Chaotic behavior in piecewise linear Linz–Sprott equations

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Abstract. This paper presents the first rigorous proof of the chaotic behavior in a piecewise linear Linz–Sprott system, which is the algebraically simplest continuous three-dimensional dynamical system with chaotic behavior. The proof is computer assisted and relies on the estimation of the numerical calculation errors. The mathematical part of the proof uses topological hyperbolicity and information on the whereabouts of the periodic trajectories.

1. Introduction
Chaotic behavior has been extensively studied in the last several decades. A classical example of chaos is the Lorenz system published in 1963, which contains as few as three variables and two quadratic nonlinearities. For continuous flows, the Poincaré–Bendixson theorem implies the necessity of three variables and chaos requires at least one nonlinearity. Yet sufficient conditions for chaos in a system of ODEs remain unknown.

In an attempt to identify algebraically simple examples of chaotic systems, Linz and Sprott embarked on an extensive numerical search for three-dimensional systems with few terms and a single nonlinearity [1, 2]. They discovered the example

$$\dot{x} + ax + bx - |x| + 1 = 0,$$

which exhibits chaos for $a = 0.6$ and $b = 1$. Chaos also occurs if the signs of the last two terms are reversed, with an attractor that is a mirror image of the original one about the $x = 0$ plane. The attractor for this case, as shown in Figure 1, resembles the folded-band structure of the Rössler attractor.

For these parameters, the Lyapunov exponents (base-e) are $(0.035, 0, -6.35)$, and the Kaplan–Yorke dimension is $D_{KY} = 2.055$. The abrupt change in the direction of the flow at $x = 0$ is not evident in Figure 1 because the discontinuity occurs only in the fourth time derivative of $x$.

Linz and Sprott claim it is the most elementary piecewise linear chaotic flow [2] and point out that the piecewise linear nature of the nonlinearity allows for an analytic solution to (1) by solving two linear equations and matching these solutions at $x = 0$. However, they note that they assume chaos to exist if the largest Lyapunov exponent exceeds 0.005 after a large number of fourth-order Runge–Kutta iterations, and suggest that the analytical motivation for the appearance of chaos in (1) should be a subject of future research [1].

This paper presents the first rigorous proof of existence of chaos for the equation (1) with coefficients $a = 0.6$ and $b = 1$. The proof is computer assisted and relies on the estimation of the
numerical integration error. The mathematical part of the proof uses topological hyperbolicity and some information on the whereabouts of the periodic trajectories. The proof follows the guidelines suggested in [3, 4].

2. Integration error estimates
Since chaotic behavior is typically studied in terms of discrete dynamical systems rather than the trajectories of differential equations, we construct the Poincaré (or first return) mapping. It is not possible to find an analytical representation of the Poincaré mapping of (1), so it is necessary to study its properties numerically. We could not use Runge–Kutta methods to calculate the Poincaré mapping for our piecewise linear equations, because all rigorous estimates of the numerical integration error with these methods rely on the smoothness of the system. However, in each of the smoothness sets there exist analytical solutions to (1), and the Poincaré mapping can be constructed by matching the solutions at $x = 0$ and localizing the first return time by following the trajectories with a varying step.

This section contains the solutions to (1) and rigorous error estimates.

2.1. Analytical solution
First let us rewrite (1) in a equivalent form of a three-dimensional system:

$$
\begin{align*}
\dot{x}^{(1)} &= x^{(2)}, \\
\dot{x}^{(2)} &= x^{(3)}, \\
\dot{x}^{(3)} &= -ax^{(3)} - bx^{(2)} + |x^{(1)}| - 1.
\end{align*}
$$

Figure 1. Attractor of (1) for $a = 0.6$ and $b = 1$.

where $a = 0.6$ and $b = 1$. We are interested in initial conditions $x(0) = \left(x^{(1)}_0, x^{(2)}_0, x^{(3)}_0\right)$.

The solutions to (2) in the sets $\{x: x^{(1)} \geq 0\}$ and $\{x: x^{(1)} \leq 0\}$ are given by functions $f$ and $g$ respectively:

$$
f(t, x_0) = \begin{pmatrix}
c_1 e^{\mu t} + c_2 e^{\rho t} \cos \sigma t + c_3 e^{\rho t} \sin \sigma t + 1 \\
c_1 \mu e^{\mu t} + c_2 e^{\rho t} (\rho \cos \sigma t - \sigma \sin \sigma t) + c_3 e^{\rho t} (\sigma \cos \sigma t + \rho \sin \sigma t) \\
c_1 \mu^2 e^{\mu t} + c_2 e^{\rho t} ((\rho^2 - \sigma^2) \cos \sigma t - 2\rho \sigma \sin \sigma t) + c_3 e^{\rho t} (2\rho \sigma \cos \sigma t + (\rho^2 - \sigma^2) \sin \sigma t)
\end{pmatrix},
$$

$$
g(t, x_0) = \begin{pmatrix}
c_\bar{1} e^{\bar{\mu} t} + \bar{c}_2 e^{\bar{\rho} t} \cos \bar{\sigma} t + \bar{c}_3 e^{\bar{\rho} t} \sin \bar{\sigma} t - 1 \\
c_\bar{1} \bar{\mu} e^{\bar{\mu} t} + \bar{c}_2 e^{\bar{\rho} t} (\bar{\rho} \cos \bar{\sigma} t - \bar{\sigma} \sin \bar{\sigma} t) + \bar{c}_3 e^{\bar{\rho} t} (\bar{\sigma} \cos \bar{\sigma} t + \bar{\rho} \sin \bar{\sigma} t) \\
c_\bar{1} \bar{\mu}^2 e^{\bar{\mu} t} + \bar{c}_2 e^{\bar{\rho} t} ((\bar{\rho}^2 - \bar{\sigma}^2) \cos \bar{\sigma} t - 2\bar{\rho} \bar{\sigma} \sin \bar{\sigma} t) + \bar{c}_3 e^{\bar{\rho} t} (2\bar{\rho} \bar{\sigma} \cos \bar{\sigma} t + (\bar{\rho}^2 - \bar{\sigma}^2) \sin \bar{\sigma} t)
\end{pmatrix},
$$
where \(c_i, \bar{c}_i\) are determined from the initial conditions \(f(0, x(0)) = x(0)\) and \(g(0, x(0)) = x(0)\) to give

\[
\begin{align*}
\bar{c}_1 &= -(x_0^{(3)} + 2x_0^{(2)} \rho + (\sigma^2 + \rho^2)(1 - x_0^{(1)})/((\rho - \mu)^2 + \sigma^2) \\
\bar{c}_2 &= x_0^{(1)} - 1 - c_1, \quad c_3 = (x_0^{(2)} - c_1 \mu - c_2 \rho)/\sigma.
\end{align*}
\]

The other coefficients are

\[
\begin{align*}
\rho &= -\frac{1}{2} \alpha - \frac{1}{3} \bar{a}, \\
\mu &= \alpha - \frac{1}{3} \bar{a}, \\
\sigma &= \sqrt{\frac{3}{4} \alpha^2 - p}, \\
\bar{\sigma} &= \sqrt{\frac{3}{4} \alpha^2 - p}, \\
\gamma &= \frac{1}{3} \bar{q} - \frac{1}{2} q^3, \\
\mu &= \sqrt{\frac{3}{4} q^2 - \frac{1}{2} \bar{q}^3}, \\
q &= 1 + \frac{1}{3} a b - \frac{2}{27} a^3, \\
p &= \frac{1}{4} a^2 - b.
\end{align*}
\]

2.2. Lipschitz continuity of the translation operator

Denote by \(\varphi(t, x_0)\) the translation operator along trajectories of the differential equation (2). In this subsection we estimate the Lipschitz constants of \(\varphi(t, x_0)\) using the concept of the logarithmic matrix norm \(\mu(\cdot)\) [5]. We reformulate the theorem about the precision of the numerical integration (see [5]) in the notation of this paper.

Theorem 2.1. Let \(\psi(t)\) be a differentiable function, let \(\varphi(t, x_0)\) denote the translation operator along the trajectories of differential equation (2) and let \(\mathcal{F}\) denote the right-hand side system matrix of this equation. Suppose that we have the estimate \(\mu(d\mathcal{F}) \leq L\) and

\[
\|\psi'(t) - \mathcal{F}(\varphi(t))\| \leq \delta(t), \quad |\psi(0) - u_0| \leq \varepsilon.
\]

Then for every \(t > 0\) we have

\[
\|\varphi(t, u_0) - \psi(t)\| \leq e^{Lt} \left( \varepsilon + \int_0^t e^{-Ls}\delta(s)\,ds \right).
\]

For the Euclidean matrix norm the corresponding logarithmic norm \(\mu(Q)\) is equal to the largest eigenvalue of symmetric matrix \(R = \frac{1}{2}(Q^T + Q)\) (see [5]). For the system (2) we have the following matrix:

\[
Q = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\pm 1 & -1 & -0.6
\end{pmatrix}, \quad R = \begin{pmatrix}
0 & 0.5 & 0.5 \cdot \alpha \\
0.5 & 0 & 0 \\
0.5 \cdot \alpha & 0 & -0.6
\end{pmatrix},
\]

where \(\alpha = \pm 1\).

Lemma 2.1.

\[
\mu \left( \frac{\partial\mathcal{F}}{\partial u} \right) < L = \frac{2}{3}.
\]
The characteristic equation $f(\lambda) = 0$ for the matrix $R$ is

$$f(\lambda) = |R - \lambda I| = -\lambda^3 - 0.6\lambda^2 + 0.5\lambda + 0.15,$$

$$f'(\lambda) = 0.5 - 1.2\lambda - 3\lambda^2,$$

It is sufficient to prove that $\lambda \geq \frac{2}{3}$ implies $f(\lambda) < 0$. This follows from the fact that for $\lambda \in (\frac{2}{3}, \infty)$ we have $f'(\lambda) < 0$, and $f\left(\frac{2}{3}\right) < 0$.

Now we can estimate the Lipschitz constant for the $\varphi(t, x)$ operator in the $x$ domain:

**Proposition 2.1.** Let $0 < t \leq T$. Then

$$\|\varphi(T, x_0) - \varphi(T, y_0)\| < e^{\frac{2}{3}T} \|x_0 - y_0\|.$$

**Proof.** Define $\psi(t) = \varphi(t, y_0)$. Then from the Theorem 2.1 we have

$$\|\varphi(T, x_0) - \varphi(T, y_0)\| \leq e^{LT} \left(\varphi_0 + \int_0^T e^{-Ls} \delta(s) \, ds\right) < e^{\frac{2}{3}T} \|x_0 - y_0\|,$$

because $\delta(t) = 0$ by choice of $\psi(t)$, and $L = \frac{2}{3}$ by the Lemma 2.1.

**Proposition 2.2.** Define $\Omega = \{x : |x| < 3\}$, and suppose that $\forall t \in [t_1, t_2] : \varphi(t, x_0) \in \Omega$. Then

$$\|\varphi(t_1, x_0) - \varphi(t_2, x_0)\| < 10 |t_1 - t_2|.$$

**Proof.**

$$\|\varphi(t_1, x_0) - \varphi(t_2, x_0)\| = \|\varphi(\tilde{t}, x_0)\| \cdot |t_1 - t_2| =$$

$$= \|A\varphi(\tilde{t}, x_0)\| \cdot |t_1 - t_2| \leq$$

$$\leq \|A\| \cdot \|\varphi(\tilde{t}, x_0)\| \cdot |t_1 - t_2| <$$

$$< \|A\| \cdot 3\sqrt{3} \cdot |t_1 - t_2| < 10 |t_1 - t_2|,$$

because $3\sqrt{3} \|A\| < 10$. 

**2.3. Error estimates**

The following propositions give guaranteed error estimates for the calculation of values of $f$ and $g$ functions with the computer.

**Proposition 2.3.** Let $0 \leq t \leq T$, $-3 \leq x_0^{(i)} \leq 3$. Denote by $\tilde{f}$ and $\tilde{g}$ computer implementations of $f$ and $g$ functions respectively. Then

$$|\tilde{f}(t, x_0) - f(t, x_0)|, |\tilde{g}(t, x_0) - g(t, x_0)| < \tilde{\varepsilon} = 10^{-10}.$$

**Proof.** The proof is performed with a computer and uses a special implementation of interval arithmetic, which calculates guaranteed error estimates on each step of the algorithm using the IEEE standard for floating-point arithmetic [6] (see also Chapter 1 in [7]).

**Proposition 2.4.** Define $\Omega(\varepsilon) = \{x : |x| < 3 - \varepsilon\}$. Suppose that the values $\tilde{f}(t_1, x_0)$ and $\tilde{f}(t_1 + h, x_0)$ are calculated, and let $t = t_1 + \theta h \in [t_1, t_1 + h]$, $\theta \in [0, 1]$. Define the following function:

$$\psi(\theta) = (1 - \theta)\tilde{f}(t_1, x_0) + \theta\tilde{f}(t_1 + h, x_0).$$

Then the inclusion $\tilde{f}(t_1, x_0), \tilde{f}(t_1 + h, x_0) \in \Omega(\varepsilon + 6h^2)$ implies

$$\|f(t, x_0) - \psi(\theta)\| < \tilde{\varepsilon} + 6h^2,$$

and $f(t, x_0) \in \Omega$ for all $t_1 \leq t \leq t_1 + h$. A similar inequality also holds for $g$. 


Proof. Let us first assume that \( f(t, x_0) \in \Omega \) for all \( t \in [t_1, t_1 + h] \). Under this assumption

\[
\|f(t, x_0) - \psi(\theta)\| < \varepsilon + \frac{h^2}{2} \sup_{u \in \Omega} \|\tilde{f}(u, x_0)\|,
\]

(see proof of Proposition 2.4.1 in [4]), where \( \tilde{f} \) is the second derivative of \( f \) with respect to \( t \).

Now we have to show that \( \|\tilde{f}_{max}\| < 12 \). This can be done by writing \( \tilde{f} \) from the system (2):

\[
\tilde{f} = \left( f^{(3)}, f^{(1)} - f^{(2)} - 0.6f^{(3)}, -0.6f^{(1)} + 1.6f^{(2)} - 0.64f^{(3)} \right),
\]

from which, recalling that \( f(t, x_0) \in \Omega \), we obtain \( |\tilde{f}^{(1)}| < 3, |\tilde{f}^{(2)}| < 7.8, |\tilde{f}^{(1)}| < 8.52 \). This gives \( \|\tilde{f}\| < 12 \).

It remains to prove that \( f(t, x_0) \in \Omega \) for all \( t \in [t_1, t_1 + h] \). Suppose that this is not true, and there exists \( \theta \in [0, h] \) such that \( f(t_1 + \theta h, x_0) \notin \Omega \). Then continuity of \( f \) implies the existence of \( \theta_0 \in [0, h] \) such that \( f(t_1 + \theta_0 h, x_0) \in \partial \Omega \) and \( f(t, x_0) \in \Omega \) for all \( t_1 \leq t < t_1 + \theta_0 h \). By the conditions of the proposition \( \tilde{f}(t_1, x_0) \) and \( \tilde{f}(t_1 + h, x_0) \) belong to \( \Omega(\varepsilon + 6h^2) \), as well as \( \psi(\theta_0) \), because \( \Omega(\varepsilon + 6h^2) \) is a convex set. Thus \( \|\psi(\theta_0) - f(t_1 + \theta_0 h, x_0)\| > \varepsilon + 6h^2 \).

On the other hand, for all \( t_1 \leq t < t_1 + \theta_0 h \) the solution \( f(t, x_0) \) belongs to \( \Omega \) and (3) holds. From the continuity of \( f(t, x_0) \), it follows that the inequality \( \|f(t_1 + \theta_0 h) - \psi(\theta_0)\| \leq \varepsilon + 6h^2 \) holds, and we arrive at a contradiction. This means that \( f(t, x_0) \in \Omega \) for all \( t \in [t_1, t_1 + h] \) and the proposition is proved. \( \square \)

3. Poincaré mapping

In this section we will numerically study the properties of the Poincaré mapping for the equation (2) on the plane \( x^{(2)} = 0 \). In particular, it will be shown that the Poincaré mapping is well defined on the simple rectangular set of \( x^{(2)} = 0 \). Moreover, it will be proved that the rectangular set is invariant which implies, in particular, the existence of a compact attracting set both for the Poincaré mapping and for the original semiflow generated by the differential equation (2).

We are interested in the properties of the Poincaré mapping because it is natural to consider chaotic behavior in terms of mappings rather than trajectories of differential equations. Since it is not possible to find an analytical representation of the Poincaré mapping of (2), all the proofs of this section are computer-assisted and rely on rigorous numerical integration error estimates obtained in Section 2. To avoid cumbersome estimates of the first intersection time, the Poincaré mapping is computed using a combination of a certain bracketing principle with integration error estimates to ensure that the required properties hold.

3.1. Definitions and theorems

Let us denote the plane \( \{x: x^{(2)} = 0\} \) by \( \tilde{S} \). The transversality condition \( \hat{x}^{(2)} \neq 0 \) is broken only for the set defined by \( x^{(3)} = 0 \). Denote the half-plane \( \{x: x^{(2)} = 0, x^{(3)} > 0\} \) by \( S \), so that we can correctly define the Poincaré mapping \( \mathcal{P} \) as a (partial) mapping from \( S \) to itself, obtained by following trajectories from one intersection with \( S \) to another.

Let us denote by \( \Pi \) the following parallelogram:

\[
\Pi = \{u_0 + \alpha u_1 + \beta u_2 : |\alpha|, |\beta| \leq 1\},
\]

where

\[
\begin{align*}
u_0 &= (-1.975, 1.265), \\
u_1 &= (0.05, 0.05), \\
u_2 &= (0.535, -0.535).
\end{align*}
\]
The geometry of the mapping $\mathcal{P}$ on $\Pi$ is illustrated by the Figure 2, which shows that $\mathcal{P}(\Pi) \subset \Pi$. However, the detailed justification of this figure is cumbersome. It requires rigorous estimates of the precision of numerical methods and accuracy of discrete computer arithmetic.

The main result of this section can be written as the following theorem:

**Theorem 3.1.** The Poincaré mapping $\mathcal{P}$ is defined for all $u \in \Pi$, and $\mathcal{P}(\Pi) \subset \Pi$.

It is technically convenient to split the theorem into two propositions.

**Proposition 3.1.** The mapping $\mathcal{P}$ is defined for all $u \in \Pi$, $\mathcal{P}(\Pi) \subset S$ and $\mathcal{P}$ is continuous on $\Pi$.

**Proposition 3.2.** The inclusion $\mathcal{P}(\Pi) \subset \Pi$ holds.

The proofs presented are computer-based and use guaranteed error estimates for floating-point calculations. The algorithms used are given in the following subsection. To cut down on calculation times, different parameters were used to prove the two propositions, which gave a sufficiently small guaranteed error for each.

The details of the numerical algorithms are given in Section 5.

4. Chaotic behavior

In this section we describe chaotic properties of the Poincaré mapping for (2). In particular it will be shown that the mapping $\mathcal{P}$ has some sort of sensitive dependence on initial conditions, has orbits of all minimal periods greater than 10, and has a homoclinic fixed point.

The proof follows the ideas suggested in [3]. The main idea is the localization of a pseudo homoclinic orbit and the pinpointing of dynamics similar to those of Smale’s horseshoe. The procedure of finding the sets with the desired properties is not formalized and relies on preliminary numerical simulations.

Hyperbolicity, that is successfully used to describe the dynamics of the horseshoe, could not be shown for $\mathcal{P}$ since computer integration provides us only with estimates for the mapping. Thus it is more natural to use the notions of $(V,W)$-hyperbolicity whose properties can easily be verified with the help of a computer. Similar geometric approaches were given in [8, 9].

In this paper we use the variation of chaos called $(U,k)$-chaos. It is described in [3], as well as $(V,W)$-hyperbolicity. We repeat the definitions for convenience.
4.1. Definition of chaos

Let \( f: S \to S \) be a mapping in a subset \( S \subset \mathbb{R}^d \). Important attributes of chaotic behavior include sensitive dependence on initial conditions, an abundance of periodic trajectories and an irregular mixing effect describable informally by the existence of a finite number of separated subsets \( U_1, \ldots, U_m \) of \( \mathbb{R}^d \) which can be visited by trajectories of some fixed iterate \( f^k \) of \( f \) in any prescribed order. Let \( \mathcal{U} = \{U_0, \ldots, U_{m-1}\}, m > 0 \) be a family of disjoint subsets of \( \mathbb{R}^d \), and let us denote the set of one-sided sequences \( \omega = \omega_0, \omega_1, \ldots, \omega_i \in \{0, 1, \ldots, m-1\} \) by \( \Omega^R_m \). Sequences in \( \Omega^R_m \) will be used to prescribe the order in which sets \( U_i \) are to be visited. For \( x \in \bigcup_{i=1}^m U_i \) we denote by \( I(x) \) the number \( i \) satisfying \( x \in U_i \).

**Definition 4.1.** A mapping \( f \) is called \( (\mathcal{U}, k) \)-chaotic \((k \text{ is a positive integer})\) if there exists a compact \( f \)-invariant set \( I \subset \bigcup_i U_i \) with the following properties:

1. \( (p1) \) for any \( \omega \in \Omega^R_m \) there exists \( x \in I \) such that \( f^i(x) = (f^k)^i \in U_{\omega_i} \) for \( i \geq 1 \);
2. \( (p2) \) for any \( p \)-periodic sequence \( \omega \in \Omega^R_m \) there exists a \( pk \)-periodic point \( x \in I \) which satisfies \( f^{pk}(x) \in U_{\omega_i} \);
3. \( (p3) \) for each \( \eta > 0 \) there exists an uncountable subset \( I(\eta) \subset S \) such that the simultaneous relationships

\[
\limsup_{i \to \infty} |I(f^{ik}(x_1)) - I(f^{ik}(x_2))| \geq 1,
\]

\[
\liminf_{i \to \infty} |f^{ik}(x_1) - f^{ik}(x_2)| < \eta.
\]

hold for all \( x_1, x_2 \in I(\eta), x_1 \neq x_2 \).

4.2. Chaos in the Linz–Sprott system

Introduce the vectors on the plane \( \{x: x(2) = 0\} \):

\[
x^{(u)} = (-0.704521, 0.709683), \quad x^{(s)} = (0.878136, 0.478412).
\]

that are close to the eigenvectors of the linearization of \( P \) at the approximate fixed point \( x_* = (-2.11274, 1.47615) \). (The upper indices \( u \) and \( s \) relate to the adjectives ‘unstable’ and ‘stable’."

We construct a sequence of points \( x_i = (x_i^{(1)}, x_i^{(2)}) \), \( i = 0, \ldots, 10 \), and the parallelograms

\[
X_i = \{x = x_i + \alpha a_i^{(u)} x^{(u)} + \beta a_i^{(s)} x^{(s)}: |\alpha|, |\beta| \leq 1\}.
\]
The coordinates of the points \( x_i \) and the sizes \((a_i^{(u)}, a_i^{(s)})\) of the corresponding parallelograms are presented in Table 1.

**Table 1.** Numerical values for \( x_i^{(1)}, x_i^{(2)}, a_i^{(u)} \) and \( a_i^{(s)} \).

| \( i \) | \( x_i^{(1)} \) | \( x_i^{(2)} \) | \( a_i^{(u)} \) | \( a_i^{(s)} \) |
|-----|-----------|-----------|-----------|-----------|
| 0   | -2.11403  | 1.47796   | 0.03      | 0.01      |
| 1   | -2.1043   | 1.46761   | 0.03      | 0.01      |
| 2   | -2.12726  | 1.49073   | 0.03      | 0.01      |
| 3   | -2.08722  | 1.45031   | 0.03      | 0.01      |
| 4   | -2.15576  | 1.51911   | 0.03      | 0.015     |
| 5   | -2.03497  | 1.39663   | 0.035     | 0.015     |
| 6   | -2.23616  | 1.59732   | 0.03      | 0.015     |
| 7   | -1.87378  | 1.22453   | 0.045     | 0.02      |
| 8   | -2.41427  | 1.75531   | 0.015     | 0.02      |
| 9   | -1.51996  | 0.790372  | 0.02      | 0.02      |
| 10  | -2.19079  | 1.42661   | 0.02      | 0.02      |

We chose \( x_i \) to be close to the elements of a pseudo-periodic orbit of the Poincaré mapping, which passes close to the fixed point \( x_* \). To find a suitable orbit the broken orbit method was used, which is described in [4, 10]. We used a modification of a C++ program from [4].

**Lemma 4.1.** The family \( \mathcal{U} \) of connected components of the set \( \bigcup_{i=0}^{10} X_i \) has 7 elements.

**Proof.** This is clear from the definition of \( X_i \). See also Figure 3. \( \square \)

**Theorem 4.1.** The mapping \( P \) is \((\mathcal{U}, 20)\)-chaotic, where \( \mathcal{U} \) is the family of connected components of the set \( U = \bigcup_{i=0}^{10} X_i \).

The proof relies on the results from [3] and will be given in Subsection 5.2.

Recall that a trajectory \( x = \{ x_i \}_{i=-\infty}^{\infty} \) of a continuous bounded mapping \( g \) is called a homoclinic trajectory if its elements are not all identical and there exists a point \( x_* \) such that \( \lim_{n \to -\infty} x_i = \lim_{n \to \infty} x_i = x_* \). The point \( x_* \) is a homoclinic fixed point.

**Corollary 4.1.** There exists the unique homoclinic fixed point \( x_* \in X_0 \).

**Proof.** Follows directly from corollary 2.1 in [3]. \( \square \)

We also note the following

**Proposition 4.1.** For any positive integer \( p \) greater than 10 the mapping \( P \) has a periodic point \( x \in X_0 \) which has the minimal period \( p \), and which is not asymptotically stable.

**Proof.** See the proof of the Proposition 5.2.1 in [4]. \( \square \)
5. Numerical algorithms

In this subsection we will introduce an algorithm, which calculates the Poincaré mapping of the point \(x_0 \in S\). Since the analytical solution of the system (2) changes when the point crosses the plane \(\{x: x^{(1)} = 0\}\), this algorithm is complex. To simplify it, three auxiliary algorithms will be described and their properties will be stated and proved, and then the main algorithm will be based on these three algorithms.

We introduce the notations

\[
\Psi^{(i)}_+(y, \delta) = \{x: x^{(i)} < y + \delta\}, \quad \Psi^{(i)}_-(y, \delta) = \{x: x^{(i)} > y - \delta\},
\]

\[
\Psi^{(i)}_0(\delta) = \Psi^{(i)}_+(0, \delta).
\]

The first algorithm follows the trajectory \(\varphi(t, x_0)\) starting from the point \(x_0\) while the trajectory is inside the set \(\{x: x^{(1)} < 0\}\), and terminates when the value of the \(\varphi\) function gets close to the boundary of this set, or close to the plane \(\{x: x^{(2)} = 0\}\).

**Algorithm1**\((x, \varepsilon, \zeta)\)

1. \(t \leftarrow 0, h \leftarrow 0.05, x_0 \leftarrow x\)
2. while \((x^{(1)} \leq -\varepsilon \text{ and } x^{(2)} \geq \varepsilon)\) or \(t < 0.5\) do
3. \(x_2 \leftarrow \tilde{g}(t+h, x_0)\)
4. if \(-\varepsilon < x_2 \in \Psi^{(1)}_0(6h^2 + \varepsilon)\) or \(x_2^{(2)} \leq -\varepsilon\) then
5. \(h \leftarrow h/2\)
6. else
7. if \(t \leq 0.5\) then
8. \(\text{Assert}((x, x_2 \in \Psi^{(3)}_+ (\zeta + 6h^2 + \varepsilon))\)
9. if \(t + h \geq 0.5\) then
10. \(\text{Assert}((x, x_2 \in \Psi^{(3)}_+ (\zeta + 6h^2 + \varepsilon)) \lor (x_2 \in \Psi^{(3)}_+ (\zeta + 6h^2 + \varepsilon)))\)
11. \(\text{Assert}((x, x_2 \in \Omega (6h^2 + \varepsilon)) \land (t \leq 10))\)
12. \(x \leftarrow x_2, t \leftarrow t + h\)
13. \(\text{Return}(x, t)\)

**Proposition 5.1.** Suppose that **Algorithm1**\((x, \varepsilon, \zeta)\) is successfully terminated for the point \(x_0 \in \Pi\), and the loop in the lines 2–12 was executed \(n\) times, and the result of the algorithm is a pair \((\tilde{x}, \tilde{t})\). Denote by \(0 = t_0, t_1, \ldots, t_n = \tilde{t}\) the sequence of values of \(t\) from line 12 of the algorithm, including the initial value \(t = 0\), and as \(x_i\) the corresponding values of \(x\) from the same line. Then the following properties hold:

- (a1) \(\forall i = 0, \ldots, n: x_i = \tilde{g}(t_i, x_0), \|x_i - g(t_i, x_0)\| < \tilde{\varepsilon};\) (and \(\tilde{g}(\tilde{t}, x_0) = \tilde{x}\))
- (a2) \((-\varepsilon < \tilde{x}^{(1)} < -\tilde{\varepsilon}) \lor (\tilde{x}^{(1)} < -\varepsilon \land \tilde{x}^{(2)} < \varepsilon;\)
- (a3) \(\forall i \in [0, \tilde{t}]: g(t, x_0)^{(1)} < 0;\)
- (a4) \(\|\tilde{x} - \varphi(\tilde{t}, x_0)\| < \varepsilon;\)
- (a5) \(\forall t \in [0.5, \tilde{t}]: (\varphi(t, x_0)^{(2)} > \zeta) \lor (\varphi(t, x_0)^{(3)} < -\zeta);\)
- (a6) \(\forall t \in [0, 0.5]: \varphi(t, x_0)^{(3)} > \zeta).\)

**Proof.**

(a1) The first part follows from the fact, that in line 2 the property \(x = \tilde{g}(t, x_0)\) holds, which in turn follows from the assignments in lines 1, 3, 12 and successful termination of the algorithm. The second part is guaranteed by the assertion in line 11, which fulfills the conditions of Proposition 2.3.

(a2) Follows from the fact that, for the pair \((\tilde{x}, \tilde{t})\), the conditions in lines 2 and 4 are both false.
(a3) The condition in line 4 guarantees that \( \forall i = 0, \ldots, n-1, \theta \in [0,1] \) and the following inequality holds:

\[
\psi(\theta)^{(1)} = (1-\theta)\tilde{g}(t_i, x_0)^{(1)} + \theta \tilde{g}(t_{i+1}, x_0)^{(1)} < -(\bar{\varepsilon} + 6(t_{i+1} - t_i)^2).
\]

From Proposition 2.4 we have

\[
\forall t \in [t_i, t_{i+1}]: (t, x_0)^{(1)} < \psi(\theta)^{(1)} + (\bar{\varepsilon} + 6(t_{i+1} - t_i)^2) < 0,
\]

which proves this property.

(a4) From property (a3) it follows that the point \( x \) does not leave the set \( \{x: x^{(1)} < 0\} \) when moving along the trajectory from \( t = 0 \) to \( \tilde{t} \), and the solution in this set is defined by the function \( g \). Thus \( \varphi(\tilde{t}, x_0) = g(\tilde{t}, x_0) \), and by the property (a3)

\[
\|\tilde{x} - \varphi(\tilde{t}, x_0)\| = \|\tilde{g}(\tilde{t}, x_0) - g(\tilde{t}, x_0)\| < \bar{\varepsilon}.
\]

(a5) The successful termination of the algorithm means that \( \forall i: t_{i+1} \in [0.5, \tilde{t}] \) and the assertion in line 10 holds, which implies

\[
\psi(\theta) = (1-\theta)\tilde{g}(t_i, x_0) + \theta \tilde{g}(t_{i+1}, x_0) \in \Psi^{(2)}_+ (\zeta + 6h^2 + \bar{\varepsilon}) \text{ or}
\]

\[
\varphi(\theta) \in \Psi^{(3)}_+ (\zeta + 6h^2 + \bar{\varepsilon}),
\]

and, since the Proposition 2.4 gives us \( g(t, x_0) \in O(\psi(\theta), \bar{\varepsilon} + 6h^2) \), we have

\[
(g(t, x_0) \in \Psi^{(2)}_+ (\zeta)) \lor (g(t, x_0) \in \Psi^{(3)}_+ (\zeta)).
\]

(a6) Follows in a similar way from the assertion in line 8. \( \square \)

The second algorithm moves along the trajectory \( \varphi(t, x_0) \) from the point \( x_0 \) while the point does not leave the set \( \{x: x^{(1)} \geq 0\} \), and terminates when the value of \( \varphi \) gets close to the boundary of this set, and the condition \( \dot{x}^{(1)} = \varphi^{(2)} < 0 \) holds (so that the point will leave this set when \( t \) is further increased).

**Algorithm 2** \((x, \varepsilon, \zeta)\)

1. \( t \leftarrow 0, \ h \leftarrow 0.05, \ x_0 \leftarrow x \)
2. \( \textbf{while } x^{(2)} \geq -\bar{\varepsilon} \text{ or } x^{(1)} \geq \varepsilon \textbf{ do} \)
3. \( x_2 \leftarrow \tilde{f}(t + h, x_0) \)
4. \( \textbf{if } (x_2^{(2)} < -\bar{\varepsilon} \text{ and } x_2^{(1)} \leq \varepsilon) \textbf{ then} \)
5. \( h \leftarrow h/2 \)
6. \( \textbf{else} \)
7. \( \textbf{assert}((x, x_2) \in \Psi^{(2)}_+ (-0.1, \zeta + 6h^2 + \bar{\varepsilon})) \lor (x, x_2) \in \Psi^{(3)}_+ (\zeta + 6h^2 + \bar{\varepsilon})) \)
8. \( \textbf{assert}((x, x_2) \in \Omega(6h^2 + \bar{\varepsilon})) \land (t \leq 10)) \)
9. \( x \leftarrow x_2, \ t \leftarrow t + h \)
10. \( \textbf{return} (x, t) \)

**Proposition 5.2.** Suppose that Algorithm 2 \((x, \varepsilon, \zeta)\) is successfully terminated for the point \( x_0 \), that the loop in lines 2–9 was executed \( n \) times, and that the result of the algorithm is a pair \((\tilde{x}, \tilde{t})\). Denote as \( 0 = t_0, t_1, \ldots, t_n = \tilde{t} \) the sequence of values of \( t \) from line 9, including \( t_0 = 0 \), and by \( x_i \) the corresponding values of \( x \). Then the following properties hold:

(b1) \( \forall i = 0, \ldots, n: x_i = \tilde{f}(t_i, x_0), \|x_i - \tilde{f}(t_i, x_0)\| < \bar{\varepsilon} \) (and \( \tilde{f}(\tilde{t}, x_0) = \tilde{x} \))

(b2) \( \varepsilon < \tilde{x}^{(1)} < \varepsilon, \ \tilde{x}^{(2)} < -\bar{\varepsilon}; \)
(b3) ∀t ∈ [0, \hat{t}]: f(t, x_0)^{(1)} \geq 0;
(b4) \|\hat{x} - \varphi(t, x_0)\| < \varepsilon;
(b5) ∀t ∈ [0, \hat{t}]: (\varphi(t, x_0)^{(2)} < -0.1 - \zeta) \lor (\varphi(t, x_0)^{(3)} < -\zeta).

**Proof.** Follows largely the proof of the Proposition 5.1.

(b1) Is proved similarly to the property (a1) from the Proposition 5.1.

(b2) Follows from the fact, that for \hat{x} the conditions in lines 2 and 4 are both false.

(b3) From the conditions of the proposition and from the property (b2) we have \( f(0, x_0)^{(2)} > \hat{f}(0, x_0)^{(2)} - \varepsilon > 0 \), and \( f(\hat{t}, x_0)^{(2)} < \hat{f}(\hat{t}, x_0)^{(2)} + \varepsilon < 0 \). By the continuity of \( f \) in the time domain, this implies that \( \exists t \in [0, \hat{t}]: f(t, x_0)^{(2)} = 0 \). Let \( \hat{t} \) be the smallest of all such \( t \). Then ∀t ∈ (0, \hat{t}): f(t, x_0)^{(2)} > 0, which together with \( \hat{f}(1) = f^{(2)} \) and \( f(0, x_0)^{(1)} = 0 \) gives us

\[
\forall t \in (0, \hat{t}]: f(t, x_0)^{(1)} > 0. \tag{4}
\]

Now we show that ∀t ∈ (\hat{t}, \hat{\hat{t}}): f(t, x_0)^{(2)} < 0. Suppose that this is not true and \( \exists t \in (\hat{t}, \hat{\hat{t}}): f(t, x_0) = 0 \) and denote as \( \hat{\hat{t}} \) the smallest of such \( t \). It is easy to see that \( t \in (\hat{t}, \hat{\hat{t}}): f(t, x_0)^{(2)} < 0 \) implies \( f(\hat{\hat{t}}, x_0)^{(2)} = f(\hat{\hat{t}}, x_0)^{(3)} \geq 0 \). But (b5) shows that this is not true (more precisely, a variation of (b5), in which \( \varphi \) is substituted by \( f \) — its proof does not rely on (b3)). This contradiction shows that ∀t ∈ (\hat{t}, \hat{\hat{t}}): f(t, x_0)^{(2)} < 0, which together with \( \hat{f}(1) = f^{(2)} \) and \( f(\hat{t}, x_0)^{(1)} < 0 \) gives

\[
\forall t \in [\hat{t}, \hat{\hat{t}}]: f(t, x_0)^{(1)} > 0. \tag{5}
\]

Now it is sufficient to combine (4) and (5).

Thus, when moving along the trajectory from \( t = 0 \) to \( \hat{\hat{t}} \) the point does not leave the set \( \{x: x^{(1)} \geq 0\} \), where the solution is defined by the function \( f \), which means that \( \varphi(\hat{\hat{t}}, x_0) = f(\hat{\hat{t}}, x_0) \).

(b4) Follows from (b3) and Proposition 2.3

(b5) Is guaranteed by the assertion in line 7. \( \square \)

The third algorithm localizes a time interval, which necessarily contains the first intersection point of the trajectory of the system and the \( S \) half-plane while moving along the trajectory starting from point \( x_0 \), and guarantees that the point does not leave the set \( \{x: x^{(1)} \leq 0\} \).

**Algorithm** 3(\( x, \varepsilon, \zeta, \delta \))

```plaintext
1  t ← 0, h ← 0.05, x_0 ← x
2  while \( x^{(2)} \leq -\varepsilon - \zeta \lor x^{(3)} < \varepsilon + \zeta \) do
3    x_2 ← \( \hat{g}(t + h, x_0) \)
4    if \( \neg(x, x_2 \in \Psi_-^{(2)}(\zeta + 6h^2 + \varepsilon)) \) and \( \neg(x, x_2 \in \Psi_-^{(3)}(\zeta + 6h^2 + \varepsilon)) \) then
5      h ← h/2
6    else
7      assert((x, x_2 \in \Omega(6h^2 + \varepsilon)) ∧ (t ≤ 10))
8    x ← x_2, t ← t + h
9  \( x_1 \leftarrow x, t_1 \leftarrow t, h ← 0.05 \)
10  while \( x^{(2)} \leq \varepsilon + \zeta \) do
11    x_2 ← \( \hat{g}(t + h, x_0) \)
12    if \( x^{(2)} \geq \varepsilon + \zeta \) then
13      h ← h/2
14    else
15      assert((x, x_2 \in \Psi_+^{(3)}(\zeta + 6h^2 + \varepsilon)) ∧ (x, x_2 \in \Psi_-^{(1)}(\zeta + 6h^2 + \varepsilon)))
16      assert((x, x_2 \in \Omega) ∧ (t ≤ 10))
```
17 \[ x \leftarrow x_2, \ t \leftarrow t + h \]
18 \[ \text{ASSERT}(|t - t_1| < \delta) \]
19 \[ \text{RETURN}(x_1, t_1, x, t) \]

**Proposition 5.3.** Suppose that ALGORITHM3(\(x, \varepsilon, \zeta\)) is successfully terminated for the point \(x_0\), and the loop in lines 2–8 was executed \(n\) times, and in lines 11–17 — \(m\) times, and the result of the algorithm is a tuple \((\tilde{x}_1, \tilde{t}_1, x_2, \tilde{t}_2)\). Denote by \(0 = t_0, t_1, \ldots, t_n = \tilde{t}_1, t_{n+1}, \ldots, t_{n+m} = \tilde{t}_2\) the sequence of values of \(t\) from lines 8 and 17, including \(t = 0\), and by \(x_i\) the corresponding values of \(x\). Then the following properties hold:

\begin{itemize}
  \item[(c1)] \[ \forall i = 0, \ldots, n + m: x_i = \tilde{g}(t_i, x_0), \|x_i - g(t_i, x_0)\| < \tilde{\varepsilon}; \]
  \item[(c2)] \[ -\varepsilon - \zeta < \tilde{x}_1^{(2)} < -\tilde{\varepsilon} - \zeta, \quad \tilde{\varepsilon} + \zeta < \|\tilde{x}_2^{(2)}\| < \varepsilon + \zeta; \]
  \item[(c3)] \[ \forall t \in [0, \tilde{t}_1]: (g(t, x_0)^{(2)} < -\zeta) \lor (g(t, x_0)^{(3)} < -\zeta); \]
  \item[(c4)] \[ \forall i \in [0, \tilde{t}_2]: \|g(t_i, x_0)^{(1)}\| < 0; \]
  \item[(c5)] \[ \|\tilde{x}_1 - \varphi(t_1, x_0)\|, \|\tilde{x}_2 - \varphi(t_2, x_0)\| < \tilde{\varepsilon}; \]
  \item[(c6)] \[ \forall t \in [\tilde{t}_1, \tilde{t}_2]: \varphi(t, x_0)^{(3)} > \zeta. \]
\end{itemize}

**Proof.**

(c1), (c2) Are proved similarly to corresponding properties from the Proposition 5.1.

(c3) Follows from the successful termination of the algorithm and the condition in line 4.

(c4) For \(t \in [0, \tilde{t}_1]\) is proved similarly to the property (b3) from Proposition 5.2; for \(t \in [\tilde{t}_1, \tilde{t}_2]\) follows from line 15 of the algorithm.

Thus while moving along the trajectory from \(t = 0\) to \(\tilde{t}_2\) the point does not leave the set \(\{x: x^{(1)} \leq 0\}\), where the solution is defined by \(g\) function, and we have \(\varphi(t_1, x_0) = g(\tilde{t}_1, x_0)\).

(c5) Follows from (c4).

(c6) Follows from the assertion in line 15.

Now we are ready to introduce the main algorithm. We will need the following notation:

\[ \varepsilon_1 = 20(\varepsilon + \tilde{\varepsilon}), \quad \varepsilon_2 = 10(\varepsilon + \varepsilon + \tilde{\varepsilon}), \quad \varepsilon_3 = \varepsilon_2 + \varepsilon + \tilde{\varepsilon}. \]

**ALGORITHM4(\(x, \varepsilon, \zeta, \delta\))**

1 \[ (\tilde{x}_1, t_1) \leftarrow \text{ALGORITHM1}(x, \varepsilon, \zeta) \]
2 \[ \text{if } \tilde{x}_1^{(2)} \geq \varepsilon \text{ then} \]
3 \[ \quad x_1 \leftarrow (0, \tilde{x}_1^{(2)}, \tilde{x}_1^{(3)}) \]
4 \[ \quad (\tilde{x}_2, t_2) \leftarrow \text{ALGORITHM2}(x_1, \varepsilon, \zeta + \varepsilon_1) \]
5 \[ \quad \text{ASSERT}(t_2 < 4.2) \]
6 \[ \quad x_2 \leftarrow (0, \tilde{x}_2^{(2)}, \tilde{x}_2^{(3)}) \]
7 \[ \text{else} \]
8 \[ \quad x_1 \leftarrow (\tilde{x}_1^{(1)}, 0, \tilde{x}_1^{(3)}), \quad x_2 \leftarrow x_1, \quad t_2 \leftarrow 0 \]
9 \[ \quad (\tilde{x}_3, t_3, \tilde{x}_4, t_4) \leftarrow \text{ALGORITHM3}(x_2, \varepsilon, \zeta + \varepsilon_2, \delta) \]
10 \[ \quad \text{ASSERT}((t_4 < 3) \land (t_1 + t_2 + t_4 < 9)) \]
11 \[ \text{RETURN}(\tilde{x}_3, t_3 + t_1 + t_2 + t_4, \tilde{x}_4, t_1 + t_2 + t_4) \]

**Proposition 5.4.** Suppose that ALGORITHM4(\(x, \varepsilon, \zeta, \delta\)) is successfully terminated from the point \(x_0 \in \Pi\), and its result is a tuple \((\tilde{x}_3, t_3, \tilde{x}_4, \tilde{t}_4)\). Let \(y_0 \in O(x_0, \zeta e^{-9L})\). Then the following properties hold:

\begin{itemize}
  \item[(d1)] \[ \|\varphi(t_1, x_0) - x_1\| < \varepsilon + \tilde{\varepsilon}; \]
  \item[(d2)] \[ \|\varphi(t_1 + t_2, x_0) - x_2\| < \varepsilon_1 + \varepsilon + \tilde{\varepsilon}; \]
\end{itemize}
(d3) \( \| \varphi(\tilde{t}_3, x_0) - \tilde{x}_3 \|, \| \varphi(\tilde{t}_4, x_0) - \tilde{x}_4 \| < \varepsilon_2 + \varepsilon \);

(d4) \( \forall t \in (0, \tilde{t}_2]: \varphi(t, y_0) \not\in S \); 

(d5) \( \varphi(\tilde{t}_3, y_0)(2) < 0, \varphi(\tilde{t}_4, y_0)(2) > 0 \);

(d6) \( \forall t \in [\tilde{t}_3, \tilde{t}_4]: \varphi(t, x_0)(3) > 0 \);

Proof.

(d1) From the property (a2) of the Proposition 5.1 it follows that \( \|x_1 - \tilde{x}_1\| < \varepsilon \), and property (a4) gives \( \|\tilde{x}_1 - \varphi(t_1, x_0)\| < \tilde{\varepsilon} \), which gives us

\[
\|\varphi(t_1, x_0) - x_1\| \leq \|\varphi(t_1, x_0) - \tilde{x}_1\| + \|\tilde{x}_1 - x_1\| < \tilde{\varepsilon} + \varepsilon.
\]

(d2) If the condition in line 2 is not true, then \( t_2 = 0 \), \( \varphi(t_1 + t_2, x_0) = \varphi(t_1, x_0) \) and \( x_2 = x_1 \), and this property follows from (d1).

If the condition in line 2 is true, then the successful termination of the algorithm guarantees \( t_2 < 4.2 \) by the assertion in line 5. This means that \( \|\varphi(t_2, z_0) - \varphi(t_2, z_1)\| < e^{4.2L} \|z_0 - z_1\| \). 

Besides, from the property (b2) of the Proposition 5.2 it follows that \( \|x_2 - \tilde{x}_2\| < \varepsilon \), and from property (b4) — that \( \|\tilde{x}_2 - \varphi(t_2, x_1)\| < \tilde{\varepsilon} \). Then

\[
\|\varphi(t_1 + t_2, x_0) - x_2\| = \|\varphi(t_2, \varphi(t_1, x_0)) - x_2\| \leq
\|\varphi(t_2, \varphi(t_1, x_0)) - \varphi(t_2, x_1)\| +\|\varphi(t_2, x_1) - \tilde{x}_2\| +\|\tilde{x}_2 - x_2\| <
\tilde{\varepsilon} + \varepsilon + \varepsilon < \tilde{\varepsilon} + \varepsilon + \varepsilon.
\]

because \( e^{4.2L} = e^{4.2 \frac{\varepsilon}{\varepsilon}} < 20 \).

(d3) We will prove this property for \( t_3 \). The assertion in line 10 guarantees that \( t_3 < 3 \), which gives us \( \|\varphi(t_3, z_0) - \varphi(t_3, z_1)\| < e^{3L} \|z_0 - z_1\| \). The property (c5) from Proposition 5.3 gives us the inequality \( \|\tilde{x}_3 - \varphi(t_3, x_2)\| < \tilde{\varepsilon} \). Then

\[
\|\varphi(\tilde{t}_3, x_0) - \tilde{x}_3\| = \|\varphi(t_3, \varphi(t_1 + t_2, x_0)) - \tilde{x}_3\| \leq
\|\varphi(t_3, \varphi(t_1 + t_2, x_0)) - \varphi(t_3, x_2)\| + \|\varphi(t_3, x_2) - \tilde{x}_3\| <
\tilde{\varepsilon} + \varepsilon + \varepsilon < \tilde{\varepsilon} + \varepsilon \).
\]

because \( e^{3L} = e^{3 \frac{\varepsilon}{\varepsilon}} < 10 \). The second inequality for \( t_4 \) is proved in the same way.

(d4) We split the interval \((0, \tilde{t}_2)\) into four parts: \((0, 0.5] \), \([0.5, \tilde{t}_2), [\tilde{t}_1, t_1 + t_2], [t_1 + t_2, \tilde{t}_3] \).

Property (a6) of the Proposition 5.1 implies that \( \forall t \in [0, 0.5]: \varphi(3)(t, y_0) > 0 \), which together with the conditions \( \varphi(2)(0, y_0) = 0 \) and \( \tilde{\varphi}(2) = \varphi(3) \) guarantees that

\[
\varphi(t, y_0)(2) > 0 \text{ for } t \in (0, 0.5].
\]

For \( t \in [0.5, t_1] \) Proposition 5.1 gives us

\[
(\varphi(t, x_0)(2) > \zeta) \lor (\varphi(t, x_0)(3) < -\zeta).
\]

Because \( t_1 < 9 \), we have \( \|\varphi(t, x_0) - \varphi(t, y_0)\| < e^{9L} \|x_0 - y_0\| < e^{9L} \cdot \zeta e^{-9L} = \zeta \), and we obtain

\[
(\varphi(t, y_0)(2) > 0) \lor (\varphi(t, x_0)(3) < 0) \text{ for } t \in [0.5, t_1].
\]

For \( t \in [0, t_2] \) Proposition 5.2 gives us

\[
(\varphi(t, x_1)(2) < -0.1 - \zeta - \varepsilon_1) \lor (\varphi(t, x_1)(3) < -\zeta - \varepsilon_1).
\]
Because
\[
\|\varphi(t_1 + t, y_0) - \varphi(t, x_1)\| \leq \|\varphi(t_1 + t, y_0) - \varphi(t_1 + t, x_0)\| + \\
+ \|\varphi(t, \varphi(t_1, x_0)) - \varphi(t, x_1)\| < e^{9L} \cdot \zeta e^{-9L} + e^{4.2L}(\varepsilon + \bar{\varepsilon}) < \zeta + \varepsilon_1,
\]
we obtain
\[
(\varphi(t, y_0)^{(2)} < -0.1) \lor (\varphi(t, y_0)^{(3)} < 0) \text{ for } t \in [t_1, t_1 + t_2]. \tag{8}
\]
In a similar way Proposition 5.3 gives us
\[
(\varphi(t, y_0)^{(2)} < 0) \lor (\varphi(t, y_0)^{(3)} < 0) \text{ for } t \in [t_1 + t_2, \bar{t}_3]. \tag{9}
\]
Recall that \(S = \{x; x^{(2)} = 0, x^{(3)} > 0\}\), so the inequalities (6), (7), (8) and (9) together guarantee, that for \(t \in (0, \bar{t}_3]\) the trajectory \(\varphi(t, y_0)\) of the system does not cross the half-plane \(S\).

(d5) Follows from the property (c2) of the Proposition 5.3.
(d6) Follows from the property (c6) of the Proposition 5.3. \(\square\)

**Theorem 5.1.** Suppose that Algorithm4\((x, \varepsilon, \zeta, \delta)\) is successfully terminated for the point \(x_0 \in \Pi\), and its result is a tuple \((\tilde{x}_3, \tilde{t}_3, \tilde{x}_4, \tilde{t}_4)\). Then \(\forall y_0 \in O(x_0, \zeta e^{-9L})\) the Poincaré mapping \(\mathcal{P}(y)\) is defined at the point \(y_0\), and
\[
\|\mathcal{P}(y_0) - \tilde{x}\| < 2\zeta + 10\delta + \varepsilon_3, \tag{10}
\]
where \(\tilde{x} = (\tilde{x}_3^{(1)}, \tilde{x}_3^{(3)})\).

**Proof.** From Proposition 5.4, \(\varphi(\tilde{t}_3, y_0)^{(2)} < 0, \varphi(\tilde{t}_4, y_0)^{(2)} > 0\), and the continuity of \(\varphi\) implies \(\exists \tilde{t} \in (\tilde{t}_3, \tilde{t}_4); \varphi(\tilde{t}, y_0)^{(2)} = 0\). Besides, we have

\[
\forall t \in [\tilde{t}_3, \tilde{t}_4]; \varphi(t, y_0)^{(3)} > 0,
\]
thus the trajectory crosses \(S\) at the moment \(\tilde{t}\). Additionally, \(\varphi^{(3)} = \dot{\varphi}^{(2)}\), which implies the uniqueness of \(\tilde{t}\) in the time interval \((\tilde{t}_3, \tilde{t}_4)\). Property (d4) from Proposition 5.4 guarantees that there are no other intersection points in the interval \((0, \tilde{t}_3]\).

Inequality (10) is obtained from the following calculations:
\[
\|\mathcal{P}(y_0) - \tilde{x}\| = \|\varphi(\tilde{t}, y_0) - \tilde{x}\| \leq \\
\leq \|\varphi(\tilde{t}, y_0) - \varphi(\tilde{t}, x_0)\| + \|\varphi(\tilde{t}, x_0) - \varphi(\tilde{t}_3, x_0)\| + \\
+ \|\varphi(\tilde{t}_3, x_0) - x_3\| + \|x_3 - \tilde{x}\| < \\
< \zeta + 10\delta + \varepsilon_2 + \bar{\varepsilon} + \zeta + \varepsilon = 2\zeta + 10\delta + \varepsilon_3. \tag{10}
\]
\(\square\)

### 5.1. Proof of Theorem 3.1

The theorem follows from the Propositions 3.1 and 3.2.

**Proof of the Proposition 3.1.** Recall that we have to prove that the Poincaré mapping is defined on \(\Pi\). To do this, we cover the parallelogram \(\Pi\) with a finite number of balls with radii equal to \(h = \zeta e^{-9L}\) such that the mapping is defined on each ball. We define a grid on the set \(\Pi\):
\[
\mathcal{G}_\Pi = \{\xi(i, j) = u_0 + ihu_1/\|u_1\| + jhu_2/\|u_2\|, i = -N_1, \ldots, N_1, j = -N_2, \ldots, N_2,\}
\]
where \(N_1 = \lceil\|u_1\|/h\rceil + 1, N_2 = \lceil\|u_2\|/h\rceil + 1\).

It is easy to see that the set \(\Pi\) is fully covered by the balls of radii \(h\) with center points in \(\xi(i, j) \in \mathcal{G}\).

Introduce the following algorithm:

**Algorithm5**\((\varepsilon, \zeta, \delta)\)
after 8 minutes of computations on a Pentium 4 computer with a frequency of 2.26 GHz.

Algorithm5

All the algorithms described above were implemented in C++, and the computer
Proof.

δ

= 0

h

1

P

inclusion

3

for

3

= 4

ζ

invariance principle it is sufficient to prove that

Algorithm6

lines which contain boundaries of Π. Consider the following algorithm:

Algorithm4(ξ(i, j), ε, ζ, δ)

By Theorem 5.1 the successful termination of Algorithm5 implies that \( P \) is defined on each
of the balls \( O(\xi(i, j), \zeta e^{-9L}) \), which in turn implies that \( P \) is defined on \( \Pi \). The continuity of \( P \)
follows from the properties of the system.

Lemma 5.1. Algorithm5 terminated successfully for parameter values \( \varepsilon = 2 \cdot 10^{-10}, \zeta = 0.1, \delta = 0.5 \).

Proof. All the algorithms described above were implemented in C++, and the computer
calculations using these algorithms serve as a proof. Algorithm5 terminated successfully
after 8 minutes of computations on a Pentium 4 computer with a frequency of 2.26 GHz.

Thus Proposition 3.1 is proved.

Proof of Proposition 3.2. Recall that we have to prove the inclusion \( P(\Pi) \subset \Pi \). By the domain
invariance principle it is sufficient to prove that \( P(\partial \Pi) \subset \Pi \). To do this, we cover the set \( \partial \Pi \)
with a finite number of balls of radii \( h \) by introducing a grid

\[
\mathcal{Y}_\Pi = \left\{ \xi^1_\pm(i) = u_0 + ihu_1/\|u_1\| \pm u_2, \ i = -N_1, \ldots, N_1 \right\} \cup \\
\left\{ \xi^2_\pm(j) = u_0 \pm u_1 + jhu_2/\|u_2\|, \ j = -N_2, \ldots, N_2 \right\}.
\]

It is easy to see that the set \( \partial \Pi \) is fully covered by the balls of radii \( h \) with center points in \( \xi^1_\pm(i), \xi^2_\pm(j) \in \mathcal{Y} \).

To check if a point falls inside a rectangle \( \Pi \) we will compute distances from a point to the
lines which contain boundaries of \( \Pi \). Consider the following algorithm:

Algorithm6(\( \varepsilon, \zeta, \delta \))

1 \( h \leftarrow \zeta e^{-9L} \)

2 \( N_1 \leftarrow \left\lfloor \|u_1\|/h \right\rfloor + 1, \ N_2 = \left\lfloor \|u_2\|/h \right\rfloor + 1 \)

3 for \( i = -N_1, \ldots, N_1 \) do

4 \( \bar{x} \leftarrow \text{Algorithm4}(\xi^1_\pm(i), \varepsilon, \zeta, \delta), \text{Assert}(O(\bar{x}, 2\zeta + \varepsilon + 10\delta) \subset \Pi) \)

5 \( \bar{x} \leftarrow \text{Algorithm4}(\xi^2_\pm(i), \varepsilon, \zeta, \delta), \text{Assert}(O(\bar{x}, 2\zeta + \varepsilon + 10\delta) \subset \Pi) \)

6 for \( j = -N_2, \ldots, N_2 \) do

7 \( \bar{x} \leftarrow \text{Algorithm4}(\xi^1_\pm(j), \varepsilon, \zeta, \delta), \text{Assert}(O(\bar{x}, 2\zeta + \varepsilon + 10\delta) \subset \Pi) \)

8 \( \bar{x} \leftarrow \text{Algorithm4}(\xi^2_\pm(j), \varepsilon, \zeta, \delta), \text{Assert}(O(\bar{x}, 2\zeta + \varepsilon + 10\delta) \subset \Pi) \)

By Theorem 5.1 the successful termination of Algorithm6 implies that for all \( x \) from the
balls \( O(\xi^1_\pm(i), \zeta e^{-9L}) \) and \( O(\xi^2_\pm(j), \zeta e^{-9L}) \) the condition \( P(x) \in \Pi \) holds, which implies the
inclusion \( P(\partial \Pi) \subset \Pi \).

Lemma 5.2. Algorithm6 terminated successfully for parameter values \( \varepsilon = 2 \cdot 10^{-10}, \zeta = 4 \cdot 10^{-4}, \delta = 1.2 \cdot 10^{-3} \).

Proof. Computer calculations by the algorithm described above serve as a proof. The C++
implementation of Algorithm6 terminated successfully after 7.5 minutes of calculations on a
Pentium 4 2.26 GHz computer.

Thus Proposition 3.2 is proved, and Theorem 3.1 is also proved.
5.2. Proof of Theorem 4.1
(V,W)-hyperbolicity

We fix two positive integers $d_u$ and $d_s$ with $d_u + d_s = d$. Let $V$ and $W$ be bounded, open and convex product-sets

$$V = V^{(u)} \times V^{(s)} \subset \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}, \quad W = W^{(u)} \times W^{(s)} \subset \mathbb{R}^{d_u} \times \mathbb{R}^{d_s},$$

satisfying the conditions $0 \notin V, W$ and let $g: V \to \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$ be a continuous mapping. It is convenient to treat $g$ as a pair $(g^{(u)}, g^{(s)})$, where $g^{(u)}: V \to \mathbb{R}^{d_u}$ and $g^{(s)}: V \to \mathbb{R}^{d_s}$.

Let symbol $\text{deg}(g, \Omega, y)$ denote the topological degree of $g$ at $y$ with respect to $\Omega$ [11]. If $0 \notin f(\partial \Omega)$, then the number $\gamma(f, \Omega) = \text{deg}(f, \Omega, 0)$ is well defined and is called the rotation of the vector field $f$ at $\partial \Omega$. The properties of the number $\gamma(f, \Omega)$ are described in detail in [12].

**Definition 5.1.** The continuous mapping $g$ is $(V, W)$-hyperbolic if the equations

$$g^{(u)} \left( \partial V^{(u)} \times \mathbb{V}^{(s)} \right) \cap \mathbb{V}^{(u)} = \emptyset, \quad g(V) \cap \left( \mathbb{W}^{(u)} \times (\mathbb{R}^{d_s} \setminus W^{(s)}) \right) = \emptyset$$

hold, and

$$\text{deg}(g^{(u)}(\cdot, 0), V^{(u)}, 0) \neq 0.$$  \hspace{1cm} (11)

The first part of (11) means geometrically that the image of the ‘$u$-boundary’ $\partial V^{(u)} \times \mathbb{V}^{(s)}$ of $V$ does not intersect the infinite cylinder $C = \mathbb{V}^{(u)} \times \mathbb{R}^{d_s}$; similarly, the second part of (11) means that the image of the whole set $g(V)$ can intersect the cylinder $C$ only by its central fragment $\mathbb{W}^{(u)} \times W^{(s)}$. Thus the first equation in (11) means that the mapping expands in a weak sense along the first coordinate in the Cartesian product $\mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$, whereas the second one confers a type of contraction along the second coordinate.

Introduce $\mathcal{H}_i: (\alpha^{(u)}, \alpha^{(s)}) \mapsto x_i + \alpha^{(u)} x^{(u)} + \alpha^{(s)} x^{(s)}$, where $i = 0, \ldots, 10$ and $x_i = (x_i^{(1)}, x_i^{(2)})$ are given in Table 1. It is easy to see that each $\mathcal{H}_i$ is a homeomorphism so we can introduce $g_{ij} = \mathcal{H}_j^{-1} \mathcal{P} \mathcal{H}_i$. Let us denote $V_i = \{(\alpha^{(u)}_i, \beta^{(s)}_i); |\alpha|, |\beta| \leq 1\}$ where $i = 0, \ldots, 10$ and $a_i^{(u)}, a_i^{(s)}$ are given in Table 1. Note that according to this notation, $\mathcal{H}_i(V_i) = X_i$. Finally let us define a $(11 \times 11)$ matrix $A = (a_{ij})$ $i, j = 0, \ldots, 10$:

$$A = \begin{pmatrix}
1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{pmatrix}.$$

Recall that a matrix $M$ is called $k$-transitive, if all entries of its $k$-th power $M^k$ are positive.

**Lemma 5.3.** The matrix $A$ is $20$-transitive and is not $k$-transitive for any $k < 20$.

**Proof.** Straightforward calculations. \hfill \Box

To prove Theorem 4.1 we need to satisfy the conditions of Corollary 3.1 from [3], which are given below.
Figure 4. First picture shows $g_{0,0}(V_0)$ and $V_0$; other 10 show $g_{i,i+1}(V_i)$ and $V_{i+1}$ for $i = 0, \ldots, 9$, and the last one shows $g_{10,0}(V_{10})$ and $V_0$.

Corollary 5.1. Let a matrix $A$ be $k$-transitive and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous mapping. Let there exist homeomorphisms $H_i : \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} \to \mathbb{R}^d$ and product sets $V_i \subset \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$ such that $H_i^{-1}fH_i$ is $(V_i, V_j)$-hyperbolic if $a_{i,j} = 1$, and let the family $\mathcal{U}$ of connected components of the union set $\mathcal{U} = \bigcup \mathcal{H}_i(V_i)$ have more than one element. Then any mapping $\tilde{f}$, which is sufficiently close to $f$ in the uniform metric, is $(\mathcal{U}, k)$-chaotic.

Note that according to Lemma 4.1 there are more than two connected components of $\mathcal{U}$. Also the matrix $A$ is 20-transitive according to Lemma 5.3, and the mapping $\mathcal{P}$ is a continuous homeomorphism. To finalize the proof we need the following lemma:

**Lemma 5.4.** The mappings $g_{i,j} = H_j^{-1} \mathcal{P}H_i$ are $(V_i, V_j)$-hyperbolic for all $i, j$ with $a_{i,j} = 1$.

**Proof.** Note that the mappings $g_{i,j}$ are defined and are continuous homeomorphisms on $V_i$, because $H_i(V_i) = X_j \subset \Pi$ for all $i = 0, \ldots, 10$.

We now describe an algorithm that, upon successful termination, establishes $(V_i, V_j)$-hyperbolicity of $g_{i,j}$ for fixed $i, j = 0, \ldots, 10$. We define two auxiliary sets

$$A_1(j) = \mathcal{H}_j((\mathbb{R} \setminus V_j^{(u)}) \times \mathbb{R}),$$
and cover $\partial V_i$ with a finite number of balls of radii $h$ with center points in

$$
\xi_{\pm}^u(i, k) = (kh, \pm a_i(s)), \ k = -N_{i,1}, \ldots, N_{i,1},
\xi_{\pm}^s(i, q) = (\pm a_i^u(qh), q = -N_{i,2}, \ldots, N_{i,2},
$$

where $N_{i,1} = \lfloor a_i^u(h)/h \rfloor + 1$, $N_{i,2} = \lfloor a_i^s(h)/h \rfloor + 1$.

**Algorithm 7** $(i, j, \varepsilon, \zeta, \delta)$

1. $h \leftarrow \zeta e^{-9L}$
2. $N_1 = \lfloor a_i^u(h)/h \rfloor + 1$, $N_2 = \lfloor a_i^s(h)/h \rfloor + 1$
3. for $k = -N_1, \ldots, N_1$ do
4.  for $p \in \{+, -\}$ do
5.     $\hat{x} \leftarrow$ Algorithm 4($\mathcal{H}_i(\xi_{p}^u(i, k)), \varepsilon, \zeta, \delta$)
6.     Assert($O(\hat{x}, 2\zeta + \varepsilon 3 + 10\delta) \subset \mathcal{A}_2(j)$)
7. for $q = -N_2, \ldots, N_2$ do
8.  for $p \in \{+, -\}$ do
9.     $\hat{x} \leftarrow$ Algorithm 4($\mathcal{H}_i(\xi_{p}^s(i, q)), \varepsilon, \zeta, \delta$)
10.    Assert($O(\hat{x}, 2\zeta + \varepsilon 3 + 10\delta) \subset \mathcal{A}_2(j), \mathcal{A}_4(j)$)

Suppose that Algorithm 7 terminated successfully. Then the assertion in line 10, according to Theorem 5.1, implies

$$
g_{i,j}^{u}(\partial V_i^{u}) (\mathcal{V}^{u}(s)) \subset \bigcup g_i(j(\xi_{i}^{u}(\zeta e^{-9L}))) \subset \mathcal{A}_1,
$$

and the assertions in lines 6 and 10 also imply

$$
g_{i,j}^{u}(\partial V_i) \subset \bigcup g_i(j(\xi_{i}^{u}, \zeta e^{-9L})) \subset \mathcal{A}_2.
$$

From the inclusion (13) the first part of (11) follows, and from (14) and the domain invariance principle the second part of (11) follows. Thus (11) is proved.

Now consider the second condition (12) for $(V, W)$-hyperbolicity. Since $d_u = 1$, $g_{i,j}^{u}(x^{u}(i), 0)$ is one dimensional, $V^{u}$ is an interval $(-a^{u}_i, a^{u}_i)$, and verifying the inequality (12) is trivial: it holds if and only if

$$
g_{i,j}^{u}(-a^{u}_i, 0) \cdot g_{i,j}^{u}(a^{u}_i, 0) < 0.
$$

**Proposition 5.5.** The implementation of Algorithm 7 terminated successfully for all $i, j$ such that $a_{i,j} = 1$, with parameter values $\varepsilon = 2 \cdot 10^{-10}$ and $\zeta = \zeta(i), \ \delta = \delta(i)$ from the Table 2. An additional verification showed that for all such $i, j$ the condition (15) also holds.

Thus, Proposition 5.5 proves Lemma 5.4, and Theorem 4.1 is also proved.

On the intuitive level we proved that the intersections seen on Figure 4 do take place. However to justify this picture we needed to calculate $P$ at about 3 million points on the boundaries of different $V_i$. The C++ implementation of the algorithm runs for 7 minutes on a Pentium IV 2.26 GHz processor.
Table 2. Values for $\zeta(i)$ and $\delta(i)$.

| $i$ | 0, ..., 6 | 7 | 8 | 9 | 10 |
|-----|-----------|---|---|---|----|
| $\zeta(i)$ | $4 \cdot 10^{-4}$ | $2 \cdot 10^{-4}$ | $1.1 \cdot 10^{-4}$ | $3 \cdot 10^{-4}$ | $2.5 \cdot 10^{-4}$ |
| $\delta(i)$ | $8 \cdot 10^{-4}$ | $4 \cdot 10^{-4}$ | $3 \cdot 10^{-4}$ | $6 \cdot 10^{-4}$ | $5 \cdot 10^{-4}$ |

6. Conclusion

We considered a three-dimensional continuous piecewise linear system of ODEs which had been identified by Linz and Sprott and which is believed to be the most elementary three-dimensional dynamical system which still exhibits a chaotic evolution in time for some control parameter values. The non-smoothness of this system required the application of special techniques to establish guaranteed estimates for computer calculations.

The definition of chaos used in this paper includes sensitivity to initial conditions, regular mixing and an abundance of periodic trajectories. Following the traditional approach, chaotic properties were considered for a Poincaré mapping generated by the original model. The construction of such a mapping involved introducing computer algorithms which allowed us to obtain rigorous estimates of the numerical error. It was shown that the Poincaré mapping is well defined on an invariant polygonal set.

The proof of chaotic behavior uses the information about pseudo-orbits of Poincaré mapping and is based on so called $(U,V)$-hyperbolicity which is similar to “usual” hyperbolicity. The required pseudo-orbits were localized with the help of the broken orbit method. The $(U,V)$-hyperbolicity of the Poincaré mapping was proved using the numerical counterpart of the mapping together with rigorous error estimates.

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