CONTINUOUSLY TRIANGULATING THE CONTINUOUS CLUSTER CATEGORY

MATTHEW GARCIA AND KIYOSHI IGUSA

Abstract. In [4], the continuous cluster category was introduced. This is a topological category whose space of isomorphism classes of indecomposable objects forms a Möbius band. It was found in [4] that, in order to have a continuously triangulated structure on this category, one needs at least two copies of each indecomposable object forming a 2-fold covering space of the Möbius band. This paper classifies all continuous triangulations of finite coverings of the basic continuous cluster category. This includes the connected 2-fold covering of Igusa-Todorov [4], the disconnected 2-fold covering of Orlov [6] and a third unexpected continuously add-triangulated 2-fold covering of the Möbius strip category.

CONTENTS

Introduction 1
1. Topological $K$-categories and examples 5
2. Equivalence coverings 13
3. Skew-continuous natural transformations 19
4. Classification of add-triangulated equivalence coverings of $\mathcal{M}_0$ 22
5. Automorphisms of $\mathcal{C}_n$ 28
6. Continuous Frobenius categories 36
Acknowledgements 46
References 47

INTRODUCTION

Following the footsteps of the Pythagoreans, we examine the following quiver:

This is the quiver, called $A_1$, with one vertex and no arrows. This quiver represents a $K$-category with one object with endomorphism ring equal to the field $K$. We consider an equivalent $K$-category $\mathcal{C}_n$ with $n$ isomorphic objects $1, 2, \ldots, n$ so that $\mathcal{C}_n(i, j) = K$ for all $i, j$. Continuous triangulations of the continuous cluster category will be given by triples $(\sigma, \tau, \varphi)$ of discrete structures on this finite category $\mathcal{C}_n$ for some $n \geq 2$.

Briefly, $\sigma, \tau$ are commuting autoequivalences of $\mathcal{C}_n$, with $\sigma$ being an automorphism, and $\varphi : \sigma \rightarrow \tau$ a “skew-continuous” natural isomorphism (Def. 3.3) the existence of which puts

---

2010 Mathematics Subject Classification. 18E30:16G20.

Key words and phrases. Frobenius categories, triangulated categories, topological categories, equivalence coverings.
a severe restriction on $\tau$ (by Prop. 3.4). These discrete structures on the finite category $C_n$ will be classified at the end of the paper assuming that $K$ is an algebraically closed field of characteristic not equal to 2. The main body of this paper describes how a continuously triangulated topological $K$-category which is algebraically equivalent to the continuous cluster category is given by such a triple of discrete structures.

0.1. **Topological $K$-categories.** The paper begins with a precise description of topological $K$-categories. Basically, it is a $K$-category together with a topology on the set of objects and the set of morphisms so that all structure maps are continuous. We restrict attention to the topological full subcategory of indecomposable objects and assume that the rest of the category is constructed in a canonical way using the “James construction” (Def. 1.9).

The two main examples we consider are the circle category $S^1$ and open Moebius band category $M_0$ (Def. 1.12, 1.15). These are topological $K$-categories where the endomorphism ring of each object is $K$ (making each object indecomposable) and the space of objects is the circle or open Moebius band, respectively. During this discussion, we recall the definition of the continuous cluster category from [4] and the topological difficulties we are working to overcome. Basically the problem is the negative sign in the rotation axiom for a triangulated category and the fact that distinguished triangles can be continuously rotated which avoids the negative sign. Since we are taking $K$ to have the discrete topology, we cannot move continuously from 1 to $-1$ (since $\text{char } K \neq 2$). So, the only resolution of this problem is to replace the category with an equivalent covering category.

0.2. **Equivalence coverings.** To do covering theory, we assume that the object space of our topological category $B$ is connected and locally simply-connected. Then, an $n$-fold “equivalence coverings” (Def. 2.6) $\tilde{B}$ of $B$ gives a homomorphism

$$\sigma : \pi_1(B, X_0) \to \text{Aut}(C_n),$$

well-defined up to conjugation, where $\text{Aut}(C_n)$ is the automorphism group of the $n$-point category of type $A_1$ discussed earlier. This is called the “holonomy” of $\tilde{B}$. Conversely, for any homomorphism $\sigma$ as above, there is a equivalence covering $\tilde{B}_\sigma$ with holonomy $\sigma$. And, by Lemma 2.10, any two equivalence coverings with conjugate holonomies are continuously isomorphic.

Since homomorphisms $\mathbb{Z} \to \text{Aut}(C_n)$ are given by the image of the generator of $\mathbb{Z}$, which we call the “holonomy functor” of the covering, we get the following.

**Theorem A** (Theorem 2.11). $n$-fold equivalence coverings of $S^1$ and $M_0$ are classified by their holonomy functors which are automorphisms of $C_n$ well-defined up to conjugation.

Given $\sigma \in \text{Aut}(C_n)$, the corresponding $n$-fold equivalence coverings of $S^1$ and $M_0$ are denoted $\tilde{S}_\sigma^1$ and $\tilde{M}_\sigma$. More generally, we denote by $\tilde{B}_\sigma$ the equivalence covering of $B$ with holonomy $\sigma : \pi_1 B \to \text{Aut}(C_n)$.

0.3. **Autoequivalences of equivalence coverings.** Next we consider continuous autoequivalences of equivalence coverings $\tilde{B}_\sigma$ which cover the identity functor on $B$. Using an explicit construction of $S_\sigma^1$ and $M_\sigma$ we prove the following.
Theorem B (special case of Theorem 2.13). Continuous autoequivalences $F_\tau$ of $\tilde{S}_\sigma^1$ and $\tilde{M}_\sigma$ are given by autoequivalences $\tau$ of $\mathcal{C}_n$ which commute with $\sigma$.

Since $\sigma$ commutes with itself, we have, in particular, continuous automorphisms of $\tilde{S}_\sigma^1$ and $\tilde{M}_\sigma$ given by the automorphism $\sigma$. Both are denoted $F_\sigma$.

0.4. Skew-continuous natural isomorphism. The last factor needed to construct a continuous triangulation of $\text{add } \tilde{M}_\sigma$ is a “skew-continuous” natural isomorphism $\varphi: \sigma \to \tau$. We explain briefly how this is used and why $\varphi$ cannot be continuous. We are constructing a triangulated category with distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\psi} TX$$

where $X, Y, Z$ are direct sums of objects in $\tilde{M}_\sigma$. The shift functor is $T = F_\tau$. But the morphism $\psi: Z \to TX = F_\tau X$ is the composition of two morphisms: a morphism $h: Z \to F_\sigma X$ followed by a “skew-continuous” isomorphism $F_\varphi X: F_\sigma X \cong F_\tau X$.

By continuity, $F_\sigma$ takes distinguished triangles to distinguished triangles, but $T = F_\tau$ does not (by the rotation axiom) unless a negative sign is inserted. But $F_\sigma$ and $F_\tau$ are naturally (but discontinuously) isomorphic by $F_\varphi$. This gives the following commuting diagram in which both horizontal sequences are distinguished triangles.

$$
\begin{array}{cccccccc}
& F_\sigma X & \xrightarrow{F_\sigma f} & F_\sigma Y & \xrightarrow{F_\sigma g} & F_\sigma Z & \xrightarrow{F_\sigma h} & F_\sigma (F_\varphi X) \\
F_\tau X & \xrightarrow{F_\tau f} & F_\tau Y & \xrightarrow{F_\tau g} & F_\tau Z & \xrightarrow{F_\tau h} & F_\tau (F_\varphi X) & \cong F_\tau F_\sigma X \\
& F_\sigma (X) & \xrightarrow{F_\sigma (\psi)} & F_\sigma (Y) & \xrightarrow{F_\sigma (\psi)} & F_\sigma (Z) & \xrightarrow{F_\sigma (\psi)} & F_\sigma (F_\varphi X) \\
& F_\tau (X) & \xrightarrow{F_\tau (\psi)} & F_\tau (Y) & \xrightarrow{F_\tau (\psi)} & F_\tau (Z) & \xrightarrow{F_\tau (\psi)} & F_\tau (F_\varphi X)
\end{array}
$$

The first three squares commute since $F_\varphi$ is a natural transformation. This forces the last square to commute. Since $F_\tau (F_\varphi X)$ is an isomorphism this forces

$$F_\sigma F_\varphi = -F_\varphi F_\sigma.$$

So, $F_\varphi$ is not a continuous functor (since it does not commute with holonomy $F_\sigma$). Because of this minus sign we say that $F_\varphi$ and $\varphi$ are “skew-continuous”. Proposition 3.4 implies that the existence of $\varphi$ imposes the following restriction on $\sigma$ and $\tau$.

Lemma C (Proposition 3.4). The following are equivalent.

1. There exists a skew-continuous natural isomorphism $\varphi: \sigma \to \tau$.
2. Every natural isomorphism $\varphi: \sigma \to \tau$ is skew-continuous.
3. $\sigma, \tau$ are anti-compatible (Def. 3.2).

Lemma C puts many conditions on $\sigma, \tau$ and $n$:

(a) Since $\sigma$ is compatible with itself, $\sigma \neq \tau$.
(b) Since the identity automorphism of $\mathcal{C}_n$ is compatible with every autoequivalence of $\mathcal{C}_n$, we have $\sigma, \tau \neq id$. (However, the underlying permutations of the objects of $\mathcal{C}_n$ are allowed to be the identity.)
(c) For $n = 1$, any two autoequivalences of $\mathcal{C}_1$ are compatible. So, we must have $n \geq 2$. 3
0.5. **Classification Theorem.** Using Lemma C, the main theorem can be phrased as follows. We use the term “continuous add-triangulation” to refer to a continuous triangulation of the topological additive category generated by an equivalence covering $\tilde{M}_\sigma$ of $M_0$ (Def. 1.3, 4.1).

**Theorem D (Theorem 4.2).** Continuous add-triangulations, $\tilde{M}_n(\sigma, \tau, \varphi)$, of $n$-fold equivalence coverings of $M_0$ are classified by triples $(\sigma, \tau, \varphi)$ where:

1. $\sigma$ is a $K$-linear automorphism of $C_n$.
2. $\tau$ is a $K$-linear autoequivalence of $C_n$ which commutes with $\sigma$ but which is anti-compatible with $\sigma$.
3. $\varphi : \sigma \to \tau$ is a skew-continuous natural isomorphism.

Furthermore, continuous (strong) isomorphisms of triangulated categories $\tilde{M}_n(\sigma, \tau, \varphi) \cong \tilde{M}_n(\sigma', \tau', \varphi')$ are given by $\rho \in \text{Aut}(C_n)$ so that $\sigma'\rho = \rho\sigma$, $\tau'\rho = \rho\tau$. In particular, $\tilde{M}_n(\sigma, \tau, \varphi) \cong \tilde{M}_n(\sigma, \tau, \varphi')$ for any $\varphi, \varphi'$.

To prove this theorem we show that the set of all distinguished triangles is determined by a single continuous family of distinguished triangles which we call the “universal triangle” (4.1). This is determined by $\sigma, \tau, \varphi$ and we show that such a triple determines a “continuous Frobenius categories” $\text{add} \tilde{M}_n(\sigma, \tau, \varphi)$ whose underlying subcategory of indecomposable objects is an $n$-fold covering of the “closed Moebius category” $\tilde{M}$ and whose stable category is $\text{add} \tilde{M}_n(\sigma, \tau, \varphi)$. (Theorem 6.22.)

One consequence of the classification theorem, together with Lemma C and the fact that anti-compatibility is a symmetric relation (Corollary 5.14), is the following duality.

**Corollary E1 (Corollary 4.4).** If $\tilde{M}_n(\sigma, \tau, \varphi)$ is a continuous add-triangulation of an $n$-fold equivalence covering of $M_0$ then so is $\tilde{M}_n(\tau, \sigma, \varphi^{-1})$ (provided $\tau$ is an isomorphism).

We call $\tilde{M}_n(\tau, \sigma, \varphi^{-1})$ the dual of $\tilde{M}_n(\sigma, \tau, \varphi)$. Finally, we classify the minimal examples given by $n = 2$.

**Corollary E2 (Corollary 4.18).** Up to isomorphism there are three add-triangulated 2-fold equivalence covering of $M_0$: The continuous cluster category $\tilde{C}$ which is self-dual, Orlov’s construction $\tilde{C}$ (as a special case of a more general construction) and one new triangulated category which is the dual of Orlov’s construction.

The last two sections deal with some of the technical details: the combinatorics of the category $C_n$ and construction of the continuous Frobenius categories needed to complete the proof of the classification Theorem D.

0.6. **Details of the category $C_n$.** In Section 5 we discuss the combinatorics of the finite category $C_n$. This is the discrete category with object set $[n] = \{1, 2, \ldots, n\}$ with $C_n(i, j) = K$ for all $i, j$ and composition given by multiplication in $K$. The structures $\sigma, \tau, \varphi$ have easy descriptions in terms of these scalars.

Any $(K$-linear) automorphism $\sigma$ of $C_n$ is given on objects by a permutation of $n$, which we also call $\sigma$, and on morphisms $C_n(i, j) \to C_n(\sigma(i), \sigma(j))$ by multiplication by $a_{ji} \in K^*$ where $(a_{ji})$ is any system of nonzero scalars satisfying the following.
(1) \( a_{ii} = 1 \) for all \( i \in [n] \)
(2) \( a_{ij}a_{jk} = a_{ik} \) for all \( i, j, k \in [n] \)

We call such a family of scalars \((a_{ji})\) a **system of parameters**.

An autoequivalence \( \tau \) of \( \mathcal{C}_n \) is given by another system of parameters \((b_{ji})\) and any set map \( \tau : [n] \to [n] \). \( \sigma, \tau \) commute if the set maps commute and if the parameters satisfy:

\[
(0.1) \quad b_{\sigma(j)\sigma(i)}a_{ji} = a_{\tau(j)\tau(i)}b_{ji}
\]

for all \( i, j \in [n] \).

A natural isomorphism \( \varphi : \sigma \to \tau \) is given at each \( i \in [n] \) by a nonzero scalar \( c_i \in \mathbb{K} \) which are any set of scalars \( c_i \in \mathbb{K}^* \) satisfying the condition

\[
(0.2) \quad c_j a_{ji} = b_{ji} c_i
\]

for all \( i, j \in [n] \). This natural isomorphism is skew-continuous (Def. 3.3) if the following diagram anti-commutes for all \( i \).

\[
\begin{array}{ccc}
\sigma^2(i) & \overset{\sigma(\varphi(i))}{\longrightarrow} & \sigma(\tau(i)) \\
\downarrow & & \downarrow \\
\sigma^2(i) & \overset{\varphi_{\sigma(i)}}{\longrightarrow} & \tau(\sigma(i))
\end{array}
\]

In terms of our scalars \( a_{ji}, b_{ji}, c_i \) this condition is:

\[
(0.3) \quad -c_{\sigma(i)} = a_{\tau(i)\sigma(i)}c_i
\]

for all \( i \in [n] \).

One easy case is when \( \tau(i) = i \). Then Equations (0.2) and (0.3) imply that \( \sigma(i) \neq i \) and \( b_{\sigma(i)} = -1 \) which describes the new triangulation in Corollary (2). (See Fig. 3.)

### 0.7. Continuous Frobenius categories

In the last section, Section 6, we show the existence of the objects \( \overline{M}_n(\sigma, \tau, \varphi) \) of the classification Theorem D by constructing a continuous Frobenius category \( \text{add} \overline{M}_n(\mathbb{K}[[t]]) \).

Recall from Happel [1] that the stable category of a Frobenius category is triangulated. Following [3], we show that the stable category of the continuous Frobenius category \( \text{add} \overline{M}_n(\mathbb{K}[[t]]) \) is continuously triangulated in several ways determined by \( \tau \) and \( \varphi : \sigma \to \tau \).

Topologically, the subcategory \( \overline{M}_n(\mathbb{K}[[t]]) \) of indecomposable objects is a covering of the closed Moebius strip category. The stable category is the additive category of the corresponding open Moebius strip since the boundary consists of the projective-injective objects which become zero in the stable category.

### 1. Topological \( K \)-categories and examples

In this section we construct the key examples of topological \( K \)-categories: the continuous path categories \( \mathcal{P}_J \) with object space \( J \subseteq \mathbb{R} \), the quotient category \( \mathcal{R}_R = \mathcal{P}_{\mathbb{R}/\mathbb{Z}} \), the circle category \( S^1 \) with object space \( S^1 = \mathbb{R}/\pi\mathbb{Z} \) (the circle with diameter 1) and the Moebius band category \( \mathcal{M}_0 \). We begin with the basic definitions.
1.1. **Topological $K$-categories in general.**

**Definition 1.1.** By a *topological category* we mean a small category $\mathcal{R}$ so that the set of objects $\text{Ob}(\mathcal{R})$ and the set of morphisms $\text{Mor}(\mathcal{R})$ are topological spaces and the four structure maps of the category are continuous maps:

1. source and target maps: $s,t : \text{Mor} \to \text{Ob}$
2. identity map: $\text{id} : \text{Ob} \to \text{Mor}$
3. composition: $c : \text{Mor} \oplus \text{Mor} \to \text{Mor}$ where $\text{Mor} \oplus \text{Mor}$ is the subset of $\text{Mor}^2$ on which composition is defined.

Since $\text{Ob}(\mathcal{R})$ is a retract of $\text{Mor}(\mathcal{R})$, the topology on the object space is uniquely determined by the topology on the morphism space. A *continuous functor* between topological categories is a functor $F : \mathcal{B} \to \mathcal{C}$ which is a continuous mapping morphism spaces (and, consequently, also on object spaces). A natural transformation $\psi : F \to G$ is *continuous* if it is given by a continuous function $\psi : \text{Ob}(\mathcal{B}) \to \text{Mor}(\mathcal{C})$.

Recall that a *$K$-category* is an additive category $\mathcal{A}$ so that each hom set $\text{Hom}(A,B)$ is a vector space over $K$ and composition is $K$-bilinear. By a *topological $K$-category* we mean a $K$-category $\mathcal{A}$ which is also a topological category so that

1. The action of $K$ on hom sets gives a continuous mapping: $K \times \text{Mor}(\mathcal{A}) \to \text{Mor}(\mathcal{A})$.
2. $\text{Hom}(A,B)$ has the discrete topology for all $A,B$.

**Example 1.2.** For any topological space $X$, let $KX$ be the topological $K$-category with object space $X$ and morphisms being scalar multiples of identity morphisms and zero morphisms, i.e., the morphism space is $K \times X \cup X^2$ where $(0,x) \in K \times X$ is identified with $(x,x) \in X^2$. Composition is given in the obvious way. Any continuous mapping $f : X \to Y$ induces a continuous $K$-linear functor $Kf : KX \to KY$. Also, for any topological $K$-category $\mathcal{A}$ there is a continuous $K$-linear embedding $KX \to \mathcal{A}$ where $X$ is the space of nonzero objects of $\mathcal{A}$. We call $KX$ the *trivial $K$-category* generated by $X$.

**Definition 1.3.** Let $\mathcal{R}$ be a topological $K$-category. By a *continuous triangulation* of $\mathcal{R}$ we mean the structure of a triangulated category on $\mathcal{R}$ so that:

1. The shift functor $TX = X[1]$ is a continuous linear functor $T : \mathcal{B} \to \mathcal{B}$.
2. The set of distinguished triangles $X \to Y \to Z \to TX$ forms a closed subspace $\Delta \subset \text{Ob}(\mathcal{R})^3 \times \text{Mor}(\mathcal{R})^3$.

A *continuous triangle functor* between continuously triangulated $K$-categories $(\mathcal{B},T,\Delta)$, $(\mathcal{B}',T',\Delta')$ is a pair $(F,\psi)$ where $F : \mathcal{B} \to \mathcal{B}'$ is a continuous linear functor and $\psi : FT \cong T'F$ is a continuous natural $K$-linear isomorphism so that for every distinguished triangle $(X,Y,Z,f,g,h) \in \Delta$, $(FX,FY,FZ,Ff,Fg,\psi_X \circ Fh) \in \Delta'$. We say $(F,\psi)$ is a (strong) isomorphism of continuously triangulated categories if $F$ is an isomorphism and $FT = T'F$ (but $\psi : FT \cong T'F$ is not necessarily the identity).

**Remark 1.4.** If we drop the assumption $FT = F'T$ we get a notion of isomorphism which is too coarse for our purposes. For example, all continuous triangulations of a covering category $\widetilde{\mathcal{M}}_{\mathcal{P}}$ would be isomorphic, the classification would be reduced to covering theory and we would losing any concept of what are the distinguished triangles in each case.
Example 1.5. Given a topological triangulation \((T, \Delta)\) of a topological \(K\)-category \(\mathcal{R}\) and any nonzero scalar \(a \in K^*\), let \(\Delta_a\) be the set of all sextuples \((X, Y, Z, f, g, h)\) so that \((X, Y, Z, f, ag, h) \in \Delta\). Then \((T, \Delta_a)\) gives another triangulation of \(\mathcal{R}\) which is strongly isomorphic to the first triangulation. The strong isomorphism is the identity functor \(F = id_{\mathcal{R}}\) together with \(\psi_X = a \cdot id_{TX} : TX \to TX\). We say that the triangulation \((T, \Delta_a)\) is a rescaling of the triangulation \((T, \Delta)\).

In the construction of variations of the continuous cluster category we will construct only the full topological subcategory of indecomposable objects with 0 attached by one-point compactification. We recall that the one-point compactification \(X_+\) of a locally compact Hausdorff space \(X\) is given by adding a disjoint point \(*\), defining the open neighborhoods of \(*\) to be the complements in \(X_+\) of compact subsets of \(X\) and the other open subsets of \(X_+\) are defined to be the open subsets of \(X\). Recall that a continuous mapping \(f : X \to Y\) is called proper if the inverse image of every compact subset of \(Y\) is a compact subset of \(X\). Then the following basic fact follows directly from the definitions.

Lemma 1.6. Any proper mapping \(f : X \to Y\) between locally compact space \(X, Y\) induces a continuous mapping on one-point compactifications \(f_+ : X_+ \to Y_+\). □

Definition 1.7. Let \(\mathcal{B}\) be a topological \(K\)-category with a locally compact space of objects and no zero object. Then the one-point compactification category \(\mathcal{B}_+\) is defined to be the topological \(K\)-category whose object space is the one-point compactification of the object space of \(\mathcal{B}\) with the additional point being 0 and with morphism space

\[
\text{Mor}(\mathcal{B}_+) = \text{Mor}(\mathcal{B}) \cup 0 \times \text{Ob}(\mathcal{B}) \cup \text{Ob}(\mathcal{B}) \times 0
\]

with topology given as follows. The subspace \(\text{Mor}(\mathcal{B})\) is an open subspace of \(\text{Mor}(\mathcal{B}_+)\) with the same topology as before. A basic open neighborhood of any zero morphism \(0 : x_0 \to y_0\) where either \(x_0\) or \(y_0\) is zero is defined to be \(\mathcal{B}_+(U, V) := \{f : x \to y \mid x \in U, y \in V\}\) where \(U, V\) are open neighborhoods of \(x_0, y_0\) respectively in \(\text{Ob}(\mathcal{B}_+) = \text{Ob}(\mathcal{B})_+\). The algebraic structure of \(\mathcal{B}_+\) is the obvious one given by the fact that composition of any morphism with a zero morphism is zero.

A continuous \(K\)-linear functor \(F : \mathcal{B} \to \mathcal{C}\) between locally compact \(K\)-categories is proper if the induced map on object spaces is proper. Since the topology on \(\text{Mor}(\mathcal{B}_+)\) is given in terms of the topology on the object space, we have the following extension of the basic Lemma 1.6.

Proposition 1.8. A continuous proper \(K\)-linear functor \(F : \mathcal{B} \to \mathcal{C}\) between locally compact topological \(K\)-categories induces a unique continuous \(K\)-linear functor on one-point compactification categories: \(F_+ : \mathcal{B}_+ \to \mathcal{C}_+\). □

In all examples, the space of indecomposable objects will be locally compact. The entire category can then be canonically constructed using the James construction given as follows.

Definition 1.9. Let \(\mathcal{B}\) be a topological \(K\)-category with no zero object and with locally compact space of objects. Let \(\mathcal{B}_+\) be the one-point compactification category. Then we define the topological additive category generated by \(\mathcal{B}\) to be the category \(\text{add}^{\text{top}} \mathcal{B}\) (denoted
category $B$ in [4]) with object space given by the James construction with 0 as base point [2 p.224], i.e., it is a quotient of the union of all products

$$\text{Ob}(\text{add}^\text{top} B) = \coprod\text{Ob}(B_+)^n / \sim$$

with the quotient topology where the objects are ordered formal direct sums $X_1 \oplus \cdots \oplus X_n$ of nonzero objects of $B_+$ with summands deleted when they converse to 0. Morphisms spaces are given by products of morphism spaces: $\text{Hom}(\oplus X_i, \oplus Y_j) = \prod_{ij} \text{Hom}(X_i, Y_j)$, again with the quotient topology:

$$\text{Mor}(\text{add}^\text{top} B) = \coprod\text{Mor}(B_+)^{nm} / \sim$$

This is a strictly monoidal category which is not strictly symmetric since $f \oplus g \neq g \oplus f$ in general. See [4] for more details.

1.1.1. Quotient and orbit categories. Recall that an ideal in a topological $K$-category $B$ is a subset $\mathcal{I}$ of the morphism space of $B$ so that $\mathcal{I}(x, y) := \mathcal{I} \cap B(x, y)$ is a vector subspace of $B(x, y)$ for every $x, y \in B$ and so that, $f \circ g \in \mathcal{I}$ if either $f$ or $g$ is in $\mathcal{I}$. The quotient category $B/\mathcal{I}$ is the topological $K$-category with the same object space as $B$ but with hom-spaces $(B/\mathcal{I})(x, y) = B(x, y)/\mathcal{I}(x, y)$ so that the entire morphism space is given the quotient topology with respect to the surjective map $\text{Mor}(B) \rightarrow \text{Mor}(B/\mathcal{I})$. Any set of morphisms $X$ generates an ideal $\mathcal{I}_X$, namely the intersection of all ideals containing that set. Then $\mathcal{I}_X(x, y)$ is the vector space spanned by all morphisms $x \rightarrow y$ factoring through some element in $X$. The ideal generated by a set of objects is defined to be the ideal generated by the identity morphisms of those objects.

Any continuous linear functor $F : B \rightarrow B'$ which is zero on every morphism in $\mathcal{I}$ induces a unique continuous linear functor $B/\mathcal{I} \rightarrow B'$. Also, the kernel of $F$, the set of all morphisms in $B$ which go to zero in $B'$, is always an ideal.

Recall that the action of a discrete group $G$ on a space $X$ is called properly discontinuous if every $x_0 \in X$ has an open neighborhood $U$ so that $gU \cap hU = \emptyset$ when $g \neq h$ in $G$. In that case $X$ will be a covering space of the orbit space $X/G$.

Suppose that $B$ is a topological $K$-category and $F$ is a continuous linear automorphism of $B$ which acts properly discontinuously on object and morphism spaces of $B$, i.e., the action of the group $G$ of automorphism of $B$ generated by $F$ is properly discontinuous. The orbit category $B/F$ is the topological $K$-category with object and morphism spaces given by the orbits of the action of this group $G$. Note that any automorphism or endomorphism of $B$ which commutes with $F$ induces an automorphism/endomorphism of $B/F$.

1.2. Outline of construction of circle and Moebius band categories. First, we construct the continuous path category $\mathcal{P}_R$. This has object space $\mathbb{R}$ and morphisms $\mathcal{P}_R(x, y) = K$ if $x \leq y$ and $\mathcal{P}_R(x, y) = 0$ otherwise. The generator of $\mathcal{P}_R(x, y)$ (corresponding to $1 \in K$) is called the basic morphism. Composition of basic morphisms is defined to be a basic morphism. For any $J \subseteq \mathbb{R}$, $\mathcal{P}_J$ is the full subcategory with object set $J$.

Nonzero morphisms $f : x \rightarrow y$ in the path category have length $\ell(f) = y - x \geq 0$. For any $c > 0$, the truncated path category is given by $\mathcal{P}_c = \mathcal{P}_R/\mathcal{I}_c$ where $\mathcal{I}_c$ is the ideal of all
morphisms of length $\geq c$ (and all zero morphisms). Figure 1 illustrates the morphism set of this category.

We take $c = \pi$ and define the circle category to be $S^1 = \mathcal{R}_\pi / G_\pi = \mathcal{R}_\pi / G_{-\pi}$ where $G_t$, $t \in \mathbb{R}$, is the continuous family of continuous automorphisms of $\mathcal{R}_\pi$ given by sending $x$ to $x + t$ and basic morphisms to basic morphisms.

Let $\mathcal{P}^2_{\mathbb{R}} = \mathcal{P}_{\mathbb{R}} \otimes \mathcal{P}_{\mathbb{R}}$. This is the $K$-category with object space $\mathbb{R}^2$ and morphisms $\mathcal{P}^2_{\mathbb{R}}(x,y) = K$ if $x_1 \leq y_1$ and $x_2 \leq y_2$ and $\mathcal{P}^2_{\mathbb{R}}(x,y) = 0$ otherwise. Any composition of basic morphisms is a basic morphism. For any $c > 0$ this category has an ideal $J_c$ consisting of all morphisms which factor through an object $z = (z_1, z_2)$ with $|z_2 - z_1| \geq c$. Let $\mathcal{D}_c$ be the full subcategory of nonzero objects in the quotient category $\mathcal{P}^2_{\mathbb{R}} / J_c$. Then $\mathcal{D}_c$ is a locally compact $K$-category with object space equal to the set of all $(x, y) \in \mathbb{R}^2$ with $|x - y| < c$.

The topological $K$-category $\mathcal{D}_c$ has a continuous family of automorphisms $G^2_1(x) = x + (t, t)$ and taking basic morphisms to basic morphisms. Another continuous automorphism of $\mathcal{D}_c$ is $S(x_1, x_2) = (x_2, x_1)$. The Moebius band category is $\mathcal{M}_0 = \mathcal{D}_\pi / SG^2_\pi = \mathcal{D}_\pi / SG^2_{-\pi}$. The object space of $\mathcal{M}_0$ is the set of all $(x, y) \in \mathbb{R}^2$ so that $|x - y| < \pi$ modulo the equivalence relation $(x, y) \sim (y, \pi + x, x + \pi)$. This is homeomorphic to the space of all unordered pairs of distinct points on the circle $\tilde{S}^1 = \mathbb{R} / 2\pi \mathbb{Z}$ where $(x, y) \in \mathcal{M}_0$ corresponds to the pair $\{x, y + \pi\} \subset \tilde{S}^1$. These two elements of $\tilde{S}^1$ will be called the ends of the object $(x, y) \in \mathcal{M}_0$.

1.3. **Continuous path categories.** We construct two topological categories with object space $\mathbb{R}$.

**Definition 1.10.** Let $\mathcal{P}_\mathbb{R} = \mathcal{P}_\mathbb{R}(K)$ be the topological $K$-category with object space $\mathbb{R}$ and morphism space

$$\mathcal{M}or(\mathcal{P}_\mathbb{R}) := K \times \{(x, y) \in \mathbb{R}^2 \mid x \leq y\} \coprod 0 \times \mathbb{R}^2$$

where $K$ has the discrete topology. Composition of morphisms is defined by

$$(b, y, z) \circ (a, x, y) = (ab, x, z).$$

For any subset $J \subseteq \mathbb{R}$, $\mathcal{P}_J$ is the full subcategory of $\mathcal{P}_\mathbb{R}$ with objects space $J$. We refer to each $\mathcal{P}_J$ as a continuous path category. The morphism $(1, x, y)$ is called a basic morphism.

For any $c > 0$, let $\mathcal{I}_c$ be the ideal in $\mathcal{P}_\mathbb{R}$ generated by the morphisms $(1, x, x + c)$ for all $x \in \mathbb{R}$. Then $\mathcal{I}_c$ consists of all morphism of length $\geq c$ and all zero morphisms where the length of a morphism is defined to be $\ell(a, x, y) = y - x$.

The truncated path category $\mathcal{R}_c = \mathcal{R}_c(K)$ is defined to be the quotient category $\mathcal{R}_c = \mathcal{P}_\mathbb{R} / \mathcal{I}_c$. This is the topological $K$-category with object space $\text{Ob}(\mathcal{R}_c) = \mathbb{R}$ and morphisms

$$\mathcal{R}_c(x, y) = \begin{cases} K & \text{if } x \leq y < x + c \\ 0 & \text{otherwise} \end{cases}$$

The set of all morphism is:

$$\mathcal{M}or(\mathcal{R}_c) = 0 \times \mathbb{R}^2 \coprod K^* \times \{(x, y) \in \mathbb{R}^2 : y \in [x, x + c]\}$$
This has the quotient topology $\mathcal{M}or = \mathcal{M}or' / \sim$ where

$$\mathcal{M}or' = \mathbb{R}_+ \times \bigsqcup_{(x,y) \in \mathbb{R}^2} (x, y) \in \mathbb{R}^2 : y \in [x, x+c]$$

under the identification $(a, x, x+c) \sim (0, x, x+c) \in K^* \times \mathbb{R}^2$.

**Remark 1.11.** Quotient topology means the morphism $f : x \rightarrow y$ given by a fixed nonzero scalar $a \in K^*$ will converge to 0 when $y \rightarrow x + c$. However, a sequence of morphisms $f_i : x \rightarrow y_i$ with distinct scalars $a_i$ will not converge to anything, even if $y_i \rightarrow x + c$. The reason is that an open neighborhood of $(0, x, x+c)$ is given by a union over all $a \in K$ of open sets $(a, U_\varepsilon(x), U_\varepsilon(x+c))$ where $U_\varepsilon(z) = (z - \varepsilon, z + \varepsilon)$. By choosing $\varepsilon = |y_i - x|/2$ we get an open neighborhood of $(0, x, x+c)$ which avoid all of the points $f_i = (a_i, x, y_i)$.

![Diagram](A) Morphism space of $\mathcal{R}_c(K)$.

**(B)** $\mathcal{R}_c(x, y)$ goes to 0 when $y \rightarrow x + c$.

**Figure 1.** The space of nonzero morphisms of $\mathcal{R}_c(K)$ is a covering space of the contractible submanifold of $\mathbb{R}^2$ shaded in (A).

In [4, 3], the value $c = 2\pi$ was taken. In this paper, we will take $c = \pi$ and define $S^1 := \mathbb{R}/\pi\mathbb{Z}$, the “circle with diameter 1” (with the standard circle denoted $\tilde{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$). The purpose of this will be apparent later.

1.4. The circle category $S^1(K)$.

**Definition 1.12.** For any $t \in \mathbb{R}$ let $G_t$ be the continuous linear automorphism of $\mathcal{R}_\pi(K)$ given on objects by $G_t(x) = x + t$ and on morphisms by $G_t(a, x, y) = (a, x + t, y + t)$. (So, $G_0$ is the identity functor.) The circle category is defined to be the quotient of $\mathcal{R}_\pi(K)$ modulo the continuous automorphism $G_\pi$.

$$S^1 = S^1(K) := \mathcal{R}_\pi(K)/G_\pi$$

This is a topological $K$-category with object space the circle $S^1 = \mathbb{R}/\pi\mathbb{Z}$. Elements of $S^1$ will be denoted $[x] = x + \pi\mathbb{Z}$.

The fundamental domain of $G_\pi$ on objects is the closed interval $[0, \pi]$. (However, on morphisms, we should take $[0, 2\pi]$.) By definition, $G_\pi$ sends the right point $\pi$ of the interval $[0, \pi]$ to its left endpoint 0. So, $G_\pi$ gives the “holonomy” of the covering map $p : \mathcal{R}_\pi \rightarrow S^1$.

**Proposition 1.13.** The quotient functor $p : \mathcal{R}_\pi(K) \rightarrow S^1(K)$ is a universal covering map on object spaces and morphism spaces and all four of these spaces are Hausdorff.
Proof. The functor $G_\pi$ gives a properly discontinuous free $\mathbb{Z}$ action on object and morphism spaces of $\mathcal{R}_\pi(K)$ both of which are contractible and locally simply connected.

The object spaces $\text{Ob}((\mathcal{R}_\pi)) = \mathbb{R}$ and $\text{Ob}(S^1) = S^1$ are clearly Hausdorff as is $\text{Mor}(\mathcal{P}_\mathbb{R})$.

So, it suffices to consider $\text{Mor}(S^1)$. If $f \neq g$ have distinct source or target, they can be separated by disjoint open sets in $\text{Ob}(S^1)^2$. So, assume $f \neq g \in S^1(X,Y)$ and $f \neq 0$. Then $S^1(X,Y) = K$ and $g = af$ for some $a \in K$. Then $f,g$ lift to distinct nonzero morphisms $\tilde{f},\tilde{g} \in \mathcal{P}_\mathbb{R}(y,x)$. Since $\tilde{f}$ does not lie in the closed set $I_\pi$, there is a small connected open neighborhood $U$ of $\tilde{f}$ disjoint from $I_\pi$. Then $U,aU$ will be disjoint open neighborhoods of $\tilde{f},\tilde{g}$ giving disjoint open neighborhoods of $f,g$. □

For any two objects $x,y \in \mathbb{R} = \text{Ob}(\mathcal{R}_\pi)$ there is a unique integer $k$ so that $x \leq (G_\pi)^k(y) = y + \pi k < x + \pi$ and $\mathcal{R}_\pi(x,(G_\pi)^k(y)) = K$ in that case and $\mathcal{R}_\pi(x,(G_\pi)^j(y)) = 0$ for $j \neq k$. Thus, $S^1(X,Y) = K$ for all $X,Y \in S^1$. So, the morphism set is in bijection with $K \times S^1 \times S^1$. However, this bijection is not a homeomorphism since $(a,x,y)$ converges to $(0,x,x+\pi) \sim (0,x,x)$ as $y \to x + \pi$ (Fig [1] of \cite{1}). The space of nonzero morphisms of $S^1(K)$ is homeomorphic to $K^* \times S^1 \times [0,\pi]$.

Define the support of an object $x$ in a topological $K$-category $\mathcal{B}$ to be the set of all $y \in \text{Ob}(\mathcal{B})$ so that $\mathcal{B}(x,y) \neq 0$ with the quotient topology with respect to the target map $t : \text{Mor}(\mathcal{B}) \to \text{Ob}(\mathcal{B})$ restricted to the nonzero morphisms with source $x$. Then the support of $[x] \in S^1(K)$ is homeomorphic to $[x,x+\pi)$ which maps onto $S^1$ by a continuous bijection.  

1.5. Moebius strip category $\mathcal{M}_0(K)$. Let $\mathcal{P}_\mathbb{R}^2 = \mathcal{P}_\mathbb{R} \otimes \mathcal{P}_\mathbb{R}$ be the topological $K$-category with object space $\mathbb{R}^2$ and morphisms given by $\mathcal{P}_\mathbb{R}^2(x,y) = \mathcal{P}_\mathbb{R}(x_1,y_1) \otimes \mathcal{P}_\mathbb{R}(x_2,y_2)$. This is $K$ if $x_1 \leq y_1$, $x_2 \leq y_2$ and $0$ otherwise. We take the topology induced by the inclusion $\text{Mor}(\mathcal{P}_\mathbb{R}) \subseteq K \times \mathbb{R}^2$.

For any $c > 0$, let $\mathcal{J}_c$ be the ideal in $\mathcal{P}_\mathbb{R}^2$ generated by all objects $x = (x_1,x_2)$ so that $|x_1 - x_2| \geq c$. Let $\mathcal{D}_c = \mathcal{D}_c(K)$ be the full subcategory of nonzero objects in $\mathcal{P}_\mathbb{R}^2/\mathcal{J}_c$. This is the topological $K$-category with object space

$$\text{Ob}(\mathcal{D}_c) = \{x \in \mathbb{R}^2 \mid |x_2 - x_1| < c\}.$$

Note that, if $y_1 \geq x_2 + c$ then any morphism $(x_1,x_2) \to (y_1,y_2)$ will factor through $(x_2 + c,x_2) \in \mathcal{J}_c$. Therefore, the space of nonzero morphisms of $\mathcal{D}_c$ is given by

$$\text{Mor}^*(\mathcal{D}_c) = K^* \times \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 \leq y_1 < x_2 + c, x_2 \leq y_2 < x_1 + c\}.$$

As in the case of $\mathcal{R}_c(K)$, there is a zero morphism between any two objects and a morphism $(a,x,y) : x \to y$ converges to zero when either $y_1 \to x_2 + c$ or $y_2 \to x_1 + c$.

Proposition 1.14. $\mathcal{R}_c$ is isomorphic as topological $K$-category to the full subcategory of $\mathcal{D}_c$ of diagonal objects $x \in \mathbb{R}^2$ with $x_1 = x_2$. □

As before, we take $c = \pi$. See Figure [2].

Definition 1.15. To construct $\mathcal{M}_0$ we need two automorphisms of $\mathcal{D}_\pi$.

1. Let $S$ be the automorphism of $\mathcal{D}_\pi$ given by switching the coordinates: $S(x_1,x_2) = (x_2,x_1)$, $S(a,x,y) = (a,Sx,Sy)$.

2. For any $t \in \mathbb{R}$ let $G_t^2 = G_t \otimes G_t$ be the automorphism of $\mathcal{D}_\pi$ given by $G_t^2(x) = x + (t,t) = (G_t(x_1),G_t(x_2))$, $G_t^2(a,x,y) = (a,G_t^2(x),G_t^2(y))$.  

11
These are continuous $K$-linear automorphisms of $\mathcal{D}_\pi$ and $G^2_t, SG^2_t$ are fixed point free and properly discontinuous for $t \neq 0$. The Moebius band category is the orbit category

$$\mathcal{M}_0 := \mathcal{D}_\pi/SG^2_\pi.$$  

We take as fundamental domain of the action of $SG^2_\pi$ the set of all $x \in \mathbb{R}^2$ so that $0 \leq x_1 + x_2 \leq 2\pi$ (and morphism from such $x$ to those $y$ where $0 \leq y_1 + y_2 \leq 4\pi$). The point $SG^2_\pi(x) = (x_2 + \pi, x_1 + \pi)$ is identified with $x = (x_1, x_2)$.

![Figure 2. Support of $x$ in $\mathcal{D}_\pi(K)$ is shaded.](image)

The objects of $\mathcal{M}_0$ can be viewed as unordered pairs of point on the circle $\tilde{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ of radius 1. The pair $(a, b) \sim (b, a+2\pi)$ with $a < b < a+2\pi$ corresponds to $(a, b-\pi) \sim (b, a+\pi)$ in $\mathcal{M}_0$ since $|a + \pi - b| < \pi$.

Analogous to Proposition 1.13 we have the following.

**Proposition 1.16.** The quotient functor $p : \mathcal{D}_\pi(K) \to \mathcal{M}_0(K)$ is a universal covering map on object spaces and morphism spaces and all four spaces are Hausdorff.

1.6. **Contractible morphisms.** The topological $K$-categories $\mathcal{S}^1$ and $\mathcal{M}_0$ share the important property that every nonzero morphism is a scalar multiple of a “contractible morphism”.

**Definition 1.17.** A nonzero morphism $f : X \to Y$ in a topological $K$-category $\mathcal{C}$ will be called contractible if there is a continuous path $Z_t, t \in [0, 1]$, in the object space of $\mathcal{C}$ so that $Z_0 = X, Z_1 = Y$ and continuous families of morphisms $f_t : X \to Z_t, g_t : Z_t \to Y$ so that $f_0 = id_X, g_1 = id_Y$ and $g_t \circ f_t = f$ for all $t$. In particular, $f_1 = f$.

We observe that every faithful (no nonzero morphism goes to zero) continuous functor $\mathcal{C} \to \mathcal{D}$ sends contractible morphisms to contractible morphisms.

**Proposition 1.18.** Every nonzero morphism in $\mathcal{S}^1$ and $\mathcal{M}_0$ is a scalar multiple of a uniquely determined contractible morphism.

**Proof.** Any nonzero morphism in $\mathcal{S}^1$ or $\mathcal{M}_0$ can be lifted up to a nonzero morphism $(a, x, y)$ in its universal covering $\tilde{\mathcal{R}}_\pi$ or $\tilde{\mathcal{D}}_\pi$ which is unique determined up to deck transformations. This is $a$ times the basic morphism $(1, x, y)$ which is contractible since

$$(1, x, y) = (1, z_t, y)(1, x, z_t)$$

where $z_t = ty + (1-t)x$. 

\[\square\]
Definition 1.19. The support of any nonzero morphism \( f : X \to Y \) in a topological \( K \)-category \( C \) is defined to be the space of all \( Z \in \text{Ob}(C) \) so that \( f \) factors through \( Z \).

Proposition 1.20. The categories \( S^1 \) and \( \mathcal{M}_0 \) have the property that every nonzero morphism has contractible support.

Proof. This is immediate: Any morphism \( f \) in \( S^1 \) lifts to a morphism \( \tilde{f} : x \to y \in \mathbb{R} \) where \( x \leq y < x + 2\pi \). The support of \( f \) is the image in \( S^1 \) of the closed interval \([x, y]\) \( \subseteq \mathbb{R} \). This is contractible since \( |y - x| < 2\pi \). Similarly a nonzero morphism \( f \) in \( \mathcal{M}_0 \) lifts to a morphism \( \tilde{f} : (x_1, x_2) \to (y_1, y_2) \) and the support of \( f \) is the image of \([x_1, y_1] \times [x_2, y_2]\) which is a contractible subset of the support of \( x = (x_1, x_2) \). (See Figure 2.)

2. Equivalence coverings

We will define \( n \)-fold equivalence coverings of a topological \( K \)-category \( B \) and show that they are classified by their holonomy which is a homomorphism from the fundamental group of the object space of \( B \) to the group of automorphism of \( \mathcal{C}_n \). We assume that all objects of \( B \) are nonzero.

2.1. Basic topology, projective bundles. To do covering theory we assume that the object space of \( B \) is locally simply connected which means it has a basis for its topology of simply connected open sets. We recall that a basis for a topology on \( X \) is a collection of open subsets \( U_\alpha \) called basic open sets satisfying the following.

1. \( X \) is the union of all \( U_\alpha \).
2. The intersection of any two, say \( U_\alpha \cap U_\beta \) is a union of other basic open sets \( U_\gamma \).
3. A subset \( V \) of \( X \) is open if and only if it is the union of basic open sets \( U_\alpha \).

We all know that \( n \)-fold covering spaces \( p : \tilde{X} \to X \) of a connected and locally simply connected Hausdorff space \( X \) are classified by homomorphisms

\[ \sigma : \pi_1(X, x_0) \to S_n \]

for any \( x_0 \in X \) where \( S_n \) is the symmetric group on \( n \) “letters” which are taken to be the elements of \([n] = \{1, 2, \cdots, n\}\). We call \( \sigma \) the holonomy of the covering.

The categorical version of the \( n \)-point set \([n]\) is the finite category \( \mathcal{C}_n \). In good cases, an “equivalence covering” of \( B \) will be classified by a homomorphism

\[ \sigma : \pi_1(\text{Ob}(B), x_0) \to \text{Aut}(\mathcal{C}_n) \]

where homomorphisms \( \sigma, \sigma' \) give the same equivalence covering if and only if they differ by conjugation, i.e., there exists an element \( \tau \in \text{Aut}(\mathcal{C}_n) \) so that \( \sigma'(x) = \tau \sigma(x) \tau^{-1} \) for all \( x \in \pi_1(\text{Ob}(B), x_0) \).

By Proposition 2.3 below, \( \text{Aut}(\mathcal{C}_n) \) is a subgroup of \( PGL_n(K) \), the quotient of \( GL_n(K) \) modulo its center which is the group \( K^* I_n \) of nonzero scalar multiples of the identity matrix \( I_n \). (In particular, \( \text{Aut}(\mathcal{C}_1) \) is trivial, being a subgroup of \( PGL_1(K) = 1 \).) So, such a homomorphism \( \sigma \) also classifies a projective bundle over \( \text{Ob}(B) \). We recall the definition.
**Definition 2.1.** Given a topological space $X$ with basic open sets $\{U_\alpha\}$, we define a system of projective transition matrices for $X$ to be a family of “projective matrices” $g_{\beta\alpha} \in PGL_n(K)$ for all pairs of basic open sets $U_\alpha \subset U_\beta$ having the property that

\begin{equation}
 g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}
\end{equation}

whenever $U_\alpha \subset U_\beta \subset U_\gamma$. This implies, in particular, that $g_{\alpha\alpha}$ is the identity in $PGL_n(K)$. Two systems of projective matrices $\{g_{\beta\alpha}\}$ and $\{g′_{\beta\alpha}\}$ are equivalent if there exist elements $h_\alpha \in PGL_n(K)$ so that $g′_{\beta\alpha} = h_\beta g_{\beta\alpha} h_\alpha^{-1}$ for all $U_\alpha \subset U_\beta$. If $h = h_\alpha$ is independent of $\alpha$ we say that $\{g′_{\beta\alpha}\}$ is conjugate to $\{g_{\beta\alpha}\}$ by $h$.

If there exist liftings $\tilde{g}_{\beta\alpha}$ of $g_{\beta\alpha}$ to $GL_n(K)$ so that $\tilde{g}_{\gamma\beta} \tilde{g}_{\beta\alpha} = \tilde{g}_{\gamma\alpha}$ whenever $U_\alpha \subset U_\beta \subset U_\gamma$, we say the system $\{g_{\beta\alpha}\}$ is linearizable. A system of projective matrices $g_{\beta\alpha} \in PGL_n(K)$ for $X$ determines a (flat) projective bundle over $X$. If the system is linearizable, this bundle will be the projectivization of a flat vector bundle $E$ with projection $p : E \to X$ given by the construction:

$$E = \coprod_{\alpha} U_\alpha \times K^n / \sim$$

where we identify $(x, v) \in U_\alpha \times K^n$ with $(x, \tilde{g}_{\beta\alpha}(x)(v)) \in U_\beta \times K^n$ whenever $x \in U_\alpha \subset U_\beta$ with projection $p : E \to X$ given by $p(x, v) = x$.

An alternate description of such a vector bundle $p : E \to X$ is a family of vector spaces $E_x := p^{-1}(x)$ parametrized by $x \in X$ together with isomorphisms $h_\alpha : E_x \cong K^n$ for all basic open neighborhoods $U_\alpha$ of $x$ so that

$$h_\beta = \tilde{g}_{\beta\alpha} h_\alpha : E_x \cong K^n$$

whenever $x \in U_\alpha \subset U_\beta$. Then we give $E$ a topology so that $h_\alpha$ gives a homeomorphism $\pi^{-1}(U_\alpha) \cong U_\alpha \times K^n$.

**Example 2.2.** Suppose that $X$ is a connected space with a basis of simply connected open sets $U_\alpha$. Let $\sigma$ be a homomorphism $\pi_1(X, x_0) \to PGL_n(K)$. Then a system of projective transition matrices for $X$ can be given as follows.

1. For each $\alpha$, choose a point $x_\alpha \in U_\alpha$ and a path $\lambda_\alpha$ from $x_0$ to $x_\alpha$.
2. For any $U_\alpha \subset U_\beta$, let $g_{\beta\alpha} = \sigma(\lambda_\beta \gamma \lambda_\alpha^{-1})$ where $\gamma$ is any path from $x_\beta$ to $x_\alpha$ in $U_\beta$.

We call this a standard system of (projective) transition matrices given by $\sigma$.

2.2. **The category $C_n$.** An $n$-fold equivalence covering category of a topological $K$-category $B$ will be a topological $K$-category $\tilde{B}$ which is locally isomorphic to $B \otimes C_n$ (Def. 2.5) where $C_n$ is a finite category with $n$ isomorphic objects.

The properties of the category $C_n$, its endofunctors and natural transformations are discussed in detail in the Appendix. Here we give the essential definitions.

For every $n \geq 1$ let $C_n$ be the discrete $K$-category with object set $[n] = \{1, 2, \cdots, n\}$ and morphism sets $C_n(i,j) = K$ with composition given by multiplication of scalars. The morphism $i \to j$ corresponding to 1 is denoted $x_{ij}$. These satisfy the equation $x_{kj} x_{ji} = x_{ki}$ for all $i, j, k$ and the collection $(x_{ji})$ is called a multiplicative basis for $C_n$.

The object of $C_n$ are isomorphic to each other and the purpose of this category is to “fatten” a $K$-category $B$ by create $n$ isomorphic copies of each object.
Recall that the monomial matrix group $M_n(K^*)$ is the subgroup of $GL_n(K)$ of all matrices which are products $PD$ where $P$ is a permutation matrix and $D$ is a diagonal matrix with diagonal entries in $K^*$.

**Proposition 2.3.** There is an epimorphism $M_n(K^*) \to \text{Aut}(C_n)$ with kernel the center of $M_n(K^*)$ which is $K^*I_n$. Thus:

$$\text{Aut}(C_n) \cong M_n(K^*)/K^*I_n \subset PGL_n(K).$$

**Proof.** The homomorphism $M_n(K^*) \to \text{Aut}(C_n)$ is given by sending $PD$ to the automorphism $\sigma$ whose underlying permutation is given by $P$ in the sense that $Pe_i = e_{\sigma(i)}$ and whose transition coefficients are $a_{ij} = d_i/d_j$ where $d_i$ are the entries of $D$. For $\sigma$ to be the identity, $P$ must be the identity permutation and $d_i = d_j$ for all $i,j$, i.e., $PD \in K^*I_n$. An easy calculation shows that $PD \mapsto \sigma$ is a homomorphism. \hfill $\Box$

**Remark 2.4.** The monomial matrix $PD$ gives a natural isomorphism $id \cong \sigma$ which, on object $i$, is $d_ia_{ij} : i \to \sigma(i)$. This is natural, i.e., the following diagram commutes, since $d_ja_{ij} = d_i$.

$$\begin{array}{ccc}
j & \xrightarrow{d_j} & \sigma(j) \\
\downarrow x_{ij} & & \downarrow a_{ij}x_{\sigma(i)\sigma(j)} \\
i & \xrightarrow{d_i} & \sigma(i)
\end{array}$$

**Definition 2.5.** For a topological $K$-category $B$ without zero objects, we define $B \otimes_K C_n$ to be the topological $K$-category with object space $\text{Ob}(B) \times [n]$, the disjoint union of $n$ copies of $\text{Ob}(B)$, and morphisms

$$(B \otimes C_n)(X \otimes i, Y \otimes j) = B(X, Y) \otimes C_n(i, j) = B(X, Y) \times x_{ji}.$$  

Since $C_n(i, j) = Kx_{ji}$, morphism $X \otimes i \to Y \otimes j$ can be written uniquely as $f \otimes x_{ji}$ where $f \in B(X,Y)$ and we give the morphism space of $B \otimes C_n$ the topology of $\text{Mor}(B) \times \{x_{ij}\}$, the disjoint union of $n^2$ copies of $\text{Mor}(B)$.

For each $i \in [n]$, $B \otimes i \subseteq B \otimes C_n$ is a full subcategory isomorphic to $B$ and $id_X \otimes x_{ji}$ gives a continuous natural isomorphism $X \otimes i \cong X \otimes j$ for any $i,j \in [n]$.

A faithful $K$-linear functor $\tau : C_n \to C_m$ is given by a mapping $\tau : [n] \to [m]$ and nonzero scalars $b_{ji}$ so that

$$\tau(x_{ji}) = y_{\tau(i)}x_{\tau(j)\tau(i)}$$

where $(y_{pq})$ is a multiplicative basis for $C_m$. The scalars $b_{ji}, i,j \in [n]$ are called the transition coefficients of $\tau$. They satisfy the equation $b_{kj}b_{ji} = b_{ki}$. For any topological $K$-category $B$ we get an induced continuous faithful linear functor

$$id \otimes \tau : B \otimes C_n \rightarrow B \otimes C_m.$$  

2.3. **Equivalence coverings.** An equivalence covering of $B$ will be a category $\tilde{B}$ which looks locally like $B \otimes C_n$ with comparison maps of the form $id \otimes \sigma$ for $\sigma \in \text{Aut}(C_n)$.

In the sequel, we use the notation $B(U, V)$ for the space of all morphisms $x \to y$ in $B$ where $x \in U$ and $y \in V$. Given continuous mappings $h_1 : U \rightarrow U'$, $h_2 : V \rightarrow V'$ where $U', V' \subset \text{Ob}(D)$, a mapping of morphism sets $F : B(U, V) \rightarrow D(U', V')$ will be called
fiberwise linear over $h_1, h_2$ if $F$ sends $B(x, y)$ to $D(h_1(x), h_2(y))$ by a $K$-linear map for all $(x, y) \in U \times V$. For example, the morphism map of any linear functor is fiberwise linear over its object map.

**Definition 2.6.** An $n$-fold equivalence covering of $B$ is a topological $K$-category $\tilde{B}$ with maps $p, F_{\beta\alpha}, g_{\beta\alpha}$ satisfying the following for $\{U_\alpha\}$ a basis of open sets for $X = Ob(B)$.

1. $p : \tilde{X} = Ob(\tilde{B}) \to X = Ob(B)$ is an $n$-fold covering map. For each $U_\alpha$, let $\tilde{U}_\alpha = p^{-1}(U_\alpha)$ and assume there is $h_\alpha : \tilde{U}_\alpha \to U_\alpha \times [n]$ a homeomorphism over $U_\alpha$, i.e., $h_\alpha$ commutes with projection to $U_\alpha$.

2. For each $U_\alpha \subset U_\beta$,
   $$F_{\beta\alpha} : \tilde{B}(\tilde{U}_\alpha, \tilde{U}_\beta) \to (B \otimes C_n)(U_\alpha \times [n], U_\beta \times [n]) = B(U_\alpha, U_\beta) \times \{x_{ji}\}$$
   is a continuous fiberwise isomorphism over $h_\alpha, h_\beta$, i.e., $F_{\beta\alpha}$ gives a $K$-linear isomorphism $F_{\beta\alpha} : \tilde{B}(\tilde{X}, \tilde{Y}) \cong B(X, Y) \times x_{ji}$ if $h_\alpha(X) = X \otimes i$ and $h_\beta(Y) = Y \otimes j$, which also satisfies the following.
   (a) $F_{\alpha\alpha}(id_x) = id_y \times x_{ji}$ if $h_\alpha(x) = (y, i)$.
   (b) $F_{\gamma\beta}(g)F_{\beta\alpha}(f) = F_{\gamma\alpha}(gf)$ if $f \in \tilde{B}(\tilde{U}_\alpha, \tilde{U}_\beta)$ and $g \in \tilde{B}(\tilde{U}_\alpha, \tilde{U}_\gamma)$ are composable.

3. There are automorphisms $g_{\beta\alpha} \in Aut(C_n)$ for all $U_\alpha \subset U_\beta$ satisfying the compatibility condition (2.1) whose underlying permutation matrix $\pi_{\beta\alpha}$ gives
   $$h_\beta h_\alpha^{-1} = (inc : U_\alpha \hookrightarrow U_\beta) \times \pi_{\beta\alpha} : U_\alpha \times [n] \to U_\beta \times [n]$$
   and so that the following diagram commutes whenever $U_\alpha \subset U_\alpha' \subset U_\beta' \subset U_\beta$.

\[
\begin{array}{ccc}
\tilde{B}(\tilde{U}_\alpha, \tilde{U}_\beta) & \xrightarrow{F_{\beta\alpha}} & B(U_\alpha, U_\beta) \times \{x_{ji}\} \\
\downarrow & & \downarrow \\
\tilde{B}(\tilde{U}_\alpha', \tilde{U}_\beta') & \xrightarrow{F_{\beta'\alpha'}} & B(U_\alpha', U_\beta') \times \{x_{ji}\}
\end{array}
\]

where, if the automorphism $g_{\beta'\beta}g_{\alpha'\alpha}^{-1}$ of $C_n$ sends $x_{ji}$ to the scalar multiple $c x_{\pi(j)\pi'(i)}$ of $x_{\pi(j)\pi'(i)}$, then $inc \otimes g_{\beta'\beta}g_{\alpha'\alpha}^{-1}$ sends $(f, x_{ji})$ to $(cf, x_{\pi(j)\pi'(i)})$.

**Theorem 2.7.** An $n$-fold equivalence covering of $B$ is completely determined up to continuous isomorphism by the family of compatible automorphisms $(g_{\beta\alpha})$ of $Aut(C_n)$. When $Ob(B)$ is connected and locally 1-connected, this system gives a homomorphism
$$\sigma : \pi_1(Ob(B), X_0) \to Aut(C_n)$$
well-defined up to conjugation for any object $X_0$. Conversely, any such homomorphism gives an $n$-fold covering $\tilde{B}_\sigma$ of $B$.

**Proof.** It is standard covering theory that, for any discrete group $G$ and system of elements $g_{\beta\alpha} \in G$ satisfying the compatibility condition $g_{\gamma\beta}g_{\beta\alpha} = g_{\gamma\alpha}$ gives a covering space of $X$ and a homomorphism $\pi_1(X, x_0) \to G$ for any $x_0 \in X$. For $X$ connected and locally 1-connected any such homomorphism $\sigma$ is realized by taking $\{g_{\beta\alpha}\}$ to be the standard system given by $\sigma$ as in Example 2.22. Thus the only thing to show is that the other structure elements of an $n$-fold equivalence cover of $B$ are determined up to isomorphism by $\{g_{\beta\alpha}\}$.
The permutation part of $g_{βα}$, denoted $π_{βα} ∈ S_n$, determines the covering space $\tilde{X}$ and covering map $p : \tilde{X} → X = Ob(\mathcal{B})$ up to isomorphism. We can also choose the local trivializations $h_α : \tilde{U}_α ≃ U_α × [n]$ arbitrarily. Then (1) is satisfied.

The morphism set $\text{Mor}(\tilde{\mathcal{B}})$ is given by pasting together the sets $\mathcal{B}(U_α, U_β) × \{x_{ji}\}$:

$$\text{Mor}(\tilde{\mathcal{B}}) = \coprod \mathcal{B}(U_α, U_β) × \{x_{ji}\} / \sim$$

and giving this the quotient topology where the isomorphisms $id ⊗ g_{β'β}g_{α'α}^{-1}$ are used to paste together the different copies of each morphism set. The compatibility condition (2.1) insures that these identifications are consistent. This gives (3).

It remains to prove (2)(b) which implies (2)(a). But, composable morphisms $f ∈ \tilde{\mathcal{B}}(U_α, U_β), g ∈ \tilde{\mathcal{B}}(U_β, U_γ)$ are given by $f = (f_0 ⊗ x_{ji}), g = (g_0 ⊗ x_{kj})$ with composition $gf = (g_0f_0 ⊗ x_{ki})$. □

**Example 2.8.** Let $X$ be a connected and locally simply connected topological space and let $KX$ be the trivial $K$-category of $X$ (Example 1.2). For any homomorphism $σ : π_1(X, x_0) → Aut(\mathcal{C}_n)$ consider the corresponding equivalence covering $KX_σ$ of $KX$.

Let $\mathcal{K}_σX$ be the topological subcategory of $KX_σ$ containing all of the objects but only those morphisms lying over the same point in $X$. The fiber $\mathcal{K}_σX_{x_0}$ of $\mathcal{K}_σX$ over $x_0 ∈ X$ is isomorphic to $\mathcal{C}_n$. For any loop $γ$ in $X$ starting and ending at $x_0$, unique lifting of paths in $X$ to the covering spaces of objects and morphisms of $\mathcal{K}_σX$ gives an automorphism of $\mathcal{K}_σX_{x_0}$. This gives a automorphism $τ[γ]$ of $\mathcal{K}_σX_{x_0}$ which induces a well-defined homomorphism

$$τ : π_1(X, x_0) → Aut(\mathcal{K}_σX_{x_0}).$$

Conjugating with an isomorphism $\mathcal{K}_σX_{x_0} ≃ \mathcal{C}_n$ gives a homomorphism $π_1(X, x_0) → Aut(\mathcal{C}_n)$ well-defined up to conjugation. By construction, this must be conjugate to $σ$.

### 2.4. Classification of equivalence coverings.

We will show that, under certain conditions, equivalence coverings of a topological $K$-category $\mathcal{B}$ are classified by their holonomy up to conjugation. The statement is that $\tilde{\mathcal{B}}_σ$ and $\tilde{\mathcal{B}}_{σ'}$ are continuously isomorphic “over $\mathcal{B}$” if and only if $σ, σ'$ are conjugate. We need the following definition to overcome the problem that there is, in general, no continuous functor $\tilde{\mathcal{B}}_σ → \mathcal{B}$.

**Definition 2.9.** Let $\tilde{\mathcal{B}}_σ, \tilde{\mathcal{B}}_{σ'}$ be two equivalence coverings of the same topological $K$-category $\mathcal{B}$. A continuous linear functor $F : \tilde{\mathcal{B}}_σ → \tilde{\mathcal{B}}_{σ'}$ will be called a continuous equivalence over $\mathcal{B}$ if the object map of $F$ is a map of covering spaces over $Ob\mathcal{B}$, i.e., the following diagram commutes.

$$\begin{array}{ccc}
Ob\tilde{\mathcal{B}}_σ & \xrightarrow{F} & Ob\tilde{\mathcal{B}}_{σ'} \\
\downarrow & & \downarrow \\
Ob\mathcal{B} & & Ob\mathcal{B}
\end{array}$$

If the functor $F$ is also an isomorphism, it will be called a continuous isomorphism over $\mathcal{B}$.

**Lemma 2.10.** Suppose $Ob\mathcal{B}$ is connected and locally simply connected and let $\tilde{\mathcal{B}}_σ$ and $\tilde{\mathcal{B}}_{σ'}$ be $n$ and $m$-fold equivalence coverings with holonomies $σ$ and $σ'$ respectively. Let $τ : \mathcal{C}_n → \mathcal{C}_m$
be a faithful linear functor with the property that \( \sigma'(\gamma) \circ \tau = \tau \circ \sigma(\gamma) : C_n \rightarrow C_m \) for any \( \gamma \in \pi_1(\text{Ob}\mathcal{B}, X_0) \). Then there is a continuous equivalence

\[ F_\tau : \tilde{\mathcal{B}}_\sigma \rightarrow \tilde{\mathcal{B}}_{\sigma'} \]

over \( \mathcal{B} \) given on \( \mathcal{B}(U_\alpha, U_\beta) \otimes C_n \), using standard coordinates, by \( \text{id} \otimes \tau \). Thus, \( F_\tau F_{\tau'} = F_{\tau \tau'} \).

In particular, when \( \tau \) is an isomorphism, \( F_\tau \) is an isomorphism with inverse \( F_{\tau^{-1}} \).

**Proof.** It suffices to show that the mappings \( \text{id} \otimes \tau \) are compatible with the standard identifications defining \( \tilde{\mathcal{B}}_\sigma \) and \( \tilde{\mathcal{B}}_{\sigma'} \), i.e., that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{B}(U_\alpha, U_\beta) \otimes C_n & \xrightarrow{\text{id} \otimes \tau} & \mathcal{B}(U_\alpha, U_\beta) \otimes C_m \\
\downarrow \text{inc} \otimes g_{\beta \alpha}^{-1} & & \downarrow \text{inc} \otimes g_{\beta \alpha}^{-1} \\
\mathcal{B}(U_\alpha', U_{\beta'}) \otimes C_n & \xrightarrow{\text{id} \otimes \tau} & \mathcal{B}(U_\alpha', U_{\beta'}) \otimes C_m
\end{array}
\]

Assuming \( \{g_{\beta \alpha}\}, \{g'_{\beta \alpha}\} \) are standard (Example 2.2), this is the following equation:

\[
\tau_\sigma(\lambda_{\beta' \gamma} \lambda_{\beta}^{-1}) \sigma(\lambda_{\alpha' \gamma} \lambda_{\alpha}^{-1}) = \sigma'(\lambda_{\beta' \gamma} \lambda_{\beta}^{-1}) \sigma'(\lambda_{\alpha' \gamma} \lambda_{\alpha}^{-1}) \tau
\]

which follows from the assumption that \( \sigma'(\gamma) = \tau \sigma(\gamma) \tau^{-1} \) for any loop \( \gamma \) in \( \text{Ob}\mathcal{B} \).

\[ F_\tau F_{\tau'} = F_{\tau \tau'} \] since, in standard coordinates, it is given by \( (\text{id} \otimes \tau)(\text{id} \otimes \tau') = \text{id} \otimes \tau \tau' \). \( \square \)

**Theorem 2.11 (Thm A).** Suppose that \( \mathcal{B} \) is a topological \( K \)-category so that every object is nonzero and every automorphism of every object is a scalar multiple of the identity. Suppose also that \( \text{Ob}\mathcal{B} \) is connected and locally simply connected. Then, for any homomorphism

\[ \sigma : \pi_1(\text{Ob}\mathcal{B}, X_0) \rightarrow \text{Aut}(C_n) \]

there exists an equivalence covering \( \tilde{\mathcal{B}}_\sigma \) of \( \mathcal{B} \) with holonomy \( \sigma \). Furthermore, any two equivalence coverings of \( \mathcal{B} \) are continuously isomorphic over \( \mathcal{B} \) if and only if their holonomies are conjugate.

**Proof.** By Lemma 2.10 two equivalence coverings of \( \mathcal{B} \) with conjugate holonomies are continuously isomorphic over \( \mathcal{B} \).

Conversely, suppose \( \tilde{\mathcal{B}}_\sigma \) and \( \tilde{\mathcal{B}}_{\sigma'} \) are continuously isomorphic over \( \mathcal{B} \). Let \( \tilde{\mathcal{B}}_\sigma \) be the \( n \)-fold equivalence covering of \( \mathcal{B} \) with holonomy \( \sigma \). The conditions on \( \mathcal{B} \) imply that \( \tilde{\mathcal{B}}_\sigma \) contains a topological \( K \)-subcategory \( \tilde{\mathcal{B}}_\sigma^* \) isomorphic to the equivalence covering \( \tilde{K}X_\sigma \) from Example 2.8 with \( X = \text{Ob}\mathcal{B} \). This subcategory is characterized by the property that any nonzero morphism \( x \rightarrow y \) in \( \tilde{\mathcal{B}}_\sigma \) lies in \( \tilde{\mathcal{B}}_\sigma^* \) if and only if \( x, y \in \text{Ob}\mathcal{B}_\sigma \) lie over the same point in \( X = \text{Ob}\mathcal{B} \). Therefore, the isomorphism \( \tilde{\mathcal{B}}_\sigma \cong \tilde{\mathcal{B}}_\tau \) restricts to an isomorphism \( \tilde{\mathcal{B}}_\sigma^* \cong \tilde{\mathcal{B}}_\tau^* \) giving an isomorphism \( \tilde{K}X_\sigma \cong \tilde{K}X_\tau \). The formula for the holonomy of \( \tilde{K}X_\sigma \) given in Example 2.8 then implies that \( \sigma \) and \( \tau \) are conjugate as claimed. \( \square \)

**Theorem 2.12.** Let \( \mathcal{B} \) be as in Theorem 2.11 above and suppose that every nonzero morphism in \( \mathcal{B} \) is a scalar multiple of a contractible morphism. Then any continuous equivalence \( F : \tilde{\mathcal{B}}_\sigma \rightarrow \tilde{\mathcal{B}}_{\sigma'} \) is equal to \( F_\tau \) for some discrete equivalence \( \tau : C_n \rightarrow C_m \) satisfying \( \sigma'(\gamma) \circ \tau = \tau \circ \sigma(\gamma) \) for all loops \( \gamma \) in \( \text{Ob}\mathcal{B} \).
Proof. As in the proof of Theorem 2.11 let $\tilde{B}_\sigma^*$ and $\tilde{B}_\sigma'^*$ denote the subcategories of $\tilde{B}_\sigma$ and $\tilde{B}_\sigma'$ isomorphic to the equivalence coverings $\tilde{KX}_\sigma$ and $\tilde{KX}_{\sigma'}$, respectively, where $X = \text{Ob}\mathcal{B}$.

The equivalence $F$ induces an equivalence $F' : \tilde{KX}_\sigma \to \tilde{KX}_{\sigma'}$ which, over the object $X_0 \in \text{Ob}\mathcal{B}$, restricts to an equivalence $\tau : \mathcal{C}_n \to \mathcal{C}_m$. By the unique path lifting property for covering space, the morphism $F'$ must be equal to $F'_\tau$, the equivalence induced by $\tau$. The assumption that every nonzero morphism in $\mathcal{B}$ is a scalar multiple of a contractible morphism implies that every nonzero morphism in every equivalence covering of $\mathcal{B}$ is homotopic through nonzero morphisms to an isomorphism. This implies that $F$ is uniquely determined by its restriction to $\tilde{B}_\sigma^*$. Therefore, $F = F_\tau$ as claimed. □

In the special cases $\mathcal{B} = \mathcal{S}^1$ and $\mathcal{M}_0$, the fundamental group is $\mathbb{Z}$. So, the holonomy morphism $\pi_1 \text{Ob}\mathcal{B} = \mathbb{Z} \to \text{Aut}(\mathcal{C}_n)$ is determined by the image of the generator which is a single element $\sigma \in \text{Aut}(\mathcal{C}_n)$. We denote the corresponding equivalence coverings of $\mathcal{S}^1, \mathcal{M}_0$ by $\tilde{S}_\sigma, \tilde{1}_\sigma$. By Proposition 1.18 both categories satisfy the conditions of Theorem 2.12. Therefore, we have the following.

Corollary 2.13 (Thm B). Continuous equivalences $F : \tilde{S}_\sigma^1 \to \tilde{S}_{\sigma'}^1$ and $\tilde{M}_\sigma \to \tilde{M}_{\sigma'}$ are equal to $F_\tau$ where $\tau$ is a linear equivalence $\mathcal{C}_n \to \mathcal{C}_m$ so that $\tau \circ \sigma = \sigma' \circ \tau$. □

Remark 2.14. Similar to the observation in Example 2.8 we note that $\tau$ is not uniquely determined in Theorem 2.12 and Corollary 2.13. For example, the functor $F$ determines a linear equivalence from a subcategory of $\tilde{S}_\sigma^1$ isomorphic to $\mathcal{C}_n$ to a subcategory of $\tilde{S}_{\sigma'}^1$ isomorphic to $\mathcal{C}_m$. The choices of these isomorphisms determines $\tau$.

3. Skew-continuous natural transformations

For any two autoequivalences $\sigma, \tau$ of $\mathcal{C}_n$ there is a natural isomorphism $\varphi : \sigma \to \tau$. This natural isomorphism will be unique up to one scalar which we call the “rescaling factor” and “continuous” up to another scalar which we call the “$\sigma$-continuity factor”. The natural isomorphism is “skew-continuous” if its $\sigma$-continuity factor is $-1$.

We give the definitions and a brief explanation why the axioms of a triangulated category require $\varphi : \sigma \to \tau$ to be skew-continuous.

3.1. Definition of skew-continuity. We will show that a natural isomorphism $\varphi : \sigma \to \tau$ always exists and is uniquely determined up to a rescaling factor $r \in K^*$. For example, the only natural isomorphisms $\sigma \to \sigma$ are scalar multiples of the identity.

Proposition 3.1. Given autoequivalences $\sigma, \tau$ of $\mathcal{C}_n$ with transition coefficients $a_{ji}, b_{ji}$, there is a unique $\tilde{c} = [c_1, \cdots, c_n] \in K^{n-1}$, with $c_i \in K^*$, so that

$$c_ja_{ji} = b_{ji}c_i$$

for all $i, j \in \mathcal{C}_n$. Each lifting $(c_i) \in (K^*)^n$ of $\tilde{c}$ gives a natural isomorphism $\varphi : \sigma \to \tau$ by

$$\varphi_i = c_ix_{\tau(i)\sigma(i)} : \sigma(i) \to \tau(i).$$

Conversely, all natural isomorphisms $\varphi : \sigma \to \tau$ are given in this way.
Proof. Existence of $c_i$ so that $c_j a_{ji} = b_{ji} c_i$ is clear: Take $c_i = a_{1i} b_{11} r$ for any $r \in K^*$. Then $b_{ji} c_i = a_{1i} b_{1j} b_{11} r = a_{1j} a_{ji} b_{11} r = c_j a_{ji}$.

Conversely, the equation $c_j a_{ji} = b_{ji} c_i$ implies, for $j = 1$, that $c_i = c_1 a_{1i} b_{11}^{-1} = a_{1i} b_{11} c_1$. So, these are all the choices. So, $\tau \in K P^{n-1}$ is uniquely determined.

Any natural isomorphism $\varphi : \sigma \to \tau$ is, by definition, given by $\varphi_i = c_i x_{\tau(i)\sigma(i)}$ for $c_i \in K^*$ making the following diagram commute for all $i, j \in C_n$:

\[
\begin{array}{ccc}
\sigma(i) & \xrightarrow{c_i x_{\tau(i)\sigma(i)}} & \tau(i) \\
\downarrow a_{ji} x_{\sigma(j)\sigma(i)} & & \downarrow b_{ji} x_{\tau(j)\tau(i)} \\
\sigma(j) & \xrightarrow{c_j x_{\tau(j)\sigma(j)}} & \tau(j)
\end{array}
\]

In other words, $b_{ji} c_i = c_j a_{ji}$, i.e., $(c_i)$ is a representative of $\tau$. \hfill $\square$

If $\sigma, \tau$ are autoequivalences of $C_n$ which commute and $\varphi : \sigma \to \tau$ is a natural isomorphism then $\sigma(\varphi)$ and $\varphi(\sigma)$ are two natural isomorphisms

$$\sigma(\varphi) \to \sigma(\tau) = \tau(\sigma)$$

By Proposition 3.1 there is a unique scalar $s \in K^*$ so that

$$\sigma(\varphi) = s \varphi(\sigma).$$

We call $s$ the $\sigma$-continuity factor of $\varphi$. Note that any scalar multiple of $\varphi$ has the same $\sigma$-continuity factor. So, $s$ depends only on $\sigma$ and $\tau$ and not on $\varphi$. We use the notation:

$$s = \{\sigma, \tau\}$$

We show in the appendix that $\{\tau, \sigma\} = \{\sigma, \tau\}^{-1}$.

**Definition 3.2.** Commuting autoequivalences $\sigma, \tau$ of $C_n$ will be called compatible, resp. anti-compatible, if $\{\sigma, \tau\} = 1$, resp., $\{\sigma, \tau\} = -1$.

**Definition 3.3.** A natural transformation $\varphi : \sigma \to \tau$ between commuting autoequivalences of $C_n$ is continuous or skew-continuous (with respect to $\sigma$) if its $\sigma$-continuity factor is 1 or $-1$, respectively. In other words, for any $i \in C_n$, the square in the following diagram commutes or anticommutes, respectively.

\[
\begin{array}{ccc}
\sigma(i) & \xrightarrow{\sigma^2(i)} & \sigma^2(i) \\
\varphi_i & \downarrow \sigma(\varphi_i) & \downarrow \varphi(\sigma(i)) \\
\tau(i) & \xrightarrow{\tau(\sigma(i))} & \tau(\sigma(i))
\end{array}
\]

Equivalently, using the scalars $c_i$ so that $\varphi_i = c_i x_{\tau(i)\sigma(i)}$,

\[
(3.1) \quad c_{\sigma(i)} = c_i a_{\tau(i)\sigma(i)}, \quad \text{resp.} \quad c_{\sigma(i)} = -c_i a_{\tau(i)\sigma(i)}.
\]

The following summary of the above definitions and observations implies Lemma $C$ in the introduction.

**Proposition 3.4.** Let $\sigma, \tau$ be commuting autoequivalences of $C_n$. Then any natural isomorphism $\varphi : \sigma \to \tau$ has $\sigma$-continuity factor equal to $\{\sigma, \tau\}$. In particular, $\varphi$ is continuous, resp. skew-continuous if and only if $\sigma, \tau$ are compatible, resp. anti-compatible. \hfill $\square$
3.2. Interpretation of skew-continuity. Given an automorphism $\sigma$ of $\mathcal{C}_n$ and an autoequivalence $\tau$ on $\mathcal{C}_n$ which commutes with $\sigma$ we have, by Corollary 2.13, continuous linear functors:

$$F_{\tau} : \widetilde{S}^1_{\sigma} \to \widetilde{S}^1_{\sigma}, \quad F_{\sigma} : \widetilde{\mathcal{M}}_{\sigma} \to \widetilde{\mathcal{M}}_{\sigma}.$$ 

Since $\sigma$ obviously commutes with $\sigma$ we also have continuous automorphisms $F_{\sigma} : \widetilde{S}^1_{\sigma} \to \widetilde{S}^1_{\sigma}$ and $F_{\sigma} : \widetilde{\mathcal{M}}_{\sigma} \to \widetilde{\mathcal{M}}_{\sigma}$.

A natural transformation $\varphi : \sigma \to \tau$ induces a continuous natural transformation $F_{\varphi} : F_{\sigma} \to F_{\tau}$ if and only if $\varphi$ is "continuous", i.e., if $\{\sigma, \tau\} = 1$. More precisely:

**Theorem 3.5.** Let $\sigma_1, \sigma_2$ be automorphisms of $\mathcal{C}_n, \mathcal{C}_m$ respectively. Let $\tau_1, \tau_2$ be faithful $K$-linear functors $\mathcal{C}_n \to \mathcal{C}_m$ so that $\sigma_j \tau_j = \tau_j \sigma_1$ for $j = 1, 2$. Then, a natural transformation $\varphi : \tau_1 \to \tau_2$ extends to continuous natural transformations $F_{\varphi} : F_{\tau_1} \to F_{\tau_2}$, i.e., continuous maps $\text{Ob}(\widetilde{S}^1_{\sigma_1}) \to \text{Mor}(\widetilde{S}^1_{\sigma_2})$ and $\text{Ob}(\widetilde{\mathcal{M}}_{\sigma_1}) \to \text{Mor}(\widetilde{\mathcal{M}}_{\sigma_2})$ giving natural transformations $F_{\tau_1} \to F_{\tau_2}$, if and only if $\sigma_2(\varphi_i) = \varphi_{\sigma_1(i)}$ for every $i \in \mathcal{C}_n$.

**Proof.** (for $\widetilde{S}^1$) For each $i \in \mathcal{C}_n$, $j \in \{1, 2\}$ and $t \in [0, 2\pi)$ we have $F_{\tau_j}(t \otimes i) = t \otimes \tau_j(i)$ and

$$F_{\varphi}(t \otimes i) = \text{id}_{[0,2\pi]} \otimes \varphi_i : F_{\tau_1}(t \otimes i) = t \otimes \tau_1(i) \xrightarrow{\sigma_{\tau_1(i)}} t \otimes \tau_2(i) = F_{\tau_2}(t \otimes i).$$

In order for $F_{\varphi}$ to be continuous, we must have

$$F_{\varphi} \lim_{t \to 2\pi} (t \otimes i) = \lim_{t \to 2\pi} F_{\varphi}(t \otimes i).$$

But, the left hand side is equal to

$$F_{\varphi}(0 \otimes \sigma_1(i)) = \text{id}_{[0,2\pi]} \otimes \varphi_{\sigma_1(i)} : 0 \otimes \tau_1(\sigma_1(i)) \xrightarrow{\sigma(\tau_1(i))} 0 \otimes \tau_2(\sigma_1(i))$$

and the right hand side is equal to

$$\text{id}_{[0,2\pi]} \otimes \sigma_2(\varphi_i) : 0 \otimes \tau_2(\tau_1(i)) \xrightarrow{\sigma(\tau_1(i))} 0 \otimes \tau_2(\tau_1(\sigma_1(i))).$$

These are equal if and only if $\sigma_2 \tau_j = \tau_j \sigma_1$ for $j = 1, 2$ and $\sigma_2(\varphi_i) = \varphi_{\sigma_1(i)}$. \qed

We explain why the rotation axiom for a triangulated category forces us to use a skew-continuous natural transformation $\varphi : \sigma \to \tau$ in the definition of distinguished triangles in $\text{add} \widetilde{\mathcal{M}}_{\sigma}$. We take a "basic positive triangle" which is the simplest example of a distinguished triangle in $\text{add} \widetilde{\mathcal{M}}_{\sigma}$ where all terms are indecomposable (and thus lie in $\widetilde{\mathcal{M}}_{\sigma}$):

$$X \otimes i \xrightarrow{f} Y \otimes i \xrightarrow{g} Z \otimes i \xrightarrow{\psi} T(X \otimes i) = X \otimes \tau(i)$$

(3.2)\]

Here $X = (x, y)$, $Y = (x, z)$, $Z = (y + \pi, z)$, $f, g$ are the unique contractible morphisms (Def. 1.17) with the given sources and targets and $\psi$ is the composition:

$$\psi : Z \otimes i \xrightarrow{h} \text{SG}^2 Z \otimes i = X \otimes \sigma(i) \xrightarrow{\text{id}_X \otimes \varphi_i} X \otimes \tau(i)$$

of the contractible morphism $h : (y + \pi, z) \otimes i \to (y + \pi, x + \pi) \otimes i = (x, y) \otimes \sigma(i) = X \otimes \sigma(i)$ and a natural isomorphism $F_{\varphi} = \text{id}_X \otimes \varphi_i : F_\sigma(X \otimes i) = X \otimes \sigma(i) \cong F_\tau(X \otimes i) = X \otimes \tau(i)$. 

\[21\]
Now consider the following diagram.

\[
\begin{array}{c}
X \otimes \sigma(i) \xrightarrow{\sigma(f)} Y \otimes \sigma(i) \xrightarrow{\sigma(g)} Z \otimes \sigma(i) \xrightarrow{\sigma(h)} SG^2 \otimes X \otimes \sigma(i) \xrightarrow{id \otimes \tau \sigma(i)} X \otimes \sigma^2(i) \xrightarrow{id \otimes \varphi \sigma(i)} X \otimes \tau \sigma(i) \\
\end{array}
\]

The top row is the distinguished triangle (3.2) at \( \sigma(i) \) instead of \( i \). The morphisms \( \sigma(f), \sigma(g), \sigma(h) \) are the required contractible morphisms since \( \sigma \), being continuous, takes contractible morphisms to contractible morphisms. The second row is \( F_\tau \) applied to (3.2).

Since \( \varphi \) is a natural transformation, the first three squares in this diagram commute. The fourth square commutes since \( SG^2 \otimes id = id \otimes \sigma \) in \( \widetilde{\mathcal{M}}_\sigma \). By the rotation axiom for triangulated categories this implies that the last square must anti-commute, i.e.,

\[ \sigma(\varphi_i) = -\varphi_{\sigma(i)}. \]

This is because the shift functor \( T = F_\tau \) applied to the distinguished triangle (3.2) gives a distinguished triangle only when the sign of one of the arrow is reversed.

Thus, the axioms of a triangulated category require \( \varphi \) to be skew-continuous.

4. Classification of add-triangulated equivalence coverings of \( \mathcal{M}_0 \)

This section contains the main result of this paper: the classification of all continuously add-triangulated equivalence coverings of \( \mathcal{M}_0 \) assuming that they exist. The construction of these categories and the proof that they are continuously triangulated is a generalization of the construction of the continuous Frobenius category given in [3]. We leave the details of that construction to Section 6, the last section, so as not to interrupt the narrative.

4.1. Classification. Recall that an equivalence covering of a topological category \( \mathcal{X} \) with a unique zero object is a topological category \( \widetilde{\mathcal{X}} \) with unique zero object and a functor \( \widetilde{\mathcal{X}} \to \mathcal{X} \) which is an equivalence of \( K \)-categories and a continuous finite covering map on nonzero object spaces. (Unfortunately, the functor is not continuous on morphisms.)

**Definition 4.1.** When \( \mathcal{X} \) is a topological \( K \)-category, a *continuous add-triangulation* of an equivalence covering \( \widetilde{\mathcal{X}} \) is defined to be a continuous triangulation of the additive category \( add^{\text{top}} \widetilde{\mathcal{X}} \).

We recall from Example 1.5 that, for any nonzero scalar \( a \in K^* \) and any triangulated \( K \)-category with \( \Delta \), the set of distinguished triangles, \( \Delta_a \) is defined to be the set of triangles \( (X,Y,Z,f,g,h) \) so that \( (X,Y,Z,f,ag,h) \in \Delta \). We say that \( \Delta, \Delta' \) are *sign equivalent* if \( \Delta' = \Delta \) or \( \Delta' = \Delta_{-1} \). We also use the notion of “strong isomorphism” from Definition 1.3.

**Theorem 4.2 (Theorem D).** Continuously add-triangulated \( n \)-fold equivalence coverings of \( \mathcal{M}_0 \) are given, up to sign equivalence, by triples \( (\sigma, \tau, \varphi) \) where

1. \( \sigma \) is a linear automorphism of the \( n \) point \( K \)-category \( \mathcal{C}_n \).
2. \( \tau \) is a continuous autoequivalence of \( \mathcal{C}_n \), i.e., one which commutes with \( \sigma \).
(3) \( \varphi: \sigma \to \tau \) is a skew-continuous natural isomorphism, i.e., so that 
\[ \varphi_{\sigma(i)} = -\sigma(\varphi_i) \]
for all \( i \in C_n \).

Given such a triple \( (\sigma, \tau, \varphi) \), the corresponding add-triangulated equivalence covering, which we call \( \widetilde{M}_n(\sigma, \tau, \varphi) \), has underlying continuous \( K \)-category \( \widetilde{M}_\sigma \) with holonomy functor \( F_\sigma \), shift functor \( F_\tau \) and universal virtual triangle (defined in section 4.2 below) given by

\[
\begin{array}{c}
X \xrightarrow{[1]} I_1 X \oplus I_2 X \xrightarrow{[-1,1]} F_\sigma X \xrightarrow{\varphi_X} F_\tau X.
\end{array}
\]

Furthermore, \( \widetilde{M}_n(\sigma, \tau, \varphi) \) is strongly isomorphic to \( \widetilde{M}_n(\sigma', \tau', \varphi') \) if and only if there is an automorphism \( \rho \) of \( C_n \) so that \( \rho \circ \sigma = \sigma' \circ \rho \) and \( \rho \circ \tau = \tau' \circ \rho \).

Before proving this result, we point out some immediate consequences.

Corollary 4.3. Add-triangulated \( n \)-fold equivalence covering of \( M_0 \) are given up to strong isomorphism by conjugacy classes of pairs \( (\sigma, \tau) \) where \( \sigma, \tau \) are commuting, skew-compatible, self-equivalences of \( C_n \) and \( \sigma \) is an automorphism of \( C_n \).

Corollary 4.4 (Corollary E1). If \( \widetilde{M}_n(\sigma, \tau, \varphi) \) is an add-triangulated equivalence covering of \( M_0 \) where \( \tau \) is an automorphism of \( C_n \), then \( \widetilde{M}_n(\tau, \sigma, \varphi^{-1}) \) is also an add-triangulated equivalence covering of \( M_0 \).

We call \( \widetilde{M}_n(\tau, \sigma, \varphi^{-1}) \) the dual of \( \widetilde{M}_n(\sigma, \tau, \varphi) \).

The proof of Theorem 4.2 relies on the following lemmas.

Lemma 4.5 (Theorem 2.11). The continuous \( n \)-fold equivalence coverings of \( M_0 \) are given by \( \widetilde{M}_\sigma \) where \( \sigma \) is an automorphism of \( C_n \).

Lemma 4.6 (Corollary 2.13). Continuous functors \( \widetilde{M}_\sigma \to \widetilde{M}_{\sigma'} \) are given by \( K \)-linear functors \( \tau: C_n \to C_m \) so that \( \tau \circ \sigma = \sigma' \circ \tau \). In particular, continuous autoequivalences of \( \widetilde{M}_\sigma \) are given by autoequivalences \( \tau \) of \( C_n \) which commute with \( \sigma \).

Lemma 4.7. Continuous triangulations are induced from a universal virtual triangle.

We show in section 6 below that each universal virtual triangle comes from a continuous Frobenius category with continuous choice of universal exact sequences. Here we consider the converse: Given a continuous triangulation, show that it comes from a universal virtual triangle and that, consequently, it is given as the stable category of a continuous Frobenius category. The first statement is Lemma 4.7 which is more precisely stated as Theorem 4.10 below. The second statement is Theorem 6.22 proved in the last section.

4.2. Universal virtual triangle. Let \( X \) be an indecomposable object of some \( \widetilde{M}_\sigma \) with chosen shift functor \( F_\tau \). Suppose \( p(X) = E(x, y) \in M_0 \) where \( x < y < x + 2\pi \) and \( 0 \leq x < y < 2\pi \). For any \( 0 < \varepsilon_1 < x + 2\pi - y \) and \( 0 < \varepsilon_2 < y - x \), let \( I_1^{\varepsilon_1} X, I_2^{\varepsilon_2} X \) be the unique objects over \( E(x, x + 2\pi - \varepsilon_1) \) and \( E(y - \varepsilon_2, y) \) respectively so that there are contractible morphisms \( f_1: X \to I_1^{\varepsilon_1} X, f_2: X \to I_2^{\varepsilon_2} X \). We denote these morphisms by
“1” or, more generally, we denote by a scalar $a$ a morphism which is $a$ times a contractible morphism.

Up to rescaling, we are given the add-triangulated structure on $\mathcal{M}_0$. (The whole point is to find an equivalence covering $\widetilde{\mathcal{M}}_\sigma$ with a continuous triangulation which is algebraically equivalent to the given algebraic triangulation of $\text{add}\, \mathcal{M}_0$.) So, we know from [4] how to complete the triangle starting with the contractible morphisms $X \to I_1^{\varepsilon_1}X \oplus I_2^{\varepsilon_2}X$ to get:

$$X \xrightarrow{\left(\begin{smallmatrix} 1 \\ \end{smallmatrix}\right)} I_1^{\varepsilon_1}X \oplus I_2^{\varepsilon_2}X \xrightarrow{[g_1,g_2]} \overline{Y} \xrightarrow{h} F_\tau X$$

We can choose $Y$ to be the object in $\widetilde{\mathcal{M}}_\sigma$ in the contractible support of $X$ which lies over $\varepsilon(y - \varepsilon_2, x + 2\pi - \varepsilon_1)$. The morphisms $g_i$ will be scalar multiples of the unique contractible morphisms $I_1^{\varepsilon_1}X \to Y$. Dividing $g_1$ and $g_2$ by the scalar for $g_2$ and multiplying $h$ by the same scalar we may assume that $g_2$ is contractible, i.e. its scalar is 1. Then $-g_1$ will be also be contractible. The morphism $h : Y \to F_\tau X$ factors uniquely through the unique contractible morphism $Y \to F_\sigma X$ inducing a morphism

$$\varphi_X : F_\sigma X \to F_\tau X.$$  

**Lemma 4.8.** $\varphi_X$ is induced by a skew-continuous natural transformation $\sigma \to \tau$.

**Proof.** We have discussed this point many times. The object space of the equivalence covering $\mathcal{M}_\sigma$ is an $n$-fold covering of the Moebius strip. We number the sheets $1, \ldots, n$ (choosing a fundamental domain). As we go around the Moebius strip, we go from sheet $i$ to sheet $\sigma(i)$. So, when $X$ lies in sheet $i$, $F_\sigma X$ lies in sheet $\sigma(i)$ and $F_\tau X$ lies in sheet $\tau(i)$. Since we have a continuous triangulated structure, the morphism $F_\sigma X \to F_\tau X$ is given by $c_i x_{\tau(i)\sigma(i)}$ where $c_i$ is a fixed scalar for all $X$ in sheet $i$. When we move $X$ continuously to the next sheet, sheet $\sigma(i)$, $X$ moves to $Y = F_\sigma X$ by definition and, by continuity, $\varphi_X$ moves to the morphism

$$c_i \sigma(x_{\tau(i)\sigma(i)}) = c_i a_{\tau(i)\sigma(i)} x_{\sigma\tau(i)\sigma^2(i)}.$$

However, we are over a Moebius strip. So, the positions of $I_1^{\varepsilon_1}X, I_2^{\varepsilon_2}X$ have switched. So, the sign of $\varphi_{F_\sigma X}$ is the negative of the continuation of $\varphi_X$. In other words,

$$c_i a_{\tau(i)\sigma(i)} x_{\sigma\tau(i)\sigma^2(i)} = -c_i a_{\sigma(i)} x_{\sigma\tau(i)\sigma^2(i)}.$$

Comparing this with (3.11) we see that the scalars $c_i$ define a skew-continuous natural transformation $\varphi : \sigma \to \tau$ as claimed. 

By continuity, as $X$ varies throughout all of $\widetilde{\mathcal{M}}_\sigma$ and $\varepsilon_1, \varepsilon_2$ vary through all admissible real numbers (i.e. values so that the following diagram makes sense), the scalars $c_i$ must remain fixed on sheet $i$. So, we have a continuous family of distinguished triangles

$$X \xrightarrow{\left(\begin{smallmatrix} 1 \\ \end{smallmatrix}\right)} I_1^{\varepsilon_1}X \oplus I_2^{\varepsilon_2}X \xrightarrow{[g_1,g_2]} Y \xrightarrow{h} F_\tau X$$

where $\hat{\varphi}$ is the composition of the unique contractible morphism $Y \to F_\sigma X$ with $\varphi_X : F_\sigma X \to F_\tau X$. 

24
Definition 4.9. We denote this family of distinguished triangles by its virtual limit (when \( \varepsilon_1, \varepsilon_2 \) are infinitesimal, making \( Y = F_\sigma X \))

\[
X \xrightarrow{(1)} I_1^0 X \oplus I_2^0 X \xrightarrow{(-1,1)} F_\sigma X \xrightarrow{\phi} F_\tau X
\]

We call this the universal virtual triangle. This is just a fancy way to denote the continuous family of distinguished triangles given by (4.1).

4.3. Construction of all distinguished triangles. So far we have shown that a continuous triangulation of \( \text{add} \, \widetilde{\mathcal{M}}_\sigma \) has parameters \( \sigma, \tau \) and these give rise to a “universal virtual triangle”. In this subsection we prove the following theorem.

Theorem 4.10. The universal virtual triangle defined above determines all distinguished triangles. In particular, the continuous triangulation of \( \text{add} \, \widetilde{\mathcal{M}}_\sigma \) is completely determined by \( \tau \) and \( \phi : \sigma \to \tau \).

First, consider morphisms \( f : X \to Y \) in \( \text{add} \, \widetilde{\mathcal{M}}_\sigma \) which are generic in the sense that \( X, Y \) do not share any ends. (Recall that the ends of \( E(x,y) \) are \( x, y \in \mathbb{S}^1 = \mathbb{R}/2\pi \mathbb{Z} \).)

Lemma 4.11. Let \( X \to Y \to Z \to TX \) be a distinguished triangle. If \( X \to Y \) is generic, then \( Z \) has the same number of components as \( X \oplus Y \).

Proof. We know that distinguished triangles are exact on ends. So, \( Z \) must have the same number of ends as \( X \oplus Y \). But the number of components is half the number of ends. \( \square \)

Lemma 4.12. Every distinguished triangle is isomorphic to a continuous limit of distinguished triangles in which the first morphism is generic.

Proof. If \( f : X \to Y \) is not generic then, by a small perturbation of the components of \( Y \) (moving points over \( E(x,y) \) to points over \( E(x+\varepsilon,y+\varepsilon) \) but staying on the same sheet of the covering) and by composing \( f : X \to Y \) with short contractible morphisms to the new \( Y \), we obtain a new distinguished triangle with more components in \( Z \) so that, as \( Y \) moves back to its original position, some of these new components go to zero. The limit is a distinguished triangle since the space of distinguished triangles is assumed to be closed. The limit must be equivalent to the distinguished triangle we started with since it has the same first morphism \( f : X \to Y \). So, every distinguished triangle is equivalent to one which is a limit of distinguished triangles with generic first morphism. \( \square \)

Proof of Theorem 4.10. By Lemma 4.12 we only need to prove the theorem in the case of when \( f : X \to Y \) is generic. Given any generic morphism \( f : X \to Y \), consider the distinguished triangles given by:

\[
(4.2) \quad X \xrightarrow{(f,1,1)} Y \oplus I_1^{\varepsilon_1} X \oplus I_2^{\varepsilon_2} X \to Z \to TX.
\]

Here, the first morphism is a monomorphism on ends and, therefore, the last morphism \( Z \to TX \) is generic. By Lemma 4.11 the number of components of \( Y \oplus I_1^{\varepsilon_1} X \oplus I_2^{\varepsilon_2} X \) is equal to the number of components of \( X \oplus Z \). Since \( I_1^{\varepsilon_1} X, I_2^{\varepsilon_2} X \) each have the same number of components as \( X \) this implies that \( Z \) has the same number of components as \( X \oplus Y \).
As $\varepsilon_1, \varepsilon_2$ both go to zero, the distinguished triangle (4.2) converges to a distinguished triangle $X \to Y \to Z' \to TX$ where, by Lemma 4.11, $Z'$ has the same number of components as $X \oplus Y$. Since $Z$ converges to $Z'$ and they have the same number of components we conclude that the components of $Z$ converge to the components of $Z'$. In particular, they do not vanish. So, we get all distinguished triangles of the form

$$X \xrightarrow{f} Y \to Z \to TX$$

with $f$ generic as a limit (or equivalent to a limit) of distinguished triangles of the form (4.2). Thus, we are reduced to showing that all distinguished triangles of the form (4.2) are determined by the universal virtual triangle.

As the values of $\varepsilon_1, \varepsilon_2$ change, the distinguished triangle (4.2) varies continuously with $Z$ having a fixed number of components. Thus, $X, Y$ are fixed and the components of $Z$ move around continuously. By continuity, the set of triangles for each value of the parameters $\varepsilon_1, \varepsilon_2$ determines the set of triangles for all values of these parameters.

Now deform the parameters $\varepsilon_1, \varepsilon_2$ to an infinitesimal amount less than their maximum possible values. Then the morphism $f : X \to Y$ will factor through $I_{\varepsilon_1}X \oplus I_{\varepsilon_2}X$. This will cause the induced map $Y \to Z$ to be a split monomorphism in the category $\tilde{\mathcal{M}}$. So, the distinguished triangle (4.2) becomes a direct sum of distinguished triangles of the form (4.1) and trivial triangles $0 \to Y \xrightarrow{=} Y \to 0$. These are given by the universal virtual triangle by definition concluding the proof of the theorem.

4.4. Consequences.

**Lemma 4.13.** Any subset of $[n]$ invariant under both $\sigma$ and $\tau$ generates a $(\sigma, \tau)$-invariant full subcategories of $\mathcal{C}_n$ and therefore of $\tilde{\mathcal{M}}_n(\sigma, \tau, \varphi)$. □

**Corollary 4.14.** Let $\tilde{\mathcal{M}}_n(\sigma, \tau, \varphi)$ be an add-triangulated equivalence covering of $\mathcal{M}_0$ so that $\tau$ is an automorphism of $\mathcal{C}_n$. Then $n$ is even.

**Proof.** Suppose $n$ is odd. Then, as permutations of $n$, $\sigma$ must have an odd number of odd cycles and the action of $\tau$ on these odd cycles must have an odd cycle. This gives a full subcategory $\mathcal{C}_{pq}$ of $\mathcal{C}_n$, with $p, q$ odd, whose object set we identify with $[p] \times [q]$ on which $\sigma$ acts by cyclically permuting the first factor and $\tau$ acts by permuting both facts but acting cyclically the second: $\sigma(i, j) = (i + 1, j), \tau(i, j) = (\tau_1(i), j + 1)$.

For each $(i, j) \in [p] \times [q]$ we have the natural transformation $\varphi_{ij}$ which is a nonzero morphism

$$\varphi_{ij} : (i + 1, j) \to (\tau_1(i), j + 1)$$

Since $\varphi$ is skew commutative, $\sigma(\varphi_{ij}) = -\varphi_{i+1,j}$. Doing this $p$ times we get $\sigma^p(\varphi_{ij}) = -\varphi_{ij}$. This means that the $p$th power of the transition factor of $\sigma$ from $A_j = [p] \times j$ to $A_{j+1}$ is $(a_{A_j, A_{j+1}})^p = -1$ for every $j$. Composing these, the $pq$th power of this transition factor will also be $-1$. But this is impossible since this is the transition factor $a_{A_j, A_j} = 1$. □

**Corollary 4.15.** If $\tilde{\mathcal{M}}_n(\sigma, \tau, \varphi)$ is a minimal add-triangulated equivalence covering of $\mathcal{M}_0$ then $\tau$ is an automorphism of $\mathcal{C}_n$ and $n$ is even.
Proof. This follows from Corollary 4.14 since $\tau$ must be an isomorphism: otherwise, the image of $\tau^n$ is a smaller $\sigma, \tau$ invariant full subcategory of $C_n$ which generates a proper equivalence subcover of $\tilde{M}_n(\sigma, \tau, \varphi)$ contradicting its minimality. 

In particular, $\tilde{M}_n(\sigma, \tau, \varphi)$ is minimal when $n = 2$ since any proper subcovering would have $n = 1$.

4.5. 2-fold equivalence coverings of $M_0$. We will describe all 2-fold equivalence coverings $\tilde{M}_n(\sigma, \tau, \varphi)$ of $M_0$. There are four cases for the underlying permutations of $\sigma, \tau$, although Case (0) is not possible. See Figure 3.

Finally, recall that “rescaling” of the distinguished triangles means multiplying all $c_i$ by the same nonzero scalar $r$. This preserves the $c$-homogeneous relations (a), (b). Up to rescaling, only the ratio $c_1/c_2$ is well-defined (when $n = 2$).

Our goal is to show that, in Cases 1, 2, 3, an add-triangulated category $\tilde{M}_n(\sigma, \tau, \varphi)$ exists and is unique up to continuous isomorphism and rescaling.

We need the following trivial observation.

**Lemma 4.16.** If $\sigma$ is the identity permutation then its transition coefficients $a_{ij}$ are well-defined, i.e., independent of the choice of $x_{ij}$.

Proof. $a_{ij}$ is the unique eigenvalue of the action of $\sigma$ on $C_n(j, i) \cong K$. 

**Case 0:** $\sigma = \tau = id$. The minimal invariant subcategory of $C_2$ is $C_1$ which contradicts Corollary 4.15. So, this case is not possible.

**Case 1:** $\sigma = id, \tau = (12)$. Since $\{1, 2\}$ is a single $\tau$-orbit, there is a multiplicative basis $x_{ij}$ for $C_2$ so that $\tau(x_{12}) = x_{21}$ and $\tau(x_{21}) = x_{12}$. I.e., $b_{ij} = 1$ for all $i, j$.

Since $\sigma, \tau$ commute, we must have $a_{12} = a_{21}$ with product $a_{11} = 1$. So, $a_{12} = \pm 1$.

The skew-commutativity of $\varphi$ implies that $a_{21}c_1 = -c_1$. So, $a_{12} = a_{21} = -1$.

Naturality of $\varphi$ implies $b_{21}c_1 = c_2a_{21}$. So, $c_1/c_2 = -a_{21}/b_{21} = -1$.

To summarize: $(a_{12}, b_{12}, c_1/c_2) = (-1, 1, -1)$ and $\tilde{M}_n(\sigma, \tau, \varphi)$ is well-defined up to rescaling of triangles as defined in Example 1.5. I.e., the triangulated categories in Case 1 are, up to continuous isomorphism, given by the single parameter $c_1 \in K^*$.

**Case 2:** $\sigma = (12), \tau = id$. As in Case 1, we have: $a_{ij} = 1$ for all $i, j$ and $b_{12} = b_{21} = \pm 1$.

By skew-commutativity of $\varphi$ we have $a_{12}c_1 = -c_2$. So, $c_1/c_2 = -1$.

By naturality of $\varphi$ we have $b_{21}c_1 = c_2a_{21}$. So, $b_{21} = b_{12} = -1$. 

27
To summarize: \((a_{12}, b_{12}, c_1/c_2) = (1, -1, -1)\) and \(\tilde{M}_n(\sigma, \tau, \varphi)\) is well-defined up to the rescaling factor \(c_1\).

**Case 3:** \(\sigma = \tau = (12)\). In this case we have a choice of normalizing either \(\sigma\) or \(\tau\). To help decide, we first evaluate \(c_1/c_2\).

By skew-commutativity of \(\varphi\) we have \(a_{22}c_1 = -c_2\). So, \(c_1/c_2 = -1\).

By naturality of \(\varphi\) we have \(b_{21}c_1 = c_2a_{21}\). So, \(a_{21} = -b_{21}\).

We still have two choices. We choose to normalize \(\sigma\) and take a multiplicative basis so that \(a_{ij} = 1\) for all \(i, j\) and \(b_{12} = b_{21} = -1\).

To summarize: \((a_{12}, b_{12}, c_1/c_2) = (1, -1, -1)\), as in Case 2 and \(\tilde{M}_n(\sigma, \tau, \varphi)\) is well-defined up to the rescaling factor \(c_1\). However, with a different choice of \(x_{ij}\), we would have another description of the same category: \((a_{12}, b_{12}, c_1/c_2) = (-1, 1, -1)\) as in Case 1.

**Remark 4.17.** Case 3, with parameters \(c_1 = 1, c_2 = -1\), is the continuous cluster category constructed in [4],[3]. The objects of the 2-fold equivalence cover of \(\mathcal{M}_0\) are ordered pairs of distinct points on the circle \(\mathbb{R}/2\pi\mathbb{Z}\) with holonomy given by rotation by \(\pi\) (making \(a_{12} = 1\)) and with the distinguished triangles rescaled by \(-1\) in the holonomy. This gives \(b_{12} = -1\). The details given in the present paper show why this is in fact continuous.

Case 1 is a variation of a special case of a construction Orlov [6]. In this case, Orlov actually constructs a 4-fold equivalence covering of \(\mathcal{M}_0\) but we interpret it as a recipe to construct a 2-fold covering. The indecomposable objects of Orlov’s category are matrix factorizations on an ordered pair of points in \(\mathbb{R}/2\pi\mathbb{Z}\):

\[
P_1 \xrightarrow{d_0} P_0 \xleftarrow{d_1}
\]

However, we modify this by identifying \((P_1, P_0, d_*)\) with \((P_0, P_1, d_*)\), i.e., by taking unordered pairs of points. In our setting we take \(d_0, d_1\) to be the unique contractible morphisms between the two points in \(\mathcal{R}_{2\pi}/G_{2\pi}\). This gives one object for every unordered pair of points on a circle of radius 1, i.e., for every object in \(\mathcal{M}_0\).

Orlov takes the shift functor to be given by changing the sign of both \(d_0\) and \(d_1\). This gives another isomorphic copy of each object. So, there are two copies of each object. Changing the sign of \(d_i\) gives a recognizably distinct component in the space of objects. So, \(\sigma\) is the identity permutation and \(\tau = (12)\).

Case 2 is a new construction. The underlying topological category \(\tilde{\mathcal{M}}_\sigma\) is the same as in Case 3, but the continuous triangulated structure is different. Note that \(b_{12} = -1\) implies that, although the shift functor \(\tau\) is the identity on objects, it is not the identity functor.

**Corollary 4.18 (Corollary E2).** Up to rescaling (Example 1.5), there are exactly three two-fold equivalence coverings of \(\mathcal{M}_0\) as shown in Figure 3.

5. **Automorphisms of \(C_n\)**

Now we set about the task of defining the structures that will allow us to classify the continuous triangulations of equivalence coverings of the Moebius category \(\mathcal{M}_0\). We begin by first investigating the \(K\)-linear structure of certain functors and natural transformations. We assume that \(K\) is algebraically closed.
5.1. Fattened Categories and the category $C_n$. By “fattening” a category we mean taking several isomorphic copies of each object. This is the discrete version of an equivalence covering. We will concentrate on fattening the category $C_1$ having one object with endomorphism ring $K$. All categories will be $K$-categories and all functors will be $K$-linear.

**Definition 5.1.** For every $n \geq 1$ let $C_n$ be the $K$-category with object set $[n] := \{1, 2, \ldots, n\}$ and $C_n(j, i) = Kx_{ij}$. I.e., every hom set is one dimensional over $K$ and is generated by $x_{ij}$:

$$x_{ij} : j \rightarrow i.$$ 

We define composition $K$-bilinearly by

$$x_{ij}x_{jk} = x_{ik} \quad \forall i, j, k.$$ 

In particular, $x_{ii} = id_i$ and $x_{ji} = x_{ij}^{-1}$. Any choice of basis elements, i.e., any system of nonzero morphisms $x_{ij} \in C_n(j, i)$ satisfying (5.1) will be called a multiplicative basis for $C_n$.

**Proposition 5.2.** Let $C$ be any category with $n$ objects, all of which are isomorphic with endomorphism ring $K$. Then $C$ is isomorphic to $C_n$.

**Proof.** Label the objects of $C$ as $v_1, \ldots, v_n$. For each $i < n$ choose an isomorphism $f_i : v_i \cong v_{i+1}$ with inverse $g_i$. For any $i, j \in [n]$ let $y_{ji} : v_i \rightarrow v_j$ be given by the unique reduced composition of the morphism $f_k, g_k$ (reduced means $f_k \circ g_k, g_k \circ f_k$ do not occur in the composition). Then $i \mapsto v_i$ and $x_{ji} \mapsto y_{ji}$ gives an isomorphism $C_n \cong C$. □

The next question is: How unique is the multiplicative basis? Equivalently: What are the automorphisms of $C_n$?

Elements of $C_n(j, i) = Kx_{ij}$ are all scalar multiples of $x_{ij}$. So, any other generator has the form $x_{ij} = a_{ij}x_{ij}$ where $a_{ij} \in K^*$. 

---

**Figure 3.** Schematic drawings of the three possible add-triangulated 2-fold equivalence coverings of $M_0$. 

**Orlov [6]** 

- $\sigma = id, a_{12} = -1$
- $\tau = (12)$
- $\varphi : c_2 = -c_1$

**Igusa-Todorov [3]** 

- $\sigma = id, a_{12} = -1$
- $\tau = (12), b_{12} = -1$
- $\varphi : c_2 = -c_1$

**not a triangulation** 

- $X = \sigma(X)$
- $\sigma = id$
- $\tau = id$
- $\varphi = ?$

**a third triangulation** 

- $X = \tau(X)$
- $\sigma = (12)$
- $\tau = id, b_{12} = -1$
- $\varphi : c_2 = -c_1$
Proposition 5.3. The following conditions on scalars \( a_{ij} \neq 0 \) are equivalent.

1. The elements \( x'_{ij} = a_{ij} x_{ij} \) form a multiplicative basis, i.e., satisfy (5.1).
2. \( a_{ij} a_{jk} = a_{ik} \) for all \( i, j, k \).
3. There are \( c_i \in K^* \) so that \( a_{ij} = c_i/c_j \).

Furthermore, the scalars \( c_i \) in (3) are uniquely determined up to multiplication by a nonzero scalar, i.e., the \( n \)-tuple \((c_1, \ldots, c_n)\) forms a well-defined element of projective space \( KP^{n-1} \).

Proof. It is clear that (1), (2) are equivalent and that (3) implies (2). To see that (2) implies (3), let \( c_j = a_{j1} \).

We say that \((a_{ij})\) is a multiplicative system of scalars if it satisfies these equivalent conditions. Note that (3) implies \( a_{ii} = 1 \) for all \( i \) and \( a_{ji} = a_{ij}^{-1} \).

5.2. Automorphisms of \( C_n \).

Proposition 5.4. A \( K \)-automorphism \( \sigma \) of \( C_n \) is given on objects and morphisms as follows.

1. On objects, \( \sigma \) is a permutation of \( n: \sigma \in S_n \).
2. On morphisms we have

\[
\sigma(x_{ij}) = a_{ij} x_{\sigma(i)\sigma(j)}
\]

where \( a_{ij} \in K^* \) is a multiplicative system of scalars.

Conversely, any permutation of \( n \) and any multiplicative system \((a_{ij})\) defines an automorphism of \( C_n \). \( \square \)

We call \((a_{ij})\) the transition coefficients of \( \sigma \) with respect to \((x_{ij})\). We will show that, if the underlying permutation of \( \sigma \) is an \( n \) cycle, then we can arrange for \( a_{ij} \) to be equal to 1 for all \( i, j \) by changing the basis for the hom sets. In particular, \( \sigma^n \) is the identity functor on \( C_n \). More generally, we can arrange for \( a_{ij} \) to be equal to 1 when \( i, j \) are in the same orbit of \( \sigma \).

Example 5.5. Suppose \( \sigma = (123 \cdots n) \) is a single \( n \)-cycle on objects. Given \( \sigma(x_{ij}) = a_{ij} x_{i+1,j+1} \), choose \( c_i \in K^* \) so that \( a_{ij} = c_i/c_j \). Assuming that \( K \) is algebraically closed, there exists a \( d \in K^* \) so that

\[
d^n = c_1 c_2 \cdots c_n
\]

Thus \( d \) is the geometric mean of the \( c_i \) and \( d \) is unique up to multiplication by an \( n \) th root of unity. Then we can choose another multiplicative basis \( x'_{ij} = b_{ij} x_{ij} \) depending on the transition coefficients of \( \sigma \) with respect to \( x_{ij} \) by:

1. \( b_{ii} = 1 \)
2. \( b_{ij} = d^{j-i}(c_i c_{i+1} \cdots c_{j-1})^{-1} \) if \( i < j \). In particular \( b_{i,i+1} = d/c_i \) if \( i \leq n-1 \).
3. \( b_{ij} = b_{ji}^{-1} \) if \( j < i \). In particular \( b_{n1} = b_{1n}^{-1} = c_1 \cdots c_{n-1} d^{1-n} = d/c_n \).

Therefore, \( b_{i,\sigma(i)} = d/c_i \) for all \( i \). It is easy to see that \((b_{ij})\) is multiplicative. And:

\[
\sigma(x'_{i,i+1}) = \sigma(b_{i,i+1} x_{i,i+1}) = \frac{d}{c_i} \sigma(x_{i,i+1}) = \frac{d}{c_i} \frac{c_i}{c_{i+1}} x_{i+1,i+2} = \frac{d}{c_{i+1}} x_{i+1,i+2} = x'_{i+1,i+2}
\]

This implies that

\[
\sigma(x'_{ij}) = \sigma(x'_{i,i+1} x'_{i+1,j}) = x'_{i+1,i+2} \cdots x'_{j,j+1} = x'_{i+1,j+1}
\]
for all $i, j$. In fraction form, $b_{ij} = s_i / s_j$ where $s_i = c_1 \cdots c_{i-1} d^{-i}$.

This proves the following.

**Lemma 5.6.** If $\sigma \in \text{Aut}(C_n)$ is an $n$ cycle then $C_n$ has a multiplicative basis $(x_{ij})$ so that $\sigma(x_{ij}) = x_{\sigma(i)\sigma(j)}$ for all $i, j \in C_n$. In particular, $\sigma^n$ is the identity automorphism of $C_n$. \hfill $\square$

**Proposition 5.7.** Given any automorphism $\sigma$ of $C_n$, there exists a multiplicative basis $(x_{ij})$ so that the transition coefficients $(a_{ij})$ of $\sigma$ with respect to $(x_{ij})$ satisfy the following.

1. $a_{ij} = 1$ if $i, j$ are in the same orbit of $\sigma$
2. $a_{ij} = a_{kl}$ if $i, k$ are in the same cycle of $\sigma$ and $j, \ell$ are in the same cycle of $\sigma$.

Furthermore, (1) implies (2).

**Proof.** Choose any multiplicative basis $(y_{ij})$, for which $\sigma$ has transition coefficients $b_{ij} = s_i / s_j$. Since any $i$ lies in some cycle of $\sigma$, it suffices to choose a cycle, say $A$, and repeat the same argument for all cycles of $\sigma$. Let $d_A \in K^*$ be the geometric mean of the $s_i$ for $i \in A$. If $i_0$ is the smallest element of $A$ then $i = \sigma^m(i_0)$ for some $m \geq 0$. Let $c_{i_0} = 1$ and let

$$c_i = s_{i_0} s_{\sigma(i)} \cdots s_{\sigma^{m-1}(i_0)}/d_A^m$$

if $m > 0$. Let $a_{ij} = c_i / c_j$. Then $a_{i,\sigma(i)} = d_A/s_i$ for all $i$. Also, $(a_{ij})$ is a multiplicative system and $x_{ij} = a_{ij} y_{ij}$ is a new multiplicative basis. We have:

$$\sigma(x_{i,\sigma(i)}) = d_A/s_i \sigma(y_{i,\sigma(i)}) = d_A/s_i s_{\sigma(i)} y_{\sigma(i),\sigma^2(i)} = d_A s_{\sigma(i)} y_{\sigma(i),\sigma^2(i)} = x_{\sigma(i),\sigma^2(i)}$$

As in the example, this implies that $\sigma(x_{ij}) = x_{\sigma(i),\sigma(j)}$ when $i, j$ are in the same $\sigma$ orbit.

To see that (1) implies (2), let $i, k \in A$ and $j, \ell \in B$ for two disjoint cycles $A, B$ of $\sigma$. Since $(a_{ij})$ is a multiplicative system, we have $a_{ij} = a_{ik} a_{kl} a_{ij}$. By (1), $a_{ik} = 1 = a_{jl}$, and so $a_{ij} = a_{kl}$. \hfill $\square$

**Definition 5.8.** Given $C_n$ with an automorphism $\sigma$, a multiplicative basis $(x_{ij})$ for $C_n$ will be called a good multiplicative basis (with respect to $\sigma$) if it satisfies the proposition above.

If $A, B, C, \cdots$ denote the orbits of the action of $\sigma$ on $n$, then let $a_{AB}$ be the element of $K^*$ so that $\sigma(x_{ij}) = a_{AB} x_{\sigma(i)\sigma(j)}$ when $i \in A$ and $j \in B$. We call $a_{AB}$ the transition factors of $\sigma$. Let $|A|$ denote the number of elements in the orbit $A$ of $\sigma$.

We need to know to what extent the good basis and transition factors are uniquely determined by $(C_n, \sigma)$.

**Proposition 5.9.** Let $(x_{ij})$ be a good basis for $(C_n, \sigma)$ with transition factors $a_{AB}$ and suppose that $(x_{ij}')$ is another good basis giving new transition factors $b_{AB}$ for $\sigma$. Then there are unique roots of unity $\delta_A$ for each orbit $A$ of $\sigma$ so that

1. $\delta_A^{|[A]} = 1$ for all $A$.
2. $b_{AB} = \delta_A a_{AB} \delta_B^{-1}$ for all orbits $A, B$ of $\sigma$ and
3. $x_{i,\sigma(i)'} = \delta_A x_{i,\sigma(i)}$ whenever $i \in A$.

Furthermore, any collection of roots of unity $\delta_A$ satisfying (1) will occur.
Proof. By Proposition 5.3, $x'_{ij} = \frac{c_i}{c_j} x_{ij}$. When $i, j$ are in the same $\sigma$ orbit this gives

$$x'_{\sigma(i)\sigma(j)} = \sigma(x'_{ij}) = \sigma\left(\frac{c_i}{c_j} x_{ij}\right) = \frac{c_i}{c_j} \sigma(x_{\sigma(i)\sigma(j)}) = \frac{c_{\sigma(i)}}{c_{\sigma(j)}} x_{\sigma(i)\sigma(j)}$$

Therefore, $\frac{c_i}{c_{\sigma(i)}} = \frac{c_i}{c_{\sigma(j)}}$ when $i, j$ are in the same orbit $A$ of $\sigma$. Denote this fraction by $\delta_A$.

Then (3) will be satisfied. When $i, j$ are in different orbits, say $i \in A, j \in B$ we get:

$$b_{AB} x'_{\sigma(i)\sigma(j)} = \sigma(x'_{ij}) = \frac{c_i}{c_j} \sigma(x_{ij}) = \frac{c_i}{c_j} a_{AB} x_{\sigma(i)\sigma(j)} = b_{AB} \frac{c_{\sigma(i)}}{c_{\sigma(j)}} x_{\sigma(i)\sigma(j)}$$

Therefore,

$$b_{AB} = \frac{c_i c_{\sigma(j)}}{c_j c_{\sigma(i)}} a_{AB} = \delta_A a_{AB} \delta_B^{-1}$$

To see that condition (1) is satisfied suppose that $A = (12 \cdots m)$ is one of the cycles of the permutation $\sigma$. Then

$$1 = \frac{c_1}{c_1} = \frac{c_1}{c_2} \frac{c_2}{c_3} \cdots \frac{c_m}{c_1} = \delta_A^m$$

Conversely, suppose $(\delta_A)$ is a collection of scalars satisfying (1). Then, for each orbit $A$ of $\sigma$, choose an element $i_0 \in A$. Let $c_{i_0} = 1$ and let $c_{\sigma^k(i_0)} = \delta_A^{-k}$. Then $x'_{ij} = \frac{c_i}{c_j} x_{ij}$ will satisfy (2) and (3).

\[\square\]

Corollary 5.10. If orbits $A, B$ of $\sigma$ have the same size, say $m$, then the $m$th power of $a_{AB}$, the $AB$ transition factor of $\sigma$, is well-defined.

The properties we have developed in this section will be revisited in later sections.

5.3. Other quivers with multiplicity. Now we consider the case of two vertices. A $K$-representation of the quiver:

$$1 \xrightarrow{\tau} 2$$

consists of two vector spaces $V_1, V_2$ and a linear map $\tau : V_1 \to V_2$. We fatten this quiver, replacing the vertices with triples $(C_n, \sigma_1, (x_{ij}))$ and $(C_m, \sigma_2, (y_{kl}))$ where $(x_{ij}), (y_{kl})$ are good multiplicative bases so that in the corresponding multiplicative systems, $a_{ij} = 1$ when $i, j$ are in the same $\sigma_1$ orbit and similarly $a_{kl}^{ij} = 1$ if $k, l$ are in the same $\sigma_2$ orbit. Our goal is to describe how $\tau$ behaves after this fattening.

A morphism

$$\tau : (C_n, \sigma_1, (x_{ij})) \to (C_m, \sigma_2, (y_{kl}))$$

is defined to be a faithful linear functor $\tau : C_n \to C_m$ so that $\tau \sigma_1 = \sigma_2 \tau$. In other words:

1. On objects, $\tau$ is a set map $[n] \to [m]$ where $[n] = \{1, 2, \cdots, n\}$.
2. On morphisms, $\tau(x_{ij}) = b_{ij} y_{\tau(i), \tau(j)}$ where $(b_{ij})$ is a multiplicative system for $C_n$ which we call the comparison coefficients for $\tau$.
3. The equation $\tau \sigma_1 = \sigma_2 \tau$ becomes:

$$a_{ij} b_{\sigma_1(i), \sigma_1(j)} = b_{ij} a_{\tau(i), \tau(j)}'$$
The proof of (3) is straightforward. Just expand both sides of \( \tau \sigma_1(x_{ij}) = \sigma_2\tau(x_{ij}) \):

\[
\tau \sigma_1(x_{ij}) = a_{ij} \tau(x_{\sigma_1(i)\sigma_1(j)}) = a_{ij}b_{\sigma_1(i),\sigma_1(j)}y_{\tau\sigma_1(i)\tau\sigma_1(j)}
= \sigma_2\tau(x_{ij}) = b_{ij}\sigma_2(y_{\tau(i)\tau(j)})
= b_{ij}a'_{\tau(i)\tau(j)}y_{\sigma_2\tau(i)\sigma_2\tau(j)}
\]

Then compare the coefficients of \( y_{\tau\sigma_1(i)\tau\sigma_1(j)} = y_{\sigma_2\tau(i)\sigma_2\tau(j)} \).

**Lemma 5.11.** Given any good basis \((y_{ki})\) for \((C_m, \sigma_2)\) there is a unique good basis \((x_{ij})\) for \((C_n, \sigma_1)\) so that all comparison coefficients \(b_{ij} = 1\) or, equivalently, \(\tau(x_{ij}) = y_{\tau(i)\tau(j)}\) for all \(i, j \in C_n\).

**Proof.** For every \(i, j \in C_n\), the functor \(\tau\) gives an isomorphism \(C_n(j, i) \cong C_m(\tau(j), \tau(i))\). Let \(x_{ij} : j \to i\) be the morphism which maps to \(y_{\tau(i)\tau(j)} : \tau(j) \to \tau(i)\). For each \(k \in C_n\), \(x_{ij}x_{jk} = x_{ik}\) since both morphisms map to the same morphism \(y_{\tau(i)\tau(j)}y_{\tau(j)\tau(k)} = y_{\tau(i)\tau(k)}\) in \(C_m\). So, \((x_{ij})\) is a multiplicative basis for \(C_n\). To see that it is a good basis, suppose that \(i, j\) lie in the same orbit of \(\sigma_1\). Then \(\tau(i), \tau(j)\) lie in the same orbit of \(\sigma_2\). So, \(\sigma_2(y_{\tau(i)\tau(j)}) = y_{\sigma_2\tau(i)\sigma_2\tau(j)}\). This implies that \(\sigma_1(x_{ij}) = x_{\sigma_1(i)\sigma_1(j)}\) since both map to the same morphism \(\tau(j) \to \tau(i)\). \(\square\)

Suppose that \((x_{ij})\) and \((y_{ki})\) are as given in Lemma 5.11 above. Suppose \(i, j \in C_n\) map to \(\tau(i), \tau(j)\) in the same orbit of \(\sigma_2\). Then, when \(\tau\) is applied to the equation \(\sigma_1(x_{ij}) = a_{ij}x_{\sigma_1(i)\sigma_1(j)}\), we get:

\[
\tau \sigma_1(x_{ij}) = \tau(a_{ij}x_{\sigma_1(i)\sigma_1(j)}) = a_{ij}y_{\tau\sigma_1(i)\tau\sigma_1(j)} = \sigma_2\tau(x_{ij}) = y_{\sigma_2\tau(i)\sigma_2\tau(j)} = \sigma_2(y_{\tau(i)\tau(j)})
\]

which implies that \(a_{ij} = 1\). This proves the second part of the following lemma.

**Lemma 5.12.** If \(i, j\) are in different orbits, say \(A, B\), of \(\sigma_1\) and \(\tau(i), \tau(j)\) are in the same orbit of \(\sigma_2\) then \(a_{ij}^k = 1\) for any \(k\) which is a common multiple of both \(|A|\) and \(|B|\). Furthermore, there is a good basis for \(C_n\) so that \(a_{ij} = 1\).

**Proof.** The good basis for \(C_n\) given in the previous lemma has the property that \(a_{ij} = 1\) whenever \(\tau(i), \tau(j)\) lie in the same \(\sigma_2\) orbit in \(C_n\). But the transition factors \(a_{AB}\) are well defined up to roots of unity. So, \(a_{ij}\) will be a product of an \(|A|\)-th root of unity and a \(|B|\)-th root of unity when \(\tau(i), \tau(j)\) lie in the same \(\sigma_2\) orbit. Then \(a_{ij}^k = 1\) as claimed. \(\square\)

5.4. **Continuous automorphisms of** \((C_n, \sigma)\). We describe the group of all automorphisms \(\tau\) of \((C_n, \sigma)\). First notice that in the discussion above, we now set \(n = m\). Hence, \(\tau\) acts on objects as a permutation of \(n\) vertices; \(\tau \in S_n\). The condition \(\sigma_2\tau = \tau\sigma_1\) becomes \(\sigma_2 = \tau\sigma_1\). So given \(\sigma \in S_n\), the centralizer of \(\sigma\) gives us the group of set maps that may underlie an automorphism of \((C_n, \sigma)\):

\[
C(\sigma) = \{ \tau \in S_n | \tau\sigma = \sigma\tau \}
\]

This group is known as the group of generalized permutation matrices or monomial matrices. Supposing \(\sigma\) has cycle decomposition \(\sigma = A_{1,1} \cdots A_{1,e_1} \cdots A_{r,1} \cdots A_{r,e_r}\), where \(A_{i,1}, \ldots, A_{i,e_i}\) are cycles of length \(\lambda_i\) for \(1 \leq i \leq r\), \(C(\sigma)\) has the structure of a product of semidirect products:

\[
C(\sigma) = \prod_{i=1}^{r} (C_{\lambda_i} \rtimes S_{e_i}),
\]
where \( C_n \) is the cyclic group on \( n \) elements. Then \( \tau \) sends each orbit of \( \sigma \) to another orbit of \( \sigma \) with the same size. So, \( \tau \) induces a permutation of the set of orbits of \( \sigma \). Consider one cycle of this action. We define a \( \sigma\tau \)-orbit to be the set of objects of \( C_n \) which lie in such a cycle. In other words, a \( \sigma\tau \)-orbit is a collection of \( \sigma \) orbits that is cyclically permuted under the action of \( \tau \). This is a minimal subset of the set of objects of \( C_n \) which is closed under \( \sigma \) and \( \tau \).

Suppose that \( C_n \) is a single \( \sigma\tau \)-orbit. Then, the orbits of \( \sigma \): \( A_1, \ldots, A_s \) have the same size, say \( |A_i| = m \), and \( \tau \) cyclically permutes these orbits: \( \tau(A_i) = A_{i+1} \) where the indices are taken modulo \( s \). Choose a good multiplicative basis \((x_{ij})\) for \( \sigma \) on \( A_1 \). Thus, \( \sigma(x_{ij}) = x_{\sigma(i)\sigma(j)} \) for all \( i, j \in A_1 \). This extends to a good basis for all of \( C_n \) by taking the set of all \( \tau^k(x_{ij}) \) where \( i, j \in A_1 \) and \( 1 \leq k \leq s \).

The functor \( \tau \) will play the role of the shift functor for the triangulated structure of a covering category, whose topology is determined by \( \sigma \). We can by the previous lemma choose a good basis so that the comparison coefficients \((b_{ij})\) associated to \( \tau \) are all 1.

However, since we dealing with an automorphism of \((C_n, \sigma, (x_{ij}))\), we consider the good basis \((x_{ij})\) fixed and so we are not able to ensure that the comparison coefficients \((b_{ij})\) are 1. Thus, the commutativity condition \( \sigma \tau = \tau \sigma \) translates into a condition on the comparison coefficients of \( \tau \) as follows:

\[
(5.2) \quad a_{A_{\tau(i)}A_{\tau(j)}} b_{ij} = b_{\sigma(i)\sigma(j)} a_{A_i A_j},
\]

where by \( A_i \) we mean the cycle of \( \sigma \) which contains \( i \) and \( a_{A_i A_j} \) denote the transition factor of \( \sigma \) between cycles \( A_i \) and \( A_j \). Notice that this means in particular that when \( i \) and \( j \) are in the same cycle of \( \sigma \) and \( \tau(i) \) and \( \tau(j) \) also in the same cycle of \( \sigma \), we have \( b_{ij} = b_{\sigma(i)\sigma(j)} \).

These are the only constraints on the comparison coefficients of \( \tau \).

### 5.5. Skew-continuity of \( \varphi : \sigma \to \tau \)

We recall that, if \( \sigma, \tau \) are commuting autoequivalences of \( C_n \), the continuity factor \( s = \{\sigma, \tau\} \) is defined to be the scalar \( s \in K^* \) so that

\[
\sigma(\varphi_i) = s \varphi_{\sigma(i)}
\]

for all \( i \in C_n \). We give another characterization of this scalar.

**Proposition 5.13.** The continuity factor \( s = \{\sigma, \tau\} \) is the unique nonzero scalar so that, for every \( i \) and nonzero \( f : i \to \tau(i) \), \( g : i \to \sigma(i) \), the following diagram commutes.

\[
\begin{array}{ccc}
i & \xrightarrow{f} & \sigma(i) \\
g \downarrow & & \downarrow \sigma(g) \\
\tau(i) & \xrightarrow{s\tau(f)} & \sigma\tau(i)
\end{array}
\]
Proof. This diagram is a combination of the following two commuting diagrams:

\[ \begin{array}{ccc}
\sigma(i) & \xrightarrow{\sigma(f)} & \sigma^2(i) \\
\parallel & & \parallel \\
\varphi_i & \xrightarrow{\tau(i)} & \tau(i) \\
\end{array} \quad \begin{array}{ccc}
\sigma(i) & \xrightarrow{\sigma(f)} & \sigma^2(i) \\
\parallel & & \parallel \\
\varphi_i & \xrightarrow{\tau(i)} & \tau(i) \\
\end{array} \]

Since \( C_n(\sigma(i), \tau(i)) = K \), \( gf^{-1} : \sigma(i) \to \tau \) is a scalar multiple of \( \varphi_i \). Thus \( t\varphi_i \circ f = g \) for some \( t \in K^* \). Since \( \sigma \) is a functor, \( t\sigma(\varphi_i) \circ \sigma(f) = \sigma(g) \). So, the right hand diagram commutes. The left hand diagram commutes since \( \varphi \) is a natural transformation. By definition of \( s = \{\sigma, \tau\} \) we get:

\[ t\sigma(\varphi_i) \circ \sigma(f) = t\varphi_i(\sigma^2) \circ \sigma(f) = t\tau(f) \circ \varphi_i \]

In other words, the dotted arrow in the right hand diagram can be filled in with the morphism \( s\tau(f) \) as claimed. \( \square \)

Corollary 5.14. Let \( \sigma, \tau \) be commuting autoequivalences of \( C_n \). Then

\[ \{\tau, \sigma\} = \{\sigma, \tau\}^{-1} \]

There is one additional piece of structure that is required to define a triangulation. Assume that we fix a system of good bases for \( (C_n, \sigma) \) and a permutation on \( n \) elements \( \tau \) that commutes with \( \sigma \). For any object \( i \) of \( C_n \), we need a natural \( K \)-linear isomorphism \( \varphi_i : \sigma(i) \cong \tau(i) \). We write:

\[ \varphi_i = c_i x_{\tau(i)\sigma(i)} : \sigma(i) \to \tau(i) \]

for each object \( i \) in \( C_n \), where \( c_i \in K^* \) and \( \cdot c_i \) indicates multiplication by \( c_i \). The outline of the construction, describing the roles of \( \sigma, \tau, \varphi \) is as follows.

1. \( n \) will be the number of sheets in the equivalence covering category \( M_\sigma \to M_0 \).
2. \( \sigma \) will be the holonomy functor which we use as the “clutching functor” to construct the \( n \)-fold equivalence covering category \( M_\sigma \).
3. The shift functor will be given by \( \tau \). Since \( TX \cong X \) in the underlying algebraic category, we must have \( \sigma(i) \cong \tau(i) \) and this is used to define the last morphism in the distinguished triangle (from \( (Z, i) \) to \( (X, \tau(i)) \)):

\[ (X, i) \to (Y, i) \to (Z, i) \to (X, \sigma(i)) \xrightarrow{\varphi_i} (X, \tau(i)) = T(X, i) \]

4. The naturality of the isomorphism \( \varphi \) is required by the axiom of triangulated category which says that any morphism from \( f : X \to Y \) to \( f' : X' \to Y' \) can be completed to a map of distinguished triangles. This will give three commuting squares, where the commutativity of the last square is the naturality of \( \varphi \).
5. The relation \( \sigma \tau = \tau \sigma \) is the requirement that the shift functor \( T(X, i) = (X, \tau(i)) \) should be continuous, i.e., it should commute with holonomy.
These conditions can be rephrased in terms of the transition coefficients $a_{ij}, b_{ij}, c_i$ of $\sigma, \tau, \varphi$ as follows. First, naturality of $\varphi$ is:
\begin{equation}
\varphi_i \circ \sigma(x_{ij}) = \tau(x_{ij}) \circ \varphi_j \Rightarrow c_i a_{ij} = b_{ij} c_j
\end{equation}
for all $i, j$. In a triangulated category, the shift of a distinguished triangle is a distinguished triangle with the sign of the last arrow reversed. Since $\sigma(i) \cong \tau(i)$ we can replace $\tau$ with $\sigma$ to get the distinguished triangle:
\[ (X, \sigma(i)) \to (Y, \sigma(i)) \to (Z, \sigma(i)) \to (X, \sigma^2(i)) \xrightarrow{-\varphi_{\sigma(i)}} T(X, \sigma(i)) = (X, \tau\sigma(i)) \]
So, we get the condition (same as (3.1))
\begin{equation}
\varphi_{\sigma(i)} = \varphi_i \circ \sigma(x_{\sigma^{-1}(i)j}) \Rightarrow c_{\sigma(i)} = -c_i a_{\sigma(i)\sigma(i)}
\end{equation}
Finally, we have the commutativity of $\sigma$ and $\tau$ which translates to the condition:
\begin{equation}
\sigma \tau(x_{ij}) = \tau \sigma(x_{ij}) \Rightarrow a_{\tau(i)\tau(j)} b_{ij} = a_{ij} b_{\sigma(i)\sigma(j)}
\end{equation}
The topological interpretation of these conditions, outlined above, is presented in detail in the next section.

6. Continuous Frobenius categories

In Section 4 we classified all possible continuously triangulated finite coverings of the Mobius strip category and showed that, if they exist, they are classified by the three parameters $\sigma, \tau, \varphi : \sigma \to \tau$. In this section we complete the classification by showing that each of these structures exist. The proof is by explicit construction of the corresponding continuous Frobenius categories. This is a generalization of the construction of the continuous Frobenius category from [3].

6.1. Circle categories. We construct covering categories $\tilde{\mathcal{T}}_n(S^1)$ for the circle category with coefficients in $K[[u]]$. Consider the power series rings $K[[t]] \subset K[[u]]$, where $t = u^2$ with the discrete topology. Let $\mathcal{C}_n(K[[u]])$ be the category with object set $[n] = \{1, 2, \ldots, n\}$ and hom sets $\text{Hom}(i, j) = K[[u]]$ for all $i, j$ with composition given by mulitplication. (Thus $\mathcal{C}_n(K[[u]]) = \mathcal{C}_n \otimes_K K[[u]]$.) $K[[u]]$-linear automorphisms of $\mathcal{C}_n(K[[u]])$ are given by pairs $(\sigma, (a_{ij}(u)))$ where $\sigma \in S_n$ is a permutation of $n$ and $a_{ij}(u) \in K[[u]]$ are power series transitions coefficients, just as in the case of $\text{Aut}(\mathcal{C}_n)$. Then $\text{Aut}(\mathcal{C}_n)$ is the subgroup of $\text{Aut}(\mathcal{C}_n(K[[u]]))$ of automorphism where the transition coefficients $a_{ji}(u) \in K$ are all constant. Given any $\sigma \in \text{Aut}(\mathcal{C}_n)$ we will choose a system of constants $c_i \in K^*$ so that $a_{ji} = c_j c_i^{-1}$. This is equivalent to choosing a lifting of $\sigma$ to an element $PD \in GL_n(K) \subset GL_n(K[[u]])$ where $P$ is a permutation matrix and $D$ is a diagonal matrix with diagonal entries $c_i$.

Consider the circle $S^1 = \mathbb{R}/\pi \mathbb{Z}$ from Section 4. This is the meridian circle of the Moebius band. The boundary of the Moebius band is $S^1 = \mathbb{R}/2\pi \mathbb{Z}$, a two fold covering of $S^1$.

**Definition 6.1.** [3] Let $\mathcal{P}(S^1)$ denote the topological $K[[u]]$-category with objects $P_x$ for all $x \in S^1$ with the topology given by the circle: $\text{Ob}(\mathcal{P}(S^1)) = S^1$. Every hom set $\text{Hom}(P_x, P_y)$
will be a free $K[[u]]$-module generated by a basic morphism $f_{yx}$ which we give the weight $\alpha(x, y) \in [0, \pi)$ the forward distance from $x$ to $y$ with composition given by

$$f_{zy} \circ f_{yx} = \begin{cases} f_{xz} & \text{if } \alpha(x, y) + \alpha(y, z) < \pi \\ uf_{zx} & \text{otherwise} \end{cases}$$

In particular, $f_{xx}$ is the identity on $P_x$ and $f_{xy}f_{yx} = uf_{xx}$ if $x \neq y$.

As $y$ converges to $x$ from below, $f_{yx}$ converges to $uf_{xx}$. It will be convenient to use a different, continuous notation. For every pair of real number $a \leq b$ with corresponding elements of $S^1$ given by $[a] = a + \pi\mathbb{Z}$ and $[b] = b + \pi\mathbb{Z}$, let $\tilde{f}_{ba} : P_{[a]} \rightarrow P_{[b]}$ be the morphism $\tilde{f}_{ba} = u^n f_{ca}$ where $n \geq 0$ is maximal so that $c = b - n\pi \geq a$. Then we have

(6.1) \[ \tilde{f}_{b+\pi,a+\pi} = \tilde{f}_{ba} \]

(6.2) \[ \tilde{f}_{b+\pi,a} = uf_{ba}. \]

The main point is that composition is given by the continuous formula:

$$(rs\tilde{f}_{cd})(s\tilde{f}_{ba}) = rs\tilde{f}_{ca}$$

for all $a \leq b \leq c \in \mathbb{R}$ and all $r, s \in R = K[[u]]$.

The topological space of basic morphisms is homeomorphic to $S^1 \times [0, \infty)$ where the homeomorphism sends $\tilde{f}_{ba}$ to $([a], b - a)$. The topological space of all morphisms is given as a quotient of $\mathbb{R} \times [0, \infty) \times K[[u]]$ modulo the relations (6.1), (6.2).

**Definition 6.2.** Given any automorphism $\sigma$ of $C_n$ and a choice of lifting to $GL_n(K)$ given by scalars $c_i$, we can construct an $n$-fold covering category $\tilde{P}_\sigma(S^1)$ with holonomy $\sigma$ as follows. The object space of $\tilde{P}_\sigma(S^1)$ is defined to be $\tilde{S}_\sigma$, the $n$-fold covering space of $S^1$ with holonomy $\sigma$:

$$Ob(\tilde{P}_\sigma(S^1)) = \tilde{S}_\sigma := \mathbb{R} \times [n]/ \sim$$

where $(x + \pi, i) \sim (x, \sigma(i))$. Equivalence classes will be denoted $[x, i]$. Morphism are $K[[u]]$-linear combinations of basic morphisms:

$$\tilde{f}_{yx} \otimes x_{ji} : [x, i] \rightarrow [y, j]$$

for all $x \leq y \in \mathbb{R}$ and $i, j \in [n]$. These are given to be continuously varying with respect to $x, y \in \mathbb{R}$ and satisfy the following relations.

(6.3) \[ \tilde{f}_{y+\pi,x+\pi} \otimes x_{ji} = a_{ji} \tilde{f}_{yx} \otimes x_{\sigma(j)\sigma(i)} : [x, \sigma(i)] \rightarrow [y, \sigma(j)] \]

(6.4) \[ \tilde{f}_{y+\pi,x} \otimes x_{ji} = c_{ji} u \tilde{f}_{yx} \otimes x_{\sigma(j)i} : [x, i] \rightarrow [y + \pi, j] = [y, \sigma(j)] \]

Here is an example to see that composition is well defined. $(x \leq y \leq z$ in this example.)

$$(\tilde{f}_{z+2\pi,y+\pi} \otimes x_{kj})(\tilde{f}_{y+\pi,x} \otimes x_{ji}) = (c_{\sigma(k)} a_{kj} u \tilde{f}_{zy} \otimes x_{\sigma^2(k)\sigma(j)})(c_{ji} u \tilde{f}_{yx} \otimes x_{\sigma(j)i})$$

$$\tilde{f}_{z+2\pi,x} \otimes x_{ki} = c_{\sigma(k)} u \tilde{f}_{z+\pi,x} \otimes x_{\sigma(k)i} = c_{\sigma(k)} c_k u^2 \tilde{f}_{zx} \otimes x_{\sigma^2(k)i}. $$

In the first line, we simplify each factor then compose. In the second line we compose first. The results are equal since $a_{kj} c_j = c_k$. 37
Proposition 6.3. \( \widetilde{P}_\sigma(S^1) \) is \( K[[u]] \)-linearly equivalent to \( P(S^1) \).

In fact \( \widetilde{P}_\sigma(S^1) \) is a \( K[[u]] \)-equivalence cover of \( P(S^1) \) but we don’t need to show this.

Proof. Let \( F : P(S^1) \to \widetilde{P}_\sigma(S^1) \) and left inverse \( G : \widetilde{P}_\sigma(S^1) \to P(S^1) \) be given as follows. Every object of \( P(S^1) \) can be represented as \( P_x \) where \( 0 \leq x < \pi \). Let \( F(P_x) = [x, k] \in \widetilde{P}_\sigma(S^1) \) where \( k = \sigma(1) \). For every \( [x], [y] \in S^1 \), \( \text{Hom}(P_x, P_y) \) is the free \( K[[u]] \)-module generated by \( \tilde{f}_{yx} \) if \( 0 \leq x \leq y < \pi \) and by \( \tilde{f}_{y+x,x} \) if \( 0 \leq y < x < \pi \). Let

\[
F(\tilde{f}_{yx}) = \tilde{f}_{yx} \otimes x_{kk} : [x, k] \to [y, k]
\]

and for any morphism of the form \( u^n \tilde{f}_{y+x,x} \otimes x_{ji} : [x, i] \to [y + \pi, j] \) where \( 0 \leq x, y < \pi \), let \( G(u^n \tilde{f}_{y+x,x} \otimes x_{ji}) = c_j u^n \tilde{f}_{y+x,i} \).

We let \( F \) be given as follows. \( G[x, i] = P_x \) and for any morphism of the form \( u^n \tilde{f}_{y+x,i} \otimes x_{ji} : [x, i] \to [y + \pi, j] \) where \( 0 \leq x, y < \pi \), let \( G(u^n \tilde{f}_{y+x,i} \otimes x_{ji}) = c_j u^n \tilde{f}_{y+x,i} \).

We let \( n \geq -1 \) when \( y \geq x \). It is straightforward to show that \( G \circ F \) is the identity functor on \( P(S^1) \) and \( F \circ G \) is equivalent to the identity functor by the isomorphism \( \tilde{f}_{xx} \otimes x_{ki} : [x, i] \cong [x, k] \).

Comparing definitions, we have the following easy theorem.

Theorem 6.4. The quotient category of \( \widetilde{P}_\sigma(S^1) \) modulo the ideal of all morphisms divisible by \( u \) is continuously isomorphic to the equivalence covering \( \widetilde{S}^1_\sigma \) of the circle category \( S^1 \) from Definition 1.12 and Theorem 2.11.

Now we need to consider a double covering of \( \widetilde{P}_\sigma(S^1) \). For the circle \( S^1 \), the double cover is \( \widetilde{S}^1 = \mathbb{R}/2\pi \mathbb{Z} \). For \( S^1 \), the double cover is:

\[
\widetilde{S}^1_\sigma := \mathbb{R} \times [n] \times \{+, -\} / \sim
\]

where \( (x + \pi, i, \varepsilon) \sim (x, \sigma(i), -\varepsilon) \). Points in \( \widetilde{S}^1_\sigma \) are the equivalence classes \( [x, i, \varepsilon] \). We view \( \widetilde{S}^1_\sigma \) as a \( 2n \)-fold covering of the small circle \( S^1 \). The sheets of the cover are labeled \( (i, \varepsilon) \). As the continuous parameter \( x \) increases to \( x + \pi \), the point \( (x, i, +) \) moves to sheet \( (\sigma(i), -) \). Then as \( x \) keeps increasing to \( x + 2\pi \), the point comes back to the same point \( [x] \in S^1 \) on sheet \( (\sigma^2(i), +) \). In particular, the path will only return to the same point after going around the circle an even number of times, moving a distance of \( 2\pi n \) where \( \sigma^{2n}(i) = i \).

Note that every point in \( \widetilde{S}^1_\sigma \) is equivalent to a positive point since

\[
[x, i, -] = [x - \pi, \sigma(i), +].
\]

As a quotient of the space of positive points, we have

\[
\widetilde{S}^1_\sigma = \mathbb{R} \times [n] / \sim
\]

where \( (x + 2\pi, i) \sim (x, \sigma^2(i)) \).
Definition 6.5. For any $\sigma \in \text{Aut}(C_n)$ with lifting to $GL_n(K)$ given by scalars $c_i$, let $\tilde{P}_\sigma(S^1)$ be the topological $K[[t]]$ category with object space $S^1$ and morphisms

$$r\tilde{f}_{yx} \otimes x_{ji} : [x, i, +] \to [y, j, +]$$

for $x \leq y \in \mathbb{R}$ and $r \in K[[t]]$, considered as elements of

$$\text{Mor} \left( \tilde{P}_\sigma(S^1) \right) := K[[t]] \times \{(x, y) \in \mathbb{R}^2 | x \leq y\} \times [n]^2 / \sim$$

(with the quotient topology), where the equivalence relation is given by

$$\tilde{f}_{y+2\pi,x+2\pi} \otimes x_{ji} = b_{ji}\tilde{f}_{yx} \otimes x_{\sigma_{2}(j),\sigma_{2}(i)} : [x, \sigma^{2}(i), +] \to [y, \sigma^{2}(j), +]$$

$$\tilde{f}_{y+2\pi,x} \otimes x_{ji} = d_{j}t\tilde{f}_{yx} \otimes x_{\sigma^{2}(j)i} : [x, i, +] \to [y + 2\pi, j, +] = [y, \sigma^{2}(j), +]$$

where $d_{j} = c_{\sigma(j)}c_{j}$ and $b_{ji} = a_{\sigma(j)\sigma(i)}a_{ji} = d_{j}d_{i}^{-1}$ are the scalars associated with $\sigma^{2}$ and its lifting $PDPD = P^{2}(P^{-1}DP)D \in GL_n(K)$.

When $n = 1$ and $\sigma \in \text{Aut}(C_1)$ is the identity element, the topological $K[[t]]$-category is continuously isomorphic to the category $\mathcal{P}(\tilde{S}^1)$, where $\tilde{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, which is used in [3] to construct the continuous Frobenius category. Following the same proof as in Proposition 6.3, we have the following.

Proposition 6.6. The $K[[t]]$-category $\tilde{P}_\sigma(S^1)$ is equivalent to the category $\mathcal{P}(\tilde{S}^1)$ from [3].

Corollary 6.7. An isomorphism $X \to Y$ between indecomposable objects of $\tilde{P}_\sigma(S^1)$ cannot factor through an object $Z$ in add $\tilde{P}_\sigma(S^1)$ none of whose components is isomorphic to $X$. □

6.2. Matrix factorizations. Following [3], we construct continuous Frobenius categories as categories of matrix factorizations for objects in add $\tilde{P}_\sigma(S^1)$.

By a matrix factorization of $t$ in an additive $K[[t]]$-category $\mathcal{P}$, we mean a pair $(P, d)$ where $P \in \mathcal{P}$ and $d : P \to P$ is an $K[[t]]$-linear endomorphism whose square is $t$ times the identity on $P$. A morphism of matrix factorizations $(P, d) \to (P', d)$ is a morphism $f : P \to P'$ in $\mathcal{P}$ which commutes with $d$, i.e. $fd = df$.

Definition 6.8. For $\mathcal{P} = \text{add } \tilde{P}_\sigma(S^1)$, the basic example of a matrix factorization is the object $M(x, y, i) = \left( [x, i, -] \oplus [y, i, +], \begin{bmatrix} 0 & d_+ \\ d_- & 0 \end{bmatrix} \right)$ for real numbers $x, y$ with $|y - x| \leq \pi$ and $i \in [n]$ which we visualize as:

$$M(x, y, i) : \quad [x, i, -] \xrightarrow{d_-} \quad \xrightarrow{d_+} [y, i, +]$$

where

$$d_- = c_{i}^{-1}f_{y-x,\pi} \otimes x_{i\sigma(i)} : [x, i, -] = [x - \pi, \sigma(i), +] \to [y, i, +]$$

$$d_+ = c_{\sigma^{-1}(i)}^{-1}f_{x+\pi,y} \otimes x_{\sigma^{-1}(i)i} : [y, i, +] \to [x, i, -] = [x + \pi, \sigma^{-1}(i), +]$$

39
Computation shows that $d_+d_-$ and $d_-d_+$ are both $t$ times identity morphisms of $[x, i, -]$ and $[y, i, +]$, respectively. Since we are using the James construction for $\widetilde{\text{add}} \mathcal{P}_\sigma(S^1)$, $\oplus$ is strictly commutative. So,

$$[x, i, -] \oplus [y, i, +] = [x - \pi, \sigma(i), +] \oplus [y - \pi, \sigma(i), -] = [y - \pi, \sigma(i), -] \oplus [x - \pi, \sigma(i), +]$$

and we have:

$$M(x, y, i) = M(y - \pi, x - \pi, \sigma(i)).$$

**Proposition 6.9.** The category of finitely generated matrix factorizations of $t$ in $\text{add} \widetilde{\mathcal{P}}_\sigma(S^1)$ is Krull-Schmidt and each indecomposable object is isomorphic to some $M(x, y, i)$.

**Proof.** This follows from Proposition 6.6 since the analogous statement for $\text{add} \mathcal{P}(\tilde{S}^1)$ was shown in [3].

**Remark 6.10.** Note that, in the special case when $y = x - \pi$, $d_+: [x, i, -] = [y, \sigma(i), +] \to [y, i, +]$ is an isomorphism. When $y = x + \pi$, $d_-: [y, i, +] \to [x, i, -] = [y, \sigma^{-1}(i), +]$ is an isomorphism. Furthermore, these give the same object when $i$ is shifted:

$$M(x, x + \pi, i) = M(x, x - \pi, \sigma(i)) = M(x + 2\pi, x + \pi, \sigma^{-1}(i))$$

We will denote this object $I_{\sigma(i)}(x - \pi) = I_{\sigma^{-1}(i)}(x + \pi)$. Thus $I_i(x) = M(x + \pi, x, i)$.

**Theorem 6.11.** Let $\overline{\mathcal{M}}_\sigma(K[[t]])$ denote the full subcategory of the matrix factorization category of $t$ in $\text{add} \widetilde{\mathcal{P}}_\sigma(S^1)$ with objects $M(x, y, i)$ for all $x, y \in \mathbb{R}$ with $|y - x| \leq \pi$ and all $i \in [n]$ modulo the relation (6.5). Then the object space of $\overline{\mathcal{M}}_\sigma(K[[t]])$ is an $n$-fold covering space of the closed Moebius strip.

Let $\overline{\mathcal{M}}(K[[t]])$ denote $\overline{\mathcal{M}}_1(K[[t]])$ where $\sigma = 1$ is the identity element of Aut$(\mathcal{C}_1)$. We call $\overline{\mathcal{M}}(K[[t]])$ the closed Moebius strip category.

**Proposition 6.12.** The topological full subcategory of $\overline{\mathcal{M}}_\sigma(K[[t]])$ with objects $M(x, x, i)$ for all $(x, i) \in \mathbb{R} \times [n]$ is continuously isomorphic to $\widetilde{\mathcal{P}}_\sigma(S^1)$.

**Proof.** There is a continuous embedding $J: \widetilde{\mathcal{P}}_\sigma(S^1) \to \overline{\mathcal{M}}_\sigma(K[[t]])$ given by sending $[x, i]$ to $M(x, x, i)$ and the morphism $\tilde{f}_{yx} \otimes x_{ji}: [x, i] \to [y, j]$ to the horizontal arrows:

$$[x - \pi, \sigma(i), +] \xrightarrow{\alpha_{ji} \tilde{f}_{yx - \pi, x - \pi} \otimes x_{j\sigma}(i)} [y - \pi, \sigma(j), +]$$

$$[x, i, +] \xrightarrow{\tilde{f}_{yx} \otimes x_{ji}} [y, j, +]$$

giving a morphism $M(x, x, i) \to M(y, y, j)$. 

Recall [3] that an exact sequence of matrix factorizations for an additive $K[[t]]$-category $\mathcal{P}$ is defined to be a short exact sequence $0 \to (A, d) \to (B, d) \to (C, d) \to 0$ so that the underlying sequence in $\mathcal{P}$ is split exact.
Theorem 6.13. \( \overline{\mathcal{M}}_\sigma(K[[t]]) \) is a Frobenius category with exact structure described above with indecomposable projective-injective objects given by \( I_i(x) = M(x + \pi, x, i) \).

Proof. In [3] it is shows that the category of matrix factorizations of \( t \) in \( \mathcal{P}(\overline{S^1}) \) is a Frobenius category equivalent to the subcategory \( \overline{\mathcal{M}}(K[[t]]) \). By Proposition 6.6 the same is true for \( \overline{\mathcal{M}}_\sigma(K[[t]]) \).

Proposition 6.14. A isomorphism between indecomposable objects in \( \overline{\mathcal{M}}_\sigma(K[[t]]) \) cannot be written as a sum of compositions of morphisms none of which are isomorphisms.

Proof. By [3] this holds in the category of matrix factorizations of \( t \) in \( \mathcal{P}(\overline{S^1}) \) which is a Frobenius category equivalent to the subcategory \( \overline{\mathcal{M}}(K[[t]]) \).

6.3. Continuous triangulation of the stable category. By [1], the stable category of the Frobenius category \( \overline{\mathcal{M}}_\sigma(K[[t]]) \) is a triangulated category. The distinguished triangles are given by pushing out “universal exact sequences” of the following form for each objects \( X \):

\[
(6.6) \quad X \to I(X) \to T(X)
\]

where \( I(X) \) is a projective-injective object and \( X \to I(X) \) is an admissible monomorphism, i.e. a split monomorphism on the underlying objects in \( \overline{\mathcal{M}}_\sigma(S^1) \). We would like this exact sequence, objects and morphisms, to be continuous in \( X \) so that the resulting triangulated category is continuously triangulated.

Since our task in this section is the construction of the continuously triangulated categories \( \overline{\mathcal{M}}_n(\sigma, \tau, \varphi) \) of the Classification Theorem 4.2, we take as given the structures \( \sigma, \tau, \varphi \).

For any \( K \)-linear autoequivalence \( \tau \) of \( \mathcal{C}_n \) which commutes with \( \sigma \), let \( F_\tau \) be the continuous autoequivalence of \( \overline{\mathcal{M}}_\sigma(S^1) \) given on objects by \( F_\tau[x, i, +] = [x, \tau(i), +] \) and on morphisms by \( F_\tau = id \otimes \tau \), i.e. \( F_\tau \) sends \( f_{xy} \otimes x_{ji} \) to \( f_{xy} \otimes b_{ji} x_{\tau(j)\tau(i)} \) where \( b_{ji} \) are the transition coefficients of \( \tau \) given by \( \tau(x_{ji}) = b_{ji} x_{\tau(j)\tau(i)} \). So far, this requires only that \( \tau \) commutes with \( \sigma^2 \). The fact that \( \tau \) commutes with \( \sigma \) implies that \( F_\tau \) has the same formula on the negative points:

\[
F_\tau[x, i, -] = F_\tau[x - \pi, \sigma(i), +] = [x - \pi, \tau\sigma(i), +] = [x, \tau(i), -]
\]

and, similarly for morphisms by the equation \( a_{\tau(i)\tau(j)} b_{ij} = a_{ij} b_{\sigma(i)\sigma(j)} \) from [5.5]:

\[
F_\tau(f_{xy} \otimes x_{ji} : [x, i, -] \to [y, j, -]) = F_\tau(a_{ji} f_{y-x-x-\pi} \otimes x_{\tau(j)\tau(i)} = b_{ji} f_{xy} \otimes x_{\tau(j)\tau(i)} : [x, \tau(i), -] \to [y, \tau(j), -].
\]

Thus, \( F_\tau \) induces an autoequivalence of \( \overline{\mathcal{M}}_\sigma(K[[t]]) \) given on objects by \( F_\tau(M(x, y, i)) = M(x, y, \tau(i)) \) and on morphisms by the equation above. Since \( \sigma \) commutes with itself, we also have the automorphism \( F_\sigma \) of \( \overline{\mathcal{M}}_\sigma(K[[t]]) \).

Definition 6.15. Let \( \sigma, \tau, \varphi \) be given as in Theorem 4.2. For every object \( M(x, y, i) \) in \( \overline{\mathcal{M}}_\sigma(K[[t]]) \) we define the universal exact sequence of \( M(x, y, i) \) to be the following

\[
(6.7) \quad M(x, y, i) \xrightarrow{\delta} M(x, x + \pi, i) \oplus M(y + \pi, y, i) \xrightarrow{\rho} M(x, y, \tau(i))
\]
where \( j = (j_1, j_2) \) with \( j_1 = id \oplus (f_{x+\pi, y} \otimes x_{ii}) : [x, i, -] \oplus [y, i, +] \to [x, i, -] \oplus [x + \pi, i, +] \) and similarly, \( j_2 = (f_{y+\pi, x} \otimes x_{ii}) \oplus id \). The morphism \( p \) is the composition of two morphisms:

\[
M(x, x + \pi, i) \oplus M(y + \pi, y, i) \overset{q}{\rightarrow} M(y + \pi, x + \pi, i) = M(x, y, \sigma(i)) \overset{\phi}{\rightarrow} M(x, y, \tau(i))
\]

where \( q = (-q_1, q_2) \) with \( q_1 = id \oplus (f_{x+\pi, y} \otimes x_{ii}) \) and \( q_2 = (f_{x+\pi, y} \otimes x_{ii}) \oplus id \). Finally, \( \varphi = \varphi_1 \oplus \varphi_2 \) is the isomorphism given on the summands of \( M(x, y, \sigma(i)) \) by \( \varphi_1 = f_{xy} \otimes c_i \tau(i) \sigma(i) \) and \( \varphi_2 = f_{xy} \otimes c_i \tau(i) \sigma(i) \) where \( c_i \) are the structure constants of \( \varphi \) defined by \( \varphi_i = c_i \tau(i) \sigma(i) : \sigma(i) \to \tau(i) \) and satisfying:

\[
\begin{align*}
\alpha_i j_1 = b_{ji} c_i, & \quad \alpha_i \sigma(i) = -c_i a_{\tau(i) \sigma(i)}
\end{align*}
\]

from \((5.3)\) and \((5.4) = (3.1)\).

To see that \((6.7)\) is exact, note that \( j_1 \) is the identity map on the first fact, \( j_2 \) is the identity map on the second factor and \( p \) sends the remaining factors of the middle term isomorphically to the two summands of \( M(x, y, \tau(i)) \). Also, a key required property is that the middle terms in the sequence \((6.7)\) is projective-injective. To emphasize this we rewrite the sequence as:

\[
M(x, y, i) \to I_{\sigma(i)}(x - \pi) \oplus I_i(y) \to TM(x, y, i)
\]

where \( TM(x, y, i) = M(x, y, \tau(i)) \). We also use the notation \( SM(x, y, i) = M(x, y, \sigma(i)) \).

**Proposition 6.16.** The sequence \((6.7)\) is well-defined, i.e. we obtain the same sequence if we use the notation \( M(x, y, i) = M(x - \pi, x - \pi, \sigma(i)) \):

\[
(6.8) \quad M(y + \pi, x - \pi, \sigma(i)) \overset{q'}{\rightarrow} M(y - \pi, x - \pi, \sigma(i)) \oplus M(y - \pi, x - \pi, \tau \sigma(i)) \overset{p'}{\rightarrow} M(y - \pi, x - \pi, \tau \sigma(i)).
\]

In particular we have a well defined functor \( I \) giving the middle term: \( IM(x, y, i) = I_{\sigma(i)}(x - \pi) \oplus I_i(y) \), although the order of the components is not well-defined.

**Proof.** To see that \((6.8)\) is the same as \((6.7)\), we note first that the terms are the same except that the terms in the middle are switched. But that is OK since, by our definitions, direct sum is strictly commutative! Since the summands are switched, \( q = (-q_1, q_2) \) changes to \( q' = (-q_2, q_1) = -q \). But the sign of \( \varphi \) also changes since \( \varphi \) is skew-commutative. Therefore, \( p' = p \). So, the sequences are equal and the universal sequence \((6.7)\) is well-defined. \( \square \)

The key point is the continuity of the universal sequence.

**Lemma 6.17.** The exact sequence \((6.7)\) is a continuous function of \( M(x, y, i) \) as it varies in the compact Hausdorff space \( \text{Ob}(\overline{M}_\sigma(K[[t]])) \) which is an \( n \)-fold covering of the closed Möbius strip. \( \square \)

Recall from \([1]\) that the stable category of a Frobenius category is defined to be the quotient category with the same objects but modding out morphisms which factor through projective-injective objects. As in \([1]\) we get the following.

**Theorem 6.18.** The stable category of the Frobenius category \( \text{add} \overline{M}_\sigma(K[[t]]) \) is \( \text{add} \overline{M}_\sigma \), the additive category of the equivalence covering \( \overline{M}_\sigma \) of \( M_0 \). \( \square \)

We recall from \([1]\) the construction of all distinguished triangles in the stable category.
Definition 6.19. Given any morphism $f : X \to Y$ in $\text{add} \overline{M}_\sigma(K[[t]])$, a distinguished triangle $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} TX$ is given as the pushout of the direct sum of universal exact sequences for the summands of $X$:

$$\begin{array}{ccc}
X & \overset{j}{\to} & IX \\
\downarrow f & & \downarrow p \\
Y & \overset{g}{\to} & Z \\
\downarrow h & & \downarrow TX \\
\end{array}$$

Here $X \to IX \to TX$ is a direct sum of the universal sequences (6.7) for each component of $X$. In the stable category, we delete all the projective-injective objects in $X,Y,Z$ and replace maps $f,g,h$ by the stable maps $\overline{f},\overline{g},\overline{h}$ which are $f,g,h$ modulo morphisms which factor through projective-injective objects.

Example 6.20. Suppose $0 < x < y < z < \pi$ Take $X = M(x,y,i)$, $Y = M(x,z,i)$ and $f = id \oplus (f_{zx} \otimes x_{ii}) : [x,i,-] \oplus [y,i,+] \to [x,i,-] \oplus [z,i,+]$. Then

$$IX = I_{\varphi(i)}(x - \pi) \oplus I_z(y) = [x,i,-] \oplus [x + \pi,i,+] \oplus [y + \pi,i,-] \oplus [y,i,+]$$

So, the $I_{\varphi(i)}(x - \pi)$ term remains and

$$Z = I_{\varphi(i)}(x - \pi) \oplus M(y + \pi,z,i).$$

Since the negative sign in the universal sequence (6.7) is on the first summand $I_{\varphi(i)}(x - \pi)$, when we go to the stable category, the scalars for the first two maps are +1:

$$M(x,y,i) \xrightarrow{1} M(x,z,i) \xrightarrow{1} M(y + \pi,z,i) \xrightarrow{c} M(x,y,\tau(i)).$$

Remark 6.21. We observe that, after applying the forgetful functor $\text{add} \overline{M}_\sigma(K[[t]]) \to \text{add} \overline{P}_\sigma(S^1)$, the diagram (6.9) can be rewritten in the following form.

$$\begin{array}{ccc}
X & \overset{(1)}{\to} & X \oplus TX \\
\downarrow f & & \downarrow \cong \\
Y & \overset{(1)}{\to} & Y \oplus TX \\
\downarrow f \circ id & & \downarrow id \\
\end{array}$$

where $r = s \circ f : X \to TX$. The operator $d$ which acts compatibly on $X,Y,TX$ gives an induced action of $d$ on $Z = Y \oplus TX$. Thus $(Z,d)$ is uniquely determined up to isomorphism by $f : X \to Y$. But we should keep in mind that $\text{add} \overline{M}_\sigma(K[[t]])$ contains $n$ isomorphic copies of each indecomposable object.

As in the main theorem 4.2, the stable category of $\text{add} \overline{M}_\sigma(K[[t]])$ together with the distinguished triangles given above is denoted $\text{add} \overline{M}(\sigma,\tau,\varphi)$.

Theorem 6.22. $\text{add} \overline{M}(\sigma,\tau,\varphi)$ is a continuously triangulated category.
6.4. Proof of Theorem 6.22. Since \( \tau \) commutes with \( \sigma \), the shift functor \( T = F_\tau \) is continuous. It remains to show that the set of distinguished triangles is a closed set. To do this we analyze the limiting behavior of morphisms and objects. There are two cases: when some objects or morphisms go to zero and when they don’t. One thing is clear: in a limit the number of objects can only decrease and the number of nonzero morphisms can only decrease. We recall that our objects have fixed direct sum decompositions and thus morphisms are given by matrices whose entries are morphisms between the indecomposable objects \( M(x, y, i) \).

The components \([x, i, –] = [x – \pi, \sigma(i), +]\) and \([y, i, +]\) of \( M(x, y, i) \) will be called the ends of \( M(x, y, i) \). Sometime, we will call \([x, i, –]\) the negative end and \([y, i, +]\) the positive end. These terms are not well-defined. They depend on the notation used to describe the objects. (However, the unordered pair of ends is well-defined.)

Recall that the support of any object \( X \) is the set of all objects \( Y \) for which there is a nonzero morphism \( X \to Y \). We already know that the support of \( M(x, y, i) \) is the set of all \( M(x', y', j) \) where \( x \leq x' < y + \pi \) and \( y \leq y' < x + \pi \). We also recall that every nonzero morphism is a scalar multiple of a basic morphism. In particular, there is a well defined scalar \( a \in K^* \) associated to every nonzero morphism and this scalar is constant on families of morphism (except when the morphism goes to zero).

Definition 6.23. There are four kinds of morphisms in \( \widetilde{M}(\sigma, \tau, \varphi) = \widetilde{M}_\sigma \):

1. a nonzero morphism \( f : X \to Y \) is stably nonzero if there exist open neighborhoods, \( U, V \) of \( X, Y \) so that \( \text{Hom}(A, B) \neq 0 \) for all \( A \in U, B \in V \). Equivalently, \( X = M(x, y, i) \) and \( Y = M(x', y', j) \) where \( x < x' < y + \pi \) and \( y < y' < x + \pi \). (See Figure 2)

2. a nonzero morphism \( f : X \to Y \) is marginal if it is not stably nonzero. Equivalently, \( X, Y \) share an end and \( f \) is an isomorphism at that end. For example, any isomorphism \( f : X \cong Y \) is marginal since \( X, Y \) have the same two ends and \( f \) is an isomorphism at each end.

3. a zero morphism \( f : X \to Y \) is stably zero if it is not a limit of nonzero morphisms.

4. a limiting null morphism is a zero morphism which is a limit of nonzero morphisms.

Proposition 6.24. (1) If \( f : X \to Y \) is stably nonzero then there are contractible open neighborhoods \( U, V \) of \( X, Y \) so that, for all \( A \in U, B \in V \) there is a unique \( f' : A \to B \) which is nonzero and homotopic to \( f \) through homotopies \( f_t : A_t \to B_t \) where \( A_t \in U, B_t \in V \). Furthermore, the set of all such maps \( f' \) forms an open neighborhood of \( f \) in \( \text{Mor}(\widetilde{M}_\sigma) \).

(2) \( f : X \to Y \) is marginal if \( X = M(x, y, i) \) and either \( Y = M(x', y', j) \) for some \( y \leq y' < x + \pi \) or \( Y = M(x', y, j) \) for some \( x \leq x' < y + \pi \) (or both).

(3) Nonzero morphisms are either marginal or stably nonzero.

(4) Every nonzero morphism has a contractible neighborhood in \( \text{Mor}(\widetilde{M}_\sigma) \) consisting of nonzero maps with the same scalar.

(5) Limiting null morphisms have the form \( M(x, y, i) \to M(y + \pi, z, j) \) where \( y < z \leq x + \pi \) or \( M(x, y, i) \to M(w, x + \pi, j) \) where \( x < w \leq y + \pi \). Any zero morphism with domain and range not of this form is stably zero.
Proof. All of this follows directly from a description of the support of any objects $M(x, y, i)$ and the fact that we are taking the discrete topology on $K$. □

To prove Theorem 6.22 we need to show that, when a family of distinguished triangles

$$(6.10) \quad X_\alpha \xrightarrow{f_\alpha} Y_\alpha \xrightarrow{g_\alpha} Z_\alpha \xrightarrow{h_\alpha} TX_\alpha$$

converges to a sequence

$$(6.11) \quad X_\infty \xrightarrow{f_\infty} Y_\infty \xrightarrow{g_\infty} Z_\infty \xrightarrow{h_\infty} TX_\infty$$

the limit sequence is a distinguished triangle.

**Case 1:** First consider the case when the terms $X_\infty, Y_\infty, Z_\infty$ in the limiting sequence (6.11) has the same number of summands as (almost all of) the terms in (6.10). Suppose further that all morphisms in (6.11) are stable: either nonzero or stably zero. In that case, we lift each distinguished triangle in (6.10) to a push-out diagram 6.9. By Proposition 6.24 the scalars associated to nonzero morphisms become constant. By the following lemma, the limit is a push-out diagram of the universal exact sequence $X_\infty \rightarrow IX_\infty \rightarrow TX_\infty$ making (6.11) into a distinguished triangle.

**Lemma 6.25.** If a family of exact sequence in $\text{add}\overline{\mathcal{M}}_\sigma(K[[t]])$, say $A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha$, converges to a sequence $A_\infty \xrightarrow{f_\infty} B_\infty \xrightarrow{g_\infty} C_\infty$, the limiting sequence is also exact.

**Proof.** In the topological $K[[t]]$-category $\text{add}\overline{\mathcal{M}}_\sigma(K[[t]])$, morphisms cannot go to zero and objects cannot go to zero. Therefore, in any converging family of objects and morphisms, the number of objects becomes stable (constant) and the scalars associated to each morphism between indecomposables also becomes constant. So, zero morphisms cannot become nonzero and $g_\infty \circ f_\infty = 0$.

Also, isomorphisms between indecomposables in $\text{add}\overline{\mathcal{M}}_\sigma(K[[t]])$ do not factor through nonisomorphism by Proposition 6.14 and the same holds in $\overline{\mathcal{P}}_\sigma(S^1)$ by Corollary 6.7. Thus, the property of being exact, which is equivalent to certain component morphism in the category $\overline{\mathcal{P}}_\sigma(S^1)$ being isomorphisms, is preserved in the limit and the limit sequence is exact. □

This lemma implies that the family of pushout diagrams in $\text{add}\overline{\mathcal{M}}_\sigma(K[[t]])$ associated to the family of distinguished triangles (6.10) converges to a pushout diagram and the limiting sequence (6.11) is a distinguished triangle.

**Case 2:** Suppose that the objects $X_\alpha, Y_\alpha, Z_\alpha$ do not converge to zero objects but some of the morphisms between some components go to zero. In other words some of the morphism between components of $X_\infty, Y_\infty, Z_\infty$ are limiting null morphisms (Definition 6.23(4)).

In this case, we apply the concept from Remark 1.11 which implies that the scalar associated to morphisms converging to zero can take only finitely many values. If we choose an ultrafilter for the set of parameters $\{\alpha\}$ we can assume that there is only one limiting scalar. This implies that, when we lift objects and morphisms in the stable category to $\text{add}\overline{\mathcal{M}}_\sigma(K[[t]])$, the limit is well-defined. So, Lemma 6.25 applies and the push-out diagrams in $\text{add}\overline{\mathcal{M}}_\sigma(K[[t]])$ converge to a pushout diagram and therefore the limiting sequence (6.11) is a distinguished triangle.

45
Case 3: Suppose that some components of $X_\alpha, Y_\alpha, Z_\alpha$ converge to zero so that the number of components in the limit sequence (6.11) is strictly smaller than the number of components in the family of triangles (6.10).

In this case, when the triangles (6.10) are lifted to pushout diagrams, some of the components converge to projective-injective objects. As in Case 2, using Remark 1.11, we may assume that the scalars associated to the morphisms in this diagram are constant and, therefore, the morphisms have well-defined limits. Again Lemma 6.25 applies to show that the limit diagram is a pushout diagram and the limiting sequence (6.11) is a distinguished triangle.

This resolves all the cases and concludes the proof of Theorem 6.22.

6.5. Universal virtual triangles. To conclude the proof of the main Theorem 4.2, we need to show that in the stable category $\text{add } M(\sigma, \tau, \varphi)$ the universal triangles given in (4.1) are distinguished triangles up to sign equivalence. This straightforward calculation will conclude this paper.

Let $X = M(x, y, i)$ where $|y - x| < \pi$. Let

$$f = (f_1, f_2) : X \rightarrow I_1^x X \oplus I_2^x X = M(y + \pi - \varepsilon, y, i) \oplus M(x, x + \pi - \varepsilon, i)$$

where $f_1 = f_{y+\pi-\varepsilon} x \otimes x \iota \oplus \text{id}$ and $f_2 = \text{id} \oplus f_{x+\pi-\varepsilon, y} x \iota$. We denote these as $f = (1, 1)$.

Since the morphism $X \rightarrow I_{\sigma(i)}(x - \pi) \oplus I_i(y)$ factors through this map, the pushout is $M(y + \pi - \varepsilon, x + \pi - \varepsilon, i) \oplus I_{\sigma(i)}(x - \pi) \oplus I_i(y)$ and the pushout diagram is the following.

$$\begin{align*}
M(x, y, i) &\xrightarrow{(1)} I_{\sigma(i)}(x - \pi) \oplus I_i(y) \xrightarrow{(-c_i, c_i)} M(x, y, \tau(i)) \\
I_1^x X \oplus I_2^x X &\xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} Y \oplus I_{\sigma(i)}(x - \pi) \oplus I_i(y) \xrightarrow{(-c_i, -c_i, c_i)} M(x, y, \tau(i))
\end{align*}$$

Thus, we obtain the distinguished triangle

$$X \xrightarrow{(1)} I_1^x X \oplus I_2^x X \xrightarrow{(-1, 1)} Y \xrightarrow{-c_i} TX$$

which is sign equivalent to the desired system of distinguished triangles.

Acknowledgements

Research for this project was funded by the National Security Agency NSA Grant #H98230-13-1-0247. The second author also acknowledges support from GAANN during this research period. Both authors are very grateful for the support and encouragement of Gordana Todorov. These results were presented by the first author at the Auslander Conference at Woods Hole in May, 2015 which was supported by the National Science Foundation. This revised version was presented by the second author at a workshop at
Tsinghua University in Beijing in July 2017. Both of these conferences were very fruitful and enjoyable events and the authors would like to thank the organizers of both events.

References

[1] Dieter Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Mathematical Society Lecture Note Series 119, Cambridge University Press (1988).
[2] Allen Hatcher, Algebraic Topology, Cambridge Univ. Press, Cambridge, 2002.
[3] Kiyoshi Igusa and Gordana Todorov, Continuous Frobenius categories, Proceedings of the Abel Symposium 2011 (2013), 115–143.
[4] Kiyoshi Igusa and Gordana Todorov, Continuous cluster categories I, Algebras and Representation Theory, vol 18, no 1 (2015), 65–101.
[5] Kiyoshi Igusa and Gordana Todorov, Cluster categories coming from cyclic posets, Communications in Algebra, vol 43 (2015), 4367–4402.
[6] Dmitri Orlov, Triangulated categories of singularities and D-branes in Landau-Ginzburg models. (Russian) Tr. Mat. Inst. Steklova 246 (2004), 3, 240–262.; transl. Proc. Steklov Inst. Math. 246 (2004), 3, 227–248.

Department of Mathematics, Brandeis University, Waltham, MA 02454
E-mail address: mcgarcia@brandeis.edu

Department of Mathematics, Brandeis University, Waltham, MA 02454
E-mail address: igusa@brandeis.edu