The Mirror Map for Invertible LG Models

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ABSTRACT

Calculating the (a,c) ring of the maximal phase orbifold for ‘invertible’ Landau–Ginzburg models, we show that the Berglund–Hübsch construction works for all potentials of the relevant type. The map that sends a monomial in the original model to a twisted state in the orbifold representation of the mirror is constructed explicitly. Via this map, the OP selection rules of the chiral ring exactly correspond to the twist selection rules for the orbifold. This shows that we indeed arrive at the correct point in moduli space, and that the mirror map can be extended to arbitrary orbifolds, including non-abelian twists and discrete torsion, by modding out the appropriate quantum symmetries.
1 Introduction

At the level of the effective 4-dimensional field theory of a compactified heterotic string, mirror symmetry \[1, 2, 3, 4\] – a redefinition of the left-moving \(U(1)\) charge of the internal \(N = 2\) superconformal field theory – boils down to charge conjugation. Nevertheless, this transformation has attracted considerable interest, because it allows for the computation of Yukawa couplings \[5\] that are otherwise inaccessible in the Calabi–Yau compactification scheme. From this geometrical point of view, mirror symmetry, which now changes the topology of a manifold, is much less obvious. And, in addition to its computational use, it may serve as an indicator for non-completeness of classes of compactifications.

Much of what we said about Calabi–Yau manifolds is also true for Landau–Ginzburg models \[6\], which provide sort of a bridge between the geometrical framework and superconformal field theory \[7\]. Here an \(N = 2\) theory is described by a quasi-homogeneous superpotential, which, according to non-renormalization theorems, contains the complete information about the chiral ring \[2\]. In order to use such \(N = 2\) theories for constructing supersymmetric string vacua we first have to project to integral (left) charges \(J_0\) \[8\]. This modding by \(j = \exp 2\pi i J_0\) leads to new states from twisted sectors, some of which may end up in the \((a,c)\) ring, i.e. the ring of primary fields whose left-moving components are anti-chiral and thus have negative left charge. All of these rings are connected among each other and with the Ramond ground states by spectral flow \[2\].

Of course we can twist the model by more general symmetries of the potential, and it is only if the resulting orbifold has integral left and right charges that we may have a geometrical interpretation, with the chiral ring corresponding to the cohomology ring of a Kähler manifold. The resulting class of (2,2) models lacks mirror symmetry in a significant way \[9, 10\], but a large subclass of the potentials, which we call ‘invertible’, leads to perfectly symmetric spectra. These potentials are defined by the property that the number of monomials is equal to the number of fields. Non-degeneracy then implies \[11\] that all these monomials are either of Fermat type \(X_i^{a_i}\), or of the form \(X_i^{a_i}X_j\). Invertible potentials thus consist of connected closed loops of ‘pointers’ from \(X_i\) to \(X_j\), or of chains that terminate with a Fermat type monomial,

\[
W_{\text{loop}} = X_1^{a_1} X_2 + \ldots + X_{n-1}^{a_{n-1}} X_n + X_n^{a_n} X_1, \\
W_{\text{chain}} = X_1^{a_1} X_2 + \ldots + X_{n-1}^{a_{n-1}} X_n + X_n^{a_n}.
\]

This implies a natural notion of ‘transposition’ by inverting the direction of the ‘pointers’ while keeping the exponents \(a_i\) attached to the fields \(X_i\), or the other way round \[12\].

It is exactly for this class of models that a construction of the mirror as a particular orbifold of the ‘transposed’ potential has been suggested some time ago by Berglund and Hübsch (BH) on the basis of systematic observation \[12\]. In that paper the correct twist group was found by demanding that the so called geometrical and quantum symmetries \[13\] be exchanged when one goes to the mirror. More recently, the BH construction has found an interpretation and a generalization for Calabi–Yau hypersurfaces of toric varieties, which are described by certain families of rational convex cones \[14\]. In a recent paper \[15\], the matching of the elliptic genera of the proposed mirror pairs has been checked. Assuming that the charges of all states that contribute to this index are anti-symmetric for the orbifold partner, this implies that at least the charge degeneracies are indeed correct.
A general proof for this construction in the Landau–Ginzburg framework, and an understanding of the mechanism by which it works, however, is still missing. In the present note we try to fill this gap. Our approach is basically along the lines of the first proof of mirror symmetry by Greene and Plesser [3], who used the results of ref. [17] on modular invariants of parafermionic theories to obtain a construction for tensor products of minimal models, i.e. the case of Fermat type potentials. All we have to do is to compute the charge degeneracies of the \((a,c)\) ring for the maximal phase orbifold of an arbitrary connected component of an invertible potential. To this end we use the formulas for Landau–Ginzburg orbifolds that were derived by Intriligator and Vafa [17].

A necessary condition for arriving at the mirror of the original \(N = 2\) theory is that all \((c,c)\) states are projected out. It is therefore not surprising that the construction works just for the invertible potentials, which have the minimal number of monomials and thus a maximal symmetry. We will find that the states in the \((a,c)\) ring turn out correctly and we will explicitly construct the map from monomials to twisted states for the two cases of chains and loops. The consistency of this map with the original ring structure and the twist selection rules will then establish that we indeed arrive at the correct point in moduli space of the correct conformal field theory.

This result immediately implies that the construction can be extended to arbitrary orbifolds of tensor products of such models, including non-abelian twists and discrete torsion\(^1\). All we need to do is to mod out the corresponding symmetries of the mirror partners. In the orbifold representation, the phase part of these symmetries is realized as a quantum symmetry, which can be implemented by introducing appropriate discrete torsions with transformations that may act trivially otherwise (effectively, this often just undoes part of the orbifolding). Permutation symmetries are, of course, ‘geometrical’ in both cases.

Assuming that the BH construction gives the mirror at the correct point in moduli space, it is not only natural to expect that it should also apply to the (left-right symmetric) original Landau–Ginzburg model, but it even has to be so. The reason is that, as discussed above, we can undo the projection to integral charges, and we can do the analogous thing for the mirror partner, which is an isomorphic conformal field theory by assumption. But then also the mirror theory has to factorize, and we conclude that the construction should work for each connected component of the potential individually.

As a final, technical point, note that we can do the calculation either in the Ramond or in the Neveu-Schwarz sector. Once we have established that the charges are anti-symmetric in the proposed mirror model, we can get the other sector by spectral flow. The Ramond ground states may seem to be the easier choice, because then we have a single formula for all states and need not worry about two different rings. Nevertheless, we prefer the NS sector, because the ring structure and the selection rules that we expect will help us find the mirror map more easily. Furthermore, the identification of twisted states of the maximal orbifold with monomials in a different LG model (with asymmetric charges) may provide new information about the \((a,c)\) ring of arbitrary phase orbifolds.

In the next section we will recall some results of ref. [17], which provide the basis for our computation of the \((a,c)\) rings for \(W_{\text{loop}}/G\) and \(W_{\text{chain}}/G\) in sections 3 and 4, respectively. In the final section we summarize our results and discuss some implications.

\(^1\)For abelian orbifolds of minimal models this was verified in ref. [18]
2 Orbifold setup: the (a,c) ring

In their semiclassical analysis Vafa and Intriligator \cite{8,17} derived all ingredients that are necessary for calculating the charges and transformation properties of the Ramond ground states and of their (anti-)chiral relatives for arbitrary Landau–Ginzburg orbifolds. In this section we briefly recall from their results what we will need in the following.

As we decided to work in the NS sector, we first need to make sure that no states survive the projection in the (c,c) ring. Setting the discrete torsions $\varepsilon(g, h)$ and the factor $(-1)^{K_g}$, which determines the sign of the group action in the Ramond sector, all equal to 1, it can be shown that the $h$-twisted vacuum $|h\rangle_{(c,c)}$ transforms with a phase $(\det g_{kh})/(\det g)$ under a generator $g$ of the twist group. Here $\det g_{kh}$ is the determinant of $g$ when restricted to the superfields $X_i$ that are invariant under the twist $h$. These fields are also the ones that generate the chiral ring of the unprojected ring in the sector $h$. These fields are also the ones that generate the chiral ring of the unprojected $h$-twisted sector, with the vanishing relations derived from their effective Landau–Ginzburg potential. We thus have to make sure that no monomial in the resulting chiral ring transforms with a phase that compensates the above phase factor, thereby forming an invariant state. Let us anticipate that this is indeed the case, which is a simple consequence of the results that we will derive below.

The situation in the (a,c) ring is more complicated. In ref. \cite{8} it has been observed that the asymmetric spectral flow operator $U_{(-1,0)}$ is identical to the field that introduces a $j$-twist. Therefore

$$|h\rangle_{(a,c)} = U_{(-1,0)}|h\rangle_{(c,c)} = U_{(-\frac{1}{2}, \frac{1}{2})}|h\rangle_R \quad \text{with} \quad h' = hj^{-1},$$

and the unprojected ring in the sector $h$ is generated by the fields that are invariant under $h'$. The spectral flow operator $U_{(-1,0)}$ shifts the left charge by $-c/3$, where $c = 3 \sum_i (1 - 2q_i)$ is the central charge. Let $0 < \theta_i^h < 1$ be the phases of the fields $X_i$ that are not invariant under the (diagonal) action of a symmetry transformation $h$, i.e. $hX_i = \exp(2\pi i \theta_i^h)X_i$. Then the left and right charges $Q_+$ and $Q_-$ of the vacuum in the $h$-twisted sector of the (a,c) ring are given by

$$Q_{\pm}|h\rangle_{(a,c)} = \left(\pm \frac{c}{6} \pm \sum_{tw} (\theta_i^h - \frac{1}{2}) - \sum_{inv} \left(\frac{1}{2} - q_i\right)\right)|h\rangle_{(a,c)},$$

where the second and the third term are the contributions of the fields that are twisted and invariant under $h'$, respectively. The first term is the charge of $U_{(-1/2,1/2)}$ and thus shifts the result for the Ramond ground states to the (a,c) ring. As an example, it is easy to check that the vacuum $|0\rangle_{(a,c)} = |0\rangle$ has vanishing charges: All fields are twisted, and $\theta_i^h = 1 - q_i$. In the sector twisted by $j$, on the other hand, all fields are invariant and we find $Q_+ = -\frac{c}{3}$ and $Q_- = 0$.

In the following we will always be interested in the situation that the states in the (a,c) sector should have anti-symmetric charges. To check this, we will first have to show that the only monomials surviving the projection are the ones that have charge $Q_{\pm} = \sum_{inv}(\frac{1}{2} - q_i)$. This will imply that $J_0 = -\bar{J}_0$ in the (a,c) ring. Using $c/6 = (\sum_{tw} + \sum_{inv})(\frac{1}{2} - q_i)$, we can simplify the formula (4) for the right charge $Q_-$ and obtain

$$\bar{J}_0|h\rangle_{(a,c)} = \sum_{tw} (1 - q_i - \theta_i^h)|h\rangle_{(a,c)}.$$  \hspace{1cm} (5)

For the projection to invariant states we need, as our final ingredient, the action of an arbitrary
group element $g$ on the twisted vacuum in the sector $h$,

$$g|_h(a, c) = (-1)^{K_g K_h} \varepsilon(g, h) \det g_{i, i'}|_h(a, c),$$

(6)

which was derived in [17] using the modular invariance of the index $\text{tr}_R(-1)^F$ and spectral flow. As long as we do not consider general orbifolds and the modding of quantum symmetries, we can set $\varepsilon(g, h) = (-1)^{K_g} = 1$.

3 Loop potentials

Because of its more symmetric form we first construct the mirror map for the potential

$$W_{\text{loop}} = \sum_{i=1}^{n} X_i^{a_i} X_{i+1},$$

(7)

where all indices are defined modulo $n$. Quasi-homogeneity implies $a_i q_i + q_{i+1} = 1$ for the $U(1)$ charges $q_i$ of the $N = 2$ superfields $X_i$ (we restrict the exponents to $a_i > 1$ so that $0 < q_i < \frac{1}{2}$). For any phase symmetry $\rho$, acting as

$$\rho X_j = e^{2\pi i \varphi_j} X_j,$$

(8)

the phase $\varphi_j$ of the field $X_j$ determines the phase $\varphi_{j+1} \equiv -a_j \varphi_j$ modulo 1. Therefore all diagonal symmetry groups of the potential are cyclic. Of course the charges $q_i$ solve these equations, but in general the corresponding symmetry, generated by $j = \exp(2\pi i J_0)$, is only a subgroup of the maximal phase symmetry.

To obtain a generator $\rho_i$ of the maximal group $G$ with order $\Gamma = |G|$, we choose the phase to be minimal, i.e. $\pm 1/\Gamma$, for some field $X_i$. It will be useful to choose the sign as $(-1)^n$, because then the phase of the determinant of $\rho_i$ is negative (see below). In this way we obtain a family of generators $\rho_i$ with corresponding phases $\varphi_j^{(i)}$ given by

$$\rho_i X_j = e^{2\pi i \varphi_j^{(i)}} X_j, \quad \varphi_{i+j}^{(i)} = \frac{(-1)^{n-j} a_i \ldots a_{i+j-1}}{\Gamma} \quad \text{for} \quad 0 \leq j < n$$

(9)

(recall that indices are identified modulo $n$). If we set $j = n$ in this formula we should get a phase that is $(-1)^n/\Gamma$ modulo 1, thus we obtain

$$\Gamma = A - (-1)^n, \quad A = a_1 a_2 \ldots a_n.$$

(10)

This is close to, but not exactly the dimension of the chiral ring

$$|\mathcal{R}| = \prod \frac{1 - q_i}{q_i} = A,$$

(11)

where we have used $a_i = (1 - q_{i+1})/q_i$. We will see later on how the counting of states work out correctly, in spite of this apparent mismatch. A convenient basis for the chiral ring is given by the monomials $X_i^{\alpha_i}$ with $0 \leq \alpha_i < a_i$. This set has the correct number of elements, and it is easy to see that, using $\partial W/\partial X_i = 0$, all non-vanishing monomials can be brought into this form.
Since all $\rho_i$ generate the same group, they must be powers of one another, and we find

$$\rho_i = (\rho_{i+1})^{-a_i}, \quad \varphi_i^{(i)} + a_i\varphi_i^{(i+1)} = -\delta_i^i. \quad (12)$$

Summing over $l$ we see that the quantities

$$\bar{q}_i = -\sum_{l=1}^n \varphi_i^{(i)} = -\frac{1}{2\pi i} \ln \det \rho_i \quad (13)$$

satisfy the relations $\bar{q}_i + a_i\bar{q}_{i+1} = 1$ and thus they are the weights of the chiral fields $X_i$ in the $N = 2$ theory with the ‘transposed’ potential

$$W_{\text{loop}} = X_1^aX_2^{a_1} + \ldots + X_{n-1}^aX_n^{a_{n-1}} + X_nX_1^{a_n}. \quad (14)$$

This is our first hint that, as is well-known [12], the orbifold $W/G$ should be compared to the model described by the transposed potential $W$.

It is now easy to see how $j$ is related to the generators $\rho_i$. Here we use the relation

$$\varphi_i^{(i)} + a_i\varphi_i^{(i+1)} = -\delta_i^{i+1}. \quad (15)$$

Summing over all $i$ we find

$$q_i = -\sum_{i=1}^n \varphi_i^{(i)}, \quad j^{-1} = \rho_1 \ldots \rho_n. \quad (16)$$

Without calculation this could have been concluded also from the fact that the phases $\varphi_i^{(i)}$ of the phase symmetries $\bar{\rho}_i$, which act on the transposed potential, coincide with $\varphi_i^{(i)}$.

With these formulas it is now easy to construct the mirror map explicitly. The natural candidate for the image of $X_i$ is the twisted ground state $|\rho_i\rangle_{(a,c)}$. It is obvious from (13) that its right charge (3) coincides with $\bar{q}_i$ up to an integer. Exact equality then follows from the inequalities $q_j - 1 \leq \varphi_j^{(i)} \leq q_j$, which will be derived in the next section: All contribution to the right charge are of the form $1 - q_i - \theta_1^{i+1} = 1 - q_i - (1 + \varphi_j^{(i)} - q_i) = -\varphi_j^{(i)}$, because $\varphi_i^{(i)} = q_i$ is always between $-1$ and 0. Invariance of these states under the group $G$ is obvious from eq. (3) because all fields are twisted.

From the ring structure associated to $W$ we thus conclude that a monomial $\prod X_i^{\alpha_i}$ should be mapped into a sector twisted by $h = \prod \rho_i^{\alpha_i}$, where $0 \leq \alpha_i < a_i-1$. To see what twists we get in this way, it is sufficient to calculate the phase of $X_1$, which is given by $\sum \alpha_i\varphi_i^{(i)} = n/\Gamma$ for some integer $n$. It is important to know this number exactly, and not just modulo $\Gamma$. Inserting the above formulas for $\varphi_i^{(i)}$ and $q_1$ we see that $n$ takes all values between $n_-$ and $n_+$ exactly once, where $n_- = \Gamma q_1 = n_- + \Gamma$ if $n$ is even, and $n_+ + 1 = \Gamma q_1 = n_+ + \Gamma - 1$ if $n$ is odd. If $n$ is even we thus get the twist $h = j$ twice, whereas for $n$ odd this twist does not occur at all in the image of the mirror map. All other twists arise exactly once. Since $X_1$ is in no way distinguished, it follows that $q_i - 1 \leq \varphi_i \leq q_i$, where the particular representation of a group element $\rho$ in terms of certain powers of the generators $\rho_i$ is, of course, crucial.

To see whether this map is one-to-one we now have to look at what happens to the sector twisted by $j$. Using the generator $\rho_1$ of $G$ for the projection, the phase $-\bar{q}_1$ of $q_1$ will
have to be compensated by the phase of a monomial $\prod X_i^{a_i}$ if a state in that sector should survive. Fortunately, we can determine the possible phases without further calculations if we use the correspondence to the transposed potential. Observing that $\varphi_{i}^{(1)} = \tilde{\varphi}_{i}^{(1)}$, the above result for the phase $n/\Gamma$ of the field $X_1$ under a general twist, when applied to the potential $\overline{W}$, now shows that there are two/zero monomials that transform with a phase $(\det \rho_1)^{-1}$ if $n$ is even/odd. It is also easy to see directly that, for even $n$, the two monomials $\prod X_i^{a_i-1}$ and $\prod X_i^{a_1-1}$ have that property. In particular, both monomials have the same charge $c/6$, and therefore the charges of the resulting $(a,c)$ states are indeed asymmetric and of the correct size. It is not clear to me, however, how to distinguish between these two monomials, as they have the same transformation properties under all symmetries. We only know that, for even $n$, the two states that they generate in the sector $j$ must be the mirror partners of the two states $\prod X_i^{a_i-1} |0\rangle$ and $\prod X_i^{a_1-1} |0\rangle$.\footnote{Consideration of selection rules in appropriate orbifolds of the mirror pair might settle this question.}

As a by-product of our results we now see that the $(c,c)$ ring is indeed empty, except for the identity, which is the only invariant monomial. It only remains to check that the charges of all states are correct, not only up to an integer. But this follows like before from the inequality $-1 \leq (\sum_j \bar{\alpha}_j \varphi_{i}^{(j)} - q_i) \leq 0$. It is also easy to check that, up to normalization of the twist fields, the orbifold selection rules are consistent with the ring relations $\partial \overline{W}/X_i = \overline{X}_i^{a_i} + a_{i-1} \overline{X}_{i-1} \overline{X}^{a_{i-1}} = 0$. The presence of the quantum symmetry, which is isomorphic to $G$ and acts with the correct phases, then implies that we indeed have constructed the mirror of the chiral ring at the correct point in moduli space.

## 4 Chain potentials

We now use our experience to find the mirror map for the second type of invertible potential,

$$W_{\text{chain}} = X_1^{a_1} X_2 + \ldots + X_n^{a_n-1} X_n + X_n^{a_n}. \quad (17)$$

Defining $q_{n+1} = 0$ we have $a_i q_i + q_{i+1} = 1$, with the solution

$$q_i = \sum_{j=1}^{n} \frac{(-1)^{i-j}}{a_i \ldots a_j}, \quad \Gamma = |G| = a_1 \ldots a_n. \quad (18)$$

This time $j$ generates the whole group $G$, but it is again useful to define a dependent set of group elements which have smaller phases and whose determinants are related to the weights of the transposed potential. So we define the transformations $\rho_l X_i = \exp(2\pi i \varphi_i^{(l)})$ with

$$\varphi_i^{(l)} = \frac{(-1)^{l-i+1}}{a_i \ldots a_l} \quad \text{for } 1 \leq i \leq l \quad (19)$$

and $\varphi_{l+1}^{(l)} = \ldots = \varphi_n^{(l)} = 0$. Of course only $\rho_n$ generates the complete group $G$, and we find

$$(\rho_l)^{q_l} \rho_{l-1} = 1, \quad \bar{q}_l \equiv -\sum_{j=1}^{l} \varphi_j^{(l)} = -\frac{\ln \det \rho_l}{2\pi i}. \quad (20)$$
As the quantities $\tilde{q}_i$, defined in this way, obey $a_i \tilde{q}_i + \tilde{q}_{i-1} = 1$ with $\tilde{q}_0 = 0$, they coincide with the charges corresponding to the transposed potential

\[ W = X_1^{a_1} + X_1 X_2^{a_2} + \ldots + X_{n-1} X_n^{a_n}. \]  

(21)

Furthermore, $q_j = -\sum_{i=j}^n \varphi_j^{(i)}$, implying that $j^{-1}$ is again the product of all $\rho_i$. The twisted ground state $|j^{-2}\rangle_{(a,c)}$ is the unique state with $Q_R = c/3 = -Q_L$.

From the generating relations $\partial W/\partial X_i = 0$ for the chiral ring it is easy to see that $X_n^{a_n} = X_1^{a_1} X_{i+1} = 0$ and that we may choose $\alpha_i < a_i$ for all monomials. Because of the relations (20) between the transformations $\rho_i$, whose orders are $a_1 \ldots a_i$, we may represent any twist $\rho \in \mathcal{G}$ uniquely in the form $\rho = \prod \rho_i^{\alpha_i}$ with $0 \leq \alpha_i < a_i$. A simple calculation shows that the corresponding phases $\varphi_i = \sum_{l=i}^n \alpha_l \varphi_i^{(l)}$ obey $-1 \leq (\varphi_i - q_i) \leq 0$, where the upper/lower bound is reached iff $n - i$ is odd/even.

So far the situation is very similar to the previous case. For the present type of potentials, however, the dimension of the chiral ring is $|\mathcal{R}| = (1 - q_i) \Gamma$. This number is in general different from

\[ |\mathcal{R}| = (1 - \tilde{q}_n) \Gamma = \sum_{i=0}^n (-1)^i a_1 \ldots a_{n-1}, \]  

(22)

which is the number of states that we should identify in the orbifold $W/\mathcal{G}$. In contrast to the previous case, these dimensions are always smaller than the order $\Gamma$ of the phase symmetry. Several of the allowed phases thus cannot be realized in the chiral ring, and, in turn, several twisted sectors will not contribute any invariant state. A possible basis for the chiral ring derived from $W_{\text{chain}}$ has been described recursively in ref. [19]. It consists of all monomials $\prod X_i^{\alpha_i}$ with $\alpha_1 \leq a_1 - 2$ and $\alpha_i \leq a_i - 1$ for $2 \leq i \leq n$ or with $\alpha_1 = a_1 - 1$, $a_2 = 0$ and all other exponents fulfilling the same relation for the potential $W$ with the first two fields set to 0. From the analysis of the mirror map we will now recover the same basis, which, as we will see, is the unique choice with $\alpha_i < a_i$.

Because of the formula for the determinant of $\rho_i$, it is not difficult to guess that the correct mirror map should send $\prod X_i^{\alpha_i}$ into the sector twisted by $\prod \rho_i^{\alpha_i}$. This covers the group $\mathcal{G}$ exactly once if let $0 \leq \bar{\alpha}_i < a_i$. We therefore only have to check that exactly one element of the $(a,c)$ ring survives in the correct sectors, and that it has the correct charges.

The chain-like structure of our potential implies that $X_{i+1}$ is invariant under a twist whenever $X_i$ is invariant. Let, therefore, $\{X_s, \ldots, X_n\}$ be the sets of fields invariant under $h' = h/j$. $X_s$ transforms with a phase

\[ \varphi_s' = -\frac{\bar{\alpha}_s + 1}{a_s} + \frac{\bar{\alpha}_{s+1} + 1}{a_s a_{s+1}} - \ldots - \frac{(-1)^{n-s} \bar{\alpha}_n + 1}{a_s \ldots a_n}, \]  

(23)

which should be integer. Together with $0 \leq \bar{\alpha}_j < a_j$ this implies that $\bar{\alpha}_{n-2i} = a_{n-2i} - 1$ and $\bar{\alpha}_{n-2i+1} = 0$ for all $\bar{\alpha}_j$ with $j \geq s$, and that $\bar{\alpha}_{j-1}$ must not follow that pattern. So we have to consider states of the form

\[ X_s^{\alpha_s} \ldots X_n^{\alpha_n} \rho_1^{\bar{\alpha}_1} \ldots \rho_{s-1}^{\bar{\alpha}_{s-1}} \ldots \rho_{n-2}^{\bar{\alpha}_{n-2}} \rho_n^{\bar{\alpha}_n - 1}. \]  

(24)

The determinant of the action of $g = \rho_n$ on the invariant fields is given by

\[ \frac{1}{2\pi i} \ln \det g_{h'} = -\frac{1}{a_n} + \frac{1}{a_n a_{n-1}} - \ldots - \frac{(-1)^{n-s}}{a_s \ldots a_n}. \]  

(25)
which must be compensated modulo 1 by the phase $\sum_{j \geq s} \alpha_j \phi_j^{(n)}$ of the monomial under the transformation $g$. As before, we conclude that $\alpha_{s+2i} = a_{s+2i} - 1$ and $\alpha_{s+2i+1} = 0$ for $i \geq 0$. If, however, $n - s$ is even, then the monomial $X_s^{a_s-1}X_{s+2}^{a_{s+2}-1}\ldots X_n^{a_n-1}$ vanishes in the effective Landau–Ginzburg theory with potential $W_{1'} = X_s^{a_s}X_{s+1} + \ldots + X_n^{a_n}$.

We conclude that the projected $(a,c)$ ring consists of all states of the form

$$X_s^{a_s-1}X_{s+2}^{a_{s+2}-1}\ldots X_n^{a_n-1}|\rho_1^{\bar{a}_1}\ldots \rho_{s-1}^{\bar{a}_{s-1}}\rho_{s+1}^{a_{s+1}-1}\rho_{s+3}^{a_{s+3}-1}\ldots \rho_n^{a_n-1},$$

with $n - s$ odd and $0 \leq \bar{a}_j < a_j - \delta_{j,s-1}$ for $1 \leq j < s$. Their charges are indeed anti-symmetric, because $X_i^{a_i}X_{i+1}$ is homogeneous of degree 1, so that the charge of $X_i^{a_i-1}$ is $(a_i - 1)q_i = 1 - q_i - q_{i+1}$, in agreement with the criterion given in the paragraph of eq. (5). To check that the value of the right charge is correct, we observe that the contribution $\sum_{\text{inv}}(\frac{1}{2} - q_i)$ of the monomial exactly matches the ‘missing’ contribution $\sum_{\text{inv}}(1 - q_i - \theta_i^{h_i})$ in eq. (5), because the value $\theta_i^{h_i}$ for an untwisted fields $X_i$ is $-1$ if $n - i$ is even and 0 if $n - i$ odd. This implies that the eigenvalue of $\tilde{J}_0$ is $\sum \bar{a}_i \bar{q}_i$ for all invariant states, regardless of the number of untwisted fields.

For the surviving set of exponents we obtain the following description: Choose some number $1 \leq s \leq n + 1$ with $n - s$ odd. Then let $0 \leq \bar{a}_j < a_j - 1$ for $j \leq s - 2$, $\bar{a}_{s+2i} = 0$ and $\bar{a}_{s+2i+1} = a_{s+2i+1} - 1$ for $i \geq 0$, and $0 \leq \bar{a}_{s-1} < a_{s-1} - 2$. The corresponding basis is indeed identical to the one given in [13]. It is also straightforward to check that the defining relations of the chiral ring are consistent with the twist selection rules, which completes our proof for the second type of polynomials.

## 5 Summary and discussion

We gave a proof for the BH construction of the mirror by explicitly constructing the mirror map for invertible potentials. Technically, the transposition can be traced to the fact that the twist group can be generated in terms of symmetry transformations $\rho_i$, the determinants of which are related to the weights of the transposed potential by $\bar{q}_i = \frac{1}{2\pi} \ln \det \rho_i$. Consistency of the operator products with the twist selection rules then implies that the mirror map sends a monomial $\prod \bar{X}_i^{\bar{a}_i}$ into a sector twisted by $\prod \rho_i^{\bar{a}_i}$.

A necessary condition for the construction to work is that the projection makes the $(c,c)$ ring trivial. This, in particular, implies that all moduli are fixed by discrete symmetries. The same should be true for the mirror model, so that the presence of the quantum symmetry, together with the matching of the charge degeneracies and the selection rules, provide strong evidence that the conformal field theories are indeed isomorphic. As a simple consequence, the BH construction can then be extended to more general orbifolds.

A remarkable feature of our models is that, except for loops with an even number of fields, not all representations of the twist group are present in the chiral ring. In turn, some of the twisted sectors do not contain any invariant anti-chiral states. For the remaining sectors, however, we have an identification between twist fields and certain monomials in the dual theory with flipped left charge. Using the information on operator products that is encoded in the dual ring relations, it should be possible to extract non-trivial information on Yukawa couplings, thereby extending the results of ref. [11].
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