Compactified Twistor Fibration and Topology of Ward Unitons

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Abstract

We use the compactified twistor correspondence for the (2 + 1)-dimensional integrable chiral model to prove a conjecture of Ward. In particular, we construct the correspondence space of a compactified twistor fibration and use it to prove that the second Chern numbers of the holomorphic vector bundles, corresponding to the uniton solutions of the integrable chiral model, equal the third homotopy classes of the restricted extended solutions of the unitons. Therefore we deduce that the total energy of a time-dependent uniton is proportional to the second Chern number.

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1 Introduction

The integrable chiral model, also known as the Ward chiral model, was first introduced by Ward in [18] as a rare example of an integrable system in (2 + 1) dimensions that admits soliton solutions, and yet is close to being Lorentz invariant. Many time-dependent soliton solutions - solutions which have smooth energy density concentrated in some finite region of space - have been constructed explicitly [18, 21, 12, 14, 5]. Some represent multi-soliton configurations with no scattering, while some represent solitons which do scatter. Analytic and algebraic Bäcklund transformations which can be used to generate all soliton solutions have also been developed [14, 5].

The integrability of the Ward chiral model comes from the fact that the equation can be realised as a symmetry reduction of the anti-self-dual Yang-Mills (ASDYM) equation in (2 + 2) dimensions. Hence, it is integrable by twistor method (see, for example, [23, 16, 6]). Ward gave a one-to-one correspondence between solutions of the Ward chiral model to certain holomorphic vector bundles over the minitwistor space [19, 20]. The minitwistor space is a 2-dimensional complex manifold. It is in fact the holomorphic tangent bundle, $T\mathbb{P}^1$, of the Riemann sphere $\mathbb{C}\mathbb{P}^1$.

This paper deals with a particular class of solutions of the Ward chiral model. These solutions correspond to the holomorphic vector bundles which extend to a fibrewise compactification $\overline{T\mathbb{P}^1}$ of $T\mathbb{P}^1$. In [20, 22] Ward discussed this class of solutions and the boundary conditions they have to satisfy, in order for the corresponding vector bundles to extend to $\overline{T\mathbb{P}^1}$. This correspondence was formulated as a theorem and proved by Anand [2, 3]. The aim of this paper is to answer a question posed by Ward in [20], of what topological invariant in spacetime corresponds to the second Chern number of the holomorphic vector bundle over the compact space $\overline{T\mathbb{P}^1}$. It was stated in both [22, 3] that the second Chern number corresponds to a topological degree of the Ward chiral field, taking values in $\pi^3(U(N))$, however no proof was presented. The goal of this paper is then to provide a proof of this conjecture, thus giving an interpretation of the second Chern number in spacetime.

The Ward chiral model is given by

$$ \left( J^{-1}J_t \right)_t - \left( J^{-1}J_x \right)_x - \left( J^{-1}J_y \right)_y - [J^{-1}J_t, J^{-1}J_y] = 0, \quad (1.1) $$

where $J : \mathbb{R}^3 \rightarrow U(N)$, $(x, y, t)$ are coordinates on $\mathbb{R}^3$ with the line element $\eta = dx^2 + dy^2 - dt^2$, and $J_x := \partial_x J$, etc.

The model is in fact equivalent to the Yang-Mills-Higgs (YMH) system in (2 + 1) dimensions, with a gauge fixing. Let $A = A_t dt + A_x dx + A_y dy$ and $\Phi$ be a one-form and
a function on $\mathbb{R}^{2,1}$ respectively, with values in the Lie algebra $\mathfrak{u}(N)$. The YMH system is given by

\[
\begin{align*}
\partial_x \Phi + [A_x, \Phi] &= \partial_y A_t - \partial_t A_y + [A_y, A_t], \\
\partial_y \Phi + [A_y, \Phi] &= \partial_t A_x - \partial_x A_t + [A_t, A_x], \\
\partial_t \Phi + [A_t, \Phi] &= \partial_y A_x - \partial_x A_y + [A_y, A_x],
\end{align*}
\] (1.2)

where the gauge potential $A$ and the Higgs field $\Phi$ are determined up to gauge transformations

\[
A \rightarrow b^{-1} A b + b^{-1} db, \quad \Phi \rightarrow b \Phi b^{-1}, \quad b = b(x^\mu) \in U(N).
\]

The system (1.2) reduces to the Ward chiral model (1.1) under a gauge choice

\[
A_t = A_y = \frac{1}{2} J^{-1} (J_t + J_y), \quad A_x = -\Phi = \frac{1}{2} J^{-1} J_x,
\] (1.3)

where $J$ is a $U(N)$-valued function.

Now, since the YMH system is a reduction of the ASDYM equation by a non-null translation, it inherits a twistor correspondence, established by Ward in [19]. This gives a one-to-one correspondence between solutions of the YMH system (1.2) and holomorphic vector bundles over the minitwistor space. As mentioned earlier, the minitwistor space is the holomorphic tangent bundle $TP^1$ of the Riemann sphere. It is considered to be the space of null planes in $\mathbb{C}^3$ - a complexification of $\mathbb{R}^{2,1}$. That is, a point of $TP^1$ corresponds to a null plane in $\mathbb{C}^3$. On the other hand, a point in $\mathbb{C}^3$, including points in $\mathbb{R}^{2,1}$ as a real slice, corresponds a holomorphic section, a $\mathbb{CP}^1$ line, of $TP^1$. We shall refer to a holomorphic section $\hat{p}$ associated with a point $p \in \mathbb{R}^{2,1}$ as a real section.

The twistor correspondence for the Ward chiral model follows readily from the correspondence for the YMH system. However, since the field $J$ is obtained from $(A, \Phi)$ by integration, it contains more information than the YMH fields. This additional data is a holomorphic framing of the vector bundle along two fibres of $TP^1$ [20].

Ward also discussed in particular the static solutions of (1.1). These are harmonic maps from $\mathbb{R}^2$ to $U(N)$. The finite energy condition allows the harmonic maps to extend to $S^2$. Moreover, it was shown, for the gauge group $SU(2)$, that finite energy static solutions correspond to holomorphic vector bundles which extend to a fibrewise compactification $\overline{TP^1}$ of $TP^1$. The generalisation of this to the gauge group $U(N)$ was established by Anand in [1].

A class of time-dependent solutions of (1.1) was also shown in [20] to give rise to
holomorphic vector bundles over $\mathbb{T}P^1$. To discuss this class of solutions, one needs to consider a Lax formulation of (1.1).

The integrability of the Ward chiral model can be realised from the fact that it is the compatibility condition for a system of overdetermined linear equations, the so-called Lax pair. The Lax pair for (1.1) comes naturally from the Lax pair of the YMH system (1.2):

$$L_0\Psi := (D_y + D_t - \lambda(D_x + \Phi))\Psi = 0, \quad L_1\Psi := (D_x - \Phi - \lambda(D_t - D_y))\Psi = 0, \quad (1.4)$$

where $D_x := \partial_x + A_x$ is the covariant derivative with respect to the gauge field $A_x$, similarly for $D_y$ and $D_t$, and $\Psi$ is a $GL(N, \mathbb{C})$-valued function of the spacetime coordinates $(x, y, t)$ and a complex parameter $\lambda \in \mathbb{CP}^1$. The YMH system (1.2) arises as the compatibility condition $[L_0, L_1] = 0$.

The gauge freedom of (1.2) allows us to choose the gauge (1.3), in which the Ward chiral model becomes the compatibility condition for the Lax pair (1.4). That is, if the map $J$ in (1.3) is a solution of (1.1), then the overdetermined system (1.4) admits a matrix solution $\Psi(x, y, t, \lambda)$ which satisfies the unitary reality condition

$$\Psi(x, y, t, \lambda)^* \Psi(x, y, t, \lambda) = I, \quad (1.5)$$

where $I$ is the identity matrix. On the other hand, a solution $\Psi$ of (1.4) which satisfies (1.5) gives rise to a solution of (1.1) via

$$J(x, y, t) = \Psi^{-1}(x, y, t, 0), \quad (1.6)$$

and all solutions of (1.1) can be constructed in this way. (See, for example, [11].) The matrix solution $\Psi$ is called an extended solutions.

The sufficient conditions for the holomorphic vector bundles corresponding to Ward chiral fields to extend to $\mathbb{T}P^1$, was first discussed by Ward in [22]. In addition to the finite energy condition, ensured by the boundary condition (valid for all $t$

$$J = J_0 + J_1(\varphi)r^{-1} + O(r^{-2}) \quad \text{as} \quad r \to \infty, \quad x + iy = re^{i\varphi}, \quad (1.7)$$

one also needs a global boundary condition on the extended solution $\Psi$ of the Lax pair (1.4). Recall that $\Psi(x, y, t, \lambda)$ is defined on $\mathbb{R}^{2,1} \times \mathbb{CP}^1$. Let $\psi$ be the restriction of the map $\Psi$ to the spacelike $t = 0$ plane and the real equator $S^1 \subset \mathbb{CP}^1$ of the space of spectral parameter $\lambda$, i.e.

$$\psi(x, y, \theta) := \Psi \left( x, y, 0, -\cot \left( \frac{\theta}{2} \right) \right), \quad (1.8)$$
where we have made change of variable for real \( \lambda = -\cot \left( \frac{\theta}{2} \right) \). Then the global boundary condition, the so-called “trivial scattering” boundary condition\(^1\), is given by

\[
\psi(x, y, \theta) \rightarrow \psi_0(\theta) \quad \text{as} \quad r = \sqrt{x^2 + y^2} \rightarrow \infty,
\]

(1.9)

where \( \psi_0(\theta) \) is a \( U(N) \)-valued function on \( S^1 \).

It was described in [22], particularly for the gauge group \( SU(2) \), that the class of Ward chiral fields, for which the corresponding vector bundles \( E \) extend to \( \overline{T\mathbb{P}^1} \), are those which satisfy the boundary conditions (1.7) and (1.9). This was then formulated as a correspondence by Anand for a general gauge group \( U(N) \). The compactified minitwistor space \( \overline{T\mathbb{P}^1} \) was defined in [20] to be the fibrewise compactification of \( T\mathbb{P}^1 \) where each \( \mathbb{C} \)-fibre becomes a copy of \( \mathbb{C}\mathbb{P}^1 \). One can also think of \( \overline{T\mathbb{P}^1} \) as a cone in \( \mathbb{C}\mathbb{P}^3 \) with blown-up vertex.

**Ward-Anand correspondence** [22, 2] There is a one-to-one correspondence between

(i) Real-analytic solutions \( J : \mathbb{R}^{2,1} \rightarrow U(N) \) of the Ward chiral model (1.1) which satisfy the boundary conditions (1.7) and (1.9), and

(ii) Holomorphic rank-\( N \) vector bundles \( E \) over the compactified minitwistor space \( \overline{T\mathbb{P}^1} \), such that \( E \) satisfies certain reality conditions and when restricted to real sections and to the fibres of \( \overline{T\mathbb{P}^1} \) over the real equator \( S^1 \subset \mathbb{C}\mathbb{P}^1 \) of the base, \( E \) is trivial with a fixed framing.

The holomorphic vector bundle \( E \rightarrow \overline{T\mathbb{P}^1} \), which now fibres over a compact manifold, has Chern numbers as topological invariants. The fact that the bundle is trivial when restricted to real sections implies that the first Chern number vanishes. The next non-trivial invariant is the second Chern number. On the other hand, the finite energy Ward chiral fields which satisfy the trivial scattering condition admit a well defined topological degree, associated with their extended solutions. The boundary conditions (1.7) and (1.9) enables \( \psi \) to extend to the suspension \( S S^2 = S^3 \) of \( S^2 \). (See [7].) The restricted extended solutions \( \psi \), now as maps from \( S^3 \) to \( U(N) \), are classified by the third homotopy class [4]

\[
[\psi] = \frac{1}{24\pi^2} \int_{S^3} \text{Tr}( (\psi^{-1}d\psi)^3 ),
\]

(1.10)

\(^1\)The restricted extended solution \( \psi \) satisfies \((u^\mu D_\mu - \Phi)\psi = 0\), where the operator anihilating \( \psi \) is the spatial part of the Lax pair (1.4), given by

\[
\frac{\lambda L_0 + L_1}{1 + \lambda^2} = u^\mu D_\mu - \Phi,
\]

where \( u = \left( 0, \frac{1 - \lambda^2}{1 + \lambda^2}, \frac{2\lambda}{1 + \lambda^2} \right) = (0, -\cos \theta, -\sin \theta) \).

The trivial scattering condition (1.9) implies that the differential operator \( u^\mu D_\mu - \Phi \) has trivial monodromy along the compactification \( S^1 \) of a straight line \((x, y) = (x_0 - \sigma \cos \theta, y_0 - \sigma \sin \theta), \sigma \in \mathbb{R} \).
which is an integer taking values in $\pi_3(U(N)) = \mathbb{Z}$ and invariant under continuous deformations of $\psi$.

The identification between the topological degree of the extended solutions and the second Chern number of the corresponding vector bundles was stated in both [22] and [3], however the proof was not presented. This conjecture is supported by the works in [1, 7]. In [1], Anand showed that the energy of a static solution of the $U(N)$ Ward chiral model is proportional to the second Chern number of the corresponding vector bundle. Later in [7], a class of Ward chiral fields called time-dependent unitons, which includes the static solutions, was shown to have their total energy proportional to the third homotopy class of the extended solutions. This explains an observation of discrete total energy of time-dependent unitons in [13]. It also gives the identification between the third homotopy class of the extended solutions and the second Chern number of the holomorphic vector bundles for finite energy static Ward chiral fields.

The time-dependent unitons form a class of solutions which satisfy the boundary conditions (1.7) and (1.9). These are soliton solutions for which the extended solutions have a pole of arbitrary order multiplicity in the complex plane of the spectral parameter $\lambda$. In [5], Dai and Terng have demonstrated that an extended solution satisfying the trivial scattering condition has poles at non-real points $\mu_1, ..., \mu_r$, with multiplicities $n_1, ..., n_r$, and is a product of $r \ N \times N$ matrices, called simple elements. The identification of the two topological invariant is valid for any such solutions. However, only the time-dependent unitons, where $r = 1$, have their energy directly proportional to the third homotopy class $[\psi]$.

The main result in this paper is the following theorem.

**Theorem 1.1** Let $\mathbb{T}P^1$ be the fibrewise compactification of $\mathbb{T}P^1$ where each fibre becomes $\mathbb{C}P^1$, and $E \to \mathbb{T}P^1$ be the holomorphic vector bundle, corresponding to a solution of the $U(N)$ Ward chiral model (1.1) which satisfies the boundary conditions (1.7), (1.9). Let $c_2(E)$ be the second Chern number of $E$, given by

$$c_2(E) = -\frac{1}{8\pi^2} \int_{\mathbb{T}P^1} \text{Tr}(F \wedge F),$$

where $F$ is the curvature two-form of an arbitrary connection on $E$, and $[\psi]$ be the third homotopy class of the restricted extended solution, defined in (1.10).

Then

$$c_2(E) = [\psi].$$
Our proof of Theorem 1.1 is based on the existence of a double fibration from a space we shall call the restricted correspondence space $F$, to a compactification of a real spacelike plane $(t = 0) \mathbb{R}^2 \subset \mathbb{R}^{2,1}$ (where the restricted extended solution $\psi$ is defined) and the compactified minitwistor space $T \mathbb{P}^1$. Consequently, the vector bundle $E \to T \mathbb{P}^1$ can be pulled back to a bundle $E^* \to F$, for which the second Chern number can be calculated and related to the topological degree of $\psi$. A related problem has been considered by Mason in [15], where an initial value problem for the Ward chiral model was formulated on a null hypersurface.

The structure of the paper is as follows. In Section 2 we give a detailed exposition of the twistor correspondence in the holomorphic setting, which was described briefly in [22], between the compactified spacetime $\overline{M}_C = \mathbb{C}P^3$ and $\overline{T \mathbb{P}^1}$. Starting from the identification between the minitwistor space $T \mathbb{P}^1$ and a cone $C$ minus its vertex in another complex projective 3-space $\mathbb{C}P^3$, we explain how the compactification $\overline{T \mathbb{P}^1}$ of $T \mathbb{P}^1$ is equivalent to the cone $C$ with blown-up vertex.

Then in Section 3, we construct the restricted correspondence space for a double fibration over $\overline{M}_C$ and $\overline{T \mathbb{P}^1}$. A double fibration picture was discussed in [1], where the correspondence space was taken to be a singular variety in the direct product $\overline{M}_C \times \mathbb{C}P^3$ and one of the target spaces is the cone $C$ instead of $\overline{T \mathbb{P}^1}$. Here we explore a double fibration where the correspondence space is a blow up of the singular variety, which fibres over $\overline{M}_C$ and $\overline{T \mathbb{P}^1}$. Then we define the restricted correspondence space which fibres over an $\mathbb{R}P^2$, regarded as a compactification of a spacelike plane $\mathbb{R}^2 \subset \mathbb{R}^{2,1}$, and show that it admits a surjective map to $\overline{T \mathbb{P}^1}$. Finally we give a proof of Theorem 1.1 in Section 4.

## 2 Compactified minitwistor space

The twistor geometry for $(2 + 1)$ dimensional flat spacetime and its use to construct the YMH fields in $\mathbb{R}^{2,1}$ was introduced in [19]. The minitwistor space can be thought of as the space of null planes in $\mathbb{C}^3$, considered as a complexification of $\mathbb{R}^{2,1}$. Let $(x, y, t)$ be complex coordinates on the complexified spacetime $M_C = \mathbb{C}^3$. A null plane in $M_C$ is given by

$$\omega = 2x \lambda + y(\lambda^2 - 1) + t(1 + \lambda^2),$$

where $\omega \in \mathbb{C}$ and $\lambda \in \mathbb{C}P^1$ are complex parameters.

Thus the space of null planes is a 2-dimensional complex manifold, which is actually the holomorphic tangent bundle of the Riemann sphere $\mathbb{C}P^1$. Hence, we denote the minitwistor space by $T \mathbb{P}^1$, with $\omega$ and $\lambda$ as fibre and base coordinates, respectively. See Appendix A for details. (The minitwistor space $T \mathbb{P}^1$ was used to construct static YMH monopoles on Euclidean space $\mathbb{R}^3$ [10].)
Another picture of the minitwistor space was given in [20, 22] as a cone minus its vertex in a complex 3-dimensional projective space. The cone picture proves to be convenient in the study of the compactified double fibrations, which will be essential to our proof of Theorem 1.1. Therefore in this section, we shall start with a detailed explanation of how to identify $T\mathbb{P}^1$ with a cone $C$, without the vertex. Then we shall discuss the correspondence between points in spacetime $M_C$ and points on the cone $C$. This will lead to a natural compactification of spacetime, $\overline{M_C} = \mathbb{C}P^3$, and the identification of the blow up of the cone $C$ with the compactified minitwistor space $\overline{T\mathbb{P}^1}$.

Let us first define the cone $C$. Let $Z_\alpha = (Z_0, Z_1, Z_2, Z_3) \in \mathbb{C}^4 - \{0\}$ be homogeneous coordinates of a complex projective 3-space, denoted by $\mathbb{C}P^3$. A cone $C$ in $\mathbb{C}P^3$ is given by

$$(Z_1)^2 + (Z_2)^2 - (Z_3)^2 = 0. \quad (2.12)$$

Note our convention of one minus sign. The vertex is the point $z_0 = [Z_0, 0, 0, 0]$, $Z_0 \neq 0$. To see that there is a bijection between $C - \{z_0\}$ and $T\mathbb{P}^1$, note that for any point on the cone except the vertex, one can parametrise $Z_i := (Z_1, Z_2, Z_3) \neq (0, 0, 0)$ by $\pi^A \in \mathbb{C}^2 - \{0\}$ as follows. Let

$$Z^{AB} = \left( \begin{array}{c} Z_1 + Z_2 \\ Z_1 \\ Z_2 \\ Z_1 - Z_2 \end{array} \right).$$

Then equation (2.12), which is the same as $-4 \det(Z^{AB}) = 0$, implies that rank $(Z^{AB}) = 1$. Hence

$$Z^{AB} = \pi^A \pi^B = \left( \begin{array}{cc} (\pi^0)^2 & \pi^0 \pi^1 \\ \pi^0 \pi^1 & (\pi^1)^2 \end{array} \right),$$

where $\pi^A \in \mathbb{C}^2 - \{0\}$. In other words, one can parametrise solutions of (2.12) with $Z_i \neq (0, 0, 0)$ by

$$Z_\alpha = (\hat{\omega}, -2\pi_0 \pi_1, \pi_1^2 - \pi_0^2, \pi_0^2 + \pi_1^2), \quad (2.13)$$

where $\hat{\omega} \in \mathbb{C}$ is arbitrary, $\pi_A = \pi^B \varepsilon_{BA}$ and $\varepsilon_{BA}$ is the alternating symbol. This shows that points $[Z_\alpha] \in C - \{z_0\}$ can be parametrised inhomogeneously in two patches, where $\pi_1 \neq 0$ and where $\pi_0 \neq 0$, by

$$[\omega, -2\lambda, 1 - \lambda^2, \lambda^2 + 1] \quad \text{and} \quad [\hat{\omega}, -2\hat{\lambda}, \hat{\lambda}^2 - 1, 1 + \hat{\lambda}^2], \quad (2.14)$$

respectively, where $\omega := \frac{\hat{\omega}}{\pi_1^2}$, $\lambda := \frac{\pi_0}{\pi_1}$ and $\hat{\omega} := \frac{\hat{\omega}}{\pi_0^2}$, $\hat{\lambda} := \frac{\pi_1}{\pi_0}$. In the overlap, the inhomogeneous coordinates are related by $\hat{\lambda} = \frac{1}{\lambda}$ and $\hat{\omega} = \frac{\omega}{\lambda^2}$. This gives the equivalence between $C - \{z_0\}$ and $T\mathbb{P}^1$. That is, there exists a bijection from $T\mathbb{P}^1$ to $C - \{z_0\}$ given
locally in the two patches by

\[(\omega, \lambda) \mapsto [\omega, -2\lambda, 1 - \lambda^2, 1 + \lambda^2] \quad \text{and} \quad (\tilde{\omega}, \tilde{\lambda}) \mapsto [\tilde{\omega}, -2\tilde{\lambda}, \tilde{\lambda}^2 - 1, 1 + \tilde{\lambda}^2]. \quad (2.15)\]

In fact, (2.15) gives a biholomorphism between \(T\mathbb{P}^1\) and \(\mathcal{C} - \{z_0\} \subset \mathbb{C}\mathbb{P}^{3*}\).

Let us now describe the correspondence between the complexified spacetime \(M_C = \mathbb{C}^3\) and the minitwistor space in the cone picture. For convenience, let us think of \(M_C\) as embedded in a \(\mathbb{C}\mathbb{P}^3\). Let \(P^\alpha = (P^0, P^1, P^2, P^3) \in \mathbb{C}^4 - \{0\}\) be homogeneous coordinates on the \(\mathbb{C}\mathbb{P}^3\) and take the open set \(P^0 \neq 0\) to be our spacetime \(M_C\). A plane in \(\mathbb{C}\mathbb{P}^3\) is defined to be the projection of a 3-dimensional subspace of the associated \(\mathbb{C}^4\), given by

\[Z_0P^0 + Z_1P^1 + Z_2P^2 - Z_3P^3 = 0. \quad (2.16)\]

Note again our convention of one minus sign. Each plane is thus labelled by \(Z_\alpha \in \mathbb{C}^4 - \{0\}\) up to a constant multiplication. That is, the space of planes in \(\mathbb{C}\mathbb{P}^3\) is another complex projective 3-space, which we shall denote \(\mathbb{C}\mathbb{P}^{3*}\). Then in this setting, 2-planes in \(M_C\) are the \(\mathbb{C}^2\)-intersections of planes in \(\mathbb{C}\mathbb{P}^3\) with \(M_C\).

This picture suggests a natural compactification of the spacetime \(M_C\) to \(\overline{M_C} = \mathbb{C}\mathbb{P}^3\). One can think of \(\overline{M_C}\) as \(M_C + \mathbb{C}\mathbb{P}^2\), where \(\mathbb{C}\mathbb{P}^2\) is the complement region \(P^0 = 0\). Let

\[x = \frac{P_1}{P_0}, \quad y = \frac{P_2}{P_0}, \quad t = \frac{P_3}{P_0} \quad (2.17)\]

be coordinates on \(M_C\). Then, one can interpret the complement \(\mathbb{C}\mathbb{P}^2\) as the infinity boundary, which will be denoted by \(\mathbb{C}\mathbb{P}^2_{\infty}\). To make contact with a real setting, note that since \(\mathbb{C}\mathbb{P}^2 \cong S^5/S^1\), the \(\mathbb{C}\mathbb{P}^2_{\infty}\) can then be thought of as the \(S^5\) infinity boundary of \(M_C \cong \mathbb{R}^6\) with the points on \(S^1\) orbits identified.

**Definition 2.1** A plane (2.16) in \(\overline{M_C} = \mathbb{C}\mathbb{P}^3\) is called a null plane if \([Z_\alpha] \in \mathbb{C}\mathbb{P}^{3*}\) lies in the cone \(\mathcal{C}\) (2.12).

Let us now show that null planes in \(\overline{M_C}\) give rise to null planes in \(M_C\), as defined in Appendix A. Since in \(M_C\), \(P^0 \neq 0\), one can divide (2.16) by \(P^0\) and use the coordinates (2.17). By substituting in the first parametrisation of (2.14) for \(Z_\alpha\), equation (2.16) becomes the null plane equation (2.11)

\[\omega = 2x\lambda + y(\lambda^2 - 1) + t(1 + \lambda^2). \quad \text{Note that the parametrisation (2.14) is only valid for the points on } \mathcal{C} - \{z_0\}. \text{ From the} \]
plane equation (2.16), one sees that the vertex $z_0 = [Z_0, 0, 0, 0]$ corresponds to the infinity boundary $\mathbb{CP}^2_\infty$, which we shall regard as a null plane by definition. Hence, the natural extension from $M_C$ to $\overline{M}_C$ makes the inclusion of the vertex $z_0$.

The correspondence between points in the compactified spacetime $\overline{M}_C = \mathbb{CP}^3$ and points on the cone $\mathcal{C} \subset \mathbb{CP}^{3*}$, including the vertex $z_0$, is summarised in Lemma 2.3 below. First, let us define what we mean by a conic section of $\mathcal{C}$.

**Definition 2.2** A conic section of a cone $\mathcal{C} \subset \mathbb{CP}^{3*}$ is given by the intersection of a plane in $\mathbb{CP}^{3*}$ with $\mathcal{C}$.

**Lemma 2.3** There is a one-to-one correspondence between points on the cone minus the vertex, $\mathcal{C} - \{z_0\} \subset \mathbb{CP}^{3*}$, and null planes in $M_C = \mathbb{CP}^3$. The vertex $z_0$ corresponds to the infinity boundary $\mathbb{CP}^2_\infty \subset \overline{M}_C$.

On the other hand, there is a one-to-one correspondence between points in $\overline{M}_C$ and conic sections of $\mathcal{C}$, where

1. points in $M_C$ correspond to the conic sections that do not intersect $z_0$
2. points in $\mathbb{CP}^2_\infty$ correspond to the conic sections, each of which consists of two $\mathbb{C}$-lines, counting multiplicity, meeting at $z_0$.

**Proof.** We have already established the first part of the lemma. The second part can be proved by considering equation (2.16). By fixing $[P^\alpha]$ and varying $[Z_\alpha]$ instead, one sees that (2.16) is also the equation for planes in $\mathbb{CP}^{3*}$. That is, a point $[P^\alpha] \in \overline{M}_C = \mathbb{CP}^3$ labels a plane in $\mathbb{CP}^{3*}$. Moreover, for a given $[P^\alpha]$ it is always possible to find common solutions $[Z_\alpha]$ to (2.12) and (2.16), which means that any plane in $\mathbb{CP}^{3*}$ intersects $\mathcal{C}$.

Now, since $P^0 \neq 0$ for a point in $M_C$, no plane labelled by $[P^\alpha] \in M_C$ passes through $z_0$. Hence we have that each point in $M_C$ corresponds to a conic section on $\mathcal{C} - \{z_0\}$. For a point on $\mathbb{CP}^2_\infty$, with $P^0 = 0$ the corresponding plane in $\mathbb{CP}^{3*}$ is given by

$$P^1Z_1 + P^2Z_2 - P^3Z_3 = 0. \tag{2.18}$$

Equation (2.18) admits the vertex $[Z_0, 0, 0, 0]$ as a solution. Thus, the plane passes through the vertex $z_0$. Thinking analogously of a cone in $\mathbb{R}^3$, one would expect the conic section to consist of two lines coming together at the vertex. This is indeed the case. For $(Z_1, Z_2, Z_3) \neq (0, 0, 0)$, we can use the parametrisation (2.13) to label $Z_i$. For
concreteness, let us consider the patch where $\pi_1 \neq 0$ and use the first parametrisation in (2.14). Equation (2.18) becomes

$$(P^2 + P^3)\lambda^2 + 2P_1 \lambda + (P^3 - P^2) = 0. \quad (2.19)$$

This is a quadratic equation for $\lambda$. Since $\omega$ (corresponding to $Z_0$) is arbitrary, it implies that a conic section corresponding to a point on $\mathbb{CP}_2^\infty$ consists of two $\mathbb{C}$-lines of constant $\lambda$, whose values are given by the two roots of (2.19) counting multiplicity. In the limit where $\omega$ approaches infinity, the two lines meet at $z_0$.

\[\square\]

The cone $C$ is however not equivalent to the compactified minitwistor space defined in [20]. The compact space $\overline{T\mathbb{P}^1}$ is defined to be the fibrewise compactification of $T\mathbb{P}^1$, where each fibre is extended from $\mathbb{C}$ to $\mathbb{CP}^1$. This can be regarded as adding a $\mathbb{CP}^1$ at $\omega = \infty$. We shall denote the additional $\mathbb{CP}^1$ by $L_\infty$. In the cone picture, the compactified minitwistor space is the cone $C$ with the vertex blown up to a $\mathbb{CP}^1$.

**Proposition 2.4** There exists a bijection from $T\mathbb{P}^1$ to the blow-up $\tilde{C}$ of the cone $C \subset \mathbb{CP}^3$ at the vertex $z_0$, where the blow-up of $z_0$ is identified with $L_\infty$.

A detailed calculation of the blow up and a proof of Proposition 2.4 can be found in Appendix B.

Lastly, let us recall briefly the construction of holomorphic vector bundles over the minitwistor space from solutions of the Ward chiral model. This can be done in holomorphic setting where $\mathbb{R}^{2,1}$ is complexified to $M_\mathbb{C} = \mathbb{C}^3$. Consider the Ward chiral model (1.1) in $M_\mathbb{C}$. Suppose $J$ in (1.3) satisfies (1.1). Then, for a fixed $\lambda$, there exists $N$ linearly independent column vector solutions of the Lax pair (1.4), forming the fundamental $N \times N$ matrix solution $\Psi$. These column vector solutions are covariantly constant sections of the trivial $\mathbb{C}^N$ bundle $V \rightarrow M_\mathbb{C}$ restricted to null planes, with respect to $(A_\mu, \Phi)$ in (1.4). Let $z \in T\mathbb{P}^1$ corresponds to a null plane $Z \subset M_\mathbb{C}$. Then, one defines a holomorphic rank $N$ vector bundle over $T\mathbb{P}^1$ by taking the fibre over each point $z \in T\mathbb{P}^1$ to be the space of covariantly constant sections of $V|_Z$.

The Ward chiral fields in $\mathbb{R}^{2,1}$ then correspond to such holomorphic vector bundles with reality conditions. This is described in details in [20], where patching matrices of $E \rightarrow T\mathbb{P}^1$ are given explicitly for static $SU(2)$ 1-uniton solutions. It was explained then how the patching matrices extend to $\overline{T\mathbb{P}^1}$, thus defining the bundle $E \rightarrow \overline{T\mathbb{P}^1}$. This was generalised to the Ward-Anand correspondence [2].
3 The Correspondence spaces

The main aim of this paper is to prove Theorem 1.1, which identifies the second Chern number of the holomorphic vector bundle $E$ over the minitwistor space $\overline{T\mathbb{P}^1}$ and the third homotopy class of the restricted extended solution $\psi$ to the Lax pair (1.4)

$$
(D_y + D_t - \lambda(D_x + \Phi))\Psi = 0, \quad (D_x - \Phi - \lambda(D_t - D_y))\Psi = 0,
$$
on $\mathbb{R}^{2,1}$.

In the holomorphic setting, the extended solution $\Psi(x, y, t, \lambda)$ is a function on the correspondence space $F = \mathbb{C}^3 \times \mathbb{C}\mathbb{P}^1$ of a double fibration over the spacetime $M_C = \mathbb{C}^3$ and the minitwistor space $T\mathbb{P}^1$. Then $\psi$ is the restriction of $\Psi$ to certain real slice $\mathbb{R}^2 \subset \mathbb{R}^{2,1} \subset \mathbb{C}^3$ and the real equator $S^1 \subset \mathbb{C}\mathbb{P}^1$ of the space of spectral parameter $\lambda$. To relate the topological invariants on $\overline{T\mathbb{P}^1}$ and that of $\psi$, one could think of pulling back the bundle over $\overline{T\mathbb{P}^1}$ to $F$. However, as we cannot find a required surjective map from $F$ to $\overline{T\mathbb{P}^1}$, we construct another correspondence space, adapted to the compactified setting, which is the blown up version of that presented in [1, 2]. Then we consider the restriction of the correspondence space to some real and 't = 0' slice.

3.1 Compactified double fibration

Recall that the correspondence space in the non-compact double fibration is the space of pairs of a spacetime point in $\mathbb{C}^3$ and a null plane on which the point lies, where the null plane corresponds to a point on the minitwistor space $T\mathbb{P}^1$. Hence $F$ is a subset of $\mathbb{C}^3 \times T\mathbb{P}^1$ defined locally by

$$
F := \{(p, z) \in \mathbb{C}^3 \times T\mathbb{P}^1 : \omega = 2x\lambda + y(\lambda^2 - 1) + t(1 + \lambda^2)\},
$$

where $(x, y, t)$ are coordinates of a point $p \in \mathbb{C}^3$ and $(\omega, \lambda)$ are local coordinates of a point $z \in T\mathbb{P}^1$. Given $(x, y, t) \in \mathbb{C}^3$ and $\lambda \in \mathbb{C}\mathbb{P}^1$, $\omega$ is determined uniquely by the incidence relation, and hence $F$ is biholomorphic to $\mathbb{C}^3 \times \mathbb{C}\mathbb{P}^1$.

\[\text{For the compactified case, consider a singular algebraic variety in } \mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^{3*} \text{ given by}\]

$$
\hat{f} := \{(p, z) \in \mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^{3*} : Z_1^2 + Z_2^2 - Z_3^2 = 0, \ P_0^0 Z_0 + P_1^1 Z_1 + P_2^2 Z_2 - P_3^3 Z_3 = 0\}. \quad (3.20)
$$

This is effectively a subset of $\mathbb{C}\mathbb{P}^3 \times \mathcal{C}$ which consists of pairs of a point $p \in \mathbb{C}\mathbb{P}^3$ and a point $z \in \mathcal{C}$ corresponding to a null plane passing through $p$. Equivalently, it is the set of
pairs of a point \( z \in \mathbb{C} \) and a point \( p \in \mathbb{C}P^3 \) that corresponds to a plane in \( \mathbb{C}P^3 \) passing through \( z \). Hence, \( \hat{f} \) has a natural double fibration

\[
\hat{r} : \hat{f} \longrightarrow \overline{M}_C = \mathbb{C}P^3 \quad \text{and} \quad \hat{q} : \hat{f} \longrightarrow \mathcal{C} \subset \mathbb{C}P^{3*},
\]

(3.21)

where \( \hat{q} \circ \hat{r}^{-1}(p) \) is the conic section \( l_p \subset \mathcal{C} \) and \( \hat{r} \circ \hat{q}^{-1}(z) \) is the null plane in \( \overline{M}_C \) which is a \( \mathbb{C}P^2 \), and \( z_0 \) corresponds to \( \mathbb{C}P^2_\infty \). This is the double fibration discussed in [1, 2].

In the double fibration (3.21), every point \( z \in \mathcal{C} \) is on an equal footing: each point corresponds to a \( \mathbb{C}P^2 \) plane, including \( z_0 \). This is not the case if one were to consider a similar fibration to \( T\overline{P}^1 \).

**Lemma 3.1** A point on \( L_\infty \subset T\overline{P}^1 \cong \tilde{\mathcal{C}} \) corresponds to a \( \mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \overline{M}_C \).

**Proof.** First, note that since the finite points on \( M_C \) are holomorphic sections (2.11) in \( T\overline{P}^1 \subset T\tilde{\mathcal{C}} \) which do not intersect \( L_\infty \), a point on \( L_\infty \) must correspond to a subset of \( \mathbb{C}P^2_{\infty} \). This subset is determined by equation (2.19) for a fixed \( \lambda \). Given a value of \( \lambda \), (2.19) is one linear equation for 3 unknowns, \( P^1, P^2, P^3 \). Since it is not possible for all coefficients to vanish at the same time, one can always determine one variable in terms of the other two. Hence, there are two degrees of freedom in the homogeneous coordinates in \( \mathbb{C}^2 - \{0\} \), and we conclude that each point in \( L_\infty \) corresponds to a \( \mathbb{C}P^1 \) line in \( \mathbb{C}P^2_{\infty} \).

\[ \square \]

We shall now present a double fibration which fibres over the compactified minitwistor space \( T\overline{P}^1 \), where each point of \( T\overline{P}^1 \) has an equal footing, i.e. a point on \( L_\infty \) is also a \( \mathbb{C}P^2 \). This is achieved simply by defining the correspondence space to be the blow-up of \( \hat{f} \) along its singularity. The singularity of \( \hat{f} \) comes from the conic singularity \( z_0 \in \mathcal{C} \), which corresponds to \( \mathbb{C}P^2_{\infty} \times \{z_0\} \subset \hat{f} \).

That is, we define the correspondence space \( \hat{F} \) of a double fibration to the compactified spacetime \( \overline{M}_C = \mathbb{C}P^3 \) and the compactified twistor space \( T\overline{P}^1 \cong \tilde{\mathcal{C}} \) as

\[
\hat{F} = \text{the blow-up of the algebraic variety } \hat{f} \text{ (3.20) along } \mathbb{C}P^2_{\infty} \times \{z_0\}.
\]

The details of the blow up are given in Appendix C.

There exists a projection \( \rho : \hat{F} \longrightarrow \hat{f} \) such that, away from \( \mathbb{C}P^2_{\infty} \times \{z_0\} \), \( \rho \) is a one-to-one and onto. We find that the preimage of \( \mathbb{C}P^2_{\infty} \times \{z_0\} \) under the map \( \rho \) is \( \mathbb{C}P^2_{\infty} \times L_\infty \), where \( L_\infty \in \tilde{\mathcal{C}} \) is the blow up of \( z_0 \). Let us denote the preimage by \( e := \mathbb{C}P^2_{\infty} \times L_\infty \).
We define a surjective map \( q : \hat{\mathcal{F}} \rightarrow \hat{\mathcal{C}} \) by its action on two disjoint regions. First, define

\[
q|_{\hat{\mathcal{F}} - e} : \hat{\mathcal{F}} - e \rightarrow \hat{\mathcal{C}} - L_{\infty} \tag{3.22}
\]

such that it is equivalent to the composition \( \hat{q} \circ \rho \), where \( \hat{q} : \hat{\mathcal{F}} - \mathbb{C}P^2_\infty \times \{z_0\} \rightarrow \mathcal{C} - \{z_0\} \) is the fibration in (3.21). Then, define

\[
q|_e : \mathbb{C}P^2_\infty \times L_{\infty} \rightarrow L_{\infty} \tag{3.23}
\]

to be the right projection. Thus, by definition, \( q \) is onto.

Therefore, we have a double fibration

\[
r : \hat{\mathcal{F}} \rightarrow \overline{M_C} = \mathbb{C}P^3 \quad \text{and} \quad q : \hat{\mathcal{F}} \rightarrow \hat{\mathcal{C}} = \overline{T\mathbb{P}^1},
\]

where a finite spacetime point \( p \in M_C \subset \overline{M_C} \) corresponds to a holomorphic section in \( T\mathbb{P}^1 \subset \overline{T\mathbb{P}^1} \) and a point \( p \in \mathbb{C}P^2_\infty \) corresponds to the union of \( L_{\infty} \) and two \( \mathbb{C} \) lines of constant \( \lambda \) (counting multiplicity). On the other hand, a point \( z \in T\mathbb{P}^1 \subset \overline{T\mathbb{P}^1} \) corresponds to a null plane in \( \overline{M_C} \) (extension of a null plane in \( M_C \)) and \( z \in L_{\infty} \) corresponds to \( \mathbb{C}P^2_\infty \).

### 3.2 The restricted correspondence space

Recall that the topological degree of a Ward chiral field \( J \), satisfying the trivial scattering condition, comes from the third homotopy class of the restricted extended solution \( \psi(x, y, \theta) \). With this in mind we shall now define a ‘restricted’ correspondence space such that it gives rise to the domain of \( \psi \).

Consider a ‘constant time’ slice \( \tau \), which is the \( \mathbb{C}P^2 \subset \overline{M_C} \) obtained by setting \( P^3 = 0 \). It is clear that the intersection of \( \tau \) with the noncompact spacetime \( M_C = \mathbb{C}^3 \), where \( P^0 \neq 0 \), is the \( t = 0 \) \( \mathbb{C}^2 \)-plane in the \( (x, y, t) \) coordinates (2.17). We will also consider the ‘real slice’ \( \mathbb{R}P^3 \subset \overline{M_C} \), which consists of the points \([P^0]\) whose homogeneous representatives can be chosen to be in \( \mathbb{R}^4 - \{0\} \). Since we write the line element on \( M_C \) as \( ds^2 = dx^2 + dy^2 - dt^2 \), the finite part of this \( \mathbb{R}P^3 \) is an \( \mathbb{R}^{2,1} \). Then, the \( \mathbb{R}P^2 \) intersection, denoted by \( \tau_R \), of \( \tau \) with this \( \mathbb{R}P^3 \) can be thought of as the extension of the \( t = 0 \) \( \mathbb{R}^2 \)-plane to the compactified space.

We define the restricted correspondence space \( \mathcal{F} \) to be the restriction of \( \hat{\mathcal{F}} \) to \( \tau_R \). This
means that away from the singularity, $F$ is the same as the algebraic variety

$$f := \{(p, z) \in \mathbb{RP}^3 \times \mathbb{CP}^3 : Z_1^2 + Z_2^2 - Z_3^2 = 0, P^0 Z_0 + P^1 Z_1 + P^2 Z_2 - P^3 Z_3 = 0, P^3 = 0\}$$

(3.24)

minus its singularity. The singularity of $f$ consists of the points $\{(P^\alpha, z_0)\}$ for all $[P^\alpha]$ such that $P^0 = 0$. Since $P^3 = 0$, this is an $\mathbb{RP}^1 \subset \mathbb{CP}^2$, which will be denoted by $\mathbb{RP}^1_{\infty}$. Under the usual projection map $\rho : F \to f$, the preimage of the singularity $\mathbb{RP}^1_{\infty} \times \{z_0\}$ is $e_{\tau_R} := \mathbb{RP}^1_{\infty} \times L_{\infty}$.

It is not immediate that the map $q$ defined by (3.22), (3.23) is still onto $\overline{T_\mathbb{P}^1} \cong \tilde{\mathcal{C}}$ when the domain of $q$ is restricted to $F$. However, this turns out to be the case.

**Proposition 3.2** The restriction of the map $q : \hat{F} \to \tilde{\mathcal{C}}$ to $F$,

$$q|_F : F \to \tilde{\mathcal{C}},$$

(3.25)

is surjective.

**Proof.** First, it follows readily from (3.23) that

$$q|_{e_{\tau_R}} : \mathbb{RP}^1_{\infty} \times L_{\infty} \to L_{\infty}$$

is onto as a right projection. However, it is not obvious that

$$q|_{\hat{F} - e_{\tau_R}} : \hat{F} - e_{\tau_R} \to \tilde{\mathcal{C}} - L_{\infty}$$

(3.26)

is also surjective. The map (3.26) is equivalent to the restriction of the map $\hat{q}$ in (3.21) to $f - \mathbb{RP}^1_{\infty} \times \{z_0\}$,

$$\hat{q} : f - \mathbb{RP}^1_{\infty} \times \{z_0\} \to \mathcal{C} - \{z_0\}.$$  

(3.27)

Therefore, the question whether $q|_{\hat{F} - e_{\tau_R}}$ is onto comes down to whether, given a point $[Z_\alpha] \in \mathcal{C} - \{z_0\}$, one can find a point in $\tau_R$ which lies on the corresponding null plane. In other words, whether all null planes in $\overline{\mathcal{M}_C} = \mathbb{CP}^3$ intersect $\tau_R$. If this is the case, then we can always find a (non-empty) preimage of every point in $\mathcal{C} - \{z_0\}$ under (3.27), which implies that (3.27) is onto, and hence so are (3.26) and (3.25).

The intersection of a null plane in $\mathbb{CP}^3$ with $\tau_R$ consists of points $[P^\alpha] \in \mathbb{RP}^3$ with $P^3 = 0$ satisfying

$$Z_0 P^0 + Z_1 P^1 + Z_2 P^2 = 0,$$

(3.28)
where \([Z^\alpha] \in \mathbb{CP}^3\) labels the null plane, i.e. satisfies (2.12). We proceed by direct calculation. By looking for \([P^\alpha] \in \tau_R\) that satisfy (3.28), we find that such solutions exist for all \([Z_\alpha] \in \mathcal{C} - \{z_0\}\). Therefore we conclude that the map (3.25) is surjective. The detailed calculation can be found in Appendix D.

\[\square\]

What is crucial to our proof of Theorem 1.1 is that now we have a surjective map (3.25), which can be used to pull back the holomorphic vector bundle over \(\tilde{C}\) to \(\mathcal{F}\).

We note here that the map \(q|_\mathcal{F}\) is not one-to-one everywhere on \(\mathcal{F}\). The preimage of a point \(z \in L_\infty \subset \tilde{C}\) is the \(\mathbb{RP}^1_\infty \times \{z\}\). Then \(\tilde{C} - L_\infty \cong \mathcal{C} - \{z_0\}\) can be divided into two disjoint regions. Let \(\mathcal{C}_R\) denote the set of points \(z := [Z_\alpha] \in \mathcal{C} - \{z_0\}\) whose representatives can be chosen to be in \(\mathbb{R}^4 - \{0\}\). We shall call the planes corresponding to \(z \in \mathcal{C}_R\) real null planes. Note that a real null plane with \((Z_1, Z_2, Z_3) \neq (0, 0, 0)\) corresponds to a null plane in \(\mathbb{R}^{2,1}\). It can be shown that the preimage in \(\mathcal{F}\) of a point \(z \in \mathcal{C}_R\) is an \(\mathbb{RP}^1\). Finally, the map \(q|_\mathcal{F}\) is one-to-one and onto \(\mathcal{C} - \{z_0\} - \mathcal{C}_R\). See Appendix D for details.

For the purpose of proving Theorem 1.1 in the next section, we shall spend the final part of this section describing \(\mathcal{F}\) in 3 disjoint regions: First,

- \(e_{\tau_R} = \mathbb{RP}^1_\infty \times L_\infty\).

Then we divide the complement \(\mathcal{F} - e_{\tau_R}\), which is identified with \(f - \mathbb{RP}^1_\infty \times \{z_0\}\), into two regions:

- \(\mathcal{R} = \{(p, z) \in f : P^0 \neq 0, z \neq z_0\}\): This is the finite part where \(p \in \mathbb{R}^2 \subset \tau_R\). In fact since \(P^0 \neq 0\), we have that \((Z_1, Z_2, Z_3) \neq (0, 0, 0)\). Hence, \((Z_1, Z_2, Z_3) \in \mathcal{R}\) can be parametrised by \([\pi_A] \in \mathbb{CP}^1\). Now, since \(P^0 \neq 0\), one can use (3.28) to determine \(Z_0\). That is, \(\mathcal{R}\) can be parametrised by \((P^1, P^2) \in \mathbb{R}^2 \subset \mathbb{RP}^2\) and \([\pi_A] \in \mathbb{CP}^1\). It can thus be deduced that \(\mathcal{R} = \mathbb{R}^2 \times \mathbb{CP}^1\). This indicates, as we expect, that there are \(\mathbb{CP}^1\)-worth of null planes passing through each point \(p\).

- \(\mathcal{R}_\infty = \{(p, z) \in f : P^0 = 0, z \neq z_0\}\): This region corresponds the extension of null planes to \(\mathbb{RP}^1_\infty\).

**Remark.** The constant time slice \(\tau\) and the real slice \(\mathbb{RP}^3\) used to define \(\tau_R\) can be regarded as the sets of fixed points of a holomorphic involution and an anti-holomorphic involution, respectively, on the complexified spacetime \(\overline{M_C}\). The maps induce the corresponding involutions on the minitwistor space \(\overline{TP^1}\). For details, see Appendix E.
4 Chern numbers and topological degrees

For a Ward chiral field $J$ satisfying the boundary conditions (1.7) and (1.9), the corresponding holomorphic vector bundle $E \to \mathbb{T} \mathbb{P}^1$ is a complex vector bundle of rank $N$ with the structure group $U(N)$. Since $\mathbb{T} \mathbb{P}^1$ is a 2-dimensional complex manifold, the only non-vanishing Chern characters are the first and the second Chern characters, given by

$$C_1(E) = \frac{i}{2\pi} \text{Tr} F, \quad C_2(E) = -\frac{1}{8\pi^2} \text{Tr}(F \wedge F)$$

respectively, where $F$ is the curvature two-form of an arbitrary connection on $E$. (See for example [17].)

The Chern number is an integer-valued topological invariant of $E$, obtained by integrating the Chern characters over the base space $\mathbb{T} \mathbb{P}^1$. The condition that the bundle $E$ is trivial when restricted to the real sections of $\mathbb{T} \mathbb{P}^1$ implies that the first Chern number vanishes. To relate the second Chern number $c_2(E) = \int_{\mathbb{T} \mathbb{P}^1} C_2(E)$ to the topological degree of the restricted extended solution $\psi$ given by (1.10), we consider the bundle $E^* \to \mathcal{F}$ over the restricted correspondence space, defined by pulling back the bundle $E \to \mathbb{T} \mathbb{P}^1$ by the map $q : \mathcal{F} \to \mathbb{T} \mathbb{P}^1$ in (3.25), i.e. $E^* := q^* E$.

An extended solution $\Psi(x, y, t, \lambda)$ of the Lax pair (1.4) is a function on $\mathbb{R}^{2,1} \times \mathbb{C}\mathbb{P}^1$. Restricting $\Psi$ to the spacelike $t = 0$ plane, the matrix

$$\psi_\lambda(x, y, \lambda) := \Psi(x, y, 0, \lambda)$$

is a function on the finite region $\mathcal{R} = \mathbb{R}^2 \times \mathbb{C}\mathbb{P}^1$ of the the restricted correspondence space $\mathcal{F}$. The trivial scattering condition (1.9) is that the restriction of $\psi_\lambda$ to the real equator $S^1 \subset \mathbb{C}\mathbb{P}^1$ of the space of spectral parameter $\lambda$ has the limit at spatial infinity

$$\psi_\lambda|_{S^1 \subset \mathbb{C}\mathbb{P}^1} \equiv \psi(x, y, \theta) \longrightarrow \psi_0(\theta) \quad \text{as} \quad r = \sqrt{x^2 + y^2} \longrightarrow \infty,$$

where $\lambda = -\cot(\frac{\theta}{2})$. We can use a residual freedom in $\psi$ so that $\psi(x, y, \theta) \longrightarrow \mathbf{I}$ as $r \rightarrow \infty$. The triviality of the vector bundle $E \to \mathbb{T} \mathbb{P}^1$ over $L_\infty$ guarantees that it is possible to choose the extended solution $\psi_\lambda$ such that

$$\psi_\lambda(x, y, \lambda) \longrightarrow \mathbf{I} \quad \text{as} \quad r = \sqrt{x^2 + y^2} \longrightarrow \infty. \quad (4.29)$$

This ensures that $\psi_\lambda(x, y, \lambda)$ extends to the region $\mathcal{R}_\infty$ and $e_{\tau \psi}$ of $\mathcal{F}$. 
Proof of Theorem 1.1

Let $E^* := q^* E$ denote the pull back of the vector bundle $E \to \mathbb{T} \mathbb{P}^1$ by the map $q : \mathcal{F} \to \mathbb{T} \mathbb{P}^1$ defined in (3.25). Then, the second Chern number $c_2(E)$ of $E$ is given by

$$c_2(E) = \int_{\mathbb{T} \mathbb{P}^1} C_2(E) = \int_{q(\mathcal{F})} C_2(E)$$

$$= \int_{\mathcal{F}} q^* C_2(E)$$

$$= \int_{\mathcal{F}} C_2(q^* E)$$

$$= c_2(E^*),$$

where we have used a property of integration of differential forms in (4.30), and that of the Chern characters in (4.31). Note that (4.30) is valid because the region where $q$ is not a bijection is of codimension one, hence it does not contribute to the integral.

So now we reduce the problem to finding the second Chern number of $E^* \to \mathcal{F}$. Let $\mathcal{F}_+$ and $\mathcal{F}_-$ denote two open sets covering $\mathcal{F}$, given by

$$\mathcal{F}_+ := \{(p, z) \in \mathcal{F} : \text{Im}(\lambda) > -\varepsilon\} \quad \text{and} \quad \mathcal{F}_- := \{(p, z) \in \mathcal{F} : \text{Im}(\lambda) < \varepsilon\},$$

for some real constant $\varepsilon > 0$.

The second Chern number $c_2(E^*)$ is given by

$$c_2(E^*) = -\frac{1}{8\pi^2} \int_{\mathcal{F}} \text{Tr}(F \wedge F),$$

where $F$ is the curvature two-form of an arbitrary connection on $E^*$. In the limit $\varepsilon \to 0$, (4.32) becomes

$$c_2(E^*) = -\frac{1}{8\pi^2} \left[ \int_{\mathcal{F}_+} \text{Tr}(F \wedge F) + \int_{\mathcal{F}_-} \text{Tr}(F \wedge F) \right].$$

Choose a connection one-form $A$ such that $A$ vanishes on $\mathcal{F}_+$. Then, we only need to consider the integral over $\mathcal{F}_-$. Now, since $\text{Tr}(F \wedge F)$ is closed, there exists a local three-form $Y$ such that

$$\text{Tr}(F \wedge F) = dY, \quad \text{where} \quad Y = \text{Tr}(F \wedge A - \frac{1}{3} A \wedge A \wedge A).$$
Then by Stokes’ Theorem
\[
c_2(E^*) = -\frac{1}{8\pi^2} \int_{\mathcal{F}_-} dY = -\frac{1}{8\pi^2} \int_{\partial\mathcal{F}_-} Y
= -\frac{1}{8\pi^2} \int_{\partial\mathcal{F}_-} \text{Tr}(F \wedge A - \frac{1}{3} A^3).
\]

The only boundary of \(\mathcal{F}_-\) is the common boundary it has with \(\mathcal{F}_+\). Hence \(\partial\mathcal{F}_- \subset \mathcal{F}_+ \cap \mathcal{F}_-\).

Now, let \(A_+\) and \(A_-\) denote the connection \(A\) in local trivialisations over \(\mathcal{F}_+\) and \(\mathcal{F}_-\) respectively. Then, over \(\mathcal{F}_+ \cap \mathcal{F}_-\)
\[
A_- = g^{-1}A_+g + g^{-1}dg = g^{-1}dg,
\]
since \(A_+ \equiv 0\), and where \(g\) denotes the transition function of \(E^*\) in the overlap.

Now the columns of \(\psi_\lambda\) can be used as meromorphic frame fields over \(\mathcal{F}_-\). Note that \(\psi_\lambda\) is holomorphic and invertible in the overlap \(\mathcal{F}_+ \cap \mathcal{F}_-\) for sufficiently small \(\varepsilon\), because \(\psi_\lambda\) have poles only at non-real values of \(\lambda\). In its own trivialisation, \(\psi_\lambda\) takes the form of the identity matrix. Now over \(\mathcal{F}_+\), choose a local frame field such that \(\psi_\lambda\) takes the form \(\psi_\lambda(x, y, \lambda)\) itself. In these local trivialisations, the transition function \(g\) is given by
\[
I = g^{-1}\psi_\lambda.
\]

Hence \(g = \psi_\lambda\). Therefore, in the overlap \(\mathcal{F}_+ \cap \mathcal{F}_-\), the connection \(A_-\) is given by
\[
A_- = \psi^{-1}_\lambda d\psi_\lambda.
\]

This implies that the curvature \(F_-\) vanishes in \(\mathcal{F}_+ \cap \mathcal{F}_-\) and
\[
c_2(E^*) = \frac{1}{24\pi^2} \int_{\partial\mathcal{F}_-} A_-^3 = \frac{1}{24\pi^2} \int_{\partial\mathcal{F}_-} \text{Tr} \left( (\psi^{-1}_\lambda d\psi_\lambda)^3 \right).
\]

Now, since over \(\partial\mathcal{F}_- \cap (e_{\tau_R} \cup \mathcal{R}_\infty)\) the spacetime points \(p\) belong to \(\mathbb{RP}_\infty^1\), the boundary condition (4.29) implies that
\[
A_- \big|_{\partial\mathcal{F}_- \cap (e_{\tau_R} \cup \mathcal{R}_\infty)} = 0, \quad (4.33)
\]
thus
\[
c_2(E^*) = \frac{1}{24\pi^2} \int_{\partial\mathcal{F}_- \cap \mathcal{R}} \text{Tr} \left( (\psi^{-1}_\lambda d\psi_\lambda)^3 \right).
\]
Then, since $\partial \mathcal{F}_- = \{(p,z) \in \mathcal{F} : \text{Im}(\lambda) = 0\}$, i.e. $\lambda \in S^1 \subset \mathbb{CP}^1$, we have

$$\partial \mathcal{F}_- \cap \mathcal{R} = \mathbb{R}^2 \times S^1$$

and

$$\psi|_{\partial \mathcal{F}_- \cap \mathcal{R}} = \psi(x,y,\theta).$$

Therefore

$$c_2(E^*) = \frac{1}{24\pi^2} \int_{\mathbb{R}^2 \times S^1} \text{Tr}((\psi^{-1}d\psi)^3) = \frac{1}{24\pi^2} \int_{S^3} \text{Tr}((\psi^{-1}d\psi)^3) = [\psi]$$

where we have used the fact the $\psi$ satisfies the trivial scattering condition, thus its domain extends to $S^3$.

\[\square\]

**Remark.** Theorem 1.1 holds for any Ward chiral field which satisfies the finite energy condition (1.7) and the trivial scattering condition (1.9). A subclass of such solutions are the so-called time-dependent $n$-unitons, where the extended solutions $\Psi(x,y,t,\lambda)$ have a pole of order $n$ at $\lambda = \mu$, where $\mu \in \mathbb{C} \setminus \mathbb{R}$. The total energy of a time-dependent uniton is known [7] to be directly proportional to $[\psi]$. Hence, we deduce that the total energy of a time-dependent uniton is proportional to $c_2(E)$, consistent with the result for statistic Ward chiral fields in [1].

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**Appendix A** Null planes in $\mathbb{C}^3$ and Ward correspondence

A null plane in a complexified spacetime $M_\mathbb{C} = \mathbb{C}^3$, with the flat metric

$$ds^2 = -dt^2 + dx^2 + dy^2,$$

(A1)
is defined as a 2-plane whose normal vector is null with respect to the metric \((A1)\). Hence, the equation for a null plane is given by

\[
\eta_{\mu\nu} k^\mu x^\nu = -\frac{1}{2} \hat{\omega},
\tag{A2}
\]

where \(x^\mu = (x^0 = t, x^1 = x, x^2 = y)\), \(\eta_{\mu\nu} = \text{diag}(-1, 1, 1)\), \(k^\mu\) is the normal null vector field and \(\hat{\omega}\) is a constant. The factor of \(-\frac{1}{2}\) is introduced for convenience, the reason for which will become apparent shortly. To parametrise null vector fields, it is useful to use the spinor formalism based on the identification

\[
TM_C = S \otimes S,
\]

where \(TM_C\) is the holomorphic tangent bundle of \(M_C\) and \(S\) is a rank two vector bundle over \(M_C\). A tangent vector field \(k\) can be written as a symmetric two-spinor

\[
k^{AB} = \begin{pmatrix}
  k_0 + k^2 & k^1 \\
  k^1 & k_0 - k^2
\end{pmatrix},
\]

such that \(\eta_{\mu\nu} k^\mu k^\nu = -\det(k^{AB}) = -\frac{1}{2} \varepsilon_{AC} \varepsilon_{BD} k^{AB} k^{CD}\). It follows that a null vector field corresponds to a symmetric two-spinor of rank 1. That is, every null vector field is given by \(k^{AB} = \pi^A \pi^B\) for \(\pi^A\) \(\neq (0, 0)\), where \(\pi^A\), \(A = 0, 1\), denotes the fibre coordinates of \(S\). Now, writing the spacetime coordinates also as a symmetric two-spinor

\[
x^{AB} = \begin{pmatrix}
  t + y & x \\
  x & t - y
\end{pmatrix},
\]

the null plane equation \((A2)\) becomes

\[
\hat{\omega} = x^{AB} \pi_A \pi_B.
\tag{A3}
\]

Moreover, let us assume that \(\pi_1 \neq 0\). Defining \(\omega = \hat{\omega}/\pi_1^2\) and \(\lambda = \frac{\pi_0}{\pi_1}\), equation \((A3)\) now reads

\[
\omega = (t + y)\lambda^2 + 2x\lambda + (t - y).
\tag{A4}
\]

The null planes with \(\pi_1 = 0\) can also be captured by \((A4)\) by allowing \(\lambda\) to go to infinity. This implies that every null plane in \(C^3\) is labelled by \((\omega, \lambda)\), where \(\omega \in C\) and \(\lambda \in CP^1\). The minitwistor space, which is the space of null planes in \(M_C = C^3\), is therefore a line bundle over \(CP^1\). It is in fact the tangent bundle \(TP^1\) of \(CP^1\). To see this, note that under the change of coordinate \(\lambda \to \tilde{\lambda} = \lambda^{-1}\), the fibre coordinate changes by \(\omega \to \tilde{\omega} = \omega \lambda^{-2}\).
It follows from (A4) that a point \( p \in M_C \) corresponds to a holomorphic section \( \hat{p} \) of \( TP^1 \). We can define the correspondence space \( F \) to be the space of pairs \( (p, Z) \), of a point \( p \in M_C \) and a null plane \( Z \) passing through \( p \). There is a \( \mathbb{CP}^1 \) worth of null planes passing through each point \( p \in M_C \), and thus \( F = \mathbb{C}^3 \times \mathbb{CP}^1 \). Note that the two vector fields in the Lax pair (1.4)

\[
  l_0 = \partial_x - \lambda(\partial_t - \partial_y), \quad l_1 = \partial_t + \partial_y - \lambda\partial_x
\]

span a null plane, as they annihilate \( \omega = (t + y)\lambda^2 + 2x\lambda + (t - y) \). The minitwistor space \( TP^1 \) can therefore be regarded as the quotient space of \( \mathbb{C}^3 \times \mathbb{CP}^1 \) by the distribution \( \{l_0, l_1\} \).

Since the Ward chiral model (1.1) is the compatibility condition of the Lax pair (1.4), there exist \( N \) linearly independent column vector solutions of (1.4) if \( J \) in (1.3) is a solution of (1.1). These column vector solutions are the covariantly constant sections with respect to \((A, \Phi)\), of the trivial \( \mathbb{C}^N \) bundle \( V \to M_C \) restricted to null planes. One can construct a holomorphic rank \( N \) vector bundle over the minitwistor space \( TP^1 \) by taking the fibre over each point \( z \in TP^1 \) to be the space of covariantly constant sections of \( V|_Z \), where \( Z \) is the null plane corresponding to the point \( z \in TP^1 \).

**Appendix B  The blow-up of the cone**

A reference for the blow-up can be found, for example, in [8, 9]. First, let us consider the blow-up of an open set \( U = \mathbb{C}^3 \subset \mathbb{CP}^3^* \) with \( Z_0 \neq 0 \). Let

\[
  z_1 = \frac{Z_1}{Z_0}, \quad z_2 = \frac{Z_2}{Z_0}, \quad z_3 = \frac{Z_3}{Z_0} \tag{B1}
\]

be coordinates on \( U \), so that the vertex \( z_0 \) coincides with the origin. By definition, the blow-up \( \widetilde{U} \) of \( U \) at the origin is given by

\[
  \{(z, l) \in U \times \mathbb{CP}^2 : z_i l_j = z_j l_i, \ i \neq j\}, \tag{B2}
\]

where \( \{l_i\} \) are homogeneous coordinates of the \( \mathbb{CP}^2 \), \( i = 1, 2, 3 \). In other words, \( \widetilde{U} \) is a 3-dimensional subspace of \( U \times \mathbb{CP}^2 \) defined by the relation in (B2). Geometrically, \( z \) lies on a line labelled by \( l \in \mathbb{CP}^2 \) passing through the origin in \( \mathbb{C}^3 \). One can consider \( \widetilde{U} \) in three coordinate neighbourhoods: \( \widetilde{U}^k \), where \( l_k \neq 0 \). A point in \( \widetilde{U}^k \) is labelled by \( (z_k, \frac{l_j}{l_k}) \) where \( j \neq k \).

There exists a surjective map from \( \widetilde{U} \) to \( U \) which is given locally in a coordinate patch.
\( \tilde{U}^k \) by
\[
\pi : \left( z_k, \frac{l_j}{l_k} \right) \mapsto \left( z_k, z_j = \frac{l_j}{l_k} \right).
\]

One sees that for a point \((z_k, z_j)\) with \(z_k \neq 0\) there is a unique preimage \(\left( z_k, \frac{l_j}{l_k} = \frac{z_j}{z_k} \right)\) in \(\tilde{U}\). However, if \(z_k = 0\) all points with coordinates \(\left( 0, \frac{l_j}{l_k} \right)\) are mapped to the origin. Hence, the preimage \(E\) of the origin is isomorphic to \(\mathbb{CP}^2\). The set \(E\) is called the exceptional divisor. It is important to note that the map \(\pi : \tilde{U} - E \rightarrow U - \{z_0\}\) is one-to-one.

Now, let us look at the blow-up \(\tilde{C}\) at the vertex of the cone \(C \subset \mathbb{CP}^{3*}\). We are only interested in the region around the vertex, as the projection from \(\tilde{C}\) to \(C\) is 1 : 1 elsewhere. The blow-up \(\tilde{C}_U = \tilde{C} \cap \tilde{U}\) is obtained from \(\tilde{U}\) by imposing the cone equation (2.12) on \(\tilde{U}\). In a coordinate patch, say \(\tilde{U}^1\) where \(l_1 \neq 0\), (2.12) becomes
\[
z_1^2 \left( 1 + \left( \frac{l_2}{l_1} \right)^2 - \left( \frac{l_3}{l_1} \right)^2 \right) = 0.
\]
The continuity implies that \(\tilde{C}_U \cap E\) are given locally in \(\tilde{U}^1\) by the points with
\[
z_1 = 0 \quad \text{and} \quad 1 + \left( \frac{l_2}{l_1} \right)^2 - \left( \frac{l_3}{l_1} \right)^2 = 0.
\]
Since \(l_1 \neq 0\) in \(\tilde{U}^1\), the second condition can be written as
\[
l_1^2 + l_2^2 - l_3^2 = 0. \tag{B4}
\]
One obtains similar descriptions in patches \(\tilde{U}^2\) and \(\tilde{U}^3\). Then it follows from (B4) that \(\tilde{C}_U \cap E = \tilde{C} \cap E = \{z_0\} \times \mathbb{CP}^1\), where the \(\mathbb{CP}^1\) is embedded in the \(\mathbb{CP}^2 \ni [l_i]\) by
\[
[l_1, l_2, l_3] = [-2\pi_0\pi_1, \pi_1^2 - \pi_0^2, \pi_0^2 + \pi_1^2] \tag{B5}
\]
with \(\pi_A \in \mathbb{C}^2 - \{0\}\), where we have used the same parametrisation as for null vectors. The \(\mathbb{CP}^1\) in (B5) can be parametrised by a single variable as
\[
[-2\lambda, 1 - \lambda^2, 1 + \lambda^2] \quad \text{and} \quad [2\tilde{\lambda}, \tilde{\lambda}^2 - 1, 1 + \tilde{\lambda}^2], \tag{B6}
\]
in the patches with \(\pi_1 \neq 0\) and \(\pi_0 \neq 0\) respectively, where \(\tilde{\lambda} = \frac{1}{\lambda}\) in the overlap.

Note that we deliberately denote the inhomogeneous coordinate of the \(\mathbb{CP}^1\) by \(\lambda\), to be the same as the base coordinate of \(TP^1\). We shall now show that the \(\tilde{C} \cap E\) indeed corresponds to \(L_\infty\) - the additional \(\mathbb{CP}^1\) at \(\omega = \infty\) of \(TP^1\).
Proof of Proposition 2.4. A bijection from $\mathbb{T}P^1$ to $\mathbb{C} - z_0$ is already given by (2.15). Here, we shall extend the map (2.15) to a bijection from $\mathbb{T}P^1$ to $\tilde{\mathbb{C}}$. Although the fibre of $\mathbb{T}P^1$ is a $\mathbb{CP}^1$, we shall avoid using two fibre-coordinate patches, but rather we will define a map by taking the limit $\omega \to \infty$. Since the projection $\pi : \tilde{\mathbb{C}} - (\tilde{\mathbb{C}} \cap E) \longrightarrow \mathbb{C} - \{z_0\}$ is one-to-one, we only need to consider the map locally in a neighbourhood of $z_0$. Assuming $\omega \neq 0$, then (2.15) can be written as

$$\left(\omega, \lambda\right) \mapsto \left[1, \frac{-2\lambda}{\omega}, \frac{1 - \lambda^2}{\omega}, \frac{1 + \lambda^2}{\omega}\right], \quad \text{and} \quad \left(\tilde{\omega}, \tilde{\lambda}\right) \mapsto \left[1, \frac{-2\tilde{\lambda}}{\tilde{\omega}}, \frac{\tilde{\lambda}^2 - 1}{\tilde{\omega}}, \frac{1 + \tilde{\lambda}^2}{\tilde{\omega}}\right]. \quad \text{(B7)}$$

To extend the domain of (B7) to $\mathbb{T}P^1$ minus the $\omega = 0$ section, we shall take the limit $\omega \to \infty$. For concreteness, let us consider the first local map of (B7). In the inhomogeneous coordinates $z_1, z_2, z_3$ (B1) of $\mathbb{C}U - \{z_0\}$, the first map of (B7) is given by

$$\left(\omega \neq 0, \lambda\right) \longmapsto (z_1, z_2, z_3) = \left(\frac{-2\lambda}{\omega}, \frac{1 - \lambda^2}{\omega}, \frac{1 + \lambda^2}{\omega}\right). \quad \text{(B8)}$$

We can now define another map from the image of (B8), which is $\mathbb{C}U - \{z_0\}$, to the blow-up $\tilde{\mathbb{C}}_U$ in terms of three local maps from the three regions $U^1 = \{\lambda \neq 0\}$, $U^2 = \{\lambda \neq \pm 1\}$ and $U^3 = \{\lambda \neq \pm i\}$ to the blow-up neighbourhoods $\tilde{U}^1 = \{l_1 \neq 0\}$, $\tilde{U}^2 = \{l_2 \neq 0\}$ and $\tilde{U}^3 = \{l_3 \neq 0\}$ in $\tilde{\mathbb{U}}$ respectively.

In $U^1$ for example, the local map is defined by

$$(z_1, z_2, z_3) \longmapsto \left(\frac{z_1}{l_1}, \frac{l_2}{l_1} = \frac{z_2}{z_1}, \frac{l_3}{l_1} = \frac{z_3}{z_1}\right).$$

Composing it with the map (B8), we have

$$(\omega \neq 0, \lambda) \longmapsto \left(\frac{z_1}{l_1}, \frac{l_2}{l_1}, \frac{l_3}{l_1}\right) = \left(\frac{-2\lambda}{\omega}, \frac{1 - \lambda^2}{-2\lambda}, \frac{1 + \lambda^2}{-2\lambda}\right). \quad \text{(B9)}$$

One sees that this is consistent with the parametrisation of $[l_i]$ in (B6). Since at this point $\omega$ is still finite and $\lambda \neq 0$ in $U^1$, then $z_1 \neq 0$. Therefore (B9) is a one-to-one map from $U^1$ to $\tilde{U}^1$, whose image is $(\tilde{\mathbb{C}} \cap \tilde{U}^1) - (\tilde{U}^1 \cap E)$. The local maps $U^2 \to \tilde{U}^2$ and $U^3 \to \tilde{U}^3$ are defined similarly from the inverse of the projection (B3).

Now, consider the limit $\omega \to \infty$ of the map (B9)

$$(\omega \neq 0, \lambda) \longmapsto \left(\frac{z_1}{l_1}, \frac{l_2}{l_1}, \frac{l_3}{l_1}\right) = \left(0, \frac{1 - \lambda^2}{-2\lambda}, \frac{1 + \lambda^2}{-2\lambda}\right) \text{ as } \omega \to \infty. \quad \text{(B10)}$$
We define a bijection from $\overline{T\mathbb{P}^1} \cap U^1$ to $\tilde{C} \cap \tilde{U}^1$ to be the extension of the map (B9) by the limit (B10). Comparing this with the local expression of $\tilde{C} \cap E \cap \tilde{U}^1$ obtained from (B6), and similarly for the other two neighbourhoods, one deduces that $L_\infty$ is mapped onto the restricted exceptional divisor $\tilde{C} \cap E$. 

\[ \square \]

**Remark.** Since every map is given in holomorphic coordinates, one deduces that $T\mathbb{P}^1$ is biholomorphic to $\tilde{C}$.

### Appendix C  The correspondence space

We define the correspondence space $\hat{F}$ of a double fibration to the compactified spacetime $\overline{M}_C = \mathbb{C}\mathbb{P}^3$ and the compactified twistor space $\overline{T\mathbb{P}^1} = \tilde{C}$ to be the blow-up of the algebraic variety $\hat{f}$ (3.20) along $\mathbb{C}\mathbb{P}^2 \times \{z_0\}$. The correspondence space $\hat{F}$ has the following properties.

1. **The blow-up of $\mathbb{C}\mathbb{P}^2 \times \{z_0\}$ is $\mathbb{C}\mathbb{P}^2 \times L_\infty$.**

   This can be derived from the direct construction of the blow-up as follows. Let us first consider the blow-up of $\mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^3$ along $\mathbb{C}\mathbb{P}^2 \times \{z_0\}$ locally in each coordinate patch. Recall that $\mathbb{C}\mathbb{P}^2 \times \{z_0\} = \{[P^0] : P^0 = 0\}$ and $z_0 = [1,0,0,0]$. Since we know that away from the singularity, the projection $\rho : \hat{F} \to \hat{f}$ is a 1 : 1 and onto, we need to consider only three coordinate patches of $\mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^3$ that include the singularity, namely $U_i = \{Z_0 \neq 0, P^i \neq 0\}, i = 1,2,3$. Then, the blow-up of $\hat{F} \subset \mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^3$ is obtained by imposing the incidence relations in (3.20). Note the lower index of $U_i$, to be distinguished from $U^i$ in Appendix B.

   First, consider the patch $U_1 = \{Z_0 \neq 0, P^1 \neq 0\} = \mathbb{C}^3 \times \mathbb{C}^3 = \mathbb{C}^6$ with coordinates

   $$(y_i) = (y_0 = p^0, y_1 = z_1, y_2 = z_2, y_3 = z_3, y_4 = p^2, y_5 = p^3),$$

   where $z_j = \frac{Z_j}{Z_0}$ and $p^i = \frac{P^i}{P^0}$. The intersection $(\mathbb{C}\mathbb{P}^2 \times \{z_0\}) \cap U_1$ is then given by $\mathbb{C}^2(1) := \{(0,0,0,0,p^2,p^3)\} = \mathbb{C}^2$. The blow-up of $U_1$ along $\mathbb{C}^2(1)$ is by definition (see for example [8]) given by

   $$\tilde{U}_1 := \{(y,l) \in \mathbb{C}^6 \times \mathbb{C}\mathbb{P}^3 : y_il_j = y_jl_i, i \neq j \in \{0,1,2,3\}\},$$

   where $\{l_i\}$ are homogeneous coordinates of the $\mathbb{C}\mathbb{P}^3$. The projection $\rho : (y,l) \mapsto y$ is
bijective onto the region away from $\mathbb{C}^2_{\infty(1)}$. If $y \in \mathbb{C}^2_{\infty(1)}$, then $l$ is arbitrary, and hence the preimage of $\mathbb{C}^2_{\infty(1)}$ is $\hat{E}_1 := \rho^{-1}(\mathbb{C}^2_{\infty(1)}) = \mathbb{C}^2_{\infty(1)} \times \mathbb{C}P^3$.

The blow-up $\hat{\mathcal{F}} \cap \hat{U}_1$ is obtained from $\hat{U}_1$ by imposing the incidence relations in (3.20)

$$z_1^2 + z_2^2 - z_3^2 = 0, \text{ and } p^0 + z_1 + p^2 z_2 - p^3 z_3 = 0. \tag{C2}$$

Lifting the relations (C2) to $\tilde{U}_1$, they can be written locally in the four coordinate neighbourhoods of $\tilde{U}_1$. First, in the patch $l_0 \neq 0$, with the coordinates $(p^0, \frac{l_1}{l_0}, \frac{l_2}{l_0}, \frac{l_3}{l_0}, p^2, p^3)$, equation (C2) becomes

$$(p^0)^2 \left( \left( \frac{l_1}{l_0} \right)^2 + \left( \frac{l_2}{l_0} \right)^2 - \left( \frac{l_3}{l_0} \right)^2 \right) = 0 \text{ and } p^0 \left( 1 + \frac{l_1}{l_0} + \frac{l_2}{l_0} p^2 - \frac{l_3}{l_0} p^3 \right) = 0. \tag{C3}$$

Recall that the exceptional divisor of $\tilde{U}_1$ is given by

$$\hat{E}_1 = \{(y, l) \in \tilde{U}_1 : y_0 = y_1 = y_2 = y_3 = 0\},$$

which is the preimage of $\mathbb{C}^2_{\infty(1)}$. The continuity of (C3) implies that $\hat{\mathcal{F}} \cap \hat{E}_1$ is given locally in the patch $l_0 \neq 0$ by the points with $p^0 = 0$ and

$$\left( \frac{l_1}{l_0} \right)^2 + \left( \frac{l_2}{l_0} \right)^2 - \left( \frac{l_3}{l_0} \right)^2 = 0, \quad 1 + \frac{l_1}{l_0} + \frac{l_2}{l_0} p^2 - \frac{l_3}{l_0} p^3 = 0.$$

Since $l_0 \neq 0$, the last two equations can be written as

$$l_1^2 + l_2^2 - l_3^2 = 0 \tag{C4}$$
$$l_0 + l_1 + l_2 p^2 - l_3 p^3 = 0. \tag{C5}$$

Similarly, in the patch $l_1 \neq 0$, with coordinates $(\frac{l_1}{l_1}, z_1, \frac{l_2}{l_1}, \frac{l_3}{l_1}, p^2, p^3)$, equation (C2) becomes

$$(z_1)^2 \left( 1 + \left( \frac{l_2}{l_1} \right)^2 - \left( \frac{l_3}{l_1} \right)^2 \right) = 0, \quad z_1 \left( \frac{l_0}{l_1} + 1 + \frac{l_2}{l_1} p^2 - \frac{l_3}{l_1} p^3 \right) = 0,$$

and $\hat{\mathcal{F}} \cap \hat{E}_1$ is locally given in this neighbourhood by the points with $z_1 = 0$ and

$$1 + \left( \frac{l_2}{l_1} \right)^2 - \left( \frac{l_3}{l_1} \right)^2 = 0 \text{ and } \frac{l_0}{l_1} + 1 + \frac{l_2}{l_1} p^2 - \frac{l_3}{l_1} p^3 = 0.$$

Now, since $l_1 \neq 0$, the above equations can also be written in the homogeneous coordinates $\{l_i\}$ as (C4, C5). The equations in the other two patches $l_2 \neq 0$ and $l_3 \neq 0$ are similar to those in the patch $l_1 \neq 0$. 

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This shows that \( \hat{\mathcal{F}} \cap \hat{E}_1 \) is the subset of

\[
\hat{E}_1 = \{(0, 0, 0, p^2, p^3, l_0, l_1, l_2, l_3) \} = (\mathbb{C}_{\infty}^2 \times \mathbb{C}^3) \subset (\mathbb{C}^6 \times \mathbb{C}^3)
\]

given by (C4, C5). Now, given \((p^2, p^3), l_0\) is uniquely determined from \((l_1, l_2, l_3)\) by (C5).

This, together with (C4), defines a \(\mathbb{C}P^1 \subset \mathbb{C}P^3\) given by

\[
[l_0, -2\alpha_0\alpha_1, \alpha_1^2 - \alpha_0^2, \alpha_0^2 + \alpha_1^2],
\]

where \(l_0 = 2\alpha_0\alpha_1 + (\alpha_0^2 - \alpha_1^2)p^2 + (\alpha_0^2 + \alpha_1^2)p^3\) and \(\alpha_A \in \mathbb{C} - \{0\}\). Hence, we conclude that \(\hat{\mathcal{F}} \cap \hat{E}_1 = \mathbb{C}_{\infty}^2 \times \mathbb{C}P^1\). Note that the local equations for \(\hat{\mathcal{F}} \cap \hat{E}_1\) are smooth and in fact holomorphic in each of the four patches of \(\hat{U}_1\).

In the other two patches \(U_2 = \{Z_0 \neq 0, P^2 \neq 0\}\) and \(U_3 = \{Z_0 \neq 0, P^3 \neq 0\}\), the blow-up follows similarly. Let \(\hat{E} = \hat{E}_1 \cup \hat{E}_2 \cup \hat{E}_3\) denotes the union of the exceptional divisors of \(\hat{U}_1, \hat{U}_2\) and \(\hat{U}_3\). The blow-up is defined such that the coordinate patches glue naturally, therefore we conclude that \(\hat{\mathcal{F}} \cap \hat{E} = \mathbb{C}P^2 \times \mathbb{C}P^1\). Note that since \(\{U_i\}, i = 1, 2, 3,\) are the only patches that include the singularity \(\mathbb{C}_{\infty}^2 \times \{Z_0\}\), then \(\hat{E}\) is also the exceptional divisor of the blow-up of \(\mathbb{C}P^3 \times \mathbb{C}P^3\) along \(\mathbb{C}_{\infty}^2 \times \{Z_0\}\).

The \(\mathbb{C}P^1\) in \(\hat{\mathcal{F}} \cap \hat{E}\) is precisely \(L_\infty\) of \(\hat{\mathcal{C}}\). To see this, consider the incidence relation (C1) in \(\hat{U}_1\). Equation \(y_il_j = y_jl_i\) implies \(z_il_j = z_jl_i, i = 1, 2, 3\), and the same holds for \(\hat{U}_2\) and \(\hat{U}_3\). This is the same expression for the blow up of \(\mathcal{C}\) along \(Z_0\).

2. \(\hat{\mathcal{F}}\) is a \(\mathbb{C}P^2\) bundle over \(\hat{\mathcal{C}}\).

This feature gives another direct way to show that \(\hat{\mathcal{F}} \cap \hat{E} = \mathbb{C}_{\infty}^2 \times L_\infty\). Let us start with the fact that the algebraic variety \(\hat{f} \subset \mathbb{C}P^3 \times \mathcal{C}\) given by (3.20) is a \(\mathbb{C}P^2\) bundle over \(\mathcal{C}\). To see this, consider the followings. Let us denote the neighbourhood \(\{Z_0 \neq 0\} \subset \mathbb{C}P^3 \times \mathbb{C}P^3\) by \(\mathcal{C}\), the same as the neighbourhood \(\{Z_0 \neq 0\} \subset \mathbb{C}P^3\). Locally in \(\mathcal{C}\), with coordinates \(z_i = \frac{z_{2i}}{z_0}\), the incidence relations in (3.20) become

\[
\begin{align*}
z_1^2 + z_2^2 - z_3^2 & = 0 \quad \text{(C6)} \\
P^0 + P^1z_1 + P^2z_2 - P^3z_3 & = 0. \quad \text{(C7)}
\end{align*}
\]

Equation (C7) is homogeneous, i.e. given a point \((z_1, z_2, z_3)\) on \(\mathcal{C}\) satisfying (C6), a solution \([P^0] \in \mathbb{C}P^3\) is given by

\[
[-P^1z_1 - P^2z_2 + P^3z_3, P^1, P^2, P^3],
\]

which is determined by \((P^1, P^2, P^3)\) up to a constant multiplication. This implies that
\( \hat{f}_U := \hat{f} \cap U = \mathbb{CP}_\infty^2 \times C_U \), where \( \mathbb{CP}_\infty^2 \) is the \( \mathbb{CP}^2 \) corresponding to \( z_0 \). In the other neighbourhoods of \( C \), for example \( W = \{ z \in C : Z_1 \neq 0 \} \), the subset \( \hat{f}_W := \hat{f} \cap W \) is also biholomorphic to \( \mathbb{CP}^2 \times \check{C}_W \), where in this case the \( \mathbb{CP}^2 \) is given by \( [P^0, (-P^0 Z_0 - P^2 Z_2 + P^3 Z_3), P^2, P^3] \). This shows that \( \hat{f} \) is a \( \mathbb{CP}^2 \) bundle over \( C \).

Now, \( \hat{F} \) is the blow-up of \( \hat{f} \) along \( \mathbb{CP}_\infty^2 \times z_0 \). Locally the blow-up \( \hat{F}_U \) is biholomorphic to \( \mathbb{CP}^2 \times \check{C}_U \), where \( \check{C}_U \) is the blow-up of \( C_U \) along \( z_0 \). To see this, consider the blow-up locally in the three regions of \( \hat{f}_U \), namely \( U_1 = \{ Z_0 \neq 0, P^1 \neq 0 \} \), \( U_2 = \{ Z_0 \neq 0, P^2 \neq 0 \} \), and \( U_3 = \{ Z_0 \neq 0, P^3 \neq 0 \} \) as discussed previously. Note that \( \hat{f}_U \) is completely covered by these three open sets. The points in \( U \) which are omitted by \( U_1, U_2, U_3 \) are the ones with \( (P^1, P^2, P^3) = (0, 0, 0) \) and are not solutions of (C7).

We have already done the blow-up \( \hat{F}_{U_1} \) of \( \hat{f}_{U_1} \) explicitly, where we describe it in coordinate patches, \( \{ l_0 \neq 0 \}, \{ l_1 \neq 0 \}, \{ l_2 \neq 0 \}, \) and \( \{ l_3 \neq 0 \} \). However, we note here that \( \mathcal{F}_{U_1} \) can in fact be described completely in the patches \( \{ l_i \neq 0 \}, \) because the point \( (l_0 \neq 0, 0, 0, 0) \) is not a solution of (C5). In the patch \( l_1 \neq 0 \), with coordinates \( (l_0, l_1, l_2, l_3, p^2, p^3) \) we can label a point in \( \hat{F}_{U_1, l_1 \neq 0} \) by

\[
(-1 - \frac{l_2}{l_1} p^2 + \frac{l_3}{l_1} p^3, z_1, \frac{l_2}{l_1}, \frac{l_3}{l_1} p^2, p^3),
\]

as a consequence of (C5). The set \( (z_1, \frac{l_2}{l_1}, \frac{l_3}{l_1}) \) can be identified with a point in \( \check{C}_U \). Hence, given a point \( z \in \check{C}_U \) we only have freedom in \( (p^2, p^3) \). Let \( \mathbb{C}^2_{(1)} \) denote the \( \mathbb{C}^2 \) defined by \( (p^2, p^3) \). Then, we have that \( \hat{F}_{U_1, l_1 \neq 0} = \mathbb{C}^2_{(1)} \times \check{C}_U \). One can deduce the same result for the patches \( l_2 \neq 0, l_3 \neq 0 \), and therefore \( \mathcal{F}_{U_1} = \mathbb{CP}^2 \times \check{C}_{U_1} \). This, together with similar results from the neighbourhood \( U_2 \) and \( U_3 \), imply that

\[
\hat{F}_U = \mathbb{CP}^2 \times \check{C}_U,
\]

where it follows that \( \hat{F} \cap \hat{E} = \mathbb{CP}_\infty^2 \times L_\infty \). Moreover, since \( \hat{F} - \hat{F}_U \) is biholomorphic to \( \hat{f} - \hat{f}_U \), we conclude that \( \hat{F} \) is a \( \mathbb{CP}^2 \) bundle over \( \check{C} \).

**Appendix D  The restricted double fibration**

The restricted correspondence space \( \mathcal{F} \) defined in Section 3.2 admits a surjective map to \( \overline{T\mathbb{P}^1} \). This is due to the following proposition.

**Proposition D1** Let \((P^0, P^1, P^2, P^3) \in \mathbb{C}^4 - \{0\}\) be homogeneous coordinates of a compactified complexified spacetime \( \overline{M_C} = \mathbb{CP}^3 \), and let \( \tau_\mathbb{R} \) denote an \( \mathbb{R}^2 \subset \overline{M_C} \) defined by
$(P^0, P^1, P^2, P^3) \in \mathbb{R}^4 - \{0\}$ and $P^3 = 0$. Recall that a null plane in $\overline{M_C}$ is defined to be a $\mathbb{C}P^2$ given by

$$Z_0P^0 + Z_1P^1 + Z_2P^2 - Z_3P^3 = 0,$$

with $(Z_0, Z_1, Z_2, Z_3) \in \mathbb{C}^4 - \{0\}$ satisfying $(Z_1)^2 + (Z_2)^2 - (Z_3)^2 = 0$.

Then, every null plane in $\overline{M_C}$ intersects $\tau_{\mathbb{R}}$.

**Proof of Proposition D1.** Let us first consider the class of real null planes. Let $\mathbb{R}P^3^*$ be the subset of $\mathbb{C}P^3^*$ that consists of points $[Z_0]$ whose representatives can be chosen to be in $\mathbb{R}^4 - \{0\}$, and let $C_\mathbb{R}$ be the intersection $C \cap \mathbb{R}P^3^*$. We call the planes corresponding to $z \in C_\mathbb{R}$ real null planes. To see the intersection of real null planes with $\tau_{\mathbb{R}}$, we first look at the real null planes with $Z_0 \neq 0$.

Such a null plane is given by

$$P^0 + P^1z_1 + P^2z_2 = 0,$$

with $(z_1, z_2, z_3)$ all real. Since everything is real, given $(z_1, z_2, z_3)$, $P^0$ is determined in terms of $P^1, P^2$ by (D1). Thus, the intersection of a null plane with $Z_0 \neq 0$ with $\tau_{\mathbb{R}}$ is an $\mathbb{R}P^1$. For a real null plane with $Z_0 = 0$, either $Z_1$ or $Z_2$ must be non-zero. Similar calculation for these planes shows that their intersections with $\tau_{\mathbb{R}}$ are also $\mathbb{R}P^1$. Therefore, one concludes that each real null plane intersections $\tau_{\mathbb{R}}$ in an $\mathbb{R}P^1$. Note that $\mathbb{C}P^2^\infty$ is a real null plane, whose intersection with $\tau_{\mathbb{R}}$ is $\mathbb{R}P^1^\infty$.

Let us call the real null planes with $(Z_1, Z_2, Z_3) \neq (0, 0, 0)$ finite real null planes. For finite real null planes we have the parametrisation (2.13), where the coordinates $[\hat{\omega}, \pi_A]$ can be chosen such that $\hat{\omega} \in \mathbb{R}$, $\pi_A \in \mathbb{R}^2 - \{0\}$. We will now show that these planes intersect $\tau_{\mathbb{R}}$ in oriented lines which are the extension of straight lines in $t = 0$ $\mathbb{R}^2$-plane. First, note that a finite real null plane corresponds to a null plane in $\mathbb{R}^{2,1}$ given by

$$\hat{\omega} = 2x\pi_0\pi_1 + y(\pi_0^2 - \pi_1^2) + t(\pi_0^2 + \pi_1^2),$$

where we use $(x, y, t)$ in (2.17) as coordinates on $\mathbb{R}^{2,1}$. Using the diffeomorphism between $\mathbb{R}P^1$ and $S^1$

$$\pi_0 = \cos \left( -\frac{\theta}{2} \right), \quad \pi_1 = \sin \left( -\frac{\theta}{2} \right),$$

the null plane equation becomes

$$\hat{\omega} = t - x \sin \theta + y \cos \theta.$$ (D2)
Now, the intersection with $\tau_R$ is obtained by restricting (D2) to the $t = 0 \mathbb{R}^2$-plane, which results in

\[
\hat{\omega} = -\sin \theta \ x + \cos \theta \ y.
\]  
(D3)

This is the equation for oriented lines in $\mathbb{R}^2$. Hence, we have that the space of finite real null planes (in spacetime $\mathbb{RP}^3$ or $\mathbb{CP}^3$) is the space of oriented lines in $\mathbb{R}^2$, which is $S^1 \times \mathbb{R}$. From (D3) we note that $(\hat{\omega}, \theta)$ and $(-\hat{\omega}, \theta + \pi)$ give the same unoriented line. This means that the two orientations of a line correspond to a pair of null planes labelled by $(\hat{\omega}, \pi_0, \pi_1)$ and $(-\hat{\omega}, -\pi_1, \pi_0)$, or $[Z_0, Z_1, Z_2, Z_3]$ and $[Z_0, Z_1, Z_2, -Z_3]$.

Now, let us consider non-real null planes, given by the points $[Z_\alpha] \in \mathbb{C} - \mathbb{C}_R$, and again first look at $[Z_\alpha]$ with $Z_0 \neq 0$. Such a null plane must have either $z_1$ or $z_2$ non-real, or both. Writing $z_1 = l + im$ and $z_2 = k + in$, the plane equation

\[
P^0 Z_0 + P^1 Z_1 + P^2 Z_2 = 0,
\]  
(D4)

becomes

\[
P^0 = -(P^1 l + P^2 k) \quad \text{and} \quad P^1 m + P^2 n = 0.
\]

There are two cases. If $m \neq 0$, then

\[
P^1 = -P^2 \frac{n}{m} \quad \text{and} \quad P^0 = P^2 \left( \frac{n}{m} l - k \right),
\]

and if $n \neq 0$,

\[
P^2 = -P^1 \frac{m}{n}, \quad \text{and} \quad P^0 = P^1 \left( \frac{m}{n} k - l \right).
\]

In other words, each non-real null plane with $Z_0 \neq 0$ intersects $\tau_R$ in a single point given by

\[
\left[ \frac{n}{m} l - k, \ -\frac{n}{m}, \ 1 \right] \quad \text{and} \quad \left[ \frac{m}{n} k - l, \ 1, \ -\frac{m}{n} \right]
\]

for $m \neq 0$ and $n \neq 0$ respectively. Note that if $m \neq 0$ it follows that $P^2 \neq 0$ and if $n \neq 0$ then $P^1 \neq 0$.

Now consider non-real null planes with $Z_0 = 0$. First, note that since $Z_0 = 0$, both $Z_1, Z_2$ must be non-zero, otherwise, for example if $Z_2 = 0$ the cone equation $(Z_1)^2 - (Z_3)^2 = 0$ implies that the plane labelled by $[0, Z_1, 0, Z_3]$ is a real null plane. Now, let us write $Z_1 = L + iM$, $Z_2 = K + iN$. Then (D4) implies that

\[
LP^1 + KP^2 = 0, \quad MP^1 + NP^2 = 0.
\]
For a non-real null plane, at least one of \(M\) or \(N\) must be non-zero. Suppose \(M \neq 0\) then \(P^1 = -\frac{N}{M}P^2\). There are 2 cases.

(i) \(L \neq 0\): Then \(P^1 = -\frac{K}{L}P^2\). Since either \(N\) or \(K\) must be non-zero, in a generic case where \(\frac{N}{M} \neq \frac{K}{L}\) we have \(P^1 = 0 = P^2\). Therefore the plane intersects \(\tau_\mathbb{R}\) at a single point \([P^0, P^1, P^2] = [1, 0, 0]\). It can be shown using the cone equation (2.12) that, if \(\frac{N}{M} = \frac{K}{L}\), then the null planes are real null planes.

(ii) \(L = 0\): Then \(KP^2 = 0\). If \(K \neq 0\) we again have \(P^1 = 0 = P^2\). On the other hand \(K = 0\) means \(Z_1, Z_2\) are pure imaginary, which implies that \(Z_3\) is also pure imaginary. Thus the plane is a real null plane.

If we suppose \(L \neq 0\) at the beginning, interchanging the roles of \((L, M)\) and \((K, N)\) yields the same result. Hence we conclude that each non-real null plane intersects \(\tau_\mathbb{R}\) at a single point.

Therefore, every null plane in \(
\overline{M}_\mathbb{C} = \mathbb{C}P^3\) intersects \(\tau_\mathbb{R}\).

\(\square\)

**Remark.** The surjectivity of the restricted map (3.25) is due to the fact that under the map (3.23) each point on \(L_\infty\) corresponds to \(\mathbb{C}P^2_\infty\). We have (3.23) essentially because we take the correspondence space \(\hat{F}\) of the fibration to be the blow-up of the variety \(\hat{f}\) along \(\mathbb{C}P^2_\infty \times \{z_0\}\).

Recall Lemma 3.1 which states that each point on \(L_\infty\) corresponds to a \(\mathbb{C}P^1\) line in \(\mathbb{C}P^2_\infty\) under (2.19). We note here that not every point in \(L_\infty\) gives a \(\mathbb{C}P^1\) that intersects \(\tau_\mathbb{R}\). Consider equation (2.19) with \(P^3 = 0\) for the intersection of such a \(\mathbb{C}P^1\) with the constant time slice \(\tau\):

\[-2\lambda P^1 + (1 - \lambda^2)P^2 = 0.\]  
(D5)

Since the coefficients of \(P^1\) and \(P^2\) cannot be zero at the same time, we have one degree of freedom in \((P^1, P^2)\). Hence, the \(\mathbb{C}P^1\) intersects \(\tau\) at a single point. For example if \(\lambda \neq 0\), the intersection point is given by

\([P^1, P^2] = \left[\frac{(1 - \lambda^2)}{2\lambda}, 1\right].\]  
(D6)

Note that the map is 2 : 1 as \(\lambda\) and \(-\frac{1}{\lambda}\) give the same point in \(\tau\).

Now assume that \([P^1, P^2] \in \tau_\mathbb{R}\). From (D6), we need \(\frac{(1 - \lambda^2)}{2\lambda}\) to be real. Writing \(\lambda = l + im\), the imaginary part of \(\frac{(1 - \lambda^2)}{2\lambda}\) is \(-m/\sqrt{l^2 + m^2}\). This vanishes if and only if \(m = 0\). Therefore, we conclude that a point \(\lambda\) in \(L_\infty\) gives rise to a \(\mathbb{C}P^1\) in \(\mathbb{C}P^2_\infty\) which intersects \(\tau_\mathbb{R}\) if and only if \(\lambda \in \mathbb{R}P^1 \subset L_\infty\). Then the intersection is a single point determined by (D5).
Appendix E Involution maps

Time reversal

In the noncompact spacetime $M_C = \mathbb{C}^3$ with coordinates $(x, y, t)$, we define the time reversal map as usual as

$$\sigma : (x, y, t) \mapsto (x, y, -t).$$  \hfill (E1)

The fixed points of (E1) are of course those with $t = 0$. The map (E1) induces a holomorphic involution on $T\mathbb{P}^1$ via the null plane equation

$$\dot{\omega} = 2x\pi_0\pi_1 + y(\pi_0^2 - \pi_1^2) + t(\pi_0^2 + \pi_1^2).$$  \hfill (E2)

Under (E1), equation (E2) becomes

$$\dot{\omega} = 2x\pi_0\pi_1 + y(\pi_0^2 - \pi_1^2) - t(\pi_0^2 + \pi_1^2).$$  \hfill (E3)

We now want to define a map $\sigma$ (keeping the same name) acting on a point $(\dot{\omega}, \pi_A) \in T\mathbb{P}^1$ such that the image $(\dot{\omega}', \pi_A')$ corresponds to the null plane defined by (E3). Multiplying (E3) by $-1$ on both sides does not change the plane and we have

$$\sigma : (\dot{\omega}, \pi_0, \pi_1) \mapsto (\dot{\omega}' = -\dot{\omega}, \pi_0' = -\pi_1, \pi_1' = \pi_0).$$

We could equally well define the map $\sigma$ with $\pi_A' = (\pi_1, -\pi_0)$, but since $\pi_A$ are homogeneous coordinates of $\mathbb{C}\mathbb{P}^1$, the choice does not matter. In the inhomogeneous coordinates $(\omega = \frac{\dot{\omega}}{\pi_1^2}, \lambda = \frac{\pi_0}{\pi_1})$ we have

$$\sigma : (\omega, \lambda) \mapsto (-\omega, -\lambda),$$

where we recall that $\dot{\omega} = \frac{\omega}{\lambda^2}$ and $\lambda = \frac{1}{\lambda}$ in the overlap. It is immediate that a holomorphic section (E2) labelled by $(x, y, t)$ is preserved by $\sigma$ if and only if $t = 0$.

For the compact case, we want to extend the map (E1) to $\overline{M_C} = \mathbb{C}\mathbb{P}^3$ and define the corresponding involutions on the cone $C \subset \mathbb{C}\mathbb{P}^3$ and its blow-up $\tilde{C} \cong \mathbb{C}\mathbb{P}^1$. Recall our convention that the extension of $t = 0$ $\mathbb{C}^2$-plane to $\overline{M_C}$ is $\tau = \mathbb{C}\mathbb{P}^2$ defined by $P^3 = 0$, and $(x, y, t)$ is given by (2.17). Then the extension of (E1) to $\overline{M_C}$ is

$$\sigma : [P^0, P^1, P^2, P^3] \mapsto [P^0, P^1, P^2, -P^3].$$
This induces a map \( \sigma \) on \( \mathbb{CP}^3 \) via (2.16), given by

\[
\sigma : [Z_0, Z_1, Z_2, Z_3] \mapsto [Z_0, Z_1, -Z_2, -Z_3],
\]

(E4)

and hence a map on the cone \( \mathcal{C} \).

By generalising the discussion in Appendix D, we can show that each null plane in \( \overline{M_C} \) intersects \( \tau \) in an \( \mathbb{CP}^1 \). Moreover, two null planes labelled by \( [Z_0, Z_1, Z_2, Z_3] \) and \( [Z_0, Z_1, Z_2, -\sqrt{Z_1^2 + Z_2^2}] \) have the same intersection line. Hence, geometrically \( \sigma \) interchanges the two members of such pair. The fixed points of the map are the planes which do not form a pair. A special case is the vertex of the cone \( Z_0 \) with \( \mathbb{CP}^1 \) intersection. The rest are those with \( Z_2 = \pm iZ_1 \). These are the points \( [\frac{Z_0}{Z_1}, 1, \pm i, 0] \in \mathcal{C} \), and there are two \( \mathbb{C} \)-worth sets of these points. Choosing new representatives as \( [\pm 2i\frac{Z_0}{Z_1}, \pm 2i, 2, 0] \), we see that these are the fibres \( \lambda = \pm i \) of \( \overline{T\mathbb{P}^1} \).

We can extend the map (E4) to the compactified twistor space \( \overline{T\mathbb{P}^1} \), which is biholomorphic to \( \tilde{\mathcal{C}} \), by demanding that it gives back (E4) under the projection \( \pi : \tilde{\mathcal{C}} \to \mathcal{C} \). Locally, say in the blow-up \( \tilde{\mathcal{C}}_U \subset U \times \mathbb{CP}^2 \) of the patch \( U \) with \( Z_0 \neq 0 \), the map is defined by its action on \( U \times \mathbb{CP}^2 \) as

\[
\sigma : (z_1, z_2, z_3) \times [l_1, l_2, l_3] \mapsto (z_1, z_2, -z_3) \times [l_1, l_2, -l_3],
\]

where \( z_i = \frac{Z_i}{Z_0} \) are local coordinates on \( U \). Note that the blow-up vertex \( L_\infty \subset \overline{T\mathbb{P}^1} \) is fixed by \( \sigma \), although it is not the set of fixed points.

**Reality condition**

Define a map \( \varphi : M_C \to M_C \) by complex conjugation

\[
\varphi : (x, y, t) \mapsto (\bar{x}, \bar{y}, \bar{t}).
\]

(E5)

The set of fixed points of (E5) is the real slice \( \mathbb{R}^{2,1} \subset M_C \). Now, considering the null plane equation (E2) one sees that the map \( \varphi \) induces an anti-holomorphic involution on \( \overline{T\mathbb{P}^1} \) which maps each point to its complex conjugate

\[
\varphi : (\tilde{\omega}, \pi_A) \mapsto (\bar{\tilde{\omega}}, \bar{\pi}_A).
\]

(E6)

The fixed points of (E6) corresponds to real null planes discussed in Section 3.2. Hence
the set of fixed point is $TS^1 \subset TP^1$.

There are unique extensions of (E6) to $\mathcal{M} = \mathbb{C}P^3$ and $\mathbb{C}P^3^*$, sending

$$[P^0, P^1, P^2, P^3] \mapsto [\bar{P}^0, \bar{P}^1, \bar{P}^2, \bar{P}^3] \quad \text{and} \quad [Z_0, Z_1, Z_2, Z_3] \mapsto [\bar{Z}_0, \bar{Z}_1, \bar{Z}_2, \bar{Z}_3],$$

respectively. The cone $\mathcal{C} \subset \mathbb{C}P^3^*$ is preserved by the map, with the vertex $z_0$ being another fixed point in addition to the set $TS^1$.

The extension to the blow-up $\tilde{U}$ of the neighbourhood $U$ around $z_0$ is obtained similarly to the case of the time reversal, where the map is given locally by the complex conjugation of the coordinates of $\tilde{U}$. The involution $\varphi$ maps the blow-up vertex

$$L_\infty = \{ [l_i] \in \mathbb{C}P^2 : l_1^2 + l_2^2 - l_3^2 = 0 \}$$

to itself, and the fixed points are those with $[l_i] \in \mathbb{R}P^2 \subset \mathbb{C}P^2$. In the coordinate $\lambda \in L_\infty$, these are the points with real $\lambda$.

Finally, $\varphi$ preserves the sections in $\overline{TP^1}$ corresponding to $[P^\alpha] \in \mathbb{R}P^3$. For finite-point sections it follows readily from (2.11). For the sections corresponding to the points at infinity, the pairs of lines of constant $\lambda$ are determined by (2.19). We see that for $\{P^\alpha\}$ real, the two roots can either be both real or complex conjugates, and thus the pairs of lines are preserved by $\varphi$.

References

[1] Anand, C. (1995) Uniton bundles. Commun. Anal. Geom. 3, 371.

[2] Anand, C. (1997) Ward’s solitons. Geom. Topol. 1, 9–20 (electronic).

[3] Anand, C. (1998) Ward’s Solitons II: Exact solutions. Can. J. Math. 50, 1119–1137.

[4] Bredon, G. E. (1993) Topology and geometry, Berlin: Springer-Verlag.

[5] Dai, B. and Terng, C.-L. (2007) Bäcklund transformations, Ward solitons, and unitons. J. Differential Geom. 75, 57–108.

[6] Dunajski, M. (2009) Solitons, Instantons and Twistors, Oxford: Oxford University Press.

[7] Dunajski, M. and Plansangkate, P. (2007) Topology and energy of time dependent unitons. Proc. Roy. Soc. Lond A463, 945–959.
[8] Griffiths, P. and Harris, J. (1978) *Principles of algebraic geometry*, New York: Wiley.

[9] Hartshorne, R. (1977) *Algebraic geometry*, Berlin: Springer-Verlag.

[10] Hitchin, N. J. (1982) Monopoles and geodesics. Commun. Math. Phys. **82**, 579–602.

[11] Hitchin, N. J., Segal, G. and Ward, R. S. (1999) *Integrable systems: twistors, loop groups, and Riemann surfaces*, Oxford: Clarendon Press.

[12] Ioannidou, T. (1996) Soliton solutions and nontrivial scattering in an integrable chiral model in (2 + 1) dimensions. J. Math. Phys. **37**, 3422–3441.

[13] Ioannidou, T. and Manton, N. S. (2005) The energy of scattering solitons in the Ward Model. Proc. Royal Soc. A **461**, 1965-1973.

[14] Ioannidou, T. and Zakrzewski, W. J. (1998) Solutions of the modified chiral model in (2 + 1) dimensions. J. Math. Phys. **39**, 2693–2701.

[15] Mason, L. (2004) Transparent connections on the plane and sphere and their relation to integrable systems. (Workshop on Integrable Systems, Leeds 2004)

[16] Mason, L. and Woodhouse, N. J. (1999) *Integrability Self-Duality, and Twistor Theory*, Oxford, UK: Clarendon Press.

[17] Nakahara, M. (2003) *Geometry, Topology and Physics*, Bristol: IOP.

[18] Ward, R. S. (1988) Soliton solutions in an integrable chiral model in 2 + 1 dimensions. J. Math. Phys. **29**, 386–389.

[19] Ward, R. S. (1989) Twistors in 2 + 1 dimensions. J. Math. Phys. **30**, 2246–2251.

[20] Ward, R. S. (1990) Classical solutions of the chiral model, unitons, and holomorphic vector bundles. Commun. Math. Phys. **128**, 319–332.

[21] Ward, R. S. (1995) Nontrivial scattering of localized solitons in a (2 + 1)-dimensional integrable system. Phys. Lett. A **208**, 203–208.

[22] Ward, R. S. (1996) Twistors, geometry, and integrable systems. (Oxford 1996), *The Geometric Universe*, 99–108, Oxford: Oxford Univ. Press (1998)

[23] Ward, R. S. and Wells, R. O. (1990) *Twistor geometry and field theory*, Cambridge: Cambridge University Press.