Index Policy for A Class of Partially Observable Markov Decision Processes

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This paper addresses an important class of restless multi-armed bandit (RMAB) problems that finds a broad application area in operations research, stochastic optimization, and reinforcement learning. There are $N$ independent Markov processes that may be operated, observed and offer rewards. Due to the resource constraint, we can only choose a subset of $M$ ($M < N$) processes to operate and accrue reward determined by the states of selected processes. We formulate the problem as an RMAB with an infinite state space and design an algorithm that achieves a near-optimal performance with low complexity. Our algorithm is based on Whittle’s original idea of index policy but can be implemented under more general scenarios, including continuous state space, relaxed indexability, online computations, etc.

Key words: restless multi-armed bandit, partial observation, index policy

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1. Introduction

The first multi-armed bandit (MAB) problem was proposed in 1933 in the context of clinical trial for adaptively selecting the best treatment over time [Thompson 1933]. In the classical Bayesian model of MAB, there are $N$ arms and a single player. At each discrete time (decision epoch), a player chooses one arm to operate and accrues certain amount of reward determined by the state of the arm. The state of the chosen arm transits to a new one according to a known Markovian rule while the states of other arms remain frozen. The observation model is assumed to be complete, i.e., the states of all arms can be observed before deciding which arm to choose. The objective is to maximize the total discounted reward over the infinite horizon [Gittins et al. 2011]. About 40 years later, Gittins (1979) solved the problem by showing that the optimal policy has an index structure, i.e., at each time one can compute an index (a real number) solely based on the current
state of an arm and choosing the arm associated with the highest index is optimal. Besides Gittins’ original proof of optimality based on an interchange argument, Whittle (1980) gave a proof by introducing retirement option which was further generalized to the restless MAB model. Weber (1992) gave a beautiful proof without any mathematical equation by an argument of fair charge, while Bertsimas and Niño-Mora (1996) took the achievable region approach for a proof based on linear programming and the duality theory. These four classical proofs of the optimality of Gittins index were elegantly summarized and extended by Frostig and Weiss (2016).

Whittle (1988) generalized the classical MAB to the restless bandit model, where each unselected arm can also change state (accordingly to another known Markovian rule) and offer reward. Furthermore, the player is not restricted to select only one but $M$ ($M < N$) arms at each time. Either extension of the above makes Gittins index suboptimal in general. Whittle introduced an index policy based on the idea of subsidy, i.e., by focusing on a single-armed bandit one can attach a fixed amount of reward (subsidy) to the arm when it is unselected (made passive) and Whittle index is defined as the minimum subsidy that makes selecting (activating) the arm or not equally optimal at its current state. This subsidy is reduced to Gittins index in the classical MAB model and decouples arms for computing Whittle index. Whittle showed that the subsidy is essentially the Lagrangian multiplier associated with a relaxed constraint on the average number of arms to activate over the infinite horizon, thus providing an upper bound for the original problem. By considering a large deviation theory applied on Markov jump processes under the time-average reward criterion, Weber and Weiss (1990) showed that Whittle index policy implemented under the strict constraint (i.e., choosing exactly $M$ arms with the highest indices at each time) converges to the upper bound with the relaxed constraint per-arm-wise as $N \to \infty$ with $M/N$ fixed under a sufficient condition that characterizes the global stability of a deterministic fluid dynamic system approximating the stochastic state evolution processes of all arms. Beside this asymptotic optimality result, Whittle index policy has shown a strong performance for various restless MAB models (see, e.g., Glazebrook et al. 2009, Liu and Zhao 2010). Various index policies have been
proposed for restless bandits with finite state spaces and finite time horizons with strong performance (see, e.g., Hu and Frazier 2017, Zayas-Cabán et al. 2019, Brown and Smith 2020, Gast et al. 2021). It is worth noticing that the general restless MAB with a finite state space is PSPACE-HARD (Papadimitriou and Tsitsiklis 1999), making it unlikely to discover an efficient algorithm in general. For Whittle index policy, the main complexity lies in verifying its existence (indexability) and the computation of the index as a function of each state of each arm. In this paper, we show that such computation can be done efficiently even that our problem has an infinite state space, thus yielding a low-complexity implementation of Whittle index policy with its near-optimality demonstrated through extensive numerical experiments.

The restless MAB (RMAB) is a special class of Markov Decision Processes (MDP) where system state vector is completely observed at the beginning of each decision epoch. However, many problems do not possess such a perfect observation model. Instead, only the selected arms will reveal their states to the player after arm selection is determined. This category of problems belongs to the class of Partially Observable MDP (POMDP), which encompasses a much wider application range than MDP (Sondik 1978). In this paper, the $N$ processes (arms) are modeled as Markov chains evolving over time, according to potentially different rules for state transitions and reward offering. At each time, the player chooses only $M$ ($M < N$) arms to observe and obtain reward determined by the observed states of chosen arms. The states of other unchosen arms remain unknown. To formulate the problem as an RMAB, we can use information state as a sufficient statistics for optimal control that characterizes the probability distribution of arm states based on past observations. For the case that each Markov chain has only 2 states, the problem was solved near-optimally by Whittle index policy (Liu and Zhao 2010, Liu et al. 2011). This paper extends those results to the case of $K$–state Markov chains for $K > 2$. As shown in the rest of this paper, this extension makes the problem fundamentally more complex. Our approach is to embrace a family of threshold policies that significantly simplifies the system dynamics while keeping the major benefits from the fundamental structure of Whittle’s relaxation. In the following, we first extend
Whittle’s relaxation from finite state space to the model with a continuous state space and then establish a general algorithmic framework to solve for an approximated Whittle index by analyzing the family of threshold polices. Finally, we will focus on the case of $K = 3$ and build a detailed algorithm to demonstrate its near-optimality.

2. RMAB Formulation and Whittle Index

In this section, we will formulate the multi-armed bandit problem as a partially observable Markov decision process and introduce the concept of Whittle Index. Consider a bandit machine with totally $N$ independent arms, each of which is modelled as a Markov process. For the $n$th arm ($n \in \{1, ..., N\}$), let $P(n) = \{p_{i,j}^{(n)}\}_{i,j \in \{0,1,2,\ldots,K_n-1\}}$ denote its state transition matrix and $B_{n,i}$ ($i \in \{0,1,2,\ldots,K_n-1\}$) the reward that can be obtained when the arm is observed in state $i$. Let $B_n = [B_{n,0}, B_{n,1}, B_{n,2},\ldots, B_{n,K_n-1}]$ be the reward vector for arm $n$. At each discrete time $t$, $M$ arms will be selected for observation (activated). Let $U(t) \subseteq \{1, ..., N\}$ ($|U(t)| = M$) be the set of arms that are observed at time $t$. The (random) reward obtained at time $t$ is given by

$$R_{U(t)}(t) = \sum_{n \in U(t)} B_{n,S_n(t)},$$

where $S_n(t) \in \{0,1,2,\ldots,K_n-1\}$ denotes the state of arm $n$ at time $t$. Our objective is to decide an optimal policy $\pi^*$ of choosing $M$ arms at each time such that the long-term reward is maximized in expectation. In this paper, we will focus on the expected total discounted reward objective function:

$$\pi^* = \arg\max_{\pi \in \Pi} \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \beta^{t-1} R_{U(t)}(t) \right],$$

where $\beta \in (0,1)$ is the discount factor for the convergence of the sum in the right-hand side of (2) and $\Pi$ the set of all feasible policies satisfying $|U(t)| = M$ at each time $t$.

2.1. Belief Vector as System State

Since no arm state is observable before $U(t)$ is decided at time $t$, we need an alternative representation of information for decision making. According to the general POMDP theory, the conditional probability distribution of the underlying Markovian states given all past knowledge is a sufficient
statistics (Sondik 1978). In our model, each arm will be associated with such a belief vector (or belief state), i.e., the conditional distribution of arm state given all past observations from this arm. Due to the Markovian property, it is sufficient to know the last state observation and the time it happened from this arm. Denoted by $\omega_n(t)$ the belief vector of arm $n$ at time $t$, we have

$$
\omega_n(t) = \left( \begin{array}{c} 
\Pr(S_n(t) = 0|\omega_n(1), S_n(\tau_n)) \\
\Pr(S_n(t) = 1|\omega_n(1), S_n(\tau_n)) \\
\vdots \\
\Pr(S_n(t) = K_n - 1|\omega_n(1), S_n(\tau_n)) 
\end{array} \right), \\
\Omega(t) = \left( \begin{array}{c} 
\omega_1(t) \\
\vdots \\
\omega_N(t) 
\end{array} \right),
$$

where $A'$ denotes the transpose of $A$ and $\tau_n$ is the time of last observation on arm $n$ and if the arm has never been observed, we set $\tau_n = -\infty$ and remove $S_n(\tau_n)$ from the condition. Initially, the belief vector $\omega_n(1)$ can be set as the stationary distribution $\bar{\omega}_n$ of the underlying Markov chain (corresponding to the case of $\tau_n = -\infty$):

$$
\omega_n(1) = \bar{\omega}_n = \lim_{k \to \infty} p^{(n)}(P^{(n)})^k,
$$

where $\bar{\omega}_n$ is the unique solution to $\omega P^{(n)} = \omega$ and $p$ an arbitrary probability distribution over the underlying state space of arm $n$. The limit in (3) can be taken under any norm since belief vectors are in a finite-dimensional vector space. It is also convenient to update the belief vector of each arm at each time according to the following rule:

$$
\omega_n(t + 1) = \left\{ \begin{array}{ll}
[p^{(n)}_{S_n(t),0}, p^{(n)}_{S_n(t),1}, \ldots, p^{(n)}_{S_n(t),K_n-1}], & n \in U(t) \\
\omega_n(t)P^{(n)}, & n \notin U(t)
\end{array} \right. ,
$$

Note that the belief update is deterministic if the arm is not chosen for observation at the time. For the case where the arm is not being observed for a consecutive sequence of time, we define the

1 Here we assume the underlying Markov chain with transition matrix $P^{(n)}$ is irreducible and aperiodic.
following operator for updating the belief vector continuously over \( k \) consecutive slots without any observation:

\[
\mathcal{T}_n^k(\omega_n(t)) = \\begin{pmatrix}
\Pr(S_n(t+k) = 0|\omega_n(t)) \\
\Pr(S_n(t+k) = 1|\omega_n(t)) \\
\vdots \\
\Pr(S_n(t+k) = K_n - 1|\omega_n(t))
\end{pmatrix}^{'}
\]

\[
= \omega_n(t)(P_n^{(n)})^k. \tag{5}
\]

Now the decision problem has a countable state space as modelled by the belief vector for a fixed initial \( \Omega(1) \) and an uncountable space for an arbitrarily chosen \( \Omega(1) \). This infinite-dimensional optimization problem can be formulated as

\[
\max_{\pi \in \Pi} \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \beta^{t-1} \sum_{n=1}^{N} \mathbb{1}(n \in U(t))B_{n,S_n(t)}|\Omega(1) \right] \tag{6}
\]

s. t. \( \sum_{n=1}^{N} \mathbb{1}(n \in U(t)) = M, \quad \forall \ t \geq 1. \tag{7} \)

It is clear that as the number of arms increases, the number of choices at each time grows geometrically. Furthermore, different choices lead to different updates of the belief vector, yielding a high complexity in solving the problem. In the following, we will extend Whittle’s original idea of arm decoupling for an index policy to our model which has an infinite state space consisting of belief vectors.

### 2.2. Whittle Index Policy

Whittle relaxed the strict constraint on the exact number of arms to choose at each time to requiring only \( M \) arms are chosen in an average sense. Particularly, we consider the following \textit{relaxed} form of problem (6):

\[
\max_{\pi \in \Pi} \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \beta^{t-1} \sum_{n=1}^{N} \mathbb{1}(n \in U(t))B_{n,S_n(t)}|\Omega(1) \right] \tag{8}
\]

s. t. \( \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \beta^{t-1} \sum_{n=1}^{N} \mathbb{1}(n \notin U(t))|\Omega(1) \right] = \frac{N - M}{1 - \beta}. \tag{9} \)
The relaxation from (7) to (9) allows us to analyze the dual problem with arms decoupled as detailed below. Applying the Lagrangian multiplier $\lambda$ to (9), we arrive at the following unconstrained optimization problem:

$$\max_{\pi \in \Pi} \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \beta^{t-1} \sum_{n=1}^{N} [1(n \in U(t))B_{n,S_n(t)} + \lambda 1(n \notin U(t))] | \Omega(1) \right].$$  \hspace{1cm} (10)

The above unconstrained optimization is equivalent to $N$ independent optimization problems as shown below:

$$\max_{\pi \in \Pi} \mathbb{E}_\pi \left[ \sum_{t=1}^{\infty} \beta^{t-1} [1(n \in U(t))B_{n,S_n(t)} + \lambda 1(n \notin U(t))] | \omega_n(1) \right], \quad n = 1, 2, \ldots, N.$$  \hspace{1cm} (11)

Therefore, it is sufficient to consider a single arm for solving problem (10). Note that the action applied on a single arm is either “selected (activated)” or “unselected (made passive)” at each time. We can thus focus on the single-armed problem (with Lagrangian multiplier $\lambda$) with state space consisting of all probability measures on the underlying Markov chain and a binary action space.

For simplicity, we will drop the subscript $n$ in consideration of a single-armed bandit without loss of generality. Let $V_{\beta,m}(\omega)$ denote the value of (11) with $\lambda = m$ and $\omega_n(1) = \omega$, it is straightforward to write out the dynamic equation of the single-armed bandit problem as follows:

$$V_{\beta,m}(\omega) = \max\{V_{\beta,m}(\omega; u = 1); V_{\beta,m}(\omega; u = 0)\},$$  \hspace{1cm} (12)

where $V_{\beta,m}(\omega; u = 1)$ and $V_{\beta,m}(\omega; u = 0)$ denote, respectively, the maximum expected total discounted reward that can be obtained if the arm is activated or made passive at the current belief state $\omega$, followed by an optimal policy in subsequent slots. Since we consider the infinite-horizon problem, a stationary optimal policy can be chosen and the time index $t$ is not needed in (12).

Let $p_i = [p_{i0}, p_{i1}, \ldots, p_{i(K-1)}], (i = 0, 1, \ldots, K - 1)$ denote the $i$-th row of $P$, we have

$$V_{\beta,m}(\omega; u = 1) = \omega B' + \beta \omega \begin{pmatrix} V_{\beta,m}(p_0) \\ V_{\beta,m}(p_1) \\ \vdots \\ V_{\beta,m}(p_{K-1}) \end{pmatrix},$$  \hspace{1cm} (13)
where $T^1(\omega)$ is the one-step state distribution as defined in (5). Without loss of generality, we assume $0 = B_0 \leq B_1 \leq \cdots \leq B_{K-1}$.

Define passive set $P(m)$ as the set of all belief states such that taking the passive action $u = 0$ is optimal:

$$P(m) \Delta= \{\omega : V_{\beta,m}(\omega; u = 1) \leq V_{\beta,m}(\omega; u = 0)\}. \hspace{1cm} (15)$$

It is clear that $P(m)$ changes from the empty set to the whole space of probability measures as $m$ increases from $-\infty$ to $\infty$. However, such change may not be monotonic as $m$ increases. If the passive set $P(m)$ increases monotonically with $m$, then for each value $\omega$ of the belief state, one can define the unique $m$ that makes it join $P(m)$ and stay in the set forever. Intuitively, such $m$ measures how attractive it is to activate the arm at the belief state $\omega$ compared to other belief states in a well-ordered manner: the larger $m$ required for it to be passive, the more incentives to activate at the belief state without $m$. This Lagrangian multiplier $m$ is thus called 'subsidy for passivity' by Whittle who formalized the following definition of indexability and Whittle index (Whittle 1988).

DEFINITION 1. A restless multi-armed bandit is indexable if for each single-armed bandit with subsidy, the passive set of arm states increases monotonically from $\emptyset$ to the whole state space as $m$ increases from $-\infty$ to $+\infty$. Under indexability, the Whittle index of an arm state is defined as the infimum subsidy $m$ such that the state remains in the passive set.

For our model in which the arm state is given by the belief vector, the indexability is equivalent to the following:

If $V_{\beta,m}(\omega; u = 1) \leq V_{\beta,m}(\omega; u = 0)$, then $\forall \ m' > m, \ V_{\beta,m'}(\omega; u = 1) \leq V_{\beta,m'}(\omega; u = 0). \hspace{1cm} (16)$

Under indexability, the Whittle index $W(\omega)$ of arm state $\omega$ is defined as

$$W(\omega) \Delta= \inf\{m : V_{\beta,m}(\omega; u = 1) \leq V_{\beta,m}(\omega; u = 0)\}. \hspace{1cm} (17)$$

Now we present a lemma that gives some fundamental properties of the value function $V_{\beta,m}(\omega)$.  

**Lemma 1.** The value function $V_{\beta,m}(\omega)$ for the single-armed bandit with subsidy is convex and Lipschitz continuous in both $\omega$ and $m$.

**Proof of Lemma [1]**. Consider a horizon of $T$ ($T \geq 1$) time slots and define $V_{\beta,m,1}(\omega)$ as the maximum expected total discounted reward over $T$ slots that can be obtained starting from initial state $\omega$ at $t = 1$:

$$V_{\beta,m,1}(\omega) = \max_{\pi \in \Pi_{sa}(T)} \mathbb{E}_\pi \left[ \sum_{t=1}^{T} \beta^{t-1} \sum_{n=1}^{N} \left\{ B_n, S_n(t) \mathbb{1}(u(t) = 1) + m \mathbb{1}(u(t) = 0) \right\} \right],$$

where $\Pi_{sa}(T)$ is the set of single-arm policies that map the belief state $\omega(t)$ to the action $u(t) \in \{1 \text{ (active)}, 0 \text{ (passive)}\}$ for $t = 1, 2, \cdots, T$. Note that $\omega(1) = \omega$ and an optimal policy $\pi^*_{sa}(T)$ achieving $V_{\beta,m,T}(\omega)$ is generally non-stationary, i.e., the mapping from $\omega(t)$ to $u(t)$ is dependent on $t$. Especially when $t = T$, we have only one more step to go and the myopic policy that maximizes the immediate reward is obviously optimal:

$$u^*(T) = \arg \max_{u \in \{0,1\}} \{ u \cdot \omega(T) B' + (1 - u) \cdot m \}.$$  

Let $V_{\beta,m,t}$ denote the maximum expected total discounted reward accumulated from slot $t$ to $T$ under $\pi^*_{sa}(T)$. We have the following dynamic equations:

$$V_{\beta,m,t}(\omega(t)) = \max \left\{ \omega B' + \beta \omega B_{t+1}(p_0), \cdots, \omega B_{t+1}(p_{K-1}), m + V_{\beta,m,t+1}(T^1(\omega(t))) \right\}, \quad t = 1, \cdots, T,$$

$$V_{\beta,m,T+1}(\cdot) \equiv 0.$$  

We first prove the properties of $V_{\beta,m}(\omega)$ regarding to $\omega$ with $m$ fixed. Our approach is based on backward induction on $t$ with $T$ fixed and then taking the limit $T \to \infty$. When $t = T$, it is clear that $V_{\beta,m,T}(\omega)$ is the maximum of a linear function of $\omega$ and a constant function ($m$), and is thus continuous, convex and piecewise linear. By the induction hypothesis that $V_{\beta,m,t+1}(\omega)$ is continuous, convex and piecewise linear, we again have that $V_{\beta,m,t}(\omega)$ is the maximum of two
continuous, convex and piecewise linear functions and is thus continuous, convex and piecewise linear. Therefore $V_{\beta,m,t}(\omega)$ is continuous, convex and piecewise linear in $\omega$ for all $t \in \{1,2,\cdots,T\}$. Using $\| \cdot \|_1$ norm on $\mathbb{R}^K$ and consider two states $\omega_1, \omega_2$ such that $\|\omega_1 - \omega_2\| > 0$. At $t = T$, we have

$$|V_{\beta,m,T}(\omega_1) - V_{\beta,m,T}(\omega_2)| = \max\{\omega_1 B', m\} - \max\{\omega_2 B', m\}. \quad (22)$$

Without loss of generality, assume $\omega_1 B' \leq \omega_2 B'$. We consider the following 3 cases:

i) if $m < \omega_1 B'$, then

$$|V_{\beta,m,T}(\omega_1) - V_{\beta,m,T}(\omega_2)| = |\omega_1 B' - \omega_2 B'| \leq B_K \|\omega_1 - \omega_2\|;$$

if $\omega_1 B' \leq m \leq \omega_2 B'$, then

$$|V_{\beta,m,T}(\omega_1) - V_{\beta,m,T}(\omega_2)| \leq B_K \|\omega_1 - \omega_2\|;$$

if $m > \omega_2 B'$, then

$$|V_{\beta,m,T}(\omega_1) - V_{\beta,m,T}(\omega_2)| = 0.$$

From the above, we have that

$$|V_{\beta,m,T}(\omega_1) - V_{\beta,m,T}(\omega_2)| \leq B_K \|\omega_1 - \omega_2\|. \quad (23)$$

At time $t + 1$, we make the following induction hypothesis that

$$|V_{\beta,m,t+1}(\omega_1) - V_{\beta,m,t+1}(\omega_2)| \leq \frac{1 - \beta^{t-1}}{1 - \beta} B_K \|\omega_1 - \omega_2\|.\quad (24)$$

At time $t$, we have, by a similar case analysis as above, that

$$|V_{\beta,m,t}(\omega_1) - V_{\beta,m,t}(\omega_2)| \leq \frac{1 - \beta^{t-1}}{1 - \beta} B_K \|\omega_1 - \omega_2\|.$$

Note that we have used the fact that $\|T^1(\omega_1) - T^1(\omega_2)\| \leq \|\omega_1 - \omega_2\|$. Therefore, we have that

$$|V_{\beta,m,1}(\omega_1) - V_{\beta,m,1}(\omega_2)| \leq \frac{1 - \beta T}{1 - \beta} B_K \|\omega_1 - \omega_2\| \leq \frac{1}{1 - \beta} B_K \|\omega_1 - \omega_2\|. \quad (25)$$
Furthermore, for all \( t \in \{1, 2, \cdots, T\} \), we have that

\[
|V_{\beta,m,t}(\omega_1) - V_{\beta,m,t}(\omega_2)| \leq \frac{1}{1 - \beta} B_K \|\omega_1 - \omega_2\|. \tag{26}
\]

This proves that the finite-horizon value function \( V_{\beta,m,t}(\omega) \) is Lipschitz continuous in \( \omega \) with constant \( \frac{1}{1 - \beta} B_K \), independent of horizon length \( T \) and starting point \( t \). Fix \( t = 1 \), if we can show as \( T \) goes to infinity \( V_{\beta,m,1}(\cdot) \) converges to \( V_{\beta,m}(\cdot) \) pointwise, then \( V_{\beta,m}(\cdot) \) must be Lipschitz continuous with the same constant. This is because that given any two states \( \omega_1, \omega_2 \) and any \( \epsilon > 0 \), there exists a positive integer \( T_0 \) such that

\[
|V_{\beta,m}(\omega_1) - V_{\beta,m}(\omega_2)| \leq 2\epsilon + |V_{\beta,m,1}(\omega_1) - V_{\beta,m,1}(\omega_2)| \leq 2\epsilon + \frac{1}{1 - \beta} B_K \|\omega_1 - \omega_2\|.
\]

Since \( \epsilon > 0 \) is arbitrary, the Lipschitz continuity of \( V_{\beta,m}(\cdot) \) follows. To prove the convergence of \( V_{\beta,m,1}(\cdot) \) with \( T \), we first apply the optimal policy \( \pi^*_{sa}(T) \) to the first \( T \) slots followed by an (stationary) optimal policy \( \pi^*_{sa} \) for the infinite-horizon problem in subsequent time slots \( t > T \), then

\[
V_{\beta,m}(\omega) \geq V_{\beta,m,1}(\omega) + \beta^T \mathbb{E}[V_{\beta,m}(\omega(T + 1))], \tag{27}
\]

where the expectation is taken with respect to \( \omega(T + 1) \) which is determined by the past observations and actions in the first \( T \) slots. It is clear that \( V_{\beta,m}(\cdot) \) is bounded:

\[
0 \leq V_{\beta,m}(\cdot) \leq \frac{\max \{B_K, m\}}{1 - \beta}. \tag{28}
\]

From (27) and (28), we know that

\[
V_{\beta,m,1}(\omega) - V_{\beta,m}(\omega) \leq 0. \tag{29}
\]

Now we apply \( \pi^*_{sa} \) to the finite-horizon problem with length \( T \) and compare the reward accumulated in the \( T \) slots:

\[
V_{\beta,m,1}(\omega) \geq V_{\beta,m}(\omega) - \beta^T \mathbb{E}[V_{\beta,m}(\omega(T + 1))]. \tag{30}
\]

From (28), (29) and (30), we have, for any initial value of \( \omega \) at \( t = 1 \),

\[
-\beta^T \frac{\max \{B_K, m\}}{1 - \beta} \leq V_{\beta,m,1}(\omega) - V_{\beta,m}(\omega) \leq 0. \tag{31}
\]
Taking the limit $T \to \infty$, we proved the (uniform) convergence of $V_{\beta,m,1}(\cdot)$ to $V_{\beta,m}(\cdot)$. Consequently $V_{\beta,m}(\cdot)$ is Lipschitz continuous. Its convexity is clear as a limiting function of convex functions.

Now we consider the properties of $V_{\beta,m}(\omega)$ regarding to $m$ with $\omega$ fixed. By a similar argument as above, we have that $V_{\beta,m,1}(\omega)$ is convex, continuous and piecewise linear in $m$. Furthermore, it is Lipschitz continuous in $m$ with constant $\frac{1}{1-\beta}$, i.e., for any $m_1$ and $m_2$,

$$|V_{\beta,m,1}(\omega) - V_{\beta,m,2}(\omega)| \leq \frac{1}{1-\beta}|m_1 - m_2|.$$

It remains to show the pointwise convergence of $V_{\beta,m,1}(\omega)$ to $V_{\beta,m}(\omega)$ for every fixed $m$ as $T \to \infty$. However, it is a direct result of \([31]\). □

**Remark**

- Note that if $m \leq 0$, it is optimal to always activate the arm (since all extreme points of a convex function under the passive action are below those of a linear one under the active action) and $V_{\beta,m}(\omega)$ does not depend on $m$ and is thus Lipschitz continuous in $m$. If $m \geq BK_1$, it is optimal to always make the arm passive so $V_{\beta,m}(\omega) = \frac{m}{1-\beta}$ and is thus Lipschitz continuous as well.

The interesting case is when $0 < m < BK_1$. In this bounded region, we can find a finite cover of $[0, BK_1]$ and show that pointwise convergence leads to uniform convergence under the condition of Lipschitz continuity. The monotone property of $V_{\beta,m}(\omega)$ as a nondecreasing function of $m$ is clear.

- Since $V_{\beta,m}(\omega)$ is Lipschitz continuous in $m$, it is also absolutely continuous and differentiable almost everywhere in $m$. Assume $m_0$ is a point where the derivative exists, a small increase to $m_0 + \Delta m$ should cause $V_{\beta,m_0}(\omega)$ to boost at a ratio at least equal to the expected total discounted time of being passive, since the subsidy $m_0$ for passivity is being paid for such a duration of time (passive time in short). The passive time is not necessarily unique and we will give a rigorous formulation of its relation to the (right) derivative of $V_{\beta,m}(\omega)$ in Theorem \([1]\).

- Since $V_{\beta,m}(\omega)$ is also Lipschitz continuous in $\omega$, for sufficiently small $\beta$, a change of $\omega$ that makes the immediate reward $\omega B'$ vary may play a dominating role in determining the order
of (13) and (14) as the value function $V_{\beta,m}(\omega)$ varies in bounded ratios with $\omega$. This motivates us to consider the family of threshold policies: follow the trajectory of $T^{k}(\omega)$ and when certain mapping $r(\cdot) : \mathbb{R}^{K} \rightarrow \mathbb{R}$ (e.g., the simple projection $r(\omega) = \omega B'$) on $T^{k}(\omega)$ exceeds certain threshold, dependent of $m$, we activate the arm and reset the value function to one of $V_{\beta,m}(p_{0}), V_{\beta,m}(p_{1}), \ldots, V_{\beta,m}(p_{K-1})$. Threshold policies are not necessarily optimal, especially when $\beta$ is large. However, they provide an efficient way in solving the approximated value functions and yield a computable Whittle index function of belief state with low-complexity and near-optimal performance even when $\beta$ is close to 1, as elaborated in Sec. 4.

Consider an extreme point $\omega = [0, 0, \cdots, 1, \cdots, 0]$ of the belief state space where it is known that the arm’s underlying state is $k$ for some $k \in \{0, 1, \cdots, K-1\}$. In this case, the next belief state is deterministically $p_{k}$, independent of the current action, i.e.,

$$\arg \max \{V_{\beta,m}(\omega; u = 1); V_{\beta,m}(\omega; u = 0)\} = \arg \max \{B_{k}, m\}. \quad (32)$$

From the above, each extreme point successively joins the passive set as $m$ increases from 0 to $B_{K-1}$.

Consider an $m \in (0, B_{K-1})$ such that $0 = B_{0} \leq \cdots \leq m < B_{j} \leq \cdots \leq B_{K-1}$. The first $j$ states are in the passive set while states $j, \cdots, K-1$ are in the active set defined as

$$A(m) \triangleq \tilde{P}(m) = \{\omega : V_{\beta,m}(\omega; u = 1) > V_{\beta,m}(\omega; u = 0)\}. \quad (33)$$

From (12), (13), (14) and Lemma 1 given any $\omega_{1}, \omega_{2} \in A(m)$ and $\lambda \in (0, 1)$, we have

$$V_{\beta,m}(\lambda \omega_{1} + (1 - \lambda)\omega_{2}; u = 1) = \lambda V_{\beta,m}(\omega_{1}; u = 1) + (1 - \lambda) V_{\beta,m}(\omega_{2}; u = 1) \quad (34)$$

$$> \lambda V_{\beta,m}(\omega_{1}; u = 0) + (1 - \lambda) V_{\beta,m}(\omega_{2}; u = 0) \quad (35)$$

$$\geq V_{\beta,m}(\lambda \omega_{1} + (1 - \lambda)\omega_{2}; u = 0). \quad (36)$$

The first equality in the above is due to the linearity of $V_{\beta,m}(\cdot; u = 1)$, the second last inequality is by Definition (33), and the last inequality is due to the convexity of $V_{\beta,m}(\cdot)$ and the linearity of $T^{1}(\cdot)$. 
Therefore, the active set $A(m)$ is convex, formed by a subset of extreme points of the $(K - 1)$-dimensional probability measure and (not including) a continuous boundary $C(m)$ between $A(m)$ and $P(m)$:

$$C(m) \triangleq \{ \omega : V_{\beta,m}(\omega; u = 1) = V_{\beta,m}(\omega; u = 0) \}.$$  \hspace{1cm} (37)

Under indexability, the boundary $C(m)$ should (continuously) move in a direction such that $A(m)$ shrinks as $m$ increases. For each $\omega$, there exists an $m$ such that $C(m)$ reaches $\omega$ for the first time and this $m$ is the Whittle index $W(\omega)$ of $\omega$:

$$W(\omega) \triangleq \inf\{ m : V_{\beta,m}(\omega; u = 1) \leq V_{\beta,m}(\omega; u = 0) \} = \min\{ m : \omega \in C(m) \}.  \hspace{1cm} (38)$$

In the above, we have used the minimization operator instead of the infimum by observing that the closure of the nontrivial region $(0, B_{K-1})$ for the subsidy $m$ is compact. A sufficient and necessary condition of indexability for our model with an infinite state space is given in the following theorem.

**Theorem 1.** Let $\Pi_{sa}^*(m)$ denote the set of all optimal single-arm policies achieving $V_{\beta,m}(\omega)$ with initial belief state $\omega$. Define the passive time

$$D_{\beta,m}(\omega) \triangleq \max_{\pi_{sa}(m) \in \Pi_{sa}(m)} \mathbb{E}_{\pi_{sa}(m)} \left[ \sum_{t=1}^{\infty} \beta^{t-1} 1(u(t) = 0) | \omega(1) = \omega \right]. \hspace{1cm} (39)$$

The right derivative of the value function $V_{\beta,m}(\omega)$ with $m$, denoted by \( \frac{dV_{\beta,m}(\omega)}{(dm)^+} \), exists at every value of $m$ and

$$\left. \frac{dV_{\beta,m}(\omega)}{(dm)^+} \right|_{m=m_0} = D_{\beta,m_0}(\omega). \hspace{1cm} (40)$$

Furthermore, the single-armed bandit is indexable if and only if for all values of $\omega$ and $m_\omega$ such that $\omega \in C(m_\omega)$, we have

$$\left. \frac{dV_{\beta,m}(\omega; u = 0)}{(dm)^+} \right|_{m=m_\omega} \geq \left. \frac{dV_{\beta,m}(\omega; u = 1)}{(dm)^+} \right|_{m=m_\omega}, \hspace{1cm} (41)$$

and for any $\omega \in C(m_\omega)$ with the equality true in (41), there exists a $\Delta m(\omega) > 0$ such that

$$V_{\beta,m}(\omega; u = 0) \geq V_{\beta,m}(\omega; u = 1), \hspace{1cm} \forall \ m \in (m_\omega + \Delta m(\omega)). \hspace{1cm} (42)$$
Proof of Theorem 2. The existence of the right (or left) derivative follows directly from the convexity of $V_{\beta,m}(\omega)$. Fix an $m_0$ and apply a change $\Delta m$ to the single-armed bandit, we have

$$V_{\beta,m_0+\Delta m}(\omega) \geq V_{\beta,m_0}(\omega) + D_{\beta,m}(\omega)\Delta m. \quad (43)$$

Now if we apply an optimal policy for the arm with subsidy $m = m_0 + \Delta m$ to the case of $m = m_0$, we have

$$V_{\beta,m}(\omega) \geq V_{\beta,m_0+\Delta m}(\omega) - D_{\beta,m+\Delta m}(\omega)\Delta m. \quad (44)$$

From (43) and (44), it is clear that

$$D_{\beta,m}(\omega) \leq \frac{V_{\beta,m_0+\Delta m}(\omega) - V_{\beta,m_0}(\omega)}{\Delta m} \leq D_{\beta,m+\Delta m}(\omega), \quad \forall \Delta m > 0. \quad (45)$$

Note that the above implies the monotonically nondecreasing property of $D_{\beta,m}(\omega)$ with $m$. To prove (40), we only need to show $D_{\beta,m}(\omega)$ is right continuous in $m$. Assume this is not true so there exists a decreasing sequence $\{m_k\}$ converging to $m_0$ and an $\epsilon > 0$ such that

$$D_{\beta,m_k}(\omega) - D_{\beta,m_0}(\omega) > \epsilon. \quad (46)$$

Since $D_{\beta,m_k}(\omega)$ has a value ranging in the compact set $[0, \frac{1}{1-\beta}]$, we can find a convergent subsequence $\{m_{k_i}\}$ of $\{m_k\}$ such that

$$\lim_{i \to \infty} D_{\beta,m_{k_i}}(\omega) = D > D_{\beta,m_0}(\omega), \quad (47)$$

where $D \in (0, \frac{1}{1-\beta}]$ is the limit of the passive time as $m_{k_i} \to m_0$. If we can show that $D$ can be achieved by a policy $\pi^* \in \Pi_{w_0}(m_0)$, then we have a contradiction to (39).

To construct $\pi^*$ with passive time $D$ and achieving $V_{\beta,m_0}(\omega)$, we look at a finite horizon $T$. Starting from the initial belief state $\omega$, the possible belief states within $T$ must be finite, leading to a finite set of possible policies. Specifically, if the number of possible states to observe is $h(T)$, the number of policies up to time $T$ is at most $2^{h(T)}$ as each state is applied with either $u = 0$ or $u = 1$. We can thus choose a subsequence $\{m_j(T)\}$ of $\{m_{k_i}\}$ such that the optimal policy achieving $D_{\beta,m_{m_j}(T)}$
under \( m_j(T) \) is the same for all \( j \) within the first \( T \) slots. Repeat the process for slots \( T + 1 \) up to \( 2T \) and keep doubling the time horizon, we arrive at a policy for all states that may happen at any time. For any time horizon \( T' \), this policy coincides with the optimal policies for a subsequence \( \{m_j(T'')\} \) of \( \{m_k\} \) for some \( T'' > T' \) and by taking \( T' \) large enough, this policy achieves a passive time at least \( D - \epsilon_1 \) and a total reward \( V_{\beta,m_0}(\omega) - \epsilon_1 \) for any arbitrarily small \( \epsilon_1 > 0 \) due to (47) and the continuity of \( V_{\beta,\cdot}(\omega) \). This policy is thus optimal for the infinite-horizon single-armed bandit problem with subsidy \( m_0 \) with passive time \( D \), as desired for \( \pi^* \).

To prove the sufficiency of (41) and (42), we assume that the arm is not indexable, i.e., there exists \( m_0 \) and \( \omega \in C(m_0) \subset P(m_0) \) such that for any \( \epsilon > 0 \), we can find an \( m_1 \) \((m_0 < m_1 < m_0 + \epsilon)\) with \( \omega \in A(m_1) \). This means that as the boundary \( C(m) \) moves (continuously) as \( m \) increases, some belief state moves from the passive set to the active set. Under this scenario, we have

\[
V_{\beta,m_0}(\omega; u = 1) = V_{\beta,m_0}(\omega; u = 0).
\]

(48)

\[
V_{\beta,m_1}(\omega; u = 1) > V_{\beta,m_1}(\omega; u = 0).
\]

(49)

According to (13) and (14), both \( V_{\beta,m}(\omega; u = 1) \) and \( V_{\beta,m}(\omega; u = 0) \) are right differentiable with \( m \) for any belief state \( \omega \), so is their difference. Therefore, by (48) and (49),

\[
\frac{dV_{\beta,m}(\omega; u = 1)}{(dm)^+} \bigg|_{m = m_0} = \lim_{m_1 \to m_0} \frac{V_{\beta,m_1}(\omega; u = 1) - V_{\beta,m_0}(\omega; u = 1)}{m_1 - m_0} \geq \lim_{m_1 \to m_0} \frac{V_{\beta,m_1}(\omega; u = 0) - V_{\beta,m_0}(\omega; u = 0)}{m_1 - m_0} = \frac{dV_{\beta,m}(\omega; u = 0)}{(dm)^+} \bigg|_{m = m_0}.
\]

(50)

(51)

This would contradict (41) unless the equality in (51) holds, which would contradict (42) given (49) and that \( \epsilon \) can be chosen arbitrarily small.

To prove the necessity of (41) and (42), assume there exists an \( \omega \in C(m_\omega) \) such that

\[
\frac{dV_{\beta,m}(\omega; u = 0)}{(dm)^+} \bigg|_{m = m_\omega} < \frac{dV_{\beta,m}(\omega; u = 1)}{(dm)^+} \bigg|_{m = m_\omega},
\]

(52)

and when

\[
\frac{dV_{\beta,m}(\omega; u = 0)}{(dm)^+} \bigg|_{m = m_\omega} = \frac{dV_{\beta,m}(\omega; u = 1)}{(dm)^+} \bigg|_{m = m_\omega},
\]

(53)
for any $\epsilon_1 > 0$, there exists an $m_2$ ($m_\omega < m_2 < m_\omega + \epsilon_1$) such that

$$V_{\beta,m_2}(\omega; u = 0) < V_{\beta,m_2}(\omega; u = 1).$$

(54)

By (52), there exists $\Delta m > 0$ such that

$$V_{\beta,m}(\omega) + \Delta m(\omega; u = 0) - V_{\beta,m}(\omega; u = 0) < V_{\beta,m}(\omega) + \Delta m(\omega; u = 1) - V_{\beta,m}(\omega; u = 1).$$

(55)

Together with the fact that

$$V_{\beta,m}(\omega; u = 0) = V_{\beta,m}(\omega; u = 1),$$

(56)

we have that $\omega \in A(m(\omega) + \Delta m)$ and obtained a contradiction to indexability as $\omega \in C(\omega) \subset P(m(\omega))$. Furthermore, when (53) holds, it is straightforward that (54) contradicts (52). □

**Remark**

- Theorem 1 establishes a crucial relation between the value function $V_{\beta,m}(\omega)$ and the passive time $D_{\beta,m}(\omega)$ as its right derivative. The convexity established in Lemma 1 then implies the monotonic property of $D_{\beta,m}(\omega)$ as $m$ increases. However, the increase of $D_{\beta,m}(\omega)$ needs not to be continuous. In the proof of Theorem 1, we have shown the right continuity of $D_{\beta,m}(\omega)$ but not the left one. These jumping points are essentially caused by the case where the points in the belief state space may not join the passive set $P(m)$ in a continuous sense. Specifically, if we fix the initial belief state $\omega$, the arm state will move in a countable set as a discrete process. Under the optimal policy that achieves the passive time defined in (39) and indexability, it is possible that when $m$ increases by a sufficiently small amount, the policy remains unchanged, i.e., the partition of active and passive sets for the countable state space is the same. Consequently, the passive time $D_{\beta,m}(\omega)$ remains a constant during this increasing period of the subsidy. However, as $m$ keeps increasing, new states would join the passive set and cause a jump in $D_{\beta,m}(\omega)$. The discontinuity of $D_{\beta,m}(\omega)$ poses a question: how should one make the continuation of $D_{\beta,m}(\omega)$ such that constraint (9) must be satisfied for the relaxed version of the multi-armed bandit problem? The technique is to use nondeterministic optimal policies: for believe states in the continuous boundary $C(m)$ that causes
discontinuities $D_{\beta,m}(\omega)$, we activate the arm with certain probability $\rho$ and make it passive with probability $1 - \rho$. As $\rho$ decreases from 1 to 0, the corresponding policies provide a continuation of $D_{\beta,m}(\omega)$. For a detailed exposition of this randomization technique in solving the original multi-armed bandit problem under the relaxed constraint, see Liu 2010 that considers a more general model of infinite arm state spaces.

- Note that (41) is equivalent to

$$\beta \sum_{k=0}^{K-1} \omega_k D_{\beta,m}(p_k) \leq 1 + \beta D_{\beta,m}(T^1(\omega)), \quad \forall \omega \in C(m).$$

The above clearly holds if $\beta \leq 0.5$ as $D_{\beta,m}(\cdot)$ is lower and upper bounded by 0 and $1 - \beta$, respectively. The strict inequality in (57) holds if $\beta < 0.5$, satisfying (41). When $\beta = 0.5$, the equality in (57) holds if $\beta \sum_{k=0}^{K-1} \omega_k D_{\beta,m}(p_k) = \frac{\beta}{1 - \beta} = 1$ and $D_{\beta,m}(T^1(\omega)) = 0$. In this case, as $m$ keeps increasing, the left-hand side of (57) can not increase while the right-hand side cannot decrease. Apply any $\Delta m > 0$ to $\omega \in C(m)$, we have

$$V_{\beta,m+\Delta m}(\omega; u = 0) \geq \Delta m(1 + \beta D_{\beta,m}(T^1(\omega))) + V_{\beta,m}(\omega; u = 0)$$

$$= \Delta m(1 + \beta D_{\beta,m}(T^1(\omega))) + V_{\beta,m}(\omega; u = 1)$$

$$= \Delta m(1 + \beta D_{\beta,m}(T^1(\omega))) + \omega B' + \beta \frac{m}{1 - \beta}$$

$$= \omega B' + \beta \frac{m + \Delta m}{1 - \beta}$$

$$= V_{\beta,m+\Delta m}(\omega; u = 1),$$

where the second last equality is due to $D_{\beta,m}(T^1(\omega)) = 0$ and that $\beta = 0.5$; while the last equality is due to the fact that any future state $p_k$ after activating at $\omega$ must remain in the passive set as $m$ increases due to the monotonic nondecreasing property of $D_{\beta,m}(p_k)$ with $m$ and that $D_{\beta,m}(p_k)$ is already equal to the upper bound $\frac{1}{1 - \beta}$. Therefore (62) is satisfied as well.

**Corollary 1.** The restless bandit is indexable if $\beta \leq 0.5$.

However, it is difficult to verify (41) and (42) when $\beta > 0.5$. This requires further analysis on the passive time $D_{\beta,m}(\omega)$ as well as the value function $V_{\beta,m}(\omega)$. If we can characterize the boundary
function $C(m)$ of subsidy $m$, then for each $\omega$, we may obtain the first crossing time $L(\omega, C(m))$ when it enters the active set under consecutive passive actions:

$$L(\omega, C(m)) \triangleq \min_{0 \leq k < \infty} \{ k : \mathcal{T}^k(\omega) \in A(m) \}. \quad (63)$$

Define $\mathcal{T}^0(\omega) \triangleq \omega$ and if $\mathcal{T}^k(\omega) \notin A(m)$ for all $k \geq 0$, we set $L(\omega, C(m)) = +\infty$. It is clear that for any $\omega \in C(m)$, we have

$$V_{\beta,m}(\omega) = \omega B' + \beta \omega (V_{\beta,m}(p_0), \cdots, V_{\beta,m}(p_{K-1}))',$$

$$= \frac{1 - \beta^{L(\omega,C(m))}}{1 - \beta} m + \beta^{L(\omega,C(m))} V_{\beta,m}(\mathcal{T}^{L(\omega,C(m))}(\omega); u = 1), \quad (64)$$

$$V_{\beta,m}(\mathcal{T}^{L(\omega,C(m))}(\omega); u = 1) = \mathcal{T}^{L(\omega,C(m))}(\omega) B' + \beta \mathcal{T}^{L(\omega,C(m))}(\omega)(V_{\beta,m}(p_0), \cdots, V_{\beta,m}(p_{K-1}))', \quad (65)$$

$$V_{\beta,m}(p_k) = \frac{1 - \beta^{L(p_k,C(m))}}{1 - \beta} m + \beta^{L(p_k,C(m))} V_{\beta,m}(\mathcal{T}^{L(p_k,C(m))}(p_k); u = 1),$$

$$\forall \, k \in \{0, \cdots, K-1\}, \quad (66)$$

$$V_{\beta,m}(\mathcal{T}^{L(p_k,C(m))}(p_k); u = 1) = \mathcal{T}^{L(p_k,C(m))}(p_k) B' + \beta \mathcal{T}^{L(p_k,C(m))}(\omega)(V_{\beta,m}(p_0), \cdots, V_{\beta,m}(p_{K-1}))',$$

$$\forall \, k \in \{0, \cdots, K-1\}. \quad (67)$$

With $\omega$ and $L(\cdot, C(m))$ fixed and known, the above equation set is linear and has $2K + 3$ equations with $2K + 3$ unknowns (value functions and $m$), so an exact solution for the value function $V_{\beta,m}(\omega)$ is possible to obtain, as well as the passive time $D_{\beta,m}(\omega)$, and the subsidy $m$, in terms of $\omega$ and $L(\cdot, C(m))$. However, even if $L(\cdot, C(m))$ is known, such a way in checking indexability and solving for Whittle index is complex. Furthermore, the function $L(\cdot, C(m))$ may be solved only if $C(m)$ is sufficiently analyzed which involves dynamic programming on an uncountable state space. To circumvent these difficulties, we consider a family of threshold policies that simply the analysis of the value function and establish an approximation of Whittle index under a relaxed requirement for indexability.

### 3. Threshold Policies and Relaxed Indexability

A 1-dimensional threshold policy is defined by a mapping $r(\cdot)$ from the system state space to the real line and a fixed state $\omega^*_{\beta}(m)$ such that the binary action at any state $\omega$ depends only on the
order between $r(\omega)$ and $r(\omega^*_\beta(m))$. We say that $\omega^*_\beta(m)$ is the threshold in the system state space with respect to $r(\cdot)$. In our model, we define the threshold function $r(\cdot)$ as

$$r(\omega) \overset{\Delta}{=} \omega B'.$$

(68)

This is a simple and intuitive definition which is identical to the immediate reward function by activating the arm. As the belief state $\omega$ varies in the $(K-1)$-dimensional probability space, we measure the attractiveness of activating the arm by the expected reward that can be immediately obtained under activation. For the original problem with multiple arms, if they share the same parameters (homogeneous arms), i.e., the transition matrix $P$ and reward vector $B$ are arm-independent, this threshold policy on a single arm with subsidy corresponds to the myopic policy: at each time we activate the $M$ arms that will yield the highest expected reward. However, for inhomogeneous arms, the myopic policy may yield a significant performance loss (see Sec. 4.4). It is thus important to precisely characterize the attractiveness of a state as a function of the arm parameters. Our attempt is to solving for the subsidy $m$ that makes a belief state $\omega$ as the threshold and define this $m$ as its (approximated) Whittle index $W(\omega)$. Given the initial belief state $\omega$ as the threshold, the action $u^*_{\beta,m}(\omega(t))$ to take at $t \geq 1$ is given by:

$$u^*_{\beta,m}(\omega(t)) = \begin{cases} 1 \text{ (active),} & \text{if } r(\omega(t)) > r(\omega^*_\beta(m)) \\ 0 \text{ (passive),} & \text{if } r(\omega(t)) \leq r(\omega^*_\beta(m)) \end{cases}. \quad (69)$$

It is important to observe that when the current arm state is equal to the threshold, e.g., at $t = 1$, we always make the arm passive (for now). This is because activating the arm does not necessarily yield the same performance when confined in the family of threshold policies. Nevertheless, the suboptimality of a threshold policy is allievated if the belief update has a sharp slope projected onto the real line by (68) and the discount factor $\beta$ is small, in which case the comparison in (12) is dominated by the order between the expected immediate reward $\omega B'$ and the subsidy $m$. Note that the continuous boundary $C(m)$ defined in (37) can be considered as a nonlinear $(K-1)$-dimensional threshold and the optimal policy is in general a $(K-1)$-dimensional threshold policy as well.
3.1. The Value Function

Consider the threshold policy $\pi_{\beta,m}$ with threshold $\omega_\beta^*(m) = \omega$ fixed and the belief points $p_0, p_1, \ldots, p_{K-1}$. Define

$$L(\omega_1, \omega_2) = \min_{0 \leq k < \infty} \{ k : \mathcal{T}^k(\omega_1)B' > \omega_2B' \}. \quad (70)$$

If $\mathcal{T}^k(\omega_1)B' \leq \omega_2B'$ for all $k \geq 0$, we set $L(\omega_1, \omega_2) = +\infty$. Under $\pi_{\beta,m}$, we have

$$\hat{V}_{\beta,m}(p_k) = \frac{1 - \beta^{L(p_k,\omega)}}{1 - \beta}m + \beta^{L(p_k,\omega)} \hat{V}_{\beta,m}(\mathcal{T}^{L(p_k,\omega)}(p_k); u = 1), \quad \forall k \in \{0, \ldots, K-1\}, \quad (71)$$

$$\hat{V}_{\beta,m}(\mathcal{T}^{L(p_k,\omega)}(p_k); u = 1) = \mathcal{T}^{L(p_k,\omega)}(p_k)B' + \beta \mathcal{T}^{L(p_k,\omega)}(\omega)(\hat{V}_{\beta,m}(p_0), \ldots, \hat{V}_{\beta,m}(p_{K-1}))', \quad \forall k \in \{0, \ldots, K-1\}, \quad (72)$$

where the value function $\hat{V}_{\beta,m}(\omega_1)$ denotes the expected total discounted reward under $\pi_{\beta,m}$, starting from a belief state $\omega_1$. If $L(\cdot, \omega)$ is solved, the above equation set is linear and has $2K$ unknowns with $2K$ equations. In this case, we show (71) and (72) yield a unique solution consisting of $\{\hat{V}_{\beta,m}(p_k)\}_{k=0}^{K-1}$.

**Lemma 2.** Given the first crossing time function $L(\cdot, \omega)$ with the threshold $\omega_\beta^*(m) = \omega$ fixed, the linear equation set (71) and (72) has a unique solution consisting of the value functions $\{\hat{V}_{\beta,m}(p_k)\}_{k=0}^{K-1}$ in terms of $\omega$ and $m$.

**Proof of Lemma 2** It is helpful to rewrite (71) and (72) in the following matrix form $AX = b$:

$$\begin{bmatrix}
\beta^{L(p_0,\omega)}p_0\mathbf{P}^{L(p_0,\omega)} \\
\beta^{L(p_1,\omega)}p_1\mathbf{P}^{L(p_1,\omega)} \\
\vdots \\
\beta^{L(p_{K-1},\omega)}p_K\mathbf{P}^{L(p_{K-1},\omega)}
\end{bmatrix} \times
\begin{bmatrix}
\hat{V}_{\beta,m}(p_0) \\
\hat{V}_{\beta,m}(p_1) \\
\vdots \\
\hat{V}_{\beta,m}(p_{K-1})
\end{bmatrix} =
\begin{bmatrix}
\frac{1 - \beta^{L(p_0,\omega)}}{1 - \beta}m + \beta^{L(p_0,\omega)}p_0\mathbf{P}^{L(p_0,\omega)} \\
\frac{1 - \beta^{L(p_1,\omega)}}{1 - \beta} \beta^{L(p_1,\omega)}p_1\mathbf{P}^{L(p_1,\omega)} \\
\vdots \\
\frac{1 - \beta^{L(p_{K-1},\omega)}}{1 - \beta} \beta^{L(p_{K-1},\omega)}p_K\mathbf{P}^{L(p_{K-1},\omega)}
\end{bmatrix}B'. \quad (73)$$

To prove the claim, we only need to show the coefficient matrix is invertible. By the Perron-Frobenius theorem, the eigenvalues $\{\lambda_i\}_{i=1}^K$ of the following matrix satisfy $|\lambda_i| \leq 1$ for all $i \in \mathbb{N}$.
\{1, \cdots, h\} since it is a transition matrix with nonnegative elements and the sum of each row is equal to 1:

\[
\begin{pmatrix}
    p_0 P_{L(p_0, \omega)} \\
p_1 P_{L(p_1, \omega)} \\
\vdots \\
p_{K-1} P_{L(p_{K-1}, \omega)}
\end{pmatrix} = Q \begin{pmatrix}
    \lambda_1 \\
    \vdots \\
    \lambda_i \\
    \vdots \\
    \lambda_h
\end{pmatrix} Q^{-1}, \tag{74}
\]

where the above equation shows the Jordan canonical form of the matrix and the square matrix \(Q\) has full rank \(K\). Therefore we can rewrite the coefficient matrix \(A\) as

\[
A = I_K - \beta \begin{pmatrix}
    \beta L(p_0, \omega) p_0 P_{L(p_0, \omega)} \\
    \beta L(p_1, \omega) p_1 P_{L(p_1, \omega)} \\
    \vdots \\
    \beta L(p_{K-1}, \omega) p_{K-1} P_{L(p_{K-1}, \omega)}
\end{pmatrix} = Q \begin{pmatrix}
    1 - \beta L(p_0, \omega) \lambda_1 \\
    \vdots \\
    1 - \beta L(p_j, \omega) \lambda_i \\
    -\beta L(p_j, \omega) + 1 \\
    \vdots \\
    1 - \beta L(p_{K-1}, \omega) \lambda_h
\end{pmatrix} Q^{-1}
\]

\[
= Q J Q^{-1}. \tag{75}
\]

It is easy to see that no eigenvalue of \(J\) can be zero so it has a full rank, leading to the full rank of \(A\) as it is similar to \(J\). \(\square\)

**Remark**

- The proof of Lemma (2) does not require any particular form of \(L(\cdot, \cdot)\) so for any optimal single-arm policy, equations (66) and (67) have a unique solution consisting of \(\{V_{\beta, m}(p_k)\}_{k=0}^{K-1}\) in
terms of $m$ and $\omega$. For equation (64) that solves for the Whittle index $m$ for a given belief state $\omega$, the existence of the continuous boundary $C(m)$ under the optimal policy implies its validity. For the 1-D threshold policy with threshold $\omega^*_\beta(m) = \omega$, if we add to (71) and (72) the following additional constraint similar to (64), then the subsidy $m$ might be solved as an approximated Whittle index under the threshold policy.

$$
\hat{V}_{\beta,m}(\omega) = \omega B' + \beta \omega (\hat{V}_{\beta,m}(p_0), \cdots, \hat{V}_{\beta,m}(p_{K-1}))'
= m + \beta \hat{V}_{\beta,m}(T(1)(\omega))
$$

Equation (76) essentially requires that there exists a subsidy $m$ such that taking active and passive actions at the threshold $\omega$ achieve the same performance. In general, this might not hold and we need to redefine the approximated Whittle index as detailed in Sec. 3.2

- The value function $\hat{V}_{\beta,m}(\omega)$ with the fixed threshold $\omega^*_\beta(m) = \omega$ is a linear function of $m$ as

$$
\hat{V}_{\beta,m}(\omega) = \frac{1 - \beta L(\omega, \omega)}{1 - \beta} m + \beta L(\omega, \omega)(T(\omega, \omega)(\omega)B' + \beta T(\omega, \omega)(\omega)(\hat{V}_{\beta,m}(p_0), \cdots, \hat{V}_{\beta,m}(p_{K-1})))'
$$

and $\{\hat{V}_{\beta,m}(p_k)\}_{k=0}^{K-1}$ are linear in $m$ as well, since $L(\cdot, \omega)$ is independent of $m$ with $\omega$ fixed as the threshold. Furthermore, the coefficient of $m$ in the linear expression of $\hat{V}_{\beta,m}(\omega)$ is equal to the expected total discounted passive time starting from $\omega$ under the threshold policy. Without the ambiguity at $\omega$ as given in (76), the passive time must be unique and equal to the derivative of $\hat{V}_{\beta,m}(\omega)$ with $m$ under the threshold policy with the fixed threshold $\omega^*_\beta(m) = \omega$.

### 3.2. Relaxed Indexability and Whittle Index

To calculate the approximated Whittle index of a belief state $\omega$ by making active and passive actions indistinguishable at it as the threshold, we introduce the following definition of relaxed indexability.

**Definition 2.** A restless multi-armed bandit satisfies the relaxed indexability with respect to a threshold policy if for each arm with subsidy, when making any of its state as the threshold, then taking the passive and active actions at this state followed by the threshold policy achieve the same performance.
For our model, relaxed indexability is equivalent to the unique solution of (71), (72) and (76). The solution, if exists, must be unique by the linearity of the value function \( \hat{V}_{\beta,m}(\omega_1) \) with \( m \) for any belief state \( \omega_1 \), given that the threshold is fixed.

**Theorem 2.** Define the passive time \( \hat{D}_{\beta,m}(\omega_1) \) under a threshold policy \( \pi_{\beta,m} \) as

\[
\hat{D}_{\beta,m}(\omega_1) \triangleq \mathbb{E}_{\pi_{\beta,m}} \left[ \sum_{t=1}^{\infty} \beta^{t-1} \mathbb{1}(u(t) = 0) | \omega(1) = \omega_1 \right].
\]

By fixing the threshold \( \omega^*(\beta)(m) = \omega \) and \( m \)-independent function \( r(\cdot) \) in (69), the passive time starting from any initial belief state \( \omega_1 \) is independent of \( m \) and denoted by \( \hat{D}_{\beta}(\omega_1) \). The restless bandit of POMDP satisfies the relaxed indexability if and only if for any arm, the corresponding single-armed bandit problem with subsidy and with any belief state \( \omega \) fixed as the threshold on the arm state space, we have

\[
\beta \omega(\hat{D}(p_0), \ldots, \hat{D}(p_{K-1}))' \neq 1 + \beta \hat{D}(T^1(\omega)).
\]

Under the relaxed indexability, the approximated Whittle index \( \hat{W}(\omega) \) for a belief state \( \omega \) is given by

\[
\hat{W}(\omega) = \frac{\omega B' - \beta g(\omega P) \left[ I_K + \beta H(P)G(P) \right] B' + \beta \omega H(P)G(P) B'}{1 + \beta f(\omega P) + \beta \left[ \beta g(\omega P) - \omega \right] H(P)F(P)},
\]

where \( L(\cdot) := L(\cdot, \omega), f(\cdot) := \frac{1 - \beta L(\cdot)}{1 - \beta}, g(\omega) := \beta L(\omega)T^L(\omega)(\omega) = \beta L(\omega)\omega P^L(\omega) \), and

\[
F(P) := \begin{pmatrix}
  f(p_0) \\
  f(p_1) \\
  \vdots \\
  f(p_{K-1})
\end{pmatrix}, \quad G(P) := \begin{pmatrix}
  g(p_0) \\
  g(p_1) \\
  \vdots \\
  g(p_{K-1})
\end{pmatrix}, \quad H(P) = \left( I_K - \beta G(P) \right)^{-1}.
\]

**Proof of Theorem** According to the remark following Lemma 2, the value function \( \hat{V}_{\beta,m}(\omega_1) \) is linear in \( m \) for any \( \omega_1 \) because the threshold policy is independent of \( m \) if \( \omega^*(\beta)(m) \) is fixed. Since the subsidy \( m \) is paid if and only if the arm is made passive, the linear coefficient of \( m \) in \( \hat{V}_{\beta,m}(\omega_1) \) is simply \( \hat{D}_{\beta}(\omega_1) \). The passive time \( \hat{D}_{\beta}(\omega_1) \) is clearly independent of \( m \) conditional on the fixed
threshold. Since (76) has a solution for \( m \) if and only if its left and right hand sides have different coefficients of \( m \), we proved the equivalence of (79) to the relaxed indexability. The expression (80) of the approximated Whittle index follows directly from the unique solution of (71), (72) and (76) under the relaxed indexability. □

**Remark**

- If there exists a belief state \( \omega \) such that (79) does not hold, *i.e.*, the denominator of the expression for Whittle index in (80) is zero, we can simply use \( \omega B' \) as a substitute for its Whittle index to measure the attractiveness of activating the arm. For the original multi-armed bandit problem (6) and (7), starting from the initial belief states for all arms \( \Omega(1) \), we only need to calculate the (approximated) Whittle index for the state of each arm and select the \( M \) arms with the highest Whittle indices at each time. Since solving for the Whittle index as well as checking condition (79) for each arm has a complexity determined by the process of solving the set of linear equations (71), (72) and (76), we have an efficient *online* algorithm for arm selections with a polynomial running time of the underlying arm state size \( K \) and a linear running time of the number of arms \( N \) at each time \( t \) which is independent of the algorithm complexity \( O(K^3N) \), given that the first-crossing function \( L(\cdot, \omega) \) is solved for any threshold \( \omega \).

- Recall the definition of \( L(\cdot, \omega) \) in (70). Since \( \mathcal{T}^k(\omega_1) = \omega_1 P^k = \omega_1 QJQ^{-1} \) with \( P = QJQ^{-1} \) in its Jordan canonical form, it is possible to have an analytical solution for \( L(\cdot, \omega) \). In Sec 4 we focus on the case of \( K = 3 \) and obtain detailed forms of \( L(\cdot, \omega) \) in various scenarios. In general, one could use the exhaustion method to search for the first crossing time with an upper bound on the number of steps. If the number of search steps exceeds the upper bound, we simply set \( L(\cdot, \omega) = \infty \). As \( \beta^k \) decreases geometrically with \( k \) and \( \omega_1 P^k \) with any belief state \( \omega_1 \) also converges geometrically with \( k \) for regular transition matrices, such a numerical exhaustion has its practical convenience.

### 4. The Case of \( K = 3 \)

In this section, we consider the case that an arm has a 3D belief state space, *i.e.*, the underlying Markov chain has 3 states. For simplicity of presentation, we assume that the Markov chain is irreducible and aperiodic, thus having a unique stationary (limiting) distribution.
4.1. The Jordan Canonical Form

To compute $L(\cdot, \omega)$, it is crucial to analyze the form of $P^k$ with $k$. A general approach is to use the Jordan canonical form of the stochastic matrix when computing its power. It is well known that any $K \times K$ square matrix $P$ can be written in its Jordan form as

$$P = Q J Q^{-1} = Q \begin{pmatrix} J_0 & & \\ & \ddots & \\ & & J_{V-1} \end{pmatrix} Q^{-1}, \quad J_v = \begin{pmatrix} \lambda_v & 1 \\ & \lambda_v & \ddots \\ & & \ddots & 1 \\ & & & \lambda_v \end{pmatrix},$$

(81)

where $Q$ is a square matrix of full rank $K$ and the upper diagonal $K_v \times K_v$ matrix $J_v$ is the $v$th Jordan block of size $K_v$ ($1 \leq K_v \leq K$) with the eigenvalue $\lambda_v$ and $\sum_{v=0}^{V-1} = K$. Note that if $K_v = 1$ then the Jordan block is simply a scalar ($\lambda_v$) and different blocks can have the same eigenvalue, i.e., there may exist $0 \leq v_1 \neq v_2 \leq V - 1$ such that $\lambda_{v_1} = \lambda_{v_2}$. The $k$th power of $P$ can thus be computed as

$$P^k = Q \begin{pmatrix} J_0^k & & \\ & \ddots & \\ & & J_{V-1}^k \end{pmatrix} Q^{-1}, \quad J_v^k = \begin{pmatrix} \lambda_v^k \binom{k}{1} \lambda_v^{k-1} \binom{k}{2} \lambda_v^{k-2} \cdots \binom{k}{K_v-1} \lambda_v^{k-K_v+1} \\ 0 \lambda_v^k \binom{k}{1} \lambda_v^{k-1} \cdots \binom{k}{K_v-2} \lambda_v^{k-K_v+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & & \vdots & \ddots & \lambda_v^k \binom{k}{1} \lambda_v^{k-1} \\ 0 & & & 0 & \lambda_v^k \end{pmatrix},$$

(82)

For a finite irreducible and aperiodic Markov chain, the transition matrix $P$ is regular, i.e., there exists an integer $k \geq 1$ such that $P^k > 0$ (element-wise). Therefore its Perron-Frobenius eigenvalue $\lambda_{pf} = 1$ has algebraic multiplicity 1, i.e., the Jordan block associated with the eigenvalue 1 is unique and of size 1. Furthermore, any other eigenvalue $\lambda_v \neq 1$ ($1 \leq v \leq K$) of $P$ satisfies $|\lambda_v| < 1$. In this case, the Markov chain has a unique stationary distribution to which $\omega_1 P^k$ converges at a geometric rate as $k \to \infty$ for any belief state $\omega_1$.

In the case of $K = 3$, the Jordan canonical form of the transition matrix $P$ takes one of the following two forms (assuming irreducible and aperiodic Markov chains):
1. \( P \) has 3 linearly independent eigenvectors: there exist \( \lambda_1, \lambda_2 \in \mathbb{R} \) or \( \lambda_1 = \bar{\lambda}_2 \in \mathbb{C} \) with \( |\lambda_1|, |\lambda_2| \in [0, 1) \),

\[
J_{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix};
\]

(83)

2. \( P \) has 2 linearly independent eigenvectors: there exists \( \lambda_1 \in \mathbb{R} \), \( |\lambda_1| \in [0, 1) \),

\[
J_{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}.
\]

(84)

Since the eigenvalue 1 corresponds to a single Jordan block of size 1 under our assumption, the matrix \( P \) has at least 2 linearly independent eigenvectors.

4.2. The \( k \)-Step Reward Function

Fix a belief state \( \omega \). Define the \( k \)-step reward function as

\[
h(k) = T^k(\omega)B' = \omega P^kB', \quad k \geq 0.
\]

(85)

To analyze \( L(\omega, \omega^*) \) for any threshold \( \omega^* \), we only need to find out the maximum of \( h(k) \) and if it exceeds \( \omega^*B' \), the first \( k \) that makes \( h(k) > \omega^*B' \). In the following lemma, we show that \( h(k) \) can only take three forms and then establish a detailed form of \( L(\omega, \omega^*) \) for the three cases respectively in Sec. 4.3.

**Lemma 3.** The \( k \)-step reward function \( h(k) \) takes one of the following three forms:

1. \( P \) has only real eigenvalues and 3 linearly independent eigenvectors:

\[
h(k) = a_1b_1^k + a_2b_2^k + c, \quad a_1, b_1, a_2, b_2, c \in \mathbb{R}, \quad |b_1|, |b_2| < 1;
\]

2. \( P \) has only real eigenvalues and 2 linearly independent eigenvectors:

\[
h(k) = ab^k + ckb^{k-1} + d, \quad a, b, c, d \in \mathbb{R}, \quad |b| < 1;
\]
3. \( P \) has a pair of conjugate complex eigenvalues:

\[ h(k) = a' A^k \sin(k\theta + b') + c', \quad a', A, b', c', \theta \in \mathbb{R}, \ A \in (0, 1), \ a' \geq 0, \ \theta \in (0, 2\pi), \ b' \in [0, 2\pi). \]

Proof of Lemma 3. Case 1 and 2 follow directly from the power of Jordan matrices with \( b_1 = \lambda_1, b_2 = \lambda_2, \) or \( b = \lambda_1 = \lambda_2: \)

\[
\begin{align*}
J^k_{(1)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^k_1 & 0 \\ 0 & 0 & \lambda^k_2 \end{pmatrix}, & J^k_{(2)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^k_1 & k \lambda^{k-1}_1 \\ 0 & 0 & \lambda^k_1 \end{pmatrix}.
\end{align*}
\]

For Case 3, write \( P = Q J_{(1)} Q^{-1} \). We have \( \lambda_2 = \overline{\lambda}_1 \) and let \( Q = \{q_{ij}\}_{i,j=0,1,2}, \ Q^{-1} = \{\overline{q}_{ij}\}_{i,j=0,1,2} \) and \( Q_i = (q_{0i}, q_{1i}, q_{2i}), \ \overline{Q}_i = (\overline{q}_{0i}, \overline{q}_{1i}, \overline{q}_{2i}), \ i = 0, 1, 2. \) Then \( Q_2 = \overline{Q}_1, \overline{Q}_2 = \overline{Q}_1 \) and \( Q_0, \overline{Q}_0 \in \mathbb{R}^3 \) (since they are respectively the right and left eigenvectors of \( P \) corresponding to the eigenvalue 1):

\[
h(k) = \omega \left( Q_0 \overline{Q}_0 + \lambda^k_1 Q_1 \overline{Q}_1 + \overline{\lambda}^k_1 \overline{Q}_2 \overline{Q}_2 \right) B' \]
\[
= \omega Q_0 \overline{Q}_0 B' + 2Re(\lambda^k_1 \omega Q_1 \overline{Q}_1 B') \quad \text{(Let } r + si = \omega Q_1 \overline{Q}_1 B', \ \lambda_1 = Ae^{i\theta})
\]
\[
= \omega Q_0 \overline{Q}_0 B' + 2A^k (r \cos k\theta - s \sin k\theta) \quad \text{(Let } a' \sin(k\theta + b') = 2(r \cos k\theta - s \sin k\theta), \ c' = \omega Q_0 \overline{Q}_0 B')
\]
\[
= a' A^k \sin(k\theta + b') + c',
\]

where \( \lambda_1 = Ae^{i\theta}, \ A \in (0, 1), \ \theta \in (0, 2\pi) \) and without loss of generality, we choose \( a' \geq 0, \ b' \in [0, 2\pi). \)

\[ \square \]

4.3. The Computation of \( L(\omega, \omega^*) \)

In the following theorem, we give the forms of the first crossing time \( L(\omega, \omega^*) \) for various cases mentioned in Lemma 3.

Theorem 3. Fix \( \omega \) and \( \omega^* \). Let \( r^* \overset{\Delta}{=} \omega^* B'. \) The first crossing time \( L(\omega, \omega^*) \) takes following forms:

\[
L(\omega, \omega^*) = \begin{cases} 
0, & \text{if } h(0) > r^* \\
1, & \text{if } h(1) > r^* \geq h(0) \\
2, & \text{if } h(2) > r^* \geq \max\{h(0), h(1)\}
\end{cases}
\]

(86)

where \( h(k) \) is the \( k \)-step reward function that depends on \( \omega \). The other cases are summarized below.
1. \( P \) has only real eigenvalues and 3 linearly independent eigenvectors: \( h(k) = a_1 b_1^k + a_2 b_2^k + c. \)

1.1 \( b_1 = b_2 \neq 0 \; \&\& \; b_1 > 0 \; \&\& \; a_1 + a_2 < 0: \)

\[
L(\omega, \omega^*) = \begin{cases} 
\left\lfloor \log_{b_1} \left( \frac{e-r^*}{a_1 + a_2} \right) + 1, \right. & \text{if } h(0) \leq r^* < c \\
\infty, & \text{if } r^* \geq c 
\end{cases}
\]

(87)

1.2 \( b_1 = b_2 \neq 0 \; \&\& \; (b_1 < 0 \; \| \; a_1 + a_2 \geq 0): L(\omega, \omega^*) = \infty \; \text{if } r^* \geq \max\{h(0), h(1)\}; \)

1.3 \( a_1 b_1 = 0 \; \&\& \; b_2 > 0 \; \&\& \; a_2 < 0: \)

\[
L(\omega, \omega^*) = \begin{cases} 
\left\lfloor \log_{b_2} \left( \frac{e-r^*}{a_1 + a_2} \right) + 2, \right. & \text{if } \max\{h(0), h(1)\} \leq r^* < c \\
\infty, & \text{if } r^* \geq \max\{h(0), c\} 
\end{cases}
\]

(88)

1.4 \( a_1 b_1 = 0 \; \&\& \; (b_2 \leq 0 \; \| \; a_2 \geq 0): L(\omega, \omega^*) = \infty \; \text{if } r^* \geq \max\{h(0), h(1), h(2)\}; \)

1.5 \( a_2 b_2 = 0 \; \&\& \; b_1 > 0 \; \&\& \; a_1 < 0: \)

\[
L(\omega, \omega^*) = \begin{cases} 
\left\lfloor \log_{b_1} \left( \frac{e-r^*}{a_1 + a_2} \right) + 2, \right. & \text{if } \max\{h(0), h(1)\} \leq r^* < c \\
\infty, & \text{if } r^* \geq \max\{h(0), c\} 
\end{cases}
\]

(89)

1.6 \( a_2 b_2 = 0 \; \&\& \; (b_1 \leq 0 \; \| \; a_1 \geq 0): L(\omega, \omega^*) = \infty \; \text{if } r^* \geq \max\{h(0), h(1), h(2)\}; \)

1.7 \( a_1, a_2, b_1, b_2 > 0: L(\omega, \omega^*) = \infty \; \text{if } r^* \geq h(0), h(1); \)

1.8 \( a_1 < 0, a_2 > 0, b_1 > b_2 > 0: \)

\[
L(\omega, \omega^*) = \begin{cases} 
\min\{k: k > \max\left\{ \left\lfloor \log_{b_2} \left( \frac{a_2 (1-b_2)}{a_1 (1-b_1)} \right) \right\rfloor, 0 \right\}, h(k) > r^* \}, & \text{if } h(0) \leq r^* < c \\
\infty, & \text{if } r^* \geq \max\{h(0), c\} 
\end{cases}
\]

(90)

1.9 \( a_1 < 0, a_2 > 0, b_2 > b_1 > 0: \)

\[
L(\omega, \omega^*) = \begin{cases} 
\min\{k: 0 < k \leq \left\lfloor \log_{b_2} \left( \frac{a_2 (1-b_2)}{a_1 (1-b_1)} \right) \right\rfloor + 1, h(k) > r^* \}, & \text{if } z_0 < 1 \; \&\& \; h\left( \left\lfloor \log_{b_2} \left( \frac{a_2 (1-b_2)}{a_1 (1-b_1)} \right) \right\rfloor + 1 \right) > r^* \geq h(0) \\
\infty, & \text{if } \left( z_0 \geq 1 \&\& r^* \geq h(0) \right) \; \| \; \left( z_0 < 1 \&\& h\left( \left\lfloor \log_{b_2} \left( \frac{a_2 (1-b_2)}{a_1 (1-b_1)} \right) \right\rfloor + 1 \right) \leq r^* \), \end{cases}
\]

(91)

where \( z_0 = -\frac{a_2 (1-b_2)}{a_1 (1-b_1)} \).

1.10 \( b_1 < 0, a_1, a_2, b_2 > 0: L(\omega, \omega^*) = \infty \; \text{if } r^* \geq h(0); \)

1.11 \( a_1, b_1 < 0, a_2, b_2 > 0: L(\omega, \omega^*) = \infty \; \text{if } r^* \geq \max\{h(0), h(1)\}; \)
1.12 \( a_2, b_1 < 0, a_1, b_2 > 0, |b_1| > b_2 \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min\{k: 0 < k \leq \left\lfloor \log_{b_2}^{z_1} \right\rfloor + 2, h(k) > r^*\}, & \text{if } z_1 \geq 1 \text{ \&\& } h(\left\lfloor \log_{b_2}^{z_1} \right\rfloor + 2) > r^* \geq h(0) \\
\infty, & \text{if } (z_1 < 1 \text{\&\& } r^* \geq h(0)) \text{ \&\& } (z_1 \geq 1 \text{ \&\& } h(\left\lfloor \log_{b_2}^{z_1} \right\rfloor + 2) \leq r^*),
\end{cases}
\]

where \( z_1 = -a_2(1-b_2^a) / a_1(1-b_2^a) \).

1.13 \( a_2, b_1 < 0, a_1, b_2 > 0, |b_1| < b_2 \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min\{k: k \geq \max\{\left\lfloor \log_{b_2}^{z_1} \right\rfloor, 1\}, h(k) > r^*\}, & \text{if } h(0) \leq r^* < c \\
\infty, & \text{if } \max\{h(0), c\} \leq r^*
\end{cases}
\]

1.14 \( a_2, b_1 < 0, a_1, b_2 > 0, |b_1| = b_2 \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min\{k: k \geq 1, h(k) > r^*\}, & \text{if } h(0) \leq r^* < c \\
\infty, & \text{if } \max\{h(0), c\} \leq r^*
\end{cases}
\]

1.15 \( b_1, b_2 < 0, a_1, a_2 > 0 \): \( L(\omega, \omega^*) = \infty \text{ if } r^* \geq h(0) \).

1.16 \( b_1, b_2 > 0, a_1, a_2 < 0 \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min\{k: k \geq 0, h(k) > r^*\}, & \text{if } r^* < c \\
\infty, & \text{if } c \leq r^*
\end{cases}
\]

1.17 \( a_1, a_2, b_1 < 0, b_2 > 0, |b_1| > b_2 \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min\{k: 0 < k \leq \left\lfloor \log_{b_2}^{z_2} \right\rfloor + 2, h(k) > r^*\}, & \text{if } z_2 > -b_1 \text{ \&\& } h(\left\lfloor \log_{b_2}^{z_2} \right\rfloor + 2) > r^* \geq h(0) \\
\infty, & \text{if } (z_2 \leq -b_1 \text{\&\& } r^* \geq h(1)) \text{ \&\& } (z_2 > -b_1 \text{ \&\& } h(\left\lfloor \log_{b_2}^{z_2} \right\rfloor + 2) \leq r^*),
\end{cases}
\]

where \( z_2 = a_2(b_2^a-1) / a_1(b_2^a-1) \).

1.18 \( a_1, a_2, b_1 < 0, b_2 > 0, |b_1| < b_2 \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min\{k: k \geq \max\{\left\lfloor \log_{b_2}^{z_2} \right\rfloor, 1\}, h(k) > r^*\}, & \text{if } h(1) \leq r^* < c \\
\infty, & \text{if } \max\{h(1), c\} \leq r^*,
\end{cases}
\]

where \( z_2 = a_2(b_2^a-1) / a_1(b_2^a-1) \).
1.19 \( a_1, a_2, b_1 < 0, b_2 > 0, |b_1| = b_2 \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min \{ k : k > 1, \ h(k) > \omega^* \}, & \text{if } h(1) \leq r^* < c \\
\infty, & \text{if } \max \{ h(1), c \} \leq r^*
\end{cases}
\tag{98}
\]

1.20 \( a_2, b_1, b_2 < 0, a_1 > 0, |b_1| > |b_2| \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min \{ k : 1 \leq k \leq \lceil \log_{b_1}^z \rceil + 2, \ h(k) > r^* \}, & \text{if } z_1 > 1 \&\& h(0) \leq r^* < h(\lceil \log_{b_1}^z \rceil + 2) \\
\infty, & \text{if } (z_1 > 1 \&\& \max \{ h(0), h(1), h(\lceil \log_{b_1}^z \rceil + 2) \} \leq r^* \} \| (z_1 \leq 1 \&\& h(0) \leq r^*)
\end{cases}
\tag{99}
\]

where \( z_1 = \frac{a_2(1-b_2^2)}{a_1(1-b_1^2)} \) and \( \lfloor n \rfloor \) denotes the maximum even integer not exceeding \( n \);  

1.21 \( a_2, b_1, b_2 < 0, a_1 > 0, |b_1| < |b_2| \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min \{ k : 1 \leq k \leq \lceil \log_{b_1}^z \rceil + 2, \ h(k) > r^* \}, & \text{if } z_1 > \frac{b_1}{b_2} \&\& h(0) \leq r^* < h(\lceil \log_{b_1}^z \rceil + 2) \\
\infty, & \text{if } (z_1 \geq \frac{b_1}{b_2} \&\& \max \{ h(0), h(1) \} \leq r^* \} \| (z_1 < \frac{b_1}{b_2} \&\& \{ h(0), h(1), h(\lceil \log_{b_1}^z \rceil + 2) \} \leq r^*)
\end{cases}
\tag{100}
\]

where \( z_1 = \frac{a_2(1-b_2^2)}{a_1(1-b_1^2)} \) and \( \lceil n \rceil \) denotes the maximum odd integer not exceeding \( n \);  

1.22 \( a_2, b_1, b_2 < 0, a_1 < 0 \): \( L(\omega, \omega^*) = \infty \) if \( \omega^* \geq h(1) \);  

1.23 any other case, which is symmetric to one of the above and omitted here.

2. \( P \) has only real eigenvalues and 2 linearly independent eigenvectors: \( h(k) = ab^k + ckb^{k-1} + d \).

2.1 \( b, c > 0 \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min \{ k : 0 \leq k < \lceil z_3 \rceil + 1, \ h(k) > r^* \}, & \text{if } z_3 > 0 \&\& h(0) \leq r^* < h(z_3 + 1) \\
\infty, & \text{if } (z_3 > 0 \&\& h(z_3 + 1) \leq r^*) \| (z_3 \leq 0 \&\& h(0) \leq r^*)
\end{cases}
\tag{101}
\]

where \( z_3 = \frac{ab - ab^2 - cb}{c(6-1)} \);

2.2 \( b > 0, c < 0 \):

\[
L(\omega, \omega^*) = \begin{cases} 
\min \{ k : k \geq \max \{ \lceil z_3 \rceil + 1, 0 \}, \ h(k) > r^* \}, & \text{if } h(0) \leq r^* < d \\
\infty, & \text{if } \max \{ h(0), d \} \leq r^*
\end{cases}
\tag{102}
\]

where \( z_3 = \frac{ab - ab^2 - cb}{c(6-1)} \);
2.3 $b < 0, c < 0$:

$$L(\omega, \omega^*) = \begin{cases} 
\min\{k : 1 \leq k \leq \lfloor z_4 \rfloor + 2, \ h(k) > r^*, \text{ if } z_4 > 0 \land h(0) \leq r^* < h(\lfloor z_4 \rfloor + 2) \} \\
\infty, \text{ if } (z_4 > 0 \land \max\{h(0), h(1), h(\lfloor z_4 \rfloor + 2)\} \leq r^*) \land (z_4 \leq 0 \land h(0) \leq r^*), 
\end{cases}$$

(103)

where $z_4 = \frac{ab - ab^3 - 2c^2}{c(b - 1)}$ and $\lfloor n \rfloor$ denotes the maximum even integer not exceeding $n$;

2.4 $b < 0, c > 0$:

$$L(\omega, \omega^*) = \begin{cases} 
\min\{k : 1 \leq k \leq \lfloor z_4 \rfloor + 2, \ h(k) > r^*, \text{ if } z_4 > 1 \land h(0) \leq r^* < h(\lfloor z_4 \rfloor + 2) \} \\
\infty, \text{ if } (z_4 \leq 1 \land \max\{h(0), h(1)\} \leq r^*) \land (z_4 > 1 \land \max\{h(0), h(1), h(\lfloor z_4 \rfloor + 2)\} \leq r^*), 
\end{cases}$$

(104)

where $z_4 = \frac{ab - ab^3 - 2c^2}{c(b - 1)}$ and $\lfloor n \rfloor$ denotes the maximum odd integer not exceeding $n$;

2.5 $bc = 0$: $h(k)$ is reduced to forms similar to those in (1.1)-(1.6) so details are omitted.

3. $P$ has a pair of conjugate complex eigenvalues: $h(k) = a' A^k \sin(k\theta + b') + c'$.

3.1 $d' = \frac{r^* - c'}{a'} > 0$:

$$L(\omega, \omega^*) = \begin{cases} 
\min\{k : 0 \leq k < \lceil \log_A d' \rceil, \ h(k) > r^*, \text{ if } \log_A d' > 0 \land \max\{h(k) : 0 \leq k < \lceil \log_A d' \rceil\} > r^* \} \\
\infty, \text{ if } \log_A d' \leq 0 \land (\log_A d' > 0 \land \max\{h(k) : 0 \leq k < \lceil \log_A d' \rceil\} \leq r^*); 
\end{cases}$$

(105)

3.2 $d' = \frac{r^* - c'}{a'} < 0$: $L(\omega, \omega^*) = \min\{k : k \geq 0, \ h(k) > r^*\}$;

3.3 $d' = \frac{r^* - c'}{a'} = 0, b' \in (0, \pi)$: $L(\omega, \omega^*) = 0$;

3.4 $d' = \frac{r^* - c'}{a'} = 0, \theta = \pi, b' \in \{0, \pi\}$: $L(\omega, \omega^*) = \infty$;

3.5 $d' = \frac{r^* - c'}{a'} = 0, \theta \neq \pi, b' = 0$: $L(\omega, \omega^*) = \lfloor \frac{\pi}{2\pi - \theta} \rfloor + 1$;

3.6 $d' = \frac{r^* - c'}{a'} = 0, \theta \neq \pi, b' = \pi$: $L(\omega, \omega^*) = \max\{0, \pi - \theta\} + 1$;

3.7 $d' = \frac{r^* - c'}{a'} = 0, \theta \in (0, \pi), b' \in (\pi, 2\pi)$: $L(\omega, \omega^*) = \lfloor \frac{2\pi - \theta}{2\pi - \theta} \rfloor + 1$;

3.8 $d' = \frac{r^* - c'}{a'} = 0, \theta \in (\pi, 2\pi), b' \in (\pi, 2\pi)$: $L(\omega, \omega^*) = \lfloor \frac{b - \pi}{2\pi - \theta} \rfloor + 1$;

Proof of Theorem[3] The base case [30] is clear. We prove the rest case by case in the same order as appeared in the theorem.

1. $P$ has only real eigenvalues and 3 linearly independent eigenvectors: $h(k) = a_1 b_1^k + a_2 b_2^k + c$. 


1.1 \( b_1 = b_2 \neq 0 \&\& b_1 > 0 \&\& a_1 + a_2 < 0 \): \( h(k) = (a_1 + a_2)b_1^k + c \) is monotonically increasing over \( k \geq 0 \) and the result follows.

1.2 \( b_1 = b_2 \neq 0 \&\& (b_1 < 0 \mid a_1 + a_2 \geq 0) \): \( L(\omega, \omega^*) \) achieves the maximum value at either \( h(0) \) or \( h(1) \) and the result follows.

1.3 \( a_1b_1 = 0 \&\& b_2 > 0 \&\& a_2 < 0 \): \( h(k) = a_2b_2^k + c \) which is monotonically increasing over \( k \geq 1 \) and the result follows.

1.4 \( a_1b_1 = 0 \&\& (b_2 \leq 0 \mid a_2 \geq 0) \): \( L(\omega, \omega^*) \) achieves the maximum value at one of \( \{h(0), h(1), h(2)\} \) and the result follows.

1.5 \( a_2b_2 = 0 \&\& b_1 > 0 \&\& a_1 < 0 \): similar to (1.3).

1.6 \( a_2b_2 = 0 \&\& (b_1 \leq 0 \mid a_1 \geq 0) \): similar to (1.4).

1.7 \( a_1, a_2, b_1, b_2 > 0 \): \( h(k) \) achieves the maximum value at \( h(0) \) and the result follows.

1.8 \( a_1 < 0, a_2 > 0, b_1 > b_2 > 0 \): observe that

\[
h(k + 1) - h(k) > 0 \iff \left( \frac{b_1}{b_2} \right)^k > -\frac{a_2(b_2 - 1)}{a_1(b_1 - 1)} > 0.
\]

If there exists a \( k_1 \geq 0 \) satisfying the above, then \( h(k) \) is monotonically decreasing until \( k_1 \) after which it increases. So the supremum of \( h(k) \) is achieved at either 0 or \( \infty \). If such \( k_1 \) does not exist, \( h(k) \) is monotonically increasing for all \( k \geq 0 \) and achieves its supremum at \( \infty \). The result thus follows.

1.9 \( a_1 < 0, a_2 > 0, b_2 > b_1 > 0 \): contrary to (1.8), if there exists a \( k_1 \geq 0 \) such that \( h(k_1 + 1) - h(k_1) > 0 \), then \( h(k) \) is monotonically increasing until \( k_1 + 1 \) after which it decreases to the stationary reward \( c \) (see Fig. [10]). So the maximum of \( h(k) \) is achieved at either 0 or \( k_1 + 1 \). If such \( k_1 \) does not exist, \( h(k) \) is monotonically decreasing for all \( k \geq 0 \) and achieves its maximum at 0. The result thus follows.

1.10 \( b_1 < 0, a_1, a_2, b_2 > 0 \): since

\[
h(k + 1) < h(k), \quad h(k + 2) < h(k), \quad \forall \text{even number } k \geq 0,
\]

so \( h(k) \) achieves its maximum at 0 and the result follows.
Figure 1 \( h(k) = -0.5 \times 0.5^k + 0.4 \times 0.7^k \), \( z(k) = \left(\frac{0.5}{0.7}\right)^k \)

\[ h(k) = -0.5 \times 0.5^k + 0.4 \times 0.7^k \]

Figure 2 \( h(k) = -0.5 \times (-0.5)^k + 0.4 \times 0.7^k \)

1.11 \( a_1, b_1 < 0, a_2, b_2 > 0 \): observe that

\[
h(k+1) - h(k) > 0 \iff z(k) = \left(\frac{b_1}{b_2}\right)^k > -\frac{a_2(b_2-1)}{a_1(b_1-1)}(> 0)
\]

\[
h(k+2) - h(k) > 0 \iff z(k) = \left(\frac{b_1}{b_2}\right)^k > -\frac{a_2(b_2^2-1)}{a_1(b_1^2-1)}(> 0)
\]

which directly lead to the following properties:

\[
f(k+1) < f(k), \ f(k+2) < f(k) \ \forall \text{odd number } k \geq 1.
\]

So \( h(k) \) achieves its maximum at 0 or 1 and the result follows. See Fig. 2 for an example.
1.12 $a_2, b_1 < 0, a_1, b_2 > 0, |b_1| > b_2$: observe that

$$h(k+1) - h(k) > 0 \iff z(k) = \left(\frac{b_1}{b_2}\right)^k < -\frac{a_2(b_2-1)}{a_1(b_1-1)} \geq 0$$

$$h(k+2) - h(k) > 0 \iff z(k) = \left(\frac{b_1}{b_2}\right)^k < -\frac{a_2(b_2^2-1)}{a_1(b_1^2-1)} \geq 0.$$ 

Let $k_1$ and $k_2$ be the maximum even integers achieving the above inequalities, respectively. Note that $k_2 \geq k_1$. If both of them are nonnegative, then $h(k)$ is monotonically increasing until $k_1 + 2$, then moving up with oscillations until $k_2 + 2$ and finally moving downward to converge to $c$. If $k_1 < 0 \leq k_2$, then $h(k)$ still achieves its maximum $k_2 + 2$. Finally, if $k_2 < 0$, $h(k)$ has its maximum at 0. The result thus follows. See Figs. 3 and 4 for an example.

![Figure 3](image1.png)

**Figure 3** $h(k) = 0.2 \times (-0.7)^k - 2 \times 0.4^k$

1.13 $a_2, b_1 < 0, a_1, b_2 > 0, |b_1| = b_2$: this case is sort of the reversed version to (1.12). Let $k_1 \geq 0$ and $k_2 \geq 0$ be the minimum even integers achieving the two inequalities in (1.12), respectively. In this case, $h(k)$ moves down with oscillations until $k_2$, then it moves up with oscillations until $k_1$ and finally increases to the stationary reward $c$. Therefore $h(k)$ achieves its supremum at 0 or $\infty$. The result thus follows.

1.14 $a_2, b_1 < 0, a_1, b_2 > 0, |b_1| = b_2$: under this case, the following holds

$$h(k+1) - h(k) > 0 \iff (-1)^k < -\frac{a_2(b_2 - 1)}{a_1(b_1 - 1)} \geq 0.$$ 


Figure 4 \[ z(k) = \left( -\frac{0.7}{0.14} \right)^k \]

\[ h(k+2) - h(k) > 0 \Leftrightarrow (-1)^k < -\frac{a_2(b_2^2 - 1)}{a_1(b_1^2 - 1)} < 0. \]

If \(-\frac{a_2(b_2 - 1)}{a_1(b_1 - 1)} \geq 1\), then \(h(k)\) is monotonically increasing to the stationary reward \(c\). If \(-\frac{a_2(b_2 - 1)}{a_1(b_1 - 1)} < 1\) and \(-\frac{a_2(b_2^2 - 1)}{a_1(b_1^2 - 1)} < 1\), then \(h(k)\) oscillates but its maximum value cannot exceed \(h(0)\). If \(-\frac{a_2(b_2 - 1)}{a_1(b_1 - 1)} < 1\) and \(-\frac{a_2(b_2^2 - 1)}{a_1(b_1^2 - 1)} > 1\), then \(h(k)\) moves up with oscillations to its supremum \(c\). The result thus follows.

1.15 \(b_1, b_2 < 0, a_1, a_2 > 0\): the maximum of \(h(k)\) clearly happens at 0 and the result thus follows.

1.16 \(b_1, b_2 > 0, a_1, a_2 < 0\): \(h(k)\) monotonically converges to \(c\) from below and the result thus follows.

1.17 \(a_1, a_2, b_1 < 0, b_2 > 0, |b_1| > b_2\): under this case, the following holds

\[ h(k+1) - h(k) > 0 \Leftrightarrow z(k) = \left( \frac{b_1}{b_2} \right)^k > -\frac{a_2(b_2 - 1)}{a_1(b_1 - 1)} < 0 \]

\[ h(k+2) - h(k) > 0 \Leftrightarrow z(k) = \left( \frac{b_1}{b_2} \right)^k > -\frac{a_2(b_2^2 - 1)}{a_1(b_1^2 - 1)} < 0. \]

Any even \(k\) clearly satisfies the above two inequalities. Let \(k_1 \geq 1\) and \(k_2 \geq 1\) be the maximum odd integers achieving the above, respectively. If \(k_1, k_2\) exist, then \(k_1 \leq k_2\) and \(h(k)\) monotonically increases until \(k_1 + 2\) after which it goes up with oscillations until \(k_2 + 2\), and finally it falls with oscillations and converges to \(c\). As long as \(k_2\) exists, \(h(k)\) has its maximum at \(k_2 + 2\). When \(k_2\) does not exist, it is clear that \(h(k)\) achieves its maximum at 1 and the result follows.
\[ 1.18 \quad a_1, a_2, b_1 < 0, b_2 > 0, |b_1| < b_2: \text{let } k_1 \geq 1 \text{ and } k_2 \geq 1 \text{ be the minimum odd integers achieving the two inequalities in (1.17), respectively. Note that } k_2 \leq k_1. \text{ Then } h(k) \text{ moves down with oscillations until } k_2 \text{ after which it goes up with oscillations until } k_1 \text{ and finally } h(k) \text{ monotonically increases to } c. \text{ The result thus follows.}

\[ 1.19 \quad a_1, a_2, b_1 < 0, b_2 > 0, |b_1| = b_2: \text{similar to (1.14), if } -\frac{a_2(b_2-1)}{a_1(b_1-1)} \leq -1, \text{ then } h(k) \text{ is monotonically increasing to the stationary reward } c. \text{ If } -\frac{a_2(b_2-1)}{a_1(b_1-1)} > -1 \text{ and } -\frac{a_2(b_2-1)}{a_1(b_1-1)} < -1, \text{ then } h(k) \text{ moves up with oscillations to } c. \text{ If } -\frac{a_2(b_2-1)}{a_1(b_1-1)} > -1 \text{ and } -\frac{a_2(b_2-1)}{a_1(b_1-1)} \geq -1, \text{ then } h(k) \text{ achieves its maximum at } 1. \text{ The result thus follows.}

\[ 1.20 \quad a_2, b_1, b_2 < 0, a_1 > 0, |b_1| > |b_2|: \text{let}

\[ k_{o1} = \min\{k: h(k+1) > h(k), k \text{ is positive and odd}\} - 2,
\]

\[ k_{o2} = \min\{k: h(k+2) > h(k), k \text{ is positive and odd}\} - 2,
\]

\[ k_{e1} = \max\{k: h(k+1) > h(k), k \text{ is nonnegative and even}\},
\]

\[ k_{e2} = \max\{k: h(k+2) > h(k), k \text{ is nonnegative and even}\}.
\]

If \( k_{e1} \) exists, then \(|k_{o1} - k_{e1}| = 1 \) and \(|k_{o2} - k_{e2}| = 1. \) Furthermore, we have that \( k_{o1} \leq k_{o2} \) and \( k_{e1} \leq k_{e2} \) (see Fig. 5 for an example). Let \( k_1 = \max\{k_{o1}, k_{e1}\}. \) From the origin 0 to \( k_1 + 1 \), we have that \( \max_{0 \leq k \leq k_1+1} h(k) = h(1) \). Then from \( k_1 + 1 \) to \( k_{e2} + 2 \), it reaches a local maximum \( \max_{k>1} h(k) = h(k_{e2}+2) \) after which it moves down to the stationary reward \( c \) (see Fig. 6 for an example). If \( k_{e1} \) does not exist but \( k_{e2} \) does, \( h(k) \) attains its maximum value at either 0 or \( h(k_{e2}+2) \).

If \( k_{e2} \) does not exist, then \( h(k) \) attains its maximum value at 0. The result thus follows.

\[ 1.21 \quad a_2, b_1, b_2 < 0, a_1 > 0, |b_1| < |b_2|: \text{let}

\[ k_{o1} = \max\{k: h(k+1) > h(k), k \text{ is positive and odd}\},
\]

\[ k_{o2} = \max\{k: h(k+2) > h(k), k \text{ is positive and odd}\},
\]

\[ k_{e1} = \min\{k: h(k+1) > h(k), k \text{ is nonnegative and even}\} - 2,
\]

\[ k_{e2} = \min\{k: h(k+2) > h(k), k \text{ is nonnegative and even}\} - 2.
\]
Figure 5 \[ z(k) = \left(\frac{-0.8}{-0.4}\right)^k \]

Figure 6 \[ h(k) = 0.2 \times (-0.8)^k - 2 \times (-0.4)^k \]

Similar to (1.20), if \( k_{o1} \) exists, then \( |k_{o1} - k_{e1}| = 1 \), \( |k_{o2} - k_{e2}| = 1 \), \( k_{o1} \leq k_{o2} \), and \( k_{e1} \leq k_{e2} \). Let \( k_1 = \max\{k_{o1}, k_{e1}\} \). We have that \( h(0) = \max_{0 \leq k \leq k_1 + 1} h(k) \) and \( \max_{k \geq k_1 + 2} h(k) = h(k_{o2}) \). If \( k_{o1} \) does not exist but \( k_{o2} \) does, then arg max \( k \) satisfies \( h(k) \) is one of \( \{0, 1, k_{o2} + 2\} \) (see Fig. 5 for an example). If \( k_{o2} \) does not exist, then arg max \( k \) is either 0 or 1. The result thus follows.

1.22 \( a_2, b_1, b_2 < 0, a_1 < 0 \): obviously \( h(k) \) achieves its maximum at 1 and the result follows.

2. \( P \) has only real eigenvalues and 2 linearly independent eigenvectors: \( h(k) = ab^k + ckb^{k-1} + d \).

2.1 \( b, c > 0 \): observe that

\[ h(k + 1) > h(k) \iff k < \frac{ab - ab^2 - cb}{c(b - 1)} \]
Figure 7 \( h(k) = 0.7 \times (-0.4)^k - 0.6 \times (-0.8)^k \), \( g(k) = \left(\frac{-0.4}{-0.8}\right)^k \)

Figure 8 \( h(k) = 2 \times 0.7^k + 3k \times 0.7^{k-1} \)

Let \( k_1 \geq 0 \) be the maximum integer satisfying the above inequality. If it exists, then \( h(k) \) will keep increasing until \( (k_1 + 1) \) after which it turns to be monotonically decreasing to the stationary reward \( d \). Hence, \( k_1 + 1 = \arg \max_k h(k) \) (see Fig. 8 for an example). If \( k_1 \) does not exist, then \( h(k) \) is monotonically decreasing and \( \arg \max_k h(k) = 0 \). The result thus follows.

2.2 \( b > 0, c < 0 \): observe that

\[
h(k + 1) > h(k) \iff k > \frac{ab - ab^2 - cb}{c(b - 1)}
\]
Let \( k_1 \geq 0 \) be the minimum integer satisfying the above inequality. Clearly \( h(k) \) is monotonically decreasing until \( k_1 \) after which it keeps increasing to \( d \). So \( h(k) \) achieves its supremum at either 0 or \( \infty \) and the result follows.

2.3 \( b < 0 \), \( c < 0 \): the proof is similar to that of (1.20) and omitted here (see Fig. 9 for an example)

2.4 \( b < 0 \), \( c > 0 \): the proof is similar to that of (1.21) and omitted here.

3. \( \mathbf{P} \) has a pair of conjugate complex eigenvalues: \( h(k) = a' A^k \sin(k \theta + b') + c' \).
   
3.1 \( d' = \frac{r' - c'}{a'} > 0 \): clearly \( h(k) \) will be smaller than \( d' \) as \( k \) becomes sufficiently large and the result follows.

3.2 \( d' = \frac{r' - c'}{a'} < 0 \): clearly \( h(k) \) will be larger than \( d' \) as \( k \) becomes sufficiently large and the exhaustion stops in finite time.

3.3-3.8 These cases follow directly by finding the first \( k \geq 0 \) such that \( \sin(k \theta + b') > 0 \) and we omit the details here.

□

The general algorithmic framework for Whittle index policy is given in Algorithm 1.

4.4. Numerical Examples

In this section, we demonstrate the near-optimality of Whittle index policy. Through extensive numerical examples, we compute the performance of the optimal policy by dynamic programming
Algorithm 1 Whittle Index Policy

**Input:** discount factor $\beta$, time period $T$, arm number $N$, active arm number $K$

**Input:** initial belief state $\omega_n(1)$, transition matrix $P^{(n)}$, reward vector $B_n$, $n = 1, \ldots, N$

1: for $t = 1, 2, \ldots, T$ do
2:     for $n = 1, \ldots, N$ do
3:         Compute $L\left(\omega_n(t)P^{(n)}, \omega_n(t)\right)$, $L\left(p_i^{(n)}, \omega_n(t)\right)$, $i = 0, 1, 2, \ldots, K - 1$
4:         Compute $f(p_i^{(n)}) = \frac{1 - \beta L\left(p_i^{(n)}, \omega_n(t)\right)}{1 - \beta}$, $i = 0, 1, 2, \ldots, K - 1$
5:         $F(P^{(n)}) = \left[f(p_0^{(n)}), f(p_1^{(n)}), f(p_2^{(n)}), \ldots, f(p_K^{(n)})\right]$′
6:         Compute $g(p_i^{(n)}) = \beta L\left(p_i^{(n)}, \omega_n(t)\right)\omega_n(t)(P^{(n)})L\left(p_i^{(n)}, \omega_n(t)\right)$, $i = 0, 1, 2, \ldots, K - 1$
7:         $G(P^{(n)}) = \left[g(p_0^{(n)}), g(p_1^{(n)}), g(p_2^{(n)}), \ldots, g(p_K^{(n)})\right]$′, $H(P^{(n)}) = \left(I_K - \beta G(P^{(n)})\right)^{-1}$
8:         Compute $f(\omega_n(t)P^{(n)}) = \frac{1 - \beta L(\omega_n(t)P^{(n)}, \omega_n(t))}{1 - \beta}$
9:         $g(\omega_n(t)P^{(n)}) = \beta L(\omega_n(t)P^{(n)}, \omega_n(t))\omega_n(t)(P^{(n)})L(\omega_n(t)P^{(n)}, \omega_n(t))1$
10: Define $A \triangleq 1 + \beta f(\omega_n(t)P^{(n)}) + \beta \left[\beta g(\omega_n(t)P^{(n)}) - \omega_n(t)\right] H(P^{(n)}) F(P^{(n)})$
11: Set $W(\omega_n(t)) = \omega_n(t)B_n'$ and skip Step 12 if $A = 0$
12: Compute $W(\omega_n(t)) = \frac{\omega_n(t)B_n' - \beta g(\omega_n(t)P^{(n)})1_{K + \beta H(P^{(n)})G(P^{(n)})}B_n' + \beta \omega_n(t)H(P^{(n)})G(P^{(n)})B_n'}{1 + \beta f(\omega_n(t)P^{(n)}) + \beta \left[\beta g(\omega_n(t)P^{(n)}) - \omega_n(t)\right] H(P^{(n)}) F(P^{(n)})}$
13: Choose the top $K$ arms with the largest Whittle Indices $W(\omega_n(t))$
14: Observe the states $S_n(t)$ of the selected $K$ arms, and accrue the reward
15:     for $n = 1, \ldots, N$ do
16:         if arm $n$ is active then
17:             $\omega_n(t + 1) = \left[p_{S_n(t), 0}^{(n)}, p_{S_n(t), 1}^{(n)}, p_{S_n(t), 2}^{(n)}, \ldots, p_{S_n(t), K - 1}^{(n)}\right]$
18:         else
19:             $\omega_n(t + 1) = \omega_n(t)P^{(n)}$

and simulate the low-complexity Whittle index policy by Monte-Carlo runs. The performance of Whittle index policy has been observed to be quite close to the optimal one from all these numerical
trials. Here we list a few examples with parameters shown in Tables 1-3 and their performance in Figures 10-15. Furthermore, we compare Whittle index policy to the myopic policy in Figures 14 and 15 and illustrate the superiority of the former.

| Table 1 | Experiment setting 1 ($\beta = 0.9999, B_i = [0, 2, 3], i = 1, \ldots, 7$) |
|---------|-----------------|
| arm     | machine 1       | machine 2       |
| 1       | $p^{(1)} = (0.514, 0.321, 0.165)$, $\omega_1 = (0.279, 0.618, 0.103)$, $\omega_2 = (0.103, 0.633, 0.264)$, $\omega_3 = (0.417, 0.301, 0.282)$ | $p^{(1)} = (0.519, 0.445, 0.036)$, $\omega_1 = (0.354, 0.164, 0.482)$ |
| 2       | $p^{(2)} = (0.312, 0.543, 0.085)$, $\omega_1 = (0.058, 0.689, 0.253)$, $\omega_2 = (0.088, 0.024, 0.208)$, $\omega_3 = (0.489, 0.048, 0.103)$ | $p^{(2)} = (0.193, 0.334, 0.215)$, $\omega_1 = (0.275, 0.485, 0.240)$, $\omega_2 = (0.234, 0.694, 0.072)$ |
| 3       | $p^{(3)} = (0.413, 0.328, 0.259)$, $\omega_1 = (0.489, 0.259, 0.103)$, $\omega_2 = (0.088, 0.024, 0.208)$, $\omega_3 = (0.489, 0.048, 0.103)$ | $p^{(3)} = (0.600, 0.242, 0.158)$, $\omega_1 = (0.333, 0.498, 0.169)$ |
| 4       | $p^{(4)} = (0.461, 0.272, 0.267)$, $\omega_1 = (0.058, 0.689, 0.253)$, $\omega_2 = (0.088, 0.024, 0.208)$, $\omega_3 = (0.489, 0.048, 0.103)$ | $p^{(4)} = (0.711, 0.279, 0.137)$, $\omega_1 = (0.354, 0.424, 0.224)$ |
| 5       | $p^{(5)} = (0.330, 0.427, 0.231)$, $\omega_1 = (0.058, 0.689, 0.253)$, $\omega_2 = (0.088, 0.024, 0.208)$, $\omega_3 = (0.489, 0.048, 0.103)$ | $p^{(5)} = (0.161, 0.445, 0.394)$, $\omega_1 = (0.330, 0.427, 0.231)$, $\omega_2 = (0.058, 0.689, 0.253)$ |
| 6       | $p^{(6)} = (0.330, 0.427, 0.231)$, $\omega_1 = (0.058, 0.689, 0.253)$, $\omega_2 = (0.088, 0.024, 0.208)$, $\omega_3 = (0.489, 0.048, 0.103)$ | $p^{(6)} = (0.080, 0.279, 0.643)$, $\omega_1 = (0.330, 0.427, 0.231)$, $\omega_2 = (0.058, 0.689, 0.253)$ |
| 7       | $p^{(7)} = (0.427, 0.324, 0.249)$, $\omega_1 = (0.058, 0.689, 0.253)$, $\omega_2 = (0.088, 0.024, 0.208)$, $\omega_3 = (0.489, 0.048, 0.103)$ | $p^{(7)} = (0.130, 0.346, 0.444)$, $\omega_1 = (0.330, 0.427, 0.231)$, $\omega_2 = (0.058, 0.689, 0.253)$ |

| Table 2 | Experiment setting 2-1 ($\beta = 0.9999$) |
|---------|-----------------|
| arm     | machine 1       | machine 2       |
| 1       | $p^{(1)} = (0.356, 0.607, 0.357)$, $\omega_1 = (0.284, 0.494, 0.312)$, $\omega_1 = (0.058, 0.689, 0.253)$, $\omega_2 = (0.489, 0.048, 0.103)$ | $p^{(1)} = (0.538, 0.305, 0.157)$, $\omega_1 = (0.621, 0.418, 0.120)$ |
| 2       | $p^{(2)} = (0.445, 0.261, 0.633)$, $\omega_1 = (0.297, 0.361, 0.342)$, $\omega_1 = (0.297, 0.361, 0.342)$, $\omega_1 = (0.297, 0.361, 0.342)$ | $p^{(2)} = (0.798, 0.089, 0.113)$, $\omega_1 = (0.367, 0.354, 0.279)$ |
| 3       | $p^{(3)} = (0.407, 0.200, 0.393)$, $\omega_1 = (0.643, 0.421, 0.536)$, $\omega_1 = (0.643, 0.421, 0.536)$, $\omega_1 = (0.643, 0.421, 0.536)$ | $p^{(3)} = (0.218, 0.15, 0.767)$, $\omega_1 = (0.405, 0.151, 0.444)$ |
| 4       | $p^{(4)} = (0.087, 0.454, 0.459)$, $\omega_1 = (0.642, 0.026, 0.332)$, $\omega_1 = (0.642, 0.026, 0.332)$, $\omega_1 = (0.642, 0.026, 0.332)$ | $p^{(4)} = (0.428, 0.294, 0.278)$, $\omega_1 = (0.113, 0.499, 0.388)$ |
| 5       | $p^{(5)} = (0.331, 0.181, 0.488)$, $\omega_1 = (0.066, 0.017, 0.977)$, $\omega_1 = (0.066, 0.017, 0.977)$, $\omega_1 = (0.066, 0.017, 0.977)$ | $p^{(5)} = (0.376, 0.333, 0.291)$, $\omega_1 = (0.351, 0.203, 0.446)$ |
| 6       | $p^{(6)} = (0.331, 0.181, 0.488)$, $\omega_1 = (0.066, 0.017, 0.977)$, $\omega_1 = (0.066, 0.017, 0.977)$, $\omega_1 = (0.066, 0.017, 0.977)$ | $p^{(6)} = (0.320, 0.649, 0.031)$, $\omega_1 = (0.112, 0.037, 0.851)$ |
| 7       | $p^{(7)} = (0.067, 0.485, 0.498)$, $\omega_1 = (0.258, 0.483, 0.259)$, $\omega_1 = (0.258, 0.483, 0.259)$, $\omega_1 = (0.258, 0.483, 0.259)$ | $p^{(7)} = (0.046, 0.213, 0.741)$, $\omega_1 = (0.483, 0.050, 0.467)$ |
general POMDP models, e.g., relaxed indexability was satisfied. Future work includes the extensions of Whittle index to more general POMDP models, e.g., those with observation errors or different state transition dynamics.

Table 3  Experiment setting 2-2 ($\beta = 0.9999$)

| arm | machine | 3 | 4 |
|-----|---------|---|---|
| 1   | $p(1) = (0.488, 0.258, 0.254)$ | $p(1) = (0.413, 0.329, 0.258)$ | $p(1) = (0.309, 0.408, 0.283)$ |
|     | $\omega(1) = (0.681, 0.208, 0.111)$ | $\omega(1) = (0.486, 0.028, 0.486)$ | $\omega(1) = (0.233, 2.853)$ |
| 2   | $p(2) = (0.341, 0.349, 0.349)$ | $p(2) = (0.679, 0.143, 0.188)$ | $p(2) = (0.250, 2.358, 2.612)$ |
|     | $\omega(1) = (0.551, 0.320, 0.121)$ | $\omega(2) = (0.408, 0.496, 0.096)$ | $\omega(2) = (0.250, 2.358, 2.612)$ |
| 3   | $p(3) = (0.341, 0.349, 0.349)$ | $p(3) = (0.250, 2.358, 2.612)$ | $p(3) = (0.250, 2.358, 2.612)$ |
|     | $\omega(1) = (0.471, 0.073, 0.456)$ | $\omega(3) = (0.250, 2.358, 2.612)$ | $\omega(3) = (0.250, 2.358, 2.612)$ |
| 4   | $p(4) = (0.304, 0.369, 0.057)$ | $p(4) = (0.250, 2.358, 2.612)$ | $p(4) = (0.250, 2.358, 2.612)$ |
|     | $\omega(1) = (0.495, 0.117, 0.388)$ | $\omega(4) = (0.250, 2.358, 2.612)$ | $\omega(4) = (0.250, 2.358, 2.612)$ |
| 5   | $p(5) = (0.404, 0.282, 0.314)$ | $p(5) = (0.358, 0.501, 0.141)$ | $p(5) = (0.358, 0.501, 0.141)$ |
|     | $\omega(1) = (0.474, 0.239, 0.287)$ | $\omega(5) = (0.358, 0.501, 0.141)$ | $\omega(5) = (0.358, 0.501, 0.141)$ |
| 6   | $p(6) = (0.586, 0.024, 0.390)$ | $p(6) = (0.358, 0.501, 0.141)$ | $p(6) = (0.358, 0.501, 0.141)$ |
|     | $\omega(1) = (0.413, 0.388, 0.199)$ | $\omega(6) = (0.358, 0.501, 0.141)$ | $\omega(6) = (0.358, 0.501, 0.141)$ |
| 7   | $p(7) = (0.612, 0.385, 0.003)$ | $p(7) = (0.358, 0.501, 0.141)$ | $p(7) = (0.358, 0.501, 0.141)$ |
|     | $\omega(1) = (0.483, 0.513, 0.004)$ | $\omega(7) = (0.358, 0.501, 0.141)$ | $\omega(7) = (0.358, 0.501, 0.141)$ |

5. Conclusions

In this paper, we proposed an efficient algorithm to achieve a strong performance for a class of restless multi-armed bandits arisen in the general POMDP framework. By formulating an uncountable belief state space, we extended Whittle index policy previously studied for finite-state models and introduced the concept of relaxed indexability. An interesting finding is that through the online computation process for the first crossing time, all our numerical studies have shown that the relaxed indexability was satisfied. Future work includes the extensions of Whittle index to more general POMDP models, e.g., those with observation errors or different state transition dynamics.

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Figure 10  Experiment 1: machine 1
Figure 11  Experiment 1: machine 2
Figure 12  Experiment 2: machine 1
Figure 13  Experiment 2: machine 2
Figure 14  Experiment 2: machine 3
Figure 15  Experiment 2: machine 4
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