REGULARITY OF THE DERIVATIVES OF $p$-ORTHOTROPIC FUNCTIONS IN THE PLANE FOR $1 < p < 2$

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Abstract. We present a proof of the $C^1$ regularity of $p$-orthotropic functions in the plane for $1 < p < 2$, based on the monotonicity of the derivatives. Moreover we achieve an explicit logarithmic modulus of continuity.

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1. Introduction

In this work we investigate the regularity of $p$-orthotropic functions in the plane for $1 < p < 2$. Let $\Omega \subset \mathbb{R}^2$ be an open set. A weak solution of the orthotropic $p$-Laplace equation (also known as pseudo $p$-Laplace equation) is a function $u \in W^{1,p}(\Omega)$ such that

$$
\sum_{i=1}^{2} \int_{\Omega} |\partial_i u|^{p-2} \partial_i u \partial_i \phi \, dx = 0 \quad \text{for all } \phi \in W^{1,p}_0(\Omega).
$$

Equation (1.1) arises as the Euler-Lagrange equation for the functional

$$
I_\Omega(v) = \sum_{i=1}^{2} \int_{\Omega} \frac{|\partial_i v|^p}{p} \, dx.
$$

The equation is singular when either one of the derivatives vanishes, and does not fall into the category of equations with $p$-Laplacian structure. It was proved by Bousquet and Brasco in [11] that weak solutions of (1.1) for $1 < p < \infty$ are $C^1(\Omega)$. A simple proof which gives a logarithmic modulus of continuity for the derivatives is contained in [6] for the case $p \geq 2$. The latter relies on a lemma on the oscillation of monotone functions due to Lebesgue [5] and the fact that derivatives of solutions are monotone (in the sense of Lebesgue). The purpose of this work is to extend this result to the case $1 < p < 2$ employing methods developed in [6]. We obtain the following:

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Theorem 1.1. Let $\Omega \subset \mathbb{R}^2$ and $u \in W^{1,p}(\Omega)$ be a solution of the equation (1.1) for $1 < p < 2$. Fix a ball $B_R \subset \subset \Omega$. Then, for all $j \in \{1, 2\}$ and $B_r \subset \subset B_{R/2}$, we have
\[
\text{osc}_{B_r}(\partial_j u) \leq C_p \left( \log \left( \frac{R}{r} \right) \right)^{1/2} \left( \int_{B_R} |\nabla u|^p \, dx \right)^{1/2},
\]
where $C_p$ is a constant depending only on $p$.

Notation. We indicate balls by $B_r = B_r(a) = \{ x \in \mathbb{R}^2 : |x - a| < r \}$ and we omit the center when not relevant. Whenever two balls $B_r \subset B_{R}$ appear in a statement they are implicitly assumed to be concentric. The variable $x$ denotes the vector $(x_1, x_2)$ and we denote the partial derivatives of a function $f$ with respect to $x_j$ as $\partial_j f$.

2. Regularization

We will consider a regularized problem by introducing a non degeneracy parameter $\epsilon > 0$.

Fix $B_R \subset \subset \Omega \subset \mathbb{R}^2$ and consider the regularized Dirichlet problem
\[
\begin{cases}
\sum_{i=1}^{2} \int_{B_R} (|\partial_i u|^2 + \epsilon) \frac{p-2}{2} \partial_i u \partial_i \phi \, dx = 0 \\
u^\epsilon - u \in W^{1,p}_0(B_R).
\end{cases}
\]
(2.1)

Note that $u^\epsilon$ is the unique minimizer of the regularized functional
\[
I^\epsilon_{B_R}(v) = \sum_{i=1}^{2} \int_{B_R} \frac{1}{p} (|\partial_i u|^2 + \epsilon)^{\frac{p}{2}} \, dx
\]
(2.2)
among $W^{1,p}(B_R)$ functions $v$ such that $v - u \in W^{1,p}_0(B_R)$. By elliptic regularity theory, the unique solution $u^\epsilon$ of (2.1) is smooth in $B_R$.

Fix an index $j \in \{1, 2\}$. Then, replacing $\phi$ by $\partial_j \phi$ in equation (2.1) and integrating by parts, we find that the derivative $\partial_j u^\epsilon$ satisfies the following equation
\[
\sum_{i=1}^{2} \int_{B_R} (e + |\partial_i u^\epsilon|^2)^{\frac{p-4}{2}} (e + (p - 1)|\partial_i u^\epsilon|^2) \partial_i \partial_j u^\epsilon \partial_i \phi \, dx = 0
\]
(2.3)
for all $\phi \in C^\infty_0(B_R)$.

We now collect some uniform estimates and convergences (see also [1]).

Lemma 2.1. Let $u \in W^{1,p}(\Omega)$ be a solution of (1.1) and $u^\epsilon$ be a solution of (2.1) for $1 < p < 2$. Then we have
\[
\int_{B_R} |\nabla u|^p \, dx \leq C_p \left( \int_{B_R} |\nabla u|^p \, dx + \epsilon^{\frac{p}{2}} R^2 \right)
\]
(2.4)
where $C_p$ is a constant depending only on $p$.

Proof. The estimate follows from
\[
I^\epsilon_{B_R}(u^\epsilon) \leq I^\epsilon_{B_R}(u).
\]
**Proposition 2.2.** Let \( u \in W^{1,p}(\Omega) \) be a solution of (1.1) and \( u^e \) be a solution of (2.1) for \( 1 < p < 2 \). Then, for all \( j \in \{1, 2\} \), we have

\[
\sup_{B_{R/2}}(\epsilon + |\nabla u|^2) \leq C_p \left( \int_{B_R} (\epsilon + |\nabla u|^2)^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}} \tag{2.5}
\]

\[
\int_{B_{R/2}} |\nabla \partial_i u^e|^2 \, dx \leq C_p \left( \int_{B_R} (|\nabla u|^p + \epsilon^p) \, dx \right)^{\frac{2}{p}} \tag{2.6}
\]

where \( C_p \) is a constant depending only on \( p \).

**Proof.** The proof of the Lipschitz bound can be found in [4] while (2.6) appears in [1]. We provide details for completeness. Note that by a change of variables, the function \( u^e(x) = e^e(x_0 + Rx) \) satisfies the equation

\[
\sum_{i=1}^2 \int_{B_1} \left( |\partial_i u^e| + R^2 \epsilon \right)^{\frac{p-2}{2}} \partial_i u^e \partial_i \phi \, dx = 0 \quad \text{for all} \quad \phi \in W^{1,p}_0(B_1). \tag{2.7}
\]

Introduce the notation \( w = eR^2 + |\nabla u^e|^2 \) and \( a_i(z_i) = a_i(z_i) = (eR^2 + |z_i|^2)^{\frac{p-2}{2}}z_i \) so that equation (2.7) rewrites as

\[
\sum_{i=1}^2 \int_{B_1} a_i(\partial_i u^e) \partial_i \phi \, dx = 0 \quad \text{for all} \quad \phi \in W^{1,p}_0(B_1).
\]

For \( j \in \{1, 2\} \) and \( \alpha \geq 0 \) take \( \phi = \partial_j(\partial_i u^e \partial_i \xi) \) so that \( \partial_j \phi = \partial_j(\partial_i \partial_j u^e \partial_i \xi + \frac{\alpha}{2} \partial_j w w^{\frac{p-2}{2}} \partial_j u^e \xi + 2 \partial_j(\xi \partial_i \xi \partial_j \xi \partial_j \xi \partial_j u^e) \). Sum in \( j \) to get

\[
A + B = \sum_{i,j=1}^2 \int_{B_1} a_i(\partial_i u^e) \partial_j(\partial_i \partial_j u^e \partial_i \xi + \frac{\alpha}{2} \partial_j w w^{\frac{p-2}{2}} \partial_j u^e \xi) \, dx + 2 \sum_{i,j=1}^2 \int_{B_1} a_i(\partial_i u^e) \partial_j(\xi \partial_i \xi \partial_j \xi \partial_j \xi \partial_j u^e) \, dx = 0.
\]

Note that \( \partial_j w = 2 \sum_{j=1}^2 \partial_i \partial_j u^e \partial_j u^e \) and \( \partial_i a_i(\partial_i u^e) \geq c_p w^{\frac{p-2}{2}} \) since \( 1 < p < 2 \). Integrate by parts in \( A \). We get \( A = A_1 + A_2 \) where

\[
A_1 := \sum_{i,j=1}^2 \int_{B_1} \partial_i a_i(\partial_i u^e)(\partial_i \partial_j u^e)^2 w^{\frac{p}{2}} \xi^2 \, dx \geq c_p \sum_{i,j=1}^2 \int_{B_1} w^{\frac{p-2}{2}} |\nabla \partial_j u^e|^2 \xi^2 \, dx,
\]

\[
A_2 := c\alpha \sum_{i,j=1}^2 \int_{B_1} \partial_i a_i(\partial_i u^e) \partial_i \partial_j u^e \partial_j u^e \partial_i w w^{\frac{p-2}{2}} \xi^2 \, dx = c \sum_{i=1}^2 \int_{B_1} \partial_i a_i(\partial_i u^e)(\partial_i w)^2 w^{\frac{p-2}{2}} \xi^2 \, dx 
\geq c_p \alpha \int_{B_1} w^{\frac{p-2}{2}} |\nabla w|^2 \xi^2 \, dx.
\]
Now we estimate $B = B_1 + B_2 + B_3$.

$$B_1 := \sum_{i,j=1}^{2} \int_{B_1} a_i (\partial_i u^e_R) w^{\frac{\alpha}{2}} |\partial_i \mu_R^e | |\partial_j (\xi \partial_i \xi)| \ dx \leq C_p \int_{B_1} w^{\frac{\alpha}{2}} (|\nabla \xi|^2 + |\nabla^2 \xi|) \ dx,$$

$$B_2 := \frac{\alpha}{2} \sum_{i,j=1}^{2} \int_{B_1} a_i (\partial_i u^e_R) w^{\frac{\alpha}{2}} |\partial_j w| |\partial_j \mu_R^e | |\partial_i \xi| \ dx \leq C \alpha \int_{B_1} w^{\frac{\alpha+2}{2}} |\nabla w| |\nabla \xi| \ dx \leq \eta \alpha \int_{B_1} w^{\frac{\alpha-2}{2}} |\nabla w|^2 \xi^2 \ dx + \frac{C \alpha}{\eta} \int_{B_1} |\nabla \xi|^2 w^{\frac{\alpha}{2}} \ dx,$$

$$B_3 := \sum_{i,j=1}^{2} \int_{B_1} a_i (\partial_i u^e_R) w^{\frac{\alpha}{2}} |\partial_j \partial_i u^e_R| |\partial_j \mu_R^e | |\partial_i \xi| \ dx \leq \sum_{j=1}^{2} \int_{B_1} w^{\frac{\alpha-1}{2}} |\nabla \partial_j \mu_R^e | |\nabla \xi| \ dx \leq \eta \sum_{j=1}^{2} \int_{B_1} w^{\frac{\alpha-2}{2}} |\nabla \partial_j \mu_R^e | |\nabla \xi| \ dx$$

where we used $a_i (\partial_i u^e_R) \leq w^{\frac{\alpha}{2}}$ and Young’s inequality with a parameter $\eta$ to be chosen suitably small. We get

$$c_p \sum_{j=1}^{2} \int_{B_1} w^{\frac{\alpha-2}{2}} |\nabla \partial_j \mu_R^e | |\nabla \xi| \ dx + c_p \alpha \int_{B_1} w^{\frac{\alpha-4}{2}} |\nabla w|^2 \xi^2 \ dx \leq C_p (\alpha+1) \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) w^{\frac{\alpha}{2}} \ dx. \tag{2.8}$$

Note that for $\alpha = 0$ we get for all $j \in \{1, 2\}$

$$\int_{B_1} w^{\frac{\alpha}{2}} |\nabla \partial_j \mu_R^e | |\nabla \xi| \ dx \leq C_p \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) w^{\frac{\alpha}{2}} \ dx, \tag{2.9}$$

and since $|\nabla w|^2 \leq c \sum_j |\nabla \partial_j \mu_R^e |^2 |\nabla \xi|^2$ we have

$$\int_{B_1} w^{\frac{\alpha}{2}} |\nabla w|^2 \xi^2 \ dx \leq c \sum_{j=1}^{2} \int_{B_1} w^{\frac{\alpha}{2}} |\nabla \partial_j \mu_R^e | |\nabla \partial_j \mu_R^e | \xi^2 \ dx \leq c \sum_{j=1}^{2} \int_{B_1} w^{\frac{\alpha}{2}} |\nabla \partial_j \mu_R^e | \xi^2 \ dx$$

$$\leq C_p \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) w^{\frac{\alpha}{2}} \ dx. \tag{2.10}$$

Now for $\alpha \geq 1$, (2.8) implies

$$\int_{B_1} w^{\frac{\alpha-4}{2}} |\nabla w|^2 \xi^2 \ dx \leq C_p \frac{\alpha+1}{\alpha} \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) w^{\frac{\alpha}{2}} \ dx \tag{2.11}$$

and combining with (2.10) we get

$$\int_{B_1} (|\nabla w|^2 \xi) \ dx \leq C (p + \alpha)^2 \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) w^{\frac{\alpha}{2}} \ dx$$

for all $\alpha \geq 0$. Using Sobolev’s embedding $W^{1,2}_0(B_1) \hookrightarrow L^{2q}(B_1)$ for a fixed $q > 1$ we get

$$\left( \int_{B_1} w^{\frac{\alpha}{2}} \xi^{2q} \ dx \right)^{\frac{1}{q}} \leq C_p (p + \alpha)^2 \int_{B_1} (|\nabla \xi|^2 + |\nabla^2 \xi|) w^{\frac{\alpha}{2}} \ dx. \tag{2.12}$$

Now choose a sequence of radii $r_i = 1/2^i + (1 - 1/2^i) \frac{1}{2^i}$, cut-off functions $\xi_i$ between $r_i$ and $r_{i+1}$ and $\alpha_i = q \beta^i - p$ so that $\frac{\alpha+1}{\alpha} = \frac{q}{2} \beta^i$. Using these in (2.12), raising to the power $1/q^i$ and
iterating we get for all $i \in \mathbb{N}$

$$
\left( \int_{B_{2i+1}^R} w^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \leq \left( \sum_{j=0}^{i} (C_{p,q} 2^j)^\frac{1}{q} \int_{B_i} w^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \leq \prod_{j=0}^{i} (C_{p,q} 2^j)^\frac{1}{q} \int_{B_1} w^{\frac{p}{q}} dx.
$$

Observe that $\prod_{i=0}^{\infty} (C_{p,q} 2^i)^\frac{1}{q} = C_{p,q} < \infty$ so passing to the limit as $i \to \infty$ we get

$$
\sup_{B_{1/2}} w^{\frac{p}{q}} \leq C_{p,q} \int_{B_1} w^{\frac{p}{q}} dx
$$

which, after rescaling, proves (2.5). Now going back to (2.9), choosing a cut-off function between $B_{R/2}$ and $B_R$ and using $1 < p < 2$ we get

$$
\int_{B_{R/2}} |\nabla \partial_i u|^2 dx \leq C_p \sup_{B_{R/2}} (\epsilon + |\nabla u|^2)^\frac{2-p}{p} \int_{B_R} (\epsilon + |\nabla u|^2)^\frac{p}{p} dx.
$$

Using (2.5) and (2.4) we obtain (2.6). □

Next we collect some facts about the convergence of $u^\epsilon$ to the solution of the degenerate equation. These are sufficient for our purposes.

**Proposition 2.3.** Let $u^\epsilon$ be the solution of (2.1) for $1 < p < 2$ and $u \in W^{1,p}(\Omega)$ the solution of (1.1). We have

- $u^\epsilon$ converges to $u$ locally uniformly in $B_R$,
- $\nabla u^\epsilon$ converges to $\nabla u$ in $L^p(B_R)$.

**Proof.** From the energy estimate (2.4) we obtain a uniform bound for the $L^p$ norm of $\nabla u^\epsilon$. Therefore (up to a subsequence) $u^\epsilon$ converges to some $v \in W^{1,p}(B_R)$ weakly in $W^{1,p}(B_R)$ and strongly in $L^p(B_R)$. Note that we have $v - u \in W^{1,p}_0(B_R)$. By weakly lower semicontinuity we get

$$
I_{B_R}(v) = \sum_{i=1}^{2} \int_{B_R} \frac{|\partial_i v|^p}{p} dx \leq \liminf_{\epsilon \to 0} \sum_{i=1}^{2} \int_{B_R} \frac{|\partial_i u^\epsilon|^p}{p} dx
$$

$$
\leq \liminf_{\epsilon \to 0} \sum_{i=1}^{2} \int_{B_R} \frac{1}{p} (|\partial_i u|^2 + \epsilon)^\frac{p}{2} dx
$$

$$
\leq \liminf_{\epsilon \to 0} \sum_{i=1}^{2} \int_{B_R} \frac{1}{p} (|\partial_i u|^2 + \epsilon)^\frac{p}{2} dx
$$

$$
= \sum_{i=1}^{2} \int_{B_R} \frac{1}{p} |\partial_i u|^p dx = I_{B_R}(u).
$$

Note that in the third inequality we used the minimality of $u^\epsilon$ subject to the boundary condition $u^\epsilon - u \in W^{1,p}_0(B_R)$. By uniqueness of the minimizer of $I_{B_R}$ among functions with boundary values $u$ in $B_R$, we get $v = u$. By the uniform Lipschitz estimate (2.5) and Ascoli-Arzela’ theorem we obtain that the convergence is uniform.
Now we show $L^p(B_R)$ convergence of the gradient. Use $\phi = u^\epsilon - u$ as a test function in (2.1), add and subtract the term $(|\partial_i u|^2 + \epsilon)\frac{p-2}{2} \partial_i u$ to get

$$\sum_{i=1}^2 \int_{B_R} \left((|\partial_i u|^2 + \epsilon)\frac{p-2}{2} \partial_i u^\epsilon - (|\partial_i u|^2 + \epsilon)\frac{p-2}{2} \partial_i u \right)(\partial_i u^\epsilon - \partial_i u) \, dx$$

$$= \sum_{i=1}^2 \int_{B_R} (|\partial_i u|^2 + \epsilon)\frac{p-2}{2} \partial_i u(\partial_i u^\epsilon - \partial_i u) \, dx.$$ 

Since $\partial_i u - \partial_i u^\epsilon$ converges to 0 weakly in $L^p(B_R)$, the integral in the right hand side converges to 0. We can minorize the integral in the left hand side using the inequality

$$|a - b|^2(e + |a|^2 + |b|^2)^{\frac{p-2}{2}} \leq C_p((e + |a|^2)^{\frac{p-2}{2}} a - (e + |b|^2)^{\frac{p-2}{2}} b)(a - b)$$

valid for $1 < p < 2$, and obtain that

$$\int_{B_R} (e + |\partial_i u|^2 + |\partial_i u^\epsilon|^2)\frac{p-2}{2} |\partial_i u^\epsilon - \partial_i u|^2 \, dx \longrightarrow 0 \quad (2.13)$$

as $\epsilon \to 0$, for $i = 1, 2$. Finally by Hölder’s inequality

$$\int_{B_R} |\partial_i u^\epsilon - \partial_i u|^2 \, dx = \int_{B_R} |\partial_i u^\epsilon - \partial_i u|^p \left(e + |\partial_i u|^2 + |\partial_i u^\epsilon|^2 \right)^{\frac{p-2}{2}}(e + |\partial_i u|^2 + |\partial_i u^\epsilon|^2)^{\frac{p-2}{2}} \, dx$$

$$\leq \left( \int_{B_R} |\partial_i u^\epsilon - \partial_i u|^2 \left(e + |\partial_i u|^2 + |\partial_i u^\epsilon|^2 \right)^{\frac{p-2}{2}} \, dx \right)^{\frac{2}{p}} \cdot \left( \int_{B_R} \left(e + |\partial_i u|^2 + |\partial_i u^\epsilon|^2 \right)^{\frac{p-2}{2}} \, dx \right)^{\frac{p}{2}}.$$ 

Since the last integral is uniformly bounded in $\epsilon$, using (2.13) we get that $\partial_i u^\epsilon$ converges to $\partial_i u$ in $L^p(B_R)$. 

3. Monotone functions and Lebesgue’s lemma

A continuous function $v : \Omega \to \mathbb{R}$ is monotone (in the sense of Lebesgue) if

$$\max_{\partial D} v = \max_D v \quad \text{and} \quad \min_{\partial D} v = \min_D v$$

for all subdomains $D \subset \subset \Omega$. Monotone functions are further discussed in [7].

The next Lemma is due to Lebesgue [5].

**Lemma 3.1.** Let $B_R \subset \mathbb{R}^2$ and $v \in C(B_R) \cap W^{1,2}(B_R)$ be monotone in the sense of Lebesgue. Then

$$(\text{osc}_v)_{B_r}^2 \log \left(\frac{R}{r}\right) \leq \pi \int_{B_R \setminus B_r} |\nabla v(x)|^2 \, dx$$

for every $r < R$.

**Proof.** Assume $v$ is smooth. Let $(\eta, \zeta)$ be the center of $B_R$. Let $x_1$ and $x_2$ be two points on the circle of radius $t$, and let $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $\gamma(s) = (\eta + t \cos(s), \zeta + t \sin(s))$ be a parametrization of the circle such that $\gamma(t) = x_1$ and $\gamma(\pi) = x_2$. Then we have

$$v(x_1) - v(x_2) = \int_a^b \frac{d}{ds} v(\gamma(s)) \, ds = \int_a^b \langle \nabla v(\gamma(s)), \gamma'(s) \rangle \, ds \leq \int_a^b t |\nabla v(\gamma(s))| \, ds.$$ 

Taking the supremum on angles $a$ and $b$ such that $|a - b| \leq \pi$ and using Hölder’s inequality, we get

$$(\text{osc}_v)_{\partial B_r}^2 \leq \pi t^2 \int_0^{2\pi} |\nabla v(\gamma(s))|^2 \, ds.$$
Now diving by $t$, integrating from $r$ to $R$, and using polar coordinates we get

$$
\int_r^R \left( \frac{\text{osc}_{B_r} \partial u}{t} \right)^2 \, dt \leq \pi \int_r^R \int_0^{2\pi} t \left| \nabla v(y(s)) \right|^2 \, ds \, dt = \pi \int_{B_r \setminus B_r} \left| \nabla v(x) \right|^2 \, dx.
$$

Thanks to the monotonicity of $v$, for $t \geq r$ we have

$$
\text{osc}_{B_r} \partial u^e \geq \text{osc}_{B_r} \partial u \geq \text{osc}_{B_r} \partial u^e
$$

and we get the result for a smooth function. The general statement follows by approximation. □

The following is credited to [1] (see Lemma 2.14 for the minimum principle).

**Lemma 3.2.** [Minimum and Maximum principles for the derivatives]
Let $u^e$ be the solution of (2.1). Then

$$
\min_{\partial B_r} \partial_j u^e \leq \partial_j u^e(x) \leq \max_{\partial B_r} \partial_j u^e
$$

for all $x \in B_r$, $B_r \subset B_R$ and $j = 1, 2$. In particular, $\partial_j u^e$ is monotone in the sense of Lebesgue.

**Proof.** We are going to show that given a constant $C$, if $\partial_j u^e \leq C$ (resp. $\partial_j u^e \geq C$) in $\partial B_r$, then $\partial_j u^e \leq C$ (resp. $\partial_j u^e \geq C$) in $B_r$. Let $\phi^\pm = 1_B(\partial_j u^e - C)' = 1_B(\text{max}|\pm(\partial_j u^e - C), 0|$ in the equation satisfied by the derivative (2.3). Since $u^e$ is smooth and $\partial_j u^e \geq C$ (resp. $\partial_j u^e \leq C$) on $\partial B_r$, we have $\phi^\pm \in W_{1,2}^1(\Omega)$, so they are admissible functions. We get

$$
0 = \sum_{i=1}^2 \int_{B_r} (e + |\partial_i u^e|^2)^{\frac{p-2}{2}} (e + (p-1)|\partial_i u^e|^2) |\partial_i(\partial_j u^e - C)^\pm|^2 \, dx
$$

$$
\geq e \sum_{i=1}^2 \int_{B_r} (e + |\nabla u^e|^2)^{\frac{p-2}{2}} |\partial_i(\partial_j u^e - C)^\pm|^2 \, dx
$$

$$
= e \int_{B_r} (e + |\nabla u^e|^2)^{\frac{p-2}{2}} |\nabla(\partial_j u^e - C)^\pm|^2 \, dx.
$$

This implies $(\partial_j u^e - C)^\pm$ is constant in $B_r$, and since it is 0 in $\partial B_r$ then $(\partial_j u^e - C)^\pm = 0$ in $B_r$. □

4. PROOF OF THE MAIN THEOREM

**Proof of Theorem (1.1).** Applying Lemma (3.1) and estimate (2.6) we get for all $r < R/2$

$$
(\text{osc}_{B_r} \partial u^e)^2 \frac{\log\left(\frac{R}{r}\right)}{r} \leq C \left\| \nabla \partial u^e \right\|_{L^p(B_r)}^2 \leq C \left( \int_{B_r} |\nabla u|^p \, dx + \epsilon \right)^{\frac{2}{p}}
$$

(4.1)

and hence for all $r < R/2$

$$
\text{osc}_{B_r} \partial u^e \leq C \left( \log\left(\frac{R}{r}\right) \right)^{\frac{1}{2}} \left( \int_{B_r} |\nabla u|^p \, dx + \epsilon \right)^{\frac{1}{p}}
$$

(4.2)

where $C$ is a constant independent of $\epsilon$.

Thanks to Proposition (2.3) we can pass to the limit and get (1.3). □

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