THE CONTINUITY EQUATION ON HOPF AND INOUÉ SURFACES

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Abstract. We study the continuity equation of La Nave-Tian, extended to the Hermitian setting by Sherman-Weinkove, on Hopf and Inoué surfaces. We prove a priori estimates for solutions in both cases, and Gromov-Hausdorff convergence of Inoué surfaces to a circle.

1. Introduction

A parabolic approach to proving existence of Kähler-Einstein metrics on a Kähler manifold by means of the Kähler-Ricci flow was used by Cao [9] to provide an alternative proof of the existence of Kähler-Einstein metrics on manifolds with $c_1(X) < 0$ and $c_1(X) = 0$, originally proved by Yau [38] and also independently by and Aubin [3] in the case $c_1(X) < 0$. The Kähler-Ricci flow preserves the Hermitian condition and has been shown to illuminate various complex and algebro-geometric properties of the manifold. Starting in 2007, Song-Tian [27, 28, 29] and Tian [32] proposed an Analytic Minimal Model Program to classify algebraic varieties using the Kähler-Ricci flow to “simplify” algebraic varieties to their minimal models. For example, the Kähler-Ricci flow can be set up to have the behavior of “blowing down” (-1)-curves on a complex surface. La Nave-Tian propose in [19] a continuity equation to provide an alternative method for carrying out the Analytic Minimal Model Program and show, under an assumption on the initial metric, the convergence of the metric to a weak Kähler-Einstein metric away from a subvariety. An advantage of the continuity equation is that solutions always have a Ricci lower bound which provides the groundwork to apply compactness theory by Cheeger-Colding-Tian and a partial $C^0$ estimate.

Beyond the Kähler setting, there is a classification of complex surfaces due to Enriques-Kodaira [4] which states that all minimal non-Kähler compact complex surfaces belong to one of the following classes:

- Kodaira surfaces
- Minimal properly elliptic surfaces
- Class VII surfaces with vanishing second Betti number
  - Hopf surfaces
  - Inoué surfaces
- Class VII surfaces with positive second Betti number

In particular, there is very little known about a special class of surfaces called the Class VII surfaces which are defined as complex surfaces with Kodaira dimension 1 and first Betti number 1. Hopf and Inoué surfaces are exactly those class VII surfaces with $b_2(X) = 0$ [7, 18, 20, 31].

One way to extend the Kähler-Ricci flow to the Hermitian setting to study the complex structures on these non-Kähler surfaces is by the Chern-Ricci flow introduced by Gill [16]. Let $(X, J, g_0)$ be a compact complex manifold of complex dimension $n$ and $g_0$ a Hermitian metric on $X$. The Chern-Ricci flow is given by

$$\frac{d}{dt} \omega = - \text{Ric}(\omega), \quad \omega(0) = \omega_0,$$
where $\omega = ig_{kj}dz^j \wedge d\bar{z}^k$ in local holomorphic coordinates, and $\text{Ric}(\omega)$ to be the Chern-Ricci curvature associated to $g$ which is given by

$$\text{Ric}(\omega) = -i\partial \bar{\partial} \log \det g.$$ 

For complex manifolds with $c_1^{BC}(X) = 0$, Gill [16] proved the long time existence of the flow and smooth convergence of the flow to the unique Chern-Ricci-flat metric in the $\partial \bar{\partial}$-class of the initial metric. For general complex manifolds, Tosatti-Weinkove characterize the maximal existence time of the solution as well as detail the behavior of the flow in [35]. They show that the Chern-Ricci flow can be used to contract $(-1)$ curves on a complex surface to arrive at a minimal surface. In addition, they provide a classification of complex surfaces based on the maximal existence time of the flow starting at a pluriclosed metric $\omega_0$. When $\omega_0$ is Kähler, the Chern-Ricci flow coincides with the Kähler-Ricci flow. The behavior of the Chern-Ricci flow on Hopf surfaces, Inoue surfaces and elliptic surfaces have been studied in [34, 35, 36, 12, 11, 2]. Beyond the Chern-Ricci flow, there exists a vast literature on complex flows of Hermitian metrics [1, 5, 6, 8, 13, 14, 17, 21, 22, 23, 24, 30, 35, 37].

In this paper, we study the continuity equation introduced by La Nave-Tian [19] in the Kähler setting and extended to the Chern-Ricci case by Sherman-Weinkove [25] in the case of Hopf and Inoue surfaces. The continuity equation starting at a Hermitian metric $\omega_0$ is given by

$$\omega = \omega_0 - s \text{Ric}(\omega)$$

for $s \geq 0$, where $\text{Ric}(\omega)$ is the Chern-Ricci curvature of $\omega$. This can be viewed as an elliptic alternative to the Chern-Ricci flow. One immediate advantage of this equation is a Ricci lower bound for $s \geq 0$.

The maximal existence interval for solutions to the continuity equation was shown by Sherman-Weinkove [25]. They also prove convergence of solutions of the continuity equation on minimal properly elliptic surfaces; specifically, for elliptic bundles over a curve of genus at least 2, solutions to the continuity equation starting at a Gauduchon metric converge to the base curve in the Gromov-Hausdorff topology.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$ with $|\alpha_1| = \cdots = |\alpha_n| \neq 1$. The round Hopf manifold is defined as $M_\alpha = (\mathbb{C}^n \setminus \{0\})/\sim$, where $(z^1, \ldots, z^n) \sim (\alpha_1 z^1, \ldots, \alpha_n z^n)$.

We define the metric

$$\omega_H = \frac{\delta_{kj}}{r^2} dz^j \wedge d\bar{z}^k$$

where $r^2 = \sum_{j=1}^{n} |z_j|^2$. Tosatti-Weinkove show that $\hat{\omega}_t := \omega_H - t \text{Ric}(\omega_H)$ gives an explicit solution to the Chern-Ricci flow for $t \in [0, \frac{1}{n})$ in [34] and this solution also extends to case of the continuity equation.

**Theorem 1.** Suppose that $\omega = \hat{\omega} + i\partial \bar{\partial} \varphi$ solves (11) on $M_\alpha \times [0, \frac{1}{n})$ with $\omega_0 = \omega_H + i\partial \bar{\partial} \varphi_0$ and $\varphi$ normalized appropriately. Then

1. $|\varphi| \leq C$
2. $C^{-1} \hat{\omega}^n \leq \omega^n \leq C \hat{\omega}^n$
3. $\omega \leq C \omega_H$
4. $\frac{C^{-1}}{(1-n)s \alpha_{n+1}} \leq R(\omega) \leq \frac{C}{(1-n) \alpha_{n+1}}$

In particular, $\varphi$ converges subsequentially in $C^{1+\beta}$ as $s \to 1/n$.

Using the continuity equation allows us to obtain the estimates (2) and (4) which are not known for solutions to the Chern-Ricci flow. The estimates (1) and (3) were shown for solutions to the Chern-Ricci flow by Tosatti-Weinkove in [35].
We now consider a primary Hopf surface of class 1. They are defined as $M_{\alpha,\beta} = (C^2 \setminus \{0\}) / \sim$, where $(z, w) \sim (\alpha z, \beta w)$ for $\alpha, \beta \in C$, with $1 < |\alpha| \leq |\beta|$. On $M_{\alpha,\beta}$, we describe a few explicit $(1, 1)$-forms (see [12, 11] for details). The equation

$$|z|^2 \Phi^{-2\gamma_1} + |w|^2 \Phi^{-2\gamma_2} = 1$$

uniquely defines a smooth function $\Phi(z, w)$ on $C^2 \setminus \{0\}$, where

$$\gamma_1 = \frac{\log |\alpha|}{\log |\alpha| + \log |\beta|}, \quad \gamma_2 = \frac{\log |\beta|}{\log |\alpha| + \log |\beta|}$$

The Gauduchon-Ornea metric

$$\omega_{GO} = \frac{i\partial \bar{\partial} \Phi}{\Phi^2}$$

is a well-defined Hermitian metric on $M_{\alpha,\beta}$. It was shown in [15], that $\omega_{GO}$ is locally conformally Kähler. Now define a non-negative $(1, 1)$-form

$$\Theta = \frac{i\partial \Phi \wedge \bar{\partial} \Phi}{\Phi^2}.$$

In this case, define $\hat{\omega}_s := (1 - 2s)\omega_{GO} + 2s\Theta$.

**Theorem 2.** Suppose that $\omega = \hat{\omega} + i\partial \bar{\partial} \varphi$ solves (1) on $M_{\alpha,\beta} \times [0, \frac{1}{2})$ with $\omega_0 = \omega_{GO} + i\partial \bar{\partial} \varphi_0$ and $\varphi$ normalized appropriately. Then

1. $|\varphi| \leq C$
2. $C^{-1}\hat{\omega}^2 \leq \omega^2 \leq C\hat{\omega}^2$
3. $\omega \leq C\omega_{GO}$
4. $\frac{C^{-1}}{\sqrt{1 - 2s}} \leq R(\omega) \leq \frac{C}{1 - 2s}$.

In particular, $\varphi$ converges subsequentially in $C^{1,\beta}$ as $s \to 1/2$.

Parts (1) and (3) reflect the behavior of solutions to the Chern-Ricci flow as shown by Edwards in [11]. However, the volume bound in (2) and the Chern scalar curvature bound in (4) for the continuity equation are not known for solutions to the Chern-Ricci flow.

On Inoue surfaces, we now consider the normalized continuity equation given by (2)

$$\omega = \omega_0 - s \text{Ric}(\omega) - s\omega.$$

An Inoue surface $S_M$ is the quotient of $C \times H$ by the relations $(\mu z, \lambda w) \sim (z, w)$ and $(z + m_j, w + e_j) \sim (z, w)$ where $M$ is a given matrix in $SL_3(\mathbb{Z})$ and $Mm = \mu m, M\ell = \lambda \ell$.

Let $y = \text{Im} w$. Let $\omega'$ be a Hermitian metric which is either Gauduchon or strongly flat along the leaves in the sense of [12, 2]. We prove the following result in Section 5.

**Theorem 3.** Suppose that $\omega = \hat{\omega} + i\partial \bar{\partial} \varphi$ solves (2) on $S_M \times [0, \infty)$ with $\omega_0 = \omega' + i\partial \bar{\partial} \psi$ and $\varphi$ normalized appropriately. Then

1. $|\varphi| \leq C/(s + 1)$
2. $e^{-C/s}\hat{\omega}^2 \leq \omega^2 \leq e^{C/s}\hat{\omega}^2$
3. $(1 - C(s + 1)^{-1/4})\hat{\omega} \leq \omega \leq (1 + C(s + 1)^{-1/4})\hat{\omega}$
4. $-C\omega \leq \text{Ric}(\omega) \leq C\omega$

In particular, $(S_M, \omega)$ converges to a circle $S^1$ in the Gromov-Hausdorff sense as $s \to \infty$.

The above result is analogous to the behavior of the evolving metric $\omega(t)$ along the normalized Chern Ricci flow as shown by Fang-Tosatti-Weinkove-Zheng in [12], and reflects the structure of Inoue surfaces as bundles over $S^1$. An improvement in using the continuity equation is that we are able to obtain a uniform Chern Ricci curvature bound as in (4) without the use of higher order estimates. The results carry over straightforwardly to the Inoue surfaces of type $S^+$ and $S^-$, as in Section 3 of [12]. In [2], Tosatti-Angella prove in
fact all higher order estimates for the evolving metric along the Chern-Ricci flow as well as uniform curvature bounds on the evolving metric.

We organize the paper as follows. In Section 2, we cover some notation and state some general estimates that will be used in the later sections. In Section 3 we prove Theorem 1 followed by Theorem 2 in Section 4. Lastly, we prove Theorem 3 in 5.

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2. Preliminaries

To a Hermitian metric $\chi$ we associate the Chern connection $\nabla = \partial + \Gamma$ defined by

$$\Gamma_{jr}^p = \chi^{pq} \partial_j \chi_{qr}$$

the torsion $T$ defined by

$$T_{jr}^p = \Gamma_{jr}^p - \Gamma_{rj}^p$$

$$\tau_j = T_{jp}^p$$

and the curvature $R$ defined by

$$R_{kj}^{pr} = -\partial_j \chi_{qr}$$

If $\omega$ is another Hermitian metric with $d(\omega - \chi) = 0$, by a computation due to Cherrier [10]

$$\Delta_{\omega} \log \tr_{\chi} \omega = -\chi^{pq} R_{qp}(\omega) + g^{jk} R_{kj}^{pq} g_{qp}$$

$$+ g^{jk} R^{pj}_{pkj} - g^{jk} R_{kj}^{p} + g^{jk} \chi^{pq} \chi_{sr} T_{pj}^{r} T_{qk}^{s}$$

$$+ g^{jk} \chi^{pq} g_{sr} \Phi_{pj}^{r} \Phi_{qk}^{s}$$

and

$$\Delta_{\omega} \log \tr_{\chi} \omega = 1 \tr_{\chi} \omega \left\{ -\chi^{pq} R_{qp}(\omega) + g^{jk} R_{kj}^{pq} g_{qp}$$

$$+ g^{jk} R^{pj}_{pkj} - g^{jk} R_{kj}^{p} + g^{jk} \chi^{pq} \chi_{sr} T_{pj}^{r} T_{qk}^{s}$$

$$+ 2 \Re g^{jk} \partial_j \tr_{\omega} \omega \tilde{T}_k + \frac{1}{\tr_{\omega}} g^{jk} \Psi_{j}^{r} \Psi_{k}^{s}$$

$$+ g^{jk} \chi^{pq} g_{sr} \Phi_{pj}^{r} \Phi_{qk}^{s} - \frac{1}{\tr_{\omega}} g^{jk} \Psi_{j}^{r} \Psi_{k}^{s} \right\}$$

where $\Phi_{pj}^{r} = g^{rs} \nabla_{p} g_{sj} + T_{pj}^{r}$ and $\Psi_{j} = \partial_j \tr_{\omega} + T_{j}$. We have in (3)

$$g^{jk} \chi^{pq} g_{sr} \Phi_{pj}^{r} \Phi_{qk}^{s} \geq 0$$

and by the methods used in the proof of [33] Theorem 2.1 (see also [26] Theorem 3) we have in (4)

$$g^{jk} \chi^{pq} g_{sr} \Phi_{pj}^{r} \Phi_{qk}^{s} - \frac{1}{\tr_{\omega}} g^{jk} \Psi_{j}^{r} \Psi_{k}^{s} \geq 0.$$
3. Hopf manifolds

In this section we study equation (1) on round Hopf manifolds of complex dimension \( n \). Specifically, let \( X \) be the quotient of \( \mathbb{C}^n \setminus \{0\} \) by the relation \((\alpha_1 z_1, \ldots, \alpha_n z_n) \sim (z_1, \ldots, z_n)\), where \(|\alpha_1| = \cdots = |\alpha_n| \neq 1\).

The Hopf metric \( \omega_\mathcal{H} \) is defined by

\[
\hat{g}_{kj}^\mathcal{H} = \frac{\delta_{kj}}{r^2}.
\]

It follows that the Ricci curvature of \( \omega_\mathcal{H} \) is given by

\[
R_{kj}^\mathcal{H} = n \frac{\delta_{kj}}{r^2} - n \frac{\bar{z}_j z_k}{r^4}.
\]

By the Cauchy-Schwarz inequality, \( \text{Ric}(\omega_\mathcal{H}) \geq 0 \).

As in [35], we consider the family of Hermitian metrics \( \hat{\omega} = \omega_\mathcal{H} - s \text{Ric}(\omega_\mathcal{H}) \), or

\[
\hat{g}_{kj} = (1 - ns) \frac{\delta_{kj}}{r^2} + ns \frac{\bar{z}_j z_k}{r^4}.
\]

It satisfies

\[
\hat{\omega}^n = (1 - ns)^{n-1} \omega^n_\mathcal{H}.
\]

Hence \( \text{Ric}(\hat{\omega}) = \text{Ric}(\omega_\mathcal{H}) \), and so \( \hat{\omega} \) gives an explicit solution of equation (1) with \( \omega_0 = \omega_\mathcal{H} \).

Let \( \varphi_0 \) be a given smooth function with \( \omega_0 = \omega_\mathcal{H} + i\partial\bar{\partial}\varphi_0 \). By [25, Theorem 1], the equation (1) admits a unique solution on \( X \times [0, \frac{1}{n}] \), which is necessarily of the form \( \omega = \hat{\omega} + i\partial\bar{\partial}\varphi \).

We start by proving parts (1) and (2) of Theorem 1:

**Proposition 1.** Suppose that \( \omega = \hat{\omega} + i\partial\bar{\partial}\varphi \) solves (1) on \( X \times [0, \frac{1}{n}] \) with \( \omega_0 = \omega_\mathcal{H} + i\partial\bar{\partial}\varphi_0 \), subject to the normalization

\[
\int_X e^{\frac{\varphi - \varphi_0}{s}} \hat{\omega}^n = \int_X \omega^n.
\]

Then \( |\varphi| \leq C \) and furthermore \( C^{-1} \leq \omega^n / \hat{\omega}^n \leq C \).

**Proof.** The forms of \( \omega, \omega_0 \) and \( \hat{\omega} \) show that the equation (1) is

\[
i\partial\bar{\partial} \log \frac{\omega^n}{\hat{\omega}^n} = i\partial\bar{\partial} \frac{\varphi - \varphi_0}{s}.
\]

The normalization (6) implies that \( \varphi \) solves the following scalar equation

\[
\log (\hat{\omega} + i\partial\bar{\partial}\varphi)^n = \frac{\varphi - \varphi_0}{s}.
\]

Now at a maximum point \( x_0 \) of \( \varphi \) we have \( i\partial\bar{\partial}\varphi(x_0) \leq 0 \), so that \( (\hat{\omega}(x_0) + i\partial\bar{\partial}\varphi(x_0))^n \leq \hat{\omega}(x_0)^n \). Thus by (7)

\[
\frac{\varphi(x_0) - \varphi_0(x_0)}{s} \leq 0
\]

or

\[
\varphi \leq \|\varphi_0\|_{C^0}.
\]

A similar argument gives a lower bound on \( \varphi \). The second estimate now follows from the first one and (7). \( \square \)

Now we can prove part (3) of Theorem 1:

**Proposition 2.** Suppose that \( \omega = \hat{\omega} + i\partial\bar{\partial}\varphi \) solves (1) on \( X \times [0, \frac{1}{n}] \) with \( \omega_0 = \omega_\mathcal{H} + i\partial\bar{\partial}\varphi_0 \). Then \( \text{tr}_{\omega_0} \omega \leq C \) and hence \( \omega \leq C \omega_\mathcal{H} \).
Proof. The proof follows \cite{13}. Compute
\[ g^{jk} R_{kj}^p q^p = \frac{1}{n} \operatorname{tr}_{\omega_n} \omega \operatorname{tr}_{\omega} \operatorname{Ric}(\omega) \]
and
\[ g^{jk} R^p_{pkj} - g^{jk} R^p_{kj} p - g^{pq} \tilde{q}^p H^r T^r_{kj} p^q = -\frac{2}{n} \operatorname{tr}_{\omega} \operatorname{Ric}(\omega). \]
Therefor by (3)
\[ \Delta_{\omega} \operatorname{tr}_{\omega_n} \omega \geq \left( \frac{\operatorname{tr}_{\omega_n} \omega}{n} - \frac{2}{n} \right) \operatorname{tr}_{\omega} \operatorname{Ric}(\omega_n) - \operatorname{tr}_{\omega_n} \operatorname{Ric}(\omega) \]
Let \( x_0 \) be a maximum point of \( \operatorname{tr}_{\omega_n} \omega \). If \( \operatorname{tr}_{\omega_n}(\omega(x_0)) < 2 \) then the result is proved, so we may assume that the quantity in parentheses in (8) is non-negative. But since \( \operatorname{Ric}(\omega_H) \geq 0 \), this means that the first term of (8) is non-negative. Thus we may use (1) to obtain
\[ 0 \geq - \operatorname{tr}_{\omega_n}(\omega(x_0)) \]
\[ = \operatorname{tr}_{\omega_n}(x_0) \frac{\omega(x_0) - \omega_0(x_0)}{s} \]
or
\[ \operatorname{tr}_{\omega_n} \omega \leq \| \operatorname{tr}_{\omega_n} \omega_0 \|_{C^0} \]
as desired. \( \square \)

Finally we complete the proof of Theorem 1:

Proposition 3. Suppose that \( \omega = \tilde{\omega} + i \partial \bar{\partial} \varphi \) solves (1) on \( X \times [0, \frac{1}{n}] \) with \( \omega_0 = \omega_H + i \partial \bar{\partial} \varphi_0 \). Then for \( s \) sufficiently close to \( 1/n \), the scalar curvature of \( \omega \) satisfies
\[ C^{-1} \leq R(\omega) \leq \frac{C}{(1 - ns)^{n-1}}. \]

Proof. Equation (1) implies
\[ R(\omega) = \frac{\operatorname{tr}_{\omega} \omega_0 - n}{s}. \]
The above estimates together with (13) give
\[ \operatorname{tr}_{\omega} \omega_0 \geq C^{-1} \left( \frac{\omega^n_0}{\omega^n} \right)^{\frac{1}{n}} \geq C^{-1} \left( \frac{\omega^n_H}{\omega^n} \right)^{\frac{1}{n}} = \frac{C^{-1}}{(1 - ns)^{1 - \frac{1}{n}}} \]
and
\[ \operatorname{tr}_{\omega} \omega_0 \leq C \frac{\omega^n_0}{\omega^n} (\operatorname{tr}_{\omega_0} \omega)^{n-1} \leq C \frac{\omega^n_H}{\omega^n} = \frac{C}{(1 - ns)^{n-1}}. \]
\( \square \)

4. HOPF SURFACES OF CLASS 1

In this section we focus on Hopf surfaces of class 1. Specifically, let \( X \) be the quotient of \( \mathbb{C}^2 \setminus \{0\} \) by the relation \((\alpha z, \beta w) \sim (z, w)\), where \( 1 < |\alpha| \leq |\beta| \). First we describe a few explicit \((1,1)\)-forms on \( X \) as detailed in \cite{15} (see also \cite{11}). The equation
\[ |z|^2 \Phi^{-2\gamma_1} + |w|^2 \Phi^{-2\gamma_2} = 1 \]
uniquely defines a smooth function $\Phi(z, w)$ on $\mathbb{C}^2 \setminus \{0\}$, where

$$
\gamma_1 = \frac{\log |\alpha|}{\log |\alpha| + \log |\beta|}, \quad \gamma_2 = \frac{\log |\beta|}{\log |\alpha| + \log |\beta|}.
$$

The Gauduchon-Ornea metric

$$
\omega_{GO} = \frac{i \partial \bar{\partial} \Phi}{\Phi}
$$

is a well-defined Hermitian metric on $X$. Define a non-negative $(1, 1)$-form

$$
\Theta = \frac{i \partial \Phi \wedge \bar{\partial} \Phi}{\Phi^2}.
$$

Then we have

$$
\Theta \leq \omega_{GO}. \tag{10}
$$

It is also convenient to define another Hermitian metric $\chi$ by

$$
\chi_{kj} = \Phi^{-2\gamma_j} \delta_{kj}
$$

as this satisfies

$$
\text{Ric}(\chi) = 2(\omega_{GO} - \Theta) \tag{11}
$$

and so $\text{Ric}(\chi) \geq 0$ by (10). As in [11], we consider the family of Hermitian metrics

$$
\hat{\omega} = \omega_{GO} - s \text{Ric}(\chi)
$$

(12)

In contrast to the case of the round Hopf manifolds, explicit solutions to the equation (11) are not known. It can be shown that

$$
\hat{\omega}^2 = (1 - 2s) \omega_{GO}^2 \tag{13}
$$

which implies

$$
\text{Ric}(\hat{\omega}) = \text{Ric}(\omega_{GO}). \tag{14}
$$

Let $\varphi_0$ be a smooth function with $\omega_0 = \omega_{GO} + i \partial \bar{\partial} \varphi_0 > 0$. The equation (1) admits a unique solution on $X \times [0, \frac{1}{2})$, which necessarily has the form $\omega = \hat{\omega} + i \partial \bar{\partial} \varphi$ [25, Theorem 1]. We begin by proving parts (1) and (2) of Theorem 2.

**Proposition 4.** Suppose that $\omega = \hat{\omega} + i \partial \bar{\partial} \varphi$ solves (1) on $X \times [0, \frac{1}{2})$ with $\omega_0 = \omega_{GO} + i \partial \bar{\partial} \varphi_0$, subject to the normalization

$$
\int_X e^{\varphi - \varphi_0} \chi^2 = \int_X \omega_{GO}^2. \tag{15}
$$

Then $|\varphi| \leq C$ and furthermore $C^{-1} \leq \omega^2/\hat{\omega}^2 \leq C$.

**Proof.** The forms of $\omega$, $\omega_0$, and $\hat{\omega}$ show that the equation (1) is

$$
i \partial \bar{\partial} \frac{\varphi - \varphi_0}{s} = \text{Ric}(\chi) - \text{Ric}(\omega).
$$

Set $f = \log \frac{\omega_{GO}^2}{\chi^2}$. Then by (14),

$$
i \partial \bar{\partial} \frac{\varphi - \varphi_0}{s} = \text{Ric}(\omega_{GO}) - \text{Ric}(\omega) + i \partial \bar{\partial} f
$$

$$
= \text{Ric}(\hat{\omega}) - \text{Ric}(\omega) + i \partial \bar{\partial} f
$$

$$
= i \partial \bar{\partial} \log \frac{\omega^2}{\hat{\omega}^2} + i \partial \bar{\partial} f.
$$
The normalization \((15)\) implies that \(\varphi\) solves the following scalar equation
\[
\log \left( \hat{\omega} + i \partial \bar{\partial} \varphi \right)^2 = \frac{\varphi - \varphi_0}{s} - f.
\]
(17)

Now at a maximum point \(x_0\) of \(\varphi\) we have \(i \partial \bar{\partial} \varphi(x_0) \leq 0\), so that \((\hat{\omega}(x_0) + i \partial \bar{\partial} \varphi(x_0))^2 \leq \hat{\omega}(x_0)^2\). Thus by (17)
\[
\frac{\varphi(x_0) - \varphi_0(x_0)}{s} - f(x_0) \leq 0
\]
or
\[
\varphi \leq \|\varphi_0\|_{C^0} + \frac{1}{2} \|f\|_{C^0}.
\]

A similar argument gives a lower bound on \(\varphi\). The second estimate now follows from the first one and (17). \(\square\)

From here, we can prove part (3) of Theorem 2:

**Proposition 5.** Suppose that \(\omega = \hat{\omega} + i \partial \bar{\partial} \varphi\) solves (1) on \(X \times [0, \frac{1}{2})\) with \(\omega_0 = \omega_{GO} + i \partial \bar{\partial} \varphi_0\). Then \(\text{tr}_{\omega_{GO}} \omega \leq C\) and hence \(\omega \leq C \omega_{GO}\).

**Proof.** The proof follows [11]. We will obtain a bound on \(\text{tr}_{\chi} \omega\), since this is equivalent to the desired bound. Compute
\[
g^{jk} R^{pq}_{kj} g_{qp} \text{tr}_{\omega} \text{Ric}(\chi) = \gamma_p \chi^{pq} g_{qp} \text{tr}_{\omega} \text{Ric}(\chi)
\]
by taking \(C^{-1} \leq \min \gamma_j\), and
\[
g^{jk} R^{pq}_{pj} - g^{jk} R^{pq}_{kj} - g^{jk} \chi^{pq} \chi_{rs} T^r_{pj} T^s_{qk} \geq - C \text{tr}_{\omega} \omega_{GO}.
\]

Therefor by (3)
\[
\Delta_{\omega} \text{tr}_{\chi} \omega \geq C^{-1} \text{tr}_{\chi} \omega \text{tr}_{\omega} \text{Ric}(\chi) - C \text{tr}_{\omega} \omega_{GO} - \text{tr}_{\chi} \text{Ric}(\omega)
\]
and since
\[
\Delta_{\omega} \varphi = 2 - \text{tr}_{\omega} \omega_{GO} + \frac{s}{2} \text{tr}_{\omega} \text{Ric}(\chi)
\]
also
\[
\Delta_{\omega} (\text{tr}_{\chi} \omega - A \varphi)
\]
\[
\geq \left( \frac{\text{tr}_{\chi} \omega}{C} - \frac{A}{2} \right) \text{tr}_{\omega} \text{Ric}(\chi) + (A - C) \text{tr}_{\omega} \omega_{GO} - \text{tr}_{\chi} \text{Ric}(\omega) - 2A
\]
\[
\geq \left( \frac{\text{tr}_{\chi} \omega}{C} - \frac{A}{2} \right) \text{tr}_{\omega} \text{Ric}(\chi) - \text{tr}_{\chi} \text{Ric}(\omega) - 2A
\]
after choosing \(A\) larger than \(C\).

Let \(x_0\) be a maximum point of \(\text{tr}_{\chi} \omega - A \varphi\). If
\[
\frac{\text{tr}_{\chi(x_0)} \omega(x_0)}{C} - \frac{A}{2} < 0
\]
then the result is proved, so we may assume that the coefficient of \(\text{tr}_{\omega} \text{Ric}(\chi)\) in (18) is non-negative. But since \(\text{Ric}(\chi) \geq 0\), this means that the first term of (18) is non-negative.
Thus we may use (1) to obtain
\[ 0 \geq -\operatorname{tr}_{x_0} \operatorname{Ric}(\omega(x_0)) - 2A \]
\[ = \frac{\operatorname{tr}_{x_0}(\omega(x_0) - \omega_0(x_0))}{s} - 2A \]
or
\[ \operatorname{tr}_{x_0} \omega(x_0) \leq \operatorname{tr}_{x_0} \omega_0(x_0) + A \]
which implies
\[ \operatorname{tr} \omega \leq \| \operatorname{tr}\omega_0 \|_{C^0} + 2A \| \varphi \|_{C^0} + A. \]

Finally, applying Proposition 4 proves the result. \( \square \)

We are now ready to finish the proof of Theorem 2:

**Proposition 6.** Suppose that \( \omega = \hat{\omega} + i\partial\bar{\partial}\varphi \) solves (1) on \( X \times [0, \frac{1}{2}] \) with \( \omega_0 = \omega_{\text{GO}} + i\partial\bar{\partial}\varphi_0 \). Then
\[ \operatorname{tr} \omega_{\text{GO}} \leq C \]
and hence \( \omega \leq C\omega_{\text{GO}} \). Then for \( s \) sufficiently close to \( 1/2 \), the scalar curvature of \( \omega \) satisfies
\[ C^{-1} \sqrt{1 - 2s} \leq R(\omega) \leq \frac{C}{1 - 2s}. \]

**Proof.** Equation (1) implies
\[ R(\omega) = \frac{\operatorname{tr}_0 \omega_0 - 2}{s}. \]
The above estimates together with (4) give
\[ \operatorname{tr}_0 \omega_0 \geq C^{-1} \left( \frac{\omega_0^2}{\omega'^2} \right)^{\frac{1}{2}} \geq C^{-1} \left( \frac{\omega_{\text{GO}}^2}{\omega'^2} \right)^{\frac{1}{2}} = \frac{C^{-1}}{\sqrt{1 - 2s}} \]
and
\[ \operatorname{tr}_0 \omega_0 = \frac{\omega_0^2}{\omega'^2} \operatorname{tr}_0 \omega \leq C \frac{\omega_{\text{GO}}^2}{\omega'^2} = \frac{C}{1 - 2s}. \]
\( \square \)

5. **Inoue Surfaces**

Let \( X \) be the quotient of \( \mathbb{C} \times \mathbb{H} \) by the relations \((\mu z, \lambda w) \sim (z, w) \) and \((z + m_j, w + \ell_j) \sim (z, w) \) where \( M \) is a given matrix in \( SL_3(\mathbb{Z}) \) and \( Mm = \mu m, M\ell = \lambda \ell \). On \( \mathbb{C} \times \mathbb{H} \) we use coordinates \((z, w)\) and set \( y = \text{Im} \ w \). The expressions
\[ \alpha = \frac{1}{4y^2} idw \wedge d\bar{w} \]
\[ \beta = y idz \wedge d\bar{z} \]
define \((1, 1)\)-forms on \( X \). Let \( \Omega = \alpha \wedge \beta \). Then
(19) \[ i\partial\bar{\partial} \log \Omega = \alpha. \]

By the recent work of Angella-Tosatti [2], in the \( i\partial\bar{\partial} \)-class of any Gauduchon metric, there is a metric \( \omega_{\text{LF}} \) satisfying
(20) \[ c\alpha \wedge \omega_{\text{LF}} = \Omega. \]

Using this, we may assume for the proof of Theorem 3 that \( \omega_0 = \omega_{\text{LF}} + i\partial\bar{\partial}\varphi_0 \). We consider the family of Hermitian metrics
\[ \hat{\omega} = (\omega_{\text{LF}} + s\alpha)/(s + 1). \]
For example, the Triceri metric $\omega_T = 4\alpha + \beta$ satisfies (20). It also satisfies $\text{Ric}(\omega_T) = -\alpha$. In this case $\tilde{\omega} = (\omega_T - \text{Ric}(\omega_T))/(s+1)$; the reader can check that $\text{Ric}(\tilde{\omega}) = \text{Ric}(\omega_T)$, and so $\tilde{\omega}$ gives an explicit solution of equation (2) with $\omega = \omega_T$.

Let $\varphi_0$ be a given smooth function with $\omega_0 = \omega_{LF} + i\partial\bar{\partial}\varphi_0 > 0$. By [25, Theorem 1], the equation (2) admits a unique solution on $X \times [0, \infty)$ of the form $\omega = \tilde{\omega} + i\partial\bar{\partial}\varphi$.

Here we prove parts (1) and (2) of Theorem 3:

**Proposition 7.** Suppose that $\omega = \tilde{\omega} + i\partial\bar{\partial}\varphi$ solves (2) on $X \times [0, \infty)$ with $\omega_0 = \omega_{LF} + i\partial\bar{\partial}\varphi_0$, subject to the normalization

$$\int_X e^{(s+1)\omega_0/2s} \Omega = \frac{(s+1)^2}{2s} \int_X \omega^2. \tag{21}$$

Then $|\varphi| \leq C/(s+1)$. Furthermore

$$-\frac{C}{s} \leq \log \frac{\omega^2}{\tilde{\omega}^2} \leq \frac{C}{s} \tag{22}$$

and in particular

$$C^{-1} \leq \frac{\omega^2}{\tilde{\omega}^2} \leq C. \tag{23}$$

**Proof.** The forms of $\omega$, $\omega_0$, and $\tilde{\omega}$ together with (19), show that the equation (2) is

$$i\partial\bar{\partial}\log \frac{\omega^2}{\Omega} = i\partial\bar{\partial}\frac{(s+1)\varphi - \varphi_0}{s}. \tag{24}$$

The normalization (21) implies that $\varphi$ solves the following scalar equation

$$\log \frac{(s+1)^2(\dot{\omega} + i\partial\bar{\partial}\varphi)^2}{2s\Omega} = \frac{(s+1)\varphi - \varphi_0}{s}. \tag{24}$$

Now at a maximum point $x_0$ of $\varphi$ we have $i\partial\bar{\partial}\varphi(x_0) \leq 0$, so that $(\dot{\omega}(x_0) + i\partial\bar{\partial}\varphi(x_0))^2 \leq \dot{\omega}(x_0)^2$. Since $(s+1)^2\dot{\omega}^2 = \omega_{LF}^2 + 2s\Omega$, (24) becomes

$$\frac{(s+1)\varphi(x_0) - \varphi_0(x_0)}{s} \leq \log \left(1 + \frac{f(x_0)}{s}\right)$$

where $f = \omega_{LF}^2/2\Omega$. This implies

$$(s+1)\varphi(x_0) - \varphi_0(x_0) \leq s \log \left(1 + \frac{f(x_0)}{s}\right) \leq f(x_0)$$

or

$$\varphi \leq \frac{||\varphi_0||_{C^0} + ||f||_{C^0}}{s+1}. \tag{26}$$

A similar argument gives

$$\varphi \geq -\frac{||\varphi_0||_{C^0}}{s+1}. \tag{27}$$

Now (24) gives

$$-\frac{C}{s} \leq \log \frac{(s+1)^2\omega^2}{2s\Omega} \leq \frac{C}{s}. \tag{28}$$

Since $(s+1)^2\dot{\omega}^2 = \omega_{LF}^2 + 2s\Omega$ we have $2s\Omega \leq (s+1)^2\dot{\omega}^2 \leq (2s + C)\Omega$ for $s \geq 1$. So

$$\log \frac{\omega^2}{\dot{\omega}^2} = \log \frac{(s+1)^2\omega^2}{(s+1)^2\dot{\omega}^2} \leq \log \frac{(s+1)^2\omega^2}{2s\Omega} \leq \frac{C}{s}.$$
and
\[
\log \frac{\omega^2}{\hat{\omega}^2} = \log \frac{(s + 1)^2 \omega^2}{(s + 1)^2 \hat{\omega}^2} \geq \log \frac{(s + 1)^2 \omega^2}{(2s + C)^2 \Omega} = \log \frac{(s + 1)^2 \omega^2}{2s \Omega} - \log \left(1 + \frac{C}{2s}\right) \geq -\frac{C}{s}
\]
since \(\log(1 + C/2s) \leq C/2s\). This proves (22).

Next we prove a first step towards part (3) of Theorem 3:

**Proposition 8.** Suppose that \(\omega = \hat{\omega} + i \partial \bar{\partial} \varphi\) solves (2) on \(X \times [0, \infty)\) with \(\omega_0 = \omega_{\text{LF}} + i \partial \bar{\partial} \varphi_0\). Then \(\text{tr} \hat{\omega} \omega \leq C\) and \(\text{tr} \hat{\omega} \omega \hat{\omega} \leq C\), and hence \(C^{-1} \hat{\omega} \leq \omega \leq C \hat{\omega}\).

**Proof.** The equation (2) gives
\[
-\text{tr} \hat{\omega} \text{Ric}(\omega) = \frac{(s + 1) \text{tr} \hat{\omega} \omega - \text{tr} \hat{\omega} \omega_0}{s} \geq -\frac{1}{s} \text{tr} \hat{\omega} \omega_0 \geq -C
\]
for \(s \geq 1\), since \(\text{tr} \hat{\omega} \omega_0 \leq (s + 1) \text{tr} \omega_{\text{LF}} \omega_0 \leq C(s + 1)\). Computing as in [36, Lemma 4.1], we have
\[
g^{jk} R_{k p} \bar{p} q R_{q k} p - g^{jk} R_{k p} \bar{p} q \hat{\omega} \partial_j \partial_p \bar{q} \hat{\omega} \geq -C \text{tr} \hat{\omega} \hat{\omega}
\]
and
\[
g^{jk} R^p \bar{p} q R_{q k} p - g^{jk} R_{k p} \bar{p} q \hat{\omega} \partial_j \partial_p \bar{q} \hat{\omega} \geq -C \text{tr} \hat{\omega} \hat{\omega}.
\]
Substituting (25), (26), (27) into (4) yields
\[
\Delta \omega \log \text{tr} \hat{\omega} \omega \geq \frac{1}{\text{tr} \hat{\omega} \omega} \left\{ -C \sqrt{s + 1} \text{tr} \hat{\omega} \omega \text{tr} \hat{\omega} \hat{\omega} - C \text{tr} \hat{\omega} \hat{\omega} - C + 2 \text{Re} g^{jk} \partial_j \text{tr} \hat{\omega} \hat{\omega} T_k \right\}
\]
\[
\geq -C \sqrt{s + 1} \text{tr} \hat{\omega} \hat{\omega} - C + \frac{1}{\text{tr} \hat{\omega} \omega} 2 \text{Re} g^{jk} \partial_j \text{tr} \hat{\omega} \hat{\omega} T_k.
\]
Here we have used \(\text{tr} \hat{\omega} \omega \geq C^{-1}\), which follows from the determinant estimate (23). We have
\[
\Delta \omega \varphi = 2 - \text{tr} \hat{\omega} \hat{\omega}.
\]
Let \(\psi = \sqrt{s + 1} \varphi + C\); by Proposition 7 we may assume that \(\psi \geq 1\). Then \(0 < 1/\psi \leq 1\) and
\[
\Delta \omega \frac{1}{\psi^3} = \frac{2(s + 1)}{\psi^3} |\partial \varphi|_\omega^2 - \frac{2 \sqrt{s + 1}}{\psi^2} + \sqrt{s + 1} \text{tr} \hat{\omega} \hat{\omega}
\]
\[
\geq \frac{2(s + 1)}{\psi^3} |\partial \varphi|_\omega^2 - 2 \sqrt{s + 1}.
\]
We now refine Proposition 8 as follows, to obtain part 3 of Theorem 3.

**Proposition 9.** Suppose that \( \omega = \omega + i \partial \bar{\partial} \varphi \) solves (2) on \( X \times [0, \infty) \) with \( \omega_0 = \omega_{LF} + i \partial \bar{\partial} \varphi_0 \). Then \( \tr_{\omega} \omega - 2 \leq C(s + 1)^{-\frac{1}{4}} \) and \( \tr_{\omega} \omega - 2 \leq C(s + 1)^{-\frac{1}{4}} \), and furthermore

\[
(1 - C(s + 1)^{-1/8}) \omega \leq \omega \leq (1 + C(s + 1)^{-1/8}) \omega.
\]
Proof. Substituting the estimates $\text{tr}_\omega \omega \leq C$ and $\text{tr}_\omega \hat{\omega} \leq C$ into our earlier computations (25), (26), (27) we obtain from (3)

$$\Delta_\omega \text{tr}_\omega \omega \geq -C\sqrt{s+1}. \quad (38)$$

Thus

$$\Delta_\omega (\text{tr}_\omega \omega - A\varphi) \geq A(\text{tr}_\omega \hat{\omega} - 2) - C\sqrt{s+1}. \quad (39)$$

Using

$$\text{tr}_\omega \hat{\omega} = \hat{\omega}^2 / \omega^2 \text{tr}_\omega \omega = \text{tr}_\omega \omega - \left(1 - \hat{\omega}^2 / \omega^2\right) \text{tr}_\omega \omega$$

(39) becomes

$$\Delta_\omega (\text{tr}_\omega \omega - A\varphi) \geq A\left(\text{tr}_\omega \hat{\omega} - 2\right) - A\left(1 - \hat{\omega}^2 / \omega^2\right) \text{tr}_\omega \omega - C\sqrt{s+1}. \quad (40)$$

By (22)

$$1 - \omega^2 / \hat{\omega}^2 \leq 1 - e^{-C/s} \leq C(s+1)^{-\frac{1}{4}} \quad (41)$$

for $s \geq 1$. Now (41) and Proposition 8 imply

$$\Delta_\omega (\text{tr}_\omega \omega - A\varphi) \geq A\left(\text{tr}_\omega \hat{\omega} - 2\right) - A\left(1 - \hat{\omega}^2 / \omega^2\right) \text{tr}_\omega \omega - C\sqrt{s+1}. \quad (42)$$

At a maximum point $x_0$ of $\text{tr}_\omega \omega - A\varphi$,

$$0 \geq A(\text{tr}_\omega(x_0) \omega(x_0) - 2) - CA(s+1)^{-\frac{1}{4}} - C\sqrt{s+1}$$

or

$$\text{tr}_\omega(x_0) \omega(x_0) - 2 \leq \frac{C\sqrt{s+1}}{A} + C(s+1)^{-\frac{1}{4}}. \quad (43)$$

Hence

$$\text{tr}_\omega \omega - 2 \leq A(\varphi - \inf_X \varphi) + \frac{C\sqrt{s+1}}{A} + C(s+1)^{-\frac{1}{4}} \quad (44)$$

and by Proposition 7

$$\text{tr}_\omega \omega - 2 \leq \frac{CA}{s+1} + \frac{C\sqrt{s+1}}{A} + C(s+1)^{-\frac{1}{4}}. \quad (45)$$

Choose $A = (s+1)^{\frac{3}{4}}$; then

$$\text{tr}_\omega \omega - 2 \leq C(s+1)^{-\frac{1}{4}}. \quad (46)$$

Now by (42) and (22)

$$\text{tr}_\omega \hat{\omega} - 2 = \hat{\omega}^2 / \omega^2 \text{tr}_\omega \omega - 2$$

$$= \hat{\omega}^2 / \omega^2 \left(\text{tr}_\omega \omega - 2\right) + 2\left(1 - \omega^2 / \hat{\omega}^2\right)$$

$$\leq Ce^{C/s}(s+1)^{-\frac{1}{4}} \leq C(s+1)^{-\frac{1}{4}}. \quad (47)$$

for $s \geq 1$. 

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By an elementary inequality [36, Lemma 7.4], the inequalities \( \lambda_1 + \lambda_2 \leq 2 + \epsilon \) and \( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \leq 2 + \epsilon \) imply \( |\lambda_j - 1| \leq 2\sqrt{\epsilon} \), so the estimates \( \text{tr}_\omega \omega - 2 \leq C(s+1)^{-\frac{1}{4}} \) and \( \text{tr}_\omega \omega - 2 \leq C(s+1)^{-\frac{1}{4}} \) give (37).

Finally we obtain part (4) of Theorem 3

**Proposition 10.** Suppose that \( \omega = \hat{\omega} + \partial \bar{\partial} \varphi \) solves (2) on \( X \times [0, \infty) \) with \( \omega_0 = \omega_{\text{LF}} + \partial \bar{\partial} \varphi_0 \). Then

\[
-C \omega \leq \text{Ric}(\omega) \leq C \omega.
\]

**Proof.** By (2)

\[
\text{Ric}(\omega) = \frac{\omega_0 - (s+1)\omega}{s}.
\]

This implies

\[
\left( 1 + \frac{1}{s} \right) \omega \leq \text{Ric}(\omega) \leq \frac{1}{s} \omega_0.
\]

Now \( \omega_0 \leq C \omega_{\text{LF}} \leq C(s+1)\omega \leq C(s+1)\omega \) by Proposition 8, so in fact

\[
\left( 1 + \frac{1}{s} \right) \omega \leq \text{Ric}(\omega) \leq C \left( 1 + \frac{1}{s} \right) \omega
\]

which implies (43). \( \square \)

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