The Canonical Quantization in Terms of Quantum Group and Yang-Baxter Equation

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Abstract

In this paper it is shown that a quantum observable algebra, the Heisenberg-Weyl algebra, is just given as the Hopf algebraic dual to the classical observable algebra over classical phase space and the Plank constant is included in this scheme of quantization as a compatible parameter living in the quantum double theory. In this sense, the quantum Yang-Baxter equation naturally appears as a necessary condition to be satisfied by a canonical elements, the universal R-matrix, intertwining the quantum and classical observable algebras. As a byproduct, a new "quantum group" is obtained as the quantum double of the classical observable algebra.

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The purpose of this paper is trying to understand directly physical meaning of the quantum group theory [1] in the basic quantum mechanics. We formally reproduce the canonical quantization from the quantum double of the commutative classical observable algebra (COA) A for the classical phase space. It has to be pointed out that the quantum group theory originally comes from the quantization of some non-linear problems in physics (such as the S-matrics theory in low dimensional quantum field theory[2] and quantum inverse scattering methods[3]) and exactly-solvable models in statistical mechanics [4].

Let us consider a classical observable algebra associated with the classical phase space with two canonical variables, the coordinate Q and the momentum P. The COA A is an associative algebra generated by P, Q and a central elements N. In the classical sense, P and Q commute each other, so the COA is an Abelian algebra and possesses simplest structure as an associative algebra. Fortunately, this simplest algebra can be endowed with a ‘quantum group’ structure \((\Delta, S, \epsilon)\)

\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \ x = P, Q
\]

\[
\Delta(N) = N \otimes 1 + 1 \otimes N + \mu P \otimes Q,
\]

\[
S(x) = -x, S(N) = -N + \mu P Q, S(1) = 1, x = P, Q
\]

\[
\epsilon(y) = 0, y = P, Q, N, \epsilon(1) = 1
\]

where \(\Delta, \epsilon\) are algebraic homomorphism and S an algebraic antihomomorphism; \(\mu\) is a compatible complex parameter for \((\Delta, S, \epsilon)\) satisfying the axioms of Hopf algebra. With the above structure \((\Delta, S, \epsilon)\) the COA becomes a commutative, but non-cocommutative Hopf algebra if \(\mu\) is not zero.

Now, we set to find the quantum dual (also called Hopf (algebraic) dual [1]) B of the COA A according to Drinfeld’s quantum double theory (for the reviews easy to physicists please see the refs.[5,6]). To this end, we define the generators \(\hat{P}, \hat{Q}\) and E by

\[
<P^m Q^n N^l, \hat{P}> = \delta_{m,1} \delta_{n,0} \delta_{l,0}
\]

\[
<P^m Q^n N^l, \hat{Q}> = \delta_{m,0} \delta_{n,1} \delta_{l,0},
\]

\[
<P^m Q^n N^l, E> = \delta_{m,0} \delta_{n,0} \delta_{l,1}
\]

where the bilinear form \(<,>\): \(A \times B \rightarrow \text{complex field } C\) satisfies the following conditions resulting from the duality in quantum double:
\[
\langle a, b_1 b_2 \rangle = \langle \Delta_A(a), b_1 \otimes b_2 \rangle, a \in A, b_1, b_2 \in B,
\]

\[
\langle a_1 a_2, b \rangle = \langle a_2 \otimes a_1, \Delta_B(b) \rangle, a_1, a_2 \in A, b \in A
\]

\[
\langle 1_A, b \rangle = \epsilon_B(b), b \in B,
\]

\[
\langle a, 1_B \rangle = \epsilon_A(a), a \in A
\]

\[
\langle S_A(a), S_B(b) \rangle = \langle a, b \rangle, a \in A, b \in B
\]

In this letter, without confusion, we do not specify the operation \((\Delta, S, \epsilon)\) for \(A\) and \(B\).

It follows from eqs. (2) and (3) that

\[
\langle P_m Q^n N^l \hat{P} \hat{Q} E^r, \rangle = m! n! l! \delta_{m,k} \delta_{n,l} \delta_{s,r}
\]

\[
\Delta(x) = x \otimes 1 + 1 \otimes x, S(x) = -x, \epsilon(x) = 0
\]

\[
\epsilon(1) = 1, S(1) = 1, x = \hat{P}, \hat{Q}, E
\]

The key to our study is the commutation relations between \(\hat{P}\) and \(\hat{Q}\). From eqs. (1), (2) and (3), we derive

\[
\langle P^m Q^n N^l, \hat{P} \hat{Q} \rangle = \langle \Delta(P^m Q^n N^l), \hat{P} \otimes \hat{Q} \rangle
\]

\[
\sum_{k=1}^{m} \sum_{r=1}^{n} \sum_{s=1}^{l} \sum_{t=1}^{l-s} \frac{m! n! l! \mu^s}{k! r! s!(m-k)!(n-r)!(l-s-t)! t!}
\]

\[
\langle P^{m-k+s} Q^{n-r} N^{l-s-t} \otimes P^k Q^{r+s} N^t, \hat{P} \otimes \hat{Q} \rangle = \mu \delta_{m+k,1} \delta_{n+r,1} \delta_{l+t,1}
\]

that is

\[
\langle P^m Q^n N^l, [\hat{P}, \hat{Q}] \rangle = \mu \delta_{m,1} \delta_{n,0} \delta_{l,0}
\]

Then, we obtain

\[
[\hat{P}, \hat{Q}] = \mu E,
\]

\[
[\hat{P}, E] = 0 = [\hat{Q}, E],
\]

Then, we show that the Hopf duals \(\hat{P}\) and \(\hat{Q}\) of the classical canonical coordinate \(Q\) and momentum \(P\) are just the quantum coordinate and momentum operators respectively if we can take \(\mu = i\hbar\). Therefore, the Drinfeld’s quantum double theory provides us with an algebraic scheme of the canonical
quantization in the basic quantum mechanics. In fact, the parameter \( \mu \) characterizes the degrees of the non-cocommutation of the classical observable algebra \( A \) and the non-commutation of the quantum observable algebra \( \text{QOA} B \) generated by \( \hat{P}, \hat{Q} \) and \( E \) (it is usually called Heisenberg-Weyl (HW) algebra). Since when \( \mu \) is zero, the algebra \( B \) becomes commutative, it is reasonable to take \( \mu = i\hbar \). In this way, the Plank constant \( \hbar \) automatically enters the algebra \( B \) to realize the ‘algebraic’ canonical quantization. Notice that with the the Hopf algebraic structure (4), the QOA \( B \) is a cocommutative, but non-commutative Hopf algebra.

In order to investigate the quantum Yang-Baxter equation in connection with canonical quantization we need to combine \( A \) with \( B \) to get a quasi-triangular Hopf algebra as the quantum double of the COA \( A \). Thanks to the double multiplication rule given in Drinfeld’s theory

\[
ba = \sum_{i,j} < a_i(1), S(b_j(1)) > < a_i(3), b_j(3) > a_i(2)b_j(2)
\]

\[
(id \otimes \Delta)\Delta(c) = \sum_i c_i(1) \otimes c_i(2) \otimes c_i(3)
\]

we get the commutation relations between \( A \) and \( B \)

\[
[E, \text{everything}] = 0, \\
[N, \hat{P}] = -\mu \hat{Q}, [N, \hat{Q}] = \mu \hat{P}.
\]

Then, we obtain the universal R-matrix

\[
\hat{R} = \sum a_i \otimes b_i = \sum_{m,n,s} \frac{P^m Q^n H^s \otimes \hat{P}^m \hat{Q}^n E^s}{m!n!s!}
\]

\[
= \exp[P \otimes \hat{P}] = \exp[Q \otimes \hat{Q}] \exp[N \otimes E]
\]

as a canonical elements intertwining the QOA \( B \) and COA \( A \). According to Drinfeld’s theory, the universal R-matrix constructed above must satisfies the abstract Yang-Baxter equation

\[
\hat{R}_{12}\hat{R}_{13}\hat{R}_{23} = \hat{R}_{23}\hat{R}_{13}\hat{R}_{12},
\]

where \( a_m \) and \( b_m \) are the basis elements of \( A \) and \( B \) respectively, and they are dual each other; \(< a_m, b_n > = \delta_{m,n};

\[
\hat{R}_{12} = \sum_m a_m \otimes b_m \otimes 1, \hat{R}_{13} = \sum_m a_m \otimes 1 \otimes b_m, \hat{R}_{23} = \sum_m 1 \otimes a_m \otimes b_m
\]
Then, we can conclude that the classical observable algebra $A$ can incorporate the quantum observable algebra $B$ to form a new quantum group $D$, a quasi-triangular Hopf algebra, which is generated by $P, Q, N, E, \hat{P}$ and $\hat{Q}$ with the relations (1), (4), (5) and (7); the quantum Yang-Baxter equation as a necessary condition is enjoyed in this construction of the canonical quantization. In fact, the new quantum group $D$, as an associative algebra, is the universal enveloping algebra of the non-simple Lie algebra with the basis $P, Q, N, E, \hat{P}$ and $\hat{Q}$. This means that, besides so-called q-deformations of some classical algebra, one not only endows some finite dimensional Lie algebras (strictly, their universal enveloping algebras) with a cocommutative, but also with a non-cocommutative Hopf algebra structure. The discussion of this paper is a typical example of exotic quantum doubles of non-q-deformation and another type of such exotic quantum double has been constructed in connection with infinite dimensional Lie algebra [7]. As for the quantum theory, the present studies shows that the Drinfeld’s quantum double theory can provides us with an algebraicized scheme of quantization available for both the basic quantum mechanics and the nonlinear quantum systems such as in quantum inverse scattering method.

A direct generalization of present study is the higher-dimensional case where we only need write down the non-cocommutative coproduct

$$\Delta(N) = N \otimes 1 + 1 \otimes N + \frac{i\hbar}{2} \sum_{k=1}^{M} P_k \otimes Q_k$$

(9)

for $N$ and the similar for other generators $P_k, Q_k$ (k=1,2,...,M) in the multi-COA $A(M)$. Its quantum dual is just the M-state HW algebra with the momentum and coordinate operators $\hat{P}_k, \hat{Q}_k$ and the unit element $E$ as generators, and there naturally is a new quantum double (‘group’) as a routine multi-generalization of the algebra $D$ obtained above.

Before concluding the discussion in this paper, we would like to understand the physical meaning of the universal R-matrix (8) and the operator $N$. As for the later, we can give $N$ a realization in terms of $\hat{P}$ and $\hat{Q}$

$$N = P\hat{P} + Q\hat{Q}$$

which preserves all the commutators of $N$ with other generators and where we have take a representation with $E$ to be unit. So it may imply a ‘interaction’ between the classical and quantum objects. For the former, we need consider the representation theory of the quantum group $D$. Thanks to the Schur lemma, the central elements
P, Q and E must be some scalars in an irreducible representation. In this sense, $P \otimes \hat{P}$, and $Q \otimes \hat{Q}$ act as $P\hat{P}$ and $Q\hat{Q}$ on second component of the product space $V \otimes V$ where $V$ is the representation space. Thus, the universal R-matrix has an equivalent form

$$\hat{R} = \exp[P\hat{P} + Q\hat{Q}] \otimes \exp[P\hat{P}] \exp[Q\hat{Q}]$$

(10)

where we have renormalized the scalar E to be 1. Using the Campbell-Hausdorff formula, we prove that the universal R-matrix is just the generating operator of the two-model coherent state

$$|Z\rangle = e^{i\frac{PQ}{2}} \exp[Za^+ + Z^*a] \otimes \exp[Za^+ + Z^*a] |0\rangle, Z = Q + iP$$

that has not been normalized. This shows the possible relations between the coherent state and the quantum Yang-Baxter equation. We also notice that in some representations similar to what we considered above, the R-matrices are completely factorized as the solutions for Yang-Baxter equation.

Finally, we point out that a class of new R-matrices, as the matrix representations of the universal R-matrix for the new quantum group $D$, can follow from the finite dimensional representations. To get them, we need the detailed discussions for the general representation theory of $D$. These corresponding studies with some mathematical interests will be published elsewhere.

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References

1. V.G. Drinfeld, Proc. ICM. Berkeley, 1986, (ed. by A. Gleason, AMS, 1987), p. 798
M. Jimbo, Lett. Math. Phys. 10 (1985) 63
L.D. Faddeev, N.Y. Reshetikhin, L.A. Takhtajian, Preprint LOMI E-14-87, 1987; also in Algebraic analysis, vol. 1 (1988) 01299
for a review see Braid Groups, Knot Theory and Statistical Mechanics C.N. Yang, M.L. Ge (eds.), Singapore: World Scientific, 1989.
2. C.N. Yang, Phys. Rev. Lett. 19 (1967) 1312
A.B. Zamolodchikov, A.B. Zamolodchikov, Ann. Phys. 120 (1979) 253
H.J. de Vega, Inter. J. Mod. Phys. A 4 (1989), 2371.
3. E.K. Sklyanin, L.A. Takhtajian and L.D. Faddeev, Theor. Math. Fisica 40 (1979) 194
P.P. Kulish and N.Y. Reshetikhin, J. Phys. A. 16 (1983), L591
4. R.J. Baxter, Exactly-Solved Models in Statistical Mechanics Academic Press, 1982.
5. L.A. Takhtajian, Quantum Groups, in Nankai Lectures, 1989, ed. by M.L. Ge and B.H. Zhao, World Scientific, 1990.
6. M. Jimbo, Topics from representations of $U_q(g)$, in Nankai Lectures, 1991, ed. by M.L. Ge, World Scientific, 1992.
7. C.P. Sun, X.F. Liu and M.L. Ge, J. Math. Phys. 33 (1992), in press.
W. Li, C.P. Sun and M.L. Ge, Preprints ITP-SB.92-59, 1992