Supplementary Materials for

Coexistence of a new type of bound state in the continuum and a lasing threshold mode induced by PT symmetry

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Section S1. Rigorous proof of robust BICs

From Eq. (1) of the main text, the matrix $M$ for the dimerized chain without gain and loss can be expressed as

$$
M = \begin{bmatrix}
\alpha_0^{-1} - S_{\infty} & -S_{AB} \\
-S_{BA} & \alpha_0^{-1} - S_{\infty}
\end{bmatrix},
$$

where $\alpha_0$ contains the term of radiation loss. To study the resonant mode, a unitary transformation
of the matrix $M$ can be introduced,

$$
M' = \begin{bmatrix}
e^{-i\alpha_0 / 2} & 0 & 0 & e^{i\alpha_0 / 2} \\
0 & e^{-i\alpha_0 / 2} & 0 & e^{i\alpha_0 / 2} \\
S_A & 0 & 0 & S_Ae^{i\alpha_0} \\
0 & 0 & 0 & S_Ae^{-i\alpha_0} \\
\end{bmatrix} = \begin{bmatrix}
\alpha_0^{-1} - S_{00} - S_{00}e^{i\alpha_0} \\
-S_{0A}e^{i\alpha_0} \\
\alpha_0^{-1} - S_{00} \\
0 \\
\end{bmatrix}.
$$

(S1)

The eigenstates can be expressed as $P' = \left(p'_A, p'_B\right)^T = \left(p_Ae^{i\alpha_0 / 2}, p_Be^{i\alpha_0 / 2}\right)^T$ which corresponds to the periodic part of the Bloch wave function. By applying an eigen-decomposition method, the eigenvalues and eigenstates of matrix $M'$ can be studied analytically.

A real-frequency solution of $M'(q, \omega)P'_0 = 0$ gives rise to a BIC at $q$ with $P'_0$ being the eigenstate of BIC. Since a BIC does not radiate, the two dipole moments in the cell-periodic wave function must be out of phase in order to produce complete destructive interference of the radiation in the far field. Thus, we can write (34)

$$
P'_0 = \begin{bmatrix} p'_A \\ p'_B \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
$$

(S2)

Substituting Eqs. (S1) and (S2) into $M'P'_0 = 0$, we obtain the condition for a BIC as the real-frequency solution of the following equations:

$$
M'_{11} - M'_{12} = (\alpha_0^{-1} - S_{00}) + S_{AB}e^{i\alpha_0} = f(q, \omega) = 0,
$$

$$
M'_{22} - M'_{21} = S_{BA}e^{-i\alpha_0} + (\alpha_0^{-1} - S_{00}) = h(q, \omega) = 0,
$$

(S3)

which is Eq. (3) in the main text. We have already proved the identity $h(-q, \omega) = f(q, \omega)$ from inversion symmetry, i.e., Eq. (8) of the main text. Here, we will prove the identity $f(-q, \omega) = f^*(q, \omega)$ at the BIC by applying time-reversal operation on $M'(q, \omega)P'_0 = 0$, i.e.,

$$
\Theta M'\Theta^{-1} \Theta P' = M''P'' = 0.
$$

where $\Theta$ is the time-reversal operator. Since the frequency is real at the BIC, we have $M''(q, \omega)P''_0 = 0$, where $P''_0$ now represents the cell-periodic wave function of a BIC located at $-q$. The Bloch wave function of this BIC now has the form $\frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_0 / 2} \\ -e^{-i\alpha_0 / 2} \end{bmatrix}$. As a result, we have the following identities:

$$
M''_{11}(q, \omega) - M''_{12}(q, \omega) = M'_1(-q, \omega) - M'_1(-q, \omega) = 0,
$$

$$
M''_{22}(q, \omega) - M''_{21}(q, \omega) = M'_2(-q, \omega) - M'_2(-q, \omega) = 0^*.
$$

from which, we obtain

$$
\left(\alpha_0^{-1}(q, \omega)\right)^* - S'_{00}(q, \omega) + \left(S_{AB}(q, \omega)e^{i\alpha_0}\right)^* = \alpha_0^{-1}(-q, \omega) - S_{00}(-q, \omega) + S_{AB}(-q, \omega)e^{-i\alpha_0},
$$

where $\Theta$ is the time-reversal operator.
i.e.,
\[ f(-q,\omega) = f^*(q,\omega). \tag{S4} \]

Combining Eq. (8) and Eq. (S4), we have \[ f(q,\omega) = h^*(q,\omega). \]

In fact, Eq. (S4) holds not only at BICs, but in general in the region of zeroth order diffraction, i.e., \[ |q| < \frac{\omega}{c} < \frac{2\pi}{a} - |q|. \]
To prove this, we need to prove the following theorem:

If a function is defined by \[ f(q,\omega) = M'_{11} - M'_{12}, \]
where \( M'_{ij} \) are the matrix elements of linear response matrix \( M' \), \( f(-q,\omega) = f^*(q,\omega) \) is always satisfied for \( |q| < \frac{\omega}{c} < \frac{2\pi}{a} - |q|. \)

If we substitute the lattice sum in Eq. (7) of the main text in \( f(-q,\omega) = f^*(q,\omega) \), the theorem can be restated as

\[ u(q,x) = 2/3, \tag{S5} \]

where

\[
\begin{align*}
3u(q,x) &= \left[ (S'_{00}(q) - S_{00}(-q)) - (S'_{AB}(q) - S_{AB}(-q)) \right] e^{-iq\omega} / (2ikq^3) \\
&= \sum_n \left( \sin[(n+b)x] / |(n+b)x| + \cos[(n+b)x] / |(n+b)x|^2 - \sin[(n+b)x] / |(n+b)x|^3 \right) \delta^{(n+b)} \left( \sin|nx| + \cos|nx| - \sin|nx| / |nx|^3 \right) e^{ixq}. 
\end{align*}
\]

In the above summation, \( x = ka \), \( b = d/a \). The rigorous mathematical proof of the identity \( u(q,x) = 2/3 \) is given in the Section S6. A numerical verification is shown in Fig. S1 below.

**Fig. S1.** Examples of numerical error \( \delta u = |u(q,x) - 2/3| / (2/3) \) in the region of \( |q|a < x < 2\pi - |q|a \).

**Section S2. Analysis of the divergence rate of Q-factor**

Here we derive analytically the divergent behaviors of the Q-factor near a BIC and \( pt \)-BIC. In the absence of gain or loss, let \( \lambda(q,\omega) \) be the eigenvalue of the matrix \( M \) in Eq. (1) with real-frequency \( \omega \). For a BIC point at \((q_0,\omega_0)\), we have \( \lambda(q_0,\omega) = 0 \). Close to this point, we can expand \( \lambda(q,\omega) \) at small \( \delta q = q - q_0 \) and \( \delta \omega = \omega - \omega_0 \) into
\[ \delta \lambda = \frac{\partial \lambda}{\partial q} \delta q + \frac{1}{2} \frac{\partial^2 \lambda}{\partial q^2} \delta q^2 + \frac{\partial \lambda}{\partial \omega} \delta \omega + \ldots . \]

The \( \delta \lambda = 0 \) condition for a resonate mode can only be satisfied if the frequency becomes complex, i.e.,

\[ \frac{\partial \lambda}{\partial q} \delta q + \frac{1}{2} \frac{\partial^2 \lambda}{\partial q^2} \delta q^2 + \frac{\partial \lambda}{\partial \omega} (\delta \omega^* + i \delta \omega^*) \sim 0 . \]

(S6)

In order to find the behaviors of the Q-factor, in Fig. S2A, we plot the real and imaginary parts of the function \( \lambda(q,\omega) \) for the case of no gain and loss. It can be clearly seen that \( \text{Im}(\lambda) \leq 0 \) (34,41). There are nodal lines of \( \text{Im}(\lambda) \) lying in the Brillouin zone symmetrically. Along the dispersion of the leaky channel, a resonant mode will become a BIC when \( \text{Im}(\lambda) = 0 \). Thus \( \text{Im}(\lambda) \) reaches a local maximum with \( \partial \text{Im}(\lambda)/\partial q = 0 \) and \( \partial \text{Im}(\lambda)/\partial \omega = 0 \) at the point of BIC, and hence both \( \partial \lambda/\partial q \) and \( \partial \lambda/\partial \omega \) are purely real. The imaginary part of Eq. (S6) gives

\[ \delta \omega^* \sim -\frac{1}{2} \frac{\partial^2 \text{Im}(\lambda)}{\partial q^2} \delta q^2 \left( \left| \frac{\partial \lambda}{\partial \omega} \right| \right) , \]

where the second derivative \( \partial^2 \text{Im}(\lambda)/\partial q^2 < 0 \) along the dispersion.

Fig. S2. Eigenvalue \( \lambda \) of the matrix \( M \). Re(\( \lambda \)) and Im(\( \lambda \)) are given separately. The nodal lines of Re(\( \lambda \)) (black line) exhibit the dispersion of resonant modes. They intersect the nodal lines of Im(\( \lambda \)) at three points as indicated by the circles in (A), giving rise to three BICs with \( \lambda = 0 \). Three pairs of nodal lines of Im(\( \lambda \)) appear in (B) under the PT-symmetric parameter \( \gamma = 1.7 \times 10^{-3}/\mu_0 \).

Since we have \( \delta \omega^* \propto \delta q^2 \), the Q-factor is proportional to \( 1/\delta q^2 \) for the case of no gain or loss. This is consistent with the numerical results shown in Fig. S3A as well as the previous results in Refs. (35-37). The Q-factors for the modes near the symmetry-protected and propagating BICs are respectively indicated by stars and circles in Fig. S3A.
In the presence of a PT-symmetric perturbation, we study the eigenvalue $\lambda(q, \omega)$ of the matrix $M' + i\gamma \sigma_z$ in Eq. (4) of the main text. Its real and imaginary parts are shown in Fig S2B. In this case, $\text{Im}(\lambda)$, however, is not semi-negative definite and crosses the nodal line linearly near a \textit{pt}-BIC point and $\partial \text{Im}(\lambda)/\partial q$ is not zero. Thus, we have

$$\delta \omega'' = -\text{Im} \left( \frac{\partial \lambda}{\partial q} \right) \frac{\delta q}{\delta \omega}.$$ 

Therefore, $\delta \omega' \propto \delta q$ and the Q-factor is proportional to $1/\delta q$. This shows that the \textit{pt}-BICs have a reduced Q-factor divergence rate and belong to a new class of BICs. A numerical verification is shown in Fig. S3B. The Q-factors we consider is only in the stable regions. The Q-factors for the \textit{pt}-BICs all diverge at a reduced rate, i.e., $Q \propto 1/\delta q$.

![Fig. S3. Divergence rates of Q-factors.](image)

For $\gamma = 0$, $Q \propto 1/\delta q^2$. The Q-factors for the resonant modes near the symmetry-protected and propagating BICs are respectively indicated by stars and circles in (A). The original BICs split into pairs of \textit{pt}-BICs and lasing threshold modes under the PT-symmetric parameter $\gamma = 1.7 \times 10^{-3}/\mu_0$. The divergence rate of $Q$ becomes $1/\delta q$ for all the \textit{pt}-BICs and lasing threshold modes. The polygons in (B) represent Q-factors for the resonant modes near the \textit{pt}-BICs in Fig. 3B of the main text.

\textbf{Section S3. Energy stored in a resonant mode}

The energy stored in a resonant mode can be calculated from the following formula in Gaussian units:

$$U_{\text{st}} = \frac{1}{16\pi} \int_{\gamma} \text{Re} \left\{ \frac{d(\mu \sigma)}{d\omega} |E|^2 + \frac{d(\sigma \omega)}{d\omega} |H|^2 \right\} dV,$$

where $E$ and $H$ are the electric and magnetic fields of the eigenmode, respectively. For a bound
state, \( E \) and \( H \) are localized in space and \( U_{\text{eff}} \) is integrable. However, unlike the bound state, a resonant mode will radiate to infinity, and \( U_{\text{eff}} \) becomes non-integrable. To estimate \( U_{\text{eff}} \), for a resonant mode, the radiation field should be excluded from the total field. As mentioned in the main text, we focus on the region of \( |q| < \frac{\omega}{c} < \frac{2\pi}{a} - |q| \), in which only one cylindrical wave is associated with the diffraction order \( n=0 \). The radiating part of this system

\[
A(\rho) = \hat{e}_r \frac{i \sqrt{\mu_0}}{2} e^{i\omega t} H_0^{(1)}(k_{\bot,0}\rho),
\]

where \( k_{\bot,0} = \sqrt{k_0^2 - q^2} \). From \( B = \nabla \times A \) and \( E = -\frac{c}{i\omega} \nabla \times H \), the electric field of this system can be calculated as

\[
E_\rho = -\frac{c}{i\omega \mu_0} \left( \frac{1}{2 \rho} \hat{\rho} + ik_\rho \right) \sin \phi A_y,
E_\phi = -\frac{c}{i\omega \mu_0} \left( \frac{3}{4 \rho^2} + i \frac{k_\rho}{\rho} + k_\phi^2 \right) \cos \phi A_y,
E_z = -\frac{c}{i\omega \mu_0} \left( \frac{i}{2 \rho} - ik_\rho \right) \sin \phi A_y.
\]

In the far field with the radial distance \( \rho \gg 1 \), we have \( A_y \sim \frac{i \sqrt{\mu_0}}{2} e^{i\omega t} H_0^{(1)}(k_{\bot,0}\rho) \) and

\[
H_0^{(1)}(k_{\bot,0}\rho) \sim \sqrt{\frac{2}{\pi k_{\bot,0}}} e^{i\rho} e^{-i\phi/4}.
\]

In the calculation of \( U_{\text{eff}} \), we subtract the radiating parts of the electromagnetic fields from the total fields obtained from the COMSOL simulations.

**Section S4. Polarizability of coated nanoparticles**

If the radius of nanoparticles is much smaller than the working wavelength, the polarizability of each nanoparticle can be expressed as

\[
\alpha(\omega) = n_0^3 \varepsilon(\omega) \left( \frac{\varepsilon(\omega) - 1}{\varepsilon(\omega) + 2} \right),
\]

under the quasi-static approximation.

Here \( \varepsilon(\omega) \) is the relative dielectric constant. If the radiation correction is included,

\[
\alpha^{-1} = \frac{1}{n_0^3} \frac{\varepsilon + 2}{\varepsilon - 1} - i \frac{2}{3} k_0^3 \varepsilon(\omega) (33,41)\]  

In the study of BICs, we are primarily interested in the existence of a bound state in continuum which is decoupled from the external field. Like previous works of BICs, we ignore the intrinsic loss of the materials and focus on the radiation loss, so \( \varepsilon(\omega) \) is treated as a real function. However, in a realistic experimental situation, when the PT-symmetric perturbation is added to the system, we need to consider the intrinsic loss of the nanoparticles. Thus, we can write \( \varepsilon = \varepsilon^* + i\varepsilon^" \), where \( \varepsilon^" > 0 \) denotes the intrinsic loss. Substituting this complex \( \varepsilon \)
into the above expression for $\alpha$ gives

$$\alpha^{-1} - \frac{1}{r_0} \left( \alpha' + 2i\alpha'' \right) - i\frac{2}{3}k_0^3 \approx \frac{1}{r_0} \left( \alpha' + 2 \right) \left( \alpha' - 1 \right) \left( 1 - \frac{3i\alpha''}{(\alpha' + 2)(\alpha' - 1)} \right) - i\frac{2}{3}k_0^3.$$ 

If we assume $\alpha'' \ll \alpha'$, we have $\Im(\alpha^{-1}) = \frac{1}{r_0} \left( \frac{-3\alpha''}{\alpha' - 1} \right) - \frac{2}{3}k_0^3$. Thus, $\Im(\alpha^{-1}) + 2k_0^3/3$ denotes the amount of intrinsic loss of the nanoparticles, which can also be expressed as $\Im(\alpha^{-1} - \alpha_0^{-1})$ if we use the expression of $\alpha_0^{-1}(\omega) = \frac{1}{r_0} \left( \frac{\alpha' + 2}{\alpha' - 1} \right) - i\frac{2}{3}k_0^3$ defined in the main text.

To introduce the PT-symmetric perturbation, we coat a thin spherical shell on the surface of the nanoparticle with a relative permittivity $\varepsilon_i = \varepsilon_i' + i\varepsilon_i''$. The outer and inner radii for this coated sphere are respectively $r_0$ and $r_1$. Under the quasi-static approximation, the polarizability of the coated nanoparticle can be expressed as (42)

$$\alpha(\omega) = \frac{1}{r_0} \left( \varepsilon_i - 1 \right)(\varepsilon' + 2\varepsilon_i) + f(\varepsilon - \varepsilon_1)(1 + 2\varepsilon_1)$$

$$\left( \varepsilon_i + 2 \right)(\varepsilon + 2\varepsilon_i) + 2f(\varepsilon - \varepsilon_1)(1 + 2\varepsilon_1),$$

where $f = r_1^3 / r_0^3$. If the radiation correction is considered,

$$\alpha^{-1} = \frac{1}{r_0^3} \left( \varepsilon_i - 1 \right)(\varepsilon' + 2\varepsilon_i) + f(\varepsilon - \varepsilon_1)(1 + 2\varepsilon_1) - i\frac{2}{3}k_0^3.$$ 

By substituting the expressions $\varepsilon = \varepsilon' + i\varepsilon''$ and $\varepsilon_i = \varepsilon_i' + i\varepsilon_i''$ into the above equation, we obtain $\Im(\alpha^{-1}) + 2k_0^3/3 = C_1\varepsilon'' + C_2\varepsilon_i''$, or

$$\gamma = \Im(\alpha^{-1} - \alpha_0^{-1}) = C_1\varepsilon'' + C_2\varepsilon_i'',$$  \hspace{1cm} (S7)

where

$$C_1 = 3 \left[ -4\varepsilon_i''(f - 1) + \varepsilon_i''(f + 2) + \varepsilon_i''(2f + 1) \right](f - 1)/D,$$

$$C_2 = -27\varepsilon_i'' f / D,$$

and

$$D = \left( \varepsilon_i'' - 1 \right) \left( 2\varepsilon_i' + \varepsilon_i'' \right)^2 + f^2 \left( 2\varepsilon_i' + 1 \right)^2 \left( \varepsilon_i'' - \varepsilon_i' \right)^2$$

$$- 2f \left[ \varepsilon_i'' + 2\varepsilon_i''(2\varepsilon_i' - \varepsilon_i' - 1) + \varepsilon_i''(-2 - 2\varepsilon_i'' - \varepsilon_i') + \varepsilon_i'\varepsilon_i'(1 + \varepsilon_i') \right].$$

Thus, the amount of gain or loss required for particles A and B can be tuned separately by varying the value of $\varepsilon_i'' < 0$ for particle A or $\varepsilon_i'' > 0$ for particle B. For a given value of $\gamma$ in Eq.
(4) of the main text, the values of $\varepsilon_{1A}$ and $\varepsilon_{1B}$ required are determined separately by the following equations:

$$\text{Im}(\alpha_{1A}^{-1} - \alpha_{0}^{-1}) = \gamma = C_0 \varepsilon'' + C_2 \varepsilon_{1A}'',$$

and

$$\text{Im}(\alpha_{1B}^{-1} - \alpha_{0}^{-1}) = -\gamma = C_0 \varepsilon'' + C_2 \varepsilon_{1B}''.$$

If the relative permittivity of the nanoparticles is described by the Drude model, $\varepsilon(\omega) = 1 - \omega_p^2 / (\omega + i \gamma_0)$, where $\omega_p$ is the plasma frequency and $\gamma_0 = 0.001 \omega_p$ is the collision frequency. Taking the dimensionless plasma frequency $\tilde{\omega}_p = 0.95$ as an example [see, e.g., Fig. 1E of the main text], the BICs exists in a very narrow frequency range $0.510 \leq \omega \pi/2a \leq 0.517$, which is less than 1% of $\omega_p$. Below we give an example with $\omega = 0.515 (2 \pi c/\lambda)$ to show how to realize a PT-symmetric dimerized lattice using nanoparticles coated with dielectric shells.

The refractive index of the shells is chosen as 1.6. We take the PT-symmetric parameter $\gamma = 2 \times 10^{-3}/r_0^3$ as an example. Due to the intrinsic loss of the core, this PT-symmetric parameter for the lossy particle B can be achieved by tuning the radius of the core. In fact, for $r_0 = 20$ nm if we choose $f = 86\%$, we can realized $\gamma_B = -2 \times 10^{-3}/r_0^3$ for particle B. To obtain the PT symmetry, gain should be introduced into the dielectric shell of particle A. From Eq. (S7), we find that $\gamma_A = 2 \times 10^{-3}/r_0^3$ can be achieved when $\varepsilon_{1A}' = -0.091$. In this way the PT symmetric potential $\gamma_A$ (gain) = $-\gamma_B$ (loss) is realized. Importantly, the gain is relatively small and could be experimentally realized (43,44).

**Section S5. Excitation by an external plane wave and point sources**

Now we consider the scattering of a plane wave $E_{\text{ext}} = E_0 e^{i(kx + qz - \omega t)}$ at an arbitrary $q$ in a stable region where $\omega''$ is negative. For the cases of $q = \pm 0.3$, we numerically calculate the absorption ($C_{\text{abs}}$), scattering ($C_{\text{sca}}$), and extinction ($C_{\text{ext}}$) cross-sections and plot the results in Fig. S4B. We find that the absorption and scattering cross-sections of the two modes are very different. In particular, the absorption cross-section at $q = 0.3$ ($q = -0.3$) is negative (positive), unlike the results shown in Fig. 4B of the main text where $\gamma_{\text{abs}} > 0$ ($\gamma_{\text{abs}} < 0$) for $q = 0.3$ ($q = -0.3$). This is due to the fact that both the $\lambda-$ and $\lambda+$ modes defined in Eq. (5) of the main text are excited under the illumination of a plane wave. Furthermore, the interference of these two resonant modes also
produces a Fano-like scattering behavior for the scattering cross-sections. Despite the differences in absorption ($C_{\text{abs}}$) and scattering ($C_{\text{sca}}$) cross-sections, the extinction cross-sections ($C_{\text{ext}}$) of the two modes at $q = 0.3$ and $q = -0.3$ are numerically identical as expected due to the symmetry of $\omega^s(q) = \omega^s(-q)$.

Fig. S4. Dimerized chain illuminated by a plane wave. (A) Schematic of the dimerized chain. (B) Scattering, absorption and extinction cross-sections of the two modes at $q = \pm 0.3$ in the stable regions.

We have also excited the resonant modes on the leaky channel using periodically arranged point sources (with a phase shift of $\pm qa$ between two adjacent unit cells) as shown in Fig. S5A. These point sources have the same magnitudes but different phases according to the Bloch condition. Each point source in a unit cell is located at equal distance from particles A and B, and at a distance $h$ from the axis of the chain. Here we set $h=2r_0$. The results shown below do not change sensitively to small variations of source location. We numerically evaluate the absorption ($P_{\text{abs}}$), scattering ($P_{\text{sca}}$), and extinction ($P_{\text{ext}}$) powers at five different pairs of $\pm q$ in the stable region. Their locations in the $q$ space are marked by circles in Fig. S5E. It is interesting to see that for all pairs, the extinction powers $P_{\text{ext}}$ for states at $q$ and $-q$ overlap completely as shown in Fig. S5B although the behaviors of $P_{\text{sca}}$ and $P_{\text{abs}}$ shown in Figs. S5C and S5D are very different. It is interesting to see that, contrary to the case of plane wave excitation, the absorption power is now positive (negative) for the positive (negative) $q$ modes, which is consistent with the results in Fig. 4B of the main text. This indicates that the point sources are more efficient than the plane-wave source for the excitation of the anti-symmetric plasmonic ($\lambda_-$) modes. To further confirm this finding, the dipole moments $p_A^q$ and $p_B^q$ at the peaks of $P_{\text{ext}}$ are calculated for different values of $q$. The relative amplitude and the relative phase of the two dipole moments are shown, respectively,
by circles in the left and right panels of Fig. S5E. It can be clearly seen that $|\rho_A| < |\rho_B|$ for positive $q$ modes and $|\rho_A| > |\rho_B|$ for negative $q$ modes. The analytical results obtained from Eq. (4) are also shown by solid lines. Again, we find excellent agreement between theory and simulation.

Fig. S5. Dimerized chain excited by periodically arranged point sources. (A) Schematic of a unit cell of the dimerized chain. (B) Extinction powers for both the positive $q$ modes (solid curves) and negative $q$ modes (dotted curves). The scattering and absorption powers are shown in (C) and (D), respectively. The absorption powers are positive (negative) for the positive (negative) $q$ modes, consistent with the results in Fig. 4B. (E) Ratio of dipole moments at the peaks of $P_{\text{ext}}$ (green circles) agrees well with the analytical results (black line) obtained from Eq. (4).

In order to show explicitly that a $pt$-BIC can indeed be excited, we have performed numerical simulations to obtain the asymptotic behavior of the excited dipole moments near a $pt$-BIC under the excitation of external fields. Both an array of point sources and plane waves are used as external fields. Firstly, as in the case of Fig. S5, periodically arranged point sources are used. We take the $pt$-BIC at $q=0.400$ in Fig. 3B as an example. The excited dipole moments $(\rho_A, \rho_B)$ at the peaks of $P_{\text{ext}}$ (extinction power) are simulated at different values of $q$. It is found that both $\text{Abs}(\rho_A)$ and $\text{Abs}(\rho_B)$ diverge when the $pt$-BIC is approached as shown in Fig. S6B below. The log-log plot of both the simulated $\text{Abs}(\rho_{A,B})$ and the analytically calculated $1/\lambda$ shown in Fig. S6C indicate that these two quantities are proportional to each other and they both diverge at the same rate when the
pt-BIC is approached. According to Eq. (5), this result clearly manifests a nonzero coupling strength $W_-$ for the pt-BICs.

The results of plane wave excitation are also shown in the right panel of Fig. S6, where the excited dipole moments $(p_A, p_B)$ are shown to follow the same divergence behavior as $1/\lambda_-$ as the pt-BIC is approached. In this case, due to the excitations of both $\lambda_-$ and $\lambda_+$ modes as indicated by the presence of a Fano-like spectrum shown in Fig. S4, the slopes of Log[Abs($p_A, p_B$)] and Log[Abs($1/\lambda_-$)] will gradually become parallel when the pt-BIC is approached and the $1/\lambda_-$ term becomes dominant.

**Fig. S6. Asymptotic behavior of the pt-BIC excited by external fields.** The dimerized chain is excited by an array of point sources and a plane wave in (A-C) and (D-F), respectively. (A and D) Dispersion of the leaky channel simulated by COMSOL Multiphysics (black solid line). The circles denote the frequencies of the peaks of $P_{\text{ext}}$. (C and F) The excited dipole moments Abs($p_{A,B}$) diverge at the same rate as that of Abs($1/\lambda_-$), which manifests a nonzero coupling strength at the pt-BICs.

**Section S6. Rigorous mathematical proof of $f(-q, \omega)=f^*(q, \omega)$**

Please see the next page.
If a function is defined by
\[ u(x, b; t) = \frac{\sin[(n + b)x]}{|(n + b)x|} + \frac{\cos[(n + b)x]}{|(n + b)x|^2} - \frac{\sin[(n + b)x]}{|(n + b)x|^3} \]  
where
\[ n M = \text{Riemann-Lebesgue theorem}. \]

In the above summation, \( x \) is the dimensionless frequency, \( b \) is the center-to-center distance and normalized by period. In fact, \( u(q, x) \) also depends explicitly on \( b \). We will prove that for \( b \in (0, 1) \) and \( |q| < 2\pi - |q| \), \( u(q, x) \) takes the same value.

**Lemma 1.** For \( x \in (0, 2\pi) \), \( b \in (0, 1) \), we obtain
\[
\begin{align*}
\sum_{k=1}^{\infty} \frac{\sin[(k+b)x]}{k+b} &= \int_{0}^{\pi} \left( \sum_{k=1}^{\infty} \frac{\cos((k+b)t)}{2\sin(\frac{t}{2})} \right) dt + \frac{\pi}{2}, \\
\sum_{k=1}^{\infty} \frac{\cos((k+b)x)}{(k+b)^2} &= \int_{0}^{\pi} \frac{\sin((\frac{1}{2}+b)t)}{2\sin(\frac{t}{2})} dt - \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{1}{(k+b)^2}, \\
\sum_{k=1}^{\infty} \frac{\sin((k+b)x)}{(k+b)^3} &= \int_{0}^{\pi} \sin((\frac{1}{2}+b)t) dt - \frac{\pi}{4} x^2 + x \sum_{k=1}^{\infty} \frac{1}{(k+b)^2}.
\end{align*}
\]

**Proof.** For \( b \in (0, 1) \) and \( x \in (0, 2\pi) \), we have
\[
\sum_{k=1}^{\infty} \frac{\sin[(k+b)x]}{k+b} = \int_{0}^{\pi} \left( \sum_{k=1}^{\infty} \frac{\cos((k+b)t)}{2\sin(\frac{t}{2})} \right) dt + \frac{\pi}{2} = \int_{0}^{\pi} \frac{\sin((\frac{1}{2}+b)t)}{2\sin(\frac{t}{2})} dt - \frac{\pi}{2} + \int_{0}^{\pi} \frac{\sin((n+1/2+b)t)}{t} dt.
\]

Note that the function
\[
\frac{1}{2\sin(\frac{t}{2})} - \frac{1}{t}
\]
is not singular at \( t = 0 \), the second integral tends to zero as \( n \to \infty \) by the Riemann-Lebesgue theorem.

Via a substitution \( s = (n + 1/2 + b)t \), we obtain
\[
\lim_{n \to \infty} \int_{0}^{\pi} \frac{\sin((n+1/2+b)t)}{t} dt = \lim_{n \to \infty} \int_{0}^{(n+1/2+b)x} \frac{\sin s}{s} ds = \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}.
\]

Therefore, we obtain (pointwise convergence),
\[
\sum_{k=1}^{\infty} \frac{\sin((k+b)x)}{k+b} = \int_{0}^{\pi} \frac{\sin((\frac{1}{2}+b)t)}{2\sin(\frac{t}{2})} dt + \frac{\pi}{2}, \quad x \in (0, 2\pi), \ b \in (0, 1).
\]

By the Dirichlet test, the summation of Eq. (S8) converges uniformly for \( x \in (\delta, 2\pi - \delta) \) for any \( 0 < \delta < \pi \). Integrating term by term, we derive
\[
\sum_{k=1}^{\infty} \frac{\cos((k+b)x)}{(k+b)^2} = \int_{\delta}^{\pi - \delta} \frac{\sin((\frac{1}{2}+b)t)}{2\sin(\frac{t}{2})} dt - \frac{\pi}{2} (x - \delta) + \sum_{k=1}^{\infty} \frac{\cos((k+b)x)}{(k+b)^2} = \int_{0}^{\pi} \frac{\sin((\frac{1}{2}+b)t)}{2\sin(\frac{t}{2})} dt - \frac{\pi}{2} x + \sum_{k=1}^{\infty} \frac{1}{(k+b)^2}.
\]

Note that the summation of Eq. (S9) is uniformly convergent for \( x \in [0, 2\pi] \). Analogously, through integration by part, we can prove the last equation.

Noting that the above results also hold for \( b \in (-1, 0] \). If we take \( b = 0 \), we obtain the following result.
Lemma 2. For $x \in (0, 2\pi)$, we have

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2},$$

(S10)

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{(x - \pi)^2}{4} - \frac{\pi^2}{12},$$

(S11)

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = \frac{x(2\pi - x)(\pi - x)}{12}.$$ (S12)

Theorem 1. The following results hold true for $|q| < x < 2\pi - |q|$, $b \in (0, 1)$,

$$- \sum_{n \neq 0} \frac{\sin \left( \frac{|nx|}{|nx|} \right)}{n^3} e^{iqn} + \Re \left\{ \sum_{n} \frac{\sin \left( \frac{(n + b)x}{(n + b)x} \right)}{n^3} e^{iqn} \right\} = 1,$$ (S13)

$$- \sum_{n \neq 0} \frac{\cos \left( \frac{|nx|}{|nx|} \right)}{n^2} e^{iqn} + \Re \left\{ \sum_{n} \frac{\cos \left( \frac{(n + b)x}{(n + b)x} \right)}{n^2} e^{iqn} \right\} = \frac{1}{x^2} \left[ - \frac{q^2}{2} - \frac{\pi^2}{3} + \frac{1}{b^2} + 2g(b) \right] - \frac{1}{2},$$ (S14)

$$\sum_{n \neq 0} \frac{\sin \left( \frac{|nx|}{|nx|} \right)}{n^2} e^{iqn} - \Re \left\{ \sum_{n} \frac{\sin \left( \frac{(n + b)x}{(n + b)x} \right)}{n^3} e^{iqn} \right\} = \frac{1}{x^2} \left[ \frac{\pi^2}{3} + \frac{q^2}{2} - 2g(b) - \frac{1}{b^2} \right] + \frac{1}{6},$$ (S15)

where

$$g(b) = \sum_{n=1}^{\infty} \frac{1}{(n + b)^2}.$$ (S16)

Proof. First, we reformulate the summation containing $b$ in Eq. (S13) as follows. Let us denote

$$* := \Re \left\{ \sum_{n} \frac{\sin \left( \frac{(n + b)x}{(n + b)x} \right)}{n} e^{iqn} \right\}$$

$$= \frac{1}{x} \left\{ \sin(bx) \cos(bq) + \sum_{n=1}^{\infty} \frac{\sin((n + b)x)(n + b)q}{n + b} n - b \right\}.\quad (\text{S16})$$

Since $|q| < x < 2\pi - |q|$, both $x + q$ and $x - q \in (0, 2\pi)$. By utilizing Eq. (S8), we obtain

$$\sum_{n=1}^{\infty} \frac{\sin((n + b)x)(n + b)q}{n + b} = \frac{1}{2} \left\{ \sum_{n=1}^{\infty} \frac{\sin((n + b)x)(n + b)q}{n + b} n + b \right\}$$

$$= \frac{1}{2} \left\{ \int_{0}^{x+q} - \frac{\sin \left( \frac{1}{2} + b \right)t}{2 \sin \left( \frac{1}{2} \right)} \, dt + \int_{0}^{x-q} - \frac{\sin \left( \frac{1}{2} + b \right)t}{2 \sin \left( \frac{1}{2} \right)} \, dt + \pi \right\}.$$ (S16)

Then, it is easy to compute

$$* = \frac{1}{x} \sin(bx) \cos(bq) + \frac{1}{2} \left[ \int_{0}^{x+q} \frac{\sin \left( \frac{1}{2} + b \right)t}{2 \sin \left( \frac{1}{2} \right)} \, dt + \int_{0}^{x-q} \frac{\sin \left( \frac{1}{2} + b \right)t}{2 \sin \left( \frac{1}{2} \right)} \, dt + 2\pi \right]$$

$$= \frac{1}{x} \left( \sin(bx) \cos(bq) \right) + \frac{1}{2} \left[ \int_{0}^{x+q} \cos(bt) \, dt + \int_{0}^{x-q} \cos(bt) \, dt + 2\pi \right] \right\} = \frac{\pi}{x}.\quad (\text{S17})$$

Moreover, we obtain from Eq. (S10) that

$$\sum_{n \neq 0} \frac{\sin \left( \frac{|nx|}{|nx|} \right)}{n} e^{iqn} = \frac{1}{x} \left\{ \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{iqn} + \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{-iqn} \right\}$$

$$= \frac{1}{x} \left\{ \sum_{n=1}^{\infty} \frac{2 \sin(nx) \cos(nq)}{n} \right\} \right\}= \frac{\pi}{x} \right\} = \frac{\pi}{x} - 1.$$ (S18)
By subtracting Eq. (S18) from Eq. (S17), we prove Eq. (S13). The other two equations Eqs. (S14) and (S15) can be proved analogously by using Lemma 1.

**Theorem 2.** For the imaginary part, the following results hold true for $|q| < x < 2\pi - |q|$, $b \in (0, 1)$,

\[
\Im \left\{ \sum_{n} \frac{\sin[(n + b)x]}{|(n + b)x|} e^{iq(n+b)} \right\} = 0, \quad \text{(S19)}
\]

\[
\Im \left\{ \sum_{n} \frac{\cos[(n + b)x]}{|(n + b)x|^2} e^{iq(n+b)} \right\} = \frac{1}{x^2} \left\{ \sum_{n=1}^{\infty} \frac{-2bq}{n^2 - b^2} + \frac{q}{b} \right\}, \quad \text{(S20)}
\]

\[
\Im \left\{ \sum_{n} \frac{\sin[(n + b)x]}{|(n + b)x|^3} e^{iq(n+b)} \right\} = \frac{1}{x^2} \left\{ \sum_{n=1}^{\infty} \frac{-2bq}{n^2 - b^2} + \frac{q}{b} \right\}. \quad \text{(S21)}
\]

**Proof.** By first computing the derivative and then integrating, we obtain for $x \in (0, 2\pi)$,

\[
f_N(x) := \sum_{n=1}^{N} \frac{\cos(n + b)x}{n + b} - \sum_{n=1}^{N} \frac{\cos(n - b)x}{n - b},
\]

\[
= \sum_{n=1}^{N} \frac{-2b}{n^2 - b^2} - \int_{0}^{x} \sin(bt) \frac{\sin(N + \frac{1}{2})t}{\sin(\frac{1}{2})} dt - \frac{\cos(bx) - 1}{b}. \quad \text{(S22)}
\]

Then, by rearranging the summation, we obtain

\[
\Im \sum_{n} \frac{\sin[(n + b)x]}{|(n + b)x|} e^{iq(n+b)}
\]

\[
= \frac{1}{x} \left\{ \sum_{n=1}^{\infty} \frac{\sin[(n + b)x] \sin[(n + b)q]}{n + b} - \sum_{n=1}^{\infty} \frac{\sin[(n - b)x] \sin[(n - b)q]}{n - b} \right\}
\]

\[
= \frac{1}{x} \left\{ \sum_{n=1}^{\infty} \frac{\cos(n + b)(x + q) - \cos(n + b)(x - q)}{2(n + b)} + \sum_{n=1}^{\infty} \frac{\cos(n - b)(x + q) - \cos(n - b)(x - q)}{2(n - b)} \right\}
\]

\[
= \frac{1}{x} \left\{ \frac{\sin(bx) \sin(bq)}{b} - \frac{1}{2} \lim_{N \to \infty} [f_N(x + q) - f_N(x - q)] \right\}.
\]

Since $x - q$, $x + q \in (0, 2\pi)$, the formula Eq. (S22) can be applied to $f_N(x - q)$ as well as $f_N(x + q)$. It is easy to obtain

\[
f_N(x + q) - f_N(x - q) = - \int_{x-q}^{x+q} \sin(bt) \frac{\sin(N + \frac{1}{2})t}{\sin(\frac{1}{2})} dt - \frac{\cos(b(x + q)) - \cos(b(x - q))}{b}
\]

\[
= - \int_{x-q}^{x+q} \sin(bt) \frac{\sin(N + \frac{1}{2})t}{\sin(\frac{1}{2})} dt + \frac{2\sin(bx) \sin(bq)}{b}.
\]

By the Riemann-Lebesgue lemma, the following holds true

\[
\lim_{N \to \infty} [f_N(x + q) - f_N(x - q)] = \frac{2\sin(bx) \sin(bq)}{b}.
\]

Then, Eq. (S19) is proved.

To prove Eq. (S20), we define

\[
g_N(x) := \sum_{n=1}^{N} \frac{\sin(n + b)x}{(n + b)b} - \sum_{n=1}^{N} \frac{\sin(n - b)x}{(n - b)^2}.
\]

Since $g_N(x)$ is the primitive of $f_N(x)$ and $g_N(0) = 0$, we obtain

\[
g_N(x) = \int_{0}^{x} f_N(\tau) d\tau = \sum_{n=1}^{N} \frac{-2bx}{n^2 - b^2} + \frac{x}{b} - \int_{0}^{x} \sin(bt) \frac{\sin(N + \frac{1}{2})t}{\sin(\frac{1}{2})} dt d\tau - \frac{\sin(bx)}{b^2}.
\]
Then, we obtain

\[ 3 \sum_n \cos \left( \frac{|(n+b)x|}{|(n+b)x|^2} \right) e^{i\theta(n+b)} \]

\[ = \frac{1}{x^2} \left\{ \frac{\cos(bx) \sin(bq)}{b^2} + \sum_{n=1}^{\infty} \frac{\cos((n+b)x \sin(n+b)q)}{(n+b)^2} - \sum_{n=1}^{\infty} \frac{\cos((n-b)x \sin(n-b)q)}{(n-b)^2} \right\} \]

\[ = \frac{1}{x^2} \left\{ \frac{\cos(bx) \sin(bq)}{b^2} + \sum_{n=1}^{\infty} \frac{\sin(n+b)(x+q) - \sin(n+b)(x-q)}{2(n+b)^2} - \sum_{n=1}^{\infty} \frac{\sin(n-b)(x+q) - \sin(n-b)(x-q)}{2(n-b)^2} \right\} \]

\[ = \frac{1}{x^2} \left\{ \frac{\cos(bx) \sin(bq)}{b^2} + \frac{1}{2} \lim_{N \to \infty} [g_N(x+q) - g_N(x-q)] \right\} , \]

while

\[ \lim_{N \to \infty} [g_N(x+q) - g_N(x-q)] = \sum_{n=1}^{\infty} \frac{-4bq}{n^2 - b^2} + \frac{2q}{b} - \frac{\sin(b(x+q)) - \sin(b(x-q))}{b^2} \]

\[ = \sum_{n=1}^{\infty} \frac{-4bq}{n^2 - b^2} + \frac{2q}{b} - \frac{2 \cos(bx) \sin(bq)}{b^2} . \]

Thus, we derive

\[ 3 \sum_n \cos \left( \frac{|(n+b)x|}{|(n+b)x|^2} \right) e^{i\theta(n+b)} = \frac{1}{x^2} \left\{ \sum_{n=1}^{\infty} \frac{-2bq}{n^2 - b^2} + \frac{q}{b} \right\} . \]

The last Eq. (S21) can also be proved analogously.

Finally, it is easy to deduce from Theorem 1 and Theorem 2 that for \( b \in (0, 1) \) and \( |q| < x < 2\pi - |q| \)

\[ u(q, x) = \sum_n \left\{ \frac{\sin(|(n+b)x|)}{|(n+b)x|^2} + \frac{\cos(|(n+b)x|)}{|(n+b)x|^2} - \frac{\sin(|(n+b)x|)}{|(n+b)x|^3} \right\} e^{i\theta(n+b)} \]

\[ = \sum_{n \neq 0} \left[ \frac{\sin |nx|}{|nx|^2} + \frac{\cos |nx|}{|nx|^2} - \frac{\sin |nx|}{|nx|^3} \right] e^{i\theta n} = (S13) + (S14) + (S15) + i[(S19) + (S20) - (S21)] \]

\[ = \frac{2}{3} . \]
References

1. J. von Neumann, E. Wigner, Über merkwürdige diskrete Eigenwerte. *Phys. Z* **30**, 465–467 (1929).

2. H. Friedrich, D. Wintgen, Interfering resonances and bound states in the continuum. *Phys. Rev. A* **32**, 3231 (1985).

3. E. Bulgakov, A. Sadreev, Formation of bound states in the continuum for a quantum dot with variable width. *Phys. Rev. B* **83**, 235321 (2011).

4. C. W. Hsu, B. Zhen, A. D. Stone, J. D. Joannopoulos, M. Soljačić, Bound states in the continuum. *Nat. Rev. Mater.* **1**, 16048 (2016).

5. M. I. Molina, A. E. Miroshnichenko, Y. S. Kivshar, Surface bound states in the continuum. *Phys. Rev. Lett.* **108**, 070401 (2012).

6. C. W. Hsu, B. Zhen, J. Lee, S.-L. Chua, S. G. Johnson, J. D. Joannopoulos, M. Soljačić, Observation of trapped light within the radiation continuum. *Nature* **499**, 188–191 (2013).

7. Y. Yang, C. Peng, Y. Liang, Z. Li, S. Noda, Analytical perspective for bound states in the continuum in photonic crystal slabs. *Phys. Rev. Lett.* **113**, 037401 (2014).

8. A. Cerjan, C. W. Hsu, M. C. Rechtsman, Bound states in the continuum through environmental design. *Phys. Rev. Lett.* **123**, 023902 (2019).

9. W. Chen, Y. Chen, W. Liu, Singularities and Poincaré indexes of electromagnetic multipoles. *Phys. Rev. Lett.* **122**, 153907 (2019).

10. S. W. Dai, L. Liu, D. Z. Han, J. Zi, From topologically protected coherent perfect reflection to bound states in the continuum. *Phys. Rev. B* **98**, 081405 (2018).

11. E. N. Bulgakov, A. F. Sadreev, Bloch bound states in the radiation continuum in a periodic array of dielectric rods. *Phys. Rev. A* **90**, 053801 (2014).

12. Y.-X. Xiao, G. C. Ma, Z.-Q. Zhang, C. T. Chan, Topological subspace-induced bound state in the continuum. *Phys. Rev. Lett.* **118**, 166803 (2017).
13. B. Zhen, C. W. Hsu, L. Lu, A. D. Stone, M. Soljačić, Topological nature of optical bound states in the continuum. *Phys. Rev. Lett.* **113**, 257401 (2014).

14. Y. Zhang, A. Chen, W. Liu, C. W. Hsu, B. Wang, F. Guan, X. Liu, L. Shi, L. Lu, J. Zi, Observation of optical vortices in momentum space. *Phys. Rev. Lett.* **120**, 186103 (2018).

15. H. M. Doeleman, F. Monticone, W. den Hollander, A. Alù, A. F. Koenderink, Experimental observation of a polarization vortex at an optical bound state in the continuum. *Nat. Photonics* **12**, 397–401 (2018).

16. E. N. Bulgakov, D. N. Maksimov, Topological bound states in the continuum in arrays of dielectric spheres. *Phys. Rev. Lett.* **118**, 267401 (2017).

17. Y. Guo, M. Xiao, S. Fan, Topologically protected complete polarization conversion. *Phys. Rev. Lett.* **119**, 167401 (2017).

18. Ş. K. Özdemir, S. Rotter, F. Nori, L. Yang, Parity-time symmetry and exceptional points in photonics. *Nat. Mater.* **1**, 783–798 (2019).

19. R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, D. N. Christodoulides, Non-Hermitian physics and PT symmetry. *Nat. Phys.* **14**, 11–19 (2018).

20. L. Feng, R. El-Ganainy, L. Ge, Non-Hermitian photonics based on parity-time symmetry. *Nat. Photonics* **11**, 752–762 (2017).

21. M.-A. Miri, A. Alù, Exceptional points in optics and photonics. *Science* **363**, eaar7709 (2019).

22. C. M. Bender, S. Boettcher, Real spectra in non-Hermitian Hamiltonians having \(\square\) symmetry. *Phys. Rev. Lett.* **80**, 5243 (1998).

23. C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, D. Kip, Observation of parity-time symmetry in optics. *Nat. Phys.* **6**, 192–195 (2010).

24. L. Feng, Y.-L. Xu, W. S. Fegadolli, M.-H. Lu, J. E. B. Oliveira, V. R. Almeida, Y.-F. Chen, A. Scherer, Experimental demonstration of a unidirectional reflectionless parity-time metamaterial at optical frequencies. *Nat. Mater.* **12**, 108–113 (2013).
25. B. Peng, Ş. K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, L. Yang, Parity-time symmetric whispering-gallery microcavities. *Nat. Phys.* **10**, 394–398 (2014).

26. Y.-L. Xu, W. S. Fegadelli, L. Gan, M.-H. Lu, X.-P. Liu, Z.-Y. Li, A. Scherer, Y.-F. Chen, Experimental realization of Bloch oscillations in a parity-time synthetic silicon photonic lattice. *Nat. Commun.* **7**, 11319 (2016).

27. L. Feng, Z. J. Wong, R.-M. Ma, Y. Wang, X. Zhang, Single-mode laser by parity-time symmetry breaking. *Science* **346**, 972–975 (2014).

28. B. Zhen, C. W. Hsu, Y. Igarashi, L. Lu, I. Kaminer, A. Pick, S.-L. Chua, J. D. Joannopoulos, M. Soljačić, Spawning rings of exceptional points out of Dirac cones. *Nature* **525**, 354–358 (2015).

29. S. Longhi, Bound states in the continuum in PT-symmetric optical lattices. *Opt. Lett.* **39**, 1697–1700 (2014).

30. M. I. Molina, Y. S. Kivshar, Embedded states in the continuum for $\mathcal{PT}$-symmetric systems. *Stud. Appl. Math.* **133**, 337–350 (2014).

31. Y. V. Kartashov, C. Milian, V. V. Konotop, L. Torner, Bound states in the continuum in a two-dimensional $\mathcal{PT}$-symmetric system. *Opt. Lett.* **43**, 575 (2018).

32. W. H. Weber, G. W. Ford, Propagation of optical excitations by dipolar interactions in metal nanoparticle chains. *Phys. Rev. B* **70**, 125429 (2004).

33. V. A. Markel, Antisymmetrical optical states. *J. Opt. Soc. Am. B* **12**, 1783–1791 (1995).

34. Q. Song, M. Zhao, L. Liu, J. Chai, G. He, H. Xiang, D. Z. Han, J. Zi, Observation of bound states in the continuum in the dimerized chain. *Phys. Rev. A* **100**, 023810 (2019).

35. E. N. Bulgakov, D. N. Maksimov, Bound states in the continuum and polarization singularities in periodic arrays of dielectric rods. *Phys. Rev. A* **96**, 063833 (2017).

36. J. Jin, X. Yin, L. Ni, M. Soljačić, B. Zhen, C. Peng, Topologically enabled ultra-high-$Q$ guided resonances robust to out-of-plane scattering. *Nature* **574**, 501–504 (2019).
37. L. Yuan, Y. Y. Lu, Strong resonances on periodic arrays of cylinders and optical bistability with weak incident waves. *Phys. Rev. A* **95**, 023834 (2017).

38. X. Gao, B. Zhen, M. Soljačić, H. S. Chen, C. W. Hsu, Bound states in the continuum in fiber Bragg gratings. *ACS Photonics* **6**, 2996–3002 (2019).

39. Z. J. Wong, Y.-L. Xu, J. Kim, K. O’Brien, Y. Wang, L. Feng, X. Zhang, Lasing and anti-lasing in a single cavity. *Nat. Photonics* **10**, 796–801 (2016).

40. H. Zhou, B. Zhen, C. W. Hsu, O. D. Miller, S. G. Johnson, J. D. Joannopoulos, M. Soljačić, Perfect single-sided radiation and absorption without mirrors. *Optica*, **3**, 1079–1086 (2016).

41. K. H. Fung, C. T. Chan, Plasmonic modes in periodic metal nanoparticle chains: A direct dynamic eigenmode analysis. *Opt. Lett.* **32**, 973–975 (2007).

42. C. F. Bohren, D. R. Huffman, *Absorption and Scattering of Light by Small Particles* (Wiley, New York, 1983).

43. S.-Y. Liu, J. Li, F. Zhou, L. Gan, Z.-Y. Li, Efficient surface plasmon amplification from gain-assisted gold nanorods. *Opt. Lett.* **36**, 1296–1298 (2011).

44. M. A. Noginov, G. Zhu, A. M. Belgrave, R. Bakker, V. M. Shalaev, E. E. Narimanov, S. Stout, E. Herz, T. Suteewongand, U. Wiesner, Demonstration of a spaser-based nanolaser. *Nature* **460**, 1110–1112 (2009).