Entropy and Uncertainty Analysis in Financial Markets

*Andreia Dionisio, **Rui Menezes and **Diana A. Mendes

*University of Evora, Center of Business Studies, CEFAGE-UE, Largo Colegiãis, 2, 7000 Evora, Portugal, E-mail: andreia@uevora.pt; **ISCTE, Department of Quantitative Methods, Av. Forcas Armadas, 1649-Lisboa, Portugal, E-mail: rui.menezes@iscte.pt, diana.mendes@iscte.pt

Abstract

The investor is interested in the expected return and he is also concerned about the risk and the uncertainty assumed by the investment. One of the most popular concepts used to measure the risk and the uncertainty is the variance and/or the standard-deviation. In this paper we explore the following issues: Is the standard-deviation a good measure of risk and uncertainty? What are the potentialities of the entropy in this context? Can entropy present some advantages as a measure of uncertainty and simultaneously verify some basic assumptions of the portfolio management theory, namely the effect of diversification?

Key words: Uncertainty, entropy, stock markets.

Introduction

This paper examines the adequacy of entropy as a measure of uncertainty in portfolio management in finance and its behaviour is compared with the most popular risk measure used in finance: the variance.

It is quite common to relate the variance or the standard-deviation and the VaR (Value-at-Risk) as the main measures of risk and uncertainty in finance. However, some authors [see e.g. Soofi (1997)] point out that these measures may fail in some specific situations as measures of uncertainty, since they require that the underlying probability distribution is symmetric and neglect the possibility of extreme events such as, for example, the existence of fat-tails.

The main goal of this paper is to assess the capability of entropy to measure the uncertainty in portfolio management. We describe the theoretical background of entropy and its mathematical properties, we present a comparative
analysis between the entropy and the variance/standard-deviation as measures of uncertainty in the stock market and we discuss the empirical results of a comparative analysis between the CAPM and some measures of information theory (namely entropy, conditional entropy and mutual information).

The results obtained point to the conclusion that the entropy observes the effect of diversification and is a more general measure of uncertainty than the variance, since it uses more information about the probability distribution. The mutual information and the conditional entropy show a good performance when compared with the systematic risk and the specific risk estimated through the linear Market Model.

1 Theoretical Background

According to Shannon (1948) entropy satisfies the main properties of a good measure of uncertainty. Let $p_1, ..., p_n$ be the probabilities of occurrence of a set of events. The entropy for discrete distributions is given by [Shannon (1948)]

$$H(X) = - \sum_i p_i \log p_i.$$  

For continuous distributions, where $p_X(x)$ is the density function of the random variable $X$, the entropy is given by

$$H(X) = - \int p_X(x) \log p_X(x) dx. \quad (1)$$

Note that $H(X)$ and $H(Y)$ are the entropies of the random variables $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, $H(X,Y)$ is the joint entropy, and $H(Y|X)$ and $H(X|Y)$ are the conditional entropies.

The properties of entropy for discrete and continuous distributions are mainly alike. In particular we have [Shannon (1948); Kraskov et al. (2004)]: (i) if $X$ is limited to a certain volume $v$ in its space, then $H(X)$ is a maximum and is equal to $\log v$ when $p_X(x)$ is constant, $1/v$, in the volume; (ii) for any two variables $X$ and $Y$, we have $H(X,Y) \leq H(X) + H(Y)$, where the equality holds if (and only if) $X$ and $Y$ are statistically independent, i.e. $p_{X,Y}(x,y) = p_X(x)p_Y(y)$; (iii) the joint entropy is given by $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$,since $H(X) + H(Y) \geq H(X,Y)$, then $H(Y) \geq H(Y|X)$ and $H(X) \geq H(X|Y)$.

The entropy of a normal distribution can be parametrically estimated by

$$NH(X) = \log \left(\sqrt{2\pi e}\sigma\right),$$

where $\sigma$ is the standard-deviation, and $\pi$ and $e$ are defined as usually. Notice that the assumption that the data follow a normal distribution is very common in portfolio management and regression analysis.

The mutual information is a measure of association between variables and can
be defined as follows [Shannon (1948)]:

\[ I(X, Y) = H(Y) - H(Y|X) = \int \int p_{X,Y}(x, y) \log \frac{p_{X,Y}(x, y)}{p_X(x)p_Y(y)} \, dxdy. \quad (2) \]

The mutual information is a nonnegative measure [Kullback (1968)], being equal to zero if and only if \( X \) and \( Y \) are statistically independent. In this way, the mutual information between two random variables \( X \) and \( Y \) can be seen as a measure of dependence between these variables, or even better, it can be regarded as a measure of the statistical correlation between \( X \) and \( Y \). However, we can not say that \( X \) is causing \( Y \) or vice-versa.

We also need to define a measure that can be directly comparable with the linear correlation coefficient. In equation (2), we have \( 0 \leq I(X, Y) \leq +\infty \), which hampers eventual comparisons between different samples. Some authors, namely Granger and Lin (1994), Darbellay (1998) and Soofi (1997) have used a standardized measure for the mutual information, referred to as the global correlation coefficient, defined by \( \lambda(X, Y) = \sqrt{1 - e^{-2I(X,Y)}} \). This measure varies between 0 and 1, being thus directly comparable with the linear correlation coefficient.

The function \( \lambda(X, Y) \) captures the overall linear and nonlinear dependence between \( X \) and \( Y \). This measure can be used as a measure of predictability based on an empirical probability distribution, although it is not derived from any particular model of predictability. In this particular case, the above mentioned properties assume the following form: (i) \( \lambda(X, Y) = 0 \), if and only if \( X \) contains no information on \( Y \); (ii) \( \lambda(X, Y) = 1 \), if there is a perfect relationship between the vectors \( X \) and \( Y \). This is the farthest case of determinism; (iii) when modelling the input-output pair \( (X, Y) \) by any model with input \( X \) and output \( U = f(X) \), where \( f \) is a function of \( X \), the predictability of \( Y \) by \( U \) cannot exceed the predictability of \( Y \) by \( X \), i.e., \( \lambda(X, Y) \geq \lambda(U, Y) \).

One of the difficulties to estimate the mutual information on the basis of empirical data lies on the fact that the underlying p.d.f. is unknown. To overcome this problem, there are essentially three different methods to estimate the mutual information: histogram-based estimators, kernel-based estimators and parametric methods.\(^1\) We will use the marginal equiquantization histogram-based estimation process proposed by Darbellay (1998), in order to minimize the bias that may occur.

Finally, we should note that the introduction of entropy as a measure of un-

\(^1\) The histogram-based estimators can be divided in two groups: equidistant cells and equiprobable cells, i.e. marginal equiquantisation. The second approach presents some advantages, since it allows for a better adequacy to the data and maximizes mutual information [Darbellay (1998)].
certainty in finance goes back to Philippatos and Wilson (1972), who present a comparative analysis between the behaviour of the standard-deviation and the entropy in portfolio management. They conclude that entropy is more general and has some advantages relatively to the standard-deviation, such as Dionisio et al. (2006).

2 Entropy and variance: a comparative analysis

Usually, the variance is the central measure in the risk and uncertainty analysis in financial markets. However, the entropy can be used as an alternative measure of dispersion, and some authors consider that the variance should be interpreted as a measure of uncertainty with some precaution [see, e.g. Maasoumi (1993) and Soofi (1997)].

Ebrahimi, Maasoumi and Soofi (1999) examined the role of the variance and entropy in ordering distributions and random prospects, and concluded that there is no general relationship between these measures in terms of ordering distributions. They found that, under certain conditions, the ordering of the variance and entropy is similar for transformations of continuous variables, and show that the entropy depends on many more parameters of a distribution than the variance. Indeed, a Legendre series expansion shows that the entropy is related to higher-order moments of a distribution and thus, unlike the variance, could offer a better characterization of $p_X(x)$ since it uses more information about the probability distribution than the variance [see Ebrahimi et al. (1999)].

Maasoumi and Racine (2002) argue that when the empirical probability distribution is not perfectly known, the entropy constitutes an alternative measure for assessing the uncertainty, predictability and also goodness-of-fit. Likewise, McCauley (2003) defends that the entropy represents the disorder and uncertainty of a stock market or a particular stock, since the entropy has the ability to capture the complexity of systems without requiring rigid assumptions that may bias the results obtained.

Our empirical analysis is based on daily closing prices of 23 stocks rated in the Portuguese stock market (Euronext Lisbon), covering the period from June, 28, 1995 to December, 30, 2002, which corresponds to 1858 observations per stock in order to compute the rates of return. The stock index PSI 20 is used as the market benchmark, or proxy, since it is the index that better represents the Euronext Lisbon. A preliminary statistical analysis of the rates of return reveals that the null that the empirical distributions are Gaussian should be rejected since they show high levels of kurtosis and skewness.
Firstly, we perform a comparative analysis between the entropy and the logarithm of the standard-deviation, $\ln(\sigma)$, for each stock in our data set and for the stock index PSI 20. The entropy was computed using expression (1) measured in nats. The $\ln(\sigma)$ was used instead of $\sigma$ in order to provide a correct comparison between these measures. The results are shown in Figure 1.

![Fig. 1. Entropy versus $\ln(\sigma)$.](image)

As may be seen, there is a strong positive relationship between the entropy and the $\ln(\sigma)$. There are, however, some larger deviations and the null under the Jarque-Bera test is rejected for all stocks and also for the stock index PSI 20. In Figure 1 the larger deviations from the regression line correspond to stocks that also exhibit higher levels of skewness and kurtosis.

We also compared the results between the empirical entropy and the normal entropy for each stock. The results reveal that the normal entropy takes always higher values than the empirical entropy, indicating that the uncertainty of these stocks and index is smaller than that we would observe if they were normally distributed. Thus, we can admit evidence of some predictability of the rate of returns, or at least that it is higher than the one assumed by the financial theory. Again, the main differences between the normal entropy and the empirical entropy are found in the stocks that exhibit the highest levels of kurtosis, skewness, autocorrelation and heteroskedasticity. Therefore, the empirical entropy appears to be sensitive to higher-order moments of the distribution, supplying thus more information about the stock and its probability distribution.\(^2\)

\(^2\) We also performed an analysis of the diversification effect for randomly selected portfolios [see Dionisio et al. (2006)]. This study revealed that both the entropy and the standard-deviation tend to decrease when more assets are included in the portfolio, pointing to the conclusion that entropy is sensitive to the effect of diversification.
3 Analysis of dependency between the stocks and the PSI 20 index

In the context of portfolio management researchers tend to pay more attention to the systematic risk than to the specific risk, because the latter can be minimized (and in the limit can be zero) by an effective diversification process. Usually, the systematic risk in financial theory is measured by the Beta of the CAPM. In this context, it is assumed that the rate of returns of an asset \( i \) is equal to the sum of the risk free rate of return \( (R_f) \) and the compensation for the risk \( \{ [E(R_m) - (R_f)] \beta_i \} \). In this second term, the coefficient \( \beta \) plays an important role, since it measures the sensibility of the rate of returns of the asset (or portfolio) to the risk premium, i.e. the systematic risk. Recall that the variance of an asset or portfolio \( (\sigma^2_i) \) can be decomposed in two components: the sum of squares of the regression and the residual sum of squares, that is

\[
\sigma^2_i = \beta^2_i \sigma^2_m + \sigma^2_{\epsilon_i},
\]

where \( \sigma^2_m \) is the variance of the independent variable, in this case the variance of the market benchmark, and \( \sigma^2_{\epsilon_i} \) is the residual term of the variance that can be minimized through a diversification process. In order to estimate the Beta of the CAPM we use the Market Model that can be given by

\[
R_{it} = R_{ft} + [R_{mt} - R_{ft}] \beta_i + \varepsilon_{it}.
\]

The Market Model is usually estimated by OLS. However, statistical tests and the empirical evidence show that the residuals are not white noise and thus OLS may not be appropriate.

The main goal of this section is to evaluate the level of dependence between each stock and the stock index. The measures of information theory described before, namely the entropy, the conditional entropy and the mutual information are used to evaluate the (in)dependence between the stocks and the stock index PSI 20. If the residuals are white noise, then the global correlation coefficient \( (\lambda) \) will be similar to the linear correlation coefficient \( (R) \) and the Beta is a good measure of the systematic risk. However, in case of nonlinearities and irregularities in the behaviour of the residuals, the simple linear regression model is not able to capture the existence of a global relationship between the stocks and the stock index PSI 20. In this case, the mutual information and the global correlation coefficient can be potential sources of information for the investor.

One may distinguish the "global" uncertainty from the "residual" uncertainty based on the properties of the entropy. The entropy of a stock (or any other variable) can be decomposed as follows

\[
H(X) = I(X, PSI) + H(X|PSI),
\]

where
where $X$ represents the stock and $PSI$ denotes the stock index. The expression (5) can be roughly compared with the expression (3), where the first term refers to the level of association or dependence between the asset (stock or portfolio) and the proxy $PSI$, and the second term is the variation of that asset (or stock, or portfolio) that is independent from the proxy used.

The Beta was estimated for each stock as well as the linear correlation coefficient resulting from the Market Model. In addition, we also calculate the systematic risk, $\beta^2 \sigma^2_m$, and the specific risk, $\sigma^2_{\epsilon i}$. In order to provide some comparisons between the two approaches, we calculated the mutual information between each stock and the stock index PSI 20, $[I(X, PSI)]$, the conditional entropy $[H(X|PSI)]$, and the global correlation coefficient, $\lambda$.

As we can see in Figure 2, there is a positive relationship between the systematic risk and the mutual information, and between the specific risk and the conditional entropy, although these measures are not directly comparable.

Fig. 2. Comparative analysis between the systematic risk, $\beta^2 \sigma^2_m$ (continuous line), and the mutual information $I$ and between the specific risk, $\sigma^2_{\epsilon i}$ (continuous line), and the conditional entropy, $H(X|PSI)$.

In spite of the evidence of a positive and strong relationship between the variance and the measures of the theory of the information, we performed a comparison between measures that can be directly compared. To this end, we used the global correlation coefficient or lambda ($\lambda$) and the global correlation coefficient assuming that the distribution is normal or lambda normal ($\lambda_n$).\(^3\)

We found that there are stocks whose relationship with the stock index PSI

\(^3\) The lambda normal was computed using equation the concept of normal mutual information given by: $IMN(X,Y) = -\frac{1}{2} \log \left(1 - R^2(X,Y)\right)$, where $R^2$ is the coefficient of determination.
20 exhibit strong discrepancies when analyzed from a global or purely linear perspective. In order to find the possible causes of such differences, several tests were accomplished for the residuals produced by the estimation of the linear Market Model, namely the Ljung-Box test, the Jarque-Bera test, the Engle test and the stability tests CUSUM and CUSUM-Q. The results of these tests indicate that there are precisely the residuals resulting from the application of the linear Market Model to stocks that appear to exhibit higher levels of nonlinearity that present more evidence of autocorrelation, non-normality, heteroskedasticity and nonstability, and this seems to be an indicator that linear analyses are possibly not enough to evaluate risk and uncertainty.

4 Conclusions

The results presented in this paper point to the conclusion that the entropy observes the effect of diversification and is a more general measure of uncertainty than the variance, since it uses more information about the probability distribution. The mutual information and the conditional entropy show a good performance when compared with the systematic risk and the specific risk estimated through the linear Market Model. Nevertheless, the use of the concept of entropy in risk analysis and portfolio selection needs some care, because it does not take into account the actual values of the variables and this may eventually compromise its inclusion in the context of an utility function.

Acknowledgement

Financial support from Fundacao da Ciencia e Tecnologia, Lisbon, is gratefully acknowledged by the authors, under the contract PTDC/GES/70529/2006.

References

[1] Darbellay, G. Predictability: an Information-Theoretic Perspective. In: Signal Analysis and Prediction, A. Procházka, J. Uhlíř, P.J.W. Rayner and N.G. Kingsbury, Birkhauser eds., Boston, (1998), 249-262.

[2] Dionisio, A., Menezes, R. and Mendes, D.A. The European Physical Journal B, (2006), 50, 161-164.

[3] Ebrahimi, N., Maasoumi, E., Soofi, E. Journal of Econometrics, (1999), 90, 2, 317-336.

[4] Granger, C., Lin, J. Journal of Time Series Analysis, (1994), 15, 4, 371-384.
[5] Kraskov, A., Stögbauer H., Andrzejak, R., Grassberger, P., Hierarchical Clustering Based on Mutual Information, (2004), preprint http://www.arxiv:q-bio.QM/0311039.

[6] Kullback, S. Information Theory and Statistics. Dover, New York. 1968.

[7] Maasoumi, E. Econometric Reviews, (1993), 12, 2, 137-181.

[8] Maasoumi, E., Racine, J. Journal of Econometrics, (2002), 107, 291-312.

[9] McCauley, J. Physica A, (2003), 329, 199-212.

[10] Philippatos, G., Wilson, C. Applied Economics, (1972), 4, 209-220.

[11] Shannon, C. E. Bell Systems Tech., (1948), 27: 379-423, 623-656.

[12] Soofi, E.. Information Theoretic Regression Methods. In: Advances in Econometrics - Applying Maximum Entropy to Econometric Problems, Fomby, T. and R. Carter Hill eds. Vol. 12. Jai Press Inc., London. 1997.
$y = 0.5872 \times + 1.485$

$R^2 = 0.8441$
