We study (non-renormalizable) five dimensional supersymmetric field theories. The theories are parametrized by quark masses and a gauge coupling. We derive the metric on the Coulomb branch exactly. We use stringy considerations to learn about new non-trivial interacting field theories with exceptional global symmetry $E_n$ ($E_8$, $E_7$, $E_6$, $E_5 = \text{Spin}(10)$, $E_4 = \text{SU}(5)$, $E_3 = \text{SU}(3) \times \text{SU}(2)$, $E_2 = \text{SU}(2) \times U(1)$ and $E_1 = \text{SU}(2)$). Their Coulomb branch is $\mathbb{R}^+$ and their Higgs branch is isomorphic to the moduli space of $E_n$ instantons. One of the relevant operators of these theories leads to a flow to $SU(2)$ gauge theories with $N_f = n - 1$ flavors. In terms of these $SU(2)$ IR theories this relevant parameter is the inverse gauge coupling constant. Other relevant operators (which become quark masses after flowing to the $SU(2)$ theories) lead to flows between them. Upon further compactifications to four and three dimensions we find new fixed points with exceptional symmetries.
1. Introduction

It is becoming clear that there is an interesting relation between quantum field theory on branes [1] and space-time dynamics in string theory [1-6]. This relation generalizes the more standard world-sheet/space-time relation. It is particularly interesting when the theory on the brane is a non-trivial quantum theory. In this context the work of [3] used non-trivial dynamics on 3-branes to explain the space-time results of [7]. The extension of these ideas to 2-branes has led to new results in three dimensional quantum field theory [4,5,8]. Here we continue this line of investigation by considering 4-branes.

Superficially, the dynamics on a 4-brane cannot be interesting. The relevant field theory is five dimensional and is believed to be “trivial.” However, we will show that already at one loop order some interesting effects take place. Furthermore, we will argue that certain strongly coupled non-trivial fixed points exist. They exhibit exceptional global symmetries: $E_8$, $E_7$, $E_6$, $E_5 = \text{Spin}(10)$, $E_4 = SU(5)$, $E_3 = SU(3) \times SU(2)$, $E_2 = SU(2) \times U(1)$ and $E_1 = SU(2)$.

In section 2 we review some basic facts about supersymmetric field theories in five dimensions. In section 3 we specialize to $U(1)$ and $SU(2)$ gauge theories with various numbers of flavors and study them in perturbation theory. In section 4 we use these five dimensional field theories in string theory. Our considerations lead us to the new fixed points with $E_n$ symmetry. They are further studied in section 5 where we relate them to similar strongly coupled fixed points in fewer dimensions.

2. Review of Five dimensional SUSY

The spinor representation of $SO(4,1)$ is four dimensional and is pseudoreal. Since the vector of $SO(4,1)$ is in the antisymmetric product of two spinors, the minimal SUSY algebra is generated by two charges. (Extended supersymmetry algebras also exist but will not be discussed here.) It is related by dimensional reduction to the minimal SUSY algebra in six dimensions, to the more familiar $N = 2$ algebra in four dimensions and to $N = 4$ SUSY in three dimensions. As in all these theories, an important role is played by the $SU(2)_R$ automorphism of this algebra under which the two supercharges are a doublet.

The massless representations of this algebra are the hypermultiplet (four real scalars and a spinor) and a vector (a vector, a real scalar and a spinor). The vector representation is equivalent under duality to a tensor representation (a two form, a real scalar and a spinor). Therefore, if we start from six dimensions the vector and the tensor become isomorphic representations in five dimensions.
All these theories have Coulomb branches which are parametrized by the expectation values of the scalars $\phi^i$ in vector multiplets. There can also be Higgs branches where the hypermultiplets vary. They are hyper-Kahler manifolds. The most general Lagrangian (with up to two derivatives) on the Coulomb branch is easily determined. If we reduce to four dimensions, it should satisfy special geometry and hence it is derived from a prepotential $F(A^i)$ which is locally a function of the vector superfields $A^i$. The requirements of five dimensional SUSY can be implemented as follows. In the reduction to four dimensions the fifth components of the vectors $A^i_5$ become scalars thus making $\phi^i$ complex. Invariance under $A^i_5 \to A^i_5 + a^i$ with $a^i$ arbitrary real constants translates to invariance under $A^i \to A^i + ia^i$. This fixes $F$ to be at most cubic

$$F = c_0 + c_i A^i + c_{ij} A^i A^j + c_{ijk} A^i A^j A^k.$$  \hspace{1cm} (2.1)

The constants $c_0$ and $c_i$ do not affect the Lagrangian and can be set to zero. The reality properties of the Lagrangian and the invariance mentioned above restrict the constants $c_{ij}$ and $c_{ijk}$ to be real.

In six dimensions there are both tensor and vector multiplets. It is not known how to write Lagrangians with an arbitrary number of tensor multiplets. If there is only one tensor multiplet $T$ and we add a gravity multiplet, a Lagrangian can be written. It is derived from a prepotential $F = c_{ij} A^i A^j + T^2 + \tilde{c}_{ij} T A^i A^j$ which becomes a special form of (2.1) upon reduction to five dimensions.

Consider for simplicity the case of only one vector multiplet. Then (2.1) takes the form

$$F = \frac{1}{2g^2} A^2 + \frac{c}{6} A^3$$ \hspace{1cm} (2.2)

with the constants $g$ and $c$ being real. The expression (2.2) is valid only locally (we will see this more explicitly below). Therefore, we will not use the freedom to shift $A$ by a constant to set the quadratic term to zero.

In components the first term in (2.2) yields the kinetic terms of the fields in the multiplet whose coefficient $\frac{1}{g^2}$ depends on the gauge coupling $g$. The cubic term leads to terms proportional to

$$c \left( \phi (\partial \phi)^2 + \phi F^2_{\mu \nu} + A \wedge F \wedge F + \cdots \right).$$ \hspace{1cm} (2.3)

It is amusing to note that the $A \wedge F \wedge F$ term is reminiscent of a similar term in the eleven dimensional supergravity Lagrangian. If we ignore this term we can perform a duality

$^1$ Since the $A^i$ are real in five dimensions, we find it convenient to redefine the prepotential by a factor of $i$ compared with the standard four dimensional definition.
transformation similar to the one in four dimensions. The vector multiplet becomes a tensor multiplet which includes a two form $B_{\mu\nu}$ gauge field and a scalar

$$\phi_D = \frac{\partial F}{\partial A}(\phi) = \frac{1}{g^2}\phi + \frac{c}{2}\phi^2. \quad (2.4)$$

The massive spectrum can include BPS saturated states. Their masses are given by the expectation values of $\phi$ and $\phi_D$. Particle are electrically charged and their masses are given by

$$m/\sqrt{2} = Z_e = n_e(\phi + c_e) \quad (2.5)$$

and strings are magnetically charged and their tensions are given by

$$T/\sqrt{2} = Z_m = n_m(\phi_D + c_m). \quad (2.6)$$

Note the freedom in shifting $\phi$ and $\phi_D$ by constants. If there are also global Abelian symmetries, $(2.5)$ can receive other contributions associated with the charges of these symmetries.

3. Perturbative Dynamics

In this section we study $U(1)$ gauge theories with $N_f$ “electron” hypermultiplets of charge one and $SU(2)$ gauge theories with $N_f$ “quark” hypermultiplets ($2N_f$ half-hypermultiplets) in the two dimensional representation. The Coulomb branch of the moduli space of the $U(1)$ theory is $\mathbb{R}$ while for the $SU(2)$ theory it is $\mathbb{R}/\mathbb{Z}_2 = \mathbb{R}^+$. These field theories are not renormalizable. Therefore, they should be viewed as field theories with a cutoff. Even if in the classical theory the cubic term in the prepotential vanishes, it can be generated in the quantum theory \cite{9}. We will refer to this phenomenon as an anomaly. Clearly, $c$ can be generated only at one loop (it is independent of $g$). As a finite quantity it is independent of the cutoff. It is easy to see that only chiral objects contribute to $c$. Therefore, only states in small representations can affect it and a contribution of a hypermultiplet has the same absolute value but the opposite sign to that of a vector multiplet. Also, it is clear that the contribution of any multiplet is proportional to the cube of its charge (the $CP$ conjugate representation has the opposite charge but also the opposite chirality and therefore contributes the same). Therefore, for the $U(1)$ theory $c = -aN_f$ while for the $SU(2)$ theory $c = a(8 - N_f)$ for some constant $a$ which can be determined by an explicit one loop computation. The sign of $a$ is important and turns out...
to be positive. The relevant one loop computation was performed in [9]. We will absorb $a$ in the normalization of the action and the gauge coupling $g$ and set

$$
c = \begin{cases} 
-N_f & \text{for } U(1) \\
2(8 - N_f) & \text{for } SU(2),
\end{cases}
$$

(3.1)

This anomaly term is similar to the standard anomaly in four dimensions and the anomaly discussed in [4, 5] in three dimensions. All of these receive contributions only from massive BPS states with hypermultiplets and vector multiplets contribute with opposite signs. In the $U(1)$ theories all of these anomalies are proportional to $N_f$. Since in four dimensions the contribution of each multiplet is proportional to the square of the charge and in three dimensions it is proportional to the charge, in the $SU(2)$ theory the anomaly is proportional to $2(4 - N_f)$ in four dimensions and to $2(2 - N_f)$ in three dimensions [5].

The expression (2.2) is valid only locally. At various points in the moduli space there can be singularities. For example, in the $U(1)$ theory there is a singularity at $\phi = 0$ where the electrons become massless. In order to extend the Lagrangian beyond that point we use the global symmetry which acts on the scalar as $\phi \rightarrow -\phi$ combined with parity. This is a symmetry of the underlying Lagrangian (in six dimensions it is part of the Lorentz group). The bosonic terms (2.3) are invariant only if they are

$$
c (|\phi|(\partial\phi)^2 + |\phi| F^2_{\mu
u} + \epsilon(\phi) A \wedge F \wedge F + \cdots).
$$

(3.2)

We see that the effective gauge coupling in the $U(1)$ theory is

$$
\frac{1}{g^2_{\text{eff}}} = \frac{1}{g^2} + c|\phi|.
$$

(3.3)

It is continuous but not smooth at $\phi = 0$. The discontinuity in its derivative is proportional to $c$. Since $c$ is negative, for every bare coupling $g$, the effective coupling $g_{\text{eff}}$ diverges at finite points in the moduli space $\phi_s = \pm \frac{1}{cg}$. This divergence reflects the fact that such a quantum field theory is not renormalizable and more data is needed at high energy, of order $\frac{1}{g^2}$, to define it.

It is straightforward to generalize this result to a $U(1)$ gauge theory with several massive electrons with masses $m_i$. (Clearly, we have the freedom to redefine the origin of $\phi$ and thus set one of the masses to zero.) The effective gauge coupling is

$$
\frac{1}{g^2_{\text{eff}}} = \frac{1}{g^2} - \sum_i |\phi - m_i|.
$$

(3.4)
In the $SU(2)$ theory, where the moduli space is modded out by $\phi \to -\phi$, we take $\phi \geq 0$ and there is a singularity at the end of the moduli space at $\phi = 0$. With several quarks with masses $m_i$ the effective gauge coupling is

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{g^2} + 16\phi - \sum_i |\phi - m_i| - \sum_i |\phi + m_i|.$$  \hspace{1cm} (3.5)

Note that the singularity at $\phi = m_i > 0$ is the same as in the $U(1)$ theory with one electron. This is consistent with the low energy theory at that point being $U(1)$ with one electron.

As in the $U(1)$ theories, the $SU(2)$ theories with $N_f > 8$ have singularities in the moduli space reflecting the lack of renormalizability of the field theory. On the other hand, for $N_f \leq 8$ there is no singularity. This suggests that for $N_f < 8$ we can consider the strong coupling limit $g = \infty$ and find a sensible theory. Indeed, below we will describe this theory and will show that upon a small perturbation by $\frac{1}{g^2}$ it flows to the $SU(2)$ theory.

### A Peculiar $U(1)$ Symmetry

In five dimensions the current

$$j = \ast (F \wedge F)$$  \hspace{1cm} (3.6)

is always conserved and the theory has a global $U(1)_I$ symmetry. Its charge is the instanton number, $I$. As in \cite{10}, we introduce parameters as background fields. In this case these are gauge superfields \cite{11} associated with gauging global symmetries. Coupling the conserved current (3.6) to vector superfields we identify the scalar component of this vector superfield, $m_0$, as the gauge coupling $m_0 \sim \frac{1}{g^2}$. In six dimensions $\frac{1}{g^2}$ is in a tensor multiplet and the charged objects are strings. In five dimensions it is in a vector multiplet and the charged objects are particles whose mass is determined by a BPS formula to be related to the expectation value of the background scalar $m_0$. In the vacuum of the $SU(2)$ theory with $\phi = 0$ these are the four dimensional instantons which appear as particles (zero branes) in five dimensions\footnote{These instantons have a non-compact dilation zero mode. Its quantization leads to a continuous spectrum of particles. The quantization of the fermion zero modes makes them spinors of $Spin(2N_f)$. The presence of massless charged fields makes the interpretation of this spectrum confusing.}. In the vacua with $\phi \neq 0$ the instantons tend to shrink and their detailed properties depend on the short distance physics, which depends on the way the theory is
regularized. In particular, there could be various BPS bound states. This dependence on unknown short distance physics is another manifestation of the lack of renormalizability of the theory.

Related to the fact that the global $U(1)_I$ symmetry with charged particles exists in five dimensions is the fact that only in $d = 5$ is the dimension of $m_0 \sim \frac{1}{g^2}$ exactly one. $m_0$ behaves like the quarks mass terms which can also be thought of as background vector superfields. The induced cubic term $A^3$ in (2.2) leads to interesting consequences for this symmetry. It leads to a coupling of the corresponding current (3.7) to the dynamical gauge field $A$. In other words, the instantons carry gauge charge! One way to see that is to note that locally we can change $m_0$ by shifting $A$.

The central charge in the supersymmetry algebra which determines the masses of BPS states is a linear combination of all $U(1)$ symmetries whether gauged or not. Therefore, superficially

$$Z^{(0)} = I_3 \phi + Im_0$$  \hspace{1cm} (3.7)

where $I_3$ is the electric charge (the value in the Cartan subalgebra in the $SU(2)$ theory) and $I$ is the instanton charge. The mixing of the two symmetries converts $m_0 \sim \frac{1}{g^2} \rightarrow \frac{1}{g_{eff}^2}$ and therefore

$$Z = (I_3 + cI) \phi + Im_0.$$  \hspace{1cm} (3.8)

We would also like to point out that in the $SU(2)$ theory there are strings. In four dimensions these theories have magnetic monopoles. These classical configurations become strings in five dimensions. Their tension is controlled by the BPS formula (2.6).

4. String considerations

One context in string theory where these theories are important is in compactifications of the type I theory on $S^1$. We use as a probe the ten dimensional 5-brane and wrap it around the compact circle to produce a 4-brane in nine dimensions. After dualizing the circle, the theory is the type I’ compactified on $S^1/Z_2$ with two orientifolds at the ends \[1\]. Our 4-brane probe can be understood then as the 4-brane of the IIA theory localized at a point in $S^1/Z_2$. The space-time moduli in the type I language are the size of the original $S^1$ and Wilson lines. The latter are 16 phases. After the duality, the moduli are the the size of $S^1/Z_2$ and the locations of 16 background D8-branes \[2\]. The theory on the 4-brane probe is an $SU(2)$ gauge theory in five dimensions (and a few free fields). The modulus of this theory is the location in $S^1/Z_2$. At the two endpoints the $SU(2)$ gauge symmetry is restored. It is broken to $U(1)$ in between the two points. The $SU(2)$ gauge
theory is coupled to \( N_f = 16 \) hypermultiplets. Their masses, which are parameters of the theory on the brane, are the locations of the sixteen background D8-branes.

For a generic radius, special points in the space-time moduli space occur when \( N_f \) D8-branes are at the same point on \( S^1/\mathbb{Z}_2 \). Then, on the 4-brane probe we get \( U(1) \) with \( N_f \) electrons. Another special point is when \( N_f \) D8-branes coincide with an orientifold. Then the theory on the 4-brane probe is \( SU(2) \) with \( N_f \) quarks. The global enhanced symmetry in these cases, \( SU(N_f) \) and \( SO(2N_f) \) respectively, are enhanced gauge symmetries in space-time. The Higgs branches which emanate from the singularities are the moduli spaces of space-time instantons in these enhanced gauge symmetries.

The preceding two paragraphs are completely analogous to the discussion in [3] for 3-brane probes and in [4] for 2-brane probes which arise in similar fashions. One aspect, which was not stressed in [3,4], is the space-time meaning of the anomaly – the constant \( c \). As we said above, this anomaly is similar to the ordinary anomaly in four dimensions and the anomaly discussed in [4,5] in three dimensions. All of them arise at one loop. The BPS particles which run in the loop originate in string theory from strings stretched between the probe and the orientifolds (vector multiplets) and D-branes (hypermultiplets). In string theory this diagram is an annulus diagram. As explained in [1,2], this diagram can be viewed in the cross channel as a tree level exchange between the probe and the orientifold or the background D-branes. In space-time, this tree level effect leads to a velocity dependent force between our probe and the background orientifolds and D-branes. These background branes are charged under some space-time gauge symmetries. In units where the charge of every background D-brane is \(-1\), the charge of the orientifold is 8 in nine dimensions, 4 in eight dimensions and 2 in seven dimensions. Therefore, the charge of \( N_f \) background D-branes is \(-N_f\) and the charge of \( N_f \) background D-branes coinciding with an orientifold is \( 8 - N_f \) in nine dimensions, \( 4 - N_f \) in eight dimensions and \( 2 - N_f \) in seven dimensions. This is precisely the value of the anomaly in the theory on the various probes we used – 4-brane in nine dimensions, 3-brane in eight dimension [3] and 2-brane in seven dimensions [4] (up to a trivial factor of two at the orientifolds).

The global \( U(1)_I \) symmetry associated with instanton number on the brane also has an obvious interpretation in this space-time description. Like every global symmetry on the brane, this is a gauge symmetry in space-time. In order to identify this symmetry, recall that in the type I theory the gauge coupling in the theory on the probe satisfies

\[
\frac{1}{g^2} \sim \frac{RM_s^2}{\lambda_I} \sim \frac{M_s}{\lambda_I'} \quad (4.1)
\]

where \( \lambda_I \) (\( \lambda_{I'} \)) is the dimensionless type I (type I') coupling constant, \( R \) is the radius of \( S^1 \) and \( M_s \) is the string scale. Therefore, changing \( R \) for fixed \( \lambda_I \) amounts to changing
the gauge coupling. The corresponding gauge field in space-time is the $U(1)$ superpartner of the space-time modulus $R$. A simple check of this identification is the following. An instanton on our 4-brane is a particle. In the underlying type I theory, this is an instanton on the 5-brane and therefore it is a string. This string was identified in [13] as the heterotic string. In our case it wraps the compact $S^1$. Therefore, it is charged under the $U(1)$ of winding number of the dual heterotic string.

Consider now a configuration of background D8-branes such that $n_L$ of them are at one orientifold, at $\phi = 0$, $p$ of them are between the two orientifolds at $\phi_i$ ($i = 1, ..., p$) and $n_R = 16 - p - n_L$ coincide with the other orientifold at $\phi = \frac{1}{R}$. Let us probe this system with our D4-brane probe. Near $\phi = 0$ the theory on the probe is $SU(2)$ with $n_L$ flavors. Near $\phi_i$ it is $U(1)$ with a massless electron. We could also think about it as $SU(2)$ with $n_L + p$ flavors, $p$ of them have masses $\phi_i$. The effective gauge coupling is

$$
\frac{1}{g_{\text{eff}}^2(0)} = \begin{cases} 
\frac{1}{g_{\text{eff}}^2(\phi)} + 2(8 - n_L)\phi & \text{for } 0 < \phi < \phi_1 \\
\frac{1}{g_{\text{eff}}^2(\phi_i)} + 2(8 - n_L - i)(\phi - \phi_i) & \text{for } \phi_i < \phi < \phi_{i+1} \\
\frac{1}{g_{\text{eff}}^2(\phi_p)} + 2(8 - n_L - p)(\phi - \phi_p) & \text{for } \phi_p < \phi < \frac{1}{R} 
\end{cases}
$$

Using (4.1) this change in the gauge coupling on the 4-brane probe as a function of the moduli translates to a space dependent dilaton field $\lambda_I'(\phi)$ in space-time. This variation has already been observed in [14]. For sufficiently large $R$ the effective gauge coupling $g_{\text{eff}}^2(\phi)$ and therefore also $\lambda_I'(\phi)$ remain finite. However, at some $R_0$ which depends on $g(0)$, $n_L$ and $\phi_i$ they diverge. For example, for $p = 0$ and $n_L > 8$ they diverge when

$$
\frac{1}{g_{\text{eff}}^2(\phi = \frac{1}{R_0})} = \frac{1}{g^2(0)} + 2(8 - n_L)\frac{1}{R_0} = 0
$$

or when

$$
\frac{1}{\lambda_I'(0)} \sim n_L - 8.
$$

We express it in terms of the dual heterotic string variables by using $R_0^2 = \lambda_I R_{0h}^2$, ($R_{0h}$ is the corresponding heterotic radius) to find the condition $R_{0h}^2 M_s^2 \sim n_L - 8$ for enhancement of the gauge symmetry in the heterotic theory. This derivation parallels the space-time analysis of [14]. It extends it by showing that the result is exact; i.e. there are no instanton corrections similar to those in [7].

This computation for $p = 0$ and $n_L > 8$ is easily generalized to $p$ branes at $\phi_i$. As we make $R$ smaller, the divergence first occurs at an orientifold (which we will take to be the right one) with $n_R < 8$. Let us consider the theory on the 4-brane probe for finite
\( g_0 = g_{\text{eff}}(\frac{1}{R}) \) near the orientifold at \( \phi = \frac{1}{R} \). It is an \( SU(2) \) theory with \( N_f = n_R < 8 \) flavors. Its modulus is \( \phi_R = \frac{1}{R} - \phi \geq 0 \) and its effective coupling is

\[
\frac{1}{g_{\text{eff}}^2(\phi_R)} = \frac{1}{g_0^2} + 2(8 - N_f)\phi_R.
\]

The long distance dynamics of the theory on the brane is always a local quantum field theory. Therefore, even as \( g_0 \) diverges, the theory on the brane must make sense as a quantum field theory. It must be at a fixed point of the renormalization group. As \( g_0 \) diverges, the string theory acquires an enhanced gauge symmetry. Correspondingly, the theory on the brane should acquire enhanced global symmetry. Therefore, for \( g_0 = \infty \) the theory on the brane is at a non-trivial fixed point with enhanced global symmetry.

For finite \( g_0 \) the symmetry of the five dimensional theory is \( SO(2N_f) \times U(1) \). We will argue that the fixed point corresponding to \( g_0 = \infty \) has an enhanced \( E_{N_f+1} \) symmetry (\( E_8, E_7, E_6, E_5 = \text{Spin}(10), E_4 = SU(5), E_3 = SU(3) \times SU(2), E_2 = SU(2) \times U(1) \) and \( E_1 = SU(2) \)). We will do that by examining the symmetries of the underlying string theory.

When the \( SO(32) \) heterotic string is compactified on \( S^1 \) an \( E_8 \) symmetry is obtained as follows. An \( SO(14) \times U(1) \) subgroup of \( SO(32) \) is combined with the left moving \( U(1) \) associated with the \( S^1 \). For special values of the Wilson lines and the radius, this \( SO(14) \times U(1) \times U(1) \) is enhanced to \( E_8 \times U(1) \). The new gauge bosons which become massless at that point are winding modes of the heterotic string. Upon heterotic/type I/type I' duality a similar enhancement is expected in the type I' theory \([1]\). It occurs at a point where the type I' coupling \( \lambda_{I'} \) diverges at one of the orientifolds (if it diverges at both orientifolds the symmetry is enhanced to \( E_8 \times E_8 \)). This is exactly the phenomenon we discussed above. Therefore, for \( N_f = 7 \) the strongly coupled theory on the probe has global \( E_8 \) symmetry.

Now it is easy to consider smaller values of \( N_f \). In terms of string theory Wilson lines break \( E_8 \) to a subgroup. In the type I' language this is represented by moving some of the background D8-branes from the orientifold. In terms of the theory on the 4-brane probe, this corresponds to giving masses \( m_i \) to some of the quarks. Examining the symmetry breaking pattern in string theory we see that for general \( N_f \leq 7 \) the enhanced symmetry is \( E_{N_f+1} \). For example, the lowest non-trivial point has \( SU(2) \) symmetry which occurs in string theory at the self dual radius. In the type I' theory there is no background D8-brane at the orientifold, and in the 4-brane probe theory this corresponds to a strong coupling fixed point with \( N_f = 0 \). \( U(1)_I \subset SU(2) \) is generated by the instanton charge. For \( g = \infty \) it is enhanced to \( SU(2) \).
The Higgs branch of the theory on the brane is the moduli space of space-time instantons. For the “trivial” fixed points with $SU(N_f)$ or $SO(2N_f)$ symmetry, this fact is as in [14]. Our analysis shows that for the special points with exceptional symmetries the Higgs branch again corresponds to the moduli space of instantons, this time for the exceptional groups.

These theories also appear in another way in string theory. Consider a compactification of the heterotic string on K3 with a small instanton. Then, the six dimensional theory has enhanced gauge symmetry for $SO(32)$ instantons [15] or is the mysterious tensionless string theory for small $E_8$ instantons [16,17]. Upon further compactification to five dimensions we find the five dimensional theories discussed here.

5. Strong coupling fixed points

In the previous section we found new fixed points with global $E_n$ symmetry. The parameters in these theories are background gauge fields in the Cartan subalgebra of $E_n$, $m_i$ ($i = 0, ..., n - 1$). Turning on $m_0$ we flow to the $SU(2)$ theory with $N_f = n - 1$ flavors where $m_0$ is interpreted there as the (irrelevant parameter) $\frac{1}{g}$. The other parameters $m_i$ for $i = 1, ..., N_f$ are interpreted as the quark masses in this IR theory. Other deformations of the $E_n$ fixed point theory are obtained by turning on some of $m_i$ for $i = p, ..., N_f$. The theory then flows to the $E_p$ fixed point. If we now turn on $m_0$ at this fixed point, we flow to an $SU(2)$ gauge theory with $p - 1$ flavors.

As we argued in the previous section, these $E_n$ fixed points have $E_n$ global symmetry where $E_5 = Spin(10)$, $E_4 = SU(5)$, $E_3 = SU(3) \times SU(2)$, $E_2 = SU(2) \times U(1)$ and $E_1 = SU(2)$. Their the Higgs branches are isomorphic to the moduli spaces of $E_n$ instantons. The Coulomb branches are $\mathbb{R}^+$. 

One might ask whether all these theories are new. In particular, perhaps some of them are the same as known theories with $SU(2)$ or $U(1)$ gauge symmetry. These have global symmetries $SO(2N_f)$ or $SU(N_f)$ and perhaps can be identified with some of the $E_n$ theories. Furthermore, the Higgs branches are the same. However, the Coulomb branches of the $U(1)$ gauge theories are $\mathbb{R}$ while those of $E_n$ are $\mathbb{R}^+$. This leaves only the $E_5$ theory as a potential $SU(2)$ theory. The relevant operator in $SU(2)$ with $N_f = 5$ which breaks the symmetry to $SU(5) \times U(1)$ takes it to a $U(1)$ theory whose moduli space is $\mathbb{R}$. In the $E_5$ theory such an operator takes us to the $E_4$ theory whose Coulomb branch is $\mathbb{R}^+$. Therefore, we conclude that all these theories are at interacting fixed points of the renormalization group and are new field theories.

It is interesting to examine these theories in various dimensions. In six dimensions, there is a strange theory (small $E_8$ instantons) which involves tensionless strings [14,17].
It is expected to be a non-trivial field theory [17] with global $E_8$ symmetry. Since the parameters are always background gauge superfields, and in six dimensions there is no scalar in the vector multiplet, these six dimensional theories do not have relevant operators which preserve the super-Poincare symmetry.

Upon compactification to five dimensions we find the $E_8$ theory described here. Unlike the situation in six dimensions, in five dimensions we can describe the theory along the Coulomb branch by a Lagrangian (in six dimensions there is a massless self-dual two form which does not have a Lorentz invariant Lagrangian description). Another difference is that in five dimensions we can perturb the theory by relevant operators whose coefficients are $m_i$. In terms of six dimensional background gauge fields, these are Wilson lines around the compact dimension which are scalars in five dimensions. This allows us to find the series of $E_n$ theories.

Upon further compactification to four dimensions we find new non-trivial theories with $E_n$ global symmetry. Now the parameters $m_i$ become complex. The Coulomb branch is one complex dimensional and the elliptic curve describing the gauge coupling has an $E_n$ singularity. The case with $E_6$ in four dimensions was recently discussed in [18].

Upon further compactification to three dimensions the parameters $m_i$ become three vectors. The Coulomb branch is a Hyper-Kahler manifold with an $E_n$ singularity [4,8]. As explained in [5], it is the four real dimensional auxiliary space discussed in section 17 of [19].

It is not known whether there are free field theories which flow to all these non-trivial fixed points in more than three dimensions. However, in three dimensions such free field theories are known [8]. The existence of this UV free field theory which flows to the non-trivial fixed points proves that they are indeed local quantum field theories thus strengthening the claim (based on string theory) that all these theories in all dimensions are local quantum field theories.

As we submitted this note, we received an interesting paper [20] which addresses related issues from a different point of view.

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