Coarse geometry of the fire retaining property and group splittings

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Abstract
Given a non-decreasing function \( f : \mathbb{N} \to \mathbb{N} \) we define a single player game on (infinite) connected graphs that we call fire retaining. If a graph \( G \) admits a winning strategy for any initial configuration (initial fire) then we say that \( G \) has the \( f \)-retaining property; in this case if \( f \) is a polynomial of degree \( d \), we say that \( G \) has the polynomial retaining property of degree \( d \). We prove that having the polynomial retaining property of degree \( d \) is a quasi-isometry invariant in the class of uniformly locally finite connected graphs. Henceforth, the retaining property defines a quasi-isometric invariant of finitely generated groups. We prove that if a finitely generated group \( G \) splits over a quasi-isometrically embedded subgroup of polynomial growth of degree \( d \), then \( G \) has polynomial retaining property of degree \( d - 1 \).

Some connections to other work on quasi-isometry invariants of finitely generated groups are discussed and some questions are raised.

Keywords Games on graphs · Quasi-isometry · Splitting · Coarse geometry · Growth

Mathematics Subject Classification (2000) Primary 05C57 · 20F65; Secondary 05C10 · 20F69

1 Introduction
The firefighter problem on graphs was introduced by Hartnell in 1995 and it has been studied by graph theorists ever since, see for example [5] and references therein. Recently, in [4], this problem was studied in the context of coarse geometry, leading to definition of new quasi-isometry invariants of finitely generated groups that were called fire containment properties.

The current article follows the same vein: we study a variation that we call fire retaining...
properties, we define new quasi-isometry invariants of finitely generated groups, and we exhibit a relation with the existence of group splittings.

Let \( \{ f_n \}_{n \in \mathbb{N}} \) be a sequence of non-negative integers. Suppose that a fire breaks out at a finite set of vertices \( X_0 \) of a connected graph \( G \). At each subsequent time unit \( n \in \mathbb{N} \) (called a turn), the player (called the firefighter) chooses a set \( W_n \) of at most \( f_n \) distinct vertices to become protected; then the fire spreads to all vertices which are adjacent to vertices which are on fire and are not yet protected. Once a vertex is on fire or is protected, it stays in such state for all subsequent turns. Denote by \( U \) the set of vertices which, after the game has been played, never caught fire.

- If \( U \) contains all but finitely many vertices of \( G \), we say that the sequence \( \{ W_n \} \) is a containment \( \{ f_n \} \)-strategy for the initial fire \( X_0 \).
- If the growth rate of \( U \) with respect to the edge-path distance in \( G \) is equivalent to the growth rate of \( G \), we say that the sequence \( \{ W_n \} \) is a retaining \( \{ f_n \} \)-strategy for the initial fire \( X_0 \).

If every finite subset of vertices \( X_0 \) of \( G \) admits a containment \( \{ f_n \} \)-strategy we say the graph \( G \) satisfies the \( \{ f_n \} \)-containment property. The graph \( G \) satisfies polynomial containment of degree \( d \) if there is a constant \( K > 0 \) such that \( G \) has the \( \{ f_n \} \)-containment property for \( f_n = Kn^d \). The \( \{ f_n \} \)-retaining property and polynomial retaining property of degree \( d \) for a connected graph are defined analogously.

In Fig. 1 we present a relation between various types of retaining and containment properties, along with some examples of graphs (Cayley graphs of finitely generated groups) satisfying these properties.

Observe that if a connected graph satisfies the \( \{ f_n \} \)-containment property then it satisfies the \( \{ f_n \} \)-retaining property. In [4], it was proved that satisfying the polynomial containment property of degree \( d \) is a quasi-isometry invariant in the class of uniformly locally finite graphs (uniformly locally finite means there is a constant that bounds from above the degree of any vertex in the graph). The first theorem in this article says that the analogous result holds for the polynomial retaining property.

**Theorem 1.1** (Corollary 4.3) Let \( G \) and \( H \) be uniformly locally finite connected graphs. Suppose that \( G \) is quasi-isometric to \( H \). If \( G \) has polynomial retaining property of degree \( d \) then \( H \) has polynomial retaining property of degree \( d \).

A more general version of Theorem 1.1 is the main content of Sect. 4. Since any two Cayley graphs of a finitely generated group \( G \) with respect to finite generating sets are quasi-isometric, satisfying polynomial retaining or containment of degree \( d \) is a well-defined invariant of finitely generated groups.

We search for algebraic interpretations of these properties in the class of finitely generated groups. For example, there are known relations between containment and growth for finitely generated groups. In [4], it is proved that if a group has polynomial growth of degree \( d \) then it satisfies polynomial containment of degree \( d - 2 \). Recently, joint work of Amir et al. [1] shows that any Cayley graph with growth bounded from below by a polynomial of degree \( d \) does not satisfy the \( \{ f_n \} \)-containment property if \( \{ f_n \} = o(n^{d-2}) \). It is known that finitely generated groups of intermediate growth do not satisfy polynomial containment [1]. See also [1, 6, 7] for other related results.

The second result of this article exhibits a relation between retaining properties and splittings of groups. We say that a group \( G \) splits over a subgroup \( C \) if either \( G = A \ast_C B \) and \( C \) is a proper subgroup of \( A \) and \( B \), or \( G \) is an HNN-extension \( A \ast_C \) (with no assumptions on \( C \)).
Theorem 1.2 (Theorem 5.5) Let $G$ be a finitely generated group that splits over a finitely generated subgroup $C$. If $C$ is quasi-isometrically embedded in $G$ and $C$ has polynomial growth of degree $d$, then $G$ has polynomial retaining property of degree $d - 1$.

Proof (Outline of the proof of Theorem 1.2.) The splitting ensures that the Cayley graph of $G$ can be disconnected into two unbounded components by removing an appropriate neighborhood $L$ of $C$. Using $L$ we can build a “wall” in $G$, namely by protecting all vertices of $L$.

To do this, one has to ensure that in the process of protecting $L$, one is always ahead of the spreading fire. Since $C$ is quasi-isometrically embedded, its growth inside $G$ is polynomial of degree $d$. Since $L$ and $C$ are quasi-isometric, the same holds for $L$. Therefore at time $n$, at most (roughly) $n^d$ vertices of $L$ could potentially catch fire. One verifies that one can protect $n^d$ vertices by protecting at time $k$ for $1 \leq k \leq n$ an amount of vertices that grows polynomially of degree $d - 1$ with $k$. Because $L$ disconnects the Cayley graph of $G$, we get that an entire unbounded component will never catch fire. Then the homogeneity of the Cayley graph implies that this component has the growth rate as large as the group. 

A finitely generated group $G$ has the constant retaining property if it has polynomial retaining property of degree zero. The following is a particular instance of Theorem 1.2.

Corollary 1.3 Suppose $G = A \ast_C B$ where $C$ is a proper subgroup of $A$ and $B$. If $C$ is virtually cyclic and quasi-isometrically embedded in $G$, then $G$ has the constant retaining property.

The quasi-isometry invariance of splittings of finitely presented groups over two-ended (i.e., virtually cyclic) groups was settled by deep results of Papasoglu [8]. In particular he shows that if $G$ is a one-ended, finitely presented group that is not commensurable to a surface group, then $G$ splits over a two-ended group if and only if the Cayley graph of $G$ with respect to a finite generating set is separated by a quasi-line. We refer the reader to [8] for the definitions of quasi-line and separation.

Question 1.4 Let $G$ be a one-ended, finitely presented group, not commensurable to a surface group. Suppose that $G$ has the constant retaining property. Is the Cayley graph of $G$ with respect to a finite generating set separated by a quasi-line?

While constant containment implies constant retaining, the converse does not hold, as exhibited for example by the free group of rank two [4]. More interesting examples are the free abelian groups of rank at least three; the proof that these groups do not satisfy the constant containment property is due to Develin and Hartke [3]. We suspect that these groups do not satisfy the constant retaining property either.

Question 1.5 Does the group $\mathbb{Z}^3$ have the constant retaining property?

A connected graph has the finite-step polynomial retaining property of degree $d$ if there is a polynomial $f$ of degree $d$ such that any initial fire $X_0$ admits a retaining $f$-strategy $\{W_n\}$ such that $W_n$ is empty for all but finitely many $n$. A corollary of the proof of Theorem 1.1 is that the finite-step polynomial retaining property of degree $d$ is a quasi-isometry invariant in the class of uniformly locally finite graphs, see Corollary 4.5. In regard to Question 1.5 above, the group $\mathbb{Z}^3$ does have the finite-step retaining property of degree one, see Remark 5.8.

The finite-step polynomial retaining property of degree zero is abbreviated as the finite-step retaining property. For finitely generated groups, this property essentially captures splittings over finite subgroups.
Theorem 1.6  Let $G$ be a finitely generated group.

(1) If $G$ splits over a finite group, then $G$ has the finite-step retaining property.
(2) If $G$ has the finite-step retaining property, then $G$ has the constant containment property or $G$ splits over a finite group.

The two statements of Theorem 1.6 correspond to Proposition 5.4 and Corollary 5.7, respectively, in the main body of the article.

Corollary 1.7  Let $G$ be a non-amenable finitely generated group. Then $G$ is one-ended if and only if $G$ does not have the finite-step retaining property.

Proof  Since $G$ is non-amenable, it does not satisfy the constant containment property [7, Corollary 8], and $G$ has either one or infinitely many ends [2, Part I, Theorem 8.32(1,2,3)].

Suppose that $G$ has infinitely many ends. By Stallings’ theorem [2, Part I, Theorem 8.32(5)], $G$ splits over a finite group and hence Theorem 1.6(1) implies that $G$ has the finite-step retaining property. Conversely, by Theorem 1.6(2) if $G$ has the finite-step retaining property then $G$ has infinitely many ends. \hfill \Box

Regarding the statement of Theorem 1.2, we believe that the hypothesis that the subgroup $C$ is quasi-isometrically embedded could be weakened. For example, we expect a positive answer to the following.

Question 1.8  Let $G$ be a finitely generated group isomorphic to an amalgamated product $A \ast_C B$. Suppose that $A$ is hyperbolic relative to $C$, and that $C$ has polynomial growth of degree $d > 0$. Does $G$ have polynomial retaining property of degree $d - 1$?

Organization

Section 2 contains some preliminary material on the notion of growth rate of graphs and quasi-isometry of metric spaces. Detailed definitions of the retaining property, containment property and finite-step retaining properties, as well as some preliminary results, are the content of Sect. 3. The proofs of Theorems 1.1 and 1.2 are given in Sects. 4 and 5 respectively.

2 Preliminaries

2.1 Growth rate

Given non-decreasing functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$, the relation $f \preceq g$ is defined as the existence of an integer $C > 0$ such that

$$f(n) \leq C g(Cn + C) + C$$

for every $n$. The functions $f$ and $g$ have equivalent growth rate, denoted by $f \sim g$, if $f \preceq g$ and $g \preceq f$. For a locally finite metric space $(X, \text{dist})$, the growth function $\beta_{X,A} : \mathbb{N} \to \mathbb{N}$ with respect to a non-empty finite subset $A \subset X$ is defined as

$$\beta_{X,A}(n) = |B_X(A, n)|,$$

where

$$B_X(A, n) = \{x \in X : \text{dist}(A, x) \leq n\}.$$
Observe that for any two finite subsets $A$, $B$ of $X$ the growth functions $\beta_{X,A}$ and $\beta_{X,B}$ have equivalent growth rate. The growth rate of $X$, denoted by $\text{Growth}(X)$, is the equivalence class of $\beta_{X,A}$ with respect to the equivalence relation $\sim$. For locally finite metric spaces $X$ and $Y$, define $\text{Growth}(X) \preceq \text{Growth}(Y)$ if $\beta_{X,A} \preceq \beta_{Y,B}$ for some (and hence for any) choices of finite subsets $A \subseteq X$ and $B \subseteq Y$.

**Remark 2.1** The following statements are easy to verify.

1. If $\text{Growth}(X) \preceq \text{Growth}(Y)$ and $\text{Growth}(Y) \preceq \text{Growth}(X)$ then $\text{Growth}(X) = \text{Growth}(Y)$.
2. Let $U$ be a subset of a locally finite metric space $X$. Consider $U$ as a metric space with the metric induced from $X$. Then $\text{Growth}(U) = \text{Growth}(X)$ if and only if $\text{Growth}(X) \preceq \text{Growth}(U)$.
3. A locally finite metric space $X$ is uniformly locally finite if there is a function $g : \mathbb{N} \to \mathbb{N}$ such that for any $x \in X$ the ball $B_X(x, n)$ has cardinality at most $g(n)$. If $X$ and $Y$ are quasi-isometric uniformly locally finite metric spaces then $\text{Growth}(X) = \text{Growth}(Y)$ (the definition of quasi-isometry is recalled in Sect. 2.3 below).

### 2.2 Graphs as metric spaces

Let $G$ be a graph. We say that $G$ is uniformly locally finite if there is a constant $M$ such that every vertex of $G$ is adjacent to at most $M$ vertices. A path of length $n$ is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $v_i, v_{i+1}$ are connected by an edge for each $i < n$. The graph $G$ is connected if there is a path between any two vertices of $G$. Assume that $G$ is connected. The set of vertices of $G$ is denoted by $V(G)$. The notion of path defines a metric on the set of vertices of $G$ by declaring $\text{dist}_G(x, y)$ to be the length of the shortest path from $x$ to $y$; we call this metric the edge-path metric.

Let $X$ and $Y$ be subsets of $V(G)$. The ball of radius $r$ centered at $X$, denoted by $B_G(X, r)$, is defined as the collection of vertices at distance less than or equal to $r$ from at least one vertex in $X$. The distance $\text{dist}_G(X, Y)$ is defined as the minimum of distances $\text{dist}_G(x, y)$ where $x \in X$ and $y \in Y$. The diameter of $X$ denoted by $\text{diam} X$ is defined as $\sup \{\text{dist}(x_1, x_2) : x_1, x_2 \in X\}$.

### Growth in graphs

If a graph $G$ is uniformly locally finite then the set of vertices of $G$ with the edge-path metric $\text{dist}_G$ is a uniformly locally finite metric space. Define $\text{Growth}(G)$ as the growth rate of $(V(G), \text{dist}_G)$. For any subset $U \subseteq V(G)$, we denote by $\text{Growth}(U)$ the growth rate of $U$ with the metric being the restriction of $\text{dist}_G$ to $U$.

### 2.3 Quasi-isometry

Let $(X, \text{dist}_X)$ and $(Y, \text{dist}_Y)$ be metric spaces and let $C > 0$ be a constant. A map $\phi : X \to Y$ is a $C$-quasi-isometric embedding if for all $x_1, x_2 \in X$ we have

$$\frac{1}{C} \text{dist}_X(x_1, x_2) - C \leq \text{dist}_Y(\phi(x_1), \phi(x_2)) \leq C \text{dist}_X(x_1, x_2) + C.$$  

A $C$-quasi-isometric embedding $\phi : X \to Y$ is a $C$-quasi-isometry if every point of $Y$ lies in the $C$-neighborhood of the image of $\phi$. We say that the metric spaces $X$ and $Y$ are quasi-isometric if there is a $C$-quasi-isometry from $X$ to $Y$ for some constant $C$. 
Note that the identity function on a metric space is a 1-quasi-isometry and that the composition of a \( C \)-quasi-isometry with a \( C' \)-quasi-isometry is a \( C'' \)-quasi-isometry for some \( C'' \) that depends only on \( C \) and \( C' \). Moreover, if \( \phi : X \to Y \) is a \( C \)-quasi-isometry then there is a \( C' \)-quasi-isometry \( \psi : Y \to X \) such that \( \text{dist}_X(x, \psi \circ \phi(x)) \leq C' \) for all \( x \in X \), see [2, p. 138].

Let \( G \) and \( H \) be connected graphs. A \( C \)-quasi-isometry \( \phi : G \to H \) is a \( C \)-quasi-isometry from the vertex set of \( G \) with its edge-path metric into the vertex set of \( H \) with its edge-path metric. We say that the graphs \( G \) and \( H \) are quasi-isometric if their vertex sets with their corresponding edge-path metrics are quasi-isometric metric spaces. A connected subgraph \( H \) of the connected graph \( G \) is quasi-isometrically embedded if the corresponding inclusion map on the vertex sets is a quasi-isometric embedding with respect to the edge-path metrics.

2.4 Finitely generated groups and Cayley graphs

Let \( G \) be a group with a finite generating set \( S \). The Cayley graph \( \Gamma(G, S) \) is the graph with vertex set \( G \) and edge set \( \{[g, gs] : g \in G \text{ and } s \in S\} \). The natural left action of \( G \) on \( \Gamma(G, S) \) has finitely many orbits of vertices and edges. Two finitely generated groups are quasi-isometric if their Cayley graphs with respect to some (and hence for any) chosen finite generating sets are quasi-isometric. For a fixed finite generating set of a group, the edge-path metric on the corresponding Cayley graph and the induced word-metric on the group coincide. A finitely generated subgroup \( C \) of a finitely generated group \( G \) is quasi-isometrically embedded if for some (and hence any) finite generating set \( S \) of \( G \) containing a finite generating set \( T \) of \( C \) the Cayley graph of \( C \) with respect to \( T \) is quasi-isometrically embedded in the Cayley graph of \( G \) with respect to \( S \). For a detailed discussion of these matters we refer the reader to [2].

3 The fire retaining property for graphs

In this section we define the retaining property, containment property and finite-step retaining properties. In Fig. 1 we show how these properties relate to each other and we give some examples of graphs satisfying these properties.

Let \( G \) be a connected graph. Let \( r > 0 \) be a positive integer that we shall call the fire reach. Let \( \{f_n\}_{n \geq 1} \) be a sequence of positive integers that we shall call the strategy bound. The player of the game shall be called the firefighter. Let \( X_0 \) be a finite subset of vertices of \( G \) that we shall call the initial fire.

3.1 Strategies

A \( \{f_n\}_{n \geq 1} \)-strategy is a sequence \( \{W_n\}_{n \geq 1} \) of subsets of vertices of \( G \) such that for every \( n \geq 1 \), the set \( W_n \) has cardinality at most \( f_n \). The set \( W_n \) is called the set of vertices to protect at time \( n \). If the sequence \( \{f_n\}_{n \geq 1} \) is constant, i.e., if \( f_n = f \), then an \( \{f_n\}_{n \geq 1} \)-strategy is called an \( f \)-strategy.
3.2 Vertices on fire at time $n$

Now we define the set $X_n$ of vertices on fire at time $n$ with respect to the $\{f_n\}$-strategy $\{W_n\}_{n \geq 1}$, and the initial fire $X_0$ of reach $r$. In words, the set $X_n$ consists of all the vertices of $G$ that can be reached from a vertex of $X_{n-1}$ by a path of length at most $r$, which avoids all vertices that have been protected up to time $n$. Since a vertex that is on fire at some time of the game remains on fire for the rest of the game, the set of vertices that are protected at time $n$ is

$$(W_1 \cup \cdots \cup W_n) \setminus X_{n-1}.$$  

Formally, for each integer $n \geq 0$, the subset $X_n$ consists of vertices which are connected to a vertex of $X_{n-1}$ by a path of length at most $r$ containing no vertices in $(W_1 \cup \cdots \cup W_n) \setminus X_{n-1}$.

The set $\bigcup_{n \geq 0} X_n$ shall be called the set of vertices on fire at the end of the game with respect to the $\{f_n\}$-strategy $\{W_n\}_{n \geq 1}$, and the initial fire $X_0$ of reach $r$.

3.3 Equivalent strategies

The $\{f_n\}_{n \geq 1}$-strategies $\{W_n\}_{n \geq 1}$ and $\{V_n\}_{n \geq 1}$ are equivalent for the initial fire $X_0$ of reach $r$ if the corresponding sets of vertices on fire at time $n$ for both strategies are equal for every $n \geq 0$.

**Remark 3.1** Let $\{W_n\}_{n \geq 1}$ be a strategy and let $X_0$ be an initial fire of reach $r$. Let $X_n$ denote the set of vertices on fire at time $n$ for the given data. Observe that the definitions above do not imply that $X_n \cap W_{n+1} = \emptyset$. In words, at time $n + 1$, the firefighter might be unable to protect a vertex $v$ in $W_{n+1}$ because $v$ caught fire at an earlier stage of the game. This can be avoided by passing to an equivalent strategy, as the following lemma states.

**Lemma 3.2** Let $X_n$ be the set of vertices on fire at time $n$ with respect to the $\{f_n\}$-strategy $\{W_n\}_{n \geq 1}$ and initial fire $X_0$ of reach $r$. Then there is an $\{f_n\}$-strategy $\{W'_n\}_{n \geq 1}$ equivalent to $\{W_n\}_{n \geq 1}$ such that $X_n \cap W'_{n+1} = \emptyset$ for every $n$. Specifically, $W'_{n+1} = W_{n+1} \setminus X_n$ for all $n > 0$.

The proof is straightforward and is left to the reader.

3.4 Retaining strategies

Given an $\{f_n\}$-strategy $\{W_n\}_{n \geq 1}$ and the initial fire $X_0$ of reach $r$, define $U$ to be the complement of $\bigcup_{n=0}^{\infty} X_n$ in the vertex set of $G$. Thus $U$ is the set of vertices of $G$ that at the end of the game are not on fire.

The strategy $\{W_n\}_{n \geq 1}$ is called a retaining $\{f_n\}$-strategy for the initial fire $X_0$ of reach $r$ if $\text{Growth}(U) = \text{Growth}(G)$ where the metric on $U$ is the restriction of the edge-path metric on the vertex set of $G$.

If we wish to emphasize the reach of the fire, we will write that $\{W_n\}_{n \geq 1}$ is a retaining $\{f_n\}$, $r$)-strategy for $X_0$.

3.5 Retaining property

The graph $G$ has the $(\{f_n\}, r)$-retaining property if for every finite subset $X_0$ of vertices of $G$ there is a retaining $\{f_n\}$-strategy for $X_0$ as an initial fire of reach $r$. 
We will use the following abbreviations:

1. $G$ has the $\{f_n\}$-retaining property means that $G$ has the $\{(f_n), r\}$-retaining property for $r = 1$.
2. $G$ has polynomial retaining property of degree $d$ means that there is a constant $K > 0$ such that $G$ has the $\{Kn^d\}$-retaining property.
3. $G$ has the $f$-retaining property means that $G$ has the $\{f_n\}$-retaining property for the constant sequence $f_n = f$.
4. $G$ has constant retaining property means that $G$ has the $f$-retaining property for some integer $f$, or equivalently, $G$ has polynomial retaining property of degree zero.

The following two observations are straightforward.

Remark 3.3 If for every vertex $x \in G$ and every integer $n \geq 0$ there is a retaining $\{(f_n), r\}$-strategy for the initial fire $X_0 = B_G(x, n)$, then $G$ has the $\{(f_n), r\}$-retaining property.

Remark 3.4 If $G$ has the $\{(f_n), r\}$-retaining property, then it has the $\{(f_n), 1\}$-retaining property.

The following lemma is a partial converse to Remark 3.4.

Lemma 3.5 If $G$ has the $\{(f_n), 1\}$-retaining property, then it has the $\{(a_n), r\}$-retaining property where

$$a_n = f_{(n-1)r+1} + \cdots + f_{nr}.$$  

Specifically, if $\{W_n\}_{n \geq 1}$ is a retaining $\{(f_n), 1\}$-strategy for $X_0$ then $\{V_n\}_{n \geq 1}$, where $V_n = W_{(n-1)r+1} \cup \cdots \cup W_{nr}$, is a retaining $\{(a_n), r\}$-strategy for $X_0$.

Proof Let $X_n$ denote the set of vertices on fire at time $n$ with respect to the retaining $\{(f_n), 1\}$-strategy $\{W_n\}_{n \geq 1}$ for $X_0$. Without loss of generality, assume that $X_n \cap W_{n+1} = \emptyset$ for all $n$; see Lemma 3.2. Let $Y_0 = X_0$, and let $Y_n$ denote the set of vertices on fire at time $n$ with respect to the strategy $\{V_n\}_{n \geq 1}$ for the initial fire $Y_0$ of reach $r$. It is immediate that $|V_n| \leq a_n$. Observe that

$$Y_n = X_{rn}$$

for every $n$. Hence $\bigcup_{n=0}^{\infty} X_n = \bigcup_{n=0}^{\infty} Y_n$, and therefore $\{V_n\}_{n \geq 1}$ is a retaining $\{(a_n), r\}$-strategy for $Y_0 = X_0$.

3.6 Finite-step retaining property

An $\{f_n\}$-strategy $\{W_n\}_{n \geq 1}$ is a finite-step retaining $\{f_n\}$-strategy for $X_0$ if $\{W_n\}_{n \geq 1}$ is a retaining $\{f_n\}$-strategy for the initial fire $X_0$ of reach one, and $W_n = \emptyset$ for all sufficiently large $n$. The graph $G$ has the finite-step polynomial retaining property of degree $d$ if there is a polynomial sequence $\{f_n\}$ of degree $d$ such that every finite subset $X_0$ of vertices of $G$ admits a finite-step retaining $\{f_n\}$-strategy. The finite-step polynomial retaining property of degree zero is abbreviated as the finite-step retaining property.

Observe that the finite-step retaining property of degree $d$ implies the finite-step retaining property of degree $d'$ for any $d' \geq d$. However, we will see in Remark 5.8 that the converse implication does not hold.
3.7 Containment property

An \( \{f_n\}_n \)-strategy \( \{W_n\}_n \) is a containment \( \{f_n\}_n \)-strategy for the initial fire \( X_0 \) of reach \( r \) if \( \bigcup_{n=0}^{\infty} X_n \) is finite. The graph \( G \) has the \( (\{f_n\}_n, r) \)-containment property if for every finite subset \( X_0 \) of vertices of \( G \) there is a containment \( \{f_n\}_n \)-strategy for \( X_0 \) as an initial fire of reach \( r \). The containment property on infinite graphs has been studied in [4]. Similarly as above, a graph \( G \) has polynomial containment of degree \( d \) if there is \( K > 0 \) such that \( G \) has the \( ((Kn^d), 1) \)-containment property; \( G \) has the \( f \)-containment property if it has the \( (\{f_n\}_n, 1) \)-containment property for the constant sequence \( f_n = f \); and \( G \) has the constant containment property if \( G \) has polynomial containment property of degree zero. Observe that containment strategies are in particular (finite-step) retaining strategies.

4 Quasi-isometry invariance

In this section we prove the following result.

**Theorem 4.1** Let \( G \) and \( H \) be connected graphs with degree bounded above by a constant \( \delta \). Let \( \phi : G \to H \) and \( \psi : H \to G \) be \( c \)-quasi-isometries, where \( c \) is a positive integer such that \( \text{dist}(u, \psi \circ \phi(u)) \leq c \) for every vertex \( u \) of \( G \). Suppose that \( G \) has the \( \{f_k\}_{k \geq 1} \)-retaining property. Then \( H \) has the \( \{b_k\}_{k \geq 1} \)-retaining property where

\[
b_k = (f_{2c(k-1)+1} + f_{2c(k-1)+2} + \cdots + f_{2ck}) \delta^{c^2 + 2c + 1}. \tag{4.1}
\]
Remark 4.2 If the sequence \( \{f_k\}_{k \geq 1} \) is non-decreasing then the sequences \( \{f_k\}_{k \geq 1} \) and \( \{b_k\}_{k \geq 1} \) have equivalent growth rate in the sense of Sect. 2.1, since \( f_k \leq b_k \leq 2c\delta^{c^2+2c+1}f_2c^k \) for every \( k > 0 \).

The following corollary is a direct consequence of Theorem 4.1 and Remark 4.2.

Corollary 4.3 Let \( G \) and \( H \) be uniformly locally finite connected graphs. Suppose that \( G \) is quasi-isometric to \( H \). If \( G \) has polynomial retaining property of degree \( d \) then \( H \) has polynomial retaining property of degree \( d \).

The proof of Theorem 4.1 relies on the following statement proved in [4, page 18]. The proof is transcribed below for the convenience of the reader.

Lemma 4.4 [4, Lemma 4.5] Let \( h_0 \) be a vertex of \( H \) and let \( g_0 = \psi h_0 \). Let \( q \) be a positive integer and let \( r = c^2 + 2c \). Let \( \{W_k\}_{k \geq 1} \) and \( \{X_k\}_{k \geq 0} \) be sequences of subsets of \( V(G) \) such that for all \( k \geq 0 \) we have:

1. \( X_0 = B_G(g_0, 2c(q + 2)) \),
2. the sets \( X_k \) and \( W_{k+1} \) are disjoint,
3. the set \( X_k \) consists of the vertices which are connected to a vertex in \( X_{k-1} \) by a path of length at most \( 2c \) containing no vertices in \( W_1 \cup \cdots \cup W_k \).

Let \( Y_0 = B_H(h_0, q) \), and for \( k \geq 1 \) define

\[
Q_k = \bigcup_{g \in W_k} B_H(\phi g, r) \setminus Y_{k-1} \quad \text{and} \quad Y_k = B_H(Y_{k-1}, 1) \setminus Q_k.
\]

Then for all \( k \geq 1 \) we have:

1. the sets \( Q_k \) and \( Y_{k-1} \) are disjoint and the cardinality of \( Q_k \) is at most \( \delta r |W_k| \),
2. if \( h \in Y_k \) then \( \psi h \in X_{k-1} \).

Proof Observe that the first statement is immediate. The second statement is proved by induction on \( k \). First let

\[
r_k = 2c(q + k + 2).
\]

Observe that \( X_k \) consists of vertices \( g \in G \) such that there is a path from \( g_0 \) to \( g \) of length at most \( r_k \) that does not contain vertices in \( W_1 \cup \cdots \cup W_k \).

Base case: If \( h \in Y_1 \) then \( \text{dist}_H(h_0, h) \leq q + 1 \) and hence

\[
\text{dist}_G(g_0, \psi h) \leq c(q + 1) + c \leq 2c(q + 2).
\]

It follows that \( \psi h \) belongs to \( X_0 = B_G(g_0, r_0) \).

Induction step: Suppose \( 2 \leq k \). The induction hypothesis is that \( h \in Y_j \) implies \( \psi h \in X_{j-1} \) for all \( j < k \). Suppose \( h \in Y_k \). Then there exists a path

\[
h_0, h_1, h_2, \ldots, h_\ell = h
\]

such that \( \ell \leq q + k \) and no \( h_i \) is in \( Q_1 \cup \cdots \cup Q_k \). Consider the sequence of vertices

\[
\psi h_0, \psi h_1, \psi h_2, \ldots, \psi h_\ell.
\]

Since \( \text{dist}_G(\psi h_{i-1}, \psi h_i) \leq c \text{dist}_H(h_i, h_{i+1}) + c = 2c \), there is a path \( \gamma_i \) of length at most \( 2c \) from \( \psi h_{i-1} \) to \( \psi h_i \). Consider the path \( \gamma \) from \( \psi h_0 \) to \( \psi h \) resulting from the concatenation
\(\gamma_1 \cdots \gamma_t\). Observe that the length of \(\gamma\) is at most \(2c\ell \leq 2c(q + k) \leq r_{k-1}\). To conclude that 
\(\psi h \in X_{k-1}\), it is enough to show that no vertex of \(\gamma\) is in the set \(W_1 \cup \cdots \cup W_{k-1}\).

Suppose there are vertices of \(\gamma\) in \(W_1 \cup \cdots \cup W_{k-1}\). By construction, each vertex of \(\gamma\) is at distance at most \(c\) from a vertex of the form \(\psi h_j \in \gamma\). Choose a vertex \(g\) of \(\gamma\) and a vertex of the form \(\psi h_j\) of \(\gamma\) (they might be the same vertex) with the following properties:

1. \(g \in W_1 \cup \cdots \cup W_{k-1}\),
2. the subpath of \(g\) and \(\psi h_j\) has length at most \(c\) and it has only one vertex in \(W_1 \cup \cdots \cup W_{k-1}\), namely \(g\).

Let \(t \leq k - 1\) be the smallest integer such that \(g \in W_t\). Since
\[\text{dist}_G(\phi g, h_j) \leq \text{dist}_H(\phi g, \phi \psi h_j) + \text{dist}_H(\phi \psi h_j, h_j) \leq c^2 + 2c = r,\]
either \(h_j \in Q_t\) or \(h_j \in Y_{t-1}\). The former case is impossible by the assumption on the path from \(h_0\) to \(h\). Therefore \(h_j \in Y_{t-1}\) and then the induction hypothesis implies that \(\psi h_j \in X_{t-2}\). Since the subpath of \(\gamma\) between \(\psi h_j\) and \(g\) has no vertices in \(W_1 \cup \cdots \cup W_{t-1}\) and \(\psi h_j \in X_{t-2}\), it follows that \(g \in X_{t-1}\). This implies that \(g \notin W_t\) which is a contradiction.

\[\square\]

**Proof of Theorem 4.1** Suppose that \(G\) has the \(\{f_k\}_{k \geq 1}\)-retaining property. By Lemma 3.5, the graph \(G\) has the \((\{a_k\}_{k \geq 1}, 2c)\)-retaining property where \(a_k = f_{2c(k-1)+1} + f_{2c(k-1)+2} + \cdots + f_{2c_k}\). Let
\[b_k = a_k \delta^{c^2+2c+1}.\]
We claim that \(H\) has the \(\{b_k\}_{k \geq 1}\)-retaining property as a consequence of Lemma 4.4. Let \(h_0\) be a vertex of \(H\) and consider the initial fire \(Y_0 = B_H(h_0, q)\) where \(q\) is a positive integer.

Let \(g_0 = \psi h_0\) and consider the initial fire \(X_0 = B_G(g_0, 2c(q + 2))\) of reach \(2c\) in \(G\). By assumption, there is a retaining \((\{a_k\}, 2c)\)-strategy \(\{W_k\}_{k \geq 1}\) for \(X_0\). Let \(X_n\) be the set of vertices on fire at time \(n\) with respect to this retaining strategy and let \(X = \bigcup_{n \geq 0} X_n\).

Now consider the sequences \(\{Q_k\}_{k \geq 1}\) and \(\{Y_k\}_{k \geq 1}\) defined in the statement of Lemma 4.4, and observe that \(Y_0\) corresponds to the set of vertices on fire in \(H\) at time \(k\) with respect to the \(\{b_k\}\)-strategy \(\{Q_k\}_{k \geq 1}\) and initial fire \(Y_0\) of reach one.

Let \(U = G \setminus \bigcup_{n \geq 0} X_n\) and \(V = H \setminus \bigcup_{n \geq 0} Y_n\). Since \(\{W_k\}_{k \geq 1}\) is a retaining \((\{a_k\}, 2c)\)-strategy for \(X_0\), it follows that
\[\text{Growth}(G) = \text{Growth}(U).\]
Since \(\psi: H \rightarrow G\) is a quasi-isometry, in particular, the restriction \(\psi: \psi^{-1}(U) \rightarrow U\) is a quasi-isometry (with metrics induced from \(H\) and \(G\) respectively). It follows that
\[\text{Growth}(H) = \text{Growth}(G)\] and \[\text{Growth}(U) = \text{Growth}(\psi^{-1}(U)).\]
By Lemma 4.4(2), we have the inclusion \(\psi^{-1}(U) \subseteq V\), and hence
\[\text{Growth}(\psi^{-1}(U)) \leq \text{Growth}(V) \leq \text{Growth}(H).\]
From the above relations, it follows that
\[\text{Growth}(V) = \text{Growth}(H).\]
Therefore \(\{Q_k\}_{k \geq 1}\) is a retaining \(\{b_k\}\)-strategy for \(Y_0\) in \(H\).

\[\square\]

As a corollary of the proof of Theorem 4.1 we obtain the following.
Corollary 4.5 Let $G$ and $H$ be connected graphs with degree bounded above by a constant $\delta$. Let $\phi : G \to H$ and $\psi : H \to G$ be $c$-quasi-isometries, where $c$ is a positive integer, and such that $\text{dist}(u, \psi \phi u) \leq c$ for every vertex $u$ of $G$. Suppose that $G$ has the finite-step $\{f_k\}_{k \geq 1}$-retaining property. Then $H$ has the finite-step $\{b_k\}_{k \geq 1}$-retaining property where $b_k$ is given by Eq. (4.1).

Proof Suppose that $G$ has the finite-step $\{f_k\}_{k \geq 1}$-retaining property. Then by Lemma 3.5, the graph $G$ has the $\{(a_n), r\}$-retaining property where $r = 2c$ and

$$a_n = f_{(n-1)r+1} + \cdots + f_{nr}.$$ 

Lemma 3.5 further implies that for any initial fire $X_0$ of reach $r$ there is a retaining $\{a_n\}$-strategy $\{W_k\}_{k \geq 1}$ with the additional property that $W_k = \emptyset$ for all $k$ large enough.

Consider the initial fire $B_H(h_0, q)$ for some vertex $h_0 \in H$ and some integer $q > 0$. Let $g_0 = \psi h_0$ and choose an $\{a_n\}$-strategy $\{W_k\}_{k \geq 1}$ for the initial fire $X_0 = B_G(g_0, 2c(q+2))$ in $G$ such that $W_k = \emptyset$ for $k$ large enough. Then define $\{Q_k\}_{k \geq 1}$ as in the proof of Theorem 4.1 by using Lemma 4.4. Note that the choice of $\{W_k\}_{k \geq 1}$ implies that $Q_k = \emptyset$ for all $k$ large enough. Then the argument proving Theorem 4.1 shows that $\{Q_k\}_{k \geq 1}$ is a finite-step retaining $\{b_k\}$-strategy for $Y_0$. □

5 Splittings over quasi-isometrically embedded subgroups

A group $G$ splits over a subgroup $C$ if either $G = A \ast_C B$ and $C$ is a proper subgroup of $A$ and $B$, or $G$ is an HNN-extension $A \ast_C$ (with no assumptions on $C$, nor on the isomorphism $\varphi : C \to \varphi(C) \subset A$).

5.1 Coarse separation in graphs

Let $\Gamma$ be a connected graph with the weak topology and the edge-path metric on its vertex set. If $K$ is a subset of vertices of $\Gamma$, a connected component of $\Gamma \setminus K$ is deep if its set of vertices is not contained in $B_{\Gamma}(K, r)$ for any $r > 0$. We shall say that $K$ coarsely separates $\Gamma$ if there is $R > 0$ such that $\Gamma \setminus B_{\Gamma}(K, R)$ has at least two deep connected components.

Lemma 5.1 [9, Lemma 2.2] If a finitely generated group $G$ splits over a finitely generated subgroup $C$, then $C$ coarsely separates any Cayley graph of $G$ with respect to a finite generating set. In particular, $C$ has infinite index in $G$.

Lemma 5.2 Let $\Gamma$ be a Cayley graph of $G$ with respect to a finite generating set. Let $C$ be a subgroup and let $l > 0$ be such that $\Gamma \setminus B_{G}(C, l)$ has at least two deep components. If $U$ is a subset of vertices of $\Gamma$ that contains $\bigcup_{g \in C} gD$ where $D$ is the set of vertices of a deep component $D$ of $\Gamma \setminus B_{G}(C, l)$, then $\text{Growth}(U) = \text{Growth}(G)$.

Proof Let $L$ denote $B_{\Gamma}(C, l)$. By hypothesis, $L$ separates $\Gamma$ into at least two deep connected components. Let $v_0$ be a vertex in $U$ such that $\text{dist}(v_0, L) = 1$; observe that such vertex always exists. Since the action of $C$ on $L$ has finitely many orbits of vertices, there exists a constant $K > 0$ such that any vertex of $L$ can be moved by an element of $C$ into $B_{\Gamma}(v_0, K) \cap L$. To prove the lemma, we will show that

$$|B_{\Gamma}(v_0, 2n + K + 1) \cap U| \leq |B_{\Gamma}(v_0, n)| \quad (5.1)$$

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for every $n \geq 0$. Therefore $\text{Growth}(G) = \text{Growth}(\Gamma) \leq \text{Growth}(U)$.

Since $U$ contains the vertices of a deep component $D$ of $\Gamma \setminus L$, for every $n \geq 1$ there is a vertex $v_n \in D$ such that $\text{dist}(L, v_n) = n + 1$. Let $u_n \in L$ be the vertex realizing this distance, i.e., $\text{dist}(u_n, v_n) = n + 1$. Since $\bigcup_{g \in C} gD \subset U$, by multiplying $v_n$ by an element of $C$ if necessary, we can assume that $u_n \in \overline{B_{\Gamma}(v_0, K)} \cap L$ and $v_n \in U$. Hence,

$$\text{dist}(v_0, u_n) \leq K, \quad v_n \in U, \quad \text{dist}(u_n, v_n) = n + 1.$$  

Notice that

$$v_n B_{\Gamma}(e, n) = B_{\Gamma}(v_n, n) \subseteq B_{\Gamma}(v_0, 2n + K + 1)$$

and

$$v_n B_{\Gamma}(e, n) = B_{\Gamma}(v_n, n) \subseteq U,$$  

where the last statement follows from the assumptions that $v_n \in U$ and that $\text{dist}(v_n, L) = n + 1$. Putting these two statements together yields

$$v_n B_{\Gamma}(e, n) \subseteq B_{\Gamma}(v_0, 2n + K + 1) \cap U$$

which verifies inequality (5.1).

\[ \square \]

**Proposition 5.3** Let $G$ be a finitely generated group that splits over a finitely generated subgroup $C$. Then there is a finite generating set $S$ of $G$ with the following property. Let $\Gamma = \Gamma(G, S)$ be the Cayley graph. If $\Gamma \setminus B_{\Gamma}(C, l)$ has at least two deep components and $X$ is a connected subgraph of $\Gamma \setminus B_{\Gamma}(C, l)$, then there is a deep component $D$ of $\Gamma \setminus B_{\Gamma}(C, l)$ such that the $\bigcup_{g \in C} gD$ has no vertex in $X$.

**Proof** If $G = A \ast_C B$ where $C$ is a proper subgroup of both factors, let $S$ be the union of finite generating sets of $A$ and $B$. If $G = A \ast_C = (A \ast \langle t \rangle)/\langle tat^{-1} \varphi(a) : a \in A \rangle$, let $S$ be a generating set for $A$ together with the stable letter $t$. Denote by $\text{dist}$ the word-metric on $G$ induced by $S$.

To prove the statement, we construct a coloring of the vertices of $\Gamma \setminus B_{\Gamma}(C, l)$ with two colors such that

1. the coloring is $C$-equivariant,
2. any connected subgraph of $\Gamma \setminus B_{\Gamma}(C, l)$ is monochromatic, and
3. there is at least one deep component of each color.

Assuming that we have such coloring, if $D$ is a deep component of color different that the color of $X$, then by $C$-equivariance, all vertices in $\bigcup_{g \in C} gD$ have the same color, and the statement of the proposition follows.

To define the coloring, we use the barycentric subdivision of the (geometric realization of the) Bass-Serre tree $T$ of the splitting, and endow it with the edge-path metric $\text{dist}_T$. Let us recall a description of $T$, for details see [10]. If $G = A \ast_C B$, let $G/A$ denote the $G$-set of left cosets of $A$, and let $G/B$ and $B/C$ denote the analogous $G$-sets. Then $T$ is the graph with vertex set $G/A \sqcup G/B \sqcup G/C$ and edge set the disjoint union of $\{gC, gA : g \in G\}$ and $\{gC, gB : g \in G\}$. In the case that $G = A \ast_C = (A \ast \langle t \rangle)/\langle tat^{-1} \varphi(a) : a \in A \rangle$, then $T$ has vertex set $G/A \sqcup G/C$, and edge set the disjoint union of $\{gA, gC : g \in G\}$ and $\{gA, gt^{-1}Ct : g \in G\}$.

**Definition of the $G$-equivariant coloring.** Removing the degree two vertex $C$ of $T$, splits $T$ into two connected components, say the red one and the blue one. Let $\rho : G \rightarrow T$ be the
Let $G$ be a finitely generated group that splits over a finitely generated subgroup $C$. In view of Corollary 4.5 it is enough to consider $\Gamma$ to be the Cayley graph of $G$ with respect to a finite generating set $S$ provided by Proposition 5.3. Denote by $\rho: G \to T$ its edge-path metric. Since $G$ splits over $C$, by Lemma 5.1, there is a constant $l > 0$ such that the $l$-neighborhood $L$ of $C$ in $\Gamma$,
\[ L = \{ g \in G : \text{dist}(g, C) \leq l \}, \]
separates $\Gamma$ into at least two deep components. Let $f$ be the cardinality of $L$. Let $X_0$ be a finite subset of $G$, since $C$ has infinite index in $G$, there is $g \in G$ such that $\text{dist}(gL, X_0) \geq \text{diam} X_0$. The inequality $\text{dist}(gL, X_0) \geq \text{diam} X_0$ implies that there is a deep component of $\Gamma \setminus gL$ that does not intersect $X_0$. Consider the strategy $\{W_n\}_{n \geq 1}$ where $W_1 = gL$ and $W_n = \emptyset$ for $n > 1$. Let $X_n$ denote the set of vertices on fire at time $n$ with respect to this strategy and the initial fire $X_0$ of each one. Observe that $X_n \cap W_1 = \emptyset$ for all $n \geq 0$. Hence $X = \bigcup_{n \geq 0} X_n$ spans a connected subgraph of $G \setminus L$. By Proposition 5.3, there is a deep component $D$ of $\Gamma \setminus L$ such that $\bigcup_{g \in G} gD$ has no vertex in $X$. Let $U = G \setminus X$. By Lemma 5.2, we have that $\text{Growth}(G) = \text{Growth}(U)$ and hence $\{W_n\}_{n \geq 1}$ is a finite-step retaining $f$-strategy for $X_0$. □

**Theorem 5.5** Let $G$ be a finitely generated group that splits over a finitely generated subgroup $C$. Suppose that $C$ has polynomial growth of degree $d > 0$, and is quasi-isometrically embedded into $G$. If $\Gamma$ is the Cayley graph of $G$ with respect to a finite generating, then $\Gamma$ has polynomial retaining property of degree $d - 1$.

**Proof** By Corollary 4.5, it is enough to prove the statement for $\Gamma$ the Cayley graph with respect a finite generating set $S$ provided by Proposition 5.3. Let $\text{dist}$ denote the word-metric on $G$ with respect to $S$. Since $G$ splits over $C$, there is a constant $l > 0$ such that the...
l-neighborhood $L$ of $C$ in $\Gamma$,

$$L = \{g \in G : \text{dist}(g, C) \leq l\},$$

separates $\Gamma$ into at least two deep components, see Lemma 5.1.

**Step 1** There is a constant $K_1 > 0$ such that for any $g \in G$, for any $y_0 \in gL$, and for any $n > 0$ we have

$$\beta_{gL,y_0}(n) \leq K_1 n^d$$

where $\beta_{gL,y_0}$ is the growth function of the metric space $(gL, \text{dist})$.

**Proof of Step 1** Let $\text{dist}_C$ denote a word-metric on $C$ with respect to a finite generating set of $C$. The assumption that $C$ is quasi-isometrically embedded in $G$ means that the spaces $(C, \text{dist})$ and $(C, \text{dist}_C)$ are quasi-isometric. It follows that $(C, \text{dist}_C)$, $(C, \text{dist})$, $(L, \text{dist})$ are all quasi-isometric. Since they all are discrete uniformly proper metric spaces, by Remark 2.1(3) they all have polynomial growth of degree $d$. Since $C$ acts by isometries and cocompactly on $(L, \text{dist})$, there exists a constant $K_1 > 0$ such that for any choice of basepoint on $L$, the corresponding growth function of $(L, \text{dist})$ is bounded from above by $K_1 n^d$. Since the spaces $(L, \text{dist})$ and $(gL, \text{dist})$ are isometric, the statement follows. $\square$

**Step 2** Let $K = 2^d K_1$. Let $X_0$ be a finite subset of $G$, $g$ an element of $G$, and $n$ a positive integer. Define

$$M_{n,g,X_0} = \{x \in gL : \text{dist}(x, X_0) \leq n\}.$$ 

Then

$$|M_{n,g,X_0}| \leq K(n + \text{diam } X_0)^d.$$ 

Note that $M_{n,g,X_0}$ is the set of vertices of $gL$ that would be on fire by the time $n$ if the initial fire was $X_0$ and no vertices were protected.

**Proof of Step 2** By Step 1, the growth function of $(gL, \text{dist})$ with respect to any basepoint is bounded by $K_1 n^d$. Let $y_0$ be an element of $gL$ such that $\text{dist}(y_0, X_0) = \text{dist}(gL, X_0)$. The triangle inequality implies that

$$\text{diam } M_{n,g,X_0} \leq 2n + \text{diam } X_0,$$

and hence $M_{n,g,X_0}$ is contained in the ball $B_{gL}(y_0, 2n + \text{diam } X_0)$. To conclude, observe that

$$|M_{n,g,X_0}| \leq |B_{gL}(y_0, 2n + \text{diam } X_0)| \leq K_1(2n + \text{diam } X_0)^d.$$ 

$\square$

**Step 3** Let $F = K + 1$. Let $X_0$ be a finite subset of $G$. Then there is $g \in G$ such that

$$\text{diam } X_0 < \text{dist}(gL, X_0),$$

and for every $n > 0$

$$|M_{n,g,X_0}| < \sum_{k=1}^{n} d F_k^{d-1}.$$
**Proof of Step 3** By enlarging $X_0$ if necessary, we can assume that it contains the identity element of $G$. Since $X_0$ is finite and the index of $C$ in $G$ is infinite, we can choose $g \in G$ such that $\text{dist}(gL, X_0)$ is large enough to guarantee that both inequality (5.3) and the following inequality are satisfied.

$$K (\text{dist}(gL, X_0) + \text{diam } X_0)^d < F(\text{dist}(gL, X_0))^d.$$  

This inequality together with the statement of Step 2 implies that

$$|M_{\text{dist}(gL, X_0), g, X_0}| < F(\text{dist}(gL, X_0))^d.$$  

Since $M_{n,g,X_0}$ is empty for $n < \text{dist}(gL, X_0)$ and $F > K$, it follows that $|M_{n,g,X_0}| < F n^d$ for every $n \in \mathbb{N}$. A calculus exercise shows that $n^d \leq d \sum_{k=1}^{n} k^{d-1}$, and thus inequality (5.4) is satisfied. □

Inequality (5.4) allows us to define a retaining $\{d F n^{d-1}\}$-strategy for any finite subset $X_0$ of $G$; this is proved in the next step concluding the proof of the theorem. To simplify the notation define

$$p_n = \sum_{k=1}^{n} dFk^{d-1},$$

and observe that $p_n$ is the maximal number of vertices that can be protected by the time $n$ using a $\{d F n^{d-1}\}$-strategy.

**Step 4** Let $X_0$ be a finite subset of $G$. Let $g \in G$ be an element satisfying inequalities (5.3) and (5.4). Let $w_1, w_2, w_3, \ldots$ be an enumeration of the countable set $gL$ such that the sequence $\{\text{dist}(w_i, X_0)\}_{i \geq 1}$ is non-decreasing, and for each integer $n \geq 1$ let

$$W_n = \left\{w_i : p_{n-1} < i \leq p_n \text{ and } w_i \notin \bigcup_{1 \leq i < n} W_i \right\}.$$  

Then $\{W_n\}_{n \geq 1}$ is a retaining $\{d F n^{d-1}\}$-strategy for $X_0$.

**Proof of Step 4** Observe that

$$|W_n| \leq p_n - p_{n-1} = d F n^{d-1}$$

for every $n \geq 0$, and hence $\{W_n\}_{n \geq 1}$ is a $\{d F n^{d-1}\}_{n \geq 1}$-strategy. Let $X_n$ denote the set of vertices on fire at time $n$ with respect to this strategy and the initial fire $X_0$ of reach one. We claim that

$$X_n \cap W_{n+1} = \emptyset, \text{ for all } n \geq 0. \quad (5.5)$$

Indeed, observe that $X_0$ and $W_1$ are disjoint as a consequence of inequality (5.3). Suppose, by induction, that $X_{n-1}$ has been defined, and $X_{n-1} \subseteq B_1(X_0, n-1)$, and $X_{n-1}$ and $W_n$ are disjoint. Recall that $X_n$ consists of vertices $v$ such that $\text{dist}(v, X_{n-1}) \leq 1$ and $v \notin W_1 \cup \cdots \cup W_n$. Thus $X_n \subseteq B_1(X_0, n)$. Since $W_{n+1} \subseteq gL$, we have

$$X_n \cap W_{n+1} \subseteq X_n \cap gL \subseteq B_1(X_0, n) \cap gL = M_{n,g,X_0} \subseteq \bigcup_{i=1}^{n} W_i,$$

where the last inclusion is a consequence of (5.4) and the definition of the $W_i$’s. By definition, $W_{n+1} \cap \bigcup_{i=1}^{n} W_i = \emptyset$, and therefore $X_n \cap W_{n+1} = \emptyset$. This concludes the verification of equation (5.5). ☑ Springer
By definition $X_n \subset X_{n+1}$ for all $n \geq 0$ and $X_n \cap W_m = \emptyset$ if $n \geq m$. Let $X = \bigcup_{n \geq 1} X_n$ and observe that (5.5) implies that
\[X \cap L = \bigcup_{n \geq 0} X_n \cap \bigcup_{n \geq 1} W_n = \emptyset.\]
Since $X$ spans a connected subgraph of $\Gamma \setminus L$, the choice of the finite generating set $S$ given by Proposition 5.3 implies that there is a deep component $D$ of $\Gamma \setminus L$ such that $\bigcup_{g \in G} gD$ has no vertex in $X$. Let $U = G \setminus X$ and note that Lemma 5.2 implies that $\text{Growth}(G) = \text{Growth}(U)$, and hence $\{W_n\}_{n \geq 1}$ is a retaining $\{dF_n^{d-1}\}$-strategy for $X_0$. \hfill \qed

Step 4 concludes the proof of the theorem. \hfill \qed

### 5.3 Ends of groups and the finite-step retaining property

The following proposition uses the notion of an *end* of a topological space. For a definition of an end we refer the reader to [2], and we follow the convention that a graph carries the weak topology.

**Proposition 5.6** Let $G$ be a locally finite connected graph. If $G$ has the finite-step retaining property of degree $d$, then either $G$ has the containment property of degree $d$, or $G$ has at least two ends.

**Proof** Let $K > 0$ be a constant such for every finite subset of vertices of $G$ there is a finite-step retaining $\{Kn^d\}$-strategy. Suppose that $G$ does not have the containment property of degree $d$. In particular, this implies that $G$ has infinitely many vertices. Then there is an initial fire $X_0$ of reach one for which there is no finite-step retaining $\{Kn^d\}$-strategy that contains it.

Let $\{W_n\}_{n \geq 1}$ be a finite-step retaining $\{Kn^d\}$-strategy for the initial fire $X_0$. Let $X_n$ be the set of vertices on fire at time $n$ with respect to this strategy. Let
\[X = \bigcup_{n \geq 0} X_n \quad \text{and} \quad W = \bigcup_{n \geq 1} W_n,\]
and let $U$ be the complement of $X$ in the set of vertices of $G$. Since $\{W_n\}_{n \geq 1}$ is a finite-step retaining strategy for $X_0$, the set $W$ is finite. We claim that $G \setminus W$ contains at least two unbounded connected components.

Since $X_0$ is not contained by the strategy $\{W_n\}_{n \geq 1}$, the set $X$ is infinite. By definition of $X_n$, see Sect. 3.2, every vertex of $X$ is connected to a vertex of $X_0$ by a path in $G$ that contains only vertices in $X$. Since $X_0$ is finite and $X$ is infinite, the subgraph $A$ of $G$ spanned by $X$ contains an infinite connected subgraph that we denote by $A'$.

Since $\{W_n\}_{n \geq 1}$ is a retaining strategy for $X_0$, it follows that $\text{Growth}(G) = \text{Growth}(U)$. Since $G$ is connected and has infinitely many vertices, $U$ is an infinite subset of vertices. Let $U' = U \setminus W$. By definition of $X_n$ every path in $G$ between a vertex in $X$ and a vertex in $U'$ contains a vertex in $W$. Let $B$ be the subgraph of $G$ spanned by $U'$. Consider the map from the collection of connected components of $B$ to the collection of non-empty subsets of $W$, that assigns to a connected component the subset of elements of $W$ that appear in minimal length paths from a vertex in the component to a vertex in $X$. Since the graph $G$ is locally finite, this map is finite to one. Therefore the number of connected components of $B$ is finite. Since $U'$ is infinite, $B$ contains an infinite connected subgraph that we denote by $B'$.

Because paths between $X$ and $U'$ have to pass through $W$, we have that $A'$ and $B'$ are contained in different connected components of $G \setminus W$. Since $G$ is locally finite, the infinite
connected subgraphs $A'$ and $B'$ are unbounded. Therefore $G \setminus W$ contains at least two unbounded connected components which implies that $G$ has at least two ends.

\textbf{Corollary 5.7} Let $G$ be a finitely generated group. If $G$ has the finite-step retaining property, then either $G$ has the constant containment property or $G$ has infinitely many ends.

\textbf{Proof} Suppose that $G$ does not have the constant containment property. Finitely generated groups with two ends are virtually cyclic and hence they have linear growth [2, Part I, Theorem 8.32(3) and Example 8.36]. Since finitely generated groups with growth at most quadratic have the constant containment property [4, Theorem 1], it follows that $G$ does not have two ends. On the other hand, a finitely generated group has either 0, 1, 2, or infinitely many ends [2, Part I, Theorem 8.32(1)]. Therefore Proposition 5.6 implies that $G$ has infinitely many ends.

\textbf{Remark 5.8} The finite-step retaining property of degree $d + 1$ is not equivalent to the finite-step retaining property of degree $d$ for $d \geq 0$. Indeed, consider the group $G = \mathbb{Z}_{d+3}$. This group has containment property of degree $d + 1$, see [4, Theorem 3]. In particular $G$ has the finite-step retaining property of degree $d + 1$.

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