ON THE PRIMITIVITY OF LAI-MASSEY SCHEMES

RICCARDO ARAGONA AND ROBERTO CIVINO

Abstract. In symmetric cryptography, the round functions used as building blocks for iterated block ciphers are often obtained as the composition of different layers providing confusion and diffusion. The study of the conditions on such layers which make the group generated by the round functions of a block cipher a primitive group has been addressed in the past years, both in the case of Substitution Permutation Networks and Feistel Networks, giving to block cipher designers the receipt to avoid the imprimitivity attack. In this paper a similar study is proposed on the subject of the Lai-Massey scheme, a framework which combines both Substitution Permutation Network and Feistel Network features. Its resistance to the imprimitivity attack is obtained as a consequence of a more general result in which the problem of proving the primitivity of the Lai-Massey scheme is reduced to the simpler one of proving the primitivity of the group generated by the round functions of a strictly related Substitution Permutation Network.

1. Introduction

Until the selection of the Advanced Encryption Standard [DR02], Feistel Networks (FN) have probably been the most popular design framework for iterated block ciphers, whereas today they share the stage with Substitution Permutation Networks (SPN). Feistel Networks are characterized by the clever idea of splitting the message into two halves, say left part and right part, and applying in each round a key-dependent non-linear transformation called F-function to the right part, which is successively mixed with the left part, just before the two halves are swapped [Fei73]. As a notable feature, FNs do not require the F-function to be invertible in order to perform decryption. The framework of SPNs is instead composed by a sequence of carefully designed key-dependent round functions composed by confusion and diffusion invertible layers acting on the whole block. If, on the one hand, SPNs’ minimalistic design allows a simple description and consequently a more careful security assessment, on the other, the structure of FNs gives the designers more freedom in the choice of the layers intervening during the encryption, although keeping confusion and diffusion confined only in half of the block in a single round. The Lai-Massey scheme (LM) [Vau99], introduced after the design of IDEA [LM91], perfectly combines the advantages of both frameworks, splitting the message into two halves but mixing the left and right part of the state and

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consequently accelerating both diffusion and confusion. Its pseudo-randomness behavior, its security against impossible differential cryptanalysis and other generic attacks has been addressed in recent years [YPL11, GJ14, LLH15, LLZ17].

In this paper we focus on the study of a group containing the group generated by the round functions of a general Lai-Massey cipher, proving its resistance against the imprimitivity attack [Pat99] provided that its inner layers satisfy certain well-established conditions. Little is known, indeed, on the group-theoretical security of such a design strategy, whereas the one of SPNs and FNs has been addressed in several works in the last decades. One remarkable exception is a paper due to Wernsdorf [Wer01] which shows that the multiply-addition box at the center of the round of IDEA generates the alternating group on $\mathbb{F}_2^{32}$ and where it is conjectured that also the entire rounds of IDEA generate the alternating group.

The topic of our research, i.e. the group generated by the round functions, was first defined in 1975 by Coppersmith and Grossman [CG75] and gained more popularity when, in 1999, Paterson introduced the imprimitivity attack showing that in a DES-like cipher may exist a partition of the message space which is invariant under the action of the group $\Gamma$ generated by the encryption functions, i.e. a block system for $\Gamma$, whose knowledge can be exploited to attack the cipher. After this, the resistance of many known ciphers to this attack has been proved [SW08, CDVS09, SW15, ACS17, ACTT18]. In Aragona et al. [ACC+19, Theorem 4.5], the authors showed that the primitivity of the group generated by the rounds of an FN can be reduced to the primitivity of the group generated by the rounds of an SPN whose round functions are the ones implemented as F-functions within each round of the FN, proving, in fact, that the primitivity of structure of an FN, in spite of its complexity, can be inherited from a simpler design. We prove here, using a similar approach, that the primitivity of the group generated by the rounds of an SPN implies the one of a group containing the group generated by the rounds of an LM which features in its structure the same key-dependent transformation acting in the SPN. Our result is referred to the closest group containing the actual group generated by the round functions of an LM for which a convenient algebraic description of the generators can be provided.

Organization of the paper. In Section 2 we introduce the notation and the preliminary results, and present our algebraic model of Lai-Massey scheme which is the subject of the study. In Section 3 we prove the primitivity reduction from the LM to the SPN.

2. Group-theoretical cryptanalysis

2.1. Preliminaries. Let us introduce our notation and some preliminary results.

2.1.1. Spaces. Let $n$ be a non-negative integer and $V \overset{\text{def}}{=} \mathbb{F}_2^n$ be the $n$-dimensional vector space over $\mathbb{F}_2$. We denote by $\text{Sym}(V)$ the symmetric group acting on $V$ and by $1$ its identity. The map $0 : V \to V$ denotes the null function. The group of the translations on $V$, i.e. the group of the maps $\sigma_v : V \to V$, such that $x \sigma_v = x + v$, is denoted by $T_v$, whereas the group of translations on $V \times V$ is denoted by $T_{2v}$, where the translation $\sigma_{(v,w)}$ acts on $(x,y)$ as $(x,y) \sigma_{(v,w)} = (x + v, y + w)$. Let us also denote by $\text{AGL}(V)$ the group of all affine permutations of $V$ and by $\text{GL}(V)$ the group of the linear ones.
2.1.2. Groups. Let $G$ be a finite group acting on a set $M$. For each $g \in G$ and $v \in M$ we denote the action of $g$ on $v$ as $vg$. The group $G$ is said to be transitive on $M$ if for each $v, w \in M$ there exists $g \in G$ such that $vg = w$. A partition $B$ of $M$ is trivial if $B = \{M\}$ or $B = \{\{v\} \mid v \in M\}$, and $G$-invariant if for any $B \in B$ and $g \in G$ it holds $Bg \in B$. Any non-trivial and $G$-invariant partition $B$ of $M$ is called a block system for $G$. In particular any $B \in B$ is called an imprimitivity block.

The group $G$ is primitive in its action on $M$ (or $G$ acts primitively on $M$) if $G$ is transitive and there exists no block system. Otherwise, the group $G$ is imprimitive in its action on $M$ (or $G$ acts imprimitively on $M$). We recall here some well-known results that will be useful in the remainder of this paper [Cam99].

Lemma 2.1. If $T \leq G$ is transitive, then a block system for $G$ is also a block system for $T$.

Lemma 2.2. Let $M$ be a finite vector space over $\mathbb{F}_2$ and $T$ its translation group. Then $T$ is transitive and imprimitive on $M$. A block system $U$ for $T$ is composed by the cosets of a non-trivial and proper subgroup $U < (M, +)$, i.e.

$$U = \{U + v \mid v \in M\}.$$  

2.1.3. Goursat’s Lemma. To prove our results, we need to determine a block system for $V \times V$. In order to do so, we use the following characterization of subgroups of the direct product of two groups in terms of suitable sections of the direct factors [Gou89].

Theorem 2.3. Let $G_1$ and $G_2$ be two groups. There exists a bijection between

1. the set of all subgroups of the direct product $G_1 \times G_2$, and
2. the set of all triples $(A/B, C/D, \psi)$, where
   - $A$ is a subgroup of $G_1$;
   - $C$ is a subgroup of $G_2$;
   - $B$ is a normal subgroup of $A$;
   - $D$ is a normal subgroup of $C$;
   - $\psi : A/B \to C/D$ is a group isomorphism.

Then, each subgroup of $G_1 \times G_2$ can be uniquely written as

$$U_\psi = \{(a, c) \in A \times C : (a + B)\psi = c + D\}.$$  

Note that the isomorphism $\psi : A/B \to C/D$ is induced by a homomorphism $\varphi : A \to C$ such that $(a + B)\psi = a\varphi + D$ for any $a \in A$, and $B\varphi \leq D$. Such homomorphism is not unique.

Corollary 2.4 ([ACC+19]). Using notation of Theorem 2.3, given any homomorphism $\varphi$ inducing $\psi$, we have

$$U_\psi = \{(a, a\varphi + d) \mid a \in A, d \in D\}. \tag{1}$$  

2.1.4. Ciphers. A block cipher $\Phi$ is a family of key-dependent permutations

$$\{E_K \mid E_K : M \to M, K \in \mathcal{K}\},$$  

where $M$ is the message space, $\mathcal{K}$ the key space, and $|M| \leq |\mathcal{K}|$. The permutation $E_K$ is called the encryption function induced by the master key $K$. The block cipher $\Phi$ is called an iterated block cipher if there exists $r \in \mathbb{N}$ such that for each $K \in \mathcal{K}$ the encryption function $E_K$ is obtained as the composition of $r$ round functions, i.e. $E_K = \varepsilon_{1,K} \varepsilon_{2,K} \ldots \varepsilon_{r,K}$. To provide efficiency, each round function
is the composition of a public component provided by the designers, and a private component derived from the user-provided key by means of a public procedure known as key-schedule. The group
\[ \Gamma_{\infty}(\Phi) \overset{\text{def}}{=} \langle \varepsilon_{i,K} \mid K \in \mathcal{K}, 1 \leq i \leq r \rangle, \]
called the group generated by the round functions of \( \Phi \), is studied to prevent group-theoretical attacks [KRS88, Pat99, CCS17].

An iterated block cipher \( \Phi \) is called an \( r \)-round Substitution Permutation Network (SPN) if \( M = V \) and for each \( 1 \leq i \leq r \) we have
\[ \varepsilon_{i,K} \overset{\text{def}}{=} \rho \sigma_{k_i}, \]
where \( \rho \in \text{Sym}(V) \setminus \text{AGL}(V) \) is designed to provide both Shannon’s principle of confusion and diffusion [Sha49]. If \( \Phi \) is an SPN, then \( \Gamma_{\infty}(\Phi) = \langle \rho, T_n \rangle [CDVS09] \).

2.2. A model for the Lai-Massey scheme. We introduce here our algebraic description of the Lai-Massey scheme [LM91] as presented by Vaudenay [Vau99].

Definition 2.5. Let \( r \) be a non-negative integer, \( \rho \in \text{Sym}(V) \setminus \text{AGL}(V) \) and \( \pi \in \text{GL}(V) \). An \( r \)-round Lai-Massey cipher \( \text{LM}(\rho, \pi) \) is a set of encryption functions
\[ \{ E_K \mid K \in \mathcal{K} \} \subseteq \text{Sym}(V \times V) \]
such that for each \( K \in \mathcal{K} \) the map \( E_K \) is the composition of \( r \) functions, i.e.
\[ E_K = \varepsilon_{1,K} \varepsilon_{2,K} \ldots \varepsilon_{r,K}. \]
The \( i \)-th round function \( \varepsilon_{i,K} \) is defined as
\[ \varepsilon_{i,K} \overset{\text{def}}{=} \bar{\rho} \pi \sigma_{(k_i, k_i)}, \]
where
\begin{itemize}
  \item \( \bar{\rho} \) denotes the formal operator
    \[ \begin{pmatrix}
      1 & 1 \\
      1 & 0 
    \end{pmatrix}
    \begin{pmatrix}
      1 & 1 + \rho \\
      0 & 1 
    \end{pmatrix} \in \text{Sym}(V \times V); \]
  \item \( \bar{\pi} \) denotes the formal operator
    \[ \begin{pmatrix}
      \pi & 0 \\
      \pi & 1 
    \end{pmatrix} \in \text{Sym}(V \times V); \]
  \item the key-schedule \( \mathcal{K} \to V^r, K \mapsto (k_1, k_2, \ldots, k_r) \) is surjective with respect to any round.
\end{itemize}

By the assumption on the key-schedule, we can always assume without loss of generality that \( 0 \rho = 0 \), provided that, in each round, the value \((0 \rho \pi, 0 \rho)\) is added to the round key of the previous iteration. It is well known that, for security concerns, the function \( \pi \) is required to be an orthomorphism [Vau99]. However, we do not make use of this hypothesis in our analysis. We strongly use, instead, the assumption that such a function is linear.

The general round function of a Lai-Massey cipher is displayed in Fig. 1. Notice that the previous formal definition coincides with the classical definition given by Vaudenay [Vau99]. Indeed, given \((x, y) \in V \times V\) we have
\[ (x, y) \overline{\varepsilon_{i,K}} = ((x + (x + y) \rho + k_i) \pi, y + (x + y) \rho + k_i). \]
Moreover, it is easy to check that \( \overline{\varepsilon_{i,K}} \) is invertible with the following inverse
\[ \overline{\varepsilon_{i,K}}^{-1} = \pi^{-1} \bar{\rho}^{-1} \sigma_{(k_i, k_i)}, \]
where \( \bar{\rho}^{-1} = \begin{pmatrix}
      1 & \rho \\
      1 & 1 
    \end{pmatrix} \) and \( \pi^{-1} = \begin{pmatrix}
      \pi^{-1} & 0 \\
      1 & 1 
    \end{pmatrix}. \)
Note that, as in the Feistel Network case, the inverse $\overline{\varepsilon_{i,K}^{-1}}$ of the round function $\overline{\varepsilon_{i,K}}$ of a Lai-Massey cipher does not involve the inverse of $\rho$. We have nonetheless assumed that $\rho$ is bijective, since in our result it is used as the generator of a group. It is worth mentioning that even if IDEA was the starting point for the definition of the LM framework, it does not fit in the presentation of Definition 2.5, since e.g. in IDEA the round key is mixed to the state by using operations different from the XOR.

Let us define the group
\[
\Gamma(LM(\rho, \pi)) \overset{\text{def}}{=} \langle \rho, \pi, T_{2n} \rangle ,
\] (2)
which clearly contains the group $\langle \rho, \pi, T_{2n} \rangle$ generated by the round functions of a Lai-Massey cipher. Notice that considering every possible translation in $T_{2n}$ in Eq.(2) implicitly means that the study is carried out without considering the role of the particular choice of the key-schedule. This is however a common practice in the study of the primitivity of the groups generated by the rounds of a cipher, except for one recent result [ACC20].

In the following section we will prove our main contribution, i.e. that $\Gamma(LM(\rho, \pi))$ is primitive provided that $\langle \rho, T_n \rangle$ is primitive. It is worth mentioning again that $\langle \rho, T_n \rangle$ is the group generated by the round functions of the SPN whose function $\rho$ is the same that composes the building block for the round functions of the LM. In this sense, the primitivity of a Lai-Massey scheme is reduced to the one of the corresponding SPN.
3. The primitivity reduction

**Definition 3.1.** A subgroup $U \leq V \times V$ is a **linear block** for $f \in \text{Sym}(V \times V)$ if for each $(v, w) \in V \times V$ there exists $(v', w') \in V \times V$ such that

$$(U + (v, w))f = U + (v', w').$$

Notice that we can always assume $(v', w') = (v, w)f$.

In the following result we assume the existence of a linear block $U$ for $\rho$. In this case we have

$$(v', w') = (v + w, w + (v + w)\rho).$$

Moreover, it is easy to check that $U$ is a linear block also for $\rho^{-1}$, from which we obtain

$$(U + (v, w))\rho^{-1} = U + (v + w + v\rho, w + v\rho).$$

We use Theorem 2.3 and Corollary 2.4 to provide an useful decomposition of $U$. The explicit dependence of all the groups from $\rho$ is here omitted.

**Lemma 3.2.** Let $U \leq V \times V$, and let $A, B, C, D \leq V$ and $\varphi : A \to C$ an homomorphism such that $U = \{(a, a\varphi + d) \mid a \in A, d \in D\}$. Let us assume that $U$ is a linear block for $\rho$. Then the following conditions hold:

1. $D \leq A$;
2. $A\varphi \leq A$;
3. $D\varphi \leq D$.

**Proof.** Since $U$ is a linear block for $\rho$, taking $u = v = 0$ in Eq. (3) we have that for each $a \in A$ and $d \in D$ there exist $x \in A$ and $y \in D$ such that

$$(a, a\varphi + d)\rho = (a, a\varphi + d) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 + \rho \\ 0 & 1 \end{pmatrix} = (a + a\varphi + d, a\varphi + d + (a + a\varphi + d)\rho) = (x, x\varphi + y).$$

If $a = 0$, then $x = d$ and therefore $D \leq A$, so (1) is proved. If $d = 0$, then $x = a + a\varphi$ and therefore $A\varphi \leq A$, which is (2). As noticed, $U$ is also a linear block for $\rho^{-1}$, hence for each $a \in A$ and $d \in D$ there exist $x \in A$ and $y \in D$ such that

$$(a, a\varphi + d)\rho^{-1} = (a, a\varphi + d) \begin{pmatrix} 1 + \rho & \rho \\ \rho & 1 \end{pmatrix} = (a + a\varphi + d + a\rho, a\varphi + d + a\rho) = (x, x\varphi + y).$$

If $a = 0$, then $y = d\varphi + d$, and consequently $d\varphi \in D$, which proves (3). \qed

We now use the previous lemma to show our main result on the primitivity of the Lai-Massey scheme. Notice that the result is valid for any choice of $\pi$.

**Theorem 3.3.** If $\langle \rho, T_n \rangle$ is primitive, then $\Gamma_\infty(\text{LM}(\rho, \pi))$ is primitive.

**Proof.** It is enough to prove that $\langle \tilde{\rho}, T_{2n} \rangle$ is primitive. Let us assume that it is imprimitive, i.e. that there exists a block system $U$ for $\langle \tilde{\rho}, T_{2n} \rangle$. Then, from Lemma 2.2,
the block system is $U = \{U + (v, w) \mid (v, w) \in V \times V\}$ for a non-trivial proper subspace $U$ of $V \times V$. Since $U$ is a linear block for $\mathcal{P}$, we have that for each $(v, w) \in V \times V$ and for each $a \in A$ and $d \in D$ there exist $x \in A$ and $y \in D$ such that
\[
(a + v, a\varphi + d + w)\mathcal{P} = (a + a\varphi + d + v + w, (a + a\varphi + d + v + w)\rho + a\varphi + d + w) = (x + v + w, x\varphi + y + w + (v + w)\rho)
\]
If $a = 0$, then $x = d$ and
\[
y + d\varphi + d + (v + w)\rho = (d + v + w)\rho.
\]
From Lemma 3.2 we have $d\varphi \in D$, and therefore, since $\rho$ is bijective, we obtain the equality
\[
(D + v + w)\rho = D + (v + w)\rho.
\]
If $D$ is a non-trivial proper subgroup of $V$, then \{(D + v \mid v \in V)\} is a block system for $\langle \rho, T_n \rangle$, which proves our claim. In order to conclude the proof, let us prove that both the assumptions $D = \{0\}$ and $D = \mathbb{F}_2^n$ lead to contradictions.

\[D = \mathbb{F}_2^n\] Since $D \leq A$, then $A = \mathbb{F}_2^n$, and therefore $B = C = \mathbb{F}_2^n$, since from the hypothesis $A/B \cong C/D$. This proves that $U$ is not proper, a contradiction.

\[D = \{0\}\] In this case we have $U = \{(a, a\varphi) \mid a \in A\}$, and so, for each $(v, w) \in V \times V$ and for each $a \in A$ there exists $x \in A$ such that
\[
(a + v, a\varphi + w)\mathcal{P} = (a + a\varphi + v + w, (a + a\varphi + v + w)\rho + a\varphi + w) = (x + v + w, x\varphi + w + (v + w)\rho),
\]
then $x = a + a\varphi$ and
\[
(a + a\varphi + v + w)\rho = a\varphi^2 + (v + w)\rho
\]
Since $B\varphi \leq D$, then $B\varphi = \{0\}$ and so, if $a \in B$ from Eq. (4) we obtain
\[
(a + v + w)\rho = (v + w)\rho,
\]
which implies $a = 0$, i.e. $B = \{0\}$. This proves that $\varphi = \psi : A \rightarrow C$ is an isomorphism. But $A\varphi \leq A$, from Lemma 3.2, therefore $\varphi$ is an automorphism of $A$ and, from Eq. (4), we obtain
\[
(A + v + w)\rho = A + (v + w)\rho.
\]
In the case under consideration, i.e. when $D = \{0\}$, the claim is proved by showing that $\{A + v \mid v \in V\}$ is a block system. This is addressed in the remainder of the proof. Let us prove that $A$ is non-trivial and proper. If $A = \{0\}$, then $C = D = \{0\}$, and so also $B = \{0\}$, therefore $U$ is trivial, a contradiction. To conclude, let us assume $A = \mathbb{F}_2^n$. From Eq. (4), setting $v = w = 0$, we obtain that $a\varphi^2 = (a + a\varphi)\rho$.

If $a \in A$ is a fixed point of $\varphi$, i.e. $a = a\varphi$, then $(a + a\varphi)\rho = 0$, and so $a\varphi^2 = 0$. Therefore $a = 0$, since $\varphi$ is an automorphism. We have proved that $\varphi$ is fixed-point free, except for the trivial one $a = 0$, from which it follows that $1 + \varphi$ is injective and, since $A\varphi \leq A$, we have $\{a + a\varphi \mid a \in A\} = A = \mathbb{F}_2^n$. Therefore $\rho$ is linear on $\mathbb{F}_2^n$, a contradiction.

We have already observed that $\langle \rho, T_n \rangle$ is the group $\Gamma_\infty(\Phi)$ generated by the rounds of the Substitution Permutation Network $\Phi$ whose $i$-th round function is $\varepsilon_{i,K} = \rho \sigma_k$, for some $\rho = \gamma \lambda \in \text{Sym}(V) \setminus \text{AGL}(V)$. The conditions which prove $\Gamma_\infty(\Phi)$ primitive in the case of the SPNs has been extensively studied, due to the popularity of the design framework. It has been proved that the primitivity is
granted when the confusion layer $\gamma$ of $\rho$ satisfies some well-established conditions of non-linearity, provided that $\lambda$ provides sufficient diffusion. The interested reader may refer to Caranti et al. [CDVS09], where the primitivity of SPNs is studied in the larger context of translation-based ciphers.

**Different rounds.** In our analysis we have assumed that, for sake of simplicity, in both the cases of SPNs and of LMs the *same* round function is applied to each round, with the only exception of the round key. It is worth being mentioned here that almost no real cipher can exactly fit this model, due to the natural need to differentiate the encryption routine by introducing some atypical rounds, for both security and efficiency reasons (see e.g. the first and last rounds of AES [DR02]). However, this does not represent an actual limitation when it comes to evaluate the security of a design from the point of view of the group generated by the round function. If we assume, indeed, that our target SPN and LM feature different round functions for each round, then the primitivity of $\langle \rho_i, \pi_i, T_{2n} \rangle$ can be reduced to that of $\langle \rho_i, T \rangle$, proceeding as in Theorem 3.3. When this is true for one round $i$, then the full group

$$
\langle \langle \rho_i, \pi_i, T_{2n} \rangle \mid 1 \leq i \leq r \rangle
$$

is primitive.

Assuming that the definition of SPN and Definition 2.5 allow different round functions for each round, then the following consequence of Theorem 3.3 can be derived.

**Corollary 3.4.** If a given round of an SPN generates a primitive group, then the group defined as in Eq. (5) generated by the rounds of the corresponding LM is primitive.

**2-transitivity.** It is well known that every 2-transitive group is primitive. It may be natural to ask whether an alternative version of Theorem 3.3 could be obtained, where primitivity is replaced by 2-transitivity. In the case $n = 3$, we have exhaustively searched using Magma [CP96] for all the non-linear functions $\rho$ such that $\langle \rho, T_3 \rangle$ is a 2-transitive group. For all of those, $\langle \rho, T_6 \rangle$ is always 2-transitive. Setting $n = 4$, a partial search in the space led to no counterexamples. A brute-force search when $n \geq 4$ is out of the scopes of this work since requires code optimization and a faster programming language. On the other side, it is well known that the 2-transitivity of $\Gamma_\infty(\text{LM}(\rho, \pi))$ is equivalent to the transitivity of the stabilizer of $(0, 0)$ on $V \times V \setminus \{(0, 0)\}$. However, a description of $\Gamma_\infty(\text{LM}(\rho, \pi))_{(0,0)}$ is not easily obtained from $\langle \rho, T_n \rangle_0$ due to the non-linear dependence introduced by the Lai-Massey formal operator. For this reasons, at the time of writing we are not able to conjecture that the 2-transitivity of $\langle \rho, T_n \rangle$ implies that $\Gamma_\infty(\text{LM}(\rho, \pi))$ is 2-transitive, and we leave this as an open problem.

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