Analysis of the Inhomogeneous Willmore Equation

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Abstract: We study a class of fourth-order geometric problems modelling Willmore surfaces, conformally constrained Willmore surfaces, isoperimetrically constrained Willmore surfaces, bi-harmonic surfaces in the sense of Chen, among others. We prove several local energy estimates and derive a global gap lemma.

I Introduction and Main Results

Let Σ be a smooth two-dimensional closed oriented manifold, and let $g_0$ be a smooth reference metric on Σ. For any $s \geq 1$, the Sobolev space $W^{k,p}(Σ, R^s)$ is the space of measurable maps $f : Σ \to R^s$ for which

$$\sum_{j=0}^{k} \int_{Σ} |\nabla^j f|^p_{g_0} \, d\text{vol}_{g_0} < \infty.$$  

For a closed surface Σ, this space is independent of the reference metric $g_0$.

The notion of weak immersion with $L^2$-bounded second fundamental form is well-understood and has been extensively studied (the interested reader will find a detailed account in [Riv3] and the references therein). They will be the main object of study in this paper, and we now recall the main definition. Let $\tilde{Φ} : Σ \to R^m$, for $m \geq 3$, be measurable and Lipschitz. The associated pull-back metric $g := \tilde{Φ}^* g_{R^m}$ is given almost everywhere by

$$g(X,Y) := d\tilde{Φ}(X) \cdot d\tilde{Φ}(Y), \quad \forall \, X,Y \in TΣ,$$

where dot indicates the standard scalar product in $R^m$. Unless otherwise specified, we will demand that $g$ be non-degenerate, that is that there exists a constant $c > 0$ satisfying

$$c^{-1}g_0(X,X) \leq g(X,X) \leq cg_0(X,X), \quad \forall \, X \in TΣ.$$  

This makes $(Σ, \tilde{Φ}^* g_{R^m})$ a Riemannian 2-manifold with a rough metric. The Gauss map is a bounded measurable map $\tilde{n}$ taking values in the Grassmanian $Gr_{m-2}(R^m)$ of oriented $(m-2)$-planes in $R^m$ satisfying

$$\tilde{n} := \ast \frac{\partial_{x_1}\tilde{Φ} \wedge \partial_{x_2}\tilde{Φ}}{|\partial_{x_1}\tilde{Φ} \wedge \partial_{x_2}\tilde{Φ}|},$$

where $\ast$ denotes the standard Hodge star operator, and $\{x^1, x^2\}$ is an arbitrary choice of local coordinates. Finally, to say that the weak immersion $\tilde{Φ}$ has square integrable second fundamental form amounts to requiring that

$$\int_{Σ} |d\tilde{n}|^2 \, d\text{vol}_{g} < \infty.$$  

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We let 
\[ \mathcal{E}_\Sigma := \{ \tilde{\Phi} : \Sigma \to \mathbb{R}^m \text{ measurable and Lipschitz such that } (I.1) \text{ and } (I.2) \text{ hold} \} . \]

Rescaling if necessary, condition (I.2) ensures that on some local patch, let us say it is the unit-disk \( D_1(0) \), there holds
\[ \int_{D_1(0)} |\nabla \tilde{n}|^2 \, dx^1 \, dx^2 < \frac{8\pi}{3} . \tag{I.3} \]
Here \( \{x^1, x^2\} \) are local coordinates on \( D_1(0) \) and \( \nabla \) stands for the usual flat gradient in these coordinates. A well-known result ([MS, Riv2]) states that if \( \tilde{\Phi} \in \mathcal{E}_{D_1(0)} \) satisfies (I.3), then there exists a bi-Lipschitz homeomorphism \( \psi \) of \( D_1(0) \) such that the map \( \tilde{\Phi} \circ \psi : D_1(0) \to \mathbb{R}^m \) is conformal, namely
\[ \partial_{x^i}(\tilde{\Phi} \circ \psi) \cdot \partial_{x^j}(\tilde{\Phi} \circ \psi) = e^{2\lambda} \delta_{ij} , \]
for some conformal factor \( \lambda \). Without loss of generality, as we are only concerned with locally analysing the solutions to problems that are independent of parametrisation, we will henceforth suppose that \( \tilde{\Phi} \) itself is conformal.

The present paper is concerned with studying the local analytical properties of the inhomogeneous Willmore equation. To an immersion \( \tilde{\Phi} \in \mathcal{E}_\Sigma \) of an oriented two-dimensional manifold \( \Sigma \) into \( \mathbb{R}^m \), some \( m \geq 3 \), we assign the second fundamental form \( \tilde{A} := \pi_\tilde{n} D^2 \tilde{\Phi} \), where \( \pi_\tilde{n} \) denotes the projection of vectors in \( \mathbb{R}^m \) onto the \((m-2)\)-place defined by the Gauss map \( \tilde{n} \). The trace of the 2-tensor \( \tilde{A} \) with respect to \( g \) is twice the normal-valued mean curvature vector:
\[ \tilde{H} := \frac{1}{2} \text{Tr}_g \tilde{A} . \]
Willmore immersions are critical points of the Willmore energy
\[ \int_{\Sigma} |\tilde{H}|^2 \, d\text{vol}_g . \]
The study of Willmore immersions has been steadily gaining momentum over the last century. It would be impossible to give a detailed account of the various works and results that have appeared in recent years. We content ourselves with mentioning the tour de force by Marques and Neves in [MN], where they prove the celebrated Willmore conjecture [Wil]: the Clifford torus minimizes, up to Möbius transformations, the Willmore energy in the class of immersed tori in \( \mathbb{R}^3 \). Although the Willmore conjecture is now resolved, the study of Willmore immersions continues to grow in intensity.

Any critical point of the Willmore energy satisfies the following fourth-order, quasi-linear, strongly coupled system of equations [Wil, Wei]:
\[ \Delta_\perp \tilde{H} + \langle \tilde{A} \cdot \tilde{H}, \tilde{A} \rangle_g - 2|\tilde{H}|^2 \tilde{H} = \vec{0} , \tag{I.4} \]
where \( \Delta_\perp \) is the negative covariant Laplacian for the connection in the normal bundle. The dot indicates the standard scalar product of vectors in \( \mathbb{R}^m \), while the product \( \langle \cdot, \cdot \rangle_g \) is the usual contraction product with respect to the metric \( g \) for tensors. Naturally, when constraints are imposed on the problem of varying the Willmore energy, the right-hand side of (I.4) is no longer zero. Various examples are provided in [Ber2] and we will below look closer at a few specific cases of relevance in applications. Thus we are motivated to study a problem of the type
\[ \Delta_\perp \tilde{H} + \langle \tilde{A} \cdot \tilde{H}, \tilde{A} \rangle_g - 2|\tilde{H}|^2 \tilde{H} = \vec{W} , \tag{I.5} \]
where the right-hand side \( \vec{W} \) is assumed to be known. Naturally, \( \vec{W} \) has to be normal vector to make sense. It also has to be independent of parametrization. Before going any further, an important observation is in order. When \( \vec{\Phi} \) lies in \( \mathcal{E}_\Sigma \), it is clear that \( \vec{H} \) is square integrable. Even in the case when \( \vec{W} \equiv \vec{0} \), the term \( |\vec{H}|^2 \vec{H} \) is already problematic, for it lies in no space that enables us to understand the equation in a distributional sense to the equation. Nevertheless, one may study the problem and obtain estimates, as is done for example in [Whe1] and the references therein. Another approach was originally devised by Tristan Riviè\'re in [Riv1]. It relies mainly on the fact that the left-hand side of (I.5) can be factored into an exact divergence, thereby rendering possible the assignment of a distributional sense to (I.4). In [Ber2], it is shown that the divergence structure seemingly hidden in (I.4) is a direct consequence of Noether’s theorem applied to the translation invariance of the Willmore energy. The present paper should be understood as a companion to [Ber2]. While in the latter only identities were derived, the present work brings to fruition the reformulations presented in [Ber2] by obtaining local analytical results for problems of the type (I.5). The present paper should also be seen as a companion to [Whe1], where only a specific class of right-hand sides \( \vec{W} \) were considered. The class of possible right-hand sides will be here significantly expanded.

As was shown in [Riv1], any conformal immersion \( \vec{\Phi} : D_1(0) \to \mathbb{R}^m \) that satisfies the Willmore equation (I.4) also satisfies the equation
\[
\text{div}(\nabla \vec{H} - 2\pi_n \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\Phi}) = \vec{0} \quad \text{on } D_1(0),
\]
where \( \pi_n \) denotes projection on the normal bundle. The operators \( \nabla \) and \( \text{div} \) are understood in local coordinates \( \{x^1, x^2\} \) on the unit disk \( D_1(0) \). This motivates us to consider inhomogeneous Willmore problems of the type
\[
\text{div}(\nabla \vec{H} - 2\pi_n \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\Phi} + \vec{T}) = \vec{v}, \tag{I.6}
\]
for some vector field \( \vec{T} \in \Gamma(\mathbb{R}^2 \otimes \mathbb{R}^m) \) and some normal vector field \( \vec{v} \in \Gamma(\mathbb{R}^m) \). Many known classes of immersed surfaces satisfy a problem of this type.

(1) Willmore immersions with \( \vec{v} \equiv 0 \) and \( \vec{T} \equiv 0 \).

(2) Constrained Willmore immersions.

(i) Varying the Willmore energy \( \int_\Sigma |\vec{H}|^2 \, d\text{vol}_g \) in a fixed conformal class (i.e. with infinitesimal, smooth, compactly supported, conformal variations) gives rise to a more general class of surfaces called \textit{conformally-constrained Willmore surfaces} whose corresponding Euler-Lagrange equation [BPP, KS3, Sch] is expressed as follows. Let \( \vec{h}_0 \) denote the trace-free part of the second fundamental form, namely
\[
\vec{h}_0 := \vec{h} - \vec{H} g.
\]
A conformally-constrained Willmore immersion \( \vec{\Phi} \) satisfies
\[
\Delta \vec{H} + (\vec{H} \cdot \vec{h}_0^i) \vec{h}^i_j - 2|\vec{H}|^2 \vec{H} = (\vec{h}_0)_j^i q^{ij}, \tag{1.7}
\]
where \( q \) is a transverse, traceless symmetric 2-form. This tensor \( q \) plays the role of Lagrange multiplier in the constrained variational problem. It is shown in [Ber1, Ber2] that in a conformal parametrization, with conformal parameter \( \lambda \), (1.7) can be brought in the form (1.6) by setting \( \vec{v} \equiv 0 \) and
\[
\vec{T} = - e^{-2\lambda} M_q \nabla^z \vec{\Phi},
\]
where \( \nabla^z \vec{\Phi} := (\partial_{x^2} \vec{\Phi}, \partial_{x^1} \vec{\Phi}) \), and \( M_q \) is the matrix
\[
M_q := \begin{pmatrix}
-q_{12} & q_{11} \\
q_{11} & q_{12}
\end{pmatrix}
\]
\footnote{i.e. \( q \) is divergence-free: \( \nabla^i q_{ij} = 0 \forall i \).}
(ii) Bilayer models \cite{BWW, Can, Hel}. These models bear also the name of Helfrich, Canham-Helfrich, and arise in the modeling of the surface of liposomes and vesicles (see \cite{Ber2} and the references therein). One seeks to minimize the Willmore energy under the requirement that the area \( A(\Sigma) \), the volume \( V(\Sigma) \), and the total curvature
\[
M(\Sigma) := \int_\Sigma H \, dv_{\text{vol}_g}
\]
be prescribed. This leads to an equation of the type \eqref{eq:Bilayer} with
\[
\vec{W} = 2(\beta + \alpha H + \gamma K) \vec{n},
\]
where \( K \) is the Gauss curvature, and \( \alpha, \beta, \gamma \) are three given parameters acting as Lagrange multipliers.

As shown in \cite{Ber2}, this problem can be brought in the form \eqref{eq:Willmore} with \( \vec{v} \equiv \vec{0} \) and \( |\vec{T}| \lesssim 1 + |\nabla \vec{n}| \).

(iii) Another instance in which minimizing the Willmore energy arises is the isoperimetric problem \cite{KMR, Scy}, which consists in minimizing the Willmore energy under the constraint that the dimensionless isoperimetric ratio
\[
\sigma := \frac{36 \pi V^2}{A^3}
\]
be a given constant in \((0,1]\). As both the Willmore energy and the constraint are invariant under dilation, one might fix the volume \( V = 1/(6 \sqrt{\pi}) \), forcing the area to satisfy \( A = \sigma^{1/3} \). This problem is thus equivalent to the bilayer model with \( \gamma = 0 \) (no constraint imposed on the total curvature, but the volume and area are prescribed separately).

(3) Chen surfaces. An isometric immersion \( \vec{\Phi} : N^n \to \mathbb{R}^{m>n} \) of an \( n \)-dimensional Riemannian manifold \( N^n \) into Euclidean space is called \textit{biharmonic} if the corresponding mean-curvature vector \( \vec{H} \) satisfies
\[
\Delta_g \vec{H} = \vec{0}.
\]
(I.8)

The study of biharmonic submanifolds was initiated by B.-Y. Chen \cite{Che1} in the mid 1980s as he was seeking a classification of the finite-type submanifolds in Euclidean spaces. Independently, G.Y. Jiang \cite{Jia} also studied \eqref{eq:Biharmonic} in the context of the variational analysis of the biharmonic energy in the sense of Eells and Lemaire. Chen conjectures that a biharmonic immersion is necessarily minimal. Smooth solutions of \eqref{eq:Biharmonic} are known to be minimal for \( n = 1 \) \cite{Dim1}, for \( (n, m) = (2, 3) \) \cite{Dim2}, and for \( (n, m) = (3, 4) \) \cite{HV}. In \cite{Wic3}, it is shown that Chen’s conjecture holds up to a growth condition on the Willmore energy, and in \cite{BWW2} the parabolic flow with velocity given by Chen’s operator is studied. Chen’s conjecture has been solved under a variety of hypotheses (see the recent survey paper \cite{Che2}). The statement remains nevertheless open in general, and in particular for immersed surfaces in \( \mathbb{R}^m \). In \cite{Ber2b}, it is shown that Chen surfaces satisfy an equation of the type \eqref{eq:Bilayer} with
\[
|\vec{V}| \simeq |\vec{A}|^3.
\]
(4) Complete Willmore immersions in asymptotically flat spaces also satisfy a problem of the type \eqref{eq:Willmore}. Details may be found in \cite{BR4}.

(5) Equilibria of flow equations. In \cite{KS1}, stability of the sphere is proven for the Willmore flow. Global existence is obtained by contradiction: one assumes that existence time is finite, and then

\begin{footnotesize}
\footnote{The conjecture as originally stated is rather analytically vague: no particular hypotheses on the regularity of the immersion are \textit{a priori} imposed. Many authors consider only smooth immersions.}
\footnote{This paper is the precursor to the published version \cite{Ber2}, which, to the referee’s request, no longer addresses the question of Chen immersions.}
\end{footnotesize}
rescales around a point in space-time where the energy concentrates. Local estimates allow one to construct a blowup. The blowup is shown to be an entire Willmore surface with small energy. To this blowup one applies a gap lemma, that implies any such surface is a standard flat plane. This is in contradiction with the concentration of energy hypothesis, and so no such concentration points can occur, and the flow exists for all time. This argument is by now standard, having been adapted at least to constrained surface diffusion flows [Whe, Whe2], locally constrained Willmore flow [MW2], Willmore flow in Riemannian spaces [Lnk, MWW], and a geometric triharmonic heat flow [MPW].

An appropriate gap lemma combined with local regularity is crucial and so far has been established separately for each of the flows given above. As our work here holds for more general equations than what is currently available, we expect that the results in this paper will apply to a broad class of fourth-order evolution equations. It is an interesting open question to investigate higher-order cases.

Our first main result consists of local energy estimates.

**Theorem I.1** Let \( \Phi \in W^{2,2} \cap W^{1,\infty}(D_1(0), \mathbb{R}^m) \) be a conformal immersion with conformal parameter \( \lambda \) satisfying
\[
\| \nabla \lambda \|_{L^{2,\infty}(D_1(0))} < +\infty ,
\]
where \( L^{2,\infty} \) denotes the weak-L\(^2 \) Marcinkiewicz space. Suppose that
\[
\int_{D_1(0)} |\nabla \bar{n}|^2 \, dx = \varepsilon_0^2 .
\] (1.9)
Provided that \( \varepsilon_0 \) is sufficiently small, there is a universal constant \( C(\varepsilon_0, \| \nabla \lambda \|_{L^{2,\infty}(D_1(0))}) \) for which the following statements hold.

(i) Let \( p \in (1, \infty) \). Suppose that \( \Phi \) is a solution on \( D_1(0) \) of
\[
\text{div}(\nabla \bar{H} - 2\pi \bar{n} \nabla \bar{H} + |\bar{H}|^2 \nabla \Phi + \bar{T}) = 0 .
\]
Then for all \( D_p(x) \subset D_1(0) \), we have
\[
 \rho^{2-\frac{2}{p}} \| \nabla \bar{n} \|_{L^p(D_p(x))} \leq (M + 1)^2 ,
\]
where
\[
M := C(\varepsilon_0, \| \nabla \lambda \|_{L^{2,\infty}(D_1(0))}) \left[ \rho^{2-\frac{2}{p}} \| \nabla \bar{T} \|_{L^p(D_p(x))} + \| \nabla \bar{n} \|_{L^{2,\infty}(D_p(x))} \right].
\]
Moreover,
\[
\begin{align*}
\rho^{2-\frac{2}{p}} \| \nabla \bar{n} \|_{L^{2p/(2-p)}(D_p(x))} & \leq (M + 1)^2 , & \text{if } p \in (1, 2) \\
\rho^{1-\frac{2}{q}} \| \nabla \bar{n} \|_{L^q(D_p(x))} & \leq M + 1 , & \forall \, q < \infty , \text{ if } p = 2 \\
\rho \| \nabla \bar{n} \|_{L^{\infty}(D_p(x))} & \leq M + 1 , & \text{if } p > 2 ,
\end{align*}
\]
and
\[
\begin{align*}
\rho^{2-\frac{2}{p}} \| e^{\lambda} \bar{H} \|_{L^{2p/(2-p)}(D_p(x))} & \leq M(M + 1) , & \text{if } p \in (1, 2) \\
\rho^{1-\frac{2}{q}} \| e^{\lambda} \bar{H} \|_{L^q(D_p(x))} & \leq M , & \forall \, q < \infty , \text{ if } p = 2 \\
\rho \| e^{\lambda} \bar{H} \|_{L^{\infty}(D_p(x))} & \leq M , & \text{if } p > 2 .
\end{align*}
\]
Let $r \in [1, \infty)$. Suppose that $\bar{\Phi}$ is a solution on $D_1(0)$ of
\[
\div (\nabla \bar{H} - 2\pi \bar{\nu} \nabla \bar{H} + |\bar{H}|^2 \nabla \bar{\Phi}) = \bar{\nu},
\]
with $e^{\lambda} \bar{\nu} \in L^r(D_1(0))$. For all $D_\rho(x) \subset D_1(0)$, we have
\[
\begin{cases}
\rho^{-2} \| \nabla^2 \tilde{n} \|_{L^p(D_\rho/2(z))} + \rho^{-2} \| \nabla \tilde{n} \|_{L^{2q/(2-r)}(D_\rho/2(z))} \leq (M + 1)^2, & \forall \ p \in (1, 2) \text{ if } r = 1 \\
\rho^{3-2} \| \nabla^2 \tilde{n} \|_{L^{2q/(2-r)}(D_\rho/2(z))} \leq (M + 1)^2, & \forall \ q < \infty \text{ if } r = 2, \\
\rho^{2-2} \| \nabla^2 \tilde{n} \|_{L^q(D_\rho/2(z))} \leq (M + 1)^2, & \forall \ q < \infty \text{ if } r = 2,
\end{cases}
\]
with
\[
M = C(\varepsilon_0, \| \nabla \lambda \|_{L^{2q}(D_1(0))}) \left[ \rho^{3-2} \| e^{\lambda} \bar{\nu} \|_{L^r(D_\rho(z))} + \| \nabla \tilde{n} \|_{L^2(D_\rho(z))} \right].
\]
Furthermore, if $r > 1$, the following estimates hold:
\[
\rho \| \nabla \tilde{n} \|_{L^q(D_\rho/2(z))} \leq M + 1.
\]
and
\[
\rho^{3-2} \| \nabla^3 \tilde{n} \|_{L^r(D_\rho/2(z))} \leq (M + 1)^3.
\]

Theorem [1.1] is used to prove the following regularity result:

**Corollary 1.1** Let $\bar{\Phi} \in W^{2, 2} \cap W^{1, \infty}(D_1(0), \mathbb{R}^m)$ be a conformal immersion satisfying (1.10) on the disk $D_1(0)$. If $\bar{T}$ and $\bar{\nu}$ are smooth, so is $\bar{\Phi}$.

Finally, we derive an interesting geometric “gap” result, obtained using the same techniques as those leading to Theorem 1.1.

**Theorem 1.2** Let $\Sigma$ be a connected oriented complete immersed surface in $\mathbb{R}^m$ whose mean curvature vector satisfies an inhomogeneous Willmore problem of the type
\[
\Delta \bar{H} + \langle \bar{A} \cdot \bar{H}, \bar{A} \rangle_g - 2|\bar{H}|^2 \bar{H} = O(|\bar{A}|^3).
\]
There exists an $\varepsilon_0 > 0$ such that if
\[
\int_{\Sigma} |\bar{A}|^2 d\text{vol}_g < \varepsilon_0^2,
\]
then $\Sigma$ is a flat plane.

This gap result is to be compared to the one given in [Whe1] (see also [MW1]).

A word of caution is now in order. Should $\bar{\Phi}$ be a (conformal) Willmore immersion satisfying the small energy condition (1.10), then $\bar{\nu} \equiv 0$ and Theorem 1.1(i) gives the estimate
\[
\| \nabla \bar{n} \|_{L^q(D_\rho/2(z))} \leq C \rho^{-1} (1 + \| \nabla \bar{n} \|_{L^2(D_\rho(z))}).
\]
This estimate, which we will term parametric $\varepsilon$-regularity, is the one that was originally derived by Rivière in [Riv1]. In conformal parametrization, $|\nabla \bar{n}| \simeq e^{\lambda} \bar{A}$, where $\bar{A}$ is the second fundamental form, so the above reads
\[
\| e^{\lambda} \bar{A} \|_{L^q(D_\rho/2(z))} \leq C \rho^{-1} (1 + \| e^{\lambda} \bar{A} \|_{L^2(D_\rho(z))}).
\]
Knowing that our conformal immersion does not "distort" flat disks much, we can further rephrase \( (I.10) \) as
\[
\|\vec{A}\|_{L^\infty(D_{\rho/2}(x))} \leq C\rho^{-1}(1 + \|\vec{A}\|_{L^2(D_{\rho}(x))}).
\] (I.11)
where \( D_{\rho}(x) \) is the metric disk with respect to the induced metric pull-back metric \( g = \tilde{\Phi}^*g_{\mathbb{R}^m} \), and \( L^2_\rho \) is the space \((L^2, d\text{vol}_\rho)\). The estimate \( (I.11) \) is to be compared with Kuwert and Schätzle’s original estimate \([KS1]\), which we will call \emph{ambient \( \varepsilon \)-regularity}, and which states that if \( \tilde{\Phi} : \Sigma \to \mathbb{R}^m \) is a Willmore immersion with
\[
\int_{\tilde{\Phi}^{-1}(B_{\varepsilon}(p))} |\vec{A}|^2 d\text{vol}_\rho < \varepsilon_0^2
\]
for some Euclidean ball \( B_{\varepsilon}(p) \subset \mathbb{R}^m \), and \( \varepsilon_0 \) is sufficiently small, then
\[
\|\vec{A}\|_{L^\infty(\tilde{\Phi}^{-1}(B_{\varepsilon}(p)))} \leq C\varepsilon^{-1}\|\vec{A}\|_{L^2(\tilde{\Phi}^{-1}(B_{\varepsilon}(p)))}.
\] (I.12)
Remark 2.11 in \([KS1]\) and more explicitly equation (2.18) in \([KS2]\) state that this estimate implies
\[
\|\vec{A}\|_{L^\infty(D_{\rho/2}(x))} \leq C\rho^{-1}\|\vec{A}\|_{L^2(D_{\rho}(x))},
\] (I.13)
which is manifestly different from \( (I.11) \). To the authors’ knowledge, it is unclear that \( (I.13) \) follows from \( (I.12) \). The two versions of the \( \varepsilon \)-regularity, parametric and ambient, are resolutely distinct and we do not know how to recover one from the other. We suspect that \( (I.13) \) might in fact be false, although what does remain true, as given in Theorem \( (I.4) \) (i), is
\[
\|e^\lambda \vec{H}\|_{L^\infty(D_{\rho/2}(x))} \leq C\rho^{-1}\|
abla \vec{n}\|_{L^2(D_{\rho}(x))}.
\]
Fortunately, this estimate is in fact all which is required to correct the proof of \([KS2]\) and the end-results remain intact. A similar inconsequent error is found in \([BR2]\).

**II** Proofs of the Results

**II.1** Controlling the conformal factor

Using F. Hélein’s method of moving Coulomb frames \([Hel]\), a weak immersion \( \tilde{\Phi} \in W^{2,2}_{\text{imm}}(D_1(0),\mathbb{R}^m) \) of the unit disk \( D_1(0) \) into \( \mathbb{R}^m \) can be reparametrized by a diffeomorphism of \( D_1(0) \) to become conformal. Our problem being independent of parametrization, we will without loss of generality suppose that \( \tilde{\Phi} \) is conformal with parameter \( \lambda \), namely:
\[
\partial_i \tilde{\Phi} \cdot \partial_j \tilde{\Phi} = e^{2\lambda} \delta_{ij}.
\]
We will henceforth use the notation \( \nabla \), \( \text{div} \), and \( \Delta \) to denote the usual gradient, divergence, and Laplacian operators in flat local coordinates \( \{x_1, x_2\} \).

Assume
\[
\int_{D_1(0)} |\nabla \vec{n}|^2 \, dx =: \varepsilon_0^2 \leq 8\pi/3 \quad \text{and} \quad \|\nabla \lambda\|_{L^\infty(D_1(0))} < +\infty.
\]
We can call upon Lemma 5.1.4 in \([Hel]\) to deduce the existence of an orthogonal frame \( \{\vec{e}_1, \vec{e}_2\} \in W^{1,2}(D_1(0)) \) satisfying \( \star \vec{n} = \vec{e}_1 \wedge \vec{e}_2 \) and
\[
\|\nabla \vec{e}_1\|_{L^2(D_1(0))} + \|\nabla \vec{e}_2\|_{L^2(D_1(0))} \leq C \|\nabla \vec{n}\|_{L^2(D_1(0))}.
\]
As is easily verified, the conformal parameter satisfies
\[
\Delta \lambda = \nabla \vec{e}_1 \cdot \nabla \perp \vec{e}_2 \quad \text{in} \ D_1(0).
\]
Let \( \mu \) satisfy
\[
\begin{cases}
\Delta \mu = \nabla \bar{e}_1 \cdot \nabla^2 \bar{e}_2, & \text{in } D_1(0) \\
\mu = 0, & \text{on } \partial D_1(0).
\end{cases}
\]
Standard Wente estimates (cf. Theorem 3.4.1 in [Hel]) give
\[
\|\mu\|_{L^\infty(D_1(0))} + \|\nabla \mu\|_{L^2(D_1(0))} \leq \|\nabla \bar{e}_1\|_{L^2(D_1(0))} \|\nabla \bar{e}_2\|_{L^2(D_1(0))} \leq C \|\nabla \bar{H}\|_{L^2(D_1(0))}^2.
\] (II.1)
The harmonic function \( \nu := \lambda - \mu \) satisfies the usual estimate
\[
\int_D |\nu - \bar{\nu}| \, dx \leq C \|\nabla \nu\|_{L^1(D_1(0))} \leq C \|\nabla \nu\|_{L^2(D_1(0))}.
\]
where \( \bar{\nu} \) denotes the average of \( \nu \) on the proper subdisk \( D \subset D_1(0) \). Hence
\[
\|\nu - \bar{\nu}\|_{L^\infty(D)} \leq C \|\nabla \nu\|_{L^2(D_1(0))}.
\]
Combining the latter to (II.1) yields now
\[
\|\lambda - \bar{\lambda}\|_{L^\infty(D)} \leq C \|\nabla \lambda\|_{L^2(D_1(0))} \|\nabla \bar{H}\|_{L^2(D_1(0))} \leq C \|\nabla \lambda\|_{L^2(D_1(0))},
\]
where \( \bar{\lambda} \) denotes the average of \( \lambda \) on \( D \). We can summarize this subsection by stating the following lemma.

**Lemma II.1** Let \( \bar{\Phi} \in W^{2,2}_{\text{imm}}(D_1(0), \mathbb{R}^m) \) be a conformal weak immersion such that
\[
\int_{D_1(0)} |\nabla \bar{n}|^2 \, dx =: \bar{e}_0^2 \leq 8\pi/3 \quad \text{and} \quad \|\nabla \lambda\|_{L^2(D_1(0))} < +\infty,
\]
with \( \bar{e}^\lambda := |\partial_{x^1} \bar{\Phi}| = |\partial_{x^2} \bar{\Phi}|. \) Then the following estimate holds for any proper subdisk \( D \subset D_1(0) \):
\[
\|\bar{e}^\lambda\|_{L^\infty(D)} \|\bar{e}^{-\lambda}\|_{L^\infty(D)} \leq C(\bar{e}_0, \|\nabla \lambda\|_{L^2(D_1(0))}).
\] (II.2)

\[ \square \]

**II.2 Proof of Theorem I.1 (i)**

Per the discussion in the introduction and our aim to study only local properties of solutions to (I.5), we assume without loss of generality that the immersion \( \bar{\Phi} \) is conformal, i.e. in local coordinates \( \{x^1, x^2\} \) on the unit disk \( D_1(0) \) that
\[
\partial_{x^1} \bar{\Phi} \cdot \partial_{x^2} \bar{\Phi} = e^{2\lambda} \delta_{ij},
\]
with bounded conformal parameter \( \lambda \), and such that \( \bar{e}^\lambda \) satisfies the Harnack inequality (II.2). We will first begin by studying an inhomogeneous Willmore equation of the form
\[
\text{div}(\nabla \bar{H} - 2\pi \bar{n} \nabla \bar{H} + |\bar{H}|^2 \nabla \bar{\Phi} + \bar{T}) = \bar{\theta}, \quad \text{on } D_1(0),
\] (II.3)
where \( \bar{T} \) satisfies the following condition for some \( p \in (1, \infty) \):
\[
\|\bar{e}^\lambda \bar{T}\|_{L^p(D_1(0))} < \infty.
\] (II.4)

Let \( D_\rho(x) \subset D_1(0) \). As is done in [Ber2], we consider the following two problems
\[
\Delta \bar{X} = \nabla \bar{\Phi} \wedge \bar{T} \quad \text{and} \quad \Delta Y = \nabla \bar{\Phi} \cdot \bar{T} \quad \text{on } D_\rho(x)
\] (II.5)
with boundary conditions $\vec{X}|_{\partial D_\rho(x)} = \vec{0}$ and $Y|_{\partial D_\rho(x)} = 0$. Standard Calderon-Zygmund estimates give:

$$\|\nabla^2 \vec{X}\|_{L^p(D_\rho(x))} + \|\nabla^2 Y\|_{L^p(D_\rho(x))} \lesssim \|e^\lambda T\|_{L^p(D_\rho(x))},$$

up to a universal multiplicative constant. Hence

$$\|\nabla \vec{X}\|_{L^{2,\infty}(D_\rho(x))} + \|\nabla Y\|_{L^{2,\infty}(D_\rho(x))} \lesssim \rho^{2 - \frac{2}{p}} \|e^\lambda T\|_{L^p(D_\rho(x))}.$$  \hspace{1cm} (II.7)

We now follow the procedure outlined in [Ber2]. Integrating (II.3), we infer the existence of a potential $\vec{L}$ satisfying

$$\nabla^\perp \vec{L} = \nabla \vec{H} - 2\pi_\rho \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\phi} + \vec{T} \equiv -\nabla \vec{H} + 2\pi_T \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\phi} + \vec{T},$$

where $\pi_T$ is the tangential projection. An elementary computation (c.f. equation (II.6) in [Ber1]) reveals

$$|\nabla T| \lesssim \rho |\nabla \vec{n}|^2. \hspace{1cm} (II.9)$$

As $\vec{L}$ is defined up to an arbitrary constant, we are certainly free to require that

$$\int_{D_\rho(x)} \vec{L} = \vec{0}. \hspace{1cm} \text{Observe next that}$$

$$\|\nabla \vec{H}\|_{W^{1,2}(D_\rho(x))} \leq \|\vec{H}\|_{L^2(D_\rho(x))} \leq \|e^{-\lambda}\|_{L^\infty(D_\rho(x))} \|\nabla \vec{H}\|_{L^2(D_\rho(x))},$$

and, owing to (II.14) and (II.19),

$$\|\vec{T} + 2\pi_T \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\phi}\|_{L^2(D_\rho(x))} \lesssim \|e^{-\lambda}\|_{L^\infty(D_\rho(x))} \left[ \|e^\lambda T\|_{L^1(D_\rho(x))} + \|\nabla \vec{n}\|_{L^2(D_\rho(x))}^2 \right]$$

up to a multiplicative constant independent of the parametrization and of the mean curvature, and irrelevant to our purpose. Geared with these last inequalities, we call upon Lemma A.1 from the Appendix and conclude that

$$\|\vec{L}\|_{L^{2,\infty}(D_\rho(x))} \lesssim \|e^{-\lambda}\|_{L^\infty(D_\rho(x))} \left[ \|e^\lambda T\|_{L^1(D_\rho(x))} + \|\nabla \vec{n}\|_{L^2(D_\rho(x))}^2 \right], \hspace{1cm} (II.10)$$

where $L^{2,\infty}$ is the weak-$L^2$ Marcinkiewicz space, seen here as a Lorentz space [Tao]. Per Lemma II.1 $e^\lambda$ satisfies a Harnack inequality. The above then yields

$$\|e^\lambda \vec{L}\|_{L^{2,\infty}(D_\rho(x))} \lesssim \|e^\lambda T\|_{L^1(D_\rho(x))} + \|\nabla \vec{n}\|_{L^2(D_\rho(x))}. \hspace{1cm} (II.11)$$

We will use the symbol $\lesssim$ to indicate the presence of a multiplicative constant depending at most only on $\epsilon_0$ and on $\|\nabla \lambda\|_{L^{2,\infty}(D_0(\rho))}$.

It is shown in [Ber2] that two important identities hold, namely

$$\text{div} (\vec{L} \wedge \nabla^\perp \vec{\phi} + \vec{H} \wedge \nabla \vec{\phi} + \nabla \vec{X}) = \vec{0} \quad \text{and} \quad \text{div} (\vec{L} \cdot \nabla^\perp \vec{\phi} + \nabla Y) = 0.$$  

Again, we infer the existence of two potentials $\vec{R}$ and $S$ satisfying

$$\nabla \vec{R} = \vec{L} \wedge \nabla \vec{\phi} - \vec{H} \wedge \nabla^\perp \vec{\phi} - \nabla^\perp \vec{X} \quad \text{and} \quad \nabla S = \vec{L} \cdot \nabla \vec{\phi} - \nabla^\perp Y. \hspace{1cm} (II.12)$$
Owing to (II.11) and (II.7), we find that $\nabla \tilde{R}$ and $\nabla S$ lie in the weak space $L^{2,\infty}$, namely
\[
\|\nabla \tilde{R}\|_{L^{2,\infty}(D_{\rho}(x))} + \|\nabla S\|_{L^{2,\infty}(D_{\rho}(x))}
\lesssim \|e^{\lambda} \tilde{T}\|_{L^{1}(D_{\rho}(x))} + \|\nabla \tilde{n}\|_{L^{2}(D_{\rho}(x))} + \|\nabla \tilde{X}\|_{L^{2,\infty}(D_{\rho}(x))} + \|\nabla Y\|_{L^{2,\infty}(D_{\rho}(x))}
\lesssim \rho^{2-\frac{2}{p}} \|e^{\lambda} \tilde{T}\|_{L^{p}(D_{\rho}(x))} + \|\nabla \tilde{n}\|_{L^{2}(D_{\rho}(x))},
\]
where $C(\varepsilon_0)$ is a constant depending only on $\varepsilon_0$. In other words
\[
\|\nabla \tilde{R}\|_{L^{2,\infty}(D_{\rho}(x))} + \|\nabla S\|_{L^{2,\infty}(D_{\rho}(x))} \lesssim M, \tag{II.13}
\]
where for notational convenience, we have set
\[
M := \rho^{2-\frac{2}{p}} \|e^{\lambda} \tilde{T}\|_{L^{p}(D_{\rho}(x))} + \|\nabla \tilde{n}\|_{L^{2}(D_{\rho}(x))}. \tag{II.14}
\]

It is remarkable that $\tilde{R}$ and $S$ are linked together via an interesting system of equations that displays a very particular structural type. It is shown in [Ber2] that for all $s < 2/(2 - p)$, it holds
\[
\|\nabla \tilde{R}\|_{L^{s}(D_{\rho/8}(x))} + \|\nabla S\|_{L^{s}(D_{\rho/8}(x))}
\lesssim \rho^{2s-1} \left[\|\nabla \tilde{R}\|_{L^{2,\infty}(D_{\rho}(x))} + \|\nabla S\|_{L^{2,\infty}(D_{\rho}(x))} + \rho^{2-\frac{2}{p}} \|e^{\lambda} \tilde{T}\|_{L^{p}(D_{\rho}(x))}\right]
\lesssim \rho^{2s-1} M, \tag{II.16}
\]
where we have used (II.13). Note that (II.16) holds in particular for $s = 2p$.

A useful identity is derived in [Ber2]; it relays information on $\tilde{R}$ and $S$ back to the immersion $\Phi$, namely:
\[
e^{2\lambda} \tilde{H} = (\nabla \tilde{R} + \nabla \tilde{X}) \bullet \nabla \Phi + (\nabla S + \nabla \tilde{Y}) \cdot \nabla \Phi. \tag{II.17}
\]
It follows from this identity and from (II.6) and (II.16) that
\[
\|e^{2\lambda} \tilde{H}\|_{L^{2p}(D_{\rho/8}(x))} \lesssim \rho^{2s-1} \|e^{\lambda}\|_{L^{\infty}(D_{\rho}(x))} M, \tag{II.18}
\]
where as always the symbol $\lesssim$ indicates the presence of a multiplicative constant involving at most $\varepsilon_0$ and $\|\nabla \lambda\|_{L^{2,\infty}(D_{1})}$. We now use the equation $\Delta \Phi = 2e^{2\lambda} \tilde{H}$ by writing $\Phi = \Phi_1 + \Phi_0$, where
\[
\left\{\begin{array}{l}
\Delta \Phi_1 = 2e^{2\lambda} \tilde{H} \\
\Phi_1 = 0
\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l}
\Delta \Phi_0 = 0 \\
\Phi_0 = \Phi_0
\end{array}\right., \quad \text{in } D_{\rho/8}(x) \quad \text{and} \quad \text{on } \partial D_{\rho/8}(x).
\]
Estimates for $\Phi_1$ follow directly from (II.13), while $\Phi_0$ is handled with the help of standard estimates for harmonic functions. We obtain

$$\|\nabla^2 \Phi\|_{L^p(D_{\rho/16}(x))} \lesssim \rho^{\frac{1}{p}} \|\nabla \Phi\|_{L^\infty(D_{\rho/16}(x))} + \|\Phi\|_{L^\infty(D_{\rho}(x))} \rho^{\frac{1}{p} - 1} M$$

Using again that $|\nabla \tilde{n}| \leq e^{-\lambda} |\nabla^2 \Phi|$ and (II.2), the latter gives

$$\|\nabla \tilde{n}\|_{L^p(D_{\rho/16}(x))} \lesssim \rho^{\frac{1}{p} - 1}(1 + M).$$

Next, using (II.15), we have

$$|\Delta \tilde{R}| + |\Delta S| \lesssim |\Delta \tilde{X}| + |\Delta Y| + |\nabla \tilde{n}|(|\nabla \tilde{R}| + |\nabla S| + |\nabla \tilde{X}| + |\nabla Y|).$$

Hence, from (II.20), (II.6), and (II.16),

$$\|\nabla^2 \tilde{R}\|_{L^p(D_{1/32}(x))} + \|\nabla^2 \tilde{S}\|_{L^p(D_{1/32}(x))} \lesssim \rho^{\frac{2}{p} - 2}(\|\nabla \tilde{R}\|_{L^2}(D_{\rho}(x)) + \|\nabla \tilde{S}\|_{L^2}(D_{\rho}(x))) + \|\nabla^2 \Phi\|_{L^p(D_{\rho/16}(x))}$$

$$+ \|\nabla \tilde{n}\|_{L^p(D_{\rho/16}(x))}(\|\nabla \tilde{R}\|_{L^p(D_{\rho/16}(x))} + \|\nabla \tilde{S}\|_{L^p(D_{\rho/16}(x))} + \|\nabla \tilde{X}\|_{L^p(D_{\rho/16}(x))} + \|\nabla Y\|_{L^p(D_{\rho/16}(x))})$$

$$\lesssim \|\nabla^2 \tilde{R}\|_{L^p(D_{\rho}(x))} + \rho^{\frac{2}{p} - 2} M(M + 1)$$

$$\lesssim \rho^{\frac{2}{p} - 2} M(M + 1),$$

where we have used (II.13).

We can now combine (II.6), (II.19), and (II.21) into (II.17) to find

$$\|\nabla^2 \Phi\|_{L^p(D_{1/32}(x))} \lesssim \rho^{\frac{2}{p} - 2} M(M + 1),$$

where we have also used (II.10) and (II.16). From the latter and (II.2), it follows that

$$\|e^{\lambda}\tilde{H}\|_{L^{p/(2-p)}(D_{\rho/2}(x))} \lesssim \rho^{\frac{2}{p} - 2} M(M + 1), \quad p \in (1, 2)$$

$$\|e^{\lambda}\tilde{H}\|_{L^{p}}(D_{\rho/2}(x)) \lesssim \rho^{\frac{2}{p} - 1} M(M + 1), \quad \forall q < \infty, \quad p = 2$$

Owing to (II.17), we verify easily that

$$2\tilde{H} \wedge \nabla^\perp \Phi = (\nabla^\perp \tilde{R} - \nabla \tilde{X}) \bullet (e^{\lambda} \tilde{n}) + (\nabla^\perp S - \nabla Y)(e^{\lambda} \tilde{n}),$$

so that, using (II.6) and (II.15),

$$|\text{div}(\tilde{H} \wedge \nabla^\perp \Phi)| \lesssim |\Delta \tilde{X}| + |\Delta Y| + |\nabla \tilde{n}|(|\nabla \tilde{R}| + |\nabla S| + |\nabla \tilde{X}| + |\nabla Y|)$$

$$\lesssim |e^{\lambda} \tilde{T}| + |\nabla \tilde{n}|(|\nabla \tilde{R}| + |\nabla S| + |\nabla \tilde{X}| + |\nabla Y|).$$
Hence, from (II.20), (II.6), and (II.16),
\[
\| \text{div}(\vec{H} \wedge \nabla^2 \vec{\phi}) \|_{L^p(D_{9\rho/16}(x))} \lesssim \| e^{\lambda T} \|_{L^p(D_{9\rho/16}(x))} \\
+ \| \nabla \vec{n} \|_{L^p(D_{9\rho/16}(x))} \| \nabla \vec{H} \|_{L^{2p}(D_{9\rho/16}(x))} + \| \nabla S \|_{L^{2p}(D_{9\rho/16}(x))} + \| \nabla X \|_{L^{2p}(D_{9\rho/16}(x))} + \| \nabla Y \|_{L^{2p}(D_{9\rho/16}(x))} \)
\lesssim \rho^{\frac{2}{p} - 2}M(M + 1) .
\] (II.25)

As shown in [BR2], the Gauss map $\vec{n}$ satisfies a perturbed harmonic map equation, namely
\[
|\Delta \vec{n}| \leq 2|\text{div}(\vec{H} \wedge \nabla^2 \vec{\phi})| + O(|\nabla \vec{n}|^2) .
\]

Accordingly, from (II.20) and (II.25), we find
\[
\| \nabla^2 \vec{n} \|_{L^p(D_{17\rho/32}(x))} \lesssim \|\text{div}(\vec{H} \wedge \nabla^2 \vec{\phi})\|_{L^p(D_{9\rho/16}(x))} + \| \nabla \vec{n} \|_{L^{2p}(D_{9\rho/16}(x))} + \rho^{\frac{2}{p} - 2} \| \nabla \vec{n} \|_{L^2(D_{9\rho/16}(x))} \\
\lesssim \| e^{\lambda T} \|_{L^p(D_{\rho/2}(x))} \rho^{\frac{2}{p} - 2}(M + 1)^2 + \rho^{\frac{2}{p} - 2} \| \nabla \vec{n} \|_{L^2(D_{\rho/2}(x))} \\
\lesssim \rho^{\frac{2}{p} - 2}(M + 1)^2 .
\] (II.26)

From this we deduce
\[
\begin{cases}
\| \nabla \vec{n} \|_{L^{2p/(2-p)}(D_{\rho/2}(x))} \lesssim \rho^{\frac{2}{p} - 2}(M + 1)^2 , & \text{if } p \in (1, 2) \\
\| \nabla \vec{n} \|_{L^q(D_{\rho/2}(x))} \lesssim \rho^{\frac{q}{p} - 1}(M + 1)^2 , & \forall q < \infty , \text{ if } p = 2 \\
\| \nabla \vec{n} \|_{L^\infty(D_{\rho/2}(x))} \lesssim \rho^{-1}(M + 1)^2 , & \text{if } p > 2 .
\end{cases}
\] (II.27)

To complete the proof of Theorem I.1-(i), we show that the estimates (II.23) and (II.24) may be slightly improved when $p \geq 2$. Firstly, when $p > 2$, we see that (II.16) holds for all $s < \infty$. In particular, (II.18), (II.19), and (II.20) hold with any $s < \infty$ in place of $2p$. But according to (II.23) and (II.27), we know that $e^{\lambda \vec{H}}$ and $\nabla \vec{n}$ are bounded when $p > 2$. We may thus let $s$ tend to infinity to find that
\[
\| e^{\lambda \vec{H}} \|_{L^\infty(D_{\rho/2}(x))} \lesssim \rho^{-1}M \quad \text{and} \quad \| \nabla \vec{n} \|_{L^\infty(D_{\rho/2}(x))} \lesssim \rho^{-1}(M + 1) .
\] (II.28)

Similarly, when $p = 2$, we have that (II.16), and thus (II.18) and (II.20) hold for all $s < \infty$ in place of $2p$. In particular,
\[
\| e^{\lambda \vec{H}} \|_{L^q(D_{\rho/2}(x))} \lesssim \rho^{\frac{2}{p} - 1}M \quad \text{and} \quad \| \nabla \vec{n} \|_{L^q(D_{\rho/2}(x))} \lesssim \rho^{\frac{2}{p} - 1}(M + 1) , \quad \forall q < \infty .
\]

II.3 Proof of Theorem I.1-(ii)

In this section we will build upon the results previously derived in order to obtain regularity estimates for an inhomogeneous Willmore equation of the type
\[
\text{div}(\nabla \vec{H} - 2\pi \vec{n}\nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\phi}) = \vec{v} ,
\]
where we suppose that
\[
e^{\lambda \vec{v}} \in L^r(D_1(0)) \quad \text{for some } r \geq 1 .
\]

Let $D_{\rho}(x) \subset D_1(0)$. In order to recover (I.23), we let $\vec{V}$ satisfy the problem
\[
\begin{cases}
-\Delta \vec{V} = \vec{v} , & \text{in } D_{\rho}(x) \\
\vec{V} = \vec{0} , & \text{on } \partial D_{\rho}(x) .
\end{cases}
\] (II.29)
According to (II.5), we find

\[
\|e^\lambda \nabla \tilde{V}\|_{L^{2,\infty}(D_p(x))} \lesssim \|e^\lambda \tilde{v}\|_{L^1(D_p(x))}, \quad r = 1
\]

\[
\|e^\lambda \nabla \tilde{V}\|_{L^{2r/(2-r)}(D_p(x))} \lesssim \|e^\lambda \tilde{v}\|_{L^r(D_p(x))}, \quad r \in (1, 2)
\]

\[
\|e^\lambda \nabla \tilde{V}\|_{L^q(D_p(x))} \lesssim \rho^{1-\frac{q}{2}} \|e^\lambda \tilde{v}\|_{L^r(D_p(x))}, \quad \forall \ q < \infty, \ r \geq 2.
\]

On the other hand, using (II.19), we have

\[
\|\nabla \tilde{n}\|_{L^p(D_{p/2}(x))} + \|\nabla \tilde{n}\|_{L^{2p/(2-p)}(D_{p/2}(x))} \lesssim \rho^{\frac{1}{2}} \|\tilde{v}\|^{p+1} \|\iota\|^{r-1} \|e^\lambda \tilde{v}\|_{L^r(D_p(x))}, \quad \forall \ p \in (1, 2),
\]

with

\[
M = \rho \|e^\lambda \tilde{v}\|_{L^1(D_p(x))} + \|\nabla \tilde{n}\|_{L^2(D_p(x))}.
\]

We are now back in the case studied in the previous section with \( \tilde{T} := \nabla \tilde{V} \). In particular, when \( r = 1 \), we find

\[
\|e^\lambda \tilde{T}\|_{L^p(D_p(x))} \lesssim \rho^{\frac{1}{2}} \|e^\lambda \tilde{T}\|_{L^{2,\infty}(D_p(x))} \lesssim \rho^{\frac{1}{2}} \|e^\lambda \tilde{v}\|_{L^1(D_p(x))} \quad \forall \ p \in (1, 2),
\]

from which (II.27) and (II.26) yield

\[
\|\nabla^2 \tilde{n}\|_{L^p(D_{p/2}(x))} + \|\nabla \tilde{n}\|_{L^{2p/(2-p)}(D_{p/2}(x))} \lesssim \rho^{\frac{1}{2}} (M + 1)^2 \quad \forall \ p \in (1, 2),
\]

where

\[
M = \rho \|e^\lambda \tilde{v}\|_{L^1(D_p(x))} + \|\nabla \tilde{n}\|_{L^2(D_p(x))}.
\]

On the other hand, when \( r > 1 \), \( e^\lambda \tilde{T} \) lies in a space of the type \( L^{2+\delta} \) for some \( \delta > 0 \). This time, the estimates (II.28) and (II.29) give

\[
\rho \|e^\lambda \tilde{H}\|_{L^\infty(D_p(x))} \lesssim M, \quad \rho \|\nabla \tilde{n}\|_{L^\infty(D_{p/2}(x))} \lesssim 1 + M,
\]

and

\[
\rho^{\frac{1}{2}} \|\nabla^2 \tilde{n}\|_{L^{2r/(2-r)}(D_{p/2}(x))} \lesssim (M + 1)^2, \quad \text{if } r \in (1, 2)
\]

\[
\rho^{1-\frac{q}{2}} \|\nabla^2 \tilde{n}\|_{L^q(D_{p/2}(x))} \lesssim (M + 1)^2, \quad \forall \ q < \infty \text{ if } r \geq 2,
\]

with

\[
M = \rho^{1-\frac{q}{2}} \|e^\lambda \tilde{v}\|_{L^r(D_p(x))} + \|\nabla \tilde{n}\|_{L^2(D_p(x))}.
\]

We now prove third-order derivative estimates for \( \nabla \tilde{n} \). For the sake of brevity, we only consider the case \( r \in (1, 2) \). From (II.29), we find

\[
\|\nabla \tilde{V}\|_{L^{r}(D_p(x))} + \|\nabla^2 \tilde{V}\|_{L^{r}(D_p(x))} \lesssim \|\tilde{v}\|_{L^r(D_p(x))},
\]

where for notational convenience, we have set \( r^* := 2r/(2-r) \). Since \( \tilde{T} := \nabla \tilde{V} \), the latter and (II.12) yield

\[
\|e^\lambda \tilde{T}\|_{L^{r}(D_p(x))} + \|e^\lambda \nabla \tilde{T}\|_{L^{r}(D_p(x))} \lesssim \|e^\lambda \tilde{v}\|_{L^r(D_p(x))}.
\]

On the other hand, using (II.19), we have

\[
\|\nabla^2 \tilde{F}\|_{L^2(D_p/\lambda(16x))} \lesssim \|e^\lambda \|_{L^{\infty}(D_p(x))}(1 + M),
\]

where \( M \) is as in (II.32). According to (II.35), we find

\[
\rho \|\nabla^2 \tilde{X}\|_{L^2(D_p(x))} + \rho \|\nabla^2 \tilde{Y}\|_{L^2(D_p(x))} + \|\nabla \tilde{X}\|_{L^2(D_p(x))} + \|\nabla \tilde{Y}\|_{L^2(D_p(x))} \lesssim \rho^{3-\frac{q}{2}} \|e^\lambda \tilde{v}\|_{L^r(D_p(x))}.
\]

and moreover, owing to (II.34)

\[
\|\nabla \Delta \tilde{X}\|_{L^{r}(D_p/\lambda(16x))} + \|\nabla \Delta \tilde{Y}\|_{L^{r}(D_p/\lambda(16x))}
\]

\[
\lesssim \|\nabla^2 \tilde{F}\|_{L^2(D_p/\lambda(16x))} \|\tilde{T}\|_{L^{r}(D_p/\lambda(16x))} + \|\tilde{F}\|_{L^{\infty}(D_p/\lambda(16x))} \|\tilde{T}\|_{L^{r}(D_p/\lambda(16x))}
\]

\[
\lesssim (1+M) \|e^\lambda \tilde{v}\|_{L^r(D_p(x))}.
\]
Note that (II.16) and (II.33) yield
\[ \| \nabla \vec{R} \|_{L^2(D_{3\rho/4}(x))} + \| \nabla S \|_{L^2(D_{3\rho/4}(x))} \lesssim M . \]
In addition, (II.21) states
\[ \| \nabla^2 \vec{R} \|_{L^2(D_{\rho/2}(x))} + \| \nabla^2 S \|_{L^2(D_{\rho/2}(x))} \lesssim \rho^{-1} M (M + 1) . \] (II.37)
From (II.17), we easily verify that
\[ | \nabla \text{div}(\vec{H} \land \nabla^4 \vec{\Phi}) | \lesssim | \nabla \Delta \vec{X} | + | \nabla \Delta \vec{Y} | + | \nabla \vec{n} (| \nabla^2 \vec{R} | + | \nabla^2 S | + | \nabla^2 \vec{X} | + | \nabla^2 \vec{Y} | ) \]
\[ + | \nabla^2 \vec{n} (| \nabla \vec{R} | + | \nabla S | + | \nabla \vec{X} | + | \nabla \vec{Y} | ) . \]
The estimates (II.30), (II.31), and (II.35) - (II.37) then show that
\[ \| \nabla \text{div}(\vec{H} \land \nabla^4 \vec{\Phi}) \|_{L^r(D_{\rho/2}(x))} \lesssim \rho^{2-3} (M + 1)^2 . \] (II.38)
We can then proceed as in (II.26) to obtain
\[ \| \nabla^3 \vec{n} \|_{L^r(D_{\rho/2}(x))} \lesssim \rho^{2-3} (M + 1)^3 . \] (II.39)
The case \( r \geq 2 \) is handled \textit{mutatis mutandis}, and one arrives too at the third-order estimate (II.39). In particular, when \( r > 2 \), we see that \( \nabla^2 \vec{n} \) is bounded, and we can slightly improve (II.31) to
\[ \rho^2 \| \nabla^2 \vec{n} \|_{L^\infty(D_{\rho/2}(x))} \lesssim (M + 1)^2 . \]
This concludes the proof of Theorem I.1-(ii).

\section*{II.4 On smoothness of the solution: proof of Corollary I.1}
Let us suppose that \( \vec{\Phi} \in W^{2,2}_{\text{imm}}(D_1(0), \mathbb{R}^m) \) satisfies the equation
\[ \text{div}(\nabla \vec{H} - 2\pi \vec{n} \nabla \vec{H} + |\vec{H}|^2 \vec{\Phi} + \vec{T}) = \vec{v} \quad \text{on} \ D_1(0) , \]
where \( \vec{T} \) and \( \vec{v} \) are smooth. As we are interested in obtaining a local result, we may always rescale so as to guarantee that the small energy assumption
\[ \| \nabla \vec{n} \|_{L^2(D_1(0))} < \varepsilon_0 \]
holds for some \( \varepsilon_0 \) sufficiently small. We proved in the last section that \( \nabla \vec{n} \) is a bounded function. Owing to the Liouville equation
\[ -\Delta \lambda = e^{2\lambda} K = O(|\nabla \vec{n}|^2) , \]
it follows that \( e^{\pm \lambda} \) lie in \( \bigcap_{p<\infty} W^{2,p} \). The function \( \vec{V} \) defined in (II.29) is smooth. By definition, so is \( \vec{U} := \vec{T} + \nabla \vec{V} \). Using (II.15), we deduce that \( \vec{X} \) and \( \vec{Y} \) belong to \( \bigcap_{p<\infty} W^{3,p} \). We see in the paragraph following (II.17) that \( \vec{R} \) and \( S \) also belong to \( \bigcap_{p<\infty} W^{2,p} \). Then the equation (II.17) yields now that the immersion lies in \( \bigcap_{p<\infty} W^{3,p} \), and thus that \( \nabla \vec{n} \) lies in \( \bigcap_{p<\infty} W^{2,p} \). This process may now be repeated as much as required to reach the conclusion that \( \vec{\Phi} \) is smooth.

\footnote{\( K \) denotes the Gauss curvature.}
II.5 Remarks about the critical case

As its name indicates, the critical case is far more delicate to handle, and, as far as the authors know, there is no general method to prove the regularity of solutions to the inhomogeneous Willmore equation

$$\text{div}(\nabla H - 2\pi n \nabla H + |H|^2 \nabla \Phi + \vec{T}) = \vec{0},$$

(II.40)

with a generic inhomogeneity $e^\lambda \vec{T} \in L^1$, if it is only known that the second fundamental form is square integrable. There are of course special cases, such as the Willmore immersions (with $\vec{T} \equiv \vec{0}$) and more generally the conformally constrained Willmore immersions (which include Willmore and CMC immersions) whose $e^\lambda \vec{T}$ has a very specific form, see [Ber1]. The conformally constrained Willmore immersions have an inhomogeneous term $\vec{T}$ for which the solutions to (II.5) are identically vanishing. In turn, this guarantees the system (II.15) is of Wente type and can thus be made subcritical just as we have done for $e^\lambda \vec{T} \in L^{p>1}$.

But even if we assume from the onset that the solution to (II.40) is sufficiently regular, the presence of an inhomogeneity $\vec{T}$, and thus of nonzero solutions of (II.5), will in general prevent us from reaching estimates of the type appearing in Theorem I.1. This difficulty can only be resolved on a case-by-case basis. We will content ourselves in this short section with mentioning one specific type of inhomogeneities for which Theorem I.1 can be obtained. Let us write the inhomogeneity $\vec{T}$ in the form

$$\vec{T} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \partial_{x_1} \Phi + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \partial_{x_2} \Phi + \begin{pmatrix} \vec{U}_1 \\ \vec{U}_2 \end{pmatrix},$$

where $\vec{U}_1$ and $\vec{U}_2$ are two normal vectors. One easily verifies that

$$\nabla \Phi \wedge \vec{T} = e^{2\lambda} (A_2 - B_1) (\ast \vec{n}) - \vec{U}_1 \wedge \partial_{x_1} \Phi - \vec{U}_2 \wedge \partial_{x_2} \Phi$$

and

$$\nabla \Phi \cdot \vec{T} = e^{2\lambda} (A_1 + B_2).$$

Accordingly, if the functions $(A_1 + B_2)$, $(A_2 - B_1)$, and the normal projection $\pi \vec{n} \vec{T}$ lie in the space $L^{1+\delta}$ for some $\delta > 0$, we can apply to (II.15) the same technique as that used in the proof of Theorem I.1. This holds of course even if the functions $A_1$, $A_2$, $B_1$, and $B_2$ are only merely integrable.

In general, it is not possible to obtain a subcritical-type energy estimate. However, as we have seen above, there are exceptions when $\vec{T}$ has a specific form. Another important exceptional case occurs when $\vec{T}$ depends on the geometry of the problem, and if the solution is already known to be regular enough, say $\Phi \in W^{2,2+\delta}(D_1(0))$, for some positive $\delta \in (0, 1)$. We only focus on the specific situation when

$$e^\lambda \vec{T} = O(|\nabla \vec{n}|^2)$$

for an inhomogeneous Willmore problem of the type

$$\text{div}(\nabla \vec{H} - 2\pi n \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\Phi} + \vec{T}) = \vec{0},$$

and assuming as usual that

$$||\nabla \vec{n}||_{L^2(D_1(0))} < \varepsilon_0$$

for some $\varepsilon_0$ chosen sufficiently small. As $e^\lambda \vec{T} \in L^{1+\frac{\delta}{2}}$, it follows from Theorem I.1 that $\nabla \vec{n}$ lies in $W^{1,1+\frac{\delta}{2}} \subset L^{2+\delta}$, and thus $e^\lambda \vec{T}$ lies in $L^{1+\frac{\delta}{2}}$ even if ever so slightly, say $\vec{H} \in L^{2+\delta}$ for some $\delta > 0$. 

9 even if ever so slightly, say $\vec{H} \in L^{2+\delta}$ for some $\delta > 0$. 

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\( W^{1,2} \), which is a proper subset of \( L^{1+\delta} \). Calling again on Theorem I.1, the integrability of \( \nabla \vec{n} \) is improved accordingly. This procedure may be repeated until reaching that \( \nabla^2 \vec{n} \) belongs to all \( L^p \) spaces, with \( p \) finite, i.e. that \( \vec{n} \) belongs to \( C^{1,\alpha} \) for all \( \alpha < 1 \). Standard arguments then imply that \( \vec{n} \), and thus the immersion \( \vec{\Phi} \), are smooth.

II.6 Gap phenomenon: proof of Theorem I.2

Let us suppose that \( \Sigma \) is a complete, connected, non-compact, oriented, immersed surface into \( \mathbb{R}^{m \geq 3} \) satisfying an inhomogeneous Willmore equation (I.4) of the form

\[
\Delta_{\perp} \vec{H} + \langle \vec{A} \cdot \vec{H}, \vec{A} \rangle_g - 2|\vec{H}|^2 \vec{H} = \vec{W},
\]

with the same notation as before, and where \( \vec{W} \) is a normal field with the property that

\[
\vec{W} = O(|\vec{A}|^3) \quad \text{i.e.} \quad \frac{1}{c} |\vec{A}|^3 \leq |\vec{W}| \leq c |\vec{A}|^3,
\]

for some constant \( c \geq 1 \). We suppose further that

\[
\int_\Sigma |\vec{A}|^2 \, d\text{vol}_g < \varepsilon_0^2,
\]

for some \( \varepsilon_0^2 \) chosen to be small enough (at least smaller than \( 8\pi/3 \)). A well-known result of Müller and Sverak \( \text{[MS]} \) guarantees that \( \Sigma \) is embedded and conformally equivalent to \( \mathbb{R}^2 \). Accordingly, we parametrize \( \Sigma \) by a conformal immersion \( \vec{\Phi} : \mathbb{R}^2 \hookrightarrow \mathbb{R}^m \) with conformal parameter \( \lambda \), and such that \( \vec{\Phi} \in W^{2,2}(\mathbb{R}^2) \).

Just as was done in Section I in the flat coordinates of \( \mathbb{R}^2 \), the inhomogeneous Willmore equation (II.41) can be recast in the form

\[
\text{div} (\nabla \vec{H} - 2\pi \vec{n} \nabla \vec{H} + |\vec{H}|^2 \nabla \vec{\Phi}) = \vec{v} \quad \text{on} \ \mathbb{R}^2,
\]

where

\[
\vec{v} := e^{2\lambda} \vec{W}.
\]

Per (II.42), note that

\[
|e^{\lambda} \vec{v}| \simeq |e^{\lambda} \vec{A}|^3 \simeq |\nabla \vec{n}|^3,
\]

where, as always, \( \vec{n} \) is the Gauss map associated with \( \vec{\Phi} \). The smallness hypothesis (II.43) translates into

\[
\|\nabla \vec{n}\|_{L^2(\mathbb{R}^2)} < \varepsilon_0,
\]

for some \( \varepsilon_0 > 0 \) sufficiently small.

Owing to the Liouville equation

\[
-\Delta \lambda = e^{2\lambda} K = O(|\nabla \vec{n}|^2) \in L^1(\mathbb{R}^2),
\]

it follows that \( \nabla \lambda \) lies in the space \( L^{2,\infty}(\mathbb{R}^2) \) with norm controlled by \( \|\nabla \vec{n}\|_{L^2(\mathbb{R}^2)} \). We can in particular repeat the analysis leading to Lemma II.1 to deduce that

\[
\|e^{\lambda}\|_{L^\infty(\mathbb{R}^2)} \|e^{-\lambda}\|_{L^{\infty}(\mathbb{R}^2)} \leq C(\varepsilon_0).
\]

This Harnack-type inequality will be used in our argument.
As we did in the proof of Theorem II.8(ii), we let

$$-\Delta \tilde{V} = \tilde{v} \quad \text{on} \ \mathbb{R}^2,$$

and \( \tilde{T} := \nabla \tilde{V} \). As the equation

$$\text{div}(\nabla \tilde{H} - 2\pi \nabla \tilde{H} + |\tilde{H}|^2 \nabla \tilde{\phi} + \tilde{T}) = 0 \quad \text{holds on} \ \mathbb{R}^2,$$

we can repeat the analysis done in the proof of Theorem II.8(i) and deduce the existence of \( \tilde{R} \) and \( S \) satisfying

$$\begin{align*}
\Delta \tilde{R} &= \nabla ((\ast \, \tilde{n}) \ast \nabla \tilde{H} + \nabla ((\ast \, \tilde{n}) \ast \tilde{S}) + \text{div} (\tilde{H} \nabla \tilde{\phi} + \tilde{T}) + \nabla (\ast \, \tilde{n}) \nabla \tilde{Y}) \\
\Delta \tilde{S} &= \nabla ((\ast \, \tilde{n}) \ast \nabla \tilde{H} + \text{div} (\tilde{H} \nabla \tilde{\phi} + \nabla (\ast \, \tilde{n}) \nabla \tilde{X}) \quad \text{(II.47)}
\end{align*}$$

where, as before, \( \tilde{X} \) and \( \tilde{Y} \) satisfy

$$\Delta \tilde{X} = \nabla \tilde{\phi} \land \tilde{T} \quad \text{and} \quad \Delta \tilde{Y} = \nabla \tilde{\phi} \cdot \tilde{T} \quad \text{on} \ \mathbb{R}^2. \quad \text{(II.48)}$$

We have

$$\|\Delta \tilde{X}\|_{L^q(\mathbb{R}^2)} + \|\Delta \tilde{Y}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla \tilde{\phi}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla \tilde{Y}\|_{L^{q^*}(\mathbb{R}^2)} \lesssim |e^{\lambda \tilde{T}}|_{L^q(\mathbb{R}^2)}, \quad \text{(II.49)}$$

for \( q \in (1,2) \) and \( q^* := 2q/(2-q) \). Applying Wente’s inequality to \( \tilde{H} \) as in Lemma IV.2 of [BR13], we find

$$\|\nabla \tilde{R}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla S\|_{L^{q^*}(\mathbb{R}^2)} \lesssim \|\nabla \tilde{n}\|_{L^2(\mathbb{R}^2)} \left( \|\nabla \tilde{R}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla S\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla \tilde{X}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla \tilde{Y}\|_{L^{q^*}(\mathbb{R}^2)} \right),$$

which, owing to \( \|\nabla \tilde{\phi}\|_{L^{q^*}(\mathbb{R}^2)} \) and \( \|\nabla \tilde{S}\|_{L^{q^*}(\mathbb{R}^2)} \), yields

$$\|\nabla \tilde{R}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla S\|_{L^{q^*}(\mathbb{R}^2)} \leq C(\varepsilon_0) \left( \|\nabla \tilde{X}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla \tilde{Y}\|_{L^{q^*}(\mathbb{R}^2)} \right) \quad \text{(II.50)}$$

We have seen in the previous section that

$$2\tilde{H} \land \nabla \tilde{\phi} = (\nabla \tilde{H} - \nabla \tilde{\phi}) \ast (\ast \, \tilde{n}) + (\nabla \tilde{S} - \nabla \tilde{Y})(\ast \, \tilde{n}) \quad \text{on} \ \mathbb{R}^2,$$

hence

$$|\text{div}(2\tilde{H} \land \nabla \tilde{\phi})| \leq |\Delta \tilde{X}| + |\Delta \tilde{Y}| + |\nabla \tilde{n}| \left( |\nabla \tilde{R}| + |\nabla \tilde{S}| + |\nabla \tilde{X}| + |\nabla \tilde{Y}| \right).$$

This gives, using \( \|\Delta \tilde{X}\|_{L^q(\mathbb{R}^2)} + \|\Delta \tilde{Y}\|_{L^q(\mathbb{R}^2)} \),

$$\|\text{div}(2\tilde{H} \land \nabla \tilde{\phi})\|_{L^q(\mathbb{R}^2)} \leq \|\Delta \tilde{X}\|_{L^q(\mathbb{R}^2)} + \|\Delta \tilde{Y}\|_{L^q(\mathbb{R}^2)} + \|\nabla \tilde{n}\|_{L^2(\mathbb{R}^2)} \left( \|\nabla \tilde{R}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla S\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla \tilde{X}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla \tilde{Y}\|_{L^{q^*}(\mathbb{R}^2)} \right) \leq 2\|e^{\lambda \tilde{T}}\|_{L^q(\mathbb{R}^2)} + C(\varepsilon_0) \left( \|\nabla \tilde{X}\|_{L^{q^*}(\mathbb{R}^2)} + \|\nabla \tilde{Y}\|_{L^{q^*}(\mathbb{R}^2)} \right),$$

$$\leq C(\varepsilon_0) \|e^{\lambda \tilde{T}}\|_{L^q(\mathbb{R}^2)}. \quad \text{(II.51)}$$

Recall that

$$\Delta \tilde{n} = \text{div}(2\tilde{H} \land \nabla \tilde{\phi}) + O(|\nabla \tilde{n}|^2).$$

According to \( \|\nabla^2 \tilde{n}\|_{L^q(\mathbb{R}^2)} \), to \( \|\tilde{n}\|_{L^q(\mathbb{R}^2)} \), and to the Sobolev embedding theorem, we thus have

$$\|\nabla^2 \tilde{n}\|_{L^q(\mathbb{R}^2)} \leq \|\text{div}(2\tilde{H} \land \nabla \tilde{\phi})\|_{L^q(\mathbb{R}^2)} + \|\nabla \tilde{n}\|_{L^q(\mathbb{R}^2)} \|\nabla \tilde{n}\|_{L^{q^*}(\mathbb{R}^2)} \leq C(\varepsilon_0) \|e^{\lambda \tilde{T}}\|_{L^q(\mathbb{R}^2)} + \varepsilon_0 \|\nabla^2 \tilde{n}\|_{L^q(\mathbb{R}^2)}.$$
thereby yielding
\[ \| \nabla^2 \vec{n} \|_{L^p(\mathbb{R}^2)} \leq C(\varepsilon_0) \| e^\lambda \vec{T} \|_{L^p(\mathbb{R}^2)}. \]  
(II.52)

We now call upon the Gagliardo-Nirenberg interpolation inequality, (II.45), and (II.52) to find
\[ \| \nabla \vec{n} \|_{L^p(\mathbb{R}^2)} \leq \| \nabla^2 \vec{n} \|_{L^q(\mathbb{R}^2)} \| \nabla \vec{n} \|_{L^2(\mathbb{R}^2)}^{1-\alpha} \leq C(\varepsilon_0) \varepsilon_0^{1-\alpha} \| e^\lambda \vec{T} \|_{L^p(\mathbb{R}^2)}^{\alpha}, \]
for
\[ \frac{1}{p} = \frac{1}{2} + \left( \frac{1}{q} - 1 \right) \alpha \quad \text{and} \quad 0 \leq \alpha \leq 1. \]

Equivalently,
\[ \| \nabla^3 \vec{n} \|_{L^p(\mathbb{R}^2)} \leq C(\varepsilon_0) \varepsilon_0^{3(1-\alpha)} \| e^\lambda \vec{T} \|_{L^p(\mathbb{R}^2)}^{3\alpha}, \]
for
\[ \frac{1}{b} = \frac{3}{2} + 3 \left( \frac{1}{q} - 1 \right) \alpha. \]

As \( e^\lambda \Delta \vec{V} = -e^\lambda \vec{v} = O(\| \nabla \vec{n} \|^3) \) and \( \vec{T} = \nabla \vec{V} \), the latter yields
\[ \| e^\lambda \Delta \vec{V} \|_{L^p(\mathbb{R}^2)} \leq C(\varepsilon_0) \varepsilon_0^{3(1-\alpha)} \| e^\lambda \nabla \vec{V} \|_{L^p(\mathbb{R}^2)}^{3\alpha}, \]
hence, using (II.46),
\[ \| \Delta \vec{V} \|_{L^p(\mathbb{R}^2)} \leq C(\varepsilon_0) \varepsilon_0^{3(1-\alpha)} \| e^\lambda \|_{L^\infty(\mathbb{R}^2)}^{1-\alpha} \| \nabla \vec{V} \|_{L^2(\mathbb{R}^2)}^{3\alpha}. \]  
(II.53)

Let \( \delta \in (0, 2/3) \). We specialize to
\[ q = 2 - \delta \quad \text{and} \quad 3\alpha = \frac{1}{1 - \delta}. \]

This gives
\[ \frac{1}{b} = \frac{3}{2} - \frac{1}{2 - \delta}, \quad \text{so that} \quad b \in (1, 2). \]

Using the Sobolev embedding theorem in (II.53) then gives
\[ \| \nabla \vec{V} \|_{L^{\frac{2-\delta}{2\delta}}(\mathbb{R}^2)} \leq C(\varepsilon_0) \varepsilon_0^{2-3\delta} \| e^\lambda \|_{L^\infty(\mathbb{R}^2)} \| \nabla \vec{V} \|_{L^{2-\delta}(\mathbb{R}^2)}. \]  
(II.54)

Since
\[ \frac{1 - \delta}{2 - \delta} + \frac{1}{2 - \delta} = 1, \]
we interpolate (II.54) to find
\[ \| \nabla \vec{V} \|_{L^{\frac{2(1-\delta)}{2-\delta}}(\mathbb{R}^2)} \leq C(\varepsilon_0) \varepsilon_0^{2-3\delta} \| e^\lambda \|_{L^\infty(\mathbb{R}^2)} \| \nabla \vec{V} \|_{L^{2-\delta}(\mathbb{R}^2)}. \]

Letting \( \delta \searrow 0 \) reveals that
\[ \| \nabla \vec{V} \|_{L^{2}(\mathbb{R}^2)} \leq C(\varepsilon_0) \varepsilon_0 \| \nabla \vec{V} \|_{L^{2}(\mathbb{R}^2)}. \]

Since \( \varepsilon_0 \) can be adjusted at will, the latter implies that \( \nabla \vec{V} \equiv \vec{0} \), hence that \( \vec{v} = -\Delta \vec{V} \equiv \vec{0} \), and therefore that \( \nabla \vec{n} \equiv \vec{0} \). This guarantees that \( \Sigma \) is a flat plane, as announced.
A Appendix

A.1 Notational conventions

We append an arrow to all the elements belonging to $\mathbb{R}^m$. To simplify the notation, by $\vec{\Phi} \in X(D_1(0))$ is meant $\vec{\Phi} \in X(D_1(0), \mathbb{R}^m)$ whenever $X$ is a function space. Similarly, we write $\nabla \vec{\Phi} \in X(D_1(0))$ for $\nabla \vec{\Phi} \in \mathbb{R}^2 \otimes X(D_1(0), \mathbb{R}^m)$.

We let differential operators act on elements of $\mathbb{R}^m$ componentwise. Thus, for example, $\nabla \vec{\Phi} \in X(D_1(0))$ for $\nabla \vec{\Phi} \in \mathbb{R}^2 \otimes X(D_1(0), \mathbb{R}^m)$.

We let differential operators act on elements of $\mathbb{R}^m$ componentwise. Thus, for example, $\nabla \vec{\Phi}$ is the element of $\mathbb{R}^2 \otimes X(D_1(0), \mathbb{R}^m)$ with $\mathbb{R}^m$-valued components ($\partial x_1 \vec{\Phi}, \partial x_2 \vec{\Phi}$). If $S$ is a scalar and $\vec{R}$ an element of $\mathbb{R}^m$, then we let $\vec{R} \cdot \nabla \vec{\Phi} := (\vec{R} \cdot \partial x_1 \vec{\Phi}, \vec{R} \cdot \partial x_2 \vec{\Phi})$.

Analogous quantities are defined according to the same logic.

Two operations between multivectors are useful. The interior multiplication $\llcorner \llcorner$ maps a pair comprising a $q$-vector $\gamma$ and a $p$-vector $\beta$ to a $(q-p)$-vector. It is defined via $\langle \gamma \llcorner \llcorner \beta, \alpha \rangle = \langle \gamma, \beta \wedge \alpha \rangle$ for each $(q-p)$-vector $\alpha$.

Let $\alpha$ be a $k$-vector. The first-order contraction operation $\bullet$ is defined inductively through

$\alpha \bullet \beta = \alpha \llcorner \llcorner \beta$ when $\beta$ is a 1-vector,

and

$\alpha \bullet (\beta \wedge \gamma) = (\alpha \bullet \beta) \wedge \gamma + (-1)^{pq} (\alpha \bullet \gamma) \wedge \beta$,

when $\beta$ and $\gamma$ are respectively a $p$-vector and a $q$-vector.

A.2 Some useful elliptic results

The following result is established in the Appendix of [BR1].

**Lemma A.1** Let $D$ be a disk and suppose that $G = G_1 + G_2$ satisfies

$$\text{div } G = 0 \quad \text{on } \quad D,$$

where

$$G_1 \in W^{-1,2}(D, \mathbb{R}^2), \quad G_3 \in L^1(D, \mathbb{R}^2).$$

Then there exists an element $L$ in the space $L^{2,\infty}(D, \mathbb{R})$ such that

$$G = \nabla L,$$

and

$$\| L - L_D \|_{L^{2,\infty}(D)} \leq C(\| G_1 \|_{W^{-1,2}(D)} + \| G_2 \|_{L^1(D)}),$$

where $L_D$ denotes the average of $L$ on the disk $D$, and $C$ is a universal constant.
Proposition A.1 Let $D_p(x) \subset D_1(0)$, and let $u \in W^{1, (2, \infty)}(D_p(x))$ satisfy the equation
\[
\Delta u = \nabla b \cdot \nabla u + \text{div}(b \nabla f) \quad \text{on} \quad D_p(x),
\] (A.1)
where $f \in W_0^{2,p}(D_p(x))$ for some $p > 1$, with
\[
\|\nabla^2 f\|_{L^p(D_p(x))} \lesssim \|f\|_{L^p(D_p(x))}.
\]
Suppose moreover that
\[
b \in W^{1,2} \cap L^\infty(D_p(x)) \quad \text{with} \quad \|\nabla b\|_{L^2(D_p(x))} < \varepsilon_0 \quad \text{and} \quad \|b\|_{L^\infty(D_p(x))} \leq 1,
\] (A.2)
for some $\varepsilon_0$ chosen to be “small enough”. Then
\[
\|\nabla u\|_{L^p(D_{\rho/3}(x))} \leq C(\varepsilon_0) \left[ \rho^\frac{2}{3-2p} \|\nabla u\|_{L^2(D_p(x))} + \rho^{\frac{2}{3-2p}} \|f\|_{L^p(D_p(x))} \right],
\]
for some constant $C(\varepsilon_0)$ depending only on $\varepsilon_0$, and where $s < 2/(2 - p)$ if $p \in (1, 2)$, or $s < \infty$ if $p \geq 2$.

**Proof.** Suppose first that $p \in (1, 2)$. Then for every $D_p(x) \subset D_p(x)$, it holds
\[
\|\nabla f\|_{L^p(D_p(x))} \lesssim \sigma^{\frac{2}{3}} \|\nabla f\|_{L^{2/(2 - p)}(D_p(x))} \lesssim \sigma^2 \|f\|_{L^p(D_p(x))}.
\] (A.3)
Let us fix once and for all some point $x_0 \in D_{3\rho/4}(x)$ and some radius $0 < r \leq \rho/4$, so that the disk $D_r(x_0)$ of radius $r$ and centered on the point $x_0$ is contained in $D_p(x)$. With the help of the theorem of Fubini, we may always find some $r_0 \in (r/2, r)$ such that
\[
\int_{\partial D_{r_0}(x_0)} |\nabla u|^\frac{1}{2} \lesssim \frac{1}{r} \int_{D_r(x_0)} |\nabla u|^\frac{1}{2} \lesssim r^{-\frac{1}{2}} \|\nabla u\|_{L^\infty(D_p(x_0))}^\frac{2}{3} \lesssim r_0^{-\frac{1}{2}} \|\nabla u\|_{L^\infty(D_p(x))}^\frac{2}{3}.
\] (A.4)
We next define $u = u_0 + u_1$, where the new variables, in accordance with (A.1), satisfy
\[
\begin{alignat}{2}
\Delta u_0 &= \text{div} (b \nabla f), & \quad \Delta u_1 &= \nabla b \cdot \nabla u \quad \text{in} \quad D_{r_0}(x_0) \\
u_0 &= u, & \quad u_1 &= 0 \quad \text{on} \quad \partial D_{r_0}(x_0).
\end{alignat}
\]
Let
\[
\bar{u} := \frac{1}{2\pi r_0} \int_{\partial D_{r_0}(x_0)} u.
\]
Standard elliptic theory, our assumptions on $b$ and $f$, and the Sobolev embedding theorem give
\[
\|\nabla u_0\|_{L^2(D_{r_0}(x_0))} \lesssim \|\nabla u\|_{L^2(D_{r_0}(x_0))} + \|u - \bar{u}\|_{H^{1/2}(\partial D_{r_0}(x_0))}
\lesssim \|\nabla f\|_{L^2(D_{r_0}(x_0))} + r_0^{\frac{1}{2}} \|\nabla u\|_{L^\infty(D_{r_0}(x_0))}
\lesssim r_0^{-\frac{2}{3}} \|f\|_{L^p(D_p(x))} + \|\nabla u\|_{L^\infty(D_p(x))},
\] (A.5)
where (A.3) and (A.4) were used.
To handle $u_1$, we apply Wente’s inequality in the form of Lemma IV.2 in [BR3] to obtain
\[
\|\nabla u_1\|_{L^2(D_{r_0}(x_0))} \lesssim \|\nabla b\|_{L^2(D_{r_0}(x_0))} \|\nabla u\|_{L^\infty(D_{r_0}(x_0))} \leq \varepsilon_0 \|\nabla u\|_{L^\infty(D_p(x))}.
\] (A.6)
Altogether, (A.5) and (A.6) yield that $\nabla u$ belongs to $L^2(D_{r_0}(x_0))$. In particular
\[
\|\nabla u\|_{L^2(D_{r_0}(x_0))} \lesssim r_0^{-\frac{2}{3}} \|f\|_{L^p(D_p(x))} + \|\nabla u\|_{L^\infty(D_p(x))}.
\] (A.7)
Let now $k \in (0,1)$. Using again (A.3) and standard elliptic theory and growth estimates give
\[
\|\nabla u_0\|_{L^2(D_{kr_0}(x_0))} \lesssim \|b\nabla f\|_{L^2(D_{r_0}(x_0))} + k\|\nabla u_0\|_{L^2(D_{r_0}(x_0))}
\lesssim \|\nabla f\|_{L^2(D_{r_0}(x_0))} + k\|\nabla u_0\|_{L^2(D_{r_0}(x_0))}
\lesssim 2^{-\frac{2}{p}}\|F\|_{L^p(D_{kr}(x))} + k\|\nabla u_0\|_{L^2(D_{r_0}(x_0))}.
\] (A.8)

For $u_1$, we apply Wente’s inequality this time as in Theorem 3.4.1 of [Hel] so as to find
\[
\|\nabla u_1\|_{L^2(D_{kr_0}(x_0))} \lesssim \|\nabla b\|_{L^2(D_{r_0}(x_0))} \|\nabla u\|_{L^2(D_{r_0}(x_0))} \lesssim \varepsilon_0 \|\nabla u\|_{L^2(D_{r_0}(x_0))},
\] (A.9)
again up to some multiplicative constant without bearing on the sequel. Hence, combining (A.8) and (A.9) we obtain the estimate
\[
\|\nabla u\|_{L^2(D_{kr_0}(x_0))} \lesssim (k + \varepsilon_0 + k\varepsilon_0) \|\nabla u\|_{L^2(D_{r_0}(x_0))} + 2^{-\frac{2}{p}}\|F\|_{L^p(D_{kr}(x))}.
\] (A.10)

Because $\varepsilon_0$ is a small adjustable parameter, we may always choose $k$ so as to arrange for $(k + \varepsilon_0 + k\varepsilon_0)$ to be small enough. A standard controlled-growth argument (see Lemma III.2.1 in [Gia]) along with (A.7) enables us to conclude that for some constant $C(\varepsilon_0)$, there holds
\[
\|\nabla u\|_{L^2(D_{\rho}(x_0))} \leq C(\varepsilon_0)\sigma^{2-\frac{2}{p}}\left[\frac{1}{r_0^{\frac{2}{p}-2}} \|\nabla u\|_{L^2(D_{r_0}(x_0))} + \|F\|_{L^p(D_{kr}(x))}\right]
\leq C(\varepsilon_0)\sigma^{2-\frac{2}{p}}\left[\rho^{\frac{2}{p}-2} \|\nabla u\|_{L^2(D_{\rho}(x_0))} + \|F\|_{L^p(D_{\rho}(x))}\right],
\]
for $x_0 \in D_{3\rho/4}(x)$ and $\sigma \in (0, r_0)$.

In particular, for $r_0 = \rho/4$, we find
\[
\|\nabla u\|_{L^2(D_{\sigma}(x_0))} \leq C(\varepsilon_0)\sigma^{2-\frac{2}{p}}\left[\rho^{\frac{2}{p}-2} \|\nabla u\|_{L^2(D_{\rho}(x_0))} + \|F\|_{L^p(D_{\rho}(x))}\right],
\] (A.11)
for $x_0 \in D_{3\rho/4}(x)$ and $\sigma \in (0, \rho/4)$.

Consider next the maximal function
\[
Mg(y) := \sup_{\sigma > 0} \sigma^{\frac{2}{p}-2} \int_{D_{\sigma}(y)} |g(z)| \, dz.
\] (A.12)

We recast the equation (A.1) in the form
\[
-\Delta u = b \Delta f + \nabla b \cdot (\nabla^\perp u + \nabla f).
\]

Calling upon (A.2)–(A.3) and upon the estimate (A.11), we derive that for $y \in D_{3\rho/4}(x)$, there holds
\[
M\left(\chi_{D_{\rho/4}(y)} \Delta u(z)\right)(y) \leq \|b\|_{L^\infty(D_{\rho}(x))} \sup_{0<\sigma<\frac{\rho}{4}} \sigma^{\frac{2}{p}-2} \|\Delta f\|_{L^1(D_{\sigma}(y))}
+ C(\varepsilon_0)\|\nabla b\|_{L^2(D_{\rho}(x))} \sup_{0<\sigma<\frac{\rho}{4}} \sigma^{\frac{2}{p}-2} \left(\|\nabla u\|_{L^2(D_{\sigma}(y))} + \|\nabla f\|_{L^2(D_{\sigma}(y))}\right)
\leq C(\varepsilon_0)\left[\|\nabla^2 f\|_{L^1(D_{\rho}(x))} + \rho^{\frac{2}{p}-2} \left(\|\nabla u\|_{L^2(D_{\rho}(x))} + \|\nabla f\|_{L^2(D_{\rho}(x))}\right)\right]
\leq C(\varepsilon_0)\left[\|F\|_{L^p(D_{\rho}(x))} + \rho^{\frac{2}{p}-2} \|\nabla u\|_{L^2(D_{\rho}(x))}\right].
\] (A.13)
On the other hand, from [Ada3] and [Ada11], we have
\[ \|\Delta u\|_{L^1(D_{3p/4}(x))} \leq \|\Delta f\|_{L^1(D_{3p/4}(x))} + \|\nabla u\|_{L^2(D_{3p/4}(x))} + \|\nabla f\|_{L^2(D_{3p/4}(x))} \]
\[ \leq C(\varepsilon_0) \left[ \rho^{2-\frac{p}{2}} \|F\|_{L^p(D_p(x))} + \|\nabla u\|_{L^2}(D_{p}(x)) \right]. \quad (A.14) \]

Proposition 3.2 from [Ada] states that
\[ \|z^{-1} \chi_{D_{p/4}(y)} \Delta u\|_{L^{\infty}(D_{p/4}(x))} \leq \|M(\chi_{D_{p/4}(y)} \Delta u)\|_{L^{\infty}(D_{p/4}(x))} \|\Delta u\|_{L^1(D_{p/4}(x))}, \]
where \( \alpha := 2/(2-p) > 2 \). Hence, according to (A.13) and (A.14), we have
\[ \|z^{-1} \chi_{D_{p/4}(y)} \Delta u\|_{L^{\infty}(D_{p/4}(x))} \]
\[ \leq C(\varepsilon_0) \left[ \|F\|_{L^p(D_p(x))} + \rho^{2-\frac{p}{2}} \|\nabla u\|_{L^2}(D_{p}(x)) \right]^{\frac{1}{2}} \]
\[ \leq C(\varepsilon_0) \rho^{1-p} \left[ \rho^{2-\frac{p}{2}} \|F\|_{L^p(D_p(x))} + \|\nabla u\|_{L^2}(D_{p}(x)) \right]. \quad (A.15) \]

We let \( y \in D_{3p/4}(x) \) and we again decompose \( u = u_2 + u_3 \) with
\[ \begin{aligned}
u_2 & = 0 & \Delta u_3 = \chi_{D_{p/4}(y)} \Delta u & \text{in } D_{3p/4}(x) \\
u_3 & = u & u_3 = 0 & \text{on } \partial D_{3p/4}(x). 
\end{aligned} \]
Let \( s \in (2, \alpha) \). Using standard estimates for the harmonic function \( u_2 \) and the estimate (A.15) gives
\[ \|\nabla u\|_{L^s(D_{p/4}(x))} \leq \|\nabla u_2\|_{L^s(D_{p/4}(x))} + \|\nabla u_3\|_{L^s(D_{p/4}(x))} \]
\[ \leq \rho^{\frac{s}{2} - \frac{p}{2}} \|\nabla u_2\|_{L^{s/2}(D_{p/4}(x))} + \|\nabla u_3\|_{L^{s/2}(D_{p/4}(x))} \]
\[ \leq \rho^{\frac{s}{2} - \frac{p}{2}} \|\nabla u\|_{L^{s/2}(D_{p/4}(x))} + \|\nabla u_3\|_{L^s(D_{p/4}(x))} \]
\[ \leq \rho^{\frac{s}{2} - \frac{p}{2} - \frac{1}{2}} \|\nabla u\|_{L^{s/2}(D_{p/4}(x))} + \|\nabla u_3\|_{L^s(D_{p/4}(x))} \]
\[ \leq C(\varepsilon_0) \left[ \rho^{\frac{s}{2} - \frac{p}{2}} \|\nabla u\|_{L^{s/2}(D_{p/4}(x))} + \rho^{\frac{s}{2} - \frac{p}{2} + 1} \|F\|_{L^p(D_p(x))} \right]. \quad (A.16) \]
As seen above, we can choose any \( s < 2/(2-p) \).

Suppose next that \( p \in (2, \infty) \). Let \( s \in (2, \infty) \) be arbitrary. Choose \( 0 < \varepsilon < 2/s \). Then, setting \( q = 2 - \varepsilon \), we have
\[ \|F\|_{L^s(D_p(x))} \leq \rho^{\frac{s}{2} - \frac{p}{2}} \|F\|_{L^p(D_p(x))}. \]
Since \( s < 2/(2-q) \), we have per the above discussion that
\[ \|\nabla u\|_{L^s(D_{p/4}(x))} \leq C(\varepsilon_0) \left[ \rho^{\frac{s}{2} - \frac{p}{2} - \frac{1}{2}} \|\nabla u\|_{L^{s/2}(D_{p/4}(x))} + \rho^{\frac{s}{2} - \frac{p}{2} + 1} \|F\|_{L^p(D_p(x))} \right] \]
\[ \leq C(\varepsilon_0) \left[ \rho^{\frac{s}{2} - \frac{p}{2} - \frac{1}{2}} \|\nabla u\|_{L^{s/2}(D_{p/4}(x))} + \rho^{\frac{s}{2} - \frac{p}{2} + 1} \|F\|_{L^p(D_p(x))} \right]. \]
In other words, (A.16) holds for all \( p \in (1, \infty) \), with any \( s < 2/(2-p) \) if \( p \in (1, 2) \) and any \( s < \infty \) if \( p \geq 2 \). We combine these facts by writing the (A.16) holds for all \( s < 2/(2-p)_+ \).
References

[Ada] Adams, Robert “A note on Riesz potentials.” Duke Math. J. 42 (1975), no. 4, 765–778.

[Ber1] Bernard, Y. “Analysis of constrained Willmore surfaces.” Comm. PDE 41 (2016), no. 10, 1513–1552.

[Ber2] Bernard, Y. “Noether’s theorem and the Willmore functional.” Adv. Calc. Var. 9 (2016), no. 3, 217–234.

[Ber2b] Bernard, Y. “Noether’s theorem and the Willmore functional.” arXiv preprint 1409.6894.

[BR1] Bernard, Y.; Rivière, T. “Local Palais-Smale sequences for the Willmore functional.” Comm. Anal. Geom. 19 (2011), 563–599.

[BR2] Bernard, Y.; Rivière, T. “Singularity removability at branch points for Willmore surfaces.” Pacific J. Math. 265 (2013), 257–311.

[BR3] Bernard, Y.; Rivière, T. “Energy quantization for Willmore surfaces and applications.” Ann. of Math. 180 (2014), 87–136.

[BR4] Bernard, Y.; Rivière, T. “Ends of immersed minimal and Willmore surfaces in asymptotically flat spaces.” arXiv preprint 1508.01391.

[BWW] Bernard, Y.; Wheeler, G.; Wheeler, V.-M. “Rigidity and stability of spheres in the Helfrich model.” Interfaces Free Bound. 19 (2017), 495–523.

[BWW2] Bernard, Y.; Wheeler, G.; Wheeler, V.-M. “Concentration-compactness and finite-time singularities for Chen’s flow” arXiv preprint 1706.01707.

[BPP] Bohle, C.; Peters, G.P.; Pinkall, U. “Constrained Willmore surfaces.” Calc. Var. PDE 32 (2008), 263–277.

[Can] Canham, P.B. “The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell.” J. Theor. Biol. 26 (1970), 61–81.

[Che1] Chen, B.-Y. “Some open problems and conjectures on submanifolds of finite type.” Soochow J. Math. 17 (1991), 169–188.

[Che2] Chen, B.-Y. “Recent developments of biharmonic conjecture and modified biharmonic conjectures.” arXiv preprint 1307.0245.

[Dim1] Dimitrić, I. “Submanifolds of $E^n$ with harmonic mean curvature vector.” Bull. Inst. Math. Acad. Sinica 20 (1992), 53–65.

[Dim2] Dimitrić, I. “Quadric representation and submanifolds of finite type.” Ph.D. Thesis, Michigan State University (1989).

[Gia] Giaquinta, M. “Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.” PUP, Princeton (1983).

[HV] Hasanis, T.; Vlachos, T. “Hypersurfaces in $E^4$ with harmonic mean curvature vector field.” Math. Nachr. 172 (1995), 145–169.

[Hel] Hélein, F. “Harmonic Maps, Conservation Laws, and Moving Frames.” Cambridge Tracts in Mathematics, 150. Cambridge University Press (2002).
[Hef] Helfrich, W. “Elastic properties of lipid bilayers: theory and possible experiments.” Z. Naturforsch., C28 (1973), 693–703.

[Hub] Huber, Alfred “On subharmonic functions and differential geometry in the large.” Comment. Math. Helv. 32 (1957), 181–206.

[Jia] Jiang, G.Y. “2-harmonic isometric immersions between Riemannian manifolds.” Chinese Ann. Math. Ser. A, 7 (1986), 130–144.

[KMR] Keller, L.G.A.; Mondino, A.; Rivi`ere, T. “Embedded surfaces of arbitrary genus minimizing the Willmore energy under isoperimetric constraint.” Arch. Rat. Mech. Anal. 212 (2014), 645–682.

[KS1] Kuwert, E.; Schätzle, R. “The Willmore flow with small initial energy.” J. Diff. Geom. 57 (2001), 409–441.

[KS2] Kuwert, E.; Schätzle R. “Removability of point singularities of Willmore surfaces.” Ann. of Math. 160 (2004), no. 1, 315–357.

[KS3] Kuwert, E.; Schätzle, R. “Minimizers of the Willmore energy under fixed conformal class.” J. Diff. Geom. 93 (2013), 471–530.

[Lnk] Link, F. “Gradient flow for the Willmore functional in Riemannian manifolds of bounded geometry.” arXiv preprint 1308.6055.

[MN] Marques, F.C.; Neves, A. “Min-max theory and the Willmore conjecture.” Ann. of Math. 179 (2014), 683–782.

[MPW] McCoy, J; Parkins, S.; Wheeler, G. “The geometric triharmonic heat flow of immersed surfaces near spheres.” Nonlinear Anal. 161 (2017), 44–86.

[MW1] McCoy, J; Wheeler, G. “A classification theorem for Helfrich surfaces.” Math. Ann. 357:4 (2013), 1485–1508.

[MW2] McCoy, J; Wheeler, G. “Finite-time singularities for the locally constrained Willmore flow of surfaces.” Comm. Anal. Geom. 24 (2016), no. 4, 843–886.

[MWW] Metzger, J; Wheeler, G.; Wheeler, V.-M. “Willmore flow of surfaces in Riemannian spaces I: Concentration-compactness.” arXiv preprint 1308.6024.

[MS] Müller, Stefan ; ˇSver´ak, Vladim´ir “On surfaces of finite total curvature.” J. Diff. Geom. 42 (1995), no. 2, 229–258.

[Riv1] Rivi`ere, T. “Analysis aspects of the Willmore functional.” Invent. Math. 174 (2008), no. 1, 1–45.

[Riv2] Rivi`ere, T. “Variational principles for immersed surfaces with \(L^2\)-bounded second fundamental form.” J. reine angew. Math. (2013).

[Riv3] Rivi`ere, T. “Weak immersions of surfaces with \(L^2\)-bounded second fundamental form”. Parks City lecture notes, July 2013.

[Sch] Schätzle, R. “Conformally constrained Willmore immersions.” Adv. Calc. Var. 6 (2013), 375–390.

[Scy] Schygulla, J. “Willmore minimizers with prescribed isoperimetric ratio”, Arch. Rat. Mech. Anal. 203 (2012), no. 3, 901–941.

[Tar] Tartar, Luc “An Introduction to Sobolev Spaces and Interpolation Spaces.” Lectures notes of the Unione Matematica Italiana, no. 3 (2007).
[Wei] Weiner, J. “On a problem of Chen, Willmore, et al.” Indiana Univ. Math. J. 27 (1978), no. 1, 19–35.

[Whe] Wheeler, G. “Fourth order geometric evolution equations” Bull. Aust. Math. Soc. 82 (2010). PhD thesis.

[Whe1] Wheeler, G. “Gap phenomena for a class of fourth-order geometric differential operators on surfaces with boundary.” Proc. AMS 143 (2015), no. 4, 1719–1737.

[Whe2] Wheeler, G. “Surface diffusion flow near spheres.” Calc. Var. PDE 44 (2012), no 1, 131–151.

[Whe3] Wheeler, G. “Chen’s conjecture and $\varepsilon$-superbiharmonic submanifolds of Riemannian manifolds.” Internat. J. Math. 24 (2013), no. 4. 1350028.

[Wil] Willmore, T. J. “Note on embedded surfaces.” Ann. Stiint. Univ. “Al. I. Cuza” Iasi. Sect. I a Mat. (N.S.) 11B (1965), 493–496.