Relativistic invariance of Lyapunov exponents in bounded and unbounded systems

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Abstract

The study of chaos in relativistic systems has been hampered by the observer dependence of Lyapunov exponents (LEs) and of conditions, such as orbit boundedness, invoked in the interpretation of LEs as indicators of chaos. Here we establish a general framework that overcomes both difficulties and apply the resulting approach to address three fundamental questions: how LEs transform under Lorentz and Rindler transformations and under transformations to uniformly rotating frames. The answers to the first and third questions show that inertial and uniformly rotating observers agree on a characterization of chaos based on LEs. The second question, on the other hand, is an ill-posed problem due to the event horizons inherent to uniformly accelerated observers.

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The quest for an observer-independent characterization of chaos in relativistic systems [1] has been an intense area of research and promises to provide significant new insights into the properties of chaotic dynamics [2]. An important recent result [3] concerns the transformation of Lyapunov exponents (LEs) under spacetime diffeomorphisms. We recall that the dynamics of a bounded solution \( X(t) \) of a dynamical system
\[
\frac{dX}{dt} = F(X)
\]
is chaotic if it presents sensitive dependence on initial conditions [4]. The associated LEs [5] are given by
\[
\lambda_i = \limsup_{t \to \infty} \frac{1}{t} \log \| \phi_i(t) \|,
\]
where \( \phi_i(t) \) are solutions of the linearized equation
\[
\frac{d}{dt} \phi_i = [D_X F(X(t))] \phi_i.
\]
Positive LEs are related to exponential divergence of initially close trajectories and, consequently, to chaotic dynamics. For space diffeomorphisms \( X = \Psi(Y) \), the invariance of the LEs is well established under rather general conditions (see, e.g., [6, 7]). In contrast, for well-behaved \emph{spacetime} diffeomorphisms involving time changes of the form \( d\tau = \Lambda(X) dt \), it has been shown [3] that the LEs transform according to
\[
\lambda_{\tau i} = \lambda_i / \langle \Lambda \rangle_t,
\]
where \( 0 < \langle \Lambda \rangle_t < \infty \) is the time average of \( \Lambda \) along the corresponding trajectory. Therefore, although the values of the LEs are themselves non-invariant, their signs are preserved and assure an invariant criterion for chaos under spacetime transformations. This result was obtained under conditions for which LEs are known to be valid quantifiers of chaos, of which the most limiting ones are the assumptions that the system has a natural invariant probability measure and the orbits are bounded both before and after the transformation.

In this Letter, we extend this result to an important class of transformations that do not preserve the boundedness of the orbits, and address fundamental questions on relativistic chaotic dynamics that require explicit in-depth investigation due to their outstanding physical properties and the violation of conditions invoked in the derivation of Eq. (2). The first question is how the LEs transform under Lorentz transformations. This question determines whether all inertial observers agree on a LE-based characterization of chaos. We show that the answer is affirmative despite the fact that the dynamics becomes unbounded with respect to at least one of the reference frames. We use this example to establish an \emph{extended boundedness} condition for the definition of the LEs as indicators of chaos, which is formulated relative to the trajectories themselves rather than a fixed point of the phase space. The second question is how the LEs behave under Rindler transformations, a question equivalent to ask whether uniformly accelerated observers agree on an
inertial characterization of chaos based on LEs. We show that this question is ill-posed because uniformly accelerated observers do not have access to the late-time dynamics. The latter relates to the fact that chaos and LEs are asymptotic concepts whose definitions involve a limit $t \to \infty$.

We also consider transformations to uniformly rotating frames, and show that the positivity of the LEs remains invariant under such transformations.

Our principal result stems from this analysis and can be stated for any system and any spacetime diffeomorphic transformation, as follows. For the system written in autonomous form, the LEs transform according to Eq. (2) and remain invariant indicators of chaos if, as shown below, (i) our extended boundedness condition is satisfied, (ii) the Jacobian of the transformation is bounded, and (iii) $\Lambda$ is positive for all $t$ and $0 < \langle \Lambda \rangle_t < \infty$. These conditions depend not only on the transformation properties of the dynamical variables $X$ and the change of reference frames but also on the choice of spacetime coordinates. They are automatically satisfied for global nonsingular transformations of bounded orbits for which $\inf \Lambda^{\pm 1} > 0$ whether the system is conservative, dissipative, mechanical, chemical, thermodynamical, electromagnetic, or fluid dynamical. These conditions clarify previous results that seem to challenge the invariance of chaos for relativistic observers, and show that LEs lead to invariant conclusions about chaos.

We first note that under a space diffeomorphism $X = \Psi(Y)$, system (1) is mapped into $\tfrac{d}{dt} Y = [D_Y \Psi(Y)]^{-1} F(\Psi(Y))$, rendering the solutions of the new linearized dynamics to be related to those of (1) as $\varphi_i(t) = [D_Y \Psi(Y(t))] \tilde{\varphi}_i(t) \|$. Hence, the corresponding LEs satisfy

$$\liminf_{t \to \infty} \frac{1}{t} \log \frac{\| [D_Y \Psi(Y(t))] \tilde{\varphi}_i(t) \|}{\| \tilde{\varphi}_i(t) \|} \leq \lambda_i - \tilde{\lambda}_i \leq \limsup_{t \to \infty} \frac{1}{t} \log \frac{\| [D_Y \Psi(Y(t))] \tilde{\varphi}_i(t) \|}{\| \tilde{\varphi}_i(t) \|}. \quad (3)$$

Suppose the solutions $X(t)$ are limited to a compact subset of the space. Since the diffemorphism maps bounded solutions $X(t)$ into bounded solutions $Y(t)$, the matrix $D_Y \Psi(Y(t))$ is nonsingular and, besides, there are time-independent finite nonzero constants $L^\pm = \sup \| [D_Y \Psi(Y(t))]^{\pm 1} \|$ leading to

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{1}{L^\pm} \leq \lambda_i - \tilde{\lambda}_i \leq \lim_{t \to \infty} \frac{1}{t} \log L^+, \quad (4)$$

which imply $\tilde{\lambda}_i = \lambda_i \|$. This argument explores the boundedness of $X(t)$ and $Y(t)$ to ensure the existence of the constants $L^\pm$. Below we extend (4) and establish Eq. (2) for an important class of unbounded orbits.

We now consider transformations of reference frame in which (1) describes a bounded autonomous system with respect to the initial (inertial) observers. More general transformations can be obtained by a composition of such transformations. We start with single-particle systems.
While general relativity allows arbitrary spacetime coordinates, and conditions (i-iii) can be applied to any of them, we will assume that the dynamics is described in terms of physical times (i.e., the time measured by observers at rest in the reference frame at the corresponding space coordinates).

**Lorentz transformations.** We first focus on the case in which function $F$ depends only on the configuration-space coordinates, such as in the evolution of a fluid element determined by a stream function, and consider a Lorentz boost with velocity $v$ along the $x$-direction, $(ct, x, y, z) \rightarrow (ct', x', y', z') = \Psi^{-1}(ct, x, y, z)$, where

$$\Psi^{-1}(ct, x, y, z) = (\gamma(ct - vx/c), \gamma(x - vt), y, z)$$

for $\gamma = 1/\sqrt{1-(v/c)^2}$. We focus on the space spanned by the coordinates $(ct, x, y, z) \equiv (ct, x)$, where we have enlarged the configuration space in order to incorporate $ct$ as a new coordinate. The extended version of (1) then reads

$$\frac{d}{dt} \begin{pmatrix} w \\ x \end{pmatrix} = \begin{pmatrix} c \\ F(x) \end{pmatrix},$$

where $dw/dt \equiv d(ct)/dt$.

The main advantage of this formulation is that the transformed system remains autonomous and the spacetime transformation can be reduced to an ordinary space diffeomorphism; it can be split as $T \circ S(ct, x)$, where $S$ is a transformation $(w', x') = \Psi^{-1}(w, x)$ that preserves the independent variable and $T$ is a time redefinition $dt' = \Lambda(w, x)dt$. (Another advantage is that the analysis extends immediately to $F$ with explicit time-periodic dependence.) The solutions of (6) are unbounded along the $w$-direction, but this is not a problem since the nonzero LEs of system (6) are identical to those of (1).

There is a caveat, however: the spatial boundedness of the solutions is not preserved under Lorentz transformations. A trajectory confined to a bounded space-like region (sup $||x(t)|| < \infty$) of the first reference frame is seen as spatially unbounded from the other inertial reference frame. Similar problem is observed even for Galilean transformations, but in classical dynamics one can adopt a reference frame where the solutions are bounded. In relativistic dynamics such a choice would raise questions about the invariance of the LEs, which is precisely the object of this Letter.

To proceed we first make the crucial observation that the study of chaos can be extended to this class of spatially unbounded orbits, even though the same does not hold true for unbounded
systems in general. Indeed, sensitive dependence on initial conditions and LEs depend exclusively on the relative time evolution between nearby trajectories; their dependence on the reference frame is limited to the definition of the spacetime coordinates used to measure the distances between the neighboring trajectories as they evolve over identical time intervals. Therefore, chaos can be properly defined and LEs can be used as indicators of chaos on an unbounded trajectory \( y(t) \) insofar as \( ||y(t) - \hat{y}(t)|| \) remains uniformly upper bounded for all \( t \) and all trajectories \( \hat{y}(t) \) with initial conditions in a neighborhood of \( y(0) \). That is, our condition is that the evolution of a small ball of points will remain bounded with respect to the local observers at position \( y(t) \), regardless of whether it remains bounded with respect to a fixed point of the reference frame. We refer to this as the extended boundedness condition. Note that this condition is satisfied for \( y(t) \) interpreted as the extended coordinates \( (w'(t), x'(t)) \) after the transformation \( S \) whenever the original system (11) is spatially bounded.

Having shown that LEs remain valid indicators of chaos despite the spatial unboundedness of the transformed orbits, we now turn to the effect of the Lorentz transformations on the LEs. For the transformation \( T \), from Eq. (5) we have

\[
\frac{dt'}{dt} = \gamma \left( 1 - \frac{v}{c^2} \frac{F_x(x(t))}{F_x(x(t))} \right) dt \equiv \Lambda(x(t))dt,
\]

where \( F_x(x) \) stands for the \( x \)-component of \( F(x) \). For \( |F_x(x(t))| \leq c \), implying \( \inf \Lambda(x(t)) > 0 \) in the present case, we have \( 0 < \langle \Lambda \rangle_t = \lim_{t \to \infty} \frac{\mu(t)}{t} = \lim_{t \to \infty} \frac{1}{t} \int_0^t \Lambda(x(p))dp < \infty \) [10]. This allows us to factor the LEs transformed by \( T \circ S \) as

\[
\tilde{\lambda}'_i = \tilde{\lambda}_i / \langle \Lambda \rangle_t,
\]

where \( \langle \Lambda \rangle_t \) is the contribution due to \( T \) and \( \tilde{\lambda}_i \) corresponds to \( \lambda_i \) transformed by \( S \). The problem is thus reduced to the transformation of the LEs under the spatial transformation \( S \). The nonsingular nature of (5) assures the existence of the constants \( L^\pm \) necessary to establish the bounds in (4) because, irrespective of the spatial unboundedness, the Jacobian matrix of the transformation is bounded. Employing the Euclidean norm to the matrix \( D_y \Psi(y) \) of (5), we obtain \( L^+ = L^- = \sqrt{(c + |v|)/(c - |v|)} \), leading to \( \tilde{\lambda}_i = \lambda_i \). In particular, all positive LEs remain positive under this transformation.

Combined with Eq. (8), this results in \( \tilde{\lambda}'_i = \lambda_i / \langle \Lambda \rangle_t \), which is precisely the transformation (2) previously established for the case of bounded orbits [3]. Our result does not agree with the result presented in [9] for averages over local LEs [11], but that is because that study was restricted
to time dilatations and length contractions, which correspond to the transformation of dynamical
variables such as volume (or the reciprocal of a density) for the time measured at a fixed point of
the reference frame, whereas our analysis describes single-particle dynamics for the time measured
at the position of the particle.

If system (1) involves the evolution of velocities, as expected for a particle in a 3D potential,
the Lorentz transformation (5) must be extended to include the transformation of
\( u = \frac{dx}{dt} \)
into \( u' = \frac{dx'}{dt'} \), which is given by \( u'_x = \eta(u_x - v), u'_y = \eta^{-1}u_y, \) and \( u'_z = \eta^{-1}u_z \), where \( \eta = 1/(1 - u_x v/c^2) \). The resulting transformation \((w, x, u) \rightarrow (w', x', u')\) satisfies the extended boundedness condition and has constants \( 0 < L^\pm < \infty \), as long as \(|v| < c \) and \(|u_x(t)| \leq c\). This ensures that the LEs of systems obtained by order reduction of second-order differential
equations, which are the most common in particle dynamics, will be transformed as in (2) under
Lorentz transformations.

**Rindler transformations.** With respect to an inertial reference frame, an observer with con-
stant proper acceleration \( a \) along the \( x \) direction has a hyperbolic worldline given by
\[
ct(\tau) = \frac{c^2}{a} \sinh \frac{a\tau}{c}, \quad x(\tau) = \frac{c^2}{a} \cosh \frac{a\tau}{c},
\]
where \( \tau \) stands for the observer’s proper time. The corresponding Rindler transformation [12] is
deﬁned by \((ct, x, y, z) \rightarrow (c\tau(t, x), \xi(t, x), y, z)\), with
\[
ct(\tau, \xi) = c\sqrt{\frac{2\xi}{a}} \sinh \frac{a\tau}{c}, \quad x(\tau, \xi) = c\sqrt{\frac{2\xi}{a}} \cosh \frac{a\tau}{c},
\]
and positive \( \xi \) (see Fig. 1). The observer on hyperbola (9) is in the Rindler reference frame at rest
at \( \xi = c^2/2a \). In contrast with the Lorentz case, the Rindler transformations are non-linear in \( x \)
and \( ct \).

Focusing on the space deﬁned by the extended conﬁguration-space coordinates, the matrix
\( D_y \Psi(y) \) and its inverse for the Rindler transformation of (6) have unit determinants but their
largest eigenvalues diverge as \( c/\sqrt{2a\xi} \cosh a\tau/c \) for \( \xi \rightarrow 0 \). Therefore, one cannot identify fi-
nite constants \( L^\pm \) that could be used to compare \( \tilde{\lambda}'_i \) and \( \lambda'_i \). This behavior can be interpreted
in terms of our extended boundedness condition, which is not satisﬁed in this case because
\((c\tau(t, x(t)), \xi(t, x(t)), y(t), z(t))\) diverges at the light cone and is undeﬁned beyond it. Moreover,
from the inverse of Eqs. (10), we have
\[
d\tau = \frac{c^2}{a} \left[ \frac{x(t) - tF_x(x(t))}{x(t)^2 - (ct)^2} \right] dt \equiv \Lambda(ct, x(t)) dt,
\]
FIG. 1: Accessibility to the dynamics is observer dependent. The lines of fixed $\xi$ in Rindler coordinates correspond to hyperbolic trajectories in the coordinates $(ct, x)$ of inertial observers. The straight dotted lines are the lines of constant time $\tau$. The uniformly accelerated observers are unaware of all events occurring in regions II and III of the original Minkowski spacetime. They only access the dynamics of a trajectory $\Gamma$ during the time the trajectory crosses region I [13]. If the trajectory is spatially bounded with respect to the original observers, as assumed for system (1), this corresponds to an infinite time interval $\Delta \tau$ but only to a finite time interval $\Delta t$.

where $\Lambda(ct, x(t))$ diverges when the original solution $(ct, x(t))$ crosses the light cone $x^2 = c^2 t^2$. The same holds true for the physical time $dt' = \sqrt{2a\xi/c^2} d\tau$. The average $\langle \Lambda \rangle_t$ is not well defined and, as a result, the Rindler transformed system does not have a natural probability measure against which the LEs could be calculated [3]. Therefore, the question of how the LEs transform under Rindler transformations is ill-posed.

The real origin of the problem is the horizon structure (and its counterpart structure for $t \to -t$) inherent to uniformly accelerated observers [12]. The Rindler transformation (10) is not a global spacetime diffeomorphism since it maps only one quarter of the Minkowski spacetime, as shown in Fig. 1. Any event located above the component of the light cone corresponding to the bisectrix in the first and third quadrants of Minkowski spacetime will never reach the accelerated observers. While singularities can be an artifact of the coordinates, event horizons are an attribute of the reference frame. The existence of an event horizon prevents the observers from having access to the asymptotic dynamics of the original system. Therefore, without having access to the complete...
dynamics, the Rindler observers cannot formulate a criterion for chaotic behavior—based on the
observation of individual trajectories—that is valid for the original system \[13\]. It is interesting
to notice that such a problem, related to the global structure of the spacetime, manifests itself as a
violation of our conditions for the transformation of LEs.

If one insists on computing the LEs from a uniformly accelerated referential frame \[9\], one must
note that the late-time dynamics of the extremely dilated time \(τ \to ∞\) does not correspond to the
real late-time dynamics of the original system since the interval \(-∞ < τ < ∞\) is the mapping
of a finite time interval \(Δt\). Therefore even if one could compute \(\tilde{λ}_τ^{i}\) as seen from the accelerated
frame, this would be, in fact, a problem different from the originally proposed one. This situation
is analogous to the limits imposed by the cosmological singularity to the determination of chaos
in FRW cosmologies \[8\] and is also predicted for Rindler transformations of any other dynamical
system and for any choice of coordinates.

**Rotating frames.** The crucial role played by the event horizon in the Rindler case can be better
appreciated if one considers a physical situation involving a non-linear transformation that does not
introduce event horizons. This is precisely the case of uniformly rotating reference frames \[14\]:

\[
\begin{align*}
    r' &= r, \quad θ' = θ + Ωt, \quad z' = z, \quad \text{and} \quad cd t' &= \left[ g(r) + Ω^2 r^2/g(r) \right] dt + \left[ Ω^2 r^2/g(r) \right] dθ,
    \end{align*}
\]

where \(g(r) = \sqrt{c^2 − Ω^2 r^2}\), \(Ω\) is a constant, and \(t'\) is the physical time in the rotating frame \[15\]. This leads to

\[
dt' = \left( \frac{g(r(t))}{c} + \frac{Ωr^2(t)[Ω + F_θ(\mathbf{x}(t))]}{cg(r(t))} \right) dt, \quad (12)
\]

where \(F_θ(\mathbf{x}) = dθ/dt\). The transformation of the LEs of \(6\) is in this case well defined since
the extended boundedness condition is satisfied for orbits in closed sets of the physical region
\(|Ω|r < c\) for which \(-Ωr^2 F_θ(\mathbf{x}) < c^2\), where both the function \(Λ(\mathbf{x})\) and the constants \(L^±\) are
upper and lower bounded away from zero. The latter follows from the fact that the entries of the
Jacobian matrix \(Dy Ψ(y)\) and its inverse for the transformation \((ct, r, θ, z) \to (ct', r', θ', z')\) are
all continuous for \(Ωr < c\). A subtlety in this calculation is that in rotating frames the differential
d\(t'\) of the physical time is not exact and cannot be integrated globally, meaning that the Jacobian
elements involving derivatives of \(ct'\) must be determined from \(cdt'\) in the immediate neighborhood
of a given \(r\). The transformation \(t \to t'\) is defined locally but it can always be extended along any
trajectory with initial condition in that neighborhood. Therefore, the LEs transform as predicted
by \(2\) also for the case of rotating frames.

**Generalization and discussion.** Our derivation of Eq. \(8\) also demonstrates that conditions
\((i-iii)\) are sufficient (and usually necessary) for the validity of \(2\) in general. Indeed, while we
considered specific transformations and specific classes of dynamical systems in our explicit examples, these three conditions are precisely the checkpoints we have to verify for any system and any transformation. The extended boundedness condition—satisfied both before and after the transformation in the extended space, which includes \( ct \) as an additional coordinate—guarantees that the system can be kept autonomous and that LEs remain valid indicators of chaos. The condition that the Jacobian is bounded—in the sense of having positive finite constants \( L^\pm \) for the transformation in the extended space—ensures the validity of the identity \( \hat{\lambda}_i^t = \lambda_i^t \). Finally, \( \Lambda \) and \( \langle \Lambda \rangle_t \) positive and finite—again, in the extended space—guarantees that the time transformation is well defined and the signs of the LEs are conserved; it also guarantees that the time transformation is invertible, a condition we saw violated for the Rindler transformation.

These conditions are readily applicable to any system and any change of reference frame and coordinates. The latter includes the choice of the time parameter or of the observers in the reference frame with respect to which the time is measured. In the examples above, the dynamical system describes the dynamics of a single particle, the dynamical variables represents the coordinates and possibly velocities of the particle, and the time was assumed to be recorded locally—each time by the observer in the reference frame that is at the point where the particle is. However, other choices are equally valid. For a many-particle system under Lorentz transformation, for example, the time could be measured, e.g., with respect to the position of one of the particles, \( dt' = \gamma \left( 1 - \frac{v}{c} F_x(x(t)) \right) dt \), with respect to the center of mass, \( dt' = \gamma \left( 1 - \frac{v}{c} \sum_i \frac{m_i}{m} F_x(x(t)) \right) dt \), or with respect to a fixed point, \( dt' = \gamma dt \). Moreover, the dynamical system can describe physical, chemical or biological activity whose dynamical variables do not necessarily correspond to coordinates and velocities in the physical space. In this general case the system can be written as \( \frac{d}{dt} X_i = F_i(X_1, \ldots, X_n) \), \( i = 1, \ldots, n \), and the transformation is locally defined as \( (cdt, dX_1, \ldots, dX_n) \rightarrow (cdt', dX'_1, \ldots, dX'_n) \). The latter is determined by the change of reference frame and spacetime coordinates, \( (cdt, dx) \rightarrow (cdt', dx') \), and depends on the nature of the dynamical variables, i.e., whether they transform as scalars, vectors, tensors, or in a different way. The choice of observers in the new reference frame is always accounted for through the choice of \( \frac{d}{dt} x \) in the transformation formula \( dt' = \left( \frac{\partial}{\partial t} t'(ct, x) + \nabla_x t'(ct, x) \cdot \frac{d}{dt} x \right) dt \), where this term vanishes only if the time is measured (remotely) by a fixed observer.

The results presented in this Letter address all these cases and show that, if conditions (i-iii) are verified, the signs of the LEs remain valid invariant indicators of chaos. Since we have extended the use of the LEs as a valid measure of chaos to include unbounded orbits, this conclusion is general:
it applies to both inertial and non-inertial reference frames and does not involve the identification of privileged observers. These results account for properties inherent to relativistic observers, such as event horizon and spatial unboundedness, significantly extending our understanding of the relativistic invariance of LEs and chaos.

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