On the roots of the Poincare structure of asymptotically flat spacetimes

László B. Szabados

Research Institute for Particle and Nuclear Physics
H-1525 Budapest 114, P.O.Box 49, Hungary
E-mail: lbszab@rmki.kfki.hu

The analysis of canonical vacuum general relativity by R. Beig and N. Ö Murchadha (Ann. Phys. 174 463-498 (1987)) is extended in numerous ways. The weakest possible power-type fall-off conditions for the energy-momentum tensor of the matter fields, the metric, the extrinsic curvature, the lapse and the shift are determined which, together with the parity conditions, are preserved by the energy-momentum conservation law $T^{ab} = 0$ and the evolution equations for the geometry. The algebra of the asymptotic Killing vectors, defined with respect to a foliation of the spacetime, is shown to be the Lorentz Lie algebra for slow fall-off of the metric, but it is the Poincare algebra for $1/r$ or faster fall-off.

It is shown that the applicability of the symplectic formalism already requires the $1/r$ (or faster) fall-off of the metric. The connection between the Poisson algebra of the Beig–Ö Murchadha Hamiltonians (and, in particular, the constraint algebra) and the asymptotic Killing vectors is clarified. Their Hamiltonian $H[K^a]$ is shown to be constant in time modulo constraints for those asymptotic Killing vectors $K^a$ that are defined with respect to the foliation by the constant time slices.

The energy-momentum and angular momentum are defined by the boundary term $Q[K^a]$ in $H[K^a]$ even in the presence of matter. Although the energy-momentum is well defined even for slightly faster than the $r^{-1/2}$ fall-off, we show that the angular momentum and centre-of-mass are finite only if the metric falls off as $1/r$ or faster. $Q[K^a]$ is constant in time for those $K^a$’s that are asymptotic Killing vectors with respect to the foliation by the constant time slices. If the foliation corresponds to proper time evolution (i.e. its lapse tends to 1 at infinity), then $Q[K^a]$ reproduces the ADM energy, the spatial momentum and spatial angular momentum, but the centre-of-mass deviates from that of Beig and Ö Murchadha by the spatial momentum times the coordinate time. The spatial angular momentum and the new centre-of-mass form an anti-symmetric Lorentz tensor, which transforms in the expected way under Poincare transformations.

* Dedicated to Jim Nester on the occasion of his 60th birthday.
1. Introduction

The quantum field theoretical investigations of the early sixties showed that, strictly speaking, the observables of quantum fields must be associated only with finite but extended spacetime domains, i.e. they are quasi-local [1]. Quantities associated with spacetime points are not observables, and the global quantities, e.g. the total energy or electric charge, should be considered as the limit of quasi-locally defined quantities. Interestingly enough (although by different reasons, but) the situation is very similar in general relativity: energy-momentum and angular momentum cannot be associated with the points of the spacetime. Any such local expression is necessarily pseudotensorial and/or internal gauge dependent. Thus if we want to characterize the gravitational ‘field’ by observables finer than those associated with the whole (necessarily asymptotically flat) spacetime, then these observables must also be defined quasi-locally.

In the last two decades a lot of efforts was concentrated on the investigations both of the general framework in which the quasi-local energy-momentum and angular momentum should be constructed and the specific constructions themselves (and their properties). Although there is no consensus at all in the relativity community even about general questions e.g. when to consider a specific construction to be ‘reasonable’, it is naturally expected that the globally defined observables, e.g. the ADM energy-momentum, must be recoverable as an appropriate limit of the corresponding quasi-local quantities. Although the energy-momentum, both at the spatial and null infinity, is well understood, the (relativistic) angular momentum (especially at the null infinity) needs more investigations. In particular, one should clarify the limit of the spatial angular momentum and centre-of-mass of Brown and York [2] and the ones based on Bramson’s superpotential and the use of the holomorphic/anti-holomorphic spinors [3] at the spatial infinity.

One of the most elegant introduction of the ADM conserved quantities at the spatial infinity is based on the requirement of the differentiability of the Hamiltonian. This approach of Regge and Teitelboim [4] was refined later by Beig and Ó Murchadha [5], recovering the ADM energy and linear momentum and the spatial angular momentum of Regge and Teitelboim, but giving a different expression for the centre-of-mass. The recent investigations of Baskaran, Lau and Petrov [6] show that the Brown–York centre-of-mass tends to the expression of Beig and Ó Murchadha.

The traditional ADM approach of the conserved quantities and the Hamiltonian analysis of general relativity is based on the 3+1 decomposition of the fields and the geometry. Hence it is not a priori clear that the energy and the spatial momentum form a Lorentz vector, or the spatial angular momentum and centre-of-mass form an anti-symmetric Lorentz tensor. To ensure the Lorentz covariance of the conserved quantities at spatial infinity Nester developed a spacetime-covariant Hamiltonian formulation of general relativity [7]. However, the content and the results of the theory can be spacetime covariant even if its form is not. Thus, in particular, we should find a spacetime interpretation of the traditional Hamiltonian formulation and the ‘conserved’ quantities of the theory in terms of some appropriately defined asymptotic spacetime Killing vectors.

It is known that the ADM energy-momentum can be finite and well defined (i.e. independent of the background structure) and the energy non-negativity can be proven even if the metric falls off with the radial distance $r$ slightly faster than $r^{-\frac{1}{2}}$ [8-12]. This raises the question of finding the weakest possible fall-off for the metric and extrinsic curvature under which the spatial angular momentum and centre-of-mass are still finite and well defined. Furthermore, we should be able to treat not only the vacuum theory, but the matter fields should also be included.

The present paper is devoted to the investigations of the global energy-momentum and angular momentum introduced at the spatial infinity of asymptotically flat spacetimes. We extend the analysis of canonical vacuum general relativity by R. Beig and N. Ó Murchadha [5] in numerous ways: the interpretatableness of the results in the spacetime is required, the symplectic structure is considered not to be fundamental and
the emphasis is shifted to the field equations, the $1/r$ and $1/r^2$ \textit{a priori} fall-off conditions for the metric and the canonical momentum, respectively, are relaxed, the matter fields are included and the background dependence of the angular momentum is investigated. In the literature several mathematically inequivalent model for the spatial infinity have been suggested (see e.g. [13-16]). However, the notion of asymptotic flatness at the spatial infinity based on a spacelike hypersurface is expected to be the weakest possible in the sense that in every reasonable model of spatial infinity the existence of such a hypersurface is expected. Thus, in the present paper, we do not use any specific model of infinity, and the notion of asymptotic flatness that we use is based on the existence of a certain spacelike hypersurface. Since there is some recent interest in higher dimensional (Lorentzian) models (see e.g. [17-18]), and since no extra effort is needed to do the analysis in general $m = n + 1$ spacetime dimensions, we assume only that the dimension of the spacetime is $m \geq 3$.

In the traditional analysis the lapse and the shift are implicitly assumed to depend only on the spatial coordinates, but not on the time coordinate. However, the equations of motion allow their time dependence. It turns out that it is precisely this freedom that makes possible to give the \textit{spacetime} interpretation of the Poincare algebra of the Hamiltonians found by Beig and Ó Murchadha. If we excluded the time dependence of the lapse and the shift then we would not be able to recover the boost Killing vectors even of the Minkowski spacetime.

The philosophy of our analysis deviates slightly from the traditional one. It is the equations of motion that are considered to be fundamental, and the boundary conditions at infinity are required to be the slowest power-type fall-off conditions compatible with the evolution equations. Then the symplectic structure and the Hamiltonian are considered to be only as secondary structures. They are considered to be important only from the point of view of finding observables and, in particular, the conserved quantities, but not from the point of view of the boundary conditions. This departs from the philosophy of Regge and Teitelboim, where the boundary conditions and the Hamiltonian were determined in a single procedure from the regularity and differentiability of the Hamiltonian. Thus, having the boundary conditions been specified, the Hamiltonian $H$ must be chosen such that the correct field equations be recoverable as the flows corresponding to the Hamiltonian vector fields of $H$. Then the value of this Hamiltonian on the constraint surface will define the ADM quantities. Although the Beig–Ó Murchadha analysis was carried out for the \textit{vacuum} Einstein theory, the value of their Hamiltonian on the constraint surface can be used to define the energy-momentum and angular momentum even in the presence of matter fields. We clarify in what sense these quantities are conserved, and how they depend on the flat background metric.

First we determine the weakest possible power-type fall-off conditions for the energy-momentum tensor of the matter fields, the metric, the extrinsic curvature, the lapse and the shift which, together with the parity conditions of Regge and Teitelboim, are preserved by the energy-momentum conservation law $T^{ab} = 0$ and the evolution equations for the geometry. In an $n + 1$ dimensional spacetime they are of order $O(r^{-(n+1)})$, $O(r^{-k})$ for some $k > 0$, $O(r^{-(k+1)})$, $O(r)$ and $O(r)$, respectively. The spacetime vector fields built from these allowed lapses and shifts will be called the \textit{allowed time axes}. Then the asymptotic spacetime Killing vectors (with respect to an allowed time axis $\xi^a$) are defined to be those vector fields for which the Killing operator is of order $O(r^{-k})$, and the space of these asymptotic Killing vectors will be denoted by $\mathcal{A}_k^\xi$. Its factor $\mathcal{A}_k^\xi / \mathcal{G}_k^\xi$ by the subspace $\mathcal{G}_k^\xi \subset \mathcal{A}_k^\xi$ (whose elements are the asymptotic Killing vectors with $O(r^{1-k})$ asymptotic behaviour) can be endowed with a Lie algebra structure in a natural way, and this is shown to be isomorphic to the Lorentz Lie algebra for slow fall-off $k \in (0, 1)$, but for faster fall-off, $k \geq 1$, it is the Poincare algebra. Thus the structure of the Lie algebra of the asymptotic symmetries is linked to the fall-off rate $k$ of the metric.

One way of associating conserved quantities to asymptotically flat spacetimes is the use of the symplectic/Hamiltonian formalism. Since the details of the formalism depend on the type of the fields, we
concentrate only on the vacuum theory. It is shown that the applicability of the symplectic formalism to the vacuum general relativity, in particular the existence of the symplectic 2-form, already implies that $k \geq \frac{1}{2}(n - 1)$. This excludes the slow ($k < 1$) fall-off in spacetime dimensions greater than 3. The constraint functions are shown to be finite, functionally differentiable and close to a Poisson algebra precisely for those lapses and shifts that correspond to the special allowed time axes with $O(r^{1-k})$ asymptotic behaviour. We show that the generators of the gauge transformations, i.e. the functions whose Hamiltonian vector fields span the kernel of the pull back to the constraint surface of the symplectic 2-form, are precisely these special time axes. A subspace $A^0_\xi \subset A^K_\xi$ is found such that the Hamiltonian $H$ of Beig and Ó Murchadha, mapping $A^K_\xi$ into the Poisson algebra of functions, preserves the Lie product of the elements of $A^0_\xi$. It is shown that $H[K^a]$ is constant in time with respect to the allowed time axes $\xi^a$ if $K^a \in A^0_\xi$, and it is only constant modulo constraint functions for $K^a \in A^K_\xi$.

The boundary term in the Hamiltonian $H[K^a]$ is used to define the energy-momentum and angular momentum even in the presence of matter, independently of any symplectic structure. It is shown that, although the energy-momentum is well defined even for $k > \frac{1}{2}(n - 2)$, the angular momentum and centre-of-mass are finite only if $k \geq \frac{1}{2}(n - 1)$. Similar result was obtained independently by Baskaran, Lau and Petrov in 3+1 dimensions recently [6]. $H[K^a]$ reproduces the ADM energy and spatial momentum, the spatial angular momentum of Regge and Teitelboim and the centre-of-mass of Beig and Ó Murchadha for $K^a \in A^K_\xi$ if the allowed time axis $\xi^a$ corresponds to a gauge generator. However, the familiar boost Killing vectors of the Minkowski spacetime are contained in $A^K_\xi$ only if the lapse part of $\xi^a$ does not tend to zero at infinity. For $K^a \in A^K_\xi$ with time axes $\xi^a$ describing pure time translation at infinity the Hamiltonian $H[K^a]$ reproduces the energy-momentum and spatial angular momentum above, but gives an additional term (the spatial momentum times the coordinate time) to the centre-of-mass of Beig and Ó Murchadha. To derive the familiar, expected transformation law for the (relativistic) angular momentum this extra term is needed. For $K^a \in A^K_\xi$ if the value of the Hamiltonian $H[K^a]$ is shown to be constant in time with respect to the allowed time axis $\xi^a$ provided the constraint equations are satisfied. We investigate the conditions of the background (in-)dependence of the energy-momentum and angular momentum, and we found that although the former is well defined even if the diffeomorphisms representing the ambiguity of the background metric tend to rigid Euclidean transformations as $O(r^R)$, where $R \leq -k$ and $R < (3 - n)$, the latter is well defined if $R \leq (1 - k)$ and $R \leq (2 - n)$ (and in the case of the equality, $R = (2 - n)$, the generator of the diffeomorphism has odd parity). In particular, to have well defined angular momentum in 3+1 dimensions the metric must fall off at least as $O(r^{-1})$, and the allowed diffeomorphisms must tend to rigid Euclidean transformations at least as $O(r^{-1})$.

In subsection 2.1 the necessary tools are introduced and reviewed, mostly to fix the notations and conventions. The new key element here is the $n + 1$ decomposition and analysis of the Killing operator. To motivate the boundary conditions and the precise definition of the asymptotic spacetime Killing vectors we discuss the Minkowski spacetime in subsection 2.2. Then, in subsections 2.3 and 2.4, the boundary conditions and the asymptotic Killing vectors are discussed. Section 3. is devoted to the analysis of the canonical general relativity, in particular to the constraints, the gauge transformations and the Hamiltonian. In section 4. we apply the Beig–Ô Murchadha Hamiltonian to define the energy-momentum and angular momentum of general asymptotically flat spacetimes, even in the presence of the matter fields, and clarify how these quantities depend on the background metric. The appendix is the brief discussion of the boundary conditions for the matter fields at the null infinity.

We use the abstract index formalism, and only the underlined and boldface indices take numerical values. The spacetime dimension and the signature will be assumed to be $m = n + 1$ and $1 - n$, respectively. The Riemann and Ricci tensors and the curvature scalar e.g. of the spacetime connection $\nabla_a$ are defined by $\nabla_a X^a := \nabla_a (\nabla X^a) - \nabla_b (\nabla_a X^b) - \nabla_{[a,b]} X^a$, $m^R_{ab} := m^R_{abc} := m^R_{abcd} := m^R_{abcd}$. The time axis $\xi^a$ is found such that the Hamiltonian $H$ of the constraints, the angular momentum, the centre-of-mass of Beig and Ó Murchadha for $K^a \in A^K_\xi$ is shown to be constant in time with respect to the allowed time axes $\xi^a$
2. The \( m = n + 1 \) decomposition

2.1 The \( n+1 \) form of the Einstein equations and the Killing operator

Let \( \Sigma_t \) be a smooth foliation of the \( m = n + 1 \) dimensional spacetime \((M, g_{ab})\) by spacelike hypersurfaces and let \( t^a \) be its future directed unit normal such that \( t^a \nabla_a t \) is positive. Let \( P_b^a := \delta_b^a - t^a t_b \), the orthogonal projection to \( \Sigma \). The lapse function \( N \), the extrinsic curvature \( \kappa_{ab} \) and the acceleration \( a_e \) of the foliation are defined by \( Nt^a \nabla_a t := 1 \), \( \kappa_{ab} := P_b^a P_a^f \nabla_c t_f := \frac{1}{2} L_q g_{ab} \) and \( a_e := t^a \nabla_a t_e = -D_e (\ln N) \), respectively. Here \( q_{ab} \) is the induced (negative definite) metric and \( D_a \) is the corresponding intrinsic Levi-Civita covariant derivative. The corresponding Riemann and Ricci tensors and the curvature scalar will be denoted by \( R^{\alpha \beta \gamma \delta} \), \( R_{\alpha \beta} \) and \( R \), respectively. If \( \xi^a \) is any smooth vector field such that \( \xi^a \nabla_a t = 1 \) (‘evolution vector field’ or rather ‘general time axis’), then \( \xi^a \) has the form \( \xi^a = N t^a + \xi^a \) for some vector field \( \xi^a \); \( N\) being the shift part of \( \xi^a \). For any vector field \( \xi^a \) and purely spatial tensor field \( T^{a_1 \ldots a_r}_{b_1 \ldots b_s} \), let us define the ‘time derivative’ of \( T^{a_1 \ldots a_r}_{b_1 \ldots b_s} \) by

\[
\dot{T}^{a_1 \ldots a_r}_{b_1 \ldots b_s} := e^{a_1} \ldots e^{a_r} P_{b_1}^r P_{b_2}^s \ldots P_{b_s}^t Q_f \dot{T}^{e_1 \ldots e_r}_{f_1 \ldots f_s} = N P^{a_1}_{b_1} \ldots P^{a_r}_{b_r} P^f_{b_s} L_q T^{e_1 \ldots e_r}_{f_1 \ldots f_s} + L_N T^{a_1 \ldots a_r}_{b_1 \ldots b_s}.
\]

In particular, for the time derivative of the induced metric we have

\[
\dot{q}_{ab} = 2 N \kappa_{ab} + L_N q_{ab}.
\]

Let \( \varepsilon_{a_1 \ldots a_m} \) be the spacetime volume \( m \)-form. The induced volume \( n \)-form and volume element on \( \Sigma_t \) is defined by \( \varepsilon_{a_1 \ldots a_n} := t^a \varepsilon_{a_1 \ldots a_n} \) and \( d \Sigma_t := \frac{1}{n!} \varepsilon_{a_1 \ldots a_n} = \sqrt{q} d^a x^a \), respectively, and hence \( d v = N d \Sigma_t d t \). (The other convention for the orientation of the submanifolds, which would be slightly more convenient if unitary spinors were used [23], is when \( \varepsilon_{a_1 \ldots a_m} := t^a \varepsilon_{a_1 \ldots a_m} t^a = (-)^m t^a \varepsilon_{a_1 \ldots a_m} \).) If \( B \subset \Sigma \) is a compact \( n \) dimensional submanifold with smooth boundary \( B \), \( v^a \) its outward directed unit normal in \( \Sigma \), then, by the negative definiteness of \( q_{ab} \), for any vector field \( X^a \) tangent to \( \Sigma \) the Gauss law takes the form \( \int_B D_b X^b d \Sigma = -\oint_B X^b d S^a \), where \( d S := \frac{1}{(n-1)!} t^e v^f \varepsilon_{e f a_2 \ldots a_n} \).

The conservation of the energy-momentum tensor, i.e. \( T^{ab},_b = 0 \), is equivalent to

\[
\dot{\mu} = N \left(-D_c j^c + \sigma^{ab} \kappa_{ab} - \frac{2}{N} j^c D_c N - \mu \chi \right) + L_N \mu,
\]

\[
\dot{j}^b = N \left(-D_a \sigma^{ab} - \frac{1}{N} \sigma^{ba} D_a N + \mu \frac{1}{N} D^b N - 2 j_a \kappa_{ab} - \chi j^b \right) + L_N j^b,
\]

where \( \mu := T^{ab} t_a t_b \), \( j^b := T^{ce} t^a t_c P_{e}^b \) and \( \sigma^{ab} := T^{cf} P_{e}^a P_{f}^b \). The projections of the \( m \) dimensional Einstein equations, \( m G_{ab} + \Lambda g_{ab} + \kappa T_{ab} = 0 \), are the constraints

\[
\kappa c := t^b j^a \left(m G_{ab} + \Lambda g_{ab} + \kappa T_{ab} \right) = -\frac{1}{2} \left(R + (\chi^2 - \chi_{ab} \chi^{ab}) \right) + \Lambda + \kappa \mu = 0,
\]

\[
\kappa c_a := P_a^b \left(m G_{eb} + \Lambda g_{eb} + \kappa T_{eb} \right) = -\left(D_c \chi^c_a - D_a \chi \right) + \kappa j_a = 0;
\]

respectively. Thus, Einstein’s equations take the form \( m R_{ab} - \frac{1}{2} m R g_{ab} + \Lambda g_{ab} = -\kappa T_{ab} \), and we use the units in which \( c = 1 \). The terminology and formalism that we follow in the symplectic/Hamiltonian description of general relativity are mostly based on [19-22], and the level of the mathematical rigor we work at in section 3. corresponds to that of [21,22].
and the evolution equations

\[ \dot{\chi}^{cd} = N \left( -R^{cd} + 2\chi^{ce} \chi_{ed} - \chi^{cd} \right) + L_N \chi^{cd} - D_c D_d N + \frac{2}{n-1} \Lambda N q_{cd} + \kappa N \left( -\sigma_{cd} + \frac{1}{n-1} \sigma^e q_{cd} + \frac{1}{(n-1)^2} \mu q_{cd} \right). \]  

(2.1.7)

To check whether the evolution equations (2.1.2) and (2.2.7) preserve the constraints, take the time derivative of \( c \) and \( c_a \) and use (2.1.2)–(2.1.7). We get

\[ \dot{c} = -2c^a D_a N - N D_a c^a + L_N c - 2N \chi c, \]  

(2.1.8)

\[ \dot{c}_a = 2c D_a N + N D_a c + L_N c_a - N \chi c_a. \]  

(2.1.9)

Therefore, if the constraints (2.1.5), (2.1.6) are satisfied at \( t = 0 \), then any of their derivatives also vanish, and hence the constraints are preserved by the evolution equations (2.1.2), (2.1.7). Since in the present paper we are interested in asymptotically flat spacetimes, the cosmological constant \( \Lambda \) will be assumed to be zero.

If \( K^a = : Mt^a + M^a \) is any smooth vector field, where \( M^a = P^a_\nu M^\nu \), then the \( n + 1 \) decomposition of the so-called Killing operator, acting on the spacetime 1-form field \( K_a \), is

\[ N t^a t^b \nabla_{(a} K_{b)} = \dot{M} + M^a D_a N - N^a D_a M, \]  

(2.1.10)

\[ 2 N P^b_a \nabla_{(b} K_{c)} = \dot{M}_a + \left( N D_a M - M D_a N \right) - 2N \chi_{ab} M^b - L_N M_a, \]  

(2.1.11)

\[ P^b_a P^d_b \nabla_{(c} K_{d)} = D_{(a} M_{b)} + M \chi_{ab}. \]  

(2.1.12)

Clearly, while the space–space projection of the Killing operator is well defined even on a single hypersurface \( \Sigma \), the first two projections are well defined only if a foliation \( \Sigma_\nu \) of \( M \), i.e. a lapse function \( N \) on \( \Sigma \), is fixed, and, in addition, their right hand side needs a choice for the shift vector \( N^a \) too. Obviously, in a generic spacetime the Killing equation \( \nabla_{(a} K_{b)} = 0 \) has only the trivial solution. However, \( t^a t^b \nabla_{(a} K_{b)} = 0 \) and \( P^b_a \nabla_{(b} K_{c)} = 0 \) can always be solved, i.e. the initial value problem for the system

\[ \dot{M} = -M^a D_a N + N^a D_a M, \]  

(2.1.13)

\[ \dot{M}_a = - \left( N D_a M - M D_a N \right) + 2N \chi_{ab} M^b + L_N M_a, \]  

(2.1.14)

is unconstrained, and the initial value problem for \( M \) and \( M^a \) always has a solution. If \( \bar{K} = \bar{M} t^a + \bar{M}^a \) is another spacetime vector field, then the \( n + 1 \) decomposition of the Lie-bracket of \( K^a \) and \( \bar{K}^a \) is

\[ \left[ K^a, \bar{K}^a \right] = (t^a t^b + 2q^{ab}) \left( M \nabla_{(b} \bar{K}_{c)} - \bar{M} \nabla_{(b} K_{c)} \right) t^c + \left[ M, \bar{M} \right]^a + \left( \bar{M} D^a M - M D^a \bar{M} \right). \]  

(2.1.15)

Thus in this decomposition only the time-time and the time-space parts of the Killing operator appear, but not the space-space parts. Therefore, if both \( K^a \) and \( \bar{K}^a \) satisfied (2.1.13-14), then the projections of the Killing operator on the right hand side of (2.1.15) would be vanishing.

2.2 Boundary conditions I.: Matter fields in Minkowski spacetime
Let \((M, g_{ab})\) be the Minkowski spacetime and \(K\) the Lie algebra of its Killing vectors. As is well known, it contains an \(m\) dimensional commutative ideal \(I\), consisting of the constant vector fields on \(M\) and inheriting a natural Lorentzian vector space structure too, and \(K/I \approx so(1, n)\). Fixing a Cartesian coordinate system \(X^a = (\tau, X^i)\), \(a = 0, 1, ..., n\) and \(i = 1, 2, ..., n\), (i.e. adapting the coordinates \(X^a\) to an orthonormal basis of the space of the constant vector fields), the translation and boost-rotation Killing 1-forms are well known to take the form \(K^e = \nabla e X^a\) and \(K^e_{ab} = X^a \nabla e X^b - X^b \nabla e X^a\), respectively. Therefore, any Killing 1-form can be written in the form \(K^e = R_{ab}^e K^a + T_a^e K^b = R_{i j} (X^i \nabla e X^j - X^j \nabla e X^i) + 2B_i (X^i \nabla e \tau - \tau \nabla e X^i) + T_i \nabla e X^i + T \nabla e \tau\), where \(R_{ab}^e = -R_{ba}^e\) and \(T_a\) are constant, and we used the notations \(B_i := R_{i0}\) and \(T := T_0\).

Let \(\Sigma_\tau\) be a \(\tau = \text{const}\) hyperplane, \(R^2 := \delta_{ij} X^i X^j\), and for some \(R > 0\) let \(B_R\) be the solid closed ball of \(\Sigma_\tau\) with radius \(R\) and \(S_R := \partial B_R\) its boundary. If \(v^a\) is its outward directed unit normal, then \(1 = v^a D_a R = -\frac{1}{R} q_{ij} v^a D_a X^j\), where \(q_{ij}\) are the components of the induced flat metric \(g_{ab}\) on \(\Sigma_\tau\) in the coordinate system \(\{X^j\}\), and hence \(v^i = \frac{\delta^i_j}{R}\). If \(f = f(\tau, R, \frac{X}{R})\) is any function, then let us define its even and odd parity parts, respectively, by \(\pm f(\tau, R, \frac{X}{R}) := \frac{1}{2} (f(\tau, R, \frac{X}{R}) \pm f(\tau, R, -\frac{X}{R}))\). Let \(\tau_a\) be the future pointing unit timelike normal to \(\Sigma_\tau\), and define the quasi-local energy, spatial momentum, spatial angular momentum and centre-of-mass of the matter fields in the \(n\)-ball \(B_R\), respectively, by taking the flux integral of the conserved current \(K_a T^{ab}\):

\[
\begin{align*}
E_R &:= \int_{B_R} K^0_a T^{ab} \tau_b d\Sigma = \int_0^R \left( \int_S \mu dS \right) R^{n-1} dR = \int_0^R \left( \int_S +\mu dS \right) R^{n-1} dR, \quad (2.2.1.a) \\
F^i_R &:= \int_{B_R} K^i_a T^{ab} \tau_b d\Sigma = \int_0^R \left( \int_S j^a D_a X^i dS \right) R^{n-1} dR = \int_0^R \left( \int_S +j^i dS \right) R^{n-1} dR, \quad (2.2.1.b) \\
J^{ij}_R &:= \int_{B_R} K^{ij}_a T^{ab} \tau_b d\Sigma = \int_0^R \left( \int_S 2 j^a v^i D_a X^j dS \right) R^{n-1} dR = -2 \int_0^R \left( \int_S -j^i v^j dS \right) R^{n} dR(2.2.1.c) \\
J^{0i}_R &:= \int_{B_R} K^{0i}_a T^{ab} \tau_b d\Sigma = \int_0^R \left( \int_S \mu v^i dS \right) R^{n} dR - \tau P^i = \int_0^R \left( \int_S -\mu v^i dS \right) R^{n} dR - \tau P^i, \quad (2.2.1.d)
\end{align*}
\]

where \(dS\) is the area element on the unit sphere \(S\). (Strictly speaking, the traditional [non-conserved non-relativistic] centre-of-mass is \(J^{0i} + \tau P^i\).) Since \(K_a T^{ab}\) is divergence-free for Killing vectors, these quasi-local quantities are, in fact, associated with the \((n - 1)\)-surface \(S_R\) and depend only on \(K_a\); if \(\Sigma\) is any compact spacelike hypersurface whose boundary \(\partial \Sigma\) coincides with \(S_R\), then the flux integral of \(T^{ab} K_b\) on \(\Sigma\) will be that on \(B_R\). The necessary and sufficient condition of the existence of the \(R \to \infty\) limit of these integrals, respectively, is\(^1\)

\[
\begin{align*}
R \int_{S_R} \mu dS_R &= R^n \int_S +\mu dS = o(R^{-0}), \quad (2.2.2.a) \\
R \int_{S_R} j^a D_a X^i dS_R &= R^n \int_S +j^i dS = o(R^{-0}), \quad (2.2.2.b) \\
2R^2 \int_{S_R} j^a v^i D_a X^j dS_R &= -2R^{n+1} \int_S -j^i v^j dS = o(R^{-0}), \quad (2.2.2.c) \\
2R^2 \int_{S_R} \mu v^i dS_R &= 2R^{n+1} \int_S -\mu v^i dS = o(R^{-0}). \quad (2.2.2.d)
\end{align*}
\]

\(^1\) A function \(f(r)\) will be called of order \(o(r^{-k})\) if \(\lim_{r \to \infty} (f(r)r^k) = 0\), and will be called of order \(O(r^{-k})\) if \(\lim_{r \to \infty} (f(r)r^k)\) exists. In particular, \(o(r^{+0})\) will denote logarithmic divergence and \(o(r^{-0})\) logarithmic fall-off, while \(O(1)\) means that \(f(r)\) tends to a constant at infinity.
These global integral conditions can be ensured by the explicit fall-off and parity conditions

\[
\mu(\tau, R, \frac{X^k}{R}) = \frac{1}{R^m} + \mu^{(m)}(\tau, \frac{X^k}{R}) + o(R^{-m}),
\]

\[
\tau^i(\tau, R, \frac{X^k}{R}) = \frac{1}{R^m} + f(\tau, R, \frac{X^k}{R}) + \frac{1}{R^m} + j_i^{(m)}(\tau, \frac{X^k}{R}) + o(R^{-m}),
\]

where \(f(\tau, R, \frac{X^k}{R})\) is an arbitrary function with even parity. \(\mu^{(m)}\) and \(j_i^{(m)}\) contribute only to the energy and the spatial momentum but not to the angular momentum and centre-of-mass, hence we may call them the ADM mass aspect of \(T^{ab}\). Repeating this analysis on boosted hyperplanes \(\Sigma_{t'} := \{ t' := \tau \cosh \beta + X^1 \alpha_1 \sinh \beta = \text{const} \}\), where \(\beta \in \mathbb{R}\) and \(\delta^{ij}\alpha_i \alpha_j = -1\), we obtain that \(f(\tau, R, \frac{X^k}{R}) = 0\) and, in addition to (2.2.3a),

\[
\tau^i(\tau, R, \frac{X^k}{R}) = \frac{1}{R^m} + f(\tau, R, \frac{X^k}{R}) + o(R^{-m}),
\]

\[
\sigma^{ij}(\tau, R, \frac{X^k}{R}) = \frac{1}{R^m} + \sigma^{ij}(\tau, \frac{X^k}{R}) + o(R^{-m}).
\]

Note that although (2.2.2) could be ensured only by fall-off conditions that are strictly faster than those in (2.2.3), but with these conditions we would exclude e.g. the electromagnetic field from our investigations, where the typical fall-off of the energy-momentum tensor is \(R^{-m}\). Obviously, the fall-off and parity conditions (2.2.3) are only sufficient, and the global integral conditions (2.2.2) can be satisfied without the parity conditions too. The advantage of the fall-off and parity conditions is that they can be given explicitly.

However, the price that we had to pay for this is that we excluded those field configurations from our investigations that satisfy the global integral conditions (2.2.2) but not the explicit fall-off and parity conditions (2.2.3).

One can carry out a similar analysis of the fall-off and global integral conditions that can ensure the finiteness of the global energy-momentum and (relativistic) angular momentum at the future null infinity. Since, however, in the present paper primarily we are interested in the kinematical quantities defined at the spatial infinity, the null infinity case will be discussed only in the Appendix.

In the rest of this subsection we discuss the conditions under which the \(\frac{1}{2}m(m+1)\) spacetime Killing vectors can be recovered, at least asymptotically, from quantities defined on a general asymptotically flat spacelike hypersurface in the Minkowski spacetime. Since, however, many parts of the following discussion are well known from various sources, we sketch only the main points of the argumentation.

The global Cartesian coordinate system \(X^0 = (\tau, X^1)\) defines a foliation of the Minkowski spacetime by the hyperplanes \(\Sigma_\tau\) and gives the ‘time axis’ \((\frac{\partial}{\partial \tau})^a\), i.e. the corresponding lapse is one and the shift is zero. Thus our aim is to determine those conditions under which a coordinate system \((t, x^3)\), based on a more general spacelike hypersurface \(\Sigma\), ‘approaches asymptotically’ the Cartesian coordinate system ‘at infinity’. However, to do ‘a coordinate system approaches a Cartesian coordinate system asymptotically at infinity’ to be meaningful we need to use some model of the spacelike infinity of the Minkowski spacetime. We choose the classical conformal boundary of Penrose. Thus let \((\tilde{M}, \tilde{g}_{ab})\) be the conformally compactified Minkowski spacetime (e.g. the closure of the conformally embedded Minkowski spacetime in the Einstein universe, see e.g. [24]), \(\Omega\) the conformal factor on \(\tilde{M}\) such that \(\tilde{g}_{ab}|_M = \Omega^2\tilde{g}_{ab}\) and \(i^0 \in \tilde{M}\) the point on the conformal boundary of \((\tilde{M}, \tilde{g}_{ab})\) representing the spatial infinity. Then the family of hyperplanes \(\Sigma_\tau\) uniquely determines a family \(\tilde{\Sigma}_\tau\) of smooth Cauchy surfaces in \(\tilde{M}\) such that \(i^0 \in \tilde{\Sigma}_\tau\) for all \(\tau \in \mathbb{R}\), and the future directed \(\tilde{g}_{ab}\)-unit normal \(\tilde{\tau}_a\) of all the surfaces \(\Sigma_\tau\) coincide at \(i^0\). The compactification of \(\Sigma_\tau\) to
\(\Sigma_t\) can also be done by the standard inversion transformation: on some open neighbourhood \(\tilde{V}_r\) of \(i^0\) in \(\Sigma_r\) let \(\tilde{X}^1 := \frac{1}{R^2}X^1\). Then the coordinates \(\tilde{X}^i\) can be extended from \(\tilde{V}_r - \{i^0\}\) to \(\tilde{V}_r\) by \(\tilde{X}^i(i^0) = 0\), and \(\tilde{E}_a^i := \left(\frac{\partial}{\partial X^a}\right)^i|_{i^0}\) is a \(\tilde{g}_{ab}\)-orthonormal spatial basis at \(i^0\) orthogonal to \(\tilde{\tau}_a\). This basis turns out to be independent of \(r\).

Let \(\tilde{\Sigma}\) be any smooth spacelike hypersurface in \(\tilde{M}\) through \(i^0\). We say that \(\Sigma := \tilde{\Sigma} - \{i^0\}\) is approaching the leaves of the foliation \(\Sigma_r\) at infinity if the future directed \(\tilde{g}_{ab}\)-unit normal \(\tilde{t}_a\) of \(\tilde{\Sigma}\) coincides with \(\tilde{\tau}_a\) at \(i^0\). We assume that \(\Sigma\) is such a hypersurface, otherwise it is called asymptotically boosted with respect to the leaves \(\Sigma_r\). (Note that we have to assume the smoothness of \(\tilde{\Sigma}\) even at \(i^0\), because otherwise its normal would not be well defined.) Let \(t_a\) be the future directed \(g_{ab}\)-unit normal to \(\Sigma\), and let \(q_{ab}\) and \(\tilde{q}_{ab}\) be the induced metrics and \(\chi_{ab}\) and \(\tilde{\chi}_{ab}\) the extrinsic curvatures of \(\Sigma\) and \(\tilde{\Sigma}\), respectively. Then, by the construction, \((\tilde{\Sigma}, i^0, \Omega|_{\tilde{\Sigma}}, \tilde{g}_{ab}, \tilde{\chi}_{ab}\) is an asymptote for \((\Sigma, q_{ab}, \chi_{ab})\) in the sense of [13] (see also [25]), and, in addition, the normal directional derivative of the conformal factor, \(\Omega := \sum_a \nabla_a \Omega_{|\Sigma}\), has a \(C^3\) extension to \(\tilde{\Sigma}\) such that \(\Omega(i^0) = 0\), \((D_a \Omega)(i^0) = 0\), \((D_a D_b \Omega)(i^0) = 0\) and \((D_a D_b D_c \Omega)(i^0) = 0\), and \(\tilde{\chi}_{ab}|_{\Sigma} = \Omega \chi_{ab} + \tilde{\Omega} q_{ab}\) holds. Let \(\{x^k\}\) be a local coordinate system on some open neighbourhood \(U\) of \(i^0\) in \(\Sigma\) such that \(x^k(i^0) = 0\) and in these coordinates \(\partial^k\Omega|_{i^0} = \hat{E}_a^k(i^0) = 0\) hold (i.e., in particular, \(\tilde{q}_{ij}(i^0) = -\delta_{ij}\) holds, and \(\{x^k\}\) is a normal coordinate system with origin \(i^0 \in \Sigma\)). Let \(\tilde{r}^2 := \delta_{ij} \tilde{x}^i \tilde{x}^j\), and define the new coordinates \(x^i := \tilde{r} - \tilde{x}^i\) and radial coordinate distance \(r^2 := \delta_{ij} x^i x^j = \tilde{r}^2\) on \(U := \tilde{U} - \{i^0\}\). (In general the coordinates \(x^i\) are not the restrictions to \(\Sigma\) of the Cartesian coordinates \(X^1\).) The properties of \(\Omega|_{\tilde{\Sigma}}\) and \(\tilde{\Omega}\) stated above imply that in these coordinates \(\Omega = r^{-2}(1 + O(r^{-k}))\) and \(\tilde{\Omega} = O(r^{-3 + h})\) for some positive \(k\) and \(h\). (In fact, even for general asymptotically flat spacetimes when \(\tilde{\Sigma}\) is not a smooth \(i^0\), one has \(k, h \geq 1\).) Then for the metrics and the extrinsic curvatures we have

\[
q_{ij} dx^i dx^j = \Omega^{-2} \tilde{q}_{ij} d\tilde{x}^i d\tilde{x}^j = \left(\delta_{ij} + \frac{1}{\tilde{r}^2} \tilde{\delta}_{ij}^{(k)} + O(r^{-2}) + o(r^{-k})\right) dx^i dx^j,
\]

\[
\chi_{ij} dx^i dx^j = \left(\Omega^{-1} \tilde{\chi}_{ij} - \Omega^{-3} \Omega_{\tilde{q}ij}\right) d\tilde{x}^i d\tilde{x}^j = \left(\frac{1}{\tilde{r}^{1 + h}} \tilde{\chi}_{ij}^{(1 + h)} + o(r^{-1 + h})\right) dx^i dx^j
\]

for some \(\tilde{q}_{ij}^{(k)}\) and \(\tilde{\chi}_{ij}^{(1 + h)}\) depending only on \(\tilde{x}^k\). Actually, for the Minkowski spacetime, both \(k\) and \(h\) must be greater than or equal to 2. Therefore, \(\{x^1\}\) is an ‘asymptotically Cartesian’ coordinate system with respect to \(q_{ab}\), and, on \(U\), it defines a (negative definite) flat metric \(\tilde{a}_{ab}\) with respect to which \(\{x^1\}\) is Cartesian.

To complete \(\{x^1\}\) to a spacetime coordinate system \(\{t, x^1\}\) (at least on an open neighbourhood of \(U \subset \Sigma \subset M\)), we need a whole family \(\Sigma_t\) of such hypersurfaces providing a foliation of this neighbourhood. However, to ensure that this spacetime coordinate system (and not only \(x^i\)) on the single hypersurface \(\Sigma_0 = \Sigma\) approaches the Cartesian one, all the leaves \(\Sigma_t\) of the foliation must approach the leaves \(\Sigma_r\) at infinity, i.e., the future directed \(\tilde{g}_{ab}\)-unit normal \(\tilde{t}_a(t)\) of \(\Sigma_t\) must coincide with \(\tilde{\tau}_a\) at \(i^0\) for all \(t \in \mathbb{R}\). (Otherwise the foliation \(\Sigma_t\) would be asymptotically accelerating at \(i^0\) rather than being inertial.) Then there is a natural extension of the spatial coordinates from \(\Sigma\) to all the leaves \(\Sigma_t\) via the construction above: on \(\Sigma_t\) let \(\{x^i\}\) be the inversion of the normal coordinates \(\tilde{x}^3\) on \(\tilde{\Sigma}_t\) based on the basis \(\{\tilde{E}_3^i\}\) at \(i^0\) and with the origin at \(i^0\). Denoting the tangent of the curves \(\gamma(t) := (t, x^1)\), \(x^3 = \text{const.}\), by \(\xi^a\) and defining the lapses and shifts by \(\xi^a = N t^a + N^a = \tilde{N} t^a + \tilde{N}^a\), we get that \(N = \Omega^{-1} \tilde{N} = O(r^{-2})\) and \(N^a = \tilde{N}^a = O(r^p)\), where \(p \geq 2\), because the leaves \(\tilde{\Sigma}_t\) of the foliation are tangent to each other at \(i^0\). Therefore, we can write \(N = N^0(\frac{\xi}{r}) + O(r^{-1})\).

Since the leaves both of the foliations \(\tilde{\Sigma}_t\) and \(\Sigma_r\) are tangent to each other at \(i^0\) and both \(\tilde{X}^1\) and \(\tilde{x}^i\) are normal coordinates based on the same basis \(\tilde{E}_3^i\) at \(i^0\), \(\tilde{X}^1 = \tilde{x}^1 + O(\tilde{r}^p)\), where \(p \geq 2\). But since \(R^2 = r^2(1 + O(r^{-1}))\) we have \(X^1 = x^1 + O(r^{2p})\), implying that \(\left(\frac{\partial}{\partial x^1}\right)^a = \left(\frac{\partial}{\partial \tilde{x}^1}\right)^a + O(r^{-1})\). Furthermore, the angle between the normals \(\tilde{\tau}_a\) and \(\tilde{t}_a\) of \(\tilde{\Sigma}_t\) and \(\Sigma_r\) is of order \(\tilde{r}^{p-1}\) if the leaves of the foliations are tangent to each other in the \((p - 1)\)th order. Thus the (hyperbolic) cosine of this angle is \(t_a \tau^a = \tilde{t}^a \tilde{\tau}_a^{(p)} g_{ab} = 1 + O(\tilde{r}^{p-1})\),
implying that $(\frac{\partial}{\partial t})^a = \xi^a = N t^a + N^a = N(\frac{\partial}{\partial t})^a + O(r^{3-2p}) + O(r^{-p})$. Therefore, the coordinate basis vectors $(\frac{\partial}{\partial x^i})^a$ tend to $(\frac{\partial}{\partial x^i})^a$ asymptotically, but $(\frac{\partial}{\partial r})^a$ tends to $(\frac{\partial}{\partial r})^a$ only if $p = 2$, and the lapse has the form $N = 1 + O(r^{-1})$ (whenever $\tau = t + c + O(r^{-1})$, where $c$ is a constant). Expanding the 1-form basis $(\nabla_c \tau, \nabla_c X^I)$ in terms of $(t_c, D_c x^I)$, where $D_c$ is the induced derivative operator in the leaves $\Sigma_c$, we can rewrite the general Killing vector of the Minkowski spacetime given in the first paragraph of this subsection. We obtain $K_c = R_{ijk}(x^i D_c x^j - x^j D_c x^i) + 2B_1(x^i t_c - t D_c x^i) + s_1 D_c x^i + st_c$, where $s(t, x^k) = s^{(0)}(t, \frac{x^k}{r}) + O(r^{-1})$ and $s^1(t, x^k) = s^{(1)}(t, \frac{x^k}{r}) + O(r^{-1})$. Its structure is similar to that of $K_c$ in the Cartesian coordinates, but instead of the constant components of the translations, $s$ and $s_1$ are functions of $t, \frac{x^k}{r}$ and higher powers of $r^{-1}$. Therefore, they are analogous to the supertranslations in the BMS group of the null infinity, and hence it is natural to call $st^a$ and $s^3(\frac{\partial}{\partial r})^a$ temporal and spatial supertranslations, respectively.

To summarize: although the global energy-momentum of the matter fields in Minkowski spacetime can be ensured to be finite by the $R^{-m}$ fall-off conditions, both at the spatial and the null infinity, to have finite angular momentum and centre-of-mass additional global integral conditions must also be imposed on the mass aspect of $T^{ab}$. At the spatial infinity these global integral conditions can be ensured by explicit parity conditions. To be able to recover the familiar Killing vectors in their usual form on a general asymptotically flat spacelike hypersurface $\Sigma$, at least asymptotically, the lapse $N$, defining the time coordinate $t$, must tend to 1 as $r \to \infty$.

### 2.3 Boundary conditions II.: Asymptotically flat spacetimes

Suppose that $\Sigma$ is asymptotically Euclidean in the sense that for some compact subset $K \subset \Sigma$ the complement $\Sigma - K$ is diffeomorphic to a finite disjoint union of manifolds $\Sigma_{(i)}$, each of which is diffeomorphic to $\mathbb{R}^s - B$, where $B$ is a solid ball in $\mathbb{R}^s$. The pieces $\Sigma_{(i)}$ are called the asymptotic ends of $\Sigma$. Since the next analysis can be repeated on each $\Sigma_{(i)}$, for the sake of simplicity we assume that there is only one such end. Suppose that there is a (negative definite) metric $\eta_{ab}$ on $\Sigma$ such that it is flat on the asymptotic end $\Sigma - K$. Let $\{x^i\}$ be a coordinate system on $\Sigma - K$ which is Cartesian with respect to $\eta_{ab}$, $r^2 := \delta_{ij} x^i x^j$, the radial distance function with respect to $\eta_{ab}$, and let $\eta_{\nu\mu}$ be the Levi-Civita covariant derivative operator corresponding to $\eta_{ab}$. Then the quotients $\frac{x^i}{r}$ can be interpreted as coordinates both on the unit sphere $S \approx S^{n-1}$ and the sphere $S_r$ of large coordinate radius $r$ in $\Sigma - K$, and the components $\eta^{ij}$ of the outward directed $\eta_{ab}$-unit normal $\xi^a$ to $\Sigma$ in the coordinate system $\{x^k\}$ too. By a ball of radius $r$ in $\Sigma$ we mean $B_r := \{ p \in \Sigma - K \mid r(p) \leq r \} \cup K$.

Let us consider the first, intuitively obvious condition of asymptotic flatness on the components of the metric and extrinsic curvature in the coordinate system $\{x^i\}$: for some positive $k$ and $l$

\begin{align}
q_{ij}(x^k) &= \eta_{ij} + \frac{1}{r^k} \eta_{ij}(\frac{x^k}{r}) + o(r^{-k}), \\
\chi_{ij}(x^k) &= \frac{1}{r^l} \chi_{ij}(\frac{x^k}{r}) + o(r^{-l}).
\end{align}

Therefore, the coefficients $q_{ij}(k), \chi_{ij}(l)$ can be interpreted as functions defined only on the unit sphere $S$. Following [4] and [5], in addition to the fall-off conditions we impose the following global parity conditions on the leading terms of the metric and extrinsic curvature:

\begin{align}
q_{ij}(k)(-\frac{x^k}{r}) &= q_{ij}(k)(\frac{x^k}{r}), \\
\chi_{ij}(l)(-\frac{x^k}{r}) &= -\chi_{ij}(l)(\frac{x^k}{r}).
\end{align}
are satisfied. However, we will see in subsection 4.2 that this notion depends on the background metric too, thus the properties of \( m_{ab} \) and, considering \( \Sigma \) to be a leaf of the foliation defined by a lapse function \( N(t, x) \) with respect to the time axis \( \xi^a \) are of odd parity. Furthermore, we assume that the ‘rests’ \( m_{ab} := g_{ab} - o g_{ab} - r^{-k} g_{ab}^{(k)} \) and \( k_{ab} := \chi_{ab} - r^{-l} \chi_{ab}^{(l)} \) satisfy the additional conditions

\[
\begin{align*}
0 & D_c m_{ab} = o(r^{-k-1}), \\
0 & D_{c0} D_c m_{ab} = o(r^{-k-2}), \\
0 & D_{c0} D_{c0} D_c m_{ab} = o(r^{-k-3}), \\
0 & D_{c0} D_{c0} D_{c0} D_c m_{ab} = o(r^{-k-4}), \\
0 & D_{c0} D_{c0} D_{c0} D_{c0} D_c m_{ab} = o(r^{-k-5}), \ldots
\end{align*}
\]  
\tag{2.3.3a}

\[
\begin{align*}
0 & D_c k_{ab} = o(r^{-l-1}), \\
0 & D_{c0} D_c k_{ab} = o(r^{-l-2}), \\
0 & D_{c0} D_{c0} D_c k_{ab} = o(r^{-l-3}), \\
0 & D_{c0} D_{c0} D_{c0} D_c k_{ab} = o(r^{-l-4}), \\
0 & D_{c0} D_{c0} D_{c0} D_{c0} D_c k_{ab} = o(r^{-l-5}), \ldots
\end{align*}
\]  
\tag{2.3.3b}

which imply that \( 0 D_{c0} \ldots 0 D_{c1} q_{ab} = O(r^{-(k+s)}) \), \( s = 1, 2, \ldots \), and, similarly, \( 0 D_{c0} \ldots 0 D_{c1} \chi_{ab} = O(r^{-(l+s)}) \). These properties make the calculations easier. The parity of these derivatives is \((-s)\) and \((-s+1)\), respectively. The properties \( m_{ab} = o(r^{-k}) \), \( 0 D_{c0} m_{ab} = o(r^{-k-1}) \), \( 0 D_{c0} D_{c0} m_{ab} = o(r^{-k-2}) \), \ldots \) of the rest \( m_{ab} \) will be denoted by \( m_{ab} = o^s(r^{-k}) \), and, similarly, \( k_{ab} = o^s(r^{-l}) \). Although in the actual calculations we will use these (essentially technical) assumptions for some finite value of \( s \), for the sake of simplicity we assume that \( m_{ab} = o^\infty(r^{-k}) \) and \( k_{ab} = o^\infty(r^{-l}) \). We may call the asymptotic end \( (\Sigma, q_{ab}, \chi_{ab}) \) to be \((k, l)\)-asymptotically flat if for some background metric \( o g_{ab} \) the conditions (2.3.1)-(2.3.3) are satisfied. However, we will see in subsection 4.2 that this notion depends on the background metric too, thus \( (\Sigma, q_{ab}, \chi_{ab}) \) will be called \((k, l)\)-asymptotically flat with respect to the background metric \( o g_{ab} \) if (2.3.1)-(2.3.3) are satisfied.

Now let us suppose that \( \mu, \ j^a \) and \( \sigma^{ab} \) satisfy the fall-off and parity conditions (2.2.3) with respect to the Cartesian coordinates \( x^k \) on \( \Sigma \) with the additional (technical) requirement that the ‘rests’ are of order \( o^\infty(r^{-m}) \). Next ask what conditions should we impose to ensure the existence of the limit

\[
Q_m[M, M^a] := \lim_{r \to \infty} \int_{B_r} \left( \mu M + j^a M_a \right) d\Sigma
\]  
\tag{2.3.4}

and, considering \( \Sigma \) to be a leaf of the foliation defined by a lapse function \( N(t, x) \), its time derivative

\[
\dot{Q}_m[M, M^a] = \lim_{r \to \infty} \int_{B_r} \left( \dot{\mu} (M - N^a D_a M + M^a D_a N) + \right.
\]
\[
\left. + j^a (M_a - M D_a N + N D_a M - L N M_a - 2 \chi_{ab} M^b N) + \sigma^{ab} N (M \chi_{ab} + D_{(a} M_{b)}) + \right.
\]
\[
\left. + D_a \left( (\mu M + j^b M_b) N^a - (j^a M + \sigma^{ab} M_b) N \right) \right) d\Sigma
\]  
\tag{2.3.5}

with respect to the time axis \( \xi^a := N^a + \dot{N^a} \). To obtain (2.3.5) we used (2.1.2)-(2.1.4). Note that if \( K^a = M^a + M^a \), then \( Q_{m[a[M, M^a]]} \) is just the \( n + 1 \) form of \( Q_{m[a[K^a]]} := \int K_a T_{abc} b_d d\Sigma \), and \( \dot{Q}_m[M, M^a] \) is the \( n + 1 \) form of \( \int (T_{abc} \nabla_a K_b) N + D_a ((P^a_{bc} - P^a_{bc}) \xi^b T_{c d} K_d)) d\Sigma \). Thus let us suppose that \( M \) and \( M_1 \), where the latter is defined by \( M_a =: M_1 \), have the asymptotic form\(^2\)

\[
M(t, x^k) = r^A M^{(A)}(t, \frac{x^k}{r}) + o^\infty(r^A), \quad (2.3.6a)
\]
\[
M_1(t, x^k) = r^B M_1^{(B)}(t, \frac{x^k}{r}) + o^\infty(r^B) \quad (2.3.6b)
\]

for some \( A, B \). Substituting these into (2.3.4) we obtain that \( Q_m[M, M^a] \) exists precisely if

\(^2\) To be consistent with our previous notations, we would have to write \( M^{(-A)} \) instead of \( M^{(A)} \). However, we resolve this apparent inconsistency with the convention that the power of \( r \) is always a capital, while the power of \( 1/r \) is always a lower case letter. In particular, the leading term in the expansion of \( f(r, \frac{x^k}{r}) \) can be written as \( r^A f^{(A)}(\frac{x^k}{r}) \) or \( r^{-a} f^{(a)}(\frac{x^k}{r}) \).
and if the equality holds in these inequalities, then the leading terms, $M^{(A)}(t, \frac{x^k}{r})$ and $M_i^{(B)}(t, \frac{x^k}{r})$, must be odd parity functions of $\frac{x^k}{r}$, respectively. The existence of $\dot{Q}_m[M, M^a]$ restricts $N$ and $N^a$ by means of which the conservation equations (2.1.3) and (2.1.4), and the consequence (2.1.2) of the definition of the ‘time derivative’ preserve the fall-off and parity conditions (2.2.3) for the matter fields and (2.3.1a-3a) for the metric, respectively. Writing $N$ and $N_i$ in the form of $M$ and $M_i$ given by (2.3.6) with some powers $C$ and $D$, substituting them into (2.1.3), (2.1.4) and (2.1.2) and requiring their right hand side to have $O(r^{-m})$ order and even parity, $O(r^{-m})$ order and even parity and $O(r^{-k})$ order and even parity, respectively, we obtain that

\[
N(t, x^k) = r^C N^{(C)}(t, \frac{x^k}{r}) + o^\infty(r^C), \tag{2.3.8a}
\]

\[
N_i(t, x^k) = 2x^k \rho_{ki} + \tau(t) + r^F \nu_i^{(F)}(t, \frac{x^k}{r}) + o^\infty(r^F), \tag{2.3.8b}
\]

where $\tau(t)$ and $\rho_{ij}(t) = -\rho_{ji}(t)$ are independent of the coordinates $\{x^k\}$ but may be arbitrary functions of the coordinate time $t$, the powers $C$ and $F$ satisfy

\[
C \leq \min\{1, l - k\}, \quad F \leq (1 - k), \tag{2.3.9}
\]

and if the equality holds then the leading terms $N^{(C)}(t, \frac{x^k}{r})$ and $\nu_i^{(F)}(t, \frac{x^k}{r})$ must be odd parity functions of $\frac{x^k}{r}$, respectively. (The first two terms on the right of (2.3.8b) together is just the kernel of the Killing operator $\partial D_{t1} t^1 = 0$ in (2.1.2).) Note, first, that by (2.3.9) $F < 1$, because we assumed that $k > 0$, and, second, that there is no reason to keep the term $\tau(t)$ in (2.3.8b) if $F > 0$. Substituting (2.3.6) and (2.3.8) into (2.3.5) and taking into account the conditions (2.3.7) and (2.3.9), one can check that the integral exists (and, in particular, the total divergence gives zero). If $K^a = Mt^a + M^a$ is a spacetime Killing vector then, by (2.1.10-12), $\dot{Q}_m[K^a]$ is zero, as it must be since then $T^{ab}K_b$ is divergence-free.

The notion of the $(k, l)$-asymptotic flatness is referring only to the asymptotic end of a single spacelike hypersurface $\Sigma$. To ensure that the spacetime itself, i.e. the evolution of $\Sigma$, is also asymptotically flat, we must ensure the compatibility of the boundary conditions with the evolution equations for the geometry. Thus let us consider the evolution equation (2.1.7) for the extrinsic curvature and ask under what additional conditions for $N$ and $N_a$ do these equations preserve the fall-off and parity conditions (2.3.1b-3b) for the extrinsic curvature. An analysis similar to that we did above yields that (2.1.7) preserves these asymptotic conditions precisely when

\[
N(t, x^k) = 2x^k \beta_k(t) + \tau(t) + r^E \nu^{(E)}(t, \frac{x^k}{r}) + o^\infty(r^E), \tag{2.3.10a}
\]

\[
N_i(t, x^k) = 2x^k \rho_{ki}(t) + \gamma_i(t) + r^F \nu_i^{(F)}(t, \frac{x^k}{r}) + o^\infty(r^F), \tag{2.3.10b}
\]

where $\tau(t)$ and $\beta_i(t)$ are independent of $\{x^k\}$ but may depend on $t$, the powers $E$ and $F$ satisfy

\[
E \leq (2 - l), \quad F \leq (1 - k), \tag{2.3.11}
\]

and if the equality holds in (2.3.11) then $\nu^{(E)}(\frac{x^k}{r})$ and $\nu_i^{(F)}(\frac{x^k}{r})$ are of odd parity, respectively. Then by (2.3.9) $E \leq C \leq (l - k)$, implying that $E \leq (1 - \frac{1}{2}k) < 1$ and $k + E \leq l \leq 2 - E$. In addition, if $\tau(t) \neq 0$ then
dependence of the coefficients in will see, important) difference is that the evolution equations, even in the presence of matter, allow the time

In the previous subsections the time axis

2.4 Allowed time axes and the asymptotic Killing vectors

In the previous subsections the time axis $\xi^a$ with respect to which the time evolution was defined and the generator $K^a$ of the physical quantity in $Q_m[K^a]$ were treated separately: While the role of $\xi^a$ was to provide a differential topological background, e.g. a foliation of the spacetime and a shift vector to carry out the analysis (general time axis), the nature of $K^a$ told us whether the corresponding $Q_m[K^a]$ should be interpreted e.g. as the energy or a component of the spatial angular momentum of the matter fields. Indeed, the lapse and shift parts of $K^a$ were defined with respect to the foliation that $\xi^a$ defined.

However, the structure (2.3.10-11) of $N$ and $N^a$ is compatible with (2.3.6-7), or, in other words, the time axes $\xi^a$ can be considered as special generators $K^a$. (Note that the vector fields $\xi^a$ and $K^a$ obtained here might be tangent to $\Sigma$ or even vanishing, and their lapse part might be positive on some subset and negative on other subsets of $\Sigma$. In spite of this fact we call $\xi^a$ an ‘allowed time axis’.) Furthermore, these two roles are mixed in the Hamiltonian formulation of the dynamics of the matter+gravity systems: $M$ and $M^a$ in the total Hamiltonian (in particular in its matter part $Q_m[M, M^a]$) play the role of the $n+1$ form of the generator $K^a$, and, at the same time, the Hamiltonian generates the time evolution of the states with respect to the spacetime vector field $\xi^a = M t^a + M^a$. Thus we assume that all the generators $M$, $M$, $M_i$, $\overline{M}_i$ also have the structure (2.3.10-11) of the allowed time axes and we write

$$
M = 2x^k B_k (t) + T(t) + r^E \mu^E + o^\infty (r^E), \\
M = 2x^k \tilde{B}_k (t) + \tilde{T}(t) + r^G \tilde{\mu}^G + o^\infty (r^G), \\
N = 2x^k \beta_k (t) + \tau(t) + r^K \nu^K + o^\infty (r^K), \\
N = 2x^k \beta_k (t) + \tau(t) + r^K \nu^K + o^\infty (r^K), \\
M_i = 2x^k \bar{R}_{k} (t) + \bar{T}_i (t) + r^F \bar{\mu}_i^F + o^\infty (r^F), \\
M_i = 2x^k \bar{R}_{k} (t) + \bar{T}_i (t) + r^H \bar{\mu}_i^H + o^\infty (r^H), \\
N_i = 2x^k \rho_{k1} (t) + \gamma_1 (t) + r^L \rho_{1}^{(L)} + o^\infty (r^L), \\
N_i = 2x^k \rho_{k1} (t) + \gamma_1 (t) + r^L \rho_{1}^{(L)} + o^\infty (r^L),
$$

where $E, F, G, H, K, L \leq (1 - k)$ and, in the case of equality here, the corresponding coefficient has odd parity. The space of the pairs $(M, M^a)$ on $\Sigma$ given by (2.4.1) will be denoted by $\mathcal{A}$. Considering $M$ to be the lapse of a (maybe degenerate) foliation of a neighbourhood of $\Sigma$ in the spacetime, $\mathcal{A}$ can also be interpreted as the space of the spacetime vector fields $\xi^a := M t^a + M^a$, where $t^a$ is the future pointing unit normal to the leaves of the foliation.

According to the double role of the components given by (2.4.1), we form another space of spacetime vector fields. Namely, if a lapse $N$ is given on $\Sigma$, then let $\mathcal{A}_N$ be the space of those spacetime vector fields $K^a := M t^a + M^a$ that are defined with respect to the (maybe degenerate) foliation determined by $\Sigma$ and $N$. (If $N$ has zeros, then not every element $(M, M^a)$ of $\mathcal{A}$ determines a spacetime vector field in $\mathcal{A}_N$: if the leaves $\Sigma_t$ and $\Sigma_{t'}$ of the foliation intersect each other at some point $p$, then only those elements $(M, M^a)$ of $\mathcal{A}$ can determine spacetime vector fields in $\mathcal{A}_N$, for which $M(t, p) t^a_p + M^a(t, p) = M(t', p) t'^a_p + M^a(t', p)$, where
\( t^a_b \) and \( t^m_n \) are the future pointing unit normal of \( \Sigma_t \) and \( \Sigma_{t'}, \) respectively, at \( p. \) The structure of the leading two terms e.g. of \( M \) and \( M_k \) resembles to that of the \( n+1 \) decomposition of the familiar spacetime Killing vectors of the Minkowski spacetime with respect to a spacelike hypersurface. However, although the first terms are linear in, and the second terms are independent of the spatial coordinates, the third terms (which would be analogous to the supertranslations of subsection 2.2) may depend on the spatial coordinates and may even be diverging. Moreover, although by (2.1.12) \( P^r_a P^d_b \nabla_{(a}K_{d)} = O(r^{-k}), \) i.e. tends to zero at infinity, in general neither \( P^a_b \nabla_{(b}K_{c)} \) nor \( t^a_b \nabla_{(a}K_{b)} \) tend to zero. The space \( \mathcal{A}_N \) does not form a Lie algebra. In fact, if \( K^a := Mt^a + M^a \) and \( \hat{K}^a := M\chi^a + \nu^a \) are any two elements of \( \mathcal{A}_N, \) then the first two terms of their Lie bracket in (2.1.15) are given by (2.1.10) and (2.1.11), respectively. Substituting (2.4.1) into these formulae we find that their asymptotic structure is dominated by \( N^{-1}x^m x^n, \) which deviates from that of the allowed time axes. Moreover, while the time dependence of the Killing vectors in a coordinate system adapted to the translations is very specific, the time dependence of \( M \) and \( M_k \) is not specified at all. In this subsection we use this freedom to specify a class of spacetime vector fields, whose elements can naturally be interpreted as asymptotic Killing vectors. The components of the spacetime vector fields that are only allowed time axes will be denoted by Greek letters, as those of \( N \) and \( N^a \) in (2.4.1).

We noted in subsection 2.1 that the parts \( P^a_b t^c \nabla_{(b}K_{c)} \) and \( t^a_b \nabla_{(a}K_{b)} \) of the Killing operator acting on some \( K^a \in \mathcal{A}_N \) can be required to be zero. Thus let us define the allowed time axis \( K^a = Mt^a + M^a \) to be a strong asymptotic Killing vector with respect to the foliation characterized by the lapse \( N \) if its components \( M \) and \( M^a \) satisfy (2.1.13) and (2.1.14). However, to ensure that the components of these asymptotic Killing vectors be well defined, a choice for the shift vector \( N^a \) must also be made. The set of these asymptotic Killing vector fields will be denoted by \( \mathcal{A}_N \), where \( \xi^a := Nt^a + N^a \), which will be assumed to be an allowed time axis. Obviously, this notion of asymptotic Killing vectors depends sharply on the lapse \( N \) and the hypersurface \( \Sigma \) (on which the foliation is based): if a neighbourhood of \( \Sigma \) in the spacetime is foliated by another lapse \( \bar{N} \) from (maybe) another hypersurface \( \bar{\Sigma} \) with normal \( \bar{\tau}^a, \) then the corresponding parts \( \bar{P}^a_b t^c \nabla_{(b}K_{c)} \) and \( \bar{P}^a_b \nabla_{(b}K_{c)} \) of the Killing operator will not be zero. (Perhaps, the notation \( \mathcal{A}_N \) is not very fortunate, and it would have to be denoted by \( \mathcal{A}^0_{(\Sigma, N); N^a} \) to stress that this notion of asymptotic Killing vectors depends both on \( \Sigma \) and \( N, \) and the asymptotic Killing vectors themselves are parameterized by using the shift \( N^a \) too.) Therefore, this notion of the asymptotic Killing vectors appears to be unnecessarily strong, and it could be enough to require that the parts \( P^a_b t^c \nabla_{(b}K_{c)} \) and \( t^a_b \nabla_{(a}K_{b)} \) of the Killing operator be at most of order \( O(r^{-k}) \) asymptotically, i.e. explicitly

\[
\begin{align*}
t^a_b \nabla_{(a}K_{b)} &= \frac{1}{N} \left( \dot{M} + M^a D_a N - N^a D_a M \right) = r^P \kappa^{(P)}(t, \frac{x^k}{r}) + o^\infty(r^P), \\
2 P^a_b t^c \nabla_{(b}K_{c)} &= \frac{1}{N} \left( \dot{M}_a + (ND_a M - MD_a N) - 2N\chi_{ab} M^b - L_N M_a \right) = (r^Q \kappa^{(Q)}(t, \frac{x^k}{r}) + o^\infty(r^Q)) D_a x^i,
\end{align*}
\]

for some \( P, Q \leq -k \) and if \( P \) and \( Q \) are equal to \( -k \) then \( \kappa^{(P)} \) and \( \kappa^{(Q)} \) have even parity, respectively. Note that (2.4.2) and (2.4.3) can always be solved for \( M \) and \( M_a \) for any given functions \( \kappa^{(P)}(t, \frac{x^k}{r}) \) and \( \kappa^{(Q)}(t, \frac{x^k}{r}). \) (2.4.2) and (2.4.3) will be called the asymptotic Killing equations. We call the vector field \( K^a \in \mathcal{A}_N \) asymptotic Killing vector with respect to the foliation determined by the lapse \( N \) if its components \( M \) and \( M^a \) are solutions of the asymptotic Killing equations. Clearly, the notion of the asymptotic Killing vectors is less sensitive to the deformation of \( \Sigma \) and \( N \) than that of the strong asymptotic Killing vectors, but it is still not independent of the foliation that \( N \) and \( \Sigma \) define. In particular, as we will see below, the asymptotic Killing vectors defined with respect to a foliation for which the lapse \( N \) tends to zero are
different from those defined with respect to ones for which $N \to 1$ as $r \to \infty$. The set of the asymptotic Killing vector fields will be denoted by $A^K_\xi$. Obviously, $A^K_\emptyset \subset A^K_\xi \subset A_N$ for any allowed time axis $\xi^a$, and $A_N$ can be injected into $A$.

Substituting (2.4.1) into the asymptotic Killing equations (2.4.2) and (2.4.3), we obtain

\begin{align}
\dot{B}_i &= -2\left(R_{ij} \beta^j - \rho_{ij} B^j\right), \\
\dot{R}_{ij} &= 2\left(B_i \beta_j - \beta_i B_j\right) - 2\left(R_{ik} \rho^{k} j - \rho_{ik} R^{k} j\right),
\end{align}

and, if $E, F \leq 0$, we also have

\begin{align}
\dot{T} &= -2\left(T_i \beta^i - \tau_i B^i\right), \\
\dot{T}_1 &= 2\left(T^j \beta_j - \beta_j T^1\right),
\end{align}

independently of $\kappa^{(P)}$ and $\kappa^{(Q)}$. Thus the coefficients $B_i$, $R_{ij}$, $T$ and $T_1$ in both of the elements of $A^K_\emptyset$ and $A^K_\xi$ satisfy (2.4.4-7). To calculate the Lie bracket of two asymptotic Killing vectors $K^a, \tilde{K}^a \in A^K_\xi$ explicitly, we should compute only the last three terms on the right hand side of (2.1.15), because, by (2.4.2) and (2.4.3), the first two are at most of order $O(r^{1-\kappa})$ in general, and zero if $K^a, \tilde{K}^a \in A^K_\emptyset$. The last three terms are

\begin{align}
M^a D_a \tilde{M} - M^a D_a M &= 4x^k \left(R_{ki} \tilde{B}^i - \tilde{R}_{ki} B^i\right) + 2\left(T_i \tilde{B}^i - T_1 B^i\right) + \\
&+ 4r^{1-\kappa} x^k \left(\tilde{R}_{ki} B_j - R_{ki} \tilde{B}_j\right) q^{(k)ij} + o^\infty (r^{1-\kappa}), \\
\left[M, \tilde{M}\right] D_a x^i &= 4x^k \left(R_{kj} \tilde{R}^j i - \tilde{R}_{kj} R^j i\right) + 2\left(T^j \tilde{R}_j i - T_j \tilde{R}_j i\right) + \\
&+ 4r^{1-\kappa} x^k \left(\frac{\tilde{R}_{kj} m j n - R_{kj} m j n}{r}\right) (\partial_m q^{(k)n i}) - \\
&\quad - (R_{kj} \tilde{R}_m i - \tilde{R}_{kj} R_m i) q^{(k)m i} - (R_{kj} \tilde{R}_j i - \tilde{R}_{kj} R_j i) q^{(k) j i} + o^\infty (r^{1-\kappa}), \\
(M D^a M - MD^a \tilde{M}) D_a x^i &= 4x^k \left(B_k \tilde{B}^i - B_k \tilde{B}^i\right) + 2\left(\tilde{T} B^i - T \tilde{B}^i\right) + \\
&+ 2G \left(\tilde{\mu}(E) B^i - \frac{\tilde{\mu}(E)}{r} B_k \left(G \tilde{\mu}(E) \frac{x^j}{r} + (\partial_j \tilde{\mu}(E)) \left(0 q^{ji} - \frac{j x^j}{r^2}\right)\right)\right) + o^\infty (r^G) - \\
&\quad - 2E \left(\mu(E) B^i - \frac{\mu(E)}{r} B_k \left(E \mu(E) \frac{x^j}{r} + (\partial_j \mu(E)) \left(0 q^{ji} - \frac{j x^j}{r^2}\right)\right)\right) + o^\infty (r^E),
\end{align}

where e.g. $\partial_j \mu(E)$ denotes the partial derivative of $\mu(E)$ with respect to its argument $\partial_j = \frac{x^j}{r}$, $q^{(k)ij} := o q^{(k)ij} o q^{(k)ij}$, and we used that $E, F, G, H \leq (1 - \kappa)$. The right hand side of (2.4.8-10) have the form of (the components of) an asymptotic Killing vector, and, as a simple calculation shows, the coefficients in $\tilde{K}^a := [K, \tilde{K}]^a$, given explicitly by

\begin{align}
\tilde{B}_i &= 2(R_{ij} \beta^j - \rho_{ij} B^j), \\
\tilde{R}_{ij} &= 2(R_{ik} \tilde{R}^{k} j - \tilde{R}_{ik} R^{k} j + \tilde{B}_i B_j - B_i \tilde{B}_j), \\
\tilde{T}_1 &= 2(T^j \tilde{R}_j i - T_j \tilde{R}_j i + \tilde{T} B_i - T \tilde{B}_i), \\
\tilde{T} &= 2(T_i \tilde{B}^i - T_1 B^i),
\end{align}
also satisfy (2.4.4-7). However, in general neither $\mathcal{A}_K^0$ nor $\mathcal{A}_K^0$ form a Lie algebra with respect to the spacetime Lie bracket. In fact, for any two spacetime vector fields $K^a$ and $\bar{K}^a$ one has $L_{[K,\bar{K}]}g_{ab} = L_K L_{\bar{K}} g_{ab} - L_{\bar{K}} L_K g_{ab}$, and for $K^a, \bar{K}^a \in \mathcal{A}_K^a$ one can form the parts $t^a t^b L_{[K,\bar{K}]} g_{ab}, P^a a^b L_{[K,\bar{K}]} g_{ab}$ and $P^a a^b L_{[K,\bar{K}]} g_{ab}$. Then, using (2.1.1) and the asymptotic Killing equations, one can show that these parts contain terms like $M, N$ for some $G, A_{\nu t}$ and for $\bar{G}, A_{\bar{\nu} \bar{t}}$. Thus for general lapse function $N$ the order of these parts is not $O(r^{-k})$. Similarly, $\mathcal{A}_K^0$ does not close to a Lie algebra either. On the other hand, by (2.4.8-10) both $\mathcal{A}_K^0$ and $\mathcal{A}_K^0$ are ‘essentially’ Lie algebras. Next we clarify in what sense do they form Lie algebras.

Let $\mathcal{G}$ denote the set of the special elements $(\nu, \nu^0)$ of $\mathcal{A}$, where

$$\nu(t, x^k) = r^M \nu^{(M)}(t, \frac{x^k}{r}) + o^\infty(r^M), \quad \nu_1(t, x^k) = r^N \nu_1^{(N)}(t, \frac{x^k}{r}) + o^\infty(r^N),$$

(2.4.15)

for some $M, N \leq (1 - k)$ and the leading terms have odd parity if $M = 1 - k$ and $N = 1 - k$, respectively. Repeating the construction above, $\mathcal{G}$ can also be considered as the space of the spacetime vector fields $\nu t^a + \nu^b$, where $\nu$ is considered to be the lapse of a (maybe degenerate) foliation and $t^a$ is the unit normal to the leaves of this foliation. If a lapse $N$ is given on $\Sigma$ then we can define $\mathcal{G}_N$ in a quite analogous way as we did above, and introduce $\mathcal{G}_K^0 := \mathcal{A}_K^0 \cap \mathcal{G}_N$. Then by (2.4.8-10) $\mathcal{G}_K^0$ behaves as an ideal in $\mathcal{A}_K^0$: $[K, \bar{K}]^a \in \mathcal{G}_K^0$ for any $k^a \in \mathcal{G}_K^0$ and $K^a \in \mathcal{A}_K^0$. We will see in subsections 3.2 and 3.3 below that, at least in the Hamiltonian framework, the theory’s gauge transformations are generated by precisely the elements of $\mathcal{G}$. Hence we call the elements of $\mathcal{G}$ gauge generators. Since $\mathcal{G}_K^0 \subset \mathcal{A}_K^0$ is a subspace, one may form the quotient space $\mathcal{A}_K^0 / \mathcal{G}_K^0$. This is spanned by the coefficients $B_1, R_{ij}, T_1$ and $T$, and by (2.4.11-14) it can be endowed with a natural Lie algebra structure. Similarly, although $\mathcal{A}_K^0$ is not closed with respect to the Lie bracket, it almost closes: just by (2.4.4-7) and (2.4.11-14) the Lie bracket $[K, \bar{K}]^a$ of any two $K^a, \bar{K}^a \in \mathcal{A}_K^0$ deviates from an element of $\mathcal{A}_K^0$ only by an element of $\mathcal{G}_N$. To determine the structure of $\mathcal{A}_K^0 / \mathcal{G}_K^0$ and of $\mathcal{A}_K^0$, we must evaluate (2.4.4-7) and (2.4.11-14).

Applying (2.4.4-7) to $N$ and $N_i$ themselves too, we obtain that $\tau, \tau_i, \beta_i$ and $\rho_{ij}$ are constant, i.e. apart from the gauge generator contents, the coefficients of an asymptotic Killing vector with respect to the differential topological background defined by itself are time independent. In particular, $\tau = \tau_i = \beta_i = \rho_{ij} = 0$ corresponds to time axes that are pure gauge generators $\xi^a = \nu t^a + \nu^a$, whenever the components $T, T_i, B_i$ and $R_{ij}$ of $K^a \in \mathcal{A}_K^0$ are all time independent, and the corresponding $M$ and $M_i$ reduce to those given by Beig and Ö Murchadha. However, we saw at the end of subsection 2.2 that such a time axis does not provide an appropriate framework in which even the familiar Killing vectors of the Minkowski spacetime could be recovered. To be able to recover them we had to assume that, with the notations of the present subsection, $\tau = 1$. Thus, if $\tau = 1$ and $\tau_i = \beta_i = \rho_{ij} = 0$ then the coefficients $T, B_i$ and $R_{ij}$ are time independent, but $T_i(t) = T_i - 2\dot{B}_i$, where $T_i$ is constant. Thus, the corresponding asymptotic Killing vector $K^a$ has exactly the same structure as that of the general Killing vector of the Minkowski spacetime in the coordinate system based on an asymptotically flat hypersurface approaching the Cartesian one. Therefore, the notion of the asymptotic Killing vectors does depend on the foliation coming from $\Sigma$ and $N$. Naturally, by (2.3.5) $Q_m[K^a]$ is still not conserved for general asymptotic Killing vectors, but, as we will see in subsection 4.1, the total energy-momentum and angular momentum of the matter+gravity system are already conserved.

If $E, F, ... > 0$ then the translation generators $T$ and $T_i$ in (2.4.1) cannot be isolated from the diverging gauge generators, and hence by (2.4.11-12) the quotient space $\mathcal{A}_K^0 / \mathcal{G}_K^0$ endowed with the Lie product (2.4.11-14) is isomorphic to the Lorentz Lie algebra so(1, $n$). If, however, $E, F, ... \leq 0$ then $T$ and $T_i$ are well defined in (2.4.1). In particular, if $E, F, ... < 0$, then asymptotically $T$ and $T_i$ dominate the (asymptotically vanishing) gauge generators. For $E, F, ... = 0$ the gauge generators do not vanish asymptotically (which therefore can be interpreted as supertranslations), but in the limiting case $E, F, ... = (1 - k) = 0$ the gauge generators have odd parity, while $T$ and $T_i$, being independent of the spatial coordinates, have even parity.
Thus the odd parity supertranslations are proper gauge generators, while the even parity ones, which are called the translations, belong to $A^K - G^K$. Thus, by (2.4.11-14), $A^K / G^K$ is isomorphic to the Poincare Lie algebra. Therefore, the algebra of the asymptotic Killing vectors modulo gauge generators depends on the fall-off of the metric: for slow $(0 < k < 1)$ fall-off we have only the Lorentz Lie algebra, but for faster $(k \geq 1)$ fall-off translations emerge naturally and we have a Poincare structure for $A^K / G^K$. This result should be intuitively obvious: if the asymptotic end is asymptotically flat in any sense then it is becoming spherical asymptotically and hence the rotation group (and its relativistic extension, the Lorentz group) emerges naturally, but the displacements of the centre of the asymptotic rotations become asymptotic symmetries, i.e. the asymptotic translations emerge and hence the symmetry group is the Poincare group, only if the geometry falls-off rapidly enough. Finally, we note that the fact that the coefficients $B_i, R_{ij}, T_i$ and $T$ satisfy (2.4.4-7) independently of whether $K$ belongs to $A^K$ or $A^0$ implies that the factor spaces $A^K / G^K$ and $A^0 / G^0$ are isomorphic, where $G^0 := G_N \cap A^0$, and hence $A^K / G^K$ also has the Lie algebra structure that $A^K / G^K$ does.

3. The Hamiltonian phase space of the vacuum GR

3.1 The (partially reduced) phase space and the constraints

Based on the $m = n + 1$ decomposition, by the a priori configuration variables we would have to mean the fields $N$, $N^a$ and $q_{ab}$ on a connected $n$ dimensional manifold $\Sigma$ of subsection 2.1 and hence $A^K$ Hamiltonian phase space could be partially reduced to the cotangent bundle $T^*Q$ of the (partially reduced) configuration space $Q := \{ q_{ab} + \text{boundary conditions} \}$. Thus our analysis will be based on the partially reduced configuration space $Q$ and its cotangent bundle $T^*Q$. Let $q_{ab}(u), u \in (-\epsilon, \epsilon)$, be any smooth 1-parameter family of metrics on $\Sigma$ from $Q$ for some $\epsilon > 0$. Then we define $\delta q_{ab} := (dq_{ab}(u)/du)_{u=0}$, which is the tangent vector of the curve $q_{ab}(u)$ at $q_{ab} := q_{ab}(0)$, i.e. $\delta q_{ab} \in T_{q_{ab}}Q$. Obviously, $\delta q_{ab}$ satisfies the same boundary conditions that $q_{ab}$ does. The elements of $T^*Q$ are the pairs $(q_{ab}, \tilde{p}^{ab})$, where

$$\tilde{p}^{ab} : T_{q_{ab}}Q \rightarrow \mathbb{R} : \delta q_{ab} \mapsto \tilde{p}^{ab} \delta q_{ab} := \int_{\Sigma} \tilde{p}^{ab} \delta q_{ab} d^nx.$$  \hspace{1cm} (3.1.1)

Thus the canonical momentum $\tilde{p}^{ab}$ is a contravariant symmetric tensor density of weight one on $\Sigma$, which is a 1-form on $Q$, and the requirement of the finiteness of its action on the tangent vectors $\delta q_{ab}$ gives boundary conditions for $\tilde{p}^{ab}$. The symplectic 2-form on $T^*Q$ is the canonical one: for any two tangent vectors $(\delta q_{ab}, \delta \tilde{p}^{ab}), (\delta' q_{ab}, \delta' \tilde{p}^{ab}) \in T_{(q_{ab},\tilde{p}^{ab})}(T^*Q)$ the value of the symplectic 2-form $\Omega_{(q_{ab},\tilde{p}^{ab})}$ is

$$\Omega_{(q_{ab},\tilde{p}^{ab})} : T_{(q_{ab},\tilde{p}^{ab})}(T^*Q) \times T_{(q_{ab},\tilde{p}^{ab})}(T^*Q) \rightarrow \mathbb{R}$$
$$: (\delta q_{ab}, \delta \tilde{p}^{ab}), (\delta' q_{ab}, \delta' \tilde{p}^{ab}) \mapsto \int_{\Sigma} (\delta \tilde{p}^{ab} \delta' q_{ab} - \delta' \tilde{p}^{ab} \delta q_{ab}) d^nx.$$ \hspace{1cm} (3.1.2)

In terms of the metric and extrinsic curvature the canonical momentum is well known to be

$$\tilde{p}^{ab} = \frac{1}{2K} \sqrt{q} \chi^{ab} - \chi q^{ab}.$$ \hspace{1cm} (3.1.3)

Thus the extrinsic curvature (i.e. by (2.1) the velocity $\dot{q}_{ab}$) can be expressed by the momenta: $\sqrt{q} \chi^{ab} = 2K(\tilde{p}^{ab} - \frac{1}{(n-1)} \tilde{p}^{ef} q_{ef} q^{ab})$.

The analysis of the field equations in the previous section lead us to the link $l = k + 1$ between the $r^{-k}$ and $r^{-l}$ a priori fall-off of the metric and extrinsic curvature. Thus, via (3.1.3), we obtain the fall-off
\[ p^{ij}(x^k) = \frac{1}{r^{k+1}} p^{ij}(k+1) \left( \frac{x^k}{r} \right) + o(r^{-(k+1)}) \] (3.1.4)
and the parity condition
\[ \bar{p}^{ij}(k+1) \left( \frac{x^k}{r} \right) = - \bar{p}^{ij}(k+1) \left( \frac{x^k}{r} \right), \] (3.1.5)
i.e. \( \bar{p}^{ij}(k+1) \) is of odd parity. In addition, the ‘rest’ \( \pi^{ab} := \bar{p}^{ab} - r^{-(k+1)} \bar{p}^{ab(k+1)} \) also satisfies the conditions
\[ q D_a \pi^{ab} = o(r^{k-2}), \quad o D_a D_b \pi^{ab} = o(r^{k-3}), \ldots \] (3.1.6)
These imply that \( q D_a \pi^{ab} = O(r^{-k+1}), \) \( s = 1, 2, \ldots, \) and the parity of the leading term is \( (-1)^{s+1}. \)

Following the notations of subsection 2.3, this property of the rest will be denoted by \( \pi^{ab} = o^\infty(r^{-k+1}) \).

By the requirement of the finiteness of (3.1.1) \( \bar{p}^{ab} \) must satisfy
\[ \oint_{s_r} \bar{p}^{ab} \delta q_{ab} dS_r = o(r^{-1}) \] (3.1.7)
for any \( \delta q_{ab} \), which implies that \( n \leq k + l = 2k + 1, \) i.e. \( k \geq \frac{1}{2}(n - 1). \) (If we wanted (3.1.7) to be satisfied without global integral conditions, and in particular the parity condition, then we would have to require \( n < k + l, \) which turns out to be too strong.) Thus the canonical momentum can be interpreted geometrically (as the integral kernel of a 1-form on \( Q \)) precisely when \( k \geq \frac{1}{2}(n - 1). \) In particular, if \( m = n + 1 \geq 4, \) then \( k \geq 1, \) and hence the Hamiltonian framework already excludes the possibility of a slower, e.g. \( r^{-\frac{3}{2}}, \) fall-off, and yields the Poincare structure for \( A_K^3 / G^K \). Slow fall-off is allowed only in 3 spacetime dimension. Since \( \delta \bar{p}^{ab} \) satisfies the same boundary conditions that \( \bar{p}^{ab} \) does, these asymptotic and parity conditions ensure that the canonical symplectic 2-form \( \Omega \) given pointwise by (3.1.2) is also well defined.

As is well known, although the two vacuum constraints \( c = 0 \) and \( c = 0, \) given by (2.1.5) and (2.1.6) in terms of the Lagrangian variables \( q_{ab} \) and \( \dot{q}_{ab}, \) do depend on the lapse \( N \) and the shift \( N^\alpha, \) their expressions by the canonical variables \( \bar{q}_{ab} \) and \( \bar{p}^{ab}, \)
\[
\tilde{c} := \tilde{c}(N, N^r, q_{ef}; q_{ef}(N, N^r, q_{cd}, \bar{p}^{cd})) = \frac{-1}{2K} \sqrt{|q|} \left( R + \frac{4K^2}{|q|} \left( \frac{1}{(n-1)}(\bar{p}^{ab} q_{ab})^2 - \bar{p}^{ab} \bar{p}_{ab} \right) \right),
\] (3.1.8a)
\[
\tilde{c}_a := \tilde{c}_a(N, N^r, q_{ef}; q_{ef}(N, N^r, q_{cd}, \bar{p}^{cd})) = -2 q_{ab} D_a \bar{p}^{bc},
\] (3.1.8b)
are independent of \( N \) and \( N^\alpha. \) Here \( \tilde{c} := \sqrt{|q|} c \) and \( \tilde{c}_a := \sqrt{|q|} c_a, \) the density weighted Lagrangian constraints, and we used the definition \( D_a \tau^{a}_{\hat{b}} := |q| \bar{p}^{a}_{\hat{b}} T^{a}_{\hat{b}} \) of the covariant derivative of the tensor density \( \tau^{a}_{\hat{b}} := |q| \bar{p}^{a}_{\hat{b}} \) of weight \( w, \) where \( T^{a}_{\hat{b}} \) is a tensor field. Asymptotically
\[
\tilde{c}(q_{ef}, \bar{p}^{ef}) = O(r^{-(k+2)}), \quad \tilde{c}_a(q_{ef}, \bar{p}^{ef}) = O(r^{-(k+2)}),
\] (3.1.9a, b)
and the leading terms are of even parity. The constraint functions \( C : T^* Q \rightarrow \mathbb{R} \) defining the constraint ‘surface’ \( \Gamma \) in \( T^* Q \) by their vanishing are
\[
C[\nu, \nu^\alpha] := \int_{\Sigma} \left( \tilde{c} (q_{cd}, \bar{p}^{cd}) \nu + \tilde{c}_a (q_{cd}, \bar{p}^{cd}) \nu^\alpha \right) d^n x.
\] (3.1.10)
Let $(q_{ab}(u), \tilde{p}^{ab}(u))$ be a curve in $T^{*}Q$ through the point $(q_{ab}, \tilde{p}^{ab})$. Then the derivative of the constraint function $C[\nu, \nu^\alpha]$ in this direction is

$$
\delta C[\nu, \nu^\alpha] = \int_{\Sigma} \left( \frac{\delta C[\nu, \nu^\alpha]}{\delta q_{ab}} \delta q_{ab} + \frac{\delta C[\nu, \nu^\alpha]}{\delta \tilde{p}^{ab}} \delta \tilde{p}^{ab} \right) \sqrt{\left| q \right|} d^a x - 
$$

$$
- \int_{\Sigma} D_e \left( \frac{\nu}{2\kappa} \left( q^{ab} D^c \delta q_{ab} - q^{ca} D^b \delta q_{ab} \right) + \frac{1}{2\kappa} (q^{ca} D^b \nu - q^{ab} D^e \nu) \right) \delta q_{ab} + 
$$

$$
+ \left( 2\nu^{ab} \tilde{p}^{bc} \delta q_{ab} - \nu^{ab} \tilde{p}^{bc} \delta q_{ab} + 2\nu_{a} \delta \tilde{p}^{ac} \right) \sqrt{\left| q \right|} d^a x, 
$$

where

$$
\frac{\delta C[\nu, \nu^\alpha]}{\delta q_{ab}} = \frac{1}{2\kappa} \sqrt{\left| q \right|} \nu \left( R^{ab} - R q^{ab} + 8\kappa^2 \left| q \right| \left( \tilde{p}^{ca} p^{bc} - \frac{1}{(n-1)} q_{cd} \tilde{p}^{cd} \tilde{p}^{ab} \right) \right) + 
$$

$$
+ D^a D^b \nu - q^{ab} D_e D^e \nu \right) - \frac{1}{2} \nu C q^{ab} - L_{\nu} \tilde{p}^{ab}, 
$$

(3.2.2)

$$
\frac{\delta C[\nu, \nu^\alpha]}{\delta \tilde{p}^{ab}} = \frac{4\kappa}{\sqrt{\left| q \right|}} \nu \left( \tilde{p}^{ab} - \frac{1}{(n-1)} \tilde{p}^{cd} q_{cd} q_{ab} \right) + L_{\nu} q_{ab}. 
$$

(3.2.3)

Here we used the definition $L_{\chi} \tau_{a...}^{\nu} := \left| q \right| \left| \chi \right| T_{b...}^{\nu} + w(\text{div} \chi) T_{b...}^{\nu}$ of the Lie derivative of the tensor density $\tau_{a...}^{\nu} := \left| q \right| \left| \chi \right| T_{b...}^{\nu}$ of weight $w$ along the vector field $X^a$, where $\text{div} \chi$ is the divergence $D_e X^e$ of $X^a$ with respect to the natural (metric) volume form. Since we would like to recover e.g. the familiar field equations in the Hamiltonian framework in their standard form instead of some of their distributional generalizations, we must require the functional differentiability of various functions on the phase space in the strong sense of [21]. Thus, in particular, the boundary terms in (3.2.1) must yield zero. Evaluating the leading order and parity of the terms in the total divergence of (3.2.1), it is easy to check that the fall-off and parity conditions imposed on $\nu$ and $\nu_i$ in (3.1.11) already imply the vanishing of the integral of the total divergence. Thus $C[\nu, \nu^\alpha]$ are already functionally differentiable.

The analysis of Beig and Ó Murchadha given in Appendix A of their paper [5] shows that the vanishing of the functional derivatives of $C[\nu, \nu^\alpha]$ together with the constraints $C[\nu, \nu^\alpha] = 0$ themselves imply the vanishing of $\nu$ and $\nu^\alpha$. Therefore, the constraint ‘surface’ $\Gamma \subset T^{*}Q$ that $C[\nu, \nu^\alpha] = 0$ defines is ‘nondegenerate’. It might be interesting to note that in the closed case (i.e. if $\Sigma$ is compact with no boundary) $C[\nu, \nu^\alpha]$ does have critical points on the ‘surface’ $C[\nu, \nu^\alpha] = 0$, and these critical points represent flat spacetimes [23].
The Hamiltonian vector field of a (functionally differentiable) function $F : T^* \mathcal{Q} \to \mathbb{R}$ is defined to be the vector field $X_F$ on $T^* \mathcal{Q}$ given explicitly by $X_F = (\delta F/\delta q_{ab}, -\delta F/\delta \bar{p}^{ab})$, and the Poisson bracket of two differentiable functions, $F$ and $G$, is defined by $\{F, G\} := 2\Omega(X_F, X_G) = X_F(G)$. Let $\nu, \bar{\nu}$ and $\nu_1, \bar{\nu}_1$ have the structure (3.1.11a) and (3.1.11b), respectively. Then the 'components' of the Hamiltonian vector field of the constraint functions $C[\nu, \nu^a]$ are given by (3.2.2) and (3.2.3), and the Poisson bracket of the constraint functions $C[0, \nu^a]$ and $C[0, \bar{\nu}^a]$ is

$$\{C[0, \nu^a], C[0, \bar{\nu}^a]\} = -\int \Sigma \left((L_{\nu}q_{ab})(L_{\bar{\nu}}\bar{p}^{ab}) - (L_{\bar{\nu}}q_{ab})(L_{\nu}\bar{p}^{ab})\right) d^n x =$$

$$= \int \Sigma D_e \left(\nu^c \bar{p}^{ab} L_{\nu}q_{ab} - \bar{\nu}^c \bar{p}^{ab} L_{\bar{\nu}}q_{ab} - 2\bar{\nu}^c f[n, \nu]_{f} \right) d^n x - \int \Sigma \bar{C}_{\alpha} [\nu, \bar{\nu}]^a d^n x. \tag{3.2.4}$$

In general, the integral of the total divergence on the right is not zero. The condition of its vanishing is $N + \bar{N} \leq k + 3 - n$, and this also ensures the existence of the second integral and that $[\nu, \bar{\nu}]^a$ has the structure (3.1.11b). The Poisson bracket of $C[0, \nu^a]$ and $C[\nu, 0]$ is

$$\{C[0, \nu^a], C[\bar{\nu}, 0]\} = \frac{1}{\kappa} \int \Sigma D_a \left(\nu^c \bar{p}^{ab} \bar{D}_b \nu - \bar{\nu}^c \bar{p}^{ab} D_b \nu\right) d^n x - \int \Sigma \bar{C}_{\alpha} \left(\nu D^a \nu - \nu D^a \bar{\nu}\right) d^n x. \tag{3.2.5}$$

The vanishing of the integral of the total divergence can be ensured by $N + M \leq 3 - n$. Furthermore, the condition of the finiteness of the second integral is $N + M \leq k + 3 - n$, which, at the same time, ensures that $\nu D^a \bar{\nu}$ has the structure (3.1.11a). Finally, the Poisson bracket of $C[\nu, 0]$ and $C[\bar{\nu}, 0]$ is

$$\{C[\nu, 0], C[\bar{\nu}, 0]\} = 2 \int \Sigma D_a \left(\nu^c \bar{p}^{ab} D_b \bar{\nu} - \bar{\nu}^c \bar{p}^{ab} D_b \nu\right) d^n x - \int \Sigma \bar{C}_{\alpha} \left(\nu D^a \nu - \nu D^a \bar{\nu}\right) d^n x. \tag{3.2.6}$$

Here the integral of the total divergence is vanishing if $M + \bar{M} \leq k + 3 - n$, which condition, at the same time, ensures the finiteness of the second integral and that $\bar{\nu} D^a \nu - \nu D^a \bar{\nu}$ has the structure (3.1.11b). Thus, to summarize, in addition to (3.1.11), the orders of the smearing fields must also satisfy $N + \bar{N}, M + \bar{M} \leq k + 3 - n$ and $N + M \leq 3 - n$. These conditions can be satisfied by requiring that the powers $M$ and $N$ in (3.1.11) satisfy

$$M, N \leq (1 - k). \tag{3.2.7}$$

By $k \geq \frac{1}{2}(n - 1)$ we have $2 + k - n = (1 - k) + 2k + 1 - n \geq (1 - k)$, i.e. the condition (3.2.7) is stronger than $M, N \leq 2 + k - n$, and, in fact, $N + M, N + \bar{N}, M + \bar{M} \leq 2(1 - k) \leq (3 - n) < 3 + k - n$. Note that, without further restrictions on the power $k$ or the dimension $n$, this is the greatest possible bound for $M$ and $N$, because, for the allowed smallest value of the rate of the fall-off of the metric, $k = \frac{1}{2}(n - 1)$, $N, M \leq (1 - k) = 2k - n$ and $N + M \leq 2(1 - k) = 3 - n$. Thus, under the condition (3.2.7), the expressions (3.2.4-6) give the familiar Lie algebra $\mathcal{C}$ of the constraint functions with the Lie product

$$\{C[\nu, \nu^a], C[\bar{\nu}, \bar{\nu}^a]\} = -C[L_{\nu} \bar{\nu} - L_{\bar{\nu}} \nu, [\nu, \bar{\nu}]^a - (\nu D^a \bar{\nu} - \bar{\nu} D^a \nu)]\]. \tag{3.2.8}$$

In particular, the Hamiltonian vector fields of the constraint functions are tangent to $\Gamma$ on $\Gamma$, i.e. $\mathcal{C}$ is a so-called first class constrained system. Next we clarify how this constraint algebra is related to $\mathcal{G}$.

The fall-off conditions (3.2.7) show that the smearing fields $\nu$ and $\nu^a$ are precisely the elements of $\mathcal{G}$. Thus, with the notation $k^a := \nu \nu^a + \nu^a$ in the spacetime picture, we can write $C[k^a] := C[\nu, \nu^a]$, and then
\( C : \mathcal{G} \to \mathcal{C} : k^a \mapsto C[k^a] \) is surjective, which is obviously linear. If \( C[k^a] \) were the zero function in \( \mathcal{C} \) for some \( k^a \in \mathcal{G} \), then the corresponding Hamiltonian vector field \( X_{C[k^a]} \) would also be zero. Thus, by the result of Beig and Ó Murchadha above, \( k^a = 0 \) would have to be held, i.e. \( C \) is a vector space isomorphism. But in general this is not a Lie algebra isomorphism. If, however, we fix a lapse \( N \) and \( k^a, \bar{k}^a \in \mathcal{G}_N \), then by (2.1.15) and the product law (3.2.8) we have

\[
\{ C[k^a], C[\bar{k}^a] \} = -C \left[ [k, \bar{k}]^a \right] + C \left[ \left( t^{a} k^{b} + 2 q^{ab} \right) \left( \nu \nabla_{(b} k_{c)} - \bar{\nu} \nabla_{(b} k_{c)} \right) t^{c} \right].
\]

In general the second term on the right is non-zero even for asymptotic Killing vectors \( k^a, \bar{k}^a \in \mathcal{G}_\xi^K \) for some \( \xi^a \). On the other hand, if we restrict the vector fields \( k^a \) and \( \bar{k}^a \) further to be in \( \mathcal{G}_\xi^K \) too, then the second term in (3.2.9) is vanishing. Thus although \( \mathcal{G}_\xi^0 \) is not closed respect to the Lie bracket, the restriction of the constraint function \( C \) to \( \mathcal{G}_\xi^0 \) mimics the injective Lie algebra (anti-)homomorphisms: the Poisson bracket of \( C[k^a] \) and \( C[\bar{k}^a] \) for \( k^a, \bar{k}^a \in \mathcal{G}_\xi^0 \) is just (minus) the constraint function at \( [k, \bar{k}]^a \in \mathcal{G}_N \).

To discuss the linear isomorphism \( C : \mathcal{G} \to \mathcal{C} \) further, recall that the flow on the phase space generated by the Hamiltonian vector field \( X_{C[k^a]} = (\delta C[k^a]/\delta \bar{p}^{ab}, -\delta C[k^a]/\delta q_{ab}) \) is the congruence of the integral curves of the differential equations

\[
\frac{dq_{ab}(u)}{du} = \frac{\delta C[k^a]}{\delta \bar{p}^{ab}}, \quad \frac{d\bar{p}^{ab}(u)}{du} = \frac{\delta C[k^a]}{\delta q_{ab}}.
\]

By (3.2.2), (3.2.3) and (3.1.3), on the constraint surface these are precisely the vacuum evolution equations (2.1.2) and (2.1.7) with lapse \( \nu \) and shift \( t^a \) with respect to the coordinate time \( t = u \). Let \( \phi_u \) be the local 1-parameter family of diffeomorphisms (on an open neighbourhood of \( \Sigma \) in the spacetime) generated by \( k^a \in \mathcal{G} \). Then, for small enough \( u \), its action on the canonical variables as fields on this neighbourhood is \( q_{ab} \mapsto q_{ab} - u q_{ab} \) and \( \bar{p}^{ab} \mapsto \bar{p}^{ab} - u \bar{p}^{ab} \). Thus the canonical variables \( q_{ab} \) and \( \bar{p}^{ab} \) are changing along the flow generated by the Hamiltonian vector field of \( C[k^a] \) on \( \Gamma \) exactly in the same way as under the action of the diffeomorphism generated by \( -k^a \) on the spacetime. Thus one may say that the flow of \( X_{C[k^a]} \) on \( \Gamma \) is the natural lift of the flow of \( -k^a \) from the spacetime to the constraint surface. In the next subsection we show that the theory’s gauge transformations on the constraint surface are generated precisely by the elements of \( \mathcal{G} \).

### 3.3 The gauge transformations

Since the Hamiltonian vector fields \( X_{C[k^a]}, k^a \in \mathcal{G} \), are tangent to \( \Gamma \) on \( \Gamma \), they belong to the kernel distribution

\[
\ker \Omega|_\Gamma := \{ (\delta q_{ab}, \delta \bar{p}^{ab}) \in TT \mid \Omega((\delta q_{ab}, \delta \bar{p}^{ab}), (\delta' q_{ab}, \delta' \bar{p}^{ab})) = 0 \ \forall \ (\delta' q_{ab}, \delta' \bar{p}^{ab}) \in TT \} \tag{3.3.1}
\]

of the pull back to \( \Gamma \) of the canonical symplectic 2-form \( \Omega \). Since \( \Omega \) is closed, this kernel distribution is always integrable. But, by definition, the reduced phase space, representing the physical degrees of freedoms, is the pair \( (\hat{\Gamma}, \hat{\Omega}) \), where \( \hat{\Gamma} \) is the set of the integral submanifolds of \( \ker \Omega|_\Gamma \) and \( \hat{\Omega} \) is the (necessarily well defined) projection of \( \Omega \) to \( \hat{\Gamma} \). Thus any integral submanifold of \( \ker \Omega|_\Gamma \) through a given point \( (q_{ab}, \bar{p}^{ab}) \in \Gamma \) is projected to a single point of \( \hat{\Gamma} \), and hence the vector fields on \( \Gamma \) belonging to the kernel distribution should be interpreted as infinitesimal gauge motions, i.e. generators of gauge transformations. Therefore, the Hamiltonian vector fields \( X_{C[k^a]} \) generate gauge transformations on the constraint surface for any \( k^a \in \mathcal{G} \).

In this subsection we show that the converse of this statement is also true, namely that any vector field on \( \Gamma \) belonging to \( \ker \Omega|_\Gamma \) (and represented by smooth fields on \( \Sigma \)), i.e. any infinitesimal gauge motion, is necessarily a Hamiltonian vector field \( X_{C[k^a]} \) for some \( k^a \in \mathcal{G} \). Thus first let us discuss this kernel.
By the definition of $\Omega$ the kernel of $\Omega|_\Gamma$ at $(q_{ab}, \tilde{p}^{ab})$ consists of all the vectors $(\delta q_{ab}, \delta \tilde{p}^{ab})$ tangent to $\Gamma$ for which $\int_\Sigma (\delta \tilde{p}^{ab} \delta q_{ab} - \delta' \tilde{p}^{ab} \delta q_{ab}) d^n x = 0$ for any vector $(\delta' q_{ab}, \delta' \tilde{p}^{ab})$ tangent to $\Gamma$ at $(q_{ab}, \tilde{p}^{ab})$.

To evaluate this condition for $(\delta q_{ab}, \delta \tilde{p}^{ab})$, we should take into account that $\delta' q_{ab}$ and $\delta' \tilde{p}^{ab}$ are not independent. In fact, since $\Gamma = \{(q_{ab}, \tilde{p}^{ab}) | \hat{C}(q_{ef}, \tilde{p}^{ef}) = 0, \hat{C}_a(q_{ef}, \tilde{p}^{ef}) = 0\}$, the tangents $(\delta q_{ab}, \delta \tilde{p}^{ab})$ of $\Gamma$ at $(q_{ab}, \tilde{p}^{ab})$ must satisfy the ‘linearized constraint equations’ $\delta' \hat{C} : = \left(\frac{\partial}{\partial a} \hat{C}(q_{ef}(u), \tilde{p}^{ef}(u)))\right)_{u=0} = 0$ and $\delta' \hat{C}_a : = \left(\frac{\partial}{\partial a} \hat{C}_a(q_{ef}(u), \tilde{p}^{ef}(u)))\right)_{u=0} = 0$. Multiplying them by an arbitrary function $\lambda$ and spatial vector field $\lambda^a$, respectively, and adding them together we obtain

\[
0 = \lambda \delta' \hat{C} + \lambda^a \delta' \hat{C}_a = \delta C[\lambda, \lambda^c]/\delta q_{ab} \delta' q_{ab} + \frac{\delta C[\lambda, \lambda^c]}{\delta \tilde{p}^{ab}} \delta' \tilde{p}^{ab} + D_q \left\{ \frac{1}{2\kappa} \sqrt{|q|} \lambda^a D^B \lambda - q^{ab} D^B \lambda - \frac{1}{2\kappa} \sqrt{|q|} \lambda (q^{ab} D^B \lambda - q^{ab} D^B \lambda) \right\}.
\]

where $\delta C[\lambda, \lambda^c]/\delta q_{ab}$ and $\delta C[\lambda, \lambda^c]/\delta \tilde{p}^{ab}$ are given formally by (3.2.2) and (3.2.3). Adding the integral of (3.3.2) to $\int_\Sigma (\delta \tilde{p}^{ab} \delta q_{ab} - \delta' \tilde{p}^{ab} \delta q_{ab}) d^n x = 0$ we obtain

\[
\int_\Sigma \left\{ \left(\frac{\delta \tilde{p}^{ab}}{C[\lambda, \lambda^c]/\delta q_{ab}}\right) \delta' q_{ab} - \left(\delta q_{ab} - \frac{\delta C[\lambda, \lambda^c]}{\delta \tilde{p}^{ab}}\right) \delta' \tilde{p}^{ab} + D_a \dot{W}^a(\lambda, \lambda^c) \right\} d^n x = 0.
\]

where $D_a \dot{W}^a(\lambda, \lambda^c)$ denotes the total divergence in (3.3.2). But from (3.3.3) we can read off the vanishing of the coefficients of $\delta' q_{ab}$ and $\delta' \tilde{p}^{ab}$ (by the Lagrange lemma of the elementary calculus of variations) only if the integral of $D_a \dot{W}^a(\lambda, \lambda^c)$ is vanishing. From (3.3.3) it follows that it is vanishing if $\lambda, \lambda^a$ have the asymptotic form (3.1.11), and hence, in particular, $C[\lambda, \lambda^a]$ exists. Therefore, the generators $\lambda, \lambda^a$ of the infinitesimal gauge transformations are precisely the smearing fields.

Since the vanishing of the Hamiltonian vector field $X_{C[k^a]}$ implies the vanishing of the vector field $k^a$ itself, the gauge transformations generated by $C[k^a]$ are effective. The condition ensuring that the infinitesimal gauge transformations close to a Lie algebra are the stronger fall-off (3.2.7), which are precisely the fall-off properties of the elements of $G$.

### 3.4 The Hamiltonian

The aim of this subsection is a concise rederivation of the Hamiltonian of Beig and Ó Murchadha, to determine the exact and most general boundary conditions for $M$ and $M^a$ for which the Hamiltonian is well defined and differentiable, and to see that $M$ and $M^a$ may still be arbitrary functions of time. Thus let us start with the ‘basic Hamiltonian’

\[
H_0[M, M^a] : = C[M, M^a] = \int_\Sigma (\hat{C} M + \hat{C}_a M^a) d^n x
\]

and determine that total divergence $D_a \dot{Z}^a$ for which the ‘total Hamiltonian’ $H[M, M^a] : = C[M, M^a] + \int_\Sigma D_a \dot{Z}^a d^n x$ is well defined even for the pairs $(M, M^a)$ defining general allowed time axes, or at least asymptotic Killing vectors. Here $\dot{Z}^a$ is expected to be a local expression of the canonical variables, the fields $M$ and $M^a$ and their spatial derivatives up to some finite order.

Let the difference of the physical and background connections be characterized by $\Gamma^c_{ab} X^b : = (D_a - \partial D_a X^b)$, which $\Gamma^c_{ab}$ is given explicitly by $\Gamma^c_{ab} = \frac{1}{2} q^{cd}(D_a D_b q_{cd} + D_b q_{ad} + D_a q_{bd})$. Then the physical curvature scalar of $q_{ab}$ can be written as $R = q^{ab} q^{cd}(D_a D_b q_{cd} - D_a D_c q_{bd}) + q_{ab} q^{ef}(\Gamma^c_{ab} q_{ef} - \Gamma^d_{ae} q^{ef} \Gamma^d_{bf})$, and, following Beig and Ó Murchadha, we write
\[ H_0[M,0] = -\frac{1}{2\kappa} \int_S \left( M q^{ab} q^{cd} (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) + \right. \\
+ M q_{cd} q^{ab} \psi^f (\Gamma_{ab}^c \Gamma_{cf}^d - \Gamma_{ac}^d \Gamma_{bf}^e) + \frac{4\kappa^2}{\sqrt{|q|}} M \left( \frac{1}{n-1} (\tilde{p}^{ab} q_{ab})^2 - \tilde{p}^{ab} \tilde{p}_{ab} \right) \right) \sqrt{|q|} |d^n x. \]

(3.4.2)

The integral of the second and third terms on the right is finite if

\[ M(t, x^k) = 2x^k B_k(t) + T(t) + x^K \mu(t, \frac{x^k}{r}) + o^\infty(r^K), \]

(3.4.3a)

where \( K \leq 2k + 2 - n \), and if the equality holds here then \( \mu(t, \frac{x^k}{r}) \) must be an odd parity function of \( \frac{x^k}{r} \).

Here we also had to use \( k \geq \frac{1}{2}(n-1) \) if \( B_1 \neq 0 \). On the other hand, without additional restrictions on \( k \) and \( n \), the first term has finite integral only for those functions (3.4.3a) in which both \( B_1 \) and \( T \) are vanishing. Since the first term is not a pure total divergence, we should write this as the sum of a total divergence and terms that already yield finite integral even for \( M \) above. Beig and Ó Murchadha wrote this ‘wrong’ term as

\[
M q^{ab} q^{cd} (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) \sqrt{|q|} = \\
= D_a \left( M q^{ab} q^{cd} (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) \sqrt{|q|} \right) - M D_a (q^{ab} q^{cd} \sqrt{|q|}) (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) - \\
- \left( \partial_a M \right) q^{ab} q^{cd} (\partial_a (q_{cd} - q_{cd}) - D_c (q_{bd} - q_{bd})) \sqrt{|q|} = \\
= D_a \left( M q^{ab} q^{cd} (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) \sqrt{|q|} \right) - M D_a (q^{ab} q^{cd} \sqrt{|q|}) (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) - \\
- D_a \left( (\partial_a M) q^{ab} q^{cd} (q_{cd} - q_{cd}) \sqrt{|q|} - (\partial_a D_b M) q^{bc} q^{cd} (q_{cd} - q_{cd}) \sqrt{|q|} \right) + \\
+ D_b \left( (\partial_a M) q^{ab} q^{cd} \sqrt{|q|} \right) (q_{cd} - q_{cd}) - D_c \left( (\partial_a M) q^{ab} q^{cd} \sqrt{|q|} \right) (q_{bd} - q_{bd}) = \\
= - D_a \left( M q^{ab} q^{cd} (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) \right) + (\partial_a D_b M) q^{ab} q^{cd} (q_{cd} - q_{cd}) - (\partial_a D_c M) q^{ab} q^{cd} (q_{bd} - q_{bd}) \sqrt{|q|} + \\
+ D_a (q^{ab} q^{cd} \sqrt{|q|}) (-M (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) + (\partial_a D_b M) (q_{cd} - q_{cd}) - (\partial_a D_c M) (q_{bd} - q_{bd})).
\]

(3.4.4)

The integral of the second and third terms on the right of (3.4.4) is finite if \( M \) has the form (3.4.3a) where \( K \) satisfies the stronger condition \( K \leq k + 2 - n \), and if the equality holds in this inequality then \( \mu(t, \frac{x^k}{r}) \) is an odd parity function of \( \frac{x^k}{r} \). Then, since \( K \leq k + 2 - n < 2k + 2 - n \),

\[
H[M,0] := C[M,0] - \frac{1}{2\kappa} \int_S \left( M q^{ab} q^{cd} (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) + (\partial_a D_b M) q^{ab} q^{cd} (q_{cd} - q_{cd}) - \\
- (\partial_a D_c M) q^{ab} q^{cd} (q_{bd} - q_{bd}) \right) |d^n x.
\]

(3.4.5)

is already well defined. If \( (\delta q_{ab}, \delta \tilde{p}^{ab}) \) is any tangent vector at \( (q_{ab}, \tilde{p}^{ab}) \in T^* Q, \) then the derivative of \( H[M,0] \) in the direction \( (\delta q_{ab}, \delta \tilde{p}^{ab}) \) is

\[
\delta H[M,0] = \left( \delta C[M,0] \frac{\delta q_{ab}}{\delta q_{ab}} + \delta C[M,0] \frac{\delta \tilde{p}^{ab}}{\delta \tilde{p}^{ab}} \right) |d^n x + \frac{1}{2\kappa} \int_S \left( M q^{ab} q^{cd} \sqrt{|q|} (\Gamma_{de} q_{be} - \Gamma_{be} q_{de}) + \\
+ \delta (q^{ab} q^{cd} \sqrt{|q|}) (M (\partial_a D_b q_{cd} - \partial_b D_c q_{ab}) + (\partial_a D_b M) (q_{cd} - q_{cd}) - (\partial_a D_c M) (q_{bd} - q_{bd})).
\]

(3.4.6)
where $\delta C[M,0]/\delta g_{ab}$ and $\delta C[M,0]/\delta \bar{p}^{ab}$ are given formally by (3.2.2) and (3.2.3). (Strictly speaking, these are not functional derivatives of $C[M,0]$, because the constraint function $C$ is not well defined for $M$ above.) Since, however, the integral of the total divergence on the right of (3.4.6) is vanishing for the functions $M$ given by (3.4.3a), $H[M,0]$ is functionally differentiable with respect to the canonical variables too.

We can write

$$H_0[0, M^a] = -2 \int_\Sigma (D_a \bar{p}^{ab}) M_b d^n x = 2 \int_\Sigma \left( \bar{p}^{ab} D_a (\bar{p}^{bc} M_c - D_a (\bar{p}^{ab} M_b)) \right) d^n x. \quad (3.4.7)$$

The integral of the first two terms on the right is well defined even for vector fields $M^a$ of the form

$$M_i(t, x^k) = 2x^k R_{k1}(t) + T_i(t) + \nu^L \mu^{(L)}(t, \frac{x^k}{\nu}) + o^\infty(\nu^L), \quad (3.4.3b)$$

where $L \leq k + 2 - n$, and if the equality holds in this inequality then $\mu^{(L)}(t, \frac{x^k}{\nu})$ is an odd parity function of $\frac{x^k}{\nu}$. Note that, to prove the existence of the integral of the first two terms for nonzero $R_{k1}$ in (3.4.3b) we also had to use $k \geq \frac{1}{2}(n - 1)$. The integral of the third term on the right of (3.4.7) is, however, finite only for $R_{k1}(t) = 0$ and $T_i(t) = 0$. Thus

$$H[0, M^a] := C[0, M^a] + 2 \int_\Sigma D_a (\bar{p}^{ab} M_b) d^n x = \int_\Sigma \bar{p}^{ab} \mathcal{L}_M g_{ab} d^n x \quad (3.4.8)$$

is well defined even for vector fields $M^a$ with the structure (3.4.3b). The derivative of $H[0, M^a]$ in the direction $(\delta q_{ab}, \delta \bar{p}^{ab})$ is

$$\delta H[0, M^a] = \int_\Sigma \left( \mathcal{L}_M g_{ab} \delta \bar{p}^{ab} - \mathcal{L}_M \bar{p}^{ab} \delta q_{ab} + D_c (M^c \bar{p}^{ab} \delta q_{ab}) \right) d^n x. \quad (3.4.9)$$

Since for the vector fields $M^a$ above the integral of the total divergence is zero, $H[0, M^a]$ is differentiable with respect to the canonical variables too.

Therefore, the Hamiltonian $H[M, M^a] := H[M,0] + H[0, M^a]$ of Beig and Ó Murchadha is finite and functionally differentiable with respect to the canonical variables even for $M$ and $M^a$ given by (3.4.3) with $K, L \leq k + 2 - n$, and if the equality holds in these inequalities then $\mu^{(K)}$ and $\mu^{(L)}$ must be odd parity functions of $\frac{x^k}{\nu}$, respectively. However, the spacetime vector field $K^a := M^a + M^a$ is still not needed to be an asymptotic Killing vector field with respect to some foliation (and not even an allowed time axis), because the fall-off rates $K$ and $L$ are still required only to satisfy $K, L \leq k + 2 - n$ (instead of $K, L \leq (1 - k)$). Moreover, $R_{k1}, B_i, T_i$ and $T$ may still have arbitrary time dependence.

### 3.5 The algebra of the Hamiltonians and the asymptotic symmetries

On $\Gamma$ the system of equations

$$\frac{dq_{ab}}{dt} = \delta H[M, M^c], \quad \frac{d\bar{p}^{ab}}{dt} = -\frac{\delta H[M, M^c]}{\delta q_{ab}}, \quad (3.5.1)$$

defining the Hamiltonian flow on $\Gamma$, is precisely the system of the vacuum evolution equations with lapse $M$ and shift $M^a$. However, this system still does not preserve the boundary conditions (2.3.1a-3a) and (3.1.4-6) for the canonical variables, because the regularity and functional differentiability of the Hamiltonian $H[M, M^a]$ implied only $K, L \leq k + 2 - n$, which is weaker than $K, L \leq (1 - k)$. Thus, based on the analysis of subsection 2.3, we must require that the powers $K$ and $L$ satisfy

$$K, L \leq (1 - k). \quad (3.5.2)$$
Therefore, \( K^a = M t^a + M^a \) already has the asymptotic form (2.4.1), i.e. \( K^a \) must be an element of \( \mathcal{A} \), and the restriction of the Hamiltonian \( H \) to the pure gauge generators \( \mu \) and \( \mu^a \) (i.e. for which \( \mu t^a + \mu^a \in \mathcal{G} \)), is just \( C[\mu, \mu^a] \).

Repeating the analysis of subsection 3.2, one can show that (3.5.2) ensures the vanishing of the boundary terms appearing in the calculation of the Poisson brackets of two Hamiltonians, \( H[M, M^a] \) and \( H[M, \dot{M}^a] \), and, by (2.4.8-10), that \( M^a D_a \dot{M} - \dot{M}^a D_a M \) has the structure of a lapse and \( [M, \dot{M}]^a \) and \( \dot{M} D^a M - M D^a \dot{M} \) have the structure of a shift satisfying (3.5.2). The resulting Poisson algebra of the Hamiltonians \([5]\) (see also \([4,28]\)) is

\[
\left\{ H[M, M^a], H[\dot{M}, \dot{M}^a] \right\} = -H \left[ I_M \dot{M} - I_M \dot{M}, [M, \dot{M}]^a - (MD^a \dot{M} - \dot{M} D^a M) \right].
\]  

(3.5.3)

For pure gauge generators (3.5.3) reduces to (3.2.8), and the Poisson bracket of \( H[M, M^a] \) and \( C[\nu, \nu^a] \) is also a constraint function. Thus the Poisson algebra \( C \) of the constraint functions is an ideal in the Poisson algebra \( \mathcal{H} \) of the Hamiltonians parameterized by the allowed time axes \( K^a \in \mathcal{A} \). The structure of \( \mathcal{H} \) can be determined easily by considering the Hamiltonians parameterized by the special allowed time axes like \( K^a = 2 x^k B_k(t) x^a \), \( K^a = 2 x^2 R_1(t)(\frac{d}{dt})^a \), ... etc. With this parameterization of the Hamiltonians (3.5.3) shows that, for each fixed value \( t \) of the coordinate time, the factor of \( \mathcal{H} \) with the ideal \( C \) is just the Poincare algebra. Thus \( \mathcal{H}/C \) is infinite dimensional. If, however, the coefficients \( B_1, R_{ij}, T_i \) and \( T \) are restricted to be the coefficients in the asymptotic Killing vectors with respect to some \( \xi^a \) as in subsection 2.4, then the whole factor \( \mathcal{H}/C \) would be finite dimensional, and, in fact, the Poincare algebra.

As we noted in subsection 2.4, the space \( \mathcal{A} \) of the allowed time axes does not form a Lie algebra with the natural Lie bracket in general. Hence the Poisson algebra \( \mathcal{H} \) of all the Hamiltonians, indexed by the elements of \( \mathcal{A} \), does not seem to be connected in a natural way to some naturally defined Lie algebra of spacetime vector fields. However, restricting the spacetime vector fields \( K^a \) and \( \dot{K}^a \) to be from the subspace \( \mathcal{A}_K^0 \) of the space \( \mathcal{A}_K^{\xi} \) of the asymptotic Killing vectors for some \( \xi^a \) and writing \( H[K^a] := H[M, M^a] \), by (2.1.15) the product law (3.5.3) takes the remarkably simple form

\[
\left\{ H[K^a], H[\dot{K}^a] \right\} = -H \left[ [K, \dot{K}]^a \right].
\]  

(3.5.4)

If \( K^a \) and \( \dot{K}^a \) are allowed to be from \( \mathcal{A}_K^0 \), then the first two terms on the right of (2.1.15) give a constraint function with uncontrollable generators, as in (3.2.9), and we have only

\[
\left\{ H[K^a], H[\dot{K}^a] \right\} + H \left[ [K, \dot{K}]^a \right] \in C(\mathcal{G}) = C.
\]  

(3.5.5)

Therefore, the set \( \mathcal{H}_K := H(\mathcal{A}_K^0) \) of the Hamiltonians parameterized by the asymptotic Killing vectors \( K^a \) with respect to \( \xi^a \) is the Lie product preserving image of \( \mathcal{A}_K^{\xi} \) modulo constraints, and on the elements of the subspace \( \mathcal{A}_K^0 \) the Hamiltonian \( H \) preserves the Lie product.

Finally, let \( M, N, N^a \) and \( M^a \) be as in (2.4.1), and calculate the total time derivative of \( H[M, M^a] \) along \( \xi^a := N t^a + N^a \). Since \( M \) and \( M^a \) may depend on \( t \), the derivative consists of two terms:

\[
\begin{align*}
\frac{d}{dt} H[M, M^a] &= H[\dot{M}, \dot{M}^a] + \int \left( \delta H[M, M^a] \frac{\delta \rho_{ab}}{\delta \rho_{ab}} + \frac{\delta H[M, M^a]}{\delta q_{ab}} q_{ab} \right) d^a x = \\
&= H[\dot{M}, \dot{M}^a] + \left\{ H[N, N^a], H[M, M^a] \right\} = \\
&= H[\dot{M} + L_M N - L_M N, \dot{M}^a + N D^a M - M D^a N - [N, M]^a].
\end{align*}
\]  

(3.5.6)

Here first we used (3.5.1) (which, on the constraint surface, are the vacuum evolution equations), and then (3.5.3). If \( K^a \in \mathcal{A}_K^0 \), then by (2.1.13) and (2.1.14) the right hand side is zero, while for \( K^a \in \mathcal{A}_K^{\xi} \) the right
hand side is a constraint function by the asymptotic Killing equations (2.4.2) and (2.4.3). Therefore, the Hamiltonian of Beig and Ó Murchadha is constant along any allowed time axis $\xi^a$ modulo constraints for the asymptotic Killing vectors $K^a \in A^K_\xi$, and it is strictly constant for vectors $K^a$ that are strong asymptotic Killing with respect to the time axis $\xi^a$.

To summarize: First, to ensure that e.g. the symplectic 2-form be well defined, in addition to the result $l = k + 1$ of the analysis of subsection 2.3, we had to assume that $k \geq \frac{1}{2}(n-1)$. Then the constraint functions are well defined, functionally differentiable and close to a Lie algebra precisely for those smearing fields that correspond to the elements of $G$ itself. The constraint function preserves the Lie product of the elements of the space $G^0_\xi$ in the Poisson algebra. The Hamiltonian of Beig and Ó Murchadha, parameterized by the allowed time axes $K^a$, are finite, functionally differentiable, close to an infinite dimensional Lie algebra, and the Hamilton equations preserve the boundary conditions for the canonical variables. The Hamiltonian preserves the Lie product of the elements of the space $A^0_\xi$, and preserves the Lie product of the elements of the space $G^K_\xi$ modulo constraints. The Beig–Ó Murchadha Hamiltonian $H[K^a]$ is constant in time with respect to $\xi^a$ for any $K^a \in A^0_\xi$, but it is only constant modulo constraints for the asymptotic Killing vectors $K^a \in A^K_\xi$.

4. The ADM conserved quantities of matter+gravity systems

4.1 The ADM conserved quantities

In the complete Hamiltonian description the matter fields would have to be included. However, a detailed Hamiltonian analysis of the matter fields is not needed if we are interested only in the ADM energy-momentum and angular momentum, because the value of the Hamiltonian of the matter fields on the matter constraint surface (if there is any, as in the spacial case of Yang–Mills fields) is expected to be $Q_m[K^a]$ for some allowed time axis $K^a \in A$. Then the gravitational constraint, i.e. a part of Einstein’s equation, is $Q_m[K^a] + C[K^a] = 0$. Thus, the value of the total Hamiltonian on the constraint surface, $H[K^a]|_\Gamma + Q_m[K^a]$, is given by the same surface term as in the vacuum case, and it has the structure $H[K^a]|_\Gamma + Q_m[K^a] = T(t)p^0 + T_i(t)p^i + R_{ij}(t)j^{ij} + 2B_i(t)j^{i0}$, where $T(t)$, $T_i(t)$, $R_{ij}(t)$ and $B_i(t)$ are the functions appearing e.g. in the form (2.4.1) of the allowed time axis $K^a$. If, however, $K^a \in A^K_\xi$ for some $\xi^a \in A$, whenever the functions $T(t)$, $T_i(t)$, $R_{ij}(t)$ and $B_i(t)$ can be represented by the $\frac{1}{2}m(m+1)$ independent parameters $T$, $T_i$, $R_{ij}$ and $B_i$, the evaluation of $H[K^a] + Q_m[K^a]$ on $\Gamma$ above defines a linear mapping $A^K_\xi/G^K_\xi \approx A^0_\xi/G^0_\xi \rightarrow \mathbb{R}$, whose components $P^0$, $P^i$, $J^{ij}$ and $J^{i0}$ define the energy, the linear momentum, the spatial angular momentum and centre-of-mass, respectively, via $H[K^a]|_\Gamma + Q_m[K^a] =: TP^0 + T_iP^i + R_{ij}J^{ij} + 2B_iJ^{i0}$. In particular, if $K^a \in A^K_\xi$ and $\xi^a \in G$, whenever the functions $T(t)$, $T_i(t)$, $R_{ij}(t)$ and $B_i(t)$ are all constant then, as we will see, $P^0$ and $P^i$ are just the familiar ADM energy and linear momentum, respectively, $J^{ij}$ is the angular momentum of Regge and Teitelboim and $J^{i0}$ is the centre-of-mass given by Beig and Ó Murchadha. If, however, $K^a \in A^K_\xi$ but $\xi^a - t^a \in G$, whenever $T(t)$, $R_{ij}(t)$ and $B_i(t)$ are still constant but $T_i(t)$ changes in time as $T_i(t) = T_i - 2tB_i$ (just according to the expression of the $n+1$ form of the boost Killing vectors of the Minkowski spacetime), then $P^0$, $P^i$ and $J^{ij}$ remain the same but the centre-of-mass $J^{i0}$ deviates from that of Beig and Ó Murchadha by the term $P^b$. Since $P^a$ and $J^{ab}$, $a, b = 0, 1, \ldots, n$, are elements of the dual space of $A^K_\xi/G^K_\xi$, it is easy to check that under a Lorentz transformation $x^a \mapsto x^a \Lambda^a_\xi$ of the Cartesian coordinates $x^a = (t, x^i)$ the energy-momentum $P^a$ transforms as a Lorentz vector and the angular momentum $J^{ab}$ as an anti-symmetric tensor: $P^a \mapsto P^a \Lambda^a_\xi \Lambda^b_\xi$ and $J^{ab} \mapsto J^{ab} \Lambda^a_\xi \Lambda^b_\xi$, while under a translation $x^a \mapsto x^a + \eta^a$ of the Cartesian coordinates $P^a$ remains intact and $J^{ab} \mapsto J^{ab} + 2\eta^aP^b$. Thus under a Poincare transformation of the Cartesian coordinates $P^a$
transforms like the energy-momentum vector and \( \mathbf{J} \) as the angular momentum tensor of a Poincare invariant system. We emphasize that to derive these transformation properties the centre-of-mass expression of Beig and \( \hat{O} \) Murchadha had to be completed by the term \( \mathbf{P} \). It might be worth noting that in the so-called field formulation of general relativity on a given background [29,30] the centre-of-mass expression also contains the extra term \( \mathbf{P} \).

The general expression of these quantities in terms of the metric and the extrinsic curvature for any allowed time axis \( K^a \) is

\[
Q[K^a] := \mathcal{H}[K^a]|_{\Gamma} + Q_m[K^a] = -\frac{1}{2\kappa} \int_\Sigma D_a \left\{ M q^{ab} q^{cd} \left( a D_c q_{bd} - a D_b q_{cd} \right) - (a D_b M) q^{ab} q^{cd} \left( q_{cd} - a q_{cd} \right) + \right. \\
\left. + (a D_c M) q^{ab} q^{cd} \left( q_{bd} - a q_{bd} \right) - 2 M_0 \left( \chi^a - \chi^b a \right) \right\} d\Sigma.
\]

Therefore, we can define the energy-momentum and angular momentum of any asymptotic end by the surface term of (4.1.1) even in the presence of matter fields, independently of any symplectic or Hamiltonian structure or phase space. The only requirement is its existence, and we assume only that the boundary conditions obtained from the investigations of the evolution equations in subsection 2.3 hold. Apparently, for asymptotic boosts (i.e. for \( B_t \neq 0 \)) and rotations (\( R_{ij} \neq 0 \)) (4.1.1) gives finite value only if \( k = n - 1 \), and for asymptotic translations (i.e. for \( B_t = 0, R_{ij} = 0 \) but \( T \neq 0 \) or \( T \neq 0 \)) only if \( k = n - 2 \). However, it is well known that the ADM energy-momentum is finite and well defined even if the metric falls off only slightly faster than \( r^{-\frac{1}{2}(n-2)} \), because the contribution of the slow fall-off ‘part’ of the metric and extrinsic curvature to the ADM energy-momentum can always be written as a constraint: the whole Hamiltonian expressed as a volume integral is finite even for the slower fall-off asymptotic ends [10]. We show that, by the same reason, the angular momentum and centre-of-mass can be finite even for metrics with \( r^{-\frac{1}{2}(n-1)} \) fall-off. In particular, in \( m = 3 + 1 \) spacetime dimensions the a priori \( 1/r \) fall-off of Beig and \( \hat{O} \) Murchadha is the weakest possible for which, in general, we can have finite angular momentum.

To determine the weakest possible power-type boundary conditions coming from the finiteness of \( Q[K^a] \), let us rewrite it as an integral on \( \Sigma \) by the very definition (3.4.5) and (3.4.8) of the Hamiltonian:

\[
Q[K^a] = \int_\Sigma \left( \sigma^a M_a + \mu M \right) d\Sigma + \frac{1}{\kappa} \int_\Sigma \left( \chi^{ab} - \chi q^{ab} \right) \left( a D_a M_b - \Gamma^c_{ab} M_c \right) d\Sigma - \\
- \frac{1}{2\kappa} \int_\Sigma \left\{ a D_a a D_b M \left( q^{ab} q^{cd} \left( q_{cd} - a q_{cd} \right) - q^{bd} q^{ac} \left( q_{cd} - a q_{cd} \right) \right) \sqrt{|q|} + \\
\right. \\
\left. + a D_a q^{ab} q^{cd} \sqrt{|q|} \left( a D_b M \left( q_{cd} - a q_{cd} \right) - (a D_c M) \left( q_{bd} - a q_{bd} \right) \right) - \\
- M_0 a D_a q^{ab} q^{cd} \sqrt{|q|} \left( a D_b q_{bd} - a D_c q_{cd} \right) + \\
\right. \\
\left. + M q_{cd} q^{ab} q^{ef} \left( \Gamma^c_{ab} \Gamma^d_{ef} - \Gamma^e_{ac} \Gamma^d_{bf} \right) \sqrt{|q|} + M \left( \chi^2 - \chi q^a \chi^{ab} \right) \sqrt{|q|} \right\} d^nx.
\]

where \( M \) and \( M_a \) have the form (2.4.1) for some (unspecified) powers \( E \) and \( F \). Suppose that the energy density and the momentum density of the matter fields satisfy the fall-off and parity conditions of subsection 2.3. If \( B_t \neq 0 \) then the condition of the existence of the integrals involving \( M \) is \( E \leq (1-k) \) and \( k \geq \frac{1}{2}(n-1) \), and if the equality \( E = (1-k) \) holds then \( \mu^{(E)}(t, \Delta^a) \) of (2.4.1) has odd parity. Thus, in particular, \( k \geq 1 \) and \( E \leq 0 \) must hold if \( n \geq 3 \). If \( B_t = 0 \) then the rate \( k \) of the fall-off can be reduced. In fact, the condition of the existence of the integrals involving \( M \) is \( E \leq -k \) and \( k > \frac{1}{2}(n-2) \), which, for \( n = 3 \), gives the well known results \( k > \frac{1}{2} \) and \( E \leq -k < -\frac{1}{2} \). Similarly, if \( R_{ij} \neq 0 \) then the condition of the finiteness of the integrals involving \( M_a \) is \( k \geq \frac{1}{2}(n-1) \) and \( F \leq (1-k) \), and if the equality \( F = (1-k) \) holds then \( \mu^{(F)}(t, \Delta^a) \) of (2.4.1) has odd parity. If \( R_{ij} = 0 \) then the fall-off may be slower: \( k > \frac{1}{2}(n-2) \) and \( F \leq -k \).
Therefore, the energy-momentum and (relativistic) angular momentum are finite for general time axes $K^a$ precisely when $k \geq \frac{1}{2}(n - 1)$, but for the slower fall-off $k > \frac{1}{2}(n - 2)$ the finiteness of the energy-momentum is not guaranteed by $B_1 = 0$ and $R_{11} = 0$ alone, $E, F \leq -k$ must also be required.

This motivates us to consider, for some $q \leq (1 - k)$, the special time axes $K^a = Mt^a + M^a \in A$ with the asymptotic structure

$$M = T(t) + r^E \mu(E)(t, \frac{r^k}{r}) + o^\infty(r^E),$$

$$M_t = T_1(t) + r^F \mu_1(F)(t, \frac{r^k}{r}) + o^\infty(r^F), \quad E, F \leq q.$$  \hspace{1cm} (4.1.3)

A simple calculation shows that they do not form a Lie algebra. If, however, they are assumed to be asymptotic Killing vectors too (whenever the powers $P$ and $Q$ in the asymptotic Killing equations (2.4.2) and (2.4.3) should be required to satisfy $P, Q \leq q - 1$, and $T_1$ and $T$ are necessarily constant), then for certain values of $q$ they form a subspace in $A^K_{\xi}$ which behaves like an ideal of a Lie algebra. In fact, if $qT^K$ is the set of the asymptotic Killing vectors $K^a = Mt^a + M^a$ whose components satisfy (4.1.3), then by a calculation similar to (2.4.8-10) shows that the Lie bracket of $K^a \in qT^K$ and $K^a \in A^K$ (where the latter is given by (2.4.1)) contains terms of order $r^{-k}$. Thus the Lie bracket operation preserves the index $q$ of $qT^K$ and the components of $[K, \bar{K}]^q$ have the structure (2.1.3) provided $q \geq -k$. The quotient $qT^K_{\xi} / qG^K$ and $\bar{K}^a \in A^K_{\xi}$ is isomorphic to $R^m$ and inherits a commutative Lie algebra structure from $A^K_{\xi} / G^K$. Thus $qT^K_{\xi}$ may be interpreted as the space of the ‘$q$ fast fall-off’ asymptotic translations in $A^K_{\xi}$, where $-k \leq q \leq (1 - k)$. These asymptotic translations can be singled out even if $k \in (0, 1)$, whenever $A^K_{\xi} / G^K$ is only the Lorentz Lie algebra rather than the Poincare one. The results of the previous paragraph show that for $k \geq \frac{1}{2}(n - 1)$ the space of the translations could be any of $qT^K_{\xi}$, $\xi \in [-k, 1 - k]$, but for $k > \frac{1}{2}(n - 2)$ it is just the space $-kT^K_{\xi}$ whose elements yield finite energy-momentum.

For the sake of logical completeness one should note that, strictly speaking, the standard expression for the ADM energy-momentum and angular momentum (including the Beig–Ó Murchadha centre-of-mass) differs from that given by (4.1.1). However, it is easy to see that (4.1.1) coincides with the standard one. In fact, for example the first term of the integral (4.1.1) can be written as

$$\int_S \left\{ M a^{ab} a_{cd} (aD_c q_{bd} - aD_b q_{cd}) \right\} \sqrt{|g|} d^nx =$$

$$= \int_S \left\{ M (a^{ab} - r^{-k} \tilde{q}^{(k)ab} + o^\infty(r^{-k})) a^{cd} - r^{-k} \tilde{q}^{(k)cd} + o^\infty(r^{-k}) \right\} \times$$

$$\times \left\{ (aD_c q_{bd} - aD_b q_{cd}) \sqrt{1 + r^{-k} d + o^\infty(r^{-k})} \right\} \sqrt{|g|} d^nx =$$

$$= - \lim_{r \to \infty} \int_S \left\{ M a^{cd} (aD_c q_{ad} - aD_a q_{cd}) + r^{-k} F_a \right\} \sigma v^a r^{n-1} dS,$$

where $q^{(k)ab} := oq^{ac} a^{bd} q_{cd}$, $d$ is a smooth function, $\sigma v^a$ is the outward directed $\sigma q_{ab}$-orthogonal unit normal to the large sphere $S_r$ of coordinate radius $r$, $dS$ is the unit sphere volume element, and the 1-form $F_a$ is defined by the last equality of the integrands. If $K^a \in A$ and $k \geq \frac{1}{2}(n - 1)$, then $F_a \sigma v^a = O(r^{-k})$ holds and its parity is odd, while if $K^a$ has the structure (4.1.3) with $q = -k$ and $k > \frac{1}{2}(n - 2)$, then $F_a \sigma v^a = O(r^{-k-1})$. However, in both cases the $r \to \infty$ limit of the integral of the term $r^{n-k-1} F_a \sigma v^a$ is zero. Similarly, all the remaining terms in (4.1.1) can also be written into the form being linear in the physical metric $q_{ab}$ and the extrinsic curvature $\chi_{ab}$, yielding the familiar expression given in [5].

Finally, calculate the total time derivative of $Q[K^a]$ along any allowed time axis $\xi^a$, where $K^a \in A_N$. However, in the present calculations we cannot use (3.5.1), because they are the vacuum evolution equations
in the *Hamiltonian phase space*. In the presence of the matter in the *spacetime* we must use (2.1.2) and (2.1.7). They, the definitions and formulae (3.1.3) and (3.2.2) imply that

\[
\dot{p}^{ab} = \frac{\delta H[N, N^c]}{\delta q_{ab}} + \frac{1}{2} N q^{ab} \left( \frac{1}{2\kappa} (R + \chi^2 - \chi_{cd}\chi^{cd}) - \mu \right) \sqrt{|q|} - \frac{1}{2} N \sigma^{ab} \sqrt{|q|}. \tag{4.1.4}
\]

Then by (2.3.5), (3.5.3), (4.1.4) and the definitions we have

\[
\frac{d}{dt} \left( H[M, M^c] + Q_m[M, M^c] \right) = \\
= \frac{d}{dt} Q_m[M, M^c] + H[\dot{M}, \dot{M}^c] + \int_{\Sigma} \left( \frac{\delta H[M, M^c]}{\delta \dot{p}^{ab}} \dot{p}^{ab} + \frac{\delta H[M, M^c]}{\delta q_{ab}} q_{ab} \right) d^a x = \\
= Q_m[\dot{M} + L_M N - L_N M, \dot{M}^c + N D^e M - MD^e N - [N, M] + H[M, M^c] + \\
+ \left\{ H[N, N^c], H[M, M^c] \right\} + \int_{\Sigma} \left( \frac{1}{2\kappa} (R + \chi^2 - \chi_{ab}\chi^{ab} - \mu) \right) (M\chi + D_e M^e) N d\Sigma = \\
= H[\dot{M} + L_M N - L_N M, \dot{M}^c + N D^e M - MD^e N - [N, M] + \\
+ Q_m[\dot{M} + L_M N - L_N M, \dot{M}^c + N D^e M - MD^e N - [N, M] + \\
+ \int_{\Sigma} \left( \frac{1}{2\kappa} (R + \chi^2 - \chi_{ab}\chi^{ab} - \mu) \right) (M\chi + D_e M^e) N d\Sigma.
\]

Taking into account the Lagrangian constraints (2.1.5) and (2.1.6) we obtain

\[
\frac{d}{dt} Q[M, M^c] = Q[\dot{M} + L_M N - L_N M, \dot{M}^c + N D^e M - MD^e N - [N, M] + \\
+ \int_{\Sigma} \left( \frac{1}{2\kappa} (R + \chi^2 - \chi_{ab}\chi^{ab} - \mu) \right) (M\chi + D_e M^e) N d\Sigma. \tag{4.1.5}
\]

By (4.1.1) the right hand side is an \((n-1)\)-sphere integral at infinity with the generators \(\dot{M} := M + L_M N - L_N M\) and \(\dot{M}^c := M^c + ND^eM - MD^eN - [N, M]^c\). If \(K^a\) is a general asymptotic Killing vector, then by (2.4.2) and (2.4.3) the order of these generators is at most \(O(r^{1-k})\), whenever their parity is odd and \(k \geq \frac{1}{2}\). If \(K^a\) is an asymptotic translation from \(-T^K_\xi\) and \(k > \frac{1}{2}\), then the order of these generators is at most \(O(r^{-k})\). However, in both cases the right hand side of (4.1.5) is vanishing. Therefore, the quantities \(Q[K^a]\) are constant in time for the asymptotic Killing vectors \(K^a \in A^K_\xi\). This result is analogous to the fact that the \(Q_m[K^a]\) of subsection 2.3 is constant in time for any Killing vector \(K^a\) of the spacetime. Nevertheless, while for the conservation of \(Q_m[K^a]\) built from the matter field variables \(K^a\) must be a genuine Killing vector, the conservation of the analogous quantity \(Q[K^a]\) of the matter+gravity system is ensured even by the asymptotic Killing fields too.

However, if the time evolution is defined by the allowed time axis \(\xi^a\) but \(K^a\) is an asymptotic Killing vector with respect to another \(\xi^a\), i.e. \(K^a \in A^K_\xi\), then in general \(Q[K^a]\) is not conserved. In particular, if \(\xi^a\) represents pure time translation at infinity (i.e. its lapse part tends to 1), but \(K^a \in A^K_\xi\) for some \(\xi^a \in G\), (whenever \(B_i(t), R_{ij}(t), T_i(t)\) and \(T(t)\) of \(K^a\) are independent of \(t\)), then \(Q[K^a]\) is not constant in time with respect to \(\xi^a\). For example, the time derivative of the centre-of-mass of Beig and Ó Murchadha with respect to \(\xi^a\) above is not zero, that is just the spatial (linear) momentum.

### 4.2 The background-independence of \(Q[K^a]\)

Let \(\phi v^a\) be the outward directed \(q_{ab}\)-unit normal to the coordinate spheres in \(\Sigma\), and let \(\gamma\) be an integral curve of \(\phi v^a\) form some \(S_{r_\alpha}\) to \(S_r\). Then the length of \(\gamma\) in the physical metric \(q_{ab}\) is
\[ R = \int_{r_0}^{r} \sqrt{|g_{ab} \delta^a \delta^b|} \, dr' = \int_{r_0}^{r} \sqrt{1 - \frac{1}{r^2} g_{ab} \delta^a \delta^b + o(r^{-k})} \, dr' = \begin{cases} r - r_0 + \text{A ln} \frac{r}{r_0} + B + o(r^{-0}), & \text{if } k = 1; \\ r - r_0 + \text{A r}^{-k+1} + B + o(r^{-k+1}), & \text{if } k \neq 1 \end{cases} \] (4.2.1)

for some constants \( A, \bar{A}, B \) and \( \bar{B} \). This implies, in particular, that

\[ \frac{1}{R^k} - \frac{1}{r^k} = \begin{cases} o(r^{-1}), & \text{if } k = 1; \\ O(r^{-2k}), & \text{if } k \neq 1. \end{cases} \] (4.2.2)

Therefore, in the definitions (2.3.1)-(2.3.3) of the asymptotic flatness the radial distance \( r \) can be substituted by the physical radial distance \( R \) without changing the structure or the leading terms of \( q_{ij} \) and \( \chi_{ij} \).

To clarify the potential ambiguity both of the notion of asymptotic flatness and the quantities \( Q[K^n] \) coming from the non-uniqueness of the background metric \( \bar{g}_{ab} \), let \( (\Sigma, q_{ab}, \chi_{ab}) \) be \((k,l)\)-asymptotically flat with respect to \( \bar{g}_{ab} \) and let \( g_{ab} \) be another background metric, being flat on \( \Sigma - K \). (Without loss of generality we may assume that the domain of the flatness of both \( g_{ab} \) and \( \bar{g}_{ab} \) coincide.) Thus there exists a diffeomorphism \( \phi : \Sigma - K \to \Sigma - K \) such that \( g_{ab} = \phi^* \bar{g}_{ab} \). For the sake of simplicity suppose that \( \phi \) is homotopic to the identity \( \text{Id}|_{\Sigma - K} \), i.e. for some one-parameter family of diffeomorphisms \( \phi_u = \text{Id}|_{\Sigma - K} \) and \( \phi_1 = \phi \). Then we can form the one-parameter family of flat metrics \( g_{ab}(u) := \phi_u^* g_{ab} \) on \( \Sigma - K \) (which can obviously be extended to the whole \( \Sigma \) as, in general curved, negative definite metrics). If \( V^a \) is the vector field on \( \Sigma - K \) generating \( \phi_u \), and its components in the coordinates \( \{x^k\} \) are defined by \( V^a g_{ab} = : V_k o D_b x^k \), then

\[ \delta_{0} q_{ab} := \left( \frac{d}{du} q_{ab}(u) \right)|_{u=0} = L V_0 q_{ab} = \left( \partial I D V_i + \partial D_j V_i \right) \partial D_a x^i \partial D_b x^j. \] (4.2.2)

Writing \( V_i \) in the form

\[ V_i(x^k) = 2 x^k \rho_{k i} + \tau_i + r^R V_i^{(R)} \left( \frac{x^k}{r} \right) + o^\infty (r^R) \] (4.2.2)

for some power \( R \), where the first two terms together is just the kernel of the flat Killing operator \( \partial D_i V_j \) for \( g_{ab} \), we have

\[ \partial D_i V_j = r^{-1} \left( R o v^k \delta_{k}^{(i}} V_{j)} + \left( \partial k (V^{(R)}_{i)} \delta_{j)}^{(m} O v^{m} v^{k) \right) \right) + o^\infty (r^{-1}). \] (4.2.3)

Its leading term has even parity iff \( V_i^{(R)}(\frac{x^k}{r}) \) has odd parity. Since, for sufficiently small \( u \), one has \( q_{ab} - o g_{ab}(u) = q_{ab} - o g_{ab}(u) - o g_{ab}(u) - O(u^2) \), and hence, in the \( g_{ab}\)-Cartesian coordinate system \( \{x^k\} \), \( q_{ij} - o q_{ij}(u) = q_{ij} - o q_{ij} + o^\infty (u^2) = r^{-k} q_{ij} + o^\infty (r^{-k}) - 2 D_i V_j u + O(u^2) \) holds. The one-parameter family of coordinate systems \( \{x^k(u)\} \), defined in terms of the coordinates \( \{x^k\} \) by \( \frac{\delta x^k}{u} := \phi_u^*(x^k) \), is Cartesian with respect to the one-parameter family of flat metrics \( o g_{ab}(u) \), i.e. \( o g_{m n}(u) = \delta_{m n} \). (To see this it is enough to recall the definition of the pull-back of \( o g_{ab}(u) \) of the metric \( g_{ab} \) along \( \phi_u \) in the coordinates \( \{x^k\} \), viz. \( o g_{ij}(u) := \frac{\partial x^m}{\partial x^i} \frac{\partial x^n}{\partial x^j} o g_{m n} = - \delta_{m n} \frac{\partial x^m}{\partial x^i} \frac{\partial x^n}{\partial x^j} \), and to compare this with the transformation law \( o g_{ij}(u) = o g_{m n}(u) \frac{\partial x^m}{\partial x^i} \frac{\partial x^n}{\partial x^j} \) of the components \( o g_{ij}(u) \) and \( o g_{m n}(u) \) of \( g_{ab}(u) \) in the coordinates \( \{x^k\} \) and \( \{x^m(u)\} \), respectively. Then for sufficiently small \( u \) we have \( \frac{\delta x^k}{u} = x^i + V^i(x^k) + O(u^2) \). Thus the components of the physical metric and the extrinsic curvature in the \( o g_{ab}(u) \)-Cartesian coordinate system \( \{x^k(u)\} \), respectively, are
\[
\bar{q}_{ij} = q_{mn} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} = o(\bar{q}_{ij}) + \frac{1}{r^k} \left( q_{j(k)}^{(k)} - 2u q_{j(k)}^{(k)} \rho_j^k - 2u q_{j(k)}^{(k)} \rho_i^k \right) + o^\infty (r^{-k}) - 2u r^{R-1} \left( R o^v_k \chi_{(l)j} V^{(l)}_{(j)} + \bar{\partial} v_{(l)j} V^{(l)}_{(j)} - o^v (\bar{\partial}_v V^{(l)}_{(j)}) \delta_j^1 o^v j + o^\infty (r^R - 1) + O(u^2) \right),
\]
\[
\bar{\chi}_{ij} = \chi_{mn} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} = \frac{1}{r^l} \left( \chi_{ij}^{(l)} - 2u \chi_{ij}^{(l)} \rho_i^k - 2u \chi_{ij}^{(l)} \rho_j^k \right) + o^\infty (r^{-l}) - 2u r^{R-1-l} \left( R o^v_m \chi_{m(i)j} V^{(l)}_{(j)} + \bar{\partial} v_{m(i)j} V^{(l)}_{(j)} - o^v (\bar{\partial}_v V^{(l)}_{(j)}) \chi_{(k)}^{(l)} o^v j + o^\infty (r^R - 1) + O(u^2) \right).
\]

(4.2.4)

Therefore, the asymptotic end \((\Sigma, q_{ab}, \chi_{ab})\), which is \((k, l)\)-asymptotically flat with respect to the background metric \(q_{ab}\), remains \((k, l)\)-asymptotically flat with respect to the new flat metrics \(q_{ab}(u) = \delta^a_u o q_{ab}\) for sufficiently small \(u\) precisely when \(R \leq (1 - k)\) and \(V^1_{(R)}\) has odd parity for \(R = 1 - k\). These changes of the background metrics will be called allowed. (Apart from a rigid Euclidean rotation, the non-trivial leading terms of \(q_{ij}\) and \(\chi_{ij}\) are invariant with respect to the change \(q_{ab} \mapsto o q_{ab}(u)\) of the background metric iff \(R < (1 - k)\).) Therefore, the corresponding \(V^n\)'s are special, purely spatial asymptotic Killing vectors.

The change of the background metric \(q_{ab}\) yields a change of the allowed time axes. By (4.2.2) for the infinitesimal change of \(M\) and \(M_a\) given by (2.4.1) one has

\[
\delta M := \left( \frac{d}{du} M(\bar{x}^k(u)) \right)_{u=0} = 2 V^i B_i + r^{R-1} \left( E o^v \delta_{kj} \nu^{(E)} (\delta^k_i - o^v \rho^i o^v j \delta^1_i) \right) + o^\infty (r^E) = 2 r^k \left( 2 o^v \rho^j \partial_j^l \right) + 2 \tau_j^i \partial_j^i \right) + 2 r^E \left( E o^v \rho^i \delta^1_i \right) + 2 r^R \left( V_i^R + o^\infty (r^E) + o^\infty (r^R) \right),
\]
\[
\delta M_a := \left( \frac{d}{du} M_a(\bar{x}^k(u)) \right)_{u=0} = \left( 2 r^k \left( 2 o^v \rho^j \partial_j^l R_{i}^j j - R_{k}^j \rho^j \partial^j \right) + \left( 2 r^j \partial_j^i - 2 T^j \rho^j \right) + 2 r^E \left( E o^v \rho^i \delta^1_i \right) - \nu^j_i \right) + o^\infty (r^E) + o^\infty (r^R) \right) \partial D_a x_i.
\]

(4.2.5)

Thus the allowed change of the background metric acts on \(K^a = M t^a + M^a\) as the diffeomorphism generated by the asymptotic Killing vector \(V^a\): for \(E, F \leq (1 - k)\) the structure of \(\delta M\) and \(\delta M_a\) is similar to that of \(M\) and \(M_a\), respectively, if \(R < (1 - k)\) or if \(R = (1 - k)\) and \(V^1_{(R)}\) has odd parity. If, however, \(B_k = 0\), \(R_{ij} = 0\) and \(E, F \leq -k\), i.e. \(K^a \in -k T^K_{\xi}\), then the structure of \(\delta M\) and \(\delta M_a\) will be similar to that of \(M\) and \(M_a\), respectively, only if \(R \leq -k\).

To calculate the change of \(Q[K^a]\) under the allowed change of the background metric let us form \(Q_a[K^a(\bar{u})]\) by using the one-parameter family of the flat background metrics \(o q_{ab}(u)\) in (4.1.1) instead of \(o q_{ab}\). Then, with the notation \(\delta K^a := \delta M t^a + q^{ab} \delta M_b\), a straightforward calculation gives

\[
\delta Q[K^a] := \left( \frac{d}{du} Q_a[K^a(\bar{u})] \right)_{u=0} = Q[\delta K^a] + \frac{1}{2k} \int_{\Sigma} \left( M \delta_0 \Gamma_{bc}^d - \delta_0 \Gamma_{dc}^b q^{ba} \right) + \left( q^{ab} q^{cd} - q^{ac} q^{bd} \right) \left( 0 D_a M \right) \delta_{o q^c d} \right) d\Sigma,
\]

(4.2.6)

where \(\delta_0 \Gamma_{bc}^c := \frac{1}{4} q^{ad} \left( - o D_a \delta q_{bc} + o D_c \delta q_{ab} + o D_b \delta q_{ca} + o D_a \delta q_{cd} \right)\). Thus the spatial momentum and angular momentum depend on the background metric only through their generator \(M_a\), but the energy and centre-of-mass may be ambiguous as a consequence of the non-vanishing of the integral on the right of (4.2.6) too. (Using
Thus if, for the sake of brevity, we define \( \rho \) as angular momentum and centre-of-mass measured at the retarded time, it can be shown that \( \delta Q[K^a] \) is finite if \( R < (1 - k) \), or if \( R = (1 - k) \) and \( V_1^{(R)} \) has odd parity. \( K^a = M^a + \alpha^a \) is an asymptotic Killing vector, and hence \( E, F \leq (1 - k) \). Then by the results of the previous subsection \( k \geq \frac{1}{2}(n - 1) \) must hold, and by (4.2.4) and (4.2.5) \( R \leq (1 - k) \) must be required. Thus \( Q[\delta K^a] \) depends only on the first two terms of \( \delta M \) and \( \delta M_\alpha \) in (4.2.5), because the remaining terms are pure gauge generators. However, by (2.4.11-14) this is nothing but the transformation law of the energy, and the components of the spatial momentum, spatial angular momentum and centre-of-mass under the Euclidean transformation coming from the diffeomorphism generated by \( V^a \). To rule out the ambiguities in the expression of the energy and the centre-of-mass, we must require the vanishing of the integral in (4.2.6). Its vanishing can be ensured by requiring \( R < (2 - n) \), or \( R = (2 - n) \) and that \( V_1^{(R)} \) be odd parity functions. Next suppose that \( K^a \in -k \xi^a \), i.e. \( R_{ij} = 0 \), \( B_1 = 0 \) and \( E, F \leq -k \). Then by the previous subsection \( Q[K^a] \) is finite even if \( k > \frac{1}{2(n - 2)} \), and by (4.2.5) \( R \leq -k \), implying that \( Q[\delta K^a] \) is finite and describes how the components of the spatial momentum transform under the Euclidean transformation coming from the diffeomorphism generated by \( V^a \). (The energy remains intact.) The vanishing of the integral in (4.2.6) can be ensured even by \( R < (3 - n) \). Therefore, \( Q[K^a] \) is unambiguously defined for \( K^a \in A^a_\xi \), if \( R \leq (1 - k) \), \( R \leq (2 - n) \) and in the case of the equality, \( R = (2 - n) \), \( V_1^{(R)} \) has odd parity; and for \( K^a \in -k \xi^a \), if \( R \leq -k \) and \( R < (3 - n) \). In particular, in \( m = 3 + 1 \) spacetime dimensions and for \( k = 1 \) the angular momentum and centre-of-mass are well defined provided the diffeomorphisms connecting the background metrics tend to the rigid Euclidean transformations like \( O(r^{-1}) \) with odd parity generator, or faster. For the condition ensuring well defined energy and spatial momentum we recovered the known results of [8,11]. Namely, writing \( k \) in the form \( k = \frac{1}{2} + \delta \) for some \( \delta > 0 \), the energy and momentum are well defined if the diffeomorphisms tend to the Euclidean transformation like \( O(r^{-\frac{1}{2} - \delta}) \).

Appendix: Global integral conditions at the null infinity

Let us consider global quantities associated to the global state of the matter fields in the Minkowski spacetime at a given retarded time \( U \), i.e. if the spacelike hypersurface of subsection 2.2 extends to the future null infinity \( \mathcal{I}^+ \) such that its intersection with \( \mathcal{I}^+ \) is the \( U = \text{constant} \) cut. However, in the present context it seems more comfortable to choose this hypersurface to be the null hypersurface \( \mathcal{N}_U := \{ (\tau, X^i) | \tau - R = U \} \) instead of a spacelike (e.g. a hyperboloidal) one. Here \( (\tau, X^i) \), \( i = 1, \ldots, n \), are still the Cartesian coordinates introduced in subsection 2.2. For the null \( \mathcal{N}_U \) the flux integral of \( T^{ab} K_b \) takes the form \( \lim_{R \rightarrow \infty} \int_{I_0} T^{ab} K_b \rho dS R^{n-1} dR' \), where \( l_a := \nabla_a (\tau - R) = t_a + v_a \), a null normal to \( \mathcal{N}_U \). Thus if, for the sake of brevity, we define \( \rho(U, R, \frac{X^a}{R}) := t_a T^{ab} l_b = \mu(U + R, X^k) + j^a(U + R, X^k) v_a \) and \( \rho^i(U, R, \frac{X^a}{R}) := K^i_a T^{ab} l_b = K^i_a j^a(U + R, X^k) + \sigma_{ab}(U + R, X^k) v_b \), then the global energy, spatial momentum, angular momentum and centre-of-mass measured at the retarded time \( U \) at future null infinity are finite precisely when \( \int_S^+ \rho dS \rho dS = o(R^{-n}) \), \( \int_S^+ \rho^1 dS = o(R^{-n}) \), \( \int_S^+ \rho^i X^i \frac{1}{R} dS = o(R^{-n}) \) and \( \int_S^+ (\rho^i - \rho^i X^i) \frac{1}{R} dS = o(R^{-n}) \), respectively. These conditions could be satisfied if \( \rho(U, R, \frac{X^a}{R}) = \frac{1}{R^n} \rho^{(m)}(U, \frac{X^a}{R}) + o(R^{-m}) \) and \( \rho^i(U, R, \frac{X^a}{R}) = \frac{1}{R^n}(\dot{\varphi}(U, R, \frac{X^a}{R}) + \frac{1}{R} - \rho^{(m)}(U, \frac{X^a}{R}) + o(R^{-m})) \), where \( -\rho^{(m)}(U, \frac{X^a}{R}) \) is the odd parity part of \( \rho^{(m)}(U, \frac{X^a}{R}) \) and \( +\varphi(U, R, \frac{X^a}{R}) \) is an arbitrary function with even parity. \( \pm \rho^{(m)} \) contribute to the energy and spatial momentum, but do not to the angular momentum and centre-of-mass. Thus we may call them the \( BS mass aspect of T^{ab} \).

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