Abstract

We search a gravitational system which allows a non-relativistic hybrid geometry interpolating the Schrödinger and Lifshitz spacetimes as a solution, as a continuation of the previous work employing a flow equation. As such a candidate an Einstein-Maxwell-Higgs system naturally arises and we verify that this system indeed supports the hybrid geometry with the help of a gauge-fixing term for diffeomorphism. As a result, this gravitational system may be interpreted as a holographic dual of a general non-relativistic system at the boundary.

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1 Introduction

It is a new paradigm to realize quantum theory of gravity as a hologram of a quantum field theory at boundary \[1, 2\] and has attracted much interest since the discovery of the AdS/CFT correspondence \[3\], which admits non-trivial tests by explicit computation \[4, 5\]. So far, various kinds of holographic duals have been proposed. Among them, we are interested here in gravity duals for non-relativistic (NR) theories based on an NR conformal symmetry called the Schrödinger symmetry \[6, 7\] and a Lifshitz scaling symmetry \[8\]. (For a nice summary, see, for example, \[9\].)

From the point of view to construct a bulk gravitational system gradually from a boundary theory \[10\], it is important to specify a scale at the boundary which plays the role of the holographic radial direction. There are several proposals to describe how the bulk radial direction emerges. As one approach, some of the authors of the present letter have proposed that a holographic direction may be described by a flow equation \[11–14\] that coarse-grains operators in a non-local fashion \[15, 16\]. In comparison to other approaches, this flow equation method has several advantages. One of them is to make it possible to construct a metric of the bulk geometry directly. In particular, it is possible to construct an AdS geometry with a general conformally flat boundary \[17\] or with a quantum-mechanically corrected bulk cosmological constant \[18\] and a holographic geometry for a general NR system \[19\].

The resulting metric of the holographic space-time for an NR system is expressed as a three-parameter deformation of the \(d + 1\)-dimensional AdS geometry \[19\],

\[
ds^2 = \ell^2 \left[ -\alpha \left( \frac{dx^+}{\tau^1} \right)^2 + \frac{d\tau^2 + (dx^i)^2 + 2(1 + \beta)dx^+dx^-}{\tau^2} + \gamma(dx^-)^2 \right],
\]

(1.1)

where \(\ell\) is the AdS radius, \(i = 1, \cdots, d - 2\), and \(\alpha, \beta, \gamma\) are real parameters satisfying

\[
\alpha \geq 0, \quad \alpha \gamma + (1 + \beta)^2 > 0, \quad (d - 2)\alpha \gamma + d(1 + \beta)^2 > 0.
\]

(1.2)

At a general point in the parameter space, this metric describes a \(d\)-dimensional Lifshitz space-time with the dynamical critical exponent 2 times a straight line, while it enhances to the Schrödinger space-time in \(d + 1\) dimensions when \(\beta = \gamma = 0\). Therefore, this geometry would deserve to be called an NR hybrid geometry.

So far, while only the metric has been computed, a gravitational system to support this geometry has not been identified. In particular, the treatment of the matter sector in the context of the flow equation method has not been discussed. The purpose of this letter is to resolve these issues, and to stress that the flow equation method works well for the matter sector as well, not restricted to the metric. This result strongly indicates that the flow equation method can capture gravitation beyond geometry.

The organization of this letter is the following. In section 2 we review how the NR hybrid geometry is obtained by employing the flow equation method, and propose a candidate of the dual bulk theory which allows the NR hybrid geometry as a solution. In
this candidate system is verified to really support the NR hybrid geometry. The last section is devoted to some discussion.

## 2 NR hybrid geometry from boundary

### 2.1 A review of the flow equation method

In this section, we shall give a brief review of the results of Ref. [19] on how the NR hybrid geometry (1.1) can be derived starting from a non-relativistic conformal field theory (NRCFT) by the flow equation method.

Let us consider a primary scalar operator $O(\vec{x}, t)$ with a general conformal dimension $\Delta$ in a $d-1$-dimensional NRCFT. Note that a primary field in NRCFT is complex in relation to a $U(1)$ symmetry group associated with the conservation of particle number.

The two-point function of primary operators is almost completely determined by an NR conformal symmetry as

$$\langle O(\vec{x}_1, t_1)O(\vec{x}_2, t_2) \rangle = \frac{1}{(t_{12})^\Delta} f \left( \frac{\vec{x}_{12}^2}{2t_{12}} \right), \quad (2.1)$$

where $t_{12} = t_1 - t_2$, $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$, and $f(x)$ is a function depending on the given theory. The function $f$ is not necessarily an eigenfunction of the mass operator, which is the center in the non-relativistic conformal algebra. Then, by introducing an extra direction, the mass operator is realized by a differential operator along this direction, and the two-point function is now expressed as

$$\langle O(\vec{x}_1, x^+_1, x^-_1)O(\vec{x}_2, x^+_2, x^-_2) \rangle = \frac{1}{(x^+_{12})^\Delta} f \left( x^-_{12} + \frac{x^2_{12}}{2x^+_{12}} \right), \quad (2.2)$$

where $x^-$ is the extra direction and we have set $x^+ = t$.

To construct the holographic geometry, let us course-grain the conformal primary operator by a non-relativistic free flow equation of the form

$$\frac{\partial}{\partial \eta} \phi(x; \eta) = (\partial^2 + 2\partial_- \partial_+ + 2\bar{m}\partial_+) \phi(x; \eta), \quad \phi(x; 0) = O(x), \quad (2.3)$$

where $\eta$ is a smearing variable and $\bar{m}$ is a real parameter of mass dimension one. Due to the parameter $\bar{m}$, the two-point function of the flowed operator $\phi$ is free from the contact singularity in a general parametrization. Thus, the flowed field can be normalized as

$$\sigma(x; \eta) = \frac{\phi(x; \eta)}{\sqrt{\langle \phi(x; \eta)\phi^\dagger(x; \eta) \rangle}}, \quad (2.4)$$

so that $\langle \sigma(x; \eta)\sigma^\dagger(x; \eta) \rangle = 1$. Then the two-point function of normalized fields is given by

$$\langle \sigma(x_1; \eta_1)\sigma^\dagger(x_2; \eta_2) \rangle = \left( \frac{4\eta_1\eta_2}{\eta^2_+} \right)^{\Delta/2} G \left( \frac{2(x^+_1 + 2i\bar{m}\eta_+)x^-_1 + (\vec{x}_{12})^2}{\eta_+}, \frac{x^2_{12}}{\eta_+} \right), \quad (2.5)$$
where \( \eta_+ = \eta_1 + \eta_2 \) and \( G(u, v) \) is a scalar function (depending on the original \( f \)) satisfying \( G(0, 0) = 1 \).

By using the normalized field, the so-called metric operator can be defined as
\[
g_{MN}(x; \eta) \equiv \frac{1}{2} \partial_M \sigma(x; \eta) \partial_N \sigma^\dagger(x; \eta),
\]
where the bracket \( \{ , \} \) denotes symmetrizing the indices like \( \{ A, B \} := AB + BA \).

The proposal is that the vacuum expectation value of the metric operator (2.6) provides the metric of the bulk space,
\[
ds^2 = \langle \hat{g}_{MN} \rangle \, dz^M dz^N,
\]
where \( z^M \) are the coordinates describing the bulk space. Note that the flow variable \( \eta \) is related to the holographic radial coordinate. The resulting metric is given by
\[
ds^2 = \Delta \eta^2 + \frac{1}{4\eta^2} G^{(0,2)}(\vec{0}) (dx^+)^2 + 2 \frac{1}{\eta} G^{(1,0)}(\vec{0}) - 2i\bar{m} G^{(1,1)}(\vec{0}) dx^+ dx^- \\
\quad + (4\bar{m})^2 G^{(2,0)}(\vec{0}) (dx^-)^2 + \frac{\delta_{ij}}{\eta} G^{(1,0)}(\vec{0}) dx^i dx^j,
\]
where
\[
G^{(n,m)}(\vec{0}) \equiv \left. \frac{\partial^n}{\partial X^n} \frac{\partial^m}{\partial Y^m} G(X, Y) \right|_{X=Y=0}.
\]

There are constraints coming from the flow equation (2.3) like
\[
-\Delta = 2(d-1) G^{(1,0)}(\vec{0}) + 8i\bar{m} G^{(1,1)}(\vec{0}) + 2i\bar{m} G^{(0,1)}(\vec{0}).
\]

By performing a coordinate transformation
\[
\eta = - \frac{G^{(1,0)}(\vec{0})}{\Delta} \tau^2,
\]
the metric (2.8) can be rewritten to the form (1.1) under the identification
\[
\alpha = \frac{\Delta G^{(0,2)}(\vec{0})}{4G^{(1,0)}(\vec{0})^2}, \quad \beta = \frac{2i\bar{m} G^{(1,1)}(\vec{0})}{G^{(1,0)}(\vec{0})}, \quad \gamma = \frac{4\bar{m}^2 G^{(2,0)}(\vec{0})}{\Delta}, \quad \ell^2 = \Delta.
\]
Note that \( \tau \) is the radial direction in AdS.
2.2 Searching for the dual bulk theory

It is natural to ask if there exists a gravitational theory which exhibits the NR hybrid geometry (1.1) as a solution to the equations of motion. To look for the dual bulk theory it is important to find out the so-called pre-geometric operators, which convert to geometric objects after taking the expectation value \[18\]. If there exists a symmetry group such as $O(N)$ whose rank is taken to be large for the emergence of the bulk, these operators are constructed so as to be invariant under its global transformation.

In the current situation the phase rotation group plays this role, and operators invariant under the action are not only the spin-2 operator $\hat{g}_{\mu\nu}$ but also the following spin-1 operator

$$\hat{A}_\mu(x; \eta) \equiv \frac{i}{2} (\sigma(x; \eta) \partial_\mu \sigma^\dagger(x; \eta) - \partial_\mu \sigma(x; \eta) \sigma^\dagger(x; \eta)).$$

(2.13)

Due to the fact that a non-relativistic conformal primary field takes values in the complex numbers, the vacuum expectation value of the spin-1 operator is non-trivial:

$$A_-(z) = 4\bar{m}G^{(1,0)}(\vec{0}), \quad A_+(z) = \frac{i\Delta G^{(0,1)}(\vec{0})}{2G^{(1,0)}(\vec{0}) \tau^2},$$

(2.14)

where we have set

$$A_\mu(z) = \langle \hat{A}_\mu(x; \eta) \rangle,$$

(2.15)

and the other components vanish. Therefore, one can expect that not only the metric tensor $g_{\mu\nu}$ but also the gauge field $A_\mu$ may dynamically exist in the bulk theory.

In fact, this anticipation is consistent with a common understanding in the holography that a global symmetry of the boundary theory becomes a gauge symmetry of the bulk theory \[4\]. (See also \[20, 21\].) To see this, let us gauge the $U(1)$ phase rotation $O \to e^{i\lambda}O$ so that the parameter depends not only on the coordinates in the boundary but also the radial direction in the bulk. Then the spin-1 operator is transformed as

$$\hat{A}_\mu \to \hat{A}_\mu - i\partial_\mu \lambda,$$

(2.16)

which is an abelian gauge transformation. Thus, one can see that the spin-1 operator is converted to a gauge field in the bulk. This result indicates that the symmetry associated with the particle number corresponds to the abelian gauge symmetry in the bulk \[22\]. Applying this argument to the flowed primary operator $\sigma$, this physical degree of freedom converts to a charged scalar field with the gauge charge unity.

Then a spin-2 operator defined in a suitable way should be invariant under the local $U(1)$ transformation, while the one defined by (2.6) is not. There is still a possibility that extra terms can be absorbed by diffeomorphism, though our preliminary analysis suggests
that it may happen by considering a special situation. We leave the detailed analysis of this issue to a future work.

As a result, one may argue that the field contents in the bulk theory are the metric, an abelian gauge field and a charged scalar field, and thus the dual bulk theory should be a low energy effective action describing the dynamics of these fields. A natural candidate of

| Operators in NRCFT       | Bulk dynamical fields |
|--------------------------|-----------------------|
| Flowed primary operator  | σ                      |
| Gauge field operator     | ̂A_μ                   |
| Metric operator          | ̂g_μν                  |
| Charged scalar field     | Φ                      |
| Gauge field              | A_μ                    |
| Metric tensor            | g_μν                   |

Table 1: An anticipated correspondence between the bulk dynamical fields and the boundary operators.

the system may be the Einstein-Hilbert action minimally coupled to an abelian gauge field as well as a charged scalar field with a certain potential. This system has been investigated before to study holographic duals of condensed matter systems. (See for instance [23–25] and references therein.)

Indeed, one can show that the Einstein-Maxwell-Higgs system is reduced to the Einstein theory coupled to a massive vector field by choosing a wine-bottle potential and making the scalar field condensed. Then it is known that the Einstein-massive vector model admits the Schrödinger space-time [6, 7, 9]. In other words, although the NR hybrid geometry (1.1) contains three parameters, the parameter ̂α can be explained by the Einstein-Maxwell-Higgs system. However, from our preliminary analysis, one can see that the rest of the parameters ̂β and ̂γ cannot be explained no matter how the scalar potential is chosen. Then a question is how we can take account of ̂β and ̂γ without adding any physical degrees of freedom to the bulk, or whether there exists a bulk theory to support the NR hybrid geometry as a solution.

A clue to answer this question is in the origin of ̂β and ̂γ. As seen from (2.12), these parameters come from ̂m contained in the flow equation. Since a choice of flow equation fixes how to perform course-graining of operators, a parameter in a flow equation is not a physical parameter in the theory. This implies that ̂β and ̂γ should be regarded as gauge degrees of freedom.\footnote{This is also suggested by the expectation value of the spin-1 field (2.14).} For this reasoning, we repeat the same analysis by adding a

\begin{equation}
  ds_{\text{inf}}^2 := \frac{1}{2} \text{tr}(d\rho(z)d\rho(z)) = g_{MN}^{\text{inf}}(z)dz^M dz^N,
\end{equation}

where  ρ(z) is the density matrix defined by \( ρ(z) := σ(x; η)|0⟩⟨0|σ(x; η)^{\dagger} \), and an explicit form of \( g_{MN}^{\text{inf}}(z) \) can be found in the footnote 1 of [19]. This is invariant under the local U(1) transformation preferably, whereas a metric operator whose vacuum expectation value becomes \( g_{MN}^{\text{inf}}(z) \) seems to be absent, unless the large \( N \) factorization occurs, which causes a trouble, for instance, when correlation functions of gravitons are computed from the boundary by using this framework.
gauge fixing term for the abelian gauge symmetry as well as that for diffeomorphism to the Einstein-Maxwell-Higgs system, and it turns out that the Einstein-Maxwell-Higgs theory with a gravitational gauge-fixing term admits the NR hybrid geometry with the general three parameters. We will verify this argument in the next section.

3 The dual bulk theory

As argued in the previous section, let us consider an Einstein-Maxwell system minimally coupled to a charged scalar field with a certain potential accompanied with a gravitational gauge-fixing term

\[ S = \int d^{d+1}x \sqrt{|g|} \left[ \frac{1}{2\kappa^2} (R - 2\Lambda) - \frac{1}{4} g^\mu\nu g^\rho\sigma F_\mu F_\nu F_\rho F_\sigma - g^\mu\nu D_\mu \Phi D_\nu \Phi - V(|\Phi|^2) - \frac{1}{2\xi} (g^-)^2 \right]. \]  

(3.1)

The first term is the Einstein-Hilbert term in the Einstein frame with a negative cosmological constant \( \Lambda \) given by

\[ \Lambda = -\frac{d(d-1)}{2\ell^2}. \]  

(3.2)

The second term is the Maxwell term with the canonical normalization. Note that the gauge coupling constant is normalized to be unity by employing the Weyl transformation. The third one is the kinetic term of a charged scalar field and the covariant derivative is defined as

\[ D_\mu \Phi \equiv \partial_\mu \Phi + iqA_\mu \Phi, \]  

(3.3)

where \( q \) is the gauge charge of the scalar field. The fourth term is the potential of the scalar field, which is supposed to have a minimum value away from the origin so that the gauge symmetry is spontaneously broken when the scalar field condensates at the potential minimum. The last term is a gauge fixing term for diffeomorphism, where \( \xi \) is a gauge-fixing parameter.

Let us show that this system supports the non-relativistic hybrid geometry with the following configuration of the gauge field,

\[ A_+ = \frac{a_+}{\tau^2}, \quad A_- = a_-, \]  

(3.4)

\(^3\) In principle there is no way to distinguish between a cosmological constant and the expectation value of the potential physically. However in a situation where NRCFT is obtained from a parent CFT by a certain deformation, it would be natural to think that the cosmological constant is unchanged under the deformation and the deviation of the vacuum energy of NRCFT from that of CFT is accounted for by a newly added term, namely the value of the scalar potential. Thus we assume the same value of the cosmological constant of the AdS in \( \Lambda \).

\(^4\) In order to obtain our final result, it is important to fix diffeomorphism by breaking the Lorentz invariance. For example, a Fierz-Pauli-like term \( \frac{1}{2\xi} g_{\mu\nu} g^{\rho\sigma} \) or a gauge fixing term for the gauge symmetry such as \( \frac{1}{2\xi} (A_-)^2 \) are not relevant to our final result.
where $a_\pm$ are real constants, which are related to the boundary theory by

$$a_+ = \frac{i\Delta G^{(0,1)}}{2G^{(1,0)}}, \quad a_- = 4\bar{m}G^{(1,0)}.$$

(3.5)

To this end, it is sufficient to show that the metric (1.1) and the gauge field (3.4) satisfy the equations of motion.

The Einstein equation is given by

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \kappa^2 T_{\mu\nu},$$

(3.6)

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor for the matter fields computed as

$$T_{\mu\nu} = T_{\mu\nu}^A + T_{\mu\nu}^\Phi + T_{\mu\nu}^{gf},$$

(3.7)

$$T_{\mu\nu}^A = -\frac{1}{4}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} + g^{\alpha\sigma}F_{\mu\rho}F_{\nu\sigma},$$

(3.8)

$$T_{\mu\nu}^\Phi = g_{\mu\nu}(-D^\rho \Phi^* D_\rho \Phi - V(|\Phi|^2)) + D_{[\mu} \Phi^* D_{\nu]} \Phi,$$

(3.9)

$$T_{\mu\nu}^{gf} = g_{\mu\nu} \left( -\frac{1}{2\xi} (g^{\rho\gamma})^2 \right) + \frac{1}{\xi} \delta_{\mu}^\rho \delta_{\nu}^\gamma g^{\rho\gamma},$$

(3.10)

where the curly bracket for two indices denotes the symmetrization again. Note that we have included the contribution coming from the gauge fixing term in the stress-energy tensor.

The equation of motion for the gauge field is given by

$$\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} F^{\mu\nu}) = i(\Phi^* D^\nu \Phi - D^\nu \Phi^* \Phi).$$

(3.11)

The equation of motion for the scalar field is satisfied if it stays at the potential minimum. We write the vacuum expectation value as

$$\langle \Phi \rangle = r e^{i\theta}$$

(3.12)

in the form of a polar decomposition. Plugging this into the right-hand side of (3.11), we find

$$i(\Phi^* D^\nu \Phi - D^\nu \Phi^* \Phi) = 2r^2(-qA^\nu - \partial^\nu \theta).$$

(3.13)

On the other hand, by plugging (1.1) and (3.4) into the left-hand side of (3.11), it becomes

$$\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} F^{\mu\nu}) = \frac{2a_+ (\delta_\mu^\nu (d-2)\gamma \tau^2 - \delta_\mu^\nu d(1+\beta))}{\ell^4 (\alpha \gamma + (1+\beta)^2)}.$$
In the assumed parameter region (1.2), the equation of motion for the gauge field (3.11) can be solved by

\[
\begin{align*}
    r &= \sqrt{\frac{(d-2)\alpha\gamma + d(\beta + 1)^2}{\ell^2(\alpha \gamma + (1 + \beta)^2)}}, \\
    \theta &= q \left( a_+ - \frac{2a_+\gamma (1 + \beta)}{(d-2)\alpha\gamma + d(\beta + 1)^2} \right) x^-.
\end{align*}
\] (3.15)

For these configurations, the stress energy tensors are computed as

\[
\begin{align*}
T^\Phi_{\mu\nu} &= g_{\mu\nu} \left( \frac{(qa_+)^2\gamma((d-2)^2\alpha\gamma + d(d-4)(\beta + 1)^2)}{\ell^4(\alpha \gamma + (1 + \beta)^2)^2} - V(r^2) \right) \\
    &\quad + 2r^2 \left( \frac{(qa_+)^2}{\tau^4} \delta^+_{\mu} \delta^+_{\nu} + 2 \frac{(qa_+)^2}{\tau^2} \delta^{\mu}_{\nu}(\beta + 1)^2 \delta_{\nu}^+ \right) \\
    &\quad + \left( \frac{qa_+\gamma 2(\beta + 1)}{(d-2)\alpha\gamma + d(\beta + 1)^2}\alpha^2 \right) (2a_+)^2. \quad (3.16)
\end{align*}
\]

\[
\begin{align*}
T^A_{\mu\nu} &= \left( \frac{1}{2} g_{\mu\nu} \gamma \left[ \frac{\alpha}{\ell^2(\alpha \gamma + (1 + \beta)^2)^2} + \delta^+_{\mu} \delta^+_{\nu} \frac{1}{\ell^2\tau^4} + \delta^-_{\mu} \delta^-_{\nu} \frac{-\gamma}{\tau^2 \ell^2(\alpha \gamma + (1 + \beta)^2)} \right] \right) (2a_+)^2, \quad (3.17)
\end{align*}
\]

\[
T^g_{\mu\nu} = g_{\mu\nu} \left( - \frac{1}{\xi} \frac{\alpha}{\ell^2(\alpha \gamma + (1 + \beta)^2)^2} + \frac{1}{\xi} \delta^-_{\mu} \delta^-_{\nu} \frac{\alpha}{\ell^2(\alpha \gamma + (1 + \beta)^2)} \right). \quad (3.18)
\]

On the other hand, the Einstein tensor with the contribution of the cosmological constant for the metric (1.1) is computed as

\[
G_{\mu\nu} + g_{\mu\nu}\Lambda = \alpha \gamma \left( \frac{\alpha}{\ell^2(\alpha \gamma + (1 + \beta)^2)^2} + \delta^+_{\mu} \delta^+_{\nu} \frac{\gamma((d-2)(1 + \beta)^2 + d\alpha\gamma)}{\alpha \gamma + (1 + \beta)^2} + \delta^+_{\mu} \delta^-_{\nu} \frac{2(1 + \beta)\alpha\gamma}{\tau^2(\alpha \gamma + (1 + \beta)^2)^2} + \delta^+_{\mu} \delta^+_{\nu} \frac{(d+2)(1 + \beta)^2 + d\alpha\gamma}{\alpha \gamma + (1 + \beta)^2} + \delta^-_{\mu} \delta^-_{\nu} \frac{-2\alpha\gamma}{\alpha \gamma + (1 + \beta)^2} \right). \quad (3.19)
\]

Plugging these into the Einstein equation, we obtain five equations to be satisfied from the components \((\mu, \nu) = (i, j), (+, +), (+, -), (-, -), (\tau, \tau)\). The equations for the \((+, +)\) and \((+, -)\) components are satisfied if and only if the gauge charge of the scalar field is unity, \(q = 1\), and

\[
a_+^2 = \frac{\ell^2 \alpha}{2\kappa^2}, \quad (3.20)
\]

thus the gauge flux does not vanish, as expected. Then the equation for the \((\tau, \tau)\) component is automatically satisfied. The equation for the \((-,-)\) component fixes the parameter in the gauge-fixing term as

\[
\frac{1}{\xi} = \frac{d(d-2)[\alpha \gamma + (1 + \beta)^2]}{(d-2)\alpha \gamma + d(1 + \beta)^2} \frac{\gamma \ell^2}{\alpha \kappa^2}. \quad (3.21)
\]
The equation for the \((i, j)\) component fixes the potential minimum as

\[
V(r^2) = \frac{a_2^2 \gamma (\alpha \gamma (d - 1)^2 + (d + 1)(\beta (\beta + 2)(d - 4) - 2))}{\ell^4 (\alpha \gamma + (\beta + 1)^2) (\alpha \gamma (d - 1) + (\beta + 1)^2(d + 1))}.
\] (3.22)

This completes the verification of our claim. Note that from (3.20) the flow field determines the Newton constant \(\kappa^2\) as

\[
\kappa^2 = \frac{G^{(0,2)}(\bar{0})}{2(iG^{(0,1)}(\bar{0}))^2}.
\] (3.23)

It is worth commenting on the Higgs mechanism to reduce the Einstein-Maxwell-Higgs theory to the Einstein theory with a massive vector field. To this end, the matter field is expanded around the vacuum expectation value like

\[
\Phi = e^{i\theta}(r + x + iy), \quad A_\mu = \langle A_\mu \rangle + \delta A_\mu.
\] (3.24)

Here \(r, \theta\) are given in (3.15) and \(\langle A_\mu \rangle\) follows from (3.4). Then \(x, y, \delta A_\mu\) are fluctuation fields taking real values. Then the matter part of the action (3.1) can be expanded, up to the quadratic order. From the expansion of the potential term, we find that the field \(y\) is a massless mode corresponding to the Nambu-Goldstone boson and \(x\) is a massive mode with the mass squared \(4V''(r^2)r^2\). By making the potential so steep that the mass is much heavier than the energy scale of our interest, the dynamics of the field \(x\) becomes negligible, while the massless mode is absorbed by the gauge field so as to become a massive vector field. As a result, by setting

\[
\tilde{A}_\mu = \delta A_\mu + \frac{1}{r}(\partial_\mu + i \langle A_\mu \rangle + i \partial_\mu \theta)y, \quad m^2 = 2r^2,
\] (3.25)

we obtain

\[
S = \int d^{d+1}x \sqrt{|g|} \left[ \frac{1}{2\kappa^2} (R - 2\Lambda) - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} \tilde{F}_{\mu\rho} \tilde{F}_{\nu\sigma} - \frac{1}{2} m^2 g^{\mu\nu} \tilde{A}_\mu \tilde{A}_\nu - \frac{1}{2\xi} (g^{\mu\nu})^2 \right],
\] (3.26)

where \(\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu\) and the irrelevant terms have been ignored. As a result, the Einstein-Maxwell-Higgs theory (3.1) reduces to the Einstein theory with a massive vector field (3.26) by choosing the potential appropriately. Thus, it has been shown that the Einstein-massive vector theory with a suitable gauge-fixing term also supports the NR hybrid geometry (1.1) as a solution.

### 4 Discussion

In this letter we have considered the dual gravitational theory of a general NRCFT and proposed that the theory is the Einstein theory minimally coupled to an abelian gauge

\footnote{For this expansion a background dependent Lorenz gauge \(g^{\mu\nu} (\nabla_\mu - ie \langle A_\mu \rangle ) \delta A_\nu = 0\) may be useful.}
field and a Higgs field with a gravitational gauge fixing term, by confirming explicitly that
the theory admits the NR hybrid geometry.

In spite of our explicit verification, we have not fully understood the role of the gravita-
tional gauge-fixing term in the context of holography. It seems that only zero or nonzero
are meaningful about the parameter of the gauge fixing term and induce an important
effect to solve the equations of motion. This result may indicate that a particular choice
of smearing would correspond to a particular gauge choice in the bulk. It is significant to
elaborate this point in detail.

It is also interesting to test our proposal by computing correlation functions for both
sides and comparing them based on the dictionary in Tab. 2.2. An advantage of the
flow equation method is that one can perform explicit computations by using traditional
techniques of quantum field theory such as the $1/N$ expansion, which enables one to verify
his/her proposal in an analytic way.

We hope to come back to these issues in the near future.

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