ON THE DEFINITION OF STABLE TRANSFER FACTORS

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Abstract. We define stable geometric and spectral transfer factors and develop some of their basic properties. Using our definition of stable geometric transfer factors, we show that the stable transfer conjecture for orbital integrals implies the stable transfer of characters and vice versa. The latter is also implied by a basic form of the local Langlands conjecture, and in particular establishes stable transfer in the archimedean case. Moreover, we introduce a notion of primitive distributions and formulate conjectural transfer identities for the local geometric and spectral distributions that occur in the stable trace formula, suggested by primitisation.

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1. INTRODUCTION

A fundamental problem in the theory of automorphic forms is the transfer of stable distributions, or as we shall call it, stable transfer. These distributions are given by irreducible characters of representations and dual to them, orbital integrals. Whereas the problem of endoscopy has been largely solved in the last decade, the more general situation has seen little progress thus far. The goal of this paper is to lay some modest foundations for the general theory of stable transfer, with a view towards applications to the Arthur-Selberg trace formula and the study of Functoriality.

The theory of stable transfer is only beginning to be developed, in particular the required stable transfer factors generalizing Langlands-Shelstad transfer factors in the endoscopic case have only been constructed in special cases [Lan13, Joh17, JL20], while Shelstad has proposed a general approach in the archimedean case, once again based on Harish-Chandra’s work [She21]. We also note that while

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Sakellaridis has advanced a theory of transfer operators in the relative setting (c.f. [Sak21] and the references therein), we are motivated here by the so-called group case, which remains less well understood in certain senses. The present work rests in part on the observation that as in the theory of endoscopy, much work remains to be done even if the conjectured existence of stable transfer is assumed. Fundamental as the stable transfer of distributions may be to the theory of the trace formula, it is also only the starting point after stabilisation. Compared to the broad vision presented in [Sak21], ours here might be seen to a simple-minded approach, nonetheless the problems that we shall encounter appear to be serious enough to be studied on their own merit.

Let us first describe the conjectural picture. Let $G$ be a quasisplit connected reductive group over a local field $F$. We shall associate a functorial transfer datum $(G', G', \xi')$ with auxiliary datum $(\tilde{G}', \tilde{\xi}')$, defined in Section 2.4 and following, which can be viewed as a weakened or ‘beyond endoscopic’ datum. In general, one expects a stable transfer

$$f \to f' = f^{G'}$$

of suitable test functions on $G$ to the space of stable orbital integrals on $\tilde{G}'$ satisfying the stable character identity

$$f'((\phi')) = f^G(\tilde{\xi}' \circ \phi'),$$

(1.1)

where $\phi'$ is a bounded Langlands parameter for $\tilde{G}'$ and $f'((\phi'))$ is the stable character of $f'$ at $\phi'$. This can be rephrased in terms of the existence of a function $f'$ on $\tilde{G}'$ such that the character identity holds, where the function $f'$ is not uniquely determined but its stable orbital integral is. Implicit in this statement is the basic local Langlands correspondence. In this paper, we shall refer to as the ‘basic local Langlands correspondence’ the existence of a map from irreducible admissible representations of $G(F)$ to Langlands parameters, in contrast to the refined conjecture, such as in [Kal16], characterising such a map. Taking the basic local Langlands conjecture for $G, G'$ as known (due to Shelstad in the archimedean case [She82], and with major progress by Fargues and Scholze in the nonarchimedean case [FS21]), it is possible to specify the transfer in terms of Paley-Wiener functions on the spaces of tempered Langlands parameters on $G$ and $G'$, denoted $\Phi(G)$ and $\Phi(G')$ respectively. The stable transfer for functions $f$ in the Hecke algebra $\mathcal{H}(G)$ of $G$ is then a simple consequence, stated as Corollary 4.2.

**Theorem 1.** The basic local Langlands correspondence implies the stable transfer (1.1) for $f \in \mathcal{H}(G)$.

On the other hand, this transfer is also implied by the transfer of stable characters and orbital integrals (Theorem 5.6 and Conjecture 5.1 respectively), the latter being the existence of a transfer $f'$ satisfying

$$f'(\delta') = \int_{\Delta(G/Z)} \Theta_{\tilde{\xi}'}(\delta', \delta) f^G(\delta) d\delta,$$

(1.2)

where the distribution $\Theta_{\tilde{\xi}'}(\delta', \delta)$ is a stable transfer factor, the focus of our study. Here $f^G(\delta)$ is the stable orbital integral of $f$ at the stable conjugacy class $\delta$. This stable transfer was first conjectured by Langlands in [Lan13, §2.1]. For applications to the trace formula, it is necessary to have both spectral and geometric characterisations of the transfer $f'$. It follows from the definitions that the stable transfer mapping $f \to f'$ is a continuous linear mapping from $\mathcal{H}(G)$ to $SI(G')$. 


where \( S_I(G') \) is the space of stable orbital integrals on \( G' \). It can be viewed as the Paley-Wiener space of tempered (or bounded) Langlands parameters \( \Phi(G') \). Since \( f' \) only depends on the image \( f^G \) of \( f \) in \( S_I(G) \), we in fact obtain a mapping \( f^G \to f' \) from \( S_I(G) \) to \( S_I(G') \).

The reader familiar with endoscopy will find many of our definitions and constructions familiar, and much of the paper is essentially an exercise in definitions. For example, our notion of functorial transfer data (or what was called ‘beyond endoscopic data’ in [Art17]) in Section 2 is but a weaker form of endoscopic data, where the datum of a semisimple element is no longer imposed. An important question in this regard is whether there exists a refinement of our transfer data that provides a disjoint partition of \( \Phi(G) \) in terms of primitive parameters (defined in Section 6.1). This is likely to be related to the fact that Arthur packets are not necessarily disjoint, and in particular may not be possible.

Our stable geometric transfer factors, on the other hand, rely heavily on the local character relations satisfied by stable orbital integrals and stable characters developed by Arthur in [Art96], and whose required properties we develop in Section 3. The geometric transfer factors, defined in Section 4, take the form

\[
\Theta_{\tilde{\xi}, \delta'}(\delta', \delta) = \int_{\Phi(\tilde{G}', \tilde{\zeta}') \Phi(\tilde{G}, \tilde{\zeta})} S'(\delta', \phi') S(\xi', \phi' \circ \delta) d\phi',
\]

where \( S'(\delta', \phi') \) and \( S(\phi, \delta) \) are the stable character and the kernel of the Fourier inversion of stable orbital integrals respectively. In particular, as we are working with stable objects, much of the theory of endoscopy is essential to our constructions. Our definition of the stable transfer factor is validated by the following result in Section 5, which is the key result of the paper, stated as Corollary 5.8, showing that the spectral and geometric transfers agree.

**Theorem 2.** The stable transfer \( f' \) satisfies (1.1) if and only if it satisfies (1.2).

It is of course expected that the spectral and geometric transfers characterize each other; the important point here is that the distribution \( \Theta_{\tilde{\xi}, \delta}(\delta', \delta) \) satisfies this expectation in general. In particular, by the local Langlands correspondence, the archimedean transfer holds unconditionally. Though it is only the most basic form of this correspondence is needed, namely the assignment of an \( L \)-parameter to each irreducible admissible representation, and we note that to even formulate the characterisation (1.1) of the stable transfer itself requires the basic local Langlands correspondence.

The broader goal of course, is the primitisation of the stable trace formula, following Arthur’s formalisation of Langlands’ proposal. In Section 6, we formulate a notion of primitive distributions, and describe a conjectural transfer identity for local stable orbital integrals which suggests itself to us based on the expected primitisation of the trace formula. This identity parallels the main local theorem required for the stabilisation of the geometric side of the trace formula [Art02]. Unlike stable distributions, which have an equally well-defined notion both geometrically and spectrally, we do not yet have a satisfactory geometric characterisation for our primitive distributions, though their spectral definition is naturally suggested by functoriality.

Finally, in Section 7 we introduce stable spectral transfer factors rather abstractly, which is enough to formulate a parallel conjectural transfer identity for local stable characters. Importantly, their definition depends on the surjectivity...
of a certain stable transfer map (6.3), which we prove unconditionally only in the archimedean case. The proof of the nonarchimedean case requires, among other things, a study of the descent of stable geometric transfer factors along the lines of Langlands and Shelstad in the endoscopic case [LS87], which we initiate but do not complete in Section 5.3, and also a stable analogue (7.3) of Waldspurger’s kernel formula for Fourier transforms on Lie algebras [Wal97]. Assuming these two identities, we are then able to define the spectral transfer factors and describe primitive distributions on the spectral side. This makes uniform the study of distributions on either side of the stable trace formula, a phenomenon that we have come to expect with the trace formula.

2. Functorial transfer data

2.1. Preliminaries. Let $F$ be a local field of characteristic zero, with an algebraic closure $F$. Let $G$ be a connected reductive group over $F$. We denote by $\mathcal{L}(M)$ the collection of Levi subgroups of $G$ containing $M$, $\mathcal{L}^0(M)$ the subset of proper Levi subgroups in $\mathcal{L}(M)$, and $\mathcal{P}(M)$ the collection of parabolic subgroups of $G$ containing $M$. Let $A_M$ be the maximal split torus of a Levi subgroup $M$ of $G$. We then identify the Weyl group of $(G, A_M)$ with the quotient of the normaliser of $M$ by $M$, thus

$$W(M) = W^G(M) = \text{Norm}_G(M)/M.$$ 

If $M_0$ is a minimal Levi subgroup of $G$, which we shall assume to be fixed, and denote $\mathcal{L} = \mathcal{L}(M_0)$, $\mathcal{P} = \mathcal{P}(M_0)$, $\mathcal{L}^0 = \mathcal{L}^0(M_0)$, and $W^G_0 = W^G(M_0)$. Also write $P_0$ for the minimal parabolic subgroup containing $M_0$. Also, we fix a maximal compact subgroup $K$ of $G(F)$, which is hyperspecial if $F$ is nonarchimedean.

As usual, we form the real vector space $a_M = \text{Hom}(X(M), \mathbb{R})$ where $X(G)_F$ is the module of $F$-rational characters on $G$. We note by $H_G : G(F) \to a_G$ canonical homomorphism defined by

$$e^{(H_G(x), \chi)} = |\chi(x)|, \quad x \in G(F), \chi \in X(G)_F,$$

where $|\cdot|$ is the normalised valuation on $F$. We set $a_{G,F} = H_G(G(F))$ and $\tilde{a}_{G,F} = H_G(A_G(F))$, which are closed subgroups of $a_G$, with associated vector spaces $a_{G,F}^\vee = \text{Hom}(a_{G,F}, 2\pi i\mathbb{Z})$ and $\tilde{a}_{G,F}^\vee = \text{Hom}(\tilde{a}_{G,F}, 2\pi i\mathbb{Z})$, which are closed subgroups of $ia_G^\vee$. If $F$ is nonarchimedean, all four groups are lattices; if $F$ is archimedean, we have $\tilde{a}_{G,F} = a_{G,F} = a_G$ and $\tilde{a}_{G,F}^\vee = a_{G,F}^\vee = \{0\}$. Fixing a Haar measure on $a_G$, we obtain a dual Haar measure on the real vector space $ia_{G,F}^\vee$. If $F$ is nonarchimedean, we normalise measures so that $a_{G,F}/\tilde{a}_{G,F}$ and $ia_{G,F}^\vee/\tilde{a}_{G,F}^\vee$ have volume 1. It follows that the volume of the quotient $ia_{G,F}^\vee/\tilde{a}_{G,F}^\vee$ equals the index $[a_{G,F}/\tilde{a}_{G,F}]$.

Let $\Gamma = \Gamma_F$ and $W_F$ be the Galois and Weil groups of $\hat{F}/F$ respectively. Let $G$ be a connected reductive group over $F$, and let $G^*$ be a quasi-split inner form of $G$ with inner twist $\psi : G \to G^*$. In other words, $\psi$ is an isomorphism such that $\psi \circ \sigma(\psi)^{-1}$ is an inner automorphism for all $\sigma \in \Gamma$. Moreover, fix a bijection of canonical based root data $\Psi(G)\vee \to \Psi(\hat{G})$, where $\hat{G}$ is the complex dual group of $G$. Let $^L G$ be the $L$-group defined by $G \rtimes W_F$, with the action of $W_F$ given by

$$W_F \to \Gamma \to \text{Out}(\hat{G}),$$

where we presume to have fixed an action of $\Gamma$ on $\hat{G}$. The $L$-isomorphism $^L \psi : ^L G \to ^L G^*$ induced by $\psi$ allows us to identify the two groups.
Let \((B,T)\) be a Borel pair of \(G\) where \(B\) is a Borel subgroup of \(G\) and \(T\) a maximal torus of \(B\), not necessarily defined over \(F\). For each pair \((B,T)\) in \(G\) and \((B_1,T_1)\) in \(\hat{G}\) we have a canonical isomorphism \(\hat{T} \simeq T_1\). Define a pinning by \(E = (B,T,\{X_\alpha\})\) where \(\{X_\alpha\}\) runs over simple roots \(\alpha\) of \(T\) acting on the Lie algebra of the unipotent radical of \(B\), and \(X_\alpha\) is an element of the eigenspace associated to \(\alpha\). Any two pinnings are related by the adjoint action \(\text{ad}_g\) for some \(g \in G_{sc}\), the simply connected cover of \(G\), unique up to translation by the centre \(Z(G_{sc})\). The restriction of \(\text{ad}_g\) to \(B\) and \(T\) are uniquely determined, so we may define a canonical pinning by taking the inductive limit over all pinnings of \((B,T)\). If we denote by \(\rho(w)\) the action of \(\Gamma\) on \(\hat{G}\), we can define a new action \(\rho_g(w) = \text{ad}_g \rho(g) \text{ad}_{g^{-1}}\) that fixes the original pinning. Then \(L^G\) is isomorphic to \(\hat{G} \rtimes W_F\) under this action, sending \((h,w)\) to \((h \rho(w)(g) g^{-1},w)\).

2.2. \(K\)-groups. We shall work with \(K\)-groups, following [Art99, \S2], which streamlines endoscopy theory over archimedean local fields. If \(F\) is \(p\)-adic, then \(G\) is just an ordinary connected group, whereas if \(F\) is archimedean, then \(G\) can have several connected components

\[ G = \coprod_{\alpha} G_{\alpha}, \quad \alpha \in \pi_0(G), \]

a variety whose connected components \(G_{\alpha}\) are reductive groups over \(F\), equipped with an equivalence class of frames \((\psi,u) = \{(\psi_{\alpha\beta},u_{\alpha\beta}) : \alpha,\beta \in \pi_0(G)\}\) satisfying natural compatibility conditions. Here \(\psi_{\alpha\beta} : G_{\alpha} \to G_{\beta}\) is an isomorphism over \(\bar{F}\), and \(u_{\alpha\beta}\) is a locally constant function from \(\Gamma = \text{Gal}(\bar{F}/F)\) to the simply connected cover \(G_{\alpha,sc}\) of the derived group of \(G_{\alpha}\). Any connected reductive group is an element of a \(K\)-group that is unique up to weak isomorphism.

We call \(G^*\) a quasisplit inner twist of \(G\) if \(G^*\) is a connected, quasisplit group over \(F\) equipped with a \(G^*\)-inner class of compatible inner twists \(\psi_{\alpha} : G_{\alpha} \to G^*\) and a corresponding family of compatible, locally constant functions \(u_{\alpha} : \Gamma \to G_{\alpha,sc}\).

We shall call a \(K\)-group \(G\) quasisplit if one of the isomorphisms \(\psi_{\alpha}\) is defined over \(F\), i.e., \(G_{\alpha}\) is quasisplit. Unless otherwise indicated, we shall assume \(G\) to be a quasisplit \(K\)-group over \(F\).

The usual definitions for connected groups extend to \(K\)-groups in a natural way. For example, there are similar notions of a Levi \(K\)-subgroup \(M\) of \(G\), with associated sets \(\mathcal{Z}(M)\) and \(\mathcal{P}(M)\). The isomorphism \(\psi_{\alpha\beta}\) induces a bijection of Borel pinnings from \(G_{\alpha}\) to \(G_{\beta}\), and taking inverse limits their canonical pinnings are thus equivalent and Galois equivariant. A central induced torus \(Z\) of a \(K\)-group \(G\) will have central embeddings \(Z \simeq Z_\alpha \subset Z(G_{\alpha})\) for each \(\alpha\), where \(Z(G_{\alpha})\) is the center of \(G_{\alpha}\), and \(\zeta\) determines a character \(\zeta_\alpha\) for each \(\alpha\). Also we have \(Z \simeq Z^* \subset Z(G^*)\) and \(\zeta^*\) correspondingly. These isomorphisms are required to be compatible with the isomorphisms \(\psi_{\alpha\beta}\) and \(\psi_{\alpha}\) respectively. We shall call such a pair \((Z,\zeta)\) a central datum for \(G\). Finally, we note that \(G(F)/Z(F) = G/Z(F)\).

2.3. Stable conjugacy classes. Let \(G'\) be a reductive group with an embedding of semisimple conjugacy classes into that of \(G\). We recall that a semisimple element \(g' \in G'(F)\) is called \(G\)-regular if the image of its conjugacy class in \(G(F)\) consists of regular semisimple elements, and strongly \(G\)-regular if the image consists of strongly regular elements, that is, whose centralisers are tori. If \(c\) is a semisimple conjugacy class of \(G\), we write \(G_{c,+}\) for the centraliser of a representative of \(c\) in the component \(G_{\alpha}\) that contains \(c\), and write \(G_{c}\) for the identity component of
G_{c,+}. We call c elliptic if it lies in an elliptic maximal torus in G_\alpha modulo the split component A_G \simeq A_{G_\alpha} of the centre of G.

We write \Gamma_{ss}(G) = \Gamma_{ss}(G(F)) for the set of semisimple conjugacy classes of G(F), \Gamma(G) = \Gamma_{reg}(G(F)) for the subset of strongly regular, semisimple conjugacy classes in G(F), and \Gamma_{ell}(G) = \Gamma_{reg,ell}(G(F)) for the subset of elliptic conjugacy classes. That is,

\Gamma_{ell}(G) \subset \Gamma_{reg}(G) \subset \Gamma_{ss}(G).

We also write \Gamma_G(G') = \Gamma_{G-reg}(G'(F)) and \Gamma_{G,ell}(G') = \Gamma_{G-reg,ell}(G'(F)) for the set of G-regular (resp. G-regular elliptic) conjugacy classes in G'(F). Clearly each of these sets are equal to the disjoint union over \alpha of the corresponding sets for each connected component, e.g.,

\Gamma(G) = \coprod_{\alpha \in \pi_0(G)} \Gamma(G_\alpha)

and so on. The Weyl group W(M) \simeq \text{Norm}_{G_\alpha}(M_\alpha)/M_\alpha acts on \Gamma_{G,ell}(M), and we have a decomposition

\Gamma(G) = \bigoplus_{\{M\}} \Gamma_{G,ell}(M)/W(M),

where the direct sum ranges over conjugacy classes of K-Levi subgroups M in the sense of [Art99, p.221].

We say that two semisimple elements c_1 \in G_{\alpha_1} and c_2 \in G_{\alpha_2} are stably conjugate if there is a g_1 \in G_{\alpha_1}(\hat{F}) such that the mapping

\varphi = \text{Int}(g_1) \circ \psi_{\alpha_1,\alpha_2} : G_{\alpha_2} \to G_{\alpha_1},

maps c_2 to c_1, and has the property that for any \sigma \in \Gamma, the automorphism \varphi \circ \sigma(\varphi)^{-1} of G_{c_1} is inner. Let \Delta_{ss}(G) = \Delta_{ss}(G(F)) be the set of semisimple stable conjugacy classes in G(F). There is a canonical injective mapping \delta \to \delta^* from \Delta_{ss}(G) to \Delta_{ss}(G^*) which is a bijection if G is quasisplit. We also define subsets

\Delta_{ell}(G) \subset \Delta_{reg}(G) = \Delta(G) \subset \Delta_{ss}(G)

as above, \Delta_G(G') = \Delta_{G-reg}(G'(F)) and \Delta_G,ell(G') = \Delta_{G-reg,ell}(G'(F)) analogously. We also define

(2.1) \Delta(G) = \bigoplus_{\{M\}} \Delta_{G,ell}(M)/W(M)

for the W(M)-orbits as above. For any maximal torus T of G over F, we have the finite abelian group K(T) = \pi_0(T^F/Z(\hat{G})^F). Given \gamma \in \Gamma(G), there is a bijection between the set of G(F) classes in the stable conjugacy class \delta of \gamma and the set of characters on the group K_\delta = K_\gamma = K(G_\gamma), so we set \nu(\delta) = |K_\delta|. (When F is archimedean, this is only true because we are taking G to be a K-group.)

2.4. Functorial transfer datum. We caution that as was in case with endoscopy, it is likely necessary to refine the datum introduced here, which is simply a weakened version of endoscopic datum. Let us call a functorial transfer datum for a connective reductive group G over F, or transfer datum for short, a tuple (G',G',\xi), where

1. G' is a connected quasisplit reductive group over F,
Levi subgroup
\( \Gamma \)-invariants of the centres satisfy \(( Z_{2.5} \text{ over } \) datum \( G \).
As usual, we shall use \( G' \) to stand in for the triple itself. We say that a transfer datum \( G' \) is elliptic if the image of \( \xi' \) in \( L \) is not contained in \( L \) for any proper Levi subgroup \( M \) of \( G \) over \( F \), or equivalently, that the connected component of \( \Gamma \)-invariants of the centres satisfy \(( Z(G')^{\Gamma})^0 = (Z(\hat{G}))^0 \). (We shall often identify \( Z(\hat{G}') \) with its image in \( \hat{G} \).) The latter condition is also equivalent to the property that \( \text{Cent}((\xi'(G'), \hat{G}))/Z(\hat{G})^\Gamma \) is finite.

Given transfer data \( (G', \xi', \hat{G}') \) and \( (G'_1, \xi'_1, \hat{G}'_1) \), we say they are isomorphic if there exist an \( F \)-isomorphism \( \alpha : G'_1 \to G' \), an \( L \)-isomorphism \( \beta : G' \to G'_1 \), and an element \( g \in \hat{G}'_1 \) such that \( \alpha : \Psi(G'_1) \to \Psi(G') \) and \( \beta : \Psi(G') \to \Psi(\hat{G}'_1) \) are dual, and \( \text{Int}(g) \circ \xi \circ \beta = \xi' \). We denote by \( \mathcal{F}_\text{ell}(G) \) the set of isomorphism classes of elliptic transfer datum for \( G \), and let \( G' \) stand for such a representative. Let \( \text{Aut}_G(G') \) be the set of \( g \in \hat{G} \) that induce an isomorphism of transfer data, i.e., \( \text{Int}(g) \) is an \( L \)-isomorphism of \( G' \) onto itself, so that we may identify the group of outer automorphisms as

\[
\text{Out}_G(G') \simeq \text{Aut}_G(G')/\xi'(\hat{G}').
\]

Any element in \( \text{Out}_G(G') \) can be identified with an outer automorphism of \( G' \) which is defined over \( F \).

The definitions extend to \( K \)-groups in a straightforward manner. If \( G' \) is a transfer datum for the component \( G_\alpha \), then so it is also for \( G_\beta \) for any \( \beta \in \pi_0(G) \). We can therefore view \( G' \) as a transfer datum for the \( K \)-group \( G \). We shall write \( \mathcal{F}(G) \) for the set of isomorphism classes of transfer data \( G' \) for \( G \) that are relevant to \( G \), by which we mean that there is an element in \( \Delta_{G, \text{reg}}(G') \) that is an image of some element in \( \Delta_{\text{reg}}(G) \), in the sense defined below. We also write \( \mathcal{F}_\text{ell}(G) \) for the set of elliptic transfer data.

2.5. Auxiliary data. The group \( G' \) need not be an \( L \)-group, so there might not be an \( L \)-isomorphism from \( G' \) to \( L G' \) which is the identity on \( \hat{G}' \). Thus given any \( G' \in \mathcal{F}_\text{ell}(G) \), we shall fix an auxiliary datum \( (\hat{G}', \hat{\xi}') \) where \( G' \to \hat{G}' \) is a \( z \)-extension, by which we mean a split central extension of \( G' \) by an induced torus \( \hat{C}' \), and \( \hat{\xi}' : G' \to \hat{L} \hat{G}' \) is an \( L \).embedding satisfying the conditions of [Art96, Lemma 2.1]. Namely, we require that the \( z \)-extension

\[
1 \to \hat{C}' \to \hat{G}' \to \hat{G}' \to 1
\]

over \( F \) satisfies:

(1) the central subgroup \( \hat{C}' \) is an induced torus,

(2) the dual exact sequence \( 1 \to \hat{G}' \to \hat{G}' \to \hat{C}' \to 1 \) extends to a short exact sequence of \( L \)-homomorphisms

\[
1 \to G' \xrightarrow{\hat{\xi}'} \hat{G}' \to \hat{C}' \to 1,
\]

(3) every element of \( \text{Out}_G(G') \) extends uniquely to an outer automorphism of \( G' \) over \( F \) which leaves \( \hat{C}' \) pointwise fixed.
As a $K$-group, the $z$-extension $\tilde{G}'$ satisfies $\pi_0(\tilde{G}') = \pi_0(G')$, and $\tilde{G}'_\alpha$ is a $z$-extension of $G'_\alpha$ by $\tilde{C}'$ for each $\alpha \in \pi_0(G)$. Moreover, for any frame $(\psi', u')$ of $G'$ there is a corresponding frame $(\tilde{\psi}', \tilde{u}')$ for $\tilde{G}'$ such that $r_\alpha \tilde{\psi}_\alpha = \psi_\alpha r_\beta$ and $\tilde{u}_\alpha = u_\alpha$ for all $\alpha, \beta \in \pi_0(G)$.

Let $\tilde{\eta}'$ be the character dual to the Langlands parameter induced by the composition

$$W_F \to G' \xrightarrow{\tilde{\xi}} L \tilde{G}' \to L\tilde{C}'$$

where $W_F \to G'$ is any section. By condition (3), $\text{Out}_G(G')$ can be identified with a finite group of $F$-rational outer automorphisms of $G'$ which leave $\tilde{C}'$ pointwise invariant, thus fixing the central character $\tilde{\eta}'$. We write $\tilde{Z}'$ for the preimage of $Z$ in $G'$, and $\tilde{\xi}'$ for the product of $\tilde{\eta}'$ and the pullback of $\xi$. We can assume that the auxiliary datum $(\tilde{G}', \tilde{\xi}')$ is compatible under isomorphisms of transfer data, and therefore depends only on the elements $G' \in \mathcal{F}_{\text{ell}}(G)$.

Furthermore, we can also assume that $\tilde{\xi}'$ is of unitary type, in the sense that if $\phi' : W_F \to G'$ is an $L$-homomorphism such that the image of $\xi' \circ \phi'$ projects to a relatively compact subset of $G$, then the image of $\tilde{\xi}' \circ \phi'$ also projects to a relatively compact subset of $\tilde{G}'$. The analogous condition ensures that the relative endoscopic transfer factors, defined for $K$-groups in [Art99, §2], have absolute value 1. We shall return to this point later.

**Remark 2.1.** In the endoscopic case, we have an endoscopic set $\mathcal{E}(G)$ consisting of ordinary endoscopic datum $(G^e, \tilde{G}^e, s^e, \xi^e)$, where $s^e$ is a semisimple element in $G$ satisfying certain assumptions. The associated auxiliary endoscopic datum $(\tilde{G}^e, \tilde{\xi}^e)$ is defined similarly, and also required to satisfy compatibility conditions, which we refer to [Art99, §2] or [Kal16] for details. In this paper, we shall generally indicate endoscopic objects with the superscript $^e$, and ‘stable’ objects with $'$.

**2.6. Transfer data associated to Levis.** We may reduce to the elliptic data in the following manner: Let $M' \in \mathcal{F}(G)$ represent the datum $(M', M', \xi'_M)$. Let $Z_G(\xi'_M(M'))$ be the centraliser in $\tilde{G}$ of $\xi'_M(M')$, and let $\tilde{M} = Z_G(\xi'_M(M'))^0$. Then $\tilde{M}$ is a $\Gamma$-stable Levi subgroup of $\tilde{G}$ such that the corresponding $L$-subgroup $L\tilde{M}$ of $L\tilde{G}$ contains $\xi'_M(M')$. Since $M'$ is relevant to $G$, there is a Levi subgroup $M$ of $G$ for which $\tilde{M}$ is a dual Levi subgroup. Thus we may view $M'$ as an elliptic transfer datum for $M$, since

$$(Z(\tilde{M}')^\Gamma)^0 = Z_G(\xi'_M(M'))^0 = (Z(\tilde{M})^\Gamma)^0.$$  

For such $M'$, we can choose $\tilde{M}'$ to be the preimage of $M'$ in $\tilde{G}'$. Then $\tilde{M}'$ is a Levi subgroup of $\tilde{G}'$ for which $\tilde{M}' \times Z(\tilde{G}')$ is a dual Levi subgroup. We also take $\xi'_M$ to be a restriction of the corresponding embedding $\tilde{\xi}'$, giving an $L$-embedding of $M'$ into $L\tilde{M}'$. Moreover, each group $\text{Out}_G(M') = \text{Aut}_G(M')/\xi'_M(M')$ acts by outer automorphisms of $M'$ over $F$ which leave $\tilde{C}'_M$ pointwise fixed. Here $\tilde{C}'_M$ is the central torus defining the extension $M'$. The pair $(\tilde{M}', \xi'_M)$ is then an auxiliary datum for $M'$ compatible with $G'$.

Consider the set of $W(M)$-orbits of $\mathcal{F}_{\text{ell}}(M)$. The image of $M'$ in $\mathcal{F}_{\text{ell}}(M)/W(M)$ depends only on the image of $M'$ in the set $\mathcal{F}(G)$ attached to $G$. Varying $M'$, we
The data \((\tilde{M}', \tilde{\xi}')\) for \(M'\) vary with \(G'\), but they can be seen to be equivalent in the following sense. We may choose auxiliary data \(\tilde{M}' \to M'\) and \(\tilde{\xi}' : M' \to L\tilde{M}'\) for \(M\) that are independent of \(G'\) but equipped with factorisations

\[
\tilde{M}' \to \tilde{M}' \to M'
\]

for each \(G' \in \mathcal{F}(G)\) containing \(M'\). Such a factorisation can be given by taking the fibre product of extensions \(\tilde{M}' \to M'\) as \(G'\) varies. Given the choice of \((\tilde{M}', \tilde{\xi}')\), we have an admissible embedding

\[
\tilde{\varepsilon}'_M : L\tilde{M}' \to L\tilde{M}', \quad G' \in \mathcal{F}(G)
\]

such that \(\tilde{\xi}' = \tilde{\varepsilon}'_M \tilde{\xi}'\). Here \(\tilde{\varepsilon}'_M\) differs from the standard embedding \(L\tilde{M}' \to L\tilde{M}'\) by a 1-cocycle from \(G'\). This determines an character \(\tilde{\chi}_M'\) of \(\tilde{M}'\), and by identifying any function on \(\tilde{M}'\) with its pullback to \(\tilde{M}'\), we obtain a topological isomorphism of functions \(f \to \tilde{\chi}_M f\) from functions on \(\tilde{M}'\) to functions on \(M'\). Thus we may identify the associated spaces of distributions by such isomorphisms.

### 2.7. Images of semisimple elements.

Let \(G' \in \mathcal{F}(G)\). The isomorphism \(\hat{T} \cong T_1\) sends the coroots of \(T\) in \(G\) to the roots of \(T_1\) in \(G\), the \(B_1\)-simple coroots to the \(B_1\)-simple roots, and the Weyl group of \(T\) with contragredient action to the Weyl group of \(T'\). If \((B'_1, T'_1)\) is a Borel pair in \(G'\) then there is an \(x \in \hat{G}\) such that \(\text{Int}(x) \circ \xi'\) maps \(T'_1\) to \(T_1\) and \(B'_1\) to \(B_1\). If \((B', T')\) is a Borel pair in \(G'\), then we have an isomorphism \(\hat{T}' \cong \hat{T}\) defined by the composition

\[
\hat{T}' \to T'_1 \to T_1 \to \hat{T}_1,
\]

and thus also an isomorphism \(T' \cong T\). These isomorphisms map the coroots of \(T'\) in \(G'\) to a subsystem of coroots of \(T\) in \(G\), the Weyl group \(W_T\) of \(T'\) into a subgroup of the Weyl group \(W_T\) of \(T\), and the roots of \(T'\) into a subset of the roots of \(T\). Then the map \(T'/W_T \to T/W_T\) of Weyl group orbits is independent of all choices. Since these orbits classify the conjugacy classes of semisimple elements in \(G'(\hat{F})\) and \(G(\hat{F})\), we thus have a canonical map of semisimple conjugacy classes from \(G'(\hat{F})\) to \(G(\hat{F})\).

Suppose that \(T'\) is defined over \(F\). If \((B^*, T^*)\) is Borel pair in \(G^*\) such that \(T^*\) and \(T' \to T^*\) are defined over \(F\) following Steinberg’s theorem, then we say \(T' \to T^*\) is an admissible embedding of \(T'\) in \(G^*\). It is determined up to conjugation by elements in

\[
\{g \in G_{sc}(\hat{F}) : g\sigma(g^{-1}) \in T^*(\hat{F}), \sigma \in \Gamma\},
\]

where as usual \(G_{sc}^*\) denotes the simply connected cover of the derived group of \(G^*\). We say a strongly \(G\)-regular \(\gamma' \in G'(\hat{F})\) is an image of \(\gamma \in G(\hat{F})\) if \(\gamma\) lies in the image of the stable conjugacy class of \(\gamma'\). For arbitrary \(G\)-regular semisimple \(\gamma'\), we set \(T' = G_{sc}^*\) and choose an admissible embedding of \(T'\) in \(G^*\). Then if \(\gamma \in G(F)\) is regular semisimple and \(T = G_\gamma\) then we say that \(\gamma'\) is an image of \(\gamma\) if there exists \(x \in G^*\) such that \(\text{Int}(x) \circ \psi\) maps \(x\) to the image \(\gamma^*\) of \(\gamma'\) under \(T' \to T^*\) and \(T \to T^*\). The correspondence \((\gamma', \gamma)\) is independent of the choice of admissible
embedding. Moreover, a $G$-regular semisimple element of $G'(F)$ is either the image of no element or the image of exactly one stable regular semisimple conjugacy class in $G(F)$.

Lemma 2.2. Let $G'$ be the quasisplit inner form of $G$. If $G^*(F)$ is nonempty, then $G^*$ is relevant to $G$. Moreover, every element $γ^* ∈ Γ_G(G^*)$ is an image of some $γ ∈ Γ(G)$.

Proof. The second statement implies the first, so it suffices to prove the second. Since $G^*(F)$ is nonempty, we can take the semisimple part $d^*$ of some element in $G^*(F)$. We fix a maximal torus $T^*$ of $G_{d^*}$ defined over $F$. For $t^* ∈ T^*(F)$ in general position, the product $γ^* = t^*d^*$ is strongly $G$-regular. In particular, $Γ_G(G^*)$ is nonempty.

Now let $(B^*, T^*)$ be a Borel pair fixed by $ad_{γ^*}$ and $(B, T)$ a Borel pair of $G$ defined over $F$. There exists an isomorphism $ξ : T → T^*$, and a cocycle $ω : Γ → W(G, T)$ such that $ξ ∘ ω(σ) ∘ σ = σ ∘ ξ$ for all $σ ∈ Γ$. Since $G$ is quasisplit, we can find some $q ∈ G(F)$ such that $ad_q^{-1}(T)$ is defined over $F$ and $ω(σ) ∘ σ ∘ ad_q = ad_q ∘ σ$ for all $σ ∈ Γ$. If we write $(B_1, T_1) = ad_q(B, T)$, then the induced isomorphism $ξ_1 : T_1 → T^*$ is Galois equivariant, and it follows then that for each $γ^*$ in the image we denote by $γ$ its inverse image under $ξ_1$, which lies in $G(F)$. □

2.8. Unramified transfer data. Let $𝓞_F$ be the ring of integers of $F$. Recall that $G$ is unramified over $F$ if $G(𝓞_F)$ is hyperspecial and $G$ splits over an unramified extension of $F$. We shall call a transfer datum $(G', G', ξ')$ unramified if $G'$ is unramified over $F$, $G' ≃ L(G')$ is the $L$-group of $G'$, and $ξ'$ descends to some finite unramified extension $E/F$, in the sense that the following commutative diagram holds:

$$
\begin{array}{ccc}
G' & \xrightarrow{ξ'} & G \\
\downarrow{φ_{G'}} & & \downarrow{φ_G} \\
\hat{G}' × Gal(E/F) & \xrightarrow{ξ'_0} & \hat{G} × Gal(E/F)
\end{array}
$$

where the maps $φ_{G'}, φ_G$ are induced by the canonical projection $W_F → Gal(E/F)$.

Lemma 2.3. Let $G, G'$ be unramified connected reductive groups over $F$, equipped with an embedding $ξ'_0 : \hat{G}' → G$. Then there is a canonical class of admissible $L$-embeddings of $L(G') → L(G)$ such that $(G', L(G'), ξ')$ is an unramified transfer datum.

Proof. The group $L(G')$ is defined by $H × Gal(E/F)$ some unramified extension $E/F$. Let $σ$ be a generator of $Gal(E/F)$. We assume that for every $σ ∈ Gal(E/F)$ there is an element $ω(σ) × θ ∈ L(G)$ such that the action of $σ$ as an outer automorphism of $\hat{G}'$ defined by conjugation by $ω(σ) × θ$. Let $ω(σ)$ be an element in the Weyl group such that $σ$ acts on $\hat{G}'$ by $ω(σ) × θ$ with $θ$ outer in $\hat{G}$. We may also assume that $ω(σ), σ$, and $θ$ act as automorphisms of extended Dynkin diagrams and stabilise $(\hat{B}', \hat{T}')$. Then we can choose a representative $w = n(ω(σ)) × θ$ as in [LS87, §2] that also stabilises $(\hat{B}', \hat{T}')$, such that $w^{[E:F]}$ has order 2. For each orbit $Oi$ of the simple roots of $G'$ generated by $w$, we choose an element $t_i ∈ \hat{T}'$ of finite order such that $α(t_i) = 1$ for any $α ∉ O_i$, and for which there exists root vectors $\{X_α\}, α ∈ O_i$ stable under $t_iw$. Then $w = w' \prod t_i$ is of finite order, say $k$, and stabilises the set of root vectors for simple roots of $\hat{G}'$. Finally, let $E'$ be a finite extension of degree...
k and σ' a generator of Gal(E'/F) over σ. We can then extend the embedding
\[ \xi_0 : \hat{G}' \times \text{Gal}(E'/F) \to \hat{G} \times \text{Gal}(E'/F) \]
by sending \( \xi_0(\sigma') = w' \). This determines \( \xi' \) in the commutative diagram above. \( \square \)

Thus if \( G \) and \( G' \) are unramified, we can take \( \hat{G}' = G' \). But the embeddings \( \hat{\xi}' \)
must still be chosen as an \( L \)-isomorphism of \( G' \) with \( L \)-isomorphism of \( G' \) that is uniquely determined
up to the action of \( H^1(\Gamma_{un}, Z(G')) \), where \( \Gamma_{un} \) is the Galois group of the maximal
unramified extension of \( F \).

Let us briefly make note of a global consequence. We can of course define global
transfer datum in the same manner. We reserve the important for applications to the global trace formula.

Lemma 2.4. There are finitely many \( \hat{G}' \in \mathcal{F}(\hat{G}) \) unramified outside \( V \).

Proof. By the decomposition above, we may assume that \( \hat{G}' = G' \). Each
isomorphism class \( G' \) has a representative with a torus \( \hat{T}' \cong \hat{T} \). For each fixed \( \hat{T}' \),
there are only a finite number of possible Borel pairs \( (\hat{B}', \hat{T}') \) that determine the
group \( G' \). Since we are considering the datum up to isomorphism, to each Borel
pair we may assume the associated pinning to also be fixed. Then as regards the
split extension \( \hat{\xi}' \) given by the homomorphism \( W_F \to \Gamma \to \text{Out}_{\hat{G}}(\hat{G}) \), we note
that there are only finitely many homomorphisms from \( \Gamma \) to a finite group that
are unramified outside of \( V \) [Lan83, Lemme 8.13]. Finally, to each \( G' \) there exist
only finitely many admissible \( L \)-embeddings into \( L \hat{G} \) such that \( \text{Cent}(\hat{G}', \hat{G})/Z(\hat{G})^F \)
is finite. \( \square \)

Let \( v \) be any valuation of \( F \). Then \( \hat{G}' \) determines a local transfer datum \( \hat{G}' \in \mathcal{F}(G_v) \). In choosing a global auxiliary datum \( (\hat{\xi}', \hat{\xi}) \) and central datum \( (\hat{Z}, \hat{\zeta}) \),
where \( \hat{\zeta} \) is a character of \( \hat{Z}(F) \backslash \hat{Z}(A) \), which pulls back to \( \hat{Z}(A) \) on \( \hat{G} \). Suppose that
if \( \hat{G}_v \) and \( \hat{\zeta}_v \) are unramified, and \( f_v \) is the element in a hyperspecial Hecke algebra
determined by \( \hat{\zeta}_v \), then \( f'_v \) vanishes unless \( \hat{G}_v \) is also unramified. The analogous
result holds in endoscopy, and reduces the consideration to the unramified places
in the following sense. Assuming the existence of the stable transfer for every
unramified place \( v \), we obtain a map \( \hat{f}_v \to \hat{f}'_v \) from \( \mathcal{H}(\hat{G}_v, \hat{\zeta}_v) \) to \( \mathcal{S}\mathcal{I}(\hat{G}_v', \hat{\zeta}_v) \), the latter defined in Section 3.1 below. Putting the local transfer maps together, we
obtain a global transfer map \( \hat{f} \to \hat{f}' \) from \( \mathcal{H}(\hat{G}(A), \hat{\zeta}) \) to \( \mathcal{S}\mathcal{I}(\hat{G}'(A), \hat{\zeta}) \). It follows
then from Lemma 2.4 that for any \( f \), there are only finitely many \( G' \in \mathcal{F}(G) \) with
\( \hat{f}' \neq 0 \).

3. Stable kernels

We return to the local setting. In preparation for the stable geometric transfer
factors, we require several constructions related to the (inverse) Fourier transforms
of stable orbital integrals and stable characters from [Art96], and develop some
properties that we shall require.
3.1. Stable orbital integrals. Let $\mathcal{C}(G)$ be the space of Harish-Chandra Schwartz functions on $G(F)$, and let $\mathcal{C}(G, \zeta)$ be the subspace of $\zeta^{-1}$ equivariant functions, i.e., such that
\[
f(az) = \zeta(z)^{-1}f(a), \quad a \in G(F), z \in Z(F).
\]
First, if $G$ is a connective reductive group, we define the normalised orbital integral
\[
f_G(\gamma) = |D(\gamma)|^{1/2} \int_{G,\gamma(F) \setminus G(F)} f(x^{-1}\gamma x) dx, \quad f \in \mathcal{C}(G, \zeta),
\]
where $D(\gamma) = \det(1 - \text{Ad}(\gamma))_{g/p}$, is the Weyl discriminant and $dx$ a fixed invariant measure on the orbit $G,\gamma(F) \setminus G(F)$. If $G$ is a $K$-group, we set $f_G(\gamma) = f_{\alpha,G,\gamma}$ where $G,\gamma$ is the component that contains $\gamma$. Define
\[
\mathcal{I}(G, \zeta) = \{f_G : f \in \mathcal{C}(G, \zeta)\},
\]
a topological space of functions on $\Gamma_{\text{reg}}(G)$, topologised in a manner so that the map $f \to f_G$ is open and continuous. Denote by $\mathcal{I}_{\text{cusp}}(G, \zeta)$ the subspace of cuspidal functions, that is, functions that vanish on the complement of $\Gamma_{\text{reg}}$. Fixing Haar measures on $G(F)$ and $M(F)$, we obtain measures on the unipotent radical $N_P$ and maximal compact subgroup $K$ by the formula
\[
\int_{G(F)} f(g) dg = \int_{M(F)} \int_{N_P(F)} \int_{K} f(mnk) dk \, dn \, dm
\]
for all $f \in \mathcal{C}(G)$. The map $f \to f_M$ is then given by
\[
f_M(\gamma) = \int_{N_P(F)} \int_{K} f(k^{-1}n^{-1}\gamma nk) dk \, dn.
\]
In particular, the map on the level of functions depends on the choice of $P$ and $K$, but choosing measures appropriately, it can be shown that the map induced from $\mathcal{I}(G, \zeta)$ to $\mathcal{I}(M, \zeta)$ indeed independent of these choices.

There is a natural measure on $\Gamma_{\text{ell}}(G)$ given by
\[
\int_{\Gamma_{\text{ell}}(G)} \alpha(\gamma) d\gamma = \sum_{\{T\}} |W(G(F), T(F))|^{-1} \int_{T(F)} \alpha(t) dt,
\]
for any $\alpha \in C_c(\Gamma(G))$, where $\{T\}$ is a set of representatives of $G(F)$-conjugacy classes of elliptic maximal tori in $G(F)$, $W(G(F), T(F))$ is the Weyl group of $(G(F), T(F))$, and $dt$ is a fixed Haar measure on $T(F)$. The corresponding measures on $\Gamma_{\text{ell}}(M)$ determine a measure
\[
\int_{\Gamma(G)} \alpha(\gamma) d\gamma = \sum_{\{M\}} |W(M)|^{-1} \int_{\Gamma_{\text{ell}}(G)} \alpha(\gamma_M) d\gamma_M,
\]
on $\Gamma(G)$.

The stable orbital integral of $f \in \mathcal{C}(G, \zeta)$ at $\delta \in \Delta_{\text{reg}}(G)$ is given by
\[
f_G(\delta) = \sum_{\gamma} f_G(\gamma),
\]
where the sum is taken over the finite set of $\gamma \in \Gamma_{\text{reg}}(G)$ that lie in the stable class $\delta$. We then define the subspace of $\mathcal{I}(G, \zeta)$

$$ST(G, \zeta) = \{f^G : f \in \mathcal{C}(G, \zeta)\},$$

and set

$$ST_{\text{cusp}}(G, \zeta) = ST(G, \zeta) \cap \mathcal{I}_{\text{cusp}}(G, \zeta).$$

We call a tempered, $\zeta$-equivariant distribution on $G(F)$ stable if its value at any $f \in \mathcal{C}(G, \zeta)$ depends only on $f^G$. We similarly define measures on $\Delta_{\text{ell}}(G)$ and $\Delta(G)$ by

$$\int_{\Delta_{\text{ell}}(G/F)} \beta(\delta)d\delta = \sum_{\{T\}_{st}} |W_F(G, T)|^{-1} \int_{T(F)} \beta(t)dt,$$

where $\beta \in \mathcal{C}(\Delta(G), \{T\}_{st}$ is a set of representatives of stable conjugacy classes of elliptic maximal tori in $G$ over $F$, and $W_F(G, T)$ is the subgroup of elements in the absolute Weyl group of $(G, T)$ defined over $F$; and

$$\int_{\Delta(G/F)} \beta(\delta)d\delta = \sum_{\{M\}} |W(M)|^{-1} \int_{\Delta_{\text{ell}}(M)} \beta(\delta_M)d\delta_M.$$

3.2. Conjugacy classes again. We shall construct certain ‘functorial’ sets that keep track of the stable transfer mappings, parallel to $\Delta(G)$. Given $G$, the orbits of $\text{Out}_G(G')$ on $\Delta_{\text{ell}}(G')$ depend only on the isomorphism class of $G'$ in $\mathcal{F}(G)$. It makes sense then to define the set

$$\Delta_{\text{ell}}(G) = \bigotimes_{G' \in \mathcal{F}_{\text{ell}}(G)} \Delta_{G,\text{ell}}(G')/\text{Out}_G(G'),$$

which we can view as equivalence classes of pairs $(G', \delta')$. That is, if we write

$$\Delta_{G,\text{ell}}(G', G) = \Delta_{G,\text{ell}}(G')/\text{Out}_G(G'),$$

then

$$\Delta_{\text{ell}}(G) = \{(G', \delta') : G' \in \mathcal{F}_{\text{ell}}(G), \delta' \in \Delta_{G,\text{ell}}(G', G)\}.$$

We can similarly define $\Delta_{G,\text{ell}}(M)$ for any $M \in \mathcal{Z}$. Taking the union over the $W(M)$-orbits, we set

$$\Delta_{\text{reg}}(G) = \bigotimes_{\{M\}} \Delta_{G,\text{ell}}(M)/W(M).$$

Fix an auxiliary datum $(\tilde{M}', \tilde{\zeta}_M)$ for $M' \in \mathcal{F}(M)$. We then define the set

$$\tilde{\Delta}_{G,\text{ell}}(M) = \bigotimes_{M' \in \mathcal{F}_{\text{ell}}(M)} \Delta_{G,\text{ell}}(M'),$$

which fibres over $\Delta_{G,\text{ell}}(M)$, with the group $\prod_{M'}(\tilde{Z}'(F) \times \text{Out}_M(M'))$ acting transitively on the fibres. We again take the union of $W(M)$ orbits of $\tilde{\Delta}_{G,\text{ell}}(M)$, and set

$$\tilde{\Delta}_{\text{reg}}(G) = \bigotimes_{\{M\}} \tilde{\Delta}_{G,\text{ell}}(M)/W(M),$$

which fibres over $\Delta_{\text{reg}}(G)$. For brevity, we write $\Delta_{\text{reg}}(G) = \Delta(G)$ and $\tilde{\Delta}_{\text{reg}}(G) = \tilde{\Delta}(G)$. We can also view elements of $\tilde{\Delta}(G)$ as equivalence classes of tuples $(G', \tilde{G}', \tilde{\zeta}', \delta')$. These constructions are readily seen to generalise the ‘endoscopic’ sets $\Delta(G)$ and its variants in [Art02, §4], also denoted $\tilde{\Gamma}(G)$ in [Art96, §2], which we shall use in this paper without comment.
3.3. **Endoscopic geometric transfer factors.** We briefly recall some basic facts about endoscopic transfer factors, such as in [Art02, §4-5]. Given an endoscopic datum $G^e \in \hat{\mathcal{E}}(G)$, the geometric endoscopic transfer factor is a smooth function $\Delta(\cdot, \cdot)$ on $\Delta_G(G^e) \times \Gamma(G)$ such as defined in [Art99, §2]. The transfer factor determines a map

$$f \to f^e(\delta^e) = \sum_{\gamma \in \Gamma(G)} \Delta(\delta^e, \gamma) f_G(\gamma), \quad \delta^e \in \Delta_G(G^e)$$

from functions $f \in \mathcal{C}(G, \zeta)$ to $f^e = f^{G^e}$ on $\Delta_G(G^e)$. The Langlands-Shelstad transfer then implies that $f'$ belongs to $SI(G^e, \zeta^e)$. Fix an auxiliary endoscopic datum $(\hat{G}^e, \hat{\zeta}^e)$ of $G^e$, so that $\hat{G}^e$ is an extension of $G^e$ by a central induced torus $\hat{C}^e$ with associated character $\hat{\eta}^e$. The group $\hat{C}^e(F)$ acts simply transitively on the fibres of the map $\Delta_G(\hat{G}^e) \to \Delta_G(G^e)$, and $\mathcal{H}(W_F, Z(\hat{G}^e))$ acts simply transitively on the set of $Z(\hat{G}^e)$-orbits of admissible embeddings $\hat{\zeta}^e$. Then if $az\delta$ is the image in $\Delta^e(G)$ of a point $(G^e, \hat{G}^e, a\hat{\zeta}^e, z\hat{\delta}^e)$ with $a \in \mathcal{H}(W_F, Z(\hat{G}^e))$ and $z \in \hat{C}^e(F)$, then the transfer factor satisfies

$$\Delta(az\delta, \gamma) = \chi_a(\delta^e) \hat{\eta}^e(z) \Delta(\delta, \gamma),$$

where $\chi_a$ is a character on $\hat{G}^e$ determined by $a$ and the local Langlands correspondence for tori.

The transfer factors and consequently the Langlands-Shelstad transfer depend only on the image of $\delta^e$ in $\Delta^e(G)$, and we can extend the transfer factors to $\Delta^e(G) \times \Gamma(G)$ and define the extended map

$$f \to f^e_G = \bigoplus_{G^e \in \mathcal{E}_{\text{ell}}(G)} f^e.$$

That is, we define $\Delta(\delta^e, \gamma)$ to be zero unless there is an $M$ such that $(\delta^e, \gamma)$ belongs to the Cartesian product of $\Delta^e_{G, \text{ell}}(M)/W(M)$ with $\Gamma_{G, \text{ell}}(M)/W(M)$. If there is such an $M$, then $(\delta^e, \gamma)$ is the image of a pair $(\delta_M^e, \gamma_M)$ in $\Delta^e_{G, \text{ell}}(M)/\Gamma_{G, \text{ell}}(M)$, and we set

$$\Delta(\delta^e, \gamma) = \Delta_G(\delta^e, \gamma) = \sum_{w \in W(M)} \Delta_M(\delta_M^e, w\gamma_M).$$

Each sum contains at most one nonzero term, and depends only on $\delta^e$ and $\gamma$.

Define also the adjoint transfer factor

$$\Delta(\gamma, \delta^e) = |K_\gamma|^{-1} \overline{\Delta(\delta^e, \gamma)}$$

on $\Gamma(G) \times \Delta^e(G)$. Then [Art99, Lemma 2.3], we have the following adjoint relations

$$\sum_{\delta^e \in \Delta^e_{\text{reg}}(G)} \Delta(\gamma, \delta^e) \Delta(\delta^e, \gamma_1) = \delta(\gamma, \gamma_1), \quad \gamma, \gamma_1 \in \Gamma(G),$$

where $\delta(\cdot, \cdot)$ is the usual Kronecker delta, and

$$\sum_{\gamma \in \Gamma_{\text{reg}}(G)} \Delta(\delta^e, \gamma) \Delta(\gamma, \delta_1^e) = \tilde{\delta}(\delta^e, \delta_1^e), \quad \delta^e, \delta_1^e \in \Delta^e(G),$$

where $\tilde{\delta}(\delta, \delta_1) = \hat{\eta}^e(z)$ if $\delta_1 = z\delta$ for some $z \in \hat{C}^e(F)$ (or equivalently, if $\delta, \delta_1$ have the same projection onto $\Delta^e_{\text{reg}}(G)$) and equal to zero otherwise. The adjoint relations imply that $f_G \to f^e_G$ is an isomorphism from $\mathcal{I}(G, \zeta)$ onto its image.
3.4. Stable virtual characters. Recall from [Art96, §4] the set $T(G)$ of $W_0$-orbits of essential triples $\tau = (L, \pi, r)$ where $L \in \mathcal{L}$, $\pi \in \Pi_2(L)$, and $r \in R_\pi$, where $\Pi_2(L)$ is the set of equivalence classes of irreducible unitary representations of $L(F)$ which are square integrable mod center, and $R_\pi$ is the $R$-group of $\pi$. Let $T_\text{ell}(G)$ be the subset of $\tau$ such that the kernel of $(1 - r)$ acting on $a_L$ is equal to $a_G$. We define

$$T(G) = \coprod_{\{M\}} T_\text{ell}(M)/W(M).$$

We also have a decomposition with respect to any central induced torus $Z(F)$, which we assume contains the maximal $F$-split torus $A_G$,

$$T(G) = \prod_{\zeta} T(G, \zeta),$$

where $\zeta$ runs over characters of $Z(F)$, and $T(G, \zeta)$ is the subset of elements of $T(G)$ whose central character on $Z(F)$ equals $\zeta$. We also write $T_\text{ell}(G, \zeta) = T_\text{ell}(G) \cap T(G, \zeta)$. The set $T(G)$ parametrises a family of locally integrable functions

$$\gamma \to I(\tau, \gamma), \quad \gamma \in \Gamma(G),$$

such that for any $\zeta$, the functions $I(\tau, \gamma)$ for $\tau \in T_\text{ell}(G, \zeta)$ form an orthogonal basis of $I_{\text{cusp}}(G, \zeta)$. Also, we have that $I(\tau, \gamma \tau) = I(\tau, \gamma)\zeta(z)$ for any $z \in Z(F)$.

Furthermore, by [Art96, Lemma 5.1] we have a set $\Phi_2(G, \zeta)$ parametrising a family of functions

$$\delta \to n(\zeta)S(\phi, \delta), \quad \delta \in \Delta(G),$$

which forms an orthogonal basis of $\mathcal{H}_{\text{cusp}}(G, \zeta)$, where $n(\zeta) = |K_\delta|$. Parallel to $T_\text{ell}(M)$ above, the basis then provides constructions of the larger sets

$$\Phi_2(G) = \prod_{\zeta} \Phi_2(G, \zeta), \quad \Phi(G) = \coprod_{\{M\}} \Phi_2(M)/W(M).$$

The set $\Phi(G)$ comes with an action $\phi \to \phi_\lambda = \phi \cdot \rho_\lambda$ where $\rho_\lambda$ is the unramified parameter which maps the Frobenius element to the image of $\lambda$ in $(Z(G)^0)^0$ under the exponential map, noting that $a_{G, \mathbb{C}}^*$ is equal to the Lie algebra of $(Z(G)^0)^0$. The measure on $T_\text{ell}(G)$ in [Art96, §4] is chosen to be

$$\int_{T_\text{ell}(G)} \alpha(\tau)d\tau = \sum_{\tau \in T_\text{ell}(G)/ia_G^*, \tau} \int_{ia_G^*, \tau} \alpha(\tau_\lambda)d\lambda$$

for any $\alpha \in C_c(T(G))$. Here we recall that $ia_{G,F}^* = ia_G^*/a_{G,F}^*$, and also $ia_{G,F}^*, \tau = ia_G^*/a_{G,F}^*$, where $a_{G,F}^*$ is the stabiliser of $\tau$ in $a_G^*$, a lattice that lies between $a_G^*$ and $a_{G,F}^*$, and $d\lambda$ is a fixed measure on $ia_G^*$. We then define the measure on $T(G)$ to be

$$\int_{T(G)} \alpha(\tau)d\tau = \sum_{\{M\}} |W(M)|^{-1} \int_{T_\text{ell}(M)} \alpha(\tau_M)d\tau_M.$$

Similarly, we define a measure on $\Phi_2(G)$ by setting

$$\int_{\Phi_2(G)} \beta(\phi)d\phi = \sum_{\phi \in \Phi_2(G)/ia_G^*} \int_{ia_G^*, \phi} \beta(\phi_\lambda)d\lambda.$$
for any \( \beta \in C_c(F_2(G)) \), where \( i a_{G, \phi}^* = i a_{G, \phi}^* / a_{G, \phi}^* \), where \( a_{G, \phi}^* \) is the stabiliser of \( \phi \) in \( i a_{G}^* \). We then define the measure on \( \Phi(G) \) to be

\[
(3.3) \quad \int_{\Phi(G)} \beta(\phi) d\phi = \sum_{\{ M \}} |W(M)|^{-1} \int_{\Phi_2(M)} \beta(\phi_M) d\phi_M
\]

for any \( \beta \in C_c(F_2(G)) \).

3.5. **Endoscopic spectral transfer factors.** Now for each elliptic endoscopic group \( G^e \in E_{\text{ell}}(G) \), we define

\[
\Phi_2(G^e, G) = \Phi_2(\tilde{G}^e, \tilde{\zeta}) / \text{Out}_G(G^e).
\]

The spectral transfer factors \( \Delta(\phi^e, \tau) \) are then defined in [Art96, §5] to be uniquely determined functions on \( \Phi_2(G^e, G) \times \tilde{T}_{\text{ell}}(G) \), satisfying

\[
f^e(\phi^e) = \sum_{\tau \in \tilde{T}_{\text{ell}}(G)} \Delta(\phi^e, \tau) f_G(\tau),
\]

and \( \Delta(\phi^e, z_{\tau}) = \chi_{\tau}(z_{\tau}) \Delta(\phi^e, \tau) \) for \( z_{\tau} \in Z_{\tau} \), where \( Z_{\tau} = Z_{\pi} \) is a central subgroup used to define a central extension \( \tilde{R}_{\pi} \) of \( R_{\pi} \) [Art96, §4]. Define

\[
T_{\text{ell}}^G = \{ (G^e, \phi^e) : G^e \in E_{\text{ell}}(G), \phi^e \in \Phi_2(\tilde{G}^e, G) \}
\]

and

\[
T^G = \prod_{\{ M \}} T_{\text{ell}}^G / W(M).
\]

Then \( \Delta(\phi^e, \tau) \) can be extended to a function on \( T^G \times \tilde{T}(G) \) again as follows. We define \( \Delta(\phi^e, \tau) \) to be zero unless there is an \( M \) such that \( (\phi^e, \tau) \) belongs to the Cartesian product of \( T_{\text{ell}}^G / W(M) \) with \( T_{\text{ell}}(M) / W(M) \). If there is such an \( M \), then \( (\phi^e, \tau) \) is the image of a pair \( (\phi_M^e, \tau_M) \) in \( T_{\text{ell}}^G \times T(M) \), and we set

\[
\Delta(\phi^e, \tau) = \Delta_G(\phi^e, \tau) = \sum_{\tau_M} \Delta_M(\phi_M^e, \tau_M),
\]

where the sum runs over Weyl orbit \( W(M) \tau_M \).

The orthogonality relation of the stable virtual characters which is a consequence of [Art96, Lemma 5.1] and its extension to strongly regular classes \( \Delta(G) \) by (2.1), is given by

\[
(3.4) \quad \int_{\Delta_{\text{ell}}(G / Z)} n(\delta) S(\phi, \delta) S(\phi_1, \delta) d\delta = \delta(\phi, \phi_1)n(\phi),
\]

where \( \delta \) is again the Kronecker delta, and \( n(\phi) \) is simply defined to be the value

\[
n(\phi) = \int_{\Delta_{\text{ell}}(G / Z)} n(\delta) S(\phi, \delta) S(\phi, \delta) d\delta.
\]

It is parallel to the formula for the invariant virtual characters

\[
\int_{\Gamma_{\text{ell}}(G / Z)} I(\tau, \gamma) I(\tau_1, \gamma) d\gamma = \delta(\tau, \tau_1)n(\tau),
\]

where the constant \( n(\tau) \) is defined by [Art93, Theorem 6.2]. We shall later derive a dual orthogonality relation for \( S(\delta, \phi) \) in Lemma 4.5 below.
3.6. Fourier transforms. To define our stable transfer factors, we must recall some constructions relating to the (inverse) Fourier transforms of orbital integrals. We have for any \( f \in \mathcal{H}(G, \zeta) \), the relations

\[
\begin{align*}
    f_G(\gamma) &= \int_{\mathcal{T}(G, \zeta)} I(\gamma, \tau) f_G(\tau) d\tau \\
    f_G(\tau) &= \int_{\mathcal{T}(G/Z)} I(\tau, \gamma) f_G(\gamma) d\gamma,
\end{align*}
\]

where we denote by \( I(\tau, \gamma) = \frac{1}{\sqrt{D(\gamma)}} \Theta(\tau, \gamma) \) the normalised virtual character associated to \( \tau \), and \( I(\gamma, \tau) \) on the other hand can be viewed as the coefficient in the Fourier inversion of the orbital integral \( f_G(\gamma) \). They are smooth functions in both variables, described in Theorems 4.1 and 4.3 of \[\text{Art94a}\]. We shall be interested in their stable analogues. By \[\text{Art96}, \text{Lemma 6.3}\] there exist smooth functions \( S(\delta, \phi) \) and \( S(\phi, \delta) \) of \( \phi \in \Phi(G, \zeta) \) and \( \delta \in \Delta(G/Z) \), which are respectively \( \zeta \) and \( \zeta^{-1} \)-equivariant under translation by \( Z(F) \), such that

\[
\begin{align*}
    f_G(\delta) &= \int_{\Phi(G, \zeta)} S(\delta, \phi) f_G(\phi) d\phi \\
    f_G(\phi) &= \int_{\Delta(G/Z)} S(\phi, \delta) f_G(\delta) d\delta,
\end{align*}
\]

for any \( f \in \mathcal{H}(G, \zeta) \). The smooth functions are given by

\[
S(\delta, \phi) = \sum_{\gamma \in \Gamma(G)} \sum_{\tau \in \mathcal{T}(G, \zeta)} \Delta(\delta, \gamma) I(\gamma, \tau) \Delta(\tau, \phi)
\]

and

\[
S(\phi, \delta) = \sum_{\tau \in \mathcal{T}(G, \zeta)} \sum_{\gamma \in \Gamma(G)} \Delta(\phi, \tau) I(\tau, \gamma) \Delta(\gamma, \delta).
\]

Here \( \Delta(\delta, \gamma) \) is the endoscopic geometric transfer factor with adjoint \( \Delta(\gamma, \delta) \), and \( \Delta(\phi, \tau) \) is the endoscopic spectral transfer factor with adjoint \( \Delta(\tau, \phi) \), as recalled above. While it is probably best to renormalise these transfer factors according to the works of Kaletha (c.f. \[\text{Kal16}, \text{§}4\]), we neglect to do so here.

We derive the following identity which will play an important role in proving identities for our stable transfer factor. It is the analogue of the relation (3.11) in \[\text{Art94a}, \text{Theorem 4.5}\] relating \( I(\tau, \gamma) \) to \( I(\gamma, \tau) \) for elliptic virtual characters.

**Proposition 3.1.** The stable kernels satisfy the adjoint relation

\[
(3.8) \quad n(\delta) S(\phi, \delta) = n(\phi) S(\delta, \phi)
\]

for \( \phi \in \Phi_2(G, \zeta) \) and \( \delta \in \Delta_{\text{ell}}(G) \).

Before proving the lemma, we first motivate the identity as follows. Applying the inversion formulae (3.7) and (3.6) consecutively to

\[
f_G(\phi) = \int_{\Delta(G/Z)} S(\phi, \delta) \int_{\Phi(G, \zeta)} S(\delta, \phi_1) f_G(\phi_1) d\phi_1 d\delta,
\]

and then interchanging the integrals, we have

\[
\int_{\Phi(G, \zeta)} \int_{\Delta(G/Z)} S(\phi, \delta) S(\delta, \phi_1) d\delta f_G(\phi_1) d\phi_1.
\]
Then if we had the relation (3.8), this is
\[
\int_{\Phi(G,\zeta)} n(\phi)^{-1} \int_{\Delta(G/Z)} n(\delta) S(\phi, \delta) S(\phi_1, \delta) d\delta f^G(\phi_1) d\phi_1,
\]
so that the orthogonality relation (3.4) gives us the tautology
\[
\int_{\Phi(G,\zeta)} \delta(\phi, \phi_1) f^G(\phi_1) d\phi_1 = f^G(\phi),
\]
as expected.

**Proof.** For simplicity, we assume \((Z, \zeta)\) to be trivial, since it does not affect the proof. We would like to compare
\[
S(\delta, \phi) = \sum_{\gamma \in \Gamma(G)} \sum_{\tau \in \mathcal{T}(G,\zeta)} \Delta(\delta, \gamma) I(\gamma, \tau) \Delta(\tau, \phi)
\]
with
\[
S(\phi, \delta) = \sum_{\tau \in \mathcal{T}(G,\zeta)} \sum_{\gamma \in \Gamma(G)} \Delta(\phi, \tau) I(\tau, \gamma) \Delta(\gamma, \delta).
\]
We recall the identities satisfied by the endoscopic geometric and spectral transfer factors, relating them to their adjoint functions from equations (2.3) and (5.5) in [Art96],
\[
\Delta(\delta^e, \gamma) = n(\delta) \Delta(\gamma, \delta^e)
\]
and
\[
\Delta(\tau, \phi^e) = |Z(G^\vee)^G / Z(\hat{G})^G|^{-1} n(\tau) n(\phi)^{-1} \Delta(\phi^e, \tau),
\]
where
\[
n(\tau) = |R_{\pi,\tau}| |\det(1 - r)_{a_L/a_G}|,
\]
and \(R_{\pi,\tau}\) is the centraliser of \(\tau\) in the \(R\)-group \(R_{\pi}\). Recall that \(T(G)\) is the set of \(W_0^G\)-orbits of essential triplets \(\tau = (L, \pi, r)\) where \(L \in \mathcal{Z}\), \(\pi\) is an equivalence class of an irreducible unitary representation of \(L(F)\) that is square integrable modulo centre, and \(r \in R_{\pi}\). The \(R\)-group of \(\pi\) is the quotient \(R_{\pi} = W_\pi / W_0^\pi\), where \(W_\pi^0\) is the subgroup of elements of \(w \in W_\pi\) such that the normalised intertwining operator \(R(\pi, w)\) acts by a scalar. The subset \(T_{\text{ell}}(G)\) consists of \(\tau\) for which the kernel of \((1 - r)\) acting on \(a_L\) equals \(a_G\). Finally, we note that in the case at hand, we shall identify \(\delta\) with its image \(\delta^e = \delta^*\) in \(\Delta_G(G^*)\), similarly \(\phi^e = \phi^*\) in \(\Phi(G^*, \zeta^*\).

Moreover, if \(\tau \in T_{\text{ell}}(G)\) we write \(\tau^\vee = (L, \pi^\vee, r)\) for the contragredient and set
\[
i^G(\tau) = |\det(1 - r)_{a_L/a_G}|^{-1},
\]
then it follows from the special case \(M = G\) of [Art94a, Theorem 4.5] that
\[
I^{\text{old}}(\gamma, \tau) = i^G(\tau) I(\tau^\vee, \gamma),
\]
for any \(\gamma \in \Gamma_{\text{ell}}(G)\) and \(\tau \in T_{\text{disc}}(G)\), which we define below. But it is crucial that the measures on \(T(G)\) assigned in [Art94a, §4] differs from that of [Art96, §4] by a factor of \(|R_{\pi,\tau}|\), which we must reconcile. This explains our notation \(I^{\text{old}}(\gamma, \tau)\) for the kernel used in [Art94a]. We first explain the measures. We write \(T_{\text{disc}}(G)\) for the subset of \(W_0^G\)-orbits for which the set of regular elements
\[
W_\pi(r)_{\text{reg}} = \{ w \in W_\pi(r) : a_L^w = a_G \}
\]
is nonempty. Here \(W_\pi(r)\) is the subset of elements in \(W(a_L) = W_\pi^0\cdot r\) which stabilise \(\pi\) and which have the same projection onto the \(R\)-group as \(r\). For any \(w\) in this
set, we write $\varepsilon_\pi(w)$ for the sign of the element $wr^{-1}$ in the Weyl group $W_\pi^\circ$. The function $i(\tau) = i^G(\tau)$ is more generally defined on $T_{\text{disc}}(G)$ as

$$i(\tau) = |W_\pi^\circ|^{-1} \sum_{w \in W_\pi(\tau)_{\text{reg}}} \varepsilon_\pi(w) |\det(1-w)_{a_L/a_G}|^{-1}.$$ 

For $\tau \in T_{\text{ell}}(G)$, the group $W_\pi^\circ$ is trivial and $i(\tau)$ specialises to the former expression. Now the measure on $T_{\text{disc}}(G)$ is chosen in [Art94a] to be

$$\int_{T_{\text{disc}}(G)} \alpha(\tau)d\tau = \sum_{\tau \in T_{\text{disc}}(G)/ia_G^\circ} |R_{\tau,r}|^{-1} |a_{G,r}^\vee/a_{G,F}|^{-1} \int_{ia_{G,r}^\circ} \alpha(\tau_\lambda)d\lambda$$

for any $\alpha \in C_c(T_{\text{disc}}(G))$. On the other hand, recalling the measure chosen in (3.1) compatibly with [Art96], it follows that the righthand side can then be written as the sum over $\tau \in T_{\text{ell}}(G)/ia_G^\circ$ of

$$|ia_{G,r}^\circ/a_{G,F}|^{-1} \int_{ia_{G,r}^\circ} \alpha(\tau_\lambda)d\lambda = |a_{G,r}^\vee/a_{G,F}|^{-1} \int_{ia_{G,F}} \alpha(\tau_\lambda)d\lambda.$$ 

In particular, we see that the measure conversion from [Art94a] to [Art96] is given by multiplication by $|R_{\tau,r}|^{-1}$.

We shall show that the factor $i(\tau)$ in (3.11) must therefore be multiplied by the same factor in order for the choice of measures to be consistent. We shall in fact prove a slightly stronger result, that is, for the weighted kernels $I_M(\gamma, \tau)$, by which $I_M(\gamma, \tau) = I(\gamma, \tau)$ is a special case. By our choice of measure, [Art94a, Theorem 4.1] asserts the existence of a smooth function $I_M(\gamma, \tau)$ on $\gamma \in \Gamma(M) \cap G_{\text{reg}}(F)$ and $\tau \in T_{\text{disc}}(L)$ for $L \in \mathcal{L}$ such that

$$I_M(\gamma, f) = \sum_{L \in \mathcal{L}} |W_0^L| |W_G^L|^{-1} \int_{T_{\text{disc}}(L)} |R_{\tau,r}|^{-1} I_M^{\text{old}}(\gamma, \tau)f_L(\tau)d\tau,$$

where $I_M(\gamma, f)$ is the weighted orbital integral of $f \in \mathcal{C}(G)$, and $R_{\tau,r}$ is the group associated to $\tau = (L, \pi, r)$. In particular, the kernel $I_M^{\text{old}}(\gamma, \tau)$ of [Art94a] relates to the kernel $I_M(\gamma, \tau) = I_M^{\text{new}}(\gamma, \tau)$ of [Art96], which is the one we are using, by the renormalisation

$$I_M(\gamma, \tau) = |R_{\tau,r}|^{-1} I_M^{\text{old}}(\gamma, \tau).$$

We may as well verify the formula by recalling Arthur’s argument. Substituting this expression into the geometric side of the invariant local trace formula [Art94a, (5.1)],

$$\sum_{M \in \mathcal{L}} |W_0^M| |W_G^M|^{-1} (-1)^{\dim(M/A_G)} \int_{\Gamma_{\text{ell}}(M/Z)} I_M(\gamma, f)g_M(\gamma)d\gamma$$

for $g \in C_c^\infty(G_{\text{reg}}(F))$, we obtain the expression

$$\sum_{M \in \mathcal{L}} |W_0^M| |W_G^M|^{-1} (-1)^{\dim(A_L/A_G)} \int_{T_{\text{disc}}(L)} |R_{\tau,r}|^{-1} I_M^{\text{old}}(\tau, g)f_L(\tau^\vee)d\tau,$$

where

$$I'_M(\tau, g) = \sum_{M \in \mathcal{L}} |W_0^M| |W_G^M|^{-1} (-1)^{\dim(A_M \times A_L)} \int_{\Gamma_{\text{ell}}(M/Z)} I_M^{\text{old}}(\gamma, \tau^\vee)g_M(\gamma)d\gamma.$$
By the local trace formula, this is equal to the spectral expansion
\[ \sum_{L \in \mathcal{L}} \left| W_0^L \right|^n \left| W_0^G \right|^{-1} (-1)^{\dim(A_L/A_G)} \int_{T_{\text{disc}}(L)} \left| R_{\pi,r} \right|^{-1} i^L(\tau) I_L(\tau, g) f_L(\tau^\vee) d\tau. \]

The remainder of the argument follows that of [Art94a, §6]. Namely, considering the difference of the spectral and geometric expansions as distributions in \( f_G \), we see that the difference is a finite sum of smooth symmetric functions on the strata \( T_{\text{disc}}(L) \) of \( T(G) \) as \( L \) varies. Since \( f_G \) ranges over \( \mathcal{I}(G) \), we can separate the contributions of the various strata, and it follows that
\[ i^L(\tau) I_L(\tau, g) = I_L(\tau, g), \quad L \in \mathcal{L}, \tau \in T_{\text{disc}}(L). \]

From this, the same argument as Arthur’s gives the parallel expansion
\[ I_L(\tau, g) = \sum_{M \in \mathcal{L}} \left| W_0^M \right|^n \left| W_0^G \right|^{-1} \int_{\Gamma_{\text{int}}(M/Z)} I_L(\tau, \gamma) g_M(\gamma) d\gamma \]

of [Art94a, (4.1)]. Then comparing the expansions for \( I_L(\tau, g) \) and \( I_L(\tau, g) \) that we have obtained, we have again that
\[ (-1)^{\dim(A_M \times A_L)} \varphi_M^{\text{old}}(\gamma, \tau^\vee) = i^L(\tau) I_L(\tau, \gamma). \]

Using the fact that \( i^L(\tau^\vee) = i^L(\tau) \) and setting \( M = L = G \), we conclude that the identity (3.11) should be indeed multiplied by \( |R_{\pi,r}|^{-1} \) to give
\[ (3.12) \quad I(\gamma, \tau) = |R_{\pi,r}|^{-1} i^G(\tau) I(\tau^\vee, \gamma) \]
by our choice of measures.

Now we can prove the proposition. Combining the identities together (3.9), (3.10), and (3.12), it follows that \( S(\delta, \phi) \) is equal to
\[ \frac{n(\delta)}{|Z(G_G)^F/Z(G)^F| n(\phi)} \sum_{\gamma \in \Gamma(G)} \sum_{\gamma \in \Gamma(G)} n(\tau) |R_{\pi,r}|^{-1} i^G(\tau) \Delta(\gamma, \delta) I(\tau^\vee, \gamma) \Delta(\phi, \tau), \]
which simplifies to
\[ n(\delta) n(\phi)^{-1} \sum_{\tau \in \Gamma(G)} \sum_{\gamma \in \Gamma(G)} \Delta(\gamma, \delta) I(\tau^\vee, \gamma) \Delta(\phi, \tau), \]
where we have used the fact that the quotient \( Z(G_G)^F/Z(G)^F \) is trivial for \( G_G = G^* \).

Finally, we see that if
\[ I(\tau^\vee, \gamma) = \overline{I(\tau, \gamma)} , \]
then the desired formula follows. We simply deduce this from the properties that
\[ \overline{f_G(\gamma)} = \overline{f_G(\gamma)}, \quad \overline{f_G(\tau^\vee)} = \overline{f_G(\tau)}, \]
(see for example, the proof of [Art93, Theorem 6.1]) and comparing the expansions on the either side of the first identity using (3.5),
\[ \int_{T(G)} I(\gamma, \tau) f_G(\tau) d\tau = \int_{T(G)} I(\gamma, \tau) \overline{f_G(\tau)} d\tau = \int_{T(G)} I(\gamma, \tau^\vee) \overline{f_G(\tau)} d\tau, \]
and again varying \( f_G \) in \( \mathcal{I}(G) \) accordingly. From this we have that
\[ I(\gamma, \tau^\vee) = \overline{I(\gamma, \tau)}, \]
then using the fact that \( i_G(\tau^\vee) = i_G(\tau) \) and the relation (3.12), the claim follows. \( \square \)
Remark 3.2. From the proof of the theorem we also find the parallel statement for
the invariant kernels, namely, \( I(\gamma, \tau) = \hat{i}G(\tau)\hat{I}(\tau, \gamma) \) where \( \hat{i}G(\tau) = |R_{\gamma, \tau}|^{-1}iG(\tau) \).

4. Stable geometric transfer factors

4.1. **Local Langlands correspondence.** Our construction of stable transfer fac-
tors relies on the transfer of \( L \)-parameters. Thus it is necessary to assume a basic
form of the local Langlands conjecture in order to formulate that definition. The
local Langlands group \( L_F \) is defined to be \( W_F \) if \( F \) is archimedean and \( W_F \times SU(2) \)
if \( F \) is nonarchimedean. Recall that an \( L \)-homomorphism in this context is a ho-
omorphism \( \phi : L_F \to L_G \) that commutes with projections onto \( W_F \) of its source
and target. We say it is admissible if it is continuous and sends elements of \( W_F \)
to semisimple elements of \( L_G \), and relevant if its image being contained in a Levi
subgroup \( L_M \) of \( L_G \) implies that \( L_M \) is the \( L \)-group of a Levi subgroup \( M \) of \( G \)
over \( F \).

Let \( \Phi^+(G) \) be the set of \( \hat{G} \)-conjugacy classes of relevant, admissible \( L \)-homomorphisms
\( \phi \). By abuse of notation, let

\[ \Phi(G) = \Phi_{\text{temp}}(G) \]

be the subset of bounded or tempered Langlands parameters \( \phi \in \Phi(G) \), that is,
whose image projects onto a relatively compact subset of \( \hat{G} \). Their correspond-
ing \( L \)-packets \( \Pi_{\phi} \) are expected to consist of tempered representations. We also let
\( \Phi_2(G) \) be the set of cuspidal parameters \( \phi \), whose image does not lie in a proper
parabolic subgroup \( L_P \) of \( L_G \). (The sets described above by the same notation
were introduced by Arthur in order to avoid the use of the local Langlands corre-
spondence.) The local Langlands conjecture, at the most basic level, asserts that
the set \( \Pi^+(G) \) of irreducible admissible representations of \( G(F) \) can be written as
a disjoint union of finite packets \( \Pi_{\phi} \) as \( \phi \) ranges over \( \Phi^+(G) \). In other words, there
exists a surjective map

\[ \Pi(G)^+ \to \Phi^+(G) \]

with finite fibres, which restricts to a surjective map of tempered representations
to tempered parameters \( \Pi(G) = \Pi_{\text{temp}}(G) \to \Phi_{\text{temp}}(G) \).

In the present context, given \( G' \in \mathcal{F}(G) \) with auxiliary datum \( (\hat{G}', \xi') \), the
associated Langlands parameters are \( \hat{G}' \)-conjugacy classes of homomorphisms \( W_F \to
G' \), and composing with \( \xi' \) we have a map into an \( L \)-group \( L_{\hat{G}'} \). We shall write
\( \Pi(G, \zeta) \) for the subset of representations with central character equal to \( \zeta \), and
similarly \( \Phi(G, \zeta) \), whereby

\[ \Phi(G) = \prod_{\zeta} \Phi(G, \zeta). \]

By construction, for any \( G' \), the auxiliary data \( (\hat{G}', \xi') \) and central datum \( (Z, \zeta) \)
are chosen such that any \( \phi' \in \Phi(G', \zeta') \) maps to a parameter \( \phi \in \Phi(G, \zeta) \), again
with parallel restrictions to tempered parameters.

The local Langlands conjecture over archimedean fields has long been known by
the work of Shelstad, and over nonarchimedean fields, a good candidate has been
constructed by Fargues and Scholze very recently (albeit in terms of semisimplified
parameters). It is thus now possible to assign \( L \)-parameters unconditionally to
elements in \( \Pi(G) \). For our present purposes, we do not require any of the desired
properties expected by the local Langlands conjecture, simply that the mapping
exists. We state the basic local Langlands correspondence formally as follows. Let
\( \Phi^{ss}(G) \) be the semisimplification of the set \( \Phi(G) \), and \( \Phi^{ss}(G, \zeta) \) that of \( \Phi(G, \zeta) \).

**Theorem 4.1 ([She82, FS21]).** Let \( G \) be a connected reductive group over a local field \( F \). Then there exists a surjective map \( \Pi(G) \to \Phi(G) \) if \( F \) is archimedean; and \( \Pi(G) \to \Phi^{ss}(G) \) if \( F \) is nonarchimedean.

Let \( \mathcal{H}(G, \zeta) \) be the \( \zeta^{-1} \)-equivariant Hecke subspace in \( \mathcal{E}(G, \zeta) \). Let \( S\mathcal{H}(G, \zeta) \) be the space of stable orbital integrals of functions \( \mathcal{H}(G, \zeta) \) on \( \Delta(G) \), and

\[
S\mathcal{H}_{cusp}(G, \zeta) = S\mathcal{H}(G, \zeta) \cap S\mathcal{L}_{cusp}(G, \zeta).
\]

The existence of this surjective map for \( \Phi(G) \) allows us to deduce the existence stable transfer mapping.

**Corollary 4.2.** Assume that \( \Pi(G) \to \Phi(G) \) is surjective. Let any \( f \in \mathcal{H}(G, \zeta) \), there exists a unique \( f' \in S\mathcal{H}(G', \zeta') \) characterised by the property (1.1).

**Proof.** The action \( \phi \to \phi_{\lambda} \) of \( i\mathfrak{a}^*_{G, \phi} \) on \( \Phi_2(G) \) makes it into a disjoint union of compact tori of the form \( i\mathfrak{a}^*_{G, \phi} = i\mathfrak{a}^*_{G, \phi}/\mathfrak{a}^*_{G, \phi} \), where \( \mathfrak{a}^*_{G, \phi} \) is the stabiliser of \( \phi \) in \( i\mathfrak{a}^*_{G} \). The orthogonal basis \( n(\delta) S(\phi, \delta) \) makes \( S\mathcal{H}_{cusp}(G, \zeta) \) into the Paley-Wiener space on \( \Phi_2(G, \zeta) \), in the sense that it is the space of functions on \( \Phi_2(G, \zeta) \) supported on finitely many connected components, and which on the component of any \( \phi \) pullback to a finite Fourier series on \( i\mathfrak{a}^*_{G, \phi} \) [Art96, p.541]. Moreover, the larger graded vector space

\[
S\mathcal{L}_{gr}(G, \zeta) = \bigoplus_{\{M\}} S\mathcal{L}_{cusp}(M, \zeta)^{W(M)}
\]

can be identified with the natural Paley-Wiener space on \( \Phi(G, \zeta) \). It is a consequence of [Art96, Theorem 6.1] that we may identify \( S\mathcal{L}_{gr}(G, \zeta) \) with \( S\mathcal{L}(G, \zeta) \).

The space \( S\mathcal{H}(G, \zeta) \) corresponds to the Paley-Wiener space of \( \Phi(G, \zeta) \) by the map

\[
\phi \to f^G(\phi), \quad f \in \mathcal{H}(G, \zeta).
\]

Moreover, the \( L \)-embedding \( \xi' \) induces isomorphisms on the corresponding maximal tori, and it follows that the function

\[
\phi' \to f^G(\xi' \circ \phi'), \quad \phi' \in \Phi(G')
\]

belongs to the Paley-Wiener space on the set of tempered local Langlands parameters of \( G'(F) \). Then there exists a function \( f' \in \mathcal{H}(G', \zeta') \), uniquely determined up to its stable orbital integral, such that \( f'(\phi') = f^G(\xi' \circ \phi') \).

**Remark 4.3.** For \( F \) nonarchimedean, there is a natural surjective map \( \Phi(G, \zeta) \to \Phi^{ss}(G, \zeta) \) and similarly \( \Phi(G', \zeta') \to \Phi^{ss}(G', \zeta') \). To each character on \( G \) we can associate the semisimplified parameter \( \phi^{ss} \), and transfer it to \( (\phi')^{ss} \in \Phi^{ss}(G', \zeta') \). We can then choose (non-canonically) an element in the preimage of \( (\phi')^{ss} \) in \( \Phi(G) \), to which we can associate a Paley-Wiener function \( f' \) at the level of semisimplified parameters. In other words, the transfer holds unconditionally up to semisimplification.

Suppose \( F \) is a nonarchimedean field over which \( G \) is unramified. Let \( \mathcal{H}(G, K) \) denote the spherical Hecke algebra of \( G(F) \), where \( K \) is a hyperspecial maximal compact subgroup of \( G(F) \). The Satake isomorphism implies an algebra homomorphism of Hecke algebras induced by the restriction of the embedding \( \xi' : G' \to \mathcal{L} \).

compatible with this spectral mapping \( f \to f' \). The goal of the stable transfer factors is to provide a parallel construction in terms of stable orbital integrals, which we now turn to.

4.2. Stable geometric transfer factors. Suppose \( G \) is arbitrary. We write \( S'((\cdot, \cdot)) = S_{G'}((\cdot, \cdot)) \) for the kernel functions associated to \( G' \in \mathcal{F}(G) \). We can now introduce our stable geometric transfer factor as a distribution on \( \Delta_{G'}(\hat{G'}) \times \Delta(\hat{G}), \)

\[
\Theta_{\xi'}(\delta', \delta) = \int_{\Phi(\hat{G'}, \hat{\zeta'})} S'((\delta', \phi')) S((\xi') \circ \phi', \delta) d\phi',
\]

where \( \delta' \) is an image of \( \delta \) and we identify \( \xi' \circ \phi' \) with the image of \( \phi' \) in \( \Phi(G, \zeta) \) determined by \( \xi' \). If \( \delta' \) is not an image then we set \( \Theta_{\xi'} \) to be zero. By construction, it is clear that the stable transfer factor depends only on the stable conjugacy classes of its inputs. It also implicitly depends on the normalisation of the Langlands-Shelstad transfer factors. At the same time, it is not immediately clear that \( \Theta_{\xi'}(\delta', \delta) \) is well-defined in the sense of the convergence of the integral. To see this, we first expand the integrand into

\[
\sum_{\gamma' \in \Phi(G')} \sum_{\tau' \in \Phi(G', \zeta')} \sum_{\tau \in \Phi(G, \zeta)} \Delta(\delta', \gamma') \Delta(\gamma, \delta) I(\gamma', \tau') I(\tau, \gamma) \Delta(\tau', \phi') \Delta((\xi') \circ \phi', \tau).
\]

For fixed \( \delta, \delta' \), the sums over \( \gamma, \gamma' \) contribute finitely many nonzero terms. The integral then reduces to a consideration of

\[
\int_{\Phi(\hat{G'}, \hat{\zeta'})} \Delta(\tau', \phi') \Delta((\xi') \circ \phi', \tau) d\phi',
\]

where again, the spectral transfer factors have finite support in \( \phi' \) for fixed \( \tau, \tau' \) [Art96, Lemma 5.2], but the sum over \( \tau, \tau' \) itself is not finite. On the other hand, we shall see that our stable transfer factors are well-defined as kernels of integral transforms on \( SL(G, \zeta) \).

The stable transfer factors can be extended to distributions on \( \Delta_{\hat{G}}(G) \times \Delta(\hat{G}) \) as follows. Set \( \Theta_{\xi'}(\delta', \delta) \) to be zero unless there is an \( M \) such that \( (\delta', \delta) \) belongs to the Cartesian product of \( \Delta_{G, \text{ell}}(M)/W(M) \) with \( \Delta_{G, \text{ell}}(M)/W(M) \). If there is such an \( M \), then \( (\delta', \delta) \) is the image of a pair \( (\delta'_M, \delta_M) \) in \( \Delta_{G, \text{ell}}(M) \times \Delta_{G, \text{ell}}(M) \), and we set

\[
(4.3) \quad \Theta_{\xi'}(\delta', \delta) = \Theta_{\xi', M}(\delta', \delta) = \sum_{w \in W(M)} \Theta_{\xi', M}(\delta'_M, w\delta_M),
\]

where again each sum contains at most one nonzero term, and depends only on \( \delta' \) and \( \delta \).

We shall use our stable transfer factor \((4.2)\) to define the stable transfer of orbital integrals \((1.2)\), and relate it to the stable transfer of characters \((1.1)\). But first, we shall first develop some basic properties that are parallel to that of the endoscopic transfer factors.

Lemma 4.4. Let \( G' \in \mathcal{F}_{\text{ell}}(G) \).

(i) For any \( z \in \tilde{Z}'(F) \) whose image in \( Z(F) \) equals \( z_G \), we have

\[
\Theta_{\xi'}(\delta'z, \delta z_G) = \tilde{\xi}'(z)^{-1} \Theta_{\xi'}(\delta', \delta) \zeta(z_G).
\]
(ii) Given the injective linear map \( \lambda \to \lambda' \) from \( a^*_{G',\mathbb{C}} \) to \( a^*_{G,\mathbb{C}} \) we have
\[
e^{-\lambda(H_G(\delta'))} \Theta_{\xi'}(\delta', \delta) = e^{\lambda(H_G(\delta))} \Theta_{\xi}(\delta', \delta).
\]

(iii) For any \( a \in SL(G, \zeta) \) and \( \delta' \in \Delta_G(\tilde{G}') \), the function
\[
a'\delta') = a_{G'}(\delta') = \int_{\Delta(G/Z)} \Theta_{\xi}(\delta', \delta) a_{G}(\delta) d\delta
\]
is well-defined.

Proof. From the definition, we first write
\[
\Theta_{\xi}(\delta', \delta z_G) = \int_{\Phi(\tilde{G}', \zeta')} S'\left(\delta' z, \phi' \right) S\left(\xi' \circ \phi', \delta z_G\right) d\phi',
\]
and the equivariance properties of the stable kernels [Art96, Lemma 6.3] yield
\[
S'(\delta' z, \phi') S(\xi' \circ \phi', \delta z_G) = \zeta(z)^{-1} S'(\delta', \phi') S(\xi' \circ \phi', \delta) \zeta(z_G).
\]
The first result follows.

In the second place, the map from \( a^*_{G,\mathbb{C}} \) to \( a^*_{G',\mathbb{C}} \) can be viewed as a map of the complex Lie algebras of
\[
(Z(\tilde{G}))^0 \to (Z(\tilde{G}'))^0,
\]
where the isomorphism follows from the fact that \( G' \) is elliptic. It follows then that there is an injection from \( (Z(\tilde{G}'))^0 \) to \( (Z(\tilde{G}))^0 \) dual to the projection \( G' \to G \) given by property (1) of the auxiliary datum. Then the second identity follows since \( \Theta_{\xi}(\delta', \delta) \) vanishes unless the projection of \( \delta' \) onto \( G'(F) \) is an image of \( \delta \).

Third, by definition we may write \( a'(\delta') \) as
\[
\int_{\Delta(G/Z)} \int_{\Phi(\tilde{G}', \zeta')} S'(\delta', \phi') S(\xi' \circ \phi', \delta) d\phi' a_{G}(\delta) d\delta.
\]

Recall by (3.2) and (3.3) that the inner integral is given by
\[
\sum_{(M')} \left|W(M')\right|^{-1} \sum_{\phi' \in \Phi(\tilde{G}', \zeta')} \int_{ia_{G'}^{*}} S'(\delta', \phi' \lambda) S(\xi' \circ \phi' \lambda, \delta) d\lambda',
\]
where the inner sum is finite since for any fixed \( \delta' \), the function \( S'(\delta', \phi') \) is supported on finitely many connected components \( ia_{G'}^{*} \). We can choose \( \lambda \in ia_{G'}^{*} \) such that \( \xi' \circ \phi \lambda = \xi' \circ \phi \lambda' \), since \( ia_{G'}^{*} \) acts on \( \Phi(\tilde{G}', \zeta') \) and \( ia_{G'}^{*} \) on \( \Phi(G, \zeta) \) respectively, and the embedding \( \xi' \) induces an embedding of \( \Phi(\tilde{G}', \zeta') \) into \( \Phi(G, \zeta) \). Moreover, recall from [Art96, §5] the property that
\[
S(\phi, \delta) = S(\phi, \delta) e^{\lambda(H_G(\delta))}, \quad \lambda \in ia_{G}^{*}, \delta \in \Delta_{\text{ell}}(G).
\]
We see that by the adjoint relation (3.8) and the identity \( n(\phi_\lambda) = n(\phi) \) which follows from the definition of \( n(\phi) \), the same property also holds for \( S(\delta, \phi_\lambda) \). The latter integral is therefore equal to
\[
S'(\delta', \phi') S(\xi' \circ \phi, \delta) \int_{ia_{G'}^{*}} e^{\lambda(H_G(\delta'))} e^{\lambda(H_G(\delta))} d\lambda,
\]
which converges absolutely since the integration is taken over a compact torus. Up to a constant depending on $\delta'$ and $\phi'$, the integral over $\Delta(G)$ of the product of (4.4) with $a^G(\delta)$ is then bounded above by

$$\int_{\Delta(G/Z)} S(\tilde{\xi}^i \circ \phi', \delta) a^G(\delta) = f^G(\tilde{\xi}^i \circ \phi'),$$

using (3.7), and it follows that the mapping $a'(\delta')$ is well-defined. \hfill\Box

4.3. **Adjoint relations.** Define the adjoint stable geometric transfer factor

$$\Theta_{\tilde{\xi}}(\delta, \delta') = n(\delta')^{-2} \Theta_{\tilde{\xi}}(\delta', \delta).$$

The stable geometric transfer factors then satisfy adjoint relations parallel to those of the endoscopic geometric transfer factors in Section 3.3. But first, we derive an orthogonality relation for the $S(\delta, \phi)$ analogous to (3.4).

**Lemma 4.5.** Let $\delta_1, \delta_2 \in \Delta_{\text{ell}}(G)$ and $\phi \in \Phi_2(G, \zeta)$. Then

$$\int_{\Phi_2(G, \zeta)} n(\phi) S(\delta_2, \phi) S(\delta_1, \phi) d\phi = n(\delta) \delta(\delta_1, \delta_2).$$

**Proof.** The proof is based on an application of the simple local stable trace formula from [Art99, §9-10]. Let $f = f_1 \times f_2$ with $f_i \in \mathcal{E}_{\text{cusp}}(G, \zeta)$, hence $f_i(G, \gamma)$ is supported on $\Gamma_{\text{ell}}(G)$ for $i = 1, 2$. Since $G$ is quasisplit, we have a stable linear form $\mathcal{E}_{\text{cusp}}(G, \zeta)$ given by

$$S^G_{\text{disc}}(f) = \int_{\Phi_2(G, \zeta)} n(\phi)^{-1} f_1^G(\phi) f_2^G(\phi) d\phi$$

that is equal to

$$S^G(f) = \int_{\Delta_{\text{ell}}(G/Z)} n(\delta)^{-1} f_1^G(\delta) f_2^G(\delta) d\delta.$$

We first consider the spectral expansion. Applying the relation (3.8) to (3.7), we may write

$$f_i^G(\phi) = \int_{\Delta_{\text{ell}}(G/Z)} S(\phi, \delta) f_i^G(\delta) d\delta = n(\phi) \int_{\Delta_{\text{ell}}(G/Z)} n(\delta)^{-1} S(\delta, \phi) f_i^G(\delta) d\delta.$$

We then vary $f_i$ in a manner such that $f_i^G$ has compact support modulo $Z(F)$ on $\Delta(G)$ and so that $f_i^G$ approaches the $\zeta^{-1}$-equivariant Dirac measure at the image of $\delta, Z(G)$ in $\Delta_{\text{ell}}(G)$ respectively for $i = 1, 2$. The function $f_i^G(\phi)$ thus approaches $n(\phi)n(\delta_i)^{-1} \Theta(\delta_i, \phi)$, and $\Theta^G_{\text{disc}}(f)$ approaches

$$n(\delta_1)^{-1}n(\delta_2)^{-1} \int_{\Phi_2(G, \zeta)} n(\phi) S(\delta_2, \phi) S(\delta_1, \phi) d\phi.$$

On the geometric side, we see that as $f_i$ approach the Dirac measures on $\delta_i Z(F)$ respectively, the geometric expansion

$$\int_{\Delta_{\text{ell}}(G/Z)} n(\delta)^{-1} f_1^G(\delta) f_2^G(\delta) d\delta$$

approaches $n(\delta)^{-1} \delta(\delta_1, \delta_2)$, and equating both sides, the identity follows. \hfill\Box
Proposition 4.6. Given $\delta', \delta'_1 \in \Delta^c(G')$ for $G' \in \mathcal{F}_\text{ell}(G)$, we have

$$\int_{\Delta(G'/Z)} n(\delta)\Theta_{\tilde{\xi}}(\delta', \delta) \Theta_{\tilde{\xi}}(\delta, \delta'_1) d\delta = n(\delta')\delta(\delta', \delta'_1).$$

Similarly, given $\delta, \delta_1 \in \Delta(G)$, we have

$$\int_{\Delta(G')} n(\delta')\Theta_{\tilde{\xi}}(\delta', \delta) \Theta_{\tilde{\xi}}(\delta_1, \delta') d\delta' = n(\delta)\delta(\delta, \delta_1).$$

Proof. The first identity will be a consequence of the second. We will first show that for any $\delta', \delta'_1 \in \Delta^c(G')$, we have

$$\int_{\Delta^c(G'/Z)} n(\delta)\Theta_{\tilde{\xi}}(\delta', \delta) \Theta_{\tilde{\xi}}(\delta, \delta'_1) d\delta = \delta(\delta', \delta'_1).$$

Then the required formula will follow from (4.3) and the decomposition of the integral over $\Delta(G)$ into

$$\sum_{\{M\}} |W(M)|^{-1} \int_{\Delta^c(G/Z)} n(\delta_M)\Theta_{\tilde{\xi}}(\delta', \delta_M) \Theta_{\tilde{\xi}}(\delta_M, \delta'_1) d\delta_M,$$

where we note that the sum contains at most one nonzero term. From the definitions, we first write (4.8) as

$$\int_{\Delta^c(G'/Z)} n(\delta)\prod_{\Phi(\tilde{\xi}, \tilde{\zeta})} S'((\delta', \delta')) S((\tilde{\xi}', \tilde{\zeta}', \delta, \delta')) d\delta' \int_{\Phi(\tilde{\xi}', \tilde{\eta}')} \overline{S'((\delta_1', \delta'_1) S((\tilde{\eta}', \tilde{\xi}', \delta, \delta')) d\delta' \delta'_1 d\delta.$$}

The integral over $\delta$ can be evaluated using the orthogonality relation (3.4) for $S(\delta, \delta)$ and (3.3). It follows then that the latter is equal to

$$\int_{\Phi(\tilde{\xi}', \tilde{\eta}')} n(\tilde{\xi}') S((\delta', \delta')) S((\delta_1', \delta'_1)) d\delta' \delta'_1 d\delta.$$

and reducing to the terms with $\phi = \phi_1$, the two integrals combine to

$$\int_{\Phi(\tilde{\xi}', \tilde{\eta}')} n(\tilde{\xi}') S((\delta', \delta')) S((\delta_1', \delta'_1)) d\delta'.$$

We claim that $n(\tilde{\xi}') S((\delta', \delta')) S((\delta_1', \delta'_1))$ equals $n(\phi')$, so that the orthogonality relation from Lemma 4.5 yields the required identity (4.8).

To prove the claim, let us compare

$$n(\tilde{\xi}') S((\delta', \delta')) S((\delta_1', \delta'_1)) d\delta'$$

with

$$n(\phi') = \int_{\Delta^c(G')} n(\delta') S((\phi', \delta') S((\delta_1', \delta'_1)) d\delta,'$$

where we note that the latter integrand depends only on the image of $\delta' \in \Delta^c(G')$ in the set $\Delta^c(G'/Z)'(F) = \Delta^c(G'/Z)' = \Delta^c(G')$. We can define an inner product on $S\mathcal{L}(G)$ by

$$(a^G, b^G) = \int_{\Delta(G'/Z)} n(\delta)^{-1} a^G(\delta) b^G(\delta) d\delta,$$

whose restriction to $S\mathcal{L}_{\text{cusp}}(G, \zeta)$ reduces to an integral over elliptic elements $\Delta^c(G)$. Note that any function in $S\mathcal{L}(G)$ is bounded on $\Delta(G)$. Since the families of functions $\{n(\delta)S(\delta, \delta)\}$ and $\{n(\delta') S((\delta', \delta'))\}$ are orthogonal bases of $S\mathcal{L}_{\text{cusp}}(G, \zeta)$ and
\( S_{\text{cusp}}(G', \tilde{\zeta}') \) respectively, the identity will follow from showing that the stable transfer map is an isometry. Let us then consider

\[
(a^{G'}, b^{G'}) = \int_{\Delta(G')} n(\delta')^{-1} a^{G'}(\delta') b^{G'}(\delta') d\delta'.
\]

Once again the integrand is

\[
n(\delta')^{-1} \int_{\Delta(G/Z)} \int_{\Delta(G/Z)} \Theta_{\xi'}(\delta', \delta_1) \Theta_{\tilde{\xi}}(\delta', \delta_2) a^{G'}(\delta_1) b^{G'}(\delta_2) d\delta_1 d\delta_2,
\]

then by (4.5) and (4.7) we see that the inner product is equal to

\[
\int_{\Delta(G/Z)} \int_{\Delta(G/Z)} n(\delta_2)^{-1} \delta(\delta_1, \delta_2) a^{G'}(\delta_1) b^{G'}(\delta_2) d\delta_1 d\delta_2,
\]

and evaluating at \( \delta_1 = \delta_2 \), we obtain \( (a^{G'}, b^{G'}) \) as desired.

It remains to prove the second required identity (4.7). In this case, beginning the argument as above leads to

\[
\int_{\Phi(G', \tilde{\xi}')} \delta(\phi', \phi_1) n(\phi') S(\tilde{\xi}' \circ \phi', \delta) d\phi' \int_{\Phi(G', \tilde{\xi}')} S(\xi' \circ \phi_1', \delta_1) d\phi_1',
\]

and hence

\[
\int_{\Phi(G', \tilde{\xi}')} n(\phi') S(\tilde{\xi}' \circ \phi', \delta) S(\xi' \circ \phi_1', \delta_1) d\phi'.
\]

We see that this closely resembles the orthogonality relation of Lemma 4.5, and indeed we shall use a variation on the proof of the latter. In particular, recall that we may choose a suitable family of test functions \( f_i \in \mathcal{C}_{\text{cusp}}(G, \zeta) \) such that \( f_i^G(\phi) \) approaches \( n(\phi) n(\delta_i)^{-1} S(\delta_i, \phi) \) for \( i = 1, 2 \). Replacing \( \phi \) by \( \tilde{\xi}' \circ \phi' \) and choosing \( \tilde{\zeta}' \) compatibly, we thus obtain a family of functions on \( \mathcal{C}_{\text{cusp}}(G', \tilde{\zeta}') \), which we write as \( \tilde{f}_i \), so that the above equation is given as the limit of

\[
\int_{\Phi(G', \tilde{\xi}')} n(\phi')^{-1} f_i(\tilde{\xi}' \circ \phi') f_2(\tilde{\xi}' \circ \phi') d\phi' = \int_{\Phi(G', \tilde{\xi}')} n(\phi')^{-1} \tilde{f}_1(\phi') \tilde{f}_2(\phi') d\phi'
\]
as \( f_1, f_2 \) vary, where we note that the cuspidality of \( \tilde{f}_i \) follows from that of \( f_i \). That is, \( f_i \) is supported on the set \( \Gamma_{\text{ell}}(G) \) which we can identify with a subset of \( \Gamma_{G, \text{ell}}(G') \) by Lemma 2.2 and the discussion preceding it. Applying the stable trace formula in this case to \( G' \), we have that the latter is equal to

\[
\int_{\Delta_{\text{ell}}(G')} n(\delta')^{-1} \tilde{f}_1^G(\delta') \tilde{f}_2^G(\delta') d\delta'.
\]

Since \( f_i \) is chosen so that \( f_i^G \) approaches the \( \zeta^{-1} \)-equivariant Dirac measure at the image of \( \delta_i Z(G) \) in \( \Delta_{\text{ell}}(G) \), it follows that \( \tilde{f}_i \) vanishes unless \( \delta' \) is an image of some \( \delta_i \). Moreover, for such \( \delta' \) we have

\[
\pi_0((\hat{G}'_{\delta_i})^F / Z(\hat{G}'_{\delta_i})^F) = \pi_0(\hat{G}'_{\delta_i} / Z(\hat{G})_{\delta_i}),
\]

since \( \hat{G}'_{\delta_i} \) is isomorphic to \( \hat{G}_{\delta_i} \) and \( G' \) is elliptic, so that \( n(\delta') = n(\delta_i) \), thus giving \( n(\delta) \delta(\delta_1, \delta_2) \).
5. Stable transfer

5.1. The stable transfer conjecture. Among the desiderata of the stable transfer mapping is the transfer of stable orbital integrals. We can now state the stable transfer conjecture for orbital integrals, which is what we have been building up to.

**Conjecture 5.1.** For every $f \in \mathcal{C}(G, \zeta)$, there exists an $f' \in S\mathcal{T}(G', \zeta)$ such that

$$f' (\delta') = \int_{\Delta(G/Z)} \Theta_{\tilde{g}} (\delta', \delta) f^G (\delta) d\delta, \quad \delta' \in \Delta_G (\tilde{G}')$$

from $G$ to $G'$, where $f^G (\delta)$ denotes the stable orbital integral of $f$ at a strongly regular stable conjugacy class $\delta$.

The stable transfer depends on the choice of auxiliary data, transfer factors, and Haar measures. In particular, we can view the conjecture as a transfer of Haar measures from $G$ to $G'$.

As in the case of endoscopy, our transfer factors are defined only up to normalisation, so it is more appropriate to speak of families of transfer factors. We shall say that $f \in \mathcal{C}(G)$ and $f' \in \mathcal{C}(G')$ have matching (stable) orbital integrals if there exists a distribution $\Theta_{\tilde{g}} (\delta', \delta)$ on $\Delta(G') \times \Delta(G)$ such that (5.1) holds for all $\delta \in \Delta_G (\tilde{G}')$. Further, we call $\Theta_{\tilde{g}} (\delta', \delta)$ a stable transfer factor if for each $f \in \mathcal{C}(G)$ there exists $f' \in \mathcal{C}(G')$ such that $f$ and $f'$ have matching orbital integrals. We may as well require that $\Theta_{\tilde{g}} (\delta', \delta)$ be nonzero only if $\delta'$ is an image of $\delta$. Conjecture 5.1 then can be rephrased as the existence of a function $f' \in \mathcal{C}(\tilde{G}', \tilde{\zeta}')$ with matching stable orbital integrals and, implicitly, that our proposed distribution (4.2) is a stable transfer factor in the latter sense.

Of course, we remind the reader that this conjecture is by no means new. We discuss some known or simple cases.

1. When $G' = \{1\}$, it is trivially verified in [Lan13, p.178]. In that case, $f'$ is a constant, equal to the integral over $\Delta(G)$ of the product of $f^G (\delta)$ with the stable character $S(\phi, \delta)$, the latter being equal to $\Theta_{\tilde{g}} (1, \delta)$.

2. When $G = SL(2)$ and $G'$ a torus, this is again verified in [Lan13, §2] and [She21, §27] (see also [Tho20, §8]). We explain the transfer factors here in brief. In the split case $G' = GL(1)$, the representations are simply one-dimensional characters $\chi$. The stable character on $G$ is a stably invariant function on regular semisimple elements, evaluating on the split torus to

$$|D^G (\delta (t))|^{-1} (\chi (t) + \chi^{-1} (t)),$$

where we embed $GL(1)$ by the usual $\delta (t) = \text{diag} (t, t^{-1})$, and zero on elliptic classes. Then for $\delta \in G(F)$, the stable transfer factor can be computed by comparing stable characters on $G$ and $G'$,

$$\Theta (\delta', \delta) = |D^G (\delta)|^{-1} (\delta, \delta') + \delta (\delta^{-1}, \delta'),$$

where $\delta (\cdot, \cdot)$ is the Kronecker delta as in [Lan13, (2.13)].

In the nonsplit case, for the real elliptic torus $G' (\mathbb{R}) = T (\mathbb{R})$, we parametrise its elements by $s (\theta)$ with $0 \leq \theta < 2\pi$. The stable transfer factor $\Theta (s (\theta), \delta)$ is given by

$$\sum_{n \in \mathbb{Z}} e^{in\theta} \frac{\mp e^{in\theta}}{|e^{i\theta} - e^{-i\theta}|} \quad \text{or} \quad \sum_{n \in \mathbb{Z}} e^{in\theta} e^{nt} + e^{-nt} |e^t - e^{-t}|.$$
depending on whether $\delta$ lies in the elliptic or the split torus of $G$ respectively. In the $p$-adic case, the cases separate into whether $G'$ is ramified, and in both cases the stable transfer is computed explicitly in [Lan13, §2.4], where we refer the reader for explicit formulas.

(3) The toric case essentially follows from the discussion of Section 2.7 and the local Langlands correspondence for tori. Given tori $T, T'$, which we shall assume for simplicity to be quasisplit, with a fixed embedding $\xi': L^T \to L^T$. In this setting, stable conjugacy coincides with ordinary conjugacy. Given a parameter $\phi': W_F \to L^{T'}$ we obtain the parameter $\phi = \xi' \circ \phi'$ of $T$. The stable character attached to $\phi$ is

$$f^T(\phi) = \int_{T(F)} \chi(\delta) f(\delta) d\delta, \quad f \in \mathcal{C}(T),$$

where we may view the character $\chi$ associated to $\phi$ as an element of $X(T) \otimes \mathbb{C}$. It follows that $\Theta_X(\delta', \delta) = \delta_{\xi'}(\delta', \delta)$, where the Kronecker delta here is 1 if and only if $\delta'$ is an image of $\delta$ under the admissible embedding $\xi'$. Also, noting that the stable orbital integral is given by

$$f^T(\delta) = |D(\delta)|^{1/2} \mathrm{vol}(T(F)) f(\delta),$$

we see that $f^T$ transfers to a smooth function $f^{T'}$ on $T'(F)$.

Now let us examine slightly more general cases, without using the local Langlands correspondence. We call a function $f \in \mathcal{C}(G, \zeta)$ cuspidal if $f_M$ vanishes for every proper Levi subgroup $M$ of $G$ (see also Section 6.2). We denote by $\mathcal{C}_{\text{cusp}}(G, \zeta)$ the subspace of cuspidal functions. It is the subspace of $\mathcal{C}(G, \zeta)$ whose image in $S\mathcal{L}(G, \zeta)$ equals $S\mathcal{L}_{\text{cusp}}(G, \zeta)$.

**Proposition 5.2.** Conjecture 5.1 holds for $f \in \mathcal{C}_{\text{cusp}}(G, \zeta)$.

**Proof.** The property that $f$ is cuspidal implies that the image $f'$, if it exists, must vanish unless there are elliptic maximal tori $T \subset G$ and $T' \subset G'$ with admissible $L$-embeddings $L^T \subset L^G$ and $L^{T'} \subset L^{G'}$ such that $\xi'(L^T)$ is contained in $L^T$. The problem thus reduces to that of tori, which we have shown by the preceding discussion. \hfill $\square$

**Remark 5.3.** Similarly, it is also possible to consider minimal Levi subgroups $M \subset G$ and $M' \subset G'$, which are maximal tori, and restricting to stable conjugacy classes in $M(F)$ and $M'(F)$ respectively, though we will not study this here.

**Lemma 5.4.** The transfer $f \to f'$ is independent of choice of auxiliary datum.

**Proof.** Suppose $(\hat{G}_1', \hat{\xi}_1')$ and $(\hat{G}_2', \hat{\xi}_2')$ are two auxiliary data with fixed central data $(Z_1, \zeta_1)$ and $(Z_2, \zeta_2)$, and associated stable transfer factors $\Theta_{\hat{\xi}_1}$ and $\Theta_{\hat{\xi}_2}$. Let $\hat{G}_{12}'$ be the fibre product of $\hat{G}_1'$ and $\hat{G}_2'$ over $G'$. We have

$$Z(\hat{G}_{12}') = (Z(\hat{G}_1') \times Z(\hat{G}_2'))/\text{diag}_-(Z(\hat{G}'))$$

where $\text{diag}_-(Z(\hat{G}'))$ is the anti-diagonal embedding. Given $w \in W_F$, let $g_w = (g(w), w)$ be an element in $G'$ such that $\text{ad}_{g_w}$ acts by $\rho'$ on $\hat{G}'$. Also let

$$\hat{\xi}_i(g_w) = (\zeta_i(w), w), \quad \zeta_i(w) \in Z(\hat{G}_i').$$

for \(i = 1, 2\). Let \(z_{12}(w)\) be the image of \((z_1(w), z_2(w)^{-1})\) in \(Z(\hat{G}'_{12})\), which is a cocycle of \(W_F\) valued in \(Z(\hat{G}'_{12})\), and by duality determines a character \(\tilde{\eta}_{12}\) of \(\hat{G}'_{12}\). Its restriction to \(\hat{C}_1 \times \hat{C}_2\) determining the fibre product \(\hat{G}'_1 \times \hat{G}'_2\) over \(G'\) is equal to \(\tilde{\eta}'_1 \times (\tilde{\eta}'_2)^{-1}\), and pulling back the central datum \((Z, \zeta)\) we obtain the character \(\tilde{\zeta}'_{12}\) on \(\hat{G}'_{12}\). Let \(\delta'_i \in \Delta_G(\hat{G}')\) be images of \(\delta \in \Delta(G)\) for \(i = 1, 2\) such that \((\delta'_1, \delta'_2) \in \hat{G}'_{12}\). Then we claim that

\[
(5.2) \quad \Theta_{\tilde{\zeta}_i}^{(\delta'_2, \delta)} = \zeta_{12}(\delta_1, \delta_2) \Theta_{\tilde{\zeta}_i}^{(\delta'_1, \delta)}.
\]

This follows from the analogous property for the endoscopic transfer factors and the property that the Langlands-Shelstad transfer mapping is independent of choice of auxiliary datum [MW16, I.2.5].

The isomorphism \(ST(\hat{G}'_1, \tilde{\zeta}_1)\) with \(ST(\hat{G}'_2, \tilde{\zeta}_2)\) induced by the linear isomorphism \(f_1 \to f_2\) from \(\mathcal{E}(\hat{G}'_1, \tilde{\zeta}_1)\) to \(\mathcal{E}(\hat{G}'_2, \tilde{\zeta}_2)\) defined by

\[
f_2(\delta'_2) = \tilde{\zeta}(\delta'_1, \delta'_2) f_1(\delta'_1),
\]

where \(\delta_1\) is determined by \((\delta_1, \delta_2) \in \hat{G}'_{12}\). The isomorphism commutes with the transfer mappings \(f \to f_i = f^{\tilde{\zeta}_i}\). Then taking the inductive limit over such maps we see that the stable transfer mapping is independent of choice of auxiliary datum.

As a consequence, the distribution \(f'(\phi')\) depends on \(\phi'\) rather than \(\tilde{\phi}' = \tilde{\xi}' \circ \phi'\). However, \(f'(\phi')\) does still depend on the choice of transfer factor. We shall think of \(\Theta_{\tilde{\zeta}_i}\) as a family of transfer factors, one for each choice of \((\hat{G}', \tilde{\zeta})\). Let us briefly indicate this. The (absolute) Langlands-Shelstad transfer factor that we have been discussing is based on the canonical relative transfer factor

\[
\Delta(\delta^\varepsilon, \gamma, \delta^\varepsilon, \tilde{\gamma}), \quad \delta^\varepsilon, \delta^\varepsilon \in \Delta_G(\hat{G}^\varepsilon), \quad \gamma, \tilde{\gamma} \in \Gamma_G(G),
\]

associated to each \(G, G^\varepsilon\), and \((\hat{G}^\varepsilon, \tilde{\zeta}^\varepsilon)\) (see for example [Art99, §2]). Recall that our assumption that \(\tilde{\zeta}\) is of unitary type ensures that \(|\Delta(\delta^\varepsilon, \gamma, \delta^\varepsilon, \tilde{\gamma})| = 1\). The pair \((\delta^\varepsilon, \tilde{\gamma})\) are chosen base points used to define the absolute transfer factor

\[
(5.3) \quad \Delta(\delta^\varepsilon, \gamma) = \Delta(\delta^\varepsilon, \gamma, \delta^\varepsilon, \tilde{\gamma}) \Delta(\delta^\varepsilon, \tilde{\gamma}),
\]

defined to be zero unless \(\delta^\varepsilon\) is an image of \(\gamma\). If we call an absolute transfer factor any function \(\Delta(\delta^\varepsilon, \gamma)\) on \(\Delta_G(\hat{G}^\varepsilon) \times \Gamma(G)\) such that \((5.3)\) holds if \(\delta^\varepsilon\) is an image of \(\gamma\), and is zero otherwise, then the space of absolute transfer factors forms a \(U(1)\)-torsor. Following [Art06, §2], we call \(\Delta\) a transfer family for \((G, G^\varepsilon)\) that varies according to \((\hat{G}^\varepsilon, \tilde{\zeta}^\varepsilon)\), uniquely determined up to a multiplicative constant of absolute value one.

As the stable transfer factor \(\Theta_{\tilde{\zeta}_i}\) depends on the transfer family \(\Delta\), we can in particular define a stable transfer family depending on \(\Delta\). The relation \((5.2)\) allows us to relate stable transfer factors associated to different auxiliary data. Similarly, suppose \(t : G'_1 \xrightarrow{\sim} G'_2\) is an isomorphism of transfer data equipped with a dual \(L\)-isomorphism \(\hat{t} : \hat{G}'_2 \xrightarrow{\sim} \hat{G}'_1\), and \((\hat{G}'_1, \tilde{\zeta}_1)\) is an auxiliary datum for \(G'_1\), we obtain an auxiliary datum

\[
(\hat{G}'_2, \tilde{\zeta}'_2) = t(\hat{G}'_1, \tilde{\zeta}_1)
\]
for $G'_2$ such that $t$ canonically extends to an $F$-isomorphism $\tilde{G}'_1 \sim \tilde{G}'_2$. We also have canonically a corresponding $L$-isomorphism $\tilde{L} t : \tilde{L} \tilde{G}'_1 \sim \tilde{L} \tilde{G}'_2$ such that

$$\tilde{\xi}'_3 = \tilde{L} t \circ \tilde{\xi}'_1 \circ t^{-1}.$$  

Then if $\Theta_{\xi'_1}(\delta'_1, \delta)$ is a transfer factor for $(\tilde{G}'_1, \tilde{\xi}'_1)$, it follows that $\Theta_{\tilde{\xi}'_1}(t\delta'_1, \delta)$ is a transfer factor for $(\tilde{G}'_2, \tilde{\xi}'_2)$. A similar relation also holds in the endoscopic case.

As regards the general case, we have as usual the following reduction.

**Proposition 5.5.** If Conjecture 5.1 holds for $G$ with $G_{\text{der}}$ simply connected and $(Z, \zeta)$ trivial, then it holds for arbitrary $G$ and $(Z, \zeta)$.

**Proof.** Suppose that $G, G'$ and $(Z, \zeta)$ are arbitrary. Let $\tilde{G}$ be a $z$-extension of $G$ by the central induced torus $\tilde{C}$, and let $(\tilde{Z}, \tilde{\zeta})$ be the pullback of $(Z, \zeta)$ to $\tilde{G}$. Since $G(F) = \tilde{G}(F)/\tilde{C}(F)$, we can identify $\mathcal{I}(G, \zeta)$ with $\mathcal{I}(\tilde{G}, \tilde{\zeta})$. Moreover, there is a bijection between isomorphism classes of transfer data $\mathcal{F}(G)$ and $\mathcal{F}(\tilde{G})$. For any $G'$, we find an extension $\tilde{G}'$ equipped with an $L$-isomorphism $\tilde{\xi}' : G' \to \tilde{L} \tilde{G}'$ and a natural embedding of $G'$ into $\tilde{G}'$, so that $(\tilde{G}', \tilde{\xi}')$ is an auxiliary datum for both $G'$ and $\tilde{G}'$. We have a natural projection $\pi$ of $\mathcal{I}(\tilde{G})$ onto $\mathcal{I}(\tilde{G}, \tilde{\zeta}) = \mathcal{I}(G, \zeta)$ given by

$$a(\gamma) \mapsto \int_{\tilde{Z}(F)} a(z\gamma)\tilde{\zeta}(z)dz$$

for any $a \in \mathcal{I}(\tilde{G})$. Similarly, we define a projection $\pi'$ of $\mathcal{S}\mathcal{I}(\tilde{G}')$ onto $\mathcal{S}\mathcal{I}(\tilde{G}', \tilde{\zeta}') = \mathcal{S}\mathcal{I}(G', \zeta')$. It then follows from Lemma 4.4(1) that the projections commute with the stable transfer mappings from $\mathcal{I}(\tilde{G})$ to $\mathcal{S}\mathcal{I}(\tilde{G}')$ and $\mathcal{I}(G, \zeta)$ to $\mathcal{S}\mathcal{I}(G', \zeta')$ respectively. That is,

$$\begin{array}{ccc}
\mathcal{I}(\tilde{G}) & \longrightarrow & \mathcal{S}\mathcal{I}(\tilde{G}') \\
\downarrow^\pi & & \downarrow^\pi' \\
\mathcal{I}(G, \zeta) & \longrightarrow & \mathcal{S}\mathcal{I}(G', \zeta')
\end{array}$$

where the horizontal maps are the stable transfer mappings. The result follows. □

### 5.2. Geometric transfer and spectral transfer.

The stable geometric transfer leads us naturally to the following stable character identity, which is simply a restatement of [Lan13, (2.3)] in different terms.

**Theorem 5.6.** For any $\delta \in \Delta(G)$, we have

$$S(\tilde{\xi}' \circ \phi', \delta) = \int_{\Delta(G')} \Theta_{\xi'_1}(\delta', \delta)S'(\phi', \delta')d\delta'.$$

**Remark 5.7.** In [Lan13], the integration is taken over the Steinberg-Hitchin base, the variety of stable semisimple conjugacy classes of $G'(F)$. Its measure is determined by the Haar measure on $G'$, and in particular the singular locus has measure zero, and in particular coincides with $\Delta(G')$.

**Proof.** By definition, the right hand side is

$$\int_{\Delta(G')} \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta', \phi'_1)S(\tilde{\xi}' \circ \phi'_1, \delta)d\phi'_1S'(\phi', \delta')d\delta'.$$
and interchanging integrals we have
\[ \int_{\Phi(G')} S'(\delta', \phi'_1) S'((\phi', \delta') d\delta' S(\tilde{\xi}' \circ \phi'_1, \delta) d\phi'_1. \]

Then applying Proposition 3.1 to the inner integral leads to the orthogonality relation (3.4) for the stable virtual characters \( S(\phi, \delta) \),
\[ \int_{\Delta(G')} n(\delta') S(\phi', \delta') S'((\phi', \delta') d\delta' = \delta(\phi', \phi'_1)n(\phi'), \]
for a positive real number \( n(\phi') \). Then evaluating the outer integral at \( \phi' = \phi'_1 \), we obtain \( S(\tilde{\xi}' \circ \phi'_1, \delta) \) as desired. \( \square \)

The stable transfer of characters mediates between the desired stable character identity and the stable transfer of orbital integrals.

**Corollary 5.8.** Conjectures 5.1 implies the stable character identity (1.1) holds for all \( f \in \mathcal{C}(G, \zeta) \). On the other hand, (1.1) implies Conjecture 5.1.

Proof. Both claims will follow from the same identity of characters. First, using the Fourier expansions for \( f(\phi') \) in (3.7), we have
\[ f(\tilde{\xi}' \circ \phi') = \int_{\Delta(G/\mathbb{Z})} S(\tilde{\xi}' \circ \phi', \delta) f^G(\delta) d\delta. \]

We next apply the stable character identity (5.4) and interchange the order of integration to get
\[ \int_{\Delta(G')} S'(\phi', \delta') \int_{\Delta(G/\mathbb{Z})} \Theta_{\tilde{\xi}'}(\delta', \delta) f^G(\delta) d\delta d\delta', \]
it then follows from the stable transfer conjecture (5.1) that \( f(\tilde{\xi}' \circ \phi') \) equals
\[ \int_{\Delta(G')} S'(\phi', \delta') f'(\delta') d\delta' = f'(\phi'), \]
and the required results follow. \( \square \)

**Remark 5.9.** It is interesting to note that a weak form of this stable transfer was constructed from the endoscopic Langlands-Shelstad transfer in a special cases, described in [Mok18, Appendix A].

The stable transfer can be easily seen to be functorial in the following sense. Given \( G \), let \( G' \in \mathcal{F}(G) \) and \( G'' \in \mathcal{F}(G') \) with accompanying auxiliary data.

**Corollary 5.10.** Assume Conjecture 5.6 holds. Then \( f \circ G' = (f \circ G') \circ G'' \).

Proof. The result can be seen to hold for stable characters by composing maps of Paley-Wiener spaces using the argument in Corollary 4.2, but we shall prove this on the level of stable orbital integrals instead. As usual, we may assume for simplicity that \( G' \) and \( G'' \) are \( L \)-groups, which we identify as \( L^{-1}G' \) and \( L^{-1}G'' \) respectively. Then we have \( L \)-embeddings
\[ L^{-1}G'' \to L^{-1}G' \to L^{-1}G, \]
and denote by \( \xi'' \) the composition. First, we claim that
\[ (5.5) \quad \Theta_{\xi''}(\delta'', \delta) = \int_{\Delta(G')} \Theta_{\xi'}(\delta'', \delta') \Theta_{\tilde{\xi}'}(\delta', \delta) d\delta'. \]
To see this, we expand the transfer factor $\Theta_{\xi''}(\delta'', \delta')$ in the righthand side
\[
\int_{\Delta(G')} \int_{\Phi(\tilde{G}'', \tilde{\xi}'')} S''(\delta'', \phi'') S''(\tilde{\xi}'' \circ \phi'', \delta') d\phi'' \Theta_{\xi}(\delta', \delta) d\delta'.
\]
Interchanging the order of integration, this is
\[
\int_{\Phi(\tilde{G}'', \tilde{\xi}'')} S''(\delta'', \phi'') \int_{\Delta(G')} S'(\tilde{\xi}'' \circ \phi'', \delta') \Theta_{\xi}(\delta', \delta) d\delta' d\phi'',
\]
and we can apply Conjecture 5.6 to see that the inner integral is equal to $S(\tilde{\xi}' \circ \tilde{\xi}'' \circ \phi'', \delta) = S(\tilde{\xi}'' \circ \phi'', \delta)$. In other words, we have
\[
\int_{\Phi(\tilde{G}'', \tilde{\xi}'')} S''(\delta'', \phi'') S(\tilde{\xi}'' \circ \phi'', \delta) d\phi'',
\]
which is equal to $\Theta_{\xi''}(\delta'', \delta)$, the lefthand side. Finally, writing $(f^{G'})^{G''}$ as
\[
\int_{\Delta(G')} \Theta_{\xi''}(\delta'', \delta') \int_{\Delta(G'/Z)} \Theta_{\xi}(\delta', \delta) f^{G}(\delta) d\delta' d\delta,
\]
then interchanging integrals and applying (5.5) gives the result. \(\square\)

5.3. Descent principles. For the remainder of this section, we briefly explore the relationship between descent and transfer. As in the case of endoscopy [LS90], this reduces to the descent of transfer factors, which we shall have to take on as a hypothesis in this paper. Given $G' \in \mathcal{F}(G)$ and $d' \in \Delta_{ss}(G')$, let $d''$ be its preimage in $\Delta_{ss}(G')$. We shall investigate the behaviour of
\[
f'(?d') = \int_{\Delta(G'/Z)} \Theta_{\xi}(?d', \delta) f^{G}(\delta) d\delta
\]
for $?d'$ near to $d'$. If $d'$ is not the image of any semisimple element in $\Delta_{ss}(G)$, then no strongly $G$-regular element in $G_{d'}(F)$ can be the image of an element in $G(F)$, thus $f'$ vanishes on $\Delta_{G}(G_{d'}^{\nu})$. It follows then that $f'$ vanishes for all $?d'$ in a neighbourhood of $d'$ in $\tilde{G}(F)$. We shall therefore assume that $d'$ is the image of some $d \in \Delta(G)$.

We can find a representative $d_1' = x^{-1} d' x$ in the stable conjugacy class of $d'$ such that $G_{d_1'}$ is quasisplit over $F$, and we can multiply $x$ by an element of $G_{d_1'}$ if necessary so that
\[
\text{Int}(x^{-1}) : G_{d'} \to G_{d_1'}
\]
is defined over $F$. Then $x$ acts on the preimage $\tilde{G}_{d_1'}$ of $G_{d_1'}$ in $\tilde{G}$, so that $f'(x^{-1} \tilde{d}' x) = f'(\tilde{d}')$ for all $\tilde{d}' \in \Delta_{G}(G_{d_1'}^{\nu})$, which can also be seen to follow from the same property for stable orbital integrals [LS90, §1.3]. In particular, replacing $d'$ by $d_1'$ if necessary, we may assume that $G_{d_1'}$ is quasisplit over $F$.

The group $G_{d_1'}$ induces a transfer datum for $G_d$ in the following sense. More generally, we call $d'$ a $T'$-image of $d$ in $G$ if there exists an admissible embedding of tori $T' \to T^*$ sending $d'$ to $d''$ in $G^{*}$ and an $x \in G_{ss}^{*}$ such that
\[
(\text{Int}(x) \circ \psi)(d) = d''
\]
and both $\text{Int}(x) \circ \psi$ and the preimage of $T$ are defined over $F$. Varying over $T'$ we obtain all images of $d$. Let $d'$ be a $T'$-image of $d$ for some torus $T'$, and let $d''$ be the image of $d'$ under an admissible embedding of $T'$ in $T^*$, which we may choose to be such that $G_{d_1'}$ is quasisplit. Fixing $G' \in \mathcal{F}(G)$, we shall attach an extension $G'_{d_1'}$ of
by which we can identify the set of coroots $R$. Proof. Let $T$ be a maximal torus of $G = G_d$. Given an embedding $\hat{G}' \to G$, we can choose an admissible embedding of a maximal torus of $G'$ in $G_\d'$ and that $\hat{G}' \to G$ is a transfer datum for $G_d$ and $T' \to T^*$ is admissible.

Lemma 5.11. The embedding $T' \to T^*$ can be chosen to be admissible for both $(G, G')$ and $(G_d,G_d')$, unique up to isomorphism. Any admissible embedding of a maximal torus of $G'$ in $G_\d'$ is admissible as an embedding of a maximal torus of $G'$ in $G^*$ and sends $d' \to d^*$. Proof. The proof is a simple modification of [LS90, 1.4]. We supply the details here simply for the sake of completeness. We first explain how the transfer data is constructed. Given an embedding $T' \to T$, let $B'$ and $B^*$ be the associated Borel subgroups and $x \in G_\d^*$ as above. The map $\psi_x = \text{Int}(x) \circ \psi$ defines the quasi-split inner twist $G_\d^*$ of $G_d$. The embedding $T' \to T^* \xrightarrow{\psi_x} \hat{T}$ is dual to the diagram

$$\hat{T} \xrightarrow{\psi_x} \hat{T^*} \to T_1,$$

by which we can identify the set of coroots $R(G, T)^\vee$ of $T$ in $G$ with the set of roots $R(\hat{G}, T_1)$ of $T_1$ in $\hat{G}$, and hence $R(G_d, T)^\vee$ with a subset of $R(\hat{G}, T_1)$. Fix an $L$-group data, meaning a complex reductive group $\hat{G}_d$, an action $\rho_d$ of $\Gamma$ on $\hat{G}_d$, and a $\Gamma$-stable bijection $\Psi((G_d)^\vee) \to \Psi(\hat{G}_d)$, by which we define $^L\hat{G}_d = \hat{G}_d \ltimes W_F$. We may assume that $\hat{G}_d$ contains $T_1$ and that $R(\hat{G}_d, T_1)$ is equal to $R(G_d, T)^\vee$ as subsets of $R(\hat{G}, T_1)$. Let $B'_d = B^* \cap G^*$, and let $B_d$ be the Borel subgroup of $\hat{G}_d$ generated by $T_1$ and the $B_1$-positive roots of $T_1$ in $\hat{G}_d$. We can then identify the map $T_1 \to \hat{T}$ in (5.6) with the embedding $\hat{T} \to T_1$ in $\hat{G}_d$ given by $B'_d$ and $B_d$. The isomorphism

$$\hat{T} \xrightarrow{\psi_x} \hat{T^*} \to T_1$$

yields an embedding of $\hat{T}$ in $\hat{G}_d$ and extends to an admissible embedding of $^L\hat{T}$ in $^L\hat{G}_d$, whose image is independent of the choice of extension.

Given $G'$, the dual $\hat{G}'_d$ of $G'_d$ is a subgroup of $\hat{G}_d$ normalised by $^L\hat{T}$. We define $\hat{G}'_d$ to be the subgroup of $^L\hat{G}$ generated by $\hat{G}'_d$ and $^L\hat{T}$, and note that it is contained in $G'$. Then we can also define $\xi'_d$ to be the map given by restriction of $\xi'$ from $G'$ to $\hat{G}'_d$. We have a split exact sequence

$$1 \to \hat{G}'_d \to G'_d \to W_F \to 1,$$

and it follows that $(G'_d, G'_d, \xi'_d)$ is a transfer datum for $G_d$. Finally, we identify the embedding $\hat{T} \to T_1$ given by $B' \cap G'_d$ and $B_d \cap \hat{G}'_d$, with the restriction of the embedding $\hat{T} \to T_1$ above. This gives the admissible embedding $T' \to T^*$ as desired.

The choice of $\hat{G}_d$ is unique up to isomorphism of transfer datum. First suppose $B, B'$ are changed but $T' \to T^*$ remains fixed. Then the $L$-data $(\hat{G}_d, \rho_d)$ is replaced by another pair $((\hat{G}'_d, \rho'_d)$ that is $\Gamma$-isomorphic to it that sends the root datum $R(\hat{G}_d, T_1)$ to $R(\hat{G}'_d, T_1)$, the image of $^L\hat{T}$ in $^L\hat{G}_d$ to its image in $^L\hat{G}'_d$, and $\hat{G}'_d$ and $G'_d$ to the new $\hat{G}'_d$ and $G'_d$ respectively. In particular, this gives an isomorphic transfer datum for $G_d$.

If we replace $T' \to T^*$ with another admissible embedding of tori $T'^{\vee} \to T^{\vee}$, such that $d'$ lies in $T', T'^{\vee}$ and $d^*$ in $T^*, T^{\vee}$. Then we may assume that the new Borel subgroups are obtained from $B'$ and $B^*$ by conjugation in $G'_d$ and $G^*_d$, respectively, and the new data is isomorphic. Finally, it is straightforward to see that the choice
Lemma 5.12. Suppose that there exists some constant $c$ such that

$$\Theta_{\xi'}(\delta', \delta) \Theta_{\xi''}(\delta', \delta)^{-1} \rightarrow c$$

as $\tilde{\delta} \rightarrow \delta'$ and $\delta \rightarrow d$. Then $f'(\delta')$ is equal to a finite linear combination of stable orbital integrals on $G'$.  

Proof. According to the measure on $\Delta(G)$, the integral decomposes into

$$\sum_{\{M\}} |W(M)|^{-1} \sum_{\{T\}} |W_F(G,T)|^{-1} \int_{T(F)} \Theta_{\xi}(\tilde{\delta'}, t) f^G(t) dt,$$

where the inner sum is over stable conjugacy classes of elliptic maximal tori of $G$ over $F$, and $W_F(G,T)$ is the subgroup of elements in the absolute Weyl group of $(G,T)$ defined over $F$. Moreover, our hypothesis (5.7) implies

$$\int_{T(F)} \Theta_{\xi'}(\tilde{\delta'}, t) f^G(t) dt = c \int_{T(F)} \Theta_{\xi''}(\tilde{\delta'}, t) f^G(t) dt.$$

Now by Lemma 5.11, we may choose an embedding $T' \rightarrow T^*$ that is admissible for both $(G,G')$ and $(G',G)$). If $\tilde{\delta}'$ is an element in the preimage of $T'(F)$ in $\Delta_G(G')$, then the integral in $f'(\delta')$ is taken over the stable conjugacy classes defined by the composition

$$\tilde{T}' \rightarrow T' \rightarrow T^* \rightarrow T$$

in $\Delta(G)$. Note that it is possible that $\tilde{\delta}'$ is a $T'$-image for more than one tori, and varying over equivalence classes tori in $G'$ we obtain all possible images. Then using the property that $\Theta_{\xi'}(\tilde{\delta'}, t)$ vanishes unless $\tilde{\delta}'$ is an image, we conclude that the righthand side can be written as a finite linear combination $f^{G'}(\delta')$. □

It follows from Lemma 4.4(i) that for $\tilde{\delta}'$ close to the identity, $\Theta_{\xi'}(\tilde{\delta}', \delta)$ depends only on the image $\delta$ of $\delta'$ in $\Delta(G')$, so we write it as $\Theta_{\xi'}^{bc}(\delta', \delta)$. We say that $(G,G')$ admits local $\Theta_{\xi'}$-transfer at the identity if for any $f \in C_c^\infty(G(F))$ we have

$$f'(\delta') = \int_{\Delta_G(G')} \Theta_{\xi'}^{bc}(\delta', \delta) f^G(\delta) d\delta$$

for all $\delta' \in \Delta_G(G')$ near to the identity.

Corollary 5.13. Let $F$ be nonarchimedean, and assume (5.7) holds. If $(G_d, G'_d)$ have local $\Theta_{\xi_d'}$-transfer at the identity for all $d \in \Delta(G)$, then $(G,G')$ has $\Theta_{\xi'}$-transfer.
Proof. Since the assumption continue to hold if \( G \) is replaced by a \( z \)-extension \( \hat{G} \), we can assume that \( G = \hat{G} \) and \( G' \) is an \( L \)-group. By [LS90, Lemma 2.2A] it suffices to show that \( f'(\delta') \) is a local stable orbital integral on \( G'(F) \), in the sense that for every semisimple element \( d \) in \( G(F) \) there exists \( f_d \in C_c^{\infty}(G(F)) \) such that \( f'(\delta) = f'_d(\delta) \) for all regular semisimple \( \delta \) near to \( d \). Again by Lemma 4.4(i), it follows that local \( \Theta_{\delta_d}^{-} \)-transfer at the identity implies local \( \Theta_{\delta'_d}^{-} \)-transfer at \( d' \) in the sense that (5.8) holds for all \( \delta' \in \Delta_G(\hat{G}') \) near to \( d' \). By assumption, we may apply the descent formula to express \( f'(\delta') \) as a finite linear combination of stable orbital integrals on \( G'_d \). Then applying local transfer at \( d' \) to each summand, the result follows. \( \Box \)

Remark 5.14. As a historical remark, the descent of endoscopic transfer factors allowed Waldspurger to reduce questions about the Fundamental Lemma on the group to the Lie algebra, which was finally solved by transfer to positive characteristic. In the present situation, it would not be unreasonable to expect that such results would again prove useful in establishing stable transfer. As we have shown, the latter is also implied by the basic local Langlands Correspondence, which may not be as satisfying. Regardless, the descent properties will be required later to define our stable spectral transfer factors, and are therefore of independent interest.

6. Primitive distributions

6.1. Primitive parameters. We return again to the local Langlands correspondence for \( G \). Let \( \phi \in \Phi(G) \). The centraliser \( S_\phi = \text{Cent}(\hat{G}, \phi(L_F)) \) in \( \hat{G} \) of the image of \( \phi \) is a complex reductive subgroup of \( \hat{G} \). If \( \phi \) is replaced by another representative \( \phi_1 \) of its equivalence class in \( \Phi(G) \), there is an isomorphism from \( S_\phi \) onto \( S_{\phi_1} \) uniquely determined up to conjugacy in \( S_\phi \). We write \( S_{\phi, \text{ad}} = S_\phi / Z(\hat{G})^\Gamma \), which is a reductive subgroup of the adjoint group \( \hat{G}_{\text{ad}} \) of \( G \), and is related to the behaviour of \( \Delta \)-transfer families [Art06, \S2]. The group of connected components \( \pi_0(S_{\phi, \text{ad}}) \) is a finite group determined up to inner automorphism by the class of \( \phi \) in \( \Phi(G) \). If \( G \) is quasi-split over \( F \), one expects an injective map from \( \Pi_\phi \) to the set of irreducible representations \( \text{Irr}(\pi_0(S_{\phi, \text{ad}})) \) that governs the internal structure of \( \Pi_\phi \). More generally, let \( G \) be a reductive group over \( F \) with quasisplit inner form \( G^* \). Let \( G^* = G^* \times \mathbb{Z} \), and let \( S^+_0 = S^+_0 \times \hat{G}^*, \) which is the preimage of \( S_0 \) under the isogeny \( \hat{G}^* \to \hat{G}^* \). Then the refined local Langlands conjecture, formulated in [Kal16, Conjecture G] is stated in terms of \( \text{Irr}(\pi_0(S^+_0)) \).

A general element in \( \Phi^+(G) \) can be represented by \( \phi_z(w) = \phi(w) z^\text{ord}(w) \) for \( w \in L_F \) if \( F \) is nonarchimedean, where \( \phi \in \Phi(G) = \Phi_{\text{temp}}(G) \) and \( z \) is a suitable point in \( S^+_0 \). By the Langlands correspondence for split tori, we can identify \( z \) with an unramified quasicharacter of a Levi subgroup of \( G \) whose \( L \)-group contains the \( \text{Im}(\phi) \). If \( \phi \) is chosen suitably, one expects a bijection from \( \Pi_\phi \to \Pi_{\delta_z} \) obtained by taking Langlands quotients of corresponding deformations of tempered representations \( \pi \) by \( z \). It suffices then to focus our study on \( \Phi(G) \).

Let \( \Phi_2(G) \) be the set of \( G \)-orbits of parameters that are cuspidal, meaning the image of \( L_F \) is contained in no parabolic subgroup. There is a canonical decomposition

\[
\Phi(G) = \prod_{\{M\}} \Phi_2(M)/W(M).
\]
Any parameter \( \phi \in \Phi(G) \) has a central character \( \zeta \) on \( Z(F) \), whose Langlands parameter is given by the composition of \( \phi \) with the projection \( \hat{G} \to L \). The set \( \Phi(G) \) then decomposes into the union over \( \zeta \) of the subsets \( \Phi(G, \zeta) \) with central character \( \zeta \).

Now let us say \( \phi \in \Phi(G) \) is primitive if it does not factor through the image of any \( \tilde{\xi} \) in \( \hat{G} \) for any \( \xi \in \ell^{\geq 0}(\hat{G}) \). Let \( \Phi_{\text{prim}}(G) \) be the set of equivalence classes of primitive Langlands parameters. There is a parallel decomposition of \( \Phi_{\text{prim}}(G) \) into the union over \( \Phi_{\text{prim}}(G, \zeta) \) consisting of primitive parameters with central character \( \zeta \). Applying the local Langlands correspondence, we have a decomposition

\[
\Phi(G) = \bigcup_{G' \in \mathcal{F}(G)} \Phi_{\text{prim}}(G'),
\]

which we caution that the union is not necessarily disjoint, which relates to the problem of refining the transfer data mentioned in the introduction.

We write \( \Phi_{\text{prim}}(G', G) \) for the Aut\(_G(G')\)-orbits in \( \Phi_{\text{prim}}(G') \).

**Remark 6.1.** A spectral characterisation of primitive parameters is available in the global case of tempered automorphic representations, using the multiplicity of the trivial representation in \( r \circ \xi' \) as we range over \( r \) and \( \xi' \). Langlands calls an automorphic representation \( \pi \) thick or hadronic if \( m(\pi, r) \), the order of \( L(s, \pi, r) \) at \( s = 1 \) is always equal to the number of times the trivial representation of \( L \) is contained in \( r \). It is well known from [AYY13] that the dimension datum does not uniquely determine the group \( G' \) (up to isomorphism), leaving open the possibility that the same primitive form could exist on different groups \( G_1' \) and \( G_2' \). We refer to the recent work [Yu21] for a potential solution to the problem.

### 6.2. Spaces of distributions.

Let \( \mathcal{D}(G, \zeta) \) be the space of \( \zeta \)-equivariant invariant distributions that are supported on the preimage in \( G(F) \) of finitely many conjugacy classes in \( \hat{G}(F) = G(F)/Z(F) \). For \( F \) nonarchimedean, it is equal to the space of ordinary orbital integrals, whereas if \( F \) is archimedean, it also includes radial derivatives of orbital integrals.

Let \( \mathcal{F}(G, \zeta) \) be the space of \( \zeta \)-equivariant invariant distributions spanned by invariant character of \( G(F) \), hence generated by characters attached to the set \( \Pi(G, \zeta) \) of irreducible representations of \( G(F) \) whose central character restricts to \( \zeta \) on \( Z(F) \). A distribution \( D \) in either \( \mathcal{D}(G, \zeta) \) and \( \mathcal{F}(G, \zeta) \) can be regarded as a linear form

\[
D(f) = f_G(D)
\]

on either \( \mathcal{H}(G, \zeta) \) and \( \mathcal{I}(G, \zeta) \).

Let \( I \) be a continuous, invariant linear form on \( \mathcal{H}(G, \zeta) \). We say that \( I \) is supported on characters if \( I(f) = 0 \) for any \( f \) such that \( f_G = 0 \). If so, then there is a continuous linear form \( \hat{I} \) on \( \mathcal{I}(G, \zeta) \) such that

\[
\hat{I}(f_G) = I(f)
\]

for all \( f \in \mathcal{H}(G, \zeta) \). If \( D \in \mathcal{F}(G, \zeta) \), it is clear that it is supported on characters. On the other hand, if \( D \in \mathcal{D}(G, \zeta) \), it can be expressed in terms of strongly regular invariant orbital integrals, which are supported on characters by [Art88]. Together with the fact that characters are locally integrable functions, it follows that we can generate \( \mathcal{I}(G, \zeta) \) by either irreducible tempered characters or strongly regular orbital integrals, both denoted \( f_G \).
We denote by $\mathcal{I}_{\text{cusp}}(G, \zeta)$ the subspace of functions in $\mathcal{I}(G, \zeta)$ supported on $\Gamma_{\text{ell}}(G)$. If $Z$ contains the split component of the centre of $G$, there is a surjective linear map

$$\mathcal{F}(G, \zeta) \to \mathcal{I}_{\text{cusp}}(G, \zeta)$$

canonically given by the elliptic virtual characters $I(\tau, \gamma)$ associated to any $D \in \mathcal{F}(G, \zeta)$. There is also a canonical linear section defined by the set $T_{\text{ell}}(G, \zeta)$ in $\mathcal{F}(G, \zeta)$ whose image forms a basis in $\mathcal{I}_{\text{cusp}}(G, \zeta)$ [Art96, §4]. We also denote by $\mathcal{I} \mathcal{I}_{\text{cusp}}(G, \zeta)$ the image of $\mathcal{I}_{\text{cusp}}(G, \zeta)$ in $\mathcal{I}(G, \zeta)$, and $\mathcal{H}_{\text{cusp}}(G, \zeta)$ the preimage of $\mathcal{I}_{\text{cusp}}(G, \zeta)$ in $\mathcal{H}(G, \zeta)$.

Let $\mathcal{SD}(G, \zeta)$ and $\mathcal{SF}(G, \zeta)$ be the stable subspaces of stable distributions in $\mathcal{D}(G, \zeta)$ and $\mathcal{F}(G, \zeta)$ respectively. Any distribution $S$ in $\mathcal{SD}(G, \zeta)$ and $\mathcal{SF}(G, \zeta)$ can be identified with a linear form $f^G \mapsto f^G(S)$ on $\mathcal{SI}(G, \zeta)$. We say a linear form $S$ on $\mathcal{SI}(G, \zeta)$ is stable if its value at $f$ depends only on the endoscopic transfer $f^\circ$ in the case $G^\circ = G^*$. If $G$ is quasisplit, there is a unique linear form $\hat{S}$ on $\mathcal{SI}(G^*, \zeta^*)$ associated to $S$ such that

$$\hat{S}(f^\circ) = S(f)$$

for any $f \in \mathcal{H}(G, \zeta)$. In general, for any stable distribution $S$ there is a unique continuous linear form $\hat{S}$ on $\mathcal{SI}(G, \zeta)$ such that

$$\hat{S}(f^G) = S(f).$$

One also has an alternative description of the cuspidal subspaces as follows. The restriction map $a^G \to a^M$ from $\mathcal{SI}(G(F))$ to $\mathcal{SI}(M(F))$ give a filtration

$$\mathcal{F}^M(\mathcal{SI}(G)) = \{ a^G \in \mathcal{SI}(G) : a^L = 0, L \subsetneq M \}$$

of $\mathcal{SI}(G(F))$ over the partially ordered set $\mathcal{L}/W_0$. We can then identify $\mathcal{SI}_{\text{cusp}}(G) = \mathcal{F}^G(\mathcal{SI}(G))$. Then the graded component

$$\mathcal{G}^M(\mathcal{SI}(G)) = \mathcal{F}^M(\mathcal{SI}(G))/ \sum_{L \geq M} \mathcal{F}^L(\mathcal{SI}(G))$$

(6.2)

attached to $\{ M \}$ is canonically isomorphic to $\mathcal{SI}_{\text{cusp}}(M)^{W(M)}$.

6.3. **Primitive subspaces.** Let now $\mathcal{F}^0(G)$ be the complement of $G^*$ in $\mathcal{F}(G)$. In view of the character identity (5.4) that characterises the transfer mapping, we can define a subspace $\mathcal{PF}(G, \zeta)$ of stable distributions in $\mathcal{SF}(G, \zeta)$, that do not lie in the preimage of (1.1) for any datum $G' \in \mathcal{F}^0(G)$ as above, and which we shall call primitive. We shall call $\mathcal{PF}(G, \zeta)$ the set of primitive characters. By the surjective mapping of $\mathcal{F}(G, \zeta)$ onto $\mathcal{I}_{\text{cusp}}(G, \zeta)$, we denote by

$$\mathcal{SI}_{\text{prim}}(G, \zeta) = \mathcal{PI}_{\text{cusp}}(G, \zeta)$$

the image of $\mathcal{PF}(G, \zeta)$ in $\mathcal{SI}_{\text{cusp}}(G, \zeta)$, which we again call primitive distributions. The following characterisation is a simple consequence of our definitions. Recall that we may view $\mathcal{SI}(G, \zeta)$ as the space of functions on $\Delta(G)$, generated by stable orbital integrals $f^G$. We can thus also call the stable orbital integrals contained in $\mathcal{SI}_{\text{prim}}(G, \zeta)$ primitive.

**Lemma 6.2.** $\mathcal{PI}_{\text{cusp}}(G, \zeta)$ is supported on the set of primitive parameters $\Phi_{\text{prim}}(G, \zeta)$. 
Proof. The surjective map from $\mathcal{F}(G, \zeta)$ to $\mathcal{I}_{\text{cusp}}(G, \zeta)$ is given by assigning to any $D \in \mathcal{F}(G, \zeta)$ the elliptic part

$$I_{\text{ell}}(D, \gamma) = \begin{cases} I(D, \gamma), & \gamma \in \Gamma_{\text{ell}}(G) \\ 0, & \text{otherwise} \end{cases},$$

of its normalised character $I(D, \gamma) = |D|^{1/2}(G(\gamma)|^{1/2} \Theta(D, \gamma)$. By [Art96, §4], it has a canonical linear section defined by the subset $T_{\text{ell}}(G, \zeta)$ of $\mathcal{F}(G, \zeta)$ whose image in $\mathcal{I}_{\text{cusp}}(G, \zeta)$ forms a basis. Similarly, by [Art96, §5], we have a surjective map from $S\mathcal{F}(G, \zeta)$ to $S\mathcal{I}_{\text{cusp}}(G, \zeta)$ given by

$$S_{\text{ell}}(\phi, \gamma) = \begin{cases} S(\phi, \delta), & \delta \in \Delta_{\text{ell}}(G) \\ 0, & \text{otherwise} \end{cases},$$

for any $\phi$ belonging to the basis $\Phi_2(G, \zeta)$ of $S\mathcal{F}_{\text{cusp}}(G, \zeta)$.

By definition, $P\mathcal{F}(G, \zeta)$ is the subspace of $S\mathcal{F}(G, \zeta)$ defined to be the complement of preimages of the transfer mapping $f \rightarrow f'$ for $G' \in F^0(G)$. It is generated by the $\phi \in \Phi_2(G, \zeta)$ such that for any $f \in \mathcal{H}(G, \zeta)$, the stable character $f^\phi(\phi)$ is not equal to $f'(\xi \circ \phi)$ for some $G' \in F^0(G)$. It follows from the definitions that the image of $P\mathcal{F}(G, \zeta)$ in $S\mathcal{I}_{\text{cusp}}(G, \zeta)$, which is equal to $P\mathcal{I}_{\text{cusp}}(G, \zeta)$, is given by characters $S_{\text{ell}}(\phi, \gamma)$ for $\phi \in \Phi_{\text{prim}}(G, \zeta)$. In other words, $P\mathcal{I}_{\text{cusp}}(G, \zeta)$ is generated by irreducible characters supported on the set of primitive parameters $\Phi_{\text{prim}}(G, \zeta)$.

Applying (6.1), we thus have a decomposition

$$S\mathcal{I}_{\text{cusp}}(G, \zeta) = \bigcup_{G'} P\mathcal{I}_{\text{cusp}}(G', \zeta')$$

that is not necessarily disjoint. We say a linear form $P$ on $\mathcal{H}(G, \zeta)$ is primitive if its value at $f$ depends only on $f'$ for $G' = G^*$, and for $G$ quasisplit, there is a unique linear form $\hat{P}$ on $P\mathcal{I}(G^*, \zeta^*)$ associated to $P$ such that

$$\hat{P}(f^*) = P(f)$$

for any $f \in \mathcal{H}(G, \zeta)$.

### 6.4. Transfer spaces

For any $G' \in \mathcal{F}_{\text{ell}}(G)$, let $S\mathcal{I}(G', G)$ be the subspace of functions in $S\mathcal{I}(G', \zeta)$ which depend only on the image of $\Delta_G(G')$ in $\Delta_G(G)$, which we denote by $\Delta(G', G)$. We define the cuspidal subspace $S\mathcal{I}_{\text{cusp}}(G', G)$ to be the intersection

$$S\mathcal{I}(G', G) \cap S\mathcal{I}_{\text{cusp}}(G', \zeta') = S\mathcal{I}_{\text{cusp}}(G', \zeta')^{\text{Out}_G(G')}.$$

Assuming the stable transfer conjecture, it follows from the definitions that $f \rightarrow f'$ maps $\mathcal{E}(G)$ continuously to $S\mathcal{I}(G', G)$ and $\mathcal{E}_{\text{cusp}}(G)$ continuously to $S\mathcal{I}_{\text{cusp}}(G', G)$. If we define a function

$$(6.3) \quad a'(\delta') = a_{G'}(\delta') = \int_{\Delta(G/Z)} \Theta_{\zeta'}(\delta', \delta) a_G(\delta) d\delta$$

on $\Delta_G(G')$, the stable transfer then gives a continuous map from $S\mathcal{I}(G)$ to $S\mathcal{I}(G', G)$ and $S\mathcal{I}_{\text{cusp}}(G)$ to $S\mathcal{I}(G', G)$. Define the topological vector space

$$S\mathcal{I}_{\text{cusp}}^x(G) = \bigoplus_{G' \in \mathcal{F}_{\text{ell}}(G)} S\mathcal{I}_{\text{cusp}}(G', G)$$
of smooth functions on $\Delta_{\text{cyl}}(G)$. For any function $a^G \in SÎl_{\text{cusp}}(G)$, we define the direct sum of images of $a^G$,

$$a^F = a^{G,F} = \bigoplus_{G' \in F_{\text{cyl}}(G)} a'.$$

Then the map

(6.4) $\mathcal{T}^F : a^G \rightarrow a^{G,F}$

is a continuous linear map from $SÎl_{\text{cusp}}(G)$ to $SÎl_{\text{cusp}}^F(G)$. The following is a natural analogue of the endoscopic mapping in [Art96, Proposition 3.5] and [MW16, I.4.11] in the nonarchimedean case and [MW16, I.4.12] in the archimedean case. The most difficult part of lies in proving the surjectivity, which we shall return to in the next section. For now we simply take it on as an assumption.

**Theorem 6.3.** Assume that $\mathcal{T}^F$ is surjective. Then it is an isometric isomorphism.

**Proof.** It is straightforward to see using the adjoint relations (4.6) and (4.7) that $\mathcal{T}^F$ is invertible on its image, with inverse $a^{G,F} \rightarrow a^G$ given by

(6.5) $a^G(\delta) = n(\delta) \int_{\Delta_{\text{cyl}}(G)} n(\delta') \Theta_{\xi'}(\delta, \delta') a^{G,F}(\delta') d\delta'$

for any $\delta \in \Delta(G)$ and $a^{G,F} \in SÎl^F_{\text{cusp}}(G)$. The map is moreover an isometry with respect to the inner product (4.9) on $SÎl_{\text{cusp}}(G)$ and

(6.6) $(a^F, b^F) = \sum_{G' \in F_{\text{cyl}}(G)} \iota(G, G')(a', b')$

on $SÎl^F_{\text{cusp}}(G)$, where $\iota(G, G') = |\text{Out}_G(G')|^{-1}$. The inner product can first be expanded as

$$\sum_{G' \in F_{\text{cyl}}(G)} \iota(G, G') \int_{\Delta_{G_{\text{cyl}}}(G')} n(\delta')^{-1} a'(\delta') b'(\delta') d\delta'',$n(\delta')^{-1} \int_{\Delta_{\text{cyl}}(G/Z)} \Theta_{\xi'}(\delta', \delta) a^G(\delta) b(\delta') d\delta' = n(\delta') \int_{\Delta_{\text{cyl}}(G/Z)} a^G(\delta) \Theta_{\xi}(\delta, \delta') b(\delta') d\delta.

Summing the integral over $G'$, we see that the constant $|\text{Out}_G(G')|^{-1}$ normalises the measure on the quotient of $G_{\text{cyl}}(G')$ by $\text{Out}_G(G')$, then applying (6.5) to $b^G$ we have

$$(a^F, b^F) = \int_{\Delta_{\text{cyl}}(G)} \int_{\Delta_{\text{cyl}}(G/Z)} n(\delta') a^G(\delta) \Theta_{\xi'}(\delta, \delta') b^F(\delta') d\delta' d\delta' = \int_{\Delta_{\text{cyl}}(G/Z)} n(\delta')^{-1} a^G(\delta) b^G(\delta') d\delta = (a^G, b^G)$$

as required. □
6.5. A conjectural transfer identity (geometric). Let us briefly speculate on the global picture in order to motivate the local study that follows. Over a global field \( \hat{F} \), Given \( G' \in \mathcal{F}(\hat{G}) \), we define \( S_{\ell}(\xi', G') \) and \( \bar{S}_{\ell} = S_{\ell}/Z(\hat{G})^{\Gamma} \). Then \( |\bar{S}_{\ell}| \) is finite if and only if \( G' \) is elliptic. Following [Art17, §2], we would like a decomposition of the global stable trace formula

\[
S_{\ell}(f) = \sum_{G' \in \mathcal{F}(\hat{G})} \iota(G, G') \hat{P}_{\ell}(f')
\]

for appropriate coefficients \( \iota(G, G') \). Similarly, for the discrete part of the stable trace formula we expect

\[
S_{\text{disc}}(f) = \sum_{G' \in \mathcal{F}(\hat{G})} \iota(G, G') \hat{P}_{\text{disc}}(f').
\]

Whereas in the case of endoscopy, the analogous coefficient \( \iota(G, G^e) \) can be expressed as a quotient of Tamagawa numbers, the work of Mok [Mok18] suggests that \( \iota(G, G^e) \) should be related to the global stable multiplicity formula, involving the cardinality \( |\bar{S}_{\ell}| \). The function \( f' \) here is the global analogue of the local stable transfer, and the linear form \( P_{\ell}^{G'} \) will be primitive in the sense that it is supported on \( \Phi_{\text{prim}}(G') = \otimes_v \Phi_{\text{prim}}(\hat{G}_v) \). We can define inductively primitive linear forms

\[
P_{\ell}^{G}(f) = S_{\ell}(f) - \sum_{\hat{G} \neq \hat{G}'} \iota(G, G') \hat{P}_{\ell}^{G'}(f'),
\]

in the case \( \hat{G}' = \hat{G} \). This can be seen as the special case of \( r = 1 \) described in [Art17], but weakened to allow for the possibility of contribution from nontempered terms, which as discussed in [Wou], does not a priori impede a putative \( r \)-trace formula. These speculative remarks notwithstanding, the important observation here is that we should expect the stable transfer \( f' \) to be primitive.

Motivated thus, we propose the following potential strengthening of Conjecture 5.1. It can be viewed as the analogue of the requirement that the Langlands-Shelstad transfer mapping is stable. Since we are not brave enough to pose it as a conjecture, we simply state it as a question.

**Question 6.4.** Does the stable transfer \( f' \) of Conjecture 5.1 lie in \( \text{PI}(\hat{G}', \hat{\xi}') \)?

In any case, we have the following special case, which is a consequence of the definitions.

**Lemma 6.5.** If \( f \in \mathcal{H}_{\text{prim}}(G, \xi) \), then \( f^* \in \text{PI}(\hat{G}^*, \hat{\xi}^*) \).

**Proof.** By assumption \( f^G(\phi) \) is supported on \( \phi \in \Phi_{\text{prim}}(G, \xi) \). It follows then that the transfer \( f'(\phi') = f(\hat{\xi}' \circ \phi) \) must vanish for any \( G' \neq G^* \). \( \square \)

Nonetheless, we shall proceed to state a stronger conjectural transfer identity. But first, we describe the endoscopic analogue when \( G \) is quasisplit. For any Levi \( M \) of \( G \), one has the usual weighted orbital integral \( I_M(\gamma, f) \) such as defined in [Art99, §4], and the subset \( \mathcal{E}_M(G) \) of endoscopic data associated to an elliptic endoscopic datum \( M^e \in \mathcal{E}(M) \) [Art99, p.227]. Define the coefficient

\[
\iota_M(G, G^e) = |Z(M^e)^{\Gamma}/Z(M)^{\Gamma}| |Z(\hat{G}^{\ell})^{\Gamma}/Z(\hat{G})^{\Gamma}|^{-1},
\]

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and \( \mathcal{E}_0^{\ell}(G) \) the complement of \( G^* \) in \( \mathcal{E}_\ell^G(G) \). Note that \( \mathcal{E}_\ell^G(G) \) contains \( G^* \) if and only if \( M^c \simeq M^* \). Now for any \( \delta^c \in \hat{\Delta}^{\ell,\text{ell}}_G(M) \), define

\[
I_M(\delta^c, f) = \sum_{\gamma \in \mathcal{F}_G^\ell(M)} \Delta_M(\delta^c, \gamma)I_M(\gamma, f),
\]

where \( I_M(\gamma, f) \) is a generalised weighted orbital integral. Assuming inductively that for any \( M^c \in \mathcal{E}_\ell^G(M), \delta^c \in \Delta_G^{\ell,\text{ell}}(M^c) \), and \( G^c \in \mathcal{E}_\ell^G(G) \), we have defined a linear form \( \hat{\tilde{S}}_M^{G^c}(\delta^c) \) on \( S\mathcal{I}(G^c, \zeta^c) \), we then define

\[
S_M^G(M^c, \delta^c, f) = I_M(\delta^c, f) - \sum_{G^c \in \mathcal{E}_\ell^G(G)} \iota_{M^c}(G, G^c)\hat{\tilde{S}}_M^{G^c}(\delta^c, f^c).
\]

Then the main Local Theorem 1 of [Art02] states that \( S_M^G(M^c, \delta^c, f) \) is stable and vanishes unless \( M^c = M^* \), which relies on the fact that the Langlands-Shelstad transfer \( f^c \) is stable. Finally, we set

\[
S_M^G(\delta, f) = S_M^G(M^*, \delta^c, f),
\]

which defines a linear form \( \hat{\tilde{S}}_M^{G^c}(\delta^c, f^c) = S_M^G(\delta, f) \) on \( S\mathcal{I}(G^*, \zeta^*) \).

Now we turn to the primitisation. For any \( M' \in \mathcal{F}_\ell^G(M) \), we define \( \mathcal{F}_{\ell,\text{pr}}^G(M) \) to the set of transfer data \( G' \) such that \( G' = M' \tilde{G} \) and \( \xi' \) restricts to \( \xi'_M \) on \( M' \). For any \( G' \in \mathcal{F}_{\ell,\text{pr}}^G(M) \), the dual group \( \bar{M}' \) of \( M' \) comes with the structure of a Levi subgroup of \( G' \). The group \( M' \) has an embedding \( M' \subset G' \) for which \( M' \subset \bar{G}' \) is a dual Levi subgroup, determined up to \( G'(F) \)-conjugacy. Fixing such an embedding for each \( G' \), we thus identify \( M' \) with a Levi subgroup of \( G' \). Moreover, given an auxiliary datum \( M^c' \) of \( M' \), we can choose an extension \( G' \) of \( G^c \) for any \( G^c \in \mathcal{F}_{\ell,\text{pr}}^G(M) \) which contains \( M' \) as a Levi subgroup. Define now the coefficient

\[
\iota_M(G, G') = |Z(\xi_M(\bar{M}'))|/|Z(\bar{M})|^{1/2}|Z(\xi'(\bar{G}'))|/|Z(\bar{G})|^{1/2},
\]

which vanishes unless \( G' \) is elliptic, and the transform

\[
S_M(\delta', f) = \int_{\Delta_G^{\ell,\text{pr}}(M)} \Theta_{\xi}(\delta', \delta)S_M(\delta, f)d\delta, \quad \delta' \in \hat{\Delta}_{G,\ell}^G(M).
\]

Note that since \( \delta' \) is elliptic, there is a unique \( M' \in \mathcal{F}_\ell^G(M) \) such that \( \delta' \) is the image of an element in \( \Delta_G^{\ell,\text{pr}}(M) \).

Assume inductively that for any \( M' \in \mathcal{F}_\ell^G(M), \delta' \in \Delta_G^{\ell,\text{pr}}(M'), \) and \( G' \in \mathcal{F}_{\ell,\text{pr}}^G(M) \), we have defined a linear form \( \hat{\tilde{P}}_M^{G'}(\delta') \) on \( P\mathcal{I}(\bar{G}', \zeta') \). Then define

\[
P_M^G(M', \delta', f) = S_M(\delta', f) - \sum_{G' \in \mathcal{F}_{\ell,\text{pr}}^G(M)} \iota_M(G, G')\hat{\tilde{P}}_M^{G'}(\delta', f).
\]

To complete the inductive definition, we would have to show in the special case of \( G \) quasisplit and \( M' = M^* \) that the distribution

\[
P_M^G(\delta, f) = P_M^G(M^*, \delta^c, f), \quad \delta \in \Delta_G(M)
\]

is primitive. Only then would we have a linear form \( \hat{\tilde{P}}_M^{G^c}(\delta^c) \) on \( P\mathcal{I}(G^*, \zeta^*) \) with

\[
\hat{\tilde{P}}_M^{G^c}(\delta^c) = P_M^G(\delta, f), \quad f \in \mathcal{E}(G, \zeta)
\]

that is the analogue of \( \hat{\tilde{P}}_M^{G^c}(\delta^c) \) for \( (G^*, M^*) \).

Let \( \theta \) be an \( F \)-isomorphism from \( G \) to another \( K \)-group \( G_1 \). Then for any function \( f \) on \( G(F) \), we have a corresponding function \((\theta f)(x) = f(\theta^{-1}x) \) on \( G_1(F) \),
and also a bijection \( \delta \to \theta \delta \) from \( \Delta(G) \) to \( \Delta(G_1) \). If \( \hat{\theta} \) is a \( \Gamma \)-isomorphism dual to \( \theta \), we can extend it by the identity on \( W_F \) to obtain an \( L \)-isomorphism \( \hat{\theta} \cdot \ell \theta : L G \to L G_1 \). Then \( \hat{\theta} \cdot \ell \theta \) maps any transfer datum \( G' \in \mathcal{F}(G) \) to a transfer datum \( \tilde{G}' \in \mathcal{F}(G_1) \), whose isomorphism class is independent of choice of \( \hat{\theta} \). We also have an isomorphism \( \theta' : \tilde{G}' \to \tilde{G}'_1 \) between extensions, whose orbit under right translation by \( \text{Aut}_G(G') \) is also independent of choice of \( \theta' \), inducing a bijection

\[
\theta' : \Delta_{G,\text{ell}}(\tilde{G}')/\text{Out}_G(G') \to \Delta_{G_1,\text{ell}}(\tilde{G}'_1)/\text{Out}_{G_1}(G'_1),
\]

and taking the union over \( G' \) we have a bijection \( \theta_F = \coprod_{G'} \theta' \) from \( \tilde{\Delta}_{G,\text{ell}}(G) \) to \( \tilde{\Delta}_{G_1,\text{ell}}(G_1) \). Similarly, \( \theta \) maps \( M \) to the Levi subgroup \( M_1 = \theta M \) of \( G_1 \), thus giving bijections from \( \Delta_{G,\text{ell}}(M) \) to \( \Delta_{G_1,\text{ell}}(M_1) \) and \( \tilde{\Delta}_{G,\text{ell}}(M) \) to \( \tilde{\Delta}_{G_1,\text{ell}}(M_1) \) respectively.

**Lemma 6.6.** \( P_{M}(M', \delta', f) \) depends only on the image of \( \delta' \) in \( \tilde{\Delta}_{G,\text{ell}}(M) \), and satisfies

\[
P_{M}(\theta M, \delta', \theta f) = P_{M}(M', \delta', f).
\]

**Proof.** We assume inductively that the lemma holds if \( G \) is replaced by any group \( \tilde{G}' \) with \( G' \in \mathcal{F}_{M}(G) \). Any element \( \theta' \in \text{Out}_M(M') \) extends to an outer automorphism of \( \tilde{G}' \) in \( \text{Out}_G(G') \). Then by the induction hypothesis we have

\[
\hat{P}_{M}(\delta', f', \theta') = \hat{P}_{M}(\delta', \theta f', \theta') = \hat{P}_{M}(\delta', \theta f', f'),
\]

where the last equality follows from the effect of the outer automorphism. From the definition (4.2) of the stable transfer factor, we see that \( \Theta_{\tilde{\zeta}}(\delta', \delta) \) depends only on the image of \( \delta' \) in \( \tilde{\Delta}_{G,\text{ell}}(M) \) because the same holds for the function \( S(\delta', \phi') \), which describe a basis for \( S_{L,\text{cusp}}(\tilde{G}', \tilde{\zeta}') \). It follows then that \( S_{M}(\delta', f) \) and \( P_{M}(M', \delta', f) \) do also. This gives the first assertion.

Moreover, by the special case of \( M = G \) of [Art99, Lemma 3.1(i)] it follows that \( f^{\theta G}(\theta \delta) = f^{\tilde{\zeta}}(\tilde{\zeta}) \) and thus applied to \( \tilde{G}' \) we have \( S(\theta \delta', \theta \phi') = S_{\tilde{\zeta}}(\delta', \phi') \). from which we conclude that

\[
\Theta_{\tilde{\zeta}}(\theta \delta', \theta \delta) = \Theta_{\tilde{\zeta}}(\delta', \delta),
\]

where \( \theta' \tilde{\zeta} \) is the \( L \)-embedding induced by \( \theta' \), and moreover \( (\theta f)' = \theta' f' \). It follows from this and the general case of [Art99, Lemma 3.1(i)] that

\[
S_{\theta M}(\theta \delta', \theta f) = S_{M}(\delta', f).
\]

Finally, applying this together with the induction hypothesis to the definition (6.9) of \( P_{M}(M', \delta', f) \), the required identity follows. \( \square \)

We now state the conjectural transfer identity for our geometric linear forms, suggested by the primisation of the stable trace formula.

**Conjecture 6.7.** \( P_{M}(M', \delta', f) \) is primitive, and vanishes unless \( M' = M^* \).

This identity lies at the heart of the trace formula, and as such appears difficult to approach. It would likely be very useful to have a geometric characterisation of such primitive distributions.
7. Stable spectral transfer factors

7.1. Surjectivity. We now turn to the spectral analogue of our constructions so far. Our main goal will be to define stable spectral transfer factors, and formulate a transfer identity parallel to Conjecture 6.7 suggested by the expected decomposition 6.7. Assume that the spectral transfer (1.1) holds. We can then define a spectral basis parallel to $\Delta_{\text{cusp}}^F(G)$, namely,

$$\Phi^F_2(G) = \prod_{G' \in \mathcal{F}_{\text{cusp}}(G)} \Phi_2(G', \zeta')/\text{Out}_G(G'),$$

which can again be written as the set of pairs $(G', \phi')$. It parametrises a basis of $SI_{\text{cusp}}^F(G)$. Also define

$$(7.1) \quad \Phi^F(G) = \prod_{M} \Phi^F_2(M)/W(M),$$

which can also be described as the union over $W_0$-orbits $\{M\}$ in $\mathcal{L}$ and $W(M)$-orbits $\{M'\}$ in $\mathcal{F}_{\text{cusp}}(M)$ of the quotient of $\Phi_2(M', \zeta')$ by $\text{Out}_M(M') \times W(M)^{M'}$, where $W(M)^{M'}$ is the stabiliser of $M'$ in $W(M)$. We again have a decomposition according to central character,

$$(7.2) \quad \Phi^F(G) = \prod_{\zeta} \Phi^F(G, \zeta).$$

With these definitions in place, we return in earnest to the surjectivity of the map $\mathcal{F}_F^\ast$ in (6.4), which is required in order to define our stable spectral transfer factors. We shall prove it unconditionally in the archimedean case, and then discuss the nonarchimedean case.

**Lemma 7.1.** Let $F$ be an archimedean local field. Then $\mathcal{F}_F^\ast$ is surjective.

**Proof.** Again relying on the local Langlands correspondence over $F$, we can identify $SI(\check{G}', \check{\zeta}')$ with the natural Schwartz space on a basis $\Phi(\check{G}', \check{\zeta}')$ of the vector space spanned by tempered, stable, $\check{\zeta}'$-equivariant characters on $\check{G}'(F)$, and similarly for $SI(G, \zeta)$ and $\Phi(G, \zeta)$. The elements in $\Phi(\check{G}', \check{\zeta}')$ are indexed by tempered Langlands parameters $\phi'$ of $\check{G}'$, and in particular decomposes into a disjoint union of cuspidal Langlands parameters attached to Levi subgroups $\check{M}'$ of $\check{G}'$. We define a space $\tilde{\Phi}^F(G, \zeta)$ analogous to $\check{\Delta}_g(G)$ in Section 3.2, which fibres over $\Phi^F(G, \zeta)$, and $SI_{\text{cusp}}^F(G, \zeta)$ can be identified with the natural equivariant Schwartz space on it as a consequence of the trace Paley-Wiener theorem for Schwartz functions on $G$ [Art94b].

Suppose that $\phi_1$ is a finite linear combination of linear forms in $\tilde{\Phi}^F(G, \zeta)$. We can assume that $\phi_1$ can be is the image of some $\phi' \in \Phi(\check{G}', \check{\zeta}')$ for some $G' \in \mathcal{F}_{\text{cusp}}(G)$, such that $a^F(\phi) = a'(\phi')$ for any $a^F \in SI^F(G, \zeta)$. The value at $\phi_1$ of any function $a^F$ in $SI^F(G, \zeta)$ is then given by a finite linear combination

$$a^F(\phi_1) = \sum_{\phi'} c_{\phi'} a'(\phi'), \quad \phi' \in \Phi(\check{G}', \check{\zeta}'), G' \in \mathcal{F}_{\text{cusp}}(G).$$

As an invariant distribution on $\check{G}'(F)$, any $\phi'$ can be identified with a locally integrable function whose restriction to $\Delta_{G}(\check{G}')$ is smooth. The set $\Delta_{G}(\check{G}')$ maps onto
an open subset of \( \tilde{\Delta}^F(G) \) with finite fibres, and we can thus write \( a^F(\phi_1) \) as
\[
\int_{\Delta(G/Z)} I^F(\phi_1, \delta_1) a^F(\delta_1) d\delta_1
\]
for some smooth, \( \zeta \)-equivariant function \( I^F(\phi_1) \) on \( \tilde{\Delta}^F(G) \), whose integral against any \( a^G = a^{G,F} \) converges with respect to the measure \( d\delta_1 \). Applying (6.3), we have then
\[
\int_{\Delta(G/Z)} I(\phi_1, \delta) a^G(\delta) d\delta, \quad a^G \in SI(G, \zeta),
\]
where
\[
I(\phi_1, \delta) = \int_{\Delta(G/Z)} I^F(\phi_1, \delta_1) \Theta_{\tilde{\xi}}(\delta_1, \delta) d\delta_1
\]
is again a smooth, \( \zeta \)-equivariant function on \( \Delta(G) \), whose integral against any \( a^G \) converges with respect to the measure \( d\delta \). Letting \( a^G \) now approximate the \( \zeta^{-1} \)-equivariant Dirac measure at \( \delta \), it follows that if the function \( a \rightarrow a^F(\phi_1), \quad a \in \mathcal{C}(G, \zeta) \)
induced by \( \phi_1 \) vanishes, then so does \( I(\phi_1, \delta) \). By the inversion formula (4.6), it follows that \( I^F(\phi_1, \delta_1) \) also vanishes on \( \tilde{\Delta}^F(G) \), and hence \( \phi_1 \) itself vanishes. The mapping \( \mathcal{F}^\prime \) is thus locally surjective.

To see that it is surjective, we can define a spectral transfer factor describing the local mapping \( a \rightarrow a^F \),
\[
a^F(\phi_1) = a'(\phi') = \int_{\mathfrak{g}_2(G, \zeta)} \Theta_{\tilde{\xi}}(\phi_1, \phi) a(\phi) d\phi, \quad a \in \mathcal{C}(G, \zeta),
\]
compatible with the decompositions (7.1) and (7.2) as below, hence with the characterisations of \( SI(G, \zeta) \) and \( SI^F(G, \zeta) \) as Schwartz spaces of functions on \( \Phi(G, \zeta) \) and \( \tilde{\Phi}^F(G, \zeta) \) respectively. Thus the surjectivity extends. \( \square \)

The proofs of surjectivity of the analogous endoscopic map for nonarchimedean fields [Art96, Lemma 3.4] and [MW16, I.4.11] both involve a reduction to the Lie algebra at the identity element, either by passing to germs of orbital integrals or Harish-Chandra descent to the Lie algebra of unipotent subgroups. In both cases, the descent of transfer factors and properties of the Fourier transform on the Lie algebra are employed. In order to discuss the latter, we need some preparations.

Let \( \mathfrak{g} \) be the Lie algebra of \( G \), and similarly \( \mathfrak{g}' \) of \( G' \). Fix a symmetric, nondegenerate \( G \)-invariant bilinear form \( B \) on \( \mathfrak{g} \) and a nontrivial additive character \( \psi_0 \) on \( F \). For any \( \varphi \in C_c^\infty(\mathfrak{g}(F)) \), we define the Fourier transform
\[
\check{\varphi}(Y) = \int_{\mathfrak{g}(F)} \varphi(X) \psi_0(B(X, Y)) dX,
\]
which acts as a linear isomorphism from \( C_c^\infty(\mathfrak{g}(F)) \) to itself. It a well-known result of Harish-Chandra that there exists a smooth, locally-integrable function
\[
i : \Gamma(\mathfrak{g}) \times \Gamma(\mathfrak{g}) \to \mathbb{C},
\]
where \( \Gamma(\mathfrak{g}) = \Gamma_{\text{reg}}(\mathfrak{g}(F)) \) is the space of regular \( G(F) \)-orbits in \( \mathfrak{g}(F) \), such that
\[
\varphi_G(X) = \int_{\Gamma(\mathfrak{g})} i(X, Y)(\check{\varphi}) dY, \quad X \in \Gamma(\mathfrak{g})
\]
for a fixed Haar measure $dY$ on $\Gamma(g)$. As with $\Gamma(G)$, we may decompose the integral into a sum of integrals over conjugacy classes of maximal tori in $G$. Taking $G$ to be quasisplit, we define the smooth function

$$ s(S, T) = |\mathcal{K}_T|^{-1} \sum_{X \to S} \sum_{Y \to T} i(X, Y), $$

where $S, T$ are regular stable $G(F)$-orbits in $g_{\text{reg}}(F)$, the sums run over the distinct $G(F)$-orbits in each respective stable orbit, and $|\mathcal{K}_T|$ is equal to the number of $Y$ in the orbit of $T$. We also have the stable analogue

$$ \varphi^G(S) = \int_{\Delta(g)} s(S, T)(\hat{\varphi})^G(T) dT, \quad S \in \Delta(g) $$

which is a consequence of [Art96, (3.3)]. We define the Lie algebra analogue of the stable transfer factor

$$ \Theta_\xi(S', T) = \int_{\Delta(g')} s'(S', T')s(d\xi'(T'), T) dT', \quad S' \in \Delta_G(g'), T \in \Delta(g), $$

where $s'$ denotes the function associated to $G'$, and $d\xi'$ is the induced map on regular semisimple elements from $g(F)$ to $g'(F)$. Then suppose the following stable analogue of Waldspurger’s kernel formula [Wal97, 1.2] holds for any $G' \in \mathcal{F}_{\text{reg}}(G)$,

$$ \int_{\Delta(g)} \Theta_\xi(S', S)s(S, T)dS = \delta_0 \int_{\Delta(g')} s'(S', T')\Theta_\xi(T', T)dT', $$

for $T \in \Delta(g')$ and $S' \in \Delta_G(g')$, where the latter is the set of regular stable $G(F)$-orbits in $g_{\text{reg}}(F)$ and $\delta_0$ is a constant depending only on $G, G'$. For any $\varphi \in C_c^\infty(g(F))$, we define the transfer

$$ \varphi'(S') = \int_{\Delta(g)} \Theta_\xi(S', S)\varphi^G(S) dS, $$

and by the preceding formulas, it is equal to

$$ \varphi'(S') = \int_{\Delta(g)} \Theta_\xi(S', S) \int_{\Delta(g)} s(S, T)(\hat{\varphi})^G(T) dT \\
= \delta_0 \int_{\Delta(g)} \int_{\Delta_G(g')} s'(S', T')\Theta_\xi(T', T)(\hat{\varphi})^G(T')dT'dT' $$

(7.4)

$$ = \delta_0 \int_{\Delta(g')} s'(S', T')(\hat{\varphi}')(T')dT'. $$

With these considerations, the surjectivity of the map $\mathcal{F}$ in the nonarchimedean case can then be shown to follow from the proposed kernel formula and the descent of transfer factors.

**Lemma 7.2.** Let $F$ be a non-archimedean local field, and assume (5.7) and (7.3). Then $\mathcal{F}$ is surjective.

**Proof.** Assume first that restriction of $\mathcal{F}$ to $ST_{\text{cusp}}(G, \zeta)$ maps onto the corresponding cuspidal subspace $ST_{\text{cusp}}^F(G, \zeta)$ of $ST^F(G, \zeta)$. Recall that the filtration on $ST(G, \zeta)$ with respect to $\mathcal{L}/W_0$ gives a grading (6.2) of the space, whereby

$$ ST(G, \zeta) = \bigoplus_{\{M\}} ST_{\text{cusp}}(M, \zeta)^{W(M)}, $$

where $ST_{\text{cusp}}(M, \zeta)^{W(M)}$.
and similarly
\[ \mathcal{S}\mathcal{T}^F(G, \zeta) = \bigoplus_{(M)} \mathcal{S}\mathcal{T}^F_{\text{cusp}}(M, \zeta)^W(M). \]

The map \( \mathcal{F}^\mathcal{T} \) is compatible with these gradings, and the transfer mapping can be identified with the corresponding transfer mapping for cuspidal functions on each \( M \).

The surjectivity of \( \mathcal{F}^\mathcal{T} \) on \( \mathcal{S}\mathcal{T}^F_{\text{cusp}}(G, \zeta) \) will follow from the corresponding surjectivity of germs on the Lie algebra. Assume for simplicity that \( G' \) is an \( L \)-group, hence \( \hat{G}' = G' \), and moreover that the central datum \((Z, \zeta)\) is trivial. Given \( a^F \in \mathcal{S}\mathcal{T}^F_{\text{cusp}}(G) \), by the relation (4.7) we see that the function
\[ a^G(\delta) = \int_{\Delta_{\text{ell}}(G')} \Theta_{\zeta}(\delta, \delta') a^F(\delta') d\delta', \quad \delta \in \Delta_{\text{ell}}(G) \]
implies
\[ a^F(\delta') = \int_{\Delta_{\text{ell}}(G')} \Theta_{\zeta}(\delta', \delta) a^G(\delta) d\delta = (\mathcal{F}^\mathcal{T}(a^G))(\delta'). \]

Together with (4.6), we see that \( a^F \) lies in the image of \( \mathcal{S}\mathcal{T}^F_{\text{cusp}}(G) \) if and only if \( a^G \in \mathcal{S}\mathcal{T}^F_{\text{cusp}}(G) \). We may assume that the components \( a' \) of \( a^F \) are nonzero for exactly one \( G' \in \mathcal{F}_{\text{ell}}(G) \). It suffices then to show that for \( a^F \in \mathcal{S}\mathcal{T}^F_{\text{cusp}}(G) \), the function
\[ a^G(\delta) = \int_{\Delta_{\text{ell}}(G')} \Theta_{\zeta}(\delta, \delta') a'(\delta') d\delta', \quad \delta \in \Delta_{\text{ell}}(G) \]
lies in \( \mathcal{S}\mathcal{T}^F_{\text{cusp}}(G) \). Let \( d \) be a fixed elliptic semisimple conjugacy class in \( G(F) \), and let \( \mathcal{S}\mathcal{G}_{\text{cusp}}(G, d) \) be the space of germs of functions in \( \mathcal{S}\mathcal{T}^F_{\text{cusp}}(G) \) around \( d \). Let \( d'_1, \ldots, d'_n \) be representatives of \( \text{Out}_{G'}(G') \)-orbits of stable conjugacy classes in \( G(F) \) that are images of \( d \), chosen such that \( G'_{d'_j} \) is quasisplit. The image of \( a^G \) in \( \mathcal{S}\mathcal{G}_{\text{cusp}}(G, d) \) depends only on the image of \( a' \) in the spaces of germs of functions in \( \mathcal{S}\mathcal{T}^F_{\text{cusp}}(G', d'_j) \) around \( d'_j \). We may assume that these images are nonzero for exactly one \( j \), so that the integral (7.5) is supported on points close to \( d'_j \). Then for any \( \delta, \delta' \) close to \( d, d' \) respectively and assuming the descent of transfer factors, Lemma 5.12 reduces \( a^G(\delta) \) and \( a'(\delta') \) to orbital integrals on \( G_d \) and, \( G'_{d'_j} \). In particular, \( d \) is central in \( G_d \), and by Lemma 4.4(i) it suffices to show that \( a^G(\delta) \) lies in \( \mathcal{S}\mathcal{G}_{\text{cusp}}(G, 1) \) for \( \delta \) close to \( 1 \).

Let \( \mathcal{S}\mathcal{G}_{\text{cusp}}^\infty(\mathfrak{g}(F)) \) be the subspace of cuspidal functions on \( \mathfrak{g}(F) \), and let \( \mathcal{S}\mathcal{G}_{\text{cusp}}(\mathfrak{g}) \) be the space of germs of stable orbital integrals of such functions around \( 0 \). It is a finite dimensional space of germs of functions on \( \Delta(\mathfrak{g}) \). The exponential map gives a linear bijection from \( \mathcal{S}\mathcal{G}_{\text{cusp}}(\mathfrak{g}) \) to \( \mathcal{S}\mathcal{G}_{\text{cusp}}(G, 1) \). (This passage from orbital integrals on the group to germs on the Lie algebra can also be alternately described by Harish-Chandra descent, c.f. [MW16, I.4.1].) Define also the finite dimensional vector space
\[ \mathcal{S}\mathcal{G}_{\text{cusp}}^\mathcal{T}(\mathfrak{g}) = \bigoplus_{G' \in \mathcal{F}_{\text{ell}}(G)} \mathcal{S}\mathcal{G}_{\text{cusp}}(\mathfrak{g}')^\text{Out}_{G'}(G'), \]
and let
\[ \mathcal{F}^\mathcal{T} : \mathcal{S}\mathcal{G}_{\text{cusp}}(\mathfrak{g}) \to \mathcal{S}\mathcal{G}_{\text{cusp}}^\mathcal{T}(\mathfrak{g}) \]
be the Lie algebra analogue of \( \mathcal{F}^\mathcal{T} \) on germs. We would like to show that it is surjective. Fix \( G' \in \mathcal{F}_{\text{ell}}(G) \) and consider the image of an arbitrary \( \mathfrak{g}'(S') \in \mathcal{S}\mathcal{G}_{\text{cusp}}^\mathcal{T}(\mathfrak{g}) \).
$S_{\mathrm{cusp}}(g')_{\mathrm{Out}(G')}$ in $S_{\mathrm{cusp}}(g)$, where $S'$ is belongs to the $G$-regular elliptic quotient $\Delta_{G,\mathrm{cusp}}(g')/\mathrm{Out}_G(G')$. In particular, $g'$ is supported on the regular elliptic locus. By (7.4), we can express $g'(S')$ as a finite linear combination

$$g'(S') = \sum_{i=1}^{n} c_i s'(S', T_i), \quad T_i \in \Delta_{G,\mathrm{cusp}}(g').$$

We may choose the bilinear form $B'$ on $g'$ to be invariant under $\mathrm{Out}_G(G')$, so that the coefficients $c_i$ are constant on $\mathrm{Out}_G(G')$ orbits. By Howe’s finiteness theorem applied to $g'(F)$ [How74, Theorem 2], we can choose a compact neighbourhood $V_i$ of $T_i$ in $\Delta_{G,\mathrm{cusp}}(g')$ for each $i$, such that $s'(S', T') = s'(S', T'_i)$ for all $T' \in V_i$ and $S'$ sufficiently small. Altogether, this implies that we can choose a $C^\infty$-function $\alpha'$ on $\Delta_{G,\mathrm{cusp}}(g')_{\mathrm{Out}(G')}$ such that

$$g'(S') = \delta_0 \int_{\Delta_{G,\mathrm{cusp}}(g')} s'(S', T') \alpha'(T') dT'$$

for all $S'$ sufficiently close to 0. The adjoint relations of Proposition 4.6 have Lie algebra analogues. Recall that the proof in the group case relies on the stable local trace formula, and similarly one may employ a stable local trace formula for the Lie algebra, which is simpler, to deduce the necessary relations. This allows us to invert the map

$$C_c^\infty(\Delta_{\mathrm{cusp}}(g)) \to \bigoplus_{G'_i \in F_{\mathrm{cusp}}(G)} C_c^\infty(\Delta_{G,\mathrm{ell}}(g'_i)),$$

thereby giving a function $\varphi_0 \in C_c^\infty(\Phi_{\mathrm{reg},\mathrm{cusp}}(F))$ such that for any $G'_i \in F_{\mathrm{cusp}}(G)$, we have $\varphi_0^{G'_i}$ equals $\alpha'$ if $G'_i = G'$ and is trivial otherwise. Extending by zero, we can find a function $\varphi \in C_c^\infty(g(F))$ such that $\tilde{\varphi} = \varphi_0$. By (7.4), it follows that

$$\varphi'(S') = \varphi_{G'}(S') = g'(S'),$$

and $\varphi^{G'_i} = 0$ for any $G'_i \neq G'$. Thus the map $\tau^x$ is surjective.}

### 7.2. Definition

Our discussion here now parallels the geometric case, so we may be brief. Let $f \in \mathcal{C}_{\mathrm{cusp}}(G, \zeta)$. For any $G' \in F_{\mathrm{ell}}(G)$, the transfer $f'$ is a function in $\mathcal{S}_{\mathrm{cusp}}(G', \zeta')_{\mathrm{Out}(G')}$ and $f'_{(\phi')}$ is defined for every $\phi' \in \Phi_2(G', \zeta')$. If $\mathcal{F}^x$ is surjective, we may define a stable spectral transfer factor $\Theta_x(\phi', \phi)$ to be any distribution on $\Phi_2(G', \zeta') \times \Phi_2(G, \zeta)$ such that the identity

$$f'(\phi') = \int_{\Phi_2(G, \zeta)} \Theta_x(\phi', \phi)f(\phi)d\phi$$

holds. The stable transfer factors can again be extended to distributions on $\mathcal{F}(G) \times \Phi(G)$ as follows. Set $\Theta_x(\phi', \delta)$ to be zero unless there is an $M$ such that $(\phi', \phi)$ belongs to the Cartesian product of $\Delta_{G,\mathrm{ell}}(M)/W(M)$ with $\Delta_{G,\mathrm{cusp}}(M)/W(M)$. If there is such an $M$, then $(\phi', \phi)$ is the image of a pair $(\phi_M', \phi_M)$ in $\Phi_2^x(M) \times \Phi_2(M)$, and we set

$$(7.6) \quad \Theta_x(\phi', \phi) = \Theta_x,M(\phi', \phi) = \sum_{w \in W(M)} \Theta_x,M'(\phi'_M, w\phi_M),$$

where again each sum contains at most one nonzero term, and depends only on $\phi'$ and $\phi$. Finally, define the adjoint spectral transfer factor

$$\Theta_x(\phi, \phi') = n(\phi')^{-2}\Theta_x(\phi', \phi).$$
which complements the adjoint stable geometric transfer factors quite nicely. As in the geometric case, their definition is imposed upon us by the adjoint relations they satisfy, parallel to Proposition 4.6.

**Proposition 7.3.** Given \( \phi', \phi'_1 \in \Phi^F(G') \) for \( G' \in \mathcal{F}_{\text{cusp}}(G) \), we have

\[
\int_{\Phi(G)} n(\phi)\Theta_{\xi}((\phi', \phi)\Theta_{\xi}(\phi, \phi'_1))d\phi = n(\phi)\delta(\phi, \phi'_1).
\]

Similarly, given \( \phi, \phi_1 \in \Phi(G) \), we have

\[
\int_{\Phi(G')} n(\phi')\Theta_{\xi}((\phi', \phi)\Theta_{\xi}(\phi_1, \phi'))d\phi' = n(\phi)\delta(\phi, \phi_1).
\]

**Proof.** Since we do not have an explicit description of the spectral transfer factors, the proof will follow instead from interpreting the linear isometry \( \mathcal{I}^F \) spectrally. We recall the spectral form of the inner product on \( \mathcal{L}_{\text{cusp}}(G) \),

\[
(a^G, b^G) = \int_{\Phi_2(G)} n(\phi)^{-1}a^G(\phi)b^G(\phi)d\phi,
\]

whenever \( a^G, b^G \in \mathcal{L}_{\text{cusp}}(G) \). We then have the spectral form of the inner product on \( \mathcal{I}^F_{\text{cusp}}(G) \) in (6.6),

\[
(a^F, b^F) = \sum_{G' \in \mathcal{F}_{\text{cusp}}(G)} \iota(G, G') \int_{\Phi_2(G')} n(\phi)^{-1}a(\phi')b(\phi')d\phi'.
\]

Then defining the spectral analogue of the inverse of \( \mathcal{I}^F \),

\[
a^G(\phi) = n(\phi)\int_{\Phi_2(G)} n(\phi')\Theta_{\xi}(\phi', \phi)\Phi_{\xi}d\phi',
\]

and using the definition of \( \Theta_{\xi}(\phi', \phi) \) and its adjoint above, it follows by the same argument as in the proof of Proposition 6.3 that \( (a^F, b^F) = (a^G, b^G) \). In particular, \( \Theta_{\xi}(\phi', \phi) \) and \( \Theta_{\xi}(\phi, \phi') \) represent kernels of inverse transforms of each other. \( \square \)

**Remark 7.4.** The stable spectral transfer factors that we have defined are implicitly given in terms of the geometric transfer. It would be useful to have an explicit construction of these, in terms of the component group \( \pi_0(S^0) \), say.

7.3. **A conjectural transfer identity (spectral).** We first review the distributions that occur in the spectral side of the trace formula. Recall the canonically normalised weighted character introduced in [Art98, §2],

\[
J_M(\pi, f) = \text{tr}(\mathcal{M}_M(\pi, P)\mathcal{I}_P(\pi, f)),
\]

where \( \mathcal{I}_P(\pi) \) is the induced representation of \( G \) obtained from \( \pi \in \Pi_{\text{unit}}(M, \zeta) \) and \( \mathcal{M}_M(\pi, P) \) is an operator constructed in a certain way from unnormalized intertwining operators, which we shall describe below. Let \( a_{M,F} = \{H_M(x) : x \in M(F)\} \). We can then define for any pair \( (\pi, X) \) in \( \Pi(M, \zeta) \times a_{M,F} \), the distribution

\[
J_M(\pi, X, f) = \int_{ia_M^*} J_M(\pi, f)e^{-\lambda(X)}d\lambda, \quad f \in \mathcal{E}(G, \zeta)
\]

if \( J_M(\pi, f) \) is regular for \( \lambda \in ia_M^* \), for example, if \( \pi \) is unitary. Whereas for more general representations \( \pi \in \Pi(M, \zeta) \) we define

\[
J_M(\pi, X, f) = \sum_{P \in P(M)} \omega_P J_M(\pi, f)e^{-\epsilon_P(X)},
\]

where \( \omega_P \) is the weight of the representation \( \pi \) on \( P \).
where for each $P \in \mathcal{P}(M)$, $\varepsilon_P$ is a small point in the positive chamber $(\mathfrak{a}_P^*)^+$ and $\omega_P = \text{vol}(\mathfrak{a}_P^* \cap B) \text{vol}(B)^{-1}$, where $B$ is a ball in $\mathfrak{a}_M$ centered at the origin. The two definitions are compatible by a contour shift. More generally, we define the function

$$J_{M,\mu}(\pi, X, f) = J_M(\pi_\mu, X, f)e^{-\mu(X)} = \int_{\mu + i\mathfrak{a}_M} J_M(\pi_\lambda, f)e^{-\lambda(X)}d\lambda,$$

which is locally constant as a function of $\mu \in \mathfrak{a}_M^*$ on the complement of a finite set of affine hyperplanes.

The invariant weighted characters are then defined inductively by the relation

$$I_M(\pi, X, f) = J_M(\pi, X, f) - \sum_{L \in \mathfrak{L}(M)} \hat{I}_M(\pi, X, \phi_L(f)),$$

where the map $\phi_M$ is based on the construction in [Art98, §2] using normalised weighted characters, which we briefly recall here. Suppose first that $f$ belongs to the Schwartz space $\mathcal{C}(G, \zeta)$. Then $\phi_M(f)$ is defined to be the function on $\Pi_{\text{temp}}(M, \zeta)$ such that $\phi_M(f, \pi) = J_M(\pi, f) = \text{tr}(\mathcal{M}_M(\pi, P)\mathcal{I}_P(\pi, f))$ for $P \in \mathcal{P}(M)$ and $\pi \in \Pi_{\text{temp}}(M, \zeta)$. The operator

$$\mathcal{M}_M(\pi, P) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} (\mathcal{M}_Q(\Lambda, \pi, P))\theta_Q(\Lambda)^{-1},$$

with

$$\theta_Q(\Lambda) = \text{vol}(\mathfrak{a}_M^*/Z(\Delta)_{\mathfrak{q}})^{-1} \prod_{\alpha \in \Delta_Q} \Lambda(\alpha^\vee),$$

is defined as part of Arthur’s theory of $(G, M)$-families, where the relevant $(G, M)$-family is a tensor product of $(G, M)$-families

$$\mathcal{M}_Q(\Lambda, \pi, P) = \mu_Q(\Lambda, \pi, P)\mathcal{I}_Q(\Lambda, \pi, P), \quad Q \in \mathcal{P}(M), \quad \Lambda \in i\mathfrak{a}_M^*$$

defined for $\pi$ in general position. Here $\mu_Q(\Lambda, \pi, P) = \mu_Q(P(\pi))^{-1}\mu_Q(P(\pi_\Lambda))$, and the functions $\mu_Q(P(\pi_\Lambda))$ are Harish-Chandra’s canonical family of $\mu$-functions.

The invariant weighted characters are stabilised as follows. Given a pair $(\phi, X)$ in $\Phi_{\mathfrak{g}}(M, \zeta) \times \mathfrak{a}_{M, F}$, define the invariant linear form

$$I_M(\phi, X, f) = \sum_{\pi \in \Pi(M, \zeta)} \Delta_M(\phi, \pi)I_M(\pi, X, f),$$

where $\Delta_M(\phi, \pi)$ is the spectral transfer factor defined in [Art02, §5], and for $f$ in $\mathcal{C}(G, \zeta)$. If $G$ is general, define for $\phi' \in \Phi(M', \zeta')$ the linear form

$$I_M^\mathfrak{g}(\phi', X, f) = \sum_{G' \in \mathcal{F}_{M}(G)} \iota_{M'}(G, G')S_M^\mathfrak{g}(\phi', f') + \varepsilon(G)S_M^\mathfrak{g}(M', \phi', X, f),$$

with the requirement that

$$I_M^\mathfrak{g}(\phi', X, f) = I_M(\phi, X, f)$$

in the case that $G$ is quasisplit. Here $\varepsilon(G)$ is equal to 1 if $G$ is quasisplit and 0 otherwise. If $\phi'$ and $M'$ are locally relevant to $M$, Proposition 6.4 of [Art02] allows us to define the endoscopic form $I_M^\mathfrak{g}(\pi, X, f)$ by inversion of the formula

$$I_M^\mathfrak{g}(\phi', X, f) = \sum_{\pi \in \Pi(M, \zeta)} \Delta_M(\phi', \pi)I_M^\mathfrak{g}(\pi, X, f).$$
The distributions $P^G_M(\pi, X, f)$ and $S^G_M(M', \phi', X, f)$ are then the main objects appearing in the spectral side of endoscopic and stable trace formulas respectively. In the case that $G$ is quasisplit and $M' = M^*$,

$$S^G_M(\phi, X, f) = S^G_M(M^*, \phi^*, X, f).$$

It is again stable and vanishes unless $M' = M^*$ by the main Local Theorem 2 of [Art02].

We can now formulate the conjectural spectral transfer identity parallel to Conjecture 6.7. Define the spectral analogue

$$S_M(\phi', X, f) = \int_{\Phi_2(M)} \Theta_{\tilde{G}}(\phi', \phi)S_M(\phi, X, f)d\phi,$$

Assume inductively that for any $M' \in \mathcal{F}_{\text{st}}(M), \phi' \in \Phi_2(\tilde{M}'), \tilde{G}', and G' \in \mathcal{F}_{M'}^G(G)$, we have defined a linear form $\tilde{P}^G_{M'}(\phi')$ on $\Pi(\tilde{G}', \tilde{\zeta}')$. Then define

$$P^G_{M'}(M', \phi', X, f) = S_M(\phi', X, f) - \sum_{G' \in F_{M'}^G(G)} i_{M'}(G, G')\tilde{P}^G_{M'}(\phi', f).$$

To complete the inductive definition, we would have to show in the special case of $G$ quasisplit and $M' = M^*$ that the distribution

$$P^G_{M'}(\phi, f) = P^G_{M'}(M^*, \phi^*, f), \quad \phi \in \Phi(M, \zeta)$$

is primitive. Only then would we have a linear form $\tilde{P}^G_{M'}(\phi^*)$ on $\Pi(G^*, \zeta^*)$ with

$$\tilde{P}^G_{M'}(\phi^*) = P^G_{M'}(\phi, f), \quad f \in \mathcal{C}(G, \zeta),$$

that is the analogue of $\tilde{P}^G_{M'}(\phi')$ for $(G^*, M^*)$.

We then pose the following analogue of Conjecture 6.7.

**Conjecture 7.5.** $P^G_{M'}(M', \phi', f)$ is primitive, and vanishes unless $M' = M^*$.

As in the stabilisation of the trace formula in [Art02], we can expect that these two conjectures will form the local basis for the primitisation of the trace formula (6.7).

### 7.4. Concluding remarks

We conclude with some provisional observations motivated by the endoscopic classification [Art13], which we hope will guide future study. For the remainder of this paper we will let $F$ be a number field and $G$ a connected reductive group defined over $F$. Let $\pi$ be an automorphic representation of $G(\mathbb{A})$. It is a restricted tensor product $\pi = \otimes_v \pi_v$ of irreducible representations of $G_v = G(F_v)$, unramified for almost all $v$. If $\pi_v$ is unramified over $F_v$ nonarchimedean, then $G(F_v)$ is also unramified and quasisplit, and the action of $W_{F_v}$ on $\tilde{G}$ factors through the infinite cyclic quotient $W_{F_v}/I_{F_v} = \langle \text{Frob}_v \rangle$, where $I_{F_v}$ is the inertia subgroup with canonical generator $\text{Frob}_v$. The $\tilde{G}$-orbit of homomorphisms $\phi_v \in \Phi(G_v)$ to which $\pi_v$ corresponds factors through the quotient

$$L_{F_v}/(I_{F_v} \times SU(2)) = W_{F_v}/I_{F_v}.$$

The resulting mapping that sends $\pi_v$ to $c(\pi_v) = \phi_v(\text{Frob}_v)$ is a bijection from the set of unramified representations of $G(F_v)$, relative to any given hyperspecial maximal compact subgroup, and the set of semisimple $\tilde{G}$-orbits in $\mathbb{L}G_v$ that project to $\text{Frob}_v$. Let $c_v(\pi)$ be the image of $c(\pi_v)$ in $L\tilde{G}$ under the embedding of $L\tilde{G}_v$ into $L\tilde{G}$, canonical up to conjugation. Thus to any $\tilde{\pi}$, we have a family of semisimple conjugacy classes

$$c^S(\pi) = \{c_v(\pi) : v \notin S\}$$
in $L G$, where $S$ is a finite set of places of $F$ outside of which $G$ is unramified. Let $C_{\text{aut}}^S(G)$ be the set of equivalence classes of such families of conjugacy classes $c^S$, where two families $c^S, (c_1)^S$ are equivalent if if $c_v = (c_1)_v$ for almost all $v$.

Let us temporarily suppose the existence of the automorphic Langlands group $L_F$ (an assumption stronger that Functoriality). It can be described as a locally compact extension

$$1 \to K_F \to L_F \to W_F$$

of the global Weil group $L_F$ by some compact connected group $K_F$. The set $\Psi(G)$ of $L$-homomorphisms $\psi : L_F \times SU(2) \to L G$ then parametrises the automorphic spectrum of $G(\mathbb{A})$. In other words, it should be in bijection with the direct limit

$$C_{\text{aut}}(G) = \lim_{S \to} C_{\text{aut}}^S(G),$$

which we denote by $\psi \to c(\psi)$. We can decompose $\psi$ uniquely into a tensor product $\mu \otimes \nu$ of irreducible representations of each factor. As usual, the assignment

$$\phi_\psi : u \to \psi \left( u, \begin{pmatrix} |u|^{1/2} & 0 \\ 0 & |u|^{-1/2} \end{pmatrix} \right), \quad u \in L_F$$

associates an $L$-parameter $\phi_\psi \in \Phi(G)$ to any $A$-parameter $\psi \in \Psi(G)$.

Let $G' \in \mathcal{F}(G)$ with auxiliary datum $(G', \xi')$. Let $c' = (c')^S = \{c_v : v \notin S\}$ be a family of conjugacy classes in $G'$ whose image $\xi'(c')$ in $L G$ equals $c$, and whose image $\bar{c'} = \xi'(c')$ in $L G'$ equals $c^S(\pi')$ for an automorphic representation $\pi'$ parametrised by $\Phi(G')$. Any $c'$ transfers to a family of conjugacy classes $c$ of $G$, but without functoriality we cannot yet say that $c$ lies in $C_{\text{aut}}(G)$. To sidestep this, we define a larger set

$$C_h(G) = \lim_{S \to} C_h^S(G)$$

of all equivalence classes of families $c^S$ of conjugacy classes in $L G$ such that $c_v$ projects onto the Frobenius class in $W_{F_v}$ for all $v$. There is a canonical mapping from $C_h(G')$ to $C_h(G)$. In relation to Remark 6.1, to any $c^S \in C_{\text{aut}}^S(G)$ and representation $r$ of $L G$, we can associated the automorphic $L$-function $L(s,c^S,r)$. Assume that it has meromorphic continuation to $s = 1$, then it follows from standard properties of $L$-functions on $GL(n)$ that

$$\text{ord}_{s=1}(L(s,c^S,r)) = m(\pi,r) \geq [r : 1_L G],$$

where the right hand side denotes the multiplicity of the trivial representation of $L G$ in $r$. We might call $c^S$ primitive (or thick) if the inequality is an equality. But this spectral definition is difficult to employ in practice, so we prefer to call $c^S$ primitive if it is not an image of $(c')^S$ for any $G' \in \mathcal{F}(G)$.

Referring to [Art13, §3] for details, the $t$-discrete part of the invariant trace formula

$$I_{\text{disc},t}(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_G^M|^{-1} \sum_{W(M)_{\text{reg}}} |\det(w - 1)_{g_{st}}|^{-1} \text{tr}(M_{\phi}(w,\chi)\mathcal{I}_{\phi}(\chi,f)),$$

where in particular $\mathcal{I}_{\phi}(\chi,f)$ is the character of an induced representation of a parabolic $P$ with Levi $M$, composed of irreducible constituents $\pi$ whose archimedean infinitesimal character has norm $t$. Define $C_{\text{aut}}(G,\chi)$ to be the subset of classes in...
\[ \mathcal{C}_{\text{aut}}(G) \] whose central characters \( \chi \) are compatible with \( \zeta \). Then for any such \( c \), let 
\[ I_{P,\pi}(\chi) \]
be the subrepresentation of \( I_{P,t}(\chi) \) corresponding to \( \pi \), and define
\[ I_{P,t,c}(\chi, f) = \bigoplus_{\pi, c(\pi) = c} I_{P,\pi}(\chi, f). \]
This yields a decomposition
\[ I_{\text{disc}, t}(f) = \sum_{c \in \mathcal{C}_{\text{aut}}(G, \chi)} I_{\text{disc}, t,c}(f), \]
where we set \( I_{\text{disc}, t,c}(f) = 0 \) for \( c \) in the complement of \( \mathcal{C}_{\text{aut}}(G, \chi) \), and
\[ I_{\text{disc}, t,c}(f) = \sum_{c^e \rightarrow c} I_{\text{disc}, t,c^e}(f), \]
with the sum is taken over the preimage of \( c \) in \( \mathcal{C}_{\text{aut}}(G, \chi) \). By [Art13, Lemma 3.3.1], a similar decomposition holds for \( t \)-discrete part of the stable trace formula \( S_{\text{disc}, t}(f) \), and moreover
\[ S_{\text{disc}, t,c}(f) = \sum_{\psi \in \mathcal{C}_{\text{ell}}(G)} \vartheta(G, G') \sum_{c' \rightarrow c} \hat{S}_{\text{disc}, t,c^e}(f^e), \]
where \( c^e \) runs over classes in \( \mathcal{C}_{\text{ell}}(G, \chi^e) \) that map to \( c \), where the endoscopic versions of the preceding objects are defined in the natural way. Analogous to this, it is then natural to ask whether there exist explicit coefficients \( \vartheta(G, G') \) for \( G' \in \mathcal{F}_{\text{ell}}(G) \) such that the decomposition
\[ S_{\text{disc}, t,c}(f) = \sum_{\psi \in \mathcal{C}_{\text{ell}}(G)} \vartheta(G, G') \sum_{c' \rightarrow c} \hat{P}_{\text{disc}, t,c^e}(f^e) \]
holds. A priori, the pair \((G', c')\) need not exist uniquely, even up to equivalence, in which case the sum will have more than one nonzero contribution. These decompositions can be described in terms of parameters where the bijection \( \psi \rightarrow c(\psi) \) is available, but the important observation is that this last expression can be stated without assuming the existence of \( L_F \) nor the meromorphic continuation of automorphic \( L \)-functions, and represents an identity to be proved.

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