Positive line modules over the irreducible quantum flag manifolds

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Abstract
Noncommutative Kähler structures were recently introduced as a framework for studying noncommutative Kähler geometry on quantum homogeneous spaces. It was subsequently observed that the notion of a positive vector bundle directly generalises to this setting, as does the Kodaira vanishing theorem. In this paper, by restricting to covariant Kähler structures of irreducible type (those having an irreducible space of holomorphic 1-forms) we provide simple cohomological criteria for positivity, allowing one to avoid explicit curvature calculations. These general results are applied to our motivating family of examples, the irreducible quantum flag manifolds $\mathcal{O}_q(G/L_S)$. Building on the recently established noncommutative Borel–Weil theorem, every relative line module over $\mathcal{O}_q(G/L_S)$ can be identified as positive, negative, or flat, and it is then concluded that each Kähler structure is of Fano type.

Keywords Quantum groups · Noncommutative geometry · Quantum flag manifolds · Complex geometry

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1 Introduction

Positive line bundles, which is to say, line bundles whose Chern curvature is a positive definite \((1,1)\)-form, play a central role in modern complex geometry. Analogously, ample line bundles are fundamental objects of study in projective algebraic geometry. An ample line bundle is a line bundle \(E\) such that, for any coherent sheaf \(S\) and sufficiently large \(k\), the tensor product \(S \otimes E^\otimes k\) is generated by global sections. Under the GAGA (géométrie algébrique et géométrie analytique) correspondence [47], positive and ample line bundles are equivalent. The existence of positive line bundles has many remarkable implications for the structure of a complex manifold. For example, the Kodaira embedding theorem says that a compact Kähler manifold is projective if and only if it admits a positive line bundle [29, § 5.3]. Positivity also has significant cohomological implications, as evidenced by the celebrated Kodaira vanishing theorem and the subsequent slew of related vanishing theorems [23]. Given these elegant results, the natural impulse is to try to extend the concept of positivity to settings beyond ordinary complex geometry. This has been met with tremendous success in the study of varieties over fields of prime characteristic. Positivity, or rather in this case ampleness, has been key to understanding innate differences between these geometries, for example, the failure of the Kodaira vanishing theorem in prime characteristic [46]. Another striking extension has been to the setting of noncommutative projective algebraic geometry, where ample sequences and ample pairs are by now considered foundational structures [1].

The goal of this paper is to explore the idea of positivity for the noncommutative differential geometry of quantum groups. In particular, we show that the relative line modules over an irreducible quantum flag manifold, endowed with its Heckenberger–Kolb calculus, are either positive, flat, or negative, Theorems 3.5 and 4.9. Furthermore, we are able to distinguish between these three cases using cohomological information, Corollary 3.6, in the form of the recently established noncommutative Akizuki–Nakano identities [43, Corollary 7.8].

Positivity in noncommutative differential geometry is a concept that has been formulated only recently in the companion paper [43]. These two papers are part of a series exploring the noncommutative complex geometry of quantum homogeneous spaces [14, 16, 41–43] based around the notion of a noncommutative Kähler structure, as introduced in [42]. In this context, the classical Koszul–Malgrange theorem [35] allows for an obvious noncommutative generalisation of the definition of a holomorphic module. As in the classical setting, every Hermitian holomorphic module has a uniquely associated Chern connection [5, Proposition 4.4]. In [43], it is observed that the definition of a positive line bundle extends directly to the noncommutative setting, where we call them positive line modules. Building on this observation, a corresponding Kodaira vanishing theorem can be formulated and the definition of a noncommutative Kähler structure can be refined to give the definition of a noncommutative Fano structure. As shown in [43], the implied vanishing of cohomologies
for Fano structure makes it possible to calculate noncommutative holomorphic Euler characteristics.

Despite an abundance of structure, calculating the curvature of a line module in the quantum setting remains an extremely challenging task: classical tools are either not yet developed or are unavailable entirely. Any attempt at brute force calculations quickly becomes prohibitively lengthy and tedious. The complications involved can already be seen for the example of the standard Podleś sphere as discussed in Example 4.10. Fortunately, the worst of these calculations can be avoided entirely by restricting to a particularly tractable subclass, which subsumes the quantum projective spaces: those covariant Kähler structures which are irreducible.

Our motivating examples are the irreducible, or cominiscule, quantum flag manifolds $O_q(G/L_S)$. Forming a large and robust family, they are a natural class to consider when attempting to extend geometric notions from the classical to the noncommutative. Indeed, it is becoming increasingly clear that the noncommutative geometry of the quantum flag manifolds is key to understanding the noncommutative geometry of quantum groups in general. Here, the necessary cohomological information is provided by the irreducible quantum flag manifold Borel–Weil theorem [8, 9]. This allows us to prove in Theorem 4.12 that every irreducible quantum flag manifold, endowed with its Heckenberger–Kolb calculus, is of Fano type in the sense of [43, Definition 8.8].

The positivity results established in this paper have a number of important applications in other works. In [43] positivity, along with the noncommutative Kodaira vanishing theorem, is used to prove a noncommutative generalisation of the Bott–Borel–Weil theorem for positive line modules. In [15] positivity is used to prove a spectral gap for the negatively twisted Dolbeault–Dirac operators over the irreducible quantum flag manifolds. This in turn allows for a proof that the closure of each such operator is Fredholm [14]. This gives a particularly satisfying application of the machinery of classical complex geometry to the quantum world, showing how the spectrum of a noncommutative Dirac operator is shaped by the geometry of the underlying Heckenberger–Kolb $q$-deformed de Rham complex.

1.1 Summary of the paper

The paper is organised as follows. In Sect. 2, we recall necessary background material, including noncommutative Kähler structures, Hermitian and holomorphic modules, and compact quantum group algebras. In particular, we recall the recently introduced notion of a compact quantum homogeneous (CQH) Kähler space [14, Definition 3.1], which details a natural set of compatibility conditions between covariant Kähler structures and compact quantum group algebras.

In Sect. 3, we develop the general theory of the paper. In particular, we introduce the notion of an irreducible CQH-Kähler space, and show that for any such space, a covariant Hermitian holomorphic line module is either positive, negative, or flat. We then build upon this result to show that we can distinguish between these three choices by examining the degree zero Dolbeault cohomology of the line module in question.
In Sect. 4, we present our motivating family of examples, the irreducible quantum flag manifolds $\mathcal{O}_q(G/L_S)$, their relative line modules $\mathcal{E}_l$, for $l \in \mathbb{Z}$, along with their Heckenberger–Kolb calculi. We recall the irreducible CQH-Kähler structure of each $\mathcal{O}_q(G/L_S)$, and the associated noncommutative generalisation of the Borel–Weil theorem. We then build upon the general theory presented in §3, and, for each $k \in \mathbb{Z}_{>0}$, prove that $\mathcal{E}_k > 0$, and $\mathcal{E}_{-k} < 0$. As a consequence, we observe that the Kähler structure of the Heckenberger–Kolb calculus is a Fano structure.

For the reader’s convenience, we also include three short appendices. “Appendix A” presents Takeuchi’s equivalence, the natural setting for discussing homogeneous vector bundles in the noncommutative setting. “Appendix B” contains a discussion of the definition of the quantum Levi group $\mathcal{O}_q(L_S)$. Finally, “Appendix C” provides necessary technical details about the quantum flag manifolds and their canonical modules.

2 Preliminaries

We recall the basic definitions and results for differential calculi, complex structures, Hermitian structures, and Kähler structures, as well as noncommutative vector bundles over these objects. For a more detailed introduction, see [41, 42], and references therein. For an excellent presentation of classical complex and Kähler geometry, see [29].

Throughout the paper, all algebras are assumed to be unital and defined over $\mathbb{C}$, and all unadorned tensor products are defined over $\mathbb{C}$. We denote $i := \sqrt{-1}$.

2.1 Differential calculi

A differential calculus $(\Omega^* \simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \Omega^k, d)$ is a differential graded algebra which is generated as an algebra by the elements $a, db$, for $a, b \in \Omega^0$. We call an element $\omega \in \Omega^*$ a form, and if $\omega \in \Omega^k$, for some $k \in \mathbb{Z}_{>0}$, then $\omega$ is said to be homogeneous of degree $|\omega| := k$. The product of two forms $\omega, \nu \in \Omega^*$ is usually denoted by $\omega \wedge \nu$, unless one of the forms is of degree 0, whereupon the product is denoted by juxtaposition. A differential calculus over an algebra $B$ is a differential calculus such that $\Omega^0 = B$. A differential calculus is said to have total degree $m \in \mathbb{Z}_{\geq 0}$, if $\Omega^m \neq 0$, and $\Omega^k = 0$, for every $k > m$.

A differential $*$-calculus over a $*$-algebra $B$ is a differential calculus over $B$ such that the $*$-map of $B$ extends to a (necessarily unique) conjugate-linear involutive map $*: \Omega^* \to \Omega^*$ satisfying $d(\omega^*) = (d\omega)^*$, and

$$(\omega \wedge \nu)^* = (-1)^{kl} \nu^* \wedge \omega^*, \quad \text{for all } \omega \in \Omega^k, \nu \in \Omega^l.$$  

We say that $\omega \in \Omega^*$ is closed if $d\omega = 0$, and real if $\omega^* = \omega$. For further details in differential calculi, we refer to [6].
2.2 Complex structures

We now recall the definition of a complex structure as introduced in [7, 33]. This generalises the properties of the de Rham complex of a classical complex manifold [29, §2.6], and was motivating by the pioneering examples of the Podleś calculus of standard Podleś sphere as described in [38], and more generally the Heckenberger–Kolb calculi [26].

**Definition 2.1** A complex structure $Ω^\bullet$ for a differential $\ast$-calculus $(Ω^\ast, d)$ is a choice of $\mathbb{Z}_\geq 0$-algebra grading $\bigoplus_{(a,b)\in\mathbb{Z}_\geq 0^2} Ω^{(a,b)}$ for $Ω^\bullet$ such that

1. $Ω^k = \bigoplus_{a+b=k} Ω^{(a,b)}$,
2. $(Ω^{(a,b)})^\ast = Ω^{(b,a)}$,
3. $dΩ^{(a,b)} \subseteq Ω^{(a+1,b)} \oplus Ω^{(a,b+1)}$,

for all $k \in \mathbb{Z}_\geq 0$, and $(a, b) \in \mathbb{Z}_\geq 0^2$. We call an element of $Ω^{(a,b)}$ an $(a, b)$-form.

Note that for any complex structure, there is a degree $(1, 0)$-map $\partial$, and a degree $(0, 1)$-map $\overline{\partial}$, uniquely defined by $d = \partial + \overline{\partial}$. The triple $(Ω^\bullet, \partial, \overline{\partial})$ forms a double complex, which we call the Dolbeault double complex. Moreover, both $\partial$ and $\overline{\partial}$ satisfy the graded Leibniz rule, and $\partial(\omega^\ast) = (\overline{\partial}\omega)^\ast$, for all $\omega \in Ω^\bullet$.

We say that a complex structure $Ω^\bullet$ is factorisable if $B$-bimodule isomorphisms

1. $Ω^{(a,0)} \otimes_B Ω^{(0,b)} \simeq Ω^{(a,b)}$,  
2. $Ω^{(0,b)} \otimes_B Ω^{(a,0)} \simeq Ω^{(a,b)}$,  

are given by the multiplication map of $Ω^\bullet$. Recall that every complex manifold is automatically factorisable [29, §1.2], as are the Heckenberger–Kolb calculi of the irreducible quantum flag manifolds (see Sect. 4.6).

2.3 Hermitian and Kähler structures

In this subsection, we recall the general theory of Hermitian and Kähler structures as introduced in [42].

**Definition 2.2** Let $Ω^\bullet$ be a differential $\ast$-calculus over a $\ast$-algebra $B$, of even total degree $2n$. An Hermitian structure for $Ω^\bullet$ is a pair $(Ω^\bullet, σ)$ consisting of a complex structure $Ω^\bullet$ and a central real $(1, 1)$-form $σ$, called the Hermitian form, such that, for the Lefschetz operator $L : Ω^\bullet → Ω^\bullet$, defined by $L(ω) := σ \wedge ω$, isomorphisms are given by

$$L^{n-k} : Ω^k → Ω^{2n-k},$$

for all $k = 0, \ldots, n - 1$.

A Kähler structure is an Hermitian structure such that the Hermitian form is closed, in which case we call it a Kähler form.
The \((a, b)\)-primitive forms of an Hermitian structure are
\[
P^{(a, b)} := \begin{cases} 
\{ \alpha \in \Omega^{(a, b)} | L^{n-a-b+1}(\alpha) = 0 \}, & \text{if } a + b \leq n, \\
0, & \text{if } a + b > n,
\end{cases}
\]
and we write \(P^k := \bigoplus_{a+b=k} P^{(a, b)}\). The existence of an Hermitian structure implies a direct generalisation of the classical Lefschetz decomposition \([42, \text{Proposition 4.3}]\):
\[
\Omega^k \simeq \bigoplus_{j \geq 0} L^j \left( P^{k-2j} \right).
\]
In classical Hermitian geometry, the Hodge map of an Hermitian metric is related to the associated Lefschetz decomposition through the well-known Weil formula \([29, \text{Proposition 1.2.31}]\). In the noncommutative setting, we take the Weil formula for our definition of the Hodge map: The Hodge map associated to an Hermitian structure \((\Omega^{\bullet\bullet}, \sigma)\) is the \(B\)-bimodule map \(*_\sigma : \Omega^\bullet \to \Omega^\bullet\) satisfying, for any \(j \in \mathbb{Z} \geq 0\),
\[
*_\sigma \left( L^j(\omega) \right) = (-1)^{\frac{k(k+1)}{2} + a-b} \frac{j!}{(n-j-k)!} L^{n-j-k}(\omega), \quad \text{for } \omega \in P^{(a, b)}, \quad a + b = k.
\]
The Hodge map allows us to construct a sesquilinear map (conjugate in the second variable) called the Hermitian metric,
\[
g_\sigma : \Omega^\bullet \times \Omega^\bullet \to B, \quad (\omega, \nu) \mapsto \begin{cases} 
*_\sigma (\omega \wedge *_\sigma (\nu^*)), & \text{if } k = l, \\
0, & \text{if } k \neq l.
\end{cases}
\]
It follows from \([43, \text{Proposition 5.6}]\) that
\[
g_\sigma (\omega, \nu) = g_\sigma (\nu, \omega)^*, \quad \text{for all } \omega, \nu \in \Omega^\bullet.
\]
With respect to the Hermitian metric, the Lefschetz map \(L\) is adjointable, and we denote its adjoint by \(L^\dagger = \Lambda\). The map \(\Lambda\) can be explicitly presented as
\[
\Lambda = *_{\sigma}^{-1} \circ L \circ *_{\sigma}.
\]
We define the counting operator \(H : \Omega^\bullet \to \Omega^\bullet\) by \(H(\omega) := (k-n)\omega\), for \(\omega \in \Omega^k\). Together the maps \(L, \Lambda,\) and \(H\) give a representation of \(sl_2\). (See §2.6 for the general twisted version of this representation.)

2.4 Holomorphic modules

In this subsection, we present the notion of a noncommutative holomorphic module, as has been considered in various places, for example \([7, 33, 45]\). Motivated by the Serre–Swan theorem, we consider projective modules as noncommutative analogues
of vector bundles. In particular, a line module over $B$ will be an invertible $B$-bimodule $\mathcal{E}$, where invertible means that there exists another $B$-bimodule $\mathcal{E}'$ such that $\mathcal{E} \otimes_B \mathcal{E}' \simeq \mathcal{E} \otimes_B \mathcal{E} \simeq B$. Note that any such $\mathcal{E}$ is automatically projective as a left (and as a right) $B$-module.

We can define a noncommutative analogue of a holomorphic vector bundle via the classical Koszul–Malgrange characterisation of holomorphic bundles [35]. (See [43] for a more detailed discussion.) For $\Omega^\bullet$ a differential calculus over an algebra $B$, and $\mathcal{F}$ a left $B$-module, a connection on $\mathcal{F}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$ satisfying

$$\nabla(bf) = db \otimes f + b\nabla f,$$

for all $b \in B$, $f \in \mathcal{F}$.

With respect to a choice $\Omega^{(\bullet, \bullet)}$ of complex structure on $\Omega^\bullet$, a $(0, 1)$-connection for $\mathcal{F}$ is a connection with respect to the differential calculus $(\Omega^{(0, \bullet)}, \partial)$. Any connection can be extended to a map $\nabla : \Omega^\bullet \otimes_B \mathcal{F} \to \Omega^\bullet \otimes_B \mathcal{F}$ uniquely defined by

$$\nabla(\omega \otimes f) = d\omega \otimes f + (-1)^{\omega} \omega \wedge \nabla f,$$

for a homogeneous form $\omega$ with degree $|\omega|$. The curvature of a connection is the left $B$-module map $\nabla^2 : \mathcal{F} \to \Omega^2 \otimes_B \mathcal{F}$. A connection is said to be flat if $\nabla^2 = 0$. Since $\nabla^2(\omega \otimes f) = \omega \wedge \nabla^2(f)$, a connection is flat if and only if the pair $(\Omega^\bullet \otimes_B \mathcal{F}, \nabla)$ is a complex.

**Definition 2.3** Fix a differential $\ast$-calculus $\Omega^\bullet$, over a $\ast$-algebra $B$, endowed with a choice of complex structure $\Omega^{(\bullet, \bullet)}$. A holomorphic module over $B$ is then a pair $(\mathcal{F}, \partial \mathcal{F})$, where $\mathcal{F}$ is a finitely generated projective left $B$-module, and the map $\partial : \mathcal{F} \to \Omega^{(0, 1)} \otimes_B \mathcal{F}$ is a flat $(0, 1)$-connection, which we call the holomorphic structure for $(\mathcal{F}, \partial \mathcal{F})$.

Note that for any fixed $a \in \mathbb{Z}_{\geq 0}$, a holomorphic module $(\mathcal{F}, \partial \mathcal{F})$ has an associated complex

$$\partial a : \Omega^{(a, \bullet)} \otimes_B \mathcal{F} \to \Omega^{(a, \bullet)} \otimes_B \mathcal{F}.$$  

For any $b \in \mathbb{Z}_{\geq 0}$, we denote by $H_{\partial}^{(a, b)}(\mathcal{F})$ the $b$th-cohomology group of this complex.

### 2.5 Holomorphic Hermitian modules

In this subsection, we assume that $B$ is a $\ast$-algebra, and generalise to our noncommutative setting the classical notion of a holomorphic Hermitian vector bundle.

For any left $B$-module, denote by $\mathcal{F}^\ast := \text{Hom}_B(\mathcal{F}, B)$ the dual module, which is a right $B$-module with respect to pointwise multiplication

$$\phi b(f) := \phi(f)b, \quad \phi \in \mathcal{F}^\ast, \quad \text{for } b \in B, f \in \mathcal{F}.$$
Moreover, we denote by $\overline{F}$ the conjugate right $B$-module of $F$, as defined by the action

$$\overline{F} \otimes B \rightarrow \overline{F}, \quad \overline{f} \otimes b \mapsto \overline{b^* f}.$$ 

For a $\ast$-algebra $B$, the cone of positive elements $B_{\geq 0}$ is defined by

$$B_{\geq 0} := \left\{ \sum_{1 \leq i \leq l} b_i^* b_i \mid b_i \in B, \ l \in \mathbb{Z}_{\geq 0} \right\}.$$ 

We denote the nonzero positive elements of $B$ by $B_{> 0} := B_{\geq 0} \setminus \{0\}$.

**Definition 2.4** An Hermitian module over a $\ast$-algebra $B$ is a pair $(F, h_F)$, where $F$ is a finitely generated projective left $B$-module and $h_F : F \rightarrow {}^*F$ is a right $B$-module isomorphism, such that, for the associated sesquilinear pairing,

$$h_F(-, -) : F \times F \rightarrow B, \quad (f, k) \mapsto h_F(k)(f)$$

it holds that, for all nonzero $f, k \in F$,

1. $h_F(f, k) = h_F(k, f)^*$,  
2. $h_F(f, f) \in B_{> 0}$.

**Example 2.5** Let $(\Omega^\bullet(B), \sigma)$ be an Hermitian structure for a differential calculus $\Omega^\bullet(B)$. If $\Omega^\bullet$ is finitely generated and projective as a left $B$-module, and we assume that

$$g_\sigma(\omega, \omega) \in B_{> 0}, \quad \text{for all nonzero } \omega \in \Omega^\bullet,$$

then the pair $(\Omega^\bullet, g_\sigma)$ is an Hermitian module. In what follows, when we say that the pair $(\Omega^\bullet, g_\sigma)$ associated to an Hermitian structure is an Hermitian module, we mean it in this sense.

Let $(F, h_F)$ be an Hermitian module, and consider the sesquilinear map

$$h_F : \Omega^\bullet \otimes_B F \times \Omega^\bullet \otimes_B F \rightarrow \Omega^\bullet, \quad (\omega \otimes f, \nu \otimes g) \mapsto \omega h_F(f, g) \wedge \nu^*.$$ 

A connection $\nabla : F \rightarrow \Omega^1 \otimes_B F$ is Hermitian if

$$dh_F(f, g) = h_F(\nabla(f), 1 \otimes g) + h_F(1 \otimes f, \nabla(g)), \quad \text{for all } f, g \in F.$$

**Definition 2.6** A holomorphic Hermitian module is a triple $(F, h_F, \overline{\partial} F)$ such that $(F, h_F)$ is an Hermitian module and $(F, \overline{\partial} F)$ is a holomorphic module.
As established in [5] (see also [43]), for any Hermitian holomorphic module \((\mathcal{F}, h_{\mathcal{F}}, \bar{\partial}_{\mathcal{F}})\), there exists a unique Hermitian connection \(\nabla : \mathcal{F} \to \Omega^1 \otimes_A \mathcal{F}\) satisfying

\[
\bar{\partial}_{\mathcal{F}} = (\text{proj}_{\Omega(0,1)} \otimes B) \circ \nabla,
\]

where \(\text{proj}_{\Omega(0,1)} : \Omega^1 \to \Omega^{0,1}\) is the obvious projection. We call \(\nabla\) the Chern connection of \((\mathcal{F}, h_{\mathcal{F}}, \bar{\partial}_{\mathcal{F}})\), and denote

\[
\partial_{\mathcal{F}} := (\text{proj}_{\Omega(1,0)} \otimes B) \circ \nabla,
\]

where \(\text{proj}_{\Omega(1,0)} : \Omega^1 \to \Omega^{1,0}\) is the obvious projection.

We finish this subsection with the notion of positivity for a holomorphic Hermitian module, motivated by the classical notion of positivity [29, Proposition 5.3.1]. It was first introduced in [43, Definition 8.2] and details a compatibility between Hermitian holomorphic modules and Kähler structures.

**Definition 2.7** Let \(\Omega^*\) be a differential calculus over a \(*\)-algebra \(B\), and let \((\Omega^{\bullet, \bullet}, \kappa)\) be a Kähler structure for \(\Omega^*\). An Hermitian holomorphic module \((\mathcal{F}, h_{\mathcal{F}}, \bar{\partial}_{\mathcal{F}})\) is said to be positive, written \(\mathcal{F} > 0\), if there exists a \(\theta \in \mathbb{R}_{>0}\), such that the Chern connection \(\nabla\) satisfies

\[
\nabla^2(f) = -\theta i\kappa \otimes f, \quad \text{for all } f \in \mathcal{F}.
\]

Analogously, \((\mathcal{F}, h_{\mathcal{F}}, \bar{\partial}_{\mathcal{F}})\) is said to be negative, written \(\mathcal{F} < 0\), if there exists a \(\theta \in \mathbb{R}_{>0}\), such that the Chern connection \(\nabla\) of \(\mathcal{F}\) satisfies

\[
\nabla^2(f) = \theta i\kappa \otimes f, \quad \text{for all } f \in \mathcal{F}.
\]

An important point to note is that for any factorisable complex structure \(\Omega^{\bullet, \bullet}\) of total degree \(2n\), the pair \((\Omega^{n,0}, \bar{\partial})\) is a holomorphic module. Moreover, for a factorisable Hermitian structure, or factorisable Kähler structure, which is to say an Hermitian, or Kähler, structure whose constituent complex structure is factorisable, the triple \((\Omega^{n,0}, g_{\sigma}, \wedge^{-1} \circ \bar{\partial})\) is an Hermitian holomorphic module.

### 2.6 Some identities

Suppose that \((\Omega^{\bullet, \bullet}, \sigma)\) is an Hermitian structure for a differential \(*\)-calculus over a \(*\)-algebra \(B\), and that \(\mathcal{F}\) is a left \(B\)-module. Consider the triple of operators on \(\Omega^* \otimes_B \mathcal{F}\):

\[
L_{\mathcal{F}} := L \otimes \text{id}_{\mathcal{F}}, \quad H_{\mathcal{F}} := H \otimes \text{id}_{\mathcal{F}}, \quad \Lambda_{\mathcal{F}} := \Lambda \otimes \text{id}_{\mathcal{F}}.
\]

As established in [43], it holds that
\[ [H_F, L_F] = 2L_F, \quad [L_F, \Lambda_F] = H_F, \quad [H_F, \Lambda_F] = -2\Lambda_F, \]

meaning that we have a representation of \( \mathfrak{sl}_2 \).

### 3 Irreducible CQH-Hermitian spaces and positive line modules

Determining positivity, or negativity, of an Hermitian holomorphic line module ostensibly requires one to calculate the Chern curvature explicitly. In practice, this can prove to be a very challenging technical task. This is true in the classical setting, and even more so in the noncommutative world, as can be seen for the simplest case of the standard Podleś sphere as presented in Example 4.10. However, as we demonstrate in this section, by imposing an irreducibility condition on our CQH-Hermitian structure, vanishing, and non-vanishing, of zeroth cohomology groups can be used to conclude positivity.

#### 3.1 Covariant Hermitian structures

Suppose that \( A \) is a Hopf algebra and \( B \) is a left \( A \)-comodule algebra. A differential calculus \( \Omega^* \) over \( B \) is said to be covariant if the coaction \( \Delta_L : B \to A \otimes B \) extends to a (necessarily unique) map \( \Delta_L : \Omega^* \to A \otimes \Omega^* \) giving \( \Omega^* \) the structure of an \( A \)-comodule algebra, and such that \( d \) is a left \( A \)-comodule map.

A complex structure for \( \Omega^* \) is said to be covariant if the \( \mathbb{Z}_2 \geq 0 \)-decomposition of \( \Omega^* \) is a decomposition in the category of left \( A \)-comodules \( A \Mod \), which is to say, \( \Omega^{(a,b)} \) is a left \( A \)-sub-comodule of \( \Omega^* \), for each \((a,b) \in \mathbb{Z}_2 \geq 0 \). This implies that \( \partial \) and \( \bar{\partial} \) are left \( A \)-comodule maps.

If \((\Omega^{(*,*)}, \sigma)\) is an Hermitian structure for \( \Omega^* \) such that \( \Omega^{(*,*)} \) is a covariant complex structure and \( \sigma \) is left \( A \)-covariant, that is, \( \Delta_L(\sigma) = 1 \otimes \sigma \), then we say that \((\Omega^{(*,*)}, \sigma)\) is a covariant Hermitian structure. In this case, \( L, \Lambda \) and \( \Lambda \) are also left \( A \)-comodule maps. A covariant Kähler structure is a covariant Hermitian structure which is moreover a Kähler structure.

We also require the notion of covariance for holomorphic modules, and for simplicity restrict to the setting of a quantum homogeneous space \( B = A^{\co(H)} \) (see “Appendix A”). A holomorphic relative Hopf module is a holomorphic module \((F, \partial F)\) over \( B \) such that \( F \) is an object in \( A_B^{0} \) (see Appendix “A.1. Takeuchi’s equivalence”) and \( \bar{\partial} F : F \to \Omega^{(0,1)} \otimes_B F \) is a left \( A \)-comodule map.

An Hermitian relative Hopf module is an Hermitian module \((F, h_F)\) such that \( F \) is an object in \( A_B^{0} \) and such that the map

\[ F \to \forall F, \quad f \mapsto h_F(\cdot, f) \]

is a morphism in \( A_B^{0} \), where the conjugate \( \overline{F} \) and dual \( \forall F \) modules are understood as objects in \( A_B^{0} \) in the sense of “Appendix A”. Note that for any simple object \( F \), its conjugate \( \overline{F} \) and its dual \( \forall F \) will again be simple, implying that any covariant Hermitian structures will be unique up to positive scalar multiple.

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If \((\mathcal{F}, h_\mathcal{F}, \partial_\mathcal{F})\) is an Hermitian holomorphic relative Hopf module, then the Chern connection is always a left \(A\)-comodule map, see [43, §7.1]. Finally, an \textit{Hermitian}, or \textit{holomorphic}, \textit{relative line module}, is an Hermitian, or holomorphic, relative Hopf module \(\mathcal{E}\) such that \(\text{dim}(\Phi(\mathcal{E})) = 1\), where \(\Phi\) is the functor from Takeuchi’s equivalence given in Appendix “A.1. Takeuchi’s equivalence”.

\textbf{Remark 3.1} In the relative Hopf module case, the differential calculus is automatically finitely generated and projective. Hence, as discussed in Example 2.5 we automatically get an Hermitian module.

In the commutative case, the induced sesquilinear pairing on the fibre of an Hermitian relative Hopf module \((\mathcal{F}, h_\mathcal{F})\) is positive definite if and only if

\[
\varepsilon(h_\mathcal{F}(f, f)) \neq 0, \quad \text{for all nonzero } f \in \mathcal{F}.
\]

Indeed, in the noncommutative setting, Takeuchi’s equivalence allows us to make sense of fibrewise positive definiteness for an Hermitian relative Hopf module \((\mathcal{F}, h_\mathcal{F})\). From the discussion of [42, §5.2], it is clear that \(h_\mathcal{F}\) is fibrewise positive definite if and only if (1) holds.

### 3.2 CQH-Hermitian and CQH-Kähler spaces

We recall the definition of a compact quantum group algebra [17], the algebraic counterpart of Woronowicz’s \(C^*\)-algebraic notion of a compact quantum group [49]. A \textit{cosemisimple} Hopf algebra is a Hopf algebra endowed with a (necessarily unique) linear map \(h : A \to \mathbb{C}\), which we call the \textit{Haar functional}, satisfying \(h(1) = 1\), and

\[
(id \otimes h) \circ \Delta(a) = h(a)1, \quad (h \otimes id) \circ \Delta(a) = h(a)1.
\]

A \textit{compact quantum group algebra}, or a \(\text{CQGA}\), is a cosemisimple Hopf \(*\)-algebra \(A\) such that \(h(a^*a) > 0\), for all nonzero \(a \in A\). A \textit{CQGA-homogeneous space} is a quantum homogeneous space such that \(A\) and \(H\) are both CQGAs and \(\pi : A \to H\) is a \(*\)-algebra map.

Let \((\Omega^{(\bullet, \bullet)}, \sigma)\) be an Hermitian structure of total degree \(2n\), for \(n \in \mathbb{Z}_{>0}\). The linear map

\[
\int := h \circ *_\sigma : \Omega^{2n} \to \mathbb{C}
\]

is called the \textit{integral}. If \(\int \omega = 0\), for all \(\omega \in \Omega^{2n-1}\), then \((\Omega^{(\bullet, \bullet)}, \sigma)\) is said to be \(f\)-closed. Note that this is a special case of an orientable differential calculus with closed integral [42, §3.2], where the Hodge map is taken as the orientation, and it generalises the classical situation of a manifold without boundary.

\textbf{Definition 3.2} A \textit{compact quantum homogeneous Hermitian space}, or simply a \textit{CQH-Hermitian space}, is a quadruple \(H := (B = A^{\text{co}(H)}, \Omega^*, \Omega^{(\bullet, \bullet)}, \sigma)\) where

\begin{enumerate}
\item \(B = A^{\text{co}(H)}\) is a CQGA-homogeneous space,
\end{enumerate}
\(\Omega^\bullet\) is a left A-covariant differential \(*\)-calculus over \(B\), and an object in \(A_B^{\rm mod_0}\).

(iii) \((\Omega^{(\bullet, \bullet)}, \sigma)\) is a covariant \(\int\)-closed Hermitian structure such that the pair \((\Omega^\bullet, g_\sigma)\) is an Hermitian module.

We denote by \(\dim(H)\) the total degree of the constituent differential calculus \(\Omega^\bullet\), and call it the \textit{dimension} of \(H\). A CQH-Kähler space \(K := (B, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \kappa)\) is a CQH-Hermitian space such that \((\Omega^{(\bullet, \bullet)}, \kappa)\) is a Kähler structure.

Over any CQH-Hermitian space \(H := (B = A_{\text{co}}^{\text{col}(H)}, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \kappa)\), every \(F \in A_B^{\text{mod}_0}\) admits an Hermitian structure. For a given Hermitian module \((F, h_F, \partial F)\) over \(B\), an inner product is given by

\[
\langle \cdot, \cdot \rangle_{F} : F \times F \to \mathbb{C}, \quad (f, g) \mapsto h_F(f, g).
\]

In particular, for the Hermitian module \(\Omega^\bullet \otimes_B F\), we have the inner product

\[
\langle \cdot, \cdot \rangle_{\sigma, F} : \Omega^\bullet \otimes_B F \times \Omega^\bullet \otimes_B F \to \mathbb{C}, \quad (\omega \otimes f, \nu \otimes g) \mapsto h \circ g_\sigma(\omega h_F(f, g), \nu).
\]

We denote by \(\partial^\dagger_F\), and \(\overline{\partial}^\dagger_F\) the adjoint operators of \(\partial_F\) and \(\overline{\partial}_F\), respectively. That \(\partial\) and \(\overline{\partial}\) are adjointable with respect to this inner product is a consequence of the fact that they are covariant [43, Proposition 5.15]. We refer to any such operator as a \textit{codifferential}. Just as in the classical case [29, §4.1], each noncommutative codifferential admits a description in terms of the Hodge map [43, Proposition 5.15]. Since such formulae will not be needed in what follows, we recall only the case where \(F = B\), as originally established in [42, §5.3.3]:

\[
\partial^\dagger = - *_\sigma \circ \overline{\partial} \circ *_\sigma, \quad \overline{\partial}^\dagger = - *_\sigma \circ \partial \circ *_\sigma.
\]

The holomorphic, and anti-holomorphic, \textit{Laplace} operators on \((F, \overline{\partial}_F)\) are defined, respectively, by

\[
\Delta_{\overline{\partial}_F} := \overline{\partial}^\dagger_F \overline{\partial}_F + \overline{\partial}_F \overline{\partial}^\dagger_F, \quad \Delta_{\partial_F} := \partial^\dagger_F \partial_F + \partial_F \partial^\dagger_F.
\]

As established in [43, Corollary 7.8], the classical relationship between the Laplacians \(\Delta_{\overline{\partial}_F}\) and \(\Delta_{\partial_F}\) carries over to the noncommutative setting. Explicitly, the \textit{Akizuki–Nakano identity}

\[
\Delta_{\overline{\partial}_F} = \Delta_{\partial_F} + [i \nabla^2, \Lambda_F].
\]

carries over to the noncommutative setting. The \textit{harmonic elements} of \((F, \overline{\partial}_F)\) are defined by

\[
\mathcal{H}^\bullet_{\overline{\partial}}(F) := \ker(\Delta_{\overline{\partial}_F}).
\]
The following noncommutative generalisation of classical Hodge decomposition was established in [43, Theorem 6.4]: Let \((\mathcal{F}, h, \partial_{\mathcal{F}})\) be an Hermitian holomorphic module over a CQH-Hermitian space \((B, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \sigma)\). Then, an orthogonal decomposition of \(A\)-comodules with respect to the Hermitian metric is given by

\[
\Omega^{(0, \bullet)} \otimes_B \mathcal{F} = H_{\overline{\partial}}^{(0, \bullet)}(\mathcal{F}) \oplus \overline{\partial_{\mathcal{F}}} (\Omega^{(0, \bullet)} \otimes_B \mathcal{F}) \oplus \overline{\partial^\dagger_{\mathcal{F}}} (\Omega^{(0, \bullet)} \otimes_B \mathcal{F}).
\]

An isomorphism with cohomology is given by the projection

\[
H_{\overline{\partial}}^{(0, \bullet)}(\mathcal{F}) \rightarrow H_{\overline{\partial}}^{(0, \bullet)}(\mathcal{F}), \quad \alpha \mapsto [\alpha].
\]

Note that, in the untwisted case (which is to say, the case where we do not tensor \(\Omega^\bullet\) with an Hermitian module \(F\)) the Laplacian operators coincide, and hence by the Hodge identification of harmonic forms and cohomology classes, the holomorphic and anti-holomorphic cohomology groups coincide [42, Corollary 7.7].

### 3.3 Irreducible CQH-Hermitian spaces

Let \(\mathcal{H} = (B = A^{\text{co}(H)}, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \sigma)\) be a CQH-Hermitian space. Since \(\Omega^{(1,0)}\) and \(\Omega^{(0,1)}\) are objects in \(A_B^{\text{mod}0}\), we can ask if they are irreducible as objects in that category. We claim that \(\Omega^{(1,0)}\) is irreducible if and only if \(\Omega^{(0,1)}\) is irreducible. Indeed, for any proper non-trivial sub-object \(N \subset \Omega^{(1,0)}\), it is clear that \(N^* := \{\omega^* \mid \omega \in N\}\) is a proper non-trivial sub-object of \(\Omega^{(0,1)}\). Clearly, we can use the same argument in the opposite direction, which proves the claim. This leads to the next definition.

**Definition 3.3** A CQH-Hermitian space \(\mathcal{H} = (B = A^{\text{co}(H)}, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \sigma)\) is said to be **irreducible** if \(\Omega^{(1,0)}\), or equivalently \(\Omega^{(0,1)}\), is irreducible as an object in \(A_B^{\text{mod}0}\).

Irreducible Hermitian structures generalise our motivating family of examples, the irreducible quantum flag manifolds \(O_q(G/L_S)\), as presented in Sect. 4. Many of the properties of the irreducible quantum flag manifolds extend to this more general setting.

**Lemma 3.4** Let \(\mathcal{H} = (B = A^{\text{co}(H)}, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \sigma)\) be a factorisable irreducible CQH-Hermitian space, and let \(\mathcal{E}\) be a relative line module over \(B\).

(i) If \(\Omega^{(0,1)}\) and \(B\) are non-isomorphic as objects in \(A_B^{\text{mod}0}\), then there exists at most one left \(A\)-covariant \((0, 1)\)-connection for \(\mathcal{E}\).

(ii) If \(\Phi(\Omega^{(0,1)})\) is not self-dual as a left \(H\)-comodule, and \(\overline{\partial}\) is a left \(A\)-covariant \((0, 1)\)-connection for \(\mathcal{E}\), then \(\overline{\partial}\) is automatically a holomorphic structure for \(\mathcal{E}\).

(iii) The space of left \(A\)-coinvariant \((1, 1)\)-forms is a one-dimensional space spanned by \(\sigma\), that is,

\[
\text{co}(A) \left( \Omega^{(1,1)} \right) = \mathbb{C} \sigma.
\]
(iv) For any Hermitian holomorphic relative line module \((E, h_E, \overline{\partial}E)\) over \(H\), there exists a scalar \(\vartheta \in \mathbb{R}\) such that
\[
\nabla^2(e) = \vartheta i\sigma \otimes e, \quad \text{for all } e \in E,
\]
where \(\nabla\) is the Chern connection of \((E, h_E, \overline{\partial}E)\).

(v) For \(E\), \(\nabla\), and \(\vartheta\) as in (iv), and denoting \(2n := \dim(H)\), it holds that
\[
[\Lambda_E, \nabla^2](e) = \Lambda_E \circ \nabla^2(e) = \vartheta ne, \quad \text{for all } e \in E.
\]

(vi) In the Kähler setting, the Akizuki–Nakano identity (4) simplifies to
\[
\Delta_{\overline{\partial}E} = \Delta_{\overline{\partial}E} + \vartheta n \text{id}
\]
for any relative line module \(E\).

**Proof** (i) Assume that there exists a nonzero morphism \(f : E \rightarrow \Omega^{(0,1)} \otimes_B E\), or equivalently, denoting \(V := \Phi(E)\), a nonzero morphism \(\phi : V \rightarrow \Phi(\Omega^{(0,1)}) \otimes V\). This would imply the existence of a nonzero morphism
\[
\mathbb{C} \simeq V \otimes V^* \rightarrow \Phi(\Omega^{(0,1)}) \otimes V \otimes V^* \simeq \Phi(\Omega^{(0,1)}),
\]
where \(V^*\) denotes the dual left \(H\)-comodule of \(V\). By irreducibility of \(\Phi(\Omega^{(0,1)})\) this is necessarily an isomorphism. Since this contradicts our assumption that \(\Omega^{(0,1)} \not\simeq B\), we must conclude that there exists no such morphism \(f\).

Let us now assume that there exists another covariant (0, 1)-connection \(\delta_E : E \rightarrow \Omega^{(0,1)} \otimes_B E\) distinct from \(\overline{\partial}E\). Then, \(\overline{\partial}E - \delta_E\) is a non-trivial left \(B\)-module map, and hence a non-trivial morphism in \(A_B \text{mod}_0\). Since we have shown that no such morphism exists, we must conclude that no such \(\delta_E\) exists, which is to say \(\overline{\partial}E\) is the unique covariant (0, 1)-connection for \(E\).

(ii) Assuming that \(\Phi(\Omega^{(0,1)})\) is not self-dual implies that there can exist no copy of \(\mathbb{C}\) in \(\Phi(\Omega^{(0,1)}) \otimes \Phi(\Omega^{(0,1)})\). Splitting the multiplication map gives an embedding of \(\Phi(\Omega^{(0,2)})\) into \(\Phi(\Omega^{(0,1)}) \otimes \Phi(\Omega^{(0,1)})\), which means that there is no copy of \(\mathbb{C}\) in \(\Phi(\Omega^{(0,2)})\).

Let us now assume that \(\overline{\partial}E\) is not flat. Since \(\overline{\partial}^2 E\) is a left \(B\)-module map from \(E\) to \(\Omega^{(0,2)} \otimes_B E\), it is automatically a nonzero morphism in \(A_B \text{mod}_0\). This means that we have a nonzero morphism \(V \rightarrow \Phi(\Omega^{(0,2)}) \otimes V\), and hence a morphism
\[
\mathbb{C} \simeq V \otimes V^* \rightarrow \Phi(\Omega^{(0,2)}) \otimes V \otimes V^* \simeq \Phi(\Omega^{(0,2)}),
\]
Since this contradicts our assumption that \(\Phi(\Omega^{(0,1)})\) is not self-dual, we are forced to conclude that \(\overline{\partial}E\) is necessarily flat.

(iii) Since \(H \text{mod}_0\) is a rigid monoidal category, \(V\) is invertible. In particular \(V \otimes V^* \simeq \mathbb{C}\). By assumption \(\Omega^\bullet \bullet\) is factorisable, hence \(\Phi(\Omega^{(1,1)})\) is isomorphic to \(\Phi(\Omega^{(1,0)}) \otimes \Phi(\Omega^{(0,1)})\). Denote the decomposition of \(\Phi(\Omega^{(1,1)})\) into irreducible comodules by
\[ \Phi(\Omega^{(1,1)}) \simeq \Phi(\Omega^{(1,0)}) \otimes \Phi(\Omega^{(0,1)}) \simeq \bigoplus_i K_i. \]

Since \( \sigma \) is a left \( A \)-coinvariant Hermitian form, we have \([\sigma] \in \text{co}(A)(\Phi(\Omega^{(1,1)}))\), implying that one of the summands \( K_i \) must be isomorphic to the trivial comodule. Moreover, since both \( \Phi(\Omega^{(1,0)}) \) and \( \Phi(\Omega^{(0,1)}) \) are by assumption irreducible, \( \Phi(\Omega^{(1,0)}) \) and \( \Phi(\Omega^{(0,1)}) \) are dual and precisely one of the summands will be trivial.

For \( U: F \mapsto \Psi_1(\Phi(\Omega^{(1,1)})) \) the unit of Takeuchi’s equivalence, it is easily seen that

\[ U\left(\text{co}(A)(\Omega^{(1,1)})\right) = 1 \otimes \left(\text{co}(H)(\Phi(\Omega^{(1,1)}))\right) \simeq 1 \otimes \mathbb{C}, \]

giving the claimed equality.

(iv), (v) Consider the space of left \( H \)-comodule maps \( V \to \Phi(\Omega^{(1,1)}) \otimes V \), which is clearly isomorphic to the space of left \( H \)-comodule maps

\[ \mathbb{C} \simeq V \otimes V^* \to \Phi(\Omega^{(1,1)}) \otimes V \otimes V^* \simeq \Phi(\Omega^{(1,1)}). \]

Since \( \text{co}(A)(\Omega^{(1,1)}) = \mathbb{C}\sigma \), these spaces must be one-dimensional. Now the curvature operator is a morphism in \( A_B \text{mod}_0 \), and so, we can consider its image under Takeuchi’s functor \( \Phi \). As a morphism from \( V \) to \( \Phi(\Omega^{(1,1)}) \otimes V \), it must be of the form

\[ \Phi(\nabla^2)([e]) = \alpha[\sigma] \otimes [e], \quad \text{for some } \alpha \in \mathbb{C}. \]

Consider now the commutative diagram:

\[
\begin{array}{ccc}
\Omega^{(1,1)} \otimes_B E & \xleftarrow{U^{-1}} & A \square_H \Phi(\Omega^{(1,1)} \otimes_B E) \\
\nabla^2 & \xrightarrow{U} & \text{id} \otimes \Phi(\nabla^2) \\
E & \xrightarrow{\text{id}} & A \square_H \Phi(E).
\end{array}
\]

For any particular element \( e \in E \), it holds that

\[
\nabla^2(e) = U^{-1} \circ (\text{id} \otimes \Phi(\nabla^2)) \circ U(e) = \alpha U^{-1}(e_{(-1)} \otimes [\sigma \otimes e_{(0)}])
\]

\[
= \alpha e_{(-1)} S(e_{(-1)}) \sigma \otimes e_{(0)}
= \alpha \sigma \otimes e,
\]

where we have used the formula for the inverse of Takeuchi’s unit, as presented in (12).

The operators \( \Delta_{\nabla E} \) and \( \Delta_{\nabla E} \) are, by construction, self-adjoint operators on \( \Omega^* \otimes_B E \). Thus, any eigenvalue of \( \Delta_{\nabla E} \) must be a real scalar. It now follows from the Akizuki–Nakano identity that
\[
(\Delta_{\partial\bar{\partial}} - \Delta_{\partial\bar{\partial}})(e) = [i\nabla^2, \Lambda_{\partial\bar{\partial}}](e) = -i\Lambda_{\partial\bar{\partial}} \circ \nabla^2(e) = -\alpha i\Lambda_{\partial\bar{\partial}}(\sigma \otimes e) = -\alpha i\Lambda_{\partial\bar{\partial}} \circ L_{\partial\bar{\partial}}(e).
\]

Recalling now the twisted Lefschetz identities, and denoting \(2n := \dim(H)\), the above expression can be reduced to

\[
-\alpha i\Lambda_{\partial\bar{\partial}} \circ L_{\partial\bar{\partial}}(e) = \alpha i[L_{\partial\bar{\partial}}, \Lambda_{\partial\bar{\partial}}](e) = \alpha iH_{\partial\bar{\partial}}(e) = -i\theta ne.
\]

Thus, \(\alpha i \in \mathbb{R}\) and setting \(\vartheta := -\alpha i\) gives the equation in (iv) as claimed, establishing (v) in the process.

(vi) The simplified form of the Akizuki–Nakano identity follows directly from (v).

For the special case of a CQH-Kähler space, Lemma 3.4 (iv) implies the following result.

**Theorem 3.5** For any Hermitian holomorphic relative line module \(\mathcal{E}\) over a factorisable irreducible CQH-Kähler space, precisely one of the following three possibilities holds:

(i) \(\mathcal{E} > 0\),
(ii) \(\mathcal{E}\) is flat,
(iii) \(\mathcal{E} < 0\).

The general cohomological consequences of positivity presented in the Kodaira vanishing theorem now allow us to produce sufficient cohomological conditions for positivity, flatness, or negativity, of a line module over an irreducible CQH-Kähler space.

**Corollary 3.6** Let \(\mathcal{E}\) be an Hermitian holomorphic relative line module over an irreducible CQH-Kähler space \(K\).

(i) If \(H^0_{\partial}(\mathcal{E})\) and \(H^0_{\bar{\partial}}(\mathcal{E})\) are both nonzero, then \(\mathcal{E}\) is flat, and \(H^0_{\partial}(\mathcal{E}) = H^0_{\bar{\partial}}(\mathcal{E})\).
(ii) If \(H^0_{\partial}(\mathcal{E}) \neq 0\) and \(H^0_{\bar{\partial}}(\mathcal{E}) \neq H^0_{\bar{\partial}}(\mathcal{E})\), then \(\mathcal{E}\) is positive and \(H^0_{\partial}(\mathcal{E}) = 0\).
(iii) If \(H^0_{\partial}(\mathcal{E}) \neq 0\) and \(H^0_{\bar{\partial}}(\mathcal{E}) \neq H^0_{\bar{\partial}}(\mathcal{E})\), then \(\mathcal{E}\) is negative and \(H^0_{\partial}(\mathcal{E}) = 0\).

**Proof** (i) By part (vi) of Lemma 3.4, for any eigenvector \(e\) of \(\Delta_{\partial\bar{\partial}}\), with eigenvalue \(\lambda\),

\[
\Delta_{\partial\bar{\partial}}(e) = \Delta_{\partial\bar{\partial}}(e) + \vartheta ne = (\lambda + \vartheta n)e.
\]

Since \(\Delta_{\partial\bar{\partial}}\) is a positive operator \(\lambda \in \mathbb{R}_{\geq 0}\). Thus, if \(\mathcal{E} < 0\), which is to say, if \(\vartheta > 0\), we would have that \(\lambda + \vartheta n > 0\), implying that \(\Delta_{\partial\bar{\partial}}(e) \neq 0\). In particular, we see that if \(\mathcal{E} < 0\), then by the isomorphism of harmonic elements and cohomology classes \(H^0_{\partial}(\mathcal{E}) = 0\). An analogous argument implies that if \(\mathcal{E} > 0\), then \(H^0_{\partial}(\mathcal{E}) = 0\).

From these considerations, we see that if \(H^0_{\partial}(\mathcal{E})\) and \(H^0_{\bar{\partial}}(\mathcal{E})\) are both nonzero,
then $E$ can be neither positive nor negative. It now follows from Theorem 3.5 that $E$ is flat. Finally, we see that if $E$ is flat, then since the Laplacians $\Delta_{\bar{\partial}_E}$ and $\Delta_{\partial_E}$ coincide, the associated cohomology groups must also coincide.

(ii) From the argument of (i), we see that if $H^0_\partial(E) \neq 0$, then $E$ cannot be negative. Moreover, since the holomorphic and anti-holomorphic cohomology groups are not equal, $E$ cannot be flat. Theorem 3.5 now implies that $E$ is positive. Finally, it follows from (i) that if $H^0_\partial(E)$ were nonzero, then $E$ would be flat. Thus to avoid contradiction, we must assume that $H^0_\partial(E) = 0$.

(iii) The argument for this case is exactly analogous to the argument of (ii). $\square$

Example 3.7 We now consider an interesting family of examples, namely irreducible CQH-Kähler spaces $K = (B = A^{\text{co}(H)}, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \kappa)$ of dimension 4. In particular, we bear in mind the quantum projective plane $O_q(\mathbb{CP}^2)$, a member of the general family of examples discussed in Sect. 4. Note that in dimension 4 the Hodge map satisfies $*^2_{\kappa} = \text{id}$ on 2-forms. Hence $*_{\kappa}$ has eigenvalues 1 and $-1$. It follows from the definition of $*_{\kappa}$ that $*_{\kappa}(\kappa) = \kappa$. Following the classical definition, we define the first Chern class $c_1(E)$, of an Hermitian holomorphic relative line module $E$, to be $c_1(E) := \text{tr } \nabla^2$, where $\text{tr}$ is the trace of $\nabla^2$ easily defined using Takeuchi’s equivalence. By Lemma 3.4, the first Chern class $c_1(E)$ is proportional to $\kappa$, and so, it satisfies

$$*_{\kappa}(c_1(E)) = c_1(E).$$

In the classical setting, such connections are called self-dual connections. They are of interest because they satisfy the Yang–Mills equations [3]. For a more detailed discussion for the special case of $O_q(\mathbb{CP}^2)$, see the pair of papers [12, 13].

Remark 3.8 Any positive, negative, or flat relative Hopf module $(\mathcal{F}, h, \bar{\partial}\mathcal{F})$ over an irreducible CQH-Kähler space $K = (B = A^{\text{co}(H)}, \Omega^\bullet, \Omega^{(\bullet, \bullet)}, \kappa)$ satisfies

$$\Lambda_\mathcal{F} \circ \nabla^2 = \vartheta \text{id}_{\mathcal{F}}, \quad \text{for some } \vartheta \in \mathbb{R}. \quad (5)$$

This is established by a verbatim extension of the arguments of Lemma 3.4 to the relative Hopf module setting. What is interesting about (5) is that it is satisfied by a far larger class of Hermitian holomorphic modules than those which are positive, negative, or flat. Classically, this motivates the definition of an Hermite–Einstein module [29, §4.B]. We observe that this definition carries over directly to the noncommutative setting: For a CQH-Kähler space $K$, we say that an Hermitian holomorphic module $(\mathcal{F}, h, \bar{\partial}\mathcal{F})$ over $K$ is Hermite–Einstein if

$$\Lambda_\mathcal{F} \circ \nabla^2 = \gamma \text{id}_{\mathcal{F}}, \quad \text{for some } \gamma \in \mathbb{R}.$$

Hermite–Einstein vector bundles are objects of central importance in classical complex geometry. They are intimately related to the theory of Yang–Mills connections [30]. Moreover, the Donaldson–Uhlenbeck–Yau theorem relates the existence of an Hermite–Einstein metric to semi-stability of the vector bundle, see [37]. The investigation of how such results extend to the noncommutative setting presents itself as a very interesting direction for future research.
4 The Heckenberger–Kolb calculi for the irreducible quantum flag manifolds

In this section, we consider our motivating family of examples: the irreducible quantum flag manifolds endowed with their Heckenberger–Kolb calculi. We assume that the reader has some familiarity with the representation theory of Lie algebras. For standard references, see [28, 31, 44].

4.1 Drinfeld–Jimbo quantum groups

In this subsection, we recall the necessary definitions from the theory of Drinfeld–Jimbo quantum groups. We refer the reader to [34] for further details, as well to the seminal papers [19, 32]. Let \( \mathfrak{g} \) be a finite-dimensional complex semisimple Lie algebra of rank \( r \). We fix a Cartan subalgebra \( \mathfrak{h} \) with corresponding root system \( \Delta \subseteq \mathfrak{h}^* \), where \( \mathfrak{h}^* \) denotes the linear dual of \( \mathfrak{h} \). With respect to a choice of simple roots \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \), denote by \( (\cdot, \cdot) \) the symmetric bilinear form induced on \( \mathfrak{h}^* \) by the Killing form of \( \mathfrak{g} \), normalised so that any shortest simple root \( \alpha_i \) satisfies \( (\alpha_i, \alpha_i) = 2 \).

Let \( \{ \varpi_1, \ldots, \varpi_r \} \) denote the corresponding set of fundamental weights of \( \mathfrak{g} \). The coroot \( \alpha_i^\vee \) of a simple root \( \alpha_i \) is defined by \( \alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i) \). The Cartan matrix \( A = (a_{ij})_{ij} \) of \( \mathfrak{g} \) is the \( (r \times r) \)-matrix defined by \( a_{ij} := (\alpha_i^\vee, \alpha_j) \).

Let \( q \in \mathbb{R} \) such that \( q \notin \{ -1, 0, 1 \} \), and denote \( q_i := q^{(\alpha_i, \alpha_i)/2} \). The quantised enveloping algebra \( U_q(\mathfrak{g}) \) is the noncommutative associative algebra generated by the elements \( E_i, F_i, K_i, \) and \( K_i^{-1} \), for \( i = 1, \ldots, r \), subject to the relations

\[
K_i E_j = q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \quad K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,
\]

\[
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},
\]

along with the quantum Serre relations which we omit. A Hopf algebra structure is defined on \( U_q(\mathfrak{g}) \) by

\[
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,
\]

\[
S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i) = K_i^{-1},
\]

\[
\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.
\]

A Hopf \(*\)-algebra structure, called the compact real form of \( U_q(\mathfrak{g}) \), is defined by

\[
K_i^* := K_i, \quad E_i^* := K_i F_i, \quad F_i^* := E_i K_i^{-1}.
\]

Let \( \mathcal{P} \) be the weight lattice of \( \mathfrak{g} \), and \( \mathcal{P}^+ \) its set of dominant integral weights. For each \( \mu \in \mathcal{P}^+ \), there exists an irreducible finite-dimensional \( U_q(\mathfrak{g}) \)-module \( V_\mu \), uniquely defined by the existence of a vector \( v_{hw} \in V_\mu \), which we call a highest weight vector, satisfying
\[ E_i \triangleright v_{hw} = 0, \quad K_i \triangleright v_{hw} = q^{(\alpha_i, \mu)} v_{hw}, \quad \text{for all } i = 1, \ldots, r. \]

Moreover, \( v_{hw} \) is unique up to scalar multiple. We call any finite direct sum of such \( U_q(\mathfrak{g}) \)-representations a type-1 representation. A vector \( v \in V_\mu \) is called a weight vector of weight \( \text{wt}(v) \in \mathcal{P} \) if
\[ K_i \triangleright v = q^{(\alpha_i, \text{wt}(v))} v, \quad \text{for all } i = 1, \ldots, r. \]

(6)

Finally, we note that since \( U_q(\mathfrak{g}) \) has an invertible antipode, we have an equivalence between \( U_q(\mathfrak{g}) \), the category of left \( U_q(\mathfrak{g}) \)-modules, and \( \text{Mod}_{U_q(\mathfrak{g})} \), the category of right \( U_q(\mathfrak{g}) \)-modules, as induced by the antipode.

### 4.2 Quantum coordinate algebras and quantum flag manifolds

In this subsection, we recall some necessary material about quantised coordinate algebras, for further details see [34, §6 and §7] and [24]. Let \( V \) be a finite-dimensional left \( U_q(\mathfrak{g}) \)-module, \( v \in V \), and \( f \in V^* \), the \( \mathbb{C} \)-linear dual of \( V \), endowed with its right \( U_q(\mathfrak{g}) \)-module structure. An important point to note is that, with respect to the equivalence of left and right \( U_q(\mathfrak{g}) \)-modules discussed above, the left module corresponding to \( V^*_\mu \) is isomorphic to \( V_{-w_0(\mu)} \), where \( w_0 \) denotes the longest element in the Weyl group of \( \mathfrak{g} \).

Consider the function \( c^V_{f,v} : U_q(\mathfrak{g}) \to \mathbb{C} \) defined by \( c^V_{f,v}(X) := f(X \triangleright v) \). The coalgebra of matrix coefficients of \( V \) is the subspace
\[ C(V) := \text{span}_\mathbb{C}\{ c^V_{f,v} \mid v \in V, \ f \in V^* \} \subseteq U_q(\mathfrak{g})^*. \]

A \( U_q(\mathfrak{g}) \)-bimodule structure on \( C(V) \) is given by
\[ (Y \triangleright c^V_{f,v} \triangleleft Z)(X) := f ((ZXY) \triangleright v) = c^V_{f \circ Z,Y \triangleright v}(X). \]

(7)

Let \( U_q(\mathfrak{g})^\circ \) denote the Hopf dual of \( U_q(\mathfrak{g}) \). By construction \( C(V) \subseteq U_q(\mathfrak{g})^\circ \), and moreover that a Hopf subalgebra of \( U_q(\mathfrak{g})^\circ \) is given by
\[ \mathcal{O}_q(G) := \bigoplus_{\mu \in \mathcal{P}^+} C(V_\mu). \]

(8)

We call \( \mathcal{O}_q(G) \), endowed with the dual \( * \)-structure, the quantum coordinate algebra of \( G \), where \( G \) is the compact, simply-connected, simple Lie group having \( \mathfrak{g} \) as its complexified Lie algebra.

### 4.3 Quantum flag manifolds

For \( \{ \alpha_i \}_{i \in S} \) a subset of simple roots, consider the Hopf \( * \)-subalgebra
\[ U_q(S) := \{ K_i, E_j, F_j \mid i = 1, \ldots, r; j \in S \}, \]
which $q$-deforms the classical reductive Lie algebra $l_S$. We have obvious analogues of weight vectors, highest weight vectors, type-1 representations. The category of finite-dimensional type-1 modules is semisimple, and each simple object admits a highest weight vector unique to scalar multiple. Moreover, the weight of the highest weight vector determines the module up to isomorphism, giving a bijective correspondence between irreducible type-1 modules and the set of weights

$$\mathcal{P}^+ \cup \mathcal{P}^{Sc}, \quad \text{where } \mathcal{P}^{Sc} = \text{span}_{\mathbb{Z}}\{\varpi_x | x \in \Pi \setminus S\}.$$  

Just as for $O_q(G)$, we can construct the type-1 dual of $U_q(l_S)$ using matrix coefficients and we denote this Hopf algebra by $O_q(L_S)$.

Restriction of domains gives us a surjective Hopf $\ast$-algebra map

$$\pi_S : O_q(G) \to O_q(L_S),$$

dual to the inclusion of $U_q(l_S)$ of $U_q(g)$ (see also “Appendix B”). The quantum flag manifold associated to $S$ is the CQGA-homogeneous space

$$O_q(G/L_S) := O_q(G)^{\text{co}(O_q(L_S))}$$

associated to $\pi_S$. Takeuchi’s equivalence now implies that every simple relative Hopf module over $O_q(G/L_S)$ is of the form

$$F_\mu := O_q(G) \Box O_q(L_S)W_\mu, \quad \text{for } \mu \in \mathcal{P}^+ \cup \mathcal{P}^{Sc},$$

where by abuse of notation $W_\mu$ denotes the $O_q(L_S)$-comodule corresponding to $\mu$.

### 4.4 Irreducible quantum flag manifolds

Let $S = \{\alpha_1, \ldots, \alpha_r\}\setminus \{\alpha_x\}$ where $\alpha_x$ has coefficient 1 in the expansion of the highest root of $g$. Then, we say that the associated quantum flag manifold is irreducible. In the classical limit of $q = 1$, these homogeneous spaces reduce to the family of compact Hermitian symmetric spaces, as classified, for example, in [4]. Presented in Table 1 of Appendix C is a useful diagrammatic presentation of the set of simple roots defining the irreducible quantum flag manifolds, where the node corresponding to $\alpha_x$ is the coloured one.

Let us now choose for once and for all, for each irreducible $U_q(g)$-module $V_\mu$, a weight basis $\{v_i\}_{i=1}^N$, with corresponding dual basis $\{f_i\}_{i=1}^N$, where $N := \dim(V_\mu)$. As shown in [25, Proposition 3.2], for the irreducible case, a set of generators for $O_q(G/L_S)$ is given by

$$z_{ij} := c_{fi,v_N}^V c_{vj,f_N}^{V_{-w_0(\mu_x)}} \quad \text{for } i, j = 1, \ldots, N = \dim(V_{\mu_x}),$$

where $v_N$ is the highest weight basis element of $V_{\mu_x}$, and $f_N$ is the lowest weight basis element of $V_{-w_0(\mu_x)}$. 

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In the classical case $l_S$ admits a direct sum decomposition $l_S = l_S^s \oplus \mathfrak{c}$ into the semisimple part $l_S^s$ and the centre $\mathfrak{c}$, given a decomposition

$$U(l_S) \simeq U(l_S^s) \otimes U(\mathfrak{c}),$$

where $U(\mathfrak{c})$ is a commutative and cocommutative Hopf algebra generated by the distinguished sum of Chevalley generators $H_i \in \mathfrak{h}$,

$$H_{\alpha_x} := (A^{-1})_{x1}H_1 + \cdots + (A^{-1})_{xr}H_r,$$

with $A^{-1}$ is the inverse of the Cartan matrix of $\mathfrak{g}$. Now for $U_q(\mathfrak{g})$, the element

$$Z := K_{\det(A)\alpha_x} = K_1^{a_1} \cdots K_r^{a_r}, \quad \text{where} \; \det(A)\alpha_x =: a_1\alpha_1 + \cdots + a_r\alpha_r$$

belongs to the centre of $U_q(l_S)$. Note that the scaling $\det(A)$ is chosen to ensure that $\det(A)\alpha_x$ is an element of the root lattice of $\mathfrak{g}$ (see also Remark 4.2 below). The elements $Z$ and $Z^{-1}$ generate a commutative and cocommutative Hopf subalgebra, and by Schur’s lemma $Z$ acts as a scalar multiple of the identity on any irreducible $U_q(l_S)$-module. It is instructive to note that any $U_q(l_S)$-module is completely determined by its $U_q(l_S^s)$-module structure and the action of the central element $Z$.

**Example 4.1** For the case of the quantum projective plane $O_q(\mathbb{C}P^2)$, taking $S = \{\alpha_2\}$, which is to say crossing the first node of the Dynkin diagram, the central element is

$$Z = K_1^2 K_2^1.$$

Alternatively, crossing the last node, the central element is given by

$$Z = K_1^1 K_2^2.$$

For the quantum Lagrangian Grassmannian $O_q(L_3) = O_q(\text{Sp}_6/L_S)$, we have

$$Z = K_1^1 K_2^2 K_3^3.$$

We note that in all three cases, the algebra generated by $U_q(l_S^s)$ and the central element $Z$ is a proper subalgebra of $U_q(l_S)$.

**Remark 4.2** One can make sense of the symbol $K_{\alpha_x}$ as an element of $U_q(\mathfrak{g}, \mathcal{P})$, the extension of $U_q(\mathfrak{g})$ which includes generators of the form $K_\lambda$, for all $\lambda \in \mathcal{P}$. This allows one to avoid multiplication by the determinant of the Cartan matrix in the definition of the central element $Z$. See, for example, [11, §3], which deals with the special case of quantum projective space, and [36, §2D] which deals with the general case.
4.5 Relative line modules over the irreducible quantum flag manifolds

In this subsection, we recall the necessary facts about the relative line modules over the irreducible quantum flag manifolds $O_q(G/L_S)$. We first observe that the one-dimensional $U_q(l_S)$-modules correspond to the weights in $P_{Sc}$, which in turn implies that the one-dimensional $O_q(L_S)$-comodules also correspond to the weights in $P_{Sc}$. Thus by Takeuchi’s equivalence, the relative line modules are indexed by the weights $P_{Sc}$. For the special case of the irreducible quantum flag manifolds $P_{Sc} = \mathbb{Z} \omega_x$. In this case, we denote by $E_l$ the relative line module corresponding to the weight $l \omega_x$.

We make two important observations about relative line modules over the irreducible quantum flag manifolds, considered as submodules of $O_q(G/L_S)$:

- Firstly, we note that for all $l \in \mathbb{Z}$, we have $(E_l)^* \simeq E_{-l}$. Secondly, we note that for each $l \in \mathbb{Z}$,

$$h : E_l \times E_l \to O_q(G/L_S), \quad (e_1, e_2) \mapsto e_1 e_2^*,$$

gives $E_l$ the structure of an Hermitian relative line module. Since $E_l$ is a simple object in $O_q(G/L_S) \mod_0$, we see that $h$ is the unique such structure up to positive scalar multiple.

4.6 The Heckenberger–Kolb Calculi

The irreducible quantum flag manifolds are distinguished by the existence of an essentially unique $q$-deformation of their classical de Rham complex. The existence of such a canonical deformation is one of the most important results in the study of the non-commutative geometry of quantum groups, serving as a solid base from which to investigate more general classes of quantum spaces. The following theorem is a direct consequence of results established in [25, 26, 40].

**Theorem 4.3** Over any irreducible quantum flag manifold $O_q(G/L_S)$, there exists a unique finite-dimensional left $O_q(G)$-covariant differential $\ast$-calculus

$$\Omega^\ast_q(G/L_S) \in \frac{O_q(G)}{O_q(G/L_S) \mod_0},$$

of classical dimension, that is to say,

$$\dim \Phi \left( \Omega^k_q(G/L_S) \right) = \binom{2M}{k}, \quad \text{for all } k = 0, \ldots, 2M,$$

where $M$ is the complex dimension of the corresponding classical manifold, as presented in Table 2 of Appendix C.

The calculus $\Omega^\ast_q(G/L_S)$, which we call the Heckenberger–Kolb calculus of $O_q(G/L_S)$, has many remarkable properties. We begin with the existence of a unique covariant complex structure, following from the results of [25, 26, 40].
Proposition 4.4 Let \( \mathcal{O}_q(G/L_S) \) be an irreducible quantum flag manifold, and \( \mathcal{O}_q^*(G/L_S) \) its Heckenberger–Kolb differential \(*\)-calculus. Then, the following hold:

(i) \( \mathcal{O}_q^*(G/L_S) \) admits a unique left \( \mathcal{O}_q(G) \)-covariant complex structure,

\[
\mathcal{O}_q^*(G/L_S) \cong \bigoplus_{(a,b) \in \mathbb{Z}^2_{\geq 0}} \Omega^{(a,b)} =: \Omega^{(\cdot, \cdot)},
\]

(ii) \( \Omega^{(\cdot, \cdot)} \) is factorisable,

(iii) \( \Omega^{(1,0)} \) and \( \Omega^{(0,1)} \) are irreducible as objects in \( \mathcal{O}_q(G) \mod_0 \).

As observed in [42, §10.8] (using the same argument as presented in part (i) of Lemma 3.4), there exists a real left \( \mathcal{O}_q(G) \)-coinvariant form \( \kappa \in \Omega^{(1,1)} \), and it is unique up to real scalar multiple. Moreover, by extending the representation theoretic argument given in [42, §4.4] for the case \( \mathcal{O}_q(\mathbb{C}P^n) \), the form \( \kappa \) is readily seen to be a closed central element of \( \mathcal{O}_q^*(G/L_S) \). This motivated [42, Conjecture 4.25], where it was proposed that the pair \( (\Omega^{(\cdot, \cdot)}, \kappa) \) is a Kähler structure for the calculus. With suitable restrictions on the values of \( q \), the conjecture was verified by Matassa in [40, Theorem 5.10].

Theorem 4.5 Let \( \mathcal{O}_q^*(G/L_S) \) be the Heckenberger–Kolb calculus of the irreducible quantum flag manifold \( \mathcal{O}_q(G/L_S) \). The pair \( (\Omega^{(\cdot, \cdot)}, \kappa) \) is a covariant Kähler structure for all \( q \in \mathbb{R}_{>0} \setminus F \), where \( F \) is a finite, possibly empty, subset of \( \mathbb{R}_{>0} \). Moreover, any element of \( F \) is necessarily non-transcendental.

The question of when this Kähler structure gives a CQH-Kähler space was addressed [14, Theorem 6.1]. Taken together with irreducibility of the holomorphic forms (as recalled in part (iii) of Proposition 4.4 above), this gives us the following theorem. (Note that in the theorem we need to restrict to a strictly positive cone of real \( \mathcal{O}_q(G) \)-coinvariant \((1,1)\)-forms to get positivity, and we denote an arbitrary Kähler form in this cone by \( \kappa_+ \)).

Theorem 4.6 For each irreducible quantum flag manifold \( \mathcal{O}_q(G/L_S) \), there exists a real left \( \mathcal{O}_q(G) \)-coinvariant \((1,1)\)-form \( \kappa_+ \), uniquely defined up to strictly positive real multiple, such that a CQH-Kähler space is given by the quadruple

\[
K_S := \left( \mathcal{O}_q(G/L_S), \mathcal{O}_q^*(G/L_S), \Omega^{(\cdot, \cdot)}, \kappa_+ \right),
\]

for all \( q \in I \), where \( I \subseteq \mathbb{R}_{>0} \) is a sufficiently small open interval around 1.

In the rest of this section, we build upon this result, using it to apply the general framework of the paper to the study of the irreducible quantum flag manifolds.

4.7 Positive and negative line modules over \( \mathcal{O}_q(G/L_S) \)

It was shown in [16, Theorem 4.5] that each relative Hopf module over any irreducible quantum flag manifold \( \mathcal{O}_q(G/L_S) \) admits a unique relative holomorphic structure. Let us now recall the precise statement of this result for the special case of line modules.
Theorem 4.7 For every relative line module $E_k$ over $O_q(G/LS)$, there exists a unique covariant $(0, 1)$-connection

$$\overline{\partial} E_k : E_k \rightarrow \Omega^{(0,1)}_q(G/LS) \otimes_{O_q(G/LS)} E_k.$$ 

Moreover, $\overline{\partial} E_k$ is flat, which is to say, it is a holomorphic structure.

It now follows directly from Theorem 3.5 that the relative line modules over the irreducible quantum flag manifolds $O_q(G/LS)$ are either positive, flat, or negative. To differentiate between these possibilities, we will use the cohomological information given by the noncommutative Borel–Weil theorem for $O_q(G/LS)$ established in [8, 9].

Theorem 4.8 For any irreducible quantum flag manifold $O_q(G/LS)$, it holds that

(i) $H^0(E_k)$ is an irreducible $U_q(g)$-module of highest weight $k\varpi$, for all $k \in \mathbb{Z}_{\geq 0}$,
(ii) $H^0(E_{-k}) = 0$, for all $k \in \mathbb{Z}_{>0}$.

where $\Pi \backslash S$ consists of the simple root $\alpha_x$.

Using this cohomological information, we can now apply Theorem 3.5 to determine which modules are positive and which are negative.

Theorem 4.9 For any irreducible quantum flag manifold $O_q(G/LS)$, and any $k \in \mathbb{Z}_{>0}$,

(i) $E_k > 0$,
(ii) $E_{-k} < 0$,

for any Kähler structure identified in Theorem 4.6.

Example 4.10 For any differential manifold, the curvature of a connection is additive over tensor products of vector bundles. In particular, any tensor power of a positive line bundle over a complex manifold is again positive. In the noncommutative setting, tensoring two holomorphic modules is more problematic, and one needs to consider bimodule connections (in the sense of [10, 20–22]). Even in this case, curvature does not behave additively. In particular, for a bimodule holomorphic line module $E$, we cannot directly conclude positivity of $E^\otimes k$ from positivity of $E$, making the general approach of this paper all the more valuable.

For the case of the standard Podleś sphere $O_q(S^2)$, the curvature can be explicitly calculated using the theory of quantum principal bundles, see [6, Example 5.23] for details. In the conventions of this paper, for any line module $E_k$ over $O_q(S^2)$, it holds that

$$\nabla^2(e) = -(k)_{q^{-2}} i\kappa \otimes e, \quad \text{for all } e \in E_k,$$

where the quantum integer is given explicitly by $(k)_{q^{-2}} := 1 + q^{-2} + q^{-4} + \cdots + q^{-2(k-1)}$, and we have chosen the unique Kähler form $\kappa$ satisfying

$$\nabla^2(e) = -i\kappa \otimes e, \quad \text{for all } e \in E_1. \quad (9)$$
This suggests that there is some type of $q$-deformed (or braided) additivity underlying these results. Understanding this process presents itself as an interesting and important future goal.

### 4.8 A Fano structure for the irreducible quantum flag manifolds

Building on the definition of Kähler structure, the notion of a noncommutative Fano structure was introduced in [43, §8.2], directly generalising the classical definition of a Fano manifold. Moreover, for any CQH-Fano space $F$ (defined in the obvious way as a refinement of the definition of a CQH-Kähler structure) the noncommutative Kodaira vanishing theorem established in [43, Theorem 8.3] implies that the anti-holomorphic cohomology $H^{(0,\bullet)}(E)$ of any positive line module $E$ over $F$ will be concentrated in degree zero.

**Definition 4.11** A **Fano structure** for a differential $\ast$-calculus $\Omega^\bullet$, of total degree $2n$, is a factorisable Kähler structure $(\Omega^{(\bullet,\bullet)}, \kappa)$ such that the holomorphic Hermitian module $(\Omega^{(n,0)}, g_\kappa, \bar{\partial})$ is negative.

We now verify the Fano condition for the irreducible quantum flag manifolds. Besides being an interesting result in its own right, it is also a necessary step for the proof of the Bott–Borel–Weil theorem for the irreducible quantum flag manifolds [43, §9]. We note first that since a line module $E_l$ is negative if and only if $l$ is a negative integer, verifying the Fano condition amounts to showing that $\Phi_1(\Omega^{(1,0)}) \simeq E_{-k}$, for some $k > 0$, where we recall that $M$ denotes the complex dimension of the corresponding classical manifold $G/L_S$. This we do by producing a general description of $k$ in terms of the Cartan matrix of $g$, for the special case of $q \in I$, where $I$ is as defined in Theorem 4.6. (In Table 2 of Appendix C, we present the explicit values of $k$ for each series of the irreducible quantum flag manifolds.)

**Theorem 4.12** Let $\mathcal{O}_q(G/L_S)$ be an irreducible quantum flag manifold, endowed with its Heckenberger–Kolb calculus $\Omega_q(G/L_S)$. For any $q \in I$, the pair $(\Omega^{(\bullet,\bullet)}, \kappa_\pm)$ is a Fano structure.

**Proof** Since the complex structure $\Omega^{(\bullet,\bullet)}$ is factorisable, we only need to verify condition (ii) of Definition 4.11, that is, show that $(\Omega^{(M,0)}, g_{\kappa_\pm}, \bar{\partial})$ is a negative line module, where $2M$ is the total degree of $\Omega_q^\bullet(G/L_S)$. First, we will identify the unique $k \in \mathbb{Z}_{>0}$ such that

$$\Phi(\Omega^{(M,0)}) \simeq \Phi(E_{-k}).$$

To do so, we will compare the actions of the central element $Z$ on the $U_q(L_S)$-modules $\Phi(\mathcal{E}_{-k})$ and $\Phi(\Omega^{(M,0)})$. Now since $\Phi(\Omega^{(1,0)})$ is irreducible as a $U_q(L_S)$-module, $Z$ acts on $\Phi(\Omega^{(1,0)})$ as multiplication by some scalar $\gamma$. For any $z_{ij} \in \mathcal{O}_q(G/L_S)$, we see that

$$[\bar{\partial}z_{ij}] \triangleright z = \left[ \partial(c_{\sigma_x}^{(f_i \not\subset Z,v_N)}c_{v_j \not\subset Z,f_N}^{-w_0(\sigma_x)}) \right] = q^{-(\sigma_x,\text{wt}(f_i) + \text{wt}(v_j))}\det(A)\bar{\partial}z_{ij}. \quad (11)$$

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As shown in [26, Proposition 3.6], a basis of \( \Phi(\Omega^{(1,0)}) \) is given by

\[
\{ [\partial z^N_i] \mid i \in J_{(1)} \},
\]

where \( J_{(1)} \) is the subset of the index set \( J := \{1, \ldots, \dim(V_{\sigma_x})\} \) defined by

\[
J_{(1)} := \{ i \in J \mid (\sigma_x, \wt(v_i)) = (\sigma_x, \sigma_x - \alpha_x) \}.
\]

Now for any \( i \in J_{(1)} \), we have

\[
(\sigma_x, \wt(f_i) + \wt(v_N)) = (\sigma_x, -\sigma_x + \alpha_x) + (\sigma_x, \sigma_x) = (\sigma_x, \alpha_x).
\]

Thus, we see that \( \gamma = q^{-(\sigma_x, \alpha_x) \det(A)} \). Since the category of \( U_q(\mathfrak{g}) \)-modules is semisimple, the projection from \( \Phi(\Omega^{(1,0)}) \otimes M \) to \( \Phi(\Omega^{(M,0)}) \) splits, meaning that \( Z \) must act on \( \Phi(\Omega^{(M,0)}) \) as multiplication by \( q^{-(\sigma_x, \alpha_x) \det(A)} \).

The definition of \( E_k = \Psi(W_{k\sigma_x}) \) implies that \( Z \) acts on \( \Phi(E_k) \) as the scalar \( q^{k(\sigma_x, \sigma_x) \det(A)} \). Thus, it follows from (10) that \( k \) is uniquely determined by

\[
M(\sigma_x, \alpha_x) \det(A) = k(\sigma_x, \sigma_x) \det(A).
\]

Recalling the standard identity

\[
(\sigma_x, \sigma_x) = \frac{(\alpha_x, \alpha_x)}{2} (A^{-1})_{xx},
\]

where \( (A^{-1})_{xx} \) denotes the \( x \)-diagonal entry of the inverse of the Cartan matrix, we see

\[
k = \frac{M(\sigma_x, \alpha_x)}{(\sigma_x, \sigma_x)} = \frac{2M(\sigma_x, \alpha_x)}{(\alpha_x, \alpha_x)(A^{-1})_{xx}} = \frac{M(\sigma_x, \alpha_x)}{(A^{-1})_{xx}} = \frac{M}{(A^{-1})_{xx}}.
\]

(We note that since \( k \) is by definition an integer, the rational number \( M/(A^{-1})_{xx} \) is necessarily an integer, as can be confirmed by direct investigation. See Table 2 of Appendix C for explicit values.) It remains to show that \( k > 0 \). Indeed, each entry of the matrix \( (A^{-1})_{xx} \) is a positive rational number (see, for example, [44, Table 2] for an explicit presentation of the values). Thus, for every irreducible quantum flag manifold \( O_q(G/L_S) \), the scalar \( k \) must be an element of \( \mathbb{Z}_{>0} \). \( \square \)

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Appendix A: Some categorical equivalences

In this appendix, we present a number of categorical equivalences, all ultimately derived from Takeuchi’s equivalence [48]. These equivalences play a prominent role in the paper, giving us a formal framework in which to understand covariant differential calculi as relative Hopf modules.

A.1 Takeuchi’s equivalence

This subsection concisely presents Takeuchi’s equivalence for relative Hopf modules, as originally established in [48], and developed in [39]. We also present the monoidal version, as considered in [41, §4], and then restrict to the finitely generated subcategory of relative Hopf modules, as considered, for example, in [42, Corollary 2.5].

Let \( \pi : A \to H \) be a surjective Hopf algebra map between Hopf algebras \( A \) and \( H \). Then, a homogeneous right \( H \)-coaction is given by the map \( \Delta_R := (\text{id} \otimes \pi) \circ \Delta : A \to A \otimes H \). The associated quantum homogeneous space is defined to be the space of coinvariant elements, \( A^{\text{co}(H)} \), that is,

\[
A^{\text{co}(H)} := \{ a \in A \mid \Delta_R(a) = a \otimes 1 \}.
\]

For any quantum homogeneous space \( B = A^{\text{co}(H)} \), we define \( A^B \text{Mod}_B \) to be the category whose objects are left \( A \)-comodules \( F \to A \otimes F \), endowed with a \( B \)-bimodule structure such that \( \Delta_L(bf) = \Delta_L(b) \Delta_L(f) \Delta(c) \), for all \( f \in F \), \( b, c \in B \), and whose morphisms are left \( A \)-comodule, \( B \)-bimodule, maps.

Let \( H^B \text{Mod}_B \) denote the category of relative Hopf modules, that is, the category whose objects are left \( H \)-comodules \( V \to H \otimes V \), endowed with a right \( B \)-module structure, such that \( \Delta_L(vb) = v \pi(b(1)) \otimes v(0)b(2) \), for all \( v \in V \), \( b \in B \), and whose morphisms are left \( H \)-comodule maps and right \( B \)-module maps.

Consider the functor \( \Phi : A^B \text{Mod}_B \to H^B \text{Mod}_B \), given by \( \Phi(F) := F/B^+ \), where the left \( H \)-comodule structure of \( \Phi(F) \) is given by \( \Delta_L[f] := \pi(f_{(1)}) \otimes [f_{(0)}] \), with square brackets denoting the coset of an element in \( \Phi(F) \). In the other direction, we define a functor \( \Psi : H^B \text{Mod}_B \to A^B \text{Mod}_B \) by setting \( \Psi(V) := A \square_H V \), where the left \( A \)-comodule structures of \( \Psi(V) \) is defined on the first tensor factor, the right \( B \)-module structure is the diagonal one, and if \( \gamma \) is a morphism in \( H \text{mod}_B \), then \( \Psi(\gamma) := \text{id} \otimes \gamma \).

An adjoint equivalence of categories between \( A^B \text{Mod}_B \) and \( H^B \text{Mod}_B \), which we call Takeuchi’s equivalence, is given by the functors \( \Phi \) and \( \Psi \), and the unit natural isomorphism

\[
U : F \to \Psi \circ \Phi(F), \quad f \mapsto f_{(-1)} \otimes [f_{(0)}].
\]

The dimension \( \dim(F) \) of an object \( F \in A^B \text{Mod} \) is the vector space dimension of \( \Phi(F) \). As observed in [41, Corollary 2.7], the inverse of the unit \( U \) of the equivalence admits a useful explicit description:
\[ U^{-1}\left( \sum_i f_i \otimes [g_i] \right) = \sum_i f_i S((g_i)_{(-1)})(g_i)_{(0)}. \] (12)

Consider \( \mathcal{A}_B\text{Mod}_0 \) the full subcategory of \( \mathcal{A}_B\text{Mod}_B \) whose objects \( \mathcal{F} \) satisfy \( B^+ \mathcal{F} = \mathcal{F} B^+ \). The corresponding full subcategory \( \mathcal{H}_\mathcal{M}_0 \) of \( \mathcal{H}_\mathcal{M}_B \) is given by objects with the trivial right \( B \)-action. The category \( \mathcal{A}_B\text{Mod}_0 \) comes equipped with a monoidal structure given by the tensor product \( \otimes_B \). Moreover, with respect to the obvious monoidal structure on \( \mathcal{H}_\mathcal{M}_0 \), Takeuchi’s equivalence is readily endowed with the structure of a monoidal equivalence (see [41, §4]). An immediate implication is that an object \( \mathcal{E} \) is invertible (that is, it is a relative line module) if and only if \( \dim(\mathcal{E}) = 1 \).

Finally, we consider \( \mathcal{A}_B\text{mod}_0 \) the full subcategory of \( \mathcal{A}_B\text{Mod}_0 \) whose objects are finitely generated as left \( B \)-modules, and note that it is a monoidal subcategory of \( \mathcal{A}_B\text{Mod}_0 \). The corresponding full subcategory \( \mathcal{H}_\text{mod} \) of \( \mathcal{H}_\mathcal{M}_B \) has as objects the finite-dimensional left \( H \)-comodules, and its monoidal structure is the usual tensor product of comodules.

### A.2 Conjugates and duals

Let us now assume that \( A \) and \( H \) are Hopf ∗-algebras, and that \( \pi : A \rightarrow H \) is a Hopf ∗-algebra map. For any relative Hopf module \( \mathcal{F} \), let \( \overline{\mathcal{F}} \) be the relative Hopf module whose bimodule structure is defined by \( b\overline{f}c = c^* fb^* \), and whose left \( A \)-comodule structure is defined by \( \Delta_L(\overline{f}) := (f_{(-1)})^* \otimes f_{(0)} \). Note that as a right \( B \)-module, \( \overline{\mathcal{F}} \) is isomorphic to the conjugate of \( \mathcal{F} \) as defined in 2.5, justifying the choice of notation. As shown in [43, Corollary 2.11], if \( \mathcal{F} \in \mathcal{A}_B\text{Mod}_0 \), then \( \overline{\mathcal{F}} \in \mathcal{A}_B\text{Mod}_0 \). It is instructive to note that the corresponding operation, through Takeuchi’s equivalence, on any object in \( V \in \mathcal{H}_\mathcal{M}_0 \) is the usual complex conjugate of \( V \) coming from the Hopf ∗-algebra structure of \( H \).

We now restrict to the categories \( \mathcal{A}_B\text{mod}_0 \) and \( \mathcal{H}_\text{mod} \), and discuss dual objects. Since \( \mathcal{H}_\text{mod} \) is a rigid monoidal category, \( \mathcal{A}_B\text{mod}_0 \), or equivalently \( \mathcal{A}_B\text{mod}_0 \), is a rigid monoidal category. In particular, every object \( \mathcal{F} \in \mathcal{A}_B\text{mod}_0 \) admits a dual object, which we denote by \( ^\vee \mathcal{F} \). Moreover, as shown in [43, Appendix A], by extending its usual bimodule structure, we can give \( B\text{Hom}(\mathcal{F}, B) \) the structure of an object in \( \mathcal{A}_B\text{Mod}_B \), with respect to which it is right dual to \( \mathcal{F} \). Now \( \mathcal{A}_B\text{mod}_0 \) is a monoidal subcategory of \( \mathcal{A}_B\text{Mod}_B \). Thus, since right duals are unique up to unique isomorphism, \( ^\vee \mathcal{F} \) must be isomorphic to \( B\text{Hom}(\mathcal{F}, B) \), justifying the abuse of notation. Finally, we note that if \( H \) is a CQGA, then for any \( V \in \mathcal{H}_\text{mod}_0 \), its dual and conjugate are always isomorphic, see [34, Theorem 11.27] for details.

### Appendix B: A remark on the definition of the quantum Levi subgroup

In this subsection, we show the image of the restriction map \( \pi_S \) defined in §4.3 is the type-I dual of \( U_q(l_S) \), for any subset of simple nodes of a semisimple Lie algebra \( \mathfrak{g} \). In fact, this is taken as the definition of \( \mathcal{O}_q(L_S) \) in [18], and is the definition used in
[16, 25, 26]. This result is a basic exercise in representation theory, and can be directly concluded from [27, Lemma 2.1]. We include a proof for the reader’s convenience.

**Proposition B.1** Let $S$ be a subset of the simple roots $\Pi$ of $\mathfrak{g}$, then the Hopf $\ast$-algebra map $\pi_S : O_q(G) \rightarrow O_q(L_S)$ is surjective.

**Proof** Take an arbitrary finite-dimensional type-1 irreducible $U_q(\mathfrak{g})$-module $V$ and decompose it into irreducible type-1 $U_q(l_S)$-submodules:

$$V \simeq \bigoplus_{\mu \in \mathcal{P}^+ \cup \mathcal{P}_{Sc}} W_\mu.$$  

This gives a corresponding coordinate coalgebra decomposition

$$\pi_S(C(V_\lambda)) \simeq \bigoplus_{\mu \in \mathcal{P}^+ \cup \mathcal{P}_{Sc}} C(W_\mu).$$  

Thus, we see that the image of $\pi_S$ is contained in $O_q(L_S)$, the type-1 dual of $U_q(l_S)$.

To prove surjectivity, we need to show that for every weight $\nu \in \mathcal{P}^+ \cup \mathcal{P}_{Sc}$, the coordinate algebra $C(W_\nu)$ is contained in the image of $\pi_S$. By (13), this would follow from a demonstration that every $W_\nu$ appears as a $U_q(l_S)$-submodule of some $U_q(\mathfrak{g})$-module.

Every element of $\mathcal{P}^+ \cup \mathcal{P}_{Sc}$ is a sum of weights of the form $\lambda + \mu$, where

$$\lambda \in \mathcal{P}^+, \quad \text{and} \quad -\mu \in \mathcal{P}_{Sc} \cap \mathcal{P}^+.$$  

Choose a highest weight vector $v_{hw}$ in the irreducible $U_q(\mathfrak{g})$-module $V_\lambda$, and choose a lowest weight vector $v_{lw}$ in the irreducible $U_q(\mathfrak{g})$-module $V_{w_0(\mu)}$. We see that since

$$v_{hw} \otimes v_{lw} \in V_\lambda \otimes V_\mu$$

is a $U_q(l_S)$-highest weight vector, $U_q(l_S)v_{hw} \otimes v_{lw}$ is an irreducible $U_q(l_S)$-submodule of $V_\lambda \otimes V_{w_0(\mu)}$ of highest weight $\lambda + \mu$. Thus, we see that $C(W_{\lambda+\mu})$ is contained in the image of $\pi_S$, and hence that $\pi_S$ is surjective.  

**Appendix C: Tables for the irreducible quantum flag manifolds**

We recall the standard pictorial description of the quantum Levi subalgebras defining the irreducible quantum flag manifolds, given in terms of Dynkin diagrams.

For a diagram of rank $r$, to the black node $\alpha_x$ we associate the set $S := \{\alpha_1, \ldots, \alpha_r\} \setminus \{\alpha_x\}$, with corresponding Levi subgroup $L_S$. The irreducible quantum flag manifold is then given by the coinvariant subspace $O_q(G/L_S) \subseteq O_q(G)$. Note that any automorphism of a Dynkin diagram results in an isomorphic quantum flag manifold, which is not denoted in the diagram. In particular, for the case of $D_n$ and $E_6$, colouring the second spinor node, and the first node, respectively, produces an isomorphic copy of the corresponding quantum flag manifold.
We present an explicit description of the canonical line modules (see Table 2) of the irreducible quantum flag manifolds using the approach of Theorem 4.12. All line modules are indexed by the negative integers, and hence are negative in the sense of Definition 2.7. The values coincide with their classical counterparts, see, for example, [31, § II.4]. This allows us to conclude in Theorem 4.12 that the Kähler structure of each irreducible quantum flag manifold is of Fano type.

**Remark C.1** By a theorem of Atiyah, a $2m$-dimensional compact Hermitian manifold is spin if and only if its canonical line module $\Omega^{(m,0)}$ admits a holomorphic square root [2, Proposition 3.2]. Thus from Table 2, we see that the classical Grassmannians $\text{Gr}_{s,n+1}$, and the classical Lagrangian Grassmannians $\text{L}_n$, are spin for all $n \in 2\mathbb{Z}_{>0} + 1$.

### Table 1

| $A_n$ | $D_n$ | $E_6$ | $E_7$ | $O_q(\text{Gr}_{s,n+1})$ | $O_q(\text{Q}_{2n+1})$ | $O_q(\text{L}_n)$ | $O_q(\text{Q}_{2n})$ | $O_q(S_n)$ | $O_q(\mathbb{P}^2)$ | $O_q(F)$ |
|-------|-------|-------|-------|-------------------------|------------------------|-------------------|---------------------|--------------|-----------------|----------|
|       |       |       |       | quantum Grassmannian | odd quantum quadric    | quantum Lagrangian | even quantum quadric | quantum spinor variety | quantum Cayley plane | quantum Freudenthal variety |

### Table 2

| $O_q(G/L_S)$ | $M := \dim(\Omega^{(1,0)})$ | Canonical line module $\Omega^{(M,0)}$ |
|--------------|-----------------------------|---------------------------------------|
| $O_q(\text{Gr}_{s,n+1})$ | $s(n-s+1)$ | $\mathcal{E}_{-(n+1)}$ |
| $O_q(\text{Q}_{2n+1})$ | $2n - 1$ | $\mathcal{E}_{-2n+1}$ |
| $O_q(\text{L}_n)$ | $\frac{n(n+1)}{2}$ | $\mathcal{E}_{-(n+1)}$ |
| $O_q(\text{Q}_{2n})$ | $2(n - 1)$ | $\mathcal{E}_{-2(n-1)}$ |
| $O_q(S_n)$ | $\frac{n(n-1)}{2}$ | $\mathcal{E}_{-2(n-1)}$ |
| $O_q(\mathbb{P}^2)$ | 16 | $\mathcal{E}_{-12}$ |
| $O_q(F)$ | 27 | $\mathcal{E}_{-18}$ |
Moreover, the even quadrics $Q_{2n}$, and the spinor varieties $S_n$, are spin, for all $n \in \mathbb{Z}_{>0}$. For the exceptional cases, both the Cayley plane and the Freudenthal variety are spin. Atiyah’s theorem suggests a definition for noncommutative Hermitian spin structures with a substantial ready-made family of noncommutative examples.

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