Half-plane contacts subject to remote tension

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Abstract
The state of stress present in an elastic half-plane contact problem, where one or both bodies is subject to remote tension has been investigated, both for conditions of full stick and partial slip. The state of stress present near the contact edges is studied for different loading scenarios in an asymptotic form. This is of practical relevance to the study of contacts experiencing fretting fatigue, and enables the environment in which cracks nucleate to be specified.

Keywords
Bulk tension, half-plane contact, partial slip, fretting fatigue, asymptotes

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Introduction
Fretting and fretting fatigue are important problems arising in gas turbines and a wide range of mechanical assemblies – in fact wherever there are components in notionally stationary contact but where the applied loads or vibration cause very small amounts of differential motion between the contacting surfaces and which consequently may cause damage promoting cracks to nucleate. Of course it is the presence of shear tractions, $q(x)$, which induces slip and, for elastically similar bodies, this may have two origins – the application of a shear force $Q$ (Figure 1(a)) or the presence of a differential tension $s_0 = s_1 - s_2$ acting parallel with the surface, Figure 1(b). Unless the tensions parallel with the surface are equal in each body shear tractions will certainly develop, and which may add to or subtract from the effect of shear force, depending on which edge of the contact is being studied.

In some problems, such as the standard laboratory test for fretting fatigue strength, Figure 1(d), it is very easy to identify the bulk tension present in each component (here, none in the pad and simply the tension in the dogbone specimen) but in many cases it is very much harder to isolate the effect of tension, such as in the fanblade dovetail arrangement in Figure 1(e), with the blade being subject to a centrifugal and vibrational load, $F_c$ and $F_v$, respectively. Andresen et al. have recently looked at means of identifying the effective bulk tension present, and there is little doubt that it has a first order effect on both the magnitude of slip and also on the state of stress present in each body. Here we concentrate on the latter, and look at contacts represented by half-planes being a typical approximation for incomplete contacts. These contacts typically arise when bodies with convex surfaces are pressed against each other by a normal load, and show the characteristic that the contact area increases in size with an increasing normal load. In the standard laboratory test (Figure 1(d)) as well as the fanblade dovetail arrangement (Figure 1(e)) we find this type of contact. We study the differences for the application of a differential tension compared to the application of a shear load, and how the combination of $s_1$ and $s_2$ affects the state of stress around the edge of the contact.

We start from the assumption that the coefficient of friction, $f$, is sufficiently high to prevent all slip at the interface. If all slip is inhibited, the problem is equivalent to an infinite plane with two semi-infinite cracks indented into it from $\pm \infty$ to $\pm a$, Figure 1(b), and we start from the case where the upper half-plane ($y > 0$) is free of remote loads while the lower is subjected to remote tension $s_0 = 0$. It has been shown, using the
Mossakovskii-Barber approach, that the traction arising along the interface, \( y = 0 \) and \( |x| < a \), is given by

\[
\sigma_{xy}(x,0) = q(x) = -\frac{\sigma_0}{4} \frac{x}{\sqrt{a^2 - x^2}}
\]  

(1)

and the state of stress in the body is therefore the sum of the state of stress induced in a half-plane by tractions (1) applied along the surface together with, for the lower body only, uniform remote tension \( \sigma_0 \).

**State of stress present**

If we consider only the half-plane \( y < 0 \) and exert tractions (1) over the surface the Muskhelishvili complex potential is given by,

\[
\phi(z) = \frac{1}{2\pi} \int_{-a}^{a} \frac{\sigma_{xy}(x,0)}{x-z} \, dx
\]

(2)

\[
\phi(z) = -\frac{\sigma_0}{8\pi} \begin{cases} 
\frac{a}{\sqrt{a^2 - x^2}} & x < 0 \\
\frac{a}{\sqrt{a^2 - x^2}} & x > 0
\end{cases}
\]

where \( z = x + iy, \, i = \sqrt{-1} \).

The same result is obtained without the need for any integration by making use of the relationship

\[
\phi(z) = -\frac{\lambda}{2} - \frac{1}{2} i\sigma_{xy}^{ad}(z)
\]

(3)
where \( \sigma_{xy} \) is the shear stress inside the stick zone. The quantity \( \lambda \) is a load case dependent parameter defined by the singular integral equation

\[
\frac{1}{\pi} \int_{\text{Contact}} \frac{\sigma_{xy}(\xi, 0)}{x - \xi} d\xi = \lambda \quad \forall x \in \text{stick zone}
\]

Figure 2. Subsurface stress field for fully stuck contact with differential bulk tension, \( \sigma_0 \), applied to the lower body only: (a) normalised stress \( \sigma_{xx}/\sigma_0 \), (b) normalised stress \( \sigma_{yy}/\sigma_0 \), and (c) normalised stress \( \sigma_{xy}/\sigma_0 \).

where the Cauchy principal value must be considered. For a differential bulk tension \( \sigma_0 \), \( \lambda = \sigma_0/4 \), independent of the individual components \( \sigma_1 \) and \( \sigma_2 \). Here, relation (3) simply provides a different route to the same result obtained in equation (2), but later on when looking at contacts in partial slip with a comparatively complicated form of the resulting shear traction it provides a much simpler way of obtaining the complex potential compared to the evaluation of the integral over the contact interface. Independent of how the complex potential \( \phi(z) \) is obtained, the corresponding state of stress is given by

\[
\sigma_{xx}(x, y) = 2(\phi(z) + \phi(z)) + iy \left( \frac{\partial}{\partial z} \phi(z) - \frac{\partial}{\partial \bar{z}} \phi(z) \right),
\]

\[
\sigma_{yy}(x, y) = -iy \left( \frac{\partial}{\partial z} \phi(z) - \frac{\partial}{\partial \bar{z}} \phi(z) \right),
\]

\[
\sigma_{xy}(x, y) = i(\phi(z) - \phi(z)) - y \left( \frac{\partial}{\partial z} \phi(z) + \frac{\partial}{\partial \bar{z}} \phi(z) \right).
\]

Explicit expressions in real algebra can be found by separation of real and imaginary part of the potential function to \( \phi(z) = u(x, y) + iv(x, y) \). The components of stress are then obtained from the complex potential (2) and in the half-plane \( y < 0 \) are

\[
\sigma_{xx}(x, y) = -\frac{\sigma_0}{8} \frac{4c_1x^2(y^2 + \sqrt{c_1})}{c_1x \sqrt{2c_1(c_1 + c_2)}}
\]

\[
+ \frac{\sigma_0}{8} \frac{a^2(-c_1c_2 + c_2^2 - 4x^2(c_1 + y^2 \sqrt{c_1 + c_2}))}{c_1x \sqrt{2c_1(c_1 + c_2)}},
\]

\[
\sigma_{xy}(x, y) = \frac{\sigma_0}{8} \frac{a^2(16c_1^2x^4 + c_1c_2 \sqrt{c_1 + c_2})}{c_1x \sqrt{2c_1(c_1 + c_2)}}
\]

\[
- \frac{\sigma_0}{16} \frac{c_2(c_2 - \sqrt{c_1}) - 4x^2y^2}{c_1x \sqrt{2c_1(c_1 + c_2)}},
\]

\[
\sigma_{yy}(x, y) = \frac{\sigma_0}{8} \frac{a^2c_2 (\sqrt{c_1 + c_2})}{c_1x \sqrt{2c_1(c_1 + c_2)}}
\]

\[
+ \frac{\sigma_0}{8} \frac{yc_2}{c_1x \sqrt{2c_1(c_1 + c_2)}},
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in the other body (Figure 3(b)) the solution may be found by superposition of these two solutions viz a mean stress of \( \frac{1}{2}(\sigma_1 + \sigma_2) \), everywhere, together with the non-vanishing components of the full stress field along the surface \( (v \to 0) \) are shown in Figure 3. We see that for the case of equal and opposite loading, as \( \sigma_m = 0 \), the stress parallel to the surface vanishes along the interface, that is, on \( y = 0, |x| < a \). At points remote to the contact it tends to \( \sigma_1 = \sigma_0/2 \).

Furthermore, we use Figure 3 to compare the stress along the contact interface, \( y = 0 \), to the stress present for the application of a shear load as shown in Figure 1(a). We used different normalisations for both load cases to ensure that the shear stress shows the same singularity when approaching the edge of the contact (Figure 3(b)). We note from Figure 3(a) that even though the stress parallel to the surface does vanish for both load cases inside the contact, \( |x| < a \), and the resulting singularity in the stress parallel to the interface is similar for both load cases, the stress will differ remote from the contact.

**Asymptotic solutions**

In the previous section we examined the state of stress in adhered half-plane contacts subject to remote tension, in detail, and compared some results with the stress state induced by the application of a shear force. One reason for doing this was as a prelude to understanding how the different forms of load affect the behaviour of contact edges, under adhered conditions and in the presence of slip. So far we considered the stress inside the contact region, \( |x| < a \) and remote from the contact, \( |x| \gg a \), but not the singularity at the edge of the contact.

We have had some success in representing the state of stress near an incomplete contact edge in asymptotic form. The contact pressure is locally square root bounded but, if the contact is adhered the shear tractions induced are, as we have just seen, square root singular in form. One important aspect of the asymptotic representation which has been missing is an understanding of the asymptotic field when the shear tractions are excited by bulk tension; information is lost when only the surface tractions induced are considered, and here we derive the local solution when tension induced, first under conditions of full stick. If we move the origin to the left hand edge by the transformation \( s = x + a \) equation (1) becomes

\[
\sigma_{xy}(s) = -\frac{\sigma_0}{4} \frac{s - a}{\sqrt{2as - s^2}}
\]

\[\approx K_{II} \frac{1}{\sqrt{s}} \quad \text{for} \quad s \ll a \quad (7)
\]

where \( K_{II} = \sigma_0 \sqrt{a}/4\sqrt{2} \). To find the stress parallel to the surface or the subsurface stress field in an asymptotic description we can proceed in one of two ways. Either we start from the negative branch of the finite potential function (2) and again apply a shift of the origin with \( w = z + a \) followed by a binomial expansion as well as the corresponding simplifications to the derivation of the asymptotic contact shear traction which yields

\[
\phi(w) = -\frac{\sigma_0}{8} \left(1 + \frac{w - a}{\sqrt{w^2 - 2aw}}\right)
\]

\[\approx K_{II} \frac{1}{\sqrt{w}} - \frac{\sigma_0}{8} \quad \text{for} \quad \text{abs}(w) \ll a. \quad (8)
\]

The different steps with corresponding assumptions for the derivation of equations (7) and (8) are given in
Appendix 2. For the potential function the same result can be obtained by using the asymptotic expression of the shear traction (7) in equation (3) and choosing the appropriate branch of the function $f = \sqrt{w}$

$$
\phi(w) = -\frac{\lambda}{2} \frac{1}{w} \left[ i \sigma_0 \phi_{\sigma_0}(w) \right]
= -\frac{\sigma_0}{8} + \frac{1}{2} \frac{iK_{II}}{\sqrt{w}}
= K_{II} \frac{1}{2} \frac{1}{\sqrt{-w}} - \frac{\sigma_0}{8}.
$$

The resulting potential function differs from the potential function for the application of a shear load by the constant term $-\sigma_0/8$ only. From equation (5) we can immediately see that a constant real term in the potential function will contribute to the stress parallel to the surface only and we find $\sigma_{xx} = -\sigma_0/2$. The complete stress field includes the superposition of the remotely applied tension $\sigma_1$ (for the lower body). In the neighbourhood of the contact edge the stress parallel to the surface then differs by $-\sigma_0/2 + \sigma_1 = \sigma_m$ for the two load cases. All other components of stress are unchanged, that is, the shear force induced $K_{II}$ solution and the bulk stress induced $K_{II}$ solution are the same. We note that for the special case of an equal tension and compression in the two bodies, that is, $\sigma_m = 0$ as considered before, the whole stress field in the neighbourhood of the contact edge will be the same as when the stress is excited by a shear force.

**Effect of slip**

We now want to investigate the effect of a differential bulk stress on the subsurface stress field by allowing for local slip at the contact interface of an incomplete contact. We consider, here, a finite Hertzian contact with the known contact pressure

$$
\sigma_{yy}(x, 0) = p(x) = -\frac{p_0}{a} \sqrt{a^2 - x^2} \quad \text{for} \quad |x| \leq a,
$$

where $p_0$ is the peak contact pressure. After the application of the normal load we apply a bulk tension $\sigma_0$ to the lower body. Symmetric zones of slip will establish themselves at the edges of the contact. The resulting shear traction was found by Nowell and Hills, Ciavarella and Macina and Moore et al. Moore et al. use distributed dislocations along the contact interface to account for relative displacements of the two bodies and found the side condition for the determination of the stick-zone half-width, $c$, generally, as

$$
-\int_c^a \frac{p(s)ds}{\sqrt{s^2 - c^2}} = \frac{\pi \sigma_0}{8f}.
$$

where $f$ is the coefficient of friction, as before. If the Hertzian pressure distribution (10) is substituted into (11), Ciavarella and Macina’s implicit expression

$$
K(k) - E(k) = \frac{\pi \sigma_0}{8fp_0}
$$

is recovered, where $K(k)$ and $E(k)$ represent the complete elliptic integral of the first and second kind respectively with $k$ being the elliptic modulus (Note that there are different conventions for the notation of elliptic integrals in terms of the elliptic modulus $k$ or the parameter $m = k^2$). In equation (12) and the following

$$
k = \sqrt{1 - \left(\frac{c}{a}\right)^2},
$$

The shear traction inside the stick zone is then given by,

$$
\sigma_{yy}(x, 0) = \frac{2fp_0}{\pi} \frac{1}{d^2} \sqrt{c^2 - x^2} \left[ \Pi(n, k) - K(k) \right]
$$

for $|x| < c$

**Figure 4.** Non-vanishing components of stress along the contact interface for the application of equal and opposite remote stress $\sigma_0$ (solid), the application of a shear load (dashed) and mixed loading such that the contact is fully stuck at the leading edge (dotted) and a slip zone side of $d/a = 0.25$ at the trailing edge: (a) normalised stress $\sigma_{xx}$ and (b) normalised stress $\sigma_{yy}$.
where $\Pi(n,k)$ represents the complete elliptic integral of the third kind, $k$ is again given by equation (13) and

$$n = \sqrt{\frac{a^2 - c^2}{a^2}}$$

(15)

is the so-called characteristic of $\Pi$. The complex potential is then simply derived by substitution of equation (14) in expression (3), where $\lambda = \sigma/4$ as for the fully stuck case, giving

$$\phi(x) = -\frac{\sigma_0}{8} \pm \frac{2fp_0}{\pi} \sqrt{z^2 - c^2} [\Pi(n,k) - K(k)].$$

(16)

Again, the upper sign in equation (16) corresponds to $x < 0$ and the lower sign to $x > 0$. For the case of an equal tension and compression of magnitude $\sigma_0/2$ in the two bodies, the non-vanishing stresses along the contact interface are shown in Figure 4. As for the fully stuck case the stress parallel to the surface vanishes inside the stick region and has its maximum at the edge of and exterior to the contact.

We compare the results for the case of an equal tension and compression in the two bodies to the stresses resulting from two different load cases, one where we apply a shear load to the upper body and a second where we have a mix of differential bulk tension and shear load such that one edge of the contact (the right hand end) remains stuck. In each case the slip zone, $d = a - c$, at the left hand edge is of the same size and the slip displacement is in the same direction. Figure 4(b) shows the resulting shear stress which is similar for all three cases within the slip zone at the left hand edge.

We are now able to compare the maximum stress parallel to the surface for the three load cases under consideration, as this is most important in terms of fretting damage. For the assumed slip zone size of $d/a = 0.25$ we see that the stress will be higher if the relative displacement is caused by a bulk tension compared to a shear force (Figure 4(a)). Figure 5 shows the stress parallel to the interface at the edge of the contact not only for the slip-zone size of $d/a = 0.25$, considered before, but for a continuous slip zone size.

For a slip zone size of $d/a = 1$ and the application of a shear load we recover the result of $\sigma_{\text{xx}}/fp_0 = 2$ for a sliding contact\textsuperscript{12} while the consideration of the contact stress field alone with the application of a differential bulk tension leads to $\sigma_{\text{xx}}/fp_0 = 4/\pi$ as shown by Ciavarella and Macina.\textsuperscript{10} Of course the results for the full stress field are somewhat arbitrary, in the case of differential bulk tension and mixed loading, since the same slip zone size could originate from different combinations of applied bulk tension and compression in the two bodies, but the stress inside the stick zone will vanish only, for the special case of equal remote tension and compression. We find from Figure 5 that the difference in the maximum stress parallel to the surface vanishes for $d \ll a$ and we will look at this in more detail by considering the asymptotic description.

**Asymptotic solutions**

Again, the subsurface stress-field might be described in an asymptotic formulation. It is worth noting that by representing each of the elliptical integrals in equation (12) by the first two terms of the corresponding series expansion around the edge of the contact, that is, $c/a = 1$, we recover the known asymptotic expression\textsuperscript{13}

$$\frac{\pi \sigma_0}{8fp_0} = K(k) - E(k) = \frac{\pi}{2} \left(1 - \frac{c}{a}\right)$$

$$\Leftrightarrow \frac{\sigma_0a}{4fp_0} = a - c$$

(17)

$$\Leftrightarrow \frac{2K_{LI}}{fp_{LI}} = d$$

for the slip zone $d$ where the shear traction results from a tangential force and the asymptotic multiplier $L_{LI}$ is defined as

$$L_{LI} = \lim_{x \to 0} \frac{p(s)}{\sqrt{s}}.$$  

(18)

Furthermore, for an asymptotic representation of the bilateral contact tractions with $q(s) = K_{II}/\sqrt{s}$ (cf. equation (7)) and $p(s) = L_{LI}/\sqrt{s}$, Moore et al.\textsuperscript{11} derive the dislocation density to (Note that equation (29) in the original publication\textsuperscript{11} is a factor of $-2$ wrong.)

$$B_s(s) = -\kappa + \frac{1}{2\mu} \frac{1}{L_{LI} \sqrt{d - s}},$$

(19)

where $\mu$ is the shear modulus of the contacting bodies and $\kappa = 3 - 4\nu$ is Kolosov’s constant for plane strain with $\nu$ being Poisson’s ratio. The resulting shear traction inside the stick zone is then given by

![Figure 5. Maximum stress parallel to the surface plotted over the slip zone size at the left hand edge of the contact.](image-url)
The observed stress field in the neighborhood of the contact edge in the lower body for the application of an equal load, by a constant stress acting parallel with the free surface, and which is responsible for the early propagation of any crack which may form, differs by the mean value of applied bulk tensions \( \sigma_m = \frac{1}{2} (\sigma_1 + \sigma_2) \). The only scenario where the local stress parallel to the surface is similar regardless whether the local shear tractions originate from a shear force or a differential bulk tension, is the special case of an equal tension and compression of the two bodies, that is, \( \sigma_m = 0 \). Furthermore, the state of stress close to the contact edge depends on the mean value of applied bulk tensions, \( \sigma_m \), so that a comparison of the stress field in which cracks nucleate in the fanblade dovetail arrangement (Figure 1(e)) and in the standard laboratory test (Figure 1(d)) depends not only on \( \sigma_0 \) but also \( \sigma_m \).

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find a split into a real and an imaginary part.

The complex function here the complex potential, terms of the real variables \((x,y)\). In the case considered Va`zquez et al.\(^7\) describe how to derive the stress in Appendix 1

\[
\phi(z) = -\frac{\sigma_0}{8} \left( 1 + \frac{z}{\sqrt{z^2 - a^2}} \right)
\]

\[
= -\frac{\sigma_0}{8} \left( 1 + f(z) \right)
\]

\[
= -\frac{\sigma_0}{8} u(x,y) - i\frac{\sigma_0}{8} v(x,y).
\]

For \(f(z) = z/\sqrt{z^2 - a^2}\) we find the corresponding split to\(^{15}\)

\[
u(x,y) = \frac{x(x^2 + y^2 - a^2 + \sqrt{c_1})}{\sqrt{2c_1\sqrt{c_1} + 2c_1c_2}}
\]

and

\[
r(x,y) = -\frac{y(x^2 + y^2 + a^2 - \sqrt{c_1})}{\sqrt{2c_1\sqrt{c_1} + 2c_1c_2}},
\]

where \(c_1 = ((a + x)^2 + y^2)((a - x)^2 + y^2)\) and \(c_2 = x^2 - y^2 - a^2\). The derivatives of \(\phi\) are then given to\(^7\)

\[
\frac{\partial}{\partial z} \phi(z) = -\frac{\sigma_0}{8} \left( \frac{\partial}{\partial x} u(x,y) - i \frac{\partial}{\partial y} u(x,y) \right)
\]

and

\[
\frac{\partial}{\partial z} \phi(z) = -\frac{\sigma_0}{8} \left( \frac{\partial}{\partial x} u(x,y) + i \frac{\partial}{\partial y} u(x,y) \right).
\]

The derivatives of the real part, \(u(x,y)\), with respect to \(x\) and \(y\) are then given by

\[
\frac{\partial}{\partial x} u(x,y) = \frac{a^2}{\sqrt{2c_1}} \left( \sqrt{c_1} - 2c_2 \right) \sqrt{c_1(\sqrt{c_1} + c_2)}
\]

and

\[
\frac{\partial}{\partial y} u(x,y) = \frac{1}{2\sqrt{2c_1c_2}} \left( c_2(\sqrt{c_1} - 4x^2 - c_2^2) \right)
\]

\[
\sqrt{c_1(\sqrt{c_1} + c_2)}.
\]

Substitution of these relations back into equations (5) leads the expressions given by (6).

Appendix 2

Shifting the origin in equation (1) to the left contact edge with \(s = x + a\) yields

\[
\sigma_{xy}(s) = -\frac{2\sigma_0}{4} \frac{s - a}{2\sqrt{2as - s^2}}
\]

\[
= -\frac{\sigma_0}{2} \frac{s - a}{\sqrt{2a\sqrt{s\sqrt{1 - s/2a}}}}.
\]

and after a binomial expansion of the last square root term

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Appendix 1

Vázquez et al.\(^7\) describe how to derive the stress in terms of the real variables \((x,y)\). In the case considered here the complex potential, \(\phi(z)\), is defined by means of the complex function \(f(z) = z/\sqrt{z^2 - a^2}\) and we need to find a split into a real and an imaginary part \(f(z) = u(x,y) + iv(x,y)\) so that...
\[ \sigma_{xy}(s) = -\frac{\sigma_0}{2} \sqrt{\frac{a}{2}} \frac{1}{\sqrt{s}} \left( 1 - \frac{s}{4a} + \ldots \right) \]

\approx \frac{\sigma_0}{2} \sqrt{\frac{a}{2}} \frac{1}{\sqrt{s}} - \frac{\sigma_0}{2} \sqrt{\frac{s}{2a}} \quad \text{for} \quad \frac{s}{4a} \ll 1. \tag{30} \]

Close to the contact edge, \( s \to 0 \), the first term, being square root singular, will dominate and we drop the second (square root bounded) term for the asymptotic description to find the result stated in equation (7)

\[ \sigma_{xy}(s) = \frac{\sigma_0}{2} \sqrt{\frac{a}{2}} \frac{1}{\sqrt{s}} = K_{II} \frac{1}{\sqrt{s}}. \tag{31} \]

For the asymptotic description of the subsurface stress field we start from the negative branch of the complex potential function (2), again with a shift of the origin via \( w = z + a \) to the left contact edge

\[ \phi(w) = -\frac{\sigma_0}{8} \left( 1 + \frac{w - a}{\sqrt{w^2 - 2aw}} \right) \]

\[ = -\frac{\sigma_0}{8} \left( 1 + \frac{1}{\sqrt{2a}} \frac{1}{\sqrt{1 - w/2a}} \right). \tag{32} \]

Again we perform a binomial expansion of the last square root term and find

\[ \phi(w) = -\frac{\sigma_0}{8} \left( 1 + \frac{1}{\sqrt{2a}} \frac{1}{\sqrt{w - (1 - w/2a)}} \right) \]

\[ \approx -\frac{\sigma_0}{8} \left( 1 + \frac{1}{\sqrt{2a}} \frac{1}{\sqrt{w - (w/4a)}} \right) \quad w \ll 4a \tag{33} \]

\[ = \frac{1}{2} \frac{\sigma_0}{4} \sqrt{\frac{2}{\sqrt{w - w}}} - \frac{\sigma_0}{8} + \frac{\sigma_0}{8} \frac{1}{\sqrt{2a}} \frac{1}{\sqrt{w - w}}. \]

Again, dropping the square root bounded term for asymptotic description leads to equation (8)

\[ \phi(w) = \frac{1}{2} \frac{\sigma_0}{4} \sqrt{\frac{2}{\sqrt{w - w}}} - \frac{\sigma_0}{8} = K_{II} \frac{1}{\sqrt{w - w}} - \frac{\sigma_0}{8}. \tag{34} \]