**Abstract**

This paper studies the $N$-tuple noncommutative Orlicz spaces $\oplus_{j=1}^{n} L_{p,\lambda}^{(\Phi_j)}(\tilde{M}, \tau)$, where $L_{p,\lambda}^{(\Phi_j)}(\tilde{M}, \tau)$ is noncommutative Orlicz spaces and $\tilde{M}$ is the $\tau$-measurable operators. Based on the maximum principle, we give the Riesz-Thorin interpolation theorem on $\oplus_{j=1}^{n} L_{p,\lambda}^{(\Phi_j)}(\tilde{M}, \tau)$. As applications, the Clarkson inequality and some geometrical properties such as uniform convexity and uniform smooth of noncommutative Orlicz space $L^{(\Phi_s)}(\tilde{M}, \tau), 0 < s \leq 1$ are given.

**Keywords:** Noncommutative Orlicz spaces, $\tau$-measurable operator, von Neumann algebra, Orlicz function, Riesz-Thorin interpolation

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1. Preliminaries

In 1936, for study uniform convexity of $L^p$ space, Clarkson gave some famous inequalities named Clarkson inequality [3]. In [10], the author used the noncommutative Riesz-Thorin interpolation theorem get the Clarkson inequality on noncommutative $L^p$ space. The principal objective of this paper is to investigate Riesz-Thorin interpolation theorem on noncommutative Orlicz spaces which yields the Clarkson inequality of noncommutative $L^p$ space. As applications, some geometrical properties such as uniform convexity and uniform smooth of noncommutative Orlicz space $L^{(\Phi_s)}(\tilde{M}, \tau), 0 < s \leq 1$ are given.
The theory of Orlicz spaces associated to a trace was introduced by Muratov \[6\] and Kunze \[12\]. Let \(\mathcal{M}\) be a semi-finite von Neumann algebra acting on a Hilbert space \(\mathcal{H}\) with a normal semi-finite faithful trace \(\tau\). A densely-defined closed linear operator \(A : D(A) \to \mathcal{H}\) with domain \(D(A) \subseteq \mathcal{H}\) is called affiliated with \(\mathcal{M}\) if and only if \(U^*AU = A\) for all unitary operators \(U\) belonging to the commutant \(\mathcal{M}'\) of \(\mathcal{M}\). Clearly, if \(A \in \mathcal{M}\) then \(A\) is affiliated with \(\mathcal{M}\). If \(A\) is a (densely-defined closed) operator affiliated with \(\mathcal{M}\) and \(A = U|A|\) the polar decomposition, where \(|A| = (A^*A)^{\frac{1}{2}}\) and \(U\) is a partial isometry, then \(A\) said to be \(\tau\)-measurable if and only if there exists a number \(\lambda \geq 0\) such that \(\tau(e_{[\lambda,\infty]}(|A|)) < \infty\), where \(e_{[0,\lambda]}\) is the spectral projection of \(|A|\) and \(\tau\) is the trace of normal faithful and semifinite. The collection of all \(\tau\)-measurable operators is denoted by \(\widetilde{\mathcal{M}}\). The spectral decomposition implies that a von Neumann algebra \(\mathcal{M}\) is generated by its projections. Recall that an element \(A \in \mathcal{M}_+\) is a linear combination of mutually orthogonal projections if \(A = \sum_{k=1}^{n} \alpha_k e_k\) with \(\alpha_k \in \mathbb{R}_+\) and projection \(e_k \in \mathcal{M}\) such that \(e_k e_j = 0\) whenever \(k \neq j\) [10].

Next we recall the definition and some basic properties of noncommutative Orlicz spaces.

A function \(\Phi : [0, \infty) \to [0, \infty]\) is called an Orlicz function if and only if \(\Phi(u) = \int_0^{|u|} p(t)dt\), where the right derivative \(p\) of \(\Phi\) satisfies \(p\) is right-continuous and non-decreasing, \(p(t) > 0\) whenever \(t > 0\) and \(p(0) = 0\) with \(\lim_{t \to \infty} p(t) = \infty\) [11]. Further we say an Orlicz function \(\Phi\) satisfies the \(\Delta_2\)-condition for large \(t\) (for small \(t\), or for all \(t\)), written often as \(\Phi \in \Delta_2\), if there exist constants \(t_0 > 0, K > 2\) such that \(\Phi(2t) \leq K\Phi(t)\) for \(|t| \geq t_0\) [7].

If \(A \in \widetilde{\mathcal{M}}\) and \(\Phi\) is an Orlicz function, we denote \(\widetilde{\rho}_\Phi(A) = \tau(\Phi(|A|))\), hence we can define a corresponding space, which is named the noncommutative Orlicz space, as follows:

\[
L^\Phi(\widetilde{\mathcal{M}}, \tau) = \{ A \in \widetilde{\mathcal{M}} : \tau(\Phi(\lambda |A|)) < \infty \text{ for some } \lambda > 0 \}.
\]

Also we could define the subspace

\[
E^\Phi(\widetilde{\mathcal{M}}, \tau) = \{ A \in \widetilde{\mathcal{M}} : \tau(\Phi(\lambda |A|)) < \infty \text{ for any } \lambda > 0 \}.
\]

We equip these spaces with the Luxemburg norm

\[
\|A\|_{(\Phi)} = \inf\{\lambda > 0 : \tau(\Phi\left(\frac{|A|}{\lambda}\right)) \leq 1\}.
\]

In the case of \(\Phi(A) = |A|^p\), \(1 \leq p < \infty\), \(L^\Phi(\widetilde{\mathcal{M}}, \tau)\) is nothing but the noncommutative space \(L^p(\widetilde{\mathcal{M}}, \tau) = \{ A \in \widetilde{\mathcal{M}} : \tau(|A|^p) < \infty \}\) [3] and the Luxemburg norm
generated by this function is expressed by the formula

\[ \|A\|_p = \left( \tau(|A|^p) \right)^{\frac{1}{p}}. \]

One can define another norm on \( L^\Phi(\widetilde{M}, \tau) \) as follows

\[ \|A\|_\Phi = \sup \{ \tau(|AB|) : B \in L^\Psi(\widetilde{M}, \tau) \text{ and } \tau(\Psi(B)) \leq 1 \}, \]

where \( \Psi : [0, \infty) \to [0, \infty] \) is defined by \( \Psi(u) = \sup \{ uv - \Phi(v) : v \geq 0 \} \). Here we call \( \Psi \) the complementary function of \( \Phi \). In the following, we use \( L^{(\Phi)}(\mathcal{M}, \tau) \) and \( L^\Phi(\widetilde{M}, \tau) \) denote the Orlicz which equipped Luxemberg and Orlicz norm respectively.

The same as \( E^{(\Phi)}(\mathcal{M}, \tau) \) and \( E^{\Phi}(\mathcal{M}, \tau) \).

For more information on the theory of noncommutative Orlicz spaces we refer the reader to [8, 9, 2, 12, 5, 6].

2. Riesz-Thorin interpolation theorem of Noncommutative Orlicz spaces

In this section, we will give the definition of \( N \)-tuple noncommutative Orlicz spaces, also give some norm inequalities. For research the Riesz-Thorin interpolation theorem, a equivalent definition of Luxemburg norm must be given. As a corollary, the Clarkson inequality of noncommutitive \( L^p \) space could be get. The main ideas and proof ideas in this article are derived from literatures [10] and [7].

Now let \( \mathcal{N} = \mathcal{M} \oplus \mathcal{M} \oplus \cdots \oplus \mathcal{M} \) be the \( n \)-th von Neumann algebra direct sum of \( \mathcal{M} \) with it self. We know that \( \mathcal{N} \) acts on the direct sum Hilbert space \( \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) coordinatewise:

\[ (A_1, A_2, \ldots, A_n)(x_1, x_2, \ldots, x_n) = \sum_{j=1}^n A_j x_n, \]

where \( A_j \in \mathcal{M}, i = 1, 2, \ldots, n \). Then \( \mathcal{N}_+ = \mathcal{M}_+ \oplus \mathcal{M}_+ \oplus \cdots \oplus \mathcal{M}_+ \).

Define: \( v : \mathcal{N}_+ \to \mathbb{C} \) by \( v(A_1, A_2, \ldots, A_n) = \sum_{j=1}^n \lambda_j \tau(A_j) \), where \( \lambda_j \geq 0 \) and \( \tau \) is a normal faithful normal faithful normalized trace on \( \mathcal{M} \), then \( v \) is a normal faithful normal faithful normalized trace on \( \mathcal{N} \). Now we give the following definition:

**Definition 2.1.** Let \( \Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n) \) be an \( n \)-tuple of \( N \) functions \( \Phi_j \). For each \( p \geq 1, \lambda_j \geq 0, \) and \( n \)-tuple of wights \( \lambda = (\lambda_1, \ldots, \lambda_n) \) consider the direct sum space, named \( n \)-tuple of wights noncommutative Orlicz spaces as follows:

\[ \bigoplus_{j=1}^n E_{p, \lambda}^{(\Phi_j)} = \{ A = (A_1, A_2, \ldots, A_n) : A_j \in E^{(\Phi_j)}(\widetilde{M}, \tau), 1 \leq j \leq n \} \]

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with norm \( \| \cdot \|_{(\Phi),p,\lambda} \) defined for \( A_j \in E^{(\Phi_j)}(\tilde{M}, \tau) \):

\[
\| A \|_{(\Phi),p,\lambda} = \begin{cases} \\
\left[ \sum_{j=1}^{n} \lambda_j \| A_j \|^{p}_{(\Phi_j)} \right]^{\frac{1}{p}}, & 1 \leq p < \infty; \\
\max_j \| A_j \|_{(\Phi_j)}, & p = \infty.
\end{cases}
\]

or the norm \( \| \cdot \|_{\Phi,j} \) defined in the same way as before in which \( \| \cdot \|_{(\Phi_j)} \) is replaced by the Orlicz norm \( \| \cdot \|_{\Phi_j} \), denotes by \( \bigoplus E^\Phi_{p,\lambda} \). The same way, if \( \Psi_j \) is the complementary \( N \)-function of \( \Phi_j \), denotes by \( \bigoplus E^\Psi_{q,\lambda} \) which equip with \( \| \cdot \|_{(\Psi),q,\lambda} \) and \( \bigoplus E^{\Psi_j}_{q,\lambda} \) which equip with \( \| \cdot \|_{\Psi,q,\lambda} \) for the same weights \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( q = \frac{p}{p-1} \). The same way, we also could define \( \bigoplus L^\Phi_{p,\lambda} \), \( \bigoplus L^\Psi_{q,\lambda} \), \( \bigoplus L^\Phi_{p,\lambda} \), \( \bigoplus L^\Psi_{q,\lambda} \), \( \bigoplus L^\Phi_{p,\lambda} \), and the norm as before.

**Remark 1.** By the Theorem 3.4 of [5], if for any \( 1 \leq j \leq n \) with \( \Phi_j \in \Delta_2 \), we have \( \bigoplus L^\Phi_{p,\lambda} = \bigoplus E^\Phi_{p,\lambda} \).

**Lemma 2.1.** If \( A \in \bigoplus L^\Phi_{p,\lambda} \) and \( B \in \bigoplus L^\Psi_{q,\lambda} \), where \( 1 \leq p < \infty \), we have

1. If \( \| A \|_{(\Phi),p,\lambda} \leq 1 \), then we have \( \nu(\Phi(A)) \leq \| A \|_{(\Phi),p,\lambda} \cdot \delta_1 \), where \( \delta_1 = \left( \sum_{j=1}^{n} \lambda_j \right)^{\frac{1}{q}} \).

2. If \( \| A \|_{(\Phi),p,\lambda} > 1 \), then we have \( \nu(\Phi(A)) > \delta_2 \), where \( \delta_2 = \left[ \sum_{j=1}^{n} \lambda_j^p \| A_j \|_{(\Phi_j)}^p \right]^{\frac{1}{p}} \).

3. \( \nu(AB) \leq \| A \|_{(\Phi),p,\lambda} \cdot \| B \|_{(\Psi),q,\lambda} \).

**Proof.** (1) If \( \| A \|_{(\Phi),p,\lambda} \leq 1 \), then from Proposition 3.4 of [2] and classical H"{o}lder inequality, we have

\[
\nu(\Phi(A)) = \sum_{j=1}^{n} \lambda_j \tau(\Phi_j(A_j))
\]

\[
= \sum_{j=1}^{n} \lambda_j^\frac{1}{q} \cdot \lambda_j^\frac{1}{p} \tau(\Phi_j(A_j))
\]

\[
\leq \left[ \sum_{j=1}^{n} \lambda_j \left[ \tau(\Phi_j(A_j)) \right]^p \right]^{\frac{1}{p}} \cdot \left( \sum_{j=1}^{n} \lambda_j \right)^{\frac{1}{q}}
\]

\[
\leq \left[ \sum_{j=1}^{n} \lambda_j \| A_j \|_{(\Phi_j)}^p \right]^{\frac{1}{p}} \cdot \delta_1
\]

\[
= \| A \|_{(\Phi),p,\lambda} \cdot \delta_1.
\]
(2) If \( \|A\|_{\Phi(p, \lambda)} > 1 \), then from Proposition 3.4 of [2], we have

\[
[v(\Phi(A))]^p = \left[ \sum_{j=1}^{n} \lambda_j \tau(\Phi_j(A_j)) \right]^p
\]

\[
> \left[ \sum_{j=1}^{n} \lambda_j \|A_j\|_{\Phi_j} \right]^p
\]

\[
\geq \sum_{j=1}^{n} \lambda_j^p \|A_j\|_{\Phi_j}^p,
\]

which means that

\[
v(\Phi(A)) > \left[ \sum_{j=1}^{n} \lambda_j^p \|A_j\|_{\Phi_j}^p \right]^{\frac{1}{p}} = \delta_2.
\]

(3) From Theorem 3.3 of [2] and classical Hölder inequality, we get that

\[
v(AB) = \sum_{j=1}^{n} \lambda_j |\tau(A_j B_j)|
\]

\[
\leq \sum_{j=1}^{n} \lambda_j^{\frac{1}{\frac{1}{p} + \frac{1}{q}}} \|A_j\|_{\Phi_j} \|B_j\|_{\Psi_j}
\]

\[
\leq \left( \sum_{j=1}^{n} \lambda_j \|A_j\|_{\Phi_j}^p \right)^{\frac{1}{p}} \cdot \left( \sum_{j=1}^{n} \lambda_j \|B_j\|_{\Psi_j}^q \right)^{\frac{1}{q}}
\]

\[
= \|A\|_{\Phi(p, \lambda)} \cdot \|B\|_{\Psi(q, \lambda)}.
\]

**Remark 2.** If \( \Phi \) is 1-tuple N-function and \( \lambda = 1 \), the Lemma 2.1 just be the Theorem 3.3 and Proposition 3.4 of [2].

**Theorem 2.1.** If \( A \in \bigoplus_{j=1}^{n} E_{p, \lambda}^{(\Phi_j)} \), then for \( 1 \leq p < \infty \), the weighted norms \( \| \cdot \|_{\Phi(p, \lambda)} \) is given by

\[
\|A\|_{\Phi(p, \lambda)} = \sup \{ v(AB) : \|B\|_{\Psi(q, \lambda)} \leq 1 \},
\]

**Proof.** If \( \|B\|_{\Psi(q, \lambda)} \leq 1 \). One side, by (3) of the Lemma 2.1, we have

\[
v(AB) \leq \|A\|_{\Phi(p, \lambda)} \cdot \|B\|_{\Psi(q, \lambda)} \leq \|A\|_{\Phi(p, \lambda)}.
\]

The other side, for simplicity, we may take that \( \|A\|_{\Phi(p, \lambda)} = 1 \) and assume that \( A_j \geq 0 \). Let \( \{ e_{jn} \} \) be the projection of \( A_j \) and \( 0 < \tau(e_{jn}) < \infty \). By Proposition 3.4 of [2], for any \( \varepsilon > 0 \), we have \( \tau(\Phi_j((1 + \varepsilon)A_j)) \geq \|(1 + \varepsilon)A_j\|_{\Phi_j} = 1 + \varepsilon. \)
If we define the operator \( A_{jm} = A_j(e_j + e_{j2} + \cdots + e_{jm})(m \leq n) \), where \( A_j = \sum_{k=1}^{n} \alpha_k e_{jk} \), and \( e_{jk} = 0, k > n \), then \( A_{jm} \uparrow A_j \) as \( m \to \infty \), there exists an \( m_0 \) such that for \( m \geq m_0 \) one have

\[
v \left[ \frac{1}{\delta_2} \Phi ((1 + \varepsilon) A_m) \right] = \sum_{k=1}^{n} \frac{1}{\delta_2} \lambda_j \Phi_j ((1 + \varepsilon) A_{jm}) \geq \left( 1 + \frac{\varepsilon}{2} \right).
\]

If we set

\[
B_{jm} = \frac{\delta_2^{-1} p((1 + \varepsilon)\lambda_j A_{jm})}{\delta_1 (1 + \tau(\Psi_j (\delta_2^{-1} p((1 + \varepsilon)\lambda_j A_{jm})))),
\]

then \( B_{jm} \) is bounded operators and \( B_m \in \bigoplus_{j=1}^{n} E_{\Psi_j,\lambda}^{\Psi_j} \) for each \( m \). Moreover by definition 1.7 of [5] and 1.9 of [11] we have, \( \|B_{jm}\|_{\Psi_j,\lambda} \leq 1 \) since \( \|A\|_{(\Phi),p,\lambda} = 1 \).

Hence,

\[
\|B_m\|_{\Psi,\lambda} = \left( \sum_{j=1}^{n} \lambda_j \|B_{jm}\|_{\Psi_j}^q \right)^{\frac{1}{q}} \leq 1.
\]

However, one has

\[
\sup \{v(AB)\} = \sup \left\{ \sum_{j=1}^{n} \lambda_j \tau(A_j B_j) : B_j \in E_{\Psi_j}, \|B\|_{\Psi,\lambda} \leq 1 \right\}
\geq \sup \left\{ \sum_{j=1}^{n} \lambda_j \tau(A_j B_{jm}) : B_{jm} \in E_{\Psi_j}, \|B_m\|_{\Psi,\lambda} \leq 1 \right\}
\geq \frac{1}{1 + \varepsilon} \sup_{m \geq m_0} \left\{ \sum_{j=1}^{n} \tau((1 + \varepsilon)\lambda_j A_{jm} B_{jm}) \right\}
= \frac{1}{1 + \varepsilon} \sup_{m \geq m_0} \left\{ \sum_{j=1}^{n} \tau(\Phi_j (1 + \varepsilon)\lambda_j A_{jm} + \tau(\Psi_j (\delta_2^{-1} p(1 + \varepsilon)\lambda_j A_{jm}))) \right\}
\geq \frac{1}{1 + \varepsilon},
\]

since \( \varepsilon > 0 \) is arbitrary we get the desired inequality.

\[\square\]

**Definition 2.2.** [1] Let \( \Phi_1 \) and \( \Phi_2 \) be N-functions and define \( \Phi_s \) to be the inverse of \( \Phi^{-1}(u) = [\Phi_1^{-1}(u)]^{1-s}[\Phi_2^{-1}(u)]^s \) for \( 0 \leq s \leq 1, u \geq 0 \), where \( \Phi^{-1} \) is the unique inverse of the N-function \( \Phi \).

**Theorem 2.2.** Let \( \Phi_i = (\Phi_i^{(1)}, \Phi_i^{(2)}, \ldots, \Phi_i^{(n)}) \), \( Q_i = (Q_{i1}, Q_{i2}, \ldots, Q_{in}), i = 1, 2 \) be n-tuples of N-functions and \( 0 \leq r_1, r_2, t_1, t_2 \leq \infty, \lambda = (\lambda_1, \ldots, \lambda_n) \) be given positive numbers. Next let \( \Phi_s = (\Phi_{s1}, \Phi_{s2}, \ldots, \Phi_{sn}), Q_s = (Q_{s1}, Q_{s2}, \ldots, Q_{sn}) \) be the associated
intermediate $N$-functions,

\[
\frac{1}{r_s} = \frac{1 - s}{r_1} + \frac{s}{r_2}, \quad \frac{1}{t_s} = \frac{1 - s}{t_1} + \frac{s}{t_2}, \quad 0 \leq s \leq 1.
\]

If $T : \bigoplus_{j=1}^{n} E_{r_j,\lambda}^{(\Phi_{s_j})} \to \bigoplus_{j=1}^{n} L_{t_j,\lambda}^{(Q_{s_j})}$ is a bounded linear operator with bounds $K_1, K_2$, such that $\|TA\|_{(Q_1),t_i,\lambda} \leq K_i \|A\|_{(\Phi_1),r_i,\lambda}$, $A \in \bigoplus_{j=1}^{n} E_{r_j,\lambda}^{(\Phi_{s_j})}$, $i = 1, 2$.

Then $T$ is also defined on $\bigoplus_{j=1}^{n} E_{r_{s_j},\lambda}^{(\Phi_{s_j})}$ into $\bigoplus_{j=1}^{n} L_{r_{s_j},\lambda}^{(\Phi_{s_j})}$ for all $0 \leq s \leq 1$ and one have the bound

\[
\|TA\|_{(Q_1),t_s,\lambda} \leq K_1^{1-s} K_2^{s} \|A\|_{(\Phi_1),r_s,\lambda},
\]

where $A \in \bigoplus_{j=1}^{n} E_{r_{s_j},\lambda}^{(\Phi_{s_j})}$.

**Proof.** Let $A = (A_1, A_2, \ldots, A_n) \in \bigoplus_{j=1}^{n} E_{r_{s_j},\lambda}^{(\Phi_{s_j})}$, $B = (B_1, B_2, \ldots, B_n) \in \bigoplus_{j=1}^{n} E_{t_{s_j},\lambda}^{(\Phi_{s_j})}$ with polar decompositions $A_k = U_k |A_k|$, $B_k = V_k |B_k|$. Assume that $\|A\|_{(\Phi_1),r_s,\lambda} \leq 1$, $\|B\|_{\psi, t_s,\lambda} \leq 1$ where $|A_k| = \sum_{j=1}^{n} \alpha_j e_{kj}$, $|B_k| = \sum_{j=1}^{n} \beta_j e'_{kj}$.

Define for $z = \mathbb{C}$ and $k = 1, 2, \ldots, n$

\[
A(z) = (A_1(z), A_2(z), \ldots, A_n(z))
\]

and

\[
B(z) = (B_1(z), B_2(z), \ldots, B_n(z)),
\]

where

\[
A_k(z) = U_k \Phi_{sk} \left[ (\Phi_{1k}^{-1})^{1-z} (\Phi_{2k}^{-1})^{z} \right] |A_k|,
\]

\[
B_k(z) = V_k \Psi_{sk} \left[ (\Psi_{1k}^{-1})^{1-z} (\Psi_{2k}^{-1})^{z} \right] |B_k|.
\]

Then,

\[
A_k(z) = U_k \Phi_{sk} \left[ \left( \sum_{j=1}^{n} \alpha_j e_{kj} \right)^{1-z} \left( \sum_{j=1}^{n} \alpha_j e_{kj} \right)^{z} \right]
\]

\[
= \sum_{j=1}^{n} \Phi_{sk} \left[ (\Phi_{1k}^{-1}(\alpha_j))^{1-z} (\Phi_{2k}^{-1}(\alpha_j))^{z} \right] U_k e_{kj}
\]

Hence, $z \to A(z)$ is an analytic function on $\mathbb{C}$ with value in $\widetilde{\mathcal{M}}$. The same reduction applies to $B$. 

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Now we could define a bounded entire function

\[ H(z) = K_1 z^{-1} K_2 z^2 \tau(B(z)TA(z)). \]

If \( z = it \) for \( t \in \mathbb{R}, \) we have

\[
A_k(it) = \sum_{j=1}^{n} \Phi_{sk} \left[ \Phi_{sk}^{-1}(\alpha_j) \right] U_k e_{kj}
\]

\[
= \sum_{j=1}^{n} \Phi_{sk} \left( \Phi_{1k}^{-1}(\alpha_j) \right) \left( \Phi_{2k}^{-1}(\alpha_j) \right)^{it} U_k e_{kj}
\]

\[
= \sum_{j=1}^{n} \Phi_{sk} \left( \frac{\Phi_{2k}^{-1}(\alpha_j)}{\Phi_{1k}^{-1}(\alpha_j)} \right)^{it} U_k e_{kj} \cdot \sum_{j=1}^{n} \Phi_{sk} \left[ \Phi_{sk}^{-1}(\alpha_j) \right] U_k e_{kj}
\]

\[
= \left[ \Phi_{sk} \left( \frac{\Phi_{2k}^{-1}(|A_k|)}{\Phi_{1k}^{-1}(|A_k|)} \right) \right]^{it} \cdot \Phi_{sk} \left( \Phi_{1k}^{-1}(|A_k|) \right).
\]

Hence,

\[
|A_k(it)|^2 = A_k(it)^* A_k(it) = \left[ \Phi_{sk} \left( \Phi_{1k}^{-1}(|A_k|) \right) \right]^2
\]

which means

\[
|A_k(it)| = \Phi_{sk} \left( \Phi_{1k}^{-1}(|A_k|) \right).
\]

Hence for any \( 1 \leq k \leq n \) we have \( \tau(\Phi_{1k}(A_k(it))) = \tau(\Phi_{sk}(A_k)) \) which implies that

\[
\|A_j(it)\|_{(\Phi_{ij})} = \|A_j\|_{(\Phi_{ij})}
\]

and

\[
v(\Phi_1(A(it))) = \sum_{j=1}^{n} \lambda_j \tau(\Phi_{1j}[\Phi_{sk}(\Phi_{1j}^{-1}(\|A_j\|)])]
\]

\[
= \lambda_1 \tau(\Phi_{s1}(\|A_1\|)) + \lambda_2 \tau(\Phi_{s2}(\|A_2\|)) + \ldots + \lambda_n \tau(\Phi_{sn}(\|A_n\|))
\]

\[
v(\Phi_s(|A|)),
\]

we get that

\[
\|A(it)\|_{(\Phi_{1})_{r_{1},\lambda}} = \|A\|_{(\Phi_{1})_{r_{1},\lambda}} \leq 1.
\]

Similar \( \|B(it)\|_{s_{1},t_{1},\lambda} = \|B\|_{s_{1},t_{1},\lambda} \leq 1. \) Thus by (3) of the Lemma 2.1 and the assumption on \( T, \) we have

\[
|\tau(B(it)TA(it))| \leq K_1 \|B(it)\|_{s_{1},t_{1},\lambda} \|A(it)\|_{(\Phi_{1})_{r_{1},\lambda}} \leq K_1.
\]

It then follows that \( |H(it)| \leq 1 \) for any \( t \in \mathbb{R}. \) In the same way, we show \( |H(1+it)| \leq \)
1. Therefore, by the maximum principle, for any $\theta \in C$, we get

$$|H(\theta)| = |K_1^{\theta-1}K_2^{-\theta} \tau(B(\theta)TA(\theta))| \leq 1.$$ 

Hence,

$$|\tau(BTA)| \leq K_1^{1-\theta}K_2^\theta$$

By the Theorem 2.1 we could get that

$$\|TA\|_{(Q_s),r,s,\lambda} \leq K_1^{1-\theta}K_2^\theta \|A\|_{(\Phi_s),r,s,\lambda}.$$  

\hfill $\Box$

**Theorem 2.3.** Let $\Phi$ be an N-function and $\Phi_s$ be the inverse which satisfies that $\Phi^{-1}_s(u) = [\Phi^{-1}(u)]^{1-s} u\Phi_0^{-1}(u)$ where $0 < s \leq 1$ and $\Phi_0(u) = u^2$. If $L^{(\Phi)}(\tilde{M}, \tau)$ is the noncommutative Orlicz space, then we have for $A, B \in L^{(\Phi_s)}(\tilde{M}, \tau)$:

$$\left(\|A + B\|_{(\Phi_s),t_1}^{\frac{2}{t_1}} + \|A - B\|_{(\Phi_s),t_1}^{\frac{2}{t_1}}\right)^{\frac{2}{2-t_1}} \leq 2^{\frac{2}{2-t_1}} \left(\|A\|_{(\Phi_s),t_1}^{\frac{2}{t_1}} + \|B\|_{(\Phi_s),t_1}^{\frac{2}{t_1}}\right)^{\frac{2-t_1}{2-t_1}}.$$

**Proof.** Let $\Phi_1 = (\Phi, \Phi)$ be the 2-vector of N-functions, $\lambda = (1,1), 1 \leq r_1 \leq \infty$ and set

$$\bigoplus_{j=1}^{2} E_{r_1}^{(\Phi)}(\tilde{M}, \tau) = \{(A, B) : A, B \in E^{(\Phi)}(\tilde{M}, \tau), \|(A, B)\|_{(\Phi_1),r_1} < \infty\},$$

where

$$\|(A, B)\|_{(\Phi_1),r_1} = \begin{cases} \left[\|A\|_{(\Phi)}^{r_1} + \|B\|_{(\Phi)}^{r_1}\right]^{\frac{1}{r_1}}, & 1 \leq r_1 < \infty, \\ \max\{\|A\|_{(\Phi)}, \|B\|_{(\Phi)}\}, & r_1 = \infty. \end{cases}$$

Take $Q_1 = \Phi_1 = (\Phi, \Phi)$ and $Q_2 = \Phi_2 = (\Phi_0, \Phi_0)$ where $\Phi_0(u) = u^2$.

Set $r_1 = 1, r_2 = t_2 = 2$ and $t_1 = +\infty$. Define the linear operator $T : \bigoplus_{j=1}^{2} E_{r_1}^{(\Phi_i)} \rightarrow \bigoplus_{j=1}^{2} L_{t_1}^{(Q_s)}$ by the equation $T(A, B) = (A + B, A - B)$, we then have

$$\|T(A, B)\|_{(Q_1),t_1} = \max\{\|A + B\|_{(\Phi)}, \|A - B\|_{(\Phi)}\} \leq \|A\|_{(\Phi)} + \|B\|_{(\Phi)} = K_1 \|(A, B)\|_{(\Phi_1),r_1}.$$
Hence, \( K_1 = 1 \) and since \( \| \cdot \|_{(\Phi_0)} = \| \cdot \|_2 \), we find

\[
\|T(A, B)\|_{(Q_s), t_2} = \left[\|A + B\|_2^2 + \|A - B\|_2^2\right]^{\frac{1}{2}}
\]

\[
= \sqrt{2} \left[\|A\|_2^2 + \|B\|_2^2\right]^{\frac{1}{2}}
\]

\[
= K_2\|T(A, B)\|_{(\Phi_s), r_2}.
\]

Thus \( K_2 = \sqrt{2} \). Let \( r_s \) and \( t_s \) be given by

\[
\frac{1}{r_s} = \frac{1 - s}{r_1} + \frac{s}{r_2}, \quad \frac{1}{t_s} = \frac{1 - s}{t_1} + \frac{s}{t_2}
\]

then we have, \( r_s = \frac{2}{2-s}, t_s = \frac{s}{2} \).

By the results of Theorem 2.2,

\[
\|T(A, B)\|_{(Q_s), t_s} \leq 2^\frac{s}{p} \|T(A, B)\|_{(\Phi_s), r_s}
\]

since \( K_1^{1-s} K_2^s = 2^\frac{s}{p} \). Hence, we have

\[
\|T(A, B)\|_{(Q_s), t_s} = \left[\|A\|_{(\Phi_s)}^\frac{2}{p} + \|B\|_{(\Phi_s)}^\frac{2}{p}\right]^{\frac{2}{p}}
\]

and

\[
\|T(A, B)\|_{(Q_s), t_s} = \left(\|A + B\|_{(\Phi_s)}^\frac{2}{p} + \|A - B\|_{(\Phi_s)}^\frac{2}{p}\right)^{\frac{1}{p}}
\]

which we could get the result. \( \square \)

The following corollary is Clarkson inequality of noncommutative \( L^p \) space and proof the process is completely similar to the P42 of [7].

**Corollary 2.1.** Suppose that \( 1 < p < \infty \) and \( q = \frac{p}{p-1} \). Then for \( A, B \in L^p(\tilde{\mathcal{M}}, \tau) \), we have

\[
(\|A + B\|^p + \|A - B\|^p)^{\frac{1}{p}} \leq 2^\frac{1}{q} (\|A\|^p + \|B\|^p)^{\frac{1}{q}}, \quad 1 < p \leq 2,
\]

and

\[
(\|A + B\|^p + \|A - B\|^p)^{\frac{1}{p}} \leq 2^\frac{1}{q} (\|A\|^p + \|B\|^p)^{\frac{1}{q}}, \quad 2 \leq p \leq \infty.
\]

**Proof.** If \( 1 < p \leq 2 \), let \( 1 < \alpha < p \leq 2 \) and \( \Phi(u) = |u|^\alpha, \Phi_0(u) = |u|^2, s = \frac{2(p-\alpha)}{\alpha(p-1)} \). Then \( 0 < s \leq 1 \) and \( \Phi_s^{-1}(u) = |u|^\frac{1}{2}\alpha \) or \( \Phi_s(u) = |u|^p \). Hence \( \| \cdot \|_{(\Phi_s)} = \| \cdot \|_{(\Phi_0)} \) and since

\[
\lim_{s \to 0} \frac{2}{s} = \frac{\alpha}{p-1} = q; \quad \lim_{s \to \infty} \frac{2}{s} = \frac{1}{p}; \quad \lim_{s \to \infty} \frac{2}{s} = \frac{1}{q},
\]

by the Theorem 2.3 we get the first inequality.

Similar let \( 2 \leq p < \beta < \infty \) and \( \Phi(u) = |u|^\beta, \Phi_0(u) = |u|^2, s = \frac{2(\beta-p)}{\beta(\beta-2)} \). Then \( 0 \leq s \leq 1 \) and \( \Phi_s(u) = |u|^p \), \( \lim_{s \to 0} \frac{2}{s} = p; \lim_{s \to \infty} \frac{2}{s} = \frac{1}{q} \), by the Theorem 2.3 we get the second inequality. \( \square \)
3. Some geometrical properties

This section we contains some geometrical properties of noncommutative Orlicz spaces. These include uniform convexity, uniform smoothness which generalize the results of noncommutative $L^p$ spaces. All these properties are based on Clarkson inequalities.

**Definition 3.1.** Let $X$ be a Banach space. We define its modulus of convexity by
\[
\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| = \varepsilon \right\}, 0 < \varepsilon < 2
\]
and its modulus of smoothness by
\[
\rho_X(t) = \sup \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : x, y \in X, \|x\| = \|y\| = 1 \right\}, t > 0.
\]

$X$ is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $2 \geq \varepsilon > 0$, and uniformly smooth if $\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0$.

**Theorem 3.1.** Let $\Phi$ be an N-function and $\Phi_s$ be the inverse which satisfies that $\Phi_s^{-1}(u) = [\Phi^{-1}(u)]^{1-s} [\Phi_0^{-1}(u)]^s = [\Phi^{-1}(u)]^{1-s} u^s$ where $0 < s \leq 1$ and $\Phi_0(u) = u^2$, then we have for $0 < \varepsilon \leq 2$,
\[
\delta_{L(\Phi_s)}(\varepsilon) \geq 1 - \frac{1}{2} \left[ 2^\frac{s}{2} - \varepsilon^\frac{s}{2} \right]^{\frac{2}{s}}
\]
and
\[
\rho_{L(\Phi_s)}(t) \leq \left( 1 + t^{\frac{2}{2-s}} \right)^{\frac{2-s}{2}} - 1.
\]

**Proof.** First, if $\|A - B\|_{(\Phi_s)} = \varepsilon$, then theorem 2.3 implies for $A, B \in L^{(\Phi_s)}(\mathcal{M}, \tau)$,
\[
\left( \|A + B\|_{(\Phi_s)}^{\frac{2}{s}} + \varepsilon^{\frac{2}{s}} \right)^{\frac{s}{2}} \leq 2^{\frac{s}{2}} \cdot 2^{\frac{2-s}{2}} = 2.
\]
Hence,
\[
1 - \frac{1}{2} \|A + B\|_{(\Phi_s)} \geq 1 - \frac{1}{2} \left[ 2^\frac{s}{2} - \varepsilon^\frac{s}{2} \right]^{\frac{s}{2}}.
\]
Taking infimum of $\|A\|_{(\Phi_s)} = \|B\|_{(\Phi_s)} = 1$ we can get the desired result and $L^{(\Phi_s)}(\mathcal{M}, \tau)$ is uniform convexity if $0 < \varepsilon \leq 2$, and reflexive.
Second, if \( \|A\|_{\Phi_s} = \|B\|_{\Phi_s} = 1 \), then since \( \frac{2}{q} \geq 2 \),

\[
\left[ \frac{1}{2} \left( \|A + tB\|_{\Phi_s} + \|A - tB\|_{\Phi_s} \right) \right]^\frac{2}{q} \leq \frac{1}{2} \left[ \|A + tB\|_{\Phi_s}^\frac{2}{q} + \|A - tB\|_{\Phi_s}^\frac{2}{q} \right]^{\frac{2}{q}} \\
\leq \frac{1}{2} \left[ 2^{\frac{2}{q}} \left( \|A\|_{\Phi_s}^\frac{2}{q} + \|B\|_{\Phi_s}^\frac{2}{q} \right) \right]^{\frac{2}{q}} \\
= \frac{1}{2} \left[ 2^{\frac{2}{q}} \left( 1 + t^\frac{2}{2-q} \right) \right]^{\frac{2}{q}} \\
= \left( 1 + t^\frac{2}{2-q} \right)^\frac{2-q}{q}.
\]

Hence,

\[
\frac{1}{2} \left( \|A + tB\|_{\Phi_s} + \|A - tB\|_{\Phi_s} \right) - 1 \leq \left( 1 + t^\frac{2}{2-q} \right)^\frac{2-q}{q} - 1.
\]

Taking the supremum on the left we can get the conclusion. Since \( t > 0 \), we have

\[ L^p(\tilde{\mathcal{M}}, \tau) \text{ is uniformly smooth.} \]

From corollary 2.1, we can easily get the following results which appeared on [10].

**Corollary 3.1.** Suppose that \( 1 < p < \infty \), \( q = \frac{p}{p-1} \), \( 0 < \varepsilon < \varepsilon \) and \( t > 0 \). Then for \( A, B \in L^p(\tilde{\mathcal{M}}, \tau) \), we have

1. If \( 1 < p < 2 \), then
   \[ \delta_{L^p}(\varepsilon) \geq \frac{\varepsilon^q}{q \cdot 2^q} \text{ and } \rho_{L^p(t)} \leq \frac{t^p}{p}. \]

2. If \( 2 < p < \infty \), then
   \[ \delta_{L^p}(\varepsilon) \geq \frac{\varepsilon^p}{p \cdot 2^q} \text{ and } \rho_{L^p(t)} \leq \frac{t^q}{q}. \]

3. \( L^p(\tilde{\mathcal{M}}, \tau) \) is uniformly convex and uniformly smooth. Consequently its reflexive.

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