Distribution dependent SDEs for Navier-Stokes type equations*

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Abstract
To characterize Navier-Stokes type equations where the Laplacian is extended to a singular second order differential operator, we propose a class of SDEs depending on the distribution in future. The well-posedness and regularity estimates are derived for these SDEs.

Keywords: Navier-Stokes type equation; distribution dependent SDE; well-posedness.
MSC2020 subject classifications: 60H10; 35A01.

1 Introduction
Let $d \in \mathbb{N}$. Consider the following incompressible Navier-Stokes equation on $E := \mathbb{R}^d$ or $\mathbb{R}^d/\mathbb{Z}^d$:

$$\partial_t u_t = \kappa \Delta u_t - (u_t \cdot \nabla)u_t - \nabla \varphi_t, \quad t \in [0, T]$$

(1.1)

with $\nabla \cdot u_t := \sum_{i=1}^d \partial_i u_{ti}^i = 0$, where $T > 0$ is a fixed time,

$$u := (u^1, \ldots, u^d) : [0, T] \times E \to \mathbb{R}^d, \quad \varphi : [0, T] \times E \to \mathbb{R},$$

and $u_t \cdot \nabla := \sum_{i=1}^d u_{ti}^i \partial_i$. This equation describes viscous incompressible fluids, where $u$ is the velocity field of a fluid flow, $\varphi$ is the pressure, and $\kappa > 0$ is the viscosity constant.

Besides existing probabilistic characterizations on Navier-Stokes equations, see [1] and references therein, in this paper we propose a new type stochastic differential equation (SDE) depending on distributions in the future, such that the solution of (1.1) is explicitly given by the initial datum $u_0$ and the pressure $\varphi$. By proving the well-posedness of the SDE, we derive the well-posedness of (1.1) in $C^n_0 (n \geq 2)$ with given pressure (which is however a part of solution in Navier-Stokes equations), see [3] for an analytic characterization on the pressure to ensure $\nabla \cdot u_t = 0$.

Indeed, we will prove a more general result for the following Navier-Stokes type equation on $E := \mathbb{R}^d$ or $E := \mathbb{R}^d/\mathbb{Z}^d$:

$$\partial_t u_t = L_t u_t - (u_t \cdot \nabla)u_t + V_t, \quad t \in [0, T],$$

(1.2)

where

$$L_t := \text{tr}\{a_t \nabla^2\} + b_t \cdot \nabla$$

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and

\[ V, b : [0, T] \times E \to \mathbb{R}^d, \ a : [0, T] \times E \to \mathbb{R}^{d \otimes d} \]

are measurable, and \( a_t(x) \) is positive definite for \((t, x) \in [0, T] \times E\).

To characterize (1.2), we consider the following SDE on \( \mathbb{R}^d \) where differentials are in \( s \in [t, T] \):

\[
\begin{align*}
    dX^x_{t,s} &= \sqrt{2a_{T-s}(X^x_{t,s})}dW_s \\
    &\quad + \left\{ b_{T-s}(X^x_{t,s}) - \left[ E\mu_0(X^y_{s,T}) + E \int_s^T V_{T-r}(X^y_{r,s})\,dr \right]_{y=X^x_{t,T}} \right\}ds,
\end{align*}
\]

(1.3)

where \((W_s)_{s \in [0,T]}\) is a \( d \)-dimensional Brownian motion on a complete filtered probability space \((\Omega, \mathcal{F}, \{ \mathcal{F}_s \}_{s \in [0,T]}, \mathbb{P})\). When \( E = T^d := \mathbb{R}^d / \mathbb{Z}^d \), by extending a function \( f \) from domain \( E \) to domain \( \mathbb{R}^d \) as

\[ f(x + k) = f(x), \quad x \in [0,1)^d, k \in \mathbb{Z}^d, \]

we also have the SDE (1.3) for the case \( E = T^d \).

Regarding \( s \) as the present time, the SDE (1.3) depends on the distribution of \((X_{s,r})_{r \in [s,T]}\) coming from the future. So, this is a future distribution dependent equation, but is essentially different from McKean-Vlasov SDEs which depend on the distribution at present rather than future. We will use \( X := (X^x_{t,s})_{0 \leq t \leq s \leq T, x \in E} \) to formulate the solution to (1.2).

Let \( D_T := \{(t, s) : 0 \leq t \leq s \leq T\} \). We define the solution \( X \) of (1.3) as follows.

**Definition 1.1.** A family \( X := (X^x_{t,s})_{(t,s,x) \in D_T \times \mathbb{R}^d} \) of random variables on \( \mathbb{R}^d \) is called a solution of (1.3), if \( X^x_{t,s} \) is \( \mathcal{F}_s \)-measurable for all \( x \in \mathbb{R}^d \) and \( 0 \leq t \leq s \leq T \), \( \mathbb{P} \)-a.s. continuous in \((t, s, x)\),

\[
E \int_t^T \left\{ \| a_{T-r}(X^x_{t,s}) \| + \| b_{T-r}(X^x_{t,s}) - \left[ E\mu_0(X^y_{s,T}) + E \int_s^T V_{T-r}(X^y_{r,s})\,dr \right]_{y=X^x_{t,T}} \right\}ds < \infty
\]

for \((t, x) \in [0, T] \times \mathbb{R}^d\), and \( \mathbb{P} \)-a.s.

\[
X^x_{t,s} = x + \int_t^s \sqrt{2a_{T-r}(X^x_{t,r})}dW_r \\
+ \int_t^s \left\{ b_{T-r}(X^x_{t,r}) - \left[ E\mu_0(X^y_{s,T}) + E \int_r^T V_{T-r}(X^y_{r,s})\,dr \right]_{y=X^x_{t,T}} \right\}d\theta,
\]

\( (t, s, x) \in D_T \times \mathbb{R}^d \).

We will allow the operator \( L_t \) to be singular, where the drift contains a locally integrable term introduced in [4] for singular SDEs. For any \( p, q > 1 \) and \( 0 \leq t < s \), we write \( f \in \dot{L}^p_q(t, s) \) if \( f = (f_s(x))(r,x) \in [t,s] \times \mathbb{R}^d \) is a measurable function on \([t,s] \times \mathbb{R}^d\) such that

\[
\| f \|_{\dot{L}^p_q(t,s)} := \sup_{z \in \mathbb{R}^d} \left( \int_t^s \| f_{r,1}B(z,1) \|_{L^p}^q dr \right)^{\frac{1}{q}} < \infty,
\]

where \( B(z, 1) \) is the unit ball at \( z \), and \( \| \cdot \|_{L^p} \) is the \( L^p \)-norm for the Lebesgue measure.

We denote \( f \in \dot{H}^2_q(t, s) \) if \( |f| + \|\nabla f\| + \|\nabla^2 f\| \in \dot{L}^q_2(t, s) \). When \((t, s) = (0, T)\) we simply denote

\[
\dot{L}^p_q = \dot{L}^p_q(0, T), \quad \dot{H}^2_q = \dot{H}^2_q(0, T).
\]

We will take \((p, q)\) from the following class:

\[
\mathcal{K} := \{(p, q) : p, q > 2, \frac{d}{p} + \frac{2}{q} < 1\}.
\]

We now make the following assumption on the operator \( L_t \).

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(H) Let \( b_t = b_t^{(0)} + b_t^{(1)} \), and when \( E = \mathbb{T}^d \) we extend \( a_t, b_t^{(0)} \) and \( b_t^{(1)} \) to \( \mathbb{R}^d \) as in (1.4).

1. \( a \) is positive definite with

\[
\|a\|_{\infty} + \|a^{-1}\|_{\infty} := \sup_{(t,x) \in [0,T] \times E} \|a_t(x)\| + \sup_{(t,x) \in [0,T] \times E} \|a_t(x)^{-1}\| < \infty,
\]

\[
\lim_{\varepsilon \to 0} \sup_{|x-y| \leq \varepsilon, t \in [0,T]} \|a_t(x) - a_t(y)\| = 0.
\]

2. There exist \( l \in \mathbb{N}, \{\{p_i, q_i\}\}_{0 \leq i \leq l} \subset \mathcal{K} \) and \( 0 \leq f_i \in \tilde{L}_2^{q_i}, 0 \leq i \leq l \), such that

\[
|b^{(0)}| \leq f_0, \quad \|\nabla a\| \leq \sum_{i=1}^l f_i.
\]

3. \( \|b^{(1)}(0)\|_{\infty} := \sup_{(t,x) \in [0,T]} |b^{(1)}(0)| < \infty \), and

\[
\|\nabla b^{(1)}\|_{\infty} := \sup_{t \in [0,T]} \sup_{x \neq y} \frac{|b^{(1)}_t(x) - b^{(1)}_t(y)|}{|x-y|} < \infty. \tag{1.5}
\]

Under this assumption, we will prove the well-posedness of (1.3) and solve (1.2) in the class

\[
\mathcal{U}(p_0, q_0) := \left\{ u : [0, T] \times E \to \mathbb{R}^d; \quad \|u\|_{\infty} + \|\nabla u\|_{\infty} + \|\nabla^2 u\|_{L^p_{2q_0}} < \infty \right\}.
\]

Recall that \( W^{1,\infty}(E; \mathbb{R}^d) \) is the space of all weakly differentiable functions \( f : E \to \mathbb{R}^d \) with \( \|f\|_{\infty} + \|\nabla f\|_{\infty} < \infty \).

**Theorem 1.1.** Assume (H). Let \( u_0 \in W^{1,\infty}(E; \mathbb{R}^d) \) and \( \int_0^T \|V_t\|_{C^2_{\infty}}^2 dt < \infty \). Then the following assertions hold.

1. The SDE (1.3) has a unique solution \( X := (X_{t,s})_{(t,s) \in D_T \times \mathbb{R}^d} \).

2. If \( u \) solves (1.2) and \( u \in \mathcal{U}(p_0, q_0) \), then

\[
u_t(x) = E \left[ u_0(X_{T-t}^x) + \int_{T-t}^T V_{T-s}(X_{T-t}^x) ds \right], \quad (t, x) \in [0, T] \times E. \tag{1.6}\]

   Moreover, there exists a constant \( c > 0 \) such that for any \( i \in \{1, 2\} \) and \( j, j' \in \{0, 1\} \),

\[
\|\nabla^i u_t\|_{\infty} \leq c t^{-\frac{i}{2}} \|\nabla^j u_0\|_{\infty} + c \int_{T-t}^T (s+t-T)^{-\frac{i-j}{2}} \|\nabla^{j'} V_{T-s}\|_{\infty} ds, \quad t \in (0, T). \tag{1.7}
\]

3. If \( b^{(1)} = 0 \) and \( u_0, V_t \in C^2_{\infty} \) with \( \int_0^T \|V_t\|_{C^2_{\infty}}^2 dt < \infty \), then \( u \) given by (1.6) solves (1.2), and \( u \) is in the class \( \mathcal{U}(p_0, q_0) \).

In the next two sections, we prove assertions (1) and (2)-(3) of Theorem 1.1 respectively, where in Section 2 the well-posedness is proved for a more general equation than (1.3). Finally, in Section 4 we apply Theorem 1.1 to the equation (1.1).

## 2 Proof of Theorem 1.1(1)

Let \( \mathcal{P} \) be the set of all probability measures on \( \mathbb{R}^d \) equipped with the weak topology, let \( \mathcal{L}_\xi \) be the distribution of a random variable \( \xi \) on \( \mathbb{R}^d \). Let

\[
\Gamma := C(D_T \times \mathbb{R}^d; \mathcal{P})
\]
be the space of continuous maps from $D_T \times \mathbb{R}^d$ to $\mathcal{P}$. For any $\lambda > 0$, $\Gamma$ is a complete space under the metric

$$
\rho_\lambda(\gamma^1, \gamma^2) := \sup_{(t, s, x) \in D_T \times \mathbb{R}^d} e^{-\lambda(T-t)} \| \gamma^1_{t,s,x} - \gamma^2_{t,s,x} \|_{\text{var}}, \quad \gamma^1, \gamma^2 \in \Gamma,
$$

where $\| \cdot \|_{\text{var}}$ is the total variation norm defined by

$$
\| \mu - \nu \|_{\text{var}} := \sup_{|f| \leq 1} |\mu(f) - \nu(f)|, \quad \mu, \nu \in \mathcal{P}
$$

for $\mu(f) := \int_{\mathbb{R}^d} f \, d\mu$. Note that the convergence in $\| \cdot \|_{\text{var}}$ is stronger than the weak convergence.

We consider the following more general equation than (1.3):

$$
\begin{align*}
\mathrm{d}X^x_{t,s} &= \left\{ b^{(1)}_{T-s}(X^x_{t,s}) + Z_s(X^x_{t,s}, \mathcal{L}_X) \right\} \mathrm{d}s + \sqrt{2a_{T-s}(X^x_{t,s})} \mathrm{d}W_s, \\
t \in [0, T], & s \in [t, T], X^x_{t,t} = x \in \mathbb{R}^d,
\end{align*}
$$

(2.1)

where $\mathcal{L}_X \in \Gamma$ is defined by $\{ \mathcal{L}_X \}_{t, s, x} := \mathcal{L}_{X^x_{t,s}}$ and

$$
Z : [0, T] \times \mathbb{R}^d \times \Gamma \to \mathbb{R}^d
$$

is measurable.

It is easy to see that (2.1) covers (1.3) for

$$
Z_t(x, \gamma) := b^{(0)}_{T-t}(x) - \int_{\mathbb{R}^d} u_0(y) \gamma_{t,T,x}(\mathrm{d}y) - \int_t^T \int_{\mathbb{R}^d} V_{T-s}(y) \gamma_{t,s,x}(\mathrm{d}y) \mathrm{d}s, \\
(t, x, \gamma) \in [0, T] \times \mathbb{R}^d \times \Gamma.
$$

(2.2)

The solution of (2.1) is defined as in Definition 1.1 using $b^{(1)}_{T-s}(X^x_{t,s}) + Z_s(X^x_{t,s}, \mathcal{L}_X)$ replacing

$$
b_{T-s}(X^x_{t,s}) - \left[ E_{u_0}(X^y_{s,T}) + E \int_s^T V_{T-r}(X^y_{s,r}) \mathrm{d}r \right]_{y=X^x_{t,s}}.
$$

We make the following assumption.

(A) $b^{(1)}$ and $a$ satisfy (H), and there exists $(p_0, q_0) \in \mathcal{K}$ and $f_0 \in \tilde{L}^{p_0}_{q_0}$ such that

$$
|Z_t(x, \gamma)| \leq f_0(t, x), \quad (t, x, \gamma) \in [0, T] \times \mathbb{R}^d \times \Gamma.
$$

Moreover, there exists $0 \leq g \in L^2([0, T])$ such that

$$
\sup_{x \in \mathbb{R}^d} |Z_t(x, \gamma^1) - Z_t(x, \gamma^2)| \leq g_t \sup_{(s,x) \in [t,T] \times \mathbb{R}^d} \| \gamma^1_{t,s,x} - \gamma^2_{t,s,x} \|_{\text{var}}, \quad t \in [0, T], \gamma^1, \gamma^2 \in \Gamma.
$$

When $\|u_0\|_{\infty} + \int_0^T \|V_t\|_{\infty}^2 \mathrm{d}t < \infty$, (H) implies (A) for $Z$ given by (2.2). So, Theorem 1.1(1) follows from the following result, which also includes regularity estimates on the solution.

**Theorem 2.1.** Assume (A). Then the following assertions hold.

1. (2.1) has a unique solution, and the solution has the flow property

$$
X^x_{t,r} = X^x_{s,r}, \quad 0 \leq t \leq s \leq r \leq T, \quad x \in \mathbb{R}^d.
$$

(2.3)
(2) For any \( j \geq 1 \),
\[
\nabla_v X^{\tau, x}_{t,s} := \lim_{\varepsilon \downarrow 0} \frac{X^{\tau, x}_{t,s} - X^{\tau, x}_{t,s}}{\varepsilon}, \quad s \in [t, T]
\]
eexists in \( L^j(\Omega \to C([t,T]; \mathbb{R}^d), \mathbb{P}) \), and there exists a constant \( c(j) > 0 \) such that
\[
\sup_{(t,s) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \sup_{s \in [t,T]} |\nabla_v X^{\tau, x}_{t,s}| \right] \leq c(j) |v|^j, \quad v \in \mathbb{R}^d.
\]
(3) For any \( 0 \leq t < s \leq T, \, v \in \mathbb{R}^d \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \),
\[
\nabla_v \left\{ \mathbb{E} f(X^{\tau, x}_{t,s}) \right\} (x) = \frac{1}{s-t} \mathbb{E} \left[ f(X^{x}_{t,s}) \int_t^s \left\langle \left( \sqrt{2a_{T-r}} \right)^{-1} (X^{x}_{t,r}) \nabla_v X^{\tau, x}_{t,r}, \, dW_r \right\rangle \right].
\]

**Proof.** (a) We first explain the idea of proof using fixed point theorem on \( \Gamma \). For any \( \gamma \in \Gamma \), we consider the following classical SDE
\[
dX^{\gamma, x}_{t,s} = \left\{ b^{(1)}(t,s,X^{\gamma, x}_{t,s}) + Z_s(X^{\gamma, x}_{t,s}, \gamma) \right\} ds + \sqrt{2a_{T-s}(X^{\gamma, x}_{t,s})} dW_s,
\]
\( t \in [0,T], \, s \in [t,T], \, X^{\gamma, x}_{t,s} = x \in \mathbb{R}^d. \)
By [2, Theorem 2.1] for \([t,T] \) replacing \([0,T] \), see also [4] for \( b^{(1)} = 0 \), this SDE is well-posed, such that for any \( j \geq 1 \) and \( v \in \mathbb{R}^d \), the directional derivative
\[
\nabla_v X^{\gamma, x}_{t,s} := \lim_{\varepsilon \downarrow 0} \frac{X^{\gamma, x}_{t,s} - X^{\gamma, x}_{t,s}}{\varepsilon}, \quad s \in [t, T]
\]
eexists in \( L^j(\Omega \to C([t,T]; \mathbb{R}^d), \mathbb{P}) \), and there exists a constant \( c(j) > 0 \) such that
\[
\sup_{(t,s) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \sup_{s \in [t,T]} |\nabla_v X^{\gamma, x}_{t,s}| \right] \leq c(j) |v|^j, \quad v \in \mathbb{R}^d,
\]
and for any \( f \in \mathcal{B}_b(\mathbb{R}^d) \),
\[
\nabla_v \left\{ \mathbb{E} f(X^{\gamma, x}_{t,s}) \right\} (x) = \frac{1}{s-t} \mathbb{E} \left[ f(X^{x}_{t,s}) \int_t^s \left\langle \left( \sqrt{2a_{T-r}} \right)^{-1} (X^{\gamma, x}_{t,r}) \nabla_v X^{\gamma, x}_{t,r}, \, dW_r \right\rangle \right].
\]
By the pathwise uniqueness of (2.6), the solution satisfies the flow property
\[
X^{\gamma, x}_{t,s} = X^{\gamma, x}_{s,r}, \quad 0 \leq t \leq s \leq r \leq T, \, x \in \mathbb{R}^d.
\]
Moreover,
\[
\Phi(\gamma)(t,s,x) := \mathcal{L}X^{\gamma, x}_{t,s}, \quad (t,s,x) \in D_T \times \mathbb{R}^d
\]
defines a map \( \Phi : \Gamma \to \Gamma \). If \( \Phi \) has a unique fixed point \( \bar{\gamma} \in \Gamma \), then (2.6) with \( \gamma = \bar{\gamma} \) reduces to (2.1), the well-posedness of (2.6) implies that of (2.1), and the unique solution is given by
\[
X^{\tau, x}_{t,s} = X^{\bar{\gamma}, x}_{t,s}.
\]
Then (2.3), (2.4) and (2.5) follow from (2.9), (2.7) and (2.8) for \( \gamma = \bar{\gamma} \) respectively. Therefore, it remains to prove that \( \Phi \) has a unique fixed point.

(b) By the fixed point theorem, we only need to find constants \( \lambda > 0 \) and \( \delta \in (0,1) \) such that
\[
\rho_\lambda(\Phi(\gamma^1), \Phi(\gamma^2)) \leq \delta \rho_\lambda(\gamma^1, \gamma^2), \quad \gamma^1, \gamma^2 \in \Gamma.
\]
Below, we prove this estimate using Girsanov’s theorem.

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For $i = 1, 2$, consider the SDE
\[
dX_{t,s}^{i,x} = \left\{ b_{T-s}^{(i)}(X_{t,s}^{i,x}) + Z_s(X_{t,s}^{i,x}, \gamma) \right\} ds + \sqrt{2\alpha_{T-s}}(X_{t,s}^{i,x}) dW_s,
\]
$t \in [0, T], s \in [t, T], X_{t,t}^{i,x} = x \in \mathbb{R}^d$.

By the definition of $\Phi$, we have
\[
\Phi(\gamma^i)_{t,s,x} = \mathcal{L}_{X_{t,s}^{i,x}}, \ i = 1, 2, \ (t, s, x) \in D_T \times \mathbb{R}^d. \tag{2.11}
\]

Let
\[
\xi_s := (\sqrt{2\alpha_{T-s}}(X_{t,s}^{1,x}))^{-1} \{ Z_s(X_{t,s}^{1,x}, \gamma^1) - Z_s(X_{t,s}^{1,x}, \gamma^2) \}, \ s \in [t, T].
\]

By (A), there exists a constant $K > 0$ such that
\[
|\xi_s| \leq Kg_s \sup_{(r,x) \in [s,T] \times \mathbb{R}^d} ||\gamma^1_{s,r,x} - \gamma^2_{s,r,x}||_{\text{var}}. \tag{2.12}
\]

By Girsanov theorem,
\[
\tilde{W}_s := W_s - \int_t^s \xi_r dr, \ s \in [t, T]
\]
is a Brownian motion under the weighted probability $dQ_t := R_t dP$, where
\[
R_t := e^{f^2(\xi_s, dW_s) - \frac{1}{2} f^2(\xi_s)^2 ds}.
\]

With this new Brownian motion, the SDE for $X^1$ becomes
\[
dX_{t,s}^{1,x} = \left\{ b_{T-s}^{(1)}(X_{t,s}^{1,x}) + Z_s(X_{t,s}^{1,x}, \gamma^2) \right\} ds + \sqrt{2\alpha_{T-s}}(X_{t,s}^{1,x}) dW_s, \ s \in [t, T].
\]

By the (weak) uniqueness for the SDE with $i = 2$, we derive
\[
\mathcal{L}_{X_{t,s}^{1,x}}|_{Q_t} = \Phi(\gamma^2)_{t,s,x},
\]
where $\mathcal{L}_{X_{t,s}^{1,x}}|_{Q_t}$ is the distribution of $X_{t,s}^{1,x}$ under $Q_t$. Combining this with (2.11), we get
\[
||\Phi(\gamma^1)_{t,s,x} - \Phi(\gamma^2)_{t,s,x}||_{\text{var}} = \sup_{|f| \leq 1} |E[f(X_{t,s}^{1,x}) - f(X_{t,s}^{1,x}) R_t]| \leq E|R_t - 1|. \tag{2.13}
\]

By Pinsker’s inequality and the definition of $R_t$, we obtain
\[
(E|R_t - 1|)^2 \leq 2E[R_t \log R_t] = 2E_{Q_t}[\log R_t] = 2E_{Q_t} \int_t^T |\xi_s|^2 ds, \tag{2.14}
\]
where $E_{Q_t}$ is the expectation under the probability $Q_t$. Combining (2.13) and (2.14) with (2.12), and using the definition of $\rho_\lambda$, we arrive at
\[
||\Phi(\gamma^1)_{t,s,x} - \Phi(\gamma^2)_{t,s,x}||_{\text{var}} \leq \left( 2K^2 \int_t^T g^2_{s,r,y} \sup_{(r,y) \in [s,T] \times \mathbb{R}^d} ||\gamma^1_{s,r,y} - \gamma^2_{s,r,y}||_{\text{var}} ds \right)^{\frac{1}{2}}
\]
\[
\leq \rho_\lambda(\gamma^1, \gamma^2) \left( 2K^2 \int_t^T g^2 e^{2\lambda(T-s)} ds \right)^{\frac{1}{2}}, \ \ (t, x) \in [0, T] \times \mathbb{R}^d.
\]

Therefore
\[
\rho_\lambda(\Phi(\gamma^1), \Phi(\gamma^2)) \leq \varepsilon_{\lambda} \rho_\lambda(\gamma^1, \gamma^2),
\]
where
\[
\varepsilon_{\lambda} := \sup_{t \in [0, T]} \left( 2K^2 \int_t^T g^2 e^{-2\lambda(s-t)} ds \right)^{\frac{1}{2}} \downarrow 0 \text{ as } \lambda \uparrow \infty.
\]

By taking large enough $\lambda > 0$, we prove (2.10) for some $\delta < 1$. \qed
For later use we present the following consequence of Theorem 2.1.

**Corollary 2.2.** Assume (A) and let

\[ P_{t,s}f(x) := \mathbb{E}[f(X_{t,s}^x)], \quad (t, s, x) \in D_T \times \mathbb{R}^d, f \in \mathcal{B}(\mathbb{R}^d). \]

Then there exists a constant \( c > 0 \) such that for any function \( f \),

\[
\|\nabla P_{t,s}f\|_\infty \leq c \min \left\{ (s-t)^{-\frac{1}{2}} \|f\|_\infty, \|\nabla f\|_\infty \right\},
\]

\[
\|\nabla^2 P_{t,s}f\|_\infty \leq c(s-t)^{-\frac{1}{2}} \|\nabla f\|_\infty, \quad 0 \leq t < s < T.
\]

**Proof.** By (2.5) we have

\[
\|\nabla P_{t,s}f\|_\infty \leq c(t-s)^{-\frac{1}{2}} \|f\|_\infty
\]

for some constant \( c > 0 \). Next, by chain rule and (2.4),

\[
\|\nabla P_{t,s}f(x)\| = |\mathbb{E}[\nabla f(X_{t,s}^x), \nabla X_{t,s}^x)]| \leq c\|\nabla f\|_\infty, \quad (t, s, x) \in D_T \times \mathbb{R}^d
\]

holds for some constant \( c > 0 \). Moreover,

\[ \nabla P_{t,s}f(x) = \mathbb{E}[\nabla f(X_{t,s}^x), \nabla X_{t,s}^x)] = \mathbb{E}[g(X_{t,s}^x)], \]

where \( g(X_{t,s}^x) := \langle \nabla f(X_{t,s}^x), \nabla(X_{t,s}^x) \rangle \). Combining this with (2.5) and (2.4), we find a constant \( c > 0 \) such that

\[
\|\nabla^2 P_{t,s}f(x)\| \leq \|\nabla \mathbb{E}[g(X_{t,s}^x)]\|
\]

\[
\leq \frac{1}{s-t} \mathbb{E}\left[ |g(X_{t,s}^x)| \left| \int_s^t \left( \langle \sqrt{2u_{t-s}}^{-1} (X^x_{t,s}) \nabla_{t,s} \rangle dW_r \right) \right| \right]
\]

\[
\leq \frac{1}{s-t} \left( \mathbb{E}[g(X_{t,s}^x)]^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \int_s^t \|a^{-1}\|_{\infty} \|\nabla X_{t,s}^x\|^2 dr \right)^{\frac{1}{2}} \leq c \|\nabla f\|_\infty.
\]

Then the proof is finished. \( \square \)

### 3 Proofs of Theorem 1.1(2)-(3)

We will need the following lemma implied by [5, Theorem 2.1, Theorem 3.1, Lemma 3.3], see also [4] and references within for the case \( b^{(1)} = 0 \).

**Lemma 3.1.** Assume (A)(1), (A)(3) and \( \|b^{(0)}\|_{L^{\infty}_p} < \infty \) for some \((p_0, q_0) \in \mathcal{K} \). Let \( \sigma_t = \sqrt{2u_t} \). Then the following assertions hold.

1. For any \( p, q > 1, \lambda \geq 0, 0 \leq t_0 < t_1 \leq T \) and \( f \in \tilde{L}^p_q(t_0, t_1) \), the PDE

\[
(\partial_t + L_t) u_t = \lambda u_t + f_t, \quad t \in [t_0, t_1], u_{t_1} = 0,
\]

has a unique solution in \( \tilde{H}^2_q(t_0, t_1) \). If \((2p, 2q) \in \mathcal{K} \), then there exist a constant \( c > 0 \) such that for any \( 0 \leq t_0 < t_1 \leq T \) and \( f \in \tilde{L}^p_q(t_0, t_1) \), the solution satisfies

\[
\|u\|_{\infty} + \|\nabla u\|_{\infty} + \|(\partial_t + \nabla b^{(1)}) u\|_{\tilde{L}^q_p(t_0, t_1)} + \|\nabla^2 u\|_{\tilde{L}^q_p(t_0, t_1)} \leq c\|f\|_{\tilde{L}^p_q(t_0, t_1)}.
\]

2. Let \( (X_t)_{t \in [0, T]} \) be a continuous adapted process on \( \mathbb{R}^d \) satisfying

\[
X_t = X_0 + \int_0^t b_s(X_s) ds + \int_0^t \sigma_s(X_s) dW_s, \quad t \in [0, T].
\]

For any \( p, q > 1 \) with \((2p, 2q) \in \mathcal{K} \), there exists a constant \( c > 0 \) such that for any \( X_t \) satisfying (3.2),

\[
\mathbb{E}\left( \int_t^T |f_r(X_r)| dr \bigg| \mathcal{F}_t \right) \leq c\|f\|_{\tilde{L}^p_q(t, s)}, \quad (t, s) \in D_T, f \in \tilde{L}^p_q(t, s).
\]
(3) Let $p, q > 1$ with $\frac{2}{q} + \frac{2}{p} < 1$. For any $u \in \tilde{H}_{d}^{p}$ with $\| (\partial_t + b^{(i)}) u \|_{L_{q}^{p}} < \infty$, 
\{u_t(X_t)\}_{t \in [0, T]} is a semimartingale satisfying 
\[
du_t(X_t) = L_t u_t(X_t) dt + \langle \nabla u_t(X_t), \sigma_t(X_t) dW_t \rangle, \quad t \in [0, T].
\]

In the following we consider $E = \mathbb{R}^d$ and $T^d$ respectively.

3.1 $E = \mathbb{R}^d$

Proof of Theorem 1.1(2). Let $u \in \mathcal{U}(p_0, q_0)$ solve (1.2). Then
\[
u u \in \tilde{H}_{d}^{p_0}, \quad \| (\partial_t + b^{(i)}) \nabla u \|_{L_{q_0}^{p_0}} < \infty \tag{3.3}
\]
as required by Lemma 3.1(3). It remains to prove (1.6), which together with Corollary 2.2 
implies (1.7).

Let 
\[
\mathcal{L}_t := \text{tr} \{a_{T-t} \nabla^2 \} + \tilde{b}_t \cdot \nabla,
\]
\[
\tilde{b}_t(x) := b_{T-t}(x) - E u_0(X_{t,T}) - E \int_t^T V_{T-s}(X_{t,s}^x) ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d. \tag{3.4}
\]

Since $\| u_0 \|_{\infty} + \int_0^T \| V_t \|_{\infty} dt < \infty$, $\| b^{(0)} \|_{L_{p_0}^{q_0}} < \infty$ implies $\tilde{b}_t(x) := b_{T-t}^{(1)}(x) + \tilde{b}_t^{(0)}(x)$ with 
$\| \tilde{b}^{(0)} \|_{L_{p_0}^{q_0}} < \infty$. Then (A) holds for $\tilde{b}$ replacing $b$, so that by (3.3) and Lemma 3.1(3), the 
following Itô's formula holds for $X_{t,s}^x$, solving (1.3):
\[
du_{T-s}(X_{t,s}^x) = (\partial_s + \mathcal{L}_s) u_{T-s}(X_{t,s}^x) ds + \{ \nabla u_{T-s}(X_{t,s}^x) \}^T \sqrt{2u_{T-s}(X_{t,s}^x)} dW_s, \quad s \in [t, T], \tag{3.5}
\]
where $\{ \nabla u \}_{ij} := (\partial_s u^i)_{1 \leq i, j \leq d}$. By (1.2) and (3.4), we obtain
\[
(\partial_s + \mathcal{L}_s) u_{T-s}(X_{t,s}^x) + V_{T-s}(X_{t,s}^x)
\]
\[
= \left\{ u_{T-s}(y) - E u_0(X_{s,T}) - E \int_s^T V_{T-r}(X_{s,r}^y) dr \right\}_{y=X_{t,s}^x} \cdot \nabla u_{T-s}(X_{t,s}^x).
\]

Combining this with the follow property (2.3) and (3.5), we derive
\[
E u_0(X_{t,T}) - u_{T-t}(x) = E [u_{T-T}(X_{t,T}^x) - u_{T-t}(X_{t,t})]
\]
\[
= E \int_t^T \left\{ \left( u_{T-s}(y) - E u_0(X_{s,T}) - E \int_s^T V_{T-r}(X_{s,r}^y) dr \right)_{y=X_{t,s}^x} \cdot \nabla \right\} u_{T-s}(X_{t,s}^x) ds
\]
\[- E \int_t^T V_{T-s}(X_{t,s}^x) ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\]

Letting 
\[
h_t := \sup_{x \in \mathbb{R}^d} \left| u_{T-t}(x) - E u_0(X_{t,T}) - E \int_t^T V_{T-s}(X_{t,s}^x) ds \right|, \quad t \in [0, T],
\]
we arrive at 
\[
h_t \leq \int_t^T h_s \| \nabla u \|_{\infty} ds, \quad t \in [0, T].
\]

By Gronwall's inequality we prove $h_t = 0$ for $t \in [0, T]$, hence (1.6) holds. \hfill \Box

Proof of Theorem 1.1(3). (a) Let $P_t f = E [f(X_{t,T}^x)]$ for $f \in B_b(\mathbb{R}^d)$, where $X_{t,s}^x$ solves (1.3). For $u$ given by (1.6) we have 
\[
u_t = P_{T-t,T} u_0 + \int_{T-t}^T P_{T-t,s} V_{T-s} ds, \quad t \in [0, T]. \tag{3.6}
\]
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By \( \|u_0\|_\infty + \int_0^T \|V_t\|dt < \infty \) and (1.7), we find a constant \( c > 0 \) such that
\[
\|u\|_\infty + \|\nabla u\|_\infty \leq c, \quad \|\nabla^2 u_t\|_\infty \leq ct^{-\frac{1}{2}}, \quad t \in (0, T]. \tag{3.7}
\]
Moreover, the SDE (1.3) becomes
\[
\text{d}X^s_{t,s} = \sqrt{2a_{T-s}(X^s_{t,s})} \text{d}W_s + \{b_{T-s} - u_{T-s}\}(X^s_{t,s}) \text{ds},
\]
\( t \in [0, T], s \in [t, T], X^s_{t,t} = x \in \mathbb{R}^d, \tag{3.8}
\]
and the generator in (3.4) reduces to
\[
\mathcal{L}_s := \text{tr}\{a_{T-s}\nabla^2\} + \{b_{T-s} - u_{T-s}\} \cdot \nabla, \quad s \in [0, T].
\]
(b) We prove the Kolmogorov backward equation
\[
\partial_t P_{t,s} f = -\mathcal{L}_t P_{t,s} f, \quad f \in C^2_b, t \in [0, s], s \in (0, T]. \tag{3.9}
\]
For any \( f \in C^2_b \), by Itô’s formula we have
\[
P_{t,s} f(x) = f(x) + \int_t^s P_{t,r}(\mathcal{L}_r f)(x) \text{dr}, \quad (t, s) \in D_T, \tag{3.10}
\]
where \( \int_t^s P_{t,r}(\mathcal{L}_r f)(x) \text{dr} = \mathbb{E}\int_t^s \mathcal{L}_r f(X^r_{t,r}) \text{dr} \) exists, since Krylov’s estimate in Lemma 3.1(2) holds under (A) and \( \|u\|_\infty < \infty \).

By (3.10), we obtain the Kolmogorov forward equation
\[
\partial_s P_{t,s} f = P_{t,s}(\mathcal{L}_s f), \quad s \in [t, T]. \tag{3.11}
\]
On the other hand, \( b^{(1)} = 0 \) and (A) imply
\[
\|\mathcal{L} f\|_{L^\infty_{0,0}} \leq c_0 \|f\|_{C^2_b} \tag{3.12}
\]
for some constant \( c_0 > 0 \). By Lemma 3.1(1), for any \( s \in (0, T] \), the PDE
\[
(\partial_t + \mathcal{L}_t) \tilde{u}_t = -\mathcal{L}_t f, \quad t \in [0, s], \tilde{u}_s = 0 \tag{3.13}
\]
has a unique solution \( \tilde{u} \in \mathcal{U}(p_0, q_0) \), such that
\[
\|\nabla^2 \tilde{u}\|_{L^\infty_{0,0}(0,s)} \leq c_1 \|f\|_{L^\infty_{0,0}(0,s)} \tag{3.14}
\]
holds for some constant \( c_1 > 0 \) independent of \( s \). By Itô’s formula in Lemma 3.1(3),
\[
\text{d} \tilde{u}_t(X^s_{0,t}) = -\mathcal{L}_t f(X^s_{0,t}) + \langle \nabla f(X^s_{0,t}), \sqrt{2a_{T-s}}(X^s_{0,t}) \text{d}W_t \rangle, \quad t \in [0, s].
\]
This (and 3.11) imply
\[
0 = \tilde{u}_s(x) = \tilde{u}_t(x) - \int_t^s (P_{t,r}\mathcal{L}_r f)(x) \text{dr}
\]
\[
= \tilde{u}_t(x) - \int_t^s \frac{d}{dr}(P_{t,r} f) \text{dr} = \tilde{u}_t(x) - P_{t,s} f(x) + f(x), \quad t \in [0, s].
\]
Thus,
\[
\tilde{u}_t = P_{t,s} f - f, \quad t \in [0, s]. \tag{3.15}
\]
Combining this with (3.13) we derive (3.9).

(c) By (3.7) and (3.9), we see that \( u \) solves (1.6) with \( u \in \mathcal{U}(p_0, q_0) \) provided
\[
\|\nabla^2 u\|_{L^\infty_{0,0}} < \infty. \tag{3.16}
\]
By (3.12), (3.14) and (3.15), we find a constant \( c_2 > 0 \) such that
\[
\sup_{t \in [0,s]} \|\nabla^2 P_{t,s} f\|_{L^\infty_{0,0}(0,s)} \leq c_2 \|f\|_{C^2_b}, \quad s \in (0, T], f \in C^2_b.
\]
Combining this with (3.6), \( b^{(1)} = 0 \) and \( \|u_0\|_{C^2_b} + \int_0^T \|V_t\|_{C^2_b} dt < \infty \), we prove (3.16). \( \Box \)
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\section{Application to (1.1)}

For any \( n \in \mathbb{N} \), let \( C^p_n \) be the class of real functions \( f \) on \( E \) having derivatives up to order \( n \) such that
\[
\| f \|_{C^p_n} := \sum_{i=0}^{n} \| \nabla^i f \|_{\infty} < \infty,
\]
where \( \nabla^0 f := f \). Moreover, for \( \alpha \in (0, 1) \), we denote \( f \in C^{\alpha,n}_b \) if \( f \in C^p_n \) such that
\[
\| f \|_{C^{\alpha,n}_b} := \| f \|_{C^p_n} + \sup_{x \neq y} \frac{\| \nabla^\alpha f(x) - \nabla^\alpha f(y) \|}{|x - y|^\alpha} < \infty.
\]

Consider the following future distribution dependent SDE on \( \mathbb{R}^d \):
\[
dX^s_t = \left[ E \int_s^T \nabla \psi_{T-r} (X^y_{s,r}) \, dr - E u_0 (X^y_{s,T}) \right] \, \mathbf{1}_{g = X^s_x} \, ds + \sqrt{2n} dW_t, \quad X^s_x = x, s, t \in [t,T].
\]
(4.1)

See Definition 1.1 below for the definition of solution. When \( E = \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d \), we extend \( u_0 \) and \( \psi_t \) to \( \mathbb{T}^d \) periodically, i.e. for a function \( f \) on \( \mathbb{T}^d \), it is extended to \( \mathbb{R}^d \) as in (1.4). With this extension, we also have the SDE (4.1) for the case \( E = \mathbb{T}^d \).

**Theorem 4.1.** If there exists \( n \geq 2 \) such that \( u_0 \in C^p_n \) and \( \psi_t \in C^p_n \) for a.e. \( t \in [0,T] \) with
\[
\int_0^T (\| \nabla \psi_t \|_{\infty}^2 + \| \psi_t \|_{C^p_n}) \, dt < \infty.
\]
Then (4.1) is well-posed and (1.1) has a unique solution satisfying
\[
\sup_{t \in [0,T]} \| u_t \|_{C^p_n} < \infty,
\]
and the solution is given by
\[
u_t(x) = E u_0 (X^x_{T-t,T}) - E \int_{T-t}^T \nabla \psi_{T-s} (X^x_{T-t,s}) \, ds.
\]
(4.3)

We only prove for \( E = \mathbb{R}^d \) as the case for \( E = \mathbb{T}^d \) follows by extending functions from \( \mathbb{T}^d \) to \( \mathbb{R}^d \) as in (1.4).

Let \( I_d \) be the \( d \times d \) identity matrix. By Theorem 1.1 with \( b = 0, a = \kappa I_d \) and \( V = -\nabla \psi \), for any \( (p_0, q_0) \in \mathcal{K} \), (1.1) has a unique solution in the class \( \mathcal{U}(p_0, q_0) \), and by (4.3),
\[
u_t(x) := E u_0 (X^x_{T-t,T}) - E \int_{T-t}^T \nabla \psi_{T-s} (X^x_{T-t,s}) \, ds = P_{T-t,T} u_0 (x) - \int_{T-t}^T P_{T-t,s} \nabla \psi_{T-s} (x) \, ds, \quad (t, x) \in [0,T] \times \mathbb{R}^d.
\]
(4.4)
By (3.8) for the present $a$ and $b$, $X^x_{t,s}$ solves the SDE
\[
\frac{dX^x_{t,s}}{ds} = \sqrt{2\kappa}dW_s - u_{T-s}(X^x_{t,s})ds, \quad X^x_{t,t} = x, \quad t \in [0,T], \quad s \in [t,T],
\]
and the generator is
\[
\mathcal{L}_s := \kappa \Delta - u_{T-s} \cdot \nabla, \quad s \in [0,T].
\]
It remains to prove (4.2). To this end, we present the following lemma.

**Lemma 4.2.** Let $P_{t,s}f := E[f(X^x_{t,s})]$ for the SDE (4.5). Let $m \geq 1$ such that
\[
\sup_{t \in [0,T]} \| u_t \|_{C^m} + \| f \|_{C^{m+1}} < \infty,
\]
then $\sup_{(t,s) \in DT} \| P_{t,s}f \|_{C^{m+1}} < \infty$.

**Proof.** By (4.5) and $\sup_{t \in [0,T]} \| u_t \|_{C^m} < \infty$, we have
\[
\sup_{(t,s) \in DT \times R^d} E[\| \nabla_i X^x_{t,s} \|] < \infty, \quad 1 \leq i \leq m.
\]
By chain rule, this implies that for some constant $c_0 > 0$,
\[
\sup_{(t,s) \in DT} \| P_{t,s}g \|_{C^m} \leq c_0 \| g \|_{C^m}, \quad g \in C^m_b.
\]
Let $P_0^t = e^{\kappa \Delta t}$. By $\partial_t P^0_{t-s} = P^0_{t-s} \kappa \Delta$ and (3.9), we have
\[
\partial_t P^0_{t-s} P_{t,s}f = P^0_{t-s} (\nabla P_{t,s}f, u_{T-t}), \quad r \in [t,s].
\]
So,
\[
P_{t,s}f = P_{t,s}^0 f - \int_t^s P_{t-r}^0 (\nabla P_{r,s}f, u_{T-r}) dr.
\]
It is well known that for any $\alpha, \beta > 0$ there exists a constant $c_{\alpha, \beta} > 0$ such that
\[
\| P^0_t g \|_{C^\alpha} \leq c_{\alpha, \beta} t^{-\frac{\alpha}{2}} \| g \|_{C^\beta}, \quad t > 0, \quad g \in C^\beta_b.
\]
This together with (4.8) implies that for some constants $c_1, c_2 > 0$,
\[
\| P_{t,s}f \|_{C^{\alpha+\frac{1}{2}}} \leq c_1 \| f \|_{C^{\alpha+\frac{1}{2}}} + c_1 \int_t^s (t+r-s)^{-\frac{\alpha}{2}} \| \nabla P_{r,s}f, u_{T-r} \|_{C^{\beta-1}} dr.
\]
Combining this with (4.7) and $\| f \|_{C^m} + \sup_{t \in [0,T]} \| u_t \|_{C^m} < \infty$, we obtain
\[
\sup_{(t,s) \in DT} \| P_{t,s}f \|_{C^{\alpha+\frac{1}{2}}} < \infty.
\]
By this together with (4.8) and (4.6), we find a constant $c_2 > 0$ such that
\[
\sup_{(t,s) \in DT} \| P_{t,s}f \|_{C^{m+1}} \leq c_2 \| f \|_{C^{m+1}}
\]
\[
+ c_2 \sup_{(t,s) \in DT} \int_t^s (t+r-s)^{-\frac{\alpha}{2}} \| \nabla P_{r,s}f, u_{T-r} \|_{C^{\beta-1}} dr < \infty.
\]
We now prove (4.2) as follows. By $u \in \mathcal{U}(p_0, q_0)$, we have
\[
\| u \|_\infty + \| \nabla u \|_\infty < \infty.
\]
Combining this with (4.4) and Lemma 4.2, we prove (4.2) by inducing in $m$ up to $m = n$. 

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