The Drell-Hearn Sum Rule at Order $\alpha^3$

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The Drell-Hearn-Gerasimov-Iddings (DHGI) sum rule for electrons is evaluated at order $\alpha^3$ and shown to agree with the Schwinger contribution to the anomalous magnetic moment.

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The DHGI sum rule relates the anomalous magnetic moment of a particle to the integral of a difference of cross sections

$$\frac{2\pi^2\alpha^2}{m^2} = \int_0^\infty \frac{d\omega}{\omega} \left[ \sigma_P(\omega) - \sigma_A(\omega) \right]$$

(1)

$\kappa$ is the anomalous moment in units of $e/2m$, $m$ is the particle’s mass and $\sigma_P, A(\omega)$ are the total cross sections for the scattering of a circularly polarized photon of energy $\omega$ in the laboratory system with polarization parallel or antiparallel to the particle’s spin. In the derivation of (1) the polarization flip of the photon picks out the magnetic interaction and the coefficient goes as $\epsilon \times \epsilon^*$ which requires circular polarization. A basic, and as yet unproven, assumption in the derivation of the sum rule is that the difference between the amplitude for spin parallel and antiparallel obeys an unsubtracted dispersion relation.

The same assumption is also needed to derive the Adler sum rule.$^3$ Furthermore, the difference of cross sections in the right hand side of (1) is, up to an overall factor, the $Q^2 \to 0$ limit of the integrand in the Bjorken sum rule.$^4$ Thus applied to QCD the DHGI sum rule provides a connection between photoproduction and deep inelastic scattering. It provides an important constraint which complements the sum rules in high $Q^2$ polarized deep inelastic scattering.$^5$. It is worth emphasizing that the underlying physics for these sum rules is derived from the very general predictions of a gauge invariant, local, relativistic quantum field theory. Therefore, experimental verification of the Drell-Hearn-Gerasimov-Iddings (DHGI) sum rule as well as the other sum rules is of fundamental importance.

The Bjorken sum rule for the iso-vector part $g_1$ has been verified in polarized deep inelastic experiments at CERN and SLAC to within 8%.$^4$. In contrast the DHGI is on less firm ground as far as experiment is concerned.$^7$, contrary to expectations the photoproduction of pions fails to saturate the sum rule. The reasons for this discrepancy are not yet understood. Fortunately, there are a series of forthcoming or planned experiments which will directly test the DHGI sum rule$^7$ in QCD.

The DHGI sum rule also provides a useful tool in testing the standard model and in the search for physics beyond the standard model. To lowest order in QED the sum rule tells us that particles of any spin, whether composite or not, must possess a gyromagnetic ratio, $g = 2$. Schwinger’s result$^8$ for the anomalous moment in QED, $\kappa = \alpha/2\pi$, implies that to order $\alpha^2$ all contributions to the integral of $\Delta \sigma$ must vanish. Some time ago Altarelli, Cabbibo, and Maiani$^9$ verified that, for instance,

$$\int_0^\infty \frac{d\omega}{\omega} \Delta \sigma^{tree}_{\gamma e \to \nu W} = 0,$$

only for $g_W = 2$. Using these results Brodsky et al.$^{11}$ were able to determine the position of the radiation zero in $\gamma e \to W \nu$ and to investigate the sensitivity of the zero position to an anomalous trilinear gauge coupling.

In other applications Brodsky and Drell$^{12}$ showed that if a lepton (L) had substructure which could be photo-excited above a threshold $m^*$, then there would be a contribution to the sum rule of order $m_L/m^*$. Thus, deviations from the sum rule predictions could be interpreted in terms of a mass scale ($m^*$) for substructure. In reference $^{13}$ the DHGI sum rule is used, within the framework of the Chiral Quark Model in the large $N_c$ limit, to compute the proton and neutron anomalous magnetic moments. Reasonable agreement with experiment was obtained. Finally, the DHGI sum rule has also been used to test the consistency of certain extra-dimensional models of quantum gravity.$^{14}$

In view of this recent activity regarding the DHGI sum rule we undertook the task of checking the sum rule, to order $\alpha^3$, in the simplest of models, QED of the electron. To our knowledge, this is the first instance in which the lowest order non-zero contribution to the right hand side of (1) has been theoretically computed. We find that the DHGI sum rule is indeed satisfied to order $\alpha^3$.

We now proceed with an outline of the calculation of the right hand side of (1) to order $\alpha^3$. At this order there are three possible contributions, pair production, $\gamma + e \to e + e + \bar{e}$, the virtual corrections to Compton scattering, and double Compton scattering, $\gamma + e \to \gamma + \gamma + e$. The first of these is zero. We consider them in turn.
Examples of the diagrams for $\gamma + e \rightarrow e + e + \gamma$. The particles in the final state are labeled with their momenta. The diagrams with the photons crossed are not shown.

\[ \gamma + e \rightarrow e + e + \gamma \]

A representative set of diagrams for this process is shown in Fig. 1. The square of the full set of diagrams of which (1a) and (1b) are examples must contribute zero to the sum rule. The upper $e\gamma$ could be replaced by $\mu\gamma$, $\tau\gamma$, $W^+W^-$, etc. To the order in which we are working the left hand side of (4) does not involve the masses of any of these particles, therefore, these contributions must vanish. We have shown by direct calculation that this is true. The same is true for the square of the diagrams represented by (1c) plus (1d). What is not obvious is that the complete cross terms between diagrams of type (1a), (1b) and those of (1c), (1d) must be zero because this contribution exists only for electrons. Nevertheless, we have computed these cross terms and found that they also vanish.

\[ \gamma + e \rightarrow \gamma + \gamma + e \text{ at order } \alpha^3 \]

The virtual radiative corrections to unpolarized Compton scattering were calculated by Brown and Feynman 14 in a well known and classic paper. Because the sum rule involves particular spins for the initial particles we need the corrections for polarized scattering and indeed these have been carefully calculated by Tsai, DeRaad and Milton 16 for every helicity combination. A byproduct of our results is that they provide a nice check on the results reported in reference [14]. If $\sigma_1$, $\sigma_2$ and $\lambda_1$, $\lambda_2$ label the spins of the initial and final electrons and photons and the amplitudes of order $e^n$ are given by $f^{(n)}(\sigma_2\lambda_2; \sigma_1\lambda_1)$ then we need the combination

\[
2 \left[ f^{(2)}(++;++)f^{(4)}(++;++) + f^{(2)}(++;--;--)f^{(4)}(++;--;--) - f^{(2)}(--;++;--)f^{(4)(++;--;--)} - f^{(2)}(--;--;--;--)f^{(4)(--;--;--;--)} \right].
\]

These amplitudes are given in Ref. 16 in terms of invariants which can easily be expressed by $\omega$ and the scattering angle $\theta$. Thus the contribution to (1) involves a double integral over $\omega$ and $\zeta$. We express the sum rule (1), with $\kappa$ set equal to $\alpha/2\pi$, as a sum of the virtual and the double Compton contributions scaled to unity

\[ 1 = I^V + I^{DC}. \tag{2} \]

It was shown by Feynman and Brown 15 that the infrared divergence, which arises as the photon energy approaches zero, cancels between the two terms in (4). Since we performed the DHGI integral numerically, special care had to be taken to ensure an accurate cancellation. If we express the virtual contribution as

\[ I^V = A + B \ln(\omega_{Min}), \tag{3} \]

where $\omega_{Min}$ is the minimum detectable photon energy (in units of the electron mass), we find

\[ A = 9.68, \quad B = -4.74. \tag{4} \]

There is an uncertainty in these numbers due to the numerical integration. By evaluating (3) for various $\omega_{Min}$ we estimate these errors to be about 0.03 in $A$ and 0.01 in $B$.

\[ \gamma + e \rightarrow \gamma + \gamma + e \]

The contribution to (1) from double Compton scattering is difficult to do accurately because the difference of cross sections decreases slowly even at very large $\omega$. Also, in (2), we must cancel numbers on the order of 50 to 100 for $\omega_{Min} = 10^{-4}$ to $10^{-6}$. If we write $I^{DC}$ in (2) as

\[ I^{DC} = C + D \ln(\omega_{Min}), \tag{5} \]

we find,

\[ C = -8.70, \quad D = 4.74, \tag{6} \]

with errors in $C$ and $D$ comparable to those of $A$ and $B$.

Thus, within the errors, we conclude that

\[ B + D = 0 \tag{7a} \]

\[ A + C = 1, \tag{7b} \]

and the sum rule (1), as expressed in (2) with $\kappa = \alpha/2\pi$, is verified.
As mentioned above there will be attempts in the near future to resolve the discrepancy in the sum rule for nucleons. It is often mentioned that the convergence of the sum rule would be destroyed by the presence of a J=1 fixed pole. In the language of current algebra this translates into an extra term in the commutator of the charge densities; it was suggested in Ref. [7] that such a term could give an additional contribution to the left hand side of (1) that could ameliorate the nucleon discrepancy. We have shown here that there is no such extra contribution for electrons. This result agrees with those presented in reference [18].

In summary we have completed the first theoretical calculation of the DHGI sum rule to order $\mathcal{O}(\alpha^3)$ in pure QED. The results presented here support the no-subtraction assumption and affirm the validity of the DHGI sum rule.

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