Abstract. The major problem in the cosmological nucleosynthesis is the evaluation of the reaction rate. The present scenario is that the standard thermonuclear function in the Maxwell-Boltzmann form is evaluated by using various techniques. The Maxwell-Boltzmannian approach to nuclear reaction rate theory is extended to cover Tsallis statistics (Tsallis, 1988) and more general cases of distribution functions. The main purpose of this paper is to investigate in some more detail the extended reaction probability integral in the equilibrium thermodynamic argument and in the cut-off case. The extended reaction probability integrals will be evaluated in closed form for all convenient values of the parameter by means of residue calculus. A comparison of the standard reaction probability integrals with the extended reaction probability integrals is also done.

Keywords: Thermonuclear function, pathway model, reaction probability integral, residue calculus.

1 Introduction

The evolution of universe is due to the thermonuclear reactions which are taking place in hot cosmic plasma. The main concept behind the description of cosmic nucleosynthesis is the rate of nuclear reactions synthesizing light nuclei into heavier ones. If we closely examine the nuclear cross section theoretically and experimentally, we can find the analytical representations of thermonuclear reactions. Many researchers were looking for these analytic representations of the reaction rate probability integrals for the last few decades. Many approaches have been made on the study of thermonuclear reactions (Haubold and John, 1978; Haubold and Mathai, 1985, 1998; Anderson et al, 1994, Saxena et al, 2004). These studies will be effective only when the reaction probability integrals are expressed in computable series representations (Mathai and Haubold, 1988; Haubold and John, 1982). The derivations of closed-form representations of nuclear reaction rates and the approximations on them are based on the theory of generalized special functions.
In the production of neutrinos in the gravitationally stabilized solar fusion reactor, due to the memory effects and long-range forces a possible deviation of the velocity distribution of plasma particles from Maxwell-Boltzmann is noted (Coraddu et al, 1999; Lavagno and Quarati, 2002; Lavagno and Quarati, 2006). This was initiated by Tsallis' non-additive generalization of Boltzmann-Gibbs statistical Mechanics. Tsallis statistics covers Boltzmann-Gibbs statistics (Tsallis, 1988; Gell-Mann and Tsallis, 2004; Tsallis, 2004). The extension of the nuclear reaction rate theory from Maxwell-Boltzmann theory to Tsallis theory was done by Saxena et al (2004), Mathai (2005), Mathai and Haubold (2007). In this scenario Mathai introduced a more general distribution function which can be incorporated in the reaction rate theory by appealing to entropic and distributional pathways. If Mathai's pathway model (Mathai, 2005; Mathai and Haubold, 2007) is introduced in the reaction probability integrals one goes into a wider class of integrals where the standard reaction probability integrals becomes a limiting case.

The paper is organized in the following way. Section 2 contains the standard representations of the non-resonant thermonuclear reaction rates. In section 3 we give the basic definitions that we use in our discussion. Section 4 gives an outline of the extensions of the reaction probability integrals using the pathway model and also establish the series representations of the extended integrals \( I_{1\alpha} \) and \( I_{2\alpha}^{(d)} \). Section 5 gives a comparison of the extended integrals with the standard integrals.

## 2 Standard representations of non-resonant thermonuclear reaction rates

For the evaluation of the reaction rate \( r_{ij} \) of the interacting particles \( i \) and \( j \) we have to consider the energies distributed between the particles. The reacting particles in the astrophysical plasma follows a Maxwell-Boltzmann distribution. From Mathai and Haubold (1988) we can see that the expression for the reaction rate \( r_{ij} \) of the reacting particles in the non-degenerate environment is

\[
r_{ij} = n_in_j \left( \frac{8}{\pi\mu} \right)^{\frac{3}{2}} \left( \frac{1}{kT} \right)^{\frac{3}{2}} \int_0^\infty E \sigma(E) e^{-\frac{E}{kT}} dE \tag{2.1}
\]

where \( n_i \) and \( n_j \) are the number densities of the reacting particles \( i \) and \( j \), the reduced mass of the particles is denoted by \( \mu = \frac{m_i m_j}{m_i + m_j} \), \( T \) is the temperature, \( k \) is the Boltzmann constant, the reaction cross section is \( \sigma(E) \) and the kinetic energy of the particles in the center of mass system is \( E = \frac{\nu^2}{2} \) where \( \nu \) is the relative velocity of the interacting particles \( i \) and \( j \).

We write \( \langle \sigma \nu \rangle \) to indicate that it is an appropriate average of the product of the cross section and relative velocity of the interacting particles. For detailed physical
reasons see Haubold and Mathai (1984, 1986).

2.1 Standard non-resonant thermonuclear function

When two nuclei of charges \( z_i \) and \( z_j \) are colliding at low energies below the coulomb barrier, the reaction cross section for the non-resonant nuclear reactions have the form (Haubold and Mathai, 1998; Bergstroem et al, 1999, Mathai and Haubold, 2002)

\[
\sigma(E) = \frac{S(E)}{E} e^{-2\eta(E)}
\]

with

\[
\eta(E) = \left( \frac{\mu}{2} \right)^{\frac{1}{2}} \frac{z_i z_j e^2}{\hbar E^{\frac{1}{2}}}
\]

where \( \eta(E) \) is the Sommerfeld parameter, \( \hbar \) is the Planck’s quantum of action, \( e \) is the quantum of electric charge, the cross section factor \( S(E) \) is often found to be a constant or a slowly varying function of energy over a limited range of energy (Mathai and Haubold, 1988). The cross section factor \( S(E) \) can be parameterized by expanding in terms of the power series about the zero energy because of its slow energy dependence. \( S(E) \) can be expressed as

\[
S(E) = S(0) + \frac{dS(0)}{dE} E + \frac{1}{2} \frac{d^2 S(0)}{dE^2} E^2,
\]

where \( S(0) \) is the value of \( S(E) \) at zero energy and \( S'(0) \) and \( S''(0) \) are the first and second order derivatives of \( S(E) \) with respect to energy evaluated at \( E = 0 \), respectively. Then

\[
\langle \sigma \nu \rangle = \left( \frac{8}{\pi \mu} \right)^{\frac{1}{2}} \sum_{\nu=0}^{2} \left( \frac{1}{kT} \right)^{-\nu+\frac{1}{2}} \frac{S^{(\nu)}(0)}{\nu!} \times \int_0^\infty E^\nu e^{-\frac{E}{kT}} e^{-2\eta(E)} dE
\]

\[
= \left( \frac{8}{\pi \mu} \right)^{\frac{1}{2}} \sum_{\nu=0}^{2} \left( \frac{1}{kT} \right)^{-\nu+\frac{1}{2}} \frac{S^{(\nu)}(0)}{\nu!} \times \int_0^\infty x^\nu e^{-x-bx^{-\frac{1}{2}}} dx
\]

where \( x = \frac{E}{kT} \) and \( b = \left( \frac{\mu}{2 kT} \right)^{\frac{1}{2}} \frac{z_i z_j e^2}{\hbar} \). The standard case of the thermonuclear function contains the nuclear cross section \( \sigma(E) \), the energy dependent cross section factor \( S(E) \) and the steady-state Maxwell-Boltzmann distribution function.

The collision probability integral for non-resonant thermonuclear reactions in the Maxwell-Boltzmannian form is (Haubold and Mathai, 1984)

\[
I_1(\nu, 1, b, \frac{1}{2}) = \int_0^\infty x^\nu e^{-x-bx^{-\frac{1}{2}}} dx.
\]

We will consider the general integral

\[
I_1(\gamma - 1, a, b, \rho) = \int_0^\infty x^{\gamma-1} e^{-ax-bx^{-\rho}} dx, \quad a > 0, b > 0, \rho > 0.
\]
2.2 Non-resonant thermonuclear function with high energy cut-off

It is assumed that the thermodynamic fusion plasma is in exact thermodynamic equilibrium. But the cut-off of the high energy tail of the Maxwell-Boltzmann distribution function in (2.6) results in a modification of the closed form representation of the appropriate quantity \( \langle \sigma \nu \rangle \) which is given by

\[
I^2_{d}(\nu, 1, b, \frac{1}{2}) = \int_{0}^{d} x^{\nu} e^{-bx^{-\frac{1}{2}}} \, dx, \quad b > 0, \quad d < \infty. \tag{2.8}
\]

Again we consider the general form of the integral (2.8) as

\[
I^2_{d}(\gamma - 1, a, b, \rho) = \int_{0}^{d} x^{\gamma - 1} e^{-ax - bx - \rho} \, dx, \quad a > 0, \quad b > 0, \quad \rho > 0, \quad d < \infty. \tag{2.9}
\]

For physical reasons for the cut-off modification of the Maxwell-Boltzmann distribution function of the relative kinetic energy of the reacting particles refer to the paper Haubold and Haubold and Mathai (1984).

2.3 Modified non-resonant thermonuclear function with depleted tail

A depletion of the high energy tail of the Maxwell-Boltzmann distribution function of the relative kinetic energies of the nuclei in the fusion plasma is explained in Haubold and Mathai(1986); Haubold and John (1982); Kaniadakis et al (1997,1998). The ad hoc modification of the Maxwell-Boltzmann distribution for the evaluation of the non-resonant thermonuclear reaction looks like a depletion of the high energy tail of the Maxwell-Boltzmann distribution. If their exists a possibility of such a modification a remarkable change of the views of astrophysical nucleosynthesis and controlled thermonuclear fusion may arise.

The integral form \( \langle \sigma \nu \rangle \) in comparison with strict Maxwell-Boltzmannian case, we have the integral

\[
I^3_{d}(\nu, 1, \delta, b, \frac{1}{2}) = \int_{0}^{\infty} x^{\nu} e^{-\delta x - bx^{-\frac{1}{2}}} \, dx, \quad b > 0. \tag{2.10}
\]

We will consider the general integral of the type

\[
I^3_{d}(\gamma - 1, a, \delta, b, \rho) = \int_{0}^{\infty} x^{\gamma - 1} e^{-ax - bx - \rho} \, dx, \tag{2.11}
\]

where \( z > 0, a > 0, b > 0, \rho > 0 \).

3 Mathematical preliminaries
The basic quantities which we need in our discussion will be given here. The gamma function denoted by $\Gamma(z)$ for complex number $z$ is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \mathcal{R}(z) > 0 \quad (3.1)$$

where $\mathcal{R}(\cdot)$ denotes the real part of $(\cdot)$. In general $\Gamma(z)$ exists for all values of $z$, positive or negative, except at the points $z = 0, -1, -2, \cdots$. These are the poles of $\Gamma(z)$. But the integral representation holds for the real part of $z$ to be positive. Another important result that we use in our discussion is the psi function. The psi function which is denoted by $\psi(z)$ is the logarithmic derivative of a gamma function and is defined as

$$\psi(z) = \frac{d}{dz}[\ln \Gamma(z)] = \frac{d[\Gamma(z)]}{\Gamma(z)} \text{ or } \ln \Gamma(z) = \int_1^z \psi(x)dx. \quad (3.2)$$

One property of the psi function that we will use is

$$\psi(1+n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \gamma \quad (3.3)$$

where $\gamma$ is the Euler’s constant and $\gamma = 0.5772156649\cdots, n = 1, 2, 3, \cdots$. The $G$-function which is originally due to C. S. Meijer in 1936 (See Mathai, 1993; Mathai and Saxena, 1973) is defined as a Mellin-Barnes type integral as follows:

$$G_{m,n}^{p,q}(z | a_1, \cdots, a_p | b_1, \cdots, b_q) = \frac{1}{2\pi i} \int_L \left\{ \prod_{j=1}^m \Gamma(b_j + s) \right\} \left\{ \prod_{j=1}^n \Gamma(1 - a_j - s) \right\} \left\{ \prod_{j=m+1}^q \Gamma(1 - b_j + s) \right\} \left\{ \prod_{j=n+1}^p \Gamma(a_j - s) \right\} z^{-s}ds \quad (3.4)$$

where $i = \sqrt{-1}$, $L$ is a suitable contour and $z \neq 0$, $m, n, p, q$ are integers, $0 \leq m \leq q$ and $0 \leq n \leq p$, the empty product is interpreted as unity and the parameters $a_1, a_2, \cdots, a_p$ and $b_1, b_2, \cdots, b_q$ are complex numbers such that no poles of $\Gamma(b_j + s)$, $j = 1, \cdots, m$ coincides with any pole of $\Gamma(1 - a_k - s)$, $k = 1, \cdots, n$;

$$-b_j - \nu \neq 1 - a_k + \lambda, \quad j = 1, \cdots, m; \quad k = 1, \cdots, n; \quad \nu, \lambda = 0, 1, \cdots.$$ 

This means that $a_k - b_j \neq \nu + \lambda + 1$ or $a_k - b_j$ is not a positive integer for $j = 1, \cdots, m; \quad k = 1, \cdots, n$. We also require that there is a strip in the complex $s$-plane which separates the poles of $\Gamma(b_j + s)$, $j = 1, \cdots, m$ from those of $\Gamma(1 - a_k - s)$, $k = 1, \cdots, n$ (For the existance conditions and properties of $G$-functions see Mathai (1993)).

Next we need the pathway model of Mathai (2005). When fitting a model to experimental data very often one needs a model with a thicker or thinner tail than the ones available from a given parametric family, or sometimes we may have a situation of the right tail cut-off. In order to take care of these situations and going from one functional form to another, a pathway parameter is introduced, see Mathai (2005) and Mathai and Haubold (2007). By this model we can proceed from a generalized type-1 beta model to a generalized type-2 beta model to a generalized gamma model when the
variable is restricted to be positive. For the real scalar case the pathway model is the following:

\[ f(x) = c|x|^{\gamma - 1}[1 - a(1-\alpha)|x|^\delta]^{-\frac{\eta}{\alpha-1}}, \quad a > 0, \delta > 0, 1 - a(1-\alpha)|x|^\delta > 0, \gamma > 0, \eta > 0 \]  

(3.5)

where \( c \) is the normalizing constant and \( \alpha \) is the pathway parameter. When \( \alpha < 1 \) the model becomes a generalized type-1 beta model in the real case. This is a model with the right tail cut-off. When \( \alpha > 1 \) we have \( 1 - \alpha = -\alpha (1-\alpha) \) so that

\[ f(x) = c|x|^{\gamma - 1}[1 + a(1-\alpha)|x|^\delta]^{-\frac{\eta}{\alpha-1}}, \]  

(3.6)

which is a generalized type-2 beta model for real \( x \). When \( \alpha \to 1 \) the above 2 forms will reduce to

\[ f(x) = c|x|^{\gamma - 1}e^{-ax|\delta}. \]  

(3.7)

Observe that the normalizing constant \( c \) appearing in (3.5), (3.6) and (3.7) are different.

4 Extended thermonuclear function through pathway model

When Mathai’s pathway model is introduced in the reaction probability integrals we get a wider class of integrals. Then the standard reaction probability integrals become particular cases of the new family of integrals. Through the pathway parameter \( \alpha \) we move to a wider class of integrals as \( \alpha \to 1 \) we get the reaction rate probability integrals. If Maxwell-Boltzmann is the stable situation, many unstable situations where Maxwell-Boltzmann is the limiting form are covered by the extended integrals.

Extended integral in the standard non-resonant case

The extended integral in the standard non-resonant case is (Haubold and Kumar, 2007)

\[ I_{1\alpha} = \int_{0}^{\infty} x^{\gamma - 1}[1 + a(\alpha - 1)x]^{-\frac{1}{\alpha-1}}e^{-bx-\rho} dx. \]  

(4.1)

**Theorem 4.1** (Haubold and Kumar, 2007)

\[ I_{1\alpha} = \int_{0}^{\infty} x^{\gamma - 1}[1 + a(\alpha - 1)x]^{-\frac{1}{\alpha-1}}e^{-bx-\rho} dx \]

\[ = \frac{1}{\rho[a(\alpha - 1)]^{\gamma} \Gamma \left( \frac{1}{\alpha-1} \right)} H_{1,2}^{2,1} \left( a(\alpha - 1) b^{\frac{1}{\gamma}} \left( \frac{1}{(\gamma,s), (0,\frac{1}{\rho})} \right) \right) \]  

(4.2)

where \( a > 0, b > 0, \mu > 0, \alpha > 1, \Re(s) > 0, \Re(\gamma + s) > 0. \)

When \( \alpha \to 1 \), \( I_{1\alpha} \) becomes \( I_1 \). But \( I_{1\alpha} \) contains all neighborhood solutions for various values of \( \alpha \) for \( \alpha > 1 \).

If in the above result \( \frac{1}{\rho} \) is an integer then by taking \( \frac{1}{\rho} = m \) we obtain
Corollary 4.1. For $a > 0$, $b > 0$ and $\alpha > 1$, we have

$$
\int_0^\infty x^{\gamma-1}[1 + a(\alpha - 1)x]^{-\frac{1}{\alpha-1}} e^{-bx^\alpha} dx
$$

$$
= \frac{\sqrt{m(2\pi)^{\frac{1-m}{2}}}}{[a(\alpha - 1)^{-\frac{1}{\alpha-1}} \Gamma(\frac{1}{\alpha-1})]^{\frac{1}{2}}} G_{1,m+1}^{m+1,1}
\left(\frac{a(\alpha - 1)b^m}{m^m} \left[1 + \frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}\right] \cdot \frac{1}{\alpha} + \gamma\right).
$$

(4.3)

By setting $\gamma - 1 = \nu$, $a = 1$ and $\rho = \frac{1}{2}$, we obtain

Corollary 4.2. For $b > 0$, $\alpha > 1$

$$
\int_0^\infty x^{\nu}[1 + (\alpha - 1)x]^{-\frac{1}{\alpha-1}} e^{-bx^\alpha} dx
$$

$$
= \frac{(\pi)^{-\frac{1}{2}}}{(\alpha - 1)^{\nu+1} \Gamma(\frac{1}{\alpha-1})} G_{3,1}^{3,1}
\left(\frac{(\alpha - 1)b^2}{4} \left[0, \frac{1}{2} + \nu + 1\right] \cdot \frac{1}{\alpha} + \nu\right).
$$

(4.4)

Extended cut-off case

In the case of non-resonant thermonuclear reactions with high energy cut-off the extended integral is (Haubold and Kumar, 2007)

$$
I_{2(\alpha)} = \int_0^d x^{\gamma-1}[1 - a(1-\alpha)x]^{\frac{1}{1-\alpha}} e^{-bx^x} dx
$$

where $d = \frac{1}{a(1-\alpha)}$, $\alpha < 1$, $a > 0$, $\delta = 1$, $\eta = 1$, $1 - a(1-\alpha)x > 0$, $\rho > 0$, $b > 0$.

Theorem 4.2

$$
\int_0^d x^{\gamma-1}[1 - a(1-\alpha)x]^{\frac{1}{1-\alpha}} e^{-bx^x} dx
$$

$$
= \frac{\Gamma(\frac{1}{1-\alpha} + 1)}{\rho(1-\alpha)^\gamma} H_{1,2}^{2,0}
\left(a(1-\alpha)b^{\frac{1}{\gamma}} \left[0, \frac{1}{\rho} + 1\right] \cdot \frac{1}{\alpha} + \gamma\right)
= I_{2(\alpha)}
$$

(4.5)

(4.6)

where $a > 0$, $b > 0$, $\rho > 0$, $\alpha < 1$, $\Re(\gamma + s) > 0$ and $d < \infty$

When $\alpha \to 1$, $I_{2(\alpha)}$ becomes $I_2^{(d)}$. But $I_{2(\alpha)}$ contains all neighborhood solutions for various values of $\alpha$ for $\alpha < 1$. In the above result if $\frac{1}{\rho}$ is an integer then by taking $\frac{1}{\rho} = m$ we obtain

Corollary 4.3. For $a > 0$, $b > 0$, $\alpha < 1$, $d < \infty$ and $\Re(\gamma + s) > 0$

$$
\int_0^d x^{\gamma-1}[1 - a(1-\alpha)x]^{\frac{1}{1-\alpha}} e^{-bx^x} dx
$$

$$
= \sqrt{m(2\pi)^{\frac{1-m}{2}} \Gamma(\frac{1}{1-\alpha} + 1)} G_{1,m+1}^{m+1,0}
\left(a(1-\alpha)b^m \left[0, \frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}\right] \cdot \frac{1}{\alpha} + \gamma\right)
$$

(4.7)
By setting $\gamma - 1 = \nu$, $a = 1$ and $\rho = \frac{1}{2}$, we obtain

**Corollary 4.4.** For $b > 0$, $\alpha < 1$, $d < \infty$ and $\Re(\nu + 1 + s) > 0$

\[
\int_0^d x^\nu [1 - (1 - \alpha)x]^{-\frac{1}{\alpha}} e^{-bx^{-\frac{1}{2}}} dx = \frac{\Gamma \left( \frac{1}{1-\alpha} + 1 \right)}{\sqrt{\pi} (1-\alpha)^{\nu+1}} C_{1,3}^{3,0} \left( \frac{(1-\alpha)b^2}{4} \right) \left|_{0,\frac{1}{2},\nu+1}^{\nu+\frac{1}{1-\alpha}+2} \right. \right) \tag{4.8}
\]

**Extended depleted case**

Proceeding similarly as in the case of $I_{1\alpha}$ we get for the depleted case

**Theorem 4.3**

\[
I_{3\alpha} = \int_0^\infty x^{\gamma-1} [1 + a(\alpha - 1)x^\delta]^{-\frac{1}{\alpha-1}} e^{-bx^{-\frac{1}{2}}} dx = \frac{1}{\rho[a(\alpha - 1)]^{\frac{1}{2}}} \Gamma \left( \frac{1}{\alpha-1} \right) H_{1,2}^{2,1} \left( \left[ a(\alpha - 1) \right]^{\frac{1}{2}} b^\frac{1}{2} \right) \left( \frac{1}{2}, \frac{1}{2}, \frac{2}{1+\nu}, (0, 0, 2, \nu) \right). \tag{4.9}
\]

where $a > 0$, $b > 0$, $\rho > 0$, $\delta > 0$, $\alpha > 1$, $\Re(s) > 0$, $\Re(\gamma + s) > 0$.

For the non-resonant thermonuclear reactions with depleted tail $\gamma - 1 = \nu$, $a = 1$, $\rho = \frac{1}{2}$ then we get,

**Corollary 4.5**

\[
\int_0^\infty x^\nu [1 + (\alpha - 1)x^\delta]^{-\frac{1}{\alpha-1}} e^{-bx^{-\frac{1}{2}}} dx = \frac{2}{\delta(\alpha - 1)^{\frac{\nu+1}{\delta}}} \Gamma \left( \frac{1}{\alpha-1} \right) H_{1,2}^{2,1} \left( \left[ \alpha - 1 \right]^{\frac{1}{2}} b^\frac{1}{2} \right) \left( \frac{1}{2}, \frac{1}{2}, \frac{2}{1+\nu}, (0, 0, 2, \nu) \right). \tag{4.10}
\]

By setting $\frac{\nu}{\delta} = s'$ and $2\delta = m$, $m = 1, 2, \ldots$ we get

**Corollary 4.6**

\[
I_{3\alpha} = \frac{2(2\pi)^{\frac{1-m}{2}} m^{-\frac{1}{2}}}{(\alpha - 1)^{2(\nu+1)} m^{\nu}} \Gamma \left( \frac{1}{\alpha-1} \right) C_{1,m+1}^{m+1,1} \left( \frac{(\alpha - 1)b^m}{m^m} \right) \left|_{0,\frac{1}{2},\frac{2}{m}, \ldots, \frac{2(1+\nu)}{m}}^{1-\frac{1}{\alpha-1}+\frac{2(1+\nu)}{m}} \right. \right) \tag{4.11}
\]

### 4.1 Series representations

In the following we derive series representations of the right-hand side of (4.4) which will be helpful for the evaluation of the extended reaction probability integrals in the Maxwell-Boltzmannian form. Taking $\nu$ as a general parameter one can consider several situations. Then following through the process in the papers of Haubold and Mathai,
The sum of the residues corresponding to the poles \( s \) when \( \Gamma(s) \) in the integral representation in (4.12) are as follows. The right hand side is the sum of the residues of the integrand. The poles of the gammas \( \nu \) for the extended integral in (4.4).

### 4.1.1 Case (I): \( \nu \neq \pm \frac{3}{2}, \lambda = 0, 1, 2, \ldots \)

Here we apply residue calculus on the G-function for obtaining the series representation of the integrals. Consider the G-function in (4.4).

\[
G_{1,3}^{3,1} \left( \frac{(\alpha - 1) b^2}{4} |_{0, \frac{1}{2}; \nu} \right) = \frac{1}{2 \pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \Gamma \left( \frac{1}{2} + s \right) \\
\times \Gamma(1 + \nu + s) \Gamma \left( \frac{1}{\alpha - 1} - \nu - 1 - s \right) \left( \frac{(\alpha - 1) b^2}{4} \right)^{-s} \, ds \tag{4.12}
\]

The right hand side is the sum of the residues of the integrand. The poles of the gammas in the integral representation in (4.12) are as follows.

- Poles of \( \Gamma(s) : s = 0, -1, -2, \ldots \); \( \Gamma \left( \frac{1}{2} + s \right) : s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots \); \( \Gamma(1 + \nu + s) : s = -\nu - 1, -\nu - 2, -\nu - 3, \ldots \).

These are all simple poles under the conditions in case(1). Then the G-function has a simple series expansion.

We know that

\[
\lim_{s \to -r} (s + r) \Gamma(s) = \frac{(-1)^r}{r!}, \tag{4.13}
\]

\[
\Gamma(a - r) = \left( \frac{-1)^r \Gamma(a) \right), \tag{4.14}
\]

\[
\Gamma(a + m) = \Gamma(a)(a)_m \tag{4.15}
\]

when \( \Gamma(a) \) is defined, \( r = 0, 1, 2, \ldots \); \( \Gamma \left( \frac{1}{2} \right) = \pi^{\frac{1}{2}} \);

\[
(a)_r = \begin{cases} 
  a(a + 1) \cdots (a + r - 1) & \text{if } r \geq 1, \ a \neq 0 \\
  1 & \text{if } r = 0,
\end{cases}
\]

The sum of the residues corresponding to the poles \( s = -r, r = 0, 1, 2, \ldots \) is given by

\[
R_1 = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \Gamma \left( \frac{1}{2} - r \right) \Gamma \left( \frac{1}{\alpha - 1} - 1 - \nu + r \right) \Gamma(1 + \nu - r) \left[ \frac{(\alpha - 1) b^2}{4} \right]^r
\]

\[
= \pi^{\frac{1}{2}} \Gamma \left( \frac{1}{\alpha - 1} - 1 - \nu \right) \Gamma(1 + \nu) \frac{1}{\alpha - 1} - 1 - \nu; \frac{1}{2}, -\nu; -\frac{(\alpha - 1) b^2}{4} \right) \tag{4.16}
\]

where \( _pF_q(a_p; b_q; z) \) denotes the generalized hypergeometric function defined as above.

The sum of the residues corresponding to the poles \( s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots \) is

\[
R_2 = -2\pi^{\frac{1}{2}} \Gamma \left( \frac{1}{\alpha - 1} - \frac{1}{2} - \nu \right) \Gamma \left( \frac{1}{2} + \nu \right) \left[ \frac{(\alpha - 1) b^2}{4} \right]^{\frac{1}{2}}
\times _2F_2 \left( \frac{1}{\alpha - 1} - \frac{1}{2} - \nu; \frac{3}{2}, \frac{1}{2}; \frac{1}{2}, -\nu; -\frac{(\alpha - 1) b^2}{4} \right). \tag{4.17}
\]
Finally the sum of the residues corresponding to $s = -\nu - 1, -\nu - 2, -\nu - 3, \cdots$ is

$$R_3 = \Gamma(-\nu - 1) \Gamma\left(-\nu - \frac{1}{2}\right) \Gamma\left(\frac{1}{\alpha - 1}\right) \left[\frac{(\alpha - 1)b^2}{4}\right]^{1+\nu} \times _1 F_2\left(\frac{1}{\alpha - 1}; 2 + \nu, \frac{3}{2} + \nu; -\frac{(\alpha - 1)b^2}{4}\right). \quad (4.18)$$

Adding $R_1, R_2, R_3$ we obtain the final result:

**Theorem 4.4** If $\nu \neq \pm \frac{\lambda}{2}$, $\lambda = 0, 1, 2, \cdots$ is an integer, then for $b > 0$, $\alpha > 1$, we have

$$\int_0^\infty x^\nu[1 + (\alpha - 1)x]^{-\frac{1}{\alpha - 1}} e^{-bx}\frac{1}{2} \, dx = \left\{ \frac{\Gamma\left(\frac{1}{\alpha - 1} - 1 - \nu\right) \Gamma(1 + \nu)}{\left[(\alpha - 1)\right]^{\nu + 1} \Gamma\left(\frac{1}{\alpha - 1}\right)} \times _1 F_2\left(\frac{1}{\alpha - 1} - 1 - \nu, \frac{1}{2}; \nu; -\frac{(\alpha - 1)b^2}{4}\right) \right\}$$

$$- \left\{ \frac{2\Gamma\left(\frac{1}{\alpha - 1} - \frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{2} + \nu\right)}{\left[(\alpha - 1)\right]^{\nu + 1} \Gamma\left(\frac{1}{\alpha - 1}\right)} \left[\frac{(\alpha - 1)b^2}{4}\right]^{\frac{1}{2}} \times _1 F_2\left(\frac{1}{\alpha - 1} - \frac{1}{2} - \nu, \frac{3}{2}; \nu; -\frac{(\alpha - 1)b^2}{4}\right) \right\}$$

$$+ \left\{ \frac{(\pi)^{-\frac{1}{2}} \Gamma(-\nu - 1) \Gamma(-\nu - \frac{1}{2})}{\left[(\alpha - 1)\right]^{\nu + 1}} \left[\frac{(\alpha - 1)b^2}{4}\right]^{1+\nu} \times _1 F_2\left(\frac{1}{\alpha - 1}; 2 + \nu, \frac{3}{2}; \nu; -\frac{(\alpha - 1)b^2}{4}\right) \right\} \quad (4.19)$$

It is to be noted that the series on the right-hand side of the equation (4.19) are term-wise integrable over any finite range.

### 4.1.2 Case (II): $\nu$ is a positive integer

In this case some poles of $\Gamma(s)$ and $\Gamma(1 + \nu + s)$ will coincide with each other. Therefore these will be of order 2. We note that the poles $s = 0, -1, -2, \cdots, -\nu$ are each of order 1: $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$ are each of order 1: $s = -\nu - 1, -\nu - 2, -\nu - 3, \cdots$ are each of order 2. Taking the sum of residues at the poles $s = 0, -1, -2, \cdots, -\nu$; at $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$; at $s = -\nu - 1, -\nu - 2, -\nu - 3, \cdots$ we have

$$R_1 = \pi^{\frac{1}{2}} \Gamma\left(\frac{1}{\alpha - 1} - 1 - \nu\right) \Gamma(1 + \nu) \sum_{r=0}^{\nu} \frac{(-1)^r \left(\frac{1}{\alpha - 1} - 1 - \nu\right)_r}{(\frac{1}{2})_r (-\nu)_r} \left[\frac{(\alpha - 1)b^2}{4}\right]^{r} \quad (4.20)$$

$$R_2 = -2\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{\alpha - 1} - \frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{2} + \nu\right) \left[\frac{(\alpha - 1)b^2}{4}\right]^{\frac{1}{2}} \times _1 F_2\left(\frac{1}{\alpha - 1} - \frac{1}{2} - \nu, \frac{3}{2}; \frac{1}{2}; \nu; -\frac{(\alpha - 1)b^2}{4}\right) \quad (4.21)$$
\( R_3 = \left( \frac{(\alpha - 1)b^2}{4} \right)^{1+\nu} \sum_{r=0}^{\infty} \left( \frac{(\alpha - 1)b^2}{4} \right)^r \left[ -\ln \left( \frac{(\alpha - 1)b^2}{4} \right) + A_r \right] B_r, \) \quad (4.22)

where

\[ A_r = \psi\left(-\frac{1}{2} - \nu - r\right) + \psi\left(\frac{1}{\alpha - 1} + r\right) + \psi(1 + r) + \psi(2 + \nu + r) \quad (4.23) \]

and

\[ B_r = \frac{(-1)^{1+\nu+r} \Gamma\left(-\frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{\alpha - 1}\right) \left(\frac{1}{\alpha - 1}\right)_r}{\left(\frac{3}{2} + \nu\right)_r r!(1 + \nu + r)!} \quad (4.24) \]

From the above results we have the following theorem.

**Theorem 4.5** If \( \nu > 0 \) is an integer, then for \( b > 0, \alpha > 1 \), we have

\[
\int_0^\infty x^{\nu}[1 + (\alpha - 1)x]^{-\frac{1}{\nu}} e^{-bx^{-\frac{1}{2}}} \, dx \\
= \frac{(\pi)^{-\frac{1}{4}}}{[(\alpha - 1)^{\nu+1} \Gamma\left(\frac{1}{\alpha - 1}\right) \left(\frac{1}{\alpha - 1}\right)_r}} \left[ \pi^2 \Gamma\left(\frac{1}{\alpha - 1} - 1 - \nu\right) \Gamma(1 + \nu) \right] \\
\times \sum_{r=0}^{\nu} \frac{(-1)^r \left(\frac{1}{\alpha - 1} - 1 - \nu\right)_r}{r!} \left[ \left(\frac{(\alpha - 1)b^2}{4}\right)^r \right] \\
- 2\pi^2 \Gamma\left(\frac{1}{\alpha - 1} - \frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{2} + \nu\right) \left[ \left(\frac{(\alpha - 1)b^2}{4}\right)^{\frac{1}{2}} \right] \\
\times {}_1F_2\left(\frac{1}{\alpha - 1} - \frac{1}{2} - \nu; \frac{3}{2}, \frac{1}{2} - \nu; -\left(\frac{(\alpha - 1)b^2}{4}\right)\right) \\
+ \left(\frac{(\alpha - 1)b^2}{4}\right)^{1+\nu} \sum_{r=0}^{\infty} \left(\frac{(\alpha - 1)b^2}{4}\right)^r \left[ -\ln \left( \frac{(\alpha - 1)b^2}{4} \right) + A_r \right] B_r \right\} \quad (4.25)
\]

where \( A_r \) and \( B_r \) are as given in (4.23) and (4.24).

**4.1.3 Case (III): \( \nu \) a negative integer**

Let \( \nu = -\mu, \mu > 0 \). Then the poles of the G-function in (4.12) are \( s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots \) are each of order 1; \( s = -\nu - 1, -\nu - 2, -\nu - 3, \ldots, +1 \) are each of order 1; poles \( s = 0, -1, -2, \ldots \) are each of order 2.Again, following through the same stages as above we have
Theorem 4.6 For $\nu$ a negative integer,

$$\int_0^\infty x^\nu [1 + (\alpha - 1)x]^{-\frac{1}{\alpha - 1}} e^{-bx^{\frac{1}{2}}} \, dx$$

$$= \frac{(\pi)^{\frac{1}{2}}}{(\alpha - 1)^{\nu + 1} \Gamma\left(\frac{1}{\alpha - 1}\right)} \left\{ -2\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{\alpha - 1} - \frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{2} + \nu\right) \left[\frac{(\alpha - 1)b^2}{4}\right]^\frac{1}{2} \times_1 F_2 \left(\begin{array}{c}
\frac{1}{\alpha - 1} - \frac{1}{2} - \nu; \frac{3}{2}, \frac{1}{2} - \nu; \frac{(\alpha - 1)b^2}{4}\n\end{array}\right)
+ \Gamma(-1 - \nu) \Gamma\left(-\frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{\alpha - 1}\right) \left[\frac{(\alpha - 1)b^2}{4}\right]^{1+\nu}
\times \sum_{r=0}^{\nu-2} \frac{(-1)^r}{r!} \frac{1}{\Gamma\left(\frac{3}{2} + \nu\right)} \Gamma\left(\frac{1}{\alpha - 1} - \nu + 1\right) \left[\frac{(\alpha - 1)b^2}{4}\right]^r
+ \sum_{r=0}^\infty \left(\frac{(\alpha - 1)b^2}{4}\right)^r \left[-\ln \left(\frac{(\alpha - 1)b^2}{4}\right) + A'_r\right] B'_r \right\}, \quad (4.26)$$

where

$$A'_r = \psi\left(\frac{1}{2} - r\right) + \psi\left(\frac{1}{\alpha - 1} - 1 - \nu + r\right) + \psi(1 + r) + \psi(r - \nu) \quad (4.27)$$

and

$$B'_r = \frac{(-1)^{1+\nu} \Gamma\left(\frac{1}{2} - r\right) \Gamma\left(\frac{1}{\alpha - 1} - 1 - \nu - r\right)}{r!(r - \nu - 1)!} \quad (4.28)$$

4.1.4 Case (IV): $\nu$ a positive half integer

Let $\nu = m + \frac{1}{2}, m = 0, 1, 2, \cdots$. Then

$$\Gamma(s) \Gamma\left(\frac{1}{2} + s\right) \Gamma(1 + \nu + s) \Gamma\left(\frac{1}{\alpha - 1} - \nu - 1 - s\right)$$

$$= \Gamma(s) \Gamma\left(\frac{1}{2} + s\right) \Gamma(m + \frac{3}{2} + s) \Gamma\left(\frac{1}{\alpha - 1} - m - \frac{3}{2} - s\right)$$

Here the poles of $\Gamma\left(\frac{1}{2} + s\right)$ and $\Gamma(s + m + \frac{3}{2})$ coincides with each other. These will be of order 2. We note that the poles $s = 0, -1, -2, \cdots$ are each of order 1; $s = -\frac{1}{2}, -\frac{3}{2} - 1, -\frac{3}{2} - 2, \cdots, -\frac{1}{2} - m$ are each of order 1; $s = -m - \frac{3}{2}, -m - \frac{5}{2}, \cdots$ are each of order 2 and we have the following theorem
Theorem 4.7 For \( \nu \) a positive half integer, namely \( \nu = m + \frac{1}{2}, \ m = 0, 1, 2, \cdots \)

\[
\int_0^\infty x^\nu [1 + (\alpha - 1)x]^{-\frac{1}{\alpha-1}} e^{-bx^{-\frac{1}{2}}} \, dx \\
= \frac{(\pi)^{-\frac{1}{2}}}{(\alpha-1)^{\frac{3}{2} + m} \Gamma \left( \frac{1}{\alpha-1} \right)} \left\{ \pi^{\frac{1}{2}} \Gamma \left( \frac{1}{\alpha-1} - \frac{3}{2} - m \right) \Gamma \left( \frac{3}{2} + m \right) \right\} \\
\times {}_1F_2 \left( \frac{1}{\alpha-1} - \frac{3}{2} - m; \frac{1}{2}, -\frac{1}{2} - m; -\frac{(\alpha-1)b^2}{4} \right) \\
- 2\pi^{\frac{1}{2}} \Gamma \left( \frac{1}{\alpha-1} - 1 - m \right) \Gamma(1+m) \left[ \frac{(\alpha-1)b^2}{4} \right]^{\frac{1}{2}} \\
\times \sum_{r=0}^{m} \left( -1 \right)^r \left( \frac{1}{\alpha-1} - 1 - m \right)_r \left[ \frac{(\alpha-1)b^2}{4} \right]^r \\
+ \left( \frac{(\alpha-1)b^2}{4} \right)^{m+\frac{3}{2} + \sum_{r=0}^{\infty} \left( \frac{(\alpha-1)b^2}{4} \right)^r \left[ -\ln \left( \frac{(\alpha-1)b^2}{4} \right) + C_r \right] D_r \right), (4.29)
\]

where

\[ C_r = \psi \left( -m - \frac{3}{2} - r \right) + \psi \left( \frac{1}{\alpha-1} + r \right) + \psi(1+r) + \psi(2+m+r) \quad (4.30) \]

and

\[ D_r = \frac{(-1)^{1+m+r}(\alpha-1) \Gamma(\frac{3}{2}) \Gamma \left( \frac{1}{\alpha-1} \right)}{(\frac{5}{2} + m)_r (1+m+r)!} \quad (4.31) \]

4.1.5 Case (V): \( \nu \) a negative half integer

Let \( \nu = -m - \frac{1}{2}, \ m = 0, 1, 2, \cdots \). Then in this case

\[
\Gamma(s) \Gamma \left( \frac{1}{2} + s \right) \Gamma(1+\nu + s) \Gamma \left( \frac{1}{\alpha-1} - \nu - 1 - s \right) \\
= \Gamma(s) \Gamma \left( \frac{1}{2} + s \right) \Gamma(s + \frac{1}{2} - m) \Gamma \left( \frac{1}{\alpha-1} - \frac{1}{2} + m - s \right)
\]

Then the poles of Meijer’s G-function in (4.12) are \( s = -r, \ r = 0, 1, 2, \cdots \) of order 1 each; \( s = m - \frac{1}{2} - r, \ r = 0, 1, 2, \cdots, m - 1 \) of order 1 each; \( s = -\frac{1}{2} - r, \ r = 0, 1, 2, \cdots \) of order 2 each. Then we have the following theorem
Theorem 4.8 For $\nu$ a negative half integer, namely $\nu = -m - \frac{1}{2}$, $m = 0, 1, 2, \cdots$

\[
\int_0^{\infty} x^\nu [1 + (\alpha - 1)x]^{-\frac{1}{\alpha}} e^{-bx^{\frac{1}{2}}} \, dx
\]

\[
= \frac{\pi^{-\frac{1}{2}}}{(\alpha - 1)^{\frac{1}{2} + m} \Gamma \left( \frac{1}{\alpha - 1} \right)} \left\{ \sqrt{\pi \Gamma \left( \frac{1}{2} - m \right)} \Gamma \left( \frac{1}{\alpha - 1} - \frac{1}{2} + m \right) \right.
\]

\[
\times \text{\textit{F}_2} \left( \frac{1}{\alpha - 1} - \frac{1}{2} + m; \frac{1}{2} + m; -\frac{(\alpha - 1)b^2}{4} \right)
\]

\[
+ \Gamma \left( m - \frac{1}{2} \right) \Gamma (m) \Gamma \left( \frac{1}{\alpha - 1} \right) \left( \frac{(\alpha - 1)b^2}{4} \right)^{-m + \frac{1}{2}}
\]

\[
\times \sum_{r=0}^{m} \frac{(-1)^r}{r!} \left( \frac{\Gamma \left( \frac{1}{\alpha - 1} \right)}{\alpha - 1} \right)_r \left[ \frac{(\alpha - 1)b^2}{4} \right]_r
\]

\[
+ \left( \frac{(\alpha - 1)b^2}{4} \right)^{\frac{1}{2}} \sum_{r=0}^{\infty} \left( \frac{(\alpha - 1)b^2}{4} \right)^r \left[ \text{\textit{ln}} \left( \frac{(\alpha - 1)b^2}{4} \right) + C_r' \right] D_r' \right\}, \quad (4.32)
\]

where

\[
C_r' = \psi \left( -\frac{1}{2} - r \right) + \psi \left( \frac{1}{\alpha - 1} + m + r \right) + \psi(1 + r) + \psi(1 + m + r) \quad (4.33)
\]

and

\[
D_r' = \frac{\sqrt{\pi} (-1)^{m + r} \Gamma \left( \frac{1}{\alpha - 1} + m + r \right) \left( \frac{1}{\alpha - 1} \right)_r}{\left( \frac{3}{2} \right)_r r!(m + r)!} \quad (4.34)
\]

4.2 Series representation for the extended cut-off case

4.2.1 Case(I): $\nu \neq \pm \frac{1}{2}$, $\lambda = 0, 1, 2, \cdots$

Here we apply the same techniques that we applied previously. Consider the evaluation of the G-function in (4.8).

\[
\int_0^{d} x^\nu [1 - (1 - \alpha)x]^{\frac{1}{\alpha}} e^{-bx^{\frac{1}{2}}} \, dx
\]

\[
= \frac{\Gamma \left( \frac{1}{1 - \alpha} + 1 \right)}{\sqrt{\pi} (1 - \alpha)^{\nu + \frac{1}{2}} 2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s) \Gamma \left( s + \frac{1}{2} \right) \Gamma(1 + \nu + s)}{\Gamma(2 + \nu + \frac{1}{1 - \alpha} + s)} \left[ \frac{(1 - \alpha)b^2}{4} \right]^{-s} ds, \quad (4.35)
\]

The poles of the integral are as follows: $\Gamma(s)$ : $s = 0, -1, -2, \cdots$; $\Gamma \left( \frac{1}{2} + s \right)$ : $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$; $\Gamma(1 + \nu + s)$ : $s = -\nu - 1, -\nu - 2, -\nu - 3, \cdots$.

These are all simple poles under case (1). Then evaluating the sum of residues we have
Theorem 4.9 If $\nu \neq \pm \frac{1}{2}$, $\lambda = 0, 1, 2, \cdots$ is an integer, then for $b > 0$, $\alpha < 1$, we have

\[
\int_0^d x^\nu [1 - (1 - \alpha)x]^{1/\alpha} e^{-bx^{1/2}} \, dx
\]

\[
= \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\sqrt{\pi}(1-\alpha)^{\nu+1}} \left\{ \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Gamma(1+\nu) \right\} \Gamma\left(2 + \nu + \frac{1}{1-\alpha}\right) \sum_{r=0}^{\nu} \left(\frac{-1 - \nu - \frac{1}{1-\alpha} - r}{1/2}\right) (-\nu)_r \left[ \left(1-\alpha\right)b^2\right]^r
\]

\[
- \frac{2 \pi^{\frac{3}{2}} \Gamma\left(\frac{1}{2} + \nu\right)}{\Gamma\left(\frac{3}{2} + \nu + \frac{1}{1-\alpha}\right)} \left(\frac{1-\alpha}{4}\right)^{\nu} \frac{1}{2} \Gamma\left(-\frac{1}{2} - \nu - \frac{1}{\alpha - 1}\right) \] 

\[
+ \frac{(-\nu - 1)\Gamma\left(-\nu - \frac{1}{2}\right)}{\Gamma\left(1 + \frac{1}{1-\alpha}\right)} \left(\frac{1-\alpha}{4}\right)^{\nu} \left. \frac{1}{2} \Gamma\left(-\frac{1}{2} - \nu - \frac{1}{\alpha - 1}\right) \right] F_2\left(1 + \frac{1}{1-\alpha}; 2 + \nu, \frac{3}{2} + \nu; \frac{-1 - \nu - \frac{1}{1-\alpha}}{4}\right)
\]

\[
(4.36)
\]

Here the series on the right-hand side of the equation (4.36) are in computable forms.

4.2.2 Case (II): $\nu$ is a positive integer

The poles of the gammas in the integral representation of (4.35) are as follows:

$\Gamma(s)$ : $s = 0, -1, -2, \cdots$; $\Gamma\left(\frac{1}{2} + s\right)$ : $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$; $\Gamma(1 + \nu + s)$ : $s = -\nu - 1, -\nu + 2, -\nu - 3, \cdots$.

Some poles of $\Gamma(s)$ and $\Gamma(1 + \nu + s)$ will coincide with each other. Therefore these will be of order 2. We note that the poles $s = 0, -1, -2, \cdots, -\nu$ are each of order 1; $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$ are each of order 1; $s = -\nu - 1, -\nu + 2, -\nu - 3, \cdots$ are each of order 2. Evaluating the sum of the residues we have

Theorem 4.10 If $\nu > 0$ is an integer, then for $b > 0$, $\alpha < 1$, we have

\[
\int_0^d x^\nu [1 - (1 - \alpha)x]^{1/\alpha} e^{-bx^{1/2}} \, dx
\]

\[
= \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\sqrt{\pi}(1-\alpha)^{\nu+1}} \left\{ \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Gamma(1+\nu) \right\} \Gamma\left(2 + \nu + \frac{1}{1-\alpha}\right) \sum_{r=0}^{\nu} \left(\frac{-1 - \nu - \frac{1}{1-\alpha} - r}{1/2}\right) (-\nu)_r \left[ \left(1-\alpha\right)b^2\right]^r
\]

\[
- \frac{2 \pi^{\frac{3}{2}} \Gamma\left(\frac{1}{2} + \nu\right)}{\Gamma\left(\frac{3}{2} + \nu + \frac{1}{1-\alpha}\right)} \left(\frac{1-\alpha}{4}\right)^{\nu} \frac{1}{2} \Gamma\left(-\frac{1}{2} - \nu - \frac{1}{\alpha - 1}\right) \] 

\[
+ \left(\frac{1-\alpha}{4}\right)^{\nu+1} \sum_{r=0}^{\infty} \left[ \left(\frac{1-\alpha}{4}\right)^{-r} \left[- \ln \left(\frac{1-\alpha}{4}\right) + E_r\right] F_r \right]
\]

\[
(4.37)
\]

where

\[
E_r = \psi\left(-\frac{1}{2} - \nu - r\right) - \psi\left(\frac{1}{1-\alpha} + 1 - r\right) + \psi(1 + r) + \psi(2 + \nu + r)
\]

and

\[
F_r = \frac{(-1)^r(-1)^{1+r}\Gamma\left(-\frac{1}{2} - \nu\right) \Gamma\left(-\frac{1}{1-\alpha}\right)}{\left(\frac{3}{2} + \nu\right)_r r!(1+\nu + r)! \Gamma\left(\frac{1}{1-\alpha} + 1\right)}
\]

(4.39)
4.2.3 Case (III): $\nu$ a negative integer

Let $\nu = -\mu, \mu > 0$. Then proceeding as in the above cases we have the result

**Theorem 4.11**

\[
\int_0^d x^\nu [1 - (1 - \alpha)x]^{1 - \frac{1}{\alpha}} e^{-bx^\frac{1}{2}} dx = \frac{\Gamma \left( \frac{1}{1-\alpha} + 1 \right)}{\sqrt{\pi}(1-\alpha)^{\nu+1}} \left\{ -2(\pi)^{\frac{1}{2}} \Gamma \left( \frac{1}{2} + \nu \right) \left( \frac{1-\alpha}{4} \right)^{\frac{1}{2}} \right. \\
\times F_2 \left( \frac{1}{2} - \nu - \frac{1}{\alpha - 1}, \frac{3}{2}, \frac{1}{2}; -\nu; \frac{(1-\alpha)b^2}{4} \right) \\
+ \frac{\Gamma(-1 - \nu) \Gamma \left( -\frac{3}{2} - \nu \right)}{\Gamma \left( 1 + \frac{1}{1-\alpha} \right)} \left[ \frac{(1-\alpha)b^2}{} \right]^{1+\nu-\nu-2} \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{-1}{1-\alpha} \right)_r \frac{(1-\alpha)b^2}{4}^r \\
+ \sum_{r=0}^{\infty} \left( \frac{(1-\alpha)b^2}{4} \right)^r \left[ -\ln \left( \frac{(1-\alpha)b^2}{4} \right) + E'_r \right] F_r \right\}, \tag{4.40}
\]

where

\[ E'_r = \psi \left( \frac{1}{2} - r \right) - \psi \left( \frac{1}{1-\alpha} + 2 + \nu - r \right) + \psi(1 + r) + \psi(r - \nu) \tag{4.41} \]

and

\[ F'_r = \frac{(-1)^{1+\nu} \Gamma \left( \frac{1}{2} - r \right)}{r!(r - \nu - 1)! \Gamma \left( \frac{1}{1-\alpha} + 2 + \nu - r \right)} \tag{4.42} \]

4.2.4 Case (IV): $\nu$ a positive half integer

Proceeding as before, we have

**Theorem 4.12** For $\nu$ a positive half integer, namely $\nu = m + \frac{1}{2}$, $m = 0, 1, 2, \cdots$

\[
\int_0^d x^\nu [1 - (1 - \alpha)x]^{1 - \frac{1}{\alpha}} e^{-bx^\frac{1}{2}} dx = \frac{\Gamma \left( \frac{1}{1-\alpha} + 1 \right)}{\sqrt{\pi}(1-\alpha)^{\nu+1}} \left\{ \frac{(\pi)^{\frac{1}{2}} \Gamma(m + \frac{3}{2})}{\Gamma \left( m + \frac{5}{2} + \frac{1}{1-\alpha} \right)} F_2 \left( -m - \frac{3}{2} - \frac{1}{1-\alpha}; \frac{1}{2}, -m - \frac{1}{2}; \frac{(1-\alpha)b^2}{4} \right) \\
- \frac{2(\pi)^{\frac{1}{2}} \Gamma(1 + m)}{\Gamma \left( 2 + m + \frac{1}{1-\alpha} \right)} \left( \frac{(1-\alpha)b^2}{4} \right)^{\frac{1}{2}} \sum_{r=0}^{m-1} \frac{(-1)^r}{r!} \frac{(-1 - m - \frac{1}{1-\alpha})_r}{\left( \frac{3}{2} \right)_r (-m)_r} \left( \frac{(1-\alpha)b^2}{4} \right)^{r} \\
+ \left( \frac{(1-\alpha)b^2}{4} \right)^{m+\frac{1}{2}} \sum_{r=0}^{\infty} \left( \frac{(1-\alpha)b^2}{4} \right)^r \left[ -\ln \left( \frac{(1-\alpha)b^2}{4} \right) + G_r \right] H_r \right\}, \tag{4.43}
\]
where
\[ G_r = \psi \left( -m - \frac{3}{2} - r \right) + \psi \left( 1 + \frac{1}{1 - \alpha} - r \right) + \psi(1 + r) + \psi(2 + m + r) \] (4.44)

and
\[ H_r = \frac{(-1)^{1+m+r}(-1)^r \Gamma \left( -m - \frac{3}{2} \right) \left( -m - \frac{3}{2} \right)_r}{r!(1 + m + r)! \Gamma \left( 1 + \frac{1}{1 - \alpha} \right) \left( 1 + \frac{1}{1 - \alpha} \right)_r} \] (4.45)

### 4.2.5 Case (V): \( \nu \) a negative half integer

Let \( \nu = -m - \frac{1}{2}, m = 0, 1, 2, \ldots \). In this case we have

**Theorem 4.13** For \( \nu \) a negative half integer, namely \( \nu = -m - \frac{1}{2}, m = 0, 1, 2, \ldots \)

\[
\int_0^d x^\nu [1 - (1 - \alpha)x]^{\frac{1}{1-\alpha}} e^{-bx - \frac{1}{4}} dx = \frac{\Gamma \left( \frac{1}{1-\alpha} + 1 \right)}{\sqrt{\pi} (1 - \alpha)^{\nu + 1}} \left\{ \frac{(\pi)^{\frac{1}{2}} \Gamma \left( \frac{1}{2} + m \right) m^{\frac{1}{2} - m} \sum_{r=0}^{m-1} \frac{1}{r! \left( \frac{3}{2} \right)_r (m)_r \left( \frac{1 - \alpha}{4} \right)^r} \right\}
\[
-2 \sqrt{\pi} \sum_{r=0}^\infty \frac{(1 - \alpha)^{b^2}}{4^r} \left[ - \ln \left( \frac{(1 - \alpha)^{b^2}}{4} \right) + G'_r \right] H'_r \}, \quad (4.46)
\]

where
\[ G'_r = \psi \left( -\frac{1}{2} - r \right) + \psi \left( 1 + \frac{1}{1 - \alpha} - m - r \right) + \psi(1 + r) + \psi(1 + m + r) \] (4.47)

and
\[ H'_r = \frac{(-1)^m \left( -m - \frac{1}{1 - \alpha} \right)_r}{r!(m + r)! \Gamma \left( 1 + \frac{1}{1 - \alpha} - m \right)} \] (4.48)

### 5 Behaviour of the integrals \( I_{1\alpha} \) and \( I_{2\alpha}^{(d)} \)

The behaviour of the integral \( I_{1\alpha} \) is such that as the value of the pathway parameter \( \alpha \) changes the curve will move away from the stable situation ie, the strict Maxwell-Boltzmann situation(Figure 3 below). The graphs of the integral \( I_{1\alpha} \) when \( \nu = 1 \) and at \( \alpha = 1, \alpha = 1.25, \alpha = 1.35, \alpha = 1.45 \) are plotted in Figure 1. As \( \alpha \geq 1.5 \) the G-function no longer exists as it violates the conditions. We can take other value of \( \nu \) also.
Figure 1. Behaviour of $I_{1\alpha}$ for various values of $\alpha > 1$
Similarly the behaviour of the integrals $I_{2\alpha}^{(d)}$ is such that the function moves away from the stable case and comes closer to the origin.

Figure 2. Behaviour of $I_{2\alpha}^{(d)}$ for various values of $\alpha < 1$
As $\alpha \to 1$ we get the standard situation which is done in the series of papers of Mathai and Haubold. As $\alpha \to 1$ the two integrals will come close to the following limiting situation.
Figure 3. Maxwell-Boltzmann case or the limiting situation $\alpha = 1$.

In figure 4 as the value of $\delta$ moves we can see the depletion in the high energy tail of the Maxwell-Boltzmann situation. The graphs of depletion in the stable situation as well as many unstable and chaotic situations are plotted here. The cases (i),(ii) and (iii) show the depletion when $\alpha = 1.25$, $\alpha = 1.35$ and $\alpha = 1.45$ respectively and (iv) shows the depletion in the stable situation ($\alpha = 1$).

Figure 4. Depletion for $\delta = 1, 2, 3$ and $\alpha = 1.25, 1.35, 1.45, 1$

6 Conclusion

For the analytic evaluation of the probability integral for equilibrium conditions we consider a more general form of the reaction probability integral. We investigated in section 4 the series representations of the extended integral, $I_{1\alpha}$ and $I_{2\alpha}^{(d)}$ where as the pathway parameter $\alpha \rightarrow 1$ one gets the Maxwell-Boltzmann case. $I_{1\alpha}$ for various values of $\alpha$ is plotted in Figure 1, $I_{2\alpha}^{(d)}$ for various values of $\alpha$ is plotted in Figure 2 and in Figure 3 the limiting case, namely $\alpha = 1$ case or Maxwell-Boltzmann situation is plotted. The plotting is done by Maple 9.
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