Algorithms for group actions in arbitrary characteristic and a problem in singularity theory

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Abstract
Let $M_{m,n}$ denote the space of $m \times n$ matrices with entries in the formal power series ring $K[[x_1, \ldots, x_s]]$, $K$ an arbitrary field. We consider different groups $G$ acting on $M_{m,n}$ by formal change of coordinates, combined with the multiplication by invertible matrices. This includes right and contact equivalence of functions, mappings, and ideals. A matrix $A$ is called finitely $G$-determined if any matrix $B$, with entries of $A - B$ in $\langle x_1, \ldots, x_s \rangle^k$ for some $k$, is contained in the $G$-orbit of $A$. In this paper we present algorithms to check finite determinacy, to compute determinacy bounds and to compute the tangent image $\tilde{T}_A(GA)$ of the action. The tangent image is an important invariant in positive characteristic since it differs in general from the tangent space $T_A(GA)$ to the orbit of $G$ (in a subtle way). This fact was only recently discovered by the authors and is proved in the present paper by using our algorithms. Besides this application, the algorithms of this paper are of interest for the classification of singularities in arbitrary characteristic, a subject of growing interest.

Keywords No-separable group action in positive characteristic · Algorithms for determinacy · Tangent image · Tangent space · Right and contact equivalence in positive characteristic

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1 Introduction

Throughout this paper let $K$ be a fixed field of arbitrary characteristic and

$$R := K[[x]] = K[[x_1, \ldots, x_s]]$$

the formal power series ring over $K$ in $s$ variables with maximal ideal $m = \langle x_1, \ldots, x_s \rangle$. We denote by

$$M_{m,n} := \text{Mat}(m,n,R)$$

the set of all $m \times n$ matrices with entries in $R$. Let $G$ denote one of the groups $\mathcal{R}, \mathcal{G}_l, \mathcal{G}_r, \mathcal{G}_{lr}$ (defined in Sect. 2), acting on $M_{m,n}$ by formal change of coordinates and multiplication with invertible matrices from the left, the right or from both sides, respectively. Two matrices $A, B \in M_{m,n}$ are called $G$-equivalent, denoted $A \sim^G B$, if $B$ lies in the orbit of $A$. $A$ is said to be $G$ $k$-determined if for each matrix $B \in M_{m,n}$ with $B - A \in m^{k+1} \cdot M_{m,n}$, we have $B \sim^G A$, i.e. if $A$ is $G$-equivalent to every matrix which coincides with $A$ up to and including terms of order $k$. $A$ is called finitely $G$-determined if there exists a positive integer $k$ such that it is $G$ $k$-determined.

In this paper we present algorithms for checking finite determinacy and to compute determinacy bounds. Moreover, if $A$ is finitely determined, we give algorithms to compute the image $\tilde{T}_A(GA)$ of the tangent map to the orbit map $G \to GA$, which is contained in the tangent space $T_A(GA)$ of the orbit $GA$. It was discovered only recently by the authors, and announced in [10], that both spaces may be different if $K$ has positive characteristic. One of the purposes of this paper is to prove this result. For this we use the above mentioned algorithms and an algorithm to compute the codimension of $T_A(GA)$ in $M_{m,n}$. Since an algorithm to compute $T_A(GA)$ directly in positive characteristic seems to be unknown, we present algorithms to compute the orbit $GA$ and the stabilizer $G_A$.

2 Tangent spaces and tangent images

We review theoretical results from [10] on tangent images and tangent spaces of the action of the group $G \in \{ \mathcal{R}, \mathcal{G}_l, \mathcal{G}_r, \mathcal{G}_{lr} \}$, with

$$\mathcal{R} := \text{Aut}(R),$$

$$\mathcal{G}_l := GL(m,R) \ltimes \mathcal{R},$$

$$\mathcal{G}_r := GL(n,R)_{\text{op}} \ltimes \mathcal{R},$$

$$\mathcal{G}_{lr} := (GL(m,R) \times GL(n,R)_{\text{op}}) \ltimes \mathcal{R},$$

where $\text{Aut}(R)$ is the group of $K$-algebra automorphisms of $R$ and $G_{\text{op}}$ is the opposite group of a group $G$. These groups act on the space $M_{m,n}$ as follows

$$(\phi, A) \mapsto \phi(A) := [\phi(a_{ij}(x))] = [a_{ij}(\phi(x))].$$
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\[(U, \phi, A) \mapsto U \cdot \phi(A),\]
\[(V, \phi, A) \mapsto \phi(A) \cdot V,\]
\[(U, V, \phi, A) \mapsto U \cdot \phi(A) \cdot V,\]

where \( \mathbf{x} = (x_1, x_2, \ldots, x_s) \), \( A = [a_{ij}(\mathbf{x})] \in M_{m,n}, U \in GL(m, R), V \in GL(n, R), \) and \( \phi(\mathbf{x}) := (\phi_1, \ldots, \phi_s) \) with \( \phi_i := \phi(x_i) \in m \) for all \( i = 1, \ldots, s \).

For \( a \in R \) and \( k \in \mathbb{N} \), we denote by \( jet_k(a) \) the image of \( a \) in \( R/m^{k+1} \), which we identify with the power series of \( a \) up to and including order \( k \). \( jet_k(A) = [jet_k(a_{ij})] \) denotes the \( k \)-jet of \( A \), and

\[ M_{m,n}^{(k)} := M_{m,n}/m^{k+1} \cdot M_{m,n}, \]

the space of all \( k \)-jets. The \( k \)-jet of \( G \) is

\[ G^{(k)} := \{jet_k(g) \mid g \in G\}, \]

where, for example, for \( g = (U, V, \phi) \in G_{lr} \) we have \( jet_k(g) = (jet_k(U), jet_k(V), jet_k(\phi)) \), with \( jet_k(\phi)(x_i) = jet_k(\phi(x_i)) \) for all \( i = 1, \ldots, s \). Then \( G^{(k)} \) is an affine algebraic group, acting algebraically on the affine space \( M_{m,n}^{(k)} \) via

\[ G^{(k)} \times M_{m,n}^{(k)} \to M_{m,n}^{(k)}, \ (jet_k(g), jet_k(A)) \mapsto jet_k(gA), \]

i.e. we let representatives act and then take the \( k \)-jets.

For \( A \in m \cdot M_{m,n} \) we define the following submodules of \( M_{m,n} \),

\[ \tilde{T}_A(\mathcal{R}A) := m \cdot \left\{ \frac{\partial A}{\partial x_v} \right\}, \]
\[ \tilde{T}_A(\mathcal{G}_lA) := (E_{m,pq} \cdot A) + m \cdot \left\{ \frac{\partial A}{\partial x_v} \right\}, \]
\[ \tilde{T}_A(\mathcal{G}_rA) := (A \cdot E_{n,hl}) + m \cdot \left\{ \frac{\partial A}{\partial x_v} \right\}, \]
\[ \tilde{T}_A(\mathcal{G}_{lr}A) := (E_{m,pq} \cdot A) + (A \cdot E_{n,hl}) + m \cdot \left\{ \frac{\partial A}{\partial x_v} \right\}, \]

called the tangent images at \( A \) to the orbit of \( A \) under the actions of \( \mathcal{R}, \mathcal{G}_l, \mathcal{G}_r, \mathcal{G}_{lr} \) on \( M_{m,n} \), respectively. Here \( (E_{m,pq} \cdot A) \) is the \( R \)-submodule generated by \( E_{m,pq} \cdot A \), \( p, q = 1, \ldots, m \), with \( E_{m,pq} \) the \( (p, q) \)-th canonical matrix of \( \text{Mat}(m, m, R) \) (1 at place \( (p, q) \) and 0 else) and \( \left\{ \frac{\partial A}{\partial x_v} \right\} \) is the \( R \)-submodule generated by the matrices

\[ \frac{\partial A}{\partial x_v} = \left[ \frac{\partial a_{ij}}{\partial x_v}(\mathbf{x}) \right], \ v = 1, \ldots, s. \]

It was shown in [10, Proposition 2.5] that the \( k \)-jet of the tangent image

\[ \tilde{T}_A^{(k)}(GA) := jet_k(\tilde{T}_A(GA)) \]

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is the image of $T_o$, where $T_o$ is the tangent map to the orbit map $o : G^{(k)} \rightarrow G^{(k)} jet_k(A)$,

$$T_o : T_e G^{(k)} \rightarrow T_A^{(k)} (GA) := T_{jet_k(A)} \left( G^{(k)} jet_k(A) \right).$$

Then $\tilde{T}_A(GA)$ is the inverse limit of the inverse system of $R$-modules $\tilde{T}_A^{(k)}(GA)$.

Moreover, we define the inverse limit of the $k-$jets of the tangent spaces to the orbit,

$$T_A(GA) := \lim_{\leftarrow k \geq 0} T_A^{(k)}(GA) \subset M_{m,n}$$

and call it the tangent space at $A$ to the orbit $GA$. We have $\tilde{T}_A(GA) \subset T_A(GA)$, and if $K$ has characteristic zero then the equality holds (see [10, Remark 2.7 and Lemma 2.8]). $\tilde{T}_A(GA)$ is contained in $m M_{m,n}$, the tangent space of $M_{m,n}$.

**Theorem 2.1** Let $A \in m M_{m,n}$ and $G$ one of the groups $\mathcal{R}, \mathcal{G}_l, \mathcal{G}_r$, and $\mathcal{G}_{lr}$.

1. If $m^p+1 M_{m,n} \subset \tilde{T}_A(GA)$ for some $p$, then $A$ is finitely $G$-determined. Moreover, $A$ is then $G (2p - \text{ord}(A) + 2)$-determined, where $\text{ord}(A)$ is the minimum order of the entries of $A$.

2. Let $K$ be infinite. If $n = 1$, i.e. $A$ is a 1-column matrix, and $G = G_{lr}, \mathcal{G}_l$, then $A$ is finitely $G$-determined if and only if $m^p+1 M_{m,n} \subset \tilde{T}_A(GA)$ for some $p$.

For the proof of 1. we refer to [10, Prop. 4.2] and for 2. to [11, Theorem 3.5].

**Remark 2.2** The most important application of part 2 of the theorem is to ideals. Two ideals $I, J \subset R$ are contact equivalent if the local $K$-algebras $R/I$ and $R/J$ are isomorphic and $I$ is finitely contact determined if it is contact equivalent to any ideal $J$ with $I \equiv J \mod m^k$ for some $k$. In [11, Theorem 4.6] we prove: a proper ideal $I \subset R = K[[x]]$ with $K$ infinite and $\dim(R/I) > 0$ is finitely contact determined, if and only if $I$ is a complete intersection with isolated singularity. The algorithms of this paper are important tools for the (still far open) classification of matrices and ideals in positive characteristic.

### 3 Algorithms for the tangent image

By Theorem 2.1, the finiteness of the codimension of the tangent image $\tilde{T}_A(GA)$, equivalent to the existence of a power $p$ of the maximal ideal $m$ such that $m^p+1 M_{m,n} \subset \tilde{T}_A(GA)$, is a sufficient condition for $A$ to be finitely $G$-determined. The following algorithms compute a local standard basis of tangent image $\tilde{T}_A(GA)$, a vector space basis of $M_{m,n}/\tilde{T}_A(GA)$ and the codimension of $\tilde{T}_A(GA)$ in $M_{m,n}$.

Theoretically the matrix $A$ has arbitrary power series $a_{ij}$ as entries, but we assume in the algorithms below that the $a_{ij}$ are polynomials. The output is then also polynomial. If $A$ is finitely $G$-determined then $A$ is $G$-equivalent to a matrix with polynomial entries, but finite $G$-determinacy is not assumed in the algorithms below. By the following remark the result is correct over the power series ring.
Remark 3.1 Let > be a local degree ordering on (the monomials of) \( K[x] \) and \((>,c)\) a module ordering on \( K[x]^n \) giving priority to the monomials in \( K[x] \), see [9, Definition 2.3.1], which will also be denoted by >.

If \( N_1, \ldots, N_r \) is a standard basis w.r.t. > of a submodule \( M \subset K[x]^n \), then
1. \( N_1, \ldots, N_r \) generate \( M \otimes_{K[x]} K[x] \) over the localization \( K[x]_> \) of \( K[x] \) w.r.t. >.
2. \( N_1, \ldots, N_r \) is a standard basis of \( M \otimes_{K[x]} K[[x]] \) w.r.t. > and hence generates \( M = M \otimes_{K[x]} K[[x]] \) over \( K[[x]] \).

For the proof, see [9, Lemma 2.3.5 and Theorem 6.4.3] (in [9, Theorem 6.4.3] it is Remark 3.1).

For the rest of the paper we fix a local degree ordering > on \( K[x] \) and a module ordering \((>,c)\) on \( M_{m,n} \), also denoted by > (cf. [9, Definition 2.3.1]).

We start with the computation of a standard basis for the tangent image w.r.t. the group \( G_{lr} \). As the algorithms for the other groups \( R, G_l, G_r \) are simplifications they are omitted.

Algorithm 1: TangImage \( G \) (for \( G = G_{lr} \))
Input: Matrix \( A = [a_{ij}] \in m_{M_{m,n}} \) (with \( a_{ij} \in (x) K[x] \)).
Output: matrices \( N_1, \ldots, N_r \in M_{m,n} \), being a standard basis of the tangent image \( \tilde{T}_A(GA) \) w.r.t. >.

1. Compute \( M := \langle E_{m,pq} \cdot A, p, q = 1, \ldots, m \rangle + \langle A \cdot E_{n,hl}, h, l = 1, \ldots, n \rangle \).
2. Compute \( N := \left( \frac{\partial A}{\partial x_v}, \nu = 1, \ldots, s \right) \).
3. Compute a standard basis \( S = \{N_1, \ldots, N_r\} \) of \( M + \langle x_1, \ldots, x_s \rangle N \) w.r.t. >.
4. Return: \( S \).

The following algorithm computes a \( K \)-basis of \( M_{m,n}/\tilde{T}_A(GA) \) and its codimension \( c = \dim_K (M_{m,n}/\tilde{T}_A(GA)) \).

Algorithm 2: BasisCodimTangImage \( G \)
Input: Matrix \( A = [a_{ij}] \in m_{M_{m,n}} \), specification of \( G \)
Output: integer \( c = \dim_K (M_{m,n}/\tilde{T}_A(GA)) \) or -1 if \( \dim_K (M_{m,n}/\tilde{T}_A(GA)) = \infty \) and, if \( c \geq 0 \), matrices \( M_1, \ldots, M_c \in M_{m,n} \), being a \( K \)-basis of \( M_{m,n}/\tilde{T}_A(GA) \).

1. Compute a standard basis \( S = \{N_1, \ldots, N_r\} \) of \( \tilde{T}_A(GA) \) with Algorithm 1 for \( G \).
2. Let \( L_1, \ldots, L_r \) be the leading monomials of \( N_1, \ldots, N_r \), respectively.
3. If \( \dim_K (M_{m,n}/\langle L_1, \ldots, L_r \rangle) = \infty \),
   Return: -1
4. Else compute matrices \( M_1, \ldots, M_c \subset M_{m,n} \) being a \( K \)-basis of \( M_{m,n}/\langle L_1, \ldots, L_r \rangle \).
5. Return: \( c, M_1, \ldots, M_c \).

Remark 3.2 The computation of a \( K \)-basis of \( M_{m,n}/\langle L_1, \ldots, L_r \rangle \) is a combinatorial task. This basis is also a \( K \)-basis of \( M_{m,n}/\tilde{T}_A(GA) \). The SINGULAR ([4]) command \( K \)-base computes the \( K \)-basis internally along these lines and returns a \( K \)-basis consisting of matrices with entries being monomials.
The following algorithm computes the pre-determinacy bound, i.e. the minimal $p$ such that $m^{p+1}M_{m,n} \subset \tilde{T}_A(GA)$, using a normal form algorithm NF (cf. [9, Definition 1.6.4]) w.r.t. a local monomial ordering (cf. [9, Definition 1.2.4]). Then we compute a $G$-determinacy bound for $A$.

**Algorithm 3:** predeterm$^G$

**Input:** $A = [a_{ij}] \in mM_{m,n}$, specification of $G$

**Output:** integer $p$, the pre-determinacy bound for $A$ w.r.t. the group $G$, or -1 if codimension of $\tilde{T}_A(GA)$ is infinite.

1. Compute a standard basis $S$ of the tangent image $\tilde{T}_A(GA)$ by Algorithm 1 for $G$.
2. Compute the codimension $c$ of $\tilde{T}_A(GA)$ by Algorithm 2.
3. If $c = -1$
   - **Return:** $-1$.
4. Else loop
   - $p := 0$.
   - while $(\text{size}(\text{NF}(m^{p+1}M_{m,n}, S)) > 0)$
   - $p := p + 1$.
5. **Return:** $p$.

**Algorithm 4:** determ$^G$

**Input:** $A = [a_{ij}] \in mM_{m,n}$, specification of $G$

**Output:** an integer $d$, a determinacy bound of $A$ w.r.t. the group $G$, or -1 if the codimension of $\tilde{T}_A(GA)$ is infinite.

1. Compute $o = \text{ord}(A)$, the order of the matrix $A$.
2. Compute $p$, the pre-determinacy bound of $A$ w.r.t. $G$ by Algorithm 3.
3. If $p = -1$
   - **Return:** $-1$.
4. Else compute $d = 2p - o + 2$.
5. **Return:** $d$.

### 4 Algorithms for the tangent space

In this section we give an algorithm to compute equations for the closure of the orbit $G^{(k)}A \subset M^{(k)}_{m,n}$ for $G$ one of the groups $\mathcal{R}$, $\mathcal{G}_l$, $\mathcal{G}_r$, $\mathcal{G}_{lr}$, and for the codimension of $G^{(k)}A$ in $M^{(k)}_{m,n}$. The orbit $G^{(k)}A$ is a locally closed subvariety of the affine space $M^{(k)}_{m,n}$ ([5, Chapter 3, §4, Theorem 4.19]). Here an algebraic variety is, as usual, considered as a set with the Zariski topology over the algebraic closure $\bar{K}$, defined over $K$.

For the application in Sect. 5, we are only interested in the dimension of the tangent space to $G^{(k)}A$ at $A$. However, in positive characteristic the tangent space may not coincide with the tangent image and therefore we cannot use the algorithms of the previous section. We do not know any other method to compute the dimension of the tangent space to the orbit, except by computing the dimension of the orbit itself (in positive characteristic).

The following theorem is the basis for our applications.
Theorem 4.1 Let G be any of the groups $\mathcal{R}, \mathcal{G}_l, \mathcal{G}_r$, and $\mathcal{G}_{lr}$, acting on $M_{m,n}$. Assume that the tangent image $\widetilde{T}_A(GA)$ has finite codimension in $M_{m,n}$. Let $p$ be the pre-determinant bound for $A$. For $k \geq p$ the following holds:

1. $A$ is $G$ $(2k - ord(A) + 2)$-determined.
2. $\dim_K M_{m,n}/\widetilde{T}_A(GA) = \dim_K M_{m,n}/\widetilde{T}_A(1)(GA)$.
3. The tangent space $T_A(GA)$ to the orbit $GA$ has finite codimension in $M_{m,n}$.
4. $\dim_K M_{m,n}/T_A(GA) = \dim_K M_{m,n}/T_A^{(k)}(GA) = \dim M_{m,n} - \dim G^{(k)}A$.
5. $\dim G^{(k)}A = \dim G^{(k)} - \dim G^{(k)}A$, where $G^{(k)}_A$ denotes the stabilizer of $A$ in $G^{(k)}$.

Proof Statement 1. was proved in [10, Proposition 4.2].

Consider the surjective homomorphism of $K$-vector spaces

$$\varphi : M_{m,n} \twoheadrightarrow M_{m,n}/\widetilde{T}_A^{(k)}(GA)$$

$$B \mapsto \text{jet}_k(B).$$

By assumption, $m^{k+1}M_{m,n} \subset \widetilde{T}_A(GA)$. This implies that $\ker(\varphi) = \widetilde{T}_A(GA)$. Hence, the equality holds.

3. This follows by combining the assumption $m^{k+1}M_{m,n} \subset \widetilde{T}_A(GA)$ with $\widetilde{T}_A(GA) \subset T_A(GA) \subset M_{m,n}$.

4. Starting from the left, the first equality follows from the inclusion $m^{k+1}M_{m,n} \subset T_A(GA)$, while the second one holds since the orbit is a smooth algebraic variety [5, Chapter 3, §4, Corollary 4.20] and therefore has the same dimension as its tangent space.

5. This is well known (cf. [5, Chapter 6, §3, Theorem 3.1]), where the dimension of an affine variety means the Krull dimension of its coordinate ring. \qed

Since the tangent image of an algebraic group action coincides with the tangent space iff the orbit map is separable (cf. [5, Chapter 6, §3, Theorem 3.1]), we get:

Corollary 4.2 With the assumptions of Theorem 4.1 the following are equivalent:

1. The orbit map $o : G^{(k)} \rightarrow G^{(k)}A$, $g \mapsto \text{jet}_k(gA)$, is separable.
2. $\widetilde{T}_A(GA) = T_A(GA)$.
3. $\dim_K M_{m,n}/\widetilde{T}_A(GA) = \dim M^{(k)}_{m,n} - \dim G^{(k)} + \dim G^{(k)}_A$.

$M^{(k)}_{m,n}$ is an affine space of dimension $t = mn^{s+k}$ with coordinate ring $K[u] = K[u_1, \ldots, u_r]$ and $G^{(k)}A$ is a (locally closed) subvariety of $M^{(k)}_{m,n}$ of a certain dimension, which we want to know.

Our first algorithm computes polynomials $F_1, \ldots, F_r \in K[u]$ by elimination, defining $G^{(k)}A$ set theoretically. Then we compute the dimension by computing a standard basis of $\langle F_1, \ldots, F_r \rangle$. This approach has the disadvantage that the dimensions of $G^{(k)}$ and $M^{(k)}_{m,n}$ are quite big, already for small $m$, $n$, $k$, and we have to compute standard basis with respect to an elimination ordering in rings with many variables.

Our second algorithm computes polynomials defining the stabilizer $G^{(k)}_A \subset G^{(k)}$ of $A$ and its dimension can be computed by a standard basis w.r.t. any ordering. This
algorithm involves less variables and is preferred if we are only interested in the
dimension of the orbit and not its equations.

Let us first consider the right group $G^{(k)} = R^{(k)}$. An element $g \in R^{(k)}$ is given by
$s$ polynomials in $K[x]^{(k)} = jet_k(K[x_1, \ldots, x_s])$, which can be written as

$$g_i(x_1, \ldots, x_s) = \sum_{j=1}^s (\delta_{ij} + g_{ij}) x_j + \sum_{|a|=2} h_{ia} x^a, \quad a = (a_1, \ldots, a_s), \quad (*)$$

with $\delta_{ij}$ the Kronecker symbol, $1 = [\delta_{ij}]$ the identity matrix and $\text{det}(1 + g_{ij}) \neq 0$.

If $A = [a_{ij}(x)] \in M^{(k)}_{m,n}$ then $g$ acts on $A$ by substitution and taking $k$-jets, i.e.

$$gA = jet_k[a_{ij}(g)] = [jet_k(a_{ij}(g_1(x), \ldots, g_s(x)))].$$

Let $G_{ij}, H_{ia}, i, \ j = 1, \ldots, s, 2 \leq |a| \leq k$, be new variables. Then the group $R^{(k)}$ is
the affine variety defined as the complement of $\text{det}(1 + G_{ij}) = 0$ in the affine space
of dimension $\dim R^{(k)} = s{\binom{s+k}{k}} - s$ with coordinates $G = (G_{ij}, H_{ia})$.

If $G^{(k)} = G^{(k)}_{lr}$ then an element $g \in G^{(k)}$ is given by $g_i(x_1, \ldots, x_s)$ as in $(*)$ above
and in addition by matrices

$$[u_{ij}(x)] \in GL(m, K[x]^{(k)}), \quad i, \ j = 1, \ldots, m,$$

$$u_{ij} = \delta_{ij} + u_{ij0} + \sum_{|a|=1} u_{ija} x^a, \quad \text{det}(1 + [u_{ij0}]) \neq 0,$$

$$[v_{ij}(x)] \in GL(n, K[x]^{(k)}), \quad i, \ j = 1, \ldots, n,$$

$$v_{ij} = \delta_{ij} + v_{ij0} + \sum_{|a|=1} v_{ija} x^a, \quad \text{det}(1 + [v_{ij0}]) \neq 0.$$

$([u_{ij}], [v_{ij}], g)$ acts on $M^{(k)}_{m,n}$ by

$$([u_{ij}], [v_{ij}], g) = jet_k([u_{ij}][a_{ij}(g)][v_{ij}]).$$

$G^{(k)}_{lr}$ is an affine variety of dimension $N = (m^2 + n^2 + s){\binom{s+k}{k}} - s$. If $U_{ija}, i, j \in \{1, \ldots, m\}, 0 \leq |a| \leq k$ and $V_{ija}, i, j \in \{1, \ldots, n\}, 0 \leq |a| \leq k$ are new variables, then $G^{(k)}_{lr}$ is the complement of the hypersurface

$$\text{det} := \text{det}(1 + [G_{ij}]) \text{det}(1 + [U_{ij0}]) \text{det}(1 + [V_{ij0}]) = 0$$

in the affine space $K^N$ with coordinates $U = (U_{ija}), i, j \in \{1, \ldots, m\}, 0 \leq |a| \leq k, V = (V_{ija}), i, j \in \{1, \ldots, n\}, 0 \leq |a| \leq k$, and $G = (G_{ij}, H_{ia}) i, j \in \{1, \ldots, n\}, 2 \leq |a| \leq k$.

The other groups $G^{(k)}$ are special cases of $G^{(k)}_{lr}$. 
By our choice of coordinates $U_{i,j_0}$, $V_{i,j_0}$, and $G_{i,i}$, the groups $G^{(k)}$ pass through $0 \in K^N$ where 0 corresponds to the identity in $G^{(k)}$. This allows us to compute in the polynomial ring $K[U, V, G]$ as well as in the localization $K[U, V, G]_\mathfrak{m}$ with $\mathfrak{m}$ a local ordering. Note that $det$ is a unit in $K[U, V, G]_\mathfrak{m}$, hence the condition $det \neq 0$ is automatic in this ring.

The algorithm for describing the orbit of $G^{(k)} = \mathcal{G}_{lr}^{(k)}$ can be described as follows:

**Algorithm 5:** OrbitEq ($G = \mathcal{G}_{lr}$)

**Input:** integer $k$, matrix $A = [a_{ij}] \in M_{m,n}$

**Output:** polynomials $F_1, \ldots, F_r \in K[u_1, \ldots, u_t]$, $t = \dim M_{m,n}$, such that the variety $V(F_1, \ldots, F_r)$ coincides with the closure of $\mathcal{G}_{lr}^{(k)} A$.

1. In the polynomial ring $K[x, U, V, G] = K[x_1, \ldots, x_s, U_{i,j}, V_{i,j}, G_{i,j}, H_{i,a}]$ define the polynomials
   \[
   G_i = \sum_{j=1}^{s} (\delta_{ij} + G_{ij})x_j + \sum_{|a|=2}^{k} H_{i,a}x^a, \quad i = 1, \ldots, s,
   \]
   \[
   U_{ij} = \delta_{ij} + U_{i,j_0} + \sum_{|a|=1}^{k} U_{i,j,a}x^a \quad i, j = 1, \ldots, m,
   \]
   \[
   V_{ij} = \delta_{ij} + V_{i,j_0} + \sum_{|a|=1}^{k} V_{i,j,a}x^a \quad i, j = 1, \ldots, n.
   \]

2. Construct the matrix $B = [b_{ij}] \in Mat(m, n, K[x, U, V, G])$ with
   \[
   b_{ij} = jet_k \left( [U_{ij}] \cdot [a_{ij}(G_1, \ldots, G_s)] \cdot [V_{ij}] \right),
   \]
   where the $k$-jet is taken w.r.t. $x$.

3. Write the polynomials $b_{ij}$ as $\sum_{|a|=0}^{k} c_{ij,a}x^a$ with coefficients $c_{ij,a}\in K[U, V, G]$. Let $C$ denote the ideal in $K[U, V, G]$ generated by the $t$ polynomials $c_{ij,a}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $0 \leq |a| \leq k$. Denote the polynomials $c_{ij,a}$ by $c_1, \ldots, c_t$.

4. Let $D \subset K[U, V, G, u]$ be the ideal generated by the polynomials
   \[
   u_i - c_i(U, V, G), \quad i = 1, \ldots, t.
   \]
   Eliminate the variables $U_{i,j}, V_{i,j}, G_{i,j}, H_{i,a}$ from $D$ by computing a standard basis w.r.t. an elimination ordering (see [9, Definition 1.5.4 and 1.6.1]). Get finitely many polynomials $F_1, \ldots, F_r \in K[u]$.

5. Return: $F_1, \ldots, F_r$.

**Remark 4.3** The algorithm terminates since each of the five steps terminates obviously. It is correct since the $\mathcal{G}_{lr}^{(k)}$ is the open subset $det \neq 0$ in $K^N$ with coordinates $U, V, G$.
and the orbit map \( o : G^{(k)} \rightarrow M_{m,n}^{(k)} \) is given on the ring level by

\[
u_i = c_i(U, V, G), \quad i = 1, \ldots, t.
\]

It is well known ([9, 1.8.3]) that the closure of image is defined by eliminating \( U, V, G \) from the ideal \( \langle u_i - c_i \rangle \).

The computation of the equation for the orbit of the other groups \( \mathcal{R}, \mathcal{G}_l, \mathcal{G}_r \) is a special case of the computation for \( \mathcal{G}_{lr} \), by omitting \( U_{ij} \) and \( V_{ij} \) for \( \mathcal{R} \), \( V_{ij} \) for \( \mathcal{G}_l \) and \( U_{ij} \) for \( \mathcal{G}_r \).

We present now an algorithm to compute the stabilizer

\[G_A^{(k)} = \{ g \in G^{(k)} \mid gA = A \}\]

of the action of \( G^{(k)} \) on \( M_{m,n}^{(k)} \). We use the notations from Algorithm 5.

**Algorithm 6:** StabEq \((G = \mathcal{G}_{lr})\)

**Input:** integer \( k \), matrix \( A = [a_{ij}] \in M_{m,n}^{(k)} \)

**Output:** polynomials \( D_1, \ldots, D_t \in K[U, V, G] \) such that the variety \( V(D_1, \ldots, D_t) \) coincides with \( G_A^{(k)} \subset G^{(k)} \).

1. and 2. are the same as in algorithm OrbitEq, we get

\[B = [b_{ij}] \in Mat(m, n, K[x, U, V, G]).\]

3. Write

\[b_{ij} - a_{ij} = \sum_{|a|=0}^k d_{ija}x^a, \quad d_{ija} \in K[U, V, G]\]

and let \( D \) be the ideal in \( K[U, V, G] \) generated by \( d_{ija}, i = 1, \ldots, m, j = 1, \ldots, n, |a| = 0, \ldots, k \). Denote the polynomials \( d_{ija} \) by \( D_1, \ldots, D_t \).

4. Return \( D_1, \ldots, D_t \).

**Remark 4.4** The algorithm terminates and since the coefficients \( d_{ija} \) of \( b_{ij} - a_{ij} \) are the polynomials defining the set \( \{ gA - A \mid g \in G^{(k)} \} \) it is also correct.

It is now easy to compute the codimension of \( T_A(GA) \) in \( M_{m,n} \) if \( \tilde{T}_A(GA) \) has finite codimension. We have two algorithms.

**Algorithm 7:** codimTangG1

**Input:** \( A = [a_{ij}] \in M_{m,n}, \) assume \( \dim_K(M_{m,n} / \tilde{T}_A(GA)) < \infty \), specification of \( G \).

**Output:** \( c = \dim_K M_{m,n} / T_A(GA) \)

(1) Compute the pre-determinacy bound \( p \) for \( A \) with Algorithm 3.
(2) Apply Algorithm 5 to compute the orbit equations \( F_1, \ldots, F_r \).
(3) Compute a standard basis of $I = \langle F_1, \ldots, F_r \rangle$ w.r.t. any monomial ordering.
(4) Compute $\dim I$.
(5) **Return:** $c = \dim M_{m,n}^{(p)} - \dim I$ (note: $\dim M_{m,n}^{(k)} = mn^{(s+k)}$).

**Algorithm 8:** codimTang$_{G2}$
**Input:** $A = [a_{ij}] \in M_{m,n}$, assume $\dim_K M_{m,n}/\mathcal{T}_A(GA) < \infty$, specification of $G$.
**Output:** $c = \dim_K M_{m,n}/\mathcal{T}_A(GA)$.

1. Compute the pre-determinacy bound $p$ for $A$ with Algorithm 3.
2. Compute stabilizer equations $D_1, \ldots, D_t$ with Algorithm 6.
3. Compute a standard basis of $D = \langle D_1, \ldots, D_t \rangle$ w.r.t. any monomial ordering.
4. Compute $\dim D$.
5. **Return:** $c = \dim M_{m,n}^{(p)} - \dim G^{(p)} + \dim D$
   (e.g. $\dim \mathcal{G}_{tr}^{(k)} = (m^2 + n^2 + s)(s+k) - s$).

5 **A problem in singularity theory**

The algorithms of this paper are of interest for the classification of singularities in arbitrary characteristic. The classification of isolated hypersurface singularities $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ has a long tradition with contribution by many authors, most notably by V.I. Arnold and his school [1]. For formal power series $f \in R = K[[x]]$, where $K$ is a field of arbitrary characteristic, the classification started with [6] and was continued in [2,3,8,13–15]. The two most important equivalence relations are right equivalence and contact equivalence. Here $f, g \in R$ are right equivalent ($f \overset{r}{\sim} g$) if $f = \phi(g)$ for some $\phi \in \text{Aut}(R)$ and they are contact equivalent ($f \overset{c}{\sim} g$) if $f = u \cdot \phi(g)$ for some $\phi \in \text{Aut}(R)$ and a unit $u \in R^*$. An indispensable assumption for the classification is that the power series are finitely determined (for the considered equivalence relation) and that an explicit and computable determinacy bound is known.

If the field $K$ has characteristic 0 (or in the case of convergent power series over $\mathbb{C}$ and $\mathbb{R}$) such bounds are known for a long time (see [12] and e.g. [7] for references) and finite determinacy is equivalent to $f$ having an isolated singularity. Moreover, in this case the orbit map is separable and the action of the right group $\mathcal{R} = \text{Aut}(R)$ and the contact group $K = R^* \rtimes \mathcal{R}$ on $R$ can be faithfully described on the tangent level.

In positive characteristic however one has to work directly with the group actions and therefore the methods of proof must be different. It has been discovered by the authors that the tangent space to the orbit of the action of $K$ on $K[[x]]$ may be different from the tangent image, a fact that had been overlooked by several authors before. We prove this fact by computing an explicit example (announced in [10, Example 2.9]).

Note that the tangent space to the orbit coincides with the tangent image iff the orbit map $\mathcal{K}^{(k)} \to \mathcal{K}^{(k)} f \cdot (u, \phi) \mapsto u\phi(f)$, is separable for sufficiently big $k$ (see [10] for a discussion and a precise statement), which is always true in characteristic 0. It came as a surprise to us that this separability may fail, since it was shown in [3] that the map of the full action, $\mathcal{K}^{(k)} \times K[[x]]^{(k)} \to K[[x]]^{(k)}$, is always separable. Our experiments with SINGULAR [4], using the algorithms of this paper, show that also in...
positive characteristic separability of the orbit map holds in many cases and in fact, for the right group \( R \) we do not have a non-separable example with isolated singularity so far.

Right resp. contact equivalence for power series is a special case of matrices of size \( m = n = 1 \) and the groups \( R \) resp. \( K = G_l \). Our algorithms go however much further by treating matrices of arbitrary size and more equivalence relations. They provide general tools, not only to compute determinacy bounds, but also to decide in concrete cases in positive characteristic whether the tangent image coincides with the tangent space, i.e. whether separability of the orbit map holds or not.

The classification of general matrices resp. ideals with a small number of moduli is still an unsolved problem and we believe that the presented algorithms are in fact useful tools in this context.

**Example 5.1** We give an example for \( G = K \) acting on \( K[[x, y]] \), where the tangent image is strictly contained in the tangent space.

Let \( \text{char}(K) = 2 \), \( f = x^2 + y^3 \in K[[x, y]] \). We compute:

- The tangent image \( \tilde{T}_f(Kf) = \langle f \rangle + m \cdot j(f) \) is \( \langle x^2, xy^2, y^3 \rangle \), its codimension in \( M_{1,1} = K[[x, y]] \) is \( c = 5 \).

- \( f \) is \( K \) 4-determined, its \( K \)-pre-determinacy is \( p = 2 \).

- \( K[[x, y]]^{(p)} = K[[x, y]]/m^3 \) has dimension \( t = 6 \).

- The group \( K^{(p)} \) has dimension 16 and the stabilizer of \( jet_p(f) \) has dimension 14.

- The dimension of the orbit is 2, its codimension (also the codimension of the tangent space) is 4. As the codimension of the tangent image is 5, the orbit map \( K^{(p)} \to K^{(p)} f \) is not separable.

In order to prove the statements, we present the **SINGULAR** input for direct use, together with some comments.

```
ring r = 2,(x,y),ds; //local ordering, char(K)=2
poly f = x2 + y3;

//compute tangent image (Algo 1)
and its codimension (Algo 2)
ideal j = jacob(f);
ideal m = maxideal(1);
ideal T = std(m*j+ideal(f)); T;
   //T=tangent image =<x2,xy2,y3>
int c = vdim(T); c;//c=codim of tangent image =5

//Compute pre-determinacy (Algo 3)
and determinacy bound (Algo 4)
int p, d;
while (size(NF(maxideal(p+1),T)))!=0)
{ p = p+2; }
p; //p=p=pre-determinacy =2
d = 2*p-ord(f)+2; d; //d=determinacy bound =4
```
//Compute orbit equations
   (Algo 5, step 1. and 2.)
ideal km = kbase(maxideal(p+1));
int t = size(km); t; //t=dim of p-jet of M_m,n = 6
int s = t-1; //we will omit km[6] = 1
ring R = (2,u(0..s),a(1..s),
b(1..s)),(x,y),ds;
   //ring for creating contact group
poly f = imap(r,f);
ideal km = imap(r,km);
poly u,h1,h2;
int ii;
for (ii=1; ii<=s; ii++)
   { u = u+km[ii]*u(ii);
     h1 = h1+km[ii]*a(ii);
     h2 = h2+km[ii]*b(ii);
   }
u = u + u(0) + 1; //u=1+ u(0)+u(1)x+..., unit
h1 = h1 + x;
h2 = h2 + y; // (h1,h2)= id+..., coord. change
map phi = (r,h1,h2); phi;
poly F = jet(phi(f),p);
F = jet(u*F,p); F; // orbit equations

//Compute stabilizer (Algo 6)
F = jet(F-f,p);
matrix C = coef(F,xy);
int n = ncols(C);
ideal D = C[2,1..n];
   //coefficients of F-f

ring S = 2,(u(0..s),a(1..s),
b(1..s)),ds; //local ring of contact group
ideal D = imap(R,D); D; //ideal of stabilizer

//Compute codimension of tangent space (Algo 8)
D = std(D);
int c1 = dim (D); c1; // c1=dimension of stabilizer = 14
int c2 = nvars(S); c2; // c2=dimension of group = 16
int c3 = c2-c1; c3; // c3=dimension of orbit = 2
int c4 = t-c3; c4; // c4=codimension of orbit = 4
C - c4; // = 1, orbit map not separable
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References

1. Arnol’d, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of Differentiable Maps, vol. I. Birkhäuser, Basel (1985)
2. Boubakri, Y., Greuel, G.-M., Markwig, T.: Normal forms of hypersurface singularities in positive characteristic. Mosc. Math. J. 11(4), 657–683 (2011)
3. Boubakri, Y., Greuel, G.-M., Markwig, T.: Invariants of hypersurface singularities in positive characteristic. Rev. Mat. Complut. 25(1), 61–85 (2012)
4. Decker, W., Greuel, G.-M., Pfister, G., Schönemann, H.: SINGULAR 4-1-0—a computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2016)
5. Ferrer Santos, W.R., Rittatore, A.: Actions and Invariants of Algebraic Groups. Monographs and Research Notes in Mathematics, 2nd edn. CRC Press, Boca Raton, FL (2017)
6. Greuel, G.-M., Kröning, H.: Simple singularities in positive characteristic. Math. Z. 203(2), 339–354 (1990)
7. Greuel, G.-M., Lossen, C., Shustin, E.: Introduction to Singularities and Deformations. Monographs in Mathematics. Springer, Berlin (2007)
8. Greuel, G.-M., Nguyen, H.D.: Right simple singularities in positive characteristic. J. Reine Angew. Math. 712, 81–106 (2014)
9. Greuel, G.-M., Pfister, G.: A SINGULAR Introduction to Commutative Algebra, Extended ed. With Contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann. Springer, Berlin (2008)
10. Greuel, G.-M., Pham, T.H.: On finite determinacy for matrices of power series. Math. Z. 290(3–4), 759–774 (2018)
11. Greuel, G.-M., Pham, T.H.: Finite determinacy of matrices and ideals. J. Algebra 530, 195–214 (2019)
12. Mather, J.N.: Stability of $C^\infty$ mappings. III. Finitely determined map-germs. Inst. Hautes Études Sci. Publ. Math. 35, 279–308 (1968)
13. Nguyen, H.D.: The right classification of univariate power series in positive characteristic. J. Singul. 10, 235–249 (2014)
14. Nguyen, H.D.: Right unimodal and bimodal singularities in positive characteristic. Int. Math. Res. Not. IMRN 2019(6), 1612–1641 (2019)
15. Pham, T.H.: On finite determinacy of hypersurface singularities and matrices in arbitrary characteristic. Ph.D. thesis, TU Kaiserslautern (2016)

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