Arc Spaces and Chiral Symplectic Cores

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Abstract. We introduce the notion of chiral symplectic cores in a vertex Poisson variety, which can be viewed as analogs of symplectic leaves in Poisson varieties. As an application we show that any quasi-lisse vertex algebra is a quantization of the arc space of its associated variety, in the sense that its reduced singular support coincides with the arc space of its associated variety. We also show that the coordinate ring of the arc space of Slodowy slices is free over its vertex Poisson center, and the latter coincides with the vertex Poisson center of the coordinate ring of the arc space of the dual of the corresponding simple Lie algebra.

1. Introduction

Any vertex algebra is canonically filtered [Li], and hence can be viewed as a quantization of its associated graded vertex Poisson algebra. Since the structure of a vertex algebra is usually quite complicated, it is often very useful to reduce a problem of a vertex algebra to that of the geometry of the associated vertex Poisson scheme, that is, the spectrum of the associated graded vertex Poisson algebra (see e.g. [Fre, A3, A4]). Since a vertex Poisson scheme can be regarded as a chiral analogue of a Poisson scheme, it is natural to try to upgrade notions in Poisson geometry to the setting of vertex Poisson schemes. We note that the arc space $J_\infty X$ of an affine Poisson scheme $X$ is a basic example of vertex Poisson schemes ([A1]).

In [BG] Brown and Gordon introduced the notion of symplectic cores in a Poisson variety which is expected to be the finest possible algebraic stratification in which the Hamiltonian vector fields are tangent, and showed that the symplectic cores in fact coincide with the symplectic leaves if there is only finitely many numbers of symplectic leaves. In this paper we introduce the notion of chiral symplectic cores in a vertex Poisson scheme, which we expect to be the finest possible algebraic stratification in which the chiral Hamiltonian vector fields are tangent.

We have two major applications of the notion of chiral symplectic cores. First, recall that a vertex algebra $V$ is called quasi-lisse if its associated variety $X_V$ has finitely many symplectic leaves ([AK]). For instance, a simple affine vertex algebra $V$ associated with a simple Lie algebra $\mathfrak{g}$ is quasi-lisse if and only if $X_V$ is contained in the nilpotent cone of $\mathfrak{g}$. Therefore [A3], admissible affine vertex algebras are quasi-lisse. Furthermore, all the vertex algebras obtained from four-dimensional $N = 2$ superconformal field theories ([BLL$^+$]) are expected to be quasi-lisse ([A5, BR]). It is also believed in physics that there exist Higgs branch vertex algebras and Column branch vertex algebras in three-dimensional gauge theories that are expected to be quasi-lisse as well. We show that any quasi-lisse vertex
algebra $V$ is a quantization of the arc space $J_\infty X_V$ of its associated variety, in the sense that its reduced singular support $\text{Specm}(\text{gr} V)$ coincides with $J_\infty(X_V)$ (cf. Theorem 9.2). Moreover, for quasi-lisse $V$, we show that $J_\infty(X_V)$ is a finite union of chiral symplectic core closures (cf. Theorem 9.2).

Second, let $\mathfrak{g}$ be a complex simple Lie algebra with adjoint group $G$. We identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ through the Killing form of $\mathfrak{g}$. Denote by $\mathcal{F}_f$ the Slodowy slice $f + \mathfrak{g}^*$ associated with an $\mathfrak{sl}_2$-triple $(e, h, f)$ of $\mathfrak{g}$. The affine variety $\mathcal{F}_f$ has a Poisson structure obtained from that of $\mathfrak{g}^*$ by Hamiltonian reduction [GG]. Consider the adjoint quotient morphism

$$\psi_f : \mathcal{F}_f \to \mathfrak{g}^*/G.$$ 

It is known [Pre1] that any fiber $\psi_f^{-1}(\xi)$ of this morphism is the closure of a symplectic leave, which is irreducible and reduced. We show that any fiber of the induced morphism

$$J_\infty \psi_f : J_\infty \mathcal{F}_f \to J_\infty(\mathfrak{g}^*/G)$$

of vertex Poisson schemes is the closure of a chiral symplectic core, which is irreducible and reduced (cf. Proposition 10.4). This result enables us to show that the morphism $(J_\infty \psi_f)^*$ induces an isomorphism of vertex Poisson algebras between $\mathbb{C}[J_\infty \mathcal{F}_f]/J_\infty G$ and the vertex Poisson center of $\mathbb{C}[J_\infty \mathcal{F}_f]$, and that $\mathbb{C}[J_\infty \mathcal{F}_f]$ is free over its vertex Poisson center (cf. Theorem 11.1). As a consequence, we obtain that the center of the affine $W$-algebra $W^{\text{cri}}(\mathfrak{g}, f)$ associated with $(\mathfrak{g}, f)$ at the critical level is identified with the Feigin-Frenkel center $\mathfrak{j}(\hat{\mathfrak{g}})$, that is, the center of the affine vertex algebra $V^{\text{cri}}(\mathfrak{g})$ at the critical level (cf. Theorem 12.1). This later fact was claimed in [A2] but the proof was incomplete. We take the opportunity of this work to clarify this point.

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**Notations.** The topology is always the Zariski topology. So the term closure always refers to the Zariski closure.

## 2. Vertex Algebras

Let $V$ be a vector space over $\mathbb{C}$.

**Definition 2.1.** The vector space $V$ is called a vertex algebra if it is equipped with the following data:

- **(the vacuum vector)** a vector $|0\rangle \in V$,
- **(the vertex operators)** a linear map

$$V \to (\text{End } V)[[z, z^{-1}]], \quad a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

such that for all $a, b \in V$, $a_{(n)} b = 0$ for $n$ sufficiently large.
- **(the translation operator)** a linear map $T : V \to V$.

These data are subject to the following axioms:

- $|0\rangle(z) = \text{id}_V$. Furthermore, for all $a \in V$, $a(z)|0\rangle \in V[[z]]$ and $\lim_{z \to 0} a(z)|0\rangle = a$. 
• for any $a \in V$,
  
  $$[T, a(z)] = \partial z a(z),$$

  and $T|0\rangle = 0$.

• for all $a, b \in V$, $(z - w)^{N_{a,b}}[a(z), b(w)] = 0$ for some $N_{a,b} \in \mathbb{Z}_{\geq 0}$.

Assume from now that $V$ is a vertex algebra. A consequence of the definition are the following relations, called Borcherds identities:

\[
\begin{align*}
[a_{(m)}, b_{(n)}] &= \sum_{i \geq 0} \binom{m}{i} (a_{(i)})_{(m+n-i)}, \\
(a_{(m)} b)_{(n)} &= \sum_{j \geq 0} (-1)^j \binom{m}{j} (a_{(m-j)} b_{(n+j)} - (-1)^m b_{(m+n-j)} a_{(j)}),
\end{align*}
\]

for $m, n \in \mathbb{Z}$.

A vertex ideal $I$ of $V$ is a $T$-invariant subspace of $V$ such that $a_{(m)} b \in I$ for all $a \in I, b \in V$. By the skew-symmetry property which says that for all $a, b \in V$, the identity

$$a(z)b = e^{zT} b(-z)a$$

holds in $V((z))$, a vertex ideal $I$ of $V$ is also a $T$-invariant subspace of $V$ such that $b_{(n)} a \in I$ for all $a \in I, b \in V$.

The vertex algebra $V$ is called commutative if all vertex operators $a(z), a \in V$, commute each other (i.e., we have $N_{a,b} = 0$ in the locality axiom). This condition is equivalent to that

$$[a_{(m)}, b_{(n)}] = 0, \quad \forall a, b \in \mathbb{Z}, m, n \in \mathbb{Z}$$

by (1).

Hence if $V$ is a commutative vertex algebra, then $a(z) \in \text{End} V[[z]]$ for all $a \in V$. Then a commutative vertex algebra has a structure of a unital commutative algebra with the product:

$$a \cdot b = :ab := a_{(-1)} b,$$

where the unit is given by the vacuum vector $|0\rangle$. The translation operator $T$ of $V$ acts on $V$ as a derivation with respect to this product:

$$T(a \cdot b) = (Ta) \cdot b + a \cdot (Tb).$$

Therefore a commutative vertex algebra has the structure of a differential algebra, that is, a unital commutative algebra equipped with a derivation.

Conversely, there is a unique vertex algebra structure on a differential algebra $R$ with derivation $\partial$ such that:

$$a(z)b = (e^{z\partial} a) b = \sum_{n \geq 0} \frac{z^n}{n!} (\partial^n a) b,$$

for all $a, b \in R$. We take the unit as the vacuum vector. This correspondence gives that the category of commutative vertex algebras is the same as that of differential algebras [Bor].
3. Jet schemes and arc spaces

Our main references about jet schemes and arc spaces are [Mus, EM, Ish].

Denote by \( \mathbf{Sch} \) the category of schemes of finite type over \( \mathbb{C} \). Let \( X \) be an object of this category, and \( n \in \mathbb{Z}_{\geq 0} \).

**Definition 3.1.** An \( n \)-jet of \( X \) is a morphism
\[
\text{Spec } \mathbb{C}[t]/(t^{n+1}) \rightarrow X.
\]
The set of all \( n \)-jets of \( X \) carries the structure of a scheme \( J_nX \), called the \( n \)-th jet scheme of \( X \). It is a scheme of finite type over \( \mathbb{C} \) characterized by the following functorial property: for every scheme \( Z \) over \( \mathbb{C} \), we have
\[
\text{Hom}_{\mathbf{Sch}}(Z, J_nX) = \text{Hom}_{\mathbf{Sch}}(Z \times_{\text{Spec } \mathbb{C}} \text{Spec } \mathbb{C}[t]/(t^{n+1}), X).
\]
The \( \mathbb{C} \)-points of \( J_nX \) are thus the \( \mathbb{C}[t]/(t^{n+1}) \)-points of \( X \). From Definition 3.1, we have for example that \( J_0X \cong X \) and that \( J_1X \cong T_X \) where \( T_X \) denotes the total tangent bundle of \( X \).

The canonical projection \( \mathbb{C}[t]/(t^{m+1}) \rightarrow \mathbb{C}[t]/(t^{n+1}) \), \( m \geq n \), induces a truncation morphism
\[
\pi^{X}_{m,n} : J_mX \rightarrow J_nX.
\]
The canonical injection \( \mathbb{C} \hookrightarrow \mathbb{C}[t]/(t^{n+1}) \) induces a morphism \( \iota^{X}_n : X \rightarrow J_nX \), and we have \( \pi^{X}_{n,0} \circ \iota^{X}_n = \text{id}_X \). Hence \( \iota^{X}_n \) is injective and \( \pi^{X}_{n,0} \) is surjective.

Define the (formal) disc as
\[
D := \text{Spec } \mathbb{C}[[t]].
\]
The projections \( \pi^{X}_{m,n} \) yield a projective system \( \{J_mX, \pi^{X}_{m,n}\}_{m \geq n} \) of schemes.

**Definition 3.2.** Denote by \( J_\infty X \) its projective limit in the category of schemes,
\[
J_\infty X = \varprojlim J_nX.
\]
It is called the arc space, or the infinite jet scheme, of \( X \).

Thus elements of \( J_\infty X \) are the morphisms
\[
\gamma : D \rightarrow \mathbb{C}[[t]],
\]
and for every scheme \( Z \) over \( \mathbb{C} \),
\[
\text{Hom}_{\mathbf{Sch}}(Z, J_nX) = \text{Hom}_{\mathbf{Sch}}(Z \times_{\text{Spec } \mathbb{C}} \mathbb{C}[[t]], X),
\]
where \( Z \times D \) is the completion of \( Z \times D \) with respect to the subscheme \( Z \times \{0\} \). In other words, the contravariant functor
\[
\mathbf{Sch} \rightarrow \text{Set}, \quad Z \mapsto \text{Hom}_{\mathbf{Sch}}(Z \times D, X)
\]
is represented by the scheme \( J_\infty X \).

We denote by \( \pi^{X}_{\infty,n} \) the morphism:
\[
\pi^{X}_{\infty,n} : J_\infty X \rightarrow J_nX.
\]
It is surjective if \( X \) is smooth. The canonical injection \( \mathbb{C} \hookrightarrow \mathbb{C}[[t]] \) induces a morphism \( \iota^{X}_\infty : X \rightarrow J_\infty X \), and we have \( \pi^{X}_{\infty,0} \circ \iota^{X}_\infty = \text{id}_X \). Hence \( \iota^{X}_\infty \) is injective and \( \pi^{X}_{\infty,0} \) is surjective (for any \( X \)).

When the variety \( X \) is obvious, we simply write \( \pi_{m,n}, \pi_{\infty,n}, \ldots \) for \( \pi^{X}_{m,n}, \pi^{X}_{\infty,n}, \ldots \).
In the case where \( X = \text{Spec} \mathbb{C}[x^1, \ldots, x^N] \cong \mathbb{A}^N \), \( N \in \mathbb{Z}_{\geq 0} \), is an affine space, we have the following explicit description of \( J_\infty X \). Giving a morphism \( \gamma: D \to \mathbb{A}^N \) is equivalent to giving a morphism \( \gamma^*: \mathbb{C}[x^1, \ldots, x^N] \to \mathbb{C}[t] \), or to giving

\[
\gamma^*(x^i) = \sum_{j \geq 0} \gamma^i_{(-j-1)} t^j, \quad i = 1, \ldots, N.
\]

Define functions over \( J_\infty \mathbb{A}^N \) by setting for \( i = 1, \ldots, N \):

\[
x^i_{(-j-1)}(\gamma) = j! \gamma^i_{(-j-1)}.
\]

Then

\[
J_\infty \mathbb{A}^N = \text{Spec} \mathbb{C}[x^i_{(-j-1)}; i = 1, \ldots, N, j \geq 0].
\]

Define a derivation \( T \) of the algebra \( \mathbb{C}[x^i_{(-j-1)}; i = 1, \ldots, N, j \geq 0] \) by

\[
Tx^i_{(-j)} = jx^i_{(-j-1)}, \quad j > 0.
\]

Here we identify \( x^i \) with \( x^i_{(-1)} \).

More generally, if \( X \subset \mathbb{A}^N \) is an affine subscheme defined by an ideal \( I = (f_1, \ldots, f_r) \) of \( \mathbb{C}[x^1, \ldots, x^N] \), that is, \( X = \text{Spec} R \) with \( R = \mathbb{C}[x^1, x^2, \ldots, x^N]/(f_1, f_2, \ldots, f_r) \), then its arc space \( J_\infty X \) is the affine scheme \( \text{Spec}(J_\infty R) \), where

\[
J_\infty R := \frac{\mathbb{C}[x^i_{(-j-1)}; i = 1, 2, \ldots, N, j \geq 0]}{(T^j f_i; i = 1, \ldots, r, j \geq 0)}
\]

and \( T \) is as defined above.

Similarly, we have for any \( n \in \mathbb{Z}_{\geq 0} \),

\[
J_n R := \frac{\mathbb{C}[x^i_{(-j-1)}; i = 1, 2, \ldots, N, j \geq 0]}{(T^j f_i; i = 1, \ldots, r, j \geq 0)}
\]

The derivation \( T \) acts on the quotient ring \( J_\infty R \) given by (3). Hence for an affine scheme \( X = \text{Spec} R \), the coordinate ring \( J_\infty R = \mathbb{C}[J_\infty X] \) of its arc space \( J_\infty X \) is a differential algebra, hence is a commutative vertex algebra.

**Remark 3.3 ([EM])**. The differential algebra \((J_\infty R, T)\) is universal in the following sense. We have a \( \mathbb{C}\)-algebra homomorphism \( j: R \to J_\infty R \) such that if \((A, \partial)\) is another differential algebra, and if \( f: R \to A \) is a \( \mathbb{C}\)-algebra homomorphism, then there is a unique differential algebra homomorphism \( h: J_\infty R \to A \) making the following diagram commutative.

\[
\begin{array}{ccc}
R & \xrightarrow{j} & (J_\infty R, T) \\
\downarrow f & & \downarrow h \\
(A, \partial) & & &
\end{array}
\]

Let \( n \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \). The natural projection \( \pi^X_{n,0}: J_n X \to X \) corresponds to the embedding \( R \to J_n R \), \( x^i \to x^i_{(-1)} \) in the case where \( X = \text{Spec} R \) is affine. If \( m \) is a maximal ideal of \( J_n R \), note that \( \pi^X_{n,0}(m) = m \cap R \).

For \( I \) an ideal of \( R \), we denote by \( J_n I \) the smallest \( T \)-stable ideal of \( J_n R \) containing \( I \), that is, \( J_n I \) is generated by the elements \( T^j a, j = 0, \ldots, n, a \in I \). Recall that \( \iota^X_n \) denotes the embedding \( X \hookrightarrow J_n X \), and observe that \( \iota^X_n (m) = J_n (m) \), for \( m \) a maximal ideal of \( \text{Spec} R \).
The map from a scheme to its \( n \)-th jet schemes and arc space is functorial. If \( f : X \to Y \) is a morphism of schemes, then we naturally obtain a morphism \( J_n f : J_n X \to J_n Y \) making the following diagram commutative,

\[
\begin{array}{c}
J_n X \\ \downarrow \pi_{n,0} \\
X \\
\downarrow f \\
Y \\
\downarrow \pi_{n,0} \\
J_n Y
\end{array}
\]

We have also the following for every schemes \( X, Y \),

\[
J_n (X \times Y) \cong J_n X \times J_n Y.
\] (5)

If \( A \) is a group scheme over \( \mathbb{C} \), then \( J_n A \) is also a group scheme over \( \mathbb{C} \). Moreover, by (5), if \( A \) acts on \( X \), then \( J_n A \) acts on \( J_n X \).

The next results are specific to the arc space \( J_\infty X \).

**Lemma 3.4.** Denote by \( X_{\text{red}} \) the reduced scheme of \( X \). The natural morphism \( X_{\text{red}} \to X \) induces an isomorphism \( J_\infty X_{\text{red}} \cong (J_\infty X)_{\text{red}} \) of topological spaces.

Similarly, if \( X = X_1 \cup \ldots \cup X_r \), where all \( X_i \) are closed in \( X \), then

\[
J_\infty X = J_\infty (X_1) \cup \ldots \cup J_\infty (X_r).
\]

(Note that Lemma 3.4 is false for the schemes \( J_n X \).)

Moreover, we have the following result (which is false for the jet schemes \( J_n X \)).

**Theorem 3.5** (Kolchin [Kol]). The scheme \( J_\infty X \) is irreducible if \( X \) is irreducible.

More precisely, we have for any \( n \in \mathbb{Z}_{\geq 0} \),

\[
J_n X = \pi_{n,0}^{-1}(X_{\text{sing}}) \cup \pi_{n,0}^{-1}(X_{\text{reg}}),
\] (6)

and \( \pi_{n,0}^{-1}(X_{\text{reg}}) \) is an irreducible component of \( J_n X \). Kolchin’s theorem says that

\[
J_\infty X = \pi_{\infty,0}^{-1}(X_{\text{reg}}).
\]

### 4. Vertex Poisson algebras and chiral Poisson ideals

**Definition 4.1.** A commutative vertex algebra \( V \) is called a **vertex Poisson algebra** if it is also equipped with a linear operation,

\[
V \to \text{Hom}(V, z^{-1}V[z^{-1}]), \quad a \mapsto a_-(z),
\]

such that

\[
(Ta)_{(n)} = -na_{(n-1)},
\] (7)

\[
a_{(n)}b = \sum_{j \geq 0} (-1)^{n+j+1} \frac{1}{j!} T^j (b_{(n+j)} a),
\] (8)

\[
[a_{(m)}, b_{(n)}] = \sum_{j \geq 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)},
\] (9)

\[
a_{(n)}(b \cdot c) = (a_{(n)}b) \cdot c + b \cdot (a_{(n)}c)
\] (10)

for \( a, b, c \in V \) and \( n, m \geq 0 \). Here, by abuse of notations, we have set

\[
a_-(z) = \sum_{n \geq 0} a_{(n)} z^{-n-1}
\]
so that the \( a_{(n)} \), \( n \geq 0 \), are “new” operators, the “old” ones given by the field \( a(z) \) being zero for \( n \geq 0 \) since \( V \) is commutative.

The equation (10) says that \( a_{(n)} \), \( n \geq 0 \), is a derivation of the ring \( V \). Note that (8), (9) and (10) are equivalent to the “skewsymmetry”, the “Jacobi identity” and the “left Leibniz rule” in [Kac, §5.1].

It follows from the definition, that we also have the “right Leibniz rule” ([Kac, Exercise 4.2]):

\[
(a \cdot b)_{(n)}c = \sum_{i \geq 0} (b_{(-i-1)}a_{(n+i)}c + a_{(-i-1)}b_{(n+i)}c),
\]

for all \( a, b, c \in V, n \in \mathbb{Z}_{\geq 0} \), where \( a_{(-n-1)} \) is considered as an element of \( V \) for \( n \in \mathbb{Z}_{\geq 0} \), i.e.,

\[
a_{(-n-1)} = \frac{1}{n!} T^n a.
\]

Arc spaces over an affine Poisson scheme naturally give rise to a vertex Poisson algebras, as shows the following result.

**Theorem 4.2** ([A1, Proposition 2.3.1]). Let \( X \) be an affine Poisson scheme, that is, \( X = \text{Spec} R \) for some Poisson algebra \( R \). Then there is a unique vertex Poisson algebra structure on \( J_\infty R = \mathbb{C}[J_\infty X] \) such that

\[
a_{(n)}b = \begin{cases} \{a, b\} & \text{if } n = 0 \\ 0 & \text{if } n > 0, \end{cases}
\]

for \( a, b \in R \).

Let \( I \) be an ideal of \( V \) in the associative sense.

**Definition 4.3.** We say that \( I \) is a **chiral Poisson ideal** of \( V \) if \( a_{(n)}I \subset I \) for all \( a \in V, n \in \mathbb{Z}_{\geq 0} \).

A **vertex Poisson ideal** of \( V \) is a chiral Poisson ideal that is stable under the action of \( T \). The quotient space \( V/I \) inherits a vertex Poisson algebra structure from \( V \) if \( I \) is a vertex Poisson ideal.

**Lemma 4.4** ([Dix, 3.3.2]). If \( I \) is a vertex (resp. chiral) Poisson ideal of \( V \), then so is its radical \( \sqrt{I} \).

**Definition 4.5.** Let \( V \) be a vertex Poisson algebra. We denote by \( Z(V) \) the **vertex Poisson center** of \( V \), that is,

\[
Z(V) = \{ z \in V \mid z_{(n)}a = 0, \forall a \in V, n \geq 0 \} = \{ z \in V \mid a_{(n)}z = 0, \forall a \in V, n \geq 0 \},
\]

the last equality holding by (8).

The vertex Poisson center \( Z(V) \) is a vertex Poisson ideal of \( V \). Indeed, it is clearly invariant by the derivations \( a_{(n)}, a \in V, n \in \mathbb{Z}_{\geq 0} \). Moreover, it is invariant by \( T \) by the axiom (7).

We call **vertex Poisson scheme** the spectrum \( \text{Spec} V \) of any vertex Poisson algebra, and we call **chiral Poisson scheme** the spectrum \( \text{Spec}(V/I) \), where \( V \) is a vertex Poisson algebra and \( I \) is a chiral Poisson ideal of \( V \).

By Lemma 4.4, the reduced scheme of a vertex Poisson scheme (resp. chiral Poisson scheme) is also a vertex Poisson scheme. In this case, we call it a **vertex Poisson variety** or a **chiral Poisson variety**.
In the case where $V$ is $J_{\infty}R$, with $R = \mathbb{C}[X]$ the coordinate ring of some affine Poisson scheme $X$, we have the following result.

**Lemma 4.6.** Let $I$ be an ideal of $J_{\infty}R$ in the associative sense. Then $I$ is a chiral Poisson ideal of $J_{\infty}R$ if and only if $a_{(n)}I \subset I$ for all $a \in R$, $n \in \mathbb{Z}_{\geq 0}$.

**Proof.** The “only if” part is obvious.

Assume now that $a_{(n)}I \subset I$ for all $a \in R$, $n \in \mathbb{Z}_{\geq 0}$ and show that $a_{(n)}I \subset I$ for all $a \in J_{\infty}R$, $n \in \mathbb{Z}_{\geq 0}$. Let $u \in I$. First, by (7),

\[
(T^j a)_{(n)} u = \begin{cases} 
(-1)^n \frac{n!}{(n-j)!} a_{(n-j)} u & \text{if } 0 \leq j \leq n, \\
0 & \text{if } j > n,
\end{cases}
\]

for all $a \in R$, $n, j \in \mathbb{Z}_{\geq 0}$. Hence $(T^j a)_{(n)} u \in I$ for all $a \in R$, $n, j \in \mathbb{Z}_{\geq 0}$ by our assumption. Next, by (11) and the above, $(a \cdot b)_{(n)} u$ is in $I$ for all $a, b$ of the form $T^j v$, $v \in R$. Since $J_{\infty}R$ is generated as a commutative algebra by the elements $T^j v$, $v \in R$, we get the expected statement. \qed

5. Rank stratification

Let $X = \text{Spec } R$ be a reduced Poisson scheme, and $\{x^1,\ldots,x^r\}$ a generating set for $R$. Let $n \in \mathbb{Z}_{\geq 0}$. Then $\{T^j x_i \mid i = 1,\ldots,r, j = 0,\ldots,n\}$ is a generating set for $J_nR = \mathbb{C}[J_nX]$. We have by the equality (10) of [A1] that for any $x, y \in R$,

\[
x_{(k)}(T^j y) = \begin{cases} 
\frac{\binom{n}{l-k}}{l!} T^{l-k} \{x, y\} & \text{if } l \geq k, \\
0 & \text{otherwise},
\end{cases}
\]

Hence, for $x \in R$, the derivations $x_{(k)}$ of $J_{\infty}R$ acts on $J_nR$ if $k \in \{0,\ldots,n\}$ by the description (4) of $J_nR$.

Consider the $(n+1)r$-size square matrix

\[
\mathcal{M}_n = (x_{(p)}^i (T^q x_j))_{1 \leq i, j \leq r, 0 \leq p, q \leq n} \in \text{Mat}_{(n+1)r} J_nR).
\]

For $x \in J_nX$, set

\[
\mathcal{M}_n(x) = (x_{(p)}^i (T^q x_j) + m_x)_{1 \leq i, j \leq r, 0 \leq p, q \leq n} \in \text{Mat}_{(n+1)r} (\mathbb{C}),
\]

where $m_x$ is the maximal ideal of $R$ corresponding to $x$.

**Lemma 5.1.** Let $x \in J_nX$. We have

\[
\text{rank } \mathcal{M}_n(x) = (n+1) \text{rank } \mathcal{M}_0(\pi_n^X(x)).
\]

In particular, $\text{rank } \mathcal{M}_n(x)$ is independent of the choice of generators $\{x^1,\ldots,x^r\}$ and depends only on $\pi_n^X(x) \in X$. 

Proof. By definition, $\mathcal{M}_n$ is the following matrix:

$$
\begin{pmatrix}
0 \cdots 0 & 1 & \cdot & \cdots & \cdot \\
0 & 0 & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & \cdot & \cdot \\
0 & 0 & \cdots & \cdots & \cdot \\
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}
$$

so by (12) it has the form

whence the first statement. Here we identify the elements $\{x^i, x^j\} = x_{(0)}^i x_{(0)}^j$ of $R$ with elements of $J_n R$ through the embedding $R \to J_n R$. The independence of the choice of generators follows from [Van].

For $x \in J_n X$, let $\text{rk} \ x$ to be $1/(n + 1) \times \text{rank} \ M_\alpha(x)$. By Lemma 5.1, $\text{rk} \ x$ is a non-negative integer. Moreover, for $x \in J_\infty X$, $\text{rk} \pi_{x,0}^X(x)$ does not depend on $n$. So we can define $\text{rk} \ x$ to be this number. Lemma 5.1 says that $\text{rk} \ x$ is nothing but the rank of the matrix $M_\alpha$ at $x_0 := \pi_{x,0}^X(x)$.

Let $\mathcal{L}$ be a chiral Poisson subscheme of $J_\infty X$. We define the rank stratification of $\mathcal{L}$ as follows, for $j \in \mathbb{Z}_{\geq 0}$:

$$
L^0_j := \{ x \in \mathcal{L} \mid \text{rk} \ x = j \} \subset L_j := \{ x \in \mathcal{L} \mid \text{rk} \ x \leq j \},
$$

Also, set $\tilde{L} = \pi_{x,0}^X(L) \subset X$, and put

$$
\tilde{L}^0_j := \{ x \in \tilde{L} \mid \text{rank} \ M_\alpha(x) = j \} \subset \tilde{L}_j := \{ x \in \tilde{L} \mid \text{rank} \ M_\alpha(x) \leq j \}.
$$

For $\mathcal{L} = J_\infty X$,

$$
X = \bigsqcup_j \tilde{L}^0_j
$$

is precisely the rank stratification of $X$ defined by Brown and Gordon [BG]. Moreover, by Lemma 5.1, note that $L_j = (\pi_{x,0}^X)^{-1}(\tilde{L}_j)$ and $L^0_j = (\pi_{x,0}^X)^{-1}(\tilde{L}^0_j)$.

Lemma 5.2. (1) $L_j$ is a closed subset of $\mathcal{L}$ with $L_0 \subseteq L_1 \subseteq \cdots \subseteq L_d = \mathcal{L}$ for some $d \in \mathbb{Z}_{\geq 0}$.

(2) $L_j$ is a chiral Poisson subscheme of $J_\infty X$.

(3) If $L^0_j \neq \emptyset$, then $\dim \pi_{x,0}^X(Z) \geq (n + 1)j$ for each irreducible component $Z$ of $L^0_j$.

Proof. Part (1) is clear by Lemma 5.1 and [BG, Lemma 3.1 (1)].
(2) Let \( \mathcal{I} \) be the defining ideal of \( \mathcal{L}_j \). By [BG, Lemma 3.1 (2)], \( \mathcal{I} \) is a Poisson ideal of \( R \). On the other hand, observe that for \( n \geq 0 \),
\[
(\pi_{n,0}^X)^{-1}(\mathcal{L}_j) \cong \mathcal{L}_j \times_X J_n X.
\]
Hence the defining ideal of \( (\pi_{n,0}^X)^{-1}(\mathcal{L}_j) \) is \( \mathcal{I} \otimes_R J_n R \), and so the defining ideal of \( \mathcal{L}_j = (\pi_{\infty,0}^X)^{-1}(\mathcal{L}_j) \) is \( \mathcal{I} \otimes_R J_{\infty} R \). So it is enough to show that \( \mathcal{I} \otimes_R J_{\infty} R \) is chiral Poisson.

Let \( u = \sum_i b_i c_i \in \mathcal{I} \otimes_R J_{\infty} R \), with \( b_i \in \mathcal{I}, c_i \in J_{\infty} R \). Then by (10) and Theorem 4.2, for all \( a \in R \) and \( k \geq 0 \),
\[
a_{(k)} u = \sum_i \left( (a_{(k)} b_i) \cdot c_i + b_i \cdot (a_{(k)} c_i) \right) = \sum_i \left( \delta_{k,0} \{a, b_i\} \cdot c_i + b_i \cdot (a_{(k)} c_i) \right)
\]
is in \( \mathcal{I} \otimes_R J_{\infty} R \) since \( \mathcal{I} \) is a Poisson ideal of \( R \). So \( \mathcal{I} \otimes_R J_{\infty} R \) is a chiral Poisson ideal by Lemma 4.6.

(3) Let \( Z \) be an irreducible component of \( \mathcal{L}_j^0 \). Denoting by \( X_1, \ldots, X_k \) the irreducible components of \( \mathcal{L}_j^0 \), we have
\[
\mathcal{L}_j^0 = (\pi_{\infty,0}^X)^{-1}(X_1) \cup \cdots \cup (\pi_{\infty,0}^X)^{-1}(X_k),
\]
since \( \mathcal{L}_j^0 = (\pi_{\infty,0}^X)^{-1}(\mathcal{L}_j^0) \). Moreover the subsets \( (\pi_{\infty,0}^X)^{-1}(X_i) \)'s are the irreducible components of \( \mathcal{L}_j^0 \), and so \( Z = (\pi_{\infty,0}^X)^{-1}(X) \) for some irreducible component \( X \) of \( \mathcal{L}_j^0 \). By [BG, Lemma 3.1 (5)], \( X \) has dimension at least \( j \). Observing that \( \pi_{\infty,n}(Z) = (\pi_{n,0}^X)^{-1}(X) \), we deduce that \( \pi_{\infty,n}(Z) \) has dimension at least \( (n + 1)j \).

6. Chiral Poisson cores and chiral symplectic cores

Let \( X = \text{Spec} m \) be a reduced Poisson scheme. For \( I \) an ideal of \( R \), the Poisson core of \( I \) is the biggest Poisson ideal contained in \( I \). We denote it by \( \mathcal{P}_R(I) \). The symplectic core \( \mathcal{C}_R(x) \) of a point \( x \in X \) is the equivalence class of \( x \) for \( \sim \), with
\[
x \sim y \iff \mathcal{P}_R(m_x) = \mathcal{P}_R(m_y),
\]
if \( m_x \) denotes the maximal ideal of \( R \) corresponding to \( x \). We refer the reader to [BG] for more details about Poisson cores and symplectic cores.

The aim of this section is to define analogue notions in the setting of vertex Poisson algebras.

Let \( V \) be a vertex Poisson algebra. By ideal of \( V \) we mean an ideal of \( V \) in the associative sense. We will always specify vertex Poisson ideal or chiral Poisson ideal (see Section 4) if necessary. An ideal \( I \) of \( V \) is said to be prime if it prime in the associative sense.

Definition 6.1. The chiral Poisson core of an ideal \( I \) of \( V \) is the biggest chiral Poisson ideal of \( V \) contained in \( I \). It exists since the sum of two chiral Poisson ideals is chiral Poisson. We denote the chiral Poisson core of \( I \) by \( \mathcal{P}_V(I) \).

Lemma 6.2. Let \( \mathcal{D} \) be the set of all derivations \( a_{(n)} \in \text{Der}(V) \), for \( a \in V \) and \( n \in \mathbb{Z}_{\geq 0} \), and let \( I \) be an ideal of \( V \).

1. We have \( \mathcal{P}_V(I) = \{ x \in I \mid D_{i_1}^{n_1} \cdots D_{i_k}^{n_k} x \in I \text{ for all } D_i \in \mathcal{D}, n_i \geq 0 \} \).
2. If \( I \) is prime, then \( \mathcal{P}_V(I) \) is prime.
3. If \( I \) is radical, then \( \mathcal{P}_V(I) \) is radical.
Proof. Set \( J = \{ x \in I \mid D_{i_1}^{n_1} \cdots D_{i_k}^{n_k} x \in I \text{ for all } D_i \in \mathcal{D}, n_i \geq 0 \} \).

(1) By construction, \( J \subset I \) and \( J \) is chiral Poisson. Hence \( J \subset \mathcal{P}_V(I) \). But if \( K \) is a chiral Poisson ideal of \( V \) contained in \( I \), then for all \( x \in K \) and \( D \in \mathcal{D}, \ D x \in K \subset I \), whence \( x \in J \). In conclusion, \( J = \mathcal{P}_V(I) \).

(2) results from [Dix, Lemma 3.3.2(ii)].

(3) Assume that \( I \) is radical. Since \( \mathcal{P}_V(I) \) is chiral Poisson, \( \sqrt{\mathcal{P}_V(I)} \) is chiral Poisson too by Lemma 4.4, and it is contained in \( \sqrt{I} \) since \( \mathcal{P}_V(I) \) is contained in \( I \). Hence,

\[
\sqrt{\mathcal{P}_V(I)} \subset \mathcal{P}_V(\sqrt{I}) = \mathcal{P}_V(I).
\]

But clearly \( \mathcal{P}_V(I) \subset \sqrt{\mathcal{P}_V(I)} \), whence the equality \( \sqrt{\mathcal{P}_V(I)} = \mathcal{P}_V(I) \) and the statement. \( \square \)

Corollary 6.3. Assume that there are finitely many minimal prime ideals \( p_1, \ldots, p_r \) over \( I \), that is, \( I = p_1 \cap \ldots \cap p_r \), and the prime ideals \( p_1, \ldots, p_r \) are minimal. If \( I \) is chiral Poisson, then so are the prime ideals \( p_1, \ldots, p_r \).

Proof. If \( I \) is chiral Poisson, then \( I \subset \mathcal{P}_V(p_i) \subset p_i \) for all \( i \). But by Lemma 6.2 (2), the ideals \( \mathcal{P}_V(p_i), i = 1, \ldots, r \), are all prime. By minimality of the prime ideals \( p_i \), we deduce that \( p_i = \mathcal{P}_V(p_i) \) for all \( i \). In particular, the prime ideals \( p_i, i = 1, \ldots, r \), are all chiral Poisson. \( \square \)

Set

\[
\mathcal{L} := \text{Specm}(V).
\]

We define a relation \( \sim \) on \( \mathcal{L} \) by

\[
x \sim y \iff \mathcal{P}_V(m_x) = \mathcal{P}_V(m_y),
\]

where \( m_x \) is the maximal ideal corresponding to \( x \in \mathcal{L} \).

Clearly \( \sim \) is an equivalence relation. We denote the equivalence class in \( \mathcal{L} \) of \( x \) by \( \mathcal{C}_\mathcal{L}(x) \) so that

\[
\mathcal{L} = \bigsqcup_x \mathcal{C}_\mathcal{L}(x).
\]

We call the set \( \mathcal{C}_\mathcal{L}(x) \) the chiral symplectic core of \( x \) in \( \mathcal{L} \).

For \( I \) an ideal of \( V \), we denote by \( \mathcal{V}(I) \) the corresponding zero locus in \( \text{Specm} V \), that is,

\[
\mathcal{V}(I) = \{ x \in \mathcal{L} \mid m_x \supset I \}.
\]

Lemma 6.4. Let \( x \in \mathcal{L} \). Then \( \mathcal{V}(\mathcal{P}_V(m_x)) \) is the smallest chiral Poisson scheme for the inclusion containing \( \mathcal{C}_\mathcal{L}(x) \). Moreover, it is reduced and irreducible.

Proof. First of all, since \( \mathcal{P}_V(m_x) \) is a chiral Poisson, prime and radical ideal of \( V \) by Lemma 6.2, \( \mathcal{V}(\mathcal{P}_V(m_x)) \) is a reduced irreducible (closed) chiral Poisson subscheme of \( \mathcal{L} \). For any \( y \in \mathcal{C}_\mathcal{L}(x) \), we have \( m_y \supset \mathcal{P}_V(m_y) = \mathcal{P}_V(m_x) \). Hence \( \mathcal{C}_\mathcal{L}(x) \subset \mathcal{V}(\mathcal{P}_V(m_x)) \). Next, if \( I \) is a chiral Poisson ideal of \( V \) such that \( \mathcal{C}_\mathcal{L}(x) \subset \mathcal{V}(I) \), then in particular \( m_x \supset I \). Since \( I \) is chiral Poisson, we get that \( m_x \supset \mathcal{P}_V(m_x) \supset I \) by maximality of \( \mathcal{P}_V(m_x) \). Hence \( \mathcal{C}_\mathcal{L}(x) \subset \mathcal{V}(\mathcal{P}_V(m_x)) \subset \mathcal{V}(I) \). This proves the statement. \( \square \)

Lemma 6.5. Let \( \mathcal{L}' \) be a reduced closed vertex Poisson subscheme of \( \mathcal{L} \). Then for any \( x \in \mathcal{L}' \), we have \( \mathcal{C}_{\mathcal{L}'}(x) = \mathcal{C}_\mathcal{L}(x) \).
Proof. Since \( \mathcal{L}' \) is a closed vertex Poisson subscheme of \( \mathcal{L} \), \( \mathcal{L}' = \text{Specm}(V/I) \), where \( I \) is a vertex Poisson ideal of \( V \). The maximal ideals of \( V/I \) are precisely the quotients \( m/I \) where \( m \) is a maximal ideal of \( V \) containing \( I \), and \( \mathcal{P}_{V/I}(m/I) = \mathcal{P}_V(m)/I \).

Hence for \( x \in \mathcal{L}' \), we have
\[
\mathcal{C}_{\mathcal{L}'}(x) = \{ y \in \mathcal{L} \mid \mathcal{P}_{V/I}(m_y/I) = \mathcal{P}_{V/I}(m_x/I) \} = \{ y \in \mathcal{L}' \mid \mathcal{P}_V(m_y) = \mathcal{P}_V(m_x) \},
\]
where \( m_x \) is the maximal ideal of \( V \) corresponding to \( x \). The maximal ideal \( m_x \) contains \( I \) because \( x \in \mathcal{L}' \). On the other hand,
\[
\mathcal{C}_{\mathcal{L}}(x) = \{ y \in \mathcal{L} \mid \mathcal{P}_V(m_y) = \mathcal{P}_V(m_x) \}.
\]
But if \( \mathcal{P}_V(m_y) = \mathcal{P}_V(m_x) \) for some \( y \in \mathcal{L} \), then \( m_y \supset \mathcal{P}_V(m_y) = \mathcal{P}_V(m_x) \supset I \) because \( I \) is a chiral Poisson ideal of \( V \) contained in \( m_x \). This shows that if \( y \in \mathcal{C}_{\mathcal{L}}(x) \), then \( y \in \mathcal{L}' \).

Therefore
\[
\mathcal{C}_{\mathcal{L}}(x) = \{ y \in \mathcal{L}' \mid \mathcal{P}_V(m_y) = \mathcal{P}_V(m_x) \} = \mathcal{C}_{\mathcal{L}'}(x).
\]
□

Proposition 6.6. Let \( z \in \mathcal{Z}(V) \) and \( x \in \mathcal{L} \). Then \( z \) is constant on \( \mathcal{V}(\mathcal{P}_V(m_x)) \) and so on \( \mathcal{C}_{\mathcal{L}}(x) \).

Proof. Let \( y \in \mathcal{V}(\mathcal{P}_V(m_x)) \), and let \( \chi_x, \chi_y \) be the homomorphisms \( \chi_x : V \to \mathbb{C} \), \( \chi_y : V \to \mathbb{C} \), corresponding to the maximal ideals \( m_x, m_y \). It is enough to show that \( \chi_x(z) = \chi_y(z) \). Set \( \lambda := \chi_x(z) \). Then \( z - \lambda \in \ker \chi_x = m_x \). In addition since \( z \) is in the center, so is \( z - \lambda \), and then \( a(a/z - \lambda) = 0 \) for any \( a \in V \) and \( a \geq 0 \). Therefore \( z - \lambda \in V_x(m_x) \subset m_y \), whence \( \chi_y(z) = \lambda \). By Lemma 6.4, we conclude that \( z \) is constant on \( \mathcal{C}_{\mathcal{L}}(x) \). □

Lemma 6.7. Let \( X = \text{Specm } R \) be a reduced Poisson scheme, and let \( \mathcal{L} \) be a closed vertex Poisson subscheme of \( J_\infty X \). Set \( \mathcal{L}' = \pi_{X,0}^X(\mathcal{L}) \). Let \( x \in \mathcal{L}' \).

1. We have \( \pi_{X,0}^X(\mathcal{C}_{\mathcal{L}}(x)) \subset \mathcal{C}_{\mathcal{L}}(\pi_{X,0}^X(x)) \).
2. If \( \text{rk } x = j \), then \( \mathcal{C}_{\mathcal{L}}(x) \subset \mathcal{L}'^j \).

Proof. (1) We have \( \mathcal{L} = \text{Specm } J_\infty R/I \), with \( I \) a vertex Poisson ideal of \( J_\infty R \).

Let \( x \in \mathcal{L} \). Let us first show that:
\[
\mathcal{P}_{R/(I \cap R)}(m_x \cap R/(I \cap R)) = \mathcal{P}_{J_\infty R/I}(m_x/I) \cap R/(I \cap R).
\]
The inclusion \( \mathcal{P}_{J_\infty R/I}(m_x/I) \cap R/(I \cap R) \subset \mathcal{P}_{R/(I \cap R)}(m_x \cap R/(I \cap R)) \) is clear because the left-hand side is Poisson and is contained in \( m_x \cap R/(I \cap R) \). For the converse inclusion, let \( a \in R, k \in \mathbb{Z}_{\geq 0} \), and \( b \in \mathcal{P}_{R/(I \cap R)}(m_x \cap R/(I \cap R)) \). Then
\[
a_{(k)}(b + I \cap R) = \delta_{k0}(\{a, b\} + \{a, I \cap R\}) \in \mathcal{P}_{R/(I \cap R)}(m_x \cap R/(I \cap R))
\]
by Theorem 4.2 since \( \mathcal{P}_{R/(I \cap R)}(m_x \cap R/(I \cap R)) \) and \( I \cap R \) are Poisson. By Lemma 4.6, this shows that \( \mathcal{P}_{R/(I \cap R)}(m_x \cap R/(I \cap R)) \) is a chiral Poisson ideal of \( J_\infty R/I \), contained in \( m_x/I \), whence the expected equality (13). Let now \( y \in \mathcal{C}_{\mathcal{L}}(x) \). Then \( \mathcal{C}_{J_\infty R/I}(m_y/I) = \mathcal{C}_{J_\infty R/I}(m_y/I) \). From (13), we deduce that
\[
\mathcal{P}_{R/(I \cap R)}(m_y \cap R/(I \cap R)) = \mathcal{P}_{R/(I \cap R)}(m_x \cap R/(I \cap R))
\]
and so
\[
\pi_{X,0}^X(y) \in \mathcal{C}_{\mathcal{L}}(\pi_{X,0}^X(x))
\]
since $m \cap R/(I \cap R)$ is the maximal ideal of $R/(I \cap R)$ corresponding to $\pi_{n,0}^X(y) \in \bar{L}$. This proves the statement.

(2) By Lemma 5.1, $\mathcal{L}_j^0 = (\pi_{n,0}^X)^{-1}(\bar{L}_j^0)$. On the other hand, by [BG, Proposition 3.6], $\mathcal{E}_L(\pi_{n,0}^X(x)) \subset \bar{L}_j^0$. Hence by (1),
\[
\mathcal{E}_L(x) \subset (\pi_{n,0}^X)^{-1}(\mathcal{E}_L(\pi_{n,0}^X(x))) \subset (\pi_{n,0}^X)^{-1}(\bar{L}_j^0) = \mathcal{L}_j^0.
\]

By Lemma 6.7 and Lemma 5.1, note that the stratification by chiral symplectic cores is a refinement of the rank stratification.

7. n-chiral Poisson cores in n-th jet schemes

Let $R$ be a Poisson algebra, $n \in \mathbb{Z}_{\geq 0}$. The derivations $a_{(k)}$, $k \geq 0$, of $J_\infty R$ acts on $J_n R$ by (4) and (12). We say that an ideal $I$ of $J_n R$ is n-chiral Poisson if $a_{(k)} I \subset I$ for any $a \in R$ and any $k = 0, \ldots, n$.

**Lemma 7.1.** For a Poisson ideal $I$ of $R$, $J_n I$ is n-chiral Poisson. Here recall that for $I = (f_1, \ldots, f_r)$, $J_n I = (T^j f_i \mid i = 1, \ldots, r, j = 0, \ldots, n) \subset J_n R$.

**Proof.** This follows from (12) since $I$ is a Poisson ideal of $R$. 

If $I$ is an ideal of $J_n R$, we define the n-chiral Poisson core of $I$ to be the biggest n-chiral Poisson ideal contained in $I$. We denote it by $\mathcal{P}_{J_n R}(I)$.

Set $X := \text{Spec} R$. We define a relation $\sim$ on $J_n X = \text{Spec} J_n R$ by:
\[
x \sim y \iff \mathcal{P}_{J_n R}(m_x) = \mathcal{P}_{J_n R}(m_y),
\]
where $m_x$ denotes the maximal ideal of $J_n R$ corresponding to $x$.

Clearly $\sim$ is an equivalence relation. We denote the equivalence class in $J_n X$ of $x$ by $\mathcal{E}_{J_n X}(x)$ so that
\[
J_n X = \bigsqcup_x \mathcal{E}_{J_n X}(x).
\]
We call the set $\mathcal{E}_{J_n X}(x)$ the n-chiral symplectic core of $x$ in $J_n X$.

Similarly to the case of chiral Poisson cores in a vertex Poisson algebra, we obtain the following facts:

**Lemma 7.2.**

(1) Let $I$ be an ideal of $J_n R$. If $I$ is prime (resp. radical) then $\mathcal{P}_{J_n R}(I)$ is prime (resp. radical).

(2) Let $I$ be an ideal of $J_n R$. If $I$ is n-chiral Poisson, then so are the minimal prime ideals over $I$.

(3) Let $x \in J_n X$. Then $\mathcal{V}(\mathcal{P}_{J_n R}(m_x))$ is the smallest n-chiral Poisson subscheme of $J_n X$ containing $\mathcal{E}_{J_n X}(x)$.

(4) Let $Y$ be a reduced Poisson subscheme of $X$, and let $x \in J_n Y$. Then $\mathcal{E}_{J_n X}(x) = \mathcal{E}_{J_n Y}(x)$ and $\pi_{n,0}^X(\mathcal{E}_{J_n X}(x)) \subset \mathcal{E}_Y(\pi_{n,0}^X(x))$.

**Proof.** To prove Part (1), we argue as the the proof of Lemma 6.2. To prove Part (2), we argue as in the proof of Corollary 6.3. To prove Part (3), we argue as in the proof of Lemma 6.4. As for Part (4), we argue as in the proofs of Lemmas 6.5 and 6.7 (1). 

Lemma 7.3. Let \( Y \) be a reduced \( n \)-chiral Poisson subscheme of \( J_n X \), and \( x \in Y \). Let \( \{ x^1, \ldots, x^r \} \) be a generating set for \( R \), and consider the matrix \( \mathcal{M}_n(x) \) as in Section 5. Suppose that the matrix \( \mathcal{M}_n(x) \) has maximal rank \((n + 1)r\). Then the tangent space at \( x \) of \( Y \) has dimension at least \((n + 1)r\). Moreover, \( Y \) has dimension at least \((n + 1)r\).

Proof. Since \( Y \) is an \( n \)-chiral Poisson subscheme of \( J_n X \), \( Y = \text{Spec} J_n R / I \), where \( I \) is a reduced \( n \)-chiral Poisson of \( J_n R \). Moreover, \( I \subset \mathfrak{m}_x \) since \( x \in Y \), if \( \mathfrak{m}_x \) denotes the maximal ideal of \( J_n R \) corresponding to \( x \).

The hypothesis implies that the derivations \( x^i_{(k)} \), \( i = 1, \ldots, r \), \( k = 0, \ldots, n \), are linearly independent in \( \text{Der}(O_{X,x}, \mathbb{C}) \). Indeed, if for some \( \lambda^i_{(k)} \), \( i = 1, \ldots, r \), \( k = 0, \ldots, n \),

\[
\sum_{i=1}^{r} \sum_{k=0}^{n} \lambda^i_{(k)} x^i_{(k)} = 0 \quad \text{in} \quad \text{Der}(O_{X,x}, \mathbb{C}),
\]

then

\[
\sum_{i=1}^{r} \sum_{k=0}^{n} \lambda^i_{(k)} (x^i_{(k)}(T^j x^j) + \mathfrak{m}_x) = 0 \quad \text{for all} \quad j = 1, \ldots, r, \quad l = 0, \ldots, r,
\]

and so \( \lambda^i_{(k)} = 0 \) for all \( i = 1, \ldots, r \) and \( k = 0, \ldots, n \) since the matrix \( \mathcal{M}_n(x) \) has rank \((n + 1)r\).

Since \( I \) is \( n \)-chiral Poisson and is contained in \( \mathfrak{m}_x \), we get that for all \( k = 0, \ldots, n \), \( x(I) \subset I \subset \mathfrak{m}_x \). Hence, the derivations \( x^i_{(k)} \), \( i = 1, \ldots, r \), \( k = 0, \ldots, n \) are also linearly independent in \( \text{Der}(O_{Y,x}, \mathbb{C}) \) since \( O_{Y,x} = O_{X,x} / (O_{X,x} \cap I) \). This shows that the tangent space at \( x \) of \( Y \) has dimension at least \((n + 1)r\).

The set of points \( y \in Y \) such that matrix \( \mathcal{M}_n(y) \) has maximal rank \((n + 1)r\) is a nonempty open subset of \( Y \). Hence it meets the set of smooth points of \( Y \). By the first step, we deduce that for some smooth point \( y \in Y \), the tangent space \( T_y Y \) has dimension at least \((n + 1)r\). Therefore \( Y \) has dimension at least \((n + 1)r\). \( \square \)

Recall that \( \iota_n \) (resp. \( \iota_\infty \)) denotes the canonical embedding from \( X \) to \( J_n X \) (resp. \( J_\infty X \)). For \( x \in X \), we simply denote by \( x_n \) (resp. \( x_\infty \)) the element \( \iota_n(x) \) (resp. \( \iota_\infty(x) \)).

Proposition 7.4. Let \( x \in X \), and set \( Y := \mathcal{C}_X(x) \).

(1) For any \( n \in \mathbb{Z}_{\geq 0} \),

\[
(\pi^Y_n,0)^{-1}(Y_{\text{reg}}) = \mathcal{V}(\mathcal{P}_{J_n R}(\mathfrak{m}_x_n)).
\]

In particular, if \( J_n Y \) is irreducible, then

\[
J_n Y = \mathcal{V}(\mathcal{P}_{J_n R}(\mathfrak{m}_x_n)).
\]

(2) We have:

\[
J_\infty Y = \mathcal{V}(\mathcal{P}_{J_\infty R}(\mathfrak{m}_x_\infty))
\]

Proof. (1) By Lemma 7.2 (4), \( \mathcal{C}_{J_n X}(x_n) = \mathcal{C}_{J_n Y}(x_n) \), and

\[
\mathcal{C}_{J_n X}(x_n) \subset (\pi^Y_n,0)^{-1}(\mathcal{C}_X(x)) \subset (\pi^Y_n,0)^{-1}(Y_{\text{reg}})
\]

since by [BG, Lemma 3.3 (2)], \( \mathcal{C}_X(x) \) is smooth in its closure. Hence

\[
\mathcal{C}_{J_n X}(x_n) \subset (\pi^Y_n,0)^{-1}(Y_{\text{reg}}) \subset J_n Y.
\]
Since $(\pi_{n,0}^{Y})^{-1}(Y_{\text{reg}})$ is an irreducible component of $J_{n}Y$ and since $J_{n}Y$ is $n$-chiral Poisson (cf. Section 3), $(\pi_{n,0}^{Y})^{-1}(Y_{\text{reg}})$ is an $n$-chiral Poisson subscheme of $J_{n}X$. Hence by Lemma 6.4,

$$\mathcal{C}_{J_{n}X}(x_{n}) \subset \mathcal{V}(\mathcal{P}_{J_{n}A}(m_{x_{n}})) \subset (\pi_{n,0}^{Y})^{-1}(Y_{\text{reg}}).$$

Set $A = \mathbb{C}[Y]$. Let $x^{1}, \ldots, x^{r}$ be generators of $A$, where $r = \dim Y$. By [BG, Proposition 3.6] (proof of (2)), the matrix $\mathcal{M}_{0}$ has maximal rank $r$ at $m_{x}$. Hence, by Lemma 5.1, $\mathcal{M}_{n}$ has rank $(n + 1)r$ at $m_{x_{n}}$. Since $\mathcal{P}_{J_{n}A}(m_{x_{n}})$ is an $n$-chiral Poisson ideal of $J_{n}A$, it results from Lemma 7.3 that $\mathcal{V}(\mathcal{P}_{J_{n}A}(m_{x_{n}}))$ has dimension at least $(n + 1)r$. But

$$\dim (\pi_{n,0}^{Y})^{-1}(Y_{\text{reg}}) = (n + 1)r.$$

Since both $\mathcal{V}(\mathcal{P}_{J_{n}A}(m_{x_{n}}))$ and $(\pi_{n,0}^{Y})^{-1}(Y_{\text{reg}})$ are irreducible, we get the first assertion of (1). Indeed note that $\mathcal{V}(\mathcal{P}_{J_{n}A}(m_{x_{n}})) = \mathcal{V}(\mathcal{P}_{J_{n}R}(m_{x_{n}}))$ since $\mathcal{C}_{J_{n}X}(x_{n}) = \mathcal{C}_{J_{n}Y}(x_{n})$.

The second one follows from the fact that $(\pi_{n,0}^{Y})^{-1}(Y_{\text{reg}})$ is an irreducible component of $J_{n}Y$.

Part (2) follows from part (1) and Kolchin’s Theorem 3.5. \qed

8. Partial stratification by chiral symplectic leaves

Recall that there is a well-defined stratification of $X$ by symplectic leaves [BG]. We assume in this section that the Poisson bracket on $X$ is algebraic, that is, the symplectic leaves in $X$ are all locally closed. Then the symplectic leaves coincide with the symplectic cores of $X$, and the defining ideal of the symplectic core closure of a point $x \in X$ is $\mathcal{P}_{R}(m_{x})$ ([BG, Proposition 3.6]).

Let $x \in X$, and $\mathcal{L}_{X}(x)$ the symplectic leaf through $x$ in $X$. Set $Y = \mathcal{L}_{X}(x)$ and $A = \mathbb{C}[Y]$. Note that $(\pi_{n,0}^{Y})^{-1}(\mathcal{L}_{X}(x))$ is open in $J_{n}Y$ since $\mathcal{L}_{X}(x)$ is open in its closure $Y$. Moreover, $(\pi_{n,0}^{Y})^{-1}(\mathcal{L}_{X}(x)) = J_{n}\mathcal{L}_{X}(x)$ by [EM, Lemma 2.3]. In particular, $(\pi_{n,0}^{Y})^{-1}(\mathcal{L}_{X}(x))$ is a smooth analytic variety.

We consider a stratification of the set

$$(J_{n}X)^{\circ} := \bigsqcup_{x \in \mathcal{X}} J_{n}\mathcal{L}_{X}(x),$$

where $\mathcal{X}$ is an index set such that

$$X = \bigsqcup_{x \in \mathcal{X}} \mathcal{L}_{X}(x)$$

is the stratification of $X$ in symplectic leaves, as follows.

Let $x' \in (J_{n}X)^{\circ}$. Then $x' \in J_{n}\mathcal{L}_{X}(x) = (\pi_{n,0}^{Y})^{-1}(\mathcal{L}_{X}(x))$ for some $x \in \mathcal{X}$, with $Y = \mathcal{L}_{X}(x)$. We denote by $\mathcal{L}_{J_{n}X}(x')$ the set of all $y \in (\pi_{n,0}^{Y})^{-1}(\mathcal{L}_{X}(x))$ which can be reach from $x$ by traveling along the integral flows of vector fields $a_{(k)}$, $a \in A$, $k = 0, \ldots, n$, where $A = \mathbb{C}[Y]$. We call $\mathcal{L}_{J_{n}X}(x')$ the chiral symplectic leaf of $x'$ in $(J_{n}X)^{\circ} \subset J_{n}X$.

Lemma 8.1. Let $x \in (J_{n}X)^{\circ}$. Then the defining ideal of the closure of $\mathcal{L}_{J_{n}X}(x)$ is $\mathcal{P}_{J_{n}R}(m_{x})$.

Proof. First of all, by construction of $\mathcal{L}_{J_{n}X}(x)$, we have $\mathcal{L}_{J_{n}X}(x) \subset J_{n}Y$, where $Y = \mathcal{L}_{X}(\pi_{n,0}(x))$. So by Lemma 7.2 (4), $\mathcal{P}_{J_{n}R}(m_{x}) = \mathcal{P}_{J_{n}A}(m_{x})$, with $A = \mathbb{C}[Y]$. 



We now follow the ideas of the proof of [BG, Lemma 3.5]. Let $\mathcal{H}_x$ be the defining ideal of $\mathcal{L}_{J_n^X}(x)$. Let

$$J_n^Y := \text{Specm } J_n A, \quad \text{with } \widetilde{J_n A} := J_n A/\mathcal{P}_{J_n A}(m_x),$$

and denote by $\widetilde{a}$ the image of $a \in J_n A$ in $\widetilde{J_n A}$. For $r > 0$, $B(r)$ denotes the open complex analytic disc of radius $r$. Let $a \in A$ and $k \in \{0, \ldots, n\}$, and consider $\sigma_x : B(r) \to J_n Y$ and $\overline{\sigma_x} : B(r) \to J_n \overline{Y}$ be integral curves of the field $a(k)$ and $\overline{a(k)}$ respectively, with $\sigma_x(0) = x$, $\overline{\sigma_x}(0) = \overline{x}$.

Viewing $J_n^Y$ as a subset of $J_n Y$, let us show that $\overline{\sigma_x} = \sigma_x$ in a neighborhood of 0. Let $f \in J_n A$. By definition of an integral curve,

\begin{align*}
\frac{d}{dz}(f \circ \sigma_x) &= a(k)(f) \circ \sigma_x, \\
\frac{d}{dz}(\overline{f} \circ \overline{\sigma_x}) &= \overline{a(k)}(\overline{f}) \circ \overline{\sigma_x}.
\end{align*}

But the left hand side of (15) is $\frac{d}{dz}(f \circ \sigma_x)$, and the right hand side is $a(k)(f) \circ \sigma_x$ for the uniqueness of flows that $\sigma_x$ is a chiral Poisson ideal of $J_n A$. Hence by (14), we conclude that $\sigma_x = \overline{\sigma_x}$ in a neighborhood of 0. Since the chiral symplectic leaf $\mathcal{L}_{J_n^X}(x)$ is by definition obtained by traveling along integral curves to fields $a(k)$, the chiral symplectic leaf $\mathcal{L}_{J_n^X}(x)$, and so its closure, is contained in $\mathcal{V}(\mathcal{P}_{J_n A}(m_x))$, whence

$$\mathcal{P}_{J_n A}(m_x) \subset \mathcal{H}_x \subset m_x.$$

To show the equality $\mathcal{P}_{J_n A}(m_x) = \mathcal{H}_x$, it remains to prove that $\mathcal{H}_x$ is an $n$-chiral Poisson ideal of $J_n A$.

Let $f \in \mathcal{H}_x$, $a \in A$ and $k \in \{0, \ldots, n\}$. Let $\sigma_x : B(r) \to J_n Y$ be an integral curve to $a(k)$, with $\sigma_x(0) = x$. Then, by definition of an integral curve, (14) holds. On a complex analytic neighborhood of $x$, $f \circ \sigma_x = 0$ since the image of $\sigma_x$ is in $\mathcal{L}_{J_n^X}(x)$. Hence

$$\frac{d}{dz}(f \circ \sigma_x)(0) = a(k)(f) \circ \sigma_x(0) = a(k)(f)(x).$$

As a consequence, $a(k)(\mathcal{H}_x) \subset m_x$ for all $a \in A$ and all $k \in \{0, \ldots, n\}$. Repeating this argument with $x$ replaced by each of the members of $\mathcal{L}_{J_n^X}(x)$, we conclude that $a(k)(\mathcal{H}_x) \subset \mathcal{H}_x$ for all $a \in A$ and all $k \in \{0, \ldots, n\}$, that is, that $\mathcal{H}_x$ is $n$-chiral Poisson.

**Corollary 8.2.** Let $x \in (J_n X)^\circ$. Then the defining ideal of the closure of $\mathcal{G}_{J_n^X}(x)$ is $\mathcal{P}_{J_n R}(m_x)$.

**Proof.** By Lemma 8.1, $\mathcal{L}_{J_n^X}(x) \subset \mathcal{G}_{J_n^X}(x)$. Indeed, if $y \in \mathcal{L}_{J_n^X}(x)$, then $\mathcal{L}_{J_n^X}(y) = \mathcal{L}_{J_n^X}(x)$ and so $\mathcal{V}(\mathcal{P}_{J_n R}(m_y)) = \mathcal{V}(\mathcal{P}_{J_n R}(m_y))$ by Lemma 8.1, that is, $\mathcal{P}_{J_n R}(m_y) = \mathcal{P}_{J_n R}(m_x)$, whence $y \in \mathcal{G}_{J_n^X}(x)$. On the other hand, by Lemma 6.4 the defining ideal of the closure of $\mathcal{G}_{J_n^X}(x)$ contains $\mathcal{P}_{J_n R}(m_x)$. So by Lemma 8.1,

$$\mathcal{V}(\mathcal{P}_{J_n R}(m_x)) = \mathcal{L}_{J_n^X}(x) \subset \mathcal{G}_{J_n^X}(x) \subset \mathcal{V}(\mathcal{P}_{J_n R}(m_x)),$$

whence the statement. \qed
Corollary 8.3. Let \( x \in X \), and set \( Y := \overline{\mathcal{C}_X(x)} \). Then
\[
J_\infty Y = \overline{\mathcal{C}_{J_\infty X}(x_\infty)}.
\]
Moreover, for \( n \in \mathbb{Z}_{\geq 0} \), if \( J_n Y \) is irreducible, then \( J_n Y = \overline{\mathcal{C}_{J_n X}(x_n)} \).

Proof. Recall that \( \mathcal{L}_X(x) = \mathcal{C}_X(x) \) is the smooth locus of \( Y \), and that \( \pi_{n,0}^{-1}(\mathcal{C}_X(x)) \) has dimension \((n+1)\dim Y\).

Since \( \mathcal{P}_R(m_x) \) is the defining ideal of \( \overline{\mathcal{C}_X(x)} \), it results from Proposition 7.4 that for all \( n \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \), \( \mathcal{P}_{J_n R}(m_{x_n}) = K_{n,x} \), where \( K_{n,x} \) is the defining ideal of \((\pi_{n,0}^{-1}(Y_{\text{reg}}))^{-1} \). We have \( K_{n,x} = J_n \mathcal{P}_R(m_x) \) if \( J_n Y \) is irreducible. In particular, \( K_{\infty,x} = J_\infty \mathcal{P}_R(m_x) \).

By Proposition 7.4 and Corollary 8.2, we have for all \( n \in \mathbb{Z}_{\geq 0} \),
\[
\overline{\mathcal{C}_{J_n X}(x_n)} = \mathcal{V}(\mathcal{P}_{J_n R}(m_{x_n})) = \mathcal{V}(K_{n,x}) \subset \mathcal{V}(J_n \mathcal{P}_R(m_x)) = J_n Y.
\]
Taking the limit when \( n \) goes to \( +\infty \), we obtain:
\[
\overline{\mathcal{C}_{J_\infty X}(x_\infty)} = \mathcal{V}(\mathcal{P}_{J_\infty R}(m_{x_\infty})) = (\pi_{\infty,0}^{-1}(Y_{\text{reg}}))^{-1} \subset \mathcal{V}(J_\infty \mathcal{P}_R(m_x)) = J_\infty Y,
\]
whence
\[
J_\infty Y = \overline{\mathcal{C}_{J_\infty X}(x_\infty)}
\]
since \( J_\infty Y = (\pi_{\infty,0}^{-1}(Y_{\text{reg}}))^{-1} \). This concludes the proof. \( \square \)

Assume now that \( X \) has only finitely many symplectic leaves. In this case, the poisson bracket on \( X \) is algebraic ([BG, Proposition 3.6]). Moreover ([Gin]), if \( X_1, \ldots, X_r \) are the irreducible components of \( X \), then for some \( x_1, \ldots, x_r \in X \), we have
\[
X_i = \mathcal{V}(\mathcal{P}_R(m_i)) = \overline{\mathcal{C}_X(x_i)}, \quad i = 1, \ldots, r,
\]
where \( m_1, \ldots, m_r \) are the maximal ideals of \( R \) corresponding to \( x_1, \ldots, x_r \), respectively.

From the decomposition \( X = X_1 \cup \ldots \cup X_r \), we get that
\[
J_\infty X = J_\infty (X_1) \cup \ldots \cup J_\infty (X_r)
\]
since the \( X_i \) are closed (see Section 3). Moreover, \( J_\infty (X_1), \ldots, J_\infty (X_r) \) are precisely the irreducible components of \( J_\infty X \). Indeed, for \( i = 1, \ldots, r \), \( J_\infty (X_i) \) is closed in \( J_\infty X \) since \( X_i \) is closed in \( X \), and for any \( i \neq j \), we have \( J_\infty (X_i) \subsetneq J_\infty (X_j) \), otherwise, taking the image by the canonical projection \( \pi_{\infty,0} : J_\infty X \to X \), we would get \( X_i \subset X_j \).

Hence, as a consequence of Corollary 8.3, we obtain the following result.

Theorem 8.4. Let \( X \) be a Poisson scheme. Assume that \( X \) has only finitely many symplectic leaves. Then each irreducible components of \( J_\infty X \) is the closure of some chiral symplectic core.

More precisely, if \( X_1, \ldots, X_r \) are the irreducible components of \( X \), then for \( i = 1, \ldots, r \), \( X_i = \overline{\mathcal{C}_X(x_i)} \) for some \( x_i \in X_i \), and we have:
\[
J_\infty X_i = J_\infty \overline{\mathcal{C}_X(x_i)} = \overline{\mathcal{C}_{J_\infty X}(x_i)}.
\]
9. Applications to quasi-lisse vertex algebras

In this section we assume that $V$ is any vertex algebra (not necessarily commutative or Poisson).

Recall that $V$ is naturally filtered by the Li filtration ([Li] or [A1]),

$$V = F^0 V \supset F^1 V \supset \cdots \supset F^p V \supset \cdots,$$

where $F^p V$ is the subspace of $V$ spanned by the vectors

$$a_1^{(-n_1-1)} \cdots a_r^{(-n_r-1)} b,$$

with $a_i \in V$, $b \in V$, $n_i \in \mathbb{Z} \geq 0$, $n_1 + \cdots + n_r \geq p$. Let $\text{gr} V = \bigoplus_p F^p V / F^{p+1} V$ be the associated graded vector space. It is well-known that the space $\text{gr} V$ is naturally a vertex Poisson algebra $[\text{Li}, A1]$. We have $[\text{Li}]$

$$F^1 V = C_2(V) := \text{span}_C \{a^{(-2)} b \mid a, b \in V\}.$$

Let

$$R_V = V / C_2(V) = F^0 V / F^1 V \subset \text{gr} V$$

be the Zhu’s $C_2$-algebra of $V$. It is a Poisson algebra [Zhu], and the Poisson algebra structure can be obtained by restriction to $R_V$ of the vertex Poisson algebra on $\text{gr} V$. Namely,

$$\{\bar{a}, \bar{b} \} = \bar{a^{(0)}} \bar{b},$$

for $a, b \in V$, where $\bar{a} = a + C_2(V)$.

Let

$$\tilde{X}_V := \text{Spec}(R_V) \quad \text{and} \quad X_V := \text{Specm}(R_V)$$

be the associated scheme and the associated variety of $V$ respectively.

We assume that the filtration $(F^p V)_p$ is separated, that is, $\bigcap F^p V = \{0\}$ and that $V$ is strongly finitely generated, that is, $R_V$ is finitely generated.

It was shown in [Li, Lemma 4.2] that $\text{gr} V$ is generated by the subring $R_V$ as a differential algebra. Thus, we have a surjection $J_\infty(R_V) \to \text{gr} V$ of differential algebras by the universal property of $J_\infty(R_V)$ (cf. Remark 3.3) since $R_V$ generates $J_\infty(R_V)$ as a differential algebra too.

This is in fact a homomorphism of vertex Poisson algebras:

**Theorem 9.1** ([Li, Lemma 4.2], [A1, Proposition 2.5.1]). The identity map $R_V \to R_V$ induces a surjective vertex Poisson algebra homomorphism

$$J_\infty(R_V) = \mathbb{C}[J_\infty(\tilde{X}_V)] \to \text{gr} V.$$

The singular support of a vertex algebra $V$ is

$$\text{SS}(V) := \text{Spec} \text{gr} V \subset J_\infty(\tilde{X}_V).$$

Recall from the introduction that the vertex algebra $V$ is called quasi-lisse if the Poisson variety $X_V$ has finitely many symplectic leaves ([?]). We refer to [A3, AM1, AM2, AM3] for many examples of simple quasi-lisse vertex algebras.

**Theorem 9.2.** Assume that $V$ is quasi-lisse. Then $\text{SS}(V)_{\text{red}}$ is a finite union of chiral symplectic cores closures in $\text{gr} V$. Moreover, $\text{SS}(V)_{\text{red}} = J_\infty X_V$. 

and the compound map is $\iota_\infty$. The Hamiltonian reduction can also be described in terms of BRST cohomology, is no nonzero proper chiral Poisson subscheme in $L$. Furthermore, the surjective morphisms, $J_\infty(R_V) \rightarrow grV \rightarrow R_V$, induce injective morphisms of varieties, $X_V \rightarrow L \rightarrow J_\infty(X_V)$, and the compound map is $\iota_\infty$. Hence for $x \in X$, we get that $m_\infty \supset I$, where $m_\infty$ denotes the maximal ideal of $J_\infty(R_V)$ corresponding to $x_\infty$, and so $x_\infty$ is a point of $L$.

Therefore, by Lemma 6.5, $CJ_\infty(x_i,\infty) = C_L(x_i,\infty)$ for any $i = 1, \ldots, r$. Then from (16) and Theorem 9.1, we obtain that $L \subset J_\infty(X_V) = C_L(x_1,\infty) \cup \ldots \cup C_L(x_r,\infty) \subset L$, since $L$ is closed, whence the first statement and the required equality $L = J_\infty(X_V)$. □

**Corollary 9.3.** Suppose that $X_V$ is smooth, reduced and symplectic. Then $grV$ is simple as a vertex Poisson algebra, and hence, $V$ is simple.

**Proof.** If $X_V$ is a smooth symplectic variety then $J_\infty X_V$ consists of a single chiral symplectic core. So $J_\infty X_V = CJ_{j\infty,X_V}(x)$ for any $x \in J_\infty X_V$. It follows that there is no nonzero proper chiral Poisson subscheme in $J_\infty X_V$. So by Theorem 9.2, there is no nonzero proper chiral Poisson subscheme in $L = Spec grV$, too. Hence $grV$ is simple as a vertex Poisson algebra. And so $V$ is simple since any vertex ideal $I \subset V$ defined a vertex Poisson, and so chiral Poisson, ideal $gr I$ in $gr V$. □

For example, if $X$ is a smooth affine variety, then the global section of the chiral differential operators $D^{ch}_X$ ([MSV, GMS2, BD2]) is simple, because its associated scheme is canonically isomorphic to $T^* X$. In particular, the global section of the chiral differential operators $D^{ch}_{G,k}$ on $G$ ([GMS1, AG]) is simple at any level $k$.

### 10. Adjoint Quotient and Arc Space of Slodowy Slice

Recall that $\mathfrak{g}$ is simple. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$. Let $(\epsilon, h, f)$ be an $\mathfrak{sl}_2$-triple, and $\mathcal{J}_f = f + \mathfrak{g}^e$ the corresponding Slodowy slice, with $\mathfrak{g}^e$ the centralizer of $\epsilon$ in $\mathfrak{g}$. Recall that $\mathfrak{g}^e \cong \mathfrak{g}$ is a Poisson variety and that the symplectic leaves of $\mathfrak{g}^e \cong \mathfrak{g}$ are the (co)adjoint orbits. The algebra $R_f := C[\mathcal{J}_f]$ inherits a Poisson structure from $C[\mathfrak{g}]$ by Hamiltonian reduction [GG]. The Hamiltonian reduction can also be described in terms of BRST cohomology, essentially following Kostant and Sternberg [KS]. The symplectic leaves of $\mathcal{J}_f$ are precisely the intersections of the adjoint orbits of $\mathfrak{g}$ with $\mathcal{J}_f$.

Let $p_1, \ldots, p_\ell$ be homogeneous generators of $C[\mathfrak{g}^*]^G \cong C[\mathfrak{g}]^G \cong C[\mathfrak{g}]^0$. Recall that $deg p_i = m_i + 1$ where $m_1, \ldots, m_\ell$ are the exponents of $\mathfrak{g}$. 

---

**Proof.** Let $L$ be the maximal spectrum of $gr V$, that is, $L = Spec gr V = SS(V)_{red}$.

Let $X_1, \ldots, X_r$ be the irreducible components of $X_V$. By Theorem 8.4, we have

$$J_\infty(X_V) = \overline{CJ_{j\infty,X_V}(x_1,\infty) \cup \ldots \cup C_{j\infty,X_V}(x_r,\infty)},$$

where $x_i \in (X_i)_{\text{reg}}$ for $i = 1, \ldots, r$. By Theorem 9.1, $gr V$ is a vertex algebra quotient of $J_\infty(R_V)$, that is, $gr V = J_\infty(R_V)/I$ with $I$ a vertex Poisson ideal of $J_\infty(R_V)$.
Consider the adjoint quotient map
\[ \psi : \mathfrak{g} \to \mathfrak{g}/G \cong \mathbb{C}^\ell, \quad x \mapsto (p_1(x), \ldots, p_\ell(x)), \]
and its restriction \( \psi_f \) to \( \mathcal{S}_f \),
\[ \psi_f : \mathcal{S}_f \to \mathfrak{g}/G \cong \mathbb{C}^\ell. \]
We first recall some facts about \( \psi_f \) and its fibers ([Pre1]).

The morphism \( \psi_f \) is faithfully flat. As a consequence, \( \psi_f \) is surjective and all fibers have the dimension \( r - \ell \) where \( r = \dim \mathfrak{g}^s \). Furthermore the fibers of \( \psi_f \) are generically smooth, that is, contain a smooth open dense subset of dimension \( r - \ell \), and they are irreducible.

More precisely, we have the following result.

**Lemma 10.1.** Let \( \xi \in \mathfrak{g}/G \). Then \( \psi_f^{-1}(\xi) \) is a finite union of symplectic leaf closures of \( \mathcal{S}_f \). Hence it is the closure of some symplectic leaf closure.

**Proof.** First show that \( \psi^{-1}(\xi) \) is a finite union of symplectic leaf closures of \( \mathfrak{g} \). The proof is standard, we recall it for the convenience of the reader. Let \( x \in \psi^{-1}(\xi) \), and write \( x = x_s + x_n \) its Jordan decomposition. Let \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} + \mathfrak{n}_+ \) be a triangular decomposition of \( \mathfrak{g} \). One can assume that \( x_s \in \mathfrak{h} \), and that \( x_n \in \mathfrak{n}_+ \) since \( x_n \in \mathfrak{g}^{x_s} \).

Let now \( y \in \psi^{-1}(\xi) \). Then \( p_i(y) = p_i(x) \) for \( i = 1, \ldots, \ell \) and so \( y_s \) is conjugate to \( x_s \) by an element \( s \) of the Weyl group \( W(\mathfrak{g}, \mathfrak{h}) \) of \( (\mathfrak{g}, \mathfrak{h}) \). Choose a representative \( \hat{s} \) of \( s \) in \( G \) so that \( y_s = \hat{s}(x_s + \hat{s}^{-1}y_n) \). Since \( W(\mathfrak{g}, \mathfrak{h}) \) is finite, one can assume that \( y_s = x_s \). Next, \( y_n \in \mathfrak{g}^{x_s} \cap \mathcal{N} \). The set \( \mathfrak{g}^{x_s} \cap \mathcal{N} \) consists of a finite union of \( L \)-orbits, where \( L \) is the connected subgroup of \( G \) with Lie algebra \( \mathfrak{g}^{x_s} \), and so it consists of a finite union of \( G \)-orbits. In conclusion, \( \psi^{-1}(\xi) \) is a finite union of \( G \)-orbits of \( x_s \), that is, a finite union of symplectic leaves of \( \mathfrak{g} \). Since \( \psi^{-1}(\xi) \) is irreducible and \( G \)-invariant, we deduce that \( \psi^{-1}(\xi) \) is the closure of some \( G \)-orbit, that is, the closure of some symplectic leaf.

Next, since \( \psi_f^{-1}(\xi) = \psi^{-1}(\xi) \cap \mathcal{S}_f \) and since the symplectic leaves of \( \mathcal{S}_f \) are the intersections of adjoint orbits of \( \mathfrak{g} \) with \( \mathcal{S}_f \), we deduce that \( \psi_f^{-1}(\xi) \) is a finite union of symplectic leaf. By [Gin], \( \psi_f^{-1}(\xi) \) is then a symplectic leaf closure since \( \psi_f^{-1}(\xi) \) is irreducible.

By Kostant [Kos],
\[ \psi_f^{-1}(0) = \mathcal{S}_f \cap \mathcal{N}, \]
with \( \mathcal{N} \) the nilpotent cone of \( \mathfrak{g} \). In particular, \( \psi_f^{-1}(0) \) has finitely many symplectic leaves. Moreover, since \( \mathcal{N} \) is normal with rational singularities and \( \mathcal{S}_f \) is a transversal slice to \( \mathcal{N} = \overline{\mathcal{O}_{reg}} \), the intersection \( \mathcal{S}_f \cap \mathcal{N} \) enjoys the same properties as \( \mathcal{N} \) (cf. e.g. [FJLS, §3.3]). Hence \( \mathcal{S}_f \cap \mathcal{N} \) is normal with rational singularities too (see also [AKM], where the argument is slightly different). Here, \( \mathcal{O}_{reg} \) denotes the regular nilpotent orbit in \( \mathfrak{g} \).

**Proposition 10.2.** Let \( n \in \mathbb{Z}_{\geq 0} \). Then \( (J_n \psi_f)^{-1}(0) = J_n(\psi_f^{-1}(0)) \) is a reduced complete intersection, and it is irreducible. Moreover, \( (J_\infty \psi_f)^{-1}(0) = J_\infty(\psi_f^{-1}(0)) \) is irreducible.

**Proof.** Since \( \psi_f^{-1}(0) = \mathcal{S}_f \cap \mathcal{N} \) is a reduced scheme, which is irreducible and a complete intersection by the above discussion, it follows from the main results of [Mus] and its consequences that \( J_n(\psi_f^{-1}(0)) \) is also reduced, irreducible and a
complete intersection for any $n \geq 0$. Now observe that $(J_n \psi_f)^{-1}(0) = J_n(\psi_f^{-1}(0))$ by the properties of jet schemes since a fiber is a fiber product (cf. Section 3). This proves the first part of the statement.

Since $\psi_f^{-1}(0)$ is irreducible, $J_\infty(\psi_f^{-1}(0))$ is irreducible by Theorem 3.5 and, from the above, we get that $(J_\infty \psi_f)^{-1}(0) = J_\infty(\psi_f^{-1}(0))$. \hfill \Box

Next, we wish to prove that the other fibers of $J_\infty \psi_f$ are also reduced and irreducible. To this end, we use ideas of [Pre1, §§5.3 and 5.4].

The Slodowy slice has a contracting $\mathbb{C}^*$-action. Recall briefly the construction. The embedding $\operatorname{span}_\mathbb{C} \{e, h, f\} \cong \mathfrak{sl}_2 \to \mathfrak{g}$ exponentiates to a homomorphism $SL_2 \to G$. By restriction to the one-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup $\tilde{\rho}: \mathbb{C}^* \to G$. Thus $\tilde{\rho}(t)x = t^{2j}x$ for any $x \in \mathfrak{g}_j = \{y \in \mathfrak{g} \mid [h, y] = 2jy\}$. For $t \in \mathbb{C}^*$ and $x \in \mathfrak{g}$, set

$$\rho(t)x := t^2 \tilde{\rho}(t)x.$$  

So, for any $x \in \mathfrak{g}_j$, $\rho(t)x = t^{2+2j}x$. In particular, $\rho(t)f = f$ and the $\mathbb{C}^*$-action of $\rho$ stabilizes $\mathscr{F}_f$. Moreover, it is contracting to $f$ on $\mathscr{F}_f$, that is,

$$\lim_{t \to 0} \rho(t)(f + x) = f$$

for any $x \in \mathfrak{g}^e$. The $\mathbb{C}^*$-action $\rho$ induces a positive grading on $\mathscr{F}_f$, and so on $R_f = \mathbb{C}[\mathscr{F}_f] \cong \mathbb{C}[\mathfrak{g}^e]$. Namely, let $(R_f)_i$ be the set of all $\chi \in R_f$ such that $\chi$ vanishes on $(\mathscr{F}_f)_j$, for $j \neq i$, with

$$(\mathscr{F}_f)_j := \{x \in \mathscr{F}_f \mid \rho(t)x = t^{2j}x\}.$$  

Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Similarly, we define a contracting $\mathbb{C}^*$-action on $J_n \mathscr{F}_f$ and a positive grading on $J_n R_f$ as follows.

Let $x^1, \ldots, x^r$ be a basis of $\mathfrak{g}^e$ so that

$$J_n R_f \cong J_\infty \mathbb{C}[\mathfrak{g}^e] \cong \text{Spec} \mathbb{C}[x^i_{(-j-1)}]_{i = 1, \ldots, r, j = 0, \ldots, n}.$$  

One can assume that the $x^i$'s are Slodowy homogeneous. One defines a grading on $J_n R_f$ by setting

$$\deg x^i_{(-j-1)} = \deg x^i + j.$$  

Since the grading is positive, it gives a contracting $\mathbb{C}^*$-action on $J_n \mathscr{F}_f$. Indeed, consider the morphism $\mathbb{C}[J_n \mathscr{F}_f] \to \mathbb{C}[J_n \mathscr{F}_f] \otimes \mathbb{C}[t, t^{-1}], f \mapsto f \otimes t^{\deg f}$, for homogeneous $f$. Its comomorphism induces a $\mathbb{C}^*$-action

$$\mu_n: \mathbb{C}^* \times J_n \mathscr{F}_f \to J_n \mathscr{F}_f$$

which is contracting since $\deg f \geq 0$ for any homogeneous $f$.

The above grading gives an increasing filtration on $J_n R_f$ in an obvious way:

$$\mathscr{F}_p(J_n R_f) := \oplus_{j \leq p}(J_n R_f)_j, \quad p \geq 0.$$  

Given a quotient $M$ of $J_n R_f$, we define a filtration $(\mathscr{F}_p M)_p$ of $M$ by setting

$$\mathscr{F}_p M := \tau_{\mu_n}(\mathscr{F}_p(J_n R_f)),$$

where $\tau$ is the canonical quotient morphism $\tau: J_n R_f \to M$, and we denote by $\text{gr}_{\mu_n} M$ the corresponding graded space.

For $M$ a subspace of $J_n R_f$ denote by $\text{gr}_{\mu_n} M$ the homogeneous subspace of $J_n R_f$ with the property that $g \in \text{gr}_{\mu_n} M \cap (J_n R_f)_0$ if and only if there is $\tilde{g} \in M$ such that $\tilde{g} - g \in \mathscr{F}_{p-1}(J_n R_f)$. Obviously the subspace $\text{gr}_{\mu_n} M$ is invariant for the $\mathbb{C}^*$-action $\mu_n$. If $M$ is an ideal of $J_n R_f$ then so is $\text{gr}_{\mu_n} M$.  


By [EF], observe that $J_n \psi_f$ corresponds to the following adjoint quotient morphism

$$J_n \psi_f : J_\infty \mathcal{J}_f \rightarrow J_n \mathcal{C}_f, \quad x \mapsto (T^j p_i(x), i = 1, \ldots, \ell, j = 0, \ldots, n),$$

since $\mathbb{C}[J_n(\mathfrak{g}/G)] = \mathbb{C}[J_n \mathfrak{g}] / (T^j p_i, i = 1, \ldots, \ell, j = 0, \ldots, n)$.

For $f \in \mathbb{C}[J_n \mathfrak{g}]$ we denote by $\overline{f}$ its restriction to $J_n \mathcal{J}_f$. Then for $\xi = (\xi^{(j)}_i | i = 1, \ldots, \ell, j = 0, \ldots, n) \in J_n \mathcal{C}_f$, $(J_n \psi_f)^{-1}(\xi)$ is the set of common zeroes of the ideal

$$\mathcal{I}_{n, \xi} := \overline{(T^j p_i - \xi^{(j)}_i | i = 1, \ldots, \ell, j = 0, \ldots, n)}.$$  

**Lemma 10.3.** Let $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ and $\xi \in J_n \mathcal{C}_f$. Then the fiber $(J_n \psi_f)^{-1}(\xi)$ is reduced and irreducible.

**Proof.** Clearly $\text{gr}_{\mu_n} \mathcal{I}_{n, \xi} = \mathcal{I}_{n, 0}$.

Let $a \in J_n R_f / \mathcal{I}_{n, \xi}$ and suppose that $a^k = 0$ for some $k \in \mathbb{Z}_{>0}$. Then $\sigma_n(a)^k = 0$, where $\sigma_n$ is the symbol of $a$ in $\text{gr}_{\mu_n}(J_n R_f / \mathcal{I}_{n, \xi}) = J_n R_f / \mathcal{I}_{n, 0}$. Since $\mathcal{I}_{n, 0}$ is reduced, $\sigma_n(a) = 0$, and hence $a = 0$. This proves that $\mathcal{I}_{n, \xi}$ is radical.

Similarly, $J_n R_f / \mathcal{I}_{n, \xi}$ is a domain since $\text{gr}_{\mu_n}(J_n R_f / \mathcal{I}_{n, \xi}) = J_n R_f / \mathcal{I}_{n, 0}$ is. Hence $\mathcal{I}_{n, \xi}$ is prime. \hfill \Box

Note that the Poisson bracket on $\mathcal{J}_f$ is algebraic. Indeed, the Poisson bracket on $\mathfrak{g}$ is algebraic since all adjoint orbits are open in their closure. Therefore the same goes for the symplectic leaves in $\mathcal{J}_f$ since they are of the form $G.x \cap \mathcal{J}_f$, for $x \in \mathfrak{g}$. As a result, the hypothesis of Corollary 8.3 is satisfied.

**Proposition 10.4.**

(1) For any $n \in \mathbb{Z}_{\geq 0}$, the fiber $(J_n \psi_f)^{-1}(0)$ is the closure of some $n$-chiral Poisson core in $J_n R_f$, and $(J_\infty \psi_f)^{-1}(0)$ is the closure of some chiral Poisson core in $J_\infty R_f$.

(2) Let $\xi \in J_\infty(\mathfrak{g}/G)$. For any $n \in \mathbb{Z}_{\geq 0}$, the fiber $(J_n \psi_f)^{-1}(\xi)$ is the closure of some $n$-chiral Poisson core in $J_n R_f$, and $(J_\infty \psi_f)^{-1}(\xi)$ is the closure of some chiral Poisson core in $J_\infty R_f$.

**Proof.** (1) We have $\psi_f^{-1}(0) = \mathcal{N} \cap \mathcal{J}_f$ and so $\psi_f^{-1}(0)$ has finitely many symplectic leaves. Since $\psi_f^{-1}(0)$ is irreducible and reduced, $\psi_f^{-1}(0)$ is the closure of some symplectic core in $\mathcal{J}_f$. On the other hand, as already observed, $\mathcal{N} \cap \mathcal{J}_f$ is a compete intersection with rational singularities. Hence by the main result of [Mus], $J_n(\psi_f^{-1}(0))$ is irreducible for any $n \in \mathbb{Z}_{\geq 0}$. By Corollary 8.3 we get the statement.

(2) Let $\xi \in J_n(\mathfrak{g}/G)$. Then $\pi_{n, \xi}((J_n \psi_f)^{-1}(\xi)) = \psi_f^{-1}(\xi_0)$ with $\xi_0 = \pi_{\mathcal{N}, \xi}(\xi)$. Since $\psi_f^{-1}(\xi_0)$ is a finite union of symplectic leaf closures (cf. Lemma 10.1), and since $\psi_f^{-1}(\xi_0)$ is reduced and irreducible we get that $\psi_f^{-1}(\xi_0) = \overline{\mathcal{E}_n \mathcal{J}_f(x_0)} =: Y$ for some $x_0 \in (\psi_f^{-1}(\xi_0))_{\text{reg}}$.

Let $x \in (Y^n)_{\text{reg}} \cap (J_n \psi_f)^{-1}(\xi)$, this intersection being nonempty. Then by Lemma 5.1, $\text{rk} x = (n + 1)j$ if $\text{rank} \mathcal{M}_0(x_0) = j$. Hence by Corollary 8.3 and the proof of Proposition 7.4 (1),

$$\dim \overline{\mathcal{E}_n \mathcal{J}_f(x)} \geq (n + 1)j.$$

From Theorem 4.2 and the fact that the $T^j p_i$’s belongs to the Poisson center of $\mathbb{C}[g]$, we get that $(J_n \psi_f)^{-1}(\xi)$ is a closed chiral Poisson subscheme of $J_n \mathcal{J}_f$. It follows that $\overline{\mathcal{E}_n \mathcal{J}_f(x)} \subset (J_n \psi_f)^{-1}(\xi)$. \hfill \Box
Since \( J_n \psi_f \) is a flat morphism, all its fibers have the same dimension. But \((J_n \psi_f)^{-1}(0) = J_n(\psi_f^{-1}(0))\) has dimension 
\[(n + 1) \dim \psi_f^{-1}(0) = (n + 1) \dim \psi_f^{-1}(\xi_0) = (n + 1) j\]
since by \([BG, \text{ Proposition } 3.6 \,(2)]\), \(\dim \mathcal{E}_f(x_0) = j\). In conclusion, \(\dim (J_n \psi_f)^{-1}(\xi) = \dim \mathcal{E}_f(x)\) and so \((J_n \psi_f)^{-1}(\xi) = \mathcal{E}_f(x)\) since by Lemmas 7.2 and 10.3, both \((J_n \psi_f)^{-1}(\xi)\) and \(\mathcal{E}_f(x)\) are irreducible and closed.

We conclude the section by the following important result.

**Theorem 10.5.** Let \( z \) be in the vertex Poisson center of \( J_\infty R_f \), and \( \xi \in J_\infty (\mathfrak{g}/G) \). Then \( z \) is constant on \((J_\infty \psi_f)^{-1}(\xi)\).

**Proof.** The statement is clear by Proposition 6.6 and Proposition 10.4 (2). \( \square \)

**Remark 10.6.** One can also prove Theorem 10.5 using only Proposition 6.6 and Proposition 10.4 (1). Indeed, by Proposition 6.6 and Proposition 10.4 (1), any element \( z \) in the vertex Poisson center of \( J_\infty R_f \) is constant on \((J_\infty \psi_f)^{-1}(0)\). Let now \( \xi \in J_\infty (\mathfrak{g}/G) \). Then the symbol \( \sigma_{\infty}(z) \in \text{gr}_{\mu_{\infty}}(J_\infty R_f/I_{\infty, \xi}) \) belongs to the center \( Z(J_\infty R_f/I_{\infty, 0}) \) of \( \text{gr}_{\mu_{\infty}}(J_\infty R_f/I_{\infty, \xi}) \). However, \( Z(J_\infty R_f/I_{\infty, 0}) \cong \mathbb{C} \) by the \( \xi = 0 \) case. Therefore, \( z \) is constant in \( J_\infty R_f/I_{\infty, \xi} \).

11. **Vertex Poisson center and arc space of Slodowy slices**

The adjoint quotient morphism \( \psi_f : \mathcal{F}_f \to \mathfrak{g}/G \) induces an embedding from \( \mathbb{C}[\mathfrak{g}/G] \cong \mathbb{C}[\mathfrak{g}]^G \), the Poisson center of \( \mathbb{C}[\mathfrak{g}] \), into the Poisson algebra \( R_f = \mathbb{C}[\mathcal{F}_f] \).

The vertex Poisson algebra structure on \( \mathbb{C}[J_\infty R_f] \) can be described using cohomology of some dg-vertex Poisson algebras, which is a tensor product of functions over \( J_\infty \mathfrak{g} \) with fermionic-ghost vertex Poisson super-algebra \( \wedge \overline{\mathfrak{p}}(m) \), where \( m \) is a certain nilpotent algebra \( m \) ([A3, Theorem 4.6]):

\[
\mathbb{C}[J_\infty R_f] \cong H^0(\mathbb{C}[J_\infty \mathfrak{g}] \otimes \wedge \overline{\mathfrak{p}}(m), Q(0)).
\]

The canonical embedding

\[
\mathbb{C}[J_\infty \mathfrak{g}] \hookrightarrow \mathbb{C}[J_\infty \mathfrak{g}] \otimes \wedge \overline{\mathfrak{p}}(m), \quad f \mapsto f \otimes 1,
\]
induces morphisms of vertex Poisson algebras,

\[
Z(\mathbb{C}[J_\infty \mathfrak{g}]) \to Z(\mathbb{C}[J_\infty \mathfrak{g}] \otimes \wedge \overline{\mathfrak{p}}(m)) \to Z(H^0(\mathbb{C}[J_\infty \mathfrak{g}] \otimes \wedge \overline{\mathfrak{p}}(m), Q(0))).
\]

Hence we get a morphism of vertex Poisson algebras,

\[
Z(\mathbb{C}[J_\infty \mathfrak{g}]) \to Z(\mathbb{C}[J_\infty R_f]) \hookrightarrow \mathbb{C}[J_\infty R_f].
\]

Note that this morphism corresponds to the restriction map.

On the other hand, we have an isomorphism \([RsT, \text{ BD1, EF}]\): \( J_\infty (\mathfrak{g}/G) \cong J_\infty \mathfrak{g}/J_\infty G \), whence the following isomorphisms:

\[
Z(\mathbb{C}[J_\infty \mathfrak{g}]) = \mathbb{C}[J_\infty \mathfrak{g}]/J_\infty G \cong \mathbb{C}[J_\infty (\mathfrak{g}/G)].
\]

Therefore the above morphism from \( Z(\mathbb{C}[J_\infty \mathfrak{g}]) \) to \( \mathbb{C}[J_\infty R_f] \) is nothing but the comorphism

\[
(J_\infty \psi_f)^* : J_\infty (\mathfrak{g}/G) \to J_\infty R_f
\]
of \( J_\infty \psi_f \).
Let \( \mathfrak{g}_{\text{reg}} \) be the set of regular elements of \( \mathfrak{g} \), that is, those elements whose centralizer has minimal dimension \( \ell \). Since the restriction of the morphism \( \psi_f \) to \( J_f \cap \mathfrak{g}_{\text{reg}} \) is smooth and surjective (see [Kos], [Pre1, Section 5]), the restriction of \( J_n \psi_f \) to \( J_n( J_f \cap \mathfrak{g}_{\text{reg}} ) \) is smooth and surjective for any \( n \) too ([EM, Remark 2.10] or [Pre, §3.4.3]). Therefore the morphism \( ( J_n \psi_f )^* : \mathbb{C}[ J_{\infty} ]_{J^G_f} \to J_f R_f \) is an embedding for any \( n \). Moreover, the restriction of \( J_{\infty} \psi_f \) to \( J_{\infty}( J_f \cap \mathfrak{g}_{\text{reg}} ) \) is (formally) smooth and surjective, whence the morphism \( ( J_{\infty} \psi_f )^* : \mathbb{C}[ J_{\infty} ]_{J^G_f} \to J_{\infty} R_f \) is an embedding of vertex Poisson algebras. The above discussion shows that its image lies in the vertex Poisson center of \( J_{\infty} R_f \). Hence we get embedding of vertex Poisson algebras from \( \mathbb{C}[ J_{\infty} ]_{J^G_f} \) into \( Z(J_{\infty} R_f) \).

The aim of this section is to prove the following result.

**Theorem 11.1.** The morphism \( ( J_{\infty} \psi )^* \) induces an isomorphism of Poisson vertex algebras between \( \mathbb{C}[ J_{\infty} ]_{J^G_f} \) and the Poisson vertex center of \( \mathbb{C}[ J_f ]_{J^G_f} \). Moreover, \( \mathbb{C}[ J_{\infty} ]_{J_f} \) is free over its vertex Poisson center \( Z( \mathbb{C}[ J_{\infty} ]_{J_f} ) \cong \mathbb{C}[ J_{\infty} ]_{J^G_f} \).

Note that \( \mathbb{C}[ J_{\infty} ]_{J^G_f} \cap \mathbb{C}[ \mathfrak{g} ] = \mathbb{C}[ \mathfrak{g} ]^G \) and \( Z( \mathbb{C}[ J_{\infty} ]_{J_f} ) \cap \mathbb{C}[ \mathfrak{g} ] = Z( \mathbb{C}[ \mathfrak{g} ]_{J_f} ) \), the Poisson center of \( \mathbb{C}[ \mathfrak{g} ]_{J_f} \). Hence from the above theorem we recover the well-known result of Ginzburg-Premet [Pre2, Question 5.1] which states that

\[
Z( \mathbb{C}[ \mathfrak{g} ]_{J_f} ) \cong \mathbb{C}[ \mathfrak{g} ]^G.
\]

To prove Theorem Th:main-result we first state some preliminary results.

**Lemma 11.2** ([GW, Theorem A.2.9]). Let \( X, Y, Z \) be irreducible affine varieties. Assume that \( f : X \to Y \) and \( h : X \to Z \) are dominant morphisms such that \( h \) is constant on the fibers of \( f \). Then there exists a rational map \( g : Y \to Z \) making the following diagram commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
Z & & \\
\end{array}
\]

**Lemma 11.3.** Let \( X \) and \( Y \) be two normal irreducible affine varieties, and \( f : X \to Y \) a flat morphism. Then \( \mathbb{C}(Y) \cap \mathbb{C}[X] = \mathbb{C}[Y] \). Here, we view \( \mathbb{C}[Y] \) as a subalgebra of \( \mathbb{C}[X] \) using \( f^* : \mathbb{C}[Y] \to \mathbb{C}[X] \).

**Proof.** Since \( X \) is normal and the fibers of \( f \) are all of dimension \( \dim X - \dim Y \), the image of the set \( X' \) of smooth points of \( X \) is an open subset \( Y' \) of \( Y \) such that \( Y \setminus Y' \) has codimension at least 2.

Let \( y \in Y' \) and \( x \in f^{-1}(y) \subset X' \). Then we have a flat extension of the local rings \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \). Since \( \mathcal{O}_{Y,y} \) and \( \mathcal{O}_{X,x} \) are regular local rings, they are factorial. For \( a \in K(Y) \cap \mathbb{C}[X] \), write \( a = p/q \) with \( p, q \) relatively prime elements of \( \mathbb{C}[Y] \). Since \( p, q \) are relatively prime, the multiplication by \( p \) induces an injective homomorphism

\[
\mathcal{O}_{Y,y}/q\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/q\mathcal{O}_{Y,y}.
\]

Since \( \mathcal{O}_{X,x} \) is flat over \( \mathcal{O}_{Y,y} \), the base change \( \mathcal{O}_{X,x} \otimes \mathcal{O}_{Y,y} \) yields an injective homomorphism

\[
\mathcal{O}_{X,x}/q\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/q\mathcal{O}_{X,x}.
\]

Hence \( p \) and \( q \) are relatively prime in \( \mathcal{O}_{X,x} \). In addition, the image of 1 is 0 because \( a = p/q \) is regular in \( X \). As a result, \( q \) is invertible in \( \mathcal{O}_{X,x} \).

Since the maximal ideal of \( \mathcal{O}_{Y,y} \) is the intersection of \( \mathcal{O}_{Y,y} \) with the maximal ideal of \( \mathcal{O}_{X,x} \), \( q \) is invertible in \( \mathcal{O}_{Y,y} \), and so \( a \) is in \( \mathcal{O}_{Y,y} \). As a result, \( a \) is regular on \( Y' \) and then extends to a regular function on \( Y \) since \( Y \) is normal. \( \square \)
Proof of Theorem 11.1. Let us prove the first assertion of the theorem. We view the algebra $J_n\mathbb{C}[g]^G$ as a subalgebra of $J_nR_f$ for any $n$. We have already noticed that the inclusion $J_n\mathbb{C}[g]^G \subset \mathcal{Z}(J_nR_f)$ holds, with $\mathcal{Z}(J_nR_f)$ the vertex Poisson center of $J_nR_f$. Conversely, we have to prove that any element $z$ in the vertex Poisson center $\mathcal{Z}(J_nR_f)$ can be lifted to an element of $J_\infty\mathbb{C}[g]^G$.

Since $J_\infty \mathcal{F}$ is the projective limit of the projective system $(J_nR_f,\pi_{m,n})$, the algebra $J_\infty R_f$ is the inductive limit of the algebras $J_nR_f$. The injection $j_n$ from $J_nR_f$ to $J_\infty R_f$ is defined by

$$j_n(\mu_n)(\gamma) := \mu(\pi_n(\gamma)), \quad \mu_n \in J_nR_f, \gamma \in J_\infty \mathcal{F}.$$ 

Let $z \in \mathcal{Z}(J_nR_f) \subset J_nR_f$. Since $z \in J_nR_f$, $z = z_n \in J_nR_f$ for $m$ big enough, where $z_n$ is such that

$$z_n(\gamma_n) = z(j_n(\gamma)), \quad z(\gamma) = z_n(\pi_n(\gamma)), \quad \gamma_n \in J_n\mathcal{F}, \gamma \in J_\infty \mathcal{F}.$$ 

By Theorem 10.5, $z$ is constant on each fibers of $J_\infty \psi_f$. As a consequence, $z_n$ is constant on each fibers of $J_n\psi_f$. Indeed, let $\xi_n \in J_nY$, and $\gamma_n, \gamma'_n \in J_n\psi^{-1}(\xi_n)$. Then $j_n(\gamma_n), j_n(\gamma'_n)$ are in $J_\infty \psi_f^{-1}(j_n(\xi_n))$ since

$$J_\infty \psi_f(j_n(\gamma_n)) = j_n(J_n\psi_f(\gamma_n)) = j_n(\xi_n) = j_n(J_n\psi_f(\gamma'_n)) = J_\infty \psi_f(j_n(\gamma'_n)).$$

Hence,

$$z_n(\gamma_n) = z(j_n(\gamma_n)) = z(j_n(\gamma'_n)) = z(\gamma'_n)$$

since $z$ is constant on $J_\infty \psi_f^{-1}(j_n(\xi_n))$.

If $z$ is a constant function, that is, $z \in \mathbb{C}$, then clearly $z$ lies in the vertex Poisson center of $J_\infty \mathbb{C}[g]^*$. So one can assume that $z$ is not constant. Furthermore, one can assume that $z$ is homogeneous for the Slodowy grading on $J_\infty R_f$ induced from the contracting $\mathbb{C}^*$-action $\mu_\infty$ on $J_\infty \mathcal{F}$ (see Section 10) since $\mathcal{Z}(J_\infty R_f)$ is Slodowy invariant. Thus for any $t \in \mathbb{C}^*$, $t^k z$ for some $k \in \mathbb{Z}_{\geq 0}$. So one can assume that the morphisms $z: J_\infty \mathcal{F} \to \mathbb{C}$ and $z_n: J_n\mathcal{F} \to \mathbb{C}$ are dominant.

Hence by Lemma 11.2, $z_n \in \mathbb{C}[J_n\mathcal{F}]$ induces a rational morphism $\tilde{z}_n$ on $J_n(g//G)$ since $z_n$ is constant on the fibers of the dominant morphism $J_n\psi_f$.

Since $\mathcal{F}$ and $g//G$ are affine spaces, $J_n\mathcal{F}$ and $J_n(g//G)$ are affine spaces for any $m$. In particular, $J_n\mathcal{F}$ and $J_n(g//G)$ are normal and irreducible for any $m$. Hence Lemma 11.3 can be applied because the morphism $J_n\psi_f: J_\infty \mathcal{F} \to J_n(g//G)$ is flat for any $n$. So, $\tilde{z}_n \in \mathbb{C}[J_n(g//G)]$. This holds for any $n$ such that $z = z_n$. Since $z = z_n$ for $n$ big enough, we deduce that $z$ can be lifted to an element of $\mathbb{C}[J_\infty(g//G)] = \mathbb{C}[J_\infty g]^{J_\infty G}$, whence the first part of the theorem.

It remains to prove the freeness. Since $\mathcal{F} \cap \mathcal{N}$ enjoys the same geometrical properties as $\mathcal{N}$, that is, $\mathcal{F} \cap \mathcal{N}$ is a reduced, irreducible and a complete intersection subscheme of $\mathcal{F}$, the arguments of [EF, Theorem A.4] can be applied in order to get that $\mathbb{C}[J_\infty \mathcal{F}]$ is free over its vertex Poisson center. This concludes the proof of the theorem.

12. Application to W-algebras

Let $V^k(g)$ be the universal affine vertex algebra associated with $g$ at level $k$, and let $W^k(g,f)$ be the (affine) W-algebra [FF1, KRW] associated with $(g,f)$ at level $k \in \mathbb{C}$. The W-algebra $W^k(g,f)$ is defined by the quantized Drinfeld-Sokolov reduction associated with $f$. 

The embedding $Z(\mathcal{V}_k(\mathfrak{g})) \hookrightarrow \mathcal{V}_k(\mathfrak{g})$ induces a vertex algebra homomorphism

$$Z(\mathcal{V}_k(\mathfrak{g})) \rightarrow Z(\mathcal{W}_k(\mathfrak{g}, f))$$

for any $k \in \mathbb{C}$. Here, for a vertex algebra $V$, $Z(V)$ denotes the vertex center of $V$, that is,

$$Z(V) = \{ z \in V \mid a_{(n)}z = 0 \text{ for all } a \in V, n \geq 0 \}.$$ Both $Z(\mathcal{V}_k(\mathfrak{g}))$ and $Z(\mathcal{W}_k(\mathfrak{g}, f))$ are trivial unless $k = cri$ is the critical level $cri = -h^\vee$ with $h^\vee$ the dual Coxeter number of $\mathfrak{g}$. For $k = cri$, $Z(\mathcal{V}_{cri}(\mathfrak{g}))$ is known as the Feigin-Frenkel center $[\text{FF2}].$

**Theorem 12.1.** The embedding $Z(\mathcal{V}_{cri}(\mathfrak{g})) \hookrightarrow \mathcal{V}_{cri}(\mathfrak{g})$ induces an isomorphism

$$Z(\mathcal{V}_{cri}(\mathfrak{g})) \rightarrow Z(\mathcal{W}_{cri}(\mathfrak{g}, f)).$$

**Proof.** Note that there is an obvious vertex algebra homomorphism $Z(\mathcal{V}_{cri}(\mathfrak{g})) \rightarrow Z(\mathcal{W}_{cri}(\mathfrak{g}, f))$, see [A2]. Hence it is sufficient to show that the induced homomorphism $gr Z(\mathcal{V}_{cri}(\mathfrak{g})) \rightarrow gr Z(\mathcal{W}_{cri}(\mathfrak{g}, f))$ is an isomorphism.

First, we have

$$gr Z(\mathcal{V}_{cri}(\mathfrak{g})) \cong \mathbb{C}[J_{\infty}^G].$$

On the other hand, we have

$$gr Z(\mathcal{W}_{cri}(\mathfrak{g}, f)) \cong \mathbb{C}[J_{\infty}^G].$$

([A3, Theorem 4.17]), and so

$$gr Z(\mathcal{W}_{cri}(\mathfrak{g}, f)) \cong \mathbb{C}[J_{\infty}^G].$$

By Theorem 11.1, we obtain that

$$gr Z(\mathcal{V}_{cri}(\mathfrak{g})) \cong gr Z(\mathcal{W}_{cri}(\mathfrak{g}, f)).$$

This completes the proof. □

Theorem 12.1 was stated in [A2], but the proof of the surjectivity was incomplete. Note that the similar argument as above using (18) recovers Premet’s result [Pre2] stating that the center of the *finite* $W$-algebra $U(\mathfrak{g}, f)$ associated with $\mathfrak{g}, f$ is isomorphic to the center of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}.$

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