CONSTRUCTIONS OF OPTIMAL LOW HIT ZONE FREQUENCY HOPPING SEQUENCE SETS WITH LARGE FAMILY SIZE

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Abstract. Frequency hopping sequences with low hit zone is significant for application in quasi synchronous multiple-access systems. In this paper, we obtained two constructions of optimal frequency hopping sequence sets with low hit zone based on interleaving techniques. The presented low hit zone frequency hopping sequence sets are with new and flexible parameters and large family size which can meet the needs of the practical applications. Moreover, all the sequences in the proposed sets are cyclically inequivalent. Some low hit zone frequency hopping sequence sets constructed in literatures are included in our family. The proposed frequency hopping sequence sets with low hit zone are contributed for quasi-synchronous frequency hopping multiple access system to reduce or eliminate multiple-access interference.

1. Introduction

Frequency hopping (FH) communication is a common spread spectrum communication method. Frequency hopping communication has the characteristics of anti-jamming, anti-fading and security. Therefore, frequency hopping communication is widely used in code division multiple access communication system, sonar system, multi-user radar and so on [2], [15]. A pseudo-random sequence which controls the hopping of carrier frequencies is called frequency hopping sequence (FHS). Frequency hopping sequence plays an important role in frequency hopping communication system. The interferences may appear in these systems, when the receiver wants to receive the demodulation signal from the transmitter. The interferences between users correspond to the number of hits between different frequency hopping sequences, which can be described as Hamming correlation. In order to reduce the multiple access interference, frequency hopping sequences with low Hamming correlation are often selected to improve the performance.

In 2003, for quasi synchronous multiple access system, frequency hopping sequence with low hit zone (LHZ) came out [18], [17]. The quasi synchronous multiple access system tolerate relative time delay between users within a zone around the origin. The design of FHSs with low hit zone or no hit zone (NHZ) is to make Hamming correlation keep at a very low level or equal to zero within the correlation zone. For any LHZ FHS sets, its maximum Hamming correlations are relatively small as long as the time delay does not exceed the certain zone. At the same time,
it is expected that the maximum Hamming correlation will be as small as possible outside the LHZ, especially that there will be no full collision. Therefore, the quasi synchronous frequency hopping multiple-access system can effectively reduce mutual interference by using frequency hopping sequence sets with low hit zone.

The design of theoretical bound is instructive to construct optimal FHS sets. Peng and Fan [13] derived a theoretical bound about the maximum periodic Hamming correlation of optimal FHS sets, called Peng-Fan bound in 2004. In 2006, Peng et al. [14] obtained a lower bound on the maximum Hamming correlation of LHZ FHS sets, called Peng-Fan-Lee bound. Liu et al. [7] obtained an upper bound on the family size of LHZ FHS sets.

In recent years, more and more attention has been paid to the constructions of LHZ FHS.

There are some extension methods for optimal LHZ FHS sets, which generate several classes of LHZ FHS sets with composite length. In 2010, Ma and Sun [8] constructed a class of optimal LHZ FHS sets firstly. In 2013, Chung et al. [1] proposed some constructions of optimal LHZ FHS sets based on Cartesian product. In the same way, Zhou et al. [21] obtained two new classes of optimal LHZ FHS sets via Cartesian product in 2017.

Based on m-sequence and its decimated sequence, Zhou et al. [20] presented two constructions of optimal LHZ FHS sets and Han et al. [5] obtained two classes of optimal LHZ FHS sets in 2016.

Some papers employed additive group or multiplicative group of finite fields for construction of LHZ FHS sets. In 2016, Wang et al. [16] constructed two classes of optimal LHZ FHSs with new parameters based on Cyclotomy, the discrete logarithm function and the Chinese remainder theorem. In 2019, Yin et al. [19] obtained two constructions of LHZ FHS based on m-sequence and d-form functions with difference-balanced property.

By applying interleaved structure with optimal FHS sets, some optimal LHZ FHS sets with more flexible parameters are obtained. In 2012, Niu et al. [9] presented new designs of optimal LHZ FHS sets, which are shift distinct. In 2013, Niu et al. [10] used short FHSs with good Hamming correlation to construct a set of long FHSs with LHZ. In 2014, Niu et al. [12] obtained new classes of optimal LHZ FHS sets with new parameters. In 2018, Ling et al. [6] proposed a new construction of optimal LHZ FHS set with large family size. Interleaved techniques can also ensure that the sequences in proposed FHS sets are equivalent cyclically in a general way.

In this paper, we present two constructions of optimal LHZ FHS sets based on interleaved techniques. It is verified that both of the LHZ FHS sets are optimal in the sense that the parameters of the proposed FHSs meet Peng-Fan-Lee bound. All the FHSs got by the two constructions are shift distinct. The parameters of LHZ FHS sets with optimal Hamming correlation from some known results and new sets are listed in Table 1.

The rest of this paper is organized as follows. In Section 2, some notations and definitions are given. In Section 3, the interleaved techniques are introduced. In Section 4, two constructions of LHZ FHS set with large family size are proposed. Finally, some conclusions and remarks in Section 5.

2. Preliminaries

Let \( F = \{f_0, f_1, \cdots, f_{q-1}\} \) be a frequency slot set with size \( |F| = q \), and \( S \) be an FHS set of \( l \) FHSs of length \( N \). For any two FHSs, \( x = (x_0, x_1, \cdots, x_{N-1}) \),
### Table 1: Some Optimal LHZ FHS Sets with Optimal Hamming Correlation

| Parameters | Constraints | According to the bound(3) | According to the bound(4) | Cyclic equivalence | Ref. |
|------------|-------------|---------------------------|---------------------------|--------------------|------|
| (TN, q, M; ω + 1, TH_ω) | M = ⌈(2|x)|gcd(l, N)⌉ = 1, l = 0, 1, ..., L | Optimal | Not optimal family size | Inequivalent | [9] |
| (TN, q, M; ω + 1, TH_ω) | M = N, T = 2, gcd(l, N) = 1. | Optimal | Not optimal family size | Inequivalent | [10] |
| (IN, q, M; ω + 1, IH_ω) | M = ⌈(2|x)|gcd(l, N)⌉ = 1. | Optimal | Not optimal family size | Inequivalent | [12] |
| (IN, q, M; ω + 1, IH_ω) | M = ⌈(2|x)|gcd(l, N)⌉ = 1. | Optimal | Not optimal family size | Inequivalent | [6] |
| (IN, q, M; ω + 1, IH_ω) | M = ⌈(2|x)|gcd(l, N)⌉ = 1. | Optimal | Not optimal family size | Inequivalent | Thm.1 |
| (IN, q, M; ω + 1, IH_ω) | M = ⌈(2|x)|gcd(l, N)⌉ = 1. | Optimal | Not optimal family size | Inequivalent | Thm.3 |
| (IN, q, M; ω + 1, IH_ω) | M = ⌈(2|x)|gcd(l, N)⌉ = 1. | Optimal | Not optimal family size | Inequivalent | Thm.3 |
| (p^2(q - 1), p, q, min(p^2 - 1, q - 2), p) | gcd(p, q - 1) = 1, 2p <= q - 1. | Optimal | Not optimal family size | Inequivalent | [1] |
| (p^2(q - 1), p^2, p^2 - 2, p(p - 1)) | gcd(p^2, p^2 - 1) = 1. | Optimal | Not optimal family size | Inequivalent | [20] |
| (p^q(q^m - 1), q^m, nM(ω = 1), q^k, n, p^q) | m >= 1, 0 < n <= k, M = ⌈(2|x)|gcd(l, N)⌉ = 1, x <= 1. | Optimal | Not optimal family size | Inequivalent | [6] |
| (p^q(q^m - 1), q^m, nM(ω = 1), q^k, n, p^q) | m >= 1, 0 < n <= k, M = ⌈(2|x)|gcd(l, N)⌉ = 1, x <= 1. | Optimal | Not optimal family size | Inequivalent | [20] |
| (p^q(q^m - 1), q^m, nM(ω = 1), q^k, n, p^q) | m >= 1, 0 < n <= k, M = ⌈(2|x)|gcd(l, N)⌉ = 1, x <= 1. | Optimal | Not optimal family size | Inequivalent | [20] |
| (p^q(q^m - 1), q^m, nM(ω = 1), q^k, n, p^q) | m >= 1, 0 < n <= k, M = ⌈(2|x)|gcd(l, N)⌉ = 1, x <= 1. | Optimal | Not optimal family size | Inequivalent | [20] |
| (p^q(q^m - 1), q^m, nM(ω = 1), q^k, n, p^q) | m >= 1, 0 < n <= k, M = ⌈(2|x)|gcd(l, N)⌉ = 1, x <= 1. | Optimal | Not optimal family size | Inequivalent | [20] |
| (p^q(q^m - 1), q^m, nM(ω = 1), q^k, n, p^q) | m >= 1, 0 < n <= k, M = ⌈(2|x)|gcd(l, N)⌉ = 1, x <= 1. | Optimal | Not optimal family size | Inequivalent | [20] |
| (p^q(q^m - 1), q^m, nM(ω = 1), q^k, n, p^q) | m >= 1, 0 < n <= k, M = ⌈(2|x)|gcd(l, N)⌉ = 1, x <= 1. | Optimal | Not optimal family size | Inequivalent | [20] |
| (p^q(q^m - 1), q^m, nM(ω = 1), q^k, n, p^q) | m >= 1, 0 < n <= k, M = ⌈(2|x)|gcd(l, N)⌉ = 1, x <= 1. | Optimal | Not optimal family size | Inequivalent | [20] |

p is a prime, q is a prime power and lpf(y) denotes the least prime factor of an integer y > 1.

y = (y_0, y_1, ..., y_{N-1}) ∈ S, and any integer τ, 0 ≤ τ < N, the periodic Hamming correlation function H_{xy}(τ) of x and y at time delay τ is defined as follows:

\[(1) \quad H_{xy}(τ) = \sum_{t=0}^{N-1} h(x_t, y_{t+τ}), \tau = 0, 1, 2, ..., N - 1,\]

where h(a, b) = 1 if a = b, and h(a, b) = 0 otherwise. The subscript t+τ is computed modulo N, and only positive shifts are considered.

For any given FHS set S, the maximum periodic Hamming autocorrelation H_a(S) and the maximum periodic Hamming crosscorrelation H_c(S) are defined as follows, respectively:

\[H_a(S) = \max \{ H_{xx}(τ) | ∀x ∈ S, 0 < τ < N \}, \]
\[H_c(S) = \max \{ H_{xy}(τ) | ∀x, y ∈ S, x \neq y, 0 ≤ τ < N \}, \]
\[H_m(S) = \max \{ H_a(S), H_c(S) \} . \]
For convenience, let \( H_a = H_a(S) \), \( H_c = H_c(S) \) and \( H_m = H_m(S) \). \( H_m \) denotes the maximum periodic Hamming correlation.

In 2004, Peng and Fan [13] established a lower bound of FHS sets, which is given as follows:

**Lemma 1.** (Peng-Fan bound) Let \( S \) be a set of \( l \) FHSs of length \( N \) over a frequency slot set \( F \) with size \( q \), and \( H_m \) be the maximum Hamming correlation of \( S \).

\[
H_m \geq \left\lceil \frac{(lN - q)N}{(lN - 1)q} \right\rceil.
\]

The FHS set \( S \) is called optimal if the inequality (2) is met with equality.

For any given FHS set \( S \), let integers \( H_a \geq 0 \), \( H_c \geq 0 \), then, the low hit zone \( L_H \), the autocorrelation low hit zone \( L_{AH}(S) \) and the crosscorrelation low hit zone \( L_{CH}(S) \) of \( S \) are defined as follows, respectively:

\[
L_{AH}(S) = \max \{ T | H_{xx}(\tau) \leq H_a, \forall x \in S, 0 < \tau \leq T \},
\]
\[
L_{CH}(S) = \max \{ T | H_{xy}(\tau) \leq H_c, \forall x, y \in S, x \neq y, 0 \leq \tau \leq T \},
\]
\[
L_H = \min \{ L_{AH}(S), L_{CH}(S) \}.
\]

When \( H_a = H_c = 0 \), the low hit zone of \( S \) is called no hit zone. An FHS set \( S \) with \( L_H \geq 0 \) or \( N_H \geq 0 \) is called LHZ FHS set or NHZ FHS set.

In 2006, Peng, Fan and Lee [14] established a lower bound on the maximum periodic Hamming correlation of LHZ FHS sets, which is given as follows:

**Lemma 2.** (Peng-Fan-Lee bound) Let \( S \) be a set of \( l \) FHSs of length \( N \) over a frequency slot set \( F \) with size \( q \), and let \( L_H \) be the low hit zone of \( S \) with respect to constant \( H_m \). Then, for any positive integer \( Z \), \( 0 \leq Z \leq L_H \),

\[
H_m \geq \left\lceil \frac{(lZ + l - q)N}{(lZ + l - 1)q} \right\rceil.
\]

If the inequality (3) is achieved for \( Z = L_H \), the corresponding LHZ FHS set \( S \) is said to be optimal. The inequality (2) is a special case for \( Z + 1 = N \) in inequality (3).

In 2019, Liu [7] established an upper bound on the family size of LHZ FHS sets, which is given as follows:

**Lemma 3.** Let \( S \) be a set of \( l \) FHSs of length \( N \) over a frequency slot set \( F \) with size \( q \), and let \( L_H \) be the low hit zone of \( S \) with respect to constant \( H_m \).

\[
l \leq \left\lceil \frac{q^{H_m+1}}{L_H + 1} \right\rceil.
\]

The LHZ FHS set is said to have optimal family size if \( l \) is the maximum integer solution of the inequality (4).

Throughout this paper, we use \((N, q, l, H_m)\) to denote an FHS set of \( l \) FHSs of length \( N \) over a given frequency slot set \( F \) with size \( q \), and \( H_m \) is the maximum Hamming correlation. Let \((N, q, l, L_H, H_m)\) denote an LHZ FHS set of \( l \) FHSs of length \( N \) over a given frequency slot set \( F \) with size \( q \), and \( H_m \) is the maximum periodic Hamming correlation with low hit zone \( L_H \).
3. INTERLEAVED STRUCTURE

The interleaved structure of sequences was introduced by Gong ([3], [4]). Let
\[ A = \{ a^i = (a_0^i, a_1^i, \ldots, a_N^i) | 0 \leq i < l \} \]
be an \((N, q, l, H_m)\) FHS set, and let
\[ e = (e_0, e_1, \ldots, e_{l-1}) \]
be a shift sequence of length \(l\) over \(Z_N\), \(e_i \in Z_N, 0 \leq i < l\).
An \(N \times l\) matrix is formed by placing the sequences \(a_i\) and \(e\) as follows:
\[
U = \begin{pmatrix}
a_0^0 + e_0 & a_1^0 + e_1 & \cdots & a_{l-1}^0 + e_{l-1} \\
a_1^0 + e_0 & a_1^1 + e_1 & \cdots & a_{l-1}^1 + e_{l-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_N^{N-1} + e_0 & a_N^{N-1} + e_1 & \cdots & a_{l-1}^{N-1} + e_{l-1}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
u_0 & u_1 & \cdots & u_{l-1} \\
u_1 & u_{l+1} & \cdots & u_{2l-1} \\
\vdots & \vdots & \ddots & \vdots \\
u_{l(N-1)} & u_{l(N-1)+1} & \cdots & u_{lN-1}
\end{pmatrix}
\]
(5)
where the additions in subscripts are performed modulo \(N\).

By reading the elements in \(U\) row by row, we get the sequence \(u=(u_0, u_1, \ldots, u_{lN-1})\)
of period \(lN\). The FHS \(a^i\) is called the basic sequence, \(e\) is called the shift sequence
and \(u\) is referred to the interleaved sequence. The matrix \(U\) is an array form of \(u\).

For short, we write the interleaved sequence \(u\) as
\[
u = I(L^{a_0}(a^0), L^{a_1}(a^1), \ldots, L^{a_{l-1}}(a^{l-1}))
\]
where \(I\) is the interleaving operator. \(L\) is the (left cyclical) shift operator, i.e.
\[
L^3(a^i) = (a_3^i, a_4^i, \ldots, a_{N-1}^i, a_0^i, a_1^i, a_2^i).
\]

Let \(g = (g_0, g_1, \ldots, g_{l-1})\) be another shift sequence over \(Z_N\) and \(v = I(L^{g_0}(a^0), L^{g_1}(a^1), \ldots, L^{g_{l-1}}(a^{l-1}))\).
Consider its cyclical shift version \(L^\tau(v)\), where \(\tau = l\tau_1 + \tau_2 (0 \leq \tau_1 < N, 0 \leq \tau_2 < l)\). By the matrix representation, \(L^\tau(v)\) could be written as
\[
L^\tau(v) = I(L^{g_2 + \tau_1}(a^{g_2}), \ldots, L^{g_{l-1} + \tau_1}(a^{g_{l-1}}), L^{g_0 + \tau_1 + \tau_2}(a^{g_0}), \ldots, L^{g_{l-1} + \tau_1 + \tau_2}(a^{g_{l-1}}))
\]
(6)
Clearly, \(L^\tau(v)\) is just another interleaved sequence. That is to say, we have
\[
L^\tau(v) = I(L^{a_2 + \tau_1}(a^{a_2}), \ldots, L^{a_{l-1} + \tau_1}(a^{a_{l-1}}), L^{a_0 + \tau_1 + \tau_2}(a^{a_0}), \ldots, L^{a_{l-1} + \tau_1 + \tau_2}(a^{a_{l-1}})).
\]

Then, the Hamming correlation function between the interleaved sequences \(u\) and \(v\) at a shift \(\tau\) becomes the summation of the inner products between the pairwise column sequences in (5) and (6), i.e.
\[
H_{uv}(\tau) = \sum_{t=0}^{l-\tau_2-1} H_{a^i a^{a_2 + \tau_1}}(\tau_1 + g_{t+\tau_2} - e_t)
\]
\[
+ \sum_{t=\tau_2+1}^{l-1} H_{a^i a^{a_2 + \tau_1 - t}}(\tau_1 + g_{t+\tau_2 - l + 1} - e_t).
\]
(7)
Especially, when $\tau_2 = 0$, we have
\begin{equation}
H_{uv}(\tau) = \sum_{t=0}^{l-1} H_{a^t a^t}(\tau_1 + g_t - e_t).
\end{equation}

For any $\tau_2$, $0 \leq \tau_2 < l$, let
\begin{equation}
d_{t,\tau_2}^{(e,g)} = \begin{cases} e_t - g_t + \tau_2, & 0 \leq t \leq l - 1 - \tau_2 \\ e_t - g_t - \tau_2 - 1, & l - \tau_2 \leq t \leq l - 1 \end{cases},
\end{equation}
where $d_{t,\tau_2}^{(e,g)}$ is calculated in $\mathbb{Z}_N$. Then, the Hamming correlation function of $u$ and $v$ can be written as:
\begin{equation}
H_{uv}(\tau) = \sum_{t=0}^{l-1} H_{a^t a^t \tau_2}(\tau_1 - d_{t,\tau_2}^{(e,g)}).
\end{equation}

By (7) and (10), it is evidently that $u$ and $v$ are cyclically equivalent if and only if for any $\tau_2$, $0 \leq \tau_2 < l$, the following equation holds:
\begin{equation}
d_{i,\tau_2}^{(e,g)} = d_{j,\tau_2}^{(e,g)},
\end{equation}
for all $0 \leq i, j < l, i \neq j$.

**Definition 1.** Any two shift sequences $e = (e_0, e_1, \cdots, e_{l-1})$ and $g = (g_0, g_1, \cdots, g_{l-1})$ over $\mathbb{Z}_N$ are said to be inequivalent if equation (11) does not hold.

Based on the above analysis, the lemma is following naturally.

**Lemma 4.** With the above notations, if $u$ and $v$ are cyclically distinct, namely, $e$ and $g$ are inequivalent shift sequences, then
\begin{equation}
H_{uv}(\tau) \leq lH_m,
\end{equation}
for any $\tau = l\tau_1 + \tau_2$ with $0 \leq \tau_1 < \min_{0 \leq i \leq l} \left\{ d_{i,\tau_2}^{(e,g)} \right\}, 0 \leq \tau_2 < l$.

4. **Optimal FHS set with low hit zone**

In this section, we propose two classes of optimal LHZ FHS set from the optimal FHS set by interleaved techniques.

Let $F = \{f_0, f_1, \cdots, f_{q-1}\}$ be a frequency slot set with size $|F| = q$. Then, we propose the first construction of LHZ FHS set.

**Construction 1 : A general interleaved construction of LHZ FHS sets.**

Step 1: Select an optimal $(N, q, l, H_m)$ FHS set $A$ over $F$, where $l \geq 2$.
\[A = \{a^i = (a_0^i, a_1^i, \cdots, a_{N-1}^i) | 0 \leq i < l \} .\]

Step 2: Let $M$, $\omega$, $s$, be three positive integers such that $M = \lceil \frac{N}{\omega} \rceil$, $\omega > 2l$, $gcd(s, N) = 1$. Generate the shift sequences set $G$ as follows:
\[G = \{ g^k = (g_0^k, g_1^k, \cdots, g_{N-1}^k) | 0 \leq k < M(\omega - 1) \} , \]
\[g^k = (Q_0(((m + r) \omega + r) s^{-1}), Q_1(((m + r) \omega + r) s^{-1}) , \cdots, Q_{l-1}(((m + r) \omega + r) s^{-1})), \]
where $r, m$ are two integers with $0 < r < \omega$, $0 \leq m < M$ and $k = (r - 1)M + m$.
\[\{ Q_0, Q_1, \cdots, Q_{l-1} \} \] are permutations over $\{ \omega + 1, 2\omega + 1, \cdots, (M - 1)\omega + \omega^2 - 1 \}$, which satisfy that all the shift sequences in $G$ are pairwise inequivalent.

Step 3: For a given positive integer $n$ with $0 < n \leq \left\lfloor \frac{1}{2} \right\rfloor$, let $t$ be an integer with $0 \leq
$t < n$. Construct desired LHZ FHS set $S = \{ s^{(t,k)} | 0 \leq t < n, 0 \leq k < M(\omega - 1) \}$ by interleaving FHS set $\mathcal{A}$ and shift sequence set $G$,

$$s^{(t,k)} = I(L^{t}_N(a^{t}_0), L^{t}_N(a^{t}_1), \ldots, L^{t}_{N-1}(a^{t}_{l_N-1})),$$

where $B = \{ b^{t} = (b^{t}_0, b^{t}_1, \ldots, b^{t}_{l_N-1}) | 0 \leq t < n \}$ over $\{0, 1, \ldots, l - 1\}$. The sequences set $B$ satisfies that $H_a(B) = 0$ and $b^{t_1} = L^{t}(b^{t_2})$, where $0 \leq t_1 \neq t_2 < n, 0 < d < n$.

**Remark 1.** The critical part in Construction 1 is to find out some proper permutations $\{Q_0, Q_1, \ldots, Q_{l_N-1}\}$, such that all the shift sequences in $G$ are pairwise inequivalent. The Construction 2 in [12] gave four kinds of specific permutations for generating inequivalent shift sequences. By means of those constructions, we can obtain inequivalent shift sequences. What’s more, the Theorem 3 in [9] has proved that shift sequences with permutations maintain the LHZ property of FHS set.

Then, we can obtain the following theorem:

**Theorem 1.** Let $(N,q,l,H_m)$ be an optimal FHS set $\mathcal{A}$ with $l \geq 2$, $M$, $n$, and $\omega$ be three positive integers with $M = \lceil \frac{N}{2} \rceil$, $0 < n \leq \lfloor \frac{N}{2} \rfloor$, and $\omega > 2l$. $S$ is an $(lN,q,nM(\omega - 1),l-n,lH_m)$ LHZ FHS set in Construction 1.

**Proof.** According to the theory of interleaved techniques, it is clearly that $S$ is an FHS set of $nM(\omega - 1)$ FHSs of length $lN$ over a frequency slot $F$ with size $q$.

Next, let $H_a, H_c$ be the maximum Hamming autocorrelation and crosscorrelation of FHS set $\mathcal{A}$ respectively. $H_m$ denotes the maximum Hamming correlation of $\mathcal{A}$. Let $H'_{a}, H'_{c}$ be the maximum Hamming autocorrelation and crosscorrelation of LHZ FHS set $S$ respectively. $H'_{m}$ denotes the maximum Hamming correlation of $S$ within the low hit zone $L_H$. Let $t_2 \geq t_1$, for any two FHSs $s^{(t_1,k_1)}, s^{(t_2,k_2)} \in S$, which correspond to the shift sequences $g^{k_1}, g^{k_2} \in G$ respectively.

**Case 1:** $k_1 = k_2$.

**Case 1.1:** $t_1 = t_2$.

By analyzing the theory of interleaving, we know that $g^{k_1}$ and $g^{k_2}$ are cyclically equivalent if $k_1 = k_2$ and it is the special case for $\tau_2 = 0$. Thus, by equation (8)

$$H_a(s^{(t_1,k_1)}, s^{(t_1,k_1)})(\tau) = \sum_{i=0}^{l-1} H_{a^i a}(\tau_1 + q^{k_1}, \tau_1 - q^{k_1}),$$

where $0 < \tau_1 < N, 0 \leq i < l$, we have $H'_a = lH_m$ with low hit zone $L_1 = lN$.

**Case 1.2:** $t_1 \neq t_2$.

By analyzing the theory of interleaving, we know that $g^{k_1}$ and $g^{k_2}$ are cyclically equivalent if $k_1 = k_2$ and it is the special case for $\tau_2 = 0$. In this case, we can get that $s^{(t_1,k_1)}, s^{(t_2,k_1)}$ are two different sequences. Thus, by equation (8)

$$H_a(s^{(t_1,k_1)}, s^{(t_2,k_1)})(\tau) = \sum_{i=0}^{l-1} H_{a^i a^{i+t_2-t_1}}(\tau_1 + q^{k_1}, \tau_1 - q^{k_1}),$$

where $0 \leq \tau_1 < N, 0 \leq i < l, t_2 > t_1$ and the superscript $i+t_2-t_1$ is computed modulo $l$, we have $H'_c = lH_m$ with low hit zone $L_{c_1} = lN$.

**Case 2:** $k_1 \neq k_2$.

**Case 2.1:** $t_1 = t_2$.

Through analyzing the theory of interleaving, we can get $\tau_2 = 0$ and $g^{k_1} \neq g^{k_2}$.
By equation (7), (9) and Lemma 4, we have \( H'_{c_2} = lH_m \), for
\[
0 \leq \tau < \min_{g^{k_1} \neq g^{k_2} \in G} \left\{ \min_{0 \leq i < l} \left\{ l(g^{k_1}_i - g^{k_2}_i) \right\} \right\}.
\]

**Case 2.2:** \( t_1 \neq t_2 \).

Through analyzing the theory of interleaving and the constraints of set \( B \), we can get \( g^{k_1} \neq g^{k_2} \) and \( t_2 > t_1 \). In addition, to eliminate the influence caused by the order of the basic sequence, let \( t_1 - t_2 \) denote the shift between \( s^{(t_1,k_1)} \) and \( s^{(t_2,k_2)} \). Then, by equation (7), (9) and Lemma 4, we have \( H'_{c_3} = lH_m \), for
\[
0 \leq \tau < \min_{g^{k_1} \neq g^{k_2} \in G} \left\{ \min_{0 \leq i < l, 0 \leq t_1, t_2 < n} \left\{ l(g^{k_1}_i - g^{k_2}_i) + t_1 - t_2 \right\} \right\}.
\]

From Case 1.2, Case 2.1 and Case 2.2, for any \( \tau, 0 \leq \tau \leq L_2 \), the maximum Hamming crosscorrelation \( H'_c \) is given by
\[
H'_c = \max \{ H'_{c_1}, H'_{c_2}, H'_{c_3} \} = lH_m,
\]
with low hit zone: \( L_2 + 1 = \min \{ L_{c_1}, L_{c_2}, L_{c_3} \} \), where \( L_{c_3} = lN \),
\[
L_{c_2} = \min_{g^{k_1} \neq g^{k_2} \in G} \left\{ \min_{0 \leq i < l, 0 \leq t_1, t_2 < n} \left\{ l(g^{k_1}_i - g^{k_2}_i) \right\} \right\},
\]
\[
L_{c_3} = \min_{g^{k_1} \neq g^{k_2} \in G} \left\{ \min_{0 \leq i < l, 0 \leq t_1, t_2 < n} \left\{ l(g^{k_1}_i - g^{k_2}_i) + t_1 - t_2 \right\} \right\}.
\]

Obviously, \( L_{c_2} < L_{c_1} \) and \( L_{c_3} < L_{c_1} \). Let \( g^{k_1}_i = (m_1 + r_1) \omega + r_1 \), \( g^{k_2}_i = (m_2 + r_2) \omega + r_2 \). From the Theorem 1 in [12], if \( m_1 + r_1 = m_2 + r_2 \), we can get the following equation:
\[
L_2 + 1 = \min \left\{ \min_{g^{k_1} \neq g^{k_2} \in G} \{ l(r_1 - r_2) \}, \min_{g^{k_1} \neq g^{k_2} \in G} \{ l(r_1 - r_2) - (n - 1) \} \right\}.
\]

Then, we can get \( L_2 = \min \{ l, l - (n - 1) \} - 1 = l - n \).

Above all, for any \( \tau, 0 \leq \tau \leq L_H \), the maximum Hamming correlation \( H'_m \) is given by
\[
H'_m = \max \{ H'_{a'1}, H'_{a'2} \} = lH_m
\]
with low hit zone: \( L_H = \min \{ L_{a'1}, L_{a'2} \} = l - n \).

Therefore, \( S \) is an \((lN, q, nM(\omega - 1), l - n, lH_m)\) LHZ FHS set.

**Theorem 2.** The LHZ FHS set \( S \) is an optimal \((lN, q, nM(\omega - 1), l - n, lH_m)\) LHZ FHS set, if the parameters satisfy:
\[
\frac{nM(\omega - 1)(l - n + 1) - q lN}{nM(\omega - 1)(l - n + 1) - 1 - q lN} = l \left\lfloor \frac{lN - q N}{lN - 1 - q} \right\rfloor.
\]
Moreover, all the FHSs in \( S \) are pairwise cyclically distinct.

**Proof.** Let \( \mathcal{A} \) be an FHS set with parameters \((N, q, l, H_m)\). \( S \) is a LHZ FHS set with \( nM(\omega - 1) \) sequences of length \( lN \). The size of frequency slots is \( q \) and \( L_H = l - n \). According to the Peng-Fan bound (Lemma 1) and the parameters of \( \mathcal{A} \), we have \( lH_m = l \left\lfloor \frac{(lN - q)N}{lN - 1 - q} \right\rfloor \). By the Peng-Fan-Lee bound (Lemma 2), we can say \( S \) is an optimal LHZ FHS set if:
\[
lH_m = \left\lfloor \frac{nM(\omega - 1)(l - n + 1) - q lN}{nM(\omega - 1)(l - n + 1) - 1 - q lN} \right\rfloor.
\]
By the Definition 1, all the sequences in FHS set $S$ are cyclically distinct for all the shift sequences in $G$ are inequivalent. 

**Corollary 1.** Choose an optimal $(q^m-1, q^k, q^k, q^{m-k})$ FHS set in [22] as the basic sequence set in Construction 1, where $1 \leq k \leq m$. Then we can obtain an LHZ FHS set $C_1$ with parameters $(q^k(q^m-1), q^k, nM(\omega-1), q^k-n, q^m)$ where $0 < n \leq \lfloor \frac{q^k}{\omega} \rfloor$, $\omega > 2q^k$, $M = \lceil \frac{q^m}{\omega} \rceil$.

**Corollary 2.** Choose an optimal $(\lfloor \frac{q^m}{\omega} \rfloor, q^k, l, \lfloor \frac{q^{m-k}}{l} \rfloor)$ FHS set in [22] as the basic sequence set in Construction 1, where $0 < k \leq m$, $l|(q-1)$, $gcd(l, m) = 1$. Then, we can obtain an LHZ FHS set $C_2$ with parameters $(q^m-1, q^k, nM(\omega-1), l-n, q^{m-k}-1)$ where $0 < n \leq \lfloor \frac{l}{\omega} \rfloor$, $\omega > 2l$, $M = \lfloor \frac{q^m}{l\omega} \rfloor$.

**Corollary 3.** Choose an optimal $(q^m-1, q^{m-k-1}+1, q^{m-k-1}, lq^k)$ FHS set in [11] as the basic sequence set in Construction 1, where $1 \leq k \leq m-1$, $l|(q-1)$, $l \geq 2$ and $q^m-1 < (\frac{e'}{q})^2 + (\frac{e'}{q}+1)p^k - \frac{q^k}{\omega}$, $e' = q^{m-k} - 1$. Then, we can obtain an LHZ FHS set $C_3$ with parameters $(e'(q^{m-1}), \frac{e'}{q}+1, nM(\omega-1), \frac{e'}{q}-n, e'q^k)$ where $0 < n \leq \lfloor \frac{e'}{q\omega} \rfloor$, $\omega > 2\frac{e'}{q}$, $M = \lfloor \frac{q^m}{q\omega} \rfloor$.

**Remark 2.** By applying three specific optimal FHS set in [22] and [11] as the basic sequence set in Construction 1, we can get three LHZ FHS sets with larger family size. Furthermore, all the sequences in above corollaries are shift distinct. The parameters of new LHZ FHS sets and some known LHZ FHS sets are listed in Table 1.

In Corollary 1, let $k = 1$, we can get a $(q(q^m-1), q, nM(\omega-1), q-n, q^m)$ LHZ FHS set $C_1$, which has a larger family size than the LHZ FHS set got by Ma in [8]. Moreover, the parameters of LHZ FHS set in corollary 1 are more flexible.

In Corollary 2, we obtain an LHZ FHS set $C_2$ with parameters $(q^m-1, q^k, nM(\omega-1), l-n, q^{m-k}-1)$. Compared with the LHZ FHS set got by Zhou in [20], our sets are with smaller maximum Hamming correlation and larger family size when $\frac{q^{m-k}}{\omega} > \frac{q^k}{2}$.

In Corollary 3, we obtain a $(e'(q^{m-1}), \frac{e'}{q}+1, nM(\omega-1), \frac{e'}{q}-n, e'q^k)$ LHZ FHS set $C_3$ firstly, which has more new and flexible parameters.

**Example 1.** We choose an optimal $(56, 13, 3, 4)$ FHS set $A = \{a^0, a^1, a^2\}$ in [8], such that

\[
\begin{align*}
a^0 &= (2, 2, 6, 12, 1, 0, 8, 12, 4, 11, 3, 8, 10, 1, 10, \cdots, 12, 2, 8), \\
a^1 &= (4, 4, 12, 11, 2, 0, 3, 11, 8, 9, 6, 3, 7, 2, 7, \cdots, 11, 4, 3), \\
a^2 &= (8, 8, 11, 9, 4, 0, 6, 9, 3, 5, 12, 6, 1, 4, 1, \cdots, 9, 8, 6).
\end{align*}
\]

Let $n = 2$, $M = 4$, $\omega = 14$, generate the shift sequence set $G = \{g^k = (g_0^k, g_1^k, g_2^k) | 0 \leq k < 51\}$. By the matrix representation, the shift sequence set $G$ can be written as follows:

\[
G' = \begin{pmatrix}
g_0^0 & g_0^1 & g_0^2 \\
g_1^0 & g_1^1 & g_1^2 \\
\vdots & \vdots & \vdots \\
g_5^0 & g_5^1 & g_5^2
\end{pmatrix} = \begin{pmatrix}
15 & 13 & 43 \\
30 & 54 & 2 \\
13 & 15 & 15
\end{pmatrix}.
\]
Construct the LHZ FHS set $S = \{s^{(0,0)}, s^{(0,1)}, \ldots, s^{(1,50)}\}$ with FHS set $A, B = \{(1, 0, 2), (0, 2, 1)\}$ and shift sequence set $G$.

$$s^{(0,0)} = (7, 1, 12, 8, 10, 10, 3, 10, 7, 10, 4, 6, 0, 8, 0, \cdots),$$
$$s^{(0,1)} = (1, 2, 11, 2, 8, 9, 11, 2, 4, 0, 2, 0, 10, 6, 6, \cdots),$$
$$\cdots$$
$$s^{(1,49)} = (2, 2, 1, 8, 4, 2, 2, 9, 11, 2, 0, 0, 6, 7, 10, \cdots),$$
$$s^{(1,50)} = (1, 1, 7, 10, 3, 8, 10, 6, 3, 4, 7, 10, 8, 0, 0, \cdots).$$

Then, the maximum Hamming correlation of $S$ can be seen in Figure 1. It can be proved that the maximum Hamming correlation $H_m = 12$ for $0 < \tau < 2$. When $2 < \tau < 168$, the maximum Hamming correlation $H_m = 116$. Hence, $S$ is an optimal $(168, 13, 104, 1, 12)$ LHZ FHS set, and all the sequences in LHZ FHS set $S$ are cyclically distinct.

**Construction 2 : A construction of LHZ FHS set based on two kinds of interleaving**

Step 1: Select an optimal $(N, q, l, H_m)$ FHS set $A$ over $F$, where $l \geq 2$.

$$A = \{a^i = (a^i_0, a^i_1, \cdots, a^i_{N-1}) | 0 \leq i < l\}.$$  

Step 2: Let $M, \omega, x, s$ be four positive integers with $M = \lfloor \frac{N}{\omega} \rfloor$, $\omega > 2l$, $0 < x < \frac{\omega - 1}{\omega - x}$, $\gcd(s, N) = 1$. Generate the shift sequence set $H = \{E, G\}$ as follows:

$$E = \{e^j = (e^j_0, e^j_1, \cdots, e^j_{l-1}) | 0 \leq j < M\},$$

$$\begin{align*}
e^j &= (P_0(j\omega s^{-1}), P_1((j + x)\omega + x)s^{-1}), \\
&\cdots, P_{l-1}(((j + (l-1)x)\omega + (l-1)x)s^{-1})),
\end{align*}$$

$$G = \{g^k = (g^k_0, g^k_1, \cdots, g^k_{l-1}) | 0 \leq k < M(\omega - x + x - 1)\},$$

$$g^k = (Q_0(((m + r)\omega + r)s^{-1}), Q_1(((m + r)\omega + r)s^{-1}),$$

$$\cdots, Q_{l-1}(((m + r)\omega + r)s^{-1})),$$

where $r, m$ are two integers with $xl - x < r < \omega, 0 \leq m < M$ and $k = (r - xl + x - 1)M + m$. $\{P_0, P_1, \cdots, P_{l-1}\}$ and $\{Q_0, Q_1, \cdots, Q_{l-1}\}$ are two permutations, which
satisfy that all the shift sequences in \( E \) and \( G \) are pairwise inequivalent.

Step 3: Construct the desired LHZ FHS set \( S = \{ S_A, S_B \} \).

Step 3.1: Construct \( S_A = \{ s^y | 0 \leq y < Ml \} \) by interleaving FHS set \( A \) and shift sequence set \( E \),

\[
s^y = I(L_s^0(a^y), L_s^1(a^y), \ldots, L_s^{n-1}(a^y)).
\]

Step 3.2: For a given integer \( n \) with \( 0 < n \leq \left\lceil \frac{1}{2} \right\rceil \), let \( t \) be an integer with \( 0 \leq t < n \).

Construct the desired LHZ FHS set \( S_B = \{ s^{(t,k)} | 0 \leq t < n, 0 \leq k < M(\omega - xt + x - 1) \} \) by interleaving FHS set \( A \) and shift sequence set \( G \),

\[
s^{(t,k)} = I(L_s^0(a^t), L_s^1(a^t), \ldots, L_s^{n-1}(a^{t-1})),
\]

where \( B = \{ b^t = (b_t^0, b_t^1, \ldots, b_t^{l-1}) | 0 \leq t < n \} \) over \( \{ 0, 1, \ldots, l - 1 \} \). The sequences set \( B \) satisfy that all the shift sequences in \( E \) and \( \omega \) sequence set \( S \).

Remark 3. Like Construction 1, we can get various LHZ FHS sets \( S_B \) with different \( n \) and the family size is maximum when \( n = \left\lfloor \frac{1}{2} \right\rfloor \). It is obviously that the family size of FHS set in Construction 1 is larger than the FHS sets in Construction 2. By choosing different optimal FHS set as basic sequences set and parameter \( x \), we can obtain more optimal LHZ FHS sets \( S_A \) with flexible parameters. As a comparison, we have listed the parameters of other constructions in Table 1. From the Table 1, we can conclude that the optimal LHZ FHS set constructed by Construction 1 has a larger family size and more flexible parameters than others. Simultaneously, all of the FHSs constructed by this paper are cyclically inequivalent.

Then, we can obtain the following theorem:

**Theorem 3.** Let \( (N, q, l, H_m) \) be an optimal FHS set \( A \) with \( l \geq 2 \), \( M, x, \omega \) and \( n \) be four positive integers such that \( M = \left\lceil \frac{N}{2} \right\rceil \), \( 0 < x < \frac{N-1}{2} \), \( \omega > 2l \) and \( 0 < n \leq \left\lceil \frac{1}{2} \right\rceil \). The LHZ FHS set \( S \) constructed above is with parameters

\[
\begin{align*}
&\begin{cases} 
(N, q, M\omega, l - 2, lH_m) & x = 1 \text{ and } n = 1, \\
(N, q, Ml + nM(\omega - xt + x - 1), l - n, lH_m) & \text{others}. 
\end{cases}
\end{align*}
\]

**Proof.** According to the theory of interleaved techniques, it is clearly that \( S \) is an LHZ FHS set of \( Ml + nM(\omega - xt + x - 1) \) FHSs of length \( lN \) over a frequency slot with size \( q \). Next, let \( H_a \) and \( H_c \) be the maximum Hamming autocorrelation and crosscorrelation of FHS set \( A \) respectively, \( H_m \) denote the maximum Hamming correlation of \( A \).

Case 1:

Let \( H_a^A \) and \( H_c^A \) be the maximum Hamming autocorrelation and crosscorrelation of LHZ FHS set \( S_A \) respectively, \( H_m^A \) denote the maximum Hamming correlation of \( S_A \) within the low hit zone \( L_m^A \). For any two FHSs \( s^u, s^v \in S_A \), which correspond to the shift sequences \( e^{j_1}, e^{j_2} \in E \) respectively.

Case 1.1: \( u = v \). If \( u = v \), \( e^{j_1} = e^{j_2} \). By Lemma 4, we have \( H_a^A = lH_a \), for

\[
0 < \tau < \min_{e^{j_1} \in E} \left\{ \min_{0 \leq i \leq l, 0 < \tau_2 < l} \left\{ l(d(e^{j_1}, e^{j_1}) + \tau_2) \right\} \right\},
\]

when \( \tau = 0 \), this is the trivial case. Let \( e^{j_1} = (j_1 + i_1x)\omega + i_1x, e^{j_2} = (j_1 + i_2x)\omega + i_2x \), where \( 0 \leq i_1 < M, 0 \leq i_1 \neq i_2 < l, 0 < x < \frac{N-1}{2} \). From the Theorem 2 in [9], we have \( L_1 + 1 = \min_{e^{j_1} \in E} \{ l(x\omega + x - 1) + l - 1 \} \) if \( \tau_2 = l - 1 \). Thus,
we can get \( L_1 = lx(\omega + 1) - 2 \).

Case 1.2: \( u \neq v \).

i): \((\lambda - 1)M \leq u, v < \lambda M, 1 \leq \lambda \leq l \).

In this case, by Lemma 4, we have \( H_c^A = lH_a \), for

\[
0 \leq \tau < \min_{e^{11} \neq e^{12} \in E} \left\{ \min_{0 \leq t_1, t_2 < l} \{ ld_{(i_1, i_2)} + \tau \} \right\}.
\]

ii): \((\lambda - 1)M \leq u < \lambda M, (\gamma - 1)M \leq v < \gamma M, 1 \leq \lambda \neq \gamma \leq l \).

By Lemma 4, we have \( H_c^C = lH_c \), for

\[
0 \leq \tau < \min_{e^{11} \neq e^{12} \in E} \left\{ \min_{0 \leq t_1, t_2 < l} \{ ld_{(i_1, i_2)} + \tau \} \right\}.
\]

Let \( e^{11}_i = (j_1 + i_1)x(\omega + 1) - 1 \) and \( e^{12}_i = (j_2 + i_2)x(\omega + 1) - 1 \), where \( 0 \leq j_1 \neq j_2 < M, 0 \leq i_1, i_2 < l, 0 < x < \frac{\omega + 1}{2} \). From the Theorem 2 in [9], if \((j_1 + i_1)x = (j_2 + i_2)x\),

we can get

\[
L_2 + 1 = \begin{cases} \min_{e^{11} \neq e^{12} \in E} \left\{ \min_{0 \leq t_1, t_2 < l} \{ ld_{(i_1, i_2)} + \tau \} \right\} & x \neq 1, \\
\min_{e^{11} \neq e^{12} \in E} \left\{ \min_{0 \leq t_1, t_2 < l} \{ l - 1 \} \right\} & x = 1. \end{cases}
\]

Thus, we have \( H_c^A = \max \{ lH_a, lH_c \} = lH_m \) with

\[
L_2 = \begin{cases} lx - x & x \neq 1, \\
1 - 2 & x = 1. \end{cases}
\]

From the above two cases, for any \( \tau, 0 \leq \tau \leq L_H^A \), the maximum Hamming correlation is

\[
H_m^A = \max \{ H_a^A, H_c^A \} = lH_m
\]

with low hit zone \( L_H^A = \min \{ L_1, L_2 \} \). If \( x \neq 1 \), \( L_H^A = lx - x \) and if \( x = 1 \), \( L_H^A = l - 2 \).

Case 2:

Let \( H_H^B \) denote the maximum Hamming correlation of \( S_B \) within the low hit zone \( L_H^B \). Let \( t_2 \geq t_1 \), for any two FHSs \( s^{(t_1, k_1)} \), \( s^{(t_2, k_2)} \in S_B \), which correspond to the shift sequences \( g^{k_1}, g^{k_2} \in G \) respectively.

This proof is same with Theorem 1, so we omit it. From Theorem 1, we can obtain that for any \( \tau, 0 \leq \tau \leq L_H^B \), the maximum Hamming correlation \( H_m^B \) is given by \( H_m^B = lH_m \) with low hit zone \( L_H^B = l - n \).

Case 3:

For any sequence \( s^u \in S_A \) and any sequence \( s^{(t_1, k_1)} \in S_B \), which correspond to the shift sequence \( e^{11}_i \in E \) and \( g^{k_1} \in G \) respectively. Let \( H_m^C \) denote the maximum Hamming correlation between \( s^u \) and \( s^{(t_1, k_1)} \) within the low hit zone \( L_H^C \). Obviously, \( s^u \) and \( s^{(t_1, k_1)} \) are two different sequences. Due to \( lx - x < r < \omega \), we can conclude that the shift sequences in \( E \) and \( G \) are inequivalent.

Thus, by Lemma 4, we have \( H_c^C = lH_m \), for

\[
0 \leq \tau < \min_{e^{11} \in E, g^{k_1} \in G} \left\{ \min_{0 \leq t_1, t_2 < l} \{ ld_{(i_1, i_2)} + \tau \} \right\}.
\]

For any \( \tau, 0 \leq \tau \leq L_H^C \), the maximum Hamming correlation \( H_m^C \) is given by \( H_m^C = H_c^C = lH_m \) with low hit zone \( L_H^C + 1 = \omega + 1 \).
Above all, for $0 < x < \frac{\omega}{l-1}, \omega > 2l, 0 < n \leq \lceil \frac{x}{l} \rceil$, the maximum Hamming correlation of $S$ is $H'_m = \max \{H^A_m, H^B_m, H^C_m\} = \ell H_m$ with low hit zone $L_H = \min \{L^A_H, L^B_H, L^C_H\}$. When $x = 1$, we have $L_H = \min \{l - 2, l - n, \omega\}$ and we can get $L_H = l - 2$ with $n = 1$. If $x \neq 1$, we have $L_H = \min \{xl - x, l - n, \omega\} = l - n$. Thus, $S$ is a LHZ FHS set with parameters

\[
\begin{cases}
(lN, q, M\omega, l - 2, lH_m) & x = 1 \text{ and } n = 1, \\
(lN, q, Ml + nM(\omega - xl + x - 1), l - n, lH_m) & \text{others}.
\end{cases}
\]

\[\Box\]

**Theorem 4.** The LHZ FHS set $S$ is an optimal $(lN, q, Ml + nM(\omega - xl + x - 1), l - n, lH_m)$ LHZ FHS set with $x \neq 1$ or $n \neq 1$, if the parameters satisfy:

\[
\frac{M(l + n(\omega - xl + x - 1))(l - n + 1) - q lN}{M(l + n(\omega - xl + x - 1))(l - n + 1) - 1 q} = l \left\lfloor \frac{(lN - q)N}{lN - 1} \right\rfloor.
\]

Besides, all the FHSs in $S$ are pairwise cyclically distinct.

**Proof.** Let $A$ be an FHS set with parameters $(N, q, l, H_m)$, $S$ be a LHZ FHS set with $Ml + nM(\omega - xl + x - 1)$ sequences of length $lN$. The size of frequency slots is $q$ and $L_H$ is $l - n$. According to the Peng-Fan bound (Lemma 1) and the parameters of $A$, we have $H_m = l \left\lfloor \frac{(lN - q)N}{lN - 1} \right\rfloor$. By the Peng-Fan-Lee bound (Lemma 2), we can say $S$ is an optimal LHZ FHS set if:

\[
lH_m = \left\lfloor \frac{M(l + n(\omega - xl + x - 1))(l - n + 1) - q lN}{M(l + n(\omega - xl + x - 1))(l - n + 1) - 1 q} \right\rfloor.
\]

The inequivalence of shift sequence sets $E$ and $G$ ensures that all the sequences in FHS set $S$ are cyclically distinct. $\Box$

**Theorem 5.** The LHZ FHS set $S$ is an optimal $(lN, q, M\omega, l - 2, lH_m)$ LHZ FHS set with $x = 1$ and $n = 1$, if the parameters satisfy:

\[
\frac{M\omega(l - 1) - q lN}{M\omega(l - 1) - 1 q} = l \left\lfloor \frac{(lN - q)N}{lN - 1} \right\rfloor.
\]

Besides, all the FHSs in $S$ are pairwise cyclically distinct.

**Proof.** This proof is same with Theorem 4, so we omit it. $\Box$

Then, we give an example to explain this construction.

**Example 2.** We choose an optimal $(56, 13, 3, 4)$ FHS set $A = \{a^0, a^1, a^2\}$, such that:

\[
a^0 = (2, 2, 6, 12, 1, 0, 8, 12, 4, 11, 3, 8, 10, 1, 10, \cdots, 12, 2, 8),
a^1 = (4, 4, 12, 11, 2, 0, 3, 11, 8, 9, 6, 3, 7, 2, 7, \cdots, 11, 4, 3),
a^2 = (8, 8, 11, 9, 4, 0, 6, 9, 3, 5, 12, 6, 1, 4, 1, \cdots, 9, 8, 6).
\]

Let $n = 2, M = 4, \omega = 14, x = 2$, and generate the shift sequence set $E = \{e^j = (e^0_j, e^1_j, e^2_j) | 0 \leq j < 4\}$ and $G = \{g^k = (g^0_k, g^1_k, g^2_k) | 0 \leq k < 36\}$. By the matrix
representation, the shift sequence sets $G$ could be written as follows:

$$E' = \begin{pmatrix} e_0^0 & e_1^0 & e_2^0 \\ e_0^1 & e_1^1 & e_2^1 \\ e_0^2 & e_1^2 & e_2^2 \\ e_0^3 & e_1^3 & e_2^3 \end{pmatrix} = \begin{pmatrix} 0 & 16 & 46 \\ 14 & 2 & 32 \\ 28 & 44 & 45 \\ 42 & 30 & 31 \end{pmatrix},$$

$$G' = \begin{pmatrix} g_0^0 & g_1^0 & g_2^0 \\ \vdots & \vdots & \vdots \\ g_{34}^0 & g_{34}^1 & g_{34}^2 \\ g_{35}^0 & g_{35}^1 & g_{35}^2 \end{pmatrix} = \begin{pmatrix} 19 & 13 & 47 \\ \vdots & \vdots & \vdots \\ 54 & 34 & 34 \\ 13 & 19 & 19 \end{pmatrix}.$$

Construct the LHZ FHS set $S_A = \{s_0^0, s_1^1, \ldots, s_{11}^{11}\}$ with basic sequence set $A$ and shift sequence set $E$. Construct $S_B = \{s^{(0,0)}, s^{(0,1)}, \ldots, s^{(1,35)}\}$ with FHS set $A$, $B = \{(1,0,2), (0,1,2)\}$ and shift sequence set $G$.

$s_0^0 = (2,4,8,2,8,0,6,5,12,12,0,5,1,1,6,\ldots),$

$s_1^1 = (10,6,12,10,12,0,4,1,5,8,0,1,5,8,9,\ldots),$

$\ldots$

$s_{11}^{11} = (12,2,4,12,4,9,10,9,0,7,0,7,6,7,4,\ldots),$

$s^{(0,0)} = (0,1,0,2,10,9,3,10,7,1,4,11,6,8,1,\ldots),$

$s^{(0,1)} = (10,2,6,2,8,9,5,2,3,4,2,5,7,6,12,\ldots),$

$\ldots$

$s^{(1,35)} = (1,0,0,10,4,2,10,6,3,4,2,1,8,12,6,\ldots).$

Then, the maximum Hamming correlation of $S$ can be seen in Figure 2. It can be proved that the maximum Hamming correlation $H_m = 12$ for $0 \leq \tau < 2$. When $2 \leq \tau < 168$, the maximum Hamming correlation $H_m = 116$. Hence, $S$ is an optimal $(168,13,84,1,12)$ LHZ FHS set, and all the sequences in FHS set $S$ are cyclically distinct.

**Remark 4.** In this example, by choosing the identical basic sequence set $A$, $\omega$, $n$ and set $B$, we obtain a new optimal LHZ FHS set. It is easily to see that the family
size of LHZ FHS set in Example 2 is smaller than Example 1, but the Hamming correlation is steadier beyond low hit zone. Therefore, both constructions are expected to be used in different circumstances with different requirements.

5. Conclusion

In this paper, two constructions of optimal LHZ FHS set with a large family size were proposed based on interleaved techniques. By selecting diverse basic sequence sets, we can obtain many optimal LHZ FHS sets with new parameters, some of which include the existed known results. It is possible for meeting different requirements to construct LHZ FHS sets with flexible parameters by choosing distinct \( x \) and \( n \). Compared with other constructions, the LHZ FHS sets obtained by the two constructions not only satisfy Peng-Fan-Lee bound, but also have a larger family size. Additionally, all the sequences constructed in this paper are cyclically distinct. The presented LHZ FHS sets are expected to eliminate multiple-access interferences in quasi-synchronous time/frequency hopping code division multiple access systems.

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