Relativistic Hydrogen-Like Atom on a Noncommutative Phase Space

Huseyin Masum\textsuperscript{a}, Sayipjamal Dulat\textsuperscript{a,*} and Mutallip Tohti\textsuperscript{b}

\textsuperscript{a} School of Physics Science and Technology, Xinjiang University, Urumqi, 830046, China and
\textsuperscript{b} Radio Therapy center, Xinjiang Tumor Hospital, Urumqi, 830000, China

The energy levels of hydrogen-like atom on a noncommutative phase space were studied in the framework of relativistic quantum mechanics. The leading order corrections to energy levels $2P_{1/2}$, $2P_{3/2}$ and $2P_{1/2}$ were obtained by using the $\theta$ and the $\bar{\theta}$ modified Dirac Hamiltonian of hydrogen-like atom on a noncommutative phase space. The degeneracy of the energy levels $2P_{1/2}$ and $2P_{3/2}$ were removed completely by $\theta$-correction. And the $\theta$-correction shifts these energy levels.

PACS numbers: 02.40.Gh, 03.65.Pm, 03.65.Ge

I. INTRODUCTION

The approach to noncommutative quantum field theory based on star products and Seiberg-Witten maps allows for the generalization of the standard model of particle physics to the case of noncommutative spacetime. Since noncommutative quantum field theory may solve the puzzles of the standard model, there are many papers concerning the quantum field theory on a noncommutative space-time \cite{1-4}. Apart from these studies, much research has been devoted to the study of various aspects of quantum mechanics (QM) on a noncommutative space (NCS) and a noncommutative phase space (NCPS), because the main goal of noncommutative quantum mechanics (NCQM) is to find measurable effects of noncommutativity. For example, the papers \cite{5-7} was devoted to study the Aharonov-Bohm phase on a NCS and a NCPS. A lower bound $1/\sqrt{\theta} \geq 10^{-6}$GeV for the space noncommutativity parameter was obtained \cite{5}. The Aharonov-Casher phase for a spin-1/2 and a spin-1 particle on a NCS and a NCPS has been studied in Refs. \cite{8-11}, and a limit $1/\sqrt{\theta} \geq 10^{-7}$GeV for the space noncommutativity parameter was obtained \cite{8}. The noncommutative quantum Hall effect has been studied in Refs. \cite{12-14}, and the authors of Ref. \cite{12} found a lower limit of $1/\sqrt{\theta} \geq 10^{-12}$GeV on the noncommutativity parameter. Ref. \cite{15,16} discussed the noncommutative spin Hall effect (SHE), and obtained interesting results. Furthermore, a lower limit of $1/\sqrt{\theta} \geq 10^{-9}$GeV on the noncommutativity parameter was given in Ref. \cite{16}. The authors in Refs. \cite{17,18} studied hydrogen atom spectrum in the nonrelativistic quantum mechanics framework both on a NCS and a NCPS, respectively, and the authors in Ref.\cite{17} found the constraint on $\theta$ is $1/\sqrt{\theta} \geq 10^{-6}$GeV. Reference \cite{19} provided the constraint: $1/\sqrt{\theta} \geq 3$GeV by studying the transitions in the helium atom. A possibility of testing spatial noncommutativity via cold Rydberg atoms is suggested in Ref.\cite{20}. In order to refine the work in Ref. \cite{17}, the authors in Ref. \cite{21} has studied the hydrogen atom on a NCS in the framework of the $\theta$ modified Dirac equation with Coulomb potential, and they showed that the degeneracy of the energy levels $2P_{1/2}$ and $2P_{3/2}$ were removed completely.

In this paper we study the NCPS effects on the energy levels of relativistic hydrogen-like atom. To begin, we must define what we mean by “noncommutative phase space”. NCPS is a deformation of ordinary space in which the space and momentum coordinate operators satisfy the following relation($\hbar = c = 1$):

\[
[\hat{x}_i, \hat{x}_j] = i\Theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\bar{\Theta}_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij},
\]

where $\Theta_{ij}$ and $\bar{\Theta}_{ij}$ are the totally antisymmetric real tensors on a NCPS: $\hat{x}$, $\hat{p}$ are the coordinate and momentum operators on a NCPS. In three dimensional NCPS $(i,j,k = 1,2,3)$, we can define a vector $\theta = (\theta_1, \theta_2, \theta_3)$ and $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$ with $\theta_i$ and $\bar{\theta}_i$ satisfy $\Theta_{ij} = \epsilon_{ijk}\theta_k$, $\bar{\Theta}_{ij} = \epsilon_{ijk}\bar{\theta}_k$, here $\epsilon_{ijk}$ is the Levi-Civita symbol.

On a NCPS, the normal product of two arbitrary functions should be replaced by the star product(Moyal-Weyl product). For example, the time independent Schrödinger equation on a NCPS is

\[
H(p, r) \star \Psi(r) = E\Psi(r).
\]

Here $H(x,p)$ is the usual Hamiltonian operator. On a NCPS star product between two functions is defined as

\[
(f \star g)(x,p) = e^{\frac{i}{\alpha^2} \Theta_{ij} \partial_i \partial_j} e^{\frac{i}{\bar{\alpha}^2} \bar{\Theta}_{ij} \partial_i \partial_j} f(x,p)g(x,p)
\]

\[
= f(x,p)g(x,p) + \frac{i}{2\alpha^2} \Theta_{ij} \partial_i f \partial_j g |_{x_i = x_j}
\]

\[
+ \frac{i}{2\bar{\alpha}^2} \bar{\Theta}_{ij} \partial_i g \partial_j f |_{p_i = p_j} + O(\theta^2, \bar{\theta}^2),
\]

here $f(x,p)$ and $g(x,p)$ are two arbitrary functions, the scaling constant $\alpha$ is related to the noncommutativity parameters $\theta$ and $\bar{\theta}$ via $\theta \bar{\theta} = 4\alpha^2(1 - \alpha^2)$. To replace the star product in Schrödinger Eq.$(2)$ with a usual product, first we need to replace $x_i$ and $p_i$ with a generalized Bopp’s shift as

\[
\hat{x}_i = \alpha x_i - \frac{1}{2\hbar \alpha} \Theta_{ij} p_j ,
\]

\[
\hat{p}_i = \alpha p_i + \frac{1}{2\hbar \alpha} \bar{\Theta}_{ij} x_j , \quad i,j = 1,2, ..., n.
\]

*Electronic address: dulat98@yahoo.com
Thus on a NCPS the Schrödinger Eq. (2) becomes,

\[ H(\hat{p}, \hat{r})\Psi(r) = E\Psi(r), \quad H(p, r) \ast \Psi(r) \equiv H(\hat{p}, \hat{r})\Psi(r), \]

(5)

here \( r \) and \( p \) are the space and the momentum operators on an ordinary space.

This paper is organized as follows. In section II, first we give usual Dirac Hamiltonian for an electron in the Coulomb field, and we list corresponding eigenfunctions and eigenvalues. Then we provide the NCPS Dirac Hamiltonian for the hydrogen-like atom. In Section III, using the perturbation theory, leading order correction to the energy levels due to space-space and momentum-momentum noncommutativity are obtained. Conclusions are given in the last section.

II. DIRAC HAMILTONIAN WITH COULOMB FIELD ON A NCPS

Before writing Dirac Hamiltonian with Coulomb field on a NCPS, first, we list usual Dirac Hamiltonian, Dirac equation, energy spectrum and eigenfunctions by following Refs. [22]-[26]. Then we provide the \( \theta \) and \( \bar{\theta} \) modified Dirac Hamiltonian on a NCPS. On an ordinary space, the Dirac Hamiltonian for an electron with charge \(-e(e > 0)\) and mass \( m \) in the Coulomb field of a nucleus \( Ze \) is given by

\[ H = \hat{\alpha} \cdot \mathbf{p} + m\gamma^0 + V(r), \]

(6)

where \( \hat{\alpha}_i = \gamma_i \gamma_i \), \( \gamma_i \) are the Dirac matrices, \( \mathbf{p} = -i\nabla \), \( V(r) = -Ze^2/r \). The stationary Dirac equation for electron is

\[ H\Psi_{n,j,l,j_z} = E_{n,j}\Psi_{n,j,l,j_z}, \]

(7)

where energy eigenvalues of electron in atom with a Coulomb potential is

\[ E_{n,j} = \frac{m}{\sqrt{1 + \left(\frac{Ze^2}{\gamma_{n,j,l,j_z}}\right)^2}}, \]

(8)

with

\[ \gamma_{n'} = \sqrt{\kappa^2 - (Ze^2)^2}, \quad n' = n - |\kappa| = n - j - \frac{1}{2}, \]

\[ \kappa = \mp(j + \frac{1}{2}) = \begin{cases} -(j + \frac{1}{2}) & \text{for } j = l + \frac{1}{2}, \\ (j + \frac{1}{2}) & \text{for } j = l - \frac{1}{2}, \end{cases} \]

(9)

here \( n = 1, 2, 3, \ldots \) is the principal quantum number. From equation (8) one can see that the energy eigenvalues only depend on the \( n, j \) and \( Z \). The wave functions \( \Psi_{n,j,l,j_z} \) are also eigenfunction of \( \hat{J}^2 \) and \( \hat{J}_z \)

\[ j^2\Psi_{n,j,l,j_z} = j(j + 1)\Psi_{n,j,l,j_z}, \quad j_z\Psi_{n,j,l,j_z} = j_z\Psi_{n,j,l,j_z}. \]

(10)

The corresponding wave functions \( \Psi_{n,j,l,j_z} \) are given by

\[ \varphi_{n,j,l,j_z} = \frac{\varphi_{n,j,l,j_z}}{\chi_{n,j,l,j_z}}, \quad \varphi_{n,j,l,j_z} = ig(r)\Omega_{n,j,l,j_z}(\mathbf{r}), \]

(11)

\[ \chi_{n,j,l,j_z} = f(r)\Omega_{n,j,l,j_z}(\mathbf{r}), \]

where \( l' = 2j - l \). The spherical spinors \( \Omega_{n,j,l,j_z} \) are eigenfunction of the operators \( \hat{L}^2, \hat{J}^2 \) and \( \hat{S}^2 = (\frac{1}{2}\hat{\sigma})^2 \) with eigenvalues \( l(l + 1)\hbar^2, j(j + 1)\hbar^2 \) and \( \frac{1}{4}\hbar^2 \) respectively; the explicit forms of the \( \Omega_{n,j,l,j_z} \) for the cases \( j = l + \frac{1}{2} \) and \( j = l - \frac{1}{2} (j \geq \frac{1}{2}) \) are

\[ \Omega_{n,j,l,j_z} = \left\{ \begin{array}{ll} \sqrt{\frac{j + l}{2j}} Y_{l,j} - \frac{1}{2}(\theta, \varphi) & \text{for } j = l + \frac{1}{2}, \\ \sqrt{\frac{j + l + 1}{2j + 1}} Y_{l,j} + \frac{1}{2}(\theta, \varphi) & \text{for } j = l - \frac{1}{2}. \end{array} \right\} \]

(12)

here the root factors of (12) are Clebsch-Gordan coefficients, \( Y_{A,B}(\theta, \varphi) \) are the spherical harmonics; the normalized radial wave functions \( f(r) \) and \( g(r) \) in (11) are

\[ f(r) = \pm \frac{2(\lambda_{n'})^{3/2}}{2\Gamma(\beta_{n'})} \sqrt{\frac{\pi}{\kappa_{2n' - 1}}} e^{-r\lambda_{n'}}\left\{ \eta_{n'}(\lambda_{n'} - \kappa)n! \right\}^{1/2} (2r\lambda_{n'})^{\gamma_{n' - 1}} \Phi(-n', \beta_{n'}/2r\lambda_{n'}) \]

\[ + n'\Phi(1 - n', \beta_{n'}/2r\lambda_{n'}), \]

(13)

with

\[ \lambda_{n'} = \sqrt{m^2 - E_{n,j}^2}, \quad \lambda_{n'}^{\pm} = \frac{1}{m} \pm \frac{1}{m}, \]

\[ \eta_{n'} = \frac{(n' + \gamma)m}{E_{n,j}}, \quad \beta_{n'} = 2\gamma + 1, \quad \gamma_{n'} = \sqrt{\kappa^2 - (Ze^2)^2}, \quad \kappa = \mp(j + \frac{1}{2}) = \begin{cases} -(j + \frac{1}{2}) & \text{for } j = l + \frac{1}{2}, \\ (j + \frac{1}{2}) & \text{for } j = l - \frac{1}{2}, \end{cases} \]

(14)

and the hypergeometric confluent function \( \Phi(a, b; z) \) [27] is

\[ \Phi(a, b; z) = 1 + \frac{a}{b}z + \frac{a(a + 1)z^2}{b(b + 1)2!} + \ldots. \]

(15)

As well as the Gamma function [27] is

\[ \Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt, \quad \Gamma(n + 1) = n\Gamma(n). \]

(16)

On a NCPS, the Dirac Hamiltonian (6) can be written as

\[ \hat{H}(\hat{p}, \hat{r}) = \hat{\alpha} \cdot \hat{p} + m\gamma^0 - \hat{V}(\hat{r}), \]

(17)
where the Coulomb potential with noncommutative correction terms is

\[
\hat{V}(\vec{r}) = -\frac{Ze^2}{\sqrt{x_i x_i}} = -\frac{Ze^2}{\sqrt{(\alpha x_i - \frac{1}{2\alpha} \Theta_{ij} p_j)(\alpha x_i - \frac{1}{2\alpha} \Theta_{ik} p_k)}} = -\frac{Ze^2}{\alpha r} - \frac{Ze^2}{4\alpha^3 r^3} (\mathbf{L} \cdot \mathbf{\theta}) + O(\theta^2),
\]

By (4) and (18), we rewrite (17) as

\[
\hat{H} = \alpha \hat{\alpha} \cdot \mathbf{p} + m\gamma^0 - \frac{Ze^2}{\alpha r} - \frac{Ze^2}{4\alpha^3 r^3} (\mathbf{L} \cdot \mathbf{\theta}) \\
+ \frac{1}{4\alpha} \hat{\alpha} \cdot (\mathbf{r} \times \mathbf{\theta}) + O(\theta^2) + O(\theta^2) \\
= \alpha (\hat{\alpha} \cdot \mathbf{p} + m\gamma^0 - \frac{Ze^2}{r}) - \frac{Ze^2}{4\alpha} (\mathbf{L} \cdot \mathbf{\theta}) \\
+ \frac{1}{4\alpha} \hat{\alpha} \cdot (\mathbf{r} \times \mathbf{\theta}) + O(\theta^2) + O(\theta^2) \\
= H' + H'^0 + H^0 + O(\theta^2) + O(\theta^2),
\]

with

\[
H' = \alpha (\hat{\alpha} \cdot \mathbf{p} + m\gamma^0 - \frac{Ze^2}{r}), \quad H^0 = -\frac{Ze^2}{4\alpha} (\mathbf{L} \cdot \mathbf{\theta}),
\]

\[
H^0 = \frac{1}{4\alpha} \hat{\alpha} \cdot (\mathbf{r} \times \mathbf{\theta}),
\]

where \( \epsilon' = \frac{\varepsilon}{\alpha} \), \( m' = \frac{m}{\alpha} \); \( \mathbf{L} \) is the orbital angular momentum operator. Since the noncommutative corrections \( H^0 \) and \( H^0 \) are very small compared to the \( H \), the change in the hydrogen-like atom energy levels due to noncommutative parts \( H^0 \) and \( H^0 \) can always be treated as some perturbation of the commutative counter part \( H \). Up to first order in \( \theta \) and \( \hat{\theta} \), one can use the usual wave functions in our forthcoming calculation of energy levels.

### III. RELATIVISTIC NCPS CORRECTION OF ENERGY LEVELS

By using the exact eigenfunctions of Dirac Hamiltonian \( H \) and treating \( H^0 \) and \( H^0 \) as perturbations, we can calculate the modification of energy levels of the Hydrogen like atom on a NCPS in this section. Thus NCPS and NCPS effects on the energy levels are obtained by computing the eigenvalues of the secular matrix \( E^0 \) and \( E^0 \), characterized by the average values of the operators \( H^0 \) and \( H^0 \), with respect to the Dirac spinors \( \Psi_{n,j,l,j} \), with angular momentum selection rules, i.e., \( \Delta j_z = |j_z - j'_z| = 0.1 \) and \( \Delta l = |l - l'| = 0. \) Thus matrix elements of \( \hat{H} \) are

\[
E_{j,j'}(nL_j) = \langle \Psi_{n,j,l,j} | \hat{H} | \Psi_{n,j,l,j'} \rangle = \langle \Psi_{n,j,l,j} | H' + H^0 | \Psi_{n,l,j,j'} \rangle = E'_{n,j} + E^0_{j,j'}(nL_j) + E^0_{j,j'}(nL_j),
\]

where \( E'_{n,j} \) is the eigenvalue of \( H' \), from equation (8) and first equation of (20), we have

\[
E'_{n,j} = E_{n,j} + \Delta E_{n,j},
\]

with

\[
\Delta E_{n,j} = -2m \left[ 1 + \left( \frac{Ze^2}{\gamma n' + n} \right)^2 \right]^{\frac{3}{2}} \times \frac{Z^2 e^3 \left[ (\gamma n' + n') + Z^2 e^4 (\kappa^2 - Z^2 e^4)^{\frac{3}{2}} \right]}{(\gamma n' + n')^{\frac{3}{2}}} c (1 - \alpha).
\]

Therefore the relativistic NCPS energy correction for a given energy level \( nL_j \) is

\[
\Delta E_{j,j'}(nL_j) = \Delta E_{n,j} + E^0_{j,j'}(nL_j) + E^0_{j,j'}(nL_j).
\]

In leading order, the matrix elements of \( H^0 \) with respect to the Dirac spinors \( \Psi_{n,j,l,j} \), are defined as

\[
E^0_{j,j'}(nL_j) = \langle \Psi_{n,j,l,j} | H^0 | \Psi_{n,j,l,j'} \rangle = \frac{Ze^2}{4\alpha^3} \int_0^{4\pi} \frac{dr}{r} |g(r)|^2 + \frac{Ze^2}{4\alpha} \int_0^{4\pi} \frac{dr}{r} \left( g(r) \right)^2 \\
\times \int_0^{4\pi} d\Omega \left[ \gamma_{j,l,j} \left( \mathbf{L} \cdot \mathbf{\theta} \right) \Omega_{j,l,j} \right] - \frac{Ze^2}{4\alpha} \rho(nL_j) \Theta_{j,j'}(nL_j),
\]

where \( \rho(nL_j) \) is defined as

\[
\rho(nL_j) = \int_0^{\infty} \frac{dr}{r} |g(r)|^2,
\]

\[
\Theta_{j,j'}(nL_j) = \int_0^{4\pi} \frac{d\Omega}{\left( \gamma_{j,l,j} \left( \mathbf{L} \cdot \mathbf{\theta} \right) \Omega_{j,l,j'} \right)}.
\]

In leading order, matrix elements of \( H^0 \) with respect to the Dirac spinors \( \Psi_{n,j,l,j} \), are defined as

\[
E^0_{j,j'}(nL_j) = \langle \Psi_{n,j,l,j} | H^0 | \Psi_{n,j,l,j'} \rangle = \frac{1}{4\alpha} \hat{\alpha} \cdot (\mathbf{r} \times \mathbf{\theta}) | \Psi_{n,j,l,j'} >
\]

Because \( f(r)/g(r) \approx v/c \) ( \( v \) is the velocity of electron in the first Bohr orbit ), we can neglect the second term in (25), then we have

\[
E^0_{j,j'}(nL_j) = \frac{Ze^2}{4\alpha} \int_0^{\infty} \frac{dr}{r} |g(r)|^2 \\
\times \int_0^{4\pi} d\Omega \left[ \gamma_{j,l,j} \left( \mathbf{L} \cdot \mathbf{\theta} \right) \Omega_{j,l,j} \right] - \frac{Ze^2}{4\alpha} \rho(nL_j) \Theta_{j,j'}(nL_j),
\]

with

\[
\rho(nL_j) = \int_0^{\infty} \frac{dr}{r} |g(r)|^2,
\]

\[
\Theta_{j,j'}(nL_j) = \int_0^{4\pi} d\Omega \left[ \gamma_{j,l,j} \left( \mathbf{L} \cdot \mathbf{\theta} \right) \Omega_{j,l,j'} \right].
\]
\[ E^\theta_{j,j^\prime}(nL_j) = \frac{1}{4\alpha} \rho(nL_j) \bar{\Theta}_{j,j^\prime}(nL_j), \]  
(29) 

with 
\[ \bar{\rho}(nL_j) = \int_0^\infty dr r^3 
\]
where \( \sigma = (\sigma_i) \) are the Pauli matrices; \( n = r, \) \( n \) is a unit vector in a direction of \( r. \) Note that for \( 2S_{1/2}, 2P_{1/2} \) and \( 2P_{3/2} \) cases, \( g(r) \) and \( f(r) \) are real functions, thus we can simplify (28) as 
\[ E^\bar{\theta}_{j,j^\prime}(nL_j) = \frac{1}{4\alpha} \bar{\rho}(nL_j) \bar{\Theta}_{j,j^\prime}(nL_j), \]  
(30) 

In the following sections we will calculate the \( \theta \) and \( \bar{\theta} \) modifications of the \( 2S \) and \( 2P \) energy levels. In our calculation, we choose \( \theta_1 = \theta_2 = 0, \theta_3 = \theta \) as well as \( \bar{\theta}_1 = \bar{\theta}_2 = 0 \) and \( \bar{\theta}_3 = \bar{\theta} \), that can be achieved by rotational invariance or redefinition of coordinates.

A. Relativistic NCPS correction of \( 2S_{1/2} \) and \( 2P_{1/2} \)

In this subsection we calculate the \( \theta \) and \( \bar{\theta} \) corrections for the energy levels \( 2S_{1/2} \) and \( 2P_{1/2} \). From (26) the \( \theta \) correction for the level \( 2S_{1/2} \) \( (n = 2, n' = 1, j = 1/2, l = 0, j = l + 1, l' = 1, j_3 = \pm 1/2) \) is 
\[ E^\theta_{j,j^\prime}(2S_{1/2}) = -\frac{Ze^2}{4\alpha} \rho(2S_{1/2}) \Theta_{j,j^\prime}(2S_{1/2}), \]  
(31) 

where \( \rho(2S_{1/2}) \) and the \( \Theta_{j,j^\prime}(2S_{1/2}) \) are follows from (27) and by some calculation we obtain 
\[ \Theta_{j,j^\prime}(2S_{1/2}) = \int_0^{4\pi} d\Omega \bar{\Omega} L \cdot \Omega = 0. \]  
(32) 

Thus the \( \theta \) correction for the relativistic energy level \( 2S_{1/2} \) is 
\[ E^\theta_{j,j^\prime}(2S_{1/2}) = 0. \]  
(33) 

It is clear that the \( H^\theta \) does not modify the energy level \( 2S_{1/2} \).

From (29) the \( \bar{\theta} \) correction for the \( 2S_{1/2} \) level is 
\[ E^\bar{\theta}_{j,j^\prime}(2S_{1/2}) = \frac{1}{4\alpha} \bar{\rho}(2S_{1/2}) \bar{\Theta}_{j,j^\prime}(2S_{1/2}), \]  
(34) 

here the \( \bar{\rho}(2S_{1/2}) \) and the \( \bar{\Theta}_{j,j^\prime}(2S_{1/2}) \) are follows from (30):
\[ \bar{\rho}(2S_{1/2}) = \int_0^\infty dr r^3 g(r) f(r), \]  
(35) 

\[ \bar{\Theta}_{j,j^\prime}(2S_{1/2}) = \int_0^{4\pi} d\Omega \bar{\Omega} L \cdot \Omega = 0. \]  
(36) 

Thus the matrix \( \bar{\Theta}(2S_{1/2}) \) is 
\[ \bar{\Theta}(2S_{1/2}) = \frac{4}{3} \left( \begin{array}{cc} \bar{\theta}_3 & 0 \\ 0 & \bar{\theta}_3 \end{array} \right). \]  
(37) 

Corresponding eigenvalues of the matrix \( \bar{\Theta}(2S_{1/2}) \) are 
\[ \bar{\Lambda}_{\pm 1/2}(2S_{1/2}) = \frac{4}{3} | \bar{\theta} |, \]  
(38) 

where \( | \bar{\theta} | = \sqrt{\bar{\theta}_1 \bar{\theta}_1}. \) The \( \bar{\theta} \) correction to the relativistic energy level \( 2S_{1/2} \) is 
\[ E^\bar{\theta}_{\pm 1/2}(2S_{1/2}) = \frac{1}{4\alpha} \bar{\rho}(2S_{1/2}) \bar{\Lambda}_{\pm 1/2}(2S_{1/2}) \]  
(39) 

Note that the \( \bar{\theta} \) correction shifts the relativistic energy level \( 2S_{1/2}. \) From (24), (32) and (39) the relativistic NCPS correction for the energy level \( 2S_{1/2} \) is 
\[ \Delta E_{\pm 1/2}(2S_{1/2}) = \Delta E_{2,1/2} + E^\theta_{\pm 1/2}(2S_{1/2}). \]  
(40) 

In the following we calculate the \( \theta \) and the \( \bar{\theta} \) corrections for the energy level \( 2P_{1/2}. \) From (26) the \( \theta \) correction
for the level $2P_{1/2}$ ($n = 2, n' = 1, j = 1/2, l = 1, j = l = 1/2, l' = 0, j_2 = \pm 1/2$) is
\[
E_\theta^{\delta}(2P_{1/2}) = \frac{Ze^2}{4\alpha^3}\rho(2P_{1/2})\Omega_{j_2,j_2'}(2P_{1/2}),
\]
where the radial integral $\rho(2P_{1/2})$ and the matrix elements $\Omega_{j_2,j_2'}(2P_{1/2})$ can be obtained from (27)
\[
\rho(2P_{1/2}) = \int_0^{\infty} \frac{dr}{r^2} g(r)^2
\]
\[
= \frac{2\lambda_1^2(\lambda_1^2)}{\eta_1(\beta_1 - 1)(\beta_1 - 2)(\beta_1 - 3)}
\times \left\{ \frac{1}{\eta_1 - 1} - 2(\eta_1 - 2)(\beta_1 - 3) \right\},
\]
(42) here $\lambda_1, \lambda_1^2$ are given in (14) and $\beta_1, \eta_1$ are given in (35). From (27) we have
\[
\Theta_{\bar{j}_2,j_2'}(2P_{1/2}) = \int_0^{\infty} d\Omega_{j_2,j_2'}(L \cdot \theta)\Omega_{\bar{j}_2,j_2'}(2P_{1/2})
\]
By some calculation we get the following matrix
\[
\Theta(2P_{1/2}) = \frac{2}{3} \left[ \begin{array}{ccc} \theta_{3} & -\theta_{1} & \theta_{2} \\ \theta_{1} & \theta_{3} & -\theta_{2} \\ -\theta_{2} & \theta_{2} & 0 \end{array} \right],
\]
where $\theta_{\pm} = \theta_{1} \pm i\theta_{2}$. The eigenvalues of the matrix $\Theta(2P_{1/2})$ are
\[
\Lambda_{\pm 1/2}(2P_{1/2}) = \pm \eta_{1} |\theta|, \quad |\theta| = \sqrt{\eta_{1} \eta_{2}},
\]
(45) here $|\theta| = \sqrt{\eta_{1} \eta_{2}}$. Then the $\theta$ correction to the relativistic energy of the $2P_{1/2}$ level is
\[
E_{\pm 1/2}(2P_{1/2}) = \frac{Ze^2}{4\alpha^3} \rho(2P_{1/2})\Lambda_{\pm 1/2}(2P_{1/2})
\]
\[
= \frac{Ze^2}{4\alpha^3} \rho(2P_{1/2}) \left\{ \frac{\lambda_1^2(\lambda_1^2)}{\eta_1(\beta_1 - 1)(\beta_1 - 2)(\beta_1 - 3)} \right\}
\times \left\{ \frac{1}{\eta_1 - 1} - 2(\eta_1 - 2)(\beta_1 - 3) \right\},
\]
\[
+ \left\{ \frac{1}{\beta_1} \right\} (\beta_1 - 2)(\beta_1 - 3) \right\},
\]
(46) We can see that the degenerate level $2P_{1/2}$ splits into two sublevels by the $\theta$ correction.

From (29) the $\bar{\theta}$ correction for the $2P_{1/2}$ level is
\[
E_{\bar{j}_2,j_2'}^{\bar{\theta}}(2P_{1/2}) = \frac{Ze^2}{4\alpha^3} \rho(2P_{1/2})\bar{\Theta}_{\bar{j}_2,j_2'}(2P_{1/2}),
\]
(47) here the $\rho(2P_{1/2})$ and the $\bar{\Theta}_{\bar{j}_2,j_2'}(2P_{1/2})$ are follows from (30):
\[
\rho(2P_{1/2}) = \int_0^{\infty} dr r^3 g(r)^2 f(r)
\]
\[
= \frac{1}{8m_1} \left\{ \frac{1}{\eta_1 - 1} - \beta_1^2(\eta_1 - 1) + 2\beta_1(\beta_1 + 1) \times (\eta_1 - 1) - (\beta_1 + 1)(\beta_1 + 2)(\eta_1 - 1) \right\},
\]
\[
\bar{\Theta}_{\bar{j}_2,j_2'}(2P_{1/2}) = \int_0^{\infty} d\Omega_{j_2,j_2'}(\bar{\Omega} \cdot \bar{\theta})\bar{\Omega}_{\bar{j}_2,j_2'}(2P_{1/2}),
\]
\[
= -\frac{1}{24\alpha^3 m_1} \left\{ \frac{1}{\beta_1^2}(\eta_1 - 1) - 2(\beta_1 + 1) \times (\beta_1 + 2)(\eta_1 - 1) \right\},
\]
(52) From (32), (39), (40), (46), (52) and (53) one can see that the $\theta$ correction splits the original energy levels $2S_{1/2}, 2P_{1/2}$ into three sublevels $2S_{1/2}$ and $2P_{1/2}$, and the $\bar{\theta}$ correction shifts the energy levels. Modifications of these energy levels are illustrated in Fig.1 for $Z = 1$ (hydrogen atom).

B. Relativistic NCPS correction for the $2P_{3/2}$ level

In this subsection we calculate the $\theta$ and $\bar{\theta}$ corrections for the energy level $2P_{3/2}$. For $2P_{3/2}$ level ($n = 2, n' = 0, j = 3/2, l = 1, j = l + 1/2, l' = 2, j_2 = \pm 1/2, \pm 3/2)$, from (26) the $\theta$ correction is
\[
E_{\theta}^{\delta}(2P_{3/2}) = \frac{Ze^2}{4\alpha^3} \rho(2P_{3/2})\Theta_{\delta,j_3'}(2P_{3/2}),
\]
(54) here the $\rho(2P_{3/2})$ and the $\Theta_{\delta,j_3'}(2P_{3/2})$ have the following forms by (27)
\[
\rho(2P_{3/2}) = \int_0^{\infty} dr r^3 g(r)^2 f(r)
\]
\[
= \frac{1}{8m_1} \left\{ \frac{1}{\beta_0 - 1} - \beta_0^2(\beta_0 - 1) + 2\beta_0(\beta_0 + 1) \times (\beta_0 - 1) - (\beta_0 + 1)(\beta_0 + 2)(\beta_0 - 1) \right\},
\]
(55)
where \( \lambda_0 \) and \( \lambda_0^\pm \) are given in (14), and
\[
\gamma_0 = \sqrt{4 - (Ze^2)^2}, \quad \beta_0 = 2\gamma_0 + 1. \tag{56}
\]

The matrix elements \( \Theta_{j_3'j_3}(2P_{3/2}) \) are given by
\[
\Theta_{j_3'j_3}(2P_{3/2}) = \int_0^{4\pi} \, d\Omega \left[ \Omega_{3/2,1,1}^j (L \cdot \theta) \Omega_{3/2,1,1}^j \right], \tag{57}
\]
the corresponding matrix \( \Theta(2P_{3/2}) \) is
\[
\Theta(2P_{3/2}) = \frac{1}{3} \begin{pmatrix}
-3\theta_3 & \sqrt{3}\theta_+ & 0 & 0 \\
\sqrt{3}\theta_- & -\theta_3 & 2\theta_+ & 0 \\
0 & 2\theta_- & \theta_3 & \sqrt{3}\theta_+ \\
0 & 0 & \sqrt{3}\theta_- & -3\theta_3
\end{pmatrix}. \tag{58}
\]

Corresponding four nondegenerate eigenvalues \( \Lambda_{\pm 3/2}(2P_{3/2}) \) and \( \Lambda_{\pm 1/2}(2P_{3/2}) \) are
\[
\Lambda_{\pm 3/2}(2P_{3/2}) = \frac{1}{2} \big| \theta \big|, \quad \Lambda_{\pm 1/2}(2P_{3/2}) = \frac{1}{2} \big| \theta \big|, \tag{59}
\]
and hence, the \( \theta \) correction for the \( 2P_{3/2} \) level have the following form
\[
E_{\pm 3/2}(2P_{3/2}) = \frac{1}{2} \bigg| \frac{P}{\alpha^3} \bigg| \bigg| \frac{\Lambda_{\pm 3/2}(2P_{3/2})}{2} \bigg|, \tag{60}
\]
Note that the \( \theta \) correction splits the \( 2P_{3/2} \) level into four nondegenerate sublevels \( 2P_{3/2}^{\pm 3/2} \) and \( 2P_{3/2}^{\pm 1/2} \).

From (29) the \( \bar{\theta} \) correction for level \( 2P_{3/2} \) is
\[
E_{\pm 3/2}(2P_{3/2}) = \frac{1}{4\alpha} \bar{\theta}(2P_{3/2}) \Theta_{j_3'j_3}(2P_{3/2}), \tag{61}
\]
where the \( \bar{\theta}(2P_{1/2}) \) and \( \bar{\theta}_{j_3'j_3}(2P_{1/2}) \) are obtained from (30)
\[
\bar{\theta}(2P_{3/2}) = \int_0^\infty \, dr r^3 g(r) f(r) = -\frac{\beta_0 (\eta_0 + 2)}{8m\eta_0}, \tag{62}
\]
here \( \beta_0 \) is given in (56) and
\[
\eta_0 = \frac{\eta_0}{P_{\alpha,j}}, \quad \gamma_0 = \sqrt{4 - (Ze^2)^2}. \tag{63}
\]

Then the matrix \( \bar{\Theta}(2P_{3/2}) \) is
\[
\bar{\Theta}(2P_{3/2}) = \frac{2}{5} \begin{pmatrix}
\frac{4}{3}\theta_3 & -\frac{1}{3}\bar{\theta}_+ & 0 & 0 \\
-\frac{1}{3}\bar{\theta}_- & \frac{4}{3}\theta_3 & 0 & 0 \\
0 & 0 & \frac{2}{3}\theta_3 & \sqrt{3}\bar{\theta}_+ \\
0 & 0 & \frac{2}{3}\theta_3 & -\frac{4}{3}\theta_3
\end{pmatrix}, \tag{64}
\]
where \( \bar{\theta}_\pm = \bar{\theta}_1 \pm i\bar{\theta}_2 \). The eigenvalues of the matrix \( \bar{\Theta}(2P_{3/2}) \) are
\[
\bar{\Lambda}_{\pm 3/2}(2P_{3/2}) = \frac{8}{5} \big| \bar{\theta} \big|, \quad \bar{\Lambda}_{\pm 1/2}(2P_{3/2}) = \frac{16}{15} \big| \bar{\theta} \big|. \tag{65}
\]

The \( \theta \) correction for the relativistic energy level \( 2P_{3/2} \) are
\[
E_{\pm 3/2}(2P_{3/2}) = E^{\theta}_{\pm 3/2}(2P_{3/2}) + E^{\theta}_{\pm 1/2}(2P_{3/2}), \tag{66}
\]
Therefor the \( \bar{\theta} \) correction shifts the energy levels. Finally, the relativistic NCPS energy corrections for the level \( 2P_{3/2} \) are
\[
\Delta E_{\pm 3/2}(2P_{3/2}) = \Delta E_{\pm 3/2}(2P_{3/2}) + \Delta E^{\theta}_{\pm 1/2}(2P_{3/2}) = \frac{2}{3} E^{\theta}_{\pm 1/2}(2P_{3/2}). \tag{67}
\]
From (60), (66) and (67) one can see that the relativistic energy level \( 2P_{3/2} \) is split into four nondegenerate sublevels by the \( \theta \) correction and shifted by the \( \bar{\theta} \) correction. The energy levels and their modified energy levels of the hydrogen atom due to space-space, and momentum-momentum noncommutativity for the relativistic case are shown in Fig.1. The subenergy level spacings for the relativistic energy levels \( 2S_{1/2} \) and \( 2P_{1/2} \) are as follows (in units of \( eV/m^2 \))
\[
\Delta E(2P^{-1/2}_{3/2}) = 6.75 \times 10^{15} \bigg| \frac{\theta}{\alpha^3} \bigg| + 8.38 \times 10^6 \bigg| \frac{\bar{\theta}}{\alpha} \bigg|, \tag{68}
\]
\[
\Delta E(2S_{1/2} \rightarrow 2P_{1/2}) = 6.75 \times 10^{10} \bigg| \frac{\theta}{\alpha^3} \bigg| - 8.38 \times 10^6 \bigg| \frac{\bar{\theta}}{\alpha} \bigg|. \tag{69}
\]

The subenergy level spacings for the relativistic energy level \( 2P_{3/2} \) are as follows (in units of \( eV/m^2 \))
\[
\Delta E(2P_{3/2} \rightarrow 2F_{1/2}) = 6.75 \times 10^{10} \bigg| \frac{\theta}{\alpha^3} \bigg| - 8.38 \times 10^6 \bigg| \frac{\bar{\theta}}{\alpha} \bigg|, \tag{70}
\]
\[
\Delta E(2P^{-1/2}_{3/2} \rightarrow 2F_{1/2}) = 6.75 \times 10^{10} \bigg| \frac{\theta}{\alpha^3} \bigg| - 8.38 \times 10^6 \bigg| \frac{\bar{\theta}}{\alpha} \bigg|. \tag{71}
\]

\[ \text{IV. CONCLUSION} \]

Since the noncommutative parameters \( \theta \) and \( \bar{\theta} \) is very small compared to the length scales of the system, one can always treat the noncommutative effects as some perturbation of the commutative counter part. Thus we calculated the energy levels \( 2S_{1/2}, 2P_{1/2} \) and \( 2P_{3/2} \) of the

\[ \text{IV. CONCLUSION} \]

Since the noncommutative parameters \( \theta \) and \( \bar{\theta} \) is very small compared to the length scales of the system, one can always treat the noncommutative effects as some perturbation of the commutative counter part. Thus we calculated the energy levels \( 2S_{1/2}, 2P_{1/2} \) and \( 2P_{3/2} \) of the.
FIG. 1: Modifications for relativistic energy levels of hydrogen atom on a NCPS

relativist hydrogen-like atom on a NCPS by using perturbation method. We found that the space-space noncommutativity splits the degenerate energy levels $2S_{1/2}$, $2P_{1/2}$ into three sublevels $2S_{1/2}$ and $2P_{1/2}^{\pm1/2}$, as well as splits the level $2P_{3/2}$ into four nondegenerate sublevels $2P_{3/2}^{\pm1/2}$ and $2P_{3/2}^{\pm3/2}$, such that new transition channels are allowed between subenergy levels. We also show that the momentum-momentum noncommutativity shifts the energy levels. The $\theta$ and the $\bar{\theta}$ modification of the energy levels for the relativistic case are shown in Fig.1.

To impose some bounds on the value of the noncommutativity parameters $\theta$, $\bar{\theta}$, one needs experimental data with a high accuracy.

If $\bar{\theta} = 0$, it leads $\alpha = 1$, then $\Delta E_{n,j} = E^{\bar{\theta}} = 0$, we get the NCS results of Ref.[17]. If $\theta = \bar{\theta} = 0$, then $\Delta E_{n,j} = E^{\theta} = E^{\bar{\theta}} = 0$, we obtain the usual relativistic quantum mechanics results of Refs.[22]-[25]. The results in this paper suggest that high precision measurements in quantum mechanical systems may be able to reveal the noncommutativity of space and phase space.

V. ACKNOWLEDGMENTS

H. Masum is grateful to Kai Ma for many useful discussions. This work is supported by the National Natural Science Foundation of China (10965006 and 11165014).

[1] S. Godfrey and M. A. Doncheski, Phys. Rev. D65, (2001) 015005.
[2] M. Haghighat and M. M. Ettefaghi, Phys. Rev. D70 (2004) 034017.
[3] A. Devoto, S. DiChiara, and W. W. Repko, Phys. Rev. D72, (2005) 056006.
[4] X. Calmet, Eur. Phys. J. C50, (2007) 113.
[5] Chaichian M, Presnajder P, Sheikh-Jabbari M M and Tureanu A 2002 Phys. Lett. B 527 149-54
[6] H. Falomir, J. Gamboa, M. Loewe, F. Méndez, J. C. Rojas, Phys. Rev. D66, (2002) 045018.
[7] K. Li, S. Dulat, Eur. Phys. J. C 46, 825 (2006).
[8] B. Mirza and M. Zarei, Eur. Phys. J. C 32 583, 2004.
[9] K. Li, J.-H. Wang, Eur. Phys. J. C50, (2007) 1007.
[10] B. Mirza, R. Narimani, M. Zarei, Eur. Phys. J. C48, (2006)641;
[11] S. Dulat, K. Li, Eur. Phys. J. C 54, 333 (2008).
[12] B. Harms and O. Micu, Phys. J. A40, (2007) 10337.
[13] O. F. Dayi and A. Jellal, J. Math. Phys. 43, (2002) 4592; J. Math. Phys. 45, (2004) 827(E); A. Kokado, T. Oikumura, and T. Saito, Prog. Theor. Phys. 110, (2003) 975; S. Dulat and K. Li, Eur.Phys.J. C60, (2009) 163.
[14] B. Chakraborty, S. Gangopadhyay, and A. Saha, Phys. Rev. D70, (2004) 107707; F. G. Scholtz, B. Chakraborty, S. Gangopadhyay, and A. G. Hazra, Phys. Rev. D71, (2005) 085005; F. G. Scholtz, B.Chakraborty, S. Gangopadhyay, and J. Govaerts, J. Phys.A38, (2005) 9849.
[15] O. F. Dayi and M. Elbistan, Phys. Lett. A373, (2009) 131.
[16] Kai Ma, Sayipjamal Dulat, Phys.Rev. A84 (2011) 012104
[17] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. 86, (2001) 2716.
[18] Li Kang, CHAMOUN Nidal, Chin. Phys. Lett. 23, 5(2006).
[19] M. Haghighat and F. Loran, Phys. Rev. D67 (2003) 096003.
[20] J.-Z. Zhang, Phys. Rev. Lett. 93 (2004) 043002.
[21] T. C. Adorno, M. C. Baldiotti, M. Chaichian, D. M. Gitman, and A. Tureanu, Phys. Lett. B682, (2009) 235.
[22] W. Greiner, Relativistic Quantum Mechanics: Wave Equation (Springer, 2000)
[23] H. A. Bethe, Edwin E. Salpeter, Quantum Mechanics of One- and Two-Electron Atoms, Springer-Verlag, 1957.
[24] M. E. Rose, Relativistic Electron Theory, John Wiley and Sons, 1961.
[25] A. I. Akhiezer, V.B. Berestetskii, Quantum Electrodynamics, Interscience Publishers, 1965.
[26] B. L. Voronov, D. M. Gitman and I. V. Tyutin (The Dirac Hamiltonian with a superstrong Coulomb field), Theoretical and Mathematical Physics , 150, 34 (2007)
[27] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, 7th ed., Academic Press, 2007.