Closed categories vs. closed multicategories

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Abstract

We prove that the 2-category of closed categories of Eilenberg and Kelly is equivalent to a suitable full 2-subcategory of the 2-category of closed multicategories.

1 Introduction

The notion of closed category was introduced by Eilenberg and Kelly [2]. It is an axiomatization of the notion of category with internal function spaces. More precisely, a closed category is a category $\mathcal{C}$ equipped with a functor $\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, called the internal Hom-functor; an object $\mathbf{1}$ of $\mathcal{C}$, called the unit object; a natural isomorphism $i_X : X \xrightarrow{\sim} \mathcal{C}(\mathbf{1}, X)$, and natural transformations $j_X : \mathbf{1} \to \mathcal{C}(X,X)$ and $L^X_Z : \mathcal{C}(Y,Z) \to \mathcal{C}(\mathcal{C}(X,Y), \mathcal{C}(X,Z))$. These data are to satisfy five axioms; see Definition 2.1 for details.

A wide class of examples is provided by closed monoidal categories. We recall that a monoidal category $\mathcal{C}$ is called closed if for each object $X$ of $\mathcal{C}$ the functor $X \otimes -$ admits a right adjoint $\mathcal{C}(X,-)$; i.e, there exists a bijection $\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(Y, \mathcal{C}(X,Z))$ that is natural in both $Y$ and $Z$. Equivalently, a monoidal category $\mathcal{C}$ is closed if and only if for each pair of objects $X$ and $Z$ of $\mathcal{C}$ there exist an internal Hom-object $\mathcal{C}(X,Z)$ and an evaluation morphism $ev^\mathcal{C}_{X,Z} : X \otimes \mathcal{C}(X,Z) \to Z$ satisfying the following universal property: for each morphism $f : X \otimes Y \to Z$ there exists a unique morphism $g : Y \to \mathcal{C}(X,Z)$ such that $f = ev^\mathcal{C}_{X,Z} \circ (1_X \otimes g)$. One can check that the map $(X,Z) \mapsto \mathcal{C}(X,Z)$ extends uniquely to a functor $\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, which together with certain canonically chosen transformations $i_X$, $j_X$, and $L^X_Z$, turns $\mathcal{C}$ into a closed category.

While closed monoidal categories are in prevalent use in mathematics, arising in category theory, algebra, topology, analysis, logic, and theoretical computer science, there are also important examples of closed categories that are not monoidal. The author’s motivation stemmed from the theory of $A_\infty$-categories.

The notion of $A_\infty$-category appeared at the beginning of the nineties in the work of Fukaya on Floer homology [3]. However its precursor, the notion of $A_\infty$-algebra, was introduced in the early sixties by Stasheff [12]. It as a linearization of the notion of $A_\infty$-space, a topological space equipped with a product operation which is associative up to homotopy, and the homotopy which makes the product associative can be chosen so that it satisfies a collection of higher coherence conditions. Loosely speaking, $A_\infty$-categories are to $A_\infty$-algebras what linear categories are to algebras. On the other hand, $A_\infty$-categories generalize differential graded categories. Unlike in differential graded categories, in $A_\infty$-categories composition need not be associative on the nose; it is only required to be associative up to homotopy that satisfies a certain equation up to another homotopy, and so on.

Many properties of $A_\infty$-categories follow from the discovery, attributed to Kontsevich, that for each pair of $A_\infty$-categories $\mathcal{A}$ and $\mathcal{B}$ there is a natural $A_\infty$-category $A_\infty(\mathcal{A}, \mathcal{B})$ with $A_\infty$-functors from $\mathcal{A}$ to $\mathcal{B}$ as its objects. These $A_\infty$-categories of $A_\infty$-functors were also investigated by many other authors, e.g. Fukaya [4], Lefèvre-Hasegawa [9], and Lyubashenko [11]; they allow us to equip the category of $A_\infty$-categories with the structure of a closed category.

In the recent monograph by Bespalov, Lyubashenko, and the author [1] the theory of $A_\infty$-categories is developed from a slightly different perspective. Our approach is based on the observation
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tor of many arguments, and thus A∞-categories form a multicategory. The notion of multicategory (known also as colored operad or pseudo-tensor category) was introduced by Lambek [6, 7]. It is a many-object version of the notion of operad. If morphisms in a category are considered as analogous to functions, morphisms in a multicategory are analogous to functions in several variables. The most familiar example of a multicategory is the multicategory of vector spaces and multilinear maps. An arrow in a multicategory looks like X1, X2, ..., Xn \to Y, with a finite sequence of objects as the domain and one object as the codomain. Multicategories generalize monoidal categories: monoidal category C gives rise to a multicategory ˆC whose objects are those of C and whose morphisms X1, X2, ..., Xn \to Y are morphisms X1 \otimes X2 \otimes ... \otimes Xn \to Y of ˆC. The notion of closedness for multicategories is a straightforward generalization of that for monoidal categories. We say that a multicategory C is closed if for each sequence X1, ..., Xm, Z of objects of C there exist an internal Hom-object C(X1, ..., Xm; Z) and an evaluation morphism evC_{X_{1}...X_{m};Z}: X_{1},...,X_{m},C(X_{1},...,X_{m};Z) \to Z satisfying the following universal property: for each morphism f : X_{1}, ..., X_{m}, Y_{1}, ..., Y_{n} \to Z there is a unique morphism g : Y_{1}, ..., Y_{n} \to C(X_{1}, ..., X_{m}; Z) such that f = evC_{X_{1}...X_{m};Z} \circ (1_{X_{1}}, ..., 1_{X_{m}}, g). We prove that the multicategory of A∞-categories is closed, thus obtaining a conceptual explanation of the origin of the A∞-categories of A∞-functors.

The definition of closed multicategory seems to be sort of a mathematical folklore, and to the best of the author’s knowledge it did not appear in press before [1]. The only reference the author is aware of is the paper of Hyland and Power on pseudo-closed 2-categories [2–4], where the notion of closed Cat-multicategory (i.e., multicategory enriched in the category Cat of categories) is implicitly present, although not spelled out.

This paper arose as an attempt to understand in general the relation between closed categories and closed multicategories. It turned out that these notions are essentially equivalent in a very strong sense. Namely, on the one hand, there is a 2-category of closed categories, closed functors, and closed natural transformations. On the other hand, there is a 2-category of closed multicategories with unit objects, multifunctors, and multinatural transformations. Because a 2-category is the same thing as a category enriched in Cat, it makes sense to speak about Cat-functors between 2-categories; these can be called strict 2-functors because they preserve composition of 1-morphisms and identity 1-morphisms strictly. We construct a Cat-functor from the 2-category of closed multicategories with unit objects to the 2-category of closed categories, and prove that it is a Cat-equivalence; see Proposition 1.6 and Theorem 5.1.

Both closed categories and multicategories can bear symmetries. With some additional work it can be proven that the 2-category of symmetric closed categories is Cat-equivalent to the 2-category of symmetric closed multicategories with unit objects. We are not going to explore this subject here.

We should mention that the definition of closed category we adopt in this paper does not quite agree with the definition appearing in [2]. Closed categories have been generalized by Street [13] to extension systems; a closed category in our sense is an extension system with precisely one object. We discuss carefully the relation between these definitions because it is crucial for our proof of Theorem 5.1; see Remark 2.3 and Proposition 2.13. Our definition of closed category also coincides with the definition appearing in Laplaza’s paper [8], to which we would like to pay special tribute because it allowed us to give an elegant construction of a closed multicategory with a given underlying closed category.

Notation. We use interchangeably the notations g \circ f and f \cdot g for the composition of morphisms f : X \to Y and g : Y \to Z in a category, giving preference to the latter notation, which is more readable. Throughout the paper the set of nonnegative integers is denoted by \mathbb{N}, the category of sets is denoted by \mathcal{S}, and the category of categories is denoted by Cat.

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In this section we give preliminaries on closed categories. We begin by recalling the definition of closed category appearing in [13, Section 4] and [8].

### 2.1 Definition.

A *closed category* $(\mathcal{C}, \mathcal{C}(-,-), 1, i, j, L)$ consists of the following data:

- a category $\mathcal{C}$;
- a functor $\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$;
- an object $1$ of $\mathcal{C}$;
- a natural isomorphism $i : \text{Id}_{\mathcal{C}} \sim - \to \mathcal{C}(1, -) : \mathcal{C} \to \mathcal{C}$;
- a transformation $j_X : 1 \to \mathcal{C}(X, X)$, dinatural in $X \in \text{Ob} \mathcal{C}$;  
- a transformation $L_{YZ}^X : \mathcal{C}(Y, Z) \to \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Z))$, natural in $Y, Z \in \text{Ob} \mathcal{C}$ and dinatural in $X \in \text{Ob} \mathcal{C}$.

These data are subject to the following axioms.

**CC1.** The following equation holds true:

$$[1 \xrightarrow{j_Y} \mathcal{C}(Y, Y) \xrightarrow{L_{YY}^X} \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Y))] = j_{\mathcal{C}(X, Y)}.$$

**CC2.** The following equation holds true:

$$[\mathcal{C}(X, Y) \xrightarrow{L_{XY}^Z} \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Z)) \xrightarrow{\mathcal{C}(j_X, 1)} \mathcal{C}(1, \mathcal{C}(X, Y))] = i_{\mathcal{C}(X, Y)}.$$

**CC3.** The following diagram commutes:

$$\begin{array}{ccc}
\mathcal{C}(U, V) & \xrightarrow{L_{UV}^X} & \mathcal{C}(\mathcal{C}(U, V), \mathcal{C}(Y, V)) \\
\downarrow_{L_{UV}^X} & & \downarrow_{\mathcal{C}(1, L_{UV}^X)} \\
\mathcal{C}(\mathcal{C}(X, U), \mathcal{C}(X, V)) & \xrightarrow{\mathcal{C}(1, L_{XY}^V)} & \mathcal{C}(\mathcal{C}(X, U), \mathcal{C}(X, V)) \\
\downarrow_{L_{(X,Y)}^{(U,V)}} & & \downarrow_{\mathcal{C}(1, L_{XY}^V)} \\
\mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, U), \mathcal{C}(X, V)) & \xrightarrow{\mathcal{C}(L_{UV}^X, 1)} & \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, U), \mathcal{C}(X, V))
\end{array}$$

**CC4.** The following equation holds true:

$$[\mathcal{C}(Y, Z) \xrightarrow{L_{YZ}^1} \mathcal{C}(1, \mathcal{C}(Y, Z)) \xrightarrow{\mathcal{C}(1, i_Y)} \mathcal{C}(Y, \mathcal{C}(1, Z))] = \mathcal{C}(1, i_Z).$$

**CC5.** The map $\gamma : \mathcal{C}(X, Y) \to \mathcal{C}(1, \mathcal{C}(X, Y))$ that sends a morphism $f : X \to Y$ to the composite

$$1 \xrightarrow{j_X} \mathcal{C}(X, X) \xrightarrow{\mathcal{C}(1, f)} \mathcal{C}(X, Y)$$

is a bijection.

We shall call $\mathcal{C}(-,-)$ the *internal Hom-functor* and $1$ the *unit object*. 

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2.2 Example. The category $S$ of sets becomes a closed category if we set $\mathbb{S}(\cdot, \cdot) = \mathbb{S}(\cdot, \cdot)$, take for $\mathbf{1}$ a set $\{\ast\}$, chosen once and for all, consisting of a single point $\ast$; and define $i, j, L$ by:

- $i_X(x)(\ast) = x, \quad x \in X$;
- $j_X(\ast) = 1_X$;
- $L_{YZ}^X(g)(f) = f \cdot g, \quad f \in \mathbb{S}(X,Y), \quad g \in \mathbb{S}(Y,Z)$.

2.3 Remark. Definition 2.1 is slightly different from the original definition by Eilenberg and Kelly [2 Section 2]. They require that a closed category $\mathcal{C}$ be equipped with a functor $C : \mathcal{C} \to \mathbb{S}$ such that the following axioms are satisfied in addition to CC1–CC4.

CC0. The following diagram of functors commutes:

\[
\begin{array}{ccc}
\mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{\mathcal{S}(-,-)} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathbb{S} & \xrightarrow{\mathcal{C}(-,-)} & \mathcal{C}
\end{array}
\]

CC5'. The map

\[Ci_{\mathcal{C}(X,Y)} : \mathcal{C}(X,Y) = C\mathcal{C}(X,Y) \to C\mathcal{C}(\mathbf{1}, \mathcal{C}(X,Y)) = \mathcal{C}(\mathbf{1}, \mathcal{C}(X,Y))\]

sends $1_X \in \mathcal{C}(X,Y)$ to $j_X \in \mathcal{C}(\mathbf{1}, \mathcal{C}(X,Y))$.

[2] Lemma 2.2 implies that

\[\gamma = Ci_{\mathcal{C}(X,Y)} : \mathcal{C}(X,Y) = C\mathcal{C}(X,Y) \to C\mathcal{C}(\mathbf{1}, \mathcal{C}(X,Y)) = \mathcal{C}(\mathbf{1}, \mathcal{C}(X,Y))\]

so that a closed category in the sense of Eilenberg and Kelly is also a closed category in our sense. Furthermore, as we shall see later, an arbitrary closed category in our sense is isomorphic to a closed category in the sense of Eilenberg and Kelly.

2.4 Proposition ([3 Proposition 2.5]). $i_{\mathcal{C}(X,Y)} = C(1, i_X) : \mathcal{C}(\mathbf{1}, X) \to \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, X))$.

Proof. The proof given in [3 Proposition 2.5] translates word by word to our setting. \qed

2.5 Proposition ([3 Proposition 2.7]). $j_{\mathbf{1}} = i_{\mathbf{1}} : \mathbf{1} \to \mathcal{C}(\mathbf{1}, \mathbf{1})$.

Proof. The proof given in [3 Proposition 2.7] relies on the axiom CC5', and thus is not applicable here; we give an independent proof. The map $\gamma : \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, \mathbf{1})) \to \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, \mathbf{1})))$ is a bijection by the axiom CC5, so it suffices to prove that $\gamma(j_{\mathbf{1}}) = \gamma(i_{\mathbf{1}})$. We have:

\[
\gamma(i_{\mathbf{1}}) = \left[\mathbf{1} \xrightarrow{j_{\mathbf{1}}} \mathcal{C}(\mathbf{1}, \mathbf{1}) \xrightarrow{\mathcal{C}(1,j_{\mathbf{1}})} \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, \mathbf{1}))\right] = \left[\mathbf{1} \xrightarrow{j_{\mathbf{1}}} \mathcal{C}(\mathbf{1}, \mathbf{1}) \xrightarrow{\mathcal{C}(1,j_{\mathbf{1}})} \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, \mathbf{1}))\right] = \left[\mathbf{1} \xrightarrow{j_{\mathbf{1}}} \mathcal{C}(\mathbf{1}, \mathbf{1}) \xrightarrow{\mathcal{C}(1,j_{\mathbf{1}})} \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, \mathbf{1}))\right]
(\text{axiom CC2})
\]

\[
= \left[\mathbf{1} \xrightarrow{j_{\mathbf{1}}} \mathcal{C}(\mathbf{1}, \mathbf{1}) \xrightarrow{\mathcal{C}(1,j_{\mathbf{1}})} \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, \mathbf{1}))\right] = \left[\mathbf{1} \xrightarrow{j_{\mathbf{1}}} \mathcal{C}(\mathbf{1}, \mathbf{1}) \xrightarrow{\mathcal{C}(1,j_{\mathbf{1}})} \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, \mathbf{1}))\right]
(\text{dinaturnality of } j)
\]

The proposition is proven. \qed

2.6 Corollary. $[\mathcal{C}(\mathbf{1}, X) \xrightarrow{\gamma} \mathcal{C}(\mathbf{1}, \mathcal{C}(\mathbf{1}, X))] = 1_{\mathcal{C}(\mathbf{1}, X)}$. 

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Proof. An element \( f \in \mathcal{C}(1, X) \) is mapped by the left hand side to the composite

\[
1 \xrightarrow{j_1} \mathcal{C}(1, 1) \xrightarrow{(1, f)} \mathcal{C}(1, X) \xrightarrow{i_X^1} X,
\]

which is equal to

\[
[1 \xrightarrow{j_1} \mathcal{C}(1, 1) \xrightarrow{i_X^{-1}} 1 \xrightarrow{f} X] = f
\]

by the naturality of \( i_X^{-1} \), and because \( j_1 = i_1 : 1 \to \mathcal{C}(1, 1) \) by Proposition 2.3. The corollary is proven.

\[
\begin{array}{cccc}
\mathcal{C}(Y, Z) & \xrightarrow{\mathcal{C}(X, -)} & \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Z)) \\
\gamma & & \gamma \\
\mathcal{C}(1, \mathcal{C}(Y, Z)) & \xrightarrow{\mathcal{C}(1, L_{YZ}^X)} & \mathcal{C}(1, \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Z)))
\end{array}
\]

\[
\begin{align*}
\mathcal{C}(1, L_{YZ}^X)(&\gamma(f)) = \left[ 1 \xrightarrow{j_Y} \mathcal{C}(Y, Y) \xrightarrow{\mathcal{C}(1, f)} \mathcal{C}(Y, Z) \xrightarrow{L_{YZ}^X} \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Z)) \right] \\
&= \left[ 1 \xrightarrow{j_Y} \mathcal{C}(Y, Y) \xrightarrow{L_{YZ}^X} \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Y)) \xrightarrow{\mathcal{C}(1, \mathcal{C}(1, f))} \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Z)) \right] \\
&= \left[ 1 \xrightarrow{j_Y} \mathcal{C}(Y, Y) \xrightarrow{\gamma(\mathcal{C}(1, f), \mathcal{C}(1, g))} \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Z)) \right]
\end{align*}
\]

where the second equality is by the dinaturality of \( L_{YZ}^X \) in \( X \), and the third equality is by the axiom CC1.

\[
\begin{array}{cccc}
\mathcal{C}(1, f) & \xrightarrow{\gamma(f)} & \mathcal{C}(1, g) \\
\gamma(\mathcal{C}(1, f), \mathcal{C}(1, g)) & & \gamma(\mathcal{C}(1, f), \mathcal{C}(1, g)) \\
\mathcal{C}(1, f, 1) & \xrightarrow{\gamma(f, 1)} & \mathcal{C}(1, g, 1)
\end{array}
\]

\[
\begin{align*}
\gamma(f) & \cdot \mathcal{C}(1, g) = \left[ 1 \xrightarrow{j_X} \mathcal{C}(X, X) \xrightarrow{\mathcal{C}(1, f)} \mathcal{C}(X, Y) \xrightarrow{\mathcal{C}(1, g)} \mathcal{C}(X, Z) \right] \\
&= \left[ 1 \xrightarrow{j_X} \mathcal{C}(Y, Y) \xrightarrow{\mathcal{C}(1, f)} \mathcal{C}(Y, Y) \xrightarrow{\mathcal{C}(1, g)} \mathcal{C}(Y, Y) \xrightarrow{\mathcal{C}(1, f)} \mathcal{C}(X, Z) \right] \quad \text{(dinaturality of} \ j) \\
&= \left[ 1 \xrightarrow{j_X} \mathcal{C}(Y, Y) \xrightarrow{\mathcal{C}(1, g)} \mathcal{C}(Y, Y) \xrightarrow{\mathcal{C}(1, f)} \mathcal{C}(X, Z) \right] \quad \text{(functoriality of} \mathcal{C}(\mathcal{C}(1, g)) \\
&= \gamma(g) \cdot \mathcal{C}(f, 1).
\end{align*}
\]

The proposition is proven.

We now recall the definitions of closed functor and closed natural transformation following [3, Section 2].

\[
\begin{array}{cccc}
\mathcal{C} & \xrightarrow{\Phi} & \mathcal{D} \\
\phi & \xrightarrow{\hat{\phi}} & \phi^0 \\
\mathcal{C}(X, Y) & \xrightarrow{\mathcal{D}(\phi X, \phi Y)} & \mathcal{D}(\mathcal{C}(X, Y))
\end{array}
\]

\[
\begin{align*}
\text{2.9 Definition.} \quad & \text{Let} \mathcal{C} \text{ and} \mathcal{D} \text{ be closed categories. A closed functor} \ \Phi = (\phi, \hat{\phi}, \phi^0) : \mathcal{C} \to \mathcal{D} \text{ consists} \\
& \quad \text{of the following data:} \\
& \text{• a functor} \ \phi : \mathcal{C} \to \mathcal{D}; \\
& \text{• a natural transformation} \ \hat{\phi} = \hat{\phi}_{X,Y} : \phi \mathcal{C}(X, Y) \to \mathcal{D}(\phi X, \phi Y); \\
& \quad \text{a morphism} \ \phi^0 : 1 \to \phi 1.
\end{align*}
\]

These data are subject to the following axioms.
The following equation holds true:
\[
\left[ \mathbb{1} \xrightarrow{\phi^{0}} \phi \mathbb{1} \xrightarrow{\phi j_X} \phi \mathcal{C}(X, X) \xrightarrow{\phi} \mathcal{D}(\phi X, \phi X) \right] = j_{\phi X}.
\]

CF2. The following equation holds true:
\[
\left[ \phi X \xrightarrow{\phi i_X} \phi \mathcal{C}(\mathbb{1}, X) \xrightarrow{\phi} \mathcal{D}(\phi \mathbb{1}, \phi X) \xrightarrow{\mathcal{D}(\phi^{0}, 1)} \mathcal{D}(\mathbb{1}, \phi X) \right] = i_{\phi X}.
\]

CF3. The following diagram commutes:
\[
\begin{array}{ccc}
\phi \mathcal{C}(Y, Z) & \xrightarrow{\phi L_X^Y Z} & \phi \mathcal{C}(\mathcal{C}(X, Y), \mathcal{C}(X, Z)) \\
\phi & \downarrow & \phi \\
\mathcal{D}(\phi Y, \phi Z) & \xrightarrow{L_X^Y, \phi Z} & \mathcal{D}(\mathcal{D}(\phi X, \phi Y), \mathcal{D}(\phi X, \phi Z))
\end{array}
\]

2.10 Proposition. Let \( \mathcal{V} \) be a closed category. There is a closed functor \( E = (e, \hat{e}, e^0) : \mathcal{V} \to \mathcal{S} \), where:

- \( e = \mathcal{V}(\mathbb{1}, -) : \mathcal{V} \to \mathcal{S} \);
- \( \hat{e} = [\mathcal{V}(\mathbb{1}, \mathcal{V}(X, Y))] \xrightarrow{\gamma^{-1}} \mathcal{V}(X, Y) \xrightarrow{\mathcal{V}(\mathbb{1}, -)} \mathcal{S}(\mathcal{V}(\mathbb{1}, X), \mathcal{V}(\mathbb{1}, X))] \);
- \( e^0 : \{\ast\} \to \mathcal{V}(\mathbb{1}, \mathbb{1}), \ast \mapsto 1_{\mathbb{1}} \).

Proof. Let us check the axioms CF1–CF3. The reader is referred to Example 2.2 for a description of the structure of a closed category on \( \mathcal{S} \).

CF1. We must prove the following equation:
\[
\left[ \{\ast\} \xrightarrow{e^0} \mathcal{V}(\mathbb{1}, \mathbb{1}) \xrightarrow{\mathcal{V}(\mathbb{1}, jX)} \mathcal{V}(\mathbb{1}, \mathcal{V}(X, X)) \xrightarrow{\gamma^{-1}} \mathcal{V}(X, X) \xrightarrow{\mathcal{V}(\mathbb{1}, -)} \mathcal{S}(\mathcal{V}(\mathbb{1}, X), \mathcal{V}(\mathbb{1}, X))] \right] = j_{\mathcal{V}(\mathbb{1}, X)}.
\]

The image of \( \ast \) under the composite in the left hand side is \( \mathcal{V}(\mathbb{1}, \gamma^{-1}(j_X)) = \mathcal{V}(\mathbb{1}, 1_X) = 1_{\mathcal{V}(\mathbb{1}, X)} \), which is precisely \( j_{\mathcal{V}(\mathbb{1}, X)}(\ast) \).

CF2. We must prove the following equation:
\[
\left[ \mathcal{V}(\mathbb{1}, X) \xrightarrow{\mathcal{V}(\mathbb{1}, jX)} \mathcal{V}(\mathbb{1}, \mathcal{V}(\mathbb{1}, X)) \xrightarrow{\gamma^{-1}} \mathcal{V}(\mathbb{1}, X) \xrightarrow{\mathcal{V}(\mathbb{1}, -)} \mathcal{S}(\mathcal{V}(\mathbb{1}, X), \mathcal{V}(\mathbb{1}, X)) \xrightarrow{\mathcal{S}(\omega^{0}, 1)} \mathcal{S}(\{\ast\}, \mathcal{V}(\mathbb{1}, X))] \right] = i_{\mathcal{V}(\mathbb{1}, X)}.
\]

By Corollary 2.6, the left hand side is equal to
\[
\left[ \mathcal{V}(\mathbb{1}, X) \xrightarrow{\mathcal{V}(\mathbb{1}, -)} \mathcal{S}(\mathcal{V}(\mathbb{1}, X), \mathcal{V}(\mathbb{1}, X)) \xrightarrow{\mathcal{S}(\omega^{0}, 1)} \mathcal{S}(\{\ast\}, \mathcal{V}(\mathbb{1}, X))] \right],
\]

and so it maps an element \( f \in \mathcal{V}(\mathbb{1}, X) \) to the function \( \{\ast\} \to \mathcal{V}(\mathbb{1}, X), \ast \mapsto f \), which is precisely \( i_{\mathcal{V}(\mathbb{1}, X)}(f) \).
These two functions are equal by Proposition 2.8. The proposition is proven.

Let us prove that so does the remaining region. Taking an element \( f \in \mathcal{V}(Y, Z) \) and tracing it along the top-right path we obtain the function

\[
\mathcal{V}(\mathbb{1}, \mathcal{V}(X, Y)) \to S(\mathcal{V}(\mathbb{1}, X), \mathcal{V}(\mathbb{1}, Z)) ,
\]

\[
g \mapsto \left( h \mapsto h \cdot \gamma^{-1}(g \cdot \mathcal{V}(1, f)) \right),
\]

whereas pushing \( f \) along the left-bottom path yields the function

\[
\mathcal{V}(\mathbb{1}, \mathcal{V}(X, Y)) \to S(\mathcal{V}(\mathbb{1}, X), \mathcal{V}(\mathbb{1}, Z)) ,
\]

\[
g \mapsto \left( h \mapsto h \cdot \gamma^{-1}(g \cdot f) \right).
\]

These two functions are equal by Proposition 2.8. The proposition is proven. \( \square \)

**2.11 Definition.** Let \( \Phi = (\phi, \hat{\phi}, \phi^0), \Psi = (\psi, \hat{\psi}, \psi^0) : \mathcal{C} \to \mathcal{D} \) be closed functors. A closed natural transformation \( \eta : \Phi \to \Psi : \mathcal{C} \to \mathcal{D} \) is a natural transformation \( \eta : \phi \to \psi : \mathcal{C} \to \mathcal{D} \) satisfying the following axioms.

CN1. The following equation holds true:

\[
[\mathbb{1} \xrightarrow{\phi} \mathbb{1} \xrightarrow{\eta} \psi \mathbb{1}] = \psi^0.
\]

CN2. The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{C}(X, Y) & \xrightarrow{\hat{\phi}} & \mathcal{D}(\phi X, \phi Y) \\
\eta_{\mathcal{C}(X,Y)} & \xrightarrow{\psi} & \mathcal{D}(\psi X, \psi Y) \\
\end{array}
\]

Closed categories, closed functors, and closed natural transformations form a 2-category \([3, \text{Theorem } 4.2]\), which we shall denote by \( \text{CICat} \). The composite of closed functors \( \Phi = (\phi, \hat{\phi}, \phi^0) : \mathcal{C} \to \mathcal{D} \) and \( \Psi = (\psi, \hat{\psi}, \psi^0) : \mathcal{D} \to \mathcal{E} \) is defined to be \( X = (\chi, \hat{\chi}, \chi^0) : \mathcal{C} \to \mathcal{E} \), where:

- \( \chi \) is the composite \( \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \);
Compositions of closed natural transformations are defined in the usual way.

We can enrich in closed categories. Below we recall some enriched category theory for closed categories mainly following [3, Section 5].

2.12 Definition. Let \( \mathcal{V} \) be a closed category. A \( \mathcal{V} \)-category \( \mathcal{A} \) consists of the following data:

- a set \( \text{Ob} \mathcal{A} \) of objects;
- for each \( X, Y \in \text{Ob} \mathcal{A} \), an object \( \mathcal{A}(X, Y) \) of \( \mathcal{V} \);
- for each \( X \in \text{Ob} \mathcal{A} \), a morphism \( j_X : 1 \to \mathcal{A}(X, X) \) in \( \mathcal{V} \);
- for each \( X, Y, Z \in \text{Ob} \mathcal{A} \), a morphism \( L^X_{YZ} : \mathcal{A}(Y, Z) \to \mathcal{V}(\mathcal{A}(X, Y), \mathcal{A}(X, Z)) \) in \( \mathcal{V} \).

These data are to satisfy axioms [2, VC1–VC3]. If \( \mathcal{A} \) and \( \mathcal{B} \) are \( \mathcal{V} \)-categories, a \( \mathcal{V} \)-functor \( F : \mathcal{A} \to \mathcal{B} \) consists of the following data:

- a function \( \text{Ob} F : \text{Ob} \mathcal{A} \to \text{Ob} \mathcal{B} \), \( X \mapsto FX \);
- for each \( X, Y \in \text{Ob} \mathcal{A} \), a morphism \( F = F_{XY} : \mathcal{A}(X, Y) \to \mathcal{B}(FX, FY) \) in \( \mathcal{V} \).

These data are subject to axioms [2, VF1–VF2].

2.13 Example. By [3, Theorem 5.2] a closed category \( \mathcal{V} \) gives rise to a category \( \mathcal{Y} \) if we take the objects of \( \mathcal{Y} \) to be those of \( \mathcal{V} \), take \( \mathcal{V}(X, Y) \) to be the internal Hom-object, and take for \( j \) and \( L \) those of the closed category \( \mathcal{V} \). Furthermore, if \( \mathcal{A} \) is a \( \mathcal{V} \)-category and \( X \) is an object of \( \mathcal{A} \), then we get a \( \mathcal{V} \)-functor \( \mathcal{L}^X : \mathcal{A} \to \mathcal{Y} \) if we take \( \mathcal{L}^X Y = \mathcal{A}(X, Y) \) and \( (\mathcal{L}^X)_{YZ} = \mathcal{L}^X_{YZ} \). In particular, for each \( X \in \text{Ob} \mathcal{V} \), there is a \( \mathcal{V} \)-functor \( \mathcal{L}^X : \mathcal{Y} \to \mathcal{Y} \) such that \( \mathcal{L}^X Y = \mathcal{V}(X, Y) \) and \( (\mathcal{L}^X)_{YZ} = \mathcal{L}^X_{YZ} \).

There is also a notion of \( \mathcal{V} \)-natural transformation. We recall it in a particular case, namely for \( \mathcal{V} \)-functors \( \mathcal{A} \to \mathcal{Y} \).

2.14 Definition. Let \( F, G : \mathcal{A} \to \mathcal{V} \) be \( \mathcal{V} \)-functors. A \( \mathcal{V} \)-natural transformation \( \alpha : F \to G : \mathcal{A} \to \mathcal{Y} \) is a collection of morphisms \( \alpha_X : FX \to GX \) in \( \mathcal{V} \), for each \( X \in \text{Ob} \mathcal{A} \), such that the diagram

\[
\begin{array}{ccc}
\mathcal{A}(X, Y) & \xrightarrow{F_{XY}} & \mathcal{V}(FX, FY) \\
G_{XY} \downarrow & & \downarrow \mathcal{V}(1, \alpha_Y) \\
\mathcal{V}(GX, GY) & \xrightarrow{\mathcal{V}(\alpha_X, 1)} & \mathcal{V}(FX, GY)
\end{array}
\]

commutes, for each \( X, Y \in \text{Ob} \mathcal{A} \).

2.15 Example. By [2, Proposition 8.4] if \( f \in \mathcal{V}(X, Y) \), the morphisms

\[
\mathcal{V}(f, 1) : \mathcal{V}(Y, Z) \to \mathcal{V}(X, Z), \quad Z \in \text{Ob} \mathcal{V},
\]

are components of a \( \mathcal{V} \)-natural transformation \( L^f : L^Y \to L^X : \mathcal{V} \to \mathcal{V} \).

By [3, Theorem 10.2] \( \mathcal{V} \)-categories, \( \mathcal{V} \)-functors, and \( \mathcal{V} \)-natural transformations form a 2-category, which we shall denote by \( \mathcal{V}\text{-Cat} \).

2.16 Proposition (2. Proposition 6.1]). If \( \Phi = (\phi, \hat{\phi}, \phi^0) : \mathcal{V} \to \mathcal{W} \) is a closed functor and \( \mathcal{A} \) is a \( \mathcal{V} \)-category, the following data define a \( \mathcal{W} \)-category \( \Phi_* \mathcal{A} \):

- \( \hat{\chi} \) is the composite \( \psi\phi((X, Y) \xrightarrow{\psi\phi^0} \psi\hat{\phi}(\phi X, \phi Y)) \);
- \( \chi^0 \) is the composite \( 1 \xrightarrow{\psi^0} \psi 1 \xrightarrow{\psi\phi^0} \psi\phi 1 \).
2.8 Proposition. The proposition is proven.

In particular, \( W_L V \) is given by

\[
\phi \Phi(\alpha, \beta) = \phi \Phi(\alpha, \beta).
\]

Proof. Let \( \Phi_{\alpha, \beta} \) be a \( \Phi \)-category. There is an isomorphism of categories \( \Phi_{\alpha, \beta} \rightarrow \Phi_{\alpha, \beta} \) that is identical on objects and is given by the bijects \( \Phi_{\alpha, \beta} \rightarrow \Phi_{\alpha, \beta} \) on morphisms.

Proof. For each \( x \in \text{Ob} \Phi_{\alpha, \beta} \), we have \( \gamma(x) = \Phi_{\alpha, \beta} \), so \( \gamma \) preserves identities. Let us check that it also preserves composition. For each \( f \in \Phi_{\alpha, \beta} \), \( g \in \Phi_{\alpha, \beta} \), we have \( \gamma(f) \cdot \gamma(g) = \gamma(f) \cdot \gamma^{-1}(g \cdot L_{YZ}^X) \).

By Proposition 2.8, \( \gamma(g) \cdot L_{YZ}^X = \gamma(g) \cdot \gamma^{-1}(g \cdot L_{YZ}^X) \), therefore \( \gamma(f) \cdot \gamma(g) = \gamma(f) \cdot \gamma^{-1}(g \cdot L_{YZ}^X) \).

2.17 Example. Let us study the effect of the closed functor \( E \) from Proposition 2.18 on \( \mathcal{V} \)-categories. Let \( \mathcal{A} \) be a \( \mathcal{V} \)-category. Then the ordinary category \( E_{\alpha, \beta} \mathcal{A} \) has the same set of objects as \( \mathcal{A} \) and its Hom-sets are \( (E_{\alpha, \beta} \mathcal{A})(X, Y) = \mathcal{V}(\mathcal{A}(X, Y)) \). The morphism \( j_X \) for the category \( E_{\alpha, \beta} \mathcal{A} \) is given by the composite

\[
\begin{align*}
\mathcal{V}(1, \mathcal{A}(X, Z)) & \xrightarrow{\mathcal{V}(1, L_{YZ})} \mathcal{V}(1, \mathcal{A}(X, Y), \mathcal{A}(X, Z)) \xrightarrow{\gamma^{-1}} \mathcal{V}(\mathcal{A}(X, Y), \mathcal{A}(X, Z)) \\
& \xrightarrow{\mathcal{V}(1, -)} S(\mathcal{A}(1, \mathcal{A}(X, Y)), \mathcal{V}(1, \mathcal{A}(X, Z))).
\end{align*}
\]

It follows that composition in \( E_{\alpha, \beta} \mathcal{A} \) is given by

\[
\mathcal{V}(1, \mathcal{A}(X, Y)) \times \mathcal{V}(1, \mathcal{A}(Y, Z)) \rightarrow \mathcal{V}(1, \mathcal{A}(X, Z)), \quad (f, g) \mapsto f \cdot \gamma^{-1}(g \cdot L_{YZ}^X).
\]

2.18 Proposition. Let \( \mathcal{V} \) be a closed category. There is an isomorphism of categories \( \gamma: \mathcal{V} \rightarrow E_{\alpha, \beta} \mathcal{V} \) that is identical on objects and is given by the bijects \( \gamma: \mathcal{V}(X, Y) \rightarrow \mathcal{V}(1, \mathcal{V}(X, Y)) \) on morphisms.

Proof. For each \( x \in \text{Ob} \mathcal{V} \), we have \( \gamma(x) = j_x \), so \( \gamma \) preserves identities. Let us check that it also preserves composition. For each \( f \in \mathcal{V}(X, Y) \), \( g \in \mathcal{V}(Y, Z) \), we have \( \gamma(f) \cdot \gamma(g) = \gamma(f) \cdot \gamma^{-1}(g \cdot L_{YZ}^X) \).

By Proposition 2.8, \( \gamma(g) \cdot L_{YZ}^X = \gamma(g) \cdot \gamma^{-1}(g \cdot L_{YZ}^X) \), therefore \( \gamma(f) \cdot \gamma(g) = \gamma(f) \cdot \gamma^{-1}(g \cdot L_{YZ}^X) \).

The proposition is proven.

2.19 Theorem. Every closed category is isomorphic to a closed category in the sense of Eilenberg and Kelly.

More precisely, for every closed category \( \mathcal{V} \) in the sense of Definition 2.1 there is a closed category \( \mathcal{W} \) in the sense of Eilenberg and Kelly such that \( \mathcal{W} \), when viewed as a closed category in the sense of Definition 2.1, is isomorphic as a closed category to \( \mathcal{V} \).

Proof. Let \( \mathcal{V} \) be a closed category. Take \( \mathcal{W} = E_{\alpha, \beta} \mathcal{V} \). The isomorphism \( \gamma \) from Proposition 2.18 allows us to translate the structure of a closed category from \( \mathcal{V} \) to \( \mathcal{W} \). Thus the unit object of \( \mathcal{W} \) is that of \( \mathcal{V} \), the internal Hom-functor is given by the composite

\[
\begin{align*}
\mathcal{W}(-, -) & = \left[ \mathcal{W}^{\text{op}} \times \mathcal{W} \xrightarrow{(\gamma^{\text{op}} \times \gamma^{-1})} \mathcal{V}^{\text{op}} \times \mathcal{V} \xrightarrow{\mathcal{V}(-, -)} \mathcal{V} \xrightarrow{\mathcal{W}} \mathcal{W} \right].
\end{align*}
\]

In particular, \( \mathcal{W}(X, Y) = \mathcal{V}(X, Y) \) for each pair of objects \( X \) and \( Y \). The transformations \( i_X, j_X, L_{YZ}^X \) for \( \mathcal{W} \) are just \( \gamma(i_X), \gamma(j_X), \gamma(L_{YZ}^X) \) respectively. The category \( \mathcal{W} \) admits a functor \( W: \mathcal{W} \rightarrow \mathcal{S} \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{W}^{\text{op}} \times \mathcal{W} & \xrightarrow{\mathcal{W}(-, -)} & \mathcal{W} \\
\downarrow \mathcal{W}(-, -) & & \downarrow W \\
\mathcal{S} & & \mathcal{S}
\end{array}
\]
from Example 2.15 is uniquely determined by the condition \((\gamma^{-1}(f), \gamma^{-1}(h)))\) for each \(n \in \mathbb{N}\), \(g \in \mathcal{V}(\gamma^{-1}(f), \gamma^{-1}(h))) : \mathcal{V}(\mathbf{1}, \mathcal{V}(Y, U)) \to \mathcal{V}(\mathbf{1}, \mathcal{V}(X, Y)), \ g \mapsto g \cdot \mathcal{V}(\gamma^{-1}(f), \gamma^{-1}(h)).

We have:

\[
g \cdot \mathcal{V}(\gamma^{-1}(f), \gamma^{-1}(h)) = \gamma(\gamma^{-1}(g)) \cdot \mathcal{V}(\gamma^{-1}(f), 1) \cdot (1, \gamma^{-1}(h)) \quad \text{(functoriality of } \mathcal{V}(-, -))
\]

\[
= \gamma(\gamma^{-1}(f) \cdot \gamma^{-1}(g)) \cdot (1, \gamma^{-1}(h)) \quad \text{(Proposition 2.8)}
\]

\[
= \gamma(\gamma^{-1}(f) \cdot \gamma^{-1}(g) \cdot \gamma^{-1}(h)) \quad \text{(Proposition 2.8)}
\]

\[
= f \cdot g \cdot h, \quad \text{(Proposition 2.18)}
\]

hence the assertion. The functor \(W\) also satisfies the axiom \(CC5'\). Indeed, we need to show that

\[
W(i_{\mathcal{V}(X, X)}) = \mathcal{V}(\mathbf{1}, i_{\mathcal{V}(X, X)}) : \mathcal{V}(\mathbf{1}, \mathcal{V}(X, X)) \to \mathcal{V}(\mathbf{1}, \mathcal{V}(\mathbf{1}, \mathcal{V}(X, X)))
\]

maps \(j_X \in \mathcal{V}(\mathbf{1}, \mathcal{V}(X, X))\) to \(\gamma(j_X) \in \mathcal{V}(\mathbf{1}, \mathcal{V}(\mathbf{1}, \mathcal{V}(X, X)))\). In other words, we need to show that the diagram

\[
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{j_X} & \mathcal{V}(X, X) \\
\downarrow j_X & & \downarrow j_{\mathcal{V}(X, X)} \\
\mathcal{V}(\mathbf{1}, \mathbf{1}) & \xrightarrow{\mathcal{V}(1, j_X)} & \mathcal{V}(\mathbf{1}, \mathcal{V}(X, X))
\end{array}
\]

commutes. However \(j_{\mathbf{1}} = i_{\mathbf{1}} : \mathbf{1} \to \mathcal{V}(\mathbf{1}, \mathbf{1})\) by Proposition 2.7, so the above diagram is commutative by the naturality of \(i\). The theorem is proven.

Finally, let us recall from [2] the representation theorem for \(\mathcal{V}\)-functors \(A \to \mathcal{V}\).

2.20 Proposition ([2 Corollary 8.7]). Suppose that \(\mathcal{V}\) is a closed category in the sense of Eilenberg and Kelly; i.e., it is equipped with a functor \(V: \mathcal{V} \to \mathbb{S}\) satisfying the axioms \(CC0\) and \(CC5'\). Let \(T : A \to \mathcal{V}\) be a \(\mathcal{V}\)-functor, and let \(W\) be an object of \(A\). Then the map

\[
\Gamma : \mathcal{V}\text{-Cat}(A, \mathcal{V})(L^W, T) \to VTW, \quad p \mapsto (V(p_W))_1\,\,
\]

is a bijection.

2.21 Example. For each \(f \in VL^XY = V(\mathcal{V}(X, Y)) = \mathcal{V}(X, Y)\), the \(\mathcal{V}\)-natural transformation \(L^f : L^Y \to L^X : \mathcal{V} \to \mathcal{V}\) from Example 2.15 is uniquely determined by the condition \((V(L^f)_Y)1_Y = f\).

## 3 Closed multicategories

We begin by briefly recalling the notions of multicategory, multifunctor, and multinatural transformation. The reader is referred to the excellent book by Leinster [10] or to [1, Chapter 3] for a more elaborate introduction to multicategories.

3.1 Definition. A multigraph \(C\) is a set \(\text{Ob } C\), whose elements are called objects of \(C\), together with a set \(C(X_1, \ldots, X_n; Y)\) for each \(n \in \mathbb{N}\) and \(X_1, \ldots, X_n, Y \in \text{Ob } C\). Elements of \(C(X_1, \ldots, X_n; Y)\) are called morphisms and written as \(X_1, \ldots, X_n \to Y\). If \(n = 0\), elements of \(C(; Y)\) are written as () \(\to Y\). A morphism of multigraphs \(F : C \to D\) consists of a function \(\text{Ob } F : \text{Ob } C \to \text{Ob } D\), \(X \mapsto FX\), and functions

\[
F = F_{X_1, \ldots, X_n; Y} : C(X_1, \ldots, X_n; Y) \to D(FX_1, \ldots, FX_n; FY), \quad f \mapsto Ff,
\]

for each \(n \in \mathbb{N}\) and \(X_1, \ldots, X_n, Y \in \text{Ob } C\).

\(^1\)It is denoted by \(\Gamma'\) in [2 Corollary 8.7].
3.2 Definition. A \textit{multicategory} \( C \) consists of the following data:

- a multigraph \( C \);
- for each \( n, k_1, \ldots, k_n \in \mathbb{N} \) and \( X_{ij}, Y_i, Z \in \text{Ob} \, C \), \( 1 \leq i \leq n, 1 \leq j \leq k_i \), a function
  \[
  \prod_{i=1}^{n} C(X_{i1}, \ldots, X_{ik_i}; Y_i) \times C(Y_1, \ldots, Y_n; Z) \to C(X_{11}, \ldots, X_{1k_1}, \ldots, X_{nk_n}; Z),
  \]
  called \textit{composition} and written \( (f_1, \ldots, f_n, g) \mapsto (f_1 \cdots f_n) \cdot g \);
- for each \( X \in \text{Ob} \, C \), an element \( 1^C_X \in C(X; X) \), called the \textit{identity} of \( X \).

These data are subject to the obvious associativity and identity axioms.

3.3 Example. A strict monoidal category \( C \) gives rise to a multicategory \( \hat{C} \) as follows:

- \( \text{Ob} \, \hat{C} = \text{Ob} \, C \);
- for each \( n \in \mathbb{N} \) and \( X_1, \ldots, X_n, Y \in \text{Ob} \, C \), \( \hat{C}(X_1, \ldots, X_n; Y) = C(X_1 \otimes \cdots \otimes X_n, Y) \); in particular \( \hat{C}(; Y) = C(1, Y) \), where \( 1 \) is the unit object of \( C \);
- for each \( n, k_1, \ldots, k_n \in \mathbb{N} \) and \( X_{ij}, Y_i, Z \in \text{Ob} \, C \), \( 1 \leq i \leq n, 1 \leq j \leq k_i \), the composition map
  \[
  \prod_{i=1}^{n} C(X_{i1}, \ldots, X_{ik_i}; Y_i) \times C(Y_1, \ldots, Y_n; Z) \to C(X_{11}, \ldots, X_{1k_1}, \ldots, X_{nk_n}; Z)
  \]
  is given by \( (f_1, \ldots, f_n, g) \mapsto (f_1 \cdots f_n) \cdot g \);
- for each \( X \in \text{Ob} \, C \), \( 1^C_X = 1_X^\hat{C} \in \hat{C}(X; X) = C(X, X) \).

3.4 Definition. Let \( C \) and \( D \) be multicategories. A \textit{multifunctor} \( F : C \to D \) is a morphism of the underlying multigraphs that preserves composition and identities.

3.5 Definition. Suppose that \( F, G : C \to D \) are multifunctors. A \textit{multinatural transformation} \( r : F \to G : C \to D \) is a family of morphisms \( r_X \in D(FX; GX), X \in \text{Ob} \, C \), such that
  \[
  F \hat{f} \cdot r_Y = (r_{X_1}, \ldots, r_{X_n}) \cdot G \hat{f},
  \]
  for each \( f \in C(X_1, \ldots, X_n; Y) \).

Multicategories, multifunctors, and multinatural transformations form a 2-category, which we shall denote by \textbf{Multicat}.

3.6 Definition \((\PageIndex{2}, \text{Definition 4.7})\). A multicategory \( C \) is called \textit{closed} if for each \( m \in \mathbb{N} \) and \( X_1, \ldots, X_m, Z \in \text{Ob} \, C \) there exist an object \( C(X_1, \ldots, X_m; Z) \), called \textit{internal Hom-object}, and an \textit{evaluation} morphism
  \[
  \text{ev}^C = \text{ev}^C_{X_1, \ldots, X_m; Z} : X_1, \ldots, X_m, C(X_1, \ldots, X_m; Z) \to Z
  \]
  such that, for each \( Y_1, \ldots, Y_n \in \text{Ob} \, C \), the function
  \[
  \varphi^C = \varphi^C_{X_1, \ldots, X_m; Y_1, \ldots, Y_n; Z} : C(Y_1, \ldots, Y_n; C(X_1, \ldots, X_m; Z)) \to C(X_1, \ldots, X_m, Y_1, \ldots, Y_n; Z)
  \]
  that sends a morphism \( f : Y_1, \ldots, Y_n \to C(X_1, \ldots, X_m; Z) \) to the composite
  \[
  X_1, \ldots, X_m, Y_1, \ldots, Y_n \xrightarrow{1_{X_1} \cdots 1_{X_m} \cdot f} X_1, \ldots, X_m, C(X_1, \ldots, X_m; Z) \xrightarrow{\text{ev}^C_{X_1, \ldots, X_m; Z}} Z
  \]
  is bijective. Let \( \textbf{ClMulticat} \) denote the full 2-subcategory of \textbf{Multicat} whose objects are closed multicategories.
3.7 Remark. Notice that for $m = 0$ an object $\mathbb{C}(Z)$ and a morphism $ev_Z^C$ with the required property always exist. Namely, we may (and we shall) always take $\mathbb{C}(Z) = Z$ and $ev_Z^C = 1_Z^C : Z \to Z$. With these choices $\varphi^C_{Y_1, \ldots, Y_n; Z} : \mathbb{C}(Y_1, \ldots, Y_n; Z) \to \mathbb{C}(Y_1, \ldots, Y_n, Z)$ is the identity map.

3.8 Example. Let $\mathcal{C}$ be a strict monoidal category, and let $\hat{\mathcal{C}}$ be the associated multicategory, see Example 3.3. It is easy to see that the multicategory $\hat{\mathcal{C}}$ is closed if and only if $\mathcal{C}$ is closed as a monoidal category.

3.9 Proposition. Suppose that for each pair of objects $X, Z \in \text{Ob} \mathcal{C}$ there exist an object $\mathbb{C}(X; Z)$ and a morphism $ev^C_{X; Z} : X, \mathbb{C}(X; Z) \to Z$ of $\mathcal{C}$ such that the function $\varphi^C_{X; Y_1, \ldots, Y_n; Z}$ is a bijection, for each finite sequence $Y_1, \ldots, Y_n$ of objects of $\mathcal{C}$. Then $\mathcal{C}$ is a closed multicategory.

Proof. Define internal Hom-objects $\mathbb{C}(X_1, \ldots, X_m; Z)$ and evaluations

$$ev^C_{X_1, \ldots, X_m; Z} : X_1, \ldots, X_m, \mathbb{C}(X_1, \ldots, X_m; Z) \to Z$$

by induction on $m$. For $m = 0$ choose $\mathbb{C}(Z) = Z$ and $ev^C_{Z} = 1^C_Z : Z \to Z$ as explained above. For $m = 1$ we are already given $\mathbb{C}(X; Z)$ and $ev^C_{X; Z}$. Assume that we have defined $\mathbb{C}(X_1, \ldots, X_k; Z)$ and $ev^C_{X_1, \ldots, X_k; Z}$ for each $k < m$, and that the function

$$\varphi^C_{X_1, \ldots, X_k; Y_1, \ldots, Y_n; Z} : \mathbb{C}(Y_1, \ldots, Y_n; \mathbb{C}(X_1, \ldots, X_k; Z)) \to \mathbb{C}(X_1, \ldots, X_k, Y_1, \ldots, Y_n; Z)$$

is a bijection, for each $k < m$ and for each finite sequence $Y_1, \ldots, Y_n$ of objects of $\mathcal{C}$. For $X_1, \ldots, X_m, Z \in \text{Ob} \mathcal{C}$ define

$$\mathbb{C}(X_1, \ldots, X_m; Z) \overset{\text{def}}{=} \mathbb{C}(X_m; \mathbb{C}(X_1, \ldots, X_m-1; Z)).$$

The evaluation morphism $ev^C_{X_1, \ldots, X_m; Z}$ is given by the composite

$$\begin{array}{ccc}
X_1, \ldots, X_m, \mathbb{C}(X_m; \mathbb{C}(X_1, \ldots, X_m-1; Z)) & \xrightarrow{1^C_{X_1, \ldots, X_m-1}, ev^C_{X_m, \mathbb{C}(X_1, \ldots, X_m-1; Z)}} & X_1, \ldots, X_m-1, \mathbb{C}(X_1, \ldots, X_m-1; Z) \\
\downarrow & & \downarrow ev^C_{X_1, \ldots, X_m-1; Z} \\
\mathbb{C}(X_1, \ldots, X_m-1; Z) & \overset{\text{ev}}{\xrightarrow{ev^C_{X_1, \ldots, X_m-1; Z}}} & Z,
\end{array}$$

It is easy to see that with these choices the function $\varphi^C_{X_1, \ldots, X_m; Y_1, \ldots, Y_n; Z}$ decomposes as

$$\begin{array}{ccc}
\mathbb{C}(Y_1, \ldots, Y_n; \mathbb{C}(X_1, \ldots, X_m; Z)) & \xrightarrow{i^C_{Y_1, \ldots, Y_n; \mathbb{C}(X_1, \ldots, X_m; Z)}} & \mathbb{C}(X_m, Y_1, \ldots, Y_n; \mathbb{C}(X_1, \ldots, X_m-1; Z)) \\
\downarrow & & \downarrow i_{Y_1, \ldots, Y_n; \mathbb{C}(X_1, \ldots, X_m-1; Z)}^C \\
\mathbb{C}(X_m, Y_1, \ldots, Y_n; \mathbb{C}(X_1, \ldots, X_m-1; Z)) & \xrightarrow{i_{X_1, \ldots, X_m-1; X_m, Y_1, \ldots, Y_n}^C} & \mathbb{C}(X_1, \ldots, X_m, Y_1, \ldots, Y_n; Z),
\end{array}$$

hence it is a bijection, and the induction goes through. \hfill \Box

Notation. For each morphism $f : X_1, \ldots, X_n \to Y$ with $n \geq 1$, denote by $\langle f \rangle$ the morphism $(\varphi^C_{X_1, X_2, \ldots, X_n; Z})^{-1}(f) : X_2, \ldots, X_n \to \mathbb{C}(X_1; Y)$. In other words, $\langle f \rangle$ is uniquely determined by the equation

$$\left[ X_1, X_2, \ldots, X_n \overset{1^C_{X_1}}{\xrightarrow{\varphi^C_{X_1}(f)}} X_1, \mathbb{C}(X_1; Y) \overset{ev^C_{X_1; Y}}{\xrightarrow{\text{ev}} Y} \right] = f.$$

Clearly we can enrich in multicategories. We leave it as an easy exercise for the reader to spell out the definitions of categories and functors enriched in a multicategory $\mathcal{V}$. 

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3.10 Proposition. A closed multicategory \( \mathcal{C} \) gives rise to a \( \mathcal{C} \)-category \( \mathcal{C} \) as follows. The objects of \( \mathcal{C} \) are those of \( \mathcal{C} \). For each pair \( X, Y \in \text{Ob} \mathcal{C} \), the Hom-object \( \mathcal{C}(X, Y) \) is the internal Hom-object of \( \mathcal{C} \). For each \( X, Y, Z \in \text{Ob} \mathcal{C} \), the composition morphism \( \mu_{\mathcal{C}} : \mathcal{C}(X, Y), \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z) \) is uniquely determined by requiring the commutativity in the diagram

\[
\begin{array}{ccc}
X, \mathcal{C}(X, Y), \mathcal{C}(Y, Z) & \xrightarrow{\mathcal{C}(X, Y), \mathcal{C}(Y, Z)} & X, \mathcal{C}(X, Z) \\
\xrightarrow{\text{ev}_{X,Y}, \mathcal{C}(Y, Z)} & & \xrightarrow{\text{ev}_{X,Z}} \\
Y, \mathcal{C}(Y, Z) & \xrightarrow{\text{ev}_{Y,Z}} & Z
\end{array}
\]

The identity of an object \( X \in \text{Ob} \mathcal{C} \) is \( 1^X_X = \langle 1^X_X \rangle : () \to \mathcal{C}(X, X) \).

Proof. The proof is similar to that for a closed monoidal category. \( \Box \)

Notation. For each morphism \( f : X_1, \ldots, X_n \to Y \) and object \( Z \) of a closed multicategory \( \mathcal{C} \), there exists a unique morphism \( \mathcal{C}(f; Z) : \mathcal{C}(Y, Z) \to \mathcal{C}(X_1, \ldots, X_n; Z) \) such that the diagram

\[
\begin{array}{ccc}
X_1, \ldots, X_n, \mathcal{C}(Y, Z) & \xrightarrow{\mathcal{C}(f; Z)} & X_1, \ldots, X_n, \mathcal{C}(X_1, \ldots, X_n; Z) \\
\xrightarrow{f, \mathcal{C}(Y, Z)} & & \xrightarrow{\mathcal{C}(f, \mathcal{C}(Y, Z))} \\
Y, \mathcal{C}(Y, Z) & \xrightarrow{\text{ev}_{Y,Z}} & Z
\end{array}
\]

in \( \mathcal{C} \) is commutative. In particular, if \( n = 0 \), then \( \mathcal{C}(f; Z) = (f, \mathcal{C}(Y, Z)) \cdot \text{ev}_{Y,Z}^C \). If \( n = 1 \), then \( \mathcal{C}(f; Z) = (\langle f, \mathcal{C}(Y, Z) \rangle \cdot \text{ev}_{Y,Z}^C) \). For each sequence of morphisms \( f_1 : X_1 \to Y_1, \ldots, f_n : X_n \to Y_n \) in \( \mathcal{C} \) there is a unique morphism \( \mathcal{C}(f_1, \ldots, f_n; Z) : \mathcal{C}(Y_1, \ldots, Y_n; Z) \to \mathcal{C}(X_1, \ldots, X_n; Z) \) such that the diagram

\[
\begin{array}{ccc}
X_1, \ldots, X_n, \mathcal{C}(Y_1, \ldots, Y_n; Z) & \xrightarrow{\mathcal{C}(f_1, \ldots, f_n; Z)} & X_1, \ldots, X_n, \mathcal{C}(X_1, \ldots, X_n; Z) \\
\xrightarrow{f_1, \ldots, f_n, \mathcal{C}(Y_1, \ldots, Y_n; Z)} & & \xrightarrow{\mathcal{C}(f_1, \ldots, f_n, \mathcal{C}(Y_1, \ldots, Y_n; Z))} \\
Y_1, \ldots, Y_n, \mathcal{C}(Y_1, \ldots, Y_n; Z) & \xrightarrow{\text{ev}_{Y_1,\ldots,Y_n;Z}} & Z
\end{array}
\]

in \( \mathcal{C} \) is commutative. Similarly, for each morphism \( g : Y \to Z \) in \( \mathcal{C} \), there exists a unique morphism \( \mathcal{C}(X_1, \ldots, X_n; g) : \mathcal{C}(X_1, \ldots, X_n; Y) \to \mathcal{C}(X_1, \ldots, X_n; Z) \) such that the diagram

\[
\begin{array}{ccc}
X_1, \ldots, X_n, \mathcal{C}(X_1, \ldots, X_n; Y) & \xrightarrow{\mathcal{C}(X_1, \ldots, X_n; g)} & X_1, \ldots, X_n, \mathcal{C}(X_1, \ldots, X_n; Z) \\
\xrightarrow{\text{ev}_{X_1,\ldots,X_n;Y}} & & \xrightarrow{\text{ev}_{X_1,\ldots,X_n;Z}} \\
Y & \xrightarrow{g} & Z
\end{array}
\]

in \( \mathcal{C} \) is commutative. In particular, if \( n = 0 \), then our conventions force \( \mathcal{C}; g = g \). If \( n = 1 \), then \( \mathcal{C}(X; g) = \langle \text{ev}_{X,Y}^C \cdot g \rangle \).

3.11 Lemma. Suppose that \( f_1 : X_1, \ldots, X_{k_1} \to Y_1, \ldots, f_n : X_{n_1}, \ldots, X_{n_n} \to Y_n, g : Y_1, \ldots, Y_n \to Z \) are morphisms in a closed multicategory \( \mathcal{C} \).

(a) If \( k_1 = 0 \), i.e., \( f_1 \) is a morphism \( () \to Y_1 \), then \( (f_1, \ldots, f_n) \cdot g \) is equal to the composite

\[
X_2, \ldots, X_{k_2}, \ldots, X_{n_1}, \ldots, X_{n_n} \xrightarrow{f_2, \ldots, f_n} Y_2, \ldots, Y_n \xrightarrow{g} \mathcal{C}(Y_1; Z) \xrightarrow{\mathcal{C}(f_1; Z)} \mathcal{C}(Z; Z) = Z.
\]
(b) If $k_1 = 1$, i.e., $f$ is a morphism $X_1 \to Y_1$, then $\langle (f_1, \ldots, f_n) \cdot g \rangle$ is equal to the composite
\[X_1^1, \ldots, X_2^k, \ldots, X_n^{k_n} \xrightarrow{f_2, \ldots, f_n} Y_2, \ldots, Y_n \xrightarrow{g} \mathfrak{C}(Y_1; Z) \xrightarrow{\mathfrak{C}(f_1; Z)} \mathfrak{C}(X_1; Z).\]
(c) If $k_1 \geq 1$, then $\langle (f_1, \ldots, f_n) \cdot g \rangle$ is equal to the composite
\[X_2^1, \ldots, X_1^{k_1}, X_2^1, \ldots, X_2^k, \ldots, X_n^{k_n} \xrightarrow{f_1, f_2, \ldots, f_n} Y_1, Y_2, \ldots, Y_n \xrightarrow{\mu_C} \mathfrak{C}(X_1; Y_1), \mathfrak{C}(Y_1; Z) \xrightarrow{\mathfrak{C}(f_1; Z)} \mathfrak{C}(X_1; Z).\]
(d) if $n = 1$, then $\langle f_1 \cdot g \rangle = \left[ X_2^1, \ldots, X_1^{k_1} \xrightarrow{(f_1)} \mathfrak{C}(X_1; Y_1) \xrightarrow{\mathfrak{C}(X_1^1; g)} \mathfrak{C}(X_1^1; Z) \right].$

**Proof.** The proofs are easy and consist of checking the definitions. For example, in order to prove (a) note that
\[\mathfrak{C}(f_1; Z) = \left[ \mathfrak{C}(Y_1; Z) \xrightarrow{f_1 \cdot \mathfrak{C}(g_{Y_1; Z})} Y_1, \mathfrak{C}(Y_1; Z) \xrightarrow{\mathfrak{C}(\mathfrak{e}_{Y_1; Z})} Z \right],\]
therefore the composite in (a) is equal to
\[\left[ X_2^1, \ldots, X_2^k, \ldots, X_n^{k_n} \xrightarrow{f_2, \ldots, f_n} Y_2, \ldots, Y_n \xrightarrow{g} \mathfrak{C}(Y_1; Z) \xrightarrow{f_1 \cdot \mathfrak{C}(g_{Y_1; Z})} Y_1, \mathfrak{C}(Y_1; Z) \xrightarrow{\mathfrak{C}(\mathfrak{e}_{Y_1; Z})} Z \right] = \left[ X_2^1, \ldots, X_2^k, \ldots, X_n^{k_n} \xrightarrow{f_1, f_2, \ldots, f_n} Y_1, Y_2, \ldots, Y_n \xrightarrow{\mathfrak{C}(\mathfrak{e}_{Y_1; Z})} Y_1, \mathfrak{C}(Y_1; Z) \xrightarrow{\mathfrak{C}(\mathfrak{e}_{Y_1; Z})} Z \right].\]
The last two arrows compose to $\varphi_{Y_1, Y_2, \ldots, Y_n; Z}(g) = g : Y_1, \ldots, Y_n \to Z$, hence the whole composite is equal to $(f_1, \ldots, f_n) \cdot g$. \qed

**3.12 Lemma.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in a closed multicategory $\mathbb{C}$. Then for each $W \in \text{Ob} \mathbb{C}$ holds $\mathfrak{C}(W; f \cdot g) = \mathfrak{C}(W; f) \cdot \mathfrak{C}(W; g)$.

**Proof.** The composite $\mathfrak{C}(W; f) \cdot \mathfrak{C}(W; g)$ can be written as
\[\mathfrak{C}(W; X) \xrightarrow{(\mathfrak{e}_{W; X} \cdot f)} \mathfrak{C}(W; Y) \xrightarrow{\mathfrak{e}_{W; Y} \cdot g} \mathfrak{C}(W; Z),\]
which is equal to $\langle \mathfrak{e}_{W; X} \cdot f \cdot g \rangle = \mathfrak{C}(W; f \cdot g)$ by Proposition 3.11. (d). \qed

**3.13 Lemma.** Let $f : W \to X$ and $g : X \to Y$ be morphisms in a closed multicategory $\mathbb{C}$. Then for each $Z \in \text{Ob} \mathbb{C}$ holds $\mathfrak{C}(f \cdot g; Z) = \mathfrak{C}(g; Z) \cdot \mathfrak{C}(f; Z)$.

**Proof.** The composite $\mathfrak{C}(g; Z) \cdot \mathfrak{C}(f; Z)$ can be written as
\[\mathfrak{C}(Y; Z) \xrightarrow{(g \cdot \mathfrak{C}(1^X_{Y; Z}) \cdot \mathfrak{e}_{Y; Z} \cdot f)} \mathfrak{C}(X; Z) \xrightarrow{\mathfrak{e}_{X; Z} \cdot f} \mathfrak{C}(W; Z),\]
which is equal to $\langle (f \cdot g \cdot \mathfrak{C}(1^X_{Y; Z}) \cdot \mathfrak{e}_{Y; Z} \cdot f) \rangle = \langle (f \cdot g, \mathfrak{C}(1^X_{Y; Z}) \cdot \mathfrak{e}_{Y; Z}) \rangle = \langle f \cdot g, \mathfrak{C}(1^X_{Y; Z}) \rangle = \langle f \cdot g ; Z \rangle$ by Proposition 3.11. (b). \qed

**3.14 Lemma.** Let $f : W \to X$ and $g : Y \to Z$ be morphisms in a closed multicategory $\mathbb{C}$. Then $\mathfrak{C}(f; Y) \cdot \mathfrak{C}(W; g) = \mathfrak{C}(X; g) \cdot \mathfrak{C}(f; Z)$.

**Proof.** Both sides of the equation are equal to $\langle (f \cdot g, \mathfrak{C}(1^X_{Y; Z}) \cdot \mathfrak{e}_{X; Y}) \cdot g \rangle$ by Proposition 3.11. (b),(d). \qed

Lemmas 3.12, 3.14 imply that there exists a functor $\mathfrak{C}(-, -) : \mathbb{C}^{\text{op}} \times \mathbb{C} \to \mathbb{C}$, $(X, Y) \mapsto \mathfrak{C}(X; Y)$, defined by the formula $\mathfrak{C}(f; g) = \mathfrak{C}(f; Y) \cdot \mathfrak{C}(W; g) = \mathfrak{C}(X; g) \cdot \mathfrak{C}(f; Z)$ for each pair of morphisms $f : W \to X$ and $g : Y \to Z$ in $\mathbb{C}$.

For each $X, Y, Z \in \text{Ob} \mathbb{C}$ there is a morphism $L^Y_Z : \mathfrak{C}(Y; Z) \to \mathfrak{C}(\mathfrak{C}(X; Y); \mathfrak{C}(X; Z))$ uniquely determined by the equation
\[\left[ \mathfrak{C}(X; Y), \mathfrak{C}(Y; Z) \xrightarrow{1 \cdot L^Y_Z \mathfrak{C}} \mathfrak{C}(X; Y) \cdot \mathfrak{C}(X; Y); \mathfrak{C}(X; Y; \mathfrak{C}(X; Y); \mathfrak{C}(X; Z)) \xrightarrow{\mathfrak{e}_{\mathbb{C}}} \mathfrak{C}(X; Z) \right] = \mu_{\mathbb{C}}.\quad (3.1)\]
3.15 Proposition. There is a C-functor $L^X : C \to D, Y \mapsto C(X; Y)$, with the action on hom-objects given by $L^X_{YZ} : C(Y; Z) \to C(C(X; Y); C(X; Z)).$

Proof. That so defined $L^X$ preserves identities is a consequence of the identity axiom. The compatibility with composition is established as follows. Consider the diagram

\[
\begin{array}{cccccc}
C(Y; Z), C(Z; W) & \xrightarrow{L^X_{YZ}, L^X_{ZW}} & C(X; Y), C(C(X; Y); C(X; Z)), C(C(X; Z); C(X; W)) & \xrightarrow{ev_{C,1}} & C(X; Z), C(C(X; Z); C(X; W)) \\
\downarrow{\mu_C} & & \downarrow{1, \mu_C} & & \downarrow{ev_C} \\
C(Y; W) & \xrightarrow{L^X_{YW}} & C(C(X; Y); C(X; W)) & & C(X; W)
\end{array}
\]

By the definition of $L^X$ the exterior expresses the associativity of $\mu_C$. The right square is the definition of $\mu_C$. By the closedness of $C$ the square

\[
\begin{array}{cccc}
C(Y; Z), C(Z; W) & \xrightarrow{L^X_{YZ}, L^X_{ZW}} & C(C(X; Y); C(X; Z)), C(C(X; Z); C(X; W)) & \xrightarrow{\mu_C} \\
\downarrow{\mu_C} & & \downarrow{1, \mu_C} & \\
C(Y; W) & \xrightarrow{L^X_{YW}} & C(C(X; Y); C(X; W)) & 
\end{array}
\]

is commutative, hence the assertion. \[\square\]

3.16 Definition ([I, Section 4.18]). Let $C, D$ be multicategories. Let $F : C \to D$ be a multifunctor. For each $X_1, \ldots, X_m, Z \in \text{Ob } C$, define a morphism in $D$

\[
F_{X_1, \ldots, X_m; Z} : FC(X_1, \ldots, X_m; Z) \to D(FX_1, \ldots, FX_m; FZ)
\]

as the only morphism that makes the diagram

\[
\begin{array}{ccc}
FX_1, \ldots, FX_m, D(FX_1, \ldots, FX_m; FZ) & \xrightarrow{1_{FX_1, \ldots, FX_m}, 1_{F_{X_1, \ldots, X_m; Z}}} & FX_1, \ldots, FX_m, FC(X_1, \ldots, X_m; Z) \\
\downarrow{Fev_{X_1, \ldots, X_m; Z}} & & \downarrow{ev_{F_{X_1, \ldots, X_m; Z}}} \\
FZ & & FZ
\end{array}
\]

commute. It is called the closing transformation of the multifunctor $F$. The following properties of closing transformations can be found in [I, Section 4.18]. To keep the exposition self-contained we include their proofs here.

3.17 Proposition ([I, Lemma 4.19]). The diagram

\[
\begin{array}{ccc}
C(Y_1, \ldots, Y_n; C(X_1, \ldots, X_m; Z)) & \xrightarrow{F} & D(FY_1, \ldots, FY_n; FC(X_1, \ldots, X_m; Z)) \\
\downarrow{\varphi_{X_1, \ldots, X_m; Y_1, \ldots, Y_n; Z}} & & \downarrow{D(1_{E_{X_1, \ldots, X_m; Z}})} \\
C(X_1, \ldots, X_m, Y_1, \ldots, Y_n; Z) & \xrightarrow{F} & D(FX_1, \ldots, FX_m, FY_1, \ldots, FY_n; FZ)
\end{array}
\]

commutes, for each $m, n \in \mathbb{N}$ and objects $X_i, Y_j, Z \in \text{Ob } C$, $1 \leq i \leq m$, $1 \leq j \leq n$. 

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Proof. Pushing an arbitrary morphism \( g : Y_1, \ldots, Y_n \to X_1, \ldots, X_m, Z \) along the top-right path produces the composite

\[
FX_1, \ldots, FX_m, FY_1, \ldots, FY_n \xrightarrow{1_{FX_1, \ldots, FX_m, Fg}} FX_1, \ldots, FX_m, F\mathcal{C}(X_1, \ldots, X_m; Z) \xrightarrow{1_{FX_1, \ldots, FX_m} \mathcal{E}(X_1); Z} FX_1, \ldots, FX_m, \mathcal{D}(FX_1, \ldots, FX_m; FZ) \xrightarrow{\text{ev}^D_{FX_1, \ldots, FX_m; FZ}} FZ.
\]

The composition of the last two arrows is equal to \( F\text{ev}_{X_1, \ldots, X_m; Z}^\mathcal{C} \) by the definition of \( F_{X_1, \ldots, X_m; Z} \). Since \( F \) preserves composition and identities, the above composite equals

\[
F\left( (1_{X_1}^\mathcal{C}, \ldots, 1_{X_m}^\mathcal{C}, g) \cdot \text{ev}_{X_1, \ldots, X_m; Z}^\mathcal{C} \right) = F\left( \varphi_{X_1, \ldots, X_m; Y_1, \ldots, Y_n; Z}(g) \right),
\]

hence the assertion. \( \square \)

Let \( F : \mathcal{V} \to \mathcal{W} \) be a multifunctor, and let \( \mathcal{C} \) be a \( \mathcal{V} \)-category. Then we obtain a \( \mathcal{W} \)-category \( F_* \mathcal{C} \) with the same set of objects if we define its Hom-objects by \( (F_* \mathcal{C})(X,Y) = F\mathcal{C}(X,Y) \), and composition and identities by respectively \( \mu_{F_* \mathcal{C}} = F(\mu_{\mathcal{C}}) : F\mathcal{C}(X,Y), F\mathcal{C}(Y,Z) \to F\mathcal{C}(X,Z) \) and \( 1^\mathcal{C}_X = F(1^\mathcal{C}_X) : () \to F\mathcal{C}(X,X) \).

3.18 Proposition \((\text{cf. } [\text{1}, \text{Proposition 4.21}])\). Let \( F : \mathcal{C} \to \mathcal{D} \) be a multifunctor between closed multicategories. There is \( \mathcal{D} \)-functor \( F_* : F_\mathcal{C} \to F_* \mathcal{D} \), \( X \mapsto FX \), such that

\[
F_{X,Y} : (F_\mathcal{C})(X,Y) = F\mathcal{C}(X,Y) \to \mathcal{D}(FX; FY)
\]

is the closing transformation, for each \( X,Y \in \text{Ob} \mathcal{C} \).

Proof. First, let us check that \( F_* \) preserves identities. In other words, we must prove the equation

\[
\left[ () \xrightarrow{F1^\mathcal{C}_X} F\mathcal{C}(X;X) \xrightarrow{F_{X,X}} \mathcal{D}(FX;FX) \right] = 1^\mathcal{D}_{FX}.
\]

Let us check that the left hand side solves the equation that determines the right hand side. We have:

\[
[FX \xrightarrow{1^\mathcal{C}_X \cdot F1^\mathcal{C}_X} FX, F\mathcal{C}(X;X) \xrightarrow{1^\mathcal{C}_X \cdot \mathcal{E}_{X,X}} FX, \mathcal{D}(FX;FX) \xrightarrow{\text{ev}^D} FX] = [FX \xrightarrow{1^\mathcal{C}_X \cdot F1^\mathcal{C}_X} FX, F\mathcal{C}(X;X) \xrightarrow{\text{ev}^\mathcal{C}} FX] = F1^\mathcal{C}_X = 1^\mathcal{D}_{FX}.
\]

To show that \( F_* \) preserves composition, we must show that the diagram

\[
\begin{align*}
\begin{array}{ccc}
F\mathcal{C}(X;X), F\mathcal{C}(Y;Z) & \xrightarrow{F\mu_{\mathcal{C}}} & F\mathcal{C}(X;Z) \\
\mathcal{E}_{X,Y} \mathcal{E}_{Y,Z} & & \mathcal{E}_{X,Z} \\
\mathcal{D}(FX;FY), \mathcal{D}(FY;FZ) & \xrightarrow{\mu_{\mathcal{D}}} & \mathcal{D}(FX;FZ)
\end{array}
\end{align*}
\]

(3.3)

commutes. This follows from Diagram 3.1. The lower diamond is the definition of \( \mu_{\mathcal{D}} \). The exterior commutes by the definition of \( \mu_{\mathcal{C}} \) and because \( F \) preserves composition. The left upper diamond and both triangles commute by the definition of the closing transformation. \( \square \)

3.19 Lemma \((\text{cf. } [\text{1}, \text{Lemma 4.25}])\). Let \( \mathcal{C}, \mathcal{D}, \mathcal{E} \) be closed multicategories, and let \( \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \) be multifunctors. Then

\[
G \circ F_{X_1, \ldots, X_m; Y} = [G F\mathcal{C}(X_1, \ldots, X_m; Y) \xrightarrow{G E_{X_1,\ldots,X_m,Y}} G\mathcal{D}(FX_1, \ldots, FX_m; FY) \xrightarrow{G F_{X_1,\ldots,X_m; FY}} G\mathcal{E}(GFX_1, \ldots, GFX_m; GFY)].
\]
Diagram 3.1

**Proof.** This follows from the commutative diagram

\[
\begin{array}{c}
\xymatrix{
GFX_1, \ldots, GFX_m, GDP(FX_1, \ldots, FX_m; FY) \ar[r] & GFX_1, \ldots, GFX_m, E(GFX_1, \ldots, GFX_m; GFY) \\
GFX_1, \ldots, GFX_m, GF\mathcal{C}(X_1, \ldots, X_m; Y) \ar[u] \ar[r] & GF\mathcal{C}(X_1, \ldots, X_m; Y) \\
F C(X_1, \ldots, X_m; Y) \ar[u] \ar[r] & D(FX_1, \ldots, FX_m; FY) \\
GX_1, \ldots, GX_m, GDP(GX_1, \ldots, GX_m; GY) \ar[u] \ar[r] & D(GX_1, \ldots, GX_m; GY) \\
G C(X_1, \ldots, X_m; Y) \ar[u] \ar[r] & D(FX_1, \ldots, FX_m; FY) \\
FX_1, \ldots, FX_m, F \mathcal{C}(Y_1, \ldots, Y_m; Z) \ar[u] & \end{array}
\]

The upper triangle is the definition of $G_{FX_1, \ldots, FX_m; FY}$, the lower triangle commutes by the definition of $F_{X_1, \ldots, X_m; Y}$ and because $G$ preserves composition. \hfill \Box

3.20 Proposition ([ Lemma 4.24]). Let $\nu : F \rightarrow G : C \rightarrow D$ be a multinatural transformation of multifunctors between closed multicategories. Then the diagram

\[
\begin{array}{c}
\xymatrix{
F \mathcal{C}(X_1, \ldots, X_m; Y) \ar[r]^E \ar[d] & D(FX_1, \ldots, FX_m; FY) \\
G \mathcal{C}(X_1, \ldots, X_m; Y) \ar[r]^{-E} & D(FX_1, \ldots, FX_m; FY) \\
FX_1, \ldots, FX_m, FY \ar[u] \ar[r] & FX_1, \ldots, FX_m, GFY \\
}\end{array}
\]

is commutative.

**Proof.** The claim follows from Diagram 3.2. Its exterior commutes by the multinaturality of $\nu$. The quadrilateral in the middle is the definition of $D(\nu_{X_1}, \ldots, \nu_{X_m}; GY)$. The trapezoid on the right is the definition of $D(FX_1, \ldots, FX_m; \nu_Y)$. The triangles commute by the definition of closing transformation. \hfill \Box
Diagram 3.2
A closed category comes equipped with a distinguished object \( \mathbf{1} \). We want to produce a closed category out of a closed multicategory, so we need a notion of a closed multicategory with a unit object. We introduce it in somewhat ad hoc fashion, which is sufficient for our purposes though. Similarly to closedness, possession of a unit object is a property of a closed multicategory rather than additional data.

4.1 Definition. Let \( \mathcal{C} \) be a closed multicategory. A unit object of \( \mathcal{C} \) is an object \( \mathbf{1} \in \text{Ob} \mathcal{C} \) together with a morphism \( u : () \rightarrow \mathbf{1} \) such that, for each \( X \in \text{Ob} \mathcal{C} \), the morphism

\[
\mathcal{C}(u; 1) : \mathcal{C}(\mathbf{1}; X) \rightarrow \mathcal{C}(; X) = X
\]

is an isomorphism.

4.2 Remark. If \( \mathbf{1} \) is a unit object of a closed multicategory \( \mathcal{C} \), then \( \mathcal{C}(u; X) : \mathcal{C}(\mathbf{1}; X) \rightarrow \mathcal{C}(; X) \) is a bijection. This follows from the equation

\[
[C(; C(\mathbf{1}; X)) \xrightarrow{\varphi_C} C(\mathbf{1}; X) \xrightarrow{C(u; X)} C(; X)] = C(; C(u; X)),
\]

which is an immediate consequence of the definitions. The bijectivity of \( \mathcal{C}(u; X) \) can be stated as the following universal property: for each morphism \( f : () \rightarrow X \), there exists a unique morphism \( f : \mathbf{1} \rightarrow X \) such that \( u \cdot f = f \). In particular, a unit object, if it exists, is unique up to isomorphism.

4.3 Proposition. A closed multicategory \( \mathcal{C} \) with a unit object gives rise to a closed category \( (\mathcal{C}, \mathcal{C}(−, −), \mathbf{1}, i, j, L) \), where:

- \( \mathcal{C} \) is the underlying category of the multicategory \( \mathcal{C} \);
- \( \mathcal{C}(X, Y) = \mathcal{C}(X; Y) \), for each \( X, Y \in \text{Ob} \mathcal{C} \);
- \( \mathbf{1} \) is the unit object of \( \mathcal{C} \);
- \( i_X = (\mathcal{C}(u; X))^{-1} : X = \mathcal{C}(; X) \rightarrow \mathcal{C}(\mathbf{1}; X) \);
- \( j_X = 1^\mathcal{C}_X : \mathbf{1} \rightarrow \mathcal{C}(X; X) \) is a unique morphism such that \( [(() \xrightarrow{u} \mathbf{1} \xrightarrow{j_X} \mathcal{C}(X; X)] = 1^\mathcal{C}_X \);
- \( L^X_{YZ} : \mathcal{C}(Y; Z) \rightarrow \mathcal{C}(\mathcal{C}(X; Y); \mathcal{C}(X; Z)) \) is determined uniquely by equation (3.1).

We shall call \( \mathcal{C} \) the underlying closed category of \( \mathcal{C} \). Usually we do not distinguish notationally between a closed multicategory and its underlying closed category; this should lead to minimal confusion.

Proof. We leave it as an easy exercise for the reader to show the naturality of \( i_X, j_X \), and \( L^X_{YZ} \), and proceed directly to checking the axioms.

CC1. By Remark 4.2 the equation

\[
[\mathbf{1} \xrightarrow{j_Y} \mathcal{C}(Y; Y) \xrightarrow{L^Y_{XY}} \mathcal{C}(\mathcal{C}(X; Y); \mathcal{C}(X; Y))] = j_{\mathcal{C}(X; Y)}
\]

is equivalent to the equation

\[
[(()) \xrightarrow{u} \mathcal{C}(Y, Y) \xrightarrow{L^Y_{XY}} \mathcal{C}(\mathcal{C}(X; Y); \mathcal{C}(X; Y))] = u \cdot j_{\mathcal{C}(X; Y)} = 1^\mathcal{C}_X(\mathcal{C}(X; Y)),
\]

which expresses the fact that the \( \mathcal{C} \)-functor \( L^X \) preserves identities.
The equation in question
\[
[C(X; Y) \xrightarrow{L^X_{Y;V}} C(C(X; X); C(X; Y))] \xrightarrow{C(j_{Y;1};1)} C(1; C(X; Y)) = i_{C(X; Y)} = (C(u; 1))^{-1}
\]
is equivalent to
\[
[C(X; Y) \xrightarrow{L^X_{Y;Z}} C(C(X; X); C(X; Y))] \xrightarrow{C(u_{Y;1};1)} C(1; C(X; Y)) = C(X; Y) = 1_{C(X; Y)}.
\]
The left hand side is equal to
\[
[C(X; Y) \xrightarrow{L^X_{Y;V}} C(C(X; X); C(X; Y))] \xrightarrow{C(1; L^X_{Y;V};1)} C(X; X) \xrightarrow{\text{ev}_C} C(X; Y)
\]
\[
= [C(X; Y) \xrightarrow{L^X_{Y;Z}} C(X; X), C(X; Y)] \xrightarrow{\text{ev}_C} C(X; Y)
\]
\[
= [C(X; Y) \xrightarrow{L^X_{Y;1}} C(X; X), C(X; Y)] \xrightarrow{\mu_C} C(X; Y) = 1_{C(X; Y)}
\]
by the identity axiom in the C-category C.

**CC3.** The commutativity of the diagram
\[
\begin{array}{ccc}
C(U; V) & \xrightarrow{L^Y_{V;U}} & C(C(Y; U); C(Y; V)) \\
\mid & & \mid \\
L^X_{U;V} & & L^X_{C(U; V); C(Y; V)} \\
\downarrow & & \downarrow \\
C(C(X; U); C(X; V)) & \xrightarrow{C(1; L^X_{Y;V};1)} & C(C(Y; U); C(C(X; Y); C(X; V)) & \xrightarrow{C(L^X_{Y;U};1)} & C(C(Y; U); C(C(X; Y); C(X; V)))
\end{array}
\]
is equivalent by closedness to the commutativity of the exterior of Diagram 4.1, which just expresses the fact that the C-functor \(L^X : C \to C\) preserves composition and which is part of the assertion of Proposition 3.17.

**CC4.** The equation in question
\[
[C(Y; Z) \xrightarrow{L^1_{Y;Z}} C(C(1; Y); C(1; Z))] \xrightarrow{C(i_{Y;1};1)} C(Y; C(1; Z)) = C(1; i_Z)
\]
is equivalent to the equation
\[
[C(Y; Z) \xrightarrow{L^1_{Y;Z}} C(C(1; Y); C(1; Z))] \xrightarrow{C(1; C(u;1);1)} C(C(1; Y); Z) = C(C(u; 1); Z).
\]
The latter follows by closedness from the commutative diagram
CC5. A straightforward computation shows that the composite
\[ C(X; Y) \xrightarrow{\gamma} C(1; C(X; Y)) \xrightarrow{\sim} C(1; C(1; C(X; Y))) \xrightarrow{\varphi^C} C(X; Y) \]
is the identity map, which readily implies that \( \gamma \) is a bijection.

The proposition is proven. \( \square \)

4.4 Proposition. Let \( C \) and \( D \) be closed multicategories with unit objects. Let \( C \) and \( D \) denote the corresponding underlying closed categories. A multifunctor \( F : C \to D \) gives rise to a closed functor \( \Phi = (\phi, \hat{\phi}, \phi^0) : C \to D \), where:

- \( \phi : C \to D \) is the underlying functor of the multifunctor \( F \);
- \( \hat{\phi} = \hat{\phi}_{X,Y} : FC(X; Y) \to DF(X; FY) \) is the closing transformation;
- \( \phi^0 = Fu : 1 \to F1 \) is a unique morphism such that \( [() ^u 1 \xrightarrow{\phi^0} F1] = Fu \).

Proof. Let us check the axioms.

CF1. By Remark 4.2 the equation
\[ [1 \xrightarrow{\phi^0} F1 \xrightarrow{FjX} FC(1; X) \xrightarrow{F} DF(X; FX)] = jFX \]
is equivalent to the equation
\[ [() ^u 1 \xrightarrow{\phi^0} F1 \xrightarrow{FjX} FC(1; X) \xrightarrow{F} DF(X; FX)] = u \cdot jFX = 1_{DFX}. \]

Since \( u \cdot \phi^0 \cdot FjX = Fu \cdot FjX = F(u \cdot JX) = F1 \), the above equation simply expresses the fact that the \( D \)-functor \( F : C \to D \) preserves identities, which is part of Proposition 3.18.

CF2. The equation in question
\[ [FX \xrightarrow{F_{jX}} FC(1; X) \xrightarrow{F} DF(1; FX) \xrightarrow{D(\phi^0; 1)} DF(1; FX)] = iFX = DF(u; 1)^{-1} \]
is equivalent to
\[ [FC(1; X) \xrightarrow{F} DF(1; FX) \xrightarrow{D(\phi^0; 1)} DF(1; FX)] = FC(u; 1). \]

The composition of the last two arrows is equal to \( DF(Fu; 1) \). Hence the left hand side of the above equation is equal to
\[ [FC(1; X) \xrightarrow{F} DF(1; FX) \xrightarrow{DFu} DF(1; FX)] = FC(u; 1). \]

CF3. We must prove that the diagram
\[
\begin{CD}
FC(Y; Z) @>{FLX}>> FC(C(X; Y); C(X; Z)) @>{F}>> DF(C(X; Y); FFX) @>{DFu}>> DF(1; FX) \\
\downarrow{E} @. @. @. \downarrow{DF(1; E)} \\
DF(FY; FZ) @>{LFX}>> DF(DFX; FY) @>{DF}>> DF(DFX; FZ) @>{DFe}>> DF(1; FX)
\end{CD}
\]
Diagram 4.2
commutes. By closedness, this is equivalent to the commutativity of the exterior of Diagram 4.2, which expresses the fact that the $D$-functor $F : F_C \to D$ preserves composition and which is part of Proposition 3.18.

The proposition is proven. □

4.5 Proposition. A multinatural transformation $t : F \to G : C \to D$ of multifunctors between closed multicategories with unit objects gives rise to a closed natural transformation given by the same components.

Proof. Let $\Phi = (\phi, \hat{\phi}, \phi^0), \Psi = (\psi, \hat{\psi}, \psi^0) : C \to D$ be closed functors induced by the multifunctors $F$ and $G$ respectively. The axiom CN1 reads

$$\left[ 1 \xrightarrow{\phi^0} F 1 \xrightarrow{t_1} G 1 \right] = \psi^0.$$

It is equivalent to the equation

$$\left[ () \xrightarrow{u_1} 1 \xrightarrow{\phi^0} F 1 \xrightarrow{t_1} G 1 \right] = u \cdot \psi^0,$$

i.e., to the equation $Fu \cdot t_1 = Gu$, which is a consequence of the multinaturality of $t$. The axiom CN2 is a particular case of Proposition 3.20. □

Let $\text{ClMulticat}^u$ denote the full 2-subcategory of $\text{ClMulticat}$ whose objects are closed multicategories with a unit object. Note that a 2-category is the same thing as a $\text{Cat}$-category. Thus we can speak about $\text{Cat}$-functors between 2-categories. These are sometimes called strict 2-functors; they preserve composition of 1-morphisms and identity 1-morphisms on the nose.

4.6 Proposition. Propositions 4.3, 4.4, and 4.5 define a $\text{Cat}$-functor $U : \text{ClMulticat}^u \to \text{ClCat}$.

Proof. It is obvious that composition of 2-morphisms and identity 2-morphisms are preserved. It is also clear that the identity multifunctor induces the closed identity functor. Finally, composition of 1-morphisms is preserved by Lemma 3.19. □

5 From closed categories to closed multicategories

In this section we prove our main result.

5.1 Theorem. The $\text{Cat}$-functor $U : \text{ClMulticat}^u \to \text{ClCat}$ is a $\text{Cat}$-equivalence.

We have to prove that $U$ is bijective on 1-morphisms and 2-morphisms, and that it is essentially surjective; the latter means that for each closed category $V$ there is a closed multicategory with a unit object such that its underlying closed category is isomorphic (as a closed category) to $V$.

5.2 The surjectivity of $U$ on 1-morphisms. Let $C$ and $D$ be closed multicategories with unit objects. Denote their underlying closed categories by the same symbols. Let $\Phi = (\phi, \hat{\phi}, \phi^0) : C \to D$ be a closed functor. We are going to define a multifunctor $F : C \to D$ whose underlying closed functor is $\Phi$. Define $FX = \phi X$, for each $X \in \text{Ob} C$. For each $Y \in \text{Ob} C$, the map $F_Y : C(\cdot; Y) \to D(\cdot; \phi Y)$ is defined via the diagram

$$\begin{array}{ccc}
C(\cdot; Y) & \xrightarrow{F_Y} & D(\cdot; \phi Y) \\
\downarrow \Phi & & \downarrow D(\cdot; \phi Y) \\
C(1; Y) & \xrightarrow{\phi} & D(1; \phi Y) \\
\end{array}$$
Recall that for a morphism $f : () \to Y$ we denote by $\bar{f} : 1 \to Y$ a unique morphism such that $u \cdot \bar{f} = f$. Then the commutativity in the above diagram means that

$$Ff = [() \to 1 \xrightarrow{\phi^0} 1 \xrightarrow{\phi(f)} \phi Y],$$

for each $f : () \to Y$. For $n \geq 1$ and $X_1, \ldots, X_n, Y \in \text{Ob} \ C$, the map

$$F_{X_1,\ldots,X_n,Y} : C(X_1, \ldots, X_n; Y) \to D(\phi X_1, \ldots, \phi X_n; \phi Y)$$

is defined inductively by requesting the commutativity in the diagram

$$\begin{array}{ccc}
C(X_2, \ldots, X_n; C(X_1; Y)) & \xrightarrow{F_{X_2,\ldots,X_n,Y}} & D(\phi X_2, \ldots, \phi X_n; \phi C(X_1; Y)) \\
\varphi^C \downarrow & & \downarrow D(1; \phi) \\
C(X_1, \ldots, X_n; Y) & \xrightarrow{F_{X_1,\ldots,X_n,Y}} & D(\phi X_1, \ldots, \phi X_n; \phi Y)
\end{array}$$

5.3 Lemma. The following diagram commutes

$$\begin{array}{ccc}
C(\phi C(X; Y)) & \xrightarrow{F_{\phi C(X; Y)}} & D(\phi \phi C(X; Y)) \\
\varphi^C \downarrow & & \downarrow D(\phi) \\
C(X; Y) & \xrightarrow{\phi} & D(\phi X; \phi Y)
\end{array}$$

In particular, $F_{X,Y} = \phi_{X,Y} : C(X; Y) \to D(\phi X; \phi Y)$.

Proof. Equivalently, the exterior of the diagram

commutes. The upper pentagon is the definition of $F_{\phi C(X; Y)}$. The bottom hexagon commutes. Indeed, taking $f \in C(X; Y)$ and tracing it along the left-top path yields

$$\phi^0 \cdot \phi(j_X) \cdot \phi C(1; f) \cdot \hat{\phi} = \phi^0 \cdot (j_X) \cdot \hat{\phi} \cdot D(1; \phi(f)) \quad \text{(naturality of } \hat{\phi})$$

$$= j_{\phi X} \cdot D(1; \phi(f)), \quad \text{(axiom CF1)}$$

which is precisely the image of $f$ along the bottom-right path. \hfill \Box

5.4 Lemma. For each $f : () \to Y$ and $Z \in \text{Ob} \ C$, the diagram

$$\begin{array}{ccc}
\phi C(Y; Z) & \xrightarrow{\phi C(f; 1)} & \phi C(Z; Z) = \phi Z \\
\downarrow \phi & & \downarrow \phi \\
D(\phi Y; \phi Z) & \xrightarrow{D(F f; 1)} & D(\phi Z; \phi Z) = \phi Z
\end{array}$$

commutes.
Proof. By definition,
\[ Ff = \left[ (\cdot) \overset{\mu}{\rightarrow} 1 \overset{\phi^0}{\rightarrow} \phi 1 \overset{\phi(f)}{\rightarrow} \phi Y \right]. \]

The diagram
\[
\begin{array}{ccc}
\phi \mathbb{C}(Y; Z) & \xrightarrow{\phi \mathbb{C}(f; 1)} & \phi \mathbb{C}(1; Z) \\
& \phi \downarrow & \phi \downarrow \\
\mathbb{D}(\phi Y; \phi Z) & \xrightarrow{\mathbb{D}(\phi(f); 1)} & \mathbb{D}(\phi 1; \phi Z) \\
\end{array}
\]

commutes. Indeed, the left square commutes by the naturality of \( \hat{\phi} \), while the commutativity of the right square is a consequence of the axiom CF2, see (4.1). \( \square \)

With the notation of Lemma 3.11, we can rewrite the commutativity condition in diagram (5.2) as a recursive formula for the multigraph morphism \( F \):
\[ Ff = \varphi^D(F((\varphi^\mathbb{C})^{-1}(f)) \cdot \hat{\phi}) = \varphi^D(F(f) \cdot \hat{\phi}), \]
for each \( f : X_1, \ldots, X_n \rightarrow Y \) with \( n \geq 1 \), or equivalently
\[ \langle Ff \rangle = \left[ \phi X_2, \ldots, \phi X_n \xrightarrow{F(f)} \phi \mathbb{C}(X_1; Y) \xrightarrow{\hat{\phi}} \mathbb{D}(\phi X_1; \phi Y) \right]. \] (5.3)

5.5 Lemma. For each \( X, Y, Z \in \text{Ob} \mathbb{C} \), the diagram
\[
\begin{array}{ccc}
\phi \mathbb{C}(X; Y), \phi \mathbb{C}(Y; Z) & \xrightarrow{F\mu_\mathbb{C}} & \phi \mathbb{C}(X; Z) \\
& \hat{\phi}, \hat{\phi} \downarrow & \hat{\phi} \downarrow \\
\mathbb{D}(\phi X; \phi Y), \mathbb{D}(\phi Y; \phi Z) & \xrightarrow{\mu_\mathbb{D}} & \mathbb{D}(\phi X; \phi Z) \\
\end{array}
\]
commutes.

Proof. It suffices to prove the equation
\[ \langle F\mu_\mathbb{C} \cdot \hat{\phi} \rangle = \langle (\hat{\phi}, \hat{\phi}) \cdot \mu_\mathbb{D} \rangle. \]

By Lemma 3.11(c), the left hand side is equal to
\[ \phi \mathbb{C}(Y; Z) \xrightarrow{(\phi \mu_\mathbb{C})} \mathbb{D}(\phi \mathbb{C}(X; Y); \phi \mathbb{C}(X; Z)) \xrightarrow{\mathbb{D}(1; \hat{\phi})} \mathbb{D}(\phi \mathbb{C}(X; Y); \mathbb{D}(\phi X; \phi Z)), \]
while the right hand side is equal to
\[ \phi \mathbb{C}(Y; Z) \xrightarrow{\hat{\phi}} \mathbb{D}(\phi Y; \phi Z) \xrightarrow{\mu_\mathbb{D}} \mathbb{D}(\mathbb{D}(\phi X; \phi Y); \mathbb{D}(\phi Y; \phi Z)) \xrightarrow{\mathbb{D}(\hat{\phi}; 1)} \mathbb{D}(\phi \mathbb{C}(X; Y); \mathbb{D}(\phi X; \phi Z)) \]
by Lemma 3.11(b). Note that \( \langle \mu_\mathbb{D} \rangle = (\varphi^\mathbb{D})^{-1}(\mu_\mathbb{D}) = L^{\phi X} \). Furthermore, by (5.3),
\[ \langle F\mu_\mathbb{C} \rangle = \left[ \phi \mathbb{C}(Y; Z) \xrightarrow{\phi \mu_\mathbb{C}} \phi \mathbb{C}(\phi \mathbb{C}(X; Y); \mathbb{C}(X; Z)) \xrightarrow{\hat{\phi}} \mathbb{D}(\phi \mathbb{C}(X; Y); \phi \mathbb{C}(X; Z)) \right] \]
\[ = \left[ \phi \mathbb{C}(Y; Z) \xrightarrow{\phi L^X} \phi \mathbb{C}(\phi \mathbb{C}(X; Y); \phi \mathbb{C}(X; Z)) \xrightarrow{\hat{\phi}} \mathbb{D}(\phi \mathbb{C}(X; Y); \phi \mathbb{C}(X; Z)) \right], \]

therefore the equation in question is simply the axiom CF3. \( \square \)
5.6 Proposition. The multigraph morphism $F : C \to D$ is a multifunctor, and its underlying closed functor is $\Phi$.

Proof. Trivially, $F$ preserves identities since so does $\phi$. Let us prove that $F$ preserves composition. The proof is in three steps.

5.7 Lemma. $F$ preserves composition of the form $X_1, \ldots, X_k \xrightarrow{f} Y \xrightarrow{g} Z$.

Proof. The proof is by induction on $k$. There is nothing to prove in the case $k = 1$. Suppose that $k = 0$ and we are given composable morphisms

$$() \xrightarrow{f} X \xrightarrow{g} Y.$$

Then since $u \cdot f g = f \cdot g = (u \cdot f) \cdot g = u \cdot (f \cdot g)$, it follows that $f g = f \cdot g$. By formula (5.4),

$$F(f \cdot g) = u \cdot \phi^0 \cdot \phi(f \cdot g) = u \cdot \phi^0 \cdot \phi(f) \cdot \phi(g) = Ff \cdot Fg.$$

Suppose that $k > 1$. Then

$$\langle F(f \cdot g) \rangle = F(f \cdot g) \cdot \hat{\phi} \quad \text{(formula (5.3))}$$
$$= F(\langle f \rangle \cdot \mathcal{C}(1; g)) \cdot \hat{\phi} \quad \text{(Lemma 3.11(c))}$$
$$= F(f) \cdot \phi \mathcal{C}(1; g) \cdot \hat{\phi} \quad \text{(induction hypothesis)}$$
$$= F(f) \cdot \hat{\phi} \cdot D(1; \phi(g)) \quad \text{(naturality of $\hat{\phi}$)}$$
$$= \langle Ff \rangle \cdot D(1; Fg) \quad \text{(formula (5.3))}$$
$$= \langle Ff \cdot Fg \rangle, \quad \text{(Lemma 3.11(c))}$$

and induction goes through. \hfill \square

5.8 Lemma. $F$ preserves composition of the form $X_1^1, \ldots, X_1^{k_1}, X_2^1, \ldots, X_2^{k_2} \xrightarrow{f_1 \cdot f_2} Y_1, Y_2 \xrightarrow{g} Z$.

Proof. The proof is by induction on $k_1$. If $k_1 = 0$, then by Lemma 3.11(a),

$$(f_1, f_2) \cdot g = [X_2^1, \ldots, X_2^{k_2} \xrightarrow{f_2} Y_2 \xrightarrow{g} \mathcal{C}(Y_1; Z) \xrightarrow{\mathcal{C}(f_1; 1)} \mathcal{C}(Z) = Z],$$

therefore

$$F((f_1, f_2) \cdot g) = F(f_2) \cdot \phi(g) \cdot \phi \mathcal{C}(f_1; 1) \quad \text{(Lemma 5.7)}$$
$$= F(f_2) \cdot \phi(g) \cdot \hat{\phi} \cdot D(\phi(f_1); 1) \quad \text{(Lemma 5.4)}$$
$$= F(f_2) \cdot (Fg) \cdot D(Ff_1; 1) \quad \text{(formula (5.3))}$$
$$= (Ff_1, Ff_2) \cdot Fg. \quad \text{(Lemma 3.11(a))}$$

If $k_1 = 1$, then by Lemma 3.11(b),

$$\langle (f_1, f_2) \cdot g \rangle = [X_2^1, \ldots, X_2^{k_2} \xrightarrow{f_2} Y_2 \xrightarrow{g} \mathcal{C}(Y_1; Z) \xrightarrow{\mathcal{C}(f_1; 1)} \mathcal{C}(X_1^1; Z)],$$

therefore

$$\langle F((f_1, f_2) \cdot g) \rangle = F((f_1, f_2) \cdot g) \cdot \hat{\phi} \quad \text{(formula (5.3))}$$
$$= F(f_2) \cdot \phi(g) \cdot \phi \mathcal{C}(f_1; 1) \cdot \hat{\phi} \quad \text{(Lemma 5.7)}$$
$$= F(f_2) \cdot \phi(g) \cdot \hat{\phi} \cdot D(\phi(f_1); 1) \quad \text{(naturality of $\hat{\phi}$)}$$
$$= F(f_2) \cdot (Fg) \cdot D(Ff_1; 1) \quad \text{(formula (5.3))}$$
$$= \langle (Ff_1, Ff_2) \cdot Fg \rangle, \quad \text{(Lemma 3.11(b))}$$

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therefore
\[ \langle F((f_1, f_2) \cdot g) \rangle = F\langle (f_1, f_2) \cdot g \rangle \cdot \hat{\phi} \quad \text{(formula (5.3))} \]

and hence \( F((f_1, f_2) \cdot g) = (Ff_1, Ff_2) \cdot Fg \). Suppose that \( k_1 > 1 \). Then by Lemma 3.11(c) \( (f_1, f_2) \cdot g \) is equal to

\[ \left[ X_1^{2}, \ldots, X_1^{k_1}, X_1^{1}, \ldots, X_1^{k_2}, \ldots, X_2^{k_n} \right] \xrightarrow{f_1, \ldots, f_n} \mathcal{C}(X_1^{1}, Y_1), \mathcal{C}(Y_1; Z) \xrightarrow{\mu_{\mathcal{C}}}, \mathcal{C}(X_1^{1}; Z) \],

therefore

\[ F((f_1, f_2) \cdot g) = (Ff_1, Ff_2) \cdot Fg, \quad \text{and the lemma is proven.} \]

5.9 Lemma. \( F \) preserves composition of the form

\[ X_1^{1}, \ldots, X_1^{k_1}, \ldots, X_n^{1}, \ldots, X_n^{k_n} \xrightarrow{f_1, \ldots, f_n} Y_1, \ldots, Y_n \xrightarrow{g} Z. \] \hspace{1cm} (5.4)

Proof. The proof is by induction on \( n \), and for a fixed \( n \) by induction on \( k_1 \). We have worked out the cases \( n = 1 \) and \( n = 2 \) explicitly in Lemmas 5.7 and 5.8. Assume that \( F \) preserves an arbitrary composition of the form

\[ U_1^{1}, \ldots, U_1^{1}, \ldots, U_n^{1}, \ldots, U_n^{1} \xrightarrow{p_1, \ldots, p_n} V_1, \ldots, V_n \xrightarrow{g} W, \]

and suppose we are given composite (5.4). We do induction on \( k_1 \). If \( k_1 = 0 \), then by Lemma 3.11(a) \( (f_1, \ldots, f_n) \cdot g \) is equal to

\[ \left[ X_2^{1}, \ldots, X_2^{k_1}, \ldots, X_n^{1}, \ldots, X_n^{k_n} \xrightarrow{f_2, \ldots, f_n} Y_2, \ldots, Y_n \xrightarrow{g} \mathcal{C}(Y_1; Z) \right] \xrightarrow{\mu_{\mathcal{C}}}, \mathcal{C}(X_1^{1}; Z) = Z, \]

therefore

\[ F((f_1, \ldots, f_n) \cdot g) = (Ff_2, \ldots, Ff_n) \cdot F\langle (g) \cdot \mathcal{C}(Y_1; 1) \rangle \xrightarrow{\text{(induction hypothesis)}} \]

\[ = (Ff_2, \ldots, Ff_n) \cdot (Fg) \cdot \mathcal{C}(f_1; 1) \quad \text{(Lemma 5.7)} \]

\[ = (Ff_2, \ldots, Ff_n) \cdot (Fg) \cdot \mathcal{C}(f_1; 1) \quad \text{(Lemma 5.4)} \]

\[ = (Ff_2, \ldots, Ff_n) \cdot (Ff_1) \cdot Fg) \quad \text{(formula (5.3))} \]

\[ = (Ff_1, \ldots, Ff_n) \cdot Fg. \quad \text{(Lemma 3.11(a))} \]

Suppose that \( k_1 = 1 \). Then by Lemma 3.11(b) \( (f_1, \ldots, f_n) \cdot g \) is equal to

\[ \left[ X_2^{1}, \ldots, X_2^{k_1}, \ldots, X_n^{1}, \ldots, X_n^{k_n} \xrightarrow{f_2, \ldots, f_n} Y_2, \ldots, Y_n \xrightarrow{g} \mathcal{C}(Y_1; Z) \right] \xrightarrow{\mu_{\mathcal{C}}}, \mathcal{C}(X_1^{1}; Z) \],

therefore

\[ \langle F((f_1, \ldots, f_n) \cdot g) \rangle = F\langle (f_1, \ldots, f_n) \cdot g \rangle \cdot \hat{\phi} \quad \text{(formula (5.3))} \]

\[ = (Ff_2, \ldots, Ff_n) \cdot F\langle (g) \cdot \mathcal{C}(f_1; 1) \rangle \cdot \hat{\phi} \quad \text{(induction hypothesis)} \]

\[ = (Ff_2, \ldots, Ff_n) \cdot F\langle g \rangle \cdot \mathcal{C}(f_1; 1) \cdot \hat{\phi} \quad \text{(Lemma 5.7)} \]

\[ = (Ff_2, \ldots, Ff_n) \cdot F\langle g \rangle \cdot \mathcal{C}(f_1; 1) \quad \text{(Lemma 5.7)} \]

\[ = (Ff_2, \ldots, Ff_n) \cdot (Fg) \cdot \mathcal{D}(f_1; 1) \quad \text{(naturality of \( \hat{\phi} \))} \]

\[ = (Ff_2, \ldots, Ff_n) \cdot (Fg) \cdot \mathcal{D}(Ff_1; 1) \quad \text{(formula (5.3))} \]

\[ = \langle (Ff_1, \ldots, Ff_n) \cdot Fg \rangle, \quad \text{(Lemma 3.11(b))} \]
and hence \( F((f_1, \ldots, f_n) \cdot g) = (Ff_1, \ldots, Ff_n) \cdot Fg \). Suppose that \( k_1 > 1 \), then by Lemma 3.11(c) \(((f_1, \ldots, f_n) \cdot g)\) is equal to

\[
X_1^{k_1}, X_2^1, \ldots, X_2^{k_2}, \ldots, X_n^1, \ldots, X_n^{k_n} \xrightarrow{(f_1, f_2, \ldots, f_n)} \mathbb{C}(X_1^1; Y_1), Y_2, \ldots, Y_n \\
\xrightarrow{1/(g)} \mathbb{C}(X_1^1; Y_1), \mathbb{C}(Y_1; Z) \\
\xrightarrow{\mu_\square} \mathbb{C}(X_1^1; Z),
\]

therefore

\[
\langle F((f_1, \ldots, f_n) \cdot g) \rangle = F((f_1, \ldots, f_n) \cdot g) \cdot \hat{\phi} \tag{5.3}
\]

(formula (5.3))

\[
= (F(f_1), F(f_2), \ldots, F(f_n)) \cdot F((1, (g)) \mu_\square) \cdot \hat{\phi} \tag{induction hypothesis}
\]

\[
= (F(f_1), F(f_2), \ldots, F(f_n)) \cdot (1, F[g]) \cdot F \mu_\square \cdot \hat{\phi} \tag{Lemma 5.8}
\]

\[
= (F(f_1), F(f_2), \ldots, F(f_n)) \cdot (1, F(g) \cdot \hat{\phi}) \cdot \mu_\Box \tag{Lemma 5.3}
\]

\[
= ((Ff_1), F(f_2), \ldots, F(f_n)) \cdot (1, (Fg) \cdot \hat{\phi}) \cdot \mu_\Box \tag{formula (5.3)}
\]

\[
= \langle (Ff_1), F(f_2), \ldots, F(f_n) \rangle \cdot Fg, \tag{Lemma 3.11(c)}
\]

hence \( F((f_1, \ldots, f_n) \cdot g) = (Ff_1, \ldots, Ff_n) \cdot Fg \), and induction goes through. \( \square \)

Thus we have proven that \( F : \mathbb{C} \to \mathbb{D} \) is a multifunctor. By construction, its underlying functor is \( \phi \). Furthermore, the closing transformation \( F_{X,Y} \) coincides with \( \hat{\phi}_{X,Y} : \phi\mathbb{C}(X; Y) \to \mathbb{D}(\phi X; \phi Y) \). Indeed, notice that \( F_{X,Y} = \langle F\text{ev}^\mathbb{C} \rangle \), where \( \text{ev}^\mathbb{C} : X, \mathbb{C}(X; Y) \to Y \) is the evaluation morphism. Further, by formula (5.3),

\[
F_{X,Y} = \langle F\text{ev}^\mathbb{C} \rangle = \phi(\text{ev}^\mathbb{C}) \cdot \hat{\phi}_{X,Y} = \hat{\phi}_{X,Y},
\]

since \( \langle \text{ev}^\mathbb{C} \rangle = 1 : \mathbb{C}(X; Y) \to \mathbb{C}(X; Y) \). Finally,

\[
Fu = [(\cdot) \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} \hat{\phi}\mathbb{1}].
\]

Indeed, by formula (5.1),

\[
Fu = [(\cdot) \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1} \xrightarrow{\phi(\square)} \hat{\phi}\mathbb{1}] = [(\cdot) \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} \phi\mathbb{1}],
\]

since \( \square = 1 : \mathbb{1} \to \mathbb{1} \). Thus we conclude that \( F : \mathbb{C} \to \mathbb{D} \) is a multifunctor whose underlying closed functor is \( \Phi \). The proposition is proven. \( \square \)

### 5.10 The injectivity of \( U \) on 1-morphisms.

The following proposition shows that the \textbf{Cat}-functor \( U \) is injective on 1-morphisms.

#### 5.11 Proposition.

Let \( F, G : \mathbb{C} \to \mathbb{D} \) be multifunctors between closed multicategories with unit objects. Suppose that \( F \) and \( G \) induce the same closed functor \( \Phi = (\phi, \hat{\phi}, \phi^0) \) between the underlying closed categories. Then \( F = G \).

**Proof.** By assumption, the underlying functors of the multifunctors \( F \) and \( G \) are the same and are equal to the functor \( \phi \). Let us prove that \( Ff = Gf \), for each \( f : X_1, \ldots, X_n \to Y \). The proof is by induction on \( n \). There is nothing to prove if \( n = 1 \). Suppose that \( n = 0 \), i.e., \( f \) is a morphism \( () \to Y \). Then since \( F \) and \( G \) are multifunctors,

\[
Ff = F(u \cdot \mathcal{J}) = Fu \cdot F\mathcal{J}, \quad Gf = G(u \cdot \mathcal{J}) = Gu \cdot G\mathcal{J}.
\]

Since \( F \) and \( G \) coincide on morphisms with one source object, it follows that \( F\mathcal{J} = G\mathcal{J} \). Furthermore,

\[
Fu = [(\cdot) \xrightarrow{u} \mathbb{1} \xrightarrow{\phi^0} F\mathbb{1} = G\mathbb{1}] = Gu,
\]

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that the required equation follows then from the axiom CN2.

\[ V \text{ category in the sense of Eilenberg and Kelly; i.e., that } \]

\[ \text{The axiom CN2} \]

\[ F \to \text{to} \]

\[ \text{implies} \]

\[ \text{Let us prove that for each closed category} \]

\[ \text{By Lemma 3.11,} \]

\[ \text{holds true. The proof is by induction on } n. \text{ Suppose that } n = 0, \text{ and that } f \text{ is a morphism } (\cdot) \to Y. \]

\[ \text{The axiom CN1} \]

\[ \text{implies} \]

\[ \text{It follows that} \]

\[ \text{where the second equality is due to the naturality of } r. \text{ There is nothing to prove in the case } n = 1. \]

\[ \text{Suppose that } n > 1. \text{ It suffices to prove that} \]

\[ \text{By Lemma 3.11(c), the left hand side expands out as } (Ff) \cdot D(1; r_Y), \text{ which by formula (6.3)} \]

\[ \text{is equal to} \]

\[ \langle Ff \rangle \cdot r_{(X_1; Y)} \]

\[ \text{By Lemma 3.11(b), the right hand side of the equation in question is equal to} \]

\[ (r_{X_2}, \ldots, r_{X_n}) \cdot D(r_{X_1}; 1), \text{ which by formula (6.3)} \]

\[ \text{is equal to} \]

\[ (r_{X_2}, \ldots, r_{X_n}) \cdot D(r_{X_1}; 1). \]

\[ \text{By the induction hypothesis, the latter is equal to} \]

\[ Ff \cdot r_{(X_1; Y)} \cdot D(r_{X_1}; 1). \]

\[ \text{The required equation follows then from the axiom CN2.} \]

\[ 5.14 \text{ The essential surjectivity of } U. \text{ Let us prove that for each closed category } \]

\[ V \text{ there is a} \]

\[ \text{a closed multicategory } \]

\[ \text{with a unit object whose underlying closed category is isomorphic to } V. \text{ First} \]

\[ \text{of all, notice that by Theorem 2.19 we may (and we shall) assume in what follows that } V \text{ is a closed} \]

\[ \text{category in the sense of Eilenberg and Kelly; i.e., that } V \text{ is equipped with a functor} V : V \to S \]

\[ \text{such that} \]

\[ V V(-,-) = V(-,-) : V^{op} \times V \to S \text{ and the axiom CC5' is satisfied. In particular, we can} \]

\[ \text{use the whole theory of closed categories developed in [3] without any modifications. We are now} \]

\[ \text{going to construct a closed multicategory } V \text{ with a unit object whose underlying closed category is} \]

\[ \text{isomorphic to } V. \text{ The construction is based on ideas of Laplaza's paper [5].} \]

\[ \text{C}(X_2, \ldots, X_n; \overline{C}(X_1; Y)) \xrightarrow{F} \text{D}(\phi X_2, \ldots, \phi X_n; \phi \overline{C}(X_1; Y)) \]

\[ \varphi^c \]

\[ \text{D}(\phi X_2, \ldots, \phi X_n; D(\phi X_1; \phi Y)) \]

\[ \varphi^d \]

\[ \text{C}(X_1, \ldots, X_n; Y) \xrightarrow{F} \text{D}(\phi X_1, \ldots, \phi X_n; \phi Y) \]

and a similar diagram for \( G \), which are particular cases of Proposition 3.17. \( \boxdot \)

\[ 5.12 \text{ The bijectivity of } U \text{ on 2-morphisms.} \text{ The following proposition implies that } U \text{ is bijec-} \]

\[ \text{tive on 2-morphisms.} \]

\[ 5.13 \text{ Proposition. Let } F, G : C \to D \text{ be multifunctors between closed multicategories with unit} \]

\[ \text{objects. Denote by } \Phi = (\phi, \hat{\phi}, \phi^0) \text{ and } \Psi = (\psi, \hat{\psi}, \psi^0) \text{ the corresponding closed} \]

\[ \text{functors. Let } \hat{r} : \Phi \to \Psi \text{ be a closed natural transformation. Then } \hat{r} \text{ is also a} \]

\[ \text{multinatural transformation } F \to G : C \to D. \]

\[ \text{Proof. We must prove that, for each } f : X_1, \ldots, X_n \to Y, \text{ the equation} \]

\[ \begin{align*}
  Ff \cdot r_Y &= (r_{X_1}, \ldots, r_{X_n}) \cdot Gf \\

  \text{holds true. The proof is by induction on } n. \text{ Suppose that } n = 0, \text{ and that } f \text{ is a morphism } (\cdot) \to Y. \end{align*} \]

\[ \text{The axiom CN1} \]

\[ \begin{align*}
  \left[ 1 \xrightarrow{\phi^0} F1 \xrightarrow{\hat{r}} G1 \right] &= \psi^0 \\\n
  \left[ (\cdot) \xrightarrow{Fu} F1 \xrightarrow{\hat{r}} G1 \right] &= Gu. \end{align*} \]

\[ \text{It follows that} \]

\[ \begin{align*}
  Ff \cdot r_Y &= Fu \cdot \hat{f} \cdot r_Y = Fu \cdot r_1 \cdot \hat{f} = Gu \cdot \hat{f} = Gf, \end{align*} \]

\[ \text{where the second equality is due to the naturality of } r. \text{ There is nothing to prove in the case } n = 1. \]

\[ \text{Suppose that } n > 1. \text{ It suffices to prove that} \]

\[ \langle Ff \rangle \cdot r_{(X_1; Y)} = (r_{X_1}, \ldots, r_{X_n}) \cdot Gf : FX_2, \ldots, FX_n \to D(FX_1; GY). \]

\[ \text{By Lemma 3.11(c), the left hand side expands out as } (Ff) \cdot D(1; r_Y), \text{ which by formula (6.3)} \]

\[ \text{is equal to} \]

\[ (r_{X_2}, \ldots, r_{X_n}) \cdot D(r_{X_1}; 1), \text{ which by formula (6.3)} \]

\[ \text{is equal to} \]

\[ (r_{X_2}, \ldots, r_{X_n}) \cdot D(r_{X_1}; 1). \]

\[ \text{By the induction hypothesis, the latter is equal to} \]

\[ F\langle f \rangle \cdot r_{(X_1; Y)} \cdot \hat{\psi} \cdot D(r_{X_1}; 1). \]

\[ \text{The required equation follows then from the axiom CN2.} \]
We recall that for each object $X$ of the category $\mathcal{V}$ we have a $\mathcal{V}$-functor $L^X: \mathcal{V} \to \mathcal{V}$, and for each $f \in \mathcal{V}(X, Y)$ there is a $\mathcal{V}$-natural transformation $L^f: L^Y \to L^X: \mathcal{V} \to \mathcal{V}$, uniquely determined by the condition $(V(L^f)_Y)_1 = f$, see Examples 2.13, 2.15, 2.21, or [2, Section 9]. Moreover, by [2, Proposition 9.2] the assignments $X \mapsto L^X$ and $f \mapsto L^f$ determine a fully faithful functor from the category $\mathcal{V}^{\text{op}}$ to the category $\mathcal{V}\text{-Cat}(\mathcal{V}, \mathcal{V})$ of $\mathcal{V}$-functors $\mathcal{V} \to \mathcal{V}$ and their $\mathcal{V}$-natural transformations. For us it is more convenient to write it as a $\mathcal{V}$-functor from $\mathcal{V}$ to $\mathcal{V}\text{-Cat}(\mathcal{V}, \mathcal{V})^{\text{op}}$. Note that the latter category is strict monoidal with the tensor product given by composition of $\mathcal{V}$-functors. More precisely, the tensor product of $F$ and $G$ in the given order is $FG = F \circ G = G \circ F$. Consider the multicategory associated with $\mathcal{V}\text{-Cat}(\mathcal{V}, \mathcal{V})^{\text{op}}$ (see Example 3.3) and consider its full submulticategory whose objects are $\mathcal{V}$-functors $L^X$, $X \in \text{Ob } \mathcal{V}$. That is, in essence, our $\mathcal{V}$. More precisely, $\text{Ob } \mathcal{V} = \text{Ob } \mathcal{V}$ and

$$\mathcal{V}(X_1, \ldots, X_n; Y) = \mathcal{V}\text{-Cat}(\mathcal{V}, \mathcal{V})^{\text{op}}(L^{X_1}, \ldots, L^{X_n}, L^Y) = \mathcal{V}\text{-Cat}(\mathcal{V}, \mathcal{V})(L^Y, L^{X_n} \circ \cdots \circ L^{X_1}).$$

Identities and composition coincide with those of the multicategory associated with the strict monoidal category $\mathcal{V}\text{-Cat}(\mathcal{V}, \mathcal{V})^{\text{op}}$. Note that by Proposition 2.20 there is a bijection

$$\Gamma: \mathcal{V}(X_1, \ldots, X_n; Y) \to (V \circ L^{X_n} \circ \cdots \circ L^{X_1})_Y, \quad f \mapsto (Vf)_Y 1_Y.$$

5.15 Theorem. The multicategory $\mathcal{V}$ is closed and has a unit object. The underlying closed category of $\mathcal{V}$ is isomorphic to $\mathcal{V}$.

Proof. First, let us check that the multicategory $\mathcal{V}$ is closed. By Proposition 3.3, it suffices to prove that for each pair of objects $X$ and $Z$ there exist an internal Hom-object $\mathcal{V}(X; Z)$ and an evaluation morphism $\text{ev}_{X,Z}^\mathcal{V}: \mathcal{V}(X; Z) \to Z$ such that the map

$$\varphi: \mathcal{V}(Y_1, \ldots, Y_n; \mathcal{V}(X; Z)) \to \mathcal{V}(X, Y_1, \ldots, Y_n; Z), \quad f \mapsto (1_X, f) \cdot \text{ev}_{X,Z}^\mathcal{V},$$

is bijective, for each sequence of objects $Y_1, \ldots, Y_n$. We set $\mathcal{V}(X; Z) = \mathcal{V}(X, Z)$. The evaluation map $\text{ev}_{X,Z}^\mathcal{V}: X, \mathcal{V}(X; Z) \to Z$ is by definition a $\mathcal{V}$-natural transformation $L^Z \to L^{\mathcal{V}(X; Z)} \circ L^X$. We define it by requesting $(V(\text{ev}_{X,Z}^\mathcal{V})_Z)_1 = 1_{\mathcal{V}(X; Z)}$ (we extensively use the representation theorem for $\mathcal{V}$-functors in the form of Proposition 2.20). Let us check that the map $\varphi$ is bijective. Note that the codomain of $\varphi$ identifies via the map $\Gamma$ with the set $(V \circ L^{Y_n} \circ \cdots \circ L^{Y_1} \circ L^X)_Z$, and that the domain of $\varphi$ identifies via $\Gamma$ with the set

$$(V \circ L^{Y_n} \circ \cdots \circ L^{Y_1})_Z = (V \circ L^{Y_n} \circ \cdots \circ L^{Y_1} \circ L^X)_Z.$$

The bijectivity of $\varphi$ follows readily from the diagram

$$\begin{array}{ccc}
\mathcal{V}(Y_1, \ldots, Y_n; \mathcal{V}(X; Z)) & \xrightarrow{\varphi} & \mathcal{V}(X, Y_1, \ldots, Y_n; Z) \\
\downarrow{\Gamma} & & \downarrow{\Gamma} \\
(V \circ L^{Y_n} \circ \cdots \circ L^{Y_1} \circ L^X)_Z & & (V \circ L^{Y_n} \circ \cdots \circ L^{Y_1} \circ L^X)_Z
\end{array}$$

whose commutativity we are going to establish. Indeed, take an element $f \in \mathcal{V}(Y_1, \ldots, Y_n; \mathcal{V}(X; Z))$, i.e., a $\mathcal{V}$-natural transformation $f: L^{\mathcal{V}(X; Z)} \to L^{Y_n} \circ \cdots \circ L^{Y_1}$. Then $\varphi(f)$ is given by the composite

$$L^Z \xrightarrow{\text{ev}_{X,Z}^\mathcal{V}} L^{\mathcal{V}(X; Z)} \circ L^X \xrightarrow{fL^X} L^{Y_n} \circ \cdots \circ L^{Y_1} \circ L^X.$$

Therefore, $\Gamma \varphi(f)$ is equal to

$$(V((fL^X) \circ \text{ev}_{X,Z}^\mathcal{V})_Z)_1 = (V(fL^X)_Z)(V \text{ev}_{X,Z}^\mathcal{V})_1 = (V(fL^X)_Z)(V \text{ev}_{X,Z}^\mathcal{V})_1 = (Vf)_Z(1_{\mathcal{V}(X; Z)}) = \Gamma(f).$$

Thus we conclude that $\mathcal{V}$ is a closed multicategory.
Let us check that \( \mathbf{1} \in \text{Obj} \mathcal{V} \) is a unit object of \( \mathcal{V} \). By definition, a morphism \( u : () \rightarrow \mathbf{1} \) is a \( \mathcal{V} \)-natural transformation \( L^\mathbf{1} \rightarrow \text{Id} \). We let it be equal to \( i^{-1}_X \), which is a \( \mathcal{V} \)-natural transformation by \([3\text{ Prop. } 8.5]\). Then for each object \( X \) of \( \mathcal{V} \) holds
\[
\mathcal{V}(u; 1) = (u, 1) \cdot \text{ev}^\mathcal{V}_{\mathbf{1}, X} : \mathcal{V}(\mathbf{1}; X) \rightarrow X,
\]
i.e., \( \mathcal{V}(u; 1) \) is the \( \mathcal{V} \)-natural transformation
\[
L^X \xrightarrow{\text{ev}^\mathcal{V}_{\mathbf{1}, X}} L^\mathcal{V}(\mathbf{1}; X) \circ L^\mathbf{1} \xrightarrow{L^\mathcal{V}(u; X)} L^\mathcal{V}(\mathbf{1}, X).
\]
We claim that it coincides with \( L^{i^*_X} \) and hence is invertible. Indeed, applying \( \Gamma \) to the above composite we obtain
\[
(V((L^\mathcal{V}(\mathbf{1}, X) u) \circ \text{ev}^\mathcal{V}_{\mathbf{1}, X})_X) 1_X = (V(L^\mathcal{V}(\mathbf{1}, X) u)_X)(V(\text{ev}^\mathcal{V}_{\mathbf{1}, X})_X) 1_X = \mathcal{V}(\mathcal{V}(\mathbf{1}, X), u_X) 1_{\mathcal{V}(\mathbf{1}, X)} = u_X = i^{-1}_X.
\]
Let us now describe the underlying closed category of the closed multicategory \( \mathcal{V} \). Its objects are those of \( \mathcal{V} \), and for each pair of objects \( X \) and \( Y \) the set of morphisms from \( X \) to \( Y \) is \( \mathcal{V}(X; Y) = \mathcal{V}\text{-Cat}(\mathcal{V}(X), Y, L^X) \). The unit object is \( \mathbf{1} \) and the internal Hom-object \( \mathcal{V}(X; Y) \) coincides with \( \mathcal{V}(X, Y) \). For each object \( X \), the identity morphism \( 1^X_X : () \rightarrow \mathcal{V}(X; X) \), i.e., a \( \mathcal{V} \)-natural transformation \( L^\mathcal{V}(X; X) \rightarrow \text{Id} \), is found from the equation
\[
[X \xrightarrow{1_X} X] \xrightarrow{\mathcal{V}(X; X)} X = [\mathcal{V}(X; X) \xrightarrow{\text{ev}^\mathcal{V}_{X, X}} X] = 1_X,
\]
or equivalently from the equation
\[
[L^X \xrightarrow{\text{ev}^\mathcal{V}_{X, X}} L^\mathcal{V}(X, X) \circ L^X] \xrightarrow{id} L^X = [L^\mathcal{V}(X, X) \xrightarrow{j_X} L^\mathbf{1} \xrightarrow{u} \text{Id}] = 1^X_X.
\]
Applying \( \Gamma \) to both sides we find that
\[
(V((1^X_X L^X) \circ \text{ev}^\mathcal{V}_{X, X})_X) 1_X = (V(1^X_X)_{\mathcal{V}(X, X)}) (V(\text{ev}^\mathcal{V}_{X, X})_X) 1_X = V(1^X_X)_{\mathcal{V}(X, X)} 1_{\mathcal{V}(X, X)} = 1_X.
\]
Here \( V(1^X_X)_{\mathcal{V}(X, X)} : \mathcal{V}(\mathcal{V}(X, X), \mathcal{V}(X, X)) \rightarrow \mathcal{V}(X, X) \). The morphism \( j_X \) of the underlying closed category of \( \mathcal{V} \) is a \( \mathcal{V} \)-natural transformation \( L^\mathcal{V}(X, X) \rightarrow L^\mathbf{1} \); it is found from the equation
\[
[L^\mathcal{V}(X, X) \xrightarrow{j_X} L^\mathbf{1} \xrightarrow{u} \text{Id}] = 1^X_X.
\]
Applying \( \Gamma \) to both sides we obtain
\[
(V(u \circ j_X)_{\mathcal{V}(X, X)}) 1_{\mathcal{V}(X, X)} = V(1^X_X)_{\mathcal{V}(X, X)} 1_{\mathcal{V}(X, X)},
\]
i.e.,
\[
(Vi^{-1}_{\mathcal{V}(X, X)})(V(j_X)_{\mathcal{V}(X, X)} 1_{\mathcal{V}(X, X)}) = 1_X,
\]
or equivalently
\[
(V(j_X)_{\mathcal{V}(X, X)}) 1_{\mathcal{V}(X, X)} = (Vi_{\mathcal{V}(X, X)}) 1_X = j_X,
\]
where the last equality is the axiom CC5’. Therefore, \( j_X = L^{j_X} : L^\mathcal{V}(X, X) \rightarrow L^\mathbf{1} \). It follows from the construction that \( i_X \) for the underlying closed category of \( \mathcal{V} \) is \( (\mathcal{V}(u; 1))^{-1} = (L^{i_X})^{-1} = L^{j_X} \).

Let us compute the morphism \( L^X_{Y, Z} : \mathcal{V}(Y; Z) \rightarrow \mathcal{V}(\mathcal{V}(X; Y); \mathcal{V}(X; Z)) \). Before we do that, note that \( \text{ev}^\mathcal{V}_{X, Y} : X, \mathcal{V}(X; Y) \rightarrow Y \) is the \( \mathcal{V} \)-natural transformation \( L^X \rightarrow L^\mathcal{V}(X, Y) \circ L^X \) with components
\[
(\text{ev}^\mathcal{V}_{X, Y})_Z = L^X_{Y, Z} : \mathcal{V}(Y; Z) \rightarrow \mathcal{V}(\mathcal{V}(X, Y), \mathcal{V}(X; Z)).
\]
In other words, \( \text{ev}^\mathcal{V}_{X, Y} = L^X_{Y, -} \). Indeed, applying \( \Gamma \) to both side of the equation in question we obtain an equivalent equation
\[
(V(\text{ev}^\mathcal{V}_{X, Y})_Y) 1_Y = (V L^X_{Y, Y}) 1_Y.
\]
Since \( VL^X = V(X, -) \), it follows that \((V L^X)_Y = L^X_{Y,Y}\), so that the obtained equation is just the definition of \( ev^X_{Y,Y} \).

The morphism \( L^X_{Y,Z} : V(Y; Z) \to V(V(X; Y); V(X; Z)) \) is found from the equation

\[
[X, V(X; Y), V(Y; Z)] \xrightarrow{1_1, L^X_{Y,Z}} X, V(X; Y), V(V(X; Y); V(X; Z)) \xrightarrow{1_{ev^X_{Y(Y,Y)}}, 1_{ev^Y_{X,X}} \circ 1_{L^X_{Y,Z}}} X, V(X; Z) \xrightarrow{ev^X_{Y,Z}} Z,
\]

or equivalently

\[
[L^Z \circ L^X, L^X \circ L^X \circ L^X \circ L^X \circ L^X] = [L^Z, L^Z \circ L^X, L^X] = [L^Z, L^Z, L^X, L^X, L^X].
\]

Applying \( \Gamma \) to both side of the above equation we obtain

\[
(V(L^X_{Y,Z}) \circ V(X, Y, Z)) (V(ev^X_{Y(Y,Y)}, V(X; Z)) (V(ev^Y_{X,X}) \circ V(Y; Z))) 1_Z
\]

or equivalently

\[
(V(L^X_{Y,Z}) \circ V(X, Y, Z)) 1_{V(X, Y, Z)} = (ev^Y_{X,X}) 1_{V(X; Z)},
\]

In other words, \( L^X_{Y,Z} \) for the underlying closed category of \( V \) is \( L^Y_{Y,Z} \).

Let us denote the underlying closed category of the multicategory \( V \) by the same symbol. There is a closed functor \( (L, 1, 1) : V \to V \), where \( L : V \to V \) is given by \( X \mapsto X, f \mapsto Lf \), and the morphisms \( V(X, Y) \to V(X; Y) \) and \( 1 \mapsto 1 \) are the identities. The axioms CF1–CF3 follow readily from the above description of the closed category \( V \). Clearly, the functor \( L \) is an isomorphism. The theorem is proven.

\[\square\]

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