Kneading determinants of infinite order linear recurrences

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Abstract

Infinite order linear recurrences are studied via kneading matrices and kneading determinants.

The concepts of kneading matrix and kneading determinant of an infinite order linear recurrence, introduced in this work, are defined in a purely linear algebraic context. These concepts extend for infinite order linear recurrences the classical notions of Frobenius companion matrix and associated discriminant of a finite order linear recurrence.

1 Introduction

In the literature we find regularly finite linear recurrences which are generalizations of the eight centuries old finite linear recurrences of Leonardo de Pisa, or Fibonacci [8], which gave us the illustrious example of the multiplication of rabbit couples

\[ q_{n+1} = q_n + q_{n-1}, \text{ with } q_0 = 1, q_1 = 1 \text{ and } n \geq 1. \]

In the 19th century Jacques Philippe Marie Binet popularized a formula, earlier known to De Moivre, solving the Fibonacci recurrence as a function of \( n \).

In a series of papers, [7] or [9], M. Rashidi and other authors studied linear infinite order scalar recurrences. Giving an infinite sequence of coefficients \( \{a_i\}_{i=0,1,2...} \) with some possible conditions on the sequence like periodicity [3] or evanescence, the problem was to find a solution of the infinite order linear scalar recurrences

\[ q_{n+1} = \sum_{i=0}^{+\infty} a_i q_{n-i}, \text{ for } n \geq 0, \quad (1) \]

with an infinite set of initial conditions \( \{q_i\}_{i=0,-1,-2,...}. \)
Let $N = \{0, 1, 2, \ldots\}$ be the set of non-negative integers, $p$ a positive integer and \( \{A_n\}_{n \in N} \) an infinite sequence of $p \times p$ matrices with complex entries. In this paper we are interested in vectorial (or matricial) homogeneous linear recurrences of the type

$$x_{n+1} = \sum_{i=0}^{+\infty} A_i x_{n-i}, \text{ for } n \in N,$$

where $x_n \in \mathbb{C}^p$ for all $n \in \mathbb{Z}$ and $x_n = 0$ for almost all negative integers $n$.

We call this type of homogeneous linear recurrences of generalized Fibonacci recurrences on $\mathbb{C}^p$, for short $\text{Fib}_p$ recurrences. So, a $\text{Fib}_p$ recurrence is completely determined by a sequence of matrices $\{A_n\}_{n \in N}$. If there exists $k \in \mathbb{Z}^+$ such that $A_{k-1} \neq 0$ and $A_n = 0$ for $n \geq k$, the $\text{Fib}_p$ recurrence is said to be of finite order $k$ (for short a $\text{Fib}_k^p$ recurrence). If the set $\{n \in N : A_n \neq 0\}$ is infinite, the $\text{Fib}_p$ recurrence is said to be of infinite order (for short a $\text{Fib}_\infty^p$ recurrence). With this notation the recurrence (1) is of type $\text{Fib}_\infty^1$ and the original Fibonacci recurrence is of type $\text{Fib}_2^1$.

The concepts of kneading matrix and kneading determinant of a $\text{Fib}_p$ recurrence, introduced in this work, will be defined in a purely linear algebraic context. These concepts extend for $\text{Fib}_\infty^p$ recurrences the classical notions of Frobenius companion matrix and associated discriminant of a $\text{Fib}_k^p$ recurrence.

The main goal of this work is to show that these two concepts play a relevant role in the study of a $\text{Fib}_\infty^p$ recurrence.

The concept of kneading determinant was introduced by Milnor and Thurston [6] in the late eighties of the last century in the completely different context of one-dimensional dynamics. Later on, was proved that the kneading determinant of an interval map can be regarded as the determinant of a pair of linear endomorphisms with finite rank, see [1] and [2]. This last point of view, purely linear algebraic, is the link between Milnor and Thurston notion and our definition of kneading determinant of a linear recurrence. Indeed, as we will see, the kneading determinant of a $\text{Fib}_p$ recurrence is a particular case of the above mentioned determinant of a pair of linear endomorphisms with finite rank.

We present the main results in the next section and the technical details and proofs in the last two sections.

2 Main results

Let us begin with some comments about notation. Throughout the paper $\mathbb{C}[z]$ and $\mathbb{C}[[z]]$ denote respectively the commutative rings of polynomials and formal power series with complex coefficients. Matrices with entries in $\mathbb{C}$, $\mathbb{C}[z]$ and $\mathbb{C}[[z]]$ will be denoted respectively as elements of $\mathbb{C}^{m \times n}$, $\mathbb{C}[z]^{m \times n}$ and $\mathbb{C}[[z]]^{m \times n}$. The $m \times m$ identity matrix $I_m$ will be usually written $I$ keeping in mind that its order is always well defined from the context.

The infinite-dimensional vector spaces over $\mathbb{C}$

$$U = \bigoplus_{n \in \mathbb{N}} \mathbb{C}^p \text{ and } V = \prod_{n \in \mathbb{Z}} \mathbb{C}^p$$

\[1\] In the sense of all but except finitely many.
will play an important role in this discussion. We write \( u \) and \( v \) for denoting the vectors of \( U \) and \( V \) with components \( u_n \in \mathbb{C}^p \) and \( v_n \in \mathbb{C}^p \), i.e.,

\[
u = (u_n)_{n \in \mathbb{N}} = (u_0, u_1, \ldots) \text{ with } u_n \in \mathbb{C}^p\]

and

\[
v = (v_n)_{n \in \mathbb{Z}} = (\ldots, v_{-1}, v_0, v_1, \ldots) \text{ with } v_n \in \mathbb{C}^p.
\]

In contrast with \( V \), the space \( U \) admits a countably infinite basis. From now on we reserve the symbols \( e_1, \ldots, e_p \) for denoting the vectors of the standard basis of \( \mathbb{C}^p \) and \( e_\beta \), with \( \beta \in \mathbb{Z}^+ \), for denoting the vectors of the standard basis of \( U \):

\[
e_1 = (e_1, 0, 0, \ldots), \ e_2 = (e_2, 0, 0, \ldots), \ldots, \ e_p = (e_p, 0, 0, \ldots),
\]

\[
e_{p+1} = (0, e_1, 0, \ldots), \ e_{p+2} = (0, e_2, 0, \ldots), \ldots, \ e_{2p} = (0, e_p, 0, \ldots),
\]

where 0 denotes the zero vector of \( \mathbb{C}^p \).

After these basic remarks we now set the terminology of linear recurrences.

A vector \( v = (v_n)_{n \in \mathbb{Z}} \in V \) is said to be a solution of a Fib\(_p\) recurrence (2) if the set \( \{ n < 0 : v_n \neq 0 \} \) is finite and

\[
v_{n+1} = \sum_{i=0}^{+\infty} A_i v_{n-i}, \text{ for all } n \geq 0.
\]

The subspace of \( V \) whose vectors are the solutions of the Fib\(_p\) recurrence is denoted by \( S \).

Naturally, there exists an isomorphism

\[
\Theta : \quad U \to \quad S
\]

\[
u = (u_n)_{n \in \mathbb{N}} \to v = (v_n)_{n \in \mathbb{Z}},
\]

where \( v = (v_n)_{n \in \mathbb{Z}} \) is the unique vector of \( S \) satisfying \( v_n = u_{-n} \) for all \( n \leq 0 \). The vector \( \Theta(u) \in S \) is called the solution of the Fib\(_p\) recurrence for the initial condition \( u \in U \). The vector space \( U \) is called space of initial conditions.

In order to analyze the asymptotic behavior of a solution \( \Theta(u) \)

we define the generating function \( G(u) \) as the formal power series with coefficients in \( \mathbb{C}^p \)

\[
G(u) = \sum_{n \geq 0} v_n z^n.
\]

Alternatively, \( G(u) \) can be defined as the element of the \( \mathbb{C} \)-vector space \( \mathbb{C}[[z]]^p \)

\[
G(u) = (G_1(u), \ldots, G_p(u)),
\]

with

\[
G_\alpha(u) = \sum_{n \geq 0} v_{n}^{(\alpha)} z^n \in \mathbb{C}[[z]], \alpha = 1, \ldots, p,
\]

(4)

where \( v_{n}^{(\alpha)} \) denotes the \( \alpha \)-th component of \( v_n \) with respect the standard base of \( \mathbb{C}^p \).
Now we introduce the main ingredients of this work: the notions of kneading matrix and kneading determinant of a Fib\(_p\) recurrence. The idea is to look at the sequence \((A_n)_{n \in \mathbb{N}}\) as a generating function \(\sum_{n \geq 0} A_n z^n\) with coefficients in \(\mathbb{C}^{p \times p}\). Naturally, this generating function can be identified with the \(p \times p\) matrix of formal power series

\[
K = \begin{pmatrix} K(1, 1) & \cdots & K(1, p) \\
\vdots & \ddots & \vdots \\
K(p, 1) & \cdots & K(p, p) \end{pmatrix},
\]

(5)
defined by

\[
K(i, j) = \sum_{n \geq 0} A_n(i, j) z^n \in \mathbb{C}[[z]].
\]

The matrix \(K\) is called the kneading matrix of the Fib\(_p\) recurrence and the invertible formal power series

\[
\Delta = \det(I - zK) \in \mathbb{C}[[z]].
\]

receives the name of kneading determinant of the Fib\(_p\) recurrence.

Trivially, the entries of the kneading matrix are polynomials if and only if the Fib\(_p\) recurrence is of finite order. So the kneading determinant of a Fib\(_k\) recurrence is actually a polynomial.

The first result of this work concerns the particular case of linear recurrences of finite order and shows that the definition of kneading determinant extends the usual definition of discriminant of a finite order linear recurrence as defined in classical textbooks [4] or [5].

Recall that the Frobenius companion matrix of a Fib\(_k\) recurrence is the \(kp \times kp\) matrix

\[
F = \begin{pmatrix} A_0 & \cdots & A_{k-2} & A_{k-1} \\
I & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & I & 0 \end{pmatrix},
\]

(7)
where \(I\) and \(0\) denote respectively the \(p \times p\) identity matrix and the \(p \times p\) zero matrix. As the next result shows, the classical discriminant \(\det(I - zF)\) coincides with the kneading determinant of the Fib\(_k\) recurrence.

**Theorem 1** For any Fib\(_k\) recurrence the relation \(\det(I - zF) = \Delta\) holds in \(\mathbb{C}[z]\).

The previous theorem is not merely conceptual. As a matter of fact it can be useful to compute explicitly the discriminant of a vectorial finite recurrence Fib\(_k\). A simple example illustrates this feature.

**Example 2** Except for the case \(p = 1\), the computation of the discriminant \(\det(I - zF)\) by standard methods requires in general a large number of tedious computations. As an example, consider the Fib\(_3\) recurrence defined by

\[
A_0 = \begin{pmatrix} 1 & 1 \\
1 & 1 \end{pmatrix}, \ A_1 = \begin{pmatrix} -1 & -1 \\
-1 & -1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix}.
\]
The companion matrix is
\[
F = \begin{pmatrix}
1 & 1 & -1 & -1 & 0 & 1 \\
1 & 1 & -1 & -1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}.
\]

After some cumbersome computations, one gets
\[
\det(I - zF) = (1 - z) (1 + z) (1 - z + z^2)^2.
\]

On the other hand, as the kneading matrix is
\[
K = \begin{pmatrix}
1 - z & 1 - z + z^2 \\
1 - z + z^2 & 1 - z
\end{pmatrix},
\]
a simple computation gives
\[
\Delta = \det(I - zK) = (1 - z) (1 + z) (1 - z + z^2)^2.
\]
which agrees with the previous value of \(\det(I - zF)\) obtained above.

Next we focus on the main result of this work. Our goal is to provide explicit formulas for computing the generating functions \(G(u)\) of a finite or infinite order \(Fib_p\) recurrence.

First of all observe that from the linearity of the map
\[
G : U \rightarrow \mathbb{C}[[z]]^p \\
u \rightarrow G(u),
\]
one has
\[
G(u) = \sum_{\beta \geq 1} c_\beta G(e_\beta) = \sum_{\beta \geq 1} c_\beta (G_1(e_\beta), ..., G_p(e_\beta)),
\]
where \((e_\beta)_{\beta \in \mathbb{Z}^+}\) denotes the coordinates of \(u\) with respect to the standard basis \((e_\beta)_{\beta \in \mathbb{Z}^+}\) of \(U\). Therefore, to accomplish this task we just need to focus on the generating functions \(G_\alpha(e_\beta)\).

For this purpose, we define for each \(\alpha = 1, ..., p\) and each \(\beta \in \mathbb{Z}^+\) the extended kneading matrix \(K_\alpha(\beta)\) adding one more row and one more column to the kneading matrix \(K\) of the \(Fib_p\) recurrence. More precisely we define \(K_\alpha(\beta) \in \mathbb{C}[[z]]^{(p+1) \times (p+1)}\) by setting
\[
K_\alpha(\beta) = \begin{pmatrix}
K(1,1) & \cdots & K(1,p) & K(1,\beta) \\
\vdots & \ddots & \vdots & \vdots \\
K(p,1) & \cdots & K(p,p) & K(p,\beta) \\
\delta(\alpha,1) & \cdots & \delta(\alpha,p) & \delta(\alpha,\beta)
\end{pmatrix},
\]
where \(\delta(i,j)\) is the usual Kronecker delta function. For the last column of \(K_\alpha(\beta)\) we consider the quotient \(q\) and the reminder \(r\) of the division of \(\beta\) by \(p\) to introduce
\[
K(i,\beta) = \begin{cases}
\sum_{n \geq 0} A_{n+q-1}(i,p) z^n & \text{if } p \text{ divides } \beta \\
\sum_{n \geq 0} A_{n+q}(i,r) z^n & \text{otherwise}
\end{cases}.
\]
Finally we define the extended kneading determinant
\[ \Delta_\alpha(\beta) = \det(I - zK_\alpha(\beta)). \]

Now we can state the main result of this work.

**Theorem 3** For every \( \alpha = 1, \ldots, p \) and every vector \( e_\beta \) of the standard basis of \( U \), the generating function \( G_\alpha(e_\beta) \) of a Fib\( p \) recurrence satisfies the following equality in \( \mathbb{C}[[z]] \)
\[ zG_\alpha(e_\beta) = 1 - \Delta^{-1} \Delta_\alpha(\beta). \]

**Example 4** In order to illustrate Theorem 3 we compute the generating functions \( G(e_\beta) \) of the Fib\( 2^{\infty} \) recurrence defined by
\[ A_n = \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right), \text{ for } n \in \mathbb{N}. \]

We have
\[ K = \left( \begin{array}{ccc} 1 & -1 & -1 \\ 1 & -z & 0 \\ 0 & 1 & z \end{array} \right) \text{ and } \Delta = \det(I - zK) = 1 - 3z + z^2 / (1 - z)^2. \]

On the other hand, as the extended kneading matrices \( K_1(\beta) \) and \( K_2(\beta) \) are defined by:
\[ K_\alpha(\beta) = \left\{ \begin{array}{cl} \left( \begin{array}{ccc} 1 & -1 & -1 \\ 1 & -z & 0 \\ \delta(\alpha,1) & \delta(\alpha,2) & \delta(\alpha,\beta) \end{array} \right) & \text{if } \beta \text{ is odd,} \\
\left( \begin{array}{ccc} 1 & -1 & -1 \\ 1 & -z & 0 \\ \delta(\alpha,1) & \delta(\alpha,2) & \delta(\alpha,\beta) \end{array} \right) & \text{if } \beta \text{ is even;} \end{array} \right. \]

the extended kneading determinants \( \Delta_\alpha(\beta) = \det(I - zK_\alpha(\beta)) \) are
\[ \Delta_\alpha(\beta) = \left\{ \begin{array}{cl} (1 - \beta(\alpha,\beta)z)(1 - 3z + z^2) + \delta(\alpha,2)z^2 - z^2 & \text{if } \beta \text{ is odd}, \\
(1 - \beta(\alpha,\beta)z)(1 - 3z + z^2) + \delta(\alpha,1)z^2(1 - 2z) - z^2 & \text{if } \beta \text{ is even} \end{array} \right. \]

and by Theorem 3
\[ G(e_\beta) = (G_1(e_\beta), G_2(e_\beta)) \]
\[ = \left\{ \begin{array}{cl} (\delta(1,\beta) + \frac{z^2 - z}{1 - 3z + z^2}, \frac{z^2 - z}{1 - 3z + z^2}) & \text{if } \beta \text{ is odd,} \\
\delta(2,\beta) + \frac{z^2 - z}{1 - 3z + z^2}, \frac{z^2 - z}{1 - 3z + z^2} & \text{if } \beta \text{ is even.} \end{array} \right. \]

We finish this section with a corollary of the previous theorem showing that under suitable convergence conditions the zeros of \( \Delta \) play a central role in the study of the analytical properties of the generating functions.

As usual, a matrix \( \mathbf{M} \in \mathbb{C}[[z]]^{p \times p} \) is said to be rational if all of its entries are rational functions of \( z \). Similarly, one says that \( \mathbf{M} \) is holomorphic if \( z \). A formal power series \( a = \sum_{n \geq 0} a_n z^n \) is said to be a rational function of \( z \) if there exist polynomials \( p, q \) such that \( q \) is invertible in \( \mathbb{C}[[z]] \) and \( a = q^{-1}p \).

\( \Delta \) is rational if \( \alpha \geq 1 \) and every vector \( e_\beta \) is holomorphic on \( D_p \). Similarly, one says that \( \alpha \in \mathbb{C}[[z]] \) is meromorphic on \( D_p \) if there exist \( b, c \in \mathbb{C}[[z]] \) such that \( b \) and \( c \) are holomorphic on \( D_p \), \( c \) is invertible in \( \mathbb{C}[[z]] \) and \( a = c^{-1}b \).
meromorphic) on the open disk $D_\rho = \{ z \in \mathbb{C} : |z| < \rho \}$, with $\rho \in [0, +\infty]$, if the entries of $M$ are holomorphic (resp. meromorphic) functions on $D_\rho$. If $M$ is rational (resp. meromorphic on $D_\rho$), one says that $w \in \mathbb{C}$ (resp. $w \in D_\rho$) is a pole of the $M$ if $w$ is a pole of some entry $M(i, j)$ of $M$.

So, if the kneading matrix $K$ of a $Fib_p$ recurrence is rational (resp. meromorphic on $D_\rho$), then the kneading determinant $\Delta$ is a rational function (resp. a meromorphic function on $D_\rho$) and each one of its poles is a pole of $K$. Hence, we can look at $\Delta$ as an analytic function on $\mathbb{C} \setminus \mathbb{P}$ (resp. $D_\rho \setminus \mathbb{P}$), where $\mathbb{P}$ denotes the set of poles of the kneading matrix $K$.

**Corollary 5** If the kneading matrix, $K$, of a $Fib_p$ recurrence is rational (resp. meromorphic on $D_\rho$), then the generating function $G(u)$ is rational (resp. meromorphic on $D_\rho$) and each one of its poles is either a pole of $K$ or a zero of $\Delta$. Moreover, if $p = 1$, then each pole of $G(u)$ is a zero of $\Delta$.

The rest of the paper will be devoted to the proofs of Theorem 1, Theorem 3 and Corollary 5.

### 3 Pairs of linear endomorphisms

The proofs of the Theorems 1 and 3 in the next section are based in the main Theorem of [1], as well other related results, proved here, concerning the determinant of a pair of linear endomorphisms with finite rank.

In order to improve the readability of the paper we present a brief description of these results.

The idea is to extend to a wider context the well known relationship between discriminant and traces for a matrix $X \in \mathbb{C}^{m \times m}$

$$
\det(I - zX) = \exp \sum_{n \geq 1} \frac{\text{tr}(X^n)}{n} z^n. \tag{10}
$$

Throughout this section, $U$ denotes an arbitrary (finite or infinite dimensional) vector space over $\mathbb{C}$; the space of linear forms on $U$ will be denoted by $U^*$ and the space of linear endomorphisms on $U$ will be denoted by $L(U)$. If $\psi \in L(U)$ and $n$ is a nonnegative integer, the $n$-th iterate $\psi^n$ is defined recursively by $\psi^0 = Id_U \in L(U)$, $\psi^n = \psi \circ \psi^{n-1} \in L(U)$, for $n \geq 1$.

Recall that a linear endomorphism $\psi \in L(U)$ is said to have finite rank if there exist vectors $u_1, \ldots, u_p \in U$ and linear forms $\omega_1, \ldots, \omega_p \in U^*$ such that

$$
\psi = \omega_1 \otimes u_1 + \omega_2 \otimes u_2 + \cdots + \omega_p \otimes u_p,
$$

with the usual notation

$$
\omega \in U^*, \ u \in U : (\omega \otimes u)(x) = \omega(x) u, \ x \in U.
$$

The subspace of $L(U)$ whose elements are the linear endomorphisms on $U$ with finite rank will be denoted by $L_{FR}(U)$.

The importance of $L_{FR}(U)$, in this context, lies in the existence of the trace for any $\psi \in L_{FR}(U)$, trace that is not evidently defined for an arbitrary $\psi \in L(U)$.

Let us then introduce the following definition.
Definition 6 A pair of endomorphisms \((\varphi, \psi) \in L(U) \times L(U)\) is said to have finite rank if 
\[\psi - \varphi \in L_{FR}(U).\]

Notice that if a pair \((\varphi, \psi)\) has finite rank, then the pair \((\varphi^n, \psi^n)\) has finite rank for all \(n \geq 0\), therefore the trace of \(\varphi^n - \psi^n\) is defined. Thus, we may define:

Definition 7 For any pair \((\varphi, \psi) \in L(U) \times L(U)\) with finite rank the determinant of \((\varphi, \psi)\) is defined by

\[\Delta (\varphi, \psi) = \exp \sum_{n \geq 1} \frac{\text{tr}(\varphi^n - \psi^n)}{n} z^n \in \mathbb{C}[[z]].\]

If a pair \((\varphi, \psi)\) has finite rank then \((\psi, \varphi)\) has finite rank too and \(\Delta (\varphi, \psi) \Delta (\psi, \varphi) = 1\). Thus, \(\Delta (\varphi, \psi)\) is invertible in \(\mathbb{C}[[z]]\) and

\[\Delta (\varphi, \psi)^{-1} = \Delta (\psi, \varphi).\]

More generally:

Proposition 8 If \((\varphi, \psi) \in L(U) \times L(U)\) and \((\psi, \chi) \in L(U) \times L(U)\) have both finite rank, then \((\varphi, \chi) \in L(U) \times L(U)\) has finite rank and \(\Delta (\varphi, \chi) = \Delta (\varphi, \psi) \Delta (\psi, \chi)\) holds in \(\mathbb{C}[[z]]\).

Notice that if the space \(U\) is finite dimensional, then every pair \((\varphi, \psi)\) has finite rank and by \((10)\) one gets

\[\Delta (\varphi, \psi) = \frac{\det(I - zY)}{\det(I - zX)}\]

where \(X\) (respectively \(Y\)) is the matrix that represents \(\varphi\) (respectively \(\psi\)) with respect to some basis of \(U\). So, in this particular case \(\Delta (\varphi, \psi)\) is a rational function of \(z\).

The situation becomes entirely different if the space \(U\) is infinite dimensional. In this case the rationality of \(\Delta (\varphi, \psi)\) fails in general. This fact is a simple consequence of the next result which enables us to express \(\Delta (\varphi, \psi)\) in terms of determinants. To state it we have to introduce some additional notation.

Observe that if a pair \((\varphi, \psi) \in L(U) \times L(U)\) has finite rank, then there exist vectors \(u_1, \ldots, u_p \in U\) and linear forms \(\omega_1, \ldots, \omega_p \in U^*\) such that

\[\psi - \varphi = \omega_1 \otimes u_1 + \omega_2 \otimes u_2 + \cdots + \omega_p \otimes u_p \quad \text{(11)}\]

and so, we can define the matrix \(M \in \mathbb{C}[[z]]^{p \times p}\) by setting

\[M = \begin{pmatrix} \sum_{n \geq 0} \omega_1 \varphi^n(u_1) z^n & \cdots & \sum_{n \geq 0} \omega_1 \varphi^n(u_p) z^n \\ \vdots & \ddots & \vdots \\ \sum_{n \geq 0} \omega_p \varphi^n(u_1) z^n & \cdots & \sum_{n \geq 0} \omega_p \varphi^n(u_p) z^n \end{pmatrix}. \quad \text{(12)}\]

Now we can state the main Theorem of [1] which establishes a fundamental relationship between \(\Delta (\varphi, \psi)\) and the determinant of the matrix \(I - zM \in \mathbb{C}[[z]]^{p \times p}\).
Lemma 9 Let \( (\varphi, \psi) \in \mathrm{L}(U) \times \mathrm{L}(U) \) be a pair of endomorphisms with finite rank. If the vectors \( u_1, \ldots, u_p \in U \) and the linear forms \( \omega_1, \ldots, \omega_p \in U^* \) satisfy (11), then \( \Delta(\varphi, \psi) = \det(I - zM) \) holds in \( \mathbb{C}[[z]] \).

Two consequences of this result are needed for our purposes.

The first one can be thought as an alternative method for computing the discriminant \( \det(I - zY) \) of a complex matrix \( Y \in \mathbb{C}^{m \times m} \).

The idea is to consider a nilpotent matrix \( X \in \mathbb{C}^{m \times m} \) and to look at \( (X, Y) \) as a pair of linear endomorphism on \( \mathbb{C}^m \) with finite rank. So we can consider column matrices \( C_1, \ldots, C_p \in \mathbb{C}^{m \times 1} \) and row matrices \( R_1, \ldots, R_p \in \mathbb{C}^{1 \times m} \) satisfying

\[
Y - X = C_1 R_1 + C_2 R_2 + \cdots + C_p R_p \tag{13}
\]

and by Lemma 9 we can write

\[
\exp \sum_{n \geq 1} \frac{\text{tr}(X^n - Y^n)}{n} z^n = \det(I - zM), \tag{14}
\]

with

\[
M = \begin{pmatrix}
\sum_{n \geq 0} R_1 X^n C_1 z^n & \cdots & \sum_{n \geq 0} R_1 X^n C_p z^n \\
\vdots & \ddots & \vdots \\
\sum_{n \geq 0} R_p X^n C_1 z^n & \cdots & \sum_{n \geq 0} R_p X^n C_p z^n
\end{pmatrix}. \tag{15}
\]

Moreover, being \( X \) nilpotent \( M \) is a \( p \times p \) matrix of polynomials. Hence \( \det(I - zM) \) is a polynomial too. As we will see in the next result this polynomial is actually the discriminant of \( Y \).

Lemma 10 Let \( X \in \mathbb{C}^{m \times m}, Y \in \mathbb{C}^{m \times m}, C_1, \ldots, C_p \in \mathbb{C}^{m \times 1} \) and \( R_1, \ldots, R_p \in \mathbb{C}^{1 \times m} \) satisfying (13). If \( X \) is nilpotent, then the equality \( \det(I - zY) = \det(I - zM) \) holds in \( \mathbb{C}[z] \).

Proof. As \( X \) is nilpotent one has \( \text{tr}(X^n) = 0 \) for \( n \geq 1 \). Combining this with (10) and (14) one gets

\[
\det(I - zY) = \exp \sum_{n \geq 1} -\frac{\text{tr}(Y^n)}{n} z^n,
\]

\[
= \exp \sum_{n \geq 1} \frac{\text{tr}(X^n - Y^n)}{n} z^n
\]

\[
= \det(I - zM),
\]

as desired. \( \blacksquare \)

A second consequence of Lemma 9 concerns the general and difficult problem of studying the analytic properties of the generating function

\[
\sum_{n \geq 0} \omega \psi^n(u) z^n, \tag{16}
\]

where \( \psi \in \mathrm{L}(U), \omega \in U^* \) and \( u \in U \) are arbitrary.
An idea that can be useful is to consider a pair \((\varphi, \psi) \in L(U) \times L(U)\) with finite rank and write (16) in terms of determinants with the desired analytic properties.

Notice that if a pair \((\varphi, \psi)\) has finite rank, then \((\varphi, \psi + \omega \otimes u)\) has finite rank too. In fact, if the vectors \(u_1, \ldots, u_p \in U\) and the linear forms \(\omega_1, \ldots, \omega_p \in U^*\) satisfy (11), then
\[
(\psi + \omega \otimes u) - \varphi = \omega_1 \otimes u_1 + \omega_2 \otimes u_2 + \cdots + \omega_p \otimes u_p + \omega \otimes u
\]
and by Lemma 9
\[
\Delta (\varphi, \psi + \omega \otimes u) = \det(I - zM_\omega(u)),
\]
where \(M_\omega(u) \in \mathbb{C}[z]^{(p+1) \times (p+1)}\) is the extended matrix defined by
\[
M_\omega(u) = \left(\begin{array}{ccc}
\sum_{n \geq 0} \omega_1 \varphi^n(u_1) z^n & \cdots & \sum_{n \geq 0} \omega_1 \varphi^n(u_p) z^n \\
\sum_{n \geq 0} \omega_2 \varphi^n(u_1) z^n & \cdots & \sum_{n \geq 0} \omega_2 \varphi^n(u_p) z^n \\
\vdots & \ddots & \vdots \\
\sum_{n \geq 0} \omega_p \varphi^n(u_1) z^n & \cdots & \sum_{n \geq 0} \omega_p \varphi^n(u_p) z^n
\end{array}\right).
\]

Now it is easy to establish a simple relationship between the generating function of (16) and the matrices \(M\) and \(M_\omega(u)\) of (17) and (18).

**Lemma 11** Let \((\varphi, \psi) \in L(U) \times L(U)\) be a pair with finite rank, \(u \in U\) and \(\omega \in U^*\). If the vectors \(u_1, \ldots, u_p \in U\) and the linear forms \(\omega_1, \ldots, \omega_p \in U^*\) satisfy (17), then the equality
\[
z \sum_{n \geq 0} \omega \psi^n(u) z^n = 1 - \frac{\det(I - zM_\omega(u))}{\det(I - zM)}
\]
holds in \(\mathbb{C}[z]\).

**Proof.** Combining Lemma 9 with (17) and Proposition 8 one gets
\[
\frac{\det(I - zM_\omega(u))}{\det(I - zM)} = \frac{\Delta (\varphi, \psi + \omega \otimes u)}{\Delta (\varphi, \psi)} = \Delta (\psi, \varphi) \Delta (\varphi, \psi + \omega \otimes u) = \Delta (\psi, \psi + \omega \otimes u).
\]

But, again by Lemma 9 and because \((\psi + \omega \otimes u) - \psi = \omega \otimes u\) we can write
\[
\Delta (\psi, \psi + \omega \otimes u) = 1 - z \sum_{n \geq 0} \omega \psi^n(u) z^n.
\]
Hence
\[
1 - z \sum_{n \geq 0} \omega \psi^n(u) z^n = \frac{\det(I - zM_\omega(u))}{\det(I - zM)},
\]
as desired. ■

At this stage we have all the ingredients to prove the main theorems 1, 5 and corollary 5.
4 Proofs

Theorem 1 is a simple consequence of Lemma 10.

Let $F \in \mathbb{C}^{kp \times kp}$ be the Frobenius companion matrix of a $Fib^k$ recurrence as defined in (7). For each $i = 1, \ldots, p$, let $R_i \in \mathbb{C}^{1 \times kp}$ be the $i$-th row of $F$ and $C_i \in \mathbb{C}^{kp \times 1}$ the $i$-th vector of the standard basis of $\mathbb{C}^{kp \times 1}$. Evidently, the $kp \times kp$ matrix

$$X = \begin{pmatrix}
0 & \cdots & 0 & 0 \\
1 & \cdots & 0 & 0 \\
\vdots & \cdots & \vdots & \vdots \\
0 & \cdots & 1 & 0
\end{pmatrix}$$

is nilpotent and

$$F - X = C_1 R_1 + \cdots + C_p R_p.$$

So, as the matrices $X, F, C_1, \ldots, C_p, R_1, \ldots, R_p$ satisfy the assumptions of Lemma 10 we can write

$$\det(I - zF) = \det(I - zM), \quad (19)$$

with

$$M = \begin{pmatrix}
\sum_{n \geq 0} R_1 X^n C_1 z^n & \cdots & \sum_{n \geq 0} R_1 X^n C_p z^n \\
\vdots & \cdots & \vdots \\
\sum_{n \geq 0} R_p X^n C_1 z^n & \cdots & \sum_{n \geq 0} R_p X^n C_p z^n
\end{pmatrix}.$$

But by (5) and because $A_n = 0$ for $n \geq k$, the entry $(i, j)$ of $M$ is

$$\sum_{n \geq 0} R_i X^n C_j z^n = \sum_{n=0}^{k-1} A_n(i, j) z^n = \sum_{n \geq 0} A_n(i, j) z^n = K(i, j).$$

Hence $M = K$ and by (5) and (19) we finally arrive at

$$\Delta = \det(I - zK) = \det(I - zM) = \det(I - zF).$$

This last relation is precisely what is stated in Theorem 1.

We now tackle the proof of Theorem 3. The idea is to regard a $Fib_p$ recurrence, determined by a sequence of matrices $(A_n)_{n \in \mathbb{N}}$, as a pair $(\varphi, \psi)$ of linear endomorphisms on the infinite dimensional vector space $U$ defined in (3).

This pair $(\varphi, \psi) \in L(U) \times L(U)$ is now defined as follows:

$$\varphi(u_0, u_1, u_2, \ldots) = (0, u_0, u_1, u_2, \ldots) \text{ for all } (u_n)_{n \in \mathbb{N}} \in U,$$

where 0 denotes the zero vector of $\mathbb{C}^p$, and

$$\psi(u_0, u_1, u_2, \ldots) = (w, u_0, u_1, u_2, \ldots) \text{ for all } (u_n)_{n \in \mathbb{N}} \in U,$$

with

$$w = \sum_{n \geq 0} A_n u_n \in \mathbb{C}^p.$$

Clearly, $(\varphi, \psi)$ is a pair of finite rank. Let us begin by proving that $\Delta(\varphi, \psi)$ is actually the kneading determinant $\Delta$ of the linear recurrence, as defined in (6).
Indeed, from the definitions of $\varphi$ and $\psi$ one has
\[
\psi - \varphi = \omega_1 \otimes e_1 + \omega_2 \otimes e_2 + \cdots + \omega_p \otimes e_p,
\]
where $e_i \in U$ denotes the $i$-th vector of the standard basis of $U$ and $\omega_i \in U^*$ is the linear form defined by
\[
\omega_i(u_0, u_1, u_2, \ldots) = \sum_{n \geq 0} R_n(i) u_n,
\]
where $R_n(i) = (A_n(i, 1) \cdots A_n(i, p)) \in \mathbb{C}^{1 \times p}$ denotes the $i$-th row of $A_n$. By Lemma 9 we have
\[
\Delta(\varphi, \psi) = \det(I - zM), \quad (20)
\]
with
\[
M = \begin{bmatrix}
\sum_{n \geq 0} \omega_1 \varphi^n(e_1) z^n & \cdots & \sum_{n \geq 0} \omega_1 \varphi^n(e_p) z^n \\
\vdots & \ddots & \vdots \\
\sum_{n \geq 0} \omega_p \varphi^n(e_1) z^n & \cdots & \sum_{n \geq 0} \omega_p \varphi^n(e_p) z^n
\end{bmatrix}.
\]
But by (5), the entry $(i, j)$ of $M$ is
\[
\sum_{n \geq 0} \omega_i \varphi^n(e_j) z^n = \sum_{n \geq 0} A_n(i, j) z^n = K(i, j).
\]
Hence $K = M$ and by (6) and (20) we arrive at
\[
\Delta = \det(I - zK) = \det(I - zM) = \Delta(\varphi, \psi). \quad (21)
\]
This formula is the first step in the proof of Theorem 3. The second step deals with the generating functions $G(u)$ of a $Fib_p$ recurrence.

Let $\pi : U \to \mathbb{C}^p$ be the projection defined by $\pi(u_0, u_1, u_2, \ldots) = u_0$. For each $\alpha = 1, \ldots, p$, define the linear form $\pi_\alpha \in U^*$, where $\pi_\alpha(u)$ is the $\alpha$-th coordinate of $\pi(u)$ with respect the standard basis of $\mathbb{C}^p$.

Now let $\Theta(u) = (v_n)_{n \in \mathbb{Z}}$ be the solution of the $Fib_p$ recurrence for the initial condition $u = (u_n)_{n \in \mathbb{N}} \in U$. Observe that from the definition of $\psi$ one has
\[
\pi \psi^n(u) = v_n \quad \text{for } n \in \mathbb{N}.
\]
Thus, the equalities
\[
G(u) = \sum_{n \geq 0} v_n z^n = \sum_{n \geq 0} \pi \psi^n(u) z^n
\]
and
\[
G_\alpha(u) = \sum_{n \geq 0} v_n^{(\alpha)} z^n = \sum_{n \geq 0} \pi_\alpha \psi^n(u) z^n, \quad (22)
\]
hold for all $u \in U$ and $\alpha = 1, \ldots, p$.

Finally, we have all the elements to conclude the proof of Theorem 3.
Let $e_\beta$ be a vector of the standard basis of $U$ and $\alpha = 1, \ldots, p$. From Lemma 11 and equality 21 one has

$$z \sum_{n \geq 0} \pi_\alpha \psi^n(e_\beta)z^n = 1 - \frac{\det(I - zM_{\pi_\alpha}(e_\beta))}{\det(I - zM)} = 1 - \Delta^{-1} \det(I - zM_{\pi_\alpha}(e_\beta)),$$

with

$$M_{\pi_\alpha}(e_\beta) = \begin{pmatrix}
\sum_{n \geq 0} \omega_1 \varphi^n(e_1) z^n & \cdots & \sum_{n \geq 0} \omega_1 \varphi^n(e_p) z^n & \sum_{n \geq 0} \omega_1 \varphi^n(e_\beta) z^n \\
\vdots & \ddots & \vdots & \vdots \\
\sum_{n \geq 0} \omega_p \varphi^n(e_1) z^n & \cdots & \sum_{n \geq 0} \omega_p \varphi^n(e_p) z^n & \sum_{n \geq 0} \omega_p \varphi^n(e_\beta) z^n \\
\sum_{n \geq 0} \pi_\alpha \varphi^n(e_1) z^n & \cdots & \sum_{n \geq 0} \pi_\alpha \varphi^n(e_p) z^n & \sum_{n \geq 0} \pi_\alpha \varphi^n(e_\beta) z^n
\end{pmatrix}.$$

On the other hand, it is easy to see that $M_{\pi_\alpha}(e_\beta)$ is actually the extended kneading matrix $K_\alpha(\beta)$ defined in 9, hence

$$z \sum_{n \geq 0} \pi_\alpha \psi^n(e_\beta)z^n = 1 - \Delta^{-1} \det(I - zM_{\pi_\alpha}(e_\beta)) = 1 - \Delta^{-1} \det(I - zK_\alpha(\beta)) = 1 - \Delta^{-1} \Delta_\alpha(\beta)$$

and by (22) we finally arrive at

$$zG_\alpha(e_\beta) = 1 - \Delta^{-1} \Delta_\alpha(\beta).$$

This last relation is precisely what is stated in Theorem 3.

For proving Corollary 5 we need a last (and simple) auxiliary result.

**Proposition 12** If the kneading matrix, $K$, of a Fib$_p$ recurrence is rational (resp. meromorphic on $D_p$), then every extending kneading matrix $K_\alpha(\beta)$ is rational (resp. meromorphic on $D_p$) and each of its poles is a pole of $K$.

**Proof.** Assume that $\beta = qp$ is a multiple of $p$ (if $p$ does not divide $\beta$ the proof is analogous). By (21) we just need to focus on the entries $K(i, \beta)$ of $K_\alpha(\beta)$. As

$$z^{q-1}K(i, \beta) = K(i, p) - \sum_{n=0}^{q-2} A_n(i, p)z^n$$

and $K(i, p)$ is rational (resp. meromorphic on $D_p$), it follows that $K(i, \beta)$ is rational (resp. meromorphic on $D_p$) and each one of its poles is a pole of $K(i, p)$ with the same order. Hence, every pole of $K_\alpha(\beta)$ is a pole of $K$, as desired. ■

So, if the kneading matrix $K$ of a Fib$_p$ recurrence is rational (resp. meromorphic on $D_p$), then the kneading determinant $\Delta$ as well every extended kneading determinant $\Delta_\alpha(\beta) = \det(I - zK_\alpha(\beta))$ are rational (resp. meromorphic on $D_p$) and each one of its poles is a pole of $K$. Combining this with Theorem 3 and formula (3) the proof of the first part of Corollary 5 is immediate.
Now set \( p = 1 \) and let \( w \) be a \( m \)-order pole of \( K = K(1, 1) \). As
\[
\Delta = 1 - zK(1, 1), \quad \Delta_1 (\beta) = \det \begin{pmatrix}
1 - zK(1, 1) & -zK(1, \beta) \\
-z & 1 - z\delta(1, \beta)
\end{pmatrix},
\]
and \( w \) is a \( m \)-order pole of \( K(1, \beta) \), both functions \( \Delta \) and \( \Delta_1 (\beta) \) have a \( m \)-order pole at \( w \). Hence, the function \( \Delta^{-1} \Delta_1 (\beta) \) has a removable singularity at \( w \) and Theorem 3 assures that \( G_1(e_3) \) does not have a pole at \( w \). This together with \( \S\) proves that any pole of \( G(u) \) is a zero of \( \Delta \) as desired.

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