ISOGENIES OF JACOBIANS

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Abstract. We prove by means of the study of the infinitesimal variation of Hodge structure and a generalization of the classical Babbage-Enriques-Petri theorem that the Jacobian variety of a generic element of a $k$ codimensional subvariety of $M_g$ is not isogenous to a distinct Jacobian if $g > 3k + 4$. We extend this result to $k = 1, g \geq 5$ by using degeneration methods.

1. Introduction

Let $Z$ be a subvariety of the moduli space $M_g$ of complex smooth curves of genus $g$ of codimension $k > 0$. We want to show that under some numerical restrictions, the Jacobian of a generic element of $Z$ is not isogenous to a distinct Jacobian. In other words, all the curves of genus $g$ contained in $JC$, with $C$ generic in $Z$, are birationally equivalent. This is an extension of the Theorem proved by Bardelli and Pirola (see [BP]) for the whole $M_g$ and can be seen as a Noether-Lefschetz locus problem for surfaces which are the product of two curves of the same genus (see Corollary (6.2)). More precisely, our result is as follows:

**Theorem 1.1.** Let $Z \subset M_g$ a codimension $k > 0$ subvariety. Assume that $g > 3k + 4$ (in particular $g > 7$), then the Jacobian of a generic curve $C$ of $Z$ is not isogenous to another Jacobian. The same is true for $k = 1$ and $g \geq 5$.

Observe that the Theorem fails for $g = 4$ and $k = 1$: in this case $M_4$ is a divisor in $A_4$, therefore intersecting in the Siegel upper space $H_4$ the Jacobian locus $J_4$ with the image $j(J_4)$ by the action of a fixed isogeny $j$, we get a divisor in $M_4$ where the Jacobian of a generic element is isogenous to a different Jacobian.

For $g > 3k + 4$, our strategy is as follows: after a base change we have two families of smooth complex curves of genus $g$ on a base variety $W$, $\pi : C \to W$ and $\pi' : C' \to W$, and a family of isogenies of the associated family of Jacobians, that is

$\chi : J(C') \to J(C)$.

This means that for $t \in W$ the map $\chi_t : J(C'_t) \to J(C_t)$ is an isogeny, here $C'_t = \pi'^{-1}(t)$ and $C_t = \pi^{-1}(t)$.

It follows that the associated rational Hodge structures are isomorphic. Consider the local (polarized) systems $\Lambda_Z = R^1\pi_*Z$ and $\Lambda'_Z = R^1\pi'_*Z$ and tensoring by $C$ $\Lambda_C = R^1\pi_*C$ and $\Lambda'_C = R^1\pi'_*C$.

In particular the infinitesimal variation of Hodge structure associated to the Hodge filtration of $\Lambda^{1,0} \subset \Lambda_C$ and $\Lambda'^{1,0} \subset \Lambda'_C$ are isomorphic. We borrowed this basic observation from Claire Voisin (see Remark (4.2.5) in [BP]). It is well known

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that the infinitesimal invariant of Hodge structure of curves determines the quadrics that contains a canonical curve (see [CGGH]). This allows to translate our problem to a geometric one. Let $I(2)$ and $J(2)$ be the space of quadrics that contains the canonical curve associated to $C_t$ and $D_t$. It follows under a choice suitable canonical embedding that $I(2) \cap J(2)$ has codimension $\geq c$, where $c$ is the codimension of $m(W)$ in $M_g$ and $m$ is the modular mapping $m : W \to M_g$. We can bound the codimension $c$ by using the Clifford index. For this we prove a result that give an interesting (at least in our opinion) reconstruction result of the curve from partial system of quadrics. It is a generalization of the Babbage-Enriques-Petri theorem (see e.g. chapter 3, section 3 in [ACGH]).

**Theorem 1.2.** Let $C$ be a generic curve in a codimension $k$ subvariety $Z$ of $M_g$. Let $I_2 \subset \text{Sym}^2 H^0(C, \omega_C)$ be the vector space of the equations of the quadrics containing $C$. Let $K \subset I_2$ be a linear subspace of codimension $k$. If $g > 2k + 3$ and the Clifford index $c$ of $C$ satisfies $c > k + 1$, then $C$ is the only irreducible non-degenerate curve contained in the intersection of the quadrics of $K$.

**Corollary 1.3.** Let $C$ be a generic curve in a codimension $k$ subvariety $Z$ of $M_g$. Let $I_2 \subset \text{Sym}^2 H^0(C, \omega_C)$ be the vector space of the equations of the quadrics containing $C$. Let $K \subset I_2$ be a linear subspace of codimension $k$. If $g > 3k + 4$, then $C$ is the only irreducible non-degenerate curve contained in the intersection of the quadrics of $K$.

The Corollary is an easy consequence of the Theorem 1.2. Indeed, let $c$ be the Clifford index of a generic element of $Z$. The locus $Z$ can have several components and the minimal codimension is attained when $c$ is realized by a $g_1^d$ linear series, with $c = d - 2$. Then, by Riemann-Hurwitz the codimension of this component is $3g - 3 - (2g - 2 + 2d - 3) = g - 2c - 2$. Hence, since we assume $g > 3k + 4$, we have

$$k \geq g - 2c - 2 > 3k + 4 - 2c - 2 = 3k - 2c + 2.$$ 

Therefore $c > k + 1$ and the result follows from 1.2.

In section 2 we start the proof of the Theorem 1.2 under the hypothesis $g > 3k + 4$ by reducing it to the Corollary 1.3 following Voisin’s observation indicated above. The Theorem 1.2 will be proved in the section 3. The idea of the proof is as follows: assuming the existence of a second curve non degenerate in the intersection of the quadrics we select linearly independent points $x_i$ in this curve. Then, by constructing a suitable 2-rank vector bundle on $C$ we are able to find points $p_j \in C$ such that the linear span of the points $x_i$ is contained in the linear span of the points $p_j$. From this is easy to obtain a contradiction by using of theorem of Ran [R].

To prove the divisorial case of the main theorem, we use the original approach in [BP], based on the analysis of the map

$$\chi_Z : \Lambda_Z \to \Lambda'_Z$$

(in fact we will work with the dual lattices, that is, with the homology groups). If we can prove that $\chi_Z(\Lambda_Z) = n\Lambda'_Z$, $n \in \mathbb{Z}$ we will get that the $C'_t$ is isomorphic to $C_t$ and $\chi_t$ is given by the multiplication by $n$. We use degeneration to $\Delta_0$, and study the monodromy action on $\Lambda_Z$, the basic geometric information is now encoded on the generalized Jacobians. Roughly speaking one has to prove that
part of the limit map $\chi_0$ is the multiplication by $n$. This gives that the map $\chi_t$ is the multiplication by $n$ on a part of the invariant cycles. We need finally to have degenerations with independent monodromy to complete the proof. It is clear that to follow this strategy one needs to control the degeneration type. Using the theory of divisors on $M_g$ and following a precious suggestion of Gavril Farkas, we realize the above program when $c = 1$, the divisors case of $M_g$. The degeneration procedure is performed in sections 4, 5 and 6. We will prove the existence of a type of degeneration (if $c = 1$) to the union of a curve of genus $g - 2$ and to two elliptic generic tails. The independent degenerations to $\Delta_0$, are obtained by letting the elliptic tails becomes singular. To extract more information from the degeneration we analyze the type of involved monodromy that we classify in three cases (a,b,c of section 5). Then we analyze the geometric insight of the generalized Jacobians by comparing their extension classes. In section 6 we complete the proof by comparing the invariant of the two degenerations.

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2. Reduction to a problem on quadrics through the canonical curve

The aim of this section is to prove that the Corollary 1.3 implies the Theorem 1.1 under the hypothesis $g > 3k + 4$ and $k > 0$.

Remember that an isogeny $\chi : A' \rightarrow A$ between principally polarized Abelian varieties $(A', L_{A'})$ and $(A, L_A)$ such that $\chi^* L_A \cong L_{A'}\otimes m$ is determined by a subgroup $H$ of the group of $m$-torsion points $A'_m$ totally isotropic with respect to the Riemann bilinear form

$$e_m : A'_m \times A'_m \rightarrow \mu_m$$

(being $A = A'/H$) and a level subgroup $\tilde{H}$ of the theta group $G(L_{A'}^\otimes m)$, see ([M], chapter 23). Then the moduli space of those isogenies can be rewritten as

$$\tilde{A}_g^m = \{ \chi : A' \rightarrow A, \chi^* L_A \cong L_{A'}\otimes m \} / \cong = \{(A', L_{A'}; H, \tilde{H})\} / \cong$$

and the forgetful map is a finite covering $\varphi : \tilde{A}_g^m \rightarrow A_g$. Moreover the map $\psi : \tilde{A}_g^m \rightarrow A_g$ sending $\chi : A' \rightarrow A$ to $(A, L_A)$ is another covering space.

Given a generic isogeny $\chi : A' \rightarrow A$ we consider tangent spaces in the following diagram:

$$\tilde{A}_g^m \xrightarrow{\varphi} A_g$$

$$\psi$$

$$\tilde{A}_g$$
and we get an isomorphism $\lambda$ as follows:

$$\begin{align*}
T_{\tilde{A}_g, \chi} & \xrightarrow{d\psi} T_{A_g, \lambda} \cong \text{Sym}^2 H^0(A, T_A) \\
& \downarrow \lambda \\
T_{A_g, A'} & \xrightarrow{d\varphi} S\text{ym}^2 H^0(A', T_{A'})
\end{align*}$$

Coming back to our problem let us assume that the locus of curves in $\mathcal{M}_g$ with Jacobian isogenous to the Jacobian of some curve in $Z$ contains a codimension $k$ component $\mathcal{Z}' \subset \mathcal{M}_g$. Our hypothesis on $k$ implies that a generic element $C' \subset Z$ satisfies $\text{End}(JC) \cong \mathbb{Z}$ (see [CGT] or [P]). Therefore an isogeny $\chi : JC' \longrightarrow JC$ must satisfy that the pull-back of the principal polarization in $JC$ is a multiple of the principal polarization in $JC'$. Hence there exists an integer $m$ and an irreducible variety $\mathcal{R} \subset \tilde{A}_g^m$ dominating $\mathcal{Z}'$ and $Z$ through $\varphi$ and $\psi$ respectively. Set $\mathcal{M} := \varphi^{-1}(\mathcal{M}_g)$ and $\mathcal{M}' := \psi^{-1}(\mathcal{M}_g)$. Then $\mathcal{R} \subset \mathcal{M} \cap \mathcal{M}'$. Fix a generic element $\chi : JC' \longrightarrow JC$ in $\mathcal{R}$. In the following diagram we consider in the first row the natural inclusions of tangent spaces at $\chi$ and we put in the second row its image by $d\varphi$:

$$\begin{align*}
T_{\mathcal{R}, \chi} & \xrightarrow{\xi} T_{\mathcal{M}, \chi} \cong T_{\mathcal{M}, \chi} + T_{\mathcal{M}', \chi} \cong T_{\tilde{A}_g, \chi} \\
& \cong T_{\mathcal{Z}, JC} \cong T_{\tilde{A}_g, JC} = H^0(C, \omega_C^2)^* \longrightarrow \text{Sym}^2 H^0(C, \omega_C)^*
\end{align*}$$

Observe that, by the Grassmann formula, the dimension of $\bar{T}$ is at most $3g - 3 + k$. Set $K(C) := \text{Kernel}(\text{Sym}^2 H^0(C, \omega_C) \longrightarrow \bar{T}^*)$, this is a subspace of the vector space $I_2(C)$ of the quadrics containing the image of $C$ by the canonical map. The codimension of $K(C)$ in $I_2(C)$ is at most $k$. By using $\psi$ instead of $\varphi$ we get the corresponding vector space $K(C') \subset I_2(C')$ and we obtain a canonical isomorphism $K(C) \cong K(C')$. Then the Corollary [13] implies that $C$ and $C'$ are isomorphic and, since $\text{End}(JC) = \mathbb{Z}$, the isogeny is a multiple of the identity.

3. High codimension family of quadrics through the canonical curve

This section is devoted to the proof of the Theorem [12]. We fix the notation $K \subset I_2$ of the statement. We assume that the intersection of all the quadrics of $K$ contains an irreducible non-degenerate curve different from $C$. In particular we can select $k + 1$ linearly independent points $x_i \in \bigcap_{Q \in K} Q \subset \mathbb{P} H^0(C, \omega_C)^*$ such that $x_i \notin C$. We choose a representative of $x_i$ in $H^0(C, \omega_C)^*$ and we denote it with the same symbol. Then $x_i \otimes x_i \in \text{Sym}^2 H^0(C, \omega_C)^*$. We denote by $L$ the linear variety spanned by these points.
Let $R, R'$ be the quotients $I_2/K$ and $\text{Sym}^2 H^0(C, \omega_C)/K$ respectively. Then we have the diagram of vector spaces

$\begin{array}{ccc}
0 & \to & K \\
\downarrow & & \downarrow \\
0 & \to & I_2 \\
\downarrow & & \downarrow \\
S\text{ym}^2 H^0(C, \omega_C) & \longrightarrow & S\text{ym}^2 H^0(C, \omega_C) \\
\downarrow & & \downarrow \\
0 & \to & R \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}$

and its dual

$\begin{array}{ccc}
0 & \to & R_* \\
\downarrow & & \downarrow \\
0 & \to & I_2^* \\
\downarrow & & \downarrow \\
S\text{ym}^2 H^0(C, \omega_C)^* & \longrightarrow & S\text{ym}^2 H^0(C, \omega_C)^* \\
\downarrow & & \downarrow \\
0 & \to & K^* \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}$

Since all the quadrics of $K$ vanish on $x_i$ then the image of $L$ in $K^*$ is zero, hence $L \subset R^*$. Since $L$ has dimension $k + 1$ and, by the hypothesis on $K$, $\dim R = k$ we get that $H^1(C, T_C) \cap L \neq (0)$. Let $\alpha$ be a non-trivial element in this intersection. Looking at $H^1(C, T_C) = \text{Ext}^1(\omega_C, \mathcal{O}_C)$ as classes of extensions we get attached to $\alpha$ a two rank vector bundle $E_\alpha$ and a short exact sequence:

$0 \longrightarrow \mathcal{O}_C \longrightarrow E_\alpha \longrightarrow \omega_C \longrightarrow 0.$

The coboundary map $H^0(C, \omega_C) \longrightarrow H^1(C, \mathcal{O}_C)$ is the cup-product with $\alpha$. Since $\alpha \in L$, then $\alpha = \sum_{i=1}^{k+1} a_i x_i \otimes x_i$. Therefore, denoting by $H_i$ the kernel of the form $x_i : H^0(C, \omega_C) \longrightarrow \mathbb{C}$, the intersection $H_1 \cap \cdots \cap H_{k+1}$ is contained in $Ker(\cdot \cup \alpha)$, in fact

\begin{equation}
Ker(\cdot \cup \alpha) = \bigcap_{i \text{ with } a_i \neq 0} H_i.
\end{equation}

We can assume that $x_1, \ldots, x_{k'}, k' \leq k + 1$, are the points such that $a_i \neq 0$. Then there are $g - k'$ sections of $H^0(C, \omega_C)$ lifting to $E_\alpha$. Since the injection $\mathcal{O}_C \hookrightarrow E_\alpha$ provides a new section linearly independent with the previous ones we get:

$\dim H^0(C, E_\alpha) \geq g - k' + 1.$
We consider the wedge product of sections of $E_\alpha$:

$$\psi: \Lambda^2 H^0(C, E_\alpha) \to H^0(C, \det E_\alpha) = H^0(C, \omega_C).$$

The hypothesis $g > 2k + 3$ implies that the projectivization of the kernel of $\psi$ (which has codimension at most $g$) intersects in $P(\Lambda^2 H^0(C, E_\alpha))$ the Grassmannian of the decomposable elements. Hence there are two sections $s_1, s_2 \in H^0(C, E_\alpha)$ such that $s_1 \wedge s_2 = 0$. This means that they generate a rank 1 torsion free sheaf $L_\alpha \subset E_\alpha$, hence a line bundle. By construction $h^0(C, L_\alpha) \geq 2$. We get a diagram:

```
(2) 0 → L_\alpha → 0
    ↓                ↓
    E_\alpha → C → 0
    ↓                ↓
    \omega_C ∩ L_\alpha^{-1} → \omega_C
    ↓                ↓
    0
```

Observe that the existence of the map $\rho$ implies that $h^0(C, \omega_C \otimes L_\alpha^{-1})$ is positive. We distinguish two cases:

Case 1: $h^0(C, \omega_C \otimes L_\alpha^{-1}) \geq 2$. Then we can use $L_\alpha$ to compute the Clifford index of the curve. We have:

$$h^0(C, L_\alpha) + h^0(C, \omega_C \otimes L_\alpha^{-1}) \geq h^0(C, E_\alpha) \geq g - k' + 1$$

that combined with Riemann-Roch gives $2h^0(C, L_\alpha) \geq \deg(L_\alpha) + 2 - k'$. Therefore

$$\deg(L_\alpha) - 2h^0(C, L_\alpha) + 2 \leq k' \leq k + 1 < c$$

which contradicts the definition of $c$.

Case 2: $h^0(C, \omega_C \otimes L_\alpha^{-1}) = 1$. Then $h^0(C, L_\alpha) \geq g - k'$. Let $e$ be the degree of $\omega_C \otimes L_\alpha^{-1} \cong O_C(p_1 + \cdots + p_e)$.

Claim: we have that $e \leq k'$.

Indeed, since $\rho$ induces an isomorphism $H^0(C, O_C) \cong H^0(C, \omega_C \otimes L_\alpha^{-1})$, then the map $H^0(C, E_\alpha) \to H^0(C, \omega_C \otimes L_\alpha^{-1})$ is surjective and we get:

$$g - k' \leq h^0(C, L_\alpha) = h^0(C, \omega_C \otimes L_\alpha^{-1}) + 2g - 2 - e + 1 - g = g - e,$$

the claim follows.

Coming back to the diagram (2) we get that

$$H^0(C, L_\alpha) = H^0(C, \omega_C(-p_1 - \cdots - p_e)) \subset Ker(\cup \alpha) = \bigcap_{i=1, \ldots, k'} H_i.$$
By dualizing, we get that the inclusion of linear spans
\[ \langle x_1, \ldots, x_{k'} \rangle \subset \langle p_1, \ldots, p_e \rangle. \]

We denote by \( \widetilde{C} \) a non-degenerate irreducible curve, \( \widetilde{C} \neq C \), contained in all the quadrics parametrized by \( K \). To simplify we assume that \( \widetilde{C} \) is also smooth, otherwise \( \widetilde{C} \) would be the normalization of this second curve and the forthcoming argument has to be modified slightly.

By choosing generically the \( k + 1 \) points \( x_i \in \widetilde{C} \), we can assume that \( k' \) and \( e \) are constant, so the correspondence:
\[
\Gamma = \{(x_1 + \cdots + x_{k'}, p_1 + \cdots + p_e) \mid \langle x_1, \ldots, x_{k'} \rangle \subset \langle p_1, \ldots, p_e \rangle \} \subset \widetilde{C}^{(k')} \times C^{(e)}
\]
dominates \( \widetilde{C}^{(k')} \). Moreover, since \( \widetilde{C} \) is non-degenerate the fibers of \( \pi_2 : \Gamma \to C^{(e)} \) must be finite. Since \( e \leq k' \leq \dim \Gamma = \dim \pi_2(\Gamma) \leq e \) we obtain that \( e = k' \).

On the other hand the natural rational maps
\[
C^{(e)} \dashrightarrow Sec^e(C) \subset Grass(e - 1, \mathbb{P}^{g-1})
\]
\[
\widetilde{C}^{(e)} \dashrightarrow Sec^e(\widetilde{C}) \subset Grass(e - 1, \mathbb{P}^{g-1})
\]
are generically injective by the uniform position theorem (remember \( e = k' \leq k + 1 \) and \( 2k + 3 \leq g \)), hence the correspondence \( \Gamma \) is of bidegree \((1, 1)\) and therefore \( C^{(e)} \) and \( C^{(e)} \) are birational. In particular \( g(C) = g(\widetilde{C}) = g \) (the induced map on Jacobians \( J\widetilde{C} \to JC \) has to be dominant since the image generates and the same in the opposite direction). By Ran’s Theorem on symmetric products, see [R], we get \( C \cong \widetilde{C} \). Observe that both curves have to be canonical, hence the isomorphism is a projectivity \( \varphi \). Coming back to our argument we get that choosing \( k \) generic points \( x_i \in \widetilde{C} \) we have that \( \langle x_1, \ldots, x_k \rangle = \langle \varphi(x_1), \ldots, \varphi(x_k) \rangle \), hence \( \varphi \) leaves invariant \( Sec^k(C) \) and must be the identity, so \( \widetilde{C} = C \) which is a contradiction.

4. Divisor case, intersection with the boundary

Now we start the proof of the codimension 1 case of the main Theorem assuming \( g \geq 5 \). We put now \( D \) instead of \( Z \). The initial step of our degeneration procedure is to show that the intersection of \( D \) with the boundary contains appropriate stable curves. These curves have to contain enough information to deduce from them the main result for the general smooth curve. The goal of this section is to define a family of convenient reducible curves and to prove that they appear in the closure of \( D \).

We start by remembering the following well-known facts on the rational Picard group of the compactified moduli space \( \overline{M}_g \) of stable curves (see for instance [ACG]):
\[
Pic_{\mathbb{Q}} \overline{M}_g = \lambda \mathbb{Q},
\]
where \( \lambda \) is the Hodge class. Moreover:
\[
\overline{M}_g \setminus M_g = \bigcup_{i=0}^{\frac{g-1}{2}} \Delta_i
\]
and
\[
Pic_{\mathbb{Q}}(\overline{M}_g) = \langle \lambda, \delta_0, \delta_1, \ldots, \delta_{\frac{g-1}{2}} \rangle \mathbb{Q}
\]
where \( \delta_i, i > 0 \) is the class of the divisor \( \Delta_i \) which general point represents a nodal curve \( C_1 \cup C_2 \), \( C_1, C_2 \) being integral, smooth curves of genus \( i \) and \( g - i \) intersecting in one point. And \( \delta_0 \) is the class of \( \Delta_0 \) which general point represents an irreducible curve with exactly one node.

We denote by \( d \) the class of \( \mathcal{D} \) in the rational Picard group. Then we can write:

\[
d = a\lambda + \sum_{i \geq 0} a_i \delta_i.
\]

Now we consider a complete integral curve \( B \) in \( \mathcal{M}_{g-2} \) (it exists because \( g-2 \geq 3 \)) and we fix two elliptic curves \( E_1, E_2 \) with arbitrary moduli class \( j_1, j_2 \in \mathcal{M}_1 \). Denote by \( \Gamma_b \) the smooth curve of genus \( g-2 \) corresponding to \( b \). We consider the set of the stable curves obtained by glueing to \( \Gamma_b \) the two elliptic curves in two distinct points \( p_1 \) and \( p_2 \) of \( \Gamma_b \). This does not depend on the choice of the points on the elliptic curves. This family is parametrized by the symmetric product \( \Gamma_b^{(2)} \setminus \Delta \Gamma_b \) out of the diagonal.

So we have a well-defined map

\[
\Gamma_b^{(2)} \setminus \Delta \Gamma_b \to \Delta_1 \subset \overline{\mathcal{M}_g},
\]

\[
p_1 + p_2 \mapsto E_1 \cup_{p_1} \Gamma \cup_{p_2} E_2,
\]

which extends to the whole symmetric product by sending \( 2p \) to the following curve: glue the infinity point of a \( \mathbb{P}^1 \) with the point \( p \) and then glue \( E_1, E_2 \) to other two points in the line.

We notice that these curves also belong to \( \Delta_2 \).

Finally, by moving \( b \) in the curve \( B \) we obtain a complete threefold \( T \subset \Delta_1 \). In other words, this threefold can be seen as the image in \( \overline{\mathcal{M}_g} \) of the relative symmetric product over \( B \):

\[
T = \bigcup_{b \in B} \Gamma_b^{(2)}.
\]
Our aim is to study the restriction of the divisor $\overline{D}$ to $T$. To do this we make a computation in $\text{Pic}_{\mathcal{Q}}(T)$. Denote by $S$ the surface in $T$ obtained as the union of all the diagonals:

$$S = \bigcup_{b \in B} \Delta_{\Gamma_b}.$$ 

We will need the following vanishing results.

**Lemma 4.1.** The restriction of the class $\delta_1$ to $S$ is zero: $\delta_1|_S = 0$.

**Proof.** We fix a smooth curve $C$ of genus 2 with a marked point $x$. We glue $C$ with $\Gamma_b$ identifying $x$ with $p \in \Gamma_b$. Then, by moving $p$ in $\Gamma_b$ and $b$ in $B$, we construct an algebraic surface $S_C$ such that $\Delta_1 \cap S_C = \emptyset$. Therefore $\delta_1 \cdot S_C = 0$. Now we degenerate $C$ to a genus 2 curve with a marked point consisting in the two elliptic curves $E_1, E_2$ glued to a $\mathbb{P}^1$ in 0 and 1 respectively and being the infinity point the marked one. Therefore by adding the curve $\Gamma_b$ identifying $\infty$ with $p$ we get our surface $S$ as a limit of a family of algebraic surfaces $S_C$ as above. We get that $\delta_1 \cdot S = 0$. □

**Lemma 4.2.** For each $b \in B$, $\lambda|_{\Delta_{\Gamma_b}} = 0$.

**Proof.** The Hodge structure is constant along the diagonal. □

We also will use the following basic observation:

**Lemma 4.3.** Let $N$ be a complete curve in $\overline{\mathcal{M}}_g$. It holds $\rho|_N \neq 0$ for at least one class $\rho \in \{\lambda, \delta_0, \ldots, \delta_{\frac{g-1}{2}}\}$.

The main result of this section is the following

**Proposition 4.4.** The restriction $d|_T$ is not a multiple of the class of $S$ in $\text{Pic}_{\mathcal{Q}}(T)$, i.e. $d|_T \neq mS$ for all $m \in \mathbb{Q}$. In particular $\overline{D} \cap T \neq \emptyset$ and this intersection contains elements out of $S$.

**Proof.** We use the notation introduced in (3). By contradiction, assume that $d|_T = mS$. Notice that $\Delta_i$ does not intersect $T$ for $i = 0$ and $i \geq 3$ and that $\Delta_2 \cap T = S$, so we get:

$$d|_T = mS = a\lambda|_T + a_1\delta_1|_T + a_2kS,$$

for some $k$. Therefore

$$(m - a_2k)S = a\lambda|_T + a_1\delta_1|_T.$$ 

Restricting to one diagonal $\Delta_{\Gamma_b}$ and using the lemmas 4.1 and 4.2 we deduce that $m - a_2k = 0$. Restricting now to $S$ we get $a\lambda|T = 0$. Since $\lambda$ is not trivial on $T$ we obtain that $a = 0$. This implies that $\overline{D}$ is contained in the border of $\overline{\mathcal{M}}_g$ which is a contradiction. □

**Remark 4.5.** Observe that the isomorphism classes of $E_1$ and $E_2$ are arbitrary, hence they could represent the $\infty$ class. Then the limit curves we were looking for are:

$$\mathcal{L} = \{E_1 \cup p_1 \Gamma \cup p_2 E_2 \in \overline{D} | \Gamma \in \mathcal{M}_{g-2}, p_1 \neq p_2, E_1, E_2 \in \overline{\mathcal{M}}_1 \} \subset \overline{D}.$$ 

A consequence of the proposition 4.4 is

**Corollary 4.6.** There is a subvariety $\mathcal{R} \subset \mathcal{M}_{g-2}$ of codimension at most 1 such that for any $\Gamma \in \mathcal{R}$ and for any $E_1, E_2 \in \overline{\mathcal{M}}_1$, there are curves $E_1 \cup p_1 \Gamma \cup p_2 E_2$ in $\mathcal{L}$. 

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Observe that all these elements belong to $\Delta_1$ and they belong to $\Delta_0$ if and only if at least one of the elliptic curves represents the infinity class.

**Remark 4.7.** With the same techniques one can prove that the divisor $\overline{D}$ contains irreducible curves with only one node. Moreover, by considering a surface in $\Delta_i$, $i > 0$ given by fixed smooth curves of genus $g - i$ and $i$ and moving the intersection point, one also shows with the same procedure that the divisor contain “generic” elements of $\Delta_i$. We do not use these facts in the rest of the paper.

### 5. Limits of isogenies

As in section 2 we assume the existence of isogenies $J\mathcal{C}' \rightarrow JC$ for generic elements $C \in D$. In order to glue all these maps together to provide a family $\chi : J' \rightarrow J$ we need to perform a suitable branching covering of our divisor. Since our calculations will be mainly of local nature we still denote this space of parameters as $D$. Our goal is to get as much information as possible from the specialization of the family of isogenies to the curves of $\mathcal{L}$.

Let $L = E_1 \cup_{p_1} \Gamma \cup_{p_2} E_2 \in \mathcal{L} \subset D$ a fixed limit curve (see Remark (4.5) and Corollary (4.6)). We assume that one of the elliptic curves has a node.

Observe that $L \in \Delta_0 \cap \Delta_1$. Let $\overline{D}_0$ be a component of $\overline{D} \cap \Delta_0$ containing $L$. Since we can assume that the generic element of this component does not belong to $\Delta_i$, $i \geq 2$, we obtain the following four cases:

a) The generic element of $\overline{D}_0$ is an irreducible curve with only one node.

b) The generic element of $\overline{D}_0$ consists of an irreducible curve with only one node with an elliptic curve attached in a smooth point.

c) The generic element of $\overline{D}_0$ consists of a smooth irreducible curve with an elliptic nodal curve attached in a smooth point.

d) The generic element of $\overline{D}_0$ consists of two irreducible smooth curves glued in two different points.

e) The generic element of $\overline{D}_0$ is an irreducible curve with two nodes.

By topological reasons the cases d) and e) can not occur. Indeed, the number of nodal points in a stable curve such that the curve remains connected when we
remove the point cannot decrease under specialization. Since there are two such points in a generic point of type d) and e) and only one in our limit curve L we can ignore these cases.

In this section we will specialize the isogeny χ to a generic curve of the component $\mathcal{D}_0$ that contains our limit curve L. The information we get is different according to the three cases explained above.

**Case a.** We consider the normalization map $p : \Delta_0 \rightarrow \mathcal{M}_{g-1}$ restricted to $\mathcal{D}_0$. There are two possibilities according to the dimension of the generic fiber of $p_0 := p|_{\mathcal{D}_0}$.

**Case a.1:** Assume that $p_0$ is dominant, therefore the generic fiber has dimension $\dim \mathcal{D}_0 - \dim \mathcal{M}_{g-1} = 3g - 5 - 3g + 6 = 1$. For a generic element $t_0 \in \mathcal{D}_0$ the limit map $\chi_{t_0} : JC_{t_0}' \rightarrow JC_{t_0}$ gives a diagram of extensions:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{C}^* & \rightarrow & JC_{t_0}' & \rightarrow & J\hat{C}_{t_0} & \rightarrow & 0 \\
& & | & & | & & | & & \\
& & \gamma & & \chi_{t_0} & & \hat{\chi}_{t_0} & & \\
& & & & & & & & \\
0 & \rightarrow & \mathbb{C}^* & \rightarrow & JC_{t_0} & \rightarrow & J\hat{C}_{t_0} & \rightarrow & 0 \\
\end{array}
$$

where $\hat{C}_{t_0}$ and $\hat{C}_{t_0}$ stand for the normalizations of $C_{t_0}'$ and $C_{t_0}$ respectively. Since $\chi_{t_0}$ has finite kernel, $r$ must be one and $\gamma(z) = z^m$ for some non-zero integer $m$.

Since $g - 1 \geq 4$ we can apply the main result in [BP] and we get that $\hat{C}_{t_0} = \hat{C}_{t_0}$ and the isogeny $\hat{\chi}_{t_0}$ is $n$ times the identity. Assume that $C_{t_0}$ (resp. $C_{t_0}'$) is obtained from $\hat{C}_{t_0}$ by pinching two distinct points $p, q$ (resp. $p', q'$). As in [BP], section 2, to compare the extension classes of each horizontal short exact sequence, we decompose the last diagram into

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{C}^* & \rightarrow & JC_{t_0}' & \rightarrow & J\hat{C}_{t_0} & \rightarrow & 0 \\
& & | & & | & & | & & \\
& & \gamma & & \chi_{t_0} & & \hat{\chi}_{t_0} & & \\
& & & & & & & & \\
0 & \rightarrow & \mathbb{C}^* & \rightarrow & E & \rightarrow & J\hat{C}_{t_0} & \rightarrow & 0 \\
& & & & & & & & \\
0 & \rightarrow & \mathbb{C}^* & \rightarrow & JC_{t_0} & \rightarrow & J\hat{C}_{t_0} & \rightarrow & 0 \\
\end{array}
$$

We identify (up to the sign) the extension class

$$[JC_{t_0}'] \in Ext(J\hat{C}_{t_0}, \mathbb{C}^*) \cong Pic^0(J\hat{C}_{t_0}) \cong J\hat{C}_{t_0}$$

with $p' - q'$ and analogously $[JC_{t_0}]$ with $p - q$. Then the equality $[E] = \gamma_*([JC_{t_0}']) = \hat{\chi}_{t_0}^*([JC_{t_0}])$ provides the following relation in $J\hat{C}_{t_0}$:

$$n(p - q) = \pm m(p' - q').$$

Hence we can assume that $np + mq' = mp' + nq$ in $J\hat{C}_{t_0}$. We assume also that the points are different. Since the dimension of the generic fiber of $p_0$ is 1 we have a one dimensional family of maps $C_{t_0} \rightarrow \mathbb{P}^1$ of degree $n + m$ with two fibers as above. Riemann-Hurwitz Theorem implies that

$$2g(\hat{C}_{t_0}) - 2 = 2g - 4 = (n + m)(2g(\mathbb{P}^1) - 2) + 2(n - 1) + 2(m - 1) + r = -4 + r,$$
so the number \( r \) of the ramification points out of the special fibers \( np + mq' \) and \( mp' + nq \) is \( r = 2g \). Then the Hurwitz scheme of maps of degree \( n + m \) into \( \mathbb{P}^1 \) with \( r + 2 = 2g + 2 \) discriminant points must cover \( \mathcal{M}_{g-1} \) with generic fibers of dimension 1. Comparing dimensions:

\[
2g + 2 - \dim \text{Aut}(\mathbb{P}^1) - \dim \text{generic fiber} = 2g + 2 - 4 = 2g - 2 \geq \dim \mathcal{M}_{g-1} = 3g - 6,
\]

which contradicts the hypothesis \( g \geq 5 \). Hence we get that the extension is the same and \( n = m = 1 \).

**Case a.2):** Assume that the generic fiber of \( p_0 \) has dimension 2. As before we get a diagram:

\[
\begin{array}{cccc}
0 & \to & \mathbb{C}^* & \to & JC'_{t_0} & \to & \tilde{J}C'_{t_0} & \to & 0 \\
& & m & & \chi_{t_0} & & \tilde{\chi}_{t_0} & & \\
0 & \to & \mathbb{C}^* & \to & JC_{t_0} & \to & \tilde{J}C_{t_0} & \to & 0
\end{array}
\]

but we have not now the genericity of \( \tilde{C}_{t_0} \) so we can not apply directly the main result in \([BP]\). The relation between extension classes is in this case

\[
m(p' - q') = \tilde{\chi}_{t_0}^*(p - q),
\]

in \( \tilde{J}C'_{t_0} \). In other words, the isogeny \( \tilde{\chi}_{t_0} \) induces a map between the surfaces \( \tilde{C}_{t_0} - \tilde{C}_{t_0} \) and \( m(\tilde{C}_{t_0} - \tilde{C}_{t_0}) \). By using the arguments of the section 3 in \([BP]\) one easily checks that, as before, the curves are the same and the map is a multiplication by an integer. So we have proved the following result:

**Proposition 5.1.** Let \( \chi : \mathcal{J} \to \mathcal{J} \) be a family of isogenies parametrized by \( \mathcal{D} \) and let \( \chi_{t_0} : JC'_{t_0} \to JC_{t_0} \) be a specialization to a generic point \( t_0 \) of a component of the boundary \( \mathcal{D} \cap \Delta_0 \), where the curve \( C_{t_0} \) is an irreducible curve with only one node. Then \( C'_{t_0} \cong C_{t_0} \) and \( \chi_{t_0} \) is the multiplication by a non zero integer.

**Case b):** We assume now that the limit curve \( L \) belongs to an irreducible component \( \mathcal{D}_0 \) of \( \mathcal{D} \cap \Delta_0 \) which generic element consists in a smooth curve \( \Gamma \) of genus \( g - 1 \) and a nodal elliptic curve \( E_\infty \) (i.e. a \( \mathbb{P}^1 \) with the points 0 and 1 identified) glued to \( \Gamma \) in a point \( p \in \Gamma \). We denote as above by \( C_{t_0} = \Gamma \cup_p E_\infty \) a generic curve in \( \mathcal{D}_0 \). Observe that the natural map \( \mathcal{D}_0 \to \mathcal{M}_{g-1} \) must be dominant by a count of dimensions (the fibre has dimension at most 1), therefore we can assume that \( \Gamma \) is generic in \( \mathcal{M}_{g-1} \).

We consider the specialization of the family of isogenies to our curve \( \chi_{t_0} : JC'_{t_0} \to JC_{t_0} = J\Gamma \times \mathbb{C}^* \) which fits in a diagram of extensions:

\[
\begin{array}{cccc}
0 & \to & \mathbb{C}^* & \to & JC'_{t_0} & \to & \tilde{J}C'_{t_0} & \to & 0 \\
& & \gamma & & \chi_{t_0} & & \tilde{\chi}_{t_0} & & \\
0 & \to & \mathbb{C}^* & \to & J\Gamma \times \mathbb{C}^* & \to & J\Gamma & \to & 0
\end{array}
\]

where \( \tilde{C}'_{t_0} \) stands for the normalization of \( C'_{t_0} \). Since \( \chi_{t_0} \) has finite kernel, \( r \) must be one and \( \gamma(z) = z^m \) for some non-zero integer \( m \). Since \( g - 1 \geq 4 \) we can apply the main result in \([BP]\) and we get that \( \tilde{C}'_{t_0} = \Gamma \) and the isogeny \( \tilde{\chi}_{t_0} \) is \( n \) times the
identity. So the diagram above becomes:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{C}^* & \rightarrow & JC'_t & \rightarrow & J\Gamma & \rightarrow & 0 \\
\downarrow & & m & \downarrow & \chi_t & \downarrow & n \\
0 & \rightarrow & \mathbb{C}^* & \rightarrow & J\Gamma \times \mathbb{C}^* & \rightarrow & J\Gamma & \rightarrow & 0 \\
\end{array}
\]

Since the extension class of the first row corresponds to a generalized Jacobian, there exist points \(q_1, q_2 \in \Gamma\) such that this class correspond (up to the sign) to \(q_1 - q_2 \in J\Gamma\). Therefore, since the class of the second row is zero we get that \(m(q_1 - q_2) = 0\). Since the \(m\) torsion points are rigid in the Jacobian of \(\Gamma\) we get that \(q_1 = q_2\). Hence the extension given by the first row is also trivial: \(JC'_t \cong J\Gamma \times \mathbb{C}^*\)

and \(\chi_t = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}\). We get the following result:

**Proposition 5.2.** Let \(\chi : J' \rightarrow J\) be a family of isogenies parametrized by \(D\) and let \(\chi_t : JC'_t \rightarrow JC_t\) be a specialization to a generic point \(t_0\) of a component of the boundary \(\overline{D} \cap D_0\), where the curve \(C_t\) is a reducible curve consisting in a smooth curve of genus \(g - 1\) with a nodal elliptic curve \(E_\infty\) attached in one point. Then \(C'_t \cong C_t\) and the isogeny in the compact part is the multiplication by a non-zero integer.

**Case c:** Finally we assume that the limit curve \(L\) belongs to an irreducible component \(\overline{D_0} \cap D_0\) which generic element consists in a nodal curve \(\Gamma_0\) of genus \(g - 1\) and an elliptic curve \(E\) glued to \(\Gamma_0\) in a smooth point \(p \in \Gamma_0\). We denote as above by \(C_t = \Gamma_0 \cup_p E\) a generic curve in \(\overline{D}_0\). Observe that the natural map \(\overline{D}_0 \rightarrow \Delta_0(M_{g-1})\) must be dominant (here \(\Delta_0(M_{g-1})\) denotes the \(\Delta_0\) divisor in the moduli space \(M_{g-1}\)). Indeed the generic fibre of this map has dimension at most 2 parametrized by the smooth point \(p\) in \(\Gamma_0\) and the moduli of the elliptic curves. Since \(\dim \overline{D}_0 = 3g - 5\) and \(\dim \Delta_0(M_{g-1}) = 3g - 7\), the dominance follows. Therefore we can assume that \(\Gamma_0\) is generic in \(\Delta_0(M_{g-1})\), and also is generic its normalization \(\overline{\Gamma_0}\) in \(M_{g-2}\). Moreover all the curves \(\Gamma_0 \cup_p E\) are contained in \(\overline{D}_0\) for a generic \(\Gamma_0\).

As in the other cases we consider the specialization of the family of isogenies to our curve \(\chi_t : JC'_t \rightarrow JC_t = J\Gamma_0 \times E\) which fits in a diagram of extensions:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{C}^* & \rightarrow & JC'_t & \rightarrow & JC_t & \rightarrow & 0 \\
\downarrow & & \gamma & \downarrow & \chi_t & \downarrow & \tilde{\chi}_t \\
0 & \rightarrow & \mathbb{C}^* & \rightarrow & J\Gamma_0 \times E & \rightarrow & J\Gamma_0 \times E & \rightarrow & 0 \\
\end{array}
\]

where \(\tilde{C}'_t\) stands for the normalization of \(C'_t\). Since \(\chi_t\) has finite kernel, \(r\) must be one and \(\gamma(z) = z^m\) for some non-zero integer \(m\).

We claim that \(J\tilde{C}'_t\) must be a product of Jacobians (in other words, the smooth curve \(\tilde{C}'_t\) reduces). We prove this by contradiction, assume that \(\tilde{C}'_t\) is irreducible and compare the extension classes. This gives (up to the sign) a relation \(m(p' - q') = \tilde{\chi}_t(p - q)\) in \(JC'_t\). Moving the points \(p\) and \(q\) in the fixed curve \(\overline{\Gamma}_0\) we get that the image of \(\overline{\Gamma}_0 - \overline{\Gamma}_0\) by the isogeny is the surface \(\tilde{C}'_t - \tilde{C}'_t\) which is impossible since it is contained in the proper abelian subvariety \(\tilde{\chi}_t^*(J\overline{\Gamma}_0)\).
By the genericity of $\tilde{\Gamma}_0$ we can assume that $J\tilde{\Gamma}_0$ is simple and then $J\tilde{C}'_{t_0} \cong J\tilde{C}''_{t_0} \times E'$, where $\tilde{C}''_{t_0}$ is an irreducible curve of genus $g - 2$ and $E'$ stands for a smooth elliptic curve. So $C'_{t_0}$ is a nodal curve $C''_{t_0}$ with the elliptic curve attached in a smooth point. The extension class is the difference of two points $p''$, $q'' \in \tilde{C}''_{t_0}$. The diagram above becomes:

\[
\begin{array}{c}
0 \rightarrow \tilde{\mathcal{C}} \rightarrow J\tilde{C}''_{t_0} \times E' \rightarrow J\tilde{\mathcal{C}}''_{t_0} \times E' \rightarrow 0 \\
\Bigm| m \\
0 \rightarrow \tilde{\mathcal{C}} \rightarrow J\mathcal{C}''_{t_0} \times E \rightarrow J\mathcal{C}''_{t_0} \times E \rightarrow 0,
\end{array}
\]

where $\varphi : E' \rightarrow E$ is a non-constant map of elliptic curves. The relation between extension classes is in this case (up to the sign):

\[m(p'' - q'') = (\tilde{\chi}''_{t_0})^*(p - q),\]

in $J\tilde{C}''_{t_0}$. In other words, the isogeny $\tilde{\chi}''_{t_0}$ induces a map between the surfaces $\tilde{\Gamma}_0 - \tilde{\Gamma}_0$ and $m(\tilde{C}''_{t_0} - C''_{t_0})$. By using the arguments of the section 3 in [BP] one easily checks the following facts: the curves $C''_{t_0}$ and $\Gamma_0$ are isomorphic, the isogeny $\tilde{\chi}''_{t_0}$ is a non-zero multiple $n$ of the identity, $n = m$ and then $\chi_{t_0} = n \times \varphi$.

**Proposition 5.3.** Let $\chi : J' \rightarrow J$ be a family of isogenies parametrized by $\mathcal{D}$ and let $\chi_{t_0} : J\tilde{C}'_{t_0} \rightarrow J\tilde{C}'_{t_0}$ be a specialization to a generic point $t_0$ of a component of the boundary $\mathcal{D} \cap \Delta_0$, where the curve $C_{t_0}$ is a reducible curve consisting in a nodal curve $\Gamma_0$ of genus $g - 1$ with an elliptic curve $E$ attached in one smooth point, then also $C'_{t_0}$ is of the form $\Gamma_0 \cup_p E'$, for some elliptic curve $E'$, and the isogeny induces the multiplication by $n$ on $J\tilde{\Gamma}_0$.

6. **End of the proof**

Let us go back to our family of isogenies $\chi : J' \rightarrow J$ parametrized by (some covering of) the divisor $\mathcal{D}$.

Consider a generic point $t \in \mathcal{D}$ corresponding to smooth curves $C'_t$ and $C_t$. Observe that for all $t$ the isogeny is determined by the map at the level of homology groups

\[\chi_{t,Z} : H_1(C'_t, \mathbb{Z}) \rightarrow H_1(C_t, \mathbb{Z})\]

which we still denote by $\chi_t$. We set $\Lambda_t \subset H_1(C_t, \mathbb{Z})$ for the image of $\chi_t$. This is a sublattice of maximal range $2g$. We first notice that the proof of the Proposition (4.2.1) in [BP] applies verbatim to obtain the following result:

**Proposition 6.1.** Assume that $\Lambda_t = nH_1(C_t, \mathbb{Z})$ for some positive integer $n$. Then $C'_t \cong C_t$ and $\chi_t$ is the multiplication by $n$.

Therefore to finish the proof of the Theorem we have to show the equality $\Lambda_t = nH_1(C_t, \mathbb{Z})$. To do this, the main idea is to get information on $\Lambda_t$ from the homology group of some convenient limits $C_0$. In the previous sections we have shown the existence of certain limits and we have seen in Propositions 5.1, 5.2 and 5.3 how is the image of the limit of $\chi_t$ when $t$ goes to one of these degenerations.

To pass information from some $C_0$ to the smooth point we use the following principle: We can assume that there exists a disc $\mathcal{D} \subset \overline{\mathcal{D}} \subset \overline{\mathcal{M}_g}$ centered at the class of the curve $C_0$ such that the curves $C_t, C'_t$ corresponding to $\mathcal{D} \setminus \{0\}$ are
of the theta divisor is a multiple of the theta divisor in $JC$ to cycles which become a basis of the homology of $E$ for some integers $a_1$, $b_1, \ldots, a_g, b_g$ in $H_1(C_t, Z)$ in such a way that:

\[ \chi_i(a'_i) = na_i + s_i a_1, \quad \chi_i(b'_i) = nb_1 + t_1 a_1, \quad \chi_i(b'_i) = nb_i + t_i a_1 \]

for some integers $s_i$ and $t_i$ and $i \geq 2$. We also can assume that $a_g, b_g$ correspond to cycles which become a basis of the homology of $E_2 \subset L$.

By the genericity of the curve $C_t$ in a divisor of the moduli space the pull-back of the theta divisor is a multiple of the theta divisor in $JC_t$. This translates into the existence of a non-zero integer $m$ such that the cuproduct satisfies

\[ \chi_l(x) \cup \chi_l(y) = mx \cup y. \]

Then one gets that

\[ m = \chi_l(a'_i) \cup \chi_l(b'_i) = la_1 \cup (nb_1 + t_1 a_1) = ln \]

\[ m = \chi_l(a'_i) \cup \chi_l(b'_i) = n^2, \]

so $n = l$. With similar computations it is easy to prove that $s_i = t_i = 0$ for $i \geq 2$, hence:

\[ \chi_l(a'_i) = na_i, \quad \chi_l(a'_i) = na_i, \quad \chi_l(b'_i) = nb_i \quad \text{for } i \geq 2. \quad (4) \]

To get the piece of information which still is unknown we need to consider a second limit. We have seen in the Corollary (1.10) that fixing $\Gamma$ we can move freely the two elliptic curves $E_1, E_2$, that is $E_1 \cup \Gamma \cup E_2 \in \bar{\mathcal{D}}$ for all $E_1, E_2$. We select a second limit curve $\hat{L} = E_1 \cup \Gamma \cup E_\infty$ in such a way that the corresponding vanishing cycle is now $a_g$. Again, to simplify, we assume that $\hat{L}$ belongs to the case $a$ of section 5. Then, with the same argument, we get that $\chi_l$ satisfies (for the same simplectic basis):

\[ \chi_l(a'_i) = \hat{n} a_i, \quad \chi_l(b'_i) = \hat{n} b_i \quad \text{for } i \leq g - 1 \]

and $\chi_l(a'_g) = \hat{n} a_g$. Therefore $\hat{n} = n$ and $\Lambda_t = nH_1(C_t, Z)$ for all $s \neq 0$. Hence we have finished (under the assumption that $L$ and $\hat{L}$ are limit cycles of nodal curves).

The rest of the proof consists in the description of the small modifications that have to be done to take care of the rest of the cases. Observe that the information on
the limit given in the case b) (see Proposition (5.2)) is the same to that given in the case a), that is we have again the relations (4). So we only have to take care of the situation when at least one of the limit curves \( L, \hat{L} \) belong to the case c).

Assume for example that \( L \) does. Using the first limit as above we know that

\[
\chi_t(a'_1) = na_1, \quad \chi_t(a'_i) = na_i, \quad \chi_t(b'_i) = nb_i \quad \text{for} \quad 2 \leq i \leq g - 1.
\]

The difference with the precedent cases is that we have no control on \( \chi_t(b'_1) \). Remember that \( L \) is the limit of a curve \( \Gamma_0 \cup E \) where \( \Gamma_0 \) is a nodal curve intersecting the elliptic curve \( E \) in a smooth point. Being \( \tilde{\Gamma}_0 \) the normalization of \( \Gamma_0 \) we noticed that all the nodal curves with this normalization belong to the divisor \( D_0 \). So we change freely the node in such a way that the vanishing cycle of the node becomes \( a_2 \) (instead of \( a_1 \)). By using the two vanishing cycles we get that \( \chi_t(b'_1) = nb'_1 \) so we recover again the relations (4). This finishes the proof of the theorem. \( \square \)

Our theorem can be interpreted as a kind of Noether-Lefschetz problem in the following way: consider in \( \mathcal{M}_g \times \mathcal{M}_g \) the set

\[
\mathcal{NL}_g = \{ (C', C) \mid \text{rang } NS(C \times C') \geq 3 \}.
\]

A consequence of what we have proved is the following result:

**Corollary 6.2.** For \( g \geq 5 \) all the components of \( \mathcal{NL}_g \) out of the diagonal have dimension less or equal to \( 3g - 5 \).

The first natural problem one could face up in this context is to investigate the existence of dimension 10 components in \( \mathcal{NL}_5 \). Similar problems on isogenies can be considered for other families of abelian varieties (see for example \([NP]\)).

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**References**

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris: Geometry of algebraic curves. Vol. I. Grundlehren der Mathematischen Wissenschaften, 267. Springer-Verlag, New York, 1985.

[ACG] E. Arbarello, M. Cornalba, P. A. Griffiths: Geometry of algebraic curves. Vol. II. Grundlehren der Mathematischen Wissenschaften, 268. Springer-Verlag, New York, 2011.

[BP] F. Bardelli, G.P. Pirola: Curves of genus \( g \) lying on a \( g \)-dimensional Jacobian variety, Invent. math., 95 (1989), pp. 263–276.

[CGGH] J. Carlson, M. Green, P. Griffiths, M. Cornalba: Infinitesimal variations of Hodge structure. I, Compositio Math. 50 (1983), 109205.

[CGT] C. Ciliberto, G. van der Geer, M. Teixidor: On the number of parameters of curves whose Jacobians possess nontrivial endomorphisms, J. Algebraic Geom. 1 (1992), 215-229.

[M] D. Mumford: Abelian Varieties. Oxford University Press, Bombay, 1974.

[NP] J.C. Naranjo and G.P. Pirola: On the genus of curves in the generic Prym variety, Indag. Math. (N.S.) 5 (1994), 101-105.

[P] G.P. Pirola: Base number theorem for abelian varieties. An infinitesimal approach, Math. Ann. 282 (1988), 361-368.

[R] Z. Ran: On a Theorem of Martens, Rend. Sem. Mat. Univ. Politec. Torino, 44 (1986), pp. 287–291.
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