Polaron in the dilute critical Bose condensate

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Abstract

The properties of an impurity immersed in a dilute $D$-dimensional Bose gas at temperatures close to its second-order phase transition point are considered. Particularly by means of the $1/N$-expansion, we calculate the leading-order polaron energy and the damping rate in the limit of vanishing boson–boson interaction. It is shown that the perturbative effective mass and the quasiparticle residue diverge logarithmically in the long-length limit, signalling the non-analytic behavior of the impurity spectrum and pole-free structure of the polaron Green’s function in the infrared region, respectively.

Keywords: Bose polaron, critical behavior, $1/N$-expansion

(Some figures may appear in colour only in the online journal)

1. Introduction

Recently, the Bose polaron problem has attracted much attention from researchers—both experimentalists [1–4] and theorists [5–16]. Although the most characteristic properties of a single impurity immersed in the superfluid were studied in the context of a dilute $^3$He–$^4$He solution [17] in the late forties, the problem is still of current interest. Moreover, the experimental possibility to fine-tune the value and even the sign of the boson–impurity coupling parameter has stimulated further research and made it possible to predict numerous phenomena including localization–delocalization transition [18, 19], non-typical dynamics [20, 21] and thermalisation [22] of a moving Bose polaron, emergence of Efimov physics [23–27], etc. Low-dimensional [28–33] and mixed-dimensional [34] systems also possess very non-trivial behavior. However, less is known about the finite-temperature phase diagram of the Bose polarons, especially in a narrow region around the Bose–Einstein condensation (BEC) transition point, where only the equal-mass limit [35, 36] has been studied. In particular, in [36], the perturbative second-order energy correction and the damping were found to increase enormously near the BEC temperature. On the other hand, from the theory of critical Bose
systems it is known [37] that reliable results for one-particle spectra can be obtained only by taking an infinite series of diagrams into account. In the present article, by means of the \(1/N\)-expansion—which effectively incorporates bosonic ‘particle-hole’ bubbles—we address the problem of an impurity weakly coupled to a dilute \(D\)-dimensional \((D > 2)\) bath that undergoes the BEC transition.

2. Model and method

The possibility of studying the properties of a single impurity atom immersed in the Bose condensate as a Fermi–Bose mixture with vanishing density of fermions opens up new opportunities for the application of field-theoretical methods to this problem. For instance, adopting a path-integral formulation, one readily writes down the Euclidean action of our model

\[
A = A_0 + A_B + A_{\text{int}}.
\]

where the first term describes the ideal Fermi gas with dispersion: \(\varepsilon_f(p) = \hbar^2 p^2/2m_f\)

\[
A_0 = \sum_p \{v_p - \varepsilon_f(p) + \mu_f\} \psi^*_p \psi_p,
\]

and the second,

\[
A_B = \sum_K \{i\omega_k - \varepsilon_k + \mu\} \phi^*_{\sigma K} \phi_{\sigma K} - \frac{gT}{2NV} \sum_{K,Q,S} \phi^*_{\sigma Q} \phi^*_{\sigma' S} \phi_{\sigma' S-K} \phi_{\sigma Q+K},
\]

is the action of the \(N\)-component Bose system \((\varepsilon_k = \hbar^2 k^2/2m, V, T\) are a volume and a temperature of the system respectively and summations over indices \(\sigma, \sigma' = 1, 2, \ldots, N\) are understood). The third term \(A_{\text{int}}\) of action (1) takes into account interaction of the impurity atom with the bosonic bath:

\[
A_{\text{int}} = -\frac{\tilde{g}T}{NV} \sum_{K,Q,P} \phi^*_{\sigma Q} \phi_{\sigma Q+K} \psi^*_p \psi_{-K}.
\]

Let us stress once again that the introduction of \(N\)-component model of the environment is a formal tool which helps to classify diagrams in further calculations and only the \(N = 1\) choice leads to the physically relevant case, while results are applicable for \(N \gg 1\). The chemical potentials \(\mu\) and \(\mu_f\) presented here denote the number of Bose and Fermi particles, respectively. Other notations are also typical: the arguments of fermionic Grassmann \(\psi^*_p, \psi_p\) and bosonic complex \(\phi^*_{\sigma K}, \phi_{\sigma K}\) fields are the \(D+1\) ‘wave-vectors’ \(P = (\nu_p, p)\) and \(K = (\omega_k, k)\) with \(\nu_p\) and \(\omega_k\) standing for odd (fermionic) and even (bosonic) Matsubara frequencies. The coupling constants controlling the interaction strength can be related to the \(s\)-wave scattering lengths in \(D\) dimensions:

\[
g = 4\pi^{D/2}a^D/\Gamma(D/2 - 1)m',
\]

\[
\tilde{g} = \frac{2\pi^{D/2}d^{D-2}(m + m_f)}{\Gamma(D/2 - 1)mm_f}.
\]

Introducing the auxiliary field \(\varphi_K\), which decouples by means of the Hubbard–Stratonovich transformation quartic term in equation (3), and making use of change of variables in the path integral \(\varphi_K = \varphi'_{K} + \frac{\tilde{g}T}{N\sqrt{V}} \sum_P \psi^*_p \psi_{P+K}\), we finally rewrite action (1) as follows:
\[ A = \sum_{K} \{ i\omega_k - \varepsilon_k + \tilde{\mu} \} \phi_{\sigma K}^* \phi_{\sigma K} + \sum_{P} \{ i\nu_p - \varepsilon_f(p) + \tilde{\mu}_f \} \psi_p^* \psi_p \]

\[ - \frac{N}{2g} \sum_{K} \tilde{\varphi}_{K} c_{-K} - i \sqrt{\frac{T}{V}} \sum_{K,Q} \varphi_{K} \phi_{\sigma Q}^* \phi_{\sigma Q - K} - i \frac{g}{\sqrt{V}} \sum_{K,P} \varphi_{K} \psi_{P}^* \psi_{P + K}, \]

(6)

where prime near field \( \varphi_K \) is dropped and \( \varphi_{|k=0} = 0 \) and \( \tilde{\mu} = \mu - ng, \tilde{\mu}_f = \mu_f - n\tilde{g} \) (\( n \) is the density of each boson component) denote the Hartree-shifted chemical potentials. After all transformations, equation (6) constitutes the action of a mixture of ideal Bose and Fermi gases coupled to the ‘phonon’ field \( \varphi_K \). In such a formulation, the properties of the impurity atom are fixed by this collective mode, which in turn is fully controlled by bosonic degrees of freedom in the thermodynamic limit. The second feature of our path-integral approach is more technical, because it allows us to simplify the further calculations of the Bose polaron properties in terms of inverse powers of \( N \).

The impurity single-particle Green’s function \( G_f(P) = \langle \psi_p^* \psi_p \rangle \), where \( \langle \ldots \rangle \) denotes the statistical averaging with action (6), is uniquely determined by the self-energy part \( \Sigma_f(P) \):

\[ G_f^{-1}(P) = i\nu_p - \xi_f(p) - \Sigma_f(P), \]

(7)

here, the notation \( \xi_f(p) = \varepsilon_f(p) - \tilde{\mu}_f \) is used. In the large-\( N \) limit, singling out the Fock term (which is equal to zero in the one-fermion limit) one arrives at the self energy (see figures 1 and 2)

\[ \Sigma_f(P) = \frac{T}{NV} \sum_{K} \frac{g^2 S(K)}{i(\omega_k + \nu_p) - \xi_f([k + p])}, \]

(8)

with \( S(K) = \Pi(K)/[1 + g\Pi(K)] \) being the density–density correlation function for each sort of boson. Close to the phase-transition point, the bosonic polarization bubble reads

\[ \Pi(K) = \frac{T}{V} \sum_{Q} \frac{1}{i\omega_q - \varepsilon_q} \frac{1}{i(\omega_q + \omega_k) - \varepsilon_{[q+k]}}. \]

(9)

The evaluation of the leading-order self-energy correction (8) is rather simple, but nevertheless requires precision during the calculation. In particular, the summation over the Matsubara frequencies in this equation can be performed by using spectral theorem for the density–density correlation function

\[ S(K) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{S(\omega, K)}{\omega - i\omega_k}, \]

(10)
\[
S_I(\omega, k) = \text{Im} S(K)|_{\omega \to \omega + i0},
\]
and gives for the self energy in the one-fermion limit \((\mu_f \to -\infty)\)
\[
\Sigma_f(P) = \tilde{g}^2 NV \sum_k \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{S_I(\omega, k)n(\omega/T)}{\omega + i\nu_p - \xi_f(|k + p|)}, \tag{11}
\]
where \(n(x) = 1/(e^x - 1)\) is the Bose distribution. It should be noted that for a Bose gas this \(1/N\)-result is equivalent to the well-known Random Phase Approximation (see, for instance [38]), which in turn leads to the Bogoliubov theory at low \(T\). Despite its simplicity, this approximation incorporates many important features of the finite-temperature properties of Bose systems, such as the presence of second-order phase transition, universal power-law behavior of thermodynamic and structure functions, etc. Performing an analytical continuation of \(\Sigma_f(P)|_{\omega \to \nu + i0} = \Sigma_R^f(\nu, p) + i\Sigma_I^f(\nu, p)\) in the upper complex half-plane, one obtains the \(1/N\)-corrected polaron energy
\[
\varepsilon_f^\ast(p) = \varepsilon_f(p) + n\tilde{g} + \Sigma_R^f(\xi_f(p), p), \tag{12}
\]
and damping
\[
\Gamma_f(p) = -\Sigma_I^f(\xi_f(p), p). \tag{13}
\]
The structure of the Green’s function near the singularity is determined by the quasiparticle residue calculated with the same accuracy
\[
Z_f^{-1}(p) = 1 - \frac{\partial}{\partial \xi_f(p)} \Sigma_R^f(\xi_f(p), p). \tag{14}
\]

At zero temperature, the result (11) reproduces to the first-order beyond mean-field correction for the self energy [39] obtained within the Rayleigh–Schrödinger perturbation theory. The \(1/N\)-expansion alters only the finite temperature region non-trivially. For a dilute Bose gas at non-zero temperatures, the physics of the system is dictated by the infrared behavior of functions standing under the integral in equation (8), therefore one may replace \(n(x)\) by its asymptote \(1/x\). This observation allowed [40] to predict qualitatively correct behavior [41] of the BEC transition temperature for a weakly-interacting Bose gas. The integration over \(\omega\) in formula (11) then yields for the energy correction
\[
\Sigma_R^f(\xi_f(p), p) = \frac{\tilde{g}^2 T}{NV} \sum_k \frac{1}{\varepsilon_f(|k + p|) - \varepsilon_f(p)} \times [S_R(\varepsilon_f(|k + p|) - \varepsilon_f(p), k) - S_R(0, k)], \tag{15}
\]

\[\text{Figure 2.} \text{ Boson-mediated effective two-body impurity potential. Dashed line is the Green’s function of an ideal Bose gas; wavy line and white dot denote} \frac{\Delta}{\pi} \text{and vertex} i \text{respectively.}\]
where only the infrared asymptote of the real part $S_R(\omega, k) = \text{Re} S(K)|_{\omega, k \to \omega + i0}$ of the retarded density–density correlator should be taken into account. Thus, the problem is reduced to the calculation of the above integral and function $S_R(\omega, k)$ (see appendix A).

3. Results

A slightly surprising aspect of result (15) is that for a dilute Bose bath ($an^{1/D} \to 0$) the leading-order binding energy correction, as well as the damping of a motionless ($p = 0$) impurity, are calculated analytically. The same conclusions can be drawn for the effective mass and for the quasiparticle residue. However, the spatial dimensionality impacts the behavior of a Bose polaron crucially; therefore, in the following, we will consider the cases $2 < D < 3$ and $D = 3$ separately.

3.1. $2 < D < 3$

It is well known that for Bose systems in $D > 2$ the condensation phenomenon occurs at finite temperatures. This second-order transition point is characterized by the developed density fluctuations, which influence the impurity properties enormously. Even for our model, with a short-ranged two-body potential, the leading-order induced boson–fermion effective interaction (see figure 2) has a power law behavior at large interparticle spacing. Although this large-$N$ approach effectively sums up infinite series of diagrams, it still possesses the perturbative nature in terms of the boson–impurity coupling parameter $\tilde{g}$. As a consequence, we assume throughout the text that the ratio $\tilde{a}/a$ is of order unity while $g \to 0$. The second restriction is more technical, and is set on the mass of the impurity atom: our leading-order calculations require the bare polaron mass to be not too close to the mass of surrounding Bose particles.

Substituting formulae from the appendix A into equation (15), and calculating a simple integral for the correction to energy of a motionless polaron, we obtain

$$\Sigma^R_f(-\mu_f, 0) = -\frac{1}{n g} \left( \frac{\tilde{a}}{a} \right)^{D-2} \left( an^{1/D} \right)^{\frac{(D-2)^2}{D}} \epsilon_D(\gamma),$$

(16)

where $\gamma = m/m_f$. The applicability of the formulae obtained here and below is not restricted to the integer dimension Bose polarons, but can be easily used for impurities immersed in Bose condensates on fractals. In particular, in figure 3 the dimensionless function $\epsilon_D(\gamma)$ is depicted for a fractional spatial dimension $D = \ln(20)/\ln(3)$ of the Menger sponge. It should be noted that the limits of applicability of the above result are restricted to $an^{1/D} \ll |1 - \gamma|^{1/D}$. This limitation is related to the simplifications performed during the calculation of the integrals. Here, we mean the replacement $n(x) \to 1/x$ and account of the leading-order infrared asymptote of the real and imaginary parts of the polarization operator. Actually, close to $\gamma = 1$ we have to go beyond the approximation for functions $I(\omega, k)$ and $R(\omega, k)$ presented in the appendix A, which will necessarily change the exponent in the power-law dependence on the gas parameter $an^{1/D}$ in equation (16). In exactly the same manner, we have calculated the leading-order damping (see figure 3):

$$\Gamma_f(0) = -\frac{1}{n g} \left( \frac{\tilde{a}}{a} \right)^{D-2} \left( an^{1/D} \right)^{\frac{(D-2)^2}{D}} \Delta_D(\gamma).$$

(17)

We recall again that our calculations are incorrect in the vicinity of the equal-mass limit and the exact damping to the order $1/N$ is equal to zero when $\gamma = 1$. A totally different situation
is met in the finite-momentum dependence of the self energy. In particular, trying to calculate the correction to the impurity effective mass by expanding \( \Sigma_R(\xi_f(p), p) \) to quadratic order in \( p \), we find out that the appropriate integral is logarithmically divergent in any dimension. This problem also appears during the calculation [42] of effective mass of Bose particles at the critical point where these divergences are treated as a signature of the power-law behavior \( p^{2-\eta} \) of the one-particle spectrum. The critical exponent \( \eta \) in that case is universal, i.e. dependent on the global parameters only. If we assume qualitatively the same behavior for the polaron renormalised dispersion relation, we immediately obtain the result—valid for any \( g \) in the adopted approximation:

\[
\lim_{p \to 0} [\epsilon_f^*(p) - \epsilon_f^*(0)] \propto p^{2-\eta_e},
\]

with the non-universal exponent, which is given by

\[
\eta_e = \frac{1}{N} \left( \frac{\bar{a}}{a} \right)^{2(D-2)} \eta_e(\gamma),
\]

on the 1/N-level and depicted in figure 4 for various mass ratios. It is no surprise that the quasiparticle residue (14) demonstrates non-analytic dependence on the wave-vector in the long-length limit too. This suggests the following behavior of the retarded impurity Green’s function at the BEC point:

\[
\lim_{p \to 0} G_R^{\nu}(\nu, p)|_{\nu \to \epsilon_f^*(p)} \propto \frac{1}{[\nu - \epsilon_f^*(p)]^{1+\eta_k}},
\]

where exponent \( \eta_k \) differs from \( \eta_e \):

\[
\eta_k = \frac{1}{N} \left( \frac{\bar{a}}{a} \right)^{2(D-2)} \eta_k(\gamma).
\]
The curves obtained (see figures 1 and 2) clearly indicate the difference in the behavior of heavy ($\gamma < 1$) and light ($\gamma > 1$) impurities when $2 < D < 3$.

### 3.2. $D = 3$

This difference is even more drastic in the three-dimensional case where the dependence on gas parameter $an^{1/3}$ of the leading-order correction to the energy of a motionless polaron is different for $\gamma < 1$ and for $\gamma > 1$. Particularly for light impurities, we have found (with logarithmic accuracy)

$$
\Sigma(-\tilde{\mu}_f, 0) = \frac{1}{N} an^{1/3} \ln[an^{1/3}] \varepsilon_3(\gamma),
$$

(22)

($\gamma > 1$). The second correction to the energy of a heavy atom is given by equation (16) with function $\varepsilon_3(\gamma)$ (see appendix B)

$$
\varepsilon_3(\gamma) = \frac{4\pi(1 + 1/\gamma)}{[\zeta(3/2)]^{4/3}} \left\{ \delta_3(\gamma)[\pi/2 - \arctan(1/\delta_3(\gamma))] - \ln \sqrt{1 + \delta_3^2(\gamma)} \right\},
$$

where $\delta_3(\gamma) = \delta_3(\gamma)/r_3(\gamma) = \frac{1}{\pi} \ln \left| \frac{1 + \gamma}{1 - \gamma} \right|$. Instead, the leading-order damping does not change its behavior thoroughly with the deviation of the mass ratio from $\gamma = 1$, and equals

$$
\frac{\Gamma_f(0)}{n^2} = \frac{4\pi(1 + 1/\gamma)\delta_3(\gamma)}{[\zeta(3/2)]^{4/3}} \tilde{a} n^{1/3} \ln[an^{1/3}],
$$

(23)

In the three-dimensional case, the formulae obtained due to the simplifications performed in the computation of integrals are valid only if $an^{1/3} \ll 1/\ln(1 - \gamma)$. Similarly to the general consideration from the previous section, we have determined the critical exponents controlling the behavior of the impurity Green’s function. The peculiarity of the spectrum in the long-wavelength limit is contained in (see figure 5).
The leading-order $1/N$-expansion is known to be insensitive to the many-body statistical effects. For instance, parameters of the impurity spectrum at low temperatures in the equal-mass limit and with $\tilde{a}$ replaced by $a$ reproduce the one-particle spectrum of medium bosons (compare the first-order correction to effective mass in [12] and the low-temperature (large-$g$) result for the one-particle spectrum in [40]). Interestingly, the same correspondence is observed in the critical temperatures. Indeed, identifying impurity with the environmental particle ($\gamma \to 1 + 0$ and $\tilde{a} = a$) and calculating the zero-frequency Green’s function at small $p$, taking into account equations (18), (20) and imposing $\tilde{\mu} = 0$ (recall that the inverse one-particle Green’s function of bosons satisfies identity $G^{-1}(0) = 0$ at the BEC temperature) we immediately find $\lim_{p\to 0} G^{-1}_f(P)|_{p=0} \propto (p^{2-\eta_1})^{1+\eta} = p^{2-\eta_1}$, where in the adopted approximation $\eta = \eta_c - 2\eta_t = \frac{16}{3\pi^2 N} - 2\frac{2}{\pi^2 N} = \frac{4}{3\pi^2 N}$, which is in agreement with the large-$N$ evaluations of the Fisher exponent [43–45] for $D = 3$.

4. Conclusions

Summarizing, we have considered the properties of a mobile impurity immersed in a $D$-dimensional superfluid across the BEC temperature. Using the large-$N$ expansion, we have calculated the leading-order (beyond trivial Hartree term) correction to the polaron spectrum in the limit of a very dilute bosonic bath. It was shown that the results for the energy and damping strongly depend on the mass ratio of the impurity atom and medium-forming particles.
Our findings also revealed the infrared structure of the impurity Green’s function, which was found to exhibit a branch point singularity in the critical region.

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### Appendix A. Polarization operator

The evaluation of the retarded density–density correlator requires the knowledge of analytically continued ‘particle-hole’ bubble $\Pi(K)|_{\omega \rightarrow \omega+i0} = R(\omega, k) + i\tilde{R}(\omega, k)$. After the Matsubara frequency summation in equation (9), the imaginary part $I(\omega, k)$ is easily obtained:

$$I(\omega, k) = \frac{\pi}{V} \sum_q n(\varepsilon_q/T)[\delta(\varepsilon_{q+k} - \varepsilon_q - \omega) - (\omega \rightarrow -\omega)].$$

Due to the presence of the $\delta$-function, the integration is relatively simple yielding, at the critical temperature,

$$I(T_0, k_0) = \frac{\sqrt{\pi}n}{2\zeta(D/2)Tk} \left[ G_{\alpha} \left( e^{-(k/2-u/2k)^2} \right) - (u \rightarrow -u) \right],$$

where $k_0 = \sqrt{2mT}/h = 2\sqrt{\pi}[n/\zeta(D/2)]^{1/D}$ denotes the characteristic inverse length scale, $\zeta(x)$ is the Riemann zeta function and $G_{\alpha}(z) = \sum_{l \geq 1} z/l^\alpha$. In the infrared region ($u, k \rightarrow 0$), the leading-order terms in square brackets read

$$\Gamma(3/2 - D/2) \left[ (2k)^{3-D}/|k^2 - u|^{3-D} - (u \rightarrow -u) \right],$$

($\Gamma(x)$ is the gamma function), and by applying the spectral representation $R(\omega, k) = P.V. \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \rho(\omega') I(\omega, k)$ we obtained the real part of polarization operator with the same accuracy

$$R(T_0, k_0) = \frac{\sqrt{\pi}n}{2\zeta(D/2)Tk} \frac{\Gamma(3/2 - D/2)}{\cot[(3/2 - D/2)\pi]} \times \left[ \frac{(2k)^{3-D}}{|k^2 - u|^{3-D}} \text{sign}(k^2 - u) + (u \rightarrow -u) \right].$$

### Appendix B. Parameters of the low-energy impurity Green’s function

In the same approximation as was used for the evaluation of equation (15), we calculated the imaginary part of self energy, which reduces to the following integral:

$$\Gamma_f(0) = \frac{1}{n \tilde{g}} \frac{\Omega_D \lambda_D^{(D-2)/2}}{\zeta(D/2)\gamma \g} \int_0^\infty \frac{dk d\gamma}{[k^{D-2} + rD(\gamma)]^{1/2} + \gamma^2(\gamma)},$$

while the real part is given by:
\[
\Sigma_R^\rho(-\mu,0)/n_{\bar{g}} = \frac{1}{N \pi^{D/2}} \frac{\Omega_D}{\pi^{D/2}} \frac{g}{g} \int_0^\infty dk k^{D-3} \times \left[ \frac{r_D(\gamma)[k^{4-D} + r_D(\gamma)] + \gamma^2}{[k^{4-D} + r_D(\gamma)]^2 + \gamma^2} - \frac{r_D(0)}{k^{4-D} + r_D(0)} \right],
\]

where, for convenience, parameter \( \lambda_{D} = \frac{\pi^{D/2}}{\zeta(D/2)M} \) is introduced; \( \Omega_D \) is the area of a \( D \)-dimensional sphere with unit radius, and shorthand notations are used for functions

\[
i_0(\gamma) = 2^{3-D} \Gamma(3/2-D/2) \left\{ \frac{1}{1-\gamma} - (\gamma \rightarrow -\gamma) \right\},
\]

and

\[
r_D(\gamma) = \frac{2^{3-D} \Gamma(3/2-D/2)}{\cot((3/2-D/2)\pi)} \left[ \frac{\text{sign}(1-\gamma)}{|1-\gamma|^{3-D}} + (\gamma \rightarrow -\gamma) \right],
\]

explicitly related to \( I(\omega,k) \) and \( R(\omega,k) \) respectively. Note that the above integrals for arbitrary spatial dimension \( D \) can be calculated exactly with the help of the residue theorem. In three dimensions, the integrals are divergent, providing the logarithmic dependence on the parameter \( \lambda_D \). Both the quasiparticle residue and effective mass by means of parameter differentiation can be expressed via a single integral, which is logarithmically divergent in the long-length limit at critical temperature. Actually, this divergence determines exponents \( \eta_e \)

\[
\eta_e = -\frac{1}{N} \frac{4\Omega_D}{\pi^{D/2}} \left( \frac{g}{g} \right)^2 \frac{\partial^2}{\partial \gamma^2} \left\{ \frac{r_D(\gamma)}{r_D^2(\gamma) + \gamma^2} - \frac{1}{r_D(0)} \right\},
\]

and \( \eta_{\pi} \)

\[
\eta_{\pi} = \frac{1}{N} \frac{\Omega_D}{\pi^{D/2}} \left( \frac{g}{g} \right)^2 \frac{\partial}{\partial \gamma} \left\{ \frac{r_D(\gamma)}{r_D^2(\gamma) + \gamma^2} - \frac{1}{r_D(0)} \right\}.
\]

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