INTRINSIC NATURE OF THE STEIN-WEISS $H^1$-INEQUALITY

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ABSTRACT. This paper explores the intrinsic nature of the celebrated Stein-Weiss $H^1$-inequality

$$\|I_s u\|_{L^n} \leq \|u\|_{L^n} + \|\vec{R} u\|_{L^n} = \|u\|_H$$

through the tracing and duality laws based on Riesz’s singular integral operator $I_s$. Surprisingly, under $n \geq 2$ we discover that $f \in \text{BMO} = [H^1]^*$ (Fefferman-Stein’s duality) if and only if $\exists \vec{g} = (g_1, \ldots, g_n) \in (L^\infty)^n$ such that $f = \vec{R} \cdot \vec{g} = \sum_{j=1}^n R_j g_j$ where $\vec{R} = (R_1, \ldots, R_n)$ is the vector-valued Riesz transform - this improves Fefferman-Stein’s decomposition $\text{BMO} = L^\infty + \vec{R} (L^\infty)^n$ (established in their 1972 Acta Math paper [7]) and yet reveals that BMO is the unique answer to Bourgain-Brezis’ question under $n \geq 2$: “What are the function spaces $X, W^{1,n} \subset X \subset \text{BMO}$, such that every $F \in X$ has a decomposition $F = \sum_{j=1}^n R_j Y_j$ where $Y_j \in L^{\infty,n}$?” (posed in their 2003 J. Amer. Math. Soc. paper [4]).

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1. INTRODUCTION

1.1. The Stein-Weiss $H^p$-inequalities. For $(n, p) \in \mathbb{N} \times [1, \infty)$, denote by $H^p$ the real Hardy space on the Euclidean space $\mathbb{R}^n$, consisting of all functions $f$ in the Lebesgue space $L^p$ with

$$\|u\|_{H^p} = \|u\|_{L^p} + \|\vec{R} u\|_{L^p} < \infty,$$

where

$$\vec{R} = (R_1, \ldots, R_n)$$
is the vector-valued Riesz transform on \( \mathbb{R}^n \), with
\[
\tilde{R}u = (R_1 u, ..., R_n u) \quad \text{and} \quad R_j u(x) = \left( \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \right) \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} u(y) \, dy \quad \text{a.e.} \ x \in \mathbb{R}^n
\]
and \( \Gamma \) being the Gamma function. Also, for a vector-valued function
\[
f = (f_1, \ldots, f_n)
\]
let
\[
\|f\|_{L^p} = \sum_{j=1}^n \|f_j\|_{L^p}.
\]
Note that \( H^p \) coincides with the classical Lebesgue space \( L^p \) whenever \( p \in (1, \infty) \) and the \( (0, 1) \ni s \)-th order Riesz singular integral operator \( L_s \) acting on a function
\[
u \in \bigcup_{p \in [1, \frac{1}{s}]} L^p
\]
is defined by
\[
L_s u(x) = \left( \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+2}{2}}2^{\frac{n+1}{2}}(\frac{1}{2})} \right) \int_{\mathbb{R}^n} |x - y|^{s-n} u(y) \, dy \quad \text{a.e.} \ x \in \mathbb{R}^n.
\]
We refer the reader to Stein’s seminal texts [26, 27] for more about these basic notions. The well-known Stein-Weiss \( H^p \)-inequality (cf. [28]) states that under
\[
0 < s < 1 \quad \text{and} \quad 1 \leq p < \frac{n}{s},
\]
the Riesz-Hardy potential space \( L_s^m(H^p) \) can be continuously embedded into \( L_s^\infty \), that is,
\[
\left\| L_s u \right\|_{L_s^\infty} \leq \|u\|_{L^p} + \|\tilde{R}u\|_{L^p} \approx \|u\|_{H^p} \quad \forall \ u \in H^p.
\]
Let \( C_c^\infty \) be the collection of all infinitely differentiable functions compactly supported in \( \mathbb{R}^n \). Note that \( C_c^\infty \cap H^p \) is dense in \( H^p \) for any \( p \in [1, \infty) \). For any \( u \in C_c^\infty \) let
\[
(-\Delta)^{\frac{s}{2}} u(x) = \begin{cases} 
L_s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x+y)}{|y|^{n+s}} \, dy & \text{as} \ s \in (-1, 0) \\
u(x) & \text{as} \ s = 0 \\
c_{n,s,+} \int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|y|^{n+s}} \, dy & \text{as} \ s \in (0, 1)
\end{cases}
\]
and
\[
D^j u(x) = \left( \frac{\partial^j u}{\partial x^j} \right)_{j=1}^n = \tilde{R}(-\Delta)^{\frac{s}{2}} u(x) = c_{n,s,-} \int_{\mathbb{R}^n} \frac{y(u(x) - u(x-y))}{|y|^{n+1+s}} \, dy,
\]
where (cf. [5, Definition 1.1, Lemma 1.4] for \( c_{n,s,+} \) and §2 below for \( c_{n,s,-} \))
\[
\begin{cases} 
c_{n,s} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}2^{\frac{n+1}{2}}(\frac{1}{2})} \\
c_{n,s,+} = \frac{c_{n,s} \tilde{\Delta}^{s-1}}{\pi^{\frac{n+1}{2}}(\frac{1}{2})} \\
c_{n,s,-} = \frac{2c_{n,s} \Delta^s}{\pi^2 \Gamma(\frac{1}{2})} \end{cases}
\]
In particular, if $0 < s < n = 1$ then there are two $s$-dependent constants $c_x$ to make the following Liouville fractional derivative formulas (cf. [22]):
\[
\begin{align*}
(-\Delta)^s u(x) &= c_x \left( \frac{d^s}{dx^s} + \frac{d^s}{dC} \right) u(x) \\
D^s u(x) &= c_x \left( \frac{d^s}{dx^s} - \frac{d^s}{dC} \right) u(x) \\
\frac{d^s}{dx^s} u(x) &= \frac{s}{\Gamma(1-s)} \int_{-\infty}^{t} \frac{u(x+s)-u(x)}{|t-s|^{s+1}} \, dt.
\end{align*}
\]

Hence it is natural and reasonable to adopt the notations
\[
\nabla^s u = (-\Delta)^s u \quad \& \quad \nabla^s u = D^s u = \tilde{R}(-\Delta)^s u.
\]

The operators $\nabla^s$ and $\nabla^s$ can be viewed as the fractional extensions of the gradient operator
\[
\nabla = (\partial_{x_1}, \ldots, \partial_{x_n}).
\]

Accordingly, for any $s \in (0, 1)$, the Stein-Weiss inequality (1.1) (cf. [20]) amounts to
\[
\tag{1.2}
\|u\|_{L^{\frac{mp}{m+n}}} \lesssim \|\nabla^s u\|_{L^p} + \|\nabla^s u\|_{L^p} \quad \forall \ u \in I_s(C^\infty_c \cap H^p).
\]

Of course, it is appropriate to mention the following basic facts:
\[
\begin{itemize}
\item If $0 < s < 1 < p < n/s$, then the right-hand-side of (1.2) can be replaced by $\|\nabla^s u\|_{L^p}$.
\item More precisely, on the one hand, the boundedness of $R$ on $L^{p,1}$ and (1.2) give (cf. [21, Lemma 2.4])
\[
\|u\|_{L^{\frac{mp}{m+n}}} \lesssim \|\nabla^s u\|_{L^p} \quad \forall \ u \in I_s(C^\infty_c \cap H^p).
\]
\item One the other hand, [21, Theorem 1.8] derives
\[
\|u\|_{L^{\frac{mp}{m+n}}} \lesssim \|\nabla^s u\|_{L^p} \quad \forall \ u \in I_s(C^\infty_c \cap H^p).
\]
\item If $0 < s < p = 1 < n$, then the right-hand-side of (1.2) cannot be replaced by either $\|\nabla^s u\|_{L^1}$ or $\|\nabla^s u\|_{L^1}$. A counterexample is given in [20, Section 3.3].
\item If $0 < s < p = 1 \leq n$, then instead of the strong-type estimates, one has the following weak-type inequality:
\[
\|u\|_{L^{\frac{mp}{m+n},\infty}} = \sup_{t > 0} \# \{ x \in \mathbb{R}^n : |u(x)| > t \} \lesssim \|\nabla^s u\|_{L^1} \quad \forall \ u \in I_s(C^\infty_c \cap H^1),
\]
\end{itemize}

\[\text{while the case for } \|\nabla^s u\|_{L^1} \text{ is due to the boundedness of } I_s \text{ from } L^1 \text{ to } L^{\frac{mp}{m+n},\infty} \text{ (cf. [1] or [26, p.119]) and for } \|\nabla^s u\|_{L^1} \text{ follows further from [16, (1.5)] showing}
\]
\[
id = - \sum_{j=1}^{n} R_j^2 \quad \& \quad \|R_j u\|_{L^{\frac{mp}{m+n},\infty}} \lesssim \|u\|_{L^{\frac{mp}{m+n},\infty}} \quad \forall \ (j, u) \in \{1, 2, \ldots, n\} \times L^{\frac{mp}{m+n},\infty}.
\]

1.2. Overview of the principal results. The above analysis has driven us to take a fractional-

geometrical-functional look at the most important case $p = 1$ of the Stein-Weiss inequality

(1.1).
Dense subspaces of $H^{s,1} \& H^{s,1}_x$. Denote by $S$ the Schwartz class on $\mathbb{R}^n$ consisting of functions $f \in C^\infty$ such that

$$\rho_{N,\alpha}(f) = \sup_{x \in \mathbb{R}^n} (1 + |x|^N)|D^\alpha f(x)| < \infty$$

holds for $N \in \mathbb{Z}_+$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Also, write $S'$ for the Schwartz tempered distribution space - the dual of $S$ endowed with the weak-* topology.

As detailed in §2, given $s \in (0, 1)$, if we let

$$S_s = \left\{ f \in C^\infty : \rho_{n+s,\alpha}(\phi) = \sup_{x \in \mathbb{R}^n} (1 + |x|^{n+s})|D^\alpha f(x)| < \infty \; \forall \; \alpha \in \mathbb{Z}_+^n \right\},$$

then for any $u \in S'_s \subset S'$ we can define $\nabla_s^\alpha u$ as a distribution in $S'$. This definition and the case $p = 1$ of (1.2) motivate us to consider the fractional Hardy-Sobolev space

$$H^{s,1} = \left\{ u \in S'_s : [u]_{H^{s,1}} = \|(-\Delta)^{\frac{s}{2}} u\|_{H^1} < \infty \right\}.$$

Note that

$$u_1 - u_2 = \text{constant} \Rightarrow [u_1]_{H^{s,1}} = [u_2]_{H^{s,1}}.$$

So, $[\cdot]_{H^{s,1}}$ is properly a norm on quotient space of $H^{s,1}$ modulo the space of all real constants, and consequently this quotient space is a Banach space.

Upon introducing

$$H^{s,1}_x = \left\{ u \in S'_s : [u]_{H^{s,1}_x} = \|\nabla_s^\alpha u\|_{L^1} < \infty \right\},$$

we find immediately

$$H^{s,1} = H^{s,1}_x \cap H^{s,1}.$$  

Also, since $S$ is dense in $H^{s,1}$ but it is hard to see the density of $S$ in $H^{s,1}_x$, we are induced to introduce

$$\hat{H}^{s,1}_x = \text{closure of } S \text{ in } H^{s,1}_x \text{ under } [\cdot]_{H^{s,1}_x},$$

and yet still have

$$H^{s,1} = \hat{H}^{s,1}_x \cap \hat{H}^{s,1}_x$$

whose $\hat{H}^{s,1}_x$ is a Banach space modulo the space of all real constants.

Correspondingly, for $s \in (0, 1)$ let $W^{s,1}$ be the collection of all locally integrable functions $u$ on $\mathbb{R}^n$ obeying

$$[u]_{W^{s,1}} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} \, dy \, dx < \infty.$$  

Then the quotient space of $W^{s,1}$ modulo the space of all real constants is equal to the homogeneous Besov space $\dot{A}^{s,1}_{1,1}$ (cf. [31]) and is also called Sobolev-Slobodeckij space (cf. [29, p. 36]) or fractional Sobolev space (cf. [18]), and hence

$$S_{\infty} = \left\{ f \in S : D^\alpha \hat{f}(0) = 0 \; \forall \; \alpha \in \mathbb{Z}_+^n \right\}$$

is dense in $W^{s,1}$. In accordance with [6, Appendix], any function

$$f \in L^1 \cap W^{s,1}$$

can be written as

$$f = u + \sum_{\alpha \in \mathbb{Z}_+^n} \int_{\mathbb{R}^n} \partial_\alpha u(y) \, dy \in \dot{A}^{s,1}_{1,1}.$$
can also be approximated by functions in $C^\infty$. Since (cf. [21, 22])

\begin{equation}
|\nabla_s^x u(x)| \leq \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} \, dy \quad \forall \, (u, x) \in S \times \mathbb{R}^n,
\end{equation}

it follows that

\begin{equation}
[u]_{H^{s,1}} = ||\nabla_s^x u||_{L^1} + ||\nabla_s^x u||_{L^1} \leq [u]_{W^{s,1}} \quad \forall \, u \in S.
\end{equation}

Thus, both $H^{s,1}$ and $H^{s,1}_x$ contain $W^{s,1}$. More information on $\{\nabla_s^x, H^{s,1}, H^{s,1}_x\}$ is demonstrated in Propositions 2.11-2.12-2.13-2.14.

**Tracing laws for $H^{s,1}$ & $H^{s,1}_x$.** The previous discussions derive that

\begin{equation}
\|u\|_{L^{s,1}} \leq \begin{cases} [u]_{H^{s,1}} & \text{under } 0 < s < 1 \leq n \\ [u]_{H^{s,1}_x} & \text{under } 0 < s < 1 < n \end{cases} \quad \forall \, u \in S
\end{equation}

and

\begin{equation}
\|u\|_{L^{s,1,\infty}} \leq \begin{cases} [u]_{H^{s,1}} & \text{under } 0 < s < 1 = n \\ [u]_{H^{s,1}_x} & \text{under } 0 < s < 1 \leq n \end{cases} \quad \forall \, u \in S
\end{equation}

are valid, but

\begin{equation}
\|u\|_{L^{s,1,\infty}} \leq \begin{cases} [u]_{H^{s,1}} & \text{under } 0 < s < 1 = n \\ [u]_{H^{s,1}_x} & \text{under } 0 < s < 1 \leq n \end{cases} \quad \forall \, u \in S
\end{equation}

is not true. In order to understand an essential reason for the truth of (1.5) or (1.6) and the fault of (1.7), we investigate under what condition of a given nonnegative Radon measure $\mu$ (restricting/tracing a function to a lower dimensional manifold) in $\mathbb{R}^n$ one has

\begin{equation}
[u]_X \geq \begin{cases} [u]_{L^{s,1,\infty}(\mu)} & \text{as } X \in \{H^{s,1}, H^{s,1}_x(n > 1)\} \\ [u]_{L^{s,1,\infty}(\mu)} & \text{as } X \in \{H^{s,1}, H^{s,1}_x(n = 1)\} \end{cases} \quad \forall \, u \in S.
\end{equation}

Accordingly, we discover such a tracing law that (1.8) is valid if and only if the isocapacitary inequality

\begin{equation}
(\mu(K))^{\frac{1}{s}} \leq \text{Cap}_X(K) \quad \forall \text{ compact } K \subset \mathbb{R}^n
\end{equation}

holds, where the right quantity of (1.9) is called $\{H^{s,1}, H^{s,1}_x\} \ni X$-capacity of $K$ and defined by

\[ \inf \{[f]_X : 1 \leq f \text{ on } K \text{ & } f \in S\}. \]

In §3, we utilize the fractional Sobolev capacity $\text{Cap}_{W^{s,1}}$ and the Hausdorff capacity $A_{\text{loc}}^{s,1}$ to handle $\text{Cap}_{X \in \{H^{s,1}, H^{s,1}_x\}}$ and its strong or weak capacitary inequality through Theorems 3.3 & 3.6-3.7. Then, we verify (1.8) $\iff$ (1.9) in Theorem 3.8.

**Duality laws for $H^{s,1}$ & $H^{s,1}_x$.** As proved in [17, Theorem 3.5], the BV space of all $L^1$-functions with bounded variation on $\mathbb{R}^n$ exists as the duality to the space comprising all tempered distributions

\[ f = \nabla \cdot (U_1, \ldots, U_n) = \sum_{j=1}^n \partial_{x_j} U_j \quad \text{for some } (U_1, \ldots, U_n) \in (L^\infty)^n. \]

This resolves the open problem in [3, Remark 3.12]. In analogy to this matter, as a by-product of (1.3)-(1.4) and the capacity analysis developed within §3, Theorem 4.3 shows that not only the duals of

$H^{s,1}$ & $W^{s,1}$
are the same but also this duality can be characterized by the bounded solutions
\((U_0, U_1, ..., U_n) \in (L^\infty)^{1+n}\)
of the fractional differential equation
\[ [\nabla^s_+]^* U_0 + [\nabla^s_-]^*(U_1, ..., U_n) = T, \]
where \([\nabla^s_+]^* = (-\Delta)^{s/2} \) and \([\nabla^s_-]^* = -(-\Delta)^{s/2} R \).

Also, a similar characterization for
\([\dot{H}^{s,1}_+]^* \) or \([\dot{H}^{s,1}_-]^* \)
is presented in Theorem 4.3 in terms of the bounded solutions to the fractional differential
equation
\[ [\nabla^s_+]^* U_0 = T \] or \([\nabla^s_-]^*(U_1, ..., U_n) = T \).

Furthermore, suppose that BMO is the John-Nirenberg class of all locally integrable functions \(f\) on \(\mathbb{R}^n\) with bounded mean oscillation (cf. [11]):
\[ \|f\|_{\text{BMO}} = \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty \]
where
\[ f_B = \frac{1}{|B|} \int_B f(x) \, dx \]
and the supremum is taken over all Euclidean balls \(B \subset \mathbb{R}^n\) with volume \(|B|\). Surprisingly and yet naturally, the argument for Theorem 4.3, plus the intrinsic structure of
\([H^{s,1}]^* \) and \([\dot{H}^{s,1}_-]^* \) under \(n \geq 2\),
reveals (cf. Theorem 4.4(iii))
\[ (1.10) \quad \text{BMO} = \bar{R} \cdot (L^\infty)^n \text{ under } n \geq 2, \]
which is surely testified by the following decomposition of the canonical BMO-function (cf. [24, 11]):
\[ \ln |x| = \sum_{j=1}^n R_j \left( \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{2-n}{2}\right)}{(n-1) \Gamma\left(\frac{n-2}{2}\right)} \right) \left( \frac{x_j}{|x|} \right) \text{ under } n \geq 2. \]

Nevertheless, the importance of (1.10) can be also seen below:

\(\triangleright\) Via a totally different argument, (1.10) improves upon the Fefferman-Stein decomposition (cf. [7, Theorems 2&3], [30] for a constructive proof, and [10, 8] for some related discussions):
\[ \text{BMO} = L^\infty + \bar{R} \cdot (L^\infty)^n. \]

\(\triangleright\) (1.10) is a unique solution to the Bourgain-Brezis question (cf. [4, p.396]) - What are the function spaces \(X, W^{1,n} \subset X \subset \text{BMO}\), such that every \(F \in X\) has a decomposition \(F = \sum_{j=1}^n R_j Y_j \) where \(Y_j \in L^\infty\)? Here \(W^{1,n}\) is the conformal Sobolev space of all functions \(f\) with
\[ \int_{\mathbb{R}^n} |\nabla f(x)|^n \, dx < \infty \]
and obeys the following decomposition ([4, p.305])
\[ W^{1,n} = \bar{R} \cdot (L^\infty \cap W^{1,n})^n \text{ under } n \geq 2. \]
(1.10) derives that (cf. Theorem 4.4(iv)) for
\[(Y_0, n - 1) \in \text{BMO} \times \mathbb{N}\]
one can get a vector-valued function\[(Y_1, ..., Y_n) \in (L^\infty)^n \text{ solving } \text{div}((-\Delta)^\frac{1}{2}Y_1, ..., (-\Delta)^\frac{1}{2}Y_n) = Y_0.\]
Consequently, this divergence-equation-result is valid for\[(Y_0, n - 1) \in (W^{1,n} \cup L^\infty) \times \mathbb{N}.

But, this consequence cannot be strengthened in the sense that (cf. [4, p.394] or [15])
\[\exists F_0 \in L^\infty \text{ such that } \text{div} F = F_0 \text{ has a solution } F = (F_1, ..., F_n) \in (W^{1,\infty})^n,\]
where\[n - 1 \in \mathbb{N} \land W^{1,\infty} = \left\{ f : f \in L^\infty \land \nabla f \in (L^\infty)^n \right\}.

**Notation.** In the foregoing and forthcoming discussions, \(U \lesssim V\) (resp. \(U \gtrsim V\)) means \(U \leq cV\) (resp. \(U \geq cV\)) for a positive constant \(c\) and \(U \approx V\) amounts to \(U \gtrsim V \lesssim U\). Moreover, \(1_E\) stands for the characteristic function of a set \(E \subset \mathbb{R}^n\), and
\[
\begin{align*}
\mathbb{N} &= \{1, 2, \ldots\} \\
\mathbb{Z}_+ &= \{0, 1, 2, \ldots\} \\
\mathbb{Z} &= \{0, \pm1, \pm2, \ldots\}.
\end{align*}
\]

2. Dense Subspaces of \(H^{s,1} \& H^{s,1}_x\)

2.1. **Initial definitions of \(\nabla_s^\phi\).** Note that any \(f \in \mathcal{S}\) has its Fourier transform
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \xi} \, dx \quad \forall \ \xi \in \mathbb{R}^n.
\]
So the Fourier transform can be naturally extended to \(\mathcal{S}'\) by the dual paring
\[
\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle \quad \forall \ f \in \mathcal{S}' \land \varphi \in \mathcal{S}.
\]

**Definition 2.1.** For \((s, \phi) \in (-n, 1] \times \mathcal{S}\) let \((-\Delta)^{s/2} \phi\) be determined by the Fourier transform
\[
((-\Delta)^{s/2} \phi)(\xi) = (2\pi|\xi|)^s \hat{\phi}(\xi) \quad \forall \ \xi \in \mathbb{R}^n.
\]
Then we have the following comments.

(i) Since \(|\xi|^s\) has singularity at the origin, it is not true that \((-\Delta)^{s/2} \phi \in \mathcal{S}\) for general \(\phi \in \mathcal{S}\). However, if
\[
S_s = \left\{ f \in C^\infty : \rho_{n+s,0}(\phi) = \sup_{x \in \mathbb{R}^n} (1 + |x|^{n+s})|D^\alpha f(x)| < \infty \ \forall \ \alpha \in \mathbb{Z}_+^n \right\},
\]
then (cf. [23, Section 2] or [5])
\[(-\Delta)^{s/2} \phi \in S_s \quad \forall \ \phi \in \mathcal{S}.
\]

(ii) Recall that
\[
S_\infty = \left\{ f \in \mathcal{S} : D^\alpha \hat{f}(0) = 0 \ \forall \ \alpha \in \mathbb{Z}_+^n \right\}.
\]
Then
\[(-\Delta)^{s/2} \phi \in S_\infty \quad \forall \ \phi \in S_\infty.
\]
(iii) As the dual space of $S'_\infty$ let $S'/\mathcal{P}$ be the space $S'$ modulo the space $\mathcal{P}$ of all real-valued polynomials. Then, for any $f \in S'/\mathcal{P}$ we can define $(-\Delta)^{\frac{s}{2}} f$ as a distribution in $S'/\mathcal{P}$:

$$
((-\Delta)^{\frac{s}{2}} f)^\wedge(\xi) = (2\pi|\xi|)^s f(\xi).
$$

Evidently, $(-\Delta)^{\frac{s}{2}}$ maps $S'/\mathcal{P}$ onto $S'/\mathcal{P}$ (cf. [29, pp. 241-242]).

The $(0, n) \ni \alpha$-th order Riesz potential $I_\alpha$ is defined by

$$
I_\alpha f = (-\Delta)^{-\frac{\alpha}{2}}.
$$

If $f \in S$, then $I_\alpha f$ has the integral expression (cf. [26, p. 117])

$$
I_\alpha f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} |x - y|^{\alpha - n} f(y) \, dy \quad \text{with} \quad c_{n,\alpha} = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{\frac{n}{2}} 2^{\alpha} \Gamma\left(\frac{\alpha}{2}\right)}.
$$

Based on Definition 2.1(iii), the definition of $I_\alpha f$ is extendable to $f \in S'/\mathcal{P}$ and so $I_\alpha$ maps $S'/\mathcal{P}$ onto $S'/\mathcal{P}$.

About $\nabla^s_+$. Upon following [23, Section 2.1], we can extend the definition of $\nabla^s_+$ to more general distributions.

**Definition 2.2.** For $(s, f) \in (0, 1) \times S$ set

$$
\nabla^s_+ \phi = (-\Delta)^{\frac{s}{2}} \phi.
$$

Then we have the following comments.

(i) If $f \in S'_s$, then $\nabla^s_+ f$ is defined as a distribution in $S'$:

$$
\langle \nabla^s_+ f, \phi \rangle = \langle f, \nabla^s_+ \phi \rangle \quad \forall \ \phi \in S.
$$

(ii) According to [23, Proposition 2.4], if $f$ belongs to the weighted-$L^1$ space

$$
\mathbb{L}_s = L^1_{\text{loc}} \cap S'_s = \left\{ f : \mathbb{R}^n \to \mathbb{R} \text{ obeys } \|f\|_{\mathbb{L}_s} = \int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{s+n}} \, dx < \infty \right\}
$$

and the Hölder space $C^{s+\varepsilon}$ in a neighborhood of $x \in \mathbb{R}^n$ for some $\varepsilon \in (0, 1 - s]$, then $\nabla^s_+ f$ is continuous at $x$ and it has the integral expression (cf. [5, Definition 1.1 & Lemma 1.4])

$$
\nabla^s_+ f(x) = c_{n,s,+} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{s+n}} \, dy \quad \text{with} \quad c_{n,s,+} = \frac{s2^{s-1} \Gamma\left(\frac{n+s}{2}\right)}{\pi^{\frac{n}{2}} \Gamma(1 - \frac{s}{2})}.
$$

Evidently, this integral expression holds for any $f \in S$.

The next lemma shows that $I_s$ is the inverse of $\nabla^s_+$ on $S$, and vice versa.

**Lemma 2.3.** If $(s, \phi, x) \in (0, 1) \times S \times \mathbb{R}^n$, then

$$
I_s(-\Delta)^{\frac{s}{2}} \phi(x) = \phi(x) = (-\Delta)^{\frac{s}{2}} I_s \phi(x).
$$

**Proof.** On the one hand, [26, p. 117, Lemma 1(a)] and Definition 2.2 derive

$$
I_s(-\Delta)^{\frac{s}{2}} \phi(x) = c_{n,s} \int_{\mathbb{R}^n} |y|^{s-n} (-\Delta)^{\frac{s}{2}} \phi(x-y) \, dy
$$

$$
= \int_{\mathbb{R}^n} (2\pi|y|)^{-s} \left((-\Delta)^{\frac{s}{2}} \phi(x-\cdot)\right)^\wedge(y) \, dy
$$

$$
= \int_{\mathbb{R}^n} (2\pi|y|)^{-s} e^{2\pi i x \cdot y} \left((-\Delta)^{\frac{s}{2}} \phi\right)^\wedge(y) \, dy
$$
\[ = \int_{\mathbb{R}^n} e^{2\pi i x y} \hat{\phi}(y) \, dy = \phi(x). \]

On the other hand, for any \( \alpha \in \mathbb{Z}^n_+ \) we use
\[
D^\alpha I_s \phi(x) = I_s D^\alpha \phi(x) = c_{\alpha,s} \int_{\mathbb{R}^n} |y|^{n-n} D^\alpha \phi(x-y) \, dy
\]
to get
\[
|D^\alpha I_s \phi(x)| \leq \int_{\mathbb{R}^n} |y|^{n-n}(1 + |x-y|)^{-(n+1)} \, dy
\]
\[
\leq \int_{|y| < 1} |y|^{n-n} \, dy + \int_{|y| \geq 1} (1 + |x-y|)^{-(n+1)} \, dy
\]
\[
\leq 1,
\]
which implies
\[
D^\alpha I_s \phi \in L^\infty.
\]
Accordingly, \( I_s \phi \in \mathcal{L}_s \) and \( I_s \phi \) locally satisfies the Lipschitz condition. Now, an application of Definition 2.2(ii) gives that \((-\Delta)^{\frac{s}{2}} I_s \phi\) is continuous on \(\mathbb{R}^n\). Furthermore, since
\[
I_s \phi \in L^n \Rightarrow I_s \phi \in S^*_s,
\]
we have
\[
\langle (-\Delta)^{\frac{s}{2}} I_s \phi, \psi \rangle = \langle I_s \phi, (-\Delta)^{\frac{s}{2}} \psi \rangle = \langle \phi, I_s (-\Delta)^{\frac{s}{2}} \psi \rangle = \langle \phi, \psi \rangle \quad \forall \ \psi \in S,
\]
where the second identity is from the Fubini theorem and the last identity is due to the already-proved identification
\[
I_s (-\Delta)^{\frac{s}{2}} \psi = \psi \quad \forall \ \psi \in S.
\]
Accordingly,
\[
(-\Delta)^{\frac{s}{2}} I_s \phi = \phi \quad \text{in} \ S'.
\]
But nevertheless,
\[
(-\Delta)^{\frac{s}{2}} I_s \phi \ & \phi
\]
are continuous on \(\mathbb{R}^n\), so we arrive at
\[
(-\Delta)^{\frac{s}{2}} I_s \phi(x) = \phi(x) \quad \forall \ x \in \mathbb{R}^n.
\]

\[\square\]

About \(\nabla_s u\). We begin with the following

**Definition 2.4.** For \((s, j, \phi) \in (0, 1) \times \{1, 2, \ldots, n\} \times S\) let
\[
\nabla_s^j \phi = (\nabla_1^j \phi, \nabla_2^j \phi, \ldots, \nabla_n^j \phi),
\]
where each \(\nabla_j^s \phi\) is defined via the Fourier transform:
\[
\left(\tilde{\nabla}_j^s \phi\right)(\xi) = (-2\pi i \xi_j)(2\pi i |\xi|)^{s-1} \hat{\phi}(\xi) \quad \forall \ \xi \in \mathbb{R}^n.
\]

**Lemma 2.5.** If \((s, \phi, x) \in (0, 1) \times S \times \mathbb{R}^n\), then
\[
(2.1) \quad \nabla_s^j \phi(x) = I_{1-s} \nabla \phi(x) = c_{n,s} \int_{\mathbb{R}^n} \left( \frac{x-y}{|x-y|} \right) \left( \frac{(\phi(x) - \phi(y))}{|x-y|^{n+s}} \right) \, dy \quad \text{with} \quad c_{n,s} = \frac{2^{1-s}(\frac{n+s}{2})}{\pi^s \Gamma(1+s/2)}.
\]
Proof. Since $\phi \in S$, it follows from Definition 2.4 that if
\[
\epsilon_{n,1-s} = \frac{2^{s-1} \Gamma \left( \frac{n+s-1}{2} \right)}{\pi^{\frac{n}{2}} \Gamma \left( \frac{s}{2} \right)}
\]
then
\[
\nabla^s_\phi(x) = \left( 2\pi |\xi| \right)^{s-1} (\nabla \phi)^\vee(x)
\]
\[
= \epsilon_{n,1-s} \int_{\mathbb{R}^n} \frac{\nabla \phi(x-y)}{|y|^{n-(1-s)}} \, dy
\]
\[
= \epsilon_{n,1-s} \lim_{\epsilon \to 0, N \to \infty} \int_{|y| < \epsilon < N} \frac{\nabla \phi(x-y)}{|y|^{n-(1-s)}} \, dy.
\]
Note that the second equality in the above formula also implies
\[
\nabla^s_\phi(x) = I_{1-s} \nabla \phi(x) \quad \forall \ x \in \mathbb{R}^n.
\]
Moreover, the integration by parts formula gives
\[
\int_{\epsilon < |y| < N} \frac{\nabla \phi(x-y)}{|y|^{n-(1-s)}} \, dy = \int_{|y| \in (\epsilon, N]} \left( \frac{\phi(x-y)}{|y|^{n-(1-s)}} \right) \bar{v}(y) \, d\mathcal{H}^{n-1}(y)
\]
\[
+ (s+n) \int_{\epsilon < |y| < N} \left( \frac{y}{|y|} \right) \left( \frac{\phi(x-y)}{|y|^{n+s}} \right) \, dy,
\]
where $\bar{v}$ is the outward unit vector on the surface of the ring $\{y \in \mathbb{R}^n : \epsilon < |y| < N\}$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. An application of
\[
\left\{ \begin{array}{ll}
\bar{v}(y) = -\frac{y}{|y|} & \text{when } |y| = \epsilon \\
\bar{v}(y) = \frac{y}{|y|} & \text{when } |y| = N
\end{array} \right.
\]
derives
\[
\left| \int_{|y|=\epsilon} \left( \frac{\phi(x-y)}{|y|^{n-(1-s)}} \right) \bar{v}(y) \, d\mathcal{H}^{n-1}(y) \right| = \left| \int_{|y|=\epsilon} \left( \frac{\phi(x-y) - \phi(x)}{|y|^{n-(1-s)}} \right) \left( \frac{y}{|y|} \right) \, d\mathcal{H}^{n-1}(y) \right| \leq \epsilon^{1-s} \|

\nabla \phi \|

_{L^\infty}

\]
and
\[
\left| \int_{|y|=N} \left( \frac{\phi(x-y)}{|y|^{n-(1-s)}} \right) \bar{v}(y) \, d\mathcal{H}^{n-1}(y) \right| \leq N^{-s} \|

\nabla \phi \|

_{L^\infty}

\]
Consequently, the Lebesgue dominated convergence theorem, along with
\[
\int_{\mathbb{R}^n} \frac{\phi(x) - \phi(y)}{|x-y|^{n+s}} \, dy < \infty,
\]
yields the desired integral expression in (2.1):
\[
\nabla^s_\phi(x) = (s+n) \epsilon_{n,1-s} \lim_{\epsilon \to 0, N \to \infty} \int_{\epsilon < |y| < N} \left( \frac{y}{|y|} \right) \left( \frac{\phi(x-y)}{|y|^{n+s}} \right) \, dy
\]
\[
= \epsilon_{n,s,-} \lim_{\epsilon \to 0, N \to \infty} \int_{\epsilon < |x-y| < N} \frac{x-y}{|x-y|} \left( \frac{\phi(x) - \phi(y)}{|x-y|^{n+s}} \right) \, dy
\]
\[
= \epsilon_{n,s,-} \int_{\mathbb{R}^n} \left( \frac{x-y}{|x-y|} \right) \left( \frac{\phi(x) - \phi(y)}{|x-y|^{n+s}} \right) \, dy.
\]
\]
**Lemma 2.6.** If \((s, j) \in (0, 1) \times \{1, 2, \ldots, n\}\), then \(\nabla_j^s\) maps \(S\) into \(S_s\).

**Proof.** Suppose 
\[
\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n.
\]
Since 
\[
\phi \in S \Rightarrow D^\alpha \phi \in S,
\]
the Fourier transform gives 
\[
\nabla_j^s(D^\alpha \phi) = \nabla_j^s \phi
\]
This, combined with the integral representation of \(\nabla_j^s D^\alpha \phi\) given in Lemma 2.5, yields 
\[
|D^\alpha \nabla_j^s \phi(x)| \approx \left| \int_{\mathbb{R}^n} \left( \frac{x_j - y_j}{|x - y|} \right) \frac{D^\alpha \phi(x) - D^\alpha \phi(y)}{|x - y|^{n+s}} \, dy \right| \quad \forall \ x \in \mathbb{R}^n.
\]
Clearly, 
\[
\left| \int_{|x - y| \geq (1+|x|)/2} \left( \frac{x_j - y_j}{|x - y|} \right) \frac{D^\alpha \phi(x) - D^\alpha \phi(y)}{|x - y|^{n+s}} \, dy \right| = \left| \int_{|x - y| \geq (1+|x|)/2} \left( \frac{x_j - y_j}{|x - y|} \right) \frac{D^\alpha \phi(y)}{|x - y|^{n+s}} \, dy \right| 
\leq (1 + |x|)^{-(n+s)} \|D^\alpha \phi\|_{L^1}.
\]
Also, the mean-value theorem derives 
\[
\left| \int_{|x - y| < (1+|x|)/2} \left( \frac{x_j - y_j}{|x - y|} \right) \frac{D^\alpha \phi(x) - D^\alpha \phi(y)}{|x - y|^{n+s}} \, dy \right| 
\leq \int_{|x - y| < (1+|x|)/2} |x - y|^{1-s-n} \sup_{\theta \in (0, 1)} \sup_{|y| = |x| + 1} |D^\alpha \phi(\theta x + (1 - \theta)y)| \, dy 
\leq (1 + |x|)^{-(n+s)}.
\]
Combining the above two estimates gives 
\[
|D^\alpha \nabla_j^s \phi(x)| \leq (1 + |x|)^{-(n+s)} \quad \forall \ x \in \mathbb{R}^n,
\]
and so 
\[
\nabla_j^s \phi \in S_s.
\]

Lemma 2.6 can be used to extend the definition of \(\nabla_j^s\) to all distributions in \(S'_s\).

**Definition 2.7.** For \((s, f) \in (0, 1) \times S'_s\) let 
\[
\nabla_j^s f = (\nabla_1^s f, \nabla_2^s f, \ldots, \nabla_n^s f),
\]
where \(\nabla_j^s\phi\) is defined by 
\[
\langle \nabla_j^s f, \phi \rangle = -\langle f, \nabla_j^s \phi \rangle \quad \forall \ \phi \in S.
\]

Like Definition 2.2 made for \(\nabla_j^s\), we have also the integral representing of \(\nabla_j^s f\) whenever \(f \in L_s\) has local H"older regularity.

**Lemma 2.8.** Let \((s, f, x) \in (0, 1) \times L_s \times \mathbb{R}^n\). If \(f\) has the H"older continuity of order \(s + \epsilon\) in a neighborhood \(\Omega\) of \(x\) for some \(\epsilon \in (0, 1 - s)\), then \(\nabla_j^s f\) is continuous at \(x\) and 
\[
\nabla_j^s f(x) = c_{n,s,-} \int_{\mathbb{R}^n} \left( \frac{x - y}{|x - y|} \right) \left( \frac{f(x) - f(y)}{|x - y|^{n+s}} \right) \, dy \quad \text{with} \quad c_{n,s,-} = \frac{2^s \Gamma(\frac{n+s+1}{2})}{\pi^s \Gamma(\frac{1}{2})}.
\]
Proof. Without loss of generality, we may assume that \( \Omega \) is bounded and naturally \( \Omega^C = \mathbb{R}^n \setminus \Omega \) is unbounded. An application of both
\[
\int_\Omega \frac{|f(x) - f(y)|}{|x - y|^{n+s}} \, dy \leq \int_\Omega |x - y|^{-n} \, dy < \infty
\]
and
\[
\int_{\Omega^C} \frac{|f(x) - f(y)|}{|x - y|^{n+s}} \, dy \leq \int_{\Omega^C} \frac{|f(x)| + |f(y)|}{(\text{dist}(x, \Omega^C) + |y|)^{n+s}} \, dy \leq |f(x)| + \|f\|_{L^\infty} < \infty
\]
derives
\[
\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|^{n+s}} \, dy < \infty,
\]
and that the integral in the right-hand-side of (2.2) converges absolutely.

To show (2.2), we take an arbitrary open set \( \Omega_0 \ni x \) compactly contained in \( \Omega \). According to the proof of [23, Proposition 2.4], there exists a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset \mathcal{S} \) uniformly bounded in \( C^{s+\epsilon}(\Omega) \), converging uniformly to \( f \) in \( \Omega_0 \) and also converging to \( f \) in the norm of \( L_s \). For any \( k \in \mathbb{N} \), since \( f_k \in \mathcal{S} \), we utilize Lemma 2.5 to write
\[
\nabla^s f_k(x) = c_{n,s} - \int_{\mathbb{R}^n} \left( \frac{x - y}{|x - y|} \right) \left( f_k(x) - f_k(y) \right) \frac{dy}{|x - y|^{n+s}} \forall x \in \mathbb{R}^n.
\]
From the uniform bound on the \( C^{s+\epsilon} \)-norm of \( f_k \) in \( \Omega_0 \) it follows that
\[
\int_{\mathbb{R}^n} \left( \frac{x - y}{|x - y|} \right) \left( f_k(x) - f_k(y) \right) \frac{dy}{|x - y|^{n+s}} \to \int_{\mathbb{R}^n} \left( \frac{x - y}{|x - y|} \right) \left( f(x) - f(y) \right) \frac{dy}{|x - y|^{n+s}}
\]
uniformly in \( \Omega_0 \) as \( k \to \infty \). Since \( \{f_k\}_{k \in \mathbb{N}} \) converges to \( f \) in the norm of \( L_s \), it follows easily that
\[
\nabla^s f_k \to \nabla^s f \quad \text{in} \quad \mathcal{S}'_0.
\]
Accordingly, \( \nabla^s f(x) \) must coincide with
\[
c_{n,s} - \int_{\mathbb{R}^n} \left( \frac{x - y}{|x - y|} \right) \left( f(x) - f(y) \right) \frac{dy}{|x - y|^{n+s}}
\]
in \( \Omega_0 \) by the uniqueness of the limits. So, (2.2) holds. \( \square \)

Below is more information on \( \nabla^s \).

Lemma 2.9. Let \( s \in (0, 1) \).

(i) If \( \phi \in \mathcal{S}_\infty \), then it holds pointwisely on \( \mathbb{R}^n \) that
\[
\nabla^s \phi = I_{1-s} \nabla \phi = \nabla I_{1-s} \phi = \tilde{R}(-\Delta)^{\frac{s}{2}} \phi = (-\Delta)^{\frac{s}{2}} \tilde{R} \phi.
\]
(ii) If \( \phi \in \mathcal{S}'_s \), then \( \nabla^s \phi \in \mathcal{S}' / \mathcal{P} \).

(iii) If \( \phi \in \mathcal{S} \), then the identity in (i) holds almost everywhere on \( \mathbb{R}^n \).

Proof. (i) Via the Fourier transform, we see that
\[
I_{1-s}, \quad (-\Delta)^{\frac{s}{2}} \quad \text{and} \quad R_{1 \leq j \leq n}
\]
map \( \mathcal{S}_\infty \) into \( \mathcal{S}_\infty \). Then, taking the inverse Fourier transform verifies the assertion in (i).

(ii) Let
\[
(\phi, j, \psi) \in \mathcal{S}'_s \times \{1, 2, \ldots, n\} \times \mathcal{S}_\infty.
\]
Then by the just-checked (i) and Definition 2.7 we have
\[
\langle \nabla^s \phi, \psi \rangle = -\langle \phi, \nabla^s \psi \rangle = -\langle \phi, \partial_{x_j} I_{1-s} \psi \rangle
\]
Further, since
\[ \phi \in S' \Rightarrow \phi \in S' \Rightarrow \partial_x \phi \in S' \subset S' / \mathcal{P}, \]
this implication, along with the fact that \( I_{1-} \) maps \( S' / \mathcal{P} \) onto \( S' / \mathcal{P} \), derives
\[ -\langle \phi, \partial_x I_{1-} \psi \rangle = \langle \partial_x \phi, I_{1-} \psi \rangle = \langle I_{1-} \partial_x \phi, \psi \rangle, \]
namely,
\[ \nabla_j \phi = I_{1-} \partial_x \phi \quad \text{in} \quad S' / \mathcal{P}. \]

(iii) Observe that \( S_\infty \) is dense in \( L^p \) whenever \( p \in (1, \infty) \). Indeed, this follows easily from the fact that the Calderón reproducing formula of an \( L^p \)-function
\[ f = \int_0^\infty \varphi_\ast \psi_\ast f \frac{dt}{t} \]
holds in \( L^p \) (cf. [9, p.8, Theorem (1.2)] for \( p = 2 \) and [19] for general \( p \)), with \( \psi, \varphi \in S_\infty \) satisfying
\[
\begin{cases}
\supp \hat{\varphi}, \supp \hat{\psi} \subset \{ \xi \in \mathbb{R}^n : 1/4 \leq \xi \leq 4 \} \\
|\hat{\varphi}(\xi)|, |\hat{\psi}(\xi)| > c \text{ on } \{ \xi \in \mathbb{R}^n : 1/2 \leq \xi \leq 2 \} \\
\int_{0}^{\infty} \hat{\varphi}(t\xi) \hat{\psi}(t\xi) \frac{dt}{t} = 1 \text{ for } \xi \neq 0.
\end{cases}
\]
Thus, if \( \phi \in S \), then a discussion similar to (ii) yields that the identity in (i) holds in \( S' / \mathcal{P} \). Moreover, by the density of \( S_\infty \) in \( L^2 \) and the duality equality
\[ \|f\|_{L^2} = \sup \{ \|\langle f, \phi \rangle\| : \phi \in S_\infty, \|\phi\|_{L^2} \leq 1 \}, \]
we obtain that the identity in (i) holds in \( L^2 \) and hence almost everywhere on \( \mathbb{R}^n \).

2.2. Dense subspaces of \( H^{s,1} \) and \( H^{s,1}_+ \). Note that \( S_\infty \) is dense in \( H^1 \). So, instead of \( S_\infty \) we may consider the following larger space
\[ S_{0, s, \infty, 0} = \left\{ f \in C^\infty : \int_{\mathbb{R}^n} f(x) \, dx = 0 \& \sup_{x \in \mathbb{R}^n}(1 + |x|)^{n+1} |f(x)| < \infty \right\}. \]

A dense subspace of \( H^1 \). As showed in the coming-up-next Lemma 2.10 whose argument relies on the radial maximal function characterization of the Hardy space \( H^1 \) (cf. [27]), the class defined by (2.3) is a dense subspace of \( H^1 \). To see this, recall that if
\[
\begin{cases}
0 \leq \phi \in S \\
\int_{\mathbb{R}^n} \phi(x) \, dx = 1 \\
\phi_j(x) = t^{-n} \phi(t^{-1}x) \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^n,
\end{cases}
\]
then
\[ H^1 = \left\{ f \in S' : f^+ = \sup_{t \in (0, \infty)} |\phi_t \ast f| \in L^1 \right\} \quad \text{with} \quad \|f\|_{H^1} \approx \|f^+\|_{L^1}. \]

We are led to discover the following density for \( H^1 \).

Lemma 2.10. Let \( s \in (0, \infty) \). Then any locally integrable function \( f \) on \( \mathbb{R}^n \) with
\[ \int_{\mathbb{R}^n} f(x) \, dx = 0 \& \sup_{x \in \mathbb{R}^n}(1 + |x|)^{n+1} |f(x)| < \infty \]
belong to the Hardy space \( H^1 \). Consequently, \( S_{s, 0} \) is dense in \( H^1 \). Moreover,
\[ \left\{ (-\Delta)^s \phi : \phi \in S \right\} \subset S_{s, 0} \quad \forall s \in (0, 1). \]
Proof. Let \( \phi \) and \( \{ \phi_t \}_{t \in (0, \infty)} \) be as in (2.4). By the radial maximal function characterization of \( H^1 \), we only need to show that
\[
(2.5) \quad |\phi_t \ast f(x)| \lesssim (1 + |x|)^{-(n+\varepsilon)} \quad \forall \ (t, x) \in (0, \infty) \times \mathbb{R}^n
\]
holds for some \( \varepsilon \in (0, 1) \). Indeed,
\[
(2.5) \Rightarrow \ f^+(x) \lesssim (1 + |x|)^{-(n+\varepsilon)} \quad \forall \ x \in \mathbb{R}^n \Rightarrow f^+ \in L^1.
\]
However, (2.5) is verified by handling two situations: \( |x| \leq 1 \) and \( |x| > 1 \).

If \( |x| \leq 1 \), then
\[
|\phi_t \ast f(x)| \lesssim \|f\|_{L^\infty} \int_{\mathbb{R}^n} \phi_t(x - y) \, dy \lesssim 1 \approx (1 + |x|)^{-(n+\varepsilon)}.
\]

If \( |x| \geq 1 \), then by the conditions of \( f \) we write
\[
|\phi_t \ast f(x)| = \left| \int_{\mathbb{R}^n} (\phi_t(x - y) - \phi_t(x)) f(y) \, dy \right| \lesssim \int_{\mathbb{R}^n} \frac{|\phi_t(x - y) - \phi_t(x)|}{(1 + |y|)^{(n+\varepsilon)}} \, dy.
\]

On the one hand, the mean value theorem gives
\[
\int_{|y| \leq |x|/2} |\phi_t(x - y) - \phi_t(x)| (1 + |y|)^{-(n+\varepsilon)} \, dy
\]
\[
\leq \int_{|y| \leq |x|/2} r^{n-1} \sup_{\theta \in (0, 1)} |\nabla \phi \left( r^{-1}(x - \theta y) \right)| (1 + |y|)^{-(n+\varepsilon)} \, dy
\]
\[
\leq \int_{|y| \leq |x|/2} r^{n-1} (1 + r^{-1}|x|)^{-(n+1)} (1 + |y|)^{-(n+\varepsilon)} \, dy
\]
\[
\leq |x|^{-(n+1)} \int_{|y| \leq |x|/2} (1 + |y|)^{-(n+\varepsilon)} \, dy
\]
\[
\leq |x|^{-(n+1)}.
\]

On the other hand,
\[
\int_{|y| \geq |x|/2} |\phi_t(x - y) - \phi_t(x)| (1 + |y|)^{-(n+\varepsilon)} \, dy
\]
\[
\lesssim (1 + |x|)^{-(n+\varepsilon)} \int_{|y| \geq |x|/2} |\phi_t(x - y)| \, dy + (1 + |x|)^{-(n+\varepsilon)} \int_{|y| \geq |x|/2} \frac{r^n (1 + r^{-1}|x|)^{-n}}{1 + |y|^{n+\varepsilon-\varepsilon}} \, dy
\]
\[
\lesssim (1 + |x|)^{-(n+\varepsilon)} + |x|^{-(n+\varepsilon)} \int_{|y| \geq |x|/2} (1 + |y|)^{-(n+\varepsilon)} \, dy
\]
\[
\lesssim (1 + |x|)^{-(n+\varepsilon)} + |x|^{-(n+\varepsilon)}.
\]

Via combining the last three formula we obtain
\[
|\phi_t \ast f(x)| \lesssim |x|^{-(n+\varepsilon)} \approx (1 + |x|)^{-(n+\varepsilon)} \quad \forall \ |x| \geq 1,
\]
thereby reaching (2.5).

The remaining part of Lemma 2.10 is obvious. \( \Box \)

The first and second dense subspaces of \( H^{s,1} \). Lemma 2.10 produces the following property.

Proposition 2.11. Let \( s \in (0, 1) \). Then
(i) \( H^{s,1} \cap S_\infty = I_s(S_\infty \cap H^1) \).
(ii) \( I_s(H^1) \subset H^{s,1} \).
(iii) For any \( f \in H^{s,1} \) there exists \( g \in H^1 \) such that \( f = I_s g \) in \( S'/\mathcal{P} \).
Proof. (i) For any $\phi \in S_{\infty}$, by the invariant property of $S_{\infty}$ under the action of $I_s$ or $(-\Delta)^{\frac{s}{2}}$, we get

$$\phi \in H^{s,1} \iff (-\Delta)^{\frac{s}{2}} \phi \in H^{1},$$

as desired.

(ii) If $f \in I_s(H^1)$, then

$$f = I_s g \text{ for some } g \in H^1,$$

and hence

$$(1.1) \implies f \in L^{\frac{2n}{n+s}}.$$  

Of course, any function in $L^{\frac{2n}{n+s}}$ belongs to $S'$. Accordingly, for any $\phi \in S$, by Lemma 2.3 we have

$$\langle (-\Delta)^{\frac{s}{2}} f, \phi \rangle = \langle f, (-\Delta)^{\frac{s}{2}} \phi \rangle = \langle I_s g, (-\Delta)^{\frac{s}{2}} \phi \rangle = \langle g, I_s (-\Delta)^{\frac{s}{2}} \phi \rangle = \langle g, \phi \rangle,$$

where in the penultimate equality the Fubini theorem has been applied due to the implication that if

$$g \in H^1 \subset L^1 \ & (-\Delta)^{\frac{s}{2}} \phi \in S,$$

then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^s |g(y)||(-\Delta)^{\frac{s}{2}} \phi(x)| \, dx \, dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^s |g(y)| \frac{1}{1+|x|^{n+s}} \, dx \, dy \leq \|g\|_{L^1}.$$  

Therefore, we obtain

$$(-\Delta)^{\frac{s}{2}} f = g \quad \text{in } S'.$$

Since $g$ belongs to $H^1$, so does $(-\Delta)^{\frac{s}{2}} f$. This proves

$$I_s(H^1) \subset H^{s,1}.$$  

(iii) Recall that both $I_s$ and $(-\Delta)^{\frac{s}{2}}$ are one-to-one maps from $S'/P$ to $S'/P$. Thus, we have

$$f \in S'_s \subset S'/P \quad \forall \ f \in H^{s,1},$$

thereby getting

$$f = I_s((-\Delta)^{\frac{s}{2}} f) \quad \text{in } S'/P$$

and so

$$f = I_s g \quad \text{in } S'/P \quad \text{with } g = (-\Delta)^{\frac{s}{2}} f \in H^1.$$

$\square$

Next, we have the following density result.

**Proposition 2.12.** If $s \in (0, 1)$, then

$$S_{\infty} \subset S \subset H^{s,1} \subset H^{s,1}. $$

Moreover, both $S_{\infty}$ and $S$ are dense in $H^{s,1}$.

**Proof.** For any $u \in S$, we use Lemma 2.10 to derive

$$(-\Delta)^{\frac{s}{2}} u \in H^1 \quad \text{i.e.} \quad u \in H^{s,1}.$$  

This proves $S \subset H^{s,1}$; the other inclusions are obvious.

It remains to show the density of $S_{\infty}$ in $H^{s,1}$. If $f \in H^{s,1}$, then

$$f \in S'_s \ & (-\Delta)^{\frac{s}{2}} f \in H^1.$$  

Due to the density of $S_{\infty}$ in $H^1$,

$$\exists \{g_j\}_{j \in \mathbb{N}} \subset S_{\infty} \text{ such that } \lim_{j \to \infty} \|g_j - (-\Delta)^{\frac{s}{2}} f\|_{H^1} \to 0.$$
For any \( j \in \mathbb{N} \), let
\[
f_j = I_s g_j \quad (\text{which actually belongs to } S_{\infty}).
\]
Noticing that
\[
g_j = (-\Delta)^{\frac{s}{2}} f_j,
\]
we have
\[
\lim_{j \to \infty} \| f_j - f \|_{H^{s,1}} = \lim_{j \to \infty} \| (-\Delta)^{\frac{s}{2}} (f_j - f) \|_{H^1} = \lim_{j \to \infty} \| g_j - (-\Delta)^{\frac{s}{2}} f \|_{H^1} = 0.
\]
Thus, \( f \in H^{s,1} \) can be approximated by the \( S_{\infty} \)-functions \( \{f_j\}_{j \in \mathbb{N}} \).

A dense subspace of \( H^{s,1}_+ \). It is difficult to determine the density of \( S \) in \( H^{s,1}_- \). However, we have

**Proposition 2.13.** If \( s \in (0, 1) \), then \( I_s(S) \) is a dense subspace of \( H^{s,1}_+ \) but
\[
I_s(S) \not\subset H^{s,1}_-.
\]

**Proof.** On the one hand, if \( f \in I_s(S) \) then
\[
\exists \phi \in S \text{ such that } f = I_s \phi,
\]
but Lemma 2.3 implies
\[
(-\Delta)^{\frac{s}{2}} f = (-\Delta)^{\frac{s}{2}} I_s \phi = \phi \in S \subset L^1,
\]
that is, \( f \in H^{s,1}_+ \). To show the density of \( I_s(S) \) in \( H^{s,1}_+ \), given any \( f \in H^{s,1}_+ \) we utilize
\[
(-\Delta)^{\frac{s}{2}} f \in L^1
\]
and the density of \( S \) in \( L^1 \) to find a sequence
\[
\{g_j\}_{j \in \mathbb{N}} \subset S
\]
such that
\[
\lim_{j \to \infty} \| g_j - (-\Delta)^{\frac{s}{2}} f \|_{L^1} = 0.
\]
Upon defining
\[
f_j = I_s g_j \in I_s(S)
\]
and using Lemma 2.3, we gain the representation
\[
g_j = (-\Delta)^{\frac{s}{2}} f_j
\]
and the desired convergence
\[
\lim_{j \to \infty} \| f_j - f \|_{H^{s,1}_+} = \lim_{j \to \infty} \| (-\Delta)^{\frac{s}{2}} (f_j - f) \|_{H^1} = \lim_{j \to \infty} \| g_j - (-\Delta)^{\frac{s}{2}} f \|_{L^1} = 0.
\]
In other words, \( I_s(S) \) is a dense subspace of \( H^{s,1}_+ \).

On the other hand, \( I_s(S) \) is not a subspace of \( H^{s,1}_- \) - otherwise - if \( I_s(S) \subset H^{s,1}_- \), then this, along with \( I_s(S) \subset H^{s,1}_+ \), would imply \( I_s(S) \subset H^{s,1} \) and hence \( S \subset H^1 \) which is impossible. \( \square \)
The third dense subspace of $H^{s,1}$. In addition to Proposition 2.12, we obtain

**Proposition 2.14.** If $s \in (0, 1)$, then

$$
\mathcal{D}_0 = \left\{ f \in C^{\infty}_c : \int_{\mathbb{R}^n} f(x) \, dx = 0 \right\}
$$

is a dense subspace of $H^{s,1}$.

**Proof.** Proposition 2.12 implies

$$
\mathcal{D}_0 \subset S \subset H^{s,1}.
$$

So, it suffices to show the density of $\mathcal{D}_0$ in $H^{s,1}$. Let $f \in H^{s,1}$. Based on Proposition 2.11(iii),

$$
\exists g \in H^1 \text{ such that } f = I_s g \text{ in } S'/\mathcal{P}.
$$

Note that $H^1$ is nothing but the homogeneous Triebel-Lizorkin space $\dot{F}^1_{1,2}$. So, the lifting property of $I_s$ on the Triebel-Lizorkin spaces (cf. [29, p. 242]) shows

$$
I_s(H^1) = I_s(\dot{F}^0_{1,2}) = \dot{F}^s_{1,2}.
$$

Therefore,

$$
\exists \tilde{f} \in \dot{F}^s_{1,2} \text{ such that } f = I_s g = \tilde{f} \text{ in } S'/\mathcal{P}.
$$

Recall that [12, Theorem 1] yields that any element in $\dot{F}^s_{1,2}$ can be written as the linear combinations of $\dot{F}^s_{1,2}$-atoms, just as the atomic decomposition of the Hardy space $H^1$. To be precise, since $\tilde{f} \in \dot{F}^s_{1,2}$, it follows that

$$
\tilde{f} = \sum_{j \in \mathbb{N}} \lambda_j a_j \text{ in } S'/\mathcal{P},
$$

where

$$
\|\tilde{f}\|_{\dot{F}^s_{1,2}} \approx \sum_{j \in \mathbb{N}} |\lambda_j|
$$

and, based on the remark after [12, Definition (1.6)], every $a_j$ is a locally integrable function on $\mathbb{R}^n$ with the following three properties:

(i) $a_j$ is supported on a ball $B_j$;
(ii) $\|a_j\|_{\dot{F}^s_{1,2}} \leq |B_j|^{-\frac{s}{2}}$;
(iii) $\int_{\mathbb{R}^n} a_j \, dx = 0$.

Again using the lifting property of $I_s$ (cf. [29, p. 242]) gives

$$
\|a_j\|_{\dot{F}^s_{1,2}} = \|I_s a_j\|_{\dot{F}^0_{2,2}}.
$$

By [29, p. 242, (2)], any element in $\dot{F}^0_{2,2}$ coincides with a function in $L^2$ in the sense of $S'/\mathcal{P}$. Thus, we know from $I_s a_j \in \dot{F}^0_{2,2}$ that $I_s a_j$ coincides with some $L^2$-function, denoted by $\tilde{I}_s a_j$, in the sense of $S'/\mathcal{P}$. So, by the density of $\mathcal{S}_\infty$ in $L^2$ (cf. the proof of Lemma 2.9(iii)) and the duality we get

$$
\|I_s a_j\|_{L^2} = \sup \left\{ |\langle I_s a_j, \phi \rangle| : \phi \in \mathcal{S}_\infty, \|\phi\|_{L^2} \leq 1 \right\}
$$

$$
= \sup \left\{ |\langle \tilde{I}_s a_j, \phi \rangle| : \phi \in \mathcal{S}_\infty, \|\phi\|_{L^2} \leq 1 \right\}
$$

$$
= \|\tilde{I}_s a_j\|_{L^2}
$$

$$
\approx \|I_s a_j\|_{\dot{F}^0_{2,2}}
$$

$$
\approx \|a_j\|_{\dot{F}^s_{1,2}}.
$$

(2.6)
Let $\psi \in C^\infty_c$ satisfy
\[ \int_{\mathbb{R}^n} \psi \, dx = 1 \quad \text{and} \quad \text{supp } \psi \subset B(0,1). \]

For any $\epsilon \in (0, \infty)$, define
\[ \psi_{\epsilon}(\cdot) = \epsilon^{-n} \psi(\epsilon^{-1} \cdot). \]

Fix an arbitrary small number $\eta \in (0, \infty)$. For any $j \in \mathbb{N}$, an application of $I_s a_j \in L^2$ produces a sufficiently small $\epsilon_j \in (0, \infty)$ such that
\[ \text{supp } \psi_{\epsilon_j} \ast a_j \subset 2B_j \quad \text{and} \quad \| I_s a_j - \psi_{\epsilon_j} \ast (I_s a_j) \|_{L^2} < \eta |2B_j|^{-\frac{1}{2}}. \]

By (2.6), the last inequality is equivalent to that
\[ \| a_j - \psi_{\epsilon_j} \ast a_j \|_{\dot{F}^{s}_{1,2}} < \eta |2B_j|^{-\frac{1}{2}}. \]

Choose $N$ large enough such that
\[ \sum_{j=N+1}^{\infty} |\lambda_j| < \eta, \]

and define
\[ f_{\epsilon,N} = \sum_{j=1}^{N} \lambda_j \psi_{\epsilon_j} \ast a_j. \]

Evidently,
\[ \psi_{\epsilon_j} \ast a_j \in D_0 \quad \text{and} \quad f_{\epsilon,N} \in D_0. \]

By the argument in [12, p. 239], we know that any $\dot{F}^{s}_{1,2}$-atom $a_j$ satisfies $\|a_j\|_{\dot{F}^{s}_{1,2}} \lesssim 1$. The choice of $\epsilon_j$ implies that
\[ \eta^{-1}(a_j - \psi_{\epsilon_j} \ast a_j) \]

is also an $\dot{F}^{s}_{1,2}$-atom, thereby yielding
\[ \|a_j - \psi_{\epsilon_j} \ast a_j\|_{\dot{F}^{s}_{1,2}} \lesssim \eta. \]

Upon recalling
\[ f = \bar{f} \in S'/\mathcal{P}, \]

we obtain
\[ \|f_{\epsilon,N} - f\|_{\dot{F}^{s}_{1,2}} = \|f_{\epsilon,N} - \bar{f}\|_{\dot{F}^{s}_{1,2}} \]
\[ \lesssim \sum_{j=1}^{N} |\lambda_j| \|\psi_{\epsilon_j} \ast a_j - a_j\|_{\dot{F}^{s}_{1,2}} + \sum_{j=N+1}^{\infty} |\lambda_j| \|a_j\|_{\dot{F}^{s}_{1,2}} \]
\[ \lesssim \eta \sum_{j=1}^{N} |\lambda_j| + \sum_{j=N+1}^{\infty} |\lambda_j| \]
\[ \lesssim \eta. \]

Finally, using the lifting property of $I_s$ (cf. [29, p. 242]) yields
\[ [f_{\epsilon,N} - f]_{H^{s,1}} = \|(-\Delta)^{\frac{s}{2}}(f_{\epsilon,N} - f)\|_{H^1} \approx \|f_{\epsilon,N} - \bar{f}\|_{\dot{F}^{s}_{1,2}} \lesssim \eta. \]

Due to the arbitrariness of $\eta$, we obtain that $f \in H^{s,1}$ can be approximated by functions in $D_0$. \qed
3. Tracing Laws for $H^{s,1}_\pm$ & $H^{s,1}_0$

3.1. Strong/Weak Estimates for $\text{Cap}_{X_0(H^{s,1}_\pm, H^{s,1}_0)}$. This section is devoted to a measure-theoretic study of the capacity living on $X \in \{W^{s,1}, H^{s,1}_\pm, H^{s,1}_0\}$.

**Capacitory Concepts.** For $\alpha \in (0, n)$, denote by $\Lambda_\alpha^{(\infty)}$ the $\alpha$-dimensional Hausdorff capacity:

$$
\Lambda_\alpha^{(\infty)}(E) = \inf \left\{ \sum_i r_i^\alpha : E \subset \bigcup_i B(x_i, r_i), (x_i, r_i) \in \mathbb{R}^n \times (0, \infty) \right\}
$$

for any set $E \subset \mathbb{R}^n$ which is covered by a sequence of balls

$$
B(x_i, r_i) = \{x \in \mathbb{R}^n : |x - x_i| < r_i\}.
$$

Classically, $\Lambda^{(\infty)}_\alpha(\cdot)$ is a monotone, countably subadditive set function on the class of all subsets of $\mathbb{R}^n$, and enjoys $\Lambda^{(\infty)}_\alpha(\emptyset) = 0$.

**Definition 3.1.** For $s \in (0, 1)$ and any compact set $K \subset \mathbb{R}^n$ define (cf. [31, 2])

$$
(3.1) \quad \text{Cap}_X(K) = \begin{cases} 
\inf \{ |u|_{X} : u \in C^\infty_0 & u \geq 1_K \} & \text{as } X = W^{s,1} \\
\inf \{ |u|_{X} : u \in \mathcal{S} \& u \geq 1 \text{ on } K \} & \text{as } X \in \{H^{s,1}_\pm, H^{s,1}_0\}.
\end{cases}
$$

Furthermore, $\text{Cap}_X(\cdot)$ is extendable from compact sets to general sets as seen below:

(i) If $O \subset \mathbb{R}^n$ is open, then

$$
\text{Cap}_X(O) = \sup_{K \text{ compact}, K \subset O} \text{Cap}_X(K)
$$

(ii) For an arbitrary set $E \subset \mathbb{R}^n$ set

$$
\text{Cap}_X(E) = \inf_{O \text{ open}, O \supset E} \text{Cap}_X(O).
$$

Thus, the definition of $\text{Cap}_X$ on any compact/open set is consistent (cf. [14, Lemma 3.2.4]).

**Lemma 3.2.** Let

$$
\begin{cases} 
s \in (0, 1) \\
(x, r) \in \mathbb{R}^n \times (0, \infty) \\
B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\} \\
X \in \{H^{s,1}_\pm, H^{s,1}_0\}.
\end{cases}
$$

Then

(i) $\text{Cap}_X(\emptyset) = 0$ \& $\text{Cap}_X(B(x, r)) = r^{n-s} \text{Cap}_X(B(0, 1))$.

(ii) $\text{Cap}_X(E_1) \leq \text{Cap}_X(E_2)$ whenever $E_1 \subset E_2 \subset \mathbb{R}^n$.

(iii) $\max \{ \text{Cap}_{H^{s,1}_+}(\cdot), \text{Cap}_{H^{s,1}_-}(\cdot) \} \leq \text{Cap}_{H^{s,1}}(\cdot) \lesssim \Lambda^{n-s}_\alpha(\cdot) \approx \text{Cap}_{W^{s,1}}(\cdot)$.

(iv) $\text{Cap}_{W^{s,1}}(\cdot)$ is countably subadditive, but $\text{Cap}_{H^{s,1}_\pm}(\cdot)$ and $\text{Cap}_{H^{s,1}_\pm}(\cdot)$ may not be countably subadditive.

**Proof.** Both (i) and (ii) follow from (3.1).

(iii) First, according to Definition 3.1, we only need to consider these capacities on compact sets. For any $u \in \mathcal{S}$, by (1.4), we get

$$
[u]_{H^{s,1}_+} \leq [u]_{H^{s,1}} \leq [u]_{W^{s,1}} \Rightarrow \text{Cap}_{H^{s,1}_+}(\cdot) \leq \text{Cap}_{H^{s,1}}(\cdot) \leq \text{Cap}_{W^{s,1}}(\cdot).
$$

Noting that

$$
\text{Cap}_{W^{s,1}}(\cdot) \approx \Lambda^{n-s}_\alpha(\cdot)
$$
is given in [18, Theorem 2.1] and [31, (2.1)], we are left to verify
\begin{equation}
\Lambda_{(\infty)}^{n-s}(\cdot) \lesssim \text{Cap}_{H^{s,1}}(\cdot).
\end{equation}

According to [2, Proposition 3], for any compact set $K$ in $\mathbb{R}^n$, the capacity
\[ R_s(K) = \inf \{\| f\|_{H^1} : f \in S_\infty \& I_s f \geq 1 \text{ on } K \} \]
satisfies
\[ \Lambda_{(\infty)}^{n-s}(K) \approx R_s(K). \]

By Lemma 2.10, we have
\[ S_\infty \subset S_{s,0} \subset H^1 \]
and the density of $S_{s,0}$ in $H^1$. Meanwhile, for any $(f, x) \in S_{s,0} \times \mathbb{R}^n$, it is obvious that $I_s f(x)$ is well defined and $I_s f$ is continuous on $\mathbb{R}^n$. Thus, instead of using $S_\infty$, we have
\[ R_s(K) = \inf \{\| f\|_{H^1} : f \in S_{s,0} \& I_s f \geq 1 \text{ on } K \}. \]

For any $u \in S$ satisfying $u \geq 1$ on $K$ let
\[ f_s = (-\Delta)^{\frac{s}{2}} u, \]
which belongs to $S_{s,0}$ in terms of Lemma 2.10. Then, by Lemma 2.3 we have
\[ I_s f_s = u \geq 1 \text{ on } K, \]
thereby achieving
\[ R_s(K) \leq \| f_s \|_{H^1} = \|(-\Delta)^{\frac{s}{2}} u\|_{H^1} = [u]_{H^{s,1}}. \]
Taking the infimum over all such $u \in S$ satisfying $u \geq 1$ on $K$ yields
\[ R_s(K) \leq \inf \{[u]_{H^{s,1}} : S \ni u \geq 1 \text{ on } K \} = \text{Cap}_{H^{s,1}}(K). \]
Thus,
\[ \Lambda_{(\infty)}^{n-s}(K) \lesssim \text{Cap}_{H^{s,1}}(K). \]

This proves (3.2).

(iv) The countable subadditivity of $\text{Cap}_{W^{s,1}}(\cdot)$ follows from [32, Theorem 1(iii)]. Since the test functions used in
\[ \text{Cap}_X(\cdot) \text{ for } X \in \{H^{s,1}, H^{s,1}_+\} \]
are not assumed to be nonnegative, the capacities under consideration may not be countably subadditive as mentioned in [2].

\textbf{Strong estimates for Cap}_{X \in \{H^{s,1}, H^{s,1}_+\}}. First of all, an application of Proposition 3.2(iii) and [31, Theorem 1.1] or [18, Theorem 1.3] gives the following strong inequality for $\text{Cap}_{W^{s,1}}$ (cf. [31, Theorem 2.2]):
\begin{equation}
\int_0^\infty \text{Cap}_{W^{s,1}}(\{x \in \mathbb{R}^n : |u(x)| > t\}) \, dt \lesssim [u]_{W^{s,1}} \quad \forall \ u \in C_\infty^\infty.
\end{equation}
Next, we are led by (3.3) to get the strong inequality for $\text{Cap}_{H^{s,1}}$ as seen below.

\textbf{Theorem 3.3.} If $s \in (0, 1)$, then
\[ \int_0^\infty \text{Cap}_{H^{s,1}}(\{x \in \mathbb{R}^n : |u(x)| > t\}) \, dt \lesssim [u]_{H^{s,1}} \quad \forall \ u \in S. \]
Proof. Note that Proposition 3.2(iii) implies
\[ \text{Cap}_{H^1} (\cdot) \leq \text{Cap}_{W^{1,1}} (\cdot) \approx \Lambda_{(\infty)}^{n-\alpha} (\cdot) \]
and [2, Proposition 5] gives that
\[ \int_0^\infty \Lambda_{(\infty)}^{n-\alpha} ([x \in \mathbb{R}^n : |I_s f(x)| > t]) dt \leq \| f \|_{H^1} \quad \forall \ f \in \mathcal{S}_{s,0}. \]
In particular, given \( u \in \mathcal{S} \), we can take
\[ f = \nabla^*_s u = (-\Delta)^s u, \]
which belongs to \( \mathcal{S}_{s,0} \) via the Fourier transform. Noting that Lemmas 2.3 and 2.9(iii) imply
\[
\begin{align*}
\nabla^*_s u &= R(-\Delta)^s u = Rf \text{ almost everywhere on } \mathbb{R}^n \\
[u]_{H^1} &= ||\nabla^*_s u||_{L^1} + ||\nabla^*_s u||_{L^1} = ||f||_{L^1} + ||Rf||_{L^1} = ||f||_{H^1},
\end{align*}
\]
we obtain
\[ \int_0^\infty \text{Cap}_{H^1} ([x \in \mathbb{R}^n : |u(x)| > t]) dt \leq \int_0^\infty \Lambda_{(\infty)}^{n-\alpha} ([x \in \mathbb{R}^n : |I_s f(x)| > t]) dt \leq ||f||_{H^1} \approx [u]_{H^1}, \]
as desired. \( \square \)

To establish the strong inequality for \( \text{Cap}_{H^1 (s>1)} \), we require two more lemmas.

Lemma 3.4. For \((p, \alpha) \in [1, \infty) \times (0, n)\) let \( L^{p, \alpha} \) be the Morrey space of all functions \( f \) on \( \mathbb{R}^n \) with
\[ \|f\|_{L^{p, \alpha}} = \sup_{(x, r) \in \mathbb{R}^n \times (0, \infty)} \left( \int_{B(x, r)} \left| f(y) \right|^p dy \right)^{\frac{1}{p}} < \infty. \]
If \( 0 < \beta < \alpha < n \) and \( \mu \) is a nonnegative Radon measure on \( \mathbb{R}^n \) with
\[ \||\mu||_{n-\alpha} = \sup_{(x, r) \in \mathbb{R}^n \times (0, \infty)} r^{\alpha-n} \mu(B(x, r)) < \infty, \]
then for \( p \in (1, \frac{n}{n-\beta}) \) the function
\[ \mathbb{R}^n \ni x \mapsto I_{\beta} \mu (x) = c_{n, \beta} \int_{\mathbb{R}^n} |x - y|^{\beta-n} d\mu(y) \]
belongs to \( L^{p, p(\alpha-\beta)}. \)

Proof. Fix \((x_0, r) \in \mathbb{R}^n \times (0, \infty)\) and write
\[ \int_{B(x_0, r)} |I_{\beta} \mu (x)|^p dx \leq \int_{B(x_0, r)} \left( \int_{|y-x_0| \geq 2r} |x - y|^{\beta-n} d\mu(y) \right)^p dx \]
\[ + \int_{B(x_0, r)} \left( \int_{|y-x_0| < 2r} |x - y|^{\beta-n} d\mu(y) \right)^p dx. \]
On the one hand, we have
\[ \int_{B(x_0, r)} \left( \int_{|y-x_0| \geq 2r} |x - y|^{\beta-n} d\mu(y) \right)^p dx \]
On the other hand, by the Hölder inequality, the Fubini theorem and \((\beta - n)p + n > 0\), we also have

\[
\int_{B(x_0, r)} \left( \int_{|y - x| < 2r} |x - y|^{\beta - n} \, d\mu(y) \right)^p \, dx 
\leq \int_{B(x_0, r)} \left( \int_{|y - x| < 2r} |x - y|^{\beta - n} \, d\mu(y) \right) \mu(B(x_0, 2r))^{p-1} \, dx 
\leq \|\mu\|_{n-\alpha}^{p-1} r^{(n-\alpha)(p-1)} \int_{|y - x| < 2r} \left( \int_{B(x_0, r)} |x - y|^{\beta - n} \, dx \right) \, d\mu(y) 
\leq \|\mu\|_{n-\alpha}^{p-1} r^{(n-\alpha)(p-1)} \int_{|y - x| < 2r} \left( \int_{B(x_0, r)} |x - y|^{\beta - n} \, dx \right) \, d\mu(y) 
\leq \|\mu\|_{n-\alpha}^{p} r^{-(\alpha - \beta)p + n}.
\]

Combining the above three estimates gives

\[
\int_{B(x_0, r)} |I_{\beta\mu}(x)|^p \, dx \leq \|\mu\|_{n-\alpha}^{p} r^{-(\alpha - \beta)p + n},
\]
that is,

\[
I_{\beta\mu} \in L^{p, p(\alpha - \beta)}.
\]

The following result improves upon [2, Proposition 5].

**Lemma 3.5.** If \((n - 1, \alpha) \in \mathbb{N} \times (0, n)\), then

\[
\int_0^\infty \Lambda_{(\infty)}^{n-\alpha} \left( \{x \in \mathbb{R}^n : |I_a f(x)| > t\} \right) \, dt \leq \|\vec{R} f\|_{L^1} \quad \forall \ f \in S_{\alpha, 0}.
\]

**Proof.** Let \(f \in S_{\alpha, 0}\). Note that Lemma 2.3 implies

\[
f \in S_{\alpha, 0} \subset H^1.
\]

So, by this and the boundedness of each Riesz transform \(R_j\) from \(H^1\) to \(L^1\), we derive \(\|\vec{R} f\|_{L^1} < \infty\). Upon applying [2, p. 118, Corollary] we have

\[
\int_0^\infty \Lambda_{(\infty)}^{n-\alpha} \left( \{x \in \mathbb{R}^n : |I_a f(x)| > t\} \right) \, dt 
\approx \sup \left\{ \int_{\mathbb{R}^n} |I_a f(x)| \, d\mu(x) : \mu \text{ nonnegative Radon measure, } \|\mu\|_{n-\alpha} \leq 1 \right\}.
\]

Thus, the desired result follows from showing that

\[
\int_{\mathbb{R}^n} |I_a f| \, d\mu \leq \|\vec{R} f\|_{L^1}.
\]
Meanwhile, the Stein-Weiss inequality (1.1) and the fact and utilizing Lemma 3.4 we achieve

\[ I_\beta \mu \in L^{p, p(\alpha - \beta)} \] with \( ||I_\beta \mu||_{L^{p, p(\alpha - \beta)}} \leq ||\mu||_{L^1} \leq 1. \)

Via \( f \in S_{\alpha, 0} \) and the Fubini theorem we write

\[
\int_{\mathbb{R}^n} |I_\alpha f(x)| \, d\mu(x) \leq \int_{\mathbb{R}^n} I_\beta (|I_{\alpha-\beta} f|) \, d\mu(x)
\]

\[
= \int_{\mathbb{R}^n} I_{\alpha-\beta} f(x) \text{sgn}(I_{\alpha-\beta} f(x)) I_\beta \mu(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} I_\varepsilon f(x) \left( \text{sgn}(I_{\alpha-\beta} f) I_\beta \mu \right)(x) \, dx
\]

\[ = \langle I_\varepsilon f, g \rangle, \]

where

\[ 0 < \varepsilon < \min\{1, \alpha - \beta\} \quad \& \quad g = I_{\alpha-\beta} \varepsilon \left( \text{sgn}(I_{\alpha-\beta} f) I_\beta \mu \right). \]

According to [1, Theorem 3.1], \( I_{\alpha-\beta} \varepsilon \) maps continuously \( L^{p, p(\alpha - \beta)} \) to \( L^2 \), and consequently,

\[ g \in L^2 \quad \& \quad ||g||_{L^2} \leq ||I_{\alpha-\beta} \varepsilon (I_\beta \mu)||_{L^2} \leq ||I_\beta \mu||_{L^{p, p(\alpha - \beta)}} \leq ||\mu||_{L^1} \leq 1. \]

Meanwhile, the Stein-Weiss inequality (1.1) and the fact \( f \in H^1 \) show that \( I_\varepsilon f \in L^{n-\varepsilon} \). Thus, the pairing \( \langle I_\varepsilon f, g \rangle \) makes sense.

Before proceeding with the argument, we claim that

\[
\text{(3.4)} \quad I_\varepsilon f = - \sum_{j=1}^n R_j^2 I_\varepsilon f = - \sum_{j=1}^n R_j I_\varepsilon R_j f \quad \text{in } L^{n-\varepsilon} \quad \& \quad \text{almost everywhere on } \mathbb{R}^n.
\]

It suffices to show the validity of (3.4) in \( L^{n-\varepsilon} \). Indeed, the first equality of (3.4) follows from

\[ I_\varepsilon f \in L^{n-\varepsilon} \quad \& \quad \text{id} = - \sum_{j=1}^n R_j^2 \quad \text{in } L^{n-\varepsilon}. \]

To see the second equality of (3.4), we fix \( i \in \{1, 2, \ldots, n\} \) and only need to validate

\[
\text{(3.5)} \quad R_j I_\varepsilon f = I_\varepsilon R_j f \quad \text{in } L^{n-\varepsilon}.
\]

To check (3.5), we use (1.1), the fact

\[ f \in S_{\alpha, 0} \subset H^1, \]

and the boundedness of each \( R_j \) on \( L^{\infty} \) or \( H^1 \) to derive

\[ (R_j I_\varepsilon f, I_\varepsilon R_j f) \in (L^{\infty})^2. \]

This, along with

\[ [L^{\infty}]^* = L^2 \]

and the density of \( S_{\alpha, 0} \) in \( L^{\infty} \) (cf. the proof of Lemma 2.9(iii)), yields

\[ ||R_j I_\varepsilon f - I_\varepsilon R_j f||_{L^{\infty}} = \sup \left\{ \langle R_j I_\varepsilon f - I_\varepsilon R_j f, \phi \rangle : \phi \in S_{\alpha, 0}, ||\phi||_{L^{n-\varepsilon}} \leq 1 \right\} \]

\[ = \sup \left\{ \langle f, I_\varepsilon R_j \phi - R_j I_\varepsilon \phi \rangle : \phi \in S_{\alpha, 0}, ||\phi||_{L^{n-\varepsilon}} \leq 1 \right\} \]

\[ = 0, \]
where the last step holds because the Fourier transform implies that
\[ I_\epsilon R_j \phi = R_j I_\epsilon \phi \quad \forall \phi \in S_\infty. \]

This proves (3.5) and hence (3.4).

To continue, by (3.4), we conclude that
\[
\langle I_\epsilon f, g \rangle = -\sum_{j=1}^{n} \langle R_j^2 I_\epsilon f, g \rangle = \sum_{j=1}^{n} \langle R_j I_\epsilon f, R_j g \rangle = \sum_{j=1}^{n} \langle I_\epsilon R_j f, R_j g \rangle = \sum_{j=1}^{n} \langle R_j f, I_\epsilon R_j g \rangle,
\]

whence
\[
\int_{\mathbb{R}^n} |I_\alpha f(x)| \, d\mu(x) \leq \sum_{j=1}^{n} \int_{\mathbb{R}^n} R_j f(x) I_\epsilon R_j g(x) \, dx.
\]

According to [20, (2.1)] and its proof, we have that if
\[ n \geq 2 \quad \& \quad (f, \varphi) \in S_{\alpha,0} \times L^2, \]

then
\[
\left\| \int_{\mathbb{R}^n} R_j f(x) I_\epsilon \varphi(x) \, dx \right\| \leq \| \tilde{R} f \|_{L^1} \| \varphi \|_{L^\infty}.
\]

This, along with the boundedness of \( R_j \) on the Lebesgue space \( L^2 \), further implies the wanted estimation
\[
\int_{\mathbb{R}^n} |I_\alpha f(x)| \, d\mu(x) \leq \sum_{j=1}^{n} \| R_j f \|_{L^1} \| R_j g \|_{L^\infty} \leq \| \tilde{R} f \|_{L^1} \| g \|_{L^\infty} \leq \| \tilde{R} f \|_{L^1}.
\]

Finally, we arrive at the strong inequality for \( \text{Cap}_{H_s^{1,1}} \) with \( n > 1 \).

**Theorem 3.6.** If \( 0 < s < 1 < n \), then
\[
\int_0^\infty \text{Cap}_{H_s^{1,1}} \left( \{ x \in \mathbb{R}^n : |u(x)| > t \} \right) \, dt \leq [u]_{H_s^{1,1}} \quad \forall \ u \in S.
\]

**Proof.** Given \( u \in S \). Lemmas 3.2 & 2.9 produce
\[
\begin{cases}
\text{Cap}_{H_s^{1,1}} \leq \text{Cap}_{W_s^{1,1}} \approx \Lambda_{(\infty)}^{n-s} \\
[u]_{H_s^{1,1}} = \| \nabla \tilde{\epsilon} u \|_{L^1} = \| \tilde{R} (-\Delta)^{\frac{s}{2}} u \|_{L^1}.
\end{cases}
\]

So, based on the argument for Theorem 3.3 and
\[ f = (-\Delta)^{\frac{s}{2}} u \in S_{s,0} \quad \text{or} \quad u = I_s f, \]

it is enough to show
\[
\int_0^\infty \text{Cap}_{H_s^{1,1}} (\{ x \in \mathbb{R}^n : |I_s f(x)| > t \}) \, dt \leq \| \tilde{R} f \|_{L^1} \quad \forall \ f \in S_{s,0}.
\]

However, this last estimation is established in Lemma 3.5. \( \square \)
Weak estimates for $\text{Cap}_{X \in \{H^1_+, H^1_-(n=1)\}}$. Unfortunately, we have the weak but no the strong estimation for $\text{Cap}_{X \in \{H^1_+, H^1_-(n=1)\}}$.

**Theorem 3.7.** Let $s \in (0, 1)$ and $X = H^s_+$ or $X = H^s_-(n = 1)$. Then

$$|u|_X \geq \left\{ \begin{array}{ll}
\sup_{t \in (0, \infty)} t \text{Cap}_X(\{x \in \mathbb{R}^n : u(x) > t\}) & \forall u \in S, \\
\sup_{t \in (0, \infty)} t \text{Cap}_X(\{x \in \mathbb{R}^n : u(x) < -t\}) & \forall u \in S.
\end{array} \right.$$  

But there is no constant $C > 0$ such that

$$\int_0^\infty \text{Cap}_X(\{x \in \mathbb{R}^n : |u(x)| > t\}) \, dt \leq C |u|_X \quad \forall u \in S.$$  

**Proof.** For $(t, u) \in (0, \infty) \times S$, since

$$\{x \in \mathbb{R}^n : u(x) > t\}$$

is open, by the definition of Cap$_X$, for any $\epsilon \in (0, \infty)$ there exists a compact set $K \subset \{x \in \mathbb{R}^n : u(x) > t\}$ such that

$$\text{Cap}_X(\{x \in \mathbb{R}^n : u(x) > t\}) \leq \text{Cap}_X(K) + \epsilon.$$  

Let $v = t^{-1}u$. Then

$$v \in S \quad \& \quad v > 1 \text{ on } K.$$  

Accordingly, by definition we have

$$\text{Cap}_X(K) \leq |v|_X = t^{-1}|u|_X,$$

which implies

$$\text{Cap}_X(\{x \in \mathbb{R}^n : u(x) > t\}) \leq t^{-1}|u|_X + \epsilon.$$  

Letting $\epsilon \to 0$ gives the desired estimate

$$\sup_{t \in (0, \infty)} t \text{Cap}_X(\{x \in \mathbb{R}^n : u(x) > t\}) \leq |u|_X.$$  

Since

$$u(x) < -t \Leftrightarrow -u(x) > t \quad \& \quad [-u]_X = |u|_X,$$

we get

$$\text{Cap}_X(\{x \in \mathbb{R}^n : u(x) < -t\}) \leq t^{-1}|u|_X.$$  

In order to verify the nonexistence of the capacitary strong estimate, suppose that

$$\exists C_0 > 0 \quad \text{such that} \quad \int_0^\infty \text{Cap}_X(\{x \in \mathbb{R}^n : |u(x)| > t\}) \, dt \leq C_0 |u|_X \quad \forall u \in S$$

for

$$\text{either } X = H^1_+ \text{ or } X = H^1_- \text{ with } n = 1.$$  

Note that

$$\|u\|_{L^{\infty, \infty}_X} = \sup_{t > 0} t^\frac{1}{s} \text{Cap}_X(\{x \in \mathbb{R}^n : |u(x)| > t\}) \leq |u|_X \quad \forall u \in S,$$

which follows from

$$\|I_s f\|_{L^{\infty, \infty}_X} \leq \min \{\|f\|_{L^1}, \|\tilde{K} f\|_{L^1}\} \quad \forall f \in \{(\Delta)^{\frac{s}{2}}\mathcal{S} \subset \mathcal{S}_{s,0}.$$

In fact,

$$\|I_s f\|_{L^{\infty, \infty}_X} \leq \|f\|_{L^1}.$$
can be seen from [1]. The last inequality, along with (cf. (3.4))

\[ I_s f = - \sum_{j=1}^{n} R_j^2 I_s f = - \sum_{j=1}^{n} R_j I_s(R_j f) \text{ in } L^{\frac{n}{n-\alpha}} \text{ & almost everywhere on } \mathbb{R}^n \]

and (cf. [16, (1.5)])

\[ \| R_j u \|_{L^{\frac{n}{n-\alpha}}} \leq \| u \|_{L^{\frac{n}{n-\alpha}}}, \]

in turn derives

\[ \| I_s f \|_{L^{\frac{n}{n-\alpha}}} \leq \| \tilde{R} f \|_{L^{1}}. \]

Both (3.6) and the definition of \( \text{Cap}_X \) give the iso-capacitary inequality

\[ |E|^{\frac{n-\alpha}{n}} \leq \text{Cap}_X(E) \quad \forall \ E \subset \mathbb{R}^n. \]

Accordingly, a standard layer-cake method (cf. [14, p.101]) derives a constant \( C_1 \) depending on \( C_0 \) such that if \( u \in S \) then

\[
\|u\|_{L^{\frac{n}{n-\alpha}}} = \int_0^\infty \| \{ x \in \mathbb{R}^n : |u(x)| > t \} \| dt^{\frac{\alpha}{n}} \\
\leq \left( \int_0^\infty \| \{ x \in \mathbb{R}^n : |u(x)| > t \} \|^{\frac{n}{n-\alpha}} dt \right)^{\frac{n-\alpha}{n}} \\
\leq \left( \int_0^\infty \text{Cap}_X(\{ x \in \mathbb{R}^n : |u(x)| > t \}) dt \right)^{\frac{n-\alpha}{n}} \\
\leq (C_1 |u|_X)^{\frac{n}{n-\alpha}}.
\]

This contradicts the observation (1.7) mentioned in Section 1.2. \( \square \)

3.2. Restrictions/traces of \( H^{s,1} \) & \( H_+^{s,1} \). Being motivated by [31, Theorem 1.4] for \( W^{s,1} \), we establish the coming-up-next restricting/tracing principle.

**Theorem 3.8.** Let \( 0 < s < 1 \leq n, \mu \) be a nonnegative Radon measure on \( \mathbb{R}^n \) and

\[
\left\{ \begin{array}{l}
\|u\|_{L^{\frac{n}{n-\alpha}}(\mu)} = \left( \int_{\mathbb{R}^n} |u|^{\frac{n}{n-\alpha}} d\mu \right)^{\frac{n-\alpha}{n}} \\
\|u\|_{L^{\frac{n}{n-\alpha}_{\infty}}(\mu)} = \sup_{t>0} t \mu(\{ x \in \mathbb{R}^n : |u(x)| > t \})^{\frac{n-\alpha}{n}}.
\end{array} \right.
\]

Then the following two assertions are equivalent:

(i) there exists a positive constant \( c \) such that

\[ (\mu(K))^{\frac{n-\alpha}{n}} \leq c \text{Cap}_X(K) \quad \forall \text{ compact } K \subset \mathbb{R}^n. \]

(ii) there exists a positive constant \( C \) such that

\[ C[u]_X \geq \begin{cases} 
\|u\|_{L^{\frac{n}{n-\alpha}}(\mu)} & \text{as } X \in \{ H^{s,1}, H_{+}^{s,1} \text{ (n }>1) \} \\
\|u\|_{L^{\frac{n}{n-\alpha}_{\infty}}(\mu)} & \text{as } X \in \{ H_{+}^{s,1}, H_{+}^{s,1} \text{ (n }=1) \} \quad \forall \ u \in S.
\end{cases} \]

Moreover, the constants \( c \) and \( C \) are comparable to each other. Consequently, one always has the iso-capacitary inequality

\[ |K|^{\frac{n-\alpha}{n}} \leq \text{Cap}_X(K) \quad \forall \text{ compact } K \subset \mathbb{R}^n. \]
Proof. The consequence part of Theorem 3.8 follows from (i)⇔(ii) and

\[ [u]_X \geq \begin{cases} \|u\|_{L^{\frac{n}{n-1}}} & \text{as } X \in \{H^{s,1}, H^{s,1}_-(n > 1)\} \\ \|u\|_{L^{\frac{n}{n-1}, \infty}} & \text{as } X \in \{H^{s,1}_+, H^{s,1}_-(n = 1)\} \end{cases} \forall u \in S. \]

So, we are required to validate (i)⇔(ii). Two cases are considered for

\[
\begin{aligned}
    &u \in S \\
t &\in (0, \infty) \\
    &E_i = \{x \in \mathbb{R}^n : |u(x)| > t\} \\
    &E_{i,+} = \{x \in \mathbb{R}^n : u(x) > t\} \\
    &E_{i,-} = \{x \in \mathbb{R}^n : u(x) < -t\}.
\end{aligned}
\]

**Case 1**: (i)⇔(ii) for \( X \in \{H^{s,1}_+, H^{s,1}_-(n = 1)\} \).

On the one hand, if (i) holds, then the subadditivity of \( \mu \), the decomposition

\[ E_i = E_{i,+} \cup E_{i,-}, \]

and Theorem 3.7 derive

\[
(\mu(E_i))^{\frac{n}{n-1}} \leq (\mu(E_{i,+}) + \mu(E_{i,-}))^{\frac{n}{n-1}} \leq (\mu(E_{i,+}))^{\frac{n}{n-1}} + (\mu(E_{i,-}))^{\frac{n}{n-1}} \leq \text{Cap}_X(E_{i,+}) + \text{Cap}_X(E_{i,-}) \leq r^{-1}[u]_X,
\]

thereby verifying (ii).

On the other hand, suppose that (ii) is valid. For any compact \( K \subset \mathbb{R}^n \) let

\[ u \in S \ \& \ u \geq 1 \text{ on } K. \]

Then

\[
(\mu(K))^{\frac{n}{n-1}} \leq (\mu(E_{1,+}))^{\frac{n}{n-1}} \leq (\mu(E_1))^{\frac{n}{n-1}} \leq [u]_X.
\]

Accordingly, by definition we reach (i).

**Case 2**: (i)⇔(ii) for \( X \in \{H^{s,1}, H^{s,1}_-(n > 1)\} \).

On the one hand, for any \( k \in \mathbb{Z} \) the open set \( E_{2^k} \) has a compact subset \( K_k \) such that

\[ \mu(E_{2^k}) \leq 2\mu(K_k). \]

Thus, if (i) is valid, then

\[
\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} d\mu = \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \mu(E_i) \ dt^{\frac{n}{n-1}} \leq (2^{\frac{n}{n-1}} - 1) \sum_{k \in \mathbb{Z}} 2^{\frac{kn}{n-1}} \mu(E_{2^k}) \leq 2^{\frac{n}{n-1}+1} \sum_{k \in \mathbb{Z}} 2^{\frac{kn}{n-1}} \mu(K_k) \leq c^{\frac{kn}{n-1}} 2^{\frac{kn}{n-1}+1} \sum_{k \in \mathbb{Z}} 2^{\frac{kn}{n-1}} (\text{Cap}_X(K_k))^{\frac{n}{n-1}} \leq c^{\frac{kn}{n-1}} 2^{\frac{kn}{n-1}+1} \sum_{k \in \mathbb{Z}} 2^{\frac{kn}{n-1}} (\text{Cap}_X(E_{2^k}))^{\frac{n}{n-1}}.
\]
Note that for any nonnegative sequence \( \{a_j\}_{j \in \mathbb{Z}} \),
\[
\left( \sum_{j \in \mathbb{Z}} a_j \right)^\kappa \leq \sum_{j \in \mathbb{Z}} a_j^\kappa \quad \forall \ k \in (0, 1].
\]
This in turn gives
\[
\sum_{k \in \mathbb{Z}} 2^\frac{dk}{n-s} (\text{Cap}_X(E_{2^k}))^\frac{n}{n-s} \leq \left( \sum_{k \in \mathbb{Z}} 2^k \text{Cap}_X(E_{2^k}) \right)^\frac{n}{n-s}.
\]
Moreover, by Lemma 3.2(ii) it follows that
\[
\sum_{k \in \mathbb{Z}} 2^k \text{Cap}_X(E_{2^k}) = 2 \sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} \text{Cap}_X(E_t) dt \\
\leq 2 \sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} \text{Cap}_X(E_t) dt \\
= 2 \int_0^{\infty} \text{Cap}_X(E_t) dt.
\]
Altogether, we use Theorems 3.3 & 3.6 to obtain
\[
\int_{\mathbb{R}^n} |u|^{\frac{n}{n-s}} \, d\mu \lesssim \left( \int_0^{\infty} \text{Cap}_X(E_t) dt \right)^\frac{n}{n-s} \lesssim [u]^\frac{n}{n-s}_X,
\]
which implies (ii).

On the other hand, suppose that (ii) is true. Upon letting \( K \) be a compact subset of \( \mathbb{R}^n \) we gain that for any \( u \in S \) with \( u \geq 1 \) on \( K \),
\[
(\mu(K))^{\frac{n}{n-s}} \leq \left( \int_{\mathbb{R}^n} |u|^{\frac{n}{n-s}} \, d\mu \right)^{\frac{n}{n-s}} \lesssim [u]_X.
\]
Via taking the supremum over all such \( u \in S \) with \( u \geq 1 \) on \( K \) we get (i). \( \square \)

4. Duality laws for \( H^{s, 1} \) & \( \dot{H}^{s, 1} \)

4.1. **Adjoint operators of** \( \nabla^s_\pm \) **via** \( \{S, \text{BMO}\} \). This subsection describes the adjoint operators of \( \nabla^s_\pm \) (existing as two basic notions in fractional vector calculus).

**Integration-by-parts.** Below is a two-fold computation.

\( \triangleright \) On the one hand, the dual operator \([(-\Delta)^{\frac{s}{2}}]^* \) of \((-\Delta)^{\frac{s}{2}}\) is itself, i.e.,
\[
[\nabla^s_+]^* = \nabla^s_+,
\]
in the sense of
\[
\langle [\nabla^s_+]^* f, \phi \rangle = \langle f, \nabla^s_+ \phi \rangle = \langle \nabla^s_+ f, \phi \rangle \quad \forall \ (f, \phi) \in S'_s \times S.
\]
This is reasonable because of (cf. [25])
\[
\begin{cases}
\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} f(x) \phi(x) \, dx = \int_{\mathbb{R}^n} f(x) (-\Delta)^{\frac{s}{2}} \phi(x) \, dx \\
\int_{\mathbb{R}^n} f(x) I_s \phi(x) \, dx = \int_{\mathbb{R}^n} I_s f(x) \phi(x) \, dx
\end{cases} \quad \forall \ (f, \phi) \in (C^\infty_c)^2
\]
and
\[
(-\Delta)^{\frac{s}{2}} ((-\Delta)^{\frac{s}{2}} u) = (-\Delta)^{\frac{s}{2}} u \quad \forall \ u \in C^\infty_c.
\]
On the other hand, if we define
\[ \text{div} \hat{g} = (-\Delta)^{1/2} \vec{R} \cdot \vec{g} \]
then it enjoys (cf. [21, Theorem 1.3])
\[ -\text{div}(\nabla \hat{g} u) = (-\Delta)^{1/2} u \quad \forall \ u \in C_c^\infty \]
and (cf. [6, Lemma 2.5])
\[ \int_{\mathbb{R}^n} f(x)(-\text{div} \hat{g}) (x) \, dx = \int_{\mathbb{R}^n} \hat{g}(x) \cdot \nabla f(x) \, dx \quad \forall \ (f, \hat{g}) \in C_c^\infty \times (C_c^\infty)^n. \]
Thus \(-\text{div} \hat{g}\) exists as the dual operator \([\nabla \hat{g}]^*\) of \(\nabla \hat{g}\), i.e.,
\[ [\nabla \hat{g}]^* = -\text{div} \hat{g}. \]

**Dual pairing for \(\{S, \text{BMO}\}\)**. We are required to verify that BMO can be embedded in a family of relatively big spaces.

**Lemma 4.1.** If \(s \in (0, 1)\), then \(\text{BMO} \subset S_s'\).

**Proof.** Suppose \(f \in \text{BMO}\). Then
\[ \|f\|_{\text{BMO}} = \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty \quad \text{with} \quad f_B = \frac{1}{|B|} \int_B f(x) \, dx. \]
In order to verify \(f \in S_s'\), it suffices to show that \(f\) induces a continuous linear functional on \(S_s\). To this end, we consider
\[ L_f(\phi) = \int_{\mathbb{R}^n} f(x) \phi(x) \, dx \quad \forall \ \phi \in S_s. \]

Upon writing
\[ \left| \int_{\mathbb{R}^n} f(x) \phi(x) \, dx \right| \leq \int_{B(0,1)} |f(x)\phi(x)| \, dx + \sum_{j=1}^{\infty} \int_{2^{j-1} \leq |x| < 2^j} |f(x)\phi(x)| \, dx, \]
and noting both
\[ \int_{B(0,1)} |f(x)\phi(x)| \, dx \leq \|\phi\|_{L^\infty} \int_{B(0,1)} |f(x)| \, dx \]
\[ \leq \|\phi\|_{L^\infty} \left( \frac{1}{|B(0,1)|} \int_{B(0,1)} |f(x) - f_{B(0,1)}| \, dx + |f_{B(0,1)}| \right) \]
\[ \leq \rho_{n+s,0}(\phi) (\|f\|_{\text{BMO}} + |f_{B(0,1)}|) \]
and
\[ \int_{2^{j-1} \leq |x| < 2^j} |f(x)\phi(x)| \, dx \]
\[ \leq \rho_{n+s,0}(\phi) \int_{2^{j-1} \leq |x| < 2^j} \frac{|f(x)|}{1 + |x|^{n+s}} \, dx \]
\[ \leq \rho_{n+s,0}(\phi) 2^{-js} \left( \frac{1}{|B(0,2^j)|} \int_{B(0,2^j)} |f(x)| \, dx \right) \]
\[ \leq \rho_{n+s,0}(\phi) 2^{-js} \left( \sum_{j=0}^{\infty} \frac{1}{|B(0,2^j)|} \int_{B(0,2^j)} |f(x) - f_{B(0,2^j)}| \, dx + |f_{B(0,1)}| \right) \]
\[ \leq \rho_{n+s,0}(\phi) 2^{-js} (\|f\|_{\text{BMO}} + |f_{B(0,1)}|), \]
we obtain
\[
\left| \int_{\mathbb{R}^n} f(x)\phi(x) \, dx \right| \leq \rho_{n+\varepsilon, 0}(\phi) \left( 1 + \sum_{j=1}^{\infty} j2^{-j\varepsilon} \right) (\|f\|_{\text{BMO}} + |f_{B(0,1)}|) \\
\leq \rho_{n+\varepsilon, 0}(\phi) (\|f\|_{\text{BMO}} + |f_{B(0,1)}|),
\]
as desired. \(\square\)

**Proposition 4.2.** For \(s \in (0, 1)\) one has the following two implications.

(i) If \((f, \phi) \in \text{BMO} \times \mathcal{S}\), then
\[
\langle \nabla^s f, \phi \rangle = \langle f, \nabla^s \phi \rangle.
\]

(ii) If \(\vec{U} = (U_1, \ldots, U_n) \in L^\infty\) and \(\phi \in \mathcal{S}\), then
\[
\langle [\nabla^s]^* \vec{U}, \phi \rangle = \sum_{j=1}^{n} \langle U_j, \nabla^s_j \phi \rangle.
\]

**Proof.** Note that (i) follows directly from Lemma 4.1 and Definition 2.2(i).

Now we show (ii). For any \(j \in \{1, 2, \ldots, n\}\), it is known that \(R_j\) maps \(L^\infty\) functions continuously into \(\text{BMO}\) and that \(R_jU_j \in \text{BMO} \subset \mathcal{S}'\) follows from Lemma 4.1. So, Definition 2.2(i) derives that every \((-\Delta)^{\frac{s}{2}} R_j U_j \in \mathcal{S}'\).

By this and the definition of \([\nabla^s]^*\), we have
\[
[\nabla^s]^* \vec{U} = -\text{div}^s \vec{U} = -\sum_{j=1}^{n} (-\Delta)^{\frac{s}{2}} R_j U_j \in \mathcal{S}'.
\]

Thus, for \(\phi \in \mathcal{S}\) we have
\[
\langle [\nabla^s]^* \vec{U}, \phi \rangle = -\sum_{j=1}^{n} \langle (-\Delta)^{\frac{s}{2}} R_j U_j, \phi \rangle = -\sum_{j=1}^{n} \langle R_j U_j, (-\Delta)^{\frac{s}{2}} \phi \rangle.
\]

Since \(\phi \in \mathcal{S}\), Lemma 2.10 yields
\((-\Delta)^{\frac{s}{2}} \phi \in H^1\).

By
\[
[H^1]^* = \text{BMO} \quad \text{and} \quad R_j = -R_j,
\]
we further obtain
\[
\langle R_j U_j, (-\Delta)^{\frac{s}{2}} \phi \rangle = \langle U_j, R_j (-\Delta)^{\frac{s}{2}} \phi \rangle = -\langle U_j, R_j (-\Delta)^{\frac{s}{2}} \phi \rangle = -\langle U_j, \nabla^s_j \phi \rangle,
\]
thereby finding
\[
\langle [\nabla^s]^* \vec{U}, \phi \rangle = \sum_{j=1}^{n} \langle U_j, \nabla^s_j \phi \rangle.
\]

\(\square\)

### 4.2. Dualities of \(H^{s,1}\) and \(\hat{H}^{s,1}_x\)

This subsection is divided into two parts.
**Fundamental duality.** Below is the expected duality law.

**Theorem 4.3.** Let $0 < s < 1 \leq n$ and $T \in S'$. Then:

1. **(i)**
   
   $T \in [H^{s, 1}]^*$
   
   if and only if
   
   $\exists (U_0, U_1, ..., U_n) \in (L^\infty)^{1+n}$ such that $T = [\nabla_+^s]^* U_0 + [\nabla_-^s]^* (U_1, ..., U_n)$ in $S'$
   
   if and only if
   
   $T \in (-\Delta)^{\frac{s}{2}} \text{BMO}$.

2. **(ii)**
   
   $[H^{s, 1}]^* = [W^{s, 1}]^*$.

3. **(iii)**
   
   $T \in [\tilde{H}^{s, 1}]^*$ if and only if
   
   $\exists U_0 \in L^\infty$ such that $T = [\nabla_+^s]^* U_0$ in $S'$.

4. **(iv)**
   
   $T \in [\tilde{H}^{s, 1}]^*$ if and only if
   
   $\exists U = (U_1, ..., U_n) \in (L^\infty)^n$ such that $T = [\nabla_-^s]^* U$ in $S'$.

**Proof.** (i) First of all, by using the density of $S_\infty$ in both $H^1$ and $H^{s, 1}$ (cf. Proposition 2.12) and the invariant of $S_\infty$ under $I_s$ and $(-\Delta)^{\frac{s}{2}}$, we have

\[
T \in [H^{s, 1}]^* \iff |Tf| \leq |f|_{H^{s, 1}} \forall f \in S_\infty
\]

\[
\iff |T(I_s g)| \leq \|g\|_{H^1} \forall g = (-\Delta)^{\frac{s}{2}} f \in S_\infty
\]

\[
\iff T \circ I_s \in [H^1]^*.
\]

Consequently, an application of the Fefferman-Stein duality and decomposition (cf. [7, Theorem 2 & Theorem 3])

\[
[H^1]^* = \text{BMO} = L^\infty + \tilde{R} \cdot (L^\infty)^n,
\]

produces some

\[
(U_0, U_1, ..., U_n) \in (L^\infty)^{1+n}
\]

such that

\[
T \in [H^{s, 1}]^* \iff T \circ I_s = U_0 + \sum_{j=1}^n R_j U_j.
\]

Next, we utilize (4.1) to show the equivalence in (i). Let $T \in [H^{s, 1}]^*$. For any $\phi \in S$, if we let

\[
\psi = (-\Delta)^{\frac{s}{2}} \phi,
\]

then Lemmas 2.3 & 2.10 imply

\[
\begin{cases}
\phi = I_s \psi \\
\psi \in S_s \cap H^1 \\
\langle T, \phi \rangle = T(\phi) = T(I_s \psi) = (T \circ I_s)(\psi) = \langle T \circ I_s, \psi \rangle.
\end{cases}
\]

Then applying (4.1),

\[
R_j U_j \in \text{BMO} \subset S'_s,
\]
Proposition 4.2(i) and Definition 2.2(i), we arrive at
\[
\langle T \circ I_s, \psi \rangle = \left( U_0 + \sum_{j=1}^{n} R_j U_j, (-\Delta)^{1/2} \phi \right)
\]
\[
= \left( (-\Delta)^{1/2} U_0 + \sum_{j=1}^{n} (-\Delta)^{1/2} R_j U_j, \phi \right).
\]
This in turn gives
\[
T = (-\Delta)^{1/2} U_0 + \sum_{j=1}^{n} (-\Delta)^{1/2} R_j U_j = [\nabla^+_{s}]^* U_0 + [\nabla^+_{-}]^* (U_1, \ldots, U_n) \text{ in } S'
\]
and so
\[
T \in (-\Delta)^{1/2} \text{ BMO}.
\]

Conversely, we assume that
\[
T \in (-\Delta)^{1/2} \text{ BMO} \text{ or (4.2) holds for some } (U_0, U_1, \ldots, U_n) \in (L^\infty)^{1+n}.
\]

Then
\[
\phi = I_s \psi \in S_\infty \text{ and } \psi = (-\Delta)^{1/2} \phi \forall \psi \in S_\infty.
\]

This, combined with the facts
\[
U_0 \in L^\infty \subset S'_s \text{ and } R_j U_j \in \text{BMO} \subset S'_s
\]
and Definition 2.2(i), yields
\[
\langle T \circ I_s, \psi \rangle = (T \circ I_s)(\psi)
\]
\[
= T(I_s \psi)
\]
\[
= T(\phi)
\]
\[
= \left( (-\Delta)^{1/2} U_0 + \sum_{j=1}^{n} (-\Delta)^{1/2} R_j U_j, \phi \right)
\]
\[
= \left( U_0 + \sum_{j=1}^{n} R_j U_j, (-\Delta)^{1/2} \phi \right)
\]
\[
= \left( U_0 + \sum_{j=1}^{n} R_j U_j, \psi \right).
\]

Due to the density of $S_\infty$ in $H^1$ and $[H^1]^* = \text{BMO}$, the last series of identities implies
\[
T \circ I_s = U_0 + \sum_{j=1}^{n} R_j U_j \text{ in } \text{BMO}.
\]

Combining this and (4.1) yields
\[
T \in [H^{s,1}]^*.
\]

(ii) Noting that $S_\infty$ is dense in both $H^{s,1}$ and $W^{s,1}$ as shown in Proposition 2.12, we apply (1.3) to deduce
\[
W^{s,1} \subset H^{s,1} \text{ and hence } [H^{s,1}]^* \subset [W^{s,1}]^*.
\]
To get the converse part, we use not only [13, Proposition 3.2] to derive that \([W^{s,1}]^*\) consists of all nonnegative Radon measures \(\nu\) on \(\mathbb{R}^n\) with
\[
\|\nu\|_{n-s} = \sup_{(x,r) \in \mathbb{R}^n \times (0,\infty)} r^{s-n} \nu(B(x,r)) < \infty,
\]
but also the argument for [2, Proposition 3] to achieve that for such a measure \(\nu\),
\[
(4.3) \quad \int_{\mathbb{R}^n} |f| \, d\nu = \int_{\mathbb{R}^n} |I_n(-\Delta)^{\frac{s}{2}} f| \, d\nu \leq \|\nu\|_{n-s} \|(-\Delta)^{\frac{s}{2}} f\|_{H^{s,1}} \approx \|\nu\|_{n-s} \|f\|_{H^{s,1}} \quad \forall \ f \in S_\infty.
\]
Consequently, \(\nu\) induces a bounded linear functional on \(H^{s,1}\) because of the density of \(S_\infty\) in \(H^{s,1}\) (cf. Proposition 2.12). Thus, we obtain
\[
[W^{s,1}]^* \subset [H^{s,1}]^*,
\]
thereby reaching the desired dual identification.

(iii) Let \(T \in S'\). If
\[
T = [\nabla^s_+]^* U_0 \quad \text{in} \quad S' \quad \text{for some} \quad U_0 \in L^\infty,
\]
then
\[
\langle T, \phi \rangle = \langle (-\Delta)^{\frac{s}{2}} U_0, \phi \rangle = \langle U_0, (-\Delta)^{\frac{s}{2}} \phi \rangle \quad \forall \ \phi \in S,
\]
where the second equality holds thanks to \(L^\infty \subset S'_\alpha\) and Definition 2.2(i). Thus,
\[
|\langle T, \phi \rangle| \leq \|U_0\|_{L^\infty} \|(-\Delta)^{\frac{s}{2}} \phi\|_{L^1} = \|U_0\|_{L^\infty} \|\phi\|_{H^{s,1}_+} \quad \forall \ \phi \in S,
\]
which implies that \(T\) induces a bounded linear functional on \(\hat{H}^{s,1}_+\) in terms of the density of \(S\) in \(\hat{H}^{s,1}_+\).

To obtain the converse part, assuming
\[
T \in [\hat{H}^{s,1}_+]^*,
\]
we are about to find
\[
U_0 \in L^\infty \quad \text{such that} \quad T = [\nabla^s_+]^* U_0 \quad \text{in} \quad S'.
\]
Motivated by the argument in [4, p. 399], we consider the linear operator
\[
A_+ : \hat{H}^{s,1}_+ \to L^1,
\]
\[
u \mapsto (-\Delta)^{\frac{s}{2}} \nu
\]
which is a closed operator in terms of the definition of \(\hat{H}^{s,1}_+\). If
\[
u \in \hat{H}^{s,1}_+ \quad \text{obeys} \quad \|(-\Delta)^{\frac{s}{2}} \nu\|_{L^1} = 0,
\]
then
\[
(-\Delta)^{\frac{s}{2}} \nu = 0 \quad \text{almost everywhere on} \quad \mathbb{R}^n,
\]
which implies
\[
\langle u, \phi \rangle = \langle u, (-\Delta)^{\frac{s}{2}} I_s \phi \rangle = \langle (-\Delta)^{\frac{s}{2}} u, I_s \phi \rangle = 0 \quad \forall \ \phi \in S_\infty,
\]
that is, \(u = 0\) in \(S'/\mathcal{P}\), or equivalently, \(u\) is a polynomial on \(\mathbb{R}^n\). Further, any \(u \in S'_\alpha\) being a polynomial forces \(u\) to be a constant function on \(\mathbb{R}^n\). In other words, it holds \(u = 0\) in \(\hat{H}^{s,1}_+\). Thus, the operator \(A_+\) is injective. In the meantime, \(A_+\) enjoys
\[
\|A_+ u\|_{L^1} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^1} = \|u\|_{H^{s,1}_+} \quad \forall \ u \in \hat{H}^{s,1}_+.
\]
Consequently, \(A_+\) has a continuous inverse from \(L^1\) to \(\hat{H}^{s,1}_+\). Note that
\[
A_+ : \hat{H}^{s,1}_+ \to L^1
\]
is a closed linear operator. So the closed range theorem (see [33, p. 208, Corollary 1]) derives that the transpose of $A_+$

$$A_+^* : L^\infty \to [\hat{\mathcal{H}}_{+}^{s,1}]^*,$$

defined by

$$\langle A_+^* F, u \rangle = \langle F, A_+ u \rangle \quad \forall F \in L^\infty \& u \in \hat{\mathcal{H}}_{+}^{s,1},$$

is surjective. In particular, since

$$T \in [\hat{\mathcal{H}}_{+}^{s,1}]^*,$$

we can find

$$U_0 \in L^\infty \text{ such that } A_+^* U_0 = T.$$ 

Consequently, for any $u \in S$, we have

$$\langle A_+^* U_0, u \rangle = \langle U_0, A_+ u \rangle = \langle U_0, A_+ u \rangle = \langle U_0, (-\Delta)\hat{u} \rangle = \langle [\nabla_+^s]^* U_0, u \rangle,$$

whence gives

$$T = A_+^* U_0 = [\nabla_+^s]^* U_0 \text{ in } S'.$$

(iv) Let $T \in S'$. If

$$T = [\nabla_+^s]^* \hat{U} \text{ in } S' \text{ for some } \hat{U} = (U_1, \ldots, U_n) \in (L^\infty)^n,$$

then Proposition 4.2(ii) implies

$$\langle T, \phi \rangle = \langle [\nabla_+^s]^* \hat{U}, \phi \rangle = \sum_{j=1}^n \langle (-\Delta)\hat{u}_j U_j, \phi \rangle = \sum_{j=1}^n \langle U_j, \nabla_j^s \phi \rangle \quad \forall \phi \in S,$$

and hence

$$|\langle T, \phi \rangle| \leq \sum_{j=1}^n \|U_j\|_{L^\infty} \|\nabla_j^s \phi\|_{L^1} \quad \forall \phi \in S.$$ 

Since $S$ is dense in $\hat{\mathcal{H}}_{+}^{s,1}$, $T$ induces a bounded linear functional on $\hat{\mathcal{H}}_{+}^{s,1}$.

To obtain the converse part, assuming

$$T \in [\hat{\mathcal{H}}_{+}^{s,1}]^*$$

we are about to show

$$T = [\nabla_+^s]^* \hat{U} \text{ for some } \hat{U} \in (L^\infty)^n.$$ 

To this end, we consider the operator

$$A_- : \hat{\mathcal{H}}_{-}^{s,1} \to (L^1)^n$$

$$u \mapsto \nabla_-^s u.$$

Suppose

$$u \in \hat{\mathcal{H}}_{-}^{s,1} \text{ obeys } \nabla_-^s u = 0 \text{ in } (L^1)^n.$$ 

Since $u \in \hat{\mathcal{H}}_{-}^{s,1}$, it follows that $u \in S_\omega$. For any $\psi \in S_\infty$, the Fourier transform implies that

$$\psi = -\sum_{j=1}^n \nabla_j^s I_I J_I \hat{\psi} \quad \text{with every } I_I J_I \hat{\psi} \in S_\infty \subset L^\infty,$$

thereby giving

$$|\langle u, \psi \rangle| = \left| \sum_{j=1}^n \langle u, \nabla_j^s I_I J_I \hat{\psi} \rangle \right| = \left| \sum_{j=1}^n \langle \nabla_j^s \hat{u}, I_I J_I \hat{\psi} \rangle \right| \leq \sum_{j=1}^n \|\nabla_j^s \hat{u}\|_{L^1} \|I_I J_I \hat{\psi}\|_{L^\infty} = 0.$$
This shows that $u = 0$ in $S'/\mathcal{P}$. In other words, $u$ is a polynomial on $\mathbb{R}^n$. However, if a polynomial $u$ is a bounded linear functional on $S_s$, then $u$ must be a constant function, which implies that $u = 0$ in $\dot{H}^{s,1}_\mu$. In other words,

$$A_\mu : \dot{H}^{s,1}_\mu \rightarrow (L^1)^n$$

is injective.

This last injectiveness and the next identification

$$\|A_\mu u\|_{(L^1)^n} = \|\nabla_s u\|_{L^1} = \|u\|_{\dot{H}^{s,1}_\mu} \quad \forall \, u \in \dot{H}^{s,1}_\mu,$$

derive that

$$A_\mu : \dot{H}^{s,1}_\mu \rightarrow \mathcal{R}(A_\mu) = A_\mu(\dot{H}^{s,1}_\mu)$$

has a continuous inverse sending $\mathcal{R}(A_\mu)$ to $\dot{H}^{s,1}_\mu$.

Clearly, $\mathcal{R}(A_\mu)$ is closed in $(L^1)^n$. So, from the closed range theorem it follows that the $A_\mu$’s transpose

$$A_\mu^* : [\mathcal{R}(A_\mu)]^* \rightarrow [\dot{H}^{s,1}_\mu]^*$$

defined by

$$\langle A_\mu^* \tilde{F}, \phi \rangle = \langle \tilde{F}, A_\mu \phi \rangle = \langle \tilde{F}, \nabla_s \phi \rangle \quad \forall \, \phi \in S,$$

is surjective. Consequently, for the hypothesis $T \in [\dot{H}^{s,1}_\mu]^*$ there exists

$$\tilde{U}_o \in [\mathcal{R}(A_\mu)]^*$$

such that $A_\mu^* \tilde{U}_o = T$ in $S'$.

Although it is uncertain that $\tilde{U}_o \in (L^\infty)^n$, we can utilize the inclusion

$$\mathcal{R}(A_\mu) \subseteq (L^1)^n$$

and the classical Hahn-Banach extension theorem to extend $\tilde{U}_o$ to an element

$$\tilde{U} \in [(L^1)^n]^* = (L^\infty)^n$$

such that

$$\langle \tilde{U}, \tilde{V} \rangle = \langle \tilde{U}_o, \tilde{V} \rangle \quad \forall \, \tilde{V} \in \mathcal{R}(A_\mu).$$

Accordingly, if $\phi \in S$, then

$$\langle T, \phi \rangle = \langle A_\mu^* \tilde{U}_o, \phi \rangle = \langle \tilde{U}_o, \nabla^s \phi \rangle = \langle \tilde{U}, \nabla^s \phi \rangle = \langle [\nabla^s]^* \tilde{U}, \phi \rangle,$$

and hence

$$T = [\nabla^s]^* \tilde{U} \text{ in } S'.$$

\hfill $\Box$

**Fefferman-Stein decomposition & Bourgain-Brezis question.** As a consequence of Theorem 4.3 under $n > 1$, we surprisingly discover the coming-up-next assertion whose (iii) is indeed a resolution of the Bourgain-Brezis problem (cf. [4, p.396]) asking for any function space $X$ between $W^{1,n}$ and BMO such that every $F \in X$ has a representation

$$F = \sum_{j=1}^n R_j Y_j \text{ where } (n - 1, Y_j) \in \mathbb{N} \times L^\infty.$$

**Theorem 4.4.** Let

$$(s, n - 1, T, Y_0) \in (0, 1) \times \mathbb{N} \times S' \times \text{BMO}.$$  

Then:
(i) \[ [H^{x,1}]^* = [\hat{H}^{x,1}]^* = [W^{x,1}]^* = (-\Delta)^{\frac{x}{2}} \text{BMO}. \]

(ii) \[ T \in [H^{x,1}]^* \iff \exists \hat{U} = (U_1, ..., U_n) \in (L^\infty)^n \text{ such that } T = [\nabla_x]^* \hat{U}. \]

(iii) \[
\begin{cases}
  f \in \text{BMO} \iff \exists (g_1, ..., g_n) \in (L^\infty)^n \text{ such that } f = \sum_{j=1}^n R_j g_j \\
  F \in H^1 \iff \exists (G_1, ..., G_n) \in (H^1)^n \text{ such that } F = \sum_{j=1}^n R_j G_j.
\end{cases}
\]

(iv) \[ \exists (Y_1, ..., Y_n) \in (L^\infty)^n \text{ such that } \text{div}((-\Delta)^{\frac{x}{2}} Y_1, ..., (\Delta)^{\frac{x}{2}} Y_n) = Y_0. \]

**Proof.**

(i) Since the last equality of (i) is from Theorem 4.3 and there is a simple implication \[ W^{x,1} \subset H^{x,1} \subset \hat{H}^{x,1} \Rightarrow [\hat{H}^{x,1}]^* \subset [H^{x,1}]^* \subset [W^{x,1}]^*, \]

it is enough to prove \[ [W^{x,1}]^* \subset [\hat{H}^{x,1}]^*. \]

To do so, let \( T \in [W^{x,1}]^* \). Then [13, Proposition 3.2] implies that \( T \) coincides with some nonnegative Radon measure \( \nu \) on \( \mathbb{R}^n \) satisfying

\[ \|\nu\|_{\text{loc}-s} = \sup_{(x,r) \in \mathbb{R}^n \times (0,\infty)} r^{s-n} \nu(B(x, r)) < \infty. \]

Since now \( n > 1 \), an application of the argument for Lemma 3.5 further reveals that (4.3) becomes

\[ \int_{\mathbb{R}^n} |f| \, d\nu = \int_{\mathbb{R}^n} \left| I_s (-\Delta)^{\frac{x}{2}} f \right| \, d\nu \leq \|\nu\|_{\text{loc}-s} \|R(-\Delta)^{\frac{x}{2}} f\|_{L^1} \approx \|\nu\|_{\text{loc}-s} \|f\|_{H^{x,1}_s} \quad \forall \ f \in \mathcal{S}. \]

Accordingly, \( \nu \) induces a bounded linear functional on \( \hat{H}^{x,1}_s \) due to the fact that \( \hat{H}^{x,1}_s \) is the closure of \( \mathcal{S} \) under \( [\nabla_x u]_{L^1} \). Namely, \( T \in [\hat{H}^{x,1}]^* \).

(ii) From Theorem 4.3(i), we deduce that if

\[ T = [\nabla_x]^* \hat{U} \text{ for some } \hat{U} \in (L^\infty)^n \]

then \( T \in [H^{x,1}]^* \). Conversely, if this last condition is valid, then the previously-verified identification

\[ [H^{x,1}]^* = [\hat{H}^{x,1}]^* \]

and Theorem 4.3(ii) produce a vector

\[ \hat{U} = (U_1, ..., U_n) \in (L^\infty)^n \]

such that

\[ T = [\nabla_x]^* \hat{U}. \]

(iii) Note that \( R_j : H^1 \to H^1 \) is bounded. So if

\[ F = \sum_{j=1}^n R_j G_j \text{ for some } G_j \in H^1 \]

then \( F \in H^1 \). Conversely, if \( F \in H^1 \) then an application of

\[ \text{id} = - \sum_{j=1}^n R_j^2 \]
gives
\[ F = \sum_{j=1}^{n} R_j F_j \text{ where } F_j = -R_j F \in H^1. \]

Next, let us show
\[ \text{BMO} = \hat{R} \cdot (L^\infty)^n \text{ under } n - 1 \in \mathbb{N}. \]

The Fefferman-Stein decomposition theorem in [7] ensures
\[ \hat{R} \cdot (L^\infty)^n \subset \text{BMO} \text{ under } n \in \mathbb{N}, \]
so it suffices to show the converse inclusion under \( n \geq 2 \). To this end, we utilize the above-proved assertion
\[ (-\Delta)^{\frac{s}{2}} \text{BMO} = [H^{s,1}]^* = [\nabla_s^{-\frac{n}{2}}(L^\infty)^n] \text{ in } S' \text{ under } 0 < s < 1 < n \]
to derive that if
\[ (f, s, n - 1) \in \text{BMO} \times (0, 1) \times \mathbb{N} \]
then
\[ \exists \hat{U} = (U_1, \ldots, U_n) \in (L^\infty)^n \text{ such that } \langle (-\Delta)^{\frac{s}{2}} f, \phi \rangle = \langle [\nabla_s^{-\frac{n}{2}}]\hat{U}, \phi \rangle \text{ for all } \phi \in S_{s\infty}. \]

Upon noting that \( S_{s\infty} \) is invariant under \( (-\Delta)^{\frac{s}{2}} \) and \( I_s \), we utilize Proposition 4.2 and
\[
\begin{cases}
\phi = I_s \psi \\
\psi = (-\Delta)^{\frac{s}{2}} \phi \\
[R_j]^* = -R_j \quad \forall \ j \in \{1, \ldots, n\}
\end{cases}
\]
to deduce
\[ \langle (-\Delta)^{\frac{s}{2}} f, \phi \rangle = \langle f, \psi \rangle \quad & \langle [\nabla_s^{-\frac{n}{2}}]\hat{U}, \phi \rangle = \sum_{j=1}^{n} \langle U_j, R_j \psi \rangle = -\sum_{j=1}^{n} \langle R_j U_j, \psi \rangle. \]

Consequently,
\[ \langle f, \psi \rangle = -\sum_{j=1}^{n} \langle R_j U_j, \psi \rangle \quad \forall \ \psi \in S_{s\infty}. \]

This, together with the density of \( S_{s\infty} \) in \( H^1 \) and the Fefferman-Stein duality theorem in [7]
\[ [H^1]^* = \text{BMO}, \]
yields
\[ f = -\sum_{j=1}^{n} R_j U_j \in \text{BMO} \]
and so
\[ \text{BMO} \subset \hat{R} \cdot (L^\infty)^n \text{ under } n - 1 \in \mathbb{N}. \]

(iv) Due to \( Y_0 \in \text{BMO} \), the just-verified (iii) allows us to find a vector-valued function
\[ \hat{g} = (g_1, \ldots, g_n) \in (L^\infty)^n \text{ under } n - 1 \in \mathbb{N} \]
such that
\[ Y_0 = \sum_{j=1}^{n} R_j g_j = \nabla \cdot ((-\Delta)^{-\frac{n}{2}} \hat{g}) = \text{div}((-\Delta)^{-\frac{n}{2}} g_1, \ldots, (-\Delta)^{-\frac{n}{2}} g_n). \]

\[ \square \]
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