Equality of pressures for diffeomorphisms preserving hyperbolic measures

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Received: 18 September 2007 / Accepted: 7 March 2008 / Published online: 4 April 2008 © Springer-Verlag 2008

Abstract For a diffeomorphism which preserves a hyperbolic measure the potential \( \varphi^u = -\log |\text{Jac } df|_{E^u} \) is studied. Various types of pressure of \( \varphi^u \) are introduced. It is shown that these pressures satisfy a corresponding variational principle.

Keywords Thermodynamical formalism · Non-uniformly hyperbolic systems

Mathematics Subject Classification (2000) 37D25 · 37D35 · 28D20

1 Introduction

For a uniformly hyperbolic diffeomorphism \( f \), the induced volume deformation \( \varphi^u \) in the unstable sub-bundle over a compact \( f \)-invariant set significantly characterizes the geometry of the set as well as the dynamics in its neighborhood. Under the hypothesis of uniform hyperbolicity, and particularly for hyperbolic surface diffeomorphisms, a large number of dynamical quantifiers such as, for example, fractal dimensions, Lyapunov exponents, and escape rates, are captured through the topological pressure of \( \varphi^u \). If \( f : M \to M \) is a \( C^{1+\varepsilon} \) diffeomorphism on a Riemannian manifold \( M \) (we assume that some Riemannian metric on \( M \) is fixed) and \( \Lambda \) is an \( f \)-invariant locally maximal set such that \( f|\Lambda \) is uniformly hyperbolic and satisfies specification (and hence is mixing), then the topological pressure \( P_{f|\Lambda} \) of the function \( \varphi^u(x) = -\log |\text{Jac } df_x|_{E^u_x} \) can be calculated through.

This research was supported by the grant EU FP6 ToK SPADE2. The author is grateful to IM PAN Warsaw for the hospitality and to C. Wolf for discussions about suitable concepts of pressure.

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\[ P_{f|A}(\varphi^u) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{f^n(x) = x} |\text{Jac} \, df^n|_{E^u}^{-1} \]

\[ = \sup_{\mu \in \mathcal{M}_E} \left( h_\mu(f) - \int_{A} \log |\text{Jac} \, df|_{E^u} \, d\mu \right) \quad (1) \]

(see [3] or [8]). Here the second equality (with the supremum taken over all ergodic \( f \)-invariant measures supported in \( A \)) simply follows from the variational principle by continuity of the function \( \log |\text{Jac} \, df|_{E^u} \colon A \to \mathbb{R} \). Note that the supremum in (1) is in fact a maximum (see [3]). The measure which realizes this maximum is of unique importance from several different points of view. A particular case is given when the maximizing measure is the SRB (after Sinai Ruelle Bowen) measure of \( f \), each of the terms in (1) is zero, and \( A \) is an attractor (see [16] for further details and references).

In the case of more general dynamical systems, the above quantities are likewise important, in particular for the issue of existence of SRB measures. Note that the classical thermodynamic formalism, however, requires the potential to be continuous. We point out that for a non-uniformly hyperbolic system (a system with non-zero integrated Lyapunov exponents) it is natural to consider potentials which are discontinuous: no continuous \( df \)-invariant sub-bundle \( E^u \subset T_A M \) may exist and \( x \mapsto -\log |\text{Jac} \, df|_{E^u} \) is in general only a measurable function. In this paper we study appropriate modifications of any of the terms in (1) by exploiting techniques which were developed by Katok [7] and Mendoza [10] for dynamical systems with some non-uniformly hyperbolic behavior. We combine them with an approach of a non-additive version of the thermodynamic formalism, developed by [2,5,12] in particular for non-conformal systems.

It is meaningful to consider a function \( \varphi^u \overset{\text{def}}{=} -\log |\text{Jac} \, df|_{E^u} \) which is defined only on a certain subset of \( A \). Pesin [12] developed an extension of the classical topological pressure to a pressure on sets which are not necessarily compact nor invariant, but his approach requires the potential functions to be continuous. Mummert [11] discusses for example a pressure for non-continuous potentials and provides a meaningful generalization of \( P_{f|A}(\varphi^u) \) in the case that \( f \) preserves a hyperbolic measure. In [6] the so-called saddle point pressure \( P_{f,SP} (\varphi^u) \) is introduced, which is entirely determined by the values of \( \varphi^u \) on the periodic points of saddle type (see Sect. 2 for the definition), and which generalizes the second term in (1) in the case that such periodic points do exist. Notice that, by the multiplicative ergodic theorem, \( h_\mu(f) + \int_{A} \varphi^u \, d\mu \) is well-defined for any ergodic \( f \)-invariant probability measures with a positive Lyapunov exponent.

The main result of this paper is to show that the equalities (1) extend to more general maps, including \( C^{1+\varepsilon} \) diffeomorphisms possessing hyperbolic invariant probability measures. Here we call an ergodic measure hyperbolic or say it is of saddle type if it possesses at least one negative and one positive, and no zero Lyapunov exponents. We say that \( f \) is non-uniformly hyperbolic if every ergodic \( f \)-invariant measure is hyperbolic.

Paradigms of genuinely non-uniformly hyperbolic diffeomorphisms are given, for example, within the family of Hénon maps (see [4]). Another perhaps simplest example is provided by the figure-8 attractor (composed of homoclinic loops joining a fixed point of saddle type, see for example [7, p. 140]). Even though here the maximal invariant set which is formed by the loops does not support any other invariant probability measures besides the Dirac measure supported at the saddle fixed point, it provides a basic plug for more sophisticated models. A similar attractor is used for example in [1] where it is inserted by a smooth surgery into a uniformly hyperbolic set. The resulting compact set \( A \) is invariant and locally maximal under

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