The chiral vortical effect in Wigner function approach

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We give an improved derivation of the solutions to the covariant Wigner function for chiral fermions in background electromagnetic fields to the first order in the Planck constant. We use the thermal distribution function under static-equilibrium conditions including the Killing condition for $\beta^\mu = u^\mu / T$ with $u^\mu$ and $T$ being the fluid velocity and temperature respectively. The results reproduce our previous ones except that the vorticity is replaced by the thermal vorticity. The chiral vortical effect (CVE) current is derived from the first order solution in the same way as in previous works with explicit Lorentz covariance. In the semiclassical expansion without assuming the thermal distribution function and static-equilibrium conditions, there is a freedom to choose a reference frame in which the time-component of the Wigner function is defined. So the explicit Lorentz covariance seems to be lost. Once the thermal distribution function under static-equilibrium conditions is used, the CVE current can be decomposed into two parts, the normal and and magnetization part, each of which depends on the reference frame. However it can be shown that the sum of the two does give the total CVE current which is reference frame independent.

I. INTRODUCTION AND SUMMARY

It is well known that rotation and polarization are closely correlated and can be converted to each other in materials. The same phenomena also exist in high energy heavy ion collisions (HIC): huge global angular momenta are produced in peripheral collisions and are expected to induce global polarization of hadrons. The global polarization of $\Lambda$ and $\bar{\Lambda}$ hyperons has been measured by the STAR collaboration, which provides a strong evidence for the global rotation in heavy ion collisions.

The spin-vorticity couplings in statistical-hydro models are widely used to describe the hadron polarization. The global polarization of $\Lambda$ and $\bar{\Lambda}$ hyperons have been calculated through spin-vorticity couplings with vorticity fields being given by hydrodynamic model simulations or transport model simulations. The results are in good agreement with experimental data. For other model calculations of vorticity fields, see Refs. 19–25.

The global polarization is an average effect over the whole volume of the strong interaction matter produced in HIC, so the net vorticity must be in the direction of the global orbital angular momentum (OAM). The local vorticity fields contain much more information than the global one. We assume the following coodinate system for the collisions: the beams are in the $x$-direction, the OAM is in the $y$-direction, and the impact parameter is in the $z$-direction. The quadrupole pattern of $\omega_y$ (vorticity in the $y$-direction) in the reaction plane have been recently studied in hydrodynamic or transport models. The similar quadrupole pattern of $\omega_z$, in the transverse plane also exists. Recently a systematic analysis of the quadrupole patterns of $\omega_x$, $\omega_y$ and $\omega_z$ have been made within the AMPT model.

It is still an unsettled question that how such huge global angular momenta are transferred to the strong interaction matter produced in HIC, which is under intensive investigation and debate. The question is profound and highly non-trivial: the spin is a quantum observable while the vorticity in hydrodynamic description is a classical observable and how to accomodate the two in a consistent way is not trivial. Some attempts have been made to include the spin degree of freedom into relativistic hydrodynamics. However there is an ambiguity or freedom on the definition of the spin tensor out of the total angular momentum one due to the pseudo-gauge transformation.

In recent years, the Wigner function approach has been revived to describe the chiral magnetic effect (CME) for massless fermions. The axial vector component of the Wigner function for massive fermions gives the spin phase-space density. Then the polarization of massive hadrons can be calculated from the Wigner functions of their constituent quarks.

In this paper, we will give an improved derivation of the solutions to the vector component $f^a_\mu(x, p)$ of the Wigner function for chiral fermions in background electromagnetic fields to the first order in the Planck constant $\hbar$, using the thermal distribution function under static-equilibrium conditions including the Killing condition for $\beta^\mu = u^\mu / T$, where $u^\mu$ and $T$ are the fluid velocity and temperature respectively, and $s = \pm 1$ labels the chirality of chiral fermions. The results in the improved derivation reproduce the previous ones except that the vorticity is replaced by the thermal vorticity. The chiral vortical effect (CVE) is then derived from the first order solution of $f^a_\mu(x, p)$ by integration over the four-momentum, same as in Ref. 49. The advantage of using the thermal distribution function under static-equilibrium conditions is that they can retain the explicit Lorentz covariance of the solutions including the
CVE current. Without assuming any particular form of the distribution function and static-equilibrium conditions, one can also make a semiclassical expansion for the Wigner function in powers of $\hbar$ in a systematic way [54]. It has been proved to any order of $\hbar$ that all spatial components of $J^i_{\mu}(x,p)$ can be derived from the time-component [54]. There is a freedom to choose a time-like vector $n^\mu$ with $n_\mu n^\mu = 1$ to define the time-component of $J^i_{\mu}(x,p)$, i.e. $n \cdot J^i_{\mu}$. In other words, there is a freedom to choose a reference frame in which the time-component of $J^i_{\mu}(x,p)$ is defined [54]. The values of $J^i_{\mu}(x,p)$ should not depend on $n^\mu$. Such a freedom for the choice of $n^\mu$ is like a gauge freedom. In this formalism we can also obtain the CVE current. Once we use the thermal distribution function under static-equilibrium conditions for the zeroth order $J^i_{\mu}(x,p)$, the CVE current has two parts, the normal current and magnetization current, each depends on the reference frame. However we can show that the sum of two does give the total CVE current which has frame independence, provided the distribution function is modified in accordance with the change of the Lorentz frame.

We use the sign convention for the metric tensor $g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$. We adopt the same sign convention for the fermion charge $Q$ and $\gamma_5$ as in Ref. [49, 50]. We use $s = \pm 1$ to label the chirality of chiral fermions.

**II. WIGNER FUNCTION AND ITS SOLUTIONS IN STATIC-EQUILIBRIUM CONDITIONS**

In a background electromagnetic field, the quantum mechanical analogue of a classical phase-space distribution for fermions is the gauge invariant Wigner function $W_{\alpha\beta}(x,p)$ which satisfies the equation of motion [34, 35],

$$\left(\gamma_\mu K^\mu - m\right)W(x,p) = 0$$

(1)

where $x = (x_0, \mathbf{x})$ and $p = (p_0, \mathbf{p})$ are space-time and energy-momentum 4-vectors. For the constant field strength $F_{\mu\nu}$, the operator $K^\mu$ is given by $K^\mu = p^\mu + i\frac{\hbar}{2}\nabla^\mu$ with $\nabla^\mu = \partial^\mu - QF^{\mu\nu}\partial_\nu$. The Wigner function can be decomposed in 16 independent generators of Clifford algebra,

$$W = \frac{1}{4} \left[ \mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \gamma^\nu \mathcal{V}_{\mu
u} + \gamma^5 \gamma^\mu \gamma^\nu \mathcal{A}_{\mu\nu} + \frac{1}{2} \sigma^{\mu\nu} \mathcal{J}_{\mu\nu} \right]$$

(2)

whose coefficients $\mathcal{F}$, $\mathcal{P}$, $\mathcal{V}_{\mu\nu}$, $\mathcal{A}_{\mu\nu}$ and $\mathcal{J}_{\mu\nu}$ are the scalar, pseudo-scalar, vector, axial-vector and tensor components of the Wigner function respectively.

For massless or chiral fermions, the equations for $\mathcal{V}_{\mu\nu}$ and $\mathcal{A}_{\mu\nu}$ are decoupled from other components of the Wigner function, from which one can obtain independent equations for vector components $J^i_{\mu}(x,p)$ of the Wigner function.

For right-handed ($s = +$) and left-handed ($s = -$) fermions,

$$p^\mu J^i_{\mu}(x,p) = 0,$$

$$\nabla^\mu J^i_{\mu}(x,p) = 0,$$

$$2s(p^\lambda J^i_{\rho}(x,p) - p^\rho J^i_{\lambda}(x,p)) = -\hbar\epsilon^{\mu\nu\lambda\rho} \nabla_\mu J^i_{\nu},$$

(3)

where $J^i_{\mu}(x,p)$ are defined as

$$J^i_{\mu}(x,p) = \frac{1}{2} \left[ \mathcal{V}_{\mu}(x,p) + s\mathcal{A}_{\mu}(x,p) \right].$$

(4)

Note that equations in [34] are valid for constant electromagnetic field strength.

Under the static-equilibrium condition, one can derive a formal solution of $J^i_{\mu}$ satisfying Eq. [34] by a perturbation in powers $\hbar$. Up to $O(\hbar)$, we have the following solutions to the Wigner functions [49],

$$J^i_{(0)s}(x,p) = p^\mu f_s(\delta(p^2)),$$

$$J^i_{(1)s}(x,p) = \frac{-s\Omega_{\mu\alpha}}{2} \frac{df_s}{d(\beta \cdot p)} \delta(p^2) - \frac{sQ}{p^2} F^{\alpha\lambda} p_{\lambda} f_s(\delta(p^2)).$$

(5)

The total quantity is given by $J^i_s = J^i_{(0)s} + \hbar J^i_{(1)s}$. In Eq. [34] we have used $\beta^\mu = \beta u^\mu$ with $\beta = 1/T$ being the temperature inverse and with $u^\mu$ being the fluid velocity, $\tilde{F}^{\alpha\lambda} = \frac{1}{2} \epsilon^{\alpha\mu\lambda\rho} F_{\mu\rho}$ denotes the dual of the electromagnetic field strength tensor, $\Omega^{\alpha\rho} = \frac{1}{2} \epsilon^{\alpha\mu\rho\sigma} \Omega_{\mu\sigma}$ denotes the dual of the thermal vorticity tensor, whose explicit forms are given in Eq. (A1). In Eq. (34), $f_s$ is the distribution function for chiral fermions at the zeroth order,

$$f_s(x,p) = \frac{2}{(2\pi)^3} \left[ \Theta(p_0) f_{FD}(\beta \cdot p - \beta \mu_s) + \Theta(-p_0) f_{FD}(-\beta \cdot p + \beta \mu_s) \right],$$

(6)

where $p_0 = u \cdot p$, $f_{FD}(y) \equiv 1/[\exp(y) + 1]$ is the Fermi-Dirac distribution function, and $\mu_s$ is the chemical potential for the chirality $s = \pm 1$. We can express $\mu_s$ in terms of the scalar and pseudo-scalar (or chiral) chemical potentials, $\mu_s = \mu + s\mu_5$. 
III. DERIVATION OF THE FIRST ORDER SOLUTION

In this section, we will give the derivation of the first order solution in \( E_p \). For simplicity of notation, we suppress the helicity \( s \) from now on. Multiplying the last line of Eq. \( \text{(3)} \) by \( p_\lambda \) and using the first line of Eq. \( \text{(3)} \), we obtain

\[
2s p^2 J^\rho = -\hbar \epsilon_{\mu\nu\lambda\rho} p_\lambda \nabla_\mu J_\nu .
\]

We insert the above into the last line of Eq. \( \text{(3)} \),

\[
J^\rho = J^\rho + \mathcal{X}^\rho + \frac{s \hbar}{2p^2} \epsilon_{\rho\lambda\mu\nu} p_\lambda \nabla_\mu J_\nu .
\]

It is easy to see that the second term is satisfied with the first line of Eq. \( \text{(3)} \). The first term should also be satisfied with it, which results in

\[
J^\rho = p^\rho + \mathcal{X}^\rho ,
\]

with \( p \cdot \mathcal{X} = 0 \). So the last line of Eq. \( \text{(3)} \) becomes

\[
\text{l.h.s.} = 2s (p^\lambda \mathcal{X}_\rho - p^\rho \mathcal{X}^\lambda) \delta(p^2) + \frac{1}{p^2} p^\lambda \epsilon_{\rho\lambda\mu\nu} p_\delta \nabla_\mu J_\nu ,
\]

\[
\text{r.h.s.} = -\frac{s \hbar}{2p^2} \epsilon_{\rho\lambda\mu\nu} p_\delta \nabla_\mu J_\nu .
\]

Then the last line of Eq. \( \text{(3)} \) becomes

\[
2s (p^\lambda \mathcal{X}^\rho - p^\rho \mathcal{X}^\lambda) \delta(p^2) = \frac{1}{p^2} [p^\mu \epsilon_{\nu\delta\lambda\rho} + p^\nu \epsilon_{\delta\lambda\rho\mu}] p_\delta \nabla_\mu J_\nu
\]

\[
= \frac{1}{p^2} \left( \nabla_\mu J_\nu - \nabla_\nu J_\mu \right) \epsilon_{\nu\delta\lambda\rho} p^\mu p_\delta ,
\]

where we have used

\[
\epsilon_{\lambda\mu\nu\rho} p^\delta + \epsilon_{\rho\mu\nu\delta} p^\lambda + \epsilon_{\mu\nu\delta\lambda} p^\rho + \epsilon_{\nu\delta\lambda\mu} p^\rho + \epsilon_{\delta\lambda\mu\rho} p^\nu = 0 .
\]

We see from Eq. \( \text{(11)} \) that \( \mathcal{X}^\rho \) is at least of \( O(\hbar) \). Now we can rewrite the last line of Eq. \( \text{(3)} \) to this form

\[
\hbar \nabla_{[\sigma} J_{\xi]} = 2s \epsilon_{\sigma\xi\lambda\rho} p^\lambda J^\rho .
\]

Using Eq. \( \text{(13)} \) in the last line of Eq. \( \text{(11)} \), without the \( 1/p^2 \) factor, we obtain

\[
\hbar \nabla_{[\mu} J_{\nu]} \epsilon_{\nu\delta\lambda\rho} p^\mu p_\delta = 2s \epsilon_{\mu\nu\alpha\beta} \epsilon_{\nu\delta\lambda\rho} p^\alpha p^\mu p_\delta J^\beta = 0 .
\]

Note that \( p^\mu \nabla_{[\mu} J_{\nu]} \) contains a term \( \sim p^2 \delta(p^2) \) which is vanishing. But when multiplying Eq. \( \text{(14)} \) by a prefactor \( 1/p^2 \), i.e. as in the the last line of Eq. \( \text{(11)} \), the term \( p^2 \delta(p^2) \) in \( p^\mu \nabla_{[\mu} J_{\nu]} \) gives non-vanishing value. Now we try to extract the \( p^2 \delta(p^2) \) term in \( p^\mu \nabla_{[\mu} J_{\nu]} \) by using Eq. \( \text{(10)} \),

\[
p^\mu \nabla_{[\mu} J_{\nu]} = p^\mu \nabla_{[\mu} [p_\nu f \delta(p^2)] + p^\mu \nabla_{[\mu} [\mathcal{X}_\nu] \delta(p^2)]
\]

\[
+ \frac{s \hbar}{2p^2} p^\mu \epsilon_{[\nu\alpha\beta\gamma} \nabla_{\mu]} [p^\alpha \nabla^\beta J^\gamma] .
\]

We look at the first term \( p^\mu \nabla_{[\mu} [p_\nu f \delta(p^2)] \),

\[
p^\mu \nabla_{[\mu} [p_\nu f \delta(p^2)] = p^\mu p_\nu \delta(p^2) \nabla_\mu f - Q p^\mu F_{\mu\nu} \delta(p^2) ,
\]

\[
p^\mu \nabla_{[\mu} [p_\nu f \delta(p^2)] = p^\nu \delta(p^2) \nabla_\nu f - Q p^\nu F_{\nu\mu} \delta(p^2) - 2Q p^2 F_{\nu\mu} p^\rho f^\rho \delta(p^2) .
\]
So the last line of Eq. (11) has a non-zero contribution
\[ \frac{1}{p^2} \nabla_{\mu} \mathcal{J} \epsilon^{\nu \delta \lambda \rho} p^\mu p_\delta \rightarrow -\frac{1}{p^2} \delta(p^2) \nabla_\nu f \epsilon^{\nu \delta \lambda \rho} p^\delta = -h \delta(p^2) \epsilon^{\nu \delta \lambda \rho} p^\delta \nabla_\nu f. \] (17)

The last two terms of Eq. (15) are of higher order and will not be considered at the first order.

From Eqs. (11,17), we finally obtain
\[ (p^\lambda \mathcal{F}^\rho - p^\rho \mathcal{X}^\lambda) \delta(p^2) = \frac{h^2}{2} \delta(p^2) \epsilon^{\nu \delta \lambda \rho} p^\delta \nabla_\nu f. \] (18)

We can multiply a factor \( \epsilon_{\mu \alpha \lambda \rho} \) and sum over \( \lambda \rho \),
\[ \epsilon_{\mu \alpha \lambda \rho}(p^\lambda \mathcal{F}^\rho - p^\rho \mathcal{X}^\lambda) \delta(p^2) = \frac{h^2}{2} \delta(p^2) \epsilon_{\mu \alpha \lambda \rho} \epsilon^{\nu \delta \lambda \rho} p^\delta \nabla_\nu f, \]
\[ 2\epsilon_{\mu \alpha \lambda \rho} p^\lambda \mathcal{X}^\rho \delta(p^2) = -hs\delta(p^2)(p_\mu \nabla_\alpha f - p_\alpha \nabla_\mu f). \] (19)

The distribution for the right-handed or left-handed fermions is given by Eq. (6). We will use following notations: \( \beta^\rho = \beta^\rho_\nu \) and \( \bar{\mu}_s = \beta \mu_s \) (again, in the following we suppress the helicity label \( s \) in the chemical potential). Let’s calculate \( \nabla_\nu f \),
\[ \nabla_\nu f = \frac{\partial f}{\partial (\beta \cdot p)} \left[ p^\rho \frac{\partial \beta_\sigma}{\partial x^\nu} - \frac{\partial \bar{\mu}}{\partial x^\nu} - \beta Q E_\nu \right] \]
\[ = f' \frac{1}{2} p^\rho (\partial_\nu \beta_\sigma - \partial_\sigma \beta_\nu) + \frac{1}{2} p^\sigma (\partial_\nu \beta_\sigma + \partial_\sigma \beta_\nu) - \partial_\nu \bar{\mu} - \beta Q E_\nu \]
\[ \rightarrow \frac{1}{2} f' p^\rho (\partial_\nu \beta_\sigma - \partial_\sigma \beta_\nu) = f' \Omega_{\nu \sigma} p^\rho, \] (20)

where we have used the shorthand notation \( f' = \frac{\partial f}{\partial (\beta \cdot p)} \), then \( \frac{\partial f}{\partial \beta_\sigma} = -f' \) and \( \Omega_{\nu \sigma} = \frac{1}{2} (\partial_\nu \beta_\sigma - \partial_\sigma \beta_\nu) \). Most importantly we have used the following static-equilibrium conditions to reach the last line of Eq. (20),
\[ \partial_\nu \beta_\sigma + \partial_\sigma \beta_\nu = 0, \]
\[ \partial_\nu \bar{\mu}_s = -\beta Q E_\nu, \] (21)

where the first line of Eq. (21) is the Killing condition for \( \beta_\nu \). The Killing condition leads to the solution \( \beta_\nu = b_\nu - \Omega^{\mu \nu} x_\mu \) with \( b_\nu \) and \( \Omega^{\mu \nu} \) being constants. The second line of Eq. (21) leads to equations for the fermion number and chiral chemical potential, \( \partial_\lambda \bar{\mu} = -\beta Q E_\lambda \) and \( \partial_\nu \bar{\mu}_s = 0 \). It also leads to the integrability condition for constant field strength,
\[ F^{\rho \nu} \partial_\nu \beta_\rho - F^{\nu \rho} \partial_\mu \beta_\rho = 0, \] (22)

or in a compact form
\[ F^{\rho \nu} \Omega_{\nu \rho} = F^{\nu \rho} \Omega_{\mu \rho} = 0. \] (23)

Then we can further derive the following identities
\[ \frac{1}{2} \tilde{\Omega}^{\mu \alpha} F_{\mu \alpha} p^2 = \Omega^{\mu \alpha} F_{\mu \rho} p_\alpha p^\rho + \tilde{\Omega}^{\mu \alpha} F_{\mu \rho} p_\alpha p^\rho \]
\[ \tilde{F}^{\mu \alpha} F_{\mu \rho} p_\alpha p^\rho = \frac{1}{4} b^2 \tilde{F}^{\rho \lambda} F_{\rho \lambda} \]
\[ \tilde{\Omega}^{\mu \alpha} \Omega_{\mu \rho} p^\rho = \frac{1}{4} b^2 \tilde{\Omega}^{\rho \lambda} \Omega_{\rho \lambda}. \] (24)

The proof of Eq. (24) is given in Eqs. (A4)(A5).

So Eq. (19) can be simplified as (we suppress \( \delta(p^2) \))
\[ 2\epsilon_{\mu \alpha \lambda \rho} p^\lambda \mathcal{X}^\rho = -hs(p_\mu \Omega_{\alpha \sigma} p^\sigma - p_\alpha \Omega_{\mu \sigma} p^\sigma) f', \] (25)
or put in another form by contraction with $\epsilon^{\mu\nu\alpha\delta}$,

$$\begin{align*}
\rho^\mu \mathcal{J}^\nu - p^\nu \mathcal{J}^\rho &= \frac{\hbar}{2} \epsilon^{\mu\nu\alpha\gamma} p_\alpha \mathcal{O}_{\gamma\sigma} p^\sigma f' \\
&= -\frac{\hbar}{4} \epsilon^{\mu\nu\alpha\gamma} \epsilon_\gamma\sigma\rho\xi p_\alpha p^\sigma f' \\
&= -\frac{\hbar}{4} \delta^{\mu\nu\alpha\gamma} \mathcal{O}_{\gamma}\rho\xi p_\alpha p^\rho f' \\
&= -\frac{\hbar}{2} (p^\mu p_\alpha \mathcal{O}^{\rho\rho} - p^\nu p_\rho \mathcal{O}^{\mu\rho}) f',
\end{align*}$$

(26)

where we have dropped the $p^2 \mathcal{O}^{\mu\nu}$ term inside the brackets due to $\delta(p^2)$, $\delta^{\mu\nu\alpha\gamma} \equiv -\epsilon^{\gamma\mu\alpha\nu} \epsilon_{\gamma\sigma\rho\xi}$ is given by Eq. (A2), and $\delta^{\mu\nu\alpha\gamma} \mathcal{O}_{\gamma}\rho\xi p_\alpha p^\rho$ is given by Eq. (A3). From the last line of Eq. (26) we obtain

$$\mathcal{J}^\rho = -\frac{\hbar}{2} p_\rho \mathcal{O}^{\mu\rho} f'.$$

(27)

Then Eq. (9) is in the form

$$\mathcal{J}^\rho = p^\rho f \delta(p^2) - \frac{\hbar}{2} p_\alpha \mathcal{O}^{\nu\alpha} f' \delta(p^2) + \frac{\hbar}{2\rho^2} \epsilon^{\mu\nu\lambda\sigma} p_\lambda \nabla_\mu \mathcal{J}_\nu,$$

(28)

where $f' \equiv \frac{\partial f}{\partial (p^\nu)}$. The first term is the zeroth order contribution (generally it can include higher order contributions, but here we neglect this possibility for simplicity), while the second and third term are at least the first order contribution. We can replace $\mathcal{J}_\rho$ in Eq. (28) with the zeroth order contribution $p_\nu f \delta(p^2)$ to obtain the zeroth and first order contribution given in Eq. (5).

Now we check if the solutions (5) satisfy the second line of Eq. (3). Let us look at the zeroth order solution

$$\nabla_\mu \mathcal{J}^\mu(0) = \nabla_\mu [p^\mu f \delta(p^2)] = \delta(p^2) p^\mu \nabla_\mu f = 0,$$

(29)

where we have used $\nabla_\mu f = f' \mathcal{O}_{\nu\sigma} p^\sigma$ in Eq. (20). For the first order solution in (5), following Eqs. (A6,A7), so we can verify

$$\nabla_\mu \mathcal{J}^\mu(1) = 0,$$

(30)

hold for constant vorticity and field strength tensor. So we see that the second line of Eq. (3) does hold. From Eqs. (A6,A7), we see that if there is an electromagnetic field in a system of charged fermions as shown in the electromagnetic term in $\mathcal{J}^\mu(1)$, it will make fermions rotate which results in a vorticity term in $\mathcal{J}^\mu$. Therefore both terms coexist in the first order solutions (5). This is consistent with the observation that both the vorticity and magnetic field terms in a system of charged fermions must coexist for the second law of thermodynamics to be satisfied [57–59].

### IV. CVE AND CME CURRENT

The current can be obtained from $\mathcal{J}^\rho$ in Eq. (5) by integration over $p$,

$$j^\rho_s = \int d^4 p \mathcal{J}^\rho_s.$$

(31)

In this section we recover the chirality label $s$. We can use the thermal distribution (6) to evaluate $j^\rho_s$ in equilibrium. The zeroth order contribution is given by $\mathcal{J}^\rho(0)_s$,

$$\begin{align*}
j^\rho(0)_s &= \int d^4 p p^\rho f_s \delta(p^2) = u^\rho \int d^4 p p^\rho f_s \delta(p^2) \\
&= u^\rho \int \frac{d^3 p}{(2\pi)^3} (f^+_s - f^-_s) \\
&= N_s u^\rho,
\end{align*}$$

(32)
where we have used the decomposition $p^\nu = p_0 u^\nu + \vec{p}^\nu$ with $p_0 \equiv u \cdot p$ and $\vec{p} \cdot u = 0$, we have used the shorthand notation $f_{\nu}^\pm = f_{\nu \mu}^\pm (\beta E_p \mp \beta \mu s)$ with $E_p = \sqrt{-p_0^2 m^2}$, and $N_s$ is the fermion number density of chiral fermions. In Eq. (32), we have dropped the term proportional to the spatial part $\vec{p}^\nu$ since its integral is zero for the thermal distribution. We can obtain the CVE current from the $\tilde{\Omega}^{\sigma\nu}$ term in Eq. (5),

$$j_{s,\nu}^\rho = -\frac{\hbar}{2} \tilde{\Omega}^{\sigma\nu} \int d^4p p_\sigma \frac{\partial f_s}{\partial (\beta \cdot \vec{p})} \delta(p^2)$$

$$= \hbar s \omega^{\nu} \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ f_{\nu}^{\prime\mu}(1 - f_{\nu}^\prime) + f_{\nu}^\prime(1 - f_{\nu}^{\prime\mu}) \right]$$

$$= \omega^{\nu} \frac{s \hbar}{2\pi^2} \int_0^\infty dE_p E_p \left( f_{\nu}^\prime + f_{\nu}^\prime \right)$$

$$= T \xi_s \omega^{\nu}, \quad (33)$$

where we have the contribution from $p_\sigma$ since its integral is zero for the thermal distribution. In Eq. (33) $\omega^{\nu}$ is the thermal vorticity and $\xi_s$ is the CVE coefficient for the chiral fermion with the chirality $s$. The vector and axial vector currents in the chiral vortical effect are given by,

$$j^\mu(\omega) = (\xi_+ + \xi_-) \omega^{\mu} = \xi \omega^{\mu},$$

$$j_5^\mu(\omega) = (\xi_+ - \xi_-) \omega^{\mu} = \xi_5 \omega^{\mu}, \quad (34)$$

where $\xi = \mu \nu_5 / \pi^2$ and $\xi_5 = T^2 / 6 + (\mu^2 + \nu_5^2) / (2\pi^2)$ are coefficients in Eqs. (22-23) of Ref. [56].

It is easy to check that the $F_{\mu\nu}$ term in Eq. (5) gives the CME current in equilibrium,

$$j_{s,B}^\rho = \hbar sQ \tilde{F}^{\rho\lambda} \int d^4p p_\lambda \delta(p^2) f_s$$

$$= \hbar sQ \tilde{F}^{\rho\lambda} \frac{1}{2} \int d^4p \left( \frac{p_\lambda}{p_0} \frac{\partial \delta(p^2)}{\partial p_0} \right) f_s$$

$$= -\hbar sQ \tilde{F}^{\rho\lambda} \frac{1}{2} \int d^4p \left( \frac{\partial \delta(p^2)}{\partial p_0} \frac{d}{dp_0} \right) f_s$$

$$= \hbar sQ \beta B^{\rho} \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} \left[ f_{\nu}^{\prime\mu}(1 - f_{\nu}^\prime) - f_{\nu}^\prime(1 - f_{\nu}^{\prime\mu}) \right]$$

$$= B^{\rho} \frac{sQ \hbar}{4\pi^2} \int_0^\infty dE_p \left( f_{\nu}^\prime - f_{\nu}^\prime \right)$$

$$= \xi_B^s B^{\rho}, \quad (35)$$

where we have dropped the term with spatial momentum $\vec{p}_\lambda$ and the surface term in $p_0$ in the third line. Note that $\xi_B^s$ is the CME conductivity for chiral fermions with the chirality $s$. Then one can reproduce the chiral magnetic effect for the vector and axial vector currents as

$$j_B^\mu = (\xi_B^+ + \xi_B^-) B^{\mu} = \xi_B^s B^{\mu},$$

$$j_5^\mu = (\xi_B^+ - \xi_B^-) B^{\mu} = \xi_B^s B^{\mu}, \quad (36)$$

where $\xi_B = Q \mu_5 / (2\pi^2)$ and $\xi_B^5 = Q \mu / (2\pi^2)$ are coefficients in Eqs. (22-23) of Ref. [56].

V. CVE AND REFERENCE FRAME

In Sect. [54] we have derived and applied the first order solutions of the Wigner function with the assumption of the thermal distribution function under static-equilibrium conditions. The advantage of this method is that the explicit Lorentz covariance of the solutions is retained.

Without assuming the form of the distribution function and static-equilibrium conditions, we can still solve the Wigner function $\mathcal{J}^{\nu}$ in the framework of semiclassical expansion [54]. To any order of $\hbar$ it has been shown that only one component of $\mathcal{J}^{\nu}$ is independent while other three components can be derived from the independent one. One can choose, for example, the time-component $\mathcal{J}^{0}$ as the independent one which defines the distribution function. But there is a freedom to choose any reference frame to define the time-component, so all spatial components can be derived from the time-component in this reference frame. This superficially break the Lorentz covariance [60][63] of
\( J^\mu \). The requirement that \( J_\mu \) should not depend on the choice of the reference frame in which \( J^0 \) is defined leads to the corresponding change of the distribution function [54]. In this method we can also derive the zeroth and first order solution \( J_{(0,1)}^\mu \). The CVE current can be obtained from \( J^\mu_{(1)} \).

The reference frame is characterized by a time-like 4-vector \( n^\mu \) with normalization \( n^2 = 1 \). In general cases, \( n^\mu \) can depend on space-time coordinates, but here we assume that \( n^\mu \) is a constant in space and time.

In this section we will show that the CVE current from \( J_{(1)}^\mu \) obtained in the semiclassical expansion in Ref. [54] has two parts, which can be identified as the normal current and magnetization current after using the thermal distribution and static-equilibrium conditions. Each part, the normal or magnetization current, depends on \( n^\mu \), but the sum of two is frame independent (or independent of \( n^\mu \)) provided the distribution function is changed in a given manner corresponding to the change of reference frames. So the choice of \( n^\mu \) is like to choose a gauge and any physical quantity such as \( J^\mu \) should not depend on \( n^\mu \).

For simplicity, we drop the electromagnetic field. Then we obtain the vector component of the Wigner function for chiral fermions up to \( O(\hbar) \) [54],

\[
\begin{align*}
J^\mu_{(0)} &= p^\mu f(0) \delta(p^2), \\
J^\mu_{(1)} &= p^\mu f(1) \delta(p^2) - \frac{s}{2n \cdot p} \epsilon_{\mu\nu\rho\sigma} n^\nu p^\rho (\partial^\sigma f(0)) \delta(p^2),
\end{align*}
\]

where \( f_{(0,1)} \) are arbitrary functions of space and time where the indices '(0)' and '(1)' label the orders in \( \hbar \). Obviously the second term of \( J^\mu_{(1)} \) in (38) depends on \( n^\mu \). We can extract \( f_{(0,1)} \) by \( f_{(0,1)} = (n \cdot J_{(0,1)})/(n \cdot p) \). We see that the form of \( J^\mu_{(1)} \) in Eq. (38) is different from Eq. (5) derived based on the thermal distribution function under static-equilibrium conditions.

When one changes the Lorentz frame \( n^\mu \to n'^\mu \), it is required that the vector component do not change

\[ \delta J_{(0,1)}^\mu = J'^\mu_{(0,1)} - J^\mu_{(0,1)} = 0, \]

which leads to the variations of distribution functions [54]

\[
\begin{align*}
\delta f_{(0)} &= 0, \\
\delta f_{(1)} &= -s \frac{\epsilon_{\lambda\nu\rho\sigma} n^\lambda p^\nu \partial^\rho \partial^\sigma f(0)}{2(n' \cdot p) (n \cdot p)}.
\end{align*}
\]

Note that \( \delta f_{(1)} \) in Eq. (40) is called side-jump [60] whose derivation follows Eq. (3) for the zeroth and first order terms. With Eq. (3) and (37), we can also verify that the variation of the second term in Eq. (38) does give \( -p^\mu \delta f_{(1)} \) which makes Eq. (39) hold,

\[
\begin{align*}
\delta J_{(1)}^\mu (2) &= -s \frac{(n \cdot p) n'^\nu - (n' \cdot p) n^\nu}{2(n' \cdot p) (n \cdot p)} \delta(p^2) \epsilon_{\mu\nu\rho\sigma} p^\rho \partial^\sigma f(0) \\
&= -p^\mu \left( \frac{n' \cdot J_{(1)}}{n' \cdot p} - \frac{n \cdot J_{(1)}}{n \cdot p} \right) = -p^\mu \delta f_{(1)},
\end{align*}
\]

where \( f_{(1)} \) is defined by \( (n \cdot J_{(1)})/(n \cdot p) \) in the last line.

Since \( \delta f_{(0)} = 0 \), we can assume that the scalar function \( f_{(0)} \) only depends on \( \beta \cdot p \) where \( \beta^\mu = u^\mu /T \). If we further assume the Killing condition \( \partial^\mu \beta_{\nu} + \partial^\nu \beta_{\mu} = 0 \) in Eq. (21) which implies a constant thermal vorticity tensor \( \Omega_{\mu\nu} = (\partial^\mu \beta_\nu - \partial^\nu \beta_\mu) /2 \), it is easy to show that \( f_{(0)} \) satisfies the chiral kinetic equation \( \delta(p^2) p^\mu \partial^\mu f_{(0)} = 0 \). From the form of \( \delta f_{(1)} \) in Eq. (40) we then obtain

\[
\begin{align*}
\delta f_{(1)} &= -s \frac{p^\mu \epsilon_{\lambda\nu\rho\sigma} n^\lambda p^\nu \partial^\rho \partial^\sigma \beta_{\mu}}{2 (n' \cdot p) (n \cdot p)} \frac{df_{(0)}}{d(\beta \cdot p)} \\
&= -s \frac{p^\mu \epsilon_{\lambda\nu\rho\sigma} n^\lambda p^\nu \Omega_{\rho\sigma}}{2 (n' \cdot p) (n \cdot p)} \frac{df_{(0)}}{d(\beta \cdot p)} \\
&= -s \frac{n'_{\alpha} p_{\gamma} \tilde{\Omega}_{\alpha\gamma}}{2 (n' \cdot p)} \frac{df_{(0)}}{d(\beta \cdot p)} + s \frac{n_{\alpha} p_{\gamma} \tilde{\Omega}_{\alpha\gamma}}{2 (n \cdot p)} \frac{df_{(0)}}{d(\beta \cdot p)},
\end{align*}
\]

where we have used Eqs. (A11-A2). Now let us express \( f_{(1)} \) as

\[ f_{(1)} = f_{(1)}^0 - s \frac{n_{\alpha} p_{\gamma} \tilde{\Omega}_{\alpha\gamma}}{2 (n \cdot p)} \frac{df_{(0)}}{d(\beta \cdot p)}. \]
where \( \tilde{f}(1) \) does not depend on the reference vector \( n^\mu \), i.e. \( \delta \tilde{f}(1) = 0 \). We can choose the specific solution with \( \tilde{f}(1) = 0 \). Substituting Eq. (43) into Eq. (38) yields

\[
\mathcal{J}^\mu(1) = -p^\mu \frac{s}{2(n \cdot p)} n_\alpha p_\gamma \tilde{\gamma}^{\alpha \gamma} \frac{df(0)}{d(\beta \cdot p)} \delta(p^2) - \frac{s}{2n \cdot p} e^{\mu \rho \sigma} n_\nu p_\rho (\partial_\sigma f(0)) \delta(p^2)
\]

\[
= \frac{s}{2n \cdot p} \left( p^\mu n_\alpha p_\gamma \Omega^{\alpha \gamma} + p^\lambda e^{\mu \nu \rho \sigma} n_\nu p_\rho \Omega_{\lambda} \right) \frac{df(0)}{d(\beta \cdot p)} \delta(p^2)
\]

\[
= -\frac{s}{2} \Omega^{\mu \nu} p_\nu \frac{df(0)}{d(\beta \cdot p)} \delta(p^2), \tag{44}
\]

where we used Eqs. (A1A2) to obtain the last equality. We see that \( \mathcal{J}^\mu(1) \) in the last line of Eq. (44) is independent of \( n^\mu \) and matches the first term of Eq. (33) by setting \( f(0) = f_s \).

Now we evaluate the contributions to the CVE current in (33) from the two terms of the first equality in Eq. (44) which we call \( j^\mu_{s,\omega}(1) \) and \( j^\mu_{s,\omega}(2) \),

\[
\begin{align*}
j^\mu_{s,\omega}(1) &= -\hbar \frac{s}{2} u_\alpha \tilde{\gamma}^{\alpha \gamma} \int d^4p \frac{1}{(u \cdot p)} p^\mu p_\nu \frac{df_s}{d(\beta \cdot p)} \delta(p^2) \\
&= -\hbar \frac{s}{2} \omega \int d^4p \frac{p^\mu p_\nu}{(2\pi)^3 E_p^2} \left[ f^+_{FD}(1 - f^-_{FD}) + f^-_{FD}(1 - f^+_{FD}) \right] \\
&= \frac{1}{3} T \xi_s \omega^{\mu},
\end{align*}
\]

\[
\begin{align*}
j^\mu_{s,\omega}(2) &= -\hbar \frac{s}{2} \epsilon^{\mu \rho \sigma} u_\nu \int d^4p \frac{1}{u \cdot p} p_\rho (\partial_\sigma f_s) \delta(p^2) \\
&= \frac{2}{3} T \xi_s \omega^{\mu}, \tag{45}
\end{align*}
\]

where we have assumed \( n^\mu = u^\mu \) and take \( f(0) = f_s(x, p) \) given by Eq. (39). Note that \( \Omega^{\mu \nu} \) and \( \tilde{\Omega}^{\mu \nu} \) are thermal vorticity tensors and \( \omega^{\mu} \) is the thermal vorticity vector which are all dimensionless. We should make comments that the assumption \( n^\mu = u^\mu \) is not very rigorous, since in making the decomposition of \( \mathcal{J}^\mu \) into the time-like and space-like part we do not have the space-time derivatives of \( n^\mu \) as it is a constant. Once this assumption is made, the space-time derivatives of \( n^\mu \) should be considered in principle. But for the sake of illustration here, we neglect this complexity by choosing \( n^\mu = u^\mu = (1, 0, 0, 0) \) below. We see in Eq. (45) that \( j^\mu_{s,\omega}(1) \) and \( j^\mu_{s,\omega}(2) \) contribute to the full CVE current by \( 1/3 \) and \( 2/3 \) respectively. In order to see the physical meaning of \( j^\mu_{s,\omega}(1) \) and \( j^\mu_{s,\omega}(2) \), we assume a simple case \( n^\mu = u^\mu = (1, 0, 0, 0) \), in which we obtain the explicit form of \( j^\mu_{s,\omega}(1) \) and \( j^\mu_{s,\omega}(2) \) in three spatial dimensions (3D),

\[
\begin{align*}
j_{s,\omega}(1) &= \frac{s}{2} \int \frac{d^3p}{(2\pi)^3} \frac{p}{|p|^2} (p \cdot \omega) \left[ f^+_{FD}(1 - f^-_{FD}) + f^-_{FD}(1 - f^+_{FD}) \right] \\
&\approx \int \frac{d^3p}{(2\pi)^3} \frac{p}{|p|^2} \left[ f_{FD} \left( \beta |p| - \beta \mu_s - s h \frac{p \cdot \omega}{2|p|} \right) \right. \\
&\left. + f_{FD} \left( \beta |p| + \beta \mu_s - s h \frac{p \cdot \omega}{2|p|} \right) \right], \\
j_{s,\omega}(2) &= \nabla \times \int \frac{d^3p}{(2\pi)^3} \left( \frac{sp}{2|p|^2} h \right) \left[ f_{FD} (\beta |p| - \beta \mu_s) + f_{FD} (\beta |p| + \beta \mu_s) \right]. \tag{46}
\end{align*}
\]

We can see that \( j_{s,\omega}(1) \) comes from the momentum integration of the fermion’s velocity \( p/|p| \) with the Fermi-Dirac distribution function in which the fermion’s energy is modified by the spin-vorticity coupling, while \( j_{s,\omega}(2) \) is from the magnetization due to the magnetic moment of the chiral fermion where the magnetic moment of the chiral fermion is given by \( \hbar s |p| (2|p|^2)^{-1} \). \[60, 64, 65].

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Appendix A: Some useful formulas and detailed derivations

First we list useful formulas for the field strength tensor, vorticity tensor and their duals,

\[ F^{\mu\nu} = E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma} u_\rho B_\sigma, \]
\[ \tilde{F}^{\mu\nu} = B^\mu u^\nu - B^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma} E_\rho u_\sigma, \]
\[ \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}, \]
\[ F^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\lambda} \tilde{F}_{\rho\lambda}, \]

\[ \Omega^{\mu\nu} = \epsilon^\mu u^\nu - \epsilon^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma} u_\rho \omega_\sigma, \]
\[ \tilde{\Omega}^{\mu\nu} = \omega^\mu u^\nu - \omega^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma} \epsilon_\rho u_\sigma, \]
\[ \Omega^{\mu\nu} = \frac{1}{2}(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \tilde{\Omega}_{\rho\sigma}, \]

\[ \tilde{\Omega}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \Omega_{\rho\sigma}, \quad (A1) \]

where \( \epsilon^{\mu\nu\beta\gamma} \) and \( \epsilon_{\mu\nu\beta\gamma} \) are anti-symmetric tensors with \( \epsilon^{\mu\nu\beta\gamma} = 1(-1) \) and \( \epsilon_{\mu\nu\beta\gamma} = -1(1) \) for even (odd) permutations of indices 0123, so we have \( \epsilon^{0123} = -\epsilon_{0123} = 1 \). Instead of \( \Omega_{\nu\sigma} \), \( \tilde{\Omega}_{\nu\sigma} \), \( F_{\nu\mu} \) and \( \tilde{F}^{\rho\lambda} \), usually we also use the thermal vorticity vector \( \omega^\rho = \frac{1}{2} \epsilon^{\rho\sigma\gamma\alpha} u_\sigma \partial_\alpha \beta_\gamma = \tilde{\Omega}^{\sigma\gamma} u_\sigma \), the electric field \( E^\mu = F^{\mu\nu} u_\nu \), and the magnetic field \( B^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} u_\nu F_{\lambda\rho} \).

The contraction formula of two anti-symmetric tensors are useful

\[ \epsilon^{\mu\nu\sigma\alpha} \epsilon_{\gamma\rho\delta\epsilon} = -2(\delta^\mu_\gamma \delta^\nu_\delta - \delta^\nu_\gamma \delta^\mu_\delta). \]
\[ \epsilon^{\mu\nu\sigma\alpha} \epsilon_{\gamma\rho\alpha\epsilon} = -\delta^{\mu\nu}_{\sigma\epsilon}, \quad (A2) \]

With the above formula we can evaluate \( \delta_{\sigma\rho\alpha\delta} \tilde{\Omega}^{\rho\delta} p_\alpha p^\sigma \) in Eq. (26),

\[ \delta^{\mu\nu}_{\sigma\rho\alpha\delta} \tilde{\Omega}^{\rho\delta} p_\alpha p^\sigma = \tilde{\Omega}^{\rho\delta} p_\alpha p^\sigma (\delta^{\mu\nu}_{\sigma\rho} \delta^{\alpha\delta}_{\alpha\rho} + \delta^{\mu\nu}_{\sigma\rho} \delta^{\alpha\delta}_{\alpha\rho}) + \delta^{\mu\nu}_{\sigma\rho} \delta^{\alpha\delta}_{\alpha\rho} \delta^{\beta\gamma}_{\beta\gamma} \]
\[ = 2\tilde{\Omega}^{\rho\delta} p_\alpha p^\rho + 2\tilde{\Omega}^{\rho\delta} p^\rho - 2\tilde{\Omega}^{\rho\delta} p_\rho p^\rho. \quad (A3) \]

We try to prove the identities in (24). In order to prove the first identity, we can start from the first term on the right-hand side by rewriting \( \Omega^{\mu\alpha} \) and \( \tilde{F}^{\rho\mu} \) in linear combinations of \( \tilde{\Omega}_{\lambda\gamma} \) and \( F^{\eta\delta} \) respectively,

\[ \Omega^{\mu\alpha} \tilde{F}^{\rho\mu} p_\alpha p^\rho = -\frac{1}{4} \epsilon^{\mu\alpha\lambda\gamma} \epsilon_{\rho\mu\eta\delta} \tilde{\Omega}_{\lambda\gamma} F^{\eta\delta} p_\alpha p^\rho \]
\[ = -\frac{1}{4} \epsilon^{\mu\alpha\lambda\gamma} \tilde{\Omega}_{\lambda\gamma} F^{\eta\delta} p_\alpha p^\rho \]
\[ = -\frac{1}{2} \tilde{\Omega}^{\mu\nu} F_{\mu\nu} p^2 - \tilde{F}^{\rho\mu} p_\rho p^\rho, \quad (A4) \]

where we have used Eq. (A2). For the second identity, in the same way we can rewrite \( \tilde{F}^{\mu\alpha} \) and \( F_{\mu\sigma} \) on the left-hand side in linear combinations of \( \tilde{F}^{\eta\delta}_{\lambda\gamma} \) and \( F^{\eta\delta} \) respectively,

\[ \tilde{F}^{\mu\alpha} F_{\mu\sigma} p_\sigma p^\sigma = F^{\mu\alpha} \tilde{F}^{\rho\mu} p_\rho p^\rho \]
\[ = -\frac{1}{4} \epsilon^{\mu\alpha\lambda\gamma} \epsilon_{\rho\mu\eta\delta} \tilde{F}^{\eta\delta}_{\lambda\gamma} F^{\eta\delta} p_\alpha p^\rho \]
\[ = -\frac{1}{4} \epsilon^{\mu\alpha\lambda\gamma} \tilde{F}^{\eta\delta}_{\lambda\gamma} F^{\eta\delta} p_\alpha p^\rho \]
\[ = \frac{1}{2} \tilde{F}^{\mu\nu} F_{\mu\nu} p^2 - \tilde{F}^{\rho\mu} p_\rho p^\rho. \quad (A5) \]

So we obtain the second identity of (24). In the same procedure we can prove the third identity of (24).
We now verify $\nabla_\mu J^\mu_\perp = 0$ with $J^\mu_\perp$ being given by Eq. (5). There are two parts in $J^\mu_\perp$: the vorticity part and electromagnetic field part. We evaluate the vorticity part as

$$\nabla_\mu J^\mu_\perp(\Omega) = -\hbar/2 \tilde{\Omega}^{\alpha\sigma}(\nabla_\mu p_\sigma) f'(p^2)$$

$$-\hbar/2 \tilde{\Omega}^{\alpha\sigma} p_\alpha(\nabla_\mu f')\delta(p^2)$$

$$-\hbar/2 \tilde{\Omega}^{\alpha\sigma} p_\alpha \nabla_\mu \delta(p^2)$$

$$= -\hbar^2/2 \tilde{\Omega}^{\mu\nu} F_{\mu\nu} p^2 - 2\tilde{\Omega}^{\mu\nu} F_{\mu\nu} p^2$$

$$= -\hbar^2/8 \tilde{\Omega}^{\mu\nu} \Omega_{\mu\nu} p^2 f''(p^2)$$

$$= \hbar s Q \Omega^{\mu\nu} F_{\mu\nu} p^2 f'(1/p^2)$$

where we have used the fact that $\tilde{\Omega}^{\alpha\sigma}$ is a constant due to the Killing condition in (21). Then we look at the electromagnetic field part

$$\nabla_\mu J^\mu_\perp(EM) = \hbar s Q \tilde{F}^{\mu\nu}(\nabla_\mu p_\nu) f\delta(p^2)$$

$$+ \hbar s Q \tilde{F}^{\mu\nu} p_\nu(\nabla_\mu f)\delta(p^2)$$

$$+ \hbar s Q \tilde{F}^{\mu\nu} p_\nu \nabla_\mu \delta(p^2)$$

$$= -\hbar^2 Q \tilde{F}^{\mu\nu} F_{\mu\nu} f\delta(p^2)$$

$$+ \hbar s Q \tilde{F}^{\mu\nu} \Omega_{\mu\nu} p^2 f'(p^2)$$

$$+ 4\hbar s Q \tilde{F}^{\mu\nu} F_{\mu\nu} p^2 f'(1/p^2)$$

$$= \hbar s Q \tilde{F}^{\mu\nu} \Omega_{\mu\nu} p^2 f'(p^2),$$

where we have assumed $\tilde{F}^{\mu\nu}$ is a constant and used $\nabla_\mu f = f'\Omega_{\mu\sigma} p^2_\sigma$, $\delta'(x) = -\delta(x)/x$, $\delta''(x) = -2\delta(x)/x$, $\nabla_\mu f = f'\Omega_{\mu\sigma} p^2_\sigma$ in Eq. (20), and the integrability condition (24). From Eqs. (A6,A7) we obtain $\nabla_\mu J^\mu_\perp = 0$. 

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