A note on noncommutative CW-spectra

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January 14, 2022

Abstract

We use the machinery of [BM20] to give an alternative proof of one of the main results of [ABS21]. This result states that the category of noncommutative CW-spectra can be modelled as the category of spectral presheaves on a certain category $M$, whose objects can be thought of as “suspension spectra of matrix algebras”. The advantage of our proof is that it mainly relies on well-known results on (stable) model categories.

1 Introduction

The well-known Gelfand duality theorem states that the category of pointed compact Hausdorff spaces is dual to the category of commutative $C^*$-algebras. This result motivates the philosophy that general $C^*$-algebras are dual to “noncommutative spaces”. A natural question arising from this philosophy is what the homotopy theory of $C^*$-algebras (or noncommutative spaces) should look like, and a natural follow-up question is then to describe their stable homotopy theory. There have been many approaches to answering these questions, see e.g. [Tho03], [Øst10], [Uuy13], [BJM17].

The approach taken by Arone-Barnea-Schlank in [ABS21] is similar to how one constructs the homotopy theory of (pointed) CW-complexes and its stabilization, but with the role of the 0-sphere $S^0$ as basic building block replaced by the finite-dimensional matrix algebras $\{M_n\}_{n \geq 1}$. In particular, they construct an $\infty$-category of noncommutative CW-complexes and its stabilization, the $\infty$-category of noncommutative CW-spectra. One of their main results is then the following theorem.

**Theorem A ([ABS21]).** There exists a symmetric monoidal spectrum-enriched category $M_s$, whose objects can be thought of as suspension spectra of matrix algebras, such that the category of spectral presheaves on $M_s$ models the symmetric monoidal $\infty$-category of noncommutative CW-spectra when equipped with the projective model structure and the Day convolution product.

Their proof uses spectrum-enriched $\infty$-categories and an $\infty$-categorical version of a theorem by Schwede-Shipley [SS03b, Thm. 3.9.3.(iii)]. It was suggested in [ABS21, Rem. 1.4] that the techniques of [BM20] allow for a more direct proof of this theorem, using standard results on stable model categories. The aim of this note is to present such a proof. In particular, we show that the result above can be proved without the use of enriched $\infty$-categories, avoiding the technicalities that such an approach comes with. Furthermore, our proof has the added benefit of producing convenient model categories of noncommutative CW-complexes and noncommutative CW-spectra.
A rough outline of our proof is as follows: We first show that for a suitably chosen subcategory $D^{op}$ of the category of $C^*$-algebras, such as the category of separable $C^*$-algebras, one can endow $\text{Ind}(D)$ with a model structure whose cofibrant objects are exactly the (retracts of) noncommutative CW-complexes. This model category is then be stabilized by considering a certain model structure on the category functors $\text{sSet}^{\text{fin}} \to \text{Ind}(D)$. This produces a model category enriched over Lydakis’s stable model category of simplicial functors $[\text{Lyd}98]$, and one can then apply a modified version of $[\text{SS}03b, \text{Thm. 3.9.3.(iii)}]$ to conclude Theorem A.

The techniques of $[\text{BM}20]$ are used to construct the model category of noncommutative CW-complexes in Section 2.2. For the convenience of the reader, we discuss (a simplification of) these techniques in the appendix.

Throughout this note, we use the convention that all topological spaces are compactly generated weak Hausdorff.

**Notation.** The categories considered in this paper often admit several useful enrichments. To avoid confusion, we will generally write $\text{Map}_V(\cdot, \cdot)$ to denote the hom-objects of a $V$-enriched category, including the base of enrichment in the notation. For brevity, we will denote a simplicial enrichment by $\text{Map}(\cdot, \cdot)$ and an enrichment in pointed simplicial sets by $\text{Map}_*(\cdot, \cdot)$.

**Acknowledgements.** The author would like to thank Floris Elzinga and Makoto Yamashita for bringing to his attention a counterexample to the claim that the maximal tensor product of separable $C^*$-algebras preserves pullbacks.

## 2 Noncommutative CW-complexes

We start this section with a brief introduction to the category of $C^*$-algebras and its subcategories that we will be interested in. We then show how to construct a model category out of such a subcategory that describes the homotopy theory of noncommutative CW-complexes. Finally, we discuss the symmetric monoidal structure on this model category induced by the minimal and maximal tensor product of $C^*$-algebras.

### 2.1 The category of $C^*$-algebras

Let $C^*$ denote the category of (not necessarily unital) $C^*$-algebras and $*$-homomorphisms. The Gelfand duality theorem states that the category of pointed compact Hausdorff spaces $\text{CH}_{pt}$ is dual to the category of commutative $C^*$-algebras $\text{cC}^*$, with the equivalence in the direction $\text{CH}_{pt} \to (\text{cC}^*)^{op}$ given by sending a pointed space $(X, x)$ to the algebra $C_0(X)$ of continuous basepoint preserving functions $(X, x) \to (C, 0)$.

As described in $[\text{Uuy13}, \text{Rem. 2.5}]$, the category $\text{C}^*$ of $C^*$-algebras is enriched over the category $\text{Top}$ of (pointed) compactly generated weak Hausdorff spaces and admits cotensors by pointed compact Hausdorff spaces. Explicitly, for a $C^*$-algebra $B$ and a pointed compact Hausdorff space $(X, x)$, the cotensor $B^X$ in $\text{C}^*$ is defined as the $C^*$-algebra of continuous basepoint preserving maps $(X, x) \to (B, 0)$ endowed with the supremum norm. We view $C^*$ as enriched in $\text{Top}$ by forgetting the basepoints of the hom-spaces. The cotensor $B^Y$ of a $C^*$-algebra $B$ by an unpointed compact Hausdorff space
Y also exists and is given by the $\mathcal{C}^*$-algebra of all continuous maps $Y \to B$ (cf. [Uuy13 Lem. 2.4]). Equivalently, it is the cotensor of $B$ by the pointed compact Hausdorff space $Y_+$. Throughout the rest of this paper, we let $D^{op}$ be a full subcategory of $\mathcal{C}^*$ satisfying the following properties:

(D1) The category $D^{op}$ is essentially small.

(D2) For every $n \geq 1$, the $\mathcal{C}^*$-algebra $M_n$ of $n$-by-$n$ matrices is contained in $D^{op}$.

(D3) For any $A \in D^{op}$, the cotensor $A^I$ by the unit interval $I$ is also an object of $D^{op}$.

(D4) The full subcategory $D^{op}$ is closed under finite limits.

Examples of subcategories $D^{op}$ to keep in mind are those of all separable $\mathcal{C}^*$-algebras and those of all separable $\mathcal{C}^*$-algebras that are furthermore nuclear. Note that separability of a $\mathcal{C}^*$-algebra $A$ implies that there exists a countable subset $Z \subseteq A$ together with a surjection $Z^N \to A$, hence the cardinality of a separable $\mathcal{C}^*$-algebra is at most $2^{\aleph_0}$. This shows that (D1) must hold for these two examples. Furthermore, finite dimensional $\mathcal{C}^*$-algebras are always separable and nuclear, hence (D2) holds as well. We leave it as an exercise to the reader to verify (D3). Property (D4) follows since the terminal object is clearly nuclear and separable, and both the subcategories of nuclear and of separable $\mathcal{C}^*$-algebras are closed under pullbacks by [Ped99 Rem. 3.5].

**Remark 2.1.** It is worth pointing out that, up to equivalence, there is a minimal choice of a full subcategory $\mathcal{C}^*$ satisfying (D1)-(D4). Namely, let $\mathcal{D}$ be the class of all full subcategories of $\mathcal{C}^*$ satisfying (D1)-(D4) and that are furthermore isomorphism-closed. Then the intersection

$$D^{op}_{min} := \bigcap_{D^{op} \in \mathcal{D}} D^{op}$$

again satisfies (D1)-(D4) and is up to equivalence the smallest such full subcategory.

### 2.2 The model category of noncommutative CW-complexes

We will now construct a model category describing the homotopy theory of noncommutative CW-complexes. Let $D$ denote the opposite category of a full subcategory $D^{op}$ of $\mathcal{C}^*$ satisfying (D1)-(D4). For a $\mathcal{C}^*$-algebra $A$ in $D^{op}$, we will write $\overline{A}$ for the corresponding object in $D$ in an attempt to avoid confusion. In particular, since cotensors are formally dual to tensors, we see that $\overline{A} \otimes X = \overline{A^X}$ for all cotensors $A^X$ that $D^{op}$ admits. Let $M \subseteq \text{Ob}(D)$ denote the set of matrix algebras; that is, $M = \{\overline{M_n}\}_{n \geq 1}$.

It is easy to verify that the cotensor functor $(-)^I$ preserves finite limits, so $D$ satisfies the properties spelled out in Example A.2. In particular, the category $D$ together with the set of object $M$ is a minimal cofibration test category in the sense of Definition A.1, with its simplicial hom-sets defined by $\text{Map}(\overline{A}, \overline{B}) := \text{Sing}(\text{Map}_{\text{top}}(B, A))$. By combining Theorem A.3 and Remark A.4, we obtain the following result.

**Theorem 2.2.** Let $D$ be as above. There exists a cofibrantly generated simplicial model structure on $\text{Ind}(D)$ such that

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1We write $D^{op}$ for this subcategory since we will mainly work with its opposite category $D$ below.
(i) a map $C \rightarrow D$ is a weak equivalence or fibration if and only if for every $n \geq 1$, the map

$$\text{Map}(\bar{M}_n, C) \rightarrow \text{Map}(\bar{M}_n, D)$$

is a weak equivalence or Kan fibration, respectively,

(ii) any object of $\text{Ind}(D)$ is fibrant,

(iii) a set of generating cofibrations is given by

$$\{ \bar{M}_n \otimes \partial D^k \rightarrow \bar{M}_n \otimes D^k \mid n \geq 1, \ k \geq 0 \}$$

and a set of generating trivial cofibrations by

$$\{ \bar{M}_n \otimes (D^k \times \{0\}) \rightarrow \bar{M}_n \otimes (D^k \times 1) \mid n \geq 1, \ k \geq 0 \},$$

(iv) the weak equivalences are stable under filtered colimits.

We will call this model category the model category of (pointed) non-commutative CW-complexes.

Remark 2.3. Note that the $C^*$-algebra $\{0\}$ defines both an initial and a terminal object of $D$, and hence also of $\text{Ind}(D)$. In particular, the simplicial enrichment of $\text{Ind}(D)$ can be upgraded to an enrichment in the category of pointed simplicial sets, where the basepoint of $\text{Map}_*(C, D)$ is the unique map $C \rightarrow D$ that factors through $\{0\}$. This makes $\text{Ind}(D)$ into a $\text{sSet}_*$-enriched model category.

Remark 2.4. The category $\text{Ind}(D)$ clearly depends on the choice of subcategory $D^{op}$ of $C^*$. However, for the model structure this is not really the case; at least, not up to Quillen equivalence. To see this, first note that we may assume without loss of generality that $D^{op}$ is an isomorphism-closed full subcategory. By [Remark 2.1] we have an inclusion $D^{op}_{\text{min}} \hookrightarrow D^{op}$. A proof similar to that of Proposition 7.8 of [BM20] then shows that this inclusion induces a Quillen equivalence $\text{Ind}(D_{\text{min}}) \rightleftarrows \text{Ind}(D)$ when both categories are endowed with the model structure of [Theorem 2.2].

2.3 Finite cell complexes

Recall the definition of a (finite) cell complex in a cofibrantly generated model category from [Hiro3, Def. 10.5.8]. In the model category $\text{Ind}(D)$ from [Theorem 2.2] an object $C$ is a finite cell complex if the map $\emptyset \rightarrow C$ is obtained by attaching cells of the form $\bar{M}_n \otimes \partial D^k \rightarrow \bar{M}_n \otimes D^k$ a finite number of times; these are precisely the finite $M$-cell complexes in the terminology of [Definition A.10]. In particular, they can be viewed as object in $D$. It easy to see that that these are exactly the objects called finite pointed noncommutative CW-complexes in [ABS21, Def. 2.3]. Denote the full simplicial subcategory that they span by $\text{NCW}_f$.

Remark 2.5. Since we do not assume that cells may only be attached to lower dimensional cells, it would perhaps be better to call these objects noncommutative cell complexes and reserve the name noncommutative CW-complex for objects where this extra assumption is made.
The $\infty$-category of noncommutative pointed CW-complexes is defined in [ABS21] §2 as $\mathrm{Ind}_{\infty}(N_{hc}(\mathrm{NCW}_f))$, where $N_{hc}$ denotes the homotopy coherent nerve of a simplicial or topological category and $\mathrm{Ind}_{\infty}$ the $\infty$-categorical ind-completion in the sense of [Lur09] Def. 5.3.5.1. In particular, the following is a direct consequence of [Proposition A.11]

**Proposition 2.6.** The underlying $\infty$-category of the model category $\mathrm{Ind}(D)$ from [Theorem 2.2] is equivalent to the $\infty$-category of noncommutative pointed CW-complexes defined in [ABS21].

### 2.4 Tensor products

Both the maximal and the minimal tensor product endow $C^*$ with a symmetric monoidal structure. It is natural to ask whether these tensor products extend to tensor products on $\mathrm{Ind}(D)$ and how these interact with the model structure from [Theorem 2.2]. The first of these questions is easy to answer: if the (maximal or minimal) tensor product $\otimes$ restricts to a symmetric monoidal structure on $D^{op}$ (and hence on $D$), then the canonical extension

$$C \otimes D := \{c_i \otimes d_j\}_{(i,j) \in I \times J} \cong \lim_{(i,j) \in I \times J} c_i \otimes d_j, \quad \tag{2.1}$$

defines a symmetric monoidal structure on $\mathrm{Ind}(D)$, where $C = \{c_i\}_{i \in I}$ and $D = \{d_j\}_{j \in J}$ are arbitrary objects of $\mathrm{Ind}(D)$. Furthermore, if the tensor product on $D$ preserves finite colimits in both its variable, then its extension to $\mathrm{Ind}(D)$ admits a right adjoint in both of its variables, meaning that $\mathrm{Ind}(D)$ is closed monoidal. This follows since the filtered colimit preserving extension of a finite colimit preserving functor always admits a right adjoint (cf. [BM20] §2.2).

Let us consider the cases where $D^{op}$ is the category of separable or the category of nuclear separable $C^*$-algebras. In both cases $D^{op}$ is closed under the minimal as well as the maximal tensor product, hence they extend to symmetric monoidal structures on $\mathrm{Ind}(D)$. However, in the case where $D^{op}$ is the full subcategory of all separable $C^*$-algebras, neither of these tensor products preserve pullbacks, so they don’t make $\mathrm{Ind}(D)$ into a closed symmetric monoidal category.

However, in the case of nuclear $C^*$-algebras the situation is much better: these are by definition the $C^*$-algebras for which the minimal and maximal tensor products agree, and by [Ped99] Thm. 3.9 the tensor product of nuclear $C^*$-algebras preserves pullbacks (and hence finite limits) in each of its variables. In particular, taking $D^{op}$ to be the category of nuclear separable $C^*$-algebras, we obtain a closed symmetric monoidal structure on $\mathrm{Ind}(D)$. Moreover, this tensor product turns out to interact well with the model structure.

**Proposition 2.7.** If $D^{op}$ is the category of nuclear separable $C^*$-algebras, then $\mathrm{Ind}(D)$ is a symmetric monoidal model category when equipped with the monoidal structure given in (2.1).

**Proof.** It follows from the above that the tensor product on $D$ preserves finite colimits in each of its variables separately and hence that it extends to a closed symmetric monoidal structure on $\mathrm{Ind}(D)$. To see that it is a monoidal model category, note that the pushout

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2It is claimed in [Ped99] Rem. 3.10] that the maximal tensor product preserves pullbacks in both of its variables. However, since being a monomorphism can be expressed through a pullback diagram, this would imply that the maximal tensor product preserves monomorphisms. This is wrong for (separable) $C^*$-algebras in general, as also mentioned in [Ped99] Rem. 3.10]
product of $\overline{M}_n \otimes \partial D^k \to \overline{M}_n \otimes D^k$ and $\overline{M}_{n'} \otimes \partial D^{k'} \to \overline{M}_{n'} \otimes D^{k'}$ is

$$\overline{M}_{n \times n'} \otimes \left( D^k \times \partial D^{k'} \bigcup_{\partial D^k \times \partial D^{k'}} D^k \times D^{k'} \right) \to \overline{M}_{n \times n'} \otimes (D^k \times D^{k'})$$

and that the pushout product of $\overline{M}_n \otimes \partial D^k \to \overline{M}_n \otimes D^k$ and $\overline{M}_{n'} \otimes (D^{k'} \times \{0\}) \to \overline{M}_{n'} \otimes (D^{k'} \times I)$ is

$$\overline{M}_{n \times n'} \otimes \left( D^k \times D^{k'} \bigcup_{\partial D^k \times D^{k'} \times \{0\}} \partial D^k \times D^{k'} \times I \right) \to \overline{M}_{n \times n'} \otimes (D^k \times D^{k'} \times I).$$

Here we use that $M_n \otimes M_{n'} = M_{n \times n'}$ and that the maximal tensor product of $\mathbb{C}^*$-algebras is compatible with cotensors by compact Hausdorff spaces (cf. [Uuy13 Lem. 2.2]). It is clear that these maps are cofibrations and trivial cofibrations in $\text{Ind}(D)$, respectively, so by [Hov99 Cor. 4.2.5] we conclude that $\text{Ind}(D)$ is a symmetric monoidal model category. ■

### 3 Noncommutative CW-spectra

In this section we will study the stabilization of the model category of noncommutative CW-complexes. In order to obtain a stable model category that is enriched in some category of spectra, we will work with a stabilization based on Lydakis’ stable model category of simplicial functors. We prove that this stabilization is equivalent to the category of spectral presheaves on a certain category $M_\Delta$, and then show that this model category is Quillen equivalent to the category of spectral presheaves on the category $M_\triangledown$ defined in [ABS21]. In particular, this recovers one of the main results of that paper.

#### 3.1 Stabilizing the category of noncommutative CW-complexes

Write $\text{Sp}(\text{Ind}(D))$ for the category of (sequential) spectrum objects in $\text{Ind}(D)$ as defined in [Sch97 Def. 2.1.1]. By item (ii) of Theorem 2.2 the model structure for noncommutative CW-complexes is right proper, hence by [BR14 Thm. 5.23] the stable model structure on $\text{Sp}(\text{Ind}(D))$ exists.

**Proposition 3.1.** The underlying $\infty$-category of the stable model structure on $\text{Sp}(\text{Ind}(D))$ is the stabilization of the underlying $\infty$-category of $\text{Ind}(D)$. In particular, $\text{Sp}(\text{Ind}(D))$ models the $\infty$-category of noncommutative CW-spectra from [ABS21 §6].

**Proof.** This is proved analogously to [Rob14 Prop. 4.2.4]. ■

Unfortunately this model category does not come with a spectral enrichment, so Theorem 3.9.3.(iii) of [SS03b] or the generalization of that theorem given in [CM20 Thm. 1.36] cannot be applied directly. To solve this, we will work with Lydakis’ model structure for linear functors.

Recall that a simplicial set is called *finite* if it has finitely many nondegenerate simplices. For any pointed simplicial category $\mathcal{E}$, let $\text{SF}(\mathcal{E})$ denote the category pointed simplicial functors $\textbf{sSet}^\text{fin} \to \mathcal{E}$, where $\textbf{sSet}^\text{fin}$ is the category of pointed finite simplicial sets. If $\mathcal{E}$ is equipped with a model structure, then we will call such a pointed simplicial functor a *homotopy functor* if it preserves weak (homotopy) equivalences and *linear
if it furthermore sends homotopy pushouts to homotopy pullbacks. In [Lyd98], Lydakis showed that there exists a left Bousfield localization of the projective model structure on $\text{SF} := \text{SF}(\text{sSet}_*)$ in which the fibrant objects are exactly the pointwise fibrant linear homotopy functors (called the \textit{stable model structure of simplicial functors} there). It is shown that this is a symmetric monoidal model structure under the Day convolution product and that it is Quillen equivalent to the model category of sequential spectra [Lyd98 Thm. 11.3]. Let us denote this model category by $\text{SF}_f$.

In [BR14], this construction is extended to a more general setting that also applies to $\text{Ind}(\text{D})$. In particular, it follows from Theorem 5.8 of that paper applied to the case $n = 1$ that the analogous left Bousfield localization for $\text{SF(Ind}(\text{D}))$ exists, which we will denote by $\text{SF}_f(\text{Ind}(\text{D}))$. Theorems 5.24 and 5.27 of [BR14] then imply that $\text{SF}_f(\text{Ind}(\text{D}))$ is equivalent to the stable model structure on $\text{Sp(Ind}(\text{D}))$. Unlike $\text{Sp(Ind}(\text{D}))$, it can be shown that this model structure \textit{does} come with a canonical spectral enrichment.

\textbf{Proposition 3.2.} $\text{SF}_f(\text{Ind}(\text{D}))$ is an $\text{SF}_f$-enriched model structure.

\textit{Proof.} The enrichment, tensor and cotensor over $\text{SF}$ are defined by formulas analogous to that of the Day convolution. Given a simplicial set $K$, write $K_i: \text{sSet}^\text{fin}_* \rightarrow \text{sSet}^\text{fin}_*$ for the functor defined by $K_i(M) = M \wedge K$. The tensor is defined as the coend

$$
(X \otimes F)(K) = \int_{K_1 \otimes K_2} \text{Map}^*_i(K_1 \wedge K_2, K),
$$

while the enrichment and cotensor are defined by

$$
\text{Map}_{\text{SF}}(X,Y)(K) = \text{Map}^*_i(X,Y \circ K_1) \quad \text{and} \quad (X^F)(K) = (X \circ K_1)^F.
$$

We leave it to the reader to verify that these indeed constitute to an enrichment of $\text{SF(Ind}(\text{D}))$ over $\text{SF}$ that is both tensored and cotensored.

It can be shown that this makes the projective model structure on $\text{SF(Ind}(\text{D}))$ into an $\text{SF}_f$-enriched model category (with respect to the projective model structure on $\text{SF}$) by studying pushout products of generating (trivial) cofibrations. This is similar to the proof of [Lyd98 Thm. 12.3] and left to the reader. To see that $\text{SF}_f(\text{Ind}(\text{D}))$ is furthermore an $\text{SF}_f$-enriched model category, it thus suffices to show that for any pair of cofibrations $X \rightarrow Y$ in $\text{SF}_f(\text{Ind}(\text{D}))$ and $F \rightarrow G$ in $\text{SF}_f$, one of which is trivial, the pushout product

$$
Y \otimes F \sqcup_{X \otimes F} X \otimes G \rightarrow Y \otimes G
$$

(3.1)

is a trivial cofibration in $\text{SF}_f(\text{Ind}(\text{D}))$. We treat the case where $F \rightarrow G$ is trivial, the other case is similar. Without loss of generality, assume that $X \rightarrow Y$ is generating cofibration, so in particular that $X$ and $Y$ are cofibrant. Since we already know that (3.1) is a cofibration, it suffices to show that this map has the left lifting property with respect to fibrations between fibrant objects $B \rightarrow A$ (cf. [HM Lem. 8.43]). By adjunction, this is equivalent to proving that

$$
\text{Map}(Y, B) \rightarrow \text{Map}(Y, A) \times_{\text{Map}(X, A)} \text{Map}(X, B)
$$

is a fibration in $\text{SF}_f$. We already know that this is a fibration in the projective model structure, so by [Hir03 Prop. 3.3.16] it suffices to show that its domain and codomain are fibrant in $\text{SF}_f(\text{Ind}(\text{D}))$. Since $X$ and $Y$ are cofibrant and $A$ and $B$ are fibrant, this
follows if we can show that for any cofibrant object \( Z \) and fibrant object \( L \) in \( \text{SF}_l(\text{Ind}(D)) \), the hom-object \( \text{Map}_{\text{SF}}(Z, L) \) is fibrant in \( \text{SF}_l \). It is projectively fibrant since the projective model structure is \( \text{SF} \)-enriched. It is a homotopy functor since for any weak homotopy equivalence \( K \xrightarrow{\sim} M \) in \( \sSet^{\text{fin}} \), the natural map \( L \circ K_! \rightarrow L \circ M_! \) is a pointwise equivalence between pointwise fibrant functors, hence \( \text{Map}_* (Z, L \circ K_!) \rightarrow \text{Map}_* (Z, L \circ M_!) \) is a weak equivalence. Finally, to see that \( \text{Map}_{\text{SF}}(G, L) \) is linear, it suffices to show that

\[
\text{Map}_{\text{SF}}(G, L) \rightarrow \Omega \circ \text{Map}_{\text{SF}}(G, L) \circ \Sigma
\]

is a pointwise weak equivalence. This follows since the right-hand side is isomorphic to \( \text{Map}_{\text{SF}}(G, \Omega L \Sigma) \) and, since \( L \) is linear, the map \( L \rightarrow \Omega L \Sigma \) is a pointwise equivalence between projectively fibrant functors. 

Recall from \textsection 2.4 that if we take \( D^{op} \) to be the category of nuclear separable \( C^* \)-algebras, then \( \text{Ind}(D) \) is a closed symmetric monoidal model category. In particular, the Day convolution product endows \( \text{Fun}(\sSet^{\text{fin}}, \text{Ind}(D)) \) with a closed symmetric monoidal model structure, which turns out to be compatible with the model structure.

**Proposition 3.3.** If \( D^{op} \) is the category of nuclear separable \( C^* \)-algebras, then \( \text{SF}_l(\text{Ind}(D)) \) is a symmetric monoidal model category under the Day convolution product.

**Proof.** This is almost identical to the proof of Proposition 3.2 and left to the reader. ■

Let \( S_0 \) denote the (pointed) simplicial set consisting of two points. The functor \( \text{SF}_l(\text{Ind}(D)) \rightarrow \text{Ind}(D) \) that evaluates \( X \in \text{SF}_l(\text{Ind}(D)) \) at \( S_0 \in \sSet^{\text{fin}} \) will be denoted \( \Omega^\infty \). It has a left adjoint defined by

\[
\Sigma^\infty D : \sSet^{\text{fin}} \rightarrow \text{Ind}(D); \ K \mapsto D \otimes K.
\]

We will call \( \Sigma^\infty D \) the suspension spectrum of \( D \). It is straightforward to see that this is a Quillen pair (with respect to both the stable and the linear model structure).

We conclude this section by identifying the weak equivalences between fibrant objects in \( \text{SF}_l(\text{Ind}(D)) \).

**Proposition 3.4.** Let \( f : X \rightarrow Y \) be a map between fibrant objects in \( \text{SF}_l(\text{Ind}(D)) \). Then \( f \) is a weak equivalence if and only if for every \( n \geq 1 \), the map

\[
\text{Map}_{\text{SF}}(\Sigma^n M_n, X) \rightarrow \text{Map}_{\text{SF}}(\Sigma^n M_n, Y)
\]

is a weak equivalence in \( \text{SF}_l \).

**Proof.** Note that for every \( n \geq 1 \), the suspension spectrum \( \Sigma^n M_n \) is cofibrant since \( \Sigma^\infty \) is left Quillen. The “only if” direction follows from the fact that \( \text{SF}_l(\text{Ind}(D)) \) is \( \text{SF}_l \)-enriched. For the other direction, note that the map \([3.2] \) is an equivalence between fibrant objects in \( \text{SF}_l \), hence a pointwise equivalence. Combining this with the definition of \( \text{Map}_{\text{SF}} \) given in the proof of Proposition 3.2 and the left-adjointness of \( \Sigma^\infty \), this shows that

\[
\text{Map}_*(M_n, X(K)) \rightarrow \text{Map}_*(M_n, Y(K))
\]

is a weak equivalence for every \( n \geq 1 \). By definition of the weak equivalences in \( \text{Ind}(D) \), it follows that \( X \rightarrow Y \) is a pointwise equivalence, hence an equivalence in \( \text{SF}_l(\text{Ind}(D)) \). ■
3.2 Noncommutative CW-spectra as spectral presheaves

We will now identify the model category $SF_l(\text{Ind}(D))$ of noncommutative CW-spectra with a spectral presheaf category.

**Lemma 3.5.** The suspension spectra of matrix algebras $\{\Sigma^n M_n\}_{n\geq 1}$ form a compact generating set in $SF_l(\text{Ind}(D))$.

**Proof.** Compactness follows since $\Sigma^n$ is left adjoint and the objects $M_n$ are compact in $\text{Ind}(D)$, while it follows from Proposition 3.4 that $\{\Sigma^n M_n\}$ is a generating set. □

Let $M_\Delta$ denote the full $SF$-enriched subcategory of $SF_l(\text{Ind}(D))$ spanned by the suspension spectra of matrix algebras $\{\Sigma^n M_n\}_{n\geq 1}$. The definition of the $SF$-enrichment together with the left adjointness of $\Sigma^n$ shows that the hom-objects of $M_\Delta$ are given by

$$\text{Map}_{SF}(\Sigma^n M_n, \Sigma^n M_k)(K) = \text{Map}_s(M_n, M_k \otimes K). \quad (3.3)$$

In particular, $M_\Delta$ can be viewed as a simplicial version of the topological category $M_s$ defined in [ABS21 Def. 6.3].

Now note that the restricted enriched Yoneda embedding

$$U : SF(\text{Ind}(D)) \to \text{Fun}(M_\Delta^{op}, SF_l);$$

$$U(X)(\Sigma^n M_n) = \text{Map}_{SF}(\Sigma^n M_n, X)$$

admits a left adjoint $T$ (cf. [GM20 Prop. 1.10]). As mentioned above, $\Sigma^n$ is left Quillen, hence the objects $\Sigma^n M_n$ are all cofibrant in $SF_l(\text{Ind}(D))$. In particular, the restricted Yoneda embedding $U$ sends (trivial) fibrations to pointwise (trivial) fibrations, hence it is right Quillen when $\text{Fun}(M_\Delta^{op}, SF_l)$ is endowed with the projective model structure (which exists by Theorem 7.2 of [SS03]). Theorem 1.36 of [GM20] gives conditions under which this adjunction is a Quillen equivalence. One of the conditions translates to every object of $M_\Delta$ being fibrant in $SF_l(\text{Ind}(D))$. While this does not hold, their proof still goes through in our particular case.

**Theorem 3.6.** The adjunction $T \dashv U$ is a Quillen equivalence between the model categories $\text{Fun}(M_\Delta^{op}, SF_l)$ and $SF_l(\text{Ind}(D))$. In particular, the $\infty$-category of noncommutative CW-spectra is modelled by $\text{Fun}(M_\Delta^{op}, SF_l)$.

**Proof.** We leave it to the reader to verify that except for the fibrancy condition on the objects of $M_\Delta$, all condition of Theorem 1.36 of [GM20] are satisfied.

We claim that the fibrancy condition is not needed in our particular case. A careful inspection of the proof of [GM20 Thm. 1.36] shows that the only reason why fibrancy of the objects of $M_\Delta$ would be necessary, is that $U$ is then already derived when applied to an object of $M_\Delta$ (i.e. $U(\Sigma^n M_n) \to RU(\Sigma^n M_n)$ is a weak equivalence for every $n \geq 1$). Even though the objects $\Sigma^n M_n$ are not fibrant in $SF_l(\text{Ind}(D))$, by Lemma 3.7 this property is still satisfied, hence $T \dashv U$ is a Quillen equivalence. □

**Lemma 3.7.** The map $U(\Sigma^n M_n) \to RU(\Sigma^n M_n)$ is a weak equivalence in $\text{Fun}(M_\Delta^{op}, SF_l)$ for every $n \geq 1$. 

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Proof. The suspension spectrum \( \Sigma^n \mathcal{M}_n \): s\text{Set} \to \text{Ind}(\text{D}) is a pointwise fibrant homotopy functor, hence by how the model structure on \( \text{SF}_l(\text{Ind}(\text{D})) \) is constructed in \cite[Thm. 5.8]{BR14}, an explicit fibrant replacement can be given by

\[
\Sigma^n \mathcal{M}_n \xrightarrow{\sim} \lim_{m \in \mathbb{N}} \Omega^m \circ \Sigma^n \mathcal{M}_n \circ \Sigma^m.
\]

It therefore suffices to show that the map

\[
\mathcal{U}(\Sigma^n \mathcal{M}_n)(\Sigma^n \mathcal{M}_k) \to \lim_{m \in \mathbb{N}} \Omega^m \lim_{m \in \mathbb{N}} \Omega^m \circ \Sigma^n \mathcal{M}_n \circ \Sigma^m
\]

is an equivalence in \( \text{SF}_l \) for every \( k, n \geq 1 \). Because of compactness of the object \( \Sigma^n \mathcal{M}_k \), the right-hand side is isomorphic to

\[
\lim_{m \in \mathbb{N}} \Omega^m \text{Map}_{\text{SF}}(\Sigma^n \mathcal{M}_n, \lim_{m \in \mathbb{N}} \Omega^m \circ \Sigma^n \mathcal{M}_n \circ \Sigma^m) = \text{Map}_{\text{SF}}(\Sigma^n \mathcal{M}_n, \lim_{m \in \mathbb{N}} \Omega^m \circ \Sigma^n \mathcal{M}_n \circ \Sigma^m) = \lim_{m \in \mathbb{N}} \Omega^m \text{Map}_{\text{SF}}(\Sigma^n \mathcal{M}_n \circ \Sigma^m) = \lim_{m \in \mathbb{N}} \Omega^m \mathcal{U}(\Sigma^n \mathcal{M}_n)(\Sigma^n \mathcal{M}_k) \circ \Sigma^m.
\]

In particular, the map \([3.4]\) is simply the stabilization map

\[
\mathcal{U}(\Sigma^n \mathcal{M}_n)(\Sigma^n \mathcal{M}_k) \to \lim_{m \in \mathbb{N}} \Omega^m \mathcal{U}(\Sigma^n \mathcal{M}_n)(\Sigma^n \mathcal{M}_k) \circ \Sigma^m,
\]

which is an equivalence in \( \text{SF}_l \) since \( \mathcal{U}(\Sigma^n \mathcal{M}_n)(\Sigma^n \mathcal{M}_k) \) is a pointwise fibrant homotopy functor. \hfill \Box

The symmetric monoidal structure of \( \text{SF}_l(\text{Ind}(\text{D})) \) from \textbf{Proposition 3.3} restricts to a symmetric monoidal structure on \( \mathcal{M}_\Delta \). It is not hard to see that the projective model structure on \( \text{Fun}(\mathcal{M}_{\Delta}^{\text{op}}, \text{SF}_l) \) is symmetric monoidal under the Day convolution. In the case that \( \text{D}^{\text{op}} \) is the category of nuclear separable \( C^* \)-algebras, the Quillen equivalence from above can be upgraded to a monoidal one.

\textbf{Theorem 3.8.} In the case that \( \text{D} \) is the category of separable nuclear \( C^* \)-algebras, the Quillen equivalence of \textbf{Theorem 3.6} is strong monoidal.

\textbf{Proof.} We need to show that \( \mathcal{T} \) is strong monoidal. By the universal property of Day convolution, it suffices to show that \( \mathcal{T} \) is strong monoidal when restricted to the image of the enriched Yoneda embedding \( \mathcal{M}_\Delta \hookrightarrow \text{Fun}(\mathcal{M}_{\Delta}^{\text{op}}, \text{SF}_l) \). This holds since the composition of \( \mathcal{T} \) with the enriched Yoneda embedding is simply the inclusion \( \mathcal{M}_\Delta \hookrightarrow \text{SF}_l(\text{Ind}(\text{D})) \), which is strong monoidal by construction. \hfill \Box

\subsection{3.3 Simplicial vs. topological functors}

Recall that spectra can also be modelled as pointed linear topological functors from the category of finite pointed CW-complexes \( \text{CW}_{f} \) to the category of all pointed (compactly generated weak Hausdorff) spaces \( \text{Top}_{w} \), called \( \mathcal{W} \)-spectra in \cite{MMSS01}. We will denote the model category of \( \mathcal{W} \)-spectra by \( \text{Sp}^{\mathcal{W}} \). This is a symmetric monoidal model category under the Day convolution product.

Since the category of \( C^* \)-algebras naturally comes with a \( \text{Top}_{w} \)-enrichment, it is perhaps more natural consider the \( \text{Sp}^{\mathcal{W}} \)-enriched analogue of \( \mathcal{M}_\Delta \). This is the category denoted by \( \mathcal{M}_s \) in \cite{ABS21}, but we will denote it by \( \mathcal{M}_{\text{Top}} \) to emphasize that it is the "topological" analogue of \( \mathcal{M}_\Delta \). Its objects are the positive natural numbers \( n \) (which we
think of as the suspension spectra of the matrix algebras \( M_n \) and for two objects \( n \) and \( k \), the hom-object \( \text{Map}_{\text{Sp}^*}(n, k) \) is defined by
\[
\text{CW}_f \to \text{Top}_*; \quad W \mapsto \text{Map}_{\text{Top}}(M_n, M_k \otimes W).
\]
Composition is defined as in \([\text{ABS}21]\) Def. 5.12. The tensor product of matrix algebras endows this category with a symmetric monoidal structure.

It is shown in \([\text{MMSS}01]\) Thm. 19.11 that there is a (strong monoidal) Quillen equivalence \( \mathcal{P}T : \text{SF}_l \rightleftharpoons \text{Sp}^W : \text{SU} \), where the right adjoint is defined by
\[
\text{SU}(F) : \text{sSet}_{\text{fin}}^* \to \text{sSet}; \quad \text{SU}(F)(K) = \text{Sing}(F(|K|)).
\]
It follows from the characterization of the hom-sets given in \((3.3)\) that \( \mathcal{M}_\Delta \) is isomorphic to the category \( \text{SU}(\mathcal{M}_{\text{Top}}) \) obtained by applying \( \text{SU} \) to each hom-object of \( \mathcal{M}_{\text{Top}} \), and one easily sees that their symmetric monoidal structures agree. In particular, by applying \( \text{SU} \) to values of a functor \( F \in \text{Fun}(\mathcal{M}_{\text{Top}}^{op}, \text{Sp}^W) \), we obtain a functor
\[
\text{SU} : \text{Fun}(\mathcal{M}_{\text{Top}}^{op}, \text{Sp}^W) \to \text{Fun}(\mathcal{M}_\Delta^{op}, \text{SF}_l).
\]

**Theorem 3.9.** The functor \( \text{SU} : \text{Fun}(\mathcal{M}_{\text{Top}}^{op}, \text{Sp}^W) \to \text{Fun}(\mathcal{M}_\Delta^{op}, \text{SF}_l) \) is the right adjoint of a strong monoidal Quillen equivalence, where both the domain and codomain are endowed with the projective model structure. In particular, \( \text{Fun}(\mathcal{M}_{\text{Top}}^{op}, \text{Sp}^W) \) models the \( \infty \)-category of non-commutative CW-spectra.

**Proof.** The fact that \( \text{SU} \) is the right adjoint of a Quillen equivalence follows by combining Theorems 6.5.(2) and 7.2 of \([\text{SS}03a]\). To see that it is a strong monoidal Quillen equivalence, we need to show that the left adjoint \( \mathcal{P}T_l \) is strong monoidal. By the universal property of Day convolution, it suffices to show this for the restriction of the left adjoint \( \mathcal{P}T_l \) to the image of the Yoneda embedding \( \mathcal{M}_\Delta \hookrightarrow \text{Fun}(\mathcal{M}_\Delta^{op}, \text{SF}_l) \). The left adjointness of \( \mathcal{P}T_l \) implies that it takes representables to representables, hence this reduces to showing that \( \mathcal{M}_\Delta \to \text{SU}(\mathcal{M}_{\text{Top}}) \) is symmetric monoidal. But this is just the symmetric monoidal isomorphism mentioned above.

**A Minimal cofibration test categories**

The goal of this appendix is to describe a simplification of the main construction from \([\text{BM}20]\). For two objects \( c \) and \( d \) in a simplicial category \( \mathcal{D} \), the simplicial hom-set is denoted by \( \text{Map}(c, d) \). Recall that the tensor of an object \( c \) in \( \mathcal{D} \) with a simplicial set \( M \) is defined (if it exists) as the essentially unique object \( c \otimes M \) such that there are natural isomorphisms \( \text{Map}(M, \text{Map}(c, d)) \cong \text{Map}(c \otimes M, d) \) for every \( d \) in \( \mathcal{D} \).

**Definition A.1.** A minimal cofibration test category \( (\mathcal{D}, \mathcal{T}) \) consists of an essentially small simplicial category \( \mathcal{D} \) together with a subset \( \mathcal{T} \subseteq \text{Ob}(\mathcal{D}) \) of test objects, satisfying the following properties:

(i) The category \( \mathcal{D} \) admits all finite colimits.

\(^3\)Here natural means natural in the enriched sense; cf. \([\text{Kel}82] \) §1.2.
(ii) The category \( D \) admits tensors by finite simplicial sets, which moreover commute with finite colimits of \( D \).

(iii) For any object \( c \) of \( D \) and any test object \( t \in T \), the simplicial set \( \text{Map}(t, c) \) is a Kan complex.

**Example A.2.** Let \( D \) be an (essentially) small topological category that admits all finite colimits and tensors by the unit interval \( I \), and moreover assume that the functor \( - \otimes I: D \to D \) preserves finite colimits. Applying the singular complex functor to the hom-sets of \( D \), we obtain a simplicial category that we will also denote by \( D \). Then for any set of objects \( T \subseteq \text{Ob}(D) \), the pair \((D, T)\) is a minimal cofibration test category. Items (i) and (iii) are obvious. To see that item (ii) holds, note that tensors by a simplicial set \( M \) in the simplicial category \( D \) agree with tensors by the geometric realization \( |M| \) when \( D \) is viewed as a topological category. Since any finite simplicial set is a finite colimit of representables \( \Delta[n] \), it suffices to show that \( D \), viewed as a topological category, admits tensors by the topological spaces \( |\Delta[n]| \). This follows since

\[
c \otimes |\Delta[n]| \cong c \otimes (I \times \ldots \times I) \cong c \otimes I \otimes \ldots \otimes I. \tag{12}
\]

For any simplicial category \( D \), the ind-category \( \text{Ind}(D) \) is again simplicial, with the enrichment defined by

\[
\text{Map}(\{c_i\}, \{d_j\}) = \lim_i \colim_j \text{Map}(c_i, d_j).
\]

The main result of this appendix is the following.

**Theorem A.3.** Let \((D, T)\) be a minimal cofibration test category. Then there exists a cofibrantly generated simplicial model structure on \( \text{Ind}(D) \) with the following properties:

(i) A map \( C \to D \) is a weak equivalence or fibration if and only if for any \( t \in T \), the map

\[
\text{Map}(t, C) \to \text{Map}(t, D)
\]

is a weak homotopy equivalence or Kan fibration, respectively.

(ii) Any object of \( \text{Ind}(D) \) is fibrant.

(iii) A set of generating cofibrations is given by

\[
\mathcal{I} = \{t \otimes \partial \Delta[n] \to t \otimes \Delta[n] \mid n \geq 0, \ t \in T\}
\]

and a set of generating trivial cofibrations by

\[
\mathcal{J} = \{t \otimes \Lambda^k[n] \to t \otimes \Delta[n] \mid 0 \leq k \leq n, \ n \neq 0, \ t \in T\}
\]

(iv) The weak equivalences are stable under filtered colimits.

In light of this theorem, we will call a map in \( \text{Ind}(D) \) a weak equivalence if \( \text{Map}(t, C) \to \text{Map}(t, D) \) is a weak equivalence for any \( t \in T \).
Remark A.4. Note that the maps $|\partial \Delta[n]| \hookrightarrow |\Delta[n]|$ and $|\Lambda^k[n]| \hookrightarrow |\Delta[n]|$ can be identified with the inclusions $\partial D^n \hookrightarrow D^n$ and $D^{n-1} \times \{0\} \hookrightarrow D^{n-1} \times I$, respectively. In particular, if the minimal cofibration test category $(D, T)$ comes from a topological category as in Example A.2, then one can also take

$$J = \{ t \otimes \partial D^n \to t \otimes D^n \mid n \geq 0, \ t \in T \}$$

and

$$J = \{ t \otimes (D^n \times \{0\}) \to t \otimes (D^n \times I) \mid n \geq 0, \ t \in T \}$$

as sets of generating (trivial) fibrations, where $D^n$ denotes the $n$-dimensional unit disc.

Remark A.5. The somewhat unfortunate name “minimal cofibration test category” comes from the fact that if $(D, T)$ is a minimal cofibration test category, then one can form a cofibration test category in the sense of Definition 3.1 of [BM20] by defining the category of test objects to be the full subcategory $T'$ consisting of objects of the form $t \otimes N$ for any finite simplicial set $N$ and the cofibrations the maps of the form $t \otimes N \rightarrow t \otimes M$ with $N \hookrightarrow M$ a monomorphism of finite simplicial sets. The trivial cofibrations are then defined as the cofibrations that are also weak equivalences. One can verify that the model structure of Theorem A.3 agrees with the model structure that one obtains from this cofibration test category by applying Theorem 3.8 of [BM20]. This cofibration test category $(D, T')$ is the smallest structure of a cofibration test category that one can put on $D$ that has the property that $T'$ contains the set $T$.

The proof relies on the following three lemmas. We call a map $i: C \to D$ in a simplicial category that admits tensors by $\Delta[1]$ an inclusion of a deformation retract if there exist maps $r: D \to C$ and $H: D \otimes \Delta[1] \to D$ such that

\[(DR1) \quad ri = id_C\]

\[(DR2) \quad H \text{ is a homotopy from } id_D \text{ to } ir \text{ that is constant on } C.\]

Lemma A.6. Inclusions of deformation retracts are stable under pushouts.

Proof. This is analogous to the proof of [Hov99, Prop. 2.4.9].

Lemma A.7. Any map in the set $\mathcal{J}$ of item (iii) of Theorem A.3 is an inclusion of a deformation retract.

Proof. Let the map $i: t \otimes \Lambda^k[n] \to t \otimes \Delta[n]$ in $\mathcal{J}$ be given. By adjunction, constructing a retract $r$ is equivalent to solving a lifting problem of the form

$$\Lambda^k[n] \to Map(t, t \otimes \Lambda^k[n])$$

$$\downarrow$$

$$\Delta[n]$$

and constructing the desired homotopy $H$ from $id_D$ to $ir$ is equivalent to solving a lifting problem of the form

$$\Lambda^k[n] \times \Delta[1] \cup_{\Lambda^k[n] \times \partial \Delta[1]} \Delta[n] \times \partial \Delta[1] \to Map(t, t \otimes \Delta[n])$$

$$\downarrow$$

$$\Delta[n] \times \Delta[1]$$
This is possible by item [iii] of Definition A.1.

**Lemma A.8.** Any inclusion of a deformation retract is a weak equivalence in \( \text{Ind}(\mathbf{D}) \).

**Proof.** This follows since \( \text{Map}(t, -) : \text{Ind}(\mathbf{D}) \to \mathbf{sSet} \) is a simplicial functor, hence it preserves deformation retracts. ■

**Proof of Theorem A.3.** The fact that \( \text{Ind}(\mathbf{D}) \) is a simplicial category that is complete, cocomplete and that admits all (co)tensors is explained in [BM20 §2.2]. Item [iv] follows since \( \text{Map}(t, -) \) preserves filtered colimits for every \( t \in \mathbf{T} \) and since weak equivalences in \( \mathbf{sSet} \) are stable under filtered colimits.

We now prove that the model structure exists by checking all items of Theorem 11.3.1 of [Hiro3]. It is clear that the class of weak equivalences satisfies the two out of three property and is closed under retracts. The sets \( \mathcal{J} \) and \( \mathcal{I} \) permit the small object argument since both sets consist of maps between objects that are compact in \( \text{Ind}(\mathbf{D}) \). It is furthermore clear that any map of \( \mathcal{J} \) lies in the saturation of \( \mathcal{J} \), since the saturation of \( \mathcal{J} \) includes all maps of the form \( t \otimes M \rightarrow t \otimes N \) where \( M \rightarrow N \) is a monomorphism of simplicial sets and \( t \in \mathbf{T} \).

It follows from Lemmas A.6 to A.8 that any pushout of a map in \( \mathcal{J} \) is a weak equivalence, and transfinite compositions of such maps are again weak equivalences since weak equivalences are stable under cofiltered limits. In particular, any map in the saturation of \( \mathcal{J} \) is a weak equivalence.

It follows by adjunction that a map \( C \rightarrow D \) has the right lifting property with respect to the maps in \( \mathcal{J} \) if and only if \( \text{Map}(t, C) \rightarrow \text{Map}(t, D) \) is a Kan fibration, while it has the right lifting property with respect to the maps in \( \mathcal{J} \) if and only if \( \text{Map}(t, C) \rightarrow \text{Map}(t, D) \) is a trivial Kan fibration. In particular, a map \( C \rightarrow D \) has the right lifting property with respect to the maps in \( \mathcal{J} \) if and only if it is a weak equivalence and has the right lifting property with respect to the maps in \( \mathcal{J} \).

To see that every object is fibrant, let \( C = \{ c_i \} \in \text{Ind}(\mathbf{D}) \) be given. For any \( t \in \mathbf{T} \), one has \( \text{Map}(t, C) = \text{colim} \text{Map}(t, c_i) \). Since cofiltered limits of Kan complexes are again Kan complexes, we conclude that for any \( t \in \mathbf{T} \), the simplicial set \( \text{Map}(t, C) \) is a Kan complex. In particular, \( C \) is fibrant.

Finally, to see that this model structure is simplicial, it suffices to show that for any \( t \in \mathbf{T} \), any cofibration \( M \rightarrow N \) between finite simplicial sets, and any fibration \( C \rightarrow D \) in \( \text{Ind}(\mathbf{D}) \), the map

\[
\text{Map}(t \otimes N, C) \rightarrow \text{Map}(t \otimes N, D) \times_{\text{Map}(t \otimes M, D)} \text{Map}(t \otimes M, C)
\]

is a Kan fibration, which is trivial if \( M \rightarrow N \) or \( C \rightarrow D \) is. This follows immediately by noting that this map agrees with the pullback-power of \( M \rightarrow N \) and \( \text{Map}(t, C) \rightarrow \text{Map}(t, D) \).

**Remark A.9.** In [BM20], cofibration test categories with respect to the Joyal model structure on \( \mathbf{sSet} \) were also considered. One can similarly define a minimal cofibration test category with respect to the Joyal model structure on \( \mathbf{sSet} \) by weakening item [iii] of Definition A.1 to only requiring that \( \text{Map}(t, C) \) is a quasicategory. One can then prove a theorem analogous to Theorem A.3, but the proof is slightly more complicated. We leave this as an exercise for the interested reader.
Finally, it will be useful to have a description of the underlying ∞-category of $\text{Ind}(D)$ for a minimal cofibration test category $(D, T)$. We will need the following definition for this.

**Definition A.10.** Let $(D, T)$ be a minimal cofibration test category. A finite $T$-cell complex is an object $d \in D$ such that the map $\emptyset \to d$ is a finite composition of pushouts of maps of the form $t \otimes \partial \Delta[n] \to t \otimes \Delta[n]$. In other words, one can write $\emptyset \to d$ as a finite composition $\emptyset = d_0 \to d_1 \to \ldots \to d_m = d$

such that for every $0 \leq i < m$, there exists a pushout square of the form

$$
\begin{array}{ccc}
t \otimes \partial \Delta[n] & \longrightarrow & d_i \\
\downarrow & & \downarrow \\
t \otimes \Delta[n] & \longrightarrow & d_{i+1}
\end{array}
$$

for some $t \in T$ and $n \geq 0$.

Denote the full simplicial subcategory of $D$ spanned by the finite $T$-cell complexes by $\text{cell}_{\text{fin}}(T)$. A proof similar to that of Theorem A.2 of [BM20] then shows the following.

**Proposition A.11.** Let $(D, T)$ be a minimal cofibration test category. Then the underlying ∞-category of $\text{Ind}(D)$ is equivalent to $\text{Ind}_{\infty}(\text{N}_{\text{hc}}(\text{cell}_{\text{fin}}(T)))$, where $\text{N}_{\text{hc}}$ denotes the homotopy coherent nerve of a simplicial category and $\text{Ind}_{\infty}$ the ind-completion of an ∞-category in the sense of [Lur09] Def. 5.3.5.1.

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