Correspondence between future-included and future-not-included theories

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Abstract

We briefly review the correspondence principle proposed in ref.[1], which claims that if we regard a matrix element defined in terms of the future state at time $T_B$ and the past state at time $T_A$ as an expectation value in the complex action theory whose path runs over not only past but also future, the expectation value at the present time $t$ of a future-included theory for large $T_B - t$ and large $t - T_A$ corresponds to that of a future-not-included theory with a proper inner product for large $t - T_A$. This correspondence principle suggests that the future-included theory is not excluded phenomenologically.

1 Introduction

Recently complex action theory (CAT) has been studied[2, 3] with the expectation that the imaginary part of the action would give some falsifiable predictions. Indeed many suggestions have been made for Higgs mass[4], quantum mechanical philosophy[5, 6, 7], some fine-tuning problems[8, 9], black holes[10], De Broglie-Bohm particle and a cut-off in loop diagrams[11]. Also, integration contours in the complex plane[12] [13], complex Langevin equations[14] and complexified solution set[15] have been studied. Especially in ref.[2] the authors studied a future-included theory, i.e. the theory including not only a past time but also a future time as an integration interval of time. They introduced a future state $\langle B(T_B) \rangle$ at the final time.
$T_B$ besides the ordinary past state $|A(T_A)\rangle$ at the initial time $T_A$, where $T_B$ and $T_A$ are set to be $\infty$ and $-\infty$ respectively. The states $|A(T_A)\rangle$ and $|B(T_B)\rangle$ time-develop according to the non-hermitian Hamiltonian $\hat{H}$ and $\hat{H}_B$, respectively, where $\hat{H}_B$ is set to be equal to $\hat{H}^\dagger$ [16]. They studied the matrix element $^1$ of some operator $O$, $\langle O \rangle^{BA} \equiv \frac{\langle B(t)|O|A(t)\rangle}{\langle B(t)|A(t)\rangle}$, where $t$ is the present time, and speculated a correspondence of a future-included theory to a future-not-included one, i.e. $\langle O \rangle^{BA} \simeq \langle O \rangle^{AA} \equiv \frac{\langle A(t)|O|A(t)\rangle}{\langle A(t)|A(t)\rangle}$.

In ref.[1] we examined the quantity $\langle O \rangle^{BA}$ carefully and found that if we regard it as an expectation value in a future-included theory, then we obtain the Heisenberg equation, the Ehrenfest’s theorem and a conserved probability current density. This result strongly suggests that we can regard $\langle O \rangle^{BA}$ as the expectation value in the future-included theory, though it is a matrix element in a usual sense. Furthermore improving the argument in ref.[2] on the correspondence of a future-included theory to a future-not-included one by using both the complex coordinate formalism [18] and the automatic hermiticity mechanism [19, 18], i.e., a mechanism for suppressing the anti-hermitian part of the Hamiltonian after a long time development in a system defined with a non-hermitian Hamiltonian $^2$, we have obtained a correspondence principle that $\langle O \rangle^{BA}$ for large $T_B - t$ and large $t - T_A$ is almost equivalent to $\langle O \rangle^{AA}$ for large $t - T_A$, where $Q'$ is a hermitian operator which is used to define a proper inner product $^3$. In this article, for simplicity without using the complex coordinate formalism by considering the real $q$ case, we briefly review the argument to obtain the correspondence principle proposed in ref.[1].

This paper is organized as follows. In section 2 we explain the future-included theory and give the definitions of the states $|A(t)\rangle$ and $|B(t)\rangle$. In section 3 we review the proper inner products for the Hamiltonians $\hat{H}$ and $\hat{H}_B = \hat{H}^\dagger$, and the automatic hermiticity mechanism. In section 4 we show that the expectation value of the future-included theory for large $T_B - t$ and large $t - T_A$ corresponds to that of the future-not-included theory with a proper inner product for large $t - T_A$. Section 5 is devoted to summary and outlook.

$^1$In the RAT the matrix element $\langle O \rangle^{BA}$ is called weak value [17] and has been intensively studied.

$^2$The Hamiltonian is generically non-hermitian, so it does not belong to a class of the PT symmetric non-hermitian Hamiltonians intensively studied recently [20, 21, 22].

$^3$Similar inner products were studied also in refs. [21, 22].
2 Future-included theory

A usual quantum theory is described with a real action and includes time integration from the past time to the present time. On the other hand we may be able to extend such a quantum theory so that it is described with a complex action and includes time integration from the past to the future. This is a future-included complex action theory (CAT), which we study in this article.

The future-included theory is described by introducing not only the ordinary past state $|A(T_A)\rangle$ at the initial time $T_A$ but also a future state $|B(T_B)\rangle$ at the final time $T_B$, where $T_A$ and $T_B$ are set to be $-\infty$ and $\infty$ respectively. In ref.[2] the state $|A(t)\rangle$ and the other state $|B(t)\rangle$ at the present time $t$ are introduced by

$$\langle q|A(t)\rangle = \int_{\text{path}(t)=q} e^{\pm S_{TA} \to t} D\text{path}, \quad (1)$$

$$\langle B(t)|q \rangle \equiv \int_{\text{path}(t)=q} e^{\pm S_{t \to TB}} D\text{path}, \quad (2)$$

where $\text{path}(t) = q$ means the boundary condition at the time $t$. The states $|A(t)\rangle$ and $|B(t)\rangle$ time-develop according to

$$i\hbar \frac{d}{dt}|A(t)\rangle = \hat{H}|A(t)\rangle, \quad (3)$$

$$i\hbar \frac{d}{dt}|B(t)\rangle = \hat{H}_B|B(t)\rangle, \quad (4)$$

where

$$\hat{H}_B = \hat{H}^\dagger. \quad (5)$$

We note that we explicitly derived the forms of $\hat{H}$ and $\hat{H}_B$ - for simplicity in a system with a single degree of freedom - via Feynman path integral in refs.[16][1] respectively. The authors in ref.[2] speculated that the quantity

$$\langle O \rangle^{BA} = \frac{\langle B(t)|O|A(t)\rangle}{\langle B(t)|A(t)\rangle} \quad (6)$$

corresponds to

$$\langle O \rangle^{AA} = \frac{\langle A(t)|O|A(t)\rangle}{\langle A(t)|A(t)\rangle} \quad (7)$$

In ref.[1] we have given the two slightly improved wave functions $\psi_A(q) = m\langle \text{new } q|A(t)\rangle$ and $\psi_B(q) = m\langle \text{new } q|B(t)\rangle$ based on the complex coordinate formalism[18] so that they are properly defined even for complex $q$. But in this article we consider only real $q$ case for simplicity and do not use the complex coordinate formalism.
in some approximation, i.e.
\[ \langle O \rangle^{BA} \simeq \langle O \rangle^{AA}. \] (8)

The right-handed side is just an expectation value of \( O \) in a usual future-not-included theory, while the left-hand side is not an expectation value but a matrix element of \( O \) in a usual sense, and has the same form as the weak value\[17\].

3 Proper inner products and the automatic hermiticity mechanism

We briefly review the proper inner product for the Hamiltonians \( \hat{H} \) and \( \hat{H}_B \), and the automatic hermiticity mechanism\[19, 18\], i.e., a mechanism for suppressing the anti-hermitian part of the Hamiltonian after a long time development in a system defined with a non-hermitian Hamiltonian.

3.1 A proper inner product for \( \hat{H} \) and \( \hat{H}_B \)

We introduce the eigenstates \( |\lambda_i\rangle (i = 1, 2, \ldots) \) of the Hamiltonian \( \hat{H} \) obeying \( \hat{H}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle \), where \( \lambda_i(i = 1, 2, \ldots) \) are the eigenvalues of \( \hat{H} \), and define the diagonalizing operator \( P \) of \( \hat{H} \) by
\[ P = (|\lambda_1\rangle, |\lambda_2\rangle, \ldots). \]

Next we introduce an orthonormal basis \( |e_i\rangle (i = 1, \ldots) \) satisfying
\[ \langle e_i|e_j \rangle = \delta_{ij} \] by \( D|e_i\rangle = \lambda_i|e_i\rangle \). The basis \( |e_i\rangle \) is related to \( |\lambda_i\rangle \) as \( |\lambda_i\rangle = P|e_i\rangle \).

Since \( |\lambda_i\rangle \)'s are not orthogonal to each other in the usual inner product
\[ I(|\lambda_i\rangle, |\lambda_j\rangle) = \langle \lambda_i|\lambda_j \rangle \neq \delta_{ij} \], the theory defined with \( I \) would measure unphysical transitions. To make a physically reasonable measurement, we introduce a proper inner product \( I_Q \)\[19, 18\] for arbitrary kets \( |u\rangle \) and \( |v\rangle \) as
\[ I_Q(|u\rangle, |v\rangle) = \langle u|Qv \rangle = \langle u|Q|v\rangle, \] (9)
where \( Q \) is a hermitian operator chosen as
\[ Q = (P^\dagger)^{-1}P^{-1} \] (10)
so that the eigenstates of \( \hat{H} \) get orthogonal to each other with regard to \( I_Q \), \( I_Q(|\lambda_i\rangle, |\lambda_j\rangle) = \delta_{ij} \). With this \( I_Q \) we can make a physically reasonable observation and have the orthogonality relation \( \sum_i |\lambda_i\rangle\langle\lambda_i|_Q = 1 \). We note that \( I_Q \) is different from the CPT inner product defined in the PT symmetric Hamiltonian formalism\[20\].

We define the \( Q \)-hermitian conjugate of some operator \( A \) by \( A^{\dagger_Q} = Q^{-1}A^{\dagger}Q \). This satisfies \( \langle \psi_2|Q^\dagger A|\psi_1 \rangle^* = \langle \psi_1|Q^\dagger A^{\dagger_Q}|\psi_2 \rangle \). We also define \( ^\dagger_Q \)
for kets and bras as \(|\lambda_i\rangle^Q \equiv \langle \lambda |Q| \) and \((\langle \lambda _i|^Q \rangle^Q \equiv \langle \lambda |).\) When some operator \(A\) satisfies \(A^Q = A,\) we call \(A\) \(Q\)-hermitian.\(^5\) Since

\[
\langle \lambda _1|Q\rangle \langle \lambda _2|Q\rangle \ldots = PQ = (P^\dagger)^{-1}.
\]

satisfies \(\langle P^Q \rangle \hat{P} = D\) and \(\langle P^Q \rangle \hat{P}^Q = D^\dagger,\) \(\hat{P}\) is \(Q\)-normal, \([\hat{P}, \hat{P}^Q]\) = \(P[D, D^\dagger]P^{-1} = 0.\) In other words the inner product \(I^Q\) is defined so that \(\hat{H}\) is normal with regard to it.

Since \(\hat{H}_B\) satisfies

\[
\hat{H}_B|\lambda_j\rangle_B = \lambda_j^*|\lambda_j\rangle_B,
\]

where we have introduced \(|\lambda_j\rangle_B \equiv Q|\lambda_j\rangle,\) the diagonalizing matrix of \(\hat{H}_B\) is given by \(P_B \equiv (|\lambda_1\rangle_B,|\lambda_2\rangle_B,\ldots) = QP = (P^\dagger)^{-1}.\) We introduce a proper inner product \(I_{QB}\) for arbitrary kets \(|u\rangle\) and \(|v\rangle\) as \(I_{QB}(|u\rangle, |v\rangle) = \langle u|Q_B v\rangle = \langle u|Q_B|v\rangle,\) where \(Q_B\) is a hermitian operator chosen as

\[
Q_B = (P_B^\dagger)^{-1}P_B = Q^{-1}
\]

in order that \(|\lambda_j\rangle_B\) get orthogonal to each other with regard to \(I_{QB}.\) We define \(\hat{H}_B^{Q}\) by

\[
\hat{H}_B^{Q} = Q^{-1}\hat{H}_B^Q Q_B,
\]

which obeys \(B\langle \lambda_i|Q_B \hat{H}_B^{Q}\rangle_B = B\langle \lambda_i|Q_B \lambda_i |.\)

For later convenience we decompose \(\hat{H}\) as \(\hat{H} = \hat{H}_{Q} + \hat{H}_{Qa}.\) Here \(\hat{H}_{QB} = \frac{\hat{H} + \hat{H}^Q}{2} = PD_R P^{-1}\) and \(\hat{H}_{QB} = \frac{\hat{H} - \hat{H}^Q}{2} = iPD_I P^{-1}\) are \(Q\)-hermitian and anti-\(Q\)-hermitian parts of \(\hat{H}\) respectively, where we have introduced \(D_R = \frac{D + D^\dagger}{2}\) and \(D_I = \frac{D - D^\dagger}{2}.

### 3.2 The automatic hermiticity mechanism

Following refs.\(^{[19][18]}\) we study a time development of some state \(|\psi(t)\rangle\) obeying the Schrödinger equation \(i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle.\) Since \(|\psi'(t)\rangle \equiv P^{-1}|\psi(t)\rangle = \sum_i a_i(t)|e_i\rangle\) obeys \(i\hbar \frac{d}{dt}|\psi'(t)\rangle = D|\psi'(t)\rangle,\) \(|\psi(t)\rangle = \sum_i a_i(t)|\lambda_i\rangle\) is expressed as

\[
|\psi(t)\rangle = \sum_i a_i(t_0)e^{\frac{i}{\hbar}E_{\lambda_i}(t-t_0)}|\lambda_i\rangle.
\]

Based on the assumption that the anti-hermitian part of \(\hat{H}\) is bounded from above, which is needed to avoid the FPI = \(\int e^{S_D} D\,path\) divergently

\(^5\)Similar inner products were studied also in refs.\(^{[21][22]}\).
meaningless, we can crudely imagine that some of \( \text{Im}\lambda_i \) take the maximal value \( B \). We denote the corresponding subset of \( \{i\} \) as \( A \). Then, if a long time has passed, namely for large \( t - t_0 \), the states with \( \text{Im}\lambda_i |_{i \in A} \) survive and contribute most in the sum. We introduce a diagonalized Hamiltonian \( \tilde{D}_R \) as

\[
\langle e_i | \tilde{D}_R | e_j \rangle = \begin{cases} 
\delta_{ij} \text{Re}\lambda_i & \text{for } i \in A, \\
0 & \text{for } i \notin A,
\end{cases}
\]  

and define \( \tilde{H}_{\text{eff}} \equiv P \tilde{D}_R P^{-1} \), which is \( Q \)-hermitian, and satisfies \( \tilde{H}_{\text{eff}} |\lambda_i\rangle = \text{Re}\lambda_i |\lambda_i\rangle \). Also, we introduce \( |\tilde{\psi}(t)\rangle \equiv \sum_{i \in A} a_i(t) |\lambda_i\rangle \). Then \( |\psi(t)\rangle \) is approximately estimated as

\[
|\psi(t)\rangle \simeq e^{\frac{i}{\hbar}B(t-t_0)} \sum_{i \in A} a_i(t_0) e^{-\frac{i}{\hbar}\text{Re}\lambda_i(t-t_0)} |\lambda_i\rangle \\
= e^{\frac{i}{\hbar}B(t-t_0)} e^{-\frac{i}{\hbar}\hat{H}_{\text{eff}}(t-t_0)} |\tilde{\psi}(t_0)\rangle = |\tilde{\psi}(t)\rangle.
\]  

Thus we have effectively obtained a \( Q \)-hermitian Hamiltonian \( \hat{H}_{\text{eff}} \) after a long time development. Indeed the normalized state

\[
|\psi(t)\rangle_N \equiv \frac{1}{\sqrt{\langle \tilde{\psi}(t)|Q|\psi(t)\rangle}} |\tilde{\psi}(t)\rangle \simeq \frac{1}{\sqrt{\langle \tilde{\psi}(t)|Q|\tilde{\psi}(t)\rangle}} |\tilde{\psi}(t)\rangle \equiv |\tilde{\psi}(t)\rangle_N
\]  

obeys the Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} |\tilde{\psi}(t)\rangle_N = \hat{H}_{\text{eff}} |\tilde{\psi}(t)\rangle_N.
\]

As we have seen above, the non-hermitian Hamiltonian \( \hat{H} \) has become a hermitian one \( \hat{H}_{\text{eff}} \) automatically with the proper inner product \( I_Q \) and a long time development.

4 Our analysis of \( \langle O \rangle^{BA} \)

We write eq. (6) as

\[
\langle O \rangle^{BA} = \frac{\langle A(t)|B(t)\rangle \langle B(t)|O|A(t)\rangle}{\langle A(t)|B(t)\rangle \langle B(t)|A(t)\rangle},
\]  

(20)
and analyze it carefully. Using the expanded expression \( |B(T_B)\rangle = \sum_i b_i |\lambda_i\rangle_B \) we obtain
\[
|B(t)\rangle\langle B(t)| = e^{-i\hat{H}_B(t-T_B)}|B(T_B)\rangle\langle B(T_B)|Q_B e^{i\hat{H}_B^T B (t-T_B)} Q_B^{-1}
\[
= \sum_{i,j} b_i b_j^* e^{i \text{Re}(\lambda_j - \lambda_i)(t-T_B)} e^{i \text{Im}(\lambda_j + \lambda_i)(T_B-t)} |\lambda_i\rangle_B B \langle \lambda_j|
\[
\simeq \frac{1}{2\Delta t} \int_{t-\Delta t}^{t+\Delta t} |B(t)\rangle\langle B(t)| dt
\[
\simeq \sum_i |b_i|^2 e^{2i \text{Im}(\lambda_i)(T_B-t)} |\lambda_i\rangle_B B \langle \lambda_i|
\[
\simeq e^{2i B(T_B-t)} Q_2 \text{ for large } T_B-t, \tag{21}
\]
where in the third line we have smeared the present time \( t \) a little bit, and then since the off-diagonal elements wash to 0, we are lead to the fourth line. In the last line we have used the automatic hermiticity mechanism for large \( T_B-t \), and \( Q_2 \) is given by
\[
Q_2 = \sum_{i \in A} |b_i|^2 |\lambda_i\rangle_B B \langle \lambda_i|
\[
= \sum_{i \in A} F(\hat{H}_B^\dagger)|\lambda_i\rangle_B B \langle \lambda_i|
\[
= F(\hat{H}_B^\dagger) Q \text{ for the restricted subspace,} \tag{22}
\]
where in the second equality assuming that \( \text{Re}\lambda_i \)'s are not degenerate, we have interpreted \( |b_i|^2 \) as a function of \( \text{Re}\lambda_i \), \( |b_i|^2 = F(\text{Re}\lambda_i) \). Also, \( \hat{H}_B^B \equiv P_B \tilde{D}_R P_B^{-1} = \hat{H}_B^\dagger \) is \( Q_B \)-hermitian, and obeys \( \hat{H}_B^B |\lambda_i\rangle_B = \text{Re}\lambda_i |\lambda_i\rangle_B \). In the last equality we have utilized the relation \( \sum_{i \in A} |\lambda_i\rangle_B \langle \lambda_i| = 1 \) for the subspace restricted by the subgroup \( A \). For large \( t - T_A \), since we have \( |A(t)\rangle = \sum_i a_i(t) |\lambda_i\rangle \simeq \sum_i a_i(t)|\lambda_i\rangle = |\hat{A}(t)\rangle \) by the automatic hermiticity mechanism eq.\,(20) is expressed as
\[
\langle O \rangle^{BA} \simeq \frac{\langle \hat{A}(t)|Q_2 O|\hat{A}(t)\rangle}{\langle \hat{A}(t)|Q_2 \hat{A}(t)\rangle} = \langle O \rangle^{\hat{A}\hat{A}}_Q \text{ for large } t - T_A. \tag{23}
\]

Next we point out that the operator \( P' = Pf(D) \), where \( f(D) \) is some function of \( D \), is another diagonalizing matrix of \( \hat{H} \), because \( P'DP'^{-1} = \hat{H} \). So we can define another inner product with \( Q' = (P'^\dagger)^{-1} P'^{-1}. \) Choosing the function \( f \) such that \( (P'^\dagger)^{-1} f(DD'^\dagger)^{-1} P'^\dagger = \hat{H}^\dagger \) and using the automatic hermiticity mechanism for large \( t - T_A \), we obtain \( Q' = F(\hat{H}^\dagger) Q \simeq F(\hat{H}_B^\dagger) Q = Q_2 \) for the restricted subspace. Then the expectation value with the proper inner product \( I_{Q'} \) in a future-not-included
theory, which is introduced in refs. [19, 18], is expressed as

\[ \langle O \rangle_{Q} = \frac{\langle A(t)|Q|A(t)\rangle}{\langle A(t)|A(t)\rangle} \]

\[ \simeq \frac{\langle \tilde{A}(t)|Q|\tilde{A}(t)\rangle}{\langle A(t)|Q_{\tilde{A}}A(t)\rangle} = \langle O \rangle_{Q_{\tilde{A}}} \text{ for large } t - T_{A}. \]  

(24)

Comparing eq. (23) with eq. (24), we obtain the following correspondence:

\[ \langle O \rangle_{Q} \text{ for large } T_{B} - t \text{ and large } t - T_{A} \simeq \langle O \rangle_{Q_{\tilde{A}}} \text{ for large } t - T_{A}. \]  

(25)

This relation means that the future-included theory for large \( T_{B} - t \) and large \( t - T_{A} \) is almost equivalent to the future-not-included theory with a proper inner product for large \( t - T_{A} \), and thus suggests that the future-included theory is not excluded though it seems exotic.

5 Summary and outlook

In ref. [2] a correspondence of a future-included complex action theory (CAT) to a future-not included one was speculated, \( \langle O \rangle^{BA} \simeq \langle O \rangle^{AA} \), where \( \langle O \rangle^{BA} \) and \( \langle O \rangle^{AA} \) are given in eqs. (6) (7) respectively. In ref. [1] we studied \( \langle O \rangle^{BA} \) with more care by using the complex coordinate formalism [18] and the automatic hermiticity mechanism [19], i.e., a mechanism for suppressing the anti-hermitian part of the Hamiltonian after a long time development in a system defined with a non-hermitian Hamiltonian, and obtained our correspondence principle that \( \langle O \rangle^{BA} \) for large \( t - T_{A} \) and large \( T_{B} - t \simeq \langle O \rangle_{Q_{\tilde{A}}}^{AA} \) for large \( t - T_{A} \), where \( T_{A}, T_{B} \) and \( t \) are the past initial time, the future final time and the present time, respectively. \( \langle O \rangle_{Q_{\tilde{A}}}^{AA} \) is given in eq. (21) and the \( Q' \) is a hermitian operator used to define the proper inner product.

In this article we briefly reviewed the argument to obtain the correspondence principle following ref. [1] without using the complex coordinate formalism [18] by considering the real \( q \) case for simplicity. We first defined the two states \( \langle B(t) | \) and \( | A(t) \rangle \) from their respective functional integrals over future and past following ref. [2] in section 2. In section 3 we reviewed the proper inner product and the automatic hermiticity mechanism [19]. In section 4 we derived the correspondence principle following ref. [1]. Thus the future-included theory for large \( T_{B} - t \) and large \( t - T_{A} \) is almost equivalent to the future-not-included theory with the proper inner product for large \( t - T_{A} \), so such a future-included theory is not excluded phenomenologically.

In the correspondence principle the hermitian operator \( Q' \) is a priori non-local, but it should be local phenomenologically. So we hope to invent some mechanism for getting it effectively local. Also, the other analyses in
ref. [1] suggest that the future-included theory looks more elegant in functional integral formulation than the future-not-included theory. We will study the future-included theory in more detail and hope to report some progress in the future.

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