} \otimes e_{j,i} \rangle \]
\[ = \frac{1}{4} \left( \langle \text{Id} \otimes \text{Id}, \text{Id} \otimes \text{Id} \rangle - 2 \langle \text{Id} \otimes \text{Id}, \sum_{i,j} e_{i,j} \otimes e_{j,i} \rangle + \right. \]
\[ \left. \sum_{i,j,k,l} \langle e_{i,j} \otimes e_{j,i}, e_{k,l} \otimes e_{l,k} \rangle \right) \]
\[ = \frac{1}{4} \left( n^2 - 2n + n^2 \right) = \frac{n(n-1)}{2} \]

A final calculation shows that
\[ \langle e_{1,1} \otimes e_{1,1}, \frac{1}{2} (\text{Id} \otimes \text{Id} + \sum_{i,j} e_{i,j} \otimes e_{j,i}) \rangle = \]
\[ \frac{1}{2} \left( \langle e_{1,1} \otimes e_{1,1}, \text{Id} \otimes \text{Id} \rangle + \langle e_{1,1} \otimes e_{1,1}, \sum_{i,j} e_{i,j} \otimes e_{j,i} \rangle \right) = \frac{1}{2} (1 + 1) = 1, \]
and similarly,
\[ \langle e_{1,1} \otimes e_{1,1}, \frac{1}{2} (\text{Id} \otimes \text{Id} - \sum_{i,j} e_{i,j} \otimes e_{j,i}) \rangle = \frac{1}{2} (1 - 1) = 0. \]

Hence we arrive at \( \int |u_{1,1}|^4 d\mu(U) = \frac{2}{n(n+1)}. \)

4.5. **Integrating over Corners.** Here, we consider the following problem: suppose than in the integrals from the previous section we assume that all indices \( I, J, K, L \in \{1, \ldots, k\}^d \) for \( k \leq n \). We can prove the following:
Theorem 17. Suppose that $I, J, K, L \in \{1, \ldots, k\}^d$. Then we have the following:

$$\int_{U_n} u_{i_1j_1} \ldots u_{i_dj_d} \overline{u_{k_1l_1}} \ldots \overline{u_{k_dl_d}} d\mu(U) = \sum_{\lambda \vdash L; S, T \in \text{Stab}(\lambda), l(\lambda) \leq \min\{k, d\}} \frac{\langle e_{I,L}, E^\lambda_{S,T} \rangle \langle E^\lambda_{S,T}, e_{I,K} \rangle}{||E^\lambda_{S,T}||^2}.$$ 

In other words, we need only sum over those $\lambda$ with $l(\lambda) \leq k$.

Recall that $p^d_n : \mathbb{C}[S_d] \to \text{End}((\mathbb{C}^n)^{\otimes d})$ is injective on $\mathbb{C}_n[S_d] := \bigoplus_{j(\lambda) \leq n} \text{End}(V_{\lambda})$, hence is injective when restricted to $\mathbb{C}_k[S_d]$ where $1 \leq k \leq n$. The integration question considered in this section can be reformulated in the following way: there is a natural embedding of $M_k^{\otimes d}$ in $M_n^{\otimes d}$ for $1 \leq k \leq n$, where we view a $k \times k$ matrix as the upper left corner of an $n \times n$ matrix, and extend this inclusion to tensors.

**Proof of Theorem 17.** We need to show that the image of $M_k^{\otimes d}$ under conditional expectation $E : M_n^{\otimes d} \to \mathbb{C}_n[S_d]$ is orthogonal to the sum of all $\text{End}(V_\lambda)$ where $l(\lambda) > k$. In particular, if $A \in M_k^{\otimes d}$ and $B \in \text{End}(V_{\lambda})$ for $l(\lambda) > k$ we have that $\langle A, B \rangle = 0$.

First note that the inclusion $i : (\mathbb{C}^k)^{\otimes d} \to (\mathbb{C}^n)^{\otimes d}$ commutes with the action of $S_d$, that is $p^d_n(\sigma)i(x) = i(p^d_n(\sigma)x)$ for $x \in (\mathbb{C}^k)^{\otimes d}$. From this it follows that $(\mathbb{C}^k)^{\otimes d}$ lies entirely in the irreducible component of $(\mathbb{C}^n)^{\otimes d}$ consisting of the $V_{\lambda}$ with $l(\lambda) \leq k$. In particular, if $x \in (\mathbb{C}^k)^{\otimes d}$ and $B \in \text{End}(V_{\lambda}) \subset M_k^{\otimes d}$ with $l(\lambda) > k$ we have $Bx = 0$. But the inner product $\langle A, B \rangle$ can be written as $\sum_{I} \langle Ae_I, Be_I \rangle$. For $e_I \in (\mathbb{C}^k)^{\otimes d}$ we have $Be_I = 0$. Further, for any $e_I$ not in $(\mathbb{C}^k)^{\otimes d}$ we have $Ae_I = 0$, since $A$ lies in the span of those $e_{J,K} \in M_k^{\otimes d}$. Hence every term in this sum is 0, and the inner product is 0 as required. This completes the proof.

The proof of theorem 17, together with the fact that $\langle E(e_{I,L}), e_{I,K} \rangle = \langle e_{I,L}, E(e_{I,K}) \rangle$ shows that in the integral $\langle E(E_{I,L}), E_{I,K} \rangle$ we need only sum over diagrams of length $k$ or less, where $k$ is the minimum of $\max\{j_1, \ldots, j_d, l_1, \ldots, l_d\}$ and $\max\{i_1, \ldots, i_d, k_1, \ldots, k_d\}$. In this way we have simplified formulas for integrating over not just upper left hand squares in $U_n$, but for integrating over upper left hand rectangles in $U_n$, in terms of the lengths of Young diagrams.

Note that in the orthogonal case, a formula was obtained in [BCS11]. While there is no overlap since the integration group is different, it is worth observing that this approach could be conducted to obtain a conceptual proof for the paper [BCS11].

Applying Theorem 17 to the case of one row, we obtain the following formula:

**Theorem 18.** Let $J = \{j_1, \ldots, j_d\}$ and $L = \{l_1, \ldots, l_d\}$ be arbitrary indices. Then the integral $\int_{U} u_{i_1j_1} \ldots u_{i_dj_d} \overline{u_{l_1k_1}} \ldots \overline{u_{l_dk_d}} d\mu(U)$ is equal to $\frac{k}{d||E_T||^2}$ where $T$ is the unique standard filling of $\lambda = (d)$ and $k$ is the number of permutations in $S_d$ mapping $J$ to $L$. 

Note that when \( d = 2 \) and \( J = L = (1, 1) \) we recover the formula from the calculation of \( \int_U |u_{1,1}|^4 d\mu(U) \) on the previous page.

Let us mention a related formula that was known for \( \int_U u_{1,j_1} \ldots u_{1,j_d} \bar{u}_{1,l_1} \ldots \bar{u}_{1,l_d} d\mu(U) \) (cf [Mat13], Proposition 2.4 for a direct proof with Weingarten calculus). Our proof in this paper is new and has the potential for generalization to more than one row. Let us outline below yet an other proof, of probabilistic nature:

First, observe that with probability one, the random vector \( (u_{1,1}, \ldots, u_{1,n}) \) has the same distribution as \( (X_1/(\sum_{i=1}^d |X_i|^2)^{1/2}, \ldots, X_d/(\sum_{i=1}^d |X_i|^2)^{1/2}) \), where \( (X_1, \ldots, X_d) \) are i.i.d. standard complex valued gaussians distributions. Further, \( \sum_{i=1}^n |X_i|^2 \) is independent from \( (X_1/\sum_{i=1}^d |X_i|^2, \ldots, X_d/\sum_{i=1}^d |X_i|^2) \). Putting this together, we obtain the following formula (cf [Mat13], Proposition 2.4):

\[
\int_U u_{1,j_1} \ldots u_{1,j_d} \bar{u}_{1,l_1} \ldots \bar{u}_{1,l_d} d\mu(U) = \frac{\prod_i r_i!}{n(n+1)\ldots(n+d-1)},
\]

where \( r_1, r_2 \ldots \) are the number of elements in the blocks induced by the partition \( i \to j_i \) assuming that these numbers are the same up to permutation, that is if \( i \to j_i \) is replaced by \( i \to l_i \). To make this more precise, we say that two \( d \)-indices \( K \) and \( J \) containing values in \( \{1, \ldots, n\} \) are *of the same type* if each index set contains the same number of occurrences of each of \( 1, 2, \ldots, n \). For example, the indices \( (1, 1, 2, 5, 5) \) and \( (5, 1, 2, 1, 5, 5) \) are of the same type, while \( (1, 1, 2, 5, 5) \) and \( (1, 2, 2, 5, 5) \) are not. Equivalently, \( I \) and \( J \) are of the same type iff one can be obtained by another by a permutation \( \sigma \in S_d \). This gives an equivalence relation of the set \( \{n\}^d \) of functions \( I : \{1, \ldots, d\} \to \{1, \ldots, n\} \). The proposition above says that if \( J \) and \( L \) lie in different equivalence classes, the integral \( \int_U u_{1,j_1} \ldots u_{1,j_d} \bar{u}_{1,l_1} \ldots \bar{u}_{1,l_d} d\mu(U) \) is zero, otherwise we get the formula above. Reformulating this using theorem 18 we get the following:

**Corollary 19.** If two indices \( J \) and \( L \) are not of the same type, then the integral \( \int_U u_{1,j_1} \ldots u_{1,j_d} \bar{u}_{1,l_1} \ldots \bar{u}_{1,l_d} d\mu(U) \) is zero.

Another formulation of the integral in Theorem 18 is that

\[
\int_U u_{1,j_1} \ldots u_{1,j_d} \bar{u}_{1,l_1} \ldots \bar{u}_{1,l_d} d\mu(U) = \frac{d! \sum_{\sigma, \beta} \delta_{\sigma, LL}}{\sum_{\sigma, \beta} \sum_{I} \delta_{\sigma I, \beta I}}.
\]

This follows by a simple calculation of \( |E_T|^2 = \langle E_T, E_T \rangle \) for \( E_T = \frac{1}{d!} \sum_{\sigma} \sum_{I} E_{\sigma I, I} \in M_n^{\otimes d} \).

Finally, note that a similar analysis could be performed for two rows or more. The notation to obtain a closed formula is already quite cumbersome at that level and will be studied elsewhere. Note that similar results were obtained in the orthogonal case by [BCST1].
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