Rayleigh-Sommerfeld scalar diffraction by rotating apertures

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Abstract
We have analytically explored the Rayleigh-Sommerfeld scalar diffraction for various rotating apertures such as rotating single-slit, rotating double-slit, rotating linear diffraction grating, and rotating regular polygonal aperture. Here the apertures are considered to be uniformly rotating along the axes perpendicular to the plane of the apertures and the diffracted fields are analysed in the far-field zone. We have compared the time-averaged intensity distributions for the rotating apertures with the intensity distributions for static circular apertures of the same area. We also have obtained angular speed of rotation dependent Fraunhofer diffraction formulae for the rotating apertures, in particular, the rotating single-slit, the rotating double-slit, and the rotating linear diffraction grating.

1. Introduction

The diffraction of electromagnetic waves by an aperture still continues to be a subject of great interest in science [1–3]. Rayleigh-Sommerfeld theory of scalar diffraction is considered to be the manifestly ‘consistent’ theory for the diffraction of the monochromatic electromagnetic wave by an aperture in a non-dispersive medium if the dimension (D) of the aperture is reasonably larger than the wavelength (λ) of the electromagnetic field incident on it [1, 4–6]. However, Rayleigh-Sommerfeld scalar diffraction by a moving aperture [3] naturally draws attention to the study of scalar diffraction by a rotating aperture as shown in figure 1. Diffraction by a rotating rectangular aperture has a practical application, in particular, to improve the discrimination capability of an optical system for an off-axis object [7]. Rotated rectangular aperture imaging, where diffraction plays a significant role, also has practical applications in telephoto systems [8, 9].

Scalar diffraction by an aperture (i.e. a linear diffraction grating) rotating along with its plane has already been studied in [10]. Scalar diffraction by a radial diffraction grating rotating along the axis perpendicular to the plane of the aperture has also been studied in [11]. The Radial diffraction grating, however, has a periodic structure in the azimuthal direction in polar coordinates. Scalar diffraction by the single-slit, the double-slit, the linear diffraction grating, and the regular polygonal aperture rotating along the axes perpendicular to the plane of the apertures has not been studied before. Such a study would be interesting because these apertures have no (continuous) symmetry of rotation towards the azimuthal direction. Hence we take up the discussion of scalar diffraction by the single-slit, the double-slit, the linear diffraction grating, and the regular polygonal aperture uniformly rotating along the axes perpendicular to the plane of the apertures.

Calculation in this article begins with the wave equation for the Rayleigh-Sommerfeld scalar diffraction [4]. A solution to the wave equation has been obtained in the form of a time-dependent Rayleigh-Sommerfeld diffraction integral (of the type-I) for a rotating aperture. The far-field approximation of the solution has been taken into account to determine the diffracted fields for (i) rotating single-slit, (ii) rotating linear diffraction grating, (iii) rotating double-slit, and (iv) rotating regular polygonal aperture. The intensity distributions of the diffracted fields for all these cases have been shown. The comparison between the time-averaged intensity distributions for the rotating apertures with the intensity distributions for static circular apertures of the same
area has been plotted. The plot shows the central region similarities of time-averaged intensity distributions for the rotating apertures with the intensity distributions for static circular apertures in this regard. The time-dependent Fraunhofer diffraction formulae for these cases except for the rotating regular polygonal aperture have also been obtained.

2. Rayleigh-Sommerfeld scalar diffraction by rotating apertures

Let us consider a monochromatic plane scalar wave of angular frequency $\omega$ and wave number $k = \frac{2\pi}{\lambda}$ incidents on an aperture $A$ with normal incidence from the left half-space $z < 0$ as shown in figure 1. Scalar diffraction incidentally takes place in the right-half space $z > 0$ due to the aperture rotating in the $x-y$ plane. Let us fix the origin of the Cartesian coordinate system to be at the middle of the aperture. Let a point on the aperture at time $t_0$ be given by $\vec{r}_0(t_0) = x_0(t_0)\hat{i} + y_0(t_0)\hat{j} + \hat{k}$. The scalar field $\psi(\vec{r}, t)$ at the position $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and time $t$ satisfies the wave equation \[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(\vec{r}, t) = 0 \] in the vacuum. Here the scalar field $\psi(\vec{r}, t)$ is equated with one of the components of the electric field ($\vec{E}(\vec{r}, t)$) or magnetic field ($\vec{B}(\vec{r}, t)$) of the electromagnetic field and $c$ is its propagation speed in the vacuum $[6]$. The solution to equation (1) for the scalar diffraction by a moving aperture, can be obtained in the form of time-dependent Rayleigh-Sommerfeld diffraction integral (of the type-I) as $[3]$

$$
\psi(\vec{r}, t) = -\frac{z}{2\pi} \int_{-\infty}^{\infty} \int_{A} dx_0(t_0) dy_0(t_0) \frac{\psi^{(i)}(\vec{r}_0(t_0), t_0)}{\delta(t_0 - [t - R(\vec{r}, \vec{r}_0(t_0))/c])} \times \frac{e^{i[R(\vec{r}, \vec{r}_0(t_0)) - \omega(t-t_0)]}}{R^2(\vec{r}, \vec{r}_0(t_0))} \left[ ik - \frac{1}{R(\vec{r}, \vec{r}_0(t_0))} \right] \tag{2}
$$

where $\psi^{(i)}(\vec{r}_0(t_0), t_0) = \psi_0 e^{-i\omega_0 t_0}$ is the incident scalar field at the point $\vec{r}_0(t_0)$ in the aperture at time $t_0$ and $R(\vec{r}, \vec{r}_0(t_0)) = \sqrt{(x - x_0(t_0))^2 + (y - y_0(t_0))^2 + z^2}$ is the distance between a secondary source point $\vec{r}_0(t_0)$ in the aperture and an observation point $\vec{r}$. Let the aperture be uniformly rotating in the $x-y$ plane with the angular velocity $\omega_0 = -\omega_0 \hat{k}$. For such a case, we have

$$
x_0(t_0) = x_0(0)\cos(\omega_0 t_0) - y_0(0)\sin(\omega_0 t_0) \tag{3}
$$

4 As long as the number of sides of a regular polygonal aperture is unknown, the diffracted scalar field appears as a summation of the contributions of the triangular segments of the polygon. Evaluation of this summation is possible only when the number of sides is known. This leads to difficulty in determining the Fraunhofer diffraction formula even for the static regular polygon of $n$ sides.
and
\[ y'_0(t_0) = x_0(0)\sin(\omega_0 t_0) + y_0(0)\cos(\omega_0 t_0). \]

Since the Jacobian of the transformation \( \begin{pmatrix} x_0(0) \\ y_0(0) \end{pmatrix} \rightarrow \begin{pmatrix} x_0(t_0) \\ y'_0(t_0) \end{pmatrix} \) given by equations (3) and (4) is 1, we have \( dx(t_0)dy(t_0) = dx(0)dy(0) \). Thus we recast equation (2) as
\[
\psi(\vec{r}, t) = \frac{-2\psi_0 e^{-i\lambda t}}{2\pi} \int_{-\infty}^{\infty} dt_0 \int_A dx_0(0) dy'_0(0) \\
\times \delta(t_0 - [t - R(\vec{r}, \vec{n}(t_0))/c]) \\
\times \frac{e^{i\rho(R(\vec{r}, \vec{n}(t_0)))}}{R^2(\vec{r}, \vec{n}(t_0))} \left[ i\frac{k}{c} - 1 \right] R(\vec{r}, \vec{n}(t_0)).
\]

Evaluation of the above three integrations in closed form is impossible. A proper approximation technique can be employed for an approximate evaluation of the same. The distance between the secondary source point \( \vec{n}(t_0) \) and the observation point \( \vec{r} \) can be approximated in the far-field zone (i.e. for \( \frac{2\pi r}{\lambda} \ll 1 \) and \( \lambda \ll D \)) as
\[
R(\vec{r}, \vec{n}(t_0)) \approx r \left[ 1 - \frac{xx_0(t_0) + yy_0(t_0)}{r^2} \right].
\]

Thus we approximate equation (5) for the observation point \( \vec{r} \) in the far field zone as
\[
\psi(\vec{r}, t) \approx \frac{-zik\psi_0 e^{ikr - \omega t}}{2\pi r^2} \int_A dx_0(0) dy'_0(0) \int_{-\infty}^{\infty} dt_0 \\
\times e^{i\frac{k|xx_0(t_0) + yy_0(t_0)|}{r}} \delta(t_0 - [t - r/c]) \\
= \frac{-zik\psi_0 e^{ikr - \omega t}}{2\pi r^2} \int_A dx_0(0) dy'_0(0) \\
\times e^{i\frac{k|x\cos(\omega_0 t) - y|}{2r}}
\]
where \( t' = t - r/c \) is the retarded time. Equation (7) is our key result. We can use this result to obtain the diffracted field at a point in the far-field zone for various rotating apertures such as rotating single-slit, rotating double-slit, and rotating linear diffraction grating.

2.1. Diffracted field for rotating single-slit

For a rectangular slit (single-slit) of area \( ab \) we have \(-\frac{a}{2} < x_0(0) < \frac{a}{2} \) and \(-\frac{b}{2} < y_0(0) < \frac{b}{2} \). Thus using equations (5) and (4) and evaluating the integrations on the right-hand side of equation (7) over the aperture, we get the diffracted scalar field at time \( t \) at the observation point \( \vec{r} \) in the far-field zone as
\[
\psi(\vec{r}, t) \approx \frac{-ziky_0 \psi_0 e^{ikr - \omega t}}{2\pi r^2} \\
\times \frac{ak}{2r} \left[ x \cos(\omega_0 [t - \frac{c}{2}]) + y \sin(\omega_0 [t - \frac{c}{2}]) \right] \\
\times \frac{bk}{2r} \left[ y \cos(\omega_0 [t - \frac{c}{2}]) - x \sin(\omega_0 [t - \frac{c}{2}]) \right]
\]
for the rotating single-slit. Here-from we get back the known result [12] for the non-rotating case (\( \omega_0 \rightarrow 0 \)) of the single-slit as [6]
\[
\lim_{\omega_0 \to 0} \psi(\vec{r}, t) \approx \frac{-ziky_0 \psi_0 e^{ikr - \omega t}}{2\pi r^2} \frac{a k x}{2r} \sin \left( \frac{a k x}{2r} \right) \sin \left( \frac{b k y}{2r} \right).
\]

However, from figure 1 we have \( y/r = \sin(\theta) \) where \( \theta \) is the diffraction angle along the \( y \)-axis. Similarly, we can say \( x/r = \sin(\phi) \) where \( \phi \) is the diffraction angle along the \( x \)-axis. Now we get time-dependent Fraunhofer diffraction formulae from equation (8) as
\[
b \sin(\theta) \cos(\omega_0 t') + \sin(\phi) \sin(\omega_0 t') = m\lambda \]
\[
a \sin(\theta) \cos(\omega_0 t') + \sin(\phi) \sin(\omega_0 t') = n\lambda \]
where \( n = \pm 1, \pm 2, \pm 3, \ldots \) and \( m = \pm 1, \pm 2, \pm 3, \ldots \). Together these formulae provide the location of the intensity minima for the rotating single-slit and \( \lambda \) represents the wavelength of the incident wave.
So far, we have discussed the diffraction of scalar plane waves for normal incidence on the rotating apertures. If the scalar plane wave is replaced with a scalar spherical wave which incident normally on the rotating single-slit coming from a point source located at $-\hat{s}$, then the incident scalar field at the point $r(t_0)$ in the aperture at time $t_0$ would take the form $\psi(0)(r(t_0), t_0) = \hat{V}_0 \frac{e^{ik\hat{s} \cdot r}}{\sqrt{r^2 + \hat{s}^2}}$ where $\hat{V}_0$ is a constant. If $s$ is much larger than the linear dimension of the aperture ($D$), then the diffracted scalar field in the far-field zone can be expressed, as a counter to equation (8), as

$$\psi(\mathbf{r}, t) \approx \frac{zika b \hat{V}_0 e^{ikt \cdot \mathbf{r} - ik^2 r^2}}{2\pi r^2 s} \times \text{sinc} \left( \frac{ak \left[ x \cos(\omega_0 t - \frac{z}{c}) + y \sin(\omega_0 t - \frac{s}{c}) \right]}{2r} \right)$$

$$\times \text{sinc} \left( \frac{bk \left[ y \cos(\omega_0 t - \frac{s}{c}) - x \sin(\omega_0 t - \frac{z}{c}) \right]}{2r} \right).$$

(10)

By comparing equations (8) and (10), one can easily say that the intensity distribution for the diffraction of a spherical wave reaches the intensity distribution for the diffraction of a plane wave in the limiting case of $s/D \to \infty$. Otherwise, these two intensity distributions differ significantly away from the central maxima, in particular, for $D \gg s$ [13]. However, let us consider a practical situation, $s \gg D$, for the rest of the discussion. Diffraction of the spherical waves does not draw any special attention beyond the diffraction of plane waves in such a situation.

We plot the intensity distribution corresponding to equation (8) in figure 2(a) for time $t = 10$ s and for the parameters as mentioned in the figure caption. It is clear from equation (8) that the density plot as shown in figure 2(a) would rotate in the course of time with angular frequency $\omega_0 = 10$ Hz in a counter-clockwise sense though the aperture is rotating in a clockwise sense. This is not a surprise because while the aperture is on the left side of the observer, the observation screen is on the right side of the observer. However, if we want to observe the density plot with a CCD camera of low temporal resolution, then we will observe the time-averaged density plot over a full time period $(2\pi/\omega_0)$. Such an intensity distribution is shown in figure 2(b) for the parameters as shown in the figure caption. The density plot in figure 2(b) resembles the intensity distribution for a (non-rotating) circular disk. The diffracted scalar field for the circular aperture of the area $\pi a^2 = ab$ can be obtained from equation (7) for $\omega_0 \to 0$ as [6]

$$\psi_1(\mathbf{r}, t) \approx \frac{zika b \hat{V}_0 e^{ikt \cdot \mathbf{r}}}{2\pi r^2} \frac{2f(k\sqrt{|ab/\pi|s^2 + r^2})}{r}$$

(11)

where $f_1$ is the Bessel function of the first kind of order 1. Equation (11) can be obtained from integral equation (7) in cylindrical coordinates.

The absolute difference between the time-averaged intensity distribution for the rotating single-slit and the intensity distribution for the circular aperture is shown in figure 3. The dark spot at the center in figure 3

5 By ‘low temporal resolution’, we mean that the resolution time is much longer than the time period of the rotation of the aperture.
indicates that the central region of the time-averaged intensity distribution for a rotating single-slit of length \(a\) and breadth \(b\) is similar to the central region of the intensity distribution for a circular aperture of the same area \((ab)\). This criterion of the same area comes from the fact that the intensity at the center of the fringe pattern at any instant of time depends on the geometric area of the aperture. If \(a\) is the larger dimension of the aperture, then the region over which the two central regions of the intensity distributions appear similar would be extended from \(\theta = \phi = 0\) to \(\theta = \phi \approx \frac{\pi}{2} \approx \frac{\pi}{2}\) on the screen in the \(x-y\) plane fixed at \(z > 0\). The region of the similarity, however, would increase for a regular polygon-shaped aperture of a large number of edges. The dissimilarity in the two fringe patterns is, of course, evident in figure 3, especially for \(\theta \geq \frac{\pi}{2}\). This dissimilarity is coming because (i) the two dimensions of the rectangular apertures are not the same and (ii) the average of the edge effects of a rectangular aperture over time is different from the edge effect of a circular aperture.

In figure 2(b) we present a case where the ‘intensity’ distribution is averaged out for a full rotation of the single-slit. However, this is unlikely to occur in the real-world where some arbitrary fraction of rotation may also contribute to the time-averaged ‘intensity’ distribution. In figure 4 we present two such cases—one (Figure 4(a)) for \(5/4\) times a full rotation and another (figure 4(b)) for \(41/4\) times a full rotation. A half-integral full rotation results in the same time-averaged ‘intensity’ distribution shown in figure 2(b) because of the two-fold rotational symmetry of the rectangular aperture. However, it is clear from figure 4(a) and figure 4(b) that the time-averaged ‘intensity’ distribution for a fractional rotation reaches the time-averaged ‘intensity’ distribution for a full rotation (figure 2(b)) for a large improper fraction of the full rotation.

It is evident from equation (9) that if the diffraction pattern obtained from the static aperture would be rotated with an angular speed same as that of the rotating slit, it would not produce the same result as obtained in
equation (8). For such a rotation in the counter-clockwise sense, equation (9) would be changed to the following

\[
\psi(\vec{r}, t) = -\frac{zikav_{0}e^{i[kr - \omega t]}}{2\pi r^{2}} \times \sin \left(\frac{ak[x \cos(\omega_{0}t) + y \sin(\omega_{0}t)]}{2r}\right) \quad \text{and} \quad \sin \left(\frac{bk[y \cos(\omega_{0}t) - x \sin(\omega_{0}t)]}{2r}\right).
\]

This result differs from equation (8) due to a phase factor of \(\frac{\omega_{0}t}{c}\) in the sinusoidal parts of the sinc functions. This difference, however, is negligible for long-time observation (\(t/c \ll t\)). However, relativistic treatment of the same would be more interesting, in particular, for the case of the fast rotation (\(a' \omega_{0}/c \lesssim 1\)). If we take the relativistic effect into account and consider the Lorentz-Poincaré invariance [14] for a uniformly rotating frame, then the rotating radius of a circular aperture would be contracted by the factor \(\sqrt{1 - a'^{2} \omega_{0}^{2}/c^{2}}\) [15] compared with the radius \(a'\) of the circular aperture measured at rest. Such a contraction leads to an increase in the size (area) of the image by the factor \(1/[1 - a'^{2} \omega_{0}^{2}/c^{2}]\). On the other hand, the size of the image would be reduced by the factor \(1 - (x^{2} + y^{2}) \omega_{0}^{2}/c^{2}\) when the image rotates uniformly with the same angular frequency. Thus a significant difference can be observed in the case of the fast rotation.

2.2. Diffraction field for rotating linear diffraction grating

Let us now replace the single-slit with a linear diffraction grating in the set-up of figure 1. Let the linear diffraction grating has reflectional symmetry about both the x-axis and the y-axis. Let us further consider that the linear diffraction grating has N rectangular slits of the previous type such that each rectangular slit has length \(a\) along the x-axis and length \(b\) along the y-axis and the middle points of two consecutive rectangular apertures are separated by a distance \(d\) along the y-axis. Let us now uniformly rotate this grating like the single-slit which was rotating with the angular velocity \(\omega_{0}k\).

The diffracted scalar field at the position \(\vec{r}\) in the far-field zone at time \(t\) is well known for the linear diffraction grating in the limit of non-rotating case as [16]

\[
\lim_{\omega_{0} \to 0} \psi(\vec{r}, t) \simeq -\frac{zikav_{0}e^{i[kr - \omega t]}}{2\pi r^{2}} \sin \left(\frac{Ndky}{2r}\right) \sin \left(\frac{aks}{2r}\right) \times \sin \left(\frac{bky}{2r}\right) \sin \left(\frac{akx}{2r}\right).
\]

If we compare equation (8) with equation (9), then we can say that they differ by time-dependent coefficients of \(x\) and \(y\), such that, \(x\) in equation (9) would be replaced by \(x \cos(\omega_{0}t') + y \sin(\omega_{0}t')\) and \(y\) in the same equation would be replaced by \(y \cos(\omega_{0}t') - x \sin(\omega_{0}t')\) to get equation (8). The same argument also works for a rotating linear diffraction grating and a non-rotating linear diffraction grating. Following this argument, we get the diffracted scalar field at the position \(\vec{r}\) in the far-field zone at time \(t\) from equation (13) for the linear diffraction grating as

\[
\psi(\vec{r}, t) \simeq -\frac{zikav_{0}e^{i[kr - \omega t]}}{2\pi r^{2}} \sin \left(\frac{Ndky}{2r}\right) \sin \left(\frac{aks}{2r}\right) \times \sin \left(\frac{bky}{2r}\right) \sin \left(\frac{akx}{2r}\right).
\]

Now we get time-dependent Fraunhofer diffraction formula from equation (14) as \(d[\sin(\theta) \cos(\omega_{0}t') - \sin(\phi) \sin(\omega_{0}t')] = n\lambda\) where \(n = 0, \pm 1, \pm 2, \ldots\) represent the location of the intensity maxima for the rotating linear diffraction grating.

We plot the intensity distribution corresponding to equation (14) in figure 5(a) for time \(t = 10\) s and for the parameters as mentioned in the figure caption. It is clear from equation (14) that the density plot as shown in figure 5(a) would rotate in the course of time with angular frequency \(\omega_{0} = 10\) Hz in a counter-clockwise sense. It has already been mentioned that if we want to observe the density plot with a CCD camera of low temporal resolution, then we will observe time-averaged density plot over a full time period \([2\pi/\omega_{0}]\). Such an intensity distribution is shown in figure 5(b) for the parameters as shown in the figure caption. It can be inferred from equation (14) and figure 5(b) that only the central region of the time-averaged intensity distribution for a
rotating linear diffraction grating would be similar to the central region of the intensity distribution for a circular aperture of the same area. Since $N_d$ is the largest dimension of the rotating linear diffraction grating, the region over which the central regions of the two intensity distributions appear similar, would be extended from $\theta = f = 0$ to $\theta = 0.00013$.

2.3. Diffracted field for rotating double-slit

Let us now replace the single-slit with a double-slit in the set-up of figure 1. We reduce the linear diffraction grating to a double-slit of two rectangular apertures by reducing $N$ to 2. Thus the diffracted field for the rotating double-slit follows equation (14) for $N = 2$. Fraunhofer diffraction formula for the rotating double-slit, however, would be the same as that for the rotating linear diffraction grating for $N = 2$.

We plot intensity distribution corresponding to equation (14) in figure 6(a) for time $t = 10$ s and for the parameters as mentioned in the figure caption. The density plot as shown in figure 6(a), of course, rotates in the course of time as mentioned before. The low temporal resolution leads to an averaging of the intensity distribution over one full time period, as mentioned before. The time-averaged intensity distribution is shown in

![Figure 5](image-url)

**Figure 5.** (a): Density plot for the ‘intensity’ distribution $|\psi(\hat{r}, t)|^2$ on the screen in the $x − y$ plane fixed at $z > 0$ for the scalar field diffracted by the rotating linear diffraction grating in units of the same $|\psi(\hat{z}, t)|^2$ obtained at $x = 0, y = 0, z = z$. The diffracted field follows equation (14) for the parameters $\omega_0 = 10$ Hz, $a = 0.5$ mm, $b = 0.25$ mm, $d = 0.50$ mm, $N = 9$, $c = 3 \times 10^8 \text{ m/s}$, $\lambda = 6000 \text{ Å}$ and $z = 1$ m. (b): Time-averaged density plot of the same ‘intensity’ distribution for the same parameters.
Figure 6. (a): Density plot for the ‘intensity’ distribution \(|\psi(\hat{r}, t)^2|\) on the screen in the \(x - y\) plane fixed at \(z > 0\) for the scalar field diffracted by the rotating double-slit in units of the same \(|\psi(z\hat{k}, t)^2|\) obtained at \(x = 0, y = 0, z = z_0\). The diffracted field follows equation (14) for the parameters as mentioned in the caption of figure 5 except \(N = 2\). (b): Time-averaged density plot of the same ‘intensity’ distribution for the same parameters.

Figure 6(b) for the parameters as shown in the figure caption. It can be inferred from equation (14) and figure 6(b) that only the central region of the time-averaged intensity distribution for a rotating double-slit would be similar to the central region of the intensity distribution for a circular aperture of the same area. Since \(2d\) is the largest dimension of the rotating double-slit, the region over which the central regions of the two intensity distributions appear similar, would be extended from \(\theta = \phi = 0\) to \(\theta = \phi (\approx \frac{\lambda}{2z} = \frac{l}{2}) \approx \frac{\lambda}{2d} \approx 0.0006\).

2.4. Diffracted field for rotating regular polygonal aperture

Let us now replace the single-slit with a regular polygonal aperture in the set-up of figure 1. Let us consider the regular polygon to be \(n\) sided and inscribed inside a circle of radius \(\rho\) as shown in figure 7. Let the polygon, as well as the circle, be in the \(x - y\) plane and the center of the circle be at the origin of the \(x - y\) plane. Positions of the vertices of the polygon at any arbitrary time are denoted in the polar coordinates by \((\rho, \theta_l(t))\) for \(l = 1, 2, 3, \ldots, n\). The polar angle \(\theta_l(t)\) for the \(l\)th vertex is time-dependent because the polygon is rotating with the constant angular velocity \(-\omega_0\hat{k}\). At time \(t = 0\), \(\theta_0(t)\) is chosen to be as 0 (as shown in figure 7) so that we have \(\theta_l(0) = \frac{2\pi l}{n}\).
The regular polygon can be divided into \( n \) identical isosceles triangles by joining all the vertices and the origin. The length of the perpendicular bisector of one such isosceles triangle, whose vertices are denoted by the origin, \((\rho, \theta_{j-1}(t)), \) and \((\rho, \theta_j(t)), \) is \( \rho_j = \frac{\rho \cos(\pi / n)}{\cos(\theta_j(0) - \theta_{j-1}(0))} \) where \( \theta_{j-1} = \frac{2\pi[j - 1/2]}{n} \). The perpendicular bisector intersects the line joining the last two vertices at \((\rho \cos(\pi / n), \theta_{j-1}(t))\). Here from the area of the regular polygonal aperture can be easily obtained as \( A = n\rho^2 \cos(\pi / n) \sin(\pi / n) \).

Equation (7) can still be used to determine the diffracted scalar field for the rotating regular polygon. The diffracted scalar field at the observation point \( \vec{r} \equiv (x, y, z) \) at time \( t \) can be obtained for the rotating regular polygon by adding up the individual contribution of each of the isosceles triangles within the polygon. Eventually, the integration in equation (7) over the secondary source points \((x_0(0), y_0(0))\) on the polygonal aperture results in

\[
\psi(\vec{r}, t) \approx - \frac{z\lambda k \delta e^{ikr - w t}}{2\pi r^2} \sum_{i=1}^{n} \frac{r^2 \cos(\pi / n)}{k^2[x_i(t) \cos(\theta_i(0)) - x_i(t) \sin(\theta_i(0))]}
\]

\[
\times \left[ -1 + \frac{e^{ik[x_i(t) \cos(\theta_i(0)) + x_i(t) \sin(\theta_i(0))]} [x_i(t) \cos(\theta_i(0)) + x_i(t) \sin(\theta_i(0))]}{[x_i(t) \cos(\theta_i(0)) + x_i(t) \sin(\theta_i(0))]}ight]
\]

where \( x_i(t) = x \cos(\omega_0[t - r/c]) + y \sin(\omega_0[t - r/c]) \) and \( y_i(t) = -x \sin(\omega_0[t - r/c]) + y \cos(\omega_0[t - r/c]) \).

In the limit \( \omega_0 \to 0 \), equation (15) takes the form same as that obtained in [17].

The absolute difference between the time-averaged intensity distribution for the rotating regular hexagonal aperture \((n = 6)\) and the intensity distribution for the circular aperture of the same area is shown in figure 8. The dark spot at the centre in figure 8 indicates that the central region of the time-averaged intensity distribution for a rotating regular polygonal aperture, inscribed in a circle of radius \( \rho \), is similar to the central region of the intensity distribution for a circular aperture of the same area \((3 \sqrt{3} \rho^2 / 2)\). Since \( 2\rho \) is the largest dimension of the aperture, the region over which the two central regions of the intensity distributions appear similar would be extended from \( \theta = \phi = 0 \) to \( \theta = \phi = \pi / 3 \), that is, \( \theta = \phi = 0.0014 \) on the screen in the \( x - y \) plane fixed at \( z > 0 \). By comparing figure 3 and figure 8 (including their respective plot-legends) we can say that the region of similarity is larger for the rotating regular polygonal aperture than that for the rotating rectangular aperture of the same area. The region of the similarity, of course, would increase for large values of \( n \).

It is expected that equation (15) would be exactly the same as equation (8) for the square-shaped apertures i.e. \( n = 4 \) and \( a = b = \sqrt{2} \rho \). This is, however, not true because orientations of both the square-shaped apertures corresponding to these two equations are not the same at time \( t = 0 \). Their orientations differ by an angle \( \pi / 4 \) in the \( x - y \) plane. This leads to an additional rotation of the intensity pattern, which corresponds to equation (15), by an angle \( -\pi / 4 \) in the \( x - y \) plane.
described in been explored. Our work can be further connected to the practical application of dynamic imaging systems as for all the rotating cases except for the rotating regular polygonal aperture has also been obtained. The time-dependent Fraunhofer diffraction formula similarities of time-averaged intensity distributions for the rotating apertures with the intensity distributions for rotating double-slit, rotating linear diffraction grating, and rotating regular regular polygonal aperture. Central region time for normal incidence of plane scalar wave on uniformly rotating apertures such as rotating single-slit, in equation shift in the wave-vector which results in a shift in the phase part of the Fourier transform of the aperture function on non-rotating apertures. However, the time-averaged density plots in the closed form because of the difficulties with the integrations of the time-dependent intensities over time. Hence with the help of numerical integrations, the density plots in figures 2(b), 5(b), 6(b), and 8 have been obtained. The Fraunhofer diffraction formula for the rotating regular polygonal aperture could not also be determined because the summation over l in equation (15) does not appear in a closed form for any arbitrary n.

It has been considered that the rotation of the apertures takes place in the clockwise direction. The density plots in figures 2(a), 5(a), and 6(a), would have rotated clockwise if we had considered counter-clockwise rotation of the apertures. However, the time-averaged density plots in figures 3, 5(b), 6(b), and 8 would have changed under the reversal of the rotation.

The square-shaped aperture corresponding to equation (15) gets a clockwise rotation by an amount $\pi/4$ in the x − y plane with respect to the initial $(t = 0)$ orientation of the square-shaped aperture corresponding to equation (8).

The dimension of the aperture is considered to be much larger than that of the wavelength for the applicability of the scalar diffraction theory. Vector diffraction theory [21–25] is needed for the diffraction of an electromagnetic wave if the dimension of aperture is comparable or smaller than that of the wavelength. Vector

![Figure 8. The absolute difference between the time-averaged 'intensity' distribution $|\langle \phi(\vec{r}) \rangle|^2$ for the rotating regular hexagonal aperture (n = 6) and 'intensity' distribution $|\langle \phi(\vec{r}, t) \rangle|^2$ for the circular aperture of the same area and the parameters as mentioned in the caption of figure 2 except $\theta = \frac{\alpha}{\theta_{\text{conn}} / \theta_{\text{res}}}$, $\beta \approx 0.219346$ mm. The density plot is extracted from equations (15) and (11).](image)

3. Conclusion

To conclude, we have explicitly obtained diffracted scalar field at an arbitrary point in the far-field zone at a given time for normal incidence of plane scalar wave on uniformly rotating apertures such as rotating single-slit, rotating double-slit, rotating linear diffraction grating, and rotating regular polygonal aperture. Central region similarities of time-averaged intensity distributions for the rotating apertures with the intensity distributions for static circular apertures of the same area have been shown. The time-dependent Fraunhofer diffraction formula for all the rotating cases except for the rotating regular polygonal aperture has also been obtained.

Our results are important because the dynamical behaviour of the fringe patterns for rotating apertures has been explored. Our work can be further connected to the practical application of dynamic imaging systems as described in [18]. Our work may also find application to the time-resolved femtosecond x-ray diffraction by a rotating crystal [19].

Rayleigh-Sommerfeld scalar diffraction theory of type-I [1, 6] has been employed for the derivation of all the results. Other scalar diffraction theories such as Rayleigh-Sommerfeld scalar diffraction theory of type-II [1, 6] and Fresnel-Kirchhoff scalar diffraction theory [1, 6] would not result in significant differences for small angle of incidence and small angle of diffraction. All our results can be generalised for oblique incidence by considering a shift in the wave-vector which results in a shift in the phase part of the Fourier transform of the aperture function in equation (7) [20]. Rayleigh-Sommerfeld scalar diffraction theory of type-I has been chosen because it is ‘consistent’ with the boundary conditions and matches better with the experimental data for oblique incidence on non-rotating apertures [1].

The time-averaged intensity distributions as shown in figures 2(b), 5(b), 6(b), and 8 could not be expressed in the closed form because of the difficulties with the integrations of the time-dependent intensities over time. Hence with the help of numerical integrations, the density plots in figures 2(b), 5(b), 6(b), and 8 have been obtained. The Fraunhofer diffraction formula for the rotating regular polygonal aperture could not also be determined because the summation over $\ell$ in equation (15) does not appear in a closed form for any arbitrary $n$.
diffraction theory for rotating apertures is kept an open problem. The diffraction of a Gaussian wave by the rotating apertures could have been studied instead of the same for a plane wave. The diffraction of a Gaussian wave by the rotating apertures is also kept as another open problem.

A modal theory which results in a robust solution has already been formulated to represent the propagation of the wavefield from the aperture plane to the detector [26, 27]. In reality, the effect of the modal structure of the wavefield on diffraction has also been captured for a partially coherent wavefield [28]. Modal representation of the diffraction pattern generated by a rotating aperture is also kept as an open problem.

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Data availability statement

No new data were created or analysed in this study.

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