FPRAS VIA MCMC WHERE IT MIXES TORPIDLY (and very little effort)

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ABSTRACT. Is Fully Polynomial-time Randomized Approximation Scheme (FPRAS) for a problem via an MCMC algorithm possible when it is known that rapid mixing provably fails? We introduce several weight-preserving maps for the eight-vertex model on planar and on bipartite graphs, respectively. Some are one-to-one, while others are holographic which map superpositions of exponentially many states from one setting to another, in a quantum-like many-to-many fashion. In fact we introduce a set of such mappings that forms a group in each case. Using some holographic maps and their compositions we obtain FPRAS for the eight-vertex model at parameter settings where it is known that rapid mixing provably fails due to an intrinsic barrier. This FPRAS is indeed the same MCMC algorithm, except its state space corresponds to superpositions of the given states, where rapid mixing holds. FPRAS is also given for torus graphs for parameter settings where natural Markov chains are known to mix torpidly. Our results show that the eight-vertex model is the first problem with the provable property that while NP-hard to approximate on general graphs (even \#P-hard for planar graphs in exact complexity), it possesses FPRAS on both bipartite graphs and planar graphs in substantial regions of its parameter space.
1. Introduction

Let \( G \) be any 4-regular graph. We label four incident edges of each vertex from 1 to 4. The eight-vertex model on \( G \) is defined as follows. The states consist of even orientations, i.e. all orientations having an even number of arrows into (and out of) each vertex. There are eight permitted types of local configurations around a vertex—hence the name eight-vertex model (see Figure 1).

Classically, the eight-vertex model is defined by statistical physicists on a square lattice region where each vertex of the lattice is connected by an edge to four nearest neighbors. In general, the eight configurations 1 to 8 in Figure 1 are associated with eight possible weights \( w_1, \ldots, w_8 \). Denote the set of these eight local configurations by \( \mathcal{S}_V \). By physical considerations, the total weight of a state remains unchanged if all arrows are flipped, assuming there is no external electric field. In this case we write \( w_1 = w_2 = a, w_5 = w_6 = b, w_7 = w_8 = d \). This complementary invariance is known as the arrow reversal symmetry or the zero field assumption.

Even in the zero-field setting, this model is already enormously expressive. The special case when \( d = 0 \) is the six-vertex model, which itself has sub-models such as the ice (\( a = b = c \)), KDP, and Rys F models; on the square lattice, some other important models such as the dimer and zero-field Ising models can be reduced to it [Bax72]. Together with ferromagnetic Ising and monomer-dimer models, the six-vertex and eight-vertex models are among the most studied models in statistical physics. Beyond physics, Kuperberg gave a simplified proof of the famous alternating-sign matrix (ASM) conjecture in combinatorics using a connection to the six-vertex model [Kup96]. Recently, the six-vertex model played an important role in explicating the phase transition of the Potts model and the random cluster model on the square lattice [DCGH+16, RS19]. After the eight-vertex model was introduced in 1970 by Sutherland [Sut70], and Fan and Wu [FW70], Baxter [Bax71, Bax72] achieved a good understanding of the zero-field case in the thermodynamic limit on the square lattice (in physics this understanding of the limiting case is called "exactly solved").

In this paper, we assume the arrow reversal symmetry and our algorithmic and complexity results further assume that \( a, b, c, d \geq 0 \) (as is the case in classical physics), unless otherwise explicitly stated. For any 4-regular graph \( G \) (not just the grid or even planar graph, however for plane graphs the edges are locally labeled from 1 to 4 cyclically), the partition function of the eight-vertex model on \( G \) with parameters \( (a, b, c, d) \) is defined as

\[
Z_{SV}(G; a, b, c, d) = \sum_{\tau \in \mathcal{O}_e(G)} a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6} d^{n_7+n_8},
\]

where \( \mathcal{O}_e(G) \) is the set of all even orientations of \( G \), and \( n_i \) is the number of vertices in type \( i \) in \( G \) (1 \( \leq i \leq 8 \), locally depicted as in Figure 1) under an even orientation \( \tau \in \mathcal{O}_e(G) \).

In terms of exact complexity, a dichotomy is given for the eight-vertex model on general 4-regular graphs for all eight (possibly complex) parameters [CF17]. This is studied in the context of a classification program for the complexity of counting problems [CC17], where the eight-vertex model serves as an important basic case for Holant problems defined by not necessarily symmetric constraint functions. It is shown that every

\*A search in Google Scholar for "six- and eight-vertex models" returns "About 153,000 results".
setting is either P-time computable (and some are surprising) or \#P-hard. However, most cases for (exact) P-time tractability are due to nontrivial cancellations. In our setting where \( a, b, c, d \) are nonnegative real numbers, the problem of computing the partition function of the eight-vertex model is \#P-hard unless: (1) \( a = b = c = d \) (this is equivalent to the unweighted case); (2) three of \( a, b, c, d \) are zero; or (3) two of \( a, b, c, d \) are zero and the other two are equal. The full classification of the exact complexity for the eight-vertex model on planar graphs is still open, but in the full version of this paper we will show that in our setting where \( a, b, c, d \) are nonnegative, the problem that is \#P-hard on general graphs remains \#P-hard on planar graphs except in the following cases where it becomes P-time computable: (1) \( a^2 + b^2 = c^2 + d^2 \) or (2) one of \( a, b \) is zero and one of \( c, d \) is zero.

Recently, the approximate complexity of counting and sampling of the eight-vertex model (and its special case, the six-vertex model) has been studied [GR10, Liu18, CLL19, FR19, CLLY20, CL20]. Interestingly, these results conform to the phase transition phenomenon in physics. In order to state the previous results and present our work, we adopt the following notations assuming \( a, b, c, d \geq 0 \).

- \( X = \{ (a, b, c, d) \mid a \leq b + c + d, \ b \leq a + c + d, \ c \leq b + c + d, \ d \leq a + b + c \} \);
- \( Y = \{ (a, b, c, d) \mid a + d \leq b + c, \ b + d \leq a + c, \ c + d \leq a + b \} \);
- \( Z = \{ (a, b, c, d) \mid a^2 \leq b^2 + c^2 + d^2, \ b^2 \leq a^2 + c^2 + d^2, \ c^2 \leq a^2 + b^2 + d^2, \ d^2 \leq a^2 + b^2 + c^2 \} \).

**Remark 1.1.** \( Y \subset X \) and \( Z \subset X \).

Physicists have shown an order-disorder phase transition for the eight-vertex model on the square lattice between parameter settings outside \( X \) and those inside (see Baxter’s book [Bax82] for more details). Physicists call \( X \) the disordered phase, and its complement \( \overline{X} \), which consists of 4 disjoint regions in which one of \( (a, b, c, d) \) dominates, the ordered phases. In [CLLY20] and [CL20], it was shown that: (1) approximating the partition function of the eight-vertex model on general 4-regular graphs outside \( X \) is NP-hard, (2) approximating the partition function of the eight-vertex model on general 4-regular graphs outside \( Y \) is at least as hard as approximately counting perfect matchings on general graphs (in short we say it is \#PM-hard), (3) there is an FPRAS\(^*\) for general 4-regular graphs in \( Y \cap Z \), and (4) there is an FPRAS for planar 4-regular graphs in a subregion of \( \overline{Y} \cap Z \). Note that all previous positive results are confined within \( Z \), in particular within the disordered phase \( \overline{X} \). See Figure 2.

Previous FPRAS results in [CLLY20] are based on the method of Markov chain Monte Carlo (MCMC). A nice upper bound on the mixing time of a specific Markov chain can be achieved only in \( Z \) (which is a subregion of the disordered phase \( \overline{X} \)) using a canonical path argument. The canonical path argument was introduced for perfect matchings [JS93] and extends well to problems which are believed to have totally different parametric settings. Interestingly, we

\[ \text{Suppose } f : \Sigma^* \rightarrow R \text{ is a function mapping problem instances to real numbers. A fully polynomial randomized approximation scheme (FPRAS)} \text{[KL83] for a problem is a randomized algorithm that takes as input an instance } x \text{ and } \epsilon > 0 \text{, running in time polynomial in the length } |x| \text{ and } \epsilon^{-1} \text{, and outputs a number } Y \text{ (a random variable) such that } \Pr[|1 - \epsilon| f(x) \leq Y \leq (1 + \epsilon) f(x)] \geq \frac{1}{4}. \]
show that these maps and their compositions among different parameter settings under which the partition function is preserved have group structures. For planar graphs, this group is isomorphic to the symmetry group $S_3$ on three elements (see Section 4); for bipartite graphs, this group is isomorphic to the dihedral group $D_6$ of order 12 (the symmetry group of a regular hexagon, see Section 5).

Therefore, although the Markov chain on a graph under certain parameter settings outside $\mathcal{Z}$ is not rapidly mixing, after "mixing up" the state space using a combination of two maps, the Markov chain turns out to be rapidly mixing. Indeed, as a consequence, this "indirect" MCMC leads to FPRAS for new regions in the disordered phase $\mathcal{X}$ and, for the first time, in the ordered phases $\mathcal{X}$ for planar graphs and for bipartite graphs.

Theorem 1.1. Let $G$ be a 4-regular plane graph. There is an FPRAS for $Z_8V(G; a, b, c, d)$ for $(a, b, c, d)$ in a subregion of $\mathcal{X} \cap \mathcal{Y} \cap \overline{\mathcal{Z}}$ and in a subregion of $\overline{\mathcal{X}}$.

We have proved that on general 4-regular graphs, approximating the eight-vertex model in $\mathcal{X} \cap \mathcal{Y}$ is $\#PM$-hard [CL20] and in $\overline{\mathcal{X}}$ is NP-hard [CLLY20]. Therefore, we have found a family of problems (with parameters ranging in a region of parameter space) having the following provable properties: For the eight-vertex model in the subregion of $\overline{\mathcal{X}}$ given in Theorem 1.1 (described more explicitly in Corollary 4.3), computing $Z_{8V}(a, b, c, d)$ is

1. NP-hard to approximate on general 4-regular graphs [CLLY20], and
2. has an FPRAS on planar 4-regular graphs (this paper).

This separation of complexity for general and for planar graphs should be compared and contrasted with the FKT algorithm [TF61, Kas61, Kas67] for exact counting of perfect matchings, but here for approximate counting. Previously the combined results of [GvV15] and [HVV15] proved a similar result for $k$-colorings for general versus planar graphs. We note that it was shown in [GJM15] that approximating the partition functions of many two-state spin systems remain NP-hard on planar graphs. A similar result was shown in [GJ12] for approximating the Tutte polynomial $T(G; x, y)$ in a large portion of the $(x, y)$ plane. We can also prove that for the subregion, $Z_{8V}(a, b, c, d)$ is $\#P$-complete in exact computation even on planar graphs (we will include the proof in the extended version of this paper).
Theorem 1.2. Let \( G \) be a 4-regular bipartite graph. There is an FPRAS for \( Z_{8V}(G; a, b, c, d) \) for \((a, b, c, d)\) in a subregion of \( \mathcal{X} \cap \mathcal{Y} \) and in a subregion of \( \mathcal{X} \).

Note that the subregions mentioned in Theorem 1.2 are disjoint from those mentioned in Theorem 1.1.

We have proved that on general 4-regular graphs, approximating the eight-vertex model in \( \mathcal{X} \cap \mathcal{Y} \) is \#PM-hard [CL20] and in \( \mathcal{X} \) is NP-hard [CLLY20]. Therefore, we have found a family of problems (with parameters ranging in a region of parameter space) having the following provable properties: For the eight-vertex model in the subregion of \( \mathcal{X} \) given in Theorem 1.2 (described more explicitly in Corollary 5.3), computing \( Z_{8V}(a, b, c, d) \) is

1. NP-hard to approximate on general 4-regular graphs [CLLY20], and
2. has an FPRAS on bipartite 4-regular graphs (this paper).

Previously the only problem having similar properties is the antiferromagnetic Ising model on general versus bipartite graphs [SS14, CGG+16]. We note that the problem of counting independent sets on bipartite graphs (\#BIS) is considered a canonical counting problem of intermediate approximation complexity [DGGJ04]. It is conjectured that \#BIS neither has an FPRAS nor is NP-hard to approximate. Many 2-spin systems on bipartite graphs are only known to be \#BIS-hard or \#BIS-equivalent (below NP-hard) to approximate [CGG+16].

To summarize, this paper establishes the eight-vertex model as the first problem with the provable property that while NP-hard to approximate on general graphs (even \#P-hard for planar graphs in exact complexity), it possesses FPRAS on both bipartite graphs and planar graphs in substantial regions of its parameter space.

A key property we use in our proof is that the dual of any planar 4-regular graph is bipartite [Wel69], and hence 2-colorable. Since torus graphs \((\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})\) with even \( m, n \) also have this property and are bipartite, the results in Theorem 1.1 and Theorem 1.2 also hold in torus graphs (with even side lengths).

Finally, we note that the techniques introduced in this paper have other applications. First, the maps between partition functions under different parameter settings established in this paper are not only useful for giving approximation algorithms. The same maps are useful in our understanding of the exact computational complexity of the eight-vertex model on planar graphs. Second, the techniques are useful for studying the partition functions of other edge-orientation problems (e.g. other vertex models in statistical physics) or edge-coloring problems (e.g. Holant problems) on planar/bipartite graphs.

2. The even-coloring model

We introduce the following edge-2-coloring model on 4-regular graphs called the even-coloring model: a valid configuration of this model assigns either green or red to every edge such that the number of green edges incident to any vertex is even (zero, two, or four). Similar to the eight-vertex model, there are also eight valid local configurations around a vertex (shown in Figure 3), and configurations 1 to 8 in Figure 3 can be associated with weights \( w_1', \ldots, w_8' \) respectively. Denote the set of these eight local configurations by \( S_{EC} \).

![Figure 3. Valid configurations of the even-coloring model.](image)

To set up a correspondence between the even-coloring model and the eight-vertex model satisfying arrow reversal symmetry, we consider the even-coloring model with weights that satisfy the color reversal...
symmetry. That is, the weight of a local configuration at a vertex remains unchanged if the color on every incident edge is changed. In this case we write \( w'_e = w'_r = w, w'_s = w'_t = x, w'_e = w'_r = y, \) and \( w'_s = w'_t = z. \) Given a 4-regular graph \( G, \) we label four incident edges of each vertex from 1 to 4. The \textit{partition function} of the even-coloring model with parameters \((w, x, y, z)\) on \( G \) is defined as
\[
Z_{EC}(G; w, x, y, z) = \sum_{\zeta \in C_{E}(G)} w^{n_1+n_2} x^{n_3+n_4} y^{n_5+n_6} z^{n_7+n_8},
\]
where \( C_{E}(G) \) is the set of all even colorings of \( G, \) and \( n_i \) is the number of vertices in type \( i \) in \( G \) \((1 \leq i \leq 8, \) locally depicted as in Figure 3) under the even-coloring \( \zeta \in C_{E}(G). \)

3. Holographic Transformation

Given a 4-regular graph \( G = (V, E), \) the \textit{edge-vertex incidence graph} \( G' = (U_E, U_V, E') \) is a bipartite graph where \((u_e, u_v) \in U_E \times U_V \) is an edge in \( E' \) iff \( e \in E \) in \( G \) is incident to \( v \in V. \) We model an orientation \((w \rightarrow v)\) on an edge \( e = \{w, v\} \in E \) from \( w \) into \( v \) in \( G \) by assigning 1 to \((u_e, u_w) \in E' \) and 0 to \((u_e, u_v) \in E' \) in \( G'. \) A configuration of the eight-vertex model on \( G \) is a \( 0 \)-\( 1 \) labeling on \( G', \) namely \( \sigma: E' \rightarrow \{0, 1\}, \) where for each \( u_e \in U_E \) its two incident edges are assigned 01 or 10, and for each \( u_v \in U_V \) the sum of values \( \sum_{i=1}^{4} \sigma(e_i) = 0 \) \((\mod 2), \) over the four incident edges of \( u_v. \) Thus we model the even orientation rule of \( G \) on all \( v \in V \) by requiring \( "two-0-two-1/four-0-four-1" \) locally at each vertex \( u_v \in U_V. \)

The \textquote{one-0-one-1} requirement on the two edges incident to a vertex in \( U_E \) is a binary \textit{disequality} constraint, denoted by \((\neq). \) The values of a 4-ary constraint function \( f \) can be listed in a matrix
\[
M(f) = \begin{bmatrix}
0_{00} & 0_{01} & 0_{02} & 0_{03} \\
0_{10} & 0_{11} & 0_{12} & 0_{13} \\
0_{20} & 0_{21} & 0_{22} & 0_{23} \\
0_{30} & 0_{31} & 0_{32} & 0_{33}
\end{bmatrix},
\]
called the \textit{constraint matrix} of \( f. \) For the eight-vertex model satisfying the even orientation rule and arrow reversal symmetry, the constraint function \( f \) at every vertex \( v \in U_V \) in \( G' \) has the form \( M(f) = \begin{bmatrix} d & b & c & 0 \\
a & b & c & 0 \\
a & b & c & 0 \\
a & b & c & 0\end{bmatrix}, \) if we locally index the left, down, right, and up edges incident to \( v \) by 1, 2, 3, and 4, respectively according to Figure 1. Thus computing the partition function \( Z_{SV}(G; a, b, c, d) \) is equivalent to evaluating
\[
\sum_{\sigma: E' \rightarrow \{0, 1\}} \prod_{u_e \in U_E} (\#_2 (\sigma | _E(u)) \prod_{u \in U_V} f (\sigma | _E(u)),
\]
where \( E'(u) \) denotes the incident edges of \( u \in U_E \cup U_V. \) In fact, in this way we express the partition function of the eight-vertex model as the Holant sum in the framework for Holant problems:
\[
Z_{SV}(G; a, b, c, d) = \text{Holant}(G' ; \#_2 \mid f)
\]
where we use \( \text{Holant}(H; g \mid f) \) to denote the Holant sum \( \sum_{\sigma: E \rightarrow \{0, 1\}} \prod_{u \in U} g (\sigma | _E(u)) \prod_{u \in V} f (\sigma | _E(u)) \) on a bipartite graph \( H = (U, V, E) \) for the Holant problem \( \text{Holant}(g \mid f). \) Each vertex in \( U \) (or \( V \)) is assigned the constraint function \( g \) (or \( f, \) respectively). The constraint function \( g \) is written as a row vector, whereas the constraint function \( f \) is written as a column vector, both as truth tables. (See [CC17] for more on Holant problems.)

The following proposition says that an invertible holographic transformation does not change the complexity of the Holant problem in the bipartite setting.

**Proposition 3.1 ([Val08]).** Suppose \( T \in C^2 \) is an invertible matrix. Let \( d_1 = \text{arity}(g) \) and \( d_2 = \text{arity}(f). \) Define \( g' = g \left( T^{-1} \right)^{d_1} \) and \( f' = T^{d_2} f. \) Then for any bipartite graph \( H, \) \( \text{Holant}(H; g \mid f) = \text{Holant}(H; g' \mid f'). \)

We denote \( \text{Holant}(G; f) = \text{Holant}(G' ; \#_2 \mid f). \) For the even-coloring model, if we view a green-red edge coloring by a 0-1 assignment to the edges such that an edge \( e \) is assigned 0 if it is colored green and assigned 1 if it is colored red, then the partition function of the even-coloring model \( Z_{EC}(G; w, x, y, z) \) is exactly the value of the Holant problem \( \text{Holant} \left( G; \begin{bmatrix} 0 & 0 & w & 0 & x & y & 0 & 0 & y \ 0 & x & y & 0 & w & 0 & 0 & 0 & z \end{bmatrix} \right) \).
The following two lemmas show that the eight-vertex model and the even-coloring model are connected via suitable holographic transformations in unexpected ways as Holant problems.

**Lemma 3.2.** Let $G$ be a 4-regular graph and let $M_Z = \frac{1}{2} \left[ \begin{array}{cccc} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{array} \right]$. Then $Z_{8V}(G; a, b, c, d) = Z_{EC}(G; w, x, y, z)$ where $\left[ \begin{array}{ccc} w \\ x \\ y \\ z \end{array} \right] = M_Z \left[ \begin{array}{c} a \\ b \\ c \\ d \end{array} \right]$.

**Proof.** Using the binary disequality function $(\#_2)$ for the orientation of any edge, we can express the partition function of the eight-vertex model $G$ as a Holant problem on its edge-vertex incidence graph $G'$.

$$Z_{8V}(G; a, b, c, d) = \text{Holant} \left( G'; \#_2 \mid f \right),$$

where $f$ is the 4-ary signature with $M(f) = \left[ \begin{array}{cccc} d & 0 & a & d \\ 0 & b & c & d \\ 0 & c & 0 & d \\ a & 0 & 0 & d \end{array} \right]$. Note that, writing the truth table of $(\#_2) = (0, 1, 1, 0)$ as a vector and multiplied by a tensor power of the matrix $Z^{-1}$, where $Z = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right]$ we get $(\#_2)(Z^{-1})^{\#_2} = (1, 0, 0, 1)$, which is exactly the truth table of the binary equality function $(\#_2)$. Then according to Proposition 3.1, by the $Z$-transformation, we get

$$\text{Holant} \left( G'; \#_2 \mid f \right) = \text{Holant} \left( G'; \#_2 \cdot (Z^{-1})^{\#_2} \mid Z^{\#_4} \cdot f \right)$$

$$= \text{Holant} \left( G'; =_2 \mid Z^{\#_4}f \right)$$

$$= \text{Holant} \left( G; Z^{\#_4}f \right),$$

and a direct calculation shows that $M(Z^{\#_4}f) = \frac{1}{2} \left[ \begin{array}{cccc} a+b+c+d & 0 & 0 & a+b+c+d \\ 0 & a-b-c-d & a-b-c-d & 0 \\ 0 & a-b+c-d & a-b+c-d & 0 \\ a+b-c+d & 0 & 0 & a+b-c+d \end{array} \right]$.

Readers are referred to Appendix A for a more insightful explanation on why the arity-4 constraint function $f$ is transformed to a real-valued constraint function, under the complex-valued $Z$-transformation.

**Lemma 3.3.** Let $G$ be a 4-regular graph and let $M_{HZ} = \frac{1}{2} \left[ \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{array} \right]$. Then $Z_{8V}(G; a, b, c, d) = Z_{EC}(G; w, x, y, z)$ where $\left[ \begin{array}{ccc} w \\ x \\ y \\ z \end{array} \right] = M_{HZ} \left[ \begin{array}{c} a \\ b \\ c \\ d \end{array} \right]$.

**Proof.** For the eight-vertex model as a Holant problem $\text{Holant} \left( G'; \#_2 \mid f \right)$, we perform a holographic transformation by the matrix $\frac{1}{2} \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$. We note that this is the composition of a $Z$-transformation and an $H$-transformation where $Z = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right]$ and $H = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right]$, namely $\frac{1}{2} \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] = HZ$. Then

$$Z_{8V}(G; a, b, c, d) = \text{Holant} \left( G'; \#_2 \mid f \right)$$

$$= \text{Holant} \left( G'; (\#_2) \cdot (HZ)^{-1})^{\#_2} \mid (HZ)^{\#_4} \cdot f \right)$$

$$= \text{Holant} \left( G'; =_2 \mid (HZ)^{\#_4}f \right)$$

$$= \text{Holant} \left( G; (HZ)^{\#_4}f \right).$$

Here $(\#_2) \cdot (HZ)^{-1})^{\#_2} = (\#_2) \cdot (Z^{-1})^{\#_2} \cdot (H^{-1})^{\#_2} = (=_2) \cdot (H^{-1})^{\#_2} = (=_2)$, because $H$ is orthogonal. Now a direct calculation shows that $M((HZ)^{\#_4}f) = \frac{1}{2} \left[ \begin{array}{cccc} a+b+c+d & 0 & 0 & -a+b+c+d \\ 0 & a-b-c-d & a-b-c-d & 0 \\ 0 & a-b+c-d & a-b+c-d & 0 \\ a+b-c+d & 0 & 0 & a+b-c+d \end{array} \right]$.

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4. Planar graphs

Lemma 4.1. Let $G$ be a 4-regular plane graph. Then $Z_{\delta V}(G; a, b, c, d) = Z_{EC}(G; b, a, d, c)$.

Proof. It is well known that a connected planar graph is Eulerian if and only if its dual is bipartite [Wel69]. Partition functions are multiplicative over connected components, so we may assume $G$ is connected. As $G$ is planar and 4-regular, the dual $G^*$ is bipartite. Hence, we can color the faces of $G$ using two colors, say black and white, so that any two adjacent faces (i.e., they share an edge) are of different colors. For definiteness, we assume that the outer face of $G$ is colored white. See Figure 4A for an example. Every edge separates one face colored white and another face colored black, so each edge is on a unique white face. This shows that the binary relation on the set of edges defined by being on the same white face is an equivalence relation.

![Figure 4](image)

**Figure 4.** A proper 2-coloring of the faces of a planar 4-regular graph and its canonical orientation $\tau$.

Based on the above facts, there is a canonical orientation $\tau$ which orients all the edges along any (non-outer) white face clockwise. This is also the same as to orient edges along every black face counterclockwise. See Figure 4B for a pictorial illustration. Observe that $\tau$ is an Eulerian orientation of $G$ (at every vertex the in-degree equals the out-degree) and in $\tau$ every vertex is in the 5th local configuration in Figure 6 (and equivalently the 6th local configuration in Figure 7).

![Figure 5](image)

**Figure 5.** An even orientation $\tau'$ and its corresponding even coloring.

For an arbitrary orientation $\tau'$ of $G$, we say an edge $e$ is green if $\tau'(e) = \tau(e)$, and $e$ is red otherwise. Then for any even orientation $\tau'$ of $G$, we can show that this assignment of green-red colors is an even coloring.
of $G$. (See an illustration in Figure 5: Figure 5A is an even orientation and Figure 5B is its corresponding even coloring.) Every even orientation $\tau'$ gives an even coloring because:

- Under this coloring, the canonical orientation $\tau$ receives the all-green coloring, which is an even coloring itself.
- For any even orientation $\tau'$, the local configuration of $\tau'$ at any vertex differs from the local configuration of $\tau$ at the same vertex on an even number of edges. These edges receive the red color and the others receive green. Therefore, at each vertex the color assignment will be in one of the states shown in Figure 3.

![Figure 6](image1)

**Figure 6**

![Figure 7](image2)

**Figure 7**

We claim that this color assignment on the edges of $G$ gives a bijection $M_{\text{PLANAR}}$ from $S_{8V}$ to $S_{EC}$. Given the black and white coloring of the faces of $G$, there are two types of vertices in $G$, either the one in Figure 6 or the other one in Figure 7. The correspondence of local configurations from Figure 6 to Figure 3 is $(1, 2, 3, 4, 5, 6, 7, 8) \rightarrow (3, 4, 1, 2, 7, 8, 5, 6)$; the correspondence of local configurations from Figure 7 to Figure 3 is $(1, 2, 3, 4, 5, 6, 7, 8) \rightarrow (4, 3, 2, 1, 8, 7, 6, 5)$. Consequently, $M_{\text{PLANAR}}$ induces a one-to-one correspondence between $\mathcal{O}_e(G)$ and $C_e(G)$.

Although there are two types of maps, both of them map $\{1, 2\}$ to $\{3, 4\}$, $\{3, 4\}$ to $\{1, 2\}$, $\{5, 6\}$ to $\{7, 8\}$, and $\{7, 8\}$ to $\{5, 6\}$. In terms of the weights of local configurations: the arrow reversal symmetry of the eight-vertex model induces an equivalence relation $\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}$ in Figure 1; the color reversal symmetry of the even-coloring model induces an equivalence relation $\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}$ in Figure 3. Thus there is a uniform and consistent way to assign a mapping of parameters. One can see that $M_{\text{PLANAR}}$ is weight-preserving if the even-coloring model has parameter setting $(w, x, y, z) = (b, a, d, c)$. It follows that

$$Z_{8V}(G; a, b, c, d) = Z_{EC}(G; b, a, d, c).$$

□

Now that we have set up multiple equations (Lemma 3.2, Lemma 3.3, and Lemma 4.1) between $Z_{8V}$ and $Z_{EC}$ under different mappings in terms of the parameter settings, we can combine them and obtain equations between $Z_{8V}$ under different parameter settings.

Before that, we make the following observation. Since in any even orientation of a 4-regular graph $G$, the number of sinks (Figure 1-7) must be equal to the number of sources (Figure 1-8) and thus their sum is
always even, we know that in the eight-vertex model under parameter setting \((a, b, c, d)\), the weight of a state is unchanged if we flip \(d\) to \(-d\). Therefore, we have

\[
Z_{SV}(G; a, b, c, d) = Z_{SV}(G; a, b, c, -d).
\]

(In particular, for non-negative \(a, b, c, d\), even though \(-d\) makes an appearance on the right-hand-side, this equation says the value \(Z_{SV}(G; a, b, c, -d) \geq 0\).) Obviously the approximation complexity for computing \(Z_{SV}(a, b, c, d)\) is the same as that for \(Z_{SV}(a, b, c, -d)\).

**Notation.** Given a set of 4-tuples \(S\), let \(N_d(S) = \{(a, b, c, d) \mid (a, b, c, d) \in S\}\).

**Notation.** Given two \(4 \times 4\) invertible matrices \(M_1\) and \(M_2\), denote by \(\langle M_1, M_2 \rangle\) the group of matrices generated by \(M_1\) and \(M_2\), where the group operation is matrix multiplication.

**Theorem 4.2.** Let \(G\) be a 4-regular plane graph and let \(M_{Pl Z}^P = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}\), \(M_{Pl Z}^H = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}\). Then for any \(M \in \langle M_{Pl Z}^P, M_{Pl Z}^H \rangle\), \(Z_{SV}(G; a, b, c, d) = Z_{SV}(G; d', b', c', d')\) where

\[
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = M \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix}.
\]

**Remark 4.1.** One can check that \(\langle M_{Pl Z}^P, M_{Pl Z}^H \rangle\) is isomorphic to the symmetry group \(S_4\) and the group elements are shown in Table 1.

**Proof.** First we prove the theorem for \(M_{Pl Z}^P\). From Lemma 3.2, we know that \(Z_{SV}(G; a, b, c, d) = Z_{EC}(G; w, x, y, z)\) where

\[
\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.
\]

From Lemma 4.1, we know that \(Z_{EC}(G; w, x, y, z) = Z_{SV}(G; d', b', c', d')\) where \(\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = M \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\).

The proof for \(M_{Pl Z}^H\) is similar. Instead of combining Lemma 3.2 and Lemma 4.1, we simply need to combine Lemma 3.3 and Lemma 4.1 and notice that \(M_{Pl Z}^H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\).

Since the theorem is proved for the two invertible matrices \(M_{Pl Z}^P\) and \(M_{Pl Z}^H\), it is also true for the group of matrices generated by these two matrices using their inverse and matrix multiplication.

**Notation.** In order to state the results in this section and the next section, we adopt the following notations assuming \(a, b, c, d \geq 0\).

- \(A = \{(a, b, c, d) \mid a \leq b + c + d\}\), \(B = \{(a, b, c, d) \mid b \leq a + c + d\}\), \(C = \{(a, b, c, d) \mid c \leq a + b + d\}\), \(D = \{(a, b, c, d) \mid d \leq a + b + c\}\);
- \(AD = \{(a, b, c, d) \mid a + d \leq b + c\}\), \(BD = \{(a, b, c, d) \mid b + d \leq a + c\}\), \(CD = \{(a, b, c, d) \mid c + d \leq a + b\}\).

**Remark 4.2.** \(AD \subseteq A \cap D, BD \subseteq B \cap D, CD \subseteq C \cap D, A = A \cap B \cap C \cap D, \ Y = AD \cap BD \cap CD\). In addition, we abuse the notation and use \(\overline{C} = \{(a, b, c, d) \mid c \geq a + b + d\}\), \(\overline{D} = \{(a, b, c, d) \mid d \geq a + b + c\}\), \(\overline{AD} = \{(a, b, c, d) \mid a + d \geq b + c\}\), \(\overline{BD} = \{(a, b, c, d) \mid b + d \geq a + c\}\), and \(\overline{CD} = \{(a, b, c, d) \mid c + d \geq a + b\}\) in Table 1 and Table 2.

**Corollary 4.3.** Let \(G\) be a 4-regular plane graph and let \(M_{Pl Z}^P\) and \(M_{Pl Z}^H\) be defined as in Theorem 4.2. Then for any \(M \in \langle M_{Pl Z}^P, M_{Pl Z}^H \rangle\), there is an FPRAS for \(Z_{SV}(G; a, b, c, d)\) if \(M \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathcal{Y} \cap \mathcal{Z}\).

Thus we know that for any \(M \in \langle M_{Pl Z}^P, M_{Pl Z}^H \rangle\), there is an FPRAS for \(Z_{SV}(G; a, b, c, d)\) for \((a, b, c, d)\) in a subregion of \(M^{-1}(\mathcal{Y})\). Note that \(AD \cap BD \cap \overline{CD} \cap C \subseteq \mathcal{X} \cap \overline{\mathcal{Y}}\) and \(\overline{C} \subseteq \mathcal{X}\). With the help of Table 1, one can see that Theorem 1.1 is implied by Corollary 4.3.

\[9\]

\(^\dagger\)In fact, to prove Theorem 1.1 we need \((M_{Pl Z}^H)^{-1} (\mathcal{Y} \cap \mathcal{Z}) = AD \cap BD \cap \overline{CD} \cap C \cap \overline{\mathcal{Z}}\) and one can check that this is true.
Table 1. Elements of $\langle M^Z, M^H \rangle$ and preimages of $\mathcal{Y} = AD \cap BD \cap CD$ under corresponding maps. A substantial subregion of each preimage admits FPRAS on planar 4-regular graphs. The last column lists the approximation complexity of the eight-vertex model on general 4-regular graphs in the corresponding preimage regions.

| Element | Matrix | Preimage | Approximation |
|---------|--------|----------|---------------|
| $I_4$   | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | $AD \cap BD \cap CD$ | FPRAS in $\mathcal{Z}$ |
| $M^Z$   | $\frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ | $N_d \left(AD \cap BD \cap \overline{CD} \cap C\right)$ | #PM-hard |
| $(M^Z)^2$ | $\frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ | $\overline{C}$ | NP-hard |
| $M^H$   | $\frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ | $AD \cap BD \cap \overline{CD} \cap C$ | #PM-hard |
| $M^H M^H$ | $\frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ | $N_d (\overline{C})$ | NP-hard |
| $(M^Z)^2 M^H$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | $N_d (AD \cap BD \cap CD)$ | FPRAS in $N_d (\mathcal{Z})$ |

5. Bipartite graphs

**Lemma 5.1.** Let $G$ be a bipartite graph. Let $G'$ denote its edge-vertex incidence graph. Suppose $f$ satisfies arrow reversal symmetry. Then Holant $(G'; \#_2 \mid f) = Holant (G'; \#_2 \mid f) = Holant (G; f)$. In particular, if $G$ is 4-regular, then $Z_{SV}(G; a, b, c, d) = Z_{EC}(G; a, b, c, d)$.

**Proof.** For any bipartite graph $G = (L, R, E)$, there is a *canonical orientation* $\tau$ which is to orient all the edges from $R$ to $L$. Let $G$ be a 4-regular bipartite graph. Observe that in $\tau$ every vertex in $L$ has local configuration Figure 1-7 and every vertex in $R$ has local configuration Figure 1-8.

For an arbitrary orientation $\tau'$ of $G$, we say an edge $e$ is *green* if $\tau'(e) = \tau(e)$, and $e$ is *red* otherwise. This coloring assignment on the edges of $G$ gives a bijection $M_{\text{BIPARTITE}}$ from $S_{SV}$ to $S_{\overline{EC}}$. For the vertices in $L$, this can be seen in the entry-wise correspondence (1, 2, 3, 4, 5, 6, 7, 8) $\rightarrow$ (1, 2, 3, 4, 5, 6, 7, 8) from Figure 1 to Figure 3; for the vertices in $R$, the correspondence of local configurations from Figure 1 to Figure 3 is (1, 2, 3, 4, 5, 6, 7, 8) $\rightarrow$ (2, 1, 4, 3, 6, 5, 8, 7). Consequently $M_{\text{BIPARTITE}}$ defines a one-to-one correspondence between $\mathcal{O}_d(G)$ and $C_{\overline{c}}(G)$. Again because both maps respect the same equivalence relation induced by the arrow reversal symmetry, this one-to-one correspondence $M_{\text{BIPARTITE}}$ is weight-preserving if the even-coloring model has the parameter setting $(w, x, y, z) = (a, b, c, d)$. It follows that

$$Z_{SV}(G; a, b, c, d) = Z_{EC}(G; a, b, c, d).$$

The idea of the above mapping can be easily extended to general (not necessarily 4-regular) graphs. For a bipartite graph $G = (L, R, E)$ and its edge-vertex incidence graph $G' = (V_E, L \cup R, E')$, every vertex $v_e \in V_E$ has degree 2 and connects a vertex $l \in L$ with a vertex $r \in R$. To see a one-to-one weight-preserving mapping from valid configurations in Holant $(G'; \#_2 \mid f)$ to valid configurations in Holant $(G'; \#_2 \mid f)$, one simply flips the assignment on every edge $\{v_e, r\}$ such that $v_e \in V_E$ and $r \in R$.

For any bipartite regular graph $G = (L, R, E)$, we know that $|L| = |R|$ and hence the total number of vertices is always even. In the eight-vertex model under parameter setting $(a, b, c, d)$, the weight of a state
is unchanged if we flip the sign of the weight on every vertex. Therefore, for bipartite 4-regular graphs, in addition to (4.1), we also have

\[ Z_{8V}(G; a, b, c, d) = Z_{8V}(G; -a, -b, -c, -d). \]

**Notation.** Given a set of 4-tuples \( S \), let \( N(S) = \{(-a, -b, -c, -d) \mid (a, b, c, d) \in S\} \).

**Theorem 5.2.** Let \( G \) be a 4-regular bipartite graph and let \( M_{Z}^{Bi} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \), \( M_{HZ}^{Bi} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \).

Then for any \( M \in \langle M_{Z}^{Bi}, M_{HZ}^{Bi} \rangle \), \( Z_{8V}(G; a, b, c, d) = Z_{8V}(G; a', b', c', d') \) where \( \begin{bmatrix} a' \\ b' \\ c' \\ d' \end{bmatrix} = M \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \).

**Remark 5.1.** One can check that \( \langle M_{Z}^{Bi}, M_{HZ}^{Bi} \rangle \) is isomorphic to the dihedral group \( D_6 \) and the group elements are shown in Table 2.

**Proof.** The proof is similar to that of Theorem 4.2. Instead of combining the holographic maps in Lemma 3.2, Lemma 3.3 with the planar map in Lemma 4.1, we need to combine them with the bipartite map in Lemma 5.1. \( \square \)

**Corollary 5.3.** Let \( G \) be a 4-regular bipartite graph and let \( M_{Z}^{Bi} \) and \( M_{HZ}^{Bi} \) be defined as in Theorem 5.2. Then for any \( M \in \langle M_{Z}^{Bi}, M_{HZ}^{Bi} \rangle \), there is an FPRAS for \( Z_{8V}(G; a, b, c, d) \) if \( M \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathcal{Y} \cap \mathcal{Z} \).

Thus we know that for any \( M \in \langle M_{Z}^{Bi}, M_{HZ}^{Bi} \rangle \), there is an FPRAS for \( Z_{8V}(G; a, b, c, d) \) for \( (a, b, c, d) \) in a subregion of \( M^{-1}(\mathcal{Y}) \). Note that \( \mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \cap \mathcal{D} \subset \mathcal{X} \cap \mathcal{Y} \cap \mathcal{Z} \) and \( \mathcal{B} \subset \mathcal{R} \). With the help of Table 2, one can see that Theorem 1.2 is implied by Corollary 5.3.

### 6. Concluding remarks

All the FPRAS results obtained in this paper come from the algorithm for \( \mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \cap \mathcal{Z} \). It is open if there exists an FPRAS/FPTAS for all \( (a, b, c, d) \in \mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \cap \mathcal{D} \). Assuming such an algorithm exists, our maps in Section 4 would imply that all \( \mathcal{A} \cap \mathcal{B} \) is approximable on planar graphs, and our maps in Section 5 would imply that all \( \mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \cap \mathcal{D} \) are approximable on bipartite graphs. We note that the approximation in \( \mathcal{A} \), \( \mathcal{B} \), and \( \mathcal{C} \) is proved to be NP-hard even on bipartite graphs [CLLY20].

In Section 4, the canonical orientation has the same weight \( c \) on every vertex and we are able to obtain algorithms for the eight-vertex model under parameter settings where \( c \) is relatively large, e.g. the region \( \mathcal{C} \); in Section 5, the canonical orientation has the same weight \( d \) on every vertex and we are able to obtain algorithms for parameter settings where \( d \) is relatively large, e.g. region \( \mathcal{D} \). In general, the paradigm proposed in this paper can be applied to the study of the eight-vertex model on other classes of graphs in additional to planar/bipartite/torus graphs. In particular, the methodology can be readily extended to any class of graphs with a “canonical” even orientation where every vertex has the same weight (one of \( a, b, c \), or \( d \)).
Table 2. Elements of $\langle M_{Z}^{B_{t}}, M_{HZ}^{B_{t}} \rangle$ and preimages of $\mathcal{Y} = AD \cap BD \cap CD$ under corresponding maps. A substantial subregion of each preimage admits FPRAS on bipartite 4-regular graphs. The last column lists the approximation complexity of the eight-vertex model on general 4-regular graphs in the corresponding preimage regions.

| Element | Matrix | Preimage | Approximation |
|---------|--------|----------|---------------|
| $I_4$   | \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\] | $AD \cap BD \cap CD$ | FPRAS in $\mathcal{Z}$ |
| $M_{Z}^{B_{t}}$ | \[
\frac{1}{2}
\begin{bmatrix}
-1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\] | $N_d (\overline{AD} \cap BD \cap \overline{CD} \cap D)$ | #PM-hard |
| $(M_{Z}^{B_{t}})^2$ | \[
\frac{1}{2}
\begin{bmatrix}
1 & 1 & -1 & 1 \\
-1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\] | $N (\overline{D})$ | NP-hard |
| $(M_{Z}^{B_{t}})^3$ | $-I_4$ | $N (AD \cap BD \cap CD)$ | FPRAS in $N (\overline{Z})$ |
| $(M_{Z}^{B_{t}})^4$ | $-M_{Z}^{B_{t}}$ | $N (N_d (\overline{AD} \cap BD \cap \overline{CD} \cap D))$ | #PM-hard |
| $(M_{Z}^{B_{t}})^5$ | $-(M_{Z}^{B_{t}})^2$ | $\overline{D}$ | NP-hard |
| $M_{HZ}^{B_{t}}$ | \[
\frac{1}{2}
\begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\] | $\overline{AD} \cap \overline{BD} \cap \overline{CD} \cap D$ | #PM-hard |
| $M_{Z}^{B_{t}} M_{HZ}^{B_{t}}$ | \[
\frac{1}{2}
\begin{bmatrix}
1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\] | $N (N_d (\overline{D}))$ | NP-hard |
| $(M_{Z}^{B_{t}})^2 M_{HZ}^{B_{t}}$ | \[
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\] | $N_d (AD \cap BD \cap CD)$ | FPRAS in $N_d (\overline{Z})$ |
| $(M_{HZ}^{B_{t}})^3$ | $-M_{HZ}^{B_{t}}$ | $N (\overline{AD} \cap \overline{BD} \cap \overline{CD} \cap D)$ | #PM-hard |
| $(M_{HZ}^{B_{t}})^4$ | $-M_{Z}^{B_{t}} M_{HZ}^{B_{t}}$ | $N_d (\overline{D})$ | NP-hard |
| $(M_{HZ}^{B_{t}})^5$ | $-(M_{Z}^{B_{t}})^2 M_{HZ}^{B_{t}}$ | $N_d (AD \cap BD \cap CD)$ | FPRAS in $N_d (\overline{Z})$ |
APPENDIX A.

The readers may have noticed that even though $Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is a complex-valued matrix, under the $Z$-transformation not only the binary EQUALITY function ($\equiv$) is transformed to the binary DISEQUALITY function ($\neq$), the arity 4 constraint function $f$ is also transformed to a real-valued constraint function $Z^{\otimes 4} f$.

This is not a coincidence, but a consequence of the fact that $f$ satisfies arrow reversal symmetry.

We say a real-valued constraint function $f$ satisfies arrow reversal symmetry if for all $(a_1, \ldots, a_n) \in \{0, 1\}^n$,

$$ f(a_1, \ldots, a_n) = f(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n}), $$

where $\overline{a_i} = 1 - a_i$ for all $i$.

**Lemma A.1.** A real-valued $f$ of arity $n$ satisfies arrow reversal symmetry, if and only if $Z^{\otimes n} f$ is real-valued.

**Proof.** Suppose $f$ satisfies arrow reversal symmetry. Denote by $\hat{f} = Z^{\otimes n} f$. We have $2^{n/2} \hat{f} = \left( \begin{bmatrix} 1 & 1 \end{bmatrix} \right)^{\otimes n} f$, and thus for all $(a_1, \ldots, a_n) \in \{0, 1\}^n$,

$$ 2^{n/2} \hat{f}_{a_1 \ldots a_n} = \sum_{(b_1, \ldots, b_n) \in \{0, 1\}^n} f_{b_1, \ldots, b_n} \prod_{1 \leq j \leq n} \left\{ (-1)^{a_j b_j} i^{a_j} \right\}. $$

Hence, taking complex conjugation,

$$ 2^{n/2} \overline{\hat{f}_{a_1 \ldots a_n}} = \sum_{(b_1, \ldots, b_n) \in \{0, 1\}^n} f_{b_1, \ldots, b_n} \prod_{1 \leq j \leq n} \left\{ (-1)^{a_j b_j} (-i)^{a_j} \right\} $$

$$ = \sum_{(c_1, \ldots, c_n) \in \{0, 1\}^n} f_{c_1, \ldots, c_n} \prod_{1 \leq j \leq n} \left\{ (-1)^{a_j (1-c_j)} (-i)^{a_j} \right\} $$

$$ = 2^{n/2} \hat{f}_{a_1 \ldots a_n}. $$

Now in the opposite direction, suppose $\hat{f}$ is real. We have $Z^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$, hence by the inverse transformation $2^{n/2} f = \left( \begin{bmatrix} 1 & -i \end{bmatrix} \right)^{\otimes n} \hat{f}$, and thus for all $(a_1, \ldots, a_n) \in \{0, 1\}^n$,

$$ 2^{n/2} f_{a_1 \ldots a_n} = \sum_{(b_1, \ldots, b_n) \in \{0, 1\}^n} \hat{f}_{b_1, \ldots, b_n} \prod_{1 \leq j \leq n} \left\{ (-1)^{a_j b_j} (-i)^{b_j} \right\}. $$

So

$$ 2^{n/2} f_{\overline{a_1} \ldots \overline{a_n}} = \sum_{(b_1, \ldots, b_n) \in \{0, 1\}^n} \hat{f}_{b_1, \ldots, b_n} \prod_{1 \leq j \leq n} \left\{ (-1)^{(1-a_j) b_j} (-i)^{b_j} \right\} $$

$$ = \sum_{(b_1, \ldots, b_n) \in \{0, 1\}^n} \hat{f}_{b_1, \ldots, b_n} \prod_{1 \leq j \leq n} \left\{ (-1)^{a_j b_j} i^{b_j} \right\} $$

$$ = 2^{n/2} f_{\overline{a_1} \ldots \overline{a_n}} $$

$$ = 2^{n/2} f_{a_1 \ldots a_n}. $$
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