A NOTE ON STABLE COMMUTATOR LENGTH IN BRAIDED PTOLEMY-THOMPSON GROUPS

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Abstract. In this note, we show that the sets of all stable commutator lengths in the braided Ptolemy-Thompson groups are equal to non-negative rational numbers.

1. Introduction

Let $G$ be a group and $[G, G]$ its commutator subgroup. For an element $g \in [G, G]$, its commutator length $\text{cl}(g)$ is defined by $\text{cl}(g) = \min\{l \mid g = [a_1, b_1] \cdots [a_l, b_l]\}$. For $g \in G$ with $g^k \in [G, G]$ for some positive integer $k$, its stable commutator length $\text{scl}(g)$ is defined by $\text{scl}(g) = \lim_{n \to \infty} \frac{\text{cl}(g^{kn})}{kn}$. For $g \in G$ with $g^k \notin [G, G]$ for any $k$, we put $\text{scl}(g) = \infty$. We say that the group $G$ has a gap in stable commutator length if there is a positive constant $c$ such that either $\text{scl}(g) \geq c$ or $\text{scl}(g) = 0$ holds for any $g \in G$.

The values of stable commutator length in many classes of groups has been studied. For example, all stable commutator lengths are rational in the central extension of the Thompson group $T$ corresponding to the Euler class(Ghys-Sergiescu[17]), free groups(Calegari[5]), free products of some classes of groups (Walker[22], Calegari[6], Chen[8]), amalgams of free abelian groups(Susse[21]), and certain graphs of groups (Clay-Forerster-Louwsma[12], Chen[10]). It is known that there is a finitely presented group which has an irrational stable commutator length(Zhuang[23]). The gap in many classes of groups has also been studied. For example, the above central extension of the Thompson group has no gap, and the following groups have a gap; free groups(Duncan-Howie[13]), word-hyperbolic groups(Calegari-Fujiwara[7]), finite index subgroups of the mapping class group of a closed orientable surface (Bestvina-Bromberg-Fujiwara[3]), right-angled Artin groups(Heuer[18]), and the fundamental group of closed oriented connected 3-manifolds (Chen-Heuer[11]).

In this note, we show that all stable commutator lengths in the braided Ptolemy-Thompson groups $T^*$ and $T^\sharp$ (see Section 2.3) are rational and these groups have no gap. More precisely, we show the following theorem.

Theorem 1.1. The sets $\text{scl}(T^*)$ and $\text{scl}(T^\sharp)$ of all stable commutator lengths in the braided Ptolemy-Thompson groups are equal to the non-negative rational numbers $\mathbb{Q}_{\geq 0}$.

2. Preliminaries
2.1. quasi-morphisms and Bavard’s duality theorem. Let $G$ be a group. A map $\phi: G \to \mathbb{R}$ is a quasi-morphism if

$$D(\phi) = \sup_{g,h \in G} |\phi(gh) - \phi(g) - \phi(h)|$$

is finite. This $D(\phi)$ is called the defect of $\phi$. A quasi-morphism is homogeneous if the condition $\phi(g^n) = n\phi(g)$ holds for any $g \in G$ and $n \in \mathbb{Z}$. Let $Q(G)$ denote the $\mathbb{R}$-vector space consisting of all homogeneous quasi-morphisms on $G$. The key ingredient to calculate the stable commutator length is the following Bavard’s duality theorem.

**Theorem 2.1.** [2] Let $G$ be a group and $g \in [G,G]$, then

$$\text{scl}(g) = \sup_{\phi \in Q(G)} \frac{|\phi(g)|}{2D(\phi)}.$$ 

This theorem allows us to calculate the stable commutator length in groups with few homogeneous quasi-morphisms. For example, any homogeneous quasi-morphism on the infinite braid group $B_\infty$ is equal to the abelianization homomorphism $B_\infty \to \mathbb{Z}$ up to constant multiple (see Kotschick [19]). Thus, we have $\text{scl}(g) = 0$ for $g \in [B_\infty, B_\infty]$. The central extension $\tilde{T}$ of the Thompson group $T$ corresponding to the Euler class is also a group with few quasi-morphisms. It is known that any homogeneous quasi-morphism on $\tilde{T}$ is equal to the rotation number up to constant multiple. Ghys-Sergiescu [17] showed that the set of all rotation numbers on $\tilde{T}$ coincides with the rational numbers $\mathbb{Q}$. Since $\tilde{T} = [\tilde{T}, \tilde{T}]$ and the defect of rotation number is equal to 1, we have $\text{scl}(\tilde{T}) = \mathbb{Q} \geq 0$.

2.2. Central extensions of the Thompson group $T$. Let $n$ be an integer and $0 \to \mathbb{Z} \to T_n \xrightarrow{p} T \to 1$ denote the central $\mathbb{Z}$-extension of $T$ corresponding to $n$ times the Euler class $e \in H^2(T; \mathbb{Z})$. Let $\chi$ be a bounded cocycle representing the Euler class $e \in H^2(T; \mathbb{Z})$. Then the extension $T_n$ can be obtained as a group $T \times \mathbb{Z}$ with the product

$$(s,i)(t,j) = (st,i + j + n\chi(s,t)).$$

Since the Thompson group $T$ is uniformly perfect and the Euler class $e$ is bounded, there is a unique homogeneous quasi-morphism $\phi_n \in Q(T_n)$ satisfying $-[\delta \phi_n] = np^*e_b \in H^2_b(T_n; \mathbb{R})$ (see Barge-Ghys [1]). Here $e_b \in H^2_b(T; \mathbb{R})$ is the bounded Euler class. The homogeneous quasi-morphism $\phi_n$ is obtained as the homogenization of the quasi-morphism $T_n \to \mathbb{R}; (t,j) \mapsto j$. More explicitly, the quasi-morphism $\phi_n$ is written as

$$\phi_n(t,j) = j + n \lim_{k \to \infty} \frac{a(t,k)}{k},$$

where $a(t,k) = \chi(t,t) + \chi(t^2,t) + \cdots + \chi(t^{k-1},t) \in \mathbb{Z}$. Since $T_1$ is equal to $\tilde{T}$, the homogeneous quasi-morphism $\phi_1 : T_1 \to \mathbb{R}$ coincides with the rotation number on $\tilde{T}$. It is known that the defect of the rotation number on $\tilde{T}$ is equal to 1 (see, for example, Calegari [3]), thus we have $D(\phi_1) = 1$. 
**Lemma 2.2.** Let $n$ be a non-zero integer, then $D(\phi_n) = n$.

**Proof.** Note that

$$1 = D(\phi_1) = \sup_{(s,i),(t,j)} |\phi_1(st, i + j + \chi(s, t)) - \phi_1(s, i) - \phi_1(t, j)|$$

$$= \sup_{s,t} \left| \chi(s, t) + \lim_{k \to \infty} \frac{a(st, k) - a(s, k) - a(s, k)}{k} \right|.$$ 

Thus we have

$$D(\phi_n) = \sup_{(s,i),(t,j)} |\phi_n(st, i + j + \chi(s, t)) - \phi_n(s, i) - \phi_n(t, j)|$$

$$= \sup_{s,t} \left| n\chi(s, t) + n \lim_{k \to \infty} \frac{a(st, k) - a(s, k) - a(s, k)}{k} \right|$$

$$= nD(\phi_1) = n.$$

\[\square\]

**Lemma 2.3.** Let $n$ be a non-zero integer, then $\phi_n(T_n) = \mathbb{Q}$.

**Proof.** Ghys-Sergiescu[17] showed that $\phi_1(T_1)$ is equal to the rational numbers $\mathbb{Q}$. Thus, for any $(t, j) \in T_1$, the number $\lim (a(t, k)) / k = \phi_1(t, j) - j$ is rational and therefore $\phi_n(T_n) \subset \mathbb{Q}$ follows. Take $q \in \mathbb{Q}$. Then, for any $q \in \mathbb{Q}$, there is an element $(t, j) \in T_1$ such that

$$q = \phi_n(t, j) = \phi_1(t, j) + \lim_{k \to \infty} \frac{a(t, k)}{k}.$$ 

Thus we have $q = \phi_n(t, nj)$.

Since $\dim \mathbb{Q}(T_n) = 1$ and $T_n = [T_n, T_n]$, we have $\text{scl}(T_n) = \mathbb{Q}_{\geq 0}$ for any non-zero integer $n$.

### 2.3. The braided Ptolemy-Thompson groups.

In [14], Funar and Kapoudjian introduced two extensions $T^\ast$ and $T^\sharp$ of Thompson group $T$ by the infinite braid group $B_\infty$ and showed that they are finitely presented. Moreover, it is shown that their abelianization $H_1(T^\ast)$ and $H_1(T^\sharp)$ are isomorphic to $\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ respectively, and therefore $T^\ast$ and $T^\sharp$ are not isomorphic. Let $\hat{\circ}$ denote either $\ast$ or $\sharp$. Since the abelianization homomorphism $B_\infty \to \mathbb{Z}$ is $T^\circ$-conjugation invariant, we have central $\mathbb{Z}$-extensions $T^\circ_\text{ab}$ of $T$, that is, we have the diagram

\[
\begin{array}{ccccccc}
1 & \rightarrow & B_\infty & \rightarrow & T^\circ & \rightarrow & T & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & T^\circ_\text{ab} & \rightarrow & T & \rightarrow & 1.
\end{array}
\]

The corresponding cohomology classes $e(T^\ast_\text{ab})$ and $e(T^\sharp_\text{ab})$ in $H^2(T; \mathbb{Z})$ are equal to $12e$ and $21e$ respectively, where $e \in H^2(T; \mathbb{Z})$ is the Euler class (see Funar-Sergiescu[16] and Funar-Kapoudjian-Sergiescu[15]). Thus $T^\circ_\text{ab}$ is isomorphic to $T_{12}$ and $T^\sharp_\text{ab}$ is isomorphic to $T_{21}$.
3. THE STABLE COMMUTATOR LENGTH IN THE BRAIDED PTOLEMY-THOMPSON GROUPS

In this section, we show that the stable commutator lengths in $T^\circ$ are rational and all positive rationals are realized as the stable commutator length.

At first, we show the following proposition, which is essentially proved in Shtern [20].

**Proposition 3.1.** For an exact sequence $1 \to K \to G \xrightarrow{\rho} H \to 1$, we have an exact sequence

$$0 \to Q(H) \to Q(G) \to Q(K)^G,$$

where $Q(K)^G$ is the vector space consisting of all $G$-conjugation invariant homogeneous quasi-morphisms on $K$.

**Proof.** Shtern [20] showed that, if $\phi \in Q(G)$ is equal to 0 on $K$, there exists a homogeneous quasi-morphism $\psi \in Q(H)$ such that $\phi = p^*\psi$. This implies that the above sequence is exact at $Q(G)$. \qed

Let us consider the diagram (2.1). There are homogeneous quasi-morphisms $\phi_* = \phi_{12} \in Q(T_{12}) = Q(T_{ab})$ and $\phi_2 = \phi_{21} \in Q(T_{21}) = Q(T_{ab}')$.

By Proposition 3.1 and the exact sequence $1 \to [B_\infty, B_\infty] \to T^\circ \xrightarrow{\rho} T^\circ_{ab} \to 1$, we have an injection $\rho^* : Q(T_{ab}') \to Q(T^\circ)$. Thus we have a non-trivial homogeneous quasi-morphism $\psi_\circ = \rho^*\phi_\circ \in Q(T^\circ)$ satisfying $[\delta\psi_\circ] = (p \circ \rho)^* e_b \in H^2_* (T^\circ; \mathbb{R})$.

**Lemma 3.2.** The dimension of $Q(T^\circ)$ is equal to 1.

**Proof.** Since there is a non-trivial element $\psi \in Q(T^\circ)$, it is enough to show that the inequality $\dim Q(T^\circ) \leq 1$ holds. For an exact sequence $1 \to B_\infty \to T^\circ \to T \to 1$, we have an exact sequence

$$0 \to Q(T) \to Q(T^\circ) \to Q(B_\infty)^T$$

by Proposition 3.1. Since the Thompson group $T$ is uniformly perfect, we have $Q(T) = 0$. Thus the map $Q(T^\circ) \to Q(B_\infty)^T$ is injective. The $\mathbb{R}$-vector space $Q(B_\infty)$ is spanned by the abelianization homomorphism $B_\infty \to \mathbb{Z}$ (see Kotschick [19]). Moreover, since the abelianization homomorphism is $T^\circ$-conjugation invariant, we have $Q(B_\infty) = Q(B_\infty)^T$. Thus we have $\dim Q(T^\circ) \leq \dim Q(B_\infty)^T = \dim Q(B_\infty) = 1$. \qed

**Proof of Theorem 1.1.** Note that, by Lemma 2.2, and the surjectivity of $\rho : T^\circ \to T^\circ_{ab}$, the defect $D(\psi_\circ) = D(\rho^*\phi_\circ)$ is equal to $D(\phi_\circ) = 12$ and $D(\psi_2) = D(\rho^*\phi_2)$ is equal to $D(\phi_2) = 21$. Since $H_1(T^\circ) = \mathbb{Z}/12\mathbb{Z}$, the power $g_{12}$ is in $[T^\circ, T^\circ]$ for any $g \in T^\circ$. Thus, by Lemma 3.2 and the Bavard’s duality theorem, we have

$$\text{scl}(g) = \frac{\text{scl}(g_{12})}{12} = \frac{|\psi_\circ (g_{12})|}{288} = \frac{|\phi_\circ (\rho(g))|}{24}$$

for any $g \in [T^\circ, T^\circ]$. Thus, by Lemma 2.3, we have $\text{scl}(T^\circ) = \mathbb{Q}_{\geq 0}$. We can prove $\text{scl}(T^\circ) = \mathbb{Q}_{\geq 0}$ in the same way. \qed
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