Zeta functions of projective hypersurfaces with ordinary double points

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Received: 5 August 2021 / Accepted: 4 October 2021 / Published online: 28 April 2022
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Abstract
We extend the approach of Abbott, Kedlaya and Roe to computation of the zeta function of a projective hypersurface with $\tau$ isolated ordinary double points over a finite field $\mathbb{F}_q$ given by the reduction of a homogeneous polynomial $f \in \mathbb{Z}[x_0, \ldots, x_n]$, under the assumption of equisingularity over $\mathbb{Z}_q$. The algorithm is based on the results of Dimca and Saito (over the field $\mathbb{C}$ of complex numbers) on the pole order spectral sequence in the case of ordinary double points. We give some examples of explicit computations for surfaces in $\mathbb{P}^3$.

Keywords Zeta function · $p$-Adic cohomology · Hypersurfaces with ordinary double points

Mathematics Subject Classification 11S40 · 14G10

1 Introduction

Let $X_k = Z(f) \subset Y = \mathbb{P}_k^n$ be a projective hypersurface of degree $N$ over a finite field $k = \mathbb{F}_q$ (for $q = p^a$ a prime power for $p > n - 1$) with $\tau$ ordinary double points and assume that the defining equation $f$ admits an equisingular lift (see Sect. 2 for definitions) $f_0 \in \mathbb{Z}_q[x_0, \ldots, x_n]$ where $\mathcal{O} := \mathbb{Z}_q \subset K := \mathbb{Q}_q$. For computational convenience we will actually work with polynomials over $\mathbb{Z}$ or its finite extension, although much of the theory holds over $\mathcal{O}$.

The purpose of this paper is to generalize the algorithm originally due to Kedlaya that computes the zeta function of a smooth projective hypersurface, to the case of...
ordinary double points. This generalization is based on the results of Dimca and Saito on the spectral sequence of the de Rham complex with deformed differential \( d + df \wedge (\cdot) \), see [8,9,20].

We follow the exposition in [13] with adjustments needed for our (mildly) singular case. We switch to the study of the open complement \( U_k = \mathbb{P}^n_k \setminus X_k \) and show that the equisingular deformation assumption allows to view \( U_\mathcal{O} = \mathbb{P}^n_\mathcal{O} \setminus X_\mathcal{O} \) as an open complement \( Y \setminus Z \) where \( Y \) is a smooth projective scheme over \( \text{Spec}(\mathcal{O}) \) and \( Z \) is a normal crossings divisor with irreducible components smooth over \( \text{Spec}(\mathcal{O}) \). In fact, in our case \( Y \) is simply the blowup of \( \mathbb{P}^n_\mathcal{O} \) at the singular points of \( X_\mathcal{O} \).

The presentation \( U_\mathcal{O} = Y \setminus Z \) allows us to use a result of Baldasarri–Chiarelotto, cf. [1], that identifies the rigid cohomology of \( U_k \), as a vector space over \( K \), with the de Rham cohomology of \( U_K \). Notice that a priori the de Rham cohomology space does not carry a natural action induced by the Frobenius map and we need to use the isomorphism with rigid cohomology to transfer it from the latter.

Fortunately, \( U_k, \text{ resp. } U_K \) are affine over \( \text{Spec}(k), \text{ resp. } \text{Spec}(K) \) and both de Rham and rigid cohomology are computed via fairly explicit complexes involving the ring of functions on \( U_k \) and its dagger completion. In other words, we use the Monsky–Weissnitzer model of rigid cohomology which is available for the smooth affine complement. The above isomorphism on cohomology is induced by the embedding of the usual functions into completed functions. Hence the Frobenius-induced operator on de Rham cohomology involves applying (a lift of) the Frobenius map to a usual differential form, obtaining a form with coefficients in the completion and then adjusting it by an exact form to get an equivalent element in the uncompleted de Rham complex. This procedure is covered in Sect. 5.

As in [13], computation involves the pole order filtration on differential forms over \( U_K \), except that for hypersurfaces with ordinary double points the corresponding spectral sequence degenerates not at the \( E_0 \) page (as it happens for smooth hypersurfaces) but at the \( E_1 \) page. This has been proved by Dimca and Saito for hypersurfaces with ordinary double points over \( \mathbb{C} \) (and later extended by Saito to the case of quasihomogeneous isolated singularities), and we adjust their results to our needs.

Unlike in the smooth case, the cohomology of the Koszul differential \( df \wedge (\cdot) \) can occur not just at the \( \Omega^{n+1} \) diagonal, but also at the \( \Omega^n \) diagonal. When the degree of polynomial coefficients is high enough, both groups stabilize to vector spaces of dimension \( \tau \), the number of ordinary double points. Moreover, the differential on \( E_1 \) page of the spectral sequence (induced by the de Rham differential) eventually becomes an isomorphism and at the \( E_2 \) page the spectral sequence degenerates. This reduces an a priori infinite computation on \( E_0 \) and \( E_1 \) pages to a computation involving only \( n \) rows of the \( E_1 \) page.

In Sect. 2 we give definitions related to projective hypersurfaces with ordinary double points which are equisingular over \( \mathbb{Z}_q \). We show that blowing up the singular set gives a divisor with smooth normal crossings. In Sect. 3, after recalling basic definitions and results we identify rigid of \( X_k \) and de Rham cohomology of \( X_K \) using a theorem of Baldasarri and Chiarelotto. In Sect. 4 we apply the results of Dimca and Saito on the de Rham cohomology on the principal open set. In Sect. 5 we explain a modified version of Kedlaya’s algorithm which allows to import the Frobenius action to the de Rham cohomology, via its isomorphism with rigid cohomology computed.
through the Monsky–Washnitzer approach. In Sect. 6 we briefly summarize the steps of our algorithm. Sect. 7 has a detailed discussion of a few examples in the \( n = 3 \) case (projective surfaces with ordinary double points). We tried to be explicit in some details for the benefit of a reader interested in practical implementation of our algorithm.

2 Equisingular deformation and its blowup

2.1 Definition of an equisingular ODP polynomial

Assume we are given a homogeneous degree \( N \) polynomial \( f \in \mathbb{Z}[x_0, \ldots, x_n] \) (one can also replace \( \mathbb{Z} \) by the ring of integers in a finite extension of \( \mathbb{Q} \)). For every commutative ring \( R \) it defines a relative hypersurface \( X_R \) in the projective space \( \mathbb{P}_R^n \) over \( \text{Spec}(R) \). We will be interested in the special cases when \( R \) is a finite field \( k = F_q \) with \( q \) elements where \( q = p^a \) for some prime \( p > n - 1 \), or unramified degree \( a \) extension \( K = \mathbb{Q}_q \) of the field \( \mathbb{Q}_p \) of \( p \)-adic numbers, or the ring \( \mathcal{O} = \mathbb{Z}_q \) of integers over \( \mathbb{Z}_p \) in \( K \). In the case when \( R = k, K \) is a field, we have the standard definitions:

**Definition 2.1** A point \( P \in Z(f) \subset \mathbb{P}_F^n \) is a singular point of the hypersurface \( Z(f) \) if

\[
\frac{\partial f}{\partial x_0}(P) = \cdots = \frac{\partial f}{\partial x_n}(P) = 0.
\]

Such a \( P \) is an ordinary double point (ODP) if the homogeneous \( (n + 1) \times (n + 1) \) Hessian matrix \( \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{0 \leq i, j \leq n} \) has rank \( n \) at \( P \). This is equivalent to requiring that \( P \) has a standard affine neighborhood \( U \simeq \mathbb{A}_R^n \subset \mathbb{P}_R^n \) such that the non-homogeneous polynomial defining \( Z(f) \cap U \) has a non-degenerate affine \( n \times n \) Hessian matrix at \( P \).

Yet another way to phrase it is to use the partial derivatives \( \frac{\partial f}{\partial x_i} \) to define a morphism

\[
\mathcal{O}_{\mathbb{P}^n_R}(-N + 1)^{\oplus (n+1)} \rightarrow \mathcal{O}_{\mathbb{P}^n_R} \tag{1}
\]

and the ODP condition is equivalent to saying that for \( R = F \) the cokernel of this morphism is supported on a zero-dimensional reduced subscheme of \( \mathbb{P}_F^n \). In the case when \( R = \mathcal{O} \), we need a condition that ensures that the ordinary double points of the induced hypersurface over \( k = \mathcal{O}/m \) are not “smoothed away” on the hypersurface over \( K \), the fraction field of \( \mathcal{O} \).

**Definition 2.2** A hypersurface \( Z(f) \subset \mathbb{P}^n_{\mathcal{O}} \) given by a homogeneous degree \( N \) polynomial \( f \in \mathbb{Z}[x_0, \ldots, x_n] \) has equisingular ordinary double points if the cokernel of the morphism (1) is scheme theoretically supported at a disjoint union \( W = \bigsqcup W_s \) of closed subschemes \( W_s, s = 1, \ldots, \tau \), such that the restriction of the structure morphism \( \pi : \mathbb{P}_{\mathcal{O}}^n \rightarrow \text{Spec}(\mathcal{O}) \) to each \( W_s \) is an isomorphism.

In this paper we assume that \( p > n - 1 \) (this is needed to apply truncation results in [13, Section 4], but not essential elsewhere) and that \( f \in \mathbb{Z}[x_0, \ldots, x_n] \) is a homogeneous degree \( N \) polynomial which induces a hypersurface with \( \tau \) equisingular ordinary double points over \( \text{Spec}(\mathcal{O}) \) (or a polynomial with coefficients in a finite extension of \( \mathbb{Z} \) which is unramified at \( p \)).
2.2 Blowup of the equisingular hypersurface

We continue with the assumptions on \( f \) imposed in the previous subsection.

Lemma 2.3 Let \( \rho : Y \to \mathbb{P}^n_O \) be the blowup of the closed subscheme \( W \) introduced at the end of the previous subsection. Then the preimage \( D \) of \( X = Z(f) \) (with the reduced scheme structure):

\[
D = \rho^{-1}(X) = \hat{X} \cup E_1 \cup \cdots \cup E_\tau
\]

has smooth irreducible components with normal crossings.

Proof Note that the question is local along \( D \) and we can restrict to a neighborhood of a point \( Q \in D \).

If \( \rho(Q) \) is not in the support of \( W \) then the only component of \( D \) that passes through \( Q \) is the proper transform \( \hat{X} \) which is smooth at \( Q \), since \( \rho(Q) \) is a smooth point of \( X \) and \( \rho \) is an isomorphism over some affine neighborhood of \( \rho(Q) \).

Now assume \( P = \rho(Q) \) is in the support of some \( W_s \). We can assume \( s = 1 \) and replacing \( Q \) by its specialization we can assume that \( Q \) and \( P \) are in the respective closed fibers over \( \text{Spec}(k) \subset \text{Spec}(\mathcal{O}) \). We can also assume that \( P = [1:0: \ldots :0] \in \mathbb{P}^n_k \). We can replace the projective space by \( \mathbb{A}^n_\mathcal{O} \) with affine coordinates \( y_1 = x_1/x_0, \ldots, y_n = x_n/x_0 \) and \( X \) by the zero set of \( g(y_1, \ldots, y_n) = f(1, y_1, \ldots, y_n) \).

By the Euler identity \( g \) and its partials generate the ideal \( I \) of the subscheme \( W_1 \). By assumption on \( W_1 \) this means that the constant and linear terms of \( g \) are zero, hence we can write \( g = g_2 + g_3 + \cdots + g_N \) where each \( g_j \) is a homogeneous degree \( j \) polynomial in \( y_1, \ldots, y_n \) with coefficients in \( \mathcal{O} \).

By the standard results on blowups, see e.g. [15, Chapter 1.4], the exceptional divisor \( E_1 \) of the blowup of \( W_1 \) in \( \mathbb{A}^n_\mathcal{O} \) is smooth. The proper transform of \( X \) is smooth away from the exceptional divisor so we just need to show that at every point \( Q \in \hat{X} \cap E_1 \) the proper transform is smooth at \( Q \) and the two hypersurfaces are transversal (i.e. the differentials over \( \mathcal{O} \) of the two local defining equations are independent). Both facts hold since \( g \) and its partial derivatives generate \( I \) hence the linear parts of partial derivatives freely generate the \( \mathcal{O} \) of linear polynomials in \( y_1, \ldots, y_n \) with coefficients in \( \mathcal{O} \).

\[ \square \]

2.3 A non-equisingular example

For an example of a lift that is not equisingular, for \( n = 3 \), we use [21, Example 5.6, p.22] which is a quartic with two ordinary double points over \( \mathbb{C} \) but four over the closure of \( \mathbb{P}^2_5 \). This surface is also mentioned on [4, p.7] where the variables \( x_1 \) and \( x_3 \) have been swapped and then written in a “cleaner” form

\[
x_0 x_1(x_0^2 + x_1^2 + x_2^2 + x_3^2) + x_2 x_3(x_0^2 + x_1^2 - x_2^2 - x_3^2) - 2x_2^2 x_3^2 + 2x_0^2 x_1^2 + 2x_0 x_1 x_2 x_3 = 0.
\]

(2)

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Over a characteristic zero field, the polynomial \( f \) on the left-hand side has two ordinary double points at \([1 : -1 : 0 : 0]\) and \([0 : 0 : -1 : 1]\). Consider it now over a field of characteristic 5. One must find the number of singularities over \( \mathbb{F}_5 \) and its finite extensions. Let \( \alpha \in \mathbb{F}_{25} \) be such that \( \alpha^2 = 3 \), then the singular points of the quartic in (2) are

\[
[1 : 4 : 0 : 0], \quad [0 : 0 : 4 : 1], \quad [\alpha : \alpha : 1 : 1], \quad [4\alpha : 4\alpha : 1 : 1].
\]

We conclude that our hypersurface is not equisingular in this case.

### 3 Zeta function, rigid and de Rham cohomology

#### 3.1 Zeta function and traces on rigid cohomology

We recall the standard definitions and results regarding zeta functions, as they apply to the situation considered. The projective zeta function of \( f \) is defined as

\[
\zeta(f, T) = \exp\left(\sum_{r=1}^{\infty} \frac{\# X_{k(r)}(r)}{r} \right)
\]

where \( X_{k(r)} = \{x \in \mathbb{P}^n_{k(r)} \mid f(x) = 0\} \) and \( k(r) \) stands for the finite field \( \mathbb{F}_{q^r} \).

In [12], Dwork proved that the zeta function of a variety is a rational function. Weil proposed using some type of cohomology theory to prove his conjectures however Dwork’s proof uses \( p \)-adic analysis and no cohomology. It was not until Grothendieck’s use of étale cohomology [14] that the rationality of the zeta function was proved using cohomological methods. This breakthrough led to the proofs of all the Weil conjectures, the last being the analogue of the Riemann Hypothesis by Deligne [5]. When it comes to actually computing the zeta function, Kedlaya in [16, Section 1.2] points out that, étale cohomology is not as useful as \( p \)-adic cohomology such as rigid, see also the introduction of [13]. Rigid cohomology can be defined for both affine and projective varieties whether they are smooth or singular, and in fact there are two versions: usual and with compact support. For our projective hypersurface \( X \) constructed from \( f \) we write \( H^*_{\text{rig}}(X) \), \( H^*_{\text{rig,c}}(X) \), both are finite dimensional vector spaces over the field \( K \). It is an important part of the standard theory, cf. [18], that the Frobenius automorphism of \( k \) induces a linear map \( F^* \) on both versions of cohomology, such that the following crucial trace formula holds:

\[
\# X_{k(r)} = \sum_{i=0}^{2 \dim(X)} (-1)^i \text{tr}((F^*)^r|H^i_{\text{rig,c}}(X)).
\]
Substituting equation (4) into the definition of the zeta function yields

\[ \zeta(f, T) = \prod_{i=0}^{2\dim(X)} \det(1 - TF^*|H^i_{\text{rig,c}}(X))(-1)^{i+1}, \]  

(5)

where \( \det(1 - TF^*|H^i_{\text{rig,c}}(X)) \) is the reciprocal of the characteristic polynomial of the map \( F^* \). Equivalently, in the smooth case by Poincaré duality, see \([18, \text{Proposition 6.4.18}]\), the information contained on the compactly supported rigid cohomology can be rephrased in terms of the usual rigid cohomology:

\[ \#X_{k(r)} = \sum_{i=0}^{2\dim(X)} (-1)^i q^{r(n-1)} \text{tr}((F^*)^{-r}|H^i_{\text{rig}}(X)). \]  

(6)

We now consider the complement \( U_k = \mathbb{P}^n_k \setminus X_k \). This section and the following are driven by the trivial, but important fact that over \( \mathbb{F}_q \)

\[ |X_k| + |U_k| = |\mathbb{P}^n_q| = q^n + \cdots + q^2 + q + 1. \]

Therefore finding the zeta function of \( X \) is equivalent to finding the zeta function of \( U \) and we choose to calculate the latter. The two main reasons for working with \( U \), instead of \( X \), are that \( U \) is smooth and affine. Yet another feature which makes our particular approach computable is the fact that we choose \( f \) giving a hypersurface with equisingular ordinary double points.

Using the above equation we can write the zeta function of \( f \) as

\[ \zeta(f, T) = \frac{Q_n(T)^{-n} Q_{n+1}(T)^{-n-1}}{(1 - T)(1 - qT) \cdots (1 - q^n T)} \text{ with } \]  

(7)

\[ Q_i(T) = \det(1 - q^i T(F^*)^{-1}|H^i_{\text{rig,c}}(U_k)) \]

(see e.g. \([13]\) or \([18]\)). In Sect. 4 we use results on spectral sequences to prove that

\[ \deg(Q_n(T)) - \deg(Q_{n+1}(T)) = \frac{1}{N} ((N - 1)^{n+1} + N - 1) - \tau. \]  

(8)

Moreover, for \( n \) odd \( \deg(Q_{n+1}(T)) = 0 \) while for \( n \) even \( \deg(Q_{n+1}(T)) = \tau \).

### 3.2 Isomorphism with de Rham cohomology

Lemma 2.3 has an important consequence, see \([3, \text{Theorem 5.2}]\) and \([1, \text{Corollary 2.6}]\).

**Theorem 3.1** In the situation considered in Sect. 2.1, there exists an isomorphism between the algebraic de Rham cohomology of \( U_K \) and the rigid cohomology of \( U_k \):

\[ H^*_\text{DR}(U_K) \to H^*_\text{rig}(U_k). \]
An important detail is that a priori the algebraic de Rham cohomology group does not carry an action of an operator induced by Frobenius, hence some information on the above isomorphism is needed to transport this action from rigid cohomology, in terms which are explicit enough for the computation.

At this point we use the fact that $U$ is smooth and affine: the de Rham cohomology can be computed in terms of the algebra $A$ of regular functions on $U_K$ while for the rigid cohomology we can use the Monsky–Washnitzer model, which is rather close to de Rham but uses a certain “dagger completion” $A^\dagger$. Thus, an important property of the isomorphism quoted above is that in the smooth affine case it is induced by the embedding $A \to A^\dagger$. This follows from the proofs in [1,3]. We delay the detailed discussion of this point until Sect. 5.

4 Pole order spectral sequence and degeneration

For most part of this section we will discuss results of Dimca and Saito which were originally stated over $\mathbb{C}$, eventually using an embedding $K \to \mathbb{C}$ to relate these results to our situation.

4.1 de Rham cohomology of the principal open set

Recall that $U = \mathbb{P}^n \setminus Z(f)$ is both smooth and affine so the computation of the de Rham cohomology does not need to involve a Čech covering. We follow Chapter 6 of Dimca’s book [7]. Let $S = \mathbb{C}[x_0, x_1, \ldots, x_n]$ and consider the graded $S$-module $\Omega^l$ of polynomial differential $k$-forms. Any element of $\Omega^l$ can be written as

$$\omega = \sum_I c_I dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

where the finite summation runs over $I = (0 \leq i_1 < \cdots < i_l \leq n)$ and $c_I \in S$. In this paper we only consider the trivial weights, $\deg(x_i) = \deg(dx_i) = 1$ making the total degree

$$\deg(x_0^{a_0} \cdots x_n^{a_n} dx_{i_1} \wedge \cdots \wedge dx_{i_l}) = a_0 + \cdots + a_n + l.$$ 

Now let $S_m \subset S$ be the set of homogeneous polynomials of degree $m$ and define $\Omega^l_m$ to be the set of $l$-forms whose total degree is $m$, i.e., if $\omega \in \Omega^l_m$, then using the notation above each $c_I$ is in $S_{m-I}$.

Relating polynomial differential forms to forms on $U_K$ is greatly simplified by using the map $\Delta$ which is the contraction with the Euler vector field $E = \sum_{i=0}^n x_i \partial/\partial x_i$. By [7, Lemma 1.15]

**Lemma 4.1** (A) There is a unique $S$-linear operator $\Delta: \Omega^l_m \to \Omega^{l-1}_m$ satisfying the properties:

(i) $\Delta(\omega \wedge \omega') = \Delta(\omega) \wedge \omega' + (-1)^l \omega \wedge \Delta(\omega')$ for $\omega \in \Omega^l, \omega' \in \Omega^s$;

(ii) $\Delta(df) = Nf$ for any $f \in S_N$.
(B) For a homogeneous differential form \( \omega \in \Omega^l_m \) we have

\[
\Delta d(\omega) + d \Delta(\omega) = m \omega.
\]

(C) The sequence

\[
0 \to \Omega^{n+1} \xrightarrow{\Delta} \Omega^n \xrightarrow{\Delta} \ldots \xrightarrow{\Delta} \Omega^1 \xrightarrow{\Delta} \Omega^0 \to 0
\]

is exact, except for the last term where

\[
\text{im}(\Delta : \Omega^1 \to \Omega^0) = (x_0, \ldots, x_n),
\]

the maximal ideal in \( S \) generated by \( x_0, \ldots, x_n \).

Note that property (ii) is a restatement of Euler’s Identity in terms of \( \Delta \) and \( d \). Now by [7, Proposition 1.16], any differential \( l \)-form \( \omega \) (for \( l > 0 \)) on the open set \( U \) can be written as

\[
\omega = \frac{\Delta(\gamma)}{f^s}
\]

for some integer \( s > 0 \) and \( \gamma \in \Omega^{l+1}_{s,N} \). At this point Dimca asks if \( d\omega \), from (9), can be expressed in a similar form

\[
d\omega = \frac{\Delta(\delta)}{f^{s+1}}
\]

for some \( \delta \in \Omega^{l+2}_{(s+1),N} \). Most calculations below can be found in [7, p. 181] but we include them since the formula for \( \delta \) in (10) gives rise to a double complex and the spectral sequence, on which our algorithm is based. This is covered in the next section, while now we consider \( d\omega \) from (10)

\[
d\omega = d\left(\frac{\Delta(\gamma)}{f^s}\right)
\]

\[
= d(f^{-s} \Delta(\gamma)) = -s f^{-s-1} df \wedge \Delta(\gamma) + f^{-s} \wedge d(\Delta(\gamma)).
\]

By Lemma 4.1 (A), (B)

\[
\Delta(df \wedge \gamma) = Nf \gamma - df \wedge \Delta(\gamma); \quad d\Delta(\gamma) = sN \gamma - \Delta d(\gamma), \quad \text{for} \quad \gamma \in \Omega^{l+1}_{s,N}.
\]

Substituting (12) into (11) yields

\[
d\omega = -s f^{-s-1} (N f \gamma - \Delta(df \wedge \gamma)) + f^{-s} \wedge (sN \gamma - \Delta(d\gamma)) = -\frac{\Delta(f d\gamma - s N f \wedge \gamma)}{f^{s+1}}.
\]

Finally we can write \( d\omega \), from (10), as

\[
d\omega = d\left(\frac{\Delta(\gamma)}{f^s}\right) = -\frac{\Delta(df(\gamma))}{f^{s+1}} \quad \text{where} \quad df(\gamma) := f d\gamma - \frac{\gamma}{N} df \wedge \gamma.
\]
In other words, for known \( f^s \) in the denominator, computation of the de Rham differential of a form on \( U \) reduces to computation of a “deformed” differential of a form on \( \mathbb{C}^{n+1} \), built out of \( d \) and \( df \wedge \). These differentials give rise to a double complex considered below.

The total degree of a polynomial differential form is preserved both by the de Rham differential and the Euler vector field contraction. Since differential forms on \( U \) are represented by fractions of homogeneous degree zero, we can restrict to polynomial differential forms of total degree \( s N \) for \( s \geq 0 \).

4.2 Pole order spectral sequence

The formulas of the previous section allow us to obtain information about the de Rham cohomology of \( U_{\mathbb{C}} \). On one hand, we have a dg algebra \( \Omega(A)^\bullet \) given by the global sections of the de Rham complex with the filtrations

\[
F^s \Omega(A)^l = \left\{ \frac{\Delta(y)}{f^a} \mid y \in \Omega^{l+1}_{aN}, |y| = aN \text{ and } a \leq l - s \right\}
\]

(which is compatible with the de Rham differential by the previous subsection). On the other hand we have a bicomplex \( B^{t,s} = \Omega^{s+t+1}_{sN} \) with differentials \( d' = d \) and \( d''(\gamma) = -\frac{|\gamma|}{N} df \wedge \gamma \) and the filtration

\[
F^s B^l = \bigoplus_{r \geq s} B^{r,l-r}.
\]

The two filtrations give standard spectral sequences \( E^{t,s}_r(A) \) and \( E^{t,s}_r(B) \) with \( E^1 \) terms given by \( H^{s+t}(F^s(\cdot)/F^{s+1}(\cdot)) \) and a morphism \( \{\delta_r\}_r \) of spectral sequences induced by \( \gamma \mapsto \frac{\Delta(\gamma)}{f} \) for \( \gamma \in B^{t,s} \). To formulate the next result one needs to adjust both spectral sequences by replacing \( E^0_0(A) = \mathbb{C} \) by zero and mod out \( E^0_0(B) \), \( E^0_0(B) \) by the 1-dimensional subspaces spanned by 1 and \( df \), respectively. After this modification (which kills the de Rham cohomology of \( U \) in degree zero and does not affect \( E_1(B) \)) we can state [7, Theorem 6.2.9]:

**Theorem 4.2** The morphism of reduced spectral sequences \( \{\delta_r\} : \tilde{E}^{t,s}_r(A) \to \tilde{E}^{t,s}_r(B) \) is an isomorphism for all \( r \geq 1 \).

In fact, we will see below that in our case all differentials on pages \( E_r \) with \( r \geq 2 \) are trivial.

To focus on \( E_0(B) \), as shown in Fig. 1, recall that \( \Omega^l_{j+l} = S_j \Omega^l \) is the set of all \( l \)-forms whose coefficients are homogeneous polynomials in \( S = \mathbb{C}[x_0, x_1, \ldots, x_n] \) of degree \( j \). For each module \( S_j \Omega^k \) in Fig. 1, \( j + k \) is a multiple of \( N = \deg(f) \).

We set \( S_{N-k} \Omega^k = 0 \) whenever the degree of \( N \) is less than \( k \). The vertical arrows in Fig. 1 are the Koszul differentials \( (-s) \cdot df \wedge \) (the exterior product) and the horizontal arrows are grayed out because we apply them on the \( E_1 \) page, are induced by the de Rham differentials \( d \). The two differentials \( df \wedge \) and \( d \) anti-commute.
The terms on page $E_1(B)$ are the appropriate homogeneous components of the Koszul cohomology, i.e., cohomology of the complex $K_f^\bullet$

$$0 \rightarrow \Omega^0 \overset{df}{\rightarrow} \Omega^1 \overset{df}{\rightarrow} \Omega^2 \overset{df}{\rightarrow} \cdots \overset{df}{\rightarrow} \Omega^n \overset{df}{\rightarrow} 0.$$ 

It is well known that if $f$ has isolated singularities then Koszul cohomology groups are trivial except for the top two, i.e., $H^i(K_f^\bullet) = 0$ for $i < n$, see [20, p. 2] and [7, Proposition 6.2.21, p. 195] for the statement and [19] for a proof. Let

$$H^n(K_f^\bullet)_j = \ker(S_j \Omega^n \overset{df}{\rightarrow} S_{j+N-1} \Omega^{n+1})/\text{im}(S_{j-N+1} \Omega^{n-1} \overset{df}{\rightarrow} S_j \Omega^n),$$

$$H^{n+1}(K_f^\bullet)_j = S_j \Omega^{n+1}/\text{im}(S_{j-N+1} \Omega^n \overset{df}{\rightarrow} S_j \Omega^{n+1})$$

then the $E_1$ page of the spectral sequence is shown in Fig. 2.

The $E_2$ page is defined by taking the kernel and cokernel of the de Rham differential on the spaces seen above. (Technically the maps above are only induced by the de Rham differential since the elements of $H^i(K_f^\bullet)$ are equivalence classes of forms, but we abuse terminology.)
4.3 Results of Dimca and Saito

Rather powerful results of Dimca and Saito, which we review below, claim that the differential of the $E_1$ page is injective in every position (with a single exception for even $n$) and is also surjective except perhaps in the first $n$ terms. In addition, the spectral sequence degenerates at the $E_2$ term.

**Theorem 4.3** Denote by $H^n(K_f^\bullet)_{jN-n} \xrightarrow{d} H^{n+1}(K_f^\bullet)_{jN-n-1}$ the horizontal differential for the $E_1$ page of the spectral sequence (induced by the de Rham differential). Then

(i) The spectral sequence degenerates at the $E_2$ page.

(ii) The $E_1^{t,s}$ terms have dimension $\tau$ (i.e., the number of nodes) when $s \geq n + 1$ and $s + t = n$ or $s + t = n - 1$.

(iii) The vector spaces $E_1^{t,s}$ vanish except possibly in two cases: either when $s \in \{1, \ldots, n\}$ and $t = n - s$, or when $n$ is even and the space $E_2^{n/2-1,n/2}$ has dimension $\tau$.

**Proof** The spectral sequence degenerates at the $E_2$ page by [20, Theorem 2]. We have seen above that $E_1^{t,s}$ can only be nonzero for $s + t = n - 1$ or $n$.

First consider the case $s + t = n - 1$. Then $\dim E_1^{t,s} \leq \tau$ for all $s$ and the equality holds for $s \geq n$ by [6, Theorem 1]. On the other hand, this vector space vanishes for $s < n/2$ by [10, Theorem 9] and the remark right after it. In addition $\dim E_1^{t,s} = 0$ unless $s = n/2$ (which can happen only when $n$ is even) and in the latter case the dimension is $\tau$. Both facts are proved in [9, Theorem 5.3]. We note here that our $s$ equals $p/d$ in the notation of loc.cit. and that in the case of ordinary double points each expression $\alpha_{h_k,l}$ of loc. cit. is $n/2$ hence the count in the theorem quoted simply reduces to the number of ordinary double points.

Now we turn to the case of $s + t = n$. Our goal is to show that for $s \geq n + 1$ the dimension of $\dim E_1^{t,s}$ is also equal to the number of singular points while $E_1^{t,s}$ also vanishes in that range. Indeed, for $s \geq n + 1$ and any smooth hypersurface of the same degree as $f$, the group similar to $E_1^{t,s}$ is zero by [7, Section 6.1]. Since the Euler characteristic of the complex $E_0^{s,*}$ with finite dimensional components and Koszul differential $df$ is independent on the choice of $f$, we conclude that $\dim E_1^{s,n-s} = \dim E_1^{s,n-s-1} = \tau$ for $s \geq n + 1$.

The $E_1$ page differential $d$: $E_1^{n-s,s-1} \to E_1^{n-s,s}$ is a linear map of vector spaces of the same dimension, and its kernel (i.e., $E_2^{n-s,s-1}$) vanishes since $s - 1 \geq n > n/2$. So the cokernel vanishes as well. Finally, for $s = 0$ and $t = n - 1$, $n$ the groups $E_1^{0,t}$ vanish since a polynomial $l$-form with $l > 0$ has degree $\geq l$. $\square$

**Corollary 4.4** Let

$$b(n, N) = \frac{1}{N} ((N - 1)^{n+1} + (-1)^{n+1}(N - 1))$$

be the dimension of $H^n_{dR}(U_K)$ for a smooth degree $N$ hypersurface in $\mathbb{P}_K^n$, cf. [13, Theorem 3.1], and $\mu$ be the number of ordinary double points for $f$. Then

\[\square\text{ Springer}\]
(i) If $n$ is odd then $H^n_{dR}(U_K)$ has dimension $b(n, N) - \tau$ and other reduced de Rham cohomology groups are zero.

(ii) If $n$ is even then $H^n_{dR}(U_K)$ has dimension $b(n, N)$, $H^{n-1}_{dR}(U_K)$ has dimension $\tau$, and other reduced de Rham cohomology groups are zero.

**Proof** The previous theorem (degeneration of pole order spectral sequence plus computation in part (ii)) establish the assertion for $H^{n-1}_{dR}$.

For $H^n$, let us compute $\dim H^n_{dR}(U_K) - \dim H^{n-1}_{dR}(U_K)$. By degeneration, this is equal to

$$\sum_{s=0,...,n} (\dim E_2^{n-s,s} - \dim E_2^{n-s-1,s}) = \sum_{s=0,...,n} (\dim E_1^{n-s,s} - \dim E_1^{n-s-1,s}) - \tau = b(n, N) - \tau$$

and the assertion follows. The first equality above holds since $E_2^{n-s-1,s}$ vanishes for $s = n + 1$ and $E_1^{n-s-1,s}$ has dimension $\tau$ by [10, Theorem 1], as quoted before. So the difference of dimensions for $E_2^{n-s,s}$ and $E_1^{n-s,s}$ for $s = n$ is precisely $\tau$. As for all other values of dimensions, we can relate the $E_1$ page to $E_2$ by using the Euler characteristic property for the $E_1$ page differential. Finally, the second equality follows similarly by the Euler characteristic of the $E_0$ page differential: since it is independent of the choice of differential we can replace $f$ by a homogenous polynomial defining a smooth hypersurface where the standard techniques give $b(n, N)$. □

**Remark 4.5** Observe that although all results of Dimca and Saito were obtained over the field $\mathbb{C}$, they immediately transfer to our choice of $K$. Indeed, there exists a (discontinuous) embedding $K \rightarrow \mathbb{C}$, all our complexes have filtrations with finite dimensional quotients hence all statements regarding vanishing and dimension over $K$ are equivalent to the complex-valued version by exactness of $\otimes_K \mathbb{C}$.

## 5 Frobenius action

### 5.1 Dagger algebra and Monsky–Washnitzer model of rigid cohomology

Next we compute the Frobenius action induced by the embedding, see [13, Section 2], of $A \leftrightarrow A^\dagger$ where $A$ is the ring of functions on $U = \mathbb{P}^3 \setminus Z(f)$ and $A^\dagger$ is a certain completion used in Monsky–Washnitzer cohomology theory. We will not repeat the definition here, sending the reader to *loc. cit.*, but infinite sums used below are well defined in $A^\dagger$. The important reason for introducing $A^\dagger$ is that the corresponding completed de Rham complex $(\Omega^\bullet(A^\dagger), d)$ computes the rigid cohomology $H^\bullet_{rig}(U_k)$. Although in our special case the same vector spaces are also isomorphic to the cohomology of uncompleted de Rham complex $(\Omega^\bullet(A), d)$, the operator $F^*$ used in the computation of zeta function is naturally defined on the completed de Rham complex, and we need to use the quasi-isomorphism of Theorem 3.1 to transport it to the usual de Rham complex involved in the results of Dimca and Saito.
Let us first consider the case when \( n \) is odd (so \( X \) has even dimension \( n - 1 \)). By Theorem 4.3 the \( E_2 \) page of the pole spectral sequence has at most \( n \) nonzero terms \( E_2^{n-s,s} \) for \( s \in \{1, \ldots, n\} \). In order to find the zeta function of \( f \) we must compute the Frobenius action of each basis element in these \( n \) spaces. Such basis element, coming from \( hdx_0 \wedge \cdots \wedge dx_n H^{n+1}(K_f)_{sN-n-1} \) for \( s \in \{1, \ldots, n\} \), can be written as

\[
\frac{\Delta(hdx_0 \wedge \cdots \wedge dx_n)}{f^s} = \frac{h \Omega}{f^s}
\]

where \( \Omega = \Delta(dx_0 \wedge \cdots \wedge dx_n) \). After applying the Frobenius action to the basis elements of each rigid cohomology module we then reduce in cohomology and form the (square) matrix of Frobenius. The coefficients of the characteristic polynomial of this matrix are integers and hence their \( p \)-adic representations are finite, thereby allowing us to truncate the results and recover the zeta function.

For even \( n \) there is a single nonzero term on the diagonal \( E_t^{t,s} \) with \( t + s = n - 1 \), for \( s = n/2 \) and the cohomology classes can be represented by closed forms which are linear combinations of \( \Delta(dx_0 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n) \) with coefficients \( h_i / f^{s_i} \), so the same procedure applies.

### 5.2 Formulas for the lift of Frobenius

Here we assume that \( k = \mathbb{F}_p \) to simplify the notation (the case of field with \( p^a \) elements follows by a simple adjustment as in [13]). Following [13, Section 4], denote by \( \sigma \) the Frobenius automorphism both in \( \text{Gal}(\mathbb{F}_q/\mathbb{F}_p) \) and in \( \text{Gal}(K/\mathbb{Q}_p) \). The Frobenius action is defined on coordinate functions by \( \hat{F}_p(x_i) = x_i^p \). To compute where \( 1/f \) is mapped to, use the equation

\[
1 = \hat{F}_p(f/f) = f^\sigma(x^p) \hat{F}_p(1/f)
\]

where \( f^\sigma \) is obtained by applying \( \sigma \) to each coefficient of \( f \). Equation (14) has no solution in \( A \) but in \( A^\dagger \) one can use

\[
f^{-p}(1 - pg/f^p)^{-1}, \quad \text{where} \quad pg = f^p - f^\sigma(x^p).
\]

For a homogeneous polynomial \( h \) of degree \( sN - (n + 1) \) this gives [13, formula (4.1)]:

\[
\hat{F}_p : \frac{h \Omega}{f^l} \mapsto p^n \frac{h(x(p))(x_0x_1 \cdots x_n)^{p-1}\Omega}{f^{pl}} \left( \sum_{k=0}^{\infty} p^k \frac{\alpha_k g^k}{f^{pk}} \right)
\]

where \( \alpha_k \) is the \( k \)th coefficient of the power series \( (1 - t)^{-l} = 1 + \alpha_1 t + \alpha_2 t^2 + \cdots \). This action extends to \( A^\dagger \) by continuity.

Note that the Frobenius action will increase the pole order but we can reduce it in cohomology because the de Rham complex of the algebra \( A \) has finite dimensional cohomology and hence the same holds for the Monsky–Washnitzer complex of \( A^\dagger \).

We are planning to truncate the infinite series to a finite sum, since the eigenvalues
of the Frobenius matrix have known absolute values, which allows to recover the contribution for large enough index \( k \), without computing the terms explicitly. In other words, adjusting by a series of exact forms we can reduce the output of \( \hat{F}_p \) to a finite expression.

However, as any algorithm can only compute finitely many terms, we will need to truncate the above series. We can do this because \( \zeta(f, T) \) is a rational function and the reciprocal of the characteristic polynomial of the Frobenius matrix has integer coefficients, cf. [18]. Any positive integer will have a terminating \( p \)-adic expansion while any negative integer’s \( p \)-adic expansion will trail off with \( p^{-1} \), since

\[
-1 = p - 1 + (p - 1)p + (p - 1)p^2 + \cdots .
\]

Theorem 3.2 in [13] gives the following bound allowing to recover the zeta function modulo \( p^D \) with

\[
D \geq \lceil \log_p (2\gamma + 1) \rceil \quad \text{where} \quad \gamma := \left( \frac{b}{[b/2]} \right) q^{b/2}
\]  

(16)

provided that the sign of the determinant of the Frobenius matrix is known. Here \( b \) is the number of basis elements for the de Rham cohomology of \( U \). For our case, when \( n \) is odd, \( f \) has \( \tau \) isolated ordinary double points, \( b = b(n, N) - \tau \). The above bound is based on the fact that the eigenvalues for the Frobenius matrix, in the smooth case, have absolute value \( p^{(n-1)/2} \), and the coefficients of the characteristic polynomial of the Frobenius matrix, which is a reciprocal polynomial, can be written in terms of symmetric polynomials in the eigenvalues/roots. If \( e_i \) stands for the \( i \)-th elementary symmetric function, then any monic polynomial \( g \) of degree \( k \) with roots \( r_1, r_2, \ldots, r_k \) can be written as

\[
g = x^k - e_1(r_1, \ldots, r_k)x^{k-1} + e_2(r_1, \ldots, r_k)x^{k-2} + \cdots + (-1)^{k}e_k(r_1, \ldots, r_k).
\]

Now when \( n \) is odd and \( f \) has isolated ordinary double points, the complement \( U_k = \mathbb{P}^3_k \setminus X_k \) along with the \( E_2 \) page having zero \( s + t = n - 1 \), satisfy the hypotheses of [3, Theorem 5.2, p. 174]. Therefore the eigenvalues of the Frobenius matrix have absolute value \( p^{(n-1)/2} \). Hence the bound (16) from [13] holds. The last bound needed is for truncating the series of the Frobenius action, (15). Corollary 4.2 on p. 22 of loc. cit. tells us that we can truncate the series \( \sum_k p^k \alpha_k g^k / f^{pk} \), removing the terms with \( k \geq M \), as long as

\[
k \geq D + (n + 1)[\log_p (p(k + n) - 1)] - n + 1 + r_1 \quad \text{for all} \quad k \geq M
\]

where \( r_1 = 0 \) for \( p > 2 \), see the bottom of p. 21 of loc. cit. Recall that \( p > n - 1 \) for us.

### 5.3 Reduction in top cohomology

The Frobenius action formula (15) is an infinite series while a computer can only handle finitely many terms. For our algorithm we truncate this series and reduce each
term until it is written in the cohomology classes of the basis elements for the $E_2$ page. We now explain the reduction process. Given $g\Omega/f^s$ for some $g\Omega \in S_{sN-n}\Omega^n$, we can write

$$g\Omega = \Delta(gdx_0 \wedge dx_1 \wedge \cdots \wedge dx_n)$$

where $gdx_0 \wedge dx_1 \wedge \cdots \wedge dx_n \in S_{sN-n-1}\Omega^{n+1}$. If $s > n$ then the de Rham differential induces an isomorphism on the $E_1$ page by Theorem 4.3 and $gdx_0 \wedge \cdots \wedge dx_n$ can be expressed as

$$gdx_0 \wedge dx_1 \wedge \cdots \wedge dx_n = d\alpha + df \wedge \omega$$

where

$$\alpha \in \ker(S_{sN-n}\Omega^n \xrightarrow{df\wedge} S_{(s+1)N-n-1}\Omega^{n+1}); \quad \omega \in S_{(s-1)N-3}\Omega^3.$$  

Now consider

$$\frac{g\Omega}{f^s} - d\left(\frac{N}{|\omega|} \frac{\Delta(\omega)}{f^{s-1}} - \frac{\Delta(\alpha)}{f^s}\right) = \frac{g\Omega}{f^s} + \frac{N}{|\omega|} \frac{\Delta(df(\omega))}{f^{s-1}} - \frac{\Delta(df(\alpha))}{f^s+1}, \quad \text{by } (13)$$

$$= \frac{g\Omega}{f^s} + \frac{N}{|\omega|} \frac{\Delta(df(\omega))}{f^{s-1}} - \frac{\Delta(gdx_0 \wedge \cdots \wedge dx_n - d\alpha)}{f^s} - \frac{\Delta(d\alpha)}{f^s}, \quad \text{by } (17) \text{ and } (18)$$

$$= \frac{g\Omega}{f^s} + \frac{N}{|\omega|} \frac{\Delta(df(\omega))}{f^{s-1}} - \frac{g\Omega}{f^s} + \frac{\Delta(d\alpha)}{f^s} - \frac{\Delta(d\alpha)}{f^s} = \frac{N}{|\omega|} \frac{\Delta(d\omega)}{f^{s-1}}$$

$$\Rightarrow \frac{g\Omega}{f^s} \equiv \frac{N}{|\omega|} \frac{\Delta(d\omega)}{f^{s-1}} \pmod{dR^*}. \quad (19)$$

Hence when reducing in cohomology we first check if the $g$ from (17) is in the image of Koszul differential, $df \wedge$. This can be determined by using a Groebner basis for the Jacobian ideal of $f$ (and in higher degrees by applying a lemma that follows). Now if $g$ is in the image of $df \wedge$ then $\alpha = 0$ and we can decrease the pole order of $g\Omega/f^s$ by using the reduction formula, (19). If $g$ is not in the image of $df \wedge$ and $s > n$ then we must find the $\alpha$ from (17) in order to reduce in cohomology. Technically this problem can be solved with Linear Algebra by using matrices, that is one can construct matrices for the kernel and image of $df \wedge$, however this is not practical if the polynomial part of $\alpha$ is large. For example suppose that $n = 3$, $\deg(f) = \deg(h) = 4$ and $p = 7$, then the third term ($k = 2$) in the Frobenius action formula has polynomial degree equal to $4p + 4(p - 1) + 4pk = 108$. There are $\binom{108+3}{3} = 221,815$ monomials of degree 108 in four variables and this is only the third term in the series! More importantly, even if one did construct a matrix of this size it would only reduce the pole order of $g\Omega/f^s$ by 1. Thankfully there is a quicker way to find $\alpha$ which uses bases of the subdiagonal on the $E_1$ page. We will apply the following

**Lemma 5.1** Let $J = (f_0, f_1, \ldots, f_n)$ be the Jacobian ideal of $f$ with $\deg(f) = N \geq 2$ and suppose that the projective hypersurface defined by $f$ has isolated ordinary double points. If $g \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ with $\deg(g) \geq (n + 1)(N - 1)$, then $g \in J$ if and only if $g$ vanishes at all singular points of $f$. 

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Proof Let \( I = \sqrt{J} \) be the radical of the Jacobian ideal, \( S = \mathbb{C}[x_0, x_1, \ldots, x_n] \), and assume that \( Z(f) \) has transversal intersection with \( Z(x_0) \). Note that we can always find a change of coordinates such that \( Z(x_0) \) is transversal to \( X = Z(f) \), for instance see \([10, \text{Example } 3.3]\).

Next, consider the map \( \Omega^n_{m+n} \to S_m/I_m \) defined by sending an \( n \) form \( \alpha \) to its coefficient of \( dx_1 \ldots dx_n \) reduced modulo \( I \). By transversality assumption and \([10, \text{Theorem } 1.2]\), this induces an embedding of \( H^n(K_s^*)_m \) into \( S_m/I_m \). If \( m \geq (n+1)(N-1) \) then by Theorem 4.3,

\[
\tau = \dim(H^n(K_s^*)_m) \leq \text{codim}(I_m), \quad \text{(from the embedding)}
\leq \text{codim}(J_m), \quad \text{(since } J \subseteq I \text{)}
= \tau, \quad \text{(since } \dim(H^{n+1}(K_s^*)_m) = \tau) .
\]

The above shows that, in this range, \( I_m = J_m \). Moreover, for a nodal hypersurface \( g \in I \) if and only if \( g \) vanishes at all singular points of \( f \): this is a reformulation of \([11, \text{Theorem } 1.5]\) (its proof in \textit{loc. cit.} is based on the Cayley–Bacharach Theorem). This proves the lemma.

It follows that in (17), \( g \) has the same values at the singular points (or rather their fixed lifts \( P_1, \ldots, P_t \) to vectors in the \((n+1)\)-dimensional affine space) as the polynomial \( d\alpha/(dx_0 \wedge \cdots \wedge dx_n) \), where \( \alpha \in H^n(K_s^*)_sN_{-n} \). By Theorem 4.3, for \( s > n \) we have \( \dim(H^n(K_s^*)(n+1)N_{-n} = \tau \). Let \( \{\gamma_1, \ldots, \gamma_\tau\} \) be a basis for this space. Further, suppose that the coordinate hyperplane \( x_0 = 0 \) is transversal to \( X \) then

\[
\{x_0^k \gamma_1, \ldots, x_0^k \gamma_\tau\}
\]

is a basis for \( H^n(K_s^*)_sN_{-n+k} \) for any \( k \geq 0 \) by \([2, \text{Corollary } 11]\). At this point it is clear that \( \alpha \) will be a linear combination of \( x_0^k \gamma_1, \ldots, x_0^k \gamma_\tau \), for \( k = (s-n-1)N \), and the coefficients of that linear combination are computed by looking at \( d(x_0^k \gamma_j) \) and computing their values at \( P_1, \ldots, P_t \).

More precisely, let us look at \( H^3(K_s^*)(n+1)N_{-n+k} \) with \( k \geq 0 \). We know that the de Rham differential induces an isomorphism and therefore

\[
\{d(x_0^k \gamma_1), \ldots, d(x_0^k \gamma_\tau)\}
\]

is a basis for \( H^4(K_s^*)(n+1)(N-1)+k \). In particular, \( d(x_0^k \gamma_i) \) is not in the image of \( df \wedge \), hence the polynomial part of \( d(x_0^k \gamma_i) \) is not in the Jacobian ideal and from the lemma above \( d(x_0^k \gamma_i) \) does not vanish at all singular points of \( f \). We then have the following linear system of equations:

\[
c_1d(x_0^k \gamma_1)|_{P_1} + \cdots + c_\tau d(x_0^k \gamma_\tau)|_{P_1} = g|_{P_1},
\]

\[
c_1d(x_0^k \gamma_1)|_{P_2} + \cdots + c_\tau d(x_0^k \gamma_\tau)|_{P_2} = g|_{P_2},
\]

\[
\vdots
\]

\[
c_1d(x_0^k \gamma_1)|_{P_\tau} + \cdots + c_\tau d(x_0^k \gamma_\tau)|_{P_\tau} = g|_{P_\tau} .
\]

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This system has a unique solution, for if it did not then there would be a nontrivial solution to
\[ c_1 v_1 + \cdots + c_\tau v_\tau = 0 \quad \text{where} \quad v_i = \left( \begin{array}{c} \frac{d(x_0^k y_1)}{p_1} \\ \vdots \\ \frac{d(x_0^k y_\tau)}{p_\tau} \end{array} \right). \]

This means that the polynomial part of \( c_1 d(x_0^k y_1)|_{p_1} + \cdots + c_\tau d(x_0^k y_\tau)|_{p_\tau} \) vanishes at all singular points of \( f \) and hence it is in the Jacobian ideal of \( f \) from the lemma above. But this is a contradiction since \( d(x_0^k y_i) \) are a basis for \( H^{n+1}(K^\bullet_{(n+1)(N-1)+k}) \) and by definition not in the image of \( df \wedge \cdot \). Therefore the system of equations (20) has a unique solution. Furthermore we can write the solution as
\[
\begin{pmatrix} c_1 \\ \vdots \\ c_\tau \end{pmatrix} = \left( \begin{array}{c} \frac{d(x_0^k y_1)}{p_1} \\ \vdots \\ \frac{d(x_0^k y_\tau)}{p_\tau} \end{array} \right)^{-1} \left( \begin{array}{c} g|_{p_1} \\ \vdots \\ g|_{p_\tau} \end{array} \right)
\]

where the \( \tau \times \tau \) matrix can be reused anytime one is reducing in cohomology in this degree. That is the square matrices used to reduce the pole order for one basis element, \( h \), do not have to be recalculated. This completes our method of finding an \( gdx_0 \wedge \cdots \wedge dx_3 = d\alpha + df \wedge \omega \) and this is one of the main differences between the ordinary double point case and the smooth case.

### 5.4 Reduction on the subdiagonal for even \( n \)

Since \( E_1^{t,s} \) has no nonzero terms with \( t + s = n - 2 \) and \( E_2^{t,s} \) has a unique nonzero term for \( t + s = n - 1 \), the situation is much simpler: in (17) we can simply take \( \alpha = 0 \) and then use an analogue of (19) to reduce the pole order while keeping the same cohomology class.

### 6 Steps of the algorithm

Below is a brief description of each step of our algorithm.

1. On the \( E_1 \) page compute \((H^n(K_f^\bullet)_{jN-n}, H^{n+1}(K_f^\bullet)_{jN-n-1})\) for \( j = 1, \ldots, n \).
2. Find a basis for every cohomology space in each pair from Step 1.
3. For odd \( n \), use the bases from Step 2 (each pair) to calculate a basis for the top cohomology spaces on the \( E_2 \) page of the spectral sequence
   \[ \frac{b_1 \Omega}{f^{ib_1}}, \ldots, \frac{b_L \Omega}{f^{ib_L}}, \quad \text{where} \quad L = \frac{1}{N} ((N-1)^{n+1} + (N-1)) - \tau. \]
4. For even \( n \), calculate the basis for \( E_2^{n-s,s} \) with \( s = 1, \ldots, n \) and for \( E_2^{n/2-1,n/2} \).
5. Calculate the Frobenius action for every basis element from Step 3 using formula (15) and its analogue for the subdiagonal in which \( p^{n+1}(x_0 x_1 \ldots x_n)^{p-1} \Omega \)
is replaced by Frobenius images of $\Delta(dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n)$ and the sum of (15) is applied to each of the coefficients $\frac{h_i}{f^i}$.

5. Reduce in cohomology as in Sect. 5.3.

6. Use Step 5 to compute the Frobenius matrix and calculate its characteristic polynomial.

Any algorithm must imply a finite computation. Results on spectral sequences given in Sect. 4 tell us that the $E_2$ page vanishes when the total degree is greater than $nN$. Thus we can restrict our attention to the terms of the $E_1$ page mentioned in Step 1. Finding their dimension and then a basis, if de Rham is not surjective, can be done with a computer by using Linear Algebra since the differentials $d$ and $df \wedge$ are linear operators. This also limits the number of spaces on the $E_2$ page for Step 3. Again we are only interested in $E_{2s}^t$ for $s = 1, \ldots, n$ and $t = n - s$, and for even $n$ also $s = n/2, t = n/2 - 1$. Step 4 is a formula which one can easily implement with a computer and bounds for when to truncate were given in Sect. 5.2. In the next section we explain Step 5.

At the end of Step 5 we have written the (truncated) Frobenius action of each basis element as a linear combination of all basis elements ($E_2$ page). If $f \in \mathbb{Z}[x_0, x_1, x_2, x_3]$ then this linear combination is over $\mathbb{Q}$. For practical purposes we can construct the Frobenius matrix so that its entries are these constants in $\mathbb{Q}$ or their $p$-adic expansion. In any case we must calculate the $p$-adic expansion of the coefficients of the reciprocal characteristic polynomial of the Frobenius matrix. Explicit examples are given in the next section and for more on $p$-adic numbers see [17].

7 Examples of computations for surfaces ($n = 3$)

We apply our algorithm to the Cayley cubic, a Kummer surface, a quartic with 6 ordinary double points and a quintic with 14 ordinary double points. For the Cayley cubic we go into greater detail.

7.1 Cayley cubic

The defining equation for the Cayley cubic is

$$f(x_0, x_1, x_2, x_3) = x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0. \quad (22)$$

This surface has four ordinary double points, the maximum for any surface of degree three, located at $[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0]$, and $[0 : 0 : 0 : 1]$. It is important to note that using our algorithm for the Cayley cubic is overkill. We have the following table.

For the last row of Table 1 we go into more detail. On $[1 : x_1 : x_2 : x_3]$ the surface becomes

$$x_1(x_2 + x_3 + x_2x_3) + x_2x_3 = 0. \quad (23)$$

**Case 1:** $x_2 + x_3 + x_2x_3 = 0$, then (23) reduces to $x_2x_3 = 0$. Subtracting gives $x_2 + x_3 = 0$ and $x_2x_3 = 0 \Rightarrow x_2 = x_3 = 0$ giving us $q$ roots since $f(1, x_1, 0, 0) = 0$. 

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Table 1 Rational points of the Cayley cubic

| point(s)       | number of roots of \( f \) |
|----------------|-----------------------------|
| \([0:0:0:1]\)  | 1                           |
| \([0:0:1:x_3]\) | \( q \)                     |
| \([0:1:x_2:x_3]\) | \( 2q - 1 \)                |
| \([1:x_1:x_2:x_3]\) | \( q^2 + 1 \)               |

**Case 2:** \( x_2 + x_3 + x_2x_3 \neq 0 \), then we can explicitly solve for \( x_1 \) in (23) yielding

\[
x_1 = -\frac{x_2x_3}{x_2 + x_3 + x_2x_3}.
\]

How many points \((x_2, x_3)\) are there such that \( x_2 + x_3 + x_2x_3 \neq 0 \)? This reduces to the question: how many points yield \( x_2 + x_3 + x_2x_3 = 0 \)? For \( x_3 \neq -1 \) we can rewrite \( x_2 = -\frac{x_3}{1 + x_3} \). This gives \( q - 1 \) points \((x_2, x_3)\) such that \( x_2 + x_3 + x_2x_3 = 0 \). The value \( x_3 = -1 \) is not compatible with \( x_2 + x_3 + x_2x_3 = 0 \).

This means that in Case 2 there are \( q^2 - (q - 1) \) roots of \( f \). Adding the number of roots in Cases 1 and 2 gives \( q + q^2 - (q - 1) = q^2 + 1 \). Since \( q \) was an arbitrary power of some prime \( p \), we have \# \( V(\mathbb{F} q^r) = 1 + 3q^r + q^{2r} \) (this is the sum of the second column of Table 1 with \( q \) replaced by \( q^r \)). Thus

\[
\zeta(f, T) = \exp\left(\sum_{r=1}^{\infty} (q^{2r} + 3q^r + 1) \frac{T^r}{r}\right) = \frac{1}{(1-T)(1-qT^3)(1-q^2T)}.
\]

To compare this with the output of our algorithm, first note that \( f \) is equisingular since the coordinates of the singular points consist only of ones or zeros which are in any field. Now we look at the \( E_0 \) page of the spectral sequence. We use Fig. 1 with \( N = n = 3 \). Starting with the top diagonal we have

\[
\dim(H^4(K_f^\bullet)_{2}) = 10 - \dim(\text{im}(S_0\Omega^3 \xrightarrow{df^\wedge} S_2\Omega^4)).
\] (24)

By straightforward Linear Algebra we get that

\[
\{x_0^2, x_0x_1, x_0x_2, x_1^2, x_2^2\}dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3
\] (25)

project to a basis for \( H^4(K_f^\bullet)_{2} \) (Table 2). For the subdiagonal space \( H^3(K_f^\bullet)_{3} \) we have

\[
\dim(H^3(K_f^\bullet)_{3}) = \dim(\ker(S_3\Omega^3 \xrightarrow{df^\wedge} S_5\Omega^4)) - \dim(\text{im}(S_1\Omega^2 \xrightarrow{df^\wedge} S_3\Omega^3)).
\]

Using Mathematica we find that \( \dim(H^3(K_f^\bullet)_{3}) = 4 \) and in general for \( j \geq 3 \).

The first nontrivial kernel of \( df^\wedge: S_j\Omega^3 \rightarrow S_{j+2}\Omega^4 \) occurs when \( j = 2 \) and it can be used to give a basis for higher cohomology groups by multiplying the basis

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elements by certain powers of $x_i$. For example Mathematica tells us that the following 3-forms are a basis for $H^3(K_f^*)_2$:

$$
n_1 = (2x_0x_2 + 3x_2x_3 - 2x_3^2) dx_0 \wedge dx_1 \wedge dx_2
- (2x_0x_3 - 2x_2^2 + 3x_2x_3) dx_0 \wedge dx_1 \wedge dx_3
- (6x_0 x_1 + 2x_0 x_2 + 2x_0 x_3 - 2x_3) dx_0 \wedge dx_2 \wedge dx_3
- (4x_0^2 - x_2 x_3) dx_1 \wedge dx_2 \wedge dx_3;$$

$$
n_2 = (2x_0 x_1 + 3x_1 x_3 - 2x_3^2) dx_0 \wedge dx_1 \wedge dx_2
+ (2x_0 x_1 + 3x_0 x_2 - x_0 x_3 - x_1 x_3 - 3x_2 x_3) dx_0 \wedge dx_1 \wedge dx_3
- (3x_0 x_1 + x_0 x_3 + 2x_1^2) dx_0 \wedge dx_2 \wedge dx_3
- (x_0^2 - x_1 x_3) dx_1 \wedge dx_2 \wedge dx_3;$$

$$
n_3 = (x_0 x_1 + x_0 x_2 + x_1 x_2 + 3x_1 x_3 + 3x_2 x_3) dx_0 \wedge dx_1 \wedge dx_2
+ (x_0 x_1 + 3x_0 x_2 + 2x_2^2) dx_0 \wedge dx_1 \wedge dx_3
- (3x_0 x_1 + x_0 x_2 + 2x_1^2) dx_0 \wedge dx_2 \wedge dx_3
- (4x_0^2 - x_1 x_2) dx_1 \wedge dx_2 \wedge dx_3. $$

For example, one can check that $\{x_0 n_1, x_2 n_1, x_3 n_1, x_1 n_2\}$ is a basis for $H^3(K_f^*)_3$. We are taking advantage of the fact that $df$ $\wedge$ is $\mathbb{C}[x_0, x_1, x_2, x_3]$-linear for if $df \wedge n_1 = 0$, then $df \wedge (x_0 n_1) = x_0 (df \wedge n_1) = 0$, and therefore we only need to check if this element is in the image of $df$ $\wedge$, where Lemma 5.1 applies.

Steps 1 and 2 of the algorithm have been completed. For Step 3 we compute a basis for the only cohomology space (for this example) on the $E_2$ page

$$E_{2}^{4,2} = H^4(K_f^*)_2 / \text{im}(H^3(K_f^*)_3 \xrightarrow{d} H^4(K_f^*)_2).$$

In order to find a basis for this space we must calculate the de Rham differential of each element in the basis $\{x_0 n_1, x_2 n_1, x_3 n_1, x_1 n_2\}$. Take the element $x_0 n_1$,

$$d(x_0 n_1) = (-6x_0^2 + x_0 x_2 + x_0 x_3 + x_2 x_3) dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3. $$

Recall that (25) is our basis for $H^4(K_f^*)_2$ and not all of the monomials of $d(x_0 n_1)$ are in this basis, only $x_0^2$. To solve this problem we can add an element in $\text{im}(P_0 \Omega^3 \xrightarrow{df \wedge} P_2 \Omega^4)$ to $x_0 n_1$. For example

\[ \text{im}(P_0 \Omega^3 \xrightarrow{df \wedge} P_2 \Omega^4) \]
\[
d((-x_0n_1 + df \wedge (x_3 dx_0 \wedge dx_2))/6) = x_0^2 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3,
\]
\[
d((x_2 n_1 + 2df \wedge (x_3 dx_0 \wedge dx_2))/3) = x_2^2 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3,
\]
\[
d((x_3 n_1 + 2df \wedge (x_3 dx_0 \wedge dx_2))/3) = x_3^2 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3,
\]
\[
d((x_1 n_2 + 2df \wedge (x_3 dx_0 \wedge dx_1))/3) = x_1^2 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3.
\]

It is obvious that \( E_2^{4,2} \) is spanned by (images of) \( \{x_0 x_1, x_0 x_2\} dx_0 \wedge \cdots \wedge dx_3 \). Now we calculate the action of Frobenius on \( \frac{x_0 x_1 \Omega}{f^2}, \frac{x_0 x_2 \Omega}{f^2} \). For \( h = x_0 x_1 \) the equation (15) becomes

\[
p^3 \frac{(x_0 x_1)^p (x_0 x_1 x_2 x_3)^{p-1} \Omega}{f^{2p}} \sum_{k=0}^{\infty} \frac{(k + 1)(f^p - f(x^p))^k}{f^{pk}}.
\]

As an example, let \( p = 5 \), then the first term \( (k = 0) \) in the series above is \( 125x_0^9 x_1^9 x_2^4 x_3^4 \Omega/f^{10} \). Our final goal is to write this form as a linear combination of \( x_0 x_1 \Omega/f^{2} \) and \( x_0 x_2 \Omega/f^{2} \), but first we must reduce it to \( \beta/f^9 \) where \( \beta \in P_{24} \Omega^3 \). To do this we write \( 125x_0^9 x_1^9 x_2^4 x_3^4 dx_0 \wedge \cdots \wedge dx_3 \) in the form of (17) which can be accomplished by using a Groebner basis for the Jacobian of \( f \). Indeed we find that

\[
\omega = 125x_0^9 x_1^9 x_2^3 x_3^2 \cdot \left[ \frac{1}{6} x_3 dx_0 \wedge dx_1 \wedge dx_2 - \left( \frac{2}{3} x_2 + \frac{1}{2} x_3 \right) dx_0 \wedge dx_1 \wedge dx_3 \right.
\]
\[
\left. - \left( \frac{1}{3} x_1 + \frac{1}{2} x_3 \right) dx_0 \wedge dx_2 \wedge dx_3 + \left( \frac{1}{3} x_0 + \frac{1}{2} x_3 \right) x_0 dx_1 \wedge dx_2 \wedge dx_3 \right].
\]

Two remarks are important here. One, the 3-form \( \omega \) is not unique, and two, we did not have to use the de Rham differential since the monomial \( x_0^9 x_1^9 x_2^4 x_3^4 \) is in the Jacobian ideal of \( f \), that is, the \( \alpha \) of (17) is zero in this case. Back to the reduction we have

\[
\frac{125x_0^9 + x_2^4 x_3^4 \Omega}{f^{10}} \equiv \frac{1}{9} \frac{\Delta(\omega)}{f^9} \pmod{d_{\text{DR}}}
\]

the \( \omega \) from above, (27). For the next reduction we must find an \( \omega_1 \in S_{21} \Omega^3 \) such that \( df \wedge \omega_1 = d\omega \). Again we check if \( d\omega \in \text{im}(S_{21} \Omega^3 \xrightarrow{df \wedge} S_{23} \Omega^4) \) and it is. When the 4-form is not in the image of the Koszul differential, for the case of the Cayley cubic, the remainder can be discarded because it is a pure power of \( x_i \) which is in the image of \( d \). By Lemma 5.1 any mixed monomial of degree 3 is in the Jacobian ideal of \( f \). Hence when we are reducing in cohomology, if there is a remainder, then \( \alpha \) from (17) can only be \( x_i^{3k+2} dx_0 \wedge \cdots \wedge dx_3 \) for \( i = 0, 1, 2, 3 \) and \( k \in \mathbb{Z}_{>0} \). We have already given explicit examples of 3-forms whose de Rham differential is \( x_i^2 dx_0 \wedge \cdots \wedge dx_3 \).
and the general case admits similar formulas. The table below shows the reductions of the first seven terms of the series in (26).

| $k$ | $k^{th}$ part of the reduction of $x_0 x_1$ |
|-----|-----------------------------------------|
| 0   | $\frac{1}{126} \cdot 5^2 x_0 x_1$       |
| 1   | $\frac{1}{34034} \cdot 5^5 x_0 x_1$     |
| 2   | $\frac{1013}{5819814} \cdot 5^6 x_0 x_1$|
| 3   | $\frac{1487}{38849972} \cdot 5^6 x_0 x_1$|
| 4   | $\frac{2084}{97668777} \cdot 5^6 x_0 x_1$|
| 5   | $\frac{2087}{1185579252} \cdot 5^8 x_0 x_1$|
| 6   |                                           |

For the other basis element $x_0 x_2 dx_0 \wedge \cdots \wedge dx_3$ the reductions are exactly the same; meaning that the table for $x_0 x_2$ is the same as the one above, but with $x_2$ instead of $x_1$. Let

$$r_0 = \frac{5^2}{126}, \quad r_1 = \frac{5^5}{9009}, \quad r_2 = \frac{5^6}{34034}, \quad \ldots, \quad r_6 = \frac{2087 \cdot 5^8}{1185579252}$$

and let $\phi$ be the embedding of $\mathbb{Q}$ into $\mathbb{Q}_5$. We now look at the convergence of $\phi(r_0 + \cdots + r_i)$.

$$\phi(r_0) = 1 \cdot 5^2 + 0 \cdot 5^3 + 0 \cdot 5^4 + 4 \cdot 5^5 + 4 \cdot 5^7 + 0 \cdot 5^8 + 0 \cdot 5^9 + \cdots,$$
$$\phi(r_0 + r_1) = 1 \cdot 5^2 + 0 \cdot 5^3 + 0 \cdot 5^4 + 3 \cdot 5^5 + 2 \cdot 5^6 + 0 \cdot 5^7 + 0 \cdot 5^8 + 0 \cdot 5^9 + \cdots,$$
$$\phi(r_0 + r_1 + r_2) = 1 \cdot 5^2 + 0 \cdot 5^3 + 0 \cdot 5^4 + 3 \cdot 5^5 + 1 \cdot 5^6 + 3 \cdot 5^7 + 4 \cdot 5^8 + 4 \cdot 5^9 + \cdots,$$
$$\phi(r_0 + \cdots + r_3) = 1 \cdot 5^2 + 0 \cdot 5^3 + 0 \cdot 5^4 + 0 \cdot 5^5 + 0 \cdot 5^6 + 2 \cdot 5^7 + 0 \cdot 5^8 + 0 \cdot 5^9 + \cdots,$$
$$\phi(r_0 + \cdots + r_4) = 1 \cdot 5^2 + 0 \cdot 5^3 + 0 \cdot 5^4 + 0 \cdot 5^5 + 1 \cdot 5^6 + 4 \cdot 5^7 + 4 \cdot 5^8 + 4 \cdot 5^9 + \cdots,$$
$$\phi(r_0 + \cdots + r_5) = 1 \cdot 5^2 + 0 \cdot 5^3 + 0 \cdot 5^4 + 0 \cdot 5^5 + 0 \cdot 5^6 + 0 \cdot 5^7 + 4 \cdot 5^8 + 0 \cdot 5^9 + \cdots,$$
$$\phi(r_0 + \cdots + r_6) = 1 \cdot 5^2 + 0 \cdot 5^3 + 0 \cdot 5^4 + 0 \cdot 5^5 + 0 \cdot 5^6 + 0 \cdot 5^7 + 0 \cdot 5^8 + 2 \cdot 5^9 + \cdots.$$

This sequence of numbers is 5-adically converging to 25 (we are using the estimate (16) here). Therefore the matrix of Frobenius is $25I_2$, and the “interesting” part of the zeta function is

$$\det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 5^{-1} T \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}\right) = (1 - 5T)^2.$$

This gives the zeta function of the Cayley cubic for $p = 5$ is

$$\zeta(f, T) = \frac{1}{(1 - T)(1 - 5T)^3(1 - 25T)}.$$
7.2 Kummer surfaces

A Kummer surface is a degree 4 surface in \( \mathbb{P}^3 \) with 16 ordinary double points, the maximum number of nodes for any quartic. Its family of defining polynomials are

\[
(x_0^2 + x_1^2 + x_2^2 - \mu^2 x_3^2)^2 \\
- \lambda (x_3 - x_2 - \sqrt{2}x_0)(x_3 - x_2 + \sqrt{2}x_0)(x_3 + x_2 + \sqrt{2}x_1)(x_3 + x_2 - \sqrt{2}x_1)
\]

where \( \lambda = (3\mu^2 - 1)/(3 - \mu^2) \) and \( \mu^2 \neq 1/3, \mu^2 \neq 1, \mu^2 \neq 3 \). In this section we find the zeta function of a Kummer surface with \( \mu = 2 \), that is

\[
f = x_0^4 + x_1^4 + 12x_2^4 + 27x_3^4 + x_0(46x_1^2 - 20x_2^2 - 44x_2x_3 - 30x_3^2) \\
- x_1^2(20x_2^2 - 44x_2x_3 + 30x_3^2) - 30x_2^2x_3^2.
\]

The 16 ordinary double points of \( f \) are

\[
[\pm(\sqrt{2} + \sqrt{3}) : \pm(\sqrt{2} - \sqrt{3}) : -\sqrt{6} : 2], \\
[\pm(\sqrt{2} - \sqrt{3}) : \pm(\sqrt{2} + \sqrt{3}) : \sqrt{6} : 2], \\
[\pm\sqrt{3} : 0 : -1 : 1], \quad [\pm 3\sqrt{2} : 0 : -3 : 2], \\
[0 : \pm 3\sqrt{2} : 3 : 2], \quad [0 : \pm \sqrt{3} : 1 : 1].
\]

Choose a prime \( p \) and see if the lift from \( \mathbb{F}_p \) from \( \mathbb{Z}_p \) is equisingular. Using a Groebner basis one can show that \( f \) has 16 singularities over the algebraic closures of \( \mathbb{F}_5 \) and \( \mathbb{F}_7 \), but things go wild for \( \mathbb{F}_{11} \). Take for instance the singular point \([0 : \sqrt{3} : 1 : 1]\) and define \( g(x_0, x_1, x_2, x_3) = f(x_3, x_0\sqrt{3} + x_2, x_0 + x_1, x_0) \). Then \([1 : 0 : 0 : 0]\) is a singular point of \( g \) and with the quadratic part of \( g(1, x_1, x_2, x_3) \) written as

\[
-18x_1^2 + 8\sqrt{3}x_1x_2 + 12x_2^2 + 44x_3^2 = (x_1 \quad x_2 \quad x_3) \begin{pmatrix} -18 & 4\sqrt{3} & 0 \\ 4\sqrt{3} & 12 & 0 \\ 0 & 0 & 44 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

where the determinant of the \( 3 \times 3 \) matrix above is \(-11616 = -2^5 \cdot 3 \cdot 11^2 \). This was just one of the 16 ordinary double points of \( f \), but one can check that for all 16 points the corresponding determinant of the matrix above will be divisible by \( 11^2 \) and therefore we exclude the case \( p = 11 \) when using our algorithm. However we can still calculate the zeta function of \( f \) over \( \mathbb{F}_{11} \) since

\[
f \equiv (x_0^2 + x_1^2 + x_2^2 + 7x_3^2)^2 \in \mathbb{F}_{11}[x_0, x_1, x_2, x_3].
\]

Thus, the zeta function of \( f \) equals that of the smooth quadratic \( x_0^2 + x_1^2 + x_2^2 + 7x_3^2 \) which is

\[
\zeta(f, T) = \frac{1}{(1 - T)(1 - 11T)(1 - 121T)(1 + 11T)}.
\]

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Now apply our algorithm to \( f \), \( p = 5, 7 \). First, \( \dim(K_2^{4,0}) = 1 \) and \( dx_0 \wedge \cdots \wedge dx_3 \) gives a basis for this space. Moving up one level, to total degree \( 2N = 8 \), we use the following basis for \( H^4(K_f^* )_4 \):

\[
\{ x_0^4, x_0^3x_1, x_0^3x_2, x_0^3x_3, x_0^2x_1^2, x_0^2x_1x_2, x_0^2x_1x_3, x_0x_2^2, x_0x_1x_2, x_0x_1^2x_3, x_0x_1x_2^2, x_1^4, x_1^3x_2, x_1^3x_3, x_2^3, x_2^2x_3, x_2x_3^2, x_3^4 \} \ dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3.
\] (29)

For the subdiagonal, \( H^3(K_f^* )_5 \), the authors were able to find a basis \( \{ b_1, \ldots, b_{15} \} \) such that

\[
\begin{align*}
\quad d(b_1) &= x_0^3x_1 &\quad d(b_5) &= x_0^2x_1x_3 &\quad d(b_9) &= x_1^3x_2 &\quad d(b_{13}) &= x_0^4 - x_1^4 \\
\quad d(b_2) &= x_0^2x_2 &\quad d(b_6) &= x_0x_1^3 &\quad d(b_{10}) &= x_1^3x_3 &\quad d(b_{14}) &= 3x_0^2x_1^2 + x_1^4 - 2x_2^4 \\
\quad d(b_3) &= x_0^3x_3 &\quad d(b_7) &= x_0x_1^2x_2 &\quad d(b_{11}) &= x_3^2x_3 &\quad d(b_{15}) &= 4x_2^4 - 9x_3^4 \\
\quad d(b_4) &= x_0^2x_1x_2 &\quad d(b_8) &= x_0x_1^2x_3 &\quad d(b_{12}) &= x_2x_3^3
\end{align*}
\]

(all polynomials above are multiplied by the factor \( dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \) which we omit to save room). Therefore

\[
\{ x_0^4, x_0^2x_1^2, x_0^2x_2^2, x_0x_1x_2^2 \} \ dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3
\]

project to a basis for \( E_2^{4,4} \). The degree of the interesting part of the zeta function, \( P(T) \), is \( 1/4((4 - 1)^{3+1} + 4 - 1) - 16 = 5 \) and we have our five basis elements to calculate the Frobenius action on

\[
\begin{align*}
\frac{x_0^4\Omega}{f^2}, &\quad \frac{x_0^2x_1^2\Omega}{f^2}, &\quad \frac{x_0^2x_2^2\Omega}{f^2}, &\quad \frac{x_0x_1x_2^2\Omega}{f^2}, &\quad \frac{\Omega}{f}.
\end{align*}
\]

For reducing in cohomology we use the following basis for \( H^3(K_f^* )_{jN-3} \):

\[
\begin{align*}
\{ &x_0^{4j-6}n_1, x_0^{4j-7}x_1n_1, x_0^{4j-7}x_2n_1, x_0^{4j-7}x_3n_1, \\
x_0^{4j-8}x_1^2n_1, x_0^{4j-8}x_1x_2n_1, x_0^{4j-8}x_1x_3n_1, \\
x_0^{4j-8}x_2^2n_1, x_0^{4j-8}x_2x_3n_1, x_0^{4j-9}x_3^2n_1, x_0^{4j-9}x_2x_2n_1, \\
x_0^{4j-9}x_1x_3n_1, x_1^{4j-6}n_2, x_1^{4j-7}x_2n_2, x_1^{4j-7}x_3n_2, x_1^{4j-8}x_2^2n_2 \}
\end{align*}
\]

where

\[
n_1 = -x_0(2x_0^2 + 6x_1^2 - 20x_2^2 - 24x_2x_3 - 5x_3^2) \ dx_0 \wedge dx_1 \wedge dx_2 \\
-5x_0x_3(5x_2 + 6x_3) \ dx_0 \wedge dx_1 \wedge dx_3 + 5x_0x_1(4x_2 + 5x_3) \ dx_0 \wedge dx_2 \wedge dx_3 \\
+ (x_0^2(22x_2 + 30x_3) - x_1^2(2x_2 - 5x_3) - 9x_3^2) \ dx_1 \wedge dx_2 \wedge dx_3,
\]

\[
n_2 = (x_0^2(4x_2 + 5x_3) + x_1^2(4x_2 - 5x_3)) \ dx_0 \wedge dx_1 \wedge dx_2 \\
+ (x_0^2(5x_2 + 6x_3) - x_1^2(5x_2 - 6x_3)) \ dx_0 \wedge dx_1 \wedge dx_3 \\
+ x_1(6x_2^2 - 9x_3^2) \ dx_0 \wedge dx_2 \wedge dx_3 + x_0(6x_2^2 - 9x_3^2) \ dx_1 \wedge dx_2 \wedge dx_3
\]
and \( j = 3, 4, \ldots, 42 \). We know that this is a basis for these values of \( j \) because we used a computer to check that the determinant of matrix in (21) is nonzero. The value \( j = 42 \) comes from reducing the first five terms of the Frobenius action series with \( p = 7 \), i.e., for \( k = 4, 4p + 4(p - 1) + pN = 164 = 42N - 4 \). For \( p = 5 \) our algorithm gives \( P(T) = (1 - 5T)(1 + 5T)^4 \) which is easy to guess if one does a point count using Magma. However things get more interesting for \( p = 7 \), so we explain this case in greater detail. As mentioned above, for \( p = 7 \) we can recover \( \zeta(f, T) \) with the reduction of the first five terms (technically three is enough) of the Frobenius action, however the fractions involved are too long to write below. So we give the Frobenius matrix whose \( p \)-adic entries have been truncated at 7

\[
\begin{pmatrix}
2932436 & 3752975 & 2573683 & 0 & 3187818 \\
3326797 & 160280 & 4878860 & 0 & 5469046 \\
273412 & 5678768 & 1729819 & 0 & 1682962 \\
0 & 0 & 0 & 7 + 7^7 & 0 \\
4996579 & 3315242 & 144893 & 0 & 5177634
\end{pmatrix}
\]

The reciprocal characteristic polynomial of the matrix above is

\[
\det(I_5 - MT) = 1 - 10823719T - 36173410616147T^2 + 190881663422782977071T^3 - 307702002432034842717713096T^4 + 148750558587753605666444041808300T^5.
\]

We are interested in the \( p \)-adic expansion of the coefficients of this polynomial which are

\[
1 = 1,
- 10823719 = 3 \cdot 7^0 + 5 \cdot 7^1 + 6 \cdot 7^2 + 6 \cdot 7^3 + 6 \cdot 7^4 + 6 \cdot 7^5 + 5 \cdot 7^6 + 0 \cdot 7^7 + \cdots ,
- 36173410616147 = 5 \cdot 7^1 + 6 \cdot 7^2 + 6 \cdot 7^3 + 6 \cdot 7^4 + 6 \cdot 7^5 + 1 \cdot 7^7 + 1 \cdot 7^8 + \cdots ,
\text{coeff } T^3 = 2 \cdot 7^2 + 0 \cdot 7^3 + 0 \cdot 7^4 + 0 \cdot 7^5 + 0 \cdot 7^6 + 0 \cdot 7^7 + 1 \cdot 7^8 + 5 \cdot 7^9 + \cdots ,
\text{coeff } T^4 = 4 \cdot 7^3 + 1 \cdot 7^4 + 0 \cdot 7^5 + 0 \cdot 7^6 + 0 \cdot 7^7 + 0 \cdot 7^8 + 6 \cdot 7^9 + 3 \cdot 7^{10} + \cdots ,
\text{coeff } T^5 = 6 \cdot 7^5 + 6 \cdot 7^6 + 6 \cdot 7^7 + 6 \cdot 7^8 + 6 \cdot 7^9 + 6 \cdot 7^{10} + 2 \cdot 7^{11} + 3 \cdot 7^{12} + \cdots .
\]

Truncating at the seventh digit gives

\[
P(T) = 1 + (3 + 5 \cdot 7 - 7^2)T + (5 \cdot 7 - 7^2)T^2 + 2 \cdot 7^2 T^3 + (4 \cdot 7^3 + 7^4)T^4 - 7^5 T^5 = 1 - 11T - 14T^2 + 98T^3 + 3773T^4 - 16807T^5 = (1 - 7T)^3(1 - (-5 + 2i\sqrt{6})T)(1 - (-5 - 2i\sqrt{6})T)
\]

and therefore the zeta function of \( f \) for \( p = 7 \) is

\[\zeta(f, T).\]
\[
\zeta(f, T) = \frac{1}{(1 - T)(1 - 7T)(1 - 49T)(1 - 7T)^3(1 + 10T + 49T^2)}.
\]

This took our algorithm, which is not fully automated yet, about 3 h. And similar to the Cayley cubic, a brute force point count is enough to find the zeta function of a Kummer surface so we end this section with two examples where a direct search is not practical.

### 7.3 A quartic with six ordinary double points

By computer experiments the authors found the following quartic surface:

\[
3x_0x_1x_2(x_0 + x_1) + 3x_2^4 - ((2x_0 + x_1)^2 - 6x_1x_2)x_3^2 = 0
\]

which has six ordinary double points

\[
[1 : 0 : 0 : 0], \ [0 : 1 : 0 : 0], \ [0 : 0 : 0 : 1], \ [1 : -1 : 0 : 0],
\]

\[
\left[ -\frac{1}{2} : 1 : 0 : -\sqrt{2} : 4 \right], \ \left[ -\frac{1}{2} : 1 : 0 : \sqrt{2} : 4 \right].
\]

For this surface we can find its zeta function for \( p = 5, 7, 11, 13, 19 \) because only the first three terms in the Frobenius action needed to be reduced in cohomology to recover \( \zeta(f, T) \). In particular, for \( p = 19 \),

\[
19P(T/19) = 19 + 30T - 18T^2 - 77T^3 - 48T^4
\]

\[
+ 48T^5 + 58T^6 - 12T^7 - 12T^8 + 58T^9
\]

\[
+ 48T^{10} - 48T^{11} - 77T^{12} - 18T^{13} + 30T^{14} + 19T^{15},
\]

which is a reciprocal polynomial so for direct computation one needs to know \( \# V(\mathbb{F}_{19^r}) \) for \( r = 1, 2, \ldots, 7 \). It takes Magma 1.172 s to calculate \( \# V(\mathbb{F}_{192}) = 132, 267 \) while calculations over \( \mathbb{F}_{192}^2 \) would take billions of years. Obviously, a better way to compute \( \zeta(f, T) \) is needed. Our algorithm is one such way and when we applied it to this surface we used the following basis for \( E_2^4,4 \):

\[
\{x_0^4, x_0^3x_1, x_0^3x_2, x_0^3x_3, x_0^2x_1^2, x_0^2x_1x_3, x_0^2x_2, x_0^2x_2x_3, x_0^2x_3, x_0x_1^3,
\]

\[
x_0x_1^2x_3, x_0x_1x_2^2, x_0x_2^2x_3, x_0x_3^3\} \ dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3.
\]
Table 3 $Q_3(T)$ for the quartic with six nodes

| $p$ | $Q_3(T)$ |
|-----|----------|
| 5   | $(1 - 5T)(1 + 5T)^2(1 - 4T + 10T^2 - 100T^3 + 625T^4)(1 - 5^4T^4)^2$ |
| 7   | $(1 + 7T)^2(1 + 7^3T^3)(1 - 8T + 77T^2 - 392T^3 + 7^4T^4)(1 - 7^6T^6)$ |
| 11  | $(1 - 11T)^3(1 - 4T + 66T^2 - 484T^3 + 11^4T^4)(1 - 11^4T^4)^2$ |
| 13  | $(1 - 13T)^2(1 - 13^3T^3)(1 + 6T + 143T^2 + 1014T^3 + 13^4T^4)(1 - 13^6T^6)$ |
| 17  | $(1 - 17T)^2(1 + 17T)(1 - 36T + 714T^2 - 1040T^3 + 17^4T^4)(1 - 17T^4)^2$ |
| 19  | $(1 + 19T)^2(1 - 19^3T^3)(1 - 8T - 399T^2 - 2888T^3 + 19^4T^4)(1 - 19^6T^6)$ |

The table below shows the interesting part of the zeta function for all primes between 5 and 19. For $p = 19$ it took the computer 2 days to calculate $Q_3(T)$.

### 7.4 A quintic with 14 ordinary double points

A quintic is of interest because it provides us with the smallest case where $p > 3$ divides the degree of $f$. Our algorithm still applies. Let

$$f = 3x_0^2x_1^2(x_0 + x_1) - x_2x_3(2x_0^3 + 2x_1^3 - x_2^3 - x_3^3) \in \mathbb{Z}[x_0, x_1, x_2, x_3].$$

Then $f$ has 14 ordinary double points listed below

- $[1 : 0 : 0 : 0]$, $[1 : 0 : 0 : \zeta^{k\sqrt{2}}]$, $[1 : 0 : \zeta^{k\sqrt{2}} : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 1 : 0 : \zeta^{k\sqrt{2}}]$, $[0 : 1 : \zeta^{k\sqrt{2}} : 0]$

where $\zeta = e^{2\pi i/3}$ for $k = 0, 1, 2$. Here is the interesting part of the zeta function for primes $p = 5, 7, 11$.

| $p$ | $Q_3(T)$ |
|-----|----------|
| 5   | $(1 - 5T)^9(1 + 5T)^5(1 + 25T^2)^2(1 - 6T + 25T^2)$ |
|     | $(1 + 15T^2 - 100T^3 + 375T^4 + 5^6T^6)(1 + 15T^2 + 100T^3 + 375T^4 + 5^6T^6)$ |
|     | $(1 + 6T + 45T^2 + 200T^3 + 1125T^4 + 3750T^5 + 5^6T^6)$ |
| 7   | $(1 - 7T)^10(1 + 7T)^6(1 - 10T + 49T^2)(1 - 7T + 49T^2)^4$ |
|     | $(1 + 47 + 42T^2 + 196T^3 + 7^4T^4)(1 + 9T + 91T^2 + 441T^3 + 7^4T^4)^2$ |
| 11  | $(1 - 11T)^10(1 + 11T)^4(1 + 121T^2)^2(1 - 3T + 121T^2)$ |
|     | $(1 - T - 44T^2 + 726T^3 - 5324T^4 - 11^4T^5 + 11^6T^6)$ |
|     | $(1 + T + 44T^2 - 726T^3 - 5324T^4 + 11^4T^5 + 11^6T^6)$ |
|     | $(1 + 11T + 187T^2 + 2178T^3 + 22627T^4 + 11^5T^5 + 11^6T^6)$ |

The dimensions of the $E_1$ and $E_2$ spectral sequence terms are as follows:

$$\dim(H^3(K_f^*)_2) = 0, \quad \dim(H^4(K_f^*)_1) = 4,$$
$$\dim(H^3(K_f^*)_7) = 10, \quad \dim(H^4(K_f^*)_6) = 44,$$
\[
\dim(H^3(K_f^*))_{12} = 14, \quad \dim(H^4(K_f^*))_{11} = 14,
\]
\[
\dim(E_2^{4,2}) = 4, \quad \dim(E_2^{4,6}) = 34.
\]

The code of the algorithm and the details of the above and some other examples are available at https://sites.google.com/view/stetson-odp-algorithm/home.

**Acknowledgements** We are grateful to Professor Kiran Kedlaya for sharing the code of his algorithm.

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