MULTIPLIERS AND OPERATOR SPACE STRUCTURE OF WEAK PRODUCT SPACES

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Abstract. In the theory of reproducing kernel Hilbert spaces, weak product spaces generalize the notion of the Hardy space $H^1$. For complete Nevanlinna–Pick spaces $\mathcal{H}$, we characterize all multipliers of the weak product space $\mathcal{H} \odot \mathcal{H}$. In particular, we show that if $\mathcal{H}$ has the so-called column-row property, then the multipliers of $\mathcal{H}$ and of $\mathcal{H} \odot \mathcal{H}$ coincide. This result applies in particular to the classical Dirichlet space and to the Drury–Arveson space on a finite dimensional ball. As a key device, we exhibit a natural operator space structure on $\mathcal{H} \odot \mathcal{H}$, which enables the use of dilations of completely bounded maps.

1. Introduction

Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions on a set $X$. The weak product space is defined by

$$\mathcal{H} \odot \mathcal{H} = \left\{ h = \sum_{i=1}^{\infty} f_i g_i : \sum_{i=1}^{\infty} \| f_i \| \| g_i \| < \infty \right\},$$

with norm

$$\| h \|_{\mathcal{H} \odot \mathcal{H}} = \inf \left\{ \sum_{i=1}^{\infty} \| f_i \| \| g_i \| : h = \sum_{i=1}^{\infty} f_i g_i \right\}.$$ 

This is a Banach function space on $X$, meaning in particular that the functionals of evaluation at points in $X$ are continuous on $\mathcal{H} \odot \mathcal{H}$. If $\mathcal{H} = H^2(\mathbb{D})$, the Hardy space on the unit disc $\mathbb{D}$, then $\mathcal{H} \odot \mathcal{H} = H^1(\mathbb{D}) \odot H^1(\mathbb{D})$ with equality of norms. In fact, every function in $H^1(\mathbb{D})$ is a product of two functions in $H^2(\mathbb{D})$. The notion of a weak product space has its origins in a famous paper of Coifman, Rochberg and Weiss [7]. There, it is shown that the Hardy spaces and Bergman spaces on the unit ball $\mathbb{B}_d$ satisfy $H^1(\partial \mathbb{B}_d) = H^2(\partial \mathbb{B}_d) \odot H^2(\partial \mathbb{B}_d)$ and $L^1_a(\mathbb{B}_d) = L^2_a(\mathbb{B}_d) \odot L^2_a(\mathbb{B}_d)$, with equivalence of norms. In general, one can regard weak product spaces as a replacement for the Hardy space $H^1$ in the context of an arbitrary reproducing kernel Hilbert space. In this setting, weak product spaces are closely related to boundedness of Hankel forms;
see e.g. the introduction of [4]. Weak product spaces have been concretely studied for instance for the classical Dirichlet space [4, 13, 17], the Drury–Arveson space [18] and more generally for complete Nevanlinna–Pick spaces [3, 11].

If $B$ is a Banach function space on $X$, the multiplier algebra of $B$ is defined by

$$\text{Mult}(B) = \{ \varphi : X \to \mathbb{C} : \varphi \cdot f \in B \text{ for all } f \in B \}. $$

The closed graph theorem implies that for each $\varphi \in \text{Mult}(B)$, the associated multiplication operator $M_\varphi$ on $B$ is bounded, so we may define $\|\varphi\|_{\text{Mult}(B)} = \|M_\varphi\|_{B(B)}$.

It is immediate from the definition of the weak product space that $\text{Mult}(H) \subset \text{Mult}(H \circ H)$, with $\|\varphi\|_{\text{Mult}(H \circ H)} \leq \|\varphi\|_{\text{Mult}(H)}$ for all $\varphi \in \text{Mult}(H)$. On the other hand, for the classical Hardy space, it is not hard to see that both $\text{Mult}(H^1(D))$ and $\text{Mult}(H^2(D))$ agree with $H^\infty(D)$, with equality of norms. This naturally raises the following question.

**Question 1.1.** Is $\text{Mult}(H \circ H) = \text{Mult}(H)$?

This question was studied by Richter and Wick [19], who provided a positive answer for first order weighted Besov spaces on the unit ball using function theoretic estimates. In particular, their result applies to the classical Dirichlet space and to the Drury–Arveson space $H^2_d$ for $d \leq 3$, but not to $H^2_d$ for $d \geq 4$.

In this article, we characterize multipliers of $H \circ H$ for normalized complete Nevanlinna–Pick spaces and are thus able to give a positive answer to Question 1.1 in many instances. The prototypical example of a normalized complete Nevanlinna–Pick space is the Hardy space $H^2(D)$, but there are many other examples such as the classical Dirichlet space or the Drury–Arveson space $H^2_d$ for $d \in \mathbb{N} \cup \{\infty\}$. Normalized complete Nevanlinna–Pick spaces can be defined in terms of the validity of a suitable version of the Nevanlinna–Pick interpolation theorem. Equivalently, by results of McCullough, Quiggin and Agler–McCarty, a reproducing kernel Hilbert space is a normalized complete Nevanlinna–Pick space if and only if its reproducing kernel $K$ is of the form

$$K(z, w) = \frac{1}{1 - \langle b(z), b(w) \rangle} \quad (z, w \in X),$$

where $b$ maps $X$ into the open unit ball of an auxiliary Hilbert space and satisfies $b(z_0) = 0$ for some distinguished point $z_0 \in X$. This characterization and more background information can be found in [1].

A key ingredient in our analysis of $\text{Mult}(H \circ H)$ is the observation that $H \circ H$ can be equipped with a natural operator space structure, and we will use this additional structure crucially. Briefly, $H \circ H$ can be identified as the dual of the space of compact Hankel operators on $H$ (see [3, Section 2]), hence $H \circ H$ is the dual of an operator space and thus itself an operator space. For more details, see Section 2.

In particular, for each $n \in \mathbb{N}$, the space $M_n(H \circ H)$ of $n \times n$ matrices with entries in $H \circ H$ carries a natural norm. As is customary in operator space theory, for each $n \in \mathbb{N}$ one associates with a linear map $A : H \circ H \to H \circ H$ its amplification $A^{(n)} : M_n(H \circ H) \to M_n(H \circ H)$, defined by applying $A$ entrywise. The linear map $A$ is then said to be a complete contraction if each map $A^{(n)}$ is a contraction. In
particular, given a function \( \theta : X \to \mathbb{C} \), we say that \( \theta \) is a contractive multiplier of \( \mathcal{H} \odot \mathcal{H} \) if the multiplication operator \( M_\theta : \mathcal{H} \odot \mathcal{H} \to \mathcal{H} \odot \mathcal{H} \) is a contraction, and that \( \theta \) is a completely contractive multiplier of \( \mathcal{H} \odot \mathcal{H} \) if \( M_\theta \) is a complete contraction.

Let \( \mathcal{M}_1^C(\mathcal{H}) \) be the space of sequences of multipliers of \( \mathcal{H} \) that are contractive when viewed as a column operator on \( \mathcal{H} \). In other words, a sequence \( (\varphi_n)_{n=1}^\infty \) belongs to \( \mathcal{M}_1^C(\mathcal{H}) \) if and only if the operator

\[
\begin{bmatrix}
M_{\varphi_1} \\
M_{\varphi_2} \\
\vdots
\end{bmatrix} : \mathcal{H} \to \mathcal{H} \otimes \ell^2
\]

is contractive. Our main result can now be stated as follows.

**Theorem 1.2.** Let \( \mathcal{H} \) be a normalized complete Nevanlinna–Pick space on \( X \). The following assertions are equivalent for a function \( \theta : X \to \mathbb{C} \):

(i) The function \( \theta \) is a contractive multiplier of \( \mathcal{H} \odot \mathcal{H} \).

(ii) The function \( \theta \) is a completely contractive multiplier of \( \mathcal{H} \odot \mathcal{H} \).

(iii) There exist sequences \( (\varphi_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty \) in \( \mathcal{M}_1^C(\mathcal{H}) \) such that \( \theta = \sum_{n=1}^\infty \varphi_n \psi_n \).

In particular, it follows that the norm of an element \( \theta \in \text{Mult}(\mathcal{H} \odot \mathcal{H}) \) is given by

\[
\|\theta\|_{\text{Mult}(\mathcal{H} \odot \mathcal{H})} = \inf \left\{ \left\| \begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
\vdots
\end{bmatrix} \right\| : \theta = \sum_{n=1}^\infty \varphi_n \psi_n \right\},
\]

where the norms of the columns in the infimum are taken in \( \text{Mult}(\mathcal{H}, \mathcal{H} \otimes \ell^2) \). Moreover, the infimum is attained.

The proof of this theorem will be separated in several steps. The implication (iii) \( \Rightarrow \) (i) easily follows from the definition of \( \mathcal{H} \odot \mathcal{H} \) and already appears in [3, Theorem 3.1]. We provide the short argument in Proposition 3.1. The proof of the implication (i) \( \Rightarrow \) (ii) uses a recent result of Jury and Martin [11] and is presented in Proposition 3.5. The majority of the work occurs in the proof of the implication (ii) \( \Rightarrow \) (iii), which uses dilation theory and is done in Theorem 3.8.

While not logically necessary, we also provide a direct proof of the implication (iii) \( \Rightarrow \) (ii); this is done in Proposition 3.2. This proof shows how the operator space structure of \( \mathcal{H} \odot \mathcal{H} \) enters the picture and motivates our approach to the harder implication (ii) \( \Rightarrow \) (iii).

In many instances, Theorem 1.2 implies an affirmative answer to Question 1.1. The key property is the following. A normalized complete Nevanlinna–Pick space \( \mathcal{H} \) is said to satisfy the column-row property (with constant \( \kappa \)) if whenever \( (\varphi_n)_{n=1}^\infty \) is a sequence in \( \text{Mult}(\mathcal{H}) \) with

\[
\left\| \begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
\vdots
\end{bmatrix} \right\|_{\text{Mult}(\mathcal{H}, \mathcal{H} \otimes \ell^2)} \leq 1,
\]


then

\[ \left\| \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots \end{bmatrix} \right\|_{\Mult(H \otimes \ell^2, \mathcal{H})} \leq \kappa. \]

The classical Hardy space $H^2(\mathbb{D})$ satisfies the column-row property with constant 1, as the norm of a row and of a column of multipliers are both given by the supremum norm over $\mathbb{D}$. It is a result of Trent [21] that the classical Dirichlet space satisfies the column-row property with constant at most $\sqrt{18}$. The Drury–Arveson space $H^2_d$ with $d < \infty$ also satisfies the column-row property with some finite constant, which possibly depends on $d$ [3, Theorem 1.5]. More generally, it was shown in [2] that radially weighted Besov spaces on the unit ball in finite dimensions satisfy the column-row property with a finite constant, which again possibly depends on the dimension and on the particular space. It is an open question whether every normalized complete Nevanlinna–Pick space satisfies the column-row property with a finite constant (see [14] for some work on this question). Recently, it has become clear that the column-row property is a very useful technical property when studying weak product spaces, see for instance [3]. A striking example of this is the result of Jury and Martin [11], according to which every function in $H \circ H$ factors as a product of precisely two functions in $H$, provided that $H$ satisfies the column-row property.

Theorem 1.2 combined with an argument from [3, Theorem 3.1] implies a positive answer to Question 1.1 in the presence of the column-row property.

**Corollary 1.3.** Let $\mathcal{H}$ be a normalized complete Nevanlinna–Pick space that satisfies the column-row property with constant $\kappa$. Then $\Mult(\mathcal{H}) = \Mult(\mathcal{H} \circ \mathcal{H})$ and

\[ \|\theta\|_{\Mult(\mathcal{H} \circ \mathcal{H})} \leq \|\theta\|_{\Mult(\mathcal{H})} \leq \kappa \|\theta\|_{\Mult(\mathcal{H} \circ \mathcal{H})} \]

for every $\theta \in \Mult(\mathcal{H})$.

**Proof.** It is elementary to check that every contractive multiplier of $\mathcal{H}$ is also a contractive multiplier of $\mathcal{H} \circ \mathcal{H}$; alternatively, this also follows from the implication $(iii) \Rightarrow (i)$ of Theorem 1.2.

Conversely, if $\theta$ is a contractive multiplier of $\mathcal{H} \circ \mathcal{H}$, then the implication $(i) \Rightarrow (iii)$ of Theorem 1.2 shows that there are $(\varphi_n), (\psi_n) \in M^G_1(\mathcal{H})$ so that $\theta = \sum_{n=1}^{\infty} \varphi_n \psi_n$. In other words, we have

\[ \theta = \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{bmatrix}. \]

The column is a contractive multiplier of $\mathcal{H}$ as $(\psi_n) \in M^G_1(\mathcal{H})$, and the row has norm at most $\kappa$ by the column-row property since $(\varphi_n) \in M^G_1(\mathcal{H})$. Hence $\theta \in \Mult(\mathcal{H})$ and $\|\theta\|_{\Mult(\mathcal{H})} \leq \kappa$. \qed

Matrix-valued multipliers play an important role in the theory of complete Nevanlinna–Pick spaces, and hence so does the operator space structure of $\Mult(\mathcal{H})$. For instance, this operator space structure encodes the difference between the Nevanlinna–Pick property and the complete Nevanlinna–Pick property. In the final section of the paper, we show how to equip $\Mult(\mathcal{H} \circ \mathcal{H})$ with an operator space structure,
i.e. we define norms for matrices of multipliers of $\mathcal{H} \odot \mathcal{H}$. We establish a version of Theorem 1.2 for matrices of multipliers (Theorem 4.1) and then use this result to compare the operator space structures of $\text{Mult}(\mathcal{H})$ and of $\text{Mult}(\mathcal{H} \odot \mathcal{H})$. In the case of the Drury–Arveson space and of the classical Dirichlet space, we show that while the inclusion $\text{Mult}(\mathcal{H}) \hookrightarrow \text{Mult}(\mathcal{H} \odot \mathcal{H})$ is a Banach space isomorphism (by Corollary 1.3), it is not an isomorphism of operator spaces.

2. Preliminaries

Let $\mathcal{H}$ be a normalized complete Nevanlinna–Pick space of functions on a set $X$. We assume throughout that $\mathcal{H}$ is separable.

2.1. Hankel operators. Nehari’s theorem identifies the dual space of $H^1(D)$ with the space of symbols of bounded Hankel operators on $H^2(D)$ via the standard integral pairing on the unit circle. When studying multipliers of $\mathcal{H} \odot \mathcal{H}$, we will heavily use a generalization of this fact, namely the duality between $\mathcal{H} \odot \mathcal{H}$ and the space $\text{Han}(\mathcal{H})$ of symbols of Hankel operators on $\mathcal{H}$. We now recall the necessary basics from [3, Section 2].

The conjugate Hilbert space of $\mathcal{H}$ is

$$\overline{\mathcal{H}} = \{ \overline{f} : f \in \mathcal{H} \}.$$ 

This space can be linearly and isometrically identified with the dual space $\mathcal{H}^*$ of $\mathcal{H}$ by means of the inner product of $\mathcal{H}$. Notice that $\overline{\mathcal{H}}$ is again a normalized complete Nevanlinna–Pick space on $X$ whose reproducing kernel is the complex conjugate of that of $\mathcal{H}$. The map $f \mapsto \overline{f}$ yields an anti-unitary operator between $\mathcal{H}$ and $\overline{\mathcal{H}}$ which conjugates $\text{Mult}(\mathcal{H})$ to $\text{Mult}(\overline{\mathcal{H}})$.

It was shown in [3, Section 2] that the dual space of $\mathcal{H} \odot \mathcal{H}$ can be linearly and isometrically identified with a subspace $\text{Han}(\mathcal{H}) \subset B(\mathcal{H}, \overline{\mathcal{H}})$; operators in $\text{Han}(\mathcal{H})$ are called Hankel operators on $\mathcal{H}$. Every operator $T \in \text{Han}(\mathcal{H})$ is uniquely determined by its symbol $T1 \in \overline{\mathcal{H}}$. Let

$$\text{Han}(\mathcal{H}) = \{ T1 \in \overline{\mathcal{H}} : T \in \text{Han}(\mathcal{H}) \}$$

denote the space of symbols of Hankel operators. Given $b \in \text{Han}(\mathcal{H})$, we write $H_b$ for the unique operator in $\text{Han}(\mathcal{H})$ that satisfies $H_b1 = b$. Notice that the map $b \mapsto H_b$ is conjugate linear. The operator $H_b$ is uniquely determined by the requirement that

$$\langle H_b f, \overline{\varphi f} \rangle_{\mathcal{H}} = \langle \varphi f, b \rangle_{\mathcal{H}} \quad \text{for all } f \in \mathcal{H}, \varphi \in \text{Mult}(\mathcal{H}).$$

(Since $\mathcal{H}$ is a normalized complete Nevanlinna–Pick space, the kernel functions are contained in $\text{Mult}(\mathcal{H})$ and hence $\text{Mult}(\mathcal{H})$ is densely contained in $\mathcal{H}$.)

Remark 2.1. If $\mathcal{H}$ satisfies the column-row property, then the space $\text{Han}(\mathcal{H})$ can be more concretely described as the space of all those $b \in \mathcal{H}$ for which the densely defined bilinear form on $\mathcal{H} \times \mathcal{H}$,

$$(\varphi, f) \mapsto \langle \varphi f, b \rangle \quad (\varphi \in \text{Mult}(\mathcal{H}), f \in \mathcal{H}),$$

is bounded; see [3, Theorem 2.6].
The version of Nehari’s theorem obtained in [3, Theorem 2.5] asserts that there is a conjugate linear isometric isomorphism \( \text{Han}(\mathcal{H}) \cong (\mathcal{H} \circ \mathcal{H})^* \), \( b \mapsto L_b \), satisfying

\[ L_b(\varphi f) = \langle \varphi f, b \rangle \quad \text{for all } b \in \text{Han}(\mathcal{H}), f \in \mathcal{H}, \varphi \in \text{Mult}(\mathcal{H}). \]

Here, the norm on \( \text{Han}(\mathcal{H}) \) is given by \( \| b \|_{\text{Han}(\mathcal{H})} = \| H_b \|_{B(\mathcal{H}, \mathcal{H})} \). We write \( [f, H_b] = L_b(f) \) for \( f \in \mathcal{H} \circ \mathcal{H} \) and \( b \in \text{Han}(\mathcal{H}) \). Thus, combining (1) and (2), we see that the linear duality between \( \text{HAN}(\mathcal{H}) \) and \( \mathcal{H} \circ \mathcal{H} \) is given by

\[ [fg, H] = \langle H_b f, g \rangle \quad (f, g \in \mathcal{H}, b \in \text{Han}(\mathcal{H})). \]

This duality also endows \( \text{HAN}(\mathcal{H}) \) with a weak-\(^*\) topology. It follows from the construction in [3, Section 2], or one checks directly, that this weak-\(^*\) topology agrees with the one inherited from \( B(\mathcal{H}, \mathcal{H}) \) as the dual of the space \( T(\mathcal{H}, \mathcal{H}) \) of trace class operators.

**Remark 2.2.** In this article, we find it convenient to distinguish between a Hankel operator and its symbol, as the correspondence between them is *conjugate* linear. Ultimately, we will work with the operators directly and only use Equation (3), which in principle could be understood without explicitly mentioning symbol functions.

Given a bounded linear operator \( A : \mathcal{H} \circ \mathcal{H} \to \mathcal{H} \circ \mathcal{H} \), let \( A^\dagger : \text{HAN}(\mathcal{H}) \to \text{HAN}(\mathcal{H}) \) be the Banach space adjoint of \( A \) modulo the linear isometric isomorphism \( \text{HAN}(\mathcal{H}) \cong (\mathcal{H} \circ \mathcal{H})^* \). Equation (3) shows that the action of \( A^\dagger \) is characterized by the formula

\[ \langle A^\dagger(H_b)f, \overline{g} \rangle = [A(fg), H_b] \quad (f, g \in \mathcal{H}, b \in \text{Han}(\mathcal{H})). \]

It follows from a theorem of Hartman that \( H^1(\mathbb{D}) \) can be identified with the dual of the space of all symbols of compact Hankel operators. In a similar fashion, the weak product \( \mathcal{H} \circ \mathcal{H} \) is also a dual space. More precisely, let

\[ \text{HAN}_0(\mathcal{H}) = \text{HAN}(\mathcal{H}) \cap K(\mathcal{H}, \mathcal{H}) \]

be the space of all compact Hankel operators on \( \mathcal{H} \). According to Theorems 2.1 and 2.5 of [3], the duality between \( \mathcal{H} \circ \mathcal{H} \) and \( \text{HAN}(\mathcal{H}) \) also yields an isometric isomorphism \( \text{HAN}_0(\mathcal{H})^* \cong \mathcal{H} \circ \mathcal{H} \).

If \( k_z \in \mathcal{H} \) denotes the kernel function associated with \( z \in X \), then boundedness of point evaluations on \( \mathcal{H} \circ \mathcal{H} \) and the duality between \( \mathcal{H} \circ \mathcal{H} \) and \( \text{Han}(\mathcal{H}) \) (see Equation (2)) imply that \( k_z \in \text{Han}(\mathcal{H}) \). Moreover, Equation (1) shows that \( H_{k_z} f = f(z)k_z \) for \( f \in \mathcal{H} \), so that \( H_{k_z} \) is a rank-one operator and \( H_{k_z} \in \text{HAN}_0(\mathcal{H}) \). In particular, point evaluations on \( \mathcal{H} \circ \mathcal{H} \) are continuous in the weak-\(^*\) topology given by the duality \( \mathcal{H} \circ \mathcal{H} = \text{HAN}_0(\mathcal{H})^* \). On bounded subsets of \( \mathcal{H} \circ \mathcal{H} \), the weak-\(^*\) topology agrees with the topology of pointwise convergence on \( X \); see [3, Corollary 2.2] and the remarks preceding it. It also follows from the Hahn–Banach theorem that \( \text{HAN}_0(\mathcal{H}) \) is weak-\(^*\) dense in \( \text{HAN}(\mathcal{H}) \).
2.2. Multiplication operators and duality. To distinguish multiplication operators on \( \mathcal{H} \) and on \( \mathcal{H} \odot \mathcal{H} \), we will use the following notation. Given \( \theta \in \text{Mult}(\mathcal{H} \odot \mathcal{H}) \), the corresponding multiplication operator on \( \mathcal{H} \odot \mathcal{H} \) is denoted by

\[
M_\theta : \mathcal{H} \odot \mathcal{H} \to \mathcal{H} \odot \mathcal{H}, \quad f \mapsto \theta \cdot f.
\]

Given \( \varphi \in \text{Mult}(\mathcal{H}) \), we denote the associated multiplication operator on \( \mathcal{H} \) by

\[
T_\varphi : \mathcal{H} \to \mathcal{H}, \quad f \mapsto \varphi \cdot f.
\]

(If \( \mathcal{H} = H^2(\mathbb{D}) \), then \( T_\varphi \) is an analytic Toeplitz operator, which motivates our choice of notation.)

Since every multiplier of \( \mathcal{H} \) is also a multiplier of \( \mathcal{H} \odot \mathcal{H} \), if \( b \in \text{Han}(\mathcal{H}) \) and \( \psi \in \text{Mult}(\mathcal{H}) \) then \( L_b \circ M_\psi \in (\mathcal{H} \odot \mathcal{H})^* \). This element in the dual is verified to correspond to \( T^*_\psi b \in \text{Han}(\mathcal{H}) \). Moreover, the defining equation of a Hankel operator, Equation (1), easily implies the intertwining relation

\[
H_b T_\psi = T^*_\psi H_b = H_{T^*_\psi b}
\]

for all \( \psi \in \text{Mult}(\mathcal{H}), b \in \text{Han}(\mathcal{H}); \) see also [3, Lemma 2.3]. In particular, \( \text{Han}(\mathcal{H}) \) is a \( \text{Mult}(\mathcal{H})^* - \text{Mult}(\mathcal{H}) \)-bimodule. The following lemma shows that if \( \theta \in \text{Mult}(\mathcal{H} \odot \mathcal{H}) \), then \( M_\theta \) respects this bimodule structure.

**Lemma 2.3.** Let \( \theta \in \text{Mult}(\mathcal{H} \odot \mathcal{H}) \), let \( \varphi \in \text{Mult}(\mathcal{H}) \) and let \( b \in \text{Han}(\mathcal{H}) \). Then:

(a) \( M^\dagger_\psi(H_b) = H_b T_\varphi = T^*_\varphi H_b \).

(b) \( M^\dagger_\theta(H_b T_\varphi) = M^\dagger_\theta(H_b) T_\varphi = M^\dagger_\theta(T^*_\varphi H_b) = T^*_\varphi M^\dagger_\theta(H_b) \).

(c) \( M^\dagger_\theta(H_{k_z}) = \theta(z) H_{k_z} \) for all \( z \in X \).

**Proof.** (a) For \( f, g \in \mathcal{H} \), we find using [3] and [4] that

\[
\langle M^\dagger_\psi(H_b)f, g \rangle_{\bar{\mathcal{H}}} = \langle \varphi f, g, H_b \rangle = \langle H_b T_\varphi f, g \rangle_{\bar{\mathcal{H}}},
\]

from which the first half of (a) follows. The second half follows from [3].

(b) Dualizing the commutation relation \( M_\varphi M_\theta = M_\theta M_\varphi \) and using (a), we see that

\[
M^\dagger_\theta(H_b T_\varphi) = (M^\dagger_\theta M^\dagger_\varphi)(H_b) = (M^\dagger_\varphi M^\dagger_\theta)(H_b) = M^\dagger_\theta(H_b) T_\varphi.
\]

The remaining parts of (b) follow from this and from [5].

(c) For \( z \in X \) we have that \( [h, H_{k_z}] = h(z) \) for every \( h \in \mathcal{H} \odot \mathcal{H} \). Using [3] and [4], we obtain for \( f, g \in \mathcal{H} \) the identity

\[
\langle M^\dagger_\theta(H_{k_z})f, g \rangle_{\bar{\mathcal{H}}} = \langle \theta f, g, H_{k_z} \rangle = \theta(z) \langle f, g, H_{k_z} \rangle = \theta(z) \langle H_{k_z} f, g \rangle_{\bar{\mathcal{H}}},
\]

from which (c) follows. \( \square \)

2.3. \( \mathcal{H} \odot \mathcal{H} \) as an operator space. As mentioned in the introduction, a key device in our analysis of multipliers of \( \mathcal{H} \odot \mathcal{H} \) is the observation that the weak product carries a natural operator space structure. We therefore recall the necessary background from the theory of operator spaces. For precise definitions and more information, the reader is referred to the books [6, 8, 15, 16].

Given a vector space \( V \), let \( M_n(V) \) be the vector space of all \( n \times n \) matrices with entries in \( V \). An (abstract) operator space is a normed space \( V \), together with a
norm on each $M_n(V)$ satisfying certain axioms. We will not require the precise form of the axioms and thus simply refer to [8, Section 2.1]. Perhaps the most important example of an operator space is the space $B(\mathcal{H},\mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. In this case, we can identify $M_n(B(\mathcal{H},\mathcal{K}))$ with $B(\mathcal{H}^n,\mathcal{K}^n)$, which equips $M_n(B(\mathcal{H},\mathcal{K}))$ with a norm. In the same vein, any subspace of $B(\mathcal{H},\mathcal{K})$ becomes an operator space in this way.

If $V, W$ are operator spaces, each linear map $A : V \to W$ induces for each $n \in \mathbb{N}$ a linear map $A^{(n)} : M_n(V) \to M_n(W)$ defined by applying $A$ entrywise. The map $A$ is said to be completely bounded if
\[ \|A\|_{cb} = \sup_{n \in \mathbb{N}} \|A^{(n)}\| < \infty, \]

and completely contractive if $\|A\|_{cb} \leq 1$. Similarly, $A$ is said to be a complete isometry if each $A^{(n)}$ is an isometry. We write $CB(V, W)$ for the space of all completely bounded linear maps from $V$ to $W$, endowed with the cb norm. It is a well-known phenomenon in operator space theory that completely bounded maps exhibit much better behavior than maps that are merely bounded.

If $V$ is an abstract operator space, then its dual space $V^*$ carries a natural operator space structure, called the dual operator space structure. It is defined via the identification $M_n(V^*) = CB(V, M_n)$. The dual operator space structure has the property that if $A : V \to W$ is a completely bounded map between operator spaces, then the adjoint $A^* : W^* \to V^*$ is completely bounded with $\|A^*\|_{cb} = \|A\|_{cb}$; see [8, Section 3.2].

We now apply these abstract considerations to our setting of weak products. Since $\mathrm{HAN}_0(\mathcal{H}) \subset \mathrm{HAN}(\mathcal{H}) \subset B(\mathcal{H},\mathcal{F})$, the spaces $\mathrm{HAN}_0(\mathcal{H})$ and $\mathrm{HAN}(\mathcal{H})$ carry a natural operator space structure. Since $\mathcal{H} \odot \mathcal{H}$ is isometrically isomorphic to $\mathrm{HAN}_0(\mathcal{H})^*$, we may endow $\mathcal{H} \odot \mathcal{H}$ with the corresponding dual operator space structure. Taking duals again, the resulting dual operator space structure on $(\mathcal{H} \odot \mathcal{H})^*$ agrees with the operator space structure of $\mathrm{HAN}(\mathcal{H})$ inherited from $B(\mathcal{H},\mathcal{F})$, as the identification of $K(\mathcal{H},\mathcal{F})^{**}$ with $B(\mathcal{H},\mathcal{F})$ is a complete isometry; see Theorem 1.4.11 in [6] and the discussion preceding it. In particular, it follows that $\mathcal{H} \odot \mathcal{H}$ is endowed with the unique operator space structure that makes $\mathrm{HAN}(\mathcal{H})$ the operator space dual of $\mathcal{H} \odot \mathcal{H}$ with respect to the given duality; this is also known as the predual operator space structure, see [12, Section 3] for further discussion.

In particular, we see that a linear map $A : \mathcal{H} \odot \mathcal{H} \to \mathcal{H} \odot \mathcal{H}$ is completely contractive if and only if its adjoint $A^\dagger : \mathrm{HAN}(\mathcal{H}) \to \mathrm{HAN}(\mathcal{H})$ is completely contractive. We will frequently use this fact. Indeed, for our purposes it will be more convenient to study properties of $A$ through $A^\dagger$, because the latter acts on a concrete space of operators as opposed to the space $\mathcal{H} \odot \mathcal{H}$, whose operator space structure is less explicit.

We will only use the description of the operator space structure on $\mathcal{H} \odot \mathcal{H}$ in terms of the duality given above. Nevertheless, we will provide a concrete description of the norm on $M_n(\mathcal{H} \odot \mathcal{H})$ in Lemma 4.3.
3. Proof of the main result

We continue to assume throughout that $\mathcal{H}$ is a normalized complete Nevanlinna–Pick space of functions on $X$.

3.1. Factorization implies complete contractivity. We first show that (iii) $\Rightarrow$ (i) in Theorem [1.2] that is, that every function that can be factored using a pair of elements in $M_1^c(\mathcal{H})$ is a contractive multiplier of $\mathcal{H} \odot \mathcal{H}$. As mentioned in the introduction, this result is known [3, Theorem 3.1], but we include the short argument for the convenience of the reader.

**Proposition 3.1.** Let $(\varphi_n), (\psi_n) \in M_1^c(\mathcal{H})$ and define $\theta = \sum_{n=1}^\infty \varphi_n \psi_n$. Then $\theta$ is a contractive multiplier of $\mathcal{H} \odot \mathcal{H}$.

**Proof.** Let $h = \sum_{n=1}^\infty f_n g_n \in \mathcal{H} \odot \mathcal{H}$ with $\sum_{n=1}^\infty \|f_n\|\|g_n\| < \infty$. By continuity of point evaluations on $\mathcal{H}$ and by the Cauchy–Schwarz inequality, the sums defining $h$ and $\theta$ converge pointwise absolutely on $X$, hence

$$\theta h = \sum_{k,n=1}^\infty (\varphi_k f_n) (\psi_k g_n).$$

Since $(\varphi_n)_{n=1}^\infty$ and $(\psi_n)_{n=1}^\infty \in M_1^c(\mathcal{H})$, we find that

$$\sum_{n,k=1}^\infty \|\varphi_k f_n\|\|\psi_k g_n\| \leq \sum_{n=1}^\infty \left( \sum_{k=1}^\infty \|\varphi_k f_n\|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^\infty \|\psi_k g_n\|^2 \right)^{\frac{1}{2}} \leq \sum_{n=1}^\infty \|f_n\|\|g_n\|,$$

so taking the infimum over all representations $h = \sum_{n=1}^\infty f_n g_n$, it follows that $\|\theta h\|_{\mathcal{H} \odot \mathcal{H}} \leq \|h\|_{\mathcal{H} \odot \mathcal{H}}$, so that $\theta$ is a contractive multiplier of $\mathcal{H} \odot \mathcal{H}$. \qed

While not logically necessary, we improve the preceding result by showing that functions that factor as above are actually completely contractive multipliers of $\mathcal{H} \odot \mathcal{H}$; this is the implication (iii) $\Rightarrow$ (ii) of Theorem [1.2]. We provide this proof as it shows how the operator space structure of $\mathcal{H} \odot \mathcal{H}$ and the duality between $\mathcal{H} \odot \mathcal{H}$ and HAN(\mathcal{H}) enter the picture, and it foreshadows the dilation theoretic proof of the reverse implication (ii) $\Rightarrow$ (iii) of Theorem [1.2].

**Proposition 3.2.** Let $(\varphi_n), (\psi_n) \in M_1^c(\mathcal{H})$ and define $\theta = \sum_{n=1}^\infty \varphi_n \psi_n$. Then $\theta$ is a completely contractive multiplier of $\mathcal{H} \odot \mathcal{H}$.

**Proof.** Observe that it suffices to show that for each $N \in \mathbb{N}$, the function $\theta_N = \sum_{n=1}^N \varphi_n \psi_n$ is a completely contractive multiplier of $\mathcal{H} \odot \mathcal{H}$. Indeed, $\theta_N$ converges pointwise to $\theta$. Hence, assuming that each $\theta_N$ is a completely contractive multiplier of $\mathcal{H} \odot \mathcal{H}$, we see that for all $f \in \mathcal{H} \odot \mathcal{H}$, the sequence $(\theta_N f)$ converges to $\theta f$ in the weak-* topology of $\mathcal{H} \odot \mathcal{H}$. Thus, $\theta$ is completely contractive if each $\theta_N$ is.

Therefore, we may assume that $\theta = \sum_{n=1}^N \varphi_n \psi_n$ for some $N \in \mathbb{N}$. In particular, $\theta \in \text{Mult}(\mathcal{H})$. We will show that, equivalently, the adjoint map $M_\theta^\dagger : \text{HAN}(\mathcal{H}) \to$
HAN(ℋ) is completely contractive. To this end, we apply part (a) of Lemma 2.3 to conclude that

$$M^\dagger_b(H_b) = H_bT_b = \left[ T^*_{\varphi_1} \cdots T^*_{\varphi_N} \right] (H_b \oplus \cdots \oplus H_b) \begin{bmatrix} T_{\varphi_1} \\ \vdots \\ T_{\varphi_N} \end{bmatrix}$$

for every $b \in \text{Han}(ℋ)$. This formula implies that $M^\dagger_b$ is completely contractive once we know that the row and the column are contractive, which in turn follows from the assumption $(\varphi_n), (\psi_n) \in M^C_1(ℋ)$ (see also the remarks about the conjugate Hilbert space in Subsection 2.1).

3.2. Contractive multipliers are completely contractive. The goal of this subsection is to show that every contractive multiplier of $ℋ \odot ℋ$ is completely contractive, that is, we prove the implication $(i) \Rightarrow (ii)$ of Theorem 1.2.

The key tool is the following lemma, which uses a recent result of Jury and Martin.

For notational convenience, we regard finite sequences of multipliers as infinite sequences that are eventually zero.

**Lemma 3.3.** Let $[A_{ij}] \in M_n(B(ℋ, ℋ))$. Then

$$\| [A_{ij}] \| = \sup \left\{ \left\| \sum_{i,j=1}^n T^*_{\varphi_i} A_{ij} T_{\psi_j} \right\| : (\varphi_i)_{i=1}^n, (\psi_i)_{i=1}^n \in M^C_1(ℋ) \right\}.$$

**Proof.** If $(\varphi_i), (\psi_i) \in M^C_1(ℋ)$, then

$$\left\| \sum_{i,j=1}^n T^*_{\varphi_i} A_{ij} T_{\psi_j} \right\| = \left\| \begin{bmatrix} T^*_{\varphi_1} & \cdots & T^*_{\varphi_n} \end{bmatrix} [A_{ij}] \begin{bmatrix} T_{\varphi_1} \\ \vdots \\ T_{\varphi_n} \end{bmatrix} \right\| \leq \| [A_{ij}] \|,$$

as the row and the column are contractions, hence the inequality “≥” holds in the statement of the lemma.

To prove the reverse inequality, it suffices to show that for every pair of sequences $(f_i)_{i=1}^n, (g_i)_{i=1}^n$ of elements of $ℋ$ with $\sum_{i=1}^n \| f_i \|^2 = \sum_{j=1}^n \| g_j \|^2 = 1$, there exist $(\varphi_i), (\psi_i) \in M^C_1(ℋ)$ so that

$$\left| \sum_{i,j=1}^n \langle A_{ij} f_j, \varphi_i \rangle \overline{\psi_j} \right| \leq \left\| \sum_{i,j=1}^n T^*_{\varphi_i} A_{ij} T_{\psi_j} \right\|.$$

To this end, we apply Theorem 1.1 of [1], which yields $(\varphi_i), (\psi_i) \in M^C_1(ℋ)$ and $F, G \in ℋ$ with $\| F \| \leq 1, \| G \| \leq 1$ such that $f_i = \varphi_i F, g_i = \psi_i G$ for all $i$. Then

$$\left| \sum_{i,j=1}^n \langle A_{ij} f_j, \varphi_i \rangle \overline{\psi_j} \right| = \left| \sum_{i,j=1}^n \langle A_{ij} T_{\varphi_j} F, T^*_{\psi_i} G \rangle \overline{\psi_j} \right| \leq \left\| \sum_{i,j=1}^n T^*_{\psi_i} A_{ij} T_{\varphi_j} \right\|,$$

as desired. □
In the language of operator bimodules, Lemma 3.3 says that the pair $(\text{Mult}(\mathcal{H})^*, \text{Mult}(\mathcal{H}))$ is matricially norming for $B(\mathcal{H}, \overline{\mathcal{H}})$, and in particular for $\text{HAN}(\mathcal{H})$. This property is most commonly studied for $C^*$-bimodules, see for example [15, Section 8].

Given the matricial norming property of Lemma 3.3, it is now routine to finish the proof of the implication $(i) \Rightarrow (ii)$ of Theorem 1.2; cf. [15, Proposition 8.6].

**Proposition 3.5.** Every contractive multiplier of $\mathcal{H} \odot \mathcal{H}$ is completely contractive.

**Proof.** Let $\theta \in \text{Mult}(\mathcal{H} \odot \mathcal{H})$ be a contractive multiplier. By duality, it suffices to show that the contractive map $M^\dagger_{\theta} : \text{HAN}(\mathcal{H}) \to \text{HAN}(\mathcal{H})$ is completely contractive. To this end, we use Lemma 3.3 and part (b) of Lemma 2.3 to see that for $[H_{ij}] \in M_n(\text{HAN}(\mathcal{H}))$,

$$
\|\{M^\dagger_{\theta}(H_{ij})\}\| = \sup \left\| \sum_{i,j=1}^n T_{\psi_i}^* M^\dagger_{\theta}(H_{ij}) T_{\varphi_j} \right\| = \sup \left\| M^\dagger_{\theta}\left( \sum_{i,j=1}^n T_{\psi_i}^* H_{ij} T_{\varphi_j} \right) \right\| 
\leq \sup \left\| \sum_{i,j=1}^n T_{\psi_i}^* H_{ij} T_{\varphi_j} \right\| = \|[H_{ij}]\|,
$$

where all suprema are taken over $(\varphi_i), (\psi_i) \in M^C_1(\mathcal{H})$. □

### 3.3. Completely contractive multipliers admit a factorization

In this subsection, we prove the remaining implication $(ii) \Rightarrow (iii)$ of Theorem 3.8, that is, the factorization for completely contractive multipliers $\theta$ of $\mathcal{H} \odot \mathcal{H}$. To this end, we use dilation theory to obtain a representation for the adjoint $M^\dagger_{\theta}$ as in the proof of Proposition 3.2. We emphasize that it is complete contractivity that enables this use of dilation theory. The first step is the following consequence of the Haagerup–Paulsen–Wittstock dilation theorem.

**Lemma 3.6.** Let $A : \text{HAN}(\mathcal{H}) \to \text{HAN}(\mathcal{H})$ be a completely contractive linear map that is (weak-$*$, weak-$*$) continuous. Then there exist linear contractions $V : \mathcal{H} \to \mathcal{H} \otimes \ell^2$ and $W : \overline{\mathcal{H}} \to \overline{\mathcal{H}} \otimes \ell^2$ such that

$$
A(H_b) = W^*(H_b \otimes I_{\ell^2})V
$$

for all $b \in \text{Han}(\mathcal{H})$.

**Proof.** Recall that $\text{HAN}(\mathcal{H}) \subset B(\mathcal{H}, \overline{\mathcal{H}})$ and $\text{HAN}_0(\mathcal{H}) \subset K(\mathcal{H}, \overline{\mathcal{H}})$. To be in the more familiar setting of spaces of operators on a single Hilbert space, we fix a (non-canonical) linear unitary $U : \overline{\mathcal{H}} \to \mathcal{H}$ and define $\widehat{\text{HAN}}_0(\mathcal{H}) = U \text{HAN}_0(\mathcal{H}) \subset K(\mathcal{H})$ and

$$
\widehat{A} : \widehat{\text{HAN}}_0(\mathcal{H}) \to B(\mathcal{H}), \quad \widehat{A}(UH_b) = UA(H_b).
$$

Then $\widehat{A}$ is completely contractive, so by the Haagerup–Paulsen–Wittstock dilation theorem (Theorems 8.2 and 8.4 in [15]), there exist a Hilbert space $\mathcal{F} \supset \mathcal{H}$, a $*$-representation $\pi : K(\mathcal{H}) \to B(\mathcal{F})$ and contractions $X, Y : \mathcal{H} \to \mathcal{F}$ so that

$$
\widehat{A}(UH_b) = X^* \pi(UH_b)Y \quad (H_b \in \text{HAN}_0(\mathcal{H})).
$$
Since $H$ is separable, $F$ can be chosen to be separable as well. Every $*$-representation of $K(H)$ is unitarily equivalent to a multiple of the identity representation, hence there exist contractions $V_0, W_0 : H \to H \otimes \ell^2$ so that

$$UA(H_b) = \tilde{A}(UH_b) = W_0^*(UH_b \otimes I)V_0 \quad (H_b \in \text{HAN}_0(H)).$$

Thus, defining $V = V_0$ and $W = (U^* \otimes I)W_0U$, we see that

$$A(H_b) = W^*(H_b \otimes I)V \quad (H_b \in \text{HAN}_0(H)).$$

Recall from Subsection 2.1 that $\text{HAN}_0(H)$ is weak-$*$ dense in $\text{HAN}(H)$ and that the inclusion $\text{HAN}(H) \subset B(H, \overline{H})$ is (weak-$*$, weak-$*$) continuous, so the (weak-$*$, weak-$*$) continuity of $A$ therefore implies that (6) holds whenever $H_b \in \text{HAN}(H)$. □

**Remark 3.7.** The use in the previous proof of the somewhat unnatural operator $U : \overline{H} \to H$ can be avoided by using “rectangular” dilation theory, see for example [9]. In this setting, $A$ dilates to a triple representation of the TRO $K(H, \overline{H})$, and every triple representation of $K(H, \overline{H})$ is unitarily equivalent to a multiple of the identity representation.

We are ready to prove the remaining implication (ii) $\Rightarrow$ (iii) of Theorem 1.2.

**Theorem 3.8.** Let $H$ be a normalized complete Nevanlinna-Pick space and let $\theta$ be a completely contractive multiplier of $H \ominus H$. Then there exist $(\varphi_n), (\psi_n) \in M^2_1(H)$ such that

$$\theta = \sum_{n=1}^{\infty} \varphi_n \psi_n.$$

**Proof.** Since $\theta$ is a completely contractive multiplier of $H \ominus H$, the adjoint map $M^\dagger_\theta : \text{HAN}(H) \to \text{HAN}(H)$ is a (weak-$*$, weak-$*$) continuous complete contraction. Hence, the dilation theoretic Lemma 3.6 implies that there exist contractions $V : H \to H \otimes \ell^2$ and $W : \overline{H} \to \overline{H} \otimes \ell^2$ such that

$$M^\dagger_\theta(H_b) = W^*(H_b \otimes I)V$$

for all $b \in \text{HAN}(H)$. We will show that $V$ and $W$ can be replaced with suitable multiplication operators, thus obtaining a representation as in the proof of Proposition 3.2.

To this end, let

$$M = \left( \bigcap_{b \in \text{HAN}(H)} \ker(W^*(H_b \otimes I)) \right)^\perp \subset H \otimes \ell^2.$$

Since $H_bT_{\varphi}$ is a Hankel operator for all $\varphi \in \text{Mult}(H)$, we find that $M$ is invariant under $T_{\varphi}^* \otimes I$ for all $\varphi \in \text{Mult}(H)$. Let $X = P_M V$. Then Equation (7) implies that

$$M^\dagger_\theta(H_b) = W^*(H_b \otimes I)X$$

for all $b \in \text{HAN}(H)$. Thus, by part (b) of Lemma 2.3, we obtain for all $\varphi \in \text{Mult}(H)$ and all $b \in \text{HAN}(H)$ the identity

$$W^*(H_b \otimes I)XT_{\varphi} = M^\dagger_\theta(H_b)T_{\varphi} = M^\dagger_\theta(H_bT_{\varphi}) = W^*(H_b \otimes I)(T_{\varphi} \otimes I)X.$$

Therefore, for all $\varphi \in \text{Mult}(H)$, we find that

$$W^*(H_b \otimes I)[XT_{\varphi} - (T_{\varphi} \otimes I)X] = 0.$$
for all $b \in \text{Han}({\mathcal{H}})$, so the definition of $\mathcal{M}$ shows that
\[
XT_\varphi = P_\mathcal{M}(T_\varphi \otimes I)X
\]
for all $\varphi \in \text{Mult}({\mathcal{H}})$. In this setting, the Ball–Trent–Vinnikov commutant lifting theorem (see [5, Theorem 5.1]) implies that there exists a contractive multiplier $\Phi \in \text{Mult}({\mathcal{H}}, {\mathcal{H}} \otimes \ell^2)$ with $X = P_\mathcal{M} T_\Phi$, and hence [8] shows that
\[
M^*_b(H_b) = W^*(H_b \otimes I)P_\mathcal{M} T_\Phi = W^*(H_b \otimes I)T_\Phi
\]
for all $b \in \text{Han}({\mathcal{H}})$.

A similar argument, applied to the space
\[
\mathcal{N} = \bigvee_{b \in \text{Han}({\mathcal{H}})} \text{ran}((H_b \otimes I)T_\Phi) \subset \overline{{\mathcal{H}}} \otimes \ell^2,
\]
shows that there exists a contractive multiplier $\Psi \in \text{Mult}({\mathcal{H}}, {\mathcal{H}} \otimes \ell^2)$ such that
\[
M^*_b(H_b) = T^*_\Psi(H_b \otimes I)T_\Phi
\]
for all $b \in \text{Han}({\mathcal{H}})$.

To finish the proof, we write $\Phi = (\varphi_n)$ and $\Psi = (\psi_n)$ with $(\varphi_n), (\psi_n) \in M^*_1({\mathcal{H}})$, so that
\[
\langle M^*_b(H_b)1, 1 \rangle_{\overline{{\mathcal{H}}}} = \langle T^*_\Psi(H_b \otimes I)T_\Phi 1, 1 \rangle_{\overline{{\mathcal{H}}}} = \sum_{n=1}^{\infty} \langle H_b \overline{\varphi_n}, \overline{\psi_n} \rangle_{{\mathcal{H}}} = \sum_{n=1}^{\infty} \langle \varphi_n \psi_n, b \rangle_{{\mathcal{H}}}.
\]
Choosing $b = k_z$ and using part (c) of Lemma 2.3 we see that
\[
\theta(z) = \sum_{n=1}^{\infty} \varphi_n(z)\psi_n(z)
\]
as desired. \hfill \Box

The proof above shows that Theorem 3.8 can be regarded as a dilation theorem for the completely bounded bimodule map $M^*_b : \text{HAN}({\mathcal{H}}) \to \text{HAN}({\mathcal{H}})$. In different settings, dilation theorems for completely bounded bimodule maps were obtained by several authors, see for instance [20, Theorem 3.1] and the references given there.

4. **Mult$({\mathcal{H}} \odot {\mathcal{H}})$ as an Operator Space**

In this section, we endow $\text{Mult}({\mathcal{H}} \odot {\mathcal{H}})$ with an operator space structure. Recall that if $V$ and $W$ are operator spaces, then $\text{CB}(V,W)$ is the space of all completely bounded maps from $V$ into $W$. This space becomes itself an operator space, via the identification $M_n(\text{CB}(V,W)) = \text{CB}(V,M_n(W))$; see [8] Section 3.2. It follows from Theorem 1.2 that every multiplier of ${\mathcal{H}} \odot {\mathcal{H}}$ defines a completely bounded map on ${\mathcal{H}} \odot {\mathcal{H}}$, so we can regard $\text{Mult}({\mathcal{H}} \odot {\mathcal{H}}) \subset \text{CB}({\mathcal{H}} \odot {\mathcal{H}}, {\mathcal{H}} \odot {\mathcal{H}})$ and we endow $\text{Mult}({\mathcal{H}} \odot {\mathcal{H}})$ with the resulting operator space structure.
4.1. **Factoring elements of** $M_n(\text{Mult}(\mathcal{H} \otimes \mathcal{H}))$. First, we establish a generalization of the equivalence of (ii) and (iii) of Theorem 1.2 for elements of $M_n(\text{Mult}(\mathcal{H} \otimes \mathcal{H}))$.

Given $\Phi, \Psi \in \text{Mult}(\mathcal{H} \otimes \mathbb{C}^n, \mathcal{H} \otimes \ell^2)$, say

$$
\Phi = \begin{bmatrix}
\varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\
\varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
$$

and

$$
\Psi = \begin{bmatrix}
\psi_{11} & \psi_{12} & \cdots & \psi_{1n} \\
\psi_{21} & \psi_{22} & \cdots & \psi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix},
$$

let $\Psi^T$ denote the transpose of the matrix $\Psi$ and define an $n \times n$ matrix $\Psi^T \Phi$ of functions on $X$ by

$$(\Psi^T \Phi)_{ij} = \sum_{k=1}^{\infty} \varphi_{kj} \psi_{ki} \quad (1 \leq i, j \leq n).$$

Note that the sum converges pointwise absolutely by the Cauchy–Schwarz inequality.

With this notation, the norm on $M_n(\text{Mult}(\mathcal{H} \otimes \mathcal{H}))$ can be described as follows.

**Theorem 4.1.** Let $\mathcal{H}$ be a normalized complete Nevanlinna–Pick space on $X$ and let $\Theta$ be an $n \times n$ matrix of functions on $X$. The following statements are equivalent.

(i) The matrix $\Theta$ belongs to the closed unit ball of $M_n(\text{Mult}(\mathcal{H} \otimes \mathcal{H}))$.

(ii) There exist $\Phi, \Psi$ in the closed unit ball of $\text{Mult}(\mathcal{H} \otimes \mathbb{C}^n, \mathcal{H} \otimes \ell^2)$ so that $\Theta = \Psi^T \Phi$.

The proof is closely modeled after those of Proposition 3.2 and Theorem 3.8. To use duality, we require the following result refining the fact that $\|A^*\|_{cb} = \|A\|_{cb}$ for a completely bounded map $A : V \to W$. This result is undoubtedly known, but we were not able to find an explicit reference.

**Lemma 4.2.** Let $V$ and $W$ be operator spaces. Then, the map

$$
\text{CB}(V, W) \to \text{CB}(W^*, V^*), \quad A \mapsto A^*
$$

is a complete isometry.

**Proof.** Let

$$
[A_{ij}] \in M_n(\text{CB}(V, W)) = \text{CB}(V, M_n(W)).
$$

An elementary computation shows that the norm of $[A_{ij}^*]$ in $M_n(\text{CB}(W^*, V^*)) = \text{CB}(W^*, M_n(V^*))$ is at most that of $[A_{ij}]$ in $M_n(\text{CB}(V, W)) = \text{CB}(V, M_n(W))$. Thus, the map $A \mapsto A^*$ is a complete contraction. Applying this map again, using that $A^{**}$ agrees with $A$ on $V$ and the fact that the inclusion of an operator space into its bidual is a complete isometry (see [8, Proposition 3.2.1]), we conclude that $A \mapsto A^*$ is a complete isometry. \hfill $\Box$

**Proof of Theorem 4.1.** (ii) $\Rightarrow$ (i) As in the proof of Proposition 3.2, an approximation argument allows us to assume that $\Phi, \Psi \in \text{Mult}(\mathcal{H} \otimes \mathbb{C}^n, \mathcal{H} \otimes \mathbb{C}^N)$ for some $N \in \mathbb{N}$. In particular, $\Theta \in \text{Mult}(\mathcal{H} \otimes \mathbb{C}^n)$. To compute the norm of

$$
\Theta = [\theta_{ij}] \in M_n(\text{Mult}(\mathcal{H} \otimes \mathcal{H})) \subset M_n(\text{CB}(\text{Mult}(\mathcal{H} \otimes \mathcal{H}), \text{Mult}(\mathcal{H} \otimes \mathcal{H}))),
$$
we apply Lemma 4.2 and instead compute the norm of 

\[ [M^\dagger_{\theta_{ij}}] \in M_n(CB(HAN(H), HAN(H))) = CB(HAN(H), M_n(HAN(H))). \]

So let \( b \in \text{Han}(H) \). An application of part (a) of Lemma 2.3 shows that, using notation as in the discussion preceding Theorem 4.1

\[ [M^\dagger_{\theta_{ij}}(Hb)] = \sum_{r=1}^N [T^*_{\psi_{riv}} Hb T_{\varphi_{riv}}] = T^*_{\Psi}(Hb \otimes I_{CN})T_{\Phi}. \]

Since \( T_{\Phi} \) and \( T_{\Psi} \) have norm at most 1, this formula implies that \( [M^\dagger_{\theta_{ij}}] \) is a completely contractive map, so (i) holds.

(i) \( \Rightarrow \) (ii) We merely sketch the main steps, as the proof closely follows that of Theorem 3.8. Let \( \Theta = [\theta_{ij}] \) be an element of the unit ball of \( M_n(\text{Mult}(H \odot H)) \). Using duality, more precisely Lemma 4.2, it follows that the map

\[ \text{HAN}(H) \to M_n(\text{HAN}(H)), Hb \mapsto [M^\dagger_{\theta_{ij}}(Hb)], \]

is completely contractive. With minor changes, the dilation theoretic argument in the proof of Lemma 3.6 yields linear contractions \( V : H \otimes C^n \to H \otimes \ell^2 \) and \( W : H \otimes C^n \to H \otimes \ell^2 \) such that

\[ [M^\dagger_{\theta_{ij}}(Hb)] = W^*(Hb \otimes I)V \quad (b \in \text{Han}(H)). \]

As in the proof of Theorem 3.8, the commutant lifting theorem allows us to replace \( V \) and \( W \) with multiplication operators. More precisely, there are contractive multipliers \( \Phi, \Psi \in \text{Mult}(H \otimes C^n, H \otimes \ell^2) \) so that

\[ (9) \quad [M^\dagger_{\theta_{ij}}(Hb)] = T^*_{\Psi}(Hb \otimes I)T_{\Phi} \quad (b \in \text{Han}(H)). \]

Somewhat more explicitly, to find \( \Phi, \Psi \), define \( M \subset H \otimes \ell^2 \) and \( X = P_M V \) verbatim as in the proof of Theorem 3.8. The bimodule property of \( M^\dagger_{\theta_{ij}} \) (part (b) of Lemma 2.3) implies that \( X(T_{\varphi} \otimes I_{\ell^2}) = P_M(T_{\varphi} \otimes I_{\ell^2})X \) for all \( \varphi \in \text{Mult}(H) \), hence the commutant lifting theorem applies. Finally, testing (9) for \( b = k_z \) yields \( \Theta(z) = \Psi^T(z)\Phi(z) \), so we have found the desired factorization. \( \square \)

The ideas used to prove Theorems 1.2 and 4.1 also yield a more concrete description of the norm on \( M_n(H \odot H) \). If \( n = 1 \) and \( h \in H \odot H \), then

\[ \|h\|_{H \odot H} = \inf \left\{ \|(f_k)\|_{H \otimes \ell^2} \|(g_k)\|_{H \otimes \ell^2} : h = \sum_{k=1}^\infty f_k g_k \right\}. \]

Indeed, this follows from the definition of the norm on \( H \odot H \) by trading constant factors between \( f_k \) and \( g_k \). This last formula can be generalized. The column operator space structure on \( H \) is defined by the identification \( H = B(C, H) \), and the resulting operator space is denoted by \( H_c \); see [8, Section 3.4]. Thus, \( M_n(H_c) = \)
\[ B(\mathbb{C}^n, \mathcal{H}^n). \] We also require matrices with infinitely many rows. Let \( M_{\infty,n}(\mathcal{H}_c) \) be the space of all matrices with entries in \( \mathcal{H} \) of the form
\[
 f = \begin{bmatrix}
 f_{11} & f_{12} & \cdots & f_{1n} \\
 f_{21} & f_{22} & \cdots & f_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 \end{bmatrix}
\]
satisfying \( \sum_{i=1}^{\infty} \|f_{ij}\|^2 < \infty \) for \( 1 \leq j \leq n \). As we did for finite matrices, we regard such a matrix as a bounded linear operator from \( \mathbb{C}^n \) to \( \mathcal{H} \otimes \ell^2 \), and we set
\[
 \|f\|_{M_{\infty,n}(\mathcal{H}_c)} = \|f\|_{B(\mathbb{C}^n, \mathcal{H} \otimes \ell^2)}. 
\]
Notice that if \( n = 1 \), then \( M_{\infty,1}(\mathcal{H}_c) = \mathcal{H} \otimes \ell^2 \) with equality of norms. Given \( f, g \in M_{\infty,n}(\mathcal{H}_c) \), we define as above an \( n \times n \) matrix \( g^T f \) of functions on \( X \) by
\[
 (g^T f)_{ij} = \sum_{k=1}^{\infty} f_{kj} g_{ki} \quad (1 \leq i, j \leq n). 
\]
By the Cauchy–Schwarz inequality, the sum converges pointwise absolutely on \( X \).

**Lemma 4.3.** The following assertions are equivalent for an \( n \times n \) matrix \( h \) of functions on \( X \).

(i) The matrix \( h \) belongs to the closed unit ball of \( M_n(\mathcal{H} \otimes \mathcal{H}) \).

(ii) There exist \( f \) and \( g \) in the closed unit ball of \( M_{\infty,n}(\mathcal{H}_c) \) so that \( h = g^T f \).

Thus, if \( h \in M_n(\mathcal{H} \otimes \mathcal{H}) \), then
\[
 \|h\|_{M_n(\mathcal{H} \otimes \mathcal{H})} = \inf \{ \|f\|_{M_{\infty,n}(\mathcal{H}_c)} \|g\|_{M_{\infty,n}(\mathcal{H}_c)} : h = g^T f \},
\]
and the infimum is attained.

**Proof.** (ii) \(\Rightarrow\) (i) By the Cauchy–Schwarz inequality, the sum defining each entry of \( h \) converges absolutely in the Banach space \( \mathcal{H} \otimes \mathcal{H} \). In particular, each entry of \( h \) belongs to \( \mathcal{H} \otimes \mathcal{H} \). We will show that \( h \) belongs to the unit ball of \( M_n(\mathcal{H} \otimes \mathcal{H}) \). By definition of the operator space structure on \( \mathcal{H} \otimes \mathcal{H} \) as the dual of \( \text{HAN}_0(\mathcal{H}) \), we have to show that the map
\[
 A : \text{HAN}_0(\mathcal{H}) \to M_n, \quad H_b \mapsto [h_{ij}, H_b]_{i,j},
\]
is completely contractive, where the inner brackets denote the duality between \( \mathcal{H} \otimes \mathcal{H} \) and \( \text{HAN}(\mathcal{H}) \). To this end, notice that for \( 1 \leq i, j \leq n \), Equation (2) implies that
\[
 [h_{ij}, H_b] = \sum_{k=1}^{\infty} [f_{kj} g_{ki}, H_b] = \sum_{k=1}^{\infty} \langle H_b f_{kj}, \overline{g_{ki}} \rangle_{\mathcal{H}},
\]
for all \( b \in \text{HAN}(\mathcal{H}) \). Let \( \overline{g} \) denote the entry-wise complex conjugate of \( g \), regarded as a contractive operator from \( \mathbb{C}^n \) to \( \overline{\mathcal{H}} \otimes \ell^2 \), and let \( \overline{f} : \overline{\mathcal{H}} \otimes \ell^2 \to \mathbb{C}^n \) be the Hilbert space adjoint of \( \overline{g} \). Then
\[
 A(H_b) = [h_{ij}, H_b]_{i,j} = \overline{f} (H_b \otimes I_{\ell^2}) f \quad (H_b \in \text{HAN}_0(\mathcal{H})),
\]
which implies that the map \( A \) is completely contractive.
(i) ⇒ (ii) If $h$ belongs to the unit ball of $M_n(H \otimes H)$, then by definition of the operator space structure on $H \otimes H$, the map $A$ defined in the first part of the proof is completely contractive. Applying the Haagerup–Paulsen–Wittstock dilation theorem as in the proof of Lemma 3.6 and using the fact that every $\ast$-representation of $K(H)$ is unitarily equivalent to a multiple of the identity representation, we obtain linear contractions $V : C^n \rightarrow H \otimes \ell^2$ and $W : C^n \rightarrow H \otimes \ell^2$ so that

\[ A(H_b) = W^*(H_b \otimes I_{\ell^2})V \quad (H_b \in \text{HAN}_0(H)). \]

Define $f, g \in M_{\infty,n}(H)$ by $f = V$ and $g = W$. We see that $f$ and $g$ have norm 1 and

\[ [h_{ij}, H_b] = \sum_{k=1}^{\infty} \langle H_b f_{kj}, g_{ki} \rangle \]

for $1 \leq i, j \leq n$ and $H_b \in \text{HAN}_0(H)$. Testing this equation for $b = k_z$, we conclude that $h(z) = g^T(z)f(z)$, as desired. □

Remark 4.4. Lemma 4.3 and Theorem 4.1 can be restated in terms of the Haagerup tensor product $\otimes_h$ of operator spaces (see [15, Chapter 17], [6, Paragraph 1.5.4] or [8, Chapter 9]), its weak-$\ast$ version $\otimes_{w\ast,h}$ (see [6, Paragraph 1.6.9]) and the opposite operator space structure $V^\text{op}$ of an operator space $V$ (see [6, Paragraph 1.2.25]). Concretely, Lemma 4.3 implies that

\[ H^\text{op}_c \otimes_h H_c \rightarrow H \otimes H, \quad \sum_{n=1}^{\infty} f_n \otimes g_n \mapsto \sum_{n=1}^{\infty} f_n g_n, \]

is a complete quotient mapping. Theorem 4.1 implies that

\[ \text{Mult}(H)^{\text{op}} \otimes_{w\ast,h} \text{Mult}(H) \rightarrow \text{Mult}(H \otimes H), \quad \sum_{n=1}^{\infty} \varphi_n \otimes \psi_n \mapsto \sum_{n=1}^{\infty} \varphi_n \psi_n, \]

is a complete quotient mapping. We will not use these formulations.

4.2. Comparing the operator space structures of $\text{Mult}(H)$ and $\text{Mult}(H \otimes H)$. We saw in Corollary 1.3 that if $H$ satisfies the column-row property (which is the case for instance for the Drury–Arveson space), then the inclusion $\iota : \text{Mult}(H) \hookrightarrow \text{Mult}(H \otimes H)$ is an isomorphism of Banach spaces. For any normalized complete Nevanlinna–Pick space, the implication (ii) ⇒ (i) of Theorem 4.1 shows that $\iota$ is a complete contraction. If $H = H^2(\mathbb{D})$, then the norm of $M_n(\text{Mult}(H))$ is simply the supremum norm over $\mathbb{D}$, hence Theorem 4.1 implies that the same is true for $M_n(\text{Mult}(H \otimes H))$. In other words, in the case of $H^2(\mathbb{D})$, the map $\iota$ is a completely isometric isomorphism. Note however that the entire space $\text{CB}(H \otimes H)$ is not completely boundedly isomorphic to an operator algebra unless $H \otimes H$ is isomorphic to a Hilbert space by [6, Proposition 5.1.9].

We show that the phenomenon observed above is somewhat special to the univariate Hardy space.
Proposition 4.5. Let $\mathcal{H}$ be either the Drury–Arveson space $H_d^2$ for $d \geq 2$ or the classical Dirichlet space. Then the inclusion 
$$
i : \text{Mult}(\mathcal{H}) \hookrightarrow \text{Mult}(\mathcal{H} \odot \mathcal{H})$$
does not have a completely bounded inverse.

Proof. It follows from Theorem 4.1 that for each $n \in \mathbb{N}$, the transpose map 
$$M_n(\text{Mult}(\mathcal{H} \odot \mathcal{H})) \to M_n(\text{Mult}(\mathcal{H} \odot \mathcal{H})), \quad \Theta \mapsto \Theta^T,$$
is isometric. On the other hand, there exist sequences of multipliers $(\varphi_n)$ in $\text{Mult}(\mathcal{H})$ that yield a bounded row multiplication operator, but an unbounded column multiplication operator. For the Dirichlet space, this can be seen from the discussion preceding Lemma 1 in [21]; for the Drury–Arveson space, see [3, Subsection 4.2]. In particular, the norms of the transpose maps 
$$M_n(\text{Mult}(\mathcal{H})) \to M_n(\text{Mult}(\mathcal{H})), \quad \Phi \mapsto \Phi^T,$$
are not uniformly bounded in $n$, so that the completely contractive map $\iota$ does not have a completely bounded inverse. \(\square\)

In fact, it is possible to determine explicitly the growth of the norms of $(\iota^{-1})^{(n)}$ in the case of the Drury–Arveson space. We begin with the following easy estimate.

Lemma 4.6. Let $\mathcal{H}$ be a reproducing kernel Hilbert space that satisfies the column-row property with constant $\kappa$. Then 
$$\|\Psi^T\|_{\text{Mult}(\mathcal{H} \odot \ell^2, \mathcal{H} \odot \mathbb{C}^n)} \leq \sqrt{n\kappa}\|\Psi\|_{\text{Mult}(\mathcal{H} \odot \mathbb{C}^n, \mathcal{H} \odot \ell^2)}$$
for all $\Psi \in \text{Mult}(\mathcal{H} \odot \mathbb{C}^n, \mathcal{H} \odot \ell^2)$.

Proof. Suppose that 
$$\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{nn} \end{bmatrix}$$
has multiplier norm at most 1. Then each of the columns has multiplier norm at most 1, so the column-row property shows that each row 
$$R_i = [T_{\psi_{1i}} \ T_{\psi_{2i}} \ T_{\psi_{3i}} \ \cdots]$$
has norm at most $\kappa$, hence 
$$\|\Psi^T\|_{\text{Mult}(\mathcal{H} \odot \ell^2, \mathcal{H} \odot \mathbb{C}^n)}^2 = \left\| \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix} \right\|^2 = \left\| \sum_{i=1}^n R_i^*R_i \right\| \leq n\kappa^2. \quad \square$$

If $A : V \to W$ is a bounded map between operator spaces, then $\|A^{(n)}\| \leq n\|A\|$, and this inequality is sharp in general; see for instance [15 Exercise 3.10]. In our setting, the preceding lemma, combined with the implication (i) $\Rightarrow$ (ii) of Theorem 4.1, implies the following better upper bound.
Corollary 4.7. Let $\mathcal{H}$ be a normalized complete Nevanlinna–Pick space on $X$ that satisfies the column-row property with constant $\kappa$ and let

$$\iota : \text{Mult}(\mathcal{H}) \to \text{Mult}(\mathcal{H} \odot \mathcal{H})$$

be the completely contractive inclusion. Then, $\iota$ is a bijection, and

$$\| (\iota^{-1}(n)) \| \leq \sqrt{n\kappa}$$

for all $n \in \mathbb{N}$. □

In the Drury–Arveson space, the upper bound in the preceding corollary is essentially best possible. To see this, we require a refinement of the construction in [3, Subsection 4.2]. Given $\{\varphi_1, \ldots, \varphi_n\} \subset \text{Mult}(\mathcal{H})$, the quantities

$$\| [\varphi_1 \varphi_2 \cdots \varphi_n] \|_{\text{Mult}(\mathcal{H} \odot \mathbb{C}^n, \mathcal{H})} \quad \text{and} \quad \| \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} \|_{\text{Mult}(\mathcal{H}, \mathcal{H} \odot \mathbb{C}^n)}$$

are called the row norm and column norm, respectively.

Lemma 4.8. Let $d \geq 2$. Then for all $n \geq 1$, there exists $\{\varphi_1, \ldots, \varphi_n\} \subset \text{Mult}(\mathcal{H}_d^2)$ with row norm 1 and column norm $\sqrt{n}$.

Proof. For $0 \leq k \leq n$, let

$$\psi_k = \binom{n}{k}^{\frac{1}{2}} z_1^k z_2^{n-k}.$$ 

It easily follows from the fact that the coordinate functions form a row contraction that $\sum_\alpha T_{z_\alpha} T_{z_\alpha}^* \leq I$, hence the row norm of $\{\psi_0, \ldots, \psi_n\}$ is at most 1.

On the other hand,

$$\left\| \begin{bmatrix} T_{\psi_0} \\ \vdots \\ T_{\psi_n} \end{bmatrix} \right\|_1 = \sum_{k=0}^n \| \psi_k \|^2 = \sum_{k=0}^n \binom{n}{k} \left\| z_1^k z_2^{n-k} \right\|^2 = n + 1.$$ 

Hence, the column norm of $\{\psi_0, \ldots, \psi_n\}$ is at least $\sqrt{n + 1}$. Since the column norm is also dominated by

$$\left\| \sum_{k=0}^n T_{\psi_k}^* T_{\psi_k} \right\|^{\frac{1}{2}} \leq \sqrt{n + 1} \max_{0 \leq k \leq n} \| T_{\psi_k} \| = \sqrt{n + 1},$$

the estimates for both the column and the row norm are in fact equalities. □
Thus, we obtain the exact behavior of $\|((\varepsilon^{-1})^{(n)})\|$, up to multiplicative constants, in the case of the Drury–Arveson space.

**Proposition 4.9.** Let $d \geq 2$ and consider the completely contractive inclusion
$$
i : \text{Mult}(H^2_d) \to \text{Mult}(H^2_d \odot H^2_d).$$
Then, there exists a constant $\kappa > 0$ depending only on $d$, so that
$$\sqrt{n} \leq \|((\varepsilon^{-1})^{(n)})\| \leq \kappa \sqrt{n}$$
for all $n \geq 1$.

**Proof.** The upper bound follows from Corollary 4.7 and the column-row property for $H^2_d$; see [3, Theorem 1.5].

To obtain the lower bound, we use Lemma 4.8 to find a row multiplier
$$\Psi = \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \end{bmatrix}$$
of norm 1 so that $\|\Psi^T\|_{\text{Mult}(\mathcal{H},\mathcal{H} \otimes \mathbb{C}^n)} = \sqrt{n}$. Let
$$\Phi = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$The implication (ii) $\Rightarrow$ (i) of Theorem 4.1 shows that $\Psi^T \Phi$ belongs to the closed unit ball of $M_n(\text{Mult}(\mathcal{H} \odot \mathcal{H}))$. On the other hand,
$$\|\Psi^T \Phi\|_{M_n(\mathcal{H})} = \|\Psi^T\|_{\text{Mult}(\mathcal{H},\mathcal{H} \otimes \mathbb{C}^n)} = \sqrt{n}$$thus showing that $\|((\varepsilon^{-1})^{(n)})\| \geq \sqrt{n}$. □

**Note added in proof.** The recent paper [10] shows that every complete Nevanlinna–Pick space satisfies the column-row property with constant 1. Hence the additional assumption in Corollary 1.3 is automatically satisfied

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