Geometrically nonlinear modelling of pre-stressed viscoelastic fibre-reinforced composites with application to arteries

I. I. Tagiltsev1,2 · A. V. Shutov1,2

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Abstract
Mechanical behaviour of pre-stressed fibre-reinforced composites is modelled in a geometrically exact setting. A general approach which includes two different reference configurations is employed: one configuration corresponds to the load-free state of the structure and another one to the stress-free state of each material particle. The applicability of the approach is demonstrated in terms of a viscoelastic material model; both the matrix and the fibre are modelled using a multiplicative split of the deformation gradient tensor; a transformation rule for initial conditions is elaborated and specified. Apart from its simplicity, an important advantage of the approach is that well-established numerical algorithms can be used for pre-stressed inelastic structures. The interrelation between the advocated approach and the widely used “opening angle” approach is clarified. A full-scale FEM simulation confirms the main predictions of the “opening angle” approach. A locking effect is discovered: in some cases the opening angle of the composite is essentially smaller than the opening angles of its individual layers. Thus, the standard cutting test typically used to analyse pre-stresses does not carry enough information and more refined experimental techniques are needed.

Keywords Pre-stresses · Finite strain viscoelasticity · Fibre-reinforced composites · Cutting test · Opening angle approach · Efficient numerics

1 Introduction

Biological tissues like arteries, tendons and muscles may be considered as composites, consisting of soft isotropic matrix reinforced with embedded stiff fibres (Owen et al. 2018). Such materials sustain large cyclic strains and show viscoelastic anisotropic mechanical behaviour. In recent years many material models were introduced to take numerous mechanical phenomena into account: not only anisotropic hyperelasticity (Chuong and Fung 1983; Vaishnav et al. 1973; Holzapfel et al. 2000; Shearer 2015; Von Hoegen et al. 2018), viscoelasticity (Holzapfel and Gasser 2001; Latorre and Montáns 2015; Tagiltsev et al. 2018; Liu et al. 2019) and elasto-plasticity (Shutov and Kreißig 2008; Guan and Zhu 2009; Shutov and Kreißig 2010) were modelled, but also damage accumulation (Hurschler et al. 1997; Balzani et al. 2006; Hamedzadeh et al. 2018) as well as growth and remodelling of living tissues (Cyron and Humphrey 2016; Braeu et al. 2017; Keshavarzian et al. 2018) were taken into account. From the mechanical standpoint, the main goal of these studies is to predict the stress response of the analysed tissue on the macroscopic level. Mechanical stresses are not only a measure of the local load intensity, they also
act as activators for a number of physiological mechanisms. Numerous studies indicate that any reliable model of tissue growth and remodelling requires knowledge of the applied mechanical loads (Keshavarzian et al. 2018; Braeu et al. 2017).

The primary concern of the current publication is the adequate modelling of residual stresses. The field of residual stresses which is realized in the homeostasis is an important constituent of the overall mechanical stress field. The residual stresses need to be taken into account since they superimpose with applied loads. In particular, they govern the spring-back of soft tissues upon surgical manipulations like cutting and affect the propagation of the pulse wave. In biological structures residual stresses arise due to such processes as growth, remodelling, adaptation and repair. The presence of residual hoop stresses is indicated by the classical opening angle test (Chuong and Fung 1983), and the residual axial stresses manifest themselves by a shortening of a blood vessel upon excising from the body (Vaishnav and Vossoughi 1983; Delfino et al. 1997). Some modelling approaches to residual stresses inside arteries were already considered in Holzapfel et al. (2000); Balzani et al. (2007); Cardamone et al. (2009) dealing with elastic (hyperelastic) stress response. To enable accurate and efficient simulations, a combination of constitutive assumptions and numerical schemes presented in Tagiltsev et al. (2018) will be generalized here to cover the presence of pre-stresses. In contrast to the mentioned publications, the interaction between residual stresses and viscous effects is incorporated. Moreover, since the mathematical structure of multiplicative plasticity models is identical to the considered equations of viscoelasticity (see, for instance, Simo and Miehe 1992; Lion 2000; Shutov and Tagiltsev 2019), the same approach can be transferred to elasto-plasticity in a straightforward way.

The so-called iso-strain approach, also known as the constrained mixture theory, is frequently used in material modelling. It is based on the assumption that the matrix and fibre experience the same deformation. As a consequence, the overall stress response is a sum of stress contributions provided by matrix and fibre. Thanks to simplified kinematics, it allows one to employ efficient numerical methods used to assess the overall behaviour of macroscopic structures (Holzapfel et al. 2000; Gasser et al. 2002; Tagiltsev et al. 2018). In this paper, in order to further reduce the computational costs, the viscoelastic properties are modelled using the Maxwell body approach (so-called spring-dashpot model). In general, the Maxwell approach is more popular within the FEM than the use of convolution integrals (Holzapfel and Gasser 2001): the Maxwell approach based on classical non-fractional time rate employs a limited number of internal variables, whereas the convolution integral with a general relaxation function requires the entire deformation history. Moreover, for the considered versions of the Maxwell bodies, efficient iteration-free time-stepping methods are already available (Shutov et al. 2013; Shutov 2018; Tagiltsev et al. 2018). The applicability of the advocated approach to fibre-reinforced composites was already assessed in Tagiltsev et al. (2018). Since the integration methods are non-iterational, much higher computational efficiency can be achieved, which is especially important for the analysis of clinical applications (Owen et al. 2018).

Dealing with pre-stressed tubes, the residual hoop stresses are related to the opening angle of the tube which is cut along. In particular, pre-stressed arteries were considered by Liu and Fung (Liu and Fung 1988; Fung and Liu 1989) and Holzapfel et al. (Holzapfel et al. 2000) introducing vessel’s stress-free configuration and it’s “opening angle” \( \alpha \) (note that the definition of the opening angle may differ depending on the author). The corresponding method will be referred to as the “opening angle” approach. The “opening angle” approach is an integral one, used for pre-stressed tubes; it includes the calculation of the stress distribution in the entire layer. As a result, its FEM implementation becomes over-complicated: every part of the body with different stress-free state requires individual modelling combined with simulation of bending into load-free state (Gasser et al. 2002).

In this work we use a more universal approach explained in Sect. 2. When working with pre-stressed structures, two different reference configurations naturally appear. First, the so-called load-free configuration corresponds to the state of the unloaded body which is characterized by the absence of external loads. Second, in material modelling one naturally introduces the so-called stress-free configuration associated with a zero stress field. Thus, there is a need to describe the transition between these two configurations. Stress-free and load-free configurations are connected in this study via the deformation gradient tensor (linear mapping) \( \mathbf{F}_0 \); the \( \mathbf{F}_0 \)-tensor may be considered as an internal variable of the material. The use of \( \mathbf{F}_0 \) to adjust the pre-stresses allows us to account for various stress-free configurations for different parts of material, which might be important in a number of applications (Hahn 1976; Holzapfel and Ogden 2010).

Whereas within the “opening angle” approach the initial pre-stressed state is obtained by a computationally expensive simulation of a boundary-value problem (like bending of a vessel), the \( \mathbf{F}_0 \)-approach allows us to describe the initial state by choosing a proper \( \mathbf{F}_0 \)-field. In this work we demonstrate that the advocated \( \mathbf{F}_0 \)-approach can be efficiently implemented within a nonlinear FEM. Towards that end, constitutive relations including evolution equations and initial

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1 A further discussion of different approaches to finite-strain Maxwell models can be found in Shutov (2016).
conditions are carefully elaborated accounting for the change of the reference configuration.

In Sect. 3 we show that the $F_0$-field can be modelled to describe vessel behaviour upon its cutting. We show that the $F_0$-approach includes the “opening angle” approach as a special case. A brief algorithm and corresponding formulae are given. FEM-simulations of cutting a viscous pre-stressed artery are provided as a numerical test. Section 4 contains further demonstration of the $F_0$-approach; in this section an artery consisting of two layers with different opening angles is considered and a new mechanical phenomenon is identified. It is shown by both semi-analytical and finite element methods that opening angle of two connected layers may be essentially smaller than the minimum between the opening angles of individual constituents. This effect is seen in the paper as a consequence of a mutual locking. The discovered locking effect indicates that a single opening angle of the overall composite does not carry enough information for the accurate modelling of pre-stresses. Finally, a summary of the results and concluding remarks are given in Sect. 5.

2 Model of a fibre-reinforced composite with pre-stresses

Let $\tilde{K}_f$ be a load-free configuration (lf-configuration) of the considered body and $F^lf$ be the corresponding deformation gradient which transforms line elements from $K_0$ to the current configuration $K$. Analogously, let $\tilde{K}_s$ be the local stress-free configuration (sf-configuration) which is characterized by zero stresses, $F^sf$ is the corresponding deformation gradient. By definition, load-free configuration is occupied by the unloaded body; it differs from the stress-free configuration due to the presence of residual stresses. The residual stresses are caused by the incompatibility of the stress-free configuration. The residual stresses deform the body from $K_0$ to $K_f$. During a local unloading of a certain particle, its local configuration transforms from $K_f$ to $K_s$; the corresponding deformation gradient will be denoted as $F_0$.

The stress-free deformation gradient $F^sf$ is connected with the load-free deformation gradient $F^lf$ by the relation

$$F^sf = F^lf F_0^{-1}. \quad (1)$$

Therefore, with a given constant $F_0$ the kinematics of the material particle can be described using two different reference configurations. A general commutative diagram is given in Fig. 1. In this paper we assume that $F_0$ is unimodular: $\det F_0 = 1$.

\[\text{Remark 1}\] Equation (1) has a similar structure to the multiplicative split used to model growth and remodelling, see, for example, (Rodriguez et al. 1994; Braeu et al. 2017). In our notation, $F^lf$ and $F^sf$ correspond to $F$ and $F_0$ from Braeu et al. (2017). In contrast to Braeu et al. (2017), inelastic stress response is included in the current study.

Special constitutive assumptions are essential to account for inelastic properties of material. For these purposes we use the Sidoroff multiplicative decomposition of deformation gradient into elastic and inelastic parts. Let us start using $\tilde{K}_s$ as the reference configuration. The deformation gradient $F^sf$ is decomposed into the elastic part $\tilde{F}_e$ and the inelastic part $F^sf_1$

$$F^sf = \tilde{F}_e F^sf_1. \quad (2)$$

This decomposition gives rise to the intermediate configuration $\hat{K}$, which is achieved by instantaneous unloading from the current configuration; corresponding deformation is denoted as $F^{lf}_e^{-1}$. Then, (1) and (2) yield the following:

$$F^lf = F^sf F_0 = \tilde{F}_e F^sf_1 F_0 = \hat{F}_e F^sf_1, \quad \text{where} \quad F^sf_1 = F^sf_1 F_0. \quad (3)$$

This means that the Sidoroff split is available on the lf-configuration as well, and the elastic part of the deformation gradient does not depend on the choice of the reference

\[\text{Fig. 1}\] General commutative diagram, showing the transition between two different reference configurations: $K_s$ and $K_f$

\[\text{Fig. 2}\] Commutative diagram for multiplicative decompositions of deformation gradients
configuration. The specific commutative diagram is shown in Fig. 2.

To clarify the nature of $F_0$, let us consider the following situation. Let $\tilde{K}_{\text{lf}}$ be a reference configuration. Assume that it coincides with the current configuration $K$ at the initial instance of time $t = 0$: $\tilde{K}_{\text{lf}} = K|_{t=0}$. Instantaneous unloading of the material particle is described by $\mathbf{F}^{-1}$: the operator $\mathbf{F}^{-1}$ transforms $K$ to $\tilde{K}$ and brings $\tilde{K}_{\text{lf}}$ to $\mathbf{F}_0$. Thus, $\tilde{K}_{\text{sf}} = \tilde{K}|_{F=0}$ in that case. Since $F_0 = \mathbf{F}_0|_{t=0} = (\mathbf{F}_e|_{t=0})^{-1}$. This special case is summarized in Fig. 3. Two different interpretations of $F_0$ are thus possible: it can be the initial inelastic deformation or the inverse of the initial elastic deformation.

The entire problem of modelling the initial stresses boils down to an accurate manipulation with reference configurations and initial conditions. In the current section, a procedure for obtaining the Cauchy stress tensor as a function of the deformation gradient $F$ with respect to the load-free configuration is suggested. The procedure is exemplified using the model of fibre-reinforced viscoelastic composite which was advocated in Tagiltsev et al. (2018). That model includes Mooney–Rivlin and Holzapfel potentials associated with isotropic and fibre-anisotropic hyperelastic behaviour. Isotropic and fibre-like Maxwell bodies are incorporated to take viscoelasticity into account.

2.1 Isotropic and fibre-like hyperelasticity with pre-stresses

The mechanical properties of hyperelastic materials are uniquely defined by the free energy density function. In many cases, the natural choice of the reference configuration is $K = \tilde{K}_{\text{sf}}$, since it provides zero stresses in undeformed state.

Isotropic hyperelasticity One-term neo-Hookean potential is widely used for the modelling of isotropic hyperelasticity (Holzapfel et al. 2000; Jin and Stanciulescu 2016; Mousavi and Avril 2017; Owen et al. 2018). However, two-term Mooney–Rivlin potential is also frequently used in biomechanics, especially in the modelling of the diseases’ development (Owen et al. 2018) and it also includes the neo-Hookean potential as a special case. Thus, in order to capture the equilibrium part of the stress response provided by the matrix material, we consider the two-term Mooney–Rivlin material. By definition, the right Cauchy–Green tensor with respect to the sf-configuration $\tilde{K}_{\text{sf}}$ is given by

$$C^{\text{sf}} = F^{\text{sf}T} F^{\text{sf}}.$$  (4)

The corresponding free energy per unit mass reads

$$\rho_R \Psi_{\text{MR}}(C^{\text{sf}}) = \frac{c_1}{2} (\text{tr} C^{\text{sf}} - 3) + \frac{c_2}{2} (\text{tr} C^{\text{sf}^{-1}} - 3),$$  (5)

where $c_1$ and $c_2$ are shear moduli of the material, $\rho_R$ is the mass density in both reference configurations, $C = (\det C)^{-1/3} C$ stands for the unimodular part of $C$. The second Piola–Kirchhoff stress operating on the sf-configuration is computed through

$$\mathbf{T}^{\text{sf}}_{\text{MR}} = 2\rho_R \frac{\partial \Psi_{\text{MR}}(C^{\text{sf}})}{\partial C^{\text{sf}}} = C^{\text{sf}^{-1}} \left( c_1 C^{\text{sf}} - c_2 C^{\text{sf}^{-1}} \right) D.$$  (6)

Here, $A^D$ stands for deviatoric part of a tensor.

Recalling that $F = F^{\text{sf}} F_0$ we obtain the following transformation rules

$$C^{\text{lf}} = F_0^T C^{\text{sf}} F_0, \quad C^{\text{sf}} = F_0^{-T} C^{\text{lf}} F_0^{-1}.$$  (7)

The second Piola–Kirchhoff operating on the lf-configuration thus reads

$$\mathbf{T}_{\text{MR}}^{\text{lf}} = 2\rho_R \frac{\partial \Psi_{\text{MR}}(F_0^{-T} C^{\text{lf}} F_0^{-1})}{\partial C^{\text{lf}}}$$

$$= 2\rho_R F_0^{-1} \frac{\partial \Psi_{\text{MR}}(C^{\text{lf}})}{\partial C^{\text{lf}}} F_0^{-T} = F_0^{-1} \mathbf{T}_{\text{MR}}^{\text{sf}} F_0^{-T}. $$  (8)

Obviously, stresses are zero whenever $C^{\text{sf}} = 1$.

Fibre-like hyperelasticity In order to model hyperelastic properties of a fibre family we consider the well-known Holzapfel potential (Holzapfel et al. 2000). Now it is a function of $C^{\text{sf}}$:

$$\Psi_{\text{Holzapfel}}(\lambda^2) = \frac{k_1}{2k_2} (e^{k_1(\lambda^2 - 1)} - 1),$$

$$f(\lambda^2) = 2k_1 (\lambda^2 - 1)e^{k_1(\lambda^2 - 1)},$$

where $f(\lambda^2) := \frac{d \Psi_{\text{Holzapfel}}(\lambda^2)}{d (\lambda^2)}$, $\lambda^2 = C^{\text{sf}} : (\mathbf{\tilde{a}} \otimes \mathbf{\tilde{a}})$ stands for the unit vector corresponding to the direction of fibre family in the sf-configuration, $\lambda$ is the stretch of that family (the dependence of the stretch on the choice of the sf-configuration is understood but omitted for brevity of notation), $k_1 > 0$ is a stress-like material parameter and $k_2 > 0$ is a non-dimensional parameter.
The second Piola–Kirchhoff stress operating on the sf-configuration is obtained by the chain rule
\[
\mathbf{T}_{\text{sf}}^{\text{Holzapfel}} = 2\rho R \frac{d}{d(\lambda^2)} \left( \mathbf{C}_{\text{sf}} : \mathbf{M} \right)
\]
\[
= 2\rho R (\lambda^2) \mathbf{P} \mathbf{C}_{\text{sf}} : \mathbf{M},
\]
where \(\mathbf{P} : \mathbf{X} = \mathbf{X} - \frac{1}{3} \text{tr}(\mathbf{C}_{\text{sf}} \mathbf{X}) \mathbf{C}_{\text{sf}}^{-1}\) and \(\mathbf{M} = \tilde{\mathbf{a}} \otimes \tilde{\mathbf{a}}\) stands for the structural tensor of the fibre family. Note that \(\mathbf{M}\) operates on the sf-configuration.

In case of fibres, similar relations to (8) govern the transformation of the second Piola–Kirchhoff stresses:
\[
\mathbf{T}_{\text{lf}}^{\text{Holzapfel}} = 2\rho R \frac{\partial \Psi_{\text{Holzapfel}}}{\partial \mathbf{C}_{\text{lf}}} \mathbf{F}_0^{-1} \mathbf{F}^{\text{sf}} - \mathbf{T}^{\text{sf}}_{\text{Holzapfel}} = 2\rho R \mathbf{F}_0^{-1} \frac{\partial \Psi_{\text{Holzapfel}}}{\partial \mathbf{C}_{\text{sf}}} \mathbf{F}_0^{-T} = \mathbf{F}_0^{-1} \mathbf{T}^{\text{sf}}_{\text{Holzapfel}} \mathbf{F}_0^{-T}.\]

### 2.2 Isotropic and fibre-like Maxwell bodies with pre-stresses

In the composite model advocated in Tagiltsev et al. (2018) viscoelastic material models are based on the Sidoroff multiplicative decomposition of deformation gradient tensor \(\mathbf{F}\) into the elastic part \(\mathbf{F}_e\) and the inelastic part \(\mathbf{F}_i\)
\[
\mathbf{F} = \mathbf{F}_e \mathbf{F}_i.\]

As already discussed (see Eq. (2)), respectively to the sf-configuration it takes the form \(\mathbf{F}^{\text{sf}} = \mathbf{F}_e \mathbf{F}_i^{\text{sf}}\). This decomposition gives rise to the inelastic right Cauchy–Green tensor operating on the sf-configuration
\[
\mathbf{C}_{\text{sf}} = \mathbf{F}_i^{\text{sf}} \mathbf{F}_i.\]

Two models of the Maxwell body will be considered. Both models are thermodynamically consistent and objective.

**Isotropic Maxwell body** Within the iso-strain approach, the viscous part of the stress response provided by the matrix will be described using the isotropic Maxwell model. The constitutive assumptions are taken from the work of Simo and Miehe (1992). The representation on the reference configuration follows the paper (Lion 1997). The elastic properties of the isotropic Maxwell body are described with the neo-Hookean potential. On the sf-configuration we have
\[
\psi_{\text{neo-Hooke}} = \psi_{\text{neo-Hooke}}(\mathbf{C}_{\text{sf}} \mathbf{C}_{\text{sf}}^{-1}) = \frac{\mu_{\text{matrix}}}{2\rho R} \left( \text{tr}(\mathbf{C}_{\text{sf}} \mathbf{C}_{\text{sf}}^{-1}) - 3 \right).
\]
\[
\mathbf{T}_{\text{neo-Hooke}}^{\text{sf}} = 2\rho R \frac{\partial \psi_{\text{neo-Hooke}}}{\partial \mathbf{C}_{\text{sf}}} \left|_{\mathbf{C}_{\text{sf}} = \text{const}} \right. = \mu_{\text{matrix}} \mathbf{C}_{\text{sf}}^{-1} \left( \mathbf{C}_{\text{sf}} \mathbf{C}_{\text{sf}}^{-1} \right) D.
\]

The evolution equation and the initial condition for \(\mathbf{C}_{\text{sf}}\) take the form (see eq. (47.1) in Lion (1997) and eq. (14) in Shutov et al. (2013))
\[
\mathbf{C}_{\text{sf}}(t) = \frac{1}{\eta_{\text{matrix}}} \left( \mathbf{C}_{\text{sf}}^{\text{visc}} \mathbf{T}_{\text{neo-Hooke}}^{\text{sf}} \right) \mathbf{D} \mathbf{C}_{\text{sf}}^{\text{visc}}, \quad \mathbf{C}_{\text{sf}}|_{t=0} = \mathbf{C}_{\text{0},\text{sf}}.
\]

Here, \(\mu_{\text{matrix}}\) is the shear modulus of the matrix material, \(\eta_{\text{matrix}}\) is the material viscosity.

For the second Piola–Kirchhoff stress the transformation relations similar to (8) are valid
\[
\mathbf{T}_{\text{neo-Hooke}}^{\text{sf}} = \mathbf{F}_0^{-1} \mathbf{T}_{\text{neo-Hooke}}^{\text{sf}} \mathbf{F}_0^{-T}.\]

An efficient implicit non-iterative time-stepping scheme for this model was suggested in Shutov et al. (2013). An extended model employing the Mooney–Rivlin potential can be treated using the algorithm suggested in Shutov (2018).

**Fibre-like Maxwell body** In order to account for the viscous part of the stress response provided by fibres, fibre-like Maxwell body is needed. The representation of the fibre-like Maxwell body on the reference configuration is given in Tagiltsev et al. (2018). We make a non-restrictive assumption that the fibres are not rotated by the inelastic deformation: \(\mathbf{F}_i^{\text{sf}} \mathbf{\tilde{a}} = \lambda \mathbf{\tilde{a}}\). Here, \(\lambda\) is the inelastic stretch of the fibre seen from the sf-configuration. This assumption allows us to define the elastic stretch as \(\lambda_e = \|\mathbf{F}_e \mathbf{\tilde{a}}\|\). The Sidoroff decomposition naturally yields the following split for stretches: \(\lambda = \lambda_e \lambda_i\). The model of fibre-like Maxwell body employs the aforementioned Holzapfel energy storage function (9) operating with elastic stretch of fibre:
\[
\psi_{\text{viscFibre}}(\lambda_e^2) = \frac{k_{1,\text{visc}}}{k_{2,\text{visc}}} (e^{k_{2,\text{visc}}(\lambda_e^2 - 1)} - 1),
\]
\[
f = 2k_{1,\text{visc}}(\lambda_e^2 - 1)e^{k_{2,\text{visc}}(\lambda_e^2 - 1)^2}.
\]

For the second Piola–Kirchhoff stress on the sf-configuration we obtain
while the evolution equation and the initial condition for inelastic stretch take the form (see eq. (33) in Tagiltsev et al. (2018))

\[
\dot{\lambda}_i = \frac{1}{\eta_{\text{fibre}}} f\left(\left(\frac{\lambda}{\lambda_i}\right)^2\right) \cdot \frac{\lambda^2}{\lambda_i^2} \rho_R, \quad \lambda_1 \big|_{t=0} = \lambda_0^i.
\]  

(19)

The transition between the two reference configurations is governed by

\[
\tilde{T}_{\text{viscFibre}}^{\text{sf}} = F_0^{-1} T_{\text{viscFibre}}^{\text{sf}} F_0^{-T}.
\]  

(20)

An efficient algorithm for the fibre-like Maxwell body was suggested and tested in Tagiltsev et al. (2018) and Shutov and Tagiltsev (2019); its extension to elasto-viscoplasticity was analysed in Shutov and Tagiltsev (2019). An efficient numerics is especially important when a big number of fibre families is considered for a greater accuracy (Jin and Staniciulescu 2016).

**Initial conditions** In case of a pre-stressed material the initial conditions should be chosen in a proper way. As is seen from (15) and (19) the required initial values are the inelastic right Cauchy–Green tensor \( C_0^i \) (corresponding to the isotropic Maxwell body) and the inelastic stretch \( \lambda_0^i \) (corresponding to the fibre-like Maxwell body). We assume that the viscous components are in their relaxed state in the \( \text{lf} \)-configuration, which yields \( C_0^i = 1 \) and \( \lambda_0^e = 1 \). This is equivalent to \( C_0^i = C_0^{\text{sf}} \big|_{t=0} \) and \( \lambda_0^e = \lambda \big|_{t=0} \).

Remark 2 An important aspect is a reliable identification of material parameters contained in the model, based on available experimental data. As shown in Tagiltsev et al. (2018), the parameters of the composite model studied here can be identified in a robust way by fitting the theoretical structural response to the macroscopic measurements.

3 Interrelation between the \( F_0 \)-approach and the “opening angle” approach

Consider a special case when the body under analysis is a thick-walled tube. In Holzapfel et al. (2000) a link between \( \tilde{K}_0^i \) and \( \tilde{K}_0^\text{lf} \) is considered as a deformation \( \chi_{\text{res}} \) which brings the open sector of the tube to the closed tube. The body in the \( \text{sf} \)-configuration is described in terms of cylindrical coordinates by

\[
R_i < R < R_o, \quad 0 < \Theta < (2\pi - \alpha), \quad 0 < Z < L,
\]  

(21)

where \( R_i, R_o, \alpha \) and \( L \) denote the inner and outer radii, the opening angle and the length of unstressed tube, respectively. The geometry of the \( \text{lf} \)-configuration of the tube is defined by

\[
r_i < r < r_o, \quad 0 < \theta < 2\pi, \quad 0 < z < l,
\]  

(22)

where \( r_i, r_o \) and \( l \) denote the inner and the outer radii and the length of the tube. Both configurations are schematically shown in Fig. 4. Effectively, the deformation \( \chi_{\text{res}} \) coincides with \( F_0^{-1} \). Now, our goal is to specify the form of \( F_0 \) for that case.
Remark 3 We use here the definition of the opening angle from Holzapfel et al. (2000), where it is measured from the centre of the internal radius. This is a convenient definition in case of \( \alpha < 360^\circ \) (even for negative values of the angle). However, for \( \alpha \geq 360^\circ \) additional discussion is required, even though the advocated approach is still able to model required geometry; a more conventional definition of the opening angle (where opening angle \( \omega \) is measured from the midpoint of the opened arc) should be useful. For geometric reasons we have \( 2\omega = \alpha \) when \( \alpha < 360^\circ \). For the case of \( \alpha = 360^\circ \) (\( \omega = 180^\circ \)) opened vessel is considered as a rectangular block. If the vessel opens up to the point where \( \alpha \) loses its geometrical meaning, the use of \( \omega \) is advised. From this point the vessel has the geometry of reversed opened sector, which can be described in the cylindrical coordinates with a virtual opening angle \( \alpha^* = 2 \times (2\pi - \omega) \).

### 3.1 Procedure for setting \( F_0 \)

The geometry relations for the reference configurations in the “opening angle” approach are described via

\[
 r = \sqrt{\frac{R^2 - R_i^2}{k\zeta} + r_i^2}, \quad \theta = k\Theta, \quad z = \lambda_z Z, \tag{23}
\]

where \( k = 2\pi/(2\pi - \alpha) \). In terms of cylindrical coordinates with the orthonormal basis of \( \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\} \) the deformation gradient \( F_0 \), which brings the sf-configuration to the sf-configuration takes the matrix form

\[
 F_0 = \begin{pmatrix}
 c\zeta & 0 & 0 \\
 0 & 1/\zeta & 0 \\
 0 & 0 & 1/c \\
 \end{pmatrix}, \tag{24}
\]

where \( c = l/L \) is the length ratio and the function \( \zeta \) describes the distribution of strains. Given the incompressibility condition, \( \zeta \) can be written in both reference configurations:

\[
 \zeta_{\text{ii}}(r) = k \frac{r}{\sqrt{(r^2 - r_i^2)kc + R^2}}, \tag{25}
\]

\[
 \zeta_{\text{o}}(R) = k \frac{\sqrt{(R^2 - R_i^2)/(kc) + r_i^2}}{R}. \]

As is seen from the equations, geometrical parameters in both stress-free and load-free configurations uniquely define \( F_0 \). If some of the parameters \( R_i, R_o, L \) are unknown, they can be obtained by solving an inverse problem. Such a situation occurs, for example, when it is impossible to cut the artery to inspect its stress-free configuration. Let us assume that \( l, r_i, r_o, \alpha \) and material parameters of the specimen are known. We need thus to find \( L \) and \( R_i \) in order to obtain \( F_0 \); in addition, \( R_o \) is also required to properly set the kinematics.

Dealing with tubes made of an incompressible material, the entire kinematics of the body is a function of two scalar parameters. This gives rise to the semi-analytical procedure described in Sect. 3.2 of Holzapfel et al. (2000) and in Appendix A of Tagiltsev et al. (2018). It allows one to calculate the internal pressure \( p_i(L, R_i, R_o) \) and the reduced axial force \( F(L, R_i, R_o) = N - \pi R_i^2 p_i \) (where \( N \) is a total axial force) which correspond to the deformation of the tube from its stress-free configuration.

For the inverse problem, the overall system of equations is as follows

\[
 \begin{align*}
 p_i(L, R_i, R_o) &= 0 \\
 F(L, R_i, R_o) &= 0 \\
 L(2\pi - \alpha)(R_o^2 - R_i^2) &= l \cdot 2\pi(r_o^2 - r_i^2), \tag{26}
\end{align*}
\]

where the first two equations control the absence of loads in the sf-configuration and the third equation stands for the incompressibility of the material. In the current study, system (26) is solved numerically using Newton–Raphson method. Note that the incompressibility condition may be accounted for analytically, by taking \( L = l \times k(r_i^2 - r_o^2)/(R_o^2 - R_i^2) \) in formula for \( F_0 \). This step yields a reduced system with only two equations, which can be solved more efficiently.

In case of incompressibility assumed here, the presented procedure for finding \( F_0 \) does not depend on specific material properties. Therefore, it can be used for multi-layered composite tube as well.

### 3.2 FEM simulation: cutting of an artery

To demonstrate the interrelation between approaches to residual stresses described above, FEM simulation of cutting of an artery with a scalpel is carried out. The artery is modelled as a double-layered viscoelastic composite tube. The inner and outer layers correspond to media and adventitia, respectively. Each layer is reinforced by two families of fibres. Fibres are arranged in symmetrical helices; they are inclined at the angles \( \pm \beta \) to the hoop (cylindrical) direction in stress-free configuration (see Fig. 4). Note that \( F_0 \) changes orientation of the fibre; thus if the fibre inclination is measured in load-free configuration as \( \tilde{\beta} \) the stress-free angle is as follows:

\[
 \beta = \arctan(\tan(\tilde{\beta}) \cdot F_0(z, z) / F_0(0, \theta, \theta)) = \arctan(\tan(\tilde{\beta}) \cdot \zeta/c). \]

In the absence of viscous stresses (which corresponds to the quasi-static loading) the model reduces to the purely hyperelastic one, previously considered in Holzapfel et al. (2000) to describe the behaviour of an artery. The material model of the viscoelastic composite is implemented into the commercial FEM code MSC. MARC using the Hypela2
interface for user-defined models. Efficient stress computation algorithms proposed in Shutov et al. (2013) and Tagiltsev et al. (2018) for the isotropic and fibre-like Maxwell bodies are used, respectively. The inner layer (media) is discretized by $10 \times 12 \times 3$ elements in the radial, circumferential and axial directions, respectively. The outer layer (adventitia) is subdivided into $5 \times 12 \times 3$ elements. The used elements are three-dimensional bricks of Hex20 type (twenty-node elements with a quadratic approximation of the geometry and displacements). Herrmann formulation is employed, which includes one extra degree of freedom for pressure. It allows one to model a nearly incompressible material behaviour in a consistent way thus preventing unphysical volumetric locking. Dynamic transient problem statement is considered; to suppress undesired oscillations additional damping is applied. The initial velocity field is equal to zero throughout the body. Kinematic and material parameters of the simulation are listed in Tables 1 and 2, respectively. Geometric and hyperelastic parameters for the model are taken from Holzapfel et al. (2000). Parameters for the viscoelastic constituents of the model were chosen in such a way to obtain realistic behaviour upon cutting. The initial conditions for internal variables are set according to Sect. 2.2; the $\mathbf{F}_p$-field is pre-defined in the form (24). The simulation is performed for the total time $t_{\text{total}} = 10s$ with the constant step size $\Delta t = 2.5 \times 10^{-2}s$, which requires

### Table 1 Kinematic parameters for the simulation in Sect. 3.2

| Parameter | Media | Adventitia |
|-----------|-------|------------|
| $r_c$, mm | 0.71  | 0.97       |
| $r_{\text{interface}}$, mm | 1.1   | 1.3948     |
| $l$, mm | 3.0   | 1.6589     |
| $R_c$, mm | 1.3948 | 1.8024     |
| $L$, mm | 2.9251 |            |
| $\alpha$, degrees | 160     |            |

### Table 2 Material parameters for the simulation in Sect. 3.2

| Parameter | Media | Adventitia |
|-----------|-------|------------|
| $c_1$, KPa | 3.0     | 0.3        |
| $c_2$, KPa | 2.0     | 0.2        |
| $\eta_{\text{matrix}}$, KPa $\times$ s | $5 \times 10^{-1}$ | $1 \times 10^{-1}$ |
| $\mu_{\text{matrix}}$, KPa | 5       | 1          |
| $\beta$, degrees | 29      | 62         |
| $k_1$, KPa | 2.3632  | 0.562      |
| $k_2$ | 0.8393  | 0.7112     |
| $k_{1,\text{visc}}$, KPa | 5.3     | 1.3        |
| $k_{2,\text{visc}}$ | 0.8393  | 0.7112     |
| $\eta_{\text{fibre}}$, KPa $\times$ s | $5.3 \times 10^{-1}$ | $1.3 \times 10^{-1}$ |

Fig. 5 FEM simulation of artery cutting with scalpel. The equivalent von Mises stress of the Cauchy stress tensor (MPa) for four different time instances is shown.
400 time steps. The motion of the perfect scalpel is modelled by deliberate removing links between two contacting surfaces along the cut. Each link is represented by an extra element of the same Hex20 type with material parameters of the adjacent bricks, but without Hermann formulation; its removal is simulated by a fast reduction of the element stiffness by five orders of magnitude. Deformed configurations and corresponding stress distributions at steps #1, #100, #136 and #400 are shown in Fig. 5. For the animation of the artery cutting, the reader is referred to https://youtu.be/8v_2RtHuqUg. Two more simulations are provided for the same model. First, cutting the half of the artery along the hoop direction is represented by separated nodes in the cut area \( r_i < r < r_o, \quad 0 < \theta < \pi, \quad z = l/2 \); the animation is available under https://youtu.be/iQeY9UvtflI. Second, cutting of the outer layer (adventitia) in the hoop direction (separated nodes in the area \( r_{\text{interface}} < r < r_o, \quad 0 < \theta < 2\pi, \quad z = l/2 \)) can be viewed under https://youtu.be/6FLOA-50e7k.

It is worth emphasizing that in the simulation the artery behaves smooth from the start at the first simulation step \( #1, t = \Delta t \). This indicates that the initial state is at equilibrium, as expected. The final step of the simulation corresponds to the maximum opening angle \( \alpha^* = 152^\circ \), which is close to the pre-defined angle \( \alpha = 160^\circ \). An additional FEM simulation is carried out with a finer mesh, where the element size is two times smaller. The opening angle in the refined simulation ranges up to 158°, which is even closer to the analytical value. Thus, we conclude that “opening angle” approach can be efficiently described by a more general \( F_0^0 \)-approach.

### 3.3 FEM simulation: behaviour under loading

Let us now consider the same vessel under physiological conditions which include an axial stretch and internal pressure. FEM model is utilized with the same mesh for the vessel as in the previous subsection. The simulated process can be subdivided into three stages. In the beginning, the vessel is being pre-stretched in axial direction, whereas its inner radius is decreasing due to the material incompressibility. Then, the vessel is held fixed in order to relax the viscous stresses. Finally, it is loaded by internal pressure up to 100 mmHg (13.33 KPa); the loading rate is so small that the viscous effects are negligible. The relation between the applied pressure and internal radius of the vessel is shown in Fig. 6 for three simulations with different values of axial pre-stretch. The value \( \lambda_z = 1.5 \) is physiological for that specific vessel (cf. Chuong and Fung 1983; Holzapfel et al. 2000); additionally, \( \lambda_z = 1.5 \) and \( \lambda_z = 1.9 \) are chosen for comparison. Obtained results are in good agreement with the results from Holzapfel et al. (2000).

### 3.4 Variations of the opening angle along the vessel

A general modelling of a real blood vessel should include the variation of the opening angle along the vessel (cf. Sokolis 2019). Let us fix load-free configuration and material...
parameters in the problem (26); now we need to establish a relation between stress-free configuration and opening angle for an arbitrary ring. Solving (26) we find \( R_i \) and \( L_i \) as functions of \( \alpha \in [120^\circ, 200^\circ] \); the values found from the inverse problem and their interpolations by a quadratic function are shown in Fig. 7.

Next, each Gauss point in the FEM simulation is described in user-defined model with a prescribed continuous dependence of the \( \mathbf{F}_0 \)-field on the axial coordinate \( z \) using the aforementioned quadratic interpolations (see Fig. 7). As a demonstration, we model a vessel with the same load-free radii (see Table 1) and the same material parameters (Table 2), but with doubled length \( l = 6.0 \) mm and non-constant opening angle \( \alpha(z) = 120^\circ + 80^\circ \times z/6.0 \). The whole model is left to relax after an instant cut at the very first time instance. The deformation becomes nearly static at \( t = 20 \) s. Deformed configuration and corresponding stress distribution at the last time step are shown from two different angles in Fig. 8. The simulation shows that different rings of the vessel lock one another leading to the incomplete release of residual stresses.

4 Locking of layers with different opening angles

In a general case, different layers of a composite tube may experience different strains upon unloading. Thus, the unloaded global configurations are, in general, incompatible: different layers in the load-free state do not fit together. An extra load along the interface occurs if the layers are glued together to form a connected composite. Therefore, the pre-stresses are defined by an interplay between the geometric mismatch of layers in the unstressed state and the corresponding material parameters. The current section focuses on the influence of the geometry on residual stresses.

Let us consider a composite tube consisting of two layers. We refer to them as media (M) and adventitia (A) like in the previous section. In contrast to the previous section, now each layer has different opening angles, which brings the layer to its stress-free configuration. We denote the inner and outer radii, the length and the opening angle of the layers in the sf-configuration as \( R_i^{\text{type}_i}, R_o^{\text{type}_o}, L_i^{\text{type}_i}, \) and \( \alpha_i^{\text{type}_i}, \) respectively (\( \text{type} \in \{M, A\} \)). In general, in their load-free states there is a mismatch between the layers. Per definition, the connected composite tube which is free from external loads occupies the load-free configuration. Note that this sf-configuration does not have to coincide with any of individual sf-configurations of media and adventitia. The inner and outer radii of the composite tube as well as the interface position are given by \( r_i, r_o, r_{\text{interface}} \); respectively; the length equals \( l \). The following problem arises in a natural way: for known \( R_i^{\text{type}_i}, R_o^{\text{type}_o}, L_i^{\text{type}_i}, \) and material parameters one needs to find \( r_i, r_o, r_{\text{interface}}, \) and to describe the mechanical behaviour of the pre-stressed composite.

Remark 4 Note that this problem statement differs conceptually from the one considered in Grobbel et al. (2018) and Mousavi and Avril (2017). In contrast to our case, in these papers a number of different unstressed configurations pertaining to different constituents are considered.

4.1 Finding \( r_i, r_o, r_{\text{interface}} \)

Applying the semi-analytical procedure which is already mentioned in Sect. 3.1 to both layers we formulate the problem as follows. In the absence of external loads pre-stressed composite tube remains in equilibrium state after two layers are glued together, see Fig. 9. Denote the deformation of the media-layer from its stress-free configuration
to the state of circular tube by $F_M$; let $l$ and $r_{\text{interface}}$ be the length and the outer radius of the media-layer within the tube. Analogously, $F_A$ stands for the deformation of the adventitia-layer from its sf-configuration to the tube with the length $l$ and the inner radius $r_{\text{interface}}$. Given the incompressibility of the material, the inner radius of the media and the outer radius of the adventitia (which are inner and outer radii of the composite tube) are explicitly expressed as

$$r_i = \sqrt{\frac{r^2_{\text{interface}}}{l} - \frac{L_M}{2\pi} \frac{2\pi - \alpha_M}{2} (R^2_{o,M} - R^2_{i,M})},$$

$$r_o = \sqrt{\frac{r^2_{\text{interface}}}{l} + \frac{L_A}{2\pi} \frac{2\pi - \alpha_A}{2} (R^2_{o,A} - R^2_{i,A})}.$$  

(27)

The equilibrium of the composite tube means that after the total deformation of the layers from their sf-configurations the difference between the internal and external pressure should be zero:

$$p_{i,M} - p_{o,A} = \int_{r_i}^{r_o} \frac{\mathbf{T}_{\theta\theta} - \mathbf{T}_{rr}}{r} \, dr = 0.$$  

(28)

Moreover, we require that the reduced axial force is zero as well:

$$F = \pi \int_{r_i}^{r_o} (2T_{zz} - T_{\theta\theta} - T_{rr}) \, dr = 0.$$  

(29)

The reduced force equals the applied axial force minus the force exerted by the internal pressure on the sealed ends of the tube.

The system of equations (28),(29) is solved numerically with respect to unknown length of the composite tube $l$ and the radius of contact surface between layers $r_{\text{interface}}$. The reader interested in details of numerical implementation is referred to Tagiltsev et al. (2018). This semi-analytical procedure is used to compute the geometric parameters $r_i$, $r_{\text{interface}}$, $r_o$, and $l$. After that, they are used to define prestressed state within general FEM simulations. Such an FEM simulation of a two-layered structure will be discussed in the following subsection.

### 4.2 Locking of composite layers

Kinematic parameters of the model are found using the procedure described above and are listed in Table 3. For material parameters of each layer see Table 4. Knowledge of the kinematic parameters allows one to obtain $F_0$ in the form (24); it will be used both in semi-analytical and FEM computations.
First, we solve the problem by the semi-analytical procedure. Thus, we neglect edge effects assuming that for each opening angle $\tilde{\alpha}$ the composite tube preserves its circular form. The opening of the composite tube is described by the deformation gradient $\mathbf{F} = \text{diag}(\lambda_r, \lambda_\theta, \lambda_z)$, where $\lambda_r$, $\lambda_\theta$, and $\lambda_z$ are material stretches in the radial, hoop and axial directions, respectively. For the semi-analytical approach we choose $\lambda_r = \lambda_\theta$ which means that the composite tube preserves its circular form, $\tilde{\alpha} = 90^\circ$. The opening of the composite tube is essentially smaller than the opening angles of individual layers. Indeed, the final angle approximately equals $118 \pm 3$ degrees, which is close to the predictions of the semi-analytical procedure. A minor mismatch between the semi-analytical result and the FEM solution is caused by neglected edge effects and FEM discretization. We conclude that the discovered locking effect is not a consequence of the simplified kinematics adopted in the semi-analytical procedure.

To check this counterintuitive result, additional FEM computations are carried out. In the FEM simulation the artery is cut along the axial direction at the very first time step (instant cut). The type of the elements and the applied boundary conditions are the same as in Sect. 3.2; each layer is discretized now by $10 \times 24 \times 3$ elements in the radial, hoop and axial directions, respectively. The simulation is performed for the total time $t_{\text{total}} = 5s$ with the constant step $5 \times 10^{-2}s$.

Deformed configuration and corresponding stress distribution of the composite for the last step are shown in Fig. 11 from two different viewing angles. The last step is close to the asymptotic state where the body is at rest. As is seen, stresses do not vanish completely in the final state. The stresses remain after the cut since the opening angle of the composite tube is essentially smaller than the opening angles of both layers. Indeed, the final angle approximately equals $118 \pm 3$ degrees, which is close to the predictions of the semi-analytical procedure. A minor mismatch between the semi-analytical result and the FEM solution is caused by neglected edge effects and FEM discretization. We conclude that the discovered locking effect is not a consequence of the simplified kinematics adopted in the semi-analytical procedure.

To the best of our knowledge, the locking effect was not reported in the literature before. We refer to this effect as to the mutual locking of layers. The locking appears due to the incompatible kinematics of unstressed layers.

Note that the considered problem setting does not cover all possible situations: opening angle of media may be either larger than the opening angle of adventitia (Amabili et al. 2019) or lower (Sokolis 2019).

![Fig. 10](image)

**Fig. 10** Dependence of the total free energy $E$ of the composite on the opening angle $\tilde{\alpha}$; the minimum of the free energy corresponds to the real opening angle of the composite.

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### Table 3 Kinematic parameters for the simulation in Sect. 4.2

| Parameter | Media | Adventitia |
|-----------|-------|------------|
| $R_M$, mm | 1.0   | 1.0        |
| $R_{o,M}$, mm | 1.4 | 1.3        |
| $L_M$, mm | 1.0   | 1.0        |
| $a_M$, degrees | 160 | 140        |
| $R_A$, mm | 1.5   | 1.0        |
| $R_{o,A}$, mm | 1.8 | 1.0        |
| $L_A$, mm | 1.0   | 1.0        |
| $a_A$, degrees | 140 | 120        |
| $r_c$, mm | 0.4852 | 0.4852     |
| $r_{\text{interface}}$, mm | 0.8749 | 0.8749     |
| $r_o$, mm | 1.1691 | 1.1691     |
| $l$, mm | 1.0063 | 1.0063     |

### Table 4 Material parameters for the simulation in Sect. 4.2

| Parameter | Media | Adventitia |
|-----------|-------|------------|
| $c_1$, KPa | 3.0   | 0.3        |
| $c_2$, KPa | 2.0   | 0.2        |
| $\eta_{\text{matrix}}$, KPa x s | 5     | 1          |
| $r_{\text{matrix}}$, KPa | 5     | 1          |
| $\beta$, degrees | 29 | 62        |
| $k_1$, KPa | 2.3632 | 0.562      |
| $k_2$, KPa | 0.8393 | 0.7112     |
| $k_{1,\text{visc}}$, KPa | 5.3   | 1.3        |
| $k_{2,\text{visc}}$, KPa | 0.8393 | 0.7112     |
| $\eta_{\text{fibres}}$, KPa x s | 5.3   | 1.3        |

---

\[ \alpha = \arg \min E(\tilde{\alpha}). \]  

(30)

The energy is plotted versus the angle $\tilde{\alpha}$ in Fig. 10. The optimum value of the angle is $\alpha = 120$ degrees. This is much smaller than the opening angles of individual layers: 160 degrees for the media and 140 degrees for the adventitia.
5 Conclusion

A geometrically exact approach to the modelling of viscoelastic composites is considered. A special field $\mathbf{F}_0$ is used to account for the presence of residual stresses in the considered structure. Given this field one can describe the kinematics of the material particle employing two different reference configurations (stress-free and load-free configurations are used in the current paper). The usefulness of the $\mathbf{F}_0$-approach to the multiplicative inelasticity (based on the Sidoroff decomposition of the deformation gradient) is demonstrated; the hyperelastic material behaviour is covered as a special case. The $\mathbf{F}_0$-field can be interpreted as both inelastic and elastic deformations which appear in the multiplicative decomposition. The positive feature of the $\mathbf{F}_0$-approach is that well-established numerical algorithms still can be used; the introduction of pre-stresses does not increase the complexity of the numerical schemes.

The existence of pre-stresses is especially important in modelling of biological tissues. One of the first attempts to take the pre-stresses into account was the “opening angle” approach (used in Holzapfel et al. (2000)), which is based on the kinematic difference between stress-free and load-free configurations for the whole circular segment of an artery. It is shown that the advocated $\mathbf{F}_0$-approach is capable of reproducing the “opening angle” approach as a special case. Explicit expressions for the $\mathbf{F}_0$-field in such a case are provided. As a demonstration of the interrelation between mentioned approaches, FEM simulations of cutting an artery are performed. The number of material parameters which appear in the considered material model is relatively low and they possess a clear mechanical interpretation. As a result, a simple modelling approach is obtained on the macro-scale which has a potential to become a practical clinical tool (Nappi et al. 2016).

In contrast to a wide-spread assumption that a sliced artery is stress free (Chuong and Fung 1986; Fung 1993; Humphrey and Delange 2004), our simulation indicates that the stresses do not vanish completely. This theoretical result is also confirmed by experiments reported in Greenwald et al. (1997). Moreover, semi-analytical and FEM solutions show the counterintuitive result that the opening angle of the composite tube is significantly smaller than the opening angles of each individual layer. The discovered effect is referred to as the mutual locking of layers. It is explained by incompatible kinematics upon unloading. An important implication of this locking is that the cutting test by itself does not carry enough information about constituents of the composite.

An essential advantage of the $\mathbf{F}_0$-approach over the conventional “opening angle” approach lies in its flexibility. In particular, different $\mathbf{F}_0$-fields can be used not only for different layers, but also for the different constituents like matrix and fibre. Experimental findings indicate different opening angles for different constituents, like collagen fibres and myocytes (Grobbel et al. 2018). Also, the growth and remodelling of soft tissues can be taken into account by a proper evolution equation for $\mathbf{F}_0$. In the follow-up studies the applicability of the $\mathbf{F}_0$-approach to complex biological structures will be investigated.

![Fig. 11 Deformed configuration at final step, showing the mutual locking effect. The equivalent von Mises stress of the Cauchy stress is shown (MPa)](image_url)
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