Spin chains and Gustafson’s integrals

S É Derkachov and A N Manashov

1 St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023 St. Petersburg, Russia
2 Institut für Theoretische Physik, Universität Hamburg, D-22761 Hamburg, Germany
3 Institute for Theoretical Physics, University of Regensburg, D-93040 Regensburg, Germany

E-mail: derkach@pdmi.ras.ru and alexander.manashov@desy.de

Received 28 December 2016, revised 12 April 2017
Accepted for publication 22 May 2017
Published 29 June 2017

Abstract
Gustafson’s integrals are multidimensional generalizations of the classical Mellin–Barnes integrals. We show that some of these integrals arise from relations between matrix elements in Sklyanin’s representation of separated variables in spin chain models. We also present several new integrals.

Keywords: Gustafson integrals, spin chains, Yang–Baxter equation

(Some figures may appear in colour only in the online journal)

1. Introduction

In the papers [1, 2] Gustafson has calculated integrals representing multidimensional generalization of the Mellin–Barnes integrals. The integrals associated with the classical su(N) and sp(N) Lie algebras take the form [1]

\[
\left( \prod_{n=1}^{N} \int_{-i\infty}^{i\infty} \frac{dz_n}{2\pi i} \right) \prod_{k=1}^{N+1} \prod_{j=1}^{N} \Gamma(\alpha_k - z_j) \Gamma(\beta_k + z_j) = N! \prod_{k=1}^{N+1} \Gamma(\alpha_k + \beta_k) \Gamma(\sum_{k=1}^{N+1} (\alpha_k + \beta_k))
\]

(1)

and

\[
\left( \prod_{n=1}^{N} \int_{-i\infty}^{i\infty} \frac{i^{z_n}}{2\pi i} \right) \prod_{k=1}^{2N+2} \prod_{j=1}^{N} \Gamma(\alpha_k \pm z_j) \Gamma(-\alpha_k \pm z_j) = \frac{2^N N! \prod_{k<j} \Gamma(\alpha_k + \alpha_j)}{\Gamma(\sum_{k=1}^{2N+2} \alpha_k)}
\]

(2)

where \( \Gamma(\alpha \pm \beta) \equiv \Gamma(\alpha + \beta) \Gamma(\alpha - \beta) \) and the integration contours separate the series of poles of \( \Gamma \) functions, \( \{\alpha_k + n_k\} \) and \( \{-\beta_k - n_k\}\), \( k = 1, \ldots, N\), \( n_k \in \mathbb{Z}_+\), in the first integral, and \( \{\alpha_k + n_k\} \) and \( \{-\alpha_k - n_k\}, k = 1, \ldots, 2N + 2 \) in the second one.
Gustafson’s integrals and their $q$- and $p, q$-generalization [1–6] play an important role in many topics in physics and mathematics such as theory of multivariable orthogonal polynomials [7], Selberg type integrals and constant term identities [8, 9], supersymmetric dualities in quantum field theory [10].

The aim of this paper is to demonstrate that Gustafson type integrals arise in a natural way in integrable spin chain models. Namely, the integrals (1) and (2) can be related to matrix elements of the shift operator $T_s$ (the operator of translations) in Sklyanin’s representation of separated variables (SoV) [11]. Moreover, we obtain a new identity which we were not able to derive from integrals (1) and (2). It takes the form

$$
\left( \prod_{m=1}^{N} \frac{dz_m}{2\pi i} \right) \frac{\prod_{k=1}^{N} \left( \prod_{\alpha=1}^{N+1} \Gamma(\alpha - z_k) \right) \left( \prod_{\beta=1}^{N} \Gamma(z_{\alpha} + \beta) \right)}{\prod_{1 \leq \alpha < j \leq N+1} \Gamma(z_{\alpha} + z_{j})} = N! \prod_{k=1}^{N} \prod_{\alpha=1}^{N+1} \Gamma(\alpha_k + \beta_j),
$$

(3)

where it is assumed that the series of poles $\{\alpha_k + n_k\}$ and $\{\pm \beta_k - n_k\}$ are separated by the integration contours.

We also calculate the scalar products between the eigenfunctions of elements of monodromy matrix and show that evaluation of these scalar products in the SoV representation gives rise to new integral identities.

The paper is organized as follows: section 2 contains basic facts about spin chain models. In section 3 we recall the construction of the SoV representation and provide an explicit expressions for the corresponding basis functions. Scalar products of certain eigenfunctions are calculated in sections 4 and 5. We also show that the SoV representation for matrix elements of the translation operators gives rise to the Gustafson integrals (1) and (2). In section 6 we present several new integrals which follow from relations between the eigenfunctions of the monodromy matrix for closed spin chain. The final section 7 contains a short summary and outlook. Some technical details and elements of the diagrammatic technique employed in this paper are given in the appendix.

2. Spin chain models

One dimensional quantum mechanical lattice models with dynamical variables being generators of some Lie algebra are usually called spin chain magnets. We consider a model with the $\text{SL}(2, \mathbb{R})$ symmetry. The dynamical variables are the generators of this group

$$
S_+^{(k)} = z^2_k \partial_{z_k} + 2z_k s_k, \quad S_0^{(k)} = z_k \partial_{z_k} + s_k, \quad S_-^{(k)} = -\partial_{z_k},
$$

(4)

where the index $k$ enumerates the lattice sites, $k = 1, \ldots, N$ and the spin parameter $s_k$ specifies the representation of $\text{SL}(2, \mathbb{R})$ group in the $k$th site. Henceforth we will consider homogeneous spin chains, $s_1 = s_2 = \ldots = s_N = s$. The generators (4) act on the irreducible discrete series representation of the $\text{SL}(2, \mathbb{R})$ group, $D^s$, where the spin $s$ is a positive integer or half-integer. This representation is realized on the space of functions holomorphic in the upper complex half-plane [12]. The Hilbert space of the model is given by the direct product of vector spaces of the representation $D^s$ at each site, $\mathcal{H}_N = \prod_{k=1}^{N} \otimes V_s$. Thus, the span is the space of functions of $N$ complex variables holomorphic in each variable in the upper half-plane and equipped with the invariant scalar product [12], which takes the form

$$
(f_1, f_2) = \prod_{k=1}^{N} \int Dz_k \left( f_1(z_1, \ldots, z_N) \right) \bar{f}_2(z_1, \ldots, z_N).
$$

(5)
Here the integration goes over the upper half-plane \( y \geq 0 \), \( z = x + iy \) and the integration measure is defined as
\[
Dz = \frac{2y - 1}{\pi} (2y)^{2r-2} dy \, dy.
\]

The scalar product (5) is invariant under the \( SL(2, \mathbb{R}) \) transformations
\[
f(z_1, \ldots, z_N) \mapsto [T(g)f](z_1, \ldots, z_N) = \left( \prod_{k=1}^N \frac{1}{(cz_k + d)^2} \right) f(z'_1, \ldots, z'_N),
\]
where \( g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \) and \( z'_k = (az_k + b)/(cz_k + d) \). The generators (4) are anti-hermitian with respect to this scalar product.

The quantum inverse scattering method (QISM) [13–16] allows one to define a physically meaningful Hamiltonian as a function of the dynamical variables, \( S^{(a)}_k, k = 1, \ldots, N \), and provides effective tools for solving the corresponding spectral problem. The pivotal object for the QISM machinery is the monodromy matrix, see e.g. [17, 18]. It is given by a product of Lax operators [14]
\[
L_k(u) = u + i \begin{pmatrix} S_0^{(k)} & S_+^{(k)} \\ S_-^{(k)} & -S_0^{(k)} \end{pmatrix}.
\]

For the closed and open spin chains the monodromy matrices are defined as [14, 19]:
\[
T_N^c(u) = L_1(u)L_2(u) \cdots L_N(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix},
\]
\[
T_N^{\top}(u) = T_N(-u)\sigma_2 T_N^c(u)\sigma_2 = \begin{pmatrix} A_N^*(u) & B_N^*(u) \\ C_N^*(u) & D_N^*(u) \end{pmatrix}.
\]

Here \( \sigma_2 \) is a Pauli matrix and \( T_N^c \) is the transposed matrix. The matrix elements \( A_N(u), \ldots, D_N(u) \) \((A_N(u), \ldots, D_N(u))\) are differential operators acting on the Hilbert space of the model. By construction they are polynomials in the spectral parameter \( u \).

According to the QISM the entries of \( T_N^c(u) \) form commuting families,
\[
[A_N(u),A_N(v)] = [B_N(u),B_N(v)] = [C_N(u),C_N(v)] = [D_N(u),D_N(v)] = 0.
\]

For open spin chains this property holds for the off-diagonal elements only,
\[
[B_N(u),B_N(v)] = [C_N(u),C_N(v)] = 0.
\]

It follows from equations (11) and (12) that the eigenfunctions of the operators \( A_N(u), \ldots, D_N(u) \) do not depend on the spectral parameter. At the same time the corresponding eigenvalues are polynomials in \( u \). It turns out that an eigenfunction is completely determined by its eigenvalue. Therefore it is convenient to label eigenfunctions by the roots of the corresponding eigenvalues. For example, if \( \Psi \) is the eigenfunction of \( A_N(u) \) with eigenvalue \( a_N(u) = (u - x_1) \ldots (u - x_N) \), we will denote this eigenfunction by \( \Psi_{\{x_1, \ldots, x_N\}} \).
\[
A_N(u)\Psi_{\{x_1, \ldots, x_N\}} = a_N(u)\Psi_{\{x_1, \ldots, x_N\}} = (u - x_1) \ldots (u - x_N)\Psi_{\{x_1, \ldots, x_N\}}.
\]

The eigenfunctions of the respective operators provide the convenient bases for studying spin chain models [11]. All these eigenfunctions admit an explicit representation in the form of multi-variable integrals which we discuss in the next section.
Closing this section we note that the operators, $B_N$ and $C_N$, $A_N$ and $D_N$, $B_N$ and $C_N$ are related to each other by an inversion [23], so that is is sufficient to consider the operators $B_N$, $A_N$ and $B_N$ only.

3. Sklyanin’s representation of separated variables

The eigenfunctions of the operators $B_N(u)$, $B_N(u)$ and $A_N(u)$ were constructed in [20–22], respectively. In this section we present the explicit expressions for these eigenfunctions and discuss their properties.

3.1. $B_N$-system

In order to present the result in a compact form we define an auxiliary operator $\Lambda_N(u)$ where $u \in \mathbb{C}$ is the spectral parameter. This operator maps a function of $N-1$ variables to a function of $N$ variables according the following rule

$$ [\Lambda_N(u)f](z_1, \ldots, z_N) = \left( \prod_{k=1}^{N-1} \int D\omega_k \right) \Lambda_N^{(a)}(z_1, \ldots, z_N|w_1, \ldots, w_{N-1})f(w_1, \ldots, w_{N-1}). \quad (14) $$

where

$$ \Lambda_N^{(a)}(z_1, \ldots, z_N|w_1, \ldots, w_{N-1}) = \prod_{j=1}^{N-1} D_{\alpha_j}(z_j, \bar{w}_j) D_{\bar{\alpha}_{j+1}}(\bar{z}_j+1, \bar{w}_j) \quad (15) $$

and the propagator $D_\alpha$ is defined as:

$$ D_\alpha(z, \bar{w}) = \left( \frac{i}{z-\bar{w}} \right)^\alpha. \quad (16) $$

Note, that under conjugation the propagator transforms as follows, $(D_\alpha(z, \bar{w}))^\dagger = D_{\bar{\alpha}}(w, \bar{z})$.

The operator $\Lambda_N$ has the following properties:

$$ (B_N(u)\Lambda_N(u)f)(z_1, \ldots, z_N) = 0, \quad (17) $$

$$ \Lambda_N(u_1)\Lambda_{N-1}(u_2) = \Lambda_N(u_2)\Lambda_{N-1}(u_1). \quad (18) $$

The eigenfunctions of the operator $B_N$ are obtained by a consecutive application of the operators $\Lambda_k$ to the exponential function

$$ \Psi_B^{(N)}(p, x|\bar{z}) = b_N(p)\Lambda_N(x_1)\Lambda_{N-1}(x_2) \ldots \Lambda_2(x_{N-1}) e^{ipw}, \quad (19) $$

where $p \geq 0$ and the $x_i$ are all real [21]. It is convenient to fix the normalization factor $b_N(p)$ as

$$ b_N(p) = p^{N-\frac{1}{2}}(\Gamma(2N))^{-N^2/2}. \quad (20) $$

It follows from (17) and (18) that

$$ B_N(u)\Psi_B^{(N)}(p, x|\bar{z}) = p(u-x_1) \cdots (u-x_{N-1}) \Psi_B^{(N)}(p, x|\bar{z}). \quad (21) $$

The eigenfunction is symmetric under permutation of the separated variables \{x_1, \ldots, x_{N-1}\}. Since the operator $B_N(u)$ is self-adjoint for real $u$, $(B_N(u))^\dagger = B_N(u)$, the eigenfunctions are mutually orthogonal [20].
\[(\Psi_B^{(N)}(p', x'), \Psi_B^{(N)}(p, x)) = (2\pi)^{N-1} \delta(p - p') \left( \sum_{w \in \mathbb{S}_N} \delta(x -wx') \right) \frac{\prod_{j\neq k} \Gamma(\alpha_j(x_k - x_j))}{\prod_{k=1}^{N} \Gamma(\alpha_k \Gamma(\beta_k))} \]  

(22)

where \( \alpha_x = s - ix, \beta_x = s + ix \) and

\[ \delta(x - x') = \delta(x_1 - x'_1) \cdots \delta(x_{N-1} - x'_{N-1}). \]  

(23)

3.2. \( A_N \)-system

The eigenfunctions of the operator \( A_N(u) \) is constructed in a similar manner [22]:

\[ \Psi_A^{(N)}(\vec{x}|\vec{z}) = a_N \tilde{A}_N(x_1) \tilde{A}_{N-1}(x_2) \cdots \tilde{A}_1(x_N). \]  

(24)

where

\[ a_N = (\Gamma(2s))^{-N/2} \]  

(25)

and the operator \( \tilde{A}_N \) is defined as follows

\[ [\tilde{A}_N(x)f](z_1, \ldots, z_N) = D_{s-\text{ix}}(zN, 0) [A_N(x)f](z_1, \ldots, z_N). \]  

(26)

The operator satisfies equations similar to equations (17) and (18):

\[ (A_N(u)\tilde{A}_N(u)f)(z_1, \ldots, z_N) = 0, \quad \tilde{A}_N(U(1))\tilde{A}_{N-1}(u_2) = \tilde{A}_N(u_2)\tilde{A}_{N-1}(u_1), \]  

(27)

which ensure that

\[ A_N(u)\Psi_A^{(N)}(\vec{x}|\vec{z}) = (u - x_1) \cdots (u - x_N) \Psi_A^{(N)}(\vec{x}|\vec{z}). \]  

(28)

The function \( \Psi_A^{(N)}(\vec{x}|\vec{z}) \) is symmetric under permutation of the spectral parameters \( x_k \), and for the scalar product one obtains [22]

\[ (\Psi_A^{(N)}(\vec{x}'), \Psi_A^{(N)}(\vec{x})) = (2\pi)^N \left( \sum_{w \in \mathbb{S}_N} \delta(x - wx') \right) \frac{\prod_{j\neq k} \Gamma(\alpha_j(x_k - x_j))}{\prod_{k=1}^{N} \Gamma(\alpha_k \Gamma(\beta_k))} \]  

(29)

3.3. \( B_N \) system

Since \( B_N(-i/2) = 0 \) it is convenient to redefine the operator to get rid of this zero [21],

\[ \tilde{B}_N(u) = B_N(u)/(2u + i). \]  

(30)

This new operator is a polynomial of degree \( 2N - 2 \) in the spectral parameter \( u \) and satisfies the symmetry \( \tilde{B}_N(u) = \tilde{B}_N(-u) \).

The eigenfunctions have the form (19) with a different type of ‘layer’ operator \( \Lambda_N \mapsto \tilde{A}_N \)

\[ \Psi_B^{(N)}(p, \vec{x}|\vec{z}) = c_N(p) \tilde{A}_N(x_1) \tilde{A}_{N-1}(x_2) \cdots \tilde{A}_2(x_{N-1}) e^{ip\vec{w}}, \]  

(31)

where

\[ c_N(p) = p^{N-1} (\Gamma(2s))^{-N(N-1)/2}. \]  

(32)
The operator $\tilde{\Lambda}_N(x)$ is defined as follows [21]

\[
[\tilde{\Lambda}_N(x)f](z_1, \ldots, z_N) = \left( \prod_{k=1}^{N-1} \int \mathcal{D}w_k \right) \tilde{\Lambda}_N^{(s)}(z_1, \ldots, z_N|w_1, \ldots, w_{N-1})f(w_1, \ldots, w_{N-1}),
\]

where the kernel $\tilde{\Lambda}_N^{(s)}$ has the form

\[
\tilde{\Lambda}_N^{(s)}(z_1, \ldots, z_N|w_1, \ldots, w_{N-1}) = D_{-i\pi}(\bar{z}_N, \bar{w}_{N-1}) \left( \prod_{k=1}^{N-1} \int \mathcal{D}\xi_k \right) \prod_{i=1}^{N-1} D_{-i\pi}(\bar{z}_i, \bar{\xi}_i)D_{-i\pi}(\bar{z}_{i+1}, \bar{\xi}_i) \times \left( \prod_{k=1}^{N-2} D_{i\pi}(\bar{\xi}_k, \bar{w}_k)D_{-i\pi}(\bar{\xi}_{k+1}, \bar{w}_k) \right) D_{+i\pi}(\bar{\xi}_{N-2}, \bar{w}_{N-1}).
\]

This operator satisfies equations

\[
(\tilde{B}_N(x)\tilde{\Lambda}_N^{(s)})f(z_1, \ldots, z_N) = 0, \quad \tilde{\Lambda}_N(x_1)\tilde{\Lambda}_{N-1}(x_2) = \tilde{\Lambda}_N(x_2)\tilde{\Lambda}_{N-1}(x_1), \quad \tilde{\Lambda}_N(x) = \tilde{\Lambda}_N(-x).
\]

It follows from (35) that

\[
\tilde{B}_N(u)\Psi_B^{(N)}(p, \bar{x}|\bar{z}) = p(u^2 - x_1^2) \cdots (u_2 - x_{N-1}^2)\Psi_B^{(N)}(p, \bar{x}|\bar{z})
\]

and $\Psi_B^{(N)}(p, \bar{x}|\bar{z})$ is an even function of the separated variables, $x_k$. The scalar product of two eigenfunctions takes the form [21]

\[
\left( \Psi_B^{(N)}(p', \bar{x}'), \Psi_B^{(N)}(p, \bar{x}) \right) = (2\pi)^{N-1} \delta(p - p') \left( \sum_{n \in S_N} \delta(x' - nx) \right) \times \prod_{n=1}^{N-1} \Gamma(2ix_n)\Gamma(-2ix_n) \prod_{k=1}^{N-1} \frac{\Gamma(i(x_k \pm x_{k+1}))\Gamma(-i(x_k \pm x_{k+1}))}{\prod_{k=1}^{N-1} \Gamma(x_k)\Gamma(\beta_k)2^{N}},
\]

where $x_k, x_k' \geq 0, k = 1, \ldots, N - 1$.

It appears quite helpful to use a diagrammatic representation for all objects considered above. They can be represented in the form of Feynman diagrams. The examples are shown in figures 2 and 3. In these figures the line with an arrow and the index $\alpha$ stands for the propagator, $D_\alpha$, equation (16), and the integration over all vertices with the measure (6) is implied. Identities like (18) are equivalent to the equality of the corresponding diagrams and can be proved with the help of a few diagrammatical rules.

The operators $B_N(u), A_N(u), \tilde{B}_N(u)$ are hermitian operators for real $u$. Provided that they can be extended to self-adjoint operators their eigenfunctions will form a complete system in the Hilbert space. The direct proof of completeness is also possible and will be given elsewhere. In particular, the completeness of the $B_N$ and $A_N$ systems is equivalent to the completeness of the SoV representation for the Toda spin chain which was proved by Koizumi [26]. In what follows we take for granted that each of these systems provides a basis in the Hilbert space $\mathcal{H}_N$.

Finally, we need two more identities for the eigenfunctions. Namely,

\[
A_N(x_1)\Psi_B^{(N)}(p, x_1, \ldots, x_{N-1}|\bar{z}) = (u + ix)^N\Psi_B^{(N)}(p, x_1 + i, \ldots, x_{N-1}|\bar{z}), \quad (38a)
\]

\[
B_N(x_1)\Psi_A^{(N)}(x_1, \ldots, x_N|\bar{z}) = -i(u + ix)^N\Psi_A^{(N)}(x_1 + i, \ldots, x_N|\bar{z}), \quad (38b)
\]
i.e. the operators $A_N$ and $B_N$ act as shift operators on the separated variables. Since the eigenfunctions are symmetric in $x_k$, a similar equation holds also for all others $x_k$. The equations (38) can be derived from the fundamental commutation relations (FCR) for the operators $A_N, B_N$, see e.g. [11, 18], or by the ‘gauge rotation’ trick for Lax operators, [20, 24].
4. Matrix elements

In this section we discuss the calculation of the scalar product of the eigenfunctions \( \Psi_B^{(N)}(p, \vec{u}) \) and \( \Psi_A^{(N)}(\vec{x}) \) and the matrix element of the shift operator, \( T_\gamma = \exp\{-\gamma S_-\} \), where \( S_- = \sum_{k=1}^N S_{-k} \) is shift generator

\[
T_\gamma f(z_1, \ldots, z_N) = f(z_1 + \gamma, \ldots, z_N + \gamma), \quad \gamma \in \mathbb{R}.
\]

We introduce the following notation

\[
S_B^{RA}(p, \vec{u}; \vec{x}) = \left( \Psi_B^{(N)}(p, \vec{u}), \Psi_A^{(N)}(\vec{x}) \right), \quad T_\gamma(\vec{x}, \vec{x}') = \left( \Psi_A^{(N)}(\vec{x}'), T_\gamma \Psi_A^{(N)}(\vec{x}) \right).
\]

The matrix elements (40) have been calculated in [22]. It was achieved by going over to the Feynman diagram representation for the quantities in question and subsequent evaluation of these diagrams. The calculation is straightforward and will not be repeated here. Other examples of the diagrammatic technique can be found in [20, 21, 25] and in section 5 of the present work\(^4\). Here we present some arguments explaining why these matrix elements can be calculated in the closed form.

It can be shown that both these matrix elements satisfy difference equations. In order to derive a difference equation for the scalar product \( S_B^{RA}(p, \vec{u}; \vec{x}) \) we consider the matrix element of the operator \( A_N(u_1) \) between the eigenstates \( \Psi_B^{(N)}(p, \vec{u}) \) and \( \Psi_A^{(N)}(\vec{x}) \). Since the function \( \Psi_A^{(N)}(\vec{x}|\vec{z}) \) is the eigenfunction of the operator \( A_N(u_1) \) and \( A_N(u_1) \) acts as shift operator on \( \Psi_B^{(N)}(p, \vec{u}) \), we obtain

\[
\left( \Psi_B^{(N)}(p, \vec{u}), A_N(u_1) \Psi_A^{(N)}(\vec{x}) \right) = \prod_{k=1}^N (u_1 - x_k) \left( \Psi_B^{(N)}(p, \vec{u}), \Psi_A^{(N)}(\vec{x}) \right) = \prod_{k=1}^N (u_1 - x_k) S_B^{RA}(p, \vec{u}; \vec{x})
\]

and

\[
\left( \Psi_B^{(N)}(p, \vec{u}), A_N(u_1) \Psi_A^{(N)}(\vec{x}) \right) = \left( A_N(u_1) \Psi_B^{(N)}(p, \vec{u}), \Psi_A^{(N)}(\vec{x}) \right) = (u + ix)^N \left( \Psi_B^{(N)}(p, \vec{u} + i\vec{e}_1), \Psi_A^{(N)}(\vec{x}) \right) = (u + ix)^N S_B^{RA}(p, \vec{u} + i\vec{e}_1; \vec{x}),
\]

where \( \vec{u} + i\vec{e}_1 = \{u_1 + i, u_2, \ldots, u_{N-1}\} \). Thus we get a recurrence relation for the function \( S_B^{RA}(p, \vec{u} + i\vec{e}_1; \vec{x}) \) in the variable \( u_1 \) of the form

\[
(\gamma + ix)^N S_B^{RA}(p, \vec{u} + i\vec{e}_1; \vec{x}) = \prod_{k=1}^N (u_1 - x_k) S_B^{RA}(p, \vec{u}; \vec{x}).
\]

The solution of the difference equation (up to multiplication by a periodic function of \( u_1 \)) has the form \( \prod_{k=1}^N \Gamma(i(x_k - u_1))/\Gamma(s - iu_1) \). Next, proceeding in the same way and considering the matrix element of the operator \( B_N(x_1) \) one can fix the \( x_1 \)-dependence of \( S_B^{RA}(p, \vec{u}; \vec{x}) \). Taking into account that \( S_B^{RA}(p, \vec{u}; \vec{x}) \) is symmetric in \( \{x\} \) and \( \{u\} \) one gets

\[
S_B^{RA}(p, \vec{u}; \vec{x}) = \frac{1}{\sqrt{p}} e^{-i\varepsilon} \prod_{k=1}^N \frac{1}{\Gamma(s - i\varepsilon_k)} \prod_{j=1}^{N-1} \frac{\Gamma(i(u_j - x_k))}{\Gamma(s - iu_j)\Gamma(s + iu_j)} \times \varphi(\vec{x}, \vec{u}),
\]

\(^4\) For an application of this technique to the Toda spin chain see [27].
where we put $X = \sum_{i=1}^{N} x_i$ and $\varphi(\vec{x}, \vec{u})$ is a periodic function in each variable. The $p$-dependence follows from two relations
\[
i S_0 \Psi^{(N)}_A(\vec{x}|\vec{z}) = -X \Psi^{(N)}_A(\vec{x} \varepsilon|\vec{z}) \quad \text{and} \quad i S_0 \Psi^{(N)}_B((p, \vec{u})|\vec{z}) = i \left( \frac{p \partial_p - 1/2}{2} \right) \Psi^{(N)}_B((p, \vec{u})|\vec{z}).
\]
(45)

In order to fix the periodic function $\varphi(\vec{x}, \vec{u})$ one can either analyze analytic properties of the function $S^{\beta A}$ or calculate it directly with the help of the diagrammatic technique considered here. For the matrix element (44) it gives $\varphi(\vec{x}, \vec{u}) = 1$. Nevertheless, the very possibility to obtain a matrix element by solving difference equations usually indicates that the corresponding Feynman diagram can be calculated in a closed form. Examples are considered in the next section.

One has also to take care about singularities in (44) arising when $u_j \to x_k$. All $\Gamma$-functions in the numerator of (44) come from the integration of the propagator’s chains (see [22]),
\[
\int D\vec{w} D\vec{z} \sum_{\alpha=1}^{\beta} \bar{D}_{\alpha}(\vec{w}, \vec{z}) \frac{\Gamma(2\epsilon)}{p \Gamma(\beta)} \int_0^{\infty} \frac{dp}{p^{2\beta}} \frac{e^{p(z-\vec{z})}}{p^{(s-\epsilon)}} = \frac{\Gamma(2\epsilon) \Gamma(i(\vec{u} - \vec{x}))}{\Gamma(\beta) \Gamma(\epsilon) \Gamma(s - \bar{u} \vec{z})}.
\]
(46)

For $x = u$ the momentum integral diverges at the lower limit. To make sense of this integral for $x = u$ one can introduce the regulator, $i(\vec{u} - \vec{x}) \to i(\vec{u} - \vec{x}) + \epsilon$. Technically, in order to not destroy the balance of indices that makes possible calculation of diagrams in a closed form it is preferable to ascribe a small positive imaginary part to the variables $x_k, u_k$, and replace $u_k \to \bar{u}_k = u_k^\epsilon$ in (44). Thus we assume that $\text{Im} x_k > 0$, and $\text{Im} u_k > 0$ and write (44) in the form
\[
S^{\beta A}_N((p, \vec{u}, \vec{x}) = \frac{1}{\sqrt{p}} e^{-ipX} \prod_{k=1}^{N} \frac{1}{\Gamma(s - i\epsilon_k)} \prod_{j=1}^{N-1} \frac{\Gamma(i(\bar{u}_k - x_k))}{\Gamma(s - i\epsilon_k) \Gamma(s + \bar{u}_k \epsilon)}.
\]
(47)

The matrix element $T_\gamma(\vec{x}, \vec{x'})$ was calculated in [22] with the help of the diagrammatic technique. Here we only briefly discuss the derivation of the recurrence relation. Making use of the commutation relation
\[
[S, A_N(u)] = B_N(u)
\]
which is a consequence of the FCR [18], and taking into account that $(A_N(x))^\dagger = A_N(\bar{x})$, one derives
\[
\prod_{k=1}^{N} (x_k - \bar{x}_k') T_\gamma(\vec{x}, \vec{x'}) = \left( A_N(\bar{x}) \Psi^{(N)}_A(\vec{x}), T_\gamma \Psi^{(N)}_A(\vec{x}) \right) = -i \gamma (x_1 + i\epsilon_1, \vec{x}).
\]
(49)

A direct calculation results in the following expression for $T_\gamma$
\[
T_\gamma(\vec{x}, \vec{x'}) = (\gamma + i0)^{(x-y)} e^{\frac{\pi}{2} (x-y)} \prod_{k,j=1}^{N} \frac{\Gamma(i(x_j - y_k))}{\Gamma(s - i\epsilon_k) \Gamma(s + \bar{u}_k \epsilon)},
\]
(50)

which, as can be easily checked, satisfies the above recurrence relation. Note also that the operator $e^{-\gamma S}$ is a well-defined on $\mathcal{H}_N$ provided that $\text{Im}(\gamma) \geq 0$. Indeed, if $f \in \mathcal{H}_N$, then $\varphi = e^{-\gamma S} f$, $(\varphi(z_1, \ldots, z_N) = f(z_1 + \gamma, \ldots, z_N + \gamma))$ also belongs to $\mathcal{H}_N$. 
4.1. First Gustafson integral

Expanding the eigenfunctions $\Psi_k^{(N)}$ over $\Psi_B^{(N)}$ one obtains the following integral representation for the matrix element $T_\gamma(x, \bar{x})$,

$$T_\gamma(x, \bar{x}) = \frac{1}{(N-1)!} \int_0^{\infty} dp \ e^{ipx} \int_0^{\infty} \prod_{k=1}^{N-1} \frac{du_k}{2\pi} \mu_N(u) S_B^{N}(p, u, \bar{x}) \left( S_B^{N}(p, u, \bar{x}) \right)^\dagger,$$

where the measure is defined as follows

$$\mu_N(u) = \prod_{k=1}^{N-1} \left[ \Gamma(s + in_k)\Gamma(s - in_k) \right]^N \prod_{k\neq j} \Gamma(i(u_k - u_j)).$$

Calculating the momentum integral and canceling common factors on both sides of equation (51) one finds

$$\frac{1}{(N-1)!} \left( \prod_{n=1}^{N} \int_0^{\infty} \frac{du_n}{(2\pi)^{n}} \right)^{\dagger} \prod_{k=1}^{N} \prod_{j=1}^{N-1} \Gamma(i(x_k' - u_j))\Gamma(i(u_j - x_k)) = \prod_{k=1}^{N} \prod_{j=1}^{N-1} \Gamma(i(x_k' - x_j)) \prod_{k=1}^{N} \Gamma(i(x_k' - u_k)) \Gamma(i(u_k - x_k)).$$

which is nothing but the first Gustafson integral (1).

Starting from the composition law for the shift operator, $T_{\gamma_1 + \gamma_2} = T_{\gamma_1} T_{\gamma_2}$, one derives an integral identity for the matrix elements (50). It takes the form

$$\frac{1}{N!} \prod_{n=1}^{N} \int_0^{\infty} \frac{du_n}{(2\pi)^{n}} \ x^n \prod_{k=1}^{N} \prod_{j=1}^{N-1} \Gamma(i(x_k' - u_j))\Gamma(i(u_j - x_k)) = \frac{\zeta x^N}{(1 + \zeta)^{|x|^N}} \prod_{k=1}^{N} \Gamma(i(x_k' - x_j)).$$

Note that (54) can be obtained from (53) by letting the parameters $x_N$ and $x_N'$ tend to infinity. Further, dividing both sides of (54) by $\zeta$ and integrating over $\zeta$ from zero to infinity one reproduces the integral (9.2) of [1].

5. Mixed scalar products

In order to prove the second Gustafson integral, we consider the scalar products between the functions of the $B$ system and the $B$ and $A$ systems. As a first step we derive the recurrence relations for the matrix elements

$$S_A^{N}(p, \bar{u}|\bar{x}) = \left( \Psi_A^{(N)}(p, \bar{u}), \Psi_A^{(N)}(\bar{x}) \right), \quad \delta(p - q)S_B^{N}(\bar{u}|\bar{x}) = \left( \Psi_B^{(N)}(p, \bar{u}), \Psi_B^{(N)}(q, \bar{x}) \right).$$

The analysis is almost the same for both products so we consider only $S_A^{N}(p, \bar{u}|\bar{x})$. We start by noticing that

$$B_N(u) = B_N(-u)A_N(u) - A_N(-u)B_N(u) = (-1)^{N-1}(2u + i) \left\{ i S_A - 2^{N-2} + O(u^{2N-4}) \right\}$$

and

$$B_N(u)\Psi_B^{(N)}(p, \bar{u}|\bar{z}) = (-1)^{N-1}(2u + i) \prod_{k=1}^{N-1} (u^2 - u_k^2)\Psi_B^{(N)}(p, \bar{u}|\bar{z}).$$
Then, considering the matrix element \( \left( \Psi^{(N)}_{\Omega}(p, \vec{u}), \mathcal{S}_N(x_1) \Psi^{(N)}_A(x) \right) \) and taking into account that the operator \( A_N(x_1) \) annihilates the eigenfunction \( \Psi^{(N)}_A(x) \) while \( B_N(x_1) \) shifts the separated variables as given by equation (38b) one derives

\[
p \prod_{k=1}^{N-1} (x_k^2 - u_k^2) \mathcal{S}_N^A(p, \vec{u}|x) = i(x_1 + ix)^N \prod_{k=2}^N (x_1 + x_k) \mathcal{S}_N^A(p, \vec{u}|x + ix_1). \tag{58}
\]

Solving this recurrence relation and taking into account that the function \( \mathcal{S}_N^A(p, \vec{u}|x) \) is a symmetric function of the separated variables \( x_1, \ldots, x_N \), we obtain

\[
\mathcal{S}_N^A(p, \vec{u}|x) = \frac{1}{N!} \prod_{k=1}^N (x_k - i\xi_k) \prod_{k=1}^N \Gamma(N(x - i\xi_k)) \prod_{k<j} \Gamma(-i(x_k + x_j)) \times \Phi_N(u). \tag{59}
\]

Of course, there is always the possibility to multiply this expression by a periodic function in \( x \). Let us for a moment assume that equation (59) correctly reproduces the \( x \)-dependence of the function \( \mathcal{S}_N^A \). Then one can see that the \( \Gamma \)-functions in the second product in the denominator do not arise from the integration of the propagator chains. It means that the diagram which represents the matrix element \( \mathcal{S}_N^A(p, \vec{u}|x) \), see figure 4, cannot be calculated with the help of the identities given in the appendix only.

In order to calculate \( \mathcal{S}_N^A(p, \vec{u}|x) \) we make use of the fact that the \( x \)-dependence of this function is known. Thus we have to determine the function \( \Phi_N(u) \) in equation (59) only. Therefore it is sufficient to calculate the corresponding Feynman diagram for some appropriately chosen specialization of the \( \{x_k\} \). For the choice, \( x_1 = u_1 \), the rhs in equation (59) becomes singular. However, the integration over a ‘free vertex’ in the diagram for \( \mathcal{S}_N^A(p, \vec{u}|x) \) (the leftmost gray vertex in figure 4) with only two propagators attached produces the factor (see equation (46))

\[
a(s + iu_1, s - ix_1) = \frac{\Gamma(2s)\Gamma(i(u_1 - x_1))}{\Gamma(s + iu_1)\Gamma(s - ix_1)}, \tag{60}
\]

which is also singular at \( x_1 \rightarrow u_1 \). Canceling the singular factor \( \Gamma(i(u_1 - x_1)) \) on both sides one can safely put \( x_1 = u_1 \). Since at \( x_1 \rightarrow u_1 \) the propagator arising due to the integration, \( D_{(u_1 - x_1)}(v'_1, v_1) \rightarrow 1 \), the line connecting vertices \( v_1 \) and \( v'_1 \) disappears. The resulting diagram can be simplified as follows:

(i) one integrates over the vertex \( v'_1 \) and moves the resulting line to the right with the help of the permutation relations given in appendix. Then one applies the same procedure to the vertices \( v_1 \), \( v_2 \) and so on. Each integration produces the factor \( a(\alpha, \beta) \) and the successive application of the permutation relations results in a rearrangement of indices. Namely, the integration over the vertices \( v'_1, v_1, v_2, \ldots \) gives the factors

\[
a(\alpha_1, \alpha_{i1}), \quad a(\beta_{ii}, \alpha_{1i}), \quad a(\beta_{ii}, \alpha_{2i}), \quad \ldots \text{ respectively. After these steps the upper part of} \quad
\]

the resulting diagram (the middle diagram in figure 4) corresponds to the diagram for the eigenfunction \( \Psi^{N-1}_A(x_2, \ldots, x_N) \).

(ii) The lower part of the middle diagram in figure 4 has the form

\[
\bar{A}_2^i(u_{N-1}) \cdots \bar{A}_{N-1}^i(u_2) F_{N-1}(u_1), \tag{61}
\]

where the operator \( F_N \) (constrained by the dashed rectangle in figure 4) has the diagrammatic representation shown in figure 5. The operators \( \bar{A}^i_2 \) and \( F_N \) obey the following exchange relation
\[ \Lambda^\dagger \mathcal{N}(v) F_{\mathcal{N}}(u) = a(\alpha u, \beta v) F_{\mathcal{N}}^{-1}(u) \Lambda^\dagger \mathcal{N}(v). \]  

(62)

The proof of this relation is straightforward and illustrated in figure 6. Using this relation we find

\[ \tilde{\Lambda}^\dagger_{N-1}(u_{N-1}) \ldots \tilde{\Lambda}^\dagger_1(u_2) F_{\mathcal{N}}^{-1}(u_1) = \prod_{k=2}^{N-1} a(\alpha_{u_k}, \beta_{u_k}) a(\alpha_{u_1}, \alpha_{u_1}) F_{\mathcal{N}}^{-1}(u_1) \tilde{\Lambda}^\dagger_1(u_{N-1}) \ldots \tilde{\Lambda}^\dagger_1(u_2). \]  

(63)

\[ \tilde{\Lambda}^\dagger_{N-1}(v) F_{\mathcal{N}}(u) = a(\alpha_u, \beta_v) a(\alpha_n, \alpha_{n}) F_{\mathcal{N}-1}(u) \tilde{\Lambda}^\dagger_{N}(v). \]  

Figure 4. The diagrammatic calculation of the scalar product $S^A_{\mathcal{N}}(p, d|x)$, equation (55). The leftmost diagram corresponds to the scalar product $S^A_{\mathcal{N}}(p, \{u_1, u_2, u_3\}) \{x_1, x_2, x_3, x_4\}$ and the rightmost to $S^A_{\mathcal{N}}(p, \{u_2, u_3\}) \{x_2, x_3, x_4\}$. The upper part of the diagrams, above the dotted line, corresponds to the eigenfunction $A_N$ and the lower part to the eigenfunction $B_N$. The indices are not shown explicitly. They can be easily read off the diagrams in figures 2 and 3 (notice that $\alpha_{u_1} = s - iX_1$ and $\beta_{u_1} = s + iu_1 = (\alpha_{u_1})^*$).

Figure 5. The diagrammatic representation of the operator $F_N(u)$ ($N = 4$). The indices are defined as follows $\alpha_u = s - iu_1$, $\beta_u = s + iu_1$.

Thus one expresses the $N$-sites scalar product, $S^A_{\mathcal{N}}$, for the special choice of parameters, $x_1 = u_1$, via the $N - 1$ sites scalar product. Collecting all factors and taking into account the normalization coefficients (25), (32) we obtain:
For the scalar product

\[ \Pi(p) \Pi(u) \]

we only state the final result

\[ \Phi_N(u_1, \ldots, u_{N-1}) = \prod_{k=2}^{N-1} \frac{1}{\Gamma(\alpha_{u_k}) \Gamma(\beta_{u_k})} \Phi_{N-1}(u_2, \ldots, u_{N-1}). \]  

(65)

Taking into account the initial condition \( \Phi_1 = 1 \) yields

\[ \Phi_N(u_1, \ldots, u_{N-1}) = \left( \prod_{k=1}^{N-1} \Gamma(\alpha_{u_k}) \Gamma(\beta_{u_k}) \right)^{-N} \]  

(66)

and, hence

\[ S^A_N(p, \vec{u}, \vec{x}) = \frac{1}{\sqrt{\pi^p}} e^{-ix} \prod_{k=1}^{p} \prod_{j=1}^{N-1} \Gamma(\pm iu_k - ix_k) \left( \prod_{k=1}^{p} \Gamma(\alpha_{u_k}) \Gamma(\beta_{u_k}) \right)^N \prod_{1 \leq k < j < N} \Gamma(-i(u_k + u_j)). \]  

(67)

The calculation of the scalar product \( S^B_N(p, \vec{u}, \vec{x}) \) follows along the same lines. Consequently we only state the final result

\[ S^B_N(p, \vec{u}, \vec{x}) = \frac{1}{\sqrt{\pi^p}} e^{-ix} \prod_{k=1}^{p} \prod_{j=1}^{N-1} \Gamma(\pm iu_k - ix_k) \left( \prod_{k=1}^{p} \Gamma(\alpha_{u_k}) \Gamma(\beta_{u_k}) \right)^N \prod_{1 \leq k < j < N} \Gamma(-i(u_k + u_j)). \]  

(68)

In our derivation equation (67) we did not consider the possibility of multiplying the solution (59) of the recursion (58) by a periodic function of \( x \). In order to see that equation (67) gives the right answer one can proceed slightly differently, finally arriving to the same result. Namely, it can be shown by a straightforward application of the integration rules to the diagram for the scalar product \( S^B_N(p, \vec{u}, \vec{x}) \), (the leftmost diagram in figure 4) that it can be represented.

**Figure 6.** The diagrammatic proof of the exchange relation (62). The right diagram corresponds to the product \( \Lambda^A_N(v) F_N(u) \). In the first step one integrates over the vertex \( v_1 \) and moves the resulting line to the right with the help of the permutation relations (ii) and (iii) in the appendix. It ends up as the arched line in the middle diagram. On the next step one repeats the same procedure for the vertex \( v_2 \). On the third step we insert the lines with the indices \( \pm i(u - v) \) as shown in the rightmost diagram and move them to the left and right interchanging on the way the indices of the two upper layers. On the final step one moves the arched line to the left. The resulting diagram corresponds to the product \( F_{N-1}(u) \Lambda^A_N(v) \).

\[
\frac{S^A_N(p, \vec{u}, \vec{x})}{\Gamma(i(u_1 - x_1))} \bigg|_{u_1=x_1} = S^A_{N-1}(p, \{u_2, \ldots, u_{N-1}\}; \{x_2, \ldots, x_{N}\}) \frac{\Gamma(2\pi) a(\alpha_{u_1}, \alpha_{u_1}) \prod_{k=2}^{N} a(\beta_{u_k}, \alpha_{u_k})}{\Gamma(\alpha_{u_1}) \Gamma(\beta_{u_1})} \times \prod_{k=2}^{N-1} a(\alpha_{u_k}, \beta_{u_k}) a(\alpha_{u_k}, \alpha_{u_k}) \times p^{-i\pi} \Gamma(\frac{2\pi}{\Gamma(\alpha_{u_1})}) \Gamma(2\pi)^{-3N+2}. 
\]  

(64)
in the form \( F(u_1, \bar{x}) \times S(u_2, \ldots, u_{N-1} | \bar{x}) \), where \( F(u_1, \bar{x}) \) is given by a product of \( \Gamma \)-functions. Since the function \( S_N^\alpha(p, \bar{u}, \bar{x}) \) is a symmetric function of the \( \bar{u} \) variables one concludes that \( S_N^\alpha(p, \bar{u}, \bar{x}) = \prod_{k=1}^{N-1} F(u_k, \bar{x}) \times \Psi_N(\bar{x}) \). Finally, in order to determine \( \Psi_N(\bar{x}) \) one applies the same ‘\( u_1 \to \bar{u}_1 \)’ trick described above.

**5.1. Second Gustafson’s integral**

The second Gustafson integral is also related to the matrix element \( T_\gamma(\bar{x}, \bar{x}') \), equation (40). Only this time we use an expansion in terms of the eigenfunctions \( \Psi_B^{(N)}(p, \bar{u}) \). One obtains

\[
T_\gamma(\bar{x}, \bar{x}') = \frac{1}{(N-1)!} \int_0^\infty dp \, e^{i p} \left( \prod_{k=1}^{N-1} \frac{d\mu_k}{4\pi} \right) \mu_N(\bar{u}) S_N^\alpha(p, \bar{u}, \bar{x}) \left( S_N^\alpha(p, \bar{u}, \bar{x}') \right)^\dagger, \tag{69}
\]

where the measure is defined as follows, see equation (37),

\[
\tilde{\mu}_N(\bar{u}) = \frac{\prod_{k=1}^{N-1} \Gamma(s + i u_k) \Gamma(s - i u_k)}{\prod_{1 \leq j < k \leq N} \Gamma(i(u_k \pm u_j)) \Gamma(-i(u_k \pm u_j)).} \tag{70}
\]

Substituting \( S_N^\alpha(p, \bar{u}, \bar{x}) \), equation (67), in (69) and integrating over \( p \) one gets

\[
\frac{1}{(N-1)!} \left( \prod_{k=1}^{N-1} \frac{d\mu_k}{4\pi} \right) \frac{\prod_{k=1}^N \Gamma(2i u_k) \Gamma(-2i u_k) \prod_{1 \leq j < k \leq N} \Gamma(i(u_k \pm u_j)) \Gamma(-i(u_k \pm u_j))}{\prod_{k=1}^N \Gamma(i(x'_k - x_j)) \prod_{1 \leq j < k \leq N} \Gamma(i(x'_k + x'_j)) \Gamma(-i(x_k + x_j))}. \tag{71}
\]

This integral, after redefining \( \alpha_k = i x'_k, \alpha_{N+k} = -i x_k \) and \( N - 1 \to N \), coincides with (2).

Writing down the scalar product \( S_B(p, \bar{x}, \bar{x}') \) in the \( \Psi_B^{(N)}(q, \bar{u}) \) basis one obtains the following identity

\[
S_B(p, \bar{x}, \bar{x}') = \frac{1}{(N-1)!} \left( \prod_{k=1}^{N-1} \frac{d\mu_k}{2\pi} \right) \mu_N(\bar{u}) S_N^\alpha(\bar{x}, \bar{u}) S_B^{(N)}\left( p, \bar{u}, \bar{x}' \right), \tag{72}
\]

where the measure \( \mu_N(\bar{u}) \) is defined in (52). Substituting the explicit expressions for the scalar products as given by equations (47), (67) and (68) one obtains after some redefinition the integral (3). Let us note here that an integral similar to (3) can be obtained from the elliptic integral, Theorem 5.3 in [5], by reduction to the rational case. However, this integral and the integral (3) have a different dependence on the external parameters and can not be transformed one to the other.

For \( N = 1 \) the integral (3) is a special case of (1). Conversely, for \( N > 1 \) the integral (1) follows from the integrals (2) and (3). If we let \( \alpha_{2N+2} \) tend to infinity and compare the asymptotics on both sides we find

\[
\left( \prod_{k=1}^N \frac{d\alpha_k}{2\pi i} \right) \frac{\prod_{k=1}^{2N+1} \Gamma(\pm z_k)}{\prod_{1 \leq k < j \leq 2N+1} \Gamma(z_k \pm z_j) \Gamma(-z_k \pm z_j)} = 2^{N-1} \prod_{1 \leq j < k \leq 2N+1} \Gamma(\alpha_k + \alpha_j). \tag{73}
\]
Then multiplying both sides of (3) by
\[
\prod_{i=1}^{N+1} \prod_{k=1}^{N} \frac{\Gamma(\gamma \pm \beta_k)}{\Gamma(2\beta_k) \Gamma(\beta_k \pm \beta_j)}
\] (74)
and carrying out the integration over \(\beta_k\) with the help of (2) and (73) one obtains Gustafson’s first integral.

6. \(\Psi_N \times \Psi_{N-1}\) scalar products

In this section we calculate another set of scalar products which lead to several new beta-type integrals. Namely, we consider the scalar products of the eigenfunctions \(\Psi_B^{(N)}(p, \vec{x}|\vec{z})\), \(\Psi_A^{(N)}(\vec{x}|\vec{z})\) with
\[
\Psi_B^{(N-1)}(p, x_1, \ldots, x_{N-2}|\vec{z}_{N-1}) \times M_\nu(z_N), \quad \Psi_A^{(N-1)}(p, x_1, \ldots, x_{N-1}|\vec{z}_{N-1}) \times M_\nu(z_N),
\] (75)
where \(M_\nu(z)\) is (see also the appendix)
\[
M_\nu(z) = \frac{\Gamma(2s)}{\Gamma(s + iv)}D_{x + iv}(z, 0) = \frac{\Gamma(2s)}{\Gamma(s + iv)}e^{i\pi/2(s+iv)}z^{s-iv}.
\] (76)

All four scalar products can be calculated by the diagrammatic technique. The calculation proceeds along the following lines: one starts with the diagram \(G_N\) for the \(N\)-point scalar product of two functions and transforms \(G_N\) into the form \(G_N = F_N \times G_{N-1}\) with the help of the identities given in appendix. The factor \(F_N\) depends on the spectral parameters and is given by a product of \(\Gamma\) functions. So one immediately gets that \(G_N = F_N F_{N-1} \ldots F_1 \times G_2\). In all cases, the starting point of the recursion, the diagram \(G_{N=2}\), can be easily evaluated.

We obtained the following explicit expressions for these scalar products:
\[
\left(\Psi_B^{(N-1)}(p, \vec{u}) \times M_\nu, \Psi_A^{(N)}(\vec{x})\right) = \frac{\rho^{-i(\nu+X)}}{\sqrt{\rho}} \frac{\Gamma(\alpha_\nu)}{\Gamma(\beta_\nu)} \prod_{k=1}^{N} \frac{\Gamma(i(x_k + \nu))}{\Gamma(\alpha_k)\Gamma(\beta_k)} \times \prod_{k=1}^{N-2} \frac{\Gamma(\beta_{uk})}{\Gamma(i(\nu + \nu))} \prod_{k=1}^{N-2} \frac{\Gamma(i(u_k - x_k))}{\Gamma(\alpha_k)\Gamma(\beta_k)},
\]
\[
\left(\Psi_A^{(N-1)}(\vec{u}) \times M_\nu, \Psi_A^{(N)}(\vec{x})\right) = 2\pi \delta(\nu + X - U) \frac{\Gamma(\alpha_\nu)}{\Gamma(\beta_\nu)} \prod_{k=1}^{N} \frac{\Gamma(\beta_{uk})}{\Gamma(i(u_k - x_k))} \prod_{k=1}^{N-2} \frac{\Gamma(i(u_k - x_k))}{\Gamma(\alpha_k)\Gamma(\beta_k)}
\] (77)
and
\[
\left(\Psi_B^{(N-1)}(q, \vec{u}) \times M_\nu, \Psi_B^{(N)}(p, \vec{x})\right) = \theta(p - q)\rho^{iU+\frac{1}{2}}q^{-iX-\frac{1}{2}}(p - q)^{i(X-U)} \times \prod_{k=1}^{N-1} \frac{1}{\Gamma(\alpha_k)} \prod_{j=1}^{N-2} \frac{\Gamma(i(u_j - x_k))}{\Gamma(\beta_k)\Gamma(\alpha_k)},
\]
\[
\left(\Psi_A^{(N-1)}(\vec{x}) \times M_\nu, \Psi_B^{(N)}(p, \vec{u})\right) = \frac{\rho^{i(\nu+X)}}{\sqrt{\rho}} \prod_{k=1}^{N} \frac{1}{\Gamma(\beta_{uk})\Gamma(i(x_k - \nu))} \prod_{k=1}^{N-1} \frac{\Gamma(i(x_k - u_j))}{\Gamma(\beta_k)\Gamma(\alpha_k)}
\] (78)
Here and below $X = \sum x_k$, $Y = \sum y_k$ and $U = \sum u_k$. It is tacitly assumed that the `+ 0' prescription is used for all $\Gamma$ functions in the numerators, i.e. $\Gamma(i(u - x)) \rightarrow \Gamma(i(u - x) + \epsilon)$.

The scalar product of two functions $\Psi_1(\tilde{y})$, $\Psi_2(\tilde{x})$ can be written, schematically, in the form

$$ (\Psi_1(\tilde{y}), \Psi_2(\tilde{x})) = \int d\mu(\tilde{u})(\Psi_1(\tilde{y}) \cdot \Psi_2(\tilde{u}))(\Psi_3(\tilde{u}), \Psi_2(\tilde{x})), \quad (79) $$

where $\Psi_3(\tilde{u})$ is a complete system of functions and $\mu(\tilde{u})$ is the corresponding measure. For the functions $\Psi_k$, $k = 1, 2, 3$ from the set $\{\Psi_B^{(N)}, \Psi_A^{(N)}, \Psi_B^{(N-1)}, \Psi_A^{(N-1)}\}$ all scalar products in (79) are known in an explicit form and the right-hand side of (79) has the form of a multidimensional Mellin–Barnes integral. We list below some of the integrals arising in this way. The most interesting ones are those for which the number of external parameters minus the number of integrations is maximal. In presenting our results we replace the integration variables, $iu_k \mapsto u_k$, and do the same for the external parameters.

Since the prescription for going around the poles is fixed and integrals are convergent, the corresponding integral identities hold for complex parameters as well. We will also sometimes shift $N \rightarrow N + 1, N + 2$.

- The first integral arises from (79) for the choice:

$$ \{\Psi_1, \Psi_2, \Psi_3\} = \{\Psi_A^{(N-1)}(\tilde{y}) \times M_\nu, \Psi_A^{(N)}(\tilde{x}), \Psi_B^{(N-1)}(\tilde{u}) \times M_\nu\}. $$

It takes the form

$$ \frac{1}{N!} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi i} \prod_{k=1}^{N} \frac{\Gamma(\nu + u_k - 1)}{\Gamma(\nu + u_k)} \prod_{k=1}^{N+2} \Gamma(y_k + x_k) \prod_{k=1}^{N} \Gamma(u_k - u_j) \prod_{k=1}^{N+2} \Gamma(y_k + x_k), \quad (80) $$

where $\nu = Y + X = \sum y_k + \sum x_k$, $\Re x_k > 0$, $\Re y_k > 0$ and coincides with [2, equation (3.2)].

- Considering $\{\Psi_A^{(N-1)}(\tilde{u}) \times M_\nu, \Psi_A^{(N)}(\tilde{x}), \Psi_A^{(N-1)}(\tilde{u}) \times M_\nu\}$ we get

$$ \frac{2\pi i}{N!} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi i} \left( \sum_{k=1}^{N} \frac{\Gamma(\nu - u_k)}{\Gamma(\nu - u_k)} \prod_{k=1}^{N} \Gamma(y_k - u_k) \prod_{k=1}^{N+1} \Gamma(x_k + u_j) \right) \prod_{k=1}^{N} \Gamma(\nu + u_k) \prod_{k=1}^{N+2} \Gamma(y_k + x_k), \quad (81) $$

where $X = \sum x_j$ and $\Re x_j > 0$, $\Re y_j > 0$. For $N = 2$ this identity is equivalent to the Wilson–de Branges integral [28, 29].

- The triple $\{\Psi_A^{(N-1)}(\tilde{y}) \times M_\nu, \Psi_B^{(N)}(\tilde{x}), \Psi_B^{(N-1)}(\tilde{q}) \times M_\nu\}$ gives rise to

$$ \frac{1}{N!} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi i} \frac{\Gamma(\nu - X - U)}{\Gamma(\nu + Y - U)} \prod_{k=1}^{N+1} \Gamma(y_k + x_k) \prod_{k=1}^{N+1} \Gamma(u_k - u_j) \prod_{k=1}^{N} \Gamma(y_k + x_k) \prod_{k=1}^{N} \Gamma(u_k - u_j), \quad (82) $$

where $\Re x_k > 0$, $\Re y_k > 0$, and $\Re \nu > \Re X$. For $N = 1$ it is equivalent to the second Barnes lemma, while for general $N$ it is a modification of Gustafson’s integral (1).

- For $\{\Psi_B^{(N)}(\tilde{u}), \Psi_A^{(N)}(\tilde{x}), \Psi_A^{(N-1)}(\tilde{u}) \times M_\nu\}$ one finds

- For $\{\Psi_B^{(N)}(\tilde{y}), \Psi_A^{(N)}(\tilde{x}), \Psi_A^{(N-1)}(\tilde{u}) \times M_\nu\}$ one finds


The last integral arises from \( \{ \Psi_B^{(N)}(p, \bar{y}), \Psi_A^{(N)}(\bar{x}), \Psi_B^{(N-1)}(p, \bar{u}) \times M_{\nu} \} \)

\[
\frac{1}{(N-1)!} \int_{\mathcal{R}^N} d\mu \int_{i\mathcal{R}^{N-1}} \prod_{k=1}^{N-1} \frac{d\nu_k}{2\pi i} \prod_{k=1}^{N-1} \frac{d\nu_k}{(s - X - \nu_k)} \prod_{k=1}^{N-1} \frac{d\nu_k}{(s + X + \nu_k)} \prod_{k=1}^{N-1} \frac{d\nu_k}{\Gamma(s + X + \nu_k)} \prod_{k=1}^{N-1} \frac{d\nu_k}{\Gamma(s - X - \nu_k)} \prod_{k=1}^{N-1} \frac{d\nu_k}{\Gamma(\nu + X - \nu_k)} \prod_{k=1}^{N-1} \frac{d\nu_k}{\Gamma(\nu + X + \nu_k)} \prod_{k=1}^{N-1} \frac{d\nu_k}{\Gamma(\nu + X + \nu_k)} \prod_{k=1}^{N-1} \frac{d\nu_k}{\Gamma(\nu + X - \nu_k)}
\]

(84)

where \( \text{Re} x_k > 0, \text{Re} y_k > 0, \text{Re} X > \text{Re} y_k \) and \( \text{Re} s > \text{Re} X \). Again, for \( N = 1 \) it reduces to Barnes’ second lemma.

- The last integral arises from \( \{ \Psi_B^{(N)}(p, \bar{y}), \Psi_A^{(N)}(\bar{x}), \Psi_B^{(N-1)}(p, \bar{u}) \times M_{\nu} \} \)

7. Summary

The eigenfunctions of the matrix elements of the monodromy matrix (for both the closed and open spin chains) provide convenient bases for the study of the spectral problem for the corresponding spin magnets. Remarkably these eigenfunctions can be constructed explicitly as multivariable integrals and represented by Feynman diagrams of a certain type. The scalar products between the different eigenfunctions can be calculated with the help of the diagrammatic technique and, as a rule, are given by a product of gamma functions with arguments depending on the parameters labeling the eigenfunctions (separated variables). In the SoV representations the scalar product or matrix elements take the form of multidimensional Mellin–Barnes integrals. Studying different scalar products we succeeded to reproduce all relevant integrals in [1, 2] except for [1, equation (9.6)] and [2, equation (5.4)] in [2] and derived several new integrals which did not follow from those of Gustafson’s.

In this work we have considered only the homogeneous spin chains. However, the eigenfunctions can be constructed in a similar way and for a general case of inhomogeneous spin chains with impurities [20]. Gustafson’s integrals (1) and (2) are not sensitive to all these modifications. At the same time we expect that inclusion of additional parameters (spins and impurities) into consideration could lead to modification of the integrals given in section 6.

Our approach can be extended to noncompact \( SL(2, \mathbb{C}) \) spin magnets [30] resulting in another extension of Gustafson’s integrals. Some insight into the possible structure of such integrals can be gained from [31–33]. We also expect that considering trigonometric and elliptic spin chains gives rise to new \( q \)-beta and elliptic Gustafson type integrals.

Acknowledgments

The authors are grateful to Karol Kozlowski and Vyacheslav Spiridonov for fruitful discussions and correspondence. We also express our gratitude to Hjalmar Rosengren and the referee.
for bringing the reference [5] to our attention. The authors are deeply indebted to the referees for a careful reading of the manuscript and suggestion of many amendments. This study was supported by the Russian Science Foundation (SD), project No 14-11-00598, and by Deutsche Forschungsgemeinschaft (AM), grant MO 1801/1-1.

Appendix. Diagram technique

In this appendix we present the basic elements of the diagram technique which was used throughout the paper. The propagator

\[ D_\alpha(z, \bar{w}) = \left( \frac{i}{z - \bar{w}} \right)^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^\infty dp \, e^{ip(z - \bar{w})} p^{\alpha-1} \]  

(A.1)
is shown by an arrow directed from \( \bar{w} \) to \( z \) with the index \( \alpha \) attached to it. Under complex conjugation it behaves as: \( (D_\alpha(z, \bar{w}))^* = D_{\alpha^*}(w, \bar{z}) \).

There are several useful identities involving propagators:

(i) Chain rule: the integral of two propagators is again a propagator:

\[ \int Dw \, D_\alpha(z, \bar{w}) D_\beta(w, \bar{\zeta}) = a(\alpha, \beta) D_{\alpha+\beta-2s}(z, \bar{\zeta}), \]

(A.2)

where

\[ a(\alpha, \beta) = \frac{\Gamma(2s)\Gamma(\alpha + \beta - 2s)}{\Gamma(\alpha)\Gamma(\beta)}. \]  

(A.3)

(ii) Permutation relation:

(iii) Reduced permutation relation:
(iv) The propagator identity:

\[
\begin{array}{c}
\begin{array}{c}
\beta_x \\
\alpha_x \\
0 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\beta'_x \\
\alpha'_x \\
0 \\
\end{array}
\end{array}
\end{array}
\]

\[= a(\beta_x, \beta'_x) \times \beta_x \alpha_x \alpha'_x 0 0 z_0 z_0 \]

All these identities can be easily checked by going over to the momentum representation.

There are two standard bases in the (one-particle) Hilbert space \( H \):

- The plane waves:

\[
E_p(z) = p^{1/2} e^{i(p \gamma / 2)} \Gamma(2s + 1/2) / 2\pi, \quad p > 0:
\]

\[ (E_{p'}, E_p) = \int Dz E_p(z) E_{p'}(z) = \delta(p - p'). \] (A.4)

- powers:

\[
M_{\nu}(z) = (\Gamma(2s))^{-1/2} \Gamma(s + i\nu) D_{s+i\nu}(z, 0) = (\Gamma(2s))^{-1/2} \Gamma(s + i\nu) e^{i\pi/2(s+i\nu)} z^{-s-i\nu},
\]

where \( \nu \in \mathbb{R} \),

\[ (M_{\nu'}, M_{\nu}) = \int Dz M_{\nu}(z) M_{\nu'}(z) = 2\pi \delta(\nu - \nu'). \] (A.5)

For the transition matrix element one obtains

\[ (M_{\nu}, E_p) = p^{-\nu-1/2}. \] (A.7)

References

[1] Gustafson R A 1994 Some \( q \)-beta and Mellin–Barnes integrals on compact Lie groups and Lie algebras Trans. Am. Math. Soc. 341 69–119

[2] Gustafson R A 1992 Some \( q \)-beta and Mellin–Barnes integrals with many parameters associated to the classical groups SIAM J. Math. Anal. 23 525

[3] Gustafson R A 1994 Some \( q \)-beta integrals on SU\( (n) \) and Sp\( (n) \) that generalize the Askey-Wilson and Nasrallah-Rahman integrals SIAM J. Math Anal. 25 441

[4] Spiridonov V P 2004 Theta hypergeometric integrals St. Petersburg Math. J. 15 929

[5] Spiridonov V P and Warnaar S O 2006 Inversions of integral operators and elliptic beta integrals on root systems Adv. Math. 207 91

[6] Spiridonov V P 2007 Short proofs of the elliptic beta integrals Ramanujan J. 13 265–283

[7] Stokman J V 2000 On BC type basic hypergeometric orthogonal polynomials Trans. Am. Math. Soc. 352 1527

[8] Andrews G E, Askey R and Roy R 1999 Special functions Encyclopedia of Mathematics and its Applications vol 71 (Cambridge: Cambridge University Press) p xvi + 664

[9] Forrester P J and Warnaar S O 2008 The importance of the Selberg integral Bull. Am. Math. Soc. 45 489

[10] Spiridonov V P and Vartanov G S 2011 Elliptic hypergeometry of supersymmetric dualities Commun. Math. Phys. 304 797
[11] Sklyanin E K 1995 Separation of variables—new trends Prog. Theor. Phys. Suppl. 118 35
[12] Gelfand I M, Graev M I and Vilenkin N Y 1966 Generalized Functions (Integral Geometry and Representation Theory) vol 5 (New York: Academic)
[13] Takhtajan L A and Faddeev L D 1979 The quantum method of the inverse problem and the Heisenberg XYZ model Russ. Math. Surv. 34 11
Takhtajan L A and Faddeev L D 1979 The quantum method of the inverse problem and the Heisenberg XYZ model Usp. Mat. Nauk 34 13
[14] Faddeev L D, Sklyanin E K and Takhtajan L A 1980 The quantum inverse problem method. 1 Theor. Math. Phys. 40 688
Faddeev L D, Sklyanin E K and Takhtajan L A 1979 The quantum inverse problem method. 1 Teor. Mat. Fiz. 40 194
[15] Kulish P P and Sklyanin E K 1982 Quantum spectral transform method. Recent developments Lect. Notes Phys. 151 61
[16] Kulish P P, Reshetikhin N Y and Sklyanin E K 1981 Yang–Baxter equation and representation theory. 1 Lett. Math. Phys. 5 393
[17] Sklyanin E K 1992 Quantum inverse scattering method. Selected topics Quantum Group and Quantum Integrable Systems: Nankai Institute of Mathematics (Nankai Lectures on Mathematical Physics) (Singapore: World Scientific) pp 63–97
[18] Faddeev L D 1998 How Algebraic Bethe Anstz works for integrable model Quantum Symmetries/ Symetries Qantiques (Proc. Les-Houches Summer School vol LXIV) ed A Connes et al (Amsterdam: North-Holland) 149–211
[19] Sklyanin E K 1988 Boundary conditions for integrable quantum systems J. Phys. A: Math. Gen. 21 2375
[20] Derkachov S E, Korchemsky G P and Manashov A N 2003 Separation of variables for the quantum $SL(2, \mathbb{R})$ spin chain J. High Energy Phys. JHEP07(2003)047
[21] Derkachov S E, Korchemsky G P and Manashov A N 2003 Baxter $Q$ operator and separation of variables for the open $SL(2, \mathbb{R})$ spin chain J. High Energy Phys. JHEP10(2003)053
[22] Belitsky A V, Derkachov S E and Manashov A N 2014 Quantum mechanics of null polygonal Wilson loops Nucl. Phys. B 882 303
[23] Derkachov S E and Manashov A N 2014 Iterative construction of eigenfunctions of the monodromy matrix for $SL(2, C)$ magnet J. Phys. A: Math. Theor. 47 305204
[24] Derkachov S E 1999 Baxter’s $Q$-operator for the homogeneous XXX spin chain J. Phys. A: Math. Gen. 32 5299
[25] Derkachov S E and Manashov A N 2006 Factorization of the transfer matrices for the quantum $sl(2)$ spin chains and Baxter equation J. Phys. A: Math. Gen. 39 4147
[26] Kozlowski K K 2015 Unitarity of the SoV transform for the Toda chain Commun. Math. Phys. 334 223
[27] Silantyev A V 2007 Transition function for the Toda chain Theor. Math. Phys. 150 315–31
[28] de Branges L 1972 Tensor product spaces J. Math. Anal. Appl. 38 109–48
[29] Wilson J A 1980 Some hypergeometric orthogonal polynomials SIAM J. Math. Anal. 11 690–701
[30] Derkachov S E, Korchemsky G P and Manashov A N 2001 Noncompact Heisenberg spin magnets from high-energy QCD: 1. Baxter $Q$ operator and separation of variables Nucl. Phys. B 617 375
[31] Bazhanov V V, Kels A P and Sergeev S M 2013 Comment on star-star relations in statistical mechanics and elliptic gamma-function identities J. Phys. A: Math. Theor. 46 152001
[32] Kels A P 2014 A new solution of the star-triangle relation J. Phys. A: Math. Theor. 47 055203
[33] Kels A P 2015 New solutions of the startriangle relation with discrete and continuous spin variables J. Phys. A: Math. Theor. 48 435201