FLOER HOMOLOGY ON THE EXTENDED MODULI SPACE

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Abstract. Given a Heegaard splitting of a 3-manifold, we use Lagrangian Floer homology to construct a relatively \( \mathbb{Z}/8\mathbb{Z} \)-graded abelian group, which we conjecture to be a 3-manifold invariant. Our motivation is to have a well-defined symplectic side of the Atiyah-Floer Conjecture, for arbitrary 3-manifolds. The symplectic manifold used in the construction is the extended moduli space of flat \( SU(2) \)-connections on the Heegaard surface. An open subset of this moduli space carries a symplectic form, and each of the two handlebodies in the decomposition gives rise to a Lagrangian inside the open set. In order to define their Floer homology, we compactify the open subset by symplectic cutting; the two-form on the resulting manifold has degeneracies, but we show that one can still develop a version of Floer homology in this setting.

1. Introduction

Floer’s instanton homology [14] is an invariant of integral homology three-spheres \( Y \) that serves as target for the relative Donaldson invariants of four-manifolds with boundary; see [12]. It is defined from a complex whose generators are (suitably perturbed) irreducible flat connections in a trivial \( SU(2) \)-bundle over \( Y \), and whose differentials arise from counting anti-self-dual \( SU(2) \)-connections on \( Y \times \mathbb{R} \). There is also a version of instanton Floer homology using connections in \( U(2) \)-bundles with \( c_1 \) odd ([16], [19]), an equivariant version ([4], [5]), and several variants that use both irreducible and reducible flat connections [12]. More recently, Kronheimer and Mrowka [27] have developed instanton homology for sutured manifolds; a particular case of their theory leads to a version of instanton homology that can be defined for arbitrary closed three-manifolds.

In another remarkable paper [15], Floer associated a homology theory to two Lagrangian submanifolds of a symplectic manifold, under suitable assumptions. This homology is defined from a complex whose generators are intersection points between the two Lagrangians, and whose differentials count pseudo-holomorphic strips. The Atiyah-Floer Conjecture [2] states that Floer’s two constructions are related: for any decomposition of the homology sphere \( Y \) into two handlebodies glued along a Riemann surface \( \Sigma \), instanton Floer homology should be the same as the Lagrangian Floer homology of the \( SU(2) \)-character varieties of the two handlebodies, viewed as subspaces of the character variety of \( \Sigma \).

As stated, an obvious problem with the Atiyah-Floer Conjecture is that the symplectic side is ill-defined: due to the presence of reducible connections, the \( SU(2) \)-character variety of \( \Sigma \) is not smooth. Nevertheless, some partial results in the direction of the conjecture were obtained by Salamon and Wehrheim, who studied instantons with Lagrangian boundary conditions; see [45], [52], [46]. Another approach is to avoid reducibles altogether by using nontrivial \( U(2) \)-bundles instead. This road was taken by Dostoglou and Salamon [13], who proved a variant of the conjecture for mapping tori.

The goal of this paper is to propose another candidate that could sit on the symplectic side of the (suitably modified) Atiyah-Floer Conjecture.

Here is a short sketch of the construction. Let \( \Sigma \) be a Riemann surface of genus \( h \geq 1 \), and \( z \in \Sigma \) a base point. The moduli space \( \mathcal{M}(\Sigma) \) of flat connections in a trivial \( SU(2) \)-bundle over \( \Sigma \) can be identified with the character variety \( \{ \rho : \pi_1(\Sigma) \to SU(2) \}/PSU(2) \). The moduli space

\(^{CM} \) was partially supported by the NSF grant DMS-0852439 and a Clay Research Fellowship.

\(^{CW} \) was partially supported by the NSF grant DMS-060509.
\( \mathcal{M}(\Sigma) \) is typically singular. However, Jeffrey \cite{21} and, independently, Huebschmann \cite{24}, showed that \( \mathcal{M}(\Sigma) \) is the symplectic quotient of a different space, called the extended moduli space, by a Hamiltonian \( PSU(2) \)-action. The extended moduli space is naturally associated not to \( \Sigma \), but to \( \Sigma' \), a surface with boundary obtained from \( \Sigma \) by deleting a small disk around \( z \). The extended moduli space has an open smooth stratum, which Jeffrey and Huebschmann equip with a natural closed two-form. This form is nondegenerate on a certain open set \( \mathcal{N}(\Sigma') \), which we take as our ambient symplectic manifold. In fact, \( \mathcal{N}(\Sigma') \) can also be viewed as an open subset of the Cartesian product \( SU(2)^{2h} \cong \{ \rho : \pi_1(\Sigma') \to SU(2) \} \). More precisely, if we pick \( 2h \) generators for the free group \( \pi_1(\Sigma') \), we can describe this subset as

\[
\mathcal{N}(\Sigma') = \left\{ (A_1, B_1, \ldots, A_h, B_h) \in SU(2)^{2h} \mid \prod_{i=1}^{h} [A_i, B_i] \neq -I \right\}.
\]

Consider a Heegaard decomposition of a three-manifold \( Y \) as \( Y = H_0 \cup H_1 \), where the handlebodies \( H_0 \) and \( H_1 \) are glued along their common boundary \( \Sigma \). There are smooth Lagrangians

\[
L_i = \{ \pi_1(H_i) \to SU(2) \} \subset \mathcal{N}(\Sigma'), \quad i = 0, 1.
\]

In order to take the Lagrangian Floer homology of \( L_0 \) and \( L_1 \), care must be taken with holomorphic strips going out to infinity; indeed, the symplectic manifold \( \mathcal{N}(\Sigma') \) is not weakly convex at infinity. Our remedy is to compactify \( \mathcal{N}(\Sigma') \) by (non-abelian) symplectic cutting. The resulting manifold \( \mathcal{N}^c(\Sigma') \) is the union of \( \mathcal{N}(\Sigma') \) and a codimension two submanifold \( R \). A new problem shows up here, because the natural two-form \( \omega_0 \) on \( \mathcal{N}^c(\Sigma') \) has degeneracies on \( R \). Nevertheless, \( (\mathcal{N}^c(\Sigma'), \omega_0) \) is monotone, in a suitable sense. One can deform \( \omega_0 \) into a symplectic form \( \omega_c \), at the expense of losing monotonicity. We are thus led to develop a version of Lagrangian Floer theory on \( \mathcal{N}^c(\Sigma') \) by making use of the interplay between the forms \( \omega_0 \) and \( \omega_c \). Our Floer complex uses only holomorphic disks lying in the open part \( \mathcal{N}(\Sigma') \) of \( \mathcal{N}^c(\Sigma') \). We show that, while holomorphic strips with boundary on \( L_0 \) and \( L_1 \) can go to infinity in \( \mathcal{N}(\Sigma') \), they do so only in high codimension, without affecting the Floer differential. The resulting Floer homology group is denoted

\[
HSI(\Sigma; H_0, H_1) = HF(L_0, L_1 \text{ in } \mathcal{N}(\Sigma'))
\]

and admits a relative \( \mathbb{Z}/8\mathbb{Z} \)-grading. We call this group the \textit{symplectic instanton homology} associated to the Heegaard splitting \( (\Sigma; H_0, H_1) \). Strictly speaking, the Floer groups also depend on the base point \( z \); as \( z \) varies over the Heegaard surface, the corresponding groups form a local system. However, we drop \( z \) from notation for simplicity.

We conjecture that \( HSI(\Sigma; H_0, H_1) \) is an invariant of the three-manifold \( Y \). In order to prove this, we would need to show that the groups are invariant under stabilization of the Heegaard splitting. One could hope to attack this by using the theory of Lagrangian correspondences and pseudo-holomorphic quilts developed in \cite{24}. Unfortunately, the structure of bubbles for pseudo-holomorphic quilts is not yet fully understood; in particular, a removal of singularity theorem is lacking. This makes it difficult to extend the theory to a setting like ours, where we have a non-monotone symplectic form, or a monotone closed form with degeneracies.

Let us explain how we expect \( HSI(\Sigma; H_0, H_1) \) to be related to the traditional instanton theory on 3-manifolds. We restrict our attention to the original set-up for Floer’s instanton theory \( I(Y) \) from \cite{14}, when \( Y \) is an integral homology sphere. It is then decidedly not the case that \( HSI \) coincides with Floer’s theory; for example, for the genus one Heegaard splitting of \( S^3 \) we have \( HSI \cong \mathbb{Z} \), but \( I(S^3) = 0 \). Nevertheless, in \cite{12} Section 7.3.3, Donaldson introduced a different version of instanton homology, a \( \mathbb{Z}/8\mathbb{Z} \)-graded vector field over \( \mathbb{Q} \) denoted \( \tilde{HF} \), which satisfies \( \tilde{HF}(S^3) \cong \mathbb{Q} \). (Floer’s theory \( I \) is denoted \( HF \) in \cite{12}.) We state the following variant of the Atiyah-Floer Conjecture:
Conjecture 1.1. For every integral homology sphere \( Y \) with a Heegaard splitting \((\Sigma; H_0, H_1), \) the symplectic instanton homology \( HSI(\Sigma; H_0, H_1) \otimes \mathbb{Q} \) and the Donaldson-Floer homology \( HF(Y) \) from \([12]\) are isomorphic, as relatively \( \mathbb{Z}/8\mathbb{Z} \)-graded vector spaces.

Alternatively, one could hope to relate \( HSI \) to the version of instanton Floer homology developed by Kronheimer and Mrowka in \([27]\). More open questions, and speculations along these lines, are presented in Section 6.2.

Acknowledgments. We would like to thank Yasha Eliashberg, Peter Kronheimer, Tim Perutz, and Michael Thaddeus for some very helpful discussions during the preparation of this paper.

2. Floer homology

2.1. The monotone, nondegenerate case. Lagrangian Floer homology was originally constructed in \([15]\) under some restrictive conditions, and later generalized by various authors to many different settings. We review here its definition in the monotone case, due to Oh \([30, 38]\), together with a discussion of orientations following Fukaya-Oh-Ohta-Ono \([17]\).

Let \((M, \omega)\) be a compact symplectic manifold. We denote by \( J(M, \omega) \) the space of compatible almost complex structures on \((M, \omega)\), and by \( J_\tau(M, \omega) = C^\infty([0, 1], J(M, \omega)) \) the space of time-dependent almost complex structures. Any compatible almost complex structure \( J \) defines a complex structure on the tangent bundle \( TM \). Since \( J(M, \omega) \) is contractible, the first Chern class \( c_1(TM) \in H^2(M, \mathbb{Z}) \) depends only on \( \omega \), not on \( J \). The minimal Chern number \( N_M \) of \( M \) is defined as the positive generator of the image of \( c_1(TM) : \pi_2(M) \to \mathbb{Z} \).

Definition 2.1. Let \((M, \omega)\) be a symplectic manifold. \( M \) is called monotone if there exists \( \kappa > 0 \) such that
\[
[\omega] = \kappa \cdot c_1(TM).
\]
In that case, \( \kappa \) is called the monotonicity constant.

Definition 2.2. A Lagrangian submanifold \( L \subset (M, \omega) \) is called monotone if there exists a constant \( \kappa > 0 \) such that
\[
\omega|_{\pi_2(M, L)} = \kappa \cdot \mu_L,
\]
where \( \mu_L : \pi_2(M, L) \to \mathbb{Z} \) is the Maslov index \([30\text{ Appendix C}]\).

The minimal Maslov number \( N_L \) of a monotone Lagrangian \( L \) is defined as the positive generator of the image of \( \mu_L \) in \( \mathbb{Z} \).

From now on we will assume that \( M \) is monotone and that we are given two closed, simply connected Lagrangians \( L_0, L_1 \subset M \). These conditions imply that \( L_0 \) and \( L_1 \) are monotone with
\[
N_{L_0} = N_{L_1} = 2N_M.
\]
We assume that \( N_M > 1 \), and denote \( N = 2N_M \geq 4 \). We also assume that \( w_2(L_0) = w_2(L_1) = 0 \).

After a small Hamiltonian perturbation we can arrange so that the intersection \( L_0 \cap L_1 \) is transverse. The Floer chain complex is then defined to be the abelian group
\[
CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} O_x,
\]
where \( O_x \) is the orientation group of \( x \). This is the abelian group (noncanonically isomorphic to \( \mathbb{Z} \)) which is generated by the two possible orientations of \( x \), with the relation that their sum is
zero. Our assumptions allow one to define a relative Maslov index \( \Delta \text{gr}(x, y) \in \mathbb{Z}/N\mathbb{Z} \) for every \( x, y \in L_0 \cap L_1 \). The relative index satisfies \( \Delta \text{gr}(x, y) + \Delta \text{gr}(y, z) = \Delta \text{gr}(x, z) \) and induces a relative \( \mathbb{Z}/N\mathbb{Z} \)-grading on the chain complex.

The Lagrangian Floer homology groups \( HF_*(L_0, L_1) \) are the homology groups of \( CF_*(L_0, L_1) \) with respect to the differential \( \partial \) defined on generators by

\[
\partial x = \sum_y n_{xy}y.
\]

Here \( n_{xy} \in \mathbb{Z} \) is the signed count of pseudo-holomorphic strips (Floer trajectories) from \( x \) to \( y \), i.e., isolated solutions (modulo translation in \( s \)) to Floer’s equation

\[
\begin{align*}
&u: \mathbb{R} \times [0, 1] \to M, \\
u(s, 0) \in L_0, & u(s, 1) \in L_1, \\
&\partial_s u + J_t(u)\partial_t u = 0, \\
&\lim_{s \to +\infty} u(s, \cdot) = x, & \lim_{s \to -\infty} u(s, \cdot) = y,
\end{align*}
\]

where \( (J_t)_{0 \leq t \leq 1} \) is chosen from a second category subset \( J^\text{reg}_r(L_0, L_1) \subset J_r(M, \omega) \) of \( (L_0, L_1) \)-\textit{regular}, time-dependent compatible almost complex structures. The monotonicity condition and the assumption \( N \geq 4 \) give control over the bubbling of disks and spheres, so that the moduli spaces of solutions to \( (1) \) have well-behaved compactifications; when their expected dimension (mod translation) is at most one, these compactifications only include broken strips with no bubbles. This implies that \( \partial \) is finite and \( \partial^2 = 0 \). The condition that the Lagrangians have vanishing \( w_2 \) is used in defining orientations on the moduli spaces, so that \( n_{xy} \) is integer valued. (More generally, one could define orientations provided that the pair of Lagrangians is relatively spin in the sense of \( [L7] \); in our case, we choose the relative spin structure to be zero.)

An important property of the Floer homology groups \( HF_*(L_0, L_1) \) is that they are independent of the choice of path of almost complex structures, and invariant under Hamiltonian isotopies of either \( L_0 \) and \( L_1 \). Since \( H_1(L_0) = H_1(L_1) = 0 \), any isotopy of \( L_0 \) or \( L_1 \) through Lagrangians can be embedded in an ambient Hamiltonian isotopy; see for example \( [H1] \) Section 6.1 or the discussion in \( [AS] \) Section 4(D)].

2.2. A relative version. Let \( R \) be a (compact) symplectic hypersurface in a compact symplectic manifold \((M^{2n}, \omega)\). We denote by P.D.\([\{R\}] \in H^2(M; \mathbb{Z})\) the Poincaré dual of \([R]\).

**Definition 2.3.** We say that a simply connected Lagrangian \( L \subset M \) is compatible with \( R \) if the restriction of P.D.\([\{R\}] \) to \( H^2(L; \mathbb{Z}) \) is trivial. In particular, \( L \) is compatible with \( R \) if \( L \cap R = \emptyset \).

The exact sequence in homology for the pair \((M, M \setminus L)\) reads

\[
H_{2n-1}(M, M \setminus L) \to H_{2n-2}(M \setminus L) \to H_{2n-2}(M) \to H_{2n-2}(M, M \setminus L) \cong H^2(L).
\]

Since \( H_{2n-1}(M, M \setminus L) \cong H^1(L) = 0 \), the compatibility condition between \( L \) and \( R \) means that \( R \) produces a well-defined class in \( H_{2n-2}(M \setminus L) \). In particular, there is a well-defined algebraic intersection number between \( R \) and surfaces with boundary on \( L \).

Further, assume that we have two simply connected Lagrangians \( L_0, L_1 \) compatible with \( R \). Then each pseudo-holomorphic strip \( u : \mathbb{R} \times [0, 1] \to (M; L_0, L_1) \) has a well-defined intersection number \( u \cdot R \), defined as a signed count of intersection points of generic perturbations. Since the tangent spaces to the image of \( u \) and \( R \) are symplectic, and this property holds under generic perturbations of \( R \), it follows that \( u \cdot R \) is nonnegative.

The intersection numbers \( u \cdot R \) depend only on the relative homology class of \( u \), and are additive under concatenation of trajectories:

\[
(u \# v) \cdot R = (u \cdot R) + (v \cdot R).
\]
Assume now that \( M, L_0 \) and \( L_1 \) satisfy the assumptions from the previous subsection, so that the Floer homology \( HF(L_0, L_1) \) is well-defined. We can then also define a restricted differential \( \partial_0 \) on \( CF(L_0, L_1) \):

\[
\partial_0 x = \sum_y n_{xy}^R y,
\]

where \( n_{xy}^R \) counts only those Floer trajectories \( u \) from \( x \) to \( y \) such that \( u \cdot R = 0 \). The fact that the intersection numbers are additive and nonnegative shows that \( \partial_0^2 = 0 \). Let \( HF(L_0, L_1; R) \) denote the homology of \( (CF(L_0, L_1), \partial_0) \). We call \( HF(L_0, L_1; R) \) the Lagrangian Floer homology of \( L_0, L_1 \) relative to the hypersurface \( R \). This kind of construction has previously appeared in the literature in various guises; see for example Seidel’s deformation of the Fukaya category \([47, p.8]\) or the hat version of Heegaard Floer homology \([40]\).

**Remark 2.4.** The image of \( H_2(M \setminus R) \) in \( H_2(M) \) consists exactly of those classes that have trivial intersection with \([R]\); the same is true for the second homology relative to \( L_0 \) or \( L_1 \). This means that \( HF(L_0, L_1; R) \) admits a relative \( \mathbb{Z}/N'\mathbb{Z} \)-grading, where \( N' = 2N_{M \setminus R} \) is a positive multiple of \( N \).

Let us assume for simplicity that \( L_0 \cap R = L_1 \cap R = \emptyset \). Let \( u \) be a Floer trajectory with \( u \cdot R = 0 \). If \( u \) intersects \( R \) transversely, the fact that all the intersection points contribute positively means that, in fact, \( u(\mathbb{R} \times [0, 1]) \cap R = \emptyset \). However, if the intersection is not transverse, it might be nonempty; this phenomenon does happen in high-dimensional families of trajectories. The following lemma shows that it does not happen in zero-dimensional families:

**Lemma 2.5.** Assume \( L_0 \cap R = L_1 \cap R = \emptyset \), and that the intersection \( L_0 \cap L_1 \) is transverse. Then, for a generic choice of a one-parameter family \( (J_t)_{t \in [0,1]} \) of almost complex structures compatible with \( \omega \), every isolated pseudo-holomorphic strip \( u \) satisfying \([1]\) and \( u \cdot R = 0 \) is disjoint from \( R \).

**Proof.** Fix \( x, y \in L_0 \cap L_1 \). Fix also a homotopy class \( A \) of strips \( \mathbb{R} \times [0, 1] \to M \) with boundary conditions on \( L_0, L_1 \) and limits \( x, y \) at \( \pm \infty \); we assume that the Maslov index of \( A \) is one, i.e. the expected dimension of the space of Floer trajectories in the class \( A \) (modulo translation) is zero.

We denote by \( \widetilde{\mathcal{M}}(M, L_0, L_1, A) \) the universal moduli space of data \((J_t, u)\) consisting of a time-dependent compatible almost complex structure \((J_t)_{t \in [0,1]}\) (in a suitable completion of the space \( \mathcal{J}_\tau(M, \omega) \)) and a \( J_t \)-holomorphic strip \( u \) in the class \( A \). Then \( \widetilde{\mathcal{M}}(M, L_0, L_1, A) \) is a Banach manifold, cf. \([15, 36] \).

We denote by \( p \) the natural projection from \( \widetilde{\mathcal{M}}(M, L_0, L_1, A) \) to the completion of \( \mathcal{J}_\tau(M, \omega) \). We also denote \( \widetilde{\mathcal{M}}(M, L_0, L_1, A, J_t) = p^{-1}(J_t) \).

There is a natural evaluation map

\[
ev: (\mathbb{R} \times [0,1]) \times \widetilde{\mathcal{M}}(M, L_0, L_1, A) \to M, \quad ev(z, J_t, u) = u(z).
\]

A standard argument (as in \([30]\), Section 3.4), but with strips instead of closed curves), shows that the evaluation map is transverse to \( R \). Hence \( ev^{-1}(R) \) is a Banach manifold. By Sard-Smale, the set of regular values of the restriction of \( p \) to \( ev^{-1}(R) \) is Baire second category. For any regular value \( (J_t) \), the restriction of \( ev \) to \((\mathbb{R} \times [0,1]) \times \widetilde{\mathcal{M}}(M, L_0, L_1, A, J_t) \) is transverse to \( R \). Since \( \widetilde{\mathcal{M}}(M, L_0, L_1, A, J_t) \) is zero-dimensional (after dividing out by translation), every \( J_t \)-holomorphic strip \( u \) in the class \( A \) is transverse to \( R \). Since \( u \cdot R = 0 \), by the principle of positivity of intersections we get that \( u \) is disjoint from \( R \).

The set of all possible \( x, y \) and \( A \) is countable, so the conclusion follows.

**Lemma 2.5** shows that (under the given assumptions) \( \partial_0 \) counts only trajectories disjoint from \( R \). Therefore, we can think of \( HF(L_0, L_1; R) \) as the Floer homology of \( L_0 \) and \( L_1 \) inside \( M \setminus R \).
2.3. Floer homology on semipositive manifolds. In this section we extend the definition of Floer homology to a semipositive setting. More precisely, we assume the following:

**Assumption 2.1.**

(i) $(M, \omega)$ is a compact symplectic manifold;

(ii) $\omega_0$ is a closed two-form on $M$;

(iii) The degeneracy locus $R \subset M$ of $\omega_0$ is a symplectic hypersurface with respect to $\omega_0$;

(iv) $\omega_0$ is monotone, i.e., $[\omega_0] = \kappa \cdot c_1(TM)$ for some $\kappa > 0$;

(v) The restrictions of $\omega_0$ and $\omega_1$ to $M \setminus R$ have the same cohomology class in $H^2(M \setminus R)$;

(vi) The forms $\omega_0$ and $\omega_1$ themselves coincide on an open subset $W \subset M \setminus R$;

(vii) We are given two closed submanifolds $L_0, L_1 \subset W$ which are Lagrangian with respect to $\omega_1$ (hence Lagrangians with respect to $\omega_0$ as well);

(viii) $\pi_1(L_0) = \pi_1(L_1) = 1$ and $w_2(L_0) = w_2(L_1) = 0$;

(ix) The minimal Chern number $N_{M \setminus R}$ (with respect to either $\omega_0$ or $\omega_1$) is at least 2, so that $N = 2N_{M \setminus R} \geq 4$;

(x) There exists an almost complex structure $\tilde{J}$ on $M$ that is compatible with respect to $\omega_1$ on $M$, and compatible with respect to $\omega_0$ on $M \setminus R$. We fix such a $\tilde{J}$, which we call the base almost complex structure.

Let us remark that, because $\tilde{J}$ is compatible with respect to $\omega_0$ on $M \setminus R$, by continuity it follows that $\tilde{J}$ is semipositive with respect to $\omega_0$ on all of $M$, i.e., $\omega_0(v, \tilde{J}v) \geq 0$ for any $v \in T_M M$.

Our goal is to define a relatively $\mathbb{Z}/N\mathbb{Z}$-graded Floer homology group $HF(L_0, L_1, \tilde{J}; R)$ using Floer trajectories away from $R$ and a path of almost complex structures that are small perturbations of $\tilde{J}$. The construction is similar to the one in Section 2.2 but a priori it depends on $\tilde{J}$. (We should note that defining a group $HF(L_0, L_1, \tilde{J})$ by using Floer trajectories that can intersect $R$ nontrivially seems much more difficult in this situation.)

**Definition 2.6.** Let $K > 0$.

(a) We say that $J \in \mathcal{J}(M, \omega_1)$ is $K$-spherically semipositive if every $J$-holomorphic sphere of energy $E(u) = \int u^* \omega_1 < K$ has non-negative Chern number $I(u) = c_1(TM)|u| \geq 0$.

(b) We say that $J \in \mathcal{J}(M, \omega_1)$ is $K$-hemispherically semipositive if $J$ is $K$-spherically semipositive and every $J$-holomorphic map $(D^2, \partial D^2) \to (M, L_i)$, $i \in \{0, 1\}$ of energy $E(u) = \int u^* \omega_1 < K$ has non-negative Maslov index $I(u)$; and, further, if $I(u) = 0$ then $u$ is constant.

Given a continuous map $u : (D^2, \partial D^2) \to (M, L_i)$, $i = 0, 1$, we define the canonical area of $u$ by

$$A_{\text{can}}(u) = \frac{[\omega_0](u)}{\kappa},$$

**Lemma 2.7.** We have $I(u) = A_{\text{can}}(u)$, for any $u : (D^2, \partial D^2) \to (M, L_i)$.

**Proof.** Since $L_i$ is simply connected, we can find a disk $v$ contained in $L_i$ with boundary equal to that of $u$, but with reversed orientation. Let $u \# v : S^2 \to M$ the map formed by gluing. By additivity of Maslov index

$$I(u) = I(u) + I(v) = I(u \# v) = \frac{[\omega_0](u \# v)}{\kappa} = \frac{[\omega_0](u)}{\kappa},$$

since both the index and the area of $v$ are trivial. \qed

We define a strip with decay near the ends to be a continuous map

$$u : ([\mathbb{R} \times [0, 1]], \mathbb{R} \times \{0\}, \mathbb{R} \times \{1\}) \to (M, L_0, L_1)$$

such that $\lim_{s \to \infty} u(s, t), \lim_{s \to -\infty} u(s, t) \in L_0 \cap L_1$ exist. Every strip with decay near the ends admits a relative homology class in $H_2(M, L_0 \cup L_1)$, and therefore has a well-defined canonical area

$$A_{\text{can}}(u) := \frac{[\omega_0](u)}{\kappa}.$$
and a Maslov index $I(u)$.

The following lemma is [36, Proposition 2.7]:

**Lemma 2.8.** Strips (2) satisfy an index-area relation

$$I(u) = A_{\text{can}}(u) + C,$$

for some constant $C$ depending only on the endpoints of $u$.

**Proof (sketch).** Pick $u_0$ a reference strip with the same endpoints as $u$. Using the fact that $\pi_1(L_0) = 1$, we can find a map $v : D^2 \to L_0$ such that half of its boundary is taken to the image of $u_0[\mathbb{R} \times \{0\}]$ and the other half to the image of $u(\mathbb{R} \times \{0\})$. By adjoining $v$ to $u$ and $u_0$ (the latter taken with reversed orientation), we obtain a disk $(-u_0)v\#u$ with boundary in $L_1$. Applying Lemma 2.7 to this disk, and using the additivity of the index and canonical area under gluing, we obtain

$$I(u) - I(u_0) = A_{\text{can}}(u) - A_{\text{can}}(u_0).$$

We then take $C = I(u_0) - A_{\text{can}}(u_0)$. \hfill $\square$

**Lemma 2.9.** For any $K > 0$, $\tilde{J}$ is $K$-hemispherically semipositive.

**Proof.** Nonnegativity of $I$ follows from the fact that $\omega_0(v, \tilde{J}v) \geq 0$ for any $v \in T_mM, m \in M$, together with the monotonicity of $\omega$ (for spheres) and Lemma 2.7 for disks. If a $\tilde{J}$-holomorphic disk $u$ has $I(u) = 0$, its canonical area must be zero. Since $\tilde{J}$ is compatible with respect to $\omega_0$ on $M \setminus R$, the disk should be contained in $R$. However, this is impossible, because the disk has boundary on a Lagrangian $L_i$ with $L_i \cap R = \emptyset$. (By contrast, in principle we can have $I(u) = 0$ for non-constant $\tilde{J}$-holomorphic spheres contained in $R$.)

In [30, Lemma 6.4.7] it is shown that the set of $K$-spherically semipositive almost complex structures is open in the space $\mathcal{J}(M, \omega_\epsilon)$ of all almost complex structures compatible with $\omega_\epsilon$. The proof there can be adapted to prove the following:

**Proposition 2.10.** For each $K > 0$, the set of $K$-hemispherically semipositive almost complex structures contains an open neighborhood of $\tilde{J}$ in $\mathcal{J}(M, \omega_\epsilon)$.

**Proof.** Assume the statement about the disks having nonnegative index is false. Then there exists a sequence $J_\nu \to \tilde{J}$ of almost complex structures and a sequence $u_\nu$ of $J_\nu$-holomorphic disks with boundary in $L_i$ satisfying $E(u_\nu) < K$ and $I(u_\nu) < 0$. The limit of such a sequence is a $\tilde{J}$-holomorphic stable configuration (disk with some bubbles) with boundary in $L_i$ and having negative index. By additivity of the index, at least one of the component disks or spheres must have negative index. The energy of that component has to be less than $K$, which contradicts the $K$-semipositivity of $\tilde{J}$. The same proof works to show that $c_1(TM)[u] \geq 0$ for $\tilde{J}$-holomorphic spheres of energy less than $K$.

We also need to check the statement about disks of index zero being constant. Assume otherwise. Then we have a sequence $J_\nu \to \tilde{J}$ and $J_\nu$-holomorphic nonconstant disks $u_\nu$ with $E(u_\nu) < K, I(u_\nu) = 0$. The limit must be a $\tilde{J}$-holomorphic stable configuration of index zero, hence constant. Thus, for $\nu \gg 0$, the disks $u_\nu$ are contained in the open set $\mathcal{W}$ where $\omega_0 = \omega_\epsilon$. Therefore, $E(u_\nu) = \kappa A_{\text{can}}(u_\nu) = \kappa I(u_\nu) = 0$. Since $J_\nu$ is compatible with respect to $\omega_\epsilon$, it follows that $u_\nu$ must be constant, a contradiction. \hfill $\square$

**Lemma 2.11.** Let $M, L_0, L_1, \omega_0, \omega_\epsilon, \tilde{J}$ satisfy Assumption 2.1. Suppose further that $L_0$ and $L_1$ intersect transversely. Then there exists a constant $K > 0$ such that, for any $(J_\epsilon) \in J_{\epsilon}(M, \omega_\epsilon)$, and for any $J_\epsilon$-holomorphic strip $u$ of Maslov index 1 satisfying (1) and whose image lies in $M \setminus R$, we have the energy bound $E(u) < K$. 


Proof. On the complement of $R$ we have $\omega_t - \omega_0 = da$, where $a \in \Omega^1(M \setminus R)$ satisfies $da = 0$ on the neighborhood $W$ of $L_0 \cup L_1$. Let $u$ be a $J_t$-holomorphic strip whose image is contained in $M \setminus R$. Then

$$E(u) - \kappa A_{\text{can}}(u) = \int_{[0,1]} u^*(\omega_t - \omega_0) = \int_{[0,1]} d(u^*a) = \int_{\gamma_0} u^*a - \int_{\gamma_1} u^*a,$$

where $\gamma_i$ is a path in the Lagrangian $L_i$ joining the endpoints of $u$. Since $da = 0$ on $L_i$, Stokes’ Theorem implies that $\int u^*a$ is independent of $\gamma_i$; it just depends on the endpoints. Therefore, $E(u) - \kappa A_{\text{can}}(u)$ only depends on the endpoints of $u$. Together with Lemma 2.8, this gives an energy index relation as follows: for any $J_t$-holomorphic $u$ in $M \setminus R$, we have

$$\frac{1}{\kappa} E(u) + C' = I(u) = 1,$$

where $C'$ is a constant depending on the endpoints of $u$. Since there is a finite number of possibilities for these endpoints, the conclusion follows. $\square$

**Proposition 2.12.** Let $M, L_0, L_1, \omega_0, \omega_t, \bar{J}$ satisfy Assumption 2.1, and that $L_0$ and $L_1$ intersect transversely. Then, for a generic path of almost complex structures $J_t \in \mathcal{J}(M, \omega_t)$, $0 \leq t \leq 1$ picked from a neighborhood of $\bar{J}$, the relative Floer differential counting trajectories disjoint from $R$ is finite and satisfies $\partial_0 = 0$. The resulting (relatively $\mathbb{Z}/N\mathbb{Z}$-graded) Floer homology groups $HF_*(L_0, L_1; \bar{J}; R)$ are independent of the choice of path $(J_t)$, and are preserved under isotopies of either Lagrangian, as long as Assumption 2.1 and the transversality are still satisfied.

**Proof.** Let $K$ be the constant from Lemma 2.11. If $(J_t)_{t \in [0,1]}$ is a generic time-dependent perturbation of $\bar{J}$, by Proposition 2.10, we can assume that all $J_t$’s are $K$-hemispherically semipositive.

We seek to define the Floer differential $\partial_0$ by counting $J_t$-holomorphic strips $u$ of index 1, with $u \cdot R = 0$. To show that $\partial_0$ is finite, we need to show that the moduli spaces of such strips are compact. Further, to show that $\partial_0^2 = 0$, we need to show that the moduli spaces of $J_t$-holomorphic strips $u$ of Maslov index 2 with $u \cdot R = 0$ have well-defined compactifications by broken trajectories only (without disk or sphere bubbling).

The first step in applying Gromov compactness is to have an energy bound on the strips. In the Maslov index 1 case, this follows from the fact that $R$ is symplectic with respect to $\omega_t$. Indeed, for generic $J_t$, holomorphic strips $u$ of Maslov index 1 with $u \cdot R = 0$ are actually disjoint from $R$; compare the proof of Lemma 2.6. Hence, we have the energy bound from Lemma 2.11. In the Maslov index 2 case, i.e. for the study of $\partial_0^2$, we only need an energy bound on strips in fixed homotopy classes. Since the energy is a homotopy invariant, this bound is automatic.

Next, we need to rule out sphere bubbles and disk bubbles in the boundary of our moduli spaces of strips. Assume that we have a sequence $(u_\nu)$ of pseudo-holomorphic strips of index 1 or 2 that Gromov converges to a configuration consisting of a strip and at least one disk or sphere bubble. Note that $E(u_\nu) < K$ for all $\nu$; the same must be true for the limiting configuration. Since $J_t$-holomorphic curves have nonnegative energy, and the energy is additive, we get that every disk or sphere appearing in the limiting configuration has energy less than $K$. Since the $J_t$’s are $K$-hemispherically semipositive, their indices are nonnegative. Further, since we started with index at most 2, and sphere and disk bubbles cannot have index 1, it follows that they all have index zero, except possibly one component of index 2. By the definition of $K$-hemispherical semipositivity, the index zero disk components are constant. The minimal Maslov index for disks with trivial intersection with $R$ is $N = 2N_{M \setminus R}$; compare Remark 2.4. Since $N > 2$, we conclude that there are no disk components at all.

Thus, the limiting configuration is a tree composed of one strip of index $\mu \leq 2$, and several sphere bubbles, all of index zero or two. Pick a sphere bubble $B$, one of those directly attached to the strip. ($B$ may be multiply covered.) Consider the configuration made of the strip plus the simple sphere $B'$ associated to $B$. The Chern number $d = c_1(TM)[B']$ is between 0 and 1, and
if \( d = 1 \), then \( \mu = 0 \). If \( J_t \) is a generic time-dependent perturbation, the moduli space of Floer trajectories with attached simple sphere bubbles has expected dimension \( \mu + 2d \), i.e. we can achieve transversality. (Here one needs to check that the evaluation map is transverse to the diagonal; the proof is similar to that of [30, Theorem 6.7.11 (ii)].) However, \( B' \) is attached to the trajectory at a point, so it has a four-dimensional automorphism group: the group of conformal transformations \( \mathbb{C} \), which consists of affine linear maps of the form \( z \rightarrow az + b \). This contradicts the fact that \( \mu + 2d < 4 \).

The statement about the invariance of \( HF(L_0, L_1, \tilde{J}; R) \) follows from the usual continuation arguments in Floer theory.

\[ \square \]

Remark 2.14. A smooth variation of the base almost complex structure \( \tilde{J} \) induces an isomorphism between the respective Floer homologies \( HF(L_0, L_1, \tilde{J}; R) \). However, if we are only given \( \omega_0 \) and \( \omega_\varepsilon \), it is not clear whether the space of possible \( \tilde{J} \)'s is contractible. This justifies keeping \( \tilde{J} \) in the notation \( HF(L_0, L_1, \tilde{J}; R) \).

3. Moduli spaces of flat bundles

3.1. Notation. Throughout the rest of the paper \( G \) will denote the Lie group \( SU(2) \), and \( G^{\text{ad}} = PSU(2) = SO(3) \) the corresponding group of adjoint type. We identify the Lie algebra \( \mathfrak{g} = \mathfrak{su}(2) \) with its dual \( \mathfrak{g}^* \) by using the basic invariant bilinear form

\[ \langle \cdot , \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad \langle A, B \rangle = -\text{Tr} (AB). \]

The maximal torus \( T \simeq S^1 \subset G \) consists of the diagonal matrices \( \text{diag}(e^{2\pi ti}, e^{-2\pi ti}), t \in \mathbb{R} \). We let \( T^{\text{ad}} = T/(\mathbb{Z}/2\mathbb{Z}) \subset G^{\text{ad}} \) and identify the Lie algebra \( \mathfrak{t} = \text{Lie}(T) \) with \( \mathbb{R} \) by sending \( \text{diag}(i, -i) \) to 1. Under this identification, the restriction of the inner product \( \langle \cdot , \cdot \rangle \) to \( \mathfrak{t} \) is twice the Euclidean metric. We use this inner product to identify \( \mathfrak{t} \) with \( \mathfrak{t}^* \) as well. Finally, we let \( \mathfrak{t}^{\perp} \) denote the orthogonal complement of \( \mathfrak{t} \) in \( \mathfrak{g} \).

Adjoint orbits in \( \mathfrak{g} \) are parametrized by the positive Weyl chamber \( t_+ = [0, \infty) \). Indeed, the adjoint quotient map

\[ Q : \mathfrak{g} \to [0, \infty) \]

takes \( \theta \in \mathfrak{g} \) to \( t \) such that \( \theta \) is conjugate to \( \text{diag}(ti, -ti) \).

On the other hand, conjugacy classes in \( G \) are parametrized by the fundamental alcove \( \mathfrak{A} = [0, 1/2] \). Indeed, for any \( g \in G \), there is a unique \( t \in [0, 1/2] \) such that \( g \) is conjugate to the diagonal matrix \( \text{diag}(e^{2\pi ti}, e^{-2\pi ti}) \).

3.2. The extended moduli space. We review here the construction of the extended moduli space ([24, 21]), mostly following Jeffrey’s gauge-theoretic approach from [24].

Let \( \Sigma \) be a Riemann surface of genus \( h \geq 1 \). Fix some \( z \in \Sigma \) and let \( \Sigma' \) denote the complement in \( \Sigma \) of a small disk around \( z \), so that \( S = \partial \Sigma' \) is a circle. Identify a neighborhood of \( S \) in \( \Sigma' \) with \( [0, \varepsilon) \times S \), and let \( s \in \mathbb{R}/2\pi \mathbb{Z} \) be the coordinate on the circle \( S \).

Consider the space \( \mathcal{A}(\Sigma') \cong \Omega^1(\Sigma') \otimes \mathfrak{g} \) of smooth connections on the trivial \( G \)-bundle over \( \Sigma' \), and set

\[ \mathcal{A}^\theta(\Sigma') = \{ A \in \mathcal{A}(\Sigma') | F_A = 0, \ A = \theta ds \text{ on some neighborhood of } S \text{ for some } \theta \in \mathfrak{g} \}. \]

The space \( \mathcal{A}^\theta(\Sigma') \) is acted on by the gauge group

\[ \mathcal{G}^c(\Sigma') = \{ f : \Sigma' \to G | f = I \text{ on some neighborhood of } S \}. \]
Proposition 3.1. (a) The space \( \mathcal{M}(\Sigma') = \mathcal{A}_g(\Sigma')/\mathcal{G}(\Sigma') \).

A more explicit description of the extended moduli space is obtained by fixing a collection of simple closed curves \( \alpha_i, \beta_i \) \((i = 1, \ldots, h)\) on \( \Sigma' \), based at a point in \( S \), such that \( \pi_1(\Sigma') \) is generated by their equivalence classes and the class of a curve \( \gamma \) around \( S \), with the relation: \( \prod_{i=1}^{h} [\alpha_i, \beta_i] = \gamma \).

To each connection on \( \Sigma' \) one can then associate the holonomies \( A_i, B_i \in G \) around the loops \( \alpha_i \) and \( \beta_i \), respectively, \( i = 1, \ldots, h \). This allows us to view the extended moduli space as

\[
(3) \quad \mathcal{M}_g(\Sigma') = \left\{ (A_1, B_1, \ldots, A_h, B_h, \theta) \in G^{2h} \times g \mid \prod_{i=1}^{h} [A_i, B_i] = \exp(2\pi \theta) \right\}.
\]

There is a proper map

\[
\Phi : \mathcal{M}_g(\Sigma') \to g
\]

which takes the class \( [A] \) of a connection \( A \) to the value \( \theta = \Phi(A) \) such that \( A|_S = \theta ds \). (This corresponds to the variable \( \theta \) appearing in (3).) There is also a natural \( G \)-action on \( \mathcal{M}_g(\Sigma') \) given by constant gauge transformations. With respect to the identification (3), it is

\[
(4) \quad g \in G : (A_1, B_1, \theta) \to (gA_1g^{-1}, gB_1g^{-1}, \Ad(g)\theta).
\]

Observe that this action factors through \( G^{\text{ad}} \). The map \( \Phi \) is equivariant with respect to this action on its domain, and the adjoint action on its target. Set

\[
\tilde{\Phi} : \mathcal{M}_g(\Sigma') \to [0, \infty), \quad \tilde{\Phi} = Q \circ \Phi.
\]

Now consider the subspace

\[
\mathcal{M}_s(\Sigma') = \left\{ x \in \mathcal{M}_g(\Sigma') \mid \tilde{\Phi}(x) \notin Z \setminus \{0\} \right\}.
\]

**Proposition 3.1.** (a) The space \( \mathcal{M}_s(\Sigma') \) is a smooth manifold of real dimension \( 6h \).

(b) Every nonzero element \( \theta \in g \) is a regular value for the restriction of \( \Phi \) to \( \mathcal{M}_s(\Sigma') \).

**Proof.** Part (a) is proved in [24, Theorem 2.7]. We copy the proof here, and explain how the same arguments can be used to deduce part (b) as well.

Consider the commutator map \( c : G^{2h} \to G, c(A_1, B_1, \ldots, A_h, B_h) = \prod_{i=1}^{h} [A_i, B_i] \). For \( \rho = (A_1, B_1, \ldots, A_h, B_h) \in G^{2h} \), the differential \( dc_\rho \) is surjective unless \( c(\rho) = I; \) see for example [1, Proposition 2.1].

Define the maps

\[
f_1 : G^{2h} \times g \to G, \quad f_1(\rho, \theta) = c(\rho) \cdot \exp(-2\pi \theta)
\]

and

\[
f_2 : G^{2h} \times g \to G \times g, \quad f_2(\rho, \theta) = (f_1(\rho, \theta), \theta).
\]

On the extended moduli space \( \mathcal{M}_g(\Sigma') = f_1^{-1}(I) \), we have

\[
(df_1)_{(\rho, \theta)} = (dc_\rho) \cdot \exp(-2\pi \theta) - 2\pi \exp(2\pi \theta)(d\exp(-2\pi \theta).
\]

When \( c(\rho) = \exp(2\pi \theta) \neq I \), we have that \( (dc_\rho) \) is surjective, hence so is \( (df_1)_{(\rho, \theta)} \). Also, when \( \theta = 0 \), \( (d\exp)_{-2\pi \theta} \) is just the identity, so again \( (df_1)_{(\rho, \theta)} \) is surjective. Claim (a) follows.

Next, observe that

\[
(df_2)_{(\rho, \theta)}(\alpha, \lambda) = ((df_1)_{(\rho, \theta)}(\alpha, \lambda), \lambda) = (dc_\rho(\alpha) \cdot \exp(-2\pi \theta) + l(\lambda), \lambda),
\]

where \( l(\lambda) \) does not depend on \( \alpha \). Hence, when \( c(\rho) = \exp(2\pi \theta) \neq I \), the differential \( (df_2)_{(\rho, \theta)} \) is surjective. This implies that any \( \theta \in g \) with \( Q(\theta) \notin Z \) is a regular value for \( \Phi|_{\mathcal{M}_s(\Sigma')} \). Since the values \( \theta \in g \) with \( Q(\theta) \in Z \setminus \{0\} \) are not in the image of \( \Phi|_{\mathcal{M}_s(\Sigma')} \), they are automatically regular values, and claim (b) follows. \( \square \)
Consider also the subspace
\[ \mathcal{N}(\Sigma') = \tilde{\Phi}^{-1}([0, 1/2]) \subset \mathcal{M}_g^g(\Sigma'). \]

Note that the restriction of the exponential map \( \theta \to \exp(2\pi \theta) \) to \( Q^{-1}([0, 1/2]) \) is a diffeomorphism onto its image \( G \setminus \{-I\} \). Therefore, using the identification [3], we can describe \( \mathcal{N}(\Sigma') \) as
\[
\mathcal{N}(\Sigma') = \left\{ (A_1, B_1, \ldots, A_h, B_h) \in G^{2h} \left| \prod_{i=1}^{h} [A_i, B_i] \neq -I \right. \right\}.
\]

### 3.3. Hamiltonian actions.

Let \( K \) be a compact, connected Lie group with Lie algebra \( \mathfrak{k} \). We let \( K \) act on the dual Lie algebra \( \mathfrak{k}^* \) by the coadjoint action.

A **pre-symplectic manifold** is a smooth manifold \( M \) together with a closed form \( \omega \in \Omega^2(M) \), possibly degenerate. A **Hamiltonian pre-symplectic \( K \)-manifold** \((M, \omega, \Phi)\) is a pre-symplectic manifold \((M, \omega)\) together with a smooth \( K \)-action and a \( K \)-equivariant smooth map \( \Phi : M \to \mathfrak{k}^* \), such that for any \( \xi \in \mathfrak{g} \), if \( X_\xi \) denotes the vector field on \( M \) generated by the one-parameter subgroup \( \exp(-t\xi) \to \mathbb{R} \), we have
\[
d(\langle \Phi, \xi \rangle) = -\iota(X_\xi)\omega.
\]

Under these hypotheses, the \( K \)-action on \( M \) is called **Hamiltonian**, and \( \Phi \) is called the **moment map**. The quotient
\[
M//K := \Phi^{-1}(0)/K
\]
is named the **pre-symplectic quotient** of \( M \) by \( K \). The following result is known as the Reduction Theorem ([29, 33, 38] Theorem 5.1):

**Theorem 3.2.** Let \((M, \omega, \Phi)\) be a Hamiltonian pre-symplectic \( K \)-manifold. Suppose that the level set \( \Phi^{-1}(0) \) is a smooth manifold on which \( K \) acts freely. Let \( i : \Phi^{-1}(0) \to M \) be the inclusion and \( \pi : \Phi^{-1}(0) \to M//K \) the projection. Then there exists a unique closed form \( \omega_{\text{red}} \) on the smooth manifold \( M//K \) with the property that \( i^*\omega = \pi^*\omega_{\text{red}} \). The reduced form \( \omega_{\text{red}} \) is non-degenerate on \( M//K \) if and only if \( \omega \) is non-degenerate on \( M \) at the points of \( \Phi^{-1}(0) \).

Furthermore, if \( M \) admits another Hamiltonian \( K' \)-action (for some compact Lie group \( K' \)) that commutes with the \( K \)-action, then \((M//K, \omega_{\text{red}})\) has an induced Hamiltonian \( K' \)-action.

When the form \( \omega \) is symplectic, \((M, \omega, \Phi)\) is simply called a **Hamiltonian \( K \)-manifold**. In this case we can drop the condition that \( \Phi^{-1}(0) \) is smooth from the hypotheses of Theorem 3.2; indeed, this condition is automatically implied by the assumption that \( K \) acts freely on \( \Phi^{-1}(0) \).

### 3.4. A closed two-form on the extended moduli space.

According to [24] Equation (2.7)], the tangent space to the smooth stratum \( \mathcal{M}_g^g(\Sigma') \subset \mathcal{M}_g^g(\Sigma) \) at some class \([A]\) can be naturally identified with
\[
T_{[A]}\mathcal{M}_g^g(\Sigma') = \frac{\ker(d_A : \Omega^1(\Sigma') \to \Omega^2_{\text{red}}(\Sigma') \otimes \mathfrak{g})}{\text{im}(d_A : \Omega^0(\Sigma') \otimes \mathfrak{g} \to \Omega^1(\Sigma))},
\]
where \( \Omega^p_{\text{red}}(\Sigma') \) denotes the space of \( p \)-forms compactly supported in the interior of \( \Sigma' \), and \( \Omega^1(\Sigma') \) denotes the space of \( 1 \)-forms \( A \) such that \( A = \theta ds \) near \( S = \partial \Sigma' \) for some \( \theta \in \mathfrak{g} \).

Define a bilinear form \( \omega \) on \( \Omega^1(\Sigma') \) by
\[
\omega(a, b) = \int_{\Sigma'} \langle a \wedge b \rangle,
\]
where the operation \( (a, b) \to \langle a, b \rangle \) on \( \mathfrak{g} \)-valued forms combines the usual exterior product with the inner product on \( \mathfrak{g} \). Stokes’ Theorem implies that \( \omega \) descends to a bilinear form on the tangent space to \( \mathcal{M}_g^g(\Sigma') \) described in Equation [6] above. Thus we can think of \( \omega \) as a two-form on \( \mathcal{M}_g^g(\Sigma') \).
**Theorem 3.3** (Huebschmann-Jeffrey). The two-form $\omega \in \Omega^2(\mathcal{M}_g^0(\Sigma'))$ is closed. It is nondegenerate when restricted to $\mathcal{N}(\Sigma') \subset \mathcal{M}_g^0(\Sigma')$. Moreover, the restriction of the $G^{ad}$-action (4) to $\mathcal{M}_g^0(\Sigma')$ is Hamiltonian with respect to $\omega$. Its moment map is the restriction of $\Phi$ to $\mathcal{M}_g^0(\Sigma')$.

For the proof, we refer to Jeffrey [24]; see also [31].

By abuse of notation, we henceforth denote the restrictions of $\Phi$ still by $\Phi$. Theorem 3.3 says that $(\mathcal{M}_g^0(\Sigma'), \omega, \Phi)$ is a Hamiltonian pre-symplectic $G^{ad}$-manifold in the sense of Section 3.3, and that its subset $(\mathcal{N}(\Sigma'), \omega, \Phi)$ is a (symplectic) Hamiltonian $G^{ad}$-manifold. The symplectic quotient

$$\mathcal{N}(\Sigma')/G^{ad} = \Phi^{-1}(0)/G^{ad} = \mathcal{N}(\Sigma)$$

is the usual moduli space of flat $G$-connections on $\Sigma$, with the symplectic form (on its smooth stratum) being the one constructed by Atiyah and Bott [3]. If $\Sigma$ is given a complex structure, $\mathcal{M}(\Sigma)$ can also be viewed as the moduli space of semistable bundles of rank two on $\Sigma$ with trivial determinant, cf. [35].

For an alternate (group-theoretic) description of the form $\omega$ on $\mathcal{N}(\Sigma')$, see [25], [21], or [22].

Let us mention two results about the two-form $\omega$. The first is proved in [32].

**Theorem 3.4** (Meinrenken-Woodward). $(\mathcal{N}(\Sigma'), \omega)$ is a monotone symplectic manifold, with monotonicity constant $1/4$.

The second result is:

**Lemma 3.5.** The cohomology class of the symplectic form $\omega \in \Omega^2(\mathcal{N}(\Sigma'))$ is integral.

**Proof.** The extended moduli space $\mathcal{M}^0(\Sigma')$ embeds in the moduli space $\mathcal{M}(\Sigma')$ of all flat connections on $\Sigma'$. The latter is an infinite dimensional Banach manifold with a natural symplectic form that restricts to $\omega$ on $\mathcal{M}_g^0(\Sigma')$. Moreover, Donaldson [11] showed that $\mathcal{M}(\Sigma')$ has the structure of a Hamiltonian $LG$-manifold, where $LG = \text{Map}(S^1, G)$ is the loop group of $G$.

Recall that a pre-quantum line bundle $E$ for a symplectic manifold $(M, \omega)$ is a Hermitian line bundle equipped with an invariant connection $\nabla$ whose curvature is $-2\pi i$ times the symplectic form. If $M$ is finite dimensional, this implies that $[\omega] = c_1(E) \in H^2(M; \mathbb{Z})$. In our situation, a pre-quantum line bundle on $M = \mathcal{N}(\Sigma')$ can be obtained by restricting the well-known $LG$-equivariant pre-quantum line bundle on the infinite-dimensional symplectic manifold $\mathcal{M}(\Sigma')$. We refer the reader to [34], [43] and [55] for the construction of the latter; see also [31].

**Corollary 3.6.** The minimal Chern number of the symplectic manifold $\mathcal{N}(\Sigma')$ is a positive multiple of $4$.

**Proof.** Use Theorem 3.4 and Lemma 3.5.

3.5. **Other versions.** Although our main interest lies in the extended moduli space $\mathcal{M}^0(\Sigma')$ and its open subset $\mathcal{N}(\Sigma')$, in order to understand them better we need to introduce two other moduli spaces. Both of them appeared in [24], where their main properties are spelled out. An alternative viewpoint on them is given in [31, Section 3.4.2], where they are interpreted as cross-sections of the full moduli space $\mathcal{M}(\Sigma')$.

The first auxiliary space that we consider is the toroidal extended moduli space:

$$\mathcal{M}^1(\Sigma') = \Phi^{-1}(t) \subset \mathcal{M}^0(\Sigma').$$

It has a smooth stratum

$$\mathcal{M}^1_s(\Sigma') = \{ x \in \mathcal{M}^1(\Sigma') \mid \tilde{\Phi}(x) \notin \mathbb{Z} \}.$$

The restrictions of $\omega$ and $\Phi$ to $\mathcal{M}^1_s(\Sigma')$ turn it into a Hamiltonian pre-symplectic $T^{ad}$-manifold. On the open subset $\mathcal{M}^1(\Sigma') \cap \tilde{\Phi}^{-1}(0,1/2)$, the two-form is nondegenerate.
The second space is the \textit{twisted extended moduli space} from \cite{21}, Section 5.3. In terms of coordinates, it is
\[
M_{tw}^g(\Sigma') = \left\{ (A_1, B_1, \ldots, A_h, B_h, \theta) \in G^{2h} \times \mathfrak{g} \left| \prod_{i=1}^{h} [A_i, B_i] = -\exp(2\pi \theta) \right\}.
\]

This space admits a $G^{ad}$-action just like $M_{tw}^g(\Sigma')$, and a natural projection $\Phi_{tw} : M_{tw}^g \to \mathfrak{g}$. Set $\Phi_{tw} = Q \circ \Phi_{tw}$. The smooth stratum of $M_{tw}^g(\Sigma')$ is
\[
M_{tw}^g(\Sigma') = \left\{ x \in M_{tw}^g(\Sigma') \mid \Phi_{tw}(x) \not\in \mathbb{Z} + \frac{1}{2} \right\}.
\]

Furthermore, $M_{tw, s}^g(\Sigma')$ admits a natural two-form $\omega_{tw}$, which turns it into a Hamiltonian pre-symplectic $G^{ad}$-manifold, with moment map $\Phi_{tw}$. The restriction of $\omega_{tw}$ to the subspace
\[
M_{tw}(\Sigma') = \tilde{\Phi}_{tw}^{-1}([0, 1/2])
\]
is nondegenerate.

Observe that the subspace $\tilde{\Phi}_{tw}^{-1}(t) \subset M_{tw}^g(\Sigma')$ can be identified with the toroidal extended moduli space $M_{tw}^g(\Sigma')$, via the map $(A_i, B_i, t) \to (A_i, B_i, 1/2 - t)$. This map is a diffeomorphism of the smooth strata, and is compatible with the restrictions of the pre-symplectic forms $\omega$ and $\omega_{tw}$.

\section{The structure of degeneracies of $M_{tw}^g(\Sigma')$.} Recall from Theorem \ref{thm:3.3} that the degeneracy locus of the pre-symplectic manifold $M_{tw}^g(\Sigma')$ is contained in the preimage $\tilde{\Phi}_{tw}^{-1}(1/2)$. We seek to understand the structure of the degeneracies.

Let $\mu = \text{diag}(i/2, -i/2)$. Note that the stabilizer $G^{ad}$ of $\exp(2\pi \mu) = -I$ is bigger than the stabilizer $T^{ad} = S^1$ of $\mu$. Thus, we have an obvious diffeomorphism
\[
\tilde{\Phi}^{-1}(1/2) \cong O_\mu \times \Phi^{-1}(\mu),
\]
where $O_\mu$ denotes the coadjoint orbit of $\mu$. The first factor $O_\mu$ is diffeomorphic to the flag variety $G^{ad}/T^{ad} \cong \mathbb{P}^1$. The second factor $\Phi^{-1}(\mu)$ is smooth by Proposition \ref{prop:3.1} (b).

There is a residual $T^{ad}$-action on the space $\Phi^{-1}(\mu)$. Thus $\Phi^{-1}(\mu)$ is an $S^1$-bundle over
\[
M_{\mu}(\Sigma') = \Phi^{-1}(\mu)/T^{ad}.
\]
Finally, $M_{\mu}(\Sigma')$ is a $\mathbb{P}^1$-bundle over
\[
M_{-I}(\Sigma') = \left\{ (A_1, B_1, \ldots, A_h, B_h) \in G^{2h} \mid \prod_{i=1}^{h} [A_i, B_i] = -I \right\}/G^{ad}.
\]

This last space $M_{-I}(\Sigma')$ can be identified with the moduli space $M_{tw}(\Sigma)$ of projectively flat connections on $E$ with fixed central curvature, where $E$ is a $U(2)$-bundle of odd degree over the closed surface $\Sigma = \Sigma' \cup D^2$. Alternatively, it is the moduli space of rank two stable bundles on $\Sigma$ having fixed determinant of odd degree, \textit{cf}. \cite{35, 3}. It can also be viewed as the symplectic quotient of the twisted extended moduli space from Section \ref{sec:3.5}
\[
M_{tw}(\Sigma) = M_{tw}(\Sigma')//G^{ad} = \tilde{\Phi}_{tw}^{-1}(0)/G^{ad}.
\]

We have described a string of fibrations that gives a clue to the structure of the space $\tilde{\Phi}_{tw}^{-1}(1/2)$. Let us now reshuffle these fibrations and view $\tilde{\Phi}_{tw}^{-1}(1/2)$ as a $G^{ad}$-bundle over the space $O_\mu \times M_{-I}(\Sigma')$. Its fiberwise tangent space (at any point) is $\mathfrak{g}$, which can be decomposed as $t \oplus t^\perp$, with $t^\perp \cong \mathbb{C}$.

\textbf{Proposition 3.7.} Let $x \in \tilde{\Phi}^{-1}(1/2) \subset M_{tw}^g(\Sigma')$. The null space of the form $\omega$ at $x$ consists of the fiber directions corresponding to $t^\perp \subset \mathfrak{g}$.
Our strategy for proving Proposition 3.7 is to reduce it to a similar statement for the toroidal extended moduli space $\mathcal{M}^t(\Sigma')$, and then study the latter via its embedding into the twisted extended moduli space $\mathcal{M}^0_t(\Sigma')$.

First, note that by $G^\text{ad}$-invariance, we can assume without loss of generality that $\Phi(x) = \mu$. The symplectic cross-section theorem [19] says that, near $\Phi^{-1}(\mu)$, the two-form on $\mathcal{M}^0(\Sigma')$ is obtained from the one on $\mathcal{M}^t(\Sigma') = \Phi^{-1}(t)$ by a procedure called symplectic induction. (Strictly speaking, symplectic induction is described in [19] for nondegenerate forms; however, it applies to the Hamiltonian pre-symplectic case as well.) More concretely, we have a (noncanonical) decomposition

$$T_x \mathcal{M}^0_t(\Sigma') = T_x \mathcal{M}^t(\Sigma') \oplus T_x (O_{\mu}) \oplus T^*_x (O_{\mu}),$$

such that $\omega|_x$ is the direct sum of its restriction to the first summand in (7) with the natural pairing between the other two summands.

Recall that we are viewing $\Phi^{-1}(1/2)$ as a $G$-bundle over $O_{\mu} \times \mathcal{M}^t(\Sigma')$. Its intersection with $\mathcal{M}^t(\Sigma')$ is $\Phi^{-1}(\mu)$, which is the part of the $G^\text{ad}$-bundle that lies over $\{\mu\} \times \mathcal{M}^t(\Sigma')$. The decomposition (7) implies that, in order to prove the final claim about the null space of $\omega|_x$, it suffices to show that the null space of $\omega|_{\mathcal{M}^t(\Sigma')}$ at $x$ consists of the fiber directions corresponding to $t^\perp \subset \mathfrak{g}$.

Let us use the observation in the last paragraph of Section 3.5 and view $\mathcal{M}^t(\Sigma')$ as $\Phi^{-1}_t(t) \subset \mathcal{M}^0_t(\Sigma')$. The point $x$ now lies in $\Phi^{-1}_t(0)$.

Recall from Section 3.5 that the two-form $\omega_t$ on $\mathcal{M}^0_t(\Sigma')$ is nondegenerate near $\Phi^{-1}_t(0)$. Further, it is easy to check that the action of $G^\text{ad}$ on $\Phi^{-1}_t(0)$ is free. This action is Hamiltonian; hence, the quotient $\Phi^{-1}_t(0)/G^\text{ad} = \mathcal{M}^t(\Sigma')$ is smooth, and the reduced two-form on it is nondegenerate. Further, there is a (noncanonical) decomposition:

$$T_x \mathcal{M}^0_t(\Sigma') \cong \pi^* T_{\pi(x)} \mathcal{M}^t(\Sigma') \oplus \mathfrak{g} \oplus \mathfrak{g}^*,\tag{8}$$

where $\pi : \Phi^{-1}_t(0) \to \mathcal{M}^t(\Sigma')$ is the quotient map. (See for example Equation (5.6) in [18].) The two-form $\omega_t$ at $x$ is the direct summand of the reduced form at $\pi(x)$ and the natural pairing of the two last factors in (8).

With respect to the decomposition (8), the subspace $T_x \mathcal{M}^t(\Sigma') \subset T_x \mathcal{M}^0_t(\Sigma')$ corresponds to $T_x \mathcal{M}^t(\Sigma') \cong \pi^* T_{\pi(x)} \mathcal{M}^t(\Sigma') \oplus \mathfrak{g} \oplus t^*$. Therefore, the null space of $\omega_t$ on $T_x \mathcal{M}^t(\Sigma')$ is the null space of the restriction of the natural pairing on $\mathfrak{g} \oplus \mathfrak{g}^*$ to $\mathfrak{g} \oplus t^*$. This is $\mathfrak{g}/t \cong t^\perp$, as claimed.

4. Symplectic cutting

4.1. Abelian symplectic cutting. We review here Lerman’s definition of (abelian) symplectic cutting, following [28].

Consider a symplectic manifold $(M, \omega)$ with a Hamiltonian $S^1$-action and moment map $\Phi : M \to \mathbb{R}$. Pick some $\lambda \in \mathbb{R}$. The diagonal $S^1$-action on the space $M \times \mathbb{C}^-$ (endowed with the standard product symplectic structure, where $\mathbb{C}^-$ is with negative the usual area form) is Hamiltonian with respect to the moment map

$$\Psi : M \times \mathbb{C}^- \to \mathbb{R}, \quad \Psi(m, z) = \Phi(m) + \frac{1}{2}|z|^2 - \lambda.$$

The symplectic quotient

$$M_{\leq \lambda} := \Psi^{-1}(0)/S^1 \cong \Phi^{-1}(\lambda)/S^1 \cup \Phi^{-1}(-\infty, \lambda)$$

is called the symplectic cut of $M$ at $\lambda$. If the action of $S^1$ on $\Phi^{-1}(\lambda)$ is free, then $M_{\leq \lambda}$ is a symplectic manifold, and it contains $\Phi^{-1}(\lambda)/S^1$ (with its reduced form) as a symplectic hypersurface, i.e. a symplectic submanifold of real codimension two.
Remark 4.1. The normal bundle to $\Phi^{-1}(\lambda)/S^1$ in $M_{\leq \lambda}$ is the complex line bundle whose associated circle bundle is $\Phi^{-1}(\lambda) \to \Phi^{-1}(\lambda)/S^1$.

Remark 4.2. Symplectic cutting is a local construction. In particular, if $(M, \omega)$ is symplectic and $\Phi: M \to \mathbb{R}$ is a continuous map that induces a smooth Hamiltonian $S^1$-action on an open set $U \subset M$ containing $\Phi^{-1}(\lambda)$, then we can still define $M_{\leq \lambda}$ as the union $(M \setminus U) \cup U_{\leq \lambda}$.

Remark 4.3. If $M$ has an additional Hamiltonian $K$-action (for some other compact group $K$), then $M_{\leq \lambda}$ has an induced Hamiltonian $K$-action. This follows from the similar statement for symplectic reduction, cf. Theorem 3.2.

4.2. Non-abelian symplectic cutting. An analog of symplectic cutting for non-abelian Hamiltonian actions was defined in [56]. We explain here the case of Hamiltonian $PSU(2)$-actions, since this is all we need for our purposes.

We keep the notation from Section 3.1, with $G = SU(2)$ and $G^{\text{ad}} = PSU(2)$. Let $(M, \omega, \Phi)$ be a Hamiltonian $G^{\text{ad}}$-manifold. Since $g$ and $g^*$ are identified using the bilinear form, from now on we will view the moment map $\Phi$ as taking values in $g$. Recall that

$$Q: g \to g/G^{\text{ad}} \cong [0, \infty)$$

denotes the adjoint quotient map. The map $Q$ is continuous, and is smooth outside $Q^{-1}(0)$. Set

$$\tilde{\Phi} = Q \circ \Phi.$$

On the complement $U$ of $\Phi^{-1}(0)$ in $M$, the map $\tilde{\Phi}$ induces a Hamiltonian $S^1$-action. Explicitly, $u \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ acts on $m \in U$ by

$$m \to \exp\left( u \cdot \frac{\Phi(m)}{2\Phi(m)} \right) \cdot m. \quad (9)$$

This action is well-defined because $\exp(\pi H) = I$ in $G^{\text{ad}}$. We can describe the action alternatively as follows: on $\Phi^{-1}(t) \subset M$, it coincides with the action of $T^{\text{ad}} \subset G^{\text{ad}}$; then it is extended to all of $M$ in a $G^{\text{ad}}$-equivariant manner.

Fix $\lambda > 0$. Using the local version (from Remark 4.2) of abelian symplectic cutting for the action $0$, we define the non-abelian symplectic cut of $M$ at $\lambda$ to be

$$M_{\leq \lambda} = \Phi^{-1}(0) \cup U_{\leq \lambda} = M_{< \lambda} \cup R,$$

where

$$M_{< \lambda} = \Phi^{-1}([0, \lambda)), \quad R_\lambda = \tilde{\Phi}^{-1}(\lambda)/S^1.$$

If $S^1$ acts freely on $\tilde{\Phi}^{-1}(\lambda)$, then $M_{\leq \lambda}$ is a smooth manifold. It can be naturally equipped with a symplectic form $\omega_{\leq \lambda}$, coming from the symplectic form $\omega$ on $M$. In fact, $M_{\leq \lambda}$ is a Hamiltonian $G^{\text{ad}}$-manifold, cf. Remark 4.3. With respect to the form $\omega_{\leq \lambda}$, $R$ is a symplectic hypersurface in $M_{\leq \lambda}$.

4.3. Monotonicity. Let $G^{\text{ad}} = PSU(2)$ as before. We aim to find a condition that guarantees that a non-abelian symplectic cut is monotone. As a toy model for future results, we start with a simple fact about symplectic reduction:

**Lemma 4.4.** Let $(M, \omega, \Phi)$ be a Hamiltonian $G^{\text{ad}}$-manifold that is monotone, with monotonicity constant $\kappa$. Assume that the moment map $\Phi$ is proper, and the $G^{\text{ad}}$-action on $\Phi^{-1}(0)$ is free. Then, the symplectic quotient $M\slash G^{\text{ad}} = \Phi^{-1}(0)/G^{\text{ad}}$ (with the reduced symplectic form $\omega^{\text{red}}$) is also monotone, with the same monotonicity constant $\kappa$. 


Proof. Consider the Kirwan map from [26]:
\[ H^2_{G_{\text{ad}}}(M; \mathbb{R}) \to H^2(M//G_{\text{ad}}; \mathbb{R}), \]
which is obtained by composing the map \( H^2_{G_{\text{ad}}}(M; \mathbb{R}) \to H^2_{G_{\text{ad}}}((\Phi^{-1}(0)); \mathbb{R}) \) (induced by the inclusion) with the Cartan isomorphism \( H^2_{G_{\text{ad}}}((\Phi^{-1}(0)); \mathbb{R}) \cong H^2(M//G_{\text{ad}}; \mathbb{R}). \) The Kirwan map takes the first equivariant Chern class \( c^\text{ad}_1(TM) \) to \( c^\text{ad}_1(T(M//G_{\text{ad}})) \), and the equivariant two-form \( \tilde{\omega} = \omega - \Phi \) to \( \omega_{\text{red}}. \)

Note that \( H^i(BG_{\text{ad}}; \mathbb{R}) = 0 \) for \( i = 1, 2 \), hence \( H^2_{G_{\text{ad}}}(M; \mathbb{R}) \cong H^2(M; \mathbb{R}). \) Under this isomorphism, \( c^\text{ad}_1 \) corresponds to \( c_1 \) and \( [\tilde{\omega}] \) to \([\omega] \), so the conclusion follows.

For \( \lambda \in (0, \infty) \), let \( O_{\lambda} \cong \mathbb{P}^1 \) be the coadjoint orbit of \( \text{diag}(i\lambda, -i\lambda) \), endowed with the Kostant-Kirillov-Souriau form \( \omega_{KKS}(\lambda) \). It has a Hamiltonian \( G_{\text{ad}} \)-action with moment map the inclusion \( \iota: O_{\lambda} \to \mathfrak{g} \). Let \( \gamma = \text{P.D.}(pt) \) denote the generator of \( H^2(O_{\lambda}; \mathbb{Z}) \subset H^2(O_{\lambda}; \mathbb{R}) \), so that \( c_1(O_{\lambda}) = 2\gamma \).

Then \( c_2(O_{\lambda}) = [\omega_{KKS}(1)], \) cf. [6, Section 7.5, Section 7.6]. Therefore, \( [\omega_{KKS}(\lambda)] = 2\lambda\gamma \).

If \((M, \omega, \Phi)\) is a Hamiltonian \( G_{\text{ad}} \)-manifold, let \( M \times O_{\lambda}^- \) denote the Hamiltonian manifold \((M \times O_{\lambda}, \omega \times -\omega_{KKS}(\lambda), \Phi - \iota)\). The reduction of \( M \) with respect to \( O_{\lambda} \) is defined as
\[ M_{\lambda} = (M \times O_{\lambda}^-)//G_{\text{ad}} = \Phi^{-1}(O_{\lambda})/G_{\text{ad}}. \]

If the \( G_{\text{ad}} \)-action on \( \Phi^{-1}(O_{\lambda}) \) is free, the quotient \( M_{\lambda} \) is smooth and admits a natural symplectic form \( \omega_{\lambda} \). It can be viewed as \( \Phi^{-1}(\text{diag}(i\lambda, -i\lambda))/T_{\text{ad}} \). We let \( E_{\lambda} \) denote the complex line bundle on \( M_{\lambda} \) associated to the respective \( T_{\text{ad}} \)-fibration.

Lemma 4.5. Let \((M, \omega, \Phi)\) be a Hamiltonian \( G_{\text{ad}} \)-manifold such that the moment map \( \Phi \) is proper, and the action of \( G_{\text{ad}} \) is free outside \( \Phi^{-1}(0) \). Assume that \( M \) is monotone, with monotonicity constant \( \kappa \). Then the cohomology class of the reduced form \( \omega_{\lambda} \) is given by the formula
\[ [\omega_{\lambda}] = \kappa \cdot c_1(TM_{\lambda}) + (\lambda - \kappa) \cdot c_1(E_{\lambda}). \]

Proof. Let us consider the Kirwan map for the manifold \( M \times O_{\lambda}^- \), whose symplectic reduction is \( M_{\lambda} \). Since \( H^2 = H^2_{G_{\text{ad}}} \) for all Hamiltonian \( G_{\text{ad}} \)-manifolds, we can view the Kirwan map as going from \( H^2(M \times O_{\lambda}^-; \mathbb{R}) \) into \( H^2(M_{\lambda}; \mathbb{R}). \)

By abuse of notation, we denote classes in \( H^2(M; \mathbb{R}) \) or \( H^2(O_{\lambda}^-; \mathbb{R}) \) the same as their pullbacks to \( H^2(M \times O_{\lambda}^-; \mathbb{R}). \)

Just as in the proof of Lemma 4.4, we get that the Kirwan map takes
\[ [\omega] - [\omega_{KKS}(\lambda)] = \kappa c_1(TM) - 2\lambda\gamma \]
to the reduced form \([\omega_{\lambda}]\), and \( c_1(TM) - c_1(TO_{\lambda}) = c_1(TM) - 2\gamma \) to the reduced Chern class \( c_1(TM_{\lambda}). \)

Note also that the image of \( c_1(TO_{\lambda}^-) = -2\gamma \) under the Kirwan map is \( c_1(E_{\lambda}) \). Hence:
\[ [\omega_{\lambda}] - \kappa \cdot c_1(TM_{\lambda}) = (\lambda - \kappa) \cdot c_1(E_{\lambda}), \]
as desired.

We are now ready to study monotonicity for non-abelian cuts:

Proposition 4.6. Let \( G_{\text{ad}} = PSU(2) \), and \((M, \omega, \Phi)\) be a Hamiltonian \( G_{\text{ad}} \)-manifold that is monotone with monotonicity constant \( \kappa > 0 \). Assume that the moment map \( \Phi \) is proper, and that \( G_{\text{ad}} \) acts freely outside \( \Phi^{-1}(0) \). Then the symplectic cut \( M_{\leq \lambda} \) at the value \( \lambda = 2\kappa \in (0, \infty) \) is also monotone, with the same monotonicity constant \( \kappa \).

Proof. Recall that the symplectic cut \( M_{\leq \lambda} \) is the union of the open piece \( M_{< \lambda} \) and the hypersurface \( R_{\lambda} = \Phi^{-1}(O_{\lambda})/S^1 \). Note that there is a natural symplectomorphism
\[ R_{\lambda} \xrightarrow{\cong} O_{\lambda} \times M_{\lambda}, \quad m \to (\Phi(m), [m]). \]
The inverse to this symplectomorphism is given by the map \([g], [m] \to [gm].\)
By Remark 4.1, the normal bundle to $R_{\lambda}$ is the line bundle associated to the defining $T^{\text{ad}}$-bundle on $R_{\lambda}$. We denote this $T^{\text{ad}}$-bundle by $N_{\lambda}$; it is the product of $G^{\text{ad}} \to G^{\text{ad}}/T^{\text{ad}} \cong O_{\lambda}$ on the $O_{\lambda}$ factor and the circle bundle of $E_{\lambda}$ on the $M_{\lambda}$ factor.

Let $\nu(R_{\lambda})$ be a regular neighborhood of $R_{\lambda}$, so that the intersection $M_{<\lambda} \cap \nu(R_{\lambda})$ admits a deformation retract into a copy of $N_{\lambda}$.

We have a Mayer-Vietoris sequence
\[ \cdots \to H^1(M_{<\lambda}) \oplus H^1(\nu(R_{\lambda})) \to H^1(N_{\lambda}) \to H^2(M_{<\lambda}) \to H^2(M_{<\lambda}) \oplus H^2(\nu(R_{\lambda})) \to \cdots \]

Note that the first Chern class of the bundle $N_{\lambda} \to R_{\lambda}$ is nontorsion in $H^2(R_{\lambda})$, because it is so on the $O_{\lambda}$ factor. Hence, the map $H^1(\nu(R_{\lambda}); \mathbb{R}) \to H^1(N_{\lambda}; \mathbb{R})$ is onto. The Mayer-Vietoris sequence then tells us that the map
\[ H^2(M_{<\lambda}; \mathbb{R}) \to H^2(M_{<\lambda}; \mathbb{R}) \oplus H^2(\nu(R_{\lambda}); \mathbb{R}) \]
is injective. Therefore, in order to check the monotonicity of $M_{<\lambda}$, it suffices to check it on $M_{<\lambda}$ and $\nu(R_{\lambda})$.

Since $M_{<\lambda}$ is symplectomorphic to a subset of $M$, by assumption monotonicity is satisfied there. Let us check it on $\nu(R_{\lambda})$ or, equivalently, on its deformation retract $R_{\lambda}$. We will use the symplectomorphism $[10]$ and, by abuse of notation, we will denote the objects on $O_{\lambda}$ or $M_{\lambda}$ the same as we denote their pullback to $R_{\lambda}$. Let $c_1$ be the generator of $H^2(O_{\lambda}; \mathbb{Z})$ as in the proof of Lemma 4.5.

By the result of that lemma, we have
\[ [\omega_{<\lambda}|_{R_{\lambda}}] = 2\lambda \gamma + \kappa c_1(TM_{\lambda}) + (\lambda - \kappa) c_1(E_{\lambda}). \]

On the other hand, the tangent space to $M_{<\lambda}$ at a point of $R_{\lambda}$ decomposes into the tangent and normal bundles to $R_{\lambda}$. Therefore,
\[ c_1(TM_{<\lambda}|_{R_{\lambda}}) = c_1(TR_{\lambda}) + 2 \gamma + c_1(E_{\lambda}) = 4 \gamma + c_1(TM_{\lambda}) + c_1(E_{\lambda}). \]

Taking into account Equation $[11]$, for $\lambda = 2\kappa$ we conclude that $[\omega_{<\lambda}|_{R_{\lambda}}] = \kappa \cdot c_1(TM_{<\lambda}|_{R_{\lambda}})$.

### 4.4. Extensions to pre-symplectic manifolds.

Abelian and non-abelian cutting are simply particular instances of symplectic reduction. Since the latter can be extended to the pre-symplectic setting, one can also define abelian and non-abelian cutting for Hamiltonian pre-symplectic manifolds.

In general, one cannot define $c_1(TM)$ (and the notion of monotonicity) for pre-symplectic manifolds, because there is no good notion of compatible almost complex structure. In order to fix that, we introduce the following:

**Definition 4.7.** An $\epsilon$-symplectic manifold $(M, \{\omega_t\})$ is a smooth manifold $M$ together with a smooth family of closed two-forms $\omega_t \in \Omega^2(M)$, $t \in [0, \epsilon]$ for some $\epsilon > 0$, such that $\omega_t$ is symplectic for all $t \in (0, \epsilon]$.

One should think of an $\epsilon$-symplectic manifold $(M, \{\omega_t\})$ as the pre-symplectic manifold $(M, \omega_0)$ together with some additional data given by the other $\omega_t$’s. In particular, by the degeneracy locus of $(M, \{\omega_t\})$ we mean the degeneracy locus of $\omega_0$, i.e.
\[ R(\omega_0) = \{m \in M \mid \omega_0 \text{ is degenerate on } T_mM\}. \]

If $(M, \{\omega_t\})$ is any $\epsilon$-symplectic manifold, we can define its first Chern class $c_1(TM) \in H^2(M; \mathbb{Z})$ by giving $TM$ an almost complex structure compatible with some $\omega_t$ for $t > 0$. (Note that the resulting $c_1(TM)$ does not depend on $t$.) We can then define the minimal Chern number of an $\epsilon$-symplectic manifold just as we did for symplectic manifolds. Moreover, we can talk about monotonicity:
Definition 4.8. The $\epsilon$-symplectic manifold $(M, \{\omega_t\})$ is called monotone (with monotonicity constant $\kappa > 0$) if

$$[\omega_0] = \kappa \cdot c_1(TM).$$

One source of $\epsilon$-symplectic manifolds is symplectic reduction. Indeed, suppose we have a Hamiltonian pre-symplectic $S^1$-manifold $(M, \omega, \Phi)$ with the moment map $\Phi : M \to \mathbb{R}$ being proper. The form $\omega$ may have some degeneracies on $\Phi^{-1}(0)$; however, we assume that it is nondegenerate on $\Phi^{-1}((0, \epsilon])$ for some $\epsilon > 0$. Assume also that $S^1$ acts freely on $\Phi^{-1}((0, \epsilon])$ (hence any $t \in (0, \epsilon]$ is a regular value for $\Phi$) and, further, $0$ is a regular value for $\Phi$ as well. Then the pre-symplectic quotients $M_t = \Phi^{-1}(t)/S^1$ for $t \in [0, \epsilon]$ form a smooth fibration over the interval $[0, \epsilon]$. By choosing a connection for this fiber bundle, we can find a smooth family of diffeomorphisms $\phi_t : M_0 \to M_t, t \in [0, \epsilon]$, with $\phi_0 = \text{id}_{M_0}$. We can then put a structure of $\epsilon$-symplectic manifold on $M_0$ by using the forms $\phi_t^*\omega_t, t \in [0, \epsilon]$, where $\omega_t$ is the reduced form on $M_t$. Note that the space of choices involved in this construction (i.e., connections) is contractible. Therefore, whether or not $(M_0, \phi_t^*\omega_t)$ is monotone is independent of these choices.

Since abelian and non-abelian cutting are instances of (pre-)symplectic reduction, one can also turn pre-symplectic cuts into $\epsilon$-symplectic manifolds in an essentially canonical way, provided that the form is nondegenerate on the nearby cuts. (By “nearby” we implicitly assume that we have chosen a preferred side for approximating the cut value: either from above or from below.) In this context, we have the following analog of Proposition 4.6.

Proposition 4.9. Let $G^{ad} = PSU(2)$, and $(M, \omega, \Phi)$ be a Hamiltonian pre-symplectic $G^{ad}$-manifold. Set $\tilde{\Phi} = Q \circ \Phi : M \to [0, \infty)$ as usual. Assume that:

- The moment map $\tilde{\Phi}$ is proper;
- The form $\omega$ is nondegenerate on the open subset $M_{< \lambda} = \tilde{\Phi}^{-1}([0, \lambda))$, for some value $\lambda \in (0, \infty)$;
- $G^{ad}$ acts freely on $\tilde{\Phi}^{-1}([0, \lambda])$ (hence, any $t \in (0, \lambda)$ is a regular value for $\tilde{\Phi}$);
- $\lambda$ is also a regular value for $\tilde{\Phi}$;
- As a symplectic manifold, $M_{< \lambda}$ is monotone, with monotonicity constant $\kappa = \lambda/2$.

Fix some $\epsilon \in (0, \lambda)$ and view the pre-symplectic cut $M_{\leq \lambda}$ as an $\epsilon$-symplectic manifold, with respect to forms $\phi_t^*\omega_{\leq \lambda - t}$, for a smooth family of diffeomorphisms $\phi_t : M_{\leq \lambda} \to M_{\leq \lambda - t}, t \in [0, \epsilon], \phi_0 = \text{id}$. Then, $M_{\leq \lambda}$ is monotone, with the same monotonicity constant $\kappa = \lambda/2$.

Proof. We can run the same arguments as in the proof of Proposition 4.6 as long as we apply them to the Hamiltonian manifold $M_{< \lambda}$, where $\omega$ is nondegenerate. This gives us the corresponding formulae for the cohomology classes $[\omega_{\leq \lambda - t}]$ and $c_1(TM_{\leq \lambda - t})$, for $t \in (0, \epsilon)$. In the limit $t \to 0$, we get monotonicity. \hfill $\square$

4.5. Cutting the extended moduli space. Recall from Section 3.4 that the smooth part $\mathcal{M}^\theta_\delta(\Sigma')$ of the extended moduli space is a Hamiltonian pre-symplectic $G^{ad}$-manifold. Let us consider its non-abelian cut at the value $\lambda = 1/2$:

$$\mathcal{N}^c(\Sigma') = \mathcal{M}^\theta_\delta(\Sigma')_{\leq 1/2}.$$

The notation $\mathcal{N}^c(\Sigma')$ indicates that this space is a compactification of $\mathcal{N}(\Sigma') = \mathcal{M}^\theta_\delta(\Sigma')_{< 1/2}$. Indeed, we have

$$\mathcal{N}^c(\Sigma') = \mathcal{N}(\Sigma') \cup R,$$

where

$$R \cong \left\{ (A_1, B_1, \ldots, A_h, B_h, \theta) \in G^{2h} \times \mathfrak{g} \left| \prod_{i=1}^h [A_i, B_i] = \exp(2\pi i \theta) = -1 \right. \right\}/S^1,$$

Here $u \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ acts by conjugating each $A_i$ and $B_i$ by $\exp(u\theta)$, and preserving $\theta$. 
The $G^\ad$-action on $\tilde{\Phi}^{-1}((0,1/2]) \subset \mathcal{M}^4_\theta(\Sigma')$ is free. Since $\omega$ is nondegenerate on $\tilde{\Phi}^{-1}((0,1/2])$ by Theorem 3.3, this implies that any $\theta \in \mathfrak{g}$ with $Q(\theta) \in (0,1/2]$ is a regular value for $\Phi$. The last statement also follows from Proposition 3.1 (b), which further says that the values $\theta \in \mathfrak{g}$ with $Q(\theta) = 1/2$ are also regular. Hence, any $t \in (0,1/2]$ is a regular value for $\tilde{\Phi}$. Lastly, note that Theorem 3.4 says that $\tilde{\Phi}^{-1}((0,1/2])$ is monotone, with monotonicity constant $\kappa = 1/4 = \lambda/2$. We conclude that the hypotheses of Proposition 4.9 are satisfied. We obtain:

**Proposition 4.10.** Fix $\epsilon \in (0,1/2)$. Endow $\mathcal{N}^c(\Sigma')$ with the structure of an $\epsilon$-symplectic manifold, using the forms $\phi_t^* \omega_{\leq 1/2-t}$, coming from a smooth family of diffeomorphisms

$$\phi_t : \mathcal{N}(\Sigma') = M_{\leq 1/2} \rightarrow M_{\leq 1/2-t}, \ t \in [0,\epsilon], \ \phi_0 = id.$$  

Then, $\mathcal{N}^c(\Sigma')$ is monotone with monotonicity constant $1/4$.

Thus, we have succeeded in compactifying the symplectic manifold $\mathcal{N}(\Sigma')$ while preserving monotonicity. The downside is that $\mathcal{N}^c(\Sigma')$ is only pre-symplectic. The resulting two-form has degeneracies on $R$. Their structure can be immediately deduced from Proposition 3.7.

**Lemma 4.11.** Let us view $R$ as a $\mathbb{P}^1$-bundle over the space $\mathcal{O}_1 \times \mathcal{M}_1(\Sigma')$. Then, at any point in $R$, the null space of the form $\omega_{\leq 1/2}$ consists of the fiber directions.

In a family of forms that make $\mathcal{N}^c(\Sigma')$ into an $\epsilon$-symplectic manifold (as in Proposition 4.10), the degenerate form $\omega_{\leq 1/2}$ always corresponds to $t = 0$. From now on we will denote it by $\omega_0$.

**Proposition 4.12.** In addition to the degenerate form $\omega_0$ coming from the cut, the space $\mathcal{N}^c(\Sigma') = \mathcal{N}(\Sigma') \cup R$ also admits a symplectic form $\omega_\epsilon$ with the following properties:

(i) $R$ is a symplectic hypersurface with respect to $\omega_\epsilon$;

(ii) The restrictions of $\omega_0$ and $\omega_\epsilon$ to $\mathcal{N}(\Sigma')$ have the same cohomology class in $H^2(\mathcal{N}(\Sigma'); \mathbb{R})$;

(iii) The forms $\omega_0$ and $\omega_\epsilon$ themselves coincide on the open subset $W = \tilde{\Phi}^{-1}((0,1/4]) \subset \mathcal{N}(\Sigma')$;

(iv) There exists an almost complex structure $\tilde{J}$ on $\mathcal{N}^c(\Sigma')$ that is compatible with respect to $\omega_\epsilon$ on $\mathcal{N}^c(\Sigma')$, and compatible with respect to $\omega_0$ on $\mathcal{N}(\Sigma')$.

**Proof.** As the name suggests, the form $\omega_\epsilon$ will be part of a family $(\omega_t), t \in [0,\epsilon]$ of the type used to turn $\mathcal{N}(\Sigma')$ into an $\epsilon$-symplectic manifold. In fact, it is easy to find such a form that satisfies conditions (i)-(iii) above. One needs to choose $\epsilon < 1/4$ and a smooth family of diffeomorphisms $\phi_t : \mathcal{N}(\Sigma') = M_{\leq 1/2} \rightarrow M_{\leq 1/2-t}, \ t \in [0,\epsilon], \ \phi_0 = id$, such that $\phi_t = id$ on $W$ and $\phi_t$ takes $R$ to $R_{1/2-t} = \tilde{\Phi}^{-1}(1/2-t)/S^{1}(S1)$; then set $\omega_\epsilon = \phi^* \omega_0$. Note that condition (ii) is automatic from (iii), because $W$ is a deformation retract of $\mathcal{N}(\Sigma')$.

However, in order to make sure that condition (iv) is satisfied, more care is needed in choosing the diffeomorphisms above. We will only construct $\phi = \phi_\epsilon$, since this is all we need for our purposes; however, it will be easy to see that one could interpolate between $\phi$ and the identity.

The strategy for constructing $\phi$ and $\tilde{J}$ is the same as in the proofs of Proposition 3.7 and Lemma 4.11: we construct a diffeomorphism and an almost complex structure on the toroidal extended moduli space $\mathcal{M}(\Sigma')$, by looking at it as a subset of the twisted extended moduli space $\mathcal{M}_\text{tw}(\Sigma')$; then, we lift them to $\mathcal{M}^g(\Sigma')$; finally, we show that they descend to the cut.

Let $\mu = \text{diag}(i/2, -i/2)$ as in Section 3.6. We start by carefully examining the restriction of the form $\omega$ to $\mathcal{M}(\Sigma')$, in a neighborhood of $\Phi^{-1}(\mu)$. By the remark at the end of Section 3.5 this is the same as looking at the restriction of $\omega_{\text{tw}}$ to $\Phi^{-1}(t^*)$ in a neighborhood of $\Phi^{-1}(0)$.

The zero set $Z$ of the moment map $\Phi_{\text{tw}}$ on (the smooth, symplectic part of) $\mathcal{M}^g_{\text{tw}}(\Sigma')$ is a coisotropic submanifold. Let $\omega_{\text{tw},0}$ be the reduced form on $Z/G^\ad = \mathcal{M}_{-1}(\Sigma')$. Pick a connection form $\alpha \in \Omega^1(Z \otimes \mathfrak{g})$ for the $G^\ad$-action on $Z$. By the equivariant coisotropic embedding theorem [19, Proposition 39.2], we can find a $G^\ad$-equivariant diffeomorphism between a neighborhood of $Z = \Phi_{\text{tw}}^{-1}(0)$ in $\mathcal{M}^g_{\text{tw}}(\Sigma')$ and a neighborhood of $Z \times \{0\}$ in $Z \times \mathfrak{g}^*$ such that the form $\omega_{\text{tw}}$ looks like

$$\omega_{\text{tw}} = \pi^* \omega_{\text{tw},0} + d(\alpha, \pi_2),$$
where \( \pi_1 : Z \times g \to Z \to Z/G^\text{ad} \) and \( \pi_2 : Z \times g^* \to g^* \) are projections. We can assume that \( \pi_2 \) corresponds to the moment map.

Restricting this diffeomorphism to \( \Phi_{tw}^{-1}(t^*) \), we obtain a local model \( Z \times t^* \) for that space. This implies that, locally near \( Z = Z \times \{0\} \), we get a decomposition of its tangent spaces into several (nontrivial) bundles

\[
T(\Phi_{tw}^{-1}(t^*)) \cong T(Z/S^1) \oplus g \oplus t^* \cong T(\mathcal{M}_{-1}(\Sigma')) \oplus t^1 \oplus (t \oplus t^*).
\]

(We omitted the pull-back symbols from notation for simplicity.)

The restriction of \( \omega_{tw} \) to \( \Phi_{tw}^{-1}(t^*) \) is nondegenerate in the horizontal directions \( T\mathcal{M}_{-1}(\Sigma') \) as well as on \( t \oplus t^* \). Let us compute it on the subbundle \( t^1 \subset g \). For a point \( x \) with \( \Phi_{tw}(x) = t\mu \in t^* \), and for \( \xi_1, \xi_2 \in t^1 \subset T_x\Phi_{tw}^{-1}(t^*) \), we have

\[
\omega_{tw}(\xi_1, \xi_2) = (d\alpha(\xi_1, \xi_2), t\mu) = -\frac{t}{2} \langle [\xi_1, \xi_2], \mu \rangle.
\]

Thus the restriction of the form to \( t^1 \) is nondegenerate as long as \( t \neq 0 \). (For \( t = 0 \), we already knew that it was degenerate from the proof of Proposition 3.7.)

We construct a \( G^\text{ad} \)-equivariant almost complex structure \( J \) in a neighborhood of \( Z \) in \( \Phi_{tw}^{-1}(t^*) \), such that \( J \) is split with respect to the decomposition \((13)\), and compatible with \( \omega_{tw} \) “as much as possible.” More precisely, we choose \( G^\text{ad} \)-equivariant complex structures \( J_1, J_2, J_3 \) on each of the subbundles \( T(\mathcal{M}_{-1}(\Sigma')) \) and \( t \oplus t^* \) that are compatible with respect to the restriction of \( \omega_{tw} \) on the respective subbundle. We also choose a \( G^\text{ad} \)-equivariant complex structure \( J_2 \) on \( t^1 \) that is compatible with respect to the form \( \sigma \) given by

\[
\sigma(\xi_1, \xi_2) = -\langle [\xi_1, \xi_2], \mu \rangle.
\]

By Equation \((14)\), we have \( \omega_{tw} = t\sigma/2 \) on \( t^1 \); hence, \( J_2 \) is compatible with respect to \( \omega_{tw} \) away from \( t = 0 \). We then let \( J = J_1 \oplus J_2 \oplus J_3 \) be the almost complex structure on \( \Phi_{tw}^{-1}(t^*) \) near \( Z \).

Choose \( \epsilon \in (0, 1/8) \) sufficiently small, so that \( Z \times (-3\epsilon, 3\epsilon) \) is part of the local model for \( \Phi_{tw}^{-1}(t^*) \) described above. Pick a smooth function \( f : \mathbb{R} \to \mathbb{R} \) with the following properties:

- \( f(t) = t + \epsilon \) for \( t \) in a neighborhood of \( 0 \);
- \( f(t) = t \) for \(|t| \geq 2\epsilon \);
- \( f'(t) > 0 \) everywhere.

This induces a \( G^\text{ad} \)-equivariant self-diffeomorphism of the open subset \( Z \times (-3\epsilon, 3\epsilon) \subset \Phi_{tw}^{-1}(t^*) \), given by \((z, t) \to (z, f(t))\). Note that this diffeomorphism preserves \( J \), is the identity near the boundary, and takes \( Z \times (0, 2\epsilon) \) to \( Z \times [\epsilon, 2\epsilon] \).

Now let us look at the constructions above in light of the identification between \( \Phi_{tw}^{-1}(t^*) \) and \( \mathcal{M}(\Sigma') = \Phi^{-1}(t) \subset \mathcal{M}^{\theta}(\Sigma') \). We have obtained a local model \( Z \times (-3\epsilon, 3\epsilon) \) for the neighborhood \( N = \Phi^{-1}(-3\epsilon\mu, 3\epsilon\mu) \) of \( \Phi^{-1} \) in \( \mathcal{M}(\Sigma') \), an almost complex structure on \( N \), and a self-diffeomorphism of \( N \).

The symplectic cross-section theorem \([19]\) says that locally near \( \tilde{\Phi}^{-1}(1/2) \), the extended moduli space \( \mathcal{M}^{\theta}(\Sigma') \) looks like \( G \times_T \mathcal{M}(\Sigma') \). Thus, we can lift the local model for \( \mathcal{M}(\Sigma') \) and obtain a \( G^\text{ad} \)-equivariant local model \( (G \times_T Z) \times (-3\epsilon, 3\epsilon) \) for \( \mathcal{M}^{\theta}(\Sigma') \). Projection onto the second factor corresponds to the map \( 1/2 - \tilde{\Phi} \). Further, locally we can decompose the tangent bundle to \( \mathcal{M}^{\theta}(\Sigma') \) into three subbundles, cf. Equation \((7)\). The form \( \omega \) is nondegenerate when restricted to the direct sum \( T_{\mu}(O_\mu) \oplus T_{\mu}^*(O_\mu) \). Let us choose a \( G^\text{ad} \)-equivariant complex structure on this subbundle that is compatible with the restriction of \( \omega \) there. By combining it with \( J \), we obtain an equivariant almost complex structure \( \tilde{J} \) on

\[
\tilde{N} = \tilde{\Phi}^{-1}(1/2 - 3\epsilon, 1/2 + 3\epsilon) \subset \mathcal{M}^{\theta}(\Sigma').
\]

We can also lift the self-diffeomorphism of \( N \subset \mathcal{M}(\Sigma') \) to \( \tilde{N} = G \times_T N \) in an equivariant manner. Since this self-diffeomorphism is the identity near the boundary, we can extend it by the
identity to all of \( \mathcal{M}_s^g(\Sigma') \). The result is a \( G^{\text{ad}} \)-equivariant diffeomorphism

\[
\mathcal{M}_s^g(\Sigma') \rightarrow \mathcal{M}_s^g(\Sigma')
\]

that preserves \( \tilde{J} \) on \( \tilde{N} \), takes \( \tilde{\Phi}^{-1}(1/2) \) to \( \tilde{\Phi}^{-1}(1/2 - \epsilon) \), and is the identity on \( \tilde{\Phi}^{-1}((0, 1/2 - 2\epsilon)) \).

This diffeomorphism descends to one between the corresponding cut spaces:

\[
\phi : \mathcal{N}^c(\Sigma') = \mathcal{M}_s^g(\Sigma')_{\leq 1/2} \rightarrow \mathcal{M}_s^g(\Sigma')_{\leq 1/2-\epsilon}.
\]

We set

\[
\omega_\epsilon = \phi^*(\omega_0)_{\leq 1/2-\epsilon}.
\]

Note that \( \omega_0 \) and \( \omega_\epsilon \) coincide on the subset \( \tilde{\Phi}^{-1}((0, 1/2 - 2\epsilon)) \). Since we chose \( 2\epsilon < 1/4 \), the latter subset contains \( \mathcal{W} = \tilde{\Phi}^{-1}((0, 1/4)) \).

The almost complex structure \( \tilde{J} \) on \( \tilde{N} \) descend to the cut \( \tilde{N}_{\leq 1/2} \) as well. Indeed, if \( t \subset T\tilde{N} \) denotes the line bundle in the direction of the \( T^{\text{ad}} \)-action used for cutting, by construction we have \( \tilde{J}_t \cap T(\tilde{\Phi}^{-1}(1/2)) = 0 \). Since \( \tilde{J} \) is \( G^{\text{ad}} \)-invariant, it is easy to see that it induces an almost complex structure (still denoted \( \tilde{J} \)) on the cut \( \tilde{N}_{\leq 1/2} \). We extend \( \tilde{J} \) to \( \tilde{\Phi}^{-1}((0, 1/2 - 2\epsilon)) \) by choosing it to be compatible with \( \omega_0 = \omega_\epsilon \) there. The resulting \( \tilde{J} \) and \( \omega_\epsilon \) satisfy the required conditions (i)-(iv). \( \square \)

**Remark 4.13.** There were several choices made in the construction of \( \omega_\epsilon \) and \( \tilde{J} \) in Proposition 4.12: the connection \( \alpha \), the structures \( J_1, J_2, J_3 \), the function \( f \), etc. The space of all these choices is contractible.

## 5. Symplectic instanton homology

### 5.1. Lagrangians from handlebodies

Let \( H \) be a handlebody of genus \( h \geq 1 \) whose boundary is the compact Riemann surface \( \Sigma \). We view \( \Sigma' \) and \( \Sigma \) as subsets of \( H \), with \( \Sigma' = \Sigma \setminus D^2 \).

Let \( \mathcal{A}^g(\Sigma'|H) \subset \mathcal{A}^g(\Sigma') \) be the subspace of connections that extend to flat connections on the trivial \( G \)-bundle over \( H \). Consider also \( \mathcal{A}(H) \), the space of flat connections on \( H \), which is acted on by the based gauge group \( \mathcal{G}_0(H) = \{ f : H \rightarrow G| f(z) = I \} \). Since \( \pi_1(G) = 1 \) and \( \Sigma' \) has the homotopy type of a wedge of spheres, every map \( \Sigma' \rightarrow G \) must be nullhomotopic. This implies that \( \mathcal{G}^c(\Sigma') \) preserves \( \mathcal{A}^g(\Sigma'|H) \) and, furthermore, the natural map

\[
\mathcal{A}(H)/\mathcal{G}_0(H) \rightarrow \mathcal{A}^g(\Sigma'|H)/\mathcal{G}^c(\Sigma')
\]

is a diffeomorphism.

Set

\[
L(H) = \mathcal{A}(H)/\mathcal{G}_0(H) \cong \mathcal{A}^g(\Sigma'|H)/\mathcal{G}^c(\Sigma') \subset \mathcal{M}_s^g(\Sigma') = \mathcal{A}^g(\Sigma')/\mathcal{G}^c(\Sigma').
\]

The left hand side of (15) is the moduli space of flat connections on \( H \). After choosing a set of \( h \) simple closed curves \( \alpha_1, \ldots, \alpha_h \) on \( H \) whose classes generate \( \pi_1(H) \), the space \( \mathcal{A}(H)/\mathcal{G}(H) \) can be identified with the space of homomorphisms \( \pi_1(H) \rightarrow G \) or, alternatively, with the Cartesian product \( G^h \).

In fact, if the curves \( \alpha_1, \ldots, \alpha_h \) are the same as the ones chosen on \( \Sigma' \) for the identification (3), so that the remaining curves \( \beta_i \) are nullhomotopic in \( H \), then with respect to the identification (5) we have

\[
L(H) \cong \{(A_1, B_1, \ldots, A_h, B_h) \in G^{2h} \mid B_i = I, \ i = 1, \ldots, h\} \subset \mathcal{N}(\Sigma').
\]

Let us now view \( L(H) \) as \( \mathcal{A}^g(\Sigma'|H)/\mathcal{G}^c(\Sigma') \) via (15). Note that connections \( A \) that extend to \( H \) in particular extend to \( \Sigma \), which means that the value \( \theta \in \mathfrak{g} \) such that \( A|_S = \theta ds \) is zero. In other words, \( L(H) \) lies in \( \Phi^{-1}(0) \subset \mathcal{N}(\Sigma') \).

**Lemma 5.1.** With respect to the Huebschmann-Jeffrey symplectic form \( \omega \) from Section 3.3, \( L(H) \) is a Lagrangian submanifold of \( \mathcal{N}(\Sigma') \).
Proof. Let $\tilde{A}$ be a flat connection on $H$ and $A$ its restriction to $\Sigma'$. With respect to the description \([6]\) of $T_{[A]}N'(\Sigma')$, the tangent space to $L(H)$ at $A$ consists of equivalence classes of $d_A$-closed forms $a \in \Omega^1(\Sigma')$ which extend to $d_{\tilde{A}}$-closed forms $\tilde{a} \in \Omega^1(H) \otimes g$. Let $a, b$ be two such forms and $\tilde{a}, \tilde{b}$ their extensions to $H$. We have $a|_S = b|_S = 0$. Furthermore, by the Poincaré lemma for connections, on the disk $D^2$ which is the complement of $\Sigma'$ in $\Sigma$ there exists $\lambda \in \Omega^0(D^2; g)$ such that $d_{\tilde{A}}\lambda = \tilde{a}|_{D^2}$. By Stokes’ Theorem,

$$\int_{D^2} \langle a \wedge b \rangle = \int_S \langle \lambda \wedge b \rangle = 0.$$ 

Another application of Stokes’ Theorem gives

$$\int_{\Sigma'} \langle a \wedge b \rangle = \int_{\Sigma} \langle \tilde{a} \wedge \tilde{b} \rangle = \int_H \langle d_{\tilde{A}}(\tilde{a} \wedge \tilde{b}) \rangle = 0.$$ 

This shows that $\omega$ vanishes on the tangent space to $L(H) \cong G^h$, which is half-dimensional. $\square$

5.2. Symplectic instanton homology. Let $Y = H_0 \cup H_1$ be a Heegaard decomposition of a three-manifold $Y$, where $H_0$ and $H_1$ are handlebodies of genus $h$, with $\partial H_0 = -\partial H_1 = \Sigma$. Let $L_0 = L(H_0)$ and $L_1 = L(H_1) \subset N'(\Sigma')$ be the Lagrangians associated to $H_0$ resp. $H_1$, as in Section 5.1. View $N'(\Sigma')$ as an open subset of the compactified space $N'(\Sigma')$, as in Section 5.5, with $R$ being its complement.

In Section 4.5 we gave $N'(\Sigma')$ the structure of an $\epsilon$-symplectic manifold. By Corollary 4.11 its degeneracy locus is exactly $R$. Using the variant of Floer homology described in Section 2.3 and letting $\omega_0, \omega_\epsilon, \tilde{J}$ be as in Proposition 4.12 we define

$$HSI(\Sigma'; H_0, H_1) = HF(L_0, L_1, \tilde{J}, R).$$

In order to make sure the Floer homology is well-defined, we should check that Assumptions 2.1 (i)-(ix) are satisfied. Indeed, (i), (ii), (iii), (v), (vi), and (x) are subsumed in Proposition 4.12 (iv), (vii), and (ix) follow from Proposition 4.10 Lemma 5.1 and Corollary 3.6, respectively. For (viii), the Lagrangians are simply connected and spin because they are diffeomorphic to $G^h$. Corollary 4.6 also implies that the Floer groups admit a relative $\mathbb{Z}/8\mathbb{Z}$-grading.

A priori the Floer homology depends on $\tilde{J}$; compare Remark 2.14. However, the set of choices used in our construction of $\tilde{J}$ is contractible, cf. Remark 4.13. Therefore, the corresponding Floer homology groups are canonically isomorphic.

5.3. Dependence on the base point. Recall that the surface $\Sigma'$ is obtained from a closed surface $\Sigma$ by deleting a disk around some base point $z \in \Sigma$. Let $z_0, z_1 \in \Sigma$ be two choices of base point, and $\Sigma_0, \Sigma_1$ the corresponding surfaces with boundary. A choice of path $\gamma : [0, 1] \to \Sigma, j \mapsto z_j, j = 0, 1$ induces an identification of fundamental groups $\pi_1(\Sigma, z_0) \to \pi_1(\Sigma, z_1)$, and equivariant pre-symplectomorphisms $T_{\gamma} : N'(\Sigma'_0) \to N'(\Sigma'_1)$ preserving the cut locus $R$. The pullbacks of the form $\omega_\epsilon$ and the almost complex structure $\tilde{J}$ from Proposition 4.12 (applied to $N'(\Sigma'_1)$) can act as the corresponding form and almost complex structure in Proposition 4.12 applied to $N'(\Sigma'_0)$. Moreover, if $H_0, H_1$ are handlebodies, the symplectomorphism $T_{\gamma}$ preserves the corresponding Lagrangians $L_0, L_1$, since the vanishing holonomy condition is invariant under conjugation by paths. Therefore, the continuation arguments in Floer theory show that $T_{\gamma}$ induces an isomorphism

$$HSI(\Sigma'_0; H_0, H_1) \to HSI(\Sigma'_1; H_0, H_1).$$

This isomorphism depends only on the homotopy class of $\gamma$ relative to its endpoints. We conclude that the symplectic instanton homology groups naturally form a flat bundle over $\Sigma$. In particular, there is a natural action of $\pi_1(\Sigma, z_0)$ on $HSI(\Sigma'_0; H_0, H_1)$.

When we only care about the Floer homology group up to isomorphism (not canonical isomorphism), we drop the base point from the notation and write $HSI(\Sigma'; H_0, H_1) = HSI(\Sigma; H_0, H_1)$, as in the Introduction.
5.4. The Euler characteristic. In general, the Euler characteristic of Lagrangian Floer homology is the intersection number of the two Lagrangians. In our situation, the corresponding intersection number is computed (up to a sign) in [11 Proposition 1.1 (a), (b)]:

\begin{equation}
\chi(HSI(\Sigma'; H_0, H_1)) = |L_0| \cdot |L_1| = \begin{cases} 
\pm |H_1(Y; \mathbb{Z})| & \text{if } b_1(Y) = 0; \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

5.5. Examples.

**Proposition 5.2.** Let \( \mathcal{H}_h \) denote the Heegaard decomposition \( S^3 = H_0 \cup \Sigma H_1 \) of genus \( h \geq 1 \) such that there is a system of \( 2h \) curves \( \alpha_1, \beta_1 \) on \( \Sigma' \) as in Section 3.2 with the property that the \( \beta_i \)'s are nullhomotopic in \( H_0 \) and the \( \alpha_i \)'s are nullhomotopic in \( H_1 \). Then \( HSI(\mathcal{H}_h) \cong \mathbb{Z} \).

**Proof.** With respect to the identification \( \mathcal{H} \), the Lagrangians corresponding to \( H_0 \) and \( H_1 \) are given by

\[ L_0 = \{(A_1, B_1, \ldots, A_h, B_h) \in G^{2h} | B_i = I, \ i = 1, \ldots, h\}, \]

\[ L_1 = \{(A_1, B_1, \ldots, A_h, B_h) \in G^{2h} | A_i = I, \ i = 1, \ldots, h\}. \]

These have exactly one intersection point, the reducible \( A_i = B_i = I \). Clearly \( L_0 \) and \( L_1 \) intersect transversely in \( \mathcal{N}(\Sigma') \subset G^{2h} \) at that point. It is somewhat counterintuitive that \( L_0 \) and \( L_1 \) can intersect transversely at \( I \), because they both live in the subspace \( \Phi^{-1}(0) \) of codimension three in \( \mathcal{N}(\Sigma') \). However, that subspace is not smooth, so there is no contradiction. We conclude that the Floer chain group has one generator; hence so does the homology.

**Proposition 5.3.** Let \( \mathcal{H}_h' \) be the Heegaard splitting of genus \( h \geq 1 \) for the connected sum \( \#^h(S^1 \times S^2) \), such that \( L_0 = L_1 \). Then \( HSI_*(\mathcal{H}_h'; \mathbb{Z}/2\mathbb{Z}) \cong (H_*(S^3; \mathbb{Z}/2\mathbb{Z}))^{\otimes h} \), where the grading of the latter vector space is collapsed mod 8.

**Proof.** Since \( L_0 = L_1 \cong G^h \cong (S^3)^h \), the cohomology ring of \( L_0 \) is generated by its degree \( d = 3 \) part. Under the monotonicity assumptions which are satisfied in our setting, Oh [37] constructed a spectral sequence whose \( E^1 \) term is \( H_*(L_0; \mathbb{Z}/2\mathbb{Z}) \) and which converges to \( HF_*(L_0, L_0; \mathbb{Z}/2\mathbb{Z}) \). This sequence is multiplicative by the results of Buhovski [10] and Biran-Cornea [7, 8]. A consequence of multiplicity is that the spectral sequence collapses at the \( E_1 \) stage provided that \( N_{L_i} > d + 1 \), cf. [8 Theorem 1.2.2]. This is satisfied in our case because \( N_{L_0} = N \geq 8 \). Hence \( HF_*(L_0, L_0; \mathbb{Z}/2\mathbb{Z}) \cong H_*(G^h; \mathbb{Z}/2\mathbb{Z}) \).

Note that the results of Oh, Buhovski and Biran-Cornea were originally formulated for monotone symplectic manifolds, i.e. in the setting of Section 2.1. However, they also apply (with minor modifications) to the Floer homology groups defined in Section 2.3.

**Proposition 5.4.** Denote by \( \mathcal{H}(p, q) \) the genus one Heegaard splitting of the lens space \( L(p, q) \), where g. c. d. \((p, q) = 1 \). Then \( HSI(\mathcal{H}(p, q)) \) is free abelian of rank \( p \).

**Proof.** In terms of the coordinates \( A = A_1 \) and \( B = B_1 \), the two Lagrangians are given by \( L_0 = \{B = 1\} \) and \( L_1 = \{APB^{-q} = 1\} \). Their intersection consists of the space of representations \( \pi_1(L(p, q)) \cong \mathbb{Z}/p \to SU(2) \), which has several components: when \( p \) is odd, there is the reducible point \( (A = B = I) \) and \((p - 1)/2 \) copies of \( S^2 \); when \( p \) is even, there are two reducibles \( (A = B = I \) and \( A = -I, B = I) \) and \((p - 2)/2 \) copies of \( S^2 \). It is straightforward to check that each component is a clean intersection in the sense of Poźniak [12]. Therefore, there exists a spectral sequence that starts at \( H_*(L_0 \cap L_1) \cong \mathbb{Z}^p \) and converges to \( HF(L_0, L_1) \), cf. [12]. Since the Euler characteristic of \( HF(L_0, L_1) \) is \( p \) by Equation (17), the sequence must collapse at the first stage.

**Remark 5.5.** More generally, whenever we have a Heegaard decomposition \( \mathcal{H} \) of a three-manifold \( Y \) with \( H^1(Y) = 0 \), the two Lagrangians \( L_0 \) and \( L_1 \) will intersect transversely at the reducible \( I \),

\[ \text{denote by } \mathcal{H}(p, q) \text{ the genus one Heegaard splitting of the lens space } L(p, q) \text{, where } g. c. d. (p, q) = 1 \text{. Then } HSI(\mathcal{H}(p, q)) \text{ is free abelian of rank } p. \]

**Proof.** In terms of the coordinates \( A = A_1 \) and \( B = B_1 \), the two Lagrangians are given by \( L_0 = \{B = 1\} \) and \( L_1 = \{APB^{-q} = 1\} \). Their intersection consists of the space of representations \( \pi_1(L(p, q)) \cong \mathbb{Z}/p \to SU(2) \), which has several components: when \( p \) is odd, there is the reducible point \( (A = B = I) \) and \((p - 1)/2 \) copies of \( S^2 \); when \( p \) is even, there are two reducibles \( (A = B = I \) and \( A = -I, B = I) \) and \((p - 2)/2 \) copies of \( S^2 \). It is straightforward to check that each component is a clean intersection in the sense of Poźniak [12]. Therefore, there exists a spectral sequence that starts at \( H_*(L_0 \cap L_1) \cong \mathbb{Z}^p \) and converges to \( HF(L_0, L_1) \), cf. [12]. Since the Euler characteristic of \( HF(L_0, L_1) \) is \( p \) by Equation (17), the sequence must collapse at the first stage.

**Remark 5.5.** More generally, whenever we have a Heegaard decomposition \( \mathcal{H} \) of a three-manifold \( Y \) with \( H^1(Y) = 0 \), the two Lagrangians \( L_0 \) and \( L_1 \) will intersect transversely at the reducible \( I \),
cf. [1] Proposition 1.1(c)]. We could then fix an absolute \( \mathbb{Z}/2\mathbb{Z} \)-grading on \( HSI(\mathcal{H}) \) by requiring that the \( \mathbb{Z} \) summand corresponding to \( I \) lies in grading zero.

6. Further remarks

6.1. Invariance. We sketch here a potential strategy for proving that the groups \( HSI(Y'; H_0, H_1) \) are invariants of the 3-manifold \( Y = H_0 \cup H_1 \). Unfortunately, certain technical results needed to carry through this program are lacking at the moment.

The strategy is based on the theory of Lagrangian correspondences in Floer theory, cf. [54]. We start by reviewing this theory. Let \( M_0, M_1 \) be compact symplectic manifolds. A Lagrangian correspondence from \( M_0 \) to \( M_1 \) is a Lagrangian submanifold \( L_{01} \subset M_0^- \times M_1 \). (The minus superscript means considering the same manifold equipped with the negative of the given symplectic form.) Given Lagrangian correspondences \( L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2 \), their composition is the subset of \( M_0^- \times M_2 \) defined by

\[
L_{01} \circ L_{12} = \pi_{02}(L_{01} \times_{M_1} L_{12})
\]

where \( \pi_{02} : M_0^- \times M_1 \times M_1^- \times M_2 \rightarrow M_0^- \times M_2 \) is the projection. If the intersection

\[
L_{01} \times_{M_1} L_{12} = (L_{01} \times L_{12}) \cap (M_0^- \times \Delta_{M_1} \times M_2)
\]

is transverse (hence smooth) in \( M_0^- \times M_1 \times M_1^- \times M_2 \), and the projection \( \pi_{02} : L_{01} \times_{M_1} L_{12} \rightarrow L_{01} \circ L_{12} \) is embedded, we say that the composition \( L_{02} = L_{01} \circ L_{12} \) is embedded. An embedded composition \( L_{02} \) is a smooth Lagrangian correspondence from \( M_0 \) to \( M_2 \).

Suppose now that \( M_0, M_1, M_2 \) are compact symplectic manifolds, monotone with the same monotonicity constant, and minimal Chern number at least 2. Suppose that \( L_0 \subset M_0, L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2 \), \( L_2 \subset M_2 \) are simply connected Lagrangian submanifolds. (This implies that their minimal Maslov numbers are at least 4.) The main theorem of [54] states that

**Theorem 6.1.** With \( M_0, M_1, M_2, L_0, L_{01}, L_{12}, L_2 \) monotone as above, there exists a canonical isomorphism of Lagrangian Floer homology groups

\[
HF(L_0 \times L_{12}, L_{01} \times L_2) \rightarrow HF(L_0 \times L_2, L_{01} \circ L_{12}).
\]

The isomorphism is defined using pseudo-holomorphic quilts, i.e., in this case, triples of strips in \( M_0, M_1, M_2 \) with boundary conditions in \( L_0, L_{01}, L_{12} \) and \( L_2 \). The count of such quilts is used in the left hand side of (18). In the limit when the width \( \delta \) of the middle strip goes to 0, the same count produces the right hand side.

Theorem 6.1 admits the following straightforward extension to the case of the relative Floer homology groups defined in Section 2.2. Suppose that \( R_0, R_1 \) are symplectic hypersurfaces in \( M_0, M_1 \). From them we obtain two hypersurfaces \( \tilde{R}_0 = R_0^- \times M_1, \tilde{R}_1 = M_0^- \times R_1 \) in \( M_0^- \times M_1 \). We have the following analog of Definition 2.3.

**Definition 6.2.** A simply connected Lagrangian correspondence \( L_{01} \subset M_0^- \times M_1 \) is called compatible with the pair \((R_0, R_1)\) if it is compatible with \( \tilde{R}_0 \cup \tilde{R}_1 \), that is, \( P.D.([\tilde{R}_0]) + P.D.([\tilde{R}_1]) \) has trivial restriction to \( H^2(L_{01}; \mathbb{Z}) \).

**Example 6.3.** Suppose \( \iota : C \rightarrow M_1 \) is a fibered coisotropic submanifold of \( M_1 \), with the fibration being \( \pi : C \rightarrow M_0 \). Then \( (\pi \times \iota) : C \rightarrow M_0^- \times M_1 \) defines a Lagrangian correspondence, compare [54] Example 2.0.3(b)]. If \( R_0 \subset M_0, R_1 \subset M_1 \) are symplectic hypersurfaces such that \( R_1 \) intersects \( C \) transversely and \( R_1 \cap C = \pi^{-1}(R_0) \), then \( C \) is compatible with \((R_0, R_1)\). Indeed, the restrictions of \( -P.D.([\tilde{R}_0]) \) and \( P.D.([\tilde{R}_1]) \) to \( H^2(C; \mathbb{Z}) \) are both equal to the Poincaré dual of \( R_1 \cap C \) in \( C \).

Given two correspondences \( L_{01}, L'_{01} \subset M_0^- \times M_1 \) compatible with \((R_0, R_1)\), and a Floer trajectory \( u_{01} : \mathbb{R} \times [0, 1] \rightarrow (M_0^- \times M_1, L_{01}, L'_{01}) \), the intersection number

\[
u_{01} \cdot (\tilde{R}_0 \cup \tilde{R}_1)
is well-defined and non-negative. We can then state the analog of Theorem 6.1 for relative Floer homology:

**Theorem 6.4.** Suppose that $R_0, R_1, R_2$ are symplectic hypersurfaces in $M_0, M_1, M_2$ and we have simply connected, monotone Lagrangians $L_0 \subset M_0, L_0 \subset M_0, L_1 \subset M_1, L_2 \subset M_2$, all compatible with the respective (pairs of) hypersurfaces, i.e. $L_0$ is compatible with $R_0$; $L_0$ is compatible with $(R_0, R_1)$, etc. We also assume that $L_0 \circ L_1$ is an embedded composition and is compatible with $(R_0, R_2)$. Set

$$R_{012} = (R_0 \times M_1^- \times M_2) \cup (M_0 \times R_1^- \times M_2) \cup (M_0 \times M_1^- \times R_2) \subset M_0 \times M_1^- \times M_2$$

and

$$R_{02} = (R_0^- \times M_2) \cup (M_0^- \times R_2) \subset M_0^- \times M_2.$$

Then, there exists a canonical isomorphism of relative Lagrangian Floer homology groups

$$HF(L_0 \times L_{12}, L_0 \times L_2; R_{012}) \to HF(L_0 \times L_2, L_0 \circ L_{12}; R_{02}).$$

The proof is similar to the one of Theorem 6.1 in [51], but keeping track of the intersection numbers. Since those numbers are homotopy invariants, they do not change when taking the limit $\delta \to 0$. Further, for $\delta$ small, the intersection of the middle strip in a quilt with $(M_0 \times R_1^- \times M_2)$ must be zero. Therefore, in the limit we only have contributions from $R_0$ and $R_2$.

Going back to topology, let $\Sigma_0, \Sigma_1$ be Riemann surfaces of genus $h$, resp. $h + 1$. Let $H_{01}$ be a compression body with boundary $\Sigma_0^- \times \Sigma_1$, that is, a cobordism consisting of attaching a single handle of index one. Associated to $H_{01}$ we have a Lagrangian correspondence

$$L_{01} \subset \mathcal{N}(\Sigma_0^-) \times \mathcal{N}(\Sigma_1)$$

defined as follows. Suppose that $\gamma$ is a path from the base points $z_0$ to $z_1$, equipped with a framing of the normal bundle. Let $H_{01}'$ denote the non-compact surface obtained from $H_{01}$ by removing a regular neighborhood of $\gamma$. The boundary of $H_{01}'$ then consists of $\Sigma_0', \Sigma_1'$ and a cylinder $S \times [0, 1]$. Let $\mathcal{N}(H_{01}')$ denote the moduli space of flat connections on $H_{01}$ of the form $\theta ds$ near $S \times [0, 1]$ (where $s$ is the coordinate on the circle $S$), for some $\theta \in \mathfrak{g}$, modulo gauge transformations equal to the identity in a neighborhood of $S \times [0, 1]$. The same arguments as in the proof of Lemma 5.1 show that $L_{01}$ is a Lagrangian correspondence.

The Lagrangian correspondence $L_{01}$ has the following explicit description in terms of holonomies, similar to [51] and [16]. Suppose that $H_{01}$ consists of attaching a one-handle whose meridian is the generator $B_{h+1}$ of $\pi_1(\Sigma_1)$. Then:

**Lemma 6.5.** The Lagrangian correspondence $L_{01}$ is given by

$$L_{01} = \{(A_1, \ldots, B_h) \in \mathcal{N}(\Sigma_0'), (A_1, \ldots, B_h, A_{h+1}, B_{h+1}) \in \mathcal{N}(\Sigma_1') \mid B_{h+1} = I\}.$$

**Proof.** $H_{01}'$ has the homotopy type of the wedge product of $\Sigma_0'$ with a circle, corresponding to a single additional generator $a_{h+1}$. Thus $\pi_1(H_{01}')$ is freely generated by $(a_1, \ldots, b_h, a_{h+1})$, and the lemma follows. □

Recall from Section 4.3 that $\mathcal{N}(\Sigma_0')$ admits a compactification $\mathcal{N}^c(\Sigma_0') = \mathcal{N}(\Sigma_0') \cup R_0$. We equip $\mathcal{N}^c(\Sigma')$ with the (non-monotone) symplectic form constructed in Proposition 4.12, which we denote by $\omega_{0,0}$. Then $R_0$ is a symplectic hypersurface. Similarly, we have a symplectic form $\omega_{0,1}$ on $\mathcal{N}^c(\Sigma_1') = \mathcal{N}(\Sigma_1') \cup R_1$. Let $L_{01}'$ denote the closure of $L_{01}$ in the compactification $\mathcal{N}^c(\Sigma_0') \times \mathcal{N}^c(\Sigma_1')$.

**Proposition 6.6.** $L_{01}'$ is a smooth Lagrangian correspondence from $\mathcal{N}^c(\Sigma_0')$ to $\mathcal{N}^c(\Sigma_1')$. 

Proof. The space $\mathcal{N}(\Sigma_0')^c \times \mathcal{N}(\Sigma_1')^c$ is symplectomorphic to $\mathcal{N}(\Sigma_0')_{\leq 1/2-\epsilon} \times \mathcal{N}(\Sigma_0')_{\leq 1/2-\epsilon}$, the latter endowed with the product of the natural symplectic forms on the cuts. Under this symplectomorphism, $L_{01}$ corresponds to $L_{01} \cap \tilde{\Phi}^{-1}([0, 1/2 - \epsilon])$.

Consider $L_{01}$ as a fiber bundle over $\mathcal{N}(\Sigma_0')$, invariant under the action of the one-parameter subgroups used for cutting at $1/2 - \epsilon$. The space obtained by applying symplectic cutting to $L_{01}$ at $1/2 - \epsilon$ is smooth. It contains $L_{01} \cap \tilde{\Phi}^{-1}([0, 1/2 - \epsilon])$ as an open dense subset, and therefore is equal to its compactification $L_{01}'$.

Lemma 6.7. The Lagrangian correspondence $L_{01}'$ is compatible with the pair $(R_0, R_1)$.

Proof. View $L_{01}'$ as a coisotropic submanifold of $\mathcal{N}(\Sigma_1')$, fibered over $\mathcal{N}(\Sigma_0')$ with fiber $G$. We are then exactly in the setting of Example 6.3. □

Lemma 6.8. Let $L_0 \subset \mathcal{N}(\Sigma_0')$, resp. $L_1 \subset \mathcal{N}(\Sigma_1')$, be the Lagrangian for the handlebody given by contracting the cycles $b_1, \ldots, b_h$, resp. $b_1, \ldots, b_{h+1}$. Then the composition $L_0 \circ L_{01}'$ is embedded, and equals $L_1$.

Proof. Immediate from Lemma 6.5 and the fact that $L_0$ does not meet the hypersurface $R_0$. □

Lemma 6.9. Let $L_{01}' \subset \mathcal{N}(\Sigma_0')^c \times \mathcal{N}(\Sigma_1')^c$ be the Lagrangian correspondence for attaching a handle corresponding to adding the cycle $a_{h+1}$, and $L_{10}' \subset \mathcal{N}(\Sigma_1')^c - \mathcal{N}(\Sigma_0')$ the Lagrangian correspondence corresponding to contracting the cycle $b_{h+1}$. Then the composition $L_{01}' \circ L_{10}'$ is embedded, and equals the diagonal $\Delta_0 \subset \mathcal{N}(\Sigma_0')^c \times \mathcal{N}(\Sigma_0')^c$.

Proof. Immediate from Lemma 6.5. □

Now consider the problem of showing that the Floer homology groups $HSI(\Sigma'; H_0, H_1) = HF(L_0, L_1; R)$ are independent of the choice of Heegard splitting of the 3-manifold $Y$.

By the Reidemeister-Singer theorem ([14], [49]), any two Heegaard splittings $Y = H_0 \cup_{\Sigma_0} H_1$, $Y = H'_0 \cup_{\Sigma_1} H'_1$, are related by a sequence of stabilizations and de-stabilizations. Therefore it suffices to consider the case that $H'_0, H'_1$ are obtained from $H_0, H_1$ by stabilization. That is, $H'_0 = H_0 \cup_{\Sigma_0} H_{01}, \quad H'_1 = H_1 \cup_{\Sigma_0} (-H_{10})$

where $H_{01}, H_{10}$ are the compression bodies corresponding to adding the cycle $a_{h+1}$, resp. contracting $b_{h+1}$. If the symplectic manifolds $\mathcal{N}(\Sigma_0'), \mathcal{N}(\Sigma_1')$ had been monotone, we could have defined usual Floer homology groups $HF(L_0, L_1)$ as in Section 2.1. Then, after three applications of Theorem 6.1 and taking into account Lemmas 6.8, 6.9 we would have found that

$$HF(L_0, L_1) \cong HF(L_0 \times L_1, \Delta_0) = HF(L_0 \times L_1, L_0^c \circ L_1^c) \cong HF(L_0 \times L_{10}, L_{10}^c \circ L_1) \cong HF(L_0 \circ L_{10}^c, L_1).$$

By Proposition 6.7 and Theorem 6.4, the same would have also worked for the versions relative to the hypersurfaces $R_0, R_1$; that is,

$$HF(L_0, L_1; R_0) \cong HF(L'_0, L'_1; R_1).$$

However, the Floer homology used in the definition of $HSI(\Sigma_0'; H_0, H_1)$ is the semipositive version from Section 2.3. At present, an analog of Theorems 6.1, 6.3 in the semipositive case is missing. The main difficulty consists in controlling the bubbling phenomenon, in the limit when the width $\delta$ of the middle strip in a pseudo-holomorphic quilt goes to zero. One can rule out disk and sphere bubbling as in the proof of Proposition 2.12, but not the figure eight bubbles mentioned in 5.4.
Section 5.3]. Indeed, removal of singularities, transversality, and Fredholm theory for figure eight bubbles have not yet been developed.

6.2. Comparison with other approaches. Let \( Y = H_0 \cup_\Sigma H_1 \) be a Heegaard splitting of a 3-manifold, with \( \Sigma \) of genus \( h \). Recall that the Lagrangians \( L_0 = L(H_0) \) and \( L_1 = L(H_1) \) live inside the subspace

\[
\Phi^{-1}(0) = \{(A_1, B_1, \ldots, A_h, B_h) \in G^{2h} \mid \prod_{i=1}^{h} [A_i, B_i] = I\} \subset \mathcal{N}(\Sigma').
\]

There is an alternative way of embedding \( \Phi^{-1}(0) \) inside a symplectic manifold of dimension \( 6h \). Namely, let \( \Sigma_+ \) be the closed surface (of genus \( h + 1 \)) obtained by gluing a copy of \( T^2 \setminus D^2 \) onto the boundary of \( \Sigma' = \Sigma \setminus D^2 \). Consider the moduli space \( \mathcal{M}_{tw}(\Sigma_+) \) of projectively flat connections (with fixed central curvature) in an odd-degree \( U(2) \)-bundle over \( \Sigma_+ \), as in Section 3.6.

\[
\mathcal{M}_{tw}(\Sigma_+) = \left\{ (A_1, B_1, \ldots, A_{h+1}, B_{h+1}) \in G^{2h+2} \mid \prod_{i=1}^{h+1} [A_i, B_i] = -I \right\} / G.
\]

Pick two particular matrices \( X, Y \in G \) with the property that \([X, Y] = -I\). Then we can embed \( \Phi^{-1}(0) \) into \( \mathcal{M}_{tw}(\Sigma_+) \) by the map

\[
(A_1, B_1, \ldots, A_h, B_h) \rightarrow [(A_1, B_1, \ldots, A_h, B_h, X, Y)].
\]

With respect to the natural symplectic form on \( \mathcal{M}_{tw}(\Sigma_+) \), the spaces \( L_0, L_1 \subset \Phi^{-1}(0) \) are still Lagrangians. One can take their Floer homology, and obtain a \( \mathbb{Z}/4\mathbb{Z} \)-graded abelian group. This is studied in [53, Section 4], where it is shown that it is a 3-manifold invariant. This invariant is called the torus-compactified Lagrangian Floer homology of \( Y \) and denoted \( HF^{tc}(Y) \). It is not obvious how \( HF^{tc} \) relates to \( HSI \).

The advantage of using \( \mathcal{M}_{tw}(\Sigma_+) \) instead of \( \mathcal{N}(\Sigma') \) is that the former is already compact (and monotone); therefore, the definition of Floer homology is less technical and this allows one to prove invariance. Nevertheless, the construction presented in this paper (using \( \mathcal{N}(\Sigma') \)) has certain advantages as well: first, the resulting groups are \( \mathbb{Z}/8\mathbb{Z} \)-graded rather than \( \mathbb{Z}/4\mathbb{Z} \)-graded. Second, it is better suited for defining an equivariant version of symplectic instanton homology. Indeed, unlike \( \mathcal{M}_{tw}(\Sigma_+) \), the space \( \mathcal{N}(\Sigma') \) comes with a natural action of \( G \) that preserves the symplectic form and the Lagrangians. Following the ideas of Viterbo from [51], [50], we expect that one should be able to use this action to define equivariant Floer groups \( HSI^G(Y) \) in the form of \( H^*(BG) \)-modules. For integral homology spheres, a suitable Atiyah-Floer Conjecture would relate these to the equivariant instanton homology of Austin and Braam [5].

In a different direction, it would be interesting to study the connection between our construction and the Heegaard Floer homology groups \( \widehat{HF}, HF^+ \) of Ozsváth and Szabó [41], [42]. In particular, we ask the following:

**Question 6.10.** For an arbitrary 3-manifold \( Y \) with Heegaard splitting \( (\Sigma; H_0, H_1) \), are the total ranks of \( HSI(\Sigma; H_0, H_1) \otimes \mathbb{Q} \) and \( \widehat{HF}(Y) \otimes \mathbb{Q} \) equal?

Finally, we remark that Jacobsson and Rubinsztein [23] have recently described a construction similar to the one in this paper, but for the case of knots in \( S^3 \) rather than 3-manifolds. Given a representation of a knot as a braid closure, they define two Lagrangians inside a certain symplectic manifold; this manifold was first constructed in [20] and is a version of the extended moduli space. Conjecturally, one should be able to take the Floer homology of the two Lagrangians and obtain a knot invariant.
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