Modeling Collisionless Matter in General Relativity:
A New Numerical Technique

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Abstract

We propose a new numerical technique for following the evolution of a self-gravitating collisionless system in general relativity. Matter is modeled as a scalar field obeying the coupled Klein-Gordon and Einstein equations. A phase space distribution function, constructed using covariant coherent states, obeys the relativistic Vlasov equation provided the de Broglie wavelength for the field is very much smaller than the scales of interest. We illustrate the method by solving for the evolution of a system of particles in a static, plane-symmetric, background spacetime.
1 Introduction

There now exists techniques to calculate numerically the dynamical evolution of matter in general relativity. For the most part research has focused on three very different types of systems; fluids, scalar fields, and collisionless matter. In this work we develop and exploit a connection between the latter two.

A collisionless system with a large number of particles is treated generally as a continuous fluid in phase space. The fluid is described by a distribution function $f$ which gives, for each region of the system, the density of particles and their distribution in velocity space. $f$ obeys the Vlasov equation with an appropriate force law which, in general relativity, is given by the geodesic and Einstein equations. Shapiro and Teukolsky [1] have developed a computational method for handling collisionless relativistic systems that combines N-body techniques and numerical relativity. An alternative approach [2] is to follow the evolution of $f$ directly in phase space by solving the coupled Einstein and Vlasov equations. Applications include violent relaxation and the collapse to a black hole of an unstable relativistic star cluster.

In this work we show that a massive scalar field obeying the coupled Klein-Gordon and Einstein equations provides an alternative model for collisionless relativistic systems that can be readily adapted to the computer. The technique makes use of what is essentially the scalar field analogue of geometric optics in general relativity [3]. The spirit and methodology is similar to that of Widrow and Kaiser [4] who showed that a field obeying the coupled Schrödinger and Poisson equations could be used to model collisionless, nonrelativistic matter.

Scalar fields in general relativity have been studied for some time, albeit for entirely different reasons. For the past fifteen years the motivation for much of this work has been to understand the inflationary universe paradigm. For example, numerical simulations of inhomogeneous scalar field cosmologies have been studied in an attempt to understand the onset of inflation [5]. Recently, there has been great deal of interest in the gravitational collapse to a black hole of a self-gravitating massless scalar field. This is due primarily to the discovery by Choptuik [6] that such systems exhibit scaling behaviour and critical phenomena.

Again our interest here is in simulations of collisionless matter. For our purposes previous investigations of scalar fields are important in that they exhibit the numerical techniques used to follow their dynamics. Our discussion does suggest that the results found by Choptuik might also apply to collisionless matter. Unfortunately the simulations that have shown scaling behaviour and critical phenomena involve massless scalar fields whereas our work requires a nonzero mass.

We model a collisionless relativistic system by a field configuration $\phi(x)$ where $\phi$ is a complex scalar field obeying the Klein-Gordon equation in curved space:

$$\left( g^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{m^2}{\hbar^2} \right) \phi(x) = 0 .$$

(The metric $g_{\mu\nu}$ has signature $(+ - - -)$ and we set $c = 1$.) $m/\hbar$ is a model parameter which must be large enough to guarantee that geometric optics (or rather the scalar field analogue of geometric optics) applies. In particular we require that throughout the system $\lambda_{\text{deB}} \ll \mathcal{L}$ and $\lambda_{\text{deB}} \ll \mathcal{R}$ where $\mathcal{L}$ is the scale over which the density and velocity field of the system (i.e., distribution function) vary, $\mathcal{R}$ is the typical radius of curvature for the spacetime, and $\lambda_{\text{deB}} \sim |\phi/\nabla \phi|$ is the typical de Broglie wavelength for the field. Locally $\phi$ can be regarded as the superposition of plane waves propagating along geodesics [3]. The amplitude of the field in a given region of the system tells us something about the density of particles while information about the velocity distribution of the particles is encoded in the phase structure of the field.

The discussion above suggests the following prescription for constructing a distribution function $\mathcal{F} = \mathcal{F}(x, p)$ from the field $\phi$. Fourier transform the field in a region of size $\eta (\lambda_{\text{deB}} \ll \eta$ and $\eta \ll \mathcal{L}, \mathcal{R})$ centered on a particular point in phase space and then take the absolute square. This is what is done in [4] for a nonrelativistic Schrödinger field $\psi = \psi(x, t)$. There $\mathcal{F}$ is constructed using coherent states: $\mathcal{F}(\mathbf{x}, \mathbf{p}) = |\langle e(\mathbf{x}, \mathbf{p}) | \psi \rangle|^2$ where $|e(\mathbf{x}, \mathbf{p})\rangle$ is a localized state centered
on \( \mathbf{x} \) with average momentum \( \mathbf{p} \). \(|e(\mathbf{x}, \mathbf{p})|\) is taken to be a Gaussian (minimum uncertainty) wavepacket,

\[
(e(\mathbf{x}, \mathbf{p})|\mathbf{x}') = \left( \frac{1}{2\pi \hbar} \right)^{3/2} \left( \frac{1}{\pi \eta^2} \right)^{3/4} e^{-\frac{(\mathbf{x}-\mathbf{x}')^2}{2\eta^2} - i\mathbf{p} \cdot \mathbf{x}/\hbar},
\]

(2)

having width \( \eta \) in position space and \( \hbar/\eta \) in momentum space. The normalization is chosen so that \( \int d^3x \int d^3p |F(x, p)| = 1 \) provided \( \int d^3x |\psi(x)|^2 = 1 \). It is straightforward to show that \( F \) obeys approximately the nonrelativistic Vlasov equation provided \( \lambda_{de B} \ll \eta \ll L \).

The coherent-state representation for a scalar field in special relativity is again based on a set of state vectors \(|e(x, p)|\) localized in both position and momentum. We choose minimum uncertainty wavepackets [7]:

\[
|e(x, p)\rangle = \left( \frac{\eta^2}{\pi \hbar^2} \right)^{3/4} e^{-i\mathbf{x} \cdot \mathbf{p}/\hbar - \mathbf{p} \cdot \mathbf{p}'/\hbar^2} \phi(p')
\]

(3)

where \( p = (p_0, \mathbf{p}) \) with \( p_0 > |\mathbf{p}| \) and \( \mathbf{a} \cdot \mathbf{b} \equiv a_0 b_0 - \mathbf{a} \cdot \mathbf{b} \). These wavepackets are centered about \( \mathbf{x} \) and move with average momentum \( \mathbf{p} \).

The extension to general relativity is most easily accomplished using Riemann normal coordinates (RNCs) and is similar to discussions found in the literature [8] of the Wigner function in curved space. In a RNC system curved space closely resembles flat space in the neighborhood of a particular point taken to be the origin. The metric can then be written as a Taylor series about the flat space metric \( \eta_{\mu\nu} \) in increasing orders of the Riemann tensor \( R_{\mu\nu\rho\sigma}; \ g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\nu\rho\sigma} y^\rho y^\sigma + \ldots \). In RNCs the phase factor \( x \cdot p = \eta_{\mu\nu} x^\mu p^\nu + \text{`post-geometric optics corrections'} [3] \), \( \phi \) in the curved space coherent-state representation is then

\[
\langle e(x, p) | \phi \rangle = \int dV_p e^{-i\mathbf{x} \cdot \mathbf{p}/\hbar - \mathbf{p} \cdot \mathbf{p}'/\hbar^2} \hat{\phi}(p')
\]

(4)

where

\[
dV_p = \frac{dp_0 dp_1 dp_2 dp_3}{(2\pi)^4} \frac{1}{2\pi \delta (p^2 - m^2)} \Theta(p_0)
\]

(5)

is the momentum space volume element on the mass shell, \( \Theta(x) = 1 \) for \( x \geq 0 \) and zero otherwise, and \( \hat{\phi}(p) \) is the Fourier transform of \( \phi(x) \):

\[
\phi(x) = \int dV_p e^{-ip \cdot x/\hbar} \hat{\phi}(p).
\]

(6)

Note that \( \langle e(x, p) | \phi \rangle \) reduces to \( e(\mathbf{x}, \mathbf{p})|\psi\rangle \) for \( |\mathbf{p}| \ll m \). In practice, one integrates over \( p_0 \) making use of the identity \( \int dV_p = d^3p/((2\pi)^3 2\hbar) \). For example

\[
F(x, p) = \int dV_p' dV'' e^{ix \cdot (p' - p'')/\hbar - p' \cdot (p'' + p') - \hbar^2 /2} \hat{\phi}(p') \hat{\phi}^*(p'')
\]

(7)

or

\[
\frac{d^3p'}{2\pi^3} \frac{d^3p''}{2\pi^3} \frac{1}{2\pi \delta (p^2 - m^2)} e^{ix \cdot (p' - p'')/\hbar - p' \cdot (p'' + p') - \hbar^2 /2} \hat{\phi}(p') \hat{\phi}^*(p'').
\]

(8)

In the second of these expression the integrand is evaluated “on mass-shell”, i.e., with \( p'^0 = \sqrt{p'^2 + m^2} \) and \( p''^0 = \sqrt{p''^2 + m^2} \).

Consider the function \( \sigma(p) = \delta (p^2 - m^2) \Theta(p_0) \hat{\phi}(p) \). From Eq. (1) and the definition of the RNCs we have

\[
\left( \frac{m^2 - p^2}{\hbar^2} - \frac{1}{3} R^\nu_{\rho\sigma} \frac{\partial}{\partial p_\rho} \frac{\partial}{\partial p_\sigma} p_\nu p_\sigma - \frac{2}{3} R^\nu_{\rho\sigma} \frac{\partial}{\partial p_\rho} p_\sigma \right) \sigma(p) \equiv O(p) \sigma(p) = 0
\]

(9)
from which follows immediately the identity
\[ 0 = \int d^4p' d^4p'' e^{ix(p'-p'')}/(\hbar p\cdot (p'+p''))\eta^2/\hbar^2 (\mathcal{O}(p')\sigma(p')) \sigma^*(p''). \]

A straightforward but tedious calculation leads to a complex equation for \( F \) which can be split into real and imaginary parts:
\[ (\frac{m^2-p^2}{\hbar^2} + \ldots) F = 0 \]  
\[ (\eta^\mu^\nu p_\mu \frac{\partial}{\partial x^\nu} + \frac{1}{3} R^\mu^\nu_\rho \sigma^\rho \sigma^\sigma_{\nu} \frac{\partial}{\partial p_\rho} + \ldots) F = 0. \]

The “…” in these equations refer to terms higher order in the \( \eta/L, \lambda_{\text{dB}}/\eta \) and \( \eta/R \) (i.e. post-geometric optics corrections). The first of these equations tells us that the distribution function is peaked on the \( p_\mu p^\mu = m^2 \) (mass shell) surface. The second equation is, to leading order, the relativistic Vlasov equation in a RNC system.

For self-gravitating collisionless matter, \( g_{\mu\nu} \) depends on \( f \) through the Einstein equations, \( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \), where the stress-energy tensor \( T_{\mu\nu} \) is given by
\[ T_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} p_\mu p_\nu f(x,p). \]

We can calculate \( T_{\mu\nu} \) for our model by substituting \( F \) for \( f \) in Eq. (13). This is, however, computationally expensive (one must update \( g \) at each timestep) and not really necessary. Consider instead the expression
\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi^* \]
\[ = \int dV p' dV' e^{i(p''-p')x/\hbar} p'_\mu p'_\nu \hat{\phi}(p') \phi^*(p''). \]

Eq. (14) is what one would obtain if this expression were smoothed over a length scale \( \sim \eta \) and since the quantity of interest, \( g \sim \nabla^{-2} T \), is itself smoother than \( T \) we can use either expression. Eq.(12) is just the usual expression for the stress energy tensor of a scalar field, save for a term proportional to \( \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* \). But this term is negligible in the geometric optics limits [9].

Evidently a self-gravitating scalar field behaves like collisionless matter and can be represented in phase space using the coherent-state representation described above provided \( l dB \ll \eta \ll L, R \). Of course to follow the field properly we must choose a grid spacing that is somewhat less than \( \lambda_{\text{dB}} \). In addition the timestep must be short enough to follow the rapid evolution of the phase factor for \( \phi \). The accuracy of a simulation is limited by constraints on memory and CPU time. The situation is similar to what is encountered in particle methods where a finite number of particles is used to provide a statistical description of the distribution function. One can improve the accuracy of a given simulation by increasing the number of particles but again this comes at the cost of memory and CPU time.

As an illustration of our method we consider “particles” in a static and plane symmetric background spacetime. For simplicity, we choose the line element \( ds^2 = A^2 dt^2 - A^{-2} dz^2 \) with \( A^2 = \exp(\omega^2 z^2) \) so that nonrelativistic particles near the origin will execute simple harmonic motion. It is convenient to write our equations in terms of the dimensionless quantities \( \zeta = \omega z, \tau = \omega t, \kappa = p/m \) and \( \mathcal{L} = m/\hbar \omega \). Eq. (1), for example, becomes
\[ \frac{\partial^2 \phi}{\partial \tau^2} - e^{2\zeta^2} \frac{\partial^2 \phi}{\partial \zeta^2} = e^{\zeta^2} \mathcal{L}^2 \phi. \]
\( \lambda_{\text{deB}} \) should be made as small as possible (this is done by increasing \( \mathcal{L} \)) but no smaller than several grid spacings. With this in mind, we choose \( \mathcal{L} \simeq 0.1N \) and \( \eta \simeq \mathcal{L}^{1/2} \) where \( N \) is the number of gridpoints used in the simulation. There are roughly \( N^{1/2} \) resolution elements in both position and momentum space which is roughly what one would expect for an N-body simulation with \( N \) particles.

Consider first an initially cold (zero velocity dispersion) distribution of particles. The field configuration at \( \tau = 0 \) is taken to be \( \phi(\zeta) = \exp\left(-\zeta^2/\zeta_0^2\right) \) and \( \partial\phi/\partial t = -i\mathcal{L}\phi \) with \( \zeta_0 = 0.5 \).

The corresponding \( \mathcal{F} \) is shown in the top left panel of Fig 1. The phase space configuration at \( \tau = 4\pi \) (i.e., time when particles near the center of the distribution have made two complete orbits) is shown in the bottom left panel. For comparison, the results of an N-body calculation are shown in the right-hand panels. As expected, particles near the center of the distribution execute simple harmonic motion while particles initially at \( z > \sim 1 \) follow anharmonic, relativistic orbits in phase space.

We next consider a system of “hot” particles. In particular, we assume the system is initially described by a truncated isothermal distribution function: \( f(\epsilon) = \Theta(\epsilon) \exp(\epsilon/T') \) where \( \epsilon \equiv p_0/E_{\text{max}} \) and \( T' \equiv T/E_{\text{max}} \). For our simulation, we take \( E_{\text{max}} = 2m \) and \( T = m \). \( p_0 \) is conserved for each particle in the system and therefore \( f(\epsilon) \) is constant.

The initial wavefunction for the Klein-Gordon simulation is taken to be

\[
\phi(x) = \frac{1}{N} \sum_{i=1}^{N} \langle e(x_i, p_i) | x \rangle
\]

(17)

where the pairs \( (x_i, p_i) \) are chosen at random from the distribution function. This is reminiscent of N-body simulations where a statistical representation of the initial distribution function is constructed by specifying the positions and velocities of \( N \) super-particles. Much of the art in N-body simulations is in calculating the forces and various methods (e.g., particle-particle with smoothing, particle-mesh) are used to deal with such problems as artificial two-body relaxation. Here, the discrete particles are replaced by wavepackets and so smoothing is built in. The pros and cons of our method, as compared with N-body techniques, will be explored in a future publication [10].

Fig. 2 shows the distribution function \( \mathcal{F} \) for \( \tau = 0 \) and \( \tau = 8\pi \); Fig. 3 gives \( \mathcal{F}(p_0) \) (calculated from \( \mathcal{F}(x, p) \) using a simple binning procedure) again for \( \tau = 0 \) and \( \tau = 8\pi \). While individual features of \( \mathcal{F} \) change (this also occurs in particle realizations of \( f \)) the overall structure of the distribution function remains constant.

This work illustrates that simulations of collisionless matter in general relativity can be done by following the dynamics of a massive scalar field and in so doing, bridges the gap between two very separate branches of numerical relativity. Applications to self-gravitating systems with more complicated geometries are straightforward only because of the large volume of work on relativistic scalar fields that already exists.

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References

[1] S. L. Shapiro and S. A. Teukolsky, Astrophys. J. 307, 575 (1986) and references therein.
[2] F. A. Rasio, S. L. Shapiro, and S. A. Teukolsky, Astrophys. J. 344, 146 (1989).
[3] See, for example C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, W. H. Freeman & Co., San Francisco, 1970, Section 22.5
[4] L. M. Widrow and N. Kaiser, Astrophys. J. Lett. 416 L71 (1993).
[5] See, e.g., D. S. Goldwirth and T. Piran, Phys. Rev. D 40, 3263 (1989).
[6] M. W. Choptuik, Phys. Rev. Lett. 70, 9 (1993).
[7] G. Kaiser, J. Math. Phys. 18, 952 (1977).
[8] E. Calzetta, S. Habib, and B. L. Hu, Phys. Rev. D 37, 2901 (1988) and references therein.
[9] S. R. de Groot, V. A. van Leeuwen, and Ch. G. van Weert, Relativistic Kinetic Theory (North Holland Publishing Co., Amsterdam) (1980) pp 70-73.
[10] G. Davies and L. M. Widrow, in preparation.

Figure Captions

FIG.1. Evolution in phase space of an initially cold distribution of particles. The system is set in a fixed background spacetime as described in the text. 4096 grid points are used for the Klein-Gordon simulation. Left-hand panels give the Klein-Gordon distribution function $\mathcal{F}$ while right-hand panels give the corresponding N-body distribution function $f$.

FIG.2. Evolution of a “hot” system in phase space. The initial distribution function is that of a truncated isothermal sphere and is shown in the left panel. The distribution function for $\tau = 8\pi$ is shown in the right panel. 2048 grid points are used for the simulation.

FIG.3. Distribution function $\mathcal{F}(E)$. $\mathcal{F}(E)$ is calculated from the $\mathcal{F}(x,p)$ of Fig.2 using a simple binning procedure. Solid curve is the initial $\mathcal{F}(E)$; dashed curve is the $\mathcal{F}(E)$ at $\tau = 8\pi$; dotted curve is the $\mathcal{F}(E)$ used to construct the initial wavefunction.
