AN INFINITE FAMILY OF VERTEX-PRIMITIVE 2-ARC-TRANSITIVE DIGRAPHS

MICHAEL GIUDICI, CAI HENG LI, AND BINZhou XIA

Abstract. We solve the long-standing existence problem of vertex-primitive 2-arc-transitive digraphs by constructing an infinite family of such digraphs.

Key words: digraphs; vertex-primitive; 2-arc-transitive

MSC2010: 05C20, 05C25

1. Introduction

A digraph (directed graph) \( \Gamma \) is a pair \((V, \rightarrow)\) with a set \(V\) (of vertices) and an antisymmetric irreflexive binary relation \(\rightarrow\) on \(V\). For a non-negative integer \(s\), an \(s\)-arc in \(\Gamma\) is a sequence \(v_0, v_1, \ldots, v_s\) of vertices with \(v_i \rightarrow v_{i+1}\) for each \(i = 0, \ldots, s - 1\). A 1-arc is also simply called an arc. We say \(\Gamma\) is \(s\)-arc-transitive if the group of all automorphisms (permutations on \(V\) that preserve the relation \(\rightarrow\)) of \(\Gamma\) acts transitively on the set of \(s\)-arcs. In sharp contrast with the situation for undirected graphs, where it is shown by Weiss [13] that finite undirected graphs of valency at least 3 can only be \(s\)-arc-transitive for \(s \leq 7\), there are infinite families of \(s\)-arc-transitive digraphs with unbounded \(s\) other than directed cycles. Constructions for such families of digraphs were initiated by Praeger in 1989 [11] and have stimulated a lot of research [2, 3, 4, 8, 9, 10].

A permutation group \(G\) on a set \(\Omega\) is said to be primitive if \(G\) does not preserve any nontrivial partition of \(\Omega\). We say a digraph is vertex-primitive if its automorphism group is primitive on the vertex set. Although various constructions of \(s\)-arc-transitive digraphs are known, no vertex-primitive \(s\)-arc-transitive digraph with \(s \geq 2\) has been found until now. Analysis of Praeger [11] has the shown that the most appropriate case to consider is the case where the automorphism group is an almost simple group. Here an almost simple group is a finite group whose socle (the product of the minimal normal subgroups) is nonabelian simple. Later in her survey paper [12], Praeger said “no such examples have yet been found despite considerable effort by several people” and thence asked the following question [12, Question 5.9]:

**Question 1.** Is there a finite 2-arc-transitive directed graph such that the automorphism group is primitive on vertices and is an almost simple group?

In the present paper, we answer this nearly 30 year old question in the affirmative by constructing an infinite family of vertex-primitive 2-arc-transitive digraphs that admit three-dimensional projective special linear groups as a group of automorphisms.

A digraph \((V, \rightarrow)\) is said to be \(k\)-regular if both the set \(\{u \in V \mid u \rightarrow v\}\) of in-neighbors of \(v\) and the set \(\{w \in V \mid v \rightarrow w\}\) of out-neighbors of \(v\) have size \(k\) for all \(v \in V\). Given a group \(G\), a subgroup \(H\) of \(G\) and an element \(g\) of \(G\) such

that \( g^{-1} \notin HgH \), there is a standard construction of a digraph \( \text{Cos}(G, H, g) \) whose vertices are the right cosets of \( H \) in \( G \) and two vertices satisfy \( Hx \rightarrow Hy \) if and only if \( yx^{-1} \in HgH \) (basic properties of this digraph can be found in Section \( 2 \)). Our main result is as follows.

**Theorem 1.1.** Let \( p > 3 \) be a prime number such that \( p \equiv \pm 2 \pmod{5} \), \( \varphi \) be the projection from \( \text{GL}_3(p^2) \) to \( \text{PGL}_3(p^2) \), \( G = \text{PSL}_3(p^2) < \text{PGL}_3(p^2) \), and \( a, b \in \mathbb{F}_{p^2} \) such that \( a^2 + a - 1 = 0 \) and \( b^2 + b + 1 = 0 \). Take

\[
g = \begin{pmatrix} b^{-1} & 0 & 1 \\ 0 & a-b & 0 \\ 1 & 0 & -b \end{pmatrix} \varphi, \quad x = \begin{pmatrix} a^{-1} & 1 & -a \\ -1 & a & -a^{-1} \\ -a & a^{-1} & 1 \end{pmatrix} \varphi, \quad y = \begin{pmatrix} -b^{-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & b & 0 \end{pmatrix} \varphi
\]

and \( H = \langle x, y \rangle \). Then \( H \cong A_6 \), \( g \) is an element of \( G \) such that \( g^{-1} \notin HgH \), and \( \text{Cos}(G, H, g) \) is a vertex-primitive 2-arc-transitive 6-regular digraph with automorphism group

\[
A = \begin{cases} 
\text{PSL}_3(p^2) : \langle \gamma \phi \rangle & \text{if } p \equiv 1 \pmod{3} \\
\text{PSL}_3(p^2) : \langle \phi \rangle = \text{PSL}_3(p^2) & \text{if } p \equiv 2 \pmod{3}
\end{cases}
\]

where \( \gamma \) and \( \phi \) are field and graph automorphisms of \( \text{PSL}_3(p^2) \) of order two.

The digraphs \( \text{Cos}(G, H, g) \) in Theorem 1.1 have vertex stabilizer \( H \cong A_6 \) and arc stabilizer \( H \cap g^{-1}Hg \cong A_5 \) (see Lemma 2.1). Examples of 2-arc-transitive digraphs with such vertex stabilizer and arc stabilizer are constructed in [3], but as pointed out in [3, Page 76], they are not vertex-primitive.

It is worth mentioning that, using the digraphs in Theorem 1.1, one can immediately construct 2-arc-transitive digraphs that are vertex-primitive of product action type using the construction in [11, Proposition 4.2].

We also remark that the digraphs \( \text{Cos}(G, H, g) \) in Theorem 1.1 is not 3-arc-transitive, so we ask the following question.

**Question 2.** Is there an upper bound on \( s \) for vertex-primitive \( s \)-arc-transitive digraphs that are not directed cycles?

## 2. Preliminaries

As mentioned in the introduction, there is a general construction for arc-transitive digraphs. We state the construction below along with its basic properties. The proof of these properties is elementary and the reader may consult [3].

Let \( G \) be a group, \( H \) be a subgroup of \( G \), \( V \) be the set of right cosets of \( H \) in \( G \) and \( g \) be an element of \( G \setminus H \) such that \( g^{-1} \notin HgH \). Define a binary relation \( \rightarrow \) on \( V \) by letting \( Hx \rightarrow Hy \) if and only if \( yx^{-1} \in HgH \) for any \( x, y \in G \). Then \( (V, \rightarrow) \) is a digraph, denoted by \( \text{Cos}(G, H, g) \). Right multiplication gives an action \( R_H \) of \( G \) on \( V \) which preserves the relation \( \rightarrow \), so that \( R_H(G) \) is a group of automorphisms of \( \text{Cos}(G, H, g) \). Recall that a digraph is said to be connected if and only if its underlying graph is connected. A vertex-primitive digraph is necessarily connected, for otherwise its connected components would form a partition of the vertex set that is invariant under digraph automorphisms.

**Lemma 2.1.** In the above notation, the following hold.

[Insert additional text here as needed.]
(a) $\text{Cos}(G, H, g)$ is $|H:H \cap g^{-1}Hg|$-regular.
(b) $\text{Cos}(G, H, g)$ is connected if and only if $(H, g) = G$.
(c) $R_H(G)$ is primitive on $V$ if and only if $H$ is maximal in $G$.
(d) $R_H(G)$ acts transitively on the set of arcs of $\text{Cos}(G, H, g)$.
(e) $R_H(G)$ acts transitively on the set of 2-arcs of $\text{Cos}(G, H, g)$ if and only if

\[ H = (H \cap g^{-1}Hg)(gHg^{-1} \cap H). \]

Proof. We only prove part (e) as the proof of the other parts is folklore. Let $u = Hg^{-1}$, $v = H$ and $w = Hg$ be three vertices of $\text{Cos}(G, H, g)$. Then $u \to v \to w$ since $g \in HgH$. Clearly, $G$ acts on the vertex set of $\text{Cos}(G, H, g)$ by right multiplication with the vertex stabilizer $G_v = H$. It follows that the arc stabilizer

\[ G_{vw} = H_w = \{ h \mid h \in H, Hgh = Hg \} = \{ h \mid h \in H, h \in g^{-1}Hg \} = H \cap g^{-1}Hg. \]

In the same vein, $G_{uw} = gHg^{-1} \cap H$. Now as $G$ already acts transitively on the set of arcs, $G$ is transitive on the set of 2-arcs of $\text{Cos}(G, H, g)$ if and only if $G_{uv}$ acts transitively on the set of out-neighbors of $v$, which is equivalent to $G_v = G_{uw}G_{uv}$ in light of Frattini’s argument. Thereby we deduce that $G$ is transitive on the set of 2-arcs of $\text{Cos}(G, H, g)$ if and only if

\[ H = G_v = (H \cap g^{-1}Hg)(gHg^{-1} \cap H), \]

as part (e) asserts. \hfill \Box

An expression of a group $G$ into the product of two subgroups $H$ and $K$ of $G$ is called a factorization of $G$, and is called a nontrivial factorization of $G$ if in addition $H$ and $K$ are both proper subgroups of $G$. The following lemma lists several equivalent conditions for a group factorization, whose proof is fairly easy and so is omitted.

Lemma 2.2. Let $H, K$ be subgroups of $G$. Then the following are equivalent.

(a) $G = HK$.
(b) $G = KH$.
(c) $G = (x^{-1}Hx)(y^{-1}Ky)$ for any $x, y \in G$.
(d) $|H \cap K||G| = |H||K|$.
(e) $H$ acts transitively on the set of right cosets of $K$ in $G$ by right multiplication.
(f) $K$ acts transitively on the set of right cosets of $H$ in $G$ by right multiplication.

We have seen in Lemma 2.1(e) that the transitivity of $R_H(G)$ on the set of 2-arcs is characterized by the group factorization (1). In the next lemma we shall see that if such a factorization is nontrivial then it already implies the condition $g^{-1} \notin HgH$ that is needed in the construction of the digraph $\text{Cos}(G, H, g)$. Note that the group factorization (1) is nontrivial if and only if $g \notin N_G(H)$. We also note that if the factorization (1) is nontrivial, then $H \cap g^{-1}Hg$ and $gHg^{-1} \cap H$ cannot be conjugate in $H$ by Lemma 2.2(c).

Lemma 2.3. Let $G$ be a group, $H$ be a subgroup of $G$ and $g$ be an element of $G$. If (1) holds and $g \notin N_G(H)$, then $g^{-1} \notin HgH$. 

**Proof.** Suppose that (1) holds, \( g \not\in N_G(H) \) and \( g^{-1} \in HgH \). Then \( g^{-1} = h_1gh_2 \) for some \( h_1, h_2 \in H \), so that \( H \cap g^{-1}Hg = H \cap h_1gHg \) and \( gHg^{-1} \cap H = gHgh_2 \cap H \).

Appealing to Lemma 2.2, we then deduce from (1) that
\[
H = (gHg^{-1} \cap H)(H \cap g^{-1}Hg) = (gHg^{-1} \cap H)h_1^{-1}(H \cap g^{-1}Hg)h_1
\]
\[
= (gHg^{-1} \cap H)h_1^{-1}(H \cap h_1gHg)h_1 = (gHg^{-1} \cap H)(H \cap gHgh_1).
\]

Hence
\[
H = Hh_1^{-1}h_2 = (gHg^{-1} \cap H)(H \cap gHgh_1)h_1^{-1}h_2
\]
\[
= (gHg^{-1} \cap H)(H \cap gHgh_2) = (gHg^{-1} \cap H)(gHg^{-1} \cap H) = gHg^{-1} \cap H,
\]
which implies \( g \in N_G(H) \), a contradiction. This proves the lemma. \( \Box \)

Let \( p \) be an odd prime number. Recall the Legendre symbol \( \left( \frac{\cdot}{p} \right) \) defined by
\[
\left( \frac{n}{p} \right) = \begin{cases} 
1 & \text{if } n \text{ is a square in } \mathbb{F}_p \\
-1 & \text{if } n \text{ is a non-square in } \mathbb{F}_p 
\end{cases}
\]
for any integer \( n \) coprime to \( p \). If \( q \) is also an odd prime number, then the quadratic reciprocity says that
\[
\left( \frac{q}{p} \right) \left( \frac{p}{q} \right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.
\]

### 3. Proof of Theorem 1.1

Throughout this section, let \( p, \varphi, G, a, b, x, y, H, g \) be as defined in Theorem 1.1.

\[
z = \begin{pmatrix} 0 & 0 & 1 \\
-1 & 0 & 0 \\
0 & -1 & 0 \end{pmatrix}^\varphi
\]
and \( w = zg^{-1}g^{-1} \). From the definition of \( a \) and \( b \) we derive
\[
(a - b)(a + b + 1) = (a^2 + a) - (b^2 + b) = 1 - (-1) = 2.
\]

Thus
\[
g^{-1} = \begin{pmatrix} b & 0 & 1 \\
0 & a + b + 1 & 0 \\
1 & 0 & -b^{-1} \end{pmatrix}^\varphi
\]

since
\[
\begin{pmatrix} b^{-1} & 0 & 1 \\
0 & a - b & 0 \\
1 & 0 & -b \end{pmatrix} \begin{pmatrix} b & 0 & 1 \\
0 & a + b + 1 & 0 \\
1 & 0 & -b^{-1} \end{pmatrix}
\]
\[
= \begin{pmatrix} 2 & 0 & 0 \\
0 & (a - b)(a + b + 1) & 0 \\
0 & 0 & 2 \end{pmatrix}
\]
\[
= \begin{pmatrix} 2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \end{pmatrix}.
\]
By virtue of the equalities $a^2 + a - 1 = 0$ and $b^2 + b + 1 = 0$ we can write each element of $\mathbb{Z}[a, a^{-1}, b, b^{-1}]$ as a linear combination of $ab, a, b$ and 1 with coefficients in $\mathbb{Z}$. In this way, we have

$$x^2 = \begin{pmatrix} 2 & 2a & -2a - 2 \\ -2a & -2a - 2 & -2 \\ -2a - 2 & 2 & -2a \end{pmatrix} \varphi = \begin{pmatrix} -1 & -a & a + 1 \\ a & a + 1 & 1 \\ a + 1 & -1 & a \end{pmatrix} \varphi,$$

$$x^2y = \begin{pmatrix} b + 1 & -ab - b & a \\ -ab - a & -b & a + 1 \\ -ab - a - b - 1 & -ab & 1 \end{pmatrix} \varphi,$$

$$yx = \begin{pmatrix} ab + a + b + 1 & b + 1 & -ab - a \\ -a & a + 1 & 1 \\ -b & ab & -ab - b \end{pmatrix} \varphi,$$

and

$$xyx = \begin{pmatrix} 2ab + 2b + 2 & 2ab + 2a + 2 & 0 \\ -2 & -2b & 2ab + 2a + 2b \\ -2b - 2 & 2 & -2ab + 2 \end{pmatrix} \varphi = \begin{pmatrix} ab + b + 1 & ab + a + 1 & 0 \\ -1 & -b & ab + a + b \\ -b - 1 & 1 & -ab + 1 \end{pmatrix} \varphi.$$

**Lemma 3.1.** $g$ is an element of $G$.

**Proof.** Note that $g \in G$ if and only if

$$d := \begin{vmatrix} b^{-1} & 0 & 1 \\ 0 & a - b & 0 \\ 1 & 0 & -b \end{vmatrix}$$

is a nonzero cube in $\mathbb{F}_{p^2}$, which is equivalent to $d^{(p^2 - 1)/3} = 1$. Write $\alpha = -(2a + 1)$ and $\beta = 2b + 1$. Then

$$d = -(a - b) - (a - b) = -2(a - b) = \alpha + \beta,$$

$$\alpha^2 = 4a^2 + 4a + 1 = 5$$

and $\beta^2 = 4b^2 + 4b + 1 = -3$. Since $p \equiv \pm 2 \pmod{5}$, we have

$$\left( \frac{5}{p} \right) = (-1)^{\frac{p-1}{2} \cdot \frac{5}{p}} \left( \frac{p}{5} \right) = \left( \frac{p}{5} \right) = \left( \frac{\pm 2}{5} \right) = -1,$$

so that 5 is a non-square in $\mathbb{F}_p$. It follows that the two square roots $\alpha$ and $-\alpha$ of 5 in $\mathbb{F}_{p^2}$ both lie in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$. Accordingly, the map $\sigma$ defined by $(\lambda \alpha + \mu)^\sigma = \lambda(-\alpha) + \mu$ for any $\lambda, \mu \in \mathbb{F}_p$ is a nontrivial element of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$, and then it must coincide with the Frobenius automorphism of $\mathbb{F}_{p^2}$ as $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p) = 2$. As a consequence, $\alpha^p = -\alpha$.

First assume $p \equiv 1 \pmod{3}$. Then

$$\left( \frac{-3}{p} \right) = \left( \frac{-1}{p} \right) \left( \frac{3}{p} \right) = (-1)^{\frac{p-1}{2} \cdot \frac{3}{p}} \left( \frac{3}{p} \right) = (-1)^{\frac{p-1}{2}} \cdot (-1)^{\frac{p-1}{2} \cdot \frac{3}{p}} \left( \frac{p}{3} \right) = \left( \frac{p}{3} \right) = \left( \frac{1}{3} \right) = 1,$$
which means that $-3$ is a square in $\mathbb{F}_p$. It follows that $\beta \in \mathbb{F}_p$ and so $\beta^p = \beta$. This in conjunction with (3) gives

$$d^p = (\alpha + \beta)^p = \alpha^p + \beta^p = -\alpha + \beta = 2(a + b + 1) = \frac{4}{a - b} = \frac{-8}{\alpha + \beta} = \frac{-8}{d}.$$

Therefore, $d^{p+1} = -8$, and then

$$d^{\frac{p+1}{3}} = (d^{p+1})^\frac{p+1}{3} = (-8)^\frac{p+1}{3} = ((-2)^3)^\frac{p+1}{3} = (-2)^{p-1} = 1.$$

Next assume $p \equiv 2 \pmod{3}$. Then

$$\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right) \left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}} \left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}} \cdot (-1)\frac{3-1}{2} \frac{p-1}{2} \left(\frac{p}{3}\right) = \left(\frac{p}{3}\right) = \left(\frac{2}{3}\right) = -1,$$

which means that $-3$ is a non-square in $\mathbb{F}_p$. Accordingly, $\beta \notin \mathbb{F}_p$ and hence $\beta^p \neq \beta$. This together with the observation $(\beta^p)^2 = (\beta^2)^p = (-3)^p = -3$ leads to $\beta^p = -\beta$. It follows that $d^p = (\alpha + \beta)^p = \alpha^p + \beta^p = -\alpha - \beta = -d$, whence $d^{p-1} = -1$ as $d = -2(a - b) \neq 0$ by (3). As a consequence,

$$d^{\frac{p+1}{3}} = ((d^{p-1})^2)^\frac{p+1}{3} = ((-1)^2)^\frac{p+1}{3} = 1,$$

as desired. \qed

**Lemma 3.2.** The orders of $x, y, z$ and $w$ are $5, 2, 3$ and $4$, respectively,

$$w = (x^2 y) x y x (x^2 y)^{-1} \in H \quad \text{and} \quad z = x^{-2} y x w^{-1} \in H.$$

**Proof.** To calculate the order of $x$ we consider the characteristic polynomial $\chi(\lambda)$ of

$$\begin{pmatrix}
-1 & 1 & -a \\
-1 & a & -a^{-1} \\
-a & a^{-1} & 1
\end{pmatrix} = \begin{pmatrix}
a + 1 & 1 & -a \\
-1 & a & -a - 1 \\
-a & a + 1 & 1
\end{pmatrix}.$$

Direct computation shows

$$\chi(\lambda) = \lambda^3 - (2a + 2)\lambda^2 + (a^2 + 5a + 3)\lambda - 6a^2 - 6a - 2 = \lambda^3 - (2a + 2)\lambda^2 + (4a + 4)\lambda - 8 = (\lambda - 2)(\lambda^2 - 2a\lambda + 4)$$

and then $\chi(\lambda)(\lambda^2 + 2a\lambda + 2\lambda + 4) = \lambda^5 - 32$. Therefore, $\chi(\lambda)$ has three distinct roots over $\mathbb{F}_p$, and each of them is a $5$th root of $32$. Hence $x$ has order $5$. 

It is evident that the orders of $y$ and $z$ are 2 and 3, respectively. Now we calculate the order of $w = zg^{-1}g^{-1}$. In view of (4) we have

\[
(9) \quad w = \left(0 \ 0 \ -1 \right) \varphi \left(0 \ a - b \ \ 1 \right) \varphi \left(0 \ 1 \ 0 \right) \varphi g^{-1}
\]

\[
= \left(-b \ -1 \ 0 \right) \varphi \left(0 \ 0 \ a - b \right) g^{-1}
\]

\[
= \left(-b \ -1 \ 0 \right) \varphi \left(0 \ 0 \ a - b \right) \left(0 \ a + b + 1 \ 0 \right) \varphi \left(b \ 0 \ 1 \right)
\]

\[
= \left(b + 1 \ -a - b - 1 \ -b \right) \varphi \left(-b \ -ab - a - b \ 0 \right) \varphi \left(a - b \ ab + a + 1 \right).
\]

Let

\[
\omega(\lambda) = \begin{vmatrix} \lambda - b - 1 & a + b + 1 & b \\ b & \lambda + ab + a + b & 1 \\ -a + b & 0 & \lambda - ab - a - 1 \end{vmatrix}
\]

Then

\[
\omega(\lambda) = \lambda^3 - 2\lambda^2 - (a^2b^2 + 2a^2b + ab^2 + a^2 + 2ab + 3b^2 + a + 2b - 1)\lambda
\]

\[
+ a^2b^3 + 5a^2b^2 + ab^2 + 5a^2b + 5ab^2 - b^3 + 5ab + 3b^2 + 3b
\]

\[
= \lambda^3 - 2\lambda^2 + 4\lambda - 8
\]

\[
= (\lambda - 2)(\lambda^2 + 4)
\]

Thus $\omega(\lambda)$ has three distinct roots over $\mathbb{F}_{p^2}$, and each of them is a 4th root of 16. Therefore, $w$ has order 4.

It remains to verify $w(x^2y) = (x^2y)xyx$ and $x^2zw = yx$. Writing each element of $\mathbb{Z}[a, b]$ as a linear combination of $ab, a, b$ and 1 with coefficients in $\mathbb{Z}$, we obtain

\[
\begin{pmatrix} b + 1 & -a - b - 1 & -b \\ -b & -ab - a - b & -1 \\ a - b & 0 & ab + a + 1 \end{pmatrix} \begin{pmatrix} b + 1 & -ab - b & a \\ -ab - a & -b & -a - 1 \\ -ab - a - b - 1 & -ab & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} -2a + 2b & 0 & 2ab + 2a + 2 \\ 2b + 2 & -2a - 2b - 2 & 2b \\ -2b & -2ab - 2a - 2b & 2 \end{pmatrix}
\]

\[
= \begin{pmatrix} b + 1 & -ab - b & a \\ -ab - a & -b & -a - 1 \\ -ab - a - b - 1 & -ab & 1 \end{pmatrix} \begin{pmatrix} ab + b + 1 & ab + a + 1 & 0 \\ ab + a + 1 & -1 & -b \ a + ab + b \\ -b & 1 & -ab + 1 \end{pmatrix}
\]
This together with (6), (8) and (9) shows that \( w(x^2y) = (x^2y)yx \). Similarly, one combines (5), (9), (7) and the equality
\[
\begin{pmatrix}
-1 & -a & a + 1 \\
 a & a + 1 & 1 \\
 a + 1 & -1 & a
\end{pmatrix}
\begin{pmatrix}
0 & 0 & -1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
b + 1 & -a - b - 1 & -b \\
-b & -ab - a - b & -1 \\
-a & 0 & ab + a + 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
-2ab - 2b & -2b & 2ab \\
2ab + 2a & -2ab - 2a - 2b - 2 & -2b - 2 \\
-2 & 2a & -2a - 2
\end{pmatrix}
\]
\[
= -2(b + 1) \begin{pmatrix}
ab + a + b + 1 & b + 1 & -ab - a \\
-a & a + 1 & 1 \\
-b & ab & -ab - b
\end{pmatrix}
\]
to see that \( x^2zw = yx \). Hence the lemma is true. \( \square \)

**Lemma 3.3.** \( H \) is a maximal subgroup of \( G \) isomorphic to \( A_6 \).

**Proof.** By (7), \( yx \) is the image of
\[
\begin{pmatrix}
ab + a + b + 1 & b + 1 & -ab - a \\
-a & a + 1 & 1 \\
-b & ab & -ab - b
\end{pmatrix}
\]
under \( \varphi \). Let
\[
\psi(\lambda) = \begin{vmatrix}
\lambda - ab - a - b - 1 & -b - 1 & ab + a \\
\lambda - a - 1 & -1 & \\
-\lambda - ab & \lambda + ab + b
\end{vmatrix}.
\]
We have
\[
\psi(\lambda) = \lambda^3 - (2a + 2)\lambda^2 - (a^2b^2 + a^2b + 3ab^2 - a^2 + 3ab + b^2 - 3a + b - 1)\lambda
+ 6a^2b^2 + 6a^2b + 6ab^2 + 6ab + 2b^2 + 2b
= \lambda^3 - (2a + 2)\lambda^2 + (4a + 4)\lambda - 8
= (\lambda - 2)(\lambda^2 - 2a\lambda + 4),
\]
whence \( \psi(\lambda)(\lambda^2 + 2a\lambda + 2\lambda + 4) = \lambda^5 - 32 \). Therefore, \( \chi(\lambda) \) has three distinct roots over \( \mathbb{F}_p \), and each of them is a 5th root of 32. It follows that \( (yx)^5 = 1 \) and thus \( (xy)^5 = 1 \). Recall from Lemma 3.2 that \( x, y \) and \( w = (x^2y)yx(x^2y)^{-1} \) are of order 5, 2 and 4, respectively. Accordingly, \( x^5 = y^2 = (xyx)^4 = 1 \). Now \( H \) is a factor group of
\[
\langle X, Y \mid X^5 = Y^2 = (XY)^5 = (XYX)^4 = 1 \rangle,
\]
which is isomorphic to the finite simple group \( A_6 \). Hence \( H = \langle x, y \rangle \cong A_6 \). This in turn forces \( H \subseteq G \) since \( H \) is nonabelian simple and \( G \) is a normal subgroup of index three in \( \text{PGL}_3(p^2) \).

Recall that \( p > 3 \) and \( p \equiv \pm 2 \) (mod 5). According to the classification of subgroups of \( \text{PSL}_2(p^2) \) (see for example [1]), \( \text{PSL}_2(p^2) \) has no subgroup isomorphic to \( A_6 \). Then inspecting the list of maximal subgroups of \( \text{PSL}_3(p) \) and \( \text{PSU}_3(p) \) (see for example [1]) we know that neither \( \text{PSL}_3(p) \) nor \( \text{PSU}_3(p) \) has a subgroup isomorphic to \( A_6 \). Let \( M \) be a maximal subgroup of \( G = \text{PSL}_3(p^2) \) containing \( H \). From the list of maximal subgroups of \( G \) we deduce that either \( M \cong A_6 \) or \( H \) is contained
Lemma 3.4. \( H = (H \cap g^{-1}Hg)(gHg^{-1} \cap H) \) with \( H \cap g^{-1}Hg \cong A_5 \).

Proof. Recall from Lemma 3.2 that \( x, y, z \) and \( w \) are elements of \( H \) of order 5, 2, 3 and 4, respectively. Also, Lemma 3.3 asserts that \( H \cong A_6 \) is maximal in \( G \). Hence \( N_G(H) = H \), for \( G \) is a simple group. From \( a^2 + a - 1 = 0 \) and \( b^2 + b + 1 = 0 \) we deduce that \( a^{-1} + b^{-1} = a - b \) and \( 1 - ab^{-1} = ab + a^{-1} \). Thereby we have

\[
\begin{align*}
gx &= \begin{pmatrix} b^{-1} & 0 & 1 \\ 0 & a - b & 0 \\ 1 & 0 & -b \end{pmatrix} \varphi \begin{pmatrix} a^{-1} & 1 & -a \\ -1 & a & -a^{-1} \\ -a & a^{-1} & 1 \end{pmatrix} \\
&= \begin{pmatrix} a^{-1}b^{-1} - a & a^{-1} + b^{-1} & 1 - ab^{-1} \\ -a + b & a^2 - ab & -1 + a^{-1}b \\ ab + a^{-1} & 1 - a^{-1}b & -a - b \end{pmatrix} \\
&= \begin{pmatrix} a^{-1}b^{-1} - a & a - b & ab + a^{-1} \\ -a^{-1} - b^{-1} & a^2 - ab & -1 + a^{-1}b \\ 1 - ab^{-1} & 1 - a^{-1}b & -a - b \end{pmatrix} \\
&= \begin{pmatrix} a^{-1} & 1 & -a \\ -1 & a & -a^{-1} \\ -a & a^{-1} & 1 \end{pmatrix} \varphi \begin{pmatrix} b^{-1} & 0 & 1 \\ 0 & a - b & 0 \\ 1 & 0 & -b \end{pmatrix} = xg.
\end{align*}
\]

As a consequence, \( x \in H \cap g^{-1}Hg \cap gHg^{-1} = (H \cap g^{-1}Hg) \cap (gHg^{-1} \cap H) \).

Suppose \( g \in H \). Then since \( g \) is a nontrivial element centralizing \( x \) and any subgroup of order five in \( A_6 \) is self-centralizing, we derive that \( \langle g \rangle = \langle x \rangle \). However, the fact that any element of \( \langle g \rangle \) has form

\[
\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}
\]

implies \( x \notin \langle g \rangle \), a contradiction. This shows that \( g \notin H \), whence \( H \neq g^{-1}Hg \) as \( N_G(H) = H \). Since \( w = zg^{-1}g^{-1} \in H \), we have \( gHg^{-1} = w^{-1}z \in H \) and thus \( z \in g^{-1}Hg \). Now \( H \cap g^{-1}Hg \) is a proper subgroup of \( H \cong A_6 \) that contains an element \( x \) of order five and an element \( z \) of order three. We conclude that \( H \cap g^{-1}Hg \cong A_5 \).

If \( H \cap g^{-1}Hg = gHg^{-1} \cap H \), then \( z \in gHg^{-1} \cap H \) due to \( z \in H \cap g^{-1}Hg \). But this yields \( w = z(gz^{-1}g^{-1}) \in gHg^{-1} \), contrary to the fact that \( gHg^{-1} \cap H = g(H \cap g^{-1}Hg)g^{-1} \cong A_5 \) has no element of order four. Therefore, \( H \cap g^{-1}Hg \neq gHg^{-1} \cap H \). Note that the intersection of any two distinct subgroups of \( A_6 \) that are both isomorphic to \( A_5 \) has order either 10 or 12. We infer from \( x \in (H \cap g^{-1}Hg) \cap (gHg^{-1} \cap H) \) that \( |(H \cap g^{-1}Hg) \cap (gHg^{-1} \cap H)| = 10 \). Hence \( H = (H \cap g^{-1}Hg)(gHg^{-1} \cap H) \), completing the proof.

Proof of Theorem 1.1. From Lemmas 3.1, 3.3 we know that \( g \) is an element of \( G \), \( H \) is a maximal subgroup of \( G \) isomorphic to \( A_6 \), and \( H = (H \cap g^{-1}Hg)(gHg^{-1} \cap H) \) is a nontrivial factorization of \( H \) with \( H \cap g^{-1}Hg \cong A_5 \). As a consequence,
Lemma 2.3 implies that $g^{-1} \notin HgH$. Thus by Lemma 2.1 $\cos(G, H, g)$ is a vertex-
primitive 2-arc-transitive 6-regular digraph. Let $A$ be the group of automorphisms of $\cos(G, H, g)$ and $R_H(G)$ be the subgroup of $A$ induced by the right multiplication of $G$. We have $R_H(G) \cong G$ since $G$ is simple, and $A$ does not contain the alternating group $A_{6}$ since $\cos(G, H, g)$ is not a complete graph. Since $A$ contains the primitive group $R_H(G)$. [1] implies that $A$ has socle $\mathbb{PSL}_3(2^2)$.

Let $\gamma$ and $\phi$ be field and graph automorphisms of $\mathbb{PSL}_3(2^2)$ of order two such that $[\gamma, \phi] = 1$, and let $v$ and $w$ be the two vertices $H$ and $Hg$ respectively of $\cos(G, H, g)$, so that $v \rightarrow w$. The fact that $R_H(G)_v = R_H(H)$ is maximal in $R_H(G)$ of index $|G:H|$ implies that the vertex stabilizer $A_v$ is maximal in $A$ of index $|G:H|$. Since $A_v \supset R_H(G) \cong A_6$, it follows from [1] Table 8.4 that $\mathbb{PSL}_3(2^2) \leq A \cong \mathbb{PSL}_3(2^2) : \langle \gamma, \phi \rangle$ and $A_6 \leq A_v \leq \text{PGL}_2(9)$. Moreover, the fact that $R_H(G)_{vw} = R_H(H \cap g^{-1}Hg)$ is maximal in $R_H(G)_v$ of index 6 implies that $A_{vw}$ is maximal in $A_v$ of index 6. Hence $A_6 \leq A_v \leq S_6$, and so [1] Table 8.4 again implies that $\mathbb{PSL}_3(2^2) \leq A \leq X$, where

$$X = \begin{cases} 
\mathbb{PSL}_3(2^2) : \langle \gamma, \phi \rangle & \text{if } p \equiv 1 \pmod{3} \\
\mathbb{PSL}_3(2^2) & \text{if } p \equiv 2 \pmod{3}.
\end{cases}$$

Let $\overline{H}$ and $\overline{K}$ be the full preimages of $H$ and $H \cap g^{-1}Hg$ in $\text{SL}_3(2^2)$. Then by [1] Table 8.4 we have $\overline{H} = 3 \cdot A_6$ and so $\overline{K} = 3 \times A_5$. Since $A_5$ has no irreducible representation of dimension 2 over any field of characteristic $p$ we deduce that $\overline{K}$ is an irreducible subgroup of $\text{SL}_3(2^2)$. Thus by Schur’s lemma we have $C_G(H \cap g^{-1}Hg) = 1$, and so

$$N_G(H \cap g^{-1}Hg) \cong N_G(H \cap g^{-1}Hg)/C_G(H \cap g^{-1}Hg) \cong \text{Aut}(H \cap g^{-1}Hg) = S_5.$$ 

Since $S_5$ has no irreducible representation of dimension 3 over any field of characteristic $p$, it follows that

$$N_G(H \cap g^{-1}Hg) = H \cap g^{-1}Hg \cong A_5.$$

Let $n$ be the number of conjugates of $H$ in $G$ that contain $H \cap g^{-1}Hg$. Note that $H \cong A_6$ has 12 distinct subgroups isomorphic to $A_5$. By counting the number of pairs $(N_1, N_2)$ such that $N_1$ is conjugate to $H$ in $G$ and $N_1 > N_2 \cong A_5$, one obtains

$$\frac{|G|}{|N_G(H)|} \cdot 12 = \frac{|G|}{|N_G(H \cap g^{-1}Hg)|} \cdot n.$$

Accordingly, $H \cap g^{-1}Hg$ is contained in exactly

$$n = \frac{12|N_G(H \cap g^{-1}Hg)|}{|N_G(H)|} = \frac{12|A_5|}{|A_6|} = 2$$

subgroups of $G$ that are conjugate to $H$. Hence $H$ and $g^{-1}Hg$ are the only conjugates of $H$ in $G$ that contains $H \cap g^{-1}Hg$. It follows that $v$ and $w$ are the only vertices fixed by $H \cap g^{-1}Hg$, and $u$ and $v$ are the only vertices fixed by $H \cap gHg^{-1}$. Thus $H$ has exactly 2 orbits of length 6. Let $N = N_X(H) \cong S_6$. Then $N$ either fixes each $H$ orbit of length 6 or interchanges them. Since $H \cap g^{-1}Hg$ and $H \cap gHg^{-1}$ are not conjugate in $S_6$, it follows that $X$ fixes each $H$-orbit. Therefore, $A = X$ and the proof is complete.
Remark 3.5. Note that the normalizer $M$ of $H$ in $\text{PSL}_3(p^2)\langle \gamma, \phi \rangle$ is isomorphic to $\text{PGL}_2(9)$, which interchanges the two subgroups $H \cap g^{-1} H g$ and $H \cap g H g^{-1}$. Since $H$ has only 2 orbits of length 6, it follows that $M$ interchanges the 2 orbits of length 6. Thus the underlying graph of the digraph $\text{Cos}(G, H, g)$ admits $\text{PSL}_3(p^2)\langle \gamma, \phi \rangle$ as an arc-transitive group of automorphisms. Consequently, the underlying graph of $\text{Cos}(G, H, g)$ is not half-arc-transitive of valency 12. It is shown in [5] that 12 is the smallest possible valency for a vertex-primitive half-arc-transitive graph and one infinite family of examples was given.

ACKNOWLEDGEMENTS. This research was supported by Australian Research Council grant DP150101066.

REFERENCES

[1] J. N. Bray, D. F. Holt and C. M. Roney-Dougal, The maximal subgroups of the low-dimensional finite classical groups, Cambridge University Press, Cambridge, 2013.
[2] P. J. Cameron, C. E. Praeger and N. C. Wormald, Infinite highly arc transitive digraphs and universal covering digraphs, Combinatorica 13 (1993), no. 4, 377–396.
[3] M. Conder, P. Lorimer and C. Praeger, Constructions for arc-transitive digraphs, J. Austral. Math. Soc. Ser. A 59 (1995), no. 1, 61–80.
[4] D. M. Evans, An infinite highly arc-transitive digraph, European J. Combin. 18 (1997), no. 3, 281–286.
[5] J. B. Fawcett, M. Giudici, C. H. Li, C. E. Praeger, G. Royle and G. Verret, Primitive permutation groups with a suborbit of length 5 and vertex-primitive graphs of valency 5, submitted. Available online at http://arxiv.org/abs/1606.02097.
[6] B. Huppert, Endliche Gruppen. I, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York 1967.
[7] M. W. Liebeck, C. E. Praeger and J. Saxl, A classification of the maximal subgroups of the finite alternating and symmetric groups, J. Algebra 111 (1987), no. 2, 365–383.
[8] A. Malnič, D. Marušič, N. Seifter and B. Zgrablić, Highly arc-transitive digraphs with no homomorphism onto $\mathbb{Z}$, Combinatorica 22 (2002), no. 3, 435–443.
[9] S. P. Mansilla and O. Serra, Construction of $k$-arc transitive digraphs, 17th British Combinatorial Conference (Canterbury, 1999), Discrete Math. 231 (2001), no. 1-3, 337–349.
[10] S. P. Mansilla and O. Serra, Automorphism groups of $k$-arc transitive covers, 6th International Conference on Graph Theory, Discrete Math. 276 (2004), no. 1-3, 273–285.
[11] C. E. Praeger, Highly arc transitive digraphs, European J. Combin. 10 (1989), no. 3, 281–292.
[12] C. E. Praeger, Finite primitive permutation groups: a survey, Groups-Canberra 1989, 63–84, Lecture Notes in Math. 1456, Springer, Berlin, 1990.
[13] R. Weiss, The nonexistence of 8-transitive graphs, Combinatorica 1 (1981), no. 3, 309–311.

School of Mathematics and Statistics, University of Western Australia, Crawley 6009, WA, Australia
E-mail address: michael.giudici@uwa.edu.au

Department of Mathematics, South University of Science and Technology of China, Shenzhen 518055, Guangdong, P. R. China
E-mail address: lich@sustc.edu.cn

School of Mathematics and Statistics, University of Western Australia, Crawley 6009, WA, Australia
E-mail address: binzhou.xia@uwa.edu.au