Abstract

We study equivalence classes relating to the Kazhdan-Lusztig $\mu(x, w)$ coefficients in order to help explain the scarcity of distinct values. Each class is conjectured to contain a “crosshatch” pair. We also compute the values attained by $\mu(x, w)$ for the permutation groups $S_{10}$ and $S_{11}$.

MSC: 05E10, 20F55

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1 Introduction

The Kazhdan-Lusztig polynomials, introduced in [16], arose in the context of constructing representations of the Hecke algebra associated to a Weyl group. It was soon apparent that these polynomials encode important information relating to geometry and representation theory. For example, they encode the singularities of Schubert varieties and the multiplicities of irreducibles in Verma modules [11, 9, 16]. They are also of interest from a purely combinatorial viewpoint (see [4]).

We restrict our attention to the type-$A$ case in which there is one Kazhdan-Lusztig polynomial $P_{x,w}(q)$ associated to every pair of permutations $x, w \in S_n$. Kazhdan and Lusztig give a simple recursion for these polynomials in their original paper (see Section 2.2 below). However, our combinatorial understanding of these polynomials is still far from complete. For example, there is neither a combinatorial proof that the coefficients of $P_{x,w}(q)$ are nonnegative nor a closed formula for the degree of a given polynomial. (A non-combinatorial proof of nonnegativity arises from the interpretation of the coefficients of Kazhdan-Lusztig polynomials in terms of intersection cohomology [17].) The reason these problems are still open is that there is a correction term in the recursion that is controlled by a poorly understood number, $\mu(x, w)$. While there are known to be a few simple, combinatorial necessary conditions for $\mu(x, w)$ to be nonzero, these conditions are by no means sufficient. In fact, there are no nontrivial sufficient conditions known. A combinatorial rule for the value $\mu(x, w)$ would likely lead to insights wherever Kazhdan-Lusztig polynomials arise.

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A major difficulty in the study of these $\mu$-coefficients is that (as shown in [20]) $S_{10}$ is the smallest symmetric group for which $\mu(x, w)$ can be anything other than $0$ or $1$. There is little overlap between what is computationally feasible and what is computationally illuminating. Nonetheless, there are a number of important combinatorial results regarding these polynomials. See the book by Björner and Brenti [4] for an overview and the papers of Brenti (such as [7] and [8]) in particular.

The organization of the paper is as follows. Section 2 provides the necessary definitions while Section 3 outlines the properties of $\mu(x, w)$ we will be using from the literature. The results of this paper are of two types. First, we present new data regarding the values $\mu(x, w)$ takes; how we do this is outlined in Section 4.2. Set $M(n) = \{\mu(x, w) : x, w \in S_n\} \setminus \{0\}$.

**Theorem 1.** We have
- $M(10) = \{1, 4, 5\}$,
- $M(11) = \{1, 3, 4, 5, 18, 24, 28\}$ and
- $M(12) \supseteq \{1, 2, 3, 4, 5, 6, 7, 8, 18, 23, 24, 25, 26, 27, 28, 158, 163\}$.

Particular pairs $x, w$ realizing each of these values are given in Table 2. The only $\mu$-values that have already appeared in the literature for $S_n$ are $\{0, 1, 2, 3, 4, 5\}$.

We also offer computer code [21] that can quickly produce a database of all Kazhdan-Lusztig polynomials in $S_{10}$; this code is discussed in Section 4.1. There are over one billion “extremal pairs” $(x, w)$ in $S_{10}$ for which one might hope $\mu(x, w) > 0$. More than 100 million of these pairs cannot be reduced to equivalent pairs in smaller symmetric groups. Altogether, approximately one million different polynomials appear. Even stored efficiently this yields a gigabyte of data. The comparable database for $S_{11}$ would be on the order of 50 times larger.

Second, we consider the question of why there are so few different values of $\mu(x, w)$. For example, in $S_{10}$ there are 664,752 non-covering pairs $x < w$ for which $\mu(x, w) > 0$. Yet, the only nonzero values taken are $1, 4$ and $5$. We explain this in Section 4.3 by showing that for $S_{10}$ and $S_{11}$, the $\mu$-positive pairs fall into a handful of equivalence classes. The $\mu$-coefficient is constant on each class by construction. The equivalence relation, $\sim$, is defined in Section 4.3; the corresponding class of a pair $(x, w)$ is denoted $[\left[x, w\right]]$. A class is $n$-minimal if it does not intersect $S_m$ for $m < n$. Pairs in $n$-minimal classes are also referred to as $n$-minimal themselves. As a consequence of Theorem 1 the number of 10- and 11-minimal classes is at least 2 and 4, respectively.

**Theorem 2.** The 2-minimal class $[[01, 10]]$ is the only class intersecting $S_m$ for any $m < 10$. The number of 10- and 11-minimal classes is at most 4 and 7, respectively.

Finally, in Section 5 we speculate that each $\sim$-equivalence class contains a “crosshatch” pair.

# 2 Definitions

## 2.1 The symmetric group

The symmetric group, $S_n$, has the following presentation as a Coxeter group:

$$S_n = \langle s_1, \ldots, s_{n-1} : s_i^2 = 1, \quad s_is_{i\pm1}s_i = s_{i\pm1}s_is_{i\pm1}, \quad \text{and} \quad s_is_j = s_js_i, \text{ for } |i - j| > 1 \rangle. \quad (1)$$
We write \( S \) for the set of generators \( \{s_1, \ldots, s_{n-1}\} \). The group \( S_n \) is often described as the group of bijections from \( \{0, 1, \ldots, n-1\} \) to itself (i.e., permutations) under the usual function composition. From this perspective, it is most convenient to identify the generator \( s_i \) with the adjacent transposition that switches \( i-1 \) and \( i \). For clarity in examples, we will write \( a \) for 10, \( b \) for 11, etc. One-line notation for \( \sigma \in S_n \) lists the elements \( [\sigma(0), \sigma(1), \ldots, \sigma(n-1)] \) in order.

We often omit commas and brackets. For example, the permutation \( \sigma \) there is a 1 in the 10th column from the left and \( \sigma(11) = 5 \). For clarity in examples, we will write \( [5, 4, 3, 2, 1, 0] \) or simply 543210.

The group \( S_n \) has the structure of a ranked poset as follows. An inversion of a permutation \( w = [w(0), w(1), \ldots, w(n-1)] \) is a pair \( i < j \) for which \( w(i) > w(j) \). The length of \( w \), \( \ell(w) \), is the total number of inversions. The rank of an element is then given by its length. To define the partial order under which we will be relating our elements, we first make two auxiliary definitions. Let \( x, w \in S_n \) and \( p, q \in \mathbb{Z} \). Define \( r_w(p, q) = |\{i \leq p : w(i) \geq q\}| \) and the difference function \( d_{x,w}(p,q) = r_w(p,q) - r_x(p,q) \). Then the Bruhat partial order, \( \leq \), is determined by setting \( x \leq w \) if and only if \( d_{x,w}(p,q) \geq 0 \) for all \( p,q \). This definition is equivalent to more common ones such as the tableau criterion (cf. [2] [11] [14]).

For a permutation \( w \), let \( D_w \) denote the permutation matrix oriented such that for each \( i \) there is a 1 in the \( i \)-th column from the left and \( w(i) \)-th row from the bottom. We will frequently display a pair of permutations \( x \) and \( w \) graphically using Bruhat pictures: Such a picture consists of \( D_w \) and \( D_x \) overlaid along with shading given by the difference function. An example is given in Figure 1. Entries of \( D_x \) (resp., \( D_w \)) are denoted by black disks (resp., circles). Positions corresponding to 1s of both \( D_x \) and \( D_w \) (termed capitols) are denoted by a black disk and a larger, concentric circle. Shading denotes regions in which \( d_{x,w} \geq 1 \). Successively darker shading denotes successively higher values of \( d_{x,w} \).

![Figure 1: Bruhat picture for \( x = [2,0,4,1,3,5], w = [5,2,3,1,4,0] \).](image)

Finally, there are two sets we associate to any permutation \( w \). We define the right descent set of \( w \), \( \text{rds}(w) \), as \( \{s \in S : ws < w\} \). Similarly, the left descent set is \( \text{lds}(w) = \{s \in S : sw < w\} \).

### 2.2 Kazhdan-Lusztig polynomials

We now define the Kazhdan-Lusztig polynomials \( P_{x,w}(q) \) associated to pairs of elements \( x, w \in S_n \). For motivation and more general definitions applicable to any Coxeter group, we refer the reader to [14] [16]. Set

\[
\mu(x,w) = \text{coefficient of } q^{\ell(w) - \ell(x) - 1/2} \text{ in } P_{x,w}(q)
\]

and define \( c_s(x) = 1 \) if \( xs < x \); \( c_s(x) = 0 \) if \( xs > x \). We have the following paraphrased theorem of Kazhdan and Lusztig:

**Theorem 3** ([10]). There is a unique set of polynomials \( \{P_{x,w}(q)\}_{x,w \in S_n} \) such that, for \( x, w \in S_n \):

- \( P_{w,w}(q) = 1 \),
\* $P_{x,w}(q) = 0$ when $x \not\leq w$ and

\* for $s \in \text{rds}(w)$,

$$P_{x,w}(q) = q^{c_s(x)} P_{x,ws}(q) + q^{1-c_s(x)} P_{xs,ws}(q) - \sum_{z \leq w \atop z \neq w} \mu(z, ws) q^{\ell(w) - \ell(z)} P_{x,z}(q).$$

When $x < w$ we have an upper bound on the degrees: $\deg(P_{x,w}(q)) \leq \frac{\ell(w) - \ell(x) - 1}{2}$.

Note that $\mu(x, w)$ is the coefficient of the highest possible power of $q$ in $P_{x,w}(q)$.

### 3 Properties satisfied by $\mu(x, w)$

We now proceed to describe various well-known properties satisfied by the $\mu$-coefficient. If $x \not\leq w$, then $\mu(x, w)$ is automatically zero. So assume $x \leq w$. There are two easily recognized instances in which the $\mu$-coefficient is zero. The first follows directly from the definitions since $P_{x,w}$ is a polynomial in $q$ rather than $q^{1/2}$.

**Fact 4.** If $\ell(w) - \ell(x)$ is even, then $\mu(x, w) = 0$.

We will refer to a pair $x, w$ for which which $\ell(w) - \ell(x)$ is odd as an **odd pair**.

The second follows from an important set of equalities satisfied by the Kazhdan-Lusztig polynomials (see [13, Corollary 7.14] for a proof):

$$P_{x,w}(q) = P_{xs,w}(q) \text{ if } s \in \text{rds}(w) \text{ and } P_{x,w}(q) = P_{sx,w}(q) \text{ if } s \in \text{lds}(w).$$

Define the set of **extremal pairs**

$$EP(n) = \{x \leq w \in S_n \times S_n : \text{lds}(x) \supseteq \text{lds}(w) \text{ and } \text{rds}(x) \supseteq \text{rds}(w)\}.$$

**Fact 5.** If $\ell(x) < \ell(w) - 1$ and $(x, w) \not\in EP(n)$, then $\mu(x, w) = 0$.

To see why Fact 5 is true, suppose we have a non-covering pair $x < w$ along with some $s \in S$ such that $xs > x$ and $ws < w$. The equality $P_{x,w}(q) = P_{xs,w}(q)$ combined with the degree bound of Theorem 3 implies, since $\ell(w) - \ell(xs) = \ell(w) - \ell(x) - 1$, that the coefficient of $q^{\ell(w) - \ell(x) - 1}/2$ in $P_{x,w}(q)$ must be zero.

According to computations in [12], there are approximately 800 billion comparable pairs $x, w$ in $S_{10}$. It turns out that whenever $w$ covers $x$, $P_{x,w}(q) = \mu(x, w) = 1$; ignore these pairs for the moment. Then, considering only pairs for which $\mu(x, w) > 0$, Facts 4 and 5 allow us to restrict our attention to the odd extremal pairs. The number of such pairs in $S_{10}$ is a modest 626145374, yet still much larger than $|M(10)| = 3$.

The idea of considering equivalence classes to explain the redundancy of $\mu$-values is not new. Lascoux and Schützenberger, and probably others, entertained the possibility that any pair $x, w$ with $\mu(x, w) > 0$ could be generated from a cover by applying certain operators (see the L-S operators below). By construction, all pairs generated in this way would have the same $\mu$-value. Our main contribution in this paper in this regard is to consider “compression” (and “decompression”) in conjunction with the L-S operators and symmetry. Our hope is that these classes are large enough to fully explain the scarcity of distinct values of $\mu$. The three relations from which we build these classes exist already in the literature. We now describe them.

The simplest relations (of various symmetries) can be derived from the definitions in [16].
Fact 6. Let $w_0$ denote the long word $[n-1, n-2, \ldots, 1, 0]$ in $S_n$. Then for $x, w \in S_n$,
\[ \mu(x, w) = \mu(x^{-1}, w^{-1}) = \mu(w_0 w, w_0 x) = \mu(w w_0, x w_0). \] (6)

Our second relation arises from the Lascoux-Schützenberger (L-S) operators (which, their name notwithstanding, were known to Kazhdan and Lusztig [16]). Define $R_k$ be the set of permutations $w$ for which $w s_k < w$ or $w s_k + 1 < w$, but not both. In other words, $R_k$ consists of all permutations in which $w(k), w(k+1), w(k+2)$ do not appear in increasing or decreasing order. Then $w R_k$ is defined to be the unique element in the intersection $R_k \cap \{ w s_k, w s_k + 1 \}$. The operators $R_k$ act “on the right” in the sense that they act on positions. Operators $L_k$ that act on the left” can be defined analogously by having them act on values. More precisely, we set $L_k = \{ w : w^{-1} \in R_k \}$ and $L_k w = (w^{-1} R_k)^{-1}$. (These operators, elementary Knuth transformations and their duals, are closely connected to the Robinson-Schensted correspondence; for details, see [11, 18].) For $x, w \in S_n$, set
\[ \mu[x, w] = \begin{cases} \mu(x, w), & \text{if } x \leq w, \\ \mu(w, x), & \text{if } w \leq x, \\ 0, & \text{if } x \text{ and } w \text{ are not comparable.} \end{cases} \] (7)

Fact 7 ([10]). If $x, w \in L_k$, then $\mu[x, w] = \mu[L_k x, L_k w]$. If $x, w \in R_k$, then $\mu[x, w] = \mu[x R_k, w R_k]$. (8)

Note that the L-S operators do not preserve the lower-order coefficients of Kazhdan-Lusztig polynomials. Also note that $\mu(\cdot, \cdot)$ is not invariant under the L-S operators (consider $L_1$ acting on the pair $(021, 201)$). In the rest of the paper, when we refer to $\mu$ being constant on an equivalence class, we are referring to $\mu[\cdot, \cdot]$ rather than $\mu(\cdot, \cdot)$.

![Figure 2: Example actions of $L_1$ and $R_0$ on the pair $x = 243015, w = 452310$. The simultaneous compression of $(x R_0, w R_0)$ at two capitols is displayed in the rightmost figure.](image)

Our third relation, unlike the L-S operators, has the potential to take a pair in one symmetric group into a pair in a different symmetric group.

We say that a capitol for a pair $x, w \in S_n$ is naked if it lies within an unshaded region of the corresponding Bruhat picture. The compression, $(x^i, w^i)$, of $(x, w)$ at the naked capitol $(i, x(i)) = (i, w(i))$ corresponds to deleting the $i$-th columns and $w(i)$-th rows of $D_x$ and $D_w$. Running the process in reverse is termed a decompression. The pair $(x, w)$ is uncompressible if its Bruhat picture has no naked capitols. Note that compressing a pair $x, w \in S_n$ produces a pair in $S_{n-1}$ while decompression produces one in $S_{n+1}$. In figures, compression(s) will be denoted by a “$C$” and decompressions by a “$D$.” A proof of the following can be found in [3 Lemma 39].

Fact 8. For any naked capitol $(i, x(i)) = (i, w(i))$, both $P_{x, w}(q) = P_{x^i, w^i}$ and $\ell(w) - \ell(x) = \ell(w^i) - \ell(x^i)$. Hence, $\mu(x, w) = \mu(x^i, w^i)$. 

5
4 Results

4.1 Computation of Kazhdan-Lusztig polynomials

Construction of the database encoding all Kazhdan-Lusztig polynomials for pairs \( x, w \in S_m \) with \( m \leq 10 \) proceeded by a direct application of (3). Our algorithm is basically that of the original recursion of Kazhdan and Lusztig [16] as described in [14]. However, two aspects of our algorithm merit note.

First, equation (4) allows us to focus on extremal pairs. As in du Cloux’s program [10], when required to compute \( P_{x,w}(q) \) for any pair \( (x, w) \not\in EP(n) \), we simply move \( x \) up in the Bruhat order through the action of elements of \( rds(w) \) and \( lds(w) \). Second, Fact 8 allows us to focus on uncompressible pairs. When required to compute the Kazhdan-Lusztig polynomial for a compressible pair, we take the novel approach of first compressing as much as possible to a pair \( (x', w') \). Often, this resulting pair is not extremal. Moving \( x' \) up in the Bruhat order can then lead to additional naked capitols. The process can repeat as illustrated in Figure 3.

![Figure 3: Example of how compression can lead to an extremal pair no longer being extremal.](image)

A great deal of redundancy is avoided by only keeping track of the uncompressible extremal pairs. In \( S_{10} \), for example, 90 percent of the extremal pairs are compressible.

Table 1 collects various data regarding Kazhdan-Lusztig polynomials and their computation. The first five rows list the number of extremal pairs, uncompressible extremal pairs, extremal pairs with positive \( \mu \)-value, irreducible pairs and \((n,0)\)-minimal pairs, respectively (these last two terms are defined in Sections 4.2 and 4.3 respectively). The final two rows reflect (among all \( P_{x,w}(q) \) with \( x, w \in S_n \)) the maximum coefficient encountered and the number of distinct, non-constant polynomials appearing, respectively. Due to memory constraints, we have only partial results for \( S_{11} \).

| \( n \) | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-------|---|---|---|---|---|---|----|----|
| \( |EP(n)| \) | 6 | 122 | 2220 | 45184 | 1107636 | 33487176 | 1248544230 | 56786656838 |
| \( |EP_{\text{unc}}(n)| \) | 2 | 10 | 152 | 3114 | 84624 | 2896168 | 122345174 | 6252533464 |
| \( |EP_{\mu>0}(n)| \) | 2 | 2 | 30 | 176 | 2312 | 33550 | 664752 |
| \( |\text{Irr}(n)| \) | 0 | 0 | 0 | 0 | 0 | 16 | 2663 | 54214 |
| \( |(n,0)\)-minimal| \) | 0 | 0 | 0 | 0 | 0 | 12 | 2512 | 51060 |
| max coeff. | 1 | 2 | 4 | 15 | 73 | 460 | 4176 | \( \geq 18915 \) |
| \( |\{P_{x,w}(q)\}| \) | 1 | 4 | 16 | 97 | 1118 | 24361 | 981174 |

Remark 9. It is not clear how to fully take advantage of parallel computers in the computation of collections of Kazhdan-Lusztig polynomials via equation (3). The computation of \( P_{x,w}(q) \) is
not local in the sense that it is not clear which $P_{u,v}(q)$ will be required during the recursive steps. In fact, due to the structure of the recursive branching, any given $P_{u,v}(q)$ may be required \textit{many} times. As such, the most efficient approach appears to store the intermediate $P_{u,v}(q)$ whenever possible. For $S_{11}$, however, such a database (useful in this way only if kept in RAM) would run roughly 50 gigabytes.

\section*{4.2 Computing possible $\mu$-values}

For $n \leq 10$, the possible $\mu$-values can be extracted directly from the database. For $n = 11$, the memory constraints discussed in Remark \[\text{9}\] prevented us from computing the Kazhdan-Lusztig polynomials for all uncompressible extremal pairs. Fortunately, the identities of Section \[\text{3}\] provide a simple way to filter out pairs $x, w$ for which $\mu(x, w) \not\in M(n) \setminus M(n - 1)$.

Define two pairs in $S_n$ to be $\sim_{ls}$-equivalent if they can be connected by via a finite chain of L-S operators. Denote the corresponding equivalence classes by $[[x, w]]_{ls}$. Let $x, w$ be an odd pair. Suppose $[[x, w]]_{ls}$ contains a pair $u, v$ that is either

1. compressible,
2. not extremal and with $\ell(u) < \ell(v) - 1$, or
3. not related in the Bruhat order.

In the first case, $\mu(x, w) \in M(m)$ for some $m < n$. But the following lemma already tells us that such values are contained in $M(n)$.

\textbf{Lemma 10.} For $n \geq 2$, $M(n - 1) \subseteq M(n)$.

\textit{Proof.} Any pair $x, w \in S_{n-1}$ can be decompressed by adding a capitol in the $n$-th row and $n$-th column. The lemma then follows by Fact \[\text{8}\]. \hfill \Box

In the second and third cases, $\mu(x, w)$ must be 0. So, in looking for elements of $M(n) \setminus M(n - 1)$, we can restrict our attention to odd extremal pairs in $S_n$ for which none of the three above cases apply. Such pairs will be termed irreducible. It is significantly faster to compute whether a pair is irreducible than it is to compute the corresponding Kazhdan-Lusztig polynomial.

Even though there are over half a million $\mu$-positive pairs in $S_{10}$, there are only 2,663 irreducible pairs. The computation of the Kazhdan-Lusztig polynomials for the 54,214 irreducible pairs in $S_{11}$ can be done in a few thousand hours of CPU time.

This completes the description of the worked required for the first two parts of Theorem \[\text{1}\]. The elements of $M(12)$ given there stem from individual Kazhdan-Lusztig polynomials we chose to compute guided by Conjecture \[\text{11}\]. See Table \[\text{2}\] for representative pairs yielding these $\mu$-values. (In the table, the polynomial $a_0 + a_1q + a_2q^2 + \cdots$ is described by its coefficient list: $a_0, a_1, a_2, \ldots$)

\section*{4.3 Equivalence classes of pairs}

Let $EP_{\mu>0}(n) = EP_{\mu>0}(n) \cup \{(x, w) : w \text{ covers } x\}$ denote the set of pairs $(x, w) \in S_n \times S_n$ for which $\mu(x, w) > 0$. Write $EP'_{\mu>0}$ for the union of $EP'_{\mu>0}(n)$ as $n$ runs over the positive integers. The identities in Facts \[\text{8}\] and \[\text{17}\] allow us to define the following equivalence relation on the elements of $EP'_{\mu>0}$: Two pairs in $EP'_{\mu>0}$ are $\sim$-equivalent if they can be connected by
Table 2: Known values of $\mu(x, w)$ and pairs that achieve them.

| $n$ | $\mu$ | $x$ | $w$ | $P_{x,w}(q)$ |
|-----|-------|-----|-----|-------------|
| 1   | 1     | 01  | 10  | 1           |
| 10  | 4     | 0432187659 | 4678091235 | 1,14,60,96,43,4 |
| 5   | 2106543987 | 5678901234 | 1,10,43,86,84,37,5 |
| 11  | 3     | 108765432a9 | 789a4560123 | 1,14,22,4,7208 |
| 18  | 4     | 21076543a98 | 792a4560813 | 1,16,112,442,1038,1485,1309,698,200,18 |
| 24  | 5     | 1065432a987 | 689a1345702 | 1,17,129,556,1416,2143,1919,993,269,24 |
| 28  | 6     | 21076543a98 | 6789a123450 | 1,18,145,646,2516,2516,2283,1197,325,28 |
| 12  | 6     | 107654328ba9 | b6789a123450 | 1,24,267,1772,7554,21518,41845,55849, |
| 7   | 21076543ba98 | b6789a501234 | 1,4,18,83,233,514,1045,1571,1648,1373, |
| 8   | 54321ba9876 | 9ab834567012 | 1,11,59,213,579,1216,1920,2216,1823, |
| 23  | 543210ba9876 | 9ab345678012 | 1,13,71,207,337,311,153,23 |
| 25  | 10765432ba98 | b6789a1345702 | 1,24,253,1527,5662,13109,18983,16997, |
| 26  | 10765432ba98 | 789ab123450 | 1,21,251,304,1008,3573,1735,387,26 |
| 27  | 10765432ba98 | b6789a012345 | 1,21,191,933,2561,4008,3573,1735,387,26 |
| 158 | 210876543ba9 | b6789a123450 | 1,24,266,1752,7380,20722,39703,52400, |
| 163 | 21076543ba98 | b6789a123450 | 1,23,250,1682,7564,23555,51779,80733, |
| 13  | 796   | 321087654cba9 | c789ab1234560 | 1,27,347,2808,15615,62330, |
|     |       |                   |                   | 183306,401999,658761,809295, |
|     |       |                   |                   | 721035,469418,215528,66010,12044,796 |

a finite chain consisting of LS-moves, compressions/decompressions and symmetries. (I.e., $\sim$ is the transitive closure of the union of the relations arising from Facts 6, 7 and 8.)

By construction, $\mu[\cdot, \cdot]$ is constant on $\sim$-equivalence classes. Hence, the number of classes intersecting $S_m$ for $m \leq n$ gives an upper bound on the size of $M(n)$. Unfortunately, we have no algorithm (in the precise sense of the word) for computing the equivalence classes: To show $(x, w)$ and $(y, v)$ are equivalent, we must provide a chain $(x, w) \sim (x', w') \sim \cdots \sim (y, v)$ where each successive pair is connected by either an L-S operator, a compression, a decompression or a symmetry. However, we have no bound on how large a symmetric group we might have to pass through in order to construct such a chain; we can always decompress. In other words, given pairs with the same $\mu$-value, we have no effective method for showing that they are not in the same $\sim$-equivalence class. In light of this problem, we define $\mu$-positive pairs $(x, w) \in S_m$ and $(x', w')$ in $S_n$ to be $\sim$-equivalent if they can be connected by a chain that does not pass through $S_{\max(m,n)+k+1}$. An $(n, k)$-minimal pair is one whose $k$-equivalence class does not intersect $S_m$ with $m < n$. The irreducible pairs in $S_n$ with positive $\mu$-value are the $(n, 0)$-minimal pairs.
Table 3: Coalescence of $k$-equivalence classes.

| $n$ | $\mu$ | No. | 0 | 1 | 2 |
|-----|-------|-----|---|---|---|
| 9   | 1     | 12  | 3 | s | s |
| 10  | 1     | 586 | 31| s+1|s+1|
|     | 4     | 428 | 10| 3 | 2 |
|     | 5     | 1498| 27| 2 | 1 |
| 11  | 1     | 26336| 419| s+1|s+1|
|     | 3     | 2466 | 36| 2 | 1 |
|     | 4     | 5166 | 59| s+3|s+1|
|     | 5     | 17052| 170| s | s |
| 18  | 16    | 1   | 1 | 1 | 1 |
| 24  | 16    | 1   | 1 | 1 | 1 |
| 28  | 8     | 2   | 2 | 2 | 2 |

Let $A$ be the $(|EP'_{\mu>0}(n)| + 1) \times (|EP'_{\mu>0}(n)| + 1)$ 0–1 matrix with the first row and column indexed by a “sink” and all other rows/columns indexed by the elements of $EP'_{\mu>0}(n)$. The sink will identify all pairs in $EP'_{\mu>0}(n)$ that are not $(n, k)$-minimal. There is a straightforward algorithm for determining the $(n, k)$-minimal equivalence classes.

1. Pick $k$. Initialize all entries of $A$ to 0.
2. For each pair $(x, w) \in EP'_{\mu>0}(n)$ (indexing row/column $i$), perform a breadth-first search of the members of its $k$-equivalence class by considering L-S moves, symmetries, compressions and decompressions. (Only allow decompressions in the case that the resulting pair lies in $S_m$ for some $m \leq n + k$.)
3. For each pair $(y, v)$ (indexing row/column $j$) encountered in Step 2 set $A(i, j) = 1$.
4. If $(x, w)$ is related to a pair in some $S_m$, $m < n$, then set $A(i, 1) = 1$.
5. We then compute the connected components using Matlab’s graphconncomp command. (Since $A$ may be missing edges originating at the sink, we use the ‘weak’ option.)

Table 3 illustrates how the various equivalence classes coalesce for $9 \leq n \leq 11$ as $k$ ranges from 0 to 2. An s entry (for “sink”) indicates that some of the pairs are not $(n, k)$-minimal. Theorem 2 is immediate. We computed the corresponding $(n, 3)$-minimal classes for all cases except the $\mu = 1$, $n = 11$ class for which we ran out of memory. For the computed cases, the $(n, 3)$-minimal classes equaled the $(n, 2)$-minimal classes. Figure 4 gives the Bruhat pictures for (non-canonical) representatives of each $(n, 2)$-minimal class.

We suspect that some of these classes may coalesce further as $k$ is increased. However, already at $k = 3$ computations become demanding. For example, consider the $(11, 0)$-minimal pair $x = 21076543a98$, $w = 6789a123450$. The size of its $k$-equivalence class grows from 1032 to 879316 to 331361376 as $k$ goes from 1 to 2 to 3.
As an example of coalescence, we consider one of the twelve \((9,0)\)-minimal pairs in \(S_9\). Figure 5 demonstrates the equality \([216540873, 567812340] = [01, 10]\). Any chain connecting these two pairs must pass through \(S_{10}\). This example also serves to illustrate that the Kazhdan-Lusztig polynomials are not preserved by the L-S operators; \(P_{01,10}(q) = 1\) while \(P_{216540873,567812340}(q) = 1 + 8q + 16q^2 + 11q^3 + q^4\).

5 Representatives of equivalence classes

Given a composition \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n\), let \(x_{\alpha}\) be the permutation
\([n - \alpha_1, n - \alpha_1 + 1, \ldots, n - 1, n - \alpha_1 - \alpha_2, n - \alpha_1 - \alpha_2 + 1, \ldots, n - \alpha_1 - 1, \ldots, 0, 1, \ldots, \alpha_k - 1]\).

Let \(X_n = \{x_{\alpha} : \alpha \models n\}\). We define a crosshatch pair to be a pair \(x \leq w\) for which \(xw_0, w \in X_n\).
Conjecture 11. Every $\sim$-equivalence class contains a crosshatch pair.

In particular, while we conjecture that each $n$-minimal class has a crosshatch pair, there may only be such pairs in $S_m$ with $m > n$. Even after factoring out symmetry, such putative representatives are not unique. Recall that Figure 4 gives representatives for the various $(n, 2)$-minimal equivalence classes that we have been able to compute. For five of these classes (one $n = 10$, $\mu = 4$ class and the $\mu = 1, 4, 18, 24$ classes for $n = 11$), the representative given in that figure is not a crosshatch pair. Figure 6 remedies this for four of the classes by giving crosshatch representatives lying in $S_m$ with $m$ equal to 12 or 13. The class we were unable to find a crosshatch representative for is the $n = 11$, $\mu = 4$ class. However, given our above remark about the sizes of $\sim$-equivalence classes, we do not feel this is a significant mark against Conjecture 11.

The three possibilities are that this class is not $(11, k)$-minimal for some $k > 3$, that its smallest crosshatch pair lies in $S_m$ for some $m \geq 15$, or that it does not contain a crosshatch pair at all.

In light of Conjecture 11, it is reasonable to ask if there are simple criteria for the $\mu$-value of a crosshatch pair to be nonzero. Or even more ambitiously, to ask for a simple closed formula for the value of $\mu$ on such an interval. We note here that Brenti (along with various coauthors — see [5, 6]) has closed formulas for Kazhdan-Lusztig polynomials based on alternating sums of paths that might be specialized for this purpose.

It would also be interesting to understand geometrically why such intervals appear so prevalent among pairs with $\mu$-values greater than 1; the crosshatch intervals are minimal coset representatives for certain Richardson varieties with respect to independent partial flag manifolds [19]. Of course, everything in this section may be attributable to working with values of $n$ that are too small. On the other hand, crosshatch pairs are relatively rare even for these small values of $n$. Of the 1.2 billion extremal pairs in $S_{10}$ only 4,708 are crosshatch pairs.

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