Sync and swarm: solvable model of non-identical swarmalators

S. Yoon,1 K. P. O’Keeffe,2 J. F. F. Mendes,1 and A. V. Goltsev1

1Departamento de Física da Universidade de Aveiro & I3N,
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal
2Sensible City Lab, Massachusetts Institute of Technology, Cambridge, MA 02139

We study a model of non-identical swarmalators, generalizations of phase oscillators that both sync in time and swarm in space. The model produces four collective states: asynchrony, sync clusters, vortex-like phase-waves, and a mixed state. These states occur in many real-world swarmalator systems such as biological microswimmers, chemical nanomotors, and groups of drones. A generalized Ott-Antonsen ansatz provides the first analytic description of these states and conditions for their existence. We show how this approach may be used in studies of active matter and related disciplines.

Synchronization is a universal phenomenon[1,2] seen in coupled lasers[3] and beating heart cells[4]. When in sync, the units of such systems align the rhythms of their oscillations, but do not move through space. Swarming, as in flocks of birds[5] or schools of fish[6], is a sister effect where the roles of space and time are swapped. The units coordinate their movements in space, but do not synchronize an internal oscillation.

The units of some systems coordinate themselves in both space and time concurrently. Japanese tree frogs sync their courting calls as they form packs to attract mates[8,9]. Starfish embryos sync their genetic cycles with their movements creating exotic ‘living crystals’[10]. Janus particles[11,12], Quincke rollers[13,14], and other driven colloids[17,18] lock their rotations as they self-assemble in space. The emergent ‘sync-selected’ structures have great applied power. They have been used to degrade pollutants[21,24], repair electrical circuits[25], and to shatter blood clots[26,27].

Theoretical studies of systems which mix sync with swarming are on the rise[28,32]. Tanaka et al. derived a universal model of chemotactic oscillators with diverse behavior[31,34]. Active matter researchers studied a Vicsek model with self-rotating (synchronizable) units[28,35,39] which imitate various types of colloid. O’Keeffe et al. introduced a model of ‘swarmalators’[29], whose states have been found in the lab and in nature[13,37,38], and is being further studied[39,47].

Analytic results on swarmalators are sparse. Order parameters, bifurcations, etc. are hard to compute given the systems’ nonlinearities and numerous degrees of freedom. Active matter such as the driven colloids mentioned earlier (which may be considered swarmalators) hard to analyze for the same reasons. The Vicsek model[18], for example, requires an in-depth use of statistical physics tools (dynamical renormalization groups etc) to be solved[49]. As for generalized Vicsek models, often only the stability of the simple incoherent state is analyzed, while order parameters are found purely numerically[28,32,35,50]. As such, easily and exactly solvable models of active matter are somewhat rare.

This Letter shows how this gap in active matter and swarmalator research may begin to be closed using technology from sync studies. We use Kuramoto’s classic self-consistency analysis[2] in hand with a generalized Ott-Antonsen ansatz[51] – two breakthrough tools – to study swarmalators which run on a 1D ring. This simple model captures the essential aspects of real-world swarmalators/active matter, yet is also solvable: Its order parameters and collective states may be characterized exactly. To our knowledge, exact results for the order parameters of an active matter collective are few; in this sense our work contributes to this vibrant field.

Model.— The model we study is[51]

\[
\dot{x}_i = v_i + \frac{J}{N} \sum_{j=1}^{N} \sin(x_j - x_i) \cos(\theta_j - \theta_i), \tag{1}
\]

\[
\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \cos(x_j - x_i), \tag{2}
\]

where \((x_i, \theta_i) \in (S^1, S^1)\) are the position and phase of the \(i\)-th swarmalator and \((v_i, \omega_i)\), \((J,K)\) are the associated natural frequencies and couplings. The \(v_i, \omega_i\) are drawn from a Lorentzian distribution, \(g_{L}(\omega)(x) = \Delta_{\omega}(x) / [\pi(x^2 + \Delta_{\omega}(x)^2)]\), with spreads \(\Delta_{\text{v}}, \Delta_{\text{w}}\) and mean set to zero via a change of frame.

The phase dynamics Eq. (2) are a generalized Kuramoto model where now depends on their pairwise distance \(K_{ij} = K \cos(x_j - x_i)\)[52]. So for \(K > 0\) neighbouring swarmalators synchronize more quickly than remote ones (the opposite occurs for \(K < 0\)). To treat sync and swarming on the same footing, the space dynamics Eq. (1) are identical to Eq. (2) but with \(x_i\) and \(\theta_i\) switched. Thus for \(J > 0\) synchronized swarmalator’s swarm (in the sense of aggregating) more readily than desynchronized ones (the opposite for \(J < 0\)). In short, the equations model location-dependent synchronization, and phase-dependent aggregation. One can also think of them as sync on the unit torus (Fig. 1) or as the rotational piece of the 2D swarmalator model[53].

Introducing the variables

\[
\zeta_i \equiv x_i + \theta_i, \quad \eta_i \equiv x_i - \theta_i, \tag{3}
\]
The internal symmetry results leaves Eqs. (1), (2) unchanged which means a locked swarmalator can be assigned to either cluster without changing the overall dynamic. The internal symmetry results in the formation of mirrored groups of synchronized swarmalators, see [53]. Movies of the evolution of these states and demonstrations that they are robust to local coupling (i.e. cutoff beyond a range σ) are provided in [53].

**Generalized OA ansatz.**— Now we analyze our model by deriving expressions for the order parameters $W_\pm$ in each state. Consider the probability $f(v, \omega, \zeta, \eta, t)$ to find a swarmalator with natural velocity $v$, a natural frequency $\omega$, and coordinates $\zeta$ and $\eta$ at time $t$

$$f = \frac{1}{N} \sum_{i=1}^{N} \delta(v-v_i)\delta(\omega-\omega_i)\delta(\zeta-\zeta_i)\delta(\eta-\eta_i). \tag{7}$$

Differentiating the left and right hand sides of Eq. (7) over $t$ gives the continuity equation,

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \zeta} \left\{ [v+\omega - J_+ S_+ \sin(\zeta-\Phi_+)-J_- S_- \sin(\eta-\Phi_-)]f \right\}
+ \frac{\partial}{\partial \eta} \left\{ [v-\omega - J_- S_+ \sin(\zeta-\Phi_+)-J_+ S_- \sin(\eta-\Phi_-)]f \right\} = 0. \tag{8}$$
Ott and Antonsen showed that for the Kuramoto model,

\[ W_+ = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} d\omega g_v(v)g_\omega(\omega)\alpha^*(v, \omega, t), \]
\[ W_- = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} d\omega g_v(v)g_\omega(\omega)\beta^*(v, \omega, t). \]

Equations (10)-(13) comprise a set of self-consistent equations for \( W_\pm \) in the \( N \to \infty \) limit.

**Analysis of async**— Here swarmalators are uniformly distributed in \( x \) and \( \theta \) which corresponds to the trivial fixed point \( W_\pm = 0 \). Equations (10)-(11) give \( \alpha = \exp[i(v + \omega)t], \beta = \exp[i(v - \omega)t] \). Linearizing around \( f = (4\pi)^{-2} \) reveals the state loses stability at

\[ J_{+e} = 2(\Delta_v + \Delta_\omega). \]

Fig. 2(a) plots this condition in the \((J, K)\) plane.

**Analysis of phase waves**— We analyze the \((S, 0)\) phase wave state. We look for a solution of Eqs. (10)-(13) that at large time \( t \) satisfies: \( \alpha = 0, \beta \neq 0, W_+ \neq 0 \) and \( W_- = 0 \). We find

\[ \alpha(v, \omega) = H\left(\frac{v + \omega}{S_+J_+}\right), \]
\[ \beta(v, \omega, t) = \exp\left[-i\frac{JK}{J_+} \left(\frac{v}{J} - \frac{\omega}{K}\right)t\right], \]

where we introduced a function,

\[ H(x) \equiv -ix + \sqrt{1 - x^2}. \]

Eq. (16) gives \( W_- = 0 \) as desired. Eq. (15) implies

\[ S_+ = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} d\omega g_v(v)g_\omega(\omega)H^*(\frac{v + \omega}{S_+J_+}), \]

where we assume \( \Phi_+ = 0 \) without loss of generality due to the rotational symmetry. To compute this integral, first observe that if \( v \) and \( \omega \) are drawn from the Lorentzian distribution, their sum \( v + \omega \) is drawn from a Lorentzian with spread \( \Delta_v + \Delta_\omega \). Then integrate over \( v + \omega \) using the residue theorem. There is a residue \( i(\Delta_v + \Delta_\omega) \) in the upper half complex plane where \( H^*(x) \) is analytic so \( S_+ = H^*[i(\Delta_v + \Delta_\omega)/S_+J_+] \). Thus,

\[ S_+ = \left[1 - \frac{2(\Delta_v + \Delta_\omega)}{J_+}\right]^{1/2}. \]

We see \( S_+ \) bifurcates from 0 at

\[ J_{+c} = \frac{1}{2}(J + K) = 2(\Delta_v + \Delta_\omega) \]

consistent with Eq. (14) as the system transitions from the async to the phase wave state (Fig. 3(a)), see the stability analysis in [53]. The phase wave \((0, S)\) is a solution of Eqs. (10)-(13) that at large time \( t \) satisfies: \( \alpha \neq 0, \beta = 0, W_+ = 0, W_- \neq 0 \).
Mixed state—Here \((S_1, S_2)\), where \(S_1 \neq S_2\). This state is intermediate between the phase wave and sync states, see Fig. 2(a) and compare Fig. 2(d) and (e). The state with either \(S_1 > S_2\) or \(S_1 < S_2\) bifurcates from \((S,0)\) or \((0, S)\), respectively. The corresponding order parameters and phase boundaries in \((J, K)\) plane are shown in Figs. 2(a) and 2(c) and discussed in \[53\]. The special property of the mixed state is that although \(S_1\) and \(S_2\) are time independent, both the functions \(\alpha\) and \(\beta\) are time dependent in contrast to time independent equations \[15\] and \[21\] (see below) for the phase wave and the sync states. Analytical properties of \(\alpha\) and \(\beta\) near the boundary with the phase wave are discussed in the Sec. IV, see \[53\].

Analysis of sync—Here \((S_+, S_-) = (S, S)\) so we seek fixed points of Eqs. (10)-(11) with \(W_\pm \neq 0\). We find

\[
\alpha(v, \omega) = H \left[ \frac{v}{JS_+} + \frac{\omega}{KS_+} \right], \quad \beta(v, \omega) = H \left[ \frac{v}{JS_-} - \frac{\omega}{KS_-} \right].
\]

We solve the integrals for \(W_\pm\) using the residue theorem. This time the natural frequencies combine as \(v / J \pm \omega / K\) which are Lorentzian distributed with spread \(\Delta_x = \Delta_c / J + \Delta_\omega / K\). Equations (12) and (13) reduce to

\[
S_\pm = H^* (i \Delta / S_\pm)
\]

which bifurcates from 0 at

\[
2\Delta = 2 \left( \frac{\Delta_c}{J} + \frac{\Delta_\omega}{K} \right) = 1.
\]

Figure 2(a) shows this critical curve in the \((J, K)\) plane. Notice it intersects with the critical curve of the phase wave at a point \(J = K = 2(\Delta_c + \Delta_\omega)\). This means the sync state may bifurcate from the asyn state directly, without passing through the phase (Fig. 2(b)), which occurs when \(J = K\). In this special case, Eqs. \(4\) and \(5\) for \(\dot{\zeta}, \dot{\eta}\) decouple and \(W_\pm(t)\) may be solved for all \(t\) (see \[53\]). In the generic case \(J \neq K\), however, the sync state bifurcates from the intermediate mixed state (Fig. 2(c)). As is evident from Fig. 3(c), the point \(J = K = 2(\Delta_c + \Delta_\omega)\) is a tetracritical point, at which four phases (async, sync, wave, and mixed) meet. The appearance of the sync state can be considered as the separation of dense clusters of locked swarmalators with time-independent coordinates and dilute drifting swarmalators in \((x, \theta)\) space. This phenomenon is qualitatively similar to motility induced phase separation observed in self-propelled particles and various microorganisms, see for example \[53\].

To back up these numerical tests of our results we performed four additional analyses. First, we re-derive \(S_\pm\) using a microscopic, swarmalator-level, approach (opposed to the macroscopic, density-level approach the OA ansatz is based on). In the phase wave \((S,0)\), swarmalators are partially locked in \(\zeta_i = 0\) and drift in \(\eta_i \neq 0\). Applying these conditions to Eqs. (4) and (5) yields

\[
\sin(\zeta_i - \Phi_+) = \frac{v_i + \omega_i}{S_+ J_+}
\]

\[
\eta_i(t) = \eta_i(0) + \frac{1}{J_+} (K v_i - J \omega_i) t,
\]

where \(-S_+ J_+ \leq v_i + \omega_i \leq S_+ J_+ \) and \(\eta_i(0)\) is an initial phase. Following Kuramoto \[2\], the order parameter must be self-consistent: \(S_+ := N^{-1} \sum_j e^{i \delta_j}\). Plugging Eq. (24) indeed gives the expression Eq. (19) for \(S_+\) in agreement with the generalized OA ansatz (Similarly, Eq. (25) implies \(S_- = 0\) as expected). We also attempted a macroscopic analysis of the sync state but the calculations were beyond the scope of this Letter \[53\]. Second, we checked the identical swarmalator limit which has been analyzed previously (without an OA ansatz) \[53\]. As \(\Delta_c, \Delta_\omega \to 0\), the critical curve for the phase wave Eq. (20) approaches \(J + K = 0\), while that of the sync state Eq. (23) approaches \(J + K > 0\) in agreement with \[51\]. Fig. 2(b) plots these in \((J, K)\) space to allow a visual comparison. Third, we calculated the stability of asyn using the OA equations (Eqs. (10)-(11)) and found it agreed with Eq. (14) \[53\] (derived by perturbing the continuity equation \[53\]). Fourth, we used the OA equations to derive \(S_{\text{sync}}\) and its critical coupling \(K_c\) for a simpler distribution \(g_{\omega(v)}(x) = \delta(x - \Delta) + \frac{1}{2} \delta(x + \Delta)\) which agreed with simulation perfectly \[53\]. This completes our analysis.

Hidden phase transition—We close by pointing out a curious feature of the swarmalator model. At \(J = 0\), the positions evolves at constant speed \(\dot{x}_i = v_i \Rightarrow x_i = v_i t\) which means the phases obey

\[
\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \cos[(v_j - v_i) t].
\]

One can think of this equation as a model for a group of oscillators with random, time-dependent couplings. In turn, the results presented in this letter reveal a phase transition hidden in the time-dependence of \(\theta_i\), which extends to the case where \(J = 0\). This ‘hidden’ phase transition causes incoherent oscillators to become phase-locked at \(\dot{\theta}_i = -v_i t + \zeta_i / 2\) (the \((S,0)\) state) or \(\dot{\theta}_i = v_i t - \eta_i / 2\) (the \((0,S)\)) where \(\zeta_i\) (\(\eta_i\)) is the phase from Eq. (20). Curiously, if we reinterpret \(v_i t\) as a heterogeneous field acting on the couplings, we see that the oscillators have become tuned to the field frequency \(v_i\). To the best of our knowledge, this is a novel result and may provide a useful means for tuning a population of oscillators to a prescribed set of frequencies in an experimental setting.

To conclude, we have presented a simple, solvable model of swarmalators. The model has a rich phase
diagram with a *tetracritical* point at which four phases meet. The model also captures the behavior of real-world swarmalators/active matter such as groups of sperm [57] and vinegar eels [58, 59] (which swarm in quasi-1D rings), and the rotational component of 2D, real-world swarmalators such as forced colloids [11, 12, 13]. Our simulations showed that the cutoff in the spatial interaction kernel does not qualitatively change the dynamics of swarmalators in comparison to global coupling [53]. Thus, the exact solution of the swarmalator model with all-to-all coupling should have applicability to a variety of situations with local coupling. We hope our work will be useful to the active matter community, as it provides a new toy model, and interesting to the sync community, as the first OA ansatz for oscillators which are mobile (mobile in a 1D periodic domain, at least).

Future work could study the stability of the phase wave, mixed, and sync states (note we derived criteria for their existence only). Incorporating delayed interactions or external forcing – which are analyzable with our OA ansatz – would also be interesting. Finally, our model and predictions could be experimentally tested in circularly confined colloids or robotic swarms [37, 38].

This work is funded by national funds (OE) through Portugal’s FCT Fundação para a Ciência e Tecnologia, I.P., within the scope of the framework contract foreseen in paragraphs 4.5 and 6 of article 23, of Decree-Law 57/2016, of August 29, and amended by Law 57/2017, of July 19. Code used in simulations available at [60].

[1] Arthur T Winfree, *The geometry of biological time*, Vol. 12 (Springer Science & Business Media, 2001).
[2] Yoshihi Kuramoto, *Chemical oscillations, waves, and turbulence* (Courier Corporation, 2003).
[3] Arkady Pikovsky, Jurgen Kurths, Michael Rosenblum, and Jürgen Kurths, *Synchronization: a universal concept in nonlinear sciences*, 12 (Cambridge university press, 2003).
[4] Ziping Jiang and Martin McCall, “Numerical simulation of a large number of coupled lasers,” JOSA B 10, 155–163 (1993).
[5] Charles S Peskin, “Mathematical aspects of heart physiology,” (Courant Institute of Mathematical Sciences, New York, 1975) pp. 268–278.
[6] William Bialek, Andrea Cavagna, Irene Giardina, Thierry Mora, Edmondo Silvestri, Massimiliano Viale, and Aleksandra M Walczak, “Statistical mechanics for natural flocks of birds,” Proceedings of the National Academy of Sciences 109, 4786–4791 (2012).
[7] Yael Katz, Kolbjørn Tunstrøm, Christos C Ioannou, Cristián Huepe, and Iain D Couzin, “Inferring the structure and dynamics of interactions in schooling fish,” Proceedings of the National Academy of Sciences 108, 18720–18725 (2011).
[8] Ikkyu Aihara, Takeshi Mizumoto, Takuma Otsuka, Hirokazu Saiki, Hiroshi Okuno, and Kazuyuki Aihara, “Spatio-temporal dynamics in collective frog choruses examined by mathematical modeling and field observations,” Scientific reports 4, 1–8 (2014).
[9] Kaichiro Ota, Ikkyu Aihara, and Toshio Aoyagi, “Interaction mechanisms quantified from dynamical features of frog choruses,” Royal Society open science 7, 191693 (2020).
[10] Tzer Han Tan, Alexander Mietke, Hugh Higinbotham, Junang Li, Yuchao Chen, Peter J Foster, Shreyas Ghokale, Jörn Dunkel, and Nikta Fakhri, “Development drives dynamics of living chiral crystals,” arXiv preprint arXiv:2105.07507 (2021).
[11] Jing Yan, Moses Bloom, Sung Chul Bae, Erik Luijten, and Steve Granick, “Linking synchronization to self-assembly using magnetic janus colloids,” Nature 491, 578–581 (2012).
[12] Jing Yan, Sung Chul Bae, and Steve Granick, “Rotating crystals of magnetic janus colloids,” Soft Matter 11, 147–153 (2015).
[13] Sangyeul Hwang, Trung Dac Nguyen, Srijanani Bhaskar, Jaewon Yoon, Marvin Klaiber, Kyung Jin Lee, Sharon C Glotzer, and Joerg Lahann, “Cooperative switching in large-area assemblies of magnetic janus particles,” Advanced Functional Materials 30, 1907855 (2020).
[14] Bo Zhang, Andrey Sokolov, and Alexey Snezhko, “Reconfigurable emergent patterns in active chiral fluids,” Nature communications 11, 1–9 (2020).
[15] Antoine Bricard, Jean-Baptiste Caussin, Debasis Das, Charles Savoie, Vijayakumar Chikkad, Kyohei Shitara, Oleksandr Chepizhko, Fernando Peruani, David Saintilan, and Denis Bartolo, “Emergent vortices in populations of colloidal rollers,” Nature communications 6, 1–8 (2015).
[16] Bo Zhang, Hamid Karani, Petia M Vlahovska, and Alexey Snezhko, “Persistence length regulates emergent dynamics in active roller ensembles,” Soft Matter (2021).
[17] Raj Kumar Manna, Oleg E Shklyaev, and Anna C Balazs, “Chemical pumps and flexible sheets spontaneously form self-regulating oscillators in solution,” Proceedings of the National Academy of Sciences 118 (2021).
[18] Menglin Li, Martin Brinkmann, Ignacio Pagonabarraga, Ralf Seemann, and Jean-Baptiste Fleury, “Spatiotemporal control of cargo delivery performed by programmable self-propelled janus droplets,” Communications Physics 1, 1–8 (2018).
[19] Kundan Chaudhary, Jaime J Juárez, Qian Chen, Steve Granick, and Jennifer A Lewis, “Reconfigurable assemblies of janus rods in ac electric fields,” Soft Matter 10, 1320–1324 (2014).
[20] Chao Zhou, Nobuhiyo Jessis Suematsu, Yixin Peng, Qizhang Wang, Xi Chen, Yongxiang Gao, and Wei Wang, “Coordinating an ensemble of chemical micromotors via spontaneous synchronization,” ACS nano 14, 5360–5370 (2020).
[21] Mario Urso, Martina Ussia, and Martin Pumera, “Breaking polymer chains with self-propelled light-controlled navigable hematite microrobots,” Advanced Functional Materials, 2101510.
[22] Jia Dai, Xiang Cheng, Xiaofeng Li, Zhisheng Wang, Yufeng Wang, Jing Zheng, Jun Liu, Jiawei Chen, Changjin Wu, and Jinyao Tang, “Solution-synthesized multifunctional janus nanotree microswimmer,” Advanced Functional Materials, 2106204 (2021).
[23] Kumar Vikrant and Ki-Hyun Kim, “Metal–organic
framework micromotors: perspectives for environmental applications,” Catalysis Science & Technology (2021).
[24] Jan Tesař, Martina Ussia, Osamah Alduhaish, and Martin Pumera, “Autonomous self-propelled mno2 micromotors for hormones removal and degradation,” Applied Materials Today 26, 101312 (2022).
[25] Jinxing Li, Oleg E Shklyaev, Tianlong Li, Wenjuan Liu, Henry Shum, Isaac Rozen, Anna C Balazs, and Joseph Wang, “Self-propelled nanomotors autonomously seek and repair cracks,” Nano Letters 15, 7077–7085 (2015).
[26] Rui Cheng, Weijie Huang, Lijie Huang, Bo Yang, Leidong Mao, Kunlin Jin, Qichuan ZhuGe, and Yiping Zhao, “Acceleration of tissue plasminogen activator-mediated thrombolysis by magnetically powered nanomotors,” ACS nano 8, 7746–7754 (2014).
[27] Laliphat Manamanchaiyaporn, Xiuzhen Tang, Xiaohui Yan, and Yuanyi Zheng, “Molecular transport of a magnetic nanoparticle swarm towards thrombolytic therapy,” IEEE Robotics and Automation Letters (2021).
[28] Bruno Ventejou, Hugues Chaté, Raul Montagne, and Xia-qing Shi, “Susceptibility of orientationally ordered active matter to chirality disorder,” Physical Review Letters 127, 238001 (2021).
[29] Kevin P O’Keeffe, Hyunsuk Hong, and Steven H Strogatz, “Oscillators that sync and swarm,” Nature communications 8, 1–13 (2017).
[30] Seung-Yeal Ha, Jinwook Jung, Jeongho Kim, Jinyeong Park, and Xiongtao Zhang, “Emergent behaviors of the swarmalator model for position-phase aggregation,” Mathematical Models and Methods in Applied Sciences 29, 2225–2269 (2019).
[31] Dan Tanaka, “General chemotactic model of oscillators,” Physical review letters 99, 134103 (2007).
[32] Zeng Tao Liu, Yan Shi, Yongfeng Zhao, Hugues Chaté, Xia-qing Shi, and Tian Hui Zhang, “Activity waves and freestanding vortices in populations of subcritical quincke rollers,” Proceedings of the National Academy of Sciences 118 (2021).
[33] Gourab Kumar Kumar Sar and Dibakar Ghosh, “Dynamics of swarmalators: A pedagogical review,” Europhysics Letters (2022).
[34] Masatomo Iwasa and Dan Tanaka, “Dimensionality of clusters in a swarm oscillator model,” Physical Review E 81, 066214 (2010).
[35] Demian Levis, Ignacio Pagonabarraga, and Benno Liebchen, “Activity induced synchronization: Mutual flocking and chiral self-sorting,” Physical Review Research 1, 023026 (2019).
[36] Benno Liebchen and Demian Levis, “Collective behavior of chiral active matter: Pattern formation and enhanced flocking,” Physical review letters 119, 058002 (2017).
[37] Agata Barciš, Michal Barciš, and Christian Bettstetter, “Robots that sync and swarm: A proof of concept in ros 2,” in 2019 International Symposium on Multi-Robot and Multi-Agent Systems (MRS) (IEEE, 2019) pp. 98–104.
[38] Agata Barciš and Christian Bettstetter, “Sandsbots: Robots that sync and swarm,” IEEE Access 8, 218752–218764 (2020).
[39] Hyun Keun Lee, Kangmo Yeo, and Hyunsuk Hong, “Collective steady-state patterns of swarmalators with finite-cutoff interaction distance,” Chaos: An Interdisciplinary Journal of Nonlinear Science 31, 033134 (2021).
[40] Hyunsuk Hong, “Active phase wave in the system of swarmalators with attractive phase coupling,“ Chaos: An Interdisciplinary Journal of Nonlinear Science 28, 103112 (2018).
[41] Joao UF Lizarraga and Marcus AM de Aguiar, “Synchronization and spatial patterns in forced swarmalators,” Chaos: An Interdisciplinary Journal of Nonlinear Science 30, 053112 (2020).
[42] Kevin P O’Keeffe, Joep HM Evers, and Theodore Kolokolnikov, “Ring states in swarmalator systems,” Physical Review E 98, 022203 (2018).
[43] Seung-Yeal Ha, Jinwook Jung, Jeongho Kim, Jinyeong Park, and Xiongtao Zhang, “A mean-field limit of the particle swarmalator model,” Kinetic & Related Models (2021).
[44] Kevin O’Keeffe and Hyunsuk Hong, “Swarmalators on a ring with distributed couplings,” Phys. Rev. E 105, 064208 (2022).
[45] Gourab K Sar, Sayantan Nag Chowdhury, Matjaz Perc, and Dibakar Ghosh, “Swarmalators under competitive time-varying phase interactions,” arXiv preprint arXiv:2201.01598 (2022).
[46] Kevin O’Keeffe and Christian Bettstetter, “A review of swarmalators and their potential in bio-inspired computing,” Micro-and Nanotechnology Sensors, Systems, and Applications XI 10882, 383–394 (2019).
[47] Udo Schilcher, Jorge F Schmidt, Arke Vogell, and Christian Bettstetter, “Swarmalators with stochastic coupling and memory,” in 2021 IEEE International Conference on Autonomic Computing and Self-Organizing Systems (ACSOS) (IEEE, 2021) pp. 90–99.
[48] Tamás Visek, András Czirók, Eshel Ben-Jacob, Iron Cohen, and Ofer Shochet, “Novel type of phase transition in a system of self-driven particles,” Physical review letters 75, 1226 (1995).
[49] John Toner and Yuhai Tu, “Flocks, herds, and schools: A quantitative theory of flocking,” Physical review E 58, 4828 (1998).
[50] Edward Ott and Thomas M Antonsen, “Low dimensional behavior of large systems of globally coupled oscillators,” Chaos: An Interdisciplinary Journal of Nonlinear Science 18, 037113 (2008).
[51] Kevin O’Keeffe, Steven Ceron, and Kirstin Petersen, “Collective behavior of swarmalators on a ring,” Physical Review E 105, 014211 (2022).
[52] The original Kuramoto model has ‘all-to-all’ coupling \( K_{ij} = K \).
[53] See Supplemental Material at http://link.aps.org/supplemental/.
[54] In 2D this looks like a vortex, see 29; that’s why we called it a ‘vortex-like’ phase wave in the abstract.
[55] Edward Ott and Thomas M. Antonsen, “Long time evolution of phase oscillator systems,” Chaos: An Interdisciplinary Journal of Nonlinear Science 19, 023117 (2009).
[56] Michael E. Cates and Julien Tailleur, “Motility-induced phase separation,” Annual Review of Condensed Matter Physics 6, 219–244 (2015).
[57] Adama Creppy, Franck Plouraboué, Olivier Praud, Xavier Druart, Sébastien Cazin, Hui Yu, and Pierre Degond, “Symmetry-breaking phase transitions in highly concentrated semen,” Journal of The Royal Society Interface 13, 20160575 (2016).
[58] AC Quillen, A Peshkov, Esteban Wright, and Sonia McGaffigan, “Synchronized oscillations in swarms of nematode turbatrix aceti,” arXiv preprint arXiv:2104.10316 (2021).
[59] AC Quillen, A Peshkov, Esteban Wright, and Sonia McGaffigan, “Metachronal waves in concentrations of swimming turbatrix aceti nematodes and an oscillator chain model for their coordinated motions,” arXiv preprint arXiv:2101.06809 (2021).
[60] https://github.com/Khev/swarmalators/tree/master/1D/on-ring/non-identical.