FOURIER FREQUENCIES IN AFFINE ITERATED FUNCTION SYSTEMS

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ABSTRACT. We examine two questions regarding Fourier frequencies for a class of iterated function systems (IFS). These are iteration limits arising from a fixed finite families of affine and contractive mappings in $\mathbb{R}^d$, and the “IFS” refers to such a finite system of transformations, or functions. The iteration limits are pairs $(X, \mu)$ where $X$ is a compact subset of $\mathbb{R}^d$, (the support of $\mu$) and the measure $\mu$ is a probability measure determined uniquely by the initial IFS mappings, and a certain strong invariance axiom. The two questions we study are: (1) existence of an orthogonal Fourier basis in the Hilbert space $L^2(X, \mu)$; and, when we do, (2) explicitly, what are the Fourier frequencies of these orthonormal bases in terms of the data that defines the iterated function system? Our main result, Theorem 3.8, shows that existence in (1) follows from geometric assumptions that are easy to check, and

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1. Introduction

Motivated in part by questions from wavelet theory, there has been a set of recent advances in a class of spectral problems from iterated function systems (IFS) of affine type. The geometric side of an IFS is a pair $(X, \mu)$ where $X$ is a compact subset of $\mathbb{R}^d$, (the support of $\mu$) and the measure $\mu$ is a probability measure determined uniquely by the initial IFS mappings, and a certain strong invariance property. In this paper, we examine two questions regarding Fourier frequencies for these iterated function systems (IFS): (1) When do we have existence of an orthogonal Fourier basis in the Hilbert space $L^2(X, \mu)$; and, when we do, (2) explicitly, what are the Fourier frequencies of these orthonormal bases in terms of the data that defines the iterated function system? Our main result, Theorem 3.8, shows that existence in (1) follows from geometric assumptions that are easy to check, and...
it is a significant improvement on earlier results in the literature. Our approach
uses a new idea from dynamics, and it allows us to also answer (2).

By a Fourier basis in $L^2(X, \mu)$ we mean a subset $\Lambda$ of $\mathbb{R}^d$ such that the functions
$\{e_\lambda | \lambda \in \Lambda\}$ form an orthogonal basis in $L^2(\mu)$. Here $e_\lambda(x) := \exp(2\pi i \lambda \cdot x)$. The functions $e_\lambda$ are restricted from $\mathbb{R}^d$ to $X$. (The factor $2\pi$ in the exponent is introduced for normalization purposes only.)

So far Fourier bases have been used only in the familiar and classical context of compact abelian groups; see, e.g., [Kat04]. There, as is well known, applications abound, and hence it is natural to attempt to extend the fundamental duality principle of Fourier bases to a wider category of sets $X$ which are not groups and which in fact carry much less structure. Here we focus on a particular such class of subsets $X$ in $\mathbb{R}^d$ which are IFS attractors. Our present paper focuses on the theoretical aspects which we feel are of independent interest, but we also allude to applications.

Since $X$ and its boundary are typically fractals in the sense of [Man04], their geometry and structure do not lend themselves in an obvious way to Fourier analysis. (Recall [Man04] that some fractals model chaos.) To begin with, the same set $X$ may arise in more than one way as a limiting object. It will be known typically from some constructive algorithm. While each finite algorithmic step can readily be pictured, not so for the iteration limit! And from the outset it may not even be clear whether or not a particular $X$ is the attractor of an iterated function system (IFS); see, e.g., [LaFr03, Fal03, Jor06, Bea65, BCMG04]. Moreover, far from all fractals fall in the affine IFS class. But even the affine class of IFSs has a rich structure which is not yet especially well understood.

The presence of an IFS structure for some particular set $X$ at least implies a preferred self-similarity; i.e., smaller parts of $X$ are similar to its larger scaled parts, and this similarity will be defined by the maps from the IFS in question. When $X$ is the attractor of a given contractive IFS $(\tau_i)$, then by [Hut81], there is a canonical positive and strongly invariant measure $\mu$ which supports $X$. But even in this case, a further difficulty arises, addressed in Section 4 below.

As illustrated with examples in Section 5 below, the geometric patterns for a particular $X$ might not at all be immediately transparent. For a given $X$, the problem is to detect significant patterns such as self-similarity, or other “hidden structures” (see, e.g., [CuSm02, Sma05]); and Fourier frequencies, if they can be found, serve this purpose. In addition, if $X$ does admit a Fourier basis, this allows us to study its geometry and its symmetries from the associated spectral data. In that case, standard techniques from Fourier series help us to detect “hidden” structures and patterns in $X$.

However, we caution the reader that recent work of Strichartz [Str05] shows that a number of “standard” results from classical Fourier series take a different form in the fractal case.

In the next section we give definitions and recall the basics from the theory of iteration limits; i.e., metric limits which arise from a fixed finite family of affine and contractive mappings in $\mathbb{R}^d$, and the “IFS” refers to such a finite system of transformations.

There are a number of earlier papers [JoPe98, DuJo05, LaWa02, Str00, LaWa06] which describe various classes of affine IFSs $(X, \mu)$ for which an orthogonal Fourier basis exists in $L^2(X, \mu)$. It is also known [JoPe98] that if the affine IFS $(X, \mu)$
is the usual middle-third Cantor set, then no such Fourier basis exists; in fact, in that case there can be no more than two orthogonal Fourier exponentials $e^{\lambda}$ in $L^2(X, \mu)$. Nonetheless, the present known conditions which imply the existence of an orthogonal Fourier basis have come in two classes, an algebraic one (Definition 2.3 below) and an analytic assumption. Our main result, Theorem 3.8, shows that the analytic condition can be significantly improved. We also conjecture that the algebraic condition is sufficient (see Conjecture 2.5).

2. Definitions and preliminaries

The definitions below serve to make precise key notions which we need to prove the main result (Theorem 3.8). In fact they are needed in relating the intrinsic geometric features of a given affine IFS $(X, \mu)$ to the spectral data for the corresponding Hilbert space $L^2(X, \mu)$. Our paper focuses on a class of affine IFSs which satisfies a certain symmetry condition (Definition 2.3). This condition involves a pair of IFSs in duality, and a certain complex Hadamard matrix. While these duality systems do form a restricted class, their study is motivated naturally by our recursive approach to building up a Fourier duality. Moreover, our recursive approach further suggests a certain random-walk model which is built directly on the initial IFS. We then introduce a crucial notion of invariant sets for this random walk (Definition 2.11). The corresponding transition probabilities of the random walk are defined in terms of the Hadamard matrix in Definition 2.3 and it lets us introduce a discrete harmonic analysis, a Perron–Frobenius operator and associated harmonic functions (Definition 2.8). The interplay between these functions and the invariant sets is made precise in Propositions 2.14 and 2.15 and Theorem 2.17.

**Definition 2.1.** A probability measure $\mu$ on $\mathbb{R}^d$ is called a **spectral measure** if there exists a subset $\Lambda$ of $\mathbb{R}^d$ such that the family of exponential functions \{\(e^{2\pi i \lambda \cdot x} \mid \lambda \in \Lambda\}\} is an orthonormal basis for $L^2(\mu)$. In this case, the set $\Lambda$ is called a **spectrum** of the measure $\mu$.

It was noted recently in [LaWa06] that the axiom which defines spectral measures $\mu$ implies a number of structural properties for $\mu$, as well as for the corresponding spectrum $\Lambda = \Lambda(\mu)$: e.g., properties regarding discreteness and asymptotic densities for $\mu$, and intrinsic algebraic relations on the configuration of vectors in $\Lambda$.

Our present paper deals with the subclass of spectral measures that can arise from affine IFSs.

**Definition 2.2.** Let $Y$ be a complete metric space. Following [Hut81] we say that a finite family $(\tau_i)_{i=1,N}$ of contractive mappings in $Y$ is an iterated function system (IFS). Introducing the Hausdorff metric on the set of compact subsets $K$ of $Y$, we get a second complete metric space, and we note that the induced mapping $K \mapsto \bigcup_{i=1}^{N} \tau_i(K)$, is contractive. By Banach’s theorem, this mapping has a unique fixed point, which we denote $X$; and we call $X$ the **attractor** for the IFS. It is immediate by restriction that the individual mappings $\tau_i$ induce endomorphisms in $X$, and we shall denote these restricted mappings also by $\tau_i$.

For IFSs where the mappings $\tau_i$ are affine as in (2.1) below, we talk of affine IFSs. In this case, the ambient space is $\mathbb{R}^d$. 
Let $R$ be a $d \times d$ expansive integer matrix, i.e., all entries are integers and all eigenvalues have absolute value strictly bigger than one. For a point $b \in \mathbb{Z}^d$ we define the function

$$\tau_b(x) := R^{-1}(x + b) \quad (x \in \mathbb{R}^d).$$

(2.1)

For a finite subset $B \subset \mathbb{Z}^d$ we will consider the iterated function system $(\tau_b)_{b \in B}$. We denote by $N$ the cardinality of $B$. We will assume also that $0 \in B$.

The fact that the matrix $R$ is expansive implies that there exists a norm on $\mathbb{R}^d$ for which the maps $\tau_b$ are contractions.

There exist then a unique compact set $X_B$, called the attractor of the IFS, with the property that

$$X_B = \bigcup_{b \in B} \tau_b(X_B).$$

Moreover, we have the following representation of the attractor:

$$X_B = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k \bigg| b_k \in B \text{ for all } k \geq 1 \right\}.$$

There exists a unique invariant probability measure $\mu_B$ for this IFS, i.e., for all bounded continuous functions on $\mathbb{R}^d$,

$$\int f \, d\mu_B = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b \, d\mu_B.$$  

Moreover, the measure $\mu_B$ is supported on the attractor $X_B$. We refer to [Hut81] for details.

Following earlier results from [JoPe98, Str00, LaWa02, DuJo05, LaWa06], in order to obtain Fourier bases for the measure $\mu_B$, we will impose the following algebraic condition on the pair $(R, B)$:

**Definition 2.3.** Let $R$ be a $d \times d$ integer matrix, $B \subset \mathbb{Z}^d$ and $L \subset \mathbb{Z}^d$ having the same cardinality as $B$, $\#B = \#L =: N$. We call $(R, B, L)$ a Hadamard triple if the matrix

$$\frac{1}{\sqrt{N}} (e^{2\pi i R^{-1} b \cdot l})_{b \in B, l \in L}$$

is unitary.

We will assume throughout the paper that $(R, B, L)$ is a Hadamard triple.

**Remark 2.4.** Note that if $(R, B, L)$ is a Hadamard triple, then no two elements in $B$ are congruent modulo $R \mathbb{Z}^d$, and no two elements in $L$ are congruent modulo $R^T \mathbb{Z}^d$.

Indeed, if $b, b' \in B$ satisfy $b - b' = Rk$ for some $k \in \mathbb{Z}^d$ then, since $L \subset \mathbb{Z}^d$,

$$e^{2\pi i R^{-1} b \cdot l} = e^{2\pi i R^{-1} b' \cdot l} \quad (l \in L),$$

so the rows $b$ and $b'$ of the matrix in Definition 2.3 cannot be orthogonal.

We conjecture that the existence of a set $L$ such that $(R, B, L)$ is a Hadamard triple is sufficient to obtain orthonormal bases of exponentials in $L^2(\mu_B)$.

**Conjecture 2.5.** Let $R$ be a $d \times d$ expansive integer matrix, $B$ a subset of $\mathbb{Z}^d$ with $0 \in B$. Let $\mu_B$ be the invariant measure of the associated IFS $(\tau_b)_{b \in B}$. If there exists a subset $L$ of $\mathbb{Z}^d$ such that $(R, B, L)$ is a Hadamard triple and $0 \in L$ then $\mu_B$ is a spectral measure.
We will prove in Theorem 3.8 that the conjecture is true under some extra analytical assumptions, thus extending the known results from \[\text{JoPe98, Str00, LaWa02, DuJo05, LaWa06}\].

2.1. Path measures. To analyze the measure \(\mu_B\) we will use certain random-walk (or “path”) measures \(P_x\) which are directly related to the Fourier transform \(\hat{\mu}_B\) of the invariant measure. Most of the results in Sections 2.1 and 2.2 are essentially contained in \[\text{CoRa90, CCR96, DuJo05}\]. We include them here for the convenience of the reader.

Define the function
\[
W_B(x) = \left| \frac{1}{N} \sum_{b \in B} e^{2\pi ib \cdot x} \right|^2 \quad (x \in \mathbb{R}^d).
\]

This function appears if one considers the Fourier transform of equation (2.2):
\[
|\hat{\mu}_B(x)|^2 = W_B((R^T)^{-1}x) |\hat{\mu}_B((R^T)^{-1}x)|^2, \quad (x \in \mathbb{R}^d).
\]

The elements of \(L\) and the transpose \(S := R^T\) will define another iterated function system
\[
\tau_l(x) = S^{-1}(x + l) \quad (x \in \mathbb{R}^d, l \in L).
\]

We underline here that we are interested in the measure \(\mu_B\) associated to the iterated function system \((\tau_b)_{b \in B}\), and the main question is whether this is a spectral measure. The iterated function system \((\tau_l)_{l \in L}\) will only help us in constructing the basis of exponentials.

The unitarity of the matrix in Definition 2.3 implies (see \[\text{LaWa02, DuJo05}\]) that
\[
(2.3) \quad \sum_{l \in L} W_B(\tau_l x) = 1 \quad (x \in \mathbb{R}^d).
\]

Remark 2.6. The reader will notice that in our analysis of the iteration steps, our measure \(\mu_B\) in (2.2) is chosen in such a way that each of the branches in the iterations is given equal weight \(1/N\). There are a number of reasons for this.

But first recall the following known theorem from \[\text{Hut81}\] to the effect that for every IFS \((\tau_b)_{b \in B}\), \(b \in B\), \(N = \# B\), and for every \(N\)-configuration of numerical weights \((p_b)_{b \in B}, p_b > 0\), with \(\sum_{b \in B} p_b = 1\), there is a unique \((p_b)\)-distributed probability measure \(\mu_{p,B}\) with support \(X_B\). This measure \(\mu_{p,B}\) is determined uniquely by the equation
\[
\mu_{p,B} = \sum_{b \in B} p_b \mu_{p,B} \circ \tau_b^{-1}.
\]

Since our focus is on spectral measures (Definition 2.1), it is natural to restrict attention to the case of equal weights, i.e., to \(p_b = 1/N\).

Another reason for this choice is a conjecture by Laba and Wang \[\text{LaWa02}\], as well as the following lemma.

Lemma 2.7. Set
\[
W_{p,B}(x) := \left| \sum_{b \in B} p_b e^{2\pi ib \cdot x} \right|^2,
\]
and assume that
\[
\sum_{l \in L} W_{p,B}(\tau_l(x)) = 1
\]
for some dual IFS
\[ \tau_l(x) = (R^T)^{-1}(x + l), \quad (x \in \mathbb{R}^d, l \in L), \]
with \( \#L = N \). Then \( p_b = 1/N \) for all \( b \in B \).

**Proof.** Expanding the modulus square and changing the order of sumation, we get that for all \( x \in \mathbb{R}^d \),
\[ \sum_{b,b' \in B} p_b p_{b'} e^{2\pi i R^{-1}(b-b') \cdot x} \sum_{l \in L} e^{2\pi i R^{-1}(b-b') \cdot l} = 1 \]
The constant term on the left must be equal to 1, so
\[ \sum_{b \in B} N p_b^2 = 1. \]
Since \( \sum_{b \in B} p_b = 1 \), this will imply that we have equality in a Schwarz inequality, so \( p_b = 1/N \) for all \( b \in B \). \( \square \)

The relation \( \eqref{2.4} \) can be interpreted in probabilistic terms: \( W_B(\tau_l x) \) is the probability of transition from \( x \) to \( \tau_l x \). This interpretation will help us define the path measures \( P_x \) in what follows.

Let \( \Omega = \{ (l_1 l_2 \ldots) \mid l_n \in L \text{ for all } n \in \mathbb{N} \} = L^\mathbb{N} \). Let \( \mathcal{F}_n \) be the sigma-algebra generated by the cylinders depending only on the first \( n \) coordinates.

There is a standard way due to Kolmogorov of using the system \( (\mathbb{R}^d, (\tau_l)_{b \in B}) \) to generate a path space \( \Omega \), and an associated family of path-space measures \( P_x \), indexed by \( x \in \mathbb{R}^d \). Specifically, using the weight function \( W_B \) in assigning conditional probabilities to random-walk paths, we get for each \( x \in \mathbb{R}^d \) a Borel measure \( P_x \) on the space of paths originating in \( x \). For each \( x \), we consider paths originating at \( x \), and governed by the given IFS. The transition probabilities are prescribed by \( W_B \); and passing to infinite paths, we get the measure \( P_x \). We shall refer to this \( (P_x)_{x \in \mathbb{R}^d} \) simply as the path-space measure, or the path measure for short.

For each \( x \in \mathbb{R}^d \) we can define the measures \( P_x \) on \( \Omega \) as follows. For a function \( f \) on \( \Omega \) which depends only on the first \( n \) coordinates
\[ \int_\Omega f \, dP_x = \sum_{\omega_1, \ldots, \omega_n \in L} W_B(\tau_{\omega_1} x) W_B(\tau_{\omega_2} \tau_{\omega_1} x) \cdots W_B(\tau_{\omega_n} \cdots \tau_{\omega_1} x) f(\omega_1, \ldots, \omega_n). \]
In particular, when the first \( n \) components are fixed \( l_1, \ldots, l_n \in L \),
\[ P_x(\{ (\omega_1 \omega_2 \ldots) \in \Omega \mid \omega_1 = l_1, \ldots, \omega_n = l_n \}) = \prod_{k=1}^n W_B(\tau_{l_k} \cdots \tau_{l_1} x). \]

Define the transfer operator
\[ R_W f(x) = \sum_{l \in L} W_B(\tau_l x) f(\tau_l x) \quad (x \in \mathbb{R}^d). \]

**Definition 2.8.** A measurable function \( h \) on \( \mathbb{R}^d \) is said to be \( R_W \)-harmonic if \( R_W h = h \). A measurable function \( V \) on \( \mathbb{R}^d \times \Omega \) is said to be a cocycle if it satisfies the following covariance property:
\[ V(x, \omega_1 \omega_2 \ldots) = V(\tau_{\omega_1} x, \omega_2 \omega_3 \ldots) \quad (\omega_1 \omega_2 \ldots \in \Omega). \]
In the following we give a formula for all the bounded $R_W$-harmonic functions. The result expresses the bounded $R_W$-harmonic functions in terms of a certain boundary integrals of cocycles, and it may be viewed as a version of the Fatou–Markoff–Primalov theorem.

If $h$ is a bounded measurable $R_W$-harmonic function on $\mathbb{R}^d$, then, for all $x \in \mathbb{R}^d$, the functions

$$(\omega_1, \ldots, \omega_n) \mapsto h(\tau_{\omega_n} \cdots \tau_{\omega_1} x)$$

define a bounded martingale. By Doob’s martingale theorem, one obtains that the following limit exists $P_x$-a.e.:

$$(2.6) \quad \lim_{n \to \infty} h(\tau_{\omega_n} \cdots \tau_{\omega_1} x) =: V(x, \cdot), \text{ for } P_x\text{-a.e. } \omega \in \Omega,$$

where $V(x, \cdot) : \omega \to \mathbb{C}$ is some bounded function on $\Omega$. Moreover, $V$ is a cocycle.

We formalize this conclusion in a lemma.

**Lemma 2.9.** If $h$ is a bounded $R_W$-harmonic function, then the associated function $V$ from $(2.6)$ is well defined, it is bounded and measurable; and it is a cocycle. Conversely, if $V : \mathbb{R}^d \times \Omega \to \mathbb{C}$ is a bounded measurable function satisfying $(2.6)$, then the function

$$(2.7) \quad h_V(x) := P_x(V(x, \cdot)) \quad (x \in \mathbb{R}^d),$$

defines a bounded function on $\mathbb{R}^d$ such that $R_W h_V = h_V$, and such that relation $(2.6)$ is satisfied with $h = h_V$.

Next we show that the family of measures $x \mapsto P_x$ is weakly continuous. More precisely, we have the following result.

**Proposition 2.10.** [CoRa90, Proposition 5.2] Let $U$ be a bounded measurable function on $\Omega$. Then there exists a constant $0 \leq D < \infty$ such that

$$|P_x(U) - P_y(U)| \leq D|x - y| ||U||_{\infty} \quad (x, y \in \mathbb{R}^d).$$

While the main ideas are contained in [CoRa90], we include the proof for the benefit of the reader; our version covers affine matrix operations for contraction, extending the one-dimensional dyadic case in [CoRa90].

**Proof.** Let $x, y \in \mathbb{R}^d$. For $\omega_1 \ldots \omega_n \in L^n$ and $1 \leq p \leq n$, define $W_{\omega,p}(x) := W_B(\tau_{\omega_p} \cdots \tau_{\omega_1} x)$, and

$$\delta_n(x, y) := \sum_{\omega_1 \ldots \omega_n \in L^n} |W_{\omega,n}(x) \cdots W_{\omega,1}(x) - W_{\omega,n}(y) \cdots W_{\omega,1}(y)|.$$

We have, using equation (2.3),

$$\delta_n(x, y) \leq \sum_{\omega_1 \ldots \omega_n \in L^n} |W_{\omega,n}(x) - W_{\omega,n}(y)| |W_{\omega,n-1}(x) \cdots W_{\omega,1}(x) + \delta_{n-1}(x, y)$$

$$\leq Mc^n |x - y| + \delta_{n-1}(x, y),$$

where $c$ is the contraction constant for the maps $\tau_l$, $l \in L$, and $M$ is a Lipschitz constant for $W_B$.

From this we obtain

$$\delta_n(x, y) \leq M|x - y| \sum_{k \geq 1} c^k.$$

This proves the result in the case when $U$ depends only on a finite number of coordinates.
Proposition 2.13. Let $x$ be a point in $\mathbb{R}^d$, and let $U_n$ be the conditional expectation $E_Q[U|F_n]$. The functions $U_n$, $n \geq 1$, are bounded by $\|U\|_\infty$ and the sequence converges $Q$-a.e., and so $P_x$ and $P_y$-a.e., to $U$. It follows from the previous estimate that

$$|P_x(U_n) - P_y(U_n)| \leq \|U\|_\infty \delta_n(x, y) \leq D|x - y|\|U\|_\infty.$$ 

The result is obtained by applying Lebesgue’s dominated convergence theorem. 

2.2. Invariant sets. In the following, we will work with the affine system $(\tau_l)_{l \in L}$, and with the weight function $W_B$. Given this pair, we introduce a notion of invariant sets as introduced in [CoRa90, CCR96, CHR97]. We emphasize that “invariance” depends crucially on the chosen pair. The reason for the name “invariance” is that the given affine system and the function $W_B$ naturally induce an associated random walk on points in $\mathbb{R}^d$ as described before.

Let $x$ and $y$ be points in $\mathbb{R}^d$ and suppose $y = \tau_l(x)$ for some $l \in L$. We then say that $W_B(y)$ represents the probability of a transition from $x$ to $y$. Continuing with paths of points, we then arrive at a random-walk model, and associated trajectories, or paths. An orbit of a point $x$ consists of the closure of the union of those trajectories beginning at $x$ that have positive transition probability between successive points. A closed set $F$ will be said to be invariant if it contains all its orbits starting in $F$. Note in particular that every (closed) orbit is an invariant set.

We now spell out these intuitive notions in precise definitions.

**Definition 2.11.** For $x \in \mathbb{R}^d$, we call a trajectory of $x$ a set of points

$$\{\tau_{\omega_n} \cdots \tau_{\omega_1} x | n \geq 1\}$$

where $\{\omega_n\}_n$ is a sequence of elements in $L$ such that $W_B(\tau_{\omega_n} \cdots \tau_{\omega_1} x) \neq 0$ for all $n \geq 1$. We denote by $\mathcal{O}(x)$ the union of all trajectories of $x$ and the closure $\overline{\mathcal{O}(x)}$ is called the orbit of $x$. If $W_B(\tau_l x) \neq 0$ for some $l \in L$ we say that the transition from $x$ to $\tau_l x$ is possible.

A closed subset $F \subset \mathbb{R}^d$ is called invariant if it contains the orbit of all of its points. An invariant subset is called minimal if it does not contain any proper invariant subsets.

A closed subset $F$ is invariant if, for all $x \in F$ and $l \in L$ such that $W_B(\tau_l x) \neq 0$, it follows that $\tau_l x \in F$.

Since the orbit of any point is an invariant set, a closed subset $F$ is minimal if and only if $F = \overline{\mathcal{O}(x)}$ for all $x \in F$. By Zorn’s lemma, every invariant subset contains a minimal subset.

**Proposition 2.12.** If $F_1$ is a closed invariant subset and $F_2$ is a compact minimal invariant subset of $\mathbb{R}^d$ then either $F_1 \cap F_2 = \emptyset$ or $F_2 \subset F_1$.

**Proof.** Indeed, if $x \in F_1 \cap F_2$ then $F_2 = \overline{\mathcal{O}(x)} \subset F_1$. 

**Proposition 2.13.** Let $F$ be a compact invariant subset. Define

$$N(F) := \{\omega \in \Omega | \lim_{n \to \infty} d(\tau_{\omega_n} \cdots \tau_{\omega_1} x, F) = 0\}.$$

(The definition of $N(F)$ does not depend on $x$). Define

$$h_F(x) := P_x(N(F)).$$
Then $0 \leq h_F(x) \leq 1$, $R^W h_F = h_F$, $h_F$ is continuous and for $P_x$-a.e. $\omega \in \Omega$
\[ \lim_{n \to \infty} h_F(\tau_{\omega_n} \cdots \tau_{\omega_1} x) = \begin{cases} 1, & \text{if } \omega \in N(F), \\ 0, & \text{if } \omega \notin N(F). \end{cases} \]

**Proof.** Since the maps $\tau_l$ are contractions, it follows that
\[ \lim_{n \to \infty} d(\tau_{\omega_n} \cdots \tau_{\omega_1} x, \tau_{\omega_n} \cdots \tau_{\omega_1} y) = 0 \]
for all $x, y \in \mathbb{R}^d$; hence the definition of $N(F)$ does not depend on $x$.

Consider the characteristic function $V_F(x, \omega) := \chi_{\mathbb{N}(F)}(\omega), x \in \mathbb{R}^d, \omega \in \Omega$. Then
\[ V_F(x, \omega_1 \omega_2 \cdots) = V_F(\tau_{\omega_1} x, \omega_2 \omega_3 \cdots). \]
And $h_F(x) = P_x(V_F(x, \cdot))$. The previous discussion in Section 2.1 then proves all the statements in the proposition. \qed

In conclusion, this shows that every invariant set $F$ comes along with a naturally associated harmonic function $h_F$; see also Lemma 2.9 above.

**Proposition 2.15.** [CCR96, Proposition 2.3] There exists a constant $\delta > 0$ such that for any two disjoint compact invariant subsets $F$ and $G$, $d(F, G) > \delta$. There is only a finite number of minimal compact invariant subsets.

**Proof.** The first statement is in [CCR96]. The only extra argument needed here is to prove that a minimal compact invariant subset is contained in some fixed compact set $K$. There is a norm which makes $S^{-1}$ a contraction. Define $K$ to be the closed ball centered at the origin with radius
\[ \rho := \sup_{l \in L} ||l|| S^{-1}. \]

Then $K$ is invariant for all maps $\tau_l, l \in L$, and
\[ \lim_{n \to \infty} d(\tau_{\omega_n} \cdots \tau_{\omega_1} x, K) = 0 \quad (x \in \mathbb{R}^d, \omega \in \Omega). \]
(See [CCR96, page 163]).

If $F$ is a minimal compact invariant subset then take $x \in F$, and take $y$ to be one of the accumulation points of one of the trajectories. Then $y \in F \cap K$. With Proposition 2.12 $F \subset K$. The second statement follows. \qed
Since $h$ is continuous this implies that the set $Z$ of the zeroes of $h$ is not empty. The equation $R_W h = h$ also shows that $Z$ is a closed invariant subset.

We show that $Z$ is disjoint from $\bigcup_k F_k$. If $Z \cap F_k \neq \emptyset$ for some $k \in \{1, \ldots, p\}$ then take $y \in F_k \cap Z$. There exists $\omega \in \Omega$ such that $W_B(\tau_{\omega_n} \cdots \tau_{\omega_1} y) \neq 0$ for all $n \geq 1$. (This is because $\sum_{i \in L} W_B(\tau_i z) = 1$ for all $z$, so a transition is always possible.) But then, by invariance, $\tau_{\omega_n} \cdots \tau_{\omega_1} y \in F_k \cap Z$. This implies $\omega \in \mathcal{N}(F_k)$ so, by Proposition 2.13, $\lim_n h_{F_k}(\tau_{\omega_n} \cdots \tau_{\omega_1} x) = 1$. On the other hand $\tau_{\omega_n} \cdots \tau_{\omega_1} y \in Z$ so $h(\tau_{\omega_n} \cdots \tau_{\omega_1} y) = 0$ for all $n \geq 1$. This yields the contradiction.

Thus $Z$ is disjoint from $\bigcup_k F_k$, and this contradicts the hypothesis, and the proposition is proved. $\square$

Remark 2.16. A family $F_1, \ldots, F_p$ as in Proposition 2.13 always exists because one can take all the minimal compact invariant sets. Proposition 2.14 shows that there are only finitely many such sets. And since every closed invariant set contains a minimal one, this family will satisfy the requirements.

Theorem 2.17. [[CCR96, Théorème 2.8]] Let $M$ be minimal compact invariant set contained in the set of zeroes of an entire function $h$ on $\mathbb{R}^d$.

a) There exists $V$, a proper subspace of $\mathbb{R}^d$ invariant for $S$ (possibly reduced to $\{0\}$), such that $M$ is contained in a finite union $\mathcal{R}$ of translates of $V$.

b) This union contains the translates of $V$ by the elements of a cycle 
\[ \{x_0, \tau_1 x_0, \ldots, \tau_{m-1} x_0, x_0\} \] contained in $M$, and for all $x$ in this cycle, the function $h$ is zero on $x + V$.

c) Suppose the hypothesis “(H) modulo $V$” is satisfied, i.e., for all $p \geq 0$ the equality $\tau_{k_1} \cdots \tau_{k_p} 0 - \tau_{k'_1} \cdots \tau_{k'_p} 0 \in V$, with $k_i, k'_i \in L$ implies $k_i - k'_i \in V$ for all $i \in \{1, \ldots, p\}$. Then 
\[ \mathcal{R} = \{x_0 + V, \tau_1 x_0 + V, \ldots, \tau_{m-1} x_0 + V\}, \]
and every possible transition from a point in $M \cap \tau_{q+1} \cdots \tau_{m-1} x_0 + V$ leads to a point in $M \cap \tau_q \cdots \tau_{m-1} x_0 + V$ for all $1 \leq q \leq m - 1$, where $\tau_m \cdots \tau_1 x_0 = x_0$.

d) Since the function $W_B$ is entire, the union $\mathcal{R}$ is itself invariant.

A particular example of a minimal compact invariant set is a $W_B$-cycle. In this case, the subspace $V$ in Theorem 2.17 can be taken to be $V = \{0\}$.

Definition 2.18. A cycle of length $p$ for the IFS $(\tau_i)_{i \in L}$ is a set of (distinct) points of the form $C := \{x_0, \tau_1 x_0, \ldots, \tau_{m-1} x_0\}$, such that $\tau_m \cdots \tau_1 x_0 = x_0$, with $l_1, \ldots, l_m \in L$. A $W_B$-cycle is a cycle $C$ such that $W_B(x) = 1$ for all $x \in C$.

For a finite sequence $l_1, \ldots, l_m \in L$ we will denote by $l_1 \ldots l_m$ the path in $\Omega$ obtained by an infinite repetition of this sequence 
\[ l_1 \ldots l_m := (l_1 \ldots l_m l_1 \ldots l_m \ldots) \]

3. Statement of results

In the next definition we describe a way a given affine IFS $(\mathbb{R}^d, (\tau_b)_{b \in B})$, might factor such that the Hadamard property of Definition 2.8 is preserved for the two factors. As a result we get a notion of reducibility (Definition 3.1) for this class of affine IFSs.

Definition 3.1. We say that the Hadamard triple $(R, B, L)$ is reducible to $\mathbb{R}^r$ if the following conditions are satisfied
(i) The subspace $\mathbb{R}^r \times \{0\}$ is invariant for $R^T$, so $S = R^T$ has the form

$$S = \begin{bmatrix} S_1 & C \\ 0 & S_2 \end{bmatrix},$$

with $S_1, C, S_2$ integer matrices.

(ii) The set $B$ has the form $\{ (r_i, \eta_{i,j}) | i \in \{1, \ldots, N_1 \}, j \in \{1, \ldots, N_2 \} \}$ where $r_i$ and $\eta_{i,j}$ are integer vectors;

(iii) The set $L$ has the form $\{ (\gamma_{i,j}, s_j) | j \in \{1, \ldots, N_2 \}, i \in \{1, \ldots, N_1 \} \}$ where $s_j, \gamma_{i,j}$ are integer vectors;

(iv) $(S^T_1, \{ r_i | i \in \{1, \ldots, N_1 \}, (\gamma_{i,j} | i \in \{1, \ldots, N_1 \}) \}$ is a Hadamard triple for all $j$;

(v) $(S^T_2, \{ s_j | j \in \{1, \ldots, N_2 \}, s_j | j \in \{1, \ldots, N_2 \} \})$ is a Hadamard triple for all $i$;

(vi) The invariant measure for the iterated function system

$$\tau_r(x) = (S^T_1)^{-1}(x + r_i) \quad (x \in \mathbb{R}^r), \quad i \in \{1, \ldots, N_1 \}$$

is a spectral measure, and has no overlap, i.e., $\mu_1(\tau_{r_i}(X_1) \cap \tau_{r_j}(X_1)) = 0$ for all $i \neq j$, where $X_1$ is the attractor of the IFS $(\tau_r)_{i \in \{1, \ldots, N_1 \}}$.

For convenience we will allow $r = 0$, and every Hadamard triple is trivially reducible to $\mathbb{R}^0 = \{0\}$. Note also that these conditions imply that $N = N_1 N_2$.

**Proposition 3.2.** Let $(R, B, L)$ be a Hadamard triple such that $\mathbb{R}^r \times \{0\}$ is invariant for $R^T$. Assume that for all $b_1 \in \text{proj}_{R^T}(B)$, the number of $b_2 \in \mathbb{R}^{d-r}$ such that $(b_1, b_2) \in B$ is $N_2$, independent of $b_1$, and for all $l_2 \in \text{proj}_{R^T}-(L)$, the number of $l_1 \in \mathbb{R}^r$ such that $(l_1, l_2) \in L$ for $N_1$, independent of $l_2$. Also assume that $N_1 N_2 = N$. Then the conditions $\text{(i), (vi)}$ in Definition 3.1 are satisfied.

**Proof.** We define $\{ r_1, \ldots, r_M \} := \text{proj}_{R^t}(B)$. Using the assumption, for each $i \in \{1, \ldots, M_1 \}$, we define $\{ \eta_{i,1}, \ldots, \eta_{i,N_2} \}$ to be the points in $\mathbb{R}^{d-r}$ with $(r_i, \eta_{i,j}) \in B$. Similarly we can define $\{ s_1, \ldots, s_M \}$, $\gamma_{i,j}$ for $L$. Since $M_1 N_2 = M_2 N_1 = N$ we get $N_1 = M_1$, $M_2 = N_2$.

Since the rows of the matrix $(e^{2\pi i \theta_i^T b_j})_{b \in B, j \in L}$ corresponding to $(r_{i_1}, \eta_{i_1,j_1})$ and $(r_{i_2}, \eta_{i_2,j_2})$ are orthogonal when $j_1 \neq j_2$, and $i_1$ is fixed, we obtain (with the notation in Definition 3.1):

$$\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} e^{2\pi i (\eta_{i,j_1} - \eta_{i,j_2})} S_2^{-1} s_j = 0,$$

and this implies (after dividing by $N_1$) that the rows of the matrix $(e^{2\pi i \theta_i^T s_j} S_2^{-1} s_j')$, with $j, j' \in \{1, \ldots, N_2 \}$, are orthogonal. This proves $\text{(i)}$. The statement in $\text{(vi)}$ is obtained using the dual argument (use the transpose of $R$ and interchange $B$ and $L$).

**Definition 3.3.** We say that two Hadamard triples $(R_1, B_1, L_1)$ and $(R_2, B_2, L_2)$ are **conjugate** if there exists a matrix $M \in GL_d(\mathbb{Z})$ (i.e., $M$ is invertible, and $M$ and $M^{-1}$ have integer entries) such that $R_2 = M R_1 M^{-1}$, $B_2 = MB_1$ and $L_2 = (M^T)^{-1} L_1$.

If the two systems are conjugate then the transition between the IFSs $(\tau_0)_{b \in B_1}$ and $(\tau_M b)_{b \in B_1}$ is done by the matrix $M$; and the transition between the IFSs $(\tau_0)_{l \in L_1}$ and $(\tau_{M^{-1} l})_{l \in L_1}$ is done by the matrix $(M^T)^{-1}$.
Proposition 3.4. If \((R_1, B_1, L_1)\) and \((R_2, B_2, L_2)\) are conjugate through the matrix \(M\), then

(i) \(\tau_{M_{B_1}}(Mx) = M\tau_{B_1}(x), \quad \tau_{(M'T)^{-1}l_1}((M'T)^{-1}x) = (M'T)^{-1}\tau_{l_1}(x)\), for all \(b_1 \in B_1, l_1 \in L_1\);

(ii) \(W_{B_2}(x) = W_{B_1}(M'Tx)\) for all \(x \in \mathbb{R}^d\);

(iii) For the Fourier transform of the corresponding invariant measures, the following relation holds: 
\[\hat{\mu}_{B_2}(x) = \hat{\mu}_{B_1}(M'Tx)\] for all \(x \in \mathbb{R}^d\);

(iv) The associated path measures satisfy the following relation:
\[P^2_x(E) = P^1_{M'T_x}((M'Tl_1, M'Tl_2, \ldots) | (l_1, l_2, \ldots) \in E)\].

Definition 3.5. Let \((R, B, L)\) be a Hadamard triple. We call a subspace \(V\) of \(\mathbb{R}^d\) reducing if there exists a Hadamard triple \((R', B', L')\), conjugate to \((R, B, L)\), which is reducible to \(\mathbb{R}^r\), and such that the conjugating matrix \(M\), i.e., \(R' = MRM^{-1}\), maps \(V\) onto \(\mathbb{R}^r \times \{0\}\). We allow here \(V = \{0\}\), and the trivial space is clearly reducing.

Definition 3.6. We say that the Hadamard triple \((R, B, L)\) satisfies the reducibility condition if for all minimal compact invariant subsets \(M\), the subspace \(V\) given in Theorem 2.17 can be chosen to be reducing, and, for any two distinct minimal compact invariant sets \(M_1, M_2\), the corresponding unions \(R_1, R_2\) of the translates of the associated subspaces given in Theorem 2.17 are disjoint.

Proposition 3.7. If \(V\) is a reducing subspace then the hypothesis “\((H)\) modulo \(V\)” is satisfied.

Proof. By conjugation we can assume \(V = \mathbb{R}^r \times \{0\}\). We use the notations in Definition 3.1.

Let \(k_n, k'_n \in L\), \(n \in \{1, \ldots, p\}\) such that \(\tau_{k_1} \cdots \tau_{k_p} 0 - \tau_{k'_1} \cdots \tau_{k'_p} 0 \in V\). Then we can write \(k_n = (\eta_{n_1}, j_{n_1}, s_{j_{n_1}}), k'_n = (\eta_{n_2}, j'_{n_2}, s_{j'_{n_2}})\) for all \(n \in \{1, \ldots, p\}\). Then by a computation we obtain
\[\sum_{n=1}^{p} S_2^{-n}(s_{j_n} - s_{j'_n}) = 0.\]
This implies
\[\sum_{n=1}^{p} S_2^{-n}(s_{j_n} - s_{j'_n}) = 0.\]
However, the Hadamard condition \((\nu)\) in Definition 3.1 implies, according to Remark 2.4, that \(s_{j_p}\) and \(s_{j'_p}\) are not congruent \(\mod S_2\), unless \(j_p = j'_p\). Thus \(j_p = j'_p\).

By induction we obtain that \(j_n = j'_n\) for all \(n\) and this implies the hypothesis “\((H)\) modulo \(V\)”.

\[\square\]

Theorem 3.8. Let \(R\) be an expanding \(d \times d\) integer matrix, \(B\) a subset of \(\mathbb{Z}^d\) with \(0 \in B\). Assume that there exists a subset \(L\) of \(\mathbb{Z}^d\) with \(0 \in L\) such that \((R, B, L)\) is a Hadamard triple which satisfies the reducibility condition. Then the invariant measure \(\mu_B\) is a spectral measure.

Remark 3.9. If for all minimal compact invariant sets one can take the subspace \(V\) to be \(\{0\}\), i.e., if all the minimal compact invariant subsets are \(W_B\)-cycles, then the reducibility condition is automatically satisfied, and we reobtain Theorem 7.4 from [DuJo].
4. Proofs

The idea of the proof is to use the relation $\sum_F h_F = 1$ from Proposition 2.13. The functions $h_F$ will be written in terms of $|\mu_B|^2$, and this relation will translate into the Parseval equality for a family of exponential function.

**Invariant sets and invariant subspaces.** We want to evaluate first $h_F(x) = P_x(N(F))$ for minimal invariant sets $F$. Theorem 2.17 will give us the structure of these sets and this will aid in the computation.

Consider a minimal compact invariant set $M$. Using Theorem 2.17 we can find an invariant subspace $V$ such that $M$ is contained in the union of some translates of $V$. Since the reducibility condition is satisfied, we can take $V$ reducible. Proposition 3.7 shows that the hypothesis “(H) modulo $V$” is satisfied. Therefore we can use part (c) of the theorem, and conclude that, for some cycle $C := \{x_0, \tau_1 x_0, \ldots, \tau_{n-1} \tau_1 x_0\}$, with $\tau_m \cdots \tau_1 x_0 = x_0$, $M$ is contained in the union

$$\mathcal{R} = \{x_0 + V, \tau_1 x_0 + V, \ldots, \tau_{n-1} \tau_1 x_0 + V\},$$

and $\mathcal{R}$ is an invariant subset.

By conjugation we can assume first that $V = \mathbb{R}^r \times \{0\}$, and the Hadamard triple $(R, B, L)$ is reducible to $\mathbb{R}^r$. We will use the notations in Definition 3.1. Thus $S$, $B$ and $L$ have the specific form given in this definition. Also, points in $\mathbb{R}^d$ are of the form $(x, y)$ with $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^{d-r}$. We refer to $x$ as the “first component” and to $y$ as the “second component”. For a path $(\omega_1 \ldots \omega_k \ldots)$ in $\Omega$ we will use the notation $(\omega_1, \ldots, \omega_k, \ldots)$ for the path of the first components, and $(\omega_1, 2 \ldots, \omega_k, 2 \ldots)$ for the path of the second components.

We will also consider the IFS defined on the second component:

$$\tau_{s_i}(y) = S_2^{-1}(y + s_i) \quad (y \in \mathbb{R}^{d-r}, i \in \{1, \ldots, N_2\}).$$

We want to compute $P_{x,y}(N(\mathcal{R}))$ (see Proposition 2.13 for the definition of $N(\mathcal{R})$).

**Lemma 4.1.** Let $h_1, \ldots, h_m \in \{s_i \mid i \in \{1, \ldots, N_2\}\}$ be the second components of the sequence $l_1, \ldots, l_m$ that defines the cycle $C$. A path $(\omega_1 \omega_2 \ldots)$ is in $N(\mathcal{R})$ if and only if the second component of this path is of the form $(\omega_{1,2} \ldots, \omega_{k,2})$, where $\omega_{1,2}, \ldots, \omega_{k,2}$ are arbitrary in $\{s_i \mid i \in \{1, \ldots, N_2\}\}$.

**Proof.** Since $V = \mathbb{R}^r \times \{0\}$, the path $\omega$ is in $N(\mathcal{R})$ if and only if the second component of $\tau_{\omega_n} \cdots \tau_{\omega_1}(x,y)$ approaches the set $\mathcal{C}_2$ of the second components of the cycle $C$. But note that $\tau_{(\omega_{k,1}, \omega_{k,2})}(x,y)$ has the form $(*, \tau_{\omega_{k,2}}y)$. Thus we must have

\[
\lim_n d(\tau_{\omega_{k,2}} \cdots \tau_{\omega_1}y, C_2) = 0.
\]

Also $\mathcal{C}_2 = \{y_0, \tau_{h_1} y_0, \ldots, \tau_{h_{m-1}} \cdots \tau_{h_1} y_0\}$ is a cycle for the IFS $(\tau_n)_l$, where $y_0$ is the second component of $x_0$, and $\tau_{h_m} \cdots \tau_{h_1} y_0 = y_0$. But then (4.1) is equivalent to the fact that the path $(\omega_1, 2 \omega_2, \ldots)$ ends in an infinite repetition of the cycle $h_1 \ldots h_m$ (see [DuJo05, Remark 6.9]). This proves the lemma. \hfill $\Box$

Thus the paths in $N(\mathcal{R})$ are arbitrary on the first component, and end in a repetition of the cycle on the second. We will need to evaluate the following quantity,
for a fixed \( l_2 \in \{s_1, \ldots, s_N\} \), and \((x, y) \in \mathbb{R}^d\):

\[
A := \sum_{l_1 \text{ with } (l_1, l_2) \in L} W_B(\tau(l_1, l_2))(x, y)
= \sum_{l_1} \frac{1}{N_1^2 N_2^2} \sum_{i, i', j, j'} e^{2\pi i ((r_i - r_{i'}) \cdot (S_1^{-1}(x + l_1) + D(y + l_2)) + (\eta_{i, j} - \eta_{i', j'}) \cdot (S_2^{-1}(y + l_2)))}.
\]

But, because of the Hadamard property [iv] in Definition 3.1,

\[
\frac{1}{N_1} \sum_{l_1} e^{2\pi i (r_i - r_{i'}) \cdot S_1^{-1} l_1} = \begin{cases} 1, & i = i', \\ 0, & i \neq i'. \end{cases}
\]

Therefore

\[
A = \frac{1}{N_1^2 N_2^2} \sum_{i} \sum_{j, j'} e^{2\pi i (\eta_{i, j} - \eta_{i, j'}) \cdot S_2^{-1} (y + l_2)}
\]

and

\[
(4.2) \quad \sum_{l_1} W_B(\tau(l_1, l_2))(x, y) = \frac{1}{N_1} \sum_{i=1}^{N_1} W_i(\tau_{l_2} y) =: \tilde{W}(\tau_{l_2} y),
\]

where

\[
(4.3) \quad W_i(y) = \left| \frac{1}{N_2} \sum_{j=1}^{N_2} e^{2\pi i \eta_{i, j} y} \right|^2
\]

Next we compute \( P_{(x, y)} \) for those paths that have a fixed second component \((l_1, 2l_2, 2l_3, \ldots l_{n, 2})\).

**Lemma 4.2.**

\[
P_{(x, y)}(\{(\omega_1 \ldots \omega_n) \mid \omega_{n, 2} = l_{n, 2} \text{ for all } n\}) = \prod_{k=1}^{\infty} \tilde{W}(\tau_{l_{k, 2}} \cdots \tau_{l_{1, 2}} y).
\]

**Proof.** We compute for all \( n \), by summing over all the possibilities for the first component, and using [2.3]:

\[
P_{(x, y)}(\{(\omega_1 \omega_2 \ldots) \mid \omega_{k, 2} = l_{k, 2}, 1 \leq k \leq n\})
= \sum_{l_1, 1 \ldots l_{n, 1}} \prod_{k=1}^{n} W_B(\tau(l_{k, 1}, l_{k, 2}) \cdots \tau(l_{1, 1}, l_{1, 2}))(x, y) = (*).
\]

Using [4.2] we obtain further

\[
(*) = \tilde{W}(\tau_{l_{n, 2}} \cdots \tau_{l_{1, 2}} y) \sum_{l_1, 1 \ldots l_{n-1, 1}} \prod_{k=1}^{n-1} W_B(\tau(l_{k, 1}, l_{k, 2}) \cdots \tau(l_{1, 1}, l_{1, 2}))(x, y)
\]

\[
= \cdots = \prod_{k=1}^{n} \tilde{W}(\tau_{l_{k, 2}} \cdots \tau_{l_{1, 2}} y).
\]

Then, letting \( n \to \infty \) we obtain the lemma. \( \square \)
We will show that the measure \( \mu \) can be decomposed through the invariant subspace \( V = \mathbb{R}^r \times \{0\} \).

The matrix \( R \) has the form:

\[
R = \begin{bmatrix}
A_1 & 0 \\
C^* & A_2
\end{bmatrix}, \quad \text{and} \quad R^{-1} = \begin{bmatrix}
A_1^{-1} & 0 \\
-A_1^{-1}C^*A_2^{-1} & A_2^{-1}
\end{bmatrix}.
\]

By induction,

\[
R^{-k} = \begin{bmatrix}
A_1^{-k} & 0 \\
D_k & A_2^{-k}
\end{bmatrix}, \quad \text{where} \quad D_k := -\sum_{l=0}^{k-1} A_2^{-(l+1)} C^* A_1^{-(k-l)}.
\]

We have

\[
X_B = \{ \sum_{k=1}^{\infty} R^{-k} b_k \mid b_k \in B \}.
\]

Therefore any element \((x, y)\) in \(X_B\) can be written in the following form:

\[
x = \sum_{k=1}^{\infty} A_1^{-k} r_{ik}, \quad y = \sum_{k=1}^{\infty} D_k r_{ik} + \sum_{k=1}^{\infty} A_2^{-k} \eta_{ik,jk}.
\]

Define

\[
X_1 := \{ \sum_{k=1}^{\infty} A_1^{-k} r_{ik} \mid i_k \in \{1, \ldots, N_1\} \}.
\]

Let \( \mu_1 \) be the invariant measure for the iterated function system

\[
\tau_{r_i}(x) = A_1^{-1}(x + r_i), \quad i \in \{1, \ldots, N_1\}.
\]

The set \( X_1 \) is the attractor of this iterated function system.

For each sequence \( \omega = (i_1i_2 \ldots) \in \{1, \ldots, N_1\}^{\mathbb{N}} \), define \( x(\omega) = \sum_{k=1}^{\infty} A_1^{-k} r_{ik} \).

Also, because of the non-overlap condition, for \( \mu_1 \)-a.e. \( x \in X_1 \), there is a unique \( \omega \) such that \( x(\omega) = x \). We define this as \( \omega(x) \). This establishes an a.e. bijective correspondence between \( \Omega_1 \) and \( X_1 \), \( \omega \leftrightarrow x(\omega) \).

Denote by \( \Omega_1 \) the set of all paths \((i_1i_2 \ldots i_n \ldots)\) with \( i_k \in \{1, \ldots, N_1\} \). For \( \omega = (i_1i_2 \ldots) \in \Omega_1 \) define

\[
\Omega_2(\omega) := \{ \eta_{i_1,j_1} \eta_{i_2,j_2} \ldots \eta_{i_n,j_n} \ldots \mid j_k \in \{1, \ldots, N_2\} \}.
\]

For \( \omega \in \Omega_1 \) define \( g(\omega) := \sum_{k=1}^{\infty} D_k r_{ik} \), and \( g(x) := g(\omega(x)) \). Also we denote \( \Omega_2(x) := \Omega_2(\omega(x)) \).

For \( x \in X_1 \), define

\[
X_2(x) := X_2(\omega(x)) := \left\{ \sum_{k=1}^{\infty} A_2^{-k} \eta_{i_k,j_k} \mid j_k \in \{1, \ldots, N_2\} \right\}.
\]

Note that the attractor \( X_B \) has the following form:

\[
X_B = \{(x, g(x) + y) \mid x \in X_1, y \in X_2(x)\}.
\]

We will show that the measure \( \mu_B \) can also be decomposed as a product between the measure \( \mu_1 \) and some measures \( \mu^2_\omega \) on \( X_2(\omega) \).

On \( \Omega_2(\omega) \), consider the product probability measure \( \mu(\omega) \) which assigns to each \( \eta_{i_k,j_k} \) equal probabilities \( 1/N_2 \).

Next we define the measure \( \mu^2_\omega \) on \( X_2(\omega) \). Let \( r_\omega : \Omega_2(\omega) \to X_2(\omega) \),

\[
r_\omega(\eta_{i_1,j_1} \eta_{i_2,j_2} \ldots) = \sum_{k=1}^{\infty} A_2^{-k} \eta_{i_k,j_k}.
\]
Define the measure $\mu_2^2 := \mu_{\omega(z)}^2 := \mu_\omega(z) \circ r_\omega^{-1}$.

**Lemma 4.3.** Let $\sigma$ be the shift on $\Omega_1$, $\sigma(i_1i_2\ldots) = (i_2i_3\ldots)$. Let $\omega = (i_1i_2\ldots) \in \Omega_1$. Then for all measurable sets $E$ in $\Omega_2(\omega)$,

$$\mu_\omega^2(E) = \frac{1}{N_2} \sum_{j=1}^{N_2} \mu_{\omega(\sigma)}^2(\tau_{\eta_{1,j}}^{-1}(E)).$$

The Fourier transform of the measure $\mu_\omega^2$ satisfies the equation:

$$(4.4) \quad \hat{\mu}_\omega^2(y) = m(S_2^{-1}y, i_1) \hat{\mu}_{\omega(\sigma)}^2(S_2^{-1}y),$$

where

$$m(y, i_1) = \frac{1}{N_2} \sum_{j=1}^{N_2} e^{2\pi i \eta_{1,j} \cdot y}.$$  

**Proof.** We define the maps $\xi_{\eta_{1,j}} : \Omega_2(\sigma(\omega)) \to \Omega_2(\omega)$,

$$\xi_{\eta_{1,j}}(\eta_{1,j_2\eta_{2,j_3}\ldots}) = (\eta_{1,j_1\eta_{2,j_2}\ldots}).$$

Then $r_\omega \circ \xi_{\eta_{1,j}} = \tau_{\eta_{1,j}} \circ \tau_\sigma$. The relation given in the lemma can be pulled back through $r_\omega$ to the path spaces $\Omega_2(\omega)$, and becomes equivalent to:

$$\mu_\omega^2(E) = \frac{1}{N_2} \sum_{j=1}^{N_2} \mu_{\omega(\sigma)}^2(\xi_{\eta_{1,j}}^{-1}(E)),$$

and this can be immediately be verified on cylinder sets, i.e., the sets of paths in $\Omega_2(\omega)$ with some prescribed first $n$ components.

From this it follows that

$$\int f \, d\mu_\omega^2 = \frac{1}{N_2} \sum_{j=1}^{N_2} \int f \circ \tau_{\eta_{1,j}} \, d\mu_{\omega(\sigma)}^2.$$  

Applying this to the function $s \mapsto e^{2\pi is \cdot y}$ we obtain equation (4.4). \hfill \Box

**Lemma 4.4.**

$$\int_{X_B} f \, d\mu_B = \int_{X_1} \int_{X_2(x)} f(x, y + g(x)) \, d\mu_2^2(y) \, d\mu_1(x).$$

**Proof.** We begin with a relation for the function $g$

$$(4.5) \quad g(A_1^{-1}(x + r_i)) = D_1(x + r_i) + A_2^{-1}g(x)$$

Indeed, if $\omega(x) = (i_1i_2\ldots)$, then $\omega(A_1^{-1}(x + r_i)) = (ii_1i_2\ldots)$. So

$$g(A_1^{-1}(x + r_i)) = D_1r_i + \sum_{k=1}^{\infty} D_{k+1}r_{ik} = D_1r_i - \sum_{k=1}^{\infty} \sum_{l=0}^{k} A_2^{-(l+1)} C^{*} A_1^{-(k+1-l)} r_{ik}$$

$$= D_1r_i - \sum_{k=1}^{\infty} A_2^{-1} C^{*} A_1^{-k} r_{ik} = \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} A_2^{-(l+2)} C^{*} A_1^{-(k-l)} r_{ik}$$

$$= D_1r_i + D_1x + A_2^{-1}g(x).$$
Next we show that the measure \( \mu_B \) has the given decomposition. We check the invariance of the decomposition. We denote by \( \iota_1(x) \), the first component of \( \omega(x) \), and \( \sigma(x) \) is the point in \( X_1 \) that corresponds to \( \sigma(\omega(x)) \).

\[
\int_{X_1} \int_{X_2(x)} f(x, y + g(x)) \, d\mu^2_x(y) \, d\mu_1(x)
\]

\[
= \frac{1}{N_2} \sum_{j=1}^{N_2} \int_{X_1} \int_{X_2(\sigma(x))} f(x, A_2^{-1}(y + \eta_{1}(x), j) + g(x)) \, d\mu_\sigma(x) \, d\mu_1(x)
\]

\[
= \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_{X_1} \int_{X_2(\sigma(\tau, x))} f(A_1^{-1}(x + r_i), A_2^{-1}(y + \eta_{1}(\tau, x), j))
\]

\[
+ g(A_1^{-1}(x + r_i)) \, d\mu_\sigma(\tau, x) \, d\mu_1(x)
\]

\[
= \frac{1}{N} \sum_{i,j} \int_{X_1} \int_{X_2(x)} f(A_2^{-1}(x + r_i), x)
\]

\[
= \frac{1}{N} \sum_{i,j} \int_{X_1} \int_{X_2(x)} f \circ \tau(x, y) \, d\mu_\sigma(\tau, x)(x, y) \, d\mu_1(x).
\]

Using the uniqueness of the invariant measure for an IFS, we obtain the lemma. \( \square \)

**Lemma 4.5.** If \( \Lambda_1 \) is a spectrum for the measure \( \mu_1 \), then

\[
F(y) := \sum_{\lambda_1 \in \Lambda_1} |\hat{\mu}_B(x + \lambda_1, y)|^2 = \int_{X_1} |\hat{\mu}_1^2(y)|^2 \, d\mu_1(s) \quad (x \in \mathbb{R}, y \in \mathbb{R}^{d-r}).
\]

**Proof.**

\[
F(y) = \sum_{\lambda_1} \int_{X_1} \int_{X_2(s)} e^{2\pi i((x+\lambda_1) \cdot s+y \cdot (t+g(s)))} \, d\mu_2(t) \, d\mu_1(s)
\]

\[
= \sum_{\lambda_1} \int_{X_1} \left( e^{2\pi i(x \cdot s+y \cdot g(s))} \hat{\mu}_2^2(y) \right) e^{2\pi i\lambda_1 \cdot s} \, d\mu_1(s)
\]

\[
= \int_{X_1} |\hat{\mu}_2^2(y)|^2 \, d\mu_1(s),
\]

where we used the Parseval identity in the last equality. \( \square \)

**Lemma 4.6.**

\[
F(y) = \tilde{W}(S_2^{-1} y) F(S_2^{-1} y).
\]

Also

\[
F(y) = \prod_{k=1}^{\infty} W(S_2^{-k} y) \quad (y \in \mathbb{R}^{d-r}).
\]

**Proof.** Using Lemma 4.5 and Lemma 4.3 and the fact that

\[
\frac{1}{N_1} \sum_{i=1}^{N_1} |m(y, i)|^2 = \tilde{W}(y) \quad (y \in \mathbb{R}^{d-r}),
\]
we obtain
\[ F(y) = \int_{X_1} |m(S_2^{-1}y, i_1(s))|^2 \left| \hat{\mu}_{\tau(s)}(S_2^{-1}y) \right|^2 \mu_1(s) \]
\[ = \frac{1}{N_1} \sum_{i=1}^{N_1} \int_{X_1} |m(S_2^{-1}y, i_1(\tau, s))|^2 \left| \hat{\mu}_{\tau(s), \tau(s)}(S_2^{-1}y) \right|^2 \mu_1(s) \]
\[ = \hat{W}(S_2^{-1}y)F(S_2^{-1}y). \]

We also have \( F(0) = 1 \) because \( \mu_1 \) and \( \mu_2^0 \) are probability measures. Using Lemma 4.6 it is easy to see that \( F \) is continuous. Also \( \hat{W}(0) = 1 \) and for some \( 0 < c < 1 \), \( \|S_2^{-k}\| \leq c^k \) for all \( k \) (because \( S_2 \) is expansive), and \( \hat{W} \) is Lipschitz, the infinite product is then convergent to \( F(y) \). □

Now consider the cycle associated to the minimal invariant set \( M \),
\[ C = \{x_0, \tau_1x_0, \ldots, \tau_{m-1}\ldots\tau_1x_0\} \]
as described in the beginning of the section, with \( \tau_m \ldots \tau_0x_0 = x_0 \). Consider the second components of this cycle. Let the second component of \( x_0 \) be \( y_0 \) and let \( h_1, \ldots, h_m \in \{s_i | i \in \{1, \ldots, N_2\}\} \) be the second components of \( \iota_1, \ldots, \iota_m \).

**Lemma 4.7.** The set \( C_2 := \{y_0, \tau h_1 y_0, \ldots, \tau_{m-1} \ldots \tau h_1 y_0\} \) is a \( \hat{W} \)-cycle.

**Proof.** We saw in the proof of Lemma 4.1 that \( C_2 \) is a cycle. We only need to check that \( \hat{W}(y) = 1 \) for all \( y \in C_2 \). Take the point \( y_0 \) and take some \( s_j \neq h_1 \). We claim that \( \tau s_j y_0 \) cannot be one of the points in \( C_2 \). Otherwise it would follow that \( y_0 \) is a fixed point for \( \tau_{\omega_1} \cdots \tau_{\omega_1} \), for some \( \omega_1, \omega_2, \ldots, \omega_q \in \{s_j | j \in \{1, \ldots, N_2\}\} \) with \( \omega_1 = s_j \neq h_1 \). But \( x_0 \) is also a fixed point for \( \tau h_m \cdots \tau h_1 \). It follows that \( x_0 \) is fixed also by \( (\tau h_m \cdots \tau h_1)^q \) and \( (\tau \omega \cdots \tau \omega)^m \). Writing the corresponding fixed point equations, we obtain:
\[(S_2^{mq} - I)^{-1}(h_1 + Sk) = x_0 = (S_2^{mq} - I)^{-1}(\omega_1 + Sk'),\]
for some \( k, k' \in \mathbb{Z}^{d-r} \). But this implies that \( h_1 \equiv \omega_1 \mod S_2 \mathbb{Z}^{d-r} \) and this is impossible because of the Hadamard property (v) in Definition 3.1 and Remark 2.4 This proves our claim.

Since \( \tau s_j y_0 \) is not in \( C_2 \), the invariance of the set \( R = \bigcup_{y \in C_2}(y + \mathbb{R}^r \times \{0\}) \) implies that \( \int_{\tau h_i s_j (x, y_0)} W_B(x) = 0 \) for all \( i \in \{1, \ldots, N_2\} \). But then, with equation 4.12, this implies that \( \hat{W}(\tau s_j y_0) = 0 \), for all \( s_j \neq h_1 \). And since
\[ \sum_{j=1}^{N_2} \hat{W}(\tau s_j y_0) = 1, \]
it follows that \( \hat{W}(\tau h_1 y_0) = 1 \). The same argument works for the other points in \( C_2 \), and we obtain the result. □

**Lemma 4.8.** The following relation holds for all \( k \geq 0 \):
\[ \hat{W}(y + S_2^{km}y_0) = \hat{W}(y) \quad (y \in \mathbb{R}^d). \]

**Proof.** Since \( \hat{W}(y_0) = 1 \), it follows that \( W_i(y_0) = 1 \) for all \( i \in \{1, \ldots, q_1\} \). Therefore all the terms in the sum which defines \( W_i \) must be 1 which means that \( \eta \cdot y_0 \in \mathbb{Z} \) for all \( i, j \). This implies that \( W_i(y + y_0) = W_i(y) \)
Proposition 4.10. There exists a set \( \Lambda(M) \subset \mathbb{Z}^d \) such that
\[
h_{\mathcal{R}}(x) = P_x(N(\mathcal{R})) = \sum_{\lambda \in \Lambda(M)} |\hat{\mu}_B(x + \lambda)|^2 \quad (x \in \mathbb{R}^d).
\]
Proof. First note that, with Proposition 3.8, we can assume that the Hadamard triple \((R, B, L)\) is reducible to \(\mathbb{R}^r\) and \(V = \mathbb{R}^r \times 0\).

With Lemma 4.1, we see that \(N(R)\) is the set of all paths such that the second component has the form \((\omega_0 \ldots \omega_k s_j \ldots s_{j-1})\).

We have
\[
P(x,y)(N(R)) = \sum_\omega P(x,y)(E_{\omega C})
\]
where the sum is indexed over all possible paths that end in a repetition of the cycle \(l_1 \ldots h_m\), so it can be indexed by a choice of a finite path \(\omega_1 \ldots \omega_{km-1}\) in with \(\omega_i \in \{s_j | j \in \{1, \ldots, N_2\}\}\) for all \(i\).

Using Lemma 4.9 and Lemma 4.5 we obtain further:
\[
P(x,y)(N(R)) = \sum_\omega F(y + k C(\omega))
\]
\[
= \sum_\omega \sum_{\lambda \in \Lambda_1} |\hat{\mu}_B(x + \lambda, y + k C(\omega))|^2.
\]
The proposition is proved.

\(\square\)

Remark 4.11. It might happen that for two different paths \(\omega\) the integers \(k C\) are the same. Therefore the same \(\lambda\) might appear twice in the set \(\Lambda(M)\). We make the convention to count it twice. We will show in the end that actually this will not be the case.

We are now in position to give the proof of the theorem.

Proof. (of Theorem 4.8) Let \(M_1, \ldots, M_p\) be the list of all minimal compact invariant sets. The hypothesis shows that for each \(k\) there is a reducing subspace \(V_k\) and some cycle \(C_k\) such that \(M_k \subset R_k := C_k + V_k\) and moreover the sets \(R_k\) are mutually disjoint. With Proposition 4.10 we see that there is some set \(\Lambda(M_k) \subset \mathbb{Z}^d\) such that
\[
h_{R_k}(x) = \sum_{\lambda \in \Lambda(M_k)} |\hat{\mu}_B(x + \lambda)|^2 \quad (x \in \mathbb{R}^d).
\]

With Proposition 4.15, we have
\[
1 = \sum_{k=1}^p h_{R_k}(x) = \sum_{k=1}^p \sum_{\lambda \in \Lambda(M_k)} |\hat{\mu}_B(x + \lambda)|^2.
\]

We check that a \(\lambda\) cannot appear twice in the union of the sets \(\Lambda(M_k)\). For some fixed \(\lambda_0 \in \bigcup_{k} \Lambda(M_k)\), take \(x = -\lambda_0\) in (4.6). Since \(\hat{\mu}_B(0) = 1\), it follows that one term in the sum is 1 (the one corresponding to \(\lambda_0\)) and the rest are 0. Thus \(\lambda_0\) cannot appear twice. Also for \(\lambda \neq \lambda_0\), this implies that \(\hat{\mu}_B(-\lambda_0 + \lambda) = 0\) so the functions \(e^{2\pi i \lambda_0 x}\) and \(e^{2\pi i \lambda x}\) are orthogonal in \(L^2(\mu_B)\).

With the notation \(e_x(t) = e^{2\pi i t x}\), we can rewrite (4.6) as
\[
\|e_{-x}\|^2 = \sum_{\lambda \in \bigcup_{k=1}^p \Lambda(M_k)} |\langle e_{-x} \rangle e_\lambda|^2 \quad (x \in \mathbb{R}^d).
\]

But this, and the orthogonality, implies that the closed span of family of functions \(\{e_\lambda | \lambda \in \Lambda\}\), where \(\Lambda = \bigcup_{k=1}^p \Lambda(M_k)\), contains all functions \(e_x\), and, by Stone-Weierstrass, this implies that it contains \(L^2(\mu_B)\). Thus, \(\{e_\lambda | \lambda \in \Lambda\}\) forms an orthonormal basis for \(L^2(\mu_B)\).

\(\square\)
5. Examples

Before we give the examples we will prove a lemma which helps in identifying candidates for the invariant subspaces containing minimal invariant sets.

Lemma 5.1. With the assumptions of Theorem 2.17 suppose that there is no proper subspace of \( W \) such that \( X_B \) is contained in a finite union of translates of \( W \). Let \( V \) be an invariant subspace as in 2.17. Then there is some \( x \in \mathbb{R}^d \) such that \( W_B(x + v) = 0 \) for all \( v \in V \). If in addition the hypothesis “(H) modulo \( V \)” is satisfied, and \( C := \{x_0, \tau_1 x_0, \ldots, \tau_{m-1} \cdots \tau_1 x_0\} \) is the cycle given in Theorem 2.17, then \( x \) can be taken to be any element of \( L \) such that \( l - l_{k+1} \notin V \).

Proof. Consider the invariant union \( \mathcal{R} \) of translates of \( V \), as in Theorem 2.17. Then \( \mathcal{R} \) cannot contain \( X_B \) so for some \( x \in \mathcal{R} \) and some \( l \in L \) we have \( \tau_l(x) \notin \mathcal{R} \). But then, for all \( v \in V \), \( \tau_l(x + v) = \tau_l x + S^{-1}v \) cannot be in \( \mathcal{R} \) (otherwise \( \tau_l x = \tau_l(x + v) - S^{-1}v \in \mathcal{R} = V \)). Since \( \mathcal{R} \) is invariant, it follows that \( W_B(\tau_l(x + v)) = 0 \). But \( \tau_l(x + V) = \tau_l x + S^{-1}V = \tau_l x + V \) and this proves the first assertion.

If \( V \) also satisfies the hypothesis “(H) modulo \( V \)”, then \( \mathcal{R} = C + V \). Take \( v \in V \) and \( l \in L \) such that \( l - l_1 \notin V \). If \( W_B(\tau_l(x_0 + v)) \neq 0 \) then, by Theorem 2.17, \( \tau_l(x_0 + v) \in \tau_l x_0 + V \). This implies that \( \tau_l(x_0) - \tau_l x_0 \in V \) so \( \tau_l - \tau_l 0 \in V \). With the hypothesis “(H) modulo \( V \)” we get \( l - l_1 \in V \), a contradiction. This proves the lemma.

Example 5.2. To illustrate our method, we now give a natural but non-trivial example \((R, B, L)\) in \( \mathbb{R}^2 \) for which \( \mu_B \) may be seen to be a spectral measure. In fact, we show that there is a choice for its spectrum \( \Lambda = \Lambda(\mu_B) \) which we compute with tools from Definition 3.6, Theorem 3.8, and Lemma 4.9. Moreover, for the computation of the whole spectrum \( \Lambda \), the \( W_B \)-cycles do not suffice. (There is one \( W_B \) cycle, a one-cycle, and it generates only part of \( \Lambda \).) Hence in this example, the known theorems from earlier papers regarding spectrum do not suffice. To further clarify the \( W_B \)-cycles in the example, we have graphed the two attractors \( X_B \) and \( X_L \) in Figures 1 and 2.

Take
\[
R := \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}, \quad B := \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}.
\]

One can take
\[
L := \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}.
\]

One can check that the matrix in Definition 2.3 is unitary so \((R, B, L)\) is a Hadamard triple.

We look for \( W_B \)-cycles. We have
\[
W_B(\mathbf{x}, \mathbf{y}) = \left| \frac{1}{4}(1 + e^{2\pi i x} + e^{2\pi i 3y} + e^{2\pi i (x+3y)}) \right|^2.
\]

Then \( W_B(\mathbf{x}, \mathbf{y}) = 1 \) iff \( \mathbf{x} \in \mathbb{Z} \) and \( \mathbf{y} \in \mathbb{Z}/3 \) (all the terms in the sum must be equal to 1).

If \((x_0, y_0)\) is a point of a \( W_B \)-cycle, then for some \((l_1, l_2) \in \mathcal{L}, \tau_{l_1 l_2}(x_0, y_0)\) is also in the \( W_B \)-cycle, so \( x_0, \frac{1}{4}(x_0 + l_1) - \frac{1}{16}(y_0 + l_2) \in \mathbb{Z} \) and \( y_0, \frac{1}{4}(y_0 + l_2) \in \mathbb{Z}/3 \).
Also, note that \((x_0, y_0)\) is in the attractor \(X_L\) of the IFS \((\tau_l)_{l \in L}\), so \(0 \leq y_0 \leq 2/3\), and \(-1/4 \leq x_0 \leq 2/3\). (This can be seen by checking that the rectangle \([-1/4, 2/3] \times [0, 2/3]\) is invariant for all \(\tau_l, l \in L\).)

Then, we can check these points and obtain that the only \(W_B\)-cycle is \((0, 0)\), of length 1, which corresponds to \(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\).

Now we look for the vector spaces \(V\) that might appear in connection to the minimal invariant sets (see Theorem 2.17). Since these spaces are proper, and we have eliminated the case when \(V = \{0\}\) by considering the \(W_B\)-cycles, it follows that \(V\) must have dimension 1 so it is generated by an eigenvector of 
\[
S = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}.
\]

Thus \(V = \{(x, 0) \mid x \in \mathbb{R}\}\).

This subspace is reducible, with \(r_1 = 0, r_2 = 1, \eta_{1,1} = 0, \eta_{1,2} = 3, \eta_{2,1} = 0, \eta_{2,2} = 3, s_1 = 0, s_2 = 2, \gamma_{1,1} = \gamma_{1,2} = 0, \gamma_{2,1} = \gamma_{2,2} = 2\). The measure \(\mu_1\) on the first component corresponds to the IFS \(\tau_0(x) = x/4, \tau_1(x) = (x + 1)/4\). This corresponds to \(R_1 = 4, B_1 := \{0, 1\}\) and one can take \(L_1 := \{0, 2\}\) to get \((R_1, B_1, L_1)\) a Hadamard pair. The associated function is \(W_{B_1}(x) = \left|\frac{1}{2}(1 + e^{2\pi i x})\right|^2\), the only points where \(W_{B_1}\) is 1 are \(x \in \mathbb{Z}\). Then one can see that the only \(W_{B_1}\)-cycle is \(\{0\}\). Thus the spectrum of \(\mu_1\) is \(\{\sum_{k=0}^{n} a_k \mid a_k \in \{0, 2\}, n \in \mathbb{N}\}\).

We have to find the associated cycle \(C\). As in Lemma 5.1, we must have \(W_B(\tau_l(x_0) + v) = 0\) for elements \(x_0\) in the cycle and some \(l \in L\) and all \(v \in V\). But this means that, for the second component \(y' \in \mathbb{R}\) of \(\tau_l x_0, 1 + e^{2\pi i x} + e^{2\pi i y'} + e^{2\pi i (x+3y')} = 0\). This implies that \(y' = (2k + 1)/6\) for some \(k \in \mathbb{Z}\). Moreover, we saw in Lemma 4.7 that the set of the second components of \(C\) must be a \(\hat{W}\) cycle. In our case \(\hat{W}(y) = \frac{1}{2} |1 + e^{2\pi i y'}|^2\), and the IFS in case is \(\{\tau_0, \tau_2\}\). The \(\hat{W}\)-cycles are \(\{0\}\) corresponding to \(\emptyset\), and \(\{2/3\}\) corresponding to \(\emptyset\). Thus we obtain that the invariant sets obtained as translations of \(V\) could be: \(V\) and \(R := 2/3 + V = \{(x, 2/3) \mid x \in \mathbb{R}\}\). We can discard the first one because we see
that $W_B(\tau_{(0,2)}(x,0))$ is not constant $0$. The set $2/3 + V$ is indeed invariant, and we have $\tau_{(l_2,2)}(x,2/3) = 0$ if $l_2 = 0$, and $\tau_{(l_2,2)}(x,2/3) \in 2/3 + V$ if $l_2 = 2$.

Next we want to compute the contribution of each of these invariant sets to the spectrum of $\mu_B$.

For the $W_B$-cycle $\{(0,0)\}$, of length $m = 1$, we have as in Lemma 4.9

$$k_0(\omega_1 \ldots \omega_{k-1}) = \omega_1 + S\omega_2 + \cdots + S^{k-1}\omega_{k-1}$$

for all $\omega_1, \ldots, \omega_{k-1} \in L$. By induction one can see that $S^n = \begin{bmatrix} 4^n & n4^{n-1} \\ 0 & 4^n \end{bmatrix}$. So the contribution from this $W_B$-cycle is

$$\Lambda(0) := \left\{ \left( \sum_{k=0}^n 4^k a_k + g(b_0, \ldots, b_n), \sum_{k=0}^n 4^k b_k \right) \mid b_k \in \{0,2\} \right\},$$

where $g(b_0, \ldots, b_n) = \sum_{k=0}^n k4^{k-1}b_k$.

For the invariant set $\mathcal{K} = \{(x,2/3) \mid x \in \mathbb{R}\}$, we have as in Lemma 4.9 with $\omega_1, \ldots, \omega_{k-1} \in \{0,2\}$, $k_{2/3}(\omega_1, \ldots, \omega_{k-1}) = \omega_1 + 4\omega_2 + \cdots + 4^{k-1}\omega_{k-1} - 4^{k-2}/3$, or writing $2/3 = 2/4 + 2/4^2 + \cdots + 2/4^k + 2/4^{k+1} + \cdots$, we obtain

$$k_{2/3}(\omega_1, \ldots, \omega_{k-1}) = \sum_{i=0}^{k-1} a_i4^{i} - \frac{2}{3},$$

with $a_i \in \{0, -2\}$.

As in Proposition 4.10 and its proof, using the spectrum of $\mu_1$, the contribution to the spectrum is

$$\Lambda(2/3) := \left\{ \left( \sum_{k=0}^n 4^k a_k, -\frac{2}{3} - \sum_{k=0}^m 4^k b_k \right) \mid a_k, b_k \in \{0,2\}, n, m \in \mathbb{N} \right\}.$$ 

Finally, the spectrum of $\mu_B$ is $\Lambda_B := \Lambda(0) \cup \Lambda(2/3)$.

Note also, that we can use the decomposition given in Lemma 4.4. The measure $\mu_1$ is the invariant measure for the IFS: $\tau_0(x) = x/4$, $\tau_1(x) = (x + 1)/4$. For all $x \in \mathbb{R}$, the measure $\mu_2 = \mu_1$ is the invariant measure for the IFS $\tau_0(x) = x/4$, $\tau_3(x) = (x + 3)/4$. Both $\mu_1$ and $\mu_2$ are spectral measures (one can use $L = \{0,2\}$ for both of them). We saw that the spectrum of $\mu_1$ is $\Lambda_1 := \{\sum_{k=0}^n 4^k a_k \mid a_k \in \{0,2\}, n \in \mathbb{N}\}$. The IFS $(\tau_0, \tau_3)$ has two $W_{B_0}$-cycles: $\{0\}$ and $\{2/3\}$, so, after a computation we get that the spectrum of $\mu_2$ will be $\Lambda_2 := \Lambda_1 \cup \{\frac{-2}{3} - \Lambda_1\}$.

Using the decomposition of Lemma 4.4 we obtain that a spectrum for $\mu_B$ is $\Lambda_1 \times \Lambda_2$. It is interesting to see that this is a different spectrum than the one computed before $\Lambda_B$.

**Remark 5.3.** Since in Example 5.2 the $W_B$-cycles are not sufficient to describe all invariant sets, the results from JoPe96, SuPe98; LaWa96, DnJo05 do not apply here; they give only part of the spectrum, namely the contribution of the $W_B$-cycle $\{0\}$.

**Example 5.4.** Take now $B$ to be a complete set of representatives for $\mathbb{Z}/RZ^d$. So $N = |\det R|$. To get a Hadamard triple, one can take $L$ to be any complete set of representatives for $\mathbb{Z}^d/S\mathbb{Z}^d$, because the matrix $\frac{1}{\sqrt{N}}(e^{2\pi i t \cdot \cdot \cdot \cdot})_{k,l}$ will then be the matrix of the Fourier transform on the finite group $\mathbb{Z}^d/RZ^d$, hence unitary.

The following proposition is folklore for affine IFSs; see, e.g., CHR97, JoPe94, JoPe96, LaWa96.
Proposition 5.5. Suppose the vectors in $B$ form a complete set of coset representatives for the finite group $\mathbb{Z}^d/\mathbb{R}^d$. Then the following conclusions hold:

(a) The attractor $X_B$ has non-empty interior relative to the metric from $\mathbb{R}^d$.
(b) The Borel probability measure $\mu_B$ is of the form $\mu_B = \frac{1}{p}(\text{Lebesgue measure in } \mathbb{R}^d \text{ restricted to } X_B)$, where $p$ is an integer.
(c) Moreover, $p = 1$ if and only if the attractor $X_B$ tiles $\mathbb{R}^d$ by translations with vectors in the standard lattice $\mathbb{Z}^d$; where by tiling we mean that the union of translates $\{X_B + k \mid k \in \mathbb{Z}^d\}$ cover $\mathbb{R}^d$ up to measure zero, and where different translates can overlap at most on sets of measure zero.
(d) In general, there is a lattice $\Gamma$ contained in $\mathbb{Z}^d$ such that $X_B$ tiles $\mathbb{R}^d$ with $\Gamma$; and the group index $[\mathbb{Z}^d : \Gamma]$ coincides with the number $p$.

Using Fuglede’s theorem [Fug74] it follows that $\mu_B$ is a spectral measure, with spectrum the dual lattice of $\Gamma$. (Fuglede’s theorem [Fug74] characterizes measurable subsets $X$ in $\mathbb{R}^d$ which are fundamental domains for some fixed rank-$d$ lattice $L$. First note that such subsets have positive and finite Lebesgue measure, $\mu$ = the $d$-dimensional Lebesgue measure. For measurable fundamental domains, Fuglede showed that $L^2(X, \mu)$ has $\{e_\lambda \mid \lambda \in \text{the dual lattice to } L\}$ as ONB, i.e., that the dual lattice is a set of Fourier frequencies. More importantly, he proved the converse as well: If $L^2(X, \mu)$ for some measurable subset of $\mathbb{R}^d$ is given to have an ONB consisting of a lattice of Fourier frequencies, then $X$ must be a fundamental domain for the corresponding dual lattice. Furthermore, he and the authors of [Ped87, JoPe92] also considered extensions of this theorem to sets of Fourier frequencies that are finite unions of lattice points. We should add that there is a much more general Fuglede problem which was shown recently [Tao04] by Tao to be negative.)

The relation between the lattice $\Gamma$ and the invariant sets will be the subject of another paper.

Notes on the literature. While there is, starting with [Hut81] and [BEHL86], a substantial literature of papers treating various geometric features of iterated function systems (IFS), the use of Fourier duality is of a more recent vintage. The idea of using substitutions together with duality was perhaps initiated in [JoPe92]; see also [Mas94]. However, the use of substitutions in dynamics is more general than the context of IFSs; see, for example, [LaMa98]. We further want to call attention to a new preprint [Fre06] which combines the substitution principle with duality in a different but related manner. The use of duality in [Fre06] serves to prove that the class of affine IFSs arises as model sets. It is further interesting to note (e.g., [Bar01]) that these fractals have found use in data analysis.

In the definition of reducible subspaces we added a certain non-overlapping condition for the measure $\mu_1$. This condition, which might be automatically satisfied for our affine IFSs, is part of a more general problem:

Problem. Give geometric conditions for a fixed $(X, \tau_i)$ which guarantee that the distinct sets $\tau_i(X)$ overlap at most on subsets of $\mu$-measure zero.

For related but different questions, the reader can consult [Sch94, LaWa93, LaRa03, HLR03].

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