A HOMOTOPY FOR A COMPLEX OF FREE LIE ALGEBRAS

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Abstract

Using the Guichardet construction, we compute the cohomology groups of a complex of free Lie algebras introduced by Alekseev and Torossian.

Introduction

In their study of the relation between the KV-conjecture and Drinfeld’s associators, Alekseev and Torossian [1] studied the Eilenberg-MacLane differential $\delta_A : L_n \to L_{n+1}$ where $L_n$ is the free Lie algebra in $n$ variables, and computed the cohomology groups of $\delta_A$ in dimensions 1, 2. Following the construction of Guichardet [2] (see also [3]), we remark that the complex $\delta_A$ is acyclic, except in dimensions 1, 2, where the cohomology is of dimension 1. We also identify the cohomology groups of a similar complex $\delta_A : T_n \to T_{n+1}$ where $T_n$ is the free associative algebra in $n$ variables: the cohomology is of dimension 1 in any degree. The Guichardet construction provides an explicit homotopy.

Alekseev and Torossian used the computations in dimension 2 to deduce the existence of a solution to the KV problem from the existence of an associator. A simple by-product of this computation is the existence and the uniqueness of the Campbell-Hausdorff formula. We do not have any other application of the computations of higher cohomologies.

In this note, we start with a review of the construction of Guichardet. Then we adapt it to free associative algebras and free Lie algebras.

I am thankful to the referee for his careful reading.

1. The Guichardet construction.

Let $V$ be a finite dimensional real vector space. Let $F^n$ be the space of polynomial functions $f$ on $V \oplus V \oplus \cdots \oplus V$. An element $f$ of $F^n$ is written as $f(v_1, v_2, \ldots, v_n)$.

Define

$$(\delta_n f)(v_1, \ldots, v_{n+1}) = \sum_{i=1}^{n} (-1)^i f(v_1, v_2, \ldots, v_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_n).$$
For example:
\[(\delta_1 f)(v_1, v_2) = -f(v_2) + f(v_1)\]
\[(\delta_2 f)(v_1, v_2, v_3) = -f(v_2, v_3) + f(v_1, v_3) - f(v_1, v_2).\]

We define \(F^0 = \mathbb{R}\), and embed \(F^0 \to F^1\) as the constant functions.

The complex \(0 \to F^0 \to F^1 \to \cdots\) is acyclic except in degree 0. Indeed \(s : F^n \to F^{n-1}\) given by
\[(sf)(v_1, v_2, \ldots, v_{n-1}) = f(0, v_1, v_2, \ldots, v_{n-1})\]
satisfies \(\text{Id} := s\delta + \delta s\).

Now the additive group \(V\) operates on \(F^n\) by translations: if \(\alpha \in V\), we write
\[(\tau(\alpha)f)(v_1, \ldots, v_n) = f(v_1 - \alpha, \ldots, v_n - \alpha).\]
The differential \(\delta\) commutes with translations, so that it induces a differential \(\delta_A\) on the subspace of translation invariant functions.

It is well known that the cohomology of the complex \(\delta_A\) is isomorphic with \(\Lambda^{n-1}V^*\). Here we recall Guichardet’s explicit construction of the isomorphism as we will adapt it to the ”universal case” considered by Alekseev-Torossian.

Let \(\Omega^{n-1}\) be the space of differential forms of exterior degree \(n-1\) on \(V\), with polynomial coefficients, equipped with the de Rham differential.

Consider the simplex \(S := S_{v_1, v_2, \ldots, v_n}\) in \(V\) with vertices \((v_1, v_2, \ldots, v_n)\). Thus the map \(\Omega^{n-1} \to F^n\) defined by \(\omega \to \int_S \omega\) induces a map from \(\Omega^{n-1}\) to \(F^n\). This map commutes with the differentials (as follows from Stokes formula) and with the natural action by translations.

Conversely, associate to \(f \in F^n\) a differential form \(\omega(f)\) of degree \((n-1)\) by setting for \(v_1, v_2, \ldots, v_{n-1}\) vectors in \(V\), identified with tangent vectors at \(v \in V\):
\[
\langle \omega(f)(v), v_1 \wedge v_2 \wedge \cdots \wedge v_{n-1} \rangle = \sum_{\sigma \in \Sigma_{n-1}} \epsilon(\sigma) \left| \frac{d}{d\epsilon} \right|_{\epsilon=0} f(v, v+\epsilon v_{\sigma(1)}, \ldots, v+\epsilon_{n-1}v_{\sigma(n-1)}).
\]

Here if \(\phi\) is a polynomial function of \(\epsilon_1, \ldots, \epsilon_{n-1}\), we employ the notation \(\left| \frac{d}{d\epsilon} \right|_{\epsilon=0} \phi(\epsilon)\) for the coefficient of \(\epsilon_1 \cdots \epsilon_{n-1}\) in \(\phi\).

The map \(\omega\) commutes with the differential, and with the action of \(V\) by translations. Thus the map \(P_n : F^n \to F^n\) defined by
\[
P_n(f) = \int_S \omega(f)
\]
produces a map from \(F^n \to F^n\), commuting with the action of \(V\). This map is the identity on \(F^1\).

Let us give the formulae for \(P_n\) so that we see that the map \(P_n\) is “universal”.

Given \( v := (v_1, v_2, \ldots, v_n) \in V \), consider the map \( p_v : \mathbb{R}^{n-1} \to V \) given by
\[
p_v(t_1, t_2, \ldots, t_{n-1}) = v_1 + t_1(v_2 - v_1) + \cdots + t_{n-1}(v_n - v_1).
\]
This map sends the standard simplex \( \Delta_{n-1} \) defined by
\[
t_i \geq 0, \sum_{i=1}^{n-1} t_i \leq 1
\]
to the simplex \( S \) in \( V \) with vertices \( v_1, v_2, \ldots, v_n \).

Let us consider the form
\[
p_v^* \omega(f) = f(t, v) dt_1 \wedge \cdots \wedge dt_{n-1}.
\]

The map \( P_n \) is given by
\[
(P_n f)(v) = \int_{\Delta_{n-1}} f(t, v) dt
\]
where \( f(t, v) \) is the element of \( F_n \) depending on \( t \) described as follows.

Lemma 1.1. Let
\[
v(t) = v_1 + t_1(v_2 - v_1) + \cdots + t_{n-1}(v_n - v_1).
\]
Define
\[
(2) \quad f(t, v_1, v_2, \ldots, v_n) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \sum_{\sigma \in \Sigma[\{2, \ldots, n\}]} \epsilon(\sigma)f(v(t), v(t) + \epsilon v_\sigma(2) - v_1), \ldots, v(t) + \epsilon v_{n-1}(v_\sigma(n) - v_1)).
\]

Here \( t = (t_1, t_2, \ldots, t_{n-1}) \) and \( \Sigma[\{2, \ldots, n\}] \) is the group of permutations of the set with \((n-1)\) elements \([2, \ldots, n]\).

Then we have the formula
\[
(P_n f)(v_1, v_2, \ldots, v_n) = \int_{\Delta_{n-1}} f(t, v) dt_1 dt_2 \cdots dt_{n-1}.
\]

Let \( H := \text{Id} - P \). Using the injectivity of the vector spaces \( F^n \) in the category of \( V \)-modules, it is standard, and we will review the procedure below, to produce a homotopy
\[
G : F^n \to F^{n-1}
\]
commuting with the action of \( V \) by translations and such that:
\[
H = G\delta + \delta G.
\]

We first use the following injectivity lemma.

Lemma 1.2. Let \( A, B \) be two real vector spaces provided with a structure of \( V \)-modules. Let \( u : A \to F^n \) be a \( V \)-module map from \( A \) to \( F^n \). Let \( v : A \to B \) be an injective map of \( V \)-modules, Then there exists a map \( w : B \to F^n \) of \( V \)-modules extending \( u \).
The formula for a map \( w \) (depending on a choice of retraction) is given below in the proof:

**Proof.** Denote by \( \tau \) the action of \( V \) on \( B \). Let \( s \) be a linear map from \( B \) to \( A \) such that \( su = \text{Id} \). Let \( b \in B \): we define the map \( w \) (depending on our choice of linear retraction \( s \)) by

\[
w(b)(v_1, v_2, \ldots, v_n) = u(s\tau(-v_1)b)(0, v_2 - v_1, \ldots, v_n - v_1).
\]

We verify that \( b \) satisfy the wanted conditions. The crucial point is that the map \( w \) is a map of \( V \)-modules, as we now show. Indeed

\[
w(\tau(v_0)b)(v_1, v_2, \ldots, v_n) = u(s(\tau(-v_1)\tau(v_0)b))(0, v_2 - v_1, \ldots, v_n - v_1)
= u(s(\tau(-v_1 + v_0)b))(0, v_2 - v_1, \ldots, v_n - v_1)
\]

while

\[
(\tau(v_0)w(b))(v_1, v_2, \ldots, v_n) = w(b)(v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0)
= u(\tau(v_0 - v_1)b)(0, v_2 - v_1, \ldots, v_n - v_1).
\]

\( \square \)

We now apply this lemma to define \( G \) inductively. Consider the injective map deduced from \( \delta \) from \( F^n/\delta(F^{n-1}) \) to \( F^{n+1} \).

Recall our linear map \( s : F^{n+1} \to F^n \) given by Equation (1). We may take as linear inverse (that we still call \( s \)) the map \( s : F^{n+1} \to F^n \) followed by the projection \( F^n \to F^n/\delta(F^{n-1}) \).

We define \( G^1 = 0 \) and inductively \( G^{n+1} \) as the map extending

\[
H^n - \delta G^n : F^n \to F^n
\]

to \( F^{n+1} \) constructed in Lemma 1.2. Indeed \( (H^n - \delta G^n)\delta = \delta H^{n-1} - \delta(-\delta G^{n-1} + H^{n-1}) = 0 \) so that the map \( H^n - \delta G^n \) produces a map from \( F^n/\delta(F^{n-1}) \to F^n \) and we use the fact that \( F^n/\delta(F^{n-1}) \) is embedded in \( F^{n+1} \) via \( \delta \) with inverse \( s \).

More precisely, given \( v_1 \) and \( f \in F^{n+1} \), we define the function \( \phi \) of \( n \) variables given by

\[
\phi(w_1, w_2, \ldots, w_n) = f(v_1, v_1 + w_1, \ldots, v_1 + w_n)
\]

and define

\[
(G^{n+1}f)(v_1, v_2, \ldots, v_n) = ((H^n - \delta G^n)\phi)(0, v_2 - v_1, \ldots, v_n - v_1).
\]

For example, this leads to the following formulae for the first elements \( G^i \).

We have \( G^1 = 0, G^2 = 0 \).

\[
(G^3f)(v_1, v_2) = f(v_1, v_1, v_2) - \int_0^1 \frac{d}{t} \left| f(v_1, v_1 + t(v_2 - v_1), v_1 + t(v_2 - v_1) + \epsilon(v_2 - v_1)) \right| dt.
\]

\[
(G^4f)(v_1, v_2, v_3) = G_0^4 + G_1^4 + G_2^4
\]
with
\[(G^4_0 f)(v_1, v_2, v_3) = f(v_1, v_1, v_2, v_3) - f(v_1, v_2, v_2, v_3) + f(v_1, v_1, v_1, v_3) - f(v_1, v_1, v_1, v_2),\]

\[(G^4_1 f)(v_1, v_2, v_3) = \int_{t=0}^{1} \frac{d}{dt} [0f(v_1, v_2, v_2 + t(v_3 - v_2), v_2 + (t + \epsilon)(v_3 - v_2)),
\]

\[- \int_{t=0}^{1} \frac{d}{dt} [0f(v_1, v_1, v_1 + t(v_3 - v_1), v_1 + (t + \epsilon)(v_3 - v_1))]
\]

\[+ \int_{t=0}^{1} \frac{d}{dt} [0f(v_1, v_1, v_1 + t(v_2 - v_1), v_1 + (t + \epsilon)(v_2 - v_1)).\]

\[(G^4_2 f)(v_1, v_2, v_3) = - \int_{t \in S^2} \frac{d}{dt} [0f(v_1, V(t), V(t + \epsilon_1), V(t + \epsilon_2)]
\]

\[- \int_{t \in \Delta_2} \frac{d}{dt} [0f(v_1, V(t), V(t + \epsilon_2), V(t + \epsilon_1)).\]

Here \(t = [t_1, t_2], t + \epsilon_1 = [t_1 + \epsilon_1, t_2], t + \epsilon_2 = [t_1, t_2 + \epsilon_2], V(t) = v_1 + t_1(v_2 - v_1) + t_2(v_3 - v_1)\), and \(\Delta_2 := \{[t_1, t_2], t_1 \geq 0, t_2 \geq 0; t_1 + t_2 \leq 1\} \).

Let us now consider the action of \(V\) by translations on the complex \(F^n\). The differential \(\delta\) induces a differential \(\delta_A : F^n_A \rightarrow F^n_A\) on the subspaces of invariants. We identify the space \(F^n_A\) with \(F^{n-1}\) by the map
\[R : F^{n-1} \rightarrow F^n_A\]

given by
\[(Rf)(v_1, v_2, \ldots, v_n) = f(v_2 - v_1, v_3 - v_2, \ldots, v_n - v_{n-1}).\]

Then the differential \(\delta_A\) induced by \(\delta\) becomes the Eilenberg-MacLane differential
\[(\delta_A f)(v_1, v_2, \ldots, v_{n-1})\]

\[= f(v_2, v_3, \ldots, v_{n-1}) - f(v_1 + v_2, v_3, \ldots, v_{n-1}) + f(v_1, v_2 + v_3, \ldots, v_{n-1}) + \cdots
\]

\[+ (-1)^{n-2} f(v_1, v_2, \ldots, v_{n-2} + v_{n-1}) + (-1)^{n-1} f(v_1, v_2, \ldots, v_{n-1}).\]

The map \(P : F^n \rightarrow F^n\) also commutes with translations.

**Lemma 1.3.** We have \(PR = R\text{Ant}\) where Ant is the anti-symmetrization operator of \(F^{n-1}\) on the space of \(\Lambda^{n-1}V^*\) of antisymmetric functions \(f(v_1, v_2, \ldots, v_{n-1}).\)

**Proof.** To compute \(P\), we have to compute
\[v(t) = v_1 + t_1(v_2 - v_1) + \cdots + t_{n-1}(v_{n-1} - v_1)\]

and
\[f(t, v_1, v_2, \ldots, v_{n-1})\]
we can define $T$ of $n$. We may identify it with $\delta \sigma f(y, 1, 2, \ldots, n, y) + \delta \sigma f(0, 1, 2, \ldots, n, y_1)$. It follows that we obtain on the complex $h$ the relation $G_A \delta_A + \delta_A G_A = \text{Id} - \text{Ant}$.

We thus obtain that the cohomology of the operator $\delta_A$ is isomorphic in degree $n$ to $\Lambda^{n-1} V^*$.

2. Free variables

Let $T_n$ be the free associative algebra in $n$ variables. We consider $L_n \subset T_n$ as the free Lie algebra in $n$ variables. An element $f$ of $T_n$ is written as $f(x_1, x_2, \ldots, x_n)$.

Define

$$(\delta_n f)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n} (-1)^i f(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n).$$

Consider $T_n(y)$ the free associative algebra generated by $(x_1, x_2, \ldots, x_n, y)$. An operator $h$ on $T_n$ is extended by an operator still denoted by $h$ on $T_n(y)$ where we do not operate on $y$.

We may consider the application $\tau : T_n \rightarrow T_n(y)$ defined by

$$(\tau_n f)(x_1, \ldots, x_n) = f(x_1 + y, x_2 + y, \ldots, x_n + y).$$

The application $\tau$ commutes with $\delta$. Thus the kernel of $\tau$ is a subcomplex of $T_n$. We may identify it with $T_{n-1}$ by $(R\sigma f)(x_1, x_2, \ldots, x_n) = f(x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1})$ and we obtain on $T_n$ the complex $\delta_A$ considered by Alekseev-Torossian. Here

$$(\deltaAf)(x_1, x_2, \ldots, x_{n-1})$$

$$= f(x_2, x_3, \ldots, x_{n-1}) - f(x_1 + x_2, x_3, \ldots, x_{n-1}) + f(x_1, x_2 + x_3, \ldots, x_{n-1}) + \cdots + (-1)^{n-2} f(x_1, x_2, \ldots, x_{n-2} + x_{n-1}) + (-1)^{n-1} f(x_1, x_2, \ldots, x_{n-1}).$$

It is clear that the complex $\delta : 0 \rightarrow T_0 \rightarrow T_1 \rightarrow T_2 \cdots$ is acyclic. Indeed we can define

$$(sf)(x_1, x_2, \ldots, x_n) = f(0, x_1, x_2, \ldots, x_n)$$

and it is immediate to verify that

$$s \delta + \delta s = \text{Id}.$$
If \( f \in T_n \), we define a function \( f(t, x) \in \mathbb{R}[t] \otimes T_k \) by the same formula as Formula (2):

**Definition 2.1.** Let
\[
x(t) = x_1 + t_1(x_2 - x_1) + \cdots + t_{n-1}(x_n - x_1).
\]

Define
\[
f(t, x_1, x_2, \ldots, x_n) = \frac{d}{dt} \bigg|_0 \sum_{\sigma \in \Sigma([2, \ldots, n])} \epsilon(\sigma) f(x(t), x(t) + \epsilon_1(x_{\sigma(2)} - x_1), \ldots, x(t) + \epsilon_{n-1}(x_{\sigma(n)} - x_1)).
\]

Define
\[
(P_nf)(x_1, x_2, \ldots, x_n) = \int_{\Delta_{n-1}} f(t, x_1, x_2, \ldots, x_n) dt_1 dt_2 \cdots dt_{n-1}.
\]

The following lemma is immediate.

**Lemma 2.2.** We have \( \delta P_n = P_n \delta \).

We define \( G^1 = 0 \) and inductively \( G^{n+1} \) by the same formula as before. More precisely, given \( f \in T^{n+1} \), we define the function \( \phi \) of \( T^n(x_1) \) given by
\[
\phi(w_1, w_2, \ldots, w_n) = f(x_1, x_1 + w_1, \ldots, x_1 + w_n)
\]
and define
\[
(G^{n+1}f)(x_1, x_2, \ldots, x_n) = ((H^n - \delta G^n)\phi)(0, x_2 - x_1, \ldots, x_n - x_1).
\]

Then we conclude as before that \( G\delta + \delta G = \text{Id} - P \). Restricting to the invariants, we obtain a map \( G_A \) such that \( \text{Id} - \text{Ant} = G_A \delta_A + G_A \delta_A \). Here Ant is the anti-symmetrization operator \( \sum_{\sigma} \epsilon(\sigma)x_{\sigma(1)} \cdots x_{\sigma(n)} \).

The subspace \( L_n \) of \( T_n \) is stable under the differential. The operator Ant is equal to 0 on \( L_n \), except in degree 1, 2, as there are no totally antisymmetric elements in \( L_n \) for \( n \geq 3 \). Thus we obtain

**Theorem 2.3.** • The cohomology groups \( H^n(T_n, \delta_A) \) of the complex \( \delta_A : T_n \to T_n \) are of dimension 1 and are generated by \( \sum_{\sigma} \epsilon(\sigma)x_{\sigma(1)} \cdots x_{\sigma(n)} \).
• The cohomology groups \( H^n(L_n, \delta_A) \) of the complex \( \delta_A : L_n \to L_n \) are of dimension 0 if \( n > 2 \). For \( n = 1, 2 \),
\[
H^1(L_1, \delta_A) = \mathbb{R}x_1, \quad H^2(L_2, \delta_A) = \mathbb{R}[x_1, x_2].
\]

Remark: The Guichardet construction also provides an explicit homotopy.

**References**

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Introduction

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Define

$$(\delta_n f)(v_1, \ldots, v_{n+1}) = \sum_{i=1}^{n} (-1)^i f(v_1, v_2, \ldots, v_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_n).$$
For example:

\[(\delta_1 f)(v_1, v_2) = -f(v_2) + f(v_1)\]
\[(\delta_2 f)(v_1, v_2, v_3) = -f(v_2, v_3) + f(v_1, v_3) - f(v_1, v_2).\]

We define \(F^0 = \mathbb{R}\), and embed \(F^0 \to F^1\) as the constant functions.

The complex \(0 \to F^0 \to F^1 \to \cdots\) is acyclic except in degree 0. Indeed \(s : F^n \to F^{n-1}\) given by

\[(sf)(v_1, v_2, \ldots, v_{n-1}) = f(0, v_1, v_2, \ldots, v_{n-1})\]
satisfies \(\text{Id} := s\delta + \delta s\).

Now the additive group \(V\) operates on \(F^n\) by translations: if \(\alpha \in V\), we write

\[(\tau(\alpha)f)(v_1, \ldots, v_n) = f(v_1 - \alpha, \ldots, v_n - \alpha).\]

The differential \(\delta\) commutes with translations, so that it induces a differential \(\delta_A\) on the subspace of translation invariant functions.

It is well known that the cohomology of the complex \(\delta_A\) is isomorphic with \(\Lambda^{n-1}V^*\). Here we recall Guichardet’s explicit construction of the isomorphism as we will adapt it to the "universal case" considered by Alekseev-Torossian.

Let \(\Omega^{n-1}\) be the space of differential forms of exterior degree \(n-1\) on \(V\), with polynomial coefficients, equipped with the de Rham differential.

Consider the simplex \(S := S_{v_1, v_2, \ldots, v_n}\) in \(V\) with vertices \((v_1, v_2, \ldots, v_n)\). Thus the map \(\Omega^{n-1} \to F^n\) defined by \(\omega \to \int_S \omega\) induces a map from \(\Omega^{n-1}\) to \(F^n\). This map commutes with the differentials (as follows from Stokes formula) and with the natural action by translations.

Conversely, associate to \(f \in F^n\) a differential form \(\omega(f)\) of degree \((n-1)\) by setting for \(v_1, v_2, \ldots, v_{n-1}\) vectors in \(V\), identified with tangent vectors at \(v \in V\):

\[\langle \omega(f)(v), v_1 \wedge v_2 \wedge \cdots \wedge v_{n-1} \rangle = \sum_{\sigma \in \Sigma_{n-1}} \epsilon(\sigma) \frac{d}{d\epsilon} |_{\epsilon=0} f(v, v + \epsilon_1 v_{\sigma(1)}, \ldots, v + \epsilon_{n-1} v_{\sigma(n-1)}).\]

Here if \(\phi\) is a polynomial function of \(\epsilon_1, \ldots, \epsilon_{n-1}\), we employ the notation \(\frac{d}{d\epsilon} |_{\epsilon=0} \phi(\epsilon)\) for the coefficient of \(\epsilon_1 \cdots \epsilon_{n-1}\) in \(\phi\).

The map \(\omega\) commutes with the differential, and with the action of \(V\) by translations. Thus the map \(P_n : F^n \to F^n\) defined by

\[P_n(f) = \int_S \omega(f)\]

produces a map from \(F^n \to F^n\), commuting with the action of \(V\). This map is the identity on \(F^1\).

Let us give the formulae for \(P_n\) so that we see that the map \(P_n\) is “universal”.
Given \( v := (v_1, v_2, \ldots, v_n) \in V \), consider the map \( p_v : \mathbb{R}^{n-1} \to V \) given by

\[
p_v(t_1, t_2, \ldots, t_{n-1}) = v_1 + t_1(v_2 - v_1) + \cdots + t_{n-1}(v_n - v_1).
\]

This map sends the standard simplex \( \Delta_{n-1} \) defined by

\[
t_i \geq 0, \sum_{i=1}^{n-1} t_i \leq 1
\]

to the simplex \( S \) in \( V \) with vertices \( v_1, v_2, \ldots, v_n \).

Let us consider the form

\[
p^*_v \omega(f) = f(t, v) dt_1 \wedge \cdots \wedge dt_{n-1}.
\]

The map \( P_n \) is given by

\[
(P_n f)(v) = \int_{\Delta_{n-1}} f(t, v) dt
\]

where \( f(t, v) \) is the element of \( F_n \) depending on \( t \) described as follows.

**Lemma 1.1.** Let

\[
v(t) = v_1 + t_1(v_2 - v_1) + \cdots + t_{n-1}(v_n - v_1).
\]

Define

\[
f(t, v_1, v_2, \ldots, v_n) = \frac{d}{d\epsilon}|_{\epsilon=0} \sum_{\sigma \in \Sigma([2, \ldots, n])} \epsilon(\sigma)f(v(t), v(t)+\epsilon_1(v_{\sigma(2)}-v_1), \ldots, v(t)+\epsilon_{n-1}(v_{\sigma(n)}-v_1)).
\]

Here \( t = (t_1, t_2, \ldots, t_{n-1}) \) and \( \Sigma([2, \ldots, n]) \) is the group of permutations of the set with \( (n-1) \) elements \([2, \ldots, n]\).

Then we have the formula

\[
(P_n f)(v_1, v_2, \ldots, v_n) = \int_{\Delta_{n-1}} f(t, v) dt_1 dt_2 \cdots dt_{n-1}.
\]

Let \( H := \text{Id} - P \). Using the injectivity of the vector spaces \( F^n \) in the category of \( V \)-modules, it is standard, and we will review the procedure below, to produce a homotopy

\[
G : F^n \to F^{n-1}
\]

commuting with the action of \( V \) by translations and such that:

\[
H = G\delta + \delta G.
\]

We first use the following injectivity lemma.

**Lemma 1.2.** Let \( A, B \) be two real vector spaces provided with a structure of \( V \)-modules. Let \( u : A \to F^n \) be a \( V \)-module map from \( A \) to \( F^n \). Let \( v : A \to B \) be an injective map of \( V \)-modules, then there exists a map \( w : B \to F^n \) of \( V \)-modules extending \( u \).
The formula for a map $w$ (depending on a choice of retraction) is given below in the proof:

**Proof.** Denote by $\tau$ the action of $V$ on $B$. Let $s$ be a linear map from $B$ to $A$ such that $sv = \text{Id}$. Let $b \in B$: we define the map $w$ (depending on our choice of linear retraction $s$) by

$$w(b)(v_1, v_2, \ldots, v_n) = u(s\tau(-v_1)b)(0, v_2 - v_1, \ldots, v_n - v_1).$$

We verify that $b$ satisfy the wanted conditions. The crucial point is that the map $w$ is a map of $V$-modules, as we now show. Indeed

$$w(\tau(v_0)b)(v_1, v_2, \ldots, v_n) = u(s(\tau(-v_1)\tau(v_0)b))(0, v_2 - v_1, \ldots, v_n - v_1)$$

while

$$(\tau(v_0)w(b))(v_1, v_2, \ldots, v_n) = w(b)(v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0)$$

We now apply this lemma to define $G$ inductively. Consider the injective map deduced from $\delta$ from $F^n/\delta(F^{n-1})$ to $F^{n+1}$.

Recall our linear map $s : F^{n+1} \rightarrow F^n$ given by Equation (1). We may take as linear inverse (that we still call $s$) the map $s : F^{n+1} \rightarrow F^n$ followed by the projection $F^n \rightarrow F^n/\delta(F^{n-1})$.

We define $G^1 = 0$ and inductively $G^{n+1}$ as the map extending

$$H^n - \delta G^n : F^n \rightarrow F^n$$

$F^{n+1}$ constructed in Lemma 1.2. Indeed $(H^n - \delta G^n)\delta = \delta H^{n-1} - \delta(-\delta G^{n-1} + H^{n-1}) = 0$ so that the map $H^n - \delta G^n$ produces a map from $F^n/\delta(F^{n-1}) \rightarrow F^n$ and we use the fact that $F^n/\delta(F^{n-1})$ is embedded in $F^{n+1}$ via $\delta$ with inverse $s$.

More precisely, given $v_1$ and $f \in F^{n+1}$, we define the function $\phi$ of $n$ variables given by

$$\phi(w_1, w_2, \ldots, w_n) = f(v_1, v_1 + w_1, \ldots, v_1 + w_n)$$

and define

$$(G^{n+1}f)(v_1, v_2, \ldots, v_n) = ((H^n - \delta G^n)\phi)(0, v_2 - v_1, \ldots, v_n - v_1).$$

For example, this leads to the following formulae for the first elements $G^i$.

We have $G^1 = 0, G^2 = 0$.

$$(G^3f)(v_1, v_2) = f(v_1, v_1, v_2) - \int_0^1 \frac{d}{dt}|_0 f(v_1, v_1 + t(v_2 - v_1), v_1 + t(v_2 - v_1) + \epsilon(v_2 - v_1))dt.$$  

$$(G^4f)(v_1, v_2, v_3) = G^4_0 + G^4_1 + G^4_2.$$
Proof.

To compute $f$ we have

Lemma 1.3. The differential $\delta$ induces a differential $\delta_A : F^n \to F^n$ on the subspaces of invariants. We identify the space $F_A^n$ with $F^{n-1}$ by the map

$$R : F^{n-1} \to F^n$$

given by

$$(Rf)(v_1, v_2, \ldots, v_n) = f(v_2 - v_1, v_3 - v_2, \ldots, v_n - v_{n-1}).$$

Then the differential $\delta_A$ induced by $\delta$ becomes the Eilenberg-MacLane differential

$$(\delta_A f)(v_1, v_2, \ldots, v_{n-1})$$

$$= f(v_2, v_3, \ldots, v_{n-1}) - f(v_1 + v_2, v_3, \ldots, v_{n-1}) + f(v_1, v_2 + v_3, \ldots, v_{n-1}) + \cdots + (-1)^{n-2} f(v_1, v_2, \ldots, v_{n-2} + v_{n-1}) - (-1)^{n-1} f(v_1, v_2, \ldots, v_{n-1}).$$

The map $P : F^n \to F^n$ also commutes with translations.

Lemma 1.3. We have $PR = R\text{Ant}$ where Ant is the anti-symmetrization operator of $F^{n-1}$ on the space of $\Lambda^{n-1} V^*$ of antisymmetric functions $f(v_1, v_2, \ldots, v_{n-1})$.

Proof. To compute $P$, we have to compute

$$v(t) = v_1 + t_1 (v_2 - v_1) + \cdots + t_{n-1} (v_{n-1} - v_1)$$

and

$$f(t, v_1, v_2, \ldots, v_{n-1})$$
we can define $T$ of and it is immediate to verify that $n$ as the free Lie algebra in $y$ where we do not operate on $x$.

It follows that we obtain on the complex of invariants. It follows that we obtain on the complex $\delta_A$ the relation $G_A \delta_A + \delta_A G_A = \text{Id} - \text{Ant}$.

We thus obtain that the cohomology of the operator $\delta_A$ is isomorphic in degree $n$ to $\Lambda^{n-1} V^*$.

2. Free variables

Let $T_n$ be the free associative algebra in $n$ variables. We consider $L_n \subset T_n$ as the free Lie algebra in $n$ variables. An element $f$ of $T_n$ is written as $f(x_1, x_2, \ldots, x_n)$.

Define $$(\delta_n f)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n} (-1)^i f(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n).$$

Consider $T_n(y)$ the free associative algebra generated by $(x_1, x_2, \ldots, x_n, y)$. An operator $h$ on $T_n$ is extended by an operator still denoted by $h$ on $T_n(y)$ where we do not operate on $y$.

We may consider the application $\tau : T_n \to T_n(y)$ defined by $$(\tau_n f)(x_1, \ldots, x_n) = f(x_1 + y, x_2 + y, \ldots, x_i + y, \ldots, x_n + y).$$

The application $\tau$ commutes with $\delta$. Thus the kernel of $\tau$ is a subcomplex of $T_n$. We may identify it with $T_{n-1}$ by $(Rf)(x_1, x_2, \ldots, x_n) = f(x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1})$ and we obtain on $T_n$ the complex $\delta_A$ considered by Alekseev-Torossian. Here

$$(\delta_A f)(x_1, x_2, \ldots, x_{n-1})$$

\[= f(x_2, x_3, \ldots, x_{n-1}) - f(x_1 + x_2, x_3, \ldots, x_{n-1}) + f(x_1, x_2 + x_3, \ldots, x_{n-1}) + \cdots + (-1)^{n-2} f(x_1, x_2, \ldots, x_{n-2} + x_{n-1}) + (-1)^{n-1} f(x_1, x_2, \ldots, x_{n-1}).$$

It is clear that the complex $\delta : 0 \to T_0 \to T_1 \to T_2 \cdots$ is acyclic. Indeed we can define $$(sf)(x_1, x_2, \ldots, x_n) = f(0, x_1, x_2, \ldots, x_n)$$

and it is immediate to verify that $s\delta + \delta s = \text{Id}$.
If \( f \in T_n \), we define a function \( f(t, x) \in \mathbb{R}[t] \otimes T_k \) by the same formula as Formula (2):

**Definition 2.1.** Let

\[
x(t) = x_1 + t_1(x_2 - x_1) + \cdots + t_{n-1}(x_n - x_1).
\]

Define

\[
f(t, x_1, x_2, \ldots, x_n) = \frac{d}{dt} \bigg|_0 \sum_{\sigma \in \Sigma([2, \ldots, n])} \epsilon(\sigma)f(x(t), x(t) + \epsilon_1(x_{\sigma(2)} - x_1), \ldots, x(t) + \epsilon_{n-1}(x_{\sigma(n)} - x_1)).
\]

Define

\[
(P_n f)(x_1, x_2, \ldots, x_n) = \int_{\Delta_{n-1}} f(t, x_1, x_2, \ldots, x_n) dt_1 dt_2 \cdots dt_{n-1}.
\]

The following lemma is immediate.

**Lemma 2.2.** We have \( \delta P_n = P_n \delta \).

We define \( G^1 = 0 \) and inductively \( G^{n+1} \) by the same formula as before. More precisely, given \( f \in T^{n+1} \), we define the function \( \phi \) of \( T^n(x_1) \) given by

\[
\phi(w_1, w_2, \ldots, w_n) = f(x_1, x_1 + w_1, \ldots, x_1 + w_n)
\]

and define

\[
(G^{n+1} f)(x_1, x_2, \ldots, x_n) = ((H^n - \delta G^n) \phi)(0, x_2 - x_1, \ldots, x_n - x_1).
\]

Then we conclude as before that \( G \delta + \delta G = \text{Id} - P \). Restricting to the invariants, we obtain a map \( G^A \) such that \( \text{Id} - \text{Ant} = G^A_\delta A + G^A \delta A \). Here Ant is the anti-symmetrization operator \( \sum_\sigma \epsilon(\sigma)x_{\sigma(1)} \cdots x_{\sigma(n)} \).

The subspace \( L_n \) of \( T_n \) is stable under the differential. The operator Ant is equal to 0 on \( L_n \), except in degree 1, 2, as there are no totally antisymmetric elements in \( L_n \) for \( n \geq 3 \). Thus we obtain

**Theorem 2.3.** • The cohomology groups \( H^n(T_n, \delta A) \) of the complex \( \delta A : T_n \to T_n \) are of dimension 1 and are generated by \( \sum_\sigma \epsilon(\sigma)x_{\sigma(1)} \cdots x_{\sigma(n)} \).

• The cohomology groups \( H^n(L_n, \delta A) \) of the complex \( \delta A : L_n \to L_n \) are of dimension 0 if \( n > 2 \). For \( n = 1, 2 \),

\[
H^1(L_1, \delta A) = \mathbb{R} x_1, \quad H^2(L_2, \delta A) = \mathbb{R}[x_1, x_2].
\]

Remark: The Guichardet construction also provides an explicit homotopy.

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