HAMMING SPACES AND LOCALLY MATRIX ALGEBRAS

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Abstract. We introduce an abstract definition of a Hamming space that generalizes standard Hamming spaces \((\mathbb{Z}/2\mathbb{Z})^n\). We classify countable locally standard Hamming spaces and show that each of them can be realized as the Boolean algebra of idempotents of a Cartan subalgebra of a locally matrix algebra.

INTRODUCTION

The Hamming metric was introduced in information theory as the most common tool to measure the difference between binary strings of the same length. The (standard) Hamming space \(H_n\) is the set of all \(n\)-tuples \(x^n = (x_1, \ldots, x_n)\), \(x_i \in \{0, 1\}, 1 \leq i \leq n\), with the distance \(d_{H_n}\), that is defined between two \(n\)-tuples as the number of coordinates where they differ.

There are different generalizations of finite metric Hamming space to infinite case, that are constructed as inductive limits of finite Hamming spaces [6], [11], [12].

In this paper we define an (abstract) Hamming space and introduce the operation of tensor product. We call a Hamming space locally standard (see the definition below) if locally it looks as \(H_n\). In the first part of the paper we prove that every countable locally standard Hamming space is isomorphic to an infinite tensor product of standard...
Hamming spaces $H_n$. In the second part of the paper we realize countable locally standard Hamming spaces as Cartan subalgebras of locally matrix algebras (see [1], [2], [8], [10]) and discuss conjugacy of Cartan subalgebras.

1. Steinitz numbers

A Steinitz number [14] is a infinite formal product of the form

$$\prod_{p \in \mathbb{P}} p^{r_p},$$

where $\mathbb{P}$ is the set of all primes, $r_p \in \mathbb{N} \cup \{0, \infty\}$ for all $p \in \mathbb{P}$. We can define the product of two Steinitz numbers by the rule:

$$\prod_{p \in \mathbb{P}} p^{r_p} \cdot \prod_{p \in \mathbb{P}} p^{k_p} = \prod_{p \in \mathbb{P}} p^{r_p + k_p}, \quad r_p, k_p \in \mathbb{N} \cup \{0, \infty\},$$

where we assume, that

$$r_p + k_p = \begin{cases} r_p + k_p, & \text{if } r_p < \infty \text{ and } k_p < \infty, \\ \infty, & \text{in other cases} \end{cases}.$$

By symbol $\mathbb{SN}$ we denote the set of all Steinitz numbers. Obviously, the set of all positive integers $\mathbb{N}$ is a subset of the set of all Steinitz numbers $\mathbb{SN}$. The elements of the set $\mathbb{SN} \setminus \mathbb{N}$ are called infinite Steinitz numbers.

2. Hamming spaces

Recall that a Boolean algebra is a commutative algebra over the field $\mathbb{Z}/2\mathbb{Z}$ satisfying the identity $x^2 = x$.

Definition 1. By a (unital) Hamming space $(H, r)$ (see [6], [11], [12]) we mean a Boolean algebra $H$ with 1 and a rang function $r : H \to [0, 1]$ such, that

1. $r(a) = 0$ if and only if $a = 0$;
2. $r(a) = 1$ if and only if $a = 1$;
3. if $a, b \in H$ and $ab = 0$, then $r(a + b) = r(a) + r(b)$.

Remark 1. Note, that if $(H, r)$ is a Hamming space then the function

$$d_H(a, b) = r(a - b), \quad a, b \in H,$$

makes the Hamming space $(H, r)$ a metric space.
Example 1. The Boolean algebra $H_n = (\mathbb{Z}/2\mathbb{Z})^n$ with the rang function

$$r_{H_n}(x_1, \ldots, x_n) = \frac{1}{n}(x_1 + \ldots + x_n)$$

for all $x_1, \ldots, x_n \in \{0, 1\}$ satisfy the assumptions (1), (2), (3) of Definition 1. We call the Hamming space $(H_n, r_{H_n})$ standard. For all $a, b \in H_n$ the corresponding distance $d_{H_n}(a, b) = \text{number of coordinates where } a \text{ and } b \text{ differ}.$

Let $\{0, 1\}^\mathbb{N}$ be the set of all (right-) infinite $(0, 1)$-sequences. Clearly, $\{0, 1\}^\mathbb{N}$ is a Boolean algebra under coordinate-wise addition (modulo 2) and multiplication.

Example 2. An infinite sequence $a = (a_1, a_2, \ldots) \in \{0, 1\}^\mathbb{N}$ is said to be periodic if there exists a natural number $k$ such that the equality $a_i = a_{i+k}$ holds for all $i \in \mathbb{N}$. In this case the number $k$ is called a period of the sequence $a$.

Let $u$ be a Steinitz number. A periodic sequence $a$ is called $u$-periodic if its minimal period is a divisor of $u$.

Let $\mathcal{H}(u)$ be the set of all $u$-periodic sequences. Clearly $\mathcal{H}(u)$ is a Boolean subalgebra of $\{0, 1\}^\mathbb{N}$. The rang function

$$r_{\mathcal{H}(u)}(a_1, a_2, \ldots) = \frac{1}{k}(a_1 + \ldots + a_k),$$

where $k$ is a period of the sequence $(a_1, a_2, \ldots)$, makes $(\mathcal{H}(u), r_{\mathcal{H}(u)})$ a Hamming space.

Example 3. For a sequence $a = (a_1, a_2, \ldots) \in \{0, 1\}^\mathbb{N}$ define its pseudorang function

$$\tilde{r}(a) = \lim_{n \to \infty} \sup \frac{1}{n}(a_1 + \ldots + a_n).$$

Then $I = \{a \in \{0, 1\}^\mathbb{N} \mid \tilde{r}(a) = 0\}$ is an ideal of the Boolean algebra $\{0, 1\}^\mathbb{N}$.

Consider the Boolean algebra $B = \{0, 1\}^\mathbb{N}/I$ and the rang function

$$r_B(a + I) = \tilde{r}(a), \quad a \in \{0, 1\}^\mathbb{N}.$$
Similarly to the unital case, we can define a metric
\[ d_H(a, b) = r(a - b), \quad a, b \in H, \]
that makes \((H, r)\) a metric space.

**Example 4.** Let \(X\) be an infinite set and let \(H\) be the non unital Boolean algebra of finite subsets of \(X\), including the empty one. The rang function \(r(a) = \# a, a \in H\), makes \((H, r)\) a non unital Hamming space (see [6], [13]).

**Remark 2.** For an arbitrary Hamming space \((H, r)\) and a nonzero element \(h \in H\) consider the ideal \(hH\) and the rang function \(r_h: hH \to [0, 1]\),
\[ r_h(a) = \frac{r(a)}{r(h)}, \quad a \in hH. \]
Clearly, \((hH, r_h)\) is a unital Hamming space with the identity element \(h\).

### 3. Tensor Product of Hamming Spaces

The purpose of the following proposition is to define tensor product of Hamming spaces.

**Proposition 1.** Let \((H_1, r_1), (H_2, r_2)\) be Hamming spaces. Then there exists a unique rang function \(r\) on \(H = H_1 \otimes \mathbb{Z} H_2\) such that \(r(a \otimes b) = r_1(a)r_2(b)\) for arbitrary elements \(a \in H_1, b \in H_2\).

**Proof.** Let \(S(H)\) be the set of all nonempty finite subsets of \(H \setminus \{0\}\). Let
\[ E(H) = \{ A \in S(H) \mid A = \{a_1, \ldots, a_r\}, a_i \neq 0, a_i a_j = 0 \text{ for } i \neq j, \quad 1 \leq i, j \leq r \}. \]

We say that a set \(X \in S(H)\) is covered by a set \(E \in E(H)\) if for arbitrary elements \(x \in X, e \in E\) we have \(xe = e\) or 0 and
\[ x = \sum_{e \in E, xe = e} e. \]

Let \(X = \{a_1, \ldots, a_r\} \in S(H)\). Let the set \(E\) consists of all nonzero products \(b_1 \ldots b_r, \text{ where } b_i = a_i \text{ or } 1 - a_i\). Then \(E \in E(H)\) and \(E\) covers \(X\).

For an arbitrary element \(x \in H_1 \otimes \mathbb{Z} H_2\) there exist \(E_1 = \{e_1, \ldots, e_n\} \in E(H_1)\) and \(E_2 = \{f_1, \ldots, f_m\} \in E(H_2)\) such that \(x \in (\text{Span } E_1) \otimes (\text{Span } E_2)\),
\[ x = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \alpha_{ij} e_i \otimes f_j, \quad \alpha_{ij} = 0 \text{ or } 1. \]
In this case we say that the element \( x \) is covered by subsets \( E_1 \) and \( E_2 \).

Define

\[
    r_{E_1,E_2}(x) = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \alpha_{ij} r_1(e_i)r_2(f_j).
\]

Let \( E_1' \in E(H_1) \) and \( E_2' \in E(H_2) \) such that \( E_1 \) and \( E_2 \) are covered by \( E_1' \) and \( E_2' \) respectively. Then

\[
    x = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \alpha_{ij} e_p' \otimes f_q',
\]

where the summation is done over all \( e_p' \in E_1', e_q' \in E_2' \) such that \( e_p'e_p' = e_p', f_j'f_q' = f_q' \). Hence

\[
    r_{E_1',E_2'}(x) = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \alpha_{ij} r_1(e_p')r_2(f_q').
\]

But

\[
    r_1(e_i) = \sum_{e_p'e_p' = e_p'} r_1(e_p') \quad \text{and} \quad r_2(f_j) = \sum_{f_j'f_q' = f_q'} r_2(f_q').
\]

This implies that \( r_{E_1,E_2}(x) = r_{E_1',E_2'}(x) \).

We claim that the function \( r(x) = r_{E_1,E_2}(x) \) is well defined. Let

\[
    x \in (\text{Span } (E_1) \otimes \text{Span } (E_2)) \cap (\text{Span } (E_1') \otimes \text{Span } (E_2')).
\]

There exists \( E_1'' \in E(H_1), E_2'' \in E(H_2) \) such that \( E_1 \) and \( E_1' \) are both covered by \( E_1'' \); \( E_2 \) and \( E_2' \) are both covered by \( E_2'' \). Then

\[
    r_{E_1,E_2}(x) = r_{E_1'',E_2''}(x) = r_{E_1',E_2'}(x).
\]

Define \( r(x) = r_{E_1,E_2}(x) \). Let us show that \( r(x) \) is a rang function.

Let \( E_1 = \{e_1, \ldots, e_n\} \in E(H_1), E_2 = \{f_1, \ldots, f_m\} \in E(H_2) \),

\[
    x = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \alpha_{ij} e_i \otimes f_j, \quad \alpha_{ij} = 0 \text{ or } 1.
\]

Clearly,

\[
    r(x) = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \alpha_{ij} r_1(e_i)r_2(f_j) \geq 0.
\]

If \( x \neq 0 \) then there exist indices \( i, j \) such that \( \alpha_{ij} \neq 0 \). This implies \( r(x) \geq r_1(e_i)r_2(f_j) > 0 \).

Also

\[
    r(x) = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \alpha_{ij} r_1(e_i)r_2(f_j) \leq \sum_{1 \leq i \leq n, 1 \leq j \leq m} r_1(e_i)r_2(f_j) = r_1(e_1 + \cdots + e_n)r_2(f_1 + \cdots + f_m) \leq 1.
\]
The equality is achieved only when \( e_1 + \cdots + e_n = 1, f_1 + \cdots + f_m = 1 \) and all \( \alpha_{ij} = 1 \), in which case \( x = 1 \) in \( H \). Finally, let \( x, y \in H \), \( xy = 0 \).

Let \( E_1 = \{e_1, \ldots, e_n\} \in E(H_1), E_2 = \{f_1, \ldots, f_m\} \in E(H_2) \) cover both \( x \) and \( y \). Let

\[
x = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \alpha_{ij} e_i \otimes f_j, \quad y = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \beta_{ij} e_i \otimes f_j,
\]

then

\[
x + y = \sum_{1 \leq i \leq n, 1 \leq j \leq m} (\alpha_{ij} + \beta_{ij}) e_i \otimes f_j.
\]

From \( xy = 0 \) it follows that \( \alpha_{ij} \beta_{ij} = 0 \) for all \( i, j \) and therefore \( \alpha_{ij} + \beta_{ij} = 0 \) or 1. Hence

\[
r(x + y) = \sum_{1 \leq i \leq n, 1 \leq j \leq m} (\alpha_{ij} + \beta_{ij}) r_1(e_i) r_2(f_j) = r(x) + r(y).
\]

This completes the proof of the Proposition. \qed

It is easy to see that \( H_n \otimes H_m \cong H_{nm} \).

**Lemma 1.** Let \( H_n \subset H_s \). Then there is a subspace \( 1 \in H' \subset H_s \) such that

\[
H_s = H_n H' \cong H_n \otimes H', \quad H' \cong H_{s/n}.
\]

**Proof.** The Boolean algebra \( H_n \) contains \( n \) orthogonal idempotents \( e_1, \ldots, e_n, 1 = \sum_{i=1}^n e_i \), each of them has rang \( 1/n \). Each element \( e_i \) in \( H_s \) has rang \( r/s, 0 \leq r \leq s \). Hence

\[
\frac{1}{n} = \frac{r}{s},
\]

so \( s = nr \), which implies that \( n | s \).

An arbitrary element of \( H_s \) of rang \( r/s \) is a sum of \( r \) orthogonal idempotents, each of rang \( 1/s \). Let

\[
e_i = \sum_{j=1}^{s/n} e_{ij}, \quad i = 1, \ldots, n, \quad e_j' = \sum_{i=1}^{n} e_{ij}, \quad j = 1, \ldots, s/n.
\]

The subalgebra \( H' \) of \( H_s \) generated by \( e_j', 1 \leq j \leq s/n \), has the claimed properties. \qed

**Definition 3.** We say that a Hamming space \((H, r)\) is locally standard if an arbitrary finite collection of elements \( a_1, \ldots, a_n \in H \) is contained in a subspace \( H' \subset H \) that is isomorphic to \( H_m \) for some \( m \geq 1 \).
Example 5. u-periodic Hamming space $\mathcal{H}(u)$ is locally standard for an arbitrary Steinitz number $u$. Indeed, for any finite collection of elements $a_1, \ldots, a_n \in \mathcal{H}(u)$ there exists a positive integer $m$ such that all sequences $a_1, \ldots, a_n$ are periodic with period $m$ and $m|u$. Then $a_1, \ldots, a_n$ are contained in a subspace $\mathcal{H}(u)$ that is isomorphic to $H_m$.

Example 6. Note that the Besicovitch space $(X_B, r_B)$ is not locally standard because we can construct $x = (x_1, x_2, \ldots) \in X_B$ such that $r_B(x)$ is irrational.

Let $\{0, 1\}^\mathbb{N}_p$ be the subset of $\{0, 1\}^\mathbb{N}$ that consists of periodic sequences. Clearly, $\{0, 1\}^\mathbb{N}_p \cap I = \{0\}$ (see Example 3), hence $B_p = \{0, 1\}^\mathbb{N}_p$ can be viewed as Hamming subspace of the Besicovitch space $(B, r_b)$, $B_p = \cup_{u \in \mathbb{N}^\mathbb{N}} \mathcal{H}(u)$.

The Hamming space $B_p$ is locally standard.

Theorem 1. Let $H$ be a locally standard countable Hamming space. Then $H \cong \bigotimes_{i=1}^\infty H_{p_i}$, each $p_i$ is a prime number.

Proof. There exists an ascending chain of subspaces

$$1 \in H_1 \subset H_2 \subset \cdots, \quad \bigcup_{i=1}^\infty H_i = H,$$

and each $H_i$ is isomorphic to a standard Hamming space. By Lemma 1 there exists a standard subspace $H'_i \subset H_{i+1}$, $i \geq 1$, such that $H_i \otimes H'_i \cong H_{i+1}$. Let $H'_1 = H_1$. Then $H \cong \bigotimes_{i=1}^\infty H'_i$. Suppose now that $H'_i \cong H_{n_i}$. If $n_i \geq 2$ and $n_i = p_1 \cdots p_k$ is a prime decomposition of $n_i$ then $H_{n_i} \cong H_{p_1} \otimes \cdots \otimes H_{p_k}$. This implies the assertion of the Theorem. □

Definition 4. Let $H$ be a locally standard Hamming space. Let $D(H) = \{n \geq 1 \mid H' \subset H, H' \cong H_n\}$.

The least common multiple of the set $D(H)$ is called the Steinitz number of $H$ and denoted as $\text{st}(H)$.

Let $H', H''$ be locally standard Hamming spaces. It is easy to see that $H' \otimes H''$ is locally standard and

$$\text{st}(H' \otimes H'') = \text{st}(H') \cdot \text{st}(H'').$$

If $H = \bigotimes_{i=1}^\infty H_{p_i}$ is a decomposition of Theorem 1 then

$$\text{st}(H) = \prod_{i=1}^\infty p_i^{\varepsilon_i},$$
where \( s_i \) is a number of copies of \( H_{p_i} \) in the decomposition of \( H \). In [4] it was shown that countable locally standard Hamming space are isomorphic if and only if \( \text{st}(H') = \text{st}(H'') \). This fact also easily follows from Theorem 1.

Example 7. The Steinitz number of the \( u \)-periodic Hamming space \( H(u) \) is equal to \( u \). Moreover, if \( u = \prod_{i=1}^{\infty} p_i^{s_i} \) then
\[
H(u) = \bigotimes_{i=1}^{\infty} H_{p_i},
\]
where the number of copies of \( H_{p_i} \) in the tensor products \( \bigotimes_{i=1}^{\infty} H_{p_i} \) is equal to \( s_i \).

Example 8. \( \text{st}(B_p) = \prod_{i=1}^{\infty} p_i^{\infty} \), where the product is taken over all prime numbers \( p_i \).

4. Cartan Subalgebras of Locally Matrix Algebras

Let \( \mathbb{F} \) be an algebraically closed field. An associative \( \mathbb{F} \)-algebra \( A \) with a unit 1 is said to be a unital locally matrix algebra (see [10]) if for an arbitrary finite collection of elements \( a_1, \ldots, a_s \in A \) there exists a subalgebra \( A' \subset A \) such that \( 1, a_1, \ldots, a_s \in A' \) and \( A' \cong M_n(\mathbb{F}) \) for some \( n \geq 1 \).

For a unital locally matrix algebra \( A \) let \( D(A) \) be the set of all positive integers \( n \) such that there exists a subalgebra \( A' \subset A \), \( A' \cong M_n(\mathbb{F}) \). The least common multiple of the set \( D(A) \) is called the Steinitz number \( \text{st}(A) \) of the algebra \( A \) (see [1]).

G. Köthe [8] showed that every countable dimensional unital locally matrix algebra is isomorphic to an infinite tensor product of finite dimensional matrix algebras.

J. G. Glimm [7] proved that every countable dimensional unital locally matrix algebra is uniquely determined by its Steinitz number.

For unital locally matrix algebras of uncountable dimensions the theorems above are no longer true (see [1], [2], [10]).

In what follows we consider only countable dimensional locally matrix algebras.

For an element \( a \in A \) choose a subalgebra \( A' \subset A \) such that \( 1, a \in A' \), \( A' \cong M_n(\mathbb{F}) \). Let \( r_{A'}(a) \) be the rang of the matrix \( a \) in \( M_n(\mathbb{F}) \). As shown by Kurochkin [9] (see also [3]) the ratio \( r(a) = \frac{1}{n} r_{A'}(a) \) does not depend on the choice of the subalgebra \( A' \). We call \( r(a) \) the relative rang of the element \( a \). Clearly, \( r(a) = 0 \) if and only if \( a = 0 \). If \( a \) is an idempotent (we call 0 and 1 idempotents as well) then \( r(a) = 1 \) if and only if \( a = 1 \). Moreover, if \( a \) and \( b \) are orthogonal idempotents then \( r(a + b) = r(a) + r(b) \).
Let $C$ be a commutative subalgebra of a locally matrix algebra $A$, $1 \in C$. Let $E(C)$ be the set of all idempotents from $C$ (including 0 and 1). For idempotents $e, f \in E(C)$ let $ef$, $e + f - 2ef$ be their Boolean product and Boolean sum respectively. The Boolean algebra $E(C)$ with the relative rang function $r : E(C) \to [0, 1]$ make $E(C)$ a Hamming space.

A subalgebra $H$ of the matrix algebra $M_n(F)$ is called a Cartan subalgebra if $H \cong F \oplus \ldots \oplus F^\otimes n$, in other words, $H$ is spanned by $n$ pairwise orthogonal idempotents. It is well known that every Cartan subalgebra is conjugate of the diagonal subalgebra of $M_n(F)$.

Let $1 \in A_1 \subset A_2 \subset \ldots$ be an ascending chain of matrix subalgebras such that $A = \bigcup_{i=1}^\infty A_i$. In each $A_i$ choose a Cartan subalgebra $H_i$ so that $1 \in H_1 \subset H_2 \subset \ldots$. We call

$$H = \bigcup_{i=1}^\infty H_i$$

a general Cartan subalgebra of $A$. As above, $r : A \to [0, 1]$ is a relative rang function. Then $(E(H), r)$ is a locally standard Hamming space.

A subalgebra $H \subset A$ is called a Cartan subalgebra if there exists a decomposition $A = \otimes_{i=1}^\infty A_i$ into a product of finite dimensional matrix algebras and Cartan subalgebras $H_i$ in $A_i$ such that $H = \otimes_{i=1}^\infty H_i$.

Theorem 2. Any two Cartan subalgebras of $A$ are conjugate via an automorphism of $A$.

Proof. Let $H', H''$ be Cartan subalgebras corresponding to tensor decompositions

$$A \cong \otimes_{i=1}^\infty M_{n_i}(F), \quad A \cong \otimes_{i=1}^\infty M_{m_i}(F), \quad H' = \otimes_{i=1}^\infty H'_i, \quad H'' = \otimes_{i=1}^\infty H''_i,$$

where $H'_i, H''_i$ are Cartan subalgebras in $M_{n_i}(F), M_{m_i}(F)$ respectively. Without loss of generality we can assume that all integers $n_i, m_i$ are prime. From

$$\text{st}(A) = \prod_{i=1}^\infty n_i = \prod_{i=1}^\infty m_i$$

it follows that up to renumeration we can assume $n_i = m_i$. There exist automorphisms $\varphi_i \in \text{Aut} M_{n_i}(F)$ such that $\varphi_i(H'_i) = H''_i$. Now $H'$ and $H''$ are conjugate via the automorphism $\varphi = \otimes_{i=1}^\infty \varphi_i$. \hfill \Box

It easy to see that a Cartan subalgebra is a general Cartan subalgebra. The reverse statement is not true. In particular, not all general Cartan subalgebras are conjugate.
Theorem 3. In an arbitrary countable dimensional locally matrix algebra there exists a general Cartan subalgebra that is not a Cartan subalgebra.

Let $A^*$ denote the group of invertible elements of the algebra $A$.

Lemma 2. If $H \subset A$ is a Cartan subalgebra then there exists an element $x \in A^* \setminus H$ such that $x^{-1}Hx = H$.

Proof. Let $A = A_1 \otimes A_2$, $A_1 \cong M_n(F)$, $n \geq 2$; $A_2$ is a locally matrix algebra. Let $H_1$, $H_2$ be Cartan subalgebras of the algebras $A_1$, $A_2$ respectively, $H = H_1 \otimes H_2$. There exists an element $a \in A_1^* \setminus H_1$ such that $a^{-1}H_1a = H_1$. Now it remains to choose $x = a \otimes 1$. □

Proof of Theorem 3. Let $A$ be a countable dimensional locally matrix algebra. In view of Lemma 2 it is sufficient to find a general Cartan subalgebra $H$ such that for an arbitrary invertible element $x \in A^*$ either $x \in H$ or $x^{-1}Hx \neq H$.

Choose an ascending chain of matrix subalgebras of algebra $A$ such that

$$1 \in A_1 \subset A_2 \subset \cdots, \cup_{i=1}^\infty A_i = A, A_i \cong M_{n_i}(F), \ n_i \geq 2.$$

In the case of a finite field $F$ we assume also that $n_i^2 \leq n_{i+1}$. We will use induction to construct an ascending chain of Cartan subalgebras $H_k \subset A_k$, $H_k \subset H_{k+1}$, $k \geq 1$.

The Cartan subalgebra $H_1 \subset A_1$ is selected arbitrary. Suppose that Cartan subalgebras $H_1 \subset H_2 \subset \cdots \subset H_k$ have been selected.

The algebra $H_k$ is isomorphic to a direct sum of $n_k$ copies of the field $F$. Let $e_1, \ldots, e_{n_k}$ be pairwise orthogonal idempotents, $H_k = F e_1 \oplus \cdots \oplus F e_{n_k}$.

Let $A_k'$ be the centraliser of the subalgebra $A_k$ in $A_{k+1}$. Then

$$A_k' \cong M_{n_{k+1}/n_k}(F) \text{ and } A_{k+1} \cong A_k \otimes_F A_k'.$$

If the field $F$ is infinite then the algebra $A_k'$ contains infinitely many distinct Cartan subalgebras. If the field $F$ is finite then $A_k'$ contains at least $n_k$ distinct Cartan subalgebras. In any case we choose distinct Cartan subalgebras $H_1', \ldots, H_{n_k}'$ of $A_k'$. Let

$$H_{k+1} = e_1 \otimes H_1' + \cdots + e_{n_k} \otimes H_{n_k}' .$$

It is easy to see that $H_{k+1}$ is a Cartan subalgebra of $A_{k+1}$. Since every $H_i'$ contains the identity element of $A_k'$ it follows that $H_k \subset H_{k+1}$. It is also easy to see that $H_{k+1} \cap A_k = H_k$.

The union $H = \cup_{k=1}^\infty H_k$ is a general Cartan subalgebra of $A$. For an arbitrary $k \geq 1$ we have $H \cap A_k = H_k$. 

□
Let $x \in A^*$ be an invertible element. There exists $k \geq 1$ such that $x \in A_k$. If $x \in H_k$ or $x^{-1}H_k x \neq H_k$ then we are done.

Suppose that $x \in A_k \setminus H_k$ and $x^{-1}H_k x = H_k$. Then $x^{-1}e_i x = e_{\pi(i)}$, where $\pi$ is a permutation on $1, 2, \ldots, n$. If $\pi = 1$ then $x$ lies in the centralizer of $H_k$, hence $x \in H_k$, which contradicts our assumption. Therefore $\pi \neq 1$.

Now $x^{-1}H_{k+1} x = \sum_{i=1}^{n_k} e_{\pi(i)} \otimes H_i' \neq H_{k+1}$

since all Cartan subalgebras $H_1', \ldots, H_{n_k}'$ are distinct. This implies $x^{-1} H x \neq H$ and completes the proof of the Theorem.

\[ \square \]

**Theorem 4.**

1. An arbitrary countable locally standard Hamming space $S$ is isomorphic to $E(H)$, where $H$ is a Cartan subalgebra of a countable dimensional locally matrix algebra $A$, $\text{st}(S) = \text{st}(A)$.

2. Let $A_1, A_2$ be locally matrix algebras with Cartan subalgebras $H_1, H_2$ respectively. The Hamming spaces $E(H_1), E(H_2)$ are isomorphic if and only if $A_1 \cong A_2$.

**Proof.** By Theorem 3 $S \cong \otimes_{i=1}^{\infty} H_{p_i}$, where each $p_i$ is a prime number. Consider the corresponding matrix algebras $M_{p_i}(F)$ and their diagonal Cartan subalgebras $D_i$. Let

$$A = \otimes_{i=1}^{\infty} M_{p_i}(F).$$

Then $D = \otimes_{i=1}^{\infty} D_i$ is a Cartan subalgebra of $A$. We have

$$E(D) = \otimes_{i=1}^{\infty} E(D_i) \cong \otimes_{i=1}^{\infty} H_{p_i} \cong S.$$ 

This proves the part (1).

If $E(H_1) \cong E(H_2)$ then $\text{st}(A_1) = \text{st}(E(H_1)) = \text{st}(E(H_2)) = \text{st}(A_2)$. By Glimm’s Theorem $[7] A_1 \cong A_2$. This completes the proof of the Theorem. \[ \square \]

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