Conserved Current Densities, Localization Probabilities, and a New Global Gauge Symmetry of Klein-Gordon Fields

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Abstract

For free Klein-Gordon fields, we construct a one-parameter family of conserved current densities $J_\mu^\alpha$, with $\alpha \in (-1, 1)$, and use the latter to yield a manifestly covariant expression for the most general positive-definite and Lorentz-invariant inner product on the space of solutions of the Klein-Gordon equation. Employing a recently developed method of constructing the Hilbert space and observables for Klein-Gordon fields, we then obtain the probability current density $J_\mu^\alpha$ for the localization of a Klein-Gordon field in space. We show that in the nonrelativistic limit both $J_\mu^\alpha$ and $J_\mu^a$ tend to the probability current density for the localization of a nonrelativistic free particle in space, but that unlike $J_\mu^a$, the current density $J_\mu^\alpha$ is neither covariant nor conserved. Because the total probability may be obtained by integrating either of these two current densities over the whole space, the conservation of the total probability may be viewed as a consequence of the local conservation of $J_\mu^\alpha$. The latter is a manifestation of a previously unnoticed global gauge symmetry of the Klein-Gordon fields. The corresponding gauge group is $U(1)$ if the parameter $\alpha$ is rational. It is the multiplicative group of positive real numbers if $\alpha$ is irrational. We also discuss an extension of our results to Klein-Gordon fields minimally coupled to an electromagnetic field.

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1 Introduction

The demand for devising a probabilistic interpretation for Klein-Gordon fields is among the oldest problems of modern physics. Though this problem was never fully resolved, it provided the motivation for some of the most important developments of the twentieth-century theoretical physics. The most notable of these are the discovery of the Dirac equation and the advent of the method of second-quantization which eventually led to the formulation of the quantum field theories. The latter, in turn, provided the grounds for completely disregarding the original problem of finding a probabilistic interpretation for Klein-Gordon fields as there were sufficient evidence that the first-quantized (scalar) field theories involved certain inconsistencies such as the Klein paradox [1].

There were also some general arguments suggesting that the localization of bosonic fields in space was not possible [3]. These led to the consensus that the correct physical picture was provided by second-quantized field theories and that one could safely neglect the above-mentioned problem, for it only arose if one dealt with the first-quantized fields.

Though this point of view is almost universally accepted, the fact that a logical transition from nonrelativistic to relativistic quantum mechanics requires studying the first-quantized fields compels one to appeal to the conventional interpretation of Klein-Gordon fields in terms of the Klein-Gordon current density:

\[ J_{\text{KG}}^\mu = ig \psi(x^0, \vec{x})^* \partial^\mu \psi(x^0, \vec{x}). \] (1)

Here \( g \in \mathbb{R}^+ \) is a constant, \( \psi \) is a solution of the Klein-Gordon equation

\[ [\partial_\mu \partial^\mu - M^2] \psi(x^0, \vec{x}) = 0, \] (2)

\( M := mc/\hbar \) is the inverse of the Compton’s wave length, \( m \) is the mass of \( \psi \), \( \partial_\mu \partial^\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu \), \( \eta^{\mu\nu} \) are components of the inverse of the Minkowski metric \( (\eta_{\mu\nu}) \) with signature \((-1,1,1,1)\), and for any pair of Klein-Gordon fields \( \psi_1 \) and \( \psi_2 \),

\[ \psi_1 \partial^{\mu} \psi_2 := \psi_1 \partial^\mu \psi_2 - (\partial^\mu \psi_1) \psi_2. \]

1See however [2].
As it is discussed in most textbooks on relativistic quantum mechanics, $J_{\text{KG}}^\mu$ is a conserved four-vector current density, i.e., it is a four-vector field satisfying the continuity equation

$$\partial_\mu J_{\text{KG}}^\mu = 0.$$  

Therefore, $J_{\text{KG}}^0$ may be used to defined a conserved quantity, namely

$$Q := \int_{\mathbb{R}^3} d^3\vec{x} \ J_{\text{KG}}^0(x^0, \vec{x}).$$  

The fact that $Q$ (respectively $J_{\text{KG}}^0$) takes positive as well as negative values does not allow one to identify it with a probability (respectively probability density). Instead, one identifies $Q$ with the electric charge of the field and views $J_{\text{KG}}^\mu$ as the corresponding four-vector charge density [4]. In this way the continuity equation [3] provides a differential manifestation of the electric charge conservation.

It can be shown that $Q$ takes positive values for positive-energy Klein-Gordon fields and that one can define a Lorentz-invariant positive-definite inner product on the set of positive-energy fields. This is the basis of the point of view according to which one identifies the physical Hilbert space with the subspace of positive-energy fields [4]. This approach however fails for the cases that the Klein-Gordon field is subject to a time-dependent background field, for in this case the notion of a positive-energy Klein-Gordon field is ill-defined. Even in the absence of time-dependent background fields, the above restriction to the positive-energy fields limits the choice of possible observables to those that do not mix positive- and negative-energy fields. There are also well-known (and related) problems regarding the violation of causality [6].\footnote{See however [7, 8] and references therein.} The safest way out of all these difficulties seems to be a total abandonment of the first-quantized field theories as viable physical theories [9].

The interest in the issue of finding a probabilistic interpretation for first-quantized Klein-Gordon fields was revived in the 1960s within the context of canonical quantum gravity. There it emerged as a fundamental obstacle in developing a quantum theory of gravity [10]. This time
neither Dirac’s trick of considering an associated first order field equation nor the application of the method of second-quantization could be applied satisfactorily [11]. It was also not possible to define a subset of positive-energy solutions which would serve as the underlying vector space for the ‘physical Hilbert space’ of the theory. It was then necessary to deal with the above-described problems with first-quantized fields directly.

The lack of a probabilistic interpretation for canonical quantum gravity and its simplified version known as quantum cosmology is widely referred to as the Hilbert-space problem, [11]. This terminology reflects the view that the problem of devising a probabilistic interpretation for the wave functions appearing in these theories (namely the Wheeler-DeWitt fields) is equivalent to constructing a Hilbert space to which these fields belong. In this way one can identify the observables of the theory with Hermitian operators acting in this Hilbert space and utilize Born’s probabilistic interpretation of quantum mechanics.

The Klein-Gordon charge density may be used to define an inner product on the space of solutions of the Klein-Gordon equation. This is known as the Klein-Gordon inner product:

\[
(\psi_1, \psi_2)_{\text{KG}} = \int_{\mathbb{R}^3} d^3 \vec{x} \left[ \psi_1(x^0, \vec{x})^{*} \dot{\psi}_2(x^0, \vec{x}) - \dot{\psi}_1(x^0, \vec{x})^{*} \psi_2(x^0, \vec{x}) \right],
\]  (5)

where \(\psi_1\) and \(\psi_2\) are Klein-Gordon fields, \(g\) is a nonzero positive real constant, and an overdot stands for a \(x^0\)-derivative, i.e., \(\partial_0\). Clearly, \((\psi, \psi)_{\text{KG}} = \int d^3 \vec{x} J_{\text{KG}}(x^0, \vec{x}) = Q\). As \(Q\) may be negative, the Klein-Gordon inner product is indefinite [12]. Therefore, endowing the (vector) space \(\mathcal{V}\) of solutions of the Klein-Gordon equation with the Klein-Gordon inner product does not produce a genuine inner product space. One may pursue the approach of the indefinite-metric quantum theories [13] and identify the subspace \(\mathcal{V}_+\) of positive-energy solutions as the physical space of state vectors. Restricting the Klein-Gordon inner product to this subspace one obtains a (definite) inner product space that can be extended to a separable Hilbert space \(\mathcal{H}_+\) through Cauchy completion [14]. This is the basis of developing quantum field theories in curved background spacetimes [15].

In order to ensure that the right-hand side of (5) is convergent, it is sufficient to assume
that for all \( x^0 \in \mathbb{R} \), the functions \( \psi(x^0), \dot{\psi}(x^0) : \mathbb{R}^3 \rightarrow \mathbb{C} \) defined by
\[
\psi(x^0)(\vec{x}) := \psi(x^0, \vec{x}), \quad \dot{\psi}(x^0)(\vec{x}) := \partial_0 \psi(x^0, \vec{x}),
\]
are square-integrable, i.e., \( \psi(x^0), \dot{\psi}(x^0) \in L^2(\mathbb{R}^3) \). This is an assumption that we shall make throughout this paper. It is supported by the fact that \( \psi \) tends to a solution of the free Schrödinger equation in the nonrelativistic limit \( (c \rightarrow \infty) \).

We can respectively express the Klein-Gordon equation (2), the space \( V \) of its solutions, and the Klein-Gordon inner product (5) in terms of the functions \( \psi(x^0) \) and \( \dot{\psi}(x^0) \) according to
\[
\ddot{\psi}(x^0) + D\psi(x^0) = 0, \quad (6)
\]
\[
V = \left\{ \psi : \mathbb{R} \rightarrow L^2(\mathbb{R}^3) \mid \ddot{\psi}(x^0) + D\psi(x^0) = 0 \right\}, \quad (7)
\]
\[
(\psi_1, \psi_2)_{KG} = ig \left[ \langle \psi_1(x^0) | \dot{\psi}_2(x^0) \rangle - \langle \dot{\psi}_1(x^0) | \psi_2(x^0) \rangle \right], \quad (8)
\]
where \( D : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3) \) is the operator defined by
\[
(D\phi)(\vec{x}) := (-\nabla^2 + M^2)\phi(\vec{x}) \quad \forall \phi \in L^2(\mathbb{R}^3), \quad (9)
\]
\( \psi_1, \psi_2 \in V \) are arbitrary, and \( \langle \cdot | \cdot \rangle \) stands for the inner product of \( L^2(\mathbb{R}^3) \).

Recently, it has been noticed that one can devise a systematic method of endowing \( V \) with a positive-definite inner product [16, 17, 18]. In particular, one may define a positive-definite and relativistically invariant inner product on \( V \), namely
\[
(\psi_1, \psi_2) := \frac{1}{2M} \left[ \langle \psi_1(x^0) | D^{1/2} \psi_2(x^0) \rangle + \langle \dot{\psi}_1(x^0) | D^{-1/2} \dot{\psi}_2(x^0) \rangle \right]. \quad (10)
\]
Note that \( D \) and consequently \( D^{-1} \) are positive-definite operators (Hermitian operators with strictly positive spectra) and \( D^{\pm 1/2} \) is the unique positive square root of \( D^{\pm 1} \). In addition to being positive-definite and relativistically invariant, the inner product (10) is a conserved quantity in the sense that the \( x^0 \)-derivative of the right-hand side of (10) vanishes. As explained in Ref. [17], this is required to make the inner product (10) well-defined.
The expression (10) was originally obtained in [16] using the results of the theory of pseudo-Hermitian operators [19]. To the best of our knowledge, the existence of a positive-definite and relativistically invariant inner product for Klein-Gordon fields was originally pointed out by the authors of Refs. [20] and [21]. Though these authors pursue completely different approaches\(^3\), they arrive at an expression which is an alternative form of the inner product (10). The advantage of the approach of [16] is that not only it yields an explicit expression for a positive-definite and relativistically invariant inner product, but it also allows for a construction of all such inner products: The most general positive-definite, relativistically invariant, and conserved inner product on the space \(V\) of solutions of the Klein-Gordon equation (3) is given by [16, 17]

\[
(\psi_1, \psi_2)_a = \frac{\kappa}{2M} \left\{ \langle \psi_1(x^0)|D^{1/2}\psi_2(x^0)\rangle + \langle \dot{\psi}_1(x^0)|D^{-1/2}\dot{\psi}_2(x^0)\rangle + ia \left[ \langle \psi_1(x^0)|\dot{\psi}_2(x^0)\rangle - \langle \dot{\psi}_1(x^0)|\psi_2(x^0)\rangle \right] \right\},
\]

where \(\kappa \in \mathbb{R}^+\) and \(a \in (-1, 1)\) are arbitrary constants. If we set \(g = 1/(2M)\) in (8), we can express (11) in the form

\[
(\psi_1, \psi_2)_a = \kappa[(\psi_1, \psi_2) + a(\psi_1, \psi_2)_{KG}].
\]

Note that \(\kappa\) is an unimportant multiplicative constant that can be absorbed in the definition of the Klein-Gordon fields \(\psi_1\) and \(\psi_2\).

The existence of the positive-definite inner products (12) and their relativistic invariance and conservation raise the natural question whether there is a conserved four-vector current density associated with these inner products. The main purpose of the present article is to show that such a current density exists. In fact, we will construct a one-parameter family \(J^\mu_a\) (with

\(^3\)The analysis of [20] involves the study of a certain Green’s function for the Klein-Gordon equation, whereas that of [21] employs the idea of gauge-fixing the inner product of the auxiliary Hilbert space obtained in the quantization of the classical relativistic particle within the framework of Dirac’s method of constraint quantization. A detailed application of the latter idea for Klein-Gordon fields is given in [22, 23, 7]. Its direct extension is the method of refined algebraic quantization also known as the group-averaging [24]. For a brief review see [25].
a ∈ (−1, 1)) of current densities and show by direct computation that not only they transform as vector fields but they also satisfy the continuity equation and yield the well-known Schrödinger probability current density in nonrelativistic limit. Furthermore, the current density $J^\mu_a$ may be used to yield a manifestly covariant expression for the inner products (11). Perhaps more importantly, its local conservation law is linked with the conservation of the total probability of the localization of the field in space, on the one hand, and a previously unnoticed Abelian global gauge symmetry of the Klein-Gordon equation, on the other hand.

The organization of the article is as follows. In Section 2, we outline a derivation of the current densities $J^\mu_a$ and explore their properties. In Section 3, we derive the probability current density for the localization of a Klein-Gordon field in space and show how it relates to the current densities $J^\mu_a$. In Section 4, we study the underlying gauge symmetry associated with the conservation of $J^\mu_a$. In Section 5, we discuss a generalization of our results to Klein-Gordon fields interacting with a background electromagnetic field. In Section 6 we present our concluding remarks. The appendices include some useful calculations that are, however, not of primary interest.

2 Derivation and Properties of $J^\mu_a$

Let $\psi \in \mathcal{V}$ be a Klein-Gordon field. Then in view of (11), we have

$$
(\psi, \psi)_a = \frac{\kappa}{2M} \int_{\mathbb{R}^3} d^3\vec{\alpha} \left\{ \psi(x^0, \vec{x})^* \hat{\mathcal{D}}^{1/2} \psi(x^0, \vec{x}) + \dot{\psi}(x^0, \vec{x})^* \hat{\mathcal{D}}^{-1/2} \dot{\psi}(x^0, \vec{x}) + ia \left[ \psi(x^0, \vec{x})^* \dot{\psi}(x^0, \vec{x}) - \dot{\psi}(x^0, \vec{x})^* \psi(x^0, \vec{x}) \right] \right\},
$$

with $\hat{\mathcal{D}} := -\nabla^2 + M^2$. Now, using the analogy with nonrelativistic quantum mechanics, we define the current density $J^0_a$ associated with $\psi$ as the integrand in (13). That is

$$
J^0_a(x) := \frac{\kappa}{2M} \left\{ \psi(x)^* \hat{\mathcal{D}}^{1/2} \psi(x) + \dot{\psi}(x)^* \hat{\mathcal{D}}^{-1/2} \dot{\psi}(x) + ia \left[ \psi(x)^* \dot{\psi}(x) - \dot{\psi}(x)^* \psi(x) \right] \right\},
$$

where we have set $x := (x^0, \vec{x})$. 

7
In order to obtain the spatial components \( J^i_a \) (with \( i \in \{1, 2, 3\} \)) of \( J^\mu_a \), we follow the procedure outlined in Ref. [26]. Namely, we perform an infinitesimal Lorentz boost transformation that changes the reference frame to the one moving with a velocity \( \vec{v} \). That is we consider

\[
x^0 \to x'^0 = x^0 - \vec{\beta} \cdot \vec{x}, \quad \vec{x} \to \vec{x}' = \vec{x} - \vec{\beta} x^0,
\]

where \( \vec{\beta} := \vec{v}/c \), and we ignore second and higher order terms in powers of the components of \( \vec{\beta} \). Assuming that \( J^\mu_a \) is indeed a four-vector field, we obtain the following transformation rule for \( J^0_a \).

\[
J^0_a(x) \to J'^0_a(x') = J^0_a(x) - \vec{\beta} \cdot \vec{J}_a(x).
\]

Next, we recall that we can use (14) to read off the expression for \( J'^0_a(x') \), namely

\[
J'^0_a(x') := \frac{k}{2M} \left\{ \psi'(x')^* \hat{D}'^{1/2} \psi'(x') + \dot{\psi}'(x')^* \hat{D}'^{-1/2} \dot{\psi}'(x') + ia \left[ \psi'(x')^* \dot{\psi}'(x') - \dot{\psi}'(x')^* \psi'(x') \right] \right\},
\]

where \( x' := (x'^0, \vec{x}') \), \( \hat{D}' = -\nabla'^2 + M^2 \) and \( \dot{\psi}'(x') := \partial \psi'(x')/\partial x'^0 \). This reduces the determination of \( \vec{J}_a \) to expressing the right-hand side of (17) in terms of the quantities associated with the original (unprimed) frame and comparing the resulting expression with (16).

It is obvious that \( \psi \) is a scalar field;

\[
\psi'(x') = \psi(x).
\]

A less obvious fact is that \( D^{-1/2} \dot{\psi} \) is also a scalar field. This can be directly checked by performing an infinitesimal Lorentz transformation as we demonstrate in Appendix A. Alternatively, we may appeal to the observation that the generator \( h \) of \( x^0 \)-translations, that is defined [17] as the operator \( h \psi := i\hbar \dot{\psi} \) acting in the space \( \mathcal{V} \) of Klein-Gordon fields, squares to \( \hbar^2 \hat{D} \). Hence, as noted in Ref. [18], \( \hbar^{-1} D^{-1/2} h \) is nothing but the charge-conjugation operator \( \mathcal{C} \). This in turn means that

\[
iD^{-1/2} \dot{\psi} = \mathcal{C} \psi =: \psi_c.
\]
Clearly $\psi_c$ is also a scalar field, and consequently $D^{-1/2}\dot{\psi}$ is Lorentz invariant;

$$D^{-1/2}\dot{\psi}(x') = \dot{D}^{-1/2}\dot{\psi}(x).$$

Next, we use (15) and (18) to deduce

$$\dot{\psi}'(x') = \dot{\psi}(x) + \vec{\beta} \cdot \vec{\nabla} \psi(x),$$

$$\hat{D}^{-\alpha} = \hat{D}^{-\alpha} - 2\alpha \vec{\beta} \cdot \vec{\nabla} \hat{D}^{-1/2} \partial_0 \quad \forall \alpha \in \mathbb{R}. (22)$$

In view of Eqs. (20) – (22), we then have

$$\psi'(x')^* \hat{D}^{-1/2} \psi(x') = \psi(x)^* \hat{D}^{-1/2} \psi(x) - \psi(x)^* \vec{\beta} \cdot \vec{\nabla} \hat{D}^{-1/2} \dot{\psi}(x),$$

$$\dot{\psi}'(x')^* \hat{D}^{-1/2} \dot{\psi}(x') = \dot{\psi}(x)^* \hat{D}^{-1/2} \dot{\psi}(x) + [\vec{\beta} \cdot \vec{\nabla} \psi(x)^*] \hat{D}^{-1/2} \dot{\psi}(x).$$

Now, we substitute (18), (21), (23), and (24) in (17) and make use of (16) to obtain

$$\vec{J}_a(x) = \frac{\kappa}{2M} \left\{ \psi(x)^* \vec{\nabla} \hat{D}^{-1/2} \dot{\psi}(x) - [\vec{\nabla} \psi(x)^*] \hat{D}^{-1/2} \dot{\psi}(x) - i\alpha \left[ \psi(x)^* \vec{\nabla} \psi(x) - [\vec{\nabla} \psi(x)^*] \psi(x) \right] \right\}.$$ (25)

This relation suggests

$$J_a^\mu(x) = \frac{\kappa}{2M} \left\{ \psi(x)^* \hat{\partial}^\mu \hat{D}^{-1/2} \dot{\psi}(x) - i\alpha \psi(x)^* \hat{\partial}^\mu \psi(x) \right\}. (26)$$

It is not difficult to check (using the Klein-Gordon equation) that the expression for $J_a^0$ obtained using this equation agrees with the one given in (14).

We can use (19) to further simplify (26). This yields

$$J_a^\mu(x) = -\frac{i\kappa}{2M} \left[ \psi(x)^* \hat{\partial}^\mu \tilde{\psi}_a(x) \right],$$

where

$$\tilde{\psi}_a := \psi_c + \alpha \psi.$$

Equation (27) is the main result of this article.\footnote{Note that the results of Ref. [27] pertaining the uniqueness of the Klein-Gordon current density do not rule out the existence of the current density (26) because these results are obtained under the assumption that the current involves only the field and its first derivatives. The appearance of $\hat{D}^{-1/2}$ in (20) (alternatively $\psi_a$ in (27)) is a clear indication that this assumption is violated.}

We can use (19) to further simplify (26). This yields

$$J_a^\mu(x) = -\frac{i\kappa}{2M} \left[ \psi(x)^* \hat{\partial}^\mu \tilde{\psi}_a(x) \right],$$

where

$$\tilde{\psi}_a := \psi_c + \alpha \psi.$$
The current density $J^\mu_a$ constructed above has the following remarkable properties.

1. The expression (27) for $J^\mu_a$ is manifestly covariant; since $\psi$ and $\tilde{\psi}_a$ are scalar fields, $J^\mu_a$ is indeed a four-vector field.

2. Using the fact that both $\psi$ and $\tilde{\psi}_a$ satisfy the Klein-Gordon equation (2), one can show (by a direct calculation) that the following continuity equation holds.

$$\partial_\mu J^\mu_a = 0.$$  \hspace{1cm} (29)

Hence $J^\mu_a$ is a conserved current density.

3. As we show in Appendix B, in the nonrelativistic limit as $c \rightarrow \infty$, $J^\mu_a$ tends to the Schrödinger’s probability current density for a free particle. Specifically, setting $\kappa = 1/(1 + a)$, we find

$$\lim_{c \rightarrow \infty} J^0_a(x^0, \vec{x}) = \varrho(x^0, \vec{x}),$$ \hspace{1cm} (30)

$$\lim_{c \rightarrow \infty} \vec{J}_a(x^0, \vec{x}) = \frac{1}{c} \vec{j}(x^0, \vec{x}),$$ \hspace{1cm} (31)

where $\varrho$ and $\vec{j}$ are respectively the nonrelativistic scalar and current probability densities [9]:

$$\varrho(x^0, \vec{x}) := |\psi(x^0, \vec{x})|^2,$$ \hspace{1cm} (32)

$$\vec{j}(x^0, \vec{x}) := -\frac{i\hbar}{2m} \left[ \psi(x^0, \vec{x})^* \nabla \psi(x^0, \vec{x}) - \psi(x^0, \vec{x}) \nabla \psi(x^0, \vec{x})^* \right].$$ \hspace{1cm} (33)

4. Although $J^0_a(x)$ has been constructed out of a positive-definite inner product, namely [11], it is in general not even real. This can be easily checked by computing $J^0_a(x)$ for a linear combination of two plane wave solutions of the Klein-Gordon equation with different and oppositely signed energies. Similarly $J^\mu_a$ is complex-valued. The real and imaginary parts of $J^\mu_a$ are by construction real-valued conserved 4-vector current densities. They are further studied in Appendix C.
We can use the relation (26) for the current density $J_a^\mu$ and Eq. (11) to yield a manifestly covariant expression for the most general positive-definite and Lorentz-invariant inner product on the space of solutions of the Klein-Gordon equation (2), namely

$$\langle \psi_1, \psi_2 \rangle_a = -\frac{i\kappa}{2M} \int_\sigma d\sigma(x) \, n_\mu(x) \left\{ \psi_1(x)^* \overset{\leftrightarrow}{\partial^\mu} C \psi_2(x) + a\psi_1(x)^* \overset{\leftrightarrow}{\partial^\mu} \psi_2(x) \right\},$$  

(34)

where $\sigma$ is an arbitrary spacelike (Cauchy) hypersurface of the Minkowski space with volume element $d\sigma$ and unit (future) timelike normal four-vector $n^\mu$. Note that in deriving (34) we have also made an implicit use of the polarization principle [28], namely that any inner product is uniquely determined by the corresponding norm.

3 Probability Current Density for Localization of Klein-Gordon Fields in Space

In nonrelativistic quantum mechanics, the interpretation of $|\psi(\vec{x}; t)|^2$ as the probability density for the localization of a particle in (configuration) space relies on the following basic premises.

1. The state of the particle is described by an element $|\psi(t)\rangle$ of the Hilbert space $L^2(\mathbb{R}^3)$.

2. There is a Hermitian operator $\vec{x}$ representing the position observable whose eigenvectors $|\vec{x}\rangle$ form a basis.

3. The position wave function $\psi(\vec{x}; t)$ uniquely determines the state vector $|\psi(t)\rangle$ and consequently the corresponding state, because $\psi(\vec{x}; t)$ are the coefficients of the expansion of $|\psi(t)\rangle$ in the position basis:

$$|\psi(t)\rangle = \int_{\mathbb{R}^3} d^3\vec{x} \, \psi(\vec{x}; t)|\vec{x}\rangle.$$

(35)

4. The probability of localization of the particle in a region $V \subseteq \mathbb{R}^3$ at time $t \in \mathbb{R}$ is given by

$$P_V(t) = \int_V d^3\vec{x} \, ||\Lambda_\vec{x}|\psi(t)\rangle||^2,$$

(36)
where \( \Lambda_x := |\bar{x}\rangle\langle \bar{x}| \) is the projection operator onto \(|\bar{x}\rangle\), \( \| \cdot \|^2 := \langle \cdot | \cdot \rangle \), and the state vector \(|\psi(t)\rangle\) is supposed to be normalized, \( \| |\psi(t)\rangle\| = 1 \). It is because of the orthonormality of the position eigenvectors, i.e., \( \langle \bar{x}|\bar{x}'\rangle = \delta^3(\bar{x} - \bar{x}') \), that we can write (36) in the form

\[
P_V(t) = \int_V d^3 \bar{x} \ |\psi(\bar{x}; t)|^2.
\]

(37)

It is our belief that the same ingredients are necessary for defining the probability density for the localization of the Klein-Gordon fields in space, i.e., one must first define a genuine Hilbert space and a position operator \( \vec{X} \) for the Klein-Gordon fields and then use the position eigenvectors to define a position wave function associated with each Klein-Gordon field. Refs. [17, 18] give a thorough discussion of how one can construct the Hilbert space, a position operator, and the corresponding position wave functions for the Klein-Gordon and similar fields. For completeness, here we include a brief summary of this construction, elaborate on its consequences, and present its application in our attempt to determine the probability density for the localization of a Klein-Gordon field in space.

### 3.1 The Hilbert Space

Endowing the vector space \( \mathcal{V} \) of Eq. (7) with the inner product (11) and performing the Cauchy completion of the resulting inner product space yield a separable Hilbert space \( \mathcal{H}_a \) for each choice of the parameter \( a \in (-1, 1) \). However as discussed in great detail in [17], the choice of \( a \) is physically irrelevant, because different choices yield unitarily equivalent Hilbert spaces \( \mathcal{H}_a \). In particular, for all \( a \in (-1, 1) \), there is a unitary transformation\(^5\)

\[
U_a : \mathcal{H}_a \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3).
\]

\(^5\)The unitarity of \( U_a \) means that for all \( \psi_1, \psi_2 \in \mathcal{H}_a \), \( \langle U_a \psi_1, U_a \psi_2 \rangle = (\psi_1, \psi_2)_a \) where \( \langle \cdot, \cdot \rangle \) stands for the inner product of \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3), [14] \).
In fact, we can obtain the explicit form of $U_a$ rather easily. In view of the general results of [17], for all $\psi \in \mathcal{H}_a$,

$$U_a\psi := \frac{1}{2} \sqrt{\frac{\kappa}{\mathcal{M}}} \left( \frac{\sqrt{1 + a} [D^{1/4}\psi(x_0^0) + iD^{-1/4}\dot{\psi}(x_0^0)]}{\sqrt{1 - a} [D^{1/4}\psi(x_0^0) - iD^{-1/4}\dot{\psi}(x_0^0)]} \right),$$

(38)

where $x_0^0 \in \mathbb{R}$ is a fixed initial value for $x^0$.

As far as the physical properties of the system are concerned we can confine our attention to the simplest choice for $a$, namely $a = 0$. In this way we obtain the Hilbert space $\mathcal{H} := \mathcal{H}_0$ that is mapped to $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ via the unitary operator $U := U_0$. Setting $a = 0$ in (38) we find [18]

$$U\psi := \frac{1}{2} \sqrt{\frac{\kappa}{\mathcal{M}}} \left( \frac{D^{1/4}\psi(x_0^0) + iD^{-1/4}\dot{\psi}(x_0^0)}{D^{1/4}\psi(x_0^0) - iD^{-1/4}\dot{\psi}(x_0^0)} \right) = \frac{1}{2} \sqrt{\frac{\kappa}{\mathcal{M}}} D^{1/4} \left( \begin{array}{c} \psi(x_0^0) + \psi_c(x_0^0) \\ \psi(x_0^0) - \psi_c(x_0^0) \end{array} \right).$$

(39)

We can also calculate the inverse of $U$. The result is [18]

$$[U^{-1}\xi](x^0) = \sqrt{\frac{\mathcal{M}}{\kappa}} D^{-1/4} \left[ e^{-i(x^0 - x_0^0)D^{1/2}} \xi_1 + e^{i(x^0 - x_0^0)D^{1/2}} \xi_2 \right],$$

(40)

where $\xi = \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ and $x^0 \in \mathbb{R}$ are arbitrary.

It is important to note that $U_a$ (and in particular $U$) depend on the choice of $x_0^0$. Therefore they fail to be unique.

### 3.2 Position and Momentum Operators

Let $\vec{x}$ and $\vec{p}$ be the usual position and momentum operators acting in $L^2(\mathbb{R}^3)$, respectively, $|\vec{x}\rangle$ be the position eigenvectors satisfying

$$\vec{x}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle, \quad \langle \vec{x}|\vec{x}'\rangle = \delta^3(\vec{x} - \vec{x}'), \quad \int_{\mathbb{R}^3} d^3\vec{x} |\vec{x}\rangle\langle \vec{x}| = 1,$$

(41)

$\sigma_i$ with $i \in \{1, 2, 3\}$ be the Pauli matrices

$$\sigma_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),$$

(42)
and \( \sigma_0 \) be the \( 2 \times 2 \) identity matrix. Then any observable acting in \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) is of the form \( O = \sum_{\mu=0}^{3} O_{\mu} \otimes \sigma_{\mu} \) where \( O_{\mu} \) are Hermitian operators acting in \( L^2(\mathbb{R}^3) \). This in turn implies that the general form of the observables (Hermitian operators) acting in the Hilbert space \( \mathcal{H} \) is given by \( U^{-1}OU \), for \( U : \mathcal{H} \rightarrow L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) is a unitary operator. In particular, as proposed in [18], we may identify the operators \( \vec{X}, \vec{P} : \mathcal{H} \rightarrow \mathcal{H} \), defined by \( \vec{X} := U^{-1}(x \otimes \sigma_0)U, \quad \vec{P} := U^{-1}(p \otimes \sigma_0)U \), (43) with position and momentum operators for the Klein-Gordon fields, respectively. It turns out that \( (\vec{P}\psi)(x^0) = \vec{p}\psi(x^0) \) but that \( \vec{X}\psi \) has a more complicated expression. It is determined by the initial conditions [18] \( (\vec{X}\psi)(x^0) = \vec{x}\psi(x^0), \quad \partial_0(\vec{X}\psi)(x^0) = \vec{X}^\dagger \dot{\psi}(x^0), \) where \( \vec{X} := \vec{x} + \frac{i\hbar \vec{p}}{2(\vec{p}^2 + m^2)} \) is the Newton-Wigner position operator [29].

3.3 Localized States and Position Wave Functions

Clearly the operators \( x \otimes \sigma_0 \) and \( 1 \otimes \sigma_3 \) from a maximal commuting set of observables acting in \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \). Hence their common eigenvectors

\[ \xi^{(\epsilon, \vec{x})} := |\vec{x}\rangle \otimes e_\epsilon \]  

(44)

with \( e_+ := \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) and \( e_- := \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \), form a complete orthonormal basis of \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \). This together with the fact that \( U \) is a unitary transformation imply that

\[ \psi^{(\epsilon, \vec{x})} := U^{-1}\xi^{(\epsilon, \vec{x})} \]  

(45)

form a complete orthonormal basis of \( \mathcal{H} \), i.e.,

\[ (\psi^{(\epsilon, \vec{x})}, \psi^{(\epsilon', \vec{x}')}_0 = \delta_{\epsilon, \epsilon'} \delta^3(\vec{x} - \vec{x}'), \quad \sum_{\epsilon=\pm 1} \int_{\mathbb{R}^3} d^3 \vec{x} |\psi^{(\epsilon, \vec{x})}(\psi^{(\epsilon, \vec{x})}| = 1, \]  

(46)
where for all $\psi \in \mathcal{H}$, $|\psi\rangle(\psi)$ is the projection operator defined by $|\psi\rangle(\psi| := (\psi, \phi) \psi$ and $I$ is the identity map acting in $\mathcal{H}$. Furthermore, we have for both $\epsilon = \pm 1$

$$\bar{X} \psi^{(\epsilon, \vec{x})} = \vec{x} \psi^{(\epsilon, \vec{x})}. \quad (47)$$

It is also not difficult to see [18] that the charge-conjugation transformation is given by $C = U^{-1}(1 \otimes \sigma_3)U$. Hence,

$$C \psi^{(\epsilon, \vec{x})} = \epsilon \psi^{(\epsilon, \vec{x})}. \quad (48)$$

In view of Eqs. (46) – (48), the state vectors $\psi^{(\epsilon, \vec{x})}$ represent spatially localized Klein-Gordon fields with definite charge-parity $\epsilon$. They can be employed to associate each Klein-Gordon field $\psi \in \mathcal{H}$ with a unique position wave function, namely

$$f(\epsilon, \vec{x}) := (\psi^{(\epsilon, \vec{x})}, \psi). \quad (49)$$

As shown in [18], one can use these wave functions to represent all the physical quantities associated with the Klein-Gordon fields. In particular, the transition amplitudes between two states (inner product of two state vectors) take the simple form

$$(\psi_1, \psi_2)_0 = \sum_{\epsilon = \pm 1} \int_{\mathbb{R}^3} d^3 \vec{x} \ f_1(\epsilon, \vec{x})^* f_2(\epsilon, \vec{x}), \quad (50)$$

where $\psi_1, \psi_2 \in \mathcal{H}$ and $f_1, f_2$ are the corresponding wave functions.

As suggested by [50], the wave functions $f(\pm, \vec{x})$ belong to $L^2(\mathbb{R}^3)$. Moreover due to the implicit dependence of $\psi^{(\epsilon, \vec{x})}$ on $x^0_0$ appearing in the expression for $U$, $f(\pm, \vec{x})$ depend on $x^0_0$. This dependence becomes explicit once we express $f(\epsilon, \vec{x})$ in terms of $\psi$ directly. In order to see this, we first substitute (10) and (44) in (45) to obtain

$$\psi^{(\epsilon, \vec{x})}(x^0) = \sqrt{\frac{\lambda}{\kappa}} D^{-1/4} e^{-i\epsilon(x^0-x^0_0)D^{1/2}} |\vec{x}\rangle. \quad (51)$$

We then use this equation and (11) to compute the right-hand side of (39). This yields

$$f(\epsilon, \vec{x}) = \sqrt{\frac{\kappa}{M}} \tilde{D}^{1/4} e^{i\epsilon(x^0-x^0_0)\tilde{D}^{1/2}} \psi(x^0, \vec{x}), \quad (52)$$
where
\[ \psi_\epsilon := \frac{1}{2}(1 + \epsilon C)\psi = \frac{1}{2}(\psi + \epsilon \psi_c) \] (53)
is the definite-charge (definite-energy) component of \( \psi \) with charge-parity \( \epsilon \). Note however that \( \psi_\epsilon \) satisfies the Foldy equation [30]
\[ i\partial_0 \psi_\epsilon(x^0, \vec{x}) = \epsilon \hat{D}^{1/2} \psi_\epsilon(x^0, \vec{x}). \] (54)
This in turn implies
\[ e^{i\epsilon(x^0-x_0^0)}\hat{D}^{1/2} \psi_\epsilon(x^0, \vec{x}) = \psi_\epsilon(x_0^0, \vec{x}). \]
Hence (52) takes the simple form
\[ f(\epsilon, \vec{x}) = \sqrt{\frac{\kappa}{M}} \hat{D}^{1/4} \psi_\epsilon(x_0^0, \vec{x}). \] (55)
As seen from this equation the wave functions \( f(\epsilon, \vec{x}) \) depend on \( x_0^0 \).

It is also interesting to note that one can use (51) to compute
\[ \psi^{(\epsilon, \vec{y})}(x^0, \vec{x}) := \langle \vec{x} | \psi^{(\epsilon, \vec{y})}(x^0) \rangle = \sqrt{\frac{M}{\kappa}} \frac{1}{2\pi^2|x-\vec{y}|} \int_0^\infty dk \left\{ \frac{k \sin(|x-\vec{y}|k) \exp \left[ -i\epsilon(x^0-x_0^0)\sqrt{k^2+M^2} \right]}{(k^2+M^2)^{1/4}} \right\}. \]
For \( x^0 = x_0^0 \), the integral on the right-hand side of this equation can be expressed in terms of the Bessel K-function \( K_{\frac{5}{4}} \). The result is, for both \( \epsilon = -1 \) and 1,
\[ \psi^{(\epsilon, \vec{y})}(x_0^0, \vec{x}) = \sqrt{\frac{M}{\kappa}} \left[ 2^{\frac{5}{4}} \pi^{\frac{3}{2}} \Gamma(\frac{5}{4}) \right]^{-1} \left( \frac{M}{|x-\vec{y}|} \right)^{\frac{5}{4}} K_{\frac{5}{4}}(M|x-\vec{y}|), \] (56)
where \( \Gamma \) stands for the Gamma function.

Equation (56) provides an explicit demonstration of the curious fact that \( \psi^{(+, \vec{x})} \) are indeed identical with the Newton-Wigner localized states [29] and that \( \psi^{(-, \vec{x})} \) are the negative-energy analogs of the latter. It is remarkable that we have obtained these localized states without pursuing the axiomatic approach of Ref. [29]. A perhaps more important observation is that actually one does not need to use the rather complicated expression (56) in calculating physical quantities [17, 18]. One can instead employ the corresponding wave functions which are simply delta functions: The wave function \( f(\epsilon, \vec{x})(\epsilon', \vec{x}') \) for \( \psi^{(\epsilon, \vec{y})}(x_0^0) \) has the form \( \delta_{\epsilon, \epsilon'} \delta(\vec{x} - \vec{x}') \).
3.4 Probability Density for Spatial Localization of a Field

Having obtained the expression for the position operator \( \vec{X} \) and position wave functions \( f(\epsilon, \vec{x}) \), we may proceed as in nonrelativistic quantum mechanics and identify the probability of the localization of a Klein-Gordon field \( \psi \) in a region \( V \subseteq \mathbb{R}^3 \), at time \( t_0 = x_0^0/c \), with

\[
P_V = \int_V d^3\vec{x} \| \Pi_\vec{x} \psi \|_0^2, \tag{57}
\]

where \( \Pi_\vec{x} \) is the projection operator onto the eigenspace of \( \vec{X} \) with eigenvalue \( \vec{x} \), i.e.,

\[
\Pi_\vec{x} = \sum_{\epsilon = \pm 1} |\psi(\epsilon, \vec{x})\rangle\langle \psi(\epsilon, \vec{x})|,
\]

\[\| \cdot \|_0^2 := (\cdot, \cdot)_0\] is the square of the norm of \( \mathcal{H} \), and we assume \( \| \psi \|_0 = 1 \). Substituting this relation in (57) and making use of (46) and (49), we have

\[
P_V = \sum_{\epsilon = \pm 1} \int_V d^3\vec{x} |f(\epsilon, \vec{x})|^2.
\]

Therefore, the probability density is given by

\[
\rho(x_0^0, \vec{x}) = \sum_{\epsilon = \pm 1} |f(\epsilon, \vec{x})|^2 = \frac{\kappa}{2\mathcal{M}} \left\{ |\hat{D}^{1/4}\psi(x_0^0, \vec{x})|^2 + |\hat{D}^{-1/4}\dot{\psi}(x_0^0, \vec{x})|^2 \right\}. \tag{58}
\]

To establish the second equality in (58), we have made use of (55), (53), and (19). For a position measurement to be made at time \( t = x^0/c \), we have the probability density

\[
\rho(x^0, \vec{x}) = \frac{\kappa}{2\mathcal{M}} \left\{ |\hat{D}^{1/4}\psi(x^0, \vec{x})|^2 + |\hat{D}^{-1/4}\dot{\psi}(x^0, \vec{x})|^2 \right\}. \tag{59}
\]

We can use (19) to express \( \rho \) in the following slightly more symmetrical form.

\[
\rho(x) = \frac{\kappa}{2\mathcal{M}} \left\{ |\hat{D}^{1/4}\psi(x)|^2 + |\hat{D}^{-1/4}\dot{\psi}(x)|^2 \right\}. \tag{60}
\]

Although the above discussion is based on a particular choice for the parameter \( a \), namely \( a = 0 \), it is generally valid. To see this, suppose we choose to work with the inner product (11) and hence the Hilbert space \( \mathcal{H}_a \) for some \( a \neq 0 \). Then we have a different position operator:

\[
\tilde{X}_a := U_a^{-1}(\vec{x} \otimes \sigma_0)U_a = U_a \vec{X} U_a^{-1}
\]

where

\[
U_a := U_a^{-1}U \tag{61}
\]
is a unitary operator mapping $\mathcal{H}$ onto $\mathcal{H}_a$. The eigenvectors $\psi_a^{(e, \vec{x})}$ of $\vec{X}_a$ are clearly related to $\psi^{(e, \vec{x})}$ by $\psi_a^{(e, \vec{x})} = \mathcal{U}_a \psi^{(e, \vec{x})}$. Now, given $\psi_a \in \mathcal{H}_a$, we define $\psi := \mathcal{U}_a^{-1} \psi_a$ and check that the position wave function for $\psi_a$ is given by

$$f_a(\epsilon, \vec{x}) := (\psi_a^{(\epsilon, \vec{x})}, \psi_a)_a = (\mathcal{U}_a \psi_a^{(\epsilon, \vec{x})}, \mathcal{U}_a \psi)_a = (\psi^{(\epsilon, \vec{x})}, \psi) = f(\epsilon, \vec{x}).$$  \hfill (62)

Here we made use of the fact that $\mathcal{U}_a : \mathcal{H} \rightarrow \mathcal{H}_a$ is a unitary operator. As seen from (62), the position wave functions for $\psi$ and $\psi_a$ coincide. As a result so do the corresponding probability densities.

If we are to compute the probability density $\rho_a$ of the spatial localization of a Klein-Gordon field $\psi$ with the position operator being identified with $\vec{X}_a$ for $a \neq 0$, we have, for a measurement made at $t_0 = x_0^0/c$,

$$\rho_a(x_0^0, \vec{x}) = \frac{\kappa}{2M} \left\{ |\hat{D}^{1/4} \psi_a'(x_0^0, \vec{x})|^2 + |\hat{D}^{-1/4} \dot{\psi}_a(x_0^0, \vec{x})|^2 \right\}$$  \hfill (63)

where $\psi_a' := \mathcal{U}_a^{-1} \psi$. We can compute the latter using (38), (39), and (61). This leads to

$$\psi_a'(x_0^0) = \alpha_+ \psi(x_0^0) + i \alpha_- \hat{D}^{-1/2} \dot{\psi}(x_0^0), \quad \dot{\psi}_a(x_0^0) = -i \alpha_- \hat{D}^{1/2} \psi(x_0^0) + \alpha_+ \dot{\psi}(x_0^0),$$  \hfill (64)

where

$$\alpha_\pm := \frac{1}{2} \left( \sqrt{1 + a} \pm \sqrt{1 - a} \right).$$  \hfill (65)

Now, substituting (64) and (65) in (63) and doing the necessary algebra, we find the following remarkably simple result.

$$\rho_a(x_0^0, \vec{x}) = \frac{\kappa}{2M} \left\{ |\hat{D}^{1/4} \psi(x_0^0, \vec{x})|^2 + |\hat{D}^{-1/4} \dot{\psi}(x_0^0, \vec{x})|^2 + \right.$$ 

$$\left. i a \left[ (\hat{D}^{1/4} \psi(x_0^0, \vec{x}))^* \hat{D}^{-1/4} \dot{\psi}(x_0^0, \vec{x}) - (\hat{D}^{1/4} \psi(x_0^0, \vec{x}))(\hat{D}^{-1/4} \dot{\psi}(x_0^0, \vec{x}))^* \right] \right\}$$ 

$$= \frac{\kappa}{2M} \left\{ |\hat{D}^{1/4} \psi_c(x_0^0, \vec{x})|^2 + |\hat{D}^{1/4} \dot{\psi}_c(x_0^0, \vec{x})|^2 + \right.$$ 

$$\left. a \left[ (\hat{D}^{1/4} \psi(x_0^0, \vec{x}))^* \hat{D}^{1/4} \dot{\psi}_c(x_0^0, \vec{x}) + (\hat{D}^{1/4} \psi(x_0^0, \vec{x}))(\hat{D}^{1/4} \dot{\psi}_c(x_0^0, \vec{x}))^* \right] \right\}. \hfill (66)$$
For a measurement made at \( t = x^0/c \) we therefore have

\[
\rho_a(x) = \frac{\kappa}{2M} \left\{ |\hat{D}^{1/4}\psi(x)|^2 + |\hat{D}^{-1/4}\dot{\psi}(x)|^2 + ia \left[ (\hat{D}^{1/4}\psi(x))^*\hat{D}^{-1/4}\dot{\psi}(x) - (\hat{D}^{1/4}\dot{\psi}(x))(\hat{D}^{-1/4}\dot{\psi}(x))^* \right] \right\}
\]

\[
= \frac{\kappa}{2M} \left\{ |\hat{D}^{1/4}\psi(x)|^2 + |\hat{D}^{1/4}\dot{\psi}_c(x)|^2 + a \left[ (\hat{D}^{1/4}\psi(x))^*\hat{D}^{1/4}\dot{\psi}_c(x) + (\hat{D}^{1/4}\dot{\psi}(x))(\hat{D}^{-1/4}\dot{\psi}_c(x))^* \right] \right\}.
\]

(67)

For a positive-energy Klein-Gordon field, (60) reduces to a probability density originally introduced by Rosenstein and Horwitz [26] by restricting the second-quantized scalar field theory to its one-particle sector. This coincidence may be viewed as a verification of the validity of our approach: The first-quantized theory formulated by an explicit construction of the Hilbert space and a position observable reproduces a result obtained from the second-quantized theory.

We can use the method discussed in Section 2 to also define a current density \( J_\mu^a \) such that \( J_0^a = \rho_a \). As we show in Appendix D, this yields

\[
J_\mu^a(x) = \frac{\kappa}{2M} \Im \left\{ (\hat{D}^{1/4}\psi(x))^* \partial^\mu \hat{D}^{-1/4}\dot{\psi}_c(x) - (\hat{D}^{1/4}\dot{\psi}(x))\partial^\mu (\hat{D}^{-1/4}\dot{\psi}(x))^* + a \left[ (\hat{D}^{1/4}\psi(x))^* \partial^\mu \hat{D}^{1/4}\dot{\psi}_c(x) + (\hat{D}^{1/4}\dot{\psi}(x))(\hat{D}^{-1/4}\dot{\psi}_c(x))^* \right] \right\},
\]

(68)

where \( \Im \) stands for the imaginary part of its argument. Furthermore, the probability current density \( J_\mu^a \) has the correct nonrelativistic limit: Setting \( \kappa = 1/(1 + a) \) yields

\[
\lim_{c \to \infty} J_0^a(x^0, \vec{x}) = \rho(x^0, \vec{x}), \quad \lim_{c \to \infty} \vec{J}_a(x^0, \vec{x}) = \frac{1}{c} \vec{j}(x^0, \vec{x}),
\]

(69)

where \( \rho \) and \( j \) are the classical scalar and current probability densities given by (32) and (33), respectively.

If we restrict to the positive-energy Klein-Gordon fields and set \( a = 0 \), Eq. (68) reduces to the probability current density obtained by Rosenstein and Horwitz in [26]. As also indicated by these authors, the resulting current density, namely \( J_0^a \), does not satisfy the continuity
equation. Hence it is not a conserved current. Furthermore, as we show in Appendix D, $\mathcal{J}_0^\mu$ is indeed not even a four-vector field. The same lack of covariance and conservation applies to $\mathcal{J}_a^\mu$ for $a \neq 0$. The only advantage of $\mathcal{J}_a^\mu$ over $J_a^\mu$ is that, unlike the latter which is generally complex-valued, the former is manifestly real-valued and positive-definite.

The non-conservation (respectively non-covariance) of the probability current density $\mathcal{J}_a^\mu$ raises the paradoxical possibility of the non-conservation (respectively frame-dependence) of the total probability:

$$ P_a := \int_{\mathbb{R}^3} d^3 \vec{x} \, \rho_a(x^0, \vec{x}). $$

(70)

It turns out that indeed the latter is a frame-independent conserved quantity, thanks to the covariance and conservation of the current density $J_a^\mu$ and the identity

$$ \int_{\mathbb{R}^3} d^3 \vec{x} \, \rho_a(x^0, \vec{x}) = \int_{\mathbb{R}^3} d^3 \vec{x} \, J_a^0(x^0, \vec{x}), $$

(71)

which follows from (14), (67) and the fact that $D^{\pm1/4}$ is a self-adjoint operator acting in $L^2(\mathbb{R})$. In a sense, $\rho_a(x)$ and $J_a^0(x)$ differ only by a “boundary term”.

Combining (70) and (71), we have

$$ P_a = \int_{\mathbb{R}^3} d^3 \vec{x} \, J_a^0(x^0, \vec{x}). $$

(72)

This relation implies that although the probability density $\rho_a$ is not the zero-component of a conserved four-vector current density, its integral over the whole space that yields the total probability (70) is nevertheless conserved. Furthermore, this global conservation law stems from a local conservation law, i.e., a continuity equation for a four-vector current density namely $J_a^\mu$.

4 Gauge Symmetry Associated with the Conservation of the Total Probability

The fact that the conservation of the total probability $P_a$ has its root in the local conservation of the covariant current $J_a^\mu$ suggests, by virtue of the Nöther’s theorem, the presence of an
underlying gauge symmetry. In order to determine the nature of this symmetry, we make use of the well-known fact that the conserved charge associated with any conserved current is the generator of the infinitesimal gauge transformations. The specific form of the latter is most conveniently obtained in the Hamiltonian formulation.

The Lagrangian $L$ for a free Klein-Gordon field $\psi$ and the corresponding canonical momenta $\pi(\vec{x})$, $\bar{\pi}(\vec{x})$ associated with $\psi(\vec{x}) := \psi(x^0, \vec{x})$ and $\psi^*(\vec{x}) := \psi^*(x^0, \vec{x})$ are respectively given by:

$$L := -\frac{\lambda}{2} \int_{\mathbb{R}^3} d^3\vec{x} \left\{ \partial_{\mu} \psi(\vec{x}) \partial^{\mu} \psi(\vec{x}) + M^2 \psi(\vec{x}) \psi(\vec{x}) \right\}, \quad (73)$$

$$\pi(\vec{x}) := \frac{\delta L}{\delta \dot{\psi}(\vec{x})} = \frac{\lambda}{2} \dot{\psi}(\vec{x}), \quad \bar{\pi}(\vec{x}) := \frac{\delta L}{\delta \dot{\psi}^*(\vec{x})} = \frac{\lambda}{2} \dot{\psi}(\vec{x}) = \pi^*(\vec{x}), \quad (74)$$

where $\lambda := \hbar c/M = \hbar^2/m$ and we have suppressed the $x^0$-dependence of the fields for simplicity.

In terms of the canonical phase space variables $(\psi, \pi)$ and $(\psi^*, \pi^*)$, the conserved charge for $J_a^\mu$, namely the total probability $(72)$, takes the form

$$P_a = \frac{\kappa}{2M} \int_{\mathbb{R}^3} d^3\vec{x} \left\{ \psi(\vec{x}) \hat{D}^{1/2} \psi(\vec{x}) + 4 \lambda^{-2} \pi(\vec{x}) \hat{D}^{-1/2} \pi^*(\vec{x}) + 2i \lambda^{-1} a[\psi(\vec{x})^* \pi(\vec{x})^* - \psi(\vec{x}) \pi(\vec{x})] \right\}, \quad (75)$$

where we have made use of (14) and (74).

We can obtain the infinitesimal symmetry transformation,

$$\psi \rightarrow \psi + \delta \psi, \quad (76)$$

generated by $P_a$ using

$$\delta \psi(\vec{x}) = \{\psi(\vec{x}), P_a\} \delta \xi, \quad (77)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket:

$$\{A, B\} := \int_{\mathbb{R}^3} d^3\vec{x} \left[ \frac{\delta A}{\delta \psi(\vec{x})} \frac{\delta B}{\delta \pi(\vec{x})} - \frac{\delta B}{\delta \psi(\vec{x})} \frac{\delta A}{\delta \pi(\vec{x})} + \frac{\delta A}{\delta \psi^*(\vec{x})} \frac{\delta B}{\delta \pi^*(\vec{x})} - \frac{\delta B}{\delta \psi^*(\vec{x})} \frac{\delta A}{\delta \pi^*(\vec{x})} \right], \quad (78)$$

$A, B$ are observables, and $\delta \xi$ is an infinitesimal real parameter. In view of (14) – (18), we have

$$\delta \psi(\vec{x}) = \frac{\delta P_a}{\delta \pi(\vec{x})} \delta \xi = \frac{\kappa}{M \lambda} \left[ \hat{D}^{-1/2} \dot{\psi}(\vec{x}) - ia \psi(\vec{x}) \right] \delta \xi.$$
We may employ (19) to further simplify this expression. The result is
\[ \delta \psi(\vec{x}) = -i \delta \theta (C + a) \psi(\vec{x}), \quad (79) \]
where \( \delta \theta := \kappa \delta \xi / (\mathcal{M} \lambda) = \kappa \delta \xi / (\hbar c). \)

According to (79), the symmetry transformations (76) are generated by the operator \( C + a. \) One can easily exponentiate the latter to obtain the following expression for the corresponding non-infinitesimal symmetry transformations.
\[ \psi \rightarrow e^{-i \theta (C + a)} \psi = e^{-ia \theta} e^{-i \delta C} \psi = e^{-ia \theta} [\cos \theta - i \sin \theta \, C] \psi. \quad (80) \]
where \( \theta \in \mathbb{R} \) is arbitrary and we have made use of \( C^2 = 1. \) In terms of the positive- and negative-energy components \( \psi_\pm \) of \( \psi. \) The expression (80) takes the form
\[ \psi = \psi_+ + \psi_- \rightarrow e^{-i(a + 1) \theta} \psi_+ + e^{-i(a - 1) \theta} \psi_- = \sum_{\epsilon = \pm} e^{-i(a + \epsilon) \theta} \psi_\epsilon. \quad (81) \]

It is not difficult to see from (80) and (81) that the gauge group \( G_a \) associated with these transformations is a one-dimensional connected Abelian Lie group. Therefore, it is isomorphic to either of \( U(1) \) or \( \mathbb{R}^+ \), the latter being the noncompact multiplicative group of positive real numbers, \( \mathbb{R}^+ \).

We can construct a simple model for (faithful representation of) the group \( G_a \) using the two-component representation \( \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \). Then \( C \) is represented by the diagonal Pauli matrix \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and a typical element of \( G_a \) takes the form
\[ g_a(\theta) := \begin{pmatrix} e^{-i(a + 1) \theta} & 0 \\ 0 & e^{-i(a - 1) \theta} \end{pmatrix}. \quad (82) \]
This expression suggests that the gauge group \( G_a \) is a subgroup of \( U(1) \times U(1). \) It is not difficult to show that \( G_a \) is a compact subgroup of this group and consequently isomorphic to \( U(1). \) Here we identify the gauge group with its connected component that includes the identity and is obtained by exponentiating the generator \( C + a. \)
the group $U(1)$ if and only if the parameter $a$ is a rational number. This in turn implies that for irrational $a$ the group $G_a$ is isomorphic to $\mathbb{R}^+$.\footnote{In this case, although $G_a$ is (isomorphic to) an abstract subgroup of $U(1) \times U(1)$ it fails to be a Lie subgroup of this group.}

We can easily construct a concrete example of these isomorphisms. For a rational $a$, we have $a = m/n$ where $m$ and $n$ are relatively prime integers with $n$ positive. In this case we let $u_a : G_a \to U(1)$ be defined by $u_a(e^{-i\theta(a+C)}) := e^{-i\theta/n}$. For an irrational $a$, we define $v_a : G_a \to \mathbb{R}^+$ according to $v_a(e^{-i\theta(a+C)}) := e^\theta$. Then it is an easy exercise to show that both $u_a$ and $v_a$ are (Lie) group isomorphisms.

Clearly, the $G_a$ gauge symmetry associated with the conservation of the total probability, alternatively the current density $J^\mu_a$, is a global gauge symmetry. Similarly to the $U(1)$ gauge symmetry associated with the Klein-Gordon current, namely the one responsible for the electric charge conservation, one may consider allowing for $(x^0, \vec{x})$-dependent gauge parameters: $\theta = \theta(x)$, i.e., consider local $G_a$ gauge transformations. One then expects that the imposition of this local gauge symmetry should lead to a gauged Klein-Gordon equation involving a gauge field that couples to the current $J^\mu_a$. The naive minimal coupling prescription, however, fails because it makes the generator $C + a$ of $G_a$ generally $(x^0, \vec{x})$-dependent. In this respect the local $G_a$ gauge symmetry is different from the usual local Yang-Mills-type gauge symmetries.

The group $U(1) \times U(1)$ that enters the above discussion of the gauge group $G_a$ as an embedding group is also a group of gauge transformations of the Klein-Gordon fields. It corresponds to the global $G_a$ gauge transformations supplemented with the global $U(1)$ gauge transformations associated with the conservation of the electric charge.\footnote{In this case, although $G_a$ is (isomorphic to) an abstract subgroup of $U(1) \times U(1)$ it fails to be a Lie subgroup of this group.}
5 Klein-Gordon Fields in a Background Electromagnetic Field

A scalar field $\psi$ minimally coupled to a background electromagnetic field $A_\mu$ satisfies

$$\left\{ [-i\partial_\mu - qA_\mu(x)][-i\partial^\mu - qA^\mu(x)] + M^2 \right\} \psi(x) = 0, \quad (83)$$

where $q := e/(\hbar c)$, $e$ is the electric charge, and $A_\mu$ is assumed to be real-valued.

Supposing that for all $x^0$, $\psi(x^0, \vec{x})$ and $\dot{\psi}(x^0, \vec{x})$ are square-integrable functions, we can easily write Eq. (83) as an ordinary differential equation in $L^2(\mathbb{R}^3)$. The latter takes the form

$$\ddot{\psi}(x^0) + 2iq \varphi(x^0, \vec{x}) \dot{\psi}(x^0) + D\psi(x^0) = 0, \quad (84)$$

where $\varphi := A^0$ is the scalar potential, $D : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the operator:

$$(D\phi)(\vec{x}) := \left\{ -[\vec{\nabla} - iq\vec{A}(x^0, \vec{x})]^2 + iq\dot{\varphi}(x^0, \vec{x}) - q^2\varphi(x^0, \vec{x})^2 + M^2 \right\} \phi(\vec{x}) \quad \forall \phi \in L^2(\mathbb{R}^3), \quad (85)$$

and $\vec{A} = (A^1, A^2, A^3)$ is the vector potential.

Equation (84) takes the form of the Klein-Gordon equation (6) for a free scalar field provided that we make the gauge transformation:

$$\psi(x^0) \to \chi(x^0) := u(x^0, \vec{x})\psi(x^0), \quad u(x^0, \vec{x}) := \exp \left[ iq \int_{x_0^0}^{x^0} d\tau \varphi(\tau, \vec{x}) \right], \quad (86)$$

where $x^0_0 \in \mathbb{R}$ is an arbitrary but fixed initial value of $x^0$. Substituting (85) and (86) in (84) and doing the necessary algebra, we find

$$\ddot{\chi}(x^0) + D_q \chi(x^0) = 0, \quad (87)$$

where $D_q : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the operator

$$(D_q\phi)(\vec{x}) := \hat{D}_q\phi(\vec{x}) \quad \forall \phi \in L^2(\mathbb{R}^3), \quad (88)$$

and

$$\hat{D}_q := u(x^0, \vec{x}) \left\{ -[\vec{\nabla} - iq\vec{A}(x^0, \vec{x})]^2 + M^2 \right\} u(x^0, \vec{x})^{-1}. \quad (89)$$
It is not difficult to observe that indeed $D_q$ is a positive-definite operator acting in $L^2(\mathbb{R}^3)$. Therefore, according to the terminology of Refs. [16, 17], Eq. (87) is an example of a Klein-Gordon-type field equation. The main difference between the free Klein-Gordon equation (6) and Eq. (87) is that the latter is a nonstationary Klein-Gordon-type equation unless $\varphi = 0$ and $\dot{A} = 0$. These conditions are fulfilled only if the electric field $\vec{E} = -(\dot{A} + \vec{\nabla}\varphi)$ vanishes and the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ is time-independent.

For a scalar field interacting with an arbitrary stationary magnetic field we can apply the results of Sections 2 and 3 by simply replacing the operator $D$ by $D$ (and noting that $\varphi = 0$ and $\vec{A} = \vec{A}(x)$) or equivalently by enforcing the minimal coupling prescription: $\vec{\nabla} \rightarrow \vec{\nabla} - iq\vec{A}(x)$.

If either a nonzero electric field or a nonstationary magnetic field is present, then the transformed field $\chi$ is a nonstationary Klein-Gordon-type field and one must employ the quantum mechanics of such fields as outlined in Ref. [17].

6 Concluding Remarks

The first-quantized relativistic quantum mechanics for free scalar fields may be formulated by constructing a genuine Hilbert space of the solutions of the Klein-Gordon equation. This involves endowing the solution space of this equation with a positive-definite inner product. The requirements that this inner product be well-defined, positive-definite, and relativistically invariant fix it up to an arbitrary real parameter $a \in (-1, 1)$ and an overall trivial coefficient $\kappa \in \mathbb{R}^+$. The resulting family of inner products, $(\cdot, \cdot)_a$, actually define unitarily equivalent Hilbert spaces $\mathcal{H}_a$ and therefore are physically identical, [17]. Furthermore, they may be used to construct a conserved four-vector current density $J^\mu_a$ that tends to the probability current density for position measurements in nonrelativistic limit. A by-product of the construction of $J^\mu_a$ is a manifestly covariant expression for the inner product $(\cdot, \cdot)_a$.

In view of the unitary-equivalence of $\mathcal{H}_a$ and $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, one can use the ordinary position operator for a nonrelativistic spin-1/2 particle, that acts in $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, to de-
fine a relativistic position operator for Klein-Gordon fields $\psi$. This in turn yields a position basis in which $\psi$ is uniquely determined by a set of wave functions $f(\epsilon, \vec{x})$. In terms of these wave functions the probability density for the localization of $\psi$ in space takes the same form as in nonrelativistic quantum mechanics. By expressing $f(\epsilon, \vec{x})$ directly in terms of $\psi$, one obtains a manifestly positive-definite probability density $\rho_a$. This turns out to coincide with the Rosenstein-Horwitz probability density \[26\], if one sets $a = 0$ and restricts to the positive-energy Klein-Gordon fields. One can define a current density $J^\mu_a$ whose zero-component equals $\rho_a$. But $J^\mu_a$ is neither covariant nor conserved.

The probability density $\rho_a = J^0_a$ may be linked with the zero-component $J^0_a$ of the conserved current density $J^\mu_a$ in the sense that their integrals over the whole space are identical. This in particular ensures the conservation and frame-independence of the total probability. It also allows for the interpretation of the continuity equation satisfied by $J^\mu_a$ as a local manifestation of the (global) conservation law for the total probability. The latter stems from an underlying Abelian global gauge-symmetry of the Klein-Gordon equation. The nature of the corresponding gauge group $G_a$ depends on the parameter $a$. For rational values of $a$, $G_a = U(1)$; for irrational values of $a$, $G_a = \mathbb{R}^+$. The expression for the current densities $J^\mu_a$ obtained for free Klein-Gordon fields may be easily generalized to scalar fields interacting with a stationary background magnetic field. For a more general background electromagnetic field, the scalar field may be gauge-transformed to a nonstationary Klein-Gordon-type field. Therefore, in order to understand first-quantized scalar fields interacting with such an electromagnetic field one should employ the quantum mechanics of nonstationary Klein-Gordon-type fields \[17\]. This requires a separate investigation of its own and will be dealt with elsewhere. Perhaps a more interesting subject of future study is the local analog of the above-described global $G_a$ gauge symmetry.
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Appendices

A Lorentz-Invariance of $D^{-1/2} \dot{\psi}$

Consider performing an infinitesimal Lorentz transformation

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu, \quad \Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu,$$

(90)

where $\omega^\mu_\nu$ are the antisymmetric generators of the Lorentz transformation \[ [32] \] and $|\omega^\mu_\nu| \ll 1$. The latter condition means that we can safely neglect the second and higher order terms in powers of $\omega^\mu_\nu$.

It is not difficult to show using (90) and its inverse transformation, namely

$$x'^\mu \rightarrow x^\mu = (\Lambda^{-1})^\mu_\nu x'^\nu = x^\mu - \omega^\mu_\nu x'^\nu,$$

(91)

that

$$\nabla'^2 := \partial'^i \partial'_i = \nabla^2 - 2 \omega^\mu_\nu \partial_\mu \partial'_\nu = \nabla^2 + 2 \omega^\mu_0 \partial_\mu \partial^0.$$

(92)

Here to establish the last equality we have added and subtracted $2\omega^\mu_0 \partial_\mu \partial^0$ and employed the antisymmetry of $\omega^\mu_\nu$.

Now, we substitute (92) in $D' = \mathcal{M}^2 - \nabla'^2$ to obtain

$$\hat{D}'^\alpha = \hat{D}^\alpha - 2\alpha \omega^\mu_0 \hat{D}^\alpha^{-1} \partial_\mu \partial^0 \quad \forall \alpha \in \mathbb{R}.$$

(93)

Moreover, using (91) we have

$$\dot{\psi}'(x') = \partial'_0 \psi'(x') = \dot{\psi}(x) - \omega^\mu_0 \partial_\mu \psi(x).$$

(94)
Finally, in view of (93), (94), and the Klein-Gordon equation (2), we find
\[
\hat{D}^{-1/2}\dot{\psi}^\prime(x^\prime) = \left(\hat{D}^{-1/2} + \omega_\mu^0\hat{D}^{-3/2}\partial_\mu\partial^0\right)\left(\dot{\psi}(x) - \omega_\mu^0\partial_\mu\dot{\psi}(x)\right)
\]
\[
= \hat{D}^{-1/2}\dot{\psi}(x) - \omega_\mu^0\left(\hat{D}^{-3/2}\partial_\mu\dot{\psi}(x) + \hat{D}^{-1/2}\partial_\mu\psi(x)\right)
\]
\[
= \hat{D}^{-1/2}\dot{\psi}(x).
\]
Therefore \(\hat{D}^{-1/2}\dot{\psi}(x)\) is Lorentz-invariant.

**B Nonrelativistic Limit of \(J_\alpha^\mu\)**

Let \(\psi\) be a Klein-Gordon field and define \(\chi : \mathbb{R}^4 \rightarrow \mathbb{C}\) according to
\[
\chi(x^0, \vec{x}) := e^{iMx^0}\psi(x^0, \vec{x}).
\]
(95)

Then as it is well-known [1], in the nonrelativistic limit as \(c \rightarrow \infty\), \(\chi(x)\) satisfies the nonrelativistic free Schrödinger equation: \(\dot{\chi} = i\frac{2}{M}\nabla^2\chi\), and
\[
\lim_{c \rightarrow \infty} \dot{\psi}(x) = e^{-iMx^0}\left\{-iM\chi(x) + i\frac{1}{2M}\nabla^2\chi(x)\right\},
\]
(96)

Furthermore,
\[
\lim_{c \rightarrow \infty} \hat{D}^{-1/2} = M^{-1} + \frac{1}{2}M^{-3}\nabla^2.
\]
(97)

Equations (19), (28), (96), and (97) imply
\[
\lim_{c \rightarrow \infty} \psi_c = \psi, \quad \lim_{c \rightarrow \infty} \tilde{\psi}_a = (1 + a)\psi.
\]
(98)

Substituting these relations in (27), we find
\[
\lim_{c \rightarrow \infty} J_\alpha^\mu(x) = -\frac{i\kappa(1 + a)}{2M} \lim_{c \rightarrow \infty} \left[\psi(x)^\ast \frac{\partial^\mu}{\partial\mu} \psi(x)\right],
\]
(99)

If we set \(\kappa = 1/(1 + a)\) and \(\mu = 1, 2, 3\) in this expression, we obtain (31). Similarly using (96) and (99), we arrive at (30). This ends our demonstration that \(J_\alpha^\mu\) has the correct nonrelativistic limit.
C Real and Imaginary Parts of $J^\mu_a$

Given a free Klein-Gordon field $\psi$, we can use (53) to write

$$\psi = \psi_+ + \psi_-,$$
$$\psi_c = \psi_+ - \psi_-.$$

Substituting these relations and (28) in (27) and carrying out the necessary calculations, we find the following expressions for the real and imaginary parts of $J^\mu_a$.

$$\Re(J^\mu_0) = \frac{\kappa}{M} \Im \left[ (1 + a) \psi^*_+ \partial^\mu \psi_+ - (1 - a) \psi^*_- \partial^\mu \psi_- + a(\psi^*_+ \partial^\mu \psi_- + \psi^*_- \partial^\mu \psi_+) \right],$$

$$\Im(J^\mu_0) = \frac{\kappa}{M} \Re(\psi^*_+ \partial^\mu \psi_- - \psi^*_- \partial^\mu \psi_+) = \frac{\kappa}{M} \Re(\psi^*_+ \partial^\mu \psi_-).$$

Here $\Re$ and $\Im$ stand for the real and imaginary part of their arguments, respectively.

Note that for a Klein-Gordon field with a definite charge-parity $\epsilon$, i.e., for a positive- or negative-energy field, $J^\mu_0$ is real and up to a real coefficient, namely $\pm(1 \pm a)$, coincides with the Klein-Gordon current density. However, due to the particular sign of this coefficient and the fact that $|a| < 1$, $J^0_0$ is positive-definite for both the positive- and negative-energy plane-wave solutions of the Klein-Gordon equation.

Another interesting case is that of the real Klein-Gordon fields for which $\psi_- = \psi^*_+$. Then in view of (100) and (101), $J^\mu_0 = -(i\kappa/M) \psi^*_+ \partial^\mu \psi_+$ which is again real, but unlike the Klein-Gordon current density it does not vanish. Note that in this case $J^\mu_0$ is independent of $a$. This was to be expected, because $a$ enters in the expression (26) for $J^\mu_0$ as the coefficient of a term which is essentially the Klein-Gordon current density.

D Derivation and Properties of $\mathcal{J}^\mu_a$

The derivation of $\mathcal{J}^\mu_a$ mimics that of $J^\mu_a$. First, we identify $\mathcal{J}^\mu_0$ with $\rho_a$ of Eq. (67). In order to obtain the spatial components of $\mathcal{J}^i_a$ of $\mathcal{J}^\mu_a$, we then perform an infinitesimal Lorentz boost
transformation (15) which yields

\[ \mathcal{J}_a^0(x) \rightarrow \mathcal{J}_a^0(x') = \mathcal{J}_a^0(x) - \beta \cdot \tilde{\mathcal{J}}_a(x). \]  \hfill (102)

Next, we use (67) to read off the expression for \( \mathcal{J}_a^0(x') \), namely

\[
\mathcal{J}_a^0(x') = \frac{k}{2M} \left\{ |\hat{D}^{1/4}\psi'(x')|^2 + |\hat{D}^{1/4}\psi_c'(x')|^2 + a \left[ (\hat{D}^{1/4}\psi'(x'))^* \hat{D}^{1/4}\psi_c'(x') + (\hat{D}^{1/4}\psi_c'(x'))(\hat{D}^{1/4}\psi_c'(x'))^* \right] \right\}. \hfill (103)
\]

In view of Eq. (22) and the fact that \( \psi(x) \) and \( \psi_c(x) \) are scalars, we further have

\[
|\hat{D}^{1/4}\psi'(x')|^2 = |\hat{D}^{1/4}\psi(x)|^2 - \frac{1}{2} \beta \cdot \left\{ (\hat{D}^{1/4}\psi(x))^* \nabla \hat{D}^{-3/4}\dot{\psi}(x) + \hat{D}^{1/4}\psi(x) \nabla (\hat{D}^{-3/4}\dot{\psi}(x))^* \right\}, \hfill (104)
\]

\[
|\hat{D}^{1/4}\psi_c'(x')|^2 = |\hat{D}^{1/4}\psi_c(x)|^2 - \frac{1}{2} \beta \cdot \left\{ (\hat{D}^{1/4}\psi_c(x))^* \nabla \hat{D}^{-3/4}\dot{\psi}_c(x) + \hat{D}^{1/4}\psi_c(x) \nabla (\hat{D}^{-3/4}\dot{\psi}_c(x))^* \right\}, \hfill (105)
\]

\[
(\hat{D}^{1/4}\psi'(x'))^* \hat{D}^{1/4}\psi_c'(x') = (\hat{D}^{1/4}\psi(x))^* \hat{D}^{1/4}\psi_c(x) - \frac{1}{2} \beta \cdot \left\{ (\hat{D}^{1/4}\psi(x))^* \nabla \hat{D}^{-3/4}\dot{\psi}_c(x) + \hat{D}^{1/4}\psi_c(x) \nabla (\hat{D}^{-3/4}\dot{\psi}_c(x))^* \right\}, \hfill (106)
\]

\[
(\hat{D}^{1/4}\psi'(x'))(\hat{D}^{1/4}\psi_c'(x'))^* = (\hat{D}^{1/4}\psi(x))(\hat{D}^{1/4}\psi_c(x))^* - \frac{1}{2} \beta \cdot \left\{ (\hat{D}^{1/4}\psi_c(x))^* \nabla \hat{D}^{-3/4}\dot{\psi}_c(x) + \hat{D}^{1/4}\psi_c(x) \nabla (\hat{D}^{-3/4}\dot{\psi}_c(x))^* \right\}. \hfill (107)
\]

Now, substituting (104) - (107) in (103) and making use of (19) and (102), we obtain

\[
\tilde{\mathcal{J}}_a(x) = -\frac{i\kappa}{4M} \left\{ (\hat{D}^{1/4}\psi(x))^* \nabla \hat{D}^{-1/4}\psi_c(x) - \hat{D}^{1/4}\psi(x) \nabla (\hat{D}^{-1/4}\psi_c(x))^* + (\hat{D}^{1/4}\psi_c(x))^* \nabla \hat{D}^{-1/4}\psi(x) - \hat{D}^{1/4}\psi_c(x) \nabla (\hat{D}^{-1/4}\psi(x))^* + a \left[ (\hat{D}^{1/4}\psi(x))^* \nabla \hat{D}^{-1/4}\psi_c(x) - \hat{D}^{1/4}\psi(x) \nabla (\hat{D}^{-1/4}\psi_c(x))^* + (\hat{D}^{1/4}\psi_c(x))^* \nabla \hat{D}^{-1/4}\psi_c(x) - \hat{D}^{1/4}\psi_c(x) \nabla (\hat{D}^{-1/4}\psi_c(x))^* \right] \right\}. \hfill (108)
\]
This relation suggests
\begin{align}
\mathcal{J}_a^\mu(x) &= -\frac{i\kappa}{2\mathcal{M}} \left\{ (\hat{D}^{1/4}\psi(x))^* \partial^\mu \hat{D}^{-1/4}\psi_c(x) - \hat{D}^{1/4}\psi(x) \partial^\mu (\hat{D}^{-1/4}\psi_c(x))^* + \\
&\quad (\hat{D}^{1/4}\psi_c(x))^* \partial^\mu \hat{D}^{-1/4}\psi(x) - \hat{D}^{1/4}\psi_c(x) \partial^\mu (\hat{D}^{-1/4}\psi(x))^* + \\
&\quad a \left[ (\hat{D}^{1/4}\psi(x))^* \partial^\mu \hat{D}^{-1/4}\psi_c(x) - \hat{D}^{1/4}\psi_c(x) \partial^\mu (\hat{D}^{-1/4}\psi_c(x))^* + \\
&\quad (\hat{D}^{1/4}\psi_c(x))^* \partial^\mu \hat{D}^{-1/4}\psi(x) - \hat{D}^{1/4}\psi_c(x) \partial^\mu (\hat{D}^{-1/4}\psi(x))^* \right] \right\},
\end{align}
which we can also write as
\begin{align}
\mathcal{J}_a^\mu(x) &= \frac{\kappa}{2\mathcal{M}} \left\{ (\hat{D}^{1/4}\psi(x))^* \partial^\mu \hat{D}^{-1/4}\psi_c(x) - (\hat{D}^{1/4}\psi_c(x))^* \partial^\mu (\hat{D}^{-1/4}\psi(x)) + \\
&\quad a \left[ (\hat{D}^{1/4}\psi(x))^* \partial^\mu \hat{D}^{-1/4}\psi_c(x) - (\hat{D}^{1/4}\psi_c(x))^* \partial^\mu (\hat{D}^{-1/4}\psi_c(x)) \right] \right\}.
\end{align}

Using Klein-Gordon equation, we can easily check that the expression for $\mathcal{J}_a^\mu(x) = \rho_a(x)$ obtained by setting $\mu = 0$ in (110) agrees with the one given in (67).

Next, we explore the classical limit of the probability current density (110). Following the treatment of Appendix B, we first derive
\begin{align}
\lim_{c \to \infty} \hat{D}^{1/4} &= \mathcal{M}^{1/2} - \frac{1}{4} \mathcal{M}^{-3/2}\nabla^2, \\
\lim_{c \to \infty} \hat{D}^{-1/4} &= \mathcal{M}^{-1/2} + \frac{1}{4} \mathcal{M}^{-5/2}\nabla^2.
\end{align}
Substituting these relations in (110), we find
\begin{align}
\lim_{c \to \infty} \mathcal{J}_a^\mu &= -\frac{i\kappa(1+a)}{2\mathcal{M}} \lim_{c \to \infty} \left[ \psi(x)^* \hat{\partial}^\mu \psi(x) \right].
\end{align}
Now, setting $\kappa = 1/(1+a)$, considering $\mu = 0$ and $\mu \neq 0$ separately, and making use of (55) and (96), we obtain (69):
\begin{align}
\lim_{c \to \infty} \mathcal{J}_a^0(x^0, \vec{x}) &= \rho(x^0, \vec{x}), \\
\lim_{c \to \infty} \mathcal{J}_a(x^0, \vec{x}) &= \frac{1}{c} \vec{j}(x^0, \vec{x}),
\end{align}
where $\rho$ and $\vec{j}$ are the classical scalar and current probability densities given by (32) and (33), respectively.

If we confine our attention to the positive-energy Klein-Gordon fields (for which $\psi_c = \psi$) and set $a = 0$, the expression (109) coincides with the Rosenstein-Horwitz’s current (26):
\begin{align}
\mathcal{J}_{RH}^\mu(x) &= -\frac{i\kappa}{2\mathcal{M}} \left\{ (\hat{D}^{1/4}\psi(x))^* \partial^\mu (\hat{D}^{-1/4}\psi(x)) - (\hat{D}^{1/4}\psi(x))^* \partial^\mu (\hat{D}^{-1/4}\psi(x))^* \right\}.
\end{align}
As we show below, in general $\partial_\mu J_\alpha^\mu \neq 0$. Hence $J_\alpha^\mu$ is not a conserved current density. This was noticed by Rosenstein and Horwitz [26] for the probability current (111). A more dramatic result that seems to be missed by these authors is that $J_\mu^{RH}$ is not even a four-vector field. The same holds for $J_\mu^\alpha$. This can be most conveniently shown by computing $J_\mu^\alpha$ for a superposition of a pair of positive-energy plane-wave Klein-Gordon fields:

$$\psi(x) = c_1 e^{ik_1 \cdot x} + c_2 e^{ik_2 \cdot x} = c_1 e^{-i\omega_1 x^0} e^{i\vec{k}_1 \cdot \vec{x}} + c_2 e^{-i\omega_2 x^0} e^{i\vec{k}_2 \cdot \vec{x}},$$

(112)

where for $\ell \in \{1, 2\}, c_\ell \in \mathbb{C} - \{0\}$, $k_\ell$ is a constant four-vector, $k_\ell \cdot x := (k_\ell)_\mu x^\mu$, and $\omega_\ell := \sqrt{k_\ell^2 + M^2}$. Note that for this choice of the field we have $\psi_c = \psi$ and $J_\mu^0 = J_\mu^{RH}$. Hence, in view of (112) and (111),

$$J_\mu^0(x) = J_\mu^{RH}(x) = \frac{k}{\mathcal{M}} \left\{ |c_1|^2 k_1^\mu + |c_2|^2 k_2^\mu + \Re \left( c_1 c_2^* e^{i(k_1 - k_2) \cdot x} K^\mu \right) \right\},$$

(113)

where

$$K^\mu = \sqrt{\frac{\omega_2}{\omega_1}} k_1^\mu + \sqrt{\frac{\omega_1}{\omega_2}} k_2^\mu.$$  

(114)

Moreover, setting $\psi_c = \psi$ in (110), we find

$$J_\alpha^\mu(x) = (1 + a) J_\mu^0(x).$$

(115)

A direct implication of Eqs. (113) and (115) is that $J_\alpha^\mu$ is a vector field if and only if $K^\mu$ is a four-vector. But as we show next the latter fails to be the case.

Suppose (by contradiction) that $K^\mu$ is a four-vector, then $K_\mu K^\mu$ must be a scalar. It is not difficult to show that

$$K_\mu K^\mu = 2k_1 \cdot k_2 - \mathcal{M}^2 \left( \frac{\omega_1}{\omega_1} + \frac{\omega_1}{\omega_2} \right),$$

(116)

where we have made use of $k_\ell \cdot k_\ell = -\mathcal{M}^2$. For $\omega_1 \neq \omega_2$, the term multiplying $\mathcal{M}^2$ on the right-hand side of (116) fails to be a scalar. This shows that $K_\mu K^\mu$ is not a scalar; $K^\mu$ is not a four-vector; and in general $J_\alpha^\mu$ is not a vector field.
Next, we wish to point out that computing the current density $J^\mu_a$ for the field (112) we find the following manifestly covariant expression.

$$J^\mu_a(x) = \frac{\kappa(1 + a)}{\mathcal{M}} \left\{ |c_1|^2 k_1^\mu + |c_2|^2 k_2^\mu + \Re \left( c_1 c_2^* e^{i(k_1 - k_2).x} \right) (k_1^\mu + k_2^\mu) \right\}. \quad (117)$$

Having obtained the explicit form of both $J^\mu_a$ and $J^\mu_a$ for the field (112), we can easily check their conservation property. A simple calculation shows that

$$\partial_\mu J^\mu_a(x) = (\mathcal{M}^2 + k_1 \cdot k_2) \left( \frac{\omega_1}{\omega_2} - \frac{\omega_2}{\omega_1} \right) \mathcal{F}(x), \quad (118)$$
$$\partial_\mu J^\mu_a(x) = [(k_1 - k_2) \cdot (k_1 + k_2)] \mathcal{F}(x) = 0, \quad (119)$$

where

$$\mathcal{F}(x) := -\frac{\kappa(1 + a)}{\mathcal{M}} \Im \left[ c_1 c_2^* e^{i(k_1 - k_2).x} \right],$$

and we have made use of (115), (113), (114), and (117) and the fact that the term in the square bracket on the right hand side of (119) vanishes identically by virtue of $k_\ell \cdot k_\ell = -\mathcal{M}^2$. According to Eq. (118), for $\omega_1 \neq \omega_2$, $\partial_\mu J^\mu_a(x) \neq 0$. Therefore, unlike $J^\mu_a$, the probability current density $J^\mu_a$ fails to be conserved.
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