Realization of Weyl semimetal phases in topoelectrical circuits

S M Rafi-Ul-Islam, Zhuo Bin Siu, Chi Sun and Mansoor B A Jalil
Department of Electrical and Computer Engineering, National University of Singapore, Singapore
1 Author to whom any correspondence should be addressed. 
E-mail: e0021595@u.nus.edu, elesiuz@nus.edu.sg and elembaj@nus.edu.sg

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Abstract
In this work, we demonstrate a simple and effective method to design and realize various Weyl semimetal (WSM) states in a three-dimensional periodic circuit lattice composed of passive electric circuit elements such as inductors and capacitors (LC). The experimental accessibility of such LC circuits offers a ready platform for the realization of not only various WSM phases but also for exploring transport properties in topological systems. The characteristics of such LC circuits are described by the circuit admittance matrices, which are mathematically related to the Hamiltonian of the quantum tight-binding model. The system can be switched between the Type-I and Type-II WSM phases simply by an appropriate choice of inductive or capacitive coupling between certain nodes. A peculiar phase with a flat admittance band emerges at the transition between the Type-I and Type-II Weyl phases. Impedance resonances occur in the LC circuits at certain frequencies associated with vanishing eigenvalues of the admittance matrix. The impedance readout can be used to classify the Type-I and Type-II WSM states. A Type-I WSM shows impedance peaks only at the Weyl points (WPs) whereas a Type-II WSM exhibits multiple secondary peaks near the WPs. This impedance behaviour reflects the vanishing and non-vanishing density of states at the Weyl nodes in the Type-I and Type-II WSM phases, respectively.

1. Introduction
Exotic topological phases of matter have emerged as one of the most exciting branches of condensed matter physics in the past decades due to their exceptional electronic properties [1–3]. Although gapped topological materials such as topological insulators [4–6], integer quantum Hall insulators [7, 8] and topological superconductors [9] have attracted a lot of attention, the physics of topological gapless materials [10] has recently gained more prominence due to their novel properties. A three-dimensional topological gapless system can be characterized by the nature of its band degeneracy points where two bands touch each other in momentum space. These band degeneracy points are classified as either Dirac [11] or Weyl points (WP) [12, 13] depending on their symmetries. Dirac points appear only when both time-reversal [14] and inversion symmetry [15] are conserved in a system. On the contrary, WPs emerge if either or both symmetries are broken. Although both types of band touching points appear and annihilate pair-wise, WPs are more robust against perturbations than Dirac points. One important class of topological system that hosts WPs are called Weyl semimetals (WSMs) [16–19]. WSMs disperse linearly in all three spatial directions in the vicinity of the WPs. Since the linear dispersion around the WPs can be described by the Weyl Hamiltonian involving all three Pauli matrices, small perturbations do not lift the energy degeneracy but only displace the WPs in momentum space. Besides fundamental properties such as the massless and chiral nature of the bulk carriers and large carrier mobility [20, 21], WSM states also exhibit unusual transport phenomena like the quantum anomalous Hall effect [22], large positive magnetoresistance [23], Klein tunnelling [24, 25], chiral anomaly [26, 27] and novel quantum oscillations [28, 29]. These exotic features and inherent robustness against disorder make WSMs promising candidate for future generation nanoelectronics, spintronics [30] and valleytronics [31] devices. WSMs can be
further classified as Type-I and Type-II based on the tilt of the Weyl cones around the WPs [32, 33]. Type-I WSMs are marked by the existence of point-like iso-energy surfaces at the WPs, and the simultaneous presence of carriers with both signs of group velocities near the WPs [34, 35]. In contrast, in the Type-II WSM phase, potential energy terms dominate over the kinetic energy terms in the energy relation. This modifies the dispersion relation in such a way that only one sign of group velocity exists along certain directions near the WPs. Moreover, there is a finite density of states (DOS) at the WPs. When the kinetic and potential energy contribute equally, the transition between the Type-I and Type-II phases arises where one of the bands become completely flat with vanishing group velocity. This is the so-called Type-III [36] or Critical-type WSM phase [37]. However, realizing WSM states in condensed matter systems and tailoring their properties are experimentally challenging. For instance, carrier doping in WSMs may compromise the stability of the Weyl phases due to broken translational symmetry [38]. Additionally, it is usually difficult to achieve transitions between different topological WSM phases (Types-I, II, III) in a given material, and thus to form heterojunctions of WSMs of different phases [23, 39, 40]. There is hence a need for alternative platforms to realize gapless topological states.

Researchers have studied various artificial systems such as photonic crystals [41], metamaterials [42, 43] and quantum resonators [44]. However, these methods all come with their own experimental complexities and limitations. Recently, topological states were realized in periodic electrical circuits consisting of inductors and capacitors (LC networks) known as topoelectrical (TE) circuits [36, 45]. Under the TE framework, many striking phenomena have been demonstrated such as the quantum spin Hall insulator state [46], magnet-less Floquet topological insulator state [47], topological photonic state [48], and edge modes in the SSH model [49]. A key advantage of electrical networks is the flexibility of experimental realization, as the circuits can be implemented even on simple printed circuit boards. Additionally, circuit parameters in electrical networks can be more readily adjusted and tuned, compared to lattice model properties of real materials.

In this paper, we demonstrate the realization of Type-I and Type-II WSM phases in a three-dimensional TE circuit network. We relate the admittance matrix of a TE circuit to the tight-binding Hamiltonian matrix of a condensed matter system. The resultant admittance band structure in a LC model resembles the energy dispersion [50]. The transition between Type-I and Type-II WSM phases can be readily effected by tuning the coupling between certain nodes. Moreover, we realize a distinct topological phase at the transition between the Type-I and Type-II phases, by isolating these nodes. This so-called Critical-type phase is characterized by the emergence of a flat admittance spectrum for one of the bands. We additionally derive the impedance spectrum and relate it to the circuit Green’s functions [51]. The impedance spectra of the TE circuits corresponding to different WSM phases show significant differences. The Type-I WSM impedance spectrum is marked solely by distinctive peaks at the WPs. On the contrary, the Type-II WSM impedance dispersion exhibits not only impedance peaks at the WPs but also multiple secondary peaks along the tilt direction in wave-vector space. Finally, the intermediate Critical-type phase is characterized by a high impedance region between two WPs which signifies the flat dispersion of the electron or hole bands there. In short, TE circuits provide an accessible and tunable platform to design and model distinct topological phases and transitions in WSMs, while their impedance spectra provide a signature of the different phases.

2. Topoelectrical circuit model

Here, we consider a periodic (along all directions) LC circuit, shown in figure 1, which comprises of two sublattices A and B. In an AC circuit, the admittance $Y$ of any two-terminal device component (resistor, capacitor, inductor etc) can be expressed as $Y = \frac{I}{V}$ where $I_a$ and $V_a$ are the current and the potential difference across the circuit component, respectively. In general, $I_a$ and $V_a$ are complex numbers [52]. By applying Kirchhoff’s current law (KCL) at nodes A and B in the unit cell at $\vec{r} = (x, y, z)$ of the circuit model shown in figure 1, we have:

\[ (i\omega)^{-1} Y_{x,y,z}^{A} = V_{x,y,z}^{A} (C_{1} + C_{2} - (\omega^2 L)^{-1} + 4C_1 + 2C_2 + 2C_{xy} + 2C_{xz}) - C_{y} (V_{x,y+1,z}^{A} - V_{x,y-1,z}^{A}) - C_{z} (V_{x+1,y,z}^{A} - V_{x-1,y,z}^{A}) - C_{x} (V_{x+1,y,z}^{A} - V_{x-1,y,z}^{A}), \]

\[ (i\omega)^{-1} Y_{x,y,z}^{B} = V_{x,y,z}^{B} (G_{1} + C_{2} - (\omega^2 L)^{-1} + 4C_1 + 2C_2 + 2C_{xy} + 2C_{xz}) - C_{y} (V_{x,y+1,z}^{B} - V_{x,y-1,z}^{B}) - C_{z} (V_{x+1,y,z}^{B} - V_{x-1,y,z}^{B}) - C_{x} (V_{x+1,y,z}^{B} - V_{x-1,y,z}^{B}), \]

where $\omega$ is the frequency of the alternating current, $C_2$ and $C_2$ are the intra- and inter-cell coupling capacitances along the x direction and $C_y$ is the coupling along the y direction between lattice points on the same A/B sublattice. The lattice circuit model in the x–y plane is connected symmetrically with capacitors $C_{xz}$ and $C_{xy}$, respectively, between A–A and B–B sites of adjacent layers along the z axis. $C_{xy}$ denotes the next nearest lattice interaction between A–A or B–B sub-lattice nodes along both the x and y directions. The A and B lattice points

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are grounded by capacitors $2C_R$ and $2C_{A_z}$ respectively, and an additional inductance $L$ that serves to adjust the offset ‘potential’. Note that the capacitors $C_{B_z}$ can be replaced by inductors $L_B = -(\omega^2 C_{B_z})^{-1}$ to switch the WSM phase of the circuit (see later). The schematic diagram of the various capacitive and inductive couplings in the LC circuit is depicted in figure 1. Using the Fourier transformation $V(\vec{k}) = \sum e^{i\vec{k} \cdot \vec{r}} V(r)$, equations (1) and (2) can be expressed in momentum space, in analogy to the tight binding (TB) approach in a crystal lattice model [53]. The relation between voltage and current distribution in $k$ space can then be expressed in terms of the Laplacian matrix equation [36, 49]

$$i\vec{k} = L_{\text{Weyl}}(\vec{k}) \nu(\vec{k}),$$

where $\nu(\vec{k}) = [V^A(\vec{k}), V^B(\vec{k})]^T$ and $\nu(\vec{k}) = [I^A(\vec{k}), I^B(\vec{k})]^T$ are the vectors representing the potential and current distribution at the $A$ and $B$ sublattice nodes in $k$ space, respectively. $L_{\text{Weyl}}(\vec{k})$ denotes the admittance matrix (Laplacian matrix) which is analogous to the Hamiltonian in condensed matter physics [36], and can be expressed as

$$(i\omega)^{-1} L_{\text{Weyl}}(\vec{k}) = ((C_i + C_2 + 2C_y - (\omega^2 L)^{-1} + 2C_n(1 - \cos k_x) + 2C_n(1 - \cos k_y) + (C_{A_z} + C_{B_z}) \times (2 - \cos k_z)) \sigma_0 - (C_i + C_2 \cos k_x + 2C_y \cos k_y) \sigma_x - C_2 \sin k_x \sigma_y - (C_{A_z} - C_{B_z}) \cos k_z \sigma_z,$$

where $(\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices corresponding to the $A/B$ sublattice pseudospin degree of freedom and $\sigma_0$ is the $2 \times 2$ identity matrix. The admittance spectrum, which corresponds to the energy dispersion in the quantum TB model [36], is obtained from the eigenvalues of $L_{\text{Weyl}}(\vec{k}) / (i\omega)$ in equation (4):

$$E = d_0 \pm \sqrt{(C_i + C_2 \cos k_x + 2C_y \cos k_y)^2 + (C_2 \sin k_x)^2 + (C_{A_z} - C_{B_z}) \cos k_z)^2},$$

where $d_0 = C_i + C_2 + 2C_y - (\omega^2 L)^{-1} + 2C_n(1 - \cos k_x) + 2C_n(1 - \cos k_y) + (C_{A_z} + C_{B_z})(2 - \cos k_z).$

Here, the 'energy' $E$ refers to the admittance (divided by $i\omega$) of the circuit. The energy spectrum is gapless at the charge neutrality point or WP in which the two eigenvalues of equation (3) are equal. This equality is satisfied by the following conditions

$$\sin k_x = \cos k_x = 0,$$

$$C_i + C_2 \cos k_x + 2C_y \cos k_y = 0.$$

Without loss of generality, we consider all capacitances and inductances as having non-negative values, $2C_y > (C_i - C_2)$ and $C_i + C_2 > 2C_y$. The solution set of equation (6) is then

$$(k_x, k_y) = \{(0, \pm\pi/2), (\pi, \pm\pi/2)\}$$

which reduces equation (7) to

$$C_i + \eta C_2 + 2C_y \cos k_y = 0,$$

where $\eta = 1$ for $(k_x, k_y) = (0, \pm\pi/2)$ and $\eta = -1$ for $(k_x, k_y) = (\pi, \pm\pi/2)$. However, for our parameter range, no real solution of $k_x$ exists in the first Brillouin zone for $\eta = 1$. For $\eta = -1$, we obtain two pairs of WPs in $k$ space which are located at $w_\pm = (\pi, \pm\arccos(C_2 - C_i)/2C_y), \pm\pi/2)$. For $|(C_2 - C_i)| > 2C_y$, no band touching points exist. The type of WPs hosted by the Laplacian in equation (4), either Type-I or Type-II, depends on the energy spectrum.

Figure 1. Illustration of a TE circuit lattice to realize various WSM phases. Periodic boundary conditions are considered along all directions. (a) The circuit representation in the $x$-$y$ plane, where each $A$-type (red dot) node is connected to a $B$-type node (green dot) by $C_1$, and the repeating units are connected by $C_2$ in the $x$ direction. All alternate lattice sites along the $y$ direction are connected by $C_r$. Nodes of the same type are coupled to each other by $C_0$ in the $x$-$y$ plane. (b) The circuit representation in the $x$-$z$ plane. All $A$-$A$ couplings are capacitive in nature with magnitude $C_{A_z}$ while the $B$-$B$ coupling can be tuned to be in-phase (out-of-phase) with respect to the $A$-$A$ coupling by selecting capacitive (inductive) hopping strengths $C_{BR} (L_B = |(\omega^2 C_{B_z})|^{-1})$, respectively, along the $z$ direction. (c) Additionally, all $A(B)$ nodes are connected by ground common inductor $L$ in parallel with capacitor $2C_{B_z}(2C_{A_z})$ respectively. The common inductor $L$ determines not only the resonant frequency of the circuit but also the effective offset 'potential' of the TE circuit.

\[\]
where \( \omega \) on the values of \( \sigma \) for Type-II WSM are hyperbolic and connect the WPs with different chiralities, as shown in figure 2. This denotes the flat band feature at the transition point of Type-I and Type-II WSM. (g) Admittance spectrum corresponding to Type-II WSM phase with capacitive coupling between \( B-B \) and \( A-A \) also sites along the \( k_z \) direction, \( C_{\text{un}} = 0.5 \text{ mF} \), \( C_{\text{un}} = 0.2 \text{ mF} \) and \( k_z = \pi \). Both upper and lower bands show group velocity of the same sign around the WPs in contrast to the Type-I dispersion of (a), opposite velocities. (h) Iso-admittance contour plot of the admittance bands of a Type-II WSM in (g), projected on the \( k_z-k_y \) plane. Due to the hyperbolic dispersion of Type-II WSM, the iso-admittance contour lines do not enclose the WPs. However, the number and position of WPs have not changed with respect to Type-I and Critical-type WSM. (c), (f), (i) Constant admittance cross section at \( E = 0 \), for admittance spectra in (a), (d) and (g) for the Type-I, Critical-type and Type-II WSM, respectively. Type-I WSM hosts no states at the nodal energy. Critical-type WSM phase exhibits straight line bulk-like states between WPs. In Type-II WSM, electron and hole pockets emerge at Fermi energy which touches the WPs.

on the values of \( C_{\text{un}} \), \( C_{\text{un}} \), and \( C_{\text{un}} \), which determine the \( d_4 \) term in equation (5). These parameters do not shift the WPs in the \( k \)-space but only modify the tilt of the Weyl dispersion cone. Figure 2 shows the evolution of the resonant admittance band structures and iso-admittance contour plots for different WSM phases. The resonant frequency is given by

\[
\omega = (L(C_1 + C_2 + 2C_y + 4C_{\text{un}} - 2C_{\text{un}} C_y) + (C_{\text{un}} + C_{\text{un}})(2 - \cos k_z)^{-1/2},
\]

where \( \omega \) corresponds to the frequency at which the \( \sigma \)-independent terms in the Hamiltonian vanish. As shown in figure 2(a), when the hopping parameters between the \( A \) and \( B \) nodes along the \( z \) direction have opposite signs, a Type-I WSM dispersion is obtained. In contrast, when the coupling between two similar sites along the \( z \) direction has the same signs, the WSM phase changes to Type-II (figure 2(g)). The Critical-type phase between two WSM phases is depicted in figure 2(d), corresponding to zero coupling between two adjacent \( B-B \) nodes along the \( z \) axis. As shown in figure 2(b), for the Type-I WSM phase, the constant admittance (iso-admittance) contour is a closed loop in the \( k_z-k_y \) plane that encompasses the WPs. In contrast, the iso-admittance contours for Type-II WSM are hyperbolic and connect the WPs with different chiralities, as shown in figure 2(h) for both energy bands. The contours for the transitional Critical-type phase between Type-I and Type-II WSMs are distinct from the other two types, as depicted in figure 2(e). For this phase, the WPs are connected by the zero-admittance contour lines for the electron (upper) band while for the hole (lower) admittance band, vanishing.
admittance is obtained for the rest of Brillouin zone along the \( k_z \) direction for a specific value of \( k_y \) that satisfies equation \((9)\). Another method to classify the different WSM phases in the TE circuit model is by noting the existence of finite DOS in the vicinity of WPs, as shown in figures 2(c), (f), and (i). In figure 2(f), there is zero DOS near WPs in the Type-I WSM where the \( B−B \) coupling along the \( z \) direction is comprised of an inductor \( L_{B} \). In figure 2(i), for the transitional Critical-type WSM where there is zero coupling between nearest \( B \) sites, the WPs are linked by a single arc of bulk states. Finally, in the Type-II WSM where the \( B−B \) coupling along the \( z \) direction is comprised of a capacitor \( C_{BB} \), the resulting tilt of the admittance band gives rise to non-zero local DOS for electron and hole bands in the vicinity of WPs. These are clearly visible in figure 2(i), where there is finite DOS at the nodal admittance giving rise to the electron and hole pockets in the \( k_y−k_z \) plane.

3. Low admittance general Laplacian and classification

To further examine the properties of the Laplacian equation \((4)\), we perform a linear expansion of the admittance near the band touching points in \( k \) space. The general form of the linear Laplacian can be expressed as

\[
(i\omega)^{-1}\text{L}_{\text{Weyl}}(\vec{q}) = (v_x q_x \sigma_x + \lambda v_y q_y \sigma_y + v_z q_z \sigma_z) + (t_x q_x + \lambda t_y q_y + t_z q_z) \sigma_0 + v_0 \sigma_0,
\]

where \( \vec{q} \equiv (q_x, q_y, q_z) = \vec{k} - \vec{\nu}_{q} \) is the small displacement from the WPs, \( v_x = C_{2x} \), \( v_y = 2C_{y} \), \( v_z = 2C_{z} \), \( t_x = 0 \), \( t_y = 2C_{y} \), \( t_z = C_{z} \), and \( \lambda = \pm 1 \) for the \( k \) and \( k' \) valleys respectively. \( \nu_0 \) is a constant ‘potential’ offset which can be tuned to zero or any other arbitrary value by varying the inductance \( L \). The tilt of the Weyl dispersion cones along the \( k_z \) direction increases with \(|\nu_0/|\nu|\). The Weyl phases can be classified based on this ratio: Type-I with \(|\nu_0/|\nu| < 1 \) \( (i = x, y, z) \) (figure 3(a)), Type-II with \(|\nu_0/|\nu| > 1 \) (figure 3(c)) and Type III with \(|\nu_0/|\nu| = 1 \) (figure 3(b)). In the vicinity of the WP, the slope of the dispersion cone, which represents the group velocity, has opposite (same) signs for opposite signs of \( q_z \) in a Type-I (Type-II) WSM, while at a transition value \( |\nu_0| = |\nu| \), one of the electron or hole bands has zero group velocity and hence a flat dispersion relation, which corresponds to the Critical-type phase. From equation \((10)\), the linearized admittance spectra of the two bands are given by

\[
E = (t_x q_x + \lambda t_y q_y + t_z q_z) + v_0 + \chi A,
\]

where \( \chi = \pm 1 \) for electron and hole bands respectively and \( A = \sqrt{v_x^2 q_x^2 + v_y^2 q_y^2 + v_z^2 q_z^2} \). The group velocities of the upper and lower bands are expressed by \( \nabla_{\vec{k}} \vec{V}_\pm = \nabla_k E \equiv \left( \frac{v_x^2 q_x}{A}, \lambda t_y, t_z + \frac{\chi v_z q_z}{A} \right) \). In the vicinity of WPs \( q_x \rightarrow 0, q_y \rightarrow 0, q_z \rightarrow 0 \), \( \nabla k \vec{V} \) can be further approximated to

\[
(V_x^\pm, V_y^\pm, V_z^\pm) = (\chi C_{2x}, 2(C_{y} + \chi C_{y}) \sin \theta, (C_{z} + C_{z}) + \chi (C_{az} - C_{z})).
\]
a Type-II WSM. Conversely, if the kinetic term dominates along all directions, then the two bands would exhibit group velocities of opposite signs (Type-I WSM). Let us consider the case where the tilt exists along the y direction. The corresponding velocities \( V_{y}^{+} \) and \( V_{y}^{-} \) will have the same signs if \( |C_{a}| > |C_{b}| \), giving rise to Type-II WPs. Conversely, if \( |C_{a}| < |C_{b}| \) we would obtain Type-I WPs, where the two admittance bands have opposite signs of admittance gradient along the y direction. At the critical coupling value \( |C_{a}| = |C_{b}| \), the amplitudes of the potential and kinetic hopping terms along the y axis are equal. This corresponds to the Critical-type phase, with a flat hole band close to the WPs (with vanishing velocity \( V_{y}^{±} \)), while the electron band retains a finite group velocity \( V_{y}^{±} \). Interestingly, the second-nearest hopping along the x axis via the coupling capacitance \( C_{n} \) does not contribute to the tilt in that direction (n.b. \( t_{b} = 0 \) in equation (10)), but rather contributes to the offset ‘potential’ term, which may also be adjusted by tuning the common inductor \( L \).

Thus far in our analytical derivation, we have considered an infinite LC circuit in all three directions. For the numerical verification of our analytical predictions, we consider a nanoribbon geometry in a LC circuit model with a finite width of \( L_{y} = 30 \) unit cells in the y-direction, under open boundary conditions. Here, multiple sub-bands are present due to the quantum confinement along the y direction. We consider the evolution of the admittance dispersion through the different WSM phases as the \( B-B \) coupling parameter along the x-direction, \( C_{Bx} \), is varied at a fixed \( k_{x} = \pi \) and \( C_{Ay} \), as shown in figures 3(a)–(c). With an inductive coupling between two neighbouring B sites along the z direction, we have \( C_{Bz} = -(\omega^{2}L_{z})^{-1} \). Any non-zero value of \( L_{z} \) will result in a dispersion tilt such that \( |t_{z}/v_{c}| < 1 \). The resulting Type-I WSM admittance dispersion is shown in figure 3(a). The dispersion relation consists of two symmetric bands touching each other at \( k_{z} = \pm \pi/2 \). The DOS at the WPs is zero and both signs of the admittance gradient are present in the vicinity of each WP. When the inductor \( L_{y} \) is replaced with a capacitor \( C_{Bx} \) the \( B-B \) coupling \( C_{Bx} \) is in-phase with the corresponding A–A coupling, \( C_{Ay} \). In this case, both would have the same sign, so that \( t_{z} > v_{c} \). In other words, for any positive value of \( C_{Bx} \), \((C_{Bx} + C_{Bz}) > (C_{Az} - C_{Bz})\) holds and so the Type-II WSM would result. This is reflected by the tipping over of the admittance bands into two asymmetric branches with respect to \( k_{z} \) (shown in figure 3(c)). The positions of the WPs are not shifted but the dispersion acquires a finite DOS at the nodal admittance—the line \( E = 0 \) cuts across multiple states in the \( E-k_{z} \) plot, spanning across the entire range of \(-\pi < k_{z} < \pi \), in contrast to the Type-I WSM in figure 2(a) where the \( E = 0 \) line cuts across only the WP states. Finally, for the case of zero \( B-B \) coupling \((C_{Bx} = 0)\), one of the admittance bands exhibits a flat zero admittance state between the two WPs \((-\pi/2 < k_{z} < \pi/2)\) while the other shows zero admittance state elsewhere in the Brillouin zone, i.e. \(|k_{z}| > \pi/2\), as shown in figure 3(b). This represents the intermediate Critical-type WSM system.

### 4. Impedance spectrum analysis

In the previous section, we presented the admittance dispersions for the different WSM phases of a TE model. In this section, we consider the impedance of a TE circuit and show that it offers an experimentally convenient way to distinguish the different topological phase of the TE circuit. The impedance readout of a TE circuit can be obtained simply by connecting a fixed current source to two arbitrary nodes in the circuit and measuring the potential difference between the nodes. This constitutes a more convenient measurement than a direct determination of the admittance dispersion relation of the circuit. Moreover, the comparatively large impedance readout (in the range of few \( \Omega \) to \( k\Omega \)) compared to the admittance readout (in the range of a few \( m\Omega \)) provides a better measurement accuracy. We begin by analysing the mathematical significance of the impedance between any two lattice sites in a TE circuit. Consider a 2D TE circuit shown in figure 4 consisting of a finite number of nodes along the y direction, and having an infinite number of nodes along both the positive and negative x directions. Each node is capacitively coupled to its left and right neighbours by \( C_{a} \), its upper and lower neighbours by \( C_{b} \), and to the ground by a common grounding capacitance \( C \). We also connect to every node an additional wire through which current may flow between the node and an external current supply. This wire is denoted as the dotted line with an arrow flowing into and out of the node, as shown in figure 4(b), and shall be referred to as the ‘current wire’ subsequently. We label the nodes by their \((x,y)\) coordinates so that \( V_{x,y} \) is the voltage at the node located at \((x,y)\). The KCL at each node (except for those at the top and bottom rows) reads as

\[
-I_{x,y}/k\omega + CV_{x,y} = C_{a}(V_{x,y+1} + V_{x,z-1} - 2V_{x,y}) + C_{b}(V_{x-1,y} + V_{x+1,y} - 2V_{x,y}).
\]

Let us first consider the case where \( I_{x,y} = 0 \) everywhere. Physically, this corresponds to leaving the current wire at each node unconnected to any current source or sink. Equation (13) with \( I_{x,y} = 0 \) then has a similar form to the Schrödinger equation for a quantum mechanical TB Hamiltonian [36], where the common grounding capacitance \( C \) is the analogue of the eigenenergy. More generally, the KCL for a TE circuit which has voltage nodes connected to common grounding capacitances \( C \) and current flowing through the current wires, can be written as
\[ -\frac{i}{\omega} + CV = Hv \]  
(14)

\[ \Rightarrow i = i\omega(C - H)v \equiv Lv, \]  
(15)

where \( i \) is the vector of currents flowing through the current wires at every node and \( v \) is the vector of voltages at every node. \( H \) is the matrix relating the voltages at the different nodes to one another, obtained by applying KCL at every node but excluding the contribution of the common grounding capacitance \( C \), which is moved to the left hand side of the equation (14). If \( I_{xyz} \) is set to 0 everywhere, equation (14) becomes an eigenvalue equation in \( C \) and \( v \), as mentioned earlier. In particular, if \( H \) is a finite-sized matrix, then the system becomes equivalent to an infinite potential well system with a discrete eigenspectrum of \( C \) consistent with having \( I_{xyz} = 0 \) everywhere. In equation (15), we defined \( L = i\omega(C - H) \) where \( L \) is the TE analogue of the inverse of the Green’s function \( G_{TB} \) for a TB system with Hamiltonian \( H_{TB} \), i.e. \( G_{TB} = (E - H_{TB}) \). Let us consider the situation where the TE circuit is translationally invariant along the \( x \) direction, resulting in the form of \( V(x, y) = V(y) \exp(ik_y x) \).

Equation (15) then reduces to

\[ \frac{I_{xyz}}{i\omega} \exp(-ik_y x) = CV_x - C_y (V_{x+1} - V_{x-1} - 2V_y) + 2C_x V_y (\cos(k_y) - 1) \]  
(16)

and the system is mathematically equivalent to a one-dimensional system with a \( k_y \) dependent ‘on-site potential’ \( 2C_x (\cos(k_y) - 1) \). Equation (16) can be schematically written as

\[ i_{(1)\exp(-ik_y x) = L_{(1)v(1)}}, \]  
(17)

where the \((1)\) subscript denotes a quasi one-dimensional system, where the effects of the infinite-length \( x \) direction have been incorporated as a \( k_y \) dependent on-site potential. In particular, for the nodes at \( x = 0 \), we have

\[ v(1) = L_{(1)k(1)} \]  
(18)

We then consider the situation depicted in figure 4(b) where two nodes at \( y = a \) and \( y = b \) are connected to a current source supplying a current as \( I_S \) (the subscript ‘S’ denotes source), while the other nodes are left unconnected. This corresponds to setting \( i_{(1)a} = -i_{(1)b} = I_S \), \( i_{(1)c} = 0 \), \( c = \{a, b\} \). The resulting voltages at nodes \( a \) and \( b \), \( V_{a/b} \), can be read off from equation (18) as

\[ V_a = V_{(1)a} = L_{(1)a}I_S \]  
(19)

\[ V_b = V_{(1)b} = -L_{(1)b}I_S \]  
(20)

\[ Z_{ab} = (L_{(1)a} + L_{(1)V_{a}} - L_{(1)b} - L_{(1)V_{b}}) \]  
(21)

Letting \( N_y \) be the number of lattice sites along the \( y \) direction, \( L_{(1)} \) is then a \( (N_y \times N_y) \) Hermitian matrix; the spectral decomposition of which is \( A = \sum |a_j\rangle A_j |a_j\rangle \) where \( |a_j\rangle \) is the \( j \)th eigenvector and \( A_j \) its corresponding eigenvalue, i.e. \( A|a_j\rangle = |a_j\rangle A_j \). Hence, any function \( f(A) \) of the matrix can be expressed as

\[ f(A) = \sum |a_j\rangle f(A_j) |a_j\rangle. \]  
(22)

Let us write the spectral decomposition of \( H_{(1)} \) as \( H_{(1)} = \sum |h_j\rangle c_j |h_j\rangle \) with

**Figure 4.** (a) Schematic of a TE circuit with a finite number of nodes along the \( y \) direction and infinite length along the \( x \) direction. The nodes are coupled to their left/right neighbours by \( C_x \), upper/lower neighbours by \( C_y \), and the ground by \( C \). Each node is connected to an additional wire through which current may flow in/out of the node. (b) The same circuit in (a) where we connect two arbitrary nodes to a current source.
By definition \( L_{(\ell)} \equiv \frac{1}{k_{\ell}} (C - H_{(\ell)}) \), so \( L^{-1} \) has the spectral decomposition

\[
L_{(\ell)}^{-1} = \frac{1}{k_{\ell}} \sum_j \frac{|h_{j\ell}|}{C - c_j}
\]

and substituting the above into equation (21), we have

\[
Z_{ab} = \frac{1}{k_{\omega}} \sum_j \frac{h_{a\ell}^* h_{j\ell} + h_{b\ell}^* h_{j\ell} - h_{a\ell}^* h_{j\ell} - h_{b\ell}^* h_{j\ell}}{C - c_j}
\]

\[
= \frac{1}{k_{\omega}} \sum_j \frac{(h_{a\ell}^* - h_{b\ell}^*) (h_{j\ell} - h_{j\ell})}{C - c_j}
\]

\[
= \frac{1}{k_{\omega}} \sum_j \frac{|h_{a\ell} - h_{b\ell}|^2}{C - c_j},
\]

where \( h_{j\ell} \) is the \( \ell \)th element of \( h_j \). Equations (21) and (25) are the key formulae to evaluate the impedance between any two arbitrary points. When \( C \) is one of the eigenvalues of \( H_{(\ell)} \), and provided that \( |h_{j\ell} - h_{j\ell}| \) is not simultaneously zero, the current flowing through the current wires attached to the nodes will be nearly zero, leading to a sharp spike in the impedance \( Z_{ab} \). This phenomenon is similar to that in quantum-dot (QD) systems [54, 55], in which a resonance occurs in the transmission across the system when the Fermi energy in the leads coincides with one of the discrete energy levels in the QD. Here \( C \) plays the role of the lead Fermi energy and \( c_j \) the discrete energy levels in the QD.

The impedance spectra across the whole circuit, i.e. between the first and last nodes (\( N = 2n \)) are plotted as a function of \( k_z \) at the resonant frequency for various WSM phases (see figure 5). Here, we consider \( \eta = 30 \) unit cells so that there are \( N = 60 \) alternating \( A \) and \( B \) lattice points. As can be seen in figure 5, the impedance distribution is symmetric about \( k_z = 0 \) for all types of WSM phases but assume different profiles for different phases. Figure 5(a) shows the impedance readout of a Type-I WSM TE circuit. The notable characteristic of the impedance spectrum is the presence of two clear peaks at the \( k_z \) points corresponding to the WPs in the \( k_z \) direction. These may be explained by considering equation (25) and figure 3(a). Figure 3(a) shows the hole and particle-like bands almost touching only at the WPs, where the admittance matrix is nearly zero. (The hole and particle bands do not actually touch due to the small band gap as a result of the finite width.) Equation (25) shows that the impedance between two points is dominated by the eigenvalues of the Laplacian matrix closest to \( C = 0 \) and inversely proportional to these eigenvalues. Thus, large impedance peaks are observed in the vicinity of WPs of Type-I circuit model. At \( k_z \) values away from the WPs, the eigenvalues of electron and hole bands have relatively large (non-zero) magnitudes, resulting in the decay of impedance readout (figure 5(a)). The impedance spectrum along \( k_z \) is plotted in figure 5(b) for Critical-type WSMs. As we have seen from figure 3(b), the electron and hole bands in the spectrum exhibit a flat band dispersion. The corresponding WPs are marked by two impedance minima, but the corresponding impedance value is large and of the same order of magnitude as that of the impedance peaks in Type-I WPs. At other \( k \) values between the WPs, the impedance exhibits much larger values by several orders of magnitude. The existence of large impedance over the whole of the Brillouin zone apart from the WPs is the direct consequence of the zero-admittance flat dispersion in the admittance spectrum. Finally, the resonant impedance characteristics of a Type-II WSM is illustrated in figure 5(c), which shows an
oscillatory response along the momentum direction that is parallel to the tilt direction (i.e. the \( k_z \) direction). The primary impedance peaks occur at \( k_z = \pm \pi/2 \), just as in the Type-I WSM system, but in addition there are multiple secondary peaks at other values of \( k_z \). The emergence of these secondary peaks is due to the presence of eigenvalues with small magnitude of admittance, thus indicating the existence of finite DOS in the proximity of Type-II WPs. This is in line with the presence of hole and electron pockets as can be seen in the admittance dispersion for Type-II WSM shown in figure 3(c).

5. Conclusion

In conclusion, we have realized and characterized various WSM phases in the admittance dispersion of three-dimensional LC topoelectric circuit. The characteristics of such LC circuits are described by the circuit admittance matrices, which are analogous to the Hamiltonians of the quantum tight-binding model. The different phases of the circuit can be switched between one another by adjusting the magnitude and sign of the admittance matrices, which are analogous to the Hamiltonians of the quantum tight-binding model. The LC circuit model allows ready implementation of WSM phases using basic circuit elements. The accessibility and ease of fabrication of the LC circuits make them an ideal platform for the design and characterization of topological WSM states and their transport properties.

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ORCID iDs

S M Rafi-Ul-Islam  https://orcid.org/0000-0002-2275-8889
Zhuo Bin Siu  https://orcid.org/0000-0002-7056-937X
Mansoor B A Jalil  https://orcid.org/0000-0002-9513-8680

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