Linear Properties of Generalized $n$-step Fibonacci Numbers

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Abstract
We present numerous interesting, mostly new, results involving the $n$-step Fibonacci numbers and $n$-step Lucas numbers and a generalization. Properties considered include recurrence relations, summation identities, including binomial and double binomial summation identities, partial sums and ordinary generating functions. Explicit examples are given for small $n$ values.

1 Introduction
For $n \geq 2$, the $n$-step Fibonacci numbers, $U_r (r \geq n)$, satisfy the linear recurrence relation \[ U_r = U_{r-1} + U_{r-2} + U_{r-3} + \cdots + U_{r-n} = \sum_{i=1}^{n} U_{r-i}, \] (1.1) with $n$ initial terms \[ U_k = 0, \quad -n + 2 \leq k \leq 0, \quad U_{-n+1} = 1. \] (1.2)

Well-known members of this number family include the Fibonacci numbers $F_r (n = 2, U = F)$, the Tribonacci numbers $T_r (n = 3, U = T)$, the Tetranacci numbers $M_r (n = 4, U = M)$. The reader is referred to Table 1 for notation and nomenclature.

By writing $U_{r-1} = U_{r-2} + U_{r-3} + U_{r-4} + \cdots + U_{r-n-1}$ and substracting this from relation (1.1), we see that the $n$-step Fibonacci numbers also obey the following recurrence relation:

\[ U_r = 2U_{r-1} - U_{r-n-1}. \] (1.3)

Extension of the definition of $n$-step Fibonacci numbers to negative subscripts $r < -n + 2$ is provided by writing the recurrence relation (1.3) as

\[ U_{-r} = 2U_{-r+n} - U_{-r+n+1}. \] (1.4)

From (1.1), (1.2), (1.3) and (1.4), we have the following special values:

\[ U_1 = 1, \quad U_k = \sum_{j=1}^{k-1} U_j, \quad 2 \leq k \leq n-1, \quad U_{-1} = \delta_{n,2}, \quad U_{-n} = -1, \quad U_{-n-1} = 2\delta_{n,2}, \] (1.5)
where $\delta_{i,j}$ is Kronecker delta, equals 1 when $i = j$ and equals 0 otherwise.

We also have
\[ U_n = 2^{n-2}, \quad U_{n+1} = 2^{n-1}, \quad U_{n+2} = 2^n - 1, \quad \text{(1.6)} \]
and, in fact,
\[ U_{n+k} = 2^{n+k-2} - \sum_{j=1}^{k} 2^{j-1}U_{n-j}, \quad k \in \mathbb{Z}. \quad \text{(1.7)} \]

We remark that identity (1.7) is equivalent to Theorem 3.1 of Howard and Cooper [6] without a restriction on $k$. Note that identity (1.7) is a special case of identity (3.20).

Like the $n$-step Fibonacci numbers, the $n$-step Lucas numbers [8] obey an $n$th order recurrence relation
\[ V_r = V_{r-1} + V_{r-2} + V_{r-3} + \cdots + V_{r-n} = \sum_{i=1}^{n} V_{r-i}, \quad \text{(1.8)} \]
but with the initial terms
\[ V_k = -1, \quad -n + 1 \leq k \leq -1, \quad V_0 = n. \quad \text{(1.9)} \]

The most well-known members of the $n$-step Lucas numbers are the Lucas numbers ($n = 2$), $(L_r)_{r \in \mathbb{Z}}$, and the Tribonacci-Lucas numbers ($n = 3$), $(K_r)_{r \in \mathbb{Z}}$.

The $n$-step Lucas numbers also obey the three-term recurrence relation
\[ V_r = 2V_{r-1} - V_{r-n-1}. \quad \text{(1.10)} \]

Extension of the definition of $n$-step Lucas numbers to integers $r < -n + 1$ is provided through
\[ V_{-r} = 2V_{-r+n} - V_{-r+n+1}. \quad \text{(1.11)} \]

Noe and Post [8] noted that the $n$-step Fibonacci numbers and the $n$-step Lucas numbers are connected through the identity
\[ V_r = U_r + 2U_{r-1} + \cdots + (n-1)U_{r-n+2} + nU_{r-n+1} = \sum_{j=1}^{n} jU_{r-j+1}. \quad \text{(1.12)} \]

From identities (1.1), (1.3) and (1.12), we can derive the following four-term relation
\[ V_r = V_{r-1} - (n + 1)U_{r-n} + 2U_r, \quad \text{(1.13)} \]
which can also be written in the alternative form
\[ V_r = V_{r-1} - nU_{r-n} + U_{r+1} \quad \text{(1.14)} \]
or
\[ V_r = V_{r-1} - 2nU_r + (n + 1)U_{r+1}. \quad \text{(1.15)} \]

From (1.8), (1.9), (1.10), (1.11) and (1.14), we also have the following special values for the $n$-step Lucas numbers:
\[ V_1 = 1, \quad V_{-n} = 2n - 1, \quad V_{-n-1} = -n - 2, \quad V_n = 2^n - 1. \quad \text{(1.16)} \]
The first few sequences of the $n$-step Fibonacci numbers and the $n$-step Lucas numbers are presented in Table 1.

The generalized $n$-step Fibonacci numbers, $W_r$, satisfy the same recurrence equation given in (1.1) but with arbitrary initial values. Thus,

$$W_r = W_{r-1} + W_{r-2} + W_{r-3} + \cdots + W_{r-n} = \sum_{i=1}^{n} W_{r-i}, \quad (1.17)$$

for $r \geq n$ but $W_0, W_1, \ldots, W_{n-1}$ are arbitrary. Analogous to (1.3) and (1.4), we have

$$W_r = 2W_{r-1} - W_{r-n-1} \quad (1.18)$$

and

$$W_{-r} = 2W_{-(r-n)} - W_{-(r-n-1)}. \quad (1.19)$$

| $n$ | Name         | Symbol | $n$ | Name         | Symbol |
|-----|--------------|--------|-----|--------------|--------|
| 2   | Fibonacci    | $F$    | 6   | Sextanacci   | $S$    |
|     | Fibonacci-Lucas | $L$   |     | Sextanacci-Lucas |     |
|     | Generalized Fibonacci | $F$   |     | Generalized Sextanacci | $S$ |
| 3   | Tribonacci   | $T$    | 7   | Heptanacci   | $H$    |
|     | Tribonacci-Lucas | $K$   |     | Heptanacci-Lucas |     |
|     | Generalized Tribonacci | $T$ |     | Generalized Heptanacci | $H$ |
| 4   | Tetranacci   | $M$    | 8   | Octanacci    | $O$    |
|     | Tetranacci-Lucas | $R$   |     | Octanacci-Lucas |     |
|     | Generalized Tetranacci | $M$ |     | Generalized Octanacci | $O$ |
| 5   | Pentanacci   | $P$    | 9   | Nanonacci    | $N$    |
|     | Pentanacci-Lucas | $Q$   |     | Nanonacci-Lucas |     |
|     | Generalized Pentanacci | $P$ |     | Generalized Nanonacci | $N$ |

Table 1: Notation and nomenclature for the first few members of the $n$-step Fibonacci numbers, $n$-step Lucas numbers and the generalized $n$-step Fibonacci numbers.

| $n$ | Name         | $r$  | -4 | -3 | -2 | -1 | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|--------------|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 2   | Fibonacci    | $F_r$| -3 | 2  | -1 | 1  | 0  | 1  | 1  | 2  | 3  | 5  | 8  | 13 | 21 | 34 | 55 |
|     | Lucas        | $L_r$| 7  | -4 | 3  | -1 | 2  | 1  | 3  | 4  | 7  | 11 | 18 | 29 | 47 | 76 |123 |
| 3   | Tribonacci   | $T_r$| 0  | -1 | 1  | 0  | 0  | 1  | 1  | 2  | 4  | 7  | 13 | 24 | 44 | 81 |149 |
|     | Trib-Lucas   | $K_r$| -5 | 5  | -1 | -1 | 3  | 1  | 3  | 7  | 11 | 21 | 39 | 71 |131 |241 |443 |
| 4   | Tetranacci   | $M_r$| -1 | 1  | 0  | 0  | 0  | 1  | 1  | 2  | 4  | 8  | 15 | 29 | 56 |108 |208 |
|     | Tetra-Lucas  | $R_r$| 7  | -1 | -1 | -1 | 4  | 1  | 3  | 7  | 15 | 26 | 51 | 99 |191 |367 |708 |
| 5   | Pentanacci   | $P_r$| 1  | 0  | 0  | 0  | 0  | 1  | 1  | 2  | 4  | 8  | 16 | 31 | 61 |120 |236 |
|     | Penta-Lucas  | $Q_r$| -1 | -1 | -1 | -1 | 5  | 1  | 3  | 7  | 15 | 31 | 57 |113 |223 |439 |863 |

Table 2: The first few sequences of the $n$-step Fibonacci numbers and $n$-step Lucas numbers.
Our aim in writing this paper is to discover various properties of the generalized \( n \)-step Fibonacci numbers, \( W_r \). Specifically we will develop recurrence relations, ordinary, binomial and double binomial summation identities, partial sums and generating functions.

## 2 Recurrence relations

**Theorem 1.** The following identity holds, where \( r \) and \( s \) are integers:

\[
W_{r+s} = \sum_{i=1}^{n} \left( \sum_{j=0}^{n-i} U_{s-j+1} \right) W_{r-i}.
\]

In particular, we have

\[
U_{r+s} = \sum_{i=1}^{n} \left( \sum_{j=0}^{n-i} U_{s-j+1} \right) U_{r-i} \tag{2.1}
\]

and

\[
V_{r+s} = \sum_{i=1}^{n} \left( \sum_{j=0}^{n-i} U_{s-j+1} \right) V_{r-i} \tag{2.2}
\]

**Proof.** We will keep \( r \) fixed and use induction on \( s \).

The identity is true for \( s = 0 \) because

\[
\sum_{j=0}^{n-i} U_{-j+1} = \sum_{j=i}^{n} U_{-n+j+1} = \sum_{j=i}^{n-1} U_{-n+j+1} + 1 = 1,
\]

for \( 1 \leq i \leq n \), by virtue of the initial terms (1.2).

Assume that the identity is true for some integer \( s = k \in \mathbb{Z}^+ \). Let

\[
P_{k} : \left( W_{r+k} = \sum_{i=1}^{n} \left( \sum_{j=0}^{n-i} U_{k-j+1} \right) W_{r-i} \right) \tag{2.3}
\]

We wish to prove that

\[
P_{k+1} : \left( W_{r+k+1} = \sum_{i=1}^{n} \left( \sum_{j=0}^{n-i} U_{k+1-j+1} \right) W_{r-i} \right) \tag{2.4}
\]

and

\[
P_{k-1} : \left( W_{r+k-1} = \sum_{i=1}^{n} \left( \sum_{j=0}^{n-i} U_{k-1-j+1} \right) W_{r-i} \right) \tag{2.5}
\]

are true whenever \( P_{k} \) holds.

By the identity (1.17) and the induction hypothesis \( P_{k} \) (identity (2.3)) we have

\[
W_{r+k+1} = \sum_{\lambda=1}^{n} W_{r+k+1-\lambda} = \sum_{\lambda=1}^{n} \left\{ \sum_{i=1}^{n} \left( \sum_{j=0}^{n-i} U_{k+1-\lambda-j+1} \right) W_{r-i} \right\}
\]

\[
= \sum_{i=1}^{n} \sum_{j=0}^{n-i} \left( \sum_{\lambda=1}^{n} U_{k+1-\lambda-j+1} \right) W_{r-i} \tag{2.6}
\]
By the recurrence relation (1.1) we have

\[ \sum_{\lambda=1}^{n} U_{k+1-\lambda-j+1} = U_{k+1-j+1}. \]  

(2.7)

Using (2.7) in (2.6) yields (2.4) and therefore \( P_k \Rightarrow P_{k+1} \). Following the same procedure, it is readily established that \( P_k \Rightarrow P_{k-1} \).

We remark that Gabai [4, Theorem 6] earlier proved the equivalent of Theorem 1. His proof, however, placed a restriction on the integers \( r \) and \( s \), consistent with his definition of the generalized \( n \)-step numbers.

**Corollary 2.** The following identity holds, where \( r \) and \( s \) are integers:

\[ W_{r+s} = \sum_{i=1}^{n} \left( \sum_{j=0}^{n-i} W_{s-j+1} \right) U_{r-i}. \]

In particular,

\[ V_{r+s} = \sum_{i=1}^{n} \left( \sum_{j=0}^{n-i} V_{s-j+1} \right) U_{r-i}. \]  

(2.8)

**Proof.** We require the following summation identities:

\[ \sum_{j=a}^{k-a} f_j = \sum_{j=a}^{k-a} f_{k-j} \]  

(2.9)

and

\[ \sum_{i=a}^{n} \sum_{j=0}^{n-i} A_{i,j+i} = \sum_{i=a}^{n} \sum_{j=a}^{i} A_{j,i}. \]  

(2.10)

Now,

\[ W_{r+s} = \sum_{i=1}^{n} \sum_{j=0}^{n-i} U_{s-j+1} W_{r-i} = \sum_{i=1}^{n} \sum_{j=0}^{n-i} W_{r-i} U_{s-j+1} = \sum_{i=1}^{n} \sum_{j=0}^{n-i} W_{r-i} U_{s-n+i+j+1}, \]  

(2.11)

by application of identity (2.9) to the \( j \) summation. Using identity (2.10) to re-write the sum in (2.11) gives

\[ W_{r+s} = \sum_{i=1}^{n} \sum_{j=1}^{i} W_{r-j} U_{s-n+i+1}, \]  

(2.12)

in which the application of identity (2.9) to the \( i \) summation gives

\[ W_{r+s} = \sum_{i=1}^{n} \sum_{j=1}^{n-i} W_{r-j} U_{s-n+i+1-j+1} = \sum_{i=1}^{n} \sum_{j=1}^{n-i} W_{r-j} U_{s-i+2} = \sum_{i=1}^{n} \sum_{j=0}^{n-i} W_{r-j-1} U_{s-j+2}. \]  

(2.13)

Finally, setting \( r = s + 2 \) and \( s = r - 2 \) in (2.13) gives the identity of Corollary 2.

We now give explicit examples of the identities of Theorem 1 and Corollary 2 for low \( n \) \( n \)-step generalized Fibonacci numbers.
2.1 Recurrence relations for the generalized Fibonacci numbers

With \( n = 2 \) in the identity of Theorem 1, we have

\[
F_{r+s} = F_{s+2}F_{r-1} + F_{s+1}F_{r-2},
\]

which is a variant of Formula (8) of Vajda [10], with particular instances

\[
F_{r+s} = F_{s+2}F_{r-1} + F_{s+1}F_{r-2}
\]

and

\[
L_{r+s} = F_{s+2}L_{r-1} + F_{s+1}L_{r-2}.
\]

2.2 Recurrence relations for the generalized Tribonacci numbers

Choosing \( n = 3 \) in the identity of Theorem 1 gives

\[
T_{r+s} = T_{s+2}T_{r-1} + (T_{s+1} + T_s)T_{r-2} + T_{s+1}T_{r-3},
\]

with the particular cases

\[
T_{r+s} = T_{s+2}T_{r-1} + (T_{s+1} + T_s)T_{r-2} + T_{s+1}T_{r-3}
\]

and

\[
K_{r+s} = T_{s+2}K_{r-1} + (T_{s+1} + T_s)K_{r-2} + T_{s+1}K_{r-3}.
\]

The identity (2.18) was also proved by Feng [3] and by Shah [9].

Since \( T_{-17} = 0, T_{-18} = -103 \) and \( T_{-19} = 159 \), setting \( s = -19 \) in identity (2.17) produces another three-term recurrence for the generalized Tribonacci numbers, namely

\[
T_{r-19} = 56T_{r-2} - 103T_{r-3},
\]

in addition to the relation

\[
T_r = 2T_{r-1} - T_{r-4},
\]

obtained at \( n = 3 \) in identity (1.18).

Choosing \( n = 3 \) in the identity of Corollary (2) with \( W = K, U = T \) gives

\[
K_{r+s} = K_{s+2}T_{r-1} + (K_{s+1} + K_s)T_{r-2} + K_{s+1}T_{r-3}.
\]

Setting \( s = -4 \) in identity (2.22) gives a three-term identity connecting the Tribonacci-Lucas numbers and the Tribonacci numbers:

\[
K_{r-4} = -T_{r-1} + 5T_{r-3},
\]

since \( K_{-3} = -K_{-4} = 5 \).
2.3 Recurrence relations for the generalized Tetranacci numbers

The choice \( n = 4 \) in the identity of Theorem 1 gives

\[
\mathcal{M}_{r+s} = M_{s+2}M_{r-1} + (M_{s+1} + M_s + M_{s-1})M_{r-2} + (M_{s+1} + M_s)M_{r-3} + M_{s+1}M_{r-4},
\]

(2.24)

with the special cases

\[
M_{r+s} = M_{s+2}M_{r-1} + (M_{s+1} + M_s + M_{s-1})M_{r-2} + (M_{s+1} + M_s)M_{r-3} + M_{s+1}M_{r-4}
\]

(2.25)

and

\[
R_{r+s} = M_{s+2}R_{r-1} + (M_{s+1} + M_s + M_{s-1})R_{r-2} + (M_{s+1} + M_s)R_{r-3} + M_{s+1}R_{r-4}
\]

(2.26)

Choosing \( n = 4 \) in the identity of Corollary (2) with \( W = R, U = M \) gives

\[
R_{r+s} = R_{s+2}M_{r-1} + (R_{s+1} + R_s + R_{s-1})M_{r-2} + (R_{s+1} + R_s)M_{r-3} + R_{s+1}M_{r-4}.
\]

(2.27)

Setting \( s = -9, s = -5 \) and \( s = -4 \), respectively, in (2.27), yields, in each case, a four-term relation expressing a Tetranacci-Lucas number in terms of Tetranacci numbers:

\[
R_{r-9} = -M_{r-1} - 4M_{r-3} + 15M_{r-4},
\]

(2.28)

\[
R_{r-5} = -M_{r-1} + M_{r-3} + 7M_{r-4},
\]

(2.29)

\[
R_{r-4} = -M_{r-1} + 6M_{r-3} - M_{r-4}.
\]

(2.30)

3 Summation identities

**Lemma 1.** Let

\[
Z_r = \sum_{j=1}^{\lceil n/2 \rceil} W_{r-2j+1} = \begin{cases} W_{r-1} + W_{r-3} + W_{r-5} + \cdots + W_{r-n+1}, & \text{if } n \text{ is even;} \\ W_{r-1} + W_{r-3} + W_{r-5} + \cdots + W_{r-n}, & \text{if } n \text{ is odd}, \end{cases}
\]

where \( \lceil q \rceil \) is the smallest integer greater than \( q \). Then

\[
Z_r + Z_{r-1} = W_r + (n \mod 2)W_{r-n-1}
\]

\[
= \begin{cases} W_r, & \text{if } n \text{ is even;} \\ 2W_{r-1}, & \text{if } n \text{ is odd}. \end{cases}
\]

(3.1)

**Lemma 2** ([1, Lemma 1]). Let \( \{X_r\} \) and \( \{Y_r\} \) be any two sequences such that \( X_r \) and \( Y_r, \)
\( r \in \mathbb{Z}, \) are connected by a three-term recurrence relation \( X_r = f_1X_{r-a} + f_2Y_{r-b}, \) where \( f_1 \)
and \( f_2 \) are arbitrary non-vanishing complex functions, not dependent on \( r, \) and \( a \) and \( b \) are integers. Then,

\[
f_2 \sum_{j=0}^{k} \frac{Y_{r-ka-b+aj}}{f_1^j} = \frac{X_r}{f_1^k} - f_1X_{r-(k+1)a},
\]

for \( k \) a non-negative integer.
The next theorem follows directly from Lemma 1 and Lemma 2.

**Theorem 3.** The following identity holds, where \( r \) and \( k \) are integers:

\[
\sum_{j=0}^{k} (-1)^j W_{r-k+j} + n \mod 2 \sum_{j=0}^{k} (-1)^j W_{r-k-n-1+j} = (-1)^k \sum_{j=1}^{\lceil n/2 \rceil} W_{r-2j+1} + \sum_{j=1}^{\lceil n/2 \rceil} W_{r-2j-k}.
\]

In particular,

\[
\sum_{j=0}^{k} (-1)^j U_{r-k+j} + n \mod 2 \sum_{j=0}^{k} (-1)^j U_{r-k-1+j} = (-1)^k \sum_{j=1}^{\lceil n/2 \rceil} U_{r-2j+1} + \sum_{j=1}^{\lceil n/2 \rceil} U_{r-2j-k} \quad (3.2)
\]

and

\[
\sum_{j=0}^{k} (-1)^j V_{r-k+j} + n \mod 2 \sum_{j=0}^{k} (-1)^j V_{r-k-1+j} = (-1)^k \sum_{j=1}^{\lceil n/2 \rceil} V_{r-2j+1} + \sum_{j=1}^{\lceil n/2 \rceil} V_{r-2j-k} \quad (3.3)
\]

Thus, if \( n \) is even, we have

\[
\sum_{j=0}^{k} (-1)^j W_{r-k+j} = (-1)^k \sum_{j=1}^{n/2} W_{r-2j+1} + \sum_{j=1}^{n/2} W_{r-2j-k} \quad (3.4)
\]

while if \( n \) is odd, we have

\[
2 \sum_{j=0}^{k} (-1)^j W_{r-k+j-1} = (-1)^k \sum_{j=1}^{(n+1)/2} W_{r-2j+1} + \sum_{j=1}^{(n+1)/2} W_{r-2j-k} \quad (3.5)
\]

We give explicit examples with small \( n \) values.

\[
\sum_{j=0}^{k} (-1)^j F_{r-k+j} = (-1)^k F_{r-1} + F_{r-k-2} \quad (3.6)
\]

\[
2 \sum_{j=0}^{k} (-1)^j T_{r-k-1+j} = (-1)^k (T_{r-1} + T_{r-3}) + T_{r-k-2} + T_{r-k-4} \quad (3.7)
\]

\[
\sum_{j=0}^{k} (-1)^j M_{r-k+j} = (-1)^k (M_{r-1} + M_{r-3}) + M_{r-k-2} + M_{r-k-4} \quad (3.8)
\]
\[ 2 \sum_{j=0}^{k} (-1)^j P_{r-k-1+j} = (-1)^k (P_{r-1} + P_{r-3} + P_{r-5}) + P_{r-k-2} + P_{r-k-4} + P_{r-k-6}. \] (3.9)

In particular,
\[ \sum_{j=0}^{k} (-1)^j F_j = (-1)^k F_{k-1} + F_{-2}, \] (3.10)
\[ 2 \sum_{j=0}^{k} (-1)^j T_j = (-1)^k (T_k + T_{k-2}) + T_{-1} + T_{-3}, \] (3.11)
\[ \sum_{j=0}^{k} (-1)^j M_j = (-1)^k (M_{k-1} + M_{k-3}) + M_{-2} + M_{-4}, \] (3.12)
\[ 2 \sum_{j=0}^{k} (-1)^j P_j = (-1)^k (P_k + P_{k-2} + P_{r-4}) + P_{-1} + P_{-3} + P_{-5}. \] (3.13)

**Lemma 3** ([1, Lemma 2]). Let \( \{X_r\} \) be any arbitrary sequence, where \( X_r, \ r \in \mathbb{Z} \), satisfies a three-term recurrence relation \( X_r = f_1 X_{r-a} + f_2 X_{r-b} \), where \( f_1 \) and \( f_2 \) are arbitrary non-vanishing complex functions, not dependent on \( r \), and \( a \) and \( b \) are integers. Then, the following identities hold for integer \( k \):
\[ f_2 \sum_{j=0}^{k} \frac{X_{r-ka-b+aj}}{f_1^j} = \frac{X_r}{f_1^k} - f_1 X_{r-(k+1)a}, \] (3.14)
\[ f_1 \sum_{j=0}^{k} \frac{X_{r-kb+a+bj}}{f_2^j} = \frac{X_r}{f_2^k} - f_2 X_{r-(k+1)b} \] (3.15)
and
\[ \sum_{j=0}^{k} \frac{X_{r-(a-b)k+b+(a-b)j}}{(-f_1/f_2)^j} = \frac{f_2 X_r}{(-f_1/f_2)^k} + f_1 X_{r-(k+1)(a-b)}. \] (3.16)

The next theorem is a consequence of identity (1.18) and Lemma 3.

**Theorem 4.** The following identities hold, where \( r \) and \( k \) are integers:
\[ \sum_{j=0}^{k} 2^{k-j} W_{r-k-n-1+j} = 2^{k+1} W_{r-k-1} - W_r, \] (3.17)
\[ 2 \sum_{j=0}^{k} (-1)^j W_{r-nk-k-1+(n+1)j} = (-1)^k W_r + W_{r-(k+1)(n+1)} \] (3.18)
and
\[ \sum_{j=0}^{k} 2^j W_{r-nk+1+nj} = 2^{k+1} W_r - W_{r-(k+1)n}. \] (3.19)
In particular,
\begin{equation}
\sum_{j=0}^{k} 2^{k-j} W_j = 2^{k+1} W_n - W_{k+n+1},
\end{equation}
(3.20)
\begin{equation}
2 \sum_{j=0}^{k} (-1)^j W_{(n+1)j} = (-1)^k W_{k(n+1)+1} + 2W_0 - W_1
\end{equation}
(3.21)
and
\begin{equation}
\sum_{j=0}^{k} 2^j W_{nj} = 2^{k+1} W_{kn-1} - 4W_{n-1} + 2W_n + W_0.
\end{equation}
(3.22)

We now illustrate Theorem 4 for small values of \( n \).

### 3.1 Summation identities involving the generalized Fibonacci numbers, \((n = 2)\)
\begin{equation}
\sum_{j=0}^{k} 2^{k-j} F_{r-k-3+j} = 2^{k+1} F_{r-k-1} - F_r,
\end{equation}
(3.23)
\begin{equation}
2 \sum_{j=0}^{k} (-1)^j F_{r-2k-1+3j} = (-1)^k F_r + F_{r-3(k+1)}
\end{equation}
(3.24)
and
\begin{equation}
\sum_{j=0}^{k} 2^j F_{r-2k+1+2j} = 2^{k+1} F_r - F_{r-2(k+1)}.
\end{equation}
(3.25)

In particular,
\begin{equation}
\sum_{j=0}^{k} 2^{k-j} j = 2^{k+1} F_2 - F_{k+3},
\end{equation}
(3.26)
\begin{equation}
2 \sum_{j=0}^{k} (-1)^j F_{3j} = (-1)^k F_{3k+1} + 2F_0 - F_1
\end{equation}
(3.27)
and
\begin{equation}
\sum_{j=0}^{k} 2^j F_{2j} = 2^{k+1} F_{2k-1} - 4F_1 + 2F_2 + F_0.
\end{equation}
(3.28)

### 3.2 Summation identities involving the generalized Tribonacci numbers, \((n = 3)\)
\begin{equation}
\sum_{j=0}^{k} 2^{k-j} T_{r-k-4+j} = 2^{k+1} T_{r-k-1} - T_r,
\end{equation}
(3.29)
\begin{equation}
2 \sum_{j=0}^{k} (-1)^j T_{r-4k-1+4j} = (-1)^k T_r + T_{r-4(k+1)}
\end{equation}
(3.30)
and
\[ \sum_{j=0}^{k} 2^j T_{r-3k+1+3j} = 2^{k+1} T_r - T_{r-3(k+1)}. \]  
(3.31)

In particular,
\[ \sum_{j=0}^{k} 2^j T_j = 2^{k+1} T_3 - T_{k+4}, \]  
(3.32)

\[ 2 \sum_{j=0}^{k} (-1)^j T_{4j} = (-1)^k T_{4k+1} + 2T_0 - T_1 \]  
(3.33)

and
\[ \sum_{j=0}^{k} 2^j T_{3j} = 2^{k+1} T_{3k-1} - 4T_2 + 2T_3 + T_0. \]  
(3.34)

3.3 Summation identities involving the generalized Tetranacci numbers, \((n = 4)\)

\[ \sum_{j=0}^{k} 2^{k-j} M_{r-k-5+j} = 2^{k+1} M_{r-k-1} - M_r, \]  
(3.35)

\[ 2 \sum_{j=0}^{k} (-1)^j M_{r-4k-k-1+5j} = (-1)^k M_r + M_{r-5(k+1)} \]  
(3.36)

and
\[ \sum_{j=0}^{k} 2^j M_{r-4k+1+4j} = 2^{k+1} M_r - M_{r-4(k+1)}. \]  
(3.37)

In particular,
\[ \sum_{j=0}^{k} 2^{k-j} M_j = 2^{k+1} M_4 - M_{k+5}, \]  
(3.38)

\[ 2 \sum_{j=0}^{k} (-1)^j M_{5j} = (-1)^k M_{5k+1} + 2M_0 - M_1 \]  
(3.39)

and
\[ \sum_{j=0}^{k} 2^j M_{4j} = 2^{k+1} M_{4k-1} - 4M_3 + 2M_4 + M_0. \]  
(3.40)

3.4 Further summation identities involving the generalized Fibonacci numbers

In addition to the summation identities (3.23) – (3.28), we also have the results stated in the next theorem, on account of identity (2.14) and Lemma 3.
Theorem 5. The following identities hold, where \( r \) and \( s \) are integers:

\[
F_s \sum_{j=0}^{k} F_{s+1}^{k-j} F_{r-1+s+j} = F_{r+s(k+1)} - F_{s+1}^{k+1} F_{r}, \quad (3.41)
\]

\[
\sum_{j=0}^{k} (-1)^j F_s^{k-j} F_{s+1}^{j} F_{r-k+s+j} = (-1)^k F_{s+1}^{k+1} F_{r} + F_{s}^{k+1} F_{r-k-1} \quad (3.42)
\]

and

\[
F_s \sum_{j=0}^{k} F_{s+1}^{k-j} F_{r-s-k+s+1+j} = F_{r} - F_{s+1}^{k+1} F_{r-(k+1)s}. \quad (3.43)
\]

In particular,

\[
F_s \sum_{j=0}^{k} F_{s+1}^{k-j} F_{s}^{j} = F_{s+1}^{k+1} F_{s} - F_{s+1}^{k+1} F_{1}. \quad (3.44)
\]

\[
\sum_{j=0}^{k} (-1)^j F_s^{k-j} F_{s+1}^{j} F_{j} = (-1)^k F_{s+1}^{k+1} F_{k-s} + F_{s}^{k+1} F_{-s-1} \quad (3.45)
\]

and

\[
F_s \sum_{j=0}^{k} F_{s+1}^{k-j} F_{s}^{j} = F_{s+1}^{k+1} F_{-1}. \quad (3.46)
\]

When identity (2.14) is written as

\[
F_{s-1} F_r = -F_s F_{r+1} + F_{r+s} \quad (3.47)
\]

and the identifications \( X = F \) and \( Y = F \) are made in Lemma 2 we have the result stated in the next theorem.

Theorem 6. The following identity holds where \( r, s \) and \( k \) are integers:

\[
\sum_{j=0}^{k} (-1)^j F_{s-1}^{k-j} F_{s}^{j} F_{r+s+j} = F_r F_{s+1}^{k+1} - (-1)^{k+1} F_{r+k+1} F_{s}^{k+1}. \quad (3.48)
\]

In particular,

\[
\sum_{j=0}^{k} (-1)^j F_{s-1}^{k-j} F_{s}^{j} F_{j} = (-1)^{s-1} F_s F_{s-1}^{k+1} - (-1)^{k-1} F_{k-s+1} F_{s}^{k+1}. \quad (3.49)
\]

3.5 Further summation identities involving the generalized Tribonacci numbers

The next theorem, expressing a summation involving Tribonacci-Lucas numbers in terms of Tribonacci numbers, follows from identity (2.23) and Lemma 2.

Theorem 7. The following identity holds, where \( r \) and \( k \) are integers:

\[
\sum_{j=0}^{k} 5^{k-j} K_{r-2k-3+2j} = T_r - 5^{k+1} T_{r-2k-2}. \quad (3.50)
\]
In particular,
\[ \sum_{j=0}^{k} 5^{k-j} K_{2j} = T_{2k+3} - 5^{k+1}. \]  
(3.50)

Further summation identities are obtained from identity (2.20) and Lemma 3. These are presented in the next theorem.

**Theorem 8.** The following identities hold, where \( r \) and \( k \) are integers:

\[ 103 \sum_{j=0}^{k} 56^j \mathcal{T}_{r+16+17j} = 56^{k+1} \mathcal{T}_{r+17k+17} - \mathcal{T}_r, \]  
(3.51)

\[ 56 \sum_{j=0}^{k} (-1)^j 103^j \mathcal{T}_{r+17+16j} = \mathcal{T}_r - (-103)^{k+1} \mathcal{T}_{r+16k+16}, \]  
(3.52)

and

\[ \sum_{j=0}^{k} 103^{k-j} 56^j \mathcal{T}_{r-16+j} = -103^{k+1} \mathcal{T}_r + 56^{k+1} \mathcal{T}_{r+k+1}. \]  
(3.53)

In particular,

\[ 103 \sum_{j=0}^{k} 56^j \mathcal{T}_{17j} = 56^{k+1} \mathcal{T}_{17k+1} - \mathcal{T}_{-16}, \]  
(3.54)

\[ 56 \sum_{j=0}^{k} (-1)^j 103^j \mathcal{T}_{16j} = \mathcal{T}_{-17} - (-103)^{k+1} \mathcal{T}_{16k-1}, \]  
(3.55)

and

\[ \sum_{j=0}^{k} 103^{k-j} 56^j \mathcal{T}_j = -103^{k+1} \mathcal{T}_{16} + 56^{k+1} \mathcal{T}_{k+17}. \]  
(3.56)

### 4 Binomial summation identities

**Lemma 4 ([1, Lemma 3]).** Let \( \{X_r\} \) be any arbitrary sequence. Let \( X_r, r \in \mathbb{Z} \), satisfy a three-term recurrence relation \( X_r = f_1 X_{r-a} + f_2 X_{r-b} \), where \( f_1 \) and \( f_2 \) are non-vanishing complex functions, not dependent on \( r \), and \( a \) and \( b \) are integers. Then,

\[ \sum_{j=0}^{k} \binom{k}{j} \left( \frac{f_1}{f_2} \right)^j X_{r-(b-a)j} = \frac{X_r}{f_2^k}, \]  
(4.1)

\[ \sum_{j=0}^{k} \binom{k}{j} \frac{X_{r+(a-b)k+bj}}{(-f_2)^j} = \left( \frac{-f_1}{f_2} \right)^k X_r, \]  
(4.2)

and

\[ \sum_{j=0}^{k} \binom{k}{j} \frac{X_{r+(b-a)k+a_j}}{(-f_1)^j} = \left( \frac{-f_2}{f_1} \right)^k X_r, \]  
(4.3)

for \( k \) a non-negative integer.
The next theorem is a consequence of identity (1.18) and Lemma 4.

**Theorem 9.** The following identities hold, where $k$ is any non-negative integer and $r$ is any integer:

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^j W_{r-(n+1)k+nj} = (-1)^k W_r ,
\]

\[
\sum_{j=0}^{k} \binom{k}{j} W_{r-nk+(n+1)j} = 2^k W_r
\]

and

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^{k-j} W_{r+nk+j} = W_r .
\]

In particular,

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^j W_{nj} = (-1)^k W_{(n+1)k} ,
\]

\[
\sum_{j=0}^{k} \binom{k}{j} W_{(n+1)j} = 2^k W_{nk}
\]

and

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^{k-j} W_j = W_{-nk} .
\]

We remark that identity (4.9) proves Conjecture 2 (equation (15)) of Hisert [5].

### 4.1 Further binomial summation identities involving generalized Fibonacci numbers

In addition to the summation identities obtained by setting $n = 2$ in identities (4.4) – (4.9) of Theorem 9, we also have the results stated in the next theorem, on account of identity (2.14) and Lemma 4.

**Theorem 10.** The following identities hold, where $k$ is any non-negative integer and $r$ and $s$ are any integers:

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{s-1}^{k-j} F_{r-k+s+j} = (-1)^k F_s^k F_r ,
\]

\[
\sum_{j=0}^{k} \binom{k}{j} F_{s-1}^{k-j} F_s^j F_{r-sk+j} = F_r
\]

and

\[
\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} F_{s+1}^{k-j} F_{r+k+s+j} = F_s^k F_r .
\]
In particular,
\begin{equation}
\sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{s+1-j}^k F_{s+j} = (-1)^k F_{s}^k, \tag{4.13}
\end{equation}
\begin{equation}
\sum_{j=0}^{k} \binom{k}{j} F_{s+1-j}^k F_{s-j} = F_{s}^k, \tag{4.14}
\end{equation}
and
\begin{equation}
\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} F_{s+1-j}^k F_{s+j} = F_{s}^k. \tag{4.15}
\end{equation}

### 4.2 Further binomial summation identities involving generalized Tribonacci numbers

In addition to the summation identities obtained by setting \( n = 3 \) in identities (4.4) – (4.9) of Theorem 9, we also have the results stated in the next theorem, on account of identity (2.20) and Lemma 4.

**Theorem 11.** The following identities hold, where \( k \) and \( r \) are integers:
\begin{equation}
\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} 103^{k-j} 56^j T_{r+16k+j} = T_r, \tag{4.16}
\end{equation}
\begin{equation}
\sum_{j=0}^{k} 103^j \binom{k}{j} T_{r-17k+16j} = 56^k T_r \tag{4.17}
\end{equation}
and
\begin{equation}
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 56^j T_{r-16k+17j} = (-103)^k T_r. \tag{4.18}
\end{equation}

In particular,
\begin{equation}
\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} 103^{k-j} 56^j T_{j} = T_{-16k}, \tag{4.19}
\end{equation}
\begin{equation}
\sum_{j=0}^{k} 103^j \binom{k}{j} T_{16j} = 56^k T_{17k} \tag{4.20}
\end{equation}
and
\begin{equation}
\sum_{j=0}^{k} (-1)^j \binom{k}{j} 56^j T_{17j} = (-103)^k T_{16k}. \tag{4.21}
\end{equation}

### 5 Double binomial summation identities

**Lemma 5** ([2, Lemma 5]). Let \( \{X_r\} \) be any arbitrary sequence, \( X_r \) satisfying a four-term recurrence relation \( X_r = f_1 X_{r-a} + f_2 X_{r-b} + f_3 X_{r-c} \), where \( f_1 \), \( f_2 \) and \( f_3 \) are arbitrary nonvanishing functions and \( a \), \( b \) and \( c \) are integers. Then, the following identities hold:
\begin{equation}
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_2}{f_3} \right)^j \left( \frac{f_1}{f_2} \right)^s X_{r-ck+(c-b)j+(b-a)s} = \frac{X_r}{f_3^k}, \tag{5.1}
\end{equation}
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_3}{f_2} \right)^j \left( \frac{f_1}{f_3} \right)^s \left( \frac{f_1}{f_2} \right)^{(c-a)s} = \frac{X_r f_2^k}{f_2^k}, \tag{5.2}
\]
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_3}{f_1} \right)^j \left( \frac{f_2}{f_3} \right)^s \left( \frac{f_1}{f_2} \right)^{(c-b)s} = \frac{X_r f_1^k}{f_1^k}, \tag{5.3}
\]
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_2}{f_3} \right)^j \left( \frac{f_1}{f_2} \right)^s \left( \frac{f_1}{f_3} \right)^{(c-a)s} = \frac{X_r f_3^k}{f_3^k}, \tag{5.4}
\]
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_3}{f_1} \right)^j \left( \frac{f_1}{f_3} \right)^s \left( \frac{f_1}{f_2} \right)^{(c-b)s} = \frac{X_r f_2^k}{f_2^k}, \tag{5.5}
\]
\[
\] and
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} \binom{k}{j} \binom{j}{s} \left( \frac{f_3}{f_2} \right)^j \left( \frac{f_1}{f_3} \right)^s \left( \frac{f_1}{f_2} \right)^{(b-a)s} = \frac{X_r f_3^k}{f_3^k}. \tag{5.6}
\]
Evaluating identities (3.17)–(3.19) at \( k = 1 \) produces the following recurrence relations:
\[
W_r = 4W_{r-2} - W_{r-n-1} - 2W_{r-n-2}, \tag{5.7}
\]
\[
W_r = 2W_{r-1} - 2W_{r-n-2} + W_{r-2n-2} \tag{5.8}
\]
and
\[
2W_r = 4W_{r-1} - W_{r-n} - W_{r-2n-1}. \tag{5.9}
\]
Evaluating identities (4.4)–(4.6) at \( k = 2 \) gives the following recurrence relations:
\[
W_r = 4W_{r-2} - 4W_{r-n-2} + W_{r-2n-2} \tag{5.10}
\]
\[
W_r = 4W_{r-2} - 2W_{r-n-1} - W_{r-2n-2} \tag{5.11}
\]
and
\[
W_r = 4W_{r-1} - 4W_{r-2} + W_{r-2n-2}. \tag{5.12}
\]
Each of identities (5.7)–(5.12) has six double binomial summation identities associated with it. In the next theorem we give the double binomial summation identities resulting from identity (5.12).

**Theorem 12.** The following identities hold for nonnegative integer \( k \) and any integer \( r \):
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} (-1)^{j+s} \binom{k}{j} \binom{j}{s} 4^j W_{r-(2n+2)k+2nj+s} = W_r, \tag{5.13}
\]
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} (-4)^{k-j} \binom{k}{j} \binom{j}{s} 4^s W_{r-2k-2nj+(2n+1)s} = W_r, \tag{5.14}
\]
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} (-1)^{s} \binom{k}{j} \binom{j}{s} 4^{k-j+s} W_{r-(2n+1)j+2ns} = W_r, \tag{5.15}
\]
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} (-1)^{j-k} \binom{k}{j} \binom{j}{s} 4^{j-k-s} W_{r-(2n+1)k+2nj+2s} = W_r, \tag{5.16}
\]
\[
\sum_{j=0}^{k} \sum_{s=0}^{j} (-1)^s \binom{k}{j} \binom{j}{s} 4^{j-s-k} W_{r-2nk+s} = W_r
\]  
(5.17)

and

\[
\sum_{j=0}^{k} \sum_{s=0}^{j} (-1)^{j+s} \binom{k}{j} \binom{j}{s} 4^{k-s} W_{r+2nk+j+s} = W_r .
\]  
(5.18)

6 Partial sums and generating function

**Lemma 6** ([2, Lemma 2] Partial sum of a n-term sequence). Let \(\{X_j\}\) be any arbitrary sequence, where \(X_j, j \in \mathbb{Z}\), satisfies a n-term recurrence relation \(X_j = f_1 X_{j-c_1} + f_2 X_{j-c_2} + \cdots + f_n X_{j-c_n} = \sum_{m=1}^{n} f_m X_{j-c_m}\), where \(f_1, f_2, \ldots, f_n\) are arbitrary non-vanishing complex functions, not dependent on \(j\), and \(c_1, c_2, \ldots, c_n\) are fixed integers. Then, the following summation identity holds for arbitrary \(x\) and non-negative integer \(k\):

\[
\sum_{j=0}^{k} x^j X_j = \frac{\sum_{m=1}^{n} \left\{ x^c m f_m \left( \sum_{j=1}^{c_m} x^{-j} X_j - \sum_{j=k-c_m+1}^{k} x^j X_j \right) \right\}}{1 - \sum_{m=1}^{\sum} x^c m f_m} .
\]

We note that a special case of Lemma 6 was proved in [11].

The next theorem follows directly from Lemma 6 on account of identity (1.18).

**Theorem 13.** The following identity holds for \(k\) an integer and any \(x\):

\[
(1 - 2x + x^{n+1}) \sum_{j=0}^{k} x^j W_j = 2W_{-1} - 2x^{k+1} W_k + x^{n+1} \sum_{j=k-n}^{k} x^j W_j - x^{n+1} \sum_{j=1}^{n+1} x^{-j} W_{-j} .
\]

We now work out the special cases of the identity of Theorem 13 for the \(n\)-step Fibonacci and \(n\)-step Lucas numbers.

Now,

\[
\sum_{j=1}^{n+1} x^{-j} U_{-j} = x^{-1} U_{-1} + x^{-2} U_{-2} + \cdots + x^{-n+2} U_{-n+2}
\]

\[+ x^{-n+1} U_{-n+1} + x^{-n} U_{-n} + x^{-n-1} U_{-n-1} .
\]  
(6.1)

All except the last three terms on the right hand side of the above expression vanish on account of the initial terms as given in equation (1.2). Thus,

\[
\sum_{j=1}^{n+1} x^{-j} U_{-j} = x^{-n+1} U_{-n+1} + x^{-n} U_{-n} + x^{-n-1} U_{-n-1}
\]

\[= x^{-n+1} - x^{-n} + 2 \delta_{n,2} x^{-n-1} , \quad \text{by (1.2) and (1.5)} .
\]

Using (6.2) in the identity of Theorem 13 with \(W = U\) we have

\[
(1 - 2x + x^{n+1}) \sum_{j=0}^{k} x^j U_j = x - x^2 - 2x^{k+1} U_k + x^{n+1} \sum_{j=k-n}^{k} x^j U_j .
\]  
(6.3)
Next, we find
\[ \sum_{j=1}^{n+1} x^{-j}V_{-j} = x^{-1}V_{-1} + x^{-2}V_{-2} + \cdots + x^{-n+1}V_{-n+1} + x^{-n}V_{-n} + x^{-n+1}V_{-n-1} \]  
\[ = -(x^{-1} + x^{-2} + \cdots + x^{-n+1}) + (2n - 1)x^{-n} - (n + 2)x^{-n-1}; \]  
so that,
\[ x^{n+1} \sum_{j=1}^{n+1} x^{-j}V_{-j} = -(x^n + x^{n-1} + \cdots + x^3 + x^2) + (2n - 1)x - (n + 2) \]
\[ = -\frac{x^{n+1} - x^2}{x - 1} + (2n - 1)x - (n + 2). \]

Putting (6.5) in the identity of Theorem 13 with \( W = V \) we have
\[ (1 - x)(1 - 2x + x^{n+1}) \sum_{j=0}^{k} x^jV_j = n - (3n - 1)x + 2nx^2 - x^{n+1} \]
\[ - (1 - x)x^{k+1}2V_k + (1 - x)x^{n+1} \sum_{j=k-n}^{k} x^jV_j. \]  

Note that the identity of Theorem 13 cannot be used directly to compute \( \sum_{j=0}^{k} W_j \) because
\[ \sum_{j=k-n}^{k} W_j = \sum_{j=0}^{n} W_{j+k-n} = \sum_{j=0}^{n} W_{k-j} = W_k + \sum_{j=1}^{n} W_{k-j} = 2W_k \]  
and
\[ \sum_{j=1}^{n+1} W_{-j} = \sum_{j=1}^{n} W_{-j} + W_{-n-1} = W_0 + W_{-n-1} = 2W_{-1}; \]
so that both sides of the identity of Theorem 13 evaluates to zero at \( x = 1 \). Nevertheless, the said sum can be evaluated if we divide both sides of the identity by \( 1 - 2x + x^{n+1} \) and then use L'Hospital’s rule to take the limit at \( x = 1 \), giving
\[ (n - 1) \sum_{j=0}^{k} W_j = 2(n - k)W_k - 2(n + 1)W_{-1} + \sum_{j=k-n}^{k} jW_j + \sum_{j=1}^{n+1} jW_{-j}. \]  

Since
\[ \sum_{j=1}^{n+1} jU_{-j} = \sum_{j=1}^{n-2} jU_{-j} + (n - 1)U_{-n+1} + nU_{-n} + (n + 1)U_{-n-1} \]
\[ = 2(n + 1)\delta_{n,2} - 1; \]  
and
\[ \sum_{j=1}^{n+1} jV_{-j} = \sum_{j=1}^{n-1} jV_{-j} + nV_{-n} + (n + 1)V_{-n-1} \]
\[ = -\sum_{j=1}^{n-1} j + nV_{-n} + (n + 1)V_{-n-1} \]
\[ = \frac{n^2}{2} - \frac{7n}{2} - 2, \]
we obtain the following results for the sum of the first \( k + 1 \) terms of the \( n \)-step Fibonacci numbers and the first \( k + 1 \) terms of the \( n \)-step Lucas numbers:

\[
(n - 1) \sum_{j=0}^{k} U_j = -1 + 2(n - k)U_k + \sum_{j=k-n}^{k} jU_j \tag{6.12}
\]

and

\[
2(n - 1) \sum_{j=0}^{k} V_j = n(n - 3) + 4(n - k)V_k + 2 \sum_{j=k-n}^{k} jV_j . \tag{6.13}
\]

**Lemma 7** ([2, Lemma 3](Generating function). Under the conditions of Lemma 6, if additionally \( x^kX_k \) vanishes in the limit as \( k \) approaches infinity, then

\[
G_X(x) = \sum_{j=0}^{\infty} x^jX_j = \frac{\sum_{m=1}^{n} \left( x^{c_m}f_m \sum_{j=1}^{c_m} x^{-j}X_{-j} \right)}{1 - \sum_{m=1}^{n} x^{c_m}f_m},
\]

so that \( G_X(x) \) is a generating function for the sequence \( \{X_j\} \).

**Theorem 14.** The generalized \( n \)-step Fibonacci numbers have the following generating function:

\[
G_W(x; n) = \sum_{j=0}^{\infty} x^jW_j = \frac{2W_{-1} - x^{n+1} \sum_{j=1}^{n+1} x^{-j}W_{-j}}{1 - 2x + x^{n+1}} .
\]

In particular, from (6.3) and (6.6), we see that the \( n \)-step Fibonacci and \( n \)-step Lucas numbers are generated, respectively, by

\[
G_U(x; n) = \sum_{j=0}^{\infty} x^jU_j = \frac{x(1 - x)}{1 - 2x + x^{n+1}} \tag{6.14}
\]

and

\[
G_V(x; n) = \sum_{j=0}^{\infty} x^jV_j = \frac{n - (3n - 1)x + 2nx^2 - x^{n+1}}{(1 - x)(1 - 2x + x^{n+1})} . \tag{6.15}
\]

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