ARBITRAGE THEORY WITHOUT A NUMÉRAIRE

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Abstract. This note develops an arbitrage theory for a discrete-time market model without the assumption of the existence of a numéraire asset. Fundamental theorems of asset pricing are stated and proven in this context. The distinction between the notions of investment-consumption arbitrage and pure-investment arbitrage provide a discrete-time analogue of the distinction between the notions of absolute arbitrage and relative arbitrage in the continuous-time theory.

1. Introduction

In most accounts of arbitrage theory, the concept of an equivalent martingale measure takes centre stage. Indeed, the discrete-time fundamental theorem of asset pricing, first proven by Harrison & Kreps [7] for models with finite sample spaces and by Dalang, Morton & Willinger [5] for general models, says that there is no arbitrage if and only if there exists an equivalent martingale measure.

An equivalent martingale measure is defined in terms of a given numéraire asset. Recall a numéraire is an asset, or more generally a portfolio, whose price is strictly positive at all times with probability one. Associated to a given equivalent martingale measure is a positive adapted process. This process is often called a martingale deflator, but is also known as a pricing kernel, a stochastic discount factor or a state price density. It seems that martingale deflators feature less prominently in the financial mathematics literature, although they are, in a sense, more fundamental. Indeed, they are the natural dual variables for an investor’s optimal investment problem and have the economic interpretation as the sensitivity of the maximised expected utility with respect to the current level of wealth. Furthermore, unlike the concept of an equivalent martingale measure, the concept of a martingale deflator is defined in a completely numéraire-independent manner.

Note that in order to define an equivalent martingale measure, it is necessary to assume that at least one numéraire exists. This assumption is ubiquitous in the financial mathematics literature, but as we will see, it is not strictly necessary. In particular, in this note, we consider a discrete-time arbitrage theory without the assumption of the existence of a numéraire, and we will see that fundamental theorems of asset pricing can be formulated in this setting. From an aesthetic, or possibly pedantic, perspective, we dispense with a mathematically unnecessary assumption and rephrase the characterisation of an arbitrage free market in terms of the more fundamental notion of a martingale deflator. However, there are other reasons to weaken the assumptions of the theorem.

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While it does not seem that the assumption of the existence of a numéraire is controversial, it is not entirely innocent either. Indeed, there is growing interest in robust arbitrage theory, where the assumption that there exists a single dominating measure is dropped. See, for instance, recent papers of Bouchard & Nutz [1] and Burzoni, Frittelli & Maggis [2] for robust versions of the fundamental theorem of asset pricing in discrete time. From the perspective of robust finance, the assumption of the existence of a numéraire seems rather strong. Indeed, when dealing with a family of possibly singular measures, insisting that there is an asset with strictly positive price almost surely under all such measures might be asking too much.

One possible interpretation of a market model without a numéraire is an economy in which the currency has the risk of hyperinflation. In particular, at each moment in time, prices are denominated by the currency which can be converted into a perishable consumption good. However, wealth cannot always be stored in the currency because there is the possibility that the currency collapses completely. A continuous-time market model with this property is considered in the paper of Carr, Fisher & Ruf [3].

Another benefit of our more general treatment of arbitrage theory is that it provides some analogues in the discrete-time theory that previously have been considered generally to be only continuous-time phenomena. In particular, we will see that the distinction between the notions of investment-consumption arbitrage and pure-investment arbitrage provide a discrete-time analogue of the distinction between the notions of absolute arbitrage and relative arbitrage in the continuous-time theory. In particular, when the market does not admit a numéraire, it is possible for there to exist a price bubble in discrete-time in the same spirit as the continuous-time notion of bubble popularised by Cox & Hobson [4] and Protter [13].

The remainder of the paper is arranged as follows. In section 2 we introduce the notation and basic definitions as well as the main results of this paper: a characterisation of no-arbitrage in a discrete-time model without a numéraire. We also characterise the minimal superreplication cost of a contingent claim in this context, and show that the martingale deflator serve as the dual variables for optimal investment problems, even when no numéraire is assumed to exist. In section 3 we introduce the notion of a pure-investment strategy, and characterise contingent claims which can be replicated by such strategies. In addition, we recall the notion of a numéraire strategy and recapture the classical no-arbitrage results when we assume that a numéraire exists. In section 4 we explore other notions of arbitrage and show that they are not equivalent in general. In section 5 we present the proofs of the main results, along with the key economic insight arising from the optimal investment problem. The ideas here originated in Rogers [14] proof of the Dalang–Morton–Willinger theorem. In particular, although the arguments are not especially novel, we present the full details since they are rather easy and probabilistic, and do not rely on any knowledge of convex analysis or separation theorems in function spaces. Finally, in section 6 we include a few technical lemmas regarding measurability and discrete-time local martingales. We note here that we do not rely on any general measurable selection theorems, preferring a hands-on treatment.

2. INVESTMENT-CONSUMPTION STRATEGIES

We consider a general frictionless market model where there are $n$ assets. We let $P^i_t$ denote the price of asset $i$ at time $t$, where we make the simplifying assumption that no asset pays a dividend. We use the notation $P_t = (P^1_t, \ldots, P^n_t)$ to denote the vector of asset price, and we model these prices as a $n$-dimensional adapted stochastic process $P = (P_t)_{t\geq 0}$ defined
on some probability space \((\Omega, \mathcal{F}_\infty, \mathbb{P})\) with filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\). Time is discrete so the notation \(t \geq 0\) means \(t \in \{0, 1, 2, \ldots\}\). We will also use the notation \(a \cdot b = \sum_{i=1}^{n} a^i b^i\) to denote the usual Euclidean inner product in \(\mathbb{R}^n\).

Importantly, we do not make any assumption about the sign of any of the random variables \(P_i^t\): when \(P_i^t = 0\) the asset is worthless, and when \(P_i^t < 0\) the asset is actually a liability. This flexibility allows us to handle claims, such as forward contracts, whose payouts can be either positive or negative.

To the market described by the process \(P\), we now introduce an investor. Suppose that \(H_i^t\) is the number of shares of asset \(i\) held during the interval \((t - 1, t]\). We will allow \(H_i^t\) to be either positive, negative, or zero with the interpretation that if \(H_i^t > 0\) the investor is long asset \(i\) and if \(H_i^t < 0\) the investor is short the asset. Also, we do not demand that the \(H_i^t\) are integers. As usual, we introduce a self-financing constraint on the possible dynamics of the \(n\)-dimensional process \(H = (H_1^t, \ldots, H_n^t)_{t \geq 1}\).

**Definition 2.1.** An investment-consumption strategy is an \(n\)-dimensional predictable process \(H\) satisfying the self-financing conditions

\[
H_t \cdot P_t \geq H_{t+1} \cdot P_t \text{ almost surely for all } t \geq 1.
\]

**Remark 1.** The idea is that the investor brings the initial capital \(x\) to the market. He then consumes a non-negative amount \(c_0\), and invests the remainder into the market by choosing a vector of portfolio weights \(H_1 \in \mathbb{R}^n\) such that \(H_1 \cdot P_0 = x - c_0\).

At each future time \(t \geq 1\), the investor’s pre-consumption wealth is just the market value \(H_t \cdot P_t\) of his current holdings. He again chooses a non-negative amount \(c_t\) to consume, and use the post-consumption wealth \(H_t \cdot P_t - c_t = H_{t+1} \cdot P_t\) to rebalance his portfolio to be held until time \(t + 1\) when the market clock ticks again.

The assumption that the strategy \(H\) is predictable models the fact that the investor is not clairvoyant.

Supposing that an investor in this market has a preference relation over the set of investment-consumption strategies, his goal then is to find the best strategy given his budget constraint.

To fix ideas, suppose that his preference has a numerical representation, so that strategy \(H\) is preferred to strategy \(H'\) if and only if \(U(H) > U(H')\) where \(U\) has the additive expected utility form

\[
U(H) = \mathbb{E} \left[ \sum_{t=1}^{T} u(t, c_t) \right]
\]

where \(x\) is his initial wealth and \(T > 0\) is a fixed, non-random time horizon, and where \(c_0 = x - H_1 \cdot P_0\) and \(c_t = (H_t - H_{t+1}) \cdot P_t\) for \(t \geq 1\) is the investor’s consumption. Assuming that he is not permitted to have negative wealth after time \(T\), the investor’s problem is to maximise \(U(H)\) subject to the budget constraint \(H_1 \cdot P_0 \leq x\) and the transversality condition \(H_{T+1} = 0\). For future reference, we will let

\[
\mathcal{H}_{x,T} = \{H : \text{self-financing, } H_1 \cdot P_0 \leq x, H_{T+1} = 0\}
\]

be the set of feasible solutions to this problem.

The notions of arbitrage are intimately related to whether this optimal investment problem has a solution. Furthermore, we will see that martingale deflators are the dual variables for
this optimisation problem. The fundamental theorems stated below establish the connection between the absence of arbitrage and the existence of martingale deflators.

2.1. Martingale deflators and optimal investment. We now come to the definition of a martingale deflator. For technical reasons, it will also be useful to introduce the related concept of a local martingale deflator.

Definition 2.2. A (local) martingale deflator is a strictly positive adapted process \( Y = (Y_t)_{t \geq 0} \) such that the \( n \)-dimensional process \( PY = (P_t Y_t)_{t \geq 0} \) is a (local) martingale.

Remark 2. A concept very closely related to that of a martingale deflator is that of an equivalent martingale measure, whose definition is recalled in section 3.2 below. While the definition of an equivalent martingale measure is highly asymmetric in that gives a distinguished role to one asset among the \( n \) total assets, note that the definition of a martingale deflator is perfectly symmetric in the sense that all assets are treated equally.

The key result underpinning many of the arguments to come is the following proposition:

Proposition 2.3. Let \( Y \) be a local martingale deflator and \( H \) an investment-consumption strategy. Then the process defined by

\[
M_t = H_{t+1} \cdot P_t Y_t + \sum_{s=1}^{t} (H_s - H_{s+1}) \cdot P_s Y_s
\]

is a local martingale. In particular, if \( H_{T+1} = 0 \) for some non-random \( T > 0 \), then

\[
\mathbb{E} \left( \sum_{s=1}^{T} (H_s - H_{s+1}) \cdot P_s Y_s \right) = H_1 \cdot P_0 Y_0
\]

Proof. Note that by rearranging the sum, we have the identity

\[
M_t = H_1 P_0 Y_0 + \sum_{s=1}^{t} H_s \cdot (P_s Y_s - P_{s-1} Y_{s-1}).
\]

If \( Y \) is a local martingale deflator, then \( M \) is a local martingale by Proposition 6.6.

For the second claim, note that if \( H \) is a self-financing investment-consumption strategy with \( H_{T+1} = 0 \), then

\[
M_T = \sum_{s=1}^{T} (H_s - H_{s+1}) \cdot P_s Y_s
\]

\[
\geq 0.
\]

By Proposition 6.7, the process \( (M_t)_{0 \leq t \leq T} \) is a true martingale and hence by the optional sampling theorem we have

\[
\mathbb{E}(M_T) = M_0 = H_1 \cdot P_0 Y_0.
\]
Remark 3. An important consequence of Proposition 2.3 is this. Suppose $Y$ is a local martingale deflator and $H$ is an investment-consumption strategy. If the negative part of the random variable $H_T \cdot P_T Y_T$ is integrable then $(H_t \cdot P_t Y_t)_{1 \leq t \leq T}$ is a supermartingale with Doob decomposition $H_t \cdot P_t Y_t = M_t - A_t$ where $A_t$ is the predictable increasing process defined by

\[ A_t = \sum_{s=1}^{t-1} (H_s - H_{s+1}) \cdot P_s Y_s. \]

We now turn our attention to the utility optimisation problem described above. In particular, we will see that martingale deflators play the role of a dual variable or Lagrange multiplier. The following proposition is not especially new, but it does again highlight the fact that utility maximisation theory does not depend on the existence of a numéraire. In particular, the following theorem gives us the interpretation of a martingale deflator as the marginal utility of an optimal consumption stream.

**Theorem 2.4.** Let the set of investment-consumption strategies $\mathcal{H}_{x,T}$ be defined by equation (2), and let $U(H)$ be defined by equation (1) for $H \in \mathcal{H}_{x,T}$ and such that the expectation is well defined. Suppose that the utility functions $c \mapsto u(t, c)$ are strictly increasing, strictly concave and differentiable. Let

\[ v(t, y) = \sup_{c \geq 0} u(t, c) - cy \]

for $y > 0$ be the concave dual function, and set

\[ V(Y) = \mathbb{E} \left[ \sum_{t=0}^{T} v(t, Y_t) \right] \]

for every local martingale deflator $Y$ such that each term in the sum is integrable. Then for every investment-consumption strategy $H$ such that $U(H) < \infty$ we have

\[ U(H) \leq V(Y) + xY_0 \]

with equality if and only if

\[ \frac{\partial}{\partial c} u(t, c_t) = Y_t \text{ almost surely for all } 0 \leq t \leq T, \]

where $c_0 = x - H_1 \cdot P_0$ and $c_t = (H_t - H_{t+1}) \cdot P_t$ for $t \geq 1$.

**Proof.** By definition $v(t, y) \geq u(t, c) - cy$ for any $c \geq 0$ and $y > 0$.

\[ V(Y) \geq \mathbb{E} \left[ \sum_{t=1}^{T} u(t, c_t) - c_t Y_t \right] \]

\[ = U(H) - xY_0 \]

by the definition of $U(H)$ and Proposition 2.3.

Finally note that $v(t, y) = u(t, c) - cy$ if and only if the first-order condition for a maximum

\[ \frac{\partial}{\partial c} [u(t, c) - cy] = 0 \]

holds. \qed
2.2. Arbitrage and the first fundamental theorem. We introduce the following definition:

**Definition 2.5.** An investment-consumption arbitrage is a strategy $H$ such that there exists a non-random time horizon $T > 0$ with the properties that

- $H_0 = H_{T+1} = 0$ almost surely
- $\mathbb{P}((H_t - H_{t+1}) \cdot P_t > 0 \text{ for some } 0 \leq t \leq T) > 0$.

Suppose that $H_{arb}$ is an arbitrage according to the above definition. If $H \in \mathcal{H}_{x,T}$ for some $x$ and $T$, where this notation is defined by equation (2), then $H + H_{arb} \in \mathcal{H}_{x,T}$. Furthermore, if the functions $c \mapsto u(t,c)$ are strictly increasing then $U(H + H_{arb}) > U(H)$, where the functional $U$ is defined by equation (1), and hence the strategy $H + H_{arb}$ is strictly preferred to $H$. In particular, the optimal investment problem cannot have a solution. In section 5 we show that a certain converse is true: if there is no arbitrage, then it is possible to formulate an optimal investment problem that has a maximiser.

We now come to our version of the first fundamental theorem of asset pricing for investment-consumption strategies.

**Theorem 2.6.** The following are equivalent:

1. The market has no investment-consumption arbitrage.
2. There exists a local martingale deflator.
3. There exists a martingale deflator.
4. For every non-random $T > 0$ and every positive adapted process $(\eta_t)_{0 \leq t \leq T}$, there exist a martingale deflator $(Y_t)_{0 \leq t \leq T}$ such that $Y_t \leq \eta_t$ almost surely for all $0 \leq t \leq T$.

The equivalence of (1) and (3) above is the real punchline of the story. Condition (2) is some what technical, but is useful since it is easier to check than condition (3). Condition (4) will prove very useful in the next section since it implies that for any $\mathcal{F}_T$-measurable random variable $\xi_T$ there exists martingale deflator $Y$ such that $\xi_T Y_T$ is integrable. Notice that although it is true that (4) implies (3), the argument is not as trivial as it might first seem, since condition (4) holds for each fixed time horizon $T$, while condition (3) says that $(P_t Y_t)_{t \geq 0}$ is a martingale over an infinite horizon.

We now prove the equivalence of conditions (2) and (3).

**Proof of (2) ⇔ (3) of Theorem 2.6** Since a martingale deflator is also a local martingale deflator, we need only prove (2) $\Rightarrow$ (3).

Let $Y$ be a local martingale deflator, so that $PY$ is a local martingale. Note that if we were to assume that each asset price is non-negative, so that $P_t^i \geq 0$ almost surely for all $1 \leq i \leq n$ and $t \geq 0$, we could invoke Proposition 6.7 to conclude that $PY$ is a true martingale and, hence, that $Y$ is a true martingale deflator. In the general case, we appeal to Kabanov’s theorem 6.8 quoted in section 8 which says that there exists an equivalent measure $Q$ such that $PY$ is a true martingale under $Q$. Letting

$$\hat{Y}_t = Y_t \mathbb{E}^P \left( \frac{dQ}{dP} \bigg| \mathcal{F}_t \right),$$

we see that $P\hat{Y}$ is a true martingale under $P$, and hence $\hat{Y}$ is a true martingale deflator. $\square$
We now prove that the existence of a local martingale deflator implies the absence of investment-consumption arbitrage. This is a well-known argument, but we include it here for completeness.

Proof of (2) ⇒ (1) and (4) ⇒ (1) of Theorem 2.6. Fix $T > 0$, and let $H$ be a strategy such that $H_0 = 0$ and $H_{T+1} = 0$ almost surely. If $(Y_t)_{0 \leq t \leq T}$ is a local martingale deflator then, Proposition 2.3 implies that

$$E\left[\sum_{s=0}^{T} c_s Y_s\right] = 0$$

where $c_t = (H_t - H_{t+1}) \cdot P_t$. Since $Y_t > 0$ and $c_t \geq 0$ almost surely for all $t \geq 0$, the pigeon-hole principle and the equality above implies $c_t = 0$ almost surely for all $0 \leq t \leq T$. Hence $H$ is not an arbitrage. □

The proofs of (1) ⇒ (2), that no arbitrage implies both the existence of a local martingale deflator, as well as of (1) ⇒ (4), that the existence over any finite time horizon of suitably bounded martingale deflator, are more technical and deferred to section 5.

2.3. Super-replication. We now turn to the dual characterisation of super-replication. It is a classical theorem of the field, but we include it here since it might be surprising to know that it holds without the assumption of the existence of a numéraire.

**Theorem 2.7.** Let $\xi$ be an adapted process such that $\xi Y$ is a supermartingale for all martingale deflators $Y$ for which $\xi Y$ is an integrable process. Then there exists an investment-consumption strategy $H$ such that

$$H_1 \cdot P_0 \leq \xi_0$$
$$H_{t+1} \cdot P_t \leq \xi_t \leq H_t \cdot P_t$$

The proof is deferred to section 5.

Note that given a non-negative $\mathcal{F}_T$-measurable random variable $X_T$, we can find the smallest process $(X_t)_{0 \leq t \leq T}$ such that $XY$ is a supermartingale for all $Y$ by

$$X_t = \text{ess sup} \left\{ \frac{1}{Y_t} E(X_T Y_T | \mathcal{F}_t) : Y \text{ a martingale deflator such that } X_T Y_T \text{ is integrable} \right\}$$

Theorem 2.7 says that there is a strategy $H$ such that $H_T \cdot P_T \geq X_T$ almost surely, such that $H_1 \cdot P_0 \leq X_0$. In other words, $X_0$ bounds the initial cost of super-replicating the contingent claim with payout $\xi_T$.

3. Pure-investment strategies

In constrast the investment-consumption strategies studied above, we now introduce the notion of a pure-investment strategy.

**Definition 3.1.** A strategy $H$ is called a pure investment strategy if

$$(H_t - H_{t+1}) \cdot P_t = 0 \text{ almost surely for all } t \geq 1.$$ For pure-investment strategies, we will make the convention that $H_0 = H_1$. 7
3.1. **Replication and the second fundamental theorem.** We are already prepared to characterise contingent claims which can be attained by pure investment. Again, the result is not especially new, but it is interesting to see that it holds without the assumption of the existence of a numéraire.

**Theorem 3.2.** Let $\xi$ be an adapted process such that $\xi Y$ is a martingale for all martingale deflators $Y$ such that $\xi Y$ is an integrable process. Then there exists a pure-investment strategy $H$ such that $H_t \cdot P_t = \xi_t$ almost surely for all $t \geq 0$.

In particular, if $X_T$ is an $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}(X_T Y_T) = xY_0$ for martingale deflators $Y$ for which $X_T Y_T$ is integrable, the there a pure-investment strategy $H$ such that $H_T \cdot P_T = X_T$.

**Proof.** If $\xi Y$ is a martingale for all suitably integrable martingale deflators $Y$, then by Theorem 3.2, there exists an investment-consumption strategy $H$ such that $\xi_t \leq H_t \cdot P_t$ almost surely for all $t \geq 1$. Now fix one such $Y$ and let

$$M_t = -Y_t \xi_t + H_{t+1} \cdot P_t + \sum_{s=1}^{t} (H_s - H_{s+1}) \cdot P_s Y_s$$

$$= (H_t \cdot P_t - \xi_t)Y_t + \sum_{s=1}^{t-1} (H_s - H_{s+1}) \cdot P_s Y_s.$$ 

In particular, note that $M$ is a local martingale by Proposition 2.3. Also, note that $M_t \geq 0$ for $t \geq 1$, and hence $M$ is a true martingale by Proposition 6.7. However,

$$\mathbb{E}(M_t) = (H_1 \cdot P_0 - \xi_0)Y_0 \leq 0$$

by Theorem 3.2 and hence $M_t = 0$ for all $t \geq 0$. The conclusion follows.

Now suppose $X_T$ is an $\mathcal{F}_T$-measurable random variable such $\mathbb{E}(X_T Y_T) = xY_0$ for all martingale deflators $Y$. Let

$$\hat{X}_t = \text{ess sup} \left\{ \frac{1}{Y_t} \mathbb{E}(X_T Y_T | \mathcal{F}_t) : Y \text{ a martingale deflator} \right\}$$

and note that $\hat{X}Y$ is a supermartingale for all $Y$. Similarly, let

$$\hat{X}_t = -\text{ess sup} \left\{ -\frac{1}{Y_t} \mathbb{E}(X_T Y_T | \mathcal{F}_t) : Y \text{ a martingale deflator} \right\}$$

and note that $\hat{XY}$ is a submartingale. Since for all $\hat{X}_T = X_T = \hat{X}_T$ and $\hat{X}_0 = x = \hat{X}_0$, we can conclude that $\hat{X} = \hat{X}$ and that $\hat{XY} = \hat{XY}$ is a martingale for all $Y$. □

**Remark 4.** An another proof of this theorem is given in [17] in the case where the market has a numéraire asset.

We now recall a definition.

**Definition 3.3.** A market model is complete if for every $T > 0$ and $\mathcal{F}_T$-measurable random variable $\xi_T$ there exist a pure-investment strategy $H$ such that

$$H_T \cdot P_T = \xi_T.$$ 

**Proposition 3.4.** Suppose the market has no arbitrage. The market is complete if and only if there is exactly one martingale deflator such that $Y_0 = 1$. 

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Proof. Suppose that there is a unique martingale deflator such that \( Y_0 = 1 \). Let \( \xi_T \) be any \( \mathcal{F}_T \)-measurable random variable. By implication (3) of Theorem 2.6 we can suppose that \( \xi_T Y_T \) is integrable. Let
\[
\xi_t = \frac{1}{Y_t} \mathbb{E}(Y_T \xi_T | \mathcal{F}_t).
\]
Note that \( \xi Y \) is a martingale. Hence \( \xi Y' \) is also a martingale for any martingale deflator \( Y' \) since \( Y' = Y_0' Y \). By Theorem 3.2 there exists a pure-investment strategy such that \( H \cdot P = \xi \). Hence the market is complete.

Conversely, suppose that the market is complete. Let \( Y \) and \( Y' \) be martingale deflators such that \( Y_0 = Y_0' = 1 \). Fix a \( T > 0 \). By completeness there exists a pure-investment strategy \( H \) such that
\[
H_T \cdot P_T = (Y_T - Y_T') Z
\]
where \( Z = \frac{1}{(Y_T + Y_T')^2} \). (The factor \( Z \) will be used to insure integrability later.)

Since \( H \cdot PY \) is a local martingale by Proposition 2.3 with non-negative value at time \( T \), it is a true martingale by Proposition 6.7. In particular,
\[
H_0 \cdot P_0 = \mathbb{E}[(Y_T - Y_T') Z Y_T].
\]
By the same argument with \( Y' \) we have
\[
H_0 \cdot P_0 = \mathbb{E}[(Y_T - Y_T') Z Y'_T].
\]
Subtracting yields
\[
\mathbb{E}[(Y_T - Y_T')^2 Z] = 0
\]
so by the pigeon-hole principle we have \( \mathbb{P}(Y_T = Y'_T) = 1 \) as desired. \( \square \)

In discrete-time models, complete markets have even more structure:

**Theorem 3.5.** Suppose the market model with \( n \) assets is complete. For each \( t \geq 0 \), every \( \mathcal{F}_t \)-measurable partition of the sample space \( \Omega \) has no more than \( n^t \) events of positive probability. In particular, the \( n \)-dimensional random vector \( P_t \) takes values in a set of at most \( n^t \) elements.

**Proof.** Fix \( t \geq 1 \). Let \( A_1, \ldots, A_p \) be a maximal partition of \( \Omega \) into disjoint \( \mathcal{F}_{t-1} \)-measurable events with \( \mathbb{P}(A_i) > 0 \) for all \( i \). Similarly, let \( B_1, \ldots, B_q \) be a maximal partition into non-null \( \mathcal{F}_t \)-measurable events. We will show that \( q \leq np \). The result will follow from induction.

First note that the set \( \{ 1_{B_1}, \ldots, 1_{B_q} \} \) of random variables is a linearly independent, and in particular, the dimension of its span is exactly \( q \). Assuming that the market is complete, each of the \( 1_{B_i} \) can be replicated by a pure-investment strategy. Hence
\[
\text{span}\{ 1_{B_1}, \ldots, 1_{B_q} \} \subseteq \{ H \cdot P_t : H \text{ is } \mathcal{F}_{t-1} \text{-meas.} \}.
\]
We need only show that the dimension of the space on the right is at most \( np \).

Now note that if a random vector \( H \) is \( \mathcal{F}_{t-1} \)-measurable, then it takes exactly one value on each of the \( A_j \)'s for a total of at most \( p \) values \( h_1, \ldots, h_p \). Hence
\[
\{ H \cdot P_t : H \text{ is } \mathcal{F}_{t-1} \text{-meas.} \} = \{ h_1 \cdot P_t 1_{A_1} + \ldots + h_p \cdot P_t 1_{A_p} : h_1, \ldots, h_p \in \mathbb{R}^n \}
\]
\[
= \text{span}\{ P_t^i 1_{A_j} : 1 \leq i \leq n, 1 \leq j \leq p \},
\]
concluding the argument. \( \square \)
3.2. Numéraires and equivalent martingale measures. In this section, we introduce
the concepts of a numéraire and an equivalent martingale measures. The primary purpose
of this section is to reconcile concepts and terminology used by other authors.

Definition 3.6. A numéraire strategy is a pure-investment strategy $\eta$ such that
$$\eta_t \cdot P_t > 0 \text{ for all } t \geq 0 \text{ almost surely.}$$

We now recall the definition of an equivalent martingale measure. We will now see that the
notions of martingale deflator and equivalent martingale measure are essentially the same
concept once a numéraire is specified.

Definition 3.7. Suppose there exists a numéraire strategy with $\eta$ with corresponding wealth
$\eta \cdot P = N$. An equivalent martingale measure relative to this numéraire is any probability
measure $Q$ equivalent to $P$ such that the discounted asset prices $P/N$ are martingales under
$Q$.

Proposition 3.8. Let $Y$ be a martingale deflator for the model, and let $\eta \cdot P = N$ be the
value of a numéraire. Fix a time horizon $T > 0$, and define a new measure $Q$ by the density
$$\frac{dQ}{dP} = \frac{N_T Y_T}{N_0 Y_0}.$$ 

Then $Q$ is an equivalent martingale measure for the finite-horizon model $(P_t)_{0 \leq t \leq T}$.

Conversely, let $Q$ be an equivalent martingale measure for $(P_t)_{0 \leq t \leq T}$. Let
$$Y_t = \frac{1}{N_t} \mathbb{E}^P \left( \frac{dQ}{dP} \Big| F_t \right).$$
Then $Y$ is a martingale deflator.

Proof. First we need to show that the proposed density does in fact define an equivalent
probability measure. Since $\eta$ is a pure-investment strategy and the $N$ is positive by definition,
the process $NY$ is a martingale by Proposition 2.3. In particular, the random variable $dQ/dP$
above is positive-valued. Also, since $NY$ is a martingale, we have
$$\mathbb{E}^P (N_T Y_T) = Y_0 N_0 \Rightarrow \mathbb{E}^P \left( \frac{dQ}{dP} \right) = 1.$$

Now we will show that the discounted price process $P/N$ is a martingale under $Q$. For
$0 \leq t \leq T$ Bayes’s formula yields
$$\mathbb{E}^Q \left( \frac{P_T}{N_T} \Big| F_t \right) = \frac{\mathbb{E}^P \left( \frac{dQ}{dP} \frac{P_T}{N_T} \Big| F_t \right)}{\mathbb{E}^P \left( \frac{dQ}{dP} \Big| F_t \right)} = \frac{\mathbb{E}^P \left( P_T N_T \Big| F_t \right)}{\mathbb{E}^P \left( N_T Y_T \Big| F_t \right)} = \frac{P_t}{N_t},$$
since by the definition of martingale deflator, both $PY$ and $NY$ are martingales.
Now for the converse. Let $\mathbb{Q}$ be an equivalent martingale measure and $Y$ be defined by the formula:

$$P_t Y_t = \mathbb{E}^{\mathbb{Q}}\left(\frac{P_T}{N_T} | \mathcal{F}_t\right) \mathbb{E}^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}| \mathcal{F}_t\right)$$

and hence $Y$ is a martingale deflator. □

**Corollary 3.9.** Consider finite horizon market model with a numéraire. There is no pure investment arbitrage if and only if there exists an equivalent martingale measure.

**Remark 5.** Note that an equivalent martingale measure as defined here only makes sense in the context of a market model with some fixed, finite time-horizon. In general, even if there is no arbitrage there does not exist an equivalent measure under which the discounted market prices are martingales over an infinite horizon, since the martingale $NY$ may fail to be uniformly integrable. This technicality can resolved by invoking the notion of a locally equivalent measure.

However, notice that the fundamental theorem of asset pricing, when stated in terms of the price density, holds for all time-horizons simultaneously.

### 4. Other notions of arbitrage

We now reconsider the definition of arbitrage as defined above. Indeed, there are a number of, a priori distinct, notions of arbitrage which appear naturally in financial modelling.

It is difficult to give a mathematically precise definition of a price bubble in a financial market model. One possible definition is to say that there exists a bubble if there exists a weak notion of arbitrage but not a stronger notion. We now elaborate on this point.

#### 4.1. Absolute arbitrage.

For the first notion, suppose we are given both a set of admissible trading strategies and a collection of preference relations on this set. An absolute arbitrage is a strategy $H^{\text{abs}}$ such that for any admissible $H$, the strategy $H + H^{\text{abs}}$ is also admissible and $H + H^{\text{abs}}$ is strictly preferred to $H$ for all preference relations. An absolute arbitrage is scalable in the sense that for all $k \geq 0$ the strategies $H^k = H + kH^{\text{abs}}$ are feasible and $H^{k+1}$ is preferred to strategy $H^k$.

It is usually considered desirable to consider models without this type of arbitrage. Indeed, if prices are derived from a competitive equilibrium, then all agents are holding their optimal allocation. However, if the market admits an absolute arbitrage, then there does not exist an optimal strategy for any agent: given any strategy, the agent can find another strategy that is strictly preferred.

The notion of arbitrage as defined in section 2.2 is that of absolute arbitrage. In particular, notice that it is the appropriate notion when our class of preference relations are given by utility functions of the form given by equation (1) where the functions $c \mapsto u(t, c)$ are strictly increasing.

To see how the choice of admissible strategies and preference relations affects this notion of arbitrage, suppose that we consider investors who only receive utility from consumption at a fixed date having utility functions of the form

$$\bar{U}(H) = \mathbb{E}[u(H_T \cdot P_T)].$$
An appropriate definition of arbitrage in this case is this:

**Definition 4.1.** A terminal-consumption arbitrage is an investment-consumption arbitrage $H$ over the time horizon $T > 0$ such that

$$\mathbb{P}(H_T \cdot P_T > 0) > 0.$$ 

In the above definition, we allow the investor to consume before the terminal date; however, the investor does not receive any utility for this early consumption. That is, we have modified the set of preference relations while fixing the given set $\{H_{x,T} : x,T\}$ of admissible strategies. If we also modify the set of admissible strategies by insisting that the investor does not consume before the terminal date, we have yet another type of arbitrage:

**Definition 4.2.** A pure-investment arbitrage is a terminal-consumption arbitrage $H$ over the time horizon $T > 0$ such that $(H_t)_{0 \leq t \leq T}$ is a pure-investment strategy.

The following proposition shows that these various types of arbitrages coincide when there exists a numéraire:

**Proposition 4.3.** Consider a marker for which there exists a numéraire strategy. If there exists an investment-consumption arbitrage then there exists a pure-investment arbitrage.

**Proof.** Let $\eta$ be a numéraire strategy with corresponding wealth process $\eta \cdot P = N$. Let $H$ be a self-financing investment-consumption strategy with $H_0 = 0$, and finally let $K$ be the strategy that consists of holding at time $t$ the portfolio $H_t$ but of instead of consuming the amount $(H_t - H_{t+1}) \cdot P_t$, this money instead is invested into the numéraire portfolio. In notation, $K$ is defined by

$$K_t = H_t + \eta_t \sum_{s=1}^{t-1} \frac{(H_s - H_{s+1}) \cdot P_s}{N_s}.$$ 

Note that

$$(K_t - K_{t+1}) \cdot P_t = (H_t - H_{t+1}) \cdot P_t - \eta_{t+1} \cdot P_t \frac{(H_t - H_{t+1}) \cdot P_s}{N_t}$$

$$+ (\eta_t - \eta_{t+1}) \cdot P_t \sum_{s=1}^{t-1} \frac{(H_s - H_{s+1}) \cdot P_s}{N_s} = 0$$

so $K$ is a pure investment strategy by the assumption that $\eta$ is pure-investment. Finally that if $H_{T+1} = 0$, then

$$K_T \cdot P_T = N_T \sum_{s=1}^{T} \frac{(H_s - H_{s+1}) \cdot P_s}{N_s} \geq 0.$$ 

In particular, $K$ is a pure-investment arbitrage if and only if $H$ is a investment-consumption arbitrage. $\Box$

Corresponding to these weakened notions of arbitrage are weakened notions of martingale deflator.

**Definition 4.4.** A signed martingale deflator is a (not necessarily positive) adapted process $Y$ such that $PY$ is an $n$-dimensional martingale.
A sufficient condition to rule out arbitrage can be formulated in this case.

**Theorem 4.5.** Suppose that for every $T > 0$, there exists a signed martingale deflator $Y^T = (Y^T_t)_{0 \leq t \leq T}$ such that $Y^T_T > 0$ almost surely. Then there is no pure-investment arbitrage. If in addition, $Y^T_t \geq 0$ almost surely for all $0 \leq t \leq T$, then there is no terminal-consumption arbitrage.

The proof makes use of Proposition 2.3. The details are omitted.

Just as we can consider martingale deflators as the dual variables for investment-consumption utility maximisation problem described in section 2, it is easy to see that we can consider signed martingale deflators as the dual variables for the utility maximisation problem with the pure-investment objective

$$\mathbb{E}[u(H_T \cdot P_T)] : \ H \text{ pure-investment with } H_0 \cdot P_0 = x.$$  

Now that we have several notions of arbitrage available, we return to the question of bubbles. Economically speaking, a market has a bubble if there is an asset whose current price is higher than some quantification of its fundamental value. Of course, the fundamental value should reflect in some way the future value of the asset. Therefore, it is natural to say that a discrete-time market has bubble if there exists an investment-consumption arbitrage. We will now give an example of such a market that has the additional property that there is no terminal-consumption arbitrage and hence no pure-investment arbitrage. The idea is that an agent who is obliged to be fully invested in the market, such as the manager of a fund which is required to hold assets in a certain sector, cannot take advantage of the ‘obvious’ risk-less profit opportunity. In the next section we will show that this situation is analogous to the continuous-time phenomenon when a market can have no absolute arbitrage yet have a relative arbitrage, where again, the obvious risk-less profit is impossible to lock in because of admissibility constraints.

Consider a market with one asset where the price is given by $P_t = 1_{\{t < \tau\}}$ for some positive, finite stopping time $\tau$. In some sense, the fundamental value of this asset is zero, since $P_t = 0$ for all $t \geq \tau$. The obvious strategy for an investor to employ is to sell the asset short at time 0, and consume the proceeds. At time $\tau$, the investor buys the asset back from the market at no cost.

We consider two cases. First, if $\tau$ is unbounded, this strategy is not an investment-consumption arbitrage according to our definition, since we require an arbitrage to be concluded at a non-random time $T$. Indeed, suppose that $\tau$ is not only unbounded but also that on the event $\{t - 1 < \tau\}$ the conditional probability $\mathbb{P}(t < \tau|\mathcal{F}_{t-1})$ is strictly positive almost surely for all $t \geq 1$. In this case we can find a martingale deflator $Y$ by defining

$$Y_t = \prod_{s=1}^{t \wedge \tau} \mathbb{P}(s < \tau|\mathcal{F}_{s-1})^{-1}.$$  

Now suppose that $\tau$ is bounded by a constant $T$, so that $\tau \leq T$ almost surely. There is an investment-consumption arbitrage: simply sell short one share of the asset and consume the proceeds. In notation, let $H_t = -1$ for $0 \leq t \leq T$ and $H_{T+1} = 0$. The corresponding consumption strategy is $c_0 = 1$ and $c_t = 0$ for $1 \leq t \leq T$.

On the other hand, since there is no numéraire asset, there is no way to lock in this arbitrage with a terminal-consumption strategy. Indeed, if $H_0 P_0 = 0$ then $H_t P_t \leq 0$ for all $t \geq 0$.  

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One might interpret this example as a discrete-time market with a bubble. There might be economic grounds for the existence of such bubbles if sufficiently many traders do not withdraw gains from trade from a market account before some specified time horizon.

Since there is no terminal-consumption arbitrage, we can find a signed martingale deflator, even when \( \tau \) is almost surely bounded by a non-random time \( T \). Indeed, let \( Y^T_T = 1 \) and \( Y^T_t = 0 \) for \( t < T \). It is trivial to check that \( P Y^T \) is a martingale.

4.2. **Relative arbitrage.** In this section we discuss the notion of relative arbitrage. The results here are not new, but are included to provide context to the discussion in the previous section.

In contrast to the notion of an absolute arbitrage, a relative arbitrage can be described as follows. As before, we are given both a set of admissible trading strategies and a collection of preference relations on this set. An arbitrage relative to the benchmark admissible strategy \( H \) is a strategy \( H^{rel} \) such that \( H + H^{rel} \) is admissible and preferred to \( H \) for all preference relations. Note that unlike the case of absolute arbitrage, a relative arbitrage is not necessarily scalable since there is no guarantee that \( H + kH^{rel} \) is admissible for \( k > 1 \). As in the case of absolute arbitrage, it might also be desirable to exclude relative arbitrage from a model, but the argument is weaker. For instance, if there is a relative arbitrage, then in equilibrium, no agent would implement the benchmark strategy \( H \). In particular, if \( H \) is a buy-and-hold strategy for one of the assets, then no agent in equilibrium would hold a static position in that asset.

For our discrete-time models, it would be natural to say that \( H^{rel} \) is an arbitrage relative to \( H \) if the initial cost \( H^{rel} \cdot P_0 = 0 \) vanishes and the consumption stream associated to \( H + H^{rel} \) dominates the consumption stream associated to \( H \), with strict domination with strictly positive probability. It easy to see that in this case \( H^{rel} \) is also an absolute arbitrage.

Therefore, we turn our attention briefly to continuous-time models. We consider a market with a continuous semimartingale price process \( P \).

First we define the set of self-financing investment-consumption strategies

\[
\mathcal{A}^* = \left\{ H : P\text{-integrable}, H \cdot P - \int H \cdot dP \text{ is decreasing} \right\}
\]

To avoid extraneous complication and highlight the difference between relative and absolute arbitrage, we now assume that there exists a numéraire strategy. Recall that in the discrete time setting this implies that the various notions of arbitrage discussed in the last section coincide. In particular, we consider on the case of pure-investment arbitrage for simplicity. The appropriate set of self-financing pure-investment strategies becomes

\[
\mathcal{A}^0 = \left\{ H : P\text{-integrable, } H_t \cdot P_t = H_0 \cdot P_0 + \int_0^t H_s \cdot dP_s \text{ a.s. for all } t \geq 0 \right\}
\]

Unlike the discrete-time setting, it is well known that in continuous time we must restrict the strategies available to investors in order to avoid trivial arbitrages arising from doubling strategies. Therefore, we assume that for every admissible strategy, the investor’s wealth remains non-negative. That is, we let

\[
\mathcal{A} = \{ H \in \mathcal{A}^0 : H_t \cdot P_t \geq 0 \text{ a.s. for all } t \geq 0 \}
\]
If $H$ is a given admissible strategy, then a relative arbitrage $H^{\text{rel}}$ has the property that the wealth generated by $H + H^{\text{rel}}$ is non-negative at all times; that is, we have

$$H_0^{\text{rel}} \cdot P_0 = 0 \text{ and } H_t^{\text{rel}} \cdot P_t \geq -H_t \cdot P_t \ a.s. \text{ for all } t \geq 0$$

On the other hand, a candidate absolute arbitrage $H^{\text{abs}}$ should be an arbitrage relative to any admissible $H$, and hence the wealth it generates should be non-negative:

$$H_0^{\text{abs}} \cdot P_0 = 0 \text{ and } H_t^{\text{abs}} \cdot P_t \geq 0 \ a.s. \text{ for all } t \geq 0$$

We state here a sufficient condition to rule out arbitrage.

**Proposition 4.6.** There is no absolute arbitrage if there exists a positive continuous semi-martingale $Y$ such that $YP$ is a local martingale. There is no arbitrage relative to an admissible strategy $H$ if the process $H \cdot PY$ is a true martingale.

**Proof.** Let $K$ be an admissible strategy. By the Itô’s formula, the Kunita–Watanabe formula and the self-financing condition we have

$$K_t \cdot P_t Y_t = K_0 \cdot P_0 Y_0 + \int_0^t K_s \cdot d(P_s Y_s).$$

In particular, by the integral representation on the right-hand side, we have that $K \cdot PY$ is a local martingale. And by admissibility, this local martingale is non-negative hence is a supermartingale my Fatou’s lemma. In particular,

$$\mathbb{E}(K_T \cdot P_T Y_T) \leq K_0 \cdot P_0 Y_0.$$

Now letting $K = H + H^*$ where $H \cdot PY$ is a true martingale and $H_0^* \cdot P_0 = 0$, we have

$$\mathbb{E}(H_T^* \cdot P_T Y_T) \leq 0.$$

Hence there is no arbitrage relative to $H$. Since we may let $H = 0$, there is no absolute arbitrage. \hfill \Box

**Remark 6.** The notion of an absolute arbitrage used here is closely related to the numéraire-independent property of no arbitrage of the first kind (NA1). Kardaras \cite{Kardaras2012} in recent paper proved a converse to the above proposition, that the market model has NA1 if and only if there exists a local martingale deflator.

**Remark 7.** Note that the process $M = H \cdot PY$ is always a local martingale when $Y$ is a local martingale deflator. It is a true martingale if and only if $M$ is of class DL, that is, the collection of random variables

$$\{M_{\tau_N} : \tau \text{ a stopping time }\}$$

is uniformly integrable for all $t \geq 0$. When $M$ is a true martingale and $H$ is a numéraire strategy, one can define an equivalent martingale measure relative to $H$ as described in section 3.2.

The notion of a relative arbitrage is closely related to no free lunch with vanishing risk (NFLVR). Indeed, consider the case when the reference strategy $H$ is a numéraire, so that it generates a strictly positive wealth process $N$. In this case, a candidate relative arbitrage $H^*$ is such that the discounted wealth $H^* \cdot P/N$ is bounded from below by the constant $-1$. In a celebrated paper of Delbaen & Schachermayer \cite{DelbaenSchachermayer1994} proved another converse of the above proposition, that a market model has NFLVR if and only if there exists an equivalent sigma-martingale measure.
A typical example of a market with a relative arbitrage but no absolute arbitrage has two assets. The first is cash with constant unit price, and the second is a risky stock with positive price process $S$. Suppose $S$ is a local martingale.

The process $P = (1, S)$ is a local martingale, so we can take $Y = 1$ to be a local martingale deflator. By Proposition 4.6, there cannot be an absolute arbitrage. Furthermore, since the value of holding a static position of cash is constant, there can be no arbitrage relative to the strategy $(1, 0)$.

However, suppose now that $S$ is a strictly local martingale (and hence a supermartingale), that the filtration is generated by a Brownian motion and that the volatility of $S$ is strictly positive. Then there does exist a relative arbitrage relative to the strategy $(0, 1)$ of holding one share of the stock. Indeed, for any fixed horizon $T > 0$ there exists a pure-investment trading strategy such that

$$H_{t}^{\text{rep}} \cdot P_{t} = \mathbb{E}(S_{T}|\mathcal{F}_{t}) \leq S_{t}$$

by the martingale representation theorem. Note that the strategy $H^{*} = H^{\text{rep}} - (0, 1)$, that is longing the dynamic replication strategy and shorting the stock, is a relative arbitrage. It is not an absolute arbitrage since $H^{*}$ is itself not admissible.

The phenomenon exhibited by this example has been proposed to model price bubbles, since the simple strategy of buying and holding the stock is dominated by a dynamic replication strategy $H^{\text{rep}}$. See the recent paper of Herdegen [8] or the presentation of Schweizer [16] for a discussion of this point. However, the above example uses in a fundamental way the special properties of continuous time. Indeed, if $S$ is a positive local martingale in discrete time, then $S$ is automatically a true martingale by Proposition 6.7.

**Remark 8.** There is a tantalising parallel between the continuous-time bubbles discussed above and the discrete-time bubbles of the last section. Indeed, in continuous time, bubbles arise when the positive process $M = NY$ is a strictly local martingale, where $N$ is a numéraire and $Y$ a local martingale deflator. Recall that a continuous positive strictly local martingale $M$ can be constructed as follows.

Let $X$ be a continuous non-negative true martingale with respect to a measure $\mathbb{P}$, where $X_{0} = 1$. Fix a time horizon $T > 0$ and define an absolutely continuous measure $\mathbb{Q}$ with density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = X_{T}.$$ 

Now let $\tau = \inf\{t \geq 0 : X_{t} = 0\}$ be the first time that $X$ hits zero. Finally, let $M$ be defined as

$$M_{t} = \frac{1_{\{t < \tau\}}}{X_{t}}.$$ 

The process $M$ is a $\mathbb{Q}$-local martingale. Indeed, it is easy to check that the sequence of stopping times $\tau_{n} = \inf\{t \geq 0 : X_{t} = 1/n\}$ localises $M$ to a bounded $\mathbb{Q}$-martingale. Furthermore, $M$ is strictly positive $\mathbb{Q}$-almost surely since

$$\mathbb{Q}(\tau \leq T) = \mathbb{P}(X_{T}1_{\{\tau \leq T\}})$$

$$= \mathbb{P}(X_{\tau}1_{\{\tau \leq T\}})$$

$$= 0.$$  

However, note that

$$\mathbb{E}^{\mathbb{Q}}(X_{T}) = \mathbb{P}(\tau > T).$$
In particular, $M$ is a true $\mathbb{Q}$-martingale if and only if $X$ is strictly positive $\mathbb{P}$-almost surely. See, for instance, the paper of Ruf & Rungaldier [15] for further details.

As a consequence of the above discussion, in the continuous-time story, there is a bubble if a certain process $X$ hits zero with positive probability. On the other hand, in the discrete-time theory there is no terminal consumption arbitrage if there is a non-negative martingale deflator $Y$. However, there may exist an investment-consumption arbitrage—that is, a bubble— if $Y$ hits zero.

5. The proofs

5.1. Proof of Theorem 2.6. Recall that we need only prove that (1) implies both (2) and (4). First, we need an additional equivalent, but more technical, formulation of Theorem 2.6:

**Theorem 5.1.** Conditions (1)-(4) of Theorem 2.6 is equivalent to

(5) For every $t \geq 1$ and positive $\mathcal{F}_t$-measurable $\zeta$, there exists a positive $\mathcal{F}_t$-measurable random variable $Z$ and positive $\mathcal{F}_{t-1}$-measurable random variable $R$ such that

$$Z \leq R \zeta \text{ almost surely}$$

and

$$\mathbb{E}(P_t Z | \mathcal{F}_{t-1}) = P_{t-1}$$

in the sense of generalised conditional expectation.

**Proof of (5) ⇒ (2) of Theorem 5.1.** For each $t \geq 1$, let $Z_t$ be such that

$$\mathbb{E}(P_t Z_t | \mathcal{F}_{t-1}) = P_{t-1}$$

in the sense of generalised conditional expectation. Let $Y_0 = 1$ and $Y_t = Z_1 \cdots Z_t$. Note that by Proposition 6.5 stated in section 6 below, the process $YP$ is a local martingale. Hence $Y$ is a local martingale deflator. □

**Proof of (5) ⇒ (4) of Theorem 5.1.** By replacing, the given positive process $\eta$ with $\hat{\eta}$ defined by

$$\hat{\eta}_t = \min\{\eta_t, e^{-\|P_t\|}\}.$$ we can assume that the process $P\eta$ is bounded. In particular, once we show that given $\eta = (\eta_t)_{0 \leq t \leq T}$ there exists local martingale deflator such that $Y_t \leq \eta_t$ for $0 \leq t \leq T$, we can conclude that $Y$ is a true martingale deflator since the process $YP$ is integrable.

Given the process $\eta = (\eta_t)_{0 \leq t \leq T}$ we will construct random variables $(Z_t)_{1 \leq t \leq T}$ such that the process $(Y_t)_{0 \leq t \leq T}$ is a local martingale deflator, where $Y_t = Y_0 Z_1 \cdots Z_t$. Need only show that we can do this construction in such a way that $Y_t \leq \eta_t$.

Let $\zeta_T = \eta_T / \eta_{T-1}$. By condition (5), there exists a positive random variable $Z_T$ and a positive $\mathcal{F}_{T-1}$-measurable random variable $R_{T-1}$ such that

$$Z_T \leq R_{T-1} \zeta_T.$$ Now we proceed backwards by specifying $\zeta_{T-1}, \ldots, \zeta_1$ to find $Z_{T-1}, \ldots, Z_1$ and corresponding bounds $R_{T-2}, R_{T-3}, \ldots, R_0$ such that

$$Z_t \leq R_{t-1} \zeta_t.$$
by letting
\[ \zeta_t = \frac{\eta_t}{\eta_{t-1}} \left( \frac{1}{1 + R_t} \right). \]
The process \( Y_t = \frac{m_0}{1 + r_0} Z_1 \cdots Z_t \) is a local martingale deflator such that \( Y_t \leq \eta_t \) for \( 0 \leq t \leq T \), as desired. \( \square \)

We now come the converse direction. The following proof is adapted from Rogers's proof [14] of the Dalang–Morton–Willinger theorem in the case when a numéraire asset is assumed to exist. The idea there is to show that no arbitrage implies a certain pure-investment utility optimisation problem has an optimal solution. The idea here is very similar: no arbitrage implies that a certain investment-consumption utility maximisation problem has an optimal solution.

**Proof of (1) \( \Rightarrow \) (5) of Theorem 5.1.** Fix \( t \geq 1 \), and suppose the positive \( \mathcal{F}_t \)-measurable random variable \( \zeta \) is given. By replacing \( \zeta \) with \( \zeta \wedge 1 \) there is no loss assuming \( \zeta \) is bounded.

Let \( \hat{\zeta} = \zeta e^{-\|P_t\|^2/2} \) and define a function \( F : \mathbb{R}^n \times \Omega \to \mathbb{R} \) by
\[ F(h) = e^{h \cdot P_{t-1}} + \mathbb{E}[e^{-h \cdot \hat{\zeta}} | \mathcal{F}_{t-1}]. \]
More precisely, let \( \mu \) be the regular conditional joint distribution of \((P_t, \zeta)\) given \( \mathcal{F}_{t-1} \), and let
\[ F(h, \omega) = e^{h \cdot P_{t-1}(\omega)} + \int e^{-h \cdot y - \|y\|^2/2} \mu(dy, dz, \omega). \]
Note that \( F(\cdot, \omega) \) is everywhere finite-valued, and hence is smooth. We will show that no investment-consumption arbitrage implies that for each \( \omega \), the function \( F(\cdot, \omega) \) has a minimiser \( H^*(\omega) \) such that \( H^* \) is \( \mathcal{F}_{t-1} \)-measurable. By the first order condition for a minimum, we have
\[ 0 = \nabla F(H^*) = e^{H^* \cdot P_{t-1}} P_{t-1} - \mathbb{E}[e^{-H^* \cdot \hat{\zeta}} P_t | \mathcal{F}_{t-1}] \]
and hence we may take
\[ Z = e^{-H^* \cdot P_{t-1} - H^* \cdot \hat{\zeta}}. \]
Note that
\[ Z \leq R \zeta, \]
where \( R \) is the \( \mathcal{F}_{t-1} \)-measurable random variable
\[ R = e^{-H^* \cdot P_{t-1} + \|H^*\|^2/2}. \]

With the above goal in mind, we define functions \( F_k : \mathbb{R}^n \times \Omega \to \mathbb{R} \) by
\[ F_k(h) = F(h) + \|h\|^2/k. \]
Now for fixed \( \omega \), the function \( F_k(\cdot, \omega) \) is smooth, strictly convex and
\[ F_k(h, \omega) \to \infty \text{ as } \|h\| \to \infty. \]
In particular, there exists a unique minimiser \( H_k(\omega) \), and by Proposition 6.9 \( H_k \) is \( \mathcal{F}_{t-1} \)-measurable.

We will make use of two observations. First, note that \( H_k \) enjoys a certain non-degeneracy property:
\[ \{H_k \cdot P_{t-1} = 0\} \cap \{\mathbb{P}(H_k \cdot P_t = 0 | \mathcal{F}_{t-1}) = 1\} \subseteq \{H_k = 0\}. \]
This fact is easily proven by noting $H_k$ minimises $F_k$ by definition, but we have

$$F_k(H_k) = F(0) + \|H_k\|^2/k \geq F_k(0).$$

on the event $\{H_k \cdot P_{t-1} = 0\} \cap \{\mathbb{P}(H_k \cdot P_t = 0|\mathcal{F}_{t-1}) = 1\}$.

Second, note that $F(H_k) \to \inf_h F(h)$ almost surely, since

$$\limsup_k F(H_k) \leq \inf_k F_k(H_k) \leq \inf_k F_k(h) \leq F(h)$$

for all $h \in \mathbb{R}^n$.

Now, let

$$A = \{\sup_k \|H_k\| < \infty\}$$

be the $\mathcal{F}_{t-1}$-measurable set on which the sequence $\langle H_k(\omega) \rangle_k$ is bounded. Hence, by Proposition 6.10 we can extract a measurable subsequence on which that $H_k$ converges on $A$ to a $\mathcal{F}_{t-1}$ measurable $H^*$. Note that by the smoothness of $F$, we have

$$F(H^*) = \lim_k F(H_k)$$

Hence $H^*$ is a minimiser of $F$ by the second observation, and the proof is complete once we show that $\mathbb{P}(A) = 1$.

We now show $\mathbb{P}(A^c) = 0$. Now on $A^c$ the sequence $\langle H_k \rangle_k$ is unbounded. Hence, we can find a measurable subsequence along which $\|H_k\| \to \infty$ on $A^c$. Since the sequence

$$\hat{H}_k = \frac{H_k}{\|H_k\|}$$

is bounded, and indeed $\|\hat{H}_k\| = 1$ for all $k$, there exists a further subsequence along which $\langle \hat{H}_k \rangle_k$ converges on $A^c$ to a $\mathcal{F}_{t-1}$-measurable random variable $\hat{H}$. Note that $\|\hat{H}(\omega)\| = 1$ for all $\omega \in A^c$. Letting

$$\tilde{H} = \hat{H}1_{A^c},$$

we need only show that $\tilde{H} = 0$ almost surely.

First note that on $A^c \cap \{\hat{H} \cdot P_{t-1} < 0\}$ we have

$$e^{H_k \cdot P_{t-1}} = (e^{-\hat{H}_k \cdot P_{t-1}})\|H^*_k\| \to \infty.$$ 

But since

$$\limsup_k e^{H_k \cdot P_{t-1}} \leq \limsup_k F(H_k) = \inf_h F(h) \leq F(0) = 2$$

by the second observation above, we conclude that

$$\mathbb{P}(A^c \cap \{\tilde{H} \cdot P_{t-1} > 0\}) = 0.$$
Similarly, on $A^c$ we have by Fatou's lemma and Markov's inequality
\[
P(\hat{H} \cdot P_t < 0 | \mathcal{F}_{t-1}) = \sup_{\epsilon > 0} P(\hat{H} \cdot P_t < -\epsilon | \mathcal{F}_{t-1})
\leq \sup_{\epsilon > 0} \liminf_k \mathbb{P}(\hat{H} \cdot P_k > \|H_k\|\epsilon | \mathcal{F}_{t-1})
\leq \sup_{\epsilon > 0} \liminf_k \frac{\mathbb{E}(e^{-H_k \cdot P_t} \hat{\zeta} | \mathcal{F}_{t-1})}{e^{\|H_k\|\epsilon} \mathbb{E}(\hat{\zeta} | \mathcal{F}_{t-1})}
= 0
\]
since
\[
\limsup_k \mathbb{E}(e^{-H_k \cdot P_t} \hat{\zeta} | \mathcal{F}_{t-1}) \leq \limsup_k F(H_k) \leq 2.
\]
In particular, we can conclude that
\[
(5) \quad P(A^c \cap \{\hat{H} \cdot P_t < 0\}) = 0.
\]
Now consider the investment-consumption strategy $(H_s)_{0 \leq s \leq t}$ defined by $H_s = 0$ for $0 \leq s \leq t - 1$ and $H_t = \hat{H}$ where $\hat{H}$ is defined by equation (3). Note that by equations (4) and (5) this strategy is self-financing. By assumption that there is no investment-consumption arbitrage, we now conclude that
\[
1 = \mathbb{P}(\hat{H} \cdot P_{t-1} = 0, \hat{H} \cdot P_t = 0)
= \mathbb{P}(\hat{H} \cdot P_{t-1} = 0, \mathbb{P}(\hat{H} \cdot P_t = 0 | \mathcal{F}_{t-1}) = 1)
\leq \mathbb{P}(\hat{H} = 0)
\]
by the first observation above. This concludes the proof. \qed

5.2. **Proof of Theorem 2.7.** The dual characterisation of super-replicable claims can be proven using the same utility maximisation idea as in the proof of Theorem 5.1 above. Indeed, the insight is that super-replication is the optimal hedging policy for an investor in the limit of large risk-aversion.

**Proof of Theorem 2.7.** Fix $t \geq 1$, and suppose that
\[
\mathbb{E}(Z\xi | \mathcal{F}_{t-1}) \leq \xi_{t-1}
\]
for any $\mathcal{F}_t$-measurable $Z$ such that
\[
\mathbb{E}(P_t Z | \mathcal{F}_{t-1}) = P_{t-1}.
\]
We will show that there exists an $\mathcal{F}_{t-1}$-measurable $H$ such that
\[
H \cdot P_{t-1} \leq \xi_{t-1} \quad \text{and} \quad H \cdot P_t \geq \xi_t \text{ almost surely}.
\]
For the sake of integrability, we introduce a factor $\zeta = e^{-\frac{\|P\|^2_t + \xi_t^2}{2}}$ and define a family of random functions
\[
F_\gamma(h) = e^{-\gamma(\xi_{t-1} - h \cdot P_{t-1})} + \mathbb{E}[e^{-\gamma(h \cdot P_t - \xi_t)} \zeta | \mathcal{F}_{t-1}],
\]
where $\gamma > 0$ has the role of risk-aversion parameter. Since there is no arbitrage, we can reuse the argument from the proof of the (1) $\Rightarrow$ (5) implication of Theorem 5.1 to conclude that $F_\gamma$ has a $\mathcal{F}_{t-1}$-measurable minimiser $H_\gamma$ with a corresponding $\mathcal{F}_t$-measurable random variable
\[
Z_\gamma = \frac{e^{-\gamma(H_{\gamma} \cdot P_t - \xi_t)} \zeta}{e^{-\gamma(\xi_{t-1} - H_{\gamma} \cdot P_{t-1})}}.
\]
with the property that
\[ \mathbb{E}(P_tZ_\gamma | F_{t-1}) = P_{t-1}. \]

Note that
\[
\frac{\partial}{\partial \gamma} F_\gamma(h) |_{h=H_\gamma} = e^{-\gamma(\xi_{t-1} - H_\gamma \cdot P_{t-1})} [(\mathbb{E}(\xi_t Z_\gamma | F_{t-1}) - \xi_{t-1}) + H_\gamma \cdot (P_{t-1} - \mathbb{E}(P_tZ_\gamma | F_{t-1})] 
\leq 0
\]
by assumption. Hence for any \( \varepsilon > 0 \) we have
\[
F_{\gamma+\varepsilon}(H_{\gamma+\varepsilon}) \leq F_{\gamma+\varepsilon}(H_\gamma) 
\leq F_\gamma(H_\gamma)
\]
where the first inequality follows from the fact that \( H_{\gamma+\varepsilon} \) minimises \( F_{\gamma+\varepsilon} \). In particular, we see that
\[
\sup_\gamma F_\gamma(H_\gamma) < \infty,
\]
where for the rest of the proof we will only consider positive integer values for \( \gamma \). Let
\[ A = \{ \sup_\gamma \|H_\gamma\| < \infty \} \]
be the \( F_{t-1} \) event on which the sequence \( (H_\gamma)_\gamma \) is bounded. By Proposition 6.10 there exists a measurable subsequence which converges to \( F_{t-1} \)-measurable \( H^* \) on \( A \).

Note that
\[
\mathbb{P}(A \cap \{ H^* \cdot P_{t-1} > \xi_{t-1} \} \cup \{ H^* \cdot P_{t} < \xi_{t} \}) \leq \mathbb{P}(F_\gamma(H_\gamma) \to \infty) = 0
\]
We need only show that \( \mathbb{P}(A) = 1 \).

We now prove that the event \( A^c \) on which the sequence \( (H_\gamma)_\gamma \) is unbounded has probability zero. This follows the same steps as in the proof of Theorem 5.1. Again by Proposition 6.10 we consider a subsequence such that \( \|H_\gamma\| \to \infty \) and then find a further subsequence and \( F_{t-1} \)-measurable random variable with \( \|H_\gamma\| = 1 \) such that
\[
\hat{H}_\gamma \to \hat{H}
\]
where
\[
\hat{H}_\gamma = \frac{H_\gamma}{\|H_\gamma\|}
\]
Since
\[
F_\gamma(H_\gamma) = (e^{\hat{H}_\gamma \cdot P_{t-1} - \xi_{t-1}})^{\gamma \|H_\gamma\|} + \mathbb{E}[(e^{\hat{\xi}_s - \hat{H}_\gamma \cdot P_{t}})^{\gamma \|H_\gamma\|} \xi_t | F_{t-1}],
\]
where
\[
\hat{\xi}_s = \frac{\xi_s}{\|H_\gamma\|} \to 0 \text{ for } s = t - 1, t,
\]
we can conclude by the boundedness of \( F_\gamma(H_\gamma) \) that
\[
\mathbb{P} \left( A^c \cap \{ \hat{H} \cdot P_{t-1} > 0 \} \cup \{ \hat{H} \cdot P_{t} < 0 \} \right) = 0.
\]
As before, we use the assumption of no arbitrage and the non-degeneracy of \( \hat{H} \) to conclude that \( \mathbb{P}(A^c) = 0. \) \qed
6. Appendix

6.1. Generalised conditional expectations and local martingales. In this appendix we recall some basic notions and useful facts regarding discrete-time local martingales.

Definition 6.1. A local martingale is an adapted process \( X = (X_t)_{t \geq 0} \), in either discrete or continuous time, such that there exists an increasing sequence of stopping times \( (\tau_N) \) with \( \tau_N \uparrow \infty \) such that the stopped process \( X^{\tau_N} = (X_{t \wedge \tau_N})_{t \geq 0} \) is a martingale for each \( N \).

We now briefly recall a definition of the conditional expectation:

Definition 6.2. Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), a square-integrable random variable \( X \) and sub-sigma-field \( \mathcal{G} \subseteq \mathcal{F} \), we define the conditional expectation \( \mathbb{E}(X|\mathcal{G}) \) as (a version) of the orthogonal projection of \( X \) onto the Hilbert space of square-integrable \( \mathcal{G} \)-measurable random variables. If \( X \) is non-negative, we define

\[
\mathbb{E}(X|\mathcal{G}) = \sup_{k \geq 0} \mathbb{E}(X \wedge k|\mathcal{G})
\]

and if \( \mathbb{E}(|X| |\mathcal{G}) < \infty \) almost surely, we define

\[
\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X^+|\mathcal{G}) - \mathbb{E}(X^-|\mathcal{G}).
\]

Note that it is not necessary for \( X \) to be integrable in order to define the conditional expectation \( \mathbb{E}(X|\mathcal{G}) \) as above. However, one must take care with this generalised notion of conditional expectation since it is possible to find a random variable \( X \) and sigma-fields \( \mathcal{G} \) and \( \mathcal{H} \) such that both conditional expectations \( \mathbb{E}(X|\mathcal{G}) \) and \( \mathbb{E}(X|\mathcal{H}) \) are defined, and in fact, both are integrable, and yet

\[
\mathbb{E}[\mathbb{E}(X|\mathcal{G})] \neq \mathbb{E}[\mathbb{E}(X|\mathcal{H})].
\]

Of course, such pathologies do not occur if \( X \) is integrable.

Here are some useful properties of the generalised conditional.

Proposition 6.3. Let \( \mu \) be regular conditional distribution of random variable \( X \) given a sigma-field \( \mathcal{G} \). Assuming the conditional expectation is defined, we have

\[
\mathbb{E}(X|\mathcal{G})(\omega) = \int x \mu(\omega, dx)
\]

for almost all \( \omega \in \Omega \).

Proposition 6.4. If \( K \) is \( \mathcal{G} \)-measurable and

\[
\mathbb{E}(|X| |\mathcal{G}) < \infty \text{ almost surely}
\]

then

\[
\mathbb{E}(KX|\mathcal{G}) = K\mathbb{E}(X|\mathcal{G}).
\]

Another property that will be useful relates to discrete-time local martingales: The proof can be found in the paper by Jacod & Shiryaev [9].

Proposition 6.5. A discrete-time process \( M \) is a local martingale if and only if for all \( t \geq 0 \) we have \( \mathbb{E}(|M_{t+1}| |\mathcal{F}_t) < \infty \) almost surely and \( \mathbb{E}(M_{t+1}|\mathcal{F}_t) = M_t \).
Proposition 6.6. Suppose $Q$ is a discrete-time local martingale and $K$ is a predictable process. Let

$$M_t = \sum_{s=1}^{t} K_s (Q_s - Q_{s-1})$$

for $t \geq 1$. Then $M$ is a local martingale.

Proposition 6.7. Suppose $M$ is a discrete-time local martingale such that there is a non-random time horizon $T > 0$ with the property that either $M_T \geq 0$ almost surely. Then $(M_t)_{0 \leq t \leq T}$ is a martingale.

We conclude with a useful technical result due to Kabanov [10]. Unlike the other results in this note, the proof of Kabanov’s theorem requires subtle ideas from functional analysis. Fortunately, it is only used in one place - to show that the existence of a local martingale deflator implies the existence of a true martingale deflator. It should be stressed that Kabanov’s theorem is only needed in the proof because we do not restrict our attention to a finite time horizon and because we do not assume that our assets have non-negative prices.

Theorem 6.8 (Kabanov). Suppose $M$ is a discrete-time local martingale with respect to a probability measure $\mathbb{P}$. Then there exists an equivalent measure $Q$, such that $M$ is a true martingale with respect to $Q$.

6.2. Measurability and selection. The following result allows us to assert the measurability of the minimiser of a random function. The proof is from the paper of Rogers [14].

Proposition 6.9. Let $f : \mathbb{R}^n \times \Omega \to \mathbb{R}$ is such that $f(x, \cdot)$ is measurable for all $x$, and that $f(\cdot, \omega)$ continuous and has a unique minimiser $X^*(\omega)$ for each $\omega$. Then $X^*$ is measurable.

Proof. For any open ball $B \subset \mathbb{R}^n$ we have

$$\{\omega : X^*(\omega) \in B\} = \bigcup_{p \in B \cap Q} \bigcap_{q \in B \cap Q} \{\omega : f(p, \omega) < f(q, \omega)\}$$

where $Q$ is a countable dense subset of $\mathbb{R}^n$. \hfill \Box

Finally, we include a useful measurable version of the Bolzano–Weierstrass theorem. An elementary proof is found in the paper of Kabanov & Stricker [11].

Proposition 6.10. Let $(\xi_k)_{k \geq 0}$ be a sequence of measurable functions $\xi_k : \Omega \to \mathbb{R}^n$ such that $\sup_k \|\xi_k(\omega)\| < \infty$ for all $\omega \in \Omega$. Then there exists an increasing sequence of integer-valued measurable functions $N_k$ and an $\mathbb{R}^n$-valued measurable function $\xi^*$ such that

$$\xi_{N_k(\omega)}(\omega) \to \xi^*(\omega) \text{ as } k \to \infty$$

for all $\omega \in \Omega$.

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References

[1] B. Bouchard and M. Nutz. Arbitrage and duality in nondominated discrete-time models. Pre-print. (2014)
[2] M. Burzoni, M. Frittelli and M. Maggis. Robust arbitrage under uncertainty in discrete time. arXiv:1407.0948 (2014)
[3] P. Carr, T. Fisher and J. Ruf. On the hedging of options on exploding exchange rates. Finance and Stochastics 18(1). (2014)
[4] A.M.G. Cox and D.G. Hobson. Local martingales, bubbles and option prices. Finance and Stochastics 9: 477–492. (2005)
[5] R.C. Dalang, A. Morton and W. Willinger. Equivalent martingale measures and no-arbitrage in stochastic securities markets. Stochastics and Stochastics Reports 29: 185–201. (1990)
[6] F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. Mathematische Annalen 312: 215–250. (1998)
[7] J.M. Harrison and D.M. Kreps. Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory 20: 381–408. (1979)
[8] M. Herdegen. No-arbitrage in a numéraire independent modelling framework. Pre-print. (2014)
[9] J. Jacod and A.N. Shiryaev. Local martingales and the fundamental asset pricing theorems in the discrete-time case. Finance and Stochastics 2: 259-273. (1998)
[10] Yu. Kabanov. In discrete time a local martingale is a martingale under an equivalent probability measure. Finance and Stochastics 2: 293297. (2008)
[11] Yu. Kabanov and Ch. Stricker. A teacher’s note on no-arbitrage criteria. Séminaire de probabilités de Strasbourg 35: 149–152. (2001)
[12] C. Kardaras. Market viability via absence of arbitrage of the first kind. Finance and Stochastics 16(4): 651–667. (2012)
[13] Ph. Protter. A mathematical theory of financial bubbles. Paris-Princeton Lecture Notes in Mathematical Finance 2081: 1–108. (2013)
[14] L.C.G. Rogers. Equivalent martingale measures and no-arbitrage. Stochastics and Stochastics Reports 51: 41–49. (1994)
[15] J. Ruf and W. Rungaldier. A systematic approach to constructing market models with arbitrage. In Arbitrage, Credit and Informational Risks, C. Hillairet, M. Jeanblanc and Y. Jiao, editors. World Scientific Publishing. (2014).
[16] M. Schweizer. Some ideas on bubbles. Presentation at the Swiss-Kyoto Symposium. http://www.ccfz.ch/files/schweizer1.pdf (2013)
[17] M.R. Tehranchi. Characterizing attainable claims: a new proof. Journal of Applied Probability 47 (4): 1013-1022. (2010)

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