ON THE UNIVERSALITY OF THE PROBABILITY DISTRIBUTION OF
THE PRODUCT $B^{-1}X$ OF RANDOM MATRICES

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Abstract

Consider random matrices $A$, of dimension $m \times (m+n)$, drawn from an ensemble with probability density $f(\text{tr}AA^\dagger)$, with $f(x)$ a given appropriate function. Break $A = (B, X)$ into an $m \times m$ block $B$ and the complementary $m \times n$ block $X$, and define the random matrix $Z = B^{-1}X$. We calculate the probability density function $P(Z)$ of the random matrix $Z$ and find that it is a universal function, independent of $f(x)$. The universal probability distribution $P(Z)$ is a spherically symmetric matrix-variate $t$-distribution. Universality of $P(Z)$ is, essentially, a consequence of rotational invariance of the probability ensembles we study. As an application, we study the distribution of solutions of systems of linear equations with random coefficients, and extend a classic result due to Girko.

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1 Introduction

In this note we will address the issue of universality of the probability density function (p.d.f.) of the product $B^{-1}X$ of real and complex random matrices. In order to motivate our discussion, before delving into random matrix theory, let us discuss a simpler problem. Thus, consider the random variables $x$ and $y$ drawn from the normal distribution

$$G(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}.$$  \hspace{0.5cm} (1.1)

Define the random variable $z = \frac{x}{y}$. Obviously, its p.d.f. is independent of the width $\sigma$ of $G(x, y)$, and it is a straightforward exercise to show that

$$P(z) = \frac{1}{\pi} \frac{1}{1 + z^2},$$  \hspace{0.5cm} (1.2)

i.e., the standard Cauchy distribution.

A slightly more interesting generalization of (1.1) is to consider the family of joint probability density (j.p.d.) functions of the form

$$G(x, y) = f(x^2 + y^2),$$  \hspace{0.5cm} (1.3)

where $f(u)$ is a given appropriate p.d.f., subjected to the normalization condition

$$\int_0^\infty f(u)du = \frac{1}{\pi}. \hspace{1cm} (1.4)$$

A straightforward calculation of the p.d.f. of $z = \frac{x}{y}$ leads again to (1.2). Thus, the random variable $z = \frac{x}{y}$ is distributed according to (1.2), independently of the function $f(u)$. In other words, (1.2) is a universal probability density function.\(^1\) $P(z)$ is universal, essentially, due to rotational invariance of (1.3). More generally, $P(z)$ must be independent, of course, of any common scale of the distribution functions of $x$ and $y$. We will now show that an analog of this universal behavior exists in random matrix theory. Our interest in this problem stems from the recent application of random matrix theory made in [1] to calculate the complexity of an analog computation process [2], which solves linear programming problems.

2 The universal probability distribution of the product $B^{-1}X$ of real random matrices

Consider a real $m \times (m+n)$ random matrix $A$ with entries $A_{i\alpha}$ ($i = 1, \ldots, m$; $\alpha = 1, \ldots, m+n$). We take the j.p.d. for the $m(m+n)$ entries of $A$ as

$$G(A) = f(\text{tr}AA^T) = f \left( \sum_{i,\alpha} A_{i\alpha}^2 \right), \hspace{1cm} (2.1)$$

\(^1\)We can generalize (1.3) somewhat further, by considering circularly asymmetric distributions $G(x, y) = f(ax^2 + by^2)$ (with $a, b > 0$ of course, and the r.h.s. of (1.4) changed to $\sqrt{ab}/\pi$), rendering (1.2) a Cauchy distribution of width $\sqrt{b/a}$, independently of the function $f(u)$.\)
with \( f(u) \) a given appropriate p.d.f.. From

\[
\int G(A) \, dA = 1
\]  
(2.2)

we see that \( f(u) \) is subjected to the normalization condition

\[
\int_{0}^{\infty} u^{\frac{m(m+n)}{2}} f(u) du = \frac{2}{S_{m(m+n)}},
\]  
(2.3)

where

\[
S_{d} = \frac{2\pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} \right)}
\]  
(2.4)

is the surface area of the unit sphere embedded in \( d \) dimensions. This implies, in particular, that \( f(u) \) must decay faster than \( u^{-\frac{m(m+n)}{2}} \) as \( u \to \infty \), and also, that if \( f(u) \) blows up as \( u \to 0^+ \), its singularity must be weaker than \( u^{-\frac{m(m+n)}{2}} \). In other words, \( f(u) \) must be subjected to the asymptotic behavior

\[
u^{\frac{m(m+n)}{2}} f(u) \to 0
\]  
(2.5)

both as \( u \to 0 \) and \( u \to \infty \).

We now choose \( m \) columns out of the \( m+n \) columns of \( A \), and pack them into an \( m \times m \) matrix \( B \) (with entries \( B_{ij} \)). Similarly, we pack the remaining \( n \) columns of \( A \) into an \( m \times n \) matrix \( X \) (with entries \( X_{ip} \)). This defines a partition

\[
A \to (B, X)
\]  
(2.6)

of the columns of \( A \).

The index conventions throughout this paper are such that indices

\[
i, j, \ldots \quad \text{range over} \quad 1, 2, \ldots, m,
\]

\[
p, q, \ldots \quad \text{range over} \quad 1, 2, \ldots, n,
\]  
(2.7)

and \( \alpha \) ranges over \( 1, 2, \ldots, m+n \).

In this notation we have \( \text{tr} AA^{T} = \sum_{i,j} B_{ij}^{2} + \sum_{i,p} X_{ip}^{2} = \text{tr} BB^{T} + \text{tr} XX^{T} \), and thus \( 2.1 \) reads

\[
G(B, X) = f(\text{tr} BB^{T} + \text{tr} XX^{T}).
\]  
(2.8)

We now define the random matrix \( Z = B^{-1}X \). Our goal is to calculate the j.p.d. \( P(Z) \) for the \( mn \) entries of \( Z \). \( P(Z) \) is clearly independent of the particular partitioning \( 2.6 \) of \( A \), since \( G(B, X) \) is manifestly independent of that partitioning. The main result in this section is stated as follows:

\footnote{We use the ordinary Cartesian measure \( dA = d^{m(m+n)}A = \prod_{i\alpha} dA_{i\alpha} \). Similarly, \( dB = d^{m^2}B \) and \( dX = d^{mn}X \) for the matrices \( B \) and \( X \) in \( 2.4 \) and \( 2.18 \).}
Theorem 2.1 The j.p.d. for the \( mn \) entries of the real random matrix \( Z = B^{-1}X \) is independent of the function \( f(u) \) and is given by the universal function

\[
P(Z) = \frac{C}{[\det(\mathbb{I} + ZZ^T)]^{\frac{m+n}{2}}},
\]

where \( C \) is a normalization constant.

Remark 2.1 The probability density function (2.9) is a special (spherically symmetric) case of the so-called matrix variate \( t \)-distributions [3, 4]: The \( m \times n \) random matrix \( Z \) is said to have a matrix variate \( t \)-distribution with parameters \( M, \Sigma, \Omega \) and \( q \) (a fact we denote by \( Z \sim T_{n,m}(q, M, \Sigma, \Omega) \)) if its p.d.f. is given by

\[
D(\det \Sigma)^{-\frac{n}{2}}(\det \Omega)^{-\frac{m}{2}} \left[ \det \left( \mathbb{I}_m + \Sigma^{-1}(Z - M)\Omega^{-1}(Z - M)^T \right) \right]^{-\frac{1}{2}(m+n+q-1)},
\]

where \( M, \Sigma \) and \( \Omega \) are fixed real matrices of dimensions \( m \times n, m \times m \) and \( n \times n \), respectively. \( \Sigma \) and \( \Omega \) are positive definite, and \( q > 0 \). The normalization coefficient is

\[
D = \frac{1}{\pi^{\frac{m+n}{2}}} \frac{\prod_{j=1}^{n} \Gamma \left( \frac{m+n+q-j}{2} \right)}{\prod_{j=1}^{n} \Gamma \left( \frac{n+q-j}{2} \right)}.
\]

It arises in the theory of matrix variate distributions as the p.d.f. of a random matrix which is the product of the inverse square root of a certain Wishart-distributed matrix and a matrix taken from a normal distribution, and by shifting this product by \( M \), as described in [3, 4]. Our universal distribution (2.9) corresponds to setting \( M = 0, \Sigma = \mathbb{I}_m, \Omega = \mathbb{I}_n \) and \( q = 1 \) in (2.10) and (2.11).

Remark 2.2 It would be interesting to distort the parent j.p.d. (2.1) into a non-isotropic distribution and see if the generic matrix variate \( t \)-distribution (2.10) arises as the corresponding universal probability distribution function in this case.

To prove Theorem (2.1), we need

Lemma 2.1 Given a function \( f(u) \), subjected to (2.3), the integral

\[
I = \int dBf(\text{tr} BB^T)|\det B|^n
\]

converges, and is independent of the particular function \( f(u) \).

Remark 2.3 A qualitative and simple argument, showing the convergence of (2.12), is that the measure \( d\mu(B) = dB |\det B|^n \) scales as \( d\mu(tB) = t^{m(m+n)}d\mu(B) \), and thus has the same scaling property as \( dA \) in (2.2), indicating that the integral (2.12) converges, in view of (2.5). To see that \( I \) is independent of \( f(u) \) one has to work harder.

\[^{3}\text{Our notations in Remark 2.1 are slightly different from the notations used in [4]. In particular, we interchanged their } \Sigma \text{ and } \Omega, \text{ and also denoted their } (T - M)^T \text{ by } Z - M \text{ here. Finally, we applied the identity } \det(\mathbb{I} + AB) = \det(\mathbb{I} + BA) \text{ to arrive after all these interchanges from their equation (4.2.1) to (2.10).}\]
Proof. We would like first to integrate over the rotational degrees of freedom in $dB$. Any real $m \times m$ matrix $B$ may be decomposed as \cite{5,6}

$$B = O_1 \Omega O_2$$

(2.13)

where $O_{1,2} \in O(m)$, the group of $m \times m$ orthogonal matrices, and $\Omega = \text{Diag}(\omega_1, \ldots, \omega_m)$, where $\omega_1, \ldots, \omega_m$ are the singular values of $B$. Under this decomposition we may write the measure $dB$ as \cite{5,6}

$$dB = d\mu(O_1)d\mu(O_2) \prod_{i<j} |\omega_i^2 - \omega_j^2| d^m \omega,$$

(2.14)

where $d\mu(O_{1,2})$ are Haar measures over the $O(m)$ group manifold. The measure $dB$ is manifestly invariant under actions of the orthogonal group $O(m)$

$$dB = d(BO) = d(O'B), \quad O, O' \in O(m),$$

(2.15)

as should have been expected to begin with.

Remark 2.4 Note that the decomposition (2.13) is not unique, since $O_1D$ and $DO_2$, with $D$ being any of the $2^m$ diagonal matrices $\text{Diag}(\pm 1, \ldots, \pm 1)$, is an equally good pair of orthogonal matrices to be used in (2.13). Thus, as $O_1$ and $O_2$ sweep independently over the group $O(m)$, the measure (2.14) over counts $B$ matrices. This problem can be easily rectified by appropriately normalizing the volume $V_m = \int d\mu(O_1)d\mu(O_2)$. One can show\footnote{One simple way to establish (2.16), is to calculate $\int dB \exp -\frac{1}{2} \text{tr} B^T B = (2\pi)^{\frac{m}{2}} = V_m \int d^m \omega \prod_{i<j} |\omega_i^2 - \omega_j^2| \exp -\frac{1}{2} \sum_i \omega_i^2$. The last integral is a known Selberg type integral \cite{7}.} that the correct normalization of the volume is

$$V_m = \frac{\pi^{m(m+1)}}{2^m \prod_{j=1}^m \Gamma \left(1 + \frac{j}{2}\right) \Gamma \left(\frac{j}{2}\right)}.$$

(2.16)

Let us now turn to (2.12). The integrals over the orthogonal group in (2.12) clearly factor out, and we obtain

$$I = V_m \int_{-\infty}^\infty \prod_{i=1}^m d\omega_i \prod_{j<k} |\omega_j^2 - \omega_k^2| \left(\prod_{i=1}^m \omega_i\right)^n f \left(\sum_{i=1}^m \omega_i^2\right).$$

(2.17)

Finally, we change the integration variables in (2.17) to the polar coordinates associated with the $\omega_i$. The angular part of that integral is fixed only by dimensionality and by the factor $\prod_{j<k} |\omega_j^2 - \omega_k^2| \left(\prod_{i=1}^m \omega_i\right)^n$, and is thus independent of the function $f(u)$.

To prove that $I < \infty$ we need only consider integration over the radius $r^2 = \sum_{i=1}^m \omega_i^2$, since integration over the angles obviously produces a finite result. Using (2.3), we find that the radial integral in question is

$$\int_0^\infty dr \, r^{m^2 + nm - 1} f(r^2) = \frac{1}{2} \int_0^\infty u^{\frac{m(m+n)}{2} - 1} f(u) du = \frac{2}{S_{m(n+n)}},$$

where $S_{m(n+n)}$ is the volume of an $m(n+n)$-dimensional sphere.

\[4\]
independently of \( f(u) \).

We are ready now to prove Theorem (2.1):

**Proof.** By definition \(^5\),

\[
P(Z) = \int dB \, dX \, f(\text{tr} BB^T + \text{tr} XX^T) \delta(Z - B^{-1}X)
\]

\[
= \int dB \, dX \, f(\text{tr} BB^T + \text{tr} XX^T) |\det B|^n \delta(X - BZ).
\]

Integration over \( X \) gives:

\[
P(Z) = \int dB \, f(\text{tr} B(1 + ZZ^T)B^T) |\det B|^n
\]

\[\tag{2.19}\]

The \( m \times m \) symmetric matrix \( 1 + ZZ^T \) can be diagonalized as \( 1 + ZZ^T = \mathcal{O} \Lambda \mathcal{O}^T \), where \( \mathcal{O} \) is an orthogonal matrix, and \( \Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_m) \) is the corresponding diagonal form. Obviously, all \( \lambda_i \geq 1 \), since \( ZZ^T \) is positive definite. Substituting this diagonal form into (2.19) we obtain

\[
P(Z) = \int dB \, f(\text{tr} B\Lambda B^T) |\det B|^n
\]

\[\tag{2.20}\]

From the invariance of the determinant \( |\det B\mathcal{O}| = |\det B| \) and of the volume element \( d(B\mathcal{O}) = dB \) under orthogonal transformations we have:

\[
P(Z) = \int dB \, f(\text{tr} B\Lambda B^T) |\det B|^n
\]

Let us now rescale \( B \) as \( \tilde{B} = B\sqrt{\Lambda} \). Thus,

\[
det \tilde{B} = \sqrt{\det \Lambda} \, \det B, \quad \text{and} \quad d\tilde{B} = (\det \Lambda)^{\frac{m}{2}} \, dB.
\]

\[\tag{2.21}\]

Finally, substituting (2.21) in (2.20) we obtain

\[
P(Z) = \int d\tilde{B} \left( \frac{\det \tilde{B}}{(\det \Lambda)^{\frac{m}{2}}} \right)^n f(\text{tr} \tilde{B} \tilde{B}^T)
\]

\[
= C \left( \frac{\det \tilde{B}}{(\det \Lambda)^{\frac{m}{2}}} \right)^n
\]

\[\tag{2.22}\]

where \( C \) is the normalization constant

\[
C = \int dB \, f(\text{tr} BB^T) |\det B|^n
\]

\[\tag{2.23}\]

rendering

\[
\int P(Z) dZ = 1.
\]

\[\tag{2.24}\]

\( C \) is nothing but the integral (2.22). Thus, according to Lemma (2.1), \( C < \infty \) and is also independent of the function \( f(u) \).

\(^5\)Our notation is such that \( \delta(X) = \prod_{i=1}^{m} \prod_{p=1}^{n} \delta(X_{ip}) = \prod_{p=1}^{n} \delta^{(m)}(X_p) \), \( X_p \) being the \( p \)-th column of \( X \).
Remark 2.5 The j.p.d. $P(Z)$ in (2.9) is manifestly a symmetric function only of the eigenvalues of $ZZ^T$, and thus, a symmetric function only of the singular values of $Z$.

Remark 2.6 From the normalization condition (2.24) we obtain an alternative expression for the normalization constant (2.23) as
\[
\frac{1}{C} = \frac{1}{\int dB f(\text{tr}BB^T) |\det B|^n} = \int \frac{dZ}{|\det(\mathbb{I} + ZZ^T)|^{(m+n)/2}},
\] (2.25)
which is manifestly independent of the particular function $f(u)$, in accordance with Lemma (2.1). The integral over the matrix $Z$ can be reduced to a multiple integral of the Selberg type \cite{5,7} over the singular values of the matrix $Z$, which can be carried out explicitly:
\[
\int \frac{dZ}{|\det(\mathbb{I} + ZZ^T)|^{(m+n)/2}} = \pi^m \prod_{j=1}^n \frac{\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{m+j}{2}\right)},
\] (2.26)
For particular choices of the function $f(u)$, we can use (2.25) to derive explicit integration formulas. For example, the function
\[
f(u) = \frac{e^{-u}}{\pi^{m(m+n)/2}}
\] (i.e., the entries $A_{\alpha\beta}$ in (2.1) are i.i.d. according to a normal distribution of variance $1/2$) satisfies (2.23). Thus, we obtain from (2.25) that
\[
\int dB e^{-\text{tr}BB^T} |\det B|^n = \pi^m \prod_{j=1}^n \frac{\Gamma\left(\frac{m+j}{2}\right)}{\Gamma\left(\frac{j}{2}\right)}.
\] (2.28)
Note that the integral on the left-hand side of (2.28) can also be reduced to a multiple integral of the Selberg type (this time, over the singular values of $B$), which can be carried out explicitly. The result is
\[
\pi^m \prod_{j=1}^m \frac{\Gamma\left(\frac{n+j}{2}\right)}{\Gamma\left(\frac{j}{2}\right)}.
\]
Since this must coincide with (2.28), we obtain the identity
\[
\prod_{j=1}^m \frac{\Gamma\left(\frac{n+j}{2}\right)}{\Gamma\left(\frac{j}{2}\right)} = \prod_{j=1}^n \frac{\Gamma\left(\frac{m+j}{2}\right)}{\Gamma\left(\frac{j}{2}\right)}.
\] (2.29)

Example 1 For $n = 1$, i.e., the case where $X$ and $Z$ are $m$ dimensional vectors, (2.1) simplifies into the $m$ dimensional Cauchy distribution
\[
P(Z) = \frac{C}{(1 + ZZ^T)^{(m+1)/2}}.
\] (2.30)
This is so because for $n = 1$, the matrix $ZZ^T$ has $m-1$ eigenvalues equal to 0, that correspond to the $m-1$ dimensional subspace of vectors orthogonal to $Z$, and one eigenvalue equal to $Z^T Z$. Thus, $\det(\mathbb{I} + ZZ^T) = 1 + Z^T Z$. Eq. (2.30) then follows by substituting this determinant into (2.22).
3 The universal probability distribution of the product $B^{-1}X$ of complex random matrices

The results of the previous section are readily generalized to complex random matrices. One has only to count the number of independent real integration variables correctly. In what follows we will use the notations defined in the previous section (unless specified otherwise explicitly). Thus, consider a complex $m \times (m + n)$ random matrix $A$ with entries $A_{i\alpha}$ ($i = 1, \ldots, m; \alpha = 1, \ldots, m + n$). We take the j.p.d. for the $m(m + n)$ entries of $A$ as

$$G(A) = f(\text{tr}AA^\dagger) = f \left( \sum_{i,\alpha} |A_{i\alpha}|^2 \right), \quad (3.1)$$

with $f(u)$ a given appropriate p.d.f.. From \[6\]

$$\int G(A) \, dA = 1 \quad (3.2)$$

we see that $f(u)$ is subjected to the normalization condition

$$\int_0^\infty u^{m(m+n)-1} f(u) du = \frac{2}{S_{2m(m+n)}}. \quad (3.3)$$

This implies that $f(u)$ must be subjected to the asymptotic behavior

$$u^{m(m+n)} f(u) \to 0 \quad (3.4)$$

both as $u \to 0$ and $u \to \infty$.

As in the previous section, we choose a partition

$$A \to (B, X) \quad (3.5)$$

of the columns of $A$. Thus, $\text{tr}AA^\dagger = \sum_{i,j} |B_{ij}|^2 + \sum_{i,p} |X_{ip}|^2 = \text{tr}BB^\dagger + \text{tr}XX^\dagger$, and thus (3.1) reads

$$G(B, X) = f(\text{tr}BB^\dagger + \text{tr}XX^\dagger). \quad (3.6)$$

We now define the random matrix $Z = B^{-1}X$. Our goal is to calculate the j.p.d. $P(Z)$ for the $mn$ entries of $Z$. The main result in this section is stated as follows:

**Theorem 3.1** The j.p.d. for the $mn$ entries of the complex random matrix $Z = B^{-1}X$ is independent of $f(u)$ and is given by the universal function

$$P(Z) = \frac{C}{[\det(1 + ZZ^\dagger)]^{m+n}}, \quad (3.7)$$

where $C$ is the normalization constant \[7\].

---

\[6\] We use the Cartesian measure $dA = d^{2m(m+n)}A = \prod_{i\alpha} d\text{Re}A_{i\alpha} \, d\text{Im}A_{i\alpha}$, with analogous definitions for $dB$ and $dX$ below.
Proof. The proof proceeds in a similar manner to the proof of Theorem (2.1). The only important difference is that now
\[ \delta(X) = \prod_{i=1}^{m} \prod_{p=1}^{n} \delta(\text{Re} X_{ip}) \delta(\text{Im} X_{ip}) = \prod_{p=1}^{n} \delta^{(2m)}(X_{p}), \]
where \(X_{p}\) is the \(p\)-th column of \(X\). One obtains
\[
P(Z) = \int dB dX f(\text{tr} BB^{\dagger} + \text{tr} XX^{\dagger}) \delta(Z - B^{-1} X)
= \int dB f[\text{tr} B(\mathbb{1} + ZZ^{\dagger})B^{\dagger}] |\det B|^{2n}
\] (3.8)
where we have integrated over \(X\).

The \(m \times m\) complex hermitean matrix \(\mathbb{1} + ZZ^{\dagger}\) can be diagonalized as \(\mathbb{1} + ZZ^{\dagger} = U\Lambda U^{\dagger}\), where \(U\) is a unitary matrix, and \(\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_m)\) is the corresponding diagonal form. Obviously, all \(\lambda_i \geq 1\), since \(ZZ^{\dagger}\) is positive definite. Substituting this diagonal form into (3.8) we obtain
\[
P(Z) = \int dB f(\text{tr} B U \Lambda U^{\dagger} B^{\dagger}) |\det B|^{2n}
= \int dB f(\text{tr} B \Lambda B^{\dagger}) |\det B|^{2n},
\] (3.9)
where we used the invariance of the determinant \(|\det BU| = |\det B|\) and the invariance of the volume element \(d(BU) = dB\) under unitary transformations.

As in the previous section we now rescale \(B\) as \(\tilde{B} = B\sqrt{\Lambda}\). Thus,
\[
det \tilde{B} = \sqrt{\det \Lambda} \det B, \quad \text{and} \quad d\tilde{B} = (\det \Lambda)^m dB.
\] (3.10)

Finally, substituting (3.10) in (3.9) we obtain
\[
P(Z) = \int \frac{d\tilde{B}}{(\det \Lambda)^m} f(\text{tr} \tilde{B} \tilde{B}^{\dagger}) \left( \frac{|\det \tilde{B}|}{(\det \Lambda)^{\frac{m}{2}}} \right)^{2n}
= \frac{C}{(\det \Lambda)^{m+n}} = \frac{C}{|\det (\mathbb{1} + ZZ^{\dagger})|^{m+n}},
\] (3.11)
where \(C\) is the normalization constant
\[
C = \int dB f(\text{tr} BB^{\dagger}) |\det B|^{2n}
\] (3.12)
rendering
\[
\int P(Z)dZ = 1.
\] (3.13)
Finally, one can show, in a manner analogous to Lemma (2.1), that \(C < \infty\) and that it is independent of the particular function \(f(u)\).

There are obvious analogs to the remarks made in the previous section, which follow from Theorem (3.1), which we will not write down explicitly.

4 More on the distribution of solutions of systems of linear equations with random coefficients: extension of a result due to Girko

The methods of the previous sections may be applied in studying the distribution of solutions of systems of linear equations with random coefficients. For concreteness, let us concentrate on real linear systems in real variables.
Consider a system of \( m \) real linear equations

\[
\sum_{\alpha=1}^{m+n} A_{i\alpha} \xi_\alpha = b_i, \quad i = 1, \ldots, m
\]  

(4.1)

in the \( m+n \) real variables \( \xi_\alpha \). With no loss of generality, we will treat the first \( m \) components of the vector \( \xi \) as unknowns, and the remaining \( n \) components of \( \xi \) as given parameters. Thus, we split

\[
\xi = \begin{pmatrix} z \\ u \end{pmatrix}
\]  

(4.2)

where \( z \) is the vector of unknowns \( z_i = \xi_i \ (i = 1, \ldots, m) \) and \( u \) is the vector of parameters \( u_p = \xi_{p+m} \ (p = 1, \ldots, n) \). Similarly, we split the matrix of coefficients

\[
A = (B, X),
\]

where the \( m \times m \) matrix \( B \) (with entries \( B_{ij} \)) and the \( m \times n \) matrix \( X \) (with entries \( X_{ip} \)) were defined in (2.6). Thus, we may rewrite (4.1) explicitly as a system for the \( z_i \):

\[
Bz = b - Xu.
\]  

(4.3)

If we consider an ensemble of systems (4.1), in which \( A \) and \( b \) are drawn according to some probability law, the unknowns \( z_i \) become random variables, which depend on the parameters \( u_p \). Girko proved the following theorem for a particular family of such ensembles:

**Theorem 4.1 (Girko)**

If the random variables \( A_{i\alpha} \) and \( b_i \ (i = 1, \ldots, m; \ \alpha = 1, \ldots, m+n) \) are independent, identically distributed variables, having a stable distribution law with the characteristic function

\[
g(t; \alpha, c) = e^{-c|t|^\alpha}, \quad 0 < \alpha \leq 2; \ c > 0,
\]

(4.4)

then the random variables \( z_i \ (i = 1, \ldots, m) \) are identically distributed with the probability density function

\[
p(\zeta; \alpha, \beta) = \frac{2}{\beta} \int_0^\infty r \rho \left( \frac{r \zeta}{\beta}; \alpha \right) \rho(r; \alpha) \, dr,
\]

(4.5)

where \( \rho(r; \alpha) \) is the probability density of the postulated stable distribution, and

\[
\beta = \left( 1 + \sum_{p=1}^n |u_p|^\alpha \right)^{\frac{1}{\alpha}}.
\]

(4.6)

The ratios \( z_i/z_j \ (i \neq j; i, j = 1, \ldots, m) \) have the density \( p(r; \alpha, 1) \).

In the special case \( \alpha = 2 \) in (4.4), the random variables \( A_{i\alpha} \) and \( b_i \) are normally distributed. For this case Girko obtained
**Corollary 4.1** If, under the conditions of Theorem (4.1), $\alpha = 2$, then

\[ p(\zeta; 2, \beta) = \frac{\beta}{\pi(\zeta^2 + \beta^2)}, \]  

(4.7)

i.e., $\zeta$ follows a Cauchy distribution of width $\beta$.

When $\alpha = 2$, the j.p.d. of the $A_{i\alpha}$ and the $b_i$ is

\[ G(A, b) = (2\pi \sigma^2)^{\frac{m(m+n+1)}{2}} e^{-\frac{1}{2\sigma^2} (\text{tr}A^T A + b^T b)}, \]  

(4.8)

which is a special case of the j.p.d.’s we have discussed in the previous sections. Thus, in the spirit of the discussion in the previous sections, we will study systems of linear equations (4.1) with random coefficients $A$ and inhomogeneous terms $b$ with j.p.d.’s of the form

\[ G(A, b) = f(\text{tr}A^T A + b^T b), \]  

(4.9)

with $f(u)$ a given appropriate p.d.f. subjected to the normalization condition

\[ \int_0^\infty u^{\frac{m(m+n+1)}{2} - 1} f(u) du = \frac{2}{S^{m(m+n+1)}}. \]  

(4.10)

Our goal is to calculate the j.p.d. $P(z; u)$ for the $m$ unknowns $z_i$. We summarize our main result in this section as

**Theorem 4.2** If the random variables $A_{i\alpha}$ and $b_i$ ($i = 1, \ldots, m; \alpha = 1, \ldots, m + n$) are distributed with a j.p.d. given by (4.9), with $f(u)$ being any appropriate probability density function subjected to (4.10), then the random variables $z_i$ ($i = 1, \ldots, m$) are distributed with the universal j.p.d. function

\[ P(z; u) = C \frac{\beta}{(\beta^2 + z^T z)^{\frac{m+1}{2}}}, \]  

(4.11)

independently of the function $f(u)$, where

\[ \beta = \sqrt{1 + u^T u} \]  

(4.12)

and $C$ is a normalization constant given by

\[ C = \int dA |\det B| f(\text{tr}A^T A). \]  

(4.13)

**Remark 4.1** Note that Theorem (4.2) generalizes the case $\alpha = 2$ of Girko’s result, Theorem (4.1), from the particular $f(u) \sim e^{-u}$ to a whole class of probability densities $f(u)$, and moreover, it determines for this class of distributions the (universal) j.p.d. of the $z_i$’s, and not only the distribution of a single component. Thus, it is an interesting question whether Girko’s result could be generalized also to other ensembles of systems of linear equations as well.
Proof. By definition, from (4.3),

\[ P(z; u) = \int dA db f(\text{tr} A^T A + b^T b)\delta(z - B^{-1}(b - Xu)) \]

\[ = \int dA db f(\text{tr} A^T A + b^T b)\left|\det(B)\right|\delta(Bz + Xu - b) \]

\[ = \int dA f[\text{tr} A^T A + (A\xi)^T (A\xi)]\left|\det(B)\right|, \quad (4.14) \]

where in the last step we integrated over the \( m \) dimensional vector \( b \) and used \( Bz + Xu = A\xi \).

The last expression in (4.14) is manifestly invariant under \( O(m) \times O(n) \) orthogonal transformations

\[ P(O_1 z; O_2 u) = P(z; u), \quad O_1 \in O(m), O_2 \in O(n), \quad (4.15) \]

due to the invariance of the measure \( dA = dB dX = d(BO_1) d(XO_2) \), the invariance of the determinant \( |\det(B)| = |\det(BO_1)| \), and the invariance of the trace \( \text{tr} A^T A = \text{tr} (BO_1)^T (BO_1) + \text{tr} (XO_2)^T (XO_2) \). With this symmetry at our disposal, we may simplify the calculation of \( P(z; u) \) by rotating the vectors \( z \) and \( u \) into fixed convenient directions, e.g., into the directions in which only \( z_1 \) and \( u_1 \) do not vanish:

\[ z_1^{(0)} = z_0 \delta_{11}, \quad u_1^{(0)} = u_0 \delta_{11}. \quad (4.16) \]

with

\[ z_0 = (z^T z)^{\frac{1}{2}}, \quad u_0 = (u^T u)^{\frac{1}{2}}. \quad (4.17) \]

Thus, we obtain

\[ P(z; u) = \int dA f[\text{tr} A^T A + (Bz^{(0)} + Xu^{(0)})^T (Bz^{(0)} + Xu^{(0)})] |\det(B)| \]

\[ = \int dB dX f(S) |\det(B)|, \quad (4.18) \]

where

\[ S = \sum_{j=2}^{m} B_j^T B_j + \sum_{p=2}^{n} X_p^T X_p + (1 + z_0^2) B_1^T B_1 + 2u_0z_0 B_1^T X_1 + (1 + u_0^2) X_1^T X_1, \quad (4.19) \]

in which \( B_i \) is the \( i \)-th column of \( B \), and \( X_p \) is the \( p \)-th column of \( X \).

The bilinear form involving \( B_1 \) and \( X_1 \) in (4.19) may be diagonalized as

\[ (1 + \xi^T \xi) \left( \frac{z_0 B_1 + u_0 X_1}{\sqrt{\xi^T \xi}} \right)^2 + \left( \frac{u_0 B_1 - z_0 X_1}{\sqrt{\xi^T \xi}} \right)^2, \quad (4.20) \]

where we have used \( z_0^2 + u_0^2 = \xi^T \xi \). We now perform a rotation in the \( B_1 - X_1 \) plane, followed by a scale transformation of the first term in (4.20), thus defining

\[ B'_1 = (1 + \xi^T \xi)\frac{1}{2} \frac{z_0 B_1 + u_0 X_1}{\sqrt{\xi^T \xi}}, \quad X'_1 = \frac{u_0 B_1 - z_0 X_1}{\sqrt{\xi^T \xi}}, \quad (4.21) \]
such that $d^m B_1' d^m X_1' = (1 + \xi^T \xi)^{\frac{m}{2}} d^m B_1 d^m X_1$. We will also need the inverse transformation for $B_1$

$$B_1(B_1', X_1') = \frac{1}{\sqrt{\xi^T \xi}} \left( \frac{z_0 B_1'}{(1 + \xi^T \xi)^{\frac{1}{2}}} + u_0 X_1' \right) \quad (4.22)$$

in order to express the matrix $B$ in terms of the primed column vectors:

$$\tilde{B} = (B_1(B_1', X_1'), B_2, \ldots, B_m) \quad (4.23)$$

Thus, using (4.20) - (4.23) and the trivial fact that $dA = \prod_{i=1}^m d^m B_i \prod_{i=p}^m d^m X_p$, we obtain

$$P(z; u) = \frac{1}{(1 + \xi^T \xi)^{\frac{m}{2}}} \int dA f(\text{tr} A^T A) |\det \tilde{B}|, \quad (4.24)$$

where we have removed the primes from the integration variables. We are not done yet, since $\tilde{B}_1$, the first column of $\tilde{B}$, depends on $z_0$ and $u_0$. To rectify this problem, we note from (4.22) that

$$\tilde{B}_1(B_1, X_1) = \sqrt{\frac{1 + u_0^2}{1 + \xi^T \xi}} \left( B_1 \cos \theta + X_1 \sin \theta \right), \quad \cos \theta = \frac{z_0}{\sqrt{\xi^T \xi (1 + u_0^2)}}. \quad (4.25)$$

Thus, performing one final rotation by an angle $\theta$ in the $B_1 - X_1$ plane, which leaves, of course, $dA$ and $\text{tr} A^T A$ invariant, we see that in terms of the rotated columns $|\det \tilde{B}| = \sqrt{\frac{1 + u_0^2}{1 + \xi^T \xi}} |\det B|$, and thus, finally, we obtain that

$$P(z; u) = \sqrt{1 + u_0^2} \left( \frac{1 + \xi^T \xi}{1 + \xi^T \xi} \right)^{\frac{m}{2}} \int dA f(\text{tr} A^T A) |\det B|, \quad (4.26)$$

which coincides with (4.11), due to (4.12) and (4.13). \[\square\]

**Remark 4.2** The fact that the integral $C = \int dA |\det B| f(\text{tr} A^T A)$ is convergent and independent of the function $f(u)$ can be proved by decomposing $A$ into its singular values \[5, 6\], essentially in a manner similar to our proof of Lemma (2.1), but with slight modifications in (2.13) and (2.14) due to the fact that $A$ is a rectangular matrix rather than a square matrix. We shall not get into these technicalities here, which the reader may find in \[5, 6\]. Note, however, that $C$ may be determined from the normalization of $P(z; u)$:

$$\int d^m z P(z; u) = 1 = C\beta \int \frac{d^m z}{(\beta^2 + z^T z)^{\frac{m+1}{2}}} = \frac{C\pi^{\frac{m+1}{2}}}{\Gamma \left( \frac{m+1}{2} \right)}. \quad (4.27)$$

Thus,

$$C = \frac{\Gamma \left( \frac{m+1}{2} \right)}{\pi^{\frac{m+1}{2}}} = \frac{2}{S_{m+1}} \quad (4.27)$$

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Remark 4.3 We note that for \( n = 0 \) (i.e., when \( A = B \)) and \( u = 0 \), \( Bz = b \), which is precisely the case \( n = 1 \) (and \( z \equiv Z \)) in the conditions for Theorem (2.1), which we analyzed in Example (1). Thus, (4.11), evaluated at \( n = 1 \) and \( u = 0 \) must coincide with (2.30), as one can easily check it does.

Since Theorem (4.2) states the explicit form (4.11) of \( P(z; u) \), we can now use it to derive, e.g., the probability density of the distribution of a single component \( z \) and that of the ratio of two different components, mentioned in Girko’s Theorem (4.1):

Corollary 4.2 The \( m \) components \( z_i \) are identically distributed, with the probability density of any one of the components \( z_i = \xi \) given by (4.7) of Corollary (4.1).

Proof. That the \( z_i \) are identically distributed is an immediate consequence of the rotational invariance of \( P(z; u) \) in (4.11). The proof is completed by performing the necessary integrals:

\[
p(\xi; \beta) = \int dz_2 \ldots dz_m P(z; u)_{|_{z_1 = \xi}} = C\beta \int \frac{dz_2 \ldots dz_m}{(\beta^2 + \xi^2 + \sum_{i=2}^{m} z_i^2)^{\frac{m+1}{2}}}
\]

\[
= \frac{C\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)} \frac{\beta}{\beta^2 + \xi^2}.
\]

Thus, from (4.27) we obtain the desired result that \( p(\xi; \beta) = \frac{\beta}{\pi(\beta^2 + \xi^2)} \). This result, should have been anticipated, since the universal formula (4.11) holds, in particular, for the Gaussian distribution (4.5).

Finally, we have

Corollary 4.3 The ratios \( z_i/z_j \) (\( i \neq j; i, j = 1, \ldots m \)) have the density \( P(r) = p(r; 1) \).

Proof. The ratio \( z_i/z_j \) is dimensionless, and thus its distribution cannot depend on the width \( \beta \), which is the only dimensionful quantity in (4.11). The proof amounts to performing the necessary integrals, e.g., for the random variable \( z_1/z_2 \):

\[
P(r) = \int d^m z P(z; u)\delta\left(r - \frac{z_1}{z_2}\right) = C\beta \int \frac{|z_2| dz_2 \ldots dz_m}{[\beta^2 + (r^2 + 1)z_2^2 + \sum_{i=3}^{m} z_i^2]^{\frac{m+1}{2}}}
\]

\[
= \frac{2CS_{m-2}}{r^2 + 1} \frac{\Gamma\left(\frac{m-2}{2}\right) \Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(\frac{m+1}{2}\right)} = \frac{1}{\pi(r^2 + 1)} = p(r; 1).
\]
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