BISTABILITY IN A QUANTUM NONLINEAR OSCILLATOR
EXCITED BY A STOCHASTIC FORCE

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Abstract
We present an approximate analytical method of analysis of stationary states of nonlinear quantum systems with noise. As an example, we consider a quantum nonlinear oscillator excited by a fluctuating force and obtain a range of parameters with more than one stationary solution. The existence of such a range is a necessary condition for bistability. We neglect fluctuations in the amplitude of oscillations but do not neglect fluctuations in its phase. Then, the oscillator noise power spectrum depends on the oscillator mean energy $n$, which leads to a nonlinear integral equation for $n$. We can find an analytical solution of this equation. We derive the oscillator stationary states for various spectra of fluctuations of the exciting force. Linear stability analysis of stationary states was carried out. This approach is a generalization of our previous analysis of thresholdless lasers.

Keywords: bistability, anharmonic oscillator, quantum nonlinear oscillator, stochastic force.

1. Introduction

The dynamics of nonlinear systems excited by fluctuating forces has attracted the attention for a long time. A well-known example of the interesting phenomena in such systems is a stochastic resonance, when a bistable system is excited by stochastic and regular forces simultaneously [1–3]. It is shown that there is an increase in the signal-to-noise ratio, stochastic synchronization of switches between the system states are possible, and for the maximum amplification of the regular component of the signal an optimum level of the noise is found. The increase in the signal-to-noise ratio (SNR) of an amplifier is, obviously, a very important practical problem.

Well-known nonlinear quantum systems, where regular (coherent) dynamics appears at the noisy (incoherent) pump, are lasers and related devices [4,5]. In particular, the nonlinearity, i.e., the saturation of lasing transition, leads to the narrowing of the laser line width with an increase in the intensity of the incoherent pump. The generic problem of locking of a self-oscillator (Van der Pol oscillator) by a random signal related to lasing was discussed in [6]. Optical laser systems provide convenient tools for studies of nonlinear dynamics with noise. For example, enhancement in the output SNR and a noise-induced...
switching were predicted and experimentally observed in the three-level atomic optical bistability (AOB) systems [7–11].

Correct theoretical description of nonlinear systems excited by noise sources is a difficult task.

For example, one cannot solve explicitly the Fokker-Planck equation and find correlation functions and spectral densities for a nonlinear bistable classical oscillator [1]. Many interesting questions concerning noise-induced transitions in nonlinear systems are discussed in [12]. For lasers, the Schawlow–Townes formula [4] cannot describe the line width of some kinds of lasers. Expressions different from that formula were derived for “bad cavity” lasers (i.e., lasers with low-quality cavities) [13,14], including plasmonic nanolasers [15,16], as well as for “thresholdless” lasers (with high spontaneous emission at the threshold) [17]. However, a general and relatively simple method of calculation of the line width of any laser, also with a high degree of spontaneous emission noise, has not yet been developed. Thus, the development of a simple and reliable method of treatment of quantum nonlinear systems with a high degree of noise, also with the noise in the pump, is a topical problem.

In [16,17], we presented an approximate analytical method of calculation of stationary states and the line width of a laser below and above the lasing threshold, valid also for “bad cavity” lasers. The method is based on quantum Langevin equations and uses the assumption that fluctuations of populations of the lasing active medium are much smaller than fluctuations of the laser medium dipole momenta and the field. This is a good assumption, in particular, for “bad cavity” lasers with high noise in the lasing mode, as well as in the polarization of lasing medium, as was confirmed in [17] by numerical calculations. Usually quantum Langevin equations for fluctuations are obtained by linearization around the steady state [18], so that the steady state does not depend on the noise. In our method, the steady state does depend on the noise; in fact, the energy of the system is fully provided by the noise source.

The main purpose of this paper is to demonstrate how the method of [16,17] can be applied to general quantum nonlinear oscillating systems, not necessarily lasers. For that, we use a rather simple example of a nonlinear quantum oscillator excited by random force originating from the oscillator–bath interaction. It is well known that a classical nonlinear oscillator excited by regular force near the resonance has bistability in its stationary states [19]. The necessary condition for the bistability is the existence of more than one stationary solution in some region of parameters. Here, we restrict ourselves to finding of such parameter regions. Employing the method elaborated in [16,17] and following the approach of catastrophe theory [20] we find the ranges of parameters with several (three) stationary states of the quantum oscillator excited by a random force with various spectra of fluctuations. Note that the random force does not lead to a true bistability, even if parameter ranges with many stationary solutions exist [21,22]: fluctuations of the oscillator energy and phase can lead to switching between stationary states that are stable, without fluctuations. Thus, the stationary states are metastable and there is only “quasibistability” at presence of the fluctuating force. We postpone calculations of the oscillator energy fluctuations and detailed analysis of the stability conditions and switching dynamics (switching times, lifetimes of metastable states) in the system for the future.

This paper is organized as follows.

In Sec. 1, we derive the equation for determining the mean energy of a nonlinear quantum oscillator. In Sec. 2, we solve this equation in some particular cases and find the parameter ranges with many stationary solutions. Results are summarized and discussed in the concluding Section 3.
2. Energy of the Oscillator Excited by a Random Force

The classical equation of motion for the coordinate $x$ of anharmonic oscillator reads [19]

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = F(t) \cos(\omega_p t) + \alpha x^2 - \beta x^3,$$

(1)

where $\gamma$ is the damping rate, $\omega_0$ is the frequency of linear oscillations, $m$ is the oscillator mass, and $\alpha$ and $\beta$ are the coefficients of nonlinearity.

In order to simplify our analysis, we assume $\alpha = 0$. In Eq. (1), $F(t) = f(t)/m$, with $f(t)$ being the amplitude of the external random force exciting (pumping) the oscillator. The spectrum of fluctuations of $f(t)$ is centered at $\omega_p \approx \omega_0$. It may be that $\gamma \ll \Gamma_p - \frac{1}{2}\omega_p$ - the half-width of the power spectrum of $f(t)$, however, $\Gamma_p \ll \omega_0$, i.e., $f(t)$ fluctuates slowly with respect to $\cos(\omega_p t)$.

Now we consider a quantum oscillator and assume that $x(t)$ and $f(t)$ are operators, and Eq. (1) is the Heisenberg–Langevin equation of motion for $x$. We replace $x$ in Eq. (1) by the Bose operator $ae^{-i\omega_p t}$ and simplify Eq. (1) using the resonant approximation and assuming that $a$ is changing more slowly than $e^{-i\omega_p t}$. On the left-hand side of Eq. (1), we took $\omega_0^2 - \omega_p^2 \approx -2\omega_0\delta$, where the detuning $\delta = \omega_p - \omega_0 \ll \omega_0$, omitted $\dot{a}$, neglected $\dot{a}$ in the term $\sim \gamma$, and left only terms $\sim e^{-i\omega_p t}$. Thus, instead of Eq. (1) we obtain

$$\dot{a} = i(\delta + ba^+a) - \gamma a + \sqrt{2}\gamma a^{\text{in}}(t),$$

(3)

where the normalized coefficient of nonlinearity $b = 3h\beta/(8m\omega_0^2)$. Here, we carried out normal ordering of Bose operators in $x^3 \sim (ae^{-i\omega_p t} + a^+e^{i\omega_p t})^3$ and then redefined the oscillator frequency $\omega_0$. In Eq. (3), the dumping term $-\gamma a$ and the quantum Langevin force $\sqrt{2}\gamma a^{\text{in}}(t)$ replace the random force term $\sim F$ in classical equation (1), describe the interaction of the oscillator with the bath in the Markovian approximation. The derivation of such terms from the system–bath interaction Hamiltonian can be found, for example, in [23] and references therein; it reads

$$a^{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_\omega^{\text{in}} e^{-i\omega t} d\omega,$$

where $a_\omega^{\text{in}}$ is the Bose operator of the bath mode, $[a_\omega^{\text{in}}, a_{\omega'}^{\text{in}+}] = \delta(\omega + \omega')$. The coefficient $\sqrt{2}\gamma$ is chosen in the Langevin force term in order to provide the Bose commutation relations for the oscillator operators $[a(t), a^+(t)] = 1$, as is shown, for example, in [24].

We assume that the mean number of quanta in the bath $\langle a^{\text{in}+}(t)a^{\text{in}}(t) \rangle > 0$. The energy from the bath goes to the oscillator, i.e., the bath “pumps” the oscillator.

The classical nonlinear oscillator excited by a regular force and described by Eq. (1) can have more than one stationary state near the resonance.

Let us check if many stationary states can appear in the quantum oscillator excited by a random force and described by Eq. (3). We find an approximate stationary solution of Eq. (3) neglecting by fluctuations of energy $a^+a$ of the oscillator. In this approximation, Eq. (3) reads

$$\dot{a} = [i(\delta + bn) - \gamma] a + \sqrt{2}\gamma a^{\text{in}}(t),$$

(4)
where \( n = \langle a^+ a \rangle \) is a dimensionless energy – the average number of quanta in the oscillator. In the stationary case, \( n \) is the \( c \)-number, so that Eq. (4) is a linear equation with respect to the operator \( a \); therefore, Eq. (4) can be solved by the Fourier transform, in view of the standard analysis of fluctuations in linear systems [18]. In contrast to the linear analysis of fluctuations, \( n \) in Eq. (4) is an unknown quantity, which itself depends on fluctuations and should be determined.

After applying the Fourier transforms in Eq. (4), we arrive at the following relation between the Fourier-component operators:

\[
a_\omega = \sqrt{2\gamma a_\omega^{in}}, \quad o(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} o_{\omega} e^{-i\omega t} d\omega, \quad o \equiv \{a, a^{in}\}. \tag{5}
\]

Using Eqs. (5), we obtain

\[
n = \langle a^+ (t) a(t) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \langle a_\omega^{in} a_{\omega'} \rangle e^{i(\omega + \omega')t}. \tag{6}
\]

After inserting expression (5) for \( a_\omega \) into Eq. (6) and taking into account that the pump-bath Bose operators are delta-correlated, \( \langle a_\omega^{in} a_{\omega'}^{in} \rangle = n_{in}(\omega)\delta(\omega + \omega') \), where \( n_{in}(\omega) \) is the number of quanta in the pump-bath mode of frequency \( \omega \), we obtain

\[
n = \frac{\gamma}{\pi} \int_{-\infty}^{\infty} \frac{n_{in}(\omega) d\omega}{\gamma^2 + (\delta + \omega + b n)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} n_{\omega} d\omega, \tag{7}
\]

where \( n_{\omega} \) is the noise power spectrum of the oscillator. The nonlinear integral equation (7) determines \( n \); in general, Eq. (7) can be solved numerically. In the next section, we solve Eq. (7) analytically for some particular \( n_{in}(\omega) \). It is worth noting that assuming “white noise” when \( n_{in} \) does not depend on \( \omega \), we see the absence of the resonance, \( n = n_{in} \) does not depend on \( \omega \). Thus, if we want to investigate the resonance, we should consider the power spectrum of a random force of finite width.

3. Analysis of Bistabilities

We analyze Eq. (7) for two examples of \( n_{in}(\omega) \).

First, we consider rectangular pump spectrum and approximate \( n_{in}(\omega) \approx \langle n_{in}(\omega) \rangle \equiv n_p \), that is, the average number of quanta in a single pump-bath mode:

\[
n_{in}(\omega) = \begin{cases} n_p \equiv \frac{\pi \gamma_p}{\Gamma_p}, & -\Gamma_p < \omega < \Gamma_p \\ 0, & \omega < -\Gamma_p, \quad \omega > \Gamma_p \end{cases}. \tag{8}
\]

In Eq. (9), we expressed \( n_p \) through the rate \( \gamma_p \) of the flux of quanta from the pump bath to the oscillator, \( \gamma_p = (1/2\pi) \int_{-\infty}^{\infty} n_{in}(\omega) d\omega \). We normalize Eq. (7) and replace it by

\[
z = \frac{1}{2\Gamma} \int_{-\Gamma}^{\Gamma} \frac{dx}{1 + (\Delta + x + Bz)^2}, \tag{9}
\]

where the normalized energy \( z \) and parameters are

\[
z = n\gamma/(2\gamma_p), \quad \Delta = \delta/\gamma, \quad B = 2b\gamma_p/\gamma^2, \quad \Gamma = \Gamma_p/\gamma. \tag{10}
\]
After taking the integral in Eq. (9), we arrive at

\[ 2\Gamma z = \arctan \left[ \frac{2\Gamma}{1 + (\Delta + Bz)^2 - \Gamma^2} \right] + \eta, \tag{11} \]

with \( \eta = 0 \) for \( \Gamma^2 - (\Delta + Bn)^2 < 1 \) and \( \eta = \pi \) for \( \Gamma^2 - (\Delta + Bz)^2 > 1 \). After some algebra, we obtain

\[ \tan(2\Gamma z) = \frac{2\Gamma}{1 + (\Delta + Bz)^2 - \Gamma^2}. \tag{12} \]

Equation (12) determines \( z \) in an inexplicit form; however, using Eq. (12), one can easily plot the stationary \( z(\Delta) \) and investigate the necessary conditions for bistability. For that, from Eq. (12) we express the normalized detuning \( \Delta \) as a function of \( z \), namely,

\[ \Delta(z) = -Bz \pm \left[ 2\Gamma \cot(2\Gamma z) + \Gamma^2 - 1 \right]^{1/2}. \tag{13} \]

For a regular force, when \( \Gamma \to 0 \), Eq. (13) reads

\[ \Delta(z) = -Bz \pm (1/z - 1)^{1/2}. \tag{14} \]

Using Eq. (13) and taking \( 0 < z < \pi/(2\Gamma) \), we plot \( z(\Delta) \) in Fig. 1.

From Fig. 1 one can see that fluctuations of exciting force broaden the oscillator spectrum. However, for a given value of \( B = 10 \), three stationary solutions exist for the same \( \Delta \); if the dimensionless width \( \Gamma \) of the random force power spectrum is not too large, one can observe bistability in curve 2 for \( \Gamma = 2 \). For larger \( \Gamma \), say, \( \Gamma = 6 \) for curve 3, bistability disappears.

We can find the necessary conditions for bistability in nonlinear resonance using Eq. (13) and applying the approach of catastrophe theory [20]. From Fig. 1 it is clear that bifurcation points, when the number of stationary \( z \) changes from 1 to 3, correspond to \( d\Delta/dz = 0 \), which is nothing else than

\[ dB/dz = 0, \text{ where } B(z) \text{ is given by Eq. (15), we obtain } \]

\[ B = \frac{2\Gamma^2[1 + \cot^2(2\Gamma z)]}{[2\Gamma \cot(2\Gamma z) + \Gamma^2 - 1]^{1/2}}. \tag{15} \]

Inserting \( z \) from domain \( 0 < z < \pi/(2\Gamma) \) into Eqs. (13) and (15), we obtain the ranges of bistability in the parameter space \( \{\Delta, B, \Gamma\} \); see Fig. 2.

The larger \( \Gamma \), i.e., the noisier the exciting force, the larger the values of \( B \) and \( \beta_{\gamma, p} \sim B \) necessary for bistability. With increase in \( \Gamma \), the bistability appears at larger \(|\Delta|\) and in a narrower interval of \( \Delta \).

According to [20], the bistability appears if \( B > B_{\text{bif}} \), where \( B_{\text{bif}} \) is determined from the condition \( d^2\Delta/dz^2 = 0 \) at \( d\Delta/dz = 0 \), i.e., when Eq. (15) is valid. Under the condition \( d^2\Delta/dz^2 = 0 \), that is, the same as \( dB/dz = 0 \), where \( B(z) \) is given by Eq. (15), we obtain

\[ \cot(2\Gamma z) = \frac{1}{3\Gamma} \left\{ 1 - \Gamma^2 + [(1 - \Gamma^2)^2 + 3\Gamma^2]^{1/2} \right\}. \tag{16} \]
Fig. 2. Ranges of bistability in the $B, \Delta$ parameter space for various values of $\Gamma$. Bistability exists for $\Delta, B$ from the range between curves 1 for $\Gamma = 0$ (regular force) and between curves 2-4 for $\Gamma = 1.5, 2.5, \text{and } 3.5$, respectively. Maxima $B$ in each curve (where $d\Delta/dB$ does not exist) correspond to $B = B_{\text{bif}}$.

Fig. 3. Bistability exists for values of $B$ above the solid curve for a rectangular noise power spectrum of a random force and above the dashed curve for Lorenz noise power spectrum of the force.

Inserting $\cot(2\Gamma z)$ from Eq. (16) into Eq. (15), we find the “bifurcation” curve $B_{\text{bif}}(\Gamma)$, which separates the bistability range from the region with a single stationary solution in the $B, \Gamma$ parameter space; see Fig. 3.

In Fig. 3, we see that $B_{\text{bif}}(\Gamma)$ is almost a straight line for $\Gamma > 1$. Thus, the minimum value of $b\gamma_p \sim B$ needed for bistability is $b\gamma_p \sim \Gamma$.

Stationary solutions shown by the dashed parts of the $z(\Delta)$ curves in Fig. 2 corresponding to $dz/d\Delta > 0$ are unstable. Indeed, using Eq. (4), we write the equation of motion for $n$,

$$\frac{dn}{dt} \equiv \langle da^+a + a^-da^+/dt \rangle = -2\gamma n + \sqrt{2\gamma}(\langle a^{in+}a \rangle + \langle a^+a^{in} \rangle). \quad (17)$$

Inserting $a(t)$ and $a^{in}(t)$ expressed through Fourier components [as in Eq. (5)] into Eq. (17), replacing $n$ by $z$, and using normalized parameters of Eq. (10), we obtain

$$\frac{dz}{dt} = -2\gamma \left[ z - \int_{-\Gamma}^{\Gamma} \frac{dx}{1 + (\Delta + x + Bz)^2} \right]. \quad (18)$$

Analysis of Eq. (18) with respect to the linear stability shows that $z(t)$ is unstable if $dz/d\Delta > 0$, where $z$ is the stationary solution of Eq. (18) inexplicitly determined by Eq. (13).

Analysis of stability of the upper and lower branches of $z(\Delta)$ curves ($z_{\pm}$ in Fig. 1) is not so straightforward. When the system is in the bistability range, the fluctuating force can, with some probability, produce large fluctuations, which switches the system from one stationary state to another [21, 22]. Thus, stationary states in the bistability range are metastable. The system is only in finite time $\tau_l$ in such “quasistationary” states; here, $\tau_l$ depends on how far the “quasistationary” states are from each other as well as on the fluctuating force spectrum and strengths. Assume that $z_+ > z_-$ are normalized
energies of quasistationary states: they correspond to the upper \((z_+)\) and the lower \((z_-)\) branch of, for example, curve 2 in Fig. 1 in the bistability range. If the normalized fluctuation of the oscillator energy \(\delta z \equiv \langle (z^2) - \langle z^2 \rangle \rangle^{1/2} > z_+ - z_-\), obviously the bistability will be destroyed. In such a case, \(\tau_l < \gamma^{-1}\) is the typical relaxation time of the oscillator to its stationary state. If instead of Eq. (17) to consider the Heisenberg equation for operator \(\hat{n} = a^+a\), one can calculate fluctuations of \(\hat{n}\) and obtain \(\delta z\). Fluctuations of the oscillator energy will lead to narrowing of the bistability ranges with respect to the ones shown in Fig. 2 and to metastability (final lifetime) of oscillator stationary states. We will carry out a more detailed analysis of metastable states of our quantum system and their properties in the future.

Assume now that the noise spectrum of the pump bath is very broad, \(\Gamma \gg 1\). In this case, we can approximate \((\gamma/\pi)[\gamma^2 + (\delta + \omega + bn)^2]\) in Eq. (7) by the Dirac delta-function, so that

\[
n \approx \frac{\gamma}{\pi} \lim_{\gamma \to 0} \int_{-\infty}^{\infty} \frac{n_{\text{in}}(\omega)d\omega}{\gamma^2 + (\delta + \omega + bn)^2} = n_{\text{in}}(-\delta - bn).
\]

For example, if we take the Lorenz power spectrum \(n_{\text{in}}(\omega) = (2\gamma_0\gamma_1)|\omega^2 + (\gamma_1)^2|^{-1}\), then Eq. (19) for normalized variables given by Eqs. (10) becomes \(z = \Gamma[(\Delta + Bz)^2 + \Gamma^2]^{-1}\). Replacing \(z' = \Gamma z\), \(B' = B/\Gamma^2\), and \(\Delta' = \Delta/\Gamma\), we obtain

\[
\Delta'(z') = -B'z' + \frac{1}{2}(1/\gamma - 1)^{1/2},
\]

which is the same (apart from the notation) as Eq. (14) for a nonlinear oscillator excited by a regular force. One can find the bistability conditions \(B' > 8\sqrt{3}/9\); therefore, \(B > B_{\text{bif}} = (8\sqrt{3}/9)\Gamma^2\) (shown in Fig. 3 by the dashed curve). In Fig. 3, we shifted the curve \(B_{\text{bif}} = (8\sqrt{3}/9)\Gamma^2\) up in order to be consistent with \(B = B_{\text{bif}} = 8\sqrt{3}/9\) at \(\Gamma = 0\). One can see that the conditions for bistability of the Lorenz noise power spectrum are qualitatively different from the conditions for rectangular spectrum — in the former case \(B > B_{\text{bif}} \sim \Gamma^2\), in the latter case \(B > B_{\text{bif}} \sim \Gamma\).

With the use of Eq. (16), one can investigate the conditions for bistability at \(\Gamma \gg 1\) for another power spectrum of the excitation forces, for example, for the Gaussian spectrum, etc.

### 4. Conclusions

Using the example of a nonlinear oscillator excited by random force, we demonstrated an approximate method of analysis of quantum nonlinear systems with strong noise. We neglected the fluctuations in amplitude of the oscillator and preserve fluctuations in its phase. This is a natural first-order approximation for studying an oscillating system excited by a noisy bath, when the mean energy of oscillations is not zero while the phase fluctuates are on the \([0, 2\pi]\) interval. A well-known example of such quantum systems is a laser described by the approach elaborated in [16,17]. In this approach, the oscillator power spectrum \(n_\omega\) depends on the oscillator mean energy \(n\) so that \(n \equiv (2\pi)^{-1} \int_{-\infty}^{\infty} n_\omega(n)d\omega\) is a nonlinear integral equation for \(n\). We solved this equation for some particular cases. Thus, in contrast to the traditional linear methods of analyzing the noise [18], we take into account the influence of the noise on the stationary state of the system.

We found necessary conditions for bistability — ranges of parameters, where more than one stationary solution exist for a quantum nonlinear oscillator driven by a random force with nonwhite spectrum of fluctuations. Fluctuations of the exciting force broaden the resonance, and more than one stationary solution exists at the resonance sideband, if dimensionless nonlinearity parameter \(B\) is large, \(B \sim \gamma_p b\),
where $\gamma_p$ is the excitation rate and $b$ is the coefficient of nonlinearity. The necessary condition for bistability is $B > B_{\text{bif}} \sim \Gamma$, where $\Gamma$ is the width of the rectangular power spectrum of the random force, or $B > B_{\text{bif}} \sim \Gamma^2$ for the case of the Lorenz power spectrum of the force. Thus, the necessary bistability conditions are substantially different for different noise power spectra.

We did not take into account amplitude fluctuations. When the amplitude fluctuations are of the order of the mean value of the energy of the oscillator in the bistability range, the bistability will be destroyed — the oscillator cannot stay in the lower or the upper stationary states of the bistability curve. We do not study here the dynamics of switchings between the states. In the future, we will estimate the contribution of amplitude fluctuations, which, at first approximation, can be done with the help of the Heisenberg equation, such as Eq. (17) for the operator of energy. In order to find sufficient conditions for the bistability in this quantum system, more detailed analysis should be performed as, for example, in [21,22] for the classical case.

The approach elaborated provides the possibility to study various interesting phenomena in the dynamics of quantum nonlinear systems with noise. For example, one can consider a combination of broad- and narrow-banded random-force excitation of nonlinear oscillators, as is for the case of stochastic resonance [1–3] and also in the three-level atomic optical bistability (AOB) systems [7–11]. The nonlinear oscillators considered in this paper are similar to the molecular vibration mode excited by short and, therefore, spectrally broad laser pulse. Thus, our results can be used for estimating the conditions of bistability in the laser excitation of molecules in selective laser chemistry [25].

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