THREE-DIMENSIONAL IROSHNIKOV-KRAICHNAN TURBULENCE IN A MEAN MAGNETIC FIELD

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(Dated: December 13, 2013)

Abstract

As shown by Müller et al. (2003), the 1D energy spectra deduced from structure functions in simulations of MHD turbulence with mean field \( B_0 \) show varying scalings, passing from \( k^{-5/3}_\perp \) to \( k^{-3/2}_\perp \) when \( B_0 \) increases, while the parallel scaling remains \( k^{-2}_\parallel \). To explain this, Boldyrev (2005, 2006) proposed, as an alternative to the Iroshnikov-Kraichnan (IK) scenario which predicts an isotropic \( k^{-3/2} \) scaling, a scenario with a strong perpendicular \( k^{-3/2} \) cascade constrained by small-scale dynamic alignment, and critical balance controlling the parallel extent of the spectrum (Goldreich and Sridhar 1995). Since then however, a more complete analysis of the 3D energy spectra by Grappin and Müller (2010) showed that the spectral anisotropy is in fact in between the two predictions, with a scaling as \( k^{-3/2} \) independent of the direction with respect to the mean field.

As this isotropic scaling is in contradiction with the 1D parallel scaling \( k^{-2}_\parallel \) deduced from structure functions, several authors proposed that the isotropic scaling is the mere consequence of measuring the energy spectrum in a frame of reference defined by the global mean field \( B_0 \) while the structure function scaling, being measured in the local frame, reflects the true anisotropic dynamics. We reanalyze the data and show here that this is not so: the structure function scalings actually yield the same results, whether measured in the local or global frame, as soon as \( B_0 / b_{\text{rms}} \gtrsim 5 \). We then show that the 1D parallel scaling \( k^{-2}_\parallel \) is generated by the scales outside the inertial range, and thus should disappear at larger Reynolds number.

To explain the scaling isotropy of the 3D spectrum, as well as the large oblique extent of the spectrum well outside the critical balance domain, we propose here a new theory, involving the ricochet process that generates oblique cascades coupled to a 2D IK perpendicular cascade. We thus conclude that the turbulent regime studied is a 3D extension of the 2D IK regime. We conjecture that the bifurcation between this regime and the regime with scale-dependent anisotropy ruled by the critical balance is controlled by the correlation time of the large-scale forcing.

Subject headings: Magnetohydrodynamics (MHD) — plasmas — turbulence — solar wind

1. Introduction

Plasma turbulence in a mean magnetic field corresponds to a rather ubiquitous astrophysical setting. Understanding its nonlinear dynamics which gives rise to measurable two-point statistics such as the energy spectrum is thus highly important. The current state of affairs with regard to the simplified incompressible magnetohydrodynamic approximation may be summarized as follows. MHD turbulence with strong mean field \( B_0 \) mainly forms gradients in directions perpendicular to the mean field, as the nonlinear cascade is based on interactions of Alfvén waves propagating in opposite directions along the mean field. These interactions are resonant, and thus most efficient perpendicular to the mean field (Montgomery and Turner 1981, Shebalin et al. 1983, Grappin 1986). On the contrary, couplings involving non-resonant wave vectors (mostly those with large components parallel to the mean field) are prone to be severely weakened by rapid oscillations. This situation has led to the development of several theories of turbulent MHD. They describe how the main perpendicular cascade extends (in a limited way) into the direction parallel to the mean field \( B_0 \). The limited parallel extension is described by the so-called critical balance between the perpendicular nonlin-
ear time and the parallel Alfvén propagation time, and leads to a spectral anisotropy growing with wavenumber (Goldreich and Sridhar 1995) (GS).

However, in the limit where gradients along the mean field are completely absent, that is, in pure 2D MHD turbulence, the weak coupling theory of Iroshnikov (1963) and Kraichnan (1965) (IK) has been found to hold (Grappin et al. 1983; Pouquet et al. 1988; Biskamp and Welter 1989; Ng et al. 2003), with the local mean field within the perpendicular plane playing the role of the (absent) global mean field.

In this paper, we will demonstrate that 3D MHD turbulence can show a state intermediate between the full isotropy of the 2D IK regime and a strongly anisotropic regime following the critical balance. This state actually reduces to the 2D IK regime when the mean field goes to infinity.

In a study of MHD turbulence with the mean magnetic field increasing from $B_0$ from 0 to 10 $b_{rms}$, Müller et al. (2003) have shown by using structure functions (SF) that the perpendicular 1D energy spectral scaling passes progressively from $k^{-5/3}$ to $k^{-3/2}$, with the parallel spectrum keeping a $k^{-2}$ scaling in the direction parallel to the local mean field. This indicates that the nature of the perpendicular couplings changes when increasing the mean field. To describe these results, Boldyrev (2005, 2006) proposed a new theory of strong coupling, based on small-scale dynamic alignment (SSDA) of velocity and magnetic field fluctuations increasing at small scales, resulting in the $k_{\perp}^{-3/2}$ scaling. This theory also predicts the $k_{\perp}^{-2}$ scaling, and so apparently predicted all the features found in the simulations of Müller et al. (2003).

In studies of the solar wind, much attention has been paid to turbulent scaling, dissipation and anisotropy. The anisotropy of the turbulent MHD spectrum is an important issue in understanding in particular the solar wind turbulent dynamics, as anisotropy determines the physics at the small dispersive scales, as well as the kinetic dissipative processes (e.g., Alexandrova et al. (2012); Howes et al. (2011)). While solar wind heating studies give a slight advantage to the IK regime (Ng et al. 2010), direct studies of the spectra indicate an anisotropy close to the prediction of critical balance (e.g., Horbury et al. 2008). However, other works report a quasi-isotropy (Tessier et al. 2009), and a recent study by Turner et al. 2012 shows that an isotropic scaling could hold as well. The new turbulent regime we propose here also shows isotropic scaling as in the latter work.

Since then, Grappin and Müller (2010) (GM10) have shown by re-analyzing the 3D spectrum $E_3(k_3,k_{\perp})$ in turbulence with mean field (mainly $B_0 = 5b_{rms}$), at higher resolution than in Müller et al. 2003 that the spectral anisotropy is scale-independent, thus much weaker than predicted by theories assuming critical balance. More precisely, these authors analyzed the scaling along radial directions in Fourier space, varying the angle $\theta$ with respect to the mean field. They found a 3D spectrum of the form:

$$E_3(k, \theta) = A(\theta)k^{-m-2}$$

with a scaling index $m = 3/2$, independent of the angle $\theta$. As these results seem to contradict the basic scaling anisotropy contained in the $k_{\perp}^{-2}$ scaling found via SF, several authors (Beresnyak 2012; Chen et al. 2011) proposed that the isotropic scaling shown by the 3D spectrum in our data should be the mere consequence of measuring spectra in the absolute frame attached to the global mean field $B_0$, while the true anisotropic dynamics could be revealed only by using the SF, the only tool able to cope directly with the detailed dynamics attached to the local mean field.

We show in this paper that this is not so, by reanalyzing the data of the simulations considered in GM10. We find that the spectral anisotropy measured using SF yields the same result independently of using the global or local magnetic field as a reference direction when $B_0/b_{rms}$ is sufficiently large (5 in the present work). Moreover, we show that this is not contradictory to the isotropic spectral scaling, since the $k_{\perp}^{-2}$ scaling of the SF originates from spectral intervals above and below the inertial range: the $k_{\perp}^{-2}$ disappears if we filter out these scales, while the power-law range of the 3D spectrum does not change.

To provide the necessary energy flux outside the critical balance region, we propose a mechanism for oblique cascades – the ricochet process. It is based on strong coupling but increasing selection of wavevector triads to preserve quasi-resonance and generates the radial $k_{\perp}^{-3/2}$ scaling. The cascade makes use of a preexisting perpendicular cascade, which a priori can be either the strong SSDA cascade, or the weak 2D IK cascade.

Based on an earlier theoretical investigation of the dynamical balance of kinetic and magnetic energy in MHD turbulence (Grappin et al. 1983; Müller and Grappin 2005), a scaling prediction for the residual energy spectrum (difference of magnetic and kinetic energy spectra) can be made, which has been found to be fulfilled in the present case, as well. Specifically, an inertial-range scaling of total energy $\sim k_{\perp}^{-1/2}$ and of the residual energy $\sim k_{\perp}^{-2}$ as in two-dimensional MHD turbulence (Biskamp and Welter 1989; Biskamp 1995). This indicates that Alfvénic coupling (on the Alfvén timescale) is of comparable strength or stronger even in the present case as nonlinear scrambling (on the nonlinear timescale) and is, thus, a strong indication in favour of the perpendicular cascade being the weak IK cascade.

The plan is as follows. In section 2, we recall the existing phenomenologies. In section 3, we reanalyze the properties of the numerical simulations studied in Grappin and Müller (2010). In section 4, we describe the ricochet process generating the oblique cascade. The last section is a discussion.

2. EQUATIONS AND PHENOMENOLOGIES

The MHD equations read, written in terms of the so-called Elsasser variables $\zeta^\pm = u \mp b$:

$$\partial_t \zeta^\pm \pm (B_0 \cdot \nabla) \zeta^\pm + (\zeta^\pm \cdot \nabla) \zeta^\pm + \nabla P = 0$$

(2)

with $\nabla \cdot \zeta^\pm = 0$, and where we leave the diffusive terms aside. After taking the Fourier transform, this leads to

$$\partial_k \tilde{\zeta}^\pm(k) + ik_{\parallel} B_0 \tilde{\zeta}^\pm(k) + iP_{\parallel\parallel}(k) \tilde{\zeta}^\pm(k) \tilde{\zeta}^\mp(k) = 0$$

(3)
where \( k_\parallel \) denotes the projection of the wave vector on the mean field \( B_0 \) and \( P_{ij}(k) \) is the usual projection operator \( P_{ij}(k) = k_j (\delta_{ij} - k_i k_j / k^2) \). We go on and rewrite the equations using the Heisenberg representation for the unknown amplitudes: \( \hat{z}^\pm(k) = u^\pm(k) e^{\pm ik \cdot B_0 t} \). After rearranging the different factors \( e^{\pm i(k_\parallel \cdot \eta_{ij}(\eta))] B_0 t} \), taking into account the condition \( k = p + q \) we obtain

\[
\partial_t \hat{u}_k^\pm(k) + i P_{ij}(k) \int d^3q \, \hat{u}_p^\pm(p) \hat{u}_q^\mp(q) e^{i k \cdot q B_0 t} = 0 \tag{4}
\]

where in the integrand \( p \) stands for \( k - q \), i.e., the triadic relation \( k = p + q \) is always respected. (See Grappin (1986) for a discussion of the equation in the 2D case).

This equation for the wave amplitudes \( \hat{u}_k^\pm \) has the merit of separating clearly the nonlinear coupling from the linear coupling due to the mean field \( B_0 \); while the former redistributes amplitudes all along the spectrum (i.e., coupling two wave vectors \( p \) and \( q \), so changing the amplitude of the wave vector \( k \)), the latter decreases strongly the effect of the former, as soon as the oscillating term is dominant.

In the following, we will use the following simplified version of this equation, replacing the complicated kernel \( P_{ij}(k) \) by a dimensional factor \( k \) and forgetting the field components: \( \hat{u}_k^\pm \) as:

\[
\partial_t \hat{u}_k^\pm(k) \int d^3q \, \hat{u}_p^\pm(k) \hat{u}_q^\mp(q) e^{2 i q B_0 t} = 0 \tag{5}
\]

### 2.1. The Kolmogorov cascade

We give here a short derivation of the Kolmogorov scaling in the hydrodynamic case of the Euler equation \((u^+ = u^- = u, B_0 = 0)\), which will serve as a guide in the following. Denote by \( \ell \) a scale between the largest scale and dissipation scale, and \( u_\ell \) the typical value of the velocity associated to scales \( \sim \ell \). A correct definition is the r.m.s value of the velocity subject to bandpass filtering, say of an octave around the wavenumber \( k = 1/\ell \) (Frisch 1995). A working definition (Rose and Sulem 1978) is to take

\[
u_\ell^2/2 \simeq \int_{k/\sqrt{2}}^{k\sqrt{2}} E(k') dk' \simeq k E(k).\tag{6}
\]

where \( E(k) = 4\pi k^2 |\hat{u}_k|^2 \) is the 1D spectral energy density. The eddy turnover (or non-linear) time is

\[
t_\ell = \ell / u_\ell.\tag{7}
\]

It is the typical time for a structure of size \( \sim \ell \) to undergo a significant distortion due to the relative motion \( u_\ell \), and thus, as well for energy to be transferred from scales \( \sim \ell \) to smaller scales. The energy flux from scale \( \ell \) to smaller scales thus may be estimated as

\[
F_\ell \sim u_\ell^2 / t_\ell \sim u_\ell^3 / \ell.\tag{8}
\]

In the inertial range, dissipation may be neglected, and the kinetic energy being an inviscid invariant, the energy flux does not depend on scale \( \ell \). The constraint \( F_\ell = \text{const} = \epsilon \), leads to the velocity \( u_\ell \) scaling as

\[
u_\ell \sim \epsilon^{-1/3} \ell^{1/3}\tag{9}
\]

which implies, in view of eq. 4 that the energy spectrum scales as

\[
E(k) \sim k^{2/3} k^{-5/3}\tag{10}
\]

In the following, we will use systematically the wavenumber \( k \) as an index, instead of the scale \( \ell = 1/k \).

The main hypothesis at the basis of the Kolmogorov phenomenology is the localness hypothesis, which means that the convolution integral in eq. 4 mainly couples wavevectors close one to the other:

\[
|k| \sim |p| \sim |q|\tag{11}
\]

where the sign \( \sim \) must be interpreted as “equal within a factor two”. This localness hypothesis is required to express the convolution integral in terms of the sole wavenumber \( k \), or as well, the single scale \( \ell \), leading to the simple form in eq. 5 for the energy flux. It will be used repeatedly in the following. The localness hypothesis actually allows to derive the characteristic time scale in a straightforward way from the primitive equation. The primitive eq. 5 may be rewritten formally when \( B_0 = 0 \) as

\[
\partial_t u_k = ku_k u_k\tag{12}
\]

which leads again to the eq. 7 for the nonlinear time.

#### 2.2. MHD: weak isotropic (IK) cascade

Incompressible MHD has two invariants that are separately conserved by nonlinear interactions, namely the energy in each Elsasser mode \( E_{\pm} = \langle (u^\pm)^2 \rangle / 2 \). As a consequence, there are now two energy fluxes, one for each of the two Elsasser energies. As is visible in eq. 5, nonlinear couplings involve only crossed terms \( \pm u_{\pm} \), which implies that the nonlinear time of a given field relies on the other field’s amplitude:

\[
t_{\pm} = \ell / |u_{\pm}| = 1 / (k u_{\pm}).\tag{13}
\]

The energy fluxes thus read

\[
F_{k_{\pm}} = (u_{\pm})^2 / t_{\pm} = k (u_{\pm})^2 u_{\mp}^2.\tag{14}
\]

However, not all possible triads \((k,p,q)\) in eq. 5 contribute equally to strong coherent cascades corresponding to the flux expression in eq. 14 if the vector associated with wavenumber \( q \) is not directed perpendicular to \( B_0 \) (resonance) then the oscillating exponential factor in eq. 5 weakens the nonlinear couplings and, consequently, the resulting cascade. Thus, a strictly coherent cascade, i.e. one that is not impaired by non-resonant scrambling only exists in the field-perpendicular plane.

In a search of a description valid in most of the Fourier space, Iroshnikov (1963) and Kraichnan (1965) (IK) tacitly ignore the strong cascade restricted to the perpendicular \( k_z \sim 0 \) plane and focus on the rest of the Fourier space, thus assuming that the dominant cascade process is ruled by non-resonant interactions, with the linear Alfvén time being shorter than the nonlinear terms. In this weak, non-resonant regime, nonlinear couplings drive the cascade via uncorrelated small steps, each step taking a local Alfvén time estimated by the isotropized expression

\[
t_{A} = 1 / (k B_0).\tag{15}
\]

The energy cascade is assumed to result from many uncorrelated small steps which lead to a long time scale for
energy transfer
\[ t_k^\pm = t_\pm (t_{iso}/t_A^0) = B_0/(k(u^\mp)^2). \]  
This leads finally to the expressions for the fluxes
\[ F_k^\pm \sim (u^\pm)^2/t_k^\pm \sim k(u^\mp)^2(u^\mp)^2/B_0. \]  

In the particular case \(|u^+| \approx |u^-|\), the scale-invariance of the two fluxes leads to the IK scaling
\[ u^+ \approx u^- \sim k^{-1/4} \]  
which corresponds to the energy spectrum scaling as \(E(k) \sim k^{-3/2}\). More general solutions are possible, in which the scalings of \(u^+\) and \(u^-\) differ, satisfying the condition \(k(u^\pm)^2(u^\mp)^2 = constant\), i.e., with the sum of spectral slopes being equal to 3. Such scalings (the standard IK and as well the generalized \(E^\pm\) scalings) have been found in closure calculations and in 2D MHD numerical simulations [Grappin et al. 1983; Pouquet et al. 1988]. Note that in these 2D MHD simulations, there is no mean field: the rms magnetic field plays the role of the mean field \(B_0\), in particular in eq. (15), \(B_0\) is to be replaced by \(b_{rms}\):
\[ t_{iso}^A = (kb_{rms})^{-1} \]  

2.3. MHD: strong perpendicular cascade (1)

When a strong mean field is present, it has been proposed that, at variance with the previous theory, the cascade develops only in the perpendicular direction. There are two inertial ranges corresponding to two different perpendicular scalings in two successive inertial ranges, with a weak cascade followed by a strong one [Goldreich and Sridhar (1995) (GS), Ng and Bhattacharjee (1997); Galtier et al. (2000)].

At the larger scales, the Alfvén time prevents the perpendicular nonlinear couplings to proceed coherently so that, as in the IK phenomenology, one obtains a long time scale. However, the resulting spectral slope differs from the isotropic IK slope because the Alfvén time does not vary with \(k_\perp\) during this first cascade, i.e., it is based on the initial parallel scale \(t_0^0 = 1/k_{\parallel 0}\):
\[ t_0^A = t_{\parallel 0}/B_0 = (k_{\parallel 0}B_0)^{-1} \]  
Replacing the isotropized Alfvén time with eq. (20) in the expression of the energy transfer time (eq. [16]), one gets the scaling law \(u \sim k_\perp^{-1/2}\), and hence the spectral scaling \(E(k) \sim k_{\perp}^{-2}\).

At smaller scales, the weak cascade transforms into a strong one when the nonlinear time decreases so as to become comparable or smaller than the Alfvén time (eq. [20]). In this range, the cascade proceeds as with non magnetic field, i.e., the perpendicular scaling is the same as in Kolmogorov phenomenology: the energy spectrum scales as \(k_\perp^{-5/3}\).

In the GS picture the strong perpendicular cascade is not completely confined within the perpendicular plane at \(k_{\parallel 0} = 0\): the 3D spectrum widens in the parallel direction (\(k_{\parallel 0} \neq 0\)) during the cascade, which does however not correspond to a parallel cascade properly speaking, and so leads to no specific scaling when looking at the 3D spectrum in the parallel direction. The parallel width of the spectrum results from the “critical balance” (CB) between the correlation time of eddies with scale \(1/k_{\parallel 0}\) and the time taken by Alfvén waves generated by such eddies to travel a distance \(1/k_{\parallel 0}\):
\[ t_A = (k_{\parallel 0}B_0)^{-1} \]  
The correlation time, in the case of a strong Kolmogorov-like cascade with scaling \(u = u_0(k_{\perp 0}/k_{\perp 0})^{-1/3}\), is given by the nonlinear time \(1/(k_{\perp 0}u_0)\):
\[ t_{cor}(k_{\perp}) = 1/(k_{\perp}u_0) = 1/(k_{\perp 0}/k_{\perp 0})^{-2/3}. \]  
The critical balance between correlation time and Alfvén time then reads \(t_{cor} = 1/(k_B B_0)\) or (identifying \(b_{rms}\) with \(u_0\)):
\[ k_{\parallel 0}/k_{\perp 0} = (k_{\perp 0}/k_{\perp 0})^{2/3} (b_{rms}/B_0). \]  
The \(-5/3\) spectral index and the critical balance relation between perpendicular and parallel times has been checked in numerical simulations [Cho et al. (2002)], and most recently using a reduced MHD shell model by Verdini and Grappin (2012).

2.4. MHD: strong perpendicular cascade (2)

To explain the occurrence of the \(k^{-3/2}\) scaling of the energy spectrum in the numerical simulations by Müller and Grappin (2005) and the \(k^{-2}, k^{-3/2}\) scalings obtained via SF in Müller et al. (2003), Boldyrev (2005, 2006) proposed a variant of the strong perpendicular cascade of GS.

To obtain the \(k^{-3/2}\) spectral scaling, one starts with the expression of the energy flux for the strong perpendicular cascade in eq. (13) without the assumption that the two Elsasser species are comparable. Instead, we assume that the dynamics generates spontaneously different scaling laws for a dominant Elsasser species (say \(u^+\)) and for a subdominant species (say \(u^-\)). The idea (confirmed by the numerical simulations of Mason et al. 2006) is however (i) that the largest scales show no \(u^+/u^-\) imbalance (ii) that the imbalance increases with wavenumber \(k\) (iii) that the dominant species is with equal probability either \(u^+\) or \(u^-\). To obtain the required scaling \(u^+ = k^{-1/4}\), one must assume that \(u^-/u^+ \propto k^{-1/4}\). We then have from eq. (14):
\[ F^+ = k(u^+)^2u^- = k(u^+)^3k^{-1/4} = k^{3/4}(u^+)^3 \]  
which indeed leads to the required IK scaling \(u^+ \propto k^{-1/4}\) when assuming \(F^+ = constant\).

Several remarks are in order. First, the version just given differs from the original one. The original version is formulated in terms of the velocity-magnetic field alignment, and measures the scaling law of the angle \(\alpha\) (Mason et al. 2006): 
\[ \sin \alpha = <|\delta u(L) \times \delta B(L)| > / < |\delta u(L)||\delta B(L)| > \]  
where \(\delta B_i(L) = B_i(x+l) - B_i(x)\) (and a similar expression for \(\delta u\)), and the brackets denote spatial averaging on the position \(x\). The two versions (either using the angle \(\alpha\) or the \(u^-/u^+\) ratio) are actually equivalent, inasmuch as, assuming equipartition of velocity and magnetic energies \(|u| \approx |b|\) at small scales, one has \(1 - (u^-/u^+)^2 \propto \cos \alpha\,
which implies for small $u^-/u^+$:

$$\alpha \simeq u^-/u^+$$  \hspace{1cm} (26)

The scaling

$$\alpha \sim k^{-1/4}$$  \hspace{1cm} (27)

has been found to hold in 3D MHD simulations by [Mason et al. 2006]. In these simulations, one has $|u^+| \approx |u^-|$ at the largest scales, while $u^- \simeq u^+ k^{-1/4} \simeq k^{-1/2}$; in other words, the velocity and magnetic field fluctuations become completely aligned at small enough scales. We will call in the following this process “small scale dynamic alignment” (SSDA). This is to be compared with the regime of “large scale dynamic alignment” studied by [Grappin et al. 1983] and [Pouquet et al. 1988] in which, on the contrary, the alignment lies at large scales and goes to zero at small scales.

Actually, these different scalings for $u^+$ and $u^-$ are in contradiction with the basic strong (i.e., resonant) expressions for the energy flux (eq. 14). Indeed, the invariance of the two fluxes during the cascade implies a single solution: the two scalings must both be $u^+ \propto u^- \propto k^{-1/3}$ and nothing else [Lithwick and Goldreich 2003; Beresnyak and Lazarian 2008]. The only way out of this paradox consists in assuming that the dimensional expression for the minor flux $F^-$ in eq. 14 is not valid for the minor flux.

3. RE-ANALYZING THE GM10 REGIME

The turbulent regime reported by [Müller and Grappin 2005] has a mean field $B_0 = 5b_{rms}$ and an isotropic constant forcing on scales $1 \leq k \leq 2$ (more precisely, these scales are frozen). As analyzed by [Grappin and Müller 2010] (GM10), it shows spectral properties which are distinct from the GS spectrum (as the scaling is different from $k^{−5/3}$) and as well from the SSDA spectrum, as the scaling is here isotropic and extending in the parallel and oblique directions largely outside the critical balance domain. We revisit here the properties of this turbulent regime.

3.1. 3D structure

We show in fig. 1a radial cuts of the 3D spectrum, obtained after averaging over the azimuthal angle $\phi$ around the mean field $B_0$. The left panel presents $E_3(k, \theta)$ vs wavenumber $k$ for a series of angles $\theta = \angle(k, B_0)$, as in fig.3a of GM10. In the right panel, spectral cuts are shown again, now compensated by $k^{-3/2}$, to highlight deviations of the slope with respect to the average 1D $−3/2$ slope, the dotted curve indicating the $k^{-5/3−2}$ scaling. (Note the difference of $−2$ between the 3D and 1D spectral indices).

We also indicate for each angle the half dissipative wavenumber $k_d(\theta)/2$ by plotting a plus sign on the corresponding curve, for later use in Section 3.3. We choose marking $k_d(\theta)/2$ instead of $k_d$ in order to select a level representative of the power-law range and not of the beginning of the dissipative range. The dissipative wavenumber $k_d(\theta)$ is obtained as in GM10 by determining the maximum of each radial profile of the 3D spectrum weighted by $k^4$ at each angle $\theta$.

Determining the spectral slope requires discarding two wavenumber ranges: (i) the dissipative scales with $k > k_d/2$ and (ii) the largest scales close to the forced scales. The boundary of the latter varies from $k = 5$ at small angles up to $k = 8$ or 10 when close to perpendicular (see fig. 4 in GM10). In doing so, one obtains a radial slope which is in average close to $−3/2 − 2$ (eq. 1), except for quasi-parallel directions, with also some exceptions at larger angles (see the dotted line that indicates the $−5/3−2$ slope). Note that the slope can be determined with sufficient accuracy only for directions with angle larger than $\theta \simeq 14^\circ$; for smaller angles, the signal is too noisy while the interval left between dissipative scales and forced scales ($1 \leq k \leq 2$) becomes too small to allow evaluating a slope.

3.2. From local to global frame

Structure functions (SF) are another way to measure scaling laws in turbulent systems. We consider here second-order SF built on the magnetic field fluctuations:

$$SF(L) = \langle \delta B^2(L) \rangle.$$  \hspace{1cm} (28)

As previously said, the perpendicular and parallel SF exhibit clearcut differences in their scaling (Müller et al. 2003), which reflects the anisotropy expected when measuring scaling in the local frame attached to the local mean magnetic field. It has been argued [Beresnyak 2012; Chen et al. 2011] that the isotropy of the 3D spectral scaling (and as a consequence, that of the 1D scaling, see definition below, eqs. 29−30) is the sole consequence of measuring the spectrum in the global frame.

There are two ways to check the correctness of this criticism: (i) compare SF measured in the local and global frame (ii) compare spectra measured in the local and global frame. We first consider the case of SF.
Fig. 2.— Local and global second order structure functions SF vs parallel and perpendicular separations \( L \), for the magnetic field fluctuations, compensated by \( L^{1/2} \). Thick line: local SF, perpendicular (upper curve) and parallel (lower curve). Thin line: global SF, with the same ordering. Note that plateaux are in principle equivalent to reduced 1D spectra scaling as \( k^{-3/2} \). Dotted lines indicate the \( L^{2/3} \) and \( L \) scalings (in principle equivalent to, respectively, \( k^{-5/3} \) and \( k^{-2} \) 1D reduced spectral scaling).

3.2.1. Global vs Local SF

Fig. 2 shows the two SF pairs (parallel and perpendicular), the local SF with thick lines and the global SF with thin lines. All SF are compensated with \( L^{1/2} \) scaling, which correspond in principle to a \( k^{-3/2} \) scaling for a 1D spectrum (see eq. 92 below). The dotted lines show the \( L^{2/3} \) and \( L \) scalings, in principle equivalent to spectral scalings \((k_\parallel^{-5/3}, k_\perp^{-2})\).

As seen in the figure, the perpendicular SF scaling lies in between the Kolmogorov scaling \( L^{2/3} \) (or equivalently \( k^{-5/3} \)) and the Iroshnikov-Kraichnan scaling \( L^{1/2} \) (or equivalently \( k^{-3/2} \)). The parallel scaling is close to SF \( \propto L_\parallel \) (or equivalently a 1D reduced spectrum scaling as \( k_\parallel^{-2} \), as we will see again below). This confirms the earlier results by Muller et al. (2003), and at the same time shows that there is actually no large difference between measuring SF in the local and global frames.

This result suggests that the same could hold for the energy spectrum, more precisely that the scaling properties of the “local” and “global” spectrum could be the same.

3.2.2. Local vs global 3D spectrum

Contrary to the SF that can be measured in real space, which conveniently allows to use a frame locally attached to the local mean field, energy spectra are defined in Fourier space. Strictly speaking, there is nothing like a “local” Fourier spectrum. In spite of this, we can try to convince the reader at least that if something like a local spectrum exists, it cannot be of the standard form proposed by the phenomenologies mentioned in the previous section in which the only true cascade is a perpendicular one, with no true scaling in the non-perpendicular directions.

Let us try to define such a “local” 3D spectrum. A steep energy drop for \( k_\parallel \) larger than the critical balance boundary is assumed, such as given in eq. 23 or any modified version e.g. adapted to the SSDA phenomenology.

Fig. 3.— 3D energy spectrum \( E_3(k_\parallel, k_\perp) \) with \( b_{rms}/B_0 = 1/5 \) in three different cases. (a) Theoretical spectrum measured in the “local” frame (see text) corresponding to a perpendicular cascade with parallel extension given by “critical balance” (eq. 24). (b) Same spectrum as viewed from the global frame, obtained by randomly shuffling the spectrum (a) with an rms angle \( \simeq (2/\sqrt{3})b_{rms}/B_0 \) around the parallel direction, (see text). (c) gyrotropized GM10 spectrum (Grappin and Muller (2010)). Dashed line: eq. 23.

The energy contours in the \( k_\parallel, k_\perp \) plane are shown in fig. 3b.

The “global” 3D energy spectrum is now deduced from the local one just defined by randomly shuffling it around the \( k_\parallel \) direction, this process corresponding to the random shuffling necessary to obtain the global mean field from the local mean field. The shuffling angle should be of the order of \( b_{rms}/B_0 \approx 1/5 \), where \( B_0 \) is the true global mean field. The resulting isocontours are shown in the mid panel (b), while the true GM10 3D spectrum is shown in the right panel (c). Comparing now the “shuffled” 3D spectrum with the true one shows that the shuffling process cannot introduce the isotropy of the scaling which is lacking in the local spectrum and present in the true spectrum. This supports the idea that the local spectrum should already own this property.

3.3. 1D parallel and perpendicular structures, SF vs reduced energy spectra (SP)

In an attempt to understand the origin of the intriguing difference between the 3D isotropic spectral scaling and the anisotropic SF scaling, we now propose to use the 3D spectra studied in subsection 3.1 to modify these spectra and to compare the different SF scalings, as well as the 1D reduced spectral scalings, that can be deduced from these modifications. In order to do that, we remark that, when the frame is attached to the global field, the SF can be measured directly in real space but can also be computed from the 3D spectra. Since local and global SF coincide, in the following we will compute the SF directly from the spectra, thus neglecting the (small) differences between SF computed in the local and global frame.

The 1D reduced spectra (SP) vs parallel \((k_\parallel)\) and per-
perpendicular \( (k_\perp) \) wavenumbers are obtained by integrating the 3D spectra on planes perpendicular to the chosen direction:

\[
E(k_\parallel) = \int \int E_3(k_\parallel, k_y, k_z) dk_y dk_z
\]

and

\[
E(k_\perp) = \int \int E_3(k_x, k_\perp, k_z) dk_x dk_z
\]

SF and reduced spectra (both parallel and perpendicular) are related by the following simple relation:

\[
SF(L) = 2[E - \mathcal{F}^{-1} E(k)]
\]

where \( E \) is the total energy (here the magnetic energy) and \( \mathcal{F}^{-1} \) denotes the inverse Fourier transform with respect to the variable \( k \).

Now, we first compare the SF and SP scalings for the 3D GM10 spectrum, and then consider modifying the 3D GM10 spectrum in order to test the robustness of the anisotropy of the scalings when extrapolating what the anisotropy could become at larger Reynolds number.

We consider the magnetic energy only, which makes essentially no difference as far as scaling is concerned. As the scaling range in the 3D spectrum is radial, we consider as a starting point the gyrotropized 3D spectrum \( E_3(k, \theta) \). We then consider reduced SP and SF build on successively

M1: the original gyrotropized spectrum \( E_3(k, \theta) \).

M2: the gyrotropized spectrum with the dissipative tails \( k > k_d/2 \) in fig. 4 replaced by pure \( k^{-3/2-2} \) tails up to the maximum wavenumber.

M3: the gyrotropized spectrum with the whole spectral range replaced by a \( k^{-3/2-2} \) scaling, fixing the power anisotropy \( A(\theta) \), as measured at \( k = k_d/2 \).

The results are shown in fig. 4. First consider the SP and SF drawn from the original 3D gyrotropized GM10 spectrum (M1), in the two top panels of fig. 4. Note first that the perpendicular SF scaling range is more extended by a factor two towards the largest scales than previously in fig. 4 due to the gyrotropization which clearly smooths the largest scales.

One sees that in both the parallel and perpendicular directions the scaling ranges are more extended for the SF than for the SP, and thus more easily measured. The most striking difference lies in the parallel scaling, which for the SF is clearly \( SF \propto L_\parallel \), while it is not measurable for the SP. (Note that one would expect \( E_1(k_\parallel) \simeq SF(k_\parallel)/k_\parallel \sim k_\parallel^{-5/3} \). As for the perpendicular direction, the scaling is close to \( SP \sim k_\perp^{-3/2} \), while for the SF it is in between \( SF \sim L_\perp^{1/2} \) (which would correspond to the measured SP scaling) and \( L_\parallel^{2/3} \).

The curves built from the GM10 spectrum with modified spectral tails (M2) are shown in the mid panels. The SP scaling ranges now become clearly visible: \( SP \sim k^{-3/2} \) for both perpendicular and parallel spectra, while the perpendicular and parallel SF follow flatter scalings, close to \( L_\parallel^{1/2} \), in between \( L_\parallel \) and \( L_\parallel^{2/3} \). Translated in terms of reduced energy spectrum, this would mean a SP scaling in between \( k_\parallel^{-2} \) and \( k_\parallel^{-5/3} \).

Applying the modification (M3) which replaces the whole radial scaling by \( k^{-3/2-2} \) is spectacular. This introduces no visible change except at the very largest scales in the SP, as expected, while the SF are completely modified. The scalings are now comparable for both the parallel and the perpendicular SF, with a scaling in between \( L_\perp^{2/3} \) and \( L_\parallel^{1/2} \), i.e., equivalent spectral scalings between \( k^{-5/3} \) and \( k^{-3/2} \).

Some comments are in order. First, concerning SP, we can safely conclude that, in the present case of a 3D spectrum with quasi-isotropic (radial) scaling, the 1D spectra simply reflect the radial scaling: the absence of clear parallel scaling in the top panel of fig. 4 is a plain consequence of the Reynolds number begin too small in the
parallel direction, as conjectured in GM10.

A more difficult case is that of the SF. From the physical viewpoint one would expect that the SF and SP scalings would correspond, that is:

\[ SF(L) = \delta b^2(L) \sim kE(k) \]  

with \( k \simeq 1/L \). The point is that the (exact) relation between SP and SF (eq. 31) does not imply in general that eq. 32 is satisfied. One can only expect that the two scalings should indeed satisfy asymptotically eq. 32 when increasing enough the Reynolds number as has been done virtually with the case M3. In cases M1 and M2, the small power-law range is too close to the non-power law range, either the small or the large scales.

We have thus a good indication that the difference between the SF anisotropy and the 3D spectral isotropy is not a frame issue, but very probably a finite Reynolds number effect. If the isotropic \(-3/2 - 2\) slope of the 3D spectrum is robust when increasing the Reynolds number, then is should show finally up in both the parallel and perpendicular SP, and as well (at least to a good approximation) in the parallel and perpendicular SF too.

A last remark is in order: we have attempted, when extrapolating the radial 1D slope for the cases M2 and M3, to take into account the fluctuations in slope which are visible in fig. 1. This does not change substantially the curves in fig. 4. This does not change substantially the curves in fig. 4.

3.4. Small scale dynamical alignment

In order to determine what could be the correct cascade scenario at the origin of the measured perpendicular \( k^{-3/2} \) spectral scaling, either the IK or the SSDA scenario, we have examined how the angle \( \alpha \) between velocity and magnetic field fluctuations depends on scale \( L \). In the SSDA scenario, the \( k^{-3/2} \) energy spectral scaling (or, equivalently, the \( u \sim k^{-1/4} \) amplitude scaling) results from the angle scaling as \( \alpha \sim k^{-1/4} \) (eq. 27, see Mason et al. 2006).

Fig. 5 gives the measured \( \alpha \) vs \( 1/L \) averaged over several successive times, compensated by \( L^{1/4} \), separately for the vector \( \mathbf{L} \) being in a direction perpendicular to \( B_0 \) (thick line), or (thin line) the vector \( \mathbf{L} \) being in a direction parallel to the mean field. The resulting scaling is in both directions at best \( \alpha \sim L^{1/8} \). We thus conclude that the SSDA is not, alone, at the origin of the GM10 regime.

4. A NEW MODEL FOR 3D MHD TURBULENCE

We propose now a model that describes how a main perpendicular cascade is able to fill the Fourier space in all directions, i.e. to pass from the configuration of fig. 3a-b to that of fig. 3a.

4.1. Perpendicular cascade and critical balance extension

As the dominant energy lies in the perpendicular plane at \( k \parallel = 0 \), we first consider the cascade leading to the perpendicular scaling. We assume that the perpendicular cascade is the 2D IK cascade, controlled by \( b_{rms} \) evaluated in the perpendicular plane, namely with the characteristic Alfvén time \((k_\perp b_{rms})^{-1}\).

In the case with \(|z^+| \approx |z^-|\), the energy flux reads \( F = ku^4 \), and its invariance leads to the usual IK scaling.

![Fig. 5.— Scaling of angle \( \alpha \) between magnetic and velocity fluctuations between points separated by a distance \( L \) in the GM10 data, vs \( 1/L \). thick curve: the vector \( \mathbf{L} \) is perpendicular to the mean field direction; solid curve: the vector \( \mathbf{L} \) is parallel to the mean field direction. The profiles are compensated by \( L^{1/4} \). Dotted line indicates \( L^{1/8} \) scaling for reference.](image)

\[ u \simeq k^{-1/4}, \text{i.e., an energy spectrum as } E_k \simeq k^{-3/2}. \text{ Now, we have seen that the dominant and sub-dominant fields do not follow the same scaling law, and that instead: } \]

\[ u^-/u^+ \sim k^{-\beta} \]

with \( \beta \simeq 1/8 \). The perpendicular flux \( F^+ \) for the dominant field thus reads

\[ F^+ = q(u_q^+)^2(u_q^-)^2 \]

which implies, using eq. 33 to replace the sub-dominant field \( u^- \) by \( u^+ \):

\[ u_q^+ = q^{-1/4+\beta/2} \]

or as well \( E_q^+ = q^{-3/2+\beta} \). This yields a spectral slope of 1.375 when replacing \( \beta = 1/8 \).

Assuming seeds of fluctuations with small \( k \parallel \) are present, the spectrum resulting from the perpendicular cascade will have a limited parallel extent. This parallel extent is controlled by the critical balance between the Alfvén time computed on it and the correlation time, which is the shortest available characteristic time produced by the perpendicular cascade. While in the usual strong cascade, the correlation time is given by the nonlinear time, here, in contrast, it is given by the perpendicular Alfvén time, namely \( 1/(k_\perp b_{rms}) \). This leads to a parallel extent scaling as

\[ k_\parallel B_0 \simeq k_\perp b_{rms} \]

so leading to a 3D extension of the spectrum in Fourier space which is qualitatively similar to that in fig. 3.

The IK assumption for the 2D cascade is discussed in detail in the discussion.

4.2. Oblique cascade

As we have seen in fig. 3 the parallel extension of the spectrum resulting from critical balance alone cannot reproduce the isotropic scaling of the GM10 spectrum: scaling remains perpendicular. We propose here a scenario, which uses the weak IK perpendicular cascade as
a driver for the oblique cascade. This, as we will see, leads to complete isotropy of the scaling.

4.2.1. Quasi-resonant cascade

We start with the basic eq. 4 and retain only \((k, p, q)\) triads such that the coupling term is quasi-resonant. This means that the oscillating term \(e^{2iqB_0t}\) should oscillate in a time longer than the nonlinear time \(t_+\) of the main Elsasser species \(u^+\):

\[
q || B_0 \leq 1/t_+ \simeq q u_q^- \tag{37}
\]

The largest value of \(q||\) compatible with quasi-resonance is:

\[
q^{\text{max}} || \simeq q u_q^- / B_0 \tag{38}
\]

Note that vectors \(q\) satisfying eq. 38 have to be found inside the critical balance cone \(q|| = q_{\text{rms}} / B_0\) as \(u_q^- / b_{\text{rms}} \to 0\) when \(q\) increases. This means that the perpendicular cascade alone provides the required reservoir of quasi-perpendicular modes.

The formal evolution eq. 5 becomes now for the dominant field \(u^+\):

\[
\partial_t \tilde{u}^+ = k \int d^3q \tilde{u}^+_p \tilde{u}^-_q e^{\mp 2iqB_0t} \approx k \int_{QR} d^3q \tilde{u}^+_p \tilde{u}^-_q \tag{39}
\]

where the mention \(QR\) indicates that the sum deals with only the vectors \(q\) that satisfy eq. 38.

Using the localness hypothesis (eq. 11) allows to estimate how the nonlinear time scale is modified by the quasi-resonance constraint. While in the full strong cascade with \(B_0 = 0\) the inverse characteristic nonlinear time can be estimated as in eq. 12 \((t^+)^{-1} \sim k \int d^3q \tilde{u}^-_q \sim ku_q^-\), denoting by \(R\) the proportion of active triads remaining due to the quasi-resonance constraint, we obtain now

\[
(t^+)^{-1} \sim Rk \int d^3q \tilde{u}^-_q \sim Rku_q^- \tag{40}
\]

The reduction factor will be estimated below. We first describe in some detail the cascade process.

4.2.2. The ricochet process

To fill the rest of the \(k||, k_\perp\) plane outside the perpendicular domain (and its critical balance extension), we propose now a specific model, that uses quasi-resonant triads in the sense given just above, i.e., with the third wavevector \(q\) quasi-perpendicular.

Fig. 6 sketches the model. The left part shows two elementary triads, each with one quasi-perpendicular vector \(q\), and the right part shows in the \((k||, k_\perp)\) plane two oblique lines \(A_1, A_2\) passing through the origin and along which the excitation is able to propagate using the elementary triads of growing size, all with wavevectors \(q\) satisfying the quasi-resonance condition.

Note that for clarity one of our two oblique lines (\(A_1\)) is horizontal, but there is no real necessity for that: any pair of oblique lines (with an initial large-scale seed provided by the forcing) is suitable for propagating the excitation towards small scales, as soon as the angle between them is non vanishing (as this would correspond to a vanishing contribution to the convolution integral).

We call the process sketched in fig. 6 the “ricochet” process, since excitation propagates along the two lines by successive bounces from one line to the other, thanks to the critical balance reservoir of wavevectors \(q\).

Note that we don’t give any argument showing that energy is indeed flowing along these rays. Detailed tests that this is indeed occurring is outside the scope of this work. Strictly speaking, our arguments only indicate that this is possible. However, we remark that the triads considered have no particular properties and are quite representative of triads able to drive a cascade in general: the only selection criterion required is that (i) they are “local”, that is, \(k \approx p \approx q\) (ii) the \(q\) vector satisfies quasi-resonance.

As a consequence, we claim that the energy flux in our oblique cascade can be estimated as usual by dividing the energy \((u^+_k)^2\) at a given scale \(1/k\) by its characteristic time \(t^+\) present in the nonlinear kernel of eq. 39:

\[
F_{ob}^+ \simeq (u^+_k)^2 / t^+ \simeq k(u^+_k)^2 u_q^- \tag{41}
\]

where the suffix || now denotes the oblique cascade. This is indeed the expression corresponding to a strong cascade. Two remarks are in order. (i) We don’t consider different flux expression for different directions: we estimate globally the “oblique” flux, i.e. the flux from the large scales flowing in directions other than the perpendicular plane. (ii) The flux expression is not completely standard: reflecting the structure of the nonlinear kernel, it mixes the dominant “oblique” field \(z^+_k\) with the subdominant “perpendicular” field \(z^-_q\). (iii) Actually, eq. 41 overestimates the flux, as we see now.

4.2.3. Flux reduction

Due to the quasi-resonance constraint (eq. 38), a growing number of triads are eliminated compared to a normal strong cascade (with no mean field). At a fixed wavenumber \(k \approx q\), only triads with \(q|| / q \leq u_q^- / B_0\) contribute to the cascade along the parallel and oblique directions, while the triads contributing to the cascade in the ab-
sence of mean field are characterized by $0 < q_x \lesssim q$.

As sketched in fig. 7, the contributing subset of quasi-resonant triads is thus only a fraction of the total number of triads, of order $R$:

$$R \simeq q_{||}^{\text{max}}/q \simeq u_0^-/B_0$$  \hspace{1cm} (42)

This reduction of the number of triads contributing to the flux implies that we must change the algebraic expression (eq. 11) relating the flux and the field amplitudes.

The reduction factor $R$ has been previously found to modify the raise the nonlinear time (see eq. 10). Accordingly, the energy flux in the oblique direction found previously (eq. 11) is to be modified, being reduced by the factor $R$ as:

$$F_{ob}^+ \simeq k(u_0^+)^2 u_q^- R \sim k(u_0^+)^2(u_q^-)^2/B_0$$  \hspace{1cm} (43)

Comparing eqs. 43 and 45 shows that the resulting scaling laws are identical in the perpendicular and oblique directions.

The spectral extent, which is controlled by viscosity, provides a test of our phenomenology of coupled perpendicular and oblique cascades. Indeed, at the dissipative scale $1/k_d$, one should find equal transfer and dissipative times. This leads to two equalities close to one another, depending on whether we consider the parallel or perpendicular cascade:

$$\nu k^2 \simeq k(u_q^-)^2/B_0$$  \hspace{1cm} (44)

$$\nu k^2 \simeq k(u_q^-)^2/b_{rms}$$  \hspace{1cm} (45)

hence

$$k_d/\kappa = b_{rms}/B_0$$  \hspace{1cm} (46)

This relation has been found numerically by GM10.

5. CONCLUSION

We have discussed the turbulent regime with mean field studied by GM10. We have reexamined its properties and we obtained a good indication that the difference between the SF anisotropy and the 3D spectral isotropy is not a global/local frame issue, but very probably a finite Reynolds number effect. If the isotropic $-3/2 - 2$ slope of the 3D spectrum happens to be robust when increasing the Reynolds number, then this isotropic scaling should show finally up in both the parallel and perpendicular SF, and as well (at least to a good approximation) in the parallel and perpendicular SF too.

To explain this isotropic scaling of the 3D energy spectrum, we propose a scenario combining a main perpendicular and an oblique strong cascade coupled to the perpendicular one via the ricochet process and the quasi-resonance constraint. The quasi-resonance constraint leads to a depletion of the nonlinear coupling during the oblique cascade, which leads to the same scaling in the oblique as in the perpendicular direction, if, as assumed, the perpendicular cascade follows the weak (2D) IK regime. The perpendicular IK cascade is based on the local mean magnetic field with amplitude $\simeq b_{rms}$, while the strong oblique cascade is based on the global mean field $B_0$: this leads to the ratio of dissipative wave numbers in the parallel and perpendicular directions being equal to $b_{rms}/B_0$, as found in GM10.

We have measured the small scale dynamic alignment in the GM10 regime and found that $\alpha \sim u_0^-/u_0^+ \sim k^{-\beta} \sim k^{-1/8}$, i.e. a scaling law with a slope a factor of two smaller than that necessary to obtain the $3/2$ 1D spectral slope in the SSDA theory. When incorporating this scaling of the minor Elsasser species within the dominant energy flux according to the weak IK perpendicular flux, the predicted spectral scaling remains isotropic. The resulting 1D spectral scaling is $-3/2 + \beta = -3/2 + 1/8 \simeq -1.375$. We have thus obtained a three-dimensional generalization of the IK phenomenology which is able to explain the isotropic scaling found for the energy spectrum in the $k_{\perp} - k_{||}$-plane. However the spectral slope is smaller than the measured value of $-1.5$. We attribute this deviation to the difference of the measured total energy and the Elsasser energy $\sim (z^+)^2$ and to non-ideal scaling behavior of $u_0^-/u_0^+$.

The weak assumption for the perpendicular cascade is essential here, since otherwise the scenario fails to predict an isotropic scaling (the actual prediction in that case, using the same line of arguments and taking into account the Elsasser imbalance, leads to an oblique spectral index of $\simeq -1.17$). The justification for assuming a weak cascade is twofold. First, a weak cascade (IK scaling and generalization to Elsasser imbalance) has been found in past simulations by Pouquet et al. (1988); Biskamp and Welter (1989); Ng et al. (2003). Second, the GM10 simulations studied here have been shown to obey the Alfvén-nonlinear balance developed by Müller and Grappin (2005). It relies heavily on the relaxation of the kinetic and magnetic energy difference by linear Alfvén waves being active in the whole inertial range and being of comparable (or larger) influence compared to the nonlinear coupling terms. The balance has been shown to regulate the magnetic excess in the inertial range of MHD turbulence. Denoting by $E^M$ and $E^K$ the magnetic and kinetic spectra, the associated relation between the residual ($E^R = |E^M - E^K|$) and the total ($E_k = E_k^M + E_k^K$) energy spectra, namely $E^R \sim k E_k^2$, has been found to be exact in the present GM10 regime with $B_0 = 5b_{rms}$, for decaying MHD turbulence with $B_0 = 0$.  

![Fig. 7.](image-url)
for two-dimensional MHD turbulence [Biskamp 1995], and also in the case of the ideal MHD with \( B_0 = 5b_{rms} \).

The main characteristics of the two turbulent regimes, the perpendicular cascade controlled by critical balance (left panel) and the GM10 weak IK cascade extended by the quasi-resonance constraint are sketched in fig. 8.

An important point deals with what determines turbulence to follow the GM10 regime or other regimes of the critical balance family. A characteristics of the GM10 regime is that it obtains when the forcing is isotropic and constant. Imposing an isotropic forcing with large mean field implies imposing a region around the origin in Fourier space (actually within \( 1 \leq k \leq 5 \)) where the ratio \( \chi = t_d/t_{NL} \) of parallel Alfvén time to the non-linear time is smaller than unity, i.e., where couplings are decorrelated, hence turbulence becomes weak.

The theory of weak turbulence predicts that forcing turbulence with \( \chi < 1 \) should lead to a steep \( k_{⊥}^{-2} \) scaling for the 1D reduced spectrum, and no scaling at all in the parallel direction [Galtier et al. (2000), Ng et al. (2003)]. In published numerical works, steepening of the energy spectrum in the perpendicular direction is indeed observed as the weak domain in Fourier space (actually within \( 1 \leq k \leq 5 \)) where the ratio \( \chi = t_d/t_{NL} \) of parallel Alfvén time to the non-linear time is smaller than unity, i.e., where couplings are decorrelated, hence turbulence becomes weak.

In conclusion, we propose here a new scenario for turbulent cascade based on an extension to 3D of a 2D IK cascade. The scenario is able to explain several properties of the observed GM10 regime, namely an isotropic scaling close to a \(-1.5\) spectral slope, a perpendicular weak cascade independent of the \( B_0 \) amplitude, and a ratio between inertial parallel and perpendicular ranges of order \( b_{rms}/B_0 \). The system chooses between the weak anisotropic \( k_{∥}^{-2} \) cascade satisfying the critical balance and the present cascade with isotropic scaling, depending on how forcing perturbs or not the perpendicular cascade, via the forcing correlation time.

We thank G. Belmont, Y. Dong for several fruitful discussions. A. Verdiini acknowledges partial funding from the European Commission’s Seventh Framework Programme (FP7/2007-2013) under the grant agreement SHOCK (project number 284515) and from the Interuniversity Attraction Poles Programme initiated by the Belgian Science Policy Office (IAP P7/08 CHARM).

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Freezing the large scales may seem a rather artificial mean of injecting energy in a turbulent flow. However, Solar wind turbulence is probably a natural example of such forcing. Indeed, in the solar wind, the energy spectrum is made of several ranges, with an \( f^{-1} \) on the top of the inertial range [Bruno et al. 2009]. The largest scales are made of streams which are uncoupled from the mean magnetic field and may be considered as frozen-in scales controlling the evolution of smaller scales [Roberts 1992, Grappin and Velli 1994].

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