Non Supersymmetric Vacua and the D-flatness Condition

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Abstract

Some $N = 1$ gauge theories, including SQED and $N_F = 1$ SQCD, have the property that, for arbitrary superpotentials, all stationary points of the potential $V = F + D$ are $D$-flat. For others, stationary points of $V$ are complex gauge transformations of $D$-flat configurations. As an implication, the technique to parametrize the moduli space of supersymmetric vacua in terms of a set of basic holomorphic $G$ invariants can be extended to non-supersymmetric vacua. A similar situation is found in non-gauge theories with a compact global symmetry group.

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I. INTRODUCTION

One interesting feature of supersymmetric gauge theories is the existence of multiple, physically inequivalent, $V = 0$ vacua. This brings the notion of “moduli space” $\mathcal{M}_{sv}$ of supersymmetric vacua (sv), the set of sv of a theory mod $G$ transformations, $G$ the gauge group of the theory. Classically, there is a well known construction of $\mathcal{M}_{sv}$. Let $\mathbb{C}^n = \{\bar{\phi}\}$ be the vector space of constant matter field configurations, $\hat{\phi}^i(\phi), i = 1, ..., s$ a basic set of holomorphic $G$ invariants, $\mathcal{D} \subseteq \mathbb{C}^s$ the algebraic subset of $\mathbb{C}^s$ defined by the polynomial constraints among the basic invariants. There is precisely one closed orbit of the complexification $G^c$ of the gauge group in each level set $\hat{\phi}^i(\phi) = \hat{\phi}^i_0$, and there is a unique $G$ orbit of $D$–flat points per closed $G^c$ orbit (no $D$–flat point can be found in non-closed $G^c$ orbits). Thus $\mathcal{D}$ is the moduli space of $D$–flat points, and $\mathcal{M}_{sv}$ is the subset of $\mathcal{D}$ selected by the condition $\partial W = 0$. For some theories the above picture changes drastically in the quantum regime, where all sv are lifted. For others, the quantum moduli space of sv is the same as $\mathcal{M}_{sv}$, or a deformation of $\mathcal{M}_{sv}$ in its ambient vector space $\mathbb{C}^s$. In the latter case, knowledge of $\mathcal{M}_{sv}$ plays a crucial role in the determination of the quantum moduli space of sv.

In this work we study non supersymmetric vacua (nsv) in the classical regime, as a first stage in the understanding of nsv in the quantum regime. A first look at the problem suggests that no much can be said about nsv, here defined to be $V \neq 0$ local minima of the scalar potential $V$. Firstly, there are strong limitations on a gauge or non-gauge supersymmetric theory to admit nsv. As an example, dimensionful constants are required in the superpotential $W(\phi)$ to allow terms with different powers of fields, otherwise $W(\phi)$ would be a homogeneous function on the chiral fields $\phi$, $W(x\phi) = x^d W(\phi)$, and every stationary point $\partial V = 0$ would be a sv, as $0 = \phi \partial V / \partial \phi = (d - 1) F (+2D)$. Secondly, for theories with nsv, there does not seem to be any reasonable way to parametrize its moduli space $\mathcal{M}_{nsv}$. Once the $D$–flatness condition is removed we may expect nsv in non-closed $G^c$ orbits. The basic holomorphic invariants do not separate $G^c$ orbits, they are only able to “distinguish” two different $G^c$ orbits if they are closed. We could tackle this problem by using the techniques developed in [6] to find the extrema of functions which are invariant under the action of a compact Lie group $G$. The $G$ orbits are the level sets of a complete (holomorphic and non-holomorphic) basic set of $G$ invariants $\psi^j(\phi, \phi^\dagger), j = 1, ..., k$. The $\psi^j$’s are subject to polynomial (in)equality constraints that define a semi-algebraic subset $\mathcal{O}$ of $\mathbb{R}^{2s} \simeq \mathbb{C}^s$. The extrema of $G$ invariant functions can be found by working directly in the orbit space $\mathcal{O}$. However, computations are cumbersome because a detailed knowledge of the $G$ strata in $\mathcal{O}$ is required. In this work we explore a simpler alternative which is based on the simple structure of the scalar potential $V = F + D$. Note that $F$ is the square norm of the $G^c$ “covector” (i.e., transforming as $\rho$ if $\phi$ is in the $\rho$ representation) $\partial W$, whereas $D$ is the square norm of the field
\[ \Phi^A = \phi^\dagger T_B \phi K^{BA} \in \text{Lie}(G), \quad K^{BA} \text{ the inverse Killing metric in Lie}(G). \]

For a large set of groups and representations this structure of \( V \) restricts all stationary points of \( V \) (not only sv) to closed \( G^c \) orbits. This fact not only simplifies the search of nsv, it also allows to construct the moduli space \( \mathcal{M}_v \) of all vacua, supersymmetric and non-supersymmetric, as a subset of \( \mathcal{D} \), i.e., \( \mathcal{M}_{sv} \subseteq \mathcal{M}_v \subseteq \mathcal{D} \).

In non gauge theories with a global symmetry group \( G \) the scalar potential equals the square norm \(|\partial W|^2\) of the \( G^c \) covector \( \partial W \). For a large set of groups and representations this implies that nsv are restricted to closed \( G^c \) orbits, i.e., they are \( G^c \) related to (formal) \( D \)–flat points \( \phi^\dagger T \phi = 0 \) for all \( T \in \text{Lie}(G) \). Thus, the \( D \)–flatness condition plays a rˆole in the search of nsv of theories with a global symmetry \( G \)!

\[ \mathcal{N} = 1 \text{ theories arise as the low energy effective actions of confining gauge theories, and they often break supersymmetry. A well known example is the chiral theory with one flavor of matter in the four dimensional representation of } SU(2) \]

The fact that nsv occur only in closed \( G^c \) orbits guarantees the exact “doubling” of Goldstone bosons \[.\] We have doubling when \( G^c_\phi \), the little group of \( G^c \) at the vacuum \( \phi \), is the same as \( G^c_\phi^c \), the complexification of the little group of \( G \) at \( \phi \) (in general, \( G^c_\phi^c \subseteq G^c_\phi \), see \[.\]). An equivalent condition is that \( T^\dagger \) be unbroken whenever \( T \in \text{Lie}(G^c) \) is unbroken \[1\]. This condition is satisfied if the orbit \( G^c_\phi \) is closed, i.e., if \( \phi \) is \( G^c \) related to a \( D \)–flat point. To show this we can assume that \( \phi \) is \( D \)–flat, as the \( G^c \) isotropy groups of two points in a \( G^c \) orbit are \( G^c \) conjugated. If \( \phi \) is \( D \)–flat

\[ |T^\dagger \phi|^2 = \phi^\dagger T^\dagger T \phi + \phi^\dagger [T,T^\dagger] \phi, \quad (1) \]

then \( T^\dagger \in \text{Lie}(G^c_\phi) \) if \( T \in \text{Lie}(G^c_\phi) \). We should remark that the condition that \( G^c_\phi \) be closed for nsv \( \phi \) is stronger than \( G^c_\phi^c = G^c_\phi \).

The organization of this paper is as follows: in Section \[II\] we introduce the notion of fibers, review the construction of \( \mathcal{M}_{sv} \), and state the Hilbert-Mumford criterion for non-closed \( G^c \) orbits; in Section \[III\] we study nsv of theories with a global symmetry. Section \[IV\] is devoted to gauge theories, and includes a subsection on abelian gauge groups, for which a more systematic treatment is possible. The main results are Theorem I in Section \[III\] and Theorems II and III in Section \[IV\].

II. PRELIMINARIES

Let \( G \) be a compact, connected group, \( \rho \) a unitary representation of \( G \) on \( \mathbb{C}^n \). We will consider simultaneously the cases where \( \mathbb{C}^n = \{ (\phi^1, \cdots, \phi^n) \} \) is the constant chiral field configuration space of a supersymmetric theory with global symmetry \( G \), or the constant matter chiral field configuration space of an \( \mathcal{N} = 1 \) gauge theory, \( G \)

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1\[\Phi^A \text{ transforms as an adjoint field under } G \text{ but this picture breaks after complexifying } G.\]

2\[To see the equivalence write } T = (T + T^\dagger)/2 + i(T - T^\dagger)/(2i).\]
Consider a "pure imaginary" GD between transformations. The relevance to supersymmetric gauge theories of the connection being the gauge group. Any GD in [11] was first pointed out in [2]. The supersymmetric vacua (sv) of a n hermitian Lie (or the moduli space of 
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being the gauge group. Any G invariant holomorphic polynomial p(φ) can be written in terms of a basic set of invariants ψi(φ), i = 1, ... , s as

\[ p(\phi) = \hat{p}(\phi^1, ..., \phi^s), \]  

(2)

where \( \hat{p} \) is a polynomial \( \mathbb{C}^s \rightarrow \mathbb{C} \) function [11]. In general, the basic invariants are constrained by polynomial equations \( C^\alpha(\phi) = 0 \), meaning that \( C^\alpha(\phi(\phi)) \equiv 0 \). The zero set \( \mathcal{D} = \{ \phi \in \mathbb{C}^s : C^\alpha(\phi) = 0 \} \subseteq \mathbb{C}^s \) plays an important rôle in the construction of the moduli space of supersymmetric vacua of the gauge theory with matter content φ and gauge group G. This construction is better understood if we introduce the notion of "fibers". Fibers are the level sets \( \hat{\phi}(\phi) = \hat{\phi}_0, i = 1, \ldots , s \) of the basic invariants, they are closed, disjoint sets. The configuration space \( \mathbb{C}^n = \{ \phi \} \) is partitioned into fibers, and the set of fibers is parametrized by \( \mathcal{D} \). Every fiber contains complete orbits of the complexification \( G^c \) of G, possibly infinitely many of them, only one of which is closed (in the topological sense) [3]. The only closed \( G^c \) orbit in a fiber \( f \) lies in the boundary of any other \( G^c \) orbit in \( f \), and can therefore be found by taking the intersection of the closures of the \( G^c \) orbits in \( f \). Let \( T_A \) be a basis of hermitian generators of G in the \( \rho \) representation. A G element admits the expansion \( g = \exp(iC^AT_A) \) with real \( C^A \)'s, whereas a \( G^c \) element admits a similar expansion with complex \( C^A \)'s. It follows that the \( G^c \) action on \( \mathbb{C}^n \) is non unitary. Consider a "pure imaginary" \( G^c \) one dimensional subgroup \( g(s) = \exp(sT) \), \( T \) a hermitian Lie (G) generator (note the absence of the i factor in the exponent) acting on an arbitrary \( \phi \in \mathbb{C}^n \), and define \( \phi(s) \equiv g(s)\phi \), then [3][11]

\[ \frac{d}{ds} (\phi^i(s)\phi(s)) = 2\phi^i(s)T\phi(s), \]  

(3)

\[ \frac{d^2}{ds^2} (\phi^i(s)\phi(s)) = 4(T\phi(s))^\dagger(T\phi(s)) \geq 0, \]  

(4)

equality holding only when \( T \) is a generator of the little group \( G_\phi \) of \( \phi \) (and so \( \phi(s) = \text{constant} \)). If \( T \notin \text{Lie}(G_\phi) \), \( \phi^i(s)\phi(s) \) is a convex (positive second derivative) function of \( s \). Convex \( \mathbb{R} \rightarrow \mathbb{R} \) functions \( f(s) \) are easily seen to satisfy the following three properties: (i) there is at most one stationary point of \( f \); (ii) if \( s_0 \) is a stationary point of \( f \), then it is a global minimum; (iii) if \( f' \geq 0 \) at some point, then \( \lim_{s \rightarrow \infty} f(s) = +\infty \).

From these properties, eqns. [3][11] and Cartan’s decomposition \( G^c = GTG \), \( T \) a pure imaginary maximal torus, follows that \( D- \)flat points \( \phi_D^\dagger T\phi_D = 0 \) are vectors of minimum length in a \( G^c \) orbit, and that there is at most one \( G \) orbit of such vectors in a given \( G^c \) orbit. It was found in [11] that closed \( G^c \) orbits contain a unique \( G \) orbit of \( D- \)flat points [11], that we will refer to as the “core” of the \( G^c \) orbit, whereas no \( D- \)flat point can be found in a non closed \( G^c \) orbit. These facts allow a gauge independent characterization of the \( D- \)flatness condition found in Wess-Zumino gauge: the supersymmetric vacua of a gauge theory with gauge group G lie on closed \( G^c \) orbits. They also allow to regard the set of fibers \( \mathcal{D} \) as the set of closed \( G^c \) orbits, or the moduli space of \( D- \)flat points, i.e., the set of \( D- \)flat configurations mod G transformations. The relevance to supersymmetric gauge theories of the connection between \( D- \)flat configurations, minimal length vectors and closed \( G^c \) orbits found in [11] was first pointed out in [2]. The supersymmetric vacua (sv) of an \( \mathcal{N} = 1 \) gauge

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theory satisfy two conditions: (F) the $F$–flatness condition $\partial W = 0$ and (D) the $D$–flatness condition $\hat{\phi}^i T \phi = 0 \forall T \in \text{Lie}(G)$. Condition (F) is $G^c$ invariant, every point in the orbit $G^c \phi_F$ of an $F$–flat point $\phi_F$ is $F$–flat, and, by continuity, every point in the closure $\overline{G^c \phi}$ is $F$–flat. Condition (D) imposes an additional restriction: the sv lie on the core of closed $G^c$ orbits. However, once an $F$–flat point $\phi_F$ is found, we know there is a $G$ orbit of sv in $\overline{G^c \phi_F}$, namely, the core of $D$–flat points in the only closed $G^c$ orbit in $\overline{G^c \phi_F}$. In other words, (F) selects the fibers $f$ where the sv live, (D) their location in $f$. As there is one closed $G^c$ orbit per fiber, which contains precisely one $G$ orbit of $D$–flat points, the moduli space of sv $\mathcal{M}_{sv}$ (sv mod $G$ transformations), is the same as the set of fibers containing $\partial W = 0$ $G^c$ orbits. $\mathcal{M}_{sv} \subseteq \mathcal{D}$ can be parametrized by adding to the constraint equations $C^\alpha(\hat{\phi}) = 0$ defining $\mathcal{D}$ the $G$ invariant holomorphic equations resulting from $\partial W = 0$ [3]. In the special case $W = 0$, $\mathcal{M}_{sv} = \mathcal{D}$, the moduli space of $D$–flat points.

In non-gauge theories with a global symmetry $G$, the sv satisfy only the $G^c$ invariant $F$–flatness condition. Generically, there are infinitely many $G$ orbits per $G^c$ orbit, and so there is no clear way to parametrize the moduli space of sv in non-gauge theories.

In the following sections we show that, for a large set of gauge theories, the $V \neq 0$ stationary points of the scalar potential $V = F + D$, $F = |\partial W|^2$, $D = \frac{k}{8} \sum_A (\hat{\phi}^i T_A \phi)^2$, lie all on closed $G^c$ orbits (not necessarily in their cores), there being at most one $G$ orbit of stationary points of $V$ in a closed $G^c$ orbit. This leads to a parametrization of the moduli space $\mathcal{M}_{nsv}$ of nsv as a subset of $\mathcal{D}$, the set of closed $G^c$ orbits. $\mathcal{M}_{nsv}$ is obtained by projecting onto $\mathcal{D}$ the stationary point condition $\partial V = 0$ and the condition that the boson mass matrix $\partial_i \partial_j V$ at the stationary point be positive semidefinite. This may result in non-holomorphic (in)equalities. The moduli space of vacua is then $\mathcal{M}_v = \mathcal{M}_{sv} \cup \mathcal{M}_{nsv} \subseteq \mathcal{D} \subseteq \mathbb{C}^s$. A similar situation is found in some non-gauge theories with a global symmetry $G$, their nsv are restricted to closed orbits of the complexification $G^c$ of the global symmetry group $G$, i.e., they are $G^c$ related to formal $D$–flat points. We make use of a theorem due to Mumford that says that, given a non-closed orbit $G^c \phi_0$, the closed $G^c$ orbit lying in the boundary of $G^c \phi_0$ can be reached by means of a one dimensional pure imaginary subgroup of $G^c$:

Theorem [Mumford] [4]: Assume $G^c \phi_0$ is not closed, then there is a hermitian generator $T$ of $G$ such that $\lim_{s \to \infty} \exp(-sT)\phi_0 = \phi_c$, and $G^c \phi_c$ is closed.

Remark: if $\phi_0 = \sum_\mu \phi_{0 \mu}$ is the weight decomposition of $\phi_0$ ($\phi_{0 \mu} \neq 0$), then $\mu(T) \geq 0$ $\forall \mu$ (and strictly positive for some $\mu$). This implies $|\phi_c| < |\phi|$, and also $\lim_{s \to \infty} |\exp(sT)\phi| = \infty$.

Example [4]: Consider $G = U(1)$ acting on $\mathbb{C}^2$, $\phi = (u, v)$, $u$ a charge 1 field and $v$ a charge $-1$ field. $\text{Lie}(G) = \text{span}(T), T = \text{diag}(1, -1)$. $G^c = GL(1, \mathbb{C})$ acting by $x \cdot (u, v) = (xu, x^{-1}v)$. The set of basic invariants contains a single field $z = uv$, then $\mathcal{D} = \mathbb{C}^1$. The fibers $uv = z_0 \neq 0$ contain a single (therefore closed) $G^c$ orbit, with a core of vectors of minimum length ($D$–flat points) satisfying $uv = z_0$, $|u| = |v|$. The fiber $z = 0$ contains the closed orbit $\mathcal{O}_1 = \{(0, 0)\}$ and the non-closed orbits $\mathcal{O}_2 = \{u, 0\}, u \neq 0$, $\mathcal{O}_3 = \{(0, v), v \neq 0\}$, which do not contain vectors of minimum length. Also $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2} \cap \overline{\mathcal{O}_3}$. For points in $\mathcal{O}_2 \setminus \mathcal{O}_3$, $e^{-sT} (e^{-s(-T)})$ is as in
Mumford’s theorem. If the $U(1)$ symmetry is local, and we add a superpotential $W(z)$ to this gauge theory, the sv condition $0 = \partial W = W'(z)(v, u)$ yields a single holomorphic $G$ invariant equation, namely $zW'(z) = 0$. This equations selects the fibers containing $\partial W = 0 \ G^c$ orbits. As there is a unique $G$ orbit of $D$—flat points per fiber, the moduli space of sv of this gauge theory is $\mathcal{M}_{sv} = \{ z \in \mathbb{C} \vert W'(z) = 0 \}$. If the $U(1)$ symmetry were global, every point in fibers $z_0$ satisfying $W'(z_0) = 0$ would be a sv. As every fiber contains infinitely many $G$ orbits, there is no clear way to parametrize $\mathcal{M}_{nsv}$.

**Example 1.2:** Consider a theory with a matrix $M$ of chiral fields and a superpotential invariant under $M \rightarrow gMg^{-1}, g \in SU(N)$. The configuration space is $\mathbb{C}^{N^2}$, $G = SU(N), \rho = \text{adj} + 1$, and $G^c = SL(N, \mathbb{C})$. The adjoint field is $A_\beta^\alpha = M_\beta^\alpha - \frac{1}{N} \delta_\beta^\alpha \text{Tr} M$, and the singlet is $u = \text{Tr} M$. The holomorphic invariants are $\hat{\phi}^1 = u$ and $\hat{\phi}^i = \text{Tr} A^i, i = 2, \cdots, N$, they are unconstrained and so $D = \mathbb{C}^N$. Jordan’s decomposition implies that in every $G^c$ orbit there is an element of the form $(u, A), A = S + N$, where $S$ is diagonal, $N$ strictly upper triangular, and $[S, N] = 0$, these are the semisimple and nilpotent parts of $A$. Note that $\hat{\phi}^1 = \text{Tr} S^i, i > 1$ then $(u, S + N)$ and $(u, S + N')$ belong to the same fiber. In [10], section 8.5, it is established that the $G^c$ orbit of $S + N$ is closed iff $N = 0$. As there is one closed $G^c$ orbit per fiber we conclude that if $S$ and $S'$ are semisimple and $\text{Tr} S^i = \text{Tr} S'^i, i = 2, \ldots, N$ then $S' = g S g^{-1}, g \in SL(N, \mathbb{C})$. As there is a finite number of $G^c$ orbits of nilpotent $A$’s ([10], section 8.5) every fiber $(u, \text{Tr} A^i) = (u_0, \hat{\phi}_0^i)$ contains the same (finite) number of $G^c$ orbits, a picture that differs substantially from that of Example II.1. Mumford’s curve “switches off” the nilpotent piece of the adjoint field. Take, e.g., $N = 3, \hat{\phi}_0 = (A_0, u), A_0 = S + N$,

$$A_0 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (5)$$

A choice of $T$ satisfying Mumford’s theorem is

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (6)$$

Note that $\lim_{s \rightarrow \infty} \exp(-sT)A_0 = S$ and the square length $\text{Tr} (A_0^i A_0) = 7 > \text{Tr} (S^i S) = 6$. Consider the gauge theory with superpotential $W = u A_\beta^\alpha A_\alpha^\beta + mu^2/2 + \gamma u \equiv u \hat{\phi} + mu^2/2 + \gamma u$ ($\hat{\phi} \equiv \hat{\phi}^2)$, $\partial W = (2u A, \hat{\phi} + mu + \gamma) = 0$ iff (i) $u = 0$ and $\hat{\phi} = -\gamma$ or (ii) $A = 0$ and $u = -\gamma/m$. The fibers containing sv (in the cores of their closed $G^c$ orbits) are (i) $\hat{\phi}^1 = 0, \hat{\phi}^2 = -\gamma$ and arbitrary $\hat{\phi}^j, j \geq 3,$ and (ii) $\hat{\phi}^1 = -\gamma/m, \hat{\phi}^j = 0, j \geq 2$, thus $\mathcal{M}_{sv} = \{ (\hat{\phi}^1, \cdots, \hat{\phi}^s) \in \mathbb{C}^s \vert \hat{\phi}^1 = 0, \hat{\phi}^2 = -\gamma \} \cup \{ (\hat{\phi}^1, \cdots, \hat{\phi}^s) \in \mathbb{C}^s \vert \hat{\phi}^1 = -\gamma/m, \hat{\phi}^j = 0, j \geq 2 \}$. Again, if the $SU(N)$ symmetry were global, $\mathcal{M}_{sv}$ constructed above would not be a parametrization of the moduli space of sv, as there are infinitely many $G$ orbits in each $\partial W = 0 \ G^c$ orbit of type (i).
III. NON-SUPERSYMMETRIC VACUA IN THEORIES WITH A GLOBAL SYMMETRY

If $W$ is a $G$ invariant superpotential its gradient $\partial W$ transforms as a $G^c$ “covector”

$$W(g\phi) = W(\phi), \quad \partial W(g\phi) = \partial W(\phi)g^{-1}. \tag{7}$$

It is useful to think of $\partial W(\cdot)$ as a map $\mathbb{C}^n \to \mathbb{C}^{n^*}$ commuting with the $G$ actions $\rho$ and $\pi$. The vector $\phi$ is assigned the covector $\partial W(\phi)$, $F = |\partial W|^2$ measures its square length. It follows from (7) that under this map the orbit $G^c\phi = \{g\phi | g \in G^c\} \subseteq \mathbb{C}^n$ gets mapped onto the orbit $G^c\partial W(\phi) \subseteq \mathbb{C}^{n^*}$; also $G_\phi \subseteq G_{\partial W(\phi)}$, $G_{\partial W(\phi)}$ being the little group of the $\mathbb{C}^{n^*}$ point $\partial W(\phi)$, $G_\phi$ the little group of $\phi$. We exploit the fact that eqs. (3, 4) and all the results of the previous section apply to any $G$ representation, in particular $\pi$, where $\partial W$ lives. Thus, if $F(\phi_0)$ is a local minimum of $F$, $\partial W(\phi_0)$ is a covector of minimum length in its $G^c$ orbit, then $G^c\partial W(\phi) \neq \phi_0$ must be closed, and $\partial W(\phi_0)$ satisfies the $*D$—flatness condition

$$(\partial W(\phi_0))(-T)(\partial W(\phi_0))^\dagger = 0, \quad \forall T \in \text{Lie}(G). \tag{8}$$

We prove now that, under certain assumptions, this implies that $G^c\phi_0$ itself is closed. To see this, define for any $\phi_0$ and hermitian $T$ the curve $\phi(s) \equiv e^{-sT}\phi_0$ and also $F(s) \equiv |\partial W(\phi(s))| |\partial W(\phi(s))|^\dagger = |(\partial W(\phi_0)) \exp(sT)|^2$. Applying (3, 4) to the $\pi$ representation (or just computing the second derivative of $F(s)$) we see that, whenever $T \notin \text{Lie}(G_{\partial W(\phi_0)})$, $F(s)$ is a convex $\mathbb{R} \to \mathbb{R}$ function. If $\partial F(\phi_0) = 0$, then $0 = F(0) = \partial W(\phi_0)(-T)(\partial W(\phi_0))^\dagger$, $F(0)$ is a global minimum of $F(s)$, and $\lim_{s \to \pm \infty} F = \infty$. As a consequence $G^c\phi_0$ must be closed. If it were not, we could choose $T$ as in Mumford’s theorem and get to a contradiction: $F(\phi_c) = \lim_{s \to \infty} F(s) = \infty$, where $\phi_c = \lim_{s \to \infty} \phi(s)$. We conclude that $G^c\phi_0$ being non-closed forbids $\phi_0$ from being a stationary point of $F$. The only exception is when, for any $T$ as in Mumford’s theorem, $T \in \text{Lie}(G_{\partial W(\phi_0)})$. If this is the case then $F$ is non-confining, that is $\lim_{s \to \infty} |\exp(sT)\phi_0| = \infty$ while $\lim_{s \to \pm \infty} F(\exp(sT)\phi_0) = F(\phi_0) < \infty$. For $\phi_0$ and $T$ as in Mumford’s theorem the weight decomposition $\partial W(\phi_0) = \sum_\lambda (\partial W(\phi_0))_\lambda$ is such that $\lambda(T) \leq 0 \forall \lambda$, then $F(\phi_c) < F(\phi_0)$ except in the non-confining case $\lambda(T) = 0 \forall \lambda$, where $F(\phi_c) = F(\phi_0)$.

These observations are gathered in the following theorem:

**Theorem I**: Assume $G^c\phi_0$ is non-closed and $\phi_c$ is as in Mumford’s theorem.

(a) $F(\phi_c) \leq F(\phi_0)$, a lower energy point can be found in the closed $G^c$ orbit in the

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3 Even if $W$ has singularities, it is not possible that $F$ be well defined at $\phi_0$ and singular at $\phi_c$. This is so because one can always write $W(\phi) = W(\hat{\phi}(\phi))$, then $\partial W = (\partial W/\partial \hat{\phi}^j)\partial \hat{\phi}^j$. Now $\partial W/\partial \hat{\phi}^j$ is constant on $G^c\phi_0$ and the $\partial \hat{\phi}^j$ are polynomials, so no singularity can develop along the bounded $\phi(s)$, $s \geq 0$ curve.

4 If this is the case, and we are only interested in the spectrum of vacuum energies, we can use the fact that $F(\phi_0) = F(\phi_c)$ and still restrict the search of vacua to closed $G^c$ orbits.
boundary of $G^c\phi_0$.

(b) If $G_{\phi_0} = G_{\partial W(\phi_0)}$ then: (i) $\phi_0$ cannot be a stationary point of $F$, (ii) $F(\phi_c) < F(\phi_0)$.

(c) Define $\hat{\mathbb{C}}^n = \{ \phi \in \mathbb{C}^n | G_{\phi} = G_{\partial W(\phi)} \}$. The moduli space $\mathcal{M}_{nsv}$ of non-supersymmetric vacua in $\hat{\mathbb{C}}^n$ is the subset of $\mathcal{D}$ obtained by projecting onto $\mathcal{D}$ the (in)equalities resulting from $\partial F = 0$ and $\partial_i \partial_j F$ positive semidefinite.

To prove (c), note from (b) and the above discussion that, in the sector $\hat{\mathbb{C}}^n = \{ \phi \in \mathbb{C}^n | G_{\phi} = G_{\partial W(\phi)} \}$ of the configuration space $\mathbb{C}^n$, the stationary points $\phi_s$ of $F$ lie all on closed $G^c$ orbits, satisfy the $*D$–flatness condition eq.(8) and are global minima of the restriction of $F$ to $G^c\phi_s$ (in particular, no local maximum of $F$ exists in $\hat{\mathbb{C}}^n$). Moreover there is at most one $G$ orbit of nsv per closed $G^c$ orbit. As the set of closed $G^c$ orbits is parametrized by $\mathcal{D}$, the moduli space of nsv in $\hat{\mathbb{C}}^n$ is the subset of $\mathcal{D}$ obtained by projecting onto $\mathcal{D}$ the (in general non-holomorphic) (in)equalities resulting from the conditions $\partial F = 0$ and $\partial_i \partial_j F$ positive semidefinite. Besides simplifying the search of nsv in $\hat{\mathbb{C}}^n$, theorem I shows a construction of $\mathcal{M}_{nsv}$ closely related to the parametrization of $\mathcal{M}_{sv}$ in gauge theories.

Example [III.1]: Consider the theory of example [I.1] with the $U(1)$ symmetry global. $\partial W = W'(z)(v,u)$, then $U(1)_{\partial W(u,v)} = U(1)_{(u,v)}$ except at nonzero sv, i.e., $\hat{\mathbb{C}}^2 = \mathbb{C}^2 \setminus \{(u,v) \neq (0,0) | W'(uv) = 0\}$. If such a vacuum exists, $F$ is non-confining, meaning that $F$ is constant along the $GL(1, \mathbb{C})$ orbit of the nontrivial sv, which extends to infinity. Theorem I guarantees that the nsv lie all on closed $GL(1, \mathbb{C})$ orbits, as they are all in $\hat{\mathbb{C}}^2$. In fact, $F = |W'(z)|^2(u\bar{u} + v\bar{v})$ and $\partial F = 0$ yield $0 = u\partial F/\partial u - v\partial F/\partial v = |W'(uv)|^2(u\bar{u} - v\bar{v})$. This means that every stationary point $(u,v)$ of $F$ in $\hat{\mathbb{C}}^2$ is $D$–flat, and so its $GL(1, \mathbb{C})$ orbit is closed, as predicted. To construct the moduli space of nsv we project $\partial F = 0$ and $\partial^2 F \geq 0$ onto $\mathcal{D}$. This is readily done if we replace $(u,v)$ in $\partial F = 0$ and $\partial^2 F \geq 0$ by the $D$–flat representative $u = v = \sqrt{z}$ in the $uv = z$ fiber. For details refer to example [IV.1], the result is $\mathcal{M}_{nsv} = \{ z \in \mathbb{C}^1 | W'(z) + 2zW''(z) = W'' + zW''' = 0 \}$.

Example [III.2]: Consider the theory of example [I.2], with a global $SU(N)$ symmetry. $\partial W = (2uA, \phi + mu + \gamma) = 0$ iff (i) $u = 0$ and $\phi = -\gamma$ or (ii) $A = 0$ and $u = -\gamma/m$. Condition (i) defines a fiber of sv containing non-closed $G^c$ orbits extending to infinity, i.e., $F$ is not confining and this explains the existence of stationary $F$ points in non-closed orbits. In the $u \neq 0$ sector $SU(N)_{(u,A)} = SU(N)_{\partial W(u,A)}$, therefore $\hat{\mathbb{C}}^N = \{(u,A) \in \mathbb{C}^N | u \neq 0\} \cup \{(0,0)\}$. All $\partial W \neq 0$ stationary points of $F$ lie in the $u \neq 0, A \neq 0$ sector of the configuration space, where Theorem I applies. In particular, these stationary configurations must lie on closed $G^c$ orbits. In fact, from $0 = \partial F/\partial A$ and $u \neq 0$ we obtain

$$A^\dagger = -A(\phi^* + mu^* + \gamma) = -Ae^{-i\alpha},$$

from where $[A, A^\dagger] = 0$, which implies $A$ is $SU(N)$ $D$–flat. Also $(\phi^* + mu^* + \gamma)/(2uu^\dagger) =$
\( e^{i\alpha} \), as this is an eigenvalue of the dagger operator. Adding \( \partial F/\partial u = 0 \) we get the equations selecting the fibers containing \( G \) orbits of stationary points of \( F \). There is only one such fiber: \( u = xe^{i\alpha}/m, \hat{\phi} = xe^{i\alpha}/2; e^{i\alpha} = \gamma/|\gamma| \) and \( x = 3mm/(8\sqrt{(3mm)^2 + |\gamma|mm})/2 < 0 \).

When proving (a) and (b) of Theorem 1 we showed that \( G^c\phi \) is closed if \( G^c\partial W(\phi) \) is closed and \( \phi \in \hat{C}^n \) (the reciprocal requires \( F \) to be confining in the sense described above). Yet, we should not expect the core of *\( D \)-flat points in \( G^c\partial W(\phi) \) to be the image under \( \partial W(\cdot) \) of the core of \( D \)-flat points in \( G^c\phi \), a non-generic feature exhibited by the two previous examples.

Example [III.3]: Consider an \( SO(N) \) theory with two vectors, \( \vec{\phi}_1 \) and \( \vec{\phi}_2 \), and a superpotential \( W = \vec{\phi}_1 \cdot (\vec{\phi}_1 + i\vec{\phi}_2) \). It can readily be checked that the isotropy groups \( SO(N)_{\vec{\phi}} \) and \( SO(N)_{\partial W(\vec{\phi})} \) agree for every \( \vec{\phi} = (\vec{\phi}_1, \vec{\phi}_2) \) in the configuration space \( \mathbb{C}^{2N} = \hat{\mathbb{C}}^{2\hat{N}} \). If \( G^c\partial W(\phi) \) is closed, then so is \( G^c\phi \). Moreover, \( G^c\phi \) is closed iff \( G^c\partial W(\phi) \) is closed, this superpotential also satisfies the confining condition. However, for \( D \)-flat \( \phi \), \( \partial W(\phi) \) is not *\( D \)-flat in general.

Example [III.4]: Theorem 3.9 in [2] states that a point \( \phi_0 \) is \( D \)-flat iff there is a holomorphic \( G \) invariant \( h(\phi) \) such that \( \phi_0 = \partial h(\phi_0) \). The special case where the set of basic invariants contains a single field \( \hat{\phi}(\phi) \) this theorem implies that any \( D \)-flat point satisfies the *\( D \)-flatness condition (5), as \( \partial W = W'(\phi)\partial \hat{\phi} \). Write \( \hat{\phi}(\phi) = C(i_{\alpha_1}...i_{\alpha_\delta})\phi^{j_1}...\phi^{j_\delta} \) and consider the \( \mathbb{C}^n \to \mathbb{C}^{n^*} \) map \( \phi^j \to \psi_i \equiv C(i_{\alpha_1}...i_{\alpha_\delta})\phi^{j_1}...\phi^{j_\delta} \). If \( \rho \) is real then \( d = 2, \ C^{\alpha_1\alpha_\delta}c_{k_j} = \delta^j_i, \partial_i W(\phi) = W'(\phi)c_{ij}\phi^j \), then \( \hat{\mathbb{C}}^n = \mathbb{C}^n \setminus \{ \phi \neq 0 | W'(\hat{\phi}(\phi)) = 0 \} \). Also \( F = |W'|^2\hat{\phi}^d \), and \( (\partial F)T\phi = |W'|^2\phi^d T\phi \). In the \( \hat{\mathbb{C}}^n \) sector stationary point are seen to lie in the core of closed \( G^c \) orbits. This generalizes the situation of example III.1.

IV. NON-SUPERSYMMETRIC VACUA IN GAUGE THEORIES

In many interesting examples, the \( D \) term \( \sum A(\phi^d T_A \phi)^2 \) along the orbit of a pure imaginary one dimensional subgroup \( \exp(-sT) \) of \( G^c \) is a convex function of \( s \), i.e., \( d^2 D (\exp(-sT)\phi_0)/ds^2 > 0 \forall s \in \mathbb{R} \). For \( \phi_0 \) and \( T \) as in Mumford’s theorem, this implies that \( \phi_0 \) cannot be a stationary point of the scalar potential \( V = F + D \), as \( V'' \geq D'' > 0 \). If it were, \( V \) would diverge at \( \phi_c = \lim_{s \to \infty} \phi(s) \). Assume there is a sector \( \hat{\mathbb{C}}^n \) of the configuration space where, for every point in non-closed \( G^c \) orbits there is a choice of \( T \) as in Mumford’s theorem for which \( d^2 D/|s|^2 > 0 \) for all \( s \). Stationary points of \( V \) in \( \hat{\mathbb{C}}^n \) are restricted to closed \( G^c \) orbits. If also \( d^2 D (\exp(-sT)\phi_0)/ds^2 > 0 \) for any \( \phi_0 \in \hat{\mathbb{C}}^n \) in closed \( G^c \) orbits and any \( T \in \{ \text{Lie} (G) \setminus \text{Lie} (G_c) \} \), we can show, as in sections [II] and [III] that there is at most one \( G \) orbit of stationary points of \( V \) per closed \( G^c \) orbit. The stationary point condition \( V'(0) = 0 \) reads

\[
\partial W(-T)(\partial W)^\dagger + \frac{g^2}{4} \phi^d T_A \phi^\dagger (TT_A + T_A T) \phi = 0.
\]
We gather the above observations in the following theorem: (in the aim of seeking simplicity we made some assumptions stronger than necessary).

**Theorem II:** Restrict to the sector \( \mathbb{C}^n = \{ \phi \in \mathbb{C}^n \mid d^2D(\exp(-sT)\phi)/ds^2 > 0 \text{ whenever } T \notin \text{Lie}(G_\phi) \} \) of the configuration space \( \mathbb{C}^n \), then:

(a) For any superpotential, every stationary point \( \phi_0 \) of \( V = F + D \) lies in a closed \( G^c \) orbit, (equivalently, it is \( G^c \) related to a \( D \)-flat configuration), satisfies the modified \( D \)-flatness (\( MD \)-flatness) condition eq. (10), and is a global minimum of the restriction of \( V \) to \( G^c\phi_0 \). In particular, there is no local maximum of \( V \).

(b) The moduli space of vacua \( \mathcal{M}_v \) is the subset of \( \mathcal{D} \subseteq \mathbb{C}^4 \) obtained by adding to the constraint equations among basic invariants the non-holomorphic (in)equalities resulting from the stationary point condition \( \partial V/\partial \phi = 0 \) and the condition that the boson mass matrix \( \partial_i \partial_j V \) at the stationary point be positive semidefinite.

The proof of (b) follows again from the fact that there is at most one \( G \) orbit of stationary points in a closed \( G^c \) orbit and that \( \mathcal{D} \) is the set of closed \( G^c \) orbits. For supersymmetric vacua the projection of \( \partial V = 0 \) onto \( \mathcal{D} \) reduces to the \( G \) holomorphic invariant equations obtained from \( \partial W = 0 \), and \( \partial^2 V \geq 0 \) does not add any restrictions. \( \mathcal{M}_v \) is the union of the moduli spaces of sv and nsv, \( \mathcal{M}_v = \mathcal{M}_{sv} \cup \mathcal{M}_{nsv} \subseteq \mathcal{D} \).

**Example IV.1:** Following the notation of examples [1] and [2], the \( D \) term of SQED is \( D = (u\bar{u} - v\bar{v})^2 = |\phi|^4 - 4|z|^2, \phi = (u, v). \) As \( z \) is \( G^c \) invariant, \( |z|^2 \) is a constant along any \( \phi(s) = \exp(-sT)\phi \) curve, whereas \( |\phi(s)|^4 \) is clearly a convex function (whenever \( T \notin \text{Lie}(U(1)(u,v)) \)), and so is \( D(s) \). Alternatively, we can apply eqs (3, 4) to the 1-dimensional charge 2, -2 and 0 \( U(1) \) representations \( u^2, v^2 \) and \( uv \) to show that \( D = |u|^2|v|^2 - 2|uv|^2 \) is the sum of two convex functions and a constant. In this example the configuration space \( \mathbb{C}^2 \) equals \( \mathbb{C}^2 \), and Theorem 2 holds everywhere. Given an arbitrary \( W(z), V = |W'|^2(|u|^2 + |v|^2) + \frac{g^2}{8}(|u|^2 - |v|^2)^2. \) The stationary point condition \( \partial V = 0 \) is always satisfied at the origin \( \phi = 0 \) and at no other point in the \( uv = 0 \) fiber. For nonzero \( uv \) it is equivalent to \( 0 = u\partial V/\partial u \pm v\partial V/\partial v \):

\[
0 = \left( |W'|^2 + \frac{g^2}{4}(|u|^2 + |v|^2) \right)(|u|^2 - |v|^2) \quad (11)
\]

\[
0 = \overline{W'}(W' + 2zW'')(|u|^2 + |v|^2) + \frac{g^2}{4}(|u|^2 - |v|^2)^2. \quad (12)
\]

Eq. (11) forces \( D = 0 \), showing that stationary points lie on closed \( G^c \) orbits, as predicted. Projecting (12) onto \( \mathcal{D} \) we obtain the equations characterizing the fibers containing critical points, namely \( 0 = zW'(z)(W'(z) + 2zW''(z)). \) To project \( \partial_i \partial_j V \geq 0 \) at stationary points onto \( \mathcal{D} \) we use the section \( \mathcal{D} \ni z \to (u = \sqrt{z}, v = \sqrt{z}) \in \mathbb{C}^2 \).

When replacing \( u = v = \sqrt{z} \) and \( W'(z) + 2zW''(z) = 0 \) in the equations requiring that the eigenvalues of \( \partial_i \partial_j V \) be \( \geq 0 \), the inequalities reduce to \( W'' + zW''' = 0 \). Thus \( \mathcal{M}_v = \{z \in \mathbb{C}^4 \mid |W'(z) = 0 \} \cup \{z \in \mathbb{C}^4 \mid |W'(z) + 2zW''(z) = W'' + zW''' = 0 \} = \mathcal{M}_{sv} \cup \mathcal{M}_{nsv}. \) The equations defining \( \mathcal{M}_{nsv} \) are independent of \( g \), this is also the moduli space of nsv of the non-gauge theory of example [1].
As a first step towards generalizing the ideas behind the previous example we re-write the $D$ term using the $G$ representation $\rho \otimes s \rho$. Let $\phi = \sum_r \phi_r$ be the decomposition of $\rho$ into irreps, then

$$ D = \sum_{r,s} (\phi_r^s T_A^r \phi_r) (\phi_r^s T_A^s \phi_s) = \sum_{r,s} (\phi_r \otimes \phi_s)^s (T_A^r \otimes T_A^s) (\phi_r \otimes \phi_s). $$

Using $T_A^{r \otimes s} = T_A^r \otimes I_s + I_r \otimes T_A^s$ we obtain

$$ T_A^r \otimes T_A^s = \frac{1}{2} [(T_A^{r \otimes s})^2 - (T_A^r)^2 \otimes I - I \otimes (T_A^s)^2]. $$

Combining eqs. (13,14) we arrive at

$$ D = \frac{1}{2} \sum_{r,s} \sum_{j \in \mathbb{Z}} (C_j - C_r - C_s) |\psi_j(\phi_r \otimes \phi_s)|^2, $$

where $\psi_j(\phi_r \otimes \phi_s)$ being the projector of $\phi_r \otimes \phi_s$ onto the irrep $j$ and $C_k$ the Casimir of the irrep $k$. The above equation reduces the $D$ term to a sum of square norms of irreps of the gauge group, eqs (13,14) hold for each one of the square norms $|\psi_j(\phi_r \otimes \phi_s)|^2$. If $\rho$ is free of gravitational anomalies then $0 = \text{Tr} (T_A^r \otimes T_A^s) = \sum_{j \in \mathbb{Z}} \dim(j) (C_j - C_r - C_s)$. This implies that some of the coefficients $(C_j - C_r - C_s)$ in (13) are negative.

In example IV.1 the only such term corresponds to a $G^c$ singlet and $D$ is readily seen to be convex along any $\exp(-sT)\phi$ curve.

**Example IV.2:** Consider $G = SO(N)$ with a single vector field, $\rho \otimes s \rho$ contains a symmetric tensor (for which $C - 2C^c$ is positive), and a $G^c$ singlet. In this example again, the only negative coefficient in eq. (15) accompanies a $G^c$ singlet, for any $\phi$ and $T D(\exp(-sT)\phi)$ is convex, nsv occur only in closed $G^c$ orbits, and Theorem II applies in $\mathbb{C}^N = \mathbb{C}^N$.

**Example IV.3:** In $N_F$ flavor, $N$ color SQCD (13) contains symmetric and adjoint tensors, for which $C > 2C_{fund}$, some $G^c$ singlets and antisymmetric tensors, for which $C < 2C_{fund}$. In the special case $N_F = 1$ there is no antisymmetric tensor, $D(s)$ is convex and Theorem II holds. For larger $N_F$ a more detailed analysis is required. Consider, e.g. the case $N_F = 2, N = 3$ and the configuration point $\phi_0 = (Q_1^g, Q_2^g)$ given by $Q_1^g = (x, y, 0), Q_2^g = (u, 0, 0), \hat{Q}_0^g = 0$. As $\phi_0 \neq 0$ and $\hat{\phi}(\phi_0) = 0$, $G^c \phi$ is non-closed. Eq. (13) yields $D \propto (N - 1)(|Q_1|^4 + |Q_2|^4 + |Q_1|^2 |Q_2|^2 + |Q_1^g Q_2^g|^2) - (N + 1)(|Q_1|^2 |Q_2|^2 - |Q_1^g Q_2^g|^2)$. The $SU(3)$ generator $T = \text{diag}(1, 1, -2)$ is as in Mumford’s theorem, and $D(e^{-sT} \phi_0)$ is convex, the exponentially decaying terms with negative coefficients in (13) get cancelled by positive coefficient terms with the same decaying rate. For other choices, like $T' = \text{diag}(1, 2, -3)$, the negative coefficient exponential terms persist but still $D(s)$ is convex. Note that among the normalized Lie ($G$) generators $\text{diag}(1, 1, -2)/\sqrt{6}$ is the one that steers $\phi_0$ to zero fastest.

As this example suggests, to determine the convexity of $D(\exp(-sT)\phi)$, eq. (13) should be supplemented with information on the weight decomposition $\phi = \sum_{\lambda} \phi_{\lambda}$. 

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As $G$ is compact, the $\lambda(T)$’s are rationally related, i.e., $\lambda(T) = nq$, $n$ a nonnegative integer, $q$ a “unit charge”. The problem of determining if $D(s)$ is convex reduces to a problem of existence of roots of the polynomial $p(x) \equiv D''(s)$, $x = \exp(-sq)$, in the range $0 \leq x \leq 1$. The convexity of $D$ along Mumford type curves would exclude points in non-closed $G^c$ orbits from the set of nsv. In this case ($G^c\phi$ non-closed and $\exp(-sT)\phi$ as in Mumford’s theorem) we know that the weight vectors $\lambda$ are all in the half space $\lambda(T) \geq 0$, as $\lim_{s \to \infty} \exp(-sT)\phi$ exists. For $G$ semisimple, no generic result has been obtained so far regarding the convexity of $D(s)$. The analysis is simplified in the abelian case $G = U(1)^k$, for which we have a fairly straightforward way to determine whether $D(s)$ is convex or not.

**A. $U(1)^k$ gauge groups**

From eq. (13), or more directly inserting $\phi = \sum_{\lambda} \phi_{\lambda}$ in $D(\phi) = \sum_{\lambda} (\phi^\dagger T_A \phi)^2$, $T_A$ an orthonormal basis of Lie ($G$), we obtain a simple expression for $D$ in the abelian case:

$$D = \sum_{\lambda \mu} |\phi_{\lambda}|^2 |\phi_{\mu}|^2 \lambda(T_A) \mu(T_A) = \sum_{\lambda \mu} <\lambda, \mu> |\phi_{\lambda}|^2 |\phi_{\mu}|^2,$$  

from where

$$D(\exp(-sT\phi)) = \sum_{\lambda \mu} <\lambda, \mu> |\phi_{\lambda}|^2 |\phi_{\mu}|^2 e^{-2s(\lambda(T) + \mu(T))}. \quad (17)$$

In the abelian case, we also have a simple criterion to determine whether $G^c\phi$ is closed or not: Construct the convex set

$$S_{\phi} = \left\{ \sum_{\phi_{\lambda} \neq 0} C_{\lambda} \lambda \ | \ 0 \leq C_{\lambda} \leq 1 \right\} \quad (18)$$

It can be shown that:

(a) 0 is outside $S_{\phi}$ iff $G^c\phi$ is a non-closed orbit and $\hat{\phi}(\phi) = 0$,

(b) 0 is a boundary point of $S_{\phi}$ iff $G^c\phi$ is a non-closed orbit and $\hat{\phi}(\phi) \neq 0$,

(c) 0 is an inner point of $S_{\phi}$ iff $G^c\phi$ is closed.

The proof follows trivially from propositions 5.3 and 6.15 in [10].

**Example IV.4:** In a $n$–dimensional $U(1)$ representation, the weights $\lambda$ of a point $\phi_0$ in a non-closed orbit lie all to the right of 0, all coefficients in (17) are non-negative, $D''(s) > 0$ and, for any superpotential, the stationary points of $V$ lie all on closed orbits. This generalizes example IV.1.

**Example IV.5:** Consider the $U(1) \times U(1)$ 4-dimensional representation with orthonormal generators

$$T_1 = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad T_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (19)$$

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The weight diagram is a square centered on 0 (figure 1). The weights are orthogonal, the matrix $<\lambda, \mu>$ in eqs. [13, 34] is diagonal, $D(\exp(-sT)\phi)$ is convex for any $\phi$ and $T$, and Theorem II holds in the entire configuration space. Vectors can be classified according to the number of nonzero weights. There are two classes of vectors in closed orbits: (i) 4 weight vectors and (ii) two opposite weight vectors. There are three different types of vectors in non-closed orbits: (iii) three weight vectors, which satisfy $\hat{\phi}(\phi) \neq 0$, and (iv) two adjacent weight vectors and (v) one weight vectors, for which $\phi(\phi) = 0$, i.e., they are in the same fiber as $\phi = 0$. Take, e.g., case (iii), Mumford’s curve $\phi(s)$ “shuts down” one weight leaving a case (ii) vector. The basic invariants are $\hat{\phi}^1 = \phi^1 \phi^3$ and $\hat{\phi}^2 = \phi^2 \phi^4$, they are unconstrained, then $D = C^2$. For any $W$, $M_v = M_{sv} \cup M_{nsv}$ will be a subset of $D = C^2$.

Example [IV] 6: Consider the $U(1) \times U(1)$ 6-dimensional representation

$$T_1 = \frac{\sqrt{3}}{6} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \quad T_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(20)

The weight diagram is a hexagon centered on 0 (figure 2). By excluding two adjacent weights we get a 4-weight vector in a non-closed orbit. Take e.g $\phi_0 = (\phi^1, \phi^2, \phi^3, \phi^4, 0, 0), \phi^i \neq 0, i = 1, ..., 4$, the boundary of $S_{\phi}$ appears in dotted lines in figure 2. It can readily be checked that: (1) there is a unique choice of $T$ satisfying Mumford’s theorem, (2) $e^{-sT}\phi_0$ turns off $\phi^2$ and $\phi^3$ and (3) $d^2D/ds^2$ may (i) change sign, (ii) be positive, (iii) be negative, and that $D(s)$ may even grow along this curve depending on the values of the $\phi^i$’s. Theorem II does not apply, we cannot draw any conclusions for this theory.

B. Energy bounds in core-to-core theories

There are many examples of theories for which $\partial W(\cdot)$ sends the core of $D$–flat points in closed $G^c$ orbits in $C^n$ onto the core of $^*D$–flat points of closed orbits in $C^{n*}$. For these theories, given any point $\phi_0$ in a non-closed $G^c$ orbit, the $D$–flat points in the closed orbit in the boundary of $G^c\phi_0$ have lower energy.

Theorem III: Assume $\partial W(\cdot)$ sends $D$–flat points onto $^*D$–flat points, i.e. $[\partial W(\phi)]T[\partial W(\phi)]^\dagger = 0$ $\forall T \in \text{Lie}(G)$ if $\phi^i T \phi = 0$ $\forall T \in \text{Lie}(G)$. If $G^c\phi_0$ is non-closed and $\phi_D$ is a $D$–flat point in the boundary of $G^c\phi_0$, then $V(\phi_D) < V(\phi_0)$.

Proof: Let $\phi_c$ be as in Mumford’s theorem, $\phi_D$ a $D$–flat point in the closed orbit $G^c\phi_c$. As $\partial W(\phi_D)$ is $^*D$–flat, $\phi_D$ is a global minimum of the restriction of $F$ to $G^c\phi_c$, then $F(\phi_D) \leq F(\phi_c)$. As $F$ decreases along Mumford’s curve $F(\phi_c) \leq F(\phi_0)$. Thus $F(\phi_D) \leq F(\phi_c) \leq F(\phi_0)$, and also $0 = D(\phi_D) < D(\phi_0)$, from where $V(\phi_D) < V(\phi_0)$. 

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Example IV.7: Theories having a single basic invariant satisfy the hypothesis of Theorem III (see example III.4). Table II lists all asymptotically free, anomaly free representations of simple groups having a single basic invariant, they were obtained from [12]. For all these theories $V(\phi_D)$, $G\phi_D$ the core of $D$–flat points in the boundary of the non-closed orbit $G^c\phi_0$, gives a lower bound to the energies $\{V(\phi)|\phi \in G^c\phi_0\}$.

Among these representations, the real ones have the property that, for any invariant $W$, $\partial W(\phi)(-T)(\partial W(\phi))^\dagger \propto \phi^\dagger T\phi$ (example III.4), this implies that $D$–flat points satisfy the $MD$–flat condition eq. (10). For a subset of the real $\rho$’s in Table II the tensor decomposition $\rho \otimes_s \rho$ contains only two irreps, one of which is a singlet. For them, theorem II holds in the entire configuration space, and, as happens for SQCD, the stationary points of $V$ are $D$–flat, a non generic feature among the theories satisfying the hypothesis of Theorem II.
TABLE I. All anomaly free representations of simple groups $G$ with a single basic holomorphic $G$ invariant. Entries 1-14 satisfy the hypothesis of Theorem III, entries 1,3,5,6 and 12 also satisfy the hypothesis of Theorem II. Pseudo-real representations are not checked in the fourth column, real representations are required in order that $(\partial W)^T(\partial W)^\dagger \propto \phi^T \phi$. In the last column Dynkin labels are used to avoid complicated Young diagrams.

|   | $G$ | $\rho$ | real | $\rho \otimes_s \rho$ |
|---|-----|-------|------|---------------------|
| 1 | $SU(N)$ | $\Box + \Box$ | ✓ | $\Box + \Box + Adj + \mathbb{I}$ |
| 2 | $SU(6)$ | $\Box$ | | $[0, 0, 2, 0, 0] + \text{adj}$ |
| 3 | $SU(4)$ | $\Box$ | ✓ | $[0, 2, 0] + \mathbb{I}$ |
| 4 | $SU(2)$ | $\Box + \Box$ | ✓ | $[2] + [6]$ |
| 5 | $SO(N)$ | $\Box$ | ✓ | $\Box + \mathbb{I}$ |
| 6 | $SO(7)$, spinor | | ✓ | $[0, 0, 2] + \mathbb{I}$ |
| 7 | $SO(9)$, spinor | | ✓ | $\Box + [0, 0, 0, 2] + \mathbb{I}$ |
| 8 | $SO(N), N = 11, 12, 14$, spinor | $[0, ..., 0, 2] + \left[ \begin{array}{c} \Box + \Box + \Box + \Box + \Box + \Box \end{array} \right]$ |
| 9 | $SO(10)$, 2 spinors | $[0, 0, 0, 0, 2] + [0, 0, 1, 0, 0] + \mathbb{I}$ |
| 10 | $Sp(2N)$, 4 spinors | $[0, 2, 0, 0, 0, 0] + [0, 0, 0, 0, 0] + \mathbb{I}$ |
| 11 | $Sp(6)$ | $\Box$ | | $[2, 0, 0] + [0, 0, 2]$ |
| 12 | $G_2$ | 7 | ✓ | $[2, 0] + \mathbb{I}$ |
| 13 | $E_6$ | 27 | $[2, 0, 0, 0, 0, 0] + [1, 0, 0, 0, 0, 0]$ |
| 14 | $E_7$ | 56 | $[2, 0, 0, 0, 0, 0, 0] + [1, 0, 0, 0, 0, 0, 0]$ |
There are many other examples of theories for which $\partial W(\cdot)$ sends $D$–flat points onto $*D$–flat points. Theorem III applies for all these theories.

**Example IV.8**: For $N_F < N$ ($N_F = N$) the basic SQCD holomorphic invariants are $M^i_j = Q^i_j Q^j_\alpha$ (and $B = \det Q, \tilde{B} = \det \tilde{Q}$). A straightforward calculation shows that the gradient of any flavor invariant superpotential $W(\det M)$ sends $D$–flat points onto $*D$–flat points.

**V. CONCLUSIONS**

We proved in Theorems I and II that for a large set of theories with a compact global symmetry $G$ and gauge theories with gauge group $G$, every non-supersymmetric vacuum is $D$–flat or $G^c$ related to a $D$–flat point. This not only simplifies the search of nsv but also leads to a parametrization of its moduli space $\mathcal{M}_{nsv}$ in terms of basic holomorphic invariants, extending the well known technique of constructing $\mathcal{M}_{sv}$. We also showed in Theorem I that in generic theories with a compact global symmetry $G$, if $G^c \phi_0$ is non-closed, a lower energy point exists in the closed $G^c$ orbit in the boundary of $G^c \phi_0$. This is also the case for a number of gauge theories, for which a $D$–flat point in the boundary of a non-closed orbit $G^c \phi_0$ always has lower energy than $\phi_0$ (Theorem III). To our knowledge, these are the first known results on moduli spaces of non-supersymmetric vacua. They uncover an unexpected connection between non-supersymmetric vacua and the $D$–flatness condition.

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Figure 1: weight diagram for the theory of Example IV.5.

Figure 2: weight diagram for the theory of Example IV.6