Non-orientable genus of a knot in punctured $\mathbb{C}P^2$

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Abstract

For any knot $K$ which bounds non-orientable and null-homologous surfaces $F$ in punctured $n\mathbb{C}P^2$, we construct a lower bound of the first Betti number of $F$ which consists of the signature of $K$ and the Heegaard Floer $d$-invariant of the integer homology sphere obtained by 1-surgery along $K$. By using this lower bound, we prove that for any integer $k$, a certain knot cannot bound any surface which satisfies the above conditions and whose first Betti number is less than $k$.

1 Introduction

Throughout this paper, we work in the smooth category, all 4-manifolds are orientable, oriented and simply-connected, and all surfaces are compact. If $M$ is a closed 4-manifold, $\text{punc} M$ denotes $M$ with an open 4-ball deleted.

The non-orientable 4-genus $\gamma_4(K)$ of a knot $K$ is the smallest first Betti number of any non-orientable surface in $B^4$ with boundary $K$. It has been investigated in [2], [3], [5], [12], and [13]. In this paper, we extend the definition of $\gamma_4(K)$ to any 4-manifold with boundary $S^3$.

Definition

Let $M$ be a closed 4-manifold and $K \subset \partial(\text{punc} M) \cong S^3$ a knot. The non-orientable $M$-genus $\gamma_M(K)$ of $K$ is the smallest first Betti number of any non-orientable surface $F \subset \text{punc} M$ with boundary $K$.

Moreover, we define $\gamma^0_M(K)$ to be the smallest first Betti number of any non-orientable surface $F \subset \text{punc} M$ with boundary $K$ which represents zero in $H_2(\text{punc} M, \partial(\text{punc} M); \mathbb{Z}_2)$.

We note that $\gamma_4(K) = \gamma_{S^4}(K) = \gamma^0_{S^4}(K)$ and $\gamma_M(\text{unknot}) = \gamma^0_M(\text{unknot}) = 1$ for any closed 4-manifold $M$. In this paper, we consider the following problem.

Problem 1

Can $\gamma_M$ and $\gamma^0_M$ be taken arbitrarily large?

The answer of this problem depends on the choice of $M$. For example, Suzuki [11] proved that any knot bounds a disk in $\text{punc}(S^2 \times S^2)$ and $\text{punc}(\mathbb{C}P^2 \# \mathbb{C}P^2)$. It follows that $\gamma_{S^2 \times S^2}(K) = \gamma_{\mathbb{C}P^2 \# \mathbb{C}P^2}(K) = 1$ for any knot $K$. For a long time, it had been unknown whether Problem 1 even in the case $\gamma_{S^4}$ is true or not. Most recently Batson in [2] gave the affirmative answer for the problem by using the following inequality:

Theorem 1.1 (Batson, [2]) Let $K \subset S^3$ be a knot. Then

$$\gamma_{S^4}(K) \geq \frac{-\sigma(K)}{2} + d(S^3_1(K)),$$
where $\sigma$ denotes the signature of $K$ and $d(S^3_1(K))$ the Heegaard-Floer d-invariant of the integer homology sphere obtained by 1-surgery along $K$.

The $d$-invariant is defined by Ozsváth and Szabó in [6]. We extend Theorem 1.1 to the case of $\gamma^0_{n\mathbb{C}P^2}$:

**Theorem 1.2** Let $K \subset S^3$ be a knot. Then

$$\gamma^0_{n\mathbb{C}P^2}(K) \geq \frac{-\sigma(K)}{2} + d(S^3_1(K)) - n.$$  

By applying Theorem 1.2, we give the answer of Problem 1 in the case of $\gamma^0_{n\mathbb{C}P^2}$.

**Theorem 1.3** For every $k$, there exists a knot $K$ such that $\gamma^0_{n\mathbb{C}P^2}(K) = k$.

In fact, we show that $\gamma^0_{n\mathbb{C}P^2}(\# 9_{42}) = k$, where $\# 9_{42}$ denotes the $n + k$ times connected sum of the knot $9_{42}$ in Rolfsen’s table [9]. In order to prove Theorem 1.2 we first prove the following proposition.

**Proposition 1.4** Let $K \subset \partial(punc(n\mathbb{C}P^2))$ be a knot and $F \subset punct(n\mathbb{C}P^2)$ a non-orientable surface with boundary $K$. Then

$$\beta_1(F) \geq \frac{e(F)}{2} - 2d(S^3_1(K)),$$

where $e(F)$ is the normal Euler number of $F$, and $\beta_i$ denotes the $i$-th Betti number.

This proposition is an extension of Theorem 4 in [2] and independent of the homology class of $F$. However, the normal Euler number $e(F)$ depends on $F$, hence this lower bound is not enough to determine the genus. To delete $e(F)$, we use the following theorem.

**Theorem 1.5** (Yasuhara, [13]) Let $M$ be a closed 4-manifold and $K \subset \partial(punct(M))$ a knot. If $K$ bounds a non-orientable surface $F$ in $punct(M)$ that represents zero in $H_2(punct(M), \partial(punct(M)); \mathbb{Z}_2)$, then

$$\left|\sigma(K) + \sigma(M) - \frac{e(F)}{2}\right| \leq \beta_2(M) + \beta_1(F),$$

where $\sigma(M)$ is the signature of $M$.

By applying this theorem to $n\mathbb{C}P^2$ and deleting the term of $e(F)/2$, we obtain the inequality of Theorem 1.2.

**Remark.** In our arguments in Section 2 and Section 3 it is clear that the inequalities in Theorem 1.2 and Proposition 1.4 hold for any 4-manifold $P$ with intersection form $n(1)$ instead of $n\mathbb{C}P^2$, because these homological data of the ambient space determines the inequalities. For simplicity we deal with the case of $n\mathbb{C}P^2$.

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2 Proof of Proposition 1.4

In order to prove Proposition 1.4 we first prove the following lemma.

Lemma 2.1 Under the hypothesis of Lemma 1.4 moreover if $\beta_1(F)$ is odd, then there exists an orientable surface $F' \subset \text{punc}(n\mathbb{C}P^2 \# S^2 \times S^2)$ which is still bounded by $K$, and has $\beta_1(F') = \beta_1(F) - 1$ and $e(F') = e(F) + 2$.

**Proof.** Since $\beta_1(F)$ is odd, there is a curve $C$ in $F$ such that the regular neighborhood of $C$ in $F$ is a Möbius band and $F \setminus C$ is orientable. By the simply-connectedness of $\text{punc}(n\mathbb{C}P^2)$, $C$ is null-homotopic. We note that in these dimensions (i.e., for 1-manifolds in 4-manifolds) every homotopy may be replaced with an isotopy. It follows that $C$ bounds an embedded 2-disk $D$ in $\text{punc}(n\mathbb{C}P^2)$. Without loss of generality we can assume that $D$ is transverse to $F$. Then $F \cap D$ is the disjoint union of the curve $C$ and some transversal intersections $\{p_i\} \ (i = 1, 2, \ldots, l)$. Let $V(D)$ be a small regular neighborhood of $D$ in $\text{punc}(n\mathbb{C}P^2)$. $V(D)$ is diffeomorphic to $D \times D^2$, and $F \cap V(D)$ consists of one Möbius band and $l$ 2-disks $p_i \times D^2$. If we draw $\partial V(D) \cong S^3$ with its standard decomposition into solid tori $\partial V(D) \cong \partial D \times D^2 \cup_{\partial D \times S^1} D \times S^1$, we see $\partial (F \cap V(D)) \subset \partial V(D)$ as the link $L$ in Figure 1 consisting of a $(2, 2k + 1)$-cable of the core for the first factor and $l$ parallel copies of the core for the second. It was proved by [2] and [13] that $L$ bounds $l + 1$ disjoint embedded disks $E$ in $\text{punc}(S^2 \times S^2)$ such that $e(E) = e(F \cap V(D)) + 2$. We remark that $[E, \partial E] \in H_2(\text{punc}(S^2 \times S^2); \mathbb{Z})$ is $2\alpha + b\beta$ ($b \in \mathbb{Z}$), where $\alpha$ and $\beta$ are standard generators of $H_2(\text{punc}(S^2 \times S^2), \partial(\text{punc}(S^2 \times S^2)); \mathbb{Z})$ such that $\alpha \cdot \alpha = \beta \cdot \beta = 0$, and $\alpha \cdot \beta = 1$. Let $F'' = F \setminus (F \cap V(D))$. Excising $V(D)$ from $\text{punc}(n\mathbb{C}P^2)$ and capping off the pair $(\partial V(D), L)$ with a pair $(\text{punc}(S^2 \times S^2), E)$, we obtain a new orientable surface $F' = F'' \cup E$ in $\text{punc}(n\mathbb{C}P^2 \# S^2 \times S^2)$ with boundary $K$. It is easy to check that $\beta_1(F') = \beta_1(F) - 1$, and by the additivity of the normal Euler number, $e(F') = e(F) + 2$. This completes the proof. □

We note that the homology class $[F', \partial F'] \in H_2(\text{punc}(n\mathbb{C}P^2 \# S^2 \times S^2), \partial(\text{punc}(n\mathbb{C}P^2 \# S^2 \times S^2)); \mathbb{Z})$ is

$$
\sum_{i=1}^j 2a_i \gamma_i + \sum_{i=j+1}^n (2a_i + 1)\gamma_i + 2\alpha + b\beta \ (a_i, j \in \mathbb{Z}, 0 \leq j \leq n),
$$

where $\gamma_i$, are standard generators of $H_2(\text{punc}(n\mathbb{C}P^2), \partial(\text{punc}(n\mathbb{C}P^2)); \mathbb{Z})$ such that $\gamma_i \cdot \gamma_j = -\delta_{ij}$ (Kronecker’s delta). Since $F'$ is orientable, $e(F') = [F', \partial F'] \cdot [F', \partial F'] = -\sum_{i=1}^j 4a_i^2 - \sum_{i=j+1}^n (2a_i + 1)^2 + 4b$.

We next prove the following lemma. It is a generalization of a discussion in Section 3 [2].

Lemma 2.2 Let $M$ be an integer homology 3-sphere, $X$ a simply-connected 4-manifold such that $\partial X = M$ and $\beta_2^+(X) = 1$ and $\Sigma$ an orientable closed surface in $X$ with genus $g$ and self-intersection $m$. Then for any Spin’ structure $s$ of $X$ which satisfies $\langle c_1(s), [\Sigma] \rangle = m - 2g > 0$, the following inequality holds:

$$
c_1(s)^2 + \beta_2^+(X) \leq 1 + 4d(M).
$$

**Proof of Lemma 2.2** Suppose that $X' = X \setminus v(\Sigma)$, then $X'$ is a negative semi-definite 4-manifold with disconnected boundaries $Y_{g,-m} \amalg M$, where $Y_{g,-m}$ denotes the Euler number $-m$ circle bundle over $\Sigma$. We apply the following theorem to the pair $(X', Y_{g,-m} \amalg M)$. 

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Theorem 2.1 (Ozsváth and Szabó, [6]) Let \( Y \) be a closed oriented 3-manifold (not necessarily connected) with standard \( HF^\infty \), endowed with a torsion \( \text{Spin}^c \) structure \( t \). If \( X \) is a negative semi-definite four-manifold bounding \( Y \) such that the restriction map \( H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z}) \) is trivial, and \( s \) is a \( \text{Spin}^c \) structure on \( X \) restricting to \( t \) on \( Y \), then
\[
c_1(s)^2 + \beta_2(X) \leq 4d_b(Y, t) + 2\beta_1(Y).
\]

It was proved by [2] and [6] that \( Y_{g,-m} \sqcup S^3_{-1}(K) \) has standard \( HF^\infty \). Since we can verify that \( H_1(X') = 0 \) in the same way as Section 3 in [2], it follows that the pair \( (X', Y_{g,-m} \sqcup M) \) satisfies all conditions of Theorem 2.1.

Applying Theorem 2.1 to the \( \text{Spin}^c \) structure \( s|_{X'} \) on \( X' \), we have
\[
(1) \quad c_1(s|_{X'})^2 + \beta_2(X') \leq 4d_b(Y_{g,-m}, s|_{Y_{g,-m}}) + 4d(M) + 2\beta_1(Y_{g,-m}) + 2\beta_1(M).
\]

Let us compute each term in this inequality (1). In order to compute \( c_1(s|_{X'})^2 \), we decompose the intersection form of \( X \) in terms of the \( \mathbb{Q} \)-valued intersection forms on \( \nu(\Sigma) \) and \( X' \); if \( c \in H^2(X) \), then
\[
Q_X(c) = Q_{\nu(\Sigma)}(c|_{\nu(\Sigma)}) + Q_{X'}(c|_{X'}).
\]

This gives \( c_1(s)^2 = c_1(s|_{\nu(\Sigma)})^2 + c_1(s|_{X'})^2 \). Hence we have
\[
c_1(s|_{X'})^2 = c_1(s)^2 - c_1(s|_{\nu(\Sigma)})^2 = c_1(s)^2 - \frac{(m - 2g)^2}{m}.
\]

For the above \( \text{Spin}^c \) structure \( s|_{Y_{g,-m}} \), the \( d \)-invariant of \( Y_{g,-m} \) is computed in section 9 of [6]. If \( \langle c_1(s), [\Sigma] \rangle = m - 2g > 0 \), then
\[
d_b(Y_{g,-m}, s|_{Y_{g,-m}}) = \frac{1}{4} - \frac{g^2}{m} - \frac{m}{4}.
\]

After substituting all the values computed above, (1) reduces to
\[
(2) \quad c_1(s)^2 - \frac{(m - 2g)^2}{m} + \beta_2(X') \leq 4\left(\frac{1}{4} - \frac{g^2}{m} - \frac{m}{4}\right) + 4d(S^3_{-1}(K)) + 2(2g).
\]
Since \( \beta_2'(X') = \beta_2(X) \), (2) gives the inequality
\[
c_1(s)^2 + \beta_2'(X) \leq 1 + 4d(M).
\]
\[
\square
\]

**Proof of Proposition 1.4** Note that for any knot \( K \), \( d(S^3_1(K)) \geq 0 \). Hence when \( e(F) \leq \beta_1(F) \), it is clear that this proposition holds. Therefore we assume that \( e(F) > \beta_1(F) \).

We first give the proof for the case where \( \beta_1(F) \) is odd. In Lemma 3.1 we constructed an orientable surface \( F' \subset (\text{punc}(\mathbb{C}P^2 \# S^2 \times S^2)) \) with boundary \( K \subset S^3 \). Attaching a (-1)-framed 2-handle along \( K \), we have a 4-manifold \( W \) with boundary \( S^3_1(K) \) and the intersection form
\[
Q_W = \begin{pmatrix}
-1 & 0 & \ldots & 0 & 0 \\
0 & -1 & O & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & O & -1 & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix}.
\]
We may cap off \( F' \) with the core of the 2-handle to form a closed surface \( \Sigma \) with genus \( g = (b_1(F) - 1)/2 \), homology class \( \gamma_0 + \sum_{i=1}^j 2a_i \gamma_i + \sum_{i=j+1}^n (2a_i + 1) \gamma_i + 2\alpha + b\beta \), and the self-intersection number
\[
m = -1 - \sum_{i=1}^j 4a_i^2 - \sum_{i=j+1}^n (2a_i + 1)^2 + 4b = e(F) + 1 > 0.
\]

We next choose a \( \text{Spin}^c \) structure on \( W \). Since \( H^2(W) \cong \mathbb{Z}^{m+3} \) has no 2-torsion, a \( \text{Spin}^c \) structure on \( W \) is determined by its first Chern class. Fix a \( \text{Spin}^c \) structure \( s_i \) on \( W \) satisfying
\[
PD(c_1(s_i)) = \varepsilon \gamma_0 + \sum_{i=1}^n (2a_i + 1) \gamma_i + 2\alpha + 2x\beta,
\]
where
\[
x = \frac{\sum_{i=1}^j 2a_i + 2(b - g) - 1 + \varepsilon}{4}
\]
and \( \varepsilon \in \{1, -1\} \) is chosen so as to make \( x \) an integer. Since the given vector is characteristic for \( Q_W \), it corresponds to a \( \text{Spin}^c \) structure. Furthermore, \( (c_1(s_i), [\Sigma]) = m - 2g = e(F) - \beta_1(F) + 2 > 0 \). Applying Lemma 2.2 to the pair \((\overline{W}, S^3_1(K))\), we have
\[
(3) \quad c_1(s_i)^2 + \beta_2(\overline{W}) \leq 1 + 4d(S^3_1(K)).
\]
Since \( c_1(s_i)^2 = -1 - \sum_{i=1}^n (2a_i + 1)^2 + 8x = e(F) - j - 1 + 2\varepsilon - 4g \), (3) gives
\[
(4) \quad (e(F) - j - 1 + 2\varepsilon - 4g) + (n + 2) \leq 1 + 4d(S^3_1(K)).
\]
Since \(-1 \leq \epsilon, j \leq n, \) and \(2g = \beta_1(F) - 1,\) (4) reduces to the following inequality

\[
\frac{e(F)}{2} - 2d(S_3^3(K)) \leq \beta_1(F).
\] (5)

Finally, we consider the case where \(\beta_1(F)\) is even. Taking the connected sum \(F \subset \text{punc}(nC\mathbb{P}^2)\) with the standard embedding of \(\mathbb{R}P^2 \subset S^4\) whose normal Euler number is \(+2,\) and we have a non-orientable surface \(\hat{F} \subset \text{punc}(nC\mathbb{P}^2)\) with boundary \(K\) such that \(\beta_1(\hat{F}) = \beta_1(F) + 1\) and \(e(\hat{F}) = e(F) + 2.\) Since \(\beta_1(\hat{F})\) is odd, \(\hat{F}\) satisfies the inequality (5). This implies that the inequality (5) holds in this case.

This completes the proof of Proposition 1.4.

3 Proof of Theorem 1.2

By reversing the orientation of \(M,\) we obtain the following proposition. We use this proposition to prove Theorem 1.2.

Proposition 3.1 For any 4-manifold \(M\) and any knot \(K,\) the following equality holds;

\[
\gamma_0^0(M)(K) = \gamma_0^0(M)(\overline{K}),
\]

where \(\overline{K}\) denotes the mirror image of \(K.\)

Let us prove Theorem 1.2.

Proof of Theorem 1.2 Suppose that \(F \subset \text{punc}(nC\mathbb{P}^2)\) is a non-orientable surface with boundary \(K\) which represents zero in \(H_2(\text{punc}(nC\mathbb{P}^2), \partial(\text{punc}(nC\mathbb{P}^2)), \mathbb{Z}_2).\) It follows from Theorem 1.5 that

\[
\left|\sigma(K) + (-n) - \frac{e(F)}{2}\right| \leq n + \beta_1(F).
\]

Hence we have

\[
\beta_1(F) \geq \sigma(K) - \frac{e(F)}{2} - 2n.
\]

Combining this inequality with Proposition 1.4, we have

\[
\gamma_0^0(nC\mathbb{P}^2)(K) \geq \frac{\sigma(K)}{2} - d(S_3^3(K)) - n.
\]

By using this inequality and Proposition 3.1 it follows that for any knot \(K \subset \partial(\text{punc}(nC\mathbb{P}^2)),\)

\[
\gamma^0_{nC\mathbb{P}^2}(K) = \gamma^0_{nC\mathbb{P}^2}(\overline{K}) \geq \frac{\sigma(K)}{2} - d(S_3^3(K)) - n = \frac{-\sigma(K)}{2} + d(S_3^3(K)) - n.
\]

This proves Theorem 1.2.
4 Proof of Theorem 1.3

To prove Theorem 1.3, it is necessary to show \( d(S^3_m(K)) = 0 \) for some knot \( K \). Since \( d(S^3_m(\cdot)) \) is a knot concordance invariant by the result of [8], but not a homomorphism, we must prove \( d(S^3_m(K)) = 0 \) for each \( m \) and the knot \( K \). Throughout this paper, the coefficient \( \mathbb{F} \) of any Heegaard Floer homology is the field with the order 2. The coordinate \((i, j)\), as is used below, is the same as that in [7]. We denote the whole differential in the knot Floer chain complex \( CFK^\infty(K) \) by \( \partial^\infty \) and denote the differential restricted to vertical (or horizontal) lines by \( \partial^{vert} \) (or \( \partial^{hor} \) respectively). For the other differentials, we use the same notation \( \partial^\infty \).

**Proposition 4.1** For any positive integer \( m \), \( d(S^3_m(\#9_{42})) = 0 \).

**Proof of Proposition 4.1.** Due to [8], the correction term \( d(S^3(K)) \) coincides with \( \tilde{d}(S^3_p(K), [0]) \), where \( p \) is a sufficient large integer. The correction term \( \tilde{d} \) is the unshifted correction term for \( CFK^\infty(K)[\max(i, j) \geq 0] \), namely \( \tilde{d}(S^3_p(K), [0]) = d(S^3_p(K), [0]) - \frac{p-1}{4} \).

For the generators of \( CFK^\infty(\#9_{42}) \), we use the same as those in Fig.14 in [7] (see Figure 2). Let

\[
\begin{align*}
S_1 &= \{x_i \mid 1 \leq i \leq 9\}.
\end{align*}
\]

Let \( G_i \) be a graded module \( \mathbb{F}(U^{-i} \cdot x \mid x \in S_1) \), where \( x_5 \) has the Alexander grading \( gr(x_5) = 0 \). We call the chain complex \( G_0 \) a fundamental part in \( CFK^\infty \). \( U \) is the action decreasing the grading by 2. The chain complex \( CFK^\infty(\#9_{42}) \) consists of an infinite direct sum \( \bigoplus_{i \in \mathbb{Z}} G_i \) and the class \( \alpha_i := U^{-i}(x_1 + x_5 + x_9) \) is the homological generator of \( CFK^\infty(\#9_{42}) \). By the result in [7], for the large \( p \)-Dehn surgery we obtain the graded isomorphism

\[
HF_{*+\infty}^+(S^3_p(\#9_{42}), [0]) \cong H_*(CFK^\infty(\#9_{42})[\max(i, j) \geq 0]).
\]

The right hand side is easily computed as \( H_*(CFK^\infty(\#9_{42})[\max(i, j) \geq 0]) = \bigoplus_{i \geq 0} \mathbb{F} \cdot \alpha_i \) and the minimal grading is \( gr(\alpha_0) = d(S^3_p(\#9_{42}), [0]) = d(S^3_1(\#9_{42})) = 0 \). This proves the case of \( m = 1 \).

Next, to compute \( d(S^3_m(\#9_{42})) \), we consider \( CFK^\infty(\#9_{42}) \cong \bigotimes^m CFK^\infty(\#9_{42}) \). We denote the set of generators by \( S_m = \{x_i \otimes x_j \otimes \cdots \otimes x_{i_m} \mid 1 \leq i_k \leq 9\} \) and \( G_i^{(m)} = \mathbb{F}(U^{-i} \cdot x \mid x \in S_m) \). The complex \( CFK^\infty(\#9_{42}) \) is decomposed into the sum \( \bigoplus_{i \in \mathbb{Z}} G_i^{(m)} \) of the chain complexes. Hence, we may consider each homology \( H_*(G_i^{(m)} \cap \{\max(i, j) \geq 0\}) \).

Figure 2: The differential maps of the fundamental part \( G_0 \) of \( CFK^\infty(\#9_{42}) \) (Fig.14 in [7]).
We define the tensor product $y \otimes \cdots \otimes y$ to be $y^{\otimes m}$. Since the differential $\partial^{\infty}$ in $\otimes^m CFK^{\infty}(9_{42})$ is as follows:

$$\partial^{\infty}(z_1 \otimes \cdots \otimes z_m) = \sum_{k=1}^{m} z_1 \otimes \cdots \otimes \partial^{\infty}z_k \otimes \cdots \otimes z_m,$$

$H_*(G_0^{(m)})$ is generated by $U^{-l} \cdot \alpha^{\otimes m}$. In fact, $\partial^{\infty}(\alpha^{\otimes m}) = \sum_{k=1}^{m} \alpha \otimes \cdots \otimes 0 \otimes \cdots \otimes \alpha = 0$. Since the generator $\alpha^{\otimes m}$ has the unique top grading in $G_0^{(m)}$, $\alpha^{\otimes m}$ does not lie in $\operatorname{Im}(\partial^{\infty}) \cap G_0^{(m)}$. Furthermore, since $HF^{\infty}(S^3_1(\#9_{42}))$ is standard, $H_*(G_0^{(m)})$ is isomorphic to $F \langle U^{-l} \cdot \alpha^{\otimes m} \rangle \cong F$. This homology computes $H_*(G_i^{(m)} \{\max(i, j) \geq 0\})$ for sufficiently large $l$.

We consider the generators of $H_*(G_0^{(m)} \{\max(i, j) \geq 0\})$, i.e. the $l = 0$ case. Clearly, we have

$$G_0^{(m)} \{\max(i, j) \geq 0\} = G_0^{(m)} \{\max(i, j) = 0\}.$$

In this case, the chain complex $G_0^{(m)} \{\max(i, j) \geq 0\}$ is isomorphic to $G_0^{(m)} \{i = 0, j \leq 0\} \oplus G_0^{(m)} \{j = 0, i < 0\}$. The chain complex $G_0^{(m)} \{i = 0, j \leq 0\}$ is isomorphic to the $m$-times tensor product of the chain complex defined in Figure 3 i.e. this is isomorphic to the fundamental part of $CFK^{\infty}(\#3_1)$ with the top grading 0. Figure 4 is the chain complex $G_0^{(1)} \{\max(i, j) \geq 0\}$ and the sum of indicated classes in the picture presents the homological generator.

![Figure 3: The chain complex of the fundamental part of $CFK^{\infty}(3_1)$.](image)

![Figure 4: The chain complex $G_0^{(1)} \{\max(i, j) \geq 0\}$ and the homological generator.](image)

We denote the homological generators in $G_0^{(m)} \{i = 0, j \leq 0\}$ and $G_0^{(m)} \{j = 0, i \leq 0\}$ by $\alpha_1$, and $\alpha_2$ respectively. The classes $\alpha_i$ have $\alpha^{\otimes m}_3$ as a non-zero component.

Here we claim the following:

**Lemma 4.1**

$$H_*(G_0^{(m)} \{\max(i, j) \geq 0\}) \cong F \cdot \beta,$$

where $\beta = (x_5 + x_9)^{\otimes m} + (x_5 + x_1)^{\otimes m} + y_5^{\otimes m}$. The absolute grading of $\beta$ is 0.
Proof of Lemma 4.1. We show the element $\beta$ is a homological generator in the chain complex $G_{0}(\max(i, j) \geq 0)$.

Since we have $\partial^{\infty}(x_5 + x_9) = 2x_6 + x_4 = x_4, \partial^{\infty}(x_5 + x_1) = 2x_4 + x_6 = x_6$, and $\partial^{\infty}x_5 = x_6 + x_4$, we get the following

$$
\partial^{\infty}((x_5 + x_9)^{\otimes m}) = \sum_{k=1}^{m} (x_5 + x_9) \otimes \cdots \otimes (x_5 + x_9) \otimes x_4 \otimes (x_5 + x_9) \cdots \otimes (x_5 + x_9) = \sum_{k=1}^{m} x_5 \otimes \cdots \otimes x_5 \otimes x_4 \otimes x_5 \cdots \otimes x_5
$$

$$
\partial^{\infty}((x_5 + x_1)^{\otimes m}) = \sum_{k=1}^{m} (x_5 + x_1) \otimes \cdots \otimes (x_5 + x_1) \otimes x_5 \otimes (x_5 + x_9) \cdots \otimes (x_5 + x_9) = \sum_{k=1}^{m} x_5 \otimes \cdots \otimes x_5 \otimes x_5 \otimes x_5 \cdots \otimes x_5,
$$

in $G_{0}(\max(i, j) \geq 0)$. Summing these, we get the following:

$$
\partial^{\infty}((x_5 + x_9)^{\otimes m} + (x_5 + x_1)^{\otimes m} + x_5^{\otimes m}) = 0.
$$

Thus, $\beta$ is a homological generator in $G_{0}(\max(i, j) \geq 0)$.

Any homological generator $z$ in $H_{*}(G_{0}(\max(i, j) \geq 0))$ has the non-zero component $x_5^{\otimes m}$. For, if $z$ does not have the non-zero component $x_5^{\otimes m}$, then $z$ is presented by $y_1 + y_2$, where $y_1 \in G_{0}(i = 0, j < 0)$ and $y_2 \in G_{0}(j = 0, i < 0)$. However $y_1, y_2$ are not homological generators in $G_{0}(i = 0, j \leq 0)$ or $G_{0}(j = 0, i \leq 0)$ respectively. Thus there exist $\tilde{y}_1 \in G_{0}(i = 0, j \leq 0)$ and $\tilde{y}_2 \in G_{0}(j = 0, i \leq 0)$ such that $y_1 = \partial^{\infty} \tilde{y}_1$ and $y_2 = \partial^{\infty} \tilde{y}_2$ respectively. We may assume that $\tilde{y}_2$ does not have $x_5^{\otimes m}$ as a non-zero component by reducing $\alpha_i$ if necessary. This means $z = \partial^{\infty}(\tilde{y}_1 + \tilde{y}_2)$ in $G_{0}(\max(i, j) = 0)$. This contradicts the fact that $z$ is a homological generator in $G_{0}(\max(i, j) \geq 0)$. Hence any homological generator in $G_{0}(\max(i, j) \geq 0)$ has the non-zero component $x_5^{\otimes m}$. Since $z - \beta$ does not have $x_5^{\otimes m}$ as a non-zero component, it is not a homological generator. This implies $z$ is homologous to $\beta$.

Therefore we have $H_{*}(G_{0}(\max(i, j) \geq 0)) \cong \mathbb{F} \cdot \beta$ and the grading of $\beta$ is 0. \hfill \Box

By Lemma 4.1 $\beta$ is the minimal generator in $H_{*}(CFK^{\infty}(\#9_{42}), \max(m, m))$ coming from $HF^{\infty}(S_{1}^{3}(\#9_{42}))$. Therefore we have $d(S_{1}^{3}(\#9_{42})) = d(S_{p}^{3}(\#9_{42})) = gr(\beta) = 0$ \hfill \Box

Proof of Theorem 1.3. Since $\sigma(9_{42}) = -2$ and the knot signature is additive, we have $\sigma(\#9_{42}) = -2(n + k)$. Thus, by using Theorem 1.2, we have

$$
\gamma_{n \# C}^{P_{2}}(\#9_{42}) \geq \frac{-(-2(n + k))}{2} + 0 - n = k.
$$

We next construct a non-orientable surface $F_{n,k} \subset punc(n \# C^{2})$ satisfying the following:

1. $\partial F_{n,k} = \#9_{42},$
2. $\beta_{1}(F_{n,k}) = k,$ and
3. $F_{n,k}$ represents zero in $H_{2}(punc(n \# C^{2}), \partial(punc(n \# C^{2})); \mathbb{Z}_{2}).$
The cobordisms in Figure 5 and 6 give a properly embedded Möbius band $M$ in $B^4$ with the boundary $9_{42}$, and a properly embedded disk $D$ in $\text{puncCP}^2$ such that it bounds $9_{42}$ and represents zero in $H_2(\text{puncCP}^2, \partial(\text{puncCP}^2); \mathbb{Z}_2)$. Taking the boundary connected sum of $n$ copies of $(\text{puncCP}^2, D)$ and $k$ copies of $(B^4, M)$, we have a new non-orientable surface $F_{n,k}$ satisfying the above properties from (1) to (3). This completes the proof. 

\[\square\]

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