The Real Scalar Field in Schwarzchild-de Sitter Spacetime*

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Abstract

In this paper, the real scalar field equation in Schwarzchild-de Sitter spacetime is solved numerically with high precision. A method called ‘polynomial’ approximation is introduced to derive the relation between the tortoise coordinate $x$ and the radius $r$. This method is different from the ‘tangent’ approximation [1] and leads to more accurate result. The Nariai black hole is then discussed in details. We find that the wave function is harmonic only near the horizons as I. Brevik and B. Simonsen [1] found. However, the wave function is not harmonic in the region of the potential peak, with amplitude increasing instead. Furthermore, we also find that, when cosmological constant decreases, the potential peak increases, and the maximum wave amplitude increases.

1 Introduction

I. Brevik and B. Simonsen [1] solved the real scalar field equation in Schwarzchild-de Sitter spacetime numerically. They approximated the tortoise coordinate $x = x(r)$ by the ‘tangent’ function $r = r(x)$ and replaced the potential function $v = v(r)$ by $v = v(r) = v(x)$. The tortoise coordinate $x$ was introduced to simply the real scalar field equation. Then they solved the equation by Matlab software, and found that the solution in SDS system is close to that of a harmonic wave [1].

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In the present paper, we continue the previous work and investigate the solution of the real scalar field equation in Schwarzschild-de Sitter spacetime. In section 2, we will introduce a new method, referred to as the ‘polynomial’ approximation, to get the relation between the tortoise coordinate $x$ and the radius $r$. This approximation is different from the ‘tangent’ approximation introduced by I. Brevik and B. Simonsen [1], and approximates the curve of the tortoise coordinate $x$ versus the radius $r$ more accurately. Then we solve the real scalar field equation by Matlab software. The solution illustrated in figures demonstrates that the solution of the real scalar field equation in Schwarzschild-de Sitter spacetime is not harmonic globally in contrast to the result from the work in Ref.(1).

We adopt the signature $(-+++)$, put $\hbar$, $c$, and $G$ (Newton’s gravitational constant) equal to unity, and follow the conventions of Misner et al.[2]

2 Global Solution of the Real Scalar Field Equation in Schwarzschild-de Sitter spacetime

2.1 Horizons

With our conventions Einstein’s equations read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (1)$$

For a spherically symmetric system the line element takes the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (2)$$

In our case

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \quad (3)$$

where $M$ is the mass of the black hole. I. Brevik and B. Simonsen[1] have discussed that when $\Lambda M^2 < \frac{1}{3}$, equation $f(r) = 0$ has three real solutions. They are event horizon $r_e$, cosmological horizon $r_c$ and a negative solution $r_o$, with $r_e > 0$, $r_c > 0$, $r_o = -(r_e + r_c)$. The choice $\Lambda M^2 = 0.11$ leads to $r_e = 2.8391M$ and $r_c = 3.1878M$.

2.2 The Wave Function

The real scalar field equation in this case can be written as

$$\Box \Phi = -\frac{\Phi_{,\mu}}{f(r)} + \frac{1}{r^2} \left[r^2 f(r) \Phi_{,r}\right]_{,r} + \frac{1}{r^2 \sin \theta} \left[(\sin \theta \Phi_{,\theta})_{,\theta} + \Phi_{,\varphi}^2 \frac{\Phi_{,\varphi}}{\sin \theta}\right] = 0. \quad (4)$$
Let
\[ \Phi = \frac{1}{r} \Psi \omega \text{e}^{-i\omega t} Y_{lm}(\theta, \varphi), \] (5)
where \( Y_{lm} \) is the usual spherical harmonic. We have
\[ \left[ -f(r) \frac{d}{dr} \left( f(r) \frac{d}{dr} \right) + v(r) \right] \Psi \omega l(r) = \omega^2 \Psi \omega l(r), \] (6)
where \( v(r) \) is the potential
\[ v(r) = f(r) \left[ \frac{f'(r)}{r} + \frac{l(l+1)}{r^2} \right] = \left( 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \right) \left( \frac{2M}{r^3} - \frac{2\Lambda}{3} + \frac{2}{r^2} \right). \] (7)
\( l \) is taken as unity here. We now introduce the tortoise coordinate \( x \) by the equation [3]
\[ x = \frac{1}{2M} \int \frac{dr}{f(r)}. \] (8)
This quantity is conveniently expressed in terms of the surface gravities \( \kappa_i \), defined by [4]
\[ \kappa_i = \frac{1}{2} \left| \frac{df}{dr} \right|_{r=r_i}. \] (9)
We get
\[ \kappa_e = \frac{(r_e - r_c)(r_e - r_o)}{6r_e} \Lambda, \] (10)
\[ \kappa_c = \frac{(r_c - r_e)(r_e - r_o)}{6r_c} \Lambda, \] (11)
\[ \kappa_o = \frac{(r_o - r_e)(r_e - r_o)}{6r_o} \Lambda. \] (12)
The tortoise coordinate can now be written as
\[ x = \frac{1}{2M} \left[ \frac{1}{2\kappa_e} \ln \left( \frac{r}{r_e} - 1 \right) - \frac{1}{2\kappa_c} \ln \left( 1 - \frac{r}{r_c} \right) + \frac{1}{2\kappa_o} \ln \left( 1 - \frac{r}{r_o} \right) \right]. \] (13)
From Eq. (8), we get
\[ \frac{dx}{dr} = \frac{1}{2f(r)}. \] (14)
So Eq. (6) is rewritten as

\[
\left[ -\frac{d^2}{dx^2} + 4M^2v \right] \Psi_{\omega l}(x) = 4M^2\omega^2\Psi_{\omega l}(x),
\]

(15)

where \(v\) is now a function of \(x\).

\[
v = v(x).
\]

(16)

### 2.3 Boundary Conditions and Results

To proceed further, we need the inversion \(r = r(x)\) of the tortoise coordinate \(x = x(r)\) to get \(v = v(r) = v(r(x)) = v(x)\). But it is difficult to invert Eq. (13). In order to get the function \(r = r(x)\), we turn to the numerical way. For Eq. (13) is a one to one mapping function, we use the command ‘polyfit’ in Matlab software to get a new function \(y = y(x) = \sum_{i=0}^{n} a_i x^i, n \in N, i \in N\), to approximate tortoise coordinate \(r = r(x)\). In different interval of the radius \(r\), the function \(y\) has different form. In the same interval, \(n\) is determined by the demand of our approximation accuracy. The bigger \(n\) is, the higher the accuracy is. Now we put \(M = 1, \Lambda = 0.11\) and consider Nariai black hole [1]. Thus the horizons are \(r_e = 2.8391\) and \(r_c = 3.1878\). In the interval \(r \in [2.83908, 3.18775]\), we have

\[
y = \sum_{i=0}^{20} a_i x^i.
\]

(17)

\(\{a_i\}\) are displayed in Table 1.

The curve of \(y\) versus the tortoise coordinate \(x\) and the curve of the radius \(r\) versus the tortoise coordinate \(x\) are illustrated in Figure 1, from which we know that the two curves overlap with each other in the interval \(r \in [2.83908, 3.18775]\). In figures, point A denotes the position at which the potential peak is. Now we see that the function \(y\) may be better as an approximation to Eq. (13) than the tangent function [1] \(\tilde{r} = 15 \tan[b(r - d) + 5]\), with \(b = 2.7/(r_c - r_e)\), \(d = (r_c + r_e)/2\). The curve of \(\tilde{r}\) versus the tortoise coordinate \(x\) and the curve of the radius \(r\) versus the radius \(x\) were illustrated in Ref. (1).

Now inserting \(y = y(x)\) into form (7) instead of \(r = r(x)\), we obtain the potential \(v\) as a function of \(x\)

\[
v = v(x).
\]

(18)

The curve of \(v\) versus \(x\) from equation (13) and the curve of \(v\) versus \(x\) from the approximation (17) is illustrated in Figure 2. The curve of \(v\) versus \(y\) and the curve of \(v\) versus the radius \(r\) are illustrated in Figure 3. Figure 2 and Figure 3 show that our approximation is good.

We set \(v = 0\) near the horizons. Thus Eq. (15) becomes

\[
\left[ -\frac{d^2}{dx^2} \right] \Psi_{\omega l}(x) = 4\Psi_{\omega l}(x),
\]

(19)
which has the following solution

\[ \Psi_{\omega l}(x) = \cos(2x). \]  

(20)

Therefore, we have [1]

\[ \Psi_{\omega l}(x = -100) = \Psi_{\omega l}(x = 100) = \cos(200). \]  

(21)

The above numerical values are used in the boundary conditions when Eq. (15) is solved with the Matlab software. The solution is illustrated in Figure 4 and Figure 5. These two figures show that the wave function near the horizons is harmonic, but it isn’t in the region where the potential gets its peak value. There is a largest amplitude at \( x = 3.1431 \) near the point \( x = 2.8582 \). While the potential peak \( v = 7.5 \times 10^{-4} \) appears at \( x = 2.8582 \). When \( r \) trends to the point \( x = 3.1431 \), the amplitude of the wave increases step by step. I. Brevik and B. Simonsen [1] illustrated the partial solution of Eq. (15) and mentioned that the solution is close to that of a harmonic wave. Now we find that the wave function \( \Psi_{\omega l} \) is not harmonic globally.

The case when \( \Lambda = 0.001 \) will not be discussed here again. Its result is similar to the one when \( \Lambda = 0.11 \). But one thing should be mentioned. It is that, When \( \Lambda \) decreases, the potential peak increases. So does the largest amplitude of the wave.

3 Summary

3.1 What We Have Done

In this paper, we introduce a new method called ‘polynomial’ approximation to approximate the tortoise coordinate \( x = x(r) \). And we find that the solution of the real scalar field equation in Schwarzschild-de Sitter spacetime is harmonic near the horizons as I. Brevik and B. Simonsen [1] found, and that it isn’t near the position where the potential gets its peak value, with higher wave peaks and lower troughs appearing instead.

3.2 About the Reflection and Transmission Coefficient

In this paper, we did not calculate the reflection and transmission coefficients. The reason is that there are many square potentials [1] to describe a common potential \( v = v(x) \). Each square potential gives out a reflection coefficient and a transmission coefficient by the method adopted in I. Brevik and B. Simonsen’s paper [1]. Different square potential gives out different reflection coefficient and transmission coefficient.
References

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*Table 1.* The coefficients in function $y = y(x)$

| $a_0 = 2.9817$ | $a_1 = 6.5107 \times 10^{-3}$ | $a_2 = 4.0912 \times 10^{-5}$ |
|----------------|-------------------------------|-------------------------------|
| $a_3 = -2.9913 \times 10^{-6}$ | $a_4 = -3.4895 \times 10^{-8}$ | $a_5 = 1.6009 \times 10^{-9}$ |
| $a_6 = 2.3413 \times 10^{-11}$ | $a_7 = -8.0083 \times 10^{-14}$ | $a_8 = -1.1964 \times 10^{-14}$ |
| $a_9 = 3.3845 \times 10^{-16}$ | $a_{10} = 4.3110 \times 10^{-18}$ | $a_{11} = -1.0899 \times 10^{-19}$ |
| $a_{12} = -1.0120 \times 10^{-21}$ | $a_{13} = 2.4364 \times 10^{-23}$ | $a_{14} = 1.4031 \times 10^{-25}$ |
| $a_{15} = -3.4329 \times 10^{-27}$ | $a_{16} = -9.5242 \times 10^{-30}$ | $a_{17} = 2.6439 \times 10^{-31}$ |
| $a_{18} = 1.5652 \times 10^{-34}$ | $a_{19} = -8.0402 \times 10^{-36}$ | $a_{20} = 4.3262 \times 10^{-38}$ |
Figure 1: The curve of the tortoise coordinate $x$ versus $r$ (full line), together with the curve of the tortoise coordinate $x$ versus polynomial approximation $y$ (dotted). $\Lambda = 0.11$. 
Figure 2: The curve of the potential $v$ versus the tortoise coordinate $x$ (dotted), together with the curve of the potential $v$ versus $x$ from the polynomial approximation $y = y(x)$ (full line). $\Lambda = 0.11$
Figure 3: The curve of the potential $v$ versus the radius $r$ (dotted), together with the curve of the potential $v$ versus $y$ (full line). $\Lambda = 0.11$. 


Figure 4: The curve of the wave function $\Psi_{nl}$ versus $x$ in SDS. $\Lambda = 0.11$. 
Figure 5: The curve of the wave function $\Psi_{\omega l}$ versus $r$ in SDS. $\Lambda = 0.11$. 
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