An improvement in the linear stable ranges for ordered configuration spaces

Cihan Bahran

In a recent paper, Church–Miller–Nagpal–Reinhold [CMNR17, Application A, Theorem 4.3] gave the first linear stable ranges (in the sense of representation stability) for the integral cohomology of ordered configuration spaces of manifolds. When the manifold is orientable and is of dimension at least 3, the relevant spectral sequence of Totaro [Tot96] becomes sparse. We note that this sparsity can be exploited, while using no more than the methods in [CMNR17], to improve the constants in the linear stable ranges. This note should be read after [CMNR17], for the notation is borrowed from there.

Theorem 1. Let $\mathcal{M}$ be an orientable, connected manifold of dimension $d \geq 3$. Let $A$ be an abelian group. Then we have:

1. The stable degree $\delta(H^k(\text{PConf}_\bullet(\mathcal{M}); A))$ is $\leq k$.
2. The local degree $h^\text{max}(H^k(\text{PConf}_\bullet(\mathcal{M}); A))$ is $\leq \max\left\{-1, \frac{2d}{d-1}k - 4\right\}$.
3. The generation degree $t_0(H^k(\text{PConf}_\bullet(\mathcal{M}); A))$ is $\leq \max\left\{k, \frac{3d-1}{d-1}k - 3\right\}$.
4. The presentation degree $t_1(H^k(\text{PConf}_\bullet(\mathcal{M}); A))$ is $\leq \max\left\{k, \frac{5d-1}{d-1}k - 6\right\}$.

For the class of manifolds in the hypothesis of Theorem 1, instead of the coefficients $\frac{2d}{d-1}, \frac{3d-1}{d-1}, \frac{5d-1}{d-1}$ that we have in the items (2),(3),(4), Church–Miller–Nagpal–Reinhold [CMNR17, Theorem 4.3] has the coefficients 4, 5, 9, respectively. The item (1) is included with no change for completeness.

A sequence of non-negatively graded cochain complexes $\{V_r\}_{r=1}^\infty$ in an abelian category is called a (cohomological) single-graded spectral sequence if $H^k(V_r^\bullet) \cong V_r^k$ for all $k \geq 0$ and $r \geq 1$. If the differentials $V_{r-1}^k \to V_r^k \to V_{r+1}^k$ are zero when $r \geq r_0$, this implies $V_{r_0}^k = V_{r_0+1}^k = \cdots$, and we write $V_r^\infty$ for this common value. We say the single-graded spectral sequence $\{V_r^\bullet\}$ converges to the graded object $M^\bullet$ if $M^\bullet$ has a filtration whose associated graded is $V_r^\bullet$.

The following lemma keeps consistent indexing with the proof of Church–Ellenberg–Farb [CEF15, Lemma 6.3.2].

Lemma 2. Let $\{E_r^{p,*}\}$ be a cohomological first quadrant spectral sequence bigraded in the standard way which converges to $M^\bullet$. If there exists $D \geq 2$ such that $E_2^{0,*} = 0$ unless $* = qD$, then setting

$$V_r^k := \bigoplus_{p+qD=k} E_r^{p,qD}$$
defines a single-graded spectral sequence \( \{V^\bullet_r\}^\infty_{r=1} \) which converges to \( M^\ast \). In addition, \( V^k_r = V^k_{[k/D]+1} \) for every \( k \).

**Proof.** The vanishing assumption yields that the only nontrivial differentials occur on \((rD + 1)\)-th pages, with \( E_{(r-1)D+2} = \cdots = E_{rD} = E_{rD+1} \). Thus we do have a single-graded spectral sequence \( \{V^\bullet_r\} \) as described.

For the last claim, first note that the differentials on the \((rD + 1)\)-th page of \( \{E^p, \ast\} \) are of the form
\[
E^p_{rD+1} \rightarrow E^p_{rD+1} \rightarrow E^p_{rD+1}.
\]
Observe that whenever \( p + qD = k \) (with \( p, q \geq 0 \)) and \( r \geq \lceil k/D \rceil + 1 \),

\begin{itemize}
  \item \( p - rD - 1 < p - k - 1 < 0 \),
  \item \( (q - r)D = qD - rD < qD - k \leq 0 \).
\end{itemize}

Therefore \( E^p_{rD+1} = E^p_{\infty} \) whenever \( p + qD = k \) and \( r \geq \lceil k/D \rceil + 1 \). \( \square \)

**Proposition 3** ([CMNR17, Proposition 4.1]). Let \( \{V^\bullet_r\} \) be a single-graded spectral sequence of \( \text{FI} \)-modules presented in finite degrees, such that for each \( k \) the \( \text{FI} \)-module \( V^k_1 \) is semi-induced and generated in degree at \( 0 \). Then we have
\[
\begin{align*}
1. & \quad \delta(V^k_\infty) \leq D_k, \\
2. & \quad h_{\max}(V^k_\infty) \leq \max(\{2D_\ell - 2 : \ell \leq k + r - 2\} \cup \{-1\}),
\end{align*}
\]
for every \( r \).

**Corollary 4.** Let \( \{E^p, \ast\} \) be a cohomological first quadrant spectral sequence of \( \text{FI} \)-modules converging to \( M^\ast \), such that \( E^p_{2, \ast} = 0 \) unless \( * = qD \) for some \( D \geq 2 \). Suppose in addition that for all \( p, q \), the \( \text{FI} \)-module \( E^p_{2, qD} = E^p_{2D+1} \) is semi-induced and generated in degree at most \( D_k \) whenever \( p + qD = k \). Then we have
\[
\begin{align*}
1. & \quad \delta(M^k) \leq D_k, \\
2. & \quad h_{\max}(M^k) \leq \max(\{2D_\ell - 2 : \ell \leq k + k/D - 1\}).
\end{align*}
\]

**Proof.** By the first part of Lemma 2, setting
\[
V^k_r := \bigoplus_{p+qD=k} E^p_{rD+1}
\]
yields single-graded spectral sequence \( \{V^\bullet_r\}_{r=1}^\infty \) of \( \text{FI} \)-modules that converges to \( M^\ast \). It follows from the hypotheses that \( V^k_r \) is semi-induced and generated in degree at most \( D_k \). Thus, by the second part of Lemma 2 and Proposition 3, we get
\[
\begin{align*}
1. & \quad \delta(V^k_\infty) \leq D_k, \\
2. & \quad h_{\max}(V^k_\infty) \leq \max(\{2D_\ell - 2 : \ell \leq k + k/D - 1\} \cup \{-1\}).
\end{align*}
\]
The analogous claims for \( M^k \) follow from [CMNR17, Proposition 3.2]. \( \square \)

**Proof of Theorem 1.** By Totaro’s and Church’s [Chu12] work (see [CEF15, Section 6.3] for details), there is a first quadrant spectral sequence \( E^p, \ast \) of \( \text{FI} \)-modules converging to
$H^*(\text{PConf}_\bullet(\mathcal{M}); A))$ that satisfies the hypotheses of Corollary 4 with $D = d - 1$ and $D_k = k$. Thus (1) and (2) follow from Corollary 4. And then (3) and (4) follow from [CMNR17, Proposition 3.1].

References

[CEF15] Thomas Church, Jordan S. Ellenberg, and Benson Farb, *FI-modules and stability for representations of symmetric groups*, Duke Mathematical Journal **164** (2015), no. 9, 1833–1910.

[Chu12] Thomas Church, *Homological stability for configuration spaces of manifolds*, Inventiones Mathematicae **188** (2012), no. 2, 465–504.

[CMNR17] Thomas Church, Jeremy Miller, Rohit Nagpal, and Jens Reinhold, *Linear and quadratic ranges in representation stability*, 2017, arXiv:1706.03845.

[Tot96] Burt Totaro, *Configuration spaces of algebraic varieties*, Topology **35** (1996), no. 4, 1057–1067.