MATRIX RING WITH COMMUTING
GRAPH OF MAXIMAL DIAMETER

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Abstract. The commuting graph of a matrix algebra is the set of non-scalar matrices; the edges are defined as pairs \((u,v)\) satisfying \(uv = vu\). Akbari et al. proposed the following conjecture: If the commuting graph is connected, then its diameter is at most five. We disprove this conjecture.

1. Introduction

Let \(F\) be a field and \(\text{Mat}_n(F)\) be the algebra of \(n \times n\) matrices over \(F\). The commuting graph of \(\text{Mat}_n(F)\) is the set of non-scalar matrices; the edges are defined as pairs \((u,v)\) satisfying \(uv = vu\). This graph is denoted by \(\Gamma(\mathbb{F},n)\). Our paper is devoted to the following conjecture.

Conjecture 1.1. [2, Conjecture 5] If \(\Gamma(F,n)\) is a connected graph, then its diameter does not exceed five.

Conjecture 1.1 has attracted considerable attention in recent publications. This conjecture is known to be true when \(n\) is prime [1] or when \(F\) is perfect [13]. Also, the diameter of \(\Gamma(F,n)\) is known to equal four if \(F\) is algebraically closed [2] or real closed [11][12][15]. Moreover, the set of matrix pairs realizing the maximal distance is well understood in the case of an algebraically closed field [7]. The upper bound in Conjecture 1.1 if it is true, cannot be improved as shown in [4]. Actually, it was proved that \(\Gamma(F_2,9)\) is connected and has diameter at least five. Conjecture 1.1 has been investigated extensively in the case when \(F\) is a more general structure than field; it is known to be true when \(F = \mathbb{Z}/m\) with non-prime \(m\), see [9]. The conjecture is true when \(F\) is taken from a sufficiently general class of semirings [8] which includes the tropical semiring [6].

We note that, however, Conjecture 1.1 remained an open problem. Our paper aims to provide a counterexample for this conjecture. In Section 2, we describe the field \(F\) over which it fails. In Section 3, we construct a pair of matrices which realize the maximal distance, equal to six, in \(\Gamma(F,38)\). In Section 4, we prove a technical claim used in the proof before, and finalize the proof of the main result.

Throughout our paper, we denote an \((i,j)\) entry of a matrix \(A \in \text{Mat}_n(F)\) by \(A_{ij}\). We denote by \(C(A)\) the centralizer of \(A\), that is, the set of all matrices that commute with \(A\). We denote the set of rational (or polynomial) expressions over \(F\) and \(x\) by \(F(x)\) (or \(F[x]\), respectively).

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The field

Let \( \mathcal{R} \) be a commutative ring without zero divisors; \( \mathcal{R} \) is called local if the set \( \mathcal{M}_\mathcal{R} = \mathcal{R} \setminus \mathcal{R}^\times \) of all non-units is an ideal of \( \mathcal{R} \). A local ring \( \mathcal{R} \) is called Henselian if it satisfies the condition of Hensel’s lemma. That is, for every monic polynomial \( f \in \mathcal{R}[t] \) whose image \( \overline{f} \) in \( \mathcal{R}/\mathcal{M}_\mathcal{R} \) factors into a product \( \overline{g}_1 \overline{g}_2 \) in which \( \overline{g}_1 \) and \( \overline{g}_2 \) are both monic and relatively prime, there exist monic polynomials \( g_1, g_2 \in \mathcal{R}[t] \) which are mutually prime and satisfy \( f = g_1 g_2 \) and \( g_i \in \mathcal{R} + \mathcal{M}_\mathcal{R} \mathcal{R}[t] \). The ring of formal power series \( \mathcal{F}_2[[x, y]] \) is a particular example of a Henselian ring [15], and this example will play a crucial role in our paper.

The Malcev-Neumann series field \( \mathcal{F}\langle \{t\} \rangle \) over a field \( \mathcal{F} \) is the set of all formal sums \( a(t) = \sum_{n \in \mathbb{Q}} a_n t^n \) such that the support \( \{e \in \mathbb{Q} : a_e \neq 0\} \) is a well-ordered subset of \( \mathbb{Q} \); we assume that coefficients \( a_e \) are taken from \( \mathcal{F} \) and say that \( a_0 \) is a constant term of \( a \). The result of Poonen [14] shows that, if \( \mathcal{F} \) is algebraically closed, then so is \( \mathcal{F}\langle \{t\} \rangle \). Denote by \( \mathcal{F}_2 \) an algebraic closure of \( \mathcal{F}_2 \); all fields discussed in our paper are subfields of \( \mathcal{H} = \mathcal{F}_2\langle \{x\} \rangle\langle \{y\} \rangle \cap \mathcal{F}_2\langle \{y\} \rangle\langle \{x\} \rangle \). In particular, one can note that the fraction field Quot \( \mathcal{F}_2[[x, y]] \) and its algebraic extensions are subfields of \( \mathcal{H} \). Now let \( h \in \mathcal{H}^\times \); define \( \deg_z h \) as the minimum of the support of \( h \) as an element of \( \mathcal{F}_2\langle \{y\} \rangle\langle \{x\} \rangle \). In other words, \( \deg_z h \) is the smallest \( i \) for which there is \( j \) such that \( x^i y^j \) appears in \( h \) with nonzero coefficient; the quantity \( \deg_y h \) is defined analogously. We define \( \mathcal{O}(\mathcal{H}) \) as the set of all \( h \in \mathcal{H} \) satisfying \( \deg_y h \geq 0 \) and \( \deg_z h \geq 0 \). Note that any subring \( \mathcal{R} \) of \( \mathcal{O}(\mathcal{H}) \) has the maximal ideal \( \mathcal{R} \cap \mathcal{O}(\mathcal{H}) \), so \( \mathcal{R} \) is always local. Also, let us note that the quotient field \( \mathcal{R}/\mathcal{M}_\mathcal{R} \) consists essentially of those elements of \( \mathcal{F}_2 \) which appear as constant terms of elements of \( \mathcal{R} \). An integral extension of local rings \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \) is called separable if \( \mathcal{R}_1 \)-minimal polynomial of every \( r \in \mathcal{R}_2 \) has non-zero formal derivative; we say that \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \) is totally ramified if \( \mathcal{R}_1/\mathcal{M}_{\mathcal{R}_1} = \mathcal{R}_2/\mathcal{M}_{\mathcal{R}_2} \).

**Lemma 2.1.** Every subring \( \mathcal{R}_0 \) of \( \mathcal{O}(\mathcal{H}) \) has a (not necessarily unique) maximal integral separable totally ramified extension.

**Proof.** Let \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \ldots \) be a totally ordered set of local rings each of which is an integral, separable, and totally ramified extension of \( \mathcal{R}_0 \). Assume \( \mathcal{R} = \cup \mathcal{R}_i \); then, every element of \( \mathcal{R} \) is integral and separable over \( \mathcal{R}_0 \), and the set of all constant terms of elements of \( \mathcal{R} \) is the same as the corresponding set for \( \mathcal{R}_1 \). In other words, \( \mathcal{R} \) is itself an integral, separable, and totally ramified extension of \( \mathcal{R}_1 \). Application of Zorn’s lemma completes the proof. \( \square \)

In the rest of our paper, \( \mathcal{P} \) denotes a certain maximal integral separable totally ramified extension of \( \mathcal{F}_2[[x, y]] \), and \( \mathcal{F} \) denotes the field of fractions of \( \mathcal{P} \).

**Lemma 2.2.** Let \( z \in \mathcal{H} \) be algebraic over \( \mathcal{P} \). Then, there is an element \( a \in \mathcal{P} \) such that \( az \) is integral over \( \mathcal{P} \) and \( a \in \mathcal{O}(\mathcal{H}) \).

**Proof.** The equality \( a_n z^n + \ldots + a_0 = 0 \) holds for some \( a_i \in \mathcal{P} \) with \( a_n \neq 0 \). Then, for any \( \pi \in \mathcal{P} \), the element \( \pi a_n z \) is integral over \( \mathcal{P} \). From definition of \( \mathcal{H} \) it follows that \( x^i y^j a_n z \in \mathcal{O}(\mathcal{H}) \), for sufficiently large \( i \) and \( j \). \( \square \)

**Lemma 2.3.** If \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \) are local rings, \( \mathcal{R}_1 \) is Henselian, and every element of \( \mathcal{R}_2 \) is integral over \( \mathcal{R}_1 \), then \( \mathcal{R}_2 \) is Henselian as well.

Recall that for every algebraic extension \( K \supseteq L \), the set of all elements \( l \in K \) that are separable over \( L \) forms a field \( K_{sep} \) which is the unique separable extension
of $L$ over which $K$ is purely inseparable \cite{11}. The degree of extension $K_{sep} \supset L$ is called the separable degree of $K \supset L$ and is denoted by $[K : L]_s$; similarly, the degree of $K \supset K_{sep}$ is called the inseparable degree of $K \supset L$ and is denoted by $[K : L]_i$. Clearly, $[K : L] = [K : L]_i[K : L]_s$.

**Lemma 2.4.** Let $n$ be a positive integer which is neither a prime nor a power of two. Assume $E$ is an extension of $F$ of degree $n$. Then, there exists a field $E'$ satisfying $F \subset E' \subset E$ and $E' \notin \{F, E\}$.

**Proof.** The degree of any purely inseparable extension is a power of characteristic, so we can assume $E_{sep} = E$ which means that $E \supset F$ is a separable extension.

By Lemma 2.2 there is an element $\pi \in E \setminus F$ integral over $P$ such that $\pi \in \mathcal{O}(H)$. Then, $P[\pi] \subset \mathcal{O}(H)$ is local and integral over $P$; in particular, $P[\pi]$ is Henselian by Lemma 2.3. Maximality of $P$ shows that $P[\pi] \supset P$ is not a totally ramified extension. The latter condition means that $P[\pi]/P[\pi] \supset F_2$ is a non-trivial extension of finite fields. So we see that there is an element $\pi \in P[\pi]/P[\pi]$ and a polynomial $\psi \in F_2[t]$ of prime degree $u$ irreducible over $F_2$ such that $\psi(\pi) = 0$. Hensel’s lemma shows that there is $e \in P[\pi]$ satisfying $\psi(e) = 0$; this means that $F(e) \supset F$ is an extension of degree $u < n$. \hfill \Box

Theorem 6 of \cite{1} states that the graph $\Gamma(F, n)$ is connected precisely when assertion of Lemma 2.4 holds for $F$. So have proved one of our key results.

**Theorem 2.5.** Let $n$ be a positive integer which is neither a prime nor a power of two. Then, the graph $\Gamma(F, n)$ is connected.

3. **The matrices**

In the rest of our paper we denote by $\varphi$ the polynomial

$$t^{38} + t^{10} + t^4 + (1 + x)t^2 + 1 + y \in P[t],$$

by $\overline{\varphi}$ its image in $F_2[t]$, and by $\theta$ a root of $\varphi$. One can check that $\sqrt{\varphi}$ is irreducible over $F_2$. In the following lemma, we use a standard result of field theory stating that $[K_1 : K_3] = [K_1 : K_2][K_2 : K_3]$ for any tower $K_1 \supset K_2 \supset K_3$ of algebraic extensions \cite{11}.

**Lemma 3.1.** If $K$ is a field and $F \subset K \subset F(\theta)$, then $K \in \{F, F(\theta^2), F(\theta)\}$.

**Proof.** Note that $\varphi$ is inseparable and irreducible, so that $[F(\theta) : F] = 38$. If $[K : F] = 19$, then $K$ is determined uniquely as the field of all separable elements of $F(\theta)$. In this case, $K = F(\theta^2)$.

Assume that $[K : F] = 2$. Now $F(\theta) \supset K$ is separable, and the only possibility is that $F(\theta) = K[t]/(\sqrt{\varphi})$. In particular, we have $\sqrt{\varphi} \in K[t]$ which implies $\sqrt{\varphi}, \sqrt{\overline{\varphi}} \in K$. Note that the extension $\text{Quot} F_2[[x, y]](\sqrt{\varphi}, \sqrt{\overline{\varphi}}) \supset \text{Quot} F_2[[x, y]]$ has inseparability degree four, which means that $[K : F]_i \geq 4$ because $F \supset \text{Quot} F_2[[x, y]]$ is a separable extension. Therefore, $[K : F] \neq 2$. \hfill \Box

The main result of this section relies on the following propositions. Theorem 3.2 is known, and the proof of Claim 3.3 will be given separately in Section 4.

**Theorem 3.2.** \cite{3} Theorem 2.8 Let $A \in \text{Mat}_n(F)$ be non-derogatory (that is, a matrix with equal minimal and characteristic polynomials). Then, $C(A) = F[A]$. 
Claim 3.3. Let $G$ be the companion matrix of an irreducible polynomial $g \in \mathbb{F}_2[t]$ of prime degree $p \geq 19$. Then, there is an invertible matrix $B = B(g) \in \text{Mat}_{3p}(\mathbb{F}_2)$ satisfying the following property: The distance between $G = \begin{pmatrix} \gamma & 0 \\ 0 & G \end{pmatrix}$ and $B^{-1}GB$ in $\Gamma(\mathbb{F}_2, 2p)$ is at least four.

Theorem 3.4. There is a matrix $\Phi \in \text{Mat}_{38}(\mathcal{F})$ with characteristic polynomial equal to $\varphi$ and a matrix $C \in \text{Mat}_{38}(\mathcal{F})$ such that the distance between $\Phi$ and $C^{-1}\Phi C$ in $\Gamma(\mathcal{F}, 38)$ is at least six.

Proof. Since a matrix $U^{-1}\Phi U$ has the same characteristic polynomial as $\Phi$, we can assume without a loss of generality that $\Phi^2$ is in rational normal form. We define $C$ as the matrix $B(t^{19} + t^5 + t^2 + t + 1)$ from Claim 3.3. Let $\Phi \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow M_3 \leftrightarrow M_4 \leftrightarrow C^{-1}\Phi C$ be a path in $\Gamma(\mathcal{F}, 38)$; we can assume without a loss of generality that the $(1,1)$ entry of every $M_i$ is zero.

Since $\varphi$ is irreducible, we can use Theorem 3.2. Then, the centralizer of $\Phi$ is the set of polynomials $\mathcal{F}[\Phi]$, which is isomorphic to $\mathcal{F}(\theta)$ as a field. By Lemma 3.3, the only matrices from $\mathcal{F}[\Phi]$ whose centralizer is larger are those from $\mathcal{F}[\Phi^2]$; moreover, all non-scalar matrices from $\mathcal{F}[\Phi^2]$ are polynomials in each other which means that these matrices have the same centralizer. Therefore, we can assume without a loss of generality that $M_1 = \Phi^2$ and $M_4 = C^{-1}\Phi^2 C$. The minimal polynomial of $\Phi^2$ is $\psi = t^{19} + t^5 + t^2 + (1 + x)t + 1 + y$ and $\Phi^2$ is in rational normal form; therefore, $\Phi^2$ is a block-diagonal matrix with two diagonal blocks equal to the companion matrix of $\psi$.

Let us define the mapping $\Theta : \text{Mat}_{38}(\mathcal{F}) \setminus \{0\} \to \text{Mat}_{38}(\mathbb{F}_2)$ as follows. Let $M$ be a matrix; we set $\xi = \min_{i,j}\{\deg_x M_{ij}\}$ and define $M' = x^{-\xi}M$. We get the matrix $M''$ from $M'$ by assigning the zero value to $x$ and define $\gamma = \min_{i,j}\{\deg_y M''_{ij}\}$. Finally, we define $\Theta(M)$ as the matrix obtained from $y^{-\gamma}M''$ by assigning the zero value to $y$. One can check that $M_iM_j = M_jM_i$ implies $\Theta(M_i)\Theta(M_j) = \Theta(M_j)\Theta(M_i)$ and derive a contradiction with Claim 3.3.

4. Proof of Claim 3.3

In this section, $k, n, p$ denote arbitrary integers such that $k \leq n$ and $p \geq 3$ is odd. Let $\mathcal{P}_{n,k}$ (or $\mathcal{J}_{n,k}$) be the set of all matrices $A \in \text{Mat}_n(\mathbb{F}_2)$ satisfying $A^2 = A$ (or $A^2 = 0$, respectively). Note that the number of linearly independent $k$-tuples of $n$-vectors over $\mathbb{F}_2$ equals $(2^n - 1)(2^n - 2)\ldots(2^n - 2^{k-1})$, and this quantity belongs to the interval $(2^{nk-3}, 2^{nk})$. Then, the number of $k$-dimensional subspaces of an $n$-space over $\mathbb{F}_2$ equals $(2^n - 1)\ldots(2^n - 2^{k-1})(2^{k-1})\ldots(2^{k-2^{k-1}})$. Theorem A.1

Lemma 4.1. We have $\log_2 |\mathcal{P}_{n,k}| \leq 2k(n-k) + 6$ and $\log_2 |\mathcal{J}_{n,k}| \leq 2k(n-k) + 3$.

Proof. Any idempotent is described uniquely by its eigenspaces. To determine a nilpotent $J$ of index two, it suffices to know ker $J$ and restriction of $J$ to the orthogonal complement ker $J^\perp$, which is a linear mapping ker $J^\perp \to \text{ker} J$.

Lemma 4.2. Let $P \in \mathcal{P}_{2p,p}$. Then, $\log_2 |C(P)| = 2p^2$ and $\log_2 |C(P)| \cap \mathcal{P}_{2p,p}| \leq p^2 + \log_2 p + 12$. If $J \in \mathcal{J}_{2p,p}$, then $PJ \neq JP$.

Proof. Assume without a loss of generality that $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, where $I$ denotes the $p \times p$ identity matrix. Then, a matrix $U$ commutes with $P$ only if $U = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}$. The first assertion is now immediate, and the second one follows from Lemma A.1. If $U \in \mathcal{J}_{2p,p}$, then $A$ and $B$ are nilpotent matrices of index two; their ranks cannot exceed $(p - 1)/2$, a contradiction.
Lemma 4.3. Let $J \in J_{2p,p}$. Then, $\log_2 |C(J)| = 2p^2$ and $\log_2 |C(J) \cap J_{2p,p}| \leq 1.5p^2 + \log_2 p + 3$.

Proof. Assume without a loss of generality that $J = (\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix})$. Then, a matrix $V$ commutes with $J$ only if $V = (\begin{smallmatrix} A & B \\ 0 & A \end{smallmatrix})$. The first assertion is now immediate; if $V \in J_{2p,p}$, then $A$ is a nilpotent matrix of index two, and the second assertion follows from Lemma 4.1. □

For $A_1, A_2 \in \text{Mat}_n(\mathbb{F}_2)$, we denote by $B(A_1, A_2)$ the set of all invertible matrices $B$ satisfying $B^{-1}A_1BA_2 = A_2B^{-1}A_1B$. A useful corollary of the above lemmas reads as follows in this notation.

Corollary 4.4. If $P_1, P_2 \in \mathcal{P}_{2p,p}$, then $\log_2 |B(P_1, P_2)| \leq 3p^2 + \log_2 p + 12$. If $J_1, J_2 \in J_{2p,p}$, then $\log_2 |B(J_1, J_2)| \leq 3.5p^2 + \log_2 p + 3$.

Proof. There are $|C(P_2) \cap \mathcal{P}_{2p,p}|$ ways to choose $P = B^{-1}P_1B$, and the number of matrices $B$ satisfying this equality (if $P$ is fixed) is at most $|C(P)|$. The second assertion can be proved analogously. □

We are now ready to finalize the proof of Claim 3.3.

Proof of Claim 3.3. Let $G \leftrightarrow U \leftrightarrow V \leftrightarrow B^{-1}GB$ be a path in $\Gamma(\mathbb{F}_2, 38)$. We say that a matrix $G' \in \text{Mat}_{38}(\mathbb{F}_2)$ is of $G$-type if $G' = (G_1, G_2, G_3)$, where each $G_i$ belongs to $\mathbb{F}_2[G]$. Since $g$ is irreducible, the centralizer of $G$ consists exactly of $G$-type matrices, so $U$ is a $G$-type matrix.

If $U$ is non-derogatory, then $C(U) = \mathbb{F}_2[U] \subset C(G)$, and we can assume that this is not the case. Otherwise, $\mathbb{F}_2[U]$ contains either an idempotent or a nilpotent matrix. Therefore, it suffices to consider the case when each of $U$ and $BV'B^{-1}$ is either idempotent or nilpotent $G$-type matrix. Lemma 4.2 implies that either both are nilpotent or both are idempotent. Note that the number of $G$-type nilpotent matrices (as well as that of idempotent matrices) equals $2^{4p}$.

By Corollary 4.4, there are at most $\exp \ln 2(3p^2 + \log_2 p + 12)$ ways to choose the matrix $B$, given idempotent $U$ and $V$. This shows that the total number of possibilities for $B$ is at most $\exp \ln 2(3p^2 + \log_2 p + 4p + 12)$, when $U$ and $V$ are assumed to be idempotent. Similarly, there are at most $\exp \ln 2(3.5p^2 + \log_2 p + 4p + 3)$ possibilities for nilpotent $U$ and $V$. It remains to note that the total number of invertible matrices is at least $\exp \ln 2(4p^2 - 3)$, and to check that $\exp \ln 2(3p^2 + \log_2 p + 4p + 12) + \exp \ln 2(3.5p^2 + \log_2 p + 4p + 3) < \exp \ln 2(4p^2 - 3)$. □

The proof of the main result is now complete. Theorems 2.5 and 3.4 show that $\Gamma(\mathbb{F}, 38)$ is a connected graph, and its diameter is at least six. By Theorem 17 of [2], the distances in commuting graphs of matrix algebras cannot exceed six; therefore, the diameter of $\Gamma(\mathbb{F}, 38)$ equals six.

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