A nonasymptotic law of iterated logarithm for robust online estimators

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Abstract

In this paper, we provide tight deviation bounds for \( M \)-estimators, which are valid with a
prescribed probability for every sample size. \( M \)-estimators are ubiquitous in machine learning and
statistical learning theory. They are used both for defining prediction strategies and for evaluating
their precision. Our deviation bounds can be seen as a non-asymptotic version of the law of iterated
logarithm. They are established under general assumptions such as Lipschitz continuity of the
loss function and (local) curvature of the population risk. These conditions are satisfied for most
examples used in machine learning, including those that are known to be robust to outliers and to
heavy tailed distributions. To further highlight the scope of applicability of the obtained results,
a new algorithm, with provably optimal theoretical guarantees, for the best arm identification in a
stochastic multi-arm bandit setting is presented. Numerical experiments illustrating the validity of
the algorithm are reported.

Keywords: Law of Iterated logarithm, \( M \)-estimators, Online learning, Robust estimation

1. Introduction

Perhaps the most fundamental theorems in statistics are the law of large numbers (LLN) and the
central limit theorem (CLT). Morally, they state that a sample average converges almost surely or
in probability to the population average, and if one zooms in by multiplying by a square root factor,
a much weaker form of stochastic convergence still holds, namely, convergence in distribution towards
a Gaussian law. A fine intermediate result shows what happens in between the two scales: the
law of iterated logarithm (LIL). By zooming in slightly less than in the CLT, \( i.e., \) by rescaling the
sample average with a slightly smaller factor than in the CLT, it is possible to gain a guarantee for
infinitely many sample sizes, almost surely. In practice, however, the LIL has limited applicability,
since it does not specify for which sample sizes the guarantee holds. The goals of the present work
are (a) to lift this limitation, by proving a LIL valid for every sample size, and (b) to extend the LIL
(known to be true for sample averages) to general \( M \)-estimators.

The precise statement of the LIL, discovered by Khintchine (1924); Kolmogoroff (1929) almost
a century ago, is as follows: For a sequence of iid random variables \( \{Y_i\}_{i \in \mathbb{N}} \) with mean \( \theta \)
and variance \( \sigma^2 < \infty \), the sample averages \( \bar{Y}_n = (Y_1 + \ldots + Y_n)/n \) satisfy the relations

\[
\liminf_{n \to \infty} \frac{\sqrt{n} (\bar{Y}_n - \theta)}{\sigma \sqrt{2 \ln \ln n}} = -1 \quad \text{and} \quad \limsup_{n \to \infty} \frac{\sqrt{n} (\bar{Y}_n - \theta)}{\sigma \sqrt{2 \ln \ln n}} = 1, \quad \text{almost surely.}
\]

This provides a guarantee on the deviations of the sample average as an estimator of the mean \( \theta \),
since it yields that with probability one, there is a \( n_0 \in \mathbb{N} \) such that \( |\bar{Y}_n - \theta| \leq \sigma (2 \ln \ln n) / n \) for
every \( n \geq n_0 \). As compared to the deviation guarantees provided by the central limit theorem, the
one of the last sentence has the advantage of being valid for any sample size large enough. This advantage is gained at the expense of a factor \((\ln \ln n)^{1/2}\). Akin for the classic version of the CLT, the applicability of LIL is limited by the fact that it is hard to get any workable expression of \(n_0\).

In the case of the CLT and its use in statistical learning, the drawback related to \(n_0\) was lifted by exploiting concentration inequalities, such as the Hoeffding or the Bernstein inequalities, that can be seen as non-asymptotic versions of the CLT. For random variables bounded by 1, the aforementioned concentration inequalities imply that for a prescribed tolerance level \(\delta \in (0, 1)\), for every \(n \in \mathbb{N}\), the event 1 \(A_n = \{ |\bar{Y}_n - \theta| \leq C(\ln(1/\delta)/n)^{1/2} \}\) holds with probability at least \(1 - \delta\). Such a deviation bound is largely satisfactory in a batch setting, when all the data are available in advance. In contrast, when data points are observed sequentially, as in on-line learning, the event of interest is \(\tilde{A}_N = A_1 \cap \ldots \cap A_N\), which is guaranteed, by the union bound, to have a probability at least \(1 - N\delta\). Replacing in \(A_n\) \(\delta/n^2\), we get coverage \(1 - \frac{\pi^2}{6}\delta\), valid for any sample size \(n\) for an interval of length \(O((\ln n/n)^{1/2})\). This result, derived by a straightforward application of the union bound, appears to be suboptimal. A remedy to such a suboptimality—in the form of a nonasymptotic version of the LIL—was proposed by Jamieson et al. (2014). In addition, its relevance for online learning was demonstrated by deriving guarantees for the best arm selection in a multi-armed bandit setting.

In this work, we address the problem of establishing a non-asymptotic LIL in a general setting that covers many estimators, far beyond the sample average. More precisely, we focus on the class of (penalized) \(M\)-estimators comprising the sample average but also the sample median, the quantiles, the least-squares estimator, etc. Of particular interest to us are estimators that are robust to outliers and/or to heavy tailed distributions. This is the case of the median, the quantiles, the Huber estimator and so on... (Huber et al., 1964). It is well known that under mild assumptions, \(M\)-estimators are both consistent and asymptotically normal, i.e., a suitably adapted version of the LLN and the CLT applies to them (van der Vaart, 1998; Portnoy, 1984; Collins, 1977). Moreover, some versions of the LIL were also shown for \(M\)-estimators (Arcones, 1994; He and Wang, 1995), with little impact in statistics and machine learning, because of the same limitations as those explained above for the standard LIL. Our contributions complement these studies by providing a general non-asymptotic LIL for \(M\)-estimators.

We apply the developed methodology to the problem of multi-armed bandits when the rewards are heavy tailed or contaminated by outliers. In such a context, Altschuler et al. (2018) tackled the problem of best median arm identification; this corresponds to replacing the average regret by the median regret. The relevance of this approach relies on the fact that even a small number of contaminated samples obtained from each arm may make the corresponding means arbitrarily large. The method proposed in Altschuler et al. (2018) is a suitable adaptation of the well-known upper confidence band (UCB) algorithm. In that setup, would it be possible to improve the upper bounds on the sample complexity of their algorithm—similarly to Jamieson et al. (2014)—by using some version of the uniform LIL for empirical medians or, more generally, for robust estimators? Our main results yield a positive answer to this question.

The rest of the paper is organized as follows. The next section contains the statement of the LIL in a univariate setting and provides some examples satisfying the required conditions. A multivariate version of the LIL for penalized \(M\)-estimators is presented in Section 3. An application to on-line...
learning is carried out in Section 4, while a summary of the main contributions and some future directions of research are outlined in Section 5. We provide full proofs of stated results in Section 6.

2. Uniform law of iterated logarithm for \( M \)-estimators

In this section, we focus on the case of univariate \( M \)-estimators, which are a natural extension of the empirical mean, especially in robust setups (see Huber et al. (1964); Maronna (1976) as well as the recent work by Loh (2017) and the references therein). We consider a sequence \( Y, Y_1, Y_2, Y_3, \ldots \) of i.i.d. random variables in some arbitrary space \( \mathcal{Y} \) and we let \( \phi: \mathcal{Y} \times \Theta \to \mathbb{R} \) be a given loss function, where \( \Theta \) is an open interval in \( \mathbb{R} \). We assume that for all \( \theta \in \Theta \), \( \phi(Y, \theta) \) has a finite expectation and that \( \phi(Y, \cdot) \) is convex \( \mathbb{P}_Y \)-almost surely, where \( \mathbb{P}_Y \) is the probability distribution of \( Y \). We define the population risk \( \Phi(\theta) = \mathbb{E} [\phi(Y, \theta)] \) and, for all integers \( n \geq 1 \), the empirical risk \( \hat{\Phi}_n(\theta) = \frac{1}{n} \sum_{i=1}^n \phi(Y_i, \theta). \) They are both convex functions of \( \theta \in \Theta \). We denote by \( \theta^* \) a minimizer of \( \Phi \) on \( \Theta \), and by \( \hat{\theta}_n \) a minimizer of \( \hat{\Phi}_n \) on \( \Theta \), for all \( n \geq 1 \).

Only in order to guarantee the existence of \( \theta^* \) and \( \hat{\theta}_n \) for all \( n \geq 1 \), we assume that \( \phi(Y, \theta) \to \infty \) as \( \theta \) approaches the boundary of \( \Theta \), \( \mathbb{P}_Y \)-almost surely. This assumption is generally not restrictive, as it is satisfied most of the time. We will also need some other assumptions, listed below.

**Assumption 2.1** For all \( \theta \in \Theta \), the random variable \( \phi(Y, \theta) \) has a finite expectation.

**Assumption 2.2** The minimizer \( \theta^* \) of \( \Phi \) is unique and there exist two positive constants \( r \) and \( \alpha \) such that for all \( \theta \in \Theta \) with \( |\theta - \theta^*| \leq r \), \( \Phi(\theta) \geq \Phi(\theta^*) + (\alpha/2)(\theta - \theta^*)^2 \).

**Assumption 2.3** There exists a positive constant \( \sigma^2 \) such that the random variables \( \phi(Y, \theta) - \phi(Y, \theta^*) \) are \( \sigma^2(\theta - \theta^*)^2 \)-sub-Gaussian for all \( \theta \in \Theta \).

Assumption 2.2 requires from \( \Phi \) to have a positive curvature in a neighborhood of the oracle \( \theta^* \). It is weaker than the local strong convexity of \( \Phi \). Assumption 2.3 is a smoothness condition on \( \phi(Y, \cdot) \). In particular, it is fulfilled if \( \phi(Y, \cdot) \) is \( \eta \)-Lipschitz with a sub-Gaussian variable \( \eta \). We stress that the function \( \phi \) is not assumed differentiable.

**Theorem 1** Let Assumptions 2.1, 2.2 and 2.3 hold. Let \( \varepsilon > 0 \) and \( \delta \in (0, 8) \). Then,

\[
\mathbb{P}
\left(
|\hat{\theta}_n - \theta^*| \leq c_{n, \delta} := \frac{2.2\sigma}{\alpha} \sqrt{\frac{2.2 \ln \ln n + 2 \ln (1/\delta) + 5.2}{n}}; \forall n \geq n_0 \right)
\geq 1 - 20\delta,
\]

where \( n_0 = n_0(\alpha, r, \delta) \) is the smallest integer \( n \geq 2 \) for which \( c_{n, \delta} \leq r \).

Note that the choice of the range of \( \delta \) in Theorem 1 is made to ensure that \( \ln \left((\log_{1,1} n)^{1.1}/\delta\right) \) is always non-negative.

**Remark 1** In the definition of \( \Phi \) and \( \hat{\Phi}_n \), one can replace \( \phi(Y, \theta) \) with \( \phi(Y, \theta) - \phi(Y, \theta_0) \) for any arbitrary \( \theta_0 \in \Theta \), without changing the values of \( \theta^* \) and \( \hat{\theta}_n \). Then, Assumption 2.1 becomes less restrictive for \( Y \) in general, since it only requires \( \phi(Y, \theta) - \phi(Y, \theta_0) \) to have a finite expectation. For instance, for median estimation, \( \phi(Y, \theta) = |Y - \theta| \), yet the median should be defined even if \( Y \) does not have an expectation. Taking \( \theta_0 = 0 \) yields \( \phi(Y, \theta) - \phi(Y, \theta_0) = |Y - \theta| - |Y| \), which is bounded, hence, always has an expectation.

We now give some natural examples for which all the assumptions presented above are satisfied.

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2. See, for instance, (Koltchinskii, 2011, Section 3.1) for a definition of sub-Gaussian random variables and their properties.
**Mean estimation** Let $\mathcal{Y} = \Theta = \mathbb{R}$ and $\phi(x, \theta) = (x - \theta)^2$. Assume that $Y$ is $\sigma^2$-sub-Gaussian. Then, it is easy to see that Assumptions 2.1, 2.2 and 2.3 are all satisfied with $r = +\infty$ and $\alpha = 1$. Thus, Theorem 1 recovers Lemma 1 in (Jamieson et al., 2014).

**Median and quantile estimation** Let $\mathcal{Y} = \Theta = \mathbb{R}$ and $\phi(x, \theta) = |x - \theta| - |x|$. Assume that $Y$ has a unique median $\theta^*$ and that its cumulative distribution function $F$ satisfies $|F(\theta) - 1/2| \geq \alpha |\theta - \theta^*|$, for all $\theta \in [\theta^* - r, \theta^* + r]$, where $r > 0$ is a fixed number. Then, $\theta^*$ is the unique minimizer of $\Phi$ and for all $\theta \in [\theta^* - r, \theta^* + r]$, 

$$\Phi(\theta) - \Phi(\theta^*) = 2 \int_{(\theta, \theta^*)} x dF(x) - (\theta - \theta^*) + 2(\theta F(\theta) - \theta^* F(\theta^*)) = 2 \int_{(\theta, \theta^*)} F(x) dx - (\theta - \theta^*) \geq \frac{\alpha}{2} (\theta - \theta^*)^2,$$

yielding Assumption 2.2. Moreover, since $\phi(Y, \theta)$ is bounded almost surely and 1-Lipschitz, for all $\theta \in \mathbb{R}$, Assumptions 2.1 and 2.3 are automatically true (with $\sigma = 1/2$).

The same arguments hold true if $\phi(x, \theta) = \tau_\alpha(x - \theta) - \tau_\alpha(x)$, where $\tau_\alpha(x) = \alpha x$ if $x \geq 0$, $\tau_\alpha(x) = (\alpha - 1)x$ otherwise, for which $\theta^*$ is the $\alpha$-quantile of $Y$, for $\alpha \in (0, 1)$.

**Huber’s $M$-estimators** Let $\mathcal{Y} = \Theta = \mathbb{R}$ and let $c > 0$. Denote by $g_c(x) = x^2$ if $|x| \leq c$, $g_c(x) = c(2x - c)$ if $|x| > c$ and let $\phi(x, \theta) = g_c(x - \theta) - g_c(x)$. This function $g_c$ being $2c$-Lipschitz, Assumption 2.3 is satisfied with $\sigma = c$. Assume that $Y$ has a positive density $f$ on $\mathbb{R}$. Then, it is easy to check that $\Phi$ is twice differentiable, with $\Phi''(\theta) = 2(F(\theta + c) - F(\theta - c)) > 0$, for all $\theta \in \mathbb{R}$, where $F$ is the cumulative distribution function of $Y$. Hence, $\theta^*$ is well-defined and unique, and if there exists $m > 0$ such that $f(x) \geq m$ for $x \in [\theta^* - 2c, \theta^* + 2c]$, then Assumption 2.2 is satisfied with $r = 2c$ and $\alpha = 4cm$.

**Comparison between union bound and LIL** Let $Y_1, \ldots, Y_n$ be i.i.d. centered sub-Gaussian random variables with scale parameter $\sigma$. Let $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a loss that satisfies the assumptions of Theorem 1 with $r = +\infty$, $n_0 = 2$, $\alpha = 2$ (e.g., the squared loss). Let $\hat{\theta}_n$ be the $M$-estimator associated with the samples $Y_1, \ldots, Y_n$ and the loss $\phi$. The following tail bound stems from the sub-Gaussian assumption (see also (4) in Section 6): $\forall n \geq 2$, $\mathbb{P}(|\hat{\theta}_n - \theta^*| > \sqrt{\frac{2\ln(2/\delta)}{n}}) \leq \delta$. The union bound then gives 

$$\mathbb{P}(\forall n \geq 2, |\hat{\theta}_n| \leq c_{n, \delta}^{\text{UB}} := \sigma \sqrt{\frac{2\ln(2n^{1+\varepsilon}/\delta)}{n}}) \geq 1 - \sum_{k=1}^{\infty} \frac{1}{(k+1)^{1+\varepsilon}} \geq 1 - \delta/\varepsilon. \quad (2)$$

Figure 1 shows the ratio of the LIL upper bound $c_{n, \delta}$ provided by Theorem 1 over the sub-Gaussian upper bound $c_{n, \delta}^{\text{UB}}$, for different levels of global confidence. The parameters $\delta$ and $\delta'$ are chosen to guarantee that the right hand sides in both (1) and (2) are equal to the prescribed confidence level. For $c_{n, \delta}^{\text{UB}}$, we chose $\varepsilon = 0.1$, the results for other values of $\varepsilon$ being very similar. The plots show that the threshold in (1) is smaller than the threshold furnished by the union bound for $n \geq 1000$.

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3. More precisely, for $n \geq 750$ if target confidence is $99\%$, $n \geq 511$ is target confidence is $95\%$ and $n \geq 438$ if $90\%$. 

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4
3. Uniform LIL for $M$-estimators of a multidimensional parameter

We consider here a standard setting in supervised learning, in which the goal is to predict a real valued label using a $d$-dimensional feature. More precisely, we are given $n$ independent label-feature pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$, with labels $Y_i \in \mathbb{R}$ and features $X_i \in \mathbb{R}^d$, drawn from a common probability distribution $P$. Let $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given loss function and $\rho_n : \mathbb{R}^d \to \mathbb{R}$ a given penalty. For a sample $(X_1, Y_1), \ldots, (X_n, Y_n)$, we define the penalized empirical and population risks

$$
\hat{\Phi}_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \phi(Y_i, \theta^\top X_i) + \rho_n(\theta) \quad \text{and} \quad \Phi_n(\theta) = \mathbb{E} \left[ \phi(Y_1, \theta^\top X_1) \right] + \rho_n(\theta).
$$

Note that the penalty $\rho_n$ is allowed to depend on the sample size $n$. Since our results are non-asymptotic, this dependence will be reflected in the constants appearing in the law of iterated logarithm stated below. We also define the penalized $M$-estimator $\hat{\theta}_n$ and its population counterpart $\theta^*$ by

$$
\hat{\theta}_n \in \arg \min_{\theta \in \mathbb{R}^d} \hat{\Phi}_n(\theta) \quad \text{and} \quad \theta^* \in \arg \min_{\theta \in \mathbb{R}^d} \Phi_n(\theta).
$$

Typical examples where such a formalism is applicable are the maximum a posteriori approach and penalized empirical risk minimization. Our goal is to establish a tight non-asymptotic bound on the error of $\hat{\theta}_n$, that is, with high probability, valid for every $n \in \mathbb{N}$. To this end, we consider a unit vector $a \in \mathbb{R}^d$ and we are interested in bounding the deviations of the random variable $a^\top (\hat{\theta}_n - \theta^*)$. One can think of $a$ as the feature vector of a new example, the label of which is unobserved. We aim at providing uniform non-asymptotic guarantees on the quality of the predicted label $\hat{y} = a^\top \hat{\theta}_n$.

The main result of this section is valid under the assumptions listed below. We will present some common examples in which all these assumptions are satisfied.

Assumption 3.1 (Finite expectation) The random variables $\phi(Y_1, \theta^\top X_1)$ has a finite expectation, for every $\theta$, with respect to the probability distribution $P$.

Assumption 3.2 (Convex and Lipschitz loss) The function $u \mapsto \phi(y, u)$ is $L$-Lipschitz and convex for any $y \in \mathbb{R}$.

Assumption 3.3 (Convex penalty) The penalty $\theta \mapsto \rho_n(\theta)$ is a convex function.
Remark 2  Assumptions 3.2 and 3.3 can be replaced by the assumption that the function $\hat{\Phi}_n$ is convex almost surely.

Assumption 3.4  (Curvature of the population risk)  There exists a positive non-increasing sequence $(\alpha_n)$ such that, for any $n \in \mathbb{N}^*$, for any $w \in \mathbb{R}^d$, $\Phi_n(\theta^* + w) - \Phi_n(\theta^*) \geq (\alpha_n/2)\|w\|_2^2$.

Assumption 3.5  (Boundedness of features)  There exists a positive constant $B$ such that $\|X_1\|_2 \leq B$ almost surely.

We will use the notation $\kappa_n = L/\alpha_n$ and refer to this quantity as the condition number. Note that all the foregoing assumptions are common in statistical learning, see for instance (Sridharan et al., 2009; Rakhlin et al., 2012). They are helpful not only for proving statistical guarantees but also for designing efficient computational methods for approximating $\theta_n$.

For instance, if $\rho_n(\theta) = \lambda_n\|\theta\|_2^2$ is the ridge penalty (Hoerl and Kennard, 2000) and $\phi$ is either the absolute deviation ($\phi_{abs}(y, y') = |y - y'|$, see for instance (Wang et al., 2014)), the hinge ($\phi_{abs}(y, y') = (1 - yy')_+$ with $y \in [-1, 1]$) or the logistic ($\phi_{log}(y, y') = \ln(1 + e^{-yy'})$ and $y \in [-1, 1]$) loss, the aforementioned assumptions are satisfied with $L = 1$ and $\alpha_n = \lambda_n$. One can also consider the usual squared loss $\phi(y, y') = (y - y')^2$ under the additional assumption that $Y$ is bounded by a known constant $B_y$. In this condition, if the minimization problems in (3) are constrained to the ball of radius $R$, Assumptions 3.2 and 3.4 are satisfied with $\alpha_n = 1$ and $L = 2B_y + BR$. It should be noted that Assumption 3.4 is satisfied, for instance, when $\Phi_n$ is strongly convex. Importantly, as opposed to some other papers (Hsu and Sabato, 2016), we need this assumption for the population risk only.

Theorem 2  Let Assumption 3.1, 3.2, 3.3, 3.4 and 3.5 be satisfied for every $n \in \mathbb{N}$. Assume, in addition, that the sequence $\ln\ln n/\alpha_n^2$ is decreasing and there exists a constant $C_\alpha \in (0, 1)$ such that $2 - \alpha_n/\alpha_{2n} \geq C_\alpha$. Then, for any vector $a \in \mathbb{R}^d$ and any $\delta \in (0, 0.05)$, with probability at least $1 - 18\delta$,

$$a^\top (\hat{\theta}_n - \theta^*) \leq \frac{32B\kappa_n\|a\|_2}{C_\alpha} \sqrt{\frac{2\ln n + \ln(1/\delta)}{n}}, \quad \forall n \geq 1.$$  

Conditions under which Theorem 2 holds can be further relaxed. We have namely in mind the following three extensions. First, Assumption 3.5 can be replaced by sub-Gaussianity of $\mathbf{X}$. Second, the curvature condition can be imposed on a neighborhood of $\theta^*$ only, by letting $\Phi_n$ grow linearly outside the neighborhood. Third, the Lipschitz assumption on $\phi$ can be replaced by the following one: for a constant $\beta$ and sub-Gaussian random variable $\eta$, the function $u \mapsto \phi(Y, u) - \beta u^2$ is $\eta$ Lipschitz. This last extension will allow us to cover the case of squared loss without restriction to a bounded domain. All these extensions are fairly easy to implement, but they significantly increase the complexity of the statement of the theorem. In this work, we opted for sacrificing the generality in order to get better readability of the result.

Another interesting avenue for future research is the extension of the presented results to high-dimensional on-line setting, i.e., when the sample size might be larger than the dimension, see (Negahban et al., 2012) for an in-depth discussion of the batch setting.
4. Application to Bandits

In this section, we apply the univariate uniform law of iterated logarithm that we proved in Section 2 to a problem of robust bandits in the fixed confidence setting. The Best Arm Identification (BAI) problem in the fixed confidence setting usually consists in identifying, as fast as possible, which arm produces the highest expected outcome, see e.g. Audibert and Bubeck (2010). In a robust setup, the expectation of the outcomes of each arm may not be defined, for example if the rewards are subject to some arbitrary contamination. In many cases, it is still possible to define a best arm. We assume the player chooses an arm \( I = \arg \max \theta \in R \) produces the highest expected outcome, see e.g. Audibert and Bubeck (2010). In a robust setup, the problem in the fixed confidence setting usually consists in identifying, as fast as possible, which arm is optimal, as demonstrated by the \( \text{LIL of Theorem 1} \), lead to Robust \( \text{lil'UCB} \) algorithm described in Algorithm 1.

**Robust BAI** We consider a robust version of BAI, which we call Robust BAI (RBAI). Let \((P_\theta)_{\theta \in \mathbb{R}}\) be a family of distributions on \( \mathbb{R} \) with a location parameter \( \theta \) (i.e., \( P_\theta \) is the distribution of \( Y + \theta \), where \( Y \sim P_0 \)). Suppose there are \( K \) arms, each arm \( k \in [K] \) producing i.i.d. rewards \( Y_{1,k}, Y_{2,k}, Y_{3,k}, \ldots \in \mathbb{R} \) with distribution \( P_{\theta_k} \), for some \( \theta_k \in \mathbb{R} \). At each round \( n = 1, 2, \ldots \), the player chooses an arm \( I_n \in [K] \) and receives the corresponding reward \( Y_{T_{n}(n-1), I_n} \), where \( T_k(n-1) = 1(I_1 = k) + \ldots + 1(I_{n-1} = k) \) is the number of times the arm \( k \) was pulled during the rounds 1, \ldots, \( n-1 \).

We let \( \phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be of the form \( \phi(y, \theta) = \hat{\phi}(y - \theta) \) and we assume that 0 is the minimizer of \( \mathbb{E}[\phi(Y - \theta)], \theta \in \mathbb{R} \), where \( Y \sim P_0 \). Therefore, for each arm \( k \in [K] \), \( \theta_k \) coincides with the population counterpart of the \( \theta \)-estimator defined in Section 2. In the rest of this section, we let Assumptions 2.1, 2.2 and 2.3 hold for \( P_0 \), which implies that they automatically hold for each \( P_\theta, \theta \in \mathbb{R} \). For every arm \( k \in [K] \) and every sample size \( t \geq 1 \), we let \( \hat{\theta}_{k,n} \) be a minimizer over \( \theta \in \mathbb{R} \) of \( \frac{1}{n} \sum_{i=1}^{n} \phi(Y_{i,k}, \theta) \). With this notation, after \( n \) rounds, we are able to compute the quantities \( \hat{\theta}_{k,T_k(n)} \) for \( k \in [K] \). These quantities, combined with the confidence bounds furnished by the LIL of Theorem 1, lead to Robust \( \text{lil'UCB} \) algorithm described in Algorithm 1.

To state the theoretical results, let \( k^*_n = \arg\max_{k \in [K]} \hat{\theta}_k \) be the subscript corresponding to the best arm. We assume \( k^*_n \) to be unique, and for \( k \neq k^*_n \), define the sub-optimality gaps \( \Delta_k = \theta_k - \hat{\theta}_k \). We also introduce the quantities

\[
H_1 = \sum_{k \neq k^*_n} \frac{1}{\Delta_k^2} \quad \text{and} \quad H_2 = \sum_{k \neq k^*_n} \frac{\ln(c/\Delta_k^2)}{\Delta_k^2},
\]

where \( c > \max_{k \in [K]} \Delta_k^2 \) is a constant that appears in mathematical derivations.

**Theorem 3** Let \( \delta \in (0,0.15), \varepsilon > 0, \beta \in (0,3) \) and \( \lambda \geq 12 \). Then, there exist positive constants \( C_1, C_2 \) and \( C_3 \) that depend on \( \varepsilon, \lambda, r, \alpha \) and \( \sigma^2 \) such that Algorithm 1 stops after at most \( Kn_0 + C_1H_1 + C_2H_2 \) steps and returns \( k^*_n \) with probability at least \( 1 - C_3 \sqrt{\delta} \).

The proof of this theorem follows line by line the proof of (Jamieson et al., 2014, Theorem 2). The only difference is the first step, which in our case requires \( Kn_0 \) initial samples. Note also that the order of magnitude of the number of steps, \( O(H_1 + H_2) \), is optimal, as demonstrated by the following result.
Algorithm 1 Robust lil’UCB

**input**: Confidence \( \delta > 0 \), parameters \( \lambda, C, \beta > 0, n_0 \in \mathbb{N} \)

Initialize: Sample each arm \( n_0 \) times and set \( n \leftarrow K n_0 \)

\[
\text{for } k \text{ in } 1 : K \text{ do }
\]

Set \( T_k(n) \leftarrow n_0 \)

\[
\text{while } \max_{\ell \in [K]} \left( T_k(n) - \lambda \sum_{\ell \neq k} T_\ell(n) \right) < 1 \text{ do }
\]

Sample the arm:

\[
I_n \leftarrow \arg \max_{k \in [K]} \left[ \hat{\theta}_{k,T_k}(n) + (1 + \beta)C \left\{ \frac{\ln \left( \frac{n_0}{T_k(n)} \right)}{T_k(n)} \right\}^{1/2} \right]
\]

\[
\text{for } k \text{ in } 1 : K \text{ do }
\]

\[
\text{if } I_n = k \text{ then }
\]

\[
T_k(n + 1) \leftarrow T_k(n) + 1
\]

\[
\text{else }
\]

\[
T_k(n + 1) \leftarrow T_k(n)
\]

\[
\text{end}
\]

\[
\text{end}
\]

\[
n \leftarrow n + 1
\]

\[
\text{end}
\]

**output**: \( \arg \max_{k \in [K]} T_k(n) \).

---

**Theorem 4** Consider the RBAI framework with fixed confidence \( \delta \in (0, 1/2) \) described above and assume \( K = 2 \). Let the gap \( \Delta = |\theta_1 - \theta_2| > 0 \). Let \( \tilde{\phi} \) be symmetric. Then, any algorithm that finds the best of the two arms with probability at least \( 1 - \delta \) in \( T \) steps must satisfy

\[
\limsup_{\Delta \to 0} \frac{\mathbb{E}[T]}{\Delta^{-2} \ln \ln(\Delta^{-2})} \geq 2 - 4\delta.
\]

This theorem is a direct consequence of (Jamieson et al., 2014, Corollary 1), which in turn is a consequence of (Farrell, 1964, Theorem 1). To complete this section, we report the results of some basic numerical experiments.

**Numerical experiments** The values of \( \theta_k \)'s in our experiments were chosen according to two models from (Jamieson et al., 2014). The \( l \)-sparse model corresponds to \( \theta_1 = 0.5 \) and \( \theta_k = 0 \) for \( k > 1 \). The \( \alpha \)-model imposes an exponential decrease on the weights, that is \( \theta_k = 1 - (k/K)^\alpha \). Along with these weights, we consider two reward generating processes: Gaussian rewards, \( Y_{i,k} \sim \mathcal{N}(\theta_k, \sigma^2) \) and (Huber’s) \( \epsilon \)-contaminated rewards, \( (Y_{i,k} - \theta_k) \sim 0.9\mathcal{N}(0,1) + 0.05\delta_{100} + 0.05\delta_{-100} \). Note that both of these processes are mean and median centered around the arms’ parameters.

In this set-up, we compared the original lil’UCB algorithm from (Jamieson et al., 2014) and its robust version described in Algorithm 1 where \( \hat{\theta}_{k,n} \) is the empirical median of rewards from arm \( k \) up to time \( n \) (this corresponds to the \( M \)-estimator associated with the absolute loss). In order to lead a fair comparison we assigned the same values to parameters shared by both procedures and set the values as in (Jamieson et al., 2014): \( \beta = 1, \lambda = (1 + 2/\beta)^2, \sigma = 0.5, \epsilon = 0.01 \) and confidence \( \delta = 0.1 \).

---

4. For any number \( x \), \( \delta_x \) denotes the Dirac mass at \( x \).
Robust LIL

Figure 2: Empirical percentiles of $(\hat{\theta}_n - \theta^*) \sqrt{\frac{n}{\ln \ln n + \ln(\frac{40}{\delta})}}$ over 5000 runs per number of samples $n$, where $\hat{\theta}_n$ is the empirical median ($\delta = 0.1$ and $\sigma = 0.5$).

In Algorithm 1 we treat the constant $C$ of the upper confidence bound as a parameter because we observed empirically that the theoretical constant from Theorem 1 may not always be tight. Figure 2 gives an estimation of the optimal constant for Theorem 1 for the models we consider. Following those empirical observations, we set $C = 0.35$ in Algorithm 1 for our experiments. We also treat the warm-up time $n_0$ as a parameter, which we set to 1 since we don’t observe any warm-up phase in practice.

The results, obtained by 50 independent runs of each algorithm on both settings, over several number of arms values, are depicted in Figures 3 and 4. The confidence of each procedure was adapted to reach a global confidence at least $1 - \delta$. Figure 3 shows the proportion of times that each algorithm returned the correct best arm. We observe, as expected, that lil’UCB performed poorly on Huber’s model, seldom returning the correct best arm. In contrast, robust lil’UCB does not seem to be affected by the contamination. It is sometimes wrong on one of the fifty runs but this is consistent with our result which holds with high probability.
Figure 4: Number of pulls in units of the problem hardness $H_1$, for various values of the number of arms $K$. Solid lines represent the average number of pulls over 50 runs. Colored areas show minimum and maximum number of pulls over the best 49 runs (i.e., without the run which required the largest number of pulls).

Figure 4 displays the number of pulls for each algorithm when reaching its stopping criterion as a function of the number of arms $K$. We removed the curve showing the number of pulls for lil’UCB in the contaminated setting since it was not informative. Indeed, the algorithm’s stopping criterion was contaminated and thus the procedure failed to correctly identify the best arm (as shown in Figure 3). The dotted curves represent the average number of pulls over the 50 runs while the colored areas around the curves are delimited by the maximum and the minimum number of pulls observed over the best 49 runs. We observe that the three curves have the same shape, hence the same dependence in the problem complexity $H_1$ and that the number of pulls for robust lil’UCB procedures are significantly smaller than of lil’UCB. Furthermore, the number of pulls does not seem to be affected by the contamination as robust lil’UCB performs similarly in the Gaussian model with and without contamination.

These basic numerical experiments illustrate the lack of robustness of lil’UCB against contamination and the ability of robust lil’UCB to overcome this problem while keeping the total number of pulls at a reasonably low level.

5. Conclusion

We have proved nonasymptotic law of iterated logarithm for general $M$-estimators both in one dimensional and in multidimensional setting. These results can be seen as off-the-shelf deviation bounds that are uniform in the sample size and, therefore, suitable for on-line learning problems. There are several avenues for future work. For simplicity, in the multi-dimensional case, the bounds we present require the population risk to be above an elliptic paraboloid on the whole space. First in our agenda is to replace this condition by a local curvature one. A second interesting line of future research is to prove the LIL for sequential estimators such as the on-line gradient descent. Finally, regarding applications, the multi-dimensional LIL could be used to obtain theoretical guarantees in contextual bandit problems.
6. Proofs

This section contains the proofs of the main theorems, up to some technical lemmas postponed to Section 7.

6.1. Proof of Theorem 1

For $n \geq 1$ and $t \in (0, r]$, let $Z_n(t) = \phi(Y_n, \theta^*) - \phi(Y_n, \theta^* + t) - \mathbb{E} [\phi(Y_n, \theta^*) - \phi(Y_n, \theta^* + t)]$. By convexity of $\Phi_n$,

$$
\hat{\theta}_n > \theta^* + t \Rightarrow \hat{\Phi}_n(\theta^*) \geq \hat{\Phi}_n(\theta^* + t)
$$

$$
\Rightarrow \left( \hat{\Phi}_n(\theta^*) - \Phi(\theta^*) \right) - \left( \hat{\Phi}_n(\theta^* + t) - \Phi(\theta^* + t) \right) \geq \Phi(\theta^* + t) - \Phi(\theta^*),
$$

$$
\Rightarrow \frac{1}{n} \sum_{i=1}^{n} Z_i(t) \geq \frac{\alpha}{2} t^2,
$$

(4)

by Assumption 2.2. Let $\beta \in (1, 2)$ and $\varepsilon > 0$ be two constants that we will choose to be equal to 1.1 and 0.1, respectively. For $n \geq 1$, let $t(n) = \frac{2 \beta \sigma}{\alpha \sqrt{n}} \{2(1 + \varepsilon)(\ln \ln n - \ln \ln \beta) + 2 \ln(1/\delta) \}^{1/2}$. Note that $t(n)$ is nonincreasing for $n \geq 6$. Denote by $n_0$ the smallest positive integer $n \geq 2$ such that $t(n) \leq r$ and by $k^*$ the smallest integer $k$ such that $n_0 < \beta^k$. Note that $n_0 \geq 2$ implies that $k^* \geq 2$. For $k \geq k^*$, let $n_k = \beta^k$ and let $n_{k-1} = n_0$. To ease notation, we set $t_k = t(n_k)$. We also define intervals $I_k = [n_k, n_{k+1})$. By union bound,

$$
\mathbb{P}\left( \bigcup_{n=n_0}^{\infty} \{ \hat{\theta}_n > \theta^* + t(n) \} \right) \leq \sum_{k=k^*-1}^{\infty} \mathbb{P}\left( \bigcup_{n \in I_k} \{ \hat{\theta}_n > \theta^* + t(n) \} \right) \leq \sum_{k=k^*-1}^{\infty} A_k
$$

where

$$
A_k = \mathbb{P}\left( \bigcup_{n \in I_k} \{ \hat{\theta}_n > \theta^* + t(k+1) \} \right) \leq \mathbb{P}\left( \bigcup_{n \in I_k} \left\{ \sum_{i=1}^{n} Z_i(t(k+1)) \geq (\alpha/2) n_{k+1}^2 \right\} \right).
$$

Letting $x_k = (\alpha/2) n_k^2$, we get

$$
A_k \leq \mathbb{P}\left( \sup_{n \in I_k} \sum_{i=1}^{n} Z_i(t(k+1)) \geq x_k \right) \leq \mathbb{P}\left( \sup_{n \in I_k} \exp \left\{ \lambda \sum_{i=1}^{n} Z_i(t(k+1)) \right\} \geq e^{\lambda x_k} \right).
$$

The process $(\sum_{i=1}^{n} Z_i(t(k+1)))_{n \in I_k}$ is a discrete martingale. Hence, by Jensen’s inequality, for all $\lambda \geq 0$, $(e^{\lambda \sum_{i=1}^{n} Z_i(t(k+1)))_{n \in I_k}$ is a discrete submartingale. Therefore, Markov’s inequality followed by Doob’s maximal inequality implies

$$
A_k \leq e^{-\lambda x_k} \mathbb{E}\left[ \sup_{n \in I_k} \exp \left\{ \lambda \sum_{i=1}^{n} Z_i(t(k+1)) \right\} \right] \leq e^{-\lambda x_k} \mathbb{E}\left[ \exp \left\{ \lambda \sum_{i=1}^{n_{k+1}} Z_i(t(k+1)) \right\} \right].
$$

The last expectation can be bounded using Assumption 2.3 leading to

$$
A_k \leq \exp \left\{ -\lambda x_k + (\lambda^2 \sigma^2/2) n_{k+1}^2 \right\}.
$$
Choosing $\lambda = x_k/(\sigma^2 n_k + t_{k+1}^2)$, we arrive at

$$A_k \leq \exp \left\{ - \frac{x_k^2}{(2\sigma^2 n_k + t_{k+1}^2)} \right\} = \exp \left\{ - \frac{\alpha^2 n_k + t_{k+1}^2}{(8\beta^2 \sigma^2)} \right\} = \frac{\delta}{(k + 1)^{1+\varepsilon}},$$

where the last equality is obtained by replacing $t_{k+1} = t(n_{k+1})$ by its expression given just after (4). This yields

$$\mathbb{P} \left[ \exists n \geq n_0, \hat{\theta}_n > \theta^* + t(n) \right] \leq \delta \sum_{k=k^*}^{n} \frac{1}{(k + 1)^{1+\varepsilon}} \leq \delta/\varepsilon.$$

Similarly, one obtains $\mathbb{P}(\exists n \geq n_0$ such that $\hat{\theta}_n < \theta^* - t(n)) \leq \delta/\varepsilon$, and Theorem 1 follows from the union bound combined with the two previous inequalities.

### 6.2. Proof of Theorem 2

Without loss of generality, we assume hereafter that $\alpha$ is a unit vector. Throughout the proof we consider the sequence $n_k = 2^k$ and the sequence of integer intervals $I_k = [n_k, n_{k+1} - 1] \cap \mathbb{N}$ for $k \in \mathbb{N}$. Define the sequence $(c_n)_{n \in \mathbb{N}}$ by setting

$$c_n = \frac{32\kappa_B}{C_{\alpha}} \sqrt{\frac{2\ln n + \ln(1/\delta)}{n}}, \quad \text{for } n \in 2^\mathbb{N} \tag{5}$$

and $c_n = c_{n_k}$, for $n \in I_k$. We wish to upper bound the probability of the event

$$\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n, \quad \text{where } \mathcal{A}_n = \{\alpha^\top (\theta^* - \hat{\theta}_n) > c_n\}.$$

Define the following set $\mathcal{V} = \{v \in \mathbb{R}^d, v^\top \alpha \geq 1\}$.

**Lemma 6.1** Define $\Psi_n(w) = \Phi_n(\theta^*) - \hat{\Phi}_n(\theta^* - w)$. Under Assumptions 3.2, 3.3 and 3.4, the event $\mathcal{A}_n$ is included in the event $\mathcal{B}_{n,2}$ defined as,

$$\mathcal{B}_{n,s} := \left\{ \sup_{w \in c_n \mathcal{V}} \left[ \Psi_n(w) - \mathbb{E} \Psi_n(w) - (\alpha_n/s)\|w\|^2 \right] \geq 0 \right\}, \quad s > 0.$$

The proofs of the lemmas stated in the section are postponed to Section 7.

Writing $S_n(w) = n(\Psi_n(w) - \mathbb{E} \Psi_n(w))$ and using successive union bounds we have,

$$\mathbb{P}(\mathcal{A}) \leq \mathbb{P} \left( \bigcup_{k \geq 1} \bigcup_{n \in I_k} \mathcal{B}_{n,2} \right) \leq \sum_{k \geq 1} \mathbb{P} \left( \bigcup_{n \in I_k} \mathcal{B}_{n,2} \right) \leq \sum_{k \geq 1} \mathbb{P}(\mathcal{B}_{n_k,4}) + \sum_{k \geq 1} \mathbb{P} \left( \sup_{n \in I_k} \sup_{w \in c_n \mathcal{V}} \left[ (S_n - S_{n_k})(w) - \frac{(2n\alpha_n - n_k\alpha_{n_k})\|w\|^2}{4} \right] \geq 0 \right). \tag{6}$$

We will first upper bound $\mathbb{P}(\mathcal{B}_{n_k,4})$ using symmetrization and contraction principles (First step), then the last probability in (6) will be bounded using Doob’s maximal inequality for sub-martingales (Second step). Both steps rely on the following lemma.
Since we intend to use Doob’s maximal inequality, we define the discrete process $M_m(w) = S_m(w) - (C_{\alpha}n_{2m/4})m\|w\|^2$ and the filtration $\mathcal{F}_m = \sigma(X_1, Y_1, \ldots, X_m, Y_m)$ for $m \in \mathbb{N}$.

**Lemma 6.3** For any $\lambda > 0$, $(e^{\lambda \sup_n M_m(w)})_{m \in \mathbb{N}}$ is a sub-martingale w.r.t. to $(\mathcal{F}_m)_m$. 

**Lemma 6.2** Under Assumptions 3.2 and 3.5, given a positive integer $m$, a real number $t > 0$ and positive reals $\alpha$ and $\kappa$ such that $\alpha\kappa = L$, we have for $s = (m/\lambda)t$,

$$\inf_{\lambda > 0} \mathbb{E} \left[ \sup_{w \in L} \exp \left\{ \lambda(S_m(w) - (\alpha/4)m\|w\|^2) \right\} \right] \leq \exp \left\{ -\frac{3s^2 - 40mB^2}{64mB^2} \right\}.$$ 

**First step** Let $k \geq 1$ and for ease of notation let $m = n_k$. Markov inequality and Lemma 6.2 with $t = c_m$ give,

$$\mathbb{P}(B_{m,4}) = \mathbb{P} \left( \sup_{w \in c_m \mathcal{X}} \left[ S_m(w) - \frac{\alpha_m\|w\|^2}{4} \right] \geq 0 \right) \leq \inf_{\lambda > 0} \mathbb{E} \left[ \sup_{w \in c_m \mathcal{X}} \exp \left\{ \lambda(S_m(w) - (\alpha_m/4)m\|w\|^2) \right\} \right] \leq \exp \left\{ -\frac{3(\gamma c_m)^2 - 40mB^2}{64mB^2} \right\}, \quad \gamma = \frac{m}{4\kappa m}.$$

The choice of $c_m$ in (5) is done in such a way that

$$\frac{3(\gamma c_m)^2 - 40mB^2}{64mB^2} \geq (2 \ln \ln m + \ln(1/\delta)).$$

This implies that $\mathbb{P}(B_{m,4}) \leq \delta/(\ln m)^2$.

**Second step** Let $k \geq 1$. Let us bound the second probability of the event

$$C_k = \left\{ \sup_{n \in I_k} \sup_{w \in c_n \mathcal{X}} \left[ S_n(w) - S_{n_k}(w) - \frac{2n\alpha_n - n_k\alpha_{n_k}}{4} \|w\|^2 \right] \geq 0 \right\}.$$ 

Observe that, since the sequence $c_n$ is decreasing, for $n \in I_k$, $c_n \mathcal{X} \subset c_{n+1} \mathcal{X} = c_2 \mathcal{X} = c_2 \mathcal{X}$ and since $(\alpha_n)$ is non-increasing, $\alpha_n \geq \alpha_{2n}$, $n \in I_k$. Therefore

$$\mathbb{P}(C_k) \leq \mathbb{P} \left( \sup_{n \in I_k} \sup_{w \in c_n \mathcal{X}} \left[ S_n(w) - S_{n_k}(w) - \frac{2n\alpha_n - n_k\alpha_{n_k}}{4} \|w\|^2 \right] \geq 0 \right) \leq \inf_{\lambda > 0} \mathbb{E} \left[ \sup_{m \leq 2^k} \sup_{w \in c_m \mathcal{X}} \exp \left\{ \lambda(S_m(w) - \frac{2\alpha_{2n} - \alpha_{n_k} n_k}{4} \|w\|^2) \right\} \right] \leq \inf_{\lambda > 0} \mathbb{E} \left[ \sup_{m \leq 2^k} \sup_{w \in c_m \mathcal{X}} \exp \left\{ \lambda(S_m(w) - \frac{C_{\alpha} \alpha_{2n} m\|w\|^2}{4}) \right\} \right].$$

Since we intend to use Doob’s maximal inequality, we define the discrete process $M_m(w) = S_m(w) - (C_{\alpha}n_{2m/4})m\|w\|^2$ and the filtration $\mathcal{F}_m = \sigma(X_1, Y_1, \ldots, X_m, Y_m)$ for $m \in \mathbb{N}$.
Lemma 6.3 implies that we can use Doob’s inequality to infer that
\[ P\left(C_k\right) \leq 4 \inf_{\lambda > 0} \mathbb{E} \left[ \sup_{w \in W_k} \exp \left\{ \lambda(S_{n_k}(w) - (1/4)C_\alpha \alpha_{2n_k} n_k \|w\|^2) \right\} \right]. \]

Applying Lemma 6.2 with \( t = c_{2n_k}, \alpha = C_{\alpha} \alpha_{2n_k}, m = n_k \) yields
\[ P(C_k) \leq 4 \exp \left\{ -\frac{3(\gamma c_{2n_k})^2 - 40n_k B^2}{64n_k B^2} \right\}, \quad \gamma = \frac{C_{\alpha} n_k}{4\kappa^{2n_k}} \]
Again, the choice of the sequence \((c_n)\) implies
\[ \frac{3(\gamma c_{2n_k})^2 - 40n_k B^2}{64n_k B^2} \geq 2 \ln \ln n_k + \ln(1/\delta) \]
Therefore, we have \( P(C_k) \leq 4\delta/(\ln n_k)^2 \). Combining the conclusions of the two steps with (6) we get \( P(A) \leq \sum_{k=1}^{\infty} \frac{5\delta}{(k \ln 2)^2} \leq 18\delta \).

7. Proofs of postponed lemmas

Proof of Lemma 6.1 We define the vectors
\[ v^*_n = \frac{\theta^* - \hat{\theta}_n}{a^\top(\theta^* - \hat{\theta}_n)} \quad \text{and} \quad \bar{\theta}_n = \theta^* - c_n v^*_n. \]
Clearly, if \( A_n \) is realized, then \( v^*_n \in \mathcal{V} \) and \( p_n = \frac{c_n}{a^\top(\theta^* - \theta_n)} \in (0, 1) \). Furthermore \( \hat{\Phi}_n \) is a convex function (Assumptions 3.2 and 3.3). This yields
\[ \inf_{w \in c_n \mathcal{V}} \hat{\Phi}_n(\theta^* - w) \leq \hat{\Phi}_n(\bar{\theta}_n) = \hat{\Phi}_n(p_n \hat{\theta}_n + (1 - p_n)\theta^*) \]
\[ \leq p_n \hat{\Phi}_n(\hat{\theta}_n) + (1 - p_n) \hat{\Phi}_n(\theta^*) \leq \hat{\Phi}_n(\theta^*). \]
Therefore, the event \( A_n \) implies that
\[ \sup_{w \in c_n \mathcal{V}} \Psi_n(w) \geq 0. \]

The curvature of the population risk (Assumption 3.4) implies that for any vector \( w \in \mathbb{R}^d \),
\[ \mathbb{E} \Psi_n(w) = \Phi_n(\theta^*) - \Phi_n(\theta^* - w) \leq -\frac{\alpha_n \|w\|^2}{2}. \]
This completes the proof.

In order to prove Lemma 6.2 we first state and proof three intermediate lemmas.

Lemma 7.1 Let \( X \) be a deterministic \( d \times m \) matrix and \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \) a \( m \)-dimensional vector with i.i.d. Rademacher entries. As soon as \( \|X\|_F^2 \leq 1/8 \), we have
\[ \mathbb{E} \left[ \exp \left\{ \|X\varepsilon\|^2_2 \right\} \right] \leq \exp \left\{ 10\|X\|_F^2 \right\}. \]
Proof of Lemma 7.1

Using the fact that for any positive random variable $\eta$, its expectation can be written as $E[\eta] = \int_0^\infty P(\eta > z) \, dz$, we get

\[
E[\exp(\|X\|_F^2)] \leq \exp\left(2\|X\|_F^2 \left(1 + \int_0^\infty P(\|X\|_F^2 \geq \frac{\ln(1 + z)}{4\|X\|_F^2}) \, dz\right)\right)
\]

We apply the result from (Boucheron et al., 2013, Example 6.3) on the variables $\varepsilon_1X_1, \ldots, \varepsilon_mX_m$ which are independent zero-mean random variable. Setting $c_i = 2\|X_i\|_2$, we have $\nu = \|X\|_F^2$ and therefore, for any $z > 0$,

\[
P\left(\|X\|_F^2 \geq \frac{\ln(1 + z)}{4\|X\|_F^2}\right) = (1 + z)^{-1/4\|X\|_F^2}. \tag{7}
\]

Assuming that $\|X\|_F^2 < 1/4$, we can plug this in inequality (7) to get

\[
E[\exp(\|X\|_F^2)] \leq \exp\left(2\|X\|_F^2 \left(1 + \frac{4\|X\|_F^2}{1 - 4\|X\|_F^2}\right)\right)
\]

The RHS of the inequality can be large when $\|X\|_F^2$ is close to $1/4$. Restricting $\|X\|_F^2 \leq 1/8$ we arrive at the desired inequality $E[\exp(\|X\|_F^2)] \leq \exp\left(10\|X\|_F^2\right)$. \hfill \blacksquare

Lemma 7.2

Under Assumption 3.2, given a positive integer $m$, a real number $t > 0$ and positive reals $\alpha$ and $\kappa$ such that $\alpha\kappa = L$, we have, for any real $\lambda > 0$,

\[
E\left[\sup_{w \in tV} \exp\left\{\lambda \left(\sum_{i=1}^m \varepsilon_i \left(\phi(Y_i, X_i^T \theta^*) - \phi(Y_i, X_i^T (\theta^* - w))\right)\right)\right\}\right] \leq E\left[\sup_{w \in tV} \exp\left\{\lambda' \left(2\sum_{i=1}^m \varepsilon_i \phi(Y_i, X_i^T \theta^*) - \|w\|_2^2\right)\right\}\right]
\]

where $t' = (m/4\kappa)t$ and $\lambda' = (4L\kappa/m)\lambda$.

Proof of Lemma 7.2

Using a modified version\(^5\) of the symmetrization inequality,

\[
E\left[\sup_{w \in tV} \exp\left\{\lambda \left(\sum_{i=1}^m \varepsilon_i \left(\phi(Y_i, X_i^T \theta^*) - \phi(Y_i, X_i^T (\theta^* - w))\right)\right)\right\}\right] \leq E\left[\sup_{w \in tV} \exp\left\{\lambda' \left(2\sum_{i=1}^m \varepsilon_i \phi(Y_i, X_i^T \theta^*) - \|w\|_2^2\right)\right\}\right]
\]

where $S'_m(w)$ is the symmetrized version of $S_m(w)$, defined by

\[
S'_m(w) = \sum_{i=1}^m \varepsilon_i \left(\phi(Y_i, X_i^T \theta^*) - \phi(Y_i, X_i^T (\theta^* - w))\right).
\]

---

5. The version we use here can be found, for instance, in (Lecué and Rigollet, 2014, Eq. (2.3)).
We define the set \( R = \{ tX^\top v : v \in V \} \subset \mathbb{R}^m \) and the functions \( \varphi_i : \mathbb{R} \to \mathbb{R} \) by
\[
\varphi_i : r \mapsto \left[ \phi(Y_i, X_i^\top \theta^*) - \phi(Y_i, X_i^\top \theta^* - r) \right] / L, \quad i = 1, \ldots, m.
\]
These functions \( \varphi_i \) are contractions (Assumption 3.2) such that \( \varphi_i(0) = 0 \). The contraction principle (Koltchinskii, 2011, Theorem 2.2) yields
\[
\mathbb{E} \left[ \sup_{w \in t^V} \exp \left\{ \lambda(2S_m^r(w) - (\alpha/4)m \|w\|_2^2) \right\} \right] \leq \mathbb{E} \left[ \sup_{w \in t^V} \exp \left\{ \lambda(2Lw^\top X\varepsilon - (\alpha/4)m \|w\|_2^2) \right\} \right].
\]
Setting \( t' = (m/4\kappa)t \) and \( \lambda' = (4L\kappa/m) \lambda \), we arrive at
\[
\mathbb{E} \left[ \sup_{w \in t^V} \exp \left\{ \lambda(2S_m^r(w) - (\alpha/4)m \|w\|_2^2) \right\} \right] \leq \mathbb{E} \left[ \sup_{w \in t'^V} \exp \left\{ \lambda'(2w^\top X\varepsilon - \|w\|_2^2) \right\} \right].
\]

In what follows, we denote by \( B_{a^\top X} \) the smallest constant \( B \) for which \( \mathbb{P}(|a^\top X_1| \leq B) = 1 \). It is clear that \( B_{a^\top X} \leq B_1 \|X\| \). Nevertheless, we prefer to use the constant \( B_{a^\top X} \) in some intermediate claims, in order to keep them as tight as possible.

**Lemma 7.3** Let \( I \) be a finite set of cardinality \( m \in \mathbb{N} \). Let \( (X_i)_{i \in I} \) be i.i.d. random vectors in \( \mathbb{R}^d \) satisfying Assumption 3.5 and let \( (\varepsilon_i)_{i \in I} \) be i.i.d. Rademacher variables, independent of \( (X_i)_{i \in I} \). Then, for any positive constants \( s, \mu \) such that \( 16\mu mB^2 \leq 1 \),
\[
\mathbb{E} \left[ \sup_{w \in sV} \exp \left\{ \mu(2w^\top X\varepsilon - \|w\|_2^2) \right\} \right] \leq \exp \left\{ (4ms^2B_{a^\top X}^2)^\mu + (10mB^2 - s^2)^\mu \right\}.
\]  

**Proof of Lemma 7.3** Let us define \( \Pi_{a^\perp} = I_d - aa^\top \) to be the projection matrix onto the orthogonal complement of the vector \( a \) and set 
\[
w_* = \Pi_{a^\perp} X\varepsilon + sa.
\]
One checks that \( w_* \in sV \) is the maximizer of the quadratic function \( G(w) = 2w^\top X\varepsilon - \|w\|_2^2 \). In addition,
\[
2\gamma w_*^\top X\varepsilon - \|w_*\|_2^2 = 2\|\Pi_{a^\perp} X\varepsilon\|_2^2 + 2sa^\top X\varepsilon - \|\Pi_{a^\perp} X\varepsilon\|_2^2 - s^2\|a\|_2^2
\]
\[
= \|\Pi_{a^\perp} X\varepsilon\|_2^2 + 2sa^\top X\varepsilon - s^2.
\]
Denoting by \( T(\mu) \) the left hand side of (8), we arrive at
\[
T(\mu) \leq e^{-\mu s^2} \mathbb{E} \left[ \exp \left\{ (\mu \|\Pi_{a^\perp} X\varepsilon\|_2^2 + 2\mu sa^\top X\varepsilon) \right\} \right].
\]
The fact that \( \Pi_{a^\top} \) is a contraction and the Cauchy-Schwarz inequality imply
\[
T(\mu) \leq e^{-\mu s^2} \left( \mathbb{E} \left[ \exp \left\{ 2\mu \|X\varepsilon\|_2^2 \right\} \right] \mathbb{E} \left[ \exp \left\{ 4\mu sa^\top X\varepsilon \right\} \right] \right)^{1/2}.
\]  

(9)
We bound separately the two last expectations. For the first one, since $2\mu\|X\|_F^2 \leq 2\mu mB^2 \leq 1/s$, we can apply Lemma 7.1, conditionally to $X$ and then integrate w.r.t. $X$, to get

$$\mathbb{E}[\exp\{2\mu\|X\|_2^2\}] \leq \mathbb{E}\left[\exp\left\{20\mu\|X\|_F^2\right\}\right] \leq \exp\left\{20m\mu B^2\right\}.$$  

We bound now the second expectation in the right hand side of (9). Using the fact that $\varepsilon_{1:m}$ are independent Rademacher random variables independent of $X$, as well as the inequality $\cosh(x) \leq e^{x^2/2}$, we arrive at

$$\mathbb{E}\left[\exp\left\{(4\mu s)a^\top X\varepsilon\right\}\right] \leq \mathbb{E}\left[\exp\left\{8(\mu s)^2\|X^\top a\|_2^2\right\}\right] \leq \exp\left\{8(\mu s)^2mB_a^2\right\}.$$  

Grouping the bounds on these two expectations together we obtain the stated inequality.

**Proof of Lemma 6.2** Let $\lambda > 0$. Lemma 7.2 gives

$$\mathbb{E}\left[\sup_{w \in \mathcal{W}} \exp\left\{\lambda(S_m(w) - (\alpha/4)m\|w\|_2^2)\right\}\right] \leq \mathbb{E}\left[\sup_{w \in \mathcal{W}} \exp\left\{\lambda'(2w^\top X\varepsilon - \|w\|_2^2)\right\}\right]$$

where $t' = (m/4\kappa)t$ and $\lambda' = (4L\kappa/m)\lambda$.

Applying Lemma 7.3 with $s = t'$, $\mu = \lambda'$ yields, as long as $16\lambda mB^2 \leq 1$,

$$\mathbb{E}\left[\sup_{w \in \mathcal{W}} \exp\left\{\lambda(2w^\top X\varepsilon - \|w\|_2^2)\right\}\right] \leq \exp\left\{(4ms^2B_a^2\top X)\lambda'^2 + (10mB^2 - s^2)\lambda'\right\}$$

Since $\lambda'$ is proportional to $\lambda$, taking the infimum over $\lambda$ yields

$$\inf_{\lambda > 0} \mathbb{E}\left[\sup_{w \in \mathcal{W}} \exp\left\{\lambda(S_m(w) - (\alpha/4)m\|w\|_2^2)\right\}\right] \leq \inf_{16\lambda mB^2 \leq 1} \exp\left\{(4ms^2B_a^2\top X)\lambda'^2 + (10mB^2 - s^2)\lambda'\right\}$$

We choose $\lambda' = 1/(16mB^2)$ and upper bound $B_a^\top X$ by $B$. This leads to the stated inequality.

**Proof of Lemma 6.3** Let $w_m$ be the maximizer of $M_m(w)$ over $\mathcal{W}$. It holds that

$$\mathbb{E}[e^{\lambda\sup_w M_{m+1}(w)}|\mathcal{F}_m] \geq \mathbb{E}[e^{\lambda M_{m+1}(w_m)}|\mathcal{F}_m] \geq e^{\lambda\mathbb{E}[M_{m+1}(w_m)|\mathcal{F}_m]},$$

where we have used the Jensen inequality. Since $S_m$ is a martingale and $w_m$ is $\mathcal{F}_m$ measurable, we obtain $\mathbb{E}[M_{m+1}(w_m)|\mathcal{F}_m] = M_m(w_m)$ and the claim of the lemma follows.

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