REGULARITY OF SYMBOLIC POWERS OF COVER IDEALS OF GRAPHS

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Abstract. Let $G$ be a graph which belongs to either of the following classes: (i) bipartite graphs, (ii) unmixed graphs, or (iii) claw–free graphs. Assume that $J(G)$ is the cover ideal $G$ and $J(G)^{(k)}$ is its $k$-th symbolic power. We prove that

$$k\deg(J(G)) \leq \text{reg}(J(G)^{(k)}) \leq (k - 1)\deg(J(G)) + |V(G)| - 1.$$ 

We also determine families of graphs for which the above inequalities are equality.

1. Introduction

Let $I$ be a homogeneous ideal in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$. Suppose that the minimal free resolution of $I$ is given by

$$0 \rightarrow \cdots \rightarrow \bigoplus_j S(-j)^{\beta_{1,j}(I)} \rightarrow \bigoplus_j S(-j)^{\beta_{0,j}(I)} \rightarrow I \rightarrow 0.$$

The Castelnuovo-Mumford regularity (or simply, regularity) of $I$, denoted by $\text{reg}(I)$, is defined as

$$\text{reg}(I) = \max\{j - i \mid \beta_{i,j}(I) \neq 0\},$$

and is an important invariant in commutative algebra and algebraic geometry.

Computing and finding bounds for the regularity of powers of a monomial ideal have been studied by a number of researchers (see for example [1], [2], [3], [4], [5], [7], [11], [14], [15]). This work is motivated by a recent paper of Hang and Trung [12]. In that paper, the authors study the regularity of powers of cover ideals of the so-called unimodular hypergraphs. It is well-known that the class of unimodular hypergraphs includes the family of bipartite graphs. Restricting to this family of graphs, their Theorem 3.3 (see also [12, Corollary 3.4]) says that if $G$ is a bipartite graph with cover ideal $J(G)$ (see Definition 2.1), then there is a non-negative integer $e \leq |V(G)| - \deg(J(G)) - 1$ such that

$$\text{reg}(J(G)^{(k)}) = k\deg(J(G)) + e,$$

for every integer $k \geq |V(G)| + 2$. Here, $\deg(J(G))$ denotes the maximum degree of the minimal monomial generators of $J(G)$. Consequently, for every integer $k \geq |V(G)| + 2$, we have

$$\text{reg}(J(G)^{(k)}) \leq (k - 1)\deg(J(G)) + |V(G)| - 1.$$

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It is natural to ask whether the above inequality is valid for every non-negative integer \( k \). In Theorem 3.4 we give a positive answer to this question. Note that by [9, Corollary 2.6], the ordinary and the symbolic powers of the cover ideal of a bipartite graph are the same. Therefore, it is reasonable to study the regularity of symbolic powers of cover ideals. This will be done in Section 3. The most general result of this paper is Theorem 3.2. Its statement is as follows. Let \( H \) be a family of graphs such that (i) for every graph \( G \in H \) and every vertex \( x \in V(G) \), the graph \( G \setminus N_G[x] \) belongs to \( H \), and (ii) every \( G \in H \) which has no isolated vertex, admits a minimal vertex cover with cardinality at least \( \frac{|V(G)|}{2} \). We show in Theorem 3.2 that for every graph \( G \in H \) and every integer \( k \geq 1 \),
\[
(1) \quad k \deg(J(G)) \leq \text{reg}(J(G)^{(k)}) \leq (k-1)\deg(J(G)) + |V(G)| - 1.
\]
It is easy to see that the class of bipartite graphs satisfies the assumption of Theorem 3.2, which implies the above mentioned result (see Theorem 3.4). It follows from [10] that the class of unmixed graphs also satisfies the assumption of Theorem 3.2. Hence, the inequalities (1) are true for every unmixed graph too (see Theorem 3.6). In Theorem 3.7, we show that the class of claw-free graphs also satisfies the assumption of Theorem 3.2. This means that the inequalities (1) are true for claw-free graphs too. In Corollaries 3.5 and 3.8, we determine the families of graphs for which the inequalities (1) are equality, showing that these inequalities are sharp.

2. Preliminaries

In this section, we provide the definitions and basic facts which will be used in the next section. We refer the reader to [13] for undefined terminologies.

Let \( G \) be a simple graph with vertex set \( V(G) = \{x_1, \ldots, x_n\} \) and edge set \( E(G) \) (by abusing the notation, we identify the vertices of \( G \) with the variables of \( S \)). For a vertex \( x_i \), the neighbor set of \( x_i \) is \( N_G(x_i) = \{x_j \mid \{x_i, x_j\} \in E(G)\} \) and we set \( N_G[x_i] = N_G(x_i) \cup \{x_i\} \) and call it the closed neighborhood of \( x_i \). The degree of \( x_i \), denoted by \( \deg_G(x_i) \), is the cardinality of \( N_G(x_i) \). A vertex of degree one is called a leaf. An edge \( e \in E(G) \) is a pendant edge, if it is incident to a leaf. For every subset \( A \subset V(G) \), the graph \( G \setminus A \) is the graph with vertex set \( V(G) \setminus A \) and edge set \( E(G \setminus A) = \{e \in E(G) \mid e \cap A = \emptyset\} \). A subset \( W \) of \( V(G) \) is said to be an independent subset of \( G \) if there are no edges among the vertices of \( W \). The cardinality of the largest independent subset of \( G \) is the independence number of \( G \) and is denoted by \( i(G) \). A subset \( C \) of \( V(G) \) is called a vertex cover of \( G \) if every edge of \( G \) is incident to at least one vertex of \( C \). A vertex cover \( C \) is called a minimal vertex cover of \( G \) if no proper subset of \( C \) is a vertex cover of \( G \). Note that \( C \) is a minimal vertex cover if and only if \( V(G) \setminus C \) is a maximal independent set. The graph \( G \) is called unmixed if all minimal vertex covers of \( G \) have the same cardinality. The graph \( G \) is said to be complete if each pair of vertices of \( G \) are adjacent by an edge. The graph \( G \) is bipartite if there exists a partition \( V(G) = A \cup B \) such that each edge of \( G \) is of the form \( \{x_i, x_j\} \) with \( x_i \in A \) and \( x_j \in B \). If moreover, every vertex of \( A \) is adjacent to every vertex of \( B \), then we say that \( G \) is a complete bipartite graph and...
denote it by $K_{a,b}$, where $a = |A|$ and $b = |B|$. The graph $K_{1,3}$ is called a *claw* and the graph $G$ is said to be *claw-free* if it has no claw as an induced subgraph.

We now define the main objective of this paper.

**Definition 2.1.** Let $G$ be a graph with $n$ vertices. The *cover ideal* of $G$, denoted by $J(G)$ is a squarefree monomial ideal of $S$ which is defined as follows.

$$J(G) = \left( \prod_{x_i \in C} x_i \mid C \text{ is a minimal vertex cover of } G \right)$$

It is well-known that

$$J(G) = \bigcap \{x_i, x_j\} \in E(G) (x_i, x_j).$$

In other words, $J(G)$ is the Alexander dual of the so-called edge ideal of $G$ (see [13, Section 9.1.1] for more details).

Assume that $I$ is an ideal of $S$ and Min$(I)$ is the set of minimal primes of $I$. For every integer $k \geq 1$, the $k$-th *symbolic power* of $I$, denoted by $I^{(k)}$, is defined to be

$$I^{(k)} = \bigcap_{p \in \text{Min}(I)} \text{Ker}(R \to (R/I^k)_p).$$

Let $I$ be a squarefree monomial ideal with irredundant primary decomposition

$$I = p_1 \cap \ldots \cap p_r,$$

where every $p_i$ is a prime ideal generated by a subset of the variables. It follows from [13, Proposition 1.4.4] that for every integer $k \geq 1$,

$$I^{(k)} = p_1^k \cap \ldots \cap p_r^k.$$

In particular, for every graph $G$, we have

$$J(G)^{(k)} = \bigcap_{\{x_i, x_j\} \in E(G)} (x_i, x_j)^k,$$

for every integer $k \geq 1$.

For a monomial ideal $I$, we denote the set of its minimal monomial generators by $G(I)$. The *degree* of $I$, denoted by deg$(I)$ is the maximum degree of elements of $G(I)$. Thus, in particular, deg$(J(G))$ is the cardinality of the largest minimal vertex cover of the graph $G$.

### 3. Main Results

In this section, we prove the main results of this paper. Namely, we show in Theorem 3.2 that for certain classes of graphs, including bipartite graphs, unmixed graphs and claw-free graphs, the inequalities hold. The first inequality is indeed true if one replaces $J(G)$ by any arbitrary squarefree monomial ideal. The proof of this assertion is simple and it follows immediately from the following lemma.
Lemma 3.1. Let $I$ be a squarefree monomial ideal of $S$. For every integer $k \geq 1$, we have \( \deg(I^{(k)}) \geq k \deg(I) \). 

Proof. Let $I = p_1 \cap \ldots \cap p_r$, be the irredundant primary decomposition of $I$. Choose a squarefree monomial $u \in G(I)$ with $\deg(u) = \deg(I)$. Notice that $u^k \in I^k \subseteq I^{(k)}$.

As $u \in G(I)$, for every $1 \leq i \leq n$, there exists an integer $1 \leq j \leq r$ such that $u/x_i \notin p_j$. Thus $(u/x_i)^k \notin p_j$. Since $p_j$ is a $p_j$-primary ideal and $x_i^{k-1} \notin p_j$, we conclude that $u^k/x_i = (u/x_i)^k x_i^{k-1} \notin p_j$. Consequently, $u^k/x_i \notin I^{(k)}$, for every integer $i$ with $1 \leq i \leq n$. Thus, $u^k$ belongs to the set of minimal monomial generators of $I^{(k)}$. Hence, 

\[
\deg(I^{(k)}) \geq \deg(u^k) = k \deg(I).
\]

We are now ready to prove the first main result of this paper.

Theorem 3.2. Let $\mathcal{H}$ be a family of graphs which satisfies the following conditions.

(i) For every graph $G \in \mathcal{H}$ and every vertex $x \in V(G)$, the graph $G \setminus N_G[x]$ belongs to $\mathcal{H}$.

(ii) If $G \in \mathcal{H}$ has no isolated vertex, then it admits a minimal vertex cover with cardinality at least $\frac{|V(G)|}{2}$.

Then for every graph $G \in \mathcal{H}$ and every integer $k \geq 1$, we have 

\[
k \deg(J(G)) \leq \reg(J(G)^{(k)}) \leq (k-1) \deg(J(G)) + |V(G)| - 1.
\]

Proof. The first inequality is an immediate consequence of Lemma 3.1. Therefore, we prove the second inequality. Equivalently, we prove that 

\[
\reg(S/J(G)^{(k)}) \leq (k-1) \deg(J(G)) + n - 2,
\]

where $n = |V(G)|$. By replacing $\mathcal{H}$ with $\mathcal{H} \cup \{K_2\}$, we may assume that $K_2 \in \mathcal{H}$. Let $m$ be the number of edges of $G$. We prove the assertions by induction on $m + k$.

By (i), for every graph $G \in \mathcal{H}$, the graph obtained from by $G$ deleting its isolated vertices belongs to $\mathcal{H}$. Thus, we can assume that $G$ has no isolated vertex. The assertion is well-known for $k = 1$ (it follows, for example, by looking at the Taylor resolution of $J(G)$). If $m = 1$, then $G = K_2$. In this case $J(G) = (x_1, x_2)$. Hence, $\deg(J(G)) = 1$ and $\reg(J(G)^{(k)}) = k$. Thus, the desired inequality is true for $m = 1$. Therefore, assume that $k, m \geq 2$. Let $S_1 = \mathbb{K}[x_2, \ldots, x_n]$ be the polynomial ring obtained from $S$ by deleting the variable $x_1$ and consider the ideals $J_1 = J(G)^{(k)} \cap S_1$ and $J_1' = (J(G)^{(k)} : x_1)$. It follows from [8] Lemma 2.10 that

\[
(2) \quad \reg(S/J(G)^{(k)}) \leq \max\{\reg_{S_1}(S_1/J_1), \reg_{S}(S/J'_1) + 1\},
\]

Set $u_1 = \prod_{x_j \in N_G(x_1)} x_j \in S_1$. Hence, $\deg(u_1) = \deg_G(x_1)$ and by [17, Lemma 2.2],

\[
J(G) \cap S_1 = u_1 J(G \setminus N_G[x_1]) S_1.
\]

It then follows that 

\[
J_1 = J(G)^{(k)} \cap S_1 = (J(G) \cap S_1)^{(k)} = u_1^k J(G \setminus N_G[x_1])^{(k)} S_1.
\]
Notice that if $C$ is a minimal vertex cover of $G \setminus N_G[x_1]$, then $C \cup N_G(x_1)$ is a minimal vertex cover of $G$. This shows that
\[ \deg(J(G \setminus N_G[x_1])) + \deg_G(x_1) \leq \deg(J(G)). \]
On the other hand, [18, Lemma 4.1] implies that $\text{reg}(J(G) \cap S_1) \leq \text{reg}(J(G))$ and therefore,
\[ \text{reg}_{S_1}(S_1/J(G \setminus N_G[x_1])S_1) \leq \text{reg}(S/J(G)) - \deg(u_1). \]
Since $G \setminus N_G[x_1] \in \mathcal{H}$, the induction hypothesis implies that
\[
\text{reg}_{S_1}(S_1/J_1) = \text{reg}_{S_1}(S_1/J(G \setminus N_G[x_1])^{(k)}S_1) + k\deg(u_1)
\leq (k - 1)\deg(J(G \setminus N_G[x_1])) + |V(G \setminus N_G[x_1])| - 2 + k\deg_G(x_1)
\leq (k - 1)(\deg(J(G)) - \deg_G(x_1)) + n - \deg_G(x_1) - 1 - 2 + k\deg_G(x_1)
< (k - 1)\deg(J(G)) + n - 2.
\]
Thus, using the inequality (2), it is enough to prove that
\[ \text{reg}_S(S/J_i) \leq (k - 1)\deg(J(G)) + n - 3. \]
For every integer $i$ with $2 \leq i \leq n$, let $S_i = \mathbb{K}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ be the polynomial ring obtained from $S$ by deleting the variable $x_i$ and consider the ideals $J'_i = (J_{i-1} : x_i)$ and $J_i = J_{i-1} \cap S_i$.

**Claim.** For every integer $i$ with $1 \leq i \leq n - 1$ we have
\[ \text{reg}(S/J'_i) \leq \max\{(k - 1)\deg(J(G)) + n - 3, \text{reg}_S(S/J'_{i+1}) + 1\}. \]

**Proof of the Claim.** For every integer $i$ with $1 \leq i \leq n - 1$, we know from [8, Lemma 2.10] that
\[ \text{reg}(S/J'_i) \leq \max\{\text{reg}_{S_{i+1}}(S_{i+1}/J_{i+1}), \text{reg}_S(S/J'_{i+1}) + 1\}. \]
Notice that for every integer $i$ with $1 \leq i \leq n - 1$, we have $J'_i = (J(G)^{(k)} : x_1x_2\ldots x_i)$. Thus,
\[ J_{i+1} = J'_i \cap S_{i+1} = (J(G)^{(k)} \cap S_{i+1}) : S_{i+1} x_1x_2\ldots x_i. \]
Hence, it follows from [18, Lemma 4.2] that
\[ \text{reg}_{S_{i+1}}(S_{i+1}/J_{i+1}) \leq \text{reg}_{S_{i+1}}(S_{i+1}/(J(G)^{(k)} \cap S_{i+1})). \]
Set $u_{i+1} = \prod_{x_j \in N_G(x_{i+1})} x_j \in S_{i+1}$. By Lemma [18, Lemma 2.2],
\[ J(G) \cap S_{i+1} = u_{i+1}J(G \setminus N_G[x_{i+1}])S_{i+1}. \]
Therefore,
\[ J(G)^{(k)} \cap S_{i+1} = (J(G) \cap S_{i+1})^{(k)} = u_{i+1}^k J(G \setminus N_G[x_{i+1}])^{(k)} S_{i+1}. \]
Notice that if $C$ is a minimal vertex cover of $G \setminus N_G[x_{i+1}]$, then $C \cup N_G(x_{i+1})$ is a minimal vertex cover of $G$. This shows that
\[ \deg(J(G \setminus N_G[x_{i+1}])) + \deg_G(x_{i+1}) \leq \deg(J(G)). \]
On the other hand, [18, Lemma 4.1] implies that \( \text{reg}(J(G) \cap S_{i+1}) \leq \text{reg}(J(G)) \). Therefore,

\[
\text{reg}_{S_{i+1}}(S_{i+1}/J(G \setminus N_G[x_{i+1}])S_{i+1}) \leq \text{reg}(S/J(G)) - \deg(u_{i+1}).
\]

Since \( G \setminus N_G[x_{i+1}] \) belongs to \( \mathcal{H} \), the induction hypothesis implies that

\[
\text{reg}_{S_{i+1}/J(G)^{(k)}(x_{i+1})} = \text{reg}_{S_{i+1}/J(G \setminus N_G[x_{i+1}])^{(k)}S_{i+1}} + k\deg(u_{i+1}) \\
\leq (k - 1)\deg(J(G \setminus N_G[x_{i+1}]) + |V(G \setminus N_G[x_{i+1}])| - 2 + k\deg_G(x_{i+1}) \\
\leq (k - 1)(\deg(J(G)) - \deg_G(x_{i+1})) + n - \deg_G(x_{i+1}) - 1 - 2 + k\deg_G(x_{i+1}) \\
\leq (k - 1)\deg(J(G)) + n - 3.
\]

Finally, the claim now follows by inequalities (3) and (4).

Now, \( J'_n = (J(G)^{(k)} : x_1 x_2 \ldots x_n) \) which is equal to \( J(G)^{(k-2)} \) by [16, Lemma 3.4]. Thus, by induction hypothesis we conclude that

\[
\text{reg}(S/J'_n) \leq (k - 3)\deg(J(G)) + n - 2.
\]

Therefore, using the claim repeatedly, implies that

\[
\text{reg}(S/J'_1) \leq \max\{(k - 1)\deg(J(G)) + n - 3, \text{reg}_S(S/J'_n) + n - 1\} \\
\leq \max\{(k - 1)\deg(J(G)) + n - 3, (k - 3)\deg(J(G)) + n - 2 + n - 1\}.
\]

As \( G \in \mathcal{H} \), the assumptions imply that \( G \) has minimal vertex cover with cardinality at least \( n/2 \). This means that \( 2\deg(J(G)) \geq n \). Thus, the above inequalities imply that

\[
\text{reg}(S/J'_1) \leq (k - 1)\deg(J(G)) + n - 3.
\]

This completes the proof of the theorem.

The following example from [10] shows that not every graph satisfies the condition (ii) of Theorem 3.2. However, we will see in Theorems 3.4, 3.6 and 3.7 that the condition (ii) of Theorem 3.2 is satisfied by bipartite graphs, unmixed graphs and claw–free graphs.

**Example 3.3.** For every pair of integers \( n \geq 3 \) and \( s \geq 2 \), let \( G_{n,s} \) be the graph obtained by attaching \( s \) pendant edges at each vertex of \( K_n \). The graph \( G_{3,3} \) is shown below. It is easy to check that the largest minimal vertex cover of \( G_{n,s} \) has \( n + s - 1 < \frac{|V(G_{n,s})|}{2} \) vertices (note that every vertex cover of \( G \) contains at least \( n - 1 \) vertices of \( V(K_n) \)).
As we mentioned in the introduction, Hang and Trung [12] recently proved the inequality \( \text{reg}(J(G)^k) \leq (k - 1)\text{deg}(J(G)) + |V(G)| - 1 \), for every bipartite graph \( G \) and every integer \( k \geq |V(G)| + 2 \). The following Theorem shows that this inequality is indeed true for every positive integer \( k \).

**Theorem 3.4.** Let \( G \) be a bipartite graph. Then for every integer \( k \geq 1 \), we have
\[
 k \text{deg}(J(G)) \leq \text{reg}(J(G)^k) \leq (k - 1)\text{deg}(J(G)) + |V(G)| - 1.
\]

**Proof.** Let \( \mathcal{H} \) be the family of all bipartite graphs. It is clear that for every graph \( G \in \mathcal{H} \) and every vertex \( x \in V(G) \), we have \( G \setminus N_G[x] \in \mathcal{H} \).

Let \( G \) be a bipartite graph without isolated vertices. Assume that \( V(G) = A \cup B \) is a bipartition for the vertex set of \( G \). Without loss of generality, we may suppose that \( |A| \geq |B| \). Then \( A \) is a minimal vertex cover of \( G \) with cardinality at least \( \frac{|V(G)|}{2} \). On the other hand, it follows from [9, Corollary 2.6] that for every integer \( k \geq 1 \) we have \( J(G)^k = J(G)^{(k)} \). The desired inequalities now follow from Theorem 3.2. \( \square \)

The following corollary shows that the inequalities of Theorem 3.4 are sharp.

**Corollary 3.5.** For the complete bipartite graph \( K_{1,n} \), we have
\[
 \text{reg}(J(K_{1,n})^k) = kn,
\]
for every integer \( k \geq 1 \).

**Proof.** The assertion follows from Theorem 3.4 by noticing that \( \text{deg}(J(K_{1,n})) = n \). \( \square \)

Let \( G \) be a bipartite graph. In [18, Theorem 4.3], we proved that \( k\text{deg}(J(G)) + \text{reg}(J(G)) - 1 \) is an upper bound for the regularity of \( J(G)^k \). In the same paper, [18, Remark 4.4], we mentioned that this bound is not probably the best one. In fact the upper bound of Theorem 3.4 is an improvement for the bound given by [18, Theorem 4.3]. To see this, assume that \( G \) has no isolated vertex and suppose that \( V(G) = A \cup B \) is a bipartition for the vertex set of \( G \). Without loss of generality,
assume that \(|A| \geq |B|\). As \(A\) is a minimal vertex cover of \(G\), we conclude that \(|A| \leq \deg(J(G))\) and hence, \(|A| \leq \text{reg}(J(G))\). Thus,

\[
(k-1)\deg(J(G)) + |V(G)| - 1 \leq (k-1)\deg(J(G)) + 2|A| - 1 \\
\leq k\deg(J(G)) + \text{reg}(J(G)) - 1.
\]

The second class of graphs which we consider is the class of unmixed graphs.

**Theorem 3.6.** Let \(G\) be an unmixed graph. Then for every integer \(k \geq 1\), we have

\[
k\deg(J(G)) \leq \text{reg}(J(G)^{(k)}) \leq (k-1)\deg(J(G)) + |V(G)| - 1.
\]

**Proof.** Let \(\mathcal{H}\) be the family of all unmixed graphs. Assume that \(G\) is an unmixed graph and \(x\) is an arbitrary vertex of \(G\). Then a subset \(W \subseteq V(G \setminus N_G[x])\) is a minimal vertex cover of \(G \setminus N_G[x]\) if and only if \(W \cup N_G(x)\) is a minimal vertex cover of \(G\). Hence \(G \setminus N_G[x] \in \mathcal{H}\). On the other hand, we know from [10] (see also [6, Theorem 0.1]) that if \(G\) is an unmixed graph without isolated vertices, then it has a minimal vertex cover with cardinality at least \(\frac{|V(G)|}{2}\). The desired inequalities now follow from Theorem 3.2. \(\square\)

The last class of graphs which we study is the family of claw–free graphs.

**Theorem 3.7.** Let \(G\) be a claw–free graph. Then for every integer \(k \geq 1\), we have

\[
k\deg(J(G)) \leq \text{reg}(J(G)^{(k)}) \leq (k-1)\deg(J(G)) + |V(G)| - 1.
\]

**Proof.** Let \(\mathcal{H}\) be the family of all claw–free graphs. It is clear that for every graph \(G \in \mathcal{H}\) and every vertex \(x \in V(G)\), we have \(G \setminus N_G[x] \in \mathcal{H}\). The assertion now follows from Theorem 3.2 together with the following claim.

**Claim.** If \(G\) is a claw–free graph which has no isolated vertex, then it admits a minimal vertex cover with cardinality at least \(\frac{|V(G)|}{2}\).

**Proof of the Claim.** We use induction on \(|V(G)|\). There is nothing to prove for \(|V(G)| = 2, 3\). Therefore, suppose that \(|V(G)| \geq 4\). Without loss of generality, we may assume that \(G\) is a connected graph. Let \(W\) be the subset of vertices of \(G\) with degree at least two. Assume that there is a vertex \(x \in W\) which is adjacent to at least two leaves, say \(y, z\). As \(G\) is connected and claw–free, we conclude that \(G\) has no other vertex, i.e., \(V(G) = \{x, y, z\}\), which contradicts our assumption that \(|V(G)| \geq 4\). Thus, every vertex in \(W\) is adjacent to at most one leaf. If every vertex in \(W\) is adjacent to exactly one leaf, then \(|V(G)| = 2|W|\) and \(W\) is a minimal vertex cover of \(G\). Hence, the claim follows in this case.

Therefore, assume that there is a vertex \(v \in W\) which is adjacent to no leaf in \(G\). Let \(w\) be a neighborhood of \(v\). As \(w\) is not a leaf, it follows that \(w \in W\). Set \(H = G \setminus N_G[w]\) and suppose that \(U\) is the set of isolated vertices of \(H\). Then every vertex in \(U\) is adjacent to a vertex in \(N_G(w)\). If there are to vertices \(w_1, w_2 \in U\) which are adjacent to a vertex \(w_0 \in N_G(w)\), then the vertices \(w, w_0, w_1, w_2\) form a claw which is a contradiction. Thus, every vertex in \(N_G(w)\) is adjacent to at most
one vertex in $U$. This shows that $|U| \leq |N_G(w)|$. However, we prove the following stronger inequality.

\[(5) \quad |U| \leq |N_G(w)| - 1\]

Indeed, using the above argument, the inequality (5) is obvious, if there is a vertex in $U$ which is adjacent to at least two vertices in $N_G(w)$. Hence, suppose that every vertex in $U$ is adjacent to exactly one vertex in $N_G(w)$. Thus, the vertices of $W'$ have degree one in $G$. This shows that no vertex in $U$ is adjacent to $v$. Since $v \in N_G(w)$, again the above argument implies that $|U| \leq |N_G(w)| - 1$.

Note that $H \setminus U$ is a claw–free graph which has no isolated vertex. It follows from the induction hypothesis that $H \setminus U$ has a minimal vertex cover $C$ with

$$|C| \geq \frac{|V(H \setminus U)|}{2} = \frac{|V(H)| - |U|}{2}.$$ 

Then $C \cup N_G(w)$ is a minimal vertex cover of $G$ and

$$|C \cup N_G(w)| = |C| + |N_G(w)| \geq \frac{|V(H)| - |U|}{2} + |N_G(w)|$$
$$\geq \frac{|V(H)| + |N_G(w)| + 1}{2} = \frac{|V(G)|}{2},$$

where the last inequality follows from the inequality (5).

The following corollary shows that the inequalities of Theorem 3.7 are sharp.

**Corollary 3.8.** Assume that $H$ is a graph with $i(G) \leq 2$. Let $G$ be the graph obtained from $H$ by adding a new vertex $y$ and connecting $y$ to every vertex of $H$. Then for every integer $k \geq 1$, we have

$$k \deg(J(G)) = \reg(J(G)^{(k)}) = (k - 1) \deg(J(G)) + |V(G)| - 1.$$ 

In particular,

$$\reg(J(K_n)^{(k)}) = kn - 1,$$

for every integer $n \geq 2$.

**Proof.** It is obvious from the construction of $G$ that $i(G) \leq 2$. Hence, $G$ is a claw–free graph. On the other hand $V(H)$ is a minimal vertex cover of $G$. Thus, $\deg(J(G)) = |V(H)| = |V(G)| - 1$. The assertions now follow from Theorem 3.7. \qed

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