BLOW-UP OF THE CRITICAL NORM FOR A SUPERCritical 
SEMILINEAR HEAT EQUATION 

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Abstract. We consider the scaling critical Lebesgue norm of blow-up solutions to the semilinear heat equation $u_t = \Delta u + |u|^{p-1}u$ in an arbitrary $C^{2+\alpha}_+$ domain of $\mathbb{R}^n$. In the range $p > p_S := (n+2)/(n-2)$, we show that the critical norm must be unbounded near the blow-up time, where the type I blow-up condition is not imposed. The range $p > p_S$ is optimal in view of the existence of type II blow-up solutions with bounded critical norm for $p = p_S$.

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1. Introduction

1.1. Background. We study blow-up solutions of the following semilinear heat equation:

\[
\begin{align*}
  u_t &= \Delta u + |u|^{p-1}u, \quad x \in \Omega, \quad t > 0, \\
  u(x, t) &= 0, \quad x \in \partial\Omega, \quad t > 0, \\
  u(x, 0) &= u_0(x), \quad x \in \Omega.
\end{align*}
\]

Here \( p > 1, \Omega \) is a domain in \( \mathbb{R}^n \) with \( n \geq 1 \) and \( u_0 \in L^q(\Omega) \) with \( q \geq 1 \). The boundary condition is not present if \( \Omega = \mathbb{R}^n \).

The equation in (1.1) has attracted much attention as one of the simplest model for scaling invariant nonlinear parabolic equations. For each solution \( u \), the rescaled function \( u_\lambda(x, t) := \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t) \) \((\lambda > 0)\) also satisfies the equation. This implies that \( L^{q_c}(\Omega) \) with \( q_c := n/(p-1) \) is the scaling critical Lebesgue space for (1.1). The critical space plays a crucial role in well-posedness. If \( u_0 \in L^{q_c}(\Omega) \) and \( q_c > 1 \), it is well-known \([11, 106, 107]\) that there exists a unique classical \( L^{q_c}(\Omega) \)-solution \( u \) of (1.1) with the maximal existence time \( T \in (0, \infty) \). For the definition of the classical \( L^{q_c}(\Omega) \)-solution, see Remark 1.2. The solution \( u \) is smooth for \( t \in (0, T) \), belongs to \( C([0, T); L^{q_c}(\Omega)) \cap C((0, T); L^\infty(\Omega)) \) and admits the following blow-up criterion: If \( T < \infty \), then \( \lim_{t \to T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \). This leads to the critical norm blow-up problem.

Problem. If \( T < \infty \), does the following property hold?

\[
(\text{CNB}) \quad \lim_{t \to T} \|u(\cdot, t)\|_{L^{q_c}(\Omega)} = \infty.
\]

The problem is stated in Brezis and Cazenave \([11, \text{Open problem } 7]\) and also a variant can be found in Quittner and Souplet \([90, \text{OP } 2.1, \text{Section } 55]\) which asks the existence of blow-up solutions with bounded \( L^{q_c}(\Omega) \) norm. Many sufficient conditions for (CNB) are known, see \([10, 37, 43, 63, 64, 66, 76, 101, 110]\) and \([90, \text{Section } 16]\). Recently, a significant progress was made by Mizoguchi and Souplet \([76]\), where they proved that (CNB) holds whenever the blow-up is type I. Here the blow-up of \( u \) is called type I if the blow-up rate is bounded by a spatially homogeneous solution up to the coefficient, that is, \( \limsup_{t \to T} (T-t)^{1/(p-1)}\|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty \). This rate is natural in view of the scaling. Note that the blow-up is called type II if it is not of type I. In the celebrated work of Giga and Kohn \([43]\), they showed that the blow-up is always type I if \( \Omega \) is convex and either \( p < p_S \) and \( u \) is nonnegative, or \( p < (3n+8)/(3n-8) \). Here \( p_S \) is the Sobolev critical exponent given by

\[
p_S := \begin{cases} 
  n+2 & \text{for } n \geq 3, \\
  n-2 \quad & \text{for } n = 1, 2.
\end{cases}
\]

Up to now, it has been proved that the blow-up is type I under the conditions that either \( p < p_S \) and \( \Omega \) is convex \([45, 46]\), or \( p < p_S \) and \( u \) is nonnegative \([89]\). Therefore, under such conditions, (CNB) holds for any blow-up solutions in the subcritical range \( p < p_S \). For related results concerning sufficient conditions for type I blow-up, we refer \([17, 18, 20, 38, 45, 66, 68, 69, 75, 81, 83]\) and \([90, \text{Section } 23]\).
The situation is different in the case \( p \geq p_S \). Type II blow-up solutions were constructed in a number of papers, see Remark 1.7 for a brief review. In particular, the recent development \[23, 26, 27, 47, 59, 95\] provides type II blow-up solutions satisfying
\[
\sup_{0 < t < T} \| u(\cdot, t) \|_{L^{q_c}(\Omega)} < \infty
\]
for \( p = p_S \) and \( 3 \leq n \leq 5 \), see \[76\] Section 4 for computations of the \( L^{q_c} \) norm of the solutions constructed in \[23, 95\]. This demonstrates that type I assumption in \[76\] is indeed necessary if \( p = p_S \) and \( 3 \leq n \leq 5 \). For \( p = p_S \) and \( n \geq 6 \), no counter-examples are known. Moreover, for \( n \geq 7 \), it was shown \[104\] that \((\text{CNB})\) holds for interior blow-up solutions with \( u \geq 0 \) and either \( \Omega = \mathbb{R}^n \) or \( \Omega \) bounded. In contrast with the case \( p = p_S \), all the known type II blow-up solutions do satisfy \((\text{CNB})\) in the range \( p > p_S \). Taking the above results into account, we expect that \((\text{CNB})\) holds whenever \( p > p_S \).

1.2. Main theorem. Our main result shows that, in the optimal range \( p > p_S \), the critical norm of all finite time blow-up solutions must be unbounded without assuming nonnegativity, monotonicity, symmetry, convexity or the type of blow-up.

**Theorem 1.1.** Let \( n \geq 3 \), \( p > p_S \), \( \Omega \) be any \( C^{2+\alpha} \) domain in \( \mathbb{R}^n \) with \( 0 < \alpha < 1 \) and \( u \) be a classical \( L^{q_c} \)-solution of (1.1) with \( u_0 \in L^{q_c}(\Omega) \). If the maximal existence time \( T > 0 \) is finite, then
\[
\limsup_{t \to T} \| u(\cdot, t) \|_{L^{q_c}(\Omega)} = \infty.
\]

The theorem immediately shows the nonexistence of blow-up solutions with bounded \( L^{q_c} \) norm, and so this resolves the open problem \[90, \text{OP 2.1, Section 55}\] for the supercritical case. We now give comments on the proof, and then we list remarks concerning the statement of this theorem and related results including other scaling invariant nonlinear evolution equations.

As in \[35, 102\] for related equations, our proof of Theorem 1.1 consists of two parts: (i) the blow-up (rescaling and compactness) procedure and (ii) the analysis of the blow-up limit. However, several additional difficulties appear from the differences of the nonlinear structure, in particular, the lack of the coercivity of the energy and the absence of the derivative in the nonlinear term. Compared with the earlier work \[76\], there are also some novelties in the proof. In (i), a concentration theorem of the \( L^{q_c} \) norm near a blow-up point plays a crucial role for the nondegeneracy of the blow-up limit in \[76\]. Unfortunately, we could not rely on their concentration theorem due to the absence of the type I assumption. In order to circumvent this difficulty, we will prove a new \( \epsilon \)-regularity theorem (Theorem 4.1), which guarantees the energy concentration near a blow-up point. Theorem 4.1 is motivated by similar \( \epsilon \)-regularity theorems in \[15\]. In comparison, we do not need to assume that the solution is globally defined in time or has a certain lower bound of the energy. In (ii), unlike the case of \[76\], the smoothness of our blow-up limit is no longer clear even before the final time. To overcome this issue, we invoke a monotonicity estimate similar to \[43\], which plays a key role to identify the blow-up limit. We note that the uses of the \( \epsilon \)-regularity theorem and the monotonicity estimate are partially inspired by the related works \[27, 102\] for the harmonic map heat flow, but several modifications are needed for adapting to our problem as explained above.
Corollary 2] in the case $p > p$. Proposition 23.1].

We note that the maximal existence time can be defined for classical $L^q$-solutions, see [90, Proposition 16.1 (i), (ii)] for the definition.

Remark 1.3 (Uniqueness). By [106, 107] and [11, Theorem 1], a unique classical $L^q_c(\Omega)$-solution of (1.1) exists for each $u_0 \in L^q_c(\Omega)$ with $q_c > 1$. Moreover, if we further assume $q_c > p$, the unconditional uniqueness [11, Theorem 4] holds for mild solutions, that is, solutions of the corresponding integral equation

$$u(\cdot, t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}[u(\cdot, s)]^{p-1}u(\cdot, s)ds \quad \text{for } t \in (0, T)$$

in $C([0, T); L^q_c(\Omega))$. Note that $q_c > p$ is equivalent to $p > n/(n-2)$ and is satisfied for $p > p_S$. Hence, under the assumption of Theorem 11, the condition $u_0 \in L^q_c(\Omega)$ implies the existence of a unique mild solution in $C([0, T); L^q_c(\Omega))$. This solution is also a classical $L^q$-solution, since $q_c > 1$.

Remark 1.4 (Other Lebesgue spaces). We recall the case $u_0 \in L^q(\Omega)$ with $q \neq q_c$. For $1 \leq q < q_c$, there are results of the nonexistence and nonuniqueness of solutions, see [9, 13, 19, 107] and [54, Section 6]. By [95, Theorem 2.4], there exist solutions such that the maximal existence time $T$ is finite and

$$\sup_{0 < t < T} \|u(\cdot, t)\|_{L^q(\Omega)} < \infty.$$ 

For $q_c < q \leq \infty$ with $q \geq 1$, it is known that (1.1) has a unique classical $L^q$-solution $u$ (see [90, Theorem 15.2, Proposition 54.40] for instance). By [11, Corollary 13], it is also known that if $T < \infty$, then $\lim_{t \to T} \|u(\cdot, t)\|_{L^q(\Omega)} = \infty$. More precisely, the lower estimate of the blow-up rate

$$\|u(\cdot, t)\|_{L^q(\Omega)} \geq C(T - t)^{-\frac{n}{2}(\frac{p}{2} - 1)}$$

holds for some constant $C > 0$, see [105, Section 6] and [90, Remark 16.2 (iii), Proposition 23.1].

Remark 1.5 (Application to backward self-similar solutions). Let $\Omega = \mathbb{R}^n$. We call the solution $u$ of (1.1) backward self-similar if it is of the form $u(x, t) = (T - t)^{-1/(p-1)}U(x/\sqrt{T - t})$ for some $T > 0$ and some profile function $U \in C^2(\mathbb{R}^n)$. Theorem 11 immediately shows the following corollary on the Liouville type theorem for backward self-similar solutions $u$ of (1.1) with $p > p_S$. It recovers [76, Corollary 2] in the case $p > p_S$.

Corollary 1.6. Let $n \geq 3$, $p > p_S$ and $u$ be a backward self-similar solution of (1.1). If the profile function $U$ of $u$ belongs to $L^q_c(\mathbb{R}^n)$, then $u \equiv 0$.

Remark 1.7 (Brief review of type II blow-up). Type II blow-up solutions were first found by Herrero and Velázquez [50, 51] in the Joseph–Lundgren supercritical range $p > p_{JL} := (n - 2\sqrt{n-1})/(n - 4 - 2\sqrt{n-1}) > p_S$ with $n \geq 11$. For a refined construction, see [72, 77]. See also [65, 73, 74] for the Lepin supercritical
range $p > p_L := (n - 4)/(n - 10)$ ($> p_{JL}$). The critical cases $p = p_{JL}$ and $p = p_L$
were handled in [91, 92]. In [78], the case where $p = p_S$, $n = 3$ and a suitably
shrinking $\Omega = \Omega (t)$ was handled. Remark that the above type II blow-up solutions
are radially symmetric. In the range $p > p_S$, $n = 3$ and $\Omega$ is a ball, or $\Omega = \mathbb{R}^n$
with assumptions on intersection properties, see also [64, 79].

For the existence of non-radial type II blow-up solutions, see [10, 19] for some $p > p_{JL}$, [26] for
$p = p_2 := (n + 1)/(n - 3)$ and $n \geq 7$, and [21] for $p = p_{n-3} := 3$ and $5 \leq n \leq 7$.
We note that $p_2$ and $p_{n-3}$ are the so-called second critical exponent (after [22]) and
$(n - 3)$-th critical exponent (after [21]). They satisfy $p S < p_2 < p_{JL}$ for $n \geq 4$ and
$p_2 \leq p_{n-3}$ for $n \geq 5$, where $p_{JL} := \infty$ for $n \leq 10$. One of the reasons why such
exponents appear is explained in [19 Subsection 1.4].

In the Sobolev critical case $p = p_S$, type II blow-up solutions were formally
found by Filippas, Herrero and Velázquez [37] for $3 \leq n \leq 6$ and were rigorously
constructed in [95] for $n = 4$. The recent development of the inner-outer gluing
method refined the construction, see [26] for $n = 3$, [27, 59] for $n = 4$, [23, 37] for
$n = 5$ and [18] for $n = 6$. On the other hand, it was proved [17] for $n \geq 7$ that
there is no type II blow-up solution if $u_0$ is close to the Aubin-Talenti function.
Recently, it was also shown [104] that all interior blow-up solutions are of type I
provided that $n \geq 7$, $u \geq 0$ and either $\Omega = \mathbb{R}^n$ or $\Omega$ is bounded.

**Remark 1.8** (Ancient solutions). We say that $u$ is an ancient solution if it satisfies
$u_t - \Delta u = |u|^{p-1}u$ for $t \in (-\infty, T)$ with some $T < \infty$. Classification results for such
solutions were obtained in [69] for $p < p_S$ and [83] for $p_S < p < p_{JL}$ and $p > p_L$,
see also the references given there. In our context, as far as the authors know,
the following question is open: Does there exist a nontrivial solution satisfying
$\sup_{-\infty < t < T} \|u(\cdot, t)\|_{L^p(\Omega)} < \infty$ for $p > p_S$?

**Remark 1.9** (Infinite time blow-up). Infinite time blow-up (or grow-up) solutions,
that is, global-in-time solutions satisfying $\lim_{t \to \infty} \|u(\cdot, t)\|_{L^p(\Omega)} = \infty$, were
constructed for $p \geq p_S$, see [40] for $p = p_S$, [83] for $p_S < p < p_{JL}$ and [55] for
$p \geq p_{JL}$. For $p = p_S$, possible asymptotic behavior was conjectured by Filia and
King [36]. Recently, the conjecture was confirmed by [21] for $n = 3$, [105] for $n = 4$ and
[60] for $n = 5$. Although this paper focuses on finite time blow-up solutions,
It may be interesting to study the behavior of the critical norm for infinite time
blow-up solutions.

**Remark 1.10** (Critical norm blow-up for the Navier-Stokes equations). Theorem
1.1 corresponds to the pioneering work of Escauriaza, Seregin and Šverák [35]
for the three-dimensional Navier-Stokes equations. They showed the blow-up of the
critical norm in the sense that if the maximal existence time $T$ is finite, then

$$\lim \sup_{t \to T} \|u(\cdot, t)\|_{L^3} = \infty.$$  

The limit superior condition was later improved to the limit condition in [93]. In
the case of the domains with boundary, the condition (1.2) was verified for the flat
boundary [93] and for general cases [71]. These results were also refined to the limit
condition for the flat case [3, 62] and for general cases [2]. On the other hand, the
$L^3$ norm in [1, 22] was further refined to the Lorentz norm [52] and the Besov norm
[1, 41]. Actually, our norm in Theorem 1.1 can be replaced by the Lorentz norm
$L^{q,r}$ with $r < \infty$, but we do not pursue this issue here. We also refer to [29] [30] for further developments in this subject. By analogy, it is expected that there is a general quantitative blow-up criterion for the semilinear heat equation (1.1) with $p > p_S$. This direction seems interesting and also challenging. Moreover, it remains an open problem whether the limit superior condition in Theorem 1.1 can be replaced with the limit condition. We note that the result of [76] for $p > p_S$ under the type I blow-up assumption is not a consequence of Theorem 1.1.

**Remark 1.11** (Critical norm blow-up for other equations). Wang [102] studied the critical norm of the harmonic map heat flow between compact Riemannian manifolds without boundaries in the energy supercritical dimension $n \geq 3$. It was shown that if the maximal existence time $T$ is finite, then

$$\limsup_{t \to T} \|\nabla u(\cdot, t)\|_{L^n} = \infty.$$ 

One of the key ideas in the proof is the monotonicity formula of Struwe [97]. For nonlinear dispersive equations with power nonlinearities, there are also many works on the blow-up of the critical norm of the form

$$\limsup_{t \to T} \|u(\cdot, t)\|_{H^s} = \infty.$$ 

Here $s_c$ is the scaling critical exponent for each of the equations. Kenig and Merle [54] showed that blow-up solutions of the cubic defocusing Schrödinger equation in $\mathbb{R}^3$ must satisfy the above condition with $s_c = 1/2$. Their method is based on the concentration compactness procedure and the rigidity theorem. A similar method is applicable to the defocusing supercritical nonlinearity, see [56]. In the radial case, Merle and Raphaël [67] gave an explicit lower bound of the critical norm in some energy subcritical range. See also a recent result [127] for the lower bound of the critical norm in the radial supercritical case. Similar results were also obtained for the nonlinear wave equation starting from [55]. For the focusing case, see [33, 34].

**Remark 1.12** (Related results for supercritical elliptic equations). In the proof of Theorem 1.1, the obtained blow-up limit $\overline{u}$ is a weak solution of the semilinear heat equation and satisfies a monotonicity estimate. In addition, the singular set of $\overline{u}(\cdot, t)$ consists of finitely many points for each $t$. A similar situation can be found for the so-called stationary solutions of the semilinear elliptic equation $-\Delta u = |u|^{p-1}u$ for $p > p_S$, see [31, 79, 80, 103]. In this context, a weak solution $u$ of the elliptic equation is called a stationary solution if $u$ is a critical point of the corresponding energy functional with respect to domain variations.

1.3. **Organization of this paper.** This paper is organized as follows. In Section 2 we derive a key gradient estimate. In Section 3 we define a localized weighted energy and prove its quasi-monotonicity. In Section 4 we prove an $\varepsilon$-regularity theorem by analyzing the energy. In Section 5 we construct and examine a blow-up limit with the aid of the $\varepsilon$-regularity, and then we prove Theorem 1.1. In
Appendix A, we give regularity estimates used in Section 5. In Appendix B, we recall an Aubin-Lions type compactness result also used in Section 5.

1.4. Notation. For \( x \in \mathbb{R}^n \), we often write \( x = (x', x_n) \) with \( x' \in \mathbb{R}^{n-1} \) and \( x_n \in \mathbb{R} \). Set \( \mathbb{R}_+^n := \{ x \in \mathbb{R}^n ; x' \in \mathbb{R}^{n-1}, x_n > 0 \} \). We denote by \( \chi_A \) and \(|A|\) the characteristic function and the Lebesgue measure of a measurable set \( A \), respectively. For \( r > 0 \) and \((x, t) \in \mathbb{R}^{n+1} \), we write

\[
B_r(x) := \{ y \in \mathbb{R}^n ; |x - y| < r \}, \quad B_r := B_r(0),
\]

\[
\Omega_r(x) := \Omega \cap B_r(x), \quad \Omega_r := \Omega_r(0),
\]

\[
P_r(x, t) := B_r(x) \times (t - r^2, t), \quad P_r = P_r(0, 0),
\]

\[
Q_r(x, t) := \Omega_r(x) \times (t - r^2, t), \quad Q_r = Q_r(0, 0).
\]

For \( \rho > 0 \) and \((x, t) \in \mathbb{R}^{n+1} \), we write

\[
B^+_\rho(x) := \mathbb{R}_+^n \cap B_\rho(x), \quad B^+_\rho := \mathbb{R}_+^n \cap B_\rho(0),
\]

\[
Q^+_\rho(x, t) := B^+_\rho(x) \times (t - \rho^2, t), \quad Q^+_\rho := Q^+_\rho(0, 0).
\]

We denote by \( G_\Omega = G_\Omega(x, y, t) \) the Dirichlet heat kernel in \( \Omega \). Set

\[
K_{(x, t)}(\tilde{x}, \tilde{t}) := (\tilde{t} - t)^{-\frac{n+2}{2}} e^{-\frac{|x - \tilde{x}|^2}{4(\tilde{t} - t)}}
\]

for \( x, \tilde{x} \in \mathbb{R}^n \) and \( t < \tilde{t} \). The critical exponents are defined by

\[
p_{S} := \frac{n+2}{n-2}, \quad q_c := \frac{n(p-1)}{2}, \quad q_* := \frac{n(p-1)}{p+1}.
\]

Note that each of the conditions \( q_c > p + 1 \) and \( q_* > 2 \) is equivalent to \( p > p_S \). In what follows, we always assume \( n \geq 3 \) and \( p > p_S \).

2. Gradient estimate

Let \( R > 0 \) and \( \Omega \) be any \( C^{2+\alpha} \) domain in \( \mathbb{R}^n \) with \( 0 \in \overline{\Omega} \). As will be seen in Section 5, the proof of Theorem 2.1 is based on the study of the localized problem

\[
\begin{cases}
    u_t = \Delta u + |u|^{p-1}u \quad &\text{in } \Omega_R \times (-1, 0),
    \\
    u = 0 \quad &\text{on } (\partial \Omega \cap B_R) \times (-1, 0),
    \\
    u \text{ is } C^{2,1} \quad &\text{on } \Omega_R \times (-1, 0),
\end{cases}
\]

under the assumption that there exists \( M > 0 \) satisfying

\[
\sup_{-1 < t < 0} \|u(\cdot, t)\|_{L^{p,\infty}(\Omega_R)} \leq M.
\]

Here the boundary condition in (2.1) is ignored if \( \partial \Omega \cap B_R = \emptyset \).

In this section, we show a gradient estimate in the Lorentz space \( L^{q_*,\infty} \) with \( q_* := n(p-1)/(p+1) \), which is our key tool to bound a weighted energy defined in Section 5. The method to estimate a term from \( |u|^{p-1}u \) is based on the idea due to Meyer [70, Theorem 18.1] (see also [99, Proposition 1.5]).

**Proposition 2.1.** If \( u \) satisfies (2.1) and (2.2), then there exists a constant \( C > 0 \) depending on \( R \) such that

\[
\sup_{-3/4 < t < 0} \|\nabla u(\cdot, t)\|_{L^{q_*,\infty}(\Omega_{3R/4})} \leq C(M + M^p).
\]
Proof. In the same spirit of [29] Proposition A.1, we derive a localized integral equation, and then we estimate each of the terms. Let $\phi \in C^2_0(\mathbb{R}^n)$ satisfy $0 < \phi \leq 1$ in $\mathbb{R}^n$, $\phi = 0$ in $\mathbb{R}^n \setminus B_{15R/16}$ and $\phi = 1$ in $B_{7R/8}$. Set $v(x, t) := u(x, t)\phi(x)$. Then $v$ belongs to $C^{2,1}(\overline{\Omega} \times (-1, 0))$ and satisfies

$$
\begin{cases}
    v_t - \Delta v = \phi|u|^{p-1}u - 2\nabla \phi \cdot \nabla u - u\Delta \phi & \text{in } \Omega \times (-1, 0), \\
    v = 0 & \text{on } \partial \Omega \times (-1, 0).
\end{cases}
$$

Thus,

$$u(x, t) = \int_{\Omega} G_\Omega(x, y, t + 7/8)\phi(y)u(y, -7/8)dy
+ \int_{-7/8}^{t} \int_{\Omega} G_\Omega(x, y, t - s)\phi(y)|u(y, s)|^{p-1}u(y, s)dyds
- \int_{-7/8}^{t} \int_{\Omega} G_\Omega(x, y, t - s)(2\nabla \phi \cdot \nabla u + u\Delta \phi)dyds
$$

for $x \in \Omega_{3R/4}$ and $-7/8 < t < 0$, where $G_\Omega = G_\Omega(x, y, t)$ is the Dirichlet heat kernel in $\Omega$. Since $G_\Omega(x, y, t) = 0$ for $y \in \partial \Omega$ and $u\phi = 0$ on $\partial \Omega$, integrating by parts in the third term in the right-hand side yields

$$u(x, t) = \int_{\Omega} G_\Omega(x, y, t + 7/8)\phi(y)u(y, -7/8)dy
+ \int_{-7/8}^{t} \int_{\Omega} G_\Omega(x, y, t - s)\phi(y)|u(y, s)|^{p-1}u(y, s)dyds
+ 2\int_{-7/8}^{t} \int_{\Omega} \nabla_y G_\Omega(x, y, t - s) \cdot \nabla \phi(y)u(y, s)dyds$$

for $x \in \Omega_{3R/4}$ and $-7/8 < t < 0$.

Since $\Omega$ is $C^{2+\alpha}$, the following estimate holds (see [58] Theorem IV.16.3 for instance): There exists a constant $C > 0$ such that

$$|\nabla_j G_\Omega(x, y, t)| \leq CK_j(x - y, t) \quad (j = 0, 1, 2)$$

for $x, y \in \Omega$ and $0 < t < 1$, where

$$K_j(x, t) := t^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}} \quad (j = 0, 1, 2).$$

Remark that the constant in (2.5) and (2.6) depends only on $n$, $\Omega$ and the length of the time interval $(0, 1)$. We prepare an estimate of $\partial_x, \partial_y, G_\Omega(x, y, t)$. By the semigroup property and (2.5), we have

$$|\partial_x, \partial_y G_\Omega(x, y, t)| = \left| \int_{\Omega} \partial_x G_\Omega(x, z, t/2)\partial_y G_\Omega(z, y, t/2)dz \right|
\leq C \int_{\Omega} K_1(x - z, t/2)K_1(z - y, t/2)dz
\leq Ct^{-1} \int_{\mathbb{R}^n} G(x - z, Ct/8)G(z - y, Ct/8)dz
= Ct^{-1}G(x - y, Ct/4) \leq CK_2(x - y, t),$$

where $G(x, t) := (4\pi t)^{-n/2}e^{-|x|^2/4t}$ and we changed the constant $C$ in (2.6). Then by differentiating the integral equation (2.4) and using $K_1(x, t) \leq K_2(x, t)$ for $x \in \Omega$.
and $0 < t < 1$, we see that
\[
|\nabla u(x,t)| \leq C \int_{\mathbb{R}^n} K_1(x - y, t + 7/8)|u(y, -7/8)|\chi_{\Omega_R}(y)dy \\
+ C \int_{-7/8}^t \int_{\Omega_R} K_1(x - y, t - s)|u(y, s)|^p dy ds \\
+ C \int_{-7/8}^t \int_{\mathbb{R}^n} K_2(x - y, t - s)|u|\chi_{\Omega_{15R/16} \setminus \Omega_{7R/8}}(y)dy ds
\]
(2.7)

for $x \in \Omega_{3R/4}$ and $-7/8 < t < 0$. Remark that each $U_i$ is defined for all $x \in \mathbb{R}^n$ and $-7/8 < t < 0$.

For $U_1$, from $q_* < q_c$ and the same argument to prove the $L^{q_*}-L^{q_c}$ estimate for the heat semigroup (see [12], Section 1.1.3] for instance, it follows that
\[
\| U_1(\cdot, t) \|_{L^{q_*} \rightarrow L^{q_c}(\Omega_{3R/4})} \leq C \| U_1(\cdot, t) \|_{L^{q_*}(\mathbb{R}^n)} \\
\leq C(t + 7/8)^{-1/2} \| u(\cdot, -7/8)\chi_{\Omega_R} \|_{L^{q_c}(\mathbb{R}^n)} \\
\leq C \| u(\cdot, -7/8) \|_{L^{q_c}(\Omega_R)} \leq CM
\]
for $-3/4 < t < 0$. Then,
\[
\sup_{-3/4 < t < 0} \| U_1(\cdot, t) \|_{L^{q_*} \rightarrow L^{q_c}(\Omega_{3R/4})} \leq CM.
\]

We consider $U_3$. Since $|x - y| \geq R/8$ for $x \in \Omega_{3R/4}$ and $y \in \Omega_{15R/16} \setminus \Omega_{7R/8}$, we have
\[
K_2(x - y, t - s)\chi_{\Omega_{15R/16} \setminus \Omega_{7R/8}}(y) \leq C \chi_{\Omega_R}(y) \sup_{s < t} (t - s)^{-\frac{n}{2} - 1} e^{-\frac{x^2}{16s(t - s)}} \\
\leq C \chi_{\Omega_R}(y)
\]
for $x \in \Omega_{3R/4}$, $y \in \mathbb{R}^n$ and $-7/8 < s < t < 0$. Therefore,
\[
U_3(x,t) \leq C \int_{-7/8}^t \int_{\Omega_R} |u| dy ds \leq CM
\]
for $x \in \Omega_{3R/4}$ and $-7/8 < t < 0$. Thus,
\[
\sup_{-3/4 < t < 0} \| U_3(\cdot, t) \|_{L^{q_*} \rightarrow L^{q_c}(\Omega_{3R/4})} \leq CM.
\]

We estimate $U_2$. This part is a modification of [70], Theorem 18.1]. Let $-7/8 < t < 0$. By the change of variables, we have
\[
U_2(x,t) \leq \tilde{U}(x,t); \quad \tilde{U} = \tilde{U}(x,t) := \int_0^\infty S(x,s,t)ds, \\
S = S(x,s,t) := \chi_{(0,t+7/8)}(s) \int_{\Omega_R} K_1(x - y, s)|u(y, t - s)|^p dy.
\]

For $\lambda > 0$ and $\tau > 0$, define $E := \{ x \in \Omega_R; \tilde{U}(x) > \lambda \}$ and
\[
\tilde{U}(x) = \left( \int_0^\tau + \int_{\tau}^\infty \right) S(x,s)ds =: \tilde{U}_1(x) + \tilde{U}_2(x).
\]
We estimate the Lebesgue measure $|E|$ of $E$. By the same argument to prove the $L^{q_\ast/p} L^\infty$ estimate for the heat semigroup, we have
\[
\|S(\cdot, s)\|_{L^\infty(\Omega_R)} \leq C s^{-\frac{np}{2}} \chi_{(0, t+7/8)}(s) \|u(\cdot, t-s)\|_{L^{q_\ast/p}(\Omega_R)}^p \leq C M^p s^{-\frac{np}{2} - \frac{1}{2}}
\]
for any $s > 0$, where $C > 0$ is independent of $t$. Then,
\[
\tilde{U}_2(x) \leq \int_\tau^{\infty} S(x, s) ds \leq C M^p \int_\tau^{\infty} s^{-\frac{np}{2} - \frac{1}{2}} ds = C' M^p \tau^{-\frac{np}{2} - \frac{1}{2}},
\]
where $C'$ is also independent of $t$. For $\lambda > 0$, we choose $\tau$ such that
\[
C' M^p \tau^{-\frac{np}{2} - \frac{1}{2}} = \frac{\lambda}{2}.
\]
Then $\tilde{U}_2 \leq \lambda/2$. By setting $E_1 := \{x \in \Omega_R; \tilde{U}_1(x) > \lambda/2\}$ and using $\tilde{U}_2 \leq \lambda/2$ and $\tilde{U} = \tilde{U}_1 + \tilde{U}_2$, we see that $E \subset E_1$.

From the same argument to prove the $L^{q_\ast/p} L^{q_\ast/p}$ estimate, it follows that
\[
\|S(\cdot, s)\|_{L^{p_\ast} \cap L^{q_\ast}(\Omega_R)} \leq \|S(\cdot, s)\|_{L^{p_\ast}(\Omega_R)} \leq C s^{-\frac{np}{2}} \chi_{(0, t+7/8)}(s) \|u(\cdot, t-s)\|_{L^{q_\ast}(\Omega_R)}^p \leq C M^p s^{-\frac{1}{2}}
\]
for any $s > 0$. Thus,
\[
\|\tilde{U}_1\|_{L^{p_\ast} \cap L^{q_\ast}(\Omega_R)} \leq \int_0^\tau \|S(\cdot, s)\|_{L^{p_\ast} \cap L^{q_\ast}(\Omega_R)} ds \leq C M^p \tau^{\frac{1}{2}}.
\]
This together with the Hölder inequality for the Lorentz spaces (see [57, Proposition 2.1] for instance) shows that
\[
\int_{E_1} \tilde{U}_1(x) dx \leq C |\lambda E_1|_{L^{p_\ast} \cap L^{q_\ast}(\Omega_R)} \|\tilde{U}_1\|_{L^{p_\ast} \cap L^{q_\ast}(\Omega_R)} \leq C M^p |E_1|^{1 - \frac{1}{p} - \frac{1}{2}} \tau^{\frac{1}{2}}.
\]
On the other hand, the definition of $E_1$ gives $\int_{E_1} \tilde{U}_1(x) dx \geq (\lambda/2) |E_1|$. By $E \subset E_1$ and (2.10), we obtain
\[
|E| \leq |E_1| \leq C \lambda^{-\frac{np}{2} - \frac{1}{2}} M^{np} \tau^{\frac{1}{2}} = C M^p \lambda^{-q_\ast},
\]
and so $\lambda\{x \in \Omega_R; \tilde{U}(x) > \lambda\}^{1/q_\ast} \leq C M^p$ for any $\lambda > 0$. This implies that $\|\tilde{U}(\cdot; t)\|_{L^{q_\ast}(\Omega_R)} \leq C M^p$ for $-7/8 < t < 0$, where $C > 0$ is independent of $t$. Hence by the definition of $\tilde{U}$ and $M$, we see that
\[
\sup_{-3/4 < t < 0} \|\tilde{U}_2(\cdot, t)\|_{L^{q_\ast}(\Omega_R)} \leq \sup_{-3/4 < t < 0} \|\tilde{U}(\cdot; t)\|_{L^{q_\ast}(\Omega_R)} \leq C M^p.
\]
Combining this inequality, (2.7), (2.8) and (2.9), we obtain the desired inequality. $\square$

3. LOCALIZED WEIGHTED ENERGY

Let $u$ be a solution of (2.1) satisfying the bound (2.2). In this section, we define a localized weighted energy of $u$ analogous to Giga, Matsui and Sasayama [45, 46] and prove its quasi-monotonicity without assuming the convexity of $\Omega$. Our computations to prove quasi-monotonicity are in the same spirit of Chou and Du [14], but the details are different. Among the results in this section, we will refer only Lemma 3.1 and Proposition 3.2 in the subsequent sections.
Let \( n \geq 3, p > p_S \) and \( \Omega \) be any \( C^{2+\alpha} \) domain in \( \mathbb{R}^n \) with \( 0 \in \overline{\Omega} \). We fix \( R > 0 \) as one of the following:

\[
\begin{cases}
\text{(3.1) } & \text{In the case } 0 \in \Omega, \text{ we fix } 0 < R < 1/2 \text{ such that } \overline{B_R} \subset \Omega.

\text{(3.2) } & \text{In the case } \partial \Omega \neq \emptyset \text{ and } 0 \in \partial \Omega, \text{ we fix } 0 < R < 1/2 \text{ such that there exists } f \in C_0^{2+\alpha}(\mathbb{R}^{n-1}) \text{ satisfying } f(0) = 0, \nabla'f(0) = 0,
\end{cases}
\]

\[\|\nabla'f\|_{L^\infty(\mathbb{R}^{n-1})} \leq 1/2 \text{ and } \Omega_R = \{ x \in B_R; x_n > f(x') \}\]

by relabeling and reorienting the coordinates axes if necessary.

Here \( \nabla'f \) is the gradient on \( \mathbb{R}^{n-1} \). Remark that the existence of \( f \) in (3.2) is guaranteed by the smoothness of \( \Omega \).

### 3.1. Definition and change of variables.

We define a localized weighted energy \( E \) and show its boundedness by using Proposition 2.1. To obtain quasi-monotonicity, we locally straighten the boundary. After that, we introduce backward similarity variables and derive the corresponding representation of \( E \).

Let \( \varphi \in C^\infty([0, \infty)) \) satisfy \( \varphi(z) = 1 \) for \( 0 \leq z \leq 1/2, \) \( 0 < \varphi(z) < 1 \) for \( 1/2 < z < 1, \) \( \varphi(z) = 0 \) for \( z \geq 1 \) and \( \varphi'(z) \leq 0 \) for \( z \geq 0 \). For \( x, \tilde{x} \in \mathbb{R}^n, t > 0 \) and \( r > 0 \), we set \( \phi_r = \phi_{x,r}(x) := \varphi(|x - \tilde{x}|/r) \) and

\[K = K_{(\tilde{x}, \tilde{t})}(x, t) := (\tilde{t} - t)^{-\frac{n}{p-1}} e^{-\frac{|x - \tilde{x}|^2}{4(t-\tilde{t})}}.\]

For \( \tilde{x} \in \overline{\Omega_{R/4}} \) and \( -1 < t < \tilde{t} \leq 0 \), define a localized weighted energy by

\[
E(t) = E_{(\tilde{x}, \tilde{t})}(t; \phi_{\tilde{x}, R/4}) := (\tilde{t} - t)^{\frac{n+1}{p-1}} \int_{\Omega_R} \left( \frac{\nabla u(x, t)}{2} - \frac{|u(x, t)|^{p+1}}{p+1} + \frac{|u(x, t)|^2}{2(p-1)(\tilde{t} - t)} \right)
\]

\[\times K_{(\tilde{x}, \tilde{t})}(x, t)\phi_{\tilde{x}, R/4}^2 \, dx.\]

Remark that \( u \) is defined on \( (\overline{\Omega_R} \cap B_R) \times (-1, 0) \), but we mainly consider the time interval \((-1/2, 0)\) to apply Proposition 2.1. Note that

\[\text{supp } \phi_{\tilde{x}, R/4} \subset B_{R/2} \quad \text{for } \tilde{x} \in \overline{\Omega_{R/4}}.\]

The following lemma guarantees the boundedness of \( E \).

**Lemma 3.1.** There exists \( C > 0 \) such that

\[|E_{(\tilde{x}, \tilde{t})}(t; \phi_{\tilde{x}, R/4})| \leq C(M + M^p)^2\]

for any \( \tilde{x} \in \overline{\Omega_{R/4}} \) and \( -1/2 < t < \tilde{t} \leq 0 \).

**Proof.** From the Hölder inequality for the Lorentz spaces (see [57, Proposition 2.1]), (3.4), Proposition 2.1 and a direct computation, it follows that

\[
\int_{\Omega_R} |\nabla u|^2 K_{(\tilde{x}, \tilde{t})} \phi_{\tilde{x}, R/4}^2 \, dx
\]

\[\leq C(\tilde{t} - t)^{-\frac{n}{p-1}} \|\nabla u(\cdot, t)\|_{L^{p, \infty}(\Omega_{R/2})}^2 \left\| e^{-\frac{|x - \tilde{x}|^2}{4(t-\tilde{t})}} \right\|_{L^{\frac{2(p-1)}{p-1}}(\mathbb{R}^n)}^2
\]

\[\leq C(M + M^p)^2 (\tilde{t} - t)^{-\frac{n}{p-1}}\]
for $-1/2 < t < \tilde{t} \leq 0$. The Hölder inequality and (2.2) show that
\[
\int_{\Omega_R} |u|^{p+1} K(\hat{\xi}, \tilde{t}) \partial_{x,R}^2 |\xi,t| dx \leq (\hat{t} - t)^{-\frac{p+1}{2}} \|u\|^{p+1}_{L^\infty(\Omega_{R/2})} \left( \int_{\Omega_R} e^{-\frac{|x-t|^2}{C(\alpha-1)}} dx \right)^{1-\frac{p+1}{2}}
\leq CM^{p+1}(\hat{t} - t)^{-\frac{p+1}{2}},
\]
\[
\int_{\Omega_R} |u|^2 K(\hat{\xi}, \tilde{t}) \partial_{x,R}^2 |\xi,t| dx \leq CM^2(\hat{t} - t)^{-\frac{2}{p+1}},
\]
for $-1/2 < t < \tilde{t} \leq 0$. The lemma follows from the above estimates.

We locally straighten the boundary. In the case (3.2), we define $C^2+\alpha$ maps $\Phi = (\Phi_1, \ldots, \Phi_n)$ and $\Psi = (\Psi_1, \ldots, \Psi_n)$ by
\[
\begin{align*}
\xi_i &= x_i =: \Phi_i(x), \\
\xi_n &= x_n - f(x') =: \Phi_n(x),
\end{align*}
\]
\[
\begin{align*}
x_i &= \xi_i =: \Psi_i(\xi), \\
x_n &= \xi_n + f(\xi') =: \Psi_n(\xi),
\end{align*}
\]
for $i = 1, \ldots, n-1$. To handle the case (3.1) in a unified way, we also set
\[
\begin{align*}
\xi_i &= x_i =: \Phi_i(x), \\
\xi_n &= x_n =: \Phi_n(x),
\end{align*}
\]
\[
\begin{align*}
x_i &= \xi_i =: \Psi_i(\xi), \\
x_n &= \xi_n =: \Psi_n(\xi),
\end{align*}
\]
for (3.1). We note that the maps $\Phi$ and $\Psi$ for (3.1) are identity maps and they are obtained by setting $f \equiv 0$ in the definitions of $\Phi$ and $\Psi$ for (3.2).

We write $\Phi(x) = \xi$ and $\Psi(\xi) = x$. Set
\[
\hat{u}(\xi, t) := u(\Psi(\xi), t).
\]
Then, direct computations show that
\[
\nabla_x u(x, t) = (\partial_{\xi_1} \hat{u} - (\partial_{\xi_n} \hat{u}) \partial_{\xi_1} f, \ldots, \partial_{\xi_{n-1}} \hat{u} - (\partial_{\xi_n} \hat{u}) \partial_{\xi_{n-1}} f, \partial_{\xi_n} \hat{u})
\]
\[
= (\nabla' \hat{u} - (\partial_{\xi_n} \hat{u}) \nabla' f, \partial_{\xi_n} \hat{u})
\]
and that
\[
\Delta_x u(x, t) = \Delta_\xi \hat{u} - 2 \sum_{i=1}^{n-1} (\partial_{\xi_i} \partial_{\xi_n} \hat{u}) \partial_{\xi_i} f + 2 \partial_{\xi_n} \hat{u} \sum_{i=1}^{n-1} (\partial_{\xi_i} f)^2 - \partial_{\xi_n} \hat{u} \sum_{i=1}^{n-1} \partial_{\xi_i}^2 f
\]
\[
= \Delta_\xi \hat{u} - 2 \nabla'_\xi (\partial_{\xi_n} \hat{u}) \cdot \nabla' f + (\partial_{\xi_n}^2 \hat{u}) |\nabla' f|^2 - (\partial_{\xi_n} \hat{u}) \Delta' f,
\]
where $\nabla'$ and $\Delta'$ are the gradient and Laplacian on $\mathbb{R}^{n-1}$ with respect to the first $(n-1)$ components, respectively. Remark that the terms in the right-hand sides are evaluated at $(\xi, t) = (\Phi(x), t)$. Since $u$ satisfies (2.1), we see that $\hat{u}$ satisfies
\[
\begin{align*}
\hat{u}_t - A\hat{u} &= |\hat{u}|^{p-1} \hat{u} \quad \text{in } \Phi(\Omega_R) \times (-1, 0), \\
\hat{u} &= 0 \quad \text{on } \Phi(\omega \cap B_R) \times (-1, 0).
\end{align*}
\]
Here, by abbreviations of the subscripts $\xi$ and $\xi_n$, we set
\[
\hat{A}\hat{u} := \Delta \hat{u} - 2 \nabla' (\partial_{\xi_n} \hat{u}) \cdot \nabla' f + (\partial_{\xi_n}^2 \hat{u}) |\nabla' f|^2 - (\partial_{\xi_n} \hat{u}) \Delta' f.
\]
For $\hat{x} \in \Omega_{R/4}$, we write $\hat{\xi} := \Phi(\hat{x})$. Note that $\hat{x} = \Psi(\hat{\xi})$. We perform the change of variables $x = \Psi(\xi)$ in (3.3) by using the relation (3.7), the definition of $K$ and
the fact that the Jacobian determinant equals 1 from the definition of Ψ, that is, $dx = dξ$. Then $E$ satisfies

$$E(t) = (\tilde{t} - t)^{\frac{p+1}{p}} \int_{\Phi(\Omega_R)} \left( \frac{\nabla \hat{u}^2}{2} - \frac{|\hat{u}|^{p+1}}{p+1} + \frac{\hat{u}^2}{2(p-1)}(t-t) \right)$$

(3.9)

$$\times e^{-\frac{e^r}{4}|\Psi(\xi,\eta)|^2} \phi_{\Psi(\xi, R/4)}(\Psi(\xi))dξ,$$

where $\nabla \hat{u}(ξ, t) := (\nabla' \hat{u} - (\partial_n \hat{u})\nabla' f, \partial_n \hat{u})$.

We introduce the backward similarity variables

$$η := \frac{ξ - \hat{ξ}}{(\tilde{t} - t)^{1/2}}, \quad \tau := -\log(\tilde{t} - t).$$

Then the rescaled functions are given by

$$w(η, τ) := e^{-\frac{1}{2\tau^r} \hat{u}(ξ + \frac{1}{2\tau} η, \tilde{t} - e^{-\tau})},$$

(3.10)

$$g(η', τ) := e^{\frac{τ}{2}} f(ξ' + e^{-\frac{1}{2\tau} η'}).$$

(3.11)

Note that

$$ξ = \tilde{ξ} + \frac{1}{2\tau} η, \quad \tilde{t} - t = e^{-r},$$

$$\hat{u}(ξ, t) = (\tilde{t} - t)^{-\frac{1}{2\tau}} w((\tilde{t} - t)^{-\frac{1}{2}}(ξ - \hat{ξ}), -\log(\tilde{t} - t)).$$

Since $\hat{u}$ satisfies (3.8), we see that $w$ solves

$$\begin{cases}
  w_r + \frac{1}{2} η \cdot \nabla w + \frac{1}{p-1} w - |w|^{p-1} w = 0, \\
  η ∈ Ω(τ), \quad τ ∈ (-\log(\tilde{t} + 1/2), \infty), \\
  w = 0, \quad η ∈ e^{r/2}(\Psi(\partial Ω \cap B_R) - \hat{ξ}),
\end{cases}$$

(3.12)

where the time interval $(-\log(\tilde{t} + 1), \infty)$ has been shortened to $(-\log(\tilde{t} + 1/2), \infty)$ for using Proposition 2.1 safely and

$$Aw := \Delta w - 2\nabla(\partial_n w) \cdot \nabla g + (\partial_n^2 w)|\nabla g|^2 - (\partial_n w)\Delta' g$$

by abbreviations of the subscripts $η$ and $η_n$. In addition,

$$Ω(τ) := \{ η ∈ R^n; \tilde{ξ} + e^{-r/2} η ∈ \Phi(Ω_R) \} = e^{r/2}(\Phi(Ω_R) - \hat{ξ}).$$

By using the backward similarity variables, (3.9) can be written as

$$E(t) = \int_{Ω(τ)} \left( \frac{||\nabla w||^2}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)} \right)$$

$$\times \exp \left( -\frac{e^r}{4}|Ψ(ξ + e^{-r/2} η) - Ψ(\hat{ξ})|^2 \right)$$

$$\times \phi^2 \left( \frac{4}{R} |Ψ(ξ + e^{-r/2} η) - Ψ(\hat{ξ})| \right) dη,$$

where $\nabla w := (\nabla' w - (\partial_n w)\nabla' g, \partial_n w)$. We observe that

$$|Ψ(ξ + e^{-r/2} η) - Ψ(\hat{ξ})|^2$$

(3.13)

$$= e^{-r}(|η'|^2 + 2(g(η', τ) - g(0, τ)) |η_n| + (g(η', τ) - g(0, τ))^2).$$
Then by setting

\[ E(\tau) := \int_{\Omega(\tau)} \left( \frac{|\nabla w|^2}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)} \right) \rho \psi^2 \, d\eta, \]

\[ \rho = \rho(\eta, \tau) := \exp \left( -\frac{1}{4} (|\eta|^2 + 2(g(\eta', \tau) - g(0, \tau)) \eta_n \right) + (g(\eta', \tau) - g(0, \tau))^2 \right), \]

\[ \psi = \psi(\eta, \tau) := \varphi \left( \frac{4}{R} e^{-\frac{\eta}{2}} (\eta', \eta_n + g(\eta', \tau) - g(0, \tau)) \right), \]

we see that \( E(t) = E(\tau) \) with \( \tau = -\log(\tilde{t} - t) \). By direct computations, we note that

\[
\begin{align*}
\partial_\psi \rho &= -\frac{1}{2} (\eta_i + (\eta_n + g(\eta', \tau) - g(0, \tau)) \partial_i \rho) \\
\partial^2 \rho &= -\frac{1}{2} (1 + (\partial_i \rho)^2 + (\eta_n + g(\eta', \tau) - g(0, \tau)) \partial_i \rho) \\
&\quad + \frac{1}{4} (\eta^2_n + 2(\eta_n + g(\eta', \tau) - g(0, \tau)) \eta_i \partial_i \rho) \\
&\quad + \frac{1}{4} (\eta_n + g(\eta', \tau) - g(0, \tau))^2 (\partial_i \rho)^2 \\
\partial_n \rho &= -\frac{1}{2} (\eta_n + g(\eta', \tau) - g(0, \tau)) \rho, \\
\partial^2 \rho &= -\frac{1}{2} \rho + \frac{1}{4} (\eta_n + g(\eta', \tau) - g(0, \tau))^2 \rho
\end{align*}
\]

and that

\[ \rho_\tau = -\frac{1}{2} \partial_\tau (g(\eta', \tau) - g(0, \tau)) (\eta_n + g(\eta', \tau) - g(0, \tau)) \rho. \]

3.2. **Quasi-monotonicity.** We prove the quasi-monotonicity of \( E \). This property plays a crucial role in the proof of the \( \varepsilon \)-regularity theorem and also in the blow-up analysis.

**Proposition 3.2.** Fix \( R > 0 \) such that either \[3.1\] or \[3.2\] holds. Let \( u \) be a solution of \[2.1\] satisfying \[2.2\]. Then there exists a constant \( C > 0 \) depending only on \( n, p, \Omega \) and \( R \) such that

\[
E_{(\tilde{\varepsilon}, \tilde{\xi})}(t; \tilde{\Phi}, R/4) + \frac{1}{2} \int^t_{t'} (\tilde{\xi} - s)^{p-1} \left| \frac{\tilde{\psi} - \tilde{\xi}}{|\tilde{\psi} - \tilde{\xi}|^{p-2}} \right|^2 \\
\times \int_{\tilde{\Phi}(\Omega_{\tilde{\xi})}} \left| \frac{\tilde{\psi} - \tilde{\xi}}{|\tilde{\psi} - \tilde{\xi}|^{p-1}} \right|^2 \\
\times \left( \frac{\Phi(\tilde{\xi})}{\Phi(\tilde{\xi}) - \tilde{\xi}} \right)^2 \Phi_{\tilde{\xi}, R/4}^2(\tilde{\Phi}(\xi)) d\xi ds \\
\leq E_{(\tilde{\varepsilon}, \tilde{\xi})}(t'; \tilde{\Phi}, R/4) + C(M + M^p)^2 (\tilde{t} - t')^{1/2}
\]

for any \( \tilde{\xi} \in \overline{\Omega_{R/4}} \) and \(-1/2 < t' < t < \tilde{t} \leq 0\).

To prove this, we compute the derivative of \( E \).
Lemma 3.3. The derivative of $E(\tau)$ satisfies

\begin{equation}
\frac{d}{d\tau} E(\tau) = -\int w^2 \rho \psi^2 - B(\tau) + R(\tau).
\end{equation}

Here

\begin{equation}
B(\tau) := \frac{1}{4} \int_{e^{\tau/2}(\Phi(\partial \Omega \cap B_R) - \xi)} (\partial_\nu w)^2 \rho \psi^2 \times (\nu \cdot \eta)(1 - 2\nu_n \partial_\nu g + |\nabla' g|^2 \nu_n^2) dS(\eta),
\end{equation}

$\nu = (\nu', \nu_n)$ is the outward unit normal, $\partial_\nu w := \nabla w \cdot \nu$, $\partial_\nu g := \nabla' g \cdot \nu'$, $dS$ is the surface area element and

\begin{equation}
R(\tau) := \int \frac{w^2}{p-1} \rho \psi \psi_{\tau} - \int \frac{2|w|^{p+1}}{p+1} \rho \psi \psi_{\tau} - 2 \int w_{\tau} \rho \psi \nabla w \cdot \nabla \psi \\
+ \int |\nabla w|^2 \rho \psi \psi_{\tau} + 2 \int w_{\tau} \rho \psi (\partial_\nu \psi) \nabla' w \cdot \nabla' g - 2 \int (\partial_\nu w) \rho \psi \psi_{\tau} \nabla' w \cdot \nabla' g + 2 \int w_{\tau}(\partial_\nu w) \rho \psi \nabla' \psi \cdot \nabla' g \\
- 2 \int w_{\tau}(\partial_\nu w) \rho \psi (\partial_\nu \psi)(|\nabla' g|^2 + |(\partial_\nu w)|^2) \rho \psi \psi_{\tau} \nabla' g + 2 \int (\partial_\nu w)^2 \rho \psi \nabla' g \cdot \nabla' g_{\tau} + \int (\partial_\nu w)^2 \rho \psi \nabla' g \cdot \nabla' g_{\tau} \\
+ \frac{1}{2} \int w_{\tau}(\partial_\nu w) \rho \psi^2 (g(\eta', \tau) - g(0, \tau) - \eta' \cdot \nabla' g) \\
+ \int \left( \frac{\nabla w}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)} \right) \rho \psi \psi_{\tau}
\end{equation}

with the abbreviation $\int (\cdots) = \int_{\Omega(\tau)} (\cdots) d\eta$.

Proof. In what follows, we will perform integration by parts several times, and so we need to observe the boundary value of $w\psi$. We claim that

\begin{equation}
w\psi = 0 \quad \text{on } \partial \Omega(\tau).
\end{equation}

To prove this, we note that

\begin{equation}
\partial \Omega(\tau) = e^{\tau/2}(\Phi(\partial \Omega \cap B_R) - \bar{\xi}) \cup e^{\tau/2}(\Phi(\Omega \cap \partial B_R) - \bar{\xi}).
\end{equation}

For $\eta \in e^{\tau/2}(\Phi(\partial \Omega \cap B_R) - \bar{\xi})$, by the boundary condition in (3.12), we obtain $w(\eta, \tau) = 0$. On the other hand, for $\eta \in e^{\tau/2}(\Phi(\Omega \cap \partial B_R) - \bar{\xi})$, we have $\Phi(x) = \xi = e^{-\tau/2} \eta + \bar{\xi} \in \Phi(\Omega \cap \partial B_R)$. Thus $x \in \Omega \cap \partial B_R$ and $|x| = R$. This together with $\bar{x} \in \Omega/4$ gives $|x - \bar{x}| \geq 3R/4$. Therefore $\psi(\eta, \tau) = \varphi(4|x - \bar{x}|/R) = 0$ for $\eta \in e^{\tau/2}(\Phi(\Omega \cap \partial B_R) - \bar{\xi})$. Hence (3.21) holds.

For simplicity, we write $\int (\cdots) = \int_{\Omega(\tau)} (\cdots) d\eta$ when no confusion can arise. By (3.21), we see that

\begin{equation}
\frac{d}{d\tau} \int \left( -\frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)} \right) \rho \psi^2 \\
= \int \frac{ww_{\tau}}{p-1} \rho \psi^2 - \int w_{\tau} |w|^{p-1} \rho \psi^2 \\
+ \int \left( -\frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)} \right) \rho \psi^2 + R_0,
\end{equation}

where $R_0$ is a bounded number.
where

\begin{equation}
\mathcal{R}_0 := \int \frac{w^2}{p - 1} \rho \psi \psi_{\tau} - \int \frac{2|w|^{p+1}}{p + 1} \rho \psi \psi_{\tau}.
\end{equation}

On the other hand, by taking the computation

\begin{equation}
|\nabla w|^2 = |\nabla w - (\partial_n w \nabla' g, 0)|^2
= |\nabla w|^2 - 2(\partial_n w) \nabla' w \cdot \nabla' g + (\partial_n w)^2 |\nabla' g|^2
\end{equation}

into account, we set

\begin{equation}
\frac{d}{d\tau} \int |\hat{\nabla} w|^2 \rho \psi = \frac{1}{2} \frac{d}{d\tau} \int |\nabla w|^2 \rho \psi^2 - \frac{d}{d\tau} \int (\partial_n w) \rho \psi^2 \nabla' w \cdot \nabla' g
+ \frac{1}{2} \frac{d}{d\tau} \int (\partial_n w)^2 \rho \psi^2 |\nabla' g|^2
=: \frac{1}{2} \frac{dI_1}{d\tau} - \frac{dI_2}{d\tau} + \frac{1}{2} \frac{dI_3}{d\tau}.
\end{equation}

We compute the derivatives of \(I_1\), \(I_2\) and \(I_3\) in the following way:

1. If a term contains the derivative of \(\psi\), keep the term as it is.
2. If a term contains the spatial derivative(s) of \(w_{\tau}\), perform integration by parts to remove the spatial derivative(s).
3. If a term does not contain \(w_{\tau}\) but contains the second order spatial derivatives of \(w\), perform integration by parts for lowering the order.

For \(I_1\), integration by parts and (3.21) show that

\begin{align*}
\frac{1}{2} \frac{dI_1}{d\tau} &= -\frac{d}{d\tau} \left( \int (w \psi \nabla w \cdot \nabla \psi) \rho + \frac{1}{2} \int w \psi^2 \nabla \cdot (\rho \nabla w) \right) \\
&= - \int (w \psi \nabla w \cdot \nabla \psi)_{\tau} \rho - \frac{1}{2} \int w_{\tau} \psi^2 \nabla \cdot (\rho \nabla w)
- \int w \psi \nabla \psi_{\tau} \cdot (\rho \nabla w) - \frac{1}{2} \int w \psi^2 \nabla \cdot (\rho \nabla w_{\tau})
- \frac{1}{2} \int w \psi^2 \nabla \cdot (\rho \nabla w) - \int w \rho \psi \nabla w \cdot \nabla \psi.
\end{align*}

Integrating by parts twice and (3.21) yield

\begin{align*}
-\frac{1}{2} \int w \psi^2 \nabla \cdot (\rho \nabla w_{\tau}) &= -\frac{1}{2} \int w_{\tau} \psi^2 \nabla \cdot (\rho \nabla w) - \int w_{\tau} \rho \psi \nabla w \cdot \nabla \psi
+ \int w \rho \psi \nabla w_{\tau} \cdot \nabla \psi + \frac{1}{2} \int_{\partial \Omega(\tau)} w_{\tau} (\partial_{\nu} w) \rho \psi^2 dS.
\end{align*}

In addition, we see that

\begin{align*}
-\frac{1}{2} \int w \psi^2 \nabla \cdot (\rho \nabla w) &= \frac{1}{2} \int |\nabla w|^2 \rho \psi^2 + \int w \rho \psi \nabla w \cdot \nabla \psi.
\end{align*}

The above computations imply that

\begin{equation}
\frac{1}{2} \frac{dI_1}{d\tau} = - \int w_{\tau} \psi^2 \nabla \cdot (\rho \nabla w) + \frac{1}{2} \int_{\partial \Omega(\tau)} w_{\tau} \rho \psi^2 \nabla w \cdot \nu dS
+ \frac{1}{2} \int |\nabla w|^2 \rho \psi^2 + \mathcal{R}_1,
\end{equation}
where

\[ R_1 := -\int (w\psi \nabla w \cdot \nabla \psi)_{\partial t} - \int w\psi \nabla_w \nabla \cdot (\rho \nabla w) - \int \omega \rho \nabla w \cdot \nabla \psi + \int w\rho \nabla w, \]

For \( R_1 \), expanding the first term and integrating by parts in the second term, we obtain

\[ R_1 = 2 \int w_{\tau} \rho \psi \nabla w \cdot \nabla \psi + \int |\nabla w|^2 \rho \psi \psi_{\tau}. \tag{3.27} \]

By the argument of [43, Proposition 2.1] with \( \Phi(\partial \Omega \cap B_R) \subset \mathbb{R}^n_+ \), we can see that

\[ \frac{1}{2} \int_{\partial \Omega(\tau)} w_{\tau} (\partial_{\nu} w) \rho \psi^2 dS = -\frac{1}{4} \int_{e^{\gamma/2}(\Phi(\partial \Omega \cap B_R) - \tilde{\xi})} (\partial_{\nu} w)^2 \rho \psi^2 (\nu \cdot \eta) dS. \tag{3.28} \]

Indeed, from the boundary conditions in (3.8) and (3.12), it follows that

\[ 0 = \tilde{a}_t = (i - t)^{-\frac{1}{p-1}} \left( \nabla w \cdot \frac{y}{2} + w_{\tau} \right), \]

and so \( w_{\tau} = -\nabla w \cdot (\eta/2) \) on \( e^{\gamma/2}(\Phi(\partial \Omega \cap B_R) - \tilde{\xi}) \). By the boundary condition in (3.12), we also have \( \nabla w = (\nabla w \cdot \nu) \nu = (\partial_{\nu} w) \nu \). Thus,

\[ w_{\tau} = -\frac{1}{2} (\partial_{\nu} w)(\nu \cdot \eta), \quad \eta \in e^{\gamma/2}(\Phi(\partial \Omega \cap B_R) - \tilde{\xi}). \tag{3.29} \]

Recall that \( \psi(\eta, \tau) = 0 \) for \( \eta \in e^{\gamma/2}(\Phi(\Omega \cap \partial B_R) - \tilde{\xi}) \) by the proof of (3.21). Then (3.28) follows. For later use, we note that, on the boundary,

\[ \nabla^i w = (\partial_{\nu} w) \nu', \quad \partial_{\nu} w = (\partial_{\nu} w) \nu_n, \tag{3.30} \]

which follow from \( \nabla w = (\partial_{\nu} w) \nu \).

We next consider \( I_2 \). Since

\[ -I_2 = \int w \psi^2 \nabla^i (\rho \partial_{\nu} w) \cdot \nabla^i g + 2 \int w(\partial_{\nu} w) \rho \psi \nabla^i \psi \cdot \nabla^i g + \int w(\partial_{\nu} w) \rho \psi^2 \Delta^i g, \]

we have

\[ \frac{dI_2}{d\tau} = \int w_{\tau} \psi^2 \nabla^i (\rho \partial_{\nu} w) \cdot \nabla^i g + \int w_{\tau} (\partial_{\nu} w) \rho \psi^2 \Delta^i g + \int w_{\tau} \psi \nabla^i (\rho \partial_{\nu} w) \cdot \nabla^i g + \int w_{\tau} (\partial_{\nu} w) \rho \psi^2 \Delta^i g + \int w_{\tau} \psi \nabla^i (\rho \partial_{\nu} w) \cdot \nabla^i g + \int w_{\tau} (\partial_{\nu} w) \rho \psi^2 \Delta^i g + 2 \int w_{\tau} \psi \nabla^i (\rho \partial_{\nu} w) \cdot \nabla^i g + 2 \int w(\partial_{\nu} w) \rho \psi \nabla^i \psi \cdot \nabla^i g + 2 \int w(\partial_{\nu} w) \rho \psi^2 \Delta^i g. \]
where the 8th term in the right-hand side requires the 3rd derivative of $f \in C_0^{2+\alpha}(\mathbb{R}^{n-1})$. But the computations here and below can be justified by the standard approximation procedure. Again by integrating by parts twice, we can see that

\begin{equation}
\int w\psi^2 \nabla'(\rho \partial_n w_r) \cdot \nabla' g + \int w(\partial_n w_r) \rho \psi^2 \Delta' g = \int w_r \psi^2 \partial_n (\rho \nabla' w) \cdot \nabla' g - \int_{\partial \Omega(\tau)} w_r \rho \psi^2 \nu_n \nabla' w \cdot \nabla' g dS \\
- 2 \int w(\partial_n w_r) \rho \psi \nabla' \psi \cdot \nabla' g + 2 \int w_r \rho \psi (\partial_n \psi) \nabla' w \cdot \nabla' g.
\end{equation}

Moreover, we have

\begin{align*}
\int w\psi^2 \nabla'(\rho_r \partial_n w) \cdot \nabla' g + \int w(\partial_n w) \rho_r \psi^2 \Delta' g \\
+ \int w\psi^2 \nabla'(\rho \partial_n w_r) \cdot \nabla' g_r + \int w(\partial_n w) \rho \psi^2 \Delta' g_r \\
- \int (\partial_n w) \rho \psi^2 \nabla' w \cdot \nabla' g - 2 \int w(\partial_n w) \rho \psi \nabla' \psi \cdot \nabla' g \\
- \int (\partial_n w) \rho \psi^2 \nabla' w \cdot \nabla' g_r - 2 \int w(\partial_n w) \rho \psi \nabla' \psi \cdot \nabla' g_r.
\end{align*}

These computations show that

\begin{equation}
- \frac{dI_2}{dt} = \int w_r \psi^2 \nabla'(\rho \partial_n w) \cdot \nabla' g + \int w(\partial_n w) \rho \psi^2 \Delta' g \\
+ \int w_r (\partial_n w) \rho \psi^2 \Delta' g - \int_{\partial \Omega(\tau)} w_r \rho \psi^2 \nu_n \nabla' w \cdot \nabla' g dS \\
- \int (\partial_n w) \rho \psi^2 \nabla' w \cdot \nabla' g + R_2,
\end{equation}

where

\begin{align*}
R_2 := & \quad - 2 \int w(\partial_n w_r) \rho \psi \nabla' \psi \cdot \nabla' g + 2 \int w_r \rho \psi (\partial_n \psi) \nabla' w \cdot \nabla' g \\
& - 2 \int w(\partial_n w) \rho_r \psi \nabla' \psi \cdot \nabla' g - \int (\partial_n w) \rho \psi^2 \nabla' w \cdot \nabla' g_r \\
& - 2 \int w(\partial_n w) \rho \psi \nabla' \psi \cdot \nabla' g_r + 2 \int w \rho \psi \nabla' (\rho \partial_n w) \cdot \nabla' g \\
& + 2 \int (w(\partial_n w) \rho \psi \nabla' \psi \cdot \nabla' g) + 2 \int w(\partial_n w) \rho \psi \psi \Delta' g.
\end{align*}

For $R_2$, integrating by parts in the sixth term and expanding the seventh term, we obtain

\begin{equation}
R_2 = 2 \int w \rho \psi (\partial_n \psi) \nabla' w \cdot \nabla' g - \int (\partial_n w) \rho \psi^2 \nabla' w \cdot \nabla' g_r \\
- 2 \int (\partial_n w) \rho \psi \psi \nabla' \psi \cdot \nabla' g + 2 \int w_r (\partial_n w) \rho \psi \nabla' \psi \cdot \nabla' g.
\end{equation}
In the same manner as in (3.31), we see that

\begin{align}
\int_{\partial \Omega(\tau)} w \rho \psi^2 \nu_n \nabla' w \cdot \nabla' g dS
\notag \\
= \frac{1}{2} \int_{e^{r^2/2}(\Phi(\partial \Omega \cap B_R) - \xi)} (\partial_r w)^2 \rho \psi^2 (\nu \cdot \eta) \nu_n \nabla' g \cdot \nu' dS.
\end{align}

Indeed, by (3.30), we have

\[ \nabla \] together with (3.29) shows the above relation.

We examine \( I_3 \). Again by integration by parts, we have

\[ \frac{1}{2} \frac{dI_3}{d\tau} = - \frac{d}{d\tau} \left( \frac{1}{2} \int w \rho \psi^2 \partial_n (\rho \partial_n w) |\nabla' g|^2 + \int w (\partial_n w) \rho \psi (\partial_n \psi) |\nabla' g|^2 \right) \]

\[ = - \frac{1}{2} \int w \rho \psi^2 \partial_n (\rho \partial_n w) |\nabla' g|^2 - \frac{1}{2} \int w \rho \psi^2 \partial_n (\rho \partial_n w_r) |\nabla' g|^2 \]

\[ - \frac{1}{2} \int w \rho \psi^2 \partial_n (\rho \partial_n w) |\nabla' g|^2 - \int w (\partial_n w) \rho \psi (\partial_n \psi) |\nabla' g|^2 \]

\[ = - \int w (\partial_n w_r) \rho \psi (\partial_n \psi) |\nabla' g|^2 + \int w (\partial_n w_r) \rho \psi (\partial_n \psi) |\nabla' g|^2 \]

and

\[ \frac{1}{2} \int w \rho \psi^2 \partial_n (\rho \partial_n w) |\nabla' g|^2 - \int w \rho \psi^2 \partial_n (\rho \partial_n w) \nabla' g \cdot \nabla' g_r \]

\[ = \frac{1}{2} \int (\partial_n w)^2 \rho \psi^2 |\nabla' g|^2 + \int (\partial_n w)^2 \rho \psi^2 \nabla' g \cdot \nabla' g_r \]

\[ + \int w (\partial_n w) \rho \psi (\partial_n \psi) |\nabla' g|^2 + 2 \int w (\partial_n w) \rho \psi (\partial_n \psi) \nabla' g \cdot \nabla' g_r. \]

Then we have

\begin{align}
\frac{1}{2} \frac{dI_3}{d\tau} = - \int w \rho \psi^2 \partial_n (\rho \partial_n w) |\nabla' g|^2 + \frac{1}{2} \int (\partial_n w)^2 \rho \psi^2 |\nabla' g|^2 \\
+ \frac{1}{2} \int_{\partial \Omega(\tau)} w (\partial_n w) \rho \psi^2 |\nabla' g|^2 \nu_n dS + R_3,
\end{align}

From the same computations as in the proof of (3.28), it follows that

\begin{align}
(3.34)
\int_{\partial \Omega(\tau)} w r \rho \psi^2 \nu_n \nabla' w \cdot \nabla' g dS \\
= \frac{1}{2} \int_{e^{r^2/2}(\Phi(\partial \Omega \cap B_R) - \xi)} (\partial_r w)^2 \rho \psi^2 (\nu \cdot \eta) \nu_n \nabla' g \cdot \nu' dS.
\end{align}

Indeed, by (3.30), we have \( \nabla \cdot \nabla' g = (\partial_r w)(\partial_r g) \) on \( e^{r^2/2}(\Phi(\partial \Omega \cap B_R) - \xi) \). This together with (3.29) shows the above relation.
where
\[ R_3 := - \int w_\tau (\partial_\nu w) \rho \psi (\partial_\nu \psi) |\nabla g|^2 + \int w(\partial_\nu w_\tau) \rho \psi (\partial_\nu \psi) |\nabla g|^2 \]
\[ + \int (\partial_\nu w)^2 \rho \psi^2 \nabla' g \cdot \nabla' g_\tau + \int w(\partial_\nu w) \rho \psi (\partial_\nu \psi) |\nabla g|^2 \]
\[ + 2 \int w(\partial_\nu w) \rho \psi (\partial_\nu \psi) \nabla' g \cdot \nabla' g_\tau - \int w \psi \psi_\tau \partial_\nu (\rho \partial_\nu w) |\nabla g|^2 \]
\[ - \int (w(\partial_\nu w) \rho \psi (\partial_\nu \psi) |\nabla g|^2) _\tau. \]

For \( R_3 \), integrating by parts in the sixth term and expanding the seventh term, we obtain
\[ R_3 = -2 \int w_\tau (\partial_\nu w) \rho \psi (\partial_\nu \psi) |\nabla g|^2 + \int (\partial_\nu w)^2 \rho \psi^2 \nabla' g \cdot \nabla' g_\tau \]
\[ + \int (\partial_\nu w)^2 \rho \psi \psi_\tau |\nabla g|^2. \]

By (3.29) and (3.30), we see that
\[ \frac{1}{2} \int_{\partial \Omega(\tau)} w_\tau (\partial_\nu w) \rho \psi^2 |\nabla g|^2 \nu_n dS \]
\[ = - \frac{1}{4} \int_{c^{\tau/2}(\partial \Omega \cap B_n) - \xi} (\partial_\nu w)^2 \rho \psi^2 (\nu \cdot \eta) \nu_n^2 |\nabla g|^2 dS. \]

By combining (3.22), (3.24), (3.28), (3.32), and (3.34), and then by (3.24), we obtain
\[ \frac{d}{d\tau} \mathcal{E}(\tau) = J + B + \sum_{i=0}^{3} R_i + \int \left( \frac{|\nabla w|^2}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)} \right) \rho_\tau \psi^2, \]

where \( B \) is given in the statement of this lemma and
\[ J := \int w_\tau \psi^2 \left( \frac{w \rho}{p-1} - w|w|^{p-1} \rho - \nabla \cdot (\rho \nabla w) + \nabla' (\rho \partial_\nu w) \cdot \nabla' g \right) \]
\[ + \partial_\nu (\rho \nabla' w) \cdot \nabla' g + \rho \partial_\nu w \Delta' g - \partial_\nu (\rho \partial_\nu w) |\nabla g|^2 \].

From (3.12) and \( \nabla' (\partial_\nu w) = \partial_\nu \nabla' w \), it follows that
\[ J = \int w_\tau \psi^2 \left( -w_\tau \rho - \frac{1}{2} \rho \eta \cdot \nabla w - \nabla \rho \cdot \nabla w + \partial_\nu (\rho \nabla' w) \cdot \nabla' g \right) \]
\[ + \partial_\nu w \nabla' \rho \cdot \nabla' g - \partial_\nu \nabla' w (\partial_\nu \rho) |\nabla g|^2 \].

By using (3.17), we have
\[ - \frac{1}{2} \rho \eta \cdot \nabla w - \nabla \rho \cdot \nabla w + \partial_\nu (\rho \nabla' w) \cdot \nabla' g = \frac{1}{2} (\partial_\nu w) \rho (g(\eta', \tau) - g(0, \tau)), \]
\[ \partial_\nu w \nabla' \rho \cdot \nabla' g - \partial_\nu \nabla' w (\partial_\nu \rho) |\nabla g|^2 = - \frac{1}{2} (\partial_\nu w) \rho \eta' \cdot \nabla' g. \]

Thus,
\[ J = - \int w_\tau^2 \rho \psi^2 + \frac{1}{2} \int w_\tau (\partial_\nu w) \rho \psi^2 (g(\eta', \tau) - g(0, \tau) - \eta' \cdot \nabla' g). \]
Substituting this into (3.38) and combining (3.23), (3.27), (3.33) and (3.36) yield the desired equality. The proof is complete.

We next estimate the terms $\mathcal{B}$ and $\mathcal{R}$ in (3.20).

**Lemma 3.4.** $\mathcal{B}(\tau) \geq 0$.

**Proof.** In (3.1), we have $\mathcal{B} = 0$, since the domain of integration is far from the boundary. Thus, we consider the case (3.2). Since $\Phi(\partial \Omega \cap B_R) \subset \mathbb{R}^n_+$ and $\xi_n = \Phi_n(x) = \tilde{x}_n - f(x') \geq 0$ for $\tilde{x} \in \Omega_{R/4}$ by the choice of $f$, we see that $\nu \cdot \eta \geq 0$ for $\eta \in e^{\tau/2}(\Phi(\partial \Omega \cap B_R) - \xi)$. In addition, since $|\nu'| \leq 1$, we have

$$1 - 2\nu_n\partial_\nu \eta + |\nabla \eta|^2 \nu_n^2 = (1 - (\partial_\nu \eta)\nu_n)^2 + (|\nabla \eta|^2 - (\partial_\nu \eta)^2)\nu_n^2 \geq 0.$$

Then the lemma follows. \hfill \Box

**Lemma 3.5.** There exists $C > 0$ such that

$$\mathcal{R}(\tau) \leq \frac{1}{2} \int w_n^2 \psi^2 \rho + C \tilde{R}(\tau)$$

for $\tau = -\log(\tilde{t} - t)$ with $-1/2 < t < \tilde{t} \leq 0$, where

$$\tilde{R}(\tau) := \int (w^2 + |\nabla w|^2 + |w|^{p+1})e^{-\frac{1}{|\nabla \psi|^2}} e^{-\frac{\tau}{\tilde{t}}} \chi_{[0,1]} \left( \frac{4}{R} \Psi(\xi + e^{-\frac{\tau}{\tilde{t}}} \eta) - \Psi(\tilde{\eta}) \right).$$

**Proof.** We only consider the case (3.2), since the case (3.1) is simpler. By (3.11) and the choice of $f$, we have

$$\|\nabla g\|_{L^\infty(\mathbb{R}^{n-1})} \leq \frac{1}{2}, \quad \|\nabla^2 g\|_{L^\infty(\mathbb{R}^{n-1})} \leq C e^{-\frac{\tau}{\tilde{t}}},$$

where $C > 0$ is independent of $\tau$. By Cauchy's inequality and (3.39), we obtain

$$\mathcal{R} \leq \frac{1}{2} \int w_n^2 \rho \psi^2 + C \int (w^2 + |\nabla w|^2 + |w|^{p+1})(\rho(\psi^2) + |\psi_\tau| + |\rho_\tau| \psi^2) + C \int |\nabla w|^2 \rho \psi^2 (|\eta'| e^{-\frac{\tau}{\tilde{t}}} + |\eta'|^4 e^{-\tau}).$$

From (3.15) and (3.39), it follows that

$$\rho(\eta, \tau) \leq \exp \left( -\frac{1}{4} \left( |\eta'|^2 + \frac{1}{2} \eta_n^2 - (g(\eta', \tau) - g(0, \tau))^2 \right) \right) \leq e^{-\frac{\eta_n^2}{4}}.$$

By (3.18) and (3.39), and then by (3.40), we also have

$$|\rho_\tau| \leq C |\eta'|^2 e^{-\frac{\tau}{\tilde{t}}} (|\eta_n| + |\eta'|) \rho \leq C e^{-\frac{|\eta_n|^2}{4}} e^{-\frac{\tau}{\tilde{t}}}.$$

These inequalities together with $\tau > -\log(\tilde{t} + 1/2) > 0$ show that

$$\mathcal{R} \leq \frac{1}{2} \int w_n^2 \rho \psi^2 + C \int (w^2 + |\nabla w|^2 + |w|^{p+1})(|\nabla \psi|^2 + |\psi_\tau| + \psi^2 e^{-\frac{\tau}{\tilde{t}}}) e^{-\frac{|\eta_n|^2}{4}}.$$
From (3.10) and (3.39), it follows that
\[
|\nabla \psi| = \frac{4e^{-\tau/2}}{R} \left| \frac{4}{R} e^{-\frac{x}{R}} \right| \left| \frac{4}{R} e^{-\frac{x}{R}} \right| \left| (\eta', \eta_n + g(\eta', \tau) - g(0, \tau)) \right| \\
\times \frac{|(\eta' + (\eta_n + g(\eta')) - g(0))\nabla g, \eta_n + g(\eta') - g(0)|}{|(\eta', \eta_n + g(\eta') - g(0))|} \\
\leq Ce^{-\frac{\tau}{2}} \chi(\check{\tau}, 1) \left( \frac{4}{R} e^{-\frac{x}{R}} |(\eta', \eta_n + g(\eta', \tau) - g(0, \tau))| \right)
\]
and
\[
|\psi_\tau| = \frac{4e^{-\tau/2}}{R} \left| \frac{4}{R} e^{-\frac{x}{R}} \right| \left| \frac{4}{R} e^{-\frac{x}{R}} \right| \left| (\eta', \eta_n + g(\eta', \tau) - g(0, \tau)) \right| \\
\times \left| -\frac{1}{2} (|\eta', \eta_n + g(\eta') - g(0)| + \partial_\tau (|\eta', \eta_n + g(\eta') - g(0)|) \right| \\
\leq Ce^{-\frac{\tau}{2}} |\phi'| \left( \frac{1}{2} (|\eta|^2 + 2\eta_n^2 + 2g(\eta') - g(0))^2 + |\eta'|^2 e^{-\frac{x}{R}} \right) \\
\leq C|\eta|^2 e^{-\frac{\tau}{2}} \chi(\check{\tau}, 1) \left( \frac{4}{R} e^{-\frac{x}{R}} |(\eta', \eta_n + g(\eta', \tau) - g(0, \tau))| \right),
\]
where \( g(\eta') := g(\eta', \tau) \) and \( g(0) := g(0, \tau) \). Then by \( e^{-\tau/2} (|\eta', \eta_n + g(\eta', \tau) - g(0, \tau)| = |\Psi(\check{\xi} + e^{-\tau/2}) - \Psi(\check{\xi}) | \) in (3.13), we have
\[
(\nabla |\psi|^2 + |\psi_\tau|^2 + |\psi|^2 e^{-\frac{x}{R}}) e^{-\frac{|\eta|^2}{2}} \\
\leq Ce^{-\frac{|\eta|^2}{2}} e^{-\frac{\tau}{2}} \chi(0, 1) \left( \frac{4}{R} e^{-\frac{x}{R}} |\Psi(\check{\xi} + e^{-\tau/2}) - \Psi(\check{\xi}) | \right).
\]
The lemma follows from (3.44) and (3.42).

We are now in a position to prove Proposition 3.2.

Proof of Proposition 3.2. By Lemmas 3.3, 3.4, 3.5, we see that
\[
\mathcal{E}(\tau) + \frac{1}{2} \int_{\tau}^{\tau'} \int_{\Omega(\sigma)} \frac{w_{\sigma}^2 \rho \psi^2 d\sigma d\sigma}{} \leq \mathcal{E}(\tau') + C \int_{\Omega(\sigma)} \mathcal{R}(\sigma) d\sigma
\]
for \( \tau' = -\log(\check{t} - t) \) and \( \tau = -\log(\check{t} - t) \) with \(-1/2 < t' < t < 0\). Note that this inequality holds for both (3.31) and (3.32). The change of variables and the same computations as in Lemma 3.1 yield
\[
\int_{\tau}^{\tau'} \mathcal{R}(\sigma) d\sigma = \int_{\tau}^{\tau'} (\check{t} - s)^{-\frac{1}{2}} \int_{\Omega} \left( \frac{w^2}{l - s} + |\nabla u|^2 + |u|^{p+1} \right) \\
\times (\check{t} - s)^{-\frac{1}{2}} e^{-\frac{|x|^2}{2(p-1)}} dx ds \\
\leq C(M + M^p)^2 \int_{\tau}^{\tau'} (\check{t} - s)^{-\frac{1}{2}} ds \\
\leq C(M + M^p)^2 (\check{t} - t')^{\frac{1}{2}}.
\]
Thus,
\[
\mathcal{E}(\tau) + \frac{1}{2} \int_{\tau}^{\tau'} \int_{\Omega(\sigma)} \frac{w_{\sigma}^2 \rho \psi^2 d\sigma d\sigma}{} \leq \mathcal{E}(\tau') + C(M + M^p)^2 (\check{t} - t')^{\frac{1}{2}}.
\]
On the other hand, by
\[ w_\tau = (\hat{t} - t)^{\frac{1}{p-1}} \left( - \frac{\hat{u}}{p-1} - \frac{(\xi - \tilde{\xi}) \cdot \nabla \hat{u}}{2} + (\hat{t} - t) \hat{u}_t \right) \]
and the change of variables, we can see that
\[
\int_{\tau'}^{\tau} \int_{\Omega(\sigma)} w^2 \rho \psi^2 \, d\eta d\sigma = \int_{t'}^{t} (\hat{t} - s)^{\frac{2}{p-1}} \left[ \hat{u} \left( \frac{\xi - \Phi(\tilde{x})}{p-1} \cdot \nabla \hat{u} \right) - (\hat{t} - s) \hat{u}_s \right]^2 \times \exp \left( \frac{\|u(t)\|_{L^\infty(Q_{\theta_0,\delta})}}{d_{\xi,R/4}(\Psi(\xi))} \right) d\xi ds.
\]
This together with (3.46) and \( E(\tau) = E(t) \) deduces the desired inequality. The proof is complete.

4. \( \varepsilon \)-regularity

The purpose of this section is to show the following \( \varepsilon \)-regularity theorem:

**Theorem 4.1.** Let \( n \geq 3 \), \( p > p_\Sigma \) and \( \Omega \) be any \( C^{2+\alpha} \) domain in \( \mathbb{R}^n \) with \( 0 \in \Omega \). Fix \( 0 < R < 1/2 \) such that either (3.1) or (3.2) holds. Let \( u \) be a solution of (2.1) satisfying (2.2). Then there exist constants \( \varepsilon_0 \), \( \delta_0 \) and \( \theta_0 \) with \( 0 < \varepsilon_0, \theta_0 < 1 \) and \( 0 < \delta_0 < R \) depending only on \( n \), \( p \), \( M \), \( \Omega \) and \( R \) such that the following holds: If there exists \( 0 < \delta \leq \delta_0 \) such that
\[
\delta^{\frac{1}{p-1}} \int_{Q_\delta} (|\nabla u|^2 + |u|^{p+1}) \, dx \, dt \leq \varepsilon_0,
\]
then
\[
\|u\|_{L^\infty(Q_{\theta_0,\delta})} \leq C(\theta_0 \delta)^{-\frac{1}{p-1}}.
\]
Here \( C \) is a positive constant depending only on \( n \), \( p \), \( M \), \( \Omega \) and \( R \) and independent of \( \varepsilon_0 \), \( \delta_0 \) and \( \theta_0 \).

Among the contents of this section, we will cite only Theorem 4.1, Remark 4.2 and Lemma 4.7 for proving our main result in Section 5.

**Remark 4.2.** Chou, Du and Zheng [15, Theorems 2, 2'] proved \( \varepsilon \)-regularity theorems for global-in-time solutions (or borderline solutions) of (2.1) in bounded convex domains. The theorems were applied for showing the decay of borderline solutions as \( t \to \infty \), see [95] for alternative approach. In [14, Theorem 2], the assumption on convexity was removed, see also [32, Proposition 4.2]. By the nontrivial modifications of [14, 15], we show the \( \varepsilon \)-regularity for local-in-time solutions.

We note that the proofs of the \( \varepsilon \)-regularity theorems in [14, 15] are based on a preliminary \( \varepsilon \)-regularity result [15, Lemma 3], where the time at which the regularity of the solution is concerned with should be contained in the interior of the time interval of the solution. Thus, it seems difficult to apply their argument near the final time of local-in-time solutions. To overcome this issue, we give a variant of [15, Lemma 3] with the aid of Blatt and Struwe [9] Proposition 4.1] and Giga and Kohn [43, Theorems 2.1, 2.5] to require only the estimate of solutions shortly before the reference time, see Lemma 4.4.
Remark 4.3. Theorem 1.1 remains to hold for weak solutions satisfying an estimate of the form (4.9) in Lemma 1.7. In particular, we may apply it to the blow-up limit of certain rescaled solutions in Section 5.

In this section, unless otherwise stated, $C$ denotes a constant depending only on $n$, $p$, $M$, $\Omega$, and $R$. Each $C$ may have different values also within the same line. We always assume either (3.1) or (3.2) and deal with each of the cases in a unified way. In addition, we always assume that $u$ is a solution of (2.1) satisfying (2.2).

We state a preliminary $\varepsilon$-regularity result.

Lemma 4.4. There exist $\varepsilon_1 > 0$ and $C > 0$ depending only on $n$, $p$ and $\Omega$ such that the following holds: If there exists $0 < R_1 < R/4$ such that

$$ (r/2)^n \int_{t_1-(r/4)^2}^{t_1} \int_{\Omega_{r/2}(x_1)} |u(x,t)|^{p+1} dx dt \leq \varepsilon_1 $$

for any $(x_1,t_1)$ and $r > 0$ satisfying $Q_r(x_1,t_1) \subset Q_{2R_1}$, then $\|u\|_{L^\infty(Q_{R_1/4})} \leq CR_1^{2/(p-1)}$.

Proof. Let $\varepsilon_1 > 0$ be a constant chosen later. Throughout this proof, $C$ depends only on $n$, $p$ and $\Omega$. Let $v(y,s) := R_1^2/(p-1)u(R_1y,R_1^2s)$. Since $R/R_1 > 4$ and $-1/R_1^2 < -16$, we can check that $v$ satisfies $v_t = \Delta v + |v|^{p-2}v$ in $Q'_r$, where $\Omega' := R_1^{-1}\Omega, Q'_r(x_1,t_1) := (\Omega' \cap B_r(x_1)) \times (t_1-r^2,t_1)$ and $Q'_r := Q'_r(0,0)$. From (4.3) and the change of variables, it follows that if $Q'_r(x_1,t_1) \subset Q'_2$ and $(r/4)^2 \leq -t_1$, then $Q'_r(x_1,t_1 + (r/4)^2) \subset Q'_2$ and

$$ (r/2)^n \int_{t_1-(r/4)^2}^{t_1} \int_{\Omega_{r/2}(x_1)} |v(y,s)|^{p+1} dy ds \leq \varepsilon_1. $$

Let $(\tilde{x}, \tilde{t}) \in Q'_{1/2}$. Set $\lambda := (\Delta - \tilde{t})^{1/2}$ and $\Omega'' := \lambda^{-1}(\tilde{\Omega} - \tilde{x})$. Then the rescaled function $\tilde{v}(x,t) := \lambda^2/(p-1)v(\lambda x + \tilde{x}, \lambda^2t + \tilde{t})$ satisfies $\tilde{v}_t = \Delta \tilde{v} + |\tilde{v}|^{p-2}\tilde{v}$ in $(\tilde{\Omega}'' \cap B_{2\lambda}(x_1 - \tilde{x})/\lambda) \times ((-4/\lambda^2,0)$. If $B_{2\lambda}(x_1 - \tilde{x})/\lambda \subset B_2$ and $((t_1-r^2)/\lambda^2, t_1/\lambda^2) \subset (-4,0)$, then $Q''_r(x_1,t_1 + \tilde{t}) \subset Q'_2$ and $(r/4)^2 \leq -t_1 + \tilde{t}$. Therefore, (4.3) shows that

$$ (r/4\lambda)^{n/2} \int_{(t_1-(r/4)^2)/\lambda^2}^{t_1+\tilde{t}/\lambda^2} \int_{\Omega'' \cap B_{r/4}(x_1 - \tilde{x})/\lambda)} |\tilde{v}(x,t)|^{p+1} dx dt \leq C \varepsilon_1. $$

Replacing $(x_1,t_1)$ and $r$ with $(\lambda x_1 + \tilde{x}, \lambda^2 t_1)$ and $\lambda r$, respectively, we see that

$$ (r/4)^{n/2} \int_{Q''_{r/4}(x_1,t_1)} |\tilde{v}(x,t)|^{p+1} dx dt \leq C \varepsilon_1 $$

for any $Q''_r(x_1,t_1) \subset Q''_2$, where $Q''_r(x_1,t_1) := (\tilde{\Omega}'' \cap B_r(x_1)) \times (t_1-r^2,t_1)$ and $Q''_r := Q''_r(0,0)$. Hence $\|\tilde{v}\|_{M^{p+1,p+1}(Q''_{1/2})} \leq C \varepsilon_1$ with $\mu_c := 2(p+1)/(p-1)$, where $\|\cdot\|_{M^{p+1,p+1}(Q'_{1/2})}$ is the parabolic Morrey norm on $Q'_{1/2}$. 

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Taking $\varepsilon_1$ sufficiently small, we may apply [9 Proposition 4.1] to see that
$$
\|\tilde{v}\|_{L^\infty(Q''_{1/4})} \leq C\|\tilde{v}\|_{L^{p+1,\infty}(Q'_{1/2})} \leq \varepsilon_1. 
$$
In particular, $\Lambda^{2/(p-1)}|v(\lambda x+\tilde{x}, \lambda^2 t+\tilde{t})| \leq C\varepsilon_1$ for $(x,t) \in Q''_{1/4}$. Letting $(x,t) \to (0,0)$ gives
$$
|v(\tilde{x}, \tilde{t})| \leq C\varepsilon_1 \lambda^{-\frac{2}{p-1}} = C\varepsilon_1 (-\tilde{t})^{-\frac{2}{p-1}}
$$
for $(\tilde{x}, \tilde{t}) \in Q'_{1/2}$. Since $\Omega'$ is $C^{2+\alpha}$, by applying [44 Theorems 2.1, 2.5] with $\varepsilon_1$ replaced by a smaller constant if necessary, we obtain $\|v\|_{L^\infty(Q'_{1/4})} \leq C$, and hence $|u(x,t)| \leq CR^{-2/(p-1)}$ for $(x,t) \in Q_{R/4}$. The proof is complete.

In the rest of this section, we prove Theorem 4.1 by using Lemma 4.4 and estimates of $E$. First of all, note that we may replace $\delta$ with $A\delta$ in the assumption (4.17), where $\Lambda > 1$ is a large constant and $0 < \delta < 1$ is a small constant depending on $A$. More precisely, we may assume
$$
\delta \frac{1}{\Lambda^{4/3}} \int_{Q_{4\delta}} (|\nabla u|^2 + |u|^{p+1})dxdt \leq A^{n-\frac{4}{p-1}}\varepsilon_0.
$$
Here we take the constants $A$, $\varepsilon_0$, $\delta$, and $\theta_0$ as
$$
A > 3, \quad 0 < \varepsilon_0 < 1, \quad 0 < \delta < \frac{R}{16A} < \frac{1}{16}, \quad 0 < \theta_0 < \frac{1}{32A},
$$
which will be specified later, see (4.17). Set
$$
I_r = I_r(x_1,t_1) := \frac{r}{2} \frac{1}{\Lambda^{4/3}} \int_{(r/2)^{-2}}^{(r/2)^{-2}} \int_{\Omega_{r/2}(x_1)} |u|^{p+1}dxdt.
$$
In order to show Theorem 4.1 it suffices to check the following statement:
$$
I_r(x_1,t_1) \leq \varepsilon_1 \quad \text{for any } Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta},
$$
where $\varepsilon_1$ is given in Lemma 4.4. Note that
$$
0 < r < 8A\theta_0\delta < \frac{1}{3} \delta < \delta < \frac{R}{16} < \frac{1}{16}.
$$
Indeed, once (4.7) is proved, Lemma 4.4 guarantees the desired $L^\infty$ bound of $u$: $\|u\|_{L^\infty(Q_{8A\theta_0\delta})} \leq C(A\theta_0\delta)^{-2/(p-1)}$. Therefore our temporary task is to estimate $I_r$.

**Proposition 4.5.** There exists a constant $C > 0$ depending only on $n$, $p$, $M$, $\Omega$ and $R$ such that
$$
I_r(x_1,t_1) \leq C\bar{h} \left( e^{-A^2/C} + A^{n-\frac{4}{p-1}}\varepsilon_0 + \delta \right)
$$
for any $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$, where $\bar{h}(s) := s + s^{1/(p+1)}$ for $s \geq 0$.

We prove this proposition by means of several lemmas.

**Lemma 4.6.** There exists $C > 0$ such that
$$
I_r \leq C \int_{t_1-(r/4)^2}^{t_1-(r/2)^2} (t_1-s) \frac{1}{\Lambda^{4/3}} \int_{\Omega_R} |u|^{p+1}K_{(x_1,t_1)}(x,s)\frac{1}{\rho^2}dxds
$$
for any $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$. 

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**Footnotes:**

[9]: Reference to the main paper for Proposition 4.1.
[44]: Reference to the main paper for Theorems 2.1, 2.5.

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**References:**

9. [Main Paper]

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Proof. For \((x, s) \in B_{r/2}(x_1) \times (t_1 - (r/2)^2, t_1 - (r/4)^2)\), we have \(|x - x_1| \leq r/2 < R/32\) and \(t_1 - (r^2/4) < s < t_1 - (r^2/16)\). Thus,
\[
K_{(x_1, t_1)}(x, s)\phi_{x_1, R/8}(x) = (t_1 - s)^{-\frac{n}{2}} e^{-\frac{|x-x_1|^2}{Rt_1-1}} \varphi^2 \left(\frac{8|x-x_1|}{R}\right) 
\geq (1/4)^{-\frac{n}{2}} \tau^{1-n} e^{-1} \varphi^2(1/4) \geq C\tau^{-n}.
\]
By \((t_1 - s)^2/(p-1) \geq 16^{-2/(p-1)}\), for \(t_1 - r^2/4 < s < t_1 - (r^2/16)\), the lemma follows. \(\square\)

To estimate the right-hand side of (4.8), we prepare the following lemma by using Proposition 3.2.

**Lemma 4.7.** There exists \(C > 0\) such that
\[
\int_{t'}^t (\tilde{t} - s)^{\frac{n}{2}} \int_{\Omega_R} |u(x, s)|^{p+1} K_{(\tilde{x}, \tilde{t})}(x, s) \phi_{\tilde{x}, \tilde{t}}^2(x) dx ds 
\leq C \left( \log \frac{\tilde{t} - t'}{t - t'} \right)^{\frac{n}{4}} \left( E_{(\tilde{x}, \tilde{t})}(t'; \phi_{\tilde{x}, \tilde{t}}^2) - E_{(\tilde{x}, \tilde{t})}(t; \phi_{\tilde{x}, \tilde{t}}^2) + C(\tilde{t} - t') \right)^{\frac{n}{4}} 
+ C_p \left( E_{(\tilde{x}, \tilde{t})}(t'; \phi_{\tilde{x}, \tilde{t}}^2) + C(\tilde{t} - t') \right) \log \frac{\tilde{t} - t'}{t - t'} + C(\tilde{t} - t')^{\frac{n}{4}}
\]
for any \(\tilde{x} \in \Omega_{R/4}\) and \(-1/2 < t' < t < \tilde{t} \leq 0\), where \(C_p := 2(p+1)/(p-1)\).

**Proof.** Define \(w, \mathcal{E}, \rho, \) and \(\psi\) by (3.10), (3.14), (3.15) and (3.16), respectively. Then the left-hand side of the desired inequality equals \(\int \int |w|^{p+1} \rho \psi^2\), where we write \(f(\cdots) = \int_{\Omega(t)} f(\cdots) dy\) and \(f(\cdots) = \int_t^{\tilde{t}} \int_{\Omega(s)} f(\cdots) d\sigma ds\) unless otherwise stated. From (3.12) and (3.11), it follows that
\[
\frac{1}{2} \frac{d}{dt} \int w^2 \rho \psi^2 = \int w w_t \rho \psi^2 + \frac{1}{2} \int w^2 \rho \psi_{\rho} + \int w^2 \rho \psi_{\psi} - 2\mathcal{E}(t) + 2\mathcal{E}(\tau)
= -2\mathcal{E}(\tau) + \frac{p-1}{p+1} \int |w|^{p+1} \rho \psi^2 - \frac{1}{2} \int w \rho \psi^2 \nabla w \cdot \eta + \int w \rho^2 \Delta w 
- 2 \int w \rho^2 \nabla (\nabla w) \cdot \nabla \psi + \int (\nabla^2 \rho) \rho^2 |\nabla' g|^2
- \int w (\partial_n^2 w) \rho \psi^2 \Delta g + \int |\nabla w|^2 \rho \psi^2 - 2 \int (\partial_n w) \rho^2 \nabla' \cdot \nabla' g 
+ \int (\partial_n w)^2 \rho^2 |\nabla' g|^2 + \frac{1}{2} \int w^2 \rho \psi_{\rho}^2 + \int w^2 \rho \psi_{\psi}.
\]
Integrating by parts gives
\[
\frac{1}{2} \frac{d}{dt} \int w^2 \rho \psi^2 = -2\mathcal{E}(\tau) + \frac{p-1}{p+1} \int |w|^{p+1} \rho \psi^2 + \tilde{R}_1 + \tilde{R}_2,
\]
where
\[ \tilde{R}_1 := -\frac{1}{2} \int w \psi^2 \nabla w \cdot \eta - \int w \psi^2 \nabla w \cdot \nabla \rho + 2 \int w (\partial_n w) \psi^2 \nabla' \rho \cdot \nabla' g \\
+ \int w (\partial_n w) \rho \psi^2 \Delta' g - \int w (\partial_n w)(\partial_n \rho) \psi^2 |\nabla' g|^2 + \frac{1}{2} \int w^2 \rho \tau \psi^2, \]
\[ \tilde{R}_2 := -2 \int w \psi \nabla w \cdot \nabla \psi + 4 \int w (\partial_n w) \rho \psi \nabla' \psi \cdot \nabla' g \\
- 2 \int w (\partial_n w) \rho \psi (\partial_n \psi) |\nabla' g|^2 + \int w^2 \rho \psi \tau. \]

Since \( w \nabla w = \nabla (w^2)/2 \) and \( w \partial_n w = \partial_n (w^2)/2 \), integrating by parts again shows that

\[ \tilde{R}_1 = \int w^2 \psi^2 \left( \frac{\eta}{4} \cdot \nabla \rho + \frac{n}{4} \rho + \frac{1}{2} \Delta \rho - \partial_n (\nabla' \rho) \cdot \nabla' g \\
- \frac{1}{2} (\partial_n \rho) \Delta' g + \frac{1}{2} (\partial_n \rho) |\nabla' g|^2 \right) + \frac{1}{2} \int w^2 \rho \psi \nabla \cdot \eta \\
+ \int w^2 \psi \nabla \psi \cdot \nabla \rho - 2 \int w^2 (\partial_n \psi) \nabla' \rho \cdot \nabla' g - \int w^2 \rho (\partial_n \psi) \Delta' g \\
+ \int w^2 (\partial_n \rho) (\partial_n \psi) |\nabla' g|^2 + \frac{1}{2} \int w^2 \rho \tau. \]

From (3.17), (3.39), (3.40) and direct computations, it follows that

\[ \frac{\eta}{4} \cdot \nabla \rho + \frac{n}{4} \rho + \frac{1}{2} \Delta \rho - \partial_n (\nabla' \rho) \cdot \nabla' g - \frac{1}{2} (\partial_n \rho) \Delta' g + \frac{1}{2} (\partial_n \rho) |\nabla' g|^2 \]

\[ = \frac{1}{8} (\eta_n + g(\eta') - g(0))(g(\eta') - g(0) - \eta' \cdot \nabla' g) \rho \\
+ \frac{1}{8} (\eta_n + g(\eta') - g(0))^2 (1 - 2|\nabla' g|^2) \rho \\
\geq -\frac{1}{8} (\eta_n + g(\eta') - g(0))|g(\eta') - g(0) - \eta' \cdot \nabla' g| \rho \\
\geq -C|\eta|^3 e^{-2} e^{-\frac{|n|^2}{4}} \geq -C e^{-2} e^{-\frac{|n|^2}{4}}, \]

where \( g(\eta') := g(\eta', \tau) \) and \( g(0) := g(0, \tau) \). The remainder terms in \( \tilde{R}_1 \) can be estimated by using (3.17), (3.39), (3.40) and (3.41). Then by \( \tau > -\log(t + 1/2) > 0 \), we obtain

\[ \tilde{R}_1 \geq -C \int w^2 \psi^2 e^{-\frac{|n|^2}{4}} - C \int w^2 |\nabla \psi| e^{-\frac{|n|^2}{4}} (1 + e^{-\frac{|n|^2}{4}}) \]

\[ \geq -C \int w^2 (|\nabla \psi| + \psi^2 e^{-\frac{|n|^2}{4}}) e^{-\frac{|n|^2}{4}}. \]

On the other hand, by Cauchy’s inequality, (3.39) and (3.40), we see that

\[ \tilde{R}_2 \geq -C \int (w^2 + |\nabla w|^2)(|\nabla \psi| + |\psi_\tau|) e^{-\frac{|n|^2}{4}}, \]

and so by (3.41) with (3.43), we obtain

\[ \tilde{R}_1 + \tilde{R}_2 \geq -C \int (w^2 + |\nabla w|^2)(|\nabla \psi| + |\psi_\tau| + \psi^2 e^{-\frac{|n|^2}{4}}) e^{-\frac{|n|^2}{4}} \geq -C R, \]

where \( R = \tilde{R} (\tau) \) is given in Lemma 3.5.
By $E(\tau) = E(t)$ and Proposition 3.2 we obtain
\[
\frac{1}{2} \frac{d}{d\tau} \int w^2 \rho \psi^2 \geq -2E(\tau) + \frac{p - 1}{p + 1} \int |w|^{p+1} \rho \psi^2 - C\bar{R}
\geq -2(E(t') + C\tilde{t} - t')^{\frac{2}{p}} + \frac{p - 1}{p + 1} \int |w|^{p+1} \rho \psi^2 - C\bar{R}.
\]
Set $C_p := 2(p + 1)/(p - 1)$. Then,
\[
\int |w|^{p+1} \rho \psi^2 \leq \frac{C_p}{4} \frac{d}{d\tau} \int w^2 \rho \psi^2 + C_p (E(t') + C\tilde{t} - t')^{\frac{2}{p}} + C\bar{R}.
\]
Integrating this inequality over $\sigma \in (\tau', \tau)$ with $\tau' = -\log(\tilde{t} - t')$ and $\tau = -\log(\tilde{t} - t)$, we have
\[
\left| K \int \int |w|^{p+1} \rho \psi^2 \leq \frac{C_p}{4} (K(\tau) - K(\tau')) \right.
\]
\[
+ C_p (E(t') + C\tilde{t} - t')^{\frac{2}{p}} \log \frac{\tilde{t} - t'}{\tilde{t} - t} + C \int_{\tau'}^{\tau} \bar{R} d\sigma,
\]
where
\[
K(\tau) := \int w^2(\eta, \tau) \rho(\eta, \tau) \psi^2(\eta, \tau) d\eta.
\]
We estimate $|K(\tau) - K(\tau')|$. By \([3.30]\) and \([3.41]\), we have
\[
|K(\tau) - K(\tau')| = \left| \int_{\tau'}^{\tau} \frac{dK}{d\sigma} d\sigma \right| = \left| \int \int (2ww_\sigma \rho \psi^2 + w^2 \rho_\sigma \psi^2 + 2w^2 \rho \psi_\sigma) \right|
\leq 2 \int \int |w||w_\sigma| \rho \psi^2 + C \int_{\tau'}^{\tau} \bar{R} d\sigma.
\]
The Hölder inequality and \([3.40]\) with $E(\tau) = E(t)$ yield
\[
\left| \int \int |w||w_\sigma| \rho \psi^2 \leq \left( \int \int w^2 \rho \psi^2 \right)^{\frac{1}{2}} \left( \int \int w^2_\sigma \rho \psi^2 \right)^{\frac{1}{2}}
\leq \sqrt{2} \left( \int \int w^2 \rho \psi^2 \right)^{\frac{1}{2}} (E(t') - E(t) + C(\tilde{t} - t')^{\frac{2}{p}})^{\frac{1}{2}}.
\]
Computations similar to \([3.45]\) give
\[
\int \int w^2 \rho \psi^2 \leq \int_{\tilde{t}}^{\tilde{t}} (\tilde{t} - s)^{\frac{1}{p-1}} \int_{Q_R} |u|^2 (\tilde{t} - s)^{-\frac{2}{p}} e^{-\frac{R}{\tilde{t} - s}} \tilde{\phi}^2 dx ds
\leq C \int_{\tilde{t}}^{\tilde{t}} (\tilde{t} - s)^{-1} ds \leq C \log \frac{\tilde{t} - t'}{\tilde{t} - t}.
\]
Thus,
\[
|K(\tau) - K(\tau')| \leq C \left( \log \frac{\tilde{t} - t'}{\tilde{t} - t} \right)^{\frac{1}{2}} (E(t') - E(t) + C(\tilde{t} - t')^{\frac{2}{p}}) + C \int_{\tau'}^{\tau} \bar{R}(\sigma) d\sigma.
\]
The above estimates show that
\[
\int \int |w|^{p+1} \rho \psi^2 \leq C \left( \log \frac{\tilde{t} - t'}{\tilde{t} - t} \right)^{\frac{1}{2}} (E(t') - E(t) + C(\tilde{t} - t')^{\frac{2}{p}})^{\frac{1}{2}}
+ C_p (E(t') + C(\tilde{t} - t')^{\frac{2}{p}}) \log \frac{\tilde{t} - t'}{\tilde{t} - t} + C \int_{\tau'}^{\tau} \bar{R}(\sigma) d\sigma,
\]
where
\[
\bar{R}(\sigma) := \int_{\tau'}^{\tau} \bar{R}(\sigma) d\sigma.
\]
From (3.45), it follows that \( \int |w|^{p+1} \rho \psi^2 \, d\eta d\sigma \) is bounded by the right-hand side of the desired inequality. Then the lemma follows. \qed

We estimate the right-hand side of (4.8). Assume \( Q_r(x_1, t_1) \subset Q_{8A00} \). Then by \( x_1 \in \Omega_{R/4} \) and \(-1/2 < t_1 - (r^2/4) < t_1 - (r^2/16) < t_1 \leq 0\), we see that Lemma 4.7 (with \( \phi_{x_1, R/4} \) replaced by \( \phi_{x_1, R/8} \)) gives
\[
I_r \leq C \left( E_{(x_1, t_1)} \left( t_1 - \frac{r^2}{4} \right) - E_{(x_1, t_1)} \left( t_1 - \frac{r^2}{16} \right) + Cr \right)^{\frac{1}{2}}
+ C_p \left( E_{(x_1, t_1)} \left( t_1 - \frac{r^2}{4}; \phi_{x_1, R/8} \right) + Cr \right) \log 4 + Cr
\]
for \( Q_r(x_1, t_1) \subset Q_{8A00} \). We derive an upper bound of \( E_{(x_1, t_1)} \) and a lower bound of \( E_{(x_1, t_1)} \).

**Lemma 4.8.** There exists \( C > 0 \) independent of \( t_1 \) such that
\[
E_{(x_1, t_1)} \left( t_1 - \frac{r^2}{4}; \phi_{x_1, R/8} \right) \leq E_{(x_1, t_1)} \left( -4\delta^2; \phi_{x_1, R/8} \right) + C\delta
\]
for \( Q_r(x_1, t_1) \subset Q_{8A00} \).

**Proof.** By \( x_1 \in \Omega_{R/4}; -1/2 < -4\delta^2 < t_1 - r^2/4 < t_1 \leq 0 \) and Proposition 5.2 (with \( \phi_{x_1, R/4} \) replaced by \( \phi_{x_1, R/8} \)), we see that
\[
E_{(x_1, t_1)} \left( t_1 - \frac{r^2}{4}; \phi_{x_1, R/8} \right) \leq E_{(x_1, t_1)} \left( -4\delta^2; \phi_{x_1, R/8} \right) + C(t_1 + 4\delta^2)^{\frac{1}{2}}.
\]
This implies the desired inequality. \qed

**Lemma 4.9.** There exists \( C > 0 \) independent of \( t_1 \) such that
\[
E_{(x_1, t_1)} \left( t_1 - \frac{r^2}{16}; \phi_{x_1, R/8} \right) \geq -C\delta
\]
for \( Q_r(x_1, t_1) \subset Q_{8A00} \).

**Proof.** By (4.10) and (3.45), there exists a constant \( C' > 0 \) such that
\[
\int_{\tau'}^{\tau} \int_{\Omega(\sigma)} |w|^{p+1} \rho \psi^2 \, d\eta d\sigma \leq \frac{C_p}{4} \mathcal{K}(\tau) + C_p \left( E_{(\hat{x}, \hat{t})}(t') + C'(\hat{t} - t')^{\frac{1}{2}} \right) \log \frac{\hat{t} - t'}{\hat{t} - t} + C'(\hat{t} - t')^{\frac{1}{2}}
\]
for any \( \hat{x} \in \Omega_{R/4}; \tau' = -\log(\hat{t} - t') \) and \( \tau = -\log(\hat{t} - t) \) with \(-1/2 < t' < t \leq 0\). We claim that
\[
E_{(\hat{x}, \hat{t})}(t') + C'(\hat{t} - t')^{\frac{1}{2}} \geq 0
\]
for any \( \hat{x} \in \Omega_{R/4} \) and \(-1/2 < t' < \hat{t} \leq 0\). To obtain a contradiction, we suppose that there exist \( \tilde{x}_0 \in \Omega_{R/4} \) and \(-1/2 < t_0' < \tilde{t}_0 \leq 0 \) such that \( E_{(\tilde{x}_0, \tilde{t}_0)}(t_0') + C'(\tilde{t}_0 - t_0')^{1/2} < 0 \). Then by (4.12), we have
\[
\int_{t_0'}^{\tilde{t}_0} \int_{\Omega(\sigma)} |w|^{p+1} \rho \psi^2 \, d\eta d\sigma \leq \frac{C_p}{4} \mathcal{K}(\tau) + C_p \left( E_{(\tilde{x}_0, \tilde{t}_0)}(t_0') + C'(\tilde{t}_0 - t_0')^{\frac{1}{2}} \right) \log \frac{\tilde{t}_0 - t_0'}{\tilde{t}_0 - t} + C'(\tilde{t}_0 - t_0')^{\frac{1}{2}}
\]
for any $\tau = -\log(\tilde{t}_0 - t)$ with $t'_0 < t < \tilde{t}_0$, where $\tau'_0 := -\log(t_0 - t'_0)$. Therefore, there exists $t'_0 < t_\ast < \tilde{t}_0$ such that

$$
\int_{\tau'_0}^{\tau} \int_{\Omega(\sigma)} |w|^{p+1} \rho \psi^2 d\eta d\sigma \leq \frac{C_p}{4} \int_{\Omega(\tau)} w^2 \rho \psi^2 d\eta - 1
$$

for any $\tau = -\log(\tilde{t}_0 - t)$ with $t_\ast < t < \tilde{t}_0$. From the H"older inequality and (3.40), it follows that

$$
\tau \text{ varies from } \tau'_0 \text{ to } \tau = \omega \log(\tilde{t}_0).
$$

By using the above lemmas, we can estimate $I_r$ by using $E_{(x_1, t_1)}(-4\delta^2)$.

**Lemma 4.10.** There exists $C > 0$ such that

$$
I_r \leq C h \left( E_{(x_1, t_1)}(-4\delta^2; \phi_{x_1, R/\delta}) + C\delta \right)
$$

for any $Q_r(x_1, t_1) \subset Q_{8A_0\delta}$, where $h(s) := s + s^{1/2}$ for $s \geq 0$.

**Proof.** By combining (4.11) and Lemmas 4.8 and 4.9, we see that

$$
I_r \leq C \left( E_{(x_1, t_1)}(-4\delta^2) + C\delta + Cr \right) ^{\frac{1}{2}} + C_p \left( E_{(x_1, t_1)}(-4\delta^2) + C\delta + Cr \right) \log 4 + Cr
$$

$$
\leq C h \left( E_{(x_1, t_1)}(-4\delta^2) + C(\delta + r) \right)
$$

for $Q_r(x_1, t_1) \subset Q_{8A_0\delta}$, where $h(s) := s + s^{1/2}$ for $s \geq 0$. By using $r < \delta$, we obtain the desired inequality for any $Q_r(x_1, t_1) \subset Q_{8A_0\delta}$. \hfill $\Box$

We have estimated the right-hand side of (4.8), and then obtained Lemma 4.10. To prove Proposition 4.2, we estimate $E_{(x_1, t_1)}(-4\delta^2)$ by using a functional defined by

$$
J_0(t) = J_0(t; x_1, t_1, R)
$$

$$
:= \int_{\Omega_R} \left( |\nabla u(x, t)|^2 + |u(x, t)|^{p+1} \right) K_{(x_1, t_1)}(x, t) \phi_{x_1, R/\delta}^2(x) dx.
$$

**Lemma 4.11.** There exists $C > 0$ such that

$$
E_{(x_1, t_1)}(-4\delta^2; \phi_{x_1, R/\delta}) \leq C h(\delta^{\frac{1}{p+2}} J_0(t)) + C\delta
$$

for any $-9\delta^2 \leq t \leq -4\delta^2$ and $Q_r(x_1, t_1) \subset Q_{8A_0\delta}$, where $h(s) := s + s^{2/(p+1)}$ for $s \geq 0$.

**Proof.** We first claim that

$$
E_{(x_1, t_1)}(-4\delta^2) \leq E_{(x_1, t_1)}(t) + C\delta
$$

for $-9\delta^2 \leq t \leq -4\delta^2$. This inequality clearly holds for $t = -4\delta^2$, and hence it suffices to consider the case $-9\delta^2 \leq t < -4\delta^2$. Since $x_1 \in \overline{\Omega} \cap B_{R/4}$ and $-1/2 < -9\delta^2 < t < -4\delta^2 < t_1 \leq 0$, Proposition 4.2 shows the claim. Indeed,

$$
E_{(x_1, t_1)}(-4\delta^2) \leq E_{(x_1, t_1)}(t) + C(t_1 - t)^{\frac{3}{2}} \leq E_{(x_1, t_1)}(t) + C\delta.
$$
By the definition of $E$, it follows that
\[
E_{(x_1,t_1)}(t) \leq \frac{1}{2} (t_1 - t)^{\frac{p+1}{p}} \int_{\Omega_R} |\nabla u|^2 K_{(x_1,t_1)} \phi_{x_1,R/s}^2 \, dx
\]
\[+ (t_1 - t)^{\frac{4}{p+1}} \int_{\Omega_R} |u|^2 K_{(x_1,t_1)} \phi_{x_1,R/s}^2 \, dx \]
\[\leq C \delta^{\frac{2(p+1)}{p}} \int_{\Omega_R} |\nabla u|^2 K_{(x_1,t_1)} \phi_{x_1,R/s}^2 \, dx \]
\[+ C \delta^\pi \int_{\Omega_R} |u|^2 K_{(x_1,t_1)} \phi_{x_1,R/s}^2 \, dx.
\]
By the Hölder inequality, $\phi_{x_1,R/s} \leq 1$ and $\int_{\Omega_R} K \, dx = (4\pi)^{n/2}$, we have
\[
E_{(x_1,t_1)}(t) \leq C \delta^{\frac{2(p+1)}{p}} \int_{\Omega_R} |\nabla u|^2 K_{(x_1,t_1)} \phi_{x_1,R/s}^2 \, dx \]
\[+ C \left( \delta^{\frac{4}{p+1}} \int_{\Omega_R} |u|^{p+1} K_{(x_1,t_1)} \phi_{x_1,R/s}^{p+1} \, dx \right)^{\frac{1}{p+1}} \]
\[\leq C \hat{h} \left( \delta^{\frac{4}{p+1}} \int_{\Omega_R} (|\nabla u|^2 + |u|^{p+1}) K_{(x_1,t_1)} \phi_{x_1,R/s}^2 \, dx \right),
\]
where $\hat{h}(s) := s + s^{2/(p+1)}$ for $s \geq 0$. Hence the desired inequality follows. \(\square\)

We next estimate $J_0(t; x_1, t_1, R)$ uniformly for $x_1$ and $t_1$.

**Lemma 4.12.** There exists $C > 0$ such that
\[
J_0(t; x_1, t_1, R) \leq C J_0(t; 0, 20\delta^2, 2R)
\]
for any $-9\delta^2 \leq t \leq -4\delta^2$ and $Q_r(x_1, t_1) \subset Q_{8A\theta_0\delta}$. 

**Proof.** By the definition of $J_0$, it suffices to show that
\[
K_{(x_1,t_1)}(x, t) \leq C K_{(0,20\delta^2)}(x, t),
\]
\[
\phi_{x_1,R/s}(x) \leq C \phi_{0,R/s}(x),
\]
for $x \in \Omega_R$ and $-9\delta^2 \leq t \leq -4\delta^2$. Since $(x_1, t_1) \in Q_{8A\theta_0\delta}$ and $A\theta_0 < 1/32$, we have
\[
\frac{K_{(x_1,t_1)}(x, t)}{K_{(0,20\delta^2)}(x, t)} = \left( \frac{20\delta^2 - t}{t_1 - t} \right)^\frac{2}{p} \exp \left( -\frac{|x - x_1|^2}{4(t_1 - t)} + \frac{|x|^2}{4(20\delta^2 - t)} \right)
\]
\[\leq \left( \frac{20\delta^2 - t}{-(8A\theta_0\delta)^2 - t} \right)^\frac{2}{p} \exp \left( -\frac{|x - x_1|^2}{36\delta^2} + \frac{|x|^2}{96\delta^2} \right)
\]
\[\leq C \exp \left( -\frac{|x - x_1|^2}{36\delta^2} + \frac{|x|^2}{96\delta^2} \right).
\]
If $|x| \geq 10\delta$, then
\[
|x - x_1| \geq |x| - |x_1| \geq 10\delta - 8A\theta_0\delta \geq 10\delta - \delta = 9\delta,
\]
and so
\[
\frac{|x|}{|x - x_1|} \leq \frac{|x - x_1| + |x_1|}{|x - x_1|} \leq 1 + \frac{8A\theta_0\delta}{9\delta} \leq \frac{10}{9}.
\]
Hence, if $|x| \geq 10\delta$, then
\[
\frac{K_{(x_1,t_1)}(x, t)}{K_{(0,20\delta^2)}(x, t)} \leq C \exp \left( -\frac{|x - x_1|^2}{36\delta^2} + \frac{25|x - x_1|^2}{24 \cdot 81\delta^2} \right) \leq C.
\]
Proof. Due to the mean value theorem, it suffices to prove Lemma 4.13. There exists $J/R/\square$ |

In the case $|x - x_1| \leq R/8$, we have

$$|x| \leq |x - x_1| + |x_1| \leq \frac{R}{8} + 8A\delta \leq \frac{3}{16}R.$$ 

Then by the choice of $\varphi$ in the first part of Subsection 3.1, we see that $\varphi(4|x|/R) \geq \varphi(3/4) > 0$ for $|x - x_1| \leq R/8$. Thus,

$$\varphi(8|x - x_1|/R) \leq 1 \leq \frac{\varphi(4|x|/R)}{\varphi(3/4)}.$$ 

In the case $|x - x_1| \geq R/8$, we see that $\varphi(8|x - x_1|/R) = 0 \leq \varphi(4|x|/R)$. Hence (4.14) holds.

By Lemmas 4.11 and 4.12 to obtain a bound of $E_{(x_1, t_1)}(-4\delta^2)$, it suffices to estimate $J_0(t; 0, 20\delta^2, 2R)$ at some $t = \hat{t}$.

**Lemma 4.13.** There exists $-9\delta^2 \leq \hat{t} \leq -4\delta^2$ such that

$$\delta \frac{1}{\epsilon} \int_{-9\delta^2}^{-4\delta^2} J_0(s; 0, 20\delta^2, 2R)ds \leq CA^n - \frac{1}{\epsilon} + C e^{-A^2/C}.$$ 

**Proof.** Due to the mean value theorem, it suffices to prove

$$\delta \frac{1}{\epsilon} \int_{-9\delta^2}^{-4\delta^2} J_0(s; 0, 20\delta^2, 2R)ds \leq CA^n - \frac{1}{\epsilon} + C e^{-A^2/C}.$$ 

To see this, we show that

(4.15) $$J_1 := \delta \frac{1}{\epsilon} \int_{-9\delta^2}^{-4\delta^2} \int_{\Omega \cap B_{A^3}} (|\nabla u|^2 + |u|^{p+1})K_{(0, 20\delta^2)} \phi^2_{\rho_0, R/4} dxds$$

$$\leq CA^n - \frac{1}{\epsilon} + C e^{-A^2/C},$$ 

(4.16) $$J_2 := \delta \frac{1}{\epsilon} \int_{-9\delta^2}^{-4\delta^2} \int_{\Omega \setminus B_{A^3}} (|\nabla u|^2 + |u|^{p+1})K_{(0, 20\delta^2)} \phi^2_{\rho_0, R/4} dxds$$

$$\leq C e^{-A^2/C}.$$ 

As for (4.15), we recall our assumption (1.5), that is,

$$\delta \frac{1}{\epsilon} \int_{-(A\delta)^2}^{0} \int_{\Omega_{A^3}} (|\nabla u|^2 + |u|^{p+1})dxds \leq A^n - \frac{1}{\epsilon} + C e^{-A^2/C}.$$ 

Since $A > 3$, we have

$$\delta \frac{1}{\epsilon} \int_{-(A\delta)^2}^{0} \int_{\Omega_{A^3}} (|\nabla u|^2 + |u|^{p+1})dxds \leq A^n - \frac{1}{\epsilon} + C e^{-A^2/C}.$$ 

Therefore we have

$$J_1 \leq \delta \frac{1}{\epsilon} \int_{-9\delta^2}^{-4\delta^2} \int_{\Omega_{A^3}} (|\nabla u|^2 + |u|^{p+1})(20\delta^2 - s)^{-\frac{n}{2}} e^{\frac{|x|^2}{20\delta^2 - s} - \hat{t}} dxds$$

$$\leq C \delta \frac{1}{\epsilon} \int_{-9\delta^2}^{-4\delta^2} \int_{\Omega_{A^3}} (|\nabla u|^2 + |u|^{p+1})dxds \leq CA^n - \frac{1}{\epsilon} + C e^{-A^2/C}.$$
which proves (4.15). We next show (4.16). For \( x \in \Omega_R \setminus B_{A\delta} \) and \(-9\delta^2 \leq s \leq -4\delta^2\), we see that
\[
\frac{K_{(0,20\delta^2)}(x,s)}{K_{(0,21\delta^2)}(x,s)} = \left(\frac{21\delta^2 - s}{20\delta^2 - s}\right)^{\frac{3}{2}} \exp\left(-\frac{|x|^2}{4} \frac{\delta^2}{(20\delta^2 - s)(21\delta^2 - s)}\right) \leq Ce^{-A^2/C}.
\]
This together with the same computations as in Lemma 3.1, we obtain
\[
J_2 \leq Ce^{-A^2/C} \int_{-9\delta^2}^{-4\delta^2} \delta^{\frac{3}{2}} \int_{\Omega_R \setminus B_{A\delta}} (|\nabla u|^2 + |u|^{p+1})
\times K_{(0,21\delta^2)}(0,R/4) dx ds
\leq Ce^{-A^2/C} \delta^{\frac{3}{2}} \int_{-9\delta^2}^{-4\delta^2} (21\delta^2 - s)^{-\frac{p+1}{p-1}} ds \leq Ce^{-A^2/C},
\]
and so (4.16) follows. As stated in the beginning of the proof, the lemma follows. \(\square\)

We are now in a position to prove Proposition 4.5.

**Proof of Proposition 4.5.** By combining Lemmas 4.11, 4.12 and 4.13, we obtain
\[
E(x_1,t_1)(-4\delta^2; \varphi_1, R/8) \leq Ch(A^{n-\frac{4}{p+1}} \varepsilon_0 + e^{-A^2/C}) + C\delta.
\]
for any \( Q_r(x_1,t_1) \subset Q_{8A\theta_0}\delta \). This together with Lemma 4.10 shows
\[
I_r \leq Ch \left( e^{A^{n-\frac{4}{p+1}}} \varepsilon_0 + e^{-A^2/C} \right) + C\delta
\]
for any \( Q_r(x_1,t_1) \subset Q_{8A\theta_0}\delta \). Since \( h \circ \tilde{h} + h \leq C\tilde{h} \) with \( \tilde{h}(s) := s + s^{1/(p+1)} \) for \( s \geq 0 \), the desired inequality follows. \(\square\)

Finally, we complete the proof of Theorem 4.1.

**Proof of Theorem 4.1.** As explained just before Proposition 4.5, it suffices to prove that the statement (4.7) holds under the assumption (4.5). Let \( A, \varepsilon_0, \delta_0 \) and \( \theta_0 \) satisfy (4.6) with \( \delta \) replaced by \( \delta_0 \). Let \( Q_r(x_1,t_1) \subset Q_{8A\theta_0}\delta_0 \). Then by Proposition 4.5 we have
\[
I_r(x_1,t_1) \leq C\tilde{h} \left( e^{-A^2/C} \right) + Ch \left( A^{n-\frac{4}{p+1}} \varepsilon_0 \right) + C\tilde{h}(\delta_0),
\]
where \( C > 0 \) is a constant depending only on \( n, p, M, \Omega \) and \( R \). By taking (4.6) into account, we choose constants \( A, \varepsilon_0, \delta_0 \) and \( \theta_0 \) satisfying the following conditions:
\[
\begin{cases}
A > 3 & \text{with } C\tilde{h} \left( e^{-A^2/C} \right) < \frac{\varepsilon_1}{3}, \\
0 < \varepsilon_0 < 1 & \text{with } Ch \left( A^{n-\frac{4}{p+1}} \varepsilon_0 \right) < \frac{\varepsilon_1}{3}, \\
0 < \delta_0 < \frac{R}{16A} & \text{with } C\tilde{h}(\delta_0) < \frac{\varepsilon_1}{3}, \\
0 < \theta_0 < \frac{1}{32A}.
\end{cases}
\] (4.17)

Here \( \varepsilon_1 \) is given in Lemma 4.4. Remark that \( \varepsilon_0, \delta_0 \) and \( \theta_0 \) can be chosen independently.

Finally, we assume that there exists \( 0 < \delta \leq \delta_0 \) such that (4.5) holds. Then we obtain \( I_r(x_1,t_1) \leq \varepsilon_1 \) for any \( Q_r(x_1,t_1) \subset Q_{8A\theta_0}\delta \). Hence the statement (4.7) holds under the assumption (4.5), and so Lemma 4.4 shows that \( \|u\|_{L^\infty(Q_{8A\theta_0}\delta)} \leq \)
Then we can conclude that the original assumption (4.1) implies (4.2). The proof of Theorem 4.1 is complete. □

5. Proof of main theorem

We first prove a localized statement of Theorem 1.1. At the level of strategy, the proof is based on [35] and [101, Theorem 9.8] for the Navier-Stokes equations, and [61, Chapter 8] and [97, 102] for the harmonic map heat flow. Indeed, we use compactness, backward uniqueness and unique continuation. However, the analysis in each of the steps seems to be more involved.

Theorem 5.1 (Localized statement). Let \( n \geq 3 \), \( p > p_s \), \( \Omega \) be any \( C^{2+\alpha} \) domain in \( \mathbb{R}^n \) and \( u \) be a classical \( L^q_c \)-solution of (1.1) with \( u_0 \in L^q_c(\Omega) \). If the maximal existence time \( T > 0 \) is finite and \( u \) has a blow-up point \( a \in \overline{\Omega} \), then for any \( r > 0 \),

\[
\limsup_{t \to T} \| u(\cdot, t) \|_{L^q_c(\Omega)} = \infty.
\]

Note that if \( \Omega \) is bounded, then Theorem 1.1 immediately follows from this theorem. The unbounded case will be considered in Subsection 5.4.

5.1. Existence of blow-up limit. To prove Theorem 5.1, we set up notation, and then we define rescaled solutions and take the limit. Let \( u \) be a classical \( L^q_c \)-solution of (1.1). Assume that \( a \in \overline{\Omega} \) is a blow-up point of \( u \). By translation and scaling, we may assume that \( a = 0 \) and \( u \) satisfies

\[
\begin{cases}
u_t = \Delta u + |u|^{p-1} u & \text{in } \Omega_{R_1} \times (-1, 0), \\
u = 0 & \text{on } (\partial \Omega \cap B_{R_1}) \times (-1, 0),
\end{cases}
\]

for some \( 0 < R_1 < 1 \). Here we assume that \( t = 0 \) is the blow-up time. Suppose, contrary to Theorem 5.1, that

\[
\sup_{-1 < t < 0} \| u(\cdot, t) \|_{L^q(\Omega_{R_2})} \leq M
\]

for some \( M > 0 \) and \( 0 < R_2 < 1 \). Fix \( 0 < R < \min\{R_1, R_2\} \) so small that (3.1) and (3.2) hold. Note that \( u \) satisfies (2.1) and (2.2).

From the parabolic regularity estimates in Lemma A.2, it follows that

\[
\| u \|_{W^{1, l}([-1/4, 0]; L^r(\Omega_{R/2}))} \leq C(M + M^p)
\]

for \( 1 \leq l < \infty \) and \( 1 \leq r \leq q_c/p \). Hence after a redefinition of a zero set in the time interval, we may assume \( u \in C([-1/4, 0]; L^r(\Omega_{R/2})) \). This together with the uniform bound (2.2) gives

(5.2) \[ u \in C_{\text{weak}}([-1/4, 0]; L^q_c(\Omega_{R/2})). \]

By the contraposition of Theorem 4.1 there exist \( \varepsilon_0 > 0 \) and \( 0 < \delta_k < 1/2 \) with \( \delta_k \to 0 \) as \( k \to \infty \) such that

\[
\delta_k \varepsilon_0^{n/(p-1)} \int_{\Omega_{R/2}} (|\nabla u|^2 + |u|^{p+1}) \, dx \, dt > \varepsilon_0.
\]

To take a blow-up limit of \( u \), we define rescaled solutions and derive the corresponding equations. In the case (3.1), we define

\[
u_k(x, t) := \delta_k^{2p-n} u(\delta_k x, \delta_k^2 t).
\]
Then,
\[
\begin{cases}
(u_k)_t - \Delta u_k = |u_k|^{p-1} u_k & \text{in } \delta^{-1}_k \Omega_R \times (-\delta^{-2}_k, 0), \\
u_k = 0 & \text{on } \delta^{-1}_k(\partial \Omega \cap B_R) \times (-\delta^{-2}_k, 0).
\end{cases}
\]
In the case (3.2), we define
\[
\rho > (3.1)
\]
is easier to handle than the case (3.2), we focus on (3.2). For all \(u\), equality for (3.1) is taken so that 0 < \(\delta_k < 1/2\). For \(\rho > 0\), we set \(B_\rho^+ := \mathbb{R}^+ \cap B_\rho\). To take a limit of \(u_k\), we give estimates of rescaled solutions uniformly for \(k \geq k_\rho\). Here \(k_\rho \geq 1\) is taken so that 0 < \(\delta_k < 1/2\), \(\Psi(B_{\delta_k/\rho}^+ \cap \Omega_{R/2})\) and \(\Psi(B_{\delta_k/\rho}^- \cap \Omega_{R/2})\) for all \(k \geq k_\rho\). Since \(\nabla'f(0) = 0, f \in C_0^{2+\alpha}(\mathbb{R}^n-1)\) and \(\delta_k \to 0\) as \(k \to \infty\), we have
\[
|\nabla'f_k(x')| = |\nabla'f(\delta_k x')| \to 0,
\]
\[
\|\Delta'f_k\|_{L^\infty(\mathbb{R}^{n-1})} \leq \delta_k \|\Delta'f\|_{L^\infty(\mathbb{R}^{n-1})} \to 0.
\]
We first give uniform estimates of \(u_k\) and \(\nabla u_k\).

**Lemma 5.2.** Assume (3.2). Let \(\rho > 0\). Then there exists \(C > 0\) such that the rescaled functions \(u_k\) satisfy
\[
\sup_{-1 < t < 0} \|u_k(\cdot, t)\|_{L^\infty(B_\rho^+)} \leq M,
\]
\[
\sup_{-1 < t < 0} \|\nabla u_k(\cdot, t)\|_{L^{\infty,\infty}(B_\rho^+)} \leq C(M + M^p),
\]
for all \(k \geq k_\rho\), where \(C\) depends on \(R\) and is independent for \(\rho\) and \(k\).

**Proof.** By the change of variables and the choice of \(k_\rho\), we can easily see the inequality for \(u_k\). From the following computation
\[
|\nabla u_k(x, t)| = \delta_k^{-1}|\nabla u(\Psi(\delta_k x), \delta_k^2 t) + \partial_{x_n} u(\Psi(\delta_k x), \delta_k^2 t) \nabla' f(\delta_k x')| \\
\leq \delta_k^{-1}(1 + \|\nabla f\|_{L^\infty(\mathbb{R}^{n-1})})|\nabla u(\Psi(\delta_k x), \delta_k^2 t)|
\]
and Proposition 2.1 it follows that
\[
\|\nabla u_k(\cdot, t)\|_{L^{\infty,\infty}(B_\rho^+)} \leq (1 + \|\nabla f\|_{L^\infty(\mathbb{R}^{n-1})}) \|\nabla u(\cdot, \delta_k^2 t)\|_{L^{\infty,\infty}(\Omega_{R/2})} \\
\leq C(M + M^p)
\]
for \(-1 < t < 0\). Then the lemma follows.

We next give a uniform parabolic regularity estimate.
Lemma 5.3. Assume \( \text{[5.2]} \). Let \( 1 \leq l < \infty \), \( 1 \leq r \leq q_c/p \) and \( \rho > 0 \). Then there exists \( C > 0 \) such that the rescaled functions \( u_k \) satisfy
\[
\|(u_k)\|_{L^l(-1,0;L^r(B^+_{\rho}))} + \|\nabla^2 u_k\|_{L^l(-1,0;L^r(B^+_{\rho}))} \leq C(M + MP)
\]
for all \( k \geq k_\rho \), where \( C \) depends on \( \rho \) and \( R \) and is independent of \( k \).

Proof. By direct computations, we have
\[
\|(u_k)\|_{L^l(-1,0;L^r(B^+_{\rho}))} + \|\nabla^2 u_k\|_{L^l(-1,0;L^r(B^+_{\rho}))} = \frac{2}{\rho^l} + \frac{2}{\rho^r}(\|\hat{u}_t\|_{L^l(-1,0;L^r(B^+_{\rho}))} + \|\nabla^2 \hat{u}\|_{L^l(-1,0;L^r(B^+_{\rho}))}).
\]

This together with Lemma A.1 shows the lemma. \( \square \)

Finally in this subsection, we show the existence of a blow-up limit. Let \( \rho > 0 \). To avoid technicalities due to the corner of \( \partial B^+_{\rho} \), we prepare a smooth domain \( B_{\rho} \) satisfying \( B^+_{\rho/2} \subset B_{\rho} \subset B^+_{\rho} \).

Lemma 5.4. Assume \( \text{[5.2]} \). There exist a subsequence still denoted by \( u_k \) and a function \( \overline{u} \) defined on \( \mathbb{R}^n_+ \times [-1,0] \) satisfying the following statements:

(i) \( u_k \to \overline{u} \) strongly in \( L^\infty(-1,0;W^{1,2}(B_{\rho})) \) for each \( \rho > 0 \) as \( k \to \infty \).

(ii) \( u_k \to \overline{u} \) strongly in \( L^\infty(-1,0;L^{p+1}(B_{\rho})) \) for each \( \rho > 0 \) as \( k \to \infty \).

(iii) \( \|\overline{u}\|_{L^\infty(-1,0;L^\infty(\mathbb{R}^n_+))} \leq M \).

(iv) \( \|\nabla \overline{u}\|_{L^\infty(-1,0;L^{\infty}\cap W^{-1,\infty}(\mathbb{R}^n_+))} \leq C(M + MP) \).

Here \( M \) is the constant in \( \text{(5.1)} \) and \( C > 0 \) is a constant depending on \( R \).

Proof. Let \( 1 < r < \min\{2,q_c/p\} \). By a consequence of an Aubin-Lions type compactness result (see Lemma [5.2]), we have
\[
W^{1,5}(-1,0;L^r(B_{\rho})) \cap L^5(-1,0;W^{2,r}(B_{\rho})) \hookrightarrow C([-1,0];W^{1,r}(B_{\rho}))
\]
and this embedding is compact. Therefore, the uniform bounds in Lemma 5.3 show that there exists a subsequence still denoted by \( \{u_k\} \) satisfying \( u_k \to \overline{u} \) in \( C([-1,0];W^{1,r}(B_{\rho})) \) as \( k \to \infty \). By Lemma 5.2 the rescaled functions \( u_k \) also satisfy the following uniform bounds:
\[
\sup_{-1 < t < 0} \|u_k(\cdot,t)\|_{L^\infty(B_{\rho})} \leq M,
\]
\[
\sup_{-1 < t < 0} \|\nabla u_k(\cdot,t)\|_{L^{\infty}\cap W^{-1,\infty}(B_{\rho})} \leq C(M + MP),
\]
for all \( k \geq k_\rho \), where \( C \) depends on \( R \) and is independent of \( \rho \) and \( k \). Hence the standard interpolations give
\[
\|u_k - \overline{u}\|_{L^\infty(-1,0;L^r)} \leq \|u_k - \overline{u}\|_{L^\infty(-1,0;L^r)}^{\theta_1} \|u_k - \overline{u}\|_{L^\infty(-1,0;L^r)}^{1-\theta_1} \leq CM\|u_k - \overline{u}\|_{L^\infty(-1,0;L^r)}^{\theta_1},
\]
\[
\|\nabla(u_k - \overline{u})\|_{L^\infty(-1,0;L^2)} \leq \|\nabla(u_k - \overline{u})\|_{L^\infty(-1,0;L^r)}^{\theta_2} \|\nabla(u_k - \overline{u})\|_{L^\infty(-1,0;L^r)}^{1-\theta_2} \leq C(M + MP)\|\nabla(u_k - \overline{u})\|_{L^\infty(-1,0;L^r)}^{\theta_2},
\]
where \( L^r_{\rho} := L^r(B_{\rho}) \) and \( L^{p,\infty}_{\rho} := L^{p,\infty}(B_{\rho}) \). These imply (i) and (ii). By the lower semicontinuity of the weak limit, we see that
\[
\|\overline{u}(\cdot,t)\|_{L^\infty(B_{\rho})} \leq \liminf_{k \to \infty} \|u_k(\cdot,t)\|_{L^\infty(B_{\rho})} \leq M
\]
for $-1 < t < 0$. Thus, letting $\rho \to \infty$ gives (iii). Since the constant $C$ in (5.8) is independent of $\rho$, we also obtain (iv). The proof is complete. □

**Remark 5.5.** By Lemma 5.3 up to subsequence still denoted by $u_k$, we can check that $(u_k)_{t} \to \overline{u}_t$ weakly in $L^p(B_{\rho} \times (-1, 0))$ for each $\rho > 0$ as $k \to \infty$.

5.2. **Blow-up analysis.** We continue to study the case (3.2). For (3.1), see Remark 5.12. By using Lemma 5.4, we show several properties of the blow-up limit $\overline{u}$. For $\tilde{x} \in \mathbb{R}^n_+$ and $-1 < t < \tilde{t} \leq 0$, define a global weighted energy by

$$E_{(\tilde{x}, \tilde{t})}(t) := (\tilde{t} - t) \int_{\mathbb{R}^n_+} \left( \frac{|\nabla u|^2}{2} - \frac{|\overline{u}|^{p+1}}{p+1} + \frac{\overline{u}^2}{2(p-1)(t-t)} \right) K_{(\tilde{x}, \tilde{t})}(x,t) dx.$$ 

Note that the same computations as in Lemma 5.1 together with Lemma 5.4 (iii) and (iv) yield

$$|E_{(\tilde{x}, \tilde{t})}(t)| \leq C(M + M^p)^2$$

for any $\tilde{x} \in \mathbb{R}^n_+$ and $-1 < t < \tilde{t} \leq 0$.

We first show that $E$ is a scaling limit of $E$.

**Lemma 5.6.** Assume (3.2). Then, for each $\tilde{x} \in \mathbb{R}^n_+$ and $-1 < t < \tilde{t} \leq 0$,

$$E_{(\delta_k \tilde{x}, \delta_k^2 \tilde{t}; \delta_k \tilde{x}, R/4)}(\delta_k^2 t; \delta_k \tilde{x}, R/4) \to E_{(\tilde{x}, \tilde{t})}(t) \quad \text{as } k \to \infty.$$

**Proof:** Fix $\tilde{x} \in \mathbb{R}^n_+$ and $-1 < t < \tilde{t} \leq 0$. We take $k$ so large that $\tilde{x} \in \delta_k^{-1} B_{R/4}$ and $-(1/2)\delta_k^{-2} < t < 0$. By the definition of $E$ in (5.3) and the change of variables, we have

$$E_{(\delta_k \tilde{x}, \delta_k^2 \tilde{t})}(\delta_k^2 t; \delta_k \tilde{x}, R/4) = (\tilde{t} - t) \int_{\delta_k^{-1} \Phi(\Omega_R)} \left( \frac{|\tilde{\nabla} u_k|^2}{2} - \frac{|u_k|^{p+1}}{p+1} + \frac{|u_k(x,t)|^2}{2(p-1)(t-t)} \right) \times \exp \left( -\frac{|\delta_k^{-1} \Psi(\delta_k x) - \tilde{x}|^2}{4(t-t)} \right) \varphi^2 \left( \frac{4}{R} \Psi(\delta_k x) - \delta_k \tilde{x} \right) dx,$$

where $\tilde{\nabla} u_k := (\nabla' u_k - (\partial_n u_k) \nabla' f_k, \partial_n u_k)$ and $f_k(x') = \delta_k^{-1} f(\delta_k x')$, see (5.4). In what follows, we focus on the convergence of the first term, since the other terms can be handled in the same manner. Namely, we prove that

$$\int_{\delta_k^{-1} \Phi(\Omega_R)} |\tilde{\nabla} u_k|^2 e^{-\frac{|\delta_k^{-1} \Psi(\delta_k x) - \tilde{x}|^2}{4(t-t)}} \varphi^2(x) dx - \int_{\mathbb{R}^n_+} |\nabla \overline{u}|^2 e^{-\frac{|x-\tilde{x}|^2}{4(t-t)}} dx$$

$$=: J^k_1 + J^k_2 + J^k_3 + J^k_4 \to 0$$

for $-1 < t < 0$. Thus, letting $\rho \to \infty$ gives (iii). Since the constant $C$ in (5.8) is independent of $\rho$, we also obtain (iv). The proof is complete. □
as \( k \to \infty \), where \( \varphi_k(x) := \varphi(4\delta_k|\delta_k^{-1}\Psi(\delta_k x) - \bar{x}|/R) \) and

\[
J_1^k := \int_{\delta_k^{-1}\Phi(\Omega_R)} (|\nabla u_k|^2 - |\nabla \mathbf{v}|^2) e^{-\frac{|(4\delta_k^{-1}\Psi(\delta_k x) - \bar{x})|^2}{4(t-\bar{t})}} \varphi_k^2 \, dx,
\]

\[
J_2^k := \int_{\delta_k^{-1}\Phi(\Omega_R)} |\nabla \mathbf{v}|^2 e^{-\frac{|(4\delta_k^{-1}\Psi(\delta_k x) - \bar{x})|^2}{4(t-\bar{t})}} (\varphi_k^2 - 1) \, dx,
\]

\[
J_3^k := \int_{\delta_k^{-1}\Phi(\Omega_R)} |\nabla \mathbf{w}|^2 e^{-\frac{|(4\delta_k^{-1}\Psi(\delta_k x) - \bar{x})|^2}{4(t-\bar{t})}} - e^{-\frac{|(4\delta_k^{-1}\Psi(\delta_k x) - \bar{x})|^2}{4(t-\bar{t})}} \, dx,
\]

\[
J_4^k := -\int_{R_x^+\delta_k^{-1}\Phi(\Omega_R)} |\nabla \mathbf{v}|^2 e^{-\frac{|(4\delta_k^{-1}\Psi(\delta_k x) - \bar{x})|^2}{4(t-\bar{t})}} \, dx.
\]

Remark that the resultant convergence in (3.10) is pointwise for \( \bar{x}, t \) and \( \bar{t} \), but this does not cause any problems for the proof of this lemma.

Let us estimate \( J_1^k \). For \( R_1 > 0 \), we estimate and set

\[
|J_1^k| \leq \left( \int_{B_{R_1}} + \int_{R_x^+ \setminus B_{R_1}} \right) |\nabla u_k|^2 - |\nabla \mathbf{v}|^2 e^{-\frac{|(4\delta_k^{-1}\Psi(\delta_k x) - \bar{x})|^2}{4(t-\bar{t})}} \varphi_k^2 \, dx
\]

\[
= J_1^{in} + J_1^{out}.
\]

We consider \( J_1^{out} \). From (5.2) and \(|f(\delta_k x')| \leq 2^{-1} \delta_k|x'| \), it follows that

\[
|\delta_k^{-1}\Psi(\delta_k x) - \bar{x}|^2 = |x + (0, \delta_k^{-1}f(\delta_k x')) - \bar{x}|^2 \geq \frac{1}{8} |x|^2 - |\bar{x}|^2.
\]

Thus,

\[
J_1^{out} \leq e^{\frac{|x|^2}{8(t-\bar{t})}} e^{-\frac{(R_1)^2}{8(t-\bar{t})}} \int_{R_x^+} (|\nabla u_k|^2 + |\nabla \mathbf{v}|^2) e^{-\frac{|(4\delta_k^{-1}\Psi(\delta_k x) - \bar{x})|^2}{4(t-\bar{t})}} \varphi_k^2 \, dx
\]

\[
\leq e^{\frac{|x|^2}{8(t-\bar{t})}} e^{-\frac{(R_1)^2}{8(t-\bar{t})}} \int_{\delta_k^{-1}\Phi(\Omega_R)} |\nabla u_k|^2 e^{-\frac{|\delta_k^{-1}\Phi(\delta_k x) - \bar{x}|^2}{8(t-\bar{t})}} \varphi_k^2 \, dx
\]

\[
+ e^{\frac{|x|^2}{8(t-\bar{t})}} e^{-\frac{(R_1)^2}{8(t-\bar{t})}} \int_{R_x^+} |\nabla \mathbf{v}|^2 e^{-\frac{|(4\delta_k^{-1}\Psi(\delta_k x) - \bar{x})|^2}{4(t-\bar{t})}} \varphi_k^2 \, dx
\]

for \(-1 < t - \bar{t} \leq 0\). The second term in the right-hand side can be estimated by computations similar to (3.3) with the aid of Lemma (3.4) (iv). As for the first term, we go back to the original variables. Then by supp \( \varphi^2(4|R| \cdot \delta_k \bar{x}) \subset B_{R/4}(\delta_k \bar{x}) \), \( \bar{x} \in \delta_k^{-1}B_{R/4} \), \( -(1/2)\delta_k^{-2} < t - \bar{t} \leq 0 \) and computations similar to (3.3) with (5.1), we see that

\[
\int_{\delta_k^{-1}\Phi(\Omega_R)} |\nabla u_k(x, t)|^2 e^{-\frac{|\delta_k^{-1}\Phi(\delta_k x) - \bar{x}|^2}{8(t-\bar{t})}} \varphi_k^2(x) \, dx
\]

\[
= \delta_k^{2(p+1)-n} \int_{\Omega_R} |\nabla u(y, \delta_k^2 t)|^2 e^{-\frac{|y - \delta_k \bar{x}|^2}{8(t-\bar{t})}} \varphi^2 \left( \frac{4}{R} |y - \delta_k \bar{x}| \right) \, dy
\]

\[
\leq \delta_k^{2(p+1)-n} \int_{\Omega_R/2} |\nabla u(y, \delta_k^2 t)|^2 e^{-\frac{|y - \delta_k \bar{x}|^2}{8(t-\bar{t})}} \, dy \leq C(M + M^p)^2 (i - t) \frac{-p+1}{p+1},
\]

where \( C > 0 \) is independent of \( k \). Thus,

\[
J_1^{out} \leq Ce^{\frac{|x|^2}{8(t-\bar{t})}} (i - t)^{-\frac{p+1}{p-1}} e^{-\frac{(R_1)^2}{8(t-\bar{t})}}.
\]
We examine \( J^k_{\text{in}} \). By using \( \hat{\nabla} u_k = (\nabla' u_k - (\partial_{n \cdot u_k}) \nabla f_k, \partial_{n \cdot u_k}) \) and \( f_k(x') = \delta_k^{-1} f(\delta_k x') \), we have

\[
|\hat{\nabla} u_k| = |\nabla u_k - ((\partial_{n \cdot u_k}) \nabla f(\delta_k x'), 0)| 
\leq (1 + |\nabla' f(\delta_k x')|)|\nabla u_k|, \\
|\hat{\nabla} u_k - \nabla \eta|^2 
\leq C |\nabla u_k - \nabla \eta|^2 + C |\nabla u_k|^2 |\nabla' f(\delta_k x')|^2.
\]

These estimates together with the Hölder inequality show that

\[
J^k_{\text{in}} \leq \int_{B_{R_1}^+} |\nabla u_k + \nabla \eta||\nabla u_k - \nabla \eta| \chi_{\delta_k^{-1}\Phi(\Omega_R)}^1 \, dx
\leq \left( \int_{B_{R_1}^+} ((1 + |\nabla' f(\delta_k x')|)|\nabla u_k| + |\nabla \eta|^2 \chi^2 dx \right)^{\frac{1}{2}}
\times \left( C \int_{B_{R_1}^+} |\nabla u_k - \nabla \eta|^2 \chi \, dx + C \int_{B_{R_1}^+} |\nabla u_k|^2 |\nabla' f(\delta_k x')|^2 \chi \, dx \right)^{\frac{1}{2}}
\]

where \( \chi := \chi_{\delta_k^{-1}\Phi(\Omega_R)} \). By \( f \in C^{2+\alpha}(\mathbb{R}^{n-1}) \), (5.8) and Lemma 5.4 (iv), the first integral in the right-hand side is bounded by a constant. In addition, Lemma 5.4 (i) guarantees that the second integral converges to 0 as \( k \to \infty \). Therefore, from \( \nabla' f(0) = 0 \) and the Lebesgue dominated convergence theorem, it follows that

\[
\limsup_{k \to \infty} J^k_{\text{in}} \leq C \limsup_{k \to \infty} \left( \int_{B_{R_1}^+} |\nabla u_k|^2 |\nabla' f(\delta_k x')|^2 \chi_{\delta_k^{-1}\Phi(\Omega_R)}^1 \, dx \right)^{\frac{1}{2}} = 0.
\]

Hence the above estimates for \( J^k_{\text{in}} \) and \( J^k_{\text{out}} \) show that

\[
\limsup_{k \to \infty} |J^k| \leq Ce^{\frac{|x|^2}{4(R-t)}} (t - t \frac{1}{R} + e^{-\frac{(R')^2}{4t}}) \to 0
\]
as \( R' \to \infty \), and so \( \lim_{k \to \infty} J^k = 0 \).

We consider \( J^k_2 \). For \( R'_2 > 0 \), we note that

\[
|\delta_k^{-1} \Phi(\delta_k x) - \tilde{x}| \leq |x| + \frac{1}{2}|x'| + |\tilde{x}| \leq \frac{3}{2} R'_2 + |\tilde{x}|
\]
if \( x \in B_{R_2} \). This together with (5.11), \( 0 \leq \varphi_k \leq 1 \) and \( \varphi' \leq 0 \) shows that

\[
\limsup_{k \to \infty} |J^k_2| \leq \limsup_{k \to \infty} \left( 1 - \varphi^2 \left( \frac{4\delta_k}{R} \left( \frac{3}{2} R'_2 + |\tilde{x}| \right) \right) \right) \int_{B_{R_2}^+} |\nabla u|^2 \, dx
\]
\[
+ e^{\frac{|x|^2}{4(R-t)}} \int_{R_2^+ \setminus B_{R_2}^+} |\nabla \eta|^2 e^{-\frac{|x|^2}{32(R-t)}} \, dx
\]

for \(-1 < t < \hat{t} \leq 0 \). Hence the inner part converges to 0 as \( k \to \infty \), and then letting \( R'_2 \to \infty \) yields \( \lim_{k \to \infty} J^k_2 = 0 \).

As for \( J^k_3 \), by (5.11), we have

\[
|J^k_3| \leq \int_{B_{R_3}^+} |\nabla \eta|^2 \left( e^{-\frac{|x|^2}{4(R-t)}} - e^{-\frac{|x|^2}{4(R-t)}} \right) \, dx
\]
\[
+ \int_{R_3^+ \setminus B_{R_3}^+} |\nabla \eta|^2 \left( e^{-\frac{|x|^2}{4(R-t)}} + e^{-\frac{|x|^2}{4(R-t)}} + e^{-\frac{|x|^2}{4(R-t)}} \right) \, dx.
\]
The outer part can be handled in the same way as above. We can also check that the inner part converges to 0 as \( k \to \infty \) by
\[
\delta_k^{-1} \Psi(\delta_k x) = x + (0, \delta_k^{-1} f(\delta_k x')) = x + (0, \nabla' f(\theta_x, \delta_k x')) \to x
\]
for some \( 0 \leq \theta_x \leq 1 \) as \( k \to \infty \). Therefore \( \lim_{k \to \infty} J_k^k = 0 \). By Lemma 5.7 (iv), we can easily see that \( \lim_{k \to \infty} J_k^k = 0 \). Hence we obtain (5.10).

We next estimate an integral concerning \( |\hat{w}|^{p+1} \) by using \( \hat{E}(\hat{z}, \hat{t}) \). More precisely, we prove an analog of Lemma 4.7 for the blow-up limit \( \hat{z} \).

**Lemma 5.7.** Assume (3.2). Then there exists \( C > 0 \) such that
\[
\int_t^\infty (\hat{t} - s)^{\frac{p+1}{2}} \int_{R_+^n} |\hat{w}|^{p+1} K(\hat{z}, \hat{t})(x, s) dx ds \\
\leq C \left( \log \frac{\hat{t} - t'}{\hat{t} - t} \right)^\frac{1}{2} (\hat{E}(\hat{z}, \hat{t})(t') - \hat{E}(\hat{z}, \hat{t})(t) + C(\hat{t} - t')^{\frac{1}{2}}) \\
+ C_p(\hat{E}(\hat{z}, \hat{t})(t') + C(\hat{t} - t')^{\frac{1}{2}}) \log \frac{\hat{t} - t'}{\hat{t} - t} + C(\hat{t} - t')^{\frac{1}{2}}
\]
for \( \hat{z} \in R_+^n \) and \(-1 < t' < t < \hat{t} \leq 0 \). Here \( C_p := 2(p+1)/(p-1) \) and \( h(s) := s^{s/2} \) for \( s \geq 0 \).

**Proof.** Define \( \hat{\varphi} \in C(0, \infty) \) so that \( \varphi' \leq 0, 0 \leq \hat{\varphi} \leq 1, \hat{\varphi}(z) = 0 \) for \( z > 2 \) and \( \hat{\varphi}(z) = 1 \) for \( 0 \leq z < 1 \). For \( R > 1 \), set \( \hat{\varphi}_R(z) := \hat{\varphi}(z/R) \). For \( \hat{z} \in R_+^n \) and \(-1 < t < \hat{t} \leq 0 \), we define
\[
E_k(t) := (\hat{t} - t)^{\frac{p+1}{2} - \frac{p}{4}} \int_{\delta_k^{-1} \Omega_R} \left( \frac{|\nabla u_k(\xi, t)|^2}{2} - \frac{|u_k|^{p+1}}{p+1} + \frac{(\hat{t} - t)^{-1}|u_k|^2}{2(p-1)} \right) \\
\times e^{-\frac{|\varphi_k(\xi) - \varphi_k(\xi_k)|^2}{4(\hat{t} - t)}} \varphi^2 \left( \frac{4\delta_k}{R} |\Psi_k(\xi) - \Psi_k(\tilde{\xi}_k)| \right) \hat{\varphi}_R^2(\xi) d\xi,
\]
where \( \Psi_k(\xi) := \delta_k^{-1} \Psi(\delta_k \xi), \Phi_k(\zeta) := \delta_k^{-1} \Phi(\delta_k \zeta) \) and \( \tilde{\xi}_k := \Phi_k(\hat{z}) \). We note that \( \Psi_k(\hat{\xi}_k) = \hat{\xi} \) and that the right-hand side of this definition coincides with \( E(\delta_k \hat{z}, \delta_k^2 \hat{t}) \) if we replace \( \hat{\varphi}_R \) with 1.

By using the backward similarity variables \( \eta : (\xi - \hat{\xi}_k)/(\hat{t} - t)^{1/2} \) and \( \tau := -\log(\hat{t} - t) \), we define
\[
w_k(\eta, \tau) := e^{-\frac{1}{4}} e^{-\frac{\tau}{4}} u_k(\hat{\xi}_k + e^{-\frac{\tau}{4}} \eta, \hat{t} - e^{-\tau}),
\]
\[
g_k(\eta, \tau) := e^{\frac{\tau}{4}} f_k(\hat{\xi}_k + e^{-\frac{\tau}{4}} \eta').
\]
Then \( w_k \) satisfies
\[
\begin{cases}
(w_k)_\tau + \frac{1}{2} \eta \cdot \nabla w_k + \frac{1}{2} w_k - A_k w_k - |w_k|^{p-1} w_k = 0, \\
\eta \in \Omega_k(\tau) := e^{\tau/2}(\Phi_k(\delta_k^{-1} \Omega_R) - \hat{z}), \tau \in (-\log(\hat{t} + 1), \infty), \\
w_k = 0, \quad \eta \in e^{\tau/2}(\Phi_k(\delta_k^{-1} (\partial \Omega \cap B_R)) - \hat{\xi}_k),
\end{cases}
\]
where $A_kw := \Delta w_k - 2\nabla'(\partial_n w_k) \cdot \nabla' g_k + (\partial^2_n w_k) |\nabla' g_k|^2 - (\partial_n w_k) \Delta g_k$. Setting $
abla w_k := (\nabla' w_k - (\partial_n w_k) \nabla' g_k, \partial_n w_k)$ and

$$\mathcal{E}_k(\tau) := \int_{\Omega_k(\tau)} \left( \frac{\nabla w_k^2}{2} - \frac{|w_k|^{p+1}}{p+1} + \frac{w_k^2}{2(p-1)} \right) \rho_k \psi_k^2 d\eta,$$

$$\rho_k = \rho_k(\eta, \tau) := \exp \left( - \frac{1}{4} (|\eta|^2 + 2(g_k(\eta, \tau) - g_k(0, \tau)) \eta, + (g_k(\eta, \tau) - g_k(0, \tau))^2) \right),$$

$$\psi_k = \psi_k(\eta, \tau) := \varphi \left( \frac{4|\Psi_k(\xi_k + e^{-\gamma/2} \eta) - \Psi_k(\xi_k)|}{R} \right) \hat{\varphi}_R(|\xi_k + e^{-\gamma} \eta|),$$

we can check that $E_k(t) = \mathcal{E}_k(\tau)$ with $\tau = -\log(t - \hat{t})$. In addition, we can observe that the above situation is almost the same as in Section 3 except for the appearance of $\hat{\varphi}_R$.

By the choice of $\hat{\varphi}_R$ and $\hat{R} > 1$, we have

$$|\nabla(\hat{\varphi}_R(|\xi_k + e^{-\gamma} \eta|))| \leq C|\nabla' \hat{\varphi}_R(|\xi_k + e^{-\gamma} \eta|)|e^{-\gamma} \leq C e^{-\gamma},$$

$$|\partial_\tau(\hat{\varphi}_R(|\xi_k + e^{-\gamma} \eta|))| \leq C|\hat{\varphi}_R(|\xi_k + e^{-\gamma} \eta|)|\eta| e^{-\gamma} \leq C|\eta| e^{-\gamma},$$

where $C > 0$ is independent of $k$ and $\hat{R}$. Then the same computations as in Section 3 show that there exists a constant $C > 0$ independent of $k$ and $\hat{R}$ satisfying

$$\mathcal{E}_k(\tau) + \frac{1}{2} \int_{\tau}^{\hat{t}} \int_{\Omega_k(\sigma)} (w_k^2)_{\sigma} \rho_k \psi_k^2 d\eta d\sigma \leq \mathcal{E}_k(\tau') + C(\hat{t} - t')^{1/2}.$$

Therefore, the same computations as in Lemma 3.7 yield

$$\int_{t'}^{\hat{t}} (\hat{t} - s)^{p+1/2/4} \int_{\delta_{\hat{R}}(\sigma; \Omega_R)} |u_k|^{p+1} \exp \left( -\frac{|\delta_k^{-1} \Psi(\delta_k x) - \hat{x}|^2}{4(\hat{t} - s)} \right) \times \varphi^2 \left( \frac{4}{\hat{R}} |\Psi(\delta_k x) - \delta_k \hat{x}| \right) \hat{\varphi}_R(|x|) dx ds$$

$$\leq C \left( \log \frac{\hat{t} - t'}{\hat{t} - t} \right)^{1/2} (E_k(t') - E_k(t) + C(\hat{t} - t')^{1/2})$$

$$+ C_p (E_k(t') + C(\hat{t} - t')^{1/2}) \log \frac{\hat{t} - t'}{\hat{t} - t} + C(\hat{t} - t')^{1/2},$$

where $C_p := 2(p+1)/(p-1)$ and $C > 0$ is independent of $k$ and $\hat{R}$. By Lemma 5.4 (ii), the same argument as in Lemma 5.6 and letting $k \to \infty$, we have

$$\int_{t'}^{\hat{t}} (\hat{t} - s)^{p+1/2} \int_{\mathbb{R}^n} |u_k|^{p+1} K(\xi, \hat{x}, \xi)(x, s) \hat{\varphi}_R(|x|) dx ds$$

$$\leq C \left( \log \frac{\hat{t} - t'}{\hat{t} - t} \right)^{1/2} (E\hat{\varphi}_R(t') - E\hat{\varphi}_R(t) + C(\hat{t} - t')^{1/2})$$

$$+ C_p (E\hat{\varphi}_R(t') + C(\hat{t} - t')^{1/2}) \log \frac{\hat{t} - t'}{\hat{t} - t} + C(\hat{t} - t')^{1/2},$$
where
\[
E_{\tilde{R}}(t) := (\tilde{t} - t)^{\frac{p}{p+1}} \int_{R_+^n} \left( \frac{R^{2}}{2} - \frac{C\pi^{p+1}}{p+1} + \frac{\pi^{2}}{2(p-1)(t-t)} \right) K(\tilde{z}, \tilde{t})\tilde{\varphi}_{\tilde{R}}^2 dx.
\]

Letting \( \tilde{R} \to \infty \) with the aid of the monotone convergence theorem to the left-hand side and the Lebesgue dominated convergence theorem to the right-hand side, we obtain the desired inequality. \( \square \)

**Remark 5.8.** As stated in Remark 4.3, the \( \varepsilon \)-regularity (Theorem 4.3) is also valid for \( \overline{\varphi} \), since \( \overline{\varphi} \) satisfies the analog of Lemma 4.7.

We next show a monotonicity estimate of \( E_{(0,0)} \).

**Lemma 5.9.** Assume (3.2). Then there exists \( C > 0 \) such that
\[
E_{(0,0)}(t') - E_{(0,0)}(t) \geq C^{-1} (-t')^{4(p-1)} (-t')^{2p/(p-1)}
\times \left( \int_{t}^{t'} \int_{S_{1}^{-1} \Phi(\Omega_R)} \left( \frac{\pi}{p-1} + \frac{x \cdot \nabla u}{2} + s\varphi_{s} \right) \varphi K_{(0,0)}(x,s)\xi(x,s) dxds \right)^{2}
\]
for any \(-1 < t' < t < 0 \) and \( \zeta \in C^\infty_0(R_+ \times (-1,0)) \) with \( |\zeta| \leq 1 \).

**Proof.** Let \(-1 < t' < t < 0 \). We take \( k \) so large that \(-1/2)\delta_k^{-2} < t' < t < 0 \). From (3.19) and the change of variables, it follows that
\[
(5.12) \quad E_{(0,0)}(\delta_k^2 t; \varphi_0,R/4) + \frac{1}{2} I \leq E_{(0,0)}(\delta_k^2 t'; \varphi_0,R/4) + C\delta_k(-t')^x,
\]
where
\[
I := \int_{t}^{t'} (-s)^{\frac{2}{p-1}} \int_{S_{1}^{-1} \Phi(\Omega_R)} \left| \frac{u_k}{p-1} + \frac{x \cdot \nabla u_k}{2} + s\varphi_{s} \right| \exp \left( -\frac{|\delta_k^{-1} \Psi(\delta_k x)|^2}{4(-s)} \right) \varphi^2 \left( \frac{4}{R} \Psi(\delta_k x) \right) dxds.
\]
Let \( \zeta \in C^\infty_0(R_+ \times (-1,0)) \) satisfy \( |\zeta| \leq 1 \). Then,
\[
I \geq \int_{S_{1}^{-1} \Phi(\Omega_R) \times (t',t)} |F(u_k)|^2 \zeta^2(x,s) d\mu_k(x,s).
\]
Here \( F(u_k) \) and \( \mu_k \) are defined by
\[
F(u_k) := \frac{u_k}{p-1} + \frac{x \cdot \nabla u_k}{2} + s\varphi_{s},
\]
\[
d\mu_k(x,s) := (-s)^{\frac{2}{p-1}} \varphi \varphi^{-1} K_{(0,0)}(\delta_k^{-1} \Psi(\delta_k x), s) \varphi^2 \left( \frac{4}{R} \Psi(\delta_k x) \right) dxds.
\]
By (5.11) with \( \tilde{\varphi} = 0 \) and \( \varphi \leq 1 \), we compute that
\[
\mu_k(\delta_k^{-1} \Phi(\Omega_R) \times (t',t)) \leq C \int_{t'}^{t} (-s)^{\frac{2}{p-1}} \int_{R^n} K_{(0,0)} \left( \frac{x}{\sqrt{8}s} \right) dxds
\leq C(-t')^\frac{n-1}{p-1}.
\]
where $C$ depends only on $n$ and $p$ and is independent of $k$. Then by Jensen’s inequality, we see that

$$
I \geq \frac{1}{\mu_k(\delta_k^{-1} \Phi(\Omega_R) \times (t', t))} \left( \iint_{\delta_k^{-1} \Phi(\Omega_R) \times (t', t)} F(u_k) \zeta d\mu_k(x, s) \right)^2 \\
\geq C^{-1}(-t')^{-\frac{2}{p-1}} \left( \iint_{\delta_k^{-1} \Phi(\Omega_R) \times (t', t)} F(u_k) \zeta d\mu_k(x, s) \right)^2.
$$

From this and (5.12), it follows that

$$
E(0,0)(\delta_k^2 t; \phi_0, R/4) + C^{-1}(-t')^{-\frac{2}{p-1}} \left( \iint_{\delta_k^{-1} \Phi(\Omega_R) \times (t', t)} F(u_k) \zeta d\mu_k(x, s) \right)^2 \\
\leq E(0,0)(\delta_k^2 t'; \phi_0, R/4) + C\delta_k(-t')^\frac{3}{2}.
$$

From Remark 5.5, Lemma 5.4 and computations similar to the derivation of (5.10), it follows that

$$
\iint_{\delta_k^{-1} \Phi(\Omega_R) \times (t', t)} F(u_k) \zeta d\mu_k(x, s) \rightarrow \int_{t'}^{t} (-s)^{-\frac{2}{p-1}} \int_{R_+^n} F(\bar{\Omega})K_{(0,0)} \zeta dxds
$$
as $k \to \infty$. This together with Lemma 5.6 and letting $k \to \infty$ implies that

$$
\bar{E}(0,0)(t') - \bar{E}(0,0)(t) \\
\geq C^{-1}(-t')^{-\frac{2}{p-1}} \left( \int_{t'}^{t} (-s)^{-\frac{2}{p-1}} \int_{R_+^n} F(\bar{\Omega})K_{(0,0)} \zeta dxds \right)^2 \\
\geq C^{-1}(-t')^{-\frac{2}{p-1}} (-t)^{-\frac{4}{p-1}} \left( \int_{t'}^{t} \int_{R_+^n} F(\bar{\Omega})K_{(0,0)} \zeta dxds \right)^2.
$$

Then the desired inequality follows. \hfill \Box

The monotonicity estimate gives the following equality based on the argument of [97] Theorem 8.1.

**Lemma 5.10.** Assume (5.2). Then,

$$
\bar{\nabla}(x, t) + \frac{x \cdot \nabla \bar{\nabla}}{2} + t\bar{\nabla}_t = 0 \quad a.e. \text{ in } R_+^n \times (-1, 0).
$$

**Proof.** For $0 < r < 1/2$, we set

$$
\Phi(r) := \int_{-4r^2}^{-r^2} \bar{E}(0,0)(t) \frac{dt}{-t}.
$$

We note that (5.9) guarantees that $\Phi$ is well-defined. By Lemmas 5.6 and 3.1 the Lebesgue dominated convergence theorem and the change of variables, we have

$$
\Phi(r) = \lim_{k \to \infty} \int_{-4r^2}^{-r^2} \bar{E}(0,0)(\delta_k^2 t) \frac{dt}{-t} = \lim_{k \to \infty} \int_{-4r^2}^{-r^2} \bar{E}(0,0)(s) \frac{ds}{-s}.
$$

We claim that $\Phi(r)$ is independent of $r$. To show this, we set

$$
H(\tilde{r}) := \int_{-4\tilde{r}^2}^{-\tilde{r}^2} \bar{E}(0,0)(s) \frac{ds}{-s} = \int_{-4}^{-1} \bar{E}(0,0)(\tilde{r}^2 \lambda) \frac{d\lambda}{-\lambda}. 
$$
for $0 < \tilde{r} < 1/2$ and check that $\lim_{\tilde{r} \to 0} H(\tilde{r})$ exists. Let $\varepsilon > 0$. Since Lemma 3.1 yields the boundedness of $H$, we see that $H_\varepsilon := \inf_{0 < \tilde{r} < \varepsilon} H(\tilde{r})$ is finite. Then there exists $0 < \tilde{r}_\varepsilon < \varepsilon$ such that $H(\tilde{r}_\varepsilon) \leq H_\varepsilon + \varepsilon$. From Proposition 3.2 and the negativity of $\lambda$, it follows that

$$\limsup_{\tilde{r} \to 0} H(\tilde{r}) \leq \limsup_{\tilde{r} \to 0} \int_{-4}^{-1} (E_{0,0}(\tilde{r}_\varepsilon^2 \lambda) + C(M + MP)^2(\tilde{r}_\varepsilon^2 \lambda)) \frac{d\lambda}{\lambda}$$

$$= H(\tilde{r}_\varepsilon) + 3C(M + MP)^2\tilde{r}_\varepsilon^2 \leq H_\varepsilon + \varepsilon + 3C(M + MP)^2\varepsilon^2.$$ 

Thus, letting $\varepsilon \to 0$ gives $\limsup_{\tilde{r} \to 0} H(\tilde{r}) \leq \liminf_{\tilde{r} \to 0} H(\tilde{r})$. Hence the limit of $H(\tilde{r})$ exists, and so $\Phi(\rho) = \lim_{\tilde{r} \to 0} H(\tilde{r})$. Then the claim follows.

From the claim and Lemma 5.3, it follows that

$$0 = \int_{r_1}^{r_2} \Phi'(s) ds = \int_{r_1}^{r_2} 2s \frac{E_{0,0}(-4s^2) - E_{0,0}(-s^2)}{s} ds$$

$$\geq C^{-1} \int_{r_1}^{r_2} \frac{9-5p}{p-1} \left( \int_{-4s^2}^{-s^2} \int_{\mathbb{R}^n_p} \left( \frac{\nabla \Phi}{p-1} + \frac{x \cdot \nabla \Phi}{2} + \frac{t \Phi}{4} \right) K_{0,0}(x,t) \zeta(x,t) dx dt \right)^2 ds$$

for any $0 < r_1 < r_2 < 1/2$ and $\zeta \in C_0^\infty(\mathbb{R}_+^n \times (-1, 0))$ with $|\zeta| \leq 1$. Thus,

$$\int_{-4s^2}^{-s^2} \int_{\mathbb{R}^n_p} \left( \frac{\nabla \Phi}{p-1} + \frac{x \cdot \nabla \Phi}{2} + \frac{t \Phi}{4} \right) K_{0,0}(x,t) \zeta(x,t) dx dt = 0$$

for a.e. $s \in (0, 1/2)$. Hence by the fundamental lemma of the calculus of variations, the lemma follows.

Finally in this subsection, we prove a partial regularity result for $\pi$ by using the $\varepsilon$-regularity. For $t \in (-1, 0)$, define the singular set of $\pi(\cdot, t)$ by

$$\Sigma(t) := \{ x \in \mathbb{R}^n_+ : \pi(\cdot, t) \notin L^\infty(B_\rho^+(x)) \text{ for all } \rho > 0 \}.$$ 

By modifying the argument of Wang [102, Lemma 3.3], we show that $\Sigma(t)$ consists of at most finitely many points.

**Lemma 5.11.** Assume (2.2). Then the singular set $\Sigma(t)$ consists of finitely many points for each $t \in (-1/4, 0)$. More precisely, there exists a constant $C > 0$ depending on $R$ such that the cardinality $\text{card}(\Sigma(t))$ of $\Sigma(t)$ satisfies

$$\text{card}(\Sigma(t)) \leq C(M + MP)^C \varepsilon_0^{2\varepsilon_0}$$

for each $t \in (-1/4, 0)$. Here $\varepsilon_0$ is given in Theorem 4.1 (see also Remark 5.8) and $C$ is independent of $t$ and $M$.

**Proof.** Let $t_0 \in (0, -1/4)$. Since $\text{card}(\Sigma(t_0)) = 0$ if $\Sigma(t_0) = \emptyset$, it suffices to consider the case $\Sigma(t_0) \neq \emptyset$. Let $x_0 \in \Sigma(t_0)$. We examine the $L^\infty$ norm of $\pi$ near $(x_0, t_0)$. From the contraposition of Theorem 1.1 (see also Remark 5.8), it follows that

$$\varepsilon_0 < (\rho/2)^{-\frac{1}{p-1}} \int_{Q_{\rho/2}^+(x_0, t_0)} (|\nabla \pi|^2 + |\pi|^{p+1}) dx dt$$

for $0 < \rho \leq 2\delta_0$.

Lemma 4.3 with translation gives

$$\int_{Q_{\rho/2}^+(x_0, t_0)} |\nabla \pi|^2 dx dt \leq C \rho^{n-\frac{1}{p+1}} \left( 1 + M^{p-1} \right)^2 \|\pi\|_{L^\infty(Q_{\rho/2}^+(x_0, t_0))}^2.$$
where $C > 0$ is independent of $\rho$. The Hölder inequality also gives
\[
\int \int_{Q_{2s}^+(x_0,t_0)} |\mathfrak{u}|^{p+1} \, dx \, dt \\
\leq C \rho^{(n+2)(1-\frac{q-1}{2})} \left( \int \int_{Q_{2s}^+(x_0,t_0)} |\mathfrak{u}|^q \, dx \, dt \right)^{\frac{q-1}{q}} \|\mathfrak{u}\|_{L^{q,\infty}(Q_{s}^+(x_0,t_0))}^2 \\
\leq C \rho^{(n+2)(1-\frac{q-1}{2})+\frac{2(p-1)}{q}+2} \rho^{M_{-1}} \|\mathfrak{u}\|_{L^{q,\infty}(Q_{s}^+(x_0,t_0))}^2.
\]
These estimates imply that
\[
\varepsilon_0 \leq C(1+M_{-1})^2 \rho \frac{1}{\rho} \|\mathfrak{u}\|_{L^{q,\infty}(Q_{s}^+(x_0,t_0))}^2.
\]
Hence there exists a constant $C > 0$ depending on $R$ and independent of $\rho$, $t_0$ and $M$ such that
\[
\varepsilon_0^\frac{q}{2} \leq C(1+M_{-1})^2 \rho \rho_0^{-2} \int_{t_0-\rho_0^2}^{t_0} \int_{B_{t_0}^+(x_0)} |\mathfrak{u}|^q \, dx \, dt 
\]
for $0 < \rho \leq 2\delta_0$.

Let $S$ be any finite subset of $\Sigma(t_0)$. We write $S = \{x_1, \ldots, x_N\}$ and choose $0 < \rho_0 \leq \delta_0$ ($< 1/2$) such that $\{B_{\rho_0}^+(x_i)\}_{1 \leq i \leq N}$ is pairwise disjoint. Then we have
\[
N \varepsilon_0^\frac{q}{2} \leq C(1+M_{-1})^2 \rho \rho_0^{-2} \int_{t_0-\rho_0^2}^{t_0} \int_{B_{t_0}^+(x_i)} |\mathfrak{u}|^q \, dx \, dt 
\]
for each $1 \leq i \leq N$, and so we also have
\[
N \varepsilon_0^\frac{q}{2} \leq C(1+M_{-1})^2 \rho \rho_0^{-2} \int_{t_0-\rho_0^2}^{t_0} \left( \sum_{i=1}^N \int_{B_{\rho_0}^+(x_i)} |\mathfrak{u}|^q \, dx \right) \, dt \\
\leq C(1+M_{-1})^2 \rho \rho_0^{-2} \int_{t_0-\rho_0^2}^{t_0} \int_{R_n^+} |\mathfrak{u}|^q \, dx \, dt \leq C(M+M_{-1})^q \varepsilon_0^\frac{q}{2},
\]
where $C > 0$ is independent of $\rho_0$, $t_0$, $M$ and $N$. Hence $\text{card}(S) \leq C(M+M_{-1})^q \varepsilon_0^{-q/2}$. Remark that the constant $C(M+M_{-1})^q \varepsilon_0^{-q/2}$ does not depend on the choice of $S$. Thus, the number of elements in any subset of $\Sigma(t_0)$ cannot exceed the constant, regardless of whether $\Sigma(t_0)$ contains an accumulation point or not. Therefore we can conclude that $\text{card}(\Sigma(t_0)) \leq C(M+M_{-1})^q \varepsilon_0^{-q/2}$. The proof is complete.

**Remark 5.12.** In the case $\Sigma \neq \emptyset$, Lemma 5.4 also holds with $B_{\rho_0}$ and $R_n^+$ replaced by $B_{\rho}$ and $R_n$, respectively. In particular, the blow-up limit $\mathfrak{u}$ exists. The analogs to the $\varepsilon$-regularity (Theorem 4.1), the equality in Lemma 5.10 and the partial regularity (Lemma 5.11) also hold for $\mathfrak{u}$ in the case $\Sigma = \emptyset$.  

5.3. **Proof of localized statement.** We are now in a position to prove Theorem 5.1.  

**Proof of Theorem 5.1.** We show this theorem by contradiction under the condition supposed in Subsection 5.1. We focus on the case $\Sigma \neq \emptyset$, since $\Sigma = \emptyset$ is easier. From
Recall from the proof of Lemma 5.4 that
\[ \int_0^\infty \int_{\Phi(\Omega_k)} (|\nabla \hat{u}|^2 + |\hat{u}|^{p+1}) d\xi dt = \int_{-1}^1 \int_{\Phi(\Omega_k)} (|\nabla u_k|^2 + |u_k|^{p+1}) dx dt, \]
where \( \hat{u} := (\nabla' \hat{u} - (\partial_n \hat{u}) \nabla' f, \partial_n \hat{u}) \) and \( \hat{u} := (\nabla' u_k - (\partial_n u_k) \nabla' f_k, \partial_n u_k) \). Using \( \|\nabla' f\|_{L^\infty(\mathbb{R}^{n-1})} \leq 1/2 \) gives \( \delta_k^{-1} \Phi(\Omega_k) \subset B_T^+ \). Hence by (5.6) and Lemma 5.4 we see that
\[ \int_{-1}^1 \int_{B_T^+} (|\nabla u|^2 + |u|^{p+1}) dx dt > \varepsilon_0. \]
We will show that \( \bar{u} \equiv 0 \) a.e. in \( \mathbb{R}^n_+ \times (-1, 0) \), contrary to (5.13).

By Lemma 5.10 it follows that
\[ \frac{d}{d\lambda} (\lambda^{p-1} \bar{u}(\lambda x, \lambda^2 t)) = 2\lambda^{p-2} \left( \frac{\bar{u}(y, s)}{p-1} + \frac{y \cdot \nabla y \bar{u}}{2} + s\bar{u} \right) = 0 \]
for \( y = \lambda x, s = \lambda^2 t \) and \( 0 < \lambda < 1/\sqrt{-t} \) in the weak sense. Hence \( \bar{u} \) is self-similar, and so there exists a profile function \( \overline{U} \in L^q(\mathbb{R}^n_+) \) such that
\[ \bar{u}(x, t) = (-t)^{-\frac{n}{p-1}} \overline{U}(z), \quad z := \frac{x}{\sqrt{-t}}, \]
where \( \overline{U} \) is a weak solution of the equation
\[ \Delta \overline{U} - \frac{1}{2} z \cdot \nabla \overline{U} - \frac{1}{p-1} \overline{U} + |\overline{U}|^{p-1} \overline{U} = 0, \quad z \in \mathbb{R}^n. \]
We note that the method for proving self-similarity can also be found in [61] Lemma 8.5.3 and [97, 102].

We claim that there exist constants \( \tilde{R}, C > 1 \) satisfying
\[ \begin{cases} \bar{u}(\cdot, 0) = 0 & \text{a.e. in } \mathbb{R}^n_+, \\ |\bar{u}| \leq C & \text{a.e. in } (\mathbb{R}^n_+ \setminus B_R) \times [-1/9, 0]. \end{cases} \]
Recall from the proof of Lemma 5.4 that \( u_k \to \bar{u} \) in \( C([-1, 0]; W^{1,r}(B_\rho)) \) for any \( \rho > 0 \) with some \( 1 < r < \min\{2, q_0/p\} \) as \( k \to \infty \). Then, for \( x_0 \in \mathbb{R}^n_+ \) and \( \delta > 0 \), we have
\[ \int_{B^+_T(x_0)} |\bar{u}(x, 0)|^r dx \leq 2^{r-1} \delta + 2^{r-1} \int_{B^+_T(x_0)} |u_k(x, 0)|^r dx \]
for sufficiently large \( k \). By returning to the original variables, we also have
\[ \int_{B^+_T(x_0)} |u_k(x, 0)|^r dx = \delta_k^{\frac{n}{p-1}} \int_{\Psi(B^+_T(\delta_k x_0))} |u(x, 0)|^r dx \]
\[ \leq \delta_k^{\frac{n}{p-1}} \int_{\Psi(B^+_T(\delta_k x_0))} |u(x, 0)|^r dx \]
\[ \leq C(1 + |x_0|)^{\frac{n}{p-1}} \|u(\cdot, 0)\|_{L^\infty(\Omega_T/2)}^r \to 0 \]
as \( k \to \infty \), where \( u \) belongs to \( C_{\text{weak}}([-1/4, 0]; L^q(\Omega_T/2)) \) in our situation, see (5.2). Hence we obtain the equality on the first line of (5.15).
We prove the inequality in (5.15). Let \( \varepsilon > 0 \). We claim that there exists a constant \( \bar{R} > 1 \) depending on \( \varepsilon \) and \( M \) satisfying

\[
\int_{Q_{1/2}^+(x_1,0)} \left( |\nabla u|^2 + |u|^{p+1} \right) dx dt \leq \varepsilon
\]

for any \( x_1 \in \mathbb{R}_+^n \setminus B_{\bar{R}}^+(0) \), where \( Q_{1/2}^+(x_1,0) = B_{1/2}^+(x_1) \times (-1/4, 0) \). Indeed, for \( \varepsilon' > 0 \), we can choose \( \bar{R}' > 1 \) by Lemma 5.4 (iii) so large that

\[
\int_{\mathbb{R}^n_+ \setminus B_{\bar{R}'-1}(0)} |u|^q dx dt \leq \varepsilon'.
\]

From the H"older inequality, it follows that

\[
\int_{Q_{1/2}^+(x_1,0)} |u|^{p+1} dx dt \leq C \left( \int_{Q_{1/2}^+(x_1,0)} |u|^q dx dt \right)^{p+1/q} \leq C(\varepsilon')^{p+1/q}.
\]

Thus choosing \( \varepsilon' \) small gives (5.16), and so the claim follows.

Hence by translation and Theorem 4.1 (see also Remark 5.8), we see that \( |\Phi| \leq C \delta_0^{-2/(p-1)} \) in \( B_{\delta_0}^+(x_1) \times [-1/9, 0] \) for any \( x_1 \in \mathbb{R}_+^n \setminus B_{\delta_0}^+(0) \), where \( C \) and \( \delta_0 \) are constants independent of \( x_1 \). This proves the inequality in (5.15).

From (5.15) and the backward uniqueness theorem [35, Theorem 5.1], it follows that

\[
\int_{Q_{1/2}^+(x_1,0)} |\nabla u|^2 dx dt \leq C(1 + M^{p-1})^2 \left( \int_{Q_{1/2}^+(x_1,0)} |u|^q dx dt \right)^{2/q} \leq C(1 + M^{p-1})^2 (\varepsilon')^2.
\]

This contradicts (5.13), and hence the proof of the localized statement is complete.

5.4. Completion of proof. Finally, we prove Theorem 1.1.

Proof of Theorem 1.1. If \( \Omega \) is bounded, then Theorem 1.1 immediately follows from Theorem 5.1. In what follows, we consider the case where \( \Omega \) is unbounded. To obtain a contradiction, suppose that

\[
\sup_{0 < t < T} \| u(\cdot, t) \|_{L^q(\Omega)} \leq M
\]

for some \( M > 0 \). Let \( \varepsilon > 0 \) and \( a \in \Omega \). Then by the same argument as in the derivation of (5.10), there exists a constant \( \bar{R} > 0 \) depending on \( \varepsilon \), \( a \), and \( M \) satisfying

\[
\int_{T/2}^T \int_{Q_{1/2}^+(x)} (|u|^{p+1} + |\nabla u|^2) dx dt \leq \varepsilon
\]
for any \( \bar{x} \in \Omega \setminus B_{R}(a) \). Therefore, in the same way as in the proof of \( \text{(5.15)} \), we see that \( u \) is bounded on \( (\Omega \setminus B_{R}(a)) \times (T/3, T) \) for some \( R > 0 \). This implies that there exists at least one blow-up point \( a' \in \Omega_{R'}(a) \). Hence Theorem \( \text{(5.1)} \) shows that

\[
\limsup_{t \to T} \| u(\cdot, t) \|_{L^p(\Omega_{R'}(a'))} = \infty
\]

for any \( r > 0 \), a contradiction. The proof of Theorem \( \text{1.1} \) is complete. \( \square \)

**Appendix A. Regularity estimates**

We give some parabolic regularity estimates for solutions of \( \text{(2.1)} \) and a gradient estimate for the blow-up limit obtained in Lemma \( \text{(5.4)} \). Let \( n \geq 3 \), \( p > p_{\text{es}} \) and \( \Omega \) be any \( C^{2+\alpha} \) domain in \( \mathbb{R}^n \) with \( 0 \in \overline{\Omega} \). Fix \( R > 0 \) such that either \( \text{(3.1)} \) or \( \text{(3.2)} \) holds. Let \( u \) satisfy \( \text{(2.1)} \) and \( \text{(2.2)} \). Define \( \hat{u} \) by \( \text{(3.6)} \). For \( \rho > 0 \), we take \( 0 < \delta' < 1/2 \) such that \( \Psi(B_{\rho}^+ \cap \Omega_{R/2}) \subset \Omega_{R/4} \) and \( \Psi(B_{\delta}^+) \subset B_{R/2} \). We first give parabolic regularity estimates. Remark that we mainly focus on the case \( \text{(3.2)} \), since \( \text{(3.1)} \) is easier.

**Lemma A.1.** Assume \( \text{(3.2)} \). Let \( 1 \leq l < \infty \) and \( 1 \leq r \leq q_{\text{es}}/p \). Then there exists a constant \( C > 0 \) such that

\[
\| \nabla u(t) \|_{L^l(-\delta, 0; L^r(B_{\rho}^+))} + \| \nabla^2 u(t) \|_{L^l(-\delta, 0; L^r(B_{\rho}^+))} \leq C\delta^{\frac{1}{2} + n(n+2q_{\text{es}})}(M + M^p)
\]

for any \( 0 < \delta < \delta' \), where \( C \) depends on \( R \) and \( \rho \) and is independent of \( \delta \).

**Proof.** By \( \text{(3.9)} \), we have

\[
|\nabla^2 \hat{u}(t, x)| \leq C(|\nabla u(\Psi(x), t)| + |(\nabla^2 u)(\Psi(x), t)|),
\]

where \( C > 0 \) depends on \( \| \nabla f \|_{L^\infty(\mathbb{R}^{n-1})} \) and \( \| \nabla^2 f \|_{L^\infty(\mathbb{R}^{n-1})} \). Remark that the choice of \( \delta' \) guarantees \( \Psi(B_{\rho}^+) \subset \Omega_{\delta R/4} \), and so Proposition \( \text{2.1} \) is applicable in \( \Psi(B_{\rho}^+) \). Then by \( r \leq q_{\text{es}}/p < q_{\text{es}} \), the Hölder inequality in the Lorentz spaces (see \( \text{[57]} \) Proposition 2.1) for instance) and Proposition \( \text{2.1} \) we see that

\[
\| \nabla u(\Psi(\cdot), \cdot) \|_{L^l(-\delta, 0; L^r(B_{\rho}^+) \cap \Omega_{\delta R/4})} \leq C\delta^{\frac{1}{2} + n(n+2q_{\text{es}})}(M + M^p).
\]

We estimate \( \| \nabla^2 u(\Psi(\cdot), \cdot) \|_{L^l(-\delta, 0; L^r(B_{\rho}^+) \cap \Omega_{\delta R/4})} \). Let \( \phi \in C_0^\infty(\mathbb{R}^n) \) satisfy \( 0 \leq \phi \leq 1 \) in \( \mathbb{R}^n \), \( \phi = 0 \) in \( \mathbb{R}^n \setminus B_{3\rho} \) and \( \phi = 1 \) in \( B_{2\rho} \). Set \( \bar{\phi}(x) := \phi(\delta^{-1} \Phi(x)) \) and \( v(x, t) := u(x, t) \bar{\phi}(x) \). We prepare a \( C^{2+\alpha} \) domain \( \mathcal{D} \) satisfying \( \Psi(B_{\rho}^+) \subset \mathcal{D} \subset \Omega_{R/2} \) to avoid technicalities due to the corner of \( \partial B_{3\rho}^+ \). Then \( v \) satisfies

\[
\begin{cases}
vt - \Delta v = \hat{\phi}|u|^{p-1}u - 2\nabla \hat{\phi} \cdot \nabla u - u\Delta \hat{\phi} & \text{in } \mathcal{D} \times (-1, 0), \\
v = 0 & \text{on } \partial \mathcal{D} \times (-1, 0).
\end{cases}
\]

By the same computation as in the derivation of \( \text{(2.3)} \), we have

\[
u(x, t) = \int_\mathcal{D} G_\mathcal{D}(x, y, t + 2\delta^2) \bar{\phi}(y) u(y, -2\delta^2)dy
\]

\[
+ \int_{-2\delta^2}^t \int_\mathcal{D} G_\mathcal{D}(x, y, t - s) \bar{\phi}|u|^{p-1}u dyds
\]

\[
- \int_{-2\delta^2}^t \int_\mathcal{D} G_\mathcal{D}(x, y, t - s)(2\nabla \hat{\phi} \cdot \nabla u + u\Delta \hat{\phi}) dyds
\]
for \(x \in \Psi(B_{\rho_0}^+)^\#\) and \(-2\delta^2 < t < 0\), and so
\[
|\nabla^2 u(x, t)| \leq C \int_{\mathbb{R}^n} K_2(x - y, t + 2\delta^2)|u(y, -2\delta^2)||\chi_{\Omega_R \cap \Psi(B_{\rho_0}^+)}(y)dy
\]
\[+
|\nabla^2 \int_{-2\delta^2}^t \int_{\mathcal{D}} G_D(x, y, t - s)\tilde{\phi}|u|^{p-1}udyds|
\]
\[+ C \int_{-2\delta^2}^t \int_{\mathbb{R}^n} K_2 \left(\frac{|y|}{\delta} + |\nabla u|\right)\chi_{\Omega_R \cap \Psi(B_{\rho_0}^+ \setminus B_{2\rho_0})}dyds
\]
\[=: CV_1(x, t) + |\nabla^2 V_2(x, t)| + CV_3(x, t)
\]
for \(x \in \Psi(B_{\rho_0}^+)\) and \(-2\delta^2 < t < 0\), where \(K_2\) is defined by (2.4).

First, we estimate \(V_1\). By \(\|\nabla f\|_{L^\infty(\mathbb{R}^n-1)} \leq 1/2\) in (3.2), we have
\[
|\Psi(x) - \Psi(y)|^2 \geq |x' - y'|^2 + \frac{1}{2}(x_n - y_n)^2 - (f(x') - f(y'))^2 \geq \frac{1}{4}|x - y|^2.
\]
This together with the change of variables shows that
\[
V_1(\Psi(x), t)
\]
\[= \int_{\mathbb{R}^n} K_2(\Psi(x) - \Psi(y), t + 2\delta^2)|u(\Psi(y), -2\delta^2)||\chi_{\Omega_R \cap \Psi(B_{\rho_0}^+)}dy
\]
\[\leq \int_{\mathbb{R}^n} K_2((x - y)/2, t + 2\delta^2)|u(\Psi(y), -2\delta^2)||\chi_{\Omega_R \cap \Psi(B_{\rho_0}^+)}dy.
\]
Then Young’s inequality gives
\[
\|V_1(\Psi(\cdot), t)\|_{L^\infty(\Omega_R \cap \Psi(B_{\rho_0}^+))} \leq C(t + 2\delta^2)^{-1}\|u(\Psi(\cdot), -2\delta^2)\|_{L^\infty(\Omega_R \cap \Psi(B_{\rho_0}^+)}) \leq C\delta^{-2}M
\]
for \(-2^2 < t < 0\). Therefore, the Hölder inequality shows that
\[
\|V_1(\Psi(\cdot), \cdot)\|_{L^1(-\delta^2, 0; L^p(\Omega_R \cap \Psi(B_{\rho_0}^+)}) \leq C\delta^{n+\frac{1}{p} - \frac{n}{p}}M.
\]

Let us next estimate \(\nabla^2 V_2\). From the Hölder inequality, the change of variables \(y = \Psi(x)\) with \(dy = dx\) and the choice of \(\mathcal{D}\), it follows that
\[
\|\nabla^2 V_2(\Psi(\cdot), \cdot)\|_{L^1(-\delta^2, 0; L^p(\Omega_R \cap \Psi(B_{\rho_0}^+)}) \leq C\delta^{n+\frac{1}{p} - \frac{n}{p}}\|\nabla^2 V_2\|_{L^1(-\delta^2, 0; L^p(\Omega_R \cap \Psi(B_{\rho_0}^+)}) \leq C\delta^{n+\frac{1}{p} - \frac{n}{p}}M.
\]
We observe that \(V_2\) is a solution of
\[
\begin{cases}
(V_2)_t - \Delta V_2 = \tilde{\phi}|u|^{p-1}u & \text{in } \mathcal{D} \times (-2\delta^2, 0), \\
V_2 = 0 & \text{on } \partial\mathcal{D} \times (-2\delta^2, 0), \\
V_2(\cdot, -2\delta^2) = 0 & \text{in } \mathcal{D}.
\end{cases}
\]
Since \(\mathcal{D}\) is a bounded \(C^{2+\alpha}\) domain, \(\mathcal{D}\) is also a uniformly regular domain of class \(C^2\). Therefore, we can apply the maximal regularity for inhomogeneous heat equations (see [20] Remark 5.5) and [28] Theorem 7.11 for instance), and so
\[
\|\nabla^2 V_2\|_{L^1(-\delta^2, 0; L^p(\mathcal{D}))} \leq C\|\tilde{\phi}|u|^{p-1}u\|_{L^1(-2\delta^2, 0; L^p(\mathcal{D}))} \leq C\|u\|_{L^p(-2\delta^2, 0; L^p(\Omega_R))} \leq C\delta^{\frac{n}{2}}M^p.
\]
Thus,
\[
\|\nabla^2 V_2(\Psi(\cdot, \cdot))\|_{L^1(-\delta^2, 0; L^r(\Omega_{\rho/4}))} \leq C\delta^\frac{2}{q_c} + n(\frac{1}{2} - \frac{q_c}{p} + \frac{q_c}{q}) M^p.
\]

Finally, we consider $V_3$. Again in the same way as in (A.2), we have
\[
V_3(\Psi(x, t)) \leq \int_{-2\delta^2}^t \int_{\Omega_R(\Omega_R \cap (\Omega_R \setminus \Omega_{\rho/4}))} K_2((x - y)/2, t - s) \times (\delta^{-2}|u(\Psi(y), s)| + \delta^{-1}|\nabla u(\Psi(y), s)|) dy ds.
\]
for $x \in B_{\rho\delta}^+$ and $-2\delta^2 < t < 0$. We observe that there exists $C > 0$ satisfying
\[
K_2((x - y)/2, t - s) \leq C\delta^{-n - 2}
\]
for $x \in B_{\rho\delta}^+$, $y \in \overline{B_{3\rho\delta} \setminus B_{2\rho\delta}}$ and $-2\delta^2 < s < t < 0$. Then by the change of variables, the Hölder inequality, $\Omega_R \cap \Psi(B_{3\rho\delta}) \subset \Omega_{3R/4}$ and Proposition 2.1, we see that
\[
V_3(\Psi(x, t)) \leq C\delta^{-n - 2} \int_{-2\delta^2}^t \int_{\Omega_R \cap (\Omega_R \setminus \Omega_{\rho/4})} (\delta^{-2}|u(\Psi(y), s)| + \delta^{-1}|\nabla u(\Psi(y), s)|) dy ds
\]
\[
\leq C\delta^{-n - 2}(M^p(1 - \frac{q_c}{p} + (M + M^p)\delta^{n(1 - \frac{q_c}{p}) + 1}) \leq C\delta^{-n - 2}(M + M^p)
\]
for $x \in B_{\rho\delta}^+$ and $-2\delta^2 < t < 0$. Hence we obtain
\[
\|V_3(\Psi(\cdot, \cdot))\|_{L^1(-\delta^2, 0; L^r(\Omega_{\rho/4}))} \leq C\delta^\frac{2}{q_c} + n(\frac{1}{2} - \frac{q_c}{p} + \frac{q_c}{q}) (M + M^p).
\]
By combining this inequality, (A.1), (A.3) and (A.4), we obtain the desired estimate for $\nabla^2 \hat{u}$. Then the desired estimate for $\hat{u}_t$ can be obtained by using the equation in (3.8).

Lemma A.2. Assume either (3.1) or (3.2). Let $1 \leq l < \infty$ and $1 \leq r \leq q_c/p$. Then there exists a constant $C > 0$ depending on $R$ such that
\[
\|u_t\|_{L^1(-1/4, 0; L^r(\Omega_{R/2}))} + \|\nabla^2 u\|_{L^1(-1/4, 0; L^r(\Omega_{R/2}))} \leq C(M + M^p).
\]
Proof. By easy modifications of Lemma A.1, we can see that
\[
\|\nabla^2 u\|_{L^1(-1/4, 0; L^r(\Omega_{R/2}))} \leq C(M + M^p).
\]
Then by the equation in (2.1), we obtain the desired inequality.

Let us next show a gradient estimate for the blow-up limit $\bar{\pi}$ obtained in Lemma 5.4. To estimate $\nabla \bar{\pi}$, we derive a localized integral equation for $\bar{\pi}$.

Lemma A.3. Assume (3.2). Let $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \phi \leq 1$ in $\mathbb{R}^n$, $\phi = 0$ in $\mathbb{R}^n \setminus B_{3\rho/5}$ and $\phi = 1$ in $B_{4\rho/5}$. Set $\phi_\rho(x) := \phi(x/\rho)$ for $0 < \rho < 1$. Then $\bar{\pi}$ satisfies
\[
\bar{\pi}(x, t) = \int_{\mathbb{R}^n_+} GR_{\rho}^n(x, y, \rho^2/4) \phi_\rho(y) \bar{\pi}(y, t - \rho^2/4) dy
\]
\[
+ \int_{1 - \rho^2/4}^t \int_{\mathbb{R}^n_+} GR_{\rho}^n(x, y, t - s) (\phi_\rho)^p \bar{\pi}^{p-1} \bar{\pi}^{\Delta} \phi_\rho dy ds
\]
\[
+ 2 \int_{1 - \rho^2/4}^t \int_{\mathbb{R}^n_+} \nabla_y GR_{\rho}^n(x, y, t - s) \cdot \nabla \phi_\rho(y) \bar{\pi}(y, s) dy ds
\]
for a.e. $x \in B_{\rho/2}^+$ and $-\rho^2/4 < t < 0$. 

Proof. Let us convert our problem to the one in \( \mathbb{R}^n \). Let \( \psi \in C^\infty(\mathbb{R}^n) \) satisfy \( 0 \leq \psi \leq 1 \) in \( \mathbb{R}^n \), \( \psi = 0 \) in \( \mathbb{R}^n \setminus B_{4R/5} \) and \( \psi = 1 \) in \( B_{3R/5} \). Set \( \psi_k(x) := \psi(\Phi_k(x)) \) and \( v_k := \phi_p \psi_k u_k \). Note that \( \psi_k = 0 \) in \( \mathbb{R}^n \setminus \delta_k^{-1}(B_{4R/5}) \) and \( \psi_k = 1 \) in \( \delta_k^{-1}(B_{3R/5}) \). Then by (A.3), we see that

\[
\begin{cases}
(v_k)_t - \Delta v_k = \phi_p \psi_k |u_k|^{p-1} u_k - \psi_k u_k \Delta \phi_p - 2 \psi_k \nabla \phi_p \cdot \nabla u_k + \mathcal{R}_k \\
in \mathbb{R}^n_+ \times (-\delta_k^{-2}, 0),

v_k = 0 \quad \text{on} \quad \partial \mathbb{R}^n_+ \times (-\delta_k^{-2}, 0),
\end{cases}
\]

where

\[
\mathcal{R}_k := \phi_p \psi_k (-2 \nabla'(\partial_{x_n} u_k) \cdot \nabla' f_k + (\partial^2_{x_n} u_k)|\nabla' f_k|^2 - (\partial_{x_n} u_k) \Delta' f_k) \\
- (2u_k \nabla \psi_k \cdot \nabla \phi_p + \phi_p u_k \Delta \psi_k + 2 \phi_p \nabla \psi_k \cdot \nabla u_k).
\]

Thus,

\[
u_k(x, t) = \int_{\mathbb{R}^n_+} G_{\mathbb{R}^n_+}(x, y, \rho^2/4) \rho_p(y) \psi(y) u_k(y, t - \rho^2/4) dy \\
+ \int_{t-\rho^2/4}^t \int_{\mathbb{R}^n_+} G_{\mathbb{R}^n_+}(x, y, t - s) \phi_p \psi_k |u_k|^{p-1} u_k dy ds \\
- \int_{t-\rho^2/4}^t \int_{\mathbb{R}^n_+} G_{\mathbb{R}^n_+}(\psi_k u_k \Delta \phi_p + 2 \psi_k \nabla \phi_p \cdot \nabla u_k + \mathcal{R}_k) dy ds
\]

\[=: V^k_1(x, t) + V^k_2(x, t) + V^k_3(x, t)\]

for \( x \in B^+_{\rho/2} \) and \(-\rho^2/4 < t < 0 \) and \( k \geq k_p \), where \( k_p \) is given by the first part of Subsection 5.1. Lemma 5.4 (ii) shows that \( u_k(\cdot, t) \) converges to \( \overline{u}(\cdot, t) \) in \( L^1(B^+_{\rho/2}) \) for each \(-\rho^2/4 < t < 0 \) as \( k \to \infty \).

We show that the right-hand side of the integral equation for a subsequence of \( u_k \) still denoted by \( u_k \) converges to the one in the following integral equation:

\[
\overline{u}(x, t) = \int_{\mathbb{R}^n_+} G_{\mathbb{R}^n_+}(x, y, \rho^2/4) \rho_p(y) \overline{u}(y, t - \rho^2/4) dy \\
+ \int_{t-\rho^2/4}^t \int_{\mathbb{R}^n_+} G_{\mathbb{R}^n_+}(x, y, t - s) \rho_p |\overline{u}|^{p-1} |\overline{u}| dy ds \\
- \int_{t-\rho^2/4}^t \int_{\mathbb{R}^n_+} G_{\mathbb{R}^n_+}(\overline{u} \Delta \phi_p + 2 \nabla \phi_p \cdot \nabla \overline{u}) dy ds
\]

\[=: \nabla \overline{V}_1(x, t) + \nabla \overline{V}_2(x, t) + \nabla \overline{V}_3(x, t)\]

for a.e. \( x \in B^+_{\rho/2} \) and \(-\rho^2/4 < t < 0 \). For \( V^k_1 \), from (2.5), \( 0 \leq \psi_k \leq 1 \), the Hölder inequality and Lemma 5.4 (ii) and (iii), it follows that

\[
\|V^k_1(\cdot, t) - \nabla \overline{V}_1(\cdot, t)\|_{L^1(B^+_{\rho/2})} \leq C \rho^{-n} \int_{B^+_{\rho/2}} \rho_p |\psi_k u_k - \overline{u}| dy dx
\]

\[
\leq C \|u_k(\cdot, t - \rho^2/4) - \overline{u}(\cdot, t - \rho^2/4)\|_{L^1(B^+_{\rho/2})}
\]

\[+ C \|(\psi_k - 1)\overline{u}(\cdot, t - \rho^2/4)\|_{L^1(B^+_{\rho/2})} \to 0
\]
as $k \to \infty$. For $V_2^k$, by computations similar to that of $V_1^k$ with Young’s inequality, we see that, as $k \to \infty$,

$$
\|V_2^k(\cdot,t) - \bar{V}_2(\cdot,t)\|_{L^1(B_{\rho/2}^+)} \leq C \int_{t - \rho^2/4}^t \|\psi_k|u_k|^{p-1}u_k - |\bar{\psi}|^{p-1}|\bar{\psi}|\|_{L^1(B_{\rho}^+)} ds
$$

$$
\leq C \int_{t - \rho^2/4}^t \|(|u_k|^{p-1} + |\bar{\psi}|^{p-1})|u_k - \bar{\psi}|\|_{L^1(B_{\rho}^+)}
$$

$$
+ C \int_{t - \rho^2/4}^t \|(|\psi_k - 1|)|\bar{\psi}|\|_{L^1(B_{\rho}^+)} ds \to 0.
$$

For $V_3^k$, we focus on the most subtle term $\phi_\rho \psi_k \nabla'(\partial_x u_k) \cdot \nabla' f_k$ in $R_k$, that is, we prove that

$$
\check{V}_3^k(\cdot,t) := \int_{t - \rho^2/4}^t \int_{R^+} G_{R^+}(\cdot,y,t-s) \phi_\rho \psi_k \nabla'(\partial_x u_k) \cdot \nabla' f_k dyds \to 0
$$

in $L^1(B_{\rho/2}^+)$ for each $-\rho^2/4 < t < 0$ as $k \to \infty$. From integration by parts and \textbf{(2.3)}, it follows that

$$
|\check{V}_3^k(x,t)| \leq C \int_{t - \rho^2/4}^t \int_{R^+} K_1(x - y, t-s) \phi_\rho \psi_k |\partial_x u_k| |\nabla' f_k| dyds
$$

$$
+ C \int_{t - \rho^2/4}^t \int_{R^+} K_0 |\nabla'(\phi_\rho \psi_k)| |\partial_x u_k| |\nabla' f_k| dyds
$$

$$
+ C \int_{t - \rho^2/4}^t \int_{R^+} K_0 \phi_\rho \psi_k |\partial_x u_k| |\Delta' f_k| dyds.
$$

By Young’s inequality, $|\nabla \phi_\rho| \leq C$, $|\nabla \psi_k| \leq C \delta_k \leq C$, Hölder inequality and $\int_{t - \rho^2/4}(t-s)^{-\alpha/2\cdot\beta/3} ds \leq C$, we have

$$
\|\check{V}_3^k(\cdot,t)\|_{L^1(B_{\rho/2}^+)} \leq C \left( \int_{t - \rho^2/4}^t \|\nabla u_k(\cdot,s)\| |\nabla' f_k|\|_{L^1(B_{\rho}^+)} ds \right)^{1/4}
$$

$$
+ C \int_{t - \rho^2/4}^t \|\nabla u_k(\cdot,s)\| |\nabla' f_k| + |\Delta' f_k|\|_{L^1(B_{\rho}^+)} ds.
$$

Hence from the Hölder inequality for the Lorentz spaces, \textbf{(3.8)}, \textbf{(3.6)}, \textbf{(3.7)} and the Lebesgue dominated convergence theorem, it follows that

$$
\|\check{V}_3^k(\cdot,t)\|_{L^1(B_{\rho/2}^+)} \leq C(M + M^p)\|\nabla' f_k| + |\Delta' f_k|\|_{L^{\infty}} \to 0
$$

for each $-\rho^2/4 < t < 0$ as $k \to \infty$. The other terms in $V_3^k$ can be handled more easily. Hence we obtain \textbf{(A.5)}. This implies the desired equality. □

We give a gradient estimate for $\bar{\psi}$.

**Lemma A.4.** Assume \textbf{(3.2)}. Then there exists a constant $C > 0$ independent of $\rho$ such that

$$
\|\nabla \bar{\psi}\|_{L^2(\mathbb{R}^+/2)} \leq C \rho^{\frac{2}{p} - \frac{2}{p+2} - \frac{2}{p} + \tau}(1 + M^{p-1}) |\bar{\psi}|_{L^{\infty}(\mathbb{R}^+/2)}
$$

for any $0 < \rho \leq 1$. 


Proof. By Lemma 3, \( \nabla \phi_\rho(x) = \rho^{-1} \nabla \phi(x/\rho) \), \( \nabla^2 \phi_\rho(x) = \rho^{-2} \nabla^2 \phi(x/\rho) \) and similar computations to (2.7), there exists \( C > 0 \) independent of \( \rho \) such that

\[
|\nabla \eta| \leq C \int_{B_\rho^+} K_1(x - y, \rho^2/4)|\eta(y, t - \rho^2/4)|dy \\
+ C \int_{t - \rho^2/4}^t \int_{B_\rho^+} K_1(x - y, t - s)|\eta(y, s)|^p dyds \\
+ C \int_{t - \rho^2/4}^t \int_{\mathbb{R}_+^n} \left( \frac{K_1}{\rho^p} + \frac{K_2}{\rho} \right) |\eta|_{L^p(B_{2\rho^1/5})} dyds \\
=: CW_1 + CW_2 + C\rho^{-2}W_3 + C\rho^{-1}W_4
\]

for a.e. \( x \in B_{\rho/2}^+ \) and \(-\rho^2/4 < t < 0\), where \( K_1 \) and \( K_2 \) are given by (2.10).

By the H"older inequality, we have

\[
W_1(x, t) \leq \|\eta(\cdot, t - \rho^2/4)\|_{L^{q_\rho}(B_\rho^+)} \left( \int_{B_\rho^+} K_1(x - y, \rho^2/4)\frac{dy}{\rho^{nq}} \right)^{1 - \frac{1}{q_\rho}} \\
\leq C\rho^{-1 - \frac{1}{q_\rho}} \|\eta(\cdot, t - \rho^2/4)\|_{L^{q_\rho}(B_\rho^+)},
\]

\[
\|W_1\|_{L^2(Q^+_{\rho/2})} \leq C\rho^{-\frac{1}{q_\rho} + \frac{1}{q_\rho} - \frac{1}{4q_\rho}} \left( \int_{t - \rho^2/4}^t \|\eta(\cdot, t - \rho^2/4)\|^{q_\rho} dt \right)^{\frac{1}{q_\rho}} \\
\leq C\rho^{-\frac{1}{q_\rho} + \frac{1}{q_\rho} - \frac{1}{4q_\rho}} \|\eta\|_{L^{q_\rho}(Q^+_{\rho/2})},
\]

We estimate \( W_2 \). We consider the cases \( q_\rho/p \geq 2 \) and \( q_\rho/p < 2 \), respectively. If \( q_\rho/p \geq 2 \), then the H"older inequality gives

\[
\|W_2\|_{L^2(Q^+_{\rho/2})} \leq C\rho^{\frac{1}{2} - \frac{1}{q_\rho} - \frac{2}{p}} \|W_2\|_{L^{q_\rho}(Q^+_{\rho/2})}.
\]

From the same argument to prove the \( L^{q_\rho/p}, L^{q_\rho/p} \) estimate, it follows that

\[
\|W_2(\cdot, t)\|_{L^{q_\rho/p}(B_{\rho/2}^+)} \leq C \int_{t - \rho^2/4}^t (t - s)^{-\frac{1}{4}} \|\eta(\cdot, s)\|_{L^{q_\rho/p}(\mathbb{R}^n)} ds \\
= C \int_0^{\rho^2/4} \tau^{-\frac{1}{4}} \|\eta(\cdot, t - \tau)\|_{L^{q_\rho/p}(B_\rho^+)} d\tau.
\]

Thus, by \((-\rho^2/4, -\tau) \subset (-\rho^2, 0)\) for \( 0 < \tau < \rho^2/4 \), we have

\[
\|W_2\|_{L^{q_\rho/p}(Q^+_{\rho/2})} \leq C \int_0^{\rho^2/4} \tau^{-\frac{1}{4}} \left( \int_{t - \rho^2/4}^t \|\eta(\cdot, t - \tau)\|^{q_\rho} d\tau \right)^{\frac{1}{q_\rho}} d\tau \\
\leq C \int_0^{\rho^2/4} \tau^{-\frac{1}{4}} \left( \int_{t - \rho^2/4}^t \|\eta(\cdot, s)\|^{q_\rho} ds \right)^{\frac{1}{q_\rho}} d\tau \\
\leq C\rho^{1 + \frac{2(p-1)}{q_\rho} - \frac{1}{q_\rho} + MP^{-1}} \|\eta\|_{L^{q_\rho}(Q^+_{\rho/2})},
\]

and so

\[
\|W_2\|_{L^2(Q^+_{\rho/2})} \leq C\rho^{-\frac{1}{4} - \frac{1}{q_\rho} - \frac{1}{4q_\rho} + MP^{-1}} \|\eta\|_{L^{q_\rho}(Q^+_{\rho/2})}.
\]
If \( q_c/p < 2 \), then the same argument to prove the \( L^{q_c/p}L^2 \) estimate yields
\[
\|W_2(t)\|_{L^2(B^+_x)} \leq C \int_{t-\rho^2/4}^t (t-s)^{-\frac{\gamma}{2}} \|\nabla(\cdot, s)\|_{L^p(B^+_x)} \|Q_{-\rho^2,0}(s)\|_{L^q(B^+_x)} ds
\leq C \int_{\mathbb{R}} |t-s|^{-1+\gamma} \|\nabla(\cdot, s)\|_{L^p(B^+_x)} \|Q_{-\rho^2,0}(s)\|_{L^q(B^+_x)} ds
\leq CM^{p-1} \int_{\mathbb{R}} |t-s|^{-1+\gamma} \|\nabla(\cdot, s)\|_{L^p(B^+_x)} \|Q_{-\rho^2,0}(s)\|_{L^q(B^+_x)} ds,
\]
where
\[
\gamma := \frac{1}{2} - \frac{n}{2} \left( \frac{p}{q_c} - \frac{1}{2} \right).
\]
Note that \( 0 < \gamma < 1/2 \) by \( p > p_S \) and \( q_c/p < 2 \). From the Hardy-Littlewood-Sobolev inequality and the Hölder inequality with \( 2/(2\gamma + 1) < q_c \), it follows that
\[
\|W_2\|_{L^2(Q_{\rho^2/2}^+)} = \left\|\|W_2(\cdot, t)\|_{L^2(B^+_x)} \chi(-\rho^2/4,0)\right\|_{L^2(\mathbb{R})}
\leq CM^{p-1} \left\|\|\nabla(\cdot, t)\|_{L^p(B^+_x)} \chi(-\rho^2,0)\right\|_{L^{q_c/(2\gamma+1)}(\mathbb{R})}
\leq C\rho^{n/2 - \frac{\theta}{p} - \frac{n-d}{p_c} + \frac{\theta}{q_c} - \frac{n+1}{p}} M^{p-1} \|Q\|_{L^{q_c/(2\gamma+1)}(\mathbb{R})}.
\]
We consider \( W_3 \) and \( W_4 \). By the definitions of \( K_1 \) and \( K_2 \) in (2.6), there exists a constant \( C > 0 \) independent of \( \rho \) such that
\[
\rho^{-2}K_1(x-y,t) + \rho^{-1}K_2(x-y,t) \leq C\rho^{-n-3}
\]
for \( x \in B_{\rho/2}, y \in \overline{B_{\rho/5}} \setminus B_{3\rho/5} \) and \( t > 0 \). Hence by the Hölder inequality, we have
\[
\rho^{-2}|W_3(x,t)| + \rho^{-1}|W_4(x,t)| \leq C\rho^{-n-3 + \theta(n+1)/p} \|Q\|_{L^{q_c/(2\gamma+1)}(\mathbb{R})},
\]
\[
\rho^{-2}\|W_3\|_{L^2(Q_{\rho^2/2}^+)} + \rho^{-1}\|W_4\|_{L^2(Q_{\rho^2/2}^+)} \leq C\rho^{n/2 - \frac{\theta}{p} - \frac{n-d}{p_c} + \frac{\theta}{q_c} - \frac{n+1}{p}} \|Q\|_{L^{q_c/(2\gamma+1)}(\mathbb{R})}.
\]
The above estimates show the desired inequality.

**Remark A.5.** Regularity estimates similar to the above are known for semilinear elliptic equations, see [82] and the references given there for recent developments.

**Appendix B. Compactness results**

We recall an Aubin-Lions type compactness result from [90] Proposition 51.3 and [87] Proposition 2.1. See also [3] and [11] Sections 2.7, 2.8 for more general statement. Note that a pair of Banach spaces \((E_0, E_1)\) is called an interpolation couple if there exists a locally convex space \( E \) such that \( E_0, E_1 \hookrightarrow E \).

**Proposition B.1.** Let \((E_0, E_1)\) be an interpolation couple. Assume that \( E_1 \) is compactly embedded in \( E_0 \). Let \( 1 \leq p_0, p_1 < \infty, 0 < \theta < 1, 1/p_0 = (1-\theta)/p_0 + \theta/p_1 \) and \( s < 1 - \theta \). Then,
\[
W^{1,p_0}(0, T; E_0) \cap L^{p_1}(0, T; E_1) \hookrightarrow W^{s,p_0}(0, T; (E_0, E_1)_{\theta,p_0})
\]
and this embedding is compact.

We give a consequence of this result in a form which is used to prove Lemma 7.4.
Lemma B.2. Let $1 < r < \infty$ and let $\mathcal{B}$ be a smooth bounded domain in $\mathbb{R}^n$. Then,
\[ W^{1,5}(-1,0; L^r(\mathcal{B})) \cap L^5(-1,0; W^{2,r}(\mathcal{B})) \hookrightarrow C([-1,0]; W^{1,r}(\mathcal{B})) \]
and this embedding is compact.

Proof. We write $L^r_x := L^r(\mathcal{B})$, $W^{2,r}_x := W^{2,r}(\mathcal{B})$ and so on. Since $W^{2,r}_x \hookrightarrow L^r_x$ is compact, Proposition [14] with $p_0 = p_1 = 5$, $\theta = 2/3$ and $s = 1/4$ gives
\[ W^{1,5}(-1,0; L^r_x) \cap L^5(-1,0; W^{2,r}_x) \hookrightarrow W^{1/4,5}(-1,0; (L^r_x, W^{2,r}_x)_{2/3,5}). \]
By [100, page 327], we have $(L^r_x, W^{2,r}_x)_{2/3,5} = B^{4/3}_{r,5,5} \hookrightarrow W^{1,r}_x$. Then the Sobolev embedding in time yields
\[ W^{1/4,5}(-1,0; (L^r_x, W^{2,r}_x)_{2/3,5}) \hookrightarrow C([-1,0]; W^{1,r}_x), \]
and hence the lemma follows. □

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