ON AN OLD THEOREM OF ERDÖS ABOUT AMBIGUOUS LOCUS

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Abstract. Erdös proved in 1946 that if a set $E \subset \mathbb{R}^n$ is closed and non-empty, then the set, called ambiguous locus or medial axis, of points in $\mathbb{R}^n$ with the property that the nearest point in $E$ is not unique, can be covered by countably many surfaces, each of finite $(n-1)$-dimensional measure. We improve the result by obtaining a new regularity result for these surfaces in terms of convexity and $C^2$ regularity.

Given a closed set $\emptyset \neq E \subset \mathbb{R}^n$, let $\text{Unp}(E)$ be the set of all points $x \in \mathbb{R}^n$ for which there is a unique point $y \in E$ nearest to $x$. Clearly $E \subset \text{Unp}(E)$. If we denote this nearest point by $\pi(x) := y$, the mapping $\pi : \text{Unp}(E) \to E$ is called the metric projection. In order to understand the properties of this mapping it is important to understand the structure of the set $\mathbb{R}^n \setminus \text{Unp}(E) \subset \mathbb{R}^n \setminus E$, where we lack uniqueness of the metric projection. This set is often called the ambiguous locus of the metric projection. It is also called the medial axis or the skeleton of $\mathbb{R}^n \setminus E$. Zamfirescu [23] (see also [16, 4A], [24]) proved that for most compact sets $\emptyset \neq E \subset \mathbb{R}^n$ (in the Baire category sense with respect to the Hausdorff distance on the spaces on compact subsets of $\mathbb{R}^n$), the set $\text{Unp}(E)$ has empty interior, meaning that the set of points in $\mathbb{R}^n$ without a unique nearest point in $E$ is dense. On the other hand, it is known that the Lebesgue measure of the set $\mathbb{R}^n \setminus \text{Unp}(E)$ equals zero, $|\mathbb{R}^n \setminus \text{Unp}(E)| = 0$. This result is due to Erdös [13]. For a simple folklore proof (different from that in [13]), see Lemma [12] and Remark [13] below.

Erdös [12], proved however, a much stronger result: The set $\mathbb{R}^n \setminus \text{Unp}(E)$ is contained in the sum of countably many surfaces of finite $(n-1)$-dimensional measure. His proof is based on Roger’s [20] proof of the contingent theorem (see also [21, pp. 264-266 and 304-307] and [18, Section 2.1.8]). Fifty years later, Erdös’ result was rediscovered by Fremlin [16, Theorem 1G] with a different proof, but the author was not aware of the work of Erdös. For more results about the structure of the set $\mathbb{R}^n \setminus \text{Unp}(E)$, see [1] and references therein. Another interesting and related reference is [9].

In Theorem [1] which is the main result of the paper, we substantially improve Erdös’ result by showing convexity and $C^2$ regularity properties of the surfaces covering the set $\mathbb{R}^n \setminus \text{Unp}(E)$. Our argument is different from that of Erdös. To the best of my knowledge, Theorem [1] is new. While, it can be easily deduced from the results existing in the literature (as we do it here), I believe, the result is of a substantial interest and it deserves a clear, self-contained, and easy to read proof. In addition to Theorem [1] I believe, the proof of Theorem [8] is of independent interest as I explained it in Remark [9].

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We say that the set
\[ G = \{ x \in \mathbb{R}^n : x_i = f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \} , \]
where \( 1 \leq i \leq n \) and \( f : \mathbb{R}^{n-1} \to \mathbb{R} \) is continuous, is a \( C^2 \)-graph if \( f \in C^2 \) and it is a \((c-c)\)-graph if \( f = g - h \) is the difference of convex functions \( g, h : \mathbb{R}^{n-1} \to \mathbb{R} \). If the convex functions \( g, h \) are of class \( C^2 \), we say that \( G \) is a \( C^2 - (c-c) \)-graph.

**Theorem 1.** For any closed set \( E \subset \mathbb{R}^n \) we have

(a) The set \( \mathbb{R}^n \setminus \text{Unp}(E) \) can be covered by countably many \((c-c)\)-graphs.

(b) There are countably many \( C^2 - (c-c) \)-graphs \( \{G_j\}_{j=1}^{\infty} \) such that
\[ \mathcal{H}^{n-1}\left((\mathbb{R}^n \setminus \text{Unp}(E)) \setminus \bigcup_{j=1}^{\infty} G_j \right) = 0 , \]
where \( \mathcal{H}^{n-1} \) stands for the Hausdorff measure.

**Remark 2.** Since convex functions are locally Lipschitz continuous [19, Theorem 41D], compact subsets of \((c-c)\)-graphs have finite \((n-1)\)-dimensional Hausdorff measure and hence the \((n-1)\)-dimensional Hausdorff measure of \( \mathbb{R}^n \setminus \text{Unp}(E) \) is \( \sigma \)-finite. Therefore, part (a) implies Erdös’ result [12] mentioned earlier.

**Remark 3.** It follows from (b) that \( \mathbb{R}^n \setminus \text{Unp}(E) \) is \( (\mathcal{H}^{n-1}, n) \)-rectifiable of class \( C^2 \) in the sense of [5].

The paper is organized as follows. In Section 1 we collect basic facts about convex functions. In Section 2 we prove a special case of a result of Zajiček [22], and in Section 3 we prove Theorem 1.

1. **Convex functions**

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous function. The **subdifferential of \( f \) at \( x \)**, denoted by \( \partial f(x) \), is the set of all \( v \in \mathbb{R}^n \) such that
\[ f(x + h) \geq f(x) + \langle v, h \rangle \quad \text{for all } h \in \mathbb{R}^n . \] (2)
The geometric interpretation is that each \( v \in \partial f(x) \) defines a hyperplane passing through \((x, f(x))\) such that the graph of \( f \) is above that hyperplane. Such hyperplanes are called **supporting hyperplanes** of the graph of \( f \) at \((x, f(x))\). If \( \partial f(x) \) contains more than one vector, it means that there is more than one supporting hyperplane at \((x, f(x))\).

**Lemma 4.** If \( f : \mathbb{R}^n \to \mathbb{R} \) is convex, then for every \( x \in \mathbb{R}^n \), \( \partial f(x) \neq \emptyset \). That is at every point, there is at least one supporting hyperplane. If in addition \( f \) is differentiable at \( x \), then \( \partial f(x) = \{ \nabla f(x) \} \) so that case the tangent hyperplane is a unique supporting hyperplane.

**Proof.** It easily follows from the definition of the derivative and convexity, that if \( f \) is differentiable at \( x \), then there is a unique supporting hyperplane defined by \( \nabla f(x) \) i.e., \( \partial f(x) = \{ \nabla f(x) \} \). Thus it remains to prove existence in the general case.
For $k = 1, 2, \ldots,$ let $p_k$ be the point on the graph of $f$ nearest to $q_k = (x, f(x) - k^{-1})$. Let $H_k$ be the hyperplane orthogonal to the segment $[p_k, q_k]$ and passing through its midpoint.

It follows from the convexity of $f$ that the graph of $f$ lies above the hyperplane $H_k$. Since the unit normal vectors to $H_k$ belong to the compact unit sphere $\nu_k \in S^{n-1}$, we can select a convergent sequence $\nu_k \to \nu \in S^{n-1}$ and it is easy to see that the hyperplane normal to $\nu$ and passing through $(x, f(x))$ is a supporting one. □

Lemma 5. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and partial derivatives $\partial f/\partial x_i$, $i = 1, 2, \ldots, n$ exist at a point $x \in \mathbb{R}^n$, then $f$ is (Fréchet) differentiable at $x$.

For a proof see [19, Theorem 42D]. This lemma follows from Jensen’s inequality: any vector $h \in \mathbb{R}^n$ can be expressed as a convex combination of vectors parallel to coordinate axes and this along with the Jensen inequality applied to the convex function $\varphi(h) := f(x + h) - f(x) - \langle \nabla f(x), h \rangle$ allows us to show that $\varphi(h)$ converges to 0 as $o(|h|)$. It might be more rewarding to fill missing details as an exercise rather than to read the proof from [19].

One sided partial derivatives will be denoted by

$$\frac{\partial^+ f}{\partial x_i}(x) = \lim_{t \to 0^+} \frac{f(x + te_i) - f(x)}{t}.$$  \hspace{1cm} (3)

The next lemma easily follows from the monotonicity of secants of a convex function in one variable.

Lemma 6. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then one-sided partial derivatives (8) exist at every point $x \in \mathbb{R}^n$ and $\partial^- f(x)/\partial x_i \leq \partial^+ f(x)/\partial x_i$. Moreover, for any $x \in \mathbb{R}^n$, and $1 \leq i \leq n$

$$f(x + te_i) \geq f(x) + st \text{ for all } t \in \mathbb{R} \text{ and all } s \text{ satisfying } \frac{\partial^- f}{\partial x_i}(x) \leq s \leq \frac{\partial^+ f}{\partial x_i}(x).$$  \hspace{1cm} (4)

Thus a convex function $f$ is not differentiable at $x$ if and only if there is $i \in \{1, 2, \ldots, n\}$ such that

$$\frac{\partial^- f}{\partial x_i}(x) < \frac{\partial^+ f}{\partial x_i}(x).$$  \hspace{1cm} (5)

Indeed, according to Lemma 5, $f$ is differentiable at $x$ if and only if partial derivatives exist and that is equivalent to equality of all one-sided partial derivatives.

Therefore, if $A \subset \mathbb{R}^n$ is the set of points where a convex function $f$ is not differentiable, then

$$A = \bigcup_{i=1}^{n} \bigcup_{\alpha < \beta < 0} \left\{ x : \frac{\partial^- f}{\partial x_i}(x) \leq \alpha < \beta \leq \frac{\partial^+ f}{\partial x_i}(x) \right\}. \hspace{1cm} (6)$$

We say that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is coercive if $f(x) \to \infty$ as $|x| \to \infty$, and $f$ is strongly convex if $f(x) - \mu |x|^2$ is convex for some $\mu > 0$. Clearly, if $f$ is convex, then $f(x) + |x|^2$ is strongly convex.

If $f$ is strongly convex, then it is coercive in the following stronger sense:

$$\lim_{|x| \to \infty} (f(x) - \ell(x)) = \infty \text{ for any linear function } \ell : \mathbb{R}^n \to \mathbb{R}. \hspace{1cm} (7)$$
Indeed, \( f(x) - \mu |x|^2 \) is convex and hence an affine function (supporting hyperplane) bounds it from below, \( f(x) - \mu |x|^2 \geq A(x) \) so \( f(x) - \ell(x) \geq A(x) + \mu |x|^2 - \ell(x) \to \infty \) as \( |x| \to \infty \).

**Lemma 7.** If \( f : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R} \) is convex and coercive, then \( F(x) := \inf_{y \in \mathbb{R}^\ell} f(x, y) \) defines a convex function \( F : \mathbb{R}^k \to \mathbb{R} \).

**Proof.** Let \( x_1, x_2 \in \mathbb{R}^k \) and \( \lambda \in [0, 1] \). Then for any \( y_1, y_2 \in \mathbb{R}^\ell \) we have
\[
F(\lambda x_1 + (1 - \lambda)x_2) \leq f(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2)
= f(\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)) \leq \lambda f(x_1, y_1) + (1 - \lambda)f(x_2, y_2)
\]
and the result follows upon taking the infima over \( y_1 \in \mathbb{R}^\ell \) and \( y_2 \in \mathbb{R}^\ell \). \( \square \)

2. A theorem of Luděk Zajíček

The next result is a special case of a theorem of Zajíček [22, Theorem 1].

**Theorem 8.** If \( f : \mathbb{R}^n \to \mathbb{R} \) is convex, then the set of points where \( f \) is not differentiable is contained in a countable union of \((c - c)\)-graphs.

**Remark 9.** While our proof is almost the same as the original one, the result is a slight improvement of that of Zajíček, as Proposition[10] and its proof provide a more detailed description of surfaces covering the non-differentiability sets \( A_{i,\alpha,\beta} \) in the case of strongly convex functions. Other reasons to include a proof are: (1) to make the paper self-contained; (2) the result [22, Theorem 1] whose special case we prove here is more difficult to read due to its generality; (3) while the theorem of Zajíček has been cited many times, it is a good idea to provide an easy to read proof as it might contribute to popularization of this beautiful result.

Since \( f \) is non-differentiable at \( x \) if and only if the strongly convex function \( f(x) + |x|^2 \) is non-differentiable at \( x \), Theorem 8 follows from the next result whose proof provides a more detailed description of the structure of the discontinuity sets \( A_{i,\alpha,\beta} \).

We say that graphs of the form \( f \) are graphs in the direction of \( x_i \).

**Proposition 10.** If \( f : \mathbb{R}^n \to \mathbb{R} \) is strongly convex, then each of the sets \( A_{i,\alpha,\beta} \) is contained in a \((c - c)\)-graph in the direction of \( x_i \). Therefore, the set \( A \) is contained in a countable union of \((c - c)\)-graphs.

**Proof.** Without loss of generality we may assume that \( i = 1 \). For any \( s \in \mathbb{R} \), the function \( f_s(x) = f(x) - sx_1 \) is convex and coercive by [7]. Therefore, Lemma 7 implies that the function
\[
g_s : \mathbb{R}^{n-1} \to \mathbb{R}, \quad g_s(x_2, \ldots, x_n) := \inf_{x_1 \in \mathbb{R}} f_s(x_1, x_2, \ldots, x_n)
\]
is convex. If \( a \in A_{i,\alpha,\beta} \), then [4] yields
\[
f(a + te_1) \geq f(a) + \alpha t \quad \text{for all } t \in \mathbb{R}
\]
or equivalently,
\[
f_\alpha(a + te_1) \geq f_\alpha(a) \quad \text{for all } t \in \mathbb{R}.
\]
The last condition however, means that the function $x_1 \mapsto f_\alpha(x_1, a_2, \ldots, a_n)$ attains minimum at $x_1 = a_1$ so
\[ g_\alpha(a_2, \ldots, a_n) = f_\alpha(a_1, a_2, \ldots, a_n) = f(a_1, a_2, \ldots, a_n) - \alpha a_1. \tag{8} \]
Similarly, inequality $f(a + te_1) \geq f(a) + \beta t$ for all $t \in \mathbb{R}$ (also guaranteed by (4)) implies that
\[ g_\beta(a_2, \ldots, a_n) = f(a_1, a_2, \ldots, a_n) - \beta a_1. \tag{9} \]
Now (8) and (9) yield
\[ a_1 = \frac{1}{\beta - \alpha} (g_\alpha(a_2, \ldots, a_n) - g_\beta(a_2, \ldots, a_n)) \]
and $A_{\alpha,\beta}^1$ is contained in the graph of the $(c - c)$-function
\[ g(x_2, \ldots, x_n) = \frac{1}{\beta - \alpha} (g_\alpha(x_2, \ldots, x_n) - g_\beta(x_2, \ldots, x_n)). \tag{10} \]
The proof is complete. $\square$

Remark 11. Note that in general not all points of the graph of (10) belong to the set $A_{\alpha,\beta}^1$ as otherwise we would obtain a whole surface of points of discontinuity of the derivative and $f(x) = |x|$ has only one such point.

3. Proof of Theorem 1

Proof of (a). In the proof we will need the following two lemmata. Erdös [13], proved that $|\mathbb{R}^n \setminus \text{Unp}(E)| = 0$, but Lemma [12] provides a different proof of this fact, see [11, Proposition 2.4], [10, Lemma 2.1], [15, Theorem 3.3], [8, Lemma 2.21].

Lemma 12. If a set $\emptyset \neq E \subset \mathbb{R}^n$ is closed, and the distance function $d(x) = \text{dist}(x, E)$ is differentiable at $x \in \mathbb{R}^n \setminus E$, then $x \in \text{Unp}(E)$.

Remark 13. Since $d$ is Lipschitz continuous, it is differentiable a.e. by the Rademacher theorem, and hence Lemma [12] yields $|\mathbb{R}^n \setminus \text{Unp}(E)| = 0$.

Proof. Assume that $d$ is differentiable at $x \in \mathbb{R}^n \setminus E$ and $p \in E$ is such that $|x - p| = d(x)$. We will prove that $p = x - d(x)\nabla d(x)$ and this will imply uniqueness of $p$.

It follows from the triangle inequality that if $y$ belongs to the interval with endpoints $x$ and $p$, $y \in [x, p]$, then $d(y) = |y - p|$. Therefore, the function $d$ decreases linearly with the slope 1 along the segment $[x, p]$. Since $d$ is differentiable at $x$, the directional derivative at $x$ in the direction of the unit vector $v = (p - x)/|p - x|$ satisfies
\[ \langle \nabla d(x), v \rangle = D_v d(x) = -1. \tag{11} \]
On the other hand, $d$ is 1-Lipschitz so $|\nabla d(x)| \leq 1$ and hence equality (11) implies that
\[ \nabla d(x) = -v = \frac{x - p}{d(x)} \text{ so } p = x - d(x)\nabla d(x). \]

$\square$
The next lemma was observed by Asplund [6] p. 235.

**Lemma 14.** If $\emptyset \neq E \subset \mathbb{R}^n$ is closed, and $d(x) = \text{dist}(x, E)$, then the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = |x|^2 - d(x)^2$ is convex.

**Proof.** We have

$$f(x) = |x|^2 - \inf_{y \in E} |x - y|^2 = |x|^2 + \sup_{y \in E} (- |x - y|^2) = \sup_{y \in E} (2\langle x, y \rangle - |y|^2).$$

Therefore, $f$ is a supremum of a family of affine functions, and hence it is convex. □

By Lemma [12] the set $\mathbb{R}^n \setminus \text{Unp}(E)$ is contained in the set $A_+$ of points where $d$ is strictly positive and non-differentiable. Since $d > 0$ in $A_+$, the set $A_+$ is contained in the set where the convex function $f(x) = |x|^2 - d(x)^2$ is non-differentiable and the result follows from Theorem [8]. □

**Proof of (b).** It is well known that if $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is convex, then for any $\varepsilon > 0$ there is a function $f_\varepsilon \in C^2(\mathbb{R}^{n-1})$ such that the Lebesgue measure of the set where the two functions differ satisfies $|\{ x : f(x) \neq f_\varepsilon(x) \}| < \varepsilon$. (In general, the function $f_\varepsilon \in C^2$ cannot be convex, see Example 1.9 and Proposition 1.10 in [7].) This is a consequence of the Aleksandrov theorem [3] about second order differentiability of convex functions which implies that a convex function satisfies assumptions of the $C^2$-Whitney extension theorem outside a set of measure less than $\varepsilon$. While the idea is simple, the details are rather difficult, see [2], [4], Corollary 1.5, [14], Proposition A1], [17]. This and part (a) easily imply that the set $\mathbb{R}^n \setminus \text{Unp}(E)$ can be covered up to a set of $\mathcal{H}^{n-1}$-measure zero by a sequence of graphs of $C^2$-functions. One only needs to note that any $(\varepsilon - \text{c})$-function, $g : \mathbb{R}^{n-1} \to \mathbb{R}$ is locally Lipschitz and hence for a set $A \subset \mathbb{R}^{n-1}$ of measure zero, the corresponding set on the graph of $g$ has vanishing Hausdorff measure $\mathcal{H}^{n-1}$.

Therefore, it remains to show that if $f \in C^2$, then on every bounded set, $f$ coincides with the difference of two convex functions of class $C^2$.

Given $R > 0$, let $\varphi$ be a compactly supported smooth function that equals 1 for $|x| \leq R$. Then $\varphi f \in C^2$ has bounded second order derivatives so there is $C_R > 0$ such that the matrix $D^2(\varphi(x)f(x) + C_R|x|^2) = D^2(\varphi f) + 2C_RI$ is positive definite and hence the function $\varphi(x)f(x) + C_R|x|^2$ is convex and of class $C^2$. Now, $f(x) = (\varphi(x)f(x) + C_R|x|^2) - C_R|x|^2$, $|x| \leq R$, represents $f$ for $|x| \leq R$ as a difference of convex functions of class $C^2(\mathbb{R}^n)$. This completes the proof. □

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