BOUNDARY CONTROL OF THE BEAM EQUATION BY LINEAR QUADRATIC REGULATION

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ABSTRACT. We present and solve a Linear Quadratic Regulator (LQR) for the boundary control of the beam equation. We use the simple technique of completing the square to get an explicit solution. By decoupling the spatial frequencies we are able to reduce an infinite dimensional LQR to an infinite family of two two dimensional LQRs each of which can be solved explicitly.

1. INTRODUCTION

We consider the stabilization of the linear beam equation using a Linear Quadratic Regulator (LQR). By decoupling the spatial frequencies we obtain a complete and explicit solution to the LQR including the closed loop eigenvalues. The simple technique of completing the square yields Riccati PDEs. At each spatial frequency the Riccati PDEs reduce to the algebraic Riccati equation of a two dimensional problem which is readily solvable. The sums of the optimal cost and optimal feedback of these two dimensional problems yield the optimal cost and optimal feedback of the infinite dimensional LQR. The only technical issues that arise are whether these sums are convergent. We discuss which Lagrangian yield convergence.

The study of optimal control of systems governed by partial differential equations goes back at least to Lions [18]. More recent treatises on this topic are the works of Curtain and Zwart [6], [7], Lasiecka and Trigiani [17] and Krstic and Smyshlyaev [15] who use backstepping. LQR boundary control has been used by Lasiecka and Trigiani [16], Burns and King [3], Burns and Hulsing [2], Cristofaro, DeLuca and Lanari [5], Coron, D’Andrea and Bastin [4] found Lyapunov functions for the boundary control of hyperbolic conservation laws. Guo et al. have considered boundary control of the beam equation in the presence of disturbances, [9], [8]. Other papers on boundary control of the beam equation are Morgul [20], Militec and Arnold [19], Han, Li, Xu and Liu [10].

More recently we introduced the Completing the Square technique to solve LQR problems for partial differential equations. We solved an LQR problem for the heat equation under distributed control in [14] and under boundary control in [13]. In both cases using an extension of Al’brekht’s method [1] we were able to find the higher degree terms in the Taylor polynomial expansions of the optimal cost and the optimal feedback.

In [14] we solved an LQR for the boundary control of the wave equation by decoupling the spatial frequencies. This allowed us to reduce an infinite dimensional LQR problem to an infinite family of two dimensional LQR problems each of which can be explicitly solved. In this paper we show that an LQR problem for the beam equation can also be...
reduced to an infinite family of two dimensional LQR problems each of which can be explicitly solved.

2. BOUNDARY CONTROL OF THE BEAM EQUATION

In Exercise 3.18 Curtain and Zwart consider the undamped beam equation subject to boundary control action

\[ \frac{\partial^2 f}{\partial t^2}(x, t) = -\frac{\partial^4 f}{\partial x^4}(x, t) \]
\[ f(0, t) = 0, \quad f(1, t) = 0 \]
\[ \frac{\partial^2 f}{\partial x^2}(0, t) = 0, \quad \frac{\partial^2 f}{\partial x^2}(1, t) = u(t) \]
\[ f(x, 0) = f_1(x), \quad \frac{\partial f}{\partial t}(x, 0) = f_2(x) \]

It is convenient to introduce vector notation, let
\[ z(x, t) = \begin{bmatrix} z_1(x, t) \\ z_2(x, t) \end{bmatrix} = \begin{bmatrix} f(x, t) \\ \frac{\partial f}{\partial t}(x, t) \end{bmatrix} \]

We also allow damping
\[ \frac{d}{dt}z(x, t) = Az(x, t) \]
where \( \alpha \geq 0 \) and \( A \) is the matrix differential operator
\[ A = \begin{bmatrix} 0 & 1 \\ -\frac{\partial^4}{\partial x^4} & -\alpha \end{bmatrix}. \]

The boundary conditions are
\[ z_1(0, t) = 0, \quad z_1(1, t) = 0 \]
\[ \frac{\partial^2 z_1}{\partial x^2}(0, t) = 0, \quad \frac{\partial^2 z_1}{\partial x^2}(1, t) = u(t) \]
and the initial conditions are
\[ z_1(x, 0) = f_1(x), \quad z_2(x, 0) = f_2(x) \]

If \( \alpha = 0 \) the beam is undamped and if \( \alpha > 0 \) the beam is damped. The eigenvalues of the open loop system are
\[ \lambda_n = -\alpha + \text{sign}(n)\sqrt{\alpha^2 - 4n^2\pi^4} \]
and the corresponding eigenvectors are
\[ v_n = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \frac{n}{\sqrt{\pi}} \end{bmatrix} \sin n|\pi x| \]
for \( n = \pm 1, \pm 2, \pm 3, \ldots \) If \( |n| \) is small enough the eigenvalues \( \lambda_n, \lambda_{-n} \) can be real numbers but for large \( |n| \) the eigenvalues \( \lambda_n, \lambda_{-n} \) are complex and conjugate.

We wish to find a feedback to stabilize the beam, \( z(x, t) \to 0 \) as \( t \to \infty \). Another possibility is that we wish to stabilize the beam to some open loop trajectory \( z^*(x, t) \). We define \( \tilde{z}(x, t) = z(x, t) - z^*(x, t) \) and we seek a feedback to drive \( \tilde{z}(x, t) \to 0 \) as \( t \to \infty \). Because of linearity these are equivalent problems so we only consider the first one.
We shall use a Linear Quadratic Regulator (LQR) to find the desired feedback. We choose a $2 \times 2$ nonnegative definite matrix valued function $Q(x_1, x_2)$ which is symmetric in $x_1, x_2$, $Q(x_1, x_2) = Q(x_2, x_1)$, and a positive scalar $R$. Consider the problem of minimizing

$$
\int_0^\infty \left( \int_S z'(x_1, t)Q(x_1, x_2)z(x_2, t) \, dA + Ru^2(t) \right) \, dt
$$

subject to the beam dynamics where $S = [0, 1]^2$ and $dA = dx_1dx_2$.

Let $P(x_1, x_2)$ be any $2 \times 2$ symmetric matrix valued function which is also symmetric in $x_1, x_2$, $P(x_1, x_2) = P(x_2, x_1)$. Suppose there is a control trajectory $u(t)$ such that the corresponding state trajectory $z(x, t)$ goes to 0 as $t \to \infty$ then by the Fundamental Theorem of Calculus

$$
0 = \int_S z'(x_1, t)P(x_1, x_2)z(x_2, t) \, dA \\
+ \int_0^\infty \int_S \frac{d}{dt} (z'(x_1, t)P(x_1, x_2)z(x_2, t)) \, dA \, dt
$$

We expand the integrand of the time integral.

$$
0 = \int_S z'(x_1, t)P(x_1, x_2)z(x_2, t) \, dA \\
+ \int_0^\infty \int_S \frac{dz'}{dt}(x_1, t)P(x_1, x_2)z(x_2, t) \, dA \, dt \\
+ \int_0^\infty \int_S z'(x_1, t)Q(x_1, x_2) \frac{dz}{dt}(x_2, t) \, dA \, dt
$$

(2) \quad 0 = \int_S z'(x_1, t)P(x_1, x_2)z(x_2, t) \, dA \\
+ \int_0^\infty \int_S z_2(x_1, t)P_{1,1}(x_1, x_2)z_1(x_2, t) + z_1(x_1, t)P_{1,2}(x_1, x_2)z_2(x_2, t) \\
+ z_2(x_1, t)P_{2,1}(x_1, x_2)z_2(x_2, t) + z_2(x_1, t)P_{2,2}(x_1, x_2)z_2(x_2, t) \\
- \frac{\partial^2 z_1}{\partial x_1^2}(x_1, t)P_{2,1}(x_1, x_2)z_1(x_2, t) - z_1(x_1, t)P_{1,2}(x_1, x_2) \frac{\partial^2 z_1}{\partial x_2^2}(x_2, t)
$$

(3) \quad - \frac{\partial^2 z_1}{\partial x_1^2}(x_1, t)P_{2,2}(x_1, x_2)z_2(x_2, t) - z_2(x_1, t)P_{2,2}(x_1, x_2) \frac{\partial^2 z_1}{\partial x_2^2}(x_2, t)

$$
- \alpha z_1(x_1, t)P_{1,1}(x_1, x_2)z_1(x_2, t) - \alpha z_2(x_1, t)P_{2,1}(x_1, x_2)z_2(x_2, t) \\
- \alpha z_1(x_1, t)P_{1,2}(x_1, x_2)z_2(x_2, t) - \alpha z_2(x_1, t)P_{2,2}(x_1, x_2)z_2(x_2, t) \, dA \, dt
We assume that $P(x_1, x_2)$ satisfies these boundary conditions

\begin{equation}
P(0, x_2) = P(1, x_2) = P(x_1, 0) = P(x_1, 1) = 0
\end{equation}

We integrate by parts four times to get these identities

\begin{align*}
&\int_S -\frac{\partial^4 z_1}{\partial x_1^4}(x_1, t)P_{2,1}(x_1, x_2)z_1(x_2, t) \, dA \\
&= \int_S u(t)\frac{\partial P_{2,1}}{\partial x_1}(1, x_2)z_1(x_2, t) - z_1(x_1, t)\frac{\partial^4 P_{2,1}}{\partial x_1^4}(x_1, x_2)z_1(x_2, t) \, dA \\
&\int_S -z_1(x_1, t)P_{1,2}(x_1, x_2)\frac{\partial^4 z_1}{\partial x_2^4}(x_2, t) \, dA \\
&= \int_S -z_1(x_1, t)\frac{\partial^4 P_{1,2}}{\partial x_2^4}(x_1, x_2)z_1(x_2, t) + z_1(x_1, t)\frac{\partial P_{1,2}}{\partial x_2}(x_1, 1)u(t) \, dA \\
&\int_S \frac{\partial^4 z_1}{\partial x_1^4}(x_1, t)P_{2,2}(x_1, x_2)z_2(x_2, t) \, dA \\
&= \int_S u(t)\frac{\partial P_{2,2}}{\partial x_2}(1, x_2)z_2(x_2, t) - z_1(x_1, t)\frac{\partial^4 P_{2,2}}{\partial x_1^4}(x_1, x_2)z_2(x_2, t) \, dA \\
&\int_S -z_1(x_1, t)P_{2,2}(x_1, x_2)\frac{\partial^4 z_1}{\partial x_2^4}(x_2, t) \, dA \\
&= \int_S -z_2(x_1, t)\frac{\partial^4 P_{1,2}}{\partial x_2^4}(x_1, x_2)z_1(x_2, t) + z_1(x_1, t)\frac{\partial P_{1,2}}{\partial x_2}(x_1, 1)u(t) \, dA
\end{align*}

We plug these identities into (2) and obtain

\begin{equation}
0 = \int_S z'(x_1, t)P(x_1, x_2)z(x_2, t) \, dA \\
+ \int_0^\infty \int_S z_2(x_1, t)P_{1,1}(x_1, x_2)z_1(x_2, t) + z_1(x_1, t)P_{1,1}(x_1, x_2)z_2(x_2, t) \\
+ \int_0^\infty \int_S u(t)\frac{\partial P_{2,1}}{\partial x_1}(1, x_2)z_1(x_2, t) - z_1(x_1, t)\frac{\partial^4 P_{2,1}}{\partial x_1^4}(x_1, x_2)z_1(x_2, t) \\
+ z_1(x_1, t)\frac{\partial^4 P_{1,2}}{\partial x_2^4}(x_1, x_2)z_1(x_2, t) + z_1(x_1, t)\frac{\partial P_{1,2}}{\partial x_2}(x_1, 1)u(t) \\
+ z_2(x_1, t)\frac{\partial^4 P_{2,2}}{\partial x_2^4}(x_1, x_2)z_2(x_2, t) - z_1(x_1, t)\frac{\partial^4 P_{2,2}}{\partial x_2^4}(x_1, x_2)z_2(x_2, t) \\
- \int_0^\infty \int_S \alpha z_2(x_1, t)P_{1,2}(x_1, x_2)z_1(x_2, t) - \alpha z_2(x_1, t)\frac{\partial P_{1,2}}{\partial x_2}(x_1, 1)u(t) \\
- \alpha z_2(x_1, t)P_{2,1}(x_1, x_2)z_2(x_2, t) - \alpha z_2(x_1, t)\frac{\partial P_{2,1}}{\partial x_2}(x_1, 1)u(t) \, dA \, dt
\end{equation}

We add the the right side of (6) to the criterion (1) to get an equivalent criterion
and 

Clearly the terms quadratic in $u$ so that the time integrand in (7) is a perfect square of the form

By symmetry

so we set

We would like to find a $2 \times 1$ matrix valued function

so that the time integrand in (7) is a perfect square of the form

Clearly the terms quadratic in $u(t)$ agree so we equate terms containing the product of $u(t)$ and $z(x_2,t)$. This yields the equation

so we set

By symmetry


Then we equate terms containing the product of $z(x_1, t)$ and $z(x_2, t)$ and we obtain the Riccati PDEs for the boundary control of the beam equation,

$$\frac{\partial^4 P_{1,2}}{\partial x_2^2}(x_1, x_2) + \frac{\partial^4 P_{2,1}}{\partial x_1^2}(x_1, x_2) + Q_{1,1}(x_1, x_2)$$

$$= \gamma^2 \frac{\partial P_{1,2}}{\partial x_2}(x_1, 1) \frac{\partial P_{2,1}}{\partial x_1}(1, x_2)$$

$$P_{1,1}(x_1, x_2) = -\frac{\partial^4 P_{2,2}}{\partial x_1^2}(x_1, x_2) - \alpha z_1(x_1, t) P_{1,2}(x_1, x_2) z_2(x_1, t) + Q_{1,2}(x_1, x_2)$$

$$= \gamma^2 \frac{\partial P_{2,2}}{\partial x_2}(x_1, 1) \frac{\partial P_{2,1}}{\partial x_1}(1, x_2)$$

$$P_{1,2}(x_1, x_2) = -\frac{\partial^4 P_{2,2}}{\partial x_2^2}(x_1, x_2) - \alpha z_2(x_1, t) P_{2,1}(x_1, x_2) z_1(x_2, t) + Q_{2,1}(x_1, x_2)$$

$$= \gamma^2 \frac{\partial P_{2,2}}{\partial x_2}(x_1, 1) \frac{\partial P_{2,1}}{\partial x_1}(1, x_2)$$

$$P_{2,1}(x_1, x_2) = \frac{\partial^4 P_{2,2}}{\partial x_2^2}(x_1, x_2) - 2\alpha z_2(x_1, t) P_{2,1}(x_1, x_2) z_2(x_2, t) + Q_{2,2}(x_1, x_2)$$

$$= \gamma^2 \frac{\partial P_{2,2}}{\partial x_2}(x_1, 1) \frac{\partial P_{2,1}}{\partial x_1}(1, x_2)$$

where $\gamma^2 = R^{-1}\beta^2$.

To simplify the problem we decouple the spatial frequencies by assuming that $Q(x_1, x_2)$ has the expansion

$$Q(x_1, x_2) = \sum_{n=1}^\infty \begin{bmatrix} Q_{1,1, n}^{n,n} & Q_{1,2, n}^{n,n} \\ Q_{2,1, n}^{n,n} & Q_{2,2, n}^{n,n} \end{bmatrix} \sin n\pi x_1 \sin n\pi x_2$$

and $Q_{1,2, n}^{n,n} = Q_{2,1, n}^{n,n}$.

We assume that $P(x_1, x_2)$ has a similar expansion

$$P(x_1, x_2) = \sum_{n=1}^\infty \begin{bmatrix} P_{1,1, n}^{m,n} & P_{1,2, n}^{m,n} \\ P_{2,1, n}^{m,n} & P_{2,2, n}^{m,n} \end{bmatrix} \sin n\pi x_1 \sin n\pi x_2$$

with $P_{1,2, n}^{m,n} = P_{2,1, n}^{m,n}$. Clearly any such $P(x_1, x_2)$ satisfies the boundary conditions (4) and (5).

Then (9) implies

$$K(x_2) = -R^{-1}\beta \sum_{n=1}^\infty \begin{bmatrix} P_{1,1, n}^{m,n} & P_{1,2, n}^{m,n} \\ P_{2,1, n}^{m,n} & P_{2,2, n}^{m,n} \end{bmatrix} n\pi \sin n\pi x$$

and the Riccati PDEs (10) (11) (12) (13) imply that

$$0 = -2n^4\pi^4 P_{1,2}^{n,n} + Q_{1,1, n}^{n,n} - n^2\pi^2\gamma^2 (P_{1,2}^{n,n})^2$$

$$0 = P_{1,1}^{n,n} - n^4\pi^4 P_{2,2}^{n,n} - \alpha P_{1,2}^{n,n} + Q_{1,2, n}^{n,n} - n^2\pi^2\gamma^2 P_{1,2}^{n,n} P_{2,2}^{n,n}$$

$$0 = P_{1,1}^{n,n} - n^4\pi^4 P_{2,2}^{n,n} - \alpha P_{2,1}^{n,n} + Q_{2,1, n}^{n,n} - n^2\pi^2\gamma^2 P_{2,2}^{n,n} P_{2,1}^{n,n}$$

$$0 = 2P_{1,2}^{n,n} - 2\alpha P_{2,2}^{n,n} + Q_{2,2, n}^{n,n} - n^2\pi^2\gamma^2 (P_{2,2}^{n,n})^2$$

where $\gamma^2 = R^{-1}\beta^2$. 
For each \( n = 1, 2, \ldots \) these are the Riccati equations of the two dimensional LQR with matrices

\[
F^{n,n} = \begin{bmatrix} 0 & 1 \\ -n^4 \pi^4 & -\alpha \end{bmatrix}, \quad G^{n,n} = \begin{bmatrix} 0 \\ n^2 \beta \end{bmatrix} \\
Q^{n,n} = \begin{bmatrix} Q_1^{n,n} & Q_2^{n,n} \\ Q_{2,1}^{n,n} & Q_{2,2}^{n,n} \end{bmatrix}, \quad R^{n,n} = \begin{bmatrix} R \\ R \end{bmatrix}
\]

We use the quadratic formula to solve (17)

\[
P^{n,n}_{1,2} = -n^2 \pi^2 \pm \sqrt{n^4 \pi^4 + \frac{\gamma^2 Q_{1,1}^{n,n}}{n^2 \pi^2}}
\]

then (20) implies

\[
P^{n,n}_{2,2} = -\alpha \pm \sqrt{\alpha^2 + n^2 \pi^2 \gamma^2 (Q_{2,2}^{n,n} + 2P^{n,n}_{1,2})}
\]

Since we want \( P^{n,n}_{2,2} \) to be nonnegative we take the positive sign. Then (18) implies

\[
P^{n,n}_{1,1} = \alpha P^{n,n}_{1,2} + n^4 \pi^4 P^{n,n}_{2,2} - Q_{1,2}^{n,n} + n^2 \pi^2 \gamma^2 P^{n,n}_{2,2} P^{n,n}_{2,2}
\]

If the two dimensional LQR (21) satisfies the standard conditions then the associated Riccati equation has a unique nonnegative definite solution. This implies that if we take the negative sign in (22) the resulting \( P^{n,n} \) is not nonnegative definite.

The \( 2 \times 2 \) closed loop system is

\[
F^{n,n} + GK^{n,n} = \begin{bmatrix} 0 & 1 \\ -n^4 \pi^4 & -\alpha - n^2 \pi^2 \gamma^2 P^{n,n}_{2,2} \end{bmatrix}
\]

and the closed loop eigenvalues are

\[
\mu_n = -\alpha + n^2 \pi^2 \gamma^2 P^{n,n}_{2,2} \pm \frac{\sqrt{\left(\alpha + n^2 \pi^2 \gamma^2 P^{n,n}_{2,2}\right)^2 - 4(n^4 \pi^4 + \gamma^2 P^{n,n}_{2,2})}}{2}
\]

(25)

for \( n = \pm 1, \pm 2, \pm 3, \ldots \). For \( n = 1, 2, \ldots \) the corresponding eigenvectors of the \( n \text{th} \) \( 2 \times 2 \) closed loop system are

\[
\begin{bmatrix} \frac{1}{\mu_n} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{\mu_n} \\ 1 \end{bmatrix}
\]

(26)

The corresponding eigenvectors of the infinite dimensional closed loop system are

\[
v_n(x) = \begin{bmatrix} \frac{1}{\mu_n} \\ 1 \end{bmatrix} \sin |n| \pi x, \quad v_{-n}(x) = \begin{bmatrix} \frac{1}{\mu_n} \\ 1 \end{bmatrix} \sin |n| \pi x
\]

Notice that at least for large \( |n| \), \( \mu_n \) and \( \mu_{-n} \) are complex conjugates as are \( v_n(x) \) and \( v_{-n}(x) \).

The trajectories of the infinite closed loop system are

\[
z(x, t) = \sum_{n=-\infty}^{\infty} \zeta_n(t) \begin{bmatrix} \frac{1}{\mu_n} \\ 1 \end{bmatrix} \sin |n| \pi x
\]

where

\[
\zeta_n(t) = e^{\mu_n t} \zeta_0^n
\]
If $\mu_n$ and $\mu_{-n}$ are complex and conjugate then $\zeta^n_0$ and $\zeta^{-n}_0$ must be complex conjugates for $z(x, t)$ to be real valued.

Notice we can control each spatial frequency independently. If we don’t want to damp out the $n^{th}$ spatial frequency then we set $Q^{n,n} = 0$ so that $P^{n,n} = 0$ and $K^{n,n} = 0$.

But we must address the questions of whether (14) and (15) converge. If there is an $N > 0$ and an $r > 1$ such that

$$\|Q_{i,j}\|_\infty \le \frac{q}{n^r}$$

for $i, j = 1, 2$ and $n > N$ then clearly (14) converges. We assume that we have chosen $Q^{n,n}$ such that this is true.

We apply the Mean Value Theorem to (22) to obtain

$$P^{n,n}_{1,2} = \frac{1}{2s^{1/2} n^{2\pi^2}}$$

for some $s$ between $n^2\pi^2$ and $\sqrt{n^4\pi^4 + \frac{\sqrt{4} Q^{1,n}}{n^2\pi^2}}$. Since $\frac{1}{2s^{1/2}}$ is monotonically decreasing on this interval and takes on its maximum value at $n^2\pi^2$ we conclude that

$$P^{n,n}_{1,2} \le \frac{Q^{n,n}_{1,1}}{2n^3 \pi^3} \le \frac{q}{2n^{3+r} \pi^3}$$

so clearly the sum

$$P_{1,2}(x_1, x_2) = P_{2,1}(x_1, x_2) = \sum_{n=0}^\infty P^{n,n}_{1,2} \sin nx_1 \sin nx_2$$

converges.

If $\alpha = 0$ then (23) implies that

$$P^{n,n}_{2,2} = \frac{1}{n\pi\gamma} \sqrt{Q^{n,n}_{2,2} + 2P^{n,n}_{1,2}} \le \frac{c}{n^{1+r/2}}$$

for $n > N$ and some constant $c$ so clearly the sum

$$P_{2,2}(x_1, x_2) = \sum_{n=0}^\infty P^{n,n}_{2,2} \sin nx_1 \sin nx_2$$

converges.

If $\alpha > 0$ then again by the Mean Value Theorem (23) implies that there exists an $s$ between $\alpha$ and $\sqrt{\alpha^2 + n^2\pi^2 \gamma^2} (Q^{n,n}_{2,2} + 2P^{n,n}_{1,2})$ such that

$$P^{n,n}_{2,2} = \frac{1}{2s^{1/2}} (Q^{n,n}_{2,2} + 2P^{n,n}_{1,2}) \le \frac{1}{2\alpha^{1/2} n^r}$$

so again the sum (28) converges.

But because of $n^{2\pi^2} P^{n,n}_{2,2}$ term in (23) in order for the sum

$$P_{1,1}(x_1, x_2) = \sum_{n=0}^\infty P^{n,n}_{1,1} \sin nx_1 \sin nx_2$$

(29) to converge $r$ must be larger than $8$ when $\alpha = 0$ and $r$ must be larger than $5$ when $\alpha > 0$.

If $\alpha > 0$ then all of the closed loop eigenvalues (26) are in the open left half of the complex plane. In particular for large $|n|$ the real parts of the closed loop eigenvalues (26) are more negative than $\frac{\alpha}{2}$.

Can we shift all the eigenvalues into the open left half of the complex plane if $\alpha = 0$? If $Q^{n,n} > 0$ but decays like $\frac{1}{n\rho}$ as $n \to \infty$ then the term outside the square root in (26) will
be negative but it will decay in absolute value like \( \frac{1}{n^{r/2 - 1}} \). For (29) to converge \( r \) must be greater than 8 so the term outside the square root in (26) is converging to zero faster than \( \frac{1}{n^3} \). But we are more interested in the convergence of feedback (16) than the convergence of the optimal cost(15). For (16) to converge \( r \) need only greater than 1. If we choose 1 < \( r < 2 \) then the term outside the square root in (26) will grow like \( n^{2-r} \) so the higher the mode the higher the damping. It is interesting to note that even if the optimal cost of an LQR problem does not exist, the LQR methodology may yield a stabilizing feedback.

3. Conclusion

We have used the simple and constructive technique of completing the square to solve the LQR problem for the stabilization of the linear beam equation using boundary control. The result is an explicit formula for the quadratic optimal cost and the linear optimal feedback. Our approach allows us to decouple the spatial frequencies so we can damp out all or just some frequencies.

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