Threshold resummation $S$-factor in QCD: the case of unequal masses

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Abstract

The new relativistic Coulomb-like threshold resummation $S$-factor in quantum chromodynamics is obtained. The consideration is given within the framework of quasipotential approach in quantum field theory which is formulated in relativistic configuration representation for two particles of unequal masses.

1 Introduction

As is well known, a description of quark-antiquark systems close to threshold does not permit us to cut off the perturbative series even if the expansion parameter, the QCD coupling constant $\alpha_s$, is small [1, 2]. The reason consist in that the real expansion parameter in the threshold region is $\alpha/v$, where $v = \sqrt{1 - 4m^2/s}$ is a quark velocity and $m$ is a quark mass. The problem is well known from QED [3]. To obtain a meaningful result, these threshold singularities of the form $(\alpha/v)^n$ have to be summarized. In the nonrelativistic case, for the Coulomb interaction

$$V(r) = -\frac{\alpha}{r}$$

this resummation is realized by the known Gamov–Sommerfeld–Sakharov $S$-factor [4–6]

$$S_{nr} = \frac{X_{nr}}{1 - \exp(-X_{nr})}, \quad X_{nr} = \frac{\pi \alpha}{v_{nr}},$$

which is related to the wave function of the continuous spectrum at the origin by $|\psi(0)|^2$ (see books [7, 8]). Here $2v_{nr}$ is the relative velocity of two nonrelativistic particles. The corresponding nonrelativistic expression can also be obtained for higher $\ell$ states (see, e.g., Ref. [9]).

In the relativistic theory the nonrelativistic approximation needs to be modified. The relativistic modification of the $S$-factor (2) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was executed in Ref. [10] (see also Ref. [11]) and it consisted in the change $v_{nr} \rightarrow v$. This factor was used for the description of effects close to the threshold of pair production in the processes $e^+e^- \rightarrow t\bar{t}$ and $e^+e^- \rightarrow W^+W^-$ [11]. Just the same form of the $S$-factor for the interaction of two particles of equal masses was later suggested in Ref. [12]. Another form of the relativistic generalization of the $S$-factor also in the case of two particles of equal masses was obtained in Ref. [13]. The relativistic $S$-factor for two particles of arbitrary masses ($m_1 \neq m_2$) we are interested in was presented in Ref. [14]. This factor was derived within the framework of relativistic quantum mechanics on the basis of the Schrödinger equation.
The new step to relativistic generalization of the $S$-factor in the case of two particles of equal masses was made by Milton and Solovtsov in Ref. [15]. For this purpose, the relativistic quasipotential (RQP) approach proposed by Logunov and Tavkhelidze [16] in the form suggested by Kadyshhevsky [17] turned out to be convenient. In Ref. [15], a transformation of the quasipotential (QP) equation from momentum space into a relativistic configuration representation for two particles of equal masses (see Ref. [18]) was used. Moreover, in Ref. [15], the potential (1) considered in Ref. [19] was used which takes into account its QCD-like behaviour. The solution containing arbitrary functions of $r$ with period $i$, the so-called $i$-periodic constants, with the same potential was investigated in Ref. [20]. However, the use of this kind of solution is adequate for the spectral problems only. Other forms of the QP equation with the Coulomb potential were considered in Ref. [21].

Thus, in Ref. [15], a new step to application of the quasipotential approach in QCD was made. This approach gives the following expression for the relativistic $S$-factor:

$$S(\chi) = \frac{X(\chi)}{1 - \exp[-X(\chi)]}, \quad X(\chi) = \frac{\pi \alpha}{\sinh \chi},$$  \hspace{1cm} (3)

where $\chi$ is the rapidity related to the total c. m. energy of interacting particles $\sqrt{s}$ by $2m \cosh \chi = \sqrt{s}$. The function $X(\chi)$ in Eq. (3) can be expressed in terms of $v$ as $X(\chi) = \pi \alpha \sqrt{1 - v^2/v}$.

The resummation factor appears in the parametrization of the imaginary part of the quark current correlator, the Drell ratio $R(s)$, which can be approximated in terms of the Bethe-Salpeter (BS) amplitude of two charged particles $\chi_{BS}(x)$ at $x = 0$ (see Ref. [22]). The non-relativistic replacement of this amplitude by the wave function, which obeys the Schrödinger equation with the Coulomb potential (1), leads to formula (2) with a substitution $\alpha \to 4 \alpha_s/3$ for the QCD case. The possibility of using the QP approach for our task is based on the fact that the BS amplitude, which parameterizes the physical quantity $R(s)$, is taken at $x = 0$; therefore, in particular, at the relative time $\tau = 0$. Thus, the QP wave function in the momentum space is defined as the BS amplitude at $\tau = 0$ and, therefore, $R(s)$ can be expressed in terms of the QP wave function in the momentum space, $\Psi_q(p)$, by using the relation

$$\chi_{BS}(x = 0) = \frac{1}{(2\pi)^3} \int d\Omega_p \Psi_q(p),$$  \hspace{1cm} (4)

where $d\Omega_p = (m \, dp)/E_p$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_p^2 - p^2 = m^2$.

The relativistic $P$-factor (for $\ell = 1$ state) in the case of two particles of equal masses was obtained in Ref. [23]. In that paper, a new model expression for $R(s)$, in which threshold singularities were summarized to the main potential contribution, was suggested as well.

The generalization of the relativistic $S$- and $P$-factors for arbitrary $\ell$ states in the case of two particles of equal masses was discussed in Ref. [24]. Applications of the factor (3) for describing some hadronic processes can be found in Refs. [25–27]. Recently, the relativistic $S$-factor (3) has been applied to reanalyze the mass limits obtained for magnetic monopoles which might have been produced at the Fermilab Tevatron [28].

The purpose of this paper is to generalize the previous study started in Ref. [15] to the case of the interaction of two particles of unequal masses ($m_1 \neq m_2$). The paper is organized as follows. In the next section, we present the formalism of the RQP approach in quantum field theory.
formulated in the relativistic configuration representation for the interaction of two relativistic particles of unequal masses. In Sec. III, within the framework of this approach we derive the new relativistic \( S \)-factor and analyze its behavior in the following cases: the nonrelativistic and relativistic cases, the case of equal masses, the case of one particle being at rest, and the ultrarelativistic case. Also, we compare the obtained factor with the factors considered in Refs. [10–14] and study the behavior of the function \( R(s) \) that is expressed in terms of these factors in the case of vector current for two quarks of unequal masses. Summarizing comments are given in Sec. IV.

2 The integral form of the quasipotential equation: the case of two particles of unequal masses

A starting point of our consideration is the QP equation in the momentum space constructed in Ref. [29] for the RQP wave function \( \Psi_{q'}(p') \) of two relativistic particles of unequal masses. This equation is given by \(^1\)

\[
(2E_{q'} - 2E_{p'}) \Psi_{q'}(p') = \frac{2\mu}{m'(2\pi)^3} \int d\Omega_{k'} \tilde{V}(p', k'; E_{q'}) \Psi_{q'}(k'), \tag{5}
\]

where

\[
d\Omega_{k'} = \frac{m'dk'}{E_{k'}}
\]

is the relativistic three-dimensional volume element in the Lobachevsky space, \( E_{k'} = \sqrt{m'^2 + k'^2} \), \( m' = \sqrt{m_1 m_2} \), and \( \mu = m_1 m_2/(m_1 + m_2) \) is the usual reduced mass. \(^2\)

Equation (5) represents a relativistic generalization of the Lippmann-Schwinger equation in the spirit of the Lobachevsky geometry which is realized on the upper half of the mass hyperboloid \( E_{k'}^2 - k'^2 = m'^2 \). This equation describes the scattering over the quasipotential \( \tilde{V}(p', k'; E_{q'}) \) of an effective relativistic particle having mass \( m' \) and a relative 3-momentum \( k' \), emerging instead of the system of two particles and carrying the total c. m. energy of the interacting particles, \( \sqrt{s} \), proportional to the energy \( E_{k'} \) of one effective relativistic particle of mass \( m' \) (see [29, 31]):

\[
\sqrt{s} = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2} = \frac{m'}{\mu} E_{k'}, \quad E_{k'} = \sqrt{m'^2 + k'^2}. \tag{6}
\]

The proper Lorentz transformations mean a translation in the Lobachevsky space. The role of the plane waves corresponding to these translations is played by the following functions:

\[
\xi(p', r) = \left( \frac{E_{p'} - p' \cdot n}{m'} \right)^{-1 - irm'}, \tag{7}
\]

where the module of the radius-vector, \( r \), (\( r = r_\mathbf{n}, |\mathbf{n}| = 1 \)) is a relativistic invariant \([31]\). These functions correspond to the principal series of unitary representations of the Lorentz group and

\(^1\)In the following we will use the system of units \( \hbar = c = 1 \).

\(^2\)Various definitions of the relativistic reduced mass were discussed in Ref. [30].
in the nonrelativistic limit \( p' \ll 1, r \gg 1 \) \( \xi(p', r) \to \exp(ip \cdot r) \). The functions (7) obey the following conditions of completeness and orthogonality [31]:

\[
\frac{1}{(2\pi)^3} \int d\Omega_{p'q'} \xi(p', r) \xi^*(p', r') = \delta(r - r'),
\]

\[
\frac{1}{(2\pi)^3} \int d\Omega \xi(p', r) \xi^*(q', r) = \delta(p'(-q')) ,
\]

where \( \delta(p'(-q')) = \sqrt{1 + p'^2/m'^2} \delta(p' - q') \) is the relativistic \( \delta \)-function in momentum-space. Moreover, these the functions satisfy the equation in terms of finite differences

\[
\left(2E_{p'} - \hat{H}_0\right) \xi(p', r) = 0 .
\]

Here

\[
\hat{H}_0 = 2m' \left[ \cosh \left( i\lambda' \frac{\partial}{\partial r} \right) + i\lambda' r \sinh \left( i\lambda' \frac{\partial}{\partial r} \right) - \frac{\lambda'^2 \Delta_{\theta,\phi} }{2r^2} \exp \left( i\lambda' \frac{\partial}{\partial r} \right) \right]
\]

is the operator of the free Hamiltonian, while \( \Delta_{\theta,\phi} \) is its angular part and \( \lambda' = 1/m' \) is the Compton wavelength associated with the effective relativistic particle of mass \( m' \).

The QP wave functions in the momentum space and relativistic configuration representation [29,31] are related as follows:

\[
\psi_{q'}(r) = \frac{1}{(2\pi)^3} \int d\Omega_{p'} \xi(p', r) \Psi_{q'}(p') ,
\]

\[
\Psi_{q'}(p') = \int d\Omega \xi^*(p', r) \psi_{q'}(r) .
\]

For a spherically symmetric potential the application of transformations (11) (Shapiro transformations or \( \xi \)-transformations) to Eq. (5) leads to the equation which is the integral form of the relativistic Schrödinger equation in the configuration representation:

\[
\frac{1}{(2\pi)^3} \int d\Omega_{p'} (2E_{q'} - 2E_{p'}) \xi(p', r) \int d\Omega' \xi^*(p', r') \psi_{q'}(r') = \frac{2m}{m'} V(r) \psi_{q'}(r) ,
\]

where the right-hand side is already local in the configuration representation and the transform of the potential, \( V(r) \), is given in terms of the same relativistic plane waves.

We note that the use relations (11) and Eq. (9) allows us to express the left-hand side of Eq. (12) in terms finite differences

\[
\left(2E_{q'} - \hat{H}_0\right) \psi_{q'}(r) = \frac{2m}{m'} V(r) \psi_{q'}(r) .
\]

Solutions of this equation, in principle, can contain arbitrary functions of \( r \) with period \( i \), the so-called \( i \)-periodic constants, which appear in the solutions due to the finite difference nature of the Hamiltonian (10). For some problems, such as defining the bound state spectrum, this \( i \)-periodic constant is not important. However, for the purpose of extracting resummation factors one must develop a method which avoids this ambiguity. For this purpose instead of Eq. (13) we will to use Eq. (12). This equation can be reduced to the form

\[
\frac{1}{(2\pi)^3} \int d\Omega_{p} (2E_{q} - 2E_{\rho}) \xi(p, \rho) \int d\rho' \xi^*(p', \rho') \psi_{q}(\rho') = \frac{2m}{m'} V(\rho) \psi_{q}(\rho) .
\]
We introduced the following notations:

\[ q' = m' q, \quad p' = m' p, \quad q = \sinh(\chi_q) n_q, \quad p = \sinh(\chi_p) n_p, \quad |n_q| = |n_p| = 1, \]
\[ \rho = m' r, \quad \rho' = m' r', \quad \rho = m' r, \quad \rho' = m' r', \]
\[ dq' = m' dq, \quad d\Omega_q' = m' d\Omega_q, \quad \frac{dp}{Ep} = m' E_q, \quad E_p = m' E_p, \]

\[ E_q = \sqrt{1 + q^2}, \quad E_p = \sqrt{1 + p^2}, \quad \xi(p', r) = (E_p - p \cdot n)^{-1-i\mu} \equiv \xi(p, \rho), \]

\[ V(r) = V(\rho/m') \equiv m' V(\rho), \quad \psi_{\theta'}(r) = \psi_{m' q}(\rho/m') \equiv \psi_{q}(\rho), \quad \Psi_{\theta'}(p') \equiv m'^{-3} \Psi_{q}(p). \]

By using the expansions

\[ \xi(p, \rho) = \sum_{\ell=0}^{\infty} (2\ell + 1) i^\ell P_\ell(p, \cosh \chi) P_\ell \left( \frac{p \cdot \rho}{\rho} \right), \]
\[ \psi_q(\rho) = \sum_{\ell=0}^{\infty} (2\ell + 1) i^\ell \frac{\varphi_\ell(\rho, \chi)}{\rho} P_\ell \left( \frac{q \cdot \rho}{\rho} \right), \]

and also formula (see Ref. [18])

\[ p_\ell(p, \cosh \chi) = \frac{(-1)^\ell (\sinh \chi)^\ell}{\rho^{(\ell+1)}} \left( \frac{d}{d \cosh \chi} \right)^\ell \left( \frac{\sin \rho \chi}{\sinh \chi} \right), \]

Eq. (14) is transformed to

\[ \frac{2}{\pi} \int_0^{\infty} d\chi \left( \frac{\sinh \chi'}{\rho^{(\ell+1)}} \right)^{2\ell+2} (-1)^{\ell+1} (2 \cosh \chi - 2 \cosh \chi') \left( \frac{d}{d \cosh \chi} \right)^\ell \left( \frac{\sin \rho \chi'}{\sinh \chi'} \right) \times \]
\[ \times \left( \frac{d}{d \cosh \chi'} \right)^\ell \frac{1}{\sinh \chi'} \int_0^{\infty} d\rho \rho' \sin \rho' \chi' (-\rho')^{(\ell+1)} \varphi_\ell(\rho', \chi) = \frac{2\mu}{m'} V(\rho) \varphi_\ell(\rho, \chi) = \frac{2 \mu V(\rho) \varphi_\ell(\rho, \chi)}{m' \rho} \]

Here \( P_{\mu}^\nu(z) \) is a Legendre function of the first kind, and the function

\[ p_\ell(\rho, \cosh \chi) = \frac{(-1)^{\ell+1}}{\rho} \sqrt{\frac{\pi}{2 \sinh \chi}} (-\rho)^{(\ell+1)} P_{-1/2+i\rho}^{(\ell+1)}(\cosh \chi) \]

is the solution of Eq. (13) in the case when the interaction is switched off, \( V(r) \equiv 0; \chi' \) and \( \chi \)

are the rapidities which are related to \( E_p, E_q \) as \( E_p = \cosh \chi', E_q = \cosh \chi \), and the function

\[ (-\rho)^{(\ell+1)} = i^{\ell+1} \frac{\Gamma(\ell + 1 + i\rho)}{\Gamma(i\rho)} \]

is the generalized power [18] where \( \Gamma(z) \) is the gamma-function.

Thus, Eq. (17) differs from the corresponding equation in the case of two particles of equal masses (see [33]) only by the factor \( 2\mu/m' \) turning into 1 at \( m_1 = m_2 \).
3 Relativistic $S$-factor

The $\xi$-transformation (11) as applied to the Coulomb interaction (1) gives the potential in momentum space

$$V(\Delta) \sim \frac{1}{\chi_\Delta \sinh \chi_\Delta},$$

where the relative rapidity $\chi_\Delta$ corresponds to $\Delta = p'(-)k'$ and is defined in terms of the square of the momentum transfer by $Q^2 = -(p' - k')^2 = 2(\cosh \chi_\Delta - 1)$ (see, for detail, Ref. [15]). For large $Q^2$ the potential $V(\Delta)$ behaves as $(Q^2 \ln Q^2)^{-1}$, which reproduces the principal behavior of the QCD potential proportional to $\bar{\alpha}_S(Q^2)/Q^2$ with $\bar{\alpha}_S(Q^2)$ being the QCD running coupling. This property of the potential (1) was noted in Ref. [19].

For the first time, the solution of Eq. (17) in the case of the interaction of two relativistic particles of equal masses at $\ell = 0$, not containing the $i$-periodic constants, was obtained in Ref. [15]. This approach leads to the relativistic $S$-factor (3). To solve the RQP equation (17) with the potential (1), we use the method developed in Ref. [15] (see also Refs. [32, 33]).

We will seek a solution of Eq. (17) with the potential (1) in the form

$$\varphi_\ell(\rho, \chi) = \frac{(-\rho)^{(\ell+1)}}{\rho} \int_\alpha^\beta d\zeta e^{i\rho\zeta} R_\ell(\zeta, \chi),$$

where the $\zeta$-integration is performed in the complex plane over a contour with end points $\alpha$ and $\beta$: $\alpha = -R - i\varepsilon$, $\beta = -R + i\varepsilon$, $R \to +\infty$, $\varepsilon \to +0$. Substituting (19) into (17) and taking into account that

$$\frac{1}{i\pi} \int_0^\infty d\rho' \sin(\rho' \chi') e^{i\rho'\zeta} = \frac{1}{i\pi} \frac{\chi'}{\chi'^2 - \zeta^2},$$

we obtain the equation

$$(-1)^\ell \int_\alpha^\beta d\zeta R_\ell(\zeta, \chi) \left( \frac{d}{d\cosh \zeta} \right)^\ell \left[ (\sinh \zeta)^{2\ell+1} (2 \cosh \chi - 2 \cosh \zeta) \times \right.$$

$$\left. \times \left( \frac{d}{d\cosh \zeta} \right)^\ell \frac{e^{i\rho\zeta}}{\sinh \zeta} \right] = -\frac{2 \alpha \mu}{m' \rho} \prod_{n=1}^\ell (\rho^2 + n^2) \int_\alpha^\beta d\zeta e^{i\rho\zeta} R_\ell(\zeta, \chi).$$

(20)

It should be noted that solutions of this equation, and hence Eq. (17), do not contain any more the $i$-periodic constants which appear in the solutions of Eq. (13) due to the finite difference nature of the Hamiltonian (10).

The solution of Eq. (20) at $\ell = 0$ leads to the following expression for the RQP partial wave function:

$$\varphi_0(\rho, \chi) = C_0(\chi) \frac{\rho}{\rho^{(1)}} \int_\alpha^\beta d\zeta \frac{e^{(i\rho+1)\zeta}}{(e^\zeta - e^\chi)^2} \left[ \frac{e^\zeta - e^{-\chi}}{e^\zeta - e^\chi} \right]^{-1+iA},$$

(21)

where

$$A = \frac{\alpha \mu}{m' \sinh \chi}.$$
Performing in Eq. (21) \( \zeta \)-integration in the complex plane along a contour with end points \( \alpha \) and \( \beta \) (in the same way as in Ref. [15, 32, 33]) we obtain the resulting solution which does not contain the \( i \)-periodic constant in the form

\[
\varphi_0(\rho, \chi) = C_0(\chi) \frac{2\rho \sinh(\pi \rho)}{\rho(1)} \int_{-\infty}^{\infty} dx \frac{e^{(i\rho+1)x}}{(e^x + e^\chi)^2} \left[ \frac{e^x + e^{-\chi}}{e^x + e^\chi} \right]^{-1+iA}. \tag{22}
\]

This solution can also be represented in terms of hypergeometric function as

\[
\varphi_0(\rho, \chi) = -N_0(\chi)(-\rho)^{(1)} e^{i\rho \chi + iA \chi} F \left( 1 - iA, 1 - i\rho; 2; 1 - e^{-2\chi} \right). \tag{23}
\]

The normalization constant \( N_0(\chi) \) in Eq. (23) can be obtained (see Ref. [15]) from the condition

\[
\lim_{\alpha \to 0} \varphi_0(\rho, \chi) = \frac{\rho p_0(\rho, \cosh \chi)}{\rho \to \infty} \frac{\sin(\rho \chi)}{\sinh \chi}. \tag{24}
\]

We should like to remind that the Bethe-Salpeter amplitude \( \chi_{BS}(x = 0) \) is associated with the RQP wave function in the momentum space, \( \Psi_q(p) \), by relation (4). Taking into account the transformations (11) and notations (15), the relationship of the Bethe-Salpeter amplitude with the RQP wave function, \( \psi_q(\rho) \), is

\[
\chi_{BS}(x = 0) = \psi_q(\rho)|_{\rho = i}.
\]

The generalized power (18) in the solution (19) vanishes at \( \rho = i \) for all \( \ell \neq 0 \). Thus, the expansion (16) for the wave function \( \psi_q(\rho) \) contains only \( s \)-waves (\( \ell = 0 \)). Hence, by using relations (23) and (24) we can calculate \( |\psi_q(i)|^2 \), which leads to the following expression for the relativistic \( S \)-factor in the case of two particles of unequal masses:

\[
S_{uneq}(\chi) = \lim_{\rho \to i} \left| \frac{\varphi_0(\rho, \chi)}{\rho} \right|^2 = \frac{X_{uneq}(\chi)}{1 - \exp[-X_{uneq}(\chi)]}, \quad X_{uneq}(\chi) = \frac{2\pi \alpha \mu}{m' \sinh \chi}, \tag{25}
\]

where \( \chi \) is the rapidity which is related to the total c. m. energy, \( \sqrt{s} \), as \( (m^2/\mu) \cosh \chi = \sqrt{s} \).

The function \( X_{uneq}(\chi) \) in Eq. (25) can be expressed in terms of the “velocity” \( u \) determined by the relation

\[
u = \sqrt{1 - \frac{4m'^2}{s}}, \quad \bar{s} = s - (m_1 - m_2)^2, \tag{26}
\]

in the form

\[
X_{uneq}(\chi) = \frac{\pi \alpha \sqrt{1 - u^2}}{u}. \tag{27}
\]

We note that the square of relative 3-momentum \( \mathbf{k}' \) for an effective relativistic particle, having mass \( m' \), the total c. m. energy of interacting particles, \( \sqrt{s} \), and emerging instead of the system of two particles, is defined by formula (6) and is connected with the relativistic relative velocity of interacting particles, \( v \), by the following expression (see Refs. [29, 31]):

\[
\mathbf{k}'^2 = 2\mu^2 \left( \frac{1}{\sqrt{1 - v^2}} - 1 \right). \tag{28}
\]
In turn from relations (6) and (28) follows that the relativistic relative velocity of interacting particles \( v \) can be expressed through their total energy c. m. of interacting particles \( \sqrt{s} \) by relation (exactly also as, for instance, in Refs. [13, 14])

\[
v = 2 \sqrt{\frac{s - (m_1 + m_2)^2}{s - (m_1 - m_2)^2}} \left[ 1 + \frac{s - (m_1 + m_2)^2}{s - (m_1 - m_2)^2} \right]^{-1}.
\]  

(29)

Thence, taking into consideration the determination (26), we find

\[
v = \frac{2 u}{1 + u^2}.
\]  

(30)

Then expressions (28) and (30) give

\[
k'^2 = (\mu)^2 (u'_\text{rel})^2,
\]  

(31)

where

\[
u'_\text{rel} = \frac{2u}{\sqrt{1 - u^2}}
\]  

(32)

is the relative velocity of an effective relativistic particle with mass \( m' \) emerging instead of the system of two particles. This result is found to be in full agreement with the physical meaning of Eq. (5) which is a relativistic generalization of the Lippmann-Schwinger equation in the spirit of Lobachevsky geometry. This equation describes the scattering of an effective relativistic particle on the quasipotential \( \tilde{V} (p', k'; E_q) \). The effective particle emerges instead of the system of two particles, has the mass \( m' \), the relative 3-momentum \( k' \) and carries the total c. m. energy of interacting particles, \( \sqrt{s} \). Notice that the 3-momentum \( k' \) of an effective relativistic particle and hence its relative velocity (32), according to Eqs. (28) and (31), are invariants of the Lorentz transformations.

Thus, in terms of relative velocity of an effective relativistic particle (32), the \( S \)-factor (25) is given by the expression

\[
S_{\text{uneq}}(u'_\text{rel}) = \frac{X_{\text{uneq}}(u'_\text{rel})}{1 - \exp \left[-X_{\text{uneq}}(u'_\text{rel})\right]}, \quad X_{\text{uneq}}(u'_\text{rel}) = \frac{2\pi\alpha}{u'_\text{rel}}.
\]  

(33)

The factor (33) only formally has the same form as the nonrelativistic factor (2). However, the factor (33) has an obviously relativistic nature since as the argument \( r \) (the module of radius-vector \( r \)) in the Coulomb potential (1) and the relativistic relative velocity of interacting particles \( v \) (see Ref. [31]) both are relativistic invariants and hence the relative velocity of an effective relativistic particle (32), according to Eqs. (28) and (31), possesses this property as well.

The relativistic threshold resummation factor (25) [or of form (33)] has the following important properties:

- In the nonrelativistic limit, \( u \ll 1 \), it reproduces the well-known nonrelativistic result.
- In the relativistic limit, \( u \to 1 \), the \( S \)-factor (25) [or (33)] goes to unity.
- In the case of equal masses it coincides with the \( S \)-factor (3).
The case when one of the particles is at rest means that $m_1 \to \infty$. This gives the following limiting expression for the “velocity” $u$:

$$u \to \frac{|k|}{\sqrt{m_2^2 + k^2 + m_2}}.$$ 

In the ultrarelativistic limit, as it was argued in Refs. [34, 35], the bound state spectrum vanishes since mass of an effective relativistic particle $m' \to 0$. This feature reflects an essential difference between potential models and quantum field theory where an additional dimensional parameter appears. One can conclude that within a potential model the $S$-factor which corresponds to the continuous spectrum should go to unity in the limit $m' \to 0$. Thus, in contrast to the nonrelativistic case, the relativistic resummation factor, the $S$-factor (25) [or (33)], reproduces both the known nonrelativistic and the expected ultrarelativistic limits.

To illustrate the differences between the nonrelativistic $S$-factor (2) and the new relativistic $S$-factor (33) in more detail, in Fig. 1 we plot the behavior of these factors as functions of $u$ at different values of the constant $\alpha$ (the numbers at the curves). The solid lines correspond to the $S$-factor (33) and the dashed lines to the $S$-factor (2) with a substitution $v_{nr} \to u$. From this figure one can see that in the region of nonrelativistic values of $u$, $u \leq 0.2$, where the influence of the $S$-factor is big, the difference between (32) and (2) is practically absent. However, when $\alpha$ increases, the nonrelativistic expression (2) gives a less suitable result in the region of large values $u$, in particular, as $u \to 1$.

It should be stressed that the $S$-factor (33) differs from the $S$-factor for the interaction of two relativistic particles of unequal masses

$$S_A(v) = \frac{X_A(v)}{1 - \exp[-X_A(v)]}, \quad X_A(v) = \frac{2\pi\alpha}{v},$$

which was obtained in Ref. [14], by the meaning of the relativistic relative velocity, $v$, which here is given by expression (29). Besides, in the case of equal masses ($m_1 = m_2 = m$), the new $S$-factor (33) differs also from the factor

$$S_H(v_H) = \frac{X_H(v_H)}{1 - \exp[-X_H(v_H)]}, \quad X_H(v_H) = \frac{2\pi\alpha}{v_H}, \quad v_H = \frac{2\beta}{1 + \beta^2}, \quad \beta = \sqrt{1 - \frac{4m^2}{s}},$$

which was presented in Ref. [12]. Indeed, the factors (34) and (35) in form and in the non-relativistic limit ($v, v_H, u \to 0$) coincide with the factor (33). However, the relativistic limits ($v, v_H \to 1$) of the factors (34) and (35) differ essentially from the relativistic limit of the factor (33) equal to unity as $u \to 1$. Furthermore, in the case of the interaction of two relativistic particles of equal masses the $S$-factor (33) differs from the factor

$$K = G(\eta) \kappa,$$

where

$$G(\eta) = \frac{2\pi\eta}{1 - \exp(-2\pi\eta)},$$

which was obtained in Ref. [13], not only in form but also in a different behavior in the non-relativistic ($v, u \to 0$) and the relativistic ($v, u \to 1$) limits (see Figs. 2 and 3 below). The

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3The same expression can be found in earlier paper [10].
function \( G(\eta) \) in Eq. (37) is the nonrelativistic \( S \)-factor (2) with \( \eta = \alpha / v \), and \( \kappa \) is a correction factor the expression for which contains the series and infinite products as multipliers and is rather cumbersome. According to Ref. [13], the influence of the \( \kappa \) is essential since \( \kappa \to 1 \) only as \( \alpha \to 0 \).

To illustrate the difference between the above factors, we show in Fig. 2 the behavior of the \( S \)-factors (2), (33), (34) and (36) as functions of \( u \) for a fixed value of \( \alpha = 0.16 \). The solid line represents the relativistic \( S \)-factor (33); the dashed line the nonrelativistic \( S \)-factor (2); the dotted line the factor defined by formula (36); the dash-dotted the factor (34). This figure demonstrates that the \( S \)-factors considered have an essentially different behavior as \( u \to 1 \).

To compare the relativistic factors (33) and (36) in more detail, we plot in Fig. 3 the ratio, which is denoted as \( N(\eta) \), of the relativistic factor (33) or (36) to the nonrelativistic \( S \)-factor (37) for different values of \( \alpha \) (the numbers at the curves). The solid lines correspond to the \( S \)-factor (33). The dashed lines, which are taken from Ref. [13], represent the ratio of the factor (36) to the nonrelativistic \( S \)-factor. As can be seen from Fig. 3, there is an essential difference between the relativistic factors (33) and (36). For example, in the nonrelativistic limit, when \( \eta \) increases (\( v \to 0 \)), the relativistic \( S \)-factor (36) reproduces the nonrelativistic limit only as \( \alpha \to 0 \) (see Ref. [13] for additional details).

Thus, the above analysis demonstrates that the relativistic \( S \)-factor (33), as would be expected, coincides in form with the nonrelativistic \( S \)-factor (2). However, the relative velocity of an effective relativistic particle (32) emerging instead of the system of two particles, now plays role of the parameter of velocity, but not the relativistic relative velocity of interacting particles, \( v \).

Let us briefly discuss the application of the \( S \)-factor. As noted above in the introduction, in the vicinity of the quark-antiquark threshold one cannot truncate the perturbative series and the resummation factor should be taken into account in its entirety. Involving a summation of threshold singularities by using the \( S \)-factor leads to the following modification of the dominant contribution of the function \( R(s) \) in the case of vector current

\[
R(s) \rightarrow R_V^{(0)}(s) = \left[ 1 - \frac{(m_1 - m_2)^2 s}{s} \right]^2 \left[ \frac{u(3 - u^2)}{2} + \frac{(m_1 - m_2)^2}{2s} u^3 \right] S(u, \alpha),
\]

where, according to Eq. (25), \( s = [(m_1 + m_2)^2 - (m_1 - m_2)^2 u^2]/(1 - u^2) \). By using this formula, we study the influence of the \( S \)-factor to the function \( R_V^{(0)} \).

Figure 4 demonstrates the difference in the behavior of \( R_V^{(0)} \), which appears when we use the above factors, as functions of variable \( u \) for a fixed value of \( \alpha = 0.16 \). The solid line corresponds to the \( S \)-factor (33), the dashed line to the nonrelativistic \( S \)-factor (2), and the dashed-dotted line to expression (34). The curves are built for effective quark masses \( m_1 = 250 \) MeV and \( m_2 = 500 \) MeV. These values are close to the constituent masses \( u \)- and \( s \)-quarks (see, for details, Ref. [27]). As can be seen from this figure, the behavior of curves is essentially different, especially as \( u \to 1 \).

Figure 5 demonstrates the behavior of the \( R_V^{(0)} \) calculated with the new \( S \)-factor (33) obtained here versus the dimensionless variable \( \sqrt{s}/(m_1 + m_2) \) for various values of \( \alpha \) (the numbers

\[4\] The corresponding expression without the \( S \)-factor can be found in Ref. [36].

\[5\] We plan to apply this \( S \)-factor to the hadronic \( \tau \) decay mediated by the strange current via \( W^- \to s\bar{u} \).
at the curves). The dashed line corresponds to the case without the $S$-factor (or $\alpha = 0$). This figure shows that the influence of relativistic threshold resummation is much stronger in the threshold region and with growing energy $\sqrt{s}$ weakens, and all curves approach unity.

4 Conclusions

In this paper, the new relativistic threshold resummation $S$-factor (33) for the interaction of two relativistic particles of unequal masses was obtained. For this aim the relativistic quasipotential equation in relativistic configuration representation [29] with the Coulomb potential for the interaction of two relativistic particles of unequal masses was used. The Coulomb potential only formally has the same form as the nonrelativistic potential but differs in the relativistic configuration representation since its behavior corresponds to the quark-antiquark potential $V_{q\bar{q}} \sim \bar{\alpha}_s(Q^2)/Q^2$ with the invariant charge $\bar{\alpha}_s(Q^2) \sim 1/\ln Q^2$. So, the principal effect coming from the running of the QCD coupling is accumulated.

The new relativistic $S$-factor (33) obtained reproduces both the known nonrelativistic behavior and the expected ultrarelativistic limit. The new $S$-factor coincides in form with the nonrelativistic $S$-factor (2); however, the role of the parameter of velocity is played not by the relative velocity of interacting particles, $v$, but by the relative velocity (32) of an effective relativistic particle emerging instead of the system of two particles. It was shown that there is a difference (see Figs. 2–4) between the expression (33) obtained here and other known forms (33), (34) and (35). As the new relativistic resummation factor (33) was obtained within the framework of completely covariant method, one can expect that this factor takes into account more adequately relativistic nature of interaction.

It was demonstrated that the new relativistic resummation factor has the influence on the behavior of the function $R(s)$. In some physically interesting cases the function $R(s)$ occurs as a factor in an integrand, as, for example, in the case of inclusive $\tau$ decay or in the Adler $D$-function, and the behavior of the $S$-factor at intermediate values of variable $u$ becomes important.

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Figure 1: Behavior of the $S$-factor at different values of the constant $\alpha$ (the numbers at the curves). The solid lines correspond to the new relativistic $S$-factor (33) and the dashed lines to the nonrelativistic $S$-factor (2).
Figure 2: Comparison of the $S$-factor behavior. The solid curve corresponds to the relativistic $S$-factor (33). The nonrelativistic $S$-factor (2) is presented by the dashed line, the factor (34) is presented by the dash-dotted line, and the factor (36) is shown as the dotted line.

Figure 3: Ratio of relativistic $S$-factor to nonrelativistic one, $N(\eta)$, for different values of $\alpha$ (the numbers at the curves). The solid lines correspond to the $S$-factor (33) and the dashed lines taken from Ref. [13] correspond to the factor (36).
Figure 4: Behavior of the function $R_V^{(0)}$ given by Eq. (38) as a function of variable $u$ for $\alpha = 0.16$. The result for the $S$-factor (33) is shown as the solid line, for the nonrelativistic $S$-factor (2) as the dashed line, and for the factor (34) as the dash-dotted line.

Figure 5: Behavior of the function $R_V^{(0)}$ with the $S$-factor (33) as a function of dimensionless variable $\sqrt{s}/(m_1 + m_2)$ for different values of $\alpha$ (the numbers at the curves). The dashed line represents $R_V^{(0)}$ without the $S$-factor.