Anderson localization transitions in disordered electronic systems represent a remarkable class of quantum phase transitions. Here we understand the term “Anderson transition” in a broad sense, including both the localization-delocalization transitions and the quantum Hall transitions between two phases with localized states. A hallmark of these transitions is the multifractality (MF) of electronic wave functions, describing their strong fluctuations at criticality. The wave functions are characterized by a whole set of fractal dimensions \(D_q\) different from the spatial dimensionality \(d\). While at present the wave function MF is routinely observed only in computer simulations, rapidly developing imaging techniques allow us to hope for its forthcoming experimental observation.

In this Letter we point out two exact relations satisfied by the multifractal dimensions. The first of these relations connects exponents with multifractal dimensions. The first relation implies a symmetry of the multifractal spectrum linking the multifractal exponents with indices \(q < 1/2\) to those with \(q > 1/2\). The second relation connects the wave function multifractality to that of Wigner delay times in a system with a lead attached.

Two exact relations between multifractal exponents are shown to hold at the critical point of the Anderson localization transition. The first relation implies a symmetry of the multifractal spectrum linking the multifractal exponents with indices \(q < 1/2\) to those with \(q > 1/2\). The second relation connects the wave function multifractality to that of Wigner delay times in a system with a lead attached.

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same fixed point in the infrared limit and will thus have the same critical exponents. Therefore, the relation must hold not only in the NLoM approximation, but be an intrinsic property of any generic microscopic model, even though the validity of Eq. (1) is in general only approximate.

The moments of the LDOS and of the wave function intensity, which we considered above, are properties of a closed system. An alternative method to study the local properties is to open the system by attaching a perfectly coupled single-channel lead at a point \(r\). The system can then be characterized by the Wigner delay time \(\tau_W\) (energy derivative of the scattering phase shift), whose statistical properties attracted a lot of research interest in recent years, see \(\raisebox{.1em}{\textcircled{F}}\). For convenience, we will consider below the dimensionless delay time \(\tilde{\tau}_W = t_W \Delta / 2\pi\) normalized to the mean level spacing \(\Delta\). At the Anderson transition point the corresponding distribution function, \(\mathcal{P}_W(\tilde{\tau}_W)\), will reflect the criticality of the system \(\text{\cite{15}}\).

To establish a connection between the MF of wave functions and that of delay times, we recall a relation between \(\mathcal{P}_W\) and the distribution function \(\mathcal{P}_y\) of normalized wave function intensities \(y = V|\psi^2(r)|\) \((V \sim L^d\) is the system volume),

\[
\mathcal{P}_W(\tilde{\tau}_W) = \mathcal{P}_y(\tilde{\tau}_W^{-1}).
\]

This formula was derived in Ref. \(\text{\cite{8}}\) and has the same status as Eq. (1): it is exact on the level of the NLoM. In particular, it implies the corresponding relation between the exponents \(\text{\cite{8}}\)

\[
\gamma_q = \tau_1 + q,
\]

where the indices \(\gamma_q\) characterize the scaling of moments of the inverse delay time, \((\tilde{\tau}_W^{-q}) \propto L^{-\gamma_q}\). Applying the same argumentation as used above for derivation of Eq. (2), we conclude that the relation (7) must again be exact for any generic microscopic model.

The following point should be emphasized here. Strictly speaking, the moments \((\tilde{\tau}_W^q)\) with \(q < -3/2\) are divergent for the one-channel scattering problem. To define the exponent \(\gamma_q\) for this case one should consider a lead with several conducting channels. This is analogous to the coarse-graining procedure for defining the wave function exponent \(\tau_q\) with negative \(q\) discussed below. Equation (6) holds for such negative \(q < -3/2\) as well, by analytical continuation.

We turn now to the analysis of consequences and applications of the derived relations, mainly concentrating on Eq. (2). First, we rewrite the relation (2) in terms of the exponents \(\tau_q\),

\[
\tau_q - \tau_1 - q = d(2q - 1).
\]

Second, performing the Legendre transformation, \(f(\alpha_q) = q\alpha_q - \tau_q\) with \(\alpha_q = d\tau_q/dq\), we get

\[
f(2d - \alpha) = f(\alpha) + d - \alpha.
\]

Equation (9) maps the part of the singularity spectrum with \(\alpha < d\) to that with \(\alpha > d\). A particular consequence of this is that the support of the singularity spectrum \(f(\alpha)\) (i.e. the region where it is different from \(-\infty\)) is bounded by the interval \([0, 2\alpha]\). The lower boundary, \(\alpha \geq 0\), is a trivial consequence of the wave function normalization; the upper boundary, \(\alpha \leq 2\alpha\), follows then from our relation (9).

It is worth mentioning that the results for the \(f(\alpha)\) spectrum, as obtained numerically for the 3d Anderson transition in a number of publications \(\text{\cite{10,11}}\), are in contradiction with this upper boundary [and thus with the relation (9)]. We believe that this is a consequence of an incomplete analysis of numerical data in \(\text{\cite{10,11}}\). Indeed, it was shown recently \(\text{\cite{12}}\) that the earlier numerics on the wave function MF suffered strongly from the absence of ensemble averaging and from finite-size effects. The problems become even more severe for negative moments, \(q < 0\), corresponding to the large-\(\alpha\) part of the singularity spectrum. This is evident, in particular, from Fig. 6 of Ref. \(\text{\cite{11}}\) where a strong drift of large-\(\alpha\) part of \(f(\alpha)\) (towards our upper boundary \(\alpha \leq 2\alpha = 6\)) with increasing system size is seen.

Let us analyze the implication of our relation for the weak-coupling expansion of the critical exponents that can be developed in \(2+\epsilon\) dimensions (where MF is weak). Since Eq. (2) is exact, it should hold in all orders of the \(\epsilon\)-expansion. The known results for the \(\epsilon\)-expansion of \(\Delta\), up to 4-loop order \(\text{\cite{13}}\) do satisfy this property. In particular, the result for the orthogonal symmetry class reads

\[
\Delta_q = q(1 - q)\epsilon + \frac{\zeta(3)}{4}q(q - 1)(q(q - 1) + 1)\epsilon^4 + O(\epsilon^5).
\]

It is indeed seen that \(\Delta\) depends on \(q\) via the combination \((1 - q)\) only, in agreement with the relation (2).

As a further application of Eq. (2), we consider the model of power-law random banded matrices (PRBM), \(\langle |H_{ij}|^2 \rangle = (1 + |i - j|^\beta/b^2)^{-1}\). This model (that describes a 1d system with long-range \(1/r\) random hopping) defines a family of critical theories parametrized by \(0 < b < \infty\) and allows to study the evolution of the critical system from the weak- to the strong-MF regime with decreasing \(b\) \(\text{\cite{14,15}}\). While for \(b \gg 1\) (weak MF) the PRBM model can be approximately mapped to the NLoM, for small \(b\) (strong MF) this mapping is not applicable, and the multifractal spectrum was analyzed in \(\text{\cite{15}}\) by a different method. Our statement about the exactness of Eqs. (2), (7) remains valid for the PRBM model. Indeed, we can construct a “granular” generalization of the model with \(N \gg 1\) states at each site of the 1d lattice and with hopping matrix elements between all states decaying with distance \(r\) as \((b/N)^{r-1}\). Changing the overall prefactor \(b\) in the hopping amplitude will yield a family of critical models that should flow in the infrared limit to the same line of critical points as the family of PRBM models. In
this way, the PRBM model with an arbitrary value of $b$ can be associated with an N-orbital model with some $b$ that will have the same critical properties. On the other hand, the latter model can be mapped onto the NLmM, which allows us again to derive the relations (2), (7) for the critical exponents.

We have verified the validity of the relation (2) by a numerical simulation of the PRBM model. The exponents $\tau_q$ were extracted from the scaling of the inverse participation ratios $\langle P_q \rangle$ for system sizes $L$ in the range from 512 to 4096. The number of disorder realizations was ranging from $2 \times 10^5$ for $L = 512$ to 1000 for $L = 4096$. It should be stressed that evaluation of negative moments requires special care, since the inverse participation ratio, as defined in Eq. (3), is divergent because of zeros of the wave function. These zeros, related to oscillations of the wave function on the scale of the wave length, have nothing to do with multifractal properties characterizing smooth envelopes of wave functions. To find $\tau_q$ with negative $q$, we have first smoothed $|\psi|^2$ by averaging over blocks of the size $m = 16$, and then applied Eq. (4). This makes finite-size effects (and thus a numerical inaccuracy in evaluation of $\tau_q$) for negative moments considerably more pronounced than for $q > 0$.

The results of the numerical simulations for the PRBM ensemble with several values of $b$, spanning the whole interval from the weak-MF to strong-MF regime, are shown in Figs. 1 2. The data in Fig. 1 nicely confirm the symmetry relation (2). A small difference between $\Delta_q$ and $\Delta_{1-q}$ can be considered as a measure of the numerical accuracy of evaluation of the exponents. As discussed above, the errors are mainly due to moments with negative $q$. In Fig. 2 the same numerical data are presented in the form of the singularity spectrum $f(\alpha)$. To demonstrate that the data support very well the relation (9), we also show the function $f(2 - \alpha) + \alpha - 1$.

We will now demonstrate the high utility of Eq. (2) by applying it for the analytical evaluation of exponents with $q < 1/2$ in the "non-NLmM" limit, $b \ll 1$. As was found in [15], the multifractal exponents in this regime are given for $q > 1/2$ by

$$\tau_q \simeq 2bT(q), \quad (11)$$

$$T(q) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(q-1/2)}{\Gamma(q-1)} \approx \begin{cases} \frac{1}{2q^{1/2}}, & q \to 1/2, \\ \frac{\pi(q-1/2)}{2\sqrt{\pi}}, & q \gg 1. \end{cases} \quad (12)$$

In terms of the singularity spectrum $f(\alpha)$, this means

$$f(\alpha) \simeq 2bF(\alpha/2b), \quad (13)$$

where $F(A)$ is the Legendre transform of $T(q)$ with the asymptotics

$$F(A) \simeq \begin{cases} -1/\pi A, & A \to 0, \\ A/2, & A \to \infty. \end{cases} \quad (14)$$

Equation (11), (12) was derived in [15] by a real-space renormalization-group method valid for $q > 1/2$. The relation (2) allows us now to find the multifractality spectrum for $q < 1/2$. When translated to $f(\alpha)$-language, this yields the singularity spectrum for $\alpha > 1$,

$$f(\alpha) = f(2 - \alpha) + \alpha - 1 \approx 2bF\left(\frac{2 - \alpha}{2b}\right) + \alpha - 1 \approx \begin{cases} \alpha/2, & 2 - \alpha \gg 2b, \\ 1 - \frac{b^2}{2(2-\alpha)}, & 2 - \alpha \ll 2b. \end{cases} \quad (15)$$

In Fig. 3 we show the MF spectrum of the PRBM model for $b = 0.1$. The dashed curve yields the $\alpha < 1$ behavior, Eq. (13), while the full line is the $\alpha > 1$ result, Eq. (15).

In the limit $b \to 0$ the MF reaches its extreme form (for the PRBM model the effective spatial dimensionality...
We have demonstrated that two exact relations, Eq. (2) (that can be equivalently represented in the form (5) or (10) and Eq. (7), hold for multifractal exponents at the critical point of the Anderson transition. We have applied the first of these relations to the multifractality spectrum of the PRBM model and verified its validity by numerical simulations. A further analysis of implications of these relations is of considerable interest.

Another direction of future research is to study whether these relations, derived here for three Wigner-Dyson classes, have some analogues for unconventional (chiral and superconducting) symmetry classes [18].

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\[ f(\alpha) = \begin{cases} \alpha/2, & 0 \leq \alpha \leq 2d, \\ -\infty, & \text{otherwise} \end{cases} \]  

The following remark is in order here. The earlier analysis [2, 16] of the statistics of critical wave functions on the Bethe lattice and in large dimensionality \( d \) allows us to conjecture that in the limit \( d \to \infty \) the multifractal spectrum at the Anderson transition acquires the same extreme form (16), (17). We stress, however, that this is only a hypothesis waiting for a more rigorous verification.

The second relation we claim to be exact, Eq. (7), is also supported by numerical results obtained for the PRBM model. Specifically, the numerical data [5] for the scaling of the delay time moments confirm (in combination with the results of Ref. [17] on the wave function MF) the validity of Eq. (7) even in the small-\( b \) limit where the mapping of the PRBM to the NLσM fails.

As a final remark, we note that the notion of MF was recently extended to the surface of a critical system [17]. While boundary multifractal exponents are different from their bulk counterparts, the relations (2) and (7) remain valid also for surface MF. Indeed, it is not difficult to check that the derivation of the relations for the distribution functions (that served as starting points for our analysis), (11) and (6), retain its validity independently on the position (in the bulk or near the boundary) of the observation point \( r \). The MF of delay times for a lead attached to the boundary has in fact been studied numerically in the PRBM model in Ref. [3]: an analysis of the surface MF of wave functions and the verification of the relation (7) at the boundary of this system will be presented elsewhere.

To summarize, we have demonstrated that two exact relations, Eq. (2) that can be equivalently represented in the form (5) or (10) and Eq. (7), hold for multifractal exponents at the critical point of the Anderson transition. We have applied the first of these relations to the multifractality spectrum of the PRBM model and verified its validity by numerical simulations. A further analysis of implications of these relations is of considerable interest. Another direction of future research is to study whether these relations, derived here for three Wigner-Dyson classes, have some analogues for unconventional (chiral and superconducting) symmetry classes [18].

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\[ \tau_q = \begin{cases} 0, & q \geq 1/2, \\ d(2q - 1), & q \leq 1/2, \end{cases} \]  

FIG. 3: Singularity spectrum for the PRBM model with \( b = 0.1 \). Dashed line: \( \alpha < 1 \) behavior, Eq. (13); full line: \( \alpha > 1 \) result, Eq. (15), following from the relation (9); circles: numerical data. Some mismatch between the slopes of the two curves at \( \alpha = 1 \) is related to the fact that the formula (15) is valid to the leading order in \( b \ll 1 \).