Field Theory as a Matrix Model

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A new formulation of four dimensional quantum field theories, such as scalar field theory, is proposed as a large $n$ limit of a special $n \times n$ matrix model. Our reduction scheme works beyond planar approximation and applies for QFT with finite number of fields. It uses quenched coordinates instead of quenched momenta of the old Eguchi-Kawai reduction known to yield correctly only the planar sector of quantum field theory. Fermions can be also included.

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1. Introduction

To reproduce the physical space-time out of some more fundamental variables, rather than introduce it explicitly, has been always a tempting idea in the quantum field theory. While the string theory sets up a fruitful framework in which the space-time is dynamically created out of the fluctuating coordinates of the strings, potentially not less fruitful may be the attempts to encode the space-time, together with the quantum fields themselves, into the dynamics of large fluctuating matrices. A quantum field theory or a string theory should be described in this case in terms of specific matrix integrals containing finite amount of matrices of infinite size. The earliest proposal of this kind belongs to T. Eguchi and H. Kawai [1], followed by a few important precisions and modifications [2], [3], [4], [5]. In these works the QFT with \( N \times N \) matrix valued fields can be reduced in the large \( N \) (planar, or ’t Hooft) limit to finite dimensional matrix integrals in the same limit.

Another successful enterprise of this kind was a formulation and solution of non-critical string theories (associated with the two dimensional quantum gravity in the presence of some matter fields) in terms of \( U(N) \) invariant matrix integrals [6], [7].

More recently, one of the most fruitful ideas of this kind in the superstring theory was the proposal of E. Witten [8] generalizing an old idea of Chan-Paton to describe the low energy physics of branes by various reductions of super Yang-Mills theory. The diagonal components of the effective SYM vector potentials play the role of space-time coordinates of branes there.

The concept of the physical space-time built out of discrete degrees of freedom has become especially inspiring due to fundamental questions in quantum gravity, such as microscopic explanation of the thermodynamics of black holes.

Our discussion in this paper will be confined to the matrix model formulation of quantum field theory. To set up the problem let us recall that the old Eguchi-Kawai (EK)
reduction reproduces correctly only the planar sector of a matrix field theory, whereas the non-planar corrections ($1/N$ expansion) were never incorporated into this scheme, let alone a nonperturbative formulation of a QFT with finite number of field components (finite $N$) in terms of some matrix model of a matrix or matrices of infinite size. The reason for this difficulty is mostly due to the lack of reduced momenta running around nontrivial cycles of non-planar graphs in the EK reduction scheme.

The aim of the present paper is to propose a new formulation of a finite component scalar field theory (finite $N$) in terms of an $n \times n$ matrix integral in the limit $n \to \infty$. One can say that it incorporates all orders of the $1/N$ expansion and is in principle a possible non-perturbative definition of the original scalar field theory. For example, the usual scalar $\phi^4$ theory can be formulated as a one matrix integral in external matrix sources. We will also show how to incorporate fermions into this scheme. Unfortunately, we did not find so far any natural way to formulate the four dimensional QCD in this way.

Our construction is in some sense T-dual to the old EK scheme: we use the diagonal matrix sources of quenched coordinates instead quenched momenta of the EK scheme. As a consequence of it the original scalar field “lives” on the graphs dual to feynman graphs of our matrix model. To control the parameters of these graphs (say, to make them exactly $\phi^4$ graphs) we apply the methods worked out in [9] and [10] for the so called model of dually weighted graphs (DWG).

We will be able to generalize our method to fermions and to their yukawa interactions with the scalars, but for the moment we don’t know a natural way to introduce gauge symmetry into our approach. So to formulate QCD is an interesting challenge in our framework.

1 Using this abbreviation we remember of course about the important modification of the original, not quite working, reduction scheme of Eguchi and Kawai related to the quenching of momenta and twisting.
2. Difficulty with $1/N$ corrections in the EK reduction scheme

Let us remind the essence of the old (and unsolved, to our knowledge) problem of $1/N$ corrections to the reduced version of planar field theory. We will mostly discuss a matrix version of scalar field theory in the euclidean four dimensional space described by the action

$$S = N \int d^4x \text{tr} \left((\partial_\mu \phi)^2 + V(\phi)\right)$$

where $\phi_{nm}(x)$ is an $N \times N$ hermitian matrix field and $V(\phi) = \sum_{k \geq 2} \frac{1}{k} t_k \phi_k$ is a scalar potential.

The EK reduction (in the most natural, Parisi formulation [3]) goes as follows: take the following particular dependence of $\phi$ on the coordinates:

$$\phi_{mn}(x) = e^{-i(p_m \cdot x)} \phi_{mn} e^{i(p_n \cdot x)}, \quad \text{no summation over } m, n$$

The action then takes the form

$$S = \mathcal{V} \text{tr} \left([p_\mu, \phi]^2 + V(\phi)\right)$$

where $\mathcal{V}$ is the 4D volume of the physical space, $p_\mu = \text{diag}(p^{(1)}_\mu, \ldots, p^{(N)}_\mu)$ are D diagonal matrices of quenched momenta scattered uniformly [4] in a large 4D “momentum box” of a size $\Lambda^4$, where $\Lambda$ is UV cutoff, and the original functional integral over scalar field is replaced by a single matrix integral over $x$-independent $\phi_{mn}$. The planar sectors (leading large $N$ approximation) of the two models, the original matrix scalar field theory in 4D and the reduced 0D one matrix model, coincide. It is immediately clear from the double line representation of planar graphs in the reduced theory (fig.1): each face of such a graph is associated with a closed index loop and with the momentum variable $p^{(i)}_\mu$ carrying the same index $i$; each double line propagator $D^{ii'}_{jj'} = \frac{1}{N} \frac{\delta_{ii'} \delta_{jj'}}{(p_i - p_j)^2}$ depends on the difference of the momenta of adjacent faces.
Fig. 1: A fragment of a planar diagram in the double line notation and the propagator depending on the quenched momenta $p_k, p_j, k = 1, \cdots, N$ in Eguchi-Kawai reduction scheme.

The planar part of the free energy of the reduced model $F_{\text{plan}} = \lim_{N \to \infty} \frac{1}{N^2} \log Z_N$ ($Z_N$ is the partition function) can be schematically written in the following way:

$$F_{\text{plan}} = V \sum_G \prod_v t_{k_v} \left( \prod_{\tilde{v}} \frac{1}{N} \sum_{i_{\tilde{v}}} \right) \prod_{<\tilde{v}\tilde{v}'>} \frac{1}{(p_{i_{\tilde{v}}} - p_{j_{\tilde{v}'}})^2}$$

(2.4)

where $\sum_G$ goes over planar Feynman graphs $G$ of the scalar field theory, $v, \tilde{v}$ we label respectively original and dual vertices of a graph $G$, $i_{\tilde{v}}, 1, \cdots, N$ is the index associated with the dual vertex $\tilde{v}$ and $\prod_{<\tilde{v}\tilde{v}'>}$ goes over the edges $<\tilde{v}\tilde{v}'>$ of the graph $\tilde{G}$ dual to $G$.

In the large $N$ limit only planar graphs survive in both models and the sums over indices reproduce the integrals over 4D momenta: $\frac{1}{N} \sum_{i=1}^{N} \cdots \to \int_{|p|<\Lambda} d^4p \cdots$. Hence the planar sectors of the original matrix scalar field theory and of its reduced version are equivalent. This equivalence extends to any one point $U(N)$ invariant physical quantities.
of the type $O_k = \langle \frac{t^k}{N} \phi^k(x) \rangle$ but is known to fail for the multi-point correlators since
$\langle \frac{t^k}{N} \phi^k(x) \frac{t^l}{N} \phi^l(y) \rangle = \langle \frac{t^k}{N} \phi^k(x) \rangle \langle \frac{t^l}{N} \phi^l(y) \rangle + O(1/N^2)$ corrections, different in two models. The EK reduction fails to describe correctly the higher $1/N$ (non-planar) corrections to the original scalar field theory.

The reason for this failure is well known: we cannot represent all momenta running through the propagators as differences of momenta of the adjacent loops (faces) on the graphs of a non-spherical topology. If we did so (and it is precisely the case of the topological expansion in the EK reduced model) the momenta running along topologically nontrivial cycles of a non-planar feynman graph would be zero. To see it one takes any nontrivial closed path on a dual graph (connecting dual vertices or original faces) and calculates the total momentum running through the propagators crossed by this path as $(p_i - p_j) + (p_j - p_k) + \cdots + (p_l - p_i) \equiv 0$. For example, for the torus topology we have two momenta of the original 4D theory missing in the reduced version: they flow through two nontrivial cycles of the torus.

The reason for the failure is simple but the remedy is not easy to find, at least in case of the EK reduction involving reduced momenta.

In the next section we will show that the goal of construction of a matrix model describing a finite $N$ (including the most frequent $N = 1$ case) scalar field theory can be achieved by introducing quenched coordinates instead of quenched momenta.

3. Matrix model formulation of finite $N$ scalar field theory

Now we will propose a one matrix model in external matrix fields which will be equivalent, at least perturbativly, graph by graph of any topology, to the original 4D finite $N$ matrix scalar field theory (2.1). We will show that the free energy and, with an appropriate definition, the physical quantities of the field theory (2.1) at finite $N$ coincide with those of the matrix integral over a hermitian $n \times n$ matrix $\Phi$ in the limit $n \to \infty$:

$$Z = e^{N^2 F} = \int d^n \Phi e^{-S} \quad (3.1)$$
with the action:

\[
S = N \text{Tr}_\mathcal{N} \left( [X_{\mu}, \Phi]^2 + \ln(I_{\mathcal{N}} - A\Phi) \right)
\]  

(3.2)

Here the matrix \( \Phi \) lives in the \( n = p \cdot q \) dimensional vector space \( \mathcal{N} \) which is a direct product \( \mathcal{N} = \mathcal{P} \times \mathcal{Q} \) of vector spaces of smaller dimensions \( p \) and \( q \), correspondingly. \( I_n \) is the \( n \times n \) unity matrix. Both dimensions \( p, q \) go to infinity as \( n \to \infty \); \( X_{\mu} \) and \( A \) are external (fixed) diagonal matrices of the form

\[
X_{\mu} = \hat{x}_{\mu} \times I_{\mathcal{Q}},
\]

(3.3)

where \( \hat{x}_{\mu} = \text{diag}(x^{(1)}_{\mu}, \ldots, x^{(p)}_{\mu}) \),

\[
A = I_{\mathcal{P}} \times \hat{a}
\]

(3.4)

where \( \hat{a} = \text{diag}(a_1, \ldots a_q) \).

So the matrices \( \hat{x}_{\mu} \) on the one hand and the matrix \( \hat{a} \) on the other hand live in orthogonal subspaces.

The matrices \( \hat{x}_{\mu} \) with \( \mu = 1, 2, 3, 4 \) will play the role of quenched coordinates: the points with coordinates \( x^{(i)}_{\mu} \) should be distributed uniformly in the physical space box of a size \( L^4 \), which is the size of our system. The ultraviolet cutoff is defined as \( \Lambda \sim \frac{p^{1/4}}{L} \).

Obviously the thermodynamic limit of (infinite volume) corresponds to \( p \to \infty \) with \( \Lambda \) fixed.

The matrix \( \hat{a} \) will encode the information about the couplings of scalar potential

\[
V(\phi) = \sum_{k \geq 2} \frac{1}{k} t_k \phi^k
\]

(3.5)

in the form:

\[
t_k = -\frac{p}{N} \text{tr}_\mathcal{Q} \hat{a}^k
\]

(3.6)

where the trace \( \text{tr}_\mathcal{Q} \) goes only with respect to the vector space \( \mathcal{Q} \). Here \( t_2 \equiv m^2 \) corresponds to the mass squared.
The parametrization (3.6) of the couplings reminds the so-called Miwa variables widely used in the theory of $\tau$-functions of the hierarchies of integrable differential equations. It is also used in the representation of characters of the group $GL(N)$ through Schur polynomials (see [10] for the details).

Obviously the last formula can be in general true only in the limit $q \to \infty$. Note that $N$ is kept as a finite fixed parameter here.

From (3.5) and (3.6) the potential can be also written in the form

$$V(z) = -\frac{p}{N} \sum_{j=1}^{q} \ln(1 - a_j z)$$  \hspace{1cm} (3.7)

In the limit $q \to \infty$ we can parameterize in principle any potential (including a polynomial one) by such a sum of logarithmic terms, but $a_i$'s need not be necessary real. They can be taken, say, in complex conjugate pairs (the potential becomes even in this case).

For instance we can choose $a_i$ in such a way that in the limit $q \to \infty$ they will reproduce any polynomial potential (3.5):

$$a_j = e^{i\theta_j}$$  \hspace{1cm} (3.8)

where

$$\theta_j = \frac{2\pi}{q} j + \frac{2N}{pq} \sum_{m \geq 1} t_m \sin \frac{2\pi jm}{q}$$  \hspace{1cm} (3.9)

Indeed, for $k > 0$ we have

$$\sum_{j=1}^{q} e^{ik\theta_j} \sim_{p,q \to \infty} \sum_{j=1}^{q} e^{\frac{2\pi i}{q} jk} \left( 1 + ik \frac{2N}{pq} \sum_{m \geq 1} t_m \frac{2\pi jm}{q} \right) = -\frac{N}{p} t_k.$$

The proof of the equivalence of the QFT (2.1) and the matrix model (3.2) is very simple. Let us consider any Feynman graph of the theory (3.2) (dotted line on fig. 2) together with its dual graph (solid line on fig. 2). The matrix structure of the theory prescribes to use the double line notations for the propagators, so such graph has a fixed
topology with the genus defined by the Euler formula. Each single line carries now a double index corresponding to the product of spaces $\mathcal{P} \times \mathcal{Q}$.

Fig. 2. Original and dual graphs and the dual propagator depending on the quenched coordinates $x_k, x_j$ in our one matrix integral representation of scalar QFT. We show the pairs of indices $b, k$ and $c, j$ belonging to $\mathcal{Q}, \mathcal{P}$ spaces, correspondingly, running around original faces, and hence placed at dual vertices.

The propagators of the original graph are given by:

$$\frac{1}{N} \frac{a_b a_c}{(x_k - x_j)^2} \delta_{bb'} \delta_{cc'} \delta_{kk'} \delta_{jj'}.$$  

Here we attributed the $a_k$ factors to the propagators rather than to the vertices which can be achieved by the change of the matrix variable: $\Phi \rightarrow A^{-1/2} \Phi A^{-1/2}$. It is easy
to see that we obtain at each face of the original graph a weight $\text{tr}_Q \hat{a}^k$ where $k$ is the order of this face (number of edges). It happens in the same way as in the so called matrix model of dually weighted graphs (DWG) - a one matrix model with the action $S = n\text{tr}[\Phi^2 + W(A\Phi)]$ studied in [11],[9] and solved in [10]: apart from original couplings coming from the potential $W(z)$ we also obtain in the DWG model the dual couplings $\tilde{t}_k$ weighting the faces of different orders $k$ (or, which is the same, the vertices of the dual graph with the coordination number $k$).

The factors $\frac{1}{(x_i - x_j)^2}$ can be now attributed to the dual propagators (crossing the original ones) and the indices $i,j$ are running around the faces adjacent to the original propagators or, which is the same, attributed to the corresponding dual vertices (see fig.2).

It is natural to formulate the result for the free energy in terms of feynman expansion with respect to the dual graphs $\tilde{G}$. It can be written in the following way:

$$
F = \sum_{\tilde{G}} N^{2-2g} \prod_{\tilde{v}} (t_{k_{\tilde{v}}} \frac{1}{p} \sum_{i_{\tilde{v}}=1}^{p} \prod_{<\tilde{v}'\tilde{v}''>} (x_{i_{\tilde{v}'}} - x_{i_{\tilde{v}''}})^{-2}
$$

where $\sum_{\tilde{G}}$ goes over all dual graphs of the matrix model (3.2), $g$ is the genus of a graph $\tilde{G}$, $\prod_{\tilde{v}}$ goes over all vertices $\tilde{v}$ of this graph with coordination numbers $k$, $t_{k_{\tilde{v}}} = -\frac{p}{N} \sum_{j=1}^{q} a_{j}^k$ are the couplings attached to these vertices and $\prod_{<\tilde{v}'\tilde{v}''>}$ goes with respect to all edges of $\tilde{G}$ connecting the vertices $\tilde{v}$ and $\tilde{v}'$. At each dual vertex $\tilde{v}$ there is a sum taken with respect to the index $i_{\tilde{v}}$ corresponding to the subspace $\mathcal{P}$.

Note also that due to the specific logarithmic form of the interactions in (3.2)

$$
-\text{Tr} \ln (I - A\Phi) = \sum_k \frac{1}{k} \text{Tr}(A\Phi)^k
$$

all faces of dual graphs appear weighted with the factor 1 (the factor $\frac{1}{k}$ compensates the cyclic symmetry of each dual face). So we see that (3.10) is given by the sum over connected graphs $\tilde{G}$ of all genera waited by $N^{2-2g}$, with unrestricted face order and with the vertices waited by couplings $t_{k_{\tilde{v}}} = \frac{p}{N} \text{tr}_Q \hat{a}_{\tilde{v}}^k$. Hence these are precisely the original graphs of the scalar matrix field theory (2.1) with the appropriate 4D massless propagators in the coordinate space. The mass $m$ is taken into account by the presence of the coupling $t_2 \equiv -m^2$ in the scalar potential.
It is left to add that the summations over the indices \(i_0 = 1, \cdots, p\) can be substituted by the integrations in the large \(p\) limit:

\[
\frac{1}{p} \sum_{i=1}^{p} \cdots \rightarrow_{p \to \infty} \int d^4x \cdots
\] (3.11)

In this way we can reproduce the correspondence between the matrix scalar 4D QFT (2.1) and the zero dimensional matrix integral (3.2), graph by graph of any topology. Hence they coincide, at least in any order of the perturbation theory.

Let us stress again that \(N\) is a fixed parameter in our construction. It does not even need to be integer, although integer \(N\) seems to be singled out by the fact that we can represent the corresponding determinant as an integral over an \(N\)-vector of complex \(n \times n\) matrices \(M_l, l = 1, \cdots, N\):

\[
\exp[-N\text{Tr} \log(I - A\Phi)] = \int \prod_{l=1}^{N} d^{2N^2}M_l \exp[-\text{Tr}M_l^+(I - A\Phi)M_l]
\]

In particular, for \(N = 1\) (3.2) is equivalent to the usual one component scalar field theory with the action \(S = \int d^4x \left[ (\partial_\mu \phi)^2 + V(\phi) \right] \).

To calculate the one point correlators we use the following correspondence between the averages in the scalar QFT and its matrix model (MM) formulation (we take \(N = 1\)):

\[
<\int d^4x \phi^k(x) >_{QFT} = \frac{1}{\text{tr}_\mathcal{N} A^k} <\text{tr}_\mathcal{N}(A\Phi)^k >_{MM}
\] (3.12)

For the two point correlator we have the correspondence:

\[
<\phi(x)\phi(y) >_{QFT} = \frac{1}{(\text{tr}_\mathcal{Q}\hat{a})^2} <(\text{tr}_\mathcal{N}(A\Phi))_{ii}(\text{tr}_\mathcal{N}(A\Phi))_{jj} >_{MM}
\] (3.13)

where the traces are taken only with respect to the subspace \(\mathcal{Q}\) and the matrix indices \(ii\) and \(jj\) of the subspace \(\mathcal{P}\) are fixed and chosen in such a way that \((x_i - x_j)^2 \simeq (x - y)^2\).
4. Reduction in the presence of fermions and yukawa interactions

Although we don’t know any natural way to build a reduction of Yang-Mills theory for finite $N$, the scalar field theory (2.1) is not the only interesting QFT which can be reduced to a matrix model in this way.

Let us consider as an example of application of our method the reduction of the QFT of massless Dirac fermions and massless bosons with yukawa interaction in 4D with the action:

$$S = \int d^4x \left\{ (\partial_\mu \phi)^2 + \bar{\psi} \gamma_\mu \psi - \frac{\lambda}{3} \bar{\psi} \psi \phi \right\}$$  \hspace{1cm} (4.1)

We stress that here $\phi(x)$ and $\psi(x)$ are usual bosonic and dirac fields: $\phi$ has only one component and $\psi$ is a dirac spinor. The reduced version of this model is given in terms of matrix integral over 2 hermitian $n \times n$ matrices $\Phi$ and $\Psi$ in the auxiliary linear space $\mathcal{N} = \mathcal{Q} \times \mathcal{P}$

$$Z = e^F = \int d^{n^2} \Phi d^{n^2} \Psi e^{-S}$$  \hspace{1cm} (4.2)

with the action

$$S = \text{Tr}_\mathcal{N} \left( [X_\mu, \Phi]^2 + [X_\mu, [X_\nu, \Psi]]^2 - \text{tr}_D \ln(I_\mathcal{N} - \gamma_\mu \times [X_\mu, \Psi] A) + \ln(I_\mathcal{N} - A \Phi A \Psi) \right)$$  \hspace{1cm} (4.3)

Here $\text{tr}_D$ is taken with respect to the dirac indices of $\gamma_\mu$ matrix which is in the direct product with all other matrices in the third term. The $A$ matrix is chosen in such a way that $p \sum_{j=1}^q a_j^k = \lambda \delta_{k,3}$ for $k > 0$ in the large $q$ limit. The last can be achieved for example by the following choice of $\hat{a}$: $a_j = \left( \frac{2\lambda}{pq} \right)^{1/3} e^{i \theta_j}$ with $\theta_j$ given by the equation $\frac{2\pi j}{q} = \theta + \frac{1}{3} \sin 3\theta$. To prove it it is enough to calculate the density of $\theta$’s in the large $q$ limit: $\rho(\theta) = \frac{1}{q} \frac{\partial}{\partial \theta} = \frac{1}{2\pi} (1 + \frac{1}{3} \cos 3\theta)$ which gives $\text{tr}_Q \hat{a}^k = \int_{-\pi}^{\pi} e^{ik\theta} \rho(\theta) = \lambda \delta_{k,3}$ for $k > 0$.

All other definitions are the same as in the previous section.

To verify the perturbative equivalence of the QFT (4.1) and the matrix integral (4.2)-(4.3) we compare again the feynman graphs of the former to the dual feynman graphs of the latter. Due to the choice of the matrix $\hat{a}$ dual graphs of (4.3) will contain only
triple interaction vertices and due to the log type potentials the order of dual faces will be unrestricted (see fig. 3).

Fig.3: Original and dual graphs of the QFT of scalars and fermions with yukawa interactions (in single line notations).

Note that feynman graphs of the QFT (4.1) have two types of faces: fermionic loops built out of Dirac propagators and the loops with the boundary built from interchanging dirac and bosonic propagators. The first type of loops will be generated by the third term in (4.3) and the second kind - by the last term in (4.3). The yukawa interaction vertices correspond to the loops of dual graphs of the matrix model (4.3). Note the (−) sign in front of the third term which gives the fermionic statistics ((−) sign for each
fermionic loop). Each fermionic loop of \(k\) vertices is equipped, as it should be, by the factor \(\text{tr}_D(\gamma_{\mu_1} \cdots \gamma_{\mu_k})\). Finally, the massless 4D fermionic propagator \(\frac{\gamma_{\mu}(x^\mu-y^\nu)}{|x-y|^4}\) will appear as a function of difference of quenched coordinates \(\frac{\gamma_{\mu}(x^\mu_i-y^\nu_j)}{|x_i-x_j|^4}\).

Hence in the same limit \(p,q \to \infty\) as for the scalar QFT of the previous section we reproduce graph by graph (with the weights independent on the topology of graphs) the free energy of the QFT (4.1) out of the matrix integral (4.3).

5. Conclusions and comments

We proposed a reduction of 4D quantum field theories with a finite number of fields (finite \(N\) in case of matrix fields) to a matrix integral over infinite matrices. The physical four dimensional coordinate space is encoded into the components of D=4 auxiliary diagonal matrices of quenched coordinates. The reduction is different from the old Eguchi-Kawai reduction with quenched momenta: it reproduces correctly not only the planar approximation but also, at least perturbatively graph by graph, all non planar corrections. The coordinate space is introduced in our reduction scheme in a way which reminds the description of coordinates of D-branes in [8]. The large \(n\) limit that we use is different from the usual \('t\)Hooft limit and rather similar to the one adopted in [12] for the matrix model of M-theory.

We showed that fermions can be also naturally introduced into this reduction scheme.

A few comments are in order:

1. The QFT’s in dimensions different from D=4 can be also reduced in this way but since the scalar propagator is different from \(1/(x-y)^2\) the corresponding matrix representation of it looks “nonlocal” (i.e., different from \(\text{Tr}[X_{\mu}, \Phi]^2\)). One can use the formula \(\text{Tr}[X_{\mu_1}, \cdots, [X_{\mu_k}, \Phi] \cdots]^2 = \sum_{i,j}(x_i - x_j)^{2k}\Phi_{ij}^2\) in the action of the reduced matrix model to supply the dual graphs by any propagator \(D(x_i - x_j)\).

2. Although our new matrix formulation of some QFT’s hardly helps for solving them analytically it may provide a new numerical approach to their study: our matrix model
does not deal with any 4D lattice and the approach to the thermodynamical limit might be much faster than in the conventional Monte Carlo algorithms based on the lattices. One could also envisage some real space renormalization schemes where the renormalization flow would be considered with respect to the size of matrices.

3. The authors of the papers [13], [14] propose a formulation of the non-commutative QFT using reduction to the matrix models with quenched momentum matrices $P_\mu$ obeying the Heisenberg commutation relations: $[P_\mu, P_\nu] = i B_{\mu\nu}$ where $B_{\mu\nu}$ is an antisymmetric $D \times D$ matrix of C-numbers. For our model, a natural non-commutative generalization would occur if we take the same commutation relations for the coordinate matrices $X_\mu$: $[X_\mu, X_\nu] = i C_{\mu\nu}$. Since the coordinate and momentum matrices are connected in [13] by the relation $P_\mu = B_{\mu\nu} X_\nu$ it is natural to expect that we get the same non-commutative scalar field theory if we take the matrix $C = B^{-1}$. We haven’t yet a proof of this statement.

Let us also mention the most obvious problems and questions concerning our matrix reduction of QFT’s:

1. We did not manage to find a natural matrix reduction of QCD with 3 colors. In principle we could try to do it in a particular gauge by systematically reproducing all feynman diagrams by the method we presented in this paper. But it would be much more instructive to find some general principle for such reduction arising explicitely from the gauge invariance. For the moment such a principle is missing.

2. It would be interesting to find a modification of our approach similar to the twisted EK model proposed in [5]. In [5] the momentum space appears as a classical vacuum of a large N lattice QFT with modified couplings. We can imagine a similar twisting of reduced matrix model with the classical vacuum solution generating the coordinate space.

3. A related but more ambitious question: can we build realistic models of fundamental interactions as some reduced matrix models where the physical space would emerge due to some symmetry breaking procedure? In other word, can our World be described by a Matrix Model? We know some of the attempts of this kind in the superstring theory and M-theory [12], [15]. Our construction might be useful to approach this problem from a different direction.
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