Semiclassical Theory of Many-Body Quantum Chaos and its Bound

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We introduce a direct quantum-classical correspondence for the chaotic dynamics of quantum many-body systems. The bound on chaos is understood as a “chaotic mobility edge” in the classical Lyapunov spectrum, separating the lower part of the spectrum for which a classical chaos picture applies from the higher part of the spectrum for which quantum interference effects destroy classical chaos. The dominant chaotic mode in bound-saturating systems is exactly at the “mobility edge” and therefore inherits a universal critical behavior. We provide an explicit example by introducing a classical version of the Sachdev-Ye-Kitaev model for which chaos can be understood as arising from diverging geodesics on a high-dimensional manifold with a random metric.

The last years have seen a surge of interest in the thermalization dynamics of closed quantum systems. One of the most intriguing outcomes of these activities has been a deeper and increasingly quantitative understanding of quantum chaos. A striking result, was the proof by Maldacena et al. of an upper bound on the rate at which chaos can develop in a quantum system, as characterized by a Lyapunov exponent $\lambda \leq 2\pi k_B T / h$ [1]. Soon after, Kitaev introduced a solvable model of interacting fermions, known as the Sachdev-Ye-Kitaev (SYK) model, which at low temperatures saturates the quantum bound on chaos [2][9] (see also [10] for a review of earlier work).

The recent results depart from a long history of studies of quantum chaos in a number of ways. An important new element is the focus on many-body systems rather than on the dynamics of single or few particles. Indeed a major goal of recent studies has been to relate the chaotic dynamics to the transport coefficients which govern the late time hydrodynamic behavior of thermalizing many-body systems [11][16]. Such a relation could have important implications for the puzzle of “strange metals” [17][19].

Another departure from previous work on quantum chaos concerns the relation to a classical limit. Studies of quantum chaos have long highlighted the close link to chaotic classical systems in the limit $\hbar \to 0$ [20][22]. Quantum chaos was in fact defined as the behavior of a quantum system whose classical limit is chaotic. It was later shown that near the classical limit the assumptions of random matrix theory (RMT), pertinent to quantum chaos, are indeed consistent with classical chaotic dynamics [23]. The new results on the chaos bound and the SYK model, on the other hand, are not rooted in the behavior of a limiting classical system. In particular the Lyapunov exponent $\lambda_{\text{bound}} = 2\pi k_B T / h$ characterizing fast scramblers seems not to have a finite classical limit. Nonetheless it is worth asking if these many-body systems admit a quantum-classical correspondence, which may shed new light on their chaotic properties.

In this paper we construct a direct quantum-classical correspondence for the chaotic dynamics of a class of quantum models, including the SYK model as a special case. We show that the dynamics in these systems can be mapped to the motion of a particle on a high-dimensional manifold with randomly varying curvature. The distribution of negative local curvatures on the manifold translates to a spectrum of Lyapunov exponents characterizing the classical motion. Our analysis suggests an interpretation of the quantum bound on chaos as a “mobility edge” in this spectrum of classical Lyapunov exponents. By comparing the radius of curvature governing each chaotic mode with the de Broglie wavelength, we show that the quantum bound marks the point in the Lyapunov spectrum above which the classical description is invalidated.

Thus, in systems that do not saturate the quantum bound on chaos, all the chaotic modes — including the one with the largest Lyapunov exponent — admit a classical description. On the other hand, the dominant chaotic mode in systems that saturate the bound is exactly at the “mobility edge”, which marks the breakdown of classical behavior. This leads us to conjecture that systems which saturate the bound inherit a universal critical behavior associated with the crossing of the chaotic mobility edge.

Before proceeding we note that Kurchan had previously suggested a semi-classical picture of the quantum bound on chaos, that was then applied to a single-particle toy model[24]. This model was characterized by a single negative curvature, fixed by the Hamiltonian. This is a crucial difference from the class of models we consider, including the SYK model, which exhibit an emergent continuous spectrum of Lyapunov exponents. Because there is no lower bound on the curvature scales chaos can survive down to the lowest temperatures.

As a first step toward a semiclassical picture we introduce an exact formulation of the SYK model as a $so(N)$ rotor Hamiltonian

$$\hat{H} = \frac{1}{2} \sum_{i<j,k<l} J_{ijkl} \gamma_i \gamma_j \gamma_k \gamma_l = \frac{1}{2} \sum_{a,b} J_{ab} \hat{L}_a \hat{L}_b \quad (1)$$

Here the $\gamma_i (i = 1, \ldots, N)$ are Majorana fermion operators (with anticommutation relations given by $\{\gamma_i, \gamma_j\} = \delta_{ij}$) and $J_{ijkl}$ is a completely antisymmetric tensor, whose
components are picked randomly with a Gaussian distribution with zero mean and with an energy scale set by $\mathcal{J}_{ijkl}/\hbar\omega$. We have defined generalized angular momentum components composed of fermion bilinears $\hat{L}_{ij} = i\hbar\hat{\gamma}_i\hat{\gamma}_j/2$. These operators obey the so($N$) algebra, $[\hat{L}_a,\hat{L}_b] = -i\hbar f_{abc}\hat{L}_c$, with $f_{abc}$ the so($N$) structure constants and where we used a combined index $a \equiv (i,j)$ with $i < j$ and $a = 1,\ldots,M$ with $M = N(N-1)/2$. Finally, we have defined the inverse moment of inertia tensor $\mathcal{J}_{ab} = 4\mathcal{J}_{ijkl}/\hbar^2$.

The above rotor Hamiltonian defines a much broader class of quantum models distinguished by the representation of so($N$) under which the operators $\hat{L}_{ij}$ transform. The SYK model corresponds to the spin-1/2 spinor representation of so($N$). We will also consider the generalized problem, where the $\hat{L}_a$ are in the spin-$S$ spinor representation.

We now study a classical version of the problem defined by the Hamiltonian

$$\mathcal{H}_cl = \frac{1}{2} \sum_{a,b} \mathcal{J}_{ab} \mathcal{L}_a \mathcal{L}_b$$ \hspace{1cm} (2)

where $\mathcal{L}_a$ should be understood as the components $\mathcal{L}_{ij}$ of a $N$ by $N$ anti-symmetric matrix. In order to ensure a good thermodynamic limit, we choose $\mathcal{J}_{ab} = J_0\hbar^{-2}S^{-2}\sqrt{M}(\alpha_{ab} + 2\delta_{ab})$ and $\alpha_{ab}$ a random symmetric matrix with Gaussian-distributed matrix elements: $\bar{\alpha}_{ab} = 0$ and $\bar{\alpha}_{ab}^2 = 1/M$ [29]. Note that this choice neglects correlations that should be present in $\mathcal{J}$ due to it being anti-symmetric to exchange of any two internal indices $i,j,k,l$. We have checked numerically that including these correlations does not have any significant effects on the classical dynamics.

The Hamiltonian (2), along with the Poisson bracket relations $\{\mathcal{L}_a,\mathcal{L}_b\}_P = -f_{abc}\mathcal{L}_c$ leads to the equations of motion

$$\partial_t \mathcal{L}_a = f_{abc}\mathcal{L}_b\mathcal{J}_{cd}\mathcal{L}_d.$$ \hspace{1cm} (3)

We note that Davidson et al. used similar classical equations of motion to simulate the relaxation of local observables in the the SYK model within the Truncated Wigner Approximation [26].

Intuitively, the dynamics corresponds to the rotation of an $N$-dimensional body with a random inertia tensor given by $\mathcal{J}_{ijkl}^{-1}$ and angular momentum components given by $\mathcal{L}_a$ [27]. This classical approximation to the quantum dynamics becomes exact for $S$ going to infinity.

Besides the energy, this problem has another integral of motion of note: the total angular momentum (normalized by the number of components) $P^2 = M^{-1}\sum_a \mathcal{L}_a^2$. While $P^2$ could take any value for a regular classical system, in this semiclassical limit its value is fixed by the class of representation of the quantum problem. In the spin-$S$ spinor case at hand, one finds $P^2 = NS^2\hbar^2/2M$ [28].

Finally, we note that the classical Hamiltonian (2) with the constraint on $P^2$ maps to the $p$-spin spherical model with $p = 2$ [29]. It has a paramagnetic phase for $T > T_{sg}$ and a spin glass phase for $T < T_{sg}$, with the spin-glass transition temperature given by $T_{sg}/J_0 = \sqrt{2}/2$. It is important to emphasize, however, that the Poisson-bracket dynamics we consider for this model is very different from the dissipative dynamics that is usually studied in spin-glass systems [30].

\section*{RESULTS}

We study numerically the time evolution for the conservative dynamics given in Eq. (3) averaged over an ensemble of initial states $\{\mathcal{L}_a(0)\}$ drawn from a thermal distribution at temperature $T$ and for the fixed value of $P^2$ given above.

In order to probe chaotic behavior, we compute the following correlation function:

$$C(t) = \frac{1}{M^2} \sum_{a,b} \langle \left( \frac{\partial \mathcal{L}_a(t)}{\partial \mathcal{L}_a(0)} \right)^2 \rangle,$$ \hspace{1cm} (4)

where the brackets signify average over initial conditions and the overline signifies average over disorder realizations. This quantity measures the sensitivity to perturbations in the initial condition. It is the leading term in a semiclassical expansion of the quantum out-of-time-order correlator (OTOC)[31,32]. We find that it exhibits chaotic behavior, $C(t) \sim e^{2\lambda t}$, where $\lambda$ is the largest Lyapunov exponent (see inset of Fig. 1).

In Fig. 1 we give the dependence of $\lambda$ on the temperature. We find that the low temperature behavior is
FIG. 2. Classical Lyapunov spectrum (in blue) and the quantum bound on chaos (in red). In panel (a), we have $S/N \gg 1$ and the entire classical Lyapunov spectrum is much lower than the quantum bound, consistently with the fact that semiclassics is correct in this limit. In panel (b), we show the same plot but in the case $S/N \ll 1$. In this case, the classical Lyapunov spectrum is split into two by the quantum bound. For the lower part of the spectrum, the curvature radius $R$ is much larger than the de Broglie wavelength $l_{dB}$ and a classical picture still applies. For the higher part of the spectrum, the de Broglie wavelength is larger than the radius of curvature, the semiclassical approximation breaks down, and chaos is lost.

described by the relation

$$\lambda_{cl} = \eta 2\pi \frac{k_B T}{\hbar} = \eta \lambda_{bound},$$

with $\eta = \frac{N}{25} \times 0.16(\pm 0.02)$. It is quite striking that this purely classical model reproduces the $T$-linear Lyapunov exponent, indicating that such a temperature dependence is not a specific signature of a quantum system, but more of a system with fixed total angular momentum.

Starting from this result, one can identify two regimes. In the case $S \gg N$, $\lambda_{cl}$ is much lower than the quantum bound. Indeed, this is the regime in which the semiclassical approximation is valid. Hence it should reproduce the quantum result and, in particular, be subject to the quantum bound. In the case $S \ll N$, the largest classical Lyapunov exponent far exceeds the quantum bound. We therefore expect in this case that adding quantum fluctuations will strongly change the physics. The way it does is by creating a “mobility edge” in the classical Lyapunov spectrum which separates the lower part of the spectrum, for which the semiclassical analysis still works, from the higher part for which it does not.

The “mobility edge” can be understood by using a geometrical picture of the classical problem [24]. The classical trajectories correspond to geodesics on $SO(N)$ for a metric given by $g_{ab} = J_{ab}^{-1}$ [33]. Studying chaos then amounts to study how geodesics diverge from each other, which can arise due to negative curvature. Given a geodesic, the geodesic deviation equation leads to a spectrum of local radii of curvature $R_\mu$, one for each plane spanned by the direction of the geodesic and the direction of the perturbation. One can easily relate the radii of curvature to the local Lyapunov exponents by $R_\mu \sim v/\lambda_\mu$, where $v$ is the typical (angular) velocity along the geodesic. Now, one expects classical chaos to hold if

$$l_{dB} \ll R_\mu$$

where $l_{dB}$ is the de Broglie wavelength (In this case it is really an angle). Indeed, if this inequality holds, the wavepacket looks essentially classical at the chaos length scale. Since the angular momentum scale is fixed at $P$ in this problem, the de Broglie wavelength is $T$-independent and is given by $l_{dB} \sim \hbar/P$. This leads to an inequality on the Lyapunov exponent given by

$$\lambda \ll \frac{vP}{\hbar} \sim \frac{\xi}{\hbar} \sim \frac{kT}{\hbar}$$

where $\xi$ is the energy density and where we used the fact that, in our classical problem, the energy is purely kinetic, and thus given by velocity times angular momentum. The last equality follows from the equipartition theorem.

Thus we find that the point at which the classical description breaks down is also where chaos is lost. This result reinforces the notion of chaos as an essentially classical phenomena. Whenever chaos is present it can be described classically. However, the bound on chaos is genuinely quantum as it marks the point where quantum interference effects set in. Note that this has nothing to do with the “classical to quantum” cross-over which appears at the Ehrenfest time and that is usually studied in quantum chaos [9, 21]. In particular, in our case quantum effects set an upper bound on the classical Lyapunov exponents, while the inverse Ehrenfest time sets a lower bound on them.

In the case at hand, this quantum-bounded classical chaos survives down to the lowest $T$ since there is no in-
frared cutoff on the the radii of curvature. This would not be the case for most toy models of classical chaos which are essentially few-body and have a finite number of curvature radii. For these models, at low enough \( T \), \( t_{dB} \) becomes larger than the largest \( R \), and chaos is lost altogether. A necessary condition for a quantum-bounded chaos that survives down to the lowest \( T \) is therefore to have curvature on all length scales, i.e. a “chaotic criticality” of the underlying classical problem. This condition should be useful to find other classical chaos models which, after quantization, can be expected to be quantum-bound-saturating. For example, the Fermi-Pasta-Ulam model was shown to have a full spectrum of Lyapunov exponents with no infrared cutoff and is therefore a good candidate (although there the maximal Lyapunov exponent has a sub-linear scaling with the energy density).

As explained in the Appendix, a convenient way to obtain the spectrum of local curvatures and (classical) Lyapunov exponents is through a linear stability analysis. It is easy to show that Eq. (3) has 2\( M \) fixed points given by \( \pm \mathcal{L}_\nu \), where \( \mathcal{L}_\nu \) is the (properly normalized) eigenvector of \( \mathcal{J}_{ab} \) with eigenvalue \( j_\nu \) (\( \nu = 1, \ldots, M \)). The intuition for these fixed points is clear: they correspond to the body rotating with the maximal angular momentum around one of its principal axes of inertia, defined by the \( \mathcal{L}_\nu \). The energy of the system at these fixed points is given by \( E_\nu = \frac{1}{2} j_\nu^2 M P^2 \). The \( j_\nu \) obey the semi-circle law and the classical ground state corresponds to the minimal value of \( j_\nu \). By sweeping over \( \nu \), one can therefore analyze the typical behavior over the full range of energy densities. A typical local Lyapunov spectrum is shown in Fig. 3, where it is clearly seen that the maximal Lyapunov exponent grows linearly with energy density at low energies, and that there is a full spectrum of Lyapunov exponents below that maximal value.

**DISCUSSION**

The semiclassical picture of many-body quantum chaos, outlined in this paper, offers a new interpretation of the quantum bound derived by Maldacena et. al. [1]. The bound emerges as a “mobility edge” in the classical Lyapunov spectrum, above which chaos is destroyed by quantum interference effects. Thus in systems where the global classical Lyapunov exponent is well below the bound, all the chaotic modes admit a classical description. This observation suggests that accurate classical simulation should be possible in a broad range of quantum systems. With these insights at hand, it is reasonable to speculate that even the dynamics of the entanglement entropy, though a purely quantum notion, can be reliably captured through the proxy of the Kolmogorov-Sinai entropy generated by the classical chaotic dynamics [35].

But while some models allow for a classical description, maximally chaotic systems, which saturate the bound, are inherently quantum; their dominant chaotic mode is pinned to the mobility edge at which the classical description breaks down. This behavior can persist down to zero temperature because the macroscopic system has a continuous spectrum of classical Lyapunov exponents with no lower bound. It would be interesting to explore how the universal critical behavior of the SYK model and other fast scramblers is tied to the structure of the modes on the chaotic mobility edge. Within the class of models that we considered here, it is possible to investigate the mobility edge by continuously tuning the ratio \( N/S \). For small values of this parameter the classical picture holds and the Lyapunov exponent is below the quantum bound, while for large values of \( N/S \) quantum effects become dominant and the system can saturate the bound. An interesting open question is whether the system undergoes a dynamical quantum phase transition at a critical value of \( N/S \) or rather a smooth crossover into the maximally chaotic regime.

It would be interesting to explore how these insights extend to higher dimensional systems and in particular to the physics of strange metals. For a regular metal, we can think of the classical model as being given by some form of kinetic Boltzmann dynamics (with particles following the Fermi-Dirac distribution). In this case the intrinsic “curvature” is the electron-electron mean free path \( l \sim 1/T^2 \) and the velocity is fixed at the Fermi velocity \( v_F \). One thus obtains a classical Lyapunov exponent proportional to \( T^2 \), and the quantum bound on chaos is irrelevant. It is tempting to conjecture that in strange metals the curvature scale is set instead by the dynamics of emergent bosonic degrees of freedom, similar to the quantum rotor we discussed here, and that this leads to the saturation of the quantum bound at low temperatures [11].
The physics of maximally chaotic systems, as it is seen from our discussion so far, is intimately tied to a transition from classical to quantum dynamical behavior. However, even at the purely classical level, the models we have considered bear surprising similarities to the universal physics of quantum fast scramblers, such as the SYK model. First, these models show a linear dependence of the Lyapunov exponent with the temperature even in the fully classical case. Second, they possess a time reparametrization symmetry similar to that found in the quantum SYK model [24]. Indeed extremizing the action is equivalent to extremizing the length of the trajectory in coordinate space, which is explicitly a time-independent problem.

Finally, the findings reported in this paper give a new perspective on the physics of glasses. Disordered models such as the one given in Eq. (2) are known to have a low temperature spin glass phase. Accordingly, the dissipative dynamics was shown to have weak ergodicity breaking below the glass transition temperature [30]. However, the Poisson bracket dynamics considered here exhibits fully developed chaos with a Lyapunov exponent that evolves smoothly as the temperature is lowered through the glass transition. It would be interesting to explore how this chaotic motion is related to thermalization in both the classical and the quantum SYK models.

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Appendix: Linear stability analysis

As mentioned in the main text, a convenient way to obtain the spectrum of local curvatures and (classical) Lyapunov exponents is through a linear stability analysis. It is easy to show that Eq. (3) has $2M$ fixed points given by $\pm L^\nu$, where $L^\nu$ is the (properly normalized) eigenvector of $J_{ab}$ with eigenvalue $j^\nu$ ($\nu = 1, \ldots, M$). The intuition for these fixed points is clear: they correspond to the body rotating with the maximal angular momentum around one of its principal axes of inertia, defined by the $L^\nu$. The energy of the system at these fixed points is given by $E^\nu = \frac{1}{2} j^\nu MP_2$. The $j^\nu$ obey the semi-circle law and the classical ground (resp. top) state corresponds to the minimal (resp. maximal) value of $j^\nu$.

By sweeping over $\nu$, one can therefore analyze the typical behavior over the full range of energy densities.

To perform the linear stability analysis around a given fixed point $L^\nu$, we define the deviation $l$ as $L(t) = L^\nu + l(t)$. We now linearize the equations of motion:

$$\partial_t l_a = \alpha K_{ab}^\nu l_b$$

$$K_{ab}^\nu = f_{vac}(-J + j^\nu)_{cb} \equiv f_{vac} \tilde{J}_{cb}^\nu$$

(8)

where $\alpha = \sqrt{MP}$ (no summation over $\nu$ is implied). $K_{ab}^\nu$ is the $M$ by $M$ stability matrix of the fixed point $L^\nu$ and $j^\nu$ is the eigenvalue of $J$ corresponding to the eigenvector $L^\nu$.

One should now analyze the spectrum of $K^\nu$: imaginary eigenvalues correspond to oscillatory behavior while real positive (resp. negative) eigenvalues correspond to unstable (resp. stable) directions, and can be interpreted as local Lyapunov exponents. It is instructive to first consider the case of dissipative (steepest-descent) dynamics [30], where $\mathcal{L}(t)$ follows the gradient of the energy, subject to the constraint of keeping $P^2$ fixed. This dynamics has the same fixed points as the Poisson bracket dynamics (i.e. $\pm L^\nu$), but the stability matrix is instead given by $\tilde{J}^\nu$. Using the fact that $\tilde{J}^\nu$ is a real symmetric matrix, one finds easily that there are only stable and unstable directions (no oscillatory behavior), and that the number of unstable directions goes from 0 in the ground state to $M$ in the top state. This means that the ground state is a global attractive fixed point, the top state a global repulsive fixed point, and the fixed points in the middle are saddles with an index that interpolates between the two cases. This type of dynamics, associated with random thermal noise, was solved in [30] and was shown to have glassy behavior at low $T$.

Instead, for the Poisson bracket dynamics, $\tilde{J}^\nu$ is multiplied by an antisymmetric matrix $f$ in order to obtain the stability matrix. This leads to a dramatic change in the dynamics which can be understood as follows. Let $K = AB$ with $A$ an antisymmetric matrix and $B$ a symmetric matrix. If $B$ is positive definite or negative definite then the spectrum of $K$ is purely imaginary. This means the stability matrix around the ground and top states has a purely imaginary spectrum for Poisson bracket dynamics. The motion around the ground state (and the top state) is thus purely oscillatory, in contrast to the dissipative dynamics case for which the ground (resp. top) state only had stable (resp. unstable) directions. As one considers fixed points away from the ground (or top) state, $\tilde{J}^\nu$ becomes less and less definite, leading to more and more $K$ eigenvalues with non-zero real part. These real eigenvalues, when positive, can be interpreted as local Lyapunov exponents, and their number and size increases as one moves away from the ground or top state, as shown in Fig. 3.