Hopf Structure in Nambu-Lie $n$-Algebras

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Abstract

We give a definition and study Hopf structures in ternary (and $n$-ary) Nambu-Lie algebra. The fundamental concepts of 3-coalgebra, 3-bialgebra and Hopf 3-algebra are introduced. Some examples of Hopf structures are analyzed.

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In 1973 Yoishira Nambu [1] proposed a generalization of classical Hamiltonian mechanics, using ternary and higher-order brackets (\(n\)-ary brackets or multibrackets). During the past two decades Nambu proposal has been a matter for many investigations [2] - [12] and the permanent interest for this issue is related with the recognition of the feasible physical richness and the mathematical beauty of ternary and higher algebraic systems. Recently such an algebraic structure has been analyzed and reformulated by Tachtajan [7] – [9] in an invariant geometrical form. He proposed the notion of Nambu-Lie "gebra", which is a generalization of Lie algebras for ternary (in general \(n\)-ary) case. The ternary algebra or "gebra" is a linear space in which conditions of antisymmetry and generalized Jacobi identity are fulfilled.

In this letter we investigate some new aspects of the Nambu proposal, connected with Hopf algebra concept. That is, a ternary (and \(n\)-ary) Hopf structure is introduced as generalization of the usual Hopf algebra, and some examples are presented.

Let us begin with a definition of a ternary (associative) algebra.

**Definition 1.** A ternary algebra with unit over a commutative ring \(\mathbb{C}\) is a vector space \(A\) together with a way of multiplying three elements \(a, b, c \in A\)

\[ m : A \otimes A \otimes A = A^{\otimes 3} \to A, \text{ such that } m(a \otimes b \otimes c) = abc \]  

(1)

The unit element in \(A\) is thus defined:

\[ m(1 \otimes 1 \otimes a) = m(1 \otimes a \otimes 1) = m(a \otimes 1 \otimes 1) = a; \]  

(2)

and the 3-associativity means

\[(abc)de = a(bcd)e = ab(cde).\]  

(3)

**Definition 2.** A 3-assciator can be defined by

\[ I^{(3)} = 2(abc)de - a(bcd)e - ab(cde), \]  

(4)

Let us give examples of an associative and non-associative 3-algebras.

**Example 1.** (Associative 3-algebra.) Let \(A = \{a, b, c...; m\}\) be a set of linear operators on a Hilbert space, such that \(m\) is the usual associative product, then \(I^{(3)} = 0\).

**Example 2.** (Non-associative 3-algebra. ) Consider \(A = \{a, b, c...; m\}\) a set of analytic functions defined on \(\mathbb{R}^3\), such that

\[ m(a \otimes b \otimes c)(x) = \frac{\partial a(x)}{\partial x_1} \frac{\partial b(x)}{\partial x_2} \frac{\partial c(x)}{\partial x_3}, \]  

where \(x = (x_1, x_2, x_3)\). In this case, \(I^{(3)} \neq 0\). Indeed, denoting

\[ \partial_i = \frac{\partial}{\partial x_i}, \quad \partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad i, j = 1, 2, 3, \]
we obtain the following non zero value for the 3-associator

\[
I^{(3)} = 2(\partial_1 a \partial_{12} b + \partial_{11} a \partial_2 b) \partial_3 c \partial_2 d \partial_3 e
\]

\[
+ \partial_1 a \partial_2 b \partial_{13} c \partial_2 d \partial_3 e
\]

\[
- \partial_1 a \partial_2 b \partial_1 c \partial_3 d \partial_3 e
\]

\[
- \partial_1 a \partial_1 b \partial_2 c \partial_3 d \partial_3 e
\]

\[
- \partial_1 a \partial_1 b \partial_2 c \partial_3 d \partial_3 e
\]

\[
- \partial_1 a \partial_2 b \partial_1 c \partial_2 d \partial_3 e.
\]

**Definition 3.** The unit map is defined (as usual) by

\[
I : C \to A, \quad i(\lambda) = \lambda 1; \quad 1 \in A.
\]

In a commutative diagrammatic representation the associativity of a ternary algebra \( A \) can be represented in the following way

\[
A \otimes C \otimes C \xrightarrow{id \otimes i \otimes i} A \otimes A \otimes A \xrightarrow{\mu} A
\]

\[
\xrightarrow{\sim} A \otimes A \otimes A \xrightarrow{\mu} A
\]

\[
C \otimes C \otimes A \xrightarrow{id \otimes i \otimes id} A \otimes A \otimes A \xrightarrow{\mu} A
\]

\[
\xrightarrow{\sim} A \otimes A \otimes A \xrightarrow{\mu} A
\]

On the other hand, the associativity of the multiplication is characterized by

\[
m(m \otimes id \otimes id) = m(id \otimes m \otimes id) = m(id \otimes id \otimes m).
\]

**Definition 4.** The Nambu product (a 3-commutator) is defined by \( \pi = \pi^+ - \pi^- \), where

\[
\pi^+ : A^{\otimes 3} \to A^{\otimes 3}, \quad \pi^- : A^{\otimes 3} \to A^{\otimes 3},
\]

\[
\pi^+(a, b, c) = a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b; \quad (5)
\]

\[
\pi^-(a, b, c) = c \otimes b \otimes a + a \otimes c \otimes b + b \otimes a \otimes c.
\]

\( \pi^+(\pi^-) \) is the sum over all terms with even (odd) permutation of \( a, b \) and \( c \in A \).

Using this definition of Nambu product, a generalization of the concept of abelian algebra (an abelian 3-algebra) is given by the square representing 3-commutativity.

\[
A^{\otimes 3} \xrightarrow{\pi^-} A^{\otimes 3} \xrightarrow{\mu} A
\]

\[
\pi^+ \downarrow \quad m \downarrow
\]

\[
A^{\otimes 3} \xrightarrow{\pi^+} A^{\otimes 3} \xrightarrow{\mu} A
\]
Notice that Definitions 1-4 can immediately be generalized for the \( n \)-ary case. Besides, for the particular case \( n = 2 \), we have
\[
\begin{align*}
m \circ \pi^+(a \otimes b) &= ab, \\
m \circ \pi^-(a \otimes b) &= ba.
\end{align*}
\]
In this (binary) situation, \( \pi^+ \) plays the role of the identity map \( (id) \), whilst \( \pi^- \) corresponds to the flip operator \( (\tau) \).

**Example 3**. (Abelian 3-algebra.) Consider the set of functions from Example 2, such that now
\[
m(a \otimes b \otimes c)(x) = a(x) b(x) c(x).
\]
In this case, the 3-commutator \( \pi = 0 \).

**Example 4**. (Noncommutative 3-algebra.) Consider \( A \) as given in Example 1. Then we can introduce the Nambu ternary bracket \([\cdot ,\cdot ,\cdot ] = m \circ \pi \) by the linear operator
\[
[a, b, c] = abc + bca + cab - cba - acb - bac,
\]
which satisfies the properties:

(\textit{alternation law})
\[
[a, b, c] = [b, c, a] = [c, a, b] = -[a, c, b] = -[c, b, a] = -[b, a, c], \quad (6)
\]
(\textit{derivation law})
\[
[a, b, cd] = c[a, b, d] + [a, b, c]d, \quad (7)
\]
(\textit{generalized Jacobi identity})
\[
[g, h[a, b, c]] = [[g, h, a], b, c] + [a, [g, h, b], c] + [a, b, [g, h, c]]. \quad (8)
\]
Such a generalized Jacobi identity has been analyzed by several authors[3, 4, 5, 6], in different contexts, and it was called \textit{fundamental identity} in Ref.[7].

**Example 5**. (Commutative 3-algebra.) Consider \( A \) as given by Example 2 (in this case \( m \) is a nonassociative product). Then we can introduce the Nambu bracket \( \{\cdot ,\cdot ,\cdot \} = m \circ \pi \) by
\[
\{a, b, c\} = \varepsilon^{ijk} \partial_i a \partial_j b \partial_k c.
\]
A basis for this (classical) Nambu bracket is then \( x_1, x_2, x_3 \), such that \( \{x_1, x_2, x_2\} = 1 \).

We use the definition of 3-algebra, given in terms of commutative diagrams, to explore the structure of dual coalgebra, and so to introduce a generalization of Hopf algebra [13, 14]. To do this, we proceed as usually is done with the concept of coalgebra: a 3-coproduct is defined by inverting the arrows in the definition of the 3-associative algebra. Therefore, we obtain the definition of 3-comultiplication \( \Delta \) and 3-counit \( \epsilon \),
\[
\Delta : A \to A^{\otimes 3},
\]
\[ \epsilon : A \to C, \]
such that the following diagrams commute

\[
\begin{array}{ccc}
A \otimes C \otimes C & \xrightarrow{\text{id} \otimes \epsilon \otimes \epsilon} & A^3 \\
\uparrow & & \uparrow \\
A & \leftarrow_{\text{id}} & A
\end{array}
\quad
\begin{array}{ccc}
C \otimes A \otimes C & \xrightarrow{\epsilon \otimes \text{id} \otimes \epsilon} & A^3 \\
\uparrow & & \uparrow \\
A & \leftarrow_{\text{id}} & A
\end{array}
\]

The 3-coassociativity is defined by

\[(\Delta \otimes \text{id} \otimes \text{id}) \Delta = (\text{id} \otimes \Delta \otimes \text{id}) \Delta = (\text{id} \otimes \text{id} \otimes \Delta) \Delta,\]

and 3-cocommutativity is expressed by square

\[
\begin{array}{ccc}
A^3 & \xrightarrow{\pi^+} & A^3 \\
\pi^- \uparrow & & \uparrow \Delta \\
A^3 & \leftarrow_{\Delta} & A
\end{array}
\]

A 3-algebra and a 3-coalgebra give rise to a generalization of bialgebras (that is, a 3-bialgebra). In order to define a Hopf 3-algebra, a generalization of antipode should be introduced. A natural 3-antipode \( S \) can be defined as follows,

\[
m \circ (S \otimes \text{id} \otimes \text{id}) \circ \Delta = i \circ \epsilon, \quad (9)
\]

\[
m \circ (\text{id} \otimes S \otimes \text{id}) \circ \Delta = i \circ \epsilon, \quad (10)
\]

\[
m \circ (\text{id} \otimes \text{id} \otimes S) \circ \Delta = i \circ \epsilon. \quad (11)
\]

For 4-bialgebras, a 4-antipode can be introduced by

\[
m \circ (S \otimes \text{id} \otimes \text{id} \otimes \text{id}) \circ \Delta = i \circ \epsilon, \quad (12)
\]

\[
m \circ (\text{id} \otimes S \otimes S \otimes \text{id}) \circ \Delta = i \circ \epsilon, \quad (13)
\]

\[
m \circ (S \otimes \text{id} \otimes \text{id} \otimes S) \circ \Delta = i \circ \epsilon, \quad (14)
\]

\[
m \circ (\text{id} \otimes S \otimes \text{id} \otimes S) \circ \Delta = i \circ \epsilon. \quad (15)
\]

All such relations are supposed to be satisfied simultaneously

**Example 6**. (3-bialgebra from Nambu-Lie algebra.) Considering \( A \) in Example 4, a 3-bialgebra can be derived, if we introduce a 3-coproduct by

\[ \Delta a = a \otimes a \otimes a \equiv a^3. \]

Indeed, in this case \( \Delta \) respects the algebraic relation, since

\[ [\Delta a, \Delta b, \Delta c] = \Delta[a, b, c]. \]
Notice that if we try to introduce a 3-coproduct by
\[ \Delta a = a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a, \] (16)
the algebra structure can not be respected, because \( \Delta[a, b, c] \neq [\Delta a, \Delta b, \Delta c] \). This result shows us that we can obtain a bialgebra structure attached to some (if any) universal enveloping 3-algebra of a Nambu-Lie algebra, but a Hopf 3-algebra can not be trivially introduced in this case.

**Example 7**. \( (SL(n, C)) \) Consider the group \( G = SL(n, C) \), where an element \( x = (a_{ij}) \in G \) has unit determinant, \( \det x = 1 \). It is well known that the algebra generated by the functions \( a_{ij}(x) \), with 2-coproduct defined by \( \Delta a_{ij} = \sum_k a^i_k \otimes a^j_k \) is a Hopf algebra (see, for example, [14]), with counit given by
\[ \epsilon(a_{ij}) = \delta_{ij} \] (17)
and the antipode given by inverse matrix
\[ S(a_{ij}) = (a^{-1})_{ij}. \] (18)

In order to study a generalization of this Hopf algebra, we can consider two cases, 3- and 4-algebras. First, a 3-coproduct \( \Delta \) can be defined as following
\[ \Delta a_{ij}^k(x, y, z) = a_{ij}^k(x, y, z) = \sum_{k,l} a_{ij}^k(x) a^l_k(y) a_{ij}^l(z) \] (19)
\[ = \sum_{k,l} a^i_k \otimes a^k_l \otimes a_{ij}^l(x, y, z), \]
or
\[ \Delta a_{ij}^k = \sum_{k,l} a^i_k \otimes a^k_l \otimes a_{ij}^l. \] (20)

Therefore, a 3-bialgebra can be derived, using the usual counit, Eq.(17). It is an easy matter to see that the usual antipode, Eq.(18), can not be used to define a satisfactory 3-antipode as given by Eqs.(9)–(11).

For 4-product, however, using
\[ \Delta a_{ij}^k = \sum_{k,l,m} a_{ij}^k \otimes a^l_k \otimes a^m_l \otimes a_{ij}^m, \]
together with Eqs.(12)–(13), where \( S \) is being given by Eq.(18), we get a Hopf 4-algebra.

In short, we have presented here a generalization of the concept of associative and commutative algebra. Exploring the notion of duality, and following in parallel with the usual (binary) approach, the concept of n-bialgebra could be introduced. In particular, we have presented examples of n-bialgebras, as that one derived from the Nambu-Lie algebra and that associated with the \( SL(n, C) \) group. It has also been indicated how to introduce the concept of 3- and 4-antipode, in order to obtain a generalization of Hopf algebra. As an example, a Hopf 4-algebra attached to \( SL(n, C) \) was derived. It should be interesting to study the connection of a 3-Hopf algebra with a universal enveloping algebra of a 3-noncommutative algebra. These aspects will be studied in more details elsewhere.
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