Elicitation of ambiguous beliefs with mixing bets

Patrick Schmidt
University of Zurich

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Abstract

Considering ambiguous beliefs, I show how to reveal ambiguity perception for several preference classes. In the proposed elicitation mechanism, agents mix binarized bets on the uncertain event and its complement under varying betting odds. Mixing is informative about the interval of relevant probabilities. In particular, the mechanism allows to distinguish ambiguous beliefs from point beliefs, and identifies the interval of relevant probabilities for maxmin and maxmax preferences. For smooth second order and variational preferences, the mechanism reveals inner bounds, which are sharp under additional assumptions. An experimental implementation suggests that participants perceive almost as much subjective ambiguity for the stock index and actions of other participants as for the Ellsberg urn, indicating the importance of ambiguity in real-world decision making. For the stock market, female participants perceived more ambiguity, but were neither more pessimistic nor more ambiguity averse.

Keywords — ambiguity aversion, binarized score, belief elicitation, interval probability, subjective expectation, uncertainty aversion

JEL codes: D81, D82, D83.
1 Introduction

Most economic modeling is based on subjective expected utility (SEU) (Savage, 1954). However, uncertainty often cannot be represented by a precise probability measure. Instead, the perception of uncertainty is ambiguous (Knight, 1921). Initiated by Ellsberg (1961), various experiments showed that ambiguity matters for decision making (for surveys on ambiguity sensitive decision models see Etner et al. (2012), Machina and Siniscalchi (2014), Trautmann and van de Kuilen (2015)).

While artificially generated ambiguity in experiments is well studied, there is little evidence on ambiguity for natural uncertainty, which makes the application of ambiguity sensitive preferences in real-world applications challenging.

1.1 Illustration: A simple mixing bet

In this paper, I propose a simple mechanism that is informative about the ambiguity of a natural event $E$. In its simplest form, the elementary building block of the mechanism is the choice between

- $[E_q]$ a lottery that pays with probability $q$ if the event $E$ realizes (“betting on the event”),
- $[C_q]$ a lottery that pays with probability $1 - q$ if the the event $E$ does not realize (“betting on the complement”), and
- $[M_q]$ a lottery that pays with probability $q(1 - q)$ (“mixing”).

The lottery $[M_q]$ can be interpreted as probabilistic mixture of option $[E_q]$ and option $[C_q]$ that does not dependent on the potentially ambiguous event $E$. The value associated with each of the choices depends on $q$. Figure 1 illustrates the value functions for SEU preferences, probabilistically sophisticated preferences (Machina and Schmeidler, 1992), and ambiguity averse preferences. All three examples assign the probability $p$ to the event $E$. Under all preferences, the option $[C_q]$ becomes more attractive and the option $[E_q]$ less attractive with increasing $1 - q$. The value $1 - q$ at which the decision maker switches between the choice $[E_q]$ and $[C_q]$ can be used to elicit the subjective probability $p$.

Under expected utility, the value of $[E_q]$ and $[C_q]$ is linear in $q$. Further, the decision maker is indifferent between the mixture $[M_q]$ and its elements $[E_q]$ and $[C_q]$ if those have equal value. The best response is the choice $[E_q]$ for large $q$ and $[C_q]$ for large $1 - q$. There exists no $q$ such that the choice $[M_q]$ is the unique best response.

The same holds true for probabilistically sophisticated preferences, where each choice is associated with a probability of payout that is transformed with a monotone value function.

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Footnote 1: Here, I argue within the Anscombe-Aumann framework and assume that prospects can be evaluated based on the distribution they induce on the outcome space. The main part of the paper will treat the state space as the interaction of an ambiguous source created by the unknown event and an unambiguous source created by the randomization device (compare e.g. Ergin and Gul (2009), Strzalecki (2011), Webb (2017)).
Figure 1: The value of the three choices $[E_q], [C_q], \text{and } [M_q]$ depending on $1 - q$. The left plot illustrates the utility functional under SEU, the middle plot under probabilistically sophisticated preferences, and the right plot under ambiguity averse preferences. In this example, the subjective probability is $p = 0.3$, and the probabilistically sophisticated values are based on a probability weighting function $w(p) = \exp(-(-\ln(p))^{3/4})$ (compare Prelec, 1998). Under ambiguity aversion, the ambiguous choices $[E_q]$ and $[C_q]$ are less attractive and the choice $[M_q]$ is the unique best response for any $1 - q$ in the interval $M$.

The difference in value between the three choices are subject to a monotone transformation. The best response remains unaffected and there exists no $q$ such that the agent strictly prefers the mixing choice $[M_q]$.

Under ambiguity aversion, the value of the choice $[M_q]$ remains unchanged, the ambiguous choices $[E_q]$ and $[C_q]$, however, are less attractive. Thus, the choice $[M_q]$ is the best response for some interval $M$ of values $1 - q$. For ambiguity averse preferences, the interval $M$ at which the decision maker prefers $[M_q]$ is related to ambiguity perception and ambiguity attitude. In particular, the set $M$ contains the probability $p$ associated with ambiguity neutral preferences. Further, the set $M$ is larger for more ambiguity averse preferences.

Without additional structure, ambiguity aversion cannot be separated into ambiguity attitude and perception. In the following, I define and identify ambiguity perception in the form of an interval of beliefs for specific classes of ambiguity averse preferences.

### 1.2 Separation of ambiguity perception: The belief interval

To understand the empirical content of decision models, it is crucial to separate perception and attitude (Manski, 2004). Let us consider the task of identifying the set of probabilities that potentially influence an agent’s decision. I call the range of probabilities that are necessary to describe the agents behavior the **belief interval**. Preferences are said to exhibit *ambiguous beliefs* if the belief interval is not a single point. The following representations for
preferences over acts $l$ that depend on an uncertain event $E$ allow to define a belief interval. The classical subjective expected utility (SEU) by Savage (1954) can be represented with a single probability $p$ in the unit interval and a utility function $u$ by

$$E_p[u(l)].$$

The non-ambiguous beliefs for SEU preferences reduce to a single point $p$. Other models require ambiguous beliefs for their representation. Maxmin expected utility (maxmin) by Gilboa and Schmeidler (1989) can be represented with a belief interval $B = [a, b]$ by

$$\min_{p \in B} E_p[u(l)].$$

The more general variational preferences by Maccheroni et al. (2006) can be represented with a positive cost function $c$ by

$$\min_{p \in B} E_p[u(l)] + c(p).$$

In second order decision models, ambiguity aversion is defined as aversion to uncertainty on the expected utility. Such ambiguity averse second order smooth preferences by Klibanoff et al. (2005) can be represented with a probability measure $P$ on the unit interval and a concave second order utility function $\phi$ by

$$E_{p \sim P}[\phi(E_p[u(l)])].$$

For second order preferences the belief interval $B$ is the support of the probability measure $P$. Other ambiguity averse preferences, e.g. biseparable preferences (Ghirardato and Marinacci 2001) that include $\alpha$-maxmin (Ghirardato et al. 2004, Marinacci 2002) and Choquet expected utility (Schmeidler 1989), do not allow for a similar separation of a belief interval from ambiguity attitude.

For the elicitation of the belief interval, consider the following mechanism that contains the prospects $[E_q], [C_q],$ and $[M_q]$ from Section 1.1 as special cases. The agent is endowed with lottery tickets, where each ticket represents a fixed probability to win a prize (e.g., a monetary reward). The agent has to bet each ticket on the event or its complement. The two events have different betting odds. If the event realizes, the agent obtains the tickets placed on the event multiplied by the odds of the event. Otherwise, she obtains the tickets placed on the complement multiplied by the odds of the complement. This task is called a mixing bet and it is repeated with different odds, where one instance is randomly selected for payout. The best response of the agent to a mixing bet depends on the ratio between the odds of the event and the odds of the complement denoted as odds quota $q$.

The lottery tickets guarantee robustness with respect to the unknown utility function (Smith 1961), if one is willing to assume that the randomization device is perceived as independent and objective lottery. Paying out only one mixing bet with specific odds is
meant to prevent hedging across the repeated betting tasks (see Azrieli et al. 2018; Bade, 2015, for discussions on validity and further references). I establish the following results: If the odds quota is above the belief interval, the best response is to bet all tickets on the event. Reversed, if the odds quota is below, the best response is to bet all tickets on the complement. Under the ambiguity averse preferences considered above, mixing (betting tickets on the event and the complement) is a sufficient condition for the odds quota being in the belief interval. Beliefs are ambiguous (i.e., they do not reduce to a single probability level) if and only if the agent mixes for at least two different quotas. Thus, ambiguous beliefs can be identified by eliciting mixing behavior for different betting odds.

As the interval of mixing quotas lies within the belief interval, the belief interval can be bounded from within. For maxmin preferences, the bounds are sharp. Under second order and variational preferences with sufficiently strong ambiguity aversion, the mixing interval recovers the belief interval for large utility differences between the prizes.

In a laboratory experiment, the mechanism is applied to events generated by an Ellsberg urn, by a stock index, and by another participant’s behaviour in a prisoners dilemma game. As expected, ambiguity perception is highest for the ambiguous colour in the Ellsberg urn and lowest for the risky colour. After observing additional draws from the urn, ambiguity perception reduces. Interestingly, ambiguity perception for the stock index and the social event, generated in the prisoners dilemma, is almost as large as for the ambiguous colour, which suggests the potential importance of ambiguity real-world decision making. Female participants were not more ambiguity averse but perceived more ambiguity for the Ellsberg urn and for the stock market.

1.3 Related literature

The contribution of this paper is the introduction of an applicable mechanism to elicit ambiguity perception under a wide range of ambiguity averse preferences. Related work obtains similar results at the expense of generality across decision models or simplicity of the mechanism. Bose and Daripa (2017a) extend the mechanism introduced by Karni (2009) to $\alpha$-maxmin preferences. In another paper Bose and Daripa (2017b) introduce a mechanism that identifies the distribution of beliefs for second order preferences. Baillon et al. (2018) propose indices of ambiguity attitude and perception based on matching probabilities (Dimmock et al., 2015) for three mutually exclusive events and their pairwise unions. Baillon et al. (2019) show that their index is insightful under a wide range of ambiguity sensitive models. However, their index of ambiguity perception is not applicable for binary events. Li et al. (2018) apply the method in a trust game and Anantanasuwong et al. (2019) to elicit ambiguity perception about different assets from a sample of investors.

Other work focuses on the revelation of dynamic information structures. Chambers and Lambert (2021) discuss the truth-telling mechanisms for dynamic elicitation of subjective probabilities of a potentially information receiving agent. Karni (2020) proposes elicitation with a quadratic scoring rule on the set of of arising posteriors in a dynamic context, which is
also applicable for the ambiguity model introduced in Karni and Safra (2016). Karni (2018) considers a similar approach for graded preferences (see Minardi and Savochkin, 2015).

So far, applied studies rely mostly on proxies for ambiguity. Brenner and Izhakian (2018) use the marginal distribution of intra day data, Anderson et al. (2009) the disagreement between professional forecasters, and Rossi et al. (2017) the deviation between probabilistic forecast and realization. Gallant et al. (2018) employ a Bayesian approach in a structural model that features inter-temporal second order preferences (Klibanoff et al., 2009).

Next to revealing ambiguity perception, mixing bets can empirically distinguish between some models of ambiguity. Differentiating between ambiguity sensitive preferences has been considered before with artificially designed events like the Ellsberg urn (e.g., Chew et al., 2017; Cubitt et al., 2020). In an incentivized experiment with mixing bets I provide complementary evidence considering uncertainty generated by two natural sources of uncertainty: The stock market and the other participants.

Finally, the experimental evidence on gender differences for ambiguity perception is connected to studies on gender differences in preferences (Croson and Gneezy, 2009) and stock market participation (Almenberg and Dreber, 2015; Van Rooij et al., 2011).

In the next section, the key findings of the paper are summarized. For technical details see Section 3, where the mixing behavior under different preferences is derived. Sections 3.1 to 3.3 cover maxmin, variational, and second order preferences respectively. Section 4 covers separated mixing bets, which allows to extend the identification result to ambiguity seeking preferences. Section 5 discusses extensions to beliefs about real-valued variables. Section 6 provides an experimental implementation and empirical evidence on ambiguity perception. Section 7 concludes. Proofs are provided in the appendix. A supplementary document discusses biseparable preferences, general ambiguity averse preferences (Cerreia-Vioglio et al., 2011), and the separation of ambiguity perception and attitude for \( \alpha \)-maxmin preferences.

## 2 Mixing bets and the belief interval

Consider the task of eliciting beliefs about an event \( E \) from an agent with unknown preferences. The state space is given by \( S = \{E, E^c\} \times [0, 1] \), where any state \( s \in S \) describes the realization of the event \( E \) and the independent random draw \( r \) of the elicitation mechanism. The agent’s preferences \( \succeq \) are defined on acts \( l : S \mapsto X \) that assign an outcome to each state. The set of all acts is denoted by \( F \).

Throughout the paper, some kind of aversion to ambiguity is assumed within one of several ambiguity sensitive preferences.

**Regularity Conditions 1** (ambiguity aversion). The agent has ambiguity averse smooth second order or variational preferences, where the random draw \( r \) is independent from \( E \) and uniformly distributed.
Essentially, Regularity Condition 1 imply expected utility for the lottery (risk) and ambiguity aversion for acts that depend on the event $E$. Regularity Conditions 1 contain maxmin preferences as a special case. Note that the assumption on the random draw $r$ has to be formulated differently depending on the preference class at hand. Similarly, the exact definition of ambiguity aversion depends on the class of preferences. For details see Regularity Conditions 3 for maxmin, Regularity Conditions 4 for variational, and Regularity Conditions 5 for second order preferences.

The belief interval is defined as the range of relevant probabilities.

**Definition 1** (belief interval). The belief interval $B$ is defined as the smallest closed interval that contains all relevant probability levels for a representation of the preferences.

Heuristically, the belief interval $B$ denotes the relevant probabilities $p$ that the agent considers when making decisions related to the uncertain event $E$. Section 3 provides details on the uniqueness of the belief interval. Cerreia-Vioglio et al. (2011) formulates a general representation of a wide range of ambiguity averse preferences with $U(l) = \min_{p \in B} G(\mathbb{E}_p[u(l)], p)$. Klibanoff et al. (2014) provide a behavioral definition of relevant probabilities in smooth models that coincides with the belief interval for maxmin and second order preferences.

For SEU preferences, the belief interval $B = \{p\}$ can be denoted by unique probability $p$ on the unit interval. Ambiguity averse preferences, however, take into account a range of probability levels.

**Definition 2** (ambiguous beliefs). Preferences are said to exhibit ambiguous beliefs about the event $E$ if the belief interval $B$ is not a single point.

The elementary building block of the elicitation mechanism can be described as follows.

1. The agent chooses the ratio $x$ of lottery tickets that she bets on the event $E$ (and the remainder $1 - x$ on its complement $E^c$).

2. If the event $E$ realizes, the agent receives $xq$ lottery tickets. If the event $E^c$ realizes, the agent receives $(1 - x)(1 - q)$ lottery tickets.

3. The agent is rewarded with the fixed prize $w$ if her ticket amount exceeds a random variable $r$ that is uniformly distributed on $[0, 1]$.

Note that the choices $[E_q], [C_q], \text{ and } [M_q]$ introduced in Section 1.1 can be recovered with $x = 1$, $x = 0$, and $x = 1 - q$. Formally, a mixing choice $x$ can be associated with an act in $\mathcal{F}$. Let $\mathbb{1}(A)$ denote the indicator function for an event $A$.

**Definition 3** (mixing bet with odds quota $q$ and prize $w$). The mixing bet with mixing choice $x \in [0, 1]$, odds quota $q \in [0, 1]$ and prize $w \in X$ is defined as the act

$$l(x, q, w) : S \rightarrow \mathbb{R} : (E, r) \rightarrow w \cdot \mathbb{1}(r \leq (xq\mathbb{1}(E) + (1 - x)(1 - q)\mathbb{1}(E^c))).$$

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2See Klibanoff (2001) for behavioral definition of independent randomization devices.
The two potential outcomes of this mixing bet are $w$ and 0. Throughout, it is assumed that the agent prefers to obtain the prize $w$. The mixing interval $M$ describes all odds for which the agent is mixing between the event and the complement.

**Definition 4** (mixing interval $M$). Let $x^*(q) : [0, 1] \rightarrow [0, 1]$ define an optimal mixing for the odds ratio $q$ such that

$$l(x^*(q), q, w) \geq l(x, q, w)$$

for all $x \in [0, 1]$. The mixing interval $M$ is defined as the smallest closed interval that contains

$$\{q \in [0, 1] \mid x^*(1-q) \in (0, 1)\}.$$  

Trivially, the optimal mixing ratio is $x^*(q) = 1$ (betting all lottery tickets on $E$), if the quota $q$ is large enough. The resulting act is $[E_q]$, a lottery with probability $q$ if the event realizes. Similarly, the optimal mixing is $x^*(q) = 0$ (betting all lottery tickets on $E^c$), if the quota $q$ is small enough. The resulting act is $[C_q]$, a lottery with probability $1 - q$ if the complement realizes. Both acts depend on the potentially ambiguous event $E$. If the agent bets $x^*(q) = 1 - q$ on the event, the resulting act is $[M_q]$, a lottery with probability $q(1-q)$ irrespective of the uncertain event $E$. The ambiguity cancels out. An ambiguity averse agent prefers to mix between the two events to hedge against ambiguity. The higher the ambiguity aversion, the stronger is the optimal mixing drawn towards $1 - q$. The main result of the paper establishes that such mixing implies ambiguous probabilities.

**Theorem 1** (belief interval). Under Regularity Conditions mixing for a quota $q$ implies that $1 - q$ is an element of the belief interval.

$$M \subset B.$$  

In particular, if an agent mixes for two different quotas the agent holds ambiguous beliefs.
Theorem 1 is established in Section 3 for each class of preferences separately. This result allows to bound the belief interval from within.

The set of values of $1 - q$ for which the agent chooses $[M_q]$ is a subset of the mixing interval $M$. Thus, the simple choice from Section 1.1 is sufficient to bound the belief interval. However, the mixing interval $M$ provides sharper bounds for the belief interval. Figure 2 illustrates how observed choices for different odds provide information about the beliefs. In this example, the agent holds ambiguous beliefs, as she is mixing for multiple odds. Further, the position of the interval is consistent with probabilities for the event $E$ ranging at least from 0.6 to 0.8.

Small mixing intervals can only be detected if appropriate odds are applied. For ambiguous beliefs such odds always exist.

**Theorem 2** (ambiguous beliefs). Under Regularity Conditions 1, beliefs are ambiguous if and only if there exist at least two different odds quotas $q$ for which the agent prefers to mix.

The identification result for the belief interval $B$ can be strengthened further. Additional considerations allow to separate the belief interval (ambiguity perception) from the ambiguity attitude. Under maxmin preferences, it holds that

$$M = B.$$  

For variational preferences, an unbounded utility difference $u_\Delta = u(w) - u(0)$ and a bounded first derivative of the cost function $c$ establish that there exists a sufficiently attractive prize $w$ such that

$$M_{u_\Delta} = B.$$ 

For second order preferences, a uniformly positive ambiguity aversion guarantees that the mixing interval $M_{u_\Delta}$ approximately recovers the belief interval $B$ for large utility difference $u_\Delta$. It holds that

$$M_{u_\Delta} \to B \quad \text{for} \quad u_\Delta \to \infty.$$  

### 3 Optimal mixing under ambiguity aversion

First, consider the best response to the betting mechanism for an agent with SEU preferences.

**Regularity Conditions 2 (SEU).** The agent has SEU preferences with a belief $p \in [0,1]$ about the event $E$. In particular, the preferences can be represented by

$$U(l) = \mathbb{E}_{E \sim p}[u(l(E))]$$ 

for some strictly increasing utility function $u$.  

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Throughout the paper, it is assumed that the agent holds accurate beliefs about the independent uniform draw that is used in the mechanism to induce risk neutrality. Hence, fully accurate the representation above is

\[ U(l(E, r)) = \mathbb{E}_{(E, r) \sim p \times U[0,1]}[u(E, r)] = \mathbb{E}_{E \sim p}\mathbb{E}_{r \sim U[0,1]}[u(E, r)]. \]

For notational convenience, the distribution of \( r \) is not stated explicitly. The best response under SEU is

\[ x^*(q) = \arg \max_{x \in [0,1]} \mathbb{E}_{E \sim p}[u(\mathbbm{1}(s_q(x, E) > r)w)], \]

where \( s_q(x, E) = xq \mathbbm{1}(E) + (1 - x)(1 - q) \mathbbm{1}(E^c) \).

In a first step, the analysis can be simplified with a key result from binarized scoring rules (compare e.g., Hossain and Okui [2013]).

**Lemma 1** (binarized score). For any score \( s \) in the unit interval, the expected utility of a lottery payout based on the score \( s \) is a positive affine transformation of the expected score

\[ \mathbb{E}_{E \sim p}[u(\mathbbm{1}(s > r)w)] = \mathbb{E}_{E \sim p}[s]u_\Delta + u_0, \]

with \( u_\Delta = u(w) - u(0) \) and \( u_0 = u(0) \).

Under Lemma 1, the optimal mixing behaviour for SEU preferences is independent of the agent’s utility function.

**Lemma 2** (SEU). The optimal mixing under SEU preferences (denoted under Regularity Conditions [2]) is

\[ x^*(q) = \begin{cases} 1 & \text{if } p > 1 - q \\ [0, 1] & \text{if } p = 1 - q \\ 0 & \text{if } p < 1 - q. \end{cases} \]
The proof of Lemma 2 is straightforward as the maximization problem can be rewritten with Lemma 1 as
\[ x^*(q) = \arg \max_{x \in [0,1]} E_{E \sim p}[s_q(x, E)], \]
which is linear in \( x \).

The optimal mixing for SEU preferences is illustrated in Figure 3. Mixing is optimal if and only if \( 1 - q \) equals the subjective probability \( p \). Otherwise, betting all lottery tickets on one event is optimal. If the elicitor observes \( x(q)^* = 1 \), it follows that \( p > 1 - q \). For \( x(q)^* = 0 \), it follows that \( p < 1 - q \). Thus, observing betting choices for different odds \( q \), it is possible to identify the belief \( p \).

The remainder of this section considers the best response to the betting mechanism for more general decision models.

### 3.1 Maxmin preferences

This section establishes the optimal mixing for maxmin preferences with belief interval \( B \).

**Regularity Conditions 3 (maxmin).** The agent holds maxmin preferences with belief interval \( B = [a, b] \) about the event \( E \). In particular, the preferences can be represented by
\[ U(l) = \min_{p \in B} E_{E \sim p}[u(l(E))] \]
for some strictly increasing utility function \( u \).

The set of measures \( B \) is unique [Gilboa and Schmeidler, 1989, Theorem 1] and the belief interval is well-defined. As a special case, maxmin preferences contain SEU preferences if the beliefs are unambiguous with \( B = \{p\} \).
Figure 5: Optimal response for variational preferences. The optimal mixing is continuous in $q$. Six examples are shown, where the belief interval is $B = [0.1,0.8]$ throughout. The left plot depicts multiplier preferences with $c(p) = \theta R(p||0.5)$ and $\theta = 0.1,0.5,1.5$, where $R$ is the relative entropy function. The right plot depicts variational preferences with $c(p) = \theta |p−0.5|^4$ and $\theta = 1,10,100$.

**Lemma 3** (maxmin). *The optimal answer for maxmin preferences as in Regularity Conditions 3 is*

$$x^*(q) = \begin{cases} 
1 & \text{if } 1−q < a \\
1−q & \text{if } a < 1−q < b \\
0 & \text{if } b < 1−q
\end{cases}$$

Lemma 3 follows from the more general statement for variational preferences in Lemma 4. See the supplementary document for $\alpha$-maxmin preferences.

**Interpreting betting behavior for maxmin preferences is straightforward.** If everything is betted on the complement $E^c$, the belief interval $B$ is below $1−q$. If everything is betted on $E$, the belief interval is above $1−q$. Finally, if mixing is observed, the belief interval contains $1−q$.

### 3.2 Variational preferences

This section establishes the mixing behavior under variational preferences (Maccheroni et al., 2006), which generalize multiplier preferences (Hansen, 2007; Hansen and Sargent, 2007). We assume variational preferences with belief interval $B$.

**Regularity Conditions 4** (variational preferences). *The agent has variational preferences. In particular, the preferences over acts $l(E)$ can be represented by*

$$U(l) = \min_{p \in B} \mathbb{E}_{E \sim p}[u(l(E))] + c(p)$$

*for some strictly increasing utility function $u$ and some grounded, strictly convex and twice continuously differentiable cost function $c : B \rightarrow \mathbb{R}$.*

Note that Regularity Conditions 4 covers variational preferences as defined in Maccheroni et al. (2006) if they are twice continuously differentiable and strictly convex on
\{ p \in [0, 1] \mid c(p) < \infty \}. Define a cost function \( c_c(p) \) by

\[
c_c(p) = \begin{cases} 
  c(p) & \text{if } p \in B \\
  \infty & \text{if } p \notin B
\end{cases}
\]

and \( U(l) = \min_{p \in [0,1]} \mathbb{E}_{E \sim p}[u(l(E))] + c_c(p) \). The minimal \( c_c \) is unique and the belief interval \( B \) is given by the closure of \{ \( p \in [0, 1] : c_c(p) < \infty \} \).

**Lemma 4** (variational preferences). If the agent follows variational preferences as in Regularity Conditions 4 with belief interval \( B = [a, b] \), the optimal mixing for a mixing bet with prize \( w \) is

\[
x^*(q) = \begin{cases} 
  1 & \text{if } 1 - q < a \\
  m_w(1 - q) & \text{if } a \leq 1 - q \leq b \\
  0 & \text{if } b < 1 - q
\end{cases}
\]

for a continuous function \( m_w \) and it holds that

- \( m_w(1 - q) \in (0, 1) \) for \( c'(1 - q)/u_\Delta < 1 - q < 1 + c'(1 - q)/u_\Delta \),
- \( m_w(1 - q) \) increasing if \( c''(1 - q) < u_\Delta \), and
- \( m_w(1 - q) \) decreasing if \( c''(1 - q) > u_\Delta \).

If \( c' \) is bounded and \( u_\Delta \) is unbounded there exists a prize \( w \) such that the mixing interval identifies the belief interval,

\[
M_{w \Delta} = B.
\]

### 3.3 Smooth second order preferences

This section considers the outcome dependent smooth second order preferences (compare [Ergin and Gul, 2009; Klibanoff et al., 2005; Nau, 2006; Seo, 2009]).

**Regularity Conditions 5.** The agent holds beliefs \( \mathbb{P} \) in form of a distribution over \([0, 1]\) with support \( B = [a, b] \) with \( 0 \leq a \leq b \leq 1 \) about the event \( E \) and has ambiguity averse smooth second order preferences. In particular, the preferences over acts \( l(E) \) can be represented by

\[
U(l) = \mathbb{E}_{p \sim \mathbb{P}}[\phi(\mathbb{E}_{E \sim p}[u(l(E))])]
\]

for some strictly increasing utility function \( u \) and some strictly increasing, concave, and twice continuously differentiable second order utility function \( \phi \).

The agent acts like a SEU type for linear \( \phi \) functions. The second order probabilities \( \mathbb{P} \) are almost surely unique and the belief interval is unique across representations. See [Klibanoff et al., 2014] for a theoretical discussion on capturing the perception of ambiguity under second order preferences and beyond.
Figure 6: Optimal response for second order ambiguity averse preferences. The optimal mixing is continuous in $q$ and lies in the shaded rectangle. Three examples are shown, where $u(0) = 0$, $u(w) = 1$, the second order distribution $\mathbb{P} = U[0.1, 0.8]$ and the second order utility function is $\Phi(z) = -e^{-\theta z}$ with $\theta = 1, 4, 16$ respectively.

Lemma 5. The optimal mixing for a mixing bet with prize $w$ of an ambiguity averse agent with second order preferences as in Regularity Conditions 3 is

$$x^*(q) = \begin{cases} 1 & \text{if } 1 - q < a \\ m_w(1 - q) & \text{if } a \leq 1 - q \leq b \\ 0 & \text{if } b < 1 - q, \end{cases}$$

for some increasing and continuous function $m_w(\cdot)$ such that for all $w$

- $m_w(1 - a) = 1$ and $m_w(1 - b) = 0$
- $m_w(1 - q) < 1$ if $1 - q > \mathbb{E}_{p \sim \mathbb{P}}[p]$ and $m_w(1 - q) > 0$ if $1 - q < \mathbb{E}_{p \sim \mathbb{P}}[p]$.

In particular it holds for the mixing interval that $\mathbb{E}_{p \sim \mathbb{P}}[p] \in M_{\Delta}$. Further, if the coefficient of ambiguity aversion $\alpha(z) = -\frac{\phi''(z)}{\phi'(z)}$ is bounded away from zero, it holds that

$$M_{\Delta} \to B \quad \text{for } \Delta \to \infty.$$  

The continuity of $m$ implies that the agent is mixing on an interval with positive length. For sufficiently strong ambiguity aversion second order preferences are essentially identical to maxmin preferences (Klibanoff et al. 2005, Proposition 3) and the belief interval can be identified with a high degree of accuracy. Lemma 5 shows that the same effect can be generated by increasing the utility difference $u_\Delta = u(w) - u(0)$ if one is willing to assume strictly positive ambiguity aversion.

In Figure 6, three examples with different constant absolute ambiguity aversion are shown. Bounds on the belief interval are conservative for moderate rates of ambiguity aversion and low utility difference in prizes.
Figure 7: The value of the three choices $[E_q], [C_q],$ and $[M_q]$ depending on $1 - q$. The left plot illustrates the utility functional under SEU, the middle plot under ambiguity averse, and the right plot under ambiguity seeking preferences. Under ambiguity seeking preferences, the ambiguous choices $[E_q]$ and $[C_q]$ are more attractive such that they are both preferred to the mixing choice $[M_q]$ for any $1 - q$ in the interval $N$.

4 Ambiguity seeking preferences

The mixing bets considered until here is unable to identify ambiguous beliefs for ambiguity seeking preferences. This section introduces a generalization, separated mixing bets, that extends the identification of the belief interval to ambiguity seeking preferences.

4.1 Illustration: Separated mixing bets

To build intuition, consider an extension of the illustration from Section 1.1 to ambiguity seeking preferences as depicted in Figure 7. Again, we analyze the value of the three choices $[E_q], [C_q],$ and $[M_q]$ (betting on the event, the complement, and mixing) under different preferences.

First, we observe that mixing is never the best response under ambiguity seeking preferences. The choice $[M_q]$ is dominated by $[E_q]$ (for low $1 - q$), $[C_q]$ (for high $1 - q$), or both (for intermediate $1 - q$). The behavior for standard mixing bets would be indistinguishable from SEU.

Instead, the difference between SEU and ambiguity seeking preferences arises if the choice between $[M_q]$ and its components is observed separately. An ambiguity seeking agent prefers the ambiguous bets $[E_q]$ and $[C_q]$ over the unambiguous $[M_q]$ for some values of $1 - q$. We refer to this interval as the non-mixing interval $N$. The non-mixing interval $N$ at which the decision maker prefers betting on the event and the complement to mixing is related to ambiguity perception and ambiguity attitude. In particular, the non-mixing interval $N$
contains the probability $p$ associated with ambiguity neutral preferences. Further, the non-mixing interval $N$ is larger for more ambiguity seeking preferences.

This section considers the separation of ambiguity perception and attitude under specific ambiguity seeking preferences.

4.2 Separated mixing bets

Consider the following ambiguity seeking preferences.

**Regularity Conditions 6** (ambiguity seeking). The agent has ambiguity seeking smooth second order or maxmax preferences, where the random draw $r$ is independent from $E$ and uniformly distributed.

Seperated mixing bets are a variation of the mixing bets considered before. The agent chooses twice, where in each choice the options are limited to allocations of lottery tickets that favor one of the two outcomes. For the odds quota $q$, the feasible allocations are restricted to $[1 - q, 1]$ (favoring the event) and $[0, 1 - q]$ (favoring the complement) respectively.

In this scenario, the mixing interval $M$ describes all odds for which the agent is mixing in both separated mixing bets. Additionally, the non-mixing interval $N$ describes all odds for which the agent is not mixing in either of the two separated mixing bets.

**Definition 5** (mixing interval $M$ and non-mixing interval $N$). Let $x^*_E(q) : [0, 1] \to [1 - q, 1]$ and $x^*_C(q) : [0, 1] \to [0, 1 - q]$ define optimal mixing for the odds ratio $q$ such that

\[
\begin{align*}
l(x^*_E(q), q, w) & \geq l(x, q, w) \text{ for all } x \in [1 - q, 1], \\
l(x^*_C(q), q, w) & \geq l(x, q, w) \text{ for all } x \in [0, 1 - q].
\end{align*}
\]

The mixing interval $M$ is defined as the smallest closed interval that contains

\[
\{ q \in [0, 1] \mid x^*_E(1 - q) \in (0, 1) \text{ and } x^*_C(1 - q) \in (0, 1) \}.
\]

The non-mixing interval $N$ is defined as the smallest closed interval that contains

\[
\{ q \in [0, 1] \mid x^*_E(1 - q) \notin (0, 1) \text{ and } x^*_C(1 - q) \notin (0, 1) \}.
\]

Trivially, the optimal mixing ratios are $x^*_E(q) = 1$ and $x^*_C(q) = 1 - q$ (betting as much lottery tickets on $E$ as possible), if the quota $q$ is large enough. Similarly, the optimal mixing is $x^*_E(q) = 1 - q$ and $x^*_C(q) = 0$ (betting as much lottery tickets on $E^c$ as possible), if the quota $q$ is small enough. An ambiguity averse agent prefers to mix for the two separated bets and the mixing interval is non-empty. An ambiguity seeking agent prefers the two ambiguous extremes to the mixing and the non-mixing interval $N$ is non-empty. The main result of this section is that non-mixing (the choice $x^*_E(q) = 1$ and $x^*_C(q) = 0$) implies ambiguous beliefs and ambiguity seeking preferences.
Theorem 3 (belief interval and non-mixing). Under Regularity Condition [8] non-mixing for a quota $q$ implies that $1 - q$ is an element of the belief interval.

$N \subset B$.

In particular, if an agent does not mix in the separated mixing bets for two different quotas the agent holds ambiguous beliefs.

Theorem 3 allows to bound the belief interval from within. Small mixing intervals can only be detected if appropriate odds are applied. For ambiguous beliefs such odds always exist.

Theorem 4 (ambiguous beliefs and non-mixing). Under Regularity Conditions [6], beliefs are ambiguous if and only if there exist at least two different odds quotas $q$ for which the agent prefers not to mix in the separated mixing bets.

The identification result for the belief interval $B$ can be strengthened further. Additional considerations allow to separate the belief interval (ambiguity perception) from the ambiguity attitude. Under maxmax preferences, it holds that

$N = B$.

4.3 Maxmax preferences

This section establishes the optimal mixing in separated mixing bets for maxmax preferences with belief interval $B$. Maxmax preferences arise as the most ambiguity seeking special case of $\alpha$-maxmin expected utility preferences [Ghirardato et al. 2004, Marinacci 2002].

Regularity Conditions 7 (maxmax). The agent holds maxmax preferences with belief interval $B = [a, b]$ about the event $E$. In particular, the preferences can be represented by

$U(l) = \max_{p \in B} E_{E \sim p}[u(l(E))]$

for some strictly increasing utility function $u$.

Lemma 6 (maxmax). The optimal responses for the separated mixing bets for maxmax preferences as in Regularity Conditions [7] are

$x_E^*(q) = \begin{cases} 1 & \text{if } 1 - q < b \\ 1 - q & \text{if } 1 - q > b \end{cases}$

and

$x_C^*(q) = \begin{cases} 0 & \text{if } a < 1 - q \\ 1 - q & \text{if } a > 1 - q \end{cases}$

The results of Lemma 6 are illustrated in Figure 8. The proof follows from the arguments in the general statement in the supplementary document for $\alpha$-maxmax preferences.
Figure 8: Optimal response for maxmax preferences with ambiguous belief interval $B = [0.1, 0.8]$. The shaded area marks the belief interval, which is identified by the mixing behavior.

Interpreting betting behavior for maxmax preferences is straightforward. If the decision maker is not choosing to mix for either of the two mixing bets, then $1 - q$ is in the belief interval and the preferences are ambiguity seeking. If the decision maker chooses to mix in both separated mixing bets, then $1 - q$ is in the belief interval and the preferences are ambiguity averse. If the decision maker chooses the pure event in one and mixes for the other bet, then $1 - q$ is beyond the belief interval and no statements about the ambiguity attitude can be made.

4.4 Smooth second order preferences

This section extends the results for second order preferences from Section 3.3.

Regularity Conditions 8. The agent holds beliefs $\mathbb{P}$ in form of a distribution over $[0, 1]$ with support $B = [a, b]$ with $0 \leq a \leq b \leq 1$ about the event $E$ and has ambiguity seeking smooth second order preferences. In particular, the preferences over acts $l(E)$ can be represented by

$$U(l) = \mathbb{E}_{p \sim \mathbb{P}}[\phi(\mathbb{E}_{p \sim \mathbb{P}}[u(l(E))])]$$

for some strictly increasing utility function $u$ and some strictly increasing, convex, and twice continuously differentiable second order utility function $\phi$.

Regularity Conditions 8 are identical to the assumption in Section 3.3 except for the convex second order utility function $\phi$.

Lemma 7. The optimal mixing for a mixing bet with prize $w$ of an ambiguity averse agent with second order preferences as in Regularity Conditions 8 is

$$x_E^*(q) = \begin{cases} 1 & \text{if } \mathbb{E}_{p \sim \mathbb{P}}[p] + \epsilon > 1 - q \\ 1 - q & \text{if } b \leq 1 - q \end{cases}$$

and

$$x_C^*(q) = \begin{cases} 0 & \text{if } \mathbb{E}_{p \sim \mathbb{P}}[p] - \epsilon < 1 - q \\ 1 - q & \text{if } a \geq 1 - q \end{cases}$$
for some $\epsilon > 0$. In particular it holds that $\mathbb{E}_{p \sim \mathcal{P}}[p] \in N_{u \Delta}$.

Note that the non-mixing is guaranteed in an environment around $\mathbb{E}_{p \sim \mathcal{P}}[p]$. Without additional assumption the best response may shift multiple times beyond said environment and the end of the belief interval, where mixing is the dominant strategy for $x^*_E(q)$ or $x^*_C(q)$.

5 Ambiguous beliefs on real-valued variables

The belief interval and mixing bets can be extended beyond events to real-valued outcome variables. Consider the task of eliciting beliefs about a real-valued variable $Y : \Omega \rightarrow \mathbb{R}$. The state space is given by $S = \mathbb{R} \times [0, 1]$. Let $\mathcal{P}$ denote a set of distributions over the real line. Further, let $\mathcal{P}_0 \subset \mathcal{P}$ denote the set of relevant distributions, which I call the belief set. As before, preferences are said to exhibit ambiguous beliefs about the random variable $Y$ if the belief set does not reduce to a unique probability measure.

The belief set for SEU preferences is simply the subjective probability distribution of $Y$. Other preferences, like maxmin preferences cannot be represented without taking into account the expected utility of a decision with respect to multiple probability measures. Such beliefs are called ambiguous.

The agent is confronted with mixing bets for a series of events $E_i \subset \mathbb{R}$. Let $[a_i, b_i]$ denote the bounds elicited for the belief interval of event $E_i$. Let us denote the set of probability distributions that is consistent with the obtained bounds as

$$\mathcal{P}^* = \{ \mathbb{P} \in \mathcal{P} | \mathbb{P}(E_i) \in [a_i, b_i] \text{ for all } i \}.$$ 

It follows that $\mathcal{P}_0 \subset \mathcal{P}^*$. Consider as example the situation where the random variable $Y$ is assumed to be bounded by $c, C \in \mathbb{R}$ with $c \leq Y \leq C \mathbb{P}$-almost surely for every $\mathbb{P} \in \mathcal{P}_0$. 
If mixing bets are applied to the events $E_i = \{ \omega \in \Omega | Y(\omega) \leq c_i \}$ for some constants $c < c_1 < \cdots < c_k < C$, any element of the belief set $P_0$ has a cdf that lies in the gray area in Figure 9. While unambiguous beliefs can be determined to an arbitrary degree of accuracy by increasing the number of thresholds $c_i$, the bounds on ambiguous beliefs are conservative without additional assumptions.

An ambiguous belief over an event $E = \{ Y \leq c \}$ implies ambiguous beliefs about $Y$. The reverse is generally not true. However, if the beliefs about the random variable $Y$ are ambiguous, there always exists a threshold $c$ such that the agent holds ambiguous beliefs over an event of the form $E = \{ Y \leq c \}$.

6 Implementation

This section provides empirical evidence that mixing bets are feasible, measure ambiguity, and can generate insightful evidence. This section covers two possible implementations, a simple analysis of ambiguity and probability perception, and a more complex measurement model to analyze ambiguity attitude. The focus is on mixing bets on events from Section 2 and 3, omitting ambiguity seeking preferences from Section 4 and real-valued variables from Section 5.

6.1 Experimental setup

In a pilot laboratory experiment, implemented with OTree, at the Frankfurt Laboratory for Experimental Economic Research (FLEX), 88 subjects were recruited. The average age was 25, 49% of participants were female, and 38% studied economics or business.

Participants could win a prize of 10 Euros with the mixing bets, next to 5 Euros for participation. Before the ambiguity elicitation, participants were asked to play a standard prisoners dilemma with potential payoffs of 1 Euro (both defect), 2 Euros (both cooperate) and 0/3 Euros (cooperating/defecting). Next, participants could choose between 2 Euros, 5 Euros with a 50% chance, and 10 Euros with a 30% chance to measure risk aversion and to establish the risk generating mechanism of drawing a number from a box with numbers from 1 to 100, which was also used as lottery for the mixing bets.

In the main part of the experiment, I revisit the Ellsberg urn asking participants to consider the event of a specific type of ball being drawn from an urn which contains 90 balls, where the color composition (60 red, 30 blue) was known to the participants, but the number of dotted balls (0 - 60 dotted) was unknown to the participants.

The current value of the German Stock Index was written down at the beginning of the experiment (before the German Stock Exchange opened) and at the end of the experiment (about 30 minutes after trading started). During the main experiment, mixing choices were elicited for five different domains. The event of a blue ball being drawn (risk), the event of a dotted ball being drawn (ambiguity), the event of the stock market rising (stock), the event of the assigned player in the prisoners dilemma choosing to defect (social). The order of the
aforementioned elicitation tasks was randomized. Finally, the participants were shown 10 draws from the Ellsberg urn and had to repeat the dotted ball elicitation under additional information (updated).

In the discrete elicitation participants had only the three choices \( x \in \{0, 1 - q, 1\} \) from Section 1.1. Preferences were elicited pairwise to reduce complexity. For each domain the quotas \( q \in \{0.1, 0.2, \ldots, 0.9\} \) were applied, where the order was randomized and the values above (below) were skipped if a higher (lower) quota elicited the answer \( x = 0 \) (\( x = 1 \)). In the continuous elicitation participants could distribute tickets with a slider between the event and the complement, where for each slider position the payoff for the two events was shown on the screen. Feasible allocations were reduced to \( x \in \{0, 0.1, \ldots, 0.9, 1\} \). To avoid excessive waiting times, the slider elicitation was skipped if a participant fell behind too much for the updated, stock, and social domain, but not for the risk and ambiguity domain. About 50% of participants were subject to the continuous elicitation for all domains.

At the end of the experiment the realization of the events was shown and an envelope was opened that contained the domain and question number that would determine the payout of the additional 10 Euros from the mixing bets. The number of obtained lottery tickets was shown on the screen and each participant had to draw a number from a box to determine the final payoff.

### 6.2 Experimental evidence

Figure 10 shows the median response by domain, odds quota, and elicitation type. Consider the discrete elicitation first. For the risky urn, the median response follows the best response under expected utility preferences, switching between 0.3 and 0.4. For the ambiguous urn the median response mixes from 0.3 to 0.5, but not for the more extreme odds quotas, which contradicts either the maxmin preferences or the claim that the described urn induces a belief interval of \([0, 0.6]\). After the update, which on average consists of 1 dotted ball and 9 undotted balls, the median response is to mix for 0.1, but bet on the undotted ball otherwise. For the stock market rising and the player not cooperating in the social game, we observe a median response mixing for 0.4 and 0.5, which is consistent with ambiguous beliefs and ambiguity aversion, contradicting ambiguity neutral preferences. The continuous elicitation, which allows greater flexibility, tends to much more mixing, while replicating the main patterns. The difference can be explained by ambiguity aversion that is less strong than maxmin preferences, by noisy responses, or by a tendency for non-extreme responses.

Figure 11 illustrates ambiguity perception (length of mixing interval) in panel A and probability perception (midpoint of mixing interval) in panel B. Perceived ambiguity is lowest for the risky urn, largest for the ambiguous urn (but decreasing sharply after 10 draws were shown), and inbetween for the two natural events stock and social. Notably, female participants showed higher ambiguity perception for the risky urn, the ambiguity urn (before updates), and the stock market. Male participants judged the stock market to be significantly less ambiguous than their female counterparts, but perceived almost as much
ambiguity for the action of their partner in the cooperation game as for the ambiguous urn.

Considering the probability associated with the measured mixing interval in Panel B, results are consistent with rational expectations and only small gender differences are observable. The risky urn probability is close to the true 33.3% and the ambiguous urn slightly higher. After the 10 additional draws for the ambiguously coloured balls, the average belief moves toward the true value of $\frac{1}{10}$. A rising stock market is expected in about 50% of cases, with male participants being slightly less optimistic. In the cooperation game, male participants expect a defecting partner in about 49% and female participants in 41% of cases. (The true average was close with 46% of participants defecting.)

The main goal of the elicitation mechanism is to reveal ambiguity perception beyond experimental tasks. The two examples considered here indicate that natural events, where uncertainty is generated by mechanisms beyond the experimental control, are indeed perceived as ambiguous by a considerable ratio of participants.

### 6.3 Ambiguity attitude in a measurement model

It is straightforward to extract the mixing interval as a proxy for ambiguity perception from the data. Ambiguity attitude, however, has a more complex connection to mixing as shown in Section 3. To disentangle ambiguity perception and ambiguity attitude, this subsection introduces a measurement model in the Bayesian framework which captures measurement error and ambiguity attitude for each individual as well as the probability and ambiguity perception for each domain and individual.

The Bayesian approach was chosen as the amount of answers one can elicit from any individual on one domain is naturally limited. For such small sample sizes inference with
Figure 11: **Proxies for ambiguity and probability perception.** The plot depicts estimated mean and 90%-confidence intervals for the proxies of ambiguity and probability perception. In panel A the ambiguity proxy is computed as the distance between the highest and lowest odds quota a participant chose to mix. In panel B the probability proxy is computed as the midpoint of the mixing interval.
maximum likelihood can be unreliable. Further, the Bayesian inference can handle partial identification. Partial or weak identification can arise as ambiguity neutral preferences with large belief intervals induce similar answers as ambiguity averse preferences with small belief intervals.

In most applications, one can expect measurement error in the answers beyond the predictions of decision models considered so far. Possible reasons for such behavior include hedging by random responses, changing preferences, and inattention. The measurement model in this section allows to estimate the noisiness of responses by a specific individual.

In the discrete elicitation only three mixing choices were feasible. Under weak ambiguity aversion less extreme mixing can be optimal, which is not part of the choice set. Measurement error allows to model the choice of the agent in such situations. The more complex continuous elicitation provides information on the ambiguity attitude, but potentially induces additional measurement error. If the additional noise is small enough, the experiment allows inference on individual specific ambiguity attitude.

Let us denote by $x_{j,d}(q)$ the mixing choice of individual $j \in \{1, \ldots, J\}$ for domain $d \in \{1, \ldots, D\}$ and betting quota $q$. The model assumes potential randomness in responses around an optimal response $\mu_{j,d}(q)$

$$x_{j,d}(q) \sim \mathcal{N}(\mu_{j,d}(q), \sigma_{j,d}^2),$$

where $\mu_{j,d}(q) = f(B_{j,d}, \alpha_j, q)$ is a function of the belief interval $B_{j,d}$ and individual specific ambiguity aversion $\alpha_j$. The function $f$ can be directly computed from a parameterization of the preferences considered here, i.e. $f(q) = \arg \max_{x \in [0,1]} U(l(E))$ where the preference functional is parameterized with the belief interval and ambiguity aversion. Alternatively, I propose a proxy model that contains maxmin as special case and approximates smooth and variational preferences with a linear best response in the belief interval. In particular, behavior without noise is given by

$$f(B, \alpha, q) = \begin{cases} 1 & \text{if } B > 1 - q \\ (1 - q)\alpha + \frac{1}{2} - \frac{1}{2}\alpha & \text{if } 1 - q \in B \\ 0 & \text{if } B < 1 - q \end{cases}$$

where $B$ denotes the belief interval and $\alpha$ the ambiguity aversion. Maxmin behavior arises with ambiguity aversion $\alpha = 1$ and a linear approximation of the behavior under ambiguity averse second order preferences with $\alpha < 0$. The ambiguity aversion $\alpha$ is assumed to be a constant characteristic of each person that does not depend on the domain or the elicitation type.

The belief interval is parameterized by its midpoint $p$ (probability of associated ambiguity neutral preferences) and its length $l$ (ambiguity perception) with $B = [p - \frac{l}{2}, p + \frac{l}{2}]$. The associated probability and ambiguity perception is assumed to be a constant characteristic for each person within a domain.
Figure 12: **Ambiguity aversion estimated in structural measurement model.** The plot depicts posterior median and 90%-confidence intervals for the ambiguity aversion $\alpha$ for all participants. The prior 90%-confidence intervals are approximately $[-3.3, 3.3]$.

Figure 13: **Belief interval estimates from hyper-parameters.** The plot depicts the belief interval consistent with posterior means of the hyper parameters.
The belief interval for each domain and participant is modeled via domain specific hyper-parameters $p_d$ and $l_d$ that capture the average probability and ambiguity perception for each domain with

$$p_{j,d} \sim \mathcal{N}(p_d, \sigma^2_p) \quad \text{and} \quad l_{j,d} \sim \mathcal{N}(l_d, \sigma^2_l),$$

where the hyper-parameters have an uninformative priors,

$$p_d \sim U[0, 1] \quad \text{and} \quad l_d \sim U[0, 1].$$

The variance of the measurement error term $\sigma^2_{j,d}$ is modeled as the sum of some person specific variance $\sigma^2_j$ and an additional measurement error for the more complex continuous elicitation $\sigma^2_c$. All precision parameters (inverse of variance) $\frac{1}{\sigma^2_p}, \frac{1}{\sigma^2_l}, \frac{1}{\sigma^2_j}, \frac{1}{\sigma^2_c}$ have a gamma prior with shape and rate equal to 0.1 and restricted to be larger than 0.1 (variance smaller than 10) for computational convenience.

An MCMC sampler was implemented with JAGS (Plummer, 2015). After 25,000 iterations burn-in, another 25,000 iterations were thinned to 1000 posterior draws. Visual diagnostics and an maximum Rhat statistics of 1.19 provide no evidence for convergence issues.

Figure 12 illustrates the posterior distributions of the individual specific ambiguity aversion $\alpha_j$. The parameter describes the slope of the best response, such that $\alpha = 1$ coincides with maxmin preferences and negative coefficients approximate smooth and variational preferences. The confidence intervals show that the data identifies the parameter for most individuals, moving the posterior strongly compared to prior. The 90%-confidence intervals for ambiguity aversion are consistent with maxmin preferences for 72% of participants and with a slope of $-1$ for 14% of participants. As shown in Section 3 the mixing bets cannot distinguish between ambiguity seeking and neutral preferences. Further, the identification of the mixing behavior depends on sufficiently strong ambiguity perception.

Figure 13 plots the belief intervals based on posterior means of $p_d$ and $l_d$. As with the reduced form analysis, we see that the ambiguous urn induces more ambiguity perception than the risky urn, but not the full interval of potential outcomes from 0 to 0.66. After observing the 10 draws, ambiguity reduces considerably and the range of probabilities falls below 0.2. The natural events of the stock market and the social game are more ambiguous than the risky draw from the urn and less than the ambiguous draw.

Table 1 provides least squares regressions of several individual specific posterior medians on demographics and the measured risk aversion. In column (1), ambiguity aversion $\alpha_j$ estimates were not found to be correlated with gender, risk aversion, or age. This suggests that ambiguity aversion constitutes a distinct preference feature from risk aversion. Measurement error was found to be highly relevant and larger for female participants. The additional noise introduced by the more complex continuous elicitation was large with a posterior median of 0.27 (90%-confidence interval: $[0.25, 0.28]$), which suggests that the pairwise elicitation is considerably more accurate.

Columns (3) to (7) use posterior medians of the length of the belief interval as dependent variable to explain ambiguity perception. Risk aversion is uncorrelated with ambiguity
| Dependent variable: | ambiguity aversion $\alpha_j$ | measurement error $\sigma_j$ | risk | ambiguity perception $l_{j,d}$ | stock | social |
|---------------------|-------------------------------|--------------------------------|------|-------------------------------|-------|--------|
| (1)                 | (2)                           | (3)                           | (4)  | (5)                           | (6)   | (7)    |
| risk aversion ($s$) | 0.28                          | -0.01*                         | 0.02 | -0.005                        | -0.03 | 0.07   |
|                     | (0.18)                        | (0.01)                         | (0.02)| (0.02)                        | (0.02)| (0.02) |
| female              | 0.12                          | 0.04***                        | 0.02 | 0.10**                        | 0.13***| 0.06   |
|                     | (0.36)                        | (0.01)                         | (0.04)| (0.04)                        | (0.04)| (0.04) |
| age ($s$)           | 0.05                          | -0.003                         | -0.02| -0.02                         | 0.01  |
|                     | (0.17)                        | (0.01)                         | (0.02)| (0.02)                        | (0.02)| (0.02) |
| Constant            | 1.40***                       | 0.11***                        | 0.16***| 0.24***                      | 0.19***| 0.19***| 0.22***|
|                     | (0.24)                        | (0.01)                         | (0.02)| (0.03)                        | (0.03)| (0.03) |
| Observations        | 88                            | 88                            | 88   | 88                            | 88    | 88     | 88     |
| $R^2$               | 0.04                          | 0.12                          | 0.05 | 0.09                          | 0.13  | 0.04   |

*Note:* $^*p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Table 1: **Least squares regressions of individual specific posterior medians.** The dependent variable consists of posterior median estimates for each participant. Covariates with marked with ($s$) were standardized.
perception. Age shows no statistically significant effect. As already seen with the simple proxies, female participants exhibit on average larger belief intervals for the ambiguous urn and for the stock market, but not for the social uncertainty (behavior of partner in cooperation game) nor for the risky urn were point estimates are smaller and not statistically significant.

7 Discussion

The separation of persistent attitudes and temporary perception is a potentially insightful endeavor. Mixing bets can be used to elicit belief intervals and subsequently analyze their impact on decision making as well as their development under changing information environments. Conveniently, the mechanism can be used within a whole range of ambiguity sensitive preferences.

I show that mixing bets can be implemented in laboratory experiments and that they offer relevant and applicable information on preferences and private information. In particular, the experiment showed that not all participants perceived the risky urn as purely risk and the induced ambiguity for the ambiguous did not induce as much ambiguity as the verbal explanation would suggest. Further, natural events were shown to be perceived as ambiguous, which underlines the relevance of ambiguity in economic decision making. Finally, ambiguity perception (as ambiguity attitude and probability perception) was heterogeneous. Female participants were found to perceive more ambiguity for the stock market and the ambiguous urn, but no difference in ambiguity aversion was found.

Elicitation of subjective probabilities is often done by matching probabilities (e.g., Holt, 2007) also called choice-based probabilities (Abdellaoui et al., 2011). In this approach, the probability of an event \( E \) is defined by the point of indifference between a lottery that pays a winning prize with probability \( p \) and a lottery that pays the same prize if \( E \) realizes. Conveniently, matching probabilities can be analyzed without a mixing concept or a product state space. From a behavioral perspective, however, the direct comparison with a randomization device could be argued to be more distorting than mixing with such a device. Comfortably, the experimental evidence for the Ellsberg urn provided here is broadly in line with a long line of experiments based on matching probabilities: On average participants avoided ambiguous uncertainty more than pure risk.

As already pointed out in Ramsey (1931) and de Finetti (1931) an obvious measure of belief is willingness to bet. Preceding ambiguity sensitive decision models in economics, Smith (1961) proposes to define subjective probabilities by the interval of odds that an agent agrees to bet on a certain event. By allowing the agent to mix bets, I extend Smith’s hypothetical design and establish that multiple mixing identifies ambiguity.

The introduced mixing bets have a direct connection to proper scoring rules and can be seen as an application of multiple point forecasting as introduced in Eyting and Schmidt (2018) to binary events. The betting mechanism can be restated as binarized asymmetric
piecewise linear score for a point forecast of the random variable $\mathbb{1}(E)$. The best response is a quantile of the underlying distribution, where the level of the quantile depends on the odds quota (compare e.g. Gneiting, 2011). As pointed out in Chambers (2008), the best response to proper scoring rules under maxmin preferences is equal to the best response to one element of the set of probabilities. This finding extends to mixing bets, where the quantile is 1 for large and 0 for small quantile levels.

A major concern is whether the agent acts differently if multiple odds are elicited and one is randomly selected for payout instead of just one choice being elicited. The validity of this random lottery procedure has been shown to falter for simple choices (Starmer and Sugden, 1991). In the ambiguity averse context, an additional concern is how the agent reacts when faced with multiple bets on the same uncertain outcome. The application of our results require the agent to apply the ambiguity aversion on each bet separately instead of hedging across bets. This point, however, arises necessarily in the elicitation with random devices for ambiguous averse agents (compare Bade, 2015) and similarly arises for other elicitation mechanisms (e.g., Baillon et al., 2018; Bose and Daripa, 2017a).

Another concern is that the validity of the mechanism depends on the existence of a randomization device for the lottery payout that is perceived as risk without ambiguity. Similar issues arise for mechanisms that elicit matching probabilities (Baillon et al., 2018) or that employ objective lotteries (Bose and Daripa, 2017a,b).

Instead of revealing preferences as proposed here, one can ask directly for ranges of probabilities (e.g., Giustinelli and Pavoni, 2017; Manski and Molinari, 2010). However, ambiguity averse decision models describe behavior, rather than thought processes. The belief interval may well have considerable explanatory power regarding an agent’s behavior, while the agent is unable or unwilling to articulate such an interval.

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**Appendix: Proofs**

**Proof of Lemma** The expected utility can be expressed as a linear function of the probability of winning,

$$
\mathbb{E}_{E \sim p}[u(1(s > r)w)] = \mathbb{P}[s > r]u(w) + (1 - \mathbb{P}[s > r])u(0) = \mathbb{P}[s > r](u(w) - u(0)) + u(0).
$$

Further, from the independent uniform distribution of $r$ and as $s \in [0, 1]$ it follows that

$$
\mathbb{P}[s > r] = \mathbb{E}_{E \sim p}[\mathbb{E}_{r \sim U}[1(s > r)]] = \mathbb{E}_{E \sim p}[s].
$$
Proof of Lemma 4. We apply Lemma 4 and obtain the simplified optimization problem

\[ x^* = \arg \max_{x \in [0,1]} \min_{p \in B} s_q(x, p) + c(p)/u_\Delta. \]

The decision maker acts as if more ambiguity averse for higher utility difference between prizes \( u_\Delta = u(w) - u(0) \) (Proposition 8, Maccheroni et al. 2006). Define \( c_t(p) = c(p)/u_\Delta \) for notational convenience. \( c_t \) is also grounded, strictly convex and twice continuously differentiable.

Examine the minimum of

\[ g(p) = s_q(x, p) + c_t(p) = 1 - x - q + xq + p(x - (1 - q)) + c_t(p). \]  \hspace{1cm} (1)

The function \( g \) is convex. For fixed \( x \), the minimum at \( p^* \) is characterized by the first order condition \( g'(p^*(x)) = x - (1 - q) + c'_t(p^*(x)) = 0 \). It holds that \( c' \) is increasing by the convexity assumption and it follows that \( p^*(x) \) is decreasing in \( x \).

- **First case:** \( p^*(x) = a \iff x > 1 - q - c'_t(a) \)

  The agent values the resulting bets as a function of \( x \) by

  \[ U(x) = s_q(x, a) + c_t(a) = 1 - a - q + aq + x(a - (1 - q)) + c_t(a). \]  \hspace{1cm} (2)

  Thus, \( x^* = 1 \) if \( 1 - q < a \). For \( 1 - q > a \), consider the following two sub-cases:

  - If \( c'_t(a) < 0 \), it follows that \( x^* = \min(1, 1 - q - c'_t(a)) \).
  - If \( c'_t(a) > 0 \), it follows that \( x^* = \max(0, 1 - q - c'_t(a)) \).

- **Second case:** \( p^*(x) = b \iff x < 1 - q - c'_t(b) \)

  The agent values the resulting bets as a function of \( x \) by

  \[ U(x) = 1 - b - q + bq + x(b - (1 - q)) + c_t(b). \]  \hspace{1cm} (3)

  Thus, \( x^* = 0 \) if \( 1 - q > b \). For \( 1 - q < b \), consider the following two sub-cases:

  - If \( c'_t(b) > 0 \), it follows that \( x^* = \max(0, 1 - q - c'_t(b)) \).
  - If \( c'_t(b) < 0 \), it follows that \( x^* = \min(1, 1 - q - c'_t(b)) \).

- **Third case:** \( p^*(x) \in (a, b) \iff x = 1 - q - c'_t(p^*(x)) \)

  The agent values the resulting bets as a function of \( x \) by

  \[ U(x) = \min_{p \in B} s_q(x, p) + c_t(p) \hspace{1cm} (4)\]

  \[ = \min_{p \in B} 1 - x - q + xq + p(x - (1 - q)) + c_t(p) \hspace{1cm} (5)\]

  \[ = 1 - x - q + xq + p^*(x)(x - (1 - q)) + c_t(p^*(x)). \]  \hspace{1cm} (6)

The first order condition is

\[-(1 - q) + p^*(x)'x + p^*(x) - p^*(x)'(1 - q) + c'_t(p^*(x))p^*(x)' = 0\]

\[-(1 - q) + p^*(x) + p^*(x)'(x - (1 - q)) + c'_t(p^*(x))p^*(x)' = 0\]

\[-(1 - q) + p^*(x) - p^*(x)'c'_t(p^*(x)) + c'_t(p^*(x))p^*(x)' = 0\]

\[p^*(x) - (1 - q) = 0.\]
And describes a maximum as \( U'(x) > 0 \iff p^*(x) > 1 - q \) and \( p^*(x) \) decreasing in \( x \). Thus, it follows that \( x^*(1 - q) = 1 - q - c'_t(1 - q) \). The mixing function \( x^* \) is increasing in \( q \) if
\[
c''_t(1 - q) > 1.
\]

For any point \( 1 - q \), mixing is optimal if
\[
0 < x^*(1 - q) < 1
\]
\[
c'_t(1 - q) < 1 - q < 1 + c'_t(1 - q) \quad (8)
\]
\[
\frac{c'(1 - q)}{u_\Delta} < 1 - q < 1 + \frac{c'(1 - q)}{u_\Delta}, \quad (9)
\]
which holds true for a sufficiently large \( u_\Delta \) if \( c' \) is bounded.

\[\square\]

**Proof of Lemma** 5. For notational convenience define \( s_q(x, p) = E_{E \sim p}[s_q(x, E)] \). With Lemma 1 it holds that
\[
x^*(q) = \arg \max_{x \in [0, 1]} \mathbb{E}_{p \sim P}[\phi_t(s_q(x, p))],
\]
with \( \phi_t(z) = \phi(u_\Delta z + u_0) \) increasing and concave and \( s_q(x, p) = 1 - p - q + pq + x(p - (1 - q)) \).

First, consider the case \( 1 - q \leq a \). As \( p \leq 1 - q \) implies \( \phi_t \) is increasing in \( x \), this case implies that \( \phi_t \) is \( \mathbb{P} \)-almost surely increasing in \( x \). Thus, \( \mathbb{E}_{p \sim P}[\phi_t(s_q(x, p))] \) increasing in \( x \) and \( x^* = 1 \). A similar argument shows \( x^* = 0 \) for \( 1 - q \geq b \).

The remainder of the proof considers the case \( a < 1 - q < b \). Let \( U(x, q) = \mathbb{E}_{p \sim P}[\phi_t(s_q(x, p))] \). As \( \phi_t \) is continuously differentiable, \( \phi_t \) and its first two derivatives are integrable on \( B \), it follows by the dominant convergence theorem that \( (\partial_x)^2 U(x, q) = \mathbb{E}_{p \sim P}[\phi''_t(s_q(x, p))(p - (1 - q))^2] \), which in turn implies that \( U(x, q) \) is concave in \( x \) as \( \phi''_t \leq 0 \). We conclude that for fixed \( q \) the optimal mixing \( x^*(q) \) is unique. Further, by the maximum theorem (Ok, 2007) \( x^*(q) \) is continuous as it holds that \( U(x, q) \) is continuous by the dominated convergence theorem.

If \( a \neq b \) the following argument shows that mixing is optimal for an interval that contains \( 1 - \mathbb{E}_{p \sim P}[p] \). Consider the first order condition \( \partial_x U(x, q) = \mathbb{E}_{p \sim P}[\phi'_t(s_q(x, p))(p - (1 - q))] = 0 \). For \( x = 1 \), the equation above is equivalent to \( \mathbb{E}_{p \sim P}[\phi'_t(pq)(p - (1 - q))] = 0 \). As \( \phi_t \) concave, the derivative \( \phi'_t \) is decreasing and it follows that \( \phi'_t(pq) \leq \phi'_t(bq) \) almost surely. Thus, \( \mathbb{E}_{p \sim P}[\phi'_t(pq)(p - (1 - q))] \leq \phi'_t(bq)(\mathbb{E}_{p \sim P}[p] - (1 - q)) < 0 \), for \( 1 - q > \mathbb{E}_{p \sim P}[p] \). Analogously, it can be followed that the FOC for \( x = 0 \) is positive if \( 1 - q < \mathbb{E}_{E \sim P}[p] \). As \( x^*(1 - q) \) is continuous on the belief interval \( B \), it follows that mixing is optimal in an environment of \( \mathbb{E}_{E \sim P}[p] \) if \( B \) doesn’t reduce to a single point.

Now consider a series \( w_n \) such that \( u_{\Delta,n} = u(w_n) - u(0) \to \infty \). The utility function is not unique (compare Theorem 1 [Klibanoff et al., 2005]). If preferences are represented by utility functions \( u_n(0) = 0 \) and \( u_n(u_n) = 1 \), the agent acts identical to a decision maker with transformed \( \phi_n(z) = \phi(z - u_{\Delta,n} + c_0) \). The coefficient of ambiguity aversion for this rescaled agent is
\[
\alpha_n(z) = -\frac{\phi''_n}{\phi'_n} = -\frac{u_{\Delta,n}\phi''(u_{\Delta,n}z + u(0))}{u_{\Delta,n}^2 \phi'(u_{\Delta,n}z + u(0))} = \frac{\alpha(z)}{u_{\Delta,n}},
\]

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where $\alpha(z) = -\frac{\phi''(z)}{\phi'(z)}$ is uniquely defined coefficient of ambiguity aversion. It holds that $\alpha_{n+1} > \alpha_n$. If $\alpha(z)$ is bounded away from zero, it holds that $\inf_z \alpha_n(z) \to \infty$. With Proposition 4 in Klibanoff et al. (2005) it follows that for large $n$ the preferences are essentially identical to maxmin preferences. Lemma 3 establishes that those have mixing interval $M = B$.

Proof of Lemma 7: We focus on $x^*_E$, which can fall in the interval $[1-q, 1]$. The results for $x^*_C$ follow analogously. As in Proof of Lemma 5 we have

$$(\partial_x)^2 U(x, q) = E_{p \sim P}[\phi''(s_q(x, p))(p - (1-q))^2],$$

which in turn implies that $U(x, q)$ is convex in $x$ as $\phi'' \geq 0$. As a consequence we have a corner solution with $x^*_E(q) \subset \{1-q, 1\}$.

Consider the case that $1-q \geq b$. For the same argument as in proof of Lemma 5 it follows that $x^*_E(q) = 1-q$.

Consider the case that $1-q < E_{p \sim P}[p]$. We have that

$$\partial_x U_q(1) = E_{p \sim P}[\phi'(pq)(p - (1-q))] > \phi'(bq)(E_{p \sim P}[p] - (1-q)) > 0.$$

Consequently, $x^*_E(q) = 1$.

Consider the utility difference between the two possible corner solution

$$\Delta(q) := U_q(1) - U_q(1-q) = E_{p \sim P}[\Phi(pq) - \Phi(q(1-q))].$$

If $\Phi$ is strictly convex and $P$ not a point mass, it follows by Jensen’s inequality that $\Delta(q) > 0$. So, $x^*_E(1-E_{p \sim P}[p]) = 1$. As $\Delta(q)$ is continuous in $q$, there exists a $\epsilon > 0$ such that $x^*_E(1-E_{p \sim P}[p]+\epsilon) = 1$. 

□
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