A NONLINEAR VERSION OF THE NEWHOUSE THICKNESS THEOREM

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Abstract. Let $C_1$ and $C_2$ be two Cantor sets with convex hull $[0, 1]$. Newhouse [12] proved if $\tau(C_1) \cdot \tau(C_2) \geq 1$, then the arithmetic sum $C_1 + C_2$ is an interval, where $\tau(C_i), 1 \leq i \leq 2$ denotes the thickness of $C_i$. In this paper, we generalize this thickness theorem as follows. Let $K_i \subset \mathbb{R}, i = 1, \ldots, d$, be some Cantor sets (perfect and nowhere dense) with convex hull $[0, 1]$. Suppose $f(x_1, \ldots, x_{d-1}, z) \in C^1$ is a continuous function defined on $\mathbb{R}^d$. Denote the continuous image of $f$ by

$$ f(K_1, \ldots, K_d) = \{f(x_1, \ldots x_{d-1}, z) : x_i \in K_i, z \in K_d, 1 \leq i \leq d - 1\}. $$

If for any $(x_1, \ldots, x_{d-1}, z) \in [0, 1]^d$, we have

$$ (\tau(K_i))^{-1} \leq \left| \frac{\partial f}{\partial x_i} \right| \leq \tau(K_d), 1 \leq i \leq d - 1 $$

then $f(K_1, \ldots, K_d)$ is a closed interval. We give two applications. Firstly, we partially answer some questions posed by Takahashi [16]. Secondly, we obtain various nonlinear identities, associated with the continued fractions with restricted partial quotients, which can represent real numbers.

1. Introduction

Let $K_1$ and $K_2$ be two Cantor sets with convex hull $[0, 1]$. Newhouse [12] proved that if $\tau(K_1) \cdot \tau(K_2) > 1$ (in fact we may replace this condition by $\tau(K_1) \cdot \tau(K_2) \geq 1$), then the arithmetic sum $K_1 + K_2$ is an interval, where $\tau(K_i)$ denotes the thickness of $K_i, i = 1, 2$. The arithmetic sum of Cantor sets appears naturally in bifurcation theory. Palis [13] posed the following problem which is currently known as the Palis’ conjecture. Whether it is true (at least generically) that the arithmetic sum of dynamically defined Cantor sets either has measure zero or contains an interior. This conjecture was solved by Moreira and Yoccoz [11]. The Newhouse’s thickness theorem is a very powerful result which can judge whether the arithmetic sum of two Cantor sets contains interior. Astels [1] Theorem 2.4] generalized the Newhouse’s thickness theorem by considering multiple sum of Cantor sets. He made use of this new thickness theorem to prove some identities which can represent real numbers. Astels’ thickness theorem implies many interesting results. For instance, we may prove some Waring type result as follows, see
For each $k \geq 2$, there is a number $n(k) \leq 2^k$ such that for any $x \in [0, n(k)]$, we have

$$x = \sum_{i=1}^{n(k)} x_i^k,$$

where $x_i$ is taken from the middle-third Cantor set. The Newhouse and Astels’ thickness theorems are very useful when we consider the sum of two Cantor sets. It is natural to consider a nonlinear version of Newhouse’s thickness theorem. Suppose $f(x_1, \cdots, x_{d-1}, z) \in C^1$ is a continuous function defined on $\mathbb{R}^d$. Denote the continuous image of $f$ by

$$f(K_1, \cdots, K_d) = \{f(x_1, \cdots, x_{d-1}, z) : x_i \in K_i, z \in K_d, 1 \leq i \leq d-1\},$$

where $\{K_i\}_{i=1}^d$ are general Cantor sets. To the best of our knowledge, there are very few results about $f(K_1, \cdots, K_d)$. Generally, to consider the topological structure of $f(K_1, \cdots, K_d)$ is a difficult question. As we know very little information about $\{K_i\}_{i=1}^d$. Moreover, the nonlinearity of $f(x_1, \cdots, x_{d-1}, z)$ makes the abstract set $f(K_1, \cdots, K_d)$ obscure. The main aim of this paper is to give some sufficient conditions on $f(x_1, \cdots, x_{d-1}, z)$ such that $f(K_1, \cdots, K_d)$ is a closed interval.

We now introduce some related results concerning with the continuous image of $f$ in $\mathbb{R}$. The first one, to the best of our knowledge, is due to Steinhaus [15] who proved in 1917 the following interesting results:

$$C + C = \{x + y : x, y \in C\} = [0, 2], C - C = \{x - y : x, y \in C\} = [-1, 1],$$

where $C$ is the middle-third Cantor set. It is worth pointing out that Steinhaus also proved that for any two sets with positive Lebesgue measure, their arithmetic sum contains interiors. In 2019, Athreya, Reznick and Tyson [2] proved that

$$C \div C = \left\{\frac{x}{y} : x, y \in C, y \neq 0\right\} = \bigcup_{n=-\infty}^{\infty} \left[\frac{3^{-n} \frac{2}{3}}{2}, \frac{3^{-n} \frac{3}{2}}{2}\right] \cup \{0\}.$$

In [8], Gu, Jiang, Xi and Zhao gave the topological structure of

$$C : C = \{xy : x, y \in C\}.$$

They proved that the exact Lebesgue measure of $C : C$ is about 0.80955. We give some remarks on the above results. The main idea of [2] is effective for homogeneous self-similar sets. For a general self-similar set or some general Cantor set, we may not utilize their idea directly. Fraser, Howroyd and Yu [6] studied the dimensions of sumsets and iterated sum sets, and provided natural conditions which guarantee that a set $F \subset \mathbb{R}$ satisfies $\dim_B F + F > \dim_B F$. The reader can find more related references in [6].

For higher dimensions, namely $\mathbb{R}^d, d \geq 3$, there are relatively few results. Banakh, Jabłońska and Jabłoński [3] proved under some mild conditions that the arithmetic sum of $d$ many compact connected sets in $\mathbb{R}^d$ has non-empty interior. As a consequence, every compact connected set in $\mathbb{R}^d$ not
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Lying a hyperplane is arithmetically thick. A compact set $E \subset \mathbb{R}^d$ is said to be arithmetically thick if there exists a positive integer $n$ so that the $n$-fold arithmetic sum of $E$ has non-empty interior. Recently Feng and Wu [7] defined the thickness of sets in $\mathbb{R}^d$, and proved the arithmetic thickness for several classes of fractal sets, including self-similar sets, self-conformal sets in $\mathbb{R}^d$ (with $d \geq 2$) and some self-affine sets. All these elegant results are concerning with arithmetic sum. They introduced some new ideas which are very useful to analyze the sets in $\mathbb{R}^d$.

In this paper, we consider similar problems. However, our main motivation is to generalize the Newhouse’s thickness theorem for some general functions. Before we introduce the main results of this paper, we give some definitions. First, we give a well-known method that can generate a Cantor set. For simplicity, we let $I_0 = [0, 1]$. In the first level, we delete an open interval from $[0, 1]$, denoted by $O$. Then there are two closed intervals left, we denote them by $B_1$ and $B_2$. Therefore, $[0, 1] = B_1 \cup O \cup B_2$. Let $E_1 = B_1 \cup B_2$. In the second level, let $O_0$ and $O_1$ be open intervals that are deleted from $B_1$ and $B_2$ respectively, then we clearly have

$$B_1 = B_{11} \cup O_0 \cup B_{12}, B_2 = B_{21} \cup O_1 \cup B_{22}.$$ 

Let

$$E_2 = B_{11} \cup B_{12} \cup B_{21} \cup B_{22}.$$ 

Repeating this process, we can generate $E_{n+1}$ from $E_n$ by removing an open interval from each closed interval in the union which consists of $E_n$. We assume that the deleted open intervals are arranged by the decreasing lengths, i.e. the lengths of deleted open intervals are decreasing. If for some levels, the deleted open intervals have the same length, then we can delete these open intervals in any order. To avoid triviality, we make the following rule. Let $B_\omega$ be a closed interval in some level, then we delete an open interval $O_\omega$ from $B_\omega$, i.e.

$$B_\omega = B_{\omega 1} \cup O_\omega \cup B_{\omega 2}.$$ 

We assume that the length of $O_\omega$ is positive and strictly smaller than $B_\omega$. We let

$$K = \cap_{n=1}^\infty E_n,$$

and call $K$ a Cantor set. The above rule is to ensure the Cantor set is perfect and nowhere dense.

The next definition is the famous Newhouse’s thickness. Given a Cantor set

$$K = \cap_{n=1}^\infty E_n.$$ 

Let $B_\omega$ be a closed interval in some level. Then by the construction of $K$, we have

$$B_\omega = B_{\omega 1} \cup O_\omega \cup B_{\omega 2},$$
where $O_\omega$ is an open interval while $B_{\omega 1}$ and $B_{\omega 2}$ are closed intervals. We call $B_{\omega 1}$ and $B_{\omega 2}$ bridges of $K$, and $O_\omega$ gap of $K$. Let

$$\tau_\omega(B_\omega) = \min \left\{ \frac{|B_{\omega 1}|}{|O_\omega|}, \frac{|B_{\omega 2}|}{|O_\omega|} \right\},$$

where $|\cdot|$ means length. We define the thickness of $K$ by

$$\tau(K) = \inf_{B_\omega} \tau_\omega(B_\omega).$$

Here the infimum takes over all bridges in every level.

Now we state the main result of this paper.

**Theorem 1.1.** Let $\{K_i\}_{i=1}^d$ be Cantor sets with convex hull $[0, 1]$. Suppose that $f(x_1, \cdots, x_{d-1}, z) \in \mathcal{C}^1$. If for any $(x_1, \cdots, x_{d-1}, z) \in [0, 1]^d$, we have

$$(\tau(K_i))^{-1} \leq \frac{\partial_x f}{\partial z f} \leq \tau(K_d), 1 \leq i \leq d - 1$$

then

$$f(K_1, \cdots, K_d) = H,$$

where

$$H = \left[ \min_{(x_1, \cdots, z) \in K_1 \times \cdots \times K_d} f(x_1, \cdots, z), \max_{(x_1, \cdots, z) \in K_1 \times \cdots \times K_d} f(x_1, \cdots, z) \right],$$

and $\tau(K_i), i = 1, \cdots, d$, denotes the thickness of $K_i$.

**Remark 1.2.** This result partially generalizes [1] Theorem 2.4. Theorem 1.1 can be given an explanation from geometric measure theory. If the convex hull of each $K_i$ is different, then we need to assume each $K_i$ is not contained in any other $K_j$'s gaps, where $j \neq i$. As for this case $f(K_1, \cdots, K_d)$ may have Lebesgue measure zero. When we use Theorem 1.1 we need to abide by this rule.

**Corollary 1.3.** Let $\{K_i\}_{i=1}^d$ be Cantor sets with convex hull $[0, 1]$. Suppose that $f(x_1, \cdots, x_{d-1}, z) \in \mathcal{C}^1$. If for any $(x_1, \cdots, x_{d-1}, z) \in [0, 1]^d$, we have

$$(\tau(K_i))^{-1} \leq \frac{\partial_x f}{\partial z f} \leq \tau(K_d), 1 \leq i \leq d - 1$$

then for any $w \in H$ the hypersurface $f(x_1, \cdots, x_{d-1}, z) = w$ intersects with $K_1 \times \cdots \times K_d$.

For $d = 2$ we have the following result which can be viewed as a nonlinear version of the Newhouse’s thickness theorem.

**Corollary 1.4.** Let $K_1$ and $K_2$ be two Cantor sets with convex hull $[0, 1]$. Suppose $f(x, y) \in \mathcal{C}^1$. If for any $(x, y) \in [0, 1]^2$, we have

$$(\tau(K_1))^{-1} \leq \frac{\partial_x f}{\partial y f} \leq \tau(K_2),$$
then
\[ f(K_1, K_2) = \left[ \min_{(x,y) \in K_1 \times K_2} f(x,y), \max_{(x,y) \in K_1 \times K_2} f(x,y) \right] = H, \]
where \( \tau(K_i), i = 1, 2 \) denotes the thickness of \( K_i \). In particular, if we take a linear function
\[ f(x, y) = x + y, \] and \( \tau(K_i) \geq 1, i = 1, 2, \) then
\[ K_1 + K_2 \]
is an interval.

Remark 1.5. The conditions in Corollary 1.4 imply that \( \tau(K_1) \tau(K_2) \geq 1. \) It is easy to find some Cantor sets, under the condition \( \tau(K_1) \tau(K_2) < 1, \) such that \( f(K_1, K_2) \) does not contain some interiors, see for instance in Corollary 1.7 and the remarks below. By the Newhouse’s thickness theorem, if \( \tau(K_1) \tau(K_2) \geq 1, \) then \( K_1 + K_2 \) is an interval. However, under the condition \( \tau(K_1) \tau(K_2) \geq 1, \) we may not have that
\[ K_1 \cdot K_2 = \{xy : x \in K_1, y \in K_2\} \]
is still an interval. A simple example is the middle-third Cantor set, denoted by \( C \). The thickness of \( C \) is 1. We have
\[ C + C = [0, 2]. \]
However, \( C \cdot C \subset [0, 1/3] \cup [4/9, 1], \) which yields that \( C \cdot C \) is not an interval. Therefore, for a general \( f, \) if we want \( f(K_1, K_2) \) to be some interval, we may expect more strong conditions on \( f \) besides \( \tau(K_1) \tau(K_2) \geq 1. \)

Corollary 1.6. Let \( K_1 \) and \( K_2 \) be two Cantor sets with convex hull \([0, 1]\). If \( \tau(K_1) \tau(K_2) > 1, \) then there are uncountably many nonlinear functions \( f(x, y) \in C^1 \) such that
\[ f(K_1, K_2) \]
is an interval.

The condition on partial derivatives in Theorem can be weakened when we consider some homogeneous self-similar sets. Indeed, the thickness gives little information about the relation between gaps and bridges. If we elaborately analyze their relation, we may obtain more delicate result. For instance, with a similar discussion as Theorem we may prove the following result.

Corollary 1.7. Let \( K_\lambda \) be the attractor of the IFS
\[ \{f_1(x) = \lambda x, f_2(x) = \lambda x + 1 - \lambda, 0 < \lambda < 1/2\}. \]
Suppose that \( f(x, y) \in C^1 \) is a continuous function defined on \( \mathbb{R}^2. \) If for any \((x, y) \in [0, 1]^2, \) we have
\[ \frac{1 - 2\lambda}{\lambda} \leq \left| \frac{\partial_x f}{\partial_y f} \right| \leq \frac{1}{1 - 2\lambda}, \]
then
\[ f(K_\lambda, K_\lambda) = \left[ \min_{(x,y) \in K_\lambda \times K_\lambda} f(x,y), \max_{(x,y) \in K_\lambda \times K_\lambda} f(x,y) \right]. \]

**Remark 1.8.** The conditions in Corollary 1.7 imply that
\[ \frac{1 - 2\lambda}{\lambda} \leq \frac{1}{1 - 2\lambda}, \text{ i.e. } 1/4 \leq \lambda < 1/2. \]
This condition is natural as for any \( f(x,y) \in C^1 \) and \( 0 < \lambda < 1/4 \), we have
\[ \dim_H(f(K_\lambda, K_\lambda)) \leq \dim_H(K_\lambda \times K_\lambda) = \frac{2 \log 2}{-\log \lambda} < 1. \]
In other words, if \( 0 < \lambda < 1/4 \), then \( f(K_\lambda, K_\lambda) \) cannot be an interval.

Note that \( \tau(K_\lambda) = \frac{\lambda}{1 - 2\lambda} < 1 \) if \( 0 < \lambda < 1/3 \). For this case, the Newhouse’s thickness theorem does not offer any information for \( f(K_\lambda, K_\lambda) \).
Moreover, by [1, Theorem 2.4], \( \gamma(K_\lambda) = \frac{\tau(K_\lambda)}{\tau(K_\lambda) + 1} = \frac{\lambda}{1 - \lambda} \), we cannot make use of Astels’ result to consider whether \( f(K_\lambda, K_\lambda) \) is an interval as for \( 1/4 < \lambda < 1/3 \) we have \( 2\gamma(K_\lambda) < 1 \). In fact, for the sum of two Cantor sets, the Newhouse’s thickness theorem and Astels’ thickness theorem are exactly the same. We mention some related work. In [14], Pourbarat proved under some assumptions that
\[ g_1(K_{\lambda_1}) + g_2(K_{\lambda_2}) = \{ g_1(x) + g_2(y) : x \in K_{\lambda_1}, y \in K_{\lambda_2} \} \]
contains an interval, where \( g_1, g_2 \in C^1 \). In Corollary 1.4, we prove under some conditions that \( f(K_1, K_2) \) is an interval for general Cantor sets.

In [16], Takahashi asked what is the topological structure of \( K_{\lambda_1} \cdot K_{\lambda_2} \). He also posed the question for the multiple product of some \( K_{\lambda_i} \). In fact, we can simultaneously consider multiplication and division on \( K_\lambda \). We partially answer his questions as follows.

**Corollary 1.9.** Let \( \{K_{\lambda_i}\}_{i=1}^d \) be self-similar sets with \( 0 < \lambda_i < 1/2, i = 1, \cdots, d \). If for any \( 1 \leq i \leq d - 1 \)
\[ \left\{ \begin{array}{l} \frac{1 - 2\lambda_i}{\lambda_i} \leq 1 - \lambda_d \\ \frac{1}{1 - \lambda_i} \leq \frac{\lambda_d}{1 - 2\lambda_d} \end{array} \right., \]
then
\[ \Pi_{i=1}^d K_{\lambda_i}^{e_i} = \{ \Pi_{i=1}^d x_i^{e_i} : x_i \in K_{\lambda_i}, e_i \in \{-1, 1\}, x_i \neq 0 \text{ if } e_i = -1 \} = U, \]
where
\[ U = \bigcup_{k_1,k_2,\ldots,k_d \in \mathbb{N}} \lambda_1^{e_{1,k_1}} \lambda_2^{e_{2,k_2}} \cdots \lambda_d^{e_{d,k_d}} \delta, \eta \cup \{0\}, \]
\[ \delta = \Pi_{e_i=1}(1 - \lambda_i), \eta = \Pi_{e_i=-1}(1 - \lambda_i)^{-1}. \]
Remark 1.10. To avoid triviality, in the definition of \( \Pi_{i=1}^{d} K_{\lambda_i}^{\epsilon_i} \), we assume that there exist some \( 1 \leq i, j \leq d \) such that \( \epsilon_i = -1, \epsilon_j = 1 \). For this case, \( \Pi_{i=1}^{d} K_{\lambda_i}^{\epsilon_i} \) contains 0. If \( \epsilon_i = -1 \), for any \( 1 \leq i \leq d \), then \( \Pi_{i=1}^{d} K_{\lambda_i}^{\epsilon_i} \) does not contain 0.

We may find more similar conditions, as in the above corollary, which allow us to describe the structure of \( \Pi_{i=1}^{d} K_{\lambda_i}^{\epsilon_i} \). Note that

\[
\Pi_{i=1}^{d} K_{\lambda_i}^{\epsilon_i} = \bigcup_{k_1, k_2, \ldots, k_d \in \mathbb{N}} \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_d^{k_d} (\Pi_{i=1}^{d} (K_{\lambda_i})^{\epsilon_i}) \cup \{0\},
\]

where each \( \tilde{K}_{\lambda_i} \) is the right similar copy of \( K_{\lambda_i} \). The above result only investigates \( f(x_1, \cdots, x_d) \) on \( \tilde{K}_{\lambda_1} \times \cdots \times \tilde{K}_{\lambda_d} \), see the details in the proof. Indeed, we may decompose each \( \tilde{K}_{\lambda_i} \) into two sub self-similar sets, and analyze the partial derivatives on these sub similar sets. We leave these considerations to the reader.

The following result indicates that the multiplication and division on some self-similar sets may simultaneously reach their maximal ranges.

**Corollary 1.11.** Let \( K \) be the attractor of the following IFS

\[
\{ f_1(x) = \lambda_1 x, f_2(x) = \lambda_2 x + 1 - \lambda_2, 0 < \lambda_2 \leq \lambda_1 < 1, \lambda_1 + \lambda_2 < 1 \}.
\]

Then the following conditions are equivalent:

1. \( K \cdot K = \{ x \cdot y : x, y \in K \} = [0, 1] \);
2. \( \lambda_1 \geq (1 - \lambda_2)^2 \);
3. \( K \div K = \left\{ \frac{x}{y} : x, y \in K, y \neq 0 \right\} = \mathbb{R} \).

Finally, we give an application to the continued fractions with restricted partial quotients. We first give some basic definitions. Let \( m \in \mathbb{N}_{\geq 2} \). Define

\[
F(m) = \{ [t, a_1, a_2, \cdots] : t \in \mathbb{Z}, 1 \leq a_i \leq m \text{ for } i \geq 1 \},
\]

where

\[
[t, a_1, a_2, \cdots] = t + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.
\]

For each \( l \in \mathbb{N}^+ \), define

\[
G(l) = \{ [t, a_1, a_2, \cdots] : t \in \mathbb{Z}, a_i \geq l \text{ for } i \geq 1 \}
\]

\[
\cup \{ [t, a_1, a_2, \cdots, a_k] : t, k, \in \mathbb{Z}, k \geq 0, \text{ and } a_i \geq l \text{ for } 1 \leq i \leq k \}.
\]

Generally, let \( B \) be a finite digits set, denote by \( F(B) \) the set of points which have an infinite continued fraction expansions with all partial quotients, except possibly the first, members of \( B \). When \( B \) is infinite, then we
define $F(B)$ in a similar way ($F(B)$ also includes some real numbers with finite continued expansions). For more detailed introduction, see [1, 4]. Let $F_t(B)$ be a subset of $F(B)$ with the first partial quotient $t \in \mathbb{Z}$. Therefore, $F(B) = \bigcup_{t \in \mathbb{Z}} F_t(B)$. With this notation, we have

$$F(m) = \bigcup_{t \in \mathbb{Z}} F_t(B)$$

where $B = \{1, 2, \ldots, m\}$. It is not difficult to calculate the Newhouse thickness of $F_t(B)$, see for instance [1, Lemma 4.3, Lemma 4.4].

The main motivation why we consider continued fractions with restricted partial quotients is due to some well-known results. Hall [9] proved that

$$F(4) + F(4) = \{x + y : x, y \in F(4)\} = \mathbb{R}.$$ 

Diviš [5] showed that Hall’s result is sharp in some sense as

$$F(3) + F(3) \neq \mathbb{R}.$$ 

Here we use one simple fact, i.e. $F(n) \subset F(n + 1)$ for any $n \geq 2$. Hlavka [10] generalized Hall’s result and proved that

$$F(3) + F(4) = \mathbb{R}, F(2) + F(7) = \mathbb{R}, F(2) + F(4) \neq \mathbb{R}.$$ 

Astels [1] showed

$$F(2) \pm F(5) = \mathbb{R}, F(3) - F(4) = \mathbb{R}, F(2) - F(4) \neq \mathbb{R}, F(3) - F(3) \neq \mathbb{R}.$$ 

All of the above equations are linear, i.e. the associated function

$$f(x, y) = x \pm y$$

is linear. It is natural to ask can we obtain similar results for some non-linear functions. Note that $F_t(B), t \in \mathbb{Z}$ is a Cantor set. Therefore, by Theorem [11] and the thickness of $F_t(B)$ we can obtain some nonlinear results for the arithmetic on $F(B)$ if we appropriately choose the functions $f(x, y)$. We only give the following equations. The reader may find more similar identities which can represent real numbers.

**Corollary 1.12.**

$$F^3(7) \pm F(7) = \mathbb{R}, (C + 1)^2 + 2F(6) = \mathbb{R}, f(K_1, K_2, K_3) = \mathbb{R},$$

where

$$K_1 = K_2 = C + 1, K_3 = F(6), f(x, y, z) = 0.1x + xy + z,$$

and $C$ is the middle-third Cantor set.

This paper is arranged as follows. In Section 2, we prove the main results of this paper. In Section 3, we give some identities which can represent real numbers. Finally, we give some remarks and questions.
2. Proofs of main results

2.1. Proof of Corollary 1.4. We first prove Corollary 1.4. The proof of general result, i.e. Theorem 1.1, depends on Corollary 1.4.

Clearly,

\[ f(K_1, K_2) \subset H. \]

To prove \( f(K_1, K_2) = H \), we suppose on the contrary that \( f(K_1, K_2) \neq H \), then we shall find some contradictions. Therefore, we finish the proof of Corollary 1.4.

If \( f(K_1, K_2) \neq H \), then we can find some \( z \in H \) such that \( \Phi_z \) does not intersect with \( K_1 \times K_2 \), where

\[ \Phi_z = \{ (x, y) \in [0,1]^2 : f(x, y) = z \}. \]

By virtue of the continuity of \( f \), it follows that \( \Phi_z \) is a compact set. Note that \( \Phi_z \) can be covered by countably many strips \( \Gamma \) of the form

\[ \{ (x,y) : x \in O, y \in [0,1] \} \text{ or } \{ (x,y) : y \in O, x \in [0,1] \}, \]

where \( O \)'s are the deleted open intervals when we construct \( K_i, i = 1, 2 \).

By the compactness of \( \Phi_z \), we may find finitely many strips from the above covering, i.e.

\[ \Phi_z \subset \bigcup_{i=1}^{n} \Gamma_i. \]

By the construction of Cantor sets (we mainly use the fact that the deleted open intervals are pairwise disjoint) and the continuity of \( f(x, y) \), it follows that for any \( 1 \leq i \leq n-1 \), \( \Gamma_i \) is perpendicular to \( \Gamma_{i+1} \).

Suppose that \( \Gamma_{\min} \) is the strip which has minimal width (every strip has length 1) among \( \bigcup_{i=1}^{n} \Gamma_i \). For every \( \Gamma_i, 1 \leq i \leq n \), we denote its width by \( L_i \).

Then we have the following lemma.

Lemma 2.1. The strip \( \Gamma_{\min} \) does not parallel with the \( x \)-axis.

Proof: We prove this lemma for three cases. Firstly, if the strip \( \Gamma_{\min} \) is paralleling with the \( x \)-axis, and it is closest to the origin. Then by the implicit function theorem and the minimal width of \( \Gamma_{\min} \) (we denote its width by \( L_{\min} \)), there exists some \((x_0, y_0) \in \Gamma_{\min}\) such that

\[ \left| \frac{dy}{dx}(x_0, y_0) \right| = \left| \frac{\partial_x f|_{(x_0, y_0)}}{\partial_y f|_{(x_0, y_0)}} \right| < \frac{L_{\min}}{\rho} \leq \frac{L_{\min}}{L_1 \tau(K_1)} \leq \frac{1}{\tau(K_1)}, \]

see the first graph of Figure 1. This contradicts to the condition in Corollary 1.4. Secondly, if the strip \( \Gamma_{\min} \) is paralleling with the \( x \)-axis, and it is closest to the line \( y = 1 \), then we may find a similar contradiction as the first case. Finally, suppose the strip \( \Gamma_{\min} \) is paralleling with the \( x \)-axis, and there is at least one paralleling strip below and above \( \Gamma_{\min} \), respectively. Let \( \Gamma_2 \) and \( \Gamma_3 \) be two strips that are perpendicular to \( \Gamma_{\min} \) such that \( \Phi_z \) enters and leaves the \( \Gamma_{\min} \). The entrance point is in \( \Gamma_2 \) while the leaving point is in \( \Gamma_3 \).
Let $L_{2,3}$ be the distance between $\Gamma_2$ and $\Gamma_3$. Then by the implicit function theorem there exists some $(x_0, y_0) \in \Gamma_{\min}$ such that

$$\left| \frac{dy}{dx}(x_0, y_0) \right| = \left| \frac{\partial_x f(x_0, y_0)}{\partial_y f(x_0, y_0)} \right| < \frac{L_{\min}}{L_{2,3}} \leq \frac{L_{\min}}{\min\{L_2, L_3\}} \tau(K_1) \leq \frac{1}{\tau(K_1)},$$

see the second graph of Figure 1. This contradicts to the assumption of Corollary 1.4. Hence, we have proved Lemma 2.1.

Similarly, we can prove the following lemma.

**Lemma 2.2.** The strip $\Gamma_{\min}$ does not parallel with the $y$-axis.

**Proof:** Suppose that $\Gamma_{\min}$ is parallel with the $y$-axis. We prove this lemma in three cases which are similar to Lemma 2.1. For simplicity, we only prove the following case.

Suppose there is at least one paralleling strip located on the left and right of $\Gamma_{\min}$, respectively. Then we let $\Gamma_4$ and $\Gamma_5$ be two strips that are perpendicular to $\Gamma_{\min}$ such that the $\Phi_z$ enters and leaves the $\Gamma_{\min}$. The entrance point is in $\Gamma_4$ and the leaving point is in $\Gamma_5$. Denote by $L_{4,5}$ the distance between $\Gamma_4$ and $\Gamma_5$, see the third graph of Figure 1. Therefore, by the implicit function theorem again, there exists some $(x_0, y_0) \in \Gamma_{\min}$ such that

$$\left| \frac{dy}{dx}(x_0, y_0) \right| = \left| \frac{\partial_x f(x_0, y_0)}{\partial_y f(x_0, y_0)} \right| \geq \frac{L_{4,5}}{L_{\min}} \geq \frac{\min\{L_4, L_5\} \tau(K_2)}{L_{\min}} \geq \tau(K_2).$$
This is a contradiction.

**Proof of Corollary 1.4.** Corollary 1.4 follows from Lemmas 2.1 and 2.2.

### 2.2. Proof of Theorem 1.1

Now, we prove Theorem 1.1. The main idea is exactly the same as Corollary 1.4. Firstly, we clearly have

\[ f(K_1, \ldots, K_d) \subset H. \]

If

\[ f(K_1, \ldots, K_d) \subset H, \]

then there exists some \( w \in H \) such that the hypersurface \( f(x_1, \ldots, z) = w \) does not intersect with \( K_1 \times K_2 \times \cdots \times K_d \).

Now we construct the following set

\[ \Psi_w = \{ (x_1, x_2, \ldots, z) \in [0, 1]^d : f(x_1, \ldots, z) = w \}. \]

It is a compact set by the continuity of \( f \). Hence, we can find finitely many \( d \)-dimensional cubes of the form

\[ \Lambda = \Delta_1 \times \Delta_2 \times \cdots \times \Delta_d \subset [0, 1]^d, \]

such that there is a unique \( \Delta_i \subset [0, 1] \) is an open interval for some \( 1 \leq i \leq d \), and the rest \( \Delta_j = [0, 1], j \neq i \). We call each \( \Delta_i, 1 \leq i \leq d \) an edge of \( \Delta_1 \times \Delta_2 \times \cdots \times \Delta_d \). For simplicity, we call the edge which is not equal to \([0, 1]\) the axis edge. Without loss of generality, we may assume that \( \Psi_w \subset \bigcup_{i=1}^n \Lambda_i \), \( \Lambda_i \) is perpendicular to \( \Lambda_{i+1} \) for \( 1 \leq i \leq n - 1 \), i.e. the axis edges of \( \Lambda_i \) and \( \Lambda_{i+1} \) have different subscripts. Let \( \Lambda_{\min} \) be the cube with minimal length, i.e. one edge of \( \Lambda_{\min} \) has minimal length among \( \bigcup_{i=1}^n \Lambda_i \). We shall prove that the above covering, i.e. \( \bigcup_{i=1}^n \Lambda_i \), does not exist. Therefore, we prove the desired result.

Let

\[ \Lambda_{\min} = [0, 1]^{i-1} \times (p_i, q_i) \times [0, 1]^{d-i}, (p_i, q_i) \subset [0, 1], 1 \leq i \leq d. \]

Suppose \( 1 \leq i \leq d - 1 \), for the hypersurface

\[ f(x_1, \ldots, z) = w, \]

we fix \( x_j, j \neq i, d \) (we let \( x_d = z \)). Therefore the hypersurface

\[ f(x_1, \ldots, z) = w \]

can be covered by \( \Omega_i \cup \Omega_d \), where

\[ \Omega_i = \bigcup \{ x_1 \} \times \{ x_2 \} \times \cdots \times \{ x_{i-1} \} \times (p_i, q_i) \times \{ x_{i+1} \} \times \cdots \times \{ x_{d-1} \} \times [0, 1], \]

and

\[ \Omega_d = \bigcup \{ x_1 \} \times \{ x_2 \} \times \cdots \times \{ x_{i-1} \} \times [0, 1] \times \{ x_{i+1} \} \times \cdots \times \{ x_{d-1} \} \times (p_d, q_d). \]

Here the unions in the above equations mean finite (by the compactness of \( \Psi_w \)) deleted open intervals when we construct \( K_i \) and \( K_d \).
Since we fix \( x_j, j \neq i, d \), it follows that the hypersurface \( f(x_1, \cdots, z) = w \) becomes a curve on a plane which is a translation of the \( x_iOz \) plane. We let this curve be \( \Upsilon \). By the above discussion, we have \( \Upsilon \subset \Omega_i \cup \Omega_d \). Nevertheless, by Lemmas 2.1 and 2.2, \( \Upsilon \) cannot be in \( \Omega_i \cup \Omega_d \), which is a contradiction.

If \( i = d \), then
\[
\Lambda_{\min} = [0, 1]^{d-1} \times (p_d, q_d).
\]

For this case, we can also prove similarly as above, and obtain a contradiction. Hence, we finish the proof.

2.3. Proof of Corollary 1.6. Note that
\[
\tau(K_1) \tau(K_2) > 1 \Leftrightarrow \frac{1}{\tau(K_1)} < \tau(K_2).
\]
If \( \tau(K_2) > \tau(K_1) > 1 \), then there exist some \( \alpha, \beta \in \mathbb{R}^+ \) such that
\[
\tau(K_2) > 1 + 2\alpha, \tau(K_1) > 1 + 2\beta.
\]
Now, we let \( f(x, y) = \alpha x^2 + \beta y^2 + x + y \). Since the convex hull of \( K_i, i = 1, 2 \) is \([0, 1]\), it follows that the conditions in Corollary 1.4 are satisfied. Therefore, \( f(K_1, K_2) \) is an interval.

If \( \tau(K_2) > 1 > \tau(K_1) \), then we can find some \( \gamma, \zeta \in \mathbb{R}^+ \) such that
\[
\frac{1}{\tau(K_1)} < \frac{\gamma}{2 + \zeta}, \frac{\gamma + 2}{\zeta} < \tau(K_2).
\]
We let
\[
f(x, y) = x^2 + y^2 + \gamma x + \zeta y.
\]
It is easy to check the conditions in Corollary 1.4 Hence, \( f(K_1, K_2) \) is an interval.

2.4. Proof of Corollary 1.7. The proof is almost the same as the proof of Corollary 1.4. We only need to prove Lemma 2.2 under the assumption
\[
\left| \frac{\partial_x f}{\partial_y f} \right| \leq \frac{1}{1 - 2\lambda}.
\]
For simplicity, we only prove the first case of Lemma 2.2. We still use the terminology of Lemma 2.2 and the third graph of Figure 1. By Lemma 2.1 \( \Gamma_{\min} \) cannot parallel with \( x \)-axis. By the minimality of \( \Gamma_{\min} \), we have
\[
\lambda \min\{L_4, L_5\} \geq L_{\min}.
\]
Therefore, by the implicit function theorem, there exists some \((x_0, y_0) \in \Gamma_{\min}\) such that
\[
\left| \frac{dy}{dx}(x_0, y_0) \right| = \left| \frac{\partial_x f(x_0, y_0)}{\partial_y f(x_0, y_0)} \right| > \frac{L_{4,5}}{L_{\min}} \geq \frac{\min\{L_4, L_5\} \tau(K_\lambda)}{L_{\min}} \geq \frac{\tau(K_\lambda)}{\lambda} = \frac{1}{1 - 2\lambda}.
\]
This is a contradiction.
2.5. **Proof of Corollary 1.9** Let $f(x_1, \cdots, x_d) = \prod_{i=1}^d x_i^{\epsilon_i}$. It is easy to check that

$$\left| \frac{\partial_x f}{\partial x_i} \right| = \frac{x_d}{x_i}, 1 \leq i \leq d - 1.$$ 

Note that

$$\Pi_{i=1}^d K^{\epsilon_i}_{\lambda_i} = \bigcup_{k_1, k_2, \cdots, k_d \in \mathbb{N}} \lambda_1^{\epsilon_1 k_1} \lambda_2^{\epsilon_2 k_2} \cdots \lambda_d^{\epsilon_d k_d} (\Pi_{i=1}^d K_{\lambda_i}^{\epsilon_i}) \cup \{0\},$$

where $\tilde{K}_{\lambda_i}$ is the right similar copy of $K_i$. Note that the convex hull of

$$K_{\lambda_1} \times K_{\lambda_2} \times \cdots \times K_{\lambda_d}$$

is

$$V = [1 - \lambda_1, 1] \times [1 - \lambda_2, 1] \times \cdots \times [1 - \lambda_d, 1].$$

Therefore, for any $(x_1, \cdots, x_d) \in V$, we have

$$1 - \lambda_d \leq \frac{x_d}{x_i} \leq \frac{1}{1 - \lambda_i}.$$

Then by the following conditions

$$\begin{cases} 
\frac{1 - 2\lambda_i}{\lambda_i} \leq 1 - \lambda_d \\
\frac{\lambda_i}{1 - \lambda_i} \leq \frac{\lambda_d}{1 - 2\lambda_d},
\end{cases}$$

we clearly have

$$\tau(K_{\lambda_i})^{-1} \leq \left| \frac{x_d}{x_i} \right| \leq \tau(K_{\lambda_d}), 1 \leq i \leq d - 1.$$ 

Now, Corollary 1.9 follows from Theorem 1.1.

2.6. **Proof of Corollary 1.11** We first prove (1) $\Rightarrow$ (2). This is clear as

$$K \subset [0, \lambda_1] \cup [1 - \lambda_2, 1] \Rightarrow K \cdot K \subset [0, \lambda_1] \cup [(1 - \lambda_2)^2, 1].$$

Now, we prove that (2) $\Rightarrow$ (1). Let $f(x, y) = xy$. First, we have the following equation:

$$K \cdot K = \cup_{i=0}^\infty \lambda_1^i (f_2(K) \cdot f_2(K)) \cup \{0\}.$$ 

The convex hull of $f_2(K)$ is $[1 - \lambda_2, 1]$. Hence, we consider the partial derivatives of $f$ on $[1 - \lambda_2, 1]^2$. It is easy to calculate that

$$1 - \lambda_2 \leq \left| \frac{\partial_x f}{\partial y} \right| = \frac{|y|}{|x|} \leq \frac{1}{1 - \lambda_2}$$

for any $(x, y) \in [1 - \lambda_2, 1]^2$.

Note that $\lambda_1 \geq (1 - \lambda_2)^2$ is equivalent to $\frac{1}{1 - \lambda_2} \leq \tau(K) = \frac{\lambda_2}{1 - \lambda_1 - \lambda_2}$.

Therefore, by Corollary 1.4,

$$f_2(K) \cdot f_2(K) = [(1 - \lambda_2)^2, 1].$$

Since $\lambda_1 \geq (1 - \lambda_2)^2$, it follows that

$$K \cdot K = \cup_{i=0}^\infty \lambda_1^i (f_2(K) \cdot f_2(K)) \cup \{0\} = [0, 1].$$
Now we prove (3) ⇒ (1). Note that
\[ K \div K = \bigcup_{n=-\infty}^{+\infty} \lambda_1^n \frac{f_2(K)}{f_2(K)} \cup \{0\} \subset \bigcup_{n=-\infty}^{+\infty} \lambda_1^n \left( \left[ 1 - \lambda_2, \frac{1}{1-\lambda_2} \right] \right) \cup \{0\}. \]
Therefore, if \( \lambda_1 < (1 - \lambda_2)^2 \), then
\[ \left[ 1 - \lambda_2, \frac{1}{1-\lambda_2} \right] \cap \left( \lambda_1(1 - \lambda_2), \frac{\lambda_1}{1-\lambda_2} \right) = \emptyset. \]
In other words,
\[ K \div K \neq \mathbb{R}. \]
Finally, we prove (1) ⇒ (3). This step is almost the same as (2) ⇒ (1) in terms of the equation
\[ K \div K = \bigcup_{n=-\infty}^{+\infty} \lambda_1^n \frac{f_2(K)}{f_2(K)} \cup \{0\}. \]

3. Some identities

In this section, we mainly prove Corollary 1.12. It is easy to calculate
\[ \tau(F_1(7)) = (42 + 24\sqrt{77})/91, t \in \mathbb{Z}, \]
see [1, Lemma 4.3, Lemma 4.4]. Therefore, by Corollary 1.4 it follows that
\[ F_1^3(7) + F_1(7) = \left[ \left( \frac{7 + \sqrt{77}}{14} \right)^3 + \frac{7 + \sqrt{77}}{14}, \left( \frac{-5 + \sqrt{77}}{2} \right)^3 + \frac{-5 + \sqrt{77}}{2} \right]. \]
Moreover, it is easy to check that
\[ (F_1^3(7) + F_i(7)) \cap (F_1^3(7) + F_{i+1}(7)) \neq \emptyset, i \in \mathbb{Z}. \]
Therefore,
\[ F_1^3(7) + F(7) = \mathbb{R}. \]
Similarly, we can prove
\[ F_1^3(7) - F(7) = \mathbb{R}. \]
For the second identity, we first note that
\[ \frac{1}{2}(C + 1)^2 + F(6) = \mathbb{R} \Leftrightarrow (C + 1)^2 + 2F(6) = \mathbb{R}, \]
where
\[ (C + 1)^2 + 2F(6) = \{ x^2 + 2y : x \in C + 1, y \in F(6) \}. \]
Hence, we only need to prove
\[ \frac{1}{2}(C + 1)^2 + F(6) = \mathbb{R}. \]
Let
\[ f(x, y) = \frac{1}{2}x^2 + y, x \in C + 1, y \in F_0(6). \]
Then by Corollary 1.4, we have
\[
\left(\frac{1}{2}(C + 1)^2 + F_0(6)\right) \cap \left(\frac{1}{2}(C + 1)^2 + F_i(6)\right) \neq \emptyset, \; i \in \mathbb{Z}.
\]
As such,
\[
\frac{1}{2}(C + 1)^2 + F(6) = \mathbb{R}.
\]
Finally, we consider the function
\[
f(x, y, z) = 0.1x + xy + z, \; x, y \in C + 1, \; z \in F_0(6).
\]
By Theorem 1.1 we have that
\[
f(C + 1, C + 1, F_0(6)) = \left[1.1 + \frac{-3 + \sqrt{15}}{6}, 1.2 + \sqrt{15}\right].
\]
Moreover,
\[
f(C + 1, C + 1, F_i(6)) \cap f(C + 1, C + 1, F_{i+1}(6)) \neq \emptyset, \; i \in \mathbb{Z}.
\]
Therefore, we have
\[
f(C + 1, C + 1, F(6)) = \mathbb{R}.
\]

4. Final remarks and some problems

Although in Theorem 1.1 we give a sufficient condition under which the continuous image of \(f\) is a closed interval, there are many problems left. We list some problems as follows.

1. For a given \(E \subset \mathbb{R}^d\), define a continuous function \(g : \mathbb{R}^d \to \mathbb{R}^d\). It would be interesting to consider when \(g(E)\) contains an interior or \(g(E)\) is exactly some convex hull.
2. In Theorem 1.1 we do not know whether for two concrete sets, the lower and upper bounds of the ratio of partial derivatives can be improved.
3. In Theorem 1.1 we only consider the first order partial derivatives. Can we give a similar nonlinear version of Theorem 1.1 using higher orders of partial derivatives.
4. In Corollary 1.1 we find an example such that the resonant maximum for the multiplication and division occurs. It would be interesting to find more sets which have this resonant phenomenon. Moreover, we may consider the resonant phenomenon for other arithmetic operation such as sum of squares and sum of cubes. These questions are motivated by the representations of real numbers from number theory.
Given two Cantor sets $K_1$ and $K_2$ with $\tau(K_1)\tau(K_2) < 1$, can we find some sufficient conditions such that $f(K_1, K_2)$ is still an interval.

(6) Given two Cantor sets $K_1$ and $K_2$, we do not know when $f(K_1, K_2)$ is a union of finitely many closed intervals.

(7) The Newhouse’s thickness, in some sense, is rough. As it gives a rough relation between gaps and bridges. It is deserved to define a finer thickness. Under the new thickness, we may partially improve Theorem 1.1.

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