Penrose’s quasi-local mass for asymptotically anti-de Sitter space-times

Ron Kelly
Type-set and annotated by Paul Tod

May 4, 2015

Abstract

Penrose’s quasi-local mass construction is carried through for two-surfaces at infinity in asymptotically anti-de Sitter space-times. A modification of the Witten argument is given to prove a positivity property of the resulting conserved quantities.

This work formed part of Ron Kelly’s Oxford D.Phil. thesis, and the first person pronoun refers to him. It appeared in hand-written form as ‘Asymptotically anti-de Sitter space-times’ in Twistor Newsletter 20 (1985) pp11-23 but is appearing type-set for the first time here. Footnotes marked PT have been added for this version by Paul Tod, in the hope of making this work available to a wider audience.

1 The Angular Momentum Twistor

I will first briefly review the work of Ashtekar and Magnon [1]: a space-time \((\hat{M}, \hat{g}_{ab})\) is said to be asymptotically anti-de Sitter if there exists a manifold \(M\) with boundary \(\partial M\) and metric \(g_{ab}\) and a diffeomorphism from \(\hat{M}\) to \(M - \partial M\) such that

1. there is a smooth real-valued function \(\Omega\) on \(M\) such that \(g_{ab} = \Omega^2 \hat{g}_{ab}\) on \(\hat{M}\);
2. \(\mathcal{I} := \partial M\) is topologically \(S^2 \times \mathbb{R}\) and \(\Omega = 0\) on \(\mathcal{I}\);
3. \(\hat{g}_{ab}\) satisfies
   \[
   \hat{R}_{ab} - \frac{1}{2} \hat{R}\hat{g}_{ab} + \lambda \hat{g}_{ab} = -8\pi G \hat{T}_{ab}
   \]
   with \(\lambda < 0\) and where \(\Omega^{-4} \hat{T}^{ab}\) has a smooth limit on \(\mathcal{I}\);
4. write \(B_{ab}\) for the magnetic part of the Weyl tensor of \(M\) then \(B'_{ab} := \Omega^{-1} B_{ab}\) vanishes on \(\mathcal{I}\).

In fact in [1] it was assumed instead in item 3 that \(\Omega^{-3} \hat{T}^{ab}\) has a limit on \(\mathcal{I}\); the choice made here preserves the conservation equation.

It will be convenient to introduce the notation \(\hat{=}\) to mean ‘equals at \(\mathcal{I}\)’, so that for example \(\Omega \hat{=} 0\) and \(B'_{ab} \hat{=} 0\). Examination of the Bianchi identities shows that

(i) with \(s_a = \partial_a \Omega\) we have \(s^a s_a \hat{=} \lambda/3\) so that \(\mathcal{I}\) is time-like;
(ii) by modifying \(\Omega\) one can require \(\nabla_a s_b \hat{=} 0\);
(iii) \(C_{abcd} \hat{=} 0\);

\footnote{PT: available at http://people.maths.ox.ac.uk/lmason/Tn/TN1-25}
(iv) condition 4 above is equivalent to the vanishing of the Cotton-York tensor of \( \mathcal{I} \), so that \( \mathcal{I} \) is conformally flat. It is also equivalent to the condition

\[
D_a V_{bc} = 0,
\]

where \( D_a \) is the intrinsic covariant derivative of \( \mathcal{I} \) and

\[
V_{ab} = \Phi_{ab} - \Lambda g_{ab} - E_{ab},
\]

where in turn \( \Phi_{ab} = -\frac{1}{2}(R_{ab} - \frac{1}{4}Rg_{ab}) \), \( \Lambda = R/24 \) and \( E_{ab} \) is the electric part of the Weyl tensor.

From the discussion in [2], conditions (iii) and (iv) in the second list are sufficient to conclude that \( \mathcal{I} \) may be embedded in conformally-flat space-time with the same first and second fundamental forms, and thus that 3-surface twistors exist on \( \mathcal{I} \).

We may therefore think of \( \mathcal{I} \) as the conformal infinity of anti-de Sitter space which can be embedded in the Einstein static cylinder as the product of the time axis with the equatorial 2-sphere of the 3-sphere cross-sections. In the usual way, the conformal group of \( \mathcal{I} \) is the anti-de Sitter group \( O(2,3) \) and there are ten linearly independent conformal Killing vectors of \( \mathcal{I} \).

Hawking [3] has shown that condition 4 in the first list is equivalent to the assumption that gravitational radiation satisfies a reflective boundary condition at \( \mathcal{I} \).

Following [1], given a cross-section \( C \) of \( \mathcal{I} \) and a conformal Killing vector \( \xi^a \) of \( \mathcal{I} \) one defines a conserved quantity

\[
Q_{\xi}[C] := -\frac{1}{8\pi G} \oint E'_{ab} \xi^a dS^b,
\]

where \( E'_{ab} = \Omega^{-1} E_{ab} \). This expression is conformally invariant and has a flux \( F_{\xi}[\Delta] \) through a region \( \Delta \) of \( \mathcal{I} \) bounded by two cross-sections given by

\[
F_{\xi}[\Delta] = \int_{\Delta} \left( \lim_{\Omega \to 0} \Omega^{-4} \hat{T}^b_a \right) s^a \xi_b d\Sigma.
\]

If there is no matter near to \( \mathcal{I} \) then the flux vanishes.

The Schwarzschild–anti-de Sitter metric can be written

\[
ds^2 = \left( 1 - \frac{2GM}{r} + a^2 r^2 \right) dt^2 - \left( 1 - \frac{2GM}{r} + a^2 r^2 \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

with \( \lambda = -3a^2 \). Choosing \( \xi^a = \frac{1}{a} \frac{\partial}{\partial t} \), which with \( \Omega = r^{-1} \) is unit at \( \mathcal{I} \) in the rescaled metric, we obtain

\[
Q_{\xi}[C] = M
\]
on any cross-section \( C \) of \( \mathcal{I} \).

To show that the expression [3] coincides with Penrose’s expression for quasi-local mass and angular momentum [1] (modified for asymptotically anti-de Sitter by the subtraction of the cosmological constant term) we first need the following Lemma:

\[\text{3PT:} \text{ Three-surface twistors are defined on a three-surface } \Sigma \text{ with normal } n^a \text{ in a space-time by taking those components of the twistor equation } \nabla_{AA'} \omega_B = -i \epsilon_{AB} \pi_{A'} \quad \text{for which the derivative is projected tangent to the three-surface. Writing } D_{AB} = n_{(A'} \nabla_{B)A'}, \text{ the 3-surface twistor equation is } D_{AB} \omega_C = -i \pi_{(A' \epsilon_B)C} \text{ with } \pi_A = n_A \pi_{A'}.\]
Lemma
Suppose Σ is a hypersurface in a conformally-flat space-time, with unit normal ζ_a and \( \nabla_a \zeta_b \) zero at \( \Sigma \); then the vector field
\[
\xi^a := \omega^A \zeta^A_B \omega^B
\]
(5)
is a (null) conformal Killing vector at \( \Sigma \) if and only if
\[
\nabla_{A'}(A \omega_B) = 0
\]
(6)
at \( \Sigma \).

The proof is simply by substitution of the solution of the twistor equation (6) in a conformally-flat space-time. For a general conformal Killing vector one has a sum of terms like (5).

On \( \mathcal{I} \), with unit normal \( \zeta^a = s^a/a \), take \( \xi^a = 2i\omega^A \zeta^A_B \omega^B \) then with a cross-section \( C \) of \( \mathcal{I} \) with unit normal \( t^a \) tangent to \( \mathcal{I} \) we have
\[
Q_{\xi}[C] := -\frac{1}{8\pi G} \oint_C E_{ab} \xi^a dS^b = -\frac{1}{8\pi G} \oint_C 2\phi_{ABCD} \zeta^A_C \zeta^B_D 2i\omega^A \zeta^A_B t^{BB'} dS
\]

\[
= -\frac{i}{4\pi G} \oint_C \phi_{ABCD} \omega^A \omega^B \phi^C t^D dS,
\]

where \( \phi_{ABCD} = \Omega^{-1} \psi_{ABCD} \) and \( \psi_{ABCD} \) is the Weyl spinor. This is now recognisable as Penrose’s expression for the quasi-local mass and angular momentum [4].

We shall choose our momentum and angular momentum conformal Killing vectors by embedding \( \mathcal{I} \) as the boundary of anti-de Sitter space in the Einstein cylinder and then restricting solutions of the twistor equation in conformally-flat space-time to it. This process depends on how we embed Minkowski space-time into the Einstein static cylinder with respect to anti-de Sitter space-time. We choose to do this symmetrically.

With respect to a constant spinor basis \( (\alpha^A, \beta^A) \) in Minkowski space the solution to the twistor equation (see eg [3]) is given by
\[
\omega^A = \Omega^A - ix^{AA'} \pi_{AA'},
\]
where
\[
\Omega^A = \Omega^0 \alpha^A + \Omega^1 \beta^A, \quad \pi_{AA'} = \pi^{0A} \lambda_{AA'} + \pi^{1A} \bar{\lambda}_{AA'}
\]
and \( \Omega^0, \Omega^1, \pi^0, \pi^1 \) are complex constants. Also \( x^{AA'} \) is the position vector in Minkowski space in Cartesians.

Take the Einstein cylinder to have metric
\[
ds^2 = dt^2 - dr^2 - \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2),
\]

4PT: The field \( \omega^A \) here is a 3-surface twistor on \( \mathcal{I} \) so (6) might better be written as \( D_{(AB} \omega_{C)} = 0 \)

5PT: Here there has been a calculation, that \( \zeta^A_{\lambda} \zeta^B_{\mu} \zeta^B_{\mu} t^{BB'} = o(C) \), where \( \alpha^A, \lambda^A \) are spinors representing the out and ingoing null normals to the cross-section \( C \), normalised to have \( o_{AA'} = 1 \). This formula establishes the connection between Penrose’s quasi-local kinematic quantities and the Ashtekar-Magnon charges.

6PT: This means the following: take the Minkowski metric to be
\[
g_M = dT^2 - dR^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]
then this is conformal to the metric of the Einstein static cylinder written as
\[
g_E = dt^2 - dr^2 - \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2),
\]
with \( T \pm R = \tan(t \pm r)/2 \) so that the worldlines \( r = 0 \) and \( R = 0 \) coincide. The anti-de Sitter metric with \( a = 1 \) can then be written
\[
g_{dS} = (sec^2 r) g_E,
\]
and occupies half the Einstein static cylinder, with \( r \leq \pi/2 \).
and introduce the null tetrad
\[ \ell = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right), \quad n = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial r} \right), \quad m = \frac{1}{\sqrt{2} \sin r} \left( \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right). \]
This tetrad implicitly defines a spinor dyad \((\nu^A, \nu^A)\) in the usual way and expanded in this dyad the solution of the twistor equation is
\[ \omega^A = \omega_0^A \nu^A + \omega_1^A \nu^A \]
with
\[ \omega_0 = \sqrt{2} \cos \left( \frac{1}{2} (t + r) \right) \left( \Omega^0 e^{-i \phi/2} \cos(\theta/2) + \Omega^1 e^{i \phi/2} \sin(\theta/2) \right) + i \sin \left( \frac{1}{2} (t + r) \right) \left( -\pi^{0'} e^{i \phi/2} \sin(\theta/2) + \pi^{1'} e^{-i \phi/2} \cos(\theta/2) \right), \]
\[ \omega_1 = \sqrt{2} \cos \left( \frac{1}{2} (t - r) \right) \left( -\Omega^0 e^{-i \phi/2} \sin(\theta/2) + \Omega^1 e^{i \phi/2} \cos(\theta/2) \right) + i \sin \left( \frac{1}{2} (t - r) \right) \left( \pi^{0'} e^{i \phi/2} \cos(\theta/2) + \pi^{1'} e^{-i \phi/2} \sin(\theta/2) \right). \]
Since the twistor equation is conformally invariant, this is also a solution of the twistor equation on the Einstein cylinder.

From these expressions we may calculate \(Q_\xi [C]\) with the conformal Killing vector
\[ \xi^a = 2i \zeta^A_B \omega^A \widetilde{\omega}^B \]
and the cross-section \(C\) in \(r = \pi/2\) as
\[ Q_\xi [C] = -\frac{i}{4\pi G} \int_C \phi_{ABCD} \omega^A \widetilde{\omega}^B dS^{CD} = A_{\alpha \beta} Z^\alpha \tilde{Z}^\beta, \]
with
\[ Z^\alpha = (\Omega^A, \pi^A), \quad \tilde{Z}^\alpha = (\tilde{\Omega}^A, \tilde{\pi}^A) \]
and
\[ A_{\alpha \beta} = \begin{pmatrix} 2\Phi_{A'B'} & P_{A'B'} \\ P^A_B & \Phi^{A'B'} \end{pmatrix} \]
In particular the 4 components of momentum \(P^a\) are given by
\[ Q_\xi [C] := -\frac{1}{8\pi G} \int \xi^a E^{ab} dS^b \]
with \(\xi^a\) one of \((\gamma^a, \frac{1}{2}(\eta^a + \bar{\eta}^a), \frac{1}{2i}(\eta^a - \bar{\eta}^a), \beta^a)\) where
\[ \gamma = \frac{\partial}{\partial t}, \quad \eta = e^{i \phi} \sin t \sin \theta \frac{\partial}{\partial \theta} - \cos t \cos \theta \frac{\partial}{\partial \phi} - i \cos t \frac{\partial}{\partial \phi}, \]
\[ \beta = \cos \theta \sin t \frac{\partial}{\partial \theta} + \sin \theta \cos t \frac{\partial}{\partial \phi}. \]
For the case of Schwarzschild–anti-de Sitter \(P^a = (M, 0, 0, 0)\) and \(\Phi_{AB} = 0\) and we recover \(M\) as the mass.

The angular momentum automatically obeys the Hermiticity property
\[ A_{\alpha \beta} I^{\beta \gamma} = \overline{A_{\gamma \beta} I^{\gamma \alpha}} \]
with respect to the infinity twistor[7]
\[ I^{\alpha \beta} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \epsilon_{A'B'} \end{pmatrix}. \]

7PF: Remember that, in this part of the discussion, \(a = 1\); otherwise \(a^2\) appears in (10).
2 A positive energy theorem

We first need to look at the work of Gibbons et al [6]. Consider a space-like hypersurface $\Sigma$ in an asymptotically anti-de Sitter space-time (possibly with an inner boundary on a past or future apparent horizon $H$) which asymptotically approaches the $t = 0$ cross-section of $I$. Define a supercovariant derivative on a 4-spinor $(\alpha^A, \beta^{A'})$ by

$$\hat{\nabla}_{MM'}\alpha_A = \nabla_{MM'}\alpha_A + \frac{a}{\sqrt{2}} \epsilon_{MA}\beta_{M'}$$

$$\hat{\nabla}_{MM'}\beta_{A'} = \nabla_{MM'}\beta_{A'} + \frac{a}{\sqrt{2}} \epsilon_{M'A'}\alpha_M.$$

Now suppose that $D_a$ is the projection into $\Sigma$ of $\nabla_a$ and introduce the supercovariant Witten equation on $\Sigma$:

$$\hat{D}_{AA'}\alpha^A = D_{AA'}\alpha^A + \frac{3}{2}\frac{a}{\sqrt{2}} \beta_{A'} = 0$$

$$\hat{D}_{AA'}\beta^{A'} = D_{AA'}\beta^{A'} + \frac{3}{2}\frac{a}{\sqrt{2}} \alpha_A = 0.$$

Then one can show that

$$-D_m\left(t^{AA'}(\hat{\alpha}_{A'}\hat{D}^m + \hat{\beta}_{A}\hat{D}^m\beta_{A'})\right) = -t^{BB'}(\hat{D}_m\alpha_{A}\hat{D}_m\alpha_{B} + \hat{D}_m\beta_{B'}\hat{D}_m\beta_{A'}) + 4\pi G T_{ab}\xi^b,$$

where $\xi^a = \alpha^A\hat{\alpha}^A + \beta^{A'}\hat{\beta}^{A'}$. Thus if $T_{ab}$ satisfies the Dominant Energy Condition, then the RHS is non-negative. We may choose the spinors $\alpha^A, \beta^{A'}$ to obey boundary conditions on an inner apparent horizon $H$ such that

$$t^{AA'}(\hat{\alpha}_{A'}\hat{D}^m\alpha_{A} + \hat{\beta}_{A}\hat{D}^m\beta_{A'})N_m = 0$$

at $H$,

where $N_m$ is the normal to $H$ lying in $\Sigma$.

If we use Green’s Theorem on the identity (12) we therefore obtain

$$-\oint_{t=0} t^{AA'}(\alpha_{A'}\hat{D}^m\alpha_{A} + \beta_{A}\hat{D}^m\beta_{A'})s_m dS \geq 0.$$  (13)

This integral is finite providing that

$$\hat{D}_m\alpha_A \to 0, \quad \hat{D}_m\beta_{A'} \to 0$$

in the limit at $I$.

In [6] it was proposed that the boundary term could be written as

$$4\pi G \left( P^{(5)AA'}(\hat{\alpha}_{A'}\hat{\alpha}_A + \hat{\beta}_{A'}\hat{\beta}_A) - \lambda^{AB} \hat{\alpha}_A \hat{\beta}_B - \lambda^{A'B'} \hat{\alpha}_{A'} \hat{\beta}_{B'} \right),$$

(credited as a private communication from D.Z. Freedman) where $\hat{\alpha}_{A'}, \hat{\beta}_{A'}$ are the limits on $I$ of the supercovariantly constant spinors $\alpha_{A'}, \beta_{A'}$ (after division by a suitable power of the conformal factor).

For Schwarzschild–anti-de Sitter the component $P^{(5)0} := \frac{1}{\sqrt{2}}(P^{(5)00'} + P^{(5)11'})$ is a (constant, positive) multiple of the mass parameter $M$. In general, the inequality (13) implies $P^{(5)0} \geq 0$ so that, if we identify this term with the mass, as was done in [6], then we have shown that the mass is positive. But, as shall be described, if we evaluate the integral in (13) explicitly we obtain Penrose’s expression $Q_\xi(C)$ and then we find that (13) contains much more information about $A_{\alpha\beta}$. 


We first write the metric in terms of coordinates \((u, s, \theta, \phi)\) as\(^8\)
\[
g = \frac{1}{\sin^2 s} (du^2 - \frac{2}{a}duds - \cos^2 s(d\theta^2 + \sin^2 \theta d\phi^2)) + O(s^{-1})
\]
where the \(O(s^{-1})\) terms do not contain \(ds^2\). Here the constant \(u\) surfaces are outgoing null hypersurfaces meeting \(\mathcal{I}\) which is located at \(s = 0\), \((x^2, x^3) = (\theta, \phi)\) label the null generators of the constant \(u\) surfaces, and \(s\) is a parameter (not affine) on each null generator. We shall choose a null tetrad as follows:
\[
\ell_a dx^a = \frac{du}{\sin s}, \text{ so that } \ell^a \partial_a = -a \sin s \frac{\partial}{\partial s},
\]
\(n^a\) is the ingoing null normal to the 2-spheres of constant \(u\) and \(s\), and \(m^a\) is a complex null tangent to these 2-spheres, chosen so that the NP spin coefficient \(\epsilon\) is real\(^9\). We thus have
\[
\ell^a \partial_a = -a \sin s \frac{\partial}{\partial s},
\]
\[
n^a \partial_a = \sin s \left( \frac{\partial}{\partial u} + aU \frac{\partial}{\partial s} + X^k \frac{\partial}{\partial x^k} \right),
\]
\[
m^a \partial_a = \frac{a}{\sqrt{2}} \xi^k \frac{\partial}{\partial x^k},
\]
for real \(U, X^k\) and complex \(\xi^k\). For anti-de Sitter space-time
\[
U = \frac{1}{2}, \quad X^k = 0, \quad \xi^2 = \tan s, \quad \xi^3 = -i \tan s / \sin \theta,
\]
so that these are the values at \(\mathcal{I}\) in general. Also from the assumption that \(\Omega^{-4} \hat{T}_{ab}\) is finite at \(\mathcal{I}\) we have the asymptotic behaviour of the NP curvature quantities:
\[
\Phi_{ij} = O(s^4), \quad \Lambda = -\lambda/6 = O(s^4), \quad \psi_i = O(s^3).
\]
We may therefore solve for the NP spin coefficients as power series expansions in \(s\) and then we may calculate \(\alpha_A, \beta_{A'}\) similarly. We find
\[
\alpha_A = s^{-1/2} \sum_{i=0}^{\infty} \alpha^A_i s^i, \quad \beta_{A'} = s^{-1/2} \sum_{i=0}^{\infty} \beta^A_i s^i,
\]
with \(\beta^A_{A'} = \sqrt{2} \alpha^A_0 \alpha^A_{A'}\) (this equality is an identity in the rescaled space-time since \(s^a = \frac{1}{2} a^a - n^a\)).

The integral on the LHS of (13) becomes
\[
-\frac{\sqrt{2}}{a^3} \int_{t=0, s=0} \left( \psi_1^{(3)} \alpha_{0}^0 + \psi_2^{(3)} (\alpha_{0}^0 \bar{\beta}_{0}^0 + \alpha_{0}^0 \bar{\beta}_{0}^0) + \psi_3^{(3)} \alpha_{0}^0 \bar{\beta}_{0}^0 \right) dS_0
\]
\[
= \frac{-i}{a^3} \int_{t=0, s=0} \left( \psi_1^{(3)} \omega^0 \omega^0 + \psi_2^{(3)} (\omega^0 \omega^0 + \omega^0 \omega^1) + \psi_3^{(3)} \omega^1 \omega^1 \right) dS_0
\]
\[
= \frac{4\pi \lambda}{a^3} \sum_{\alpha \beta} \hat{A}_{\alpha \beta} \hat{Z}^\alpha \hat{Z}^\beta,
\]
where \(\psi_i^{(3)}\) means the \(O(s^3)\) term in \(\psi_i\), and \(\omega^A \hat{=} \alpha^A_0, \bar{\omega}^A \hat{=} -i \sqrt{2} \beta^A_0\) which are found to be 2-surface twistors on the \(t = 0\) cross-section of \(\mathcal{I}\). But in (7) we have expressions for \(\omega^0, \omega^1\) in terms of \((\Omega^4, \pi_{A'})\) at \(t = 0\), and the relation between \(\alpha^A_0\) and \(\beta^A_0\) gives us that
\[
\hat{Z}_\alpha = 2 i \alpha^A_{\alpha} \hat{Z}_\beta,
\]
\(^8\text{PT}:\) The leading terms in this expression correspond to the metric \(g_{adS}\) of footnote 6 but with general \(a\).

\(^9\text{PT}:\) This can be accomplished by rotating \(m^a\) in the 2-plane tangent to the chosen 2-spheres.
using (10). We have therefore shown, by (13), that
\[ A_{\alpha\beta} I^{\beta\gamma} Z^{\alpha} \bar{Z}^{\gamma} \geq 0. \] (19)

In particular this implies that \( P^a \) is time-like and future-pointing. From the angular momentum twistor \( A_{\alpha\beta} \) we may calculate the associated mass \( m_P \) as
\[ m_P^2 = -\frac{1}{2} A_{\alpha\beta} \bar{A}^{\alpha\beta} = P_a P^a - \Phi_{\alpha\beta} \Phi^{\alpha\beta} - \bar{\Phi}_{\alpha\beta} \bar{\Phi}^{\alpha\beta}. \]

Inequality (19) implies that \( m_P^2 \) is non-negative and also provides a further inequality relating components of \( P^a \) and \( \Phi_{\alpha\beta} \). An alternative definition of mass is
\[ m_D^4 = 4 \det A_{\alpha\beta} = 4 \epsilon^{\alpha\beta\gamma\delta} A_{\alpha1} A_{\beta2} A_{\gamma3} A_{\delta4}, \]
and \( m_D^4 \) is also non-negative by virtue of (19). When \( \Phi_{\alpha\beta} = 0 \), we have \( P^a P_a = m_P^2 = m_D^2 \) but in general the masses are different.

**Acknowledgements**

I would like to thank Dr Paul Tod for considerable assistance in this work and Professor Penrose and William Shaw for useful discussions. I hope to publish a more detailed version of this work in the near future.

**References**

[1] A Ashtekar and A Magnon *Asymptotically anti-de Sitter space-times* Classical Quantum Gravity 1 (1984) L39-L44.

[2] K.P. Tod *Three-surface twistors and conformal embedding* Gen. Relativity Gravitation 16 (1984) 435-443.

[3] S.W. Hawking *The boundary conditions for gauged supergravity* Phys. Lett. B 126 (1983) 175-177.

[4] R. Penrose *Quasilocal mass and angular momentum in general relativity* Proc. Roy. Soc. London Ser. A 381 (1982) 53-63

[5] S.A. Huggett and K.P. Tod *An introduction to twistor theory* Second edition. London Mathematical Society Student Texts, 4. Cambridge University Press, Cambridge, 1994

[6] G.W. Gibbons, S. W. Hawking, G.T. Horowitz and M.J. Perry *Positive mass theorems for black holes* Comm. Math. Phys. 88 (1983), 295-308

[7] P.T. Chruściel, D. Maerten and P. Tod, *Rigid upper bounds for the angular momentum and centre of mass on non-singular asymptotically anti-de Sitter space-times* J. High Energy Phys. 11 (2006) 084, 42 pp. (electronic).

\[ \text{PT}: \text{Kelly didn't make the point explicitly but this result also provides } \text{rigidity } \text{in the sense that if there exists a } Z^{\alpha} \text{ giving zero in (19) then the space-time is exactly anti-de Sitter at least near } \Sigma. \text{ This follows from (12): Dominant Energy implies that } T_{ab} \text{ vanishes at } \Sigma, \text{ and therefore everywhere; and one can evolve } (A_{\alpha}, \beta^{\alpha'}) \text{ from } \Sigma \text{ to obtain a solution of (11) in the space-time, which forces the Weyl tensor to vanish.} \]

\[ \text{PT}: \text{In this connection, see [7].} \]