Isolation of $k$-cliques

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Abstract

For any positive integer $k$ and any $n$-vertex graph $G$, let $\iota(G,k)$ denote the size of a smallest set $D$ of vertices of $G$ such that the graph obtained from $G$ by deleting the closed neighbourhood of $D$ contains no $k$-clique. Thus, $\iota(G,1)$ is the domination number of $G$. We prove that if $G$ is connected, then $\iota(G,k) \leq \frac{n}{k+1}$ unless $G$ is a $k$-clique or $k = 2$ and $G$ is a 5-cycle. The bound is sharp. The case $k = 1$ is a classical result of Ore, and the case $k = 2$ is a recent result of Caro and Hansberg. Our result solves a problem of Caro and Hansberg.

1 Introduction

Unless stated otherwise, we use small letters such as $x$ to denote non-negative integers or elements of a set, and capital letters such as $X$ to denote sets or graphs. The set of positive integers is denoted by $\mathbb{N}$. For $n \in \{0\} \cup \mathbb{N}$, the set $\{i \in \mathbb{N} : i \leq n\}$ is denoted by $[n]$. Note that $[0]$ is the empty set $\emptyset$. Arbitrary sets are assumed to be finite. For a set $X$, the set of 2-element subsets of $X$ is denoted by $\binom{X}{2}$ (that is, $\binom{X}{2} = \{\{x, y\} : x, y \in X, x \neq y\}$).

If $Y$ is a subset of $\binom{X}{2}$ and $G$ is the pair $(X,Y)$, then $G$ is called a graph, $X$ is called the vertex set of $G$ and is denoted by $V(G)$, and $Y$ is called the edge set of $G$ and is denoted by $E(G)$. A vertex of $G$ is an element of $V(G)$, and an edge of $G$ is an element of $E(G)$. We call $G$ an $n$-vertex graph if $|V(G)| = n$. We may represent
an edge \{v, w\} by vw. If vw \in E(G), then we say that w is a neighbour of v in G (and vice-versa). For v \in V(G), \ N_G(v) denotes the set of neighbours of v in G, \ N_G[v] denotes \ N_G(v) \cup \{v\}, and \ d_G(v) denotes |\ N_G(v)| and is called the degree of v in G. For S \subseteq V(G), \ N_G[S] denotes \ \bigcup_{v \in S} \ N_G(v) \ (the ~closed ~neighbourhood ~of ~S), \ G[S] denotes the graph \ (S, E(G) \cap (S)^2) \ (the ~subgraph ~of ~G ~induced ~by ~S), ~and ~G − S denotes G[V(G) \setminus S] \ (the ~graph ~obtained ~from ~G ~by ~deleting ~S). We may abbreviate G − \{v\} to G − v. Where no confusion arises, the subscript G is omitted from any of the notation above that uses it; for example, \ N_G(v) is abbreviated to \ N(v).

If G and H are graphs, f : V(H) \to V(G) is a bijection, and E(G) = \{f(v)f(w) : vw \in E(H)\}, then we say that G is a copy of H, and we write G \simeq H. Thus, a copy of H is a graph obtained by relabeling the vertices of H.

For \ n \geq 1, \ the ~graphs \ ([n], \binom{[n]}{2}) ~and \ ([n], \{\{i, i + 1\} : i \in [n − 1]\}) ~are ~denoted ~by \ K_n ~and \ P_n, ~respectively. ~A ~copy ~of ~K_n ~is ~called ~a ~complete ~graph ~or ~an ~n-clique. ~A ~copy ~of ~P_n ~is ~called ~an ~n-path ~or ~simply ~a ~path.

If G and H are graphs such that V(H) \subseteq V(G) and E(H) \subseteq E(G), then H is called a subgraph of G, and we say that G contains H.

If \ F \ is ~a ~set ~of ~graphs ~and \ F \ is ~a ~copy ~of ~a ~graph ~in \ F, ~then ~we ~call \ F ~an ~\mathcal{F}-graph. ~If G is a graph and \ D \subseteq V(G) such that G − \ N[D] ~contains ~no ~\mathcal{F}-graph, ~then \ D ~is ~called ~an ~\mathcal{F}-isolating ~set ~of ~G. ~Let \ i(G, \mathcal{F}) ~denote ~the ~size ~of ~a ~smallest ~\mathcal{F}-isolating ~set ~of ~G. ~The study of isolating sets was introduced recently by Caro and Hansberg \[1\]. ~It ~is ~an ~appealing ~and ~natural ~generalization ~of ~the ~classical ~domination problem \[2, 3, 4, 5, 6, 7\]. ~Indeed, \ D ~is ~a ~\{K_1\}-isolating ~set ~of ~G ~if ~and ~only ~if ~D ~is ~a ~dominating ~set ~of ~G ~(that ~is, \ N[D] = V(G)), ~so \ i(G, \{K_1\}) ~is ~the ~domination ~number ~of ~G ~(the ~size ~of ~a ~smallest ~dominating ~set ~of ~G). ~In this paper, we obtain a sharp upper bound for \ i(G, \{K_k\}), ~and ~consequently ~we ~solve ~a ~problem ~of ~Caro ~and ~Hansberg \[1\].

We call a subset \ D ~of ~V(G) ~a ~k-clique isolating set of G if \ G − \ N[D] ~contains ~no ~k-clique. ~We ~denote ~the ~size ~of ~a ~smallest ~k-clique isolating set of G by \ i(G, k). ~Thus, \ i(G, k) = \ i(G, \{K_k\}).

If \ G_1, \ldots, G_t ~are ~graphs ~such ~that ~V(G_i) \cap V(G_j) = \emptyset ~for ~every ~i, j \in [t] ~with ~i \neq j, ~then \ G_1, \ldots, G_t ~are ~vertex-disjoint. ~A ~graph ~G ~is ~connected ~if, ~for ~every ~v, w \in V(G), ~G ~contains ~a ~path ~P ~with ~v, w \in V(P). ~A ~connected ~subgraph ~H ~of ~G ~is ~a ~component ~of ~G ~if, ~for ~each ~connected ~subgraph ~K ~of ~G ~with \ K \neq H, ~H ~is ~not ~a ~subgraph ~of ~K.

Clearly, ~two ~distinct ~components ~of ~G ~are ~vertex-disjoint.

For \ n, k \in \mathbb{N}, \ let \ a_{n,k} = \left\lfloor \frac{n}{k+1} \right\rfloor ~and ~b_{n,k} = n − ka_{n,k}. ~We ~have \ a_{n,k} \leq b_{n,k} \leq a_{n,k} + k.

If \ n \leq k, ~then ~let \ B_{n,k} = P_n. ~If \ n \geq k + 1, ~then ~let \ F_1, \ldots, F_{a_{n,k}} ~be ~copies ~of ~K_k ~such ~that \ P_{a_{n,k}}, F_1, \ldots, F_{a_{n,k}} ~are ~vertex-disjoint, ~and ~let \ B_{n,k} ~be ~the ~connected ~n-vertex ~graph ~given ~by

\[ B_{n,k} = \left( V(P_{b_{n,k}}) \cup \bigcup_{i=1}^{a_{n,k}} V(F_i), E(P_{b_{n,k}}) \cup \{iv : i \in [a_{n,k}], v \in V(F_i)\} \cup \bigcup_{i=1}^{a_{n,k}} E(F_i) \right). \]

Thus, \ B_{n,k} ~is ~the ~graph ~obtained ~by ~taking ~P_{b_{n,k}}, F_1, \ldots, F_{a_{n,k}} ~and ~joining ~i ~(a ~vertex ~of ~P_{b_{n,k}}) ~to ~each ~vertex ~of ~F_i ~for ~each ~i \in [a_{n,k}].

For \ n, k \in \mathbb{N} ~with ~k \neq 2, ~let

\[ i(n, k) = \max\{i(G, k) : G ~is ~a ~connected ~graph, V(G) = [n], G \not\simeq K_k\}. \]
For $n \in \mathbb{N}$, let

$$\iota(n, 2) = \max\{\iota(G, 2) : G \text{ is a connected graph}, V(G) = [n], G \not\cong K_2, G \not\cong C_5\}.$$  

In Section 2, we prove the following result.

**Theorem 1.1** If $G$ is a connected $n$-vertex graph, then, unless $G$ is a $k$-clique or $k = 2$ and $G$ is a 5-cycle,

$$\iota(G, k) \leq \frac{n}{k + 1}.$$  

Consequently, for any $k \geq 1$ and $n \geq 3$,

$$\iota(n, k) = \iota(B_{n,k}, k) = \left\lfloor \frac{n}{k + 1} \right\rfloor.$$  

A classical result of Ore [8] is that the domination number of a graph $G$ with $\min\{d(v) : v \in V(G)\} \geq 1$ is at most $\frac{n}{2}$ (see [4]). Since the domination number is $\iota(G, 1)$, it follows by Lemma 2.2 in Section 2 that Ore’s result is equivalent to the bound in Theorem 1.1 for $k = 1$. The case $k = 2$ is also particularly interesting; while deleting the closed neighbourhood of a $K_1$-isolating set yields the graph with no vertices, deleting the closed neighbourhood of a $K_2$-isolating set yields a graph with no edges. In [1], Caro and Hansberg proved Theorem 1.1 for $k = 2$, using a different argument. Consequently, they established that

$$\frac{1}{k + 1} \leq \limsup_{n \to \infty} \frac{\iota(n,k)}{n} \leq \frac{1}{3}.$$  

In the same paper, they asked for the value of $\limsup_{n \to \infty} \frac{\iota(n,k)}{n}$. The answer is given by Theorem 1.1.

**Corollary 1.2** For any $k \geq 1$,

$$\lim_{n \to \infty} \sup \{\frac{\iota(n,k)}{n} : p \geq n\} = \frac{1}{k + 1}.$$  

**Proof.** By Theorem 1.1 for any $n \geq 3$, $\frac{1}{k + 1} - \frac{k}{(k+1)n} = \frac{n-k}{(k+1)n} \leq \frac{\iota(n,k)}{n} \leq \frac{1}{k + 1}$, and, if $n$ is a multiple of $k + 1$, then $\frac{\iota(n,k)}{n} = \frac{1}{k + 1}$. Thus, $\lim_{n \to \infty} \sup \{\frac{\iota(p,k)}{p} : p \geq n\} = \lim_{n \to \infty} \frac{1}{k + 1} = \frac{1}{k + 1}$. \hfill $\square$

**2 Proof of Theorem 1.1**

In this section, we prove Theorem 1.1. We start with two lemmas that will be used repeatedly.

If a graph $G$ contains a $k$-clique $H$, then we call $H$ a $k$-clique of $G$. We denote the set $\{V(H) : H \text{ is a } k\text{-clique of } G\}$ by $\mathcal{C}_k(G)$.

**Lemma 2.1** If $v$ is a vertex of a graph $G$, then $\iota(G, k) \leq 1 + \iota(G - N_G[v], k)$.

**Proof.** Let $D$ be a $k$-clique isolating set of $G - N_G[v]$ of size $\iota(G - N_G[v], k)$. Clearly, $C \cap N_G[v] \neq \emptyset$ for each $C \in \mathcal{C}_k(G) \setminus \mathcal{C}_k(G - N_G[v])$. Thus, $D \cup \{v\}$ is a $k$-clique isolating set of $G$. The result follows. \hfill $\square$
Lemma 2.2 If \( G_1, \ldots, G_r \) are the distinct components of a graph \( G \), then \( \iota(G, k) = \sum_{i=1}^{r} \iota(G_i, k) \).

Proof. For each \( i \in [r] \), let \( D_i \) be a smallest \( k \)-clique isolating set of \( G_i \). Then, \( \bigcup_{i=1}^{r} D_i \) is a \( k \)-clique isolating set of \( G \). Thus, \( \iota(G, k) \leq \sum_{i=1}^{r} |D_i| = \sum_{i=1}^{r} \iota(G_i, k) \). Let \( D \) be a smallest \( k \)-clique isolating set of \( G \). For each \( i \in [r] \), \( D \cap V(G_i) \) is a \( k \)-clique isolating set of \( G_i \), so \( |D_i| \leq |D \cap V(G_i)| \). We have \( \sum_{i=1}^{r} \iota(G_i, k) = \sum_{i=1}^{r} |D_i| \leq \sum_{i=1}^{r} |D \cap V(G_i)| = |D| = \iota(G, k) \). The result follows. □

Proof of Theorem 1.1. We use induction on \( n \). If \( G \) is a \( k \)-clique, then \( \iota(G) = 1 = \frac{n+1}{k+1} \). If \( k = 2 \) and \( G \) is a 5-cycle, then \( \iota(G) = 2 = \frac{n+1}{k+1} \). Suppose that \( G \) is not a \( k \)-clique and that, if \( k = 2 \), then \( G \) is not a 5-cycle. Suppose \( n \leq 2 \). If \( k \geq 3 \), then \( \iota(G) = 0 \). If \( k = 2 \), then \( G \simeq K_1 \), so \( \iota(G) = 0 \). If \( k = 1 \), then \( G \simeq K_2 \), so \( \iota(G) = 1 = \frac{n}{k+1} \). Now suppose \( n \geq 3 \). If \( C_k(G) = \emptyset \), then \( \iota_k(G) = 0 \). Suppose \( C_k(G) \neq \emptyset \). Let \( C \in C_k(G) \). Since \( G \) is connected and \( G \) is not a \( k \)-clique, there exists some \( v \in C \) such that \( N[v] \setminus C \neq \emptyset \). Thus, \( |N[v]| \geq k + 1 \) as \( C \subset N[v] \). If \( V(G) = N[v] \), then \( \{v\} \) is a \( k \)-clique isolating set of \( G \), so \( \iota(G) = 1 \leq \frac{n}{k+1} \). Suppose \( V(G) \neq N[v] \). Let \( G' = G - N[v] \) and \( n' = |V(G')| \). Then,

\[
 n \geq n' + k + 1
\]

and \( V(G') \neq \emptyset \). Let \( H \) be the set of components of \( G' \). If \( k \neq 2 \), then let \( H' = \{H \in H' : H \simeq K_k\} \). If \( k = 2 \), then let \( H' = \{H \in H : H \simeq K_2 \text{ or } H \simeq C_4\} \). By the induction hypothesis, \( \iota(H, k) \leq \frac{|V(H)|}{k+1} \) for each \( H \in H \setminus H' \). If \( H' = \emptyset \), then, by Lemmas 2.1 and 2.2,

\[
 \iota(G, k) \leq 1 + \iota(G', k) = 1 + \sum_{H \in H} \iota(H, k) \leq 1 + \sum_{H \in H} \frac{|V(H)|}{k+1} = \frac{k+1+n'}{k+1} \leq \frac{n}{k+1}.
\]

Suppose \( H' \neq \emptyset \). For any \( H \in H \) and any \( x \in N(v) \), we say that \( H \) is linked to \( x \) if \( xy \in E(G) \) for some \( y \in V(H) \). Since \( G \) is connected, each member of \( H \) is linked to at least one member of \( N(v) \). One of Case 1 and Case 2 below holds.

Case 1: For each \( H \in H' \), \( H \) is linked to at least two members of \( N(v) \). Let \( H' \in H' \) and \( x \in N(v) \) such that \( H' \) is linked to \( x \). Let \( H_x \) be the set of members of \( H \) that are linked to \( x \) only. Then,

\[
 H_x \subseteq H \setminus H',
\]

and hence, by the induction hypothesis, each member \( H \) of \( H_x \) has a \( k \)-clique isolating set \( D_H \) with \( |D_H| \leq \frac{|V(H)|}{k+1} \).

Let \( X = \{x\} \cup V(H') \) and \( G^* = G - X \). Then, \( G^* \) has a component \( G^*_v \) with \( N[v] \setminus \{x\} \subseteq V(G^*_v) \), and the other components of \( G^* \) are the members of \( H_x \). Let \( D^*_v \) be a \( k \)-clique isolating set of \( G^*_v \) of size \( \iota(G^*_v, k) \). Since \( H' \) is linked to \( x \), \( xy \in E(G) \) for some \( y \in V(H') \). If \( H' \) is a \( k \)-clique, then let \( D' = \{y\} \). If \( k = 2 \) and \( H' \) is a 5-cycle, then let \( y' \) be one of the two vertices in \( V(H') \setminus N_{H'}[y] \), and let \( D' = \{y, y'\} \). We have \( X \subseteq N[D'] \) and \( |D'| = \frac{|X|}{k+1} \). Let \( D = D' \cup D^*_v \cup \bigcup_{H \in H_x} D_H \). Since the components of
Thus, we have induction hypothesis. By (1),

\[ \nu(G, k) \leq |D| = |D'_v| + |D'| + \sum_{H \in \mathcal{H}_x} |D_H| \leq |D'_v| + \frac{|X|}{k + 1} + \sum_{H \in \mathcal{H}_x} |V(H)|. \]  

(1)

**Subcase 1.1:** \( G^*_v \) is neither a \( k \)-clique nor a 5-cycle. Then, \( |D'_v| \leq \frac{|V(G^*_v)|}{k + 1} \) by the induction hypothesis. By (1), \( \nu(G, k) \leq \frac{|V(G^*_v)|}{k + 1} \).

**Subcase 1.2:** \( G^*_v \) is a \( k \)-clique. Since \( |N[v]| \geq k + 1 \) and \( N[v] \subseteq V(G^*_v) \), we have \( V(G^*_v) = N[v] \setminus \{x\} \). If \( H' \) is a \( k \)-clique, then let \( X' = \{y\} \) and \( D'' = \{x\} \). If \( k = 2 \) and \( H' \) is a 5-cycle, then let \( X' \) be the set whose members are \( y, y' \), and the two neighbours of \( y' \) in \( H' \), and let \( D'' = \{x, y', x\} \). Let \( Y = (X \cup V(G^*_v)) \setminus \{v, x \cup X'\} \). Let \( G_Y = G - (\{v, x \cup X'\}) \). Then, the components of \( G_Y \) are the components of \( G[Y] \) and the members of \( \mathcal{H}_x \).

If \( G[Y] \) has no \( k \)-clique, then, since \( \{v, x \cup X' \} \subseteq N[D''] \), \( D'' \cup \bigcup_{H \in \mathcal{H}_x} D_H \) is a \( k \)-clique isolating set of \( G \), and hence

\[ \nu(G, k) \leq |D''| + \sum_{H \in \mathcal{H}_x} |D_H| \leq \frac{|X \cup V(G^*_v)|}{k + 1} + \sum_{H \in \mathcal{H}_x} |V(H)| = \frac{n}{k + 1}. \]

This is the case if \( k = 1 \) as we then have \( Y = \emptyset \).

Suppose that \( k \geq 2 \) and \( G[Y] \) has a \( k \)-clique \( C_Y \). We have

\[ V(C_Y) \subseteq (V(G^*_v) \setminus \{v\}) \cup (V(H') \setminus X'). \]  

(2)

Thus, \( |V(C_Y) \cap V(G^*_v)| = |V(C_Y) \setminus (V(H') \setminus X')| \geq \sum_{x \in X'} \frac{|V(C_Y)|}{k + 1} + \sum_{H \in \mathcal{H}_x} |V(H)| = \frac{|V(C_Y)|}{k + 1} + \sum_{H \in \mathcal{H}_x} |V(H)| = \frac{|V(G^*_v)|}{k + 1} + \sum_{H \in \mathcal{H}_x} |V(H)| = \frac{|V(G^*_v)|}{k + 1} + \sum_{H \in \mathcal{H}_x} |V(H)| \geq \sum_{x \in X'} \frac{|V(C_Y)|}{k + 1} + \sum_{H \in \mathcal{H}_x} |V(H)| \geq k + 1. \]

Thus, \( Z \subseteq N[z] \).

(3)

We have

\[ |Z| = |V(G^*_v)| + |V(C_Y) \setminus V(G^*_v)| = k + |V(C_Y) \cap V(H')| \geq k + 1. \]  

(4)

Let \( G_Z = G - Z \). Then, \( V(G_Z) = \{x\} \cup (V(H') \setminus V(C_Y)) \cup \bigcup_{H \in \mathcal{H}_x} V(H) \). We have that the components of \( G_Z - x \) are \( G_Z[V(H') \setminus V(C_Y)] \) (which is a clique or a path, depending on whether \( H' \) a \( k \)-clique or a 5-cycle) and the members of \( \mathcal{H}_x \), \( y \in V(H') \setminus V(C_Y) \) (by (2)), \( y \in N_{G_Z}[x] \), and, by the definition of \( \mathcal{H}_x \), \( N_{G_Z}[x] \cap V(H) \neq \emptyset \) for each \( H \in \mathcal{H}_x \). Thus, \( G_Z \) is connected, and, if \( \mathcal{H}_x \neq \emptyset \), then \( G_Z \) is neither a clique nor a 5-cycle.

Suppose \( \mathcal{H}_x \neq \emptyset \). By the induction hypothesis, \( \nu(G, k) \leq \frac{|V(G_Z)|}{k + 1} \). Let \( D_{G_Z} \) be a \( k \)-clique isolating set of \( G_Z \) of size \( \nu(G, k) \). By (3), \( \{z\} \cup D_{G_Z} \) is a \( k \)-clique isolating set of \( G \). Thus, \( \nu(G, k) \leq 1 + \nu(G, k) \leq 1 + \frac{|V(G^*_v)|}{k + 1} \), and hence, by (1),

\[ \nu(G, k) \leq \frac{|Z|}{k + 1} + \frac{|V(G^*_v)|}{k + 1} = \frac{k}{k + 1}. \]

Now suppose \( \mathcal{H}_x = \emptyset \). Then, \( G^*_v = G^*_v \), so \( V(G) = V(G^*_v) \cup \{x\} \cup V(H') \). Recall that either \( H' \) is a \( k \)-clique or \( k = 2 \) and \( H' \) is a 5-cycle.
Suppose that $H'$ is a $k$-clique. Then, $n = 2k + 1$. By (3), $|V(G - N[z])| \leq |V(G - Z)| = n - |Z| = 2k + 1 - |Z|$. Suppose $|Z| \geq k + 2$. Then, $|V(G - N[z])| \leq k - 1$, and hence $\{z\}$ is a $k$-clique isolating set of $G$. Thus, $\iota(G, k) = 1 < \frac{n}{k+1}$. Now suppose $|Z| \leq k + 1$. Then, by (1), $|Z| = k + 1$ and $|V(C_Y) \cap V(H')| = 1$. Let $z'$ be the element of $V(C_Y) \cap V(H')$, and let $Z' = V(C_Y) \cup \{z'\}$. Since $z'$ is a vertex of each of the $k$-cliques $C_Y$ and $H'$, $Z' \subseteq N[z']$. We have $|Z'| = |V(C_Y)| + |V(H')| - |V(C_Y) \cap V(H')| = 2k - 1$ and $|V(G - N[z'])| = |V(G - Z')| = n - |Z'| = (2k + 1) - (2k - 1) = 2$. If $k \geq 3$, then $\{z'\}$ is a $k$-clique isolating set of $G$, and hence $\iota(G) = 1 < \frac{n}{k+1}$. Suppose $k = 2$. Then, $H', G_2^*$, and $C_Y$ are the 2-cliques with vertex sets $\{y, z'\}$, $\{v, z\}$, and $\{z, z'\}$, respectively. Thus, $V(G) = \{v, z, z', y, x\}$, and $G$ contains the 5-cycle with edge set $\{vx, vz, z'y, yx\}$. Since $G$ is not a 5-cycle, $d(w) \geq 3$ for some $w \in V(G)$. Since $|V(G - N[w])| = 5 - |N[w]| \leq 1$, $\{w\}$ is a $k$-clique isolating set of $G$, and hence $\iota(G, k) = 1 < \frac{n}{3} = \frac{n}{k+1}$.

Now suppose that $k = 2$ and $H'$ is a 5-cycle. Then, $V(G_2^*) = \{v, z\}$ and $E(H') = \{yy_1, y_1y_2, y_2y_3, y_3y_4, y_4y\}$ for some $y_1, y_2, y_3, y_4 \in V(G)$. Recall that $|V(C_Y) \cap V(H')| \geq 1$. Let $z' \in V(C_Y) \cap V(H')$. Since $z$ and $z'$ are vertices of $C_Y$, $z' \in E(G)$. We have $V(G) = \{v, z, x, y, y_1, y_2, y_3, y_4\}, N(v) = \{x, z\}, z' \in \{y_1, y_2, y_3, y_4\}$ (as $y \notin V(C_Y)$ by (2)), and $\{vx, vz, xy, z'y, yx\} \subseteq E(H') \subseteq E(G)$. If $z'$ is $y_1$ or $y_2$, then $V(G - N[\{y, z'\}]) = \{y, y_3\}$ or $\{y, y_2\}$. If $z'$ is $y_3$ or $y_4$, then $V(G - N[\{y, z'\}]) = \{v\}$. Thus, $\{v, z\}$ is a $k$-clique reducing set of $G$, and hence $\iota(G, k) = 2 < \frac{n}{3} = \frac{n}{k+1}$.

**Subcase 1.3: $G_2^*$ is a 5-cycle.** If $k \neq 2$, then the result follows as in Subcase 1.1. Suppose $k = 2$. We have $E(G_2^*) = \{vv_1, v_1v_2, v_2v_3, v_3v_4, v_4v\}$ for some $v_1, v_2, v_3, v_4 \in V(G)$. Let $Y = \{v_2, v_3, v_4\}$. Recall that the components of $G^*$ are $G_2^*$ and the members of $H_x$. Thus, $G - Y$ is connected and $V(G - Y) = \{v, v_1, x\} \cup V(H') \cup \bigcup_{H \in H_x} V(H)$.

Suppose that $G - Y$ is not a 5-cycle. By the induction hypothesis, $G - Y$ has a $k$-clique isolating set $D$ with $|D| \leq \frac{|V(G - Y)|}{k+1} = \frac{n^3}{4} = \frac{n}{4} - 1$. Since $Y \subseteq N[v_3], \{v_3\} \cup D$ is a $k$-clique isolating set of $G$, so $\iota(G, k) \leq \frac{n}{3} = \frac{n}{k+1}$.

Now suppose that $G - Y$ is a 5-cycle. Then, $H'$ is a 2-clique and $V(G - Y) = \{v, v_1, x, y, z\}$, where $\{z\} = V(H') \setminus \{y\}$. Since $v_1, v, x, y, z \in E(G - Y)$ and $G - Y$ is a 5-cycle, $E(G - Y) = \{v_1v, v_1x, xy, yz, zv_1\}$. We have $V(G - N[\{v, v_1\}]) \subseteq \{v_3, y\}$. If $v_3y \notin E(G)$, then $\{v, v_1\}$ is a $k$-clique isolating set of $G$. If $v_3y \in E(G)$, then $V(G - N[v, v_3]) \subseteq \{z\}$, so $\{v, v_3\}$ is a $k$-clique isolating set of $G$. Therefore, $\iota(G, k) = 2 < \frac{n}{3} = \frac{n}{k+1}$.

**Case 2:** For some $x \in N[v]$ and some $H' \in \mathcal{H}', H'$ is linked to $x$ only. Let $\mathcal{H}_1 = \{H \in H': H$ is linked to $x$ only$\}$ and $\mathcal{H}_2 = \{H \in H \setminus H': H$ is linked to $x$ only$\}$. Let $h_1 = |\mathcal{H}_1|$ and $h_2 = |\mathcal{H}_2|$. Since $H' \in \mathcal{H}_1, h_1 \geq 1$. For each $H \in \mathcal{H}_1, y_H \in N(x)$ for some $y_H \in V(H)$. Let $X = \{x\} \cup \bigcup_{H \in \mathcal{H}_1} V(H)$.

For each $k$-clique $H \in \mathcal{H}_1$, let $D_H = \{x\}$. If $k = 2$, then, for each $5$-cycle $H \in \mathcal{H}_1$, let $y_H$ be one of the two vertices in $V(H) \setminus N_H[y_H]$, and let $D_H = \{x, y_H\}$. Let $D_X = \bigcup_{H \in \mathcal{H}_1} D_H$. Then, $D_X$ is a $k$-clique isolating set of $G[X]$. If $k \neq 2$, then $D_X = \{x\}$, so $|D_X| = 1 \leq \frac{\frac{k+1}{k+1} + 1}{k+1} = \frac{|X|}{k+1}$. If $k = 2$ and we let $h_1' = |\{H \in \mathcal{H}_1: H \simeq C_5\}|$, then $|D_X| = 1 + h_1' \leq \frac{\frac{4k+2}{k+1} + 2(k_1' - h_1)}{k+1} = \frac{|X|}{k+1}$.

Let $G^* = G - X$. Then, $G^*$ has a component $G_2^*$ with $N[v] \setminus \{x\} \subseteq V(G_2^*)$, and the other components of $G^*$ are the members of $\mathcal{H}_2$. By the induction hypothesis,
\( \iota(H, k) \leq \frac{|V(H)|}{k+1} \) for each \( H \in \mathcal{H}_2 \). For each \( H \in \mathcal{H}_2 \), let \( D_H \) be a \( k \)-clique isolating set of \( H \) of size \( \iota(H, k) \).

If \( G_v^* \) is a \( k \)-clique, then let \( D_v^* = \{x\} \). If \( k = 2 \) and \( G_v^* \) is a 5-cycle, then let \( v' \) be one of the two vertices in \( V(G_v^*) \setminus N_{G_v^*}(v) \), and let \( D_v^* = \{x, v'\} \). If neither \( G_v^* \) is a \( k \)-clique nor \( k = 2 \) and \( G_v^* \) is a 5-cycle, then, by the induction hypothesis, \( G_v^* \) has a \( k \)-clique isolating set \( D_v^* \) with \( |D_v^*| \leq |V(G_v^*)| \).

Let \( D = D_v^* \cup D_X \cup \bigcup_{H \in \mathcal{H}_2} D_H \). By the definition of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), the components of \( G - x \) are \( G_v^* \) and the members of \( \mathcal{H}_1 \cup \mathcal{H}_2 \). Thus, \( D \) is a \( k \)-clique isolating set of \( G \) since \( x \in D \), \( v \in V(G_v^*) \cap N[x] \), and \( D_X \) is a \( k \)-clique isolating set of \( G[X] \). Let \( D' = D_X \cup \bigcup_{H \in \mathcal{H}_2} D_H \) and \( n^* = |V(G_v^*)| \). We have
\[
|D'| = |D_X| + \sum_{H \in \mathcal{H}_2} |D_H| \leq \frac{|X|}{k+1} + \sum_{H \in \mathcal{H}_2} \frac{|V(H)|}{k+1} = \frac{n - n^*}{k+1}.
\]

If \( G_v^* \) is a \( k \)-clique, then \( |D| = |D'| < \frac{n}{k+1} \). If \( k = 2 \) and \( G_v^* \) is a 5-cycle, then
\[
|D| = 1 + |D'| \leq 1 + \frac{n - n^*}{k+1} = 1 + \frac{n - 5}{3} < \frac{n}{k+1}.
\]

If neither \( G_v^* \) is a \( k \)-clique nor \( k = 2 \) and \( G_v^* \) is a 5-cycle, then \( |D| = |D_v^*| + |D'| \leq \frac{n^*}{k+1} + \frac{n - n^*}{k+1} = \frac{n}{k+1} \). \( \square \)

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