Further development of positive semidefinite solutions of the operator equation $\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B^*$

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Abstract In [7], T. Furuta discusses the existence of positive semidefinite solutions of the operator equation $\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B$. In this paper, we shall apply Grand Furuta inequality to study the operator equation. A generalized special type of $B$ is obtained due to [7].

Keywords: Furuta inequality; Grand Furuta inequality; operator equation; matrix equation; positive semidefinite operator; positive definite operator
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1 Introduction

A capital letter $T$ means a bounded linear operator on a Hilbert space. $T \geq 0$ and $T > 0$ mean a positive semidefinite operator and a positive definite operator, respectively.

In the middle of last century, E. Heinz et al. studied operator theory and obtained the following famous theorem:

Theorem LH (Löwner-Heinz inequality, [9] [8]). If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.

In 1987, T. Furuta proved the following operator inequality as an extension of Theorem LH:

Theorem F (Furuta inequality, [4]). If $A \geq B \geq 0$, then for each $r \geq 0$,

$$
(B^{r/2} A^p B^{r/2})^{1/2} \geq (B^{r/2} B^p B^{r/2})^{1/2}
$$

(1.1)

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\[ (A^\frac{r}{2} A^p A^\frac{r}{2})^{\frac{1}{s}} \geq (A^\frac{r}{2} B^p A^\frac{r}{2})^{\frac{1}{s}} \]  

(1.2)

hold for \( p \geq 0, q \geq 1 \) with \( (1 + r)q \geq p + r \).

K. Tanahashi, in [10], proved the conditions \( p, q \) in Theorem F are best possible if \( r \geq 0 \).

In 1995, T. Furuta showed another operator inequality which interpolates Theorem F:

**Theorem GF** (Grand Furuta inequality, [5]). If \( A \geq B \geq 0 \) with \( A > 0 \), then for each \( t \in [0, 1] \) and \( p \geq 1 \),

\[ A^{1-t+r} \geq \{ A^\frac{r}{2} (A^{-\frac{r}{2}} B^p A^{-\frac{r}{2}})^s A^\frac{r}{2} \}^{\frac{1-t+r}{p-t+r}} \]  

(1.3)

holds for \( s \geq 1 \) and \( r \geq t \).

Afterwards, some nice proof of Grand Furuta inequality are shown, such as [2], [6]. K. Tanahashi, in [11], proved that the outer exponent value of (1.3) is the best possible. Later on, the proof was improved, see [3], [12].

Recently, T. Furuta proved the following theorem by Furuta inequality:

**Theorem A** ([7]). If \( A \) is a positive definite operator and \( B \) is positive semidefinite operator. Let \( m \) and \( n \) be nature numbers. There exists positive semidefinite operator solution \( X \) of the following operator equation:

\[
\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{nr}{m+r}} \left( \sum_{i=1}^{m} A^{\frac{n(i-1)}{m+r}} B A^{\frac{n(i-1)}{m+r}} \right) A^{\frac{nr}{m+r}}
\]  

(1.4)

for \( r \) such that

\[
\begin{align*}
& \{ r \geq 0, \quad \text{if } n \geq m; \\
& r \geq \frac{m-n}{n-1}, \quad \text{if } m \geq n \geq 2.
\end{align*}
\]

In the rest of this short paper, we shall apply Grand Furuta inequality to discuss the existence of positive semidefinite solution of operator equation \( \sum_{j=1}^{n} A^{n-j} BA^{j-1} = B \), and show a generalized special type of \( B \) due to Theorem A.

2 Extension of Furuta’s result

**Lemma 2.1** ([1], [7]). Let \( A \) be a positive definite operator and \( B \) a positive semidefinite operator. Let \( m \) be a positive integer and \( x \geq 0 \). Then \( \frac{d}{dx}[(A + xB)^m]_{x=0} = \sum_{j=1}^{m} A^{m-j} BA^{j-1} \).
Theorem 2.1. Let $A$ be a positive definite operator and $B$ be a positive semidefinite operator. Let $m, n, k$ be positive integers, $t \in [0,1]$. There exists positive semidefinite operator solution $X$ which satisfies the operator equation:

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{t}{2}} \left( \sum_{i=1}^{k} \frac{m}{(m-t)(k-i)} A^{\frac{n}{2}} \right) \left( \sum_{j=1}^{m} \frac{n}{(m-t)(j-k)} X A^{\frac{n}{2}} \right).$$

(2.1)

for $r$ such that

\[
\begin{cases} 
  r \geq t, & \text{if } (1-t)n \geq (m-t)k; \\
  r \geq \max \left\{ \frac{(m-t)(k-(1-t)n)}{n-1}, t \right\}, & \text{if } (m-t)k \geq (1-t)n \text{ with } n \geq 2.
\end{cases}
\]

Proof. As in the proof of [7, Theorem 2.1], by $A+xB \geq A \geq 0$ holds for any $x \geq 0$, then $A^{-1} \geq (A+xB)^{-1} > 0$. Replace $A$ by $A^{-1}$, $B$ by $(A+xB)^{-1}$, $p$ by $m$, $s$ by $k$ in (1.3), and take reverse, we have

\[
(A^\frac{t}{2} (A+xB)^m A^{-\frac{t}{2}})^{\frac{1}{m-t}} \geq A^{1-t+r}.
\]

(2.2)

For any $\alpha \in [0,1]$, apply Löwner-Heinz inequality to (2.2), and take $\frac{1}{n} = \frac{1-t+r}{(m-t)(k+r}_n \alpha$, the following inequality is obtained:

\[
(A^\frac{t}{2} (A+xB)^m A^{-\frac{t}{2}})^\frac{1}{n} \geq A^{\frac{(m-t)(k+r)}{n}}.
\]

(2.3)

By $\alpha \in [0,1]$ and the condition of $r$ in Grand Furuta inequality, we can take $r \geq t$ if $(1-t)n \geq (m-t)k$, or $r \geq \max \left\{ \frac{(m-t)(k-(1-t)n)}{n-1}, t \right\}$ if $(m-t)k \geq (1-t)n \text{ with } n \geq 2$.

Take $Y(x) = (A^\frac{t}{2} (A+xB)^m A^{-\frac{t}{2}})^{\frac{1}{n}}$. According to (2.3), $Y(x) \geq Y(0) = A^{\frac{(m-t)(k+r)}{n}}$ for any $x \geq 0$, then $Y'(0) \geq 0$. Differentiate $Y^n(x) = A^\frac{t}{2} (A+xB)^m A^{-\frac{t}{2}})^n$, use Lemma 2.1, then take $x = 0$, the following equality holds:

\[
\frac{d}{dx} [Y^n(x)] \bigg|_{x=0} = \sum_{j=1}^{n} Y(0)^{n-j} Y'(0) Y^{j-1}
\]

\[
= \frac{d}{dx} [A^\frac{t}{2} (A+xB)^m A^{-\frac{t}{2}})^\frac{1}{n}] \bigg|_{x=0}
\]

\[
= A^\frac{t}{2} \sum_{i=1}^{k} \left[ (A^{-\frac{t}{2}} (A+xB)^m A^{-\frac{t}{2}})^{k-i} \right] \cdot \left[ (A^{-\frac{t}{2}} (A+xB)^m A^{-\frac{t}{2}}) \right] \bigg|_{x=0}
\]

\[
\cdot \left[ (A^{-\frac{t}{2}} (A+xB)^m A^{-\frac{t}{2}})^{i-1} \bigg|_{x=0} \right] A^\frac{t}{2}
\]

\[
= A^\frac{t}{2} \sum_{i=1}^{k} \left[ A^{(m-t)(k-i)} (A^{-\frac{t}{2}}) \sum_{j=1}^{m} A^{m-j} B A^{j-1} A^{-\frac{t}{2}} \right] A^{(m-t)(i-1)} A^\frac{t}{2}
\]

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Replace $Y(0)$ by $A^{(m-t)k+r}$, $Y'(0)$ by $X$, we have
\[
\sum_{j=1}^{n} A^{(m-t)k+r/2}(n-j) X A^{(m-t)k+r/2}(j-1) = A^{\frac{r}{2}} \left\{ \sum_{i=1}^{k} A^{(m-t)(k-i)} \left[ A^{-\frac{r}{2}} \left( \sum_{j=1}^{m} A^{m-j} B A^{j-1} \right) A^{-\frac{r}{2}} \right] A^{(m-t)(i-1)} \right\} A^{\frac{r}{2}}.
\]

Replace $A$ by $A^{(m-t)k+r}$ in above equality, then (2.1) is obtained. \[
\square
\]

**Remark 2.1.** If take $t = 0$ and $k = 1$ in Theorem 2.1, this theorem is just Theorem A, which is the main result of [7].

**Example 2.1.** We use the same example as [7]: For two $l \times l$ matrices $A$ and $B$, take $A = \text{diag}(a_1, a_2, \ldots, a_2)$, all entries of $B$ are 1. If $m, n, k$ are positive integers, $t \in [0, 1]$, there exists positive semidefinite matrix $X$ which satisfies:
\[
\sum_{j=1}^{n} A^{(m-t)k+r/2}(n-j) X A^{(m-t)k+r/2}(j-1) = A^{\frac{r}{2}} \left\{ \sum_{i=1}^{k} A^{(m-t)(k-i)} \left[ A^{-\frac{r}{2}} \left( \sum_{j=1}^{m} A^{m-j} B A^{j-1} \right) A^{-\frac{r}{2}} \right] A^{(m-t)(i-1)} \right\} A^{\frac{r}{2}}
\]
for $r$ such that
\[
\begin{cases}
  r \geq t, & \text{if } (1-t)n \geq (m-t)k; \\
  r \geq \text{max} \left\{ \frac{(m-t)k-(1-t)n}{n-1}, t \right\}, & \text{if } (m-t)k \geq (1-t)n \text{ with } n \geq 2.
\end{cases}
\]

It is easy to calculate the expression of $X$:
\[
X = \left( \frac{r-1}{a_p} \right)^{r-1} \left( \frac{r-1}{a_q} \right) \left( \sum_{i=1}^{k} a_p^{(m-t)(k-i)} a_q^{(m-t)(i-1)} \right) \left( \sum_{j=1}^{m} a_p^{m-j} a_q^{j-1} \right) \left( \sum_{j=1}^{n} a_p^{(m-t)k+r(j-1)} a_q^{(m-t)k+r(j-1)} \right). 
\tag{2.4}
\]

**Remark 2.2.** The condition of $r$ in Theorem 2.1 is necessary. If the condition cannot be fulfilled, the solution of the equation may be not positive semidefinite.

For example, take
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]
and $m = 2$, $n = 2$, $k = 2$, $t = \frac{1}{2}$ in Example 2.1. If we put $r = \frac{1}{2}$, then $r \neq \text{max} \left\{ \frac{(m-t)k-(1-t)n}{n-1}, t \right\}$ By (2.4), the solution of the following matrix equation
\[
A^{\frac{r}{2}} X + X A^{\frac{r}{2}} = A^{\frac{r}{2}} \left( A^{\frac{r}{2}} B A^{\frac{r}{2}} + A^{-\frac{r}{2}} B A^{\frac{r}{2}} \right) + (A^{\frac{r}{2}} B A^{-\frac{r}{2}} + A^{-\frac{r}{2}} B A^{\frac{r}{2}}) A^{\frac{r}{2}}
\]
\[
= A^{\frac{r}{2}} B + A^{\frac{r}{2}} B A + A B A^{\frac{r}{2}} + B A^{\frac{r}{2}}
\]
\[
= \begin{pmatrix} 4 & 3 + 6 \times 2^{\frac{1}{2}} \\ 3 + 6 \times 2^{\frac{1}{2}} & 16 \times 2^{\frac{1}{2}} \end{pmatrix}
\]

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is
\[
X = \begin{pmatrix}
2 & \frac{3+6\times 2^{\frac{3}{4}}}{1+2\times 2^{\frac{3}{4}}} \\
\frac{3+6\times 2^{\frac{3}{4}}}{1+2\times 2^{\frac{3}{4}}} & 2 \\
\end{pmatrix}.
\]
However, \(X\) is not a positive semidefinite matrix because its eigenvalues are \(\{5.4007\ldots, -0.0372\ldots\}\).

**Remark 2.3.** In [1], the authors showed that if \(A\) and \(Y\) are positive semidefinite matrices in matrix equation \(A^{n-1}X + A^{n-2}XA + \cdots + AXA^{n-2} + XA^{n-1} = Y\), then so is \(X\). By Theorem 2.1, in some special cases, if \(Y\) can be expressed as the right hand of (2.1), though it is not positive semidefinite, then there still exists positive semidefinite solution satisfies \(A^{n-1}X + A^{n-2}XA + \cdots + AXA^{n-2} + XA^{n-1} = Y\).

For example, take
\[
A = \begin{pmatrix}
1 & 0 \\
0 & 2 \times 2^{\frac{3}{4}}
\end{pmatrix}, \quad Y = \begin{pmatrix}
4 & 3 \times 2^{\frac{3}{4}} + 6 \times 2^{\frac{3}{4}} \\
3 \times 2^{\frac{3}{4}} + 6 \times 2^{\frac{3}{4}} & 32
\end{pmatrix}.
\]
Although \(Y\) is not a positive semidefinite matrix (because its eigenvalues are \(\{37.5589\ldots, -1.5589\ldots\}\), by simple calculation, the solution of the following matrix equation
\[
A^{2}X + AXA + XA^{2} = Y
\]
is
\[
X = \begin{pmatrix}
4 & \frac{3 \times 2^{\frac{3}{4}} + 6 \times 2^{\frac{3}{4}}}{1+2\times 2^{\frac{3}{4}}} \\
\frac{3 \times 2^{\frac{3}{4}} + 6 \times 2^{\frac{3}{4}}}{1+2\times 2^{\frac{3}{4}}} & \frac{4 \times 2^{\frac{3}{4}}}{3}
\end{pmatrix},
\]
which is still a definite matrix whose eigenvalues are \(\{2.9013\ldots, 0.1119\ldots\}\). The critical reason is that \(Y\) can be expressed as follows,
\[
Y = A^{\frac{3}{4}} \left( \sum_{i=1}^{2} A^{\frac{3}{4}(2-i)} [A^{-\frac{3}{4}} \left( \sum_{j=1}^{2} A^{\frac{3}{4}(2-j)} BA^{\frac{3}{4}(j-1)} \right) A^{-\frac{3}{4}}] A^{\frac{3}{4}(i-1)} \right) A^{\frac{3}{4}},
\]
which is the right hand of (2.1) under the condition of \(m = 2, n = 3, k = 2, t = \frac{1}{2}, r = 1\).

**Remark 2.4.** The following question remains open: How to investigate the properties of the solution of operator equation \(X^{n-1}A + X^{n-2}AX + \cdots + XAX^{n-2} + AX^{n-1} = B\)?
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