Relativistic diffusive transport

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Abstract

We discuss transport equations resulting from relativistic diffusions in the proper time. We show that a solution of the transport equation can be obtained from the solution of the diffusion equation by means of an integration over the proper time. We study the stochastic processes solving the relativistic diffusion equation and the relativistic transport equation. We show that the relativistic transport equation for massive particles in the light-cone coordinates and for the massless particles in spatial momentum coordinates are related to the (generalized) Bessel diffusion which has an analytic solution. The solution describes a particle moving in a fixed direction whose frequency distribution is the Bessel process. An approach to an equilibrium in a moving frame is discussed. We formulate the equilibrating diffusion and transport processes in a Lorentz covariant way.

1 Introduction

There are many versions of the relativistic diffusion and of the relativistic transport (see [1][2][3] for reviews). However, if we assume that the diffusing particle does not change its mass, the diffusion evolves in the proper time and that it is generated by the second order Lorentz invariant differential operator, then its definition is unique as shown by Schay [4] and Dudley [5]. The theory can be generalized to describe a particle diffusing on a manifold (pseudo-Riemannian background metric) [6][7]. We have extended these models to a diffusion of a particle with spin [8]. We have shown [9] that in the models of the relativistic diffusion the energy and the angular momentum grow exponentially fast in the proper time. Such a growth is rather unphysical in a steady world around us. In refs.[10][9] we have studied a modification of the model by an addition of some drag terms (friction) resulting from the requirement that a definite equilibrium
state is achieved at a large time. In [11] we have shown that a relativistic diffusion equilibrating to the Bose-Einstein equilibrium is a linearized version of the Kompaneets equation [12] applied for a long time to a description of photons diffusing in an electron gas [13]. There are phenomena in astrophysics [13], [14], [15] as well as in particle physics [16], [17] which can be described by relativistic kinetic and diffusion equations. The general theory of relativistic kinetic equations has been developed in [18], [19]. Various forms of relativistic kinetic equations can be derived from relativistic quantum field theory [20], [21], [22]. These kinetic equations are applied to the quark-gluon plasma [17] and to the heavy ion collisions [16] often in the diffusion approximation.

In our earlier papers [10], [8] we have concentrated on the diffusion in the proper time. The proper time is a reasonable mathematical tool (see [23] for its use in relativistic dynamics) to keep explicit Lorentz invariance. However, we should be able to describe the relativistic phenomena also in the coordinate time (we call it the laboratory time). In relativistic dynamics there is an equation relating the proper time with the coordinate time and we can exchange one into the other. The probability density as a function of the proper time trajectories should be independent of the proper time. This requirement is equivalent to an evolution equation of the probability density in the coordinate time (the kinetic equation). It is natural to interpret an analogous requirement (the diffusion probability distribution defined on the Minkowski space-time is independent of the proper time) as the kinetic equation (the transport equation) in the stochastic case.

We can obtain the transport equation by a random time change as well. This is a stochastic analogue of the change of the proper time (associated with a moving particle) into a laboratory time. In this paper the random time change (which is a standard notion in the theory of diffusion processes [24]) is formulated as a calculational tool useful for analytic as well as numerical calculations. The method involves an integration over the proper time analogous to Feynman’s proper time formulation of quantum mechanics [25].

In the case of particles of moderate velocity (non-zero mass) the meaning of the proper time diffusion can be elaborated by means of an expansion of momenta in powers of \((mc)^{-1}\). In this paper we study in detail the ultrarelativistic case of massless particles (then the proper time is a formal affine parameter on the trajectory) and concentrated beams of massive particles when only light-cone momenta \(p_+\) are relevant. We perform probabilistic calculations of the integral over the proper time by an application of some results of Yor [26], [27]. It comes out that the solution of the transport equation obtained as an integral over the proper time from the solution of the relativistic diffusion is expressed by Bessel functions. This is not accidental as the transport equation coincides with the Bessel diffusion. We compare our probabilistic proper time method of the solution of the transport equation with the theory of Bessel diffusions.

In general, the relativistic diffusion will have no limit for a large proper time (no equilibrium). For the same reason the solution of the transport equation will
have no limit as the laboratory time tends to infinity. The diffusion equation must be supplemented by some drag terms describing a friction if the diffusion is to achieve an equilibrium. The equilibrium distribution depends on the velocity of the frame where it is observed. So, the equilibrium is a covariant but not an invariant notion. In this paper we formulate the dependence of the relativistic dynamics on the equilibration reference frame in a covariant way.

The plan of the paper is the following. In sec.2 we review the mathematical scheme of the relativistic diffusion theory in the proper time. In this section we also formulate and prove the basic theorem which allows to obtain a solution of the transport equation if the diffusion in the proper time is known. In sec.3 we discuss various coordinates which will be applied in subsequent sections. In sec.4 transport equations for relativistic diffusions with drifts leading to an equilibrium are discussed. In sec.5 the relativistic diffusion process as a function of the Brownian motion is obtained in an explicit form. In sec.6 a diffusion of massless particles is discussed. Sec.7 is the main part of this paper. We study a relation between the relativistic diffusion in the proper time and the transport equation in the ultrarelativistic case of massless particles and the light-cone variables for massive particles. We solve the transport equation by an application of the theorem of sec.2. These equations are solved also by another method. The result is that the diffusing particle has the momentum whose probability distribution is the Bessel process. In sec. 8 we discuss the frame dependence of the equilibrating diffusion process. We formulate the theory in terms of relativistic invariant variables. The results are summarized in the last section of this paper.

2 The general scheme of the relativistic diffusion and relativistic transport

Following [4][5][10] we consider an evolution in the proper time of points on the phase space \((x,p)\)

\[
\frac{dx^\mu}{d\tau} = \frac{1}{m} p^\mu, \quad (1)
\]

\[
\frac{dp^\mu}{d\tau} = F^\mu, \quad (2)
\]

preserving the mass-shell \(\mathcal{H}_+\)

\[
\eta^{\mu\nu} p_\nu p_\mu = p^0 p_0 = p^2 = m^2 c^2. \quad (3)
\]

In eq.(2) \(F\) can be a random force leading to a diffusive behaviour of a test particle described by coordinates \((x,p)\). From eq.(1) and (3) (for \(m > 0\)) it follows that \(\tau\) really has the meaning of the proper time. Then, the evolution of a scalar function of the trajectory is determined by the equation

\[
\partial_\tau \phi = \mathcal{G} \phi = (p^\mu \partial_\tau p^\mu + \mathcal{A}) \phi, \quad (4)
\]
where (in the case of a diffusion) $\mathcal{A}$ is an $O(3, 1)$ invariant second order differential operator; differentiation over space-time coordinates has an index $x$ whereas differentiation without an index concerns momenta. The probability density $\Phi$ evolves according to an adjoint equation

$$\partial_\tau \Phi_\tau = G^* \Phi_\tau = (-p^\mu \partial_\tau + \mathcal{A}^*) \Phi_\tau$$

resulting from $(x \in \mathbb{R}^4, p \in \mathbb{R}^3$; fourvectors have Greek indices and threevectors Latin indices)

$$\int dxdp \phi_\tau(x, p) \Phi(x, p) = \int dxdp \phi_\tau(x, p) \Phi_\tau(x, p).$$

$\mathcal{A}$ is the second order differential operator such that $\mathcal{A}1 = 0$ and the quadratic term is negatively definite. Then, there exists a diffusion process $(x_\tau, p_\tau)$ starting from $(x, p)$ at $\tau = 0$ such that

$$\phi_\tau = E[\phi(x_\tau, p_\tau)] \equiv \mathcal{K}_\tau \phi,$$

where

$$\mathcal{K}_\tau(x, p; x', p') = E[\delta(x_\tau(x, p) - x')\delta(p_\tau(x, p) - p')].$$

From eq.(6)

$$\Phi_\tau = \mathcal{K}_\tau^* \Phi \equiv \mathcal{K}_\tau \Phi,$$

where

$$\mathcal{K}_\tau^*(x, p; x', p') = E[\delta(x_\tau(x', p') - x)\delta(p_\tau(x', p') - p)].$$

The probability distribution $\Phi$ is independent of $\tau$ if it satisfies the transport equation

$$G^* \Phi = 0.$$

We define an equilibrium distribution $\Phi_E$ as an $x^0$ independent solution of the transport equation. Let

$$\Phi = \Psi \Phi_E.$$

Then, $\Psi$ satisfies a diffusion equation

$$\partial_\tau \Psi = \hat{G} \Psi,$$

where

$$\hat{G} = -p^\mu \partial_\tau^\mu + \hat{\mathcal{A}}$$

and $\hat{\mathcal{A}} = \Phi_E^{-1} \mathcal{A} \Phi_E$ is a second order differential operator closely related to $\mathcal{A}$. The solution of eq.(13) can be expressed by a stochastic process

$$\Psi_\tau = E[\Psi(\hat{x}_\tau, \hat{p}_\tau)].$$
Then, if $\Phi$ is a solution of the transport equation (11) then $\Psi$ is a solution of a diffusion equation (we set $t = \frac{\tau}{c} \geq 0$)

$$\partial_t \Psi = \frac{c}{\hat{p}_0} \nabla_x \Psi + \frac{c}{\hat{p}_0} \hat{A} \Psi. \quad (16)$$

A solution of the diffusion equation (16) can again be expressed by a diffusion process $(\hat{x}_t, \hat{p}_t)$

$$\Psi_t = E[\Psi(\hat{x}_t, \hat{p}_t)]. \quad (17)$$

We can establish a relation between the solutions of the diffusion equations (13) and (16)

**Theorem**

Let $(\hat{x}_\tau, \hat{p}_\tau)$ be the diffusion process (15) solving the proper time diffusion equation (13) then the solution of the transport equation (16) with the initial condition $\Psi(x, p)$ reads

$$\Psi_t(x, p) = \frac{1}{mc} \int_0^\infty d\tau E\left[ \Psi(\hat{x}_\tau, \hat{p}_\tau) \delta(\tau) \delta\left(t - \frac{1}{mc} \int_0^\tau ds\hat{p}_0(s)\right) \right] \quad (18)$$

where

$$\hat{p}_0(s) = \sqrt{m^2c^2 + \hat{p}_s^2}.$$

**Proof:** Let us first check the initial condition. For small $t$ the proper time $\tau$ is also small and $\int_0^\tau \hat{p}_0(s) ds \simeq p_0 \tau$, then $x_\tau \simeq x$ and $p_\tau \simeq p$. An integration over $\tau$ gives $\Psi(x, p)$ when $t \to 0$. Next, for a derivation of the diffusion equation (16) we apply the formula $\delta(f(\tau)) = |f'|^{-1} \delta(\tau - \tau(t))$ in order to express the time $\tau$ in the diffusion process $(x_\tau, p_\tau)$ by $t$. Then, $|f'|^{-1} = p_0(\tau)^{-1}$ cancels the $p_0$ term in eq.(18). There remains a function $\Psi$ of the process $(x_\tau, p_\tau)$ at the time $\tau(t)$ in eq.(18). This is the random time change well-known from the theory of diffusion processes. It is proved in [24] that after the random change of time eq.(16) is satisfied (we shall still discuss the random time change in sec.7, eqs.(76)-(79)).

We shall apply the formula (18) for an explicit calculation of expectation values. Although in general the formula (18) may be difficult to use for an analytic treatment it still can be very useful for numerical calculations.

We can express the solution of the transport equation (16) in terms of a kernel $\tilde{K}_t$

$$\Psi_t = \tilde{K}_t \Psi. \quad (19)$$

Then, from eqs.(17)-(18) the kernel can be expressed in the form

$$\tilde{K}_t(x, p; x', p') = \frac{1}{mc} \int_0^\infty d\tau E[\delta(\hat{x}_\tau(x, p) - x')\delta(\hat{p}_\tau(p) - p')]\hat{p}_0(\tau)\delta(t - \frac{1}{mc} \int_0^\tau ds\hat{p}_0(s))]$$

$$= E[\delta(\hat{x}_t(x, p) - x')\delta(\hat{p}_t(p) - p')]. \quad (20)$$
In this section we have applied a notation suggesting the choice of spatial momenta \( p \) as coordinates on the mass-shell (3). We shall apply some other coordinates. The definition of the adjoint in eq.(6) is with respect to the Lebesgue measure which is not relativistic invariant. A density factor ensuring the invariance is contained in \( \Phi \). In the next sections we shall apply light-cone coordinates. Then, the evolution in \( x_0 \) is replaced by an evolution in \( x_\perp = x^0 - x^3 \) but the integration in eq.(6) is still with respect to a Lebesgue measure.

3 Various coordinate systems

The relativistic diffusion is generated by the Laplace-Beltrami operator on the mass-shell \( p^2 = m^2c^2 \). If we choose the spatial momenta \( p \) as coordinates on the mass-shell then the diffusion generator reads

\[
2\gamma^{-2}A = \Delta_H = (\delta_{jk} + m^{-2}c^{-2}p_jp_k)\frac{\partial^2}{\partial p_j \partial p_k} + 3m^{-2}c^{-2}p_k \frac{\partial}{\partial p_k}. \tag{21}
\]

We shall also use \( \kappa^2 = m^{-2}c^{-2}\gamma^2 \) as the diffusion constant.

The spatial momenta are convenient for a physical interpretation. However, for a derivation of explicit solutions some other coordinates are more useful. Let us consider the Poincare coordinates \((q_1, q_2, q_3)\) on \( \mathcal{H}_+ \) which are related to momenta \( p \) as follows

\[
p_3 + p_0 = \frac{mc}{q_3}, \tag{22}
\]

\[
p_3 - p_0 = -\frac{mc}{q_3}(q_1^2 + q_2^2 + q_3^2),
\]

\[
p_1 = \frac{mcq_1}{q_3}, \tag{23}
\]

\[
p_2 = \frac{mcq_2}{q_3},
\]

where \( q_3 \geq 0 \). Then, the metric is

\[
ds^2 = (mc)^2q_3^{-2}(dq_1^2 + dq_2^2 + dq_3^2) \tag{24}
\]

and

\[
(mc)^2\Delta_H = q_3^2(\partial_1^2 + \partial_2^2 + \partial_3^2) - q_3\partial_3. \tag{25}
\]

The Poincare coordinates are closely related to the light-cone coordinates \((p_+, p_a)\) (where \( a = 1, 2 \))

\[
p_+ = p_0 + p_3. \tag{26}
\]

We have \( q_a = p_ap_a^{-1} \) and \( q_3 = mc\gamma^{-1} \). Then

\[
\Delta_H = \partial_1^2 + \partial_2^2 + (mc)^{-2}p_+^2\partial_3^2 + (mc)^{-2}p_1^2\partial_1^2 + (mc)^{-2}p_2^2\partial_2^2 + 2(mc)^{-2}p_+p_1\partial_3\partial_1 + 2(mc)^{-2}p_+p_2\partial_3\partial_2 + 2(mc)^{-2}p_1p_2\partial_3 \tag{27}
\]

+ \( (mc)^{-2}3p_+\partial_3 + (mc)^{-2}3p_a\partial_a \).
where $\partial_a = \frac{\partial}{\partial p_a}$. The generator $G$ in the light-cone coordinates is

$$G = p_- \partial_+ + p_+ \partial_- - p_0 \partial^2_0 + \frac{\gamma^2}{2} \Delta_H.$$  

(28)

The time evolution is in $x_-$ instead of $x_0$. In the formulae of sec.2 we replace $p_0$ by $p_+$ and $x_0$ by $x_-$. The integration measure in eq.(6) is $dx_1 dx_2 dp_1 dp_2 dp_+ dp_- dp_0$.

4 Transport equations with friction

In order to achieve an equilibrium at a large time we add a friction term $K$ to the diffusion generator (21)

$$K = K_j \frac{\partial}{\partial p_j}.$$  

(29)

The transport equation (11) with friction reads

$$\frac{1}{m} \eta_{\mu \nu} p_\nu \frac{\partial}{\partial p_\mu} \Phi = \frac{\kappa^2}{2} m^2 c^2 \Phi - \frac{3}{2} \frac{\partial}{\partial p_k} p_k \Phi - \frac{\partial}{\partial p_k} K_k \Phi.$$  

(30)

Eq.(30) determines the drift $K$ if $\Phi_E$ is space-time independent

$$K_k = \kappa^2 m^2 c^2 \Phi_E^{-1} \left( \frac{1}{2} \frac{\partial}{\partial p_j} (\delta_{jk} + m^{-2} c^{-2} p_j p_k) \Phi - \frac{3}{2} \frac{\partial}{\partial p_k} p_k \Phi - \frac{\partial}{\partial p_k} K_k \Phi \right).$$  

(31)

Let us consider a class of solutions $\Phi_E$ depending solely on $p_0$. We write $\Phi_E$ in the form

$$\Phi_E = p_0^{-1} \exp(f(\beta c p_0)).$$  

(32)

The $p_0^{-1}$ factor ensures the Lorentz invariance of the measure $dp_0$. From eq.(31) we obtain

$$K_k = \frac{\kappa^2}{2} p_k \beta c p_0 f'(\beta c p_0).$$  

(33)

We choose $\Phi$ in the form

$$\Phi = p_0^{-1} \exp(f(c \beta p_0)) \Psi$$  

(34)

Then, the transport equation (16) reads

$$\partial_t \Psi = \frac{c}{p_0} (p \nabla_x + \hat{A}) \Psi$$  

(35)

where

$$\hat{A} = K_j \partial^j + A$$  

(36)

and $A$ is defined in eq.(21).
Explicitly
\[
\frac{\partial}{\partial x} \Psi = \frac{1}{p_0} p_j \frac{\partial}{\partial x_j} \Psi \\
+ \frac{\epsilon^2 m^2 \beta c^2}{p_0} (\delta_{jk} + m^{-2} c^{-2} p_j p_k) \frac{\partial^2}{\partial p_j \partial p_k} \Psi + \frac{mc^2}{p_0} (\frac{\epsilon}{2} + \frac{1}{2} c \beta c') (\beta c p_0) p_k \frac{\partial^2}{\partial p_k} \Psi.
\]
(37)

(the formulae (35)-(37) are not obvious but come out from calculations). The transport equation (without friction) in light-cone coordinates is
\[
p_+ \partial_+^2 \Psi + p_\alpha \partial_\alpha \Psi - \frac{\gamma^2}{2} \triangle_+ \Psi = 0
\]
(38)

where \(x_+ = x^0 + x^3\) and \(x_- = x^0 - x^3\). The adjoint of the operator (27) in eq.(38) is with respect to the Lebesgue measure \(dp_+ dp_1 dp_2\).

We look for an equilibrium solution of the transport equation in the form (it is not normalizable with respect to the integration over the transverse momenta \(p_\alpha\))
\[
\Phi_E = p_+^{-1} \exp(f(\beta c p_+)).
\]
(39)

We shall still discuss equilibrium distributions of the form (39) in sec.8. Here, we only point out that the light-cone coordinates are appropriate for particle beams moving in the direction of the third axis such that the remaining momenta \(p_-\) and \(p_\alpha\) are small. An analogue of eq.(31) gives
\[
K = K_0 \partial_+ + K_\alpha \partial_\alpha = \frac{1}{2} \beta c \kappa^2 p_+^2 f' \partial_+ + \kappa^2 p_+ \partial_+ - \frac{3}{2} \kappa^2 p_\alpha \partial_\alpha
\]
(40)

for the friction leading to the equilibrium (39). A particular (non-normalizable but relativistic invariant) distribution corresponds to \(f = 0\). We note that the change of coordinates \((p_1, p_2, p_3) \rightarrow (p_1, p_2, p_+)\) transforms the relativistic invariant measure \(dp_1 dp_2 dp_3 p_+^{-1}\) into
\[
dp_1 dp_2 dp_+ p_+^{-1}.
\]

With the friction (40) the diffusion generator (4) reads
\[
2\gamma^{-2} \mathcal{A} = \partial_+^2 + \partial_\alpha^2 + (mc)^{-2} p_+^2 \partial_+^2 + (mc)^{-2} p_\alpha^2 \partial_\alpha^2 \\
+ (mc)^{-2} p_+^2 \partial_\alpha^2 + 2(mc)^{-2} p_+ p_\alpha \partial_+ \partial_\alpha + 2(mc)^{-2} p_1 p_2 \partial_1 \partial_2
\]
(41)

and the diffusion operator \(\hat{\mathcal{A}}\) of eqs.(13)-(14) and eq.(36) in the evolution equation for the proper time probability distribution
\[
\partial_\tau \Psi = (-p_- \partial_-^2 - p_+ \partial_+^2 \Psi + p_\alpha \partial_\alpha \Psi + \hat{\mathcal{A}}) \Psi
\]
is
\[
2\gamma^{-2} \hat{\mathcal{A}} = \partial_+^2 + \partial_\alpha^2 + (mc)^{-2} p_+^2 \partial_+^2 + (mc)^{-2} p_\alpha^2 \partial_\alpha^2 \\
+ (mc)^{-2} p_+^2 \partial_\alpha^2 + 2(mc)^{-2} p_+ p_\alpha \partial_+ \partial_\alpha + 2(mc)^{-2} p_1 p_2 \partial_1 \partial_2
\]
(42)

+ (mc)^{-2} p_+ \partial_+ + (mc)^{-2} p_\alpha \partial_\alpha + \beta \frac{1}{mc^2} p_+^2 f' \partial_+ + 2 \beta \frac{1}{mc^2} p_+ p_\alpha f' \partial_\alpha.
5 Stochastic equations

In the Poincare coordinates the diffusion process is a solution of the linear stochastic differential equations \((\gamma^2 = m^2 c^2 \kappa^2)\)

\[
dq_a = \kappa q_3 db_a,
\]

\(a = 1, 2,\)

\[
dq_3 = -\frac{\kappa^2}{2} q_3 d\tau + \kappa q_3 db_3 = -\kappa^2 q_3 d\tau + \kappa q_3 \circ db_3, \tag{43}
\]

where Stratonovitch differentials are denoted by a circle and the Ito stochastic differentials without the circle (the notation is the same as in [29]). The Brownian motion appearing on the rhs of eqs.(43) is defined as the Gaussian process with the covariance

\[
E[b_a(\tau)b_c(s)] = \delta_{ac} \min(\tau, s). \tag{44}
\]

The solution of eq.(43) is

\[
q_3(\tau) = \exp(-\kappa^2 \tau + \kappa b_3(\tau))q_3
\]

and

\[
q_a(\tau) = q_a + \kappa \int_0^\tau q_3(s)db_a(s). \tag{46}
\]

In the light-cone coordinates

\[
p_\pm = p_0 \pm p_3
\]

we have

\[
dp_+ = \kappa^2 p_+ d\tau + \kappa p_+ \circ db_+ = \frac{3\kappa^2}{2} p_+ d\tau + \kappa p_+ db_+ + \gamma \circ db_a \tag{48}
\]

\[
dp_a = \kappa^2 p_a d\tau - \kappa p_a db_+ + \gamma db_a \tag{49}
\]

where \(a = 1, 2.\)

The solution of eqs.(49) is

\[
p_a(\tau) = \exp(\kappa^2 \tau - \kappa b_+(\tau))p_a + \gamma \int \exp(\kappa^2 s - \kappa b_+(s))db_a(s) \equiv e_a + \delta_a. \tag{50}
\]

The similarity between the solutions in light-cone coordinates and the Poincare coordinates is not accidental. It follows from the relation (23) between these coordinates.

A relativistic transport equation can be important in applications to high-energy plasma. In an electromagnetic field there is an additional drift term in the diffusion equation (4)

\[
K = \frac{e}{mc} F_{j\nu} p^\nu \partial^j, \tag{51}
\]
where \( F_{\mu\nu} \) denotes the electromagnetic field tensor. The stochastic equations are linear and can be solved if the only components of \( F \) are \( F_{12} = B \) and \( F_{30} = E \). In such a case the stochastic equations (48)-(49) read

\[
dp_1 = \kappa^2 p_1 \dtau + \alpha B p_2 \dtau - \kappa p_1 \db_+ + \gamma \db_1, \tag{52}
\]
\[
dp_2 = \kappa^2 p_2 \dtau - \alpha B p_1 \dtau - \kappa p_2 \db_+ + \gamma \db_2, \tag{53}
\]
\[
dp_+ = \kappa^2 p_+ \dtau + \alpha E p_+ \dtau + \kappa p_+ \db_+, \tag{54}
\]

where

\[
\alpha = \frac{e}{mc}.
\]

Eqs.(52)-(54) should be supplemented by equations determining the coordinates

\[
dx_a = m^{-1} p_a \dtau, \; dx_+ = m^{-1} p_- \dtau \quad \text{and} \quad dx_- = m^{-1} p_+ \dtau.
\]

A particular solution \( \Phi_E \) of the transport equation in an electric field (51) (\( F_{30} = E \), the remaining \( F_{\mu\nu} = 0 \) in eq.(51))

\[-p_+ \partial_+^x \Phi - p_- \partial_-^x \Phi + p_a \partial_a^x \Phi + \mathcal{A}^* \Phi - \alpha E \partial_+ p_+ \Phi = 0,
\]

where \( \mathcal{A}^* \) is the adjoint of \( \mathcal{A} \) (eq.(41) with \( f = 0 \)), is

\[
\Phi_E = p_+^{-1}. \tag{55}
\]

Let \( \Phi = p_+^{-1} \Psi \) then the proper time diffusion equation for \( \Psi \) takes the form

\[
\partial_\tau \Psi = ( -p_+ \partial_+^x \Phi - p_- \partial_-^x \Phi + p_a \partial_a^x \Phi + \hat{\mathcal{A}} \Phi + \alpha E p_+ \partial_+ \Phi ) \Psi.
\]

The transport equation reads

\[
\partial_\tau^x \Psi = -p_+^{-1} p_- \partial_-^x \Psi + p_+^{-1} p_a \partial_a^x \Psi + p_+^{-1} \hat{\mathcal{A}} \Psi + \alpha E \partial_+ \Psi, \tag{56}
\]

where \( \hat{\mathcal{A}} \) is defined in eq.(42) with \( f = 0 \). In the light-cone coordinates the stochastic equation for the process \( \hat{p}_+ \) of eq.(15) in an electric field \( E \) (but without friction) is

\[
d\hat{p}_+ = \frac{1}{2} \kappa^2 \hat{p}_+ \dtau + \alpha E \hat{p}_+ \dtau + \kappa \hat{p}_+ \db_+.
\]

It has the solution

\[
\hat{p}_+(\tau) = \exp(\alpha E \tau + \kappa b_+ (\tau)). \tag{58}
\]

If we have a friction leading to the Jüttner equilibrium distribution (without an electromagnetic field and \( f' = -1 \)) then the stochastic equation for \( \hat{p}_+ \) is

\[
d\hat{p}_+ = \frac{1}{2} \kappa^2 \hat{p}_+ \dtau - \frac{1}{2} \beta c \kappa^2 \hat{p}_+^2 \dtau + \kappa \hat{p}_+ \db_+.
\]

We discuss its solutions in sec.7.
6 Massless particles

There is a substantial simplification of stochastic equations if $m = 0$. In order to obtain the limit of zero mass we let $\delta_{jk} \to 0$ in eq. (21). The diffusion generator in the limit $m \to 0$ reads

$$\triangle H = p_j p_k \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} + 3 p_k \frac{\partial}{\partial p_k}. \quad (60)$$

The transport equation for a diffusion with a friction is (when $m = 0$ we use only the diffusion constant $\kappa^2$)

$$\eta^\mu_\nu p_\nu \frac{\partial x^\mu}{\partial x^\nu} \Phi = \kappa^2 \frac{\partial}{\partial p_j} p_j p_k \Phi - \frac{3 \kappa^2}{2} \frac{\partial}{\partial p_k} p_k \Phi - \frac{\partial}{\partial p_k} K_k \Phi. \quad (61)$$

If $\Phi_E$ depends only on momenta then from eq. (61)

$$K_k = \kappa^2 \Phi_E^{-1} \left( \frac{1}{2} \frac{\partial}{\partial p_j} p_j p_k - \frac{3}{2} p_k \right) \Phi_E. \quad (62)$$

Then, eq. (13) reads (in the massless case the proper time is just an affine time parameter without any physical meaning)

$$\partial_\tau \Psi_\tau = \frac{1}{2} \kappa^2 p_j p_k \partial^j \partial^k \Psi_\tau + 2 \kappa^2 p_j \partial^j \eta_\tau + \frac{1}{2} \kappa^2 p_j p_k (\partial^j \ln \Phi_E) \partial^k \Psi_\tau - p_\mu \partial_\mu \Psi_\tau. \quad (63)$$

The transport equation follows from eqs. (62)-(63). In an explicit form

$$\frac{\partial}{\partial x^\mu} \Psi = \frac{1}{p_0} p_j \frac{\partial}{\partial x^j} \Psi + \frac{\kappa^2}{p_0} p_j p_k \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} \Psi + 2 \kappa^2 p_j p_0^{-1} \partial^j \Psi + \frac{1}{2} \kappa^2 p_j p_k p_0^{-1} (\partial^j \ln \Phi_E) \partial^k \Psi. \quad (64)$$

We discuss in more detail the Jüttner equilibrium distribution [34]

$$\Phi_E = p_0^{-1} \exp(-c\beta p_0). \quad (65)$$

Then, from eq. (62) we obtain

$$K_k = \frac{-c\kappa^2}{2} \beta p_k p_0. \quad (66)$$

The stochastic equation for the diffusion (15) in the proper time with the friction (66) reads

$$d\hat{p}_j = \frac{3 \kappa^2}{2} \hat{p}_j d\tau - c\beta \kappa^2 \hat{p}_j |\hat{p}| d\tau + \kappa \hat{p}_j db. \quad (67)$$

From eq. (67) we obtain the stochastic equation for $\hat{p}_0 = |\hat{p}|$

$$d\hat{p}_0 = \frac{3 \kappa^2}{2} \hat{p}_0 d\tau - \frac{c\kappa^2}{2} \beta \hat{p}_0^{-2} d\tau + \kappa \hat{p}_0 db. \quad (68)$$
7 Solution of the transport equation

In this section we wish to apply the Theorem of sec.2 in order to derive a solution of the transport equation. Let us begin with the spatial momenta as coordinates on the mass-shell $\mathcal{H}_+$. The stochastic equation for the transport diffusion $(\tilde{x}_t, \tilde{p}_t)$ (17) of a massive particle solving eq.(37) in $p$ coordinates reads

$$d\tilde{p}_j = \frac{\bar{p}_j \kappa^2 m^2 c^2 \beta f}{2} dt + \frac{3 \kappa^2 mc}{2p_0} \tilde{p}_j dt + mc\sqrt{mc\tilde{p}_0} \frac{1}{2} \kappa dB_j, \quad (69)$$

$$d\tilde{x}_j = -c\tilde{p}_j \tilde{p}_0^{-1} dt, \quad (70)$$

Here we defined the square root of the metric (the second order terms in eq.(21))

$$e^n_j e^n_k = \delta_{jk} + (mc)^{-2} \tilde{p}_j \tilde{p}_k.$$

The process (69)-(70) gives the solution of the transport equation (37) in the form

$$\Psi_t(x, p) = E[\Psi(\tilde{x}_t(x, p), \tilde{p}_t(p))].$$

Unfortunately, the non-linear equations (69)-(70) are difficult to solve explicitly. We could also consider the proper time equations in these coordinates discussed extensively in [10]. Then, we could apply the Theorem of sec.2. However, such a method would be fruitful only in numerical calculations. Computing the expectation values (18) can be more efficient than solving eqs.(69)-(70) directly.

The light-cone coordinates are more useful for analytic solutions. The formula (18) for the solution of the transport equation in these coordinates takes the form

$$\Psi(x_-, x_+, x_a, p_+, p_a) = \frac{1}{m} \int_0^\infty d\tau E\left[\Psi(x_+ (\tau), \hat{p}_a (\tau)) \delta\left(x_- - \frac{1}{m} \int_0^\tau ds \hat{p}_+ (s) ds\right) \hat{p}_+ (\tau)\right], \quad (71)$$

where $\hat{p}$ is the stochastic process (15) generated by $\hat{A}$ of eq.(42). If there is no friction then the stochastic equations of the diffusion (42) are linear and can be solved explicitly (see eqs.(50) and (58)). Then, the expectation value of a function $h$ of $\hat{p}_a$ has a representation in terms of its Fourier transform

$$E[h(\hat{p}_1 (\tau), \hat{p}_2 (\tau))] = \int dk_1 dk_2 \tilde{h}(k_1, k_2) E[\exp(ik_a \hat{p}_a (\tau))]$$

$$= \int dk_1 dk_2 \tilde{h}(k_1, k_2) E\left[\exp(ik_a e_a (\tau)) \exp\left(-\frac{2}{\gamma^2} k_a k_a \int \exp(2\gamma^2 s + 2\gamma b_+(s)) ds\right)\right]$$

($e_a$ is defined in eq.(50)). The joint distribution law of $b$, $\int \exp b$, $\int \exp 2b$ can be explicitly calculated [30]. Hence, a solution of the transport equation (including the electromagnetic field (51)) can be obtained in a form of an integral kernel.
We consider here a simplified version of the Theorem of sec.2 when the initial condition $\Psi$ depends solely on $p_+$. Let us note that $p_+$ enters the generator of the diffusion with the factor $(mc)^{-1}$ (or with $\kappa = (mc)^{-1}\gamma$). Hence, it is irrelevant for moderate velocities. Our restriction to probabilities depending only on $p_+$ in fact applies to the ultrarelativistic limit when the transverse momenta can be neglected. In such a case the transport equation with the Jüttner friction (39) ($f' = -1$) but without the electric field reads

$$\partial_- \Psi = \frac{\kappa^2}{2} p_+ \partial^2_+ \Psi + \frac{\kappa^2}{2} \partial_+ \Psi - \frac{1}{2} c \beta \kappa^2 p_+ \partial_+ \Psi. \quad (72)$$

The transport equation with the electric field $E$ and without friction ($f = 0$ in eq.(39)) defined in eq.(56) can be written explicitly as

$$\partial_- \Psi = \frac{\kappa^2}{2} p_+ \partial^2_+ \Psi + \left( \frac{\kappa^2}{2} + \alpha E \right) \partial_+ \Psi. \quad (73)$$

Eq.(71) gives the solution of eqs. (72)-(73) in the form

$$\Psi(x_-, p_+) = m^{-1} \int_0^\infty d\tau E \left[ \Psi(p_+ (\tau, p_+)) \delta \left( x - \frac{1}{m} \int_0^\tau ds \partial_+ (s, p_+) \right) p_+ (\tau, p_+) \right]. \quad (74)$$

We can solve the diffusion equations (72)-(73) by means of the stochastic process (17). For eq.(72) the stochastic equation is

$$d\hat{p}_+ = \frac{\kappa^2}{2} dx_- - \frac{1}{2} \beta \kappa^2 \hat{p}_+ dx_- + \kappa \sqrt{\hat{p}_+} db. \quad (75)$$

whereas the stochastic process solving eq.(73) satisfies the equation

$$d\hat{p}_+ = \left( \frac{\kappa^2}{2} + \alpha E \right) dx_- + \kappa \sqrt{\hat{p}_+} db. \quad (76)$$

We shall discuss solutions of eqs.(72) and (73) together with the transport equation for massless particles because all these equations are related to the Bessel diffusion.

Before solving the equations let us explain the text-book method [24] of the random change of time in eq.(67) (then we show that the Theorem of sec.2 is its efficient realization). Let us treat the formula (1) for $x^0(\tau)$ as a definition of the proper time $\tau$

$$\tau = \int_0^{x^0} \left| \hat{p}_s \right|^{-1} ds \quad (77)$$

(here $\hat{p}_s$ is the solution of the proper time diffusion (67); $x^0(\tau)$ can be defined implicitly by (77)). We can see from eq.(77) that $\tau$ depends only on events earlier than $x^0$. As a consequence $\hat{p}_j(x^0) = p_j(\tau(x^0))$ is again a Markov process. Then, differentiating the momenta and coordinates according to the rules of the Ito calculus [29] we obtain the following Langevin equations (for mathematical
details of a random change of time see [24][29]; a random time change from $x^0$ to $\tau$ is discussed in [28])

$$dp_j(x^0) = \frac{3\kappa^2}{2} |p|^{-1} p_j dx^0 - \frac{\kappa^2}{2} p_j dx^0 + \kappa p_j |p|^{-\frac{4}{3}} db, \quad (78)$$

$$dx^j = -p_j |p(x^0)|^{-1} dx^0. \quad (79)$$

Let $\Psi(x, p)$ be an arbitrary function of $x$ and $p$ and $(x(x^0, x, p), p(x^0, p))$ the solution of eqs.(78)-(79) with the initial condition $(x, p)$ then

$$\Psi_t(x, p) = E[\Psi(x(x^0, x, p), p(x^0, p))] \quad (80)$$

is the solution of the transport equation

$$\frac{\partial}{\partial x^j} \Psi = \frac{1}{p_0} p_j \frac{\partial}{\partial x_j} \Psi + \frac{\kappa^2}{2} |p|^{-1} p_k \frac{\partial}{\partial p_k} \Psi - \frac{\kappa^2}{2} \beta c p_k \frac{\partial}{\partial p_k} \Psi + \frac{3\kappa^2}{2} p_0 p_k \frac{\partial}{\partial p_k} \Psi \quad (81)$$

with the initial condition $\Psi(x, p)$. We can prove that eq.(81) is satisfied by differentiation of eq.(80) and application of the rules of the Ito calculus [29] (or the well-known relation between Langevin equation and the diffusion equation).

We return to the diffusion in the proper time in order to perform the average over the random time $\tau(x^0)$ as prescribed by eq.(18) (this formula is treating the random change of time as a computational tool). From eqs.(67)-(68) it follows that

$$\hat{p}_\tau |\hat{p}_\tau|^{-1} \equiv n = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) = \text{const} \quad (82)$$

is time independent. Hence, according to eq.(18) the solution of the transport equation (64) takes the form

$$\Psi_t(x, |p|, n) = \frac{1}{mc} \int_0^\infty dt E\left[\Psi(x(\tau, x), |\hat{p}_\tau|, n) \delta\left(t - \frac{1}{mc} \int_0^\tau |\hat{p}_\tau| ds\right) |\hat{p}_\tau|\right]. \quad (83)$$

where

$$x(\tau, x) = x - \int_0^\tau ds |\hat{p}_\tau| n. \quad (84)$$

In order to calculate the expectation value (83) we apply the results of Yor [26][27] who obtained (using the Feynman-Kac formula for an exponential potential) the joint distribution of

$$B^{(\mu)}(\tau) = b_\tau + \mu \tau \quad (85)$$

and

$$A^{(\mu)}(\tau) = \int_0^\tau ds \exp(2B^{(\mu)}(s)). \quad (86)$$

We need some rescalings (using $a b(s) = b(a^2 s)$) in order to bring our formulae to the Yor's form. We write

$$\exp(\kappa^2 s + \kappa b(s)) = \exp\left(2\left(2\kappa^2 \frac{s}{4} + b\left(\kappa^2 \frac{s}{4}\right)\right)\right). \quad (87)$$
It follows that \( \mu = 2 \) in Yor's formula.

After such an rescaling an expectation value of a function of \( B_\tau = \kappa^2 \tau + \kappa b(\tau) \) and \( A(\tau) = \int_0^\tau ds \exp 2B_\tau \) is [26][27]

\[
E \left[ F \left( B(\tau), A(\tau) \right) \right] = \exp(-\frac{\tau \kappa^2}{2}) \int F(v, \frac{v}{\kappa^2}|p|) \exp(2v) \exp \left( -\frac{1}{2\kappa^2} (1 + \exp(2v)) \right) \theta(y^{-1} \exp(v), \frac{\kappa^2}{2}) y^{-1} dy dv
\]

where \( v \in \mathbb{R}, y \in \mathbb{R}^+ \)

\[
\theta(r, \tau) = r(2\pi \tau)^{-\frac{1}{2}} \exp \left( \frac{-\kappa^2}{2\tau} \right) \int_0^\infty \exp(-\frac{\kappa^2}{2\tau} - r \cosh(\xi)) \sinh(\xi) \sin(\frac{\kappa^2}{2\tau}) d\xi.
\]

The \( \tau \)-integral in the Theorem can be calculated by an application of another Yor's formula

\[
I_0^\infty d\tau \exp(-2\nu \tau) \theta(r, \tau) = I_{\sqrt{2\nu}}(r),
\]

where \( I_\alpha \) is the modified Bessel function of order \( \alpha \) [33]. Hence,

\[
\int_0^\infty d\tau E[F(B(\tau), A(\tau))] = \frac{\nu}{\sqrt{2\nu}} \int F(v, \frac{v}{\kappa^2}|p|) \exp(2v) \exp \left( -\frac{1}{2\kappa^2} (1 + \exp(2v)) \right) I_2(y^{-1} \exp(v)) y^{-1} dy dv
\]

We can apply the formula (91) only to the expectation values (18) of diffusions without friction (because only in this case the integrand (18) depends solely on \( A \) and \( B \)) corresponding to the Jüttner distribution with \( f = 0 \) (this is the Lorentz invariant but non-normalizable distribution \( dxdp\Phi_E = dxdp_0^{-1} \)).

Then, with the proper rescalings we have (we omit the index \( \mu = 2 \) in \( A \) and \( B \))

\[
p_\tau = n|p| \exp \left( 2B_\tau \kappa^4 \right),
\]

\[
x_\tau = x - \frac{4 n |p|}{\kappa^2} A(\frac{\tau \kappa^2}{4}).
\]

We can insert these formulae in eqs.(18) and (83). After the change of variables

\[
v = \frac{1}{2} \ln \left( \frac{r}{|p|} \right)
\]

we obtain from the Theorem (in eq.(18) the integrals over \( \delta \)-function can be performed directly and only the \( v \)-integral remains)

\[
\Psi_t(x, p) = t'^{-1} \int_0^\infty dr \frac{r}{|p|} \exp \left( -\frac{1}{2\sigma^2} (|p| + r) \right) I_2(t'^{-1}\sqrt{|p|r}) \Psi(x - nct, nr),
\]

where \( t' = \frac{\kappa^2}{2}t. \)
Calculations in the light-cone coordinates for a massive particle diffusing without a friction ($f = 0$ in eq.(39)) are similar. We obtain from eq.(58) that $\mu = 0$ for $E = 0$. Then, from eqs.(88)-(90) (we change variables $r = p_+ \exp(2v)$)

$$
\Psi_{t-}(p_+) = p_+ \int_0^\infty dr E \left[ \Psi \left( p_+ \exp(2B) \right) \exp(2B) \delta(t - \frac{1}{mc^2} p_+ A) \right]
$$

$$
= p_+ \int \exp(2v) \Psi(\exp(2v) p_+) I_0(\frac{\exp(v)}{y}) \delta(t - \frac{1}{mc^2} p_+ y)
$$

$$
\exp(-\frac{1}{2}\gamma(1+\exp(2v)))d\nu^{-1}dy = t_-^{-1} \int dr \Psi(r) \exp(-\frac{p_+ + r}{2t_-}) I_0(\frac{1}{t_-} \sqrt{p_+}),
$$

(96)

where $t_- = \frac{1}{4} \kappa^2 x_-$.

We can generalize this result to a particle in a constant electric field. Then, from eq.(58) after a rescaling similar to eq.(87) we obtain

$$
\mu = 2 \alpha E \kappa^2.
$$

(97)

Hence, in the transition function we shall have $I_\mu(\frac{1}{t_-} \sqrt{p_+})$. The solution of eq.(73) obtained from eq.(74) is

$$
\Psi_{t-}(p_+) = t_-^{-1} \int dr \left( \frac{r}{p_+} \right)^\mu \Psi(r) \exp(-\frac{p_+ + r}{2t_-}) I_\mu(\frac{1}{t_-} \sqrt{p_+}).
$$

(98)

We shall check the results (95)-(98) of the probabilistic averaging by solving the transport equations (73) and (81) directly. Let $\rho = \lvert p \rvert$ then from eq.(78) $\rho$ satisfies the stochastic equation

$$
d\rho = 3 \kappa^2 c dt - \frac{e^2 \beta \kappa^2}{2} \rho dt + \sqrt{\kappa \rho \chi} db.
$$

(99)

whereas

$$
dx^j = -e \lvert p \rvert^{-1} p_j dt = -cn_j dt.
$$

(100)

Eq.(100) has the solution

$$
x(t) = x - nt.
$$

(101)

The transport equation (64) for massless particles with the Jüttner friction can be expressed in the form

$$
\partial_t \Psi = \frac{\kappa^2 c^2}{2} r \partial_r^2 \Psi + \left( \frac{\sigma}{4} - \frac{\nu}{2r} \right) \partial_r \Psi - \nabla x \Psi
$$

(102)

where $\sigma = 6 \kappa^2 c$ and $\nu = \beta c^2 \kappa^2$. The transport equation (102) is related to the (generalized) Bessel diffusion

$$
\partial_\mu \chi = \frac{\kappa^2 c}{2} \partial_\mu^2 \chi + \frac{\sigma - \kappa^2 c}{2\rho} \partial_\mu \chi - \nu \rho \partial_\mu \chi.
$$

(103)

The diffusion process $\rho_t$ solving the diffusion equation (103) (i.e., $\chi_t = E[\chi(\rho_t)]$) is the solution of the stochastic equation

$$
d\rho = \frac{\sigma - \kappa^2 c}{2\rho} dt - \nu \rho dt + \kappa db.
$$

(104)
Let 
\[ r = \frac{1}{4} \rho^2 \] 
then 
\[ dr = \frac{\sigma}{4} dt - \frac{\nu}{2} r dt + \kappa \sqrt{c} \sqrt{r} db. \] 
(106)

If \( \nu = 0 \) then the Kolmogorov transition function (the probability density to go from \( x \) to \( y \) in time \( t \)) for the Bessel process is (ref. [29], sec. IV.8; see also [31][32])
\[ P(t, x, y) = \frac{1}{t} (xy)^{1 - \frac{\nu}{2}} y^{\sigma - 1} I_{\frac{\nu}{2} - 1} \left( \frac{xy}{t} \right) \] 
(107)

where \( \tilde{\sigma} = \frac{\sigma}{\kappa} \) and \( t' = \frac{\kappa^2 c}{2}. \) Eq. (107) coming from ref. [29] coincides with our results (95)-(96).

We can calculate the Laplace transform of the process \( p_+(t) \) applying the results (91), (96) and (98) of the Theorem of sec. 2 to \( \Psi = \exp(-\lambda r) \) and using the integrals 6.614 from ref. [33]
\[ E[\exp(-\lambda p_+(t-\nu))] = t^{-1} \int dr \exp(-\lambda r) \exp(-\frac{p_+(x)}{2t}) I_0 \left( \frac{1}{\sqrt{t}} \sqrt{p_+} \right) \]
\[ = (1 + \lambda t_-)^{-1} \exp \left( -\lambda p_+(1 + \lambda t_-)^{-1} \right). \] 
(108)

The diffusion process (73) in an electric field \( E \) has the Laplace transform
\[ (1 + \lambda t_-)^{-1-2\alpha E \kappa^{-2}} \exp \left( -\lambda p_+(1 + \lambda t_-)^{-1} \right). \] 
(109)

The Laplace transform (108)-(109) has been calculated earlier in [29] (sec. IV.8).

The solution of the transport equation with friction cannot be calculated with Yor’s formula (88). The diffusion process (see eqs. (117)-(118) below) depends on the variables \( A \) and \( B \) but in eq. (18) we still need its integral over time. Nevertheless, we can solve the diffusion equations (72) and (75) directly applying their relation to the (generalized) Bessel diffusion (104) with \( \nu > 0. \)

We have calculated the transition function of the process \( \rho_t \) with \( \nu \neq 0 \) in [35]
\[ P_t(x, y) = \exp(t\nu(\alpha + 1))(\sinh \nu t)^{-1} y^{\alpha+1} x^{-\alpha} I_\alpha(\nu xy(\sinh \nu t)^{-1}) \]
\[ \exp(-\frac{\nu}{2}(\coth(\nu t) + 1)y^2 - \frac{\nu}{2}(\coth(\nu t) - 1)x^2), \] 
(110)

where
\[ \alpha = \frac{(\kappa^2 c)^{-1} \sigma}{2} - 1 \]
\( (\alpha = 2 \) in the model of the diffusion of massless particles). The transition function (108) describes the imaginary time evolution in quantum mechanics with the potential \( V(x) = \frac{\nu}{2} x^2 + g x^{-2} \) (then \( \alpha = \frac{1}{2} \sqrt{1 + 8 g}, [35] \)).

The Laplace transform of the transition function of the process \( \rho_t \) (with \( \nu > 0 \)) starting from \( k \) which is the square of the generalized Bessel process can
be calculated using the transition function (110) (it has been has been derived earlier in [29]). We obtain

\[ E[\exp(-\lambda r_t)] = \int dy P_t(k, y) \exp(-\lambda y^2) = \left( -\lambda \kappa^2 \nu^{-1}(\exp(-\frac{\nu}{2}t) - 1) + 1 \right)^{-\frac{c^2}{2\kappa^2}} \exp \left( -\left( -\lambda \kappa^2 \nu^{-1}(\exp(-\frac{\nu}{2}t) - 1) + 1 \right)^{-1} \lambda \exp(-\frac{\nu}{2}t)k \right). \]  

(111)

For the massless particles \( \sigma = 6\kappa^2 \), \( \nu = \kappa^2 c\beta \); for the process \( p_+ \) with the Jüttner friction (\( f' = -1 \)) we have \( \sigma = 2\kappa^2 \) and \( \nu = \kappa^2 c\beta \). The expansion of eq.(111) for the diffusion of massless particles

\[ E[\exp(-\lambda r_t)] = \frac{1}{2} (c\beta)^3 \int dy^2 \exp(-c\beta y - \lambda y) = \sum_{n=1}^{\infty} c_n(k) \exp(-n\frac{\nu}{2}t) \]  

(112)

shows that \( d\mu = \frac{1}{2} (c\beta)^3 y^2 \exp(-c\beta y) \) is the invariant measure for the process \( \rho_t \) and that the convergence to the equilibrium is exponential with the speed \( \frac{2}{\nu} \), proportional to the temperature.

Applying the transition function (110) we can calculate the solution of the transport equation (81) as an expectation value over the stochastic process

\[ \Psi_t(x, |p|, n) = E[\Psi(x-cnt, \rho_t, n)] = \int_0^\infty dy P_t(\sqrt{|p|}, \sqrt{y}) \psi(x-cnt, y, n) \]  

(113)

From eq.(113) it follows that if \( \Psi \) as the function of \( x \) has no limit at infinity then the limit \( t \rightarrow \infty \) does not exist (this is a consequence of the wave propagation). If the limit \( t \rightarrow \infty \) in eq.(113) exists then it follows that \( \Psi_{t_\infty} \rightarrow 1 \) and \( \Phi_{t_\infty} \rightarrow \Phi_E \) (up to a normalization constant). Moreover, for the momentum probability distribution of an "observable" \( \phi \) in a "state" \( \Phi \) we obtain

\[ \lim_{t \rightarrow \infty} \int dxdp \phi(p) \Phi_t(x, p) = \left( \int dp \Phi_E(p) \phi(p) \right) \frac{1}{8\pi^2} (c\beta)^3 \int dxdx \exp(-c\beta |p|) \Psi(x, p) \]  

(114)

As an example of the solution (113) we could consider the plane wave with the initial condition \( \Psi(x, |p|, n) = \exp(i |p| nx) \). Then, from eq.(111)

\[ \Psi_t(x, |p|, n) = \left( \frac{1}{c^3} (nx - ct)(\exp(-\frac{1}{2} \kappa^2 c\beta t) - 1) + 1 \right)^{-3} \exp \left( i \left( \frac{1}{c^3} (nx - ct)(\exp(-\frac{1}{2} \kappa^2 c\beta t) - 1) + 1 \right)^{-1} \exp(-\frac{1}{2} \kappa^2 c\beta t) |p|(nx - ct) \right) \]  

(115)

As a result we obtain again a wave (note that \( |p|(x - nct)n = p_\mu x^\mu \) in the relativistic notation) moving in the direction \( n \) with a decreasing amplitude and exponentially decreasing wave vector.
We can solve the stochastic equations (68) and (75) and calculate expectation values of functions of the processes with a friction. The solution of eq.(68) reads

\[ p_\tau = |p| \exp(\kappa^2 \tau + \kappa b_\tau) \left( 1 + \frac{\kappa^2}{2} \beta |p| \int_0^\tau ds \exp(\kappa^2 s + \kappa b_s) \right)^{-1} \]  

(116)

The stochastic equation (75) for \( p_+ \) can also be solved with the result

\[ p_+(\tau) = p_+ \exp(\kappa b_\tau) \left( 1 + \frac{\kappa^2}{2} \beta p_+ \int_0^\tau ds \exp(\kappa b_s) \right)^{-1} \]  

(117)

Unfortunately, with the Yor’s formula we are unable to calculate the expectation value (18). However, we can obtain expectation values \( E[F(p_\tau)] \) of any function of the process. This may be sufficient for a calculation of some mean values with the probability distribution \( \Phi \) as a solution of the transport equation (11), for example applying eq.(18)

\[ \int d\mathbf{x} d\mathbf{p} \phi(\mathbf{x}, \mathbf{p}) \Phi(x, p) = \int d\mathbf{p} \phi(\mathbf{p}) \Phi_E(\mathbf{p}) \int_0^\infty d\tau E[\int d\mathbf{x} \Psi(\mathbf{x}, \mathbf{p}_\tau)] \]  

The rhs is of the Yor’s form (88) also for the processes (116)-(117) with a friction.

8 Transport equations in a moving frame

We consider a covariant form of the equilibrium distribution

\[ \Phi_E = p_0^{-1} \exp(f(c\beta^\mu p_\mu)). \]  

(118)

The four-vector \( \beta^\mu \) can be related to the velocity of the frame of reference in such a way that in the rest frame \( \beta = (\frac{1}{c}, 0, 0, 0) \), where \( T \) is the temperature and \( k \) is the Boltzmann constant (see [36][37][38][39][40] for a discussion of such equilibrium distributions). Then, the probability distribution \( d\sigma(x, p, \beta) = dx dp \Phi_E \) transforms in a covariant way under Lorentz transformations \( \Lambda \)

\[ d\sigma(Ax, Ap, A\beta) = d\sigma(x, p, \beta). \]

An expectation value in a "state" \( \Phi \) satisfying the transport equation (11) is

\[ \int d\mathbf{p} d\mathbf{x} \phi(\mathbf{x}, \mathbf{p}) \Phi_t(\mathbf{x}, \mathbf{p}) = \int d\sigma(x, p, \beta) \phi(\mathbf{x}, \mathbf{p}) E[\Psi(\mathbf{x}_\tau(x, p), \mathbf{p}_\tau)] \]  

(119)

We applied the formula (18) in order to express the time evolution of the solution \( \Phi_t \) of the transport equation by the proper time. In eq.(119) \( \tau(t) \) is expressed by the coordinate time from the equation \( t = (mc)^{-1} \int_0^\tau \rho_0 ds \).

If \( m > 0 \) then the drift (31) reads

\[ K_j = -\frac{m c \kappa^2}{2} \beta_j f' + \frac{\kappa^2 c}{2} \beta^\mu p_\mu p_j f'. \]  

(120)
In the massless case
\[ K_j = \frac{c\kappa^2}{2} \beta^\mu p_\mu p_j f'. \] (121)

The covariant form of the transport equation takes the form
\[ \partial \partial_t \Psi = \frac{c p_0}{p_0} p_j \partial x_j \Psi + \frac{c^2 \kappa^2}{2 p_0} (\delta_{jk} + m^{-2} c^{-2} p_j p_k) \partial p_j \partial p_k \Psi + \frac{m^2 c^2}{2 p_0} \beta_\mu p_\mu p_j f' \partial p_j \Psi. \] (122)

In the proper time formalism the diffusion process solving the diffusion equation (15) satisfies the equation
\[ d\hat{p}_j = \frac{3 \kappa^2}{2} \hat{p}_j d\tau - \frac{m^2 c^2 \kappa^2}{2} \beta_j f' d\tau + \frac{m^2 c^2 \kappa^2}{2} \beta_\mu p_\mu \hat{p}_j \hat{p}_j f' d\tau + \kappa e^j_n d\bar{b}^n. \] (124)

where \( e^j_n \) (the square root of the metric in the second order differential operator (21), see [10]) has been defined in eq.(69). The diffusion process \( \hat{p}_\tau \) (124) is the same as the process \( p_\tau \) generated by \( A \) (eq.(7)) because \( A = \Phi^{-1} A \Phi = A \). We have obtained this equality by direct calculations but there should be a deeper reason for it.

We can derive from eq.(124) the formula for \( dp_0 \). Let us define
\[ \pi = \beta^\mu p_\mu \] (125)
and
\[ X = \beta^\mu x_\mu. \] (126)

Then, a direct calculation (using the Ito stochastic calculus [29]) leads to the proper time stochastic equations
\[ d\pi = \frac{3 m^2 c^2 \kappa^2}{2} \pi d\tau + \frac{m^2 c^2 \kappa^2}{2} \pi f' d\tau - \frac{m^2 c^2 \kappa^2}{2} \beta_\mu p_\mu f' d\tau + \frac{m^2 c^2 \kappa^2}{2} \beta_\mu p_\mu \hat{p}_j \hat{p}_j f' d\tau + \kappa e^j_n d\bar{b}^n. \] (127)

In the massless case eq.(127) reads
\[ d\pi = \frac{3 m^2 c^2 \kappa^2}{2} \pi d\tau + \frac{m^2 c^2 \kappa^2}{2} \pi f' d\tau + \kappa \pi d\bar{b}. \] (129)

Using the variables \((X, \pi)\) we can write a proper time diffusion equation for an evolution of the probability distribution (13) \( \Psi(X, \pi) \) assuming that it depends solely on the variables \((X, \pi)\)
\[ \partial \partial_t \Psi = \pi \partial X \Psi + \left( \frac{3 m^2 c^2 \kappa^2}{2} \pi + \frac{m^2 c^2 \kappa^2}{2} \pi f' - \frac{m^2 c^2 \kappa^2}{2} \beta_\mu p_\mu \right) \partial \pi \Psi + \frac{m^2 c^2 \kappa^2}{2} \beta_\mu p_\mu \partial ^2 \pi \Psi \] (130)
In the massless case this diffusion equation reads

$$\partial_\tau \Psi = \pi \partial_X \Psi + \left( \frac{3\kappa^2}{2} + \frac{c\kappa^2}{2}\pi^2 f' \right) \partial_\pi \Psi + \frac{\kappa^2}{2} \pi^2 \partial^2_\pi \Psi$$  \hspace{1cm} (131)

The transport equation for massless particles is the same as the one for the Bessel diffusion

$$\partial_X \Psi = \frac{\kappa^2}{2} \pi \partial^2_\pi \Psi + \left( \frac{3\kappa^2}{2} + \frac{c\kappa^2}{2}\pi f' \right) \partial_\pi \Psi.$$  \hspace{1cm} (132)

In order to preserve an interpretation of $\beta$ as the temperature the four-vector $\beta$ should be time-like. We can see from eq.(130) that the model simplifies substantially (the transport equation reduces to the Bessel diffusion) if we let $\beta^u \beta_u \to 0$. In a formal limit $\beta = (\beta^-, 0, 0, 0)$ (when $\beta$ is on the light cone) and with $\partial_\tau \Psi = 0$ we obtain the model (72) corresponding to the light cone coordinates with the exception of the factor $\frac{3\kappa^2}{2} \partial_\pi$ in eq.(130) which is replaced by $\frac{\kappa^2}{2} \partial_\pi$ in eq.(72). The incorrect factor in the formal limit comes from the incorrect probability measure in this limit which in the light-cone coordinates should be $dp dp_1 dp_2 dp_{+1} \exp f$ and not $dp p_{+1} \exp f$. The factor $p_{+1}$ is responsible for the difference between the term $\frac{3\kappa^2}{2} \partial_\pi$ in eq.(130) and $\frac{\kappa^2}{2} \partial_\pi$ in eq.(72).

Relativistic invariance means that a Lorentz transformation of the processes $X$ and $\pi$ can be shifted into a transformation of the frame of reference, i.e., for a function $F$

$$E[F(\Lambda x(\tau, \beta, \Lambda x, \Lambda p), \Lambda p(\tau, \beta, \Lambda p))] = E[F(x(\tau, \Lambda^{-1} \beta, x, p), p(\tau, \Lambda^{-1} \beta, p))]$$
\hspace{1cm} (133)

9 Discussion

In our earlier papers we have developed a formalism of relativistic diffusions evolving in the proper time. Such a formalism is useful because the explicit relativistic invariance is a strong guiding principle when building relativistic diffusion equations. In this paper we have shown that a solution of the stochastic equations evolving in the proper time allows a construction of the solution of the transport equation (in the laboratory time) by means of an integration over the proper time. We have shown how the method works in some models which can be solved exactly. These examples are not typical for the proper time equations. For massless particles the proper time is just an affine parameter on the trajectory without a physical meaning. A function depending solely on $p_{+}$ can describe a massive particle whose motion is restricted to a straight line (in the direction of the third axis). The results concerning the transport equations described by the Bessel diffusion of momenta could find applications in astrophysics, in plasma physics and in heavy ion collisions. The momentum
\(|p|\) evolves in time as the Bessel process which has some distinguished features among all diffusion processes [31][32]. In particular, the (generalized) Bessel diffusion has the exceptional property that an exponential initial distribution remains exponential for any time.

In general, the method of proper time could be applied for approximate calculations and computer simulations. If \(m > 0\) then an expansion parameter \((mc)^{-1}\) appears in all our equations. We could use it in the Theorem (eq.(18)) in order to calculate the integral over \(\tau\) with an expanded argument of the \(\delta\)-function \(\hat{\rho}_s = b_s\) in the lowest order of the expansion in \((mc)^{-1}\)

\[
\delta \left( t - \tau - \frac{1}{2m^2c^2} \int_0^\tau dB_s^2 \right).
\]

We can now calculate the expectation value (18) in a way similar to the one in this paper because the probability distribution of the integral \(\int_0^\tau dB_s^2\) is known.

The original relativistic diffusion in the proper time of Schay [4] and Dudley [5] is explicitly Lorentz invariant because its proper time evolution is generated by a Lorentz invariant differential operator. In applications we need diffusions which have a limit for a large time. The necessary drag terms which force the process to an equilibrium cannot be Lorentz invariant. The reason is that the notion of the equilibrium is itself frame dependent. We described this frame dependence in the last section. The stochastic process transforms in a covariant way with respect to the Lorentz transformation if together with the process we transform also the frame.

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