We study a setting with a single type of resource and with several players, each associated with a single resource (of this type). Unavailability of these resources comes unexpectedly and with player-specific costs. Players can cooperate by reallocating the available resources to the ones that need the resources most and let those who suffer the least absorb all the costs. We address the cost savings allocation problem with concepts of cooperative game theory. In particular, we formulate a probabilistic resource pooling game and study them on various properties. We show that these games are not necessarily convex, do have non-empty cores, and are totally balanced. The latter two are shown via an interesting relationship with Böhm-Bawerk horse market games. Next, we present an intuitive class of allocation rules for which the resulting allocations are core members and study an allocation rule within this class of allocation rules with an appealing fairness property. Finally, we show that our results can be applied to a spare parts pooling situation.

**KEYWORDS**

allocation rules, Böhm-Bawerk horse market games, cooperative game, resource pooling, spare parts application
resource pooling (PRP) game. For this game, we assume that players are interested in minimizing expected costs (or maximizing expected cost savings) and cooperate by pooling their resources as follows: whenever there are \( k \) available resources remaining, the \( k \) players with highest unavailability costs are the ones that are allowed to make use of a resource only. We investigate PRP games on several interesting properties. First, we show that the game is not necessarily convex. Then, we show that the coalitional values of any PRP game can be recognized as a convex combination of coalitional values of Böhm-Bawerk horse market (BBHM) games (see, e.g., Núñez & Rafels, 2005). As a consequence, we can show our first main contribution: for every PRP game there exists at least one efficient allocation that cannot be improved upon by any coalition, that is, the core of this game is non-empty. It shows that reallocating the available resources to the ones that need the resources most pays off, that is, there is always an incentive for players to cooperate! In addition, we show that every subgame of a PRP game has a non-empty core as well, that is, PRP games are totally balanced. As our second main result, we present an intuitive class of allocation rules for which the resulting allocations are core members. In particular, within the class of allocation rules, we focus on the allocation rule constructed by combining midpoints of the cores of underlying BBHM games. The resulting allocations of this allocation rule coincide with the average of so-called extreme BBHM allocations. Such an allocation results from an allocation rule out of our class of allocation rules, where based on an order of the players, a player selects core allocations in its own best interest of BBHM games in which he is active, under the restriction that the players before him in the corresponding order recursively selected their best core allocations. As a third and final main result, we show that our results can be applied to a spare parts setting in the railway sector that inspired us in the first place. In this spare parts setting, several service providers face different penalty costs for spare parts stock-out. The service providers can cooperate by pooling their spare parts according to a policy that gives only the \( k \) players with highest penalty costs the right to demand for a spare part whenever there are \( k \) spare parts in stock. In the spare parts literature, such type of policy constitutes a so-called critical level policy, which typically performs well in terms of minimizing long-term average costs when penalty costs differ significantly. The game associated with such a spare parts setting, in which spare parts are pooled via a critical level policy, can be recognized as a PRP game.

In the last couple of years, there is an increasing interest in cooperative games driven by pooling of resources. Some examples are pooling of inventory in a retail setting (Sošić, 2006), pooling of emergency vehicles in the health care (Karsten, Slikker, & Van Houtum, 2015), pooling of technicians in the service industry (Anily & Haviv, 2010), and pooling of repair vans amongst contractors in a railway network (Schlicher, Slikker, & van Houtum, 2017). Next to these examples, there is a growing interest in spare parts pooling, which has started with the paper of Wong, Van Oudheusden, and Cattrysse (2007). In this work, they investigate a situation in which players maintain a large number of machines, each containing a critical component that fails once in a while. Due to the uncertain nature of these failures, players keep repairable spare parts in stock to prevent downtime of their machines. Players can cooperate by pooling their spare parts. Here, it is assumed that pooling is facilitated by (expensive) lateral transshipments. Wong et al. (2007) formulate a spare parts game and analyze four cost allocation rules in a numerical experiment. Karsten, Slikker, and Van Houtum (2012) investigate a quite similar spare parts game. However, they assume that pooling is facilitated by free and instantaneous lateral transshipments. Under this assumption, they are able to provide sufficient and necessary conditions for core non-emptiness. Karsten and Basten (2014) study a variant of the model of Karsten et al. (2012) with backordering of unmet demand. Under this assumption, they are able to derive sufficient and necessary conditions for core non-emptiness. In addition, they present an allocation rule with appealing properties. Recently, Guajardo and Rönqvist (2015) have investigated a spare parts game with several players having different target service levels. When collaborating, new target service levels are set and a jointly spare parts pool is formed. They provide a core element, under the assumption that the core is non-empty.

A major overlap in all these spare parts pooling games is the assumption of full pooling, which means that all players can demand from the joint pool of spare parts whenever needed. From Dekker, Hill, Kleijn, and Teunter (2002) and Ha (1997) we know that full pooling may be far from optimal in situations in which player-specific costs differ significantly. The optimal policy is then a critical level policy, which leads to a form of partial pooling. Notice that critical level policies are also used in practice (Kleijn & Dekker, 1999). In Section 5, in which we focus on a spare parts setting, we analyze a specific critical level policy, which may be suboptimal as well, but can outperform full pooling, for example, when the player-specific costs differ significantly. To the best of our knowledge, there exists no literature that investigates a cost savings allocation aspect in combination with spare parts pooling according to another (smart) pooling policy, in which, based on the player-specific costs and the remaining available resources, some players are allowed to make use of the joint pool of resources while others are not. In this article, we make a first step in this direction.

The remainder of this article is as follows. In Section 2, we describe probabilistic resource situations and introduce the associated probabilistic resource pooling games. In Section 3,
2 | MODEL

In this section, we introduce probabilistic resource situations and define the associated games, called probabilistic resource pooling games.

2.1 | Probabilistic resource situation

We consider an environment with a finite set \( N \subset \mathbb{N} \) of players, each associated with a single resource. For each \( S \subseteq N \), we define \( p_S \in [0, 1] \) as the probability that each player \( i \in S \) has a resource available and each player \( j \in N \setminus S \) has no resource available, with the natural restriction that \( \sum_{S \subseteq N} p_S = 1 \). Note, if resources behave independently from each other, we can use that \( p_S = \prod_{i \in S} p_i \prod_{j \in N \setminus S} (1 - p_j) \), where \( p_i \in [0, 1] \) represents the availability of the resource of player \( i \in N \). If player \( i \in N \) has no resource available, an unavailability cost \( u_i \in \mathbb{R}_+ \) per time unit is involved. Players are interested in minimizing expected costs. To analyze this setting, we define a probabilistic resource situation as a tuple \((N, p, u)\) with \( N \), \( p = (p_S)_{S \subseteq N} \), and \( u = (u_i)_{i \in N} \) as defined above. We assume, without loss of generality, that \( u_i \geq u_j \) if \( i, j \in N \) with \( i < j \).

For short, we use \( \Theta \) to refer to such a probabilistic resource situation and \( \Theta \) for the set of probabilistic resource situations.

2.2 | Probabilistic resource pooling games

Let \( \theta \in \Theta \) be probabilistic resource situation. Players can cooperate in the following way. Let \( T \subseteq N \) be a coalition and \( S \subseteq N \) the coalition of players that have an available resource initially, which occurs with probability \( p_S \). So, within coalition \( T \), each player \( i \in S \cap T \) has an available resource initially. These players will reallocate their resources to the \(|S \cap T|\) players in coalition \( T \) with highest unavailability costs. Note, sometimes there is no real reallocation of the resources as the players with the highest unavailability costs already have an available resource initially. In order to describe the corresponding expected costs per coalition formally, we need two definitions. For any coalition \( T \subseteq N \), we define bijection \( \sigma_T : T \to \{1, 2, \ldots, |T|\} \) with \( \sigma_T(i) = |\{1, 2, \ldots, i\} \cap T| \) for any \( i \in T \). So, \( \sigma_T(i) \) can be seen as the position of player \( i \in T \) in coalition \( T \) and (the inverse) function \( \sigma_T^{-1}(j) \) as the \( j \)-th player in coalition \( T \).

**Definition 1** For every probabilistic resource situation \( \theta \in \Theta \), the expected costs for coalition \( T \subseteq N \) are given by

\[
c^\theta(T) = \sum_{S \subseteq N} p_S \left[ \sum_{j=|S \cap T|+1}^{|T|} u_{\sigma_T^{-1}(j)} \right]
\]

for all \( T \subseteq N \). (1)

We illustrate the cost function by means of an example.

**Example 1** Let \( \theta \in \Theta \) be a probabilistic resource situation with \( N = \{1, 2, 3\} \), \( p_S = (\frac{3}{5})^{|S|} \cdot \left(\frac{2}{5}\right)^{|N \setminus S|} \) for all \( S \subseteq N \), and \( u = (5, 2, 1) \).

For instance, for coalition \( T = \{1, 2\} \) we have

\[
c^\theta(\{1, 2\}) = \frac{1}{27} (5 + 2) + \frac{2}{27} (2 + 2 + (5 + 2)) + \frac{4}{27} (0 + 2 + 2) + \frac{8}{27} (0) = \frac{1}{3}.
\]

Similarly, the expected costs of the other coalitions can be determined. In Table 1, the corresponding costs of all coalitions are depicted.

We proceed with associating a cost savings game to any probabilistic resource situation, which we call a probabilistic resource pooling game.

**Definition 2** For every probabilistic resource situation \( \theta \in \Theta \), the associated probabilistic resource pooling (PRP) cost savings game \((N, v^\theta)\) is defined by

\[
v^\theta(T) = \sum_{i \in T} c^\theta(\{i\}) - c^\theta(T) \text{ for all } T \subseteq N.
\]

Now, we give an explicit expression for the characteristic value function \( v^\theta \). Recall that the proofs of theorems and lemmas are relegated to the appendix.

**Lemma 1** For every probabilistic resource situation \( \theta \in \Theta \), it holds for all \( T \subseteq N \) that

\[
v^\theta(T) = \sum_{S \subseteq N} p_S \left( \sum_{\ell \in T \setminus S \cap T} u_{\ell} - \sum_{j=|S \cap T|+1}^{|T|} u_{\sigma_T^{-1}(j)} \right) \text{ for all } T \subseteq N.
\]

(2)
The part between the brackets of Equation (2) can be recognized as the cost savings between the situation with and without reallocatable resources. Note that the lemma implies that for each probabilistic resource situation \( \theta \in \Theta \) we have \( v^\theta(T) \geq 0 \) for all \( T \subseteq N \).

We conclude this section with an example of a PRP game.

**Example 2** Consider the situation of Example 1. For instance, the value of coalition \( T = \{1, 2\} \) can be obtained as follows

\[
v^\emptyset(\{1, 2\}) = c(\{1\}) + c(\{2\}) - c(\{1, 2\}) = \frac{2}{3} + \frac{2}{3} - \frac{2}{3} = \frac{2}{3}.
\]

In a similar way, all other values can be determined. The expected cost savings for every coalition of the PRP game are presented in Table 2.

### 3 | PROPERTIES OF PRP GAMES

In this section, we study several interesting properties that PRP games might satisfy. We start by considering convexity and subsequently investigate whether the core of each PRP game is non-empty. Besides, we show an interesting relationship between PRP games and Böhm-Bawerk horse market games.

#### 3.1 | Convexity

Convexity is a desirable property of cooperative games. If a cooperative game is convex, the Shapley value is a core member and as a consequence, the core is non-empty (Shapley, 1953). So, we first investigate whether PRP games are convex in general. The following example illustrates that this is not the case.

**Example 3** Consider the situation of Example 2. Observe that \( v^\emptyset(\{1, 2, 3\}) - v^\emptyset(\{1, 3\}) = 1 - \frac{2}{3} = \frac{1}{3} \), \( \frac{7}{9} = \frac{10}{12} < \frac{15}{27} = \frac{5}{9} = 0 = v^\emptyset(\{1, 2\}) - v^\emptyset(\{1\}) \) and we can conclude that the game is not convex.

Despite that PRP games are not convex in general, we can identify necessary and sufficient conditions for convexity for a subclass of PRP games. In this subclass of PRP games, the probabilities do not depend on the identities of the players, that is, \( p_S \) depends on the cardinality of \( S \) only. So, the situations in which all resources behave independently from each other, and all with the same probability, belong to this class of PRP games.

**Theorem 1** For every probabilistic resource situation \( \emptyset \in \Theta \) with \(|N| \geq 3\) and \( p_S = p_S' > 0 \) for all \( S, S' \subseteq N \) with \(|S| = |S'| \) the corresponding PRP game is convex if and only if

\[
u_i - 2u_j + u_k = 0 \text{ for all } i, j, k \in N \text{ with } i < j < k.
\]

We remark that within this subclass of PRP games with more than three players, the PRP game is convex if and only if all unavailability costs are equal, implying that the PRP game is additive.

#### 3.2 | Relationship with Böhm-Bawerk horse market games

A Böhm-Bawerk horse market (BBHM) situation is a two-sided market with homogenous goods, for example, horses. In this market, there are sellers that each have one good for sale and buyers that each want to buy one good. The finite set of sellers and buyers together is denoted by \( N \subseteq \mathbb{N} \). The set of sellers is denoted by \( S \subseteq N \) and the set of buyers is denoted by \( B = N \setminus S \). Each seller (or buyer) \( i \in N \) has a valuation \( w_i \) for its good. We assume, without loss of generality, that \( w_i \geq w_j \) if \( i, j \in N \) with \( i < j \). Now, we define such a BBHM situation as a tuple \((N, S, w)\) with \( N, S \) and \( w = (w_i)_{i \in N} \) as described above. For short, we use \( \gamma \) to refer to such a BBHM situation and \( \Gamma \) for the set of BBHM situations.

Sellers and buyers can cooperate by trading goods. For every \( \gamma \in \Gamma \), it holds for any buyer \( j \in B \) and any seller \( i \in S \) that if \( w_j < w_i \), no good will be traded between buyer \( j \) and seller \( i \), and if \( w_j \geq w_i \), buyer \( j \) and seller \( i \) can trade a good with a joint profit of \( w_j - w_i \geq 0 \). Now, let \( \mathcal{A}_{S,B} \) be defined as \( \mathcal{A}_{S,B} = (a_{ij})_{i \in S, j \in B} \), where \( a_{ij} = \max \{w_j - w_i, 0\} \). A matching between set \( S_1 \subseteq S \) and set \( B_1 \subseteq B \) is a subset of pairs \( \mu \subseteq S_1 \times B_1 \), where each seller (of \( S_1 \)) as well as each buyer (of \( B_1 \)) belongs to at most one pair in \( \mu \). Let \( \mathcal{M}(S_1, B_1) \) be the set of all such possible matchings. A matching \( \mu \in \mathcal{M}(S_1, B_1) \) is called optimal on \((B_1, S_1)\) if for all \( \mu' \in \mathcal{M}(B_1, S_1) \) it holds that \( \sum_{(i,j) \in \mu} a_{ij} \geq \sum_{(i,j) \in \mu'} a_{ij} \).

An optimal matching on \((B_1, S_1)\) always exists as the total number of possible matchings is finite. The joint profit of any coalition \( T \subseteq N \) is defined as the sum of the joint profits of

| \( T \) | \( \emptyset \) | \{1\} | \{2\} | \{3\} | \{1, 2\} | \{1, 3\} | \{2, 3\} | \{1, 2, 3\} |
|---|---|---|---|---|---|---|---|---|
| \( v^\emptyset(T) \) | 0 | 0 | 0 | 0 | \( \frac{2}{3} \) | \( \frac{8}{9} \) | \( \frac{5}{6} \) | 1 |
all pairs under an optimal matching \( \mu \in \mathcal{M}(S \cap T, B \cap T) \). Note, if \( T \) exists of buyers only or sellers only, then this joint profit equals zero. Now, we will associate a BBHM game to any BBHM situation.

**Definition 3** For any \( \gamma \in \Gamma \), the associated BBHM game \( (N, \nu^\gamma) \) is given by

\[
\nu^\gamma(T) = \max_{\mu \in \mathcal{M}(S \cap T, B \cap T)} \left\{ \sum_{i,j \in \mu} a_{ij} \right\} \quad \text{for all } T \subseteq N.
\]

BBHM games are well-studied (see, eg, Shapley & Shubik, 1971; Moulin, 1995; Núñez & Rafels, 2005). In particular, Shapley and Shubik (1971) showed that the core of a BBHM game consists of a line segment. In order to define this segment, we need to introduce some additional notation. Let \( z = \min |[S]|, |[B]| \) and \( l = \max \{i \in \{1, 2, \ldots, z\} | w_{\sigma_B^{-1}(i)} - w_{\sigma_B^{-1}(i+1)} \geq 0 \} \). So, in words, \( l \) indicates the number of sellers (or buyers) that trade a good. Those sellers (buyers) are called active sellers (buyers) and the remaining sellers (buyers) are called non-active sellers (buyers). For any BBHM situation \( (N, S, w) \in \Gamma \) with \( S \neq \emptyset, N \), that is, with at least one and at most \( |N| - 1 \) sellers, we introduce

\[
\delta = \max \left\{ w_{\sigma_B^{-1}(i)} - w_{\sigma_B^{-1}(i+1)} \right\}, \quad \text{and} \quad \delta = \min \left\{ w_{\sigma_B^{-1}(i)} - w_{\sigma_B^{-1}(i+1)} \right\},
\]

with \( w_{\sigma_B^{-1}(0)} = \infty \) and \( w_{\sigma_B^{-1}(|B|+1)} = -\infty \). For any BBHM situation \( \gamma \in \Gamma \) with \( S = \emptyset \), let \( \hat{\delta} = \delta = w_{\sigma_B^{-1}(1)} \) and for any BBHM situation \( \gamma \in \Gamma \) with \( S = N \), let \( \hat{\delta} = \delta = w_{\sigma_B^{-1}(|N|)} \).

We will refer to \( \delta \) and \( \delta \) as the extreme prices to trade a good. Now, for any \( \gamma \in \Gamma \), vector \( \mathcal{L}(\gamma, \alpha) \) defined by

\[
\mathcal{L}(\gamma, \alpha) = \begin{cases} 
\max \{ \delta + \alpha(\delta - \delta) - w_i, 0 \} & \text{if } i \in S \\
\max \{ w_i - (\delta + \alpha(\delta - \delta)), 0 \} & \text{if } i \in B
\end{cases}
\]

with \( \alpha \in [0, 1) \) for all \( i \in N \), is a core element of the corresponding BBHM game. Varying between \( \alpha = 0 \), that is, the buyer optimal core allocation, and \( \alpha = 1 \), that is, the seller optimal core allocation, describes the whole core segment.

**Theorem 2** (Shapley & Shubik, 1971) For every BBHM situation \( \gamma \in \Gamma \) the core of the associated BBHM game \( (N, \nu^\gamma) \) is non-empty. In particular, the core is given by

\[
\mathcal{C}(N, \nu^\gamma) = \left\{ \mathcal{L}(\gamma, \alpha) \in \mathbb{R}^N \mid \alpha \in [0, 1] \right\}.
\]

It turns out that PRP games can be related to BBHM games. The idea of this relationship is as follows. For each probabilistic resource situation \( \theta \in \Theta \) we introduce \( 2^{N} \) different BBHM situations, each having a unique set of sellers. Then, the value of each coalition of a PRP game can be recognized as a convex combination of the coalitional values of these BBHM games.

In order to show this relationship, we need to formalize these BBHM situations. Let \( \theta = (N, p, u) \in \Theta \) be a probabilistic resource situation. Then, for every \( S \subseteq N \) we define a corresponding BBHM situation \( \gamma_{\theta}^S = (N, S, w) \) with \( w = u \).

**Theorem 3** For every probabilistic resource situation \( \theta \in \Theta \), it holds that

\[
\nu^\theta = \sum_{S \subseteq N} p_S \nu^{S}. \tag{3}
\]

We illustrate the result of Theorem 3 in the following example.

**Example 4** Consider the situation of Example 3. The coalitional values of the BBHM games associated with all corresponding BBHM situations are depicted in Table 3. Note, in Table 3 we consider any \( i \in \{1, 2, 3\} \).

Using Table 3 and Theorem 3, it becomes relatively easy to calculate the coalitional values of a PRP game via the coalitional values of the corresponding BBHM games. For instance, for coalition \( T = \{1, 2\} \), we have

| \( S \) | \( \emptyset \) | \{1\} | \{2\} | \{1, 2\} | \{1, 3\} | \{2, 3\} | \{1, 2, 3\} |
|---|---|---|---|---|---|---|---|
| \( v^\emptyset(S) \) | 0 | 0 | 5 - 2 | 5 - 1 | 0 | 2 - 1 | 5 - 1 | 0 |
| \( v^\emptyset(S) \) | 0 | 0 | 5 - 2 | 5 | 0 | 5 - 2 | 0 | |
| \( v^\emptyset(S) \) | 0 | 0 | 5 - 1 | 0 | 0 | 5 - 1 | 0 | |
| \( v^\emptyset(S) \) | 0 | 0 | 2 - 1 | 0 | 2 - 1 | 0 | 0 | |
| \( v^\emptyset(S) \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
A CLASS OF ALLOCATION RULES

In this section, we introduce and analyze a class of allocation rules, which comes to mind intuitively, once one has identified the relationship between BBHM games and PRP games combined with the fact that the core of each BBHM game is non-empty. In particular, within this class of allocation rules, we focus on the allocation rule constructed by combining midpoints of the cores of underlying BBHM games. This rule satisfies an appealing fairness property, namely the coincidence with the average of so-called extreme BBHM allocations.

An allocation rule on probabilistic resource situations is a mapping \( f \) that assigns to any \( \theta = (N, p, u) \in \Theta \) a vector \( f(\theta) \in \mathbb{R}^N \). Recall from Section 3.2 that PRP games can be recognized as convex combinations of BBHM games. We will exploit this result to construct a class of allocation rules. Note that any core-element of a BBHM situation \( \gamma \in \Gamma \) can be described as \( L(\gamma, \alpha) \) for some \( \alpha \in [0, 1] \). Now, for any probabilistic resource situation \( \theta \in \Theta \) and all related BBHM situations, that is, all possible sets of sellers, we set \( \alpha \in [0, 1] \), that is, we split the (trading) profit between the active buyers and active sellers in such a way that it is a core member of the corresponding BBHM game. Then, we multiply these outcomes with the probability that these BBHM situations occur and finally we add these terms. As for every possible set of sellers \( S \subseteq N \) we select a parameter, we formulate a whole class of allocation rules.

**Definition 4** For all \( \hat{\alpha} \in [0, 1]^N \) allocation rule \( B^\hat{\alpha} \) assigns to any probabilistic resource situation \( \theta \in \Theta \) allocation

\[
B^\hat{\alpha}(\theta) = \sum_{S \subseteq N} p_S \left( L(\gamma^0_S, \hat{\alpha}_S) \right).
\]

(4)

We illustrate such an allocation rule in Example 5.

**Example 5** Consider the situation of Example 4. In Table 4, the extreme prices \( \delta \) and \( \bar{\delta} \) are presented for any \( \gamma^0_S \) with \( S \subseteq N \).

Let \( \hat{\alpha}_S = \frac{1}{2} \) for all \( S \subseteq N \). This fixes a core-element for all \( (N, v^{\delta}_S) \) with \( S \subseteq N \). In Table 5, all these core-elements are presented.
TABLE 5 Corresponding core elements for BBHM situations

| $S$ | $\emptyset$ | [1] | [2] | [3] | [1,2] | [1,3] | [2,3] | [1,2,3] |
|-----|------------|-----|-----|-----|------|------|------|-------|
| $\mathcal{L}_1(y_{[1,2]}^0, \frac{1}{2})$ | 0 | 0 | $1\frac{1}{2}$ | $1\frac{1}{2}$ | 0 | 0 | $3\frac{1}{2}$ | 0 |
| $\mathcal{L}_2(y_{[1,2]}^0, \frac{1}{2})$ | 0 | 0 | $1\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 0 | 0 |
| $\mathcal{L}_3(y_{[1,2]}^0, \frac{1}{2})$ | 0 | 0 | 0 | $2\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |

For example, given that $S = \{2\}$, the payoff of player 1 is

$$
\mathcal{L}_1\left(y_{[1,2]}^0, \frac{1}{2}\right) = \max \left\{ 5 - (2 + \frac{1}{2} \cdot (5 - 2)), 0 \right\} = 1\frac{1}{2}.
$$

Now, by combining terms, we obtain

$$
\mathcal{B}_1^\alpha(\theta) = \frac{1}{27}(0) + \frac{2}{27} \left( 0 + \frac{1}{2} + \frac{1}{2} \right) + \frac{4}{27} \left( 0 + 0 + \frac{3}{2} \right) + \frac{8}{27}(0) = \frac{20}{27}
$$

$$
\mathcal{B}_2^\alpha(\theta) = \frac{1}{27}(0) + \frac{2}{27} \left( 0 + \frac{1}{2} + 0 \right) + \frac{4}{27} \left( 0 + \frac{1}{2} + 0 \right) + \frac{8}{27}(0) = \frac{5}{27}
$$

$$
\mathcal{B}_3^\alpha(\theta) = \frac{1}{27}(0) + \frac{2}{27} \left( 0 + 0 + \frac{2}{2} \right) + \frac{4}{27} \left( 0 + \frac{1}{2} + \frac{1}{2} \right) + \frac{8}{27}(0) = \frac{9}{27}.
$$

It is easily checked that $(\frac{20}{27}, \frac{5}{27}, \frac{9}{27}) \in C(N, \nu^0)$.

The next result follows directly from Lemma 2, Theorem 2, and Theorem 3.

**Corollary 2** For every probabilistic resource situation $\theta \in \Theta$ it holds for any $\hat{\alpha} \in [0, 1]^{2N}$ that $\mathcal{B}^\alpha(\theta) \in C(N, \nu^0)$.

One may wonder whether every element of the core of a PRP game results from an allocation rule $\mathcal{B}^\alpha$ for some $\hat{\alpha} \in [0, 1]^{2N}$, that is, if $\{\mathcal{B}^\alpha[\hat{\alpha}] \in [0, 1]^{2N} \} = C(N, \nu^0)$. The following example shows that this is not the case in general.

**Example 6** Consider the situation of Example 5 and $x = (1\frac{1}{27}, \frac{6}{27}, 0) \in C(N, \nu^0)$. Let $\hat{\alpha} \in [0, 1]^{2N}$. Then, it holds that

$$
\mathcal{L}_1\left(y_{[1,2]}^0, \hat{\alpha}_{[3]}\right) = \hat{\delta} + \hat{\alpha}_{[3]}(\hat{\delta} - \delta) - \nu_{[3]} = 1 + 3 \cdot \hat{\alpha}_{[3]} > 0,
$$

as $\hat{\alpha}_{[3]} \in [0, 1]$. In addition, $\rho_{[3]} = \frac{2}{27} > 0$ and thus $\mathcal{B}_3^\alpha(\theta) > 0$.

We conclude that $x \neq (\mathcal{B}_3^\alpha(\theta))_{\theta \in N}$ for all $\hat{\alpha} \in [0, 1]^{2N}$. A graphical representation of $\{\mathcal{B}^\alpha[\hat{\alpha}] \in [0, 1]^{2N} \}$ and the core is represented in Figure 1.

Figure 1 shows that $\{\mathcal{B}^\alpha[\hat{\alpha}] \in [0, 1]^{2N} \}$ is a convex set and is spanned by six vectors only. (One can check easily that this set is convex by the linearity of $\mathcal{B}^\alpha$ in $\hat{\alpha}$. See also the first part of the proof of Theorem 7.) The six vectors turn out to be special allocations. For example, vector $(\frac{18}{27}, 0, \frac{16}{27})$ is the allocation where, based on order (3,1,2), a player selects core allocations in its own best interest of BBHM games in which he is active under the restriction that the players before him in the corresponding order recursively selected their related core allocations. For example, for order (3,1,2), first player 3 will select core allocations of BBHM games in which he is active under the restriction that the players before him in the corresponding order recursively selected their related core allocations. Subsequently player 1 has the right to select the core allocation of the remaining BBHM game $(N, \nu^{(2)})$. As player 3 is an active buyer in all of these BBHM games, player 3 will select three times the *seller optimal* core allocation. In a similar way, the other vectors can be obtained by considering the different orders of the player set. Note that for BBHM games without active players, and so no trading profit, there is no need to select a core allocation.

Now, we will describe these vectors formally. An order on $N$ is a bijection $\omega : \{1, 2, \ldots, |N|\} \rightarrow N$, where $\omega(j)$
Table 6  All extreme BBHM allocations

| $\omega$       | $\alpha_{[0]}^\omega$ | $\alpha_{[1]}^\omega$ | $\alpha_{[2]}^\omega$ | $\alpha_{[3]}^\omega$ | $\alpha_{[1,2]}^\omega$ | $\alpha_{[1,3]}^\omega$ | $\alpha_{[2,3]}^\omega$ | $\alpha_{[1,2,3]}^\omega$ |
|----------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| (1,2,3)        | $\frac{1}{2}$         | $\frac{1}{2}$         | 0                     | 0                     | $\frac{1}{2}$         | 0                     | 0                     | $\frac{1}{2}$         |
| (1,3,2)        | $\frac{1}{2}$         | $\frac{1}{2}$         | 0                     | 0                     | $\frac{1}{2}$         | 1                     | 0                     | $\frac{1}{2}$         |
| (2,1,3)        | $\frac{1}{2}$         | $\frac{1}{2}$         | 1                     | 0                     | $\frac{1}{2}$         | 0                     | 0                     | $\frac{1}{2}$         |
| (2,3,1)        | $\frac{1}{2}$         | $\frac{1}{2}$         | 1                     | 1                     | $\frac{1}{2}$         | 0                     | 1                     | $\frac{1}{2}$         |
| (3,1,2)        | $\frac{1}{2}$         | $\frac{1}{2}$         | 0                     | 1                     | $\frac{1}{2}$         | 1                     | 1                     | $\frac{1}{2}$         |
| (3,2,1)        | $\frac{1}{2}$         | $\frac{1}{2}$         | 1                     | 1                     | $\frac{1}{2}$         | 1                     | 1                     | $\frac{1}{2}$         |

Example 7  Consider the situation of Example 6. For order (2, 1, 3), the first active player in BBHM game $(N, v(\cdot))$ is a seller, namely player 2, and as a consequence $\alpha_{[2]}^{(2,1,3)} = 1$. In Table 6 all extreme BBHM allocations are presented.

Example 8  Consider the situation of Example 7. In Table 7 $\mathcal{B}^{\alpha}$ is represented for all $\omega \in \Omega$ as well as the average of these vectors. In addition, the Shapley value is given by $(\frac{27}{10}, \frac{27}{10}, \frac{27}{10})$ and the Alexia value by $(\frac{9}{27}, \frac{9}{27}, \frac{9}{27})$.

Note that the average of $\mathcal{B}^{\alpha}$ deviates from the Shapley value and the Alexia value. However, it coincides with the allocation of (the simple) allocation rule $\mathcal{B}^{\hat{\alpha}}$ with $\hat{\alpha} = \frac{1}{2}$ for all $S \subseteq N$. This is no coincidence.

Theorem 6  Let $\theta \in \Theta$ be a probabilistic resource situation and $\hat{\alpha} = \frac{1}{2}$ for all $S \subseteq N$. Then, it holds that

$$\mathcal{B}^{\hat{\alpha}}(\theta) = \frac{1}{n!} \sum_{\omega \in \Omega} \mathcal{B}^{\alpha}(\omega).$$

Recall that Figure 1 showed us that $\{\mathcal{B}^{\alpha} | \alpha \in [0,1]^N\}$ is spanned by the extreme BBHM allocations. This does not hold in general.

Theorem 7  For every probabilistic resource situation $\theta \in \Theta$ it holds that

$$\text{convex hull} \{\mathcal{B}^{\alpha} | \alpha \in [0,1]^N\} \subseteq \{\mathcal{B}^{\hat{\alpha}} | \hat{\alpha} \in [0,1]^N\}$$

and there exists a probabilistic resource situation $\theta \in \Theta$ for which this relation is strict.

We conclude this section by showing that the average of the allocations resulting from all allocation rules $\mathcal{B}^{\alpha}(\theta)$ with $\alpha \in [0,1]^N$ coincides with the average of the extreme BBHM allocations.

Table 7  $\mathcal{B}^{\alpha}$ for all $\omega \in \Omega$ and average

| $\omega$       | (1,2,3) | (1,3,2) | (2,1,3) | (2,3,1) | (3,1,2) | (3,2,1) | average |
|----------------|---------|---------|---------|---------|---------|---------|---------|
| $\mathcal{B}_{[0]}^{\alpha}(\theta)$ | $\frac{27}{10}$ | $\frac{27}{10}$ | $\frac{27}{10}$ | $\frac{27}{10}$ | $\frac{27}{10}$ | $\frac{27}{10}$ | $\frac{27}{10}$ |
| $\mathcal{B}_{[1]}^{\alpha}(\theta)$ | $\frac{4}{27}$ | 0 | $\frac{10}{27}$ | $\frac{10}{27}$ | 0 | $\frac{6}{27}$ | $\frac{5}{27}$ |
| $\mathcal{B}_{[2]}^{\alpha}(\theta)$ | $\frac{7}{27}$ | $\frac{7}{27}$ | $\frac{7}{27}$ | $\frac{7}{27}$ | $\frac{7}{27}$ | $\frac{7}{27}$ | $\frac{7}{27}$ |

indicates which player is in position $j$ and $\omega^{-1}(i)$ indicates the position of player $i$. (Note that order $\omega$ has an interpretation similar to $\sigma_{N^{-1}}$.) The set of all orders is denoted by $\Omega$. For any $S \subseteq N$ let $l_S$ be the number of active sellers (buyers) that trade a good in $\gamma(S)$. For any $S \subseteq N$ with at least one active seller in $\gamma(S)$, that is, with $l_S \geq 1$, let $N_S = \{\sigma_S^{-1}(i) | 1 \leq i \leq l_S \} \cup \{\sigma_S^{-1}(i) | s - l_S + 1 \leq i \leq s \}$ be the subset of $N$ with active sellers and buyers, and let $\omega(S) \in N_S$ such that $\omega^{-1}(\omega(S)) \leq \omega^{-1}(i)$ for all $i \in N_S$. So, in words, $\omega(S)$ is the first active player according to $\omega$.

Then, for any $S \subseteq N$ with $l_S \geq 1$ we define

$$\alpha^\omega_S = \begin{cases} 1 & \text{if } \omega(S) \in S \\ 0 & \text{if } \omega(S) \notin B. \end{cases}$$

Hence, if the first active player is a seller, he selects the seller optimal core allocation and if the first active player is a buyer, he selects the buyer optimal core allocation. For any $S \subseteq N$ for which $l_S = 0$, and thus $\mathcal{L}(\gamma(S), \alpha_S) = 0$ for all $i \in N$ and any $\alpha_S \in [0,1]$, we set (arbitrarily) $\alpha^\omega_S = \frac{1}{2}$. For any $\omega \in \Omega$ this fixes vector $\alpha^\omega = (\alpha^\omega_S)_{S \subseteq N}$. We call allocation $\mathcal{B}^{\alpha}$ an extreme BBHM allocation.
Theorem 8 Let \( \theta \in \Theta \) be a probabilistic resource situation and \( \hat{\alpha}_S = \frac{1}{2} \) for all \( S \subseteq N \). Then, it holds that
\[
\mathcal{B}_i^\theta (\theta) = \frac{1}{2^{|N|}} \sum_{\tau \in \{0,1\}^{|N|}} \mathcal{B}^\tau (\theta).
\]

5 | A SPARE PARTS APPLICATION

In the introduction of this article, we mentioned that the model under consideration may have many interpretations in real-life. In this section, we will discuss one of them, a spare parts situation, in more detail.

5.1 | Spare parts situation

We consider an environment with a finite set \( N \subseteq \mathbb{N} \) of service providers that keep spare parts on stock to prevent costly downtime of their machines. We limit ourselves to one critical component, that is, one stock-keeping unit, which is subject to failures. For each service provider \( i \in N \), it holds that a failure of a machine immediately leads to a demand for a spare part. This occurs according to a Poisson process with service provider independent rate \( \lambda \in \mathbb{R}_+ \). This assumption is realistic in situations in which service providers have similar amounts of high-tech machines that are used frequently. In addition, we assume that each service provider \( i \in N \) starts with one spare part in stock only. This assumption is reasonable in situations in which spare parts are (relatively) expensive. If a spare part is on hand when demand occurs, this demand is always satisfied and a replenishment order is placed immediately. The lead times of such orders are independent and identically distributed according to a general distribution function with service provider independent mean \( \mu^{-1} \in \mathbb{R}_+ \).

If no spare part is available when demand occurs, an external spare part is ordered immediately and the machine goes down until the failed component is replenished by the external spare part. The expected costs associated with the extra idleness of the machine, shipments, and so on, shortly called downtime costs, are \( d_i \in \mathbb{R}_+ \) for service provider \( i \). The service provider independent holding costs per spare part are \( h \in \mathbb{R}_+ \) per time unit and are paid over inventory on-hand only. Finally, we assume that each service provider \( i \in N \) is interested in its long-term average costs per time unit. To analyze this setting, we define a spare parts situation as a tuple \((N, \lambda, \mu, d, h)\) with \( N, \lambda, \mu, d = (d_i)_{i \in N} \) and \( h \) as defined above. For short, we use \( \psi \) to refer to a spare parts situation and \( \Psi \) for the set of spare parts situations.

5.2 | Spare parts pooling game

The service providers, from now on called players, can cooperate by pooling their spare parts. We assume that players can transship these spare parts instantaneously and at negligible costs. In many settings, such assumption is reasonable, as (1) removing a defect component of a high-tech machine may already take hours, which makes that transshipment will not affect the duration of the replacement and (2) transshipment costs are often negligible compared to (high) downtime costs. Next, we assume that each coalition \( T \subseteq N \) pools the spare parts according to a specific (smart) critical level policy, which can reduce long-term average costs significantly (Dekker et al., 2002). Under this specific policy, demand of player \( i \in T \) is filled as long as inventory on-hand is at least equal to critical level \( \sigma_T(i) \). So, the player with the highest downtime costs can satisfy demand as long as inventory on-hand is at least one, the player with the second highest downtime costs can satisfy demand as long as inventory on-hand is at least two, and so on. Moreover, we assume that after any satisfied demand, a replenishment order is placed immediately. Finally, we assume that if player \( i \in T \) faces demand, while inventory on-hand is below critical level \( \sigma_T(i) \), an external spare part is used and related downtime costs of \( d_i \) are incurred. As players are interested in the long-term average costs per time unit, we will determine the steady state probabilities of coalition \( T \) with \( i \in \{0, 1, 2, \ldots, |T|\} \) spare part(s) on stock. Let \( \pi(\{T\}, i) \) be defined as the steady state probability of coalition \( T \) with \( i \) spare part(s) on stock. The next lemma presents a closed-form description of these steady state probabilities.

Lemma 3 For every spare parts situation \( \psi \in \Psi \) the steady state probabilities are given for all \( T \subseteq N \) and all \( i \in \{0, 1, \ldots, |T|\} \) by
\[
\pi(\{T\}, i) = \binom{|T|}{i} \cdot \left( \frac{\mu}{\lambda + \mu} \right)^i \cdot \left( \frac{\lambda}{\lambda + \mu} \right)^{|T| - i}.
\]

Proof See Van Houtum and Kranenburg (2015, p. 74, eq. 4.1).

For coalition \( T \subseteq N \) the expected costs per time unit in state \( i \) are \( \pi(\{T\}, i) \cdot i \cdot h \) for holding spare parts and \( \pi(\{T\}, i) \cdot \lambda \cdot \sum_{j=i+1}^{|T|} d_{\sigma_T^{-1}(j)} \) for the related downtime costs. Summing over all \( |T| + 1 \) states results in the expected costs per time unit in steady state for coalition \( T \subseteq N \) under spare parts situation \( \psi \in \Psi \).

Definition 5 For every spare parts situation \( \psi \in \Psi \) the expected costs per time unit in steady state for any coalition \( T \subseteq N \) are denoted by
\[
\psi^\phi(T) = \sum_{i=0}^{|T|} \pi(\{T\}, i) \cdot \left( i \cdot h + \lambda \cdot \sum_{j=i+1}^{|T|} d_{\sigma_T^{-1}(j)} \right).
\]
Let $\mathcal{C} \subset \{1, 2, 3\}$ be a coalition of players with highest penalty costs and let $\mathcal{C}' \subset \{1, 2, 3\}$ be the set of players who suffer the least absorb all the costs. We addressed the cost savings allocation problem with concepts of cooperative game theory. In particular, we formulated a PRP game and showed that these games are not necessarily convex, do have non-empty cores, and are totally balanced. The latter two are shown via an interesting relationship with Böhm-Bawerk horse market games. Next, we presented an intuitive class of allocation rules for which the resulting allocation are core members and studied an allocation rule within this class of allocation rules with an appealing fairness property. Finally, we introduced a spare parts pooling setting (in the railway sector) that inspired us in the first place. In this spare parts pooling setting, several service providers face different penalty costs for spare parts stock-out. The service providers can cooperate by pooling their spare parts according to a policy that gives only the $k$ players with highest penalty costs the right to demand for a spare part whenever there are $k$ spare parts in stock. We showed that the game associated to such a spare parts situation can be recognized as a PRP game, and so, the core of such a spare parts pooling game is non-empty as well. This is an interesting result, as spare parts pooling games investigated so far, in which full pooling is applied, may have empty cores (see, eg, Karsten et al., 2012).

### 6 | CONCLUSIONS

We presented an environment with a single type of resource and with several players, each associated with a single resource (of this type). Unavailability of these resources comes unexpectedly and with player-specific costs. Players can cooperate by reallocating the available resources to the ones that need the resources most and let those who suffer the least absorb all the costs. We addressed the cost savings allocation problem with concepts of cooperative game theory. In particular, we formulated a PRP game and showed that these games are not necessarily convex, do have non-empty cores, and are totally balanced. The latter two are shown via an interesting relationship with Böhm-Bawerk horse market games. Next, we presented an intuitive class of allocation rules for which the resulting allocation are core members and studied an allocation rule within this class of allocation rules with an appealing fairness property. Finally, we introduced a spare parts pooling setting (in the railway sector) that inspired us in the first place. In this spare parts pooling setting, several service providers face different penalty costs for spare parts stock-out. The service providers can cooperate by pooling their spare parts according to a policy that gives only the $k$ players with highest penalty costs the right to demand for a spare part whenever there are $k$ spare parts in stock. We showed that the game associated to such a spare parts situation can be recognized as a PRP game, and so, the core of such a spare parts pooling game is non-empty as well. This is an interesting result, as spare parts pooling games investigated so far, in which full pooling is applied, may have empty cores.

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**Example 9** Consider spare parts situation $\psi = (N, \lambda, \mu, d, h) \in \Psi$ with $N = \{1, 2, 3\}$, $\lambda = 1, \mu = 2, d = (5, 2, 1)$, and $h = \frac{1}{7}$.

For example, for coalition $T = \{1, 2\}$ we obtain

$$c^\psi(\{1, 2\}) = \frac{1}{9} \left( 0 \cdot \frac{1}{2} + 1 \cdot (5 + 2) \right)$$

$$+ \frac{4}{9} \left( 1 \cdot \frac{1}{2} + 2 \cdot 1 \right) + \frac{4}{9} \left( 2 \cdot \frac{1}{2} + 1 \cdot 0 \right) = 2\frac{1}{3}.$$  

Similarly, the costs of the other coalitions can be determined. In Table 8, the corresponding costs for all coalitions are depicted.

We proceed with associating a spare parts pooling (cost savings) game to any spare parts situation.

**Definition 6** For every spare parts situation $\psi \in \Psi$, the associated spare parts pooling (SPP) cost savings game $(N, v^\psi)$ is defined by

$$v^\psi(T) = \sum_{i \in T} c(i) - c^\psi(T)$$

for all $T \subseteq N$.

**Example 10** Consider the situation of Example 9. The coalitional (cost savings) values of game $(N, v^\psi)$ are presented in Table 9 below.

Note that the coalitional values of Table 9 coincide with the coalitional values of the PRP game as presented in Example 2. The fact that there exists a PRP game with exactly the same coalitional values as the SPP game is non-coincide.

We will show that any SPP game can (in fact) be recognized as a PRP game. For this, we describe a new definition. Let $\psi = (N, \lambda, \mu, h, d) \in \Psi$ be a spare parts situation. Then, we formulate an associated PRP situation $\theta^\psi = (N, p, u) \in \Theta$ with $p_S = \left( \frac{\mu}{\mu + \lambda} \right)^{|S|} \cdot \left( \frac{\lambda}{\mu + \lambda} \right)^{|N| - |S|}$ for all $S \subseteq N$, and $u_i = \lambda \cdot d_i$ for all $i \in N$.

**Theorem 9** For every SPP situation $\psi \in \Psi$ it holds that $v^\psi = v^{\theta^\psi}$.

---

**Table 8** Corresponding costs per coalition

| $T$ | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1, 2\}$ | $\{1, 3\}$ | $\{2, 3\}$ | $\{1, 2, 3\}$ |
|-----|-------------|----------|----------|----------|------------|------------|------------|--------------|
| $c^\psi(T)$ | 0 | 2 | 1 | $\frac{2}{3}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{2}{3}$ | $\frac{1}{27}$ |

---

**Table 9** Cost savings per coalition

| $T$ | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1, 2\}$ | $\{1, 3\}$ | $\{2, 3\}$ | $\{1, 2, 3\}$ |
|-----|-------------|----------|----------|----------|------------|------------|------------|--------------|
| $v^\theta(T)$ | 0 | 0 | 0 | 0 | $\frac{2}{3}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{2}{3}$ | $\frac{1}{27}$ |
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APPENDIX A: PRELIMINARIES ON COOPERATIVE GAMES

In this appendix, we provide some basic elements of cooperative game theory. Consider a finite set of players \( N \subseteq \mathbb{N} \) and a function \( v : 2^N \rightarrow \mathbb{R} \) called a characteristic function, with \( v(\emptyset) = 0 \). The pair \((N, v)\) is called a cooperative game with transferable utility, shortly called game. A subset \( T \subseteq N \) is called a coalition and \( v(T) \) is the worth that coalition \( T \) can achieve by itself. The worth can be transferred freely among the players. The set \( N \) is called the grand coalition. For a given coalition \( T \subseteq N \), the subgame \((T, v_T)\) is the game with player set \( T \) and characteristic function \( v_T \) such that \( v_T(K) = v(K) \) for all \( K \subseteq T \). A game \((N, v)\) is called superadditive if the value of the union of two disjoint coalitions is more than or equal to the sum of the values of the disjoint coalitions, that is, \( v(S) + v(T) \leq v(S \cup T) \) for all \( S, T \subseteq N \) with \( S \cap T = \emptyset \). A game \((N, v)\) is called monotonic if the value of every coalition is at least the value of any of its subcoalitions, that is, \( v(S) \leq v(T) \) for all \( S, T \subseteq N \) with \( D \subseteq T \) and convex if the marginal contribution of any player to any coalition is less than his marginal contribution to a larger coalition, that is, \( v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \) for all \( i \in N \) and all \( S \subseteq T \subseteq N \setminus \{i\} \). An allocation for a game \((N, v)\) is a vector \( x \in \mathbb{R}^N \) describing the payoffs to the players, where player \( i \in N \) receives \( x_i \). An allocation is called efficient if \( \sum_{i \in N} x_i = v(N) \). This implies that all worth is divided among the players of the grand coalition \( N \). An allocation is called individually rational if \( x_i \geq v(\{i\}) \) for all \( i \in N \) and called stable if no group of players has an incentive to leave the grand coalition \( N \), that is, \( \sum_{i \in T} x_i \geq v(T) \) for all \( T \subseteq N \). The set of efficient and stable allocations of \((N, v)\), called the core of \((N, v)\), is denoted by \( \mathcal{C}(N, v) \). Following Bondareva (1963) and Shapley (1967), a game \((N, v)\) is called balanced
APPENDIX B: PROOFS

Proof of Lemma 1 Let \( \theta \in \Theta \) be a probabilistic resource situation and \( T \subseteq N \). Now, observe that

\[
v^\theta(T) = \sum_{i \in T} c^\theta(\{i\}) - c^\theta(T)
\]

\[
= \sum_{i \in T} \left( \sum_{S \subseteq N} p_S \left( \sum_{j = |S \cap \{i\}| + 1} u_{|S|}^{-1} \right) \right)
\]

\[
- \sum_{S \subseteq N} p_S \left( \sum_{j = |T \cap |S|| + 1} u_{|S|}^{-1} \right)
\]

\[
= \sum_{S \subseteq N} p_S \left( \sum_{i \in T \cap S} u_i - \sum_{j = |T \cap |S|| + 1} u_{|S|}^{-1} \right).
\]

The first and second equality hold by definition. The third equality holds by swapping the first and second summation and writing \( \sum_{S \subseteq N} p_S \) in front of all terms. In the fourth equality, we use that for each \( i \in T \cap S \) the third summation is empty and that \( d_{\theta(\{i\})} = d_i \) for all \( j \in T \setminus (T \cap S) \). This concludes the proof.

Proof of Theorem 1 We distinguish \( |N| = 3 \) and \( |N| \geq 4 \). Now, let \( \theta \in \Theta \) be a probabilistic resource situation with \( N = \{1, 2, 3\} \) and \( p_S = p_{S'} > 0 \) for all \( S, S' \subseteq N \) with \( |S| = |S'| \). Note, the same reasoning holds for any \( N \) with \( |N| = 3 \).

\( \Rightarrow \) Suppose that the game is convex. We will show that \( u_1 - 2u_2 + u_3 = 0 \). Let \( p_{\{1\}} = p_{\{2\}} = p_{\{3\}} = p_{S_1} > 0 \) and \( p_{\{1,2\}} = p_{\{1,3\}} = p_{\{2,3\}} = p_{S_2} > 0 \) for notational convenience. By convexity, it holds that

\[
v^\theta(\{1,2,3\}) - v^\theta(\{1,3\}) - (v^\theta(\{1,2\}) - v^\theta(\{1\})) \geq 0.
\]

By using Lemma 1, we obtain

\[
v^\theta(\{1,2,3\}) - v^\theta(\{1,3\}) - (v^\theta(\{1,2\}) - v^\theta(\{1\}))
\]

\[
= p_{S_1}((0 - 0 - 0 + 0) + (u_1 - u_2) - 0 - (u_1 - u_2))
\]

\[
+ p_{S_2}((0 - 0 - 0 + 0) + (u_2 - u_3) - 0 - 0 + 0)
\]

\[
+ (u_1 - u_2) - (u_1 - u_3) - (u_1 - u_2) - 0))
\]

\[
= p_{S_2}(-u_1 + 2u_2 - u_3).
\]

As \( p_{S_2} > 0 \) it should (thus) hold that

\[
u_1 - 2u_2 + u_3 \leq 0.
\]

Moreover, it holds that

\[
v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - (v^\theta(\{1,3\}) - v^\theta(\{3\})) \geq 0.
\]

By using Lemma 1, we obtain (after some rewriting)

\[
v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - (v^\theta(\{1,3\}) - v^\theta(\{3\}))
\]

\[
= p_{S_1}(u_1 - 2u_2 + u_3).
\]

As \( p_{S_1} > 0 \) it should (thus) hold that

\[
u_1 - 2u_2 + u_3 \geq 0.
\]

As (6) and (7) should hold both, we conclude that

\[
u_1 - 2u_2 + u_3 = 0.
\]

\( \Leftarrow \) Suppose that \( u_1 - 2u_2 + u_3 = 0 \). We will show that \( v^\theta(T \cup \{i\}) - v^\theta(T) - (v^\theta(S \cup \{i\}) - v^\theta(S)) \geq 0 \) for all \( S, T \subseteq \{1,2,3\} \setminus \{i\} \) and all \( i \in \{1,2,3\} \). Let \( p_{\{1\}} = p_{\{2\}} = p_{\{3\}} = p_{S_1} > 0 \) and \( p_{\{1,2\}} = p_{\{1,3\}} = p_{\{2,3\}} = p_{S_2} > 0 \) for notational convenience. From the only-if-part we know that

\[
v^\theta(\{1,2,3\}) - v^\theta(\{1,3\}) - (v^\theta(\{1,2\}) - v^\theta(\{1\})) = 0,
\]

\[
v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - (v^\theta(\{1,3\}) - v^\theta(\{3\})) = 0.
\]

By using Lemma 1, we obtain (after some rewriting)

\[
v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - (v^\theta(\{1,2\}) - v^\theta(\{2\}))
\]

\[
= p_{S_1}(u_1 - u_2) + p_{S_2}(u_2 - u_3) \geq 0,
\]

where the inequality holds as \( p_{S_1}, p_{S_2} > 0 \) and \( u_1 \geq u_2 \geq u_3 \). By Lemma 1, we also obtain (after some rewriting)

\[
v^\theta(\{1,2,3\}) - v^\theta(\{1,2\}) - (v^\theta(\{2,3\}) - v^\theta(\{2\}))
\]

\[
= p_{S_1}(u_1 - u_2) + p_{S_2}(u_2 - u_3) \geq 0,
\]
where the inequality holds as $p_{S_1}p_{S_2} > 0$ and $u_1 \geq u_2 \geq u_3$. By Lemma 1, we also obtain (after some rewriting)

\[
v^\theta((1, 2, 3)) - v^\theta((1, 2)) - (v^\theta((1, 3)) - v^\theta((1))) = -p_{S_2}(u_1 - 2u_2 + u_3) = 0,
\]

where the equality holds as $u_1 - 2u_2 + u_3 = 0$. By Lemma 1, we finally obtain (after some rewriting)

\[
v^\theta((1, 2, 3)) - v^\theta((1, 3)) - (v^\theta((2, 3)) - v^\theta((3))) = p_{S_2}(u_1 - 2u_2 + u_3) = 0,
\]

where the inequality holds as $u_1 - 2u_2 + u_3 = 0$.

Finally, let $i, j \in N$ with $i \neq j$. Then

\[
v^\theta([i, j]) - v^\theta((i)) - (v^\theta([j]) - v^\theta(\emptyset)) = v((i, j)) \geq 0,
\]

where the equality holds as $v^\theta((i)) = v^\theta((j)) = v^\theta(\emptyset) = 0$. The inequality holds as $v^\theta(T) \geq 0$ for all $T \subseteq N$. Hence, $v^\theta(T \cup \{i\}) - v^\theta(T) - (v^\theta(S \cup \{i\}) - v^\theta(S)) \geq 0$ for all $S, T \subseteq \{1, 2, 3\} \setminus \{i\}$ and all $i \in \{1, 2, 3\}$ and thus we can conclude that the game is convex.

Now, let $\theta \in \Theta$ be a probabilistic resource situation with $|N| \geq 4$.

(⇒) Suppose that the game is convex. As a consequence, any subgame with player set $N'$ where $N' \subset N$ and $|N'| = 3$ is convex as well. From this, we conclude by the first part of this proof that

\[
u_i - 2u_j + u_k = 0 \forall i, j, k \in N \text{ with } i < j < k.
\]

(⇐) Suppose that $u_i - 2u_j - u_k = 0$ for all $i, j, k \in N$ with $i < j < k$. Given that $u_{g^{-1}_N(i)} - 2 \cdot u_{g^{-1}_N(j)} - u_{g^{-1}_N(k)} = 0$ and $u_{g^{-1}_N(i)} - 2 \cdot u_{g^{-1}_N(j)} + u_{g^{-1}_N(k)} = 0$ for all $k \in \{4, 5, \ldots, |N|\}$, we can conclude that $u_{g^{-1}_N(2)} = u_{g^{-1}_N(4)} = \cdots = u_{g^{-1}_N(1)}$. From $u_{g^{-1}_N(1)} - 2 \cdot u_{g^{-1}_N(3)} + u_{g^{-1}_N(4)} = 0$ we can conclude that $u_{g^{-1}_N(1)} = u_{g^{-1}_N(2)}$ and from $u_{g^{-1}_N(1)} - 2 \cdot u_{g^{-1}_N(3)} + u_{g^{-1}_N(4)} = 0$ and $u_{g^{-1}_N(1)} - 2 \cdot u_{g^{-1}_N(3)} + u_{g^{-1}_N(4)} = 0$ we can conclude that $u_{g^{-1}_N(2)} = u_{g^{-1}_N(3)}$. Hence, for every possible solution, it should hold that $u_{g^{-1}_N(1)} = u_{g^{-1}_N(2)} = \cdots = u_{g^{-1}_N(1)}$. Now we will show that $(u_k)_{k \in N}$ with $u_i = z \in \mathbb{R}_+$ for all $i \in N$ is a feasible solution. Let $i, j, k \in N$ with $i < j < k$. Then, $u_i - 2u_j + u_k = z - 2z + z = 0$. Hence, $u_i = z \in \mathbb{R}_+$ is indeed feasible. For this feasible solution, it holds that all unavailability costs are all equal and thus no cost savings are obtained in the corresponding game, that is, $v^\theta(T) = 0$ for all $T \subseteq N$. We conclude that the game is additive and as a consequence the game is convex.

**Proof of Theorem 3** Let $\theta = (N, p, u) \in \Theta$ be a probabilistic resource situation, $S \subseteq N$, and $\gamma^\theta_S = (N, S, w)$ with $w = u$ be the corresponding BBHM situation. Let $T \subseteq N$. As $u_i \geq u_j$ for all $i, j \in N$ with $i \geq j$, there exists an optimal matching where the first $|S \cap T|$ players have a resource after trading. From this, we can conclude that

\[
v^\theta(T) = \sum_{i \in T} u_{g^{-1}_T(i)} - \sum_{i \in S \cap T} u_i = \sum_{j = 1}^{|T|} u_{g^{-1}_T(j)} - \sum_{j = |S \cap T| + 1}^{|T|} u_{g^{-1}_T(j)}.
\]

The second equality holds by adding zero. In the last equality, we use that $\sum_{j = 1}^{|T|} u_{g^{-1}_T(j)} = \sum_{j \in T} u_j$ and combine the first and second summation of the second equality.

Let $T \subseteq N$. Now, observe that

\[
v^\theta(T) = \sum_{S \subseteq N} p_S \left( \sum_{i \in T \setminus (S \cap T)} u_i - \sum_{j = |S \cap T| + 1}^{|T|} u_{g^{-1}_T(j)} \right) = \sum_{S \subseteq N} p_S \left( v^\theta(S \cap T) \right),
\]

where the first equality holds by Lemma 1 and second one by Equation (8). This concludes the proof.

**Proof of Theorem 4** Let $\theta = (N, p, u) \in \Theta$ be a probabilistic resource situation and $(N, v^\theta)$ be the associated PRP game. For each coalition $S \subseteq N$ it holds, based on Theorem 2, that $\mathcal{C}(N, v^{\theta_S}) \neq \emptyset$. Then, by Theorem 3 and Lemma 2 with $v^\theta = v^{\theta_S}$ for all $S \subseteq N$, $\alpha_S = p_S$ for all $S \subseteq N$ and $z = v^\theta$, it follows that $\mathcal{C}(N, v^\theta) \neq \emptyset$.

**Proof of Theorem 5** Let $\theta \in \Theta$ be probabilistic resource situation and $(N, v^\theta)$ be the associated PRP game. Let $M \subseteq N$ and $(M, v^\theta_M)$
with \( v^\emptyset_T(T) = v^T(T) \) for all \( T \subseteq M \) be a sub-
game. Let \( \theta' = (N', p', u') \in \Theta \) be a probabilistic 
resource situation with \( N' = M, p' = (p'_{T_1})_{T_1 \subseteq M} \)
with \( p'_{T_1} = \sum_{T_2 \subseteq M \setminus P \cap \cup T_2} \) for all \( T_1 \subseteq M, \)
\( u'_i = u_i \) for all \( i \in M \) and \( (N', v^\emptyset) \) be the 
associated PRP game. Now, let \( T \subseteq M \). Then, observe
that

\[
v^\emptyset_M(T) = \sum_{T \subseteq N} p_S \left( v^\emptyset_{T \cap \cup T_2} \right)
= \sum_{T \subseteq N} \sum_{T_2 \subseteq M \setminus P \cap \cup T_2} \left( v^\emptyset_{T \cap \cup T_2} \right)
= \sum_{T \subseteq M} \sum_{T_2 \subseteq M \setminus P \cap \cup T_2} \left( v^\emptyset_{T \cap \cup T_2} \right)
= \sum_{T \subseteq M} p'_{T_1} \left( v^\emptyset_{T \cap \cup T_2} \right)
= v^\emptyset(T).
\]

The first and last equality holds by definition. The second equality holds by some
rewriting. In the third equality, we use that \( v^\emptyset_{T \cap \cup T_2} = v^\emptyset_{T_1} \)
for all \( T \subseteq M \), which can be concluded from
Definition 3 easily. The fourth equality holds as
\( v^\emptyset_{T \cap \cup T_2} = v^\emptyset_{T_1} \) for all \( T \subseteq M = (N') \)
and \( p'_{T_1} = \sum_{T_2 \subseteq M \setminus P \cap \cup T_2} \).

Hence, \( v^\emptyset_M(T) = v^\emptyset(T) \) for all \( T \subseteq M \).

Proof of Theorem 6
First, observe that for every \( S \subseteq N \) with \( l_S \geq 1 \), that is, with at least 
one trading couple, the number of orders \( \omega \in \Omega \)
for which an active seller selects \( \alpha^w \) equals the number of orders \( \omega \in \Omega \) for which an active
buyer selects \( \alpha^w = 0 \) as we consider all possible
orders of the player set. As \( |\Omega| = n! \), the total
number of times an active seller (buyer)
selects is \( n!/2 \). Secondly, observe that for every
\( S \subseteq N \) with \( l_S = 0 \), that is, with no trading couple,
\( \mathcal{L}_i(y^w_S, \hat{\alpha}_S) = 0 \) for any \( \hat{\alpha}_S \) \( \in \{0, 1\} \)
and all \( i \in N \). Now, observe that

\[
\mathcal{R}^\emptyset(\Theta) = \sum_{S \subseteq N} p_S \left( \mathcal{L}_i(y^w_S, \frac{1}{2}) \right)
= \sum_{S \subseteq N} p_S \left( \frac{1}{2} \cdot \mathcal{L}(y^w_S, 0) + \frac{1}{2} \cdot \mathcal{L}(y^w_S, 1) \right)
= \frac{1}{n!} \sum_{S \subseteq N} p_S \left( \sum_{\omega \in \Omega} \mathcal{L}(y^w_S, \alpha^w) \right)
= \frac{1}{n!} \sum_{S \subseteq N} p_S \left( \sum_{\omega \in \Omega} \mathcal{L}(y^w_S, \alpha^w) \right)
= \frac{1}{n!} \sum_{S \subseteq N} p_S \left( \mathcal{R}^\emptyset(\Theta) \right).
\]

The first and last equality hold by definition. The second equality holds as
\( \mathcal{L}(y^w_S, \frac{1}{2}) = \frac{1}{2} \cdot \mathcal{L}(y^w_S, 0) + \frac{1}{2} \cdot \mathcal{L}(y^w_S, 1) \) for any \( S \subseteq N \). The
third and fifth equality hold by some rewriting. The
fourth equality is a result of the description
given at the start of the proof. This concludes
the proof.

Proof of Theorem 7
Let \( \theta \in \Theta \) be a probabilistic resource situation. First we will show that
\( \text{convexhull} \{ \mathcal{R}^\emptyset(\Theta) \} \subseteq \{ \mathcal{R}^\emptyset(\hat{\alpha}) \} \)
for all \( \theta \in \Theta \) and subsequently that there
exists a probabilistic resource situation \( \hat{\theta} \in \Theta \)
for which it holds that \( \{ \mathcal{R}^\emptyset(\hat{\theta}) \} \subseteq \{ \mathcal{R}^\emptyset(\Theta) \} \).

Let \( \theta \in \Theta, \hat{\alpha}, \hat{\alpha}' \in \{0, 1\}^N \) and \( \beta \in [0, 1] \).
Then observe that

\[
\beta \mathcal{R}^\emptyset + (1 - \beta) \mathcal{R}^\emptyset'
= \beta \sum_{S \subseteq N} p_S \left( \mathcal{L}(y^w_S, \hat{\alpha}_S) \right)
+ (1 - \beta) \sum_{S \subseteq N} p_S \left( \mathcal{L}(y^w_S, \hat{\alpha}'_S) \right)
= \sum_{S \subseteq N} p_S \left( \mathcal{L}(y^w_S, \beta \hat{\alpha}_S + (1 - \beta) \hat{\alpha}'_S) \right)
= \mathcal{R}^\emptyset(\beta \hat{\alpha} + (1 - \beta) \hat{\alpha}').
\]

The second equality holds as \( \mathcal{L}(y^w_S, \alpha) \) is linear
in \( \alpha \) for all \( S \subseteq N \). We conclude that
\( \{ \mathcal{R}^\emptyset(\Theta) \} \subseteq \{ \mathcal{R}^\emptyset(\hat{\alpha}) \} \) is a convex set. Moreover, observe that for all \( \omega \in \Omega \) there
exists an \( \hat{\alpha} \in [0, 1]^{N} \) such that \( \alpha^w = \hat{\alpha} \). As a consequence, we
can conclude that \( \text{convexhull} \{ \mathcal{R}^\emptyset(\Theta) \} \subseteq \{ \mathcal{R}^\emptyset(\hat{\alpha}) \} \).
Proof of Theorem 8

Consider set \( \{0, 1\}^N \) and let \( S \subseteq N \). The number of \( \tau \in \{0, 1\}^N \) for which \( \tau_S = 1 \) coincides with the number of \( \tau \in \{0, 1\}^N \) for which \( \tau_S = 0 \). As the total number of unique \( \tau \in \{0, 1\}^N \) is given by \( 2^{|S|^N} \), we conclude that the total number of \( \tau \in \{0, 1\}^N \) for which \( \tau_S = 1 \) (\( \tau_S = 0 \)) equals \( 2^{|S|^N} \).

Now, observe that

\[
B^2(0) = \sum_{S \subseteq N} p_S \left( \mathcal{L} \left( \gamma^{0}, \frac{1}{2} \right) \right)
\]

\[
= \sum_{S \subseteq N} p_S \left( \frac{1}{2} \cdot \mathcal{L}(\gamma^{0}, 0) + \frac{1}{2} \cdot \mathcal{L}(\gamma^{0}, 1) \right)
\]

\[
= \frac{1}{2^{|N|^N}} \sum_{S \subseteq N} p_S \left( \sum_{\tau \subseteq \{0,1\}^N} \mathcal{L}(\gamma^{0}, \tau_S) \right)
\]

\[
= \frac{1}{2^{|N|^N}} \sum_{\tau \subseteq \{0,1\}^N} \sum_{S \subseteq N} p_S \left( \mathcal{L}(\gamma^{0}, \tau_S) \right)
\]

The first and last equality hold by definition. The second equality holds as \( \mathcal{L}(\gamma^{0}, \frac{1}{2}) = \frac{1}{2} \cdot \mathcal{L}(\gamma^{0}, 0) + \frac{1}{2} \cdot \mathcal{L}(\gamma^{0}, 1) \) for all \( S \subseteq N \). The third and fifth equality hold by rewriting. The fourth equality is a result of the description given at the start of the proof. This concludes the proof.

Proof of Theorem 9

Proof Let \( \psi = (N, \lambda, \mu, d, h) \in \Psi \) be a spare parts situation and \( d^\theta \) be the corresponding probabilistic resource situation. Let \( T \subseteq N \). Now, observe that

\[
c^\theta(T) = \sum_{i=0}^{\left| T \right|} \left[ \pi(|T|, i) \cdot \left( i \cdot h + \lambda \cdot \sum_{j=i+1}^{\left| T \right|} d_{\sigma_t}^{-1}(j) \right) \right]
\]

\[
= h \cdot \sum_{i=0}^{\left| T \right|} \pi(|T|, i) \cdot i + \lambda \cdot \sum_{i=0}^{\left| T \right|} \sum_{j=i+1}^{\left| T \right|} \lambda \cdot d_{\sigma_t}^{-1}(j)
\]

\[
= h \cdot \sum_{i=0}^{\left| T \right|} \left( \frac{\mu}{\mu + \lambda} \cdot |T| \right) + \lambda \cdot \sum_{i=0}^{\left| T \right|} \left( \frac{\mu}{\mu + \lambda} \cdot i \right) \cdot \left( \frac{\lambda}{\mu + \lambda} \cdot \sum_{j=i+1}^{\left| T \right|} \left( \frac{\mu}{\mu + \lambda} \cdot j \right) \right)
\]

\[
= \frac{h \mu |T|}{\mu + \lambda} + \sum_{i=0}^{\left| T \right|} \left( \frac{\mu}{\mu + \lambda} \cdot i \right) \cdot \left( \frac{\lambda}{\mu + \lambda} \cdot \sum_{j=i+1}^{\left| T \right|} \left( \frac{\mu}{\mu + \lambda} \cdot j \right) \right)
\]

\[
\times \sum_{j=i+1}^{\left| T \right|} \left( \frac{\mu}{\mu + \lambda} \cdot j \right)
\]
\[
\frac{h\mu|T|}{\mu + \lambda} + \sum_{S \subseteq T} \left( \frac{\mu}{\mu + \lambda} \right)^{|S|} \lambda \left( \frac{\mu}{\mu + \lambda} \right)^{|T|-|S|} \times \sum_{j=|S'|+1}^{[T]} \lambda \cdot d_{\sigma^{-1}(j)}
\]

\[
= \frac{h\mu|T|}{\mu + \lambda} + \sum_{S \subseteq T} \sum_{S' \subseteq N \setminus T} \left( \frac{\mu}{\mu + \lambda} \right)^{|S|+|S'|} \lambda \cdot d_{\sigma^{-1}(j)}
\]

\[
= \frac{h\mu|T|}{\mu + \lambda} + \sum_{S \subseteq N} \left( \frac{\mu}{\mu + \lambda} \right)^{|S|} \lambda \cdot d_{\sigma^{-1}(j)}
\]

\[
= \frac{h\mu|T|}{\mu + \lambda} + \sum_{S \subseteq N} p_S \sum_{j=|S'|+1}^{[T]} \lambda \cdot d_{\sigma^{-1}(j)}.
\]

The first equality holds by definition. The second equality holds by some rewriting. The third equality holds as the term \(\frac{\mu}{\mu + \lambda}|T|\) can be recognized as the expectation of the binomial distribution formulated in Lemma 3. The fourth equality holds by Lemma 3. In the fifth equality, we rewrite the first summation, which is allowed as this new summation depends on the cardinality of \(S\) only. In the sixth equality, we multiply the first summation by \(1 = \sum_{S' \subseteq N \setminus T} \left( \frac{\mu}{\mu + \lambda} \right)^{|S'|} \lambda^{[N]-|S'|}\). In the seventh equality, we use that the first and second summation can be rewritten as one summation. In the last equality, we use that \(p_S = \left( \frac{\mu}{\mu + \lambda} \right)^{|S|}\).

\[
\sum_{i \in T} \lambda \cdot d_i = \frac{\lambda}{\mu + \lambda} \text{ for all } S \subseteq N, \text{ and } u_i = \lambda \cdot d_i \text{ for all } i \in N.
\]

Secondly, observe that

\[
v^\psi(T) = \sum_{i \in T} c^\psi(i) - c^\psi(T)
\]

\[
= \sum_{i \in T} \left[ \frac{h\mu}{\mu + \lambda} + \sum_{S \subseteq N} ps \sum_{j=|S'|+1}^{[T]} u_{\sigma^{-1}(j)} \right]
\]

\[
- \left( \frac{h\mu|T|}{\mu + \lambda} + \sum_{S \subseteq N} p_S \sum_{j=|S'|+1}^{[T]} u_{\sigma^{-1}(j)} \right)
\]

\[
= \sum_{i \in T} \sum_{S \subseteq N} p_S \sum_{j=|S'|+1}^{[T]} u_{\sigma^{-1}(j)} - \sum_{S \subseteq N} p_S \sum_{j=|S'|+1}^{[T]} u_{\sigma^{-1}(j)}
\]

\[
= \sum_{S \subseteq N} p_S \left[ \sum_{i \in T \setminus T \cap S} u_i - \sum_{j=|S'|+1}^{[T]} u_{\sigma^{-1}(j)} \right]
\]

\[
= v^\psi(T).
\]

The first equality holds by definition. In the second equality we use the result we obtained in the first part of this proof. In the third equality, we use that \(\sum_{i \in T} \frac{h\mu}{\mu + \lambda} = \frac{h\mu |T|}{\mu + \lambda}\). The fourth equality holds by swapping the first and second summation and by writing \(\sum_{S \subseteq N} p_S\) in front of all terms. In the fifth equality, we use that that third equality is empty for \(i \in T \cap S\) and use that \(u_{\sigma^{-1}(1)} = u_i\) for all \(i \in T \setminus (T \cap S)\). The last equality holds by definition.