Complexity and Approximation of the Continuous Network Design Problem

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Abstract

We revisit a classical problem in transportation, known as the continuous (bilevel) network design problem, CNDP for short. We are given a graph for which the latency of each edge depends on the ratio of the edge flow and the capacity installed. The goal is to find an optimal investment in edge capacities so as to minimize the sum of the routing cost of the induced Wardrop equilibrium and the investment cost for installing the capacity. While this problem is considered as challenging in the literature, its complexity status was still unknown. We close this gap showing that CNDP is strongly NP-complete and APX-hard, both on directed and undirected networks and even for instances with affine latencies.

As for the approximation of the problem, we first provide a detailed analysis for a heuristic studied by Marcotte for the special case of monomial latency functions (Mathematical Programming, Vol. 34, 1986). Specifically, we derive a closed form expression of its approximation guarantee for arbitrary sets $S$ of allowed latency functions. Second, we propose a different approximation algorithm and show that it has the same approximation guarantee. As our arguably most interesting result regarding approximation, we show that using the better of the two approximation algorithms results in a strictly improved approximation guarantee for which we give a closed form expression. For affine latencies, e.g., this algorithm achieves a $49/41 \approx 1.195$-approximation which improves on the $5/4$ that has been shown before by Marcotte. We finally discuss the case of hard budget constraints on the capacity investment.

Keywords: Bilevel optimization, Optimization under equilibrium constraints, Network design, Wardrop equilibrium, Computational complexity, Approximation algorithms
1 Introduction

Starting with the seminal works of Pigou [24] and Wardrop [32], the impact of selfish behavior in congested transportation networks has been investigated intensively over the past decades. In Wardrop’s basic model of traffic flows, the interaction between the selfish network users is modeled as a non-cooperative game. This game takes place in a directed graph with latency functions on the edges and a set of origin-destination pairs, called commodities. Every commodity has a demand associated with it, which specifies the amount of flow that needs to be sent from the respective origin to the respective destination. It is assumed that every demand represents a large population of players, each controlling an infinitesimal small amount of flow, thus, having a negligible impact on the latencies of others. The latency that a player experiences when traversing an edge is determined by a non-decreasing latency function of the edge flow on that edge. In practice, latency functions are calibrated to reflect edge specific parameters such as street length and capacity. One of the most prominent and popular functions used in actual traffic models are the ones put forward by the Bureau of Public Roads (BPR) [30]. BPR-type latency functions are of the form $	ext{Labour}(v) = t_c \cdot (1 + b_c \cdot (v_c/z_c)^4)$, where $v_c$ is the edge flow, $t_c$ represents the free-flow travel time, $b_c > 0$ is an edge-specific bias, and $z_c$ represents the street capacity. In a Wardrop equilibrium (also called Wardop flow), every player chooses a minimum-latency path from its origin to the destination; under mild assumptions on the latency functions this corresponds to a Nash equilibrium for an associated non-cooperative game [3].

It is well known that Wardrop equilibria can be inefficient in the sense that they do not minimize the total travel time in the network [13]. Prominent examples of this inefficiency include the famous Braess Paradox [7], where improving the network infrastructure by adding street capacity may result in a Wardrop equilibrium with strictly higher total travel time. This at the first sight surprising non-monotonic behavior of selfish flows illustrates that designing networks for good traffic equilibria is an important and non-trivial issue.

In this paper, we revisit one of the most classical network design problems, termed the continuous (bilevel) network design problem, CNDP for short, which has been introduced by Dafermos [10], Dantzig et al. [12], and Abdulaal et al. [1], and was later studied by Marcotte [22]. In this problem, we are given a graph for which the latency of each edge depends on the ratio of the edge flow and the capacity installed and the goal is to find an optimal investment in edge capacities so as to minimize the sum of the routing cost of the induced Wardrop equilibrium and the investment cost. From a mathematical perspective, CNDP is a bilevel optimization problem (cf. [8, 20] for an overview), where in the upper level the edge capacities are determined and, given these capacities, in the lower level the flow will settle into a Wardrop equilibrium. Clearly, the lower level reaction depends on the first level decision because altering the capacity investment on a subset of edges may result in revised route choices by users.

CNDP has been intensively studied since the late sixties (cf. [10, 21]) and several heuristic approaches have been proposed since then; see Yang et al. [33] for a comprehensive survey. Most of the proposed heuristics are numerical in nature and involve iterative computations of relaxations of the problem (for instance the iterative optimization and assignment algorithm as described in [23] and augmented Lagrangian methods or linearizations of the objective in the leader and follower problem). An exception is the work of Marcotte [22] who considered several algorithms based on solutions of associated convex optimization problems which can be solved in polynomial time [15]. He derives worst-case bounds for his heuristics and, in particular, for affine latency functions he devises an approximation algorithm with an approximation factor of $5/4$. For general monomial latency functions plus a constant (including the latency functions used by the Bureau of Public Roads [30]) he obtains a polynomial time 2-approximation.

Variants of CNDP have also been considered in the networking literature, see [16, 17, 18, 6]. These works, however, consider the case where a budget capacity must be distributed among a set of edges to improve the resulting equilibrium. Most results, however, only work for simplified network topologies (e.g., parallel links) or special latency functions (e.g., $M/M/1$ latency functions).
Our Results and Used Techniques. Despite more than forty years of research, to the best of our knowledge, the computational complexity status of CNDP is still unknown. We close this gap as we show that CNDP is strongly NP-complete and APX-hard, both on directed and undirected networks and even for instances with affine latencies of the form \( S_e(t_e/z_e) = \alpha_e + \beta_e \cdot (t_e/z_e), \alpha_e, \beta_e \geq 0 \). For the proof of the NP-hardness, we reduce from 3-SAT. The reduction has the property that in case that the underlying instance of 3-SAT has a solution the cost of an optimal solution is equal to the minimal cost of a relaxation of the problem, in which the equilibrium conditions are relaxed. The key challenge of the hardness proof is to obtain a lower bound on the optimal solution when the underlying 3-SAT instance has no solution.

Our main idea is to relax the equilibrium conditions only partially which enables us to bound the cost of an optimal solution from below by solving an associated constrained quadratic optimization problem. With a more involved construction and a more detailed analysis, we can even prove APX-hardness of the problem. Here, we reduce from a symmetric variant of MAX-3-SAT, in which all literals occur exactly twice. While all our hardness proofs rely on instances with an arbitrary number of commodities and respective sinks, we show that for instances in which all commodities share a common sink, CNDP can be solved to optimality in polynomial time.

In light of the hardness of CNDP, we focus on approximation algorithms. We first consider a polynomial time algorithm proposed by Marcotte [22]. This algorithm, which we call BRINGToEQUILIBRIUM, first computes a relaxation of CNDP by removing the equilibrium conditions. Then, it reduces the edge capacities individually such that the flow computed in the relaxation becomes a Wardrop equilibrium. We give a closed form expression of the performance of this algorithm with respect to the set \( S \) of allowed latency functions. Specifically, we show that this algorithm is a \((1 + \mu(S))\)-approximation, where \( \mu(S) = \sup_{S \in S} \sup_{\gamma \in [0,1]} \max_{\gamma \in [0,1]} \gamma \cdot (1 - S(\gamma x))/S(x) \). The value \( \mu(S) \) has been used before by Correa et al. [9] and Roughgarden [29] in the context of price of anarchy bounds for selfish routing where they showed that the routing cost of a Wardrop equilibrium is no more than a factor of \( 1/(1 - \mu(S)) \) away of the cost of a system optimum. For the special case that \( S \) is the set of polynomials with non-negative coefficients and maximal degree \( \Delta \), we derive exactly the approximation guarantees that Marcotte obtained for monomials. As an outcome of our more general analysis, we further derive that this algorithm is a 2-approximation for general convex latency functions and a \( 5/4 \)-approximation for concave latency functions.

We then propose a new algorithm which we call SCALEUNIFORMLY. This algorithm first computes an optimal solution of the relaxation (as before) and then uniformly scales the capacities with a certain parameter \( \lambda(S) \) that depends on the class of allowable latency functions \( S \). Based on well-known techniques using variational inequalities (Correa et al. [9] and Roughgarden [29]), we prove that this algorithm also yields a \((1 + \mu(S))\)-approximation. As our main result regarding approximation algorithms, we show that using the better of the two solutions returned by BRINGToEQUILIBRIUM and SCALEUNIFORMLY yields strictly better approximation guarantees. We give a closed form expression for the new approximation guarantee (as a function of \( S \)) that, perhaps interestingly, depends not only on the well-known value \( \mu(S) \) but also on the argument maximum \( \gamma(S) \) in the definition of \( \mu(S) \). We demonstrate the applicability of this general bound by showing that it achieves a \( 9/5 \)-approximation for \( S \) containing arbitrary convex latencies. For affine latencies it achieves a \( 49/41 \approx 1.195 \)-approximation improving on the \( 5/4 \) of Marcotte. An overview of our results compared to those of Marcotte can be found in Table I in the appendix.

In the final section we consider the case of arbitrary convex constraints on the capacity variables that includes global as well as individual budget constraints on edges. We show that solving the relaxed problem with removed equilibrium constraints achieves a trivial approximation ratio of \( 1/(1 - \mu(S)) \) using the well-known price of anarchy results. For affine latencies, however, we show that this is essentially best possible by giving a corresponding hardness result. All proof missing in this extended abstract can be found in the appendix.
Further Application. Our results have impact beyond the classical application of designing street capacities of road networks. In the telecommunication networking literature, Wardrop equilibria appear in networks with source-routing, where it is assumed that end-users choose least-delay paths knowing the state of all available paths. As outlined in [31], Wardrop equilibria arise even in networks with distributed delay-based routing protocols such as OSPF using delay for setting the routing weights. In telecommunication networks, the latency at switches and routers depends on the installed capacity and has been modeled by BPR-type functions of the form \( S_e(v_e/z_e) = \rho \left( 1 + 0.15 \left( v_e/z_e \right)^4 \right) \), where \( \rho \) represents the propagation delay and \( z_e \) the installed capacity [25]. These functions fit into our framework, and our analysis improves the state-of-the-art to a 1.418-approximation. Additionally, our 9/5-approximation applies to Davidson latency functions of the form \( S_e(z_e) = \frac{v_e}{z_e} / (1 - \frac{v_e}{z_e}) = v_e / (z_e - v_e) \), where \( z_e \) represents the capacity of edge \( e \). These functions behave quite similar to the frequently used \( M/M/1 \)-delay functions of the form \( S_e(v_e) = 1 / (z_e - v_e) \), cf. [16][27].

Further Related Work. Quoting [33], CDNP has been recognized to be “one of the most difficult and challenging problems in transport” and there are numerous works approaching this problem. In light of the substantial literature on heuristics for CNDP, we refer the reader to the survey papers [8, 14, 21, 33].

While to the best of our knowledge prior to this work, the complexity status of CNDP was open, there have been several papers on the complexity of the discrete (bilevel) network design problem, DNDP for short, see [19][26]. Given a network with edge latency functions and traffic demands, a basic variant of DNDP is to decide which edges should be removed from the network to obtain a Wardrop equilibrium in the resulting sub-network with minimum total travel time. This variant is motivated by the classical Braess paradox, where removing an edge from the network may improve the travel time of the new Wardrop equilibrium. Roughgarden [26] showed that DNDP is strongly NP-hard and that there is no \( \left( \lfloor n/2 \rfloor - \varepsilon \right) \)-approximation algorithm (unless \( \text{P} = \text{NP} \)), even for single-commodity instances. He further showed that for single-commodity instances the trivial algorithm of not removing any edge from the graph is essentially best possible and achieves a \( \lfloor n/2 \rfloor \)-approximation. For affine latency functions, the trivial algorithm gives a \( 4/3 \)-approximation (even for general networks) and this is also shown to be best possible. These results in comparison to ours highlight interesting differences. While DNDP is not approximable by any constant for convex latencies, for CDNP we give a 9/5-approximation. Moreover, all hardness results for DNDP already hold for single-commodity instances, while for CDNP we show that this case is solvable in polynomial time.

Bhaskar et al. [6] studied a variant of CNDP where initial edge capacities are given and additional budget must be distributed among the edges to improve the resulting equilibrium. Among other results they show that the problem is NP-complete in single-commodity networks that consist of parallel links in series. This again stands in contrast to our polynomial-time algorithm for CDNP for these instances.

2 Preliminaries

Let \( G = (V, E) \) be a directed or undirected graph, \( V \) its set of vertices and \( E \subseteq V \times V \) its set of edges. We are given a set \( K \) of commodities, where each commodity \( k \) is associated with a triple \( (s_k, t_k, d_k) \in V \times V \times \mathbb{R}_{>0} \), where \( s_k \in V \) is the source, \( t_k \in V \) the sink and \( d_k \) the demand of commodity \( k \). A multi-commodity flow on \( G \) is a collection of non-negative flow vectors \( (v^k)_{k \in K} \) such that for each \( k \in K \) the flow vector \( v^k = (v^k_e)_{e \in E} \) satisfies the flow conservation constraints \( \sum_{u \in V \setminus \{s_k, t_k\}} v^k_{(u,w)} = 0 \) for all \( u \in V \setminus \{s_k, t_k\} \) and \( \sum_{u \in V \setminus \{s_k, t_k\}} v^k_{(w,u)} = d_k \). Whenever we write \( v \) without a superscript \( k \) for the commodity, we implicitly sum over all commodities, i.e., \( v_e = \sum_{k \in K} v^k_e \) and \( v = (v_e)_{e \in E} \). We call \( v_e \) an edge flow. The set of all feasible edge flows will be denoted by \( \mathcal{F} \).

The latency of each edge \( e \) depends on the installed capacity \( z_e \geq 0 \) and the edge flow \( v_e \) on \( e \), and is given by a latency function \( S_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\} \) that maps \( v_e/z_e \) to a latency value \( S_e(v_e/z_e) \), where
we use the convention that \( S_e(v_e/z_e) = \infty \) whenever \( z_e = 0 \). Throughout this paper, we assume that the set of allowable latency functions is restricted to some set \( S \) and we impose the following assumptions on \( S \).

**Assumption 2.1.** The set \( S \) of allowable latency functions only contains continuously differentiable and semi-convex functions \( S \) such that the functions \( x \mapsto S(x) \) and \( x \mapsto x^2 S'(x) \) are strictly increasing and unbounded.

Assumption 2.1 is slightly more general than requiring that all latency functions are strictly increasing and convex. For instance, the function \( S(x) := \sqrt{x} \) satisfies Assumption 2.1 although it is concave.

Given a vector of capacities \( z = (z_e)_{e \in E} \), the latency of each edge \( e \) solely depends on the edge flow \( v_e \). Under these conditions, there exists a Wardrop flow \( v = (v_e)_{e \in E} \), i.e., a flow in which each commodity only uses paths of minimal latency. It is well known (see e.g. [3, 11, 28]) that each Wardrop flow is a solution to the optimization problem \( \min_{v \in F} \sum_{e \in E} \int_0^{v_e} S_e(t/z_e) \, dt \), and satisfies the variational inequality

\[
\sum_{e \in E} S(v_e/z_e)(v_e - v'_e) \leq 0
\]

for every feasible flow \( v' \in F \). For a vector of capacities \( z \) we denote by \( W(z) \) the corresponding set of Wardrop flows \( v(z) \). Beckmann et al. [3] showed that Wardrop flows and optimum flows are related:

**Proposition 2.2** (Beckmann et al. [3]). Denote by \( S^*_e(x) = (xS_e(x))' = S_e(x) + xS'_e(x) \) the marginal cost function of edge \( e \in E \). Then \( v^* \) is an optimum flow with respect to the latency functions \( (S_e)_{e \in E} \) if and only if it is Wardrop flow with respect to \( (S^*_e)_{e \in E} \).

In the continuous (bilevel) network design problem (CNDP) the goal is to buy capacities \( z_e \) at a price per unit \( \ell_e > 0 \) so as to minimize the sum of the construction cost \( C^Z(v, z) = \sum_{e \in E} z_e \ell_e \) and the routing cost \( C^R(v, z) = \sum_{e \in E} S_e(v_e/z_e) v_e \) of a resulting Wardrop equilibrium \( v \). Observe that \( C^R(v, z) \) is well defined as, by (2.1), it is the same for all Wardrop equilibria with respect to \( z \). Denote the combined cost by \( C(v, z) = C^R(v, z) + C^Z(v, z) \).

**Definition 2.3** (Continuous network design problem (CNDP)). Given a directed graph \( G = (V, E) \) and for each edge \( e \) a latency function \( S_e \) and a construction cost \( \ell_e > 0 \), the continuous network design problem (CNDP) is to determine a non-negative capacity vector \( z = (z_e)_{e \in E} \) that minimizes

\[
\min_{z \geq 0} \min_{v \in W(z)} \sum_{e \in E} (S_e(v_e/z_e) v_e + z_e \ell_e).
\]

(CNDP)

Relaxing the condition that \( v \) is a Wardrop equilibrium in (CNDP), we obtain the following relaxation of the continuous network design problem:

\[
\min_{z \geq 0} \min_{v \in F} \sum_{e \in E} (S_e(v_e/z_e) v_e + z_e \ell_e).
\]

(CNDP')

Marcotte [22] showed that for convex and unbounded latency functions, the relaxed problem (CNDP') can be solved efficiently by performing \(|K|\) independent shortest path computations on the graph \( G \), one for each commodity \( k \in K \). The following proposition slightly generalizes his result to arbitrary, not necessarily convex latency functions that satisfy Assumption 2.1.

**Proposition 2.4** (Marcotte [22]). The relaxation (CNDP') can be solved by performing \(|K|\) shortest path computation problems in polynomial time.

**Remark 2.5.** To speak about polynomial algorithms and hardness, we need to specify how the instances of CNDP, in particular the latency functions, are encoded, cf. [2, 17, 26]. While our hardness results hold even if all functions are linear and given by their rational coefficients, for our approximation algorithms, we require that we can solve (symbolically) equations involving a latency function and its derivative, e.g., Equation (4.4). Without this assumption, we still obtain the claimed approximation guarantees within arbitrary precision by polynomial time algorithms.
3 Hardness

As the main result of this section, we show that CNDP is APX-hard both on directed and undirected networks and even for affine latency functions. The proof of this result is technically quite involved, and we first show the weaker result that CNDP on directed networks is NP-complete. Due to space constraints, we here only sketch the proof of the NP-completeness for directed networks and the case that there are edges with zero latency. For the full proof and the discussion that the problem remains hard, even if no edges with zero latency are allowed, we refer to the appendix.

Theorem 3.1. The continuous network design problem on directed networks is NP-complete in the strong sense, even if all latency functions are affine.

Sketch of proof. We reduce from 3-SAT. Let a Boolean formula \( \phi \) in conjunctive normal form be given and for \( \kappa, \nu \in \mathbb{N} \), let \( K(\phi) = \{1, 2, \ldots, \kappa\} \) and \( V(\phi) = \{x_1, x_2, \ldots, x_\nu\} \) denote the set of its clauses and variables, respectively. For each variable \( x_i \in V(\phi) \), we introduce a variable commodity \( j_{x_i} \) with unit demand, and for each clause \( k \in K(\phi) \) we introduce a clause commodity \( j_k \) with unit demand. For each literal \( l \) and each clause \( k \), there is a literal edge \( e_{l,k} \) with latency function \( S_{e_{l,k}}(v_{e_{l,k}}) = v_{e_{l,k}} \) and construction cost \( \ell_{e_{l,k}} = 1 \). Further, for some \( \epsilon > 0 \) and for each clause \( k \), there is a clause edge \( e_k \) with \( S_{e_k}(v_{e_k}) = 4 + v_{e_k} \) and construction cost \( \ell_{e_k} = (\epsilon/2)^2 \). Every variable commodity \( j_{x_i} \) has two feasible paths, one consist of the literal edges \( \{e_{x_i,k} : k \in K(\phi)\} \) corresponding to the positive literal \( x_i \), the other one consists of the literal edges \( \{e_{x_i,k} : k \in K(\phi)\} \) corresponding to the negative literal \( \bar{x}_i \). In that way, each route choice of the variable commodities corresponds to a fractional assignment of the variables. For each clause \( k = l_{k} \lor l_k' \lor l_k'' \), the clause commodity \( j_k \) has two feasible paths as well, one consists of the clause edge \( e_k \), the other one contains the three literal edges \( e_{l_{k},k}, e_{l_k',k}, \) and \( e_{l_k'',k} \). We add some additional edges with zero latency to this path in order to obtain a network structure, see Figure 1 in the appendix.

Let us first assume that \( \phi \) has a solution \( y = (y_{x_i})_{x_i \in V(\phi)} \) and let \( \bar{y} = (\bar{y}_{x_i})_{x_i \in V(\phi)} \) be the negation of \( y \). Then, an optimal solution to the so-defined instance of CNDP is follows. For each variable commodity \( j_{x_i} \), we buy capacity 1 on the path consisting of the edges \( \{e_{y_{x_i},k} : k \in K(\phi)\} \) and we route the unit demand of variable commodity \( j_{x_i} \) over the edges of that path. For each clause commodity \( j_k \), we route the unit demand over the clause edge \( e_k \). Using that \( y \) is a solution of \( \phi \), we derive that for each clause, there is a literal \( l^* \) that occurs in \( y \), and thus, \( l^* \) does not occur in \( \bar{y} \). However, this implies that for each clause \( k = l_{k} \lor l_k' \lor l_k'' \), at least one of the three literal edges \( e_{l_{k},k}, e_{l_k',k}, \) and \( e_{l_k'',k} \) has capacity 0 and, thus, infinite latency. Thus, the clause commodity \( j_k \) has only one path with finite length and we conclude that the so-defined flow is a Wardrop equilibrium. This solution has total cost \( 2\kappa \nu + (4 + \epsilon)\kappa \) which can be shown to be minimal as it coincides with the total cost of the relaxation of the problem without the equilibrium constraints.

If \( \phi \) does not admit a solution, we show that each feasible solution has cost strictly larger than \( 2\kappa \nu + (4 + \epsilon)\kappa \). Assume by contradiction that there is a solution \( (v, z) \) with cost at most \( 2\kappa \nu + (4 + \epsilon)\kappa \). We claim that in \( v \), each clause commodity \( e_k \) uses its clause edge, i.e., \( v_{e_k} \geq 0 \). To see this, note that each unit of flow of the clause commodities that is routed over the three corresponding literal edges contributes at least 6 to the total cost of a solution while each unit of flow that is routed over a clause edge contributes at most \( (4 + \epsilon) \) to the total cost. This implies that the total cost is at least \( 2\kappa \nu + (4 + \epsilon)\kappa + (2 - \epsilon) \) if one of the clause commodities does not use its clause edge. However, since \( \phi \) does not admit a solution, we cannot prevent a clause commodity from using three of the corresponding literal edges without reducing the capacity on at least one of these edges below 1. Reducing the capacity on the literal edges below 1, however, comes at a cost, since the resulting capacities are then strictly smaller than in the relaxation of the problem. By solving an associated constrained quadratic program, we show that the total cost of any feasible solution is at least \( 2\kappa \nu + (4 + \epsilon)\kappa + 1/8 \), if \( \phi \) does not admit a solution.

With a more involved construction and a more detailed analysis, we can show that CNDP is in fact
APX-hard. For this proof, we use a similar construction as in the proof of Theorem 3.1. However, instead from 3-SAT, we reduce from a specific variant of MAX-3-SAT, which is NP-hard to approximate.

**Theorem 3.2.** The continuous network design problem on directed networks is APX-hard, even if all latency function are affine.

With a similar construction, we can also show APX-hardness for CNDP on undirected networks as well, see Theorem A.2 in the appendix. For our hardness results, we use instances with different sinks. In contrast, CNDP can be solved efficiently for networks with a single sink.

**Proposition 3.3.** In networks with only one sink vertex \( t \), the continuous network design problem can be solved in polynomial time.

**4 Approximation**

Given the APX-hardness of the problem, we study the approximation of CNDP. We first provide a detailed analysis of the approximation guarantees of two different approximation algorithms. Then, as the arguably most interesting result of this section, we provide an improved approximation guarantee for taking the better of the two algorithms. The approximation guarantees proven in this section depend on the set \( S \) of allowable cost functions and are in fact closely related to the anarchy value \( \alpha(S) \) introduced by Roughgarden [29] and Correa et al. [9]. Intuitively, the anarchy value of a set of latency functions \( S \) is the worst case ratio between the routing cost of a Wardrop equilibrium and that of a system optimum of an instance in which all latency functions are contained in \( S \). Roughgarden [29] and Correa et al. [9] show that

\[
\alpha(S) = \frac{1}{1 - \mu(S)},
\]

For a set \( S \) of latency functions, we denote by \( \gamma(S) \) the argmaximum \( \gamma \) in (4.1) for which \( \mu(S) \) is achieved. The following lemma gives an alternative representation of \( \mu(S) \) that will be useful in the remainder of this section.

**Lemma 4.1.** For a latency function \( S \),

\[
\sup_{x \geq 0} \max_{\gamma \in [0,1]} \left\{ \gamma \left( 1 - \frac{S(\gamma x)}{S(x)} \right) \right\} = \sup_{x \geq 0} \left\{ \gamma \cdot \frac{S'(x) x}{S(x) + S'(x) x} : S(x) + S'(x) x = S(x/\gamma) \right\}.
\]

**4.1 Two Approximation Algorithms**

The first algorithm that we call BRINGTOEQUILIBRIUM (cf. Algorithm 1) was already proposed by Marcotte [22, Section 4.3] and analyzed for monomial latency functions. Our contribution is a more general analysis of BRINGTOEQUILIBRIUM that works for arbitrary sets of latency functions \( S \), requiring only Assumption 2.1. The second algorithm, that we call SCALEUNIFORMLY (cf. Algorithm 2), is a new algorithm that we introduce in this paper.

For both approximation algorithms, we first compute an optimum solution \((v^*, z^*)\) to a relaxation of CNDP without the equilibrium constraints, i.e., we compute a solution \((v^*, z^*)\) to the problem

\[
\min_{z \geq 0} \min_{v \in F} \sum_{e \in E} \left( S_e(v_e/z_e) v_e + z_e \ell_e \right),
\]

which can be done in polynomial time (Proposition 2.4). Then, in both algorithms, we reduce the capacity vector \( z^* \), and determine a Wardrop equilibrium for the new capacity vector. The algorithms differ in the way we adjust the capacity vector \( z^* \). While in BRINGTOEQUILIBRIUM, we reduce the edge capacities individually such that the optimum solution to the relaxation
We are interested in bounding $C(v^*, z)$. To this end, we calculate

$$C(v^*, z) = \sum_{e \in E} \left( S_e(v_e^*/z_e^*) + S_e'(v_e^*/z_e^*) \gamma_e + S_e'(\delta_e) \delta_e \right)$$

By Proposition 2.2, the flow $v^*$ is a Wardrop flow with respect to $z$. We are interested in bounding $C(v^*, z)$. To this end, we calculate

$$C(v^*, z) \leq (1 + \mu(S)) \sum_{e \in E} \left( S_e(\delta_e) + S_e'(\delta_e) \delta_e \right) v_e^* \leq (1 + \mu(S)) C(v^*, z^*)$$

We proceed by showing that SCALEUNIFORMLY achieves the same approximation guarantee of $1 + \mu(S)$. Recall that SCALEUNIFORMLY first computes a relaxed solution $(v^*, z^*)$. Then, this relaxed solution is used to compute an optimal scaling factor $\lambda \leq 1$ with which all capacities are scaled subsequently.
The algorithm then returns the scaled capacity vector $\lambda z^*$ together with a correspond Wardrop equilibrium $v \in \mathcal{W}(\lambda z^*)$.

An (worse) approximation guarantee of 2 can be inferred directly from a bicriteria result of Roughgarden and Tardos [27] who showed that for any instance the routing cost of a Wardrop equilibrium is not worse than a system optimum that ships twice as much flow. This implies that for $\lambda = 1/2$ we have $C(v, \lambda z^*) \leq 2C(v^*, z^*$), as claimed.

For the proof of the following result, we take a different road that allows us to express the approximation guarantee of SCALEUNIFORMLY as a function of the parameter $p$ defined as the fraction of the total cost $C(v^*, z^*)$ of the relaxed solution allotted to the routing costs $C^R(v^*, z^*)$. This is an important ingredient for the analysis of the best-of-two algorithm.

**Theorem 4.3.** The approximation guarantee of SCALEUNIFORMLY is at most $(1 + \mu(S))$.

**Proof.** The algorithm first computes an optimum solution $(v^*, z^*)$ of the relaxed problem (CNDP). Then $p \in [0, 1]$ is defined as the fraction of $C(v^*, z^*)$ that corresponds to the routing cost $C^R(v^*, z^*)$, i.e.,

$$C^R(v^*, z^*) = \sum_{e \in E} S_e(v_e^*/z_e^*) v_e^* = p C(v^*, z^*) + \sum_{e \in E} \left(S_e(v_e^*/z_e^*) - S_e(v_e/z_e^*)\right) v_e^*.$$  

(4.6)

where the first inequality uses the variational inequality (2.1). We proceed to bound $\frac{v_e^* - v_e}{\lambda z_e^*} v_e$ for each edge $e \in E$ we have

$$\frac{S_e(v_e^*/x_e^*) v_e^*}{S_e(v_e^*/x_e^*) v_e} \leq \sup_{y \in [0, 1]} \frac{S((\lambda y) x - S(x))}{S(y) x} = \frac{\sup_{y \geq 0} y x - S(x) x}{S(y) y} = \frac{\sup_{y \geq 0} y x - S(x) x}{S(y) y}.$$  

This implies $y \geq x$ and we may substitute $x = \gamma y$ with $\gamma \in [0, 1]$. We then obtain for each edge $e \in E$ that

$$\frac{v_e^* - v_e}{\lambda z_e^*} v_e \leq \sup_{y \geq 0} \gamma S(y) - \gamma S(\gamma y) \leq \sup_{y \geq 0} \gamma (1 - S(y)) / S(y) \leq \frac{\mu(S)}{\lambda}.$$  

(4.7)

Plugging (4.7) in (4.6), we obtain for the routing cost $C^R(v, \lambda z^*) \leq p C(v^*, z^*) + \frac{\mu(S)}{\lambda} C^R(v, \lambda z^*)$ or, equivalently, $C^R(v, \lambda z^*) \leq \frac{p}{1 - \mu(S)/\lambda} C(v^*, z^*)$. Thus, we can bound the total cost of the outcome of SCALEUNIFORMLY by

$$C(v, \lambda z^*) = C^R(v, \lambda z^*) + C^Q(v, \lambda z^*) \leq \frac{p}{1 - \mu(S)/\lambda} C(v^*, z^*) + \lambda (1 - p) C(v^*, z^*)$$  

$$= \lambda \left(\frac{p}{1 - \mu(S)/\lambda} + 1 - p\right) C(v^*, z^*).$$

Since $\lambda = \mu(S) + \sqrt{\mu(S)} \frac{p}{1 - p}$, we obtain

$$\frac{C(v, \lambda z^*)}{C(v^*, z^*)} \leq p + 2 \sqrt{p(1 - p) \mu(S)} + \mu(S)(1 - p) = \left(\sqrt{p} + \sqrt{\mu(S)}(1 - p)\right)^2.$$  

(4.8)
Elementary calculus shows that \((\sqrt{p} + \sqrt{\mu(S)(1 - p)})^2\) attains its maximum at \(p = \frac{1}{1 + \mu(S)}\). Substituting this value into (4.8) gives \(C(v, \lambda z^*)/C(v^*, z^*) \leq 1 + \mu(S)\), as claimed. 

For particular sets \(S\) of latency functions, we compute upper bounds on \(\mu(S)\) in order to obtain an explicit upper bound on the approximation guarantees of \(\textsc{BringToEquilibrium}\) and \(\textsc{ScaleUniformly}\). We then obtain the following corollary of Theorem 4.2 and Theorem 4.3.

**Corollary 4.4.** For a set \(S\) of latency functions satisfying Assumption 2.1, the approximation guarantee of \(\textsc{BringToEquilibrium}\) and \(\textsc{ScaleUniformly}\) is at most

(a) \(2\), without further requirements on \(S\).

(b) \(\frac{5}{4}\), if \(S\) contains concave latencies only.

(c) \(1 + \frac{\Delta}{\Delta +1} (\frac{1}{\Delta +1})^{1/\Delta}\), if \(S\) contains only polynomials with non-negative coefficients and degree at most \(\Delta\), i.e., every \(S \in S\) is of the form \(S(x) = \sum_{j=0}^{\Delta} a_j x^j\) with \(a_j \geq 0\) for all \(j\).

### 4.2 Best-of-Two Approximation

In this section we show that although both \(\textsc{BringToEquilibrium}\) and \(\textsc{ScaleUniformly}\) achieve an approximation guarantee of \((1 + \mu(S))\) taking the better of the two algorithms we obtain a strictly better performance guarantee.

The key idea of the proof is to extend the analysis of the \(\textsc{BringToEquilibrium}\) algorithm in order to express its approximation guarantee as a function of the parameter \(p\) that measures the proportion of the routing cost in the total cost of a relaxed solution. This allows us to determine the worst-case \(p\) for which the approximation guarantee of the both algorithm is maximized.

**Theorem 4.5.** Taking the better solution of \(\textsc{BringToEquilibrium}\) and \(\textsc{ScaleUniformly}\) has an approximation guarantee of at most \(\frac{(\gamma(S) + \mu(S) + 1)^2}{(\gamma(S) + \mu(S) + 1)^2 - 4 \mu(S) \gamma(S)}\), which is strictly smaller than \(1 + \mu(S)\).

**Proof.** Recall from (4.8) that the approximation guarantee of the algorithm \(\textsc{ScaleUniformly}\) is \((\sqrt{p} + \sqrt{\mu(S)(1 - p)})^2\), where \(p = C^R(v^*, z^*)/C(v^*, z^*)\). We extend our analysis of \(\textsc{BringToEquilibrium}\) using this parameter \(p\). With the notation in Theorem 4.2 by (4.5), \(\textsc{BringToEquilibrium}\) returns a feasible solution \((v^*, z)\) with

\[
C(v^*, z) = \sum_{e \in E} \left( (S_e(\delta_e) + S_e^R(\delta_e) \delta_{e1} + \gamma_e S_e^L(\delta_e) \delta_{e2} v_e^*) \right) = pC(v^*, z^*) + \sum_{e \in E} S_e^L(\delta_e) \delta_{e2} v_e^*(1 + \gamma_e)
\]

\[
\leq pC(v^*, z^*) + (1 + \gamma(S)) \sum_{e \in E} S_e^L(\delta_e) \delta_{e2} v_e^* = pC(v^*, z^*) + (1 + \gamma(S))(1 - p) C(v^*, z^*)
\]

\[
= (1 + \gamma(S)(1 - p)) C(v^*, z^*).
\]

Thus, by taking the best of the two heuristics, we obtain an approximation guarantee of

\[
\max_{p \in (0,1)} \min \left\{ 1 + \gamma(S)(1 - p), \left( \sqrt{p} + \sqrt{\mu(S)(1 - p)} \right)^2 \right\}.
\]

The maximum of this expression is attained for

\[
p = p^* := \frac{(\gamma(S) - \mu(S) + 1)^2}{(\gamma(S) - \mu(S) + 1)^2 + 4 \mu(S)}
\]

which yields the claimed improved upper bound (cf. Lemma A.3 in the appendix for details). 

\[\square\]
It is not necessary to run both approximation algorithms to get this approximation guarantee. After computing the optimum solution to the relaxation (CNDP'), we can determine the value for \( p = C^R(v^*, z^*) / C(v^*, z^*) \) and proceed with SCALEUNIFORMLY if \( p \leq p^* \) (cf. (4.9)) and with BRINGTOEQUILIBRIUM otherwise.

For particular sets \( S \) of latency functions, we evaluate \( \mu(S) \) and \( \gamma(S) \) and obtain the following corollary of Theorem 4.5.

**Corollary 4.6.** For a set \( S \) of latency functions satisfying Assumption 2.1, the approximation guarantee in Theorem 4.5 is at most

\[
\begin{align*}
(a) \quad & \frac{9}{5}, \text{ without further requirements on } S, \\
(b) \quad & \frac{49}{41} \approx 1.195, \text{ if } S \text{ contains concave latencies only.} \\
(c) \quad & 1 + \frac{4\Delta(\Delta+1)}{2(2\Delta+1)(\Delta+1)+\Delta+1(\Delta+1)^{2}}\text{, if } S \text{ contains only polynomials with non-negative coefficients and degree at most } \Delta, \text{ i.e., every } S \in S \text{ is of the form } S(x) = \sum_{j=0}^{\Delta} a_j x^j \text{ with } a_j \geq 0 \text{ for all } j.
\end{align*}
\]

## 5 Conclusion

We reconsidered the classical continuous network design problem (CNDP). To the best of our knowledge, we established the first hardness results for CNDP. Specifically, we have shown the APX-hardness of CNDP both on directed and undirected networks and even if all latency functions are affine. We then turned to the approximation of the problem. First, we provided a thorough analysis of an algorithm proposed and studied by Marcotte [22] for monomial latency functions. We showed a general approximation guarantee depending on the set of allowed cost functions which is related to the anarchy value of the set of cost functions. Second, we proposed and studied a different approximation algorithm that turned out to provide the same approximation guarantee. As our arguably most interesting result concerning approximation, we then showed that taking the best of the two algorithms, we can guarantee a strictly better approximation factor.

In the transportation literature, further variants of CNDP have been investigated. One such example are situations in which the network designer is only interested in minimizing total travel time but investments are restricted, e.g., by budget constraints. More generally, suppose there is a convex function \( g : \mathbb{R}^m \rightarrow \mathbb{R}^k, k \in \mathbb{N} \) such that for any feasible solution \( z \) the condition \( g(z) \leq 0 \) must be satisfied. The function \( g \), for instance, can represent edge-specific budget constraints \( \ell_e z_e \leq B_e \) for \( e \in E \) and/or a global budget constraint \( \sum_{e \in E} \ell_e z_e \leq B \). We arrive at the following budgeted continuous network design problem (bCNDP):

\[
\min_{z \geq 0} \min_{v \in \mathcal{W}(z)} \sum_{e \in E} S_e(v_e / z_e) v_e \quad \text{s.t. } g(z) \leq 0.
\]

(bCNDP)

Using existing results from the price of anarchy literature (Roughgarden [29] and Correa et al. [9]), we can show that there is a 4/3-approximation for affine latencies and assuming \( P \neq NP \), for any \( \epsilon > 0 \), there is no polynomial time approximation algorithm with a performance guarantee better than \( 4/3 - \epsilon \), see Theorem A.4 in the appendix. For proving the lower bound, we use edge-specific budget constraints and mimic a construction from Roughgarden [26]. It is an interesting open problem whether such a lower bound can also be achieved if we allow only a global budget constraint.
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Appendix

Missing Material of Section 1

Table 1: Approximation guarantees of the algorithms BRINGTOEQUILIBRIUM, SCALEUNIFORMLY, and the best of the two for convex latency functions, concave latency functions and sets of polynomials with non-negative coefficients depending on the maximal degree $\Delta$. The approximation guarantees stated for convex latency functions even hold for sets of semi-convex latency functions as in Assumption 2.1. For BRINGTOEQUILIBRIUM, the approximation guarantees marked with (*) have been obtained before in [22].

| Functions | BRINGTOEQUILIBRIUM | ScaleUniformly | Better of the two |
|-----------|-------------------|----------------|-------------------|
| concave   | $5/4 = 1.25$      | $49/41 \approx 1.195$ |                 |
| convex    | $2$               | $9/5 = 1.8$    |                   |
| polynomials $\Delta$ | | | |
| $0$       | $1$               | $1$            |                   |
| $1/4$     | $3381/3125 \approx 1.082$ | $\approx 1.064$ | $1$ |
| $1/3$     | $283/256 \approx 1.105$   | $\approx 1.083$ | $1$ |
| $1/2$     | $31/27 \approx 1.148$    | $1849/1657 \approx 1.116$ | $1$ |
| $1$       | $5/4 = 1.25$ (*)      | $49/41 \approx 1.195$ | $1$ |
| $2$       | $1 + \frac{2}{3} \sqrt{3} \approx 1.385^*$ | $\frac{331}{259} \approx 1.300$ | $1$ |
| $3$       | $1 + \frac{3}{4} \sqrt{3} \approx 1.472^*$ | $\approx 1.369$ | $1$ |
| $4$       | $1 + \frac{4}{5} \sqrt{3} \approx 1.535^*$ | $\approx 1.418$ | $1$ |
| $\infty$  | $2$ (*)            | $9/5 = 1.8$    |                   |

Proof of Proposition 2.4

Proof. As the latency of all edges diverges to $\infty$ as the capacity approaches 0 we obtain $z_e > 0$ if and only if $v_e > 0$ for all edges $e \in E$. The Karush-Kuhn-Tucker conditions of the relaxed problem (CNDP’) imply that

$$\frac{\partial}{\partial z_e} \sum_{e \in E} (S_e(v_e/z_e) v_e + z_e \ell_e) = 0,$$

or, equivalently, $\ell_e = (v_e / z_e)^2 S'_e(v_e / z_e)$ for all $e \in E$ with $z_e > 0$. Using that $x^2 S'_e(x)$ is non-decreasing and unbounded, for each $e \in E$ there is a solution to the equation $x^2 S'_e(x) = \ell_e$ which we denote by $u_e$. Since $\ell_e > 0$, we derive that $u_e > 0$ as well. By definition, $u_e$ is the unique optimal ratio of $v_e / z_e$ for edge $e$ with $z_e > 0$ in an optimal solution of (CNDP’). Substituting $z_e = v_e / u_e$ in (CNDP’), we obtain the equivalent mathematical problem

$$\min_{v \in F} \sum_{e \in E} (S_e(u_e) + \ell_e / u_e) v_e,$$

which can be solved by performing $|K|$ independent shortest path path computations, one for each commodity $k \in K$. 

Proof of Theorem 3.1

Proof. CNDP lies in NP as a vector of capacities $z$ is a polynomial certificate. Given $z$, we can compute in polynomial time a corresponding Wardrop equilibrium and the total cost $C(v, z)$.

To show the NP-hardness of the problem, we reduce from 3-SAT. Let $\phi$ be a Boolean formula in conjunctive normal form. We denote the set of variables and clauses of $\phi$ with $V(\phi)$ and $K(\phi)$, respectively,
and set $\nu = |V(\phi)|$ and $\kappa = |K(\phi)|$. The set $L(\phi)$ of literals of $\phi$ contains for each variable $x_i \in V(\phi)$ the positive literal $\bar{x}_i$ and the negative literal $x_i$, i.e., $L(\phi) = \{x_i \in V(\phi)\} \cup \{\bar{x}_i : x_i \in V(\phi)\}$. In the following, we will associate clauses with the set of literals that they contain.

We now explain the construction of a continuous network design problem based on $\phi$ that has the property that, for some $\epsilon \in (0, 1/8)$, an optimal solution has total cost less or equal to $(4 + \epsilon)\kappa + 2\kappa\nu$ if and only if $\phi$ has a solution. Let $\epsilon \in (0, 1/8)$ be arbitrary. For each clause $k \in K(\phi)$, we introduce a clause edge $e_k$ with latency function $S_{e_k}(v_{e_k}/z_{e_k}) = 4 + v_{e_k}/z_{e_k}$ and construction cost $\ell_{e_k} = (\epsilon/2)^2$. For each literal $l \in L(\phi)$ and each clause $k \in K(\phi)$, we introduce a literal edge $e_{l,k}$ with latency function $S_{e_{l,k}}(v_{e_{l,k}}/z_{e_{l,k}}) = v_{e_{l,k}}/z_{e_{l,k}}$ and cost $\ell_{e_{l,k}} = 1$. We denote the set of clause edges and literal edges by $E_K$ and $E_L$, respectively.

For each variable $x_i \in V(\phi)$, there is a variable commodity $j_{x_i}$ with source $s_{j_{x_i}}$, sink $t_{j_{x_i}}$, and demand $d_{j_{x_i}} = 1$. This commodity has two feasible paths, one path uses exclusively the literal edges $\{e_{x_i,k} : k \in K(\phi)\}$ that correspond to the non-negated variable $x_i$, the other to the negated variable $\bar{x}_i$. In that way, each feasible path of the variable commodity $j_{x_i}$ corresponds to a true/false assignment of the variable $x_i$. For each clause $k = l_k \lor l'_k \lor l''_k$, we introduce a clause commodity $j_k$ with source $s_{j_k}$, sink $t_{j_k}$ and demand $d_{j_k} = 1$. The clause commodity may either choose its corresponding clause edge $e_k$ or the corresponding literal edges that occur in $k$, i.e., $e_{k}, e'_{k}, e''_{k}$, and $e_{l_k,k}$. For notational convenience, we set $E_k = \{e_{l_k,k}, e'_{l'_k,k}, e''_{l''_k,k}\}$. We add some additional edges with latency 0 to obtain a network; see Figure 1 where these edges are dashed. Note that the problem remains NP-hard, even if we do not allow edges with zero latency, see Remark A.1 after this proof.

First, we show that an optimal solution of the so-defined instance of the continuous network design problem $P$ has total cost less or equal to $(4 + \epsilon)\kappa + 2\kappa\nu$, if $\phi$ has a solution. To this end, let $y = (y_{x_i})_{x_i \in V(\phi)}$ be a solution of $\phi$. Then, a feasible solution of $P$ is as follows: For each positive literal $x_i$ that is selected in the solution $y$, we buy capacity 1 for the corresponding negative literal edges $\{e_{\bar{x}_i,k} : k \in K(\phi)\}$, and vice versa. Formally, we set

$$z_{a_{i,k}} = \begin{cases} 1, & \text{if } l = x_i \text{ and } y_{x_i} = \text{false}, \\ 1, & \text{if } l = \bar{x}_i \text{ and } y_{x_i} = \text{true}, \\ 0, & \text{otherwise}. \end{cases}$$

For each clause edge $e_k$, $k \in K(\phi)$, we buy capacity $2/\epsilon$. This particular capacity vector $z = (z_e)_{e \in E}$ implies that each variable commodity $j_{x_i}$ has a unique path of finite length, i.e., the path using the edges corresponding to the negation of the corresponding literal in $y$. Using that $y$ is a solution of $\phi$, we further obtain that for each clause commodity $j_k$ at least one of the edges in $E_k$ has capacity zero and, thus, infinite latency. This implies that, in the unique Wardrop equilibrium, the demand of each clause commodity $j_k$ is
routed along the corresponding clause edge $e_k$. For the total cost of this solution, we obtain

$$C(v, z) = \sum_{e \in E_K} \left((4 + v_e/z_e)v_e + (\epsilon/2)^2 z_e\right) + \sum_{e \in E_L} \left((v_e/z_e)v_e + z_e\right)$$

$$= \sum_{e \in E_K} \left((4 + \epsilon/2 + (\epsilon/2)^2) + \frac{1}{2} \sum_{e \in E_L} (1 + 1) = (4 + \epsilon)\kappa + 2\kappa\nu.\right.$$  \hspace{1cm} (A.1)

Hence, an optimal solution has cost not larger than (A.1) if $\phi$ has a solution.

We proceed to prove that the total cost of an optimal solution are strictly larger than (A.1) if $\phi$ does not admit a solution. Let $z = (z_e)_{e \in E}$ be an optimal solution of $P$ and let $v = (v_e)_{e \in E}$ be a corresponding Wardrop flow. We distinguish two cases.

First case: $v_{e_k} > 0$ for all $k \in K(\phi)$, i.e., each clause commodity $j_k$ sends flow over the corresponding clause edge $e_k$.

Before we prove the thesis for this case, we need some additional notation. For the Wardrop flow $v$ on edge $e \in E$, let $v_v^V$ and $v_v^K$ denote the flow on $e$ that is due to the variable commodities and the clause commodities, respectively.

We claim that there is a clause $\tilde{k} \in K(\phi), \tilde{k} = l_k \lor l_k' \lor l_k''$ such that the flow of the variable commodities on each of the corresponding literal edges in $E_{\tilde{k}} = \{e_{l_k,k}, e_{l_k',k}, e_{l_k'',k}\}$ is at least $1/2$, i.e.,

$$v_{e_{l_k,k}}^V \geq 1/2, \quad v_{e_{l_k',k}}^V \geq 1/2, \quad \text{and} \quad v_{e_{l_k'',k}}^V \geq 1/2. \hspace{1cm} (A.2)$$

For a contradiction, let us assume that for each clause $k = l_k \lor l_k' \lor l_k''$ there is a literal $l_k' \in \{l_k, l_k', l_k''\}$ such that $v_{e_{l_k,k}}^V < 1/2$. As each variable $x_i \in V(\phi)$ splits its unit demand between the path consisting of the positive literal edges $\{e_{x_i,k} : k \in K(\phi)\}$ and the path consisting of the negative literal edges $\{e_{\bar{x}_i,k} : k \in K(\phi)\}$, at most one of these two paths is used with a flow strictly smaller than $1/2$. Thus, the assignment vector $y$ defined as

$$y_{x_i} = \begin{cases} 
\text{true,} & \text{if } v_{e_{x_i,k}}^V < 1/2 \text{ for all } e \in \{e_{x_i,k} : k \in K(\phi)\}, \\
\text{false,} & \text{if } v_{e_{\bar{x}_i,k}}^V < 1/2 \text{ for all } e \in \{e_{\bar{x}_i,k} : k \in K(\phi)\}, \\
\text{true,} & \text{otherwise,}
\end{cases}$$

is well-defined. By construction, $y$ satisfies all clauses, which is a contradiction to the assumption that no such assignment exists. We conclude that there is a clause $\tilde{k}$ such that (A.2) holds.

We proceed to bound the total cost of a solution. As $v$ is a Wardrop equilibrium in which the clause commodity $j_k$ uses at least partially the clause edge $e_k$, we further derive that $\sum_{e \in E_k} v_e/z_e \geq v_{e_k}/z_{e_k} > 4$. We bound the total cost of the solution $(v, z)$ by observing

$$C(v, z) = \sum_{e \in E_L} \left(v_e^2/z_e + z_e\right) + \sum_{e \in E_K} \left((4 + v_e/z_e)v_e + (\epsilon/2)^2 z_e\right)$$

$$\geq \sum_{e \in E_L} \min_{z_e \geq 0} \left(v_e^2/z_e + z_e\right) + \sum_{e \in E_K} \min_{z_e \geq 0} \left((4 + v_e/z_e)v_e + (\epsilon/2)^2 z_e\right),$$

where we slightly abuse notation by writing $\min_{z_e \geq 0}$ shorthand for $\min_{z_e \geq 0 : v_e \in W(z)}$. We obtain an upper bound by relaxing $\min_{z_e \geq 0}$ to $\min_{z_e \geq 0} \text{ for the edges in } E_{L \setminus E_k}$ and $E_{K \setminus E_k}$. Hence,

$$C(v, z) \geq \sum_{e \in E_{L \setminus E_k}} \min_{z_e \geq 0} \left(v_e^2/z_e + z_e\right) + \sum_{e \in E_{E_k \setminus E_k}} \min_{z_e \geq 0} \left(v_e^2/z_e + z_e\right) + \sum_{e \in E_{K \setminus E_k}} \min_{z_e \geq 0} \left((4 + v_e/z_e)v_e + (\epsilon/2)^2 z_e\right).$$

Calculating the respective minima, we obtain

$$C(v, z) = \sum_{e \in E_{L \setminus E_k}} 2v_e + \sum_{e \in E_{E_k \setminus E_k}} \underbrace{\min_{z_e \geq 0} \left(v_e^2/z_e + z_e\right)}_{\geq 2v_e} + \sum_{e \in E_{K \setminus E_k}} (4 + \epsilon)v_e. \hspace{1cm} (A.3)$$
Each clause commodity $j_k$ can route its demand either over the clause edge $e_k$ or over the three literal edges in $E_k$. Every fraction of the demand routed over the clause edge contributes $4 + \epsilon$ to the expression on the right hand side of (A.3) while it contributes at least 6 when routed over the literal edges. Thus, the right hand side of (A.3) is minimized when the clause commodities do not use the literal edges at all. We then obtain

$$C(v, z) \geq \sum_{e \in E_L \setminus E_k} 2v_e^V + \sum_{e \in E_k} \min_{z_e \geq 0} ((v_e^V)^2 / z_e + z_e) + (4 + \epsilon)|E_K|$$

$$= 2 \left(\kappa \nu - \sum_{e \in E_k} v_e^V\right) + (4 + \epsilon)\kappa + \sum_{e \in E_k} \min_{z_e \geq 0} ((v_e^V)^2 / z_e + z_e),$$

$$= 2\kappa \nu + (4 + \epsilon)\kappa + \sum_{e \in E_k} \min_{z_e \geq 0} ((v_e^V)^2 / z_e + z_e - 2v_e^V),$$

$$> 2\kappa \nu + (4 + \epsilon)\kappa + Q,$$

where $Q$ is the solution to the constrained minimization problem

$$Q = \min_{v^V, z_e \geq 0} \sum_{e \in E_k} ((v_e^V)^2 / z_e + z_e - 2v_e^V)$$

s.t.: \( \sum_{e \in E_k} v_e^V / z_e \geq 4 \) \hspace{1cm} (A.4)

\( v_e^V \geq 1/2 \) for all $e \in E_k$, \hspace{1cm} (A.5)

Side constraint (A.4) is a relaxation of the requirement that $v$ is a Wardrop equilibrium as the latency of the literal edges is strictly larger than 4. Side constraint (A.5) is due to the fact that for clause $k$ the three corresponding literal edges $e_{k, \ell, \tilde{k}}, e_{k, \ell, \tilde{k}}$, and $e_{k, \ell, \tilde{k}}$ are used with a flow of at least 1/2 by the variable commodities. The optimal solution to the constraint optimization problem $Q$ is equal to $Q = 1/8$ and is attained for $v_e^V = 1/2$ and $z_e = 3/8$ for all $e \in E_k$. This implies that the total cost of a solution is not smaller than $(4 + \epsilon)\kappa + 2\kappa \nu + 1/8$, which finishes the first case of this proof.

**Second case:** There is a clause commodity $j_{\tilde{k}}$ that does not use its clause edge $e_{\tilde{k}}$, i.e., $v_{e_{\tilde{k}}} = 0$. As for first case, we observe

$$C(v, z) = \sum_{e \in E_L} (v_e^2 / z_e + z_e) + \sum_{e \in E_K} (4v_e + v_e^2 / z_e + (\epsilon/2)^2 z_e) \geq 2v_e + \sum_{e \in E_K} (4 + \epsilon)v_e.$$

Using that $j_{\tilde{k}}$ does not use its clause edge, we derive that the flow on the literal edges amounts to $\nu \kappa + 3$ and we obtain

$$C(v, z) \geq 2(\kappa \nu + 3) + (4 + \epsilon)(\kappa - 1) = 2\kappa \nu + (4 + \epsilon)\kappa + 2,$$

which concludes the proof. \( \square \)

In the following remark we discuss that although the hardness proof of Theorem 3.1 used edges with zero latency, the hardness result continues to hold even if edges with zero latency are not allowed.

**Remark A.1.** The continuous network design problem is NP-hard in the strong sense, even if no edges with zero latency are allowed.

**Sketch of proof.** Let $M$ be an upper bound on the total cost of an optimal solution to a continuous network design problem constructed in the proof of Theorem 3.1 and let $E_0$ be the set of edges with zero latency. We replace each edge $e \in E_0$, $e = (s, t)$, $s, t \in V$ by an edge $e' = (s, t)$ with latency function $S_{e'}(v_{e'}/z_{e'}) = v_{e'}/z_{e'}$ and construction cost $\ell_{e'} = (\frac{m}{2M})^2$. For each new edge $e'$, we introduce an additional commodity
Proof of Theorem 3.2

Proof. We reduce from a symmetric variant of 4-OCC-MAX-3-SAT which is NP-hard to approximate, see Berman et al. [4]. An instance of 4-OCC-MAX-3-SAT, is given by a Boolean formula $\phi$ in conjunctive normal form with the property that each clause contains exactly three literals and each variable occurs exactly four times. The problem to determine the maximal number of clauses that can be satisfied simultaneously is known to be NP-hard to approximate within a factor of $1016/1015 - \delta \approx 1.00099 - \delta$ for any $\delta > 0$, even for the special case that each variable occurs exactly twice as a positive literal and exactly twice as a negative literal, see a follow-up paper by the same authors [5].

Let us again denote by $V(\phi)$, $K(\phi)$, and $L(\phi)$ the set of variables, clauses and literals of $\phi$ and let $\nu = |V(\phi)|$ and $\kappa = |K(\phi)|$. It is convenient to assume that $K(\phi) = \{1, \ldots, \kappa\}$ and $V(\phi) = \{x_1, \ldots, x_{\nu}\}$. As every variable occurs exactly four times and every clause contains exactly three literals, we have $4\nu = 3\kappa$.

We slightly adjust the construction in the proof of Theorem 3.1 to make use of the information that each literal occurs in exactly two clauses. We proceed to explain the construction of an instance of CNDP relative to a fixed parameter $\epsilon \in (0, 1/8)$. For a literal $l \in L(\phi)$, let $k_l, k'_l \in K(\phi)$ be the clauses that contain the literal $l$. We introduce two literal edges $e_{l,k_l}$ and $e_{l,k'_l}$ with latency function $S_{e_l}(v_e/z_e) = v_e/z_e$ and construction cost $\ell_e = 1$. For each variable $x_i \in V(\phi)$, we introduce a corresponding variable commodity $j_{x_i}$ with source $s_{j_{x_i}}$, sink $t_{j_{x_i}}$ and demands $d_{j_{x_i}} = 1$ that may then either choose the path consisting of the edges $e_{x_i,k_{x_i}}$ and $e_{x_i,k'_{x_i}}$ that correspond to the positive literal $x_i$ or the edges $e_{\bar{x}_i,k_{x_i}}$ and $e_{\bar{x}_i,k'_{x_i}}$ that correspond to the negative literal $\bar{x}_i$. We construct the network such that in the directed path containing the edges $e_{x_i,k}$ and $e_{x_i,k'}$ the edge $e_{x_i,k}$ appears before the edge $e_{x_i,k'}$ if and only if $k < k'$, i.e., the corresponding clause $k$ has a smaller index than the respective clause $k'$. As in the proof of Theorem 3.1, for each clause $k \in K(\phi)$, let $l_k \in l_k' \cup l_k''$, we introduce a clause edge $e_k$ with latency $S_{e_k}(v_e/z_e) = 4 + v_e/z_e$ and construction cost $\ell_k = (\epsilon/2)^2$. For each clause $k \in K(\phi)$, there is a clause commodity $j_k$ with source $s_{j_k}$, sink $t_{j_k}$ and demand $d_{j_k} = 1$. The clause commodity $j_k$ may choose either the clause edge $e_k$ or a path that contains all the corresponding literal edges $e_{l_k,k}, e_{l'_k,k}, e_{l''_k,k}$. The set of literal edges and clause edges is denoted by $E_L$ and $E_K$, respectively. For notational convenience, for a clause $k \in K(\phi)$, let $l_k \in l_k' \cup l_k''$, we set $E_k = \{e_{l_k,k}, e_{l'_k,k}, e_{l''_k,k}\}$. We add some additional edges with zero latency to obtain a network, see Figure 2. Because in each path for a variable commodity the clauses appear in increasing order of their index, adding these additional edges with zero latency does not add any further paths to the literal or variable commodities.

The hardness result continues to hold, even if edges with zero latency are not allowed, see Remark A.1 after the proof of Theorem 3.1.

We claim that the so-defined instance of CNDP has a solution with total cost in the interval

$$\left[10\kappa + |K|/4, (10 + \epsilon)\kappa + (1/4 + \epsilon/2)|K|\right]$$

if and only if the minimum number of unsatisfied clauses is $|\tilde{K}|$.

First, we show that an optimal solution has total cost not larger than $(10 + \epsilon)\kappa + |\tilde{K}|/4$ if $\phi$ has a solution $y$ that violates $|\tilde{K}|$ clauses only. To this end, let $y = (y_{x_i})_{x_i \in V(\phi)}$ be such a solution and let $\tilde{K}$ be the set of
the other hand, the corresponding literal edges

\[ \text{Dashed edges have zero latency.} \]

Figure 2: Network used to show the APX-hardness of the continuous network design problem. Clause 1 is equal to \( x_1 \lor \bar{x}_2 \lor x_4 \). Dashed edges have zero latency.

clauses that is not satisfied by \( y \). Consider the tuple \((v, z)\) defined as

\[
z_{e_{l,k}} = \begin{cases} 
1, & \text{if } k \not\in \tilde{K} \text{ and } l = \bar{y}_i \text{ for some } i \in V(\phi), \\
\frac{1}{4/3+\epsilon/6}, & \text{if } k \in \tilde{K} \text{ and } l = \bar{y}_i \text{ for some } i \in V(\phi), \\
0, & \text{otherwise}, 
\end{cases} \quad \text{for all } l \in L(\phi), k \in \{k_1, k'_1\},
\]

\[
v_{e_{l,k}} = \begin{cases} 
1, & \text{if } l = \bar{y}_i \text{ for some } i \in V(\phi), \\
0, & \text{otherwise}, 
\end{cases} \quad \text{for all } l \in L(\phi), k \in \{k_1, k'_1\},
\]

\[
z_{e_k} = 2/\epsilon, \quad \text{for all } k \in K(\phi).
\]

\[
v_{e_k} = 1, \quad \text{for all } k \in K(\phi).
\]

First, we show that the tuple \((v, z)\) is a solution to CNDP. To this end, it suffices to prove that \( v \) is a Wardrop equilibrium for the latency functions defined by \( z \). We will argue for each commodity separately that it only uses shortest paths, starting with an arbitrary clause commodity \( j_k \) that corresponds to a non-satisfied clause \( k \in \tilde{K} \), \( k = l_k \lor l'_k \lor l''_k \). Such a clause uses the clause edge \( e_k \) with latency \( 4 + \epsilon/2 \). On the other hand, the corresponding literal edges \( e_{l_k,k}, e_{l'_k,k}, e_{l''_k,k} \) have capacity \( \frac{1}{4/3+\epsilon/6} \) and carry one unit of flow of the variable commodities. Thus, their latencies sum up to \( 4 + \epsilon/2 \), implying that clause commodity \( j_k \) is in equilibrium. Next, consider a clause commodity \( j_k \) that corresponds to a satisfied clause \( k \in K(\phi) \setminus \tilde{K} \), \( k = l_k \lor l'_k \lor l''_k \). As \( k \) is satisfied by \( y \), there is a literal \( l''_k \in \{l_k, l'_k, l''_k\} \) such that \( l''_k = \bar{y}_i \) for some \( x_i \in V(\phi) \). This implies that \( z_{e_{l''_k,k}} = 0 \) and, thus, edge \( e_{l''_k,k} \) has infinite latency. We derive that clause commodity \( j_k \) has a unique path of finite latency and this path is used in \( v \). Finally, consider a variable commodity \( j_{x_i}, x_i \in V(x_i) \). As we buy either the capacity for the edges \{\( e_{x_i,k_2}, e_{x_i,k'_2} \}\} corresponding to the positive literal or the edges \{\( e_{x_i,k_2}, e_{x_i,k'_2} \}\} that correspond to the negative literal, but not both, commodity \( j_{x_i} \) has only one path with finite latency, and it uses that path in \( v \).

We proceed to calculate the total cost of the solution \((v, z)\). Every literal edge that corresponds to a satisfied clause and the negation of a literal in \( y_i \) has capacity 1 and flow 1 and thus causes a total cost of 2. In contrast to this, each literal edge that corresponds to a violated clause and the negation of a literal in \( y_i \) has capacity \( \frac{1}{4/3+\epsilon/6} \) and flow 1 and, thus, causes a total cost of

\[
\frac{4}{3} + \frac{\epsilon}{6} + \frac{1}{4/3+\epsilon/6} \leq \frac{25}{12} + \frac{\epsilon}{6}.
\]

Further, each clause edge has capacity \( 2/\epsilon \) and is used by 1 unit of flow and, thus, contributes \( 4 + \epsilon \) to the total cost. We calculate

\[
C(v, z) \leq 3(2(\kappa - |\tilde{K}|) + (25/12 + \epsilon/6)|\tilde{K}|) + (4 + \epsilon)\kappa
= (10 + \epsilon)\kappa + (1/4 + \epsilon/2)|\tilde{K}|.
\]

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We proceed to prove that an optimal solution of CNDP has total cost not smaller than $10\kappa + \bar{K}/4$ if each solution $y$ of $\phi$ violates at least $\bar{K}$ clauses. To this end, we need some additional notation. For an edge flow $v$, let $v^V$ denote the edge flow that is due to the variable commodities and $v^K$ denote the edge flow that is due to the clause commodities. For a clause $k \in K(\phi)$, let $m_k(v^K) = \min_{e \in E_k} v^K_e$. In addition, we set $\bar{K}(v^K) = \{ k \in K : v^K_k > 0 \}$, i.e., $\bar{K}(v^K)$ is the set of clauses $k$ that uses (at least partially) its clause edge $e_k$.

We bound $C(v, z)$ by observing 
\[
C(v, z) = \sum_{e \in E_L} \left( v^2_e / z_e + z_e \right) + \sum_{e \in E_K} \left( (4 + v_e / z_e)v_e + (\epsilon/2)^2 z_e \right) \\
\geq \sum_{e \in E_L} \min_{z_e \geq 0} \left( v^2_e / z_e + z_e \right) + \sum_{e \in E_K} \min_{z_e \geq 0} \left( (4 + v_e / z_e)v_e + (\epsilon/2)^2 z_e \right),
\]

where we again slightly abused notation writing $\min_{z_e \geq 0}$ shorthand for $\min_{z_e \geq 0 : v \in V(z)}$. We obtain a lower bound on the total cost observing that the latency of the clause edges is at least 4. Thus, 
\[
C(v, z) \geq 4\bar{K}(v^K) + \sum_{k \in K(\phi)} \sum_{e \in E_k} \min_{z_e \geq 0} \left( (v^K_e + v^K_e)^2 / z_e + z_e \right).
\]

Every unit of flow of a clause commodity $j_k$ with $k \in \bar{K}(v^K)$, $k = l_k \lor l'_k \lor l''_k$ contributes at least 6 to the right hand side of (A.6) when routed over the corresponding literal edges $e_{l_k,k}$, $e_{l'_k,k}$, and $e_{l''_k,k}$, but contributes only 4 when routed over the corresponding clause edge $e_k$. Thus, we obtain a lower bound assuming that each clause commodities $j_k$, $k \in \bar{K}(v^K)$ exclusively uses its clause edge, i.e., 
\[
C(v, z) \geq 4|\bar{K}(v^K)| + \sum_{k \in K(\phi \setminus \bar{K}(v^K))} \sum_{e \in E_k} \min_{z_e \geq 0} \left( (v^K_e + v^K_e)^2 / z_e + z_e \right).
\]

With the same arguments, we observe that every clause commodity $j_k$ with $k \in K(\phi \setminus \bar{K}(v^K))$ contributes at least 6 to the right hand side of (A.7) when routed over the literal edges, but contributes only 4 when routed over the clause edge. Thus, we obtain a lower bound assuming that $K(\phi) = \bar{K}(v^K)$, i.e., every clause commodity $j_k$ routes its demand exclusively over the corresponding clause edge $e_k$. Then, 
\[
C(v, z) \geq 4\kappa + \sum_{k \in K(\phi \setminus \bar{K}(v^K))} \sum_{e \in E_k} \min_{z_e \geq 0} \left( (v^K_e + v^K_e)^2 / z_e + z_e \right) \\
= 4\kappa + \sum_{k \in K(\phi \setminus \bar{K}(v^K))} \sum_{m_k(v^K) = 0} \sum_{e \in E_k} \min_{z_e \geq 0} \left( (v^K_e + v^K_e)^2 / z_e + z_e \right) + \sum_{k \in K(\phi \setminus \bar{K}(v^K))} \sum_{m_k(v^K) > 0} \sum_{e \in E_k} \min_{z_e \geq 0} \left( (v^K_e + v^K_e)^2 / z_e + z_e \right)
\]

Note that for all clauses $k \in K(\phi)$ with $m_k(v^K) = 0$ at least one of the corresponding clause edges is not used by the variable commodities and, thus, we can set the capacity of this edge to 0. This implies that the corresponding clause commodity $k$ stays at its clause edge and we can optimize the capacity of the remaining edges in $E_k$ irrespective of the equilibrium constraints. We obtain 
\[
C(v, z) \geq 10\kappa + \sum_{k \in K(\phi \setminus \bar{K}(v^K))} \sum_{m_k(v^K) > 0} \sum_{e \in E_k} \min_{z_e \geq 0} \left( (v^K_e + v^K_e)^2 / z_e + z_e - 2v^K_e \right) \\
\geq 10\kappa + \sum_{k \in K(\phi \setminus \bar{K}(v^K))} \sum_{m_k(v^K) > 0} Q_k,
\]

where
where each $Q_k$ is the solution to the constrained minimization problem

$$Q_k = \min_{v^V, z_e > 0} \sum_{e \in E_k} \frac{(v_e^V)^2}{z_e} + z_e - 2v_e^V$$

s.t.: \( \sum_{e \in E_k} v_e^V / z_e \geq 4 \)

\( v_e^V \geq m_k \) for all \( e \in E_k, v \)

The optimal solution to this problem is equal to \( Q_k = m_k(v^V)/4 \) and is attained for \( v_e^V = m_k(v^V) \) and \( z_e = 3m_k(v^V)/4 \) for all \( e \in E_k, k \in K(\phi) \). We obtain

$$C(v, z) \geq 10\kappa + \sum_{k \in K(\phi)} m_k(v^V)/4.$$  

To finish the proof it suffices to show that \( \sum_{k \in K(\phi)} m_k(v^V) \geq \tilde{K} \) for each flow of the variable commodities \( v^V \). To this end, let \( v^V \) be a flow that minimizes \( \sum_{k \in K(\phi)} m_k(v^V) \). We claim that it is without loss of generality to assume that \( v^V \) is integral. To see this claim, suppose that the flow for all variable commodities except \( j_{x_i} \) is fixed and consider the variable commodity \( j_{x_i} \). Let \( p \) denote the portion of the flow sent over the path consisting of the positive literal edges \( e_{x_i,k_{x_i}} \) and \( e_{x_i,k'_{x_i}} \). By definition, only the clauses \( k_{x_i} \) and \( k'_{x_i} \) contain the literal \( x_i \) and only the clauses \( k_{\bar{x}_i} \) and \( k'_{\bar{x}_i} \) contain the literal \( \bar{x}_i \). Then, we can calculate the contribution of these four clauses to \( \sum_{k \in K(\phi)} m_k(v^V) \) as follows:

$$\sum_{k \in \{k_{x_i}, k'_{x_i}, k_{\bar{x}_i}, k'_{\bar{x}_i}\}} m_k(v^V) = \sum_{k \in \{k_{x_i}, k'_{x_i}, k_{\bar{x}_i}, k'_{\bar{x}_i}\}} \min_{e \in E_k} v_e^V$$

$$= \sum_{k \in \{k_{x_i}, k'_{x_i}\}} \min(p, e_{x_i,k_{x_i}}) v_e^V + \sum_{k \in \{k_{\bar{x}_i}, k'_{\bar{x}_i}\}} \min(1 - p, e_{x_i,k_{\bar{x}_i}} v_e^V)$$

For a fixed flow \( v^V \) on the literal edges not involving \( x_i \), this expression is concave in \( p \). Hence, the minimum is attained for either \( p = 0 \) or \( p = 1 \). Put differently, for any flow of the other variable commodities, the expression \( \sum_{k \in K(\phi)} m_k(v^V) \) is minimized when variable commodity \( j_{x_i} \) routes all of its demand on one path. Iterating this argument for all variable commodities, we conclude that is without loss of generality to assume that \( v^V \) is integral.

For an integral flow \( v^V \) of the variable commodities, consider the true/false assignment \( y = (y_{x_i})_{x_i \in V(\phi)} \) defined as \( y_{x_i} = \text{true} \) if and only if \( v_{x_i,k_{y_i}} = 0 \). As this assignment satisfies at most \( K_{\phi} \) clauses, we have that \( \sum_{k \in K(\phi)} m_k(v^V) \geq \tilde{K} \).

Plugging everything together, we obtain that the total cost of an optimal solution to CNDP lies in the range

$$\left[10\kappa + |\tilde{K}|/4, (10 + \epsilon)\kappa + (1/4 + \epsilon/2)|\tilde{K}|\right] \quad \text{(A.8)}$$

if \( |\tilde{K}| \) clauses cannot be satisfied.

Berman et al. [4, 5] construct a family of symmetric instances of 4-OCC-MAX-3-SAT with \( \kappa = 1016n \), \( n \in \mathbb{N} \) that has the property that for any \( \delta \in (0, 1/2) \) it is NP-hard to distinguish between the systems where \( (1016 - \delta)n \) clauses can be satisfied and systems where at most \( (1015 + \epsilon)n \) clauses can be satisfied. Using (A.8), the corresponding instances of CNDP have the property that they have total cost at most \( (10 + \epsilon)1016n + \delta n(1 + 1/2) \), if at least \( (1016 - \delta)n \) clauses can be satisfied, and total cost at least \( 10 \cdot 1016n + 1/4 - \delta n \), if at most \( (1015 + \delta)n \) clauses can be satisfied. As we let \( \epsilon \) and \( \delta \) go to zero, we derive that it is NP-hard to approximate CNDP by any factor better than 10160.25/10160 \( \approx 1.000024 \). This proves the APX-hardness of the problem. \[\square\]
Figure 3: Network used to show the \( \text{APX} \)-hardness of the continuous network design problem on undirected graphs. Clause 1 is equal to \( x_1 \lor \overline{x}_2 \lor x_\nu \). The clause edges (straight lines in the upper part of the graph) and the literal edges (straight edges in the lower part of the graph) are connected via different auxiliary edges (dashed). The auxiliary edges have different constant latencies dependent on their type. Type one edges are auxiliary edges adjacent to a source or a target of a variable commodity. Type two edges are auxiliary edges connecting two literal edges that correspond to the same literal. Type three edges are auxiliary edges adjacent to the a source or a target of a clause commodity. Type four edges are auxiliary edges connecting two literal edges that correspond that correspond to different literals that appear together in a clause. Type five edges connect the source of a clause commodity with the respective clause edge.

**Hardness for undirected networks**

**Theorem A.2.** The continuous network design problem on undirected networks is \( \text{APX} \)-hard, even if all latency functions are affine.

**Sketch of proof.** As in the proof of Theorem 3.2, we reduce from a symmetric variant of 4-OCC-MAX-3-SAT where each variable occurs exactly twice negated and twice unnegated. We will closely mimic the proof of Theorem 3.2 and only sketch how to adjust it to the undirected case.

We use a construction similar to the directed case, see Figure 3. We carefully choose the latency of the auxiliary edges in order to prevent the commodities from taking undesired paths. For each variable commodity \( j_{x_i} \), let us call the two dashed edges adjacent to \( s_{j_{x_i}} \) and the two edges adjacent to \( t_{j_{x_i}} \) type one edges. Further, let us call the dashed edge between the edges \( e_{x_i,k} \) and \( e_{x_i,k'} \) and between \( e_{\overline{x}_i,k} \) and \( e_{\overline{x}_i,k'} \) type two edges. For each clause commodity \( j_i \), we call the dashed edge connecting \( s_{j_i} \) the a variable gadget and the dashed edge adjacent to \( t_{j_i} \) type three edges. We call the dashed edges connecting two literal edges corresponding to different variables but the same clause type four edges. Finally, we call the dashed edges that connect the source node of a clause commodity with the respective clause edge type five edges.

We set the latency of the type one edges to 50, of the type two edges to 100, of the type three edges to 0, of the type four edges to 20, and of the type five edges to 40.

We claim that the total cost of an optimal solution to CNDP lies in the range

\[
200\kappa + |\tilde{K}|/4, (200 + \epsilon)\kappa + (1/4 + \epsilon/2)|\tilde{K}|
\]

if exactly \( |\tilde{K}| \) clauses cannot be satisfied.

To see the upper bound in (A.9), fix an assignment of the variables that satisfies \( \kappa - |\tilde{K}| \) clauses and construct a solution to CNDP analogously to the proof of the directed case, i.e., route all clause commodities along the clause edges, all variable edges along the negation of the assignment of the variable and choose the installed capacities as in the proof of Theorem 3.2. We will show that with these capacities the constructed flow is a Wardrop equilibrium. Since the auxiliary edges have non-zero latency, compared to the solution in the directed case, the latency cost of each clause commodity increased by 40 and the latency cost of each variable commodity increased by 200. Thus, the total cost increased by \( 40\kappa + \frac{3}{4} \cdot 200\kappa = 190\kappa \) giving a total cost of \( (200 + \epsilon)\kappa + (1/4 + \epsilon/2)|\tilde{K}| \). It is left to argue that this solution still constitutes a Wardrop equilibrium.
although all edges can now be used in both directions. To this end, note that each clause commodity uses its clause edge and experiences a total latency of $40 + 4 + \varepsilon/2 = 44 + \varepsilon/2$. However, each other path available to a clause commodity uses either a type two edge (with latency 100), or two type three edges, two type four edges (each with latency 20), and the three corresponding literal edges (with latencies summing up to $4 + \varepsilon/2$, as before). Thus, no clause commodity wants to deviate to another path and the constructed solution is a Wardrop equilibrium analogously to the directed case.

For the lower bound, we argue as follows. If no variable commodity uses a type three edge or a type four edge, then each variable commodity has to split its flow between the path corresponding to the positive and the negative literal, respectively, and the lower bound can be proven analogously to the directed case.

So we are left with cases that a variable commodity uses a type three edge or a type four edge. Let us first assume that we have an optimal solution, in which a variable commodity uses a type four edge. We may assume without loss of generality that every literal edge that carries flow has a latency of at most 5, because we could decrease the total cost by increasing the capacity on these edges, otherwise. (However, we may not decrease the latency below $4 + \varepsilon/2$ because this might give an incentive to the clause commodities to use these edges as well.) Every path available to a variable commodity uses at least two type one edges as these edges are adjacent to the source and target of each variable commodity. It is also not hard to see that every path available to a variable commodity has to use at least either two additional type one edges or one type two edge. Using that the variable commodity also uses a type four edge, this implies that the latency of the variable commodity is at least 200 + 20. However, it would also be feasible to route that variable along the path corresponding to the positive literal say while installing an additional capacity of 1/5 on the two literal edges of the positive literal resulting in a total cost of $200 + 10 + 2/5 < 220$. This low capacity would not prevent any of the clause commodities from using their clause edge and has a lower total cost. Thus, we may conclude that no variable commodity uses a type four edge. As any path of a variable commodity that uses a type three edge also uses a type five edge with latency 40, we may conclude that no variable commodity uses such an edge as well.

\textbf{Proof of Proposition 3.3}

\textit{Proof.} We solve the relaxed problem (CNDP$^*$). As in the proof of Proposition 2.4, for each edge $e \in E$, we find a solution to the equation $x^2 S'_e(x) = \ell_e$, which we denote by $u_e$. Then, we find an unsplittable flow that minimizes

$$
\min_{v \in F} \sum_{e \in E} (S_e(u_e) + \ell_e/u_e)v_e,
$$

(A.10)

Let $T$ be a shortest path tree routed in $t$ w.r.t. the edge weights $w_e = S_e(u_e) + \ell_e/u_e$. By construction, each commodity $k$ has a unique path in $T$ that connects the source $s_k$ to the joint sink $t$. For each $e \in T$, let $d_e$ be the sum of the demands of the commodities that use edge $e$ in $T$ along its path. For each edge $e \in T$ we buy capacity $z_e = d_e/u_e$ and route a flow of $d_e$. All other edges have zero capacity and, thus, infinite latency. By construction, the total cost of this solution equals (A.10). Also, the resulting flow is a Wardrop equilibrium as every commodity $k$ has a unique path from $s_k$ to $t$ that uses only edges with non-zero capacity.

\textbf{Proof of Lemma 4.1}

\textit{Proof.} The expression $\sup_{x \geq 0} \max_{\gamma \in [0,1]} \gamma \left(1 - \frac{S(\gamma x)}{S(x)}\right)$ is non-negative and strictly positive for $\gamma \in (0, 1)$, thus, the inner maximum is attained for $\gamma \in (0, 1)$. Hence, $\gamma$ satisfies the first order optimality conditions

$$
0 = \left(1 - \frac{S(\gamma x)}{S(x)}\right) - \gamma x \cdot \frac{S'(\gamma x)}{S(x)}
$$

$$
\iff
S(x) = S(\gamma x) + \gamma x S'(\gamma x)
$$

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By substituting $y = \gamma x$, we obtain
\[
\sup_{x \geq 0} \max_{\gamma \in [0, 1]} \gamma \left(1 - \frac{S(\gamma x)}{S(x)}\right) = \sup_{y \geq 0} \left\{ \gamma \left(1 - \frac{S(y)}{S(\gamma y)}\right) : \gamma \in [0, 1] \text{ with } S(y/\gamma) = S(y) + S'(y)y \right\}
\]
\[
= \sup_{y \geq 0} \left\{ \gamma \cdot \frac{S'(y)y}{S(y) + S'(y)y} : \gamma \in [0, 1] \text{ with } S(y/\gamma) = S(y) + S'(y)y \right\},
\]
which proves the lemma.

**Additional material for the proof of Theorem 4.5**

**Lemma A.3.** For all $\gamma, \mu \in (0, 1]$, we have
\[
\max_{p \in (0, 1)} \min \left\{ 1 + \gamma (1 - p), \left(\sqrt{p} + \sqrt{\mu(1 - p)}\right)^2 \right\} = \frac{(\gamma + \mu + 1)^2}{(\gamma + \mu + 1)^2 - 4\mu\gamma} < 1 + \mu. \quad (A.11)
\]

**Proof.** Observe that $1 + \gamma (1 - p)$ is decreasing in $p$. Elementary calculus shows that $\left(\sqrt{p} + \sqrt{\mu(1 - p)}\right)^2$ attains its maximum at $p = \hat{p} := \frac{1}{1 + \mu}$, is increasing when $p < \hat{p}$ and decreasing afterwards. Now, $\left(\sqrt{\hat{p}} + \sqrt{\mu(1 - \hat{p})}\right)^2 = 1 + \mu$ and $1 + \gamma (1 - \hat{p}) = 1 + \mu \frac{\gamma}{1 + \mu} < 1 + \mu$, the inequality in (A.11) follows.

Moreover, it follows that the maximum on the left hand side of (A.11) is attained for the unique $p^* \in (0, \hat{p})$ such that $1 + \gamma (1 - p^*) = \left(\sqrt{p^*} + \sqrt{\mu(1 - p^*)}\right)^2$. Thus, $p^*$ is a solution to the equation
\[
0 = -(1 - p^*) - \gamma (1 - p^*) + 2\sqrt{p^*(1 - p^*)} \mu + \mu(1 - p^*)
\]
\[
= (1 - p^*)\left(2\sqrt{\frac{p^*}{1 - p^*}} + \mu - \gamma - 1\right)
\]
and since $p^* < 1$
\[
0 = 2\sqrt{\frac{p^*}{1 - p^*}} + \mu - \gamma(S) - 1.
\]

The unique solution to this equation is
\[
p^* = \frac{(\gamma - \mu + 1)^2}{(\gamma - \mu + 1)^2 + 4\mu}.
\]

Plugging this into the left hand side of (A.11) gives
\[
\frac{(\gamma + \mu + 1)^2}{(\gamma + \mu + 1)^2 - 4\mu\gamma},
\]
which proves the identity in (A.11).

**Proof of Corollary 4.4 and Corollary 4.6**

**Proof.** For the proofs of Corollary 4.4 and Corollary 4.6 we give bounds on $\mu(S)$ and $\gamma(S)$ for the respective sets $S$ of allowable latency functions. Theorem 4.2, Theorem 4.3 and Theorem 4.5 then give the respective approximation guarantees.
**Arbitrary latency functions.** First, we consider case (a) of both Corollaries, where $S$ is a class of arbitrary non-negative and non-decreasing latencies. We observe that

$$\mu(S) = \sup_{S \in S} \sup_{a \geq 0} \max_{\gamma \in [0,1]} \gamma \left( 1 - \frac{S(\gamma x)}{S(x)} \right) \leq 1.$$  

By definition $\gamma(S) \leq 1$. Now Corollary 4.4(a) follows immediately and Corollary 4.6(b) follows from the fact that

$$\frac{(\gamma(S) + \mu(S) + 1)^2}{(\gamma(S) + \mu(S) + 1)^2 - 4\mu(S)\gamma(S)}$$

is strictly increasing in $\gamma(S)$ and $\mu(S)$.

**Concave latency functions.** Next, consider case (b) of both Corollaries, where $S$ contains concave latencies only. Observe that

$$\mu(S) = \sup_{S \in S} \sup_{a \geq 0} \max_{\gamma \in [0,1]} \gamma \left( 1 - \frac{S(\gamma x)}{S(x)} \right) \leq \sup_{S \in S} \sup_{a \geq 0} \max_{\gamma \in [0,1]} \gamma \left( 1 - \frac{1 - (1 - \gamma)S(0)}{S(x)} \right) \leq \max_{\gamma \in [0,1]} (1 - \gamma) = 1/4,$$

where the first inequality uses the concavity of all functions $S \in S$. Further, as shown in Lemma 4.1, the $\gamma$ for which the inner maximum is attained, satisfies the first order optimality conditions $S(x) = S(\gamma x) + \gamma x S'(\gamma x)$. As $S$ is concave, we derive that $\gamma x S'(\gamma x) \leq S(\gamma x)$, which implies

$$S(x) \geq 2S(\gamma x) \geq 2(\gamma S(x) + (1 - \gamma)S(0)) \geq 2\gamma S(x),$$

and, thus, $\gamma(S) \leq 1/2$. Again, Corollary 4.4(b) follows immediately and Corollary 4.6(b) follows from the fact that (A.12) is increasing in $\gamma(S)$ and $\mu(S)$.

**Polynomial latency functions.** Finally, consider case (c) of both Corollaries, where for some fixed maximal degree $\Delta \geq 0$, the set $S$ contains only polynomial latency functions of type $S(x) = \sum_{j=0}^{\Delta} a_j x^j$, with $a_j \geq 0$ for all $j$. Denote $a = (a_j)_{j \in [0,\Delta]}$. We calculate

$$\mu(S) = \sup_{S \in S} \sup_{a \geq 0} \max_{\gamma \in [0,1]} \gamma \left( 1 - \frac{\sum_{j=0}^{\Delta} a_j \gamma^j x^j}{\sum_{j=0}^{\Delta} a_j x^j} \right)$$

$$= \sup_{a \geq 0} \max_{\gamma \in [0,1]} \gamma \left( \frac{\sum_{j=0}^{\Delta} a_j x^j(1 - \gamma^j)}{\sum_{j=0}^{\Delta} a_j x^j} \right)$$

As $(1 - \gamma^j)$ is increasing in $j$ for every $\gamma \in (0,1)$, it follows that the supremum over $a \geq 0$ is attained if $a_{\Delta} > 0$ and $a_j = 0$ for all $j \in [0,\Delta - 1]$. We get

$$\mu(S) = \max_{\gamma \in [0,1]} \gamma (1 - \gamma^\Delta)$$

$$= \left( \frac{1}{\Delta + 1} \right)^{1/\Delta} \left( 1 - \frac{1}{\Delta + 1} \right)$$

$$= \left( \frac{1}{\Delta + 1} \right)^{1/\Delta} \left( \frac{\Delta}{\Delta + 1} \right).$$
which directly implies the statement of Corollary 4.4(c). Further, this value is attained for $\gamma(S) = \left(\frac{1}{\Delta + 1}\right)^{1/p}$. Plugging these values in (A.12) and rearranging terms, we obtain the approximation guarantee claimed in Corollary 4.6(c).

\[
\square
\]

**Convex budget constraints**

**Theorem A.4.** Let $S$ be a class of latency functions.

1. The following algorithm is a $\frac{1}{1-\mu(S)}$-approximation for $(\mathcal{S})$NDP
   (in particular a $4/3$-approximation for affine latencies):
   
   (a) Compute a solution $(v^*, z^*)$ to relaxation $\mathcal{(S)}$NDP.
   
   (b) Compute a Wardrop equilibrium $w \in \mathcal{W}(z^*)$.
   
   (c) Return $(w, z^*)$.

2. For affine latencies, there is no polynomial time approximation algorithm with a performance guarantee better than $4/3 - \epsilon$ for any $\epsilon > 0$, unless $\mathcal{P} = \mathcal{NP}$.

**Proof.** The upper bound in 1. is straight forward by using well known price of anarchy results known in the literature, cf. Correa et al. [9] and Roughgarden [29] and Roughgarden and Tardos [27]. For 2., we mimic the construction put forward in Roughgarden [26].

We reduce from the 2-Directed-Vertex-Disjoint-Paths $(\mathcal{DDP})$ problem, which is strongly NP-complete. Given a directed graph $G = (V, E)$ and two node pairs $(s_1, t_1)$, $(s_2, t_2)$ the problem is to decide whether there exist a pair of vertex-disjoint paths $P_1$ and $P_2$, where $P_1$ and $P_2$ are $(s_1, t_1)$ and $(s_2, t_2)$-paths, respectively.

We will show that a $(\frac{4}{3} - \epsilon)$-approximation algorithm can be used to differentiate between “Yes” and “No” instances of $(\mathcal{DDP})$ in polynomial time. Given an instance $\mathcal{I}$ of $(\mathcal{DDP})$ we construct a graph $G'$ by adding a super source $s$ and a super sink $t$ to the network. We connect $s$ to $s_1$ and $s_2$ and $t_1$ and $t_2$ to $t$, respectively. The latency functions of the added edges are set to $S_e(v_e/z_e) = v_e/z_e$ for $e \in \{(s, s_1), (t_2, t)\}$ and $S_e(v_e/z_e) = 1 + v_e/z_e$ for $e \in \{(s, s_2), (t_1, t)\}$. The function $g(z)$ assigns edge-specific budgets according to $B_{(s, s_1)} = 1$ and $B_{(t_2, t)} = 1$. The per-unit cost of capacities are given by $\ell_e = 1$ for $e \in \{(s, s_1), (t_2, t)\}$ and $\ell_e = 0$, otherwise.

We proceed to prove the following two statements:

1. If $\mathcal{I}$ is a “Yes” instance of $(\mathcal{DDP})$, then $G'$ admits a solution $(v, z)$ with $v \in \mathcal{W}(z)$ satisfying $C(v, z) \leq 3/2$.

2. If $\mathcal{I}$ is a “No” instance of $(\mathcal{DDP})$, then $C(v, z) \geq 2$ for all $(v, z)$ with $v \in \mathcal{W}(z)$.

To see the first statement, suppose $\mathcal{I}$ is a “Yes” instance and let $P_1$ and $P_2$ be the respective disjoint paths. For all edges contained in neither $P_1$ nor $P_2$, we install a capacity of 0 leading to infinite latency of these edges. For the edges in $P_1 \cup P_2 \cup \{(s_2, s_1), (1, t)\}$ we buy infinite capacity resulting in 0 latency on edges in $P_1 \cup P_2$ and a latency of 1 on $\{(s, s_2), (t_1, t)\}$. For the edges in $\{(s, s_1), (t_2, t)\}$ we spend the budgets of 1 each. Then, splitting the flow evenly along these paths yields a Wardrop flow with routing cost $C(z, v) = 2 \cdot ((1/2)^2 + 1/2 \cdot 1) = 3/2$.

To show the second statement, let $(v, z)$ be an optimal solution. We may assume that there is an $(s, t)$ path. We consider the following cases.

1. For exactly one $i \in \{1, 2\}$, all flow-carrying paths contain the edges $(s, s_i)$ and $(t_i, t)$. For this case it is easy to see that $C(v, z) \geq 2$ since all 4 new edges have at least latency of 1 if used with 1 unit of flow.
2. There is a flow-carrying path \( P \) containing \((s, s_2)\) and \((t_1, t)\). In this case, the latency along this path is at least 2, hence, since every flow-carrying path has the same latency, we obtain \( C(z, v) \geq 2 \).

3. There is a flow-carrying path \( P \) containing \((s, s_1)\) and \((t_2, t)\). If all flow-carrying paths from \( s \) to \( t \) contain \((s, s_1)\) and \((t_2, t)\), we obtain \( C(z, v) \geq 2 \) using the budget constraints at \\{ \((s, s_1), (t_2, t)\) \}. Suppose there is another flow-carrying path \( Q \) containing \((s, s_1)\) and \((t_1, t)\). Then the latency on the subpath \( Q[s_1, t] \) must be at least 1 and, by the Wardrop conditions, the latency of \( P[s_1, t] \) must be at least one. If the entire demand uses edge \((s, s_1)\), the minimum possible latency on this edge is 1 and the latency of \( P \) (and also \( Q \)) must be at least two, thus, we obtain \( C(z, v) \geq 2 \). Suppose, there is a flow-carrying path \( R \) containing the edge \((s, s_2)\). If \( R \) contains edge \((t_1, t)\), we are in case 2. Thus we may assume that \( R \) contains edge \((t_2, t)\). Since we are in a “No” instance of 2DDP, the path \( R \) must have one vertex with the path \( Q \) in common which implies that for \( R[s_2, t] \) the latency is at least 1 and, hence, the latency of \( R \) is at least 2 giving \( C(v, z) \geq 2 \).

4. The case that we have two flow-carrying \((s_1, t_1)\) and \((s_2, t_2)\) paths reduces to one of the cases 1., 2. or 3. since we are in a “No” instance of 2DDP.