ON THE MAXIMUM PRINCIPLE FOR THE MULTI-TERM FRACTIONAL TRANSPORT EQUATION

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Abstract. In this paper, we prove a maximum principle for the general multi-term space-
time-fractional transport equation and apply it for establishing uniqueness of solution to an
initial-boundary-value problem for this equation. We also derive some comparison principles
for solutions to the initial-boundary-value problems with different problem data. Finally,
we present a maximum principle for the Cauchy problem for a time-fractional transport
equation on an unbounded domain.

Key words: time-fractional transport equation, space-time-fractional multi-term transport
equation, initial-boundary-value problem, Cauchy problem, maximum principle, comparison
principle

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1. Introduction

Within the last few decades, fractional calculus in general and fractional partial differential equations became a very popular and important topic both in mathematics and in numerous applications. The framework of fractional calculus has been widely employed to
describe several physical phenomena including anomalous diffusion and anomalous transport processes in various areas, such as material science [15], medical engineering [5, 27], electrical engineering [21], hydrology [2], geological engineering [17, 25], and the earth systems [30].

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One of the most investigated and used fractional partial differential equations is the time-fractional diffusion equation. In the one-dimensional case and on the finite space- and time-intervals, the time-fractional diffusion equation with the convection and reaction terms is formulated as follows:

$$\frac{\partial^\alpha t}{\partial t^\alpha} u(x,t) = \frac{\partial^2 x}{\partial x^2} u(x,t) - q(x,t) \partial_x u(x,t) + r(x,t) u(x,t), \quad 0 < \alpha \leq 1, \quad 0 < x < \ell, \quad 0 < t < T.$$  

(1.1)

For $0 < \alpha < 1$, by $\partial^\alpha_t$ we denote the Caputo fractional derivative ([18]):

$$\frac{\partial^\alpha t}{\partial t^\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s} u(x,s) \, ds,$$

where $\frac{\partial}{\partial s} u(\cdot, s)$ is assumed to belong to the space $L^1(0,T)$ for any $x \in (0,\ell)$ and $\Gamma$ is the Euler gamma function. For $\alpha = 1$, $\partial^\alpha_t$ is interpreted as the conventional first order derivative. As usual, we set $\partial_x := \frac{\partial}{\partial x}$ and $\partial^2_x := \frac{\partial^2}{\partial x^2}$.

The initial-boundary-value problems for the equation (1.1) in different settings and properties of their solutions have been already intensively studied in the literature. Especially for the unique existence of solutions to the initial-boundary-value problems for the equation (1.1) and its generalizations, we refer to [3, 7, 9, 10, 20, 29] to mention only few of many relevant publications. Moreover, there are many works on numerical analysis of the fractional partial differential equations, but our focus in this paper is on their analytical treatment and we do not refer to any publications regarding numerical methods.

In this paper, we address the following time-fractional transport equation with the Caputo fractional derivative of the order $\alpha$, $0 < \alpha < 1$:

$$p(x,t)\frac{\partial^\alpha t}{\partial t^\alpha} u(x,t) + q(x,t) \partial_x u(x,t) = r(x,t) u(x,t) + F(x,t), \quad 0 < x < \ell, \quad 0 < t < T \quad (1.2)$$

along with the boundary and initial conditions

$$u(0,t) = g(t), \quad 0 < t < T,$$  

(1.3)

$$u(x,0) = a(x), \quad 0 < x < \ell,$$  

(1.4)

respectively, as well as its multi-term time-space-fractional generalizations which we formulate in Section 3.

In what follows, we assume the inclusions

$$p, q, r \in C([0,\ell] \times [0,T])$$  

(1.5)

as well as some conditions on the signs of the functions $p, q, r$ that we formulate in due time. Taking into consideration the outgoing and ingoing sub-boundaries in the case of the transport equation (1.2) with $\alpha = 1$, it is natural to prescribe the boundary condition (1.3) at the point $x = 0$, not at the point $x = \ell$. 
Throughout the paper, we assume the existence of a solution to the initial-boundary-value problem (1.2)-(1.4) that satisfies the following inclusions:

\[ u \in C([0,\ell] \times [0,T]), \quad u(\cdot, t) \in W^{1,1}(0,\ell), \quad u(x, \cdot) \in W^{1,1}(0,T), \quad (1.6) \]

where \( W^{1,1}(0,T) = \{ g; \ g, \partial_t g \in L^1(0,T) \} \) and \( W^{1,1}(0,\ell) = \{ a; \ a, \partial_x a \in L^1(0,\ell) \} \).

The time-fractional transport equation (1.2) was already employed for modeling various anomalous transport processes including the mass and heat transfer for characterizing geothermal reservoirs \([22, 23, 24, 26]\). However, compared to the comprehensive results already obtained for the time-fractional diffusion equation of type (1.1), it may be a surprise that until now only few theoretical publications were devoted to the fractional transport equations, for instance, to the problem of unique existence of solution to the initial-boundary-value problem (1.2)-(1.4). For a treatment of the viscosity solutions to the time-fractional transport equations we refer to \([16]\).

For \( \alpha = 1 \), the equation (1.2) is the classical and well studied transport equation. It is well known that the solutions to the initial-boundary-value problems of type (1.2)-(1.4) with \( \alpha = 1 \) can be analyzed by the method of characteristics. However, for \( 0 < \alpha < 1 \), the method of characteristics does not work and thus the properties of the fractional transport equation (1.2) are not yet well investigated.

Another important approach to analysis of the solution properties to the partial differential equations is the maximum principle \([19]\). For the multi-dimensional time-fractional diffusion equation of type (1.1), the maximum principle in different settings has been proved in \([8, 12, 28]\) for the case of the Caputo time-fractional derivative, in \([11]\) for the case of the Riemann-Liouville derivative, and in \([11]\) for the case of the general fractional derivative introduced in \([6]\). For more results regarding the maximum principles for the fractional partial differential equations we refer to the surveys \([13, 14]\).

However, to the best knowledge of the authors, no maximum principle for the fractional transport equations has been yet established. In this paper, we formulate and prove a maximum principle for the multi-term space-time-fractional transport equation and derive some of its useful consequences. Since the method of characteristics does not work for the fractional transport equations, the maximum and comparison principles are worth employing as an alternative methodology for their analytical treatment.

The rest of the paper is organized as follows. In Section 2 we start with a simple case of the fractional transport equation (1.2) and illustrate the main ideas behind the derivations in the general case. In Section 3, our main results are formulated. The next two sections are devoted to the proofs of two main theorems stated in Section 3. Finally, in the last section, we provide some concluding remarks and directions for further research.
Before stating and proving our main results, in this section, we address the following simple particular case of the time-fractional transport equation (1.2):
\[ \partial_t^\alpha u(x, t) + q_0 \partial_x u(x, t) = r(x, t)u(x, t), \quad 0 < \alpha \leq 1, \quad 0 < x < \ell, \quad 0 < t < T \] (1.7)
along with the boundary condition (1.3) and the initial condition (1.4). In the equation (1.7), \( q_0 > 0 \) is a constant and we assume that the condition
\[ r(x, t) < 0, \quad 0 \leq x \leq \ell, \quad 0 \leq t \leq T \] (1.8)
is satisfied. It is worth mentioning that the condition (1.8) can be replaced with a weaker condition \( r(x, t) \leq 0 \) on \([0, \ell] \times [0, T]\). However, for simplicity of the proofs, in this section we suppose that the stronger inequality (1.8) is satisfied. For the solution \( u = u(x, t) \) of the initial-boundary-value problem (1.7), (1.3) and (1.4), the following result holds true:

**Proposition 2.1.** Let \( u = u(x, t) \) satisfy the inclusion (1.6) and \( u(0, t) \geq 0 \) for \( 0 \leq t \leq T \) and \( u(x, 0) \geq 0 \) for \( 0 \leq x \leq \ell \). Then \( u = u(x, t) \) is non-negative on the whole domain \( Q_T := [0, \ell] \times [0, T] \), i.e.,
\[ u(x, t) \geq 0, \quad 0 \leq x \leq \ell, \quad 0 \leq t \leq T. \]

**Proof.** The proof of the proposition essentially relies on the extremum principle for the Caputo fractional derivative.

**Lemma 2.2 (\cite{8}).** Let \( f \in C[0, T] \cap W^{1,1}(0, T) \) attain its maximum (resp. its minimum) over the closed interval \([0, T]\) at a point \( t_0 \in (0, T) \). Then for any \( \alpha \in (0, 1] \) the inequality
\[ (\partial_t^\alpha f)(t_0) \geq 0 \quad (\text{resp. } (\partial_t^\alpha f)(t_0) \leq 0) \]
holds true.

We employ an indirect proof and assume that the conclusion of the proposition does not hold true. Then there exists a point \((x_0, t_0) \in \Omega_T\) such that
\[ u(x_0, t_0) := \min_{(x, t) \in \Omega_T} u(x, t) < 0. \]
Since \( u(0, t) \geq 0 \) and \( u(x, 0) \geq 0 \), we conclude that \( x_0 > 0 \) and \( t_0 > 0 \). By Lemma 2.2, we have \( \partial_t^\alpha u(x_0, t_0) \leq 0 \). Moreover, since \( u(x, t_0) \) as a function in \( x \) attains its minimum at the point \( x = x_0 > 0 \), we get the inequality \( \partial_x u(x_0, t_0) \leq 0 \). Hence
\[ 0 \geq \partial_t^\alpha u(x_0, t_0) = -q_0 \partial_x u(x_0, t_0) + r(x_0, t_0)u(x_0, t_0) \geq r(x_0, t_0)u(x_0, t_0) > 0 \]
because \( r(x_0, t_0) < 0 \) and \( u(x_0, t_0) < 0 \). With the last inequality, we arrived to a contradiction and the proof of Proposition 2.1 is completed. \( \square \)
Proposition 2.1 is quite preliminary and serves just for illustration of our method. In the next section, we present a maximum principle for the more general multi-term time-space-fractional transport equation that is valid under the weaker conditions on the problem data compared to the ones formulated in Proposition 2.1.

3. Main results

In this section, we address an initial-boundary-value problem for a one-dimensional multi-term time-space-fractional transport equation defined on the bounded domain \( Q_T := (0, \ell) \times (0, T), \ell > 0, T > 0 \) with the boundary \( S_T := \{(x,0); 0 < x < \ell\} \cup \{(0,t); 0 < t < T\} \).

To formulate the equation, we first introduce the functions \( p_i = p_i(x,t), i = 1, \ldots, n \) and \( q_j = q_j(x,t), j = 1, \ldots, m \) and the constants \( \alpha_i, i = 1, \ldots, n \) and \( \beta_j, j = 1, \ldots, m \) that satisfy the following conditions and inclusions:

\[
\begin{cases}
0 < \alpha_1 < \cdots < \alpha_n \leq 1, & 0 < \beta_1 < \cdots < \beta_m \leq 1, \\
p_i \in C(\overline{Q_T}), q_j \in C(\overline{Q_T}) & \text{for } 1 \leq i \leq n, 1 \leq j \leq m, \\
p_i(x,t) \geq 0, 1 \leq i \leq n, & q_j(x,t) \geq 0, 1 \leq j \leq m, \quad (x,t) \in Q_T, \\
\sum_{i=1}^n p_i(x,t) > 0 & \text{for all } (x,t) \in \overline{Q_T}.
\end{cases}
\] (2.1)

For the given functions \( r, F \in C(\overline{Q_T}) \), the one-dimensional multi-term time-space-fractional transport equation is introduced as follows:

\[
\sum_{i=1}^n p_i(x,t) \partial_t^{\alpha_i} u(x,t) + \sum_{j=1}^m q_j(x,t) \partial_x^{\beta_j} u(x,t) = r(x,t)u(x,t) + F(x,t), (x,t) \in Q_T,
\] (2.2)

where for \( 0 < \beta < 1 \) the space-fractional Caputo derivative is defined by the formula

\[
\partial_x^{\beta} u(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^x (x-y)^{-\beta} \frac{\partial}{\partial y} u(y,t) dy
\]

in analogy to the time-fractional Caputo derivative. Of course, for \( \beta = 1 \) the Caputo fractional derivative is interpreted as the conventional first order derivative.

In what follows, we always assume that the function \( r = r(x,t) \) is non-positive, i.e.,

\[
r(x,t) \leq 0, \quad (x,t) \in \overline{Q_T}
\] (2.3)

and that any solution \( u = u(x,t) \) to the equation (2.2) satisfies the regularity conditions (1.6).

Our main results are formulated in the following two theorems.

**Theorem 3.1.** (i) Let \( F(x,t) \leq 0 \) for \( (x,t) \in \overline{Q_T} \). Then

\[
\max_{(x,t)\in \overline{Q_T}} u(x,t) \leq \max_{(x,t)\in \overline{S_T}} u(x,t).
\] (2.4)
(ii) Let $F(x,t) \geq 0$ for $(x,t) \in \overline{Q}_T$. Then
\[
\min_{(x,t) \in \overline{Q}_T} u(x,t) \geq \min \{0, \min_{(x,t) \in S_T} u(x,t)\}. \tag{2.5}
\]

In the case of $r \equiv 0$ in $Q_T$, the inequalities (2.4) and (2.5) can be replaced by the equalities
\[
\max_{(x,t) \in \overline{Q}_T} u(x,t) = \max_{(x,t) \in S_T} u(x,t) \tag{2.4}'
\]
and
\[
\min_{(x,t) \in \overline{Q}_T} u(x,t) = \min_{(x,t) \in S_T} u(x,t), \tag{2.5}'
\]
respectively.

From Theorem 3.1 we readily derive the following useful consequence:

Corollary 3.2. If $F \equiv 0$ in $Q_T$, then
\[
\max_{(x,t) \in \overline{Q}_T} u(x,t) = \max_{(x,t) \in S_T} u(x,t), \quad \min_{(x,t) \in \overline{Q}_T} u(x,t) = \min_{(x,t) \in S_T} u(x,t). \tag{2.6}
\]

In its turn, this corollary immediately yields an uniqueness result.

Corollary 3.3 (uniqueness of solution). Let the functions $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ satisfy the equation (2.2) and the regularity conditions (1.6). If $u_1(x,t) = u_2(x,t)$ on the boundary $S_T$ of the domain $Q_T$, then $u_1(x,t) = u_2(x,t)$ on the whole domain $Q_T$.

Proof. Indeed, setting $u := u_1 - u_2$, we see that the function $u$ satisfies (1.6) and (2.2) with $F \equiv 0$. Therefore Corollary 3.2 implies \[\max_{(x,t) \in \overline{Q}_T} u(x,t) = \min_{(x,t) \in S_T} u(x,t) = 0,\]
which means that $u_1 \equiv u_2$ in $Q_T$. \qed

Moreover, Theorem 3.1 implicates some important comparison principles. Let the function $u = u_{r,a,g,F}(x,t)$ satisfy the regularity conditions (1.6), the equation (2.2), and the following initial and boundary conditions:
\[
u(0,t) = g(t), \quad 0 < t < T, \quad u(x,0) = a(x), \quad 0 < x < \ell. \tag{2.7}
\]

Corollary 3.4 (comparison principles). (i) Let $F_1(x,t) \geq F_2(x,t)$ for $(x,t) \in \overline{Q}_T$, $g_1(t) \geq g_2(t)$ for $t \in [0,T]$, and $a_1(x) \geq a_2(x)$ for $x \in [0,\ell]$. Then
\[
u_{r,a_1,g_1,F_1}(x,t) \geq \nu_{r,a_2,g_2,F_2}(x,t), \quad (x,t) \in \overline{Q}_T.
\]

(ii) Let $F_1(x,t) \geq F_2(x,t) \geq 0$ for $(x,t) \in \overline{Q}_T$, $g_1(t) \geq g_2(t) \geq 0$ for $t \in [0,T]$, and $a_1(x) \geq a_2(x) \geq 0$ for $x \in [0,\ell]$. If $0 \geq r_1(x,t) \geq r_2(x,t)$ for $(x,t) \in \overline{Q}_T$, then
\[
u_{r_1,a_1,g_1,F_1}(x,t) \geq \nu_{r_1,a_2,g_2,F_2}(x,t), \quad (x,t) \in \overline{Q}_T.
\]
For the similar comparison principles for the time-fractional diffusion equation we refer the readers to [12].

Now we formulate a maximum principle for the following Cauchy problem for a time-fractional transport equation of order $\alpha$, $0 < \alpha < 1$ defined on an unbounded domain $\Omega_T := \mathbb{R} \times (0, T)$:

$$\partial_t^\alpha u(x, t) + q(x)\partial_x u(x, t) = F(x, t), \quad (x, t) \in \Omega_T$$  \hfill (2.8)

along with the initial condition

$$u(x, 0) = a(x), \quad x \in \mathbb{R}.$$ \hfill (2.9)

In (2.9), we assume the inclusion $a \in W^{1,1}_{\text{loc}}(\mathbb{R})$ that means that $a|_{(-X, X)} \in W^{1,1}(-X, X)$ for any $X > 0$. Evidently, equation (2.8) is a particular case of the multi-term time-space-fractional transport equation (2.2). The following maximum principle is valid:

**Theorem 3.5.** Let $u \in C(\mathbb{R} \times [0, T]) \cap L^\infty(\mathbb{R} \times (0, T))$ satisfy (2.8) and (2.9) and the inclusions

$$u(x, \cdot) \in W^{1,1}(0, T), \quad u(\cdot, t) \in W^{1,1}_{\text{loc}}(\mathbb{R}).$$

Moreover, we assume

$$\int_{-\infty}^{0} \left| \frac{1}{q(\xi)} \right| d\xi = \infty.$$ \hfill (2.10)

If $F(x, t) \leq 0$ for $(x, t) \in Q_T$, then

$$\sup_{(x, t) \in \Omega_T} u(x, t) = \sup_{x \in \mathbb{R}} a(x).$$ \hfill (2.11)

If $F(x, t) \geq 0$ for $(x, t) \in Q_T$, then

$$\inf_{(x, t) \in \Omega_T} u(x, t) = \inf_{x \in \mathbb{R}} a(x).$$ \hfill (2.12)

To demonstrate the statement of Theorem 3.5, we consider two simple examples and address the case when the solution $u$ can be represented in the form $u(x, t) = \psi(t) + \varphi(x)$ for $x \in \mathbb{R}$ and $0 < t < T$, where $\psi \in C^1[0, T]$ and $\varphi \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

**Example 3.6.** Let us suppose that $\frac{d\psi}{dt}(t) \leq 0$ for $0 < t < T$. Then we can easily verify that $\partial_t^\alpha \psi(t) \leq 0$ for $0 < t < T$. If we choose $\varphi(x)$ such that $\frac{d\varphi}{dx}(x) \leq 0$ for $x \in \mathbb{R}$, then $u(x, t) = \psi(t) + \varphi(x)$ satisfies (2.8) with $q(x, t) \geq 0$ and $F(x, t) = \partial_t^\alpha \psi + q(x)\partial_x \varphi \leq 0$. Hence the equality (2.11) holds true. However, in this case, (2.11) is trivial because $\frac{d\psi}{dt}(t) = \frac{d\varphi}{dt}(t) \leq 0$ for $0 \leq t \leq T$ and so we immediately see that $u(x, t) \leq u(x, 0)$ for $x \in \mathbb{R}$ and $0 < t < T$. 
Example 3.7. Theorem 3.5 is less trivial if the inequalities \( \frac{d\psi}{dt}(t) \leq 0 \) and \( \frac{d\phi}{dx}(x) \leq 0 \) do not hold. For example, let \( \varphi(x) = e^{-x^2}, x \in \mathbb{R} \). Then \( \frac{d\phi}{dx}(x) > 0 \) for \( x < 0 \).

If \( \partial_t^\alpha \psi(t) \leq -\sqrt{2}e^{-\frac{t}{2}} = -\max_{x \in \mathbb{R}} \left| \frac{d\phi}{dx}(x) \right| \), then

\[
\partial_t^\alpha u(x,t) + \partial_x u(x,t) = \partial_t^\alpha \psi(t) + \frac{d\phi}{dx}(x) \leq -\max_{x \in \mathbb{R}} \left| \frac{d\phi}{dx}(x) \right| + \frac{d\phi}{dx}(x) \leq 0, \quad x \in \mathbb{R}, \ 0 \leq t \leq T,
\]

i.e., the equation (2.8) holds valid with a function \( F = F(x,t) \leq 0 \) and \( q(x) = 1, x \in \mathbb{R} \). The statement of Theorem 3.5 is that the inequality \( \partial_t^\alpha \psi(t) \leq -\sqrt{2}e^{-\frac{t}{2}} \) for \( 0 < t < T \) implies the inequality \( \frac{d\psi}{dt}(t) \leq 0 \) for \( 0 < t < T \) that is not trivial.

4. Proof of Theorem 3.1

In this section, we present a proof of Theorem 3.1 that is based on Lemma 2.2 and carried out similarly to the proof of Theorem 2 from [8]. It suffices to prove the inequality (2.4) and the equality (2.4)' because the inequality (2.5) and the equality (2.5)' can be proved by replacing \( u \) by \( -u \) and arguing in the same way.

Proof. We prove the inequality (2.4) by contradiction. Assume that (2.4) does not hold true. Then there exist \( x_0 \in [0,\ell] \) and \( t_0 \in [0,T] \) such that

\[
u(x_0,t_0) > M := \max\{0, \max_{(x,t) \in \mathbb{S}_T} u(x,t)\} \geq 0.
\]

Now we set

\[
\varepsilon := u(x_0,t_0) - M > 0 \tag{3.1}
\]

and introduce an auxiliary function \( w(x,t) \), which is the same as the one employed in [8]:

\[
w(x,t) := u(x,t) + \frac{\varepsilon}{2T}(T-t), \ (x,t) \in Q_T.
\]

It is easy to calculate that

\[
\partial_t^\alpha_i w(x,t) = \partial_t^\alpha_i u(x,t) - \frac{\varepsilon t^{1-\alpha_i}}{2T\Gamma(2-\alpha_i)}, \ 1 \leq i \leq n.
\]

Therefore we have the following equality

\[
\sum_{i=1}^n p_i(x,t) \partial_t^\alpha_i w(x,t) + \sum_{j=1}^m q_j(x,t) \partial_x^j w(x,t) = r(x,t) \left( w(x,t) - \frac{\varepsilon}{2T}(T-t) \right) - \frac{\varepsilon}{2T} \sum_{i=1}^n p_i(x,t) t^{1-\alpha_i} \Gamma(2-\alpha_i) + F(x,t), \ (x,t) \in Q_T. \tag{3.2}
\]

By definition of \( w \), the inequality

\[
w(x,t) \geq u(x,t), \ (x,t) \in \overline{Q}_T
\]
holds true. On the other hand, the condition (3.1) yields

\[ w(x_0, t_0) \geq u(x_0, t_0) = M + \varepsilon. \quad (3.3) \]

Since \( u(x, t) \leq M \) for \((x, t) \in \overline{Q}_T\), the chain of inequalities

\[ w(x_0, t_0) \geq M + \varepsilon \geq u(x, t) \geq \varepsilon + w(x, t) - \frac{\varepsilon}{2T}(T - t) \]
\[ \geq \varepsilon + w(x, t) - \frac{\varepsilon}{2} = w(x, t) + \frac{\varepsilon}{2} > w(x, t) \]

holds true for any \((x, t) \in \overline{S}_T\) that in its turn implies the inequality \( \max_{\overline{S}_T} w < w(x_0, t_0) \).

This means that if \( w \) attains its maximum over \( \overline{Q}_T \) at the point \((x_1, t_1)\), then \((x_1, t_1) \notin \overline{S}_T\) and therefore \( x_1 > 0, \quad t_1 > 0. \quad (3.4) \)

Moreover, by (3.3) and \( M \geq 0 \), we obtain the estimates

\[ w(x_1, t_1) \geq w(x_0, t_0) \geq M + \varepsilon \geq \varepsilon. \quad (3.5) \]

Because of the conditions (3.4), we may apply Lemma 2.2 and get the following inequalities (in the case \( \alpha_i = 1 \) or \( \beta_j = 1 \), these inequalities are well known in calculus):

\[ \partial_t^{\alpha_i} w(x_1, t_1) \geq 0, \quad 1 \leq i \leq n, \quad \partial_x^{\beta_j} w(x_1, t_1) \geq 0, \quad 1 \leq j \leq m. \]

Hence

\[ \sum_{i=1}^{n} p_i(x_1, t_1) \partial_t^{\alpha_i} w(x_1, t_1) + \sum_{j=1}^{m} q_j(x_1, t_1) \partial_x^{\beta_j} w(x_1, t_1) \geq 0. \quad (3.6) \]

It follows from the inequality (3.5) that

\[ w(x_1, t_1) - \frac{\varepsilon}{2T}(T - t_1) \geq \varepsilon - \frac{\varepsilon}{2T}T = \frac{\varepsilon}{2} > 0 \]

and

\[ r(x_1, t_1) \left( w(x_1, t_1) - \frac{\varepsilon}{2T}(T - t_1) \right) \leq 0 \quad (3.7) \]

because of the assumption (2.3).

Moreover, the inequality

\[ \sum_{i=1}^{n} \frac{p_i(x_1, t_1) t_1^{1-\alpha_i}}{\Gamma(2-\alpha_i)} > 0 \]

holds true. Indeed, let us assume that \( \sum_{i=1}^{n} \frac{p_i(x_1, t_1) t_1^{1-\alpha_i}}{\Gamma(2-\alpha_i)} = 0 \). Since \( \frac{p_i(x_1, t_1) t_1^{1-\alpha_i}}{\Gamma(2-\alpha_i)} \geq 0 \) for \( 1 \leq i \leq n \) by the assumptions (2.1), we get \( \frac{p_i(x_1, t_1) t_1^{1-\alpha_i}}{\Gamma(2-\alpha_i)} = 0 \) for any \( i = 1, \ldots, n \), which implies \( p_i(x_1, t_1) = 0, \quad i = 1, \ldots, n \). Therefore \( \sum_{i=1}^{n} p_i(x_1, t_1) = 0 \) that contradicts the last of the conditions (2.1).
Thus, the inequality
\[- \frac{\varepsilon}{2T} \sum_{i=1}^{n} p_i(x_1, t_1) t_1^{1-\alpha_i} \Gamma(2 - \alpha_i) < 0 \] (3.8)
holds true. Using the condition \( F(x, t) \geq 0 \) and substituting the inequalities (3.6) - (3.8) into the formula (3.2), we arrive at a contradiction that proves the inequality (2.4) (and hence the inequality (2.5)).

Now we proceed with a proof of the equality (2.4)’ and assume that \( r \equiv 0 \) in \( Q_T \). Then, instead of \( M \) as in the previous proof, we set \( M_0 := \max_{(x, t) \in \overline{S}_T} u(x, t) \). We repeat the same arguments as above to obtain the inequalities \( x_1 > 0 \) and \( t_1 > 0 \), where \( w(x_1, t_1) \) is the maximum of \( w = w(x, t) \) over \( \overline{Q}_T \), and the equation
\[
\sum_{i=1}^{n} p_i(x_1, t_1) \partial_{x_i} u(x_1, t_1) + \sum_{j=1}^{m} q_j(x_1, t_1) \partial_{x_j} w(x_1, t_1)
= - \frac{\varepsilon}{2T} \sum_{i=1}^{n} p_i(x_1, t_1) t_1^{1-\alpha_i} \Gamma(2 - \alpha_i) + F(x_1, t_1)
\]
in place of the equation (3.2). Furthermore, we can verify that the inequalities (3.6) and (3.8) hold true and then arrive at a contradiction similar to the one formulated above. The only difference to the previous proof is that we cannot use the inequality (3.5) because the case \( M_0 < 0 \) may occur. However, we do not need it this time because of the assumption \( r \equiv 0 \). The proof of Theorem 3.1 is completed.

Now we prove Corollary 3.4.

**Proof.** First we prove the part (i) of Corollary 3.4. We start by setting
\[ u := u_{r, a_1, g_1, F_1} - u_{r, a_2, g_2, F_2} \]
and
\[ F := F_1 - F_2. \]
Then \( F(x, t) \geq 0 \), \( (x, t) \in Q_T \) and the function \( u \) satisfies the equation
\[
\sum_{i=1}^{n} p_i(x, t) \partial_{x_i}^\alpha u(x, t) + \sum_{j=1}^{m} q_j(x, t) \partial_{x_j}^\beta u(x, t) = r(x, t) u(x, t) + F(x, t), \ (x, t) \in Q_T
\]
and the inequalities
\[ u(0, t) \geq 0, \ 0 \leq t \leq T, \ u(x, 0) \geq 0, \ 0 \leq x \leq \ell. \]
Thus \( \min_{(x, t) \in \overline{S}_T} u(x, t) \geq 0 \) and \( \min \{ 0, \min_{(x, t) \in \overline{S}_T} u(x, t) \} = 0 \). Since \( F(x, t) \geq 0 \), \( (x, t) \in Q_T \), we can apply the inequality (2.5) from Theorem 3.1 and get the inequality
\[ \min_{(x, t) \in \overline{Q}_T} u(x, t) \geq \min \{ 0, \min_{(x, t) \in \overline{S}_T} u(x, t) \} = 0. \]
The proof of (i) is completed.

Then we proceed with a proof of the part (ii) of Corollary 3.4.
Because \( a_2(x) \geq 0, \ x \in [0, \ell] \) and \( g_2(t) \geq 0, \ t \in [0, T] \), Theorem 3.1 yields the inequality

\[
 u_{r_2,a_2,g_2,F_2}(x, t) \geq 0, \ (x, t) \in \overline{Q}_T. \tag{3.9}
\]

Now we again use the notations \( u := u_{r,a_1,g_1,F_1} - u_{r,a_2,g_2,F_2} \) and \( F := F_1 - F_2 \). Then \( F(x, t) \geq 0, \ (x, t) \in Q_T \) and the function \( u \) satisfies the equation

\[
 \sum_{i=1}^{n} p_i(x, t) \partial_i^\alpha u(x, t) + \sum_{j=1}^{m} q_j(x, t) \partial_j^\beta u(x, t)
 = r_1(x, t)u(x, t) + (r_1(x, t) - r_2(x, t))u_{r_2,a_2,g_2,F_2}(x, t) + F(x, t), \ (x, t) \in Q_T
\]

and the inequalities

\[
u(0, t) \geq 0, \quad 0 \leq t \leq T, \quad u(x, 0) \geq 0, \quad 0 \leq x \leq \ell. \]

Because \( r_1(x, t) - r_2(x, t) \geq 0, \ (x, t) \in Q_T \) and using the inequality (3.9), we get the inequality \( (r_1(x, t) - r_2(x, t))u_{r_2,a_2,g_2,F_2}(x, t) + F(x, t) \geq 0, \ (x, t) \in Q_T \). Thus we can apply the inequality (2.5) to the equation for \( u \) that completes the proof of (ii) and thus the proof of Corollary 3.4.

\[ \square \]

5. Proof of Theorem 3.5

Proof. The main element of our proof is a suitably chosen auxiliary function (see, e.g., [3] or [19]).

First we set \( M_1 := \|u\|_{L^\infty(\mathbb{R} \times (0, T))} \) and fix \( x_0, t_0 \), and \( \delta \) that satisfy the inequalities \( x_0 < 0, \ 0 < t_0 < T, \) and \( \delta > 0 \). Now we introduce an auxiliary function in the form

\[
w(x, t) := u(x, t) - \sup_{x \in \mathbb{R}} a(x) - \delta \left( \frac{t^\alpha}{\Gamma(1+\alpha)} - \int_0^x \frac{1}{q(\xi)}d\xi \right), \quad x \in \mathbb{R}, \ 0 < t < T
\]

and choose \( L > 0 \) sufficiently large such that the inequality

\[
\int_{-L}^0 \frac{1}{q(\xi)}d\xi > \frac{M_1 + \sup_{x \in \mathbb{R}} a(x)}{\delta}, \quad -L < x_0 < 0
\]

holds true. Then, by (4.1), we get the following inequality

\[
w(x, -L) = u(-L, t) - \sup_{x \in \mathbb{R}} a(x) - \delta \left( \frac{t^\alpha}{\Gamma(1+\alpha)} + \int_{-L}^0 \frac{1}{q(\xi)}d\xi \right)
\]

\[
\leq M_1 + \sup_{x \in \mathbb{R}} |a(x)| - \delta \int_{-L}^0 \frac{1}{q(\xi)}d\xi < 0, \quad 0 < t < T. \tag{4.2}
\]

Furthermore,

\[
w(x, 0) = u(x, 0) - \sup_{x \in \mathbb{R}} a(x) + \delta \int_0^x \frac{1}{q(\xi)}d\xi \leq u(x, 0) - \sup_{x \in \mathbb{R}} a(x) \leq 0, \quad -L \leq x \leq 0,
\]
that is,

$$w(x,0) \leq 0, \ -L \leq x \leq 0. \quad (4.3)$$

On the other hand, direct calculations yield

$$\partial_t^\alpha w(x,t) + q(x)\partial_x w(x,t) = F(x,t) \leq 0, \ -L < x < 0, \ 0 < t < T. \quad (4.4)$$

The inequalities (4.2)-(4.4) allow us to apply Theorem 3.1 (formula (2.4)) that leads to the inequality

$$w(x,t) \leq 0, \ -L \leq x \leq 0, \ 0 \leq t \leq T$$

and thus we arrive at the estimate

$$u(x_0,t) \leq \sup_{x \in \mathbb{R}} a(x) + \delta \left( \frac{t^\alpha}{\Gamma(1 + \alpha)} - \int_{x_0}^{x} \frac{1}{q(\xi)} d\xi \right), \ 0 < t < T. \quad (4.5)$$

In the last formula, we let $\delta$ go to zero and get the inequality $u(x_0,t) \leq \sup_{x \in \mathbb{R}} a(x)$. Since the point $x_0 < 0$ is arbitrarily chosen, we have proved that

$$u(x,t) \leq \sup_{x \in \mathbb{R}} a(x) \quad \text{if} \ x \leq 0. \quad (4.5)$$

Introducing a new variable $y := x + x_1$ with an arbitrarily chosen $x_1 > 0$, we can transfer the previous arguments to any interval $(-L + x_1, x_1)$ and thus arrive at the inequality $u(x,t) \leq \sup_{x \in \mathbb{R}} a(x)$ for $x \leq x_1$ and $0 < t < T$. Since the point $x_1$ can be arbitrarily chosen, we have proved the relation (2.11). Because $\sup v = -\inf(-v)$, the relation (2.12) can be derived from the relation (2.11) by changing the signs in the equation (2.8) and in the initial condition (2.9) and considering $-u$ instead of $u$. The proof of Theorem 3.5 is completed. \(\square\)

6. Conclusions and directions for further research

In this paper, we proved a maximum principle for the general multi-term space-time-fractional transport equation and applied it for analysis of solutions to the initial-boundary-value problems for this equation. Here we restricted ourselves to the case of the one-dimensional fractional transport equation. However, our arguments can be transferred to the multi-dimensional case without any essential changes. Say, one can similarly treat the multi-term time-fractional transport equation

$$\sum_{i=1}^{n} p_i(x,t)\partial_t^{\alpha_i} u(x,t) - A(x,t) \cdot \nabla u(x,t) = r(x,t)u(x,t) + F(x,t), \ x \in \Omega, \ 0 < t < T,$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $A(x,t) = (a_1(x,t), \ldots, a_d(x,t))$, and $\nabla v(x) = \left( \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_d} \right)$. This equation will be considered elsewhere.

For validity of the results presented in this paper, we assumed that the zeroth order coefficient $r = r(x,t)$ of the fractional transport equation is non-positive on the whole domain
$Q_T$. However, it is not clear if this condition can be weakened or even removed. This problem is also a topic for our further research.

In this paper, we did not address any nonlinear equations. However, at least for some semilinear fractional transport equations, our arguments still work and several important results can be derived. For example, let us consider the equation

$$\sum_{i=1}^{n} p_i(x,t) \partial_t^\alpha u(x,t) + \sum_{j=1}^{m} q_j(x,t) \partial_x^\beta u(x,t) = r(x,t) u(x,t) + f(u(x,t)) \quad (5.1)$$

on the finite domain $Q_T := (0,\ell) \times (0,T)$ and assume that the conditions (2.1) and (2.3) hold true. Moreover, we suppose that the semilinear term from the equation (5.1) belongs to the following admissible set $\mathcal{F}$ of functions:

$$\mathcal{F} := \{ f \in C^1(\mathbb{R}); \frac{df}{d\xi}(\xi) \leq 0, \xi \in \mathbb{R} \}.$$

In fact, the set $\mathcal{F}$ of admissible functions can be extended, but here we do not pursue the generality and prefer to focus on the underlying ideas.

In what follows, by $u_f = u_f(x,t)$ we denote a function that satisfies the inclusions (1.6) and the equation (5.1) with the semilinear term $f$ from $\mathcal{F}$. Then the following result holds true:

**Let** $f_1, f_2 \in \mathcal{F}$. **If**

$$f_1(\xi) \leq f_2(\xi), \quad \xi \in \mathbb{R}$$

**and**

$$u_{f_1}(0,t) \leq u_{f_2}(0,t), \quad 0 < t < T, \quad u_{f_1}(x,0) \leq u_{f_2}(x,0), \quad 0 < x < \ell,$$

**then**

$$u_{f_1}(x,t) \leq u_{f_2}(x,t), \quad (x,t) \in Q_T.$$

Let us prove this statement. Setting $u := u_{f_1} - u_{f_2}$, by the mean value theorem we have the representation

$$f_1(u_{f_1}(x,t)) - f_1(u_{f_2}(x,t)) = \frac{df_1}{d\xi}(\eta)u(x,t) =: g(x,t)u(x,t),$$

where $\eta = \eta(x,t)$ is a number from the interval $[u_{f_1}(x,t), u_{f_2}(x,t)]$. Because $f_1 \in \mathcal{F}$, the function $\eta = \eta(x,t)$ is a continuous function in both variables. Now we employ this representation and the identity

$$f_1(u_{f_1}(x,t)) - f_2(u_{f_2}(x,t)) = (f_1(u_{f_1}(x,t)) - f_1(u_{f_2}(x,t))) + (f_1(u_{f_2}(x,t)) - f_2(u_{f_2}(x,t)))$$
to rewrite the equation (5.1) as follows

\[
\sum_{i=1}^{n} p_i(x,t) \partial_{t}^{\alpha_i} u(x,t) + \sum_{j=1}^{m} q_j(x,t) \partial_{x}^{3j} u(x,t) = (r(x,t) + g(x,t)) u(x,t) + (f_1(u_{f_2}(x,t)) - f_2(u_{f_2}(x,t))), \quad (x,t) \in Q_T.
\]

In the last equation, \((r + g)(x,t) \leq 0, \quad (f_1(u_{f_2}(x,t)) - f_2(u_{f_2}(x,t)) \leq 0\) for \((x,t) \in Q_T,\) \(u(0,t) \leq 0\) for \(0 \leq t \leq T\) and \(u(x,0) \leq 0\) for \(0 \leq x \leq l.\) Thus we can apply Theorem 3.4 (the formula (2.4)) and obtain the inequality \(u(x,t) \leq 0\) for \((x,t) \in \overline{Q_T},\) that is, \(u_{f_1}(x,t) \leq u_{f_2}(x,t)\) for \((x,t) \in Q_T.\)

The last remark concerns Theorem 3.5 for the Cauchy problem (2.8)-(2.9) for the time-fractional transport equation. We state that the result formulated in Theorem 3.5 is valid for a more general time-fractional transport equation in place of the equation (2.8):

\[
\partial_{t}^{\alpha_n} u(x,t) + \sum_{i=1}^{n-1} p_i(t) \partial_{t}^{\alpha_i} u(x,t) + q(x) \partial_{x} u(x,t) = F(x,t), \quad x \in \mathbb{R}, \quad 0 < t < T,
\]

where \(0 < \alpha_1 < ... < \alpha_n \leq 1\) and \(p_i(t) \geq 0\) for \(0 \leq t \leq T\) and \(1 \leq i \leq n - 1.\)

In the rest of this section, we present a short sketch of its proof. The results presented in Chapter 3 of [7] ensure existence and uniqueness of solution \(u_0 = u_0(t)\) to the initial-value problem

\[
\partial_{t}^{\alpha_n} u_0(t) + \sum_{i=1}^{n-1} p_i(t) \partial_{t}^{\alpha_i} u_0(t) = 1, \quad 0 < t < T, \quad u_0(0) = 0.
\]

Its solution \(u_0 = u_0(t)\) is employed to define an auxiliary function in the form

\[
w(x,t) := u(x,t) - \sup_{x \in \mathbb{R}} a(x) - \delta \left( u_0(t) - \int_{0}^{x} \frac{1}{q(\xi)} d\xi \right), \quad x \in \mathbb{R}, \quad 0 < t < T.
\]

Now we suitably modify the condition (4.1), choose \(L > 0\) sufficiently large, and proceed as in the proof of Theorem 3.5 from Section 5. A complete version of the proof will be presented elsewhere.

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REFERENCES

[1] M. Al-Refai and Yu. Luchko, Maximum principles for the fractional diffusion equations with the Riemann-Liouville fractional derivative and their applications, Fract. Calc. Appl. Anal. 17 (2014) 483-498.

[2] D.A. Benson, M.M. Meerschaert, J. Revielle. Fractional calculus in hydrologic modeling: A numerical perspective. Advances in water resources 51 (2013), 479–497.

[3] L.C. Evans. Partial Differential Equations, Amer. Math. Soc., Providence, Rhode Island, 1998.

[4] Y. Kian and M. Yamamoto, On existence and uniqueness of solutions for semilinear fractional wave equations, Fract. Calc. Appl. Anal. 20 (2017) 117–138.

[5] B. Li, W. Xie, Adaptive fractional differential approach and its application to medical image enhancement, Computers and Electrical Engineering, 45, pp. 324–335, 2015.

[6] A.N. Kochubei, General fractional calculus, evolution equations, and renewal processes, Integr. Equa. Oper. Theory 17 (2011), 583–600.

[7] A. Kubica, K. Ryszewska, and M. Yamamoto, Introduction to a Theory of Time-fractional Partial Differential Equations, Springer Japan, Tokyo, 2020.

[8] Yu. Luchko, Maximum principle for the generalized time-fractional diffusion equation, J. Math. Anal. Appl. 351 (2009) 218–223.

[9] Yu. Luchko, Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation, Computers and Mathematics with Applications 59 (2010), 1766–1772.

[10] Yu. Luchko, Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation, J. Math. Anal. Appl. 374 (2011), 538–548.

[11] Yu. Luchko and M. Yamamoto General time-fractional diffusion equation: Some uniqueness and existence results for the initial-boundary-value problems. Fract. Calc. Appl. Anal. 19 (2016), 676–695.

[12] Yu. Luchko and M. Yamamoto, On the maximum principle for a time-fractional diffusion equation, Frac. Calc. Appl. Anal. 20 (2017), 1131-1145.

[13] Yu. Luchko and M. Yamamoto, A Survey on the Recent Results Regarding Maximum Principles for the Time-Fractional Diffusion Equations. Chapter in: Bhalekar, Sachin (Ed.), Frontiers in Fractional Calculus, Bentham Science Publishers, Sharjah, United Arab Emirates, 2018, pp. 33-69.

[14] Yu. Luchko and M. Yamamoto, Maximum principle for the time-fractional PDEs. Chapter in: A. Kochubei, Yu. Luchko (Eds.), Handbook of Fractional Calculus with Applications. Vol.2: Fractional Differential Equations, Walter de Gruyter, Berlin/Boston, 2019, pp.299–326.

[15] R. Metzler, T. F. Nonnenmacher, Fractional relaxation processes and fractional rheological models for the description of a class of viscoelastic materials, International Journal of Plasticity, 19 (7) (2003), 941–959.
[16] T. Namba, On existence and uniqueness of viscosity solutions for second order fully nonlinear PDEs with Caputo time fractional derivatives, Nonlinear Differential Equations and Applications 25 (2018), article no. 23.

[17] A.D. Obembe, H.Y. Al-Yousef, M.E. Hosain, S.A. Abu-Khamsin, Fractional derivatives and their applications in reservoir engineering problems: a review. Journal of Petroleum Science and Engineering, 157 (2017), 312–327.

[18] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[19] M.H. Protter, H.F. Weinberger, Maximum Principles in Differential Equations, Springer-Verlag, New York, 1984.

[20] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, J. Math. Anal. Appl. 382 (2011), 426-447.

[21] A. Schmidt, L. Gaul, On the numerical evaluation of fractional derivatives in multi-degree-of-freedom systems, Signal Processing, 86, no. 10, pp. 2592–2601, 2006.

[22] A. Suzuki, Y. Niibori, S.A. Fomin, V.A. Chugunov, T. Hashida, Fractional derivative-based tracer analysis method for the characterization of mass transport in fractured geothermal reservoirs, Geothermics, 53 (2015), 125–132.

[23] A. Suzuki, Y. Niibori, S.A. Fomin, V.A. Chugunov, T. Hashida, Prediction of reinjection effects in fault-related subsidiary structures by using fractional derivative-based mathematical models for sustainable design of geothermal reservoirs, Geothermics, 57 (2015), 196–204.

[24] A. Suzuki, S.A. Fomin, V.A. Chugunov, Y. Niibori, T. Hashida, Fractional diffusion modeling of heat transfer in porous and fractured media, International Journal of Heat and Mass Transfer, 103 (2016), 611–618.

[25] S. Suzuki, S. A. Fomin, V. A. Chugunov, T. Hashida, Mathematical modeling of non-fickian diffusional mass exchange of radioactive contaminants in geological disposal formations, Water, 10 (2) (2018), 123.

[26] A. Suzuki, T. Hashida, K. Li, R. N. Horne, Experimental tests of truncated diffusion in fault damage zones, Water Resources Research, 52(11) (2016), 8578–8589.

[27] J. West, Fractional calculus in bioengineering, Journal of Statistical Physics, 126(6), (2007), 1285–1286.

[28] R. Zacher, Boundedness of weak solutions to evolutionary partial integro-differential equations with discontinuous coefficients, J. Math. Anal. Appl. 348 (2008), 137–149.

[29] R. Zacher, Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces, Funkcial. Ekvac. 52 (2009), 1-18.

[30] Y. Zhang, H. Sun, H. H. Stowell, M. Zayernouri, S. E. Hansen, S. E. A review of applications of fractional calculus in Earth system dynamics. Chaos, Solitons and Fractals, 102 (2017), 29–46.