COMPUTATIONAL PERFORMANCE STUDIES FOR SPACE-TIME PHASE-FIELD FRACTURE OPTIMAL CONTROL PROBLEMS

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Abstract. The purpose of this work are computational demonstrations for a newly developed space-time phase-field fracture optimal control framework. The optimal control solution algorithm is a Newton algorithm, which is obtained with the reduced approach by eliminating the state constraint. Due to the crack irreversibility constraint, a rate-independent problem arises, which is treated by penalization and for which we utilize a space-time approach. Therein, we deal with the state, adjoint, tangent, and adjoint Hessian equations. Our fully discretized space-time optimization algorithm is presented and extensively tested with various numerical experiments.

1. Introduction

In this paper, we present algorithms and then focus on performance studies for extensive testing of the newly developed space-time phase-field fracture optimal control setting that we have introduced in [26]. The temporal discretization (incremental loading) and the spatial discretization are both based on the Galerkin finite element method.

Phase-field fracture propagation in a variational formulation was established in [20] [10]. The interpretation as a phase-field method was given in [30] [31]. Overview articles and monographs include [11] [12] [17] [43] [19] [17]. It is well-known that the efficient and robust numerical solution of the nonlinear and linear subproblems in phase-field fracture is challenging. This is mainly due to the nonlinear structure of the coupled problem and the interaction of model, discretization and material parameters. In spite of the development of robust preconditioning and parallel, scalable, iterative algorithms, the forward solution remains costly in general for both two-dimensional and three-dimensional settings.

Due to these difficulties in the numerical solution there is much less published work on phase-field fracture as inner problem within parameter estimation, stochastic modeling, and optimal control. First results on mathematical and numerical analysis were recently established in [39] [40] [38] [23]. Computational studies were undertaken in parameter estimation [28] [48] [41] [42], stochastic modeling [21], topology optimization [16], and on optimal control in our prior work [27]. Solving phase-field fracture problems using methods from shape optimization was proposed in [1]. To the best of our knowledge a first design of a space-time framework was derived in [26] with some prototype numerical illustrations. However, computational studies for different configurations exploring the performance and capabilities remain open and constitute the principal goal of the current work.

The optimal control problem is formulated in terms of the reduced approach by eliminating the state variable with a control-to-state operator; see for instance [43] [24]. Therein, Newton’s
method requires the evaluation of the adjoint, tangent, and adjoint Hessian equations. The latter requires the evaluation of second-order derivatives; see, e.g., [7] for parabolic optimization problems. Specifically, in [26] we employ Galerkin formulations in time and discuss in detail how the crack irreversibility constraint is formulated using a penalization [37, 39] and an additional viscous regularization [40, 29]. Based on these settings, concrete time-stepping schemes are formulated. As usual, the primal and tangent problem run forward in time whereas the adjoint and adjoint Hessian equations run backward in time.

The main objective of this paper is, first, to formulate an overall algorithm for phase-field fracture optimal control utilizing the four space-time discretized equations derived in [26]. Second, we then perform extensive tests by means of six numerical experiments with different complexities. Therein, the overall goals are computational investigations of the performance of the reduced Newton algorithm (NLP), the linear conjugate gradient (CG) method, and convergence of the residuals, cost functionals, tracking parts, Tikhonov parts and optimal controls. We recall upfront that such investigations are even challenging for forward phase-field fracture problems due to the interaction of model, discretization and material parameters (see [10, 30] and closely related work on image segmentation [9], and the prior seminal work on Gamma convergence [3, 4, 13]), and possibly also penalization parameters for treating the crack irreversibility constraint [45]. All of them have an impact on mathematical well-posedness [11] (and references cited therein), and on numerical approximations and nonlinear and linear solution algorithms [11, 12, 45, 19, 17]. The parameters include: phase-field regularization \( \varepsilon \) and bulk regularization \( \kappa \), crack irreversibility penalization \( \gamma \) and viscous regularization \( \eta \), mesh size \( h \) and loading step size \( \Delta t \), critical energy release rate \( G_c \) and Lamé parameters \( \lambda \) and \( \mu \).

The extension to optimization adds further levels of complexity: the forward problem, with all its own challenges, must be solved numerous times, more parameters enter such as the Tikhonov regularization \( \alpha \), and in order to guarantee a well-posed optimization setting, the adjustment of \( \alpha \) is delicate for weighting the physical tracking functional against the Tikhonov regularization term. Having the derivations from [26] at hand, we are now in the position to investigate in detail such phase-field fracture optimal control problems for various situations. Our experiments below encompass propagating fractures, non-constant controls on one or more boundary sections, multiple (propagating) fractures, an adaptation of Winkler’s [46] L-shaped panel test, and using controls to prevent crack growth. These tests provide novel insight for both the capabilities of the phase-field method for fracture from a numerical viewpoint as well as for applications. On the other hand, limitations and opportunities for future work also become visible, such as the need to further improve the linear solver’s cost complexity (e.g., by parallel multigrid methods [25] and model order reduction [8]) as for fine meshes the forward solver becomes prohibitively expensive.

The outline of the paper is as follows. In Section 2 we present the weak formulation of our fracture model along with a standard Galerkin discretization in space and our discontinuous Galerkin time-stepping developed in [26]. Section 3 presents the tracking type control problem, its reduced form, and our time-stepping for the adjoint, tangent, and adjoint Hessian equations, resulting in our overall algorithm. The main contribution of this paper consists of six numerical experiments that we discuss in detail in Section 4. Finally Section 5 gives some conclusions and an outlook on future work.
2. Space-time phase-field fracture forward model

In this section, the weak formulation with our space-time Galerkin finite element discretization is presented.

2.1. Notation. We consider a bounded domain $\Omega \subset \mathbb{R}^2$. The boundary is partitioned as $\partial \Omega = \Gamma_N \cup \Gamma_D$ where both the Dirichlet boundary $\Gamma_D$ and the Neumann boundary $\Gamma_N$ have nonzero Hausdorff measure. Next we define two function spaces, $V := H^1_0(\Omega; \mathbb{R}^2) \times H^1(\Omega)$ for the displacement field $u$ and the phase-field $\varphi$, and $Q := L^2(\Gamma_N)$ for the control $q$, where

$$H^1(\Omega; \mathbb{R}^2) := \{ v \in L^2(\Omega; \mathbb{R}^2) : D^\alpha v \in L^2(\Omega; \mathbb{R}^2) \, \forall \alpha \in \mathbb{N}_0^2, \, |\alpha| \leq 1 \},$$

$$H^1_0(\Omega; \mathbb{R}^2) := \{ v \in H^1(\Omega; \mathbb{R}^2) : v|_{\Gamma_D} = 0 \}.$$  

Moreover we consider a bounded time interval $I = [0, T]$ and introduce the spaces

$$X := \{ u = (u, \varphi) : u \in L^2(I, V), \partial_t \varphi \in L^2(I, H^{-1}(\Omega)) \}, \quad W := C(I, Q).$$  

On $V$ respectively $X$ we define the scalar products

$$(u, v) := \int_{\Omega} u \cdot v \, dx \quad \forall u, v \in V,$$

$$(u, v)_I := \int_{I} \int_{\Omega} u \cdot v \, dx \, dt = \int_{I} (u(t), v(t)) \, dt \quad \forall u, v \in X,$$

with induced norms $\| \cdot \|$ and $\| \cdot \|_I$, and furthermore the restricted inner products

$$(u(t), v(t))_{\{ \partial_t \varphi > 0 \}, I} := \int_{I} (u(t), v(t))_{\{ \partial_t \varphi > 0 \}} \, dt \quad \forall u, v \in X,$$

with induced semi-norms $\| \cdot \|_{\{ \partial_t \varphi > 0 \}, I}$ and $\| \cdot \|_{\{ \partial_t \varphi > 0 \}, I}$. We also notice that we later work with $(\cdot, \cdot)_{\{ \varphi(t_i) > \varphi(t_j) \}}$, defined like $(\cdot, \cdot)_{\{ \partial_t \varphi > 0 \}, I}$, and with a semi-linear form $a(\cdot)(\cdot)$ in which the first argument is nonlinear and the second argument is linear.

2.2. Weak formulation. We deal with the following weak formulation: Given $u_0 \in V$ and $q \in W$, find $u \in X$ such that

$$(g(\varphi)C e(u), e(\Phi_u))_I = (g, \Phi_{u: \perp})_{\Gamma_N, I} = 0,$$

$$G_c\varepsilon(\nabla \varphi, \nabla \Phi_\varphi)_I - \frac{G_c}{\varepsilon}(1 - \varphi, \Phi_\varphi)_I + (1 - \kappa)(\varphi C e(u) : e(u), \Phi_\varphi)_I + \gamma(\partial_t \varphi, \Phi_\varphi)_{\{ \partial_t \varphi > 0 \}, I} + \eta(\partial_t \varphi, \Phi_\varphi)_{\{ \partial_t \varphi > 0 \}, I} = 0, \quad (1)$$

for every test function $\Phi = (\Phi_u, \Phi_\varphi) \in X$. Herein $\Phi_{u: \perp}$ denotes the component of $\Phi_u$ that is orthogonal to $\Gamma_N$. The critical energy release rate is denoted by $G_c > 0$. The so-called degradation function $g(\varphi) := (1 - \kappa)\varphi^2 + \kappa$ helps to extend the displacements to the entire domain $\Omega$. The bulk regularization parameter is $\kappa > 0$, the phase-field regularization parameter is $\varepsilon > 0$, the penalization parameter is $\gamma > 0$, and the viscosity parameter is $0 < \eta \ll \gamma$. Furthermore, $C$ denotes the elasticity tensor and $e(u)$ the symmetric gradient. Then, we have

$$C e(u) = \sigma(u) = 2\mu e(u) + \lambda \text{tr}(e(u))I,$$

where $\mu, \lambda > 0$ are the Lamé parameters and $I$ is the identity matrix.
2.3.1. Space-time finite element discretization.

2.3. Temporal discretization. Given $T > 0$, we define the time grid $0 = t_0 < \cdots < t_M = T$ to partition the interval $I$ into $M$ left-open subintervals $I_m = (t_{m-1}, t_m)$, $I = \{0\} \cup I_1 \cup \cdots \cup I_M$.

By using the discontinuous Galerkin method, here dG(0), we seek a solution $u$ in the space $X^0_k$ of piecewise polynomials of degree 0.

\[ X^0_k := \{ v \in X : v(0) \in V \text{ and } v|_{t_m} \in P_0(I_m, V), m = 1, \ldots, M \}. \]

Here, the subindex $k$ indicates the time-discretized function space in order to distinguish it from the continuous space $X$. For the jump terms arising in $X^0_k$ we use the standard notation

\[ v^+_m := v(t_{m+}), \quad v^-_m := v(t_{m-}) = v(t_m), \quad [v]_m := v^+_m - v^-_m. \]

Remark 2.1. The above weak formulation differs slightly from many other phase-field fracture formulations found in the literature since the crack irreversibility constraint $\partial_t \varphi \leq 0$ is kept on the time-continuous level in order to apply a Galerkin discretization in time.

Remark 2.2. Since we work with dG(0), i.e., piece-wise constant functions in time, we have

\[ \partial_t v = v^-_m - v^+_m = 0 \quad \forall v \in X^0_k \forall m = 1, \ldots, M. \]

The discretized state equation combines the two equations of \((3)\). For a concise formulation, the energy-related terms are expressed as a semi-linear form $a : Q \times V \times V \rightarrow \mathbb{R},$

\[ a(q, u)(\Phi) := g(\varphi) \cdot (\mathcal{C}e(u), e(\Phi_u)) \]

\[ + G_c\varepsilon(\nabla \varphi, \nabla \Phi_\varphi) - \frac{G_c}{\varepsilon} (1 - \varphi, \Phi_\varphi) \]

\[ + (1 - \kappa)(\varphi \cdot \mathcal{C}e(u) : e(u), \Phi_\varphi) - (q, \Phi_{u, \varphi})_{\Gamma_N}. \]

Now the fully discretized state equation determines a function $u \in X^0_k$ for a given initial value $u_0 = (u_0, \varphi_0) \in V$ and a given control $q \in W$ such that for every $\Phi \in X^0_k$

\[ 0 = \sum_{m=1}^{M} \left[ \gamma(\partial_t \varphi, \Phi_\varphi)\{\partial_t \varphi > 0\, t_m\} \right. + \eta(\partial_t \varphi, \Phi_\varphi)_{t_m} \right. \]

\[ + \sum_{m=0}^{M-1} \left[ \gamma([\varphi]_m, \Phi^+_m, \varphi^-_{m+1} > \varphi^-_m) \right. + \eta([\varphi]_m, \Phi^+_m, \varphi^-_{m+1}) \right. \]

\[ \left. + \sum_{m=1}^{M} a(q(t_m), u(t_m))(\Phi(t_m)) \Delta t_m \right. \]

\[ \left. + (u_0 - u_0, \Phi_{u, 0}) + (\varphi_0 - \varphi_0, \Phi_{\varphi, 0}) \right. \]

The time integral in \((3c)\) has been approximated by the right-sided box rule, where $\Delta t_m := t_m - t_{m-1}$. Discontinuities of the functions in $X^0_k$ are captured by the jump terms in \((3b)\) in the typical dG(0) manner. These jump terms can be rewritten as

\[ \sum_{m=1}^{M} \left[ \gamma(\varphi^+_m - \varphi^-_{m-1}, \Phi^+_m, \varphi^-_{m+1}) \right. + \eta(\varphi^+_m - \varphi^-_{m-1}, \Phi^+_m, \varphi^-_{m+1}) \right. \]

\[ + \gamma([\varphi]_m, \Phi^+_m, \varphi^-_{m+1} > \varphi^-_m) \right. + \eta([\varphi]_m, \Phi^+_m, \varphi^-_{m+1}) \right. \]

\[ \left. + \sum_{m=1}^{M} a(q(t_m), u(t_m))(\Phi(t_m)) \Delta t_m \right. \]

\[ \left. + (u_0 - u_0, \Phi_{u, 0}) + (\varphi_0 - \varphi_0, \Phi_{\varphi, 0}) \right. \]
Moreover, since we are employing a dG(0) scheme, our test functions satisfy
\[ \Phi_{m-1}^+ = \Phi_m^- \quad \forall m = 1, \ldots, M. \]

Thus (3a) vanishes entirely by Remark 2.2, and the two terms containing \( \varphi_{m-1}^+ \) in (4) become \( (\varphi_m^-, \Phi_{\varphi,m})_{\varphi_m^+ > \varphi_{m-1}^-} \) and \( (\varphi_m^-, \Phi_{\varphi,m}) \), respectively. Together with (3b) and (3d), the discrete state equation (3) is finally written as
\[
0 = \sum_{m=1}^{M} \left( \gamma \left[ (\varphi_m^-, \Phi_{\varphi,m})_{\varphi_m^+ > \varphi_{m-1}^-} \right] - (\varphi_{m-1}^-, \Phi_{\varphi,m})_{\varphi_{m-1}^- > \varphi_{m-1}^+} \right) \\
+ \eta \left[ (\varphi_m^-, \Phi_{\varphi,m}) - (\varphi_{m-1}^-, \Phi_{\varphi,m}) \right] \\
+ a(q(t_m), u(t_m))(\Phi(t_m))\Delta t_m \\
+ (u_0^- - u_0, \Phi_{u,0}^-) + (\varphi_0^- - \varphi_0, \Phi_{\varphi,0}^-) \forall \Phi \in X_k^0.
\]

To solve (5), we first obtain \( u_0^- = u(0) \) from the initial condition
\[ (u(0), \Phi_0^-) = (u_0, \Phi_0^-) \quad \forall \Phi_0^- \in V. \]

Then we compute \( u(t_m) \) for \( m = 1, \ldots, M \) from
\[
0 = \gamma(\varphi(t_m)), \Phi_{\varphi(t_m)})_{\varphi(t_m) > \varphi(t_{m-1})} + \eta(\varphi(t_m), \Phi_{\varphi(t_m)}) \\
- \gamma(\varphi(t_{m-1}), \Phi_{\varphi(t_{m-1})})_{\varphi(t_{m-1}) > \varphi(t_m)} - \eta(\varphi(t_{m-1}), \Phi_{\varphi(t_m)}) \\
+ a(q(t_m), u(t_m))(\Phi(t_m))\Delta t_m \forall \Phi \in X_k^0.
\]

2.3.2. Spatial discretization. For the spatial discretization, we employ again a Galerkin finite element scheme by introducing \( H^1 \) conforming discrete spaces. We consider two-dimensional shape-regular meshes with quadrilateral elements \( K \) forming the mesh \( T_h = \{ K \} \); see [15]. The spatial discretization parameter is the diameter \( h_K \) of the element \( K \). On the mesh \( T_h \) we construct a finite element space \( V_h := V_{uh} \times V_{\varphi h} \) as usual:
\[
V_{uh} := \{ v \in H^1_0(\Omega; \mathbb{R}^2) : v |_{K} \in Q_d(K) \text{ for } K \in T_h \}, \\
V_{\varphi h} := \{ v \in H^1(\Omega) : v |_{K} \in Q_s(K) \text{ for } K \in T_h \}.
\]

Herein \( Q_s(K) \) consists of shape functions that are obtained as bilinear transformations of functions defined on the master element \( K = (0,1)^2 \), where \( \tilde{Q}_s(K) \) is the space of tensor product polynomials up to degree \( s \) in dimension \( d \), defined as
\[ \tilde{Q}_s(K) := \text{span} \left\{ \prod_{i=1}^d x_i^{\alpha_i} : \alpha_i \in \{0,1,\ldots,s\} \right\}. \]

Specifically, for \( s = 1 \) and \( d = 2 \) we have
\[ \tilde{Q}_1(K) = \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1\hat{x}_2\}. \]

With these preparations, based on (2), we now design the fully discrete function space
\[ X_{kh}^0 := \{ v \in X : v_h(0) \in V_h \text{ and } v |_{I_m} \in P_0(I_m, V_h), \text{ } m = 1, \ldots, M \}. \]

The discrete control space \( Q_h \) is constructed like \( X_{kh}^0 \) using \( Q_1(K) \) (again \( s = 1 \)) elements, but restricted to the Neumann boundary \( \Gamma_N \). Then, the fully discrete system consists of the initial condition
\[ (u_h(0), \Phi_{h,0}^-) = (u_{h,0}, \Phi_{h,0}^-) \quad \forall \Phi_{h,0}^- \in V_h \]
and for $m = 1, \ldots, M$ of the local system

$$
0 = \gamma (\varphi_h(t_m), \Phi_{\varphi,h}(t_m)) (\varphi_h(t_m) > \varphi_h(t_{m-1})) + \eta(\varphi_h(t_m), \Phi_{\varphi,h}(t_m))
- \gamma (\varphi_h(t_{m-1}), \Phi_{\varphi,h}(t_m)) (\varphi_h(t_{m-1}) > \varphi_h(t_{m-2})) - \eta(\varphi_h(t_{m-1}), \Phi_{\varphi,h}(t_m))
+ a(q_h(t_m), u_h(t_m))(\Phi_h(t_m)) \Delta t_m \quad \forall \Phi_h \in X^0_{hk}.
$$

(9)

### 3. Optimization with phase-field fracture

In this section, we state the phase-field optimal control problem and introduce the reduced solution approach. Therein, the primal forward problem plus three additional equations must be solved. Their combination yields the final solution algorithm.

#### 3.1. Optimization problem

We consider a separable NLP (Non-Linear Program) with a cost functional of tracking type: For given $(u_0, \varphi_0) \in V$ we seek a solution $(q, u) \in W \times X$ of

$$
\min_{q, u} \ J(q, u) := \frac{1}{2} \sum_{m=1}^{M} \| \varphi(t_m) - \varphi_d(t_m) \|^2 + \frac{\alpha}{2} \sum_{m=1}^{M} \| q(t_m) - q_d(t_m) \|^2_X
$$

(10)

s.t. $(q, u)$ solves (8) and (7) for $m = 1, \ldots, M$,

where $\varphi_d \in L^n(\Omega)$ is some desired phase-field and $q_d$ is a suitable nominal control that we use for numerical stabilization. The second sum represents a common Tikhonov regularization with parameter $\alpha$. The existence of a global solution of (10) in $L^2(I, Q) \times X$ has been shown in [39, Theorem 4.3] for functions that are non-negative and weakly semi-continuous.

**Remark 3.1.** The fully discrete version of (10) is obtained by working with the equations (6) and (5). In what follows, in order to keep the notation comfortable, we omit the index $h$ indicating the spatial discretization.

#### 3.2. Reduced optimization problem

In order to handle (10) by the reduced approach, we assume that a solution operator $S: W \rightarrow X$ exists for the PDE (1). The cost functional $J(q, u)$ then reduces to $j: W \rightarrow \mathbb{R}$, $j(q) := J(q, S(q))$, and we replace (10) by the unconstrained optimization problem

$$
\min_{q} \ j(q).
$$

(11)

To solve $j'(q) = 0$ by Newton’s method, we compute representations of $j'$ and $j''$ using the established approach in [7]. It requires the solution of four equations (given below) for the Lagrangian $L: W \times X^0_h \times X^0_h \rightarrow \mathbb{R}$ which is defined within the dG(0) setting as

$$
L(q, u, z) := J(q, u) - \gamma (\partial_t \varphi, z_\varphi)_{\partial_0 \varphi > 1} - \eta(\partial_t \varphi, z_\varphi)
- \int_I a(q(t), u(t))(z(t)) \, dt
- \eta_0(u(0) - u_0, z_u(0) - \varphi(0) - \varphi_0, z_u(0)).
$$

3.3. State, adjoint, tangent, adjoint Hessian

In this section we state the four equations to be solved for computing $j'$ and $j''$.

1. **State equation:** given $q \in W$, find $u = S(q) \in X$ such that the PDE (1) holds:

$$
L'_u(q, u, z)(\Phi) = 0 \quad \forall \Phi \in X.
$$

(12)

2. **Adjoint equation:** given $q \in W$ and $u = S(q)$, find $z \in X$ such that

$$
L'_z(q, u, z)(\Phi) = 0 \quad \forall \Phi \in X.
$$

(13)
(3) **Tangent equation**: given \( q \in W, \ u = S(q) \), and a direction \( \delta q \in W \), find \( \delta u \in X \) such that
\[
L''_{qq}(q, u, z)(\delta q, \Phi) + L''_{uq}(q, u, z)(\delta u, \Phi) = 0 \quad \forall \Phi \in X. \tag{13}
\]

(4) **Adjoint Hessian equation**: given \( q \in W, \ z \in X \) from \( \ref{12} \), a direction \( \delta q \in W \), and \( \delta u \in X \) from \( \ref{13} \), find \( \delta z \in X \) such that
\[
L''_{zu}(q, u, z)(\delta q, \Phi) + L''_{uuz}(q, u, z)(\delta u, \Phi) + L''_{zzu}(q, u, z)(\delta z, \Phi) = 0 \quad \forall \Phi \in X. \tag{14}
\]

Solving these equations in a specific order (see for instance \( \ref{7, 33} \)) leads to the following representations of the derivatives that we need for Newton’s method:
\[
\begin{align*}
\gamma'&(q)(\delta q) = L'_{q}(q, u, z)(\delta q) \quad \forall \delta q \in W, \\
\gamma''&(q)(\delta q_1, \delta q_2) = L''_{qq}(q, u, z)(\delta q_1, \delta q_2) + L''_{uq}(q, u, z)(\delta u, \delta q_2) + L''_{zu}(q, u, z)(\delta z, \delta q_2) \quad \forall \delta q_1, \delta q_2 \in W.
\end{align*}
\]

### 3.4. Time-stepping schemes of the additional equations

In our prior work \( \ref{26} \), the main purpose was the rigorous derivation of time-stepping schemes utilized for the implementation. The resulting schemes are now listed for a self-contained presentation and understanding of the final algorithm. The adjoint time-stepping scheme starts by computing \( z(t_M) \) from
\[
\begin{align*}
a'_u(q(t_M), u(t_M))(\Phi(t_M), z(t_M))\Delta t_M + \gamma(\Phi_{-\varphi, m}, z_{-\varphi, m}) & \{\varphi_m > \varphi_{m-1}\} \nonumber \\
+ \eta(\Phi_{-\varphi, m}, z_{-\varphi, m}) & = J'_u(q(t_M), u(t_M))(\Phi(t_M)) \quad \forall \Phi \in X^0_k.
\end{align*}
\]

Then, we compute \( z(t_{M-1}), \ldots, z(t_1) \) backwards in time from the equation
\[
\begin{align*}
a'_u(q(t_m), u(t_m))(\Phi(t_m), z(t_m))\Delta t_m + \gamma(\Phi_{-\varphi, m}, z_{-\varphi, m}) & \{\varphi_m > \varphi_{m-1}\} \nonumber \\
+ \eta(\Phi_{-\varphi, m}, z_{-\varphi, m}) & = J'_u(q(t_m), u(t_m))(\Phi(t_m)) \quad \forall \Phi \in X^0_k.
\end{align*}
\]

Finally, \( z(t_0) \) is obtained from
\[
\gamma(\Phi_{-\varphi, 0}, z_{-\varphi, 0}) - \gamma(\Phi_{-\varphi, 1}, z_{-\varphi, 1}) + \eta(\Phi_{-\varphi, 0}, z_{-\varphi, 0}) - \eta(\Phi_{-\varphi, 1}, z_{-\varphi, 1}) = 0. \tag{17}
\]

The tangent time-stepping starts by computing \( \delta u(t_0) \) from
\[
(\delta u(t_0), \Phi_0) = 0 \quad \forall \Phi_0 \in V. \tag{18}
\]

Applying the \( X^0_k \) (respectively \( X^0_{hk} \)) property to \( \delta \varphi_{m-1}^+ \), we then obtain \( \delta u(t_1), \ldots, \delta u(t_M) \) from the equation
\[
\begin{align*}
\gamma(\delta \varphi_{-\varphi, m}, \Phi_{-\varphi, m}) & \{\varphi_m > \varphi_{m-1}\} + \eta(\delta \varphi_{-\varphi, m}, \Phi_{-\varphi, m}) \\
+ a'_u(q(t_m), u(t_m))(\delta u(t_m), \Phi(t_m))\Delta t_m & = (\delta \varphi_{m-1}, \Phi_{-\varphi, m}) + (\delta \varphi_{m-1}, \Phi_{-\varphi, m}) \{\varphi_m > \varphi_{m-1}\} \\
- a'_u(q(t_m), u(t_m))(\delta q(t_m), \Phi(t_m))\Delta t_m & = \forall \Phi \in X^0_k.
\end{align*}
\]
The time-stepping scheme for the adjoint Hessian equation is similar to the one for the adjoint equation, since both have to be solved backwards in time. First we compute \( \delta z(t_M) \) from
\[
0 = J''_{uu}(q(t_M), u(t_M))(\delta u(t_M), \Phi(t_M))
- a'_{uu}(q(t_M), u(t_M))(\Phi(t_M), \delta z(t_M)) \Delta t_M
- a''_{uu}(q(t_M), u(t_M))(\delta u(t_M), \Phi(t_M), z(t_M)) \Delta t_M
- \gamma(\Phi_{\varphi,M}, \delta z_{\varphi,M})\{\varphi_{\varphi,M} > \varphi_{\varphi,M-1}\} - \eta(\Phi_{\varphi,M}, \delta z_{\varphi,M}) \forall \Phi \in X^0_k.
\]

Then we compute \( \delta z(t_{M-1}), \ldots, \delta z(t_1) \) by solving
\[
0 = J''_{uu}(q(t_m), u(t_m))(\delta u(t_m), \Phi(t_m))
- a'_{uu}(q(t_m), u(t_m))(\Phi(t_m), \delta z(t_m)) \Delta t_m
- a''_{uu}(q(t_m), u(t_m))(\delta u(t_m), \Phi(t_m), z(t_m)) \Delta t_m
+ \gamma(\Phi_{\varphi,m}, \delta z_{\varphi,m}^+ - \delta z_{\varphi,m}^-)\{\varphi_{\varphi,m} > \varphi_{\varphi,m-1}\}
+ \eta(\Phi_{\varphi,m}, \delta z_{\varphi,m}^+ - \delta z_{\varphi,m}^-) \forall \Phi \in X^0_k.
\]

Finally we obtain \( \delta z(t_0) \) from the initial condition
\[
\gamma(\Phi_{\varphi,0}, \delta z_{\varphi,0}^+ - \delta z_{\varphi,0}^-)\{\varphi_{\varphi,0} > \varphi_{\varphi,0}\} + \eta(\Phi_{\varphi,0}, \delta z_{\varphi,0}^+ - \delta z_{\varphi,0}^-) - \eta_0(\Phi_{\varphi,0}, \delta z_{\varphi,0}^-).
\]

3.5. Final complete algorithm. Gathering the optimization problem statement and the space-time discretizations from the previous sections and resulting time-stepping schemes for the four equations yields the complete method given in Algorithm 1.

4. Numerical studies

In this section we present six numerical experiments. In these experiments we use the tracking type functional of [10] to find an optimal control force that approximately produces a desired phase-field. All numerical computations are performed with the open source software libraries deal.II [5] [6] and DOPeLIB [13] [22].

4.1. Experiment 1: horizontal fracture in right half domain. The first experiment is motivated by a standard problem: the single edge notched tension test [36] [35]. Here we consider the square domain \( \Omega = (0,1)^2 \) with a horizontal notch, see Fig. 1. The notch is in the middle of the right side of the domain, defined as \( (0.5, 1) \times (0.5) \). The boundary \( \partial \Omega \) is partitioned as \( \partial \Omega = \Gamma_N \cup \Gamma_D \cup \Gamma_{free} \), where \( \Gamma_N := [0,1] \times \{1\}, \Gamma_D := [0,1] \times \{0\}, \), and \( \Gamma_{free} := \{0,1\} \times (0,1) \). On \( \Gamma_N \) we apply the force \( q \) in orthogonal direction to the domain, on \( \Gamma_D \) we enforce homogeneous Dirichlet boundary conditions for the displacement \( u = 0 \), and on \( \Gamma_{free} \) we set homogeneous Neumann boundary conditions. We choose the time interval \( [0,1] \) with 41 equidistant time points \( t_m \), i.e., \( T = 1 \) and \( M = 40 \). The discrete control space \( Q_h \) is one-dimensional in the sense that the force is constant in time and is only applied in \( y \) direction. The spatial mesh consists of 64 \( \times \) 64 square elements, hence the element diameter is \( h = \sqrt{2}/64 \approx 0.0221 \). The initial values are given by \( u_0 = (u_0, \varphi_0) \) where \( \varphi_0 \) describes the horizontal notch:
\[
\varphi_0(x,y) := \begin{cases} 
0, & x \in (0.50, 1.00) \text{ and } y = 0.5, \\
1, & \text{else}. 
\end{cases}
\]

The desired phase-field \( \varphi_d \) is defined as a continuation of the initial notch to the left hand side of the domain, see \( \varphi_0 \) in Fig. 1. Preliminary results with Algorithm 1 are given in [26]. In
Algorithm 1: Overall space-time phase-field fracture control algorithm

Data: Domain Ω, mesh T_h, number of time intervals M, parameters ε, κ, G_c, μ, λ, γ, η, α, initial value u0, initial control guess q0.

Result: Optimal control q and admissible solution u.

1: Set k = 0 and q^k = q^0 and solve the state equation for u: L'_u(q^k, u, z)(Φ) = 0 ∀Φ. Specifically, obtain u(t_0) from [6] and then u(t_1),...,u(t_M) from [7];
2: Solve the adjoint equation for z: L'_z(q^k, u, z)(Φ) = 0 ∀Φ. Obtain z(t_M) from [15], then z(t_{M-1}),...,z(t_1) from [16], and finally z(t_0) from [17];
3: Construct the coefficient vector f ∈ R^n for the reduced gradient ∇j(q^k) by solving Gf = [j'(q^k)(q_i)]_{i=1}^n. Here q_i denotes the i-th basis function of the discrete control space Q_h and G_{ij} = (q_i, q_j) defines the mass matrix. The derivatives j'(q^k)(q_i) for the right hand side are computed from the representation [14];

while ∥f∥_2 > TOL do
4: Obtain δq from the Newton equation, j''(q^k)(δq, q_i) = −j'(q^k)(q_i) ∀q_i, by minimizing m(q^k, d) = j(q^k) + ⟨f, d⟩ + 1/2⟨Hd, d⟩ for a vector d ∈ R^n using the CG-method (matrix free). Here H ∈ R^{n×n} denotes the coefficient matrix of ∇^2j(q^k)δq;
for every CG step do
5: Solve the tangent equation for δu:
L''_{uz}(q^k, u, z)(δq, Φ) + L''_{uz}(q^k, u, z)(δu, Φ) = 0 ∀Φ. Obtain δu(t_0) from [18] and then δu(t_1),...,δu(t_M) from [19];
6: Solve the adjoint Hessian equation for δz:
L''_{uu}(q^k, u, z)(δq, Φ) + L''_{uu}(q^k, u, z)(δu, Φ) + L''_{zu}(q^k, u, z)(δz, Φ) = 0 ∀Φ. Obtain δz(t_M) from [20], then δz(t_{M-1}),...,δz(t_1) from [21], and finally δz(t_0) from [22];
7: Construct the coefficient vector h ∈ R^n for ∇^2j(q^k)δq by solving Gh = j''(q^k)(δq, q_i)^{n=1}, where j''(q^k)(δq, q_i) is represented via [14];
end
8: Choose a step length ν by an Armijo backtracking method;
9: Set q^{k+1} = q^k + νδq;
10: Repeat steps 1, 2, 3 for the new control q^{k+1} to obtain f for ∇j(q^{k+1});
11: Increment k = k + 1;
end

order to investigate the effect of ϕ_d on the optimal solution, we will use two different homotopy approaches. In approach (a) we will successively increase the length of the desired phase-field, and in approach (b) we will successively reduce the Tikhonov parameter α. In both cases the motivation is to increase the weight of the physically motivated term 1/2∥ϕ − ϕ_d∥^2 in relation to the Tikhonov term. We will perform as many homotopy steps as possible, solving one NLP per step. The common nominal parameters used in both approaches are given in Table 1.

4.1.1. Approach (a): length increment. Here we will solve the NLP (10) several times with different desired phase-fields ϕ_d^k. Formally we define a sequence of desired phase-fields ϕ_d^k with
The number of homotopy steps performed in this experiment is 21: in step 22, the iterative solution of the nonlinear state equation fails because the Newton residuals do not decrease towards zero. Our results are presented in Tables 2 and 3. The first column (Step) counts the homotopy steps. The second column (Iter) gives the number of Newton iterations for solving the associated reduced problem (11), except that Iter 0 in Step 0 refers to the initial guess from which the homotopy starts. The remaining values are the absolute Newton residual, the cost functional $J$ and its tracking part $\frac{1}{2} \sum_q \| \varphi(t_m) - \varphi_d(t_m) \|_2^2$, the maximal force $|q_{\text{max}}|$ applied on $\Gamma_N$, and the Tikhonov regularization term, $\frac{1}{2} \sum_q q(t_m) - q_d(t_m) \|_{\Gamma_N}^2$. All values are rounded to three, five, or six significant digits. For every NLP the Newton iteration terminates when the residual falls below the tolerance $5 \times 10^{-11}$.

4.1.2. Approach (b): Tikhonov iteration. The second approach is a successive reduction of $\alpha$, a so called Tikhonov iteration. In this case the length of the desired phase-field remains constant at $\varphi_d^0$ for all homotopy steps while the weight of the Tikhonov term in $J$ is successively reduced. Here we define the sequence by $\alpha_k = 0.99 \alpha_{k-1}$ with $\alpha_0 = 4.75 \times 10^{-10}$. The number of homotopy steps performed is 8: in step 9, we have terminated the computation because of
controls on iterations 8 and 9 of the initial homotopy step. A comparison of the corresponding parameter α

Table 2. Experiment 1a: number of Newton iterations, absolute residual, cost terms and maximal force during homotopy. Iter 0 in step 0 refers to initial state from which homotopy starts.

| Step | Iter | Residual | Cost   | Tracking | Tikhonov | Force |
|------|------|----------|--------|----------|----------|-------|
| 0    | 0    | 4.62e−7  | 4.1532e−3 | 3.9192e−3 | 2.3406e−4 | 1.0   |
| 0    | 9    | 2.62e−11 | 3.4863e−3 | 3.3648e−3 | 1.2150e−4 | 2379.02 |
| 1    | 3    | 2.70e−11 | 3.5681e−3 | 3.4447e−3 | 1.2337e−4 | 2388.20 |
| 2    | 4    | 1.83e−11 | 3.6503e−3 | 3.5254e−3 | 1.2489e−4 | 2395.58 |
| 3    | 0    | 1.83e−11 | 3.6503e−3 | 3.5254e−3 | 1.2489e−4 | 2395.58 |
| 4    | 4    | 2.76e−11 | 3.7823e−3 | 3.6552e−3 | 1.2708e−4 | 2405.78 |
| 5    | 0    | 2.76e−11 | 3.7822e−3 | 3.6552e−3 | 1.2708e−4 | 2405.78 |
| 6    | 2    | 2.37e−11 | 3.8645e−3 | 3.7357e−3 | 1.2883e−4 | 2414.00 |
| 7    | 0    | 2.38e−11 | 3.8645e−3 | 3.7357e−3 | 1.2883e−4 | 2414.00 |
| 8    | 2    | 1.29e−11 | 3.9466e−3 | 3.8156e−3 | 1.3099e−4 | 2424.06 |
| 9    | 0    | 1.28e−11 | 3.9466e−3 | 3.8156e−3 | 1.3099e−4 | 2424.06 |
| 10   | 2    | 4.28e−11 | 4.0779e−3 | 3.9433e−3 | 1.3465e−4 | 2439.95 |
| 11   | 0    | 4.28e−11 | 4.0779e−3 | 3.9433e−3 | 1.3465e−4 | 2439.95 |
| 12   | 2    | 2.77e−11 | 4.1605e−3 | 4.0241e−3 | 1.3640e−4 | 2447.20 |
| 13   | 6    | 4.03e−11 | 4.2438e−3 | 4.1063e−3 | 1.3748e−4 | 2451.38 |
| 14   | 0    | 4.03e−11 | 4.2438e−3 | 4.1063e−3 | 1.3748e−4 | 2451.38 |
| 15   | 0    | 4.03e−11 | 4.2438e−3 | 4.1063e−3 | 1.3748e−4 | 2451.38 |
| 16   | 2    | 4.43e−11 | 4.3760e−3 | 4.2356e−3 | 1.4043e−4 | 2464.78 |
| 17   | 0    | 4.43e−11 | 4.3760e−3 | 4.2356e−3 | 1.4043e−4 | 2464.78 |
| 18   | 2    | 3.82e−11 | 4.4585e−3 | 4.3160e−3 | 1.4258e−4 | 2472.69 |
| 19   | 7    | 4.98e−11 | 4.5360e−3 | 4.3858e−3 | 1.5012e−4 | 2498.67 |
| 20   | 0    | 4.98e−11 | 4.5360e−3 | 4.3858e−3 | 1.5012e−4 | 2498.67 |
| 21   | 0    | 4.98e−11 | 4.5360e−3 | 4.3858e−3 | 1.5012e−4 | 2498.67 |

Figure 2. Experiment 1a: cost functional of each NLP iteration in homotopy (blue: tracking part above 3.2e−3 + red: Tikhonov part).

very slow alternating convergence of the residual as is often observed for very small values of the parameter α. The results are presented in Table 2. First we notice the high sensitivity of our NLP solution with respect to the control force. In Fig. 2 we present the difference of the controls on iterations 8 and 9 of the initial homotopy step. A comparison of the corresponding
residuals shows a reduction from $6.78 \times 10^{-11}$ to $2.62 \times 10^{-11}$ (approximately 60%), even though
the maximal difference between the applied controls is only 2.1 (or 0.1%). The values of
the cost functional and the residual on all iterations of both homotopies are presented in
Figs. 2, 3, 5 and 6. The behavior of the residual values in both approaches is typical for
homotopy methods: in each homotopy step they are reduced below the tolerance, and they
increase slightly afterwards. In the final homotopy step of each approach the reduction is non-
monotonous because the maximal number of line search iterations is reached; this indicates
the difficulty of the NLP. In Fig. 2 we observe that the value of the cost functional increases
with each homotopy step. This is a consequence of the increasing length of the desired phase-
field: $\psi_{d}^{21} < \cdots < \psi_{d}^{0}$. A closer look at the results reveals that the tracking part actually
increases non-linearly with the length of the desired phase-field, which is not surprising as our
overall problem is nonlinear. Finally we observe that both approaches yield larger maximal
control forces when compared to the results without homotopy ansatz. In approach (a) the
maximal final control is 2498.67, and in approach (b) it is 2438.79. This corresponds to the
different cracks being produced: without any homotopy approach the crack has a total length
of 0.063, with the Tikhonov iteration (approach b) we obtain 0.078, and with the crack length
increment (approach a) we obtain 0.094; see Fig. 7.

4.2. Experiment 2: two-sided control for diagonal crack. Our second experiment is an
extension of the first one with the aim to create a crack that grows diagonally, in negative $x$
direction and positive $y$ direction. We consider the same domain as before, $\Omega = (0,1)^2$, but
since cracks grow orthogonal to the maximum tensile stress [19 Chapter 4], the original control
Table 3. Experiment 1b: number of Newton iterations, absolute residual, cost terms and maximal force during homotopy. Iter 0 in step 0 refers to initial state from which homotopy starts.

| Step | Iter | Residual | Cost       | Tracking   | Tikhonov    | Force       |
|------|------|----------|------------|------------|-------------|-------------|
| 0    | 0    | 4.62e-7  | 4.1532e-3  | 3.9192e-3  | 2.3406e-4  | 1.0         |
| 0    | 1    | 2.62e-11 | 3.4863e-3  | 3.3648e-3  | 1.2150e-4  | 2379.02     |
| 1    | 3    | 4.47e-11 | 3.4835e-3  | 3.3614e-3  | 1.2206e-4  | 2387.69     |
| 2    | 4    | 1.55e-11 | 3.4812e-3  | 3.3590e-3  | 1.2212e-4  | 2393.56     |
| 3    | 6    | 2.20e-11 | 3.4787e-3  | 3.3565e-3  | 1.2223e-4  | 2399.79     |
| 4    | 6    | 2.81e-11 | 3.4764e-3  | 3.3541e-3  | 1.2226e-4  | 2405.43     |
| 5    | 2    | 2.81e-11 | 3.4737e-3  | 3.3512e-3  | 1.2257e-4  | 2412.03     |
| 6    | 3    | 3.88e-11 | 3.4706e-3  | 3.3469e-3  | 1.2362e-4  | 2423.04     |
| 7    | 2    | 4.42e-11 | 3.4676e-3  | 3.3433e-3  | 1.2437e-4  | 2432.00     |
| 8    | 6    | 4.79e-11 | 3.4648e-3  | 3.3310e-3  | 1.2487e-4  | 2438.79     |

Figure 5. Experiment 1b: cost functional of each NLP iteration in homotopy (blue: tracking part above $3.2e-3$ + red: Tikhonov part).

Figure 6. Experiment 1b: absolute residual of each NLP iteration in homotopy.

boundary $\Gamma_N = [0,1] \times \{1\}$ becomes $\Gamma_{N_1}$ and we extend the control to a second boundary $\Gamma_{N_2} = \{0\} \times [0,1]$, i.e., in the PDE constraint, we have

\[
(q, \Phi_{u:y})_{\Gamma_{N_2},I} = (q, \Phi_{u:y})_{\Gamma_{N_1},I} + (q, \Phi_{u:x})_{\Gamma_{N_2},I}.
\]
The overall setting is shown in Fig. 8. Because of the second control boundary, the Tikhonov term in the cost functional now becomes an integral over the union $\Gamma_N := \Gamma_{N_1} \cup \Gamma_{N_2}$. The domain is partitioned into $128 \times 128$ square elements with diameter $h = \sqrt{2}/128$. The number of time steps is $M = 100$. The desired phase-field is given as

$$\varphi_d(x, y) := \begin{cases} 0, & x \in (0.1, 0.5) \text{ and } |y - (0.85 - 0.7x)| \leq 3h, \\ 1, & \text{else.} \end{cases}$$

In short, the desired crack goes diagonally from $(0.5, 0.5)$ to $(0.1, 0.78)$ with a vertical diameter of $6h$. The results are presented in Table 4 and Figs. 9 to 11. From Table 4 we can see that it takes 13 iterations to solve the NLP with an absolute tolerance of $2.0e-10$. Note that from now on the first two columns (Iter, CG) give the iteration index of Newton’s method on the reduced NLP and the number of CG iterations required for computing the Newton increment, respectively. The Newton iteration terminates when either the relative or the absolute residual falls below the requested tolerance. The cost functional is reduced from $4.47e-2$ to $1.28e-2$, by approximately 70%. The final phase-field is shown in Fig. 9. As one can clearly see, the desired diagonal crack propagation has been produced successfully. The displacement and adjoint fields in Fig. 10 and the control on $\Gamma_{N_1}$ in Fig. 11 are consistent with our findings from [26]. On the one hand, the crack has to propagate to the left, therefore the control on the upper boundary $\Gamma_{N_1}$ has to increase from left to right. On the other hand, the crack should propagate upwards, therefore the control on the left boundary $\Gamma_{N_2}$ has to decrease from bottom to top. In contrast to Experiment 1, no symmetry in the displacement or adjoints fields can be expected since here the notch is horizontal whereas the desired phase-field is diagonal. Note that Fig. 11 shows a kink in each control. This is a numerical artefact: at the cell in the top left corner, the control acts on two adjacent boundaries simultaneously, and the discretized quantities interact within this single cell. In Fig. 22 we finally notice a tiny crack propagation starting from the bottom left edge $(0, 0)$. This is due to the singularity caused by the Dirichlet condition on the bottom boundary $\Gamma_D$ in combination with the control acting as Neumann condition on $\Gamma_{N_2}$. Similar observations are made in Section 4.3.
**Figure 8.** Experiment 2: domain $\Omega = (0, 1)^2$ with partitioned boundary, initial notch and desired crack $\varphi_d$.

**Table 4.** Experiment 2: number of CG iterations, residuals, cost terms and maximal force during NLP iteration.

| Iter | CG Relative residual | Absolute residual | Cost | Tracking residual | Tikhonov | Force |
|------|----------------------|--------------------|------|-------------------|-----------|-------|
| 0    | –                    | 1.0                | 2.0   | 1.0e-5            | 1.3723e-2 | 0.025 |
| 1    | 2                    | 3.70e-3            | 7.36e-8 | 1.3075e-2       | 1.3075e-2 | 8.9902e-9 | 2202.36 |
| 2    | 9                    | 1.27e-3            | 2.52e-8 | 1.2879e-2       | 1.2824e-2 | 5.4707e-5 | 2443.45 |
| 3    | 6                    | 5.77e-4            | 1.15e-8 | 1.2818e-2       | 1.2719e-2 | 9.8469e-5 | 2525.62 |
| 4    | 5                    | 3.11e-4            | 6.20e-9 | 1.2790e-2       | 1.2667e-2 | 1.2261e-4 | 2565.05 |
| 5    | 4                    | 1.84e-4            | 3.67e-9 | 1.2775e-2       | 1.2638e-2 | 1.3678e-4 | 2584.23 |
| 6    | 4                    | 1.13e-4            | 2.26e-9 | 1.2766e-2       | 1.2620e-2 | 1.4553e-4 | 2596.34 |
| 7    | 3                    | 7.57e-5            | 1.51e-9 | 1.2760e-2       | 1.2609e-2 | 1.5102e-4 | 2605.50 |
| 8    | 2                    | 4.86e-5            | 9.67e-10 | 1.2756e-2    | 1.2602e-2 | 1.5474e-4 | 2609.38 |
| 9    | 3                    | 3.53e-5            | 7.03e-10 | 1.2754e-2    | 1.2597e-2 | 1.5710e-4 | 2613.69 |
| 10   | 2                    | 2.42e-5            | 4.82e-10 | 1.2752e-2    | 1.2593e-2 | 1.5886e-4 | 2614.90 |
| 11   | 2                    | 1.55e-5            | 3.09e-10 | 1.2751e-2    | 1.2591e-2 | 1.6008e-4 | 2616.43 |
| 12   | 2                    | 1.01e-5            | 2.01e-10 | 1.2750e-2    | 1.2589e-2 | 1.6086e-4 | 2617.60 |
| 13   | 2                    | 6.81e-6            | 1.36e-10 | 1.2750e-2    | 1.2588e-2 | 1.6138e-4 | 2618.43 |

**Table 5.** Experiment 2: regularization and penalty parameters (left), model and material parameters (right).

| Par. | Definition                     | Value  | Par. | Definition                     | Value  |
|------|--------------------------------|--------|------|--------------------------------|--------|
| $\varepsilon$ | Regul. (crack) $\approx 4h$ | 0.0442 | $G_c$ | Fracture toughness             | 1.0    |
| $\kappa$  | Regul. (crack)                | 1.0e-10 | $\nu_s$ | Poisson’s ratio                | 0.2    |
| $\eta$   | Regul. (viscosity)            | 1.0e3  | $E$  | Young’s modulus                | 1.0e6  |
| $\gamma$ | Penalty                       | 1.0e5  | $q_0$ | Initial control                | 1.0    |
| $\alpha$ | Tikhonov                      | 6.5e-9 | $q_d$ | Nominal control                | 2.2e3  |
Figure 9. Experiment 2: optimal phase-field $\phi$ at times 50, 75, and 100.

Figure 10. Experiment 2: optimal displacement field $u$ (top: $x$ left, $y$ right) and adjoint field $z_u$ (bottom: $x$ left, $y$ right) at time 250.
4.3. **Experiment 3: connecting horizontal cracks for a sliced domain.** Our third experiment is motivated by a simple question: Is it possible to connect some (but not all) notches in a given domain? Here we consider the rectangle $\Omega = (0, 2.2) \times (0, 0.4)$ with four horizontal notches $N_1 := (0.3, 0.5) \times \{0.2\}$, $N_2 := (0.7, 0.9) \times \{0.2\}$, $N_3 := (1.3, 1.5) \times \{0.2\}$, $N_4 := (1.7, 1.9) \times \{0.2\}$, see Fig. 12. This yields the combined notch $N := \bigcup_{i=1}^{4} N_i$ with initial phase-field

$$\varphi_0(x, y) := \begin{cases} 0, & (x, y) \in N, \\ 1, & \text{else}. \end{cases}$$

The boundary $\partial \Omega$ is partitioned as in Section 4.1. The time interval is again $[0, 1]$ but with 2001 equidistant time points, i.e., $T = 1$ and $M = 2000$. The spatial mesh now consists of $352 \times 64$ square elements with diameter $h = \sqrt{2} \times 0.4/64 \approx 0.00884$. The desired phase-field $\varphi_d$ connects $N_1$ with $N_2$ and $N_3$ with $N_4$, hence it is defined as follows:

$$\varphi_d(x, y) := \begin{cases} 0, & x \in (0.5, 0.7) \cup (1.5, 1.7) \text{ and } y \in (0.2 - 4h, 0.2 + 4h), \\ 1, & \text{else}. \end{cases}$$

All relevant parameters for this experiment are presented in Table 6. The results for the
domain $\Omega = (0,1)^2$ by the question whether it is possible to connect two horizontal notches to achieve an entirely sliced domain. Here we consider again the square domain $\Omega = (0,1)^2$, but now with two horizontal notches, see Fig. 17. The left notch is defined as $(0.0, 0.375) \times \{0.5\}$, the right notch is defined as $(0.625, 1.0) \times \{0.5\}$. The boundary $\partial \Omega$ is partitioned as in Section 4.1. We choose the time interval $[0,1]$ with 251 equidistant time steps.

### Table 6. Experiment 3: regularization and penalty parameters (left), model and material parameters (right).

| Par. | Definition | Value | Par. | Definition | Value |
|------|------------|-------|------|------------|-------|
| $\varepsilon$ | Regul. (crack) $\approx 4h$ | 0.035 | $G_c$ | Fracture toughness | 1.0 |
| $\kappa$ | Regul. (crack) | $1.0e-10$ | $\nu_s$ | Poisson’s ratio | 0.2 |
| $\eta$ | Regul. (viscosity) | 1.0e3 | $E$ | Young’s modulus | $1.00e6$ |
| $\gamma$ | Penalty | 1.0e5 | $q_0$ | Initial control | 1.0 |
| $\alpha$ | Tikhonov | $2.1e-10$ | $q_d$ | Nominal control | $6.53e3$ |

### Table 7. Experiment 3: number of CG iterations, residuals, cost terms and maximal force during NLP iteration.

| Iter | CG | Relative residual | Absolute residual | Cost | Tracking | Tikhonov | Force |
|------|----|-------------------|-------------------|------|----------|----------|-------|
| 0    | -  | 1.0               | 2.00e-6           | 2.3850e-2 | 1.4302e-2 | 9.5483e-3 | 100.0 |
| 1    | 2  | 0.110             | 2.20e-7           | 1.2839e-2 | 1.2839e-2 | 3.4223e-8 | 6550.69 |
| 2    | 2  | 3.13e-2           | 6.27e-8           | 1.2428e-2 | 1.2309e-2 | 1.1897e-4 | 7899.32 |
| 3    | 2  | 1.22e-2           | 2.44e-8           | 1.2330e-2 | 1.2139e-2 | 1.9136e-4 | 8264.47 |
| 4    | 2  | 9.34e-3           | 1.87e-8           | 1.2292e-2 | 1.2066e-2 | 2.2519e-4 | 8403.51 |
| 5    | 2  | 1.44e-3           | 2.89e-9           | 1.2263e-2 | 1.2011e-2 | 2.5205e-4 | 8501.55 |
| 6    | 2  | 9.20e-4           | 1.84e-9           | 1.2261e-2 | 1.2006e-2 | 2.5478e-4 | 8514.26 |

The optimal control force shown in Fig. 16 is rather strong and has two roughly parabolic maxima right at the two sections where notches are to be connected, which is to be expected from a mechanical point of view. The four cracks propagating from both ends of each pair of connected notches, where no cracks are desired, can be explained by the decreasing control at the end points. That decreasing control generates a different principal axis of tension which in turn produces the non-horizontal crack growth. In Fig. 15 (top) we present the optimal displacement fields at time step 2000. They are both symmetric and reach their maxima right at the two sections where notches are to be connected. This is consistent with the behavior of the control forces and again physically plausible. For comparison, before the middle cracks join, we also display the respective fields at time step 1500 in Fig. 14. These results are also consistent with the ones from [26, Example 2], where we considered a notch in the middle of the domain and a desired phase-field that continues this notch only to one side. There we have observed that the optimal force is large in the area where the crack propagation is desired and small where no crack evolution should occur.

### 4.4. Experiment 4: connecting two horizontal cracks for an entirely sliced domain.

The fourth experiment is motivated by the question whether it is possible to connect two horizontal notches to achieve an entirely sliced domain. Here we consider again the square domain $\Omega = (0,1)^2$, but now with two horizontal notches, see Fig. 17. The left notch is defined as $(0.0, 0.375) \times \{0.5\}$, the right notch is defined as $(0.625, 1.0) \times \{0.5\}$. The boundary $\partial \Omega$ is partitioned as in Section 4.1. We choose the time interval $[0,1]$ with 251 equidistant time steps.
Figure 13. Experiment 3: optimal phase-field $\varphi$ at times 1400, 1800, and 2000.

Figure 14. Experiment 3: optimal displacement field $u$ (top: $x$ left, $y$ right) and adjoint field $z_u$ (bottom: $x$ left, $y$ right) at time 1800.

Figure 15. Experiment 3: optimal displacement field $u$ (top: $x$ left, $y$ right) and adjoint field $z_u$ (bottom: $x$ left, $y$ right) at time 2000.
Figure 16. Experiment 3: optimal control force (solid) and nominal control force (dotted) on upper boundary $\Gamma_N$.

points, i.e., $T = 1$ and $M = 250$. The spatial mesh now consists of $128 \times 128$ square elements with diameter $h = \sqrt{2}/128 \approx 0.011$. The desired phase-field $\varphi_d$ connects the left notch with the right notch and is defined as follows:

$$\varphi_d(x, y) := \begin{cases} 0, & x \in (0.375, 0.625) \text{ and } y \in (0.5 - 2h, 0.5 + 2h), \\ 1, & \text{else}. \end{cases}$$

Our goal in this experiment is rather peculiar. Analytically, the PDE constraint becomes singular once the domain is entirely sliced. In the phase-field model this happens when the left and right boundaries of $\Omega$ are connected by a path along which the phase-field $\varphi$ vanishes. Numerical difficulties are to be expected even before such a path exists: the PDE becomes increasingly ill-conditioned when the transition zones with $0 < \varphi < 1$ come into contact. Nevertheless it is possible to create a domain-splitting crack with a pure forward model, see for instance the related single edge notched tension test [36, 35, 2, 14]. Yet this experiment remains numerically difficult and becomes even more challenging within our optimization setting. Since we have to expect that the solution of the forward problem might be close to singularities, it is not clear what will happen when we insert this solution into the optimization algorithm. With regard to this challenge we have observed that in many experiments the Tikhonov term acts against extreme forces and improves the solvability of the PDE for the resulting controls. In the experiment under consideration we set $\alpha$ to $2.0e-10$. By this the Tikhonov term is not the driving factor of the optimization process, but still large enough to avoid extreme forces. The choice of the other parameters is shown in Table 8, and our results are presented in Table 9 and Figs. 18 to 20.

In Table 9 we observe that the residual value is decreasing, except for the last iteration. After iteration 6 the PDE forward problem becomes unsolvable. Therefore we regard iteration 5 as the optimal solution: it has the lowest absolute residual value, $5.13e-9$, and also the lowest relative residual value, $1.57e-2$. The results presented in Table 9 and Figs. 18 to 20 refer to iteration 5. The optimal phase-field presented in Fig. 18 does not connect the two notches but reaches approximately two thirds of the length of the desired phase-field. In Fig. 20 we see that the optimal control force is nearly twice as large as the nominal control force $q_d$. This means that the optimization is primarily driven by the physical term $\|\varphi - \varphi_d\|^2$. 
4.5. **Experiment 5: L-shaped domain.** In our fifth experiment we study a modification of the L-shaped panel test within an optimization context. The L-shaped panel test was originally developed by Winkler [46] and extensively studied in [2, 34, 44, 32]. In the original test the applied force pushes upwards against a small left-most section of the upper part of the domain. In our experiment we apply a pulling force on the top boundary \( \Gamma_N \) instead. We do this in order to have a complete control boundary within the optimization context. The L-shaped domain \( \Omega = (0,1)^2 \setminus (0.5,0.5)^2 \) and its partitioning of the boundary \( \partial \Omega \) are

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**Figure 17.** Experiment 4: domain \( \Omega = (0,1)^2 \) with partitioned boundary \( \partial \Omega \), initial notches, and desired crack \( \varphi_d \).

**Table 8.** Experiment 4: regularization and penalty parameters (left), model and material parameters (right).

| Par. | Definition | Value   |
|------|------------|---------|
| \( \varepsilon \) | Regul. (crack) \( \approx 2h \) | 0.0221  |
| \( \kappa \) | Regul. (crack) | 1.0e-10 |
| \( \eta \) | Regul. (viscosity) | 1.0e3   |
| \( \gamma \) | Penalty     | 1.0e5   |
| \( \alpha \) | Tikhonov   | 2.0e-10 |

| Par. | Definition             | Value  |
|------|------------------------|--------|
| \( G_c \) | Fracture toughness     | 1.0    |
| \( \nu_s \) | Poisson’s ratio       | 0.2    |
| \( E \) | Young’s modulus       | 1.00e6 |
| \( q_0 \) | Initial control         | 1.0    |
| \( q_d \) | Nominal control         | 1.85e3 |

**Table 9.** Experiment 4: number of CG iterations, residuals, cost terms and maximal force during NLP iteration.

| Iter | CG | Relative residual | Absolute residual | Cost         | Tracking | Tikhonov | Force       |
|------|----|-------------------|-------------------|--------------|----------|----------|-------------|
| 0    |    | 1.0               | 3.26e-7           | 5.5093e-3    | 5.2936e-3 | 2.1566e-4 | 380.0       |
| 1    | 2  | 0.424             | 1.38e-7           | 4.9612e-3    | 4.9584e-3 | 2.7903e-6 | 2043.40     |
| 2    | 3  | 0.238             | 7.75e-8           | 4.7486e-3    | 4.6748e-3 | 7.3883e-5 | 2845.85     |
| 3    | 3  | 0.145             | 4.72e-8           | 4.6179e-3    | 4.4620e-3 | 1.5592e-4 | 3300.68     |
| 4    | 3  | 4.54e-2           | 1.48e-8           | 4.5391e-3    | 4.3184e-3 | 2.2067e-4 | 3573.22     |
| 5    | 2  | 1.57e-2           | 5.13e-9           | 4.5160e-3    | 4.2733e-3 | 2.4274e-4 | 3629.90     |
| 6    | 2  | 0.26              | 8.36e-8           | 4.5088e-3    | 4.2591e-3 | 2.4976e-4 | 3632.82     |
shown in Fig. 21. We choose the time interval $[0, 1]$ with 301 equidistant time points, i.e., $T = 1$ and $M = 300$. Each of the $3 \times 80 \times 80$ square spatial mesh elements has a diameter of $h = \sqrt{2}/160 \approx 0.00884$. All other parameters are shown in Table 10. From [2, 34, 44, 32] and [46] we already know that the crack will grow slightly above the horizontal line $[0.5, 1] \times \{0.5\}$. Therefore we place the desired phase-field $\varphi_d$ also slightly above that line,

$$\varphi_d(x, y) := \begin{cases} 0, & x \in (0.5, 1.0) \text{ and } y \in (0.53 - 4h, 0.53 + 4h), \\ 1, & \text{else}. \end{cases}$$

We are aware that a fracture with this phase-field cannot be produced in our setting for two reasons. First, a sharp crack along $[0.5, 1.0] \times \{0.53\}$ is physically impossible because the crack will always start to grow from the singularity in $(0.5, 0.5)$. Second, a decomposition of the stress tensor is needed in order to distinguish crack growth under tension and compression; see extensive findings and discussions for the L-shaped panel test in [2]. Since stress splittings laws introduce further nonlinearities in the forward phase-field fracture model and do not contribute to significant further insight in the current work, we have not used them, despite implemented in our software, e.g., [32]. We also tried to define $\varphi_d$ on the horizontal line $[0.5, 1] \times \{0.5\}$. However, since the crack starts propagating diagonally upwards from $(0.5, 0.5)$, the values of the residual and the cost functional did not decrease, and as a consequence the Newton iteration for the optimization problem did not converge. Our results for the tolerance $2.0e-10$ are presented in Table 11 and Figs. 22 to 24.
Figure 19. Experiment 4: optimal displacement field $u$ (top: $x$ left, $y$ right) and adjoint field $z_u$ (bottom: $x$ left, $y$ right) at time 250.

Figure 20. Experiment 4: optimal control force (solid) and nominal control force (dotted) on upper boundary $\Gamma_N$. 
Figure 21. Experiment 5: L-shaped domain $\Omega = (0,1)^2 \setminus (0.5, 0.5)^2$ with partitioned boundary and desired crack $\varphi_d$.

Table 11. Experiment 5: number of CG iterations, residuals, cost terms and maximal force during NLP iteration.

| Iter | CG | Relative residual | Absolute residual | Cost | Tracking | Tikhonov | Force |
|------|----|-------------------|-------------------|------|----------|----------|-------|
| 0    |    | 1.0               | 4.34e-6           | 2.0637e-2 | 8.9261e-3 | 1.1711e-2 | 1.0   |
| 1    | 2  | 0.117             | 5.09e-7           | 1.6652e-2 | 8.8773e-3 | 7.7750e-3 | 1600.27 |
| 2    | 2  | 4.71e-2           | 2.04e-7           | 1.6473e-2 | 8.9092e-3 | 7.5637e-3 | 1982.44 |
| 3    | 3  | 2.45e-2           | 1.06e-7           | 1.6404e-2 | 8.9210e-3 | 7.4829e-3 | 2143.55 |
| 4    | 2  | 1.18e-3           | 5.11e-9           | 1.6367e-2 | 8.9284e-3 | 7.4387e-3 | 2226.13 |
| 5    | 3  | 7.86e-4           | 3.41e-9           | 1.6344e-2 | 8.9340e-3 | 7.4103e-3 | 2279.40 |
| 6    | 4  | 5.85e-4           | 2.54e-9           | 1.6329e-2 | 8.9382e-3 | 7.3905e-3 | 2304.63 |
| 7    | 3  | 4.33e-4           | 1.88e-9           | 1.6317e-2 | 8.9419e-3 | 7.3750e-3 | 2334.29 |
| 8    | 3  | 3.37e-4           | 1.46e-9           | 1.6308e-2 | 8.9450e-3 | 7.3630e-3 | 2350.53 |
| 9    | 2  | 2.68e-4           | 1.16e-9           | 1.6301e-2 | 8.9478e-3 | 7.3534e-3 | 2360.71 |
| 10   | 3  | 2.20e-4           | 9.56e-10          | 1.6296e-2 | 8.9501e-3 | 7.3455e-3 | 2374.60 |
| 11   | 2  | 1.84e-4           | 7.96e-10          | 1.6291e-2 | 8.9521e-3 | 7.3388e-3 | 2380.56 |
| 12   | 2  | 1.58e-4           | 6.83e-10          | 1.6287e-2 | 8.9539e-3 | 7.3313e-3 | 2387.88 |
| 13   | 2  | 1.36e-4           | 5.92e-10          | 1.6284e-2 | 8.9555e-3 | 7.3281e-3 | 2394.83 |
| 14   | 3  | 1.20e-4           | 5.21e-10          | 1.6281e-2 | 8.9569e-3 | 7.3239e-3 | 2401.26 |
| 15   | 2  | 1.03e-4           | 4.46e-10          | 1.6278e-2 | 8.9583e-3 | 7.3200e-3 | 2403.82 |
| 16   | 2  | 9.21e-5           | 4.00e-10          | 1.6276e-2 | 8.9594e-3 | 7.3166e-3 | 2407.58 |
| 17   | 2  | 8.23e-5           | 3.57e-10          | 1.6274e-2 | 8.9605e-3 | 7.3135e-3 | 2411.42 |
| 18   | 2  | 7.45e-5           | 3.23e-10          | 1.6272e-2 | 8.9615e-3 | 7.3107e-3 | 2415.00 |
| 19   | 2  | 6.83e-5           | 2.96e-10          | 1.6271e-2 | 8.9624e-3 | 7.3082e-3 | 2418.35 |
| 20   | 2  | 6.17e-5           | 2.68e-10          | 1.6269e-2 | 8.9633e-3 | 7.3058e-3 | 2421.39 |
| 21   | 2  | 5.84e-5           | 2.53e-10          | 1.6268e-2 | 8.9640e-3 | 7.3037e-3 | 2424.17 |
| 22   | 2  | 5.31e-5           | 2.30e-10          | 1.6266e-2 | 8.9648e-3 | 7.3016e-3 | 2426.76 |
| 23   | 2  | 4.82e-5           | 2.09e-10          | 1.6265e-2 | 8.9658e-3 | 7.2997e-3 | 2429.09 |
| 24   | 2  | 4.35e-5           | 1.89e-10          | 1.6264e-2 | 8.9661e-3 | 7.2980e-3 | 2431.26 |
In Table 11 we see that 24 iterations were required to reach the final residual value $1.89 \times 10^{-10}$. The propagating crack, shown in Fig. 22, is very similar to the results from [32]. The corresponding optimal control is presented in Fig. 24. It decreases almost linearly, approximately from 2400 to 1500, which is plausible since this experiment has similarities to Section 4.1. Similar to Section 4.2 and Section 4.3 we notice a small crack propagation starting from the lower left corner $(0.5, 0)$. This is due to the singularity caused by the Dirichlet condition on $\Gamma_D$ in combination with the Neumann condition on $\{0.5\} \times [0, 0.5]$.

4.6. Experiment 6: inhibiting horizontal crack growth. In our final experiment we expose the domain $\Omega = (0, 1)^2$ to a time-independent external force $q_c$ which creates a growing crack (for the tiny initial control $q = 1$). Then we seek an optimal control $q$ that counteracts the external force $q_c$ to inhibit the crack growth. We choose the same partitioning of $\partial \Omega$ and the same notch as in Section 4.1. The initial phase-field is

$$\varphi_0(x, y) := \begin{cases} 0, & x \in (0.5, 1) \text{ and } y = 0.5 \\ 1, & \text{else.} \end{cases}$$

We define the external force as a linear function: $q_c(x) = 850 + 1800x$. The time interval is $[0, 1]$ with 101 equidistant time points, i.e., $T = 1$ and $M = 100$. The spatial mesh consists of $64 \times 64$ square elements with diameter $h = \sqrt{2}/64$. The desired phase-field $\varphi_d$ has the value one on the whole domain. Our findings for the tolerance $2 \times 10^{-11}$ are presented in Table 12 and Figs. 26 to 28.

In Fig. 26 we observe that no crack propagation occurs with the computed optimal control. As a result, the desired phase-field is successfully reproduced and the initial value of the cost functional is reduced by 96%. In comparison to all other experiments, where we achieved a maximum reduction of 70%, this is a remarkable result. Although the sum of the optimal control $q$ and the constant external force $q_c$ is positive everywhere, see Fig. 27, it is not large enough to create a propagating crack. This is to be expected since the Tikhonov term would penalize an unnecessarily strong control force. In Fig. 28 the control forces of iterations 1 to 4 from Table 12 are shown. We observe that the first control $q_1$ decreases almost linearly to a minimal value of $-4262.61$, which produces a relatively large Tikhonov term. The second

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig22.png}
\caption{Experiment 5: optimal phase-field $\varphi$ at times 200, 250, and 300.}
\end{figure}
Figure 23. Experiment 5: optimal displacement field $u$ (top: $x$ left, $y$ right) and adjoint field $z_u$ (bottom: $x$ left, $y$ right) at time 300.

control $q_2$ is instead a linearly increasing function that minimizes this term. The third and fourth controls lie between $q_1$ and $q_2$, and $q_4$ behaves similar to the optimal control. When studying the value of the tracking part in Table 12 we observe that it becomes almost zero on the third iteration where the Tikhonov term is more than 10 times larger. Consequently the Tikhonov term must be reduced next. Subsequently, the two terms oscillate until they are roughly balanced. Some of the former experiments have been sensitive to the choice of $\alpha$, but none of them has been as sensitive as this experiment.

5. Conclusions

In this work, we performed several detailed computational performance studies for space-time phase-field fracture optimal control problems. The optimization problem was designed
Figure 24. Experiment 5: optimal control force (solid) and nominal control force (dotted) on upper boundary $\Gamma_N$.

$$\Gamma_N \quad \uparrow q \quad \Gamma_N \quad \uparrow q$$

$\Omega$

$\Gamma_{\text{free}}$

$\sim \varphi_d$

$\text{notch}$

$\Gamma_{\text{free}}$

$\Gamma_D$

Figure 25. Experiment 6: domain $\Omega = (0, 1)^2$ with partitioned boundary, initial notch, undesired crack $\sim \varphi_d$ and constant pulling force $q_c$.

Figure 26. Experiment 6: initial phase-field $\varphi$ (left, iteration 0) and optimal phase-field (right, iteration 15) at final time 100.
the computational cost of the inner linear solver of the forward problem, which is the performance of the NLP solver (Algorithm 1) and the inner CG method as well as the fracture operator acting on the controls. Therein, a monolithic space-time representation of the phase-field fracture problem was adopted. Moreover, the crack irreversibility constraint was regularized using a penalty approach. To study the performance, we investigated six numerical experiments with single (Experiments 1, 2, 5, 6) and multiple fractures (Experiments 3, 4), single controls (Experiments 1, 3, 4, 5, 6) and two controls (Experiments 2), propagating fractures (Experiments 1, 2, 3, 4, 5) and inhibiting crack growth (Experiment 6). Therein, the performance of the NLP solver (Algorithm 1) and the inner CG method as well as the phase-field fracture PDE constraint were computationally analyzed in great detail. One main bottleneck is the computational cost of the inner linear solver of the forward problem, which is well-known and analogous in other PDE-constrained optimization problems. In ongoing work, we plan to incorporate parallel adaptive preconditioned iterative solvers which, however, is a major extension and was out of scope in this work.

### Table 12. Experiment 6: number of CG iterations, residuals, cost terms and maximal force during NLP iteration.

| Iter | CG Residual | Absolute residual | Cost | Tracking | Tikhonov | Force |
|------|-------------|--------------------|------|----------|----------|-------|
| 0    | 1.0         | 3.23e-6            | 1.7274e-3 | 1.4082e-3 | 3.1920e-4 | 1.0 |
| 1    | 3           | 5.58e-6            | 1.5649e-3 | 2.1034e-5 | 1.5438e-3 | 4262.61 |
| 2    | 3           | 0.166              | 1.9198e-4 | 1.8984e-4 | 2.1348e-6 | 824.97 |
| 3    | 3           | 0.137              | 1.2174e-4 | 7.6886e-6 | 1.1405e-4 | 1731.04 |
| 4    | 3           | 5.26e-2            | 1.70e-7  | 9.0843e-5 | 2.7177e-6 | 956.04 |
| 5    | 3           | 4.40e-2            | 1.42e-7  | 6.5026e-5 | 3.4897e-5 | 1280.33 |
| 6    | 3           | 1.25e-3            | 4.05e-9  | 7.2949e-5 | 6.3052e-5 | 9.8963e-6 | 1079.64 |
| 7    | 12          | 1.02e-3            | 3.31e-9  | 6.7607e-5 | 5.2538e-5 | 1.5069e-5 | 1142.82 |
| 8    | 12          | 3.10e-4            | 1.00e-9  | 7.0443e-5 | 5.8585e-5 | 1.1858e-5 | 1104.25 |
| 9    | 10          | 2.52e-4            | 8.16e-10 | 6.9141e-5 | 5.5993e-5 | 1.3148e-5 | 1120.23 |
| 10   | 9           | 7.94e-5            | 2.57e-10 | 6.9890e-5 | 5.7514e-5 | 1.2377e-5 | 1110.86 |
| 11   | 7           | 6.42e-5            | 2.07e-10 | 6.9560e-5 | 5.6853e-5 | 1.2707e-5 | 1114.79 |
| 12   | 7           | 2.0e-5             | 6.57e-11 | 6.9753e-5 | 5.7241e-5 | 1.2512e-5 | 1112.52 |
| 13   | 5           | 1.63e-5            | 5.28e-11 | 6.9669e-5 | 5.7073e-5 | 1.2596e-5 | 1113.45 |
| 14   | 4           | 5.26e-6            | 1.70e-11 | 6.9718e-5 | 5.7172e-5 | 1.2546e-5 | 1112.98 |
| 15   | 3           | 4.21e-6            | 1.36e-11 | 6.9697e-5 | 5.7128e-5 | 1.2568e-5 | 1113.14 |

### Table 13. Experiment 6: regularization and penalty parameters (left), model and material parameters (right).

| Par. | Definition | Value | Par. | Definition | Value |
|------|------------|-------|------|------------|-------|
| $\varepsilon$ | Regul. (crack) $\approx 2h$ | 0.0442 | $G_c$ | Fracture toughness | 1.0 |
| $\kappa$ | Regul. (crack) | 1.0e-10 | $\nu_s$ | Poisson’s ratio | 0.2 |
| $\eta$ | Regul. (viscosity) | 1.0e3 | $E$ | Young’s modulus | 1.0e6 |
| $\gamma$ | Penalty | 1.0e5 | $q_0$ | Initial control | 1.0 |
| $\alpha$ | Tikhonov | 1.0e-9 | $q_d$ | Nominal control | -8.0e2 |
Figure 27. Experiment 6: optimal control force (blue), nominal control (red, dotted), constant control (magenta) and resulting total control $q + q_c$ (green) on upper boundary $\Gamma_N$.

Figure 28. Experiment 6: control forces for iterations 1–4 (solid) and nominal control (dotted) on upper boundary $\Gamma_N$.

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