Perturbations of cosmological and black hole solutions in massive gravity and bi-gravity

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We investigate perturbations of a class of spherically symmetric solutions in massive gravity and bi-gravity. The background equations of motion for the particular class of solutions we are interested in reduce to a set of the Einstein equations with a cosmological constant. Thus, the solutions in this class include all the spherically symmetric solutions in general relativity, such as the Friedmann–Lemaître–Robertson–Walker solution and the Schwarzschild (–de Sitter) solution, though the one-parameter family of two parameters of the theory admits such a class of solutions. We find that the equations of motion for the perturbations of this class of solutions also reduce to the perturbed Einstein equations at first and second order. Therefore, the perturbative stability of the solutions coincides with that of the corresponding solutions in general relativity at least up to the second-order perturbations.

Subject Index E03

1. Introduction

Massive gravity is one of the potent candidates for a modified theory of general relativity. As early as 1939, a linear theory of massive gravity was proposed by Fierz and Pauli (FP) [1]. In order to avoid the inconsistency on massless limit of this theory [2,3], a nonlinear extension of the FP theory was considered [4]. Boulware and Deser, however, found that the nonlinear theory simply extended from the FP theory contains an unphysical ghost degree of freedom (BD ghost) [5]. Because of this ghost problem, a healthy theory of nonlinear massive gravity has not been established for a long time.

Recently, de Rham, Gabadadze, and Tolley (dRGT) proposed a mass potential that can remove the BD ghost mode in a decoupling limit [6,7], and Hassan and Rosen have finally proven that the dRGT massive gravity theory is free from the BD ghost without taking the decoupling limit [8–10]. The dRGT theory of massive gravity has three parameters: graviton mass $m$ and coupling constants of nonlinear self-interactions, $\alpha_3, \alpha_4$. Complementary approaches of the BD ghost problem are studied in Refs. [11–15].

Massive gravity includes a nondynamical tensor field called the fiducial metric, $f_{\mu\nu}$, in order to construct a mass potential. For example, the FP and the original dRGT theories are constructed by adopting the Minkowski fiducial metric. Hassan and Rosen proposed an extended theory of dRGT massive gravity with a fiducial metric being dynamical as well by introducing the Einstein–Hilbert term of the fiducial metric in the action. They proved that this theory is also free from the BD
ghost [16]. Since this theory contains two symmetric dynamical tensor fields of metrics, it is called bi-gravity theory.

Tests of massive gravity and bi-gravity using cosmological and black hole solutions have been explored intensively. In dRGT massive gravity, several types of exact, homogeneous, and isotropic solutions have been found so far. One example is the open Friedmann–Lemaître–Robertson–Walker (FLRW) solution with the flat fiducial metric in the FLRW slice [17,18]. However, nonperturbative instability of this solution is found in [19], and hence this solution is not viable, unfortunately. It should be noted that similar solutions with an anisotropic fiducial metric are known to be stable [21–23]. Another example of cosmological solutions is with a fiducial metric which is flat but expressed in terms of nontrivial coordinates. This class of solutions can be divided into two types. The first type includes the solutions found in Refs. [24–26], which exist for the whole parameter region of $\alpha_3$ and $\alpha_4$, while the other type includes the solutions found in Refs. [27,28], which exist only for a one-parameter family in the parameter space $(\alpha_3, \alpha_4)$. Though perturbations of the former type of solutions have already been studied in Refs. [29–32], perturbations of the latter type of solutions have not yet been investigated. Therefore, in this paper, we focus on the latter type of solutions. It should be noted that cosmological solutions with non-flat fiducial metrics are also studied in Refs. [33–36].

The situation for cosmological solutions in bi-gravity is similar. The cosmological solutions found thus far are divided into two classes [37–40]. The first class is a solution with diagonal metric tensors [41–44]. Perturbations of this class of solutions have been studied intensively [45–57]. On the other hand, for the other class of solutions with off-diagonal components of the physical or fiducial metric tensor, perturbations have not yet been studied, similarly to the case of massive gravity. It should be noted that coupling between matter and (bi-)metrics is a nontrivial issue in bi-gravity and is studied in Refs. [58–72].

A lot of static and spherically symmetric solutions have also been found up to now. The classification of such spherically symmetric solutions is studied in Refs. [40,73–75]. The exact Schwarzschild (–de Sitter) solutions are classified in the following three classes. The first class is a solution with diagonal metric tensors [76–79], and linear perturbations of this class of solutions are studied in Refs. [79–83] in the framework of both massive gravity and bi-gravity. The second class of solutions is a solution with an off-diagonal metric tensor and arbitrary $\alpha_3$ and $\alpha_4$ [84], and perturbation of this solution is studied in Refs. [79,85]. The last class is a solution with an off-diagonal metric tensor and a special choice of the parameters $\alpha_3$ and $\alpha_4$ [86–88], where linear perturbations have been studied only in massive gravity with a flat fiducial metric [89] and not in bi-gravity.

In the present article, we will give a unified and general analysis of the perturbative stability of solutions with an off-diagonal metric tensor in massive gravity and bi-gravity belonging to a one-parameter family of $\alpha_3$ and $\alpha_4$, which include both cosmological [27,28] and spherically symmetric black hole solutions [73,86–89]. We will find that the equations of motion for this class of solutions exactly reduce to those of general relativity (GR) with a cosmological constant not only at the background and linear (first-order) perturbation level but also at the level of quadratic (second-order) perturbations. This result shows that massive gravity and bi-gravity can allow any spherically symmetric solution of GR including its perturbative stability, the evolution of linear perturbations, and the back reaction from linear perturbations, while it simultaneously implies that one cannot distinguish massive gravity or bi-gravity from GR by using spherically symmetric solutions and their perturbations, at least up to quadratic order.
Our paper is organized as follows. In the next section, we briefly review the theory of bi-gravity (and massive gravity as a trivial case of a fixed fiducial metric) and derive the equations of motion in a general setting. In Sect. 3, we derive a generic nondiagonal spherically symmetric background solution. Then, we investigate linear perturbations around those background solutions in Sect. 4. There we will see that the terms coming from the mass potential must vanish in order to satisfy the Bianchi identity. In Sect. 5, we investigate higher-order perturbations and find that the same results as the linear perturbations apply for the quadratic perturbations. The final section is devoted to summary and discussion. Some details will be given in the appendices.

2. Review of bi-gravity

In this section, we give a brief review of bi-gravity. Bi-gravity is a theory consisting of two dynamical tensor fields, $g_{\mu\nu}$ and $f_{\mu\nu}$, called physical and fiducial metrics, respectively. Massive gravity can be understood as a special case where the fiducial metric is fixed and nondynamical. Its action is given by the Einstein–Hilbert term for each metric with the interaction term $S_{\text{mass}}$ and matter actions [16]:

$$S = \frac{1}{2} M_{\text{pl}}^2 \int d^4 x \sqrt{-g} R[g] + \frac{1}{2} \kappa^2 M_{\text{pl}}^2 \int d^4 x \sqrt{-f} R[f] + S_{\text{mass}}[g,f] + S_{\text{matter}}[g] + S_{\text{matter}}[f], \quad (1)$$

where $R[\cdot]$ is the Ricci scalar and $\kappa$ represents the ratio of the effective Planck masses for $g_{\mu\nu}$ and $f_{\mu\nu}$. In the case of $\kappa = 0$, the tensor field $f_{\mu\nu}$ does not have its kinetic term and hence is nondynamical. This case corresponds to the massive gravity theory originally proposed by de Rham, Gabadadze, and Tolley [6,7]. $S_{\text{matter}}[g]$ and $S_{\text{matter}}[f]$ are the matter actions coupled to $g$ and $f$, respectively,

$$S_{\text{matter}}[g] = \int d^4 x \sqrt{-g} \mathcal{L}_{\text{matter}}^{(g)}, \quad (2)$$

$$S_{\text{matter}}[f] = \int d^4 x \sqrt{-f} \mathcal{L}_{\text{matter}}^{(f)}. \quad (3)$$

Here we implicitly assume the matter actions possess the two general covariances with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$ separately, though the full theory does not have such a symmetry. It should be noted that another type of matter coupling, which does not possess the two general covariances, is also studied in [58–72]. The energy–momentum tensors coming from $S_{\text{matter}}[g]$ and $S_{\text{matter}}[f]$ are defined as

$$T_{(g)}^{\mu \nu} = \frac{2}{\sqrt{-g}} g^{\mu \rho} \frac{\delta S_{\text{matter}}[g]}{\delta g^{\rho \nu}}, \quad (4)$$

$$T_{(f)}^{\mu \nu} = -\frac{2}{\sqrt{-f}} \frac{\delta S_{\text{matter}}[f]}{\delta f^{\mu \rho}} f_{\rho \nu}. \quad (5)$$

Due to the two general covariance of matter actions we assumed, both energy–momentum tensors are conserved, that is, $\nabla_{(g)} T_{(g)}^{\mu \nu} = 0$ and $\nabla_{(f)} T_{(f)}^{\mu \nu} = 0$, where $\nabla_{(g)}$ and $\nabla_{(f)}$ are the covariant derivatives with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. Hereafter, we will omit the suffixes $g$ and $f$ when no confusion is expected.

Now, as given by [6,7,90], the interaction term $S_{\text{mass}}$ is tuned to be free from the BD ghost mode and given by

$$S_{\text{mass}}[g,f] = \frac{M_{\text{pl}}^2}{2} \int d^4 x \sqrt{-g} 2 m^2 \sum_{i=0}^{4} \beta_i e_i(\gamma), \quad (6)$$

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where $i$th order contributions $e_i(\gamma)$ are given by

\begin{align}
e_0(\gamma) &= 1, \\
e_1(\gamma) &= \text{Tr}[\gamma], \\
e_2(\gamma) &= \frac{1}{2} (\text{Tr}[\gamma]^2 - \text{Tr}[\gamma^2]), \\
e_3(\gamma) &= \frac{1}{3!} (\text{Tr}[\gamma]^3 - 3 \text{Tr}[\gamma] \text{Tr}[\gamma^2] + 2 \text{Tr}[\gamma^3]), \\
e_4(\gamma) &= \text{det}(\gamma),
\end{align}

with

\begin{align}
\gamma^\mu_\nu &= (\sqrt{g^{-1} f})^\mu_\nu,
\end{align}

and $m, \beta_i$ being free parameters of the interaction term. Since they always appear in the combination $m^2 \beta_i$, essentially there are five free parameters. The space of parameters corresponds to the three parameters of the dRGT theory, $m, \alpha_3, \alpha_4$, and two cosmological constants, $\Lambda(\mathcal{g}), \Lambda(\mathcal{f})$, and their relations are given by

\begin{align}
m^2 \beta_0 &= -\Lambda(\mathcal{g}) + m^2 (6 + 4\alpha_3 + \alpha_4), \\
m^2 \beta_1 &= m^2 (-3 - 3\alpha_3 - \alpha_4), \\
m^2 \beta_2 &= m^2 (1 + 2\alpha_3 + \alpha_4), \\
m^2 \beta_3 &= m^2 (-\alpha_3 - \alpha_4), \\
m^2 \beta_4 &= -\kappa^2 \Lambda(\mathcal{f}) + m^2 \alpha_4.
\end{align}

Taking the variation of the action with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$, we will obtain the equations of motion for the two tensor fields. The equations of motion for $g_{\mu\nu}$ are given by

\begin{align}
G[g]^\mu_\nu + X_{(g)}^\mu_\nu &= \frac{1}{M^2_{\text{pl}}} T_{(g)}^\mu_\nu,
\end{align}

where $G[g]^\mu_\nu$ is the Einstein tensor constructed from $g_{\mu\nu}$, and

\begin{align}
X_{(g)}^\mu_\nu &= 2m^2 \left( \tau^\mu_\nu - \frac{1}{2} \delta^\mu_\nu \sum_{i=0}^{3} \beta_i e_i(\gamma) \right), \\
\tau^\mu_\nu &= \frac{1}{2} \left[ \beta_1 \gamma^\mu_\nu + \beta_2 (e_1(\gamma) \gamma^\mu_\nu - (\gamma^2)^\mu_\nu)
+ \beta_3 (e_2(\gamma) \gamma^\mu_\nu - e_1(\gamma) (\gamma^2)^\mu_\nu + (\gamma^3)^\mu_\nu) \right].
\end{align}

1) It is useful to rewrite the action in term of these parameters as

\begin{align}
S_{\text{mass}}[g,f] &= \frac{M^2_{\text{pl}}}{2} \int d^4x \left[ \sqrt{-g} 2m^2 \sum_{i=2}^{4} \alpha_i e_i(\mathcal{K}) + \sqrt{-g} (-2\Lambda(\mathcal{g})) + \sqrt{-f} (-2\kappa^2 \Lambda(\mathcal{f})) \right],
\end{align}

with $\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \gamma^\mu_\nu$ and $\alpha_2 = 1$. 
The indices here are raised or lowered by $g_{\mu\nu}$. The equations of motion for $f_{\mu\nu}$ are given by

$$G[f]^\mu_v + X(f)^\mu_v = \frac{1}{\kappa^2 M_{\text{pl}}^2} T(f)^\mu_v,$$

(21)

where $G[f]^\mu_v$ is the Einstein tensor constructed from $f_{\mu\nu}$, and

$$X(f)^\mu_v = -\frac{m^2}{\kappa^2} \text{sgn}(\det[\gamma]) \left(\frac{2}{\det[\gamma]} \tau^\mu_v + \beta_4 \delta^\mu_v \right).$$

(22)

The indices here are raised or lowered by $f_{\mu\nu}$.

3. Bi-spherically symmetric background solutions

Here, we attempt to classify some of the spherically symmetric solutions in bi-gravity and identify those which obey the same equations of motion as in general relativity. These classes of solutions include the cosmological and black hole solutions known so far [27,37–40,73,89].

Let us consider the following bi-spherically symmetric metrics:

$$\tilde{g}_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{00}(t,r) dt^2 + 2\tilde{g}_{01}(t,r) dt dr + \tilde{g}_{11}(t,r) dr^2 + R(t,r)^2 d\Omega^2,$$

(23)

$$\tilde{f}_{\mu\nu} dx^\mu dx^\nu = \tilde{f}_{00}(t,r) dt^2 + 2\tilde{f}_{01}(t,r) dt dr + \tilde{f}_{11}(t,r) dr^2 + A^2(t,r) R^2(t,r) d\Omega^2,$$

(24)

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The matrix $\tilde{g}^{-1}\tilde{f}$ takes the form

$$(\tilde{g}^{-1}\tilde{f})^\mu_v = \begin{pmatrix}
(\tilde{g}^{-1}\tilde{f})_0^0 & (\tilde{g}^{-1}\tilde{f})_0^1 & 0 & 0 \\
(\tilde{g}^{-1}\tilde{f})_1^0 & (\tilde{g}^{-1}\tilde{f})_1^1 & 0 & 0 \\
0 & 0 & A^2(t,r) & 0 \\
0 & 0 & 0 & A^2(t,r)
\end{pmatrix},$$

(25)

and from these ansatz it is straightforward to see that the square root of the above matrix is of the form

$$\tilde{\gamma}^\mu_v = \left(\sqrt{\tilde{g}^{-1}\tilde{f}}\right)^\mu_v = \begin{pmatrix}
\tilde{\gamma}_0^0(t,r) & \tilde{\gamma}_0^1(t,r) & 0 & 0 \\
\tilde{\gamma}_1^0(t,r) & \tilde{\gamma}_1^1(t,r) & 0 & 0 \\
0 & 0 & A(t,r) & 0 \\
0 & 0 & 0 & A(t,r)
\end{pmatrix}.$$  

(26)

It should be emphasized that the following discussion does not rely on the concrete expressions of $\tilde{\gamma}^I_j(t,r)$ for $I,J = 0,1$, but rather relies only on the fact that $\tilde{\gamma}^\mu_v$ is of the form of Eq. (26).

As explained earlier, we are interested in the case where the equations of motion for both metrics reduce to the Einstein equations with cosmological constants at the background level. Therefore, in order for $X(g)^\mu_v$ to be a cosmological constant, the nontrivial off-diagonal components,

$$\tilde{X}_{(g)}^0_1 = -m^2 \tilde{\gamma}^0_1(3 - 2A + (A - 3)(A - 1)\alpha_3 + (A - 1)^2 \alpha_4),$$

(27)

$$\tilde{X}_{(g)}^1_0 = -m^2 \tilde{\gamma}^1_0(3 - 2A + (A - 3)(A - 1)\alpha_3 + (A - 1)^2 \alpha_4),$$

(28)

must vanish. We focus on the case of $\tilde{\gamma}^0_1 \neq 0$ or $\tilde{\gamma}^1_0 \neq 0$, and $A \neq 1$, since with $\tilde{\gamma}^1_0 = \tilde{\gamma}^0_1 = 0$, we will obtain diagonal metrics as mentioned in Sect. 1 and the perturbations of such diagonal solutions have already been studied.
For the nondiagonal solutions we are interested in, the condition that Eqs. (27) and (28) vanish leads to

\[ A(t, r) = \frac{2\alpha_3 + \alpha_4 + 1 \pm \sqrt{\alpha_3^2 + \alpha_3 - \alpha_4 + 1}}{\alpha_3 + \alpha_4} = \text{const.} \]  
\[ (29) \]

Another requirement necessary for \( \bar{X}(g)^\mu_\nu \) to be a cosmological constant is

\[ \bar{X}(g)^0_0 - \bar{X}(g)^2_2 = (1 - A)C(t, r) [A - 2 + (A - 1)\alpha_3] = 0, \]  
\[ (30) \]

where we have defined \( C(t, r) \) as

\[ C(t, r) = m^2 A^2 - A \text{tr}[\bar{\gamma}^I_J] + \text{det}[\bar{\gamma}^I_J], \]  
\[ (31) \]

and

\[ \text{tr}[\bar{\gamma}^I_J] = \bar{\gamma}^0_0 + \bar{\gamma}^1_1, \]  
\[ (32) \]

\[ \text{det}[\bar{\gamma}^I_J] = \bar{\gamma}^0_0 \bar{\gamma}^1_1 - \bar{\gamma}^0_1 \bar{\gamma}^1_0. \]  
\[ (33) \]

With \( C(t, r) = 0 \), at least three eigenvalues of \( \bar{\gamma}^\mu_\nu \) are equal to \( A \). This class of solutions includes the cosmological solutions found in Refs. [24–26] and the Schwarzschild solutions obtained in Ref. [85]. The perturbations of those solutions have already been studied in detail in Refs. [29–32].

In the present study, we therefore concentrate on the case with \( C(t, r) \neq 0 \). In this case, the solution to Eq. (30) is

\[ A = \frac{2 + \alpha_3}{1 + \alpha_3}. \]  
\[ (34) \]

Here we have assumed that \( \alpha_3 \neq -1 \). Equations (29) and (34) are consistent provided that the parameters of the theory, \( \alpha_3 \) and \( \alpha_4 \), satisfy

\[ 0 = 1 + \alpha_3 + \alpha_3^2 - \alpha_4 \left( = \beta_2^2 - \beta_1 \beta_3 \right). \]  
\[ (35) \]

This is equivalent to the condition that the two branches of the solution (29) degenerate. Thus, we see that only the particular one-parameter family of \( \alpha_3 \) and \( \alpha_4 \) satisfying (35) admits the class of solutions we are focusing on. Note that when Eq. (35) is fulfilled, \( A \) can also be expressed simply as \( A = -\beta_2 / \beta_3 \). Note also that the range of \( \alpha_4 \) is limited as \( \alpha_4 = (\alpha_3 + 1/2)^2 + 3/4 \geq 3/4 \).

In this one-parameter family of \( \alpha_3 \) and \( \alpha_4 \) with Eq. (35), the interaction terms for bi-spherically symmetric metrics (23) and (24) with Eq. (34) are of the form of a cosmological term. For \( g_{\mu\nu} \) the interaction term gives

\[ \bar{X}(g)^\mu_\nu = \Lambda_{\text{eff}}^{(g)} \delta^\mu_\nu, \]  
\[ (36) \]

with

\[ \Lambda_{\text{eff}}^{(g)} = m^2 (A - 1) + \Lambda^{(g)}, \]  
\[ (37) \]

while for \( f_{\mu\nu} \)

\[ \bar{X}(f)^\mu_\nu = \Lambda_{\text{eff}}^{(f)} \delta^\mu_\nu, \]  
\[ (38) \]
\[
\Lambda_{\text{eff}}^{(f)} = \text{sgn}(\det(\tilde{\gamma}_f)) \left( \frac{m^2 A - 1}{A} + \Lambda^{(f)} \right). \tag{39}
\]

In the case of dRGT massive gravity, \(f_{\mu\nu}\) is not a dynamical but a fixed metric, and hence we need not consider the equations of motion for \(f_{\mu\nu}\).

The class of solutions with \(C(t, r) \neq 0\) includes the cosmological solutions \([27,28]\), the black hole solutions \([86–89]\), the Lemaître–Tolman–Bondi (LTB) solution \([28]\), and the Reissner–Nordström (RN) solution \([87]\). Since the equations of motion for \(g_{\mu\nu}\) (and, in fact, those for \(f_{\mu\nu}\) as well) reduce to the Einstein equations with a cosmological constant, any spherically symmetric solution in GR is also a solution of the one-parameter subclass (35) of bi-gravity and massive gravity with a suitable fiducial metric. In Appendix A, we present some examples of bi-FLRW and bi-Schwarzschild–de Sitter solutions belonging to this class.

4. Linear perturbations

Now, we analyze linear perturbations around the bi-spherically symmetric solutions given in the previous section. It should be emphasized that the linear perturbations of this branch of solutions in bi-gravity have not been investigated in any literature. The two tensor fields of metrics are perturbed as

\[
\begin{align*}
  g_{\mu\nu} &= \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \\
  f_{\mu\nu} &= \bar{f}_{\mu\nu} + \delta f_{\mu\nu}.
\end{align*}
\tag{40, 41}
\]

The first-order perturbation, \(\delta \gamma^\mu_\nu\), of \(\gamma^\mu_\nu\) is defined as

\[
\sqrt{-g^{-1}f} = \gamma^\mu_\nu = \tilde{\gamma}^\mu_\nu + \delta \gamma^\mu_\nu + O(\text{second-order perturbations}), \tag{42}
\]

which can be written in terms of the metric perturbations by solving the equations

\[
\tilde{\gamma}^\mu_\rho \delta \gamma^\rho_\nu + \delta \gamma^\mu_\rho \tilde{\gamma}^\rho_\nu = -\delta g^\mu_\rho \tilde{\gamma}^\sigma_\sigma \tilde{\gamma}^\sigma_\nu + \delta f^\mu_\nu, \tag{43}
\]

where \(\delta g^\mu_\nu = \bar{g}^{\mu\rho} \delta g_{\rho\nu}\) and \(\delta f^\mu_\nu = \bar{g}^{\mu\rho} \delta f_{\rho\nu}\). For our purpose we do not need the explicit form of the solution to the above equation, though it is obtained for a general fiducial metric in Ref. [91]. Actually, without the explicit form of \(\delta \gamma^\mu_\nu\), we can directly calculate \(X^\mu_\nu\) from Eq. (42) with Eq. (26) as

\[
\delta X_{(g)}^\mu_\nu = C(t, r) \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & \delta \gamma^3_3 & -\delta \gamma^2_3 \\
  0 & 0 & -\delta \gamma^3_2 & \delta \gamma^2_2 
\end{pmatrix}. \tag{44}
\]

Since the Einstein tensor satisfies the Bianchi identity \(\nabla_\mu G^\mu_\nu[g] = 0\) and the energy–momentum tensor is conserved, the tensor \(X_{(g)}^\mu_\nu\) also satisfies \(\nabla_\mu X_{(g)}^\mu_\nu = 0\). As demonstrated in Appendix B, this requirement leads to the stronger condition

\[
\delta X_{(g)}^\mu_\nu = 0, \tag{45}
\]

which yields \(\delta \gamma^\mu_\nu = 0\) for \(a, b = 2, 3\). (Note that we are interested in the case with \(C(t, r) \neq 0\).) Since

\[
\delta X_{(f)}^\mu_\nu = -\frac{1}{\kappa^2 A^2 |\det(\tilde{\gamma}_f)|} \delta X_{(g)}^\mu_\nu, \tag{46}
\]
Eq. (45) also implies $\delta X_{(f)}^{\mu} = 0$. Thus, the equations of motion for the linear perturbations $\delta g_{\mu\nu}$ and $\delta f_{\mu\nu}$ reduce to the linearized Einstein equations.

In order to see the implications of Eq. (45) in more detail, we express $\delta \gamma^{\mu\nu}$ in terms of the metric perturbations. Since only the angular components of $\delta \gamma^{\mu\nu}$ enter Eq. (45), we only have to deal with the angular components of Eq. (43), which can easily be solved because $\bar{\gamma}^{a}{}_{b} = A \delta^{a}{}_{b}$ for $a, b = 2, 3$. In fact, Eq. (43) reduces to

$$2A \delta \gamma^{a}{}_{b} = -A^{2} \delta g^{a}{}_{b} + \delta f^{a}{}_{b} = 0. \quad (47)$$

To sum up, the equations of motion for the first-order perturbations are equivalent to the following three equations:

$$\delta G[\mathcal{G}]^{\mu}{}_{\nu} = \delta T_{(\mathcal{G})}^{\mu}{}_{\nu}, \quad (48)$$

$$\delta G[\mathcal{F}]^{\mu}{}_{\nu} = \delta T_{(\mathcal{F})}^{\mu}{}_{\nu}, \quad (49)$$

$$A^{2} \delta g_{ab} - \delta f_{ab} = 0. \quad (50)$$

This is one of the main results of this investigation. The equations of motion for the perturbations of the two metrics coincide with the perturbed Einstein equations, though $\delta g_{\mu\nu}$ and $\delta f_{\mu\nu}$ are subject to Eq. (50).

Then let us count the number of graviton degrees of freedom for this perturbed system. Each symmetric tensor field of the metric has ten components, and there are, respectively, four constraints (the Hamiltonian and momentum constraints) in Eqs. (48) and (49), since those equations are the same as the perturbed Einstein equations. Furthermore, Eq. (50) gives three constraints among the angular components of the perturbed metrics. We have four spacetime coordinates and hence there are four gauge degrees of freedom representing the choice of coordinates. In addition to those familiar gauge degrees of freedom, it turns out that there still remains another gauge transformation retaining the equations of motion (48), (49), and (50), as explicitly shown in Appendix C. Note that this gauge degree of freedom corresponds to the ambiguity of the linear perturbations mentioned in Ref. [89] for the Schwarzschild–de Sitter solution in the dRGT theory. Thus, the number of remaining degrees of freedom is $10 \times 2 - 4 \times 2 - 3 - (4 + 1) = 4$, which coincides with that of two massless gravitons. We can confirm that this is consistent with the result of the Hamiltonian analysis given in Appendix D: there are 10 first-class constraints and 12 second-class constraints, and hence there are 8 ($= 40 - 10 \times 2 - 12$) degrees of freedom in phase space.

The above analysis can be applied to dRGT massive gravity only with Eqs. (48) and (50) because the derivations of these equations do not depend on the equation of motion for $f_{\mu\nu}$. Since, in this case, $\delta f_{\mu\nu}$ is composed of Stückelberg fields, the condition (50) just determines perturbations of Stückelberg fields. The remaining variables $\delta g_{\mu\nu}$ are governed by the Einstein equations, and additional gauge symmetry appears as a gauge degree of freedom for Stückelberg fields.

### 5. Second-order perturbations

In the previous section, we have shown that the first-order perturbations obey the perturbed Einstein equations and hence the behavior of the perturbations coincides with that of GR, though $\delta g_{\mu\nu}$ and $\delta f_{\mu\nu}$ are subject to Eq. (50). One may then ask the question as to how one can discriminate this class of solutions in bi-gravity from the corresponding solutions in GR.
One possibility is to take into account the back reaction on the physical metric $g_{\mu\nu}$ from $\delta f_{\mu\nu}$ at second order. For this purpose, we incorporate second-order perturbations as follows:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} + g^{(2)}_{\mu\nu},$$  \hspace{1cm} (51)$$

$$f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu} + f^{(2)}_{\mu\nu}.\hspace{1cm} (52)$$

The perturbed metrics now give rise to the second-order perturbations of $\gamma_{\mu\nu}$ as

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} + \delta \gamma_{\mu\nu} + \gamma^{(2)}_{\mu\nu} + O(\text{third-order perturbations}), \hspace{1cm} (53)$$

where $\bar{\gamma}_{\mu\nu}$ is the background quantity defined in Eq. (26) and $\delta \gamma_{ab}$ satisfies Eq. (45), and hence $\delta \gamma^a_b = 0$ for $a, b = 2, 3$. The interaction term in the equations of motion for $g_{\mu\nu}$ can be calculated explicitly even at second order; it is given by

$$X^{(2)} g_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & X^{(2)}_{22}(t, r, \theta, \phi) & X^{(2)}_{23}(t, r, \theta, \phi) \\
0 & 0 & X^{(2)}_{32}(t, r, \theta, \phi) & X^{(2)}_{33}(t, r, \theta, \phi) \\
\end{pmatrix}. \hspace{1cm} (54)$$

This tensor satisfies the conditions assumed in Appendix B, which, together with the Bianchi identity, yield

$$X^{(2)} f_{\mu\nu} = 0. \hspace{1cm} (55)$$

Even at second order, $X^{(2)} g_{\mu\nu}$ is proportional to $X^{(2)} f_{\mu\nu}$,

$$X^{(2)} f_{\mu\nu} = -\frac{1}{k^2 A^2 |\det[\bar{\gamma}^{IJ}]|} X^{(2)} g_{\mu\nu}, \hspace{1cm} (56)$$

leading to $X^{(2)} f_{\mu\nu} = 0$ as well. These conditions provide the relation between $g^{(2)}_{ab}$ and $f^{(2)}_{ab}$ as follows:

$$\gamma^{(2)}_{ab} = -\frac{1}{A^2 - A \, \text{tr}[\bar{\gamma}^{IJ}]} \delta \gamma^a_j (\bar{\gamma}^{I J} - (\bar{\gamma}^K_L - A) \delta^I_L) \delta \gamma^J_b, \hspace{1cm} (57)$$

for $a, b = 2, 3$ and $I, J, K = 0, 1$. Thus, the metric perturbations obey the perturbed Einstein equations also at second order, and the number of graviton degrees of freedom coincides with that of two massless gravitons even at second order.

This fact indicates that one cannot discriminate this class of solutions from the corresponding solutions in GR even at second order, unfortunately. On the other hand, this fact, fortunately, implies that our solutions are free from any perturbative instability peculiar to massive and bi-gravity even in second-order perturbations.

6. Conclusions and discussion

In the present study we have investigated the perturbative stability of a class of spherically symmetric solutions in massive gravity and bi-gravity in order to find stable solutions of these theories. First, we classified spherically symmetric solutions in massive gravity and bi-gravity in order to find stable solutions of these theories.

This class of solutions includes many known
solutions, e.g., the FLRW solutions in Refs. [27,28], the Schwarzschild (–de Sitter) solutions in Refs. [86–89], the LTB solution in Ref. [28], and the RN solution in Ref. [87]. In fact, any spherically symmetric solution in GR is included in this class with a suitable choice of the fiducial metric $f_{\mu\nu}$.

Next, in order to clarify the perturbative stability of the solutions, we have investigated linear perturbations on this class of solutions. We have found that the interaction terms in the equations of motion for both metrics, $\delta X(g)_{\mu\nu}$ and $\delta X(f)_{\mu\nu}$, vanish thanks to the Bianchi identities, and hence the equations of motion reduce to Eqs. (48)–(50), which are the perturbed Einstein equations with the relation (50). This result shows our solutions do not suffer from any perturbative instability peculiar to massive and bi-gravity at linear order perturbations. We have also found that, in addition to the usual gauge symmetry associated with spacetime coordinate transformation, there is another gauge symmetry of the linear perturbations given by Eqs. (C8)–(C11), which has already been known for the perturbations of the Schwarzschild–de Sitter solution in dRGT massive gravity [89]. This implies that the time evolution of the perturbations is not uniquely determined even if the usual diffeomorphism gauge degrees of freedom are fixed. It seems to be pathological, apparently, but if we appropriately fix the additional gauge freedom or confine ourselves to gauge-invariant quantities about the additional gauge symmetry, which are the observables of this system, we believe that their time evolution can be calculated properly at least at linear order. We leave it to future work to investigate whether this additional gauge symmetry would exist even at nonlinear orders. The presence of the additional gauge symmetry implies that the number of propagating degrees of freedom is less than that expected from the full nonperturbative analysis, as was shown by Hamiltonian analysis. Thus these degrees of freedom are regarded as strongly coupled at first-order perturbations. Note that our perturbative analysis does not guarantee the stability about nonperturbative deformations of the background solution. In particular, there is the possibility that strongly coupled modes reappear around such deformed backgrounds, as in the case of Ref. [19].

In addition, we have shown that the above result applies to second-order perturbations as well. Thus, one cannot distinguish this class of solutions in massive gravity and bi-gravity from the corresponding solutions of GR and, in particular, the perturbative stability of our solutions is the same as in GR even at second order. We expect that our analysis can be extended to higher-order perturbations. Actually, brute force calculations by a computer show that our analysis can be extended to at least fifth-order perturbations, though we have not yet succeeded in proving it rigorously. It should be emphasized that the presence of solutions whose perturbations coincide with those of GR at arbitrary order does not imply the theory of massive gravity and bi-gravity itself are equivalent to GR at the nonperturbative level. Perturbative analysis, in general, does not have any information about nonperturbative theory, and it just reveals the property of the background solutions.

In this article, only spherically symmetric background solutions and their perturbative stabilities are discussed. So, it is an interesting and open question whether the results obtained in this article hold for more general background solutions, including nonperturbative deformation of our background solutions. Our analysis on the background solutions can at least be applied to any $\tilde{\gamma}$ having the form of Eq. (26) in any basis vectors, because the background equations of motion (36) can be obtained in an algebraic way from Eq. (26) irrespective of a concrete expression for $\tilde{\gamma}$. Extending the above analysis to linear and nonlinear perturbations of more general solutions, however, is a nontrivial issue simply because such analysis accompanies the derivatives. We will address these issues in a future publication.
Appendix A. Concrete examples of background solutions

A.1. Bi-cosmological solutions

First we consider a family of bi-FLRW solutions, in which the physical metric takes the following FLRW form:

\[ g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - K r^2} + r^2 d\Omega^2 \right]. \]  \hfill (A1)

Comparing this metric with Eq. (23) yields \( R(t,r) = a(t)r \). We assume that \( f_{\mu\nu} \) takes the same FLRW metric but in a coordinate \((\tilde{t}(t,r), \tilde{r}(t,r), \theta, \phi)\) different from that of \( g_{\mu\nu} \).

\[ f_{\mu\nu} dx^\mu dx^\nu = -d\tilde{t}^2 + b^2(\tilde{t}) \left[ \frac{d\tilde{r}^2}{1 - K\tilde{r}^2} + \tilde{r}^2 d\Omega^2 \right] \]
\[ = f_{00} dt^2 + 2f_{01} dt dr + f_{11} dr^2 + b^2(\tilde{t})\tilde{r}^2(t,r) d\Omega^2, \]  \hfill (A2)

where

\[ f_{00} = -\left( \frac{\partial \tilde{t}}{\partial t} \right)^2 + \frac{b^2(\tilde{t}(t,r))}{1 - K\tilde{r}^2(t,r)} \left( \frac{\partial \tilde{r}}{\partial t} \right)^2, \]  \hfill (A3)
\[ f_{01} = -\frac{\partial \tilde{t}}{\partial t} \frac{\partial \tilde{t}}{\partial r} + \frac{\partial \tilde{r}}{\partial t} \frac{\partial \tilde{r}}{\partial r} + \frac{b^2(\tilde{t}(t,r))}{1 - K\tilde{r}^2(t,r)} \frac{\partial \tilde{t}}{\partial \tilde{r}}, \]  \hfill (A4)
\[ f_{11} = -\left( \frac{\partial \tilde{r}}{\partial r} \right)^2 + \frac{b^2(\tilde{t}(t,r))}{1 - K\tilde{r}^2(t,r)} \left( \frac{\partial \tilde{r}}{\partial r} \right)^2. \]  \hfill (A5)

In order to apply the results of the main body, the radial coordinate \( \tilde{r} \) is determined to satisfy the relation

\[ \tilde{r}(t,r) = \frac{AR(t,r)}{b(\tilde{t}(t,r))} = A \frac{a(t)r}{b(\tilde{t}(t,r))}, \]  \hfill (A6)

while the time coordinate \( \tilde{t} \) is arbitrary. In this case, the equations of motion for both metrics become Einstein equations with cosmological constants so that \( a(t) \) and \( b(\tilde{t}) \) obey the Friedmann equation with respect to each (cosmic) time, \( t \) or \( \tilde{t} \). This kind of bi-FLRW solution becomes a slight generalization of that found in Ref. [37], in which a specific choice of the coordinate \( \tilde{t} \) is adopted.

For \( b = 1, K = 0 \), and \( \Lambda^{(f)}_{\text{eff}} = 0 \), the fiducial metric \( f_{\mu\nu} \) becomes the flat Minkowski one and hence this bi-cosmological solution includes that obtained in Refs. [27,28] in dRGT massive gravity with the flat fiducial metric.

A.2. Bi-Schwarzschild–de Sitter solutions

Our results are applied to the following bi-Schwarzschild–de Sitter metrics as well:

\[ g_{\mu\nu} dx^\mu dx^\nu = -\left( 1 - \frac{r_{(g)}}{r} + \Lambda^{(g)}_{\text{eff}} r^2 \right) dt^2 + \frac{dr^2}{1 - \frac{r_{(g)}}{r} + \Lambda^{(g)}_{\text{eff}} r^2} + r^2 d\Omega^2, \]  \hfill (A7)
\[ f_{\mu \nu} dx^\mu dx^\nu = - \left( 1 - \frac{\tilde{r}(f)}{\tilde{r}} + \Lambda_{\text{eff}}(f) \tilde{r}^2 \right) d\tilde{t}^2 + \frac{d\tilde{r}^2}{1 - \frac{\tilde{r}(f)}{\tilde{r}} + \Lambda_{\text{eff}}(f) \tilde{r}^2} + \tilde{r}^2 d\Omega^2, \]  

(A8)

with

\[ \tilde{r} = Ar, \]  

(A9)

where \( r(g) \) and \( \tilde{r}(f) \) represent Schwarzschild radii, and \( \Lambda_{\text{eff}}(g) \) and \( \Lambda_{\text{eff}}(f) \) are effective cosmological constants defined in Eqs. (37) and (39), respectively. Since the Schwarzschild–de Sitter metric is a solution of the Einstein equation with cosmological constant, this is a vacuum solution in our setting. As is the case with the cosmological solution, this black hole solution can be obtained with arbitrary choice of the time coordinate \( \tilde{t}(t, r) \). By tuning the parameters \( \beta_0 \) and \( \beta_4 \), we can set \( \Lambda_{\text{eff}}(g) \) and \( \Lambda_{\text{eff}}(f) \) to be zeros simultaneously, which corresponds to a bi-Schwarzschild solution.

**Appendix B. Bianchi identity**

In this appendix, we will show that a symmetric tensor satisfying a condition given below must vanish as long as it obeys the Bianchi identity and the background metric \( \tilde{g}_{\mu \nu} \) takes the matrix form of Eq. (23).

Let us consider the following symmetric tensor \( X_{\mu \nu} \):

\[ X^\mu_\nu = \Lambda \delta^\mu_\nu + \epsilon^n X^{(n) \mu}_\nu + \mathcal{O}(\epsilon^{n+1}), \]  

(B1)

with

\[ X^{(n) 0}_\mu = X^{(n) 1}_\mu = X^{(n) \mu}_0 = X^{(n) \mu}_1 = 0, \]  

(B2)

where \( \epsilon \) denotes the order of perturbations and \( \Lambda \) is a constant. The goal of this section is to show that \( X^{(n) \mu}_\nu \) vanishes if the Bianchi identity, \( \nabla_{\mu} X^{\mu}_\nu = 0 \), is imposed.

The tensor \( \tilde{g}_{\mu \rho} X^{(n) \rho}_\nu \) is symmetric because

\[ X_{\mu \nu} = \tilde{g}_{\mu \rho} X^{(n) \rho}_\nu \]  

with

\[ \tilde{g}_{\mu \rho} X^{(n) \rho}_\nu = \Lambda_{\text{eff}} g_{\mu \nu} + \tilde{g}_{\mu \rho} X^{(n) \rho}_\nu + \mathcal{O}(\epsilon^{n+1}), \]  

(B4)

and both \( X_{\mu \nu} \) and \( \Lambda g_{\mu \nu} \) are symmetric. Then, from the property of the background metric \( \tilde{g}_{\mu \nu} \), it is characterized by three arbitrary functions as follows:

\[ \tilde{g}_{\mu \rho} X^{(n) \rho}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X^{(n)}_{22}(t, r, \theta, \phi) & X^{(n)}_{23}(t, r, \theta, \phi) \\ 0 & 0 & X^{(n)}_{23}(t, r, \theta, \phi) & X^{(n)}_{33}(t, r, \theta, \phi) \end{pmatrix}, \]  

(B5)

or equivalently,

\[ X^{(n) \mu}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X^{(n)}_{22}(t, r, \theta, \phi) & X^{(n)}_{23}(t, r, \theta, \phi) \\ 0 & 0 & X^{(n)}_{23}(t, r, \theta, \phi) & X^{(n)}_{33}(t, r, \theta, \phi) \end{pmatrix} \frac{R^2(t, r) \sin^2 \theta}{R^2(t, r) \sin^2 \theta}. \]  

(B6)
On the other hand, from Eq. (B1), the Bianchi identity reads
\[
\nabla_\mu X^{\mu \nu} = \epsilon^\mu \nabla_\mu X^{(n)\mu \nu} + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (B7)
\]
where \( \nabla_\mu \) is the covariant derivative with respect to \( \tilde{g}_{\mu \nu} \). Then, the zeroth and first components of this equation are
\[
\nabla_\mu X^{\mu 0} = -(X^{(n)}_{22} + (\sin \theta)^{-2} X^{(n)}_{33}) \frac{\partial_t R}{R^3} \epsilon^n + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (B8)
\]
\[
\nabla_\mu X^{\mu 1} = -(X^{(n)}_{22} + (\sin \theta)^{-2} X^{(n)}_{33}) \frac{\partial_r R}{R^3} \epsilon^n + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (B9)
\]
which yields the following solution when \( R \) is not a constant:
\[
X^{(n)}_{22} (t, r, \theta, \phi) = - \frac{X^{(n)}_{33} (t, r, \theta, \phi)}{\sin^2 \theta}. \quad (B10)
\]
The remaining components of this equation are given by
\[
\nabla_\mu X^{\mu 2} = (R \sin \theta)^{-2} \left( \partial_\phi X^{(n)}_{23} - \partial_\theta X^{(n)}_{33} \right) \epsilon^n + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (B11)
\]
\[
\nabla_\mu X^{\mu 3} = (R \sin \theta)^{-2} \left( \partial_\phi X^{(n)}_{33} + \sin \theta \partial_\theta (\sin \theta X^{(n)}_{23}) \right) \epsilon^n + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (B12)
\]
where we have used the relation (B10). Removing \( X^{(n)}_{23} \) from these equations leads to the following equation for \( X^{(n)}_{33} \):
\[
\frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta X^{(n)}_{33} \right) + \frac{1}{\sin^2 \theta} \partial_\phi \partial_\phi X^{(n)}_{33} = 0. \quad (B13)
\]
Since this is just the Laplace equation on a sphere, its solution is constant over the sphere:
\[
X^{(n)}_{33} = f(t, r). \quad (B14)
\]
By plugging this solution into Eqs. (B11) and (B12), we obtain
\[
X^{(n)}_{23} = \frac{g(t, r)}{\sin \theta}. \quad (B15)
\]
Thus, the solution of the Bianchi identity is given by
\[
X^{(n)}_{\mu \nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{f(t, r)}{\sin^2 \theta} & \frac{g(t, r)}{\sin \theta} \\
0 & 0 & \frac{f(t, r)}{\sin^2 \theta} & f(t, r)
\end{pmatrix}. \quad (B16)
\]
However, the components with \( \sin \theta \) in their denominators are singular at \( \theta = 0, \pi \) unless
\[
f(t, r) = 0, \quad (B17)
\]
\[
g(t, r) = 0. \quad (B18)
\]
Therefore, the regular solution of \( \nabla_\mu X^{\mu \nu} = 0 \) is
\[
X^{(n)}_{\mu \nu} = 0. \quad (B19)
\]
Appendix C. Additional gauge symmetry of linear perturbations

The linear perturbations have an additional gauge symmetry, which is a combination of gauge transformation of $g_{\mu\nu}$ and $f_{\mu\nu}$ separately but keeping Eq. (50). In this appendix, we will give a concrete form of such a coordinate transformation.

For this purpose, let us consider the infinitesimal gauge transformation generated by $x^\mu \rightarrow x^\mu - \xi^\mu$ for $g_{\mu\nu}$ and $x^\mu \rightarrow x^\mu - (\xi^\mu + \delta \xi^\mu)$ for $f_{\mu\nu}$.

We denote the difference $A^2 \delta g_{ab} - \delta f_{ab}$ in Eq. (50) under this transformation by $\Delta_{ab}$; that is,

\[ A^2 \delta g_{ab} - \delta f_{ab} \rightarrow A^2 \delta g_{ab} - \delta f_{ab} + \Delta_{ab}. \quad (C1) \]

The additional gauge symmetry is characterized by $\Delta_{ab} = 0$. The $(2, 2)$ component of this condition is given by

\[ 0 = \frac{\Delta_{22}}{-2A^2R} = \delta \xi^0 \partial_t R + \delta \xi^1 \partial_r R + R \partial_\theta \delta \xi^2. \quad (C2) \]

The remaining $(2, 3)$ and $(3, 3)$ components are given by

\[ 0 = \frac{\Delta_{23}}{-2A^2R^2 \sin \theta} = \partial_\phi \left( \frac{\delta \xi^2}{\sin \theta} \right) + \sin \theta \partial_\theta \delta \xi^3, \quad (C3) \]

\[ 0 = \frac{\Delta_{33}}{-2A^2R^2 \sin^3 \theta} = -\partial_\theta \left( \frac{\delta \xi^2}{\sin \theta} \right) + \frac{\partial_\phi \delta \xi^3}{\sin \theta}, \quad (C4) \]

where we have used Eq. (C2). One can easily find, similarly to Eq. (B13), that these equations reduce to the Laplace equation on a sphere:

\[ \frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta \delta \xi^3 \right) + \frac{1}{\sin^2 \theta} \partial_\phi \partial_\phi \delta \xi^3 = 0, \quad (C5) \]

whose solution becomes

\[ \delta \xi^3 = P(t, r). \quad (C6) \]

Plugging this solution into Eqs. (C3) and (C4), we find

\[ \delta \xi^2 = Q(t, r) \sin \theta. \quad (C7) \]

To sum up, this additional gauge symmetry is characterized by $\Xi(t, r, \theta, \phi), P(t, r), Q(t, r)$ as

\[ \delta \xi^0 = \Xi(t, r, \theta, \phi), \quad (C8) \]

\[ \delta \xi^1 = -\frac{\partial_t R(t, r) \Xi(t, r, \theta, \phi) + R(t, r) Q(t, r) \cos \theta}{\partial_t R(t, r)}, \quad (C9) \]

\[ \delta \xi^2 = Q(t, r) \sin \theta, \quad (C10) \]

\[ \delta \xi^3 = P(t, r). \quad (C11) \]

---

2 To determine the gauge transformation, one establishes a bi-tangent bundle $T^2 M$, i.e. a fibre bundle locally isomorphic to $M \times T^0 \times T^0$. To have a usual tangent bundle $TM$, two horizontal lifts $\pi^{(1)}(M)$ and $\pi^{(2)}(M)$ are identified by this relation of the diffeomorphisms so that it determines a diffeomorphism group of the base manifold.
One may regard $R(t, r)$ itself as a radial coordinate and, in the new coordinates $(t, R, \theta, \phi)$, the above transformation (C8)–(C11) with $P(t, r) = Q(t, r) = 0$ simply reduces to the transformation of the time coordinate.

We can directly observe this symmetry in the action. Actually, the quadratic action of the mass term for the linear perturbations becomes

$$S^{(2)}_{\text{mass}} = \frac{M^2_{\text{pl}}}{2} \int d^4 x \sqrt{-g} \frac{2C(t, r)}{AR^2} \left( \delta \gamma^2_2 \delta \gamma^3_3 - \delta \gamma^2_3 \delta \gamma^3_2 \right)$$

$$+ \frac{M^2_{\text{pl}}}{2} \int d^4 x \sqrt{-g} \left( -2 \Lambda^{(g)}_{\text{eff}} \right) + \frac{\kappa^2 M^2_{\text{pl}}}{2} \int d^4 x \sqrt{-f} \left( -2 \Lambda^{(f)}_{\text{eff}} \right), \quad (C12)$$

where $C(t, r)$ is the function defined in Eq. (31), $\Lambda^{(g)}_{\text{eff}}$ and $\Lambda^{(f)}_{\text{eff}}$ are effective cosmological constants defined in Eqs. (37) and (39), and $\sqrt{-g}$ and $\sqrt{-f}$ are quadratic perturbations of $\sqrt{-g}$ and $\sqrt{-f}$. Clearly, this term is invariant under the transformation (C8)–(C11) because this transformation leaves $\delta \gamma^a_b$ unchanged.

**Appendix D. Hamiltonian analysis of linear perturbations**

We will count the number of graviton degrees of freedom of linear perturbations by means of the Hamiltonian analysis. So, we omit the matter action in this appendix. For this purpose, it is useful to decompose the perturbations in terms of spherical harmonics $Y^m_l$ as in Ref. [92]. Due to the spherical symmetry of the background metrics, the modes with different eigenvalues of rotation $(l, m)$ or parity (odd or even) develop independently, and the dynamics of each mode does not depend on $m$. Hence, we may suppose that $m$ is equal to zero, without loss of generality.

**D.1. Odd mode perturbations**

Non-vanishing components of the odd mode perturbations with $m = 0$ are given by

$$\delta g_{03} = \sum_{l \geq 1} h^{(g)}_{l_0}(t, r) \sin \theta \partial_\theta P_l(\cos \theta), \quad (D1)$$

$$\delta g_{13} = \sum_{l \geq 1} h^{(g)}_{l_1}(t, r) \sin \theta \partial_\theta P_l(\cos \theta), \quad (D2)$$

$$\delta g_{23} = \sum_{l \geq 2} h^{(g)}_{l_2}(t, r) \sin^2 \theta \partial_\theta \left( \frac{\partial_\theta P_l(\cos \theta)}{\sin \theta} \right), \quad (D3)$$

and

$$\delta f_{03} = \sum_{l \geq 1} h^{(f)}_{l_0}(t, r) \sin \theta \partial_\theta P_l(\cos \theta), \quad (D4)$$

$$\delta f_{13} = \sum_{l \geq 1} h^{(f)}_{l_1}(t, r) \sin \theta \partial_\theta P_l(\cos \theta), \quad (D5)$$

$$\delta f_{23} = \sum_{l \geq 2} h^{(f)}_{l_2}(t, r) \sin^2 \theta \partial_\theta \left( \frac{\partial_\theta P_l(\cos \theta)}{\sin \theta} \right), \quad (D6)$$

where $P_l$ is the Legendre polynomial. Hereafter in this subsection we omit the suffix $l$ and the summation with respect to $l$ for brevity. From the perturbed Einstein–Hilbert action with the mass
term (C12), the conjugate momenta of $h_{I}^{(g/f)}$ ($I = 0, 1, 2$) are calculated as

\[
P_{0}^{(g)} = \frac{\delta S}{\delta \dot{h}_{0}^{(g)}} = 0, \quad (D7)
\]

\[
P_{1}^{(g)} = \frac{\delta S}{\delta \dot{h}_{1}^{(g)}} = \frac{2\tilde{\mathcal{M}}_{pl}^{2}}{\sqrt{-g}} \left( 2(\ln R)'h_{0}^{(g)} - h_{0}^{(g)'} + \dot{h}_{1}^{(g)} \right), \quad (D8)
\]

\[
P_{2}^{(g)} = \frac{\delta S}{\delta \dot{h}_{2}^{(g)}} = \frac{2\lambda \tilde{\mathcal{M}}_{pl}^{2}}{R^{2}} \sqrt{-g} \left( \tilde{g}^{00} h_{0}^{(g)} + \tilde{g}^{01} h_{1}^{(g)} - \tilde{g}^{00} \dot{h}_{2}^{(g)} - \tilde{g}^{01} \dot{h}_{2}^{(g)'} \right), \quad (D9)
\]

\[
P_{0}^{(f)} = \frac{\delta S}{\delta \dot{h}_{0}^{(f)}} = 0, \quad (D10)
\]

\[
P_{1}^{(f)} = \frac{\delta S}{\delta \dot{h}_{1}^{(f)}} = \frac{2\kappa \tilde{\mathcal{M}}_{pl}^{2}}{\sqrt{-f}} \left( 2(\ln R)'h_{0}^{(f)} - h_{0}^{(f)'} + \dot{h}_{1}^{(f)} \right), \quad (D11)
\]

\[
P_{2}^{(f)} = \frac{\delta S}{\delta \dot{h}_{2}^{(f)}} = \frac{2\lambda \kappa \tilde{\mathcal{M}}_{pl}^{2}}{A^{2}R^{2}} \sqrt{-f} \left( \tilde{f}^{00} h_{0}^{(f)} + \tilde{f}^{01} h_{1}^{(f)} - \tilde{f}^{00} \dot{h}_{2}^{(f)} - \tilde{f}^{01} \dot{h}_{2}^{(f)'} \right), \quad (D12)
\]

where $\lambda := (l - 1)(l + 2)$, $\tilde{\mathcal{M}}_{pl}^{2} := \frac{l(l+1)}{1+2l} M_{pl}^{2} \pi$, and $\sqrt{-g}$ represents the determinant of only 0, 1 components:

\[
\sqrt{-g} := \sqrt{-\tilde{g}_{00} \tilde{g}_{11} + (\tilde{g}_{01})^{2}}. \quad (D13)
\]

We schematically decompose the Hamiltonian density as follows:

\[
\mathcal{H}_{\text{odd}} = \mathcal{H}_{\text{GR,(g/f)}}^{\text{odd}} + \mathcal{H}_{\text{GR,(f)}}^{\text{odd}} + \mathcal{H}_{\text{mass}}^{\text{odd}}, \quad (D14)
\]

where $\mathcal{H}_{\text{GR,(g/f)}}^{\text{odd}}$ represents the contribution from each Einstein–Hilbert term and the effective cosmological term, which is the second term (or third term) in the right-hand side of Eq. (C12). $\mathcal{H}_{\text{mass}}^{\text{odd}}$ represents the contribution from the first term in the right-hand side of Eq. (C12). This decomposition is justified because $S_{\text{mass}}$ does not include time derivative of $g_{\mu\nu}$ and $f_{\mu\nu}$. From the expression of the action (C12), $\mathcal{H}_{\text{mass}}^{\text{odd}}$ is explicitly calculated as

\[
\mathcal{H}_{\text{mass}}^{\text{odd}} = \tilde{\mathcal{M}}_{pl}^{2} \lambda C(t, r) \sqrt{-g} \left( A^{2} \dot{h}_{2}^{(g)} - \dot{h}_{2}^{(f)} \right)^{2} A^{2}. \quad (D15)
\]

It should be noted that, for $l = 1$ mode, $\mathcal{H}_{\text{mass}}^{\text{odd}}$ vanishes, which implies that the dynamics of $l = 1$ mode coincides with that of GR. Therefore, there should be additional gauge symmetry, under which each metric transforms independently. This transformation, actually, corresponds to the arbitrary function $P(t, r)$ in Eq. (C11).

From now on, we focus on $l \geq 2$ modes. The Hamiltonian density from the Einstein–Hilbert term is calculated as

\[
\mathcal{H}_{\text{GR,(g)}}^{\text{odd}} = \frac{1}{4\mathcal{M}_{pl}^{2}} \sqrt{-g} \left( P_{1}^{(g)} \right)^{2} + \frac{1}{4\mathcal{M}_{pl}^{2}} \sqrt{-g} g_{11}^{2} \left( P_{2}^{(g)} \right)^{2} + \frac{\tilde{g}_{01}}{g_{11}} (-h_{1}^{(g)} + h_{2}^{(g)'}) P_{2}^{(g)}
\]

\[
+ \tilde{M}_{pl}^{2} \sqrt{-g} g_{11}^{2} h_{2}^{(g)'} + 2 \tilde{M}_{pl}^{2} \lambda \sqrt{-g} g_{11}^{2} h_{2}^{(g)} + 4 \tilde{M}_{pl}^{2} \lambda \sqrt{-g} g_{11}^{2} R^{2} h_{1}^{(g)} h_{2}^{(g)}
\]

\[
+ M_{1}^{(g)} \left( h_{1}^{(g)} \right)^{2} + M_{2}^{(g)} \left( h_{2}^{(g)} \right)^{2} - h_{0}^{(g)} C_{(g)}^{(1)} \left[ P_{1}^{(g)}, P_{2}^{(g)}, h_{1}^{(g)}, h_{2}^{(g)} \right], \quad (D16)
\]
where \( M^{(g)}_1, M^{(g)}_2 \) are some functions of \( t, r \) and \( C^{(1)}_{(g)} \) is given by

\[
C^{(1)}_{(g)} = 2(\ln R)P^{(g)}_1 + P^{(g)}_1 - P^{(g)}_2 - 4\tilde{M}^2_{pl}\frac{(\ln R)'}{\sqrt{-g}}h^{(g)}_1,
\]

\[+ 4\lambda\tilde{M}^2_{pl}\sqrt{-g}g^{04}\partial_t(\ln R)\frac{1}{R^2}h^{(g)}_2 - 2\tilde{M}^2_{pl}\frac{1}{R^2}\partial_r\left(\frac{R^2}{\sqrt{-g}}\right)\tilde{A}_{(g)}', \tag{D17}\]

with \( A = 0, 1, H^{\text{odd}}_{GR,(f)} \) is obtained by replacing \( g \rightarrow f, R \rightarrow AR, \tilde{M}^2_{pl} \rightarrow \kappa^2\tilde{M}^2_{pl}. \)

The primary constraints of this system are

\[
C^{(0)}_{(g)} := P_0^{(g)} \approx 0, \tag{D18}\]
\[
C^{(0)}_{(f)} := P_0^{(f)} \approx 0, \tag{D19}\]

and then, the total Hamiltonian is

\[
H^\text{odd} = H^\text{odd} + \int dr \left[v^{(g)}(t, r)C^{(0)}_{(g)} + v^{(f)}(t, r)C^{(0)}_{(f)}\right], \tag{D20}\]

\[
H^\text{odd} = H^\text{odd}_{GR,(g)} + H^\text{odd}_{GR,(f)} + H^\text{odd}_{\text{mass}}, \tag{D21}\]

\[
H^\text{odd}_{GR,(g,f)} = \int dr H^{\text{odd}}_{GR,(g,f)}, \quad H^\text{odd}_{\text{mass}} = \int dr H^\text{odd}_{\text{mass}}. \tag{D22}\]

Time evolution of the primary constraints is given by

\[
\dot{C}^{(0)}_{(g,f)} = \left\{ C^{(0)}_{(g,f)}, H^\text{odd} \right\} \approx C^{(1)}_{(g,f)}\left[ P^{(g)}_1, P^{(f)}_2, h^{(g)}_1, h^{(f)}_2 \right], \tag{D23}\]

which generate the following two secondary constraints,

\[
C^{(1)}_{(g,f)} \approx 0. \tag{D24}\]

Time evolution of \( C^{(1)}_{(g)} \) is given by

\[
\dot{C}^{(1)}_{(g)} = \frac{\partial C^{(1)}_{(g)}}{\partial t} + \left\{ C^{(1)}_{(g)}, H^\text{odd} \right\} \approx -\left\{ P^{(g)}_2, H^\text{odd}_{\text{mass}} \right\}
\]

\[
= 2\lambda\tilde{M}^2_{pl}\frac{\sqrt{-g}}{AR^4}C(t, r)\left(\hat{A}^2 h^{(g)}_2 - h^{(f)}_2\right), \tag{D25}\]

and that of \( C^{(1)}_{(f)} \) is given by

\[
\dot{C}^{(1)}_{(f)} \approx -2\lambda\tilde{M}^2_{pl}\frac{\sqrt{-g}}{A^2R^4}C(t, r)\left(\hat{A}^2 h^{(g)}_2 - h^{(f)}_2\right). \tag{D26}\]

These equations impose another constraint,

\[
C^{(2)} := A^2 h^{(g)}_2 - h^{(f)}_2 \approx 0. \tag{D27}\]

From time evolution of \( C^{(2)} \),

\[
\dot{C}^{(2)} = \left\{ C^{(2)}, H^\text{odd} \right\}
\]

\[
\approx A^2\left\{ h^{(g)}_2, H^\text{odd}_{GR,(g)} \right\} - \left\{ h^{(f)}_2, H^\text{odd}_{GR,(f)} \right\}. \tag{D26}\]
\[ \approx A^2 \left( h_{0}^{(g)} - \frac{g_{01}}{g_{11}} h_{1}^{(g)} + \frac{\sqrt{g} R^2}{2\lambda M_{pl}^2 g_{11}} P^{(g)} \right) \]

\[ - \left( h_{0}^{(f)} - \frac{f_{01}}{f_{11}} h_{1}^{(f)} + \frac{\sqrt{f} A^2 R^2}{2\lambda \kappa^2 M_{pl}^2 f_{11}} P^{(f)} \right) \text{,} \quad \text{(D28)} \]

we obtain yet another constraint,

\[ C^{(3)} := A^2 \left( h_{0}^{(g)} - \frac{g_{01}}{g_{11}} h_{1}^{(g)} + \frac{\sqrt{g} R^2}{2\lambda M_{pl}^2 g_{11}} P^{(g)} \right) \]

\[ - \left( h_{0}^{(f)} - \frac{f_{01}}{f_{11}} h_{1}^{(f)} + \frac{\sqrt{f} A^2 R^2}{2\lambda \kappa^2 M_{pl}^2 f_{11}} P^{(f)} \right) \text{.} \quad \text{(D29)} \]

Since \( C^{(3)} \) includes \( h_{0}^{(g)} \) and \( h_{0}^{(f)} \) terms, the Poisson brackets of \( C^{(3)} \) and primary constraints \( C_{(g/f)}^{(0)} \) do not vanish. Thus, the consistency relation on \( C^{(3)} \),

\[ \dot{C}^{(3)} \approx \partial_t C^{(3)} + \left\{ C^{(3)}, H \right\} + A^2 v^{(g)} - v^{(f)} \approx 0, \quad \text{(D30)} \]

determines the combination of multipliers, \( A^2 v^{(g)} - v^{(f)} \). Then, no further constraints are generated.

Since one multiplier remains undetermined, one can easily find that there is gauge symmetry in this system. More explicitly, one can confirm that there are two first-class constraints (and four second-class constraints) in this system through the presence of two zero eigenvalues of \( 6 \times 6 \) matrix \( \{ C_I, C_J \} \), where \( C_I \) represent all of the six constraints. These two gauge symmetries correspond to the ones which the theory originally possesses. To summarize, the number of graviton degrees of freedom in this system is

\[ \frac{1}{2} \left( \begin{array}{ccc} \text{variables} & \text{constraints} & \text{gauge dofs} \\ 12 & 6 & 2 \end{array} \right) = 2, \quad \text{(D31)} \]

and completely coincides with the case of two massless gravitons.

For the \( l = 1 \) mode, there are four variables (eight variables in phase space), \( h_{0}^{(g/f)}, h_{1}^{(g/f)} \). As mentioned above, the interaction term \( S_{\text{mass}} \) vanishes for the \( l = 1 \) mode, and hence the action reduces to decoupled two Einstein–Hilbert action. Then, there are four first-class constraints and four gauge symmetries which correspond to the general covariance of \( g_{\mu\nu} \) and \( f_{\mu\nu} \) separately. These four gauge symmetries can be arranged into those of the full theory and the additional ones described by \( P(t, r) \) in Eq. (C11). To summarize, the number of degrees of freedom of the odd \( l = 1 \) mode is

\[ \frac{1}{2} \left( \begin{array}{ccc} \text{variables} & \text{constraints} & \text{gauge dofs} \\ 8 & 4 & 4 \end{array} \right) = 0. \quad \text{(D32)} \]
D.2. Even mode perturbations

Similarly, we consider even mode perturbations. Non-vanishing components of the even mode perturbations are given by

\[(D33)\]
\[\delta g_{00} = \sum_{l \geq 0} H_0^{(g)l}(t, r) P_l(\cos \theta),\]
\[\delta g_{01} = \sum_{l \geq 0} H_1^{(g)l}(t, r) P_l(\cos \theta),\]
\[\delta g_{02} = \sum_{l \geq 1} H_2^{(g)l}(t, r) \partial_\theta P_l(\cos \theta),\]
\[\delta g_{11} = \sum_{l \geq 0} H_3^{(g)l}(t, r) P_l(\cos \theta),\]
\[\delta g_{12} = \sum_{l \geq 1} H_4^{(g)l}(t, r) \partial_\theta P_l(\cos \theta),\]
\[\delta g_{22} = \sum_{l \geq 0} H_5^{(g)l}(t, r) P_l(\cos \theta) + \sum_{l \geq 2} H_6^{(g)l} \partial_\theta \partial_\theta P_l(\cos \theta),\]
\[\delta g_{33} = \sum_{l \geq 0} H_7^{(g)l}(t, r) \sin^2 \theta P_l(\cos \theta) + \sum_{l \geq 2} H_8^{(g)l} \sin \theta \cos \theta \partial_\theta P_l(\cos \theta),\]

and similar expansions are applied for \(\delta f_{\mu \nu}\). Hereafter in this subsection we omit the suffix \(l\) and the summation with respect to \(l\) for brevity. First we treat the \(l \geq 2\) modes, and the Hamiltonian density for even modes is decomposed into

\[(D40)\]
\[\mathcal{H}^{\text{even}} = \mathcal{H}^{\text{even}}_{\text{GR}(g)} + \mathcal{H}^{\text{even}}_{\text{GR}(f)} + \mathcal{H}^{\text{even}}_{\text{mass}}.\]

\(\mathcal{H}^{\text{even}}_{\text{GR}(g,f)}\) represents a contribution from the Einstein–Hilbert term and effective cosmological constant terms, explicitly given by

\[(D41)\]
\[\mathcal{H}^{\text{even}}_{\text{GR}(g,f)} = -H_0^{(g,f)} C_{0,(g,f)}^{(1)} \left[ P_3^{(g,f)}, P_5^{(g,f)}, H_3^{(g,f)}, H_5^{(g,f)}, H_3^{(g,f)}, H_5^{(g,f)}, H_6^{(g,f)} \right]
- H_1^{(g,f)} C_{1,(g,f)}^{(1)} \left[ P_4^{(g,f)}, P_6^{(g,f)}, P_5^{(g,f)}, H_4^{(g,f)}, H_5^{(g,f)}, H_5^{(g,f)}, H_6^{(g,f)} \right]
- H_2^{(g,f)} C_{2,(g,f)}^{(1)} \left[ P_3^{(g,f)}, P_6^{(g,f)}, P_5^{(g,f)}, H_3^{(g,f)}, H_5^{(g,f)}, H_5^{(g,f)}, H_6^{(g,f)} \right]
+ \left(\text{second-order terms of } H_3^{(g,f)}, H_4^{(g,f)}, H_5^{(g,f)}, H_6^{(g,f)}, P_3^{(g,f)}, P_4^{(g,f)}, P_5^{(g,f)}, P_6^{(g,f)} \right),\]

with

\[C_{0,(g)}^{(1)} = -R g^{0l} (\partial_t R) P_5^{(g)} + \left(\text{linear terms of } P_3^{(g)}, H_3^{(g)}, H_4^{(g)}, H_5^{(g)}, H_6^{(g)} \right),\]
\[C_{1,(g)}^{(1)} = -2 R g^{1l} (\partial_t R) P_5^{(g)} + \left(\text{linear terms of } P_3^{(g)}, P_4^{(g)}, H_3^{(g)}, H_4^{(g)}, H_5^{(g)}, H_6^{(g)} \right),\]
\[C_{2,(g)}^{(1)} = -2 P_6^{(g)} + \left(\text{linear terms of } P_4^{(g)}, H_3^{(g)}, H_4^{(g)}, H_5^{(g)}, H_6^{(g)} \right),\]
where \(p^{(g/f)}_I\) are the conjugate momenta of \(H^{(g/f)}_I\). On the other hand, \(\mathcal{H}^{\text{even}}_{\text{mass}}\) is given by

\[
\mathcal{H}^{\text{even}}_{\text{mass}} = -\hat{M}^2 \sqrt{-g} C(t, r) \frac{g^{0B}}{A^3 R^4} \left[ (A^2 H^g_5 - H^f_5)^2 - l(l+1)(A^2 H^g_6 - H^f_6)(A^2 H^g_6 - H^f_6) \right]
+ \frac{l(l+1)}{2} (A^2 H^g_6 - H^f_6)^2, \tag{D45}
\]

where \(\hat{M}^2 = M^2 \pi/(1+2l)\). Then, the following six primary constraints are imposed:

\[
\mathcal{C}^{(0)}_{I, (g/f)} := p^{(g/f)}_I \approx 0 \tag{D46}
\]

for \(I = 0, 1, 2\). The total Hamiltonian is

\[
\mathcal{H}^{\text{even}}_I = \mathcal{H}^{\text{even}} + \int dr \left[ v^{(g)}_{(g)}(t, r) p^{(g)}_I + v^{(f)}_{(f)}(t, r) p^{(f)}_I \right]. \tag{D47}
\]

Time evolution of primary constraints is

\[
\dot{\mathcal{C}}^{(0)}_{I, (g/f)} = \partial_t \mathcal{C}^{(0)}_{I, (g/f)} + \left\{ \mathcal{C}^{(0)}_{I, (g/f)}, \mathcal{H}_I \right\} \approx \left\{ p^{(g/f)}_I, \mathcal{H}^{\text{even}}_{\text{GR}, (g/f)} \right\} = \mathcal{C}^{(1)}_{I, (g/f)}, \tag{D48}
\]

which imposes six secondary constraints,

\[
\mathcal{C}^{(1)}_{I, (g/f)} \approx 0. \tag{D49}
\]

Time evolution of these constraints is given by

\[
\dot{\mathcal{C}}^{(1)}_{0, (g)} = -2\hat{M}^2 \sqrt{-g} C(t, r) g^{0B} \partial_B R \left[ (A^2 H^g_5 - H^f_5) - \frac{l(l+1)}{2} (A^2 H^g_6 - H^f_6) \right], \tag{D50}
\]

\[
\dot{\mathcal{C}}^{(1)}_{1, (g)} = -4\hat{M}^2 \sqrt{-g} C(t, r) g^{1B} \partial_B R \left[ (A^2 H^g_5 - H^f_5) - \frac{l(l+1)}{2} (A^2 H^g_6 - H^f_6) \right], \tag{D51}
\]

\[
\dot{\mathcal{C}}^{(1)}_{2, (g)} = 2\hat{M}^2 \sqrt{-g} C(t, r) \frac{R g}{A^3 R^4} l(l+1) \left[ (A^2 H^g_5 - H^f_5) - (A^2 H^g_6 - H^f_6) \right], \tag{D52}
\]

with \(B = 0, 1\), and similar terms appear in the constraints for \(f_{\mu\nu}\). Consequently, we obtain two additional constraints:

\[
\mathcal{C}^{(2)}_1 := A^2 H^g_5 - H^f_5, \tag{D53}
\]

\[
\mathcal{C}^{(2)}_2 := A^2 H^g_6 - H^f_6. \tag{D54}
\]

The time evolutions of these constraints are given by

\[
\dot{\mathcal{C}}^{(2)}_1 = A^2 \left( 2R g^{0B} \partial_B R H^g_0 + 2 R g^{1B} \partial_B RH^g_1 \right) - \left( A^2 R g^{0B} \partial_B RH^f_0 + 2 A^2 R g^{1B} \partial_B RH^f_1 \right)
+ \text{linear terms of } H^{(g/f)}_4, H^{(g/f)}_5, H^{(g/f)}_6, P^{(g/f)}_3, P^{(g/f)}_5, P^{(g/f)}_6, \tag{D55}
\]

\[
\dot{\mathcal{C}}^{(2)}_2 = 2A^2 H^g_2 - 2H^f_2 + \text{linear terms of } H^{(g/f)}_4, H^{(g/f)}_5, H^{(g/f)}_6, P^{(g/f)}_3, P^{(g/f)}_5, P^{(g/f)}_6, \tag{D56}
\]

which impose a further two constraints,

\[
\mathcal{C}^{(3)}_1 := \dot{\mathcal{C}}^{(2)}_1 \approx 0, \tag{D57}
\]

\[
\mathcal{C}^{(3)}_2 := \dot{\mathcal{C}}^{(2)}_2 \approx 0. \tag{D58}
\]
Since the above constraints include $H_{0}^{(g/f)}, H_{1}^{(g/f)}, H_{2}^{(g/f)}, \ldots$, time development of these constraints only determines two of the multipliers $v_{1}^{(g/f)}, v_{2}^{(g/f)}, \ldots$, and hence no more constraints appear. One can see that four of the multipliers $v_{1}^{(g/f)}, v_{2}^{(g/f)}, \ldots$ remain undetermined, which implies that this system has corresponding gauge symmetry. Concrete calculation shows that this system has eight first-class constraints (and eight second-class constraints) through eight non-zero eigenvalues of the $16 \times 16$ matrix $\{C_{1}, C_{2}, \ldots\}$, where $C_{i}$ represent all of the 16 constraints. These eight constraints are composed of six gauge symmetries of the full theory and two additional symmetries described by $\Xi$ in Eqs. (C8) and (C9).

To summarize, the number of graviton degrees of freedom for even modes can be estimated as

$$\frac{1}{2} \left( \frac{\text{variables}}{28} - \frac{\text{constraints}}{16} - \frac{\text{gauge dofs}}{8} \right) = 2,$$

which again coincides with that of two massless gravitons for $l \geq 2$ modes.

The structure of the Hamiltonian analysis is similar for the $l = 0, 1$ modes. For the $l = 0$ mode, initially we have eight variables (16 phase space variables), $H_{0}^{(g/f)}, H_{1}^{(g/f)}, H_{2}^{(g/f)}, H_{3}^{(g/f)}, H_{4}^{(g/f)}, H_{5}^{(g/f)}, H_{6}^{(g/f)}, H_{7}^{(g/f)}$. Similar analysis shows that there are ten constraints, $C_{0}, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}, C_{6}, C_{7}, C_{8}, C_{9}$. Similar analysis shows that there are five gauge symmetries; that is, there are five first-class constraints and four second-class constraints. Four gauge symmetries come from those of the full theory and two come from the additional ones described by $\Xi$ in Eqs. (C8) and (C9). Then, the number of dynamical degrees of freedom is

$$\frac{1}{2} \left( \frac{\text{variables}}{24} - \frac{\text{constraints}}{14} - \frac{\text{gauge dofs}}{10} \right) = 0.$$

For the $l = 1$ mode, we have 12 variables (24 phase space variables), $H_{0}^{(g/f)}, H_{1}^{(g/f)}, H_{2}^{(g/f)}, H_{3}^{(g/f)}, H_{4}^{(g/f)}, H_{5}^{(g/f)}, 14$ constraints, $C_{0}, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}, C_{6}, C_{7}, C_{8}, C_{9}, C_{10}, C_{11}, C_{12}, C_{13}$, and 10 gauge symmetries; that is, there are ten first-class constraints and four second-class constraints. Then, the number of dynamical degrees of freedom is

$$\frac{1}{2} \left( \frac{\text{variables}}{24} - \frac{\text{constraints}}{14} - \frac{\text{gauge dofs}}{10} \right) = 0.$$

It should be noted that six gauge symmetries correspond to those of the full theory, two gauge symmetries correspond to $\Xi$ in Eqs. (C8) and (C9), and the other two gauge symmetries correspond to $Q(t, r)$ in Eq. (C10).

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