An application of the partial $r$-Bell polynomials on some family of bivariate polynomials

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Abstract. The aim of this paper is to give some combinatorial relations linked polynomials generalizing those of Appell type to the partial $r$-Bell polynomials. We give an inverse relation, recurrence relations involving some family of polynomials and their exact expressions at rational values in terms of the partial $r$-Bell polynomials. We illustrate the obtained results by various comprehensive examples.

Keywords. The partial $r$-Bell polynomials; polynomials of Appell type; inverse relations; recurrence relations.

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1 Introduction

This work is motivated by the work of Mihoubi and Tiachachat [15] on the expressions of the Bernoulli polynomials at rational numbers by the whitney numbers and the work of Mihoubi and Saidi [14] on the polynomials of Appell type.

The aim of this paper is to give some combinatorial relations linked polynomials generalizing those of Appell type to the partial $r$-Bell polynomials. For given two sequences of real numbers $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$, recall that the partial $r$-Bell polynomials

$$B_{n,k}^{(r)}(\mathbf{a}; \mathbf{b}) := B_{n,k}^{(r)}((a_j); (b_j)) = B_{n,k}^{(r)}(a_1, a_2, \ldots; b_1, b_2, \ldots)$$

are defined by their generating function to be

$$\sum_{n \geq k} B_{n+r,k+r}^{(r)}(\mathbf{a}; \mathbf{b}) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{j \geq 1} a_j \frac{t^j}{j!} \right)^r \left( \sum_{j \geq 0} b_{j+1} \frac{t^j}{j!} \right)^r.$$

These polynomials present a nature extension of the partial Bell polynomials [1] and generalize the $r$-Whitney numbers of both kinds, the $r$-Lah numbers and the $r$-Whitney-Lah numbers. Mihoubi et Rahmani [12] introduced and studied these polynomials for which they gave combinatorial and probabilistic interpretations and several properties. Shattuck [19] gave more properties and Chouria and Luque [4] defined three versions of the partial $r$-Bell polynomials in three combinatorial Hopf algebras. For an application of these polynomials on a family of bivariate polynomials, let us define this family. Indeed, let $A$, $B$ and $H$ be three analytic functions around zero with $A(0) = 0$, $A'(0) = B(0) = 1$ and let $\alpha$ and $x$ be real numbers. A sequence of numbers $P_n^{(\alpha)}(A, H)$ is defined by

$$\sum_{n \geq 0} P_n^{(\alpha)}(A, H) \frac{t^n}{n!} = \left( \frac{t}{A(t)} \right)^\alpha H(t)$$

(1)
and a sequence of polynomials \( P_n^{(\alpha)} (x, y \mid A, B, H) \) is to be

\[
\sum_{n \geq 0} P_n^{(\alpha)} (x, y \mid A, B, H) \frac{t^n}{n!} = \left( \frac{t}{A(t)} \right)^{\alpha} (A'(t))^x (B(t))^y H(t) .
\]

(2)

In the second section we give an inverse relation linked to the partial \( r \)-Bell polynomials and in the third section we give recurrence relations and exact expressions at rational values for the bivariate polynomials defined above.

2 The partial \( r \)-Bell polynomials and inverse relations

For any power series \( A(t) = \sum_{j \geq 1} a_j \frac{t^j}{j!} \) with \( a_1 \neq 0 \), below, we denote by \( \overline{A}(t) = \sum_{j \geq 1} \overline{a}_j \frac{t^j}{j!} \) for the compositional inverse of \( A(t) \) and we let \( \mathbf{a} := (a_1, a_2, \ldots) \) and \( \overline{\mathbf{a}} := (\overline{a}_1, \overline{a}_2, \ldots) \).

The following theorem gives inverse relations linked to the partial \( r \)-Bell polynomials.

**Theorem 1** The following inverse relations hold

\[
U_n = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)} (\mathbf{a}; \mathbf{b}) V_k , \quad V_n = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)} (\overline{\mathbf{a}}; (\mathbf{b} \circ \overline{\mathbf{a}})^{-1}) U_k ,
\]

i.e.

\[
\sum_{k=j}^{n} B_{n+r,k+r}^{(r)} (\mathbf{a}; \mathbf{b}) B_{k+r,j+r}^{(r)} (\overline{\mathbf{a}}; (\mathbf{b} \circ \overline{\mathbf{a}})^{-1}) = \delta_{(j,n)},
\]

where \( \mathbf{b} \circ \overline{\mathbf{a}} \) is the sequence of the coefficients of the power series \( B(\overline{A}(t)) \) and \( (\mathbf{b} \circ \overline{\mathbf{a}})^{-1} \) is the sequence of the coefficients of the power series \( (B(\overline{A}(t)))^{-1} \).

**Proof.** If \( U(t) = \sum_{n \geq 0} U_n \frac{t^n}{n!} \) and \( V(t) = \sum_{n \geq 0} V_n \frac{t^n}{n!} \), then \( U(t) = V(A(t))(B(t))^r \), or equivalently, by replacing \( t \) by \( \overline{A}(t) \), it becomes \( V(t) = U(\overline{A}(t))(B(\overline{A}(t)))^{-r} \). \( \square \)

**Example 1** For \( A(t) = \frac{\exp(mt)}{m} \) and \( B(t) = \exp(t) \) we get

\[
\overline{A}(t) = \frac{\ln(1 + mt)}{m}, \quad (B(\overline{A}(t)))^{-1} = (1 + mt)^{-\frac{1}{m}}, \quad B_{n+r,k+r}^{(r)} (\mathbf{a}; \mathbf{b}) = W_{m,r}(n,k), \quad B_{n+r,k+r}^{(r)} (\overline{\mathbf{a}}; (\mathbf{b} \circ \overline{\mathbf{a}})^{-1}) = w_{m,r}(n,k).
\]

So, we obtain the inverse relations

\[
U_n = \sum_{k=0}^{n} W_{m,r}(n,k) V_k , \quad V_n = \sum_{k=0}^{n} w_{m,r}(n,k) U_k ,
\]

where \( w_{m,r}(n,k) \) and \( W_{m,r}(n,k) \) are, respectively, the whitney numbers of the first and second kind, for more information, see for example [2, 7, 8, 12, 17].

For \( B(t) = A'(t) \) in Theorem 1 we obtain:
Corollary 2  The following inverse relations hold

\[ U_n = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)}(a) V_k, \quad V_n = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)}(\mathfrak{a}) U_k, \]

where \( B_{n+r,k+r}^{(r)}(a) := B_{n+r,k+r}^{(r)}(a; a) \).

For \( A(t) = (1 - t)^{\beta} - 1 \) and \( B(t) = (1 - t)^{\alpha} \) in Theorem 1 we obtain:

Corollary 3  Let \( \alpha, \beta \) be two real numbers such that \( \beta \neq 0 \).
Then, the following inverse relations hold

\[ U_n = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)}(\beta_j; (\alpha)_{j-1}) V_k, \]
\[ V_n = \sum_{k=0}^{n} (-1)^{n-k} B_{n+r,k+r}^{(r)}((-1/\beta)_j; (\alpha/\beta)_{j-1}) U_k, \]

where \( (\alpha)_n := \alpha (\alpha - 1) \cdots (\alpha - n + 1) \) if \( n \geq 1 \), \( (\alpha)_0 := 1 \) and \( (\alpha)_n := (-1)^n (-\alpha)_n \).

For \( B(t) = A(t) + 1 \) in Theorem 1 we obtain:

Corollary 4  Then, the following inverse relations hold

\[ U_n = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)}((a_j; (a)_{j-1}) V_k, \quad \alpha_0 = 1, \]
\[ V_n = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)}(\mathfrak{a}_j; (\mathfrak{a}_j^{-1}; (j-1)!)) U_k. \]

3  Recurrence relations involving the polynomials \( P_n^{(\alpha)} \)

In [14] we are proved the following identity

\[ P_n^{(\alpha)}(A, H) = P_n^{(n+1-\alpha)}(B, (A' \circ B)^{-1}(H \circ B)), \quad (3) \]

from which we gave

\[ P_n^{(n+1+\alpha)}(A, H) = \sum_{k=0}^{n} B_{n,k}^{(\alpha)} P_k^{(\alpha)}(A, (A')^{-1} H), \]

and, in particular, if we let \( \left( \frac{t}{A(t)} \right)^{\alpha}(A'(t))x = \sum_{n \geq 0} T_n^{(\alpha)}(x \mid A)^{\alpha}_n \), we get

\[ T_n^{(n+1+\alpha)}(x + 1 \mid A) = \sum_{k=0}^{n} B_{n,k}^{(\alpha)} T_k^{(\alpha)}(x \mid A). \]

The following theorem generalizes this last identity as follows:
Theorem 5  There holds

\[ P_{n}^{(\alpha)} (x, y \mid A, B, H) = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)} (a; b) P_{k}^{(k+1+\alpha)} (x+1, y-r \mid A, B, H), \tag{4} \]

or equivalently

\[ P_{n}^{(n+1+\alpha)} (x, y \mid A, B, H) = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)} (\overline{a}; (b \circ \overline{a})^{-1}) P_{k}^{(\alpha)} (x-1, y+r \mid A, B, H). \tag{5} \]

Proof. We prove that the two sides of (5) have the same exponential generating functions. Indeed,

\[
\sum_{n \geq 0} \left( \sum_{k \geq 0} n \binom{n}{k} B_{n+r,k+r}^{(r)} (a; b) P_{k}^{(k+1+\alpha)} (x, y \mid A, B, H) \right) \frac{t^n}{n!} \\
= \sum_{k \geq 0} P_{k}^{(k+1+\alpha)} (x, y \mid A, B, H) \left( \sum_{n \geq k} B_{n+r,k+r}^{(r)} (a; b) \frac{t^n}{n!} \right) \\
= \sum_{k \geq 0} P_{k}^{(k+1+\alpha)} (x, y \mid A, B, H) \frac{(A(t))^{k}}{k!} (B(t))^r
\]

and, by identity (3), the following identity

\[
P_{k}^{(k+1+\alpha)} (x, y \mid A, B, H) = D_{t=0}^{k} \left[ \left( \frac{t}{A(t)} \right)^{k+1+\alpha} (A'(t))^x (B(t))^y H(t) \right] \\
= D_{t=0}^{k} \left[ \left( \frac{t}{A(t)} \right)^{-\alpha} (A' \circ \overline{A}(t))^{x-1} (B \circ \overline{A}(t))^y (H \circ \overline{A}(t)) \right] \\
= D_{t=0}^{k} \left[ \left( \frac{t}{A(t)} \right)^{-\alpha} (\overline{A}(t))^{1-x} (B \circ \overline{A}(t))^y (H \circ \overline{A}(t)) \right] \\
= P_{k}^{(-\alpha)} (1-x, y \mid \overline{A}, B \circ \overline{A}, H \circ \overline{A}),
\]

shows that the last expansion becomes

\[
\sum_{k \geq 0} P_{k}^{(-\alpha)} (1-x, y \mid \overline{A}, B \circ \overline{A}, H \circ \overline{A}) \frac{(A(t))^{k}}{k!} (B(t))^r \\
= \left( \frac{A(t)}{A(A(t))} \right)^{-\alpha} (\overline{A}(A(t)))^{1-x} (B(\overline{A}(A(t))))^y (H(\overline{A}(A(t)))) (B(t))^r \\
= \left( \frac{t}{A(t)} \right)^{-\alpha} (A(t))^{x-1} (B(t))^{y+r} H(t) \\
= \sum_{n \geq 0} P_{n}^{(\alpha)} (x-1, y+r \mid A, B, H) \frac{t^n}{n!}.
\]

The first identity of this theorem can be obtained from the second one by apply Theorem 1. \qed
Corollary 6 There holds

\[ L_{n}^{(\alpha,\beta)}(x + y) = \sum_{k=0}^{n} (-1)^k L_{n-k}^{(k,\beta)}(x) \left( L_{k}^{(-\alpha-2,-\beta)}(y) - 2kL_{k-1}^{(-\alpha-3,-\beta)}(y) \right). \]

Proof. For \( A(t) = t - t^2 \), \( B(t) = \exp \left( (1 - t)^{\beta} - 1 \right) \) and \( H(t) = 1 \) we get

\[ \sum_{n \geq 0} B_{n+r,k+r}^{(r)}(a; b) \frac{t^n}{n!} = \frac{1}{k!} \sum_{n \geq k} L_{n-k}^{(k,\beta)}(r) t^n, \]

\[ \sum_{n \geq 0} P_{n}^{(-\alpha)}(1, y | A, B, 1) \frac{t^n}{n!} = \left( \frac{t}{t - t^2} \right)^{-\alpha} (1 - 2t) \exp \left( y \left( (1 - t)^{\beta} - 1 \right) \right) \]

\[ \sum_{n \geq 0} P_{n}^{(-\alpha)}(0, y | A, B, 1) \frac{t^n}{n!} = \sum_{n \geq 0} L_{n}^{(\alpha,\beta)}(y) t^n. \]

So, we get

\[ B_{n+r,k+r}^{(r)}(a; b) = \frac{n!}{k!} L_{n-k}^{(k,\beta)}(r), \]

\[ P_{n}^{(-\alpha)}(1, y | A, B, 1) = n! \left( L_{n}^{(\alpha,\beta)}(y) - 2nL_{n-1}^{(\alpha,\beta)}(y) \right), \]

\[ P_{n}^{(-\alpha)}(0, y | A, B, 1) = n! L_{n}^{(\alpha,\beta)}(y). \]

Then, from [[12]] we get

\[ L_{n}^{(\alpha,\beta)}(y) = \sum_{k=0}^{n} L_{n-k}^{(k,\beta)}(r) \left( L_{k}^{(k+1+\alpha,\beta)}(y - r) - 2kL_{k-1}^{(k+1+\alpha,\beta)}(y - r) \right) \]

and by the identity \( L_{n}^{(\alpha,\beta)}(x) = (-1)^n L_{n-1}^{(n-\alpha,-\beta)}(x) \) [[13]] we get

\[ L_{n}^{(\alpha,\beta)}(r + y) = \sum_{k=0}^{n} (-1)^k L_{n-k}^{(k,\beta)}(r) \left( L_{k}^{(-\alpha-2,-\beta)}(y) - 2kL_{k-1}^{(-\alpha-3,-\beta)}(y) \right). \]

Now, since the polynomial

\[ Q_{n}(x) = L_{n}^{(\alpha,\beta)}(x + y) - \sum_{k=0}^{n} (-1)^k L_{n-k}^{(k,\beta)}(x) \left( L_{k}^{(-\alpha-2,-\beta)}(y) - 2kL_{k-1}^{(-\alpha-3,-\beta)}(y) \right) \]

vanishes on each non-negative integer \( r \), the desired identity follows.

Recall that the high order Bernoulli polynomials of the first kind \( B_{n}^{(\alpha)}(x) \) are defined by

\[ \sum_{n \geq 0} B_{n}^{(\alpha)}(x) \frac{t^n}{n!} = \left( \frac{t}{\exp(t) - 1} \right)^{\alpha} \exp(xt) \]

with \( B_{n}^{(1)}(x) = B_{n}(x) \) is the \( n \)-th Bernoulli polynomials of the first kind, see [[9]] [[16]] [[18]] [[20]].
Corollary 7 There hold

\[ B_n^{(a)} (x) = \sum_{k=0}^{n} \frac{W_m(x) \binom{n+k}{n-k} B_k^{(k+1+a)}}{m^{n-k}} \left( x + 1 - \frac{r}{m} \right), \]

\[ B_n^{(n+1+a)} (x) = \sum_{k=0}^{n} \frac{w_m(x) \binom{n+k}{n-k} B_k^{(a)}}{m^{n-k}} \left( x - 1 + \frac{r}{m} \right) \]

and

\[ B_n^{(n+p+1)} \left( 1 - \frac{r}{m} \right) = \frac{1}{m^n} \binom{n+p}{p}^{-1} w_m(n+p,p), \]

\[ B_n^{(p)} \left( \frac{r-s}{m} \right) = \frac{1}{m^n} \sum_{k=0}^{n} \binom{k+p}{p}^{-1} w_m(n,k) w_{m,s}(k+p,p). \]

Proof. For \( A(t) = \frac{1}{m} \exp(mt) - 1 \), \( B(t) = \exp(t) \) and \( H(t) = 1 \) we get

\[ \Xi(t) = \frac{1}{m} \ln(1 + mt), \]

\[ (B(\Xi(t)))^{-1} = (1 + mt)^{-\frac{1}{m}}, \]

\[ B_{n+r,k+r}^{(a)}(a:b) = W_m(n,k), \]

\[ B_{n+r,k+r}^{(r)}(a\cdot\Xi)^{-1} = w_m(n,k), \]

\[ P_n^{(a)}(x, y \mid A, B, 1) = m^n B_n^{(a)}(x + \frac{y}{m}). \]

Then, from Theorem 5 we get

\[ B_n^{(a)} (x) = \sum_{k=0}^{n} \frac{W_m(x) \binom{n+k}{n-k} B_k^{(k+1+a)}}{m^{n-k}} \left( x + 1 - \frac{r}{m} \right), \]

\[ B_n^{(n+1+a)} (x) = \sum_{k=0}^{n} \frac{w_m(x) \binom{n+k}{n-k} B_k^{(a)}}{m^{n-k}} \left( x - 1 + \frac{r}{m} \right), \]

and, since \( \frac{d}{dx} B_n^{(a)}(x) = nB_{n-1}^{(a)}(x) \), if one differenciate the two sides of this last identity \( p \) times he obtain

\[ B_n^{(n+p+1+a)} (x) = \binom{n+p}{p}^{-1} \sum_{k=0}^{n} \binom{k+p}{p} w_m(n+p,k+p) \frac{m^{n-k} B_k^{(a)}}{m^{n-k}} \left( x - 1 + \frac{r}{m} \right). \]

Then, for \( \alpha = 0 \), there holds

\[ B_n^{(n+p+1)} (x) = \binom{n+p}{p}^{-1} \sum_{k=0}^{n} \binom{k+p}{p} w_m(n+p,k+p) \frac{m^{n-k} B_k^{(a)}}{m^{n-k}} \left( x - 1 + \frac{r}{m} \right) \]

which gives for \( x = 1 - \frac{r}{m} \):

\[ B_n^{(n+p+1)} \left( 1 - \frac{r}{m} \right) = \binom{n+p}{p}^{-1} w_m(n+p,p) \frac{m^n}{m^n}. \]

Upon using this identity, identity 6 becomes when \( \alpha = p \):

\[ B_n^{(p)} \left( \frac{r-s}{m} \right) = \frac{1}{m^n} \sum_{k=0}^{n} \binom{k+p}{p}^{-1} w_m(n,k) w_{m,s}(k+p,p). \]
The high order Bernoulli polynomials of the second kind \( b_n^{(\alpha)}(x) \) can be defined by
\[
\sum_{n \geq 0} b_n^{(\alpha)}(x) \frac{t^n}{n!} = \left( \frac{t}{\ln(1+t)} \right)^\alpha (1+t)^x
\]
with \( b_n^{(1)}(x) = b_n(x) \) is the \( n \)-th Bernoulli polynomial of the second kind, see \([6, 16, 18, 20]\).

**Corollary 8** There hold
\[
b_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{w_{m,r}(n,k) b_k^{(k+1+\alpha)}}{m^{n-k}} \left( x - 1 + \frac{r}{m} \right), \quad (8)
b_n^{(n+1+\alpha)}(x) = \sum_{k=0}^{n} \frac{W_{m,r}(n,k) b_k^{(\alpha)}}{m^{n-k}} \left( x + 1 - \frac{r}{m} \right), \quad (9)
\]
and
\[
b_n^{(n+p+1)}(1 + \frac{r}{m}) = \frac{1}{m^n} \binom{n+p}{p}^{-1} W_{m,r}(n+p,p),
b_n^{(p)} \left( s - \frac{r}{m} \right) = \frac{1}{m^n} \sum_{k=0}^{n} \binom{k+p}{p}^{-1} w_{m,r}(n,k) W_{m,s}(k+p,p).
\]

**Proof.** For \( A(t) = \frac{1}{m} \ln(1+mt) \), \( B(t) = (1+mt)^{-\frac{1}{m}} \) and \( H(t) = 1 \) we get
\[
\overline{A}(t) = \frac{1}{m} (e^{mt} - 1),
\]
\[
(B(\overline{A}(t)))^{-1} = e^t,
\]
\[
B_{n+r,j+r}^{(r)}(a; b) = w_{m,r}(n,k),
\]
\[
B_{n+r,k+r}^{(r)}\left( \overline{a}; \overline{b}^{-1} \right) = W_{m,r}(n,k),
\]
\[
P_n^{(\alpha)}(x, y \mid A, B, 1) = m^n b_n^{(\alpha)} \left( -x - \frac{y}{m} \right).
\]

Then, from Theorem 5 we get
\[
b_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{w_{m,r}(n,k) b_k^{(k+1+\alpha)}}{m^{n-k}} \left( x - 1 + \frac{r}{m} \right),
b_n^{(n+1+\alpha)}(x) = \sum_{k=0}^{n} \frac{W_{m,r}(n,k) b_k^{(\alpha)}}{m^{n-k}} \left( x + 1 - \frac{r}{m} \right),
\]
and, since \( \frac{\partial}{\partial x} b_n^{(\alpha+1)}(x) = n b_n^{(\alpha)}(x) \), if one differentiate the two sides of this last identity \( p \) times he obtain
\[
b_n^{(n+p+1+\alpha)}(x) = \binom{n+p}{p}^{-1} \sum_{k=0}^{n} \binom{k+p}{p} \frac{W_{m,r}(n+p,k+p)}{m^{n-k}} b_k^{(\alpha)} \left( x + 1 - \frac{r}{m} \right).
\]
Then, for \( \alpha = 0 \), there holds
\[
b_n^{(n+p+1)}(x) = \binom{n+p}{p}^{-1} \sum_{k=0}^{n} \binom{k+p}{p} \frac{W_{m,r}(n+p,k+p)}{m^{n-k}} \left( x + 1 - \frac{r}{m} \right).
\]
which gives for $x = -1 + \frac{r}{m}$:

$$b_n^{(n+p+1)} \left(-1 + \frac{r}{m}\right) = \left(\frac{n+p}{p}\right)^{-1} W_{m,r} \left(n+p,p\right).$$

Upon using this identity, identity (9) becomes when $\alpha = p$:

$$b_n^{(p)} \left(\frac{s-r}{m}\right) = \frac{1}{m^n} \sum_{k=0}^{n} \left(\frac{k+p}{p}\right)^{-1} w_{m,r} \left(n,k\right) W_{m,s} \left(k+p,p\right).$$

The above corollaries can be written in their general case as follows.

**Proposition 9** There holds

$$D_{t=0}^{n} \left[ \left(\frac{t}{A(t)}\right)^{p} (B(t))^{r-s} \right] = \frac{1}{p!} \sum_{k=0}^{n} \left(\frac{k+p}{p}\right)^{-1} B_{n+r,k+r}^{(p)}(a;b) B_{k+p+s,p+s}^{(s)}(\overline{a}:(b\circ\overline{a})^{-1}).$$

**Proof.** By definition and by Theorem (1) we have

$$D_{t=0}^{n} \left[ \left(\frac{t}{A(t)}\right)^{p} (B(t))^{r-s} \right] = P_{n}^{(p)} \left(0,r-s \mid A,B,1\right)$$

and since

$$P_{k}^{(k+p+1)} \left(1,-s \mid A,B,1\right) = D_{t=0}^{k} \left(\frac{t}{A(t)}\right)^{k+p+1} A'(t) (B(t))^{-s}$$

$$= D_{t=0}^{k} \left(\frac{A(t)}{t}\right)^{p} (B \circ \overline{A}(t))^{-s}$$

$$= \frac{1}{p!} \left(\frac{k+p}{p}\right)^{-1} B_{k+p+s,p+s}^{(s)}(\overline{a}:(b\circ\overline{a})^{-1}),$$

it follows $P_{n}^{(p)} \left(0,r-s \mid A,B,1\right) = \frac{1}{p!} \sum_{k=0}^{n} \left(\frac{k+p}{p}\right)^{-1} B_{n+r,k+r}^{(p)}(a;b) B_{k+p+s,p+s}^{(s)}(\overline{a}:(b\circ\overline{a})^{-1}).$ $\square$

Similarly, we also have:

**Proposition 10** There holds

$$D_{t=0}^{n} \left[ \left(\frac{t}{A(t)}\right)^{p} (B'(t))^{r-s} \right] = \frac{1}{p!} \sum_{k=0}^{n} \left(\frac{k+p}{p}\right)^{-1} B_{n+r,k+r}^{(p)}(a;b) B_{k+p+s,p+s}^{(s)}(\overline{a}).$$

**Proof.** By definition and by Theorem (1) we have

$$D_{t=0}^{n} \left[ \left(\frac{t}{A(t)}\right)^{p} (B'(t))^{r-s} \right] = P_{n}^{(p)} \left(-s,r \mid A,B,1\right)$$

$$= \sum_{k=0}^{n} B_{n+r,k+r}^{(p)}(a;b) P_{k}^{(k+p+1)} \left(1-s,0 \mid A,B,1\right),$$
and since
\[
P_k^{(k+p+1)}(1-s,0 \mid A,B,1) = D^{k}_{t=0} \left( \frac{t}{A(t)} \right)^{k+p+1} (A'(t))^{1-s}
\]
\[
= D^{k}_{t=0} \left( \frac{A(t)}{t} \right)^{p} (A' \circ A(t))^{-s}
\]
\[
= D^{k}_{t=0} \left( \frac{A(t)}{t} \right)^{p} (A'(t))^s
\]
\[
= \frac{1}{p!} \binom{k+p}{p}^{-1} B^{(s)}_{n+r,k+r} (a; b) B^{(s)}_{k+p+s,p+s} (\overline{a}),
\]

it follows \(P_n^{(p)} (-s, r \mid A, B, 1) = \frac{1}{p!} \sum_{k=0}^{n} \binom{k+p}{p}^{-1} B^{(r)}_{n+r,k+r} (a; b) B^{(s)}_{k+p+s,p+s} (\overline{a})\). \(\square\)

References

[1] E. T. Bell, Exponential polynomials. Ann. Math., 35 (1934) 258–277.
[2] H. Belbachir and I. E. Bousbaa, Translated Whitney and \(r\)-Whitney numbers: a combinatorial approach. J. Integer Seq. 16 (2013) Art. 13.8.6.
[3] W. S. Chou, L. C. Hsu and P. J. S. Shiue, Application of Faà di Bruno’s formula in characterization of inverse relations. J. Comput. Appl. Math., 190 (2006) 151–169.
[4] A. Chouria and J. -G. Luque, \(r\)-Bell polynomials in combinatorial Hopf algebras. C. R. Acad. Sci. Paris, Ser. I, 355 (2017), 243–247.
[5] H. W. Gould, Higher order extensions of Melzak’s formula. Util. Math., 72 (2007) 23–32.
[6] Y. L. Luke, The Special Functions and Their Approximations, vol. I. Academic Press, New York, London, 1969.
[7] M. M. Mangontarum and J. Katriel, On \(q\)-boson operators and \(q\)-analogues of the \(r\)-Whitney and \(r\)-Dowling numbers. J. Integer Seq., 18 (2015) Art. 15.9.8.
[8] M. Merca, A note on the \(r\)-Whitney numbers of Dowling lattices. J. Integer Seq., 18 (2015) Art. 15.9.8.
[9] I. Mező, A new formula for the Bernoulli polynomials. Results. Math. 58 (2010), 329–335.
[10] M. Mihoubi, Bell polynomials and binomial type sequences. Discrete Math., 308 (2008), 2450-2459.
[11] M. Mihoubi, Bell polynomials and inverse relations. J. Integer Seq., 13 (2010), Art. 10.4.5.
[12] M. Mihoubi and M. Rahmani, The partial \(r\)-Bell polynomials. Afr. Mat., 28 (Issue 7-8) (2017) 1167–1183.
[13] M. Mihoubi and M. Sahari, On some polynomials applied to the theory of hyperbolic differential equations. Submitted.
[14] M. Mihoubi and M. Saidi, An identity on pairs of Appell-type polynomials. C. R. Acad. Sci. Paris, Ser.I, 353 (2015) 773–778.
[15] M. Mihoubi and M. Tiachachat, Some applications of the $r$-Whitney numbers. *C. R. Acad. Sci. Paris, Ser.I*, 352 (2014) 965–969.

[16] T. R. Prabhakar and S. Gupta, Bernoulli polynomials of the second kind and general order. *Indian J. pure appl. Math.*, 11 (1980) 1361-1368.

[17] M. Rahmani, Some results on Whitney numbers of Dowling lattices. *Arab J. Math. Sci.*, 20 (1) (2014) 11–27.

[18] S. Roman, The Umbral Calculus, Academic Press, INC, 1984.

[19] M. Shattuck, Some combinatorial formulas for the partial $r$-Bell polynomials. *Notes Number Th. Discr. Math.*, 23 (1) (2017) 63–76.

[20] H. M. Srivastava, An explicit formula for the generalized Bernoulli polynomials. *J. Math. Anal. Appl.*, 130 (1988) 509–513.