A study on causality in $f(R,\phi,X)$ theory

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Abstract

The $k$-essence modified $f(R)$ gravity model, i.e., $f(R,\phi,X)$ theory is studied. The question of violation of causality, in the framework of Gödel-type universes, is investigated in this gravitational model. Causal and non-causal solutions are allowed. A critical radius for non-causal solution is calculated. It is shown that the violation of causality depends on the content of matter.
I. INTRODUCTION

The discovery of accelerated cosmic expansion \cite{1,2,3} led to the search of theoretical models consistent with observational data \cite{4}. In this search, two possibilities have been developed to explain this phenomenon: (i) alternative models that modify the General Relativity (GR) and (ii) add an exotic component, called dark energy, in GR. For a review, see \cite{5}. The reasons and motivations that lead to alternatives gravity theories have changed over the years. Some models are based on theoretical reasons while others are more phenomenological, among other considerations. There are a number of theoretical models that were developed to study the accelerated expansion of the universe. Here our main interesting is in the theories of gravity: $f(R)$, $k$-essence and $k$-essence modified $f(R)$ gravity.

The $f(R)$ gravity is a modified gravitational theory in which the standard Einstein-Hilbert action is replaced by an arbitrary function of the Ricci scalar \cite{6,7}. These theories have received considerable attention in the last years motivated by the fact that they can explain the observed accelerating late expansion of the universe \cite{8,9}. There are several studies on $f(R)$ gravity, such as, solar system tests \cite{10,11}, Newtonian limit \cite{12,13}, gravitational stability \cite{14,15}, singularities \cite{16}, energy conditions have been used to place constraints on the theory \cite{17,18}, cosmological perturbations during the inflationary epoch \cite{19}, gravitoelectromagnetism formalism in the context of metric $f(R)$ theory \cite{20}, among others.

Another interesting modified gravity model is the $k$-essence gravity. It emerged as an alternative dynamical dark energy model that has been developed in the context of inflationary theory \cite{21,22}. The $k$-essence gravity also has been used with the objective to explain the accelerated expansion of the universe \cite{23,24,25,26,27}. The main idea of this theory is to minimally couple a scalar field with the gravity that depends only on the kinetic terms and does not depend on the scalar field itself. There are numerous applications with this theory. For example, the FRW $k$-essence cosmologies has been analyzed \cite{28}, exact solutions in $k$-essence for isotropic cosmology have been studied \cite{29}, dark matter to dark energy transition in $k$-essence cosmologies has been examined \cite{30}, power-law expansion in $k$-essence cosmology has been investigated \cite{31}, quantum cosmology with $k$-essence has been discussed \cite{32}, Gödel-type universe in $k$-essence gravity has been studied \cite{33}, among others.

In this paper, the approach is to study a model that is a combination of both modifications, the $f(R)$ gravity and the $k$-essence theory, called $f(R, \phi, X)$ theory, where $\phi$ is a scalar field and $X = \frac{1}{2} \partial^\mu \phi \partial^\nu \phi$. This model has been investigated in recent works such as \cite{27,34,35}. So our main aim in this paper is to investigate the question of causality in this $k$-essence modified $f(R)$ gravitational...
model. To achieve this goal, the Gödel universe and its generalizations will be considered.

In 1949, Kurt Gödel proposed an exact solution of Einstein equations for a homogeneous rotating universe. This solution leads to the possibility of Closed Timelike Curves (CTCs), which allow violation of causality and makes time-travel theoretically possible in this space-time \[36\]. CTCs are also admitted in other cosmological models, such as Kerr black hole, Van-Stockum model, cosmic string, among others \[37, 38\]. A generalization of the Gödel solution, known as Gödel-type solution, has been developed \[39\]. The Gödel-type solutions add more details to the problem of causality. In addition, in the Gödel-type universes there is a possibility of a causal rather than non-causal solution. These regions are determined from free parameters of the metric and limited by a critical radius \(r_c\). From this critical radius there is violation of causality.

The problem on violation of causality has been investigated in several modified gravity theories, such as, in \(f(R)\) theory \[40\], in \(k\)-essence theory \[33\], in Chern-Simons gravity \[41, 42\], in \(f(T)\) gravity \[43\], in \(f(R, T)\) gravity \[44\], in bumblebee gravity \[45\], in Horava-Lifshitz gravity \[46\], Brans-Dicke theory \[47\], in \(f(R, Q)\) gravity \[48\], among others. Here the consistency of Gödel-type solutions and their implications about causality are analyzed in the \(k\)-essence modified \(f(R)\) gravity.

This paper is organized as follows. In section 2, a brief introduction to \(f(R)\) gravity and \(k\)-essence theory is presented. The \(f(R, \phi, X)\) theory is introduced. In sections 3 and 4, the Gödel and Gödel-type universes are analyzed in the \(f(R, \phi, X)\) theory for different matter contents such as perfect fluid, perfect fluid plus electromagnetic field and only for electromagnetic field. In all cases, the question of causality is investigated. In section 5, some concluding remarks about the causal and non-causal solutions are discussed.

II. MODIFIED GRAVITY MODELS

In this section, the \(k\)-essence modified \(f(R)\) gravity is discussed. First, the \(f(R)\) and \(k\)-essence gravity models are briefly introduced.

A. \(f(R)\) theory

\(f(R)\) is a gravitational theory that generalizes GR. This generalization consist in replace the Ricci scalar \(R\) in the Einstein-Hilbert action by a general function of \(R\). The action that describes the \(f(R)\) theory is

\[
S = \int d^4x \sqrt{-g} \left( \frac{f(R)}{2\beta} + \mathcal{L}_m \right)
\]  

(1)
where $\beta \equiv 8\pi G$, $g$ is the determinant of the metric $g_{\mu\nu}$ and $\mathcal{L}_m$ is the lagrangian that describes the content of matter. Varying the action with respect to the metric, after some manipulations, the field equations are obtained as

$$R_{\mu\nu}f'_R - \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box\right) f'_R - \frac{1}{2} g_{\mu\nu} f = \beta T_{\mu\nu},$$

(2)

where $f = f(R)$, $f'_R \equiv df(R)/d(R)$, $\nabla_\mu$ is the covariant derivative, $\Box \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ and $T_{\mu\nu}$ represents the energy-momentum tensor associated to the matter defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} \mathcal{L}_m\right)}{\delta g^{\mu\nu}}.$$  

(3)

A constraint, often used to simplify the field equations, comes from the trace of eq. 2, which is given by

$$f'_R R - 2f + 3 \Box f'_R = \beta T,$$

(4)

where $T = g^{\mu\nu} T_{\mu\nu}$ is the trace of the energy-momentum tensor. Then the relation between $R$ and $T$ differs from the trace of the Einstein equation, i.e., $R = -\beta T$. This is an indication that the field equations of $f(R)$ theories will admit a larger variety of solutions than GR.

**B. $k$-essence theory**

The $k$-essence theory is characterized by a scalar field with a non-canonical kinetic energy. In this model a single scalar field $\phi$ interacts with gravity through non-standard kinetic terms. The action that describes the $k$-essence theory is given by

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{2\beta} - K(\phi, X) + \mathcal{L}_m\right),$$

(5)

where $K(\phi, X)$ is a function of the scalar field $\phi$ and its derivatives with $X = \partial^\mu \phi \partial_\mu \phi$. Here the scalar field depends only on time, i.e. $\phi = \phi(t)$.

Varying the action with respect to the metric $g_{\mu\nu}$, the field equations are given as

$$\frac{1}{\beta} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = Kg_{\mu\nu} - K_x \partial_\mu \phi \partial_\nu \phi + T_{\mu\nu},$$

(6)

where $K = K(\phi, X)$, $K_x = \frac{\partial K(\phi, X)}{\partial X}$ and $T_{\mu\nu}$ is the energy momentum tensor associated with the content of matter, as defined in eq. 3. In limit $K(\phi, X) \rightarrow 0$, GR is recovered.

The second field equation is obtained varying the action with respect to the scalar field. Then

$$- K(\phi, X) + 2\nabla_\mu (K_x(\phi, X) \partial^\mu \phi) = 0,$$

(7)

with $K(\phi, X) = \frac{\partial K(\phi, X)}{\partial \phi}$ and $\nabla_\mu$ being the covariant derivative.
C. \( f(R, \phi, X) \) theory

In this section, a hybrid model, composed of \( k \)-essence and \( f(R) \) gravity theories, is proposed. Here the main idea is to study the question of causality in this gravitational model, known as \( f(R, \phi, X) \) theory. A general class of \( k \)-essence \( f(R) \) gravity models is described by the action

\[
S = \int d^4 x \sqrt{-g} \left( \frac{f(R)}{2\beta} - 2\Lambda - K(\phi, X) + \mathcal{L}_m, \right),
\]

where \( \Lambda \) is the cosmological constant. It is important to note that, the question about ghost degrees of freedom in this hybrid theory has been investigated and the no-ghost constraints on a special class of \( k \)-essence \( f(R) \) gravity models have been found \[34\].

By varying the action with respect to the metric, the field equations for the hybrid model are

\[
\frac{1}{\beta} \left[ f'_R R_{\mu\nu} - \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) f'_R - \frac{1}{2} g_{\mu\nu} f - g_{\mu\nu} \Lambda \right] = -K g_{\mu\nu} + 2K_x \partial_\mu \phi \partial_\nu \phi + \mathcal{T}_{\mu\nu}. \tag{9}
\]

In order to simplify the field equations, a constraint is imposed by the trace of eq. (9) that is given as

\[
f'_R R - 2f + 3 \Box f'_R = \beta \left[ -4K + 2K_x \partial^\alpha \phi \partial_\alpha \phi + T \right]. \tag{10}
\]

Note that, it is different from the results of GR and \( f(R) \). Therefore, the \( f(R, \phi, X) \) gravity theory will admit a larger variety of solutions than GR, \( f(R) \) and \( k \)-essence theory separately. In the next sections, the compatibility of \( f(R, \phi, X) \) gravity theory with Gödel and Gödel-type universes as well as the question of causality are investigated.

III. GÖDEL UNIVERSE IN \( f(R, \phi, X) \) THEORY

The line element that represent the Gödel universe is given by

\[
d s^2 = a^2 \left( d t^2 - d x^2 + \frac{e^{2x}}{2} d y^2 - d z^2 + 2e^x d t \, d y \right), \tag{11}
\]

where \( a \) is a positive number. The non-zero Ricci tensor components are

\[
R_{00} = 1, \quad R_{02} = R_{20} = e^x, \quad R_{22} = e^{2x}, \tag{12}
\]

and the Ricci scalar is

\[
R = \frac{1}{a^2}. \tag{13}
\]
By considering that the energy-momentum tensor is given as

\[ T_{\mu\nu} = \rho u_{\mu} u_{\nu}, \]  

(14)

with \( \rho \) being the energy density of the fluid of matter, and \( u \) being a unit time-like vector whose explicit contravariant components look like \( u_{\mu} = (a, 0, ae^x, 0) \), the field equations, eq. (9), become

\[ \beta \left( 2K_\phi \dot{\phi}^2 - a^2K + a^2\rho \right) - f'_R + \frac{a^2}{2}f + a^2\Lambda = 0, \]  

(15)

\[ \beta K - \frac{1}{2}f - \Lambda = 0, \]  

(16)

\[ \beta (\rho - K) - \frac{1}{a^2}f'_R + \frac{1}{2}f + \Lambda = 0, \]  

(17)

\[ \beta \left( \rho - \frac{1}{2}K \right) - \frac{1}{a^2}f'_R + \frac{1}{4}f + \frac{1}{2}\Lambda = 0. \]  

(18)

Here it has been assumed that the field \( \phi \) is a function in the time coordinate only. This set of equations is satisfied if

\[ \rho = \frac{1}{a^2\beta}f'_R, \quad \Lambda = \beta K - \frac{1}{2}f. \]  

(19)

In addition, a restriction on the \( K(\phi, X) \) function is imposed, i.e., \( \frac{\partial K(\phi, X)}{\partial X} = 0 \). Then the \( K(\phi, X) \) function should only depend on \( \phi \). However, the scalar field equation of motion, eq. (7), it implies \( dK/d\phi = 0 \). Then \( K \) must be a constant independent of \( \phi \), which just shifts the cosmological constant by a given constant. This result implies that the Gödel metric solves the modified Einstein equations if and only if these conditions are satisfied. Therefore, in this case, the \( k \)-essence function turns out to be trivial, i.e., the \( k \)-essence \( f(R) \) theory is reduced to the usual \( f(R) \) gravity. For the case \( f = R, f' = 1 \) the Gödel universe in GR is recovered. This implies that Closed Time-like Curves (CTC) are possible and, as a consequence, violation of causality is permitted.

In the next section, a generalization of the Gödel solution will be considered and the question of causality will be investigated for different contents of matter.

IV. GÖDEL-TYPE UNIVERSE IN \( f(R, \phi, X) \) THEORY

In order to obtain more information on the issue of causality, a generalization of the Gödel solution, called Gödel-type metrics, has been proposed \[ [39]. \] In cylindrical coordinates \((r, \varphi, z)\) the line element is

\[ ds^2 = [dt + H(r)d\varphi]^2 - D^2(r)d\varphi^2 - dr^2 - dz^2, \]  

(20)
where

\[ H(r) = \frac{4\omega}{m^2} \sinh^2 \left( \frac{mr}{2} \right), \quad (21) \]

\[ D(r) = \frac{1}{m} \sinh(mr). \quad (22) \]

with \( \omega \) and \( m \) being parameters such that \( \omega^2 > 0 \) and \(-\infty \leq m^2 \leq +\infty\). When \( m^2 = 2\omega^2 \) the standard Gödel solution is obtained.

The line element (20) can be written as

\[ ds^2 = -dt^2 - 2H(r)dt\,d\varphi + dr^2 + G(r)d\varphi^2 + dz^2, \quad (23) \]

where \( G(r) = D^2(r) - H^2(r) \). If \( G(r) < 0 \) in the interval \( r_1 < r < r_2 \), a circle for \( r, t, z = \text{const} \) is obtained, i.e., \( ds^2 = G(r)d\varphi^2 \) denotes a closed time-like curve. For \( 0 < m^2 < 4\omega^2 \) non-causal Gödel circles occur for a region bigger than the critical radius \( r_c \). The critical radius is defined by

\[ \sinh^2 \left( \frac{mr_c}{2} \right) = \left( \frac{4\omega^2}{m^2} - 1 \right)^{-1}. \quad (24) \]

When \( m^2 = 4\omega^2 \) the critical radius is infinity, \( r_c = \infty \). This leads to a causal universe. There are no Gödel-type circles, and the breakdown of causality is avoided.

Using the functions defined in eqs. (21) and (22), the metric components are written, explicitly, as

\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & \frac{4w\sinh^2 \left( \frac{mr}{2} \right)}{m^2} & 0 \\ 0 & -1 & 0 & 0 \\ \frac{4w\sinh^2 \left( \frac{mr}{2} \right)}{m^2} & 0 & \frac{16w^2\sinh^4 \left( \frac{mr}{2} \right)}{m^4} - \frac{\sinh^2(mr)}{m^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (25) \]

Then the non-zero Einstein tensor components are

\[ G_{00} = 3\omega^2 - m^2, \quad (26) \]

\[ G_{02} = -\frac{4w\sinh^2 \left( \frac{mr}{2} \right)}{m^2} \left( m^2 - 3w^2 \right), \quad (27) \]

\[ G_{11} = \omega^2, \quad (28) \]

\[ G_{20} = -\frac{4w\sinh^2 \left( \frac{mr}{2} \right)}{m^2} \left( m^2 - 3w^2 \right), \quad (29) \]

\[ G_{22} = \frac{4w^2\sinh^2 \left( \frac{mr}{2} \right)}{m^4} \left( -3m^2\sinh^2 \left( \frac{mr}{2} \right) + m^2 + 12w^2\sinh^2 \left( \frac{mr}{2} \right) \right), \quad (30) \]

\[ G_{33} = m^2 - \omega^2. \quad (31) \]

And the Ricci scalar is \( R = 2(m^2 - \omega^2) \).
For simplicity, let us choose a new basis such that the metric becomes

\[ ds^2 = \eta_{AB} \theta^A \theta^B = (\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2, \]  

(32)

where \( \eta_{AB} = \text{diag}(1, -1, -1, -1) \) and \( \theta^A = e^A \mu \, dx^\mu \). Then

\[
\begin{align*}
\theta^{(0)} &= dt + H(r)d\varphi, \\
\theta^{(1)} &= dr, \\
\theta^{(2)} &= D(r)d\varphi, \\
\theta^{(3)} &= dz,
\end{align*}
\]

(33-36)

with \( e^A \mu \) being the tetrad, which the non-null components are

\[
e^0 \ (0) = e^1 \ (1) = e^3 \ (3) = 1, \quad e^0 \ (2) = - \frac{H(r)}{D(r)}, \quad e^2 \ (2) = D^{-1}(r).
\]

(37)

Then the non-vanishing components of the Einstein tensor, in the flat (local) space-time, take the form

\[
G^{(0)(0)} = 3\omega^2 - m^2, \\
G^{(1)(1)} = G^{(2)(2)} = \omega^2, \\
G^{(3)(3)} = m^2 - \omega^2.
\]

(38-40)

where \( G_{AB} = e^\mu_A e^\nu_B G_{\mu\nu} \) has been used.

Then the field equation, eq.(9), in the flat space-time becomes

\[
\frac{1}{\beta} \left[ f'_{RAB} - (\nabla_A \nabla_B - \eta_{AB} \Box) f'_R - \frac{1}{2} \eta_{AB} f \right] = -K\eta_{AB} + 2Kx\partial_A \phi \partial_B \phi + T_{AB}.
\]

(41)

Here the cosmological constant is taken as zero. Note that, as the Ricci scalar takes a constant value, the second term on the left hand side of equations (41) vanishes. The trace of this equation is

\[
f'_R R - 2f = \beta [-4K + 2Kx\partial^\alpha \phi \partial_\alpha \phi + T].
\]

(42)

Defining \( \gamma = -4K + 2Kx\partial^\alpha \phi \partial_\alpha \phi \), we get

\[
f'_R R - 2f = \beta [\gamma + T].
\]

(43)

And the eq.(41) is written as

\[
f'_R R_{AB} - \frac{1}{2} \eta_{AB} f = \beta [\sigma_{AB} + T_{AB}],
\]

(44)
where \( \sigma_{AB} \equiv -K\eta_{AB} + 2K_x \partial_A \phi \partial_B \phi \).

Combining eqs. (43) and (44) the field equation takes the form
\[
f'G_{AB} = \beta [\sigma_{AB} + T_{AB}] - \frac{1}{2} (f + \beta \gamma + \beta T) \eta_{AB}. \tag{45}
\]

Now, let us analyze this field equation for different contents of matter.

A. Perfect fluid

Here let us consider the perfect fluid as the matter content such that its energy-momentum tensor is
\[
T_{AB} = (\rho + p)u_A u_B + p\eta_{AB}, \tag{46}
\]
where \( u_A = (1, 0, 0, 0) \) is the 4-velocity of the fluid, \( p \) and \( \rho \) is the pressure and energy density, respectively. The trace of the energy-momentum tensor is \( T = \rho - 3p \). Then the components of the field equations for Gödel-type metric are
\[
\begin{align*}
2f'_R (3\omega^2 - m^2) + f &= \beta (2\sigma_{00} - \gamma + \rho + 3p), \quad (47) \\
2f'_R \omega^2 - f &= \beta (2\sigma_{11} + \gamma + \rho - p), \quad (48) \\
2f'_R (m^2 - \omega^2) - f &= \beta (2\sigma_{33} + \gamma + \rho - p). \quad (49)
\end{align*}
\]

With \( \sigma_{11} = \sigma_{22} = \sigma_{33} = K(\phi, X) \) and \( \sigma_{00} = -K(\phi, X) + 2K_x \dot{\phi}^2 \).

From eqs. (48) and (49), we get
\[
f'_R (4\omega^2 - 2m^2) = 0. \tag{50}
\]

In order to avoid ghost-like and instability, it is considered that \( \omega > 0 \) and \( f'_R > 0 [48-51] \). Then eq. (50) gives \( m^2 = 2\omega^2 \), which is the condition for Gödel metric solution. Thus, the field equations are reduced to
\[
\begin{align*}
f'_R m^2 + f &= \beta (2\sigma_{00} - \gamma + \rho + 3p), \quad (51) \\
f'_R m^2 - f &= \beta (2\sigma_{33} + \gamma + \rho - p). \quad (52)
\end{align*}
\]

The eqs. (51) and (52) lead to the relations
\[
\begin{align*}
p &= \gamma + \frac{1}{2} \left( \frac{f}{\beta} - \sigma_{00} + \sigma_{33} \right), \quad (53) \\
\rho &= \frac{f'_R m^2}{\beta} - \gamma - \frac{1}{2} \left( \frac{f}{\beta} - \sigma_{00} + 3\sigma_{33} \right). \quad (54)
\end{align*}
\]
From these results, the critical radius in the framework of \( f(R, \phi, X) \) theory is
\[
    r_c = \frac{2 \sinh^{-1}(1)}{\sqrt{\frac{1}{T_R} [\beta (\rho + \frac{1}{2} \sigma_{00} + 3 \sigma_{33} + \gamma) + f]}}.
\] (55)

This shows that it exists non-causal Gödel circles. In addition, the critical radius depends on the \( k \)-essence theory, \( f(R) \) function and matter content. An important note, the expression for the critical radius holds for any \( K(\phi, X) \) function and any \( f(R) \) theory.

In order to obtain causal Gödel-type solutions, different sources of matters will be considered. Let us consider as matter content: a combination of a perfect fluid and an electromagnetic field and a single electromagnetic field.

**B. Perfect Fluid Plus Electromagnetic Field**

Here the matter content is composed of a perfect fluid plus an electromagnetic field aligned on \( z \)-axis and dependent of \( z \). For this choice, the non-vanishing components of the electromagnetic tensor in the flat space-time is
\[
    F_{(0)(3)} = E(z) \quad \text{e} \quad F_{(1)(2)} = B(z),
\] (56)

where \( E(z) \) and \( B(z) \) are solutions of Maxwell equations given by
\[
    E(z) = E_0 \cos [2 \omega_0 (z - z_0)],
\] (57)
\[
    B(z) = E_0 \sin [2 \omega_0 (z - z_0)],
\] (58)

with \( E_0 \) being the amplitude of the electric and magnetic fields and \( z_0 \) is an arbitrary constant. Thus, the components of energy-momentum tensor of electromagnetic field are
\[
    T_{EM}^{(0)(0)} = T_{EM}^{(1)(1)} = T_{EM}^{(2)(2)} = \frac{E_0^2}{2}, \quad T_{EM}^{(3)(3)} = -\frac{E_0^2}{2}.
\] (59)

Then the energy-momentum tensor of perfect fluid plus electromagnetic field is
\[
    T_{AB} = (\rho + p)u_A u_B - p \eta_{AB} + T_{EM}^{AB}.
\] (60)

For this content of matter, the field equations, eq.(45), are
\[
    2f_R' (3 \omega^2 - m^2) + f - \frac{E_0^2}{2} = \beta (2 \sigma_{00} - \gamma + \rho + 3p),
\] (61)
\[
    2f_R' \omega^2 - f + \frac{E_0^2}{2} = \beta (2 \sigma_{11} + \gamma + \rho - p),
\] (62)
\[
    2f_R' (m^2 - \omega^2) - f - \frac{E_0^2}{2} = \beta (2 \sigma_{33} + \gamma + \rho - P).
\] (63)
The eqs. (62) and (63) give

\[ m^2 = \frac{E_0^2}{2f'_R} + 2\omega^2. \]  

(64)

By taking the amplitude of the electric field \( E_0^2 \geq 0 \) and \( f'_R > 0 \), this equation permits the condition \( m^2 = 4\omega \), which implies in \( r_c \to \infty \). Therefore, this solution leads to a causal Gödel-type universe. Moreover, this combination of matter sources does not imply causality violation for any \( k \)-essence function \( K(\phi, X) \) and \( f(R) \) theory.

C. Electromagnetic field

Now a single electromagnetic field aligned on \( z \)-axis and dependent of \( z \) is considered as a matter source. The field equations become

\[ 2f'_R(3\omega^2 - m^2) + f = \beta (2\sigma_{00} - \gamma), \]  

(65)

\[ 2f'_R\omega^2 - f = \beta (2\sigma_{11} + \gamma + 2E_0^2), \]  

(66)

\[ 2f'_R(m^2 - \omega^2) - f = \beta (2\sigma_{33} + \gamma). \]  

(67)

The field equations (65) and (66) give rise to the class of Gödel-type solutions

\[ 2\omega^2 - m^2 = \beta E_0^2. \]  

(68)

This shows that a causal universe is allowed for the condition \( E_0^2 \leq 0 \). Therefore a causal Gödel-type solution, i.e., \( m^2 = 4\omega \) that implies \( r_c \to \infty \), is permitted in \( k \)-essence theory plus \( f(R) \) theory for an electromagnetic field as matter content.

It is interesting note that, the \( f(R, \phi, X) \) theory allows both causal and non-causal Gödel-type solutions.

V. CONCLUSIONS

Here a hybrid model of \( f(R) \) gravity and \( k \)-essence theory, called \( f(R, \phi, X) \) theory, is studied. In this gravitational model the problem of causality is explored. The Gödel and Gödel-type metrics are considered. The Gödel solution is a homogeneous rotating cosmological solution of Einstein equations with pressureless matter and negative cosmological constant, which played an important role in the conceptual development of general relativity. Then is very important to verify the consistency of Gödel solution with alternatives theories of gravity. In addition, Gödel universes
could represent a theoretical laboratory for testing modified gravity theories. Here, the consistency between this solution and $f(R, \phi, X)$ gravity theory is developed. The field equations are solved for different contents of matter. Thus, it is obtained as a result: (i) the Gödel metric is solution in $f(R, \phi, X)$ theory. Then violation of causality is allowed in this theory. (ii) Considering the Gödel-type metrics with perfect fluid as matter source, the critical radius shows that causality is violated, and depends on the functions $K(\phi, X)$, $f(R)$, $f'_R$ and of the content of matter. (iii) Looking for a causal solution, the matter source is considered as a combination of a perfect fluid with an electromagnetic field, and simply a single electromagnetic field. For this content of matter, causal regions are permitted.

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