We present a probabilistic scheme for generating and purifying maximally-entangled states of two atoms inside an optical cavity via no-photon detection at the cavity output, where ideal detectors are not required. The intermediate mixed states can be continuously purified so as to violate Bell inequalities in a parametrized manner. The scheme relies on an additional strong-driving field that realizes, atypically, simultaneous Jaynes-Cummings and anti-Jaynes-Cummings interactions.

Generation and purification of maximally-entangled atomic states in optical cavities

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Entanglement, first considered by Schrödinger \cite{1}, is recognized nowadays as a cornerstone in the fundamentals of quantum physics and as a source of diverse applications in quantum information and computation \cite{2}. In particular, entangled states of discrete systems, such as two or more qubits, play an important role in testing fundamental properties of quantum theory. They allow one, for instance, to prove the nonlocal character of quantum mechanics versus local hidden-variable theories. Maximally-entangled states of two-qubit systems have already been produced experimentally in photonic systems \cite{3,4,5,6} and in the internal degrees of freedom of atoms interacting with a microwave cavity \cite{7,8,9,10}. In the case of trapped ions \cite{11,12,13}, maximally entangled states have been created through the manipulation of their collective motion, but cavity QED devices are needed for transferring the stored information. Despite the diverse and recent theoretical proposals, see Refs. \cite{8,9,10} and references therein, generation of maximally-entangled states of two atoms inside an optical cavity has not yet been accomplished in the lab. The relevance of this achievement strongly relies on the possibility of using atoms in quantum cavities as quantum networks \cite{11}, where quantum processing could take place among the entangled atoms and purities of the generated entangled state.

We consider two identical three-level atoms in Λ-configuration placed inside an optical cavity, see Fig. 1(a), where the allowed transitions $|c\rangle \leftrightarrow |g\rangle$ and $|e\rangle \leftrightarrow |g\rangle$ are excited off-resonantly by laser fields and a cavity mode, see Fig. 1(b). The metastable states $|g\rangle$ and $|e\rangle$ are resonantly coupled through level $|c\rangle$ by two effective interactions, one stemming from a laser field and the cavity mode and the other from two additional laser fields. The different frequency detunings, $\Delta$ and $\Delta'$, of these two Λ-processes prevent the system from undesired transitions. We assume that both atoms couple to the cavity mode with similar strength $g$, taken as real as all other coupling strengths $\{\Omega, \Omega'_1, \Omega'_2\}$ for the sake of simplicity. Then, the Hamiltonian for the system can be written as

\begin{equation}
H = \hbar \omega_c \sum_{j=1}^{2} |j\rangle \langle j_e| + \hbar \omega_c \sum_{j=1}^{2} |c_j\rangle \langle c_j| + \hbar \omega_f a^\dagger a
\end{equation}

\begin{equation}
+ \hbar g (a^\dagger)^2 \sum_{j=1}^{2} |e_j\rangle \langle e_j| + a \sum_{j=1}^{2} |c_j\rangle \langle e_j|
\end{equation}

\begin{equation}
+ \hbar \Omega (e^{-i(\omega_c - \Delta)t} \sum_{j=1}^{2} |c_j\rangle \langle g_j| + h.c.)
\end{equation}

\begin{equation}
+ \hbar \Omega'_1 (e^{-i(\omega_c - \Delta')t} \sum_{j=1}^{2} |c_j\rangle \langle e_j| + h.c.)
\end{equation}

\begin{equation}
+ \hbar \Omega'_2 (e^{-i(\omega_c - \Delta')t} \sum_{j=1}^{2} |e_j\rangle \langle g_j| + h.c.).
\end{equation}

Here, $\omega_c$, $\omega_e$, and $\omega_g$ are the Bohr frequencies associated with the transitions $|c\rangle \leftrightarrow |g\rangle$ and $|e\rangle \leftrightarrow |g\rangle$, respectively, while $\omega_f$ is the frequency of the cavity mode and $a$ ($a^\dagger$) the associated annihilation (creation) operator. To eliminate level $|c\rangle$ adiabatically, so as to discard spontaneous emission from our model, we require

\begin{equation}
\left\{\frac{\Omega}{\Delta}, \frac{g}{\Delta}, \frac{\Omega'_1}{\Delta'}, \frac{\Omega'_2}{\Delta'}\right\} \ll 1.
\end{equation}

In addition, and in order to avoid undesired atomic transitions, we need the following rotating-wave-approximation (RWA) inequalities

\begin{equation}
\Delta - \Delta' \gg \left\{\frac{\Omega'_1}{\Delta'}, \frac{\Omega'_2}{\Delta'}, \frac{\Omega g}{\Delta}, \frac{\Omega_g}{\Delta}\right\},
\end{equation}

turning Eq. (1) into the effective Hamiltonian

\begin{equation}
H_{\text{eff}} = -\hbar g \sum_{j=1}^{2} (a^\dagger \sigma_j^+ + a \sigma_j) - \hbar g \sum_{j=1}^{2} (\sigma_j^+ + \sigma_j),
\end{equation}

where $g_{\text{eff}} \equiv \Omega g/\Delta$, $\Omega'_{\text{eff}} \equiv \Omega'_1, \Omega'_2/\Delta'$, $\sigma_j^+ \equiv |e_j\rangle \langle g_j|$ and $\sigma_j \equiv |g_j\rangle \langle e_j|$. In Eq. (4), AC Stark shifts are assumed to be corrected by retuning the laser frequencies \cite{17}. 

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\end{flushright}
We will describe a scheme for the case of a three-level atom. Here, we have shown that a similar effective Hamiltonian can be obtained from Eq. (19) for the two-level problem. ∆ and ∆' are frequency detunings. The states are called dark states in the literature [18]. Note that the size of the coherent state, estimated by the amplitude α of Eq. (9), is proportional to the time τ of the unitary process described by the effective Hamiltonian of Eq. (4). If |α| is large enough (|α| ≥ 2), such that we can consider the states |−α⟩, |0⟩ and |α⟩ as mutually orthogonal, then a measurement of the vacuum field state with the atomic state |Ψ+⟩ through the whole unitary evolution. These states are called dark states in the literature [18]. However, one would require (unavailable) detectors with high efficiency and the (already available) strong-coupling regime of optical cavities, see Refs. [21].

It is possible to extend our method to less demanding regimes and more realistic conditions, involving field damping, weak-coupling regime and finite efficiency detection, without increasing the complexity of the experimental requirements. We then write down a master equation describing the atom-field dynamics

\[ \dot{\rho}_{at-f} = -i \hbar^{-1} [H_{eff}^{int}, \rho_{at-f}] + L \rho_{at-f}, \]

where the (field) dissipative term is described by

\[ L \rho_{at-f} = -\frac{\kappa}{2}(a a_{\rho_{at-f}} - 2 a a_{\rho_{at-f}} a + a_{\rho_{at-f}} a a_{\rho_{at-f}}). \]

The master equation in Eq. (10) can be solved analytically by means of phase-space techniques [22], representing an unusual case of a solvable master equation involving coherent driving and dissipation. When the atom-field system is prepared initially in its ground state, the steady-state solution of the master equation reads

\[ \rho_{at-f}^{ss} = \frac{1}{4} |+\rangle \langle +| \otimes |\overline{2\bar{\alpha}}\rangle \langle \overline{2\bar{\alpha}}| + \frac{1}{4} |\overline{2\bar{\alpha}}\rangle \langle \overline{2\bar{\alpha}}| \otimes |+\rangle \langle +| + \frac{1}{2} |\Psi^+\rangle \langle \Psi^+| \otimes |0\rangle \langle 0|, \]

where

\[ |+\rangle \equiv |+\rangle |+\rangle = \frac{1}{\sqrt{2}}(|g_1\rangle + |e_1\rangle) \otimes \frac{1}{\sqrt{2}}(|g_2\rangle + |e_2\rangle), \]

\[ |-\rangle \equiv |-\rangle |-\rangle = \frac{1}{\sqrt{2}}(|g_1\rangle - |e_1\rangle) \otimes \frac{1}{\sqrt{2}}(|g_2\rangle - |e_2\rangle), \]

such that the maximally-entangled state

\[ |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|-\rangle |+\rangle + |+\rangle |-\rangle), \]

and the amplitude of the coherent states generated is

\[ \alpha(\tau) = i \frac{g_{eff} \tau}{2}. \]
with $\tilde{\alpha} = i \frac{2\pi}{\lambda}$. In contrast to the pure state of Eq. (3), the steady state of the atom-cavity system is a mixed state with no quantum correlation between the atoms and the field. The remarkable feature of the state in Eq. (11) is that the atomic state $|\Psi^+\rangle$ is still correlated with the vacuum of the cavity field. Henceforth, if one performs a first no-photon measurement of the cavity field in the steady state, the projected (normalized) atomic density operator is

$$
\rho_{at}^{ss} = \frac{e^{-|2\tilde{\alpha}|^2}}{1 + e^{-|2\tilde{\alpha}|^2}} (|++\rangle\langle++| + |+-\rangle\langle-+| + |\tilde{\alpha}^\dagger\tilde{\alpha}\rangle) \langle \tilde{\alpha}^\dagger\tilde{\alpha} | \Psi^+ \rangle \langle \Psi^+ |.
$$

(12)

When $|\tilde{\alpha}| > 1$, in the strong-coupling regime $[20, 21]$, the condition $|2\tilde{\alpha}|^2 \gg 1$ is automatically fulfilled and Eq. (12) reduces to the maximally entangled atomic state

$$
\rho_{at}^{ss} = |\Psi^+\rangle \langle \Psi^+ |.
$$

(13)

with unity fidelity and success probability $1/2$.

If $|\tilde{\alpha}| < 1$, in the weak-coupling regime, the desired atomic state of Eq. (13) may be contaminated by other contributions as shown in Eq. (12). In this case, we are still able to develop a protocol which purifies the atomic state $|\Psi^+\rangle$ via a successive application of the same scheme. We repeat our procedure, shining a similar laser system on our new initial atom-cavity state

$$
\rho_{at - j} = \rho_{at}^{ss} \otimes |0\rangle \langle 0 |
$$

(14)

until it reaches a new steady state, followed by a measure of a no-photon event with a certain finite probability. By repeating this sequence of steps $N$ times, we arrive at the projected atomic density operator

$$
\rho_{at}^{ss}(N) = \frac{e^{-N|2\tilde{\alpha}|^2}}{1 + 2e^{-N|2\tilde{\alpha}|^2}} (|++\rangle\langle++| + |+-\rangle\langle-+| + |\tilde{\alpha}^\dagger\tilde{\alpha}\rangle) \langle \tilde{\alpha}^\dagger\tilde{\alpha} | \Psi^+ \rangle \langle \Psi^+ |.
$$

(15)

For a given $|\tilde{\alpha}|$, even in the weak-coupling regime, we can always choose a number of repetitions $N$ such that $e^{-N|2\tilde{\alpha}|^2} = 0$, warranting a highly pure atomic state $|\Psi^+\rangle$. In this case, the fidelity of the state $|\Psi^+\rangle$ is given by

$$
F(N) = \langle \Psi^+ | \rho_{at}^{ss}(N) | \Psi^+ \rangle = \frac{1}{1 + 2e^{-N|2\tilde{\alpha}|^2}},
$$

(16)

with success probability

$$
P_{suc} = \frac{1}{2} \frac{1}{1 + e^{-|2\tilde{\alpha}|^2}} \prod_{m=2}^{N} \frac{1}{1 + 2e^{-m|2\tilde{\alpha}|^2}}.
$$

(17)

Note that even for $|\tilde{\alpha}| \sim 1$, $N = 1, 2$, fidelity $F(N) \sim 1$.

Even in the weak-coupling regime, $|\tilde{\alpha}| < 1$, the vacuum state $|0\rangle$ can be neatly distinguished from coherent states $|\pm \tilde{\alpha}\rangle$. This is a natural consequence of the fact that if one photon is emitted by measuring $|\pm \tilde{\alpha}\rangle$, but possibly not detected due to dark counts or any other noisy effect, there will be further emission of photons due to the continuous laser pumping mechanism $[22]$. In other words, our measurement is split in two distinct outcomes, or we have repeated opportunities of detecting no photon, and the protocol continues, or we have repeated opportunities of measuring single photons, which means that we should start again. In this way, we are able to overcome the detection efficiency problem.

It is not necessary to wait until the steady state of Eq. (11) is reached to realize this protocol. For example, one could implement the following variation: i) production of state in Eq. (3) followed by disconnection of lasers, ii) photo-detection (unaffected by decay process), iii) if no-photon was detected then back to (i) until desired fidelity is reached, iv) if a photon is detected then restart the protocol. This variation minimizes the time in which the lasers are on and assures no influence of spontaneous-emission effects at all.

It is important to estimate the influence of atomic localization on the generation of entanglement, as long as our method relies strongly on the production of dark states $[18]$. The fidelity of the purified dark state in Eq. (13) follows $F = 1/1 + \epsilon^2$. Here, $\epsilon \equiv \delta g/\kappa$, where $\delta g$ is the differential variation of the atom-field coupling due to the differential variation in the localization of the two atoms. The parameter $\epsilon$ changes very slowly with possible errors in the atomic locations, due to their intrinsic cosine dependence. For realistic parameters, see $[24]$, a maximal localization error of $10\%$ of an optical wavelength yields $\epsilon \approx 0.1$. This implies a fidelity $F > 0.99$, showing the robustness of our proposal.

It was proved $[23, 26]$ that a family of mixed states, similar to that of Eq. (15),

$$
\rho = \frac{1 - \lambda}{2} |++\rangle\langle++| + |\tilde{\alpha}^\dagger\tilde{\alpha}\rangle \langle \tilde{\alpha}^\dagger\tilde{\alpha} | \Psi^+ \rangle \langle \Psi^+ | + \lambda |\Psi^+\rangle \langle \Psi^+ |.
$$

(18)

violates the Bell inequality if and only if $\lambda > \frac{1}{\sqrt{2}}$. In our case $\lambda$ is a function of two parameters, $|\tilde{\alpha}|$ and $N$. In Fig. 2, we plot the surface $\lambda(|\tilde{\alpha}|, N)$ cut by a plane.
corresponding to the boundary value $\frac{1}{\sqrt{2}}$. There, we can clearly see that it is possible to cross the threshold parameter $\lambda = 1/\sqrt{2}$ by increasing the number $N$ of repetitions and the amplitude $\tilde{a}$ in the proposed scheme. This transition shows a local parametrized evolution from a mixture (with classical correlations) to a maximally-entangled atomic state (with quantum correlations), as was discussed in Refs. [23, 24].

Finally, we illustrate our protocol with a variant for direct generation of another maximally-entangled state. We assume that in the previous scheme, see Fig. 1, the coupling strength is $\Omega = 0$ and the detuning frequencies are $\Delta_1 = \Delta_2 = \Delta$. Suppose now that the atoms couple differently to the cavity mode in such a way that the coupling strengths have the same absolute values but the opposite phases ($g_1 = |g|, g_2 = -|g|$). In this case, the adiabatic elimination conditions of Eq. (2) reduce to
\[
\left\{ \Omega_1', \Omega_2', \frac{|g|}{\Delta} \right\} \ll 1. \tag{19}
\]
Therefore, the effective Hamiltonian in the interaction picture, after imposing the strong-(external)driving regime $\Omega_{\text{eff}}' \ll |g_{\text{eff}}'|$, with $g_{\text{eff}}' = \Omega_1' g/\Delta$, reads
\[
\hat{H}_{\text{int}}^{\text{eff}} = \hbar \frac{g_{\text{eff}}'}{2} (a^\dagger + a) \sum_{j=1}^{2} (-1)^j (\sigma_j^+ + \sigma_j^-). \tag{20}
\]

The Hamiltonian of Eq. (20) is slightly different from the Hamiltonian of Eq. (5) and, when substituted in Eq. (10), yields the atom-field steady state
\[
\rho_{\text{at-f}}^* = \frac{1}{4} \left| + \right\rangle \left\langle + \right| \otimes \left| 2\beta \right\rangle \left\langle 2\beta \right| + \frac{1}{4} \left| + \right\rangle \left\langle - \right| \otimes \left| -2\beta \right\rangle \left\langle -2\beta \right| + \frac{1}{2} |\Phi^+\rangle \langle \Phi^+| \otimes |0\rangle \langle 0|, \tag{21}
\]
where $|\beta\rangle$, $|\beta\rangle$ are coherent states with $|\beta| = \frac{g_{\text{eff}}'}{\kappa}$ and
\[
|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|1\rangle |+\rangle + |2\rangle |-\rangle)
\]
is another maximally-entangled Bell state. Purification of the state $|\Phi^+\rangle$, out of the steady state in Eq. (21), can be done by following steps similar to the ones before.

The coupling of atoms to the cavity mode, with similar or opposite phase, can be achieved with ion traps [24].

In conclusion, we have proposed a protocol for generating and purifying maximally-entangled states in the internal degrees of freedom of two atoms inside an optical cavity. Our protocol is of a quantum-non-demolition kind, where successive no-photon detection of the cavity field projects and purifies sequentially the desired atomic state. In contrast to recent proposals, our scheme produces atomic Bell states as a steady state of the two-atom-field interaction in correlation with the vacuum field state, which makes our method robust to decoherence. It combines a reasonably high success probability ($\sim 1/2$) with very high fidelity ($\sim 1$) for a wide range of parameters. The proposed scheme does not rely on a high detection efficiency as long as it is based on discrimination, through projection, between a vacuum field state (zero detector clicks) and orthogonal coherent states (necessarily more than one detector click). Furthermore, the procedure described above not only allows efficient creation of maximally-entangled states in two-atom states inside optical cavities under realistic conditions, but also suggests strategies for purifying chosen entangled states with known contamination states.

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