Apportionment Methods

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Abstract

Most democratic countries use election methods to transform election results into whole numbers which usually give the number of seats in a legislative body the parties obtained. Which election method does this best can be specified by measuring the error between the allocated result and the ideal proportion. We show how to find an election method which is best suited to a given error function. We also discuss several properties of election methods that have been labelled paradoxa. In particular we explain the highly publicised “Alabama” Paradox for the Hare/Hamilton method and show that other popular election methods come with their very own paradoxa.

1 Introduction

Practically all democratic countries are faced with the problem of selecting members of their legislative bodies according to votes of their population. The method by which this selection of representatives is performed is commonly known as an election method. Its main function is to transform the
election results, which are usually the number of votes for various candidates or parties, into whole numbers which usually give the number of seats in a legislative body. Nearly every democratic country employs its own favourite election method. This immediately leads to the question of which election method is in some sense optimal and most just? This is not an easy question to answer as one can see from the vast amount of literature it has generated. History shows that at various times different election methods were in favour and many countries have changed their election method in the past.

One can distinguish between two major types of election methods, namely the majority voting methods and the proportional voting methods. The majority voting method is used, for example, in Great Britain to elect members of parliament and in the United States of America to elect members of the congress. In this method, the eligible voters are divided into voting districts and, in its purest form, each district elects one candidate by majority vote. It is well known that the percentage of members in parliament or congress belonging to a given party need not be close to the percentage of votes this party obtained overall. For this reason, we do not consider this voting method in this paper.

Among the most frequently discussed proportional voting systems are the D’Hondt or Jefferson method (e.g. used in Germany until 1983, still used in many countries), the Hare or Hamilton method (used e.g. in Germany, Tasmania, used in the US 1840 to 1890) and Sainte-Lagué or Webster method (used in New Zealand).

The main aim of this paper is to present a general treatment of all proportional voting methods. Mathematically, the accuracy of a voting method can be measured by a so-called error function, a function which measures the gravity of the error between the exact votes and the allocated seats. In practice error functions are dictated by courts or legislative bodies and these error functions need not coincide with mathematical error functions. We show in Theorem 6.2 how to find a proportional voting method which minimises a given error function. A slightly weaker version of this theorem has been announced by Niemeyer and Wolf (1984), [8]. Here we give a first complete proof. We demonstrate how most of the well known election methods can be obtained from this general result by special choices of error functions. The Hare method can be singled out by the fact that it minimises infinitely many different error functions. This is certainly not the case for the methods of D’Hondt and Sainte-Lagué. Another aim of this paper is to give a mathematical argument as to why the Hare method is the best among the
proportional voting methods.

Each voting method can lead to results which appear paradoxical. We discuss various paradoxes which have enjoyed a lot of attention in the literature, for example, the so-called Alabama Paradox for the Hare method. We argue that in many cases the "paradoxity" is only in the eye of the beholder. In particular, the methods of D’Hondt and Sainte-Laguë are blessed with their own paradoxes.

2 Background

This paper is concerned with four different but mathematically related problems. First there is the apportionment problem as seen in the example of the American House of Representatives of the United States Congress, where each state of the union is entitled to a number of seats, at least one, according to and as closely proportional as possible to its number of legal inhabitants. The latter is determined every ten years by a census of the United States. Each state is subdivided into voting districts according to how many seats are assigned to it and the representatives of each district are assigned by a majority vote.

Second there is the apportionment problem for political parties participating in an election of a country where seats in the parliament are assigned to parties as proportionally as possible to the election results. As an example we can take the German Federal election. Germany is divided into voting districts. The number of seats in parliament is twice the number of voting districts. Voters receive a ballot on which there are two votes. The second vote (Zweitstimme) is cast for a political party. The number of seats in parliament is assigned to eligible parties as proportionally as possible to these second votes. (A party is eligible if it received at least 5% of the total number of second votes or at least three seats by first vote.) The first vote (Erststimme) is to determine by simple majority a representative for the voting district in parliament. The elected candidates receive seats in parliament which are counted towards the allocated number of seats of the party they represent. The remaining seats are distributed among the participating parties according to the second votes and according to the election method used. It may happen that a party obtains more seats by the first vote (Erststimme) than is allowed according to the second vote (Zweitstimme). The parties keep those seats and they are called “Überhangmandate”. Using a
first and second vote, the Federal elections combines the important feature of the majority election methods, that each voting district is represented by a member of parliament, with the advantage of proportional election methods, by which the number of seats allocated to a party is as closely proportional to the number of votes as possible. Of course, there is always the difficulty that one or more parties get more seats directly than by the number of total votes.

The third problem is that of rounding a list of given numbers, so that the total sum of the rounded numbers equals the rounded (standard rounded) sum. The easiest example perhaps is that of rounding percentage numbers, which add up to 100%, but after (standard) rounding fail to add up to 100%. Another problem can be the amount of exports of a certain commodity given in Euro by each country of the European community, for instance. The press publishes figures representing the exports of each country rounded to millions of Euros. The total amount is also rounded to the nearest million independently of the individual figures. Now the problem is that the rounded figures should add up to the rounded total.

The fourth problem comes from Operations Research. Let us assume that a factory is producing indivisible goods, e.g. cars of the same make. The profit is to be distributed among the share holders as closely proportional to the number of shares as possible.

2.1 Describing the four problems mathematically

In all of these problems we can start with a positive integer $M$, where $M$ represents the number of seats in the first and second problem; $M$ denotes the sum of the rounded integers in the third problem; $M$ is the number of goods produced in the fourth problem. Let $n$ denote the number of states in the Union, or parties, or numbers to be rounded, or shareholders in a manufacturing company. Further, we define a real $n$-dimensional vector $\mathbf{a} = (a_1, \ldots, a_n)$ and an integer vector $\mathbf{m} = (m_1, \ldots, m_n)$. Let $A$ denote the sum of the entries of $\mathbf{a}$. In the first problem $a_j$ is the number of votes in the $j$-th state, $A$ the total number of votes and $m_j$ is the number of seats allocated to the state. Let $q_j = a_j M/A$ be the exact quota, that is the exact proportion of seats the $j$-th state may claim. In the second problem $a_j$ is the number of votes party $j$ obtains and $m_j$ is the number of seats allocated to party $j$. Let $q_j = a_j M/A$ be the exact quota, that is the exact proportion of seats the $j$-th party may claim. In the third problem $a_j$ is the $j$-th given number and
Figure 1: Election Polyhedra

$m_j$ is the integer obtained by the rounding process applied to $a_j M$. Here let $q_j = a_j$. In the fourth problem, $a_j$ is the exact quota to which the $j$-th share holder is entitled to and $m_j$ the number of goods he receives. Again we let $q_j = a_j$.

This leads to the following definitions. We are interested in sets of vectors whose entries are the exact quotas. Let $M$ be a positive integer (e.g. number of seats) and $A$ a positive real number (e.g. number of valid votes). Let

$$Q = \{(q_1, \ldots, q_n) \mid q_j \in \mathbb{R}, q_j \geq 0, \sum_{i=1}^{n} q_i = M\}.$$ 

This set contains all possible exact quotas among $n$ parties (where in problems one and two we computed the exact quotas from a total of $A$ valid votes). Note that we allow $Q$ to contain real vectors. Let

$$M = \{(m_1, \ldots, m_n) \mid m_j \in \mathbb{Z}, m_j \geq 0 \text{ and } \sum_{j=1}^{n} m_j = M\}$$

be the subset of all integer vectors whose entries are non-negative and sum to $M$. Note that $M$ is a lattice over the integers and represents the possible seat allocations.

We illustrate the set $Q$ for 3 parties and for $M = 5$ in Figure 1. The set $Q$ is a unilateral triangle in the space $\mathbb{R}^3$. Its corners are $(5, 0, 0), (0, 5, 0)$ and $(0, 0, 5)$. The black dots represent the exact quotas which are also possible seat distributions, that is exact quotas which lie in $M$. The other points in
the triangle correspond to all points in $Q$. A valid apportionment method should map every point in the triangle to a seat allocation. It should be possible to draw a region around each black dot containing only this one black dot such that all exact quotas inside this region are mapped to the seat allocation to which the black dot is mapped. We believe a good apportionment should map a black dot to the seat allocation it describes.

Finally, the apportionment method might have a choice of which seat allocation to choose for the exact quotas which lie on the borders between two or more possible regions.

An apportionment method is in principal a function which maps each possible vote distribution to a certain seat distribution. For a precise definition see p. 97 Balinski and Young (1982).

We believe a good apportionment method should satisfy

\begin{equation}
\text{If } q \in M \text{ then } f(q) = \{q\}.
\end{equation}

In most cases $a$ and $M$ are fixed. Sometimes it might be necessary to emphasise the dependence of $f$ on $a$ and $M$, in which case $f_{a,M}$ denotes an apportionment method with given $a$ and $M$.

While a mathematical definition of an apportionment method is sufficient to procure a meaningful apportionment, considerations of fairness dictate additional requirements. Especially the order in which the parties are listed on a ballot should have no effect on the outcome of the election, that is the apportionment method should be symmetric. Formally, let $\pi$ be a permutation of $\{1, \ldots, n\}$ and let $q \in Q$ be the $n$-dimensional real vector $(q_1, \ldots, q_n)$. If we define $q^\pi = (q_{1\pi}, \ldots, q_{n\pi})$. If $B$ is a subset of $Q$ then define $B^\pi = \{b^\pi \mid b \in B\}$.

**Definition 2.1. (Symmetry of an apportioning method)** An apportionment method is symmetric, if for every permutation $\pi$ of $\{1, \ldots, n\}$ we have $f(q^\pi) = (f(q))^\pi$.

The outcome of a reasonable election method should depend only on the vectors of the exact quotas, that is if two different vote distributions yield the same exact quotas, the results should be the same. Thus the apportionment method should be homogeneous.

**Definition 2.2. (Homogeneity)** An apportionment method is homogeneous if for every real $\lambda > 0$ we have $f_{\lambda a,M}(q) = f_{a,M}(q)$.

From now on all apportionment methods are symmetric and homogeneous.
3 Overview over various apportionment methods

Historically, many apportionment methods have been discussed in political assemblies and in the literature. In this paper we are concerned with two major classes: the divisor methods and the rounding methods. Examples of the divisor methods are the methods of D’Hondt and Sainte-Laguë. The best known rounding method is the Hare method, also called the method of the greatest remainders. A detailed description of these methods can be found in Balinski and Young (1982), [2]. Here we give a very brief overview in order to establish the notation we use for the various methods.

The divisor methods start with a strictly increasing sequence \((d_j)_{j \in \mathbb{N}}\) of non-negative real numbers, called the sequence of divisors. We then have to divide the exact quotas \(q_k\) for \(k = 1, \ldots, n\) by the \(d_j\) and define the \(M \times n\) matrix whose entry in row \(j\) and column \(k\) is \(q_k/d_j\). We select the \(M\) largest entries of this matrix and count the number \(m_k\) of entries selected from the \(k\)-th column. Then the party \(P_k\) will get \(m_k\) seats. A linear divisor method is given by an arithmetically increasing sequence \((d_j)_{j \in \mathbb{N}}\), that is \(d_j = d(d_0 + (j - 1))\), where \(d\) and \(d_0\) are fixed real numbers. As \(d\) divides all terms of the sequence \(d_j\), we can omit \(d\) by the homogeneity of the apportionment method, that is we can choose as sequence of divisors the sequence \(d_j = d_0 + (j - 1)\).

Linear divisor methods have been treated extensively in the literature, see for example the literature review by Heinrich et al. (2005), [5]. The following sequences yield some well known linear divisor methods (see [7, p. 124]):

| \(d_0\) | Method                          |
|--------|--------------------------------|
| 0      | Adams                          |
| 1/3    | Danish Method                  |
| 2/5    | Condorcet                      |
| 1/2    | Sainte-Laguë or Webster        |
| 2/3    | Considerant                    |
| 1      | D’Hondt or Jefferson           |
| 2      | Imperiali                      |

Note that for \(d_0 \leq 1\) the linear divisor methods satisfy Equation [1].

Non-linear divisor methods have also been considered. The most well-known ones are the Dean method, where \(d_j = j(j - 1)/(j - 1/2)\), and the
method of Hill or Huntington, where \( d_j = \sqrt{j(j-1)} \).

As we will see later, the values \( d_0 = 0 \) is biased towards small parties, \( d_0 = 1 \) is biased towards large parties, whereas the value \( d_0 = 1/2 \) is considered to be neutral.

Let \( \rho \) be real constant with \( 0 \leq \rho \leq 1 \). The \( \rho \)-rounding method (see Kopfermann [7, p. 117]) considers the exact quotas \( q_j \) and put \( q^\rho_j := \frac{q_j}{M}(M + 2\rho - 1) \). Then we have that \( \sum_{j=1}^{n} q^\rho_j = (M + 2\rho - 1) \) and let \( \mu^\rho_j = \lfloor q^\rho_j \rfloor \). Then

\[
\sum_{j=1}^{n} \mu^\rho_j \leq \begin{cases} 
M - 1 & \text{for } \rho = 0 \\
M & \text{for } 0 < \rho < 1 \\
M + 1 & \text{for } \rho = 1
\end{cases}
\] (2)

For \( 0 \leq \rho < 1 \) we have therefore \( \sum_{j=1}^{n} r^\rho_j \leq M \) and we define \( r^\rho_j = q^\rho_j - \mu^\rho_j \).

We start by allocating \( \mu^\rho_j \) seats to party \( P_j \). This leaves \( M - \sum_{j=1}^{n} \mu^\rho_j \) or, when \( \rho = 0 \), \( M - \sum_{j=1}^{n} \mu^\rho_j + 1 \), unallocated seats. Party \( P_j \) obtains an additional seat if \( r^\rho_j \) is among the \( M - \sum_{j=1}^{n} \mu^\rho_j \) (or \( M - \sum_{j=1}^{n} \mu^\rho_j + 1 \)) largest numbers of \( r^0_1, \ldots, r^0_n \). If \( \rho = 0 \) there is still a problem if all \( q_i \) are integers. In this case all \( r^0_j = 0 \) and we have to assign one additional seat. This is allocated to a party at random. In the case \( \rho = 1 \) and all \( q_i \) are integers, we have assigned one seat too many and this seat is taken from a random party among the parties who obtained more seats than their exact quota.

For \( \rho = 1/2 \) we obviously have the well-known method of the greatest remainder. This method is also known under the name Hare or Hamilton. For all other values of \( \rho \) \( 0 \leq \rho \leq 1 \) we have a general rounding method or a general greatest remainder method. We will discuss these methods especially with respect to their effect on the apportionment method a little later.

Note that for \( 0 < \rho < 1 \) the \( \rho \)-rounding methods satisfy Equation (1).

4 Constraints

In this section we examine the constraints of apportionment methods, which are necessary or beneficial for solving the four problems considered above.

There are several possibilities to enforce certain conditions. Not all of these conditions can be fulfilled exactly but it is also important to be able to compute the probability with which a condition is violated when choosing an apportionment method.

These conditions are as follows:
1. **Bias Condition:** The apportionment method should be free of bias, that is it should neither favour large nor small parties. Suppose $L, S$ are subsets of $\{1, \ldots, n\}$ such that $m_j > m_i$ whenever $j \in L$ and $i \in S$. Then an apportionment method favours large parties if

$$\frac{\sum_{i \in L} m_i}{\sum_{i \in L} a_i} > \frac{\sum_{j \in S} m_j}{\sum_{j \in S} a_j},$$

and it favours small parties if

$$\frac{\sum_{i \in L} m_i}{\sum_{i \in L} a_i} < \frac{\sum_{j \in S} m_j}{\sum_{j \in S} a_j},$$

see [2, p. 125].

2. **Monotony Condition:** If one party has a larger exact quota than another, it cannot receive less seats, that is if $q_j < q_k$ then $m_j \leq m_k$ for all $j, k$ with $j, k \in \{1, \ldots, n\}$.

3. **Lower Quota Condition:** Each party is assigned at least as many seats as the largest integer less than or equal to the exact quota, that is $\lfloor q_j \rfloor \leq m_j$ for all $j$ with $1 \leq j \leq n$.

4. **Upper Quota Condition:** Each party receives at most as many seats as the least integer greater or equal to the exact quota, that is $m_j \leq \lceil q_j \rceil$ for all $j$ with $1 \leq j \leq n$.

5. **Majority Condition:** If a party obtains the absolute majority of the votes then it receives the absolute majority of the seats, that is if for some $j$ with $1 \leq j \leq n$ we have $a_j > \frac{1}{2}A$ (and hence $q_j > \frac{1}{2}M$) then $m_j > \frac{1}{2}M$.

6. **Coalition Condition:** If a party has less than half of the total number of votes then it receives also less than half of the total number of seats, that is if for some $j$ with $1 \leq j \leq n$ we have $a_j < \frac{1}{2}A$ (and hence $q_j < \frac{1}{2}M$) then $m_j < \frac{1}{2}M$.

This condition is called Coalition Condition as it ensures that in a three party method the coalition of the two smaller parties which received together the absolute majority of the votes, will also receive the absolute majority of the seats.
7. **Independence Condition**: The number $m_j$ of seats assigned to the party $P_j$ depends only on the exact quota $q_j$ but not on the distribution of the quotas $q_k$ of other parties $P_k$ for $k \neq j$.

8. **House Monotony**: Let $a$ be a vector of votes. Let $m$ be the seat distribution obtained from $a$ in the case that there are $M$ seats and $\tilde{m}$ the seat distribution obtained from $a$ in the case that there are $M + 1$ seats. Then a House monotone apportionment method satisfies $m \leq \tilde{m}$.

The following table indicates when the conditions are satisfied for the $\rho$-rounding methods and the linear divisor methods for given $d_0$.

| Condition       | $\rho$-rounding method | $d_0$ linear divisor method |
|-----------------|-------------------------|-----------------------------|
| Homogeneity     | always                  | always                      |
| Unbiased        | $\rho = 1/2$            | $\delta_0 = 1/2$ [2] Prop. 5.3 |
| Monotony        | always                  | always                      |
| Lower Quota     | $1/2 \leq \rho \leq 1$ [7] p.120 | $\frac{n-2}{n-1} \leq d_0 \leq 1$ [7] p.131 |
| Upper Quota     | $0 \leq \rho \leq 1/2$ [7] p.120 | $0 \leq d_0 \leq \frac{1}{n-1}$ [7] p.131 |
| Majority        | $\rho = 1$, $M$ odd [7] p.121 | $d_0 = 1$, $M$ odd [7] p.131 |
| Coalition       | $\rho = 0$, $M$ odd [7] p.121 | $d_0 = 0$, $M$ odd [7] p.131 |
| Independence    | $n \leq 2$ or $m_i = m_j$ [7] p.97 | $n \leq 2$ or $m_i = m_j$ [7] p.97 |
| House Monot.    | never                   | all [2] Cor. 4.3.1, p.117 |

All non-linear divisor methods are homogeneous. None is unbiased, see [2] Prop. 5.3, all are monotone. No non-linear divisor method satisfies both the upper and lower quota condition, however it never violates both simultaneously, see [2], Prop. 6.4 and 6.5. All non-linear divisor methods are House monotone, see [2] Cor. 4.3.1, p. 117.

A method for computing the seat bias for a given apportionment method with a hurdle (e.g. the 5% hurdle) can be found in Schwingenschl"ogl and Pukelsheim (2006), [13]. Another condition, a Gentle Majority Condition, which is important for forming committees is discussed in Pukelsheim (2006), [12].

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5 A modification of Hare and the history of the Hare-Niemeyer Method in Germany

It all started with an article in the newspaper “Frankfurter Allgemeine” (FAZ) by Dr. K.F. Fromme which appeared on 14 October 1970, pointing out the difficulties in determining the number of seats each party gets in the various committees pursuant to the 1970 elections in the Federal Republic of Germany. In this election, the CDU/CSU won 253 seats in parliament, the SPD 237 and the FDP 28, giving the SPD/FDP coalition under Chancellor Helmut Schmidt a majority of 265 seats (including the members from Berlin, who did not always have a vote but were counted in the assignment of seats in the committees). The D’Hondt system was used at that time to determine the number of parliamentary seats a given party won in general elections and to determine the distribution of committee seats.

The difficulty that Fromme pointed out was the paradoxon that - in a committee with a given number of 33 seats - the distribution according to D’Hondt was as follows: the CDU/CSU was assigned 17 seats, SPD 15 and the FDP 1, thus giving the opposition party a majority. The same is also true e.g. for committees with 33, 31, 29, . . . , 9 members. This led to a discussion in parliament about changing the size of the committees because it was assumed that a method better than D’Hondt “would be hard to find”. The political question which now arose was how to keep the majority and the mathematical question concerned which method to use. After giving the matter some thought, on 16 October 1970 the first author wrote to the administration of the Bundestag suggesting the method of the largest remainders, which was subsequently adopted.

Almost seven years later, the CDU/FDP coalition in the state Niedersachsen wanted to introduce legislation which would replace the D’Hondt-system by the Hare-system. But the procedure left a couple of questions open. Therefore, the committees of the state parliament which were discussing this piece of legislation invited the first author to a joint hearing on 23 March 1977. The problem they discussed was the following: Even with the Hare method it can happen, that a coalition of parties with the majority of the seats in parliament does not get a majority of the seats in a committee.

As an example to demonstrate that the Hare method does not always satisfy the Majority Condition consider three parties competing for $M = 101$ seats. They received 50600, 40650 and 9750 votes, respectively. The exact
quotas they receive are $q_1 = 50.6; q_2 = 40.65; q_3 = 9.75$. Then, according to the Hare method, the seat distributions are $m_1 = 50, m_2 = 41$ and $m_3 = 10$ and so the first party does not have the majority of the seats, even though it received the majority of the votes.

In general, the Hare method does not always satisfy the Majority Condition when $M$ is odd. If $M = 2k + 1$, then it is possible that a party obtains more than 50% of the votes but only receives $k$ seats. However, it is not possible that the party receives less than $k$ seats as is possible using the Sainte-Laguë method. Further, it is also fairly easy to fix this situation as suggested by the first author. If a party obtains more than 50% of the votes then its quota is at least $k + 1/2$. If one gives one of the remaining seats to this party, then the Majority Condition is fulfilled and it is still possible to redistribute the remaining seats such that each party gets at least the lower quota. D’Hondt’s method favors the largest party so much so, that it can receive more than $q + 1$ seats, where $q$ is the exact quota. This is because this method does not satisfy the Upper Quota Condition.

As a result of this hearing, the parliament in Niedersachsen decided to hold the elections in Niedersachsen according to the modified Hare method, sometimes also called the Hare-Niemeyer procedure. However, in 1986 this method was again replaced by the d’Hondt method (see [14]).

From 1987 onwards the Hare-Niemeyer method has also been used to for the seat allocations in the German Bundestag until it was replaced in 2008 by the method of Sainte-Laguë (see [15]).

6 On the error function of a general apportionment method

We start with a general apportionment problem. Suppose $M$ seats in parliament are to be distributed among parties $P_1, \ldots, P_n$. Suppose further the parties received votes $a_1, \ldots, a_n$, represented by the vector $a = (a_1, \ldots, a_n)$, and let $A = \sum_{i=1}^{n} a_i$. Then the exact quotas are $q_j = \frac{Ma_j}{A}$ for $1 \leq j \leq n$ and are exactly proportional to the number of votes party $P_j$ received. Let $f$ be an apportionment method. In any apportionment method the transition from a vector $q$ with real entries to a vector $m$ with integer entries is bound to cause errors. On the one hand we can measure individual errors between $q_j$ and $m_j$ for $1 \leq j \leq n$ and on the other hand there is a global error
which comprises the individual errors. The following definition captures the properties of an error function which is decomposable into individual error functions.

**Definition 6.1.** For \(1 \leq j \leq n\) let \(\varphi_j : \mathbb{N} \to \mathbb{R}_{\geq 0}\) be functions such that the functions \(H_j : \mathbb{N}^+ \to \mathbb{R}_{\geq 0}\) defined by \(H_j(\ell) = \varphi_j(\ell) - \varphi_j(\ell - 1)\) are increasing. Then the function \(\psi : \mathcal{M} \to \mathbb{R}_{\geq 0}\) defined by \(\psi(m) = \sum_{j=1}^{n} \varphi_j(m_j)\) is called a decomposable error function. The function \(\varphi_j(x)\) is called the \(j\)th component of \(\psi\).

Further, we have

\[
\psi(m) = \psi(0) + \sum_{j=1}^{n} \sum_{\ell=1}^{m_j} H_j(\ell). \tag{3}
\]

Decomposable error functions are also considered by Gaffke and Pukelsheim (2008). For a given a decomposable error function \(\psi\) the following theorem describes an algorithm called \textsc{MinimalSolution} to determine the vector \(m_0 \in \mathcal{M}\) which minimizes the error function. Hence for a given decomposable error function the theorem can be used to give rise to an apportionment method, namely the method which assigns each vector \(a\) the vector \(m_0\) which minimizes the error function, see also Niemeyer and Wolf (1984). For divisor methods see also the Min-Max Theorem of Balinski and Young (1982), [2, p. 100].

**Theorem 6.2.** Let \(\psi\) be a decomposable error function. Then

1. There is at least one solution \(m_0 \in \mathcal{M}\) which minimizes the error function \(\psi\);

2. \(m_0 = (m_0^1, \ldots, m_0^n)\) is a minimal solution if and only if \(H_j(m_j^0 + 1) \geq H_k(m_k^0)\) for all \(j, k \in \{1, \ldots, n\}\);

3. \(m_0\) is unique if \(H_j(m_j^0 + 1) > H_k(m_k^0)\) for all \(j, k \in \{1, \ldots, n\}\) with \(j \neq k\);

4. Consider the following algorithm \textsc{MinimalSolution}: Let \(S\) denote the set of \(m = (m_1, \ldots, m_n)\) such that

   \[
   (a) \sum_{j=1}^{n} m_j = M,
   \]
(b) the multiset \( \{ H_j(\ell) \mid 1 \leq \ell \leq m_j, 1 \leq j \leq n \} \) contains \( M \) smallest elements of the matrix

\[
\begin{pmatrix}
H_1(1) & \ldots & H_n(1) \\
\vdots & & \vdots \\
H_1(M) & \ldots & H_n(M)
\end{pmatrix}.
\]

Then \( S \) consists of all minimal solutions.

**Proof.**

1. The existence follows from the fact that the image of \( \mathcal{M} \) under \( \psi \) is a finite subset of \( \mathbb{R}_{\geq 0} \) and therefore has a minimal subset. Hence there is at least one element \( m \in \mathcal{M} \) for which \( \psi(m) \) is the minimum and so \( m \) is a minimal solution.

2. Suppose first that \( m^0 = (m^0_1, \ldots, m^0_n) \) is a minimal solution and \( j, k \in \{1, \ldots, n\} \). If \( j = k \) the result follows since \( H_j(\ell) \) is increasing. Now suppose \( j \neq k \). Choose another solution \( m^1 \) which differs from \( m^0 \) in exactly two components, namely \( m^1_j = m^0_j + 1 \) and \( m^1_k = m^0_k - 1 \). Since \( m^0 \) is minimal we have \( \psi(m^0) \leq \psi(m^1) \) and hence

\[
\sum_{i=1}^{n} \varphi_i(m^0_i) \leq \sum_{i=1}^{n} \varphi_i(m^1_i)
\]

which implies

\[
\varphi_j(m^0_j) + \varphi_k(m^0_k) \leq \varphi_j(m^1_j) + \varphi_k(m^1_k)
= \varphi_j(m^0_j + 1) + \varphi_k(m^0_k - 1)
\]

and so

\[
H_k(m^0_k) = \varphi_k(m^0_k) - \varphi_k(m^0_k - 1)
\leq \varphi_j(m^0_j + 1) - \varphi_j(m^0_j) = H_j(m^0_j + 1).
\]

On the other hand suppose \( m^0 \) is a solution for which \( H_k(m^0_k) \leq H_j(m^0_j + 1) \) for all \( j, k \in \{1, \ldots, n\} \). Let \( m = (m_1, \ldots, m_n) \) be another
solution. Then $\psi(m) = \sum_{j=1}^{n} \varphi_j(m_j)$. Compare the values of $\psi$ at $m_j^0$ with those at arbitrary points $m_j$ and note

$$M = \sum_{j=1}^{n} m_j^0 = \sum_{j=1}^{n} m_j.$$  

Therefore one can divide the indices $1, \ldots, n$, for which $m_j^0 \neq m_j$ into two disjoint sets $J_1$ and $J_2$, according to whether $m_j^0 < m_j$ or $m_j^0 > m_j$. Then $\sum_{j \in J_1} (m_j - m_j^0) = \sum_{j \in J_2} (m_j^0 - m_j)$. This yields that

$$\psi(m) - \psi(m_0) = \sum_{j \in J_1} (\varphi_j(m_j) - \varphi_j(m_j^0)) + \sum_{j \in J_2} (\varphi_j(m_j) - \varphi_j(m_j^0)) = L - R$$

and that

$$L = \sum_{j \in J_1} (\varphi_j(m_j) - \varphi_j(m_j^0)) = \sum_{j \in J_1} \left( H_j(m_j) + H_j(m_j - 1) + \cdots + H_j(m_j^0 + 1) \right) \geq \sum_{j \in J_1} (m_j - m_j^0) \cdot H_j(m_j^0 + 1) \geq \min_{j \in J_1} H_j(m_j^0 + 1) \cdot \sum_{j \in J_1} (m_j - m_j^0).$$

On the other hand

$$R = \sum_{j \in J_2} (\varphi_j(m_j^0) - \varphi_j(m_j)) = \sum_{j \in J_2} \left( H_j(m_j^0) + H_j(m_j^0 - 1) + \cdots + H_j(m_j + 1) \right) \leq \sum_{j \in J_2} (m_j^0 - m_j) \cdot H_j(m_j^0) \leq \max_{j \in J_2} H_j(m_j^0) \cdot \sum_{j \in J_2} (m_j^0 - m_j) \leq \max_{j \in J_2} H_j(m_j^0) \cdot \sum_{j \in J_1} (m_j - m_j^0).$$
Then finally
\[ L - R \geq \min_{j \in J_1} H_j(m_j^0 + 1) \cdot \sum_{j \in J_1} (m_j - m_j^0) - \max_{j \in J_2} H_j(m_j^0) \cdot \sum_{j \in J_1} (m_j - m_j^0) \]
\[ \geq \left( \min_{j \in J_1} H_j(m_j^0 + 1) - \max_{j \in J_2} H_j(m_j^0) \right) \cdot \sum_{j \in J_1} (m_j - m_j^0) \]
\[ \geq \left( \min_{j \in J_1} H_j(m_j^0 + 1) - \max_{j \in J_2} H_j(m_j^0) \right). \]

By or assumption, \( H_j(m_j^0 + 1) \geq H_k(m_k^0) \) for all \( j, k \) and so \( H_j(m_j^0 + 1) - H_k(m_k^0) \geq 0 \) for \( j \in J_1 \) and \( k \in J_2 \). This shows that \( L - R \geq 0 \) and therefore \( \psi(m) \geq \psi(m^0) \) for any \( m \in M \). Therefore \( m^0 \) is a minimal solution.

3. If the condition \( H_j(m_j^0 + 1) > H_k(m_k^0) \) is also satisfied for all \( j \) and \( k \) with \( j \neq k \) then \( L - R > 0 \), that is, the minimal solution is unique.

4. Let \( m \in S \) with \( m = (m_1, \ldots, m_n) \). Suppose that there are \( j, k \) with \( H_j(m_j + 1) < H_k(m_k) \). Then the multiset \( \{ H_i(\ell) \mid 1 \leq \ell \leq m_i, 1 \leq i \leq n \} \) does not contain \( M \) smallest elements as we can replace \( H_k(m_k) \) by the smaller \( H_j(m_j + 1) \). Thus for all \( j, k \) we have \( H_j(m_j + 1) \geq H_k(m_k) \) and by (2) \( m \) is a minimal solution.

\[ \square \]

Sainte-Laguë [7, Satz 6.1.8] knew that the Sainte-Laguë method minimised the error function with \( \varphi_j(x) = \frac{1}{q_j}(x - q_j)^2 \). Balinski and Ramirez [1] show that the linear divisor methods minimize the error function with \( \varphi_j(x) = \frac{1}{q_j}(x - q_j + d_0 - \frac{1}{2})^2 \).

**Corollary 6.3.** The linear divisor method given by \( d_0 \) can be obtained from the algorithm MINIMALSOLUTION for the decomposable error function \( \psi \), with \( j \)-th component \( \varphi_j(x) = \frac{1}{q_j}(x - q_j + d_0 - \frac{1}{2})^2 \).

**Proof.** Note first that \( \varphi_j(x) \) is a convex function. The corresponding decomposable error function \( \psi \) always has a minimal solution. Algorithm MINIMALSOLUTION leads to the linear divisor method described by \( d_0 \), because we have that \( H_j(\ell) = \varphi_j(\ell) - \varphi_j(\ell - 1) = \frac{2}{q_j}(\ell - q_j + d_0 - 1) - 2/2(\ell + d_0 - 1) \) is strictly increasing. Consider the function \( K_j(\ell) = \frac{q_j}{d_0 + \ell - 1} \). This function
yields the standard algorithm for all linear divisor methods. Then \( K_j(\ell) \) is strictly decreasing and therefore has a minimal solution which can be found by the analog of algorithm \textsc{MinimalSolution} by selecting the \( M \) largest elements in a matrix whose entries are \( K_j(\ell) \). Note that \( K_j(\ell) = \frac{2}{H_j(\ell) + 2} \) and therefore we can also find a minimal solution by applying algorithm \textsc{MinimalSolution} directly to the matrix \( H_j(\ell) \).

\[ \square \]

**Corollary 6.4.** Let \( \varphi \) be a symmetric and strictly convex function with \( \varphi(0) = 0 \) and \( \varphi(x) > 0 \) for all \( x > 0 \). Let

\[ \psi(m) = \sum_{j=0}^{n} \varphi(|m_j - q_j^\rho|) = \sum_{j=0}^{n} \varphi(|m_j - \lfloor q_j^\rho \rfloor - r_j^\rho|). \]

Then the algorithm \textsc{MinimalSolution} with the decomposable error function \( \psi \) yields the \( \rho \)-rounding method.

**Proof.** Recall that for the \( \rho \)-rounding method we let \( q^\rho = (q_1^\rho, \ldots, q_n^\rho) \) by defining \( q_j^\rho = \frac{q_j}{H_j(M + 2\rho - 1)} \). Define \( r_j^\rho \) by \( q_j^\rho = \lfloor q_j^\rho \rfloor + r_j^\rho \). We shall see that the function \( \psi(m) \) yields the \( \rho \)-rounding method. For \( j \in \{1, \ldots, n\} \) define as in Theorem 6.2 \( H_j(x) = \varphi(|x - q_j^\rho|) - \varphi(|x - q_j^\rho - 1|) \). As \( \varphi(x) \) is strictly convex, it follows that \( H_j(x) \) is strictly increasing for all \( x \in \mathbb{R} \). In particular,

\[ H_j(\lfloor q_j^\rho \rfloor) \leq H_j(q_j^\rho) = -\varphi(1) < 0. \]

Also,

\[ -\varphi(1) = H_j(q_j^\rho) < H_j(\lfloor q_j^\rho \rfloor + 1) \leq H_j(\ell) \]

for all \( \ell \geq \lfloor q_j^\rho \rfloor + 1 \). Hence the union of the sets \( L_j = \{ \ell \mid H_j(\ell) \leq -\varphi(1) \} \) contain \( \sum_{j=1}^{n} \lfloor q_j^\rho \rfloor \) elements. By Equation (2) these are at most \( M \) elements for \( 0 < \rho < 1 \) and at most \( M - 1 \) elements for \( \rho = 0 \) and at most \( M + 1 \) elements for \( \rho = 1 \). Therefore algorithm \textsc{MinimalSolution} allocates to party \( P_j \) a number \( m_j^0 \geq |L_j| \) seats. If \( \rho \leq 1 \), each party obtains at least \( \lfloor q_j^\rho \rfloor \) seats and the remaining seats are allocated according to the smallest elements among \( H_j(\lfloor q_j^\rho \rfloor + 1) \) for \( 1 \leq j \leq n \). Since \( H_j(\lfloor q_j^\rho \rfloor + 1) = \varphi(1 - r_j^\rho) - \varphi(r_j^\rho) \) and \( \varphi \) is strictly convex we have \( \varphi(1 - r_j^\rho) - \varphi(r_j^\rho) \leq \varphi(1 - r_k^\rho) - \varphi(r_k^\rho) \) if and only if \( r_j^\rho \geq r_k^\rho \). Thus the remaining seats are allocated according to the greatest remainders. If \( \rho = 0 \) and all \( r_j^\rho = 0 \) the additional seat is allocated at random which corresponds to choosing a random minimal solution. If
\[ \rho = 1 \text{ and all } r_j^\rho = 0 \text{ we have assigned } M + 1 \text{ seats. One seat is taken from a random party which corresponds to choosing a random minimal solution. This yields precisely the } \rho\text{-rounding method.} \]

Note that this was proved by Pólya (1919), [10], for \( \rho = 1/2. \)

**Corollary 6.5.**

\[ \psi(m) = \sum_{j=0}^{n} |m_j - q_j^\rho| = \sum_{j=0}^{n} |m_j - \lfloor q_j^\rho \rfloor - r_j^\rho|. \]

The \( \rho\)-rounding method can be obtained from the algorithm \textsc{MinimalSolution} for the decomposable error function \( \psi. \)

**Proof.** Choose \( \varphi_j(x) = |x - q_j|. \) Note that this function is convex, though not strictly convex. Now

\[ H_j(x) = |x - q_j^\rho| - |x - q_j^\rho - 1| = \begin{cases} -1 & x < q_j^\rho \\ 2x - 2q_j^\rho - 1 & q_j^\rho \leq x \leq q_j^\rho + 1 \\ 1 & x > q_j^\rho + 1 \end{cases}. \]

Observe that \( H_j(x) \) is strictly increasing for \( q_j^\rho \leq x \leq q_j^\rho + 1. \) Thus the proof of Corollary 6.4 immediately generalises to this situation.

**Corollary 6.6.** Let \( \varphi(x) = (x - q_j^\rho)^p \) for all \( p \geq 1 \) and let \( \psi(m) = \sum_{j=1}^{n} |q_j^\rho - m_j|^p. \) Then the algorithm \textsc{MinimalSolution} with the decomposable error function \( \psi \) yields the \( \rho\)-rounding method.

Obviously one can also take the \( p \)-th root of \( \psi(m) \) as the function \( \psi \) and this shows that all \( \ell_p \)-norms can be used as error functions. In particular for \( p = 2 \) the \( \ell_2 \)-norm is the usual Euclidean distance. Thus the \( \rho\)-rounding method also minimises the Euclidean distance between the points \( q^\rho \) and \( m_0, \) and thus yields the closest integer valued lattice point for each vector \( q \) of exact quotas. Note that this was known to Pólya (1918), [9], for general convex functions \( \varphi \) and to Birkhoff (1976), [3], for the \( \ell_p \)-norms. Finally we emphasize that

\[ \lim_{p \to \infty} \psi_p(m) = \psi_\infty(m) = \text{Max}_{j=1,\ldots,n}(|m_j - q_j^\rho|) = ||m - q^\rho||_\infty \]

yields that the \( \rho\)-rounding method also minimises the maximum norm.
7 The paradox paradoxa

We finish this paper by discussing certain paradoxa. As all linear divisor methods except Sainte-Laguë’s and all $\rho$-rounding methods except Hare’s are biased, we restrict our attention to these two methods. First we show, by giving some examples, that the Sainte-Laguë method comes with its own set of paradoxa. It is very susceptible to minor variations in quotas (INSTABILITY PARADOX). We demonstrate that it violates the Majority Condition in a major way and finally show that the seat distributions for one party can vary immensely when the votes of other parties change.

We then consider the Hare method. Here we address the highly publicised NEW STATE, ALABAMA and INCREASED VOTES paradoxa and argue why we believe that this is no paradoxical behaviour. Finally, we show how a slight modification of the Hare method fulfils the majority condition.

7.1 The method of Sainte-Laguë

7.1.1 INSTABILITY PARADOX

The method of Sainte-Laguë displays the INSTABILITY PARADOX, that is small variations in the exact quotas can lead to large variations in the seat allocations. In the following two examples we see in the case where one party has the absolute majority of the votes, variations in the votes of the small parties can lead to significantly different seat allocations for the major party, without any changes in the votes of the major party. Similar examples are also discussed by Huntington (1928), [6, p. 95].

For example, suppose 5 parties, 4 of which are very small, are competing in an election for $M = 68$ seats. If the exact quotas given by the election are:

$$q_1 = 65.91, q_2 = 0.53, q_3 = 0.521, q_4 = 0.52, q_5 = 0.519,$$

then the seat allocation according to Sainte-Laguë is:

$$m_1 = 64, m_2 = m_3 = m_4 = m_5 = 1$$

whereas for a slightly different election result for the 5 parties, namely

$$q_1 = 66.075, q_2 = 0.485, q_3 = 0.481, q_4 = 0.48 \text{ and } q_5 = 0.479$$

the seat allocation according to Sainte-Laguë is now:

$$m_1 = 68, m_2 = m_3 = m_4 = m_5 = 0.$$
A change of the exact proportions by 0.165 seats enforces upon the large party a change of 4 seats. With Hare’s method in both cases the seat allocation would be $m_1 = 66$, $m_2 = m_3 = 1$, $m_4 = m_5 = 0$.

Of course these are extreme examples, but even under such conditions an election method has to be able to prove itself.

### 7.1.2 The Majority Paradox

The Sainte-Laguë method also displays the **Majority Paradox**. It can happen that a party obtains more than 50% of the votes but receives several seats less than 50% of the seats.

For example, at a community election of a town council with $M = 51$ seats the following exact quotas might occur:

\[
q_1 = 26; \quad q_2 = 7.96; \quad q_3 = 5.84; \quad q_4 = 4.78; \\
q_5 = 3.72; \quad q_6 = 1.60; \quad q_7 = 0.56; \quad q_8 = 0.54.
\]

Then the seat allocation according to Sainte-Laguë is:

\[
m_1 = 24; \quad m_2 = 8; \quad m_3 = 6; \quad m_4 = 5; \quad m_5 = 4; \quad m_6 = 2; \quad m_7 = m_8 = 1.
\]

Even though Party 1 won the absolute majority of the votes and was even allocated 26 seats by the exact quota, it looses 2 seats through the allocation method and therefore looses the absolute majority in the council.

### 7.1.3 Vote Stability Paradox

This example also displays the **Vote Stability Paradox**. A slightly different election result could have been:

\[
q_1 = 26; \quad q_2 = 8.03; \quad q_3 = 7.09; \quad q_4 = 6.12; \\
q_5 = 1.415; \quad q_6 = 1.405; \quad q_7 = 0.472; \quad q_8 = 0.468.
\]

Then the seat allocation according to Sainte-Laguë is:

\[
m_1 = 28; \quad m_2 = 8; \quad m_3 = 7; \quad m_4 = 6; \quad m_5 = m_6 = 1; \quad m_7 = m_8 = 0.
\]

Note also a variation of 4 seats for the largest party between these two elections, even though in both cases the number of votes for the largest party was the same. The smaller parties dictated the outcome. Apart from these examples, especially if a 5% hurdle is installed, the results of Sainte-Laguë
and Hare’s procedure are almost always the same. For many Federal elections Sainte-Lagüè and Hare differed in the seat allocation for the Lower House (Bundestag) only in two cases. The effect of the 5% hurdle is much larger than the change of apportionment method. However, if the 5% hurdle is abolished, the effect of the apportionment methods becomes much larger and is sometimes surprising, as the examples above point out.

Note that this cannot happen in any $\rho$-rounding method as the seat variations for a given party for constant $M$, $A$ and number of votes is at most one seat.

7.2 The Hare method

7.2.1 New State Paradox

Many people think there is a paradox in the method of the greatest remainder. Especially, the New State Paradox (or “Parteizuwachsparadox“): if a new party enters the apportionment method without changing any of the original votes of the other parties, then it can happen that the new party gets a certain number of seats but in addition, there is a redistribution of the other parties too. For instance, Pukelsheim (1989), Table 6, gives the following example. Consider parties $A, B, C$ and $D$ which each have attained 320, 238, 79, and 17 votes, respectively. If we distribute 37 seats among parties $A, B,$ and $C$, then according to the method of Hare, they receive 18, 14 and 5 seats, respectively. However, if we distribute 37+1 seats among parties $A, B, C,$ and $D$ then they receive 19, 14, 4 and 1 seats, respectively. This seems to be a paradox as nothing has changed in the votes for the parties $A, B,$ and $C$. Nevertheless, the party $A$ took one seat away from party $C$.

To understand this behaviour, we have to calculate the exact quotas $q_A, q_B, q_C$ for parties $A, B$ and $C$, which are

$$q_A = 320 \cdot 37/637 = 18.587127,$$
$$q_B = 238 \cdot 37/637 = 13.824175,$$
$$q_C = 79 \cdot 37/637 = 4.588697.$$

If we add the votes for party $D$ then we obtain at total of 654 votes and
the exact quotas \( q'_A, q'_B, q'_C, q'_D \) for parties \( A, B, C \) and \( D \) are

\[
q'_A = 320 \cdot 38/654 = 18.593272,
q'_B = 238 \cdot 38/654 = 13.828746,
q'_C = 79 \cdot 38/654 = 4.590214,
q'_D = 17 \cdot 38/654 = .987767.
\]

In our opinion this behaviour does not deserve the label *paradox* as it is easily explained. The new party \( D \) changes the exact quotas of all other parties and hence to obtain a fair apportionment of the seats as close to the exact quotas as possible, it is only fair that the seat allocation changes. The literature which label this behaviour a *paradox* avoids the exact quotas like the bubonic plague.

### 7.2.2 Alabama Paradox

The Alabama Paradox (Mandatszuwachsparadox) appeared first in the United States. If one increases the number of seats from \( M \) to \( M' \) then it may happen that a party receives more seats when \( M \) seats are distributed than when \( M' \) are distributed. When computing all seat allocations for the American House of Representatives using the \( \rho \)-rounding method for \( \rho = 1/2 \) and the election results from 1880 and varying the possible seat numbers between 275 and 350, the chief Clark of the Census office noticed that for \( M = 299 \) and \( M' = 300 \) the number of representatives for Alabama decreased from 8 to 7, see Balinski and Young (1982), [2, Table 5.1]. This cannot happen in linear divisor methods.

Since the Hare method respects quotas it may happen that the exact quotas change by increasing the number of seats. As a result the number of seats a party receives might fall back to its lower quota. Perhaps a good way to explain the situation is to think of the lower quotas as the guaranteed number of seats for each party and an additional seat as a bonus. The bonuses are then distributed according to greatest claim. They are not guaranteed. It can thus happen that with different number of seats the bonus seats are reallocated.

Consider for example three parties who received the following votes: \( a_1 = 107890192; a_2 = 197827864; \) and \( a_3 = 18986361 \). Thus the total number of votes is \( A = 324704417 \). The following table lists the exact quotas and the
seat allocations using Hare’s and Sainte-Lagué’s method for $M = 94$ seats and $M = 95$ seats.

| Exact Quotas | $q_1 = 31.2336$ | $q_2 = 57.2700$ | $q_3 = 5.4964$ |
|--------------|-----------------|-----------------|-----------------|
| $M = 94$ Hare| $m_1 = 31$      | $m_2 = 57$      | $m_3 = 6$       |
|        Sainte-Lagué | $m_1 = 31$      | $m_2 = 57$      | $m_3 = 6$       |
| $M = 95$ Hare| $m_1 = 32$      | $m_2 = 58$      | $m_3 = 5$       |
|        Sainte-Lagué | $m_1 = 31$      | $m_2 = 58$      | $m_3 = 6$       |

While at a superficial glance it seems unfair that the third party should have a seat less when more seats in total are allocated, a look at the exact quotas explains the situation. By the addition of one seat all exact quotas increase by the same percentage. The $\rho$-rounding method is closer to the ideal of being as close to exact proportion as possible.

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