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The Wentzel-Kramers-Brillouin approximation for light scattering by horizontally oriented prismatic particles

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Abstract

In this research, we propose a simple approximation technique based on the decomposition of particles and on the Wentzel-Kramers-Brillouin approximation (WKB). The latter allows us to express analytically the expression of the form factor of a particle of a regular prismatic ice crystal shape. Our focus will be on studying the light scattering by horizontally oriented prismatic particles which rotate about their principal axes. To illustrate our formalism, a few numerical examples will be analyzed.

1. Introduction

To fully understand and study the influence of various characteristics of particles such as shape, refractive index, composition or surface roughness, on the radiation-matter interaction phenomena, it is necessary to determine the scattering and absorption of light by small particles. This study advances many areas from medical technology to computer engineering, geophysics, photonics and military technology [1, 2]. In addition to that, light scattering by aerosols particles is of importance such as in the remote sensing of cirrus clouds and in studying of their influence on climate. Most atmospheric aerosols have a spherical shape, but due to various external influences (temperature, humidity, super saturation...), other forms may exist (cubical, hexagonal, prismatic...) [3].

Gustav Mie realizes an exact treatment of light scattering, he used the factorization of Maxwell’s equations in spherical coordinates and he obtained the rigorous analytical solution of the scattering of electromagnetic waves by the homogenous sphere. Mie also took advantage of the spherical morphology to solve the boundary conditions using vector spherical harmonic expansions. The exact solution for non-spherical particles that are characterized by distinct surfaces seems not to be possible because appropriate coordination systems for the separation of the Helmholtz equation cannot be defined, in addition to the constraint of surface discontinuities, contrary to the sphere which can be represented with a single continuous surface [4].

The mathematical descriptions of scattering by a non-spherical particle need an approximate method. This can be done through a particularly useful framework which is provided by what is called the volume integral equation resulting macroscopic Maxwell equations and incorporates the boundary conditions. This integral is expressed in terms of total electric field inside the particle which is usually unknown. In many practical situations, it is possible to substitute the total electric field inside the particle by an approximate electric field obtained using approximate methods such as the Rayleigh-Gans (RG) approximation, the anomalous diffraction (AD) and the WKB method [5–7].

In the WKB approximation, we postulated the rectilinear propagation of the incident wave in the particle. Moreover, there is a change of phase of the wave in proportion to the degree of penetration into the object [7]. This approach is applied earlier on spheres, cylinders and spheroids [7–10]. Recently this approach has been applied to modelling the scattering properties of cubical and hexagonal particle [11, 12].

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This paper is devoted to extend the WKB approximation to the prismatic particle. We investigate the amplitude of the light scattering by homogenous absorbing prismatic particle oriented with the axis of revolution perpendicular of the incident light direction.

2. Form factor of light scattering in WKB approximation

2.1. General formulation of the WKB approximation

Consider a small particle with complex refractive index \( m = m_r + m_i \) illuminated by a plane wave of wave number \( k = \frac{2\pi}{\lambda} \) polarized in the direction \( \vec{e}_x \) and propagating along the \( z \)-axis (Figure 1). In the literature, the expression of the amplitude of light scattering in the WKB approximation is related to the form factor \( F(\theta, \varphi) \) by [7]:

\[
|f(\vec{s}, \vec{i})| = \sin(\zeta) \frac{k^2}{2\pi} |(m - 1) F(\theta, \varphi)|
\]

where \( \vec{s} \) and \( \vec{i} \) are the unit vectors along the directions of scattering and propagation of light, respectively. \( \zeta \) is the angle between the polarization vector \( \vec{e}_x \) and the unit vector \( \vec{s} \), \( \varphi \) is the azimuthally angle and \( \theta \) is the scattering angle between \( \vec{s} \) and \( \vec{i} \) and \( F(\theta, \varphi) \) is the form factor given by [7, 8]:

\[
F(\theta, \varphi) = \int \int \int_v \exp(ik(\vec{i} - \vec{s})) \exp(ikw)dv
\]

where \( v \) is the volume of the scattered, and \( w \) is the optical path which is introduced by the scattering object in the form:

\[
w = \int_{z_o}^{z_s} [m(z') - 1]dz' = (m - 1) \times (z - z_e)
\]

The form factor can be expressed in rectangular coordinates by [11]:

\[
F(\theta, \varphi) = \int \int e^{-ik(x \cos \theta + y \sin \theta)} e^{ikz(m - \cos \theta - z_e(m - 1))}dx dy dz
\]

where \( z_o \) and \( z_s \) is the \( z \)-coordinates of the intersection between the incident light and the input and output surfaces of the body respectively.

After the integration over \( z \) we get:

\[
F(\theta, \varphi) = \frac{1}{ik(m - \cos \theta)} \int e^{-ikx \sin \theta} e^{ikz(m - \cos \theta - z_e(m - 1)} dx dy
\]

2.2. Form factor of light scattering in the WKB approximation

Ice crystals are found in the form of plates, columns and needles. Plates and needles can be considered as special cases of columns. Our study focuses on columnar prismatic particles. In this section we consider the horizontal incidence of light.

Consider a prism of length \( l \), to simplify the calculation, the two triangular sides are considered to be equilateral with side \( a \) and height \( h \). The incident rays are characterized by an elevation angle \( \alpha \). From the symmetry of the prism, \( \alpha \) is from 0 to \( \frac{\pi}{3} \), \( \alpha \) is zero when the incident ray is perpendicular to one of the three sides of the prism. For non-spherical particles, the Cartesian coordinates system \( R(O, X, Y, Z) \) is adopted, its origin...
coincide with the geometric center of the particle and the main axis of the particle is oriented along the $x$-axis (figure 2).

In this study, we apply the WKB approximation to a prismatic particle oriented horizontally, we will focus on two particular cases of horizontal incidence:

(a): $0 \leq \alpha < \frac{\pi}{6}$

(b): $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{3}$

For obliquely incident ray with angle $\alpha$, the triangular side is divided into two areas by rays 1, 2 and 3 for $\alpha$ from 0 to $\frac{\pi}{6}$ (see figure 3(a)) or by rays $1'$, $2'$ and $3'$ when $\alpha$ is between $\frac{\pi}{6}$ and $\frac{\pi}{3}$ (see figure 3(b)). To apply the WKB method, the prism is cut into infinitely thin slices of length $l$, thickness $dy$, and width $D = z_s - z_e$, where $z_e$ and $z_s$ are the $z$-coordinates of the intersection of the incident ray and the body lateral surfaces of the particle (figure 3(c)). Indices $j = 1, 2$ (or $1', 2'$) are from area 1, 2 (or $1', 2'$) respectively, the cut is made along the direction of $\alpha$.

It can be seen from figure 3 that the coordinates $z_e$ and $z_s$ are only function of the variable $y$. So From figure 3(a), we have:

$$z_{e1} = -\frac{a}{2\sqrt{3} \cos \alpha} - y \tan \alpha$$

$$z_{s1} = \frac{a}{2\sqrt{3} \sin \left(\frac{\pi}{6} + \alpha\right)} + y \cot \left(\frac{\pi}{6} + \alpha\right); y_M \leq y \leq y_N$$

$$z_{e2} = z_{e1}$$

$$z_{s2} = -\frac{a}{2\sqrt{3} \sin \left(\frac{\pi}{6} - \alpha\right)} - y \cot \left(\frac{\pi}{6} - \alpha\right); y_N \leq y \leq y_P$$

(6)

(7)
with:

\[
\begin{align*}
    y_M &= -a \cos \left( \frac{\pi}{6} - \alpha \right) \\
    y_N &= a \sin \alpha \\
    y_p &= a \cos \left( \frac{\pi}{6} + \alpha \right) \\
\end{align*}
\] (8)

And from figure 3(b), we have:

\[
\begin{align*}
    z_{4l'} &= -\frac{a}{2\sqrt{3}} \cos \alpha - y \tan(\alpha) \\
    z_{4l} &= \frac{a}{2\sqrt{3}} \sin \left( \frac{\pi}{6} + \alpha \right) + y \cot \left( \frac{\pi}{6} + \alpha \right); \; y_M' \leq y \leq y_{N'} \\
    z_{4l''} &= \frac{a}{2\sqrt{3}} \sin \left( \frac{\pi}{6} - \alpha \right) - y \cot \left( \frac{\pi}{6} - \alpha \right); \; y_{N'} \leq y \leq y_{P'} \\
    z_{4l'} &= z_{4l}
\end{align*}
\] (9)

with:

\[
\begin{align*}
    y_{M'} &= -a \cos \left( \frac{\pi}{6} - \alpha \right) \\
    y_{N'} &= a \cos \left( \frac{\pi}{6} + \alpha \right) \\
    y_{P'} &= a \sin (\alpha) \\
\end{align*}
\] (11)

It is easy to integrate the equation (5) over the variable \( x \).

So, in rectangular coordinate system \( R(o, x, y, z) \), the form factor (equation (5)) can be expressed in a simple form:

\[
F(\theta, \varphi) = \begin{cases} 
    F_1(\theta, \varphi) + F_2(\theta, \varphi); & 0 \leq \alpha < \frac{\pi}{6} \\
    F_1(\theta, \varphi) + F_2'(\theta, \varphi); & \frac{\pi}{6} \leq \alpha \leq \frac{\pi}{3} 
\end{cases}
\] (12)

\( F_1(\theta, \varphi) \) and \( F_2(\theta, \varphi) \) are the contribution from the area \( j = 1, 2 \) and \( j' = 1', 2' \) to the form factor respectively.

where:

\[
\begin{align*}
    F_1(\theta, \varphi) &= \int_{y_m}^{y_c} \text{ampl}(\theta, \varphi, \Delta z_i) \, dy \\
    F_2(\theta, \varphi) &= \int_{y_c}^{y_e} \text{ampl}(\theta, \varphi, \Delta z_i) \, dy \\
    F_1'(\theta, \varphi) &= \int_{y_m}^{y_{c'}} \text{ampl}(\theta, \varphi, \Delta z_i') \, dy \\
    F_2'(\theta, \varphi) &= \int_{y_{c'}}^{y_e} \text{ampl}(\theta, \varphi, \Delta z_i') \, dy \\
\end{align*}
\] (13) (14) (15) (16)

with \( \text{ampl}(\theta, \varphi, \Delta z_i(y)) \, dy \) the contribution of the slice to the form factor is denoted by:

\[
\text{ampl}(\theta, \varphi, \Delta z_i(y)) \, dy = A(\theta, \varphi) e^{-iky} \sin \theta \sin G(z_{(i)}, z_{(i)}) \, dy
\] (17)

where

\[
A(\theta, \varphi) = \frac{l}{ik(m - \cos \theta)} \sin \frac{d}{d}
\] (18)
and
\[ G(z_{ij}, z_{0}) = e^{ik(m - \cos \theta)x_{ij}} - e^{-ik(m - 1)x_{ij}} \]

with
\[ d = \frac{kl}{2} \sin \theta \cos \varphi \]  

Equation (17) is obtained by doing the integral in equation (4) over the variable \( x \) from \(-\frac{l}{2}\) to \( \frac{l}{2}\), and over the variable \( z \) from \( z_e \) to \( z_s \).

By doing the integration over the variable \( y \) in the equations (13), (14), (15) and (16), the from factor can be expressed in simple analytical expressions:

- when \( 0 \leq \alpha < \frac{\pi}{6} \):
  \[ F_1(\theta, \varphi, \alpha) = A(\theta, \varphi)B_x(\alpha)(I_-(\alpha) - J_-(\alpha)) \]
  \[ F_2(\theta, \varphi, \alpha) = A(\theta, \varphi)B_y(\alpha)(I_+(\alpha) - J_+(\alpha)) \]

where:
\[ B_x(\alpha) = a \sin \left( \frac{\pi}{6} + \alpha \right) e^{i \frac{\alpha}{\sqrt{3}}} e^{i2\alpha(1 \pm \sin \alpha \cos \left( \frac{\pi}{6} + \alpha \right))} \]
\[ I_\pm(\alpha) = e^{i\mu} \frac{\sin \left( t - \frac{2q \sin \alpha}{\sqrt{3}} \right) \sin \left( \frac{\pi}{6} + \alpha \right) \pm \mu}{\left( t - \frac{2q \sin \alpha}{\sqrt{3}} \right) \sin \left( \frac{\pi}{6} + \alpha \right) \pm \mu} \]

The parameters used are:
\[ t = \frac{ka}{2} \sin \theta \sin \varphi \]
\[ \mu = \frac{ka\sqrt{3} (m - \cos \theta)}{4 \cos \alpha} \]
\[ q = \frac{ka\sqrt{3} (\cos \theta - 1)}{4 \cos \alpha} \]

when \( \frac{\pi}{6} \leq \alpha \leq \frac{\pi}{3} \):
\[ F_1(\theta, \varphi, \alpha) = A(\theta, \varphi)B'_x(\alpha)(I'_-(\alpha) - J'_-(\alpha)) \]
\[ F_2(\theta, \varphi, \alpha) = A(\theta, \varphi)B'_y(\alpha)(I'_+(\alpha) - J'_+(\alpha)) \]

where:
\[ B'_x(\alpha) = a \cos \alpha e^{i \frac{\alpha}{\sqrt{3}}} e^{i2q' \cos \alpha \sin \left( \frac{\pi}{6} + \alpha \right)} \]
\[ B'_y(\alpha) = -a \cos \left( \frac{\pi}{3} + \alpha \right) e^{-i \frac{\alpha}{\sqrt{3}}} \sin \left( \frac{\pi}{6} + \alpha \right) e^{-i2q' \cos \left( \frac{\pi}{6} + \alpha \right) \sin \left( \frac{\pi}{6} + \alpha \right)} \]
\[ I'_{\pm}(\alpha) = e^{i\mu'} \frac{\sin \left( t \cos \alpha - \mu' - \frac{2}{\sqrt{3}} \sin \alpha \sin \left( \frac{\pi}{6} + \alpha \right) \right)}{\left( t \cos \alpha - \mu' - \frac{2}{\sqrt{3}} \sin \alpha \sin \left( \frac{\pi}{6} + \alpha \right) \right)} \]
\[ I'_-(\alpha) = e^{i\mu'} \frac{\sin \left( t \cos \left( \frac{\pi}{3} + \alpha \right) - \mu' - \frac{2}{\sqrt{3}} \sin \left( \frac{\pi}{6} + \alpha \right) \sin \left( \frac{\pi}{6} + \alpha \right) \right)}{\left( t \cos \left( \frac{\pi}{3} + \alpha \right) - \mu' - \frac{2}{\sqrt{3}} \sin \left( \frac{\pi}{6} + \alpha \right) \sin \left( \frac{\pi}{6} + \alpha \right) \right)} \]
The parameters used are:

\[
\mu' = \frac{\cos \alpha}{\sin \left(\frac{\pi}{6} + \alpha\right)} \mu \\
q' = \frac{\cos \alpha}{\sin \left(\frac{\pi}{6} + \alpha\right)} q
\]

The parameter \( t \) was already defined in the first case.

3. Results and discussion

We have established the analytical expressions of the form factor of a prismatic particle as a part of the WKB approximation. These analytical expressions were obtained for the horizontally incidence of light. First, we checked the accuracy of the form factor analytical expressions. For that, we numerically calculated the various double integrals which intervene here by using the method of quadrature of Gauss Legendre. By taking 16 \( \times \) 16 integration points for this quadrature, we numerically reproduced the results obtained with the analytical expressions of the form factor.

To illustrate this, we show in figure 4 the angular variation of the normalized form factor of the light scattered by horizontally oriented prismatic plates and columns, illuminated by a monochromatic wave, for three values of \( \varphi = 0^\circ, 30^\circ, 90^\circ \), at a wavelength \( \lambda = 0.55 \) m, and complex refractive index \( m = 1.311 + 0.31 \times 10^{-8} i \), and \( ka = 10 \).

These Figures show that the form factor exhibits some lobes with intensity decreasing with scattering angle and we note that the rotation around the main axis of the particles has a significant influence on the form factor of the scattered light. This is obvious because the particle does not have a cylindrical symmetry. In addition, the figure shows that the backscattering \( \varphi > 90^\circ \) of the horizontally oriented particles is smaller compared to the forward scattering. We find also that the form factor is sensitive to the aspect ratio \( l/a \) for these particles.

4. Extinction efficiency

The extinction efficiency \( Q_{ext} \) is defined as the extinction cross section divided by the projection area of the particle. By using the optical theorem, we can easily find the expression of the extinction efficiency from the forward form factor [5, 6]:

\[
Q_{ext} = \frac{2k}{P} \text{Im}((m - 1)F(0, 0, \alpha))
\]

(39)

where \( P \) is the projected area of the particle on the plane perpendicular to the direction of the incident wave, and \( \text{Im} \) is the imaginary part.

In this section we present the extinction efficiency for prismatic ice crystals at the 0.55 \( \mu \)m wavelength. The complex refractive indices of ice at these wavelength is \( m = m_r + 0.31 \times 10^{-8} i \). The projected area of the prism in the first case \( 0 \leq \alpha < \frac{\pi}{3} \) is given by \( P_1 = a l \cos \alpha \) and in the second case \( \frac{\pi}{3} \leq \alpha < \frac{\pi}{6} \) by \( P_2 = a l \sin \left(\frac{\pi}{6} + \alpha\right) \).

The extinction efficiency of these two cases of horizontally oriented incidence should be written as:

\[
Q_{ext1} = 2\text{Re} \left(1 - \frac{e^{i\varphi_1} - 1}{i\varphi_1}\right)
\]

(40)

\[
Q_{ext2} = 2\text{Re} \left(1 - \frac{e^{i\varphi_2} - 1}{i\varphi_2}\right)
\]

(41)
Figure 4. Normalized form factor versus scattering angle $\theta$ for absorbing prismatic ice column with aspect ratio $l/a = 2$ in (a), and plate ($l/a = 0.2$) in (b), at the wavelength $\lambda = 0.55\ m$ and complex refractive index $n = 1.311 + 0.31 \times 10^{-3}$, $ka = 10$ for three values of $\varphi = 0^\circ, 30^\circ, 90^\circ$. 
where the symbol \( \text{Re} \) designates the real part and the parameters used are:

\[
\rho_1 = \frac{1}{2} \sqrt{3} ka \frac{(m - 1)}{\cos \alpha} \\
\rho_2 = \frac{1}{2} \sqrt{3} ka \frac{(m - 1)}{\sin \left( \frac{\pi}{6} + \alpha \right)}
\]

Since the parameters \( \rho_1 \) and \( \rho_2 \) do not depend on height \( l \), the expressions of \( Q_{\text{ext}} \) above show that the extinction efficiency also does not depend on the height \( l \).

For real refractive index, the extinction efficiency become:

\[
Q_{\text{ext}} = 2 \left( 1 - \frac{\sin \rho_1}{\rho_1} \right) \\
Q_{\text{ext}} = 2 \left( 1 - \frac{\sin \rho_2}{\rho_2} \right)
\]

For illustration we present in figure 5, the behavior of the extinction efficiency factors for horizontally oriented prismatic particles in the WKB approximation, as functions of parameter \( X = \frac{2}{\lambda} \). This parameter allows us to compare the wavelength with the largest possible ray path within the considered particle. According to this figure, there are three phases of variation of the extinction coefficient:

- When the geometric path (the size of the diffuser) is small compared to the incident wavelength, the extinction coefficient is small.
- When the size of the diffuser is of the order of magnitude of the incident wavelength, the extinction coefficient increases rapidly up to a peak of maximum efficiency, then oscillates author of the value 2.
- When the size of the diffuser becomes large compared to the wavelength, the efficiency is totally independent of the size of the particle studied, and approach the limiting value 2, this would suggest that the particle will block off twice the light falling upon it, an effect calling the ‘extinction paradox’ van de Hulst 1957 [6], Bohren and Huffman 1983 [13].

We show that the formulas of the extinction efficiency of a prismatic particle in the WKB approximation are identical to the expression in the AD approximation found by P. Chylek and J.D. Klett [14, 15] for non-absorbing homogeneous particles.

5. Conclusion

In this paper, we have determined the analytical expression of the form factor of the scattered light for homogenous prismatic particle for horizontally incidence in the WKB approximation. According to this expression, we conclude that the physical properties of the light scattered by the small particles are affected by the orientation of the particle (angle of incidence \( \alpha \)), the geometrical shape and the aspect ratio (plate or column), as well as by the scattering angle \( \theta \) and the azimuthal angle \( \varphi \). Using our analytical formulation we have...
addressed some numerical examples for the normalized form factor. The future attentions will be given to study for arbitrary oriented prismatic particles.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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