ZIPPING TATE RESOLUTIONS AND EXTERIOR COALGEBRAS

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Abstract. We conjecture what the cone of hypercohomology tables of bounded complexes of coherent sheaves on projective spaces are, when we have specified regularity conditions on the cohomology sheaves of this complex and its dual.

There is an injection from this cone into the cone of homological data sets of squarefree modules over a polynomial ring $k[x_1, \ldots, x_n]$, and we conjecture that this is an isomorphism: The Tate resolutions of a complex of coherent sheaves on projective space $\mathbb{P}(W)$, and the exterior coalgebra on $\langle x_1, \ldots, x_n \rangle$ may be amalgamated together to form a complex of free $\text{Sym}(\oplus_i x_i \otimes W^*)$-modules, a procedure introduced by Cox and Materov. Via a reduction $\oplus_i x_i \otimes W^* \rightarrow \oplus_i x_i \otimes k$ we get a complex of free modules over $k[x_1, \ldots, x_n]$.

The extremal rays in the cone of squarefree complexes are conjecturally given by triplets of pure free squarefree complexes introduced in [15]. We describe the corresponding classes of hypercohomology tables, a class which generalizes vector bundles with supernatural cohomology.

We also show how various pure resolutions in the literature, like resolutions of modules supported on determinantal varieties, and tensor complexes, may be obtained by the first part of the procedure.

Introduction

What are the possible cohomology tables of coherent sheaves on projective spaces, or more generally hypercohomology tables of bounded complexes of coherent sheaves? And what are the possible values of the homological invariants of complexes of graded modules over the polynomial rings: Graded Betti numbers and Hilbert functions of their homology and cohomology modules? These questions have made considerable advances initiated by the conjectures of M.Boij and J.Söderberg [2], and their subsequent settling by D.Eisenbud and F.-O.Schreyer in [10]. In particular the latter achieved the complete classification of all cohomology tables of vector bundles on projective spaces, up to scalar multiple. Further advances of notice occurred in [11] where they gave a decomposition of cohomology tables of coherent sheaves, but in a non-algorithmic way since it involved an infinite number of steps. In [3] Boij and Söderberg classified all Betti diagrams of graded modules, and in [7] D.Eisenbud and D.Erman classify Betti diagrams with conditions on the codimension of the homology of the complexes. For an introduction and survey of this area see [14].

Cone of hypercohomology tables of coherent sheaves. A Betti diagram of a graded module is specified by a finite table. A cohomology table of a coherent sheaf or a bounded complex of coherent sheaves $F^\bullet$ on a projective space $\mathbb{P}(W)$ is however an infinite table. To specify the cohomology table, whose values are

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\[ \dim_k \mathbb{H}^i(\mathbb{P}(W), \mathcal{F}^\bullet(j)), \text{ one notes that its numerical linear strands are eventually} \]
\[ \text{given by the Hilbert polynomials of the cohomology sheaves of the complex when} \]
\[ j \gg 0 \text{ and of its derived dual when} j \ll 0. \] 
Thus one would specify \( l \leq r \) such that for twists \( l < j < r \) one gives the finite set of cohomological dimensions, while for \( j \geq r \) the cohomology table is given by the Hilbert polynomial of the cohomology sheaf \( H^i(\mathcal{F}^\bullet) \) in degrees \( \geq r \), and for \( j \leq l \) it is given by the Hilbert polynomial of the cohomology sheaf \( H^{-i}((\mathcal{F}^\bullet)^\vee) \) in degrees \( \geq -l \).

By twisting the complex we may as well assume that \( r = 1 \) and now write \( l = -n - 1 \). We give a precise conjecture as to what the cone of such hypercohomology tables are, with the purely numerical conditions on the cohomology sheaves \( H^i(\mathcal{F}^\bullet) \) and \( H^{-i}((\mathcal{F}^\bullet)^\vee) \) replaced by the natural algebraic conditions that these sheaves are respectively 1-regular and \( n + 1 \)-regular. (These regularity conditions implies that the hypercohomology tables of \( \mathcal{F}^\bullet \) are given by the Hilbert polynomials of the cohomology sheaves in these ranges.)

Over the polynomial ring \( k[x_1, \ldots, x_n] \) there is the category of \( \mathbb{N}^n \)-graded squarefree modules, introduced in [23]. A complex of such modules comes with three homological data, its graded Betti numbers \( B \), the Hilbert functions \( H \) of its homology sheaves, and the Hilbert functions \( C \) of the homology sheaves of the dual complex. We show that there is an injection from the cone of hypercohomology tables to the cone of homological data sets \( (B, H, C) \), and conjecture that this is an isomorphism, Conjecture 7.8. This conjecture is a consequence of two conjectures on what are the extremal rays in these cones, thereby providing the precise description of the cones. In Conjecture 7.9 we conjecture that the extremal rays of homological data sets are generated by the data sets that come from triplets of pure free squarefree complexes. In Conjecture 5.7 we propose a corresponding description of the extremal rays on the hypercohomology table side.

**Triplets of pure free squarefree modules.** In a recent paper [15] we introduced the notion of triplets of pure free squarefree complexes. A complex \( \mathcal{F}^\bullet \) of free modules in the squarefree module category is said to be pure if its terms \( F_i = S(-d_i)^{\beta_i} \) are generated in a single degree \( d_i \) when considered as \( \mathbb{Z} \)-graded modules. The sequence \( (d_0, d_1, \ldots, d_r) \) is its degree sequence. On the category of free squarefree modules there are two duality functors, standard duality \( D \) and Alexander duality \( A \). The composition functor \( A \circ D \) has order three up to translation of complexes. (It is the Auslander-Reiten translate on the derived category of squarefree modules.) If all three complexes

\[ \mathcal{F}^\bullet, \quad A \circ D(\mathcal{F}^\bullet), \quad (A \circ D)^2(\mathcal{F}^\bullet) \]

are pure, we say it is a triplet of pure free squarefree complexes. The degree sequences of these three complexes is called a degree triplet.

In [15] we conjectured the existence of such triplets of complexes for every degree triplet fulfilling certain natural necessary conditions. We showed the existence of such triplets of complexes provided two of the degree sequences were intervals. This was done using the tensor complexes introduced by C.Berkesch et. al. in [25]. These are complexes over a polynomial ring \( \text{Sym}(V \otimes W^*) \) where \( V = (x_1, \ldots, x_n) \). By taking a suitable quotient \( V \otimes W^* \rightarrow V \) equivariant for the diagonal matrices in \( GL(V) \), we obtain a free squarefree complex over \( \text{Sym}(V) \) giving rise to such a triplet.
This suggests that there may be wider classes of complexes of Sym($V \otimes W^*$)-modules which by the same procedure will give more, possibly all of the triplets of pure free squarefree complexes whose existence were conjectured in [15]. Here we introduce such a class, but only as a secondary class associated, via a procedure we call zipping, to a primary class of complexes of coherent sheaves on a projective space $\mathbb{P}(W)$.

The procedure of zipping. This was introduced by D.Cox and E.Materov in a recent paper [5]. By [8] a coherent sheaf $\mathcal{F}$ on $\mathbb{P}(W)$, or more generally a complex of coherent sheaves $\mathcal{F}^\bullet$, [13], [4] corresponds to a Tate resolution $\mathcal{T}(\mathcal{F}^\bullet)$ over the exterior algebra $E(W^*) = \oplus \wedge^i W^*$. Following [5] such a Tate resolution and the exterior coalgebra on a vector space $V$ may be amalgamated together to form a complex of free $\text{Sym}(V \otimes W^*)$-modules, the zip complex. They show in the case that $\mathcal{F}^\bullet = \mathcal{F}$ a coherent sheaf, $\text{char}(k) = 0$, and $\dim_k V \leq \dim_k W - 1$ that this corresponds exactly to the complex obtained by the method of Lascoux for constructing resolutions of sheaves supported on determinantal varieties. This method starts with a coherent sheaf $\mathcal{F}$ on $\mathbb{P}(W)$ usually a vector bundle, and via a pullback and pushdown procedure gives a complex of free $\text{Sym}(V \otimes W^*)$-modules. Here we get rid of the above restrictions in [5] and show that the result holds in arbitrary characteristic, for complexes of coherent sheaves, and most importantly, without the bound on $\dim_k V$. As a consequence we can show how various old and recent resolutions in the literature, the Eagon-Northcott complex, Buchsbaum-Rim and Buchsbaum-Eisenbud complexes [6, A2.6], pure resolutions of modules supported on determinantal varieties [9], and tensor complexes [25], may be obtained by the simple procedure of zipping the Tate resolutions of various vector bundles with supernatural cohomology on $\mathbb{P}(W)$, or more generally locally Cohen-Macaulay sheaves, with an exterior coalgebra on $V$.

Complexes of coherent sheaves associated to homology triplets. Our main objective in this paper is however to introduce the said primary class of complexes of coherent sheaves on projective spaces. To each degree triplet $T$, or an equivalent but slight variation thereof $T'$, called a homology triplet, we conjecture the existence of a complex of coherent sheaves with specified properties determined by $T'$. The coefficients of the Hilbert polynomial of this complex fulfills a number of equations which is one less than the number of unknown coefficients. Hence we expect the Hilbert polynomial to be unique up to scalar multiple. Our conditions on these complexes are so strong that we show that each entry of the hypercohomology table of the complex of coherent sheaves is uniquely determined by this Hilbert polynomial. But also additional cohomological properties of the individual cohomology sheaves of the complex are determined.

When such a complex exists for a given homology triplet $T'$, we show that by zipping its Tate resolution with the exterior coalgebra on a vector space $V = \langle x_1, \ldots, x_n \rangle$ to get a complex of free $\text{Sym}(V \otimes W^*)$-modules, and thereafter taking a suitable quotient map $V \otimes W^* \to V$ to get a complex of free $\text{Sym}(V)$-modules, we get a complex of free squarefree modules giving rise to a triplet of pure free squarefree complexes whose degree sequence is $T$. This shows, Theorem 6.8, that the conjecture in this paper implies the conjecture in [15].

The organization of the paper is as follows.
Section 1 recalls the method of Lascoux [20] as presented in Weyman [22], associating to a coherent sheaf $\mathcal{F}$ on $\mathbb{P}(W)$ a complex of free $\text{Sym}(V \otimes W^*)$-modules. We investigate in more detail the properties of the homology modules of this complex, in particular their dimension and regularity.

Section 2 gives the zipping functor, introduced in [5], associating to a complex of free $E(W^*)$-modules and a vector space $V$, a complex of free $\text{Sym}(V \otimes W^*)$-modules. We give some elementary properties of this functor.

Section 3 extends the main theorem of Cox and Materov [5] which says that the method of Lascoux by pullback and projection, and the zipping functor give the same complex of free $\text{Sym}(V \otimes W^*)$-modules. We show how various old and recent resolutions in the literature may be obtained by the zipping procedure.

Section 4 gives first a brief recollection of squarefree modules. Then a specific example of a triplet of pure free squarefree modules over $k[x_1, x_2, x_3]$ is considered. We show in two detailed examples how two of the complexes in the triplet may be obtained from two complexes of coherent sheaves on projective spaces $\mathbb{P}^2$ resp. $\mathbb{P}^3$, taking their Tate resolutions, zip them with the exterior coalgebra on $V = \langle x_1, x_2, x_3 \rangle$ to get $\text{Sym}(V \otimes W^*)$-complexes and then take reductions via a map $V \otimes W^* \to V$ to get complexes of free squarefree $k[x_1, x_2, x_3]$-modules.

Section 5 first introduces the notion of three degree sequences forming a homology triplet. Then we give our main conjecture on the existence of complexes of coherent sheaves associated to such homology triplets. We give the equations for the coefficients of the Hilbert polynomial $P$ of this complex and show that all the Hilbert polynomials of the cohomology sheaves of this complex and its dual are uniquely determined by the polynomial $P$. We also show that the hypercohomology table of the complex is determined by $P$. The section also contains a detailed example of such complexes.

Section 6. We prove that the main conjecture of Section 5 implies Conjecture 2.11 in [15] on the existence of triplets of pure free squarefree complexes.

Section 7. We show that there is an injection from the cone of hypercohomology tables of bounded complexes of coherent sheaves on projective spaces with specified regularity conditions, to the cone of homological data sets of complexes of free squarefree modules. We conjecture that this is an isomorphism of cones.

Section 8. We prove that the complexes obtained by the method of Lascoux and the zipping procedure are the same, in our extended version.

Note. We have developed a Macaulay2 package “Triplets”, [17], which computes the Betti diagrams of pure free squarefree complexes associated to degree triplets, and the hypercohomology tables of the complexes of coherent sheaves associated to homology triplets.

1. Regularity and dimension

We recall the basic setup of Lascoux [20], as presented by Weyman in [22 Section 5], for computing syzygies of modules supported on determinantal varieties. We develop some results on regularities and Krull dimensions of the modules occurring in this construction.
1.1. The basic setup. Let $W$ be a finite-dimensional vector space over a field $k$ of arbitrary characteristic and $\mathbb{P}(W)$ the projective space $\text{Proj}(\text{Sym}(W))$. On this space there is a tautological sequence of locally free sheaves:

$$0 \to \Omega_{\mathbb{P}(W)}(1) \to \mathcal{O}_{\mathbb{P}(W)} \otimes W \to \mathcal{O}_{\mathbb{P}(W)}(1) \to 0.$$ 

Let $V$ be another finite-dimensional vector space. Dualizing the sequence above and tensoring with $V$ we get a sequence:

$$(1) \quad 0 \leftarrow V \otimes_k \mathcal{Q} \leftarrow (V \otimes_k W^*) \otimes \mathcal{O}_{\mathbb{P}(W)} \leftarrow V \otimes_k \mathcal{O}_{\mathbb{P}(W)}(-1)$$

where $\mathcal{Q}$ is the dual of $\Omega_{\mathbb{P}(W)}(1)$. Let $X$ be the affine space $\mathbb{V}_{\text{Spec}k}(V \otimes W^*)$, so

$$X \times \mathbb{P}(W) = V_{\mathbb{P}(W)}(V \otimes W^* \otimes \mathcal{O}_{\mathbb{P}(W)}) = V$$

is the trivial (geometric) vector bundle on $\mathbb{P}(W)$. The quotient bundle $V \otimes \mathcal{Q}$ of $V \otimes W^* \otimes \mathcal{O}_{\mathbb{P}(W)}$ gives a geometric subbundle $Z = \mathbb{V}(V \otimes \mathcal{Q})$ of $X \times \mathbb{P}(W)$. So we have a diagram:

$$\begin{array}{ccc}
Z & \subseteq & X \times \mathbb{P}(W) \\
\downarrow q' & & \downarrow q \\
Y & \subseteq & X,
\end{array}$$

where $Y$ is the image of $Z$. When the dimension of $V$ is greater than or equal to $\mathbb{P}(W)$, the varieties $Z$ and $Y$ are birational by [22] Prop.6.1.1.

Since $Y$ is an affine variety, the coherent sheaf $q_* \mathcal{O}_Z$ is simply the sheafification of the $\text{Sym}(V \otimes W^*)$-module $H^0(Z, \mathcal{O}_Z) = H^0(\mathbb{P}(W), p_* \mathcal{O}_Z)$. Since $Z$ is the bundle $\mathbb{V}_{\mathbb{P}(W)}(V \otimes \mathcal{Q})$, the pushdown $p_* \mathcal{O}_Z = \text{Sym}(V \otimes \mathcal{Q})$ and so $q_* \mathcal{O}_Z$ is the sheafification of:

$$(2) \quad \Gamma(\mathbb{P}(W), \text{Sym}(V \otimes \mathcal{Q})) = \bigoplus_{d \geq 0} \Gamma(\mathbb{P}(W), \text{Sym}(V \otimes \mathcal{Q})_d).$$

There is a natural action of $k^* = k \setminus \{0\}$ on $V$ by scalar multiplication. Hence a natural action on the sequence \[(1)\] with trivial action on $W$. So we get an action of $k^*$ on $Z, V, Y$ and $X$, and this action on $q_* \mathcal{O}_Z$ gives the natural grading on (2) above. By the left quotient map of \[(1)\] this is a graded $\text{Sym}(V \otimes W^*)$-module.

For $\mathcal{F}$ a coherent sheaf on $\mathbb{P}(W)$ we get the pullback sheaf $\mathcal{O}_Z \otimes_{\mathcal{O}_Y} p^* \mathcal{F}$ on $V = X \times \mathbb{P}(W)$, and then the pushdown $q_*(\mathcal{O}_Z \otimes_{\mathcal{O}_Y} p^* \mathcal{F})$. Again since $Y$ is affine, this sheaf is the sheafification of its global sections. By the projection formula these are:

$$\Gamma(V, \mathcal{O}_Z \otimes_{\mathcal{O}_Y} p^* \mathcal{F}) = \Gamma(\mathbb{P}(W), (p_* \mathcal{O}_Z) \otimes_{\mathcal{O}_{p(W)}} \mathcal{F})$$

$$= \Gamma(\mathbb{P}(W), \text{Sym}(V \otimes \mathcal{Q}) \otimes_{\mathcal{O}_{p(W)}} \mathcal{F})$$

$$= \bigoplus_{d \geq 0} \Gamma(\mathbb{P}(W), \text{Sym}(V \otimes \mathcal{Q})_d \otimes_{\mathcal{O}_{p(W)}} \mathcal{F})$$

This is a graded $\Gamma(\mathbb{P}(W), \text{Sym}(V \otimes \mathcal{Q}))$-module and so a graded $\text{Sym}(V \otimes W^*)$-module.

Since we will use it a lot, for a coherent sheaf $\mathcal{F}$, or complex thereof, it is convenient to introduce the following short notation

$$\mathcal{S}(\mathcal{F}) = \text{Sym}(V \otimes \mathcal{Q}) \otimes_{\mathcal{O}_{p(W)}} \mathcal{F}.$$ 

So

$$\mathcal{S}(\mathcal{F}) = \bigoplus_{d \geq 0} \mathcal{S}_d(\mathcal{F}) = \bigoplus_{d \geq 0} \text{Sym}_d(V \otimes \mathcal{Q}) \otimes_{\mathcal{O}_{p(W)}} \mathcal{F}.$$
Lemma 1.1. Let $\text{Sym}(V)$ be taken to be equivariant for $k$. The derived complex $H^a$ becomes a complex of $k$-modules. The derived complex $S$ of quasi-coherent $O_Y$-sheaves, and exactly the same as above holds. The injective resolutions $T^*$ may be taken to be equivariant for $k^*$-action and so we also get the structure of graded $\text{Sym}(V \otimes W^*)$-module on the global sections of $(3)$.

Lemma 1.2. Let $F^*$ be a complex of coherent sheaves on $\mathbb{P}(W)$. The hypercohomology module $\mathbb{H}^i(X, \mathbb{R}q_*(O_Z \otimes p^*F^*))$ is the hypercohomology module

$$\mathbb{H}^i(\mathbb{P}(W), S(F^*)) = \oplus_{d \geq 2} \mathbb{H}^i(\mathbb{P}(W), S_d(F^*))$$

It is a graded $\text{Sym}(V \otimes W^*)$-module for this grading.

Proof. Since $X$ is affine the first hypercohomology modules are the hypercohomology modules

$$\mathbb{H}^i(V, O_Z \otimes p^*F^*)$$

which are $\Gamma(V, O_Z)$-modules. Since $p$ is an affine map, by the spectral sequence of the composition $V \xrightarrow{p} \mathbb{P}(W) \rightarrow \text{Spec} k$ this is

$$\mathbb{H}^i(\mathbb{P}(W), p_*(O_Z \otimes p^*F^*))$$

By the projection formula this is

$$\mathbb{H}^i(\mathbb{P}(W), \text{Sym}(V \otimes Q) \otimes F^*)$$

and so this is a $\Gamma(\mathbb{P}(W), \text{Sym}(V \otimes Q))$-module. The facts about the grading follow by the $k^*$-action.

1.2. Regularity. Let the vector space $V$ have dimension $n$. The sequence $(1)$ gives an exact sequence (the sheafified Koszul complex):

$$0 \leftarrow \text{Sym}(V \otimes Q) \leftarrow \text{Sym}(V \otimes W^*) \otimes O_{\mathbb{P}(W)} \leftarrow \cdots$$

$$\cdots \leftarrow \text{Sym}(V \otimes W^*)(-i) \otimes \wedge^i(V \otimes O_{\mathbb{P}(W)}(-1)) \leftarrow \cdots$$

$$\cdots \leftarrow \text{Sym}(V \otimes W^*)(-n) \otimes \wedge^n(V \otimes O_{\mathbb{P}(W)}(-1)) \leftarrow 0$$

Lemma 1.2. Suppose $F$ is a 0-regular coherent sheaf.

a. $S(F)$ is also a 0-regular coherent sheaf.

b. $S(F)$ is generated in degree 0 by $\Gamma(\mathbb{P}(W), F)$ as a $\text{Sym}(V \otimes W^*)$-module. Suppose $F$ is a 1-regular coherent sheaf on $\mathbb{P}(W)$.

c. The minimal degree of a generator of the graded $\text{Sym}(V \otimes W^*)$-module $S(F)$ is the smallest integer $e$ such that $H^e(\mathbb{P}(W), F(-e))$ is nonzero.

Proof. Tensoring $(1)$ with $F(r)$ we get a hypercohomology spectral sequence:

$$E_2^{pq} = H^p(\mathbb{P}(W), \text{Sym}_{d+p}(V \otimes W^*) \otimes F(r) \otimes \wedge^{-p}(V \otimes O_{\mathbb{P}(W)}(-1)))$$

$$\Rightarrow H^{p+q}(\mathbb{P}(W), S_d(F)(r)).$$

(5)
In order to show that $S_d(F)$ is 0-regular, we need to show that the term on the right in the spectral sequence above is zero when $p + q > 0$ and $p + q + r = 0$. So choosing any negative $r$, this will follow provided we show that $E_{pq}^2 = 0$ for all $(p, q)$ on the line $p + q = -r$.

But by our regularity assumptions, the terms on the first line of (5) may be nonzero only when i. $p \leq 0$, and ii. $q = 0$ or $p + q + r < 0$. Since we are assuming $r$ negative we have also when $q = 0$ that $p + q + r < 0$. Since $-r - 1 \geq 0$ the possible nonzero range is then:

Hence we obtain that $S_d(F)$ is a 0-regular sheaf.

When $r = 0$ the possible nonzero terms are

including the half line where $p \leq 0$ and $q = 0$. We then see that $S_d(F)$ is a quotient of $H^0(\mathbb{P}(W), \text{Sym}_d(V \otimes W^*) \otimes F)$, proving b.

As for c. let $r = 0$ in (5). Consider terms on the first line when $p + q \geq 1$. If $p > 0$ these terms are zero. When $p \leq 0$ then $q \geq 1$ and so the terms on the first line are again zero. When $p + q \leq 0$, the terms on the first line may be nonzero only when $p \geq -d$. Also when $p + q = 0$ the terms on the left may be nonzero only when $p \leq -e$. Hence the possible nonzero range of the terms on the left side is:
Therefore when \( d < e \) we see that the global sections \( H^0(\mathbb{P}(W), \mathcal{S}(\mathcal{F})_d) \) vanish. When \( d = e \) we see that this space of global sections is isomorphic to \( H^e(\mathbb{P}(W), \wedge^e V \otimes \mathcal{F}(-e)) \).

**Lemma 1.3.** Let \( \mathcal{F}^\bullet \) be a complex of coherent sheaves on \( \mathbb{P}(W) \). Fix an integer \( r \) and suppose the cohomology sheaves \( H^{r-j}(\mathcal{F}^\bullet) \) are \( j \)-regular for \( j \geq 1 \). Then the hypercohomology:

- a. \( H^r(\mathbb{P}(W), \mathcal{F}^\bullet) = H^0(\mathbb{P}(W), H^r(\mathcal{F}^\bullet)) \),
- b. \( H^r(\mathbb{P}(W), \mathcal{S}(\mathcal{F}^\bullet)) = H^0(\mathbb{P}(W), \mathcal{S}(H^r(\mathcal{F}^\bullet))) \).

**Proof.** There is a spectral sequence:

\[
E_2^{p,q} = H^p(\mathbb{P}(W), H^q(\mathcal{F}^\bullet)) \Rightarrow H^{p+q}(\mathbb{P}(W), \mathcal{F}^\bullet).
\]

By hypothesis the left side may be nonzero only in the range:

\[
p + q = r
\]

We therefore see that the right side of (6), when \( p + q = r \), becomes:

\[
H^r(\mathbb{P}(W), \mathcal{F}^\bullet) = H^0(\mathbb{P}(W), H^r(\mathcal{F}^\bullet)),
\]

proving part a.

Part b. follows from a. by: i) \( S(H^q(\mathcal{F}^\bullet)) = H^q(S(\mathcal{F}^\bullet)) \) because the \( \text{Sym}_d(V \otimes Q) \) are locally free sheaves. ii) When a coherent sheaf \( \mathcal{G} \) is \( t \)-regular then \( \mathcal{G}(t) \) is 0-regular.

By Lemma 1.2 a. \( S(\mathcal{G}(t)) = S(\mathcal{G})(t) \) is 0-regular and so \( S(\mathcal{G}) \) is \( t \)-regular. \( \square \)

### 1.3. Krull dimensions

Let \( n \) denote the dimension of the vector space \( V \), and \( w \) the dimension of \( W \). We let \( m = w - 1 \) be the dimension of the projective space \( \mathbb{P}(W) \). This subsection is devoted to prove the following.

**Proposition 1.4.** Let \( \mathcal{F} \) be a coherent sheaf on \( \mathbb{P}(W) \) whose support has dimension \( \delta \), and \( V \) a vector space of dimension \( \geq \) that of the support of \( \mathcal{F} \). Then \( S(\mathcal{F}) \) is a finitely generated \( \text{Sym}(V \otimes W^*) \)-module of Krull dimension \( nm + \delta \).

We prove this at the end of this subsection. First we develop some lemmata.

**Lemma 1.5.** The \( \text{Sym}(V \otimes W^*) \)-module \( S(\mathcal{O}_{\mathbb{P}(W)}) \) has Krull dimension \( (n + 1)m \) when \( \dim_k V \geq \dim \mathbb{P}(W) \).

**Proof.** This module is the global sections of \( q_* (\mathcal{O}_Z) \) on the affine variety \( X \). Being affine this is the dimension of the sheaf \( q_* \mathcal{O}_Z \). Since \( Z \) and its image \( Y \) in \( X \) are birational by [22, 6.1.1], \( q_* (\mathcal{O}_Z) \) has the same dimension as \( \mathcal{O}_Z \). Since the dimension of \( Z \) is \( (n + 1)m \) we are done. \( \square \)

**Lemma 1.6.** Let \( L \) be a linear subspace of \( \mathbb{P}(W) \) of codimension \( c \). The module \( S(\mathcal{O}_L) \) has Krull dimension \( (n + 1)m - c \) when \( \dim_k V \geq \dim_k L \).
Proof. On the linear subspace \( L \) we have the dual tautological sequence:

\[
0 \to \mathcal{O}_L(-1) \to H^0\mathcal{O}_L(1) \otimes \mathcal{O}_L \to \mathcal{Q}_L \to 0.
\]

Recall the bundle \( \mathcal{Q} \) of (1) and note that \( \mathcal{Q}_L = \mathcal{Q}_L \oplus \mathcal{O}_L^c \). Hence

\[
(7) \quad \text{Sym}_d(\mathcal{V} \otimes \mathcal{Q}) \otimes \mathcal{O}_L = \text{Sym}_d((\mathcal{V} \otimes \mathcal{Q}_L) \oplus (\mathcal{V} \otimes \mathcal{O}_L^c))
\]

\[
= \oplus_i \text{Sym}_{d-i}(\mathcal{V} \otimes \mathcal{Q}_L) \otimes \mathcal{O}_L^c \text{Sym}_i(\mathcal{V} \otimes \mathcal{O}_L^c).
\]

The subspace \( L \subseteq \mathbb{P}(W) \) may be considered as a projective space \( L = \mathbb{P}(W') \) for some quotient \( W \to W' \). We then get the functors \( S_L \) and \( S'_L \) on coherent sheaves on this space. Taking sections of the above equation we get:

\[
\sum_{d \geq 0} \dim_k S_d(\mathcal{O}_L)t^d = \left( \sum_{d \geq 0} \dim_k S_{d,d}(\mathcal{O}_L)t^d \right) \left( \sum_{d \geq 0} \dim_k \text{Sym}_d(\mathcal{V} \otimes \mathcal{O}_L^c)t^d \right),
\]

and so

\[
\dim S(\mathcal{O}_L) = \dim S_L(\mathcal{O}_L) + nc.
\]

By Lemma 1.5 the dimension of \( S_L(\mathcal{O}_L) \) is

\[
(n + 1)(w' - 1) = (n + 1)(w - c - 1) = (n + 1)(w - 1) - cn - c.
\]

We then get the result. \( \Box \)

Proof of Proposition 1.4. Let \( L_d \) be a subspace of \( \mathbb{P}(W) \) of dimension \( d \) for \( d = 0, \ldots, m \). The Grothendieck group of coherent sheaves on \( \mathbb{P}(W) \) has a basis consisting of classes \( \mathcal{O}_{L_d} \) for \( d = 0, \ldots, m \). So let \( \mathcal{F} = \sum_{i=0}^d a_i \mathcal{O}_{L_i} \) where \( a_\delta \neq 0 \). Since \( \text{Sym}(\mathcal{V} \otimes \mathcal{Q}) \) is locally free we get by tensoring the above that

\[
[S_d(\mathcal{F})] = \sum_{i=0}^\delta a_i [S_d(\mathcal{O}_{L_i})].
\]

The Hilbert-Poincaré polynomial \( \chi \) of coherent sheaves on \( \mathbb{P}(W) \) is an additive function on exact sequences and so descends to a function on the Grothendieck group. Hence

\[
\chi S_d(\mathcal{F}) = \sum_{i=0}^\delta a_i \chi S_d(\mathcal{O}_{L_i}).
\]

Assume that \( \mathcal{F} \) is 0-regular. By Lemma 1.2b. \( S(\mathcal{F}) \) is finitely generated and by Lemma 1.2a. the Euler-Poincaré characteristics above are simply given by the dimension of the space of global sections. Hence we obtain an equality of Hilbert functions:

\[
\sum_d \dim_k S_d(\mathcal{F})t^d = \sum_{i=0}^\delta a_i \sum_d \dim_k S_d(\mathcal{O}_{L_i})t^d.
\]

Whence \( S(\mathcal{F}) \) has dimension \( nm + \delta \) by Lemma 1.6.

In general if \( \mathcal{F} \) is not assumed to be 0-regular, we may argue by induction on the dimension of the support of \( \mathcal{F} \) as follows: If \( \dim \text{Supp}\mathcal{F} = 0 \), then \( \mathcal{F} \) is 0-regular and so we are done. If \( \dim \text{Supp}\mathcal{F} \geq 1 \) let \( \mathcal{F} \) be \( r \)-regular where \( r > 0 \). Let \( Q \) be a general form of degree \( r \) defining a hypersurface not containing any associated subscheme of \( \mathcal{F} \) (embedded or not). This gives a short exact sequence

\[
0 \to \mathcal{F} \xrightarrow{\cdot Q} \mathcal{F}(r) \to \mathcal{G} \to 0
\]
where \( \mathcal{G} \) is the cokernel, and \( \dim \text{Supp} \mathcal{G} < \dim \text{Supp} \mathcal{F} \). We then get an exact sequence

\[
0 \to S(\mathcal{F}) \to S(\mathcal{F}(r)) \to S(\mathcal{G}).
\]

By induction \( S(\mathcal{G}) \) is finitely generated of dimension \( \leq nm + \delta - 1 \). Since \( S(\mathcal{F}(r)) \) is finitely generated of dimension \( nm + \delta \) the same holds for \( S(\mathcal{F}) \).

\[\square\]

## 2. Free complexes over the exterior algebra

In this section we give elementary properties of free complexes over the exterior algebra \( E(W^*) \) where \( W \) is a finite dimensional vector space. If \( V \) is another finite dimensional vector space we define a functor associating to a complex of free \( E(W^*) \)-modules a complex of free \( \text{Sym}(V \otimes W^*) \)-modules. This functor was introduced by Cox and Materov in [5] and is here of central importance.

### 2.1. Starting with the vector space \( W \)

Let \( W \) be a finite dimensional vector space of dimension \( w \) and \( E = E(W^*) = \bigoplus_{i=0}^w \wedge^i W^* \) the exterior algebra on the dual space \( W^* \). We consider \( W \) to have degree 1, so \( \wedge^i W^* \) has degree \( -i \). Let \( \widehat{E} = \text{Hom}_k(E,k) \) be the dualizing module for \( E \). This is both a free (hence projective) and injective module over \( E \).

In this section \( E^* \) shall denote a complex of graded free left modules over the exterior algebra. In the setting we consider it will be natural to write the terms as

\[
E^p = \bigoplus_{j \in \mathbb{Z}} \widehat{E}(j - p) \otimes_k N^p_{p-j}
\]

where \( N^p_{p-j} \) is a vector space. Note that there may be a nonzero map \( \widehat{E}(j) \to \widehat{E}(j') \) iff \( j \geq j' \geq j - w \). Such a complex is linear if there is some \( k \) such that

\[
E^p = \widehat{E}(k - p) \otimes_k N^p_{p-k}
\]

for all \( p \). Let \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) be a graded \( \text{Sym}(W) \)-module. Then we may associate a linear complex of free \( E \)-modules

\[
R(M) : \cdots \to \widehat{E}(-i) \otimes_k M_i \xrightarrow{d} \widehat{E}(-i - 1) \otimes_k M_{i+1} \to \cdots.
\]

Letting \( y_1, \ldots, y_w \) be a basis for \( W \) and the \( y_i^* \) a dual basis for \( W^* \), the differential \( d \) is defined by

\[
u \otimes m \mapsto \sum_{i=1}^w uy_i^* \otimes y_i m.
\]

Any linear complex is a shift of \( R(M) \) for a graded \( \text{Sym}(W) \)-module \( M \). We define the dimension of the linear complex to be the Krull dimension of \( M \).

If \( E^* \) is a minimal complex over \( E \), meaning that all maps \( \widehat{E}(i) \otimes N^p_{i-j} \to \widehat{E}(i) \otimes N^{p+1}_{i-j} \) between terms with the same twist \( i \), are zero, then there is a filtration \( E^* \) of \( E^* \) given by

\[
E^p_{\leq k} = \bigoplus_{j \leq k} \widehat{E}(j - p) \otimes_k N^p_{p-j}.
\]

The quotient \( E^p_{\leq k}/E^p_{\leq k-1} \) is a linear complex. It is the \( k \)th linear strand of \( E^* \).

The complex \( E^* \) is a Tate resolution if it is acyclic, i.e. all homology groups \( H^p(E^*) \) vanish, and its terms are finitely generated modules. Such a complex will be unbounded to the left and right, if it is not nullhomotopic. These are central objects of the next sections. Let us note that such are easily constructed from any finitely generated \( E \)-module \( N \) by taking a free resolution of \( N \), and an injective resolution...
of $N$ by modules of the form $\oplus_{j \in \mathbb{Z}} \hat{E}(j) \otimes_{k} N_{-j}$, and then splicing these resolutions together.

Let $W' \subseteq W$ be a subspace, giving a quotient exterior algebra $E = E(W^*) \rightarrow E(W'^*) = E'$.

Note that the dualizing module $\hat{E}' = \text{Hom}_{k}(E', k) = \text{Hom}_{E}(E', \hat{E})$. Given a complex $E'$ over the exterior algebra $E$ we obtain a restricted complex $\text{Hom}_{E}(E', E^*)$ over the exterior algebra $E'$. If the term $E^p$ is given as in (8) then this complex has terms

$$\text{Hom}_{E}(E', E^p) = \oplus_{j \in \mathbb{Z}} \hat{E}'(j - p) \otimes_{k} N_{p-j}^p.$$

2.2. Introducing the vector space $V$. Let $V$ be a finite dimensional vector space and $S = \text{Sym}(V \otimes W^*)$ the symmetric algebra. Cox and Materov [5] define a $k$-linear functor $W_{V}$ from finitely generated graded free $E$-modules with homomorphisms of degree zero, to finitely generated free $S$-modules with homomorphisms of degree zero, as follows. A map

$$(10) \quad \hat{E}(j) \xrightarrow{\beta} \hat{E}(j')$$

is given on the generator of the first module as

$$\wedge^w V \rightarrow \wedge^{w+j'-j} W$$

or equivalently a map

$$k \xrightarrow{\beta} \wedge^{j'-j} W^*.$$

By the Cauchy formula [22 2.3.2, 3.2.3] there is a natural inclusion of $\wedge^d V \otimes \wedge^d W^*$ into $\text{Sym}_d(V \otimes W^*)$. This holds in any characteristic. Consider $\oplus_i \wedge^i V$ as a coalgebra and denote the comultiplication map by $\delta$. We define the map

$$\wedge^j V \otimes S(-j) \xrightarrow{W_{V}(\beta)} \wedge^{j'} V \otimes S(-j')$$

to be given by

$$\wedge^j V \otimes k \xrightarrow{\delta \otimes \beta} \wedge^j V \otimes \wedge^{j'-j} V \otimes \wedge^{j-j'} W^* \rightarrow \wedge^j V \otimes \text{Sym}(V \otimes W^*)_{j-j'}.$$

Given a complex $E^*$ of free graded $E$-modules with terms as in (8), we then obtain a complex $W_{V}(E^*)$ of free graded $S$-modules with terms

$$W_{V}(E^p) = \oplus_{j \in \mathbb{Z}} \wedge^{j-p} V \otimes S(p-j) \otimes_{k} N_{p-j}^p.$$

Lemma 2.1. (Duality.) Let $\beta$ be the map of (10) and $n = \dim_{k} V$. Then

$$(11) \quad \text{Hom}_{S}(W_{V}(\beta), S(-n) \otimes \wedge^n V) = W_{V}(\text{Hom}_{k}(\beta, \wedge^{m+1} W)(n)).$$

Hence we have a commutative diagram of functors starting from complexes of free graded $E$-modules

$$\begin{array}{ccc}
E^* & \xrightarrow{W_{V}} & \bullet \\
\text{Hom}_{k}(-, \wedge^{m+1} W)(n) \downarrow & & \downarrow \text{Hom}_{S}(-, S(-n) \otimes \wedge^n V) \\
\bullet & \xrightarrow{W_{V}} & \bullet
\end{array}$$
Proof. The left side of (11) is
\[ \wedge^{n-j} V \otimes S(-n+j) \leftarrow \wedge^{n-j'} V \otimes S(-n+j'). \]
On the other hand \( \text{Hom}_k(\beta, \wedge^{m+1} W)(n) \) is
\[ \hat{E}(n-j) \leftarrow \hat{E}(n-j'). \]
And so we get the equality. \( \Box \)

For the subspace \( W' \) of \( W \) let \( S' = \text{Sym}(V \otimes (W')^*) \). The following now relates the functor \( W^V \) and restriction.

Lemma 2.2. (Restriction.) Let \( \beta \) be the map of (10). Then
\[ \mathbb{W}^V_W(\beta) \otimes_S S' = \mathbb{W}^V_{W'}(\text{Hom}_E(E', \beta)). \]
As a consequence we have a commutative diagram of functors starting from complexes of free graded \( E \)-modules
\[
\begin{array}{ccc}
E^\bullet & \xrightarrow{\mathbb{W}^V_W} & \bullet \\
\downarrow_{\text{Hom}_E(E',-)} & & \downarrow -\otimes_S S' \\
\bullet & \xrightarrow{\mathbb{W}^V_{W'}} & \bullet
\end{array}
\]
Proof. The map \( \hat{E}(j) \rightarrow \hat{E}(j') \) is given by a map
\[ k \rightarrow \wedge^{j-j'} W^*. \]
The map \( \text{Hom}_E(E', \beta) \) is given by composing this with the natural projection \( \wedge^{j-j'} W^* \rightarrow \wedge^{j-j'} W^{**} \). The map \( \mathbb{W}^V_W(\beta) \) is given on generators by
\[ \wedge^j V \overset{\beta}{\rightarrow} \wedge^{j'} \otimes \wedge^{j-j} V \otimes \wedge^{j-j} W^* \]
and so \( \mathbb{W}^V_W(\beta) \otimes_S S' \) is given by the composition of this with \( \wedge^{j-j} W^* \rightarrow \wedge^{j-j} W^{**} \).
It is clear that this map equals \( \mathbb{W}^V_{W'}(\text{Hom}_E(E', \beta)). \) \( \Box \)

3. Zipping

In this section we recall the notion of Tate resolution, and then we extend the main result of Cox and Materov [5] on how Tate resolutions may be zipped with the exterior coalgebra on a vector space. This is applied to show how various old and recent resolutions from the Eagon-Northcott complex, 1962, to tensor complexes, 2011, may be obtained by zipping a Tate resolution with an exterior coalgebra.

3.1. Tate resolutions of coherent sheaves. The dualizing complex, [18] V §0, on the category of complexes of coherent sheaves on the projective space \( \mathbb{P}(W) \) of dimension \( m \) is \( \omega_{\mathbb{P}(W)}[m] \), where \( \omega_{\mathbb{P}(W)} = \mathcal{O}_{\mathbb{P}(W)}(-m-1) \). For a complex of coherent sheaves \( F^\bullet \) on \( \mathbb{P}(W) \) the dual complex is
\[ (F^\bullet)^\vee = \mathbb{R}\text{Hom}_{\mathcal{O}_{\mathbb{P}(W)}}(F^\bullet, \omega_{\mathbb{P}(W)}[m]). \]
If we have a complex of vector bundles \( E^\bullet \) together with a quasi-isomorphism \( E^\bullet \rightarrow F^\bullet \) this may be computed as
\[ \text{Hom}_{\mathcal{O}_{\mathbb{P}(W)}}(E^\bullet, \omega_{\mathbb{P}(W)}[m]). \]
Serre duality [18 Thm.III.5.1] gives dualities between the hypercohomology modules:
\[ \mathbb{H}^p(\mathbb{P}(W), \mathcal{F}^\bullet) = \mathbb{H}^{-p}(\mathbb{P}(W), (\mathcal{F}^\bullet)^\vee)^*. \]
We use the following notation for the graded \( \text{Sym}(W) \)-modules
\[ \mathbb{H}^p_*(\mathbb{P}(W), \mathcal{F}^\bullet) = \oplus_{i \in \mathbb{Z}} \mathbb{H}^p(\mathbb{P}(W), \mathcal{F}^\bullet(i)). \]
Since we will work much in the context of a vector space \( V \) having dimension \( n \), it will be convenient to define
\[ (\mathcal{F}^\bullet)^* = (\mathcal{F}^\bullet)^\vee(n)[-n]. \]

If \( \mathcal{F} \) is a coherent sheaf on \( \mathbb{P}(W) \) there is associated a Tate resolution \( \mathbb{T}(\mathcal{F}) \), see [8 Section 4], whose terms are given by
\begin{equation}
\mathbb{T}^p(\mathcal{F}) = \oplus_{j \in \mathbb{Z}} \hat{\mathbb{E}}(j - p) \otimes_k \mathbb{H}^j(\mathbb{P}(W), \mathcal{F}(p - j)).
\end{equation}
In particular note that when \( p \gg 0 \) this term is simply \( \hat{\mathbb{E}}(-p) \otimes_k \mathbb{H}^0(\mathbb{P}(W), \mathcal{F}(p)) \), so we say it is eventually linear.

Furthermore the \( i \)'th strand of this complex is the linear complex \( \mathbb{R}(\mathbb{H}^i(\mathbb{P}(W), \mathcal{F}))[\ldots i] \), associated to the \( i \)'th cohomology module. Sending a sheaf to its Tate resolution is a functor from the category of coherent sheaves to the homotopy category of complexes of free \( E \)-modules
\[ \mathbb{T} : \text{coh} \, \mathbb{P}(W) \to K(E - \text{free}). \]
More generally if \( \mathcal{F}^\bullet \) is a complex of coherent sheaves there is similarly associated a Tate resolution \( \mathbb{T}(\mathcal{F}^\bullet) \), see [13 Thm.3.2.1], [4 Thm.10], and whose terms are given as in (12) but now with the cohomology modules of \( \mathcal{F}(p - j) \) replaced by the hypercohomology modules of \( \mathcal{F}^\bullet(p - j) \). It gives a functor from the bounded derived category of coherent sheaves on \( \mathbb{P}(W) \)
\[ \mathbb{T} : D^b(\text{coh} \, \mathbb{P}(W)) \to K(E - \text{free}). \]
The image is in the subcategory of \( K(E - \text{free}) \) consisting of Tate resolutions. In fact \( \mathbb{T} \) gives an equivalence of categories between the bounded derived category above and the category of Tate resolutions up to homotopy, loc.cits.

**Lemma 3.1.** Let \( \mathcal{F}^\bullet \) be a complex of coherent sheaves on \( \mathbb{P}(W) \).

\begin{enumerate}
\item \( \mathbb{T}(\mathcal{F}^\bullet[k]) = \mathbb{T}(\mathcal{F}^\bullet)[k] \).
\item \( \mathbb{T}(\mathcal{F}^\bullet(n)) = \mathbb{T}(\mathcal{F}^\bullet)(n)[n] \).
\item \( \mathbb{T}((\mathcal{F}^\bullet)^\vee) = \text{Hom}_k(\mathbb{T}(\mathcal{F}^\bullet), \wedge^{m+1}W) \).
\item \( \text{Hom}_S(\mathbb{W}_V^W(\mathbb{T}(\mathcal{F}^\bullet)), S(-n) \otimes \wedge^n V) = \mathbb{W}_V^W(\mathbb{T}((\mathcal{F}^\bullet)^\vee)) \).
\end{enumerate}

**Proof.** Parts a. and b. are trivial. For part c. note that \( \mathbb{T} \) is a functor of triangulated categories, [13 Thm.3.2.1]. Let \( \mathcal{F}^\bullet \xrightarrow{\phi} \mathcal{G}^\bullet \) be a morphism of complexes. It gives a distinguished triangle
\[ \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \text{cone} \, \phi \to \mathcal{F}^\bullet[1]. \]
Then we get a distinguished triangle
\[ \mathbb{T}(\mathcal{F}^\bullet) \to \mathbb{T}(\mathcal{G}^\bullet) \to \mathbb{T}(\text{cone} \, \phi) \to \mathbb{T}(\mathcal{F}^\bullet)[1]. \]
Dualizing we get a distinguished triangle
\[ \text{Hom}_k(\mathbb{T}(\text{cone} \, \phi)[-1], \wedge^{m+1}W) \leftarrow \text{Hom}_k(\mathbb{T}(\mathcal{F}^\bullet), \wedge^{m+1}W) \leftarrow \text{Hom}_k(\mathbb{T}(\mathcal{G}^\bullet), \wedge^{m+1}W). \]
Also there is a distinguished triangle
\[ T(\text{cone } \phi^\vee) \leftarrow T((F^\bullet)^\vee) \leftarrow T((G^\bullet)^\vee). \]
Note that cone \((\phi^\vee) = (\text{cone } \phi)^\vee[1].\) If part c. holds for \(F^\bullet\) and \(G^\bullet\), we get a morphism between these two distinguished triangles where two of the maps are isomorphisms. Hence the third is also an isomorphism and so part c. holds for cone \(\phi\). Now part c. clearly holds for vector bundles. Any complex of vector bundles may be built up as cones from complexes of smaller length. This proves part c. Part d. follows by Lemma 2.1 since
\[ \text{Hom}(T(F^\bullet), \wedge^{m+1}W)(n) \cong T((F^\bullet)^\vee)(n) = T((F^\bullet)^\vee) \]
using parts a., b. and c. above. □

3.2. Zipping and the method of Lascoux.

**Definition 3.2.** Let \(F^\bullet\) be a complex of coherent sheaves on \(P(W)\) and \(V\) a finite-dimensional vector space over \(k\). The complex \(W^V_W(T(F^\bullet))\) is the *zip complex* or the *Weyman complex* (in [5]) of the Tate resolution of \(F^\bullet\) w.r.t. the exterior coalgebra on \(V\). We say that this complex is obtained by *zipping* the Tate resolution \(T(F^\bullet)\) and the exterior coalgebra of the vector space \(V\), or simply with the vector space \(V\).

Since the Tate resolution is unique up to isomorphism, so is the zip complex. It is also a minimal complex since \(T(F^\bullet)\) is minimal.

Our main general observation is the following theorem which is an extension of the main Theorem 1.4 of Cox and Materov [5]. It is also close to the Basic Theorem 5.1.2 in Weyman’s book [22]. Its proof is given in Section 8. After the statement we explain how it extends and relates to these. Recall the setup of Subsection 1.1.

**Theorem 3.3.** Let \(F^\bullet\) be a complex of coherent sheaves on the projective space \(P(W)\). The graded complexes \(\Gamma(X, \mathcal{R}_q^*(O_Z \otimes p^*F^\bullet))\) and \(F^\bullet = W^V_W(T(F^\bullet))\) are isomorphic in the derived category of graded \(\text{Sym}(V \otimes W^\vee)\)-modules.

In particular the terms of \(F^\bullet\) are given by
\[ F_p = \bigoplus \wedge^{p+j}V \otimes S(-p-j) \otimes H^j(P(W), F^\bullet(-p-j)) \]
and the homology of \(F^\bullet\) is the hypercohomology
\[ H_p(F^\bullet) = H^{-p}(P(W), S(F^\bullet)). \]

In the notation of Weyman’s book [22, Section 5.1] we work in the special case that his projective variety \(V = P(W)\), and the sequence \(0 \to S \to E \to T \to 0\) is the sequence (11). The new thing Cox and Materov noted, compared to the Basic Theorem 5.1.2 in [22] in this setting, is that the complex \(F^\bullet\) may be obtained by *first* taking the Tate resolution of the coherent sheaf \(F\) and *then* zipping it with \(V\), i.e. applying the functor \(W^V_W\). We thus have a factorization of a functor. Apart from this, the above is an extension of 5.1.2 in this setting in the sense that we phrase it for complexes of coherent sheaves, while Weyman does this only for vector bundles. This is however an extension that would not require much modification in Weyman’s arguments.

The main new feature above compared to Theorem 1.4 of Cox and Materov is that they prove it under the assumption that \(\dim_k V \leq \dim P(W)\). This is outside the range of our most significant applications where in general \(V\) has dimension larger
than $\mathbb{P}(W)$. The reason for their restriction is that their proof relies heavily on it, but they say that the statement probably holds also when $\dim_k V > \mathbb{P}(W)$. Also Cox and Materov work only in $\text{char}(k) = 0$, while the above is stated characteristic free. Their proof, using representation theory, depends heavily on characteristic zero. Cox and Materov also only consider coherent sheaves and not complexes and hypercohomology, which will be quite essential in our applications.

In [5] Prop. 1.3 they show that when $\dim_k V$ is greater than the dimension of a coherent sheaf $\mathcal{F}$ then $\mathcal{F}$ is determined by $\mathcal{W}_W^\vee(\mathcal{T}(\mathcal{F}))$. We state the following more general:

**Conjecture 3.4.** If $\dim_k V$ is greater than the dimensions of the cohomology sheaves of $\mathcal{F}^\bullet$, then $\mathcal{F}^\bullet$ is determined (up to isomorphism in the derived category) by $\mathcal{W}_W^\vee(\mathcal{T}(\mathcal{F}^\bullet))$.

3.3. **When the zip complex becomes a resolution.** Let $\mathcal{E}$ be a locally Cohen-Macaulay sheaf of pure dimension $d$. This is equivalent to $H^i(\mathbb{P}(W), \mathcal{E})$ being of finite length for $0 < i < d$. Note that if $d = \dim \mathbb{P}(W) = m$ this is equivalent to $\mathcal{E}$ being a vector bundle.

**Lemma 3.5.** Let $\mathcal{E}$ be locally Cohen-Macaulay sheaf on $\mathbb{P}(W)$ of dimension $d$. Then $\mathcal{E}^\bullet$ (forget the cohomological position) is $k$-regular iff $H^{d-i}(\mathbb{P}(W), \mathcal{E}(i - n - k)) = 0$ for $i > 0$. In this case we say that $\mathcal{E}$ is $(d - n - k)$-coregular.

**Proof.** Forgetting the cohomological position, $H^i(\mathbb{P}(W), \mathcal{E}^\bullet(k-i))$ equals $H^i(\mathbb{P}(W), \mathcal{E}^\vee(n+k-i))$. If $\mathcal{E}$ has dimension $d$, then $\mathcal{E}^\vee$ is in cohomological position $m - d$. Hence the latter cohomology group is

$$H^{i+m-d}(\mathbb{P}(W), \mathcal{E}^\vee(n+k-i)) = H^{d-i}(\mathbb{P}(W), \mathcal{E}(i-k-n))^\ast$$

by Serre duality. □

**Proposition 3.6.** Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(W)$ and $V$ a vector space of dimension $n$.

a. The zip complex $\mathcal{W}_W^\vee(\mathcal{T}(\mathcal{F}))$ is a free resolution iff $\mathcal{F}$ is a 1-regular coherent sheaf. In this case it is a free resolution of the $S$-module $S(\mathcal{F})$.

b. This zip complex is a minimal free resolution of a Cohen-Macaulay module iff $\mathcal{F}$ is a locally Cohen-Macaulay sheaf of pure dimension which is:

i. 1-regular and,

ii. $(d - n - 1)$-coregular, where $d$ is the dimension of $\mathcal{F}$.

**Proof.** a. If $\mathcal{F}$ is a 1-regular coherent sheaf, then all cohomology groups $H^i(\mathbb{P}(W), S(\mathcal{F}))$ vanish for $i \neq 0$ by Lemma 1.2. Hence by Theorem 3.3 $F_\bullet$ is a resolution. If $\mathcal{F}$ is not 1-regular. Then $H^j(\mathbb{P}(W), \mathcal{F}(1-j))$ is nonzero for some $j \geq 1$. But then $F_{-1}$ is nonzero. Since $F_\bullet$ is a minimal complex, it has nonzero homology $H_p(F^\bullet)$ for some negative $p$. But also $H_0(F^\bullet) = S(\mathcal{F})$ is nonzero. Hence $F_\bullet$ is not a resolution.

b. By part a. and Lemma 3.1 d. the dual complex Hom$(F^\bullet, S(-n) \otimes \wedge^n V)$ is a resolution iff $\mathcal{F}^\bullet$ is 1-regular. But by Lemma 3.5 this is equivalent to $\mathcal{F}$ being $(d - n - 1)$-coregular. □

Following [10] a vector bundle $\mathcal{E}$ on $\mathbb{P}(W)$ is said to have supernatural cohomology if there is a sequence

$$-\infty = r_{m+1} < r_m < \cdots < r_1 < r_0 = +\infty$$
such that the \( i \)-th cohomology \( H^i(\mathcal{P}(W), \mathcal{E}(r)) \) is nonzero only if \( r \) is in the interval \( \langle r_{i+1}, r_i \rangle \). In particular note that the Hilbert polynomial \( P \) of \( \mathcal{E} \) must be

\[
P(n) = c/m!(n - r_1)(n - r_2) \cdots (n - r_m)
\]

for some constant \( c \) (which is the rank of \( \mathcal{E} \)), and that the regularity of \( \mathcal{E} \) is \( r_1 + 1 \) and the coregularity is \( m + r_m - 1 \).

More generally a locally Cohen-Macaulay sheaf \( \mathcal{E} \) of dimension \( d \) on \( \mathbb{P}(W) \) is said to have supernatural cohomology if the above holds with the index \( m \) replaced by \( d \). Note that \( \mathcal{E} \) has supernatural cohomology iff its Tate resolution is pure, i.e. each cohomological term \( \mathbb{T}^p(\mathcal{E}) = \hat{E}(j - p) \otimes N_{p-j}^p \) has only one twist \( j - p \) occurring.

**Corollary 3.7.** Let \( \mathcal{E} \) be a locally Cohen-Macaulay sheaf of dimension \( d \) on \( \mathbb{P}(W) \) with supernatural cohomology and root sequence as above. The zip complex \( \mathbb{W}_W^d(\mathbb{T}(\mathcal{E})) \) is a resolution iff \( r_1 \leq 0 \). It is a resolution of a Cohen-Macaulay module iff \(-n \leq r_d < r_1 \leq 0\). In any case the complex is pure with degree sequence given by the complement \( [0, n] \setminus \{-r_1, \ldots, -r_d\} \).

**Proof.** That the complex is pure is clear. The other parts follow from Proposition 3.6 and the observations above on the regularity and coregularity of \( \mathcal{E} \): The sheaf \( \mathcal{E} \) is 1-regular iff \( r_1 + 1 \leq 1 \), and it is \( d - n - 1 \)-coregular iff \( d - n - 1 \leq d + r_d - 1 \). \( \square \)

### 3.4. Complexes from the literature

Here is how some notable old and recent resolutions in the literature are obtained by zipping an appropriate Tate resolution and a vector space \( V \) of dimension \( n \).

**Example 3.8.** Eagon-Northcott, 1962. Denote by \( D_i(W^*) \) the \( i \)'th divided powers of \( W^* \). (In characteristic zero this is isomorphic to \( \text{Sym}_i(W^*) \).) Let \( \hat{D}_i(W^*) = \wedge^{m+1} W^* \otimes D_i(W^*) \). The structure sheaf \( \mathcal{O}_{\mathbb{P}(W)} \) has Tate resolution:

\[
\cdots \rightarrow \hat{E}(n) \otimes \hat{D}_{n-m-1}(W^*) \rightarrow \hat{E}(n-1) \otimes \hat{D}_{n-m-2}(W^*) \rightarrow \cdots
\]

\[
\rightarrow \hat{E}(m+2) \otimes \hat{D}_1(W^*) \rightarrow \hat{E}(m+1) \otimes \hat{D}_0(W^*) \xrightarrow{\beta} \hat{E} \rightarrow \hat{E}(-1) \otimes \text{Sym}_1(W) \rightarrow \cdots
\]

Let \( S = \text{Sym}(V \otimes W^*) \) and assume \( \dim_k V \geq \dim_k W \) or equivalently \( n \geq m + 1 \). We zip with the exterior powers of \( V \) and obtain the Eagon-Northcott complex:

\[
\wedge^n V \otimes S \otimes \hat{D}_{n-m-1}(W^*) \rightarrow \cdots \rightarrow \wedge^{m+2} V \otimes S \otimes \hat{D}_1(W^*) \rightarrow \wedge^{m+1} V \otimes S \otimes \hat{D}_0(W^*) \xrightarrow{\alpha} S.
\]

From the explicit form of \( \beta \) and the explicit way zipping is done, the image of \( \alpha \) is seen to be the ideal of maximal minors of the generic map

\[
S \otimes V \rightarrow S \otimes W.
\]

Also since the structure sheaf \( \mathcal{O}_{\mathbb{P}(W)} \) has supernatural cohomology with root sequence \(-1, -2, \ldots, -m\) this complex is a free resolution of a Cohen-Macaulay ring.

**Example 3.9.** Buchsbaum-Rim, 1964, and Buchsbaum-Eisenbud, 1973. The twisted sheaf \( \mathcal{O}_{\mathbb{P}(W)}(r) \) when \( r \geq 1 \) has Tate resolution:

\[
\cdots \rightarrow \hat{E}(r + m + 1) \otimes \hat{D}_0(W^*) \rightarrow \hat{E}(r) \otimes \text{Sym}_0(W) \rightarrow \hat{E}(r - 1) \otimes \text{Sym}_1(W) \rightarrow \cdots
\]

\[
\rightarrow \hat{E} \otimes \text{Sym}_r(W) \rightarrow \cdots \rightarrow \hat{E}(-i) \otimes \text{Sym}_{r+i}(W) \rightarrow \cdots
\]
Zipping this with $V$, we obtain when $r = 1$ the Buchsbaum-Rim complex and when $r \geq 2$ the Buchsbaum-Eisenbud complexes:

$$\cdots \to \wedge^{r+m+1} V \otimes S \otimes \tilde{D}_0(W^*) \to \wedge^r V \otimes S \otimes \text{Sym}_0(W) \to \wedge^{r-1} V \otimes S \otimes \text{Sym}_1(W) \to \cdots \to \wedge^0 V \otimes S \otimes \text{Sym}_n(W).$$

The root sequence of $O_{P(W)}(r)$ is $-r - 1, -r - 2, \ldots, -r - m$. Hence these complexes are resolutions (assuming $r \geq 1$) and they are resolutions of Cohen-Macaulay modules exactly when $n \geq r + m$.

**Example 3.10. Pure resolutions of modules supported on determinantal varieties, 2007.** Let $\lambda$ be a partition into $m$ parts and $Q$ the dual of the tautological rank $m$ subbundle on $\mathbb{P}(W)$. When $\text{char}(k) = 0$ the Schur bundle $S_\lambda(Q)$ has supernatural cohomology with root sequences given by

$$-\lambda_1 - m, -\lambda_2 - m + 1, \ldots, -\lambda_m - 1,$$

see [12, Thm.5.6] Thus any root sequence may be obtained for a suitable partition $\lambda$. It is a 1-regular coherent sheaf when $\lambda_m \geq -1$. By zipping its Tate resolution with the vector space $V$ of dimension $\lambda_1 + m + 1$, we obtain the second construction in [9] of pure resolutions of Cohen-Macaulay modules supported on determinantal varieties.

**Example 3.11. Tensor complexes, 2011.** Let $W_1, \ldots, W_r$ be vector spaces of positive dimensions $w_1, \ldots, w_r$. On the Segre embedding

$$\mathbb{P}(W_1) \times \mathbb{P}(W_2) \times \cdots \times \mathbb{P}(W_r) \subseteq \mathbb{P}(W_1 \otimes \cdots \otimes W_r)$$

consider the line bundle

$$O_{\mathbb{P}(W_1)}(u_1) \boxtimes O_{\mathbb{P}(W_2)}(u_2) \boxtimes \cdots \boxtimes O_{\mathbb{P}(W_r)}(u_r).$$

When $u_i + w_i - 1 \leq u_{i+1}$, this is a locally Cohen-Macaulay sheaf on $\mathbb{P}(W_1 \otimes \cdots \otimes W_r)$ with supernatural cohomology. Its root sequence is the union of the intervals $U_{i=1}^r [-u_i - w_i + 1, -u_i - 1]$. Zipping its Tate resolution with a vector space $V$ we get a pure complex. When $u_1 \geq -1$ it becomes a pure resolution, and when $n \geq u_r + w_r - 1$ it is a pure resolution of a Cohen-Macaulay module. This gives the tensor complexes of [25], with pinching weights $(0; u_1, \ldots, u_r)$.

4. **Examples**

In this section we first briefly recall the notion of squarefree modules and the two dualities, standard duality and Alexander duality, we have on complexes of free squarefree modules.

We then consider the triplet of pure free squarefree complexes of Example 2.2 in [15] and show in two detailed examples how two of them may be obtained by i) starting with a complex of coherent sheaves on $\mathbb{P}(W)$, ii) zipping its Tate resolution with a vector space $V$ to get a complex of free $\text{Sym}(V \otimes W^*)$-modules, and iii) taking a suitable general quotient map $V \otimes W^* \to V$ to get a complex of pure free squarefree $\text{Sym}(V)$-modules.
4.1. **Squarefree modules and dualities.** We briefly recall basic notions. For more detail see Section 1 of [15]. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $e_i$ the $i$-th coordinate vector in $\mathbb{N}^n$. An $\mathbb{N}^n$-graded module $M$ over $k[x_1, \ldots, x_n]$ is squarefree if the multiplication map

$$M_d \xrightarrow{x_i} M_{d+e_i}$$

is an isomorphism whenever the $i$’th coordinate $d_i > 0$.

Let $1 = (1, 1, \ldots, 1)$. The Alexander dual $A(M)$ is the squarefree module such that when $0 \leq d \leq 1$:

- $A(M)_d = \text{Hom}_k(M_{1-d}, k)$.
- If $d_i = 0$ then

$$A(M)_d \xrightarrow{x_i} A(M)_{d+e_i}$$

is the dual of

$$M_{1-d-e_i} \xrightarrow{x_i} M_{1-d}.$$ 

Given a squarefree module $M$ there is associated a linear complex $L(M)$ of free squarefree modules with terms

$$L^i(M) = \oplus_{|d|=i} (M_d)^o \otimes_k S$$

where we sum over all 0, 1-vectors $d$ of total degree $i$, and $(M_d)^o$ is $M_d$ but considered to have multidegree $1 - d$. The differential is given by

$$m^o \otimes s \mapsto \sum_{j:d_j=0} (-1)^{\alpha(j,d)} (x_j m)^o \otimes x_j s$$

where $\alpha(j,d)$ is the number of $i < j$ with $d_i = 1$.

On the homotopy category of complexes of free squarefree modules there is the standard dual of the complex $F_*^\bullet$ of free modules

$$\mathbb{D}(F_*) = \text{Hom}_S(F_*, \omega_S).$$

The Alexander dual $A(F_*)$ is defined as a complex of free squarefree modules for which there is a quasi-isomorphic map to $A(F_*)$. Yanagawa [24] shows that the third iterate $(A \circ \mathbb{D})^3$ is isomorphic to the $n$’th iterate $[n]$ of the translation functor. The complex $F_*$ is said to belong to a *triplet of pure free squarefree complexes* if all three of

$$F_*, \quad (A \circ \mathbb{D})(F_*), \quad (A \circ \mathbb{D})^2(F_*)$$

are pure when considered as singly-graded modules. Since $(A \circ \mathbb{D})(\mathbb{D}(F_*))$ is the dual $\mathbb{D}((A \circ \mathbb{D})^2)(F_*)$ (up to translation of complexes), we see that a pure complex $F_*$ belongs to such a triplet iff $A \circ \mathbb{D}$ applied to both $F_*$ and $\mathbb{D}(F_*)$ are both pure.

The functor $A \circ \mathbb{D}$ rotates the homological invariants of $F_*$. Whether $(A \circ \mathbb{D})(F_*)$ is pure can then be checked on the homology modules of $F_*$ using the following, which is [24, Theorem 3.8].

**Lemma 4.1.** The $i$’th linear strand of $(A \circ \mathbb{D})(F_*)$ is $L(H^i(F_*))[n-i]$. 
Let $S = k[x_1, x_2, x_3]$. In [15], Example 2.2 we showed that there is a triplet of pure free squarefree complexes

\begin{equation}
G_\bullet : S \xrightarrow{[x_0 x_1, x_0 x_2, x_1 x_2]} S(-2)^3,
\end{equation}

\begin{equation}
(A \circ \mathbb{D})(G_\bullet) : S^2 \leftarrow S(-2)^3 \leftarrow S(-3),
\end{equation}

\begin{equation}
(A \circ \mathbb{D})^2(G_\bullet) : S(-1)^3 \leftarrow S(-2)^6 \leftarrow S(-3)^2.
\end{equation}

In the next example we show how to obtain $(A \circ \mathbb{D})(G_\bullet)$ by i) starting from a complex of coherent sheaves on $\mathbb{P}(W) = \mathbb{P}^2$, ii) zipper its Tate resolution with $V = \langle x_1, x_2, x_3 \rangle$, and iii) tensoring the zip complex with $- \otimes_{\text{Sym}(V \otimes W^*) \text{Sym}(V)}$, where $\text{Sym}(V)$ becomes a module via a general map $V \otimes W^* \to V$, equivariant for the diagonal matrices in $\text{GL}(V)$. In the example following thereafter we show how to obtain $(A \circ \mathbb{D})^3(G_\bullet)$ in a similar way.

In the following we let $S = \text{Sym}(V \otimes W^*)$ and for a complex $F_\bullet$ of free $S$-modules we get (recall $n$ is the dimension of $V$)

\begin{equation}
F_\bullet^\vee = \text{Hom}_S(F_\bullet, S(-n) \otimes \wedge^n V).
\end{equation}

4.2. First example. Consider the ideal sheaf $\mathcal{I}_P$ of a point $P$ in $\mathbb{P}^2 = \mathbb{P}(W)$. The resolution of the twisted sheaf $\mathcal{I}_P(1)$ is

\begin{equation}
\mathcal{F}^* : \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}^3.
\end{equation}

Its cohomology table is:

\begin{center}
\begin{tabular}{cccccccccc}
\cdots & 10 & 6 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & 2 \\
\cdots & 1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & 2 & 5 & 9 & 14 & \cdots & \cdots & 0 \\
\cdots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\
\end{tabular}
\end{center}

Letting $E$ be the exterior algebra $\bigoplus_{i=0}^3 \wedge^i W^*$, the complex $\mathcal{F}^*$ has Tate resolution:

\begin{equation}
\begin{array}{ccc}
\rightarrow & \hat{E}(6)^6 & \rightarrow \\
\oplus & \hat{E}(5) & \oplus \\
\oplus & \hat{E}(4) & \oplus \\
\oplus & \hat{E}(3) & \\
\end{array}
\end{equation}

Writing $W = \langle y_0, y_1, y_2 \rangle$ in terms of a basis, we get a dual basis $W^* = \langle y_0^*, y_1^*, y_2^* \rangle$. If $P$ is the point $y_1 = y_2 = 0$, the two maps in the Tate resolution above may be written

\begin{equation}
\alpha = \begin{bmatrix}
  y_0^* \wedge y_1^* \\
  y_0^* \wedge y_2^* 
\end{bmatrix}, \quad \beta = \begin{bmatrix}
  y_1^* \wedge y_2^* \wedge y_0 
\end{bmatrix}.
\end{equation}

Let $V = \langle x_0, x_1, x_2 \rangle$ be a vector space of dimension 3. We then zip the Tate resolution [16] with the exterior coalgebra on $V$ and get a complex of $S = \text{Sym}(V \otimes W^*)$-modules:

\begin{equation}
F_\bullet : \wedge^3 V \otimes S(-3) \xrightarrow{\psi} \wedge^2 V \otimes S(-2) \xrightarrow{\phi} S^2
\end{equation}

or simply

\begin{equation}
F_\bullet : S(-3) \xrightarrow{\psi} S(-2)^3 \xrightarrow{\phi} S^2.
\end{equation}

The first map $\psi$ is determined by $\beta$ and is given by

\begin{equation}
x_0 \wedge x_1 \wedge x_2 \mapsto (x_0 \otimes y_0^*) \wedge x_1 \wedge x_2 - (x_1 \otimes y_0^*) \wedge x_0 \wedge x_2 + (x_2 \otimes y_0^*) \wedge x_0 \wedge x_1
\end{equation}
so
\[
\psi = \begin{bmatrix}
x_0 \otimes y_0^* \\
-x_1 \otimes y_0^* \\
x_2 \otimes y_0^*
\end{bmatrix}.
\]

Also the map \( \phi \) sends
\[
x_i \wedge x_j \mapsto \left( (x_i \wedge x_j) \otimes (y_0^* \wedge y_1^*) - (x_i y_1^*) \cdot (x_j y_0^*) \right).
\]

Via the embedding of \( \wedge^2 V \otimes \wedge^2 W^* \) into \( \text{Sym}(V \otimes W^*)_2 \) this matrix becomes
\[
\left[ (x_i y_0^*) \cdot (x_j y_1^*) - (x_i y_1^*) \cdot (x_j y_0^*) \\
(x_i y_0^*) \cdot (x_j y_2^*) - (x_i y_2^*) \cdot (x_j y_0^*) \right].
\]

By Theorem 3.3 we find that \( F_\bullet \) has only one nonvanishing homology module:
\[
\bullet
\quad H_0(F_\bullet) = H^0(\mathbb{P}(W), S(\mathcal{I}_P(1))).
\]
This has dimension \( 4 \cdot 2 = 8 \) by Proposition 1.4 and its smallest degree generator is in degree 0 by Lemma 1.2c.

We now consider the dual complex:
\[
(F^\bullet)^*[2] : \mathcal{O}_2 \to \mathcal{O}_{p^2}(1)
\]
where the last term is in cohomological degree 0. Its hypercohomology table is:
\[
\begin{array}{ccccccccc}
\cdots & 14 & 9 & 5 & 2 & \cdots & \cdots & \cdots & 1 \\
\cdots & . & . & . & . & 1 & 1 & 1 & 1 & \cdots & 0 \\
\cdots & . & . & . & . & . & 1 & 3 & 6 & 10 & \cdots & -1 \\
\cdots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots & d^i
\end{array}
\]
so its Tate resolution is:
\[
\to \hat{E}(4)^5 \to \hat{E}(3)^2 \to \hat{E}(1) \to \frac{\hat{E}}{\hat{E}(-1)} \to \hat{E}(-2) \oplus \hat{E}(-3)^5 \to \cdots.
\]

Zipping this with the exterior coalgebra on \( V \) we get a complex (recall the notation (15))
\[
F^\vee : \wedge^3 V \otimes S(-3)^2 \to \wedge^1 V \otimes S(-1) \to \wedge^0 V \otimes S
\]
or simply
\[
F^\vee : S(-3)^2 \to S(-1)^3 \to S.
\]

Again using Theorem 3.3 and Lemma 1.3 its homology is:
\[
\bullet
\quad H_0(F^\vee_\bullet) = H^0(\mathbb{P}(W), S(\mathcal{O}_{p^2} \to \mathcal{O}_{p^2}(1)))
= H^0(\mathbb{P}(W), S(\mathcal{O}_P(1))) = S(\mathcal{O}_P(1)).
\]
This has dimension \( 4 \cdot 2 - 2 = 6 \) by Proposition 1.4 and its smallest degree generator is in degree 0 by Lemma 1.2c.

\[
\bullet
\quad H_1(F^\vee_\bullet) = H^{-1}(\mathbb{P}(W), S(\mathcal{O}_{p^2} \to \mathcal{O}_{p^2}(1)))
= H^0(\mathbb{P}(W), S(\mathcal{O}_{p^2}(-1))).
\]
This has dimension 8 and its smallest degree generator is of degree 2, coming from the fact that
\[ H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}^2}(-1)) = 0 = H^1(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}^2}(-2)) \]
while \( H^2(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}^2}(-3)) \) is nonzero.

We now take a general map \( V \otimes W^* \to V \), equivariant for the action of the diagonal matrices in \( GL(V) \), sending
\[ x_i \otimes y_j^* \mapsto \alpha_{ij} x_i, \]
where the \( \alpha_{ij} \in k \) are general. Letting \( \overline{S} = \text{Sym}(V) \), we get from \( F_* \) a complex:
\[ \overline{F}_* : \overline{S}(-3) \xrightarrow{\overline{\psi}} \overline{S}(-2)^3 \xrightarrow{\overline{\phi}} \overline{S}^2 \]
where
\[ \overline{\psi} = \begin{bmatrix} \alpha_{00} x_0 \\ -\alpha_{10} x_1 \\ \alpha_{20} x_2 \end{bmatrix} \]
and the map \( \overline{\phi} \) given by
\[ \begin{bmatrix} x_1 x_2 (\alpha_{10} \alpha_{21} - \alpha_{11} \alpha_{20}), x_0 x_2 (\alpha_{00} \alpha_{21} - \alpha_{01} \alpha_{20}), x_0 x_1 (\alpha_{00} \alpha_{11} - \alpha_{01} \alpha_{10}) \\ x_1 x_2 (\alpha_{10} \alpha_{22} - \alpha_{12} \alpha_{20}), x_0 x_2 (\alpha_{00} \alpha_{22} - \alpha_{02} \alpha_{20}), x_0 x_1 (\alpha_{00} \alpha_{12} - \alpha_{02} \alpha_{10}) \end{bmatrix}. \]

We show in Section 5 that the kernel of the map \( V \otimes W^* \to V \), has a basis giving a regular sequence for the homology modules of these complexes. Hence \( \overline{F}_* \) has only one nonvanishing homology module:
- \( H_0(\overline{F}_*) \) of dimension 2 with generator of degree 0.

Also \( \overline{F}_* \otimes_S \overline{S} \) (which is \( \mathbb{A}(\overline{F}_*) \)) has nonvanishing homology modules:
- \( H_0(\overline{F}_*) \) of dimension 0 and with generator of degree 0,
- \( H_1(\overline{F}_*) \) of dimension 2 and with generator of degree 2.

We now apply the functor \( \mathbb{A} \circ \mathbb{D} \) to these free squarefree complexes. The homology of \( \overline{F}_* \) is transferred to the linear strands of \( (\mathbb{A} \circ \mathbb{D})(\overline{F}_*) \). By Lemma 4.1 we easily compute:
\[ (\mathbb{A} \circ \mathbb{D})(\overline{F}_*) : \overline{S}(-3)^2 \to \overline{S}(-2)^6 \to \overline{S}(-1)^3. \]
The complex \( (\mathbb{A} \circ \mathbb{D})(\overline{F}_*) \) is the dual of \( (\mathbb{A} \circ \mathbb{D})(\overline{F}_*) \) up to translation. Since the homology of \( \overline{F}_* \) is transferred to the linear strands of \( (\mathbb{A} \circ \mathbb{D})(\overline{F}_*) \) we again easily compute
\[ (\mathbb{A} \circ \mathbb{D})(\overline{F}_*) : \overline{S}(-3) \to \overline{S}(-1)^3 \]
\[ (\mathbb{A} \circ \mathbb{D})^2(\overline{F}_*) : \overline{S}(-2)^3 \to \overline{S}. \]
Thus \( \overline{F}_*, (\mathbb{A} \circ \mathbb{D})(\overline{F}_*) \) and \( (\mathbb{A} \circ \mathbb{D})^2(\overline{F}_*) \) is a triplet of pure free squarefree complexes of the type in (14). Looking ahead to the next Section 5, Definition 5.3 and Remark 5.5 our starting complex \( F^* \) corresponds to the homology triplet
\[ B = \{0, 2, 3\}, H = \{0, 1, 2\}, C = \{0, 2\}. \]
Our triplet of free squarefree complexes above then corresponds to the degree triplet
\[ B = \{0, 2, 3\}, \overline{H} = \{1, 2, 3\}, C = \{0, 2\}. \]
4.3. From complexes on $\mathbb{P}(W)$ to squarefree complexes. Analyzing why $(\mathbb{A} \circ \mathbb{D})(\mathcal{F}_{\bullet})$ becomes a pure complex, the reason is that the dimension of $H_{0}(\mathcal{F}_{\bullet})$ which is 0, is two less than the minimal degree of a generator of the previous homology module $H_{1}(\mathcal{F}_{\bullet})$, which is 2. In general the following holds.

Lemma 4.2. Let the nonzero homology modules of a squarefree complex $\mathcal{G}_{\bullet}$ be

$$H_{i_{r}}(\mathcal{G}_{\bullet}), \ldots, H_{i_{0}}(\mathcal{G}_{\bullet})$$

where $i_{r} > i_{r-1} > \cdots > i_{1} > i_{0}$. Let $H_{i_{j}}(\mathcal{G}_{\bullet})$ have dimension $d_{j}$ and minimal degree generator of degree $e_{j}$. Then $(\mathbb{A} \circ \mathbb{D})(\mathcal{G}_{\bullet})$ is pure iff $e_{j+1} = d_{j} + (i_{j+1} - i_{j} + 1)$ for $j = 0, \ldots, r - 1$.

Proof. This follows from Lemma 4.1. \[ \square \]

The following summarizes the procedure used in Subsection 4.2 to obtain a complex $\mathcal{G}_{\bullet}$ of $\text{Sym}(V)$-modules from a complex $\mathcal{G}_{\bullet}$ of coherent sheaves on $\mathbb{P}(W)$.

Procedure 4.2.

1. Start with a complex $\mathcal{G}_{\bullet}$ of coherent sheaves on $\mathbb{P}(W)$.
2. Zip its Tate resolution with the exterior coalgebra on a vector space $V$ to get $G_{\bullet}$, a complex of free $S = \text{Sym}(V \otimes W^*)$-modules.
3. Let $V$ have basis $x_{1}, \ldots, x_{n}$ and take a general map $V \otimes W^* \rightarrow V$, equivariant for the diagonal matrices in $GL(V)$. Let $S = \text{Sym}(V)$ and $G_{\bullet} = G_{\bullet} \otimes_{S} S$.

In order for the passing from $G_{\bullet}$ to $\mathcal{G}_{\bullet}$ to behave well on homology modules, it is necessary that we divide out by a regular sequence on these homology modules. We develop conditions ensuring this in Section 5. In Section 5 we develop in detail the properties of $\mathcal{G}_{\bullet}$ such that the procedure above gives a complex $\mathcal{G}_{\bullet}$ with $\mathcal{G}_{\bullet}$, $(\mathbb{A} \circ \mathbb{D})(\mathcal{G}_{\bullet})$ and $(\mathbb{A} \circ \mathbb{D})^{2}(\mathcal{G}_{\bullet})$ a triplet of pure free squarefree complexes, for any degree triplet as conjectured in [15, Conjecture 2.11].

4.4. Second example. We shall now find a complex $\mathcal{E}_{\bullet}$ which via Procedure 4.2 gives the third complex $(\mathbb{A} \circ \mathbb{D})^{2}(\mathcal{G}_{\bullet})$ of (14).

Let $l_{0}$ and $l_{1}$ be linear forms in $k[y_{0}, y_{1}, y_{2}, y_{3}]$ and $q$ a quadratic form in this ring. We get the Koszul complex on $\mathbb{P}^{3} = \mathbb{P}(W)$.

$$O_{\mathbb{P}^{3}}[l_{0}, l_{1}, q] \leftarrow O_{\mathbb{P}^{3}}(-1)^{2} \oplus O_{\mathbb{P}^{3}}(-2) \leftarrow O_{\mathbb{P}^{3}}(-2) \oplus O_{\mathbb{P}^{3}}(-3)^{2} \leftarrow O_{\mathbb{P}^{3}}(-4).$$

Twist this complex with 1 and consider the quotient complex (look at the end)

$$\mathcal{E}_{\bullet}: O_{\mathbb{P}^{3}}(-1) \leftarrow O_{\mathbb{P}^{3}}(-1) \oplus O_{\mathbb{P}^{3}}(-2)^{2} \leftarrow O_{\mathbb{P}^{3}}(-3)$$

where we let the left term have cohomological degree 0. The cohomology is $H^{0}(\mathcal{E}_{\bullet}) = O_{L}(-1)$ where $L$ is the line in $\mathbb{P}^{3}$ defined by $l_{0}$ and $l_{1}$, and $H^{-1}(\mathcal{E}_{\bullet}) = O_{\mathbb{P}^{3}}(-1)$. Both of these are 1-regular sheaves.

Dualizing this complex we obtain

$$(\mathcal{E}_{\bullet})^{*}[2]: O_{\mathbb{P}^{3}}(2) \leftarrow O_{\mathbb{P}^{3}}(1)^{2} \oplus O_{\mathbb{P}^{3}} \leftarrow O_{\mathbb{P}^{3}}$$

where the left term is in cohomological degree 0. The cohomology is $H^{0}((\mathcal{E}_{\bullet})^{*}[2]) = O_{X}(2)$ where $X$ is the two point subscheme in $\mathbb{P}^{3}$ which is the intersection of the forms $l_{0}, l_{1}$ and $q$, and $H^{-1}((\mathcal{E}_{\bullet})^{*}[2])$ is the ideal sheaf $\mathcal{I}_{L}$. Both of these are 1-regular sheaves. From this we compute the hypercohomology table of $\mathcal{E}_{\bullet}$. 

\[ \text{GUNNAR FLØYSTAD} \]
The Tate resolution of \( E^\bullet \) is then
\[
\cdots \rightarrow \hat{E}(5)^2 \oplus \hat{E}(4)^2 \rightarrow \hat{E}(2)^2 \rightarrow \hat{E}(1) \rightarrow \hat{E}(-1)^2 \rightarrow \hat{E}(-2)^2 \rightarrow \hat{E}(-3)^{10} \rightarrow .
\]

Zipping this with the exterior coalgebra on a three dimensional vector space \( V \), we get the complex of free \( S = \text{Sym}(V \otimes W^*) \)-modules
\[
E_\bullet : S(-3)^2 \rightarrow S(-2)^6 \rightarrow S(-1)^3.
\]

Using Proposition 1.4 and Lemma 1.2c, the homology is given as follows.
- \( H_0(E_\bullet) = H^0(\mathbb{P}(W), S(O_L(-1))) \) is 10-dimensional with minimal degree generator of degree 1.
- \( H_1(E_\bullet) = H^0(\mathbb{P}(W), S(O_{\mathbb{P}^3}(-1))) \) is 12-dimensional with minimal degree generators of degree 3.

Zipping the Tate resolutions of \( (E^\bullet)^*[2] \) with the exterior coalgebra on \( V \) we get the dual of \( E_\bullet \):
\[
E^\vee_\bullet : S(-2)^3 \rightarrow S(-1)^6 \rightarrow S^2.
\]

The following is its homology:
- \( H_0(E^\vee_\bullet) = H^0(\mathbb{P}(W), S(O_X(2))) \) is 9-dimensional with minimal degree generators of degree 0.
- \( H_1(E^\vee_\bullet) = H^0(\mathbb{P}(W), S(\mathcal{I}_L)) \) is 12-dimensional with minimal degree generators of degree 2.

Take a general map \( V \otimes W^* \rightarrow V \), equivariant for the diagonal matrices in \( GL(V) \) and reduce to squarefree \( \overline{S} \)-modules
\[
\overline{E}_\bullet : \overline{S}(-3)^2 \rightarrow \overline{S}(-2)^6 \rightarrow \overline{S}(-1)^3
\]
where
- \( H_0(\overline{E}_\bullet) \) is 1-dimensional with minimal degree generator of degree 1.
- \( H_1(\overline{E}_\bullet) \) is 3-dimensional with minimal degree generator of degree 3. (Note that this degree is 2 more than the dimension of \( H_0(\overline{E}_\bullet) \).)

Also
\[
\overline{E}^\vee_\bullet : \overline{S}(-2)^3 \rightarrow \overline{S}(-1)^6 \rightarrow \overline{S}^2
\]
where
- \( H_0(\overline{E}^\vee_\bullet) \) is 0-dimensional with minimal degree generator of degree 0.
- \( H_1(\overline{E}^\vee_\bullet) \) is 3-dimensional with minimal degree generator of degree 2. (This is again 2 more than the dimension of \( H_0(\overline{E}^\vee_\bullet) \).)

It then follows that
\[
(\mathbb{A} \circ \mathbb{D})(\overline{E}_\bullet) : \overline{S}(-2)^3 \rightarrow \overline{S}
\]
and
\[
(\mathbb{A} \circ \mathbb{D})(\overline{E}^\vee_\bullet) : \overline{S}(-3)^2 \rightarrow \overline{S}(-1)^3 \rightarrow \overline{S},
\]
giving that the dual complex
\[
(\mathbb{A} \circ \mathbb{D})^2(\overline{E}_\bullet) : \overline{S}(-3) \rightarrow \overline{S}(-2)^3 \rightarrow \overline{S}^2.
\]
Hence \( \mathcal{E}_\bullet, (A \circ \mathbb{D})(\mathcal{E}_\bullet) \) and \( (A \circ \mathbb{D})^2(\mathcal{E}_\bullet) \) is a triplet of pure free squarefree complexes of the same type as in \([14]\). Looking ahead to Section \([5]\) Definition \([5.4]\) and Remark \([5.5]\), the complex \( \mathcal{E}_\bullet \) corresponds to the homology triplet

\[
B = \{1, 2, 3\}, H = \{1, 3\}, C = \{0, 2, 3\}.
\]

This gives the degree triplet

\[
B = \{1, 2, 3\}, \overline{H} = \{0\}, C = \{0, 2, 3\}.
\]

### 5. Homology triplets and complexes of coherent sheaves

First we introduce the elementary notion of three sets of natural numbers forming a homology triplet. This is (almost) the same as the notion of degree triplet in \([15]\). We then give our main conjecture concerning the existence of certain complexes of coherent sheaves associated to such homology triplets. The class of such complexes may be considered an extension of the class of vector bundles with supernatural cohomology. We also give an example and translate the conjecture to its corresponding statement for Tate resolutions.

#### 5.1. Sets of integers and their strands.

Fix an interval \([m, M]\). Let \( X \) be a subset of this interval. For \( d \in [m, M + 1] \) with \( d - 1 \not\in X \) let \( i \) be the number of elements in the interval \([m, d - 1]\) which are not in \( X \). The \( i \)’th strand of \( X \) is the interval \([d, e - 2]\) where \( e \) is the maximal number with \([d, e - 2]\) contained in \( X \).

**Example 5.1.** Let \([m, M] = [2, 12]\) and \( X = \{2, 3, 5, 9, 10, 11\} \).

- The 0’th strand is \([2, 3]\).
- The 1’th strand is \([5, 5]\).
- The 2’nd and 3’rd strands are \( \emptyset \).
- The 4’th strand is \([9, 10, 11]\).
- The 5’th strand is \( \emptyset \).

We may note the following.

1. \( e > d \) and \( e = d + 1 \) iff \( d \not\in X \). In this case the \( i \)’th strand is the empty interval.
2. The number of integers in \([m, M]\) not in \( X \), denoted \( s(X) \) is called the strand span. It is one less than the number of strands of \( X \) in \([m, M]\).
3. There are integers, called the strand starts (except the last one)

\[
m = x_0 < x_1 < \cdots < x_s < x_{s+1} = M + 2
\]

where \( s = s(X) \), such that the \( i \)’th linear strand of \( X \) is the interval \([x_i, x_{i+1} - 2]\).

**Example 5.2.** In the example above there are 6 linear strands. The strand starts are

\[
x_0 = 2, \ x_1 = 5, \ x_2 = 7, \ x_3 = 8, \ x_4 = 9, \ x_5 = 13.
\]

An element of \( X \) will be called a degree of \( X \), and an element not in \( X \) a nondegree of \( X \).

Let \( Y \) be another subset of \([m, M]\). We say that \((X, Y)\) is a balanced pair with respect to \([m, M]\) if for each \( m \leq u \leq M \) the following equivalent conditions hold:

1. The number of degrees of \( X \) in \([m, u]\) is greater than the number of nondegrees of \( Y \) in \([m, u]\).
(2) The number of degrees of \( X \) in \([m, u]\) plus the number of degrees of \( Y \) in \([m, u]\) is greater than the cardinality of \([m, u]\), which is \( u - m + 1 \).

(3) The number of degrees of \( Y \) in \([m, u]\) is greater than the number of nondegrees of \( X \) in \([m, u]\).

(4) The number of degrees of \( X \cap [m, u] \) is greater or equal to the number of strands of \( Y \cap [m, u] \) as a subset of \([m, u]\).

Note that (4) is because the number of strands of \( Y \cap [m, u] \) in \([m, u]\) is one more than the number of its nondegrees in \([m, u]\). Also note that both \( X \) and \( Y \) must contain \( m \).

**Lemma 5.3.** Let the degrees of \( X \) be \( m = d_0 < d_1 < \cdots \) and let the strand starts of \( Y \) be \( m = y_0 < y_1 < \cdots \). Then \( X \) and \( Y \) are balanced iff \( y_i > d_i \) for \( i \geq 1 \).

**Proof.** Let \( i \geq 1 \). The number of nondegrees of \( Y \) in \([m, y_i - 1]\) is \( i \). Then note that the number of degrees of \( X \) in \([m, y_i - 1]\) is greater than \( i \) if an only if \( d_i \leq y_i - 1 \). \( \Box \)

**Definition 5.4.** Let \( n \) be a natural number. For \( 0 \leq u \leq n \) let \( \overline{u} = n - u \). A triplet \((B, H, C)\) of non-empty subsets of \( \mathbb{N}_0 \) is a homology triplet of type \( n \) if there are integers \( 0 \leq b, h, c \leq n \) such that:

1. \( B \subseteq [h, \overline{c}], \quad H \subseteq [h, \overline{b}], \quad C \subseteq [c, \overline{b}] \)
   and both endpoints of each interval are in the subset. (Note the slight asymmetry, but see the remark below.)

2. Let \( i(B) \) be the number elements of \([h, \overline{c}]\setminus B\), i.e., the number of internal nondegrees of \( B \). Recall that the number of elements of \([h, \overline{b}]\setminus H\) (resp. \([c, \overline{b}]\setminus C\)) is \( s(H) \) (resp. \( s(C) \)). Then \( n = b + h + c + i(B) + s(H) + s(C) \).

3. Each of the pairs \((B, H), (\overline{B}, C), \) and \((\overline{H}, \overline{C})\) are balanced with respect to \([h, n],[c, n]\) and \([b, n]\) respectively.

We usually say only that \((B, H, C)\) is a homology triplet since the last element of \( B \) and the first of \( C \) sum to \( n \), and so determine \( n \).

**Remark 5.5.** If \((B, H, C)\) is a homology triplet, then \((\overline{B}, C, H)\) is also a homology triplet, the dual homology triplet. Also \((B, \overline{H}, C)\) is a degree triplet in the sense of [15] Definition 2.9.

Denote by \( e = i(B) + s(H) + s(C) \) the nondegree number of the homology triplet.

**Observation.** The cardinality of the interval \([h, \overline{c}]\) is \( b + e + 1 \). This follows by the equation in (2). Similarly the cardinality of \([h, \overline{b}]\) is \( c + e + 1 \) and that of \([c, \overline{b}]\) is \( h + e + 1 \).

For \( X \subseteq [0, n] \), if \( u \) is the maximum of \( X \) we define the codimension of \( X \) to be \( \overline{u} = n - u \). Note that the common codimension of \( H \) and \( C \) is \( b \).

**Lemma 5.6.** For any homology triplet \((B, H, C)\) the following equality holds for the strand spans and codimension:

\[
s(H) + s(C) + b = |B| - 1.
\]

In particular the number of (Betti) degrees in \( B \) minus one, is greater than or equal to the number of strands of \( C \) plus the codimension \( b \) of \( C \). This latter inequality is an equality iff \( H \) has only one strand or equivalently \( H \) is the interval \([h, \overline{b}]\).
Proof. The equality is because $i(B)$ is the cardinality of $[h, \overline{c}] \setminus B$ and so $|B| = n - c - h + 1 - i(B)$. Then use (2) of Definition 5.4 to get:

$$|B| - 1 = (n - c - h) - i(B) = (b + i(B) + s(H) + s(C)) - i(B) = b + s(H) + s(C).$$

The last statement is immediate from the the equality and the definition of linear strand. \qed

5.2. The main conjecture. We are now ready to state our conjecture.

Conjecture 5.7. Let $(B, H, C)$ be a homology triplet of type $n$, where the degrees of $B$ are:

$$h = d_0 < d_1 < d_2 < \cdots < d_t = c,$$

and the strand starts of $H$ and $C$ are, respectively:

$$h = h_0 < h_1 < \cdots, \quad c = c_0 < c_1 < \cdots$$

There is a complex $E^\bullet$ of coherent sheaves on some projective space $\mathbb{P}(W)$ such that the following holds:

1. **Betti degrees.** For each integer $t$ with $0 \leq t \leq n$ the hypercohomology $\mathbb{H}^t(\mathbb{P}(W), E^*(-t))$ is nonzero iff $(j, t) = (d_p - p, d_p)$ for some $d_p \in B$.

2. **Homology strands.** The homology module $H^{-p}(E^\bullet)$ of the complex is nonzero iff the $p$'th strand $[h_p, h_{p+1} - 2]$ of $H$ is nonempty. In this case:
   a. The homology is 1-regular.
   b. Its dimension is $h_{p+1} - 2$.
   c. The smallest $i$ with $H^i(\mathbb{P}(W), H^{-p}(E^\bullet)(-i))$ nonzero is $i = h_p$.

3. **Cohomology strands.** The cohomology module $H^{-p}(E^\bullet([-B] - 1))$ is nonzero iff the $p$'th strand $[c_p, c_{p+1} - 2]$ of $C$ is nonempty. In this case:
   a. The cohomology is 1-regular.
   b. Its dimension is $c_{p+1} - 2$.
   c. The smallest $i$ with $H^i(\mathbb{P}(W), H^{-p}(E^\bullet([-B] - 1))(-i))$ nonzero is $i = c_p$.

Remark 5.8. The above definition generalizes the notion of a vector bundle with supernatural cohomology in the following sense: When $H$ and $C$ are intervals, the conjecture is realized by vector bundles with supernatural cohomology and root sequence the negatives of $[0, n] \setminus B$.

5.3. Hilbert polynomials. In this subsection we calculate the Hilbert polynomial of the complex $E^\bullet$ of Conjecture 5.7. We will show that its coefficients fulfill a number of equations which is one less than the number of coefficients. Hence we expect it to be uniquely determined up to scalar multiple. We also show that the Hilbert polynomials of the homology sheaves $H^{-p}(E^\bullet)$ are uniquely determined from that of $E^\bullet$.

We seek a convenient basis for the polynomials of degree $\leq n$. For $i = 0, \ldots, n$ let

$$P_{n,i}(d) = (-1)^i \binom{d + n}{i} \binom{d + n}{n - i} = \binom{d + i - 1}{i} \binom{d + n}{n - i}.$$
These form a basis for the vector space of such polynomials since
\[
P_{n,i}(d) = \begin{cases} 
0 & d \in [-n, 0] \setminus \{-i\} \\
(-1)^i & d = -i 
\end{cases}.
\]
If \( P \) is a polynomial of degree \( \leq n \) we may then write
\[
(18) \quad P(d) = \sum_{i=0}^n \alpha_i P_{n,i}(d).
\]

**Lemma 5.9.** \( P(d) \) is a polynomial of degree \( \leq n - b \) iff the coefficients \( \alpha_i \) fulfill the equations
\[
\sum_{i=0}^{n-j} \alpha_i \binom{n-j}{i} = 0, \quad j = 0, \ldots, b - 1.
\]
Alternatively iff they fulfill
\[
\sum_{i=j}^n \alpha_i \binom{n-j}{i-j} = 0, \quad j = 0, \ldots, b - 1.
\]

**Proof.** We use induction on \( b \). Let \( b = 1 \). We have
\[
P_{n,i}(d) = \frac{1}{n!} \binom{n}{i} d^n + \text{lower terms in } d.
\]
Thus \( P \) is of degree \( \leq n - 1 \) iff \( \sum \alpha_i \binom{n}{i} = 0 \).

Suppose \( b > 0 \). We verify easily that for \( i = 0, \ldots, n - 1 \)
\[
P_{n-1,i} = P_{n,i} - \binom{n}{i} P_{n,n}.
\]
Let \( P(d) = \sum_{i=0}^{n-1} \beta_i P_{n-1,i}(d) \). Then
\[
P = \sum_{i=0}^{n-1} \beta_i P_{n,i} - \binom{n}{i} P_{n,n},
\]
so \( \alpha_i = \beta_i \) for \( i = 0, \ldots, n - 1 \). By induction \( P \) is of degree \( \leq n - 1 - (b - 1) \) iff
\[
\sum_{i=0}^{n-1-j} \beta_i \binom{n-1-j}{i} = 0, \quad j = 0, \ldots, b - 2.
\]
This translates to the conditions in the lemma. \( \square \)

For a coherent sheaf \( E \) on \( \mathbb{P}(W) \), its Hilbert polynomial is
\[
P(E, d) = \sum_{i \geq 0} (-1)^i \dim_k H^i(\mathbb{P}(W), E(d))
\]
and this is \( \dim_k H^0(\mathbb{P}(W), E(d)) \) when \( d \gg 0 \). Its degree equals the dimension of the support of \( E \). For a bounded complex of coherent sheaves \( E^\bullet \) we define
\[
P(E^\bullet, d) = \sum_{p \in \mathbb{Z}} (-1)^p P(E^p, d).
\]
This is the same as the alternating sum of the hypercohomology groups
\[
\sum_{i \in \mathbb{Z}} (-1)^i \dim_k \mathbb{H}^i(\mathbb{P}(W), E(d)).
\]
Note that the degree of $P(\mathcal{E}^\bullet, d)$ is the maximum of the dimensions of the homology sheaves, provided there is only one such attaining this maximum.

By Serre duality $P((\mathcal{E}^\bullet)^\vee, d) = P(\mathcal{E}, -d)$. Since $(\mathcal{E}^\bullet)^*= (\mathcal{E}^\bullet)^\vee (n)[n]$ we have

$$P((\mathcal{E}^\bullet)^*[|B| - 1], d) = (-1)^{|B|-1-n}P(\mathcal{E}^\bullet, -n - d).$$

Note that $P_{n, i}(-n - d) = (-1)^nP_{n, n-i}(d)$. Hence if $P(\mathcal{E}^\bullet, d)$ is given by (18) then $P((\mathcal{E}^\bullet)^*[|B| - 1], d)$ is given by

$$(-1)^{|B|-1}\sum_{i=0}^{n} \alpha_{n-i}P_{n, i}(d).$$

Let $\chi_p(d)$ be the Hilbert polynomial of $H^{-p}(\mathcal{E}^\bullet)$. Then $P(\mathcal{E}^\bullet, d) = \sum_{p=0}^{s(H)} (-1)^p \chi_p(d)$.

**Proposition 5.10.** Let $\mathcal{E}^\bullet$ be a complex corresponding to the homology triplet $(B, H, C)$.

a. The Hilbert polynomial $P(\mathcal{E}^\bullet, d)$ has degree $n - b$. We may write it as

$$\sum_{i=0}^{n} \alpha_i P_{n, i}(d) = \sum_{i \in B} \alpha_i P_{n, i}(d),$$

where $\alpha_i = 0$ for $i \notin B$. Its coefficients fulfill the following equations:

$$\alpha_0 \left( \begin{array}{c} r \\ 0 \end{array} \right) + \alpha_1 \left( \begin{array}{c} r \\ 1 \end{array} \right) + \cdots + \alpha_r \left( \begin{array}{c} r \\ r \end{array} \right) = 0$$

for each $r$ in $[h, n]$ not contained in $H$, and

$$\alpha_{n-r} \left( \begin{array}{c} r \\ r \end{array} \right) + \cdots + \alpha_{n-1} \left( \begin{array}{c} r \\ 1 \end{array} \right) + \alpha_n \left( \begin{array}{c} r \\ 0 \end{array} \right) = 0$$

for each $r$ in $[c, n]$ not contained in $C$. In the range $r \in [b, n]$ (the interval of integers excluding $b = n - b$) the set of equations (21) and (22) are equivalent by Lemma 5.9.

After this reduction we have $s(H) + s(C) + b = |B| - 1$ equations in the unknowns $\alpha_i, i \in B$.

b. The polynomials $\chi_p(d)$ are determined by $P$. More precisely

$$\sum_{i=1}^{p} (-1)^i \chi_{i-1}(d) = \sum_{i \in B \cap [0, h_p - 2]} \alpha_i P_{h_p-2, i}(d)$$

for $p = 1, \ldots, s(H) + 1$.

We expect the linear equations in a. to be independent. Hence there is a unique Hilbert polynomial up to constant, and so unique $\chi_p(d)$ up to common constant.

**Conjecture 5.11.** Let $\mathcal{E}^\bullet$ be a complex of coherent sheaves corresponding to the homology triplet $(B, H, C)$ as in Conjecture 5.7. Then its Hilbert polynomial is uniquely determined up to constant.

In Proposition 6.9 we show that Conjecture 5.7 implies the above conjecture.

**Proof of Proposition 5.10.** By Property 1. of Conjecture 5.7 $P(d) = 0$ for $-d \in [0, n] \setminus B$. This shows (20).

The last nonempty strand in $H$ is the $s(H)$'st. The homology sheaf $H^{-s(H)}(\mathcal{E}^\bullet)$ is then the one with largest dimension, $h_{s(H)+1} - 2 = b = n - b$. So this is the degree of $P(d) = P(\mathcal{E}^\bullet, d)$.
Consider the sheaf $H^{-p}(\mathcal{E}^*)$. By Properties 2.a. and 2.c. of Conjecture 5.7, no term of the Tate resolution of this sheaf can involve the modules $\mathcal{E}(i)$ when $0 \leq i < h_p$. Therefore $\chi_p(k) = 0$ when $k = 0, -1, \ldots, -(h_p - 1)$. In fact, since the $h_p$ are increasing this implies $\chi_i(k) = 0$ for all $i > p$ when $k$ is in this range. Therefore for $k = 0, -1, \ldots, -(h_p - 1)$ we have $P(k) = \sum_{i=1}^{p}(-1)^{i-1}\chi_{i-1}(k)$. For $p = 0$ this condition is already taken care of by equation (20), so we may assume $p \geq 1$. But

$$Q = \sum_{i=1}^{p}(-1)^{i-1}\chi_{i-1}(d)$$

is a polynomial of degree $\leq h_p - 2$ (with equality if the $p-1$th strand is nonempty). For such a polynomial the $(h_p - 1)$'st difference is zero and so

$$Q(0) - \binom{h_p - 1}{1}Q(-1) + \binom{h_p - 1}{2}Q(-2) + \cdots + (-1)^{p-1}\binom{h_p - 1}{h_p - 1}Q(-(h_p - 1)) = 0.$$ 

This gives the same relation if we in the above equation replace $Q$ with $P$. Since $P(-i) = (-1)^i\alpha_i$ we get

$$\alpha_0\binom{h_p - 1}{0} + \alpha_1\binom{h_p - 1}{1} + \cdots = 0, \quad p = 1, \ldots, s(H) + 1.$$ 

Since the $h_p - 1$ for $p = 1, \ldots, s(H)$ are exactly the nondegrees of $H$ in $[h, b]$, we get (21) for $r \in [h, b]$. Note that $h_{s(H) + 1} = b + 1$. For $r \in (b, n]$ we get (21) by Lemma 5.9.

Similarly $P^*(d) = P(\mathcal{E}^*|[B| - 1], d)$ which is $(-1)^{[B] - 1 - n}P(-n - d)$ will fulfill the relations

$$P^*(0) - \binom{c_p - 1}{1}P^*(-1) + \binom{c_p - 1}{2}P^*(-2) + \cdots + (-1)^{r-1}\binom{c_p - 1}{c_p - 1}P^*(-(c_p - 1))$$

for $r = 1, \ldots, s(C)$. The $c_p - 1$ in this range are precisely the nondegrees of $C$ in $[c, b]$. Since $P^*(-i) = (-1)^{[B] - 1 + n - i}\alpha_{n-i}$ we get the equations (22) for $r \in [c, b]$. For $r \in (b, n]$ we get (22) by Lemma 5.9. The equivalence of (21) and (22) when $r \in (b, n]$ also follows by Lemma 5.9.

b. Since $\sum_{i=1}^{p}(-1)^{i-1}\chi_{i-1}(d)$ is a polynomial of degree $\leq h_p - 2$, we may write it as $\sum_{k=0}^{h_p-2}\beta_kP_{h_p-2,k}(d)$. We have $P(k) = \sum_{i=1}^{p}(-1)^{i-1}\chi_{i-1}(k)$ for $k = 0, -1, \ldots, -(h_p - 1)$. Since the value of the right side of this equation is $\beta_k$ and the value of the left side is $\alpha_k$ when $k \in B$, and zero otherwise, we get part b. \qed

**Remark 5.12.** In a similar way all the Hilbert polynomials of the homology sheaves $H^{-p}((\mathcal{E}^*)^*|[B| - 1])$ are determined by the Hilbert polynomial of $\mathcal{E}^*$.

**Corollary 5.13.** Let $\mathcal{E}^*$ be a complex of coherent sheaves corresponding to the homology triplet $(B, H, C)$ as in Conjecture 5.7. The hypercohomology table of $\mathcal{E}^*$ is determined by its Hilbert polynomial.

**Proof.** Let $P$ be the Hilbert polynomial of $\mathcal{E}^*$. When $t$ is in $[-n, 0]$ the dimension of $H^p(\mathbb{P}(W), \mathcal{E}^*(t))$ is given by $(-1)^pP(t)$ by part 1. of Conjecture 5.7. When $t \geq 1$

$$\mathbb{H}^p(\mathbb{P}(W), \mathcal{E}^*(t)) = H^0(\mathbb{P}(W), H^p(\mathcal{E}^*)(t))$$

by the same argument as in Lemma 1.3. But this dimension is determined by the Hilbert polynomial of $H^p(\mathcal{E}^*)$ which is determined by $P$.

The dimensions of $\mathbb{H}^p(\mathbb{P}(W), \mathcal{E}^*(t))$ when $t \leq -n-1$ are similarly, by Serre duality, determined by the Hilbert polynomials of the homology modules of $(\mathcal{E}^*)^*$. \qed
Remark 5.14. In the Macaulay2 package Triples there are routines for computing the hypercohomology table associated to a homology triplet.

5.4. A third example. We shall construct a complex fulfilling the conjecture for the homology triplet of type \( n = 4 \) with

\[
B = \{0, 1, 2\}, \quad H = \{0, 2, 4\}, \quad C = \{2, 3, 4\}.
\]

1. Let \( X \) be three general points in \( \mathbb{P}^2 \), so the twisted ideal sheaf \( \mathcal{I}_X(2) \subseteq \mathcal{O}_{\mathbb{P}^2}(2) \) has resolution:

\[
\mathcal{E}^\bullet : \mathcal{O}_{\mathbb{P}^2}(-1)^2 \to \mathcal{O}_{\mathbb{P}^2}^3.
\]

The cohomology diagram of \( \mathcal{E}^\bullet \) is:

\[
\begin{array}{cccccccccccc}
\cdots & 6 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 2 \\
\cdots & 3 & 3 & 3 & 3 & 2 & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & 3 & 7 & 12 & \cdots & 0 & \cdot \\
\cdots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & \cdots & d \setminus i
\end{array}
\]

The dual complex is

\[
(\mathcal{E}^\bullet)^*[2] : \mathcal{O}_{\mathbb{P}^2}(1)^3 \to \mathcal{O}_{\mathbb{P}^2}(2)^2
\]

with cohomology \( \omega_X(2) \cong \mathcal{O}_X(2) \) in cohomological degree 0 and \( \mathcal{O}_{\mathbb{P}^2}(-1) \) in cohomological degree \(-1\). These are 1-regular coherent sheaves. Its hypercohomology table is:

\[
\begin{array}{cccccccccccc}
\cdots & 12 & 7 & 3 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\cdots & \cdots & 2 & 3 & 3 & 3 & 3 & \cdots & 0 \\
\cdots & \cdots & 1 & 3 & 6 & 10 & \cdots & -1 & \cdot \\
\cdots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots & d \setminus i
\end{array}
\]

2. Now we embed \( \mathbb{P}^2 \) into \( \mathbb{P}^4 = \mathbb{P}(W) \). Consider \( \mathcal{I}_X(2) \) as a sheaf on \( \mathbb{P}^4 \) via this embedding. Its resolution (obtained essentially by tensoring \( \mathcal{E}^\bullet \) with the Koszul complex \( \mathcal{O}(-2) \to \mathcal{O}(-1)^2 \to \mathcal{O} \)) is:

\[
\mathcal{G}^\bullet : \mathcal{O}_{\mathbb{P}^4}(-3)^2 \to \mathcal{O}_{\mathbb{P}^4}(-2)^7 \to \mathcal{O}_{\mathbb{P}^4}(-1)^8 \to \mathcal{O}_{\mathbb{P}^4}^3.
\]

The dual complex (also obtained essentially by tensoring \( (\mathcal{E}^\bullet)^*[2] \) with the Koszul complex \( \mathcal{O}(-2) \to \mathcal{O}(-1)^2 \to \mathcal{O} \)) is:

\[
(\mathcal{G}^\bullet)^*[2] : \mathcal{O}_{\mathbb{P}^4}(-1)^3 \to \mathcal{O}_{\mathbb{P}^4}^3 \to \mathcal{O}_{\mathbb{P}^4}(1)^7 \to \mathcal{O}_{\mathbb{P}^4}(2)^2
\]

and has cohomology

\[
H^0((\mathcal{G}^\bullet)^*[2]) = \omega_X(2) = \mathcal{O}_X(2), \quad H^{-1}((\mathcal{G}^\bullet)^*[2]) = \mathcal{O}_{\mathbb{P}^2}(-1).
\]

Its hypercohomology table is then

\[
\begin{array}{cccccccccccc}
\cdots & 12 & 7 & 3 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\cdots & \cdots & 2 & 3 & 3 & 3 & 3 & \cdots & 0 \\
\cdots & \cdots & 1 & 3 & 6 & 10 & \cdots & -1 & \cdot \\
\cdots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots & d \setminus i
\end{array}
\]

3. We now drop the last term \( \mathcal{O}_{\mathbb{P}^4}^3 \) of \( \mathcal{G}^\bullet \), and shift by 1 to get a complex

\[
\mathcal{F}^\bullet : \mathcal{O}_{\mathbb{P}^4}(-3)^2 \to \mathcal{O}_{\mathbb{P}^4}(-2)^7 \to \mathcal{O}_{\mathbb{P}^4}(-1)^8
\]

(with the last term in cohomological position 0) which is a resolution of the kernel \( \mathcal{K} \) of \( \mathcal{O}_{\mathbb{P}^4}^3 \to \mathcal{I}_X(2) \), a 1-regular sheaf. We dualize this to get:

\[
(\mathcal{F}^\bullet)^*[2] : \mathcal{O}_{\mathbb{P}^4}^8 \to \mathcal{O}_{\mathbb{P}^4}(1)^7 \to \mathcal{O}_{\mathbb{P}^4}(2)^2
\]
whose cohomology is
\[ H^0((\mathcal{F}^\bullet)^*[2]) = O_X(2), \quad H^{-1}((\mathcal{F}^\bullet)^*[2]) = O_{\mathbb{P}^2}(-1), \quad H^{-2}((\mathcal{F}^\bullet)^*[2]) = O_{\mathbb{P}^4}(-1)^3, \]
all of which are 1-regular. We see that \((\mathcal{F}^\bullet)^*[2]\) is a complex on \(\mathbb{P}^4\) fulfilling the conditions of Conjecture 5.7 for the homology triplet (23).

The hypercohomology table of \((\mathcal{F}^\bullet)^*[2]\) is:

```
|       | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | ... | d \i |
|-------|----|----|----|----|----|---|---|---|---|-----|-----|
| 87    |    |    |    |    |    |   |   |   |   |     |     |
| 33    |    |    |    |    |    |   |   |   |   |     |     |
| 8     |    |    |    |    |    |   |   |   |   |     |     |
|       | 2  | 3  | 3  | 3  | 3  | 1 | 1 | 1 | 1 |     |     |
|       | 3  | 6  | 10 | -1 |     |   |   |   |   |     |     |
|       | 15 | 45 | 105| 210| -2  |   |   |   |   |     |     |
|       |    |    |    |    |     | 2 | 3 | 6 | 10 | -15 | -45 |
|       |    |    |    |    |     | 1 | 3 | 6 | 10 | -15 | -45 |
|       |    |    |    |    |     | 1 | 3 | 6 | 10 | -15 | -45 |
|       |    |    |    |    |     | 1 | 3 | 6 | 10 | -15 | -45 |
```

and so the Tate resolution of \((\mathcal{F}^\bullet)^*[2]\) is:
\[
(24) \quad \to \hat{E}(7)^{87} \to \hat{E}(6)^{33} \to \hat{E}(5)^8 \to \hat{E}(2)^2 \to \hat{E}(1)^3 \to \hat{E}(-1)^3 \to \hat{E}(-2)^3 \to \hat{E}(-3)^{15} \to \hat{E}(-4)^{45}
\]

We zip this complex with the exterior coalgebra on a four-dimensional vector space \(V\) and get
\[
\wedge^2 V \otimes S(-2)^2 \to V \otimes S(-1)^3 \to S^3.
\]

Then we reduce to a squarefree complex of \(S = \text{Sym}(V)\)-modules
\[
F^\bullet : S(-2)^{12} \to S(-1)^{12} \to S^3.
\]

This sits in a triplet of pure free squarefree complexes where
\[
(A \circ \mathbb{D})(F^\bullet) : S(-4)^3 \to S(-2)^6 \to S^3 \quad \text{and} \quad (A \circ \mathbb{D})^2(F^\bullet) : S(-4)^3 \to S(-3)^{12} \to S(-2)^{12}.
\]

5.5. **The conjecture in terms of Tate resolutions.** Conjecture 5.7 may also be stated in terms of Tate resolutions of the complex \(E^\bullet\), which may be convenient when trying to construct such complexes.

Given a Tate resolution \(T\). Let \(T(e, -)\) be the subcomplex consisting of the terms \(\hat{E}(i)\) where \(i \leq e\), and let \(T(-, d)\) be the quotient complex consisting of the terms \(\hat{E}(i)\) where \(i \geq d\). Also let \(T(e, d)\) be the subquotient complex consisting of the \(\hat{E}(i)\) where \(e \geq i \geq d\).

**Example 5.15.** Consider the Tate resolution \(T\) in (24). Then
\[
T(-1, -) : \hat{E}(-1)^3 \to \hat{E}(-1)^3 \to \hat{E}(-2)^{15} \to \hat{E}(-2)^{15} \to \cdots \to \hat{E}(-3)^{15} \to \hat{E}(-3)^{15}
\]
and
\[
T(4, 0) : \hat{E}(2)^2 \to \hat{E}(1)^3 \to \hat{E}^3.
\]

Now given a homology triplet \((B, H, C)\) of type \(n\), where the degrees of \(B\) are:
\[
h = d_0 < d_1 < d_2 < \cdots < d_t = \overline{c},
\]
and the strand starts of $H$ and $C$ are, respectively:

$$h = h_0 < h_1 < \cdots, \quad c = c_0 < c_1 < \cdots$$

Let $\mathcal{E}^\bullet$ be a complex of coherent sheaves on $\mathbb{P}(W)$ and $T = T(\mathcal{E}^\bullet)$ its Tate resolution.

**Proposition 5.16.** (Betti invariants) Property 1. of Conjecture 5.7 is equivalent to $T(n, 0)$ being a pure complex

$$\widehat{E}(d_i)^{\beta_i} \rightarrow \cdots \rightarrow \widehat{E}(d_0)^{\beta_0}.$$  

**Proof.** This is clear. □

**Remark 5.17.** This is equivalent to the Beilinson monad of the complex $\mathcal{E}^\bullet$, [8] or see [8, Sec.6], having pure term $(\Omega^j(d_j))^{\beta_j}$ in homological degree $j$.

**Proposition 5.18.** (Homology invariants.) Property 2. of Conjecture 5.7 is equivalent to: The $p$'th linear strand of $T(-1, -)$ is nonzero iff the $p$'th homology strand $[h_p, h_{p+1} - 2]$ of $H$ is nonempty. In this case the linear strand is

$$\widehat{E}(-1)^{\alpha_p} \rightarrow \widehat{E}(-2)^{\alpha_{p+1}} \rightarrow \cdots$$

where the first term is in cohomological position $-p + 1$. Furthermore

a'. This linear strand is a resolution of $\ker d^{-p+1}$.

b'. The dimension of the linear strand is $h_{p+1} - 2$.

c'. The smallest $t$ with a nonzero map $\widehat{E}(t) \rightarrow \ker d^{-p+1}$ is for $t = h_p$. Alternatively the smallest degree generator of $\ker d^{-p+1}$ has degree $n - h_p$.

**Example 5.19.** In the example of Subsection 5.4 we see that the 0'th, 1'st, and 2'nd homology strands $H$ in (23) are respectively $[0, 0]$, $[2, 2]$, and $[4, 4]$. Considering $T(-1, -)$ its strands are:

$$T(-1, -)_{(0)} : \widehat{E}(-1)^3 \xrightarrow{d^1} \widehat{E}(-2)^3 \rightarrow \cdots.$$  

which is of dimension 0 with a nonzero map $\widehat{E} \rightarrow \ker d^1$.

$$T(-1, -)_{(1)} : \widehat{E}(-1) \xrightarrow{d^0} \widehat{E}(-2)^3 \rightarrow \widehat{E}(-3)^6 \rightarrow$$

is of dimension 2 with a nonzero map $\widehat{E}(2) \rightarrow \ker d^0$.

$$T(-1, -)_{(2)} : \widehat{E}(-1)^3 \xrightarrow{d^{1}} \widehat{E}(-2)^{15} \rightarrow \widehat{E}(-3)^{45} \rightarrow$$

is of dimension 4 with a nonzero map $\widehat{E}(4) \rightarrow \ker d^{-1}$.

**Proof of Proposition 5.18.** First note the following.

Fact 1. For $t \gg 0$ we have.

$$H^{-p}(\mathbb{P}(W), \mathcal{E}^\bullet(t)) = H^0(\mathbb{P}(W), H^{-p}(\mathcal{E}^\bullet)(t)).$$

Thus the linear strand $T(-1, -)_{(p)}$ has $t$'th term

$$\widehat{E}(-t) \otimes_k H^0(\mathbb{P}(W), H^{-p}(\mathcal{E}^\bullet)(t))$$

when $t$ is large.

Fact 2. By the form of the Tate resolution [12] of a coherent sheaf $\mathcal{F}$, its regularity $r$ is precisely the smallest cohomological index $r$ such that $T^r(\mathcal{F})$ is a linear complex.

Assume now that Property 2. of Conjecture 5.7 holds. By its part a. and Fact 2. immediately above, the $t$'th term of the linear strand is (25) for $t \geq 1$. Thus the
p’th linear strand and $H^{-p}(\mathcal{E}^\bullet)$ are nonzero at the same time, and by assumption in Conjecture 5.7 this holds iff $[h_p, h_{p+1} - 2]$ is nonempty.

If Property 2a. holds then a’. above holds by Fact 2. about the regularity. If 2b. holds then clearly b’. holds. That 2c. implies c’. follows by the form of the Tate resolution for $H^{-p}(\mathcal{E}^\bullet)[p]$ when looking at it in cohomological degree $-p$.

Assume now the statements within Proposition 5.18 above hold. Nonzero injective resolutions over the exterior algebra are infinite. Therefore a’ together with (25) for $t \gg 0$, imply that the $p$’th linear strand is nonzero iff $H^{-p}(\mathcal{E}^\bullet)$ is nonzero. And so this latter holds iff $[h_p, h_{p+1} - 2]$ is nonempty.

That 2b. follows from b’. is clear. Also 2c. implies c’. by the form of the Tate resolution for $H^{-p}(\mathcal{E}^\bullet)$ when looking at it in cohomological degree $-p$.

Let

$$T' = T((\mathcal{E}^\bullet)^\ast)[|B| - 1] = T((\mathcal{E}^\bullet)^\ast)(n)[|B| - 1] = \text{Hom}_k(T, \wedge^{m+1}W)(n)[|B| - 1].$$

In the same way as we proved the above Proposition 5.18, the following holds.

**Proposition 5.20.** (Cohomology invariants.) Property 3. of Conjecture 5.7 is equivalent to: The $p$’th linear strand of $T'(-1, -)$ is nonzero iff the $p$’th cohomology strand $[c_p, c_{p+1} - 2]$ of $C$ is nonempty. Letting this linear strand start as

$$\hat{E}(-1)^{-p+1} \to \cdots$$

where the first term must be in cohomological degree $-p + 1$, it has the following properties.

a’. The linear strand is a resolution of $\ker(d')^{-p+1}$.

b’. The dimension of the linear strand is $c_{p+1} - 2$.

c’. The smallest $t$ with a nonzero map $\hat{E}(t) \to \ker(d')^{-p+1}$ is for $t = c_p$. Alternatively the smallest degree generator of $\ker(d')^{-p+1}$ has degree $n - c_p$.

**Example 5.21.** In the example of Subsection 5.4 there is one cohomology strand $[2, 4]$. We see that

$$T'(-1, -) : \hat{E}(-1)^8 \to \hat{E}(-2)^{33} \to \cdots$$

It is of dimension 4 and 2 is the smallest $t$ with a nonzero map $\hat{E}(t) \to \ker d'_1$.

**Remark 5.22.** We see that Conjecture 5.7 is not simply equivalent to a statement on the form of the hypecohomology table of the Tate resolution. There is also the conditions that the linear strands are the resolutions of the specified differential, as well as on the degree of the minimal degree generator of its kernel.

### 6. The Conjecture on Triplets of Pure Free Squarefree Complexes

In this section we show that Conjecture 5.7 implies the Conjecture 2.11 in [15] on the existence of triplets of pure free squarefree complexes with balanced degree triplets.

The procedure for this is demonstrated in the examples of Section 4, in particular we use Procedure 4.2. The crucial thing is to prove that when passing from $S(V \otimes W^*)$ to the quotient $S(V)$, we divide out by a sequence which is regular for the homology modules of the zip complex.
Note. For the results in this section we assume that $k$ is an infinite field. This is in order to ensure the expected codimension of some degeneracy loci of maps of vector bundles.

6.1. Degeneracy loci of vector bundles.

Proposition 6.1. Let $X$ be scheme of finite type over a field $k$ (assumed infinite), and $\mathcal{E}$ a vector bundle of rank $e$ on $X$ generated by a subspace $E$ of its global sections $\Gamma(X, \mathcal{E})$. Let $E_i$ for $i = 1, \ldots, t$ be general vector subspaces of $E$, all of these subspaces of dimension $e$, the rank of $\mathcal{E}$. This gives maps between vector bundles of the same rank $E_i \otimes \mathcal{O}_X \rightarrow \mathcal{E}$. Then the locus in $X$ where $\alpha_i$ has corank $\geq q_i$ for each value $i = 1, \ldots, t$, has codimension $\geq \sum q_i^2$.

Proof. We will use induction on $t$. When $t = 1$, the codimension of $X$ is $q_1^2$, by [16, Example 14.3.2(d)]

Suppose the statement holds for $t - 1$. Let $X'$ be the locus where $\alpha_i$ has corank $\geq q_i$ for each value $i = 1, \ldots, t - 1$. It has codimension $\sum_{i=1}^{t-1} q_i^2$. Consider the restriction $E_i \otimes \mathcal{O}_{X'} \rightarrow \mathcal{E}|_{X'}$. Now the latter restricted vector bundle is generated by the image of $E$ in its global sections $\Gamma(X', \mathcal{E}|_{X'})$. Since $E_i$ is a general subspace of $E$, of dimension $e$, the map $\alpha_i$ will degenerate to corank $\geq q_i$ in codimension $q_i^2$ in $X'$. Hence the locus in $X$ where the $\alpha_i$ degenerate as prescribed, has codimension $\geq \sum q_i^2$.

Let $\mathcal{E}$ be a vector bundle, i.e. a locally free sheaf of finite rank $e$, on a $k$-scheme $X$. Let $T$ be a subspace of the sections $\Gamma(X, \mathcal{E})$. The map $T \otimes_k \mathcal{O}_X \rightarrow \mathcal{E}$ defines a map and an exact sequence

$$(26) \quad T \otimes_k \text{Sym}(\mathcal{E}) \rightarrow \text{Sym}(\mathcal{E}) \rightarrow R \rightarrow 0.$$  

where the cokernel $R$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The space $T$ gives global sections of the affine bundle $V = V_X(\mathcal{E})$ and they generate a sheaf of ideals of $\mathcal{O}_V$ defining a subscheme $X = \text{Spec}_{\mathcal{O}_X} R$.

Now we may stratify $X$ according to the rank of the map $T \otimes_k \mathcal{O}_X \rightarrow \mathcal{E}$. Let $U_c$ be the open subset where the rank is $\geq \dim_k T - c = t - c$. Then if $x \in U_c \setminus U_{c-1}$ we get an exact sequence

$$T \otimes_k \text{Sym}(\mathcal{E}_{k(x)}) \rightarrow \text{Sym}(\mathcal{E}_{k(x)}) \rightarrow \mathcal{R}_{k(x)} \rightarrow 0$$

where $\mathcal{R}_{k(x)}$ is the quotient symmetric algebra generated by a vector space of dimension $e - t + c$. Hence the fiber $X_{k(x)}$ has dimension $e - t + c$. We observe that the dimension of $X$ is less than or equal to the maximum of

$$(27) \quad \max\{\dim(X \setminus U_{c-1}) + e - t + c\}.$$

Now let $X = \mathbb{P}(W)$, let $V$ be a vector space with basis $x_1, \ldots, x_n$, and $\mathcal{E} = V \otimes \mathcal{Q}$ where $\mathcal{Q}$ is the dual of the tautological subbundle of rank $m$ on $\mathbb{P}(W)$. For each $x_i$ chose a general subspace $E_i \subseteq W^*$ of codimension one, so its dimension equals the rank of $\mathcal{Q}$. Define the subspace

$$(28) \quad T = \oplus_{i=1}^n x_i \otimes E_i \subseteq \oplus_{i=1}^n x_i \otimes W^* = V \otimes W^*.$$

Corollary 6.2. a. The locus where the composition $\alpha : T \otimes_k \mathcal{O}_{\mathbb{P}(W)} \rightarrow V \otimes_k W^* \otimes_k \mathcal{O}_{\mathbb{P}(W)} \rightarrow V \otimes_k \mathcal{Q}$
degenerates to rank \( \dim_k T - c \), has codimension \( \geq c \).

b. The dimension of \( X = \text{Spec}_{\mathcal{O}_P W)} \mathcal{R} \), the subscheme of \( \mathcal{V}_P(W)(V \otimes Q) \) defined by the vanishing of \( T \), has dimension less than or equal to the dimension of \( P(W) \).

Proof. Part a. follows by Proposition 6.1 by letting the maps \( x_i \otimes E_i \otimes \mathcal{O}_P(W) \rightarrow x_i \otimes Q \). If each \( \alpha_i \) degenerates to corank \( q_i \), then \( \alpha \) degenerates to corank \( \sum_i q_i \). If \( \sum_i q_i \geq c \), then the degeneracy locus in \( P(W) \) has codimension \( \sum_i q_i^2 \geq c \).

Part b. follows by part a. and the expression for the dimension given by (27).

6.2. Regular sequences. If \( u_1, \ldots, u_n \) in a \( k \)-algebra \( R \) form a regular sequence for the module \( M \), then any basis for the vector space they generate, \( \langle u_1, \ldots, u_n \rangle \subseteq R \), forms a regular sequence, as is easily seen by Koszul homology, [6, Theorem 17.4, 17.6]. We then call this a regular subspace for the module \( M \).

Proposition 6.3. Let \( \dim_k V \geq \dim P(W) \) and suppose \( S(\mathcal{E}) \) is a Cohen-Macaulay \( \text{Sym}(V \otimes W^*) \)-module. Then for general \( E_i \subseteq W^* \) of codimension one, the subspace \( T = \oplus_i x_i \otimes E_i \subseteq V \otimes W^* \) is a regular subspace for this module.

Proof. The subscheme \( V(V) \) of \( V(V \otimes W^*) \) is defined by the vanishing of the subspace \( T \) of \( V \otimes W^* \). Let \( Z' \) be the pullback in the diagram

\[
\begin{array}{c}
Z' \subseteq V(V) \\
\downarrow \\
Z \subseteq V(V \otimes W^*) \\
\downarrow \\
\mathcal{V}(V \otimes P(W)) \\
\end{array}
\]

Since \( Z = \mathcal{V}_P(W)(V \otimes_k Q) \) we see that \( Z' \) is the subscheme of \( Z \) defined by the vanishing of the sections \( T \) of \( V \otimes Q \) given by the composition \( \alpha \) in Corollary 6.2.a. By part b. of this, the dimension of \( Z' \) is less than or equal to \( \dim P(W) \). Since \( \dim_k T \) equals the rank of \( V \otimes Q \), the dimension of \( Z \) is \( \dim P(W) + \dim_k T \) and so \( \dim Z' \leq \dim Z - \dim_k T \).

Let \( Y' \) be the pullback in the diagram

\[
\begin{array}{c}
Y' \subseteq V(V) \\
\downarrow \\
Y \subseteq V(V \otimes W^*) \\
\end{array}
\]

Since the image of \( Z \) is \( Y \), the image of \( Z' \) is \( Y' \). Since \( \dim Y = \dim Z \) by [22, Prop.6.1.1] this gives

\[ \dim Y' \leq \dim Z' \leq \dim Z - \dim_k T = \dim Y - \dim_k T. \]

The sheaf \( \mathcal{O}_Z \otimes p^*(\mathcal{E}) \) has support \( Z \) and so the support of \( S(\mathcal{E}) \) is \( Y \). Denote by \( S(\mathcal{E})' \) the module \( S(\mathcal{E}) \otimes_{\text{Sym}(V \otimes W^*)} \text{Sym}(V) = \text{Sym}(V \otimes W^*)/(T) \). Then \( S(\mathcal{E})' \) is supported on \( Y' \) and so

\[ \dim S(\mathcal{E})' \leq \dim Y' \leq \dim Y - \dim_k T = \dim S(\mathcal{E}) - \dim_k T. \]

Since \( S(\mathcal{E}) \) is a Cohen-Macaulay module and \( S(\mathcal{E})' = S(\mathcal{E})/(T \cdot S(\mathcal{E})) \), the subspace \( T \) must be a regular subspace for the module \( S(\mathcal{E}) \).

Lemma 6.4. The module \( S(\mathcal{O}_P(-1)) \) is a Cohen-Macaulay \( \text{Sym}(V \otimes W^*) \)-module.
Proof. This module has dimension \((n + 1)\) by Proposition 3.4. Since \(\mathcal{O}_{P(W)}(-1)\) is 1-regular, by Proposition 3.6 the module has resolution given by \(\mathcal{W}^V_W(\mathcal{O}_{P(W)}(-1))\). The terms of the Tate resolution of \(\mathcal{O}_{P(W)}(-1)\) are
\[
\cdots \rightarrow \hat{E}(m + 1)^{m+1} \rightarrow \hat{E}(m) \rightarrow \hat{E}(-1) \rightarrow \cdots
\]
and so the associated zip complex is
\[
\wedge^n V \otimes S(-n)^{n-m} \rightarrow \cdots \rightarrow \wedge^{m+1} V \otimes S(-m+1)^{m+1} \rightarrow \wedge^m V \otimes S(-m).
\]
This complex has length \(n - m\). Since it resolves a module of dimension \((n + 1)m\)
and
\[
(n + 1)m + (n - m) = n(m + 1) = \dim \text{Sym}(V \otimes W^*)
\]
this module is Cohen-Macaulay. \(\square\)

Let \(W' \subseteq W\) and consider the projection \(\mathbb{P}(W) \rightarrow \mathbb{P}(W')\) with center the subspace \(\mathbb{P}(W/W') \subseteq \mathbb{P}(W)\). Let \(\mathcal{G}\) be a coherent sheaf on \(\mathbb{P}(W)\) whose support is disjoint from \(\mathbb{P}(W/W')\). Pushing forward, we get a coherent sheaf \(\pi_\ast \mathcal{G}\) on \(\mathbb{P}(W')\). It is well known that the cohomology \(H^p(\mathbb{P}(W), \mathcal{G}(i)) = H^p(\mathbb{P}(W'), (\pi_\ast \mathcal{G})(i))\). In fact we have:

**Proposition 6.5.** Let \(\mathcal{G}\) be a coherent sheaf on \(\mathbb{P}(W)\) whose support is disjoint from \(\mathbb{P}(W/W')\). Let \(E' = \bigoplus \wedge^i (W')^\ast\). The Tate resolution \(\mathbb{T}(\pi_\ast \mathcal{G}) = \text{Hom}_E(E', \mathbb{T}(\mathcal{G}))\).

**Proof.** This is [3] Thm.7.1.2]. \(\square\)

**Corollary 6.6.** Suppose \(\mathcal{G}\) is 1-regular. Then \(V \otimes (W/W')^\ast\) is a regular subspace of \(\text{Sym}(V \otimes W^*)\) for the module \(\mathcal{S}(\mathcal{G})\). Furthermore

\[
S(\pi_\ast \mathcal{G}) = \mathcal{S}(\mathcal{G}) \otimes \text{Sym}(\mathcal{V} \otimes W^*) \otimes \text{Sym}(V \otimes W'^*).
\]

**Proof.** The complex \(\mathcal{W}^V_W(\mathbb{T}(\mathcal{G}))\) is a resolution of \(\mathcal{S}(\mathcal{G})\) by Proposition 3.6. Let \(S' = \text{Sym}(V \otimes W'^*)\). Then by Lemma 2.2

\[
\mathcal{W}^V_W(\mathcal{G}) \otimes_S S' = \mathcal{W}^V_{W'}(\text{Hom}_E(E', \mathbb{T}(\mathcal{G}))).
\]

By Proposition 6.5 the latter is \(\mathcal{W}^V_{W'}(\mathbb{T}(\pi_\ast \mathcal{G}))\). But since \(\pi_\ast \mathcal{G}\) is also a 1-regular coherent sheaf on \(\mathbb{P}(W')\), (29) is a resolution of \(S(\pi_\ast \mathcal{G})\), and so

\[
\text{Tor}_i^S(S(\mathcal{G}), S') = 0, \quad i > 0.
\]
But then by Koszul homology, [3] Thm.17.4, 17.6] the space \(V \otimes (W/W')^\ast\) is a regular subspace of \(\text{Sym}(V \otimes W^*)\) for the module \(S(\pi_\ast \mathcal{G})\). \(\square\)

**Theorem 6.7.** Let \(\mathcal{F}\) be a 1-regular coherent sheaf on \(\mathbb{P}(W)\), and \(V = \langle x_1, \ldots, x_n \rangle\) a vector space of dimension \(\geq 1\) that of the support of \(\mathcal{F}\). Let \(E_i \subseteq W^*\) be general subspaces of codimension 1 for \(i = 1, \ldots, n\). The subspace \(T = \bigoplus_{i=1}^n x_i \otimes E_i \subseteq V \otimes W^*\)

is a regular subspace of \(\text{Sym}(V \otimes W^*)\) for the module \(S(\mathcal{F})\).

**Proof.** We will use induction on the dimension of \(\mathbb{P}(W)\). If \(\dim \mathbb{P}(W) = 0\) there is nothing to prove, since then \(\dim_k W = 1\) and \(T = 0\).

Suppose \(\dim \mathbb{P}(W) > 0\). If the support of \(\mathcal{F}\) is not all of \(\mathbb{P}(W)\) we project down to a projective space \(\mathbb{P}(W')\) such that \(\dim \mathbb{P}(W') = \dim \text{Supp} \mathcal{F}\), with a center of projection disjoint from the support of \(\mathcal{F}\). Then by the previous corollary \(T' = \bigoplus_{i=1}^n x_i \otimes (W/W'^*) \subseteq V \otimes W^*\) is a regular subspace for \(S(\mathcal{F})\) and

\[
S(\mathcal{F}) \otimes \text{Sym}(V \otimes W^*) \text{ Sym}(V \otimes W'^*) = S(\mathcal{F})/(T' \cdot S(\mathcal{F})) \cong S(\pi_\ast \mathcal{F}).
\]
By induction hypothesis there is a regular subspace $\oplus_i x_i \otimes E'_i$ of $V \otimes (W')^*$ for the module $S(\pi, F)$, where $E'_i \subseteq (W')^*$ has codimension 1. Letting $E_i$ be the inverse image of $E'_i$ for $W^* \to W'^*$ we see that $\oplus_i x_i \otimes E_i$ is a regular subspace.

Now suppose the support of $F$ is $\mathbb{P}(W)$. Since $F$ is 1-regular, $F(1)$ is generated by its global sections, and so if $F$ has rank $r$, there is a map and an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(W)}(-1)^r \to F \to G \to 0$$

where the cokernel $G$ is supported in a proper closed subscheme of $\mathbb{P}(W)$. By Lemma 1.2a, $H^1(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(-1))$ vanishes and so we have an exact sequence

$$0 \to S(\mathcal{O}_{\mathbb{P}(W)}(-1))^r \to S(F) \to S(G) \to 0.$$

By Proposition 6.3 and Lemma 6.4, a general subspace $T$ will be a regular subspace for $S(\mathcal{O}_{\mathbb{P}(W)}(-1))$ by the first part of the argument, and induction, such a $T$ is a regular subspace for $S(G)$. But then it must also be a regular subspace for $S(F)$.

\section{Triplets of pure free squarefree complexes}

A subspace $T = \oplus_{i=1}^n x_i \otimes E_i$ of $V \otimes W^*$ as in (28) defines a quotient map

$$V \otimes W^* = \oplus_{i=1}^n (x_i \otimes W^*) \to \oplus_{i=1}^n (W^*/E_i) \cong V$$

equivariant for the subgroup of diagonal matrices in $GL(V)$. We get the quotient map of algebras $Sym(V \otimes W^*) \to Sym(V)$.

The following shows that Conjecture 5.7 implies Conjecture 2.11 in [15] concerning the existence of triplets of pure free squarefree complexes.

\begin{theorem}
Let $E^*$ be a complex of coherent sheaves in Conjecture 5.7 associated to the homology triplet $(B, H, C)$. Let $Sym(V \otimes W^*) \to Sym(V)$ be the quotient map defined above where $T$ is sufficiently general and let

$$(30) \quad E^* = W^V_W(T(E^*)) \otimes_{Sym(V \otimes W^*)} Sym(V)$$

This is a complex of free squarefree $Sym(V)$-modules such that

$$E^*_i, \quad (A \otimes \mathbb{D})(E^*_i), \quad (A \otimes \mathbb{D})^2(E^*_i)$$

is a triplet of free squarefree complexes of $Sym(V)$-modules with degree triplet $(B, H, C)$.

\end{theorem}

\begin{proof}
The quotient map of algebras is equivariant for the action of the diagonal matrices in $GL(V)$. The terms of $E^*_i$ have the form $\oplus \wedge V \otimes Sym(V) \otimes N_j^p$ with the diagonal matrices of $GL(V)$ acting naturally on $\wedge V$ and $Sym(V)$ and trivially on $N_j^p$. Hence the terms of $E^*_i$ are pure free squarefree $S(V)$-modules.

The $p$th homology of $W^V_W(T(E^*))$ is $S(H^{-p}(E^*))$ by Theorem 6.3 and Lemma 1.3. By Theorem 6.7 the subspace $T$ of $V \otimes W^*$ is a regular space for these modules. Thus $E^*_i$ of (30) will be a complex of free $S(V)$-modules whose $p$th homology module is

$$(31) \quad S(H^{-p}(E^*)) \otimes_{Sym(V \otimes W^*)} Sym(V).$$

The dimension of $H^{-p}(E^*)$ is $h_{p+1} - 2$ and so by Proposition 1.4 the dimension of $S(H^{-p}(E^*))$ is $nm + h_{p+1} - 2$. Since the kernel of $V \otimes W^* \to V$ has dimension $nm$, the module (31) above also has dimension $h_{p+1} - 2$. Also the minimal degree generator of (31) has the same degree as the minimal degree generator of $S(H^{-p}(E^*))$, which is $h_p$ by Lemma 1.2. This gives that $(A \otimes \mathbb{D})(E^*_i)$ will be a pure free squarefree complex with degree sequence $\overline{H}$. 

\end{proof}
Applying the same procedure to $E^*[|B| - 1]$, we get a complex $E^\vee$ which is the dual of the complex $E$. Furthermore $(A \circ \mathbb{D})(E^\vee)$ will be a pure free squarefree complex with degree sequence $\overline{C}$. But this complex is the dual of $(A \circ \mathbb{D})^2(E)$ and so this is a pure free squarefree complex with degree sequence $C$. \hfill \Box

**Corollary 6.9.** If Conjecture 5.7 holds for all homology triplets, then Conjecture 5.11 holds.

**Proof.** Let $P$ be the Hilbert polynomial of $E$. The complex $E$ has terms
\[
S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_t)^{\beta_t}
\]
where
\[
\beta_i = (\dim_k \wedge^d V) \cdot (-1)^{d-i} P(-d_i).
\]
By Theorem 3.9 in [15], the existence of triplets of pure free squarefree complexes for all degree triplets $(B, \overline{H}, C)$ implies that the $\beta_i$ are uniquely determined up to constant. Hence the values of $P(-d_i)$ are also so. Since $P(-d) = 0$ for $-d$ in $[-n, 0] \backslash \{-d_0, \ldots, -d_t\}$, the polynomial $P$ is uniquely determined up to constant. \hfill \Box

7. Cones of cohomology tables of coherent sheaves, and of homology triplets of squarefree modules

The previous sections show a close relationship between cohomology tables of coherent sheaves where the cohomology sheaves have certain regularity properties, and the triplets of homological data $(B, H, C)$ for complexes of squarefree modules.

In this section we give a conjecture, Conjecture 7.8, on this relationship: There is an isomorphism between the positive rational cone generated by such cohomology tables and the positive rational cone generated by such numerical homological triples.

### 7.1. Hypercohomology tables of complexes of coherent sheaves.

**Definition 7.1.** Fix a natural number $n \geq m - 1$. We consider complexes $\mathcal{F}$ such that $H^i(\mathcal{F})$ is $1$-regular for all $i$, and $H^i((\mathcal{F}^\vee)^\vee)$ is $n + 1$-regular for all $i$. Let $\gamma$ be the cohomology table $\dim_k \mathbb{H}(\mathbb{P}(W), \mathcal{F}(j))$ which we may consider as an element of $\mathbb{Q}^{2n \times 2}$. Denote by $C(\text{coh}, n)$ the positive rational cone generated by such tables, i.e. the cone of all expressions $\sum c_i \gamma^i$ where $c_i \in \mathbb{Q}^+$ and $\gamma^i$ is the cohomology table of such a complex $\mathcal{F}$.

**Remark 7.2.** The sheaves $\mathcal{O}_{\mathbb{P}^m}(j)$ for $j = -1, \ldots, n - m$ fulfill the definition above. More generally for $i \leq m$, the sheaves $\mathcal{O}_{\mathbb{P}^i}(j)$ for $j = -1, \ldots, n - i$ fulfill this.

**Lemma 7.3.** Let $\mathcal{F}$ fulfill the conditions in the definition above. For $t \geq 1$ the hypercohomology
\[
\mathbb{H}^i(\mathbb{P}(W), \mathcal{F}(t)) = H^0(\mathbb{P}(W), H^i(\mathcal{F})(t)),
\]
and for $t \leq -n - 1$ the hypercohomology
\[
\mathbb{H}^{-i}(\mathbb{P}(W), \mathcal{F}(t)) \cong H^0(\mathbb{P}(W), H^{-i}(\mathcal{F}^\vee)(-t))^\vee.
\]
Thus for twists outside the range $[0, n]$, the hypercohomology of $\mathcal{F}$ is completely determined by the Hilbert functions of the cohomology sheaves of $\mathcal{F}$ and of $(\mathcal{F}^\vee)(n)$, in positive degrees.
Proof. The first part is by Lemma 1.3. By Serre duality
\[ \mathbb{H}^i(\mathbb{P}(W), \mathcal{F}^*(-t)) \cong \mathbb{H}^{-i}(\mathbb{P}(W), (\mathcal{F}^*)^v(t))^* \]
and so we get the second part also by Lemma 1.3. □

Now let \( V \) be a vector space of dimension \( n \). We get the Tate resolutions \( \mathbb{T}(\mathcal{F}^*) \) with
\[ \mathbb{T}^{-p}(\mathcal{F}^*) = \bigoplus_{j \in \mathbb{Z}} \mathbb{E}(p + j) \otimes \mathbb{H}^j(\mathbb{P}(W), \mathcal{F}^*(-p - j)), \]
and the zip complex \( F_* = \mathbb{W}(\mathcal{F}^*) \), where \( S = \text{Sym}(V \otimes W^*) \) and
\[ F_i = \sum_{j+i=0}^n S(-i-j) \otimes \mathbb{H}^j(\mathbb{P}(W), \mathcal{F}(-i-j)) \otimes \wedge^{i+j}V, \]
so the Betti spaces encode all the cohomology of twists of \( \mathcal{F}^* \), when the twist is in the range \([-n, 0]\).

The homology of \( \mathcal{F}^* \) is
\[ H_i(\mathcal{F}^*) = \mathbb{H}^{-i}(\mathbb{P}(W), S(\mathcal{F}^*)) \cong S(H^{-i}(\mathcal{F}^*)). \]
The last isomorphism is because the homology sheaves of \( S(\mathcal{F}^*) \) are 1-regular and then we apply Lemma 1.3. Let
\[ (\mathcal{F}^*)_v = \text{Hom}_S(\mathcal{F}_*, S(-n) \otimes \wedge^nV). \]
Similarly to the above its homology is, confer Lemma 3.1 d.
\[ H_i((\mathcal{F}^*)_v) \cong S(H^{-i}((\mathcal{F}^*)^*)). \]

**Proposition 7.4.** In the Grothendieck group of coherent sheaves write
\[ [\mathcal{F}] = \sum_{i=0}^m a_i [\mathcal{O}_{\mathbb{P}^1}(-1)]. \]
a) If \( \mathcal{F} \) is 1-regular, then all \( a_i \) are non-negative.
   Now assume this.
b) For non-negative degrees
\[ \mathcal{F} \text{ and } \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(-1)^{a_i} \]
have the same Hilbert functions.
c) The \( S(V \otimes W^*) \)-modules
\[ S(\mathcal{F}) \text{ and } \bigoplus_i S(\mathcal{O}_{\mathbb{P}^1}(-1))^{a_i} \]
have the same Hilbert functions, in all degrees.
   In particular the Hilbert functions of the two latter modules are completely determined by the Hilbert polynomial of \( \mathcal{F} \).

Proof. If the support of \( \mathcal{F} \) is all of \( \mathbb{P}(W) \), there is a short exact sequence
\[ 0 \to \mathcal{O}_{\mathbb{P}(W)}(-1)^{a_m} \to \mathcal{F} \to \mathcal{G} \to 0. \]
So
\[ [\mathcal{F}] = a_m [\mathcal{O}_{\mathbb{P}(W)}(-1)] + [\mathcal{G}] \]
where \( \mathcal{G} \) has support on a proper subspace of \( \mathbb{P}(W) \) and is also 1-regular. Since
\[ H^1(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(d-1)) = 0 \text{ for } d \geq 0, \]
\[ \mathcal{F} \text{ and } \mathcal{G} \oplus \mathcal{O}_{\mathbb{P}(W)}(-1)^{a_m} \]
have the same Hilbert function in non-negative degrees.

Now take a general projection $\pi : \mathbb{P}(W) \dashrightarrow \mathbb{P}(W')$ with center of projection $\mathbb{P}(W/W')$ disjoint from the support of $\mathcal{G}$. The sheaves $\mathcal{G}$ and $\pi_*\mathcal{G}$ have the same Hilbert polynomial, in fact exactly the same cohomology of all twists. Hence the classes $[\mathcal{G}] = [\pi_*\mathcal{G}]$. Let this class be $\sum_{i=0}^{m'} a_i [\mathcal{O}_{\mathbb{P}^i}(-1)]$ where $m' < m$.

a. By induction on the dimension all $a_i \geq 0$ for $0 \leq i \leq m'$. By equation (34) this shows part a.

b. By induction $\pi_*\mathcal{G}$ and $\sum_{i=0}^{m'} \mathcal{O}_{\mathbb{P}^i}(-1)^{a_i}$ have the same Hilbert function in non-negative degrees. Since the sheaves in (35) have the same Hilbert function in non-negative degrees, this shows part b.

c. Since $\mathcal{S}_d(V \otimes \mathcal{Q})$ is a vector bundle, we get that the classes

$$[\mathcal{S}_d(\mathcal{F})] = \sum_{i=0}^{m} a_i [\mathcal{S}_d(\mathcal{O}_{\mathbb{P}^i}(-1))]$$

Since the Hilbert polynomial is an additive function on the Grothendieck group we get

$$\chi(\mathcal{S}_d(\mathcal{F})) = \sum_{i=0}^{m} a_i \chi(\mathcal{S}_d(\mathcal{O}_{\mathbb{P}^i}(-1))).$$

But since $\mathcal{F}$ is 1-regular, $\mathcal{F}(1)$ is 0-regular, and so by Proposition 7.4 a., $\mathcal{S}_d(\mathcal{F})(1)$ is 0-regular. Then $H^i(\mathbb{P}(W), \mathcal{S}_d(\mathcal{F})) = 0$ for $i > 0$, and so

$$H^0(\mathbb{P}(W), \mathcal{S}_d(\mathcal{F})) = \sum_{i=0}^{m} a_i \chi(\mathcal{S}_d(\mathcal{O}_{\mathbb{P}^i}(-1)))$$

for $d \geq 0$, showing part c. \hfill \Box

7.2. The map of cones. Now let $\sum_{i} x_i E_i \subseteq V \otimes W^*$ be a general subspace of codimension $n = \dim_k V$, as in Theorem 6.7. We may assume this is a regular subspace for all $S(H^i(\mathcal{F}^*))$ and for all $S(H^i(\mathcal{F}^*)(n))$. The complex $\mathbb{W}(\mathcal{F}^*) \otimes_{S(V \otimes W^*)} S(V)$ then has the following properties:

- It is a complex of free squarefree $S(V) = k[x_1, \ldots, x_n]$-modules.
- Its Betti diagram is the same as that of $\mathbb{W}(\mathcal{F}^*)$ and hence is equivalent to give the hypercohomology modules
  $$\mathbb{H}^t(\mathbb{P}(W), \mathcal{F}^*(-t)), \text{ for } t \in [0, n].$$
- The Hilbert functions of its homology modules are equivalent to give the Hilbert functions of the homology modules of $\mathbb{W}(\mathcal{F}^*)$. By (32), Proposition 7.4 and Lemma 7.3 this is equivalent to give the hypercohomology
  $$\mathbb{H}^t(\mathbb{P}(W), \mathcal{F}^*(t)), \text{ for } t \geq 1.$$
- The Hilbert functions of its cohomology modules are equivalent to give the Hilbert functions of the cohomology modules of $\mathbb{W}(\mathcal{F}^*)$. By (33), Proposition 7.4 and Lemma 7.3 this is equivalent to give the hypercohomology
  $$\mathbb{H}^t(\mathbb{P}(W), \mathcal{F}^*(t)), \text{ for } t \leq -n - 1.$$
We thus get a numerical data triplet \((B, H, C)\) for a free squarefree complex which only depends on the hypercohomology table of \(F^\bullet\).

For a squarefree module \(M\), denote by \(h^\text{sq}_M(i)\) the sum of the dimensions of all squarefree degrees \(d\) of \(M\) of total degree \(i\)

\[
h^\text{sq}_M(i) = \sum_{|d|=i} \dim_k M_d.
\]

Note that this is equivalent to give the ordinary Hilbert function of \(M\) due to the following.

**Lemma 7.5.** Let \(M\) be a squarefree module over \(k[x_1, \ldots, x_n]\) and \(H_M(t)\) its Hilbert series. Then

\[
H_M(t) = \sum_i h^\text{sq}_M(i) t^i / (1 - t)^i.
\]

Thus either determine the other.

**Definition 7.6.** Given a complex of free squarefree modules \(F^\bullet\) over \(k[x_1, \ldots, x_n]\). Let \(\beta = (\beta_{ij}(F_\bullet))\) in \(Q^{Z \times [0,n]}\) be its Betti table, \(H = (h^\text{sq}_{H_i(F_\bullet)}(j))\) in \(Q^{Z \times [0,n]}\) be the Hilbert function table of homology modules, and \(C = (h^\text{sq}_{H_i(F_\bullet)^\vee}(j))\) in \(Q^{Z \times [0,n]}\) be the Hilbert function table of cohomology modules. We call \((B, H, C)\) the homological data triplet of \(F^\bullet\).

**Definition 7.7.** Let \(C(\text{SqFree}, n)\) be the positive rational cone generated by all homological data triplets of free squarefree complexes over \(k[x_1, \ldots, x_n]\), i.e. consisting of all linear combinations \(\sum c_k (B^k, H^k, C^k)\) where \(c_k \in Q^+\), and \((B^k, H^k, C^k)\) are homological data triplets.

From the bullet points in the beginning of this subsection, we see that we get an injective map

\[
C(\text{coh}, n) \xhookrightarrow{\iota} C(\text{SqFree}, n).
\]

**Conjecture 7.8.** The map \(\iota\) is an isomorphism of cones.

This conjecture would follow by Conjecture 5.7 and the following.

**Conjecture 7.9.** The extremal rays in \(C(\text{SqFree}, n)\) are precisely generated by the homology triplets of the shifts \(F_\bullet[s], s \in \mathbb{Z}\) of pure free squarefree complexes \(F_\bullet\) belonging to a triplet of pure free squarefree complexes.

The classical results in Boij-Söderberg theory concerns graded Cohen-Macaulay modules and vector bundles on projective spaces. The following shows that the map \(\iota\) restricts to an isomorphism in this case.

**Definition 7.10.** Let \(C(\text{vb}, c, n)\) be the positive rational subcone of \(C(\text{coh}, n)\) generated by vector bundles \(E\) on projective space of dimension \(c\) (we considered \(E\) to be in cohomological degree 0), and such that \(E\) is 1-regular and \(\text{Hom}_{\text{O}_P(W)}(E, \omega_{P(W)})\) is \(n + 1\)-regular.

Let \(C(\text{CM}, c, n)\) be the positive rational subcone of \(C(\text{SqFree}, n)\) generated by homology triplets of Cohen-Macaulay squarefree modules (in homological degree 0) of dimension \(c\). Note that for such triplets, the homology \(H\) and the cohomology \(C\) are determined by \(B\), so this identifies as the cone generated by Betti tables of such modules.
Theorem 7.11. The map \( \iota \) induces an isomorphism of cones

\[ C(vb,c,n) \cong C(CM,c,n). \]

Proof. That the map in \( \iota \) restricts to this map follows by Proposition 3.6. The extremal rays in \( C(vb,n,c) \) are by [10, Thm. 0.5] generated by the cohomology tables of vector bundles with supernatural cohomology and root sequences

\[ 0 \geq r_1 \geq \cdots \geq r_c \geq -n. \]

The extremal rays in \( C(CM,c,n) \) are by [10, Thm.0.2] and the construction of these in the squarefree case given by zipping Tate resolutions of vector bundles with supernatural cohomology, with the exterior co-algebra on \( V \), generated by Betti tables of Cohen-Macaulay modules of dimension \( c \) with pure resolution and degree sequences

\[ 0 \leq d_0 \leq \cdots \leq d_{n-c} \leq n. \]

These two sequences are related by the negatives of the former being the complement in \([0,n] \) of the latter. Hence the extremal rays correspond, and we get an isomorphism of cones. \( \square \)

Remark 7.12. The work of Eisenbud and Schreyer in [10], and Eisenbud and Erman in [7] suggests that the relationship between the cones of cohomology tables and Betti diagrams should be a duality. Here the correspondence is direct, an isomorphism. The cones of Betti tables are however not the same in our case and in the classical case. Another subtlety is that the correspondence here between the extremal rays are given by complementary root and degree sequences.

Why there is from one viewpoint a duality and from the viewpoint here a direct relationship is still something that might await a deeper understanding.

Remark 7.13. As Theorem 7.11 indicates, Conjecture 7.8 is quite amenable to doing special cases. For instance consider the subcone of \( C(coh,n) \) generated by the cohomology tables of coherent sheaves \( F \) of dimension \( c \) (situated in cohomological position 0) such that \( F \) is 1-regular and its derived dual has cohomology sheaves which are \( n+1 \)-regular. Also consider the subcone \( C(SqFree,n) \) generated by modules of dimension \( c \) (in homological degree 0). Then the map \( \iota \) restricts to a map between these subcones. The extremal rays should corresponds to homology triplets where \( H \) has one strand ending in \( c \).

Similarly one can consider 1-regular reflexive sheaves \( F \) on \( \mathbb{P}^3 \) such that its dual sheaf \( F^\vee \) is \( n+1 \)-regular, and the subcone generated by their cohomology tables. On the other side consider the subcone generated by homological data triplets of modules \( M \) of dimension 3 such that \( \text{Ext}^{n-1}(M,S) \) vanishes for \( i \leq 1 \) and is zero-dimensional for \( i = 2 \). Then \( \iota \) restricts to a map between these subcones. The extremal rays should correspond to homology triplets such that \( H \) has one strand ending in 3 and \( C = \{0,2,3\} \).

Remark 7.14. In [11] Eisenbud and Schreyer give a decomposition of the cohomology table of a coherent sheaf into the cohomology tables of vector bundles. However this decomposition involves an infinite number of terms, i.e. an infinite number of cohomology tables of vector bundles. It is not even known if the coefficients in this decomposition are rational. It may be that this type of decomposition arises as the limit of decompositions obtained above, by letting \( n \to \infty \).
8. Proof of Theorem 3.3

Our proof is fairly close to the proof of the Basic Theorem 5.1.2 in Weyman’s book [22], with some modifications to demonstrate how the functor $Rf$ factors into first taking the Tate resolution and then zipping with a vector space $V$.

Recall the basic setup of Subsection 1.1. The variety $Z$ is the affine bundle $\mathcal{V}(V \otimes \mathbb{Q})$. Set $\mathcal{O}_\mathcal{V}(i) = p^*(\mathcal{O}_{\mathbb{P}(W)}(i))$. By the exact sequence (11) we get a resolution of $\mathcal{O}_Z$:

$$K_\bullet : \mathcal{O}_\mathcal{V} \leftarrow V \otimes \mathcal{O}_\mathcal{V}(-1) \leftarrow \wedge^2 V \otimes \mathcal{O}_\mathcal{V}(-2) \leftarrow \cdots \leftarrow \wedge^i V \otimes \mathcal{O}_\mathcal{V}(-i) \leftarrow \cdots .$$

By the projection formula the global sections

$$\Gamma(\mathcal{V}, \mathcal{O}_\mathcal{V}(1)) = \text{Sym}(V \otimes W^*) \otimes W,$$

and the map $t(1)$ sends

$$V \otimes 1 \mapsto V \otimes W^* \otimes W \subseteq \text{Sym}_1(V \otimes W^*) \otimes W.$$

Lemma 8.1. a. $\mathcal{O}_Z \otimes p^*(-)$ is an exact functor on quasi-coherent sheaves on $\mathbb{P}(W)$.

b. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbb{P}(W)$. Then

$$\mathcal{O}_Z \otimes p^* \mathcal{F} \leftarrow K_\bullet \otimes p^* \mathcal{F}$$

is an exact sequence.

Proof. Let $p'$ be the composition

$$Z \hookrightarrow \mathcal{V}(V \otimes W^* \otimes \mathcal{O}_{\mathbb{P}(W)}) \to \mathbb{P}(W).$$

Then $\mathcal{O}_Z \otimes p^* \mathcal{F}$ identifies as $p'^* \mathcal{F}$. Since $Z \overset{p'}{\to} \mathbb{P}(W)$ is an affine bundle, the pullback $p'^*$ is exact.

For part b. note that locally on $U = \text{Spec } A \subseteq \mathbb{P}(W)$, the resolution $\mathcal{O}_Z \leftarrow K_\bullet$ on $\mathcal{V}$ is just a Koszul resolution of a polynomial ring as a quotient of a larger polynomial ring:

$$A[x_1, \ldots, x_r] \leftarrow A[x_1, \ldots, x_n] \leftarrow \langle x_{r+1}, \ldots, x_n \rangle \otimes A[x_1, \ldots, x_n] \leftarrow \cdots .$$

Since $\mathcal{F}|_U = \tilde{M}$ for an $A$-module $M$, the complex in b. is locally just tensoring the above with $\mathcal{F} = - \otimes_A M$. This is exact since the above is an exact sequence of free $A$-modules.

By the projection formula the global section of $p^* \mathcal{F}(-i)$ is

$$\Gamma(\mathcal{V}, p^* \mathcal{F}(-i)) = \Gamma(\mathbb{P}(W), p_* p^* \mathcal{F}(-i)) = S \otimes \Gamma(\mathbb{P}(W), \mathcal{F}(-i)).$$

Write $\Gamma(\mathcal{F}(-i))$ for short for the latter global sections. The complex of global sections $\Gamma(\mathcal{V}, K_\bullet \otimes p^* \mathcal{F})$ is

$$S \otimes \Gamma(\mathcal{F}) \leftarrow V \otimes S(-1) \otimes \Gamma(\mathcal{F}(-1)) \leftarrow \wedge^2 V \otimes S(-2) \otimes \Gamma(\mathcal{F}(-2)) \leftarrow \cdots \leftarrow \wedge^i V \otimes S(-i) \otimes \Gamma(\mathcal{F}(-i)) \leftarrow \cdots .$$

The module multiplication $W \otimes \Gamma(\mathcal{F}(-i)) \to \Gamma(\mathcal{F}(-i+1))$ gives a comultiplication $\Gamma(\mathcal{F}(-i)) \overset{\Delta}{\to} W^* \otimes \Gamma(\mathcal{F}(-i + 1))$ and the differential $d$ is given by

$$\wedge^i V \otimes k \otimes \Gamma(\mathcal{F}(-i)) \overset{\delta \otimes 1 \otimes \Delta}{\to} \wedge^i V \otimes V \otimes k \otimes W^* \otimes \Gamma(\mathcal{F}(-i+1)) = \wedge^{i-1} V \otimes S_1 \otimes \Gamma(\mathcal{F}(-i+1)).$$
Recall the graded global section module $\Gamma_*(\mathbb{P}(W), \mathcal{F})$. Applying the functor $R$ of (3) of Section 2 we get the linear complex $R \circ \Gamma_*(\mathcal{F})$:

$$
\cdots \leftarrow \widehat{E}(-1) \otimes \Gamma(\mathcal{F}(1)) \leftarrow \widehat{E} \otimes \Gamma(\mathcal{F}) \leftarrow \widehat{E}(1) \otimes \Gamma(\mathcal{F}(-1)) \leftarrow \cdots \\
\leftarrow \widehat{E}(i) \otimes \Gamma(\mathcal{F}(-i)) \leftarrow \cdots .
$$

Zipping this complex with the vector space $V$ we get

$$
S \otimes \Gamma(\mathcal{F}) \leftarrow V \otimes S(-1) \otimes \Gamma(\mathcal{F}(-1)) \leftarrow \cdots \leftarrow \wedge V \otimes S(-i) \otimes \Gamma(\mathcal{F}(-i)) \leftarrow \cdots 
$$

and the differentials are identified as in (38). So we obtain the following.

**Lemma 8.2.** The functors $\Gamma(V, \mathcal{K}_* \otimes p^*(-))$ and $\mathcal{W}_W^V \circ R \circ \Gamma_*$ from quasi-coherent sheaves on $\mathbb{P}(W)$ to linear $S$-complexes, are the same.

Let $C(q\text{-coh}/\mathbb{P}(W))$ be the category of complexes of quasicoherent sheaves on $\mathbb{P}(W)$. The functors in the lemma above take an object here to a double complex of free $S$-modules. Taking the total complex, Tot, of this we get a complex in $C(S - \text{free})$, the category of complexes of free $S$-modules.

**Corollary 8.3.** The functors $\text{Tot} \circ \Gamma(V, \mathcal{K}_* \otimes p^*(-))$ and $\text{Tot} \circ \mathcal{W}_W^V \circ R \circ \Gamma_*$ from $C(q\text{-coh}/\mathbb{P}(W)) \to C(S - \text{free})$ are equal.

Recall that the category of quasi-coherent sheaves on a noetherian scheme has enough injectives, [19, Ex.III.3.6].

**Lemma 8.4.** Let $\mathcal{I}$ be an injective quasi-coherent sheaf on $\mathbb{P}(W)$.

a. $\mathcal{I}(n)$ is injective.

b. $\mathcal{O}_Z \otimes p^* \mathcal{I}$ is a $q_*$-acyclic sheaf.

**Proof.** Part a. is clear. Since $Y$ is affine, the quasi-coherent sheaf $\mathbb{R}^i q_* (\mathcal{O}_Z \otimes p^* \mathcal{I})$ is the sheafification of $H^i(V, \mathcal{O}_Z \otimes p^* \mathcal{I})$. By [19, Lemma 2.10] we can compute this as the cohomology $H^i(Z, p^* \mathcal{I})$ on $Z$, where $p'$ is the restriction of $p$, see (37). Since $p'$ is an affine map, by the spectral sequence associated to the composition $Z \xrightarrow{p'} \mathbb{P}(W) \to \text{Spec} k$ we have

$$H^i(Z, p^* \mathcal{I}) = H^i(\mathbb{P}(W), p'_* p^* \mathcal{I}).$$

By the projection formula

$$p'_* p^* \mathcal{I} = \mathcal{I} \otimes \text{Sym}(V \otimes \mathcal{O}) = S(\mathcal{I}).$$

Since $\mathcal{I}(r)$ is 0-regular for all $r$, by Lemma [12] a. all the higher cohomology of $S(\mathcal{I})$ will vanish. \qed

Now we have the following.

**Fact 8.5.** Let $\mathcal{I}^*$ be an injective resolution of $\mathcal{F}^*$. The Tate resolution $T(\mathcal{F}^*)$ may be defined as the minimal complex homotopy equivalent to $\text{Tot} \circ R \circ \Gamma_*(\mathcal{I}^*)$. This is Corollary 3.2.3 of the unpublished article [13]. See also Proposition 1.6.1. therein.
Proof of Theorem 3.3. Let $\mathcal{F}^\bullet$ be a bounded complex of coherent sheaves and $\mathcal{F}^\bullet \xrightarrow{\phi} \mathcal{I}^\bullet$ an injective resolution of quasi-coherent sheaves. We get a morphism of double complexes (the first one with only one column)

$$O_Z \otimes p^* \mathcal{F}^\bullet \leftarrow K_\bullet \otimes p^* \mathcal{F}^\bullet,$$

and then morphisms

$$O_Z \otimes p^* \mathcal{F}^\bullet \leftarrow \text{Tot}(K_\bullet \otimes p^* \mathcal{F}^\bullet)$$

$$\downarrow$$

$$\text{Tot}(K_\bullet \otimes p^* \mathcal{I}^\bullet).$$

The horizontal map is a quasi-isomorphism by Lemma 8.1 and the Acyclic Assembly Lemma 2.7.3 part 3. of [21]. The vertical map is a quasi-isomorphism by applying the same lemma part 4. to the cone $K_\bullet \otimes \text{cone}(\phi)$. (Note that direct sums and products are the same here since a finite number of sheaves are involved.) Since the lower total complex consist of $q_\ast$-acyclic objects, it can be used to calculate the derived complex $\mathbb{R}q_\ast$ of the upper left complex. Hence

$$\mathbb{R}q_\ast(O_Z \otimes p^* \mathcal{F}) = \text{Tot} \circ q_\ast(K_\bullet \otimes p^* \mathcal{I}^\bullet)$$

and so

$$\Gamma(X, \mathbb{R}q_\ast(O_Z \otimes p^* \mathcal{F})) = \text{Tot} \circ \Gamma(V, K_\bullet \otimes p^* \mathcal{I}^\bullet).$$

By Corollary 8.3

$$\text{Tot} \circ \Gamma(V, K_\bullet \otimes p^* \mathcal{I}^\bullet) = \text{Tot} \circ \Gamma(W, \mathbb{W}_V^\mathcal{I}^\bullet)$$

$$= \mathbb{W}_V^\mathcal{I}^\bullet(\text{Tot} \circ \mathbb{R} \circ \Gamma(W, \mathcal{I}^\bullet)).$$

Now $\mathbb{W}_V^\mathcal{I}^\bullet$ is a functor between additive categories that takes cones to cones. It is a general fact that such functors take homotopy equivalences to homotopy equivalences. Then by the Fact 8.5 above the latter complex is homotopy equivalent to $\mathbb{W}_V^\mathcal{I}^\bullet(\Gamma(\mathcal{F}^\bullet))$ and this together with Lemma 1.1 concludes the proof of Theorem 3.3. □

References

[1] Alexander A Beilinson. Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra. Functional Analysis and its Applications, 12(3):214–216, 1978.
[2] M. Boij and J. Söderberg. Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture. Journal of the London Mathematical Society, 78(1):78–101, 2008.
[3] Mats Boij and Jonas Söderberg. Betti numbers of graded modules and the multiplicity conjecture in the non-Cohen–Macaulay case. Algebra & Number Theory, 6(3):437–454, 2012.
[4] I. Coandă. On the Bernstein-Gel’fand-Gel’fand correspondence and a result of Eisenbud, Fløystad, and Schreyer. Kyoto Journal of Mathematics, 43(2):429–439, 2003.
[5] D. Cox and E. Materov. Tate resolutions and Weyman complexes. Pacific Journal of Mathematics, 252(1):51–68, 2011.
[6] D. Eisenbud. Commutative algebra with a view toward algebraic geometry. GTM 150. Springer, 1995.
[7] D. Eisenbud and D. Erman. Categorified duality in Boij-Söderberg theory and invariants of free complexes. Arxiv preprint arXiv:1205.0449, pages 1–19, 2012.
[8] D. Eisenbud, G. Fløystad, and F.O. Schreyer. Sheaf cohomology and free resolutions over exterior algebras. Transactions of the American Mathematical Society, 355(11):4397–4426, 2003.
[9] D. Eisenbud, G. Fløystad, and J. Weyman. The existence of equivariant pure free resolutions. Annales de l’Institut Fourier, 61(3):905–926, 2011.
[10] D. Eisenbud and F.O. Schreyer. Betti numbers of graded modules and cohomology of vector bundles. *Journal of the American Mathematical Society*, 22(3):859–888, 2009.

[11] D. Eisenbud and F.O. Schreyer. Cohomology of coherent sheaves and series of supernatural bundles. *Journal of the European Mathematical Society*, 12(3):703–722, 2010.

[12] D. Eisenbud, F.O. Schreyer, and J. Weyman. Resultants and Chow forms via exterior syzygies. *Journal of the American Mathematical Society*, 16(3):537–579, 2003.

[13] G. Fløystad. Describing coherent sheaves on projective spaces via Koszul duality. *arXiv preprint math/0012263*, 2000.

[14] G. Fløystad. Boij-Söderberg theory: Introduction and survey. In C. Francisco, L. Klingler, S. Sather-Wagstaff, and J. Vassilev, editors, *Progress in Commutative Algebra 1, Combinatorics and homology*, Proceedings in mathematics, pages 1–54. de Gruyter, 2012.

[15] G. Fløystad. Triplets of pure free squarefree complexes. *Journal of Commutative Algebra*, 5(1):101–139, 2013.

[16] W. Fulton. *Intersection theory*, volume 1998. Springer-Verlag Berlin, 1984.

[17] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[18] R. Hartshorne. *Residues and duality*. Lecture Notes in mathematics, 20. Springer, New York, Berlin, Heidelberg, 1966.

[19] R. Hartshorne. *Algebraic geometry*. GTM 52. Springer Verlag, 1977.

[20] A. Lascoux. Syzygies des variétés déterminantales. *Advances in Mathematics*, 30(3):202–237, 1978.

[21] Charles A Weibel. *An introduction to homological algebra*. Number 38. Cambridge university press, 1995.

[22] J. Weyman. *Cohomology of vector bundles and syzygies*. Cambridge tracts in mathematics 149. Cambridge Univ Pr, 2003.

[23] K. Yanagawa. Alexander duality for Stanley-Reisner rings and squarefree N-graded modules. *Journal of Algebra*, 225(2):630–645, 2000.

[24] K. Yanagawa. Derived category of squarefree modules and local cohomology with monomial ideal support. *Journal of the Mathematical Society of Japan*, 56(1):289–308, 2004.

[25] Christine Berkesch Zamaere, Daniel Erman, Manoj Kummini, and Steven V Sam. Tensor complexes: Multilinear free resolutions constructed from higher tensors. *Journal of the European Mathematical Society (Print)*, 15(6):2257–2295, 2013.