Extensions of diffeomorphism and current algebras

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Abstract
Dzhumadil’daev has classified all tensor module extensions of $\text{diff}(N)$, the diffeomorphism algebra in $N$ dimensions, and its subalgebras of divergence free, Hamiltonian, and contact vector fields. I review his results using explicit tensor notation. All of his generic cocycles are limits of trivial cocycles, and many arise from the Mickelsson-Faddeev algebra for $\mathfrak{gl}(N)$. Then his results are extended to some non-tensor modules, including the higher-dimensional Virasoro algebras found by Eswara Rao/Moody and myself. Extensions of current algebras with $d$-dimensional representations are obtained by restriction from $\text{diff}(N + d)$. This gives a connection between higher-dimensional Virasoro and Kac-Moody cocycles, and between Mickelsson-Faddeev cocycles for diffeomorphism and current algebras.

1 Introduction

An extension $\hat{L}$ of a Lie algebra $L$ by a module $M$ is an exact sequence

$$0 \rightarrow M \xrightarrow{\iota} \hat{L} \xrightarrow{\pi} L \rightarrow 0.$$ 

This means that $\iota$ is injective, $\pi$ is surjective, and $M$ is an ideal in $\hat{L}$. It is precisely this situation which is of interest in physics, because if $L$ is a classical symmetry algebra (realized in terms of Poisson brackets), quantum corrections are of order $\hbar$ and thus generate an ideal. In particular, if $M = \mathbb{C}$ we say that the extension is central, which is the case that has attracted most attention in physics; suffice it to mention the ample applications of Virasoro and affine Kac-Moody algebras.
The best known non-central extension is the Mickelsson-Faddeev (MF) 
algebra \cite{8,13,14}, which is an abelian extension of the algebra $map(N,g)$ of 
maps from $N$-dimensional spacetime to a finite-dimensional Lie algebra $g$. 
$map(N,g)$ also admits higher-dimensional generalizations of the Kac-Moody 
cocycle \cite{11,16,21}, whose Fock representations were first constructed in 
\cite{6,21}. Similarly, the diffeomorphism algebra in $N$ dimensions, $diff(N)$, 
has non-central extensions analogous to the Virasoro algebra \cite{7,15,16}. The 
representation theory of these algebras was developed in \cite{1,2,7,17,18,19}. It 
appears that representations of the MF algebra, if they exist, are not 
attainable by similar methods \cite{20}.

In \cite{5}, Dzhumadil’daev classified all extensions of $L = diff(N)$ when $M$ 
is a tensor module. He also covered the cases when $L$ is one of the algebras 
of divergence free, Hamiltonian and contact vector fields. Unfortunately, 
his paper is not easy to read for a physicist (at least not for this one), so 
one purpose of the present paper is to review his classification in a more 
physicist-friendly manner, using notation from tensor calculus. Moreover, 
his results are quite bewildering, since he obtains no less than seventeen 
different cocycles. However, it turns out that all of them can be grouped 
into four classes:

1. A cubic (in derivatives) cocycle, which splits into its traceless and trace 
   parts.

2. Quartic cocycles, which follow from the MF extension for $gl(N)$ (recall 
   that tensor fields are functions with values in $gl(N)$ modules). In fact, 
   only the tensor part of the MF extension was used, which can be 
   removed by a redefinition à la Cederwall et al. \cite{3}.

3. Quintic cocycles which only exist in two dimensions.

4. Special cases in one dimension.

Dzhumadil’daev also considered cocycles for the $diff(N)$ subalgebras $svect(N)$ 
(divergence-free vector fields), $Ham(N)$ (Hamiltonian vector fields) and 
$K(N)$ (contact vector fields). All such cocycles follow directly by restriction 
from the full diffeomorphism algebra.

I extend Dzhumadil’daev’s result by constructing some extensions where $M$ 
is not a tensor module. These cases include the higher-dimensional Virasoro 
algebras of Eswara Rao and Moody \cite{7} and myself \cite{16}. Another 
generalization is found by considering the inhomogenous term in the MF 
extension.
It turns out that all of Dzhumadil’daev’s generic cocycles and several of his low-dimensional ones can be obtained as limits of trivial cocycles. One constructs a family of trivial cocycles, parametrized by a continuous parameter (the conformal weight $\lambda$), and let $\lambda$ approach a critical value $\lambda_0$. Thus, these cocycles are non-trivial in the usual cohomological sense, but belong to the closure of the space of trivial cocycles. On the other hand, the Virasoro and MF extensions do not belong to this closure, because there is no continuous parameter that can be varied. Geometrically, they involve closed chains (one- and three-, respectively), and the closedness condition is consistent for $\lambda = 1$ only.

Dzhumadil’daev [4] and Ovsienko and Roger [22] have also classified cocycles for the special case $N = 1$. I show that some of these have higher-dimensional analogues not previously considered, although this generalization is quite unnatural and uninteresting. Moreover, in one dimension these cocycles, possibly with one exception, are limits of trivial cocycles.

Dzhumadil’daev’s classification can be used to construct interesting extensions of subalgebras of the diffeomorphism algebra. To this end, I consider the inclusion $map(N, diff(d)) \subset diff(N + d)$, and further $g \subset gl(d) \subset diff(d)$. In fact, the algebra $map(N, diff(d))$ defines an interesting generalization of gauge symmetry: the replacement of a global symmetry $g$ by a gauge symmetry $map(N, g)$ amounts to a localization in base space, but the gauge transformations are still rigid in target space. Replacing $g$ by $diff(d)$ makes transformations local in target space as well. By studying the restriction of the extensions under the above inclusions, existence of extensions for subalgebras is shown, but neither non-triviality nor exhaustion. However, non-triviality can be checked by hand, and my method tautologically exhaust all extensions that can be lifted to tensor module extensions of the algebra of diffeomorphisms in total space.

2 Background

2.1 Diffeomorphism algebra

Let $\xi = \xi^\mu(x)\partial_\mu$, $x \in \mathbb{R}^N$, $\partial_\mu = \partial/\partial x^\mu$, be a vector field, with commutator $[\xi, \eta] = \xi^\mu \partial_\mu \eta^\nu \partial_\nu - \eta^\nu \partial_\nu \xi^\mu \partial_\mu$. Greek indices $\mu, \nu = 1, 2, \ldots, N$ label the spacetime coordinates and the summation convention is used on all kinds of indices. The diffeomorphism algebra $diff(N)$ is generated by Lie derivatives $\mathcal{L}_\xi$. Dzhumadil’daev denotes this algebra by $W_N$ in honour of Witt. In the literature, it is also known as the algebra of vector fields and is denoted by $Vect(N)$ or $Vect(\mathbb{R}^N)$. 
An extension of $\text{diff}(N)$ is given by a bilinear cocycle $c(\xi, \eta)$:

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]} + c(\xi, \eta).$$  \hspace{1cm} (2.1)

For convenience, some formulas are also displayed in a Fourier basis. With $L_\mu(m) = L_\xi$ for $\xi = i \exp(\text{im}_p x^\rho) \partial_\mu, m \in \mathbb{R}^N$, (2.1) is replaced by

$$[L_{\mu}(m), L_{\nu}(n)] = n_\mu L_{\mu}(m + n) - m_\nu L_{\nu}(m + n) + c_{\mu\nu}(m, n).$$  \hspace{1cm} (2.2)

We say that an extension is local if it has the form

$$c_{\mu\nu}(m, n) = \text{pol}_{\mu\nu}^a(m, n) A_a(m + n),$$  \hspace{1cm} (2.3)

where $\text{pol}_{\mu\nu}^a(m, n) = -\text{pol}_{\nu\mu}^a(n, m)$ is a polynomial and $A_a$ is some operator.

Let $T^\mu_\nu$ form a basis for $\mathfrak{gl}(N)$, with brackets

$$[T^\mu_\nu, T^\sigma_\tau] = \delta^\sigma_\nu T^\mu_\tau - \delta^\mu_\tau T^\nu_\sigma.$$  \hspace{1cm} (2.4)

Then

$$\mathcal{L}_\xi = \xi^\mu(x) \partial_\mu + \partial_\nu \xi^\mu(x) T^{\nu}_{\mu}$$  \hspace{1cm} (2.5)

satisfies (2.1) with zero cocycle. Analogously, if $T^\mu_\nu(m)$ form a basis for an extension of $\text{map}(N, \mathfrak{gl}(N))$, with brackets

$$[T^\mu_\nu(m), T^\sigma_\tau(n)] = \delta^\sigma_\nu T^\mu_\tau(m + n) - \delta^\mu_\tau T^\nu_\sigma(m + n) + k_{\mu\tau}^{\nu\sigma}(m, n),$$  \hspace{1cm} (2.6)

then

$$L'_\mu(m) = L_\mu(m) + m_\nu T^{\nu}_{\mu}(m)$$  \hspace{1cm} (2.7)

satisfies an extension of $\text{diff}(N)$ with cocycle

$$c_{\mu\nu}(m, n) = m_\sigma n_\tau k_{\mu\nu}^{\sigma\tau}(m, n).$$  \hspace{1cm} (2.8)

A tensor module is the carrying space of the $\text{diff}(N)$ representation obtained by substituting a $\mathfrak{gl}(N)$ representation into (2.5). A tensor of type $(p, q; \lambda)$ ($p$ contravariant and $q$ covariant indices and conformal weight $\lambda$) is
described by the equivalent formulas
\[
\left[ \mathcal{L}_x, \Phi^{\sigma_1, \sigma_p}_{\tau_1, \tau_q}(x) \right] = -\xi^\mu(x) \partial_\mu \Phi^{\sigma_1, \sigma_p}_{\tau_1, \tau_q}(x) - \lambda \partial_\mu \xi^\mu(x) \Phi^{\sigma_1, \sigma_p}_{\tau_1, \tau_q}(x) - \\
+ \sum_{i=1}^p \partial_\mu \xi_i^{\sigma_1}(x) \Phi^{\sigma_1, \mu, \sigma_p}_{\tau_1, \tau_q}(x) - \sum_{j=1}^q \partial_\mu \xi^\mu(x) \Phi^{\sigma_1, \mu, \sigma_p}_{\tau_1, \tau_q}(x),
\]
and
\[
\left[ L_\mu(m), \Phi^{\sigma_1, \sigma_p}_{\tau_1, \tau_q}(n) \right] = (n_\mu + (1 - \lambda)m_\mu) \Phi^{\sigma_1, \sigma_p}_{\tau_1, \tau_q}(m + n) + \\
+ \sum_{i=1}^p \delta^\mu_\nu m_\rho \Phi^{\sigma_1, \mu, \sigma_p}_{\tau_1, \tau_q}(m + n) - \sum_{j=1}^q m_\rho \Phi^{\sigma_1, \mu, \sigma_p}_{\tau_1, \tau_q}(m + n),
\]
where \( \phi^{\sigma_1, \tau_q}_{\tau_1, \sigma_p} \) is an arbitrary function on \( \mathbb{R}^N \) and
\[
\Phi^{\sigma_1, \sigma_p}_{\tau_1, \tau_q}(\phi^{\sigma_1, \tau_q}_{\tau_1, \sigma_p}) = \int d^N x \phi^{\sigma_1, \tau_q}_{\tau_1, \sigma_p}(x) \Phi^{\sigma_1, \sigma_p}_{\tau_1, \tau_q}(x).
\]

For brevity, we shall often write the rhs of (2.9) simply as \( (p, q; \lambda) \). Further, we abbreviate the action on objects which contain additional terms as
\[
[L_\mu(m), \Phi^{\sigma_1, \sigma_p}_{\tau_1, \tau_q}(n)] = (p, q; \lambda) + \text{more},
\]
etc. Tensor modules contain irreducible submodules consisting of symmetric, anti-symmetric, and traceless tensors, labelled by the irreps of \( gl(N) \). As is standard in physics, (anti-)symmetrization of indices is denoted by parentheses (brackets), and vertical bars inhibit the operation. Thus, \( \phi^{(\mu|\nu|\rho)} = \phi_{\mu\nu\rho} + \phi_{\nu\mu\rho} \) and \( \phi_{[\mu|\nu]} = \phi_{\mu|\nu} - \phi_{\nu|\mu} \).

However, not all modules are tensor modules. The following cases will be considered below:

1. A totally skew tensor \( \omega_{\sigma_1, \sigma_p} = \omega_{[\sigma_1, \sigma_p]} \) of type \( (0, p; 0) \), i.e. a \( p \)-form, contains a submodule consisting of closed \( p \)-forms, which amounts to the conditions
\[
\partial_\nu \omega_{\sigma_1, \sigma_p}(x) \equiv 0, \quad \omega_{\sigma_1, \sigma_p}(\partial_\nu \phi^{[\nu|\sigma_1, \sigma_p]}) \equiv 0, \quad m_\nu \omega_{\sigma_1, \sigma_p}(m) \equiv 0 \quad (2.12)
\]

2. Dually, a totally skew tensor \( S_{\sigma_1, \sigma_p} = S^{[\sigma_1, \sigma_p]} \) of type \( (p, 0; 1) \), can be identified as a \( p \)-chain. Closed \( p \)-chains satisfy the conditions
\[
\partial_{\sigma_1} S^{\sigma_1, \sigma_p}(x) \equiv 0, \quad S^{\sigma_1, \sigma_p}(\partial_{\sigma_1} \phi_{\sigma_2, \sigma_p}) \equiv 0, \quad m_{\sigma_1} S^{\sigma_1, \sigma_p}(m) \equiv 0 \quad (2.13)
\]
3. The connection $\Gamma^\rho_{\sigma\tau}$ transforms as a tensor field of type $(1, 2; 0)$, apart from an additional term

$$
[L_\xi, \Gamma^\rho_{\sigma\tau}(x)] = (1, 2; 0) + \partial_\sigma \partial_\tau \xi^\rho,
$$

$$
[L_\xi, \Gamma^\rho_{\sigma\tau}(\phi^\sigma_{\rho})] = (1, 2; 0) + \int d^N x \partial_\sigma \partial_\tau \xi^\rho(x) \phi^\sigma_{\rho}(x),
$$

$$
[L_\mu(m), \Gamma^\rho_{\sigma\tau}(n)] = (1, 2; 0) + m_\sigma m_\tau \delta^\rho_{\mu}(m + n),
$$

where $(1, 2; 0)$ denote regular terms as in (2.11).

### 2.2 Divergence-free vector fields

The algebra of divergence-free (or special) vector fields $svect(N) \subset diff(N)$ is generated by $L_\xi$ such that $\partial_\mu \xi^\mu = 0$, or equivalently $L_\mu(m)$ such that $m_\mu = 0$. Tensor modules are given by (2.9), except that the conformal weight $\lambda$ is irrelevant.

### 2.3 Hamiltonian vector fields

The algebra $Ham(N) \subset diff(N)$ ($N$ even) consists of vector fields $\xi$ that leave the two-form $\epsilon_{\mu\nu} dx^\mu \wedge dx^\nu$ invariant, where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ is the symplectic form, whose inverse $\epsilon^{\mu\nu}$ satisfies $\epsilon^{\mu\nu} \epsilon_{\nu\rho} = \delta^\mu_{\rho}$ and $\epsilon^{\mu\nu} \epsilon_{\nu\rho} = \delta^\mu_{\rho}$. Such Hamiltonian vector fields are of the form $\xi = \epsilon^{\mu\nu} \partial_\mu f \partial_\nu g$. Dzhumadil’daev denotes $Ham(N)$ by $H_n$, where $N = 2n$. Any extension of $Ham(N)$ takes the form

$$
[H_f, H_g] = H_{\{f, g\}} + c_H(f, g),
$$

where $\{f, g\} = \epsilon^{\mu\nu} \partial_\mu f \partial_\nu g$ is the Poisson bracket. Alternatively, in the Fourier basis

$$
[H(m), H(n)] = (m \times n) H(m + n) + c_H(m, n),
$$

where $m \times n \equiv \epsilon^{\mu\nu} m_\mu n_\nu$. Since $\epsilon^{\mu\nu} \partial_\mu \partial_\nu f \equiv 0$, any Hamiltonian vector field is divergence free. The converse is also true in two dimensions, but when $N \geq 4$ there are divergence-free vector fields that are not Hamiltonian. One checks that $[H_f, \epsilon^{\mu\nu}] \equiv 0$, so $\epsilon^{\mu\nu}$ and $\epsilon_{\mu\nu}$ can be used to raise and lower indices. A typical tensor module is hence of the form

$$
[H_f, \Phi^\sigma(\phi_\sigma)] = \Phi^\sigma(\{f, \phi_\sigma\} + \epsilon^{\mu\nu} \partial_\mu \partial_\nu f \phi_\nu)
$$

$$
[H(m), \Phi^\sigma(n)] = n_\mu \Phi^\sigma(m + n) + \epsilon^{\mu\sigma} m_\mu m_\nu \Phi^\nu(m + n).
$$
2.4 Contact vector fields

The contact algebra $K(N) \subset \text{diff}(N)$ ($N$ odd) consists of vector fields which leave the one-form $\alpha = dx^0 + \epsilon_{ij} x^i dx^j$ invariant, where $\epsilon_{ij}$ is the symplectic form in one dimension less. Here greek indices $\mu, \nu = 0, 1, 2, ..., N$ run over all $N + 1$ indices but latin indices $i, j = 1, 2, ..., N$ exclude the time index 0. Contact vector fields are of the form

$$K_f = \Delta f \partial_0 + \partial_0 f x^i \partial_i + \epsilon^{ij} \partial_j f \partial_i,$$

(2.18)

where $\Delta f = 2 f - x^i \partial_i f$. Any extension of $K(N + 1)$ has the form

$$[K_f, K_g] = K_{[f,g]_K} + c_K(f, g),$$

(2.19)

where the contact bracket reads

$$[f, g]_K = \partial_0 f \Delta g - \partial_0 g \Delta f + \{f, g\}$$

(2.20)

and $\{f, g\} = \epsilon^{ij} \partial_i f \partial_j g$ is the Poisson bracket in $N$ dimensions. Due to the explicit appearance of $x^i$ in the definition of $\Delta$, it is inconvenient to describe this algebra in a Fourier basis. Dzhumadil’daev denotes $K(N)$ by $K_n$, where $N = 2n + 1$.

2.5 Gauge algebra

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with basis $J^a$ (hermitian if $\mathfrak{g}$ is compact and semisimple), structure constants $f^{ab}_c$ and brackets $[J^a, J^b] = i f^{ab}_c J^c$. Our notation is similar to [10] or [9], chapter 13. We always assume that $\mathfrak{g}$ has a Killing metric proportional to $\delta^{ab}$. Further assume that there is a privileged vector $\delta^a \propto \text{tr} J^a$, such that $f^{ab}_c \delta^c \equiv 0$. Of course, $\delta^a = 0$ if $\mathfrak{g}$ is semisimple, but it may be non-zero if $\mathfrak{g}$ contains abelian factors. The primary example is $\mathfrak{g} = \mathfrak{gl}(N)$, where $\text{tr} (T^\mu_\nu) \propto \delta^\mu_\nu$.

Let $\text{map}(N, \mathfrak{g})$ be the algebra of maps from $\mathbb{R}^N$ to $\mathfrak{g}$, also known as the gauge or current algebra. We denote its generators by $J_X$, where $X = X_a(x) J^a, x \in \mathbb{R}^N$, is a $\mathfrak{g}$-valued function, and define $[X, Y] = i f^{ab}_c X_a Y_b J^c$. Alternatively, we use a Fourier basis with generators $J^a(m), m \in \mathbb{R}^N$. Any extension of $\text{map}(N, \mathfrak{g})$ has the form

$$[J_X, J_Y] = J_{[X,Y]} + c(X,Y),$$

$$[\mathcal{L}_\xi, J_X] = J_{\xi^a \partial_a X} + c(\xi, X),$$

$$[J^a(m), J^b(n)] = i f^{ab}_c J^c(m + n) + c^{ab}(m, n),$$

$$[L_\mu(m), J^a(n)] = n_{\mu a} J^a(m + n) + c^a_\mu(m, n).$$

(2.21)
weights, we can associate a partition \( \{ \ell , \delta , Dzhumadil'daev \} \) classified extensions of \( Dzhumadil'daev \). The analogue of tensor modules are functions with values in some \( \mathfrak{g} \) module \( M \). To the three formulas in (2.16) correspond

\[
\begin{align*}
[\mathcal{J}_X, \Phi^i(x)] & = -X_a \sigma^{ia}_j \Phi^j(x), \\
[\mathcal{J}_X, \Phi^i(\phi_i)] & = -\Phi^i(X_a \sigma^{ia}_j \phi_j), \\
[\mathcal{J}^a(m), \Phi^i(n)] & = -\sigma^{ia}_j \Phi^j(m + n),
\end{align*}
\]

(2.22)

where \( \sigma^a = (\sigma^{ia}_j) \) are the representation matrices acting on \( M \). Moreover, the intertwining action of diffeomorphisms is given by (2.9).

Among non-tensor modules we cite the connection, which is a central extension of the adjoint.

\[
\begin{align*}
[\mathcal{J}_X, A^b(x)] & = if^{ab}_c X_a(x) A^c_c(x) + \delta^{ab}_c \partial_c X_a(x), \\
[\mathcal{J}_X, A^b(\phi_i^\nu)] & = A^b_c (-if^{ac}_b X_a(\phi_i^\nu) + \int d^N x \delta^{ab}_c \partial_c X_a(\phi_i^\nu)(x), \\
[\mathcal{J}^a(m), A^b(n)] & = if^{ab}_c A^c_c(m + n) + n \delta^{ab}_c.
\end{align*}
\]

(2.23)

The best known extension of \( \text{map}(N, \mathfrak{g}), N \geq 3 \), is the MF extension:

\[
\begin{align*}
[\mathcal{J}^a(m), \mathcal{J}^b(n)] & = if^{ab}_c \mathcal{J}^c(m + n) + m_n \mathcal{H}^{ab\mu\nu}(m + n), \\
[\mathcal{J}^a(m), \mathcal{H}^{bc\mu\nu}(n)] & = if^{ab}_d \mathcal{H}^{dc\mu\nu}(m + n) + if^{ac}_d \mathcal{H}^{bd\mu\nu}(m + n) + d^{abc} m \mathcal{S}^{\mu\nu}(m + n),
\end{align*}
\]

(2.24)

and all other brackets vanish. Here, \( S^{\mu\nu}_3 \) is a closed three-chain (2.13) and \( d^{abc} = \text{tr} \mathcal{J}^a J^b J^c \) are totally symmetric. In particular, in three dimensions we can write \( \mathcal{H}^{ab\mu\nu}(m) = \delta^{\mu\nu} \mathcal{A}^c_c(m), S^{\mu\nu}_3(m) = \delta^{\mu\nu} \delta(m), \) so \( \mathcal{A}^c_c(m) \) transforms as a connection. It was found in [21] that the three-chain term constitutes an obstruction against the construction of Fock modules.

\section{Dzhumadil’daev’s classification: \textit{diff}(N) cocycles}

Dzhumadil’daev classified extensions of \textit{diff}(N) by tensor modules [3], and found 17 inequivalent ones. A tensor density can be viewed as a function with values in a \( \mathfrak{gl}(N) \) module. Since \( \mathfrak{gl}(N) \sim \mathfrak{sl}(N) \oplus \mathfrak{gl}(1) \), \( \mathfrak{gl}(N) \) irreps are labelled by an \( \mathfrak{sl}(N) \) highest weight and a conformal weight \( \lambda \). To an \( \mathfrak{sl}(N) \) highest weight \( \ell_1 \pi_1 + \ell_2 \pi_2 + \ldots + \ell_{N-1} \pi_{N-1}, \pi_i \) being the fundamental weights, we can associate a partition \( \{ \lambda_1, \lambda_2, \ldots, \lambda_{N-1} \} \), where \( \lambda_i = \ell_i + \ell_{i+1} + \ldots + \ell_{N+1} = \sum_{j=i}^{N-1} \ell_j \). This can be visualized as a Young tableaux
with $\lambda_i$ boxes in the $i$:th row. A contravariant vector corresponds to a tableaux with a single box, i.e. to the root $\pi_1$. A covariant vector can be identified with a tableaux with a single column with $N - 1$ boxes, i.e. the root $\pi_{N-1}$.

Let $\epsilon_{\mu_1..\mu_N}$ be the totally antisymmetric symbol, given by $\epsilon_{\mu_1..\mu_N} = +1$ if $\mu_1..\mu_N$ is a even permutation of $12..N$, $= -1$ if it is an odd permutation, and $= 0$ if two indices are equal. It can be regarded as a constant tensor field of type $(0, N; -1)$, since this makes the transformation law $[\mathcal{L}_\xi, \epsilon_{\mu_1..\mu_N}(x)] = 0$ consistent. Alternatively, we may view it as a constant tensor field $\epsilon_{\mu_1..\mu_N}$ of type $(N, 0; 1)$. The existence of this symbol establishes the standard isomorphism between $p$ lower indices and $N - p$ upper indices, e.g. $p$-forms and $(N-p)$-chains:

$$A_{[\mu_1..\mu_p]} = \epsilon_{\mu_1..\mu_p\nu_1..\nu_{N-p}}A^{[\nu_1..\nu_{N-p}]}.$$  \hspace{1cm} (3.1)

In particular, a covariant vector field (of weight $\lambda$) is equivalent to a skew tensor field with $N - 1$ contravariant indices and weight $\lambda + 1$.

Dzhumadil’daev’s classification is encoded in \cite{5}, Table 1. The following three tables describe the corresponding modules, both in his notation and tensor calculus notation.

1. For every $N \geq N_0$:

| $N_0$ | HW   | $\lambda$ | Type | Tensor                  |
|-------|------|-----------|------|------------------------|
| $\psi_1^W$ | 1    | $\pi_1$  | 1    | $(1,0;1)$ $S^\rho$    |
| $\psi_2^W$ | 2    | $2\pi_1 + \pi_{N-1}$ | 2    | $(2,1;1)$ $K_{\rho\sigma}$ |
| $\psi_{3,4}^W$ | 2   | $\pi_2$  | 1    | $(2,0;1)$ $F[^{\mu
u}]$ |
| $\psi_{5,6}^W$ | 3    | $\pi_1 + \pi_2 + \pi_{N-1}$ | 2    | $(3,1;1)$ $E_{\mu}^{\rho(\sigma\tau)} : E_{\mu}^{(\rho\sigma\tau)} = 0$ |
| $\psi_{7}^W$ | 2    | $3\pi_1 + \pi_{N-1}$ | 2    | $(3,1;1)$ $D_{\mu}^{(\rho\sigma\tau)}$ |
| $\psi_{8}^W$ | 3    | $2\pi_1 + \pi_2 + 2\pi_{N-1}$ | 3    | $(4,2;1)$ $B_{[^{\mu\nu}]}^{(\lambda\rho)(\sigma\tau)}$ |
| $\psi_{9}^W$ | 3    | $4\pi_1 + \pi_{N-2}$ | 2    | $(4,2;1)$ $A_{[^{\mu\nu}]}^{(\lambda\rho\sigma\tau)}$ |
| $\psi_{10}^W$ | 4 | $2\pi_2 + \pi_{N-2}$ | 2    | $(4,2;1)$ $C_{[^{\mu\nu}]}((\lambda\rho)(\sigma\tau)) : C_{[^{\mu\nu}]}^{(\lambda\rho\sigma\tau)} = 0$ |
The symmetry conditions can alternatively be written as

\[ A^\lambda_{\mu\nu} = A^\rho_{\mu\nu} = A^\tau_{\mu\nu} = -A^\lambda_{\mu\nu}, \]
\[ B^\lambda_{\mu\nu} = B^\rho_{\mu\nu} = -B^\tau_{\mu\nu} = B^\lambda_{\mu\nu}, \]
\[ C^\lambda_{\mu\nu} = C^\rho_{\mu\nu} = -C^\tau_{\mu\nu} = C^\lambda_{\mu\nu}, \]
\[ D^\rho_{\sigma\tau} = D^\sigma_{\rho\tau} = D^\rho_{\sigma\tau} = -D^\rho_{\sigma\tau}, \]
\[ E^\rho_{\sigma\tau} = E^\sigma_{\rho\tau} = -E^\sigma_{\rho\tau}, \]
\[ F^\mu_{\nu} = -F^\nu_{\mu}, \]
\[ K^\sigma_{\mu} = K^\mu_{\sigma}, \]

in addition to total tracelessness.

2. In addition for \( N = 2 \):

| Name \( \psi \) | HW \( \lambda \) | Partition | Type | Tensor |
|-----------------|-----------------|-----------|------|--------|
| \( \psi_1^W \), \( \psi_2^W \) | \( \pi_1 \) | 0 \{1\} | (1, 0; 0) | \( S^\rho \) |
| \( \psi_3^W \) | 5\( \pi_1 \) | 2 \{5\} | (5, 0; 2) | \( S^{(\rho_1\ldots\rho_5)} \) (3.4) |
| \( \psi_4^W \) | 7\( \pi_1 \) | 3 \{7\} | (7, 0; 3) | \( S^{(\rho_1\ldots\rho_7)} \) |

3. In addition for \( N = 1 \), there are three cocycles with \( \lambda = -1, -4, \) and \(-6\), respectively.

Of course, knowledge of the relevant module is not enough to uniquely describe the cocycle. Dzhumadil’daev has also given formulas for the cocycles, but it is in fact quite easy to reconstruct them from scratch. By writing down manifestly non-trivial cocycles valued in the right modules, we are guaranteed to obtain expressions that are equivalent to Dzhumadil’daev’s, without having to decipher his notation. This is the subject of the rest of this section.

3.1 \( \psi_1^W \)

\( \psi_1^W \) corresponds to the partition \{1, 0, \ldots, 0\}, i.e. a tensor density \( S^\rho \) of type \((1, 0; 1)\). This extension was first described in [15]:

\[ c(\xi, \eta) = S^\rho (\partial_\rho \partial_\mu \xi^\mu \partial_\nu \eta^\nu - \partial_\rho \xi^\mu \partial_\mu \partial_\nu \eta^\nu), \]
\[ c_\mu(\nu, m, n) = m_\mu n_\nu (m_\rho - n_\rho) S^\rho (m + n). \]

Such a tensor can be identified with a one-chain, which is reducible according to (2.13). It was noted in [16] that (3.5) still defines a cocycle when restricted to the submodule of closed one-chains. In one dimension, \( \psi_1^W \) is related to
the Virasoro algebra. We have
\[ [L_m, L_n] = (n - m)L_{m+n} + (n - m)mnS_{m+n}, \]
\[ [L_m, S_n] = (n + m)S_{m+n}. \]  
(3.6)

The closedness condition, \( mS_m = 0 \), has the unique solution \( S_m = c/24\delta_m \). Substituting this into (3.6) yields the Virasoro algebra with central charge \( c \).

Let \( X \) be a function on \( \mathbb{R}^N \) and \( E_X \) a tensor density of type \((0,0;1)\). If
\[ [L_\xi, E_X] = E_\xi X, \quad [E_X, E_Y] = S^\rho(X\partial_\rho Y - \partial_\rho XY), \]  
(3.7)
then \( L'_\xi = L_\xi + E_{\partial_\mu \xi^\mu} \) satisfies \( \text{diff}(N) \) with cocycle \( \psi_1^W \).

3.2 \( \psi_2^W \)

\( \psi_2^W \) corresponds to the partition \( \{3,1,\ldots,1\} \), i.e. a traceless tensor density \( K^{(\sigma\tau)}_{\mu} \) of type \((2,1;1)\):
\[ c(\xi,\eta) = K^{(\sigma\tau)}_{(\xi\eta)}(\partial_\mu \xi^\mu \partial_\sigma \eta^\nu - K^{(\sigma\tau)}_{\mu}(\partial_\sigma \xi^\mu \partial_\tau \eta^\nu)), \]
\[ c_{\mu\nu}(m,n) = m_\mu n_\sigma n_\tau K^{(\sigma\tau)}_{\nu}(m+n) - n_\rho m_\mu m_\tau K^{(\sigma\tau)}_{\nu}(m+n). \]  
(3.8)

Change the weight to some \( \lambda \neq 1 \), i.e. \( K^{(\sigma\tau)}_{\nu} \) is of type \((2,1;\lambda)\), and redefine the \( \text{diff}(N) \) generators by
\[ L_\mu(m) \mapsto L_\mu(m) + am_\sigma m_\tau K^{(\sigma\tau)}_{\nu}(m). \]  
(3.9)

The new generators satisfy \( \text{diff}(N) \) with the extension \( a(1 - \lambda)c_{\mu\nu}(m,n) \), which thus is trivial. If we now fix \( a = (1 - \lambda)^{-1} \) and let \( \lambda \to 1 \), \( \psi_2^W \) is recovered. Tracelessness does not play a role here; setting \( K^{(\sigma\tau)}_{\mu} = \delta^{(\sigma\tau)}_{\mu} \), we see that the trace is of type \( \psi_1^W \).

3.3 \( \psi_3^W - \psi_{10}^W \)

These eight extensions all follow from the following reducible extension
\[ c(\xi,\eta) = R^{(\sigma\tau)}_{(\lambda\rho)}(\partial_\chi \partial_\rho \xi^\mu \partial_\sigma \eta^\nu), \]
\[ c_{\mu\nu}(m,n) = m_\lambda n_\rho n_\sigma n_\tau R^{(\lambda\rho)}_{(\sigma\tau)}(m+n), \]  
(3.10)

where \( R^{(\lambda\rho)}_{(\sigma\tau)} \) is a tensor of type \((4,2;1)\), and
\[ R^{(\sigma\tau)}_{(\lambda\rho)} = -R^{(\lambda\rho)}_{(\sigma\tau)}. \]  
(3.11)
Such a tensor can be decomposed into irreducible submodules as follows.

\[ R^{(\lambda \rho)(\sigma \tau)}_{\mu \nu} = A^{(\lambda \rho \sigma \tau)}_{[\mu \nu]} + B^{(\mu \nu)(\sigma \tau)}_{(\lambda \rho)} + C^{(\lambda \rho)(\sigma \tau)}_{[\mu \nu]} + \delta^{(\lambda \rho)}_{\mu} G^{(\sigma \tau)}_{\nu} - \delta^{(\sigma \tau)}_{\nu} G^{(\lambda \rho)}_{\mu} + \delta^{(\lambda \rho)}_{\nu} G^{(\sigma \tau)}_{\mu} - \delta^{(\sigma \tau)}_{\mu} G^{(\lambda \rho)}_{\nu}, \]  \hspace{1cm} (3.12)

where

\[ G^{\rho(\sigma \tau)}_{\nu} = D^{\rho(\sigma \tau)}_{\nu} + E^{\rho(\sigma \tau)}_{\nu} + \delta^{(\sigma)}_{\nu} F^{\tau \rho} + \delta^{(\sigma)}_{\nu} H^{\tau \rho} + \delta^{\rho}_{\nu} H^{\sigma \tau}, \]

\[ C^{(\lambda \rho \sigma \tau)}_{[\mu \nu]} \equiv 0, \quad E^{(\rho \sigma \tau)}_{\nu} \equiv 0, \]  \hspace{1cm} (3.13)

\[ F^{\rho \sigma} = F^{[\rho \sigma]}, \quad H^{\rho \sigma} = H^{(\rho \sigma)}. \]
Substitution of $\text{(3.12)}$–$\text{(3.13)}$ into $\text{(3.11)}$ yields

**Cocycle** \[ c(\xi, \eta) \]

**Partition** \[ c_{\mu \nu}(m, n) \]

\[ \psi^W_3 \{1,1,0,\ldots,0\} \]
\[ F^{[\rho \tau]}(\partial_\rho \partial_\mu \xi^\mu \partial_\sigma \eta^\nu) \]
\[ m_\rho m_\mu n_\tau n_\nu F^{[\rho \tau]}(m + n) \]

\[ \psi^W_4 \{1,1,0,\ldots,0\} \]
\[ F^{[\rho \tau]}(\partial_\rho \partial_\nu \xi^\mu \partial_\sigma \eta^\nu) \]
\[ m_\rho m_\mu n_\tau n_\nu F^{[\rho \tau]}(m + n) \]

\[ \psi^W_5 \{3,2,1,\ldots,1\} \]
\[ E^{\rho(\sigma \tau)}(\partial_\rho \partial_\mu \xi^\mu \partial_\sigma \eta^\nu) - \xi \leftrightarrow \eta \]
\[ m_\rho m_\mu n_\sigma n_\tau E^{\rho(\sigma \tau)}(m + n) - m \leftrightarrow n \]

\[ \psi^W_6 \{3,2,1,\ldots,1\} \]
\[ E^{\rho(\sigma \tau)}(\partial_\rho \partial_\nu \xi^\mu \partial_\sigma \eta^\nu) - \xi \leftrightarrow \eta \]
\[ m_\rho m_\mu n_\sigma n_\tau E^{\rho(\sigma \tau)}(m + n) - m \leftrightarrow n \]

\[ \psi^W_7 \{4,1,\ldots,1\} \]
\[ D^{(\rho \sigma \tau)}(\partial_\rho \partial_\mu \xi^\mu \partial_\sigma \eta^\nu) - \xi \leftrightarrow \eta \]
\[ m_\rho m_\mu n_\sigma n_\tau D^{(\rho \sigma \tau)}(m + n) - m \leftrightarrow n \]

\[ \psi^W_8 \{5,3,2,\ldots,2\} \]
\[ B^{(\lambda \rho)(\sigma \tau)}(\partial_\lambda \partial_\mu \xi^\mu \partial_\sigma \partial_\tau \eta^\nu) \]
\[ m_\lambda m_\rho n_\sigma n_\tau B^{(\lambda \rho)(\sigma \tau)}(m + n) \]

\[ \psi^W_9 \{5,1,\ldots,1,0\} \]
\[ A^{(\lambda \rho \sigma \tau)}(\partial_\lambda \partial_\mu \xi^\mu \partial_\sigma \partial_\tau \eta^\nu) \]
\[ m_\lambda m_\rho n_\sigma n_\tau A^{(\lambda \rho \sigma \tau)}(m + n) \]

\[ \psi^W_{10} \{3,3,1,\ldots,1\} \]
\[ C^{(\lambda \rho)(\sigma \tau)}(\partial_\lambda \partial_\mu \xi^\mu \partial_\sigma \partial_\tau \eta^\nu) \]
\[ m_\lambda m_\rho n_\sigma n_\tau C^{(\lambda \rho)(\sigma \tau)}(m + n) \]

Splitting $R^{(\lambda \rho)(\sigma \tau)}_{\mu \nu}$ as in $\text{(3.12)}$ and $\text{(3.13)}$ suggests that there should be additional cocycles

\[ m_\rho m_\sigma n_\tau D^{(\rho \sigma \tau)}_\mu(m + n) - m \leftrightarrow n, \tag{3.14} \]
\[ m_\mu m_\rho n_\sigma n_\tau H^{(\sigma \tau)}(m + n) - m \leftrightarrow n, \]

but these cocycles can be removed by the redefinitions

\[ L_\mu(m) \mapsto L_\mu(m) + m_\rho m_\sigma n_\tau D^{(\rho \sigma \tau)}_\mu(m), \]
\[ L_\mu(m) \mapsto L_\mu(m) + m_\mu m_\rho n_\sigma n_\tau H^{(\sigma \tau)}(m). \tag{3.15} \]
respectively.

The extension (3.10) can be understood as a MF term (2.24) for $gl(N)$ with $S_{\lambda}^{\mu\nu} = 0$. This MF algebra is the extension of $map(N, gl(N))$ (2.6) with

$$k_{\nu\tau}^{\mu\sigma}(m, n) = m_\rho n_\lambda R_{\nu\tau}^{\mu\rho\sigma\lambda}(m + n),$$

where $R_{\nu\tau}^{\mu\rho\sigma\lambda}$ transforms as a tensor of type $(4, 2)$ under $gl(N)$. (3.10) now follows immediately from (2.8). Moreover, since $m_\mu m_\rho \propto m_{(\mu} m_{\rho)}$, only the part with symmetries (3.11) is relevant.

There is another way to arrive at the extension (3.10). An analogous construction was carried out by [3] in the case of arbitrary gauge algebras $map(N, g)$. Let $P_{\nu\rho}^\mu$ be a tensor field of type $(2, 1; 1)$. Then

$$L'_\xi = L_\xi + P^{\mu\rho}(\partial_\nu \partial_\rho \xi_\mu), \quad L'_\mu(m) = L_\mu(m) + m_\nu m_\rho P^{\mu\rho}(m),$$

satisfies an extension of $\text{diff}(N)$ given by

$$c(\xi, \eta) = \{P^{\lambda\rho}(\partial_\lambda \partial_\rho \xi_\mu), P^{\sigma\tau}(\partial_\sigma \partial_\tau \eta_\nu)\},$$

$$c_{\mu\nu}(m, n) = m_\lambda m_\rho n_\sigma n_\tau [P^{\lambda\rho}(m), P^{\sigma\tau}(n)].$$

This is of the form (3.10), if we impose the condition that the extension be local in the sense of (2.3). In particular, the symmetry condition (3.11) holds automatically.

3.4 $N = 2$: $\psi_1^W - \psi_4^W$

In two dimensions, the symplectic form $\epsilon_{\mu\nu}$ coincides with the anti-symmetric symbol, and hence it commutes with all vector fields, not just the Hamilton-
nian ones. This makes it possible to construct further cocycles:

\[
\begin{align*}
\psi^W_{11} &= S^\rho (\partial_\rho (\epsilon^{\sigma\tau} \partial_\sigma \xi^\mu \partial_\mu \eta^\nu - \epsilon_{\mu\nu} \epsilon^{\rho\lambda} \partial_\rho \xi^\mu \partial_\lambda \eta^\nu)) \\
& \quad (m_\nu n_\mu (m \times n) - \epsilon_{\mu\nu} (m \times n)^2) (m_\rho + n_\rho) S^\rho (m + n) \\
\psi^W_{12} &= S^\rho (\partial_\rho (\epsilon^{\sigma\tau} \partial_\sigma \xi^\mu \partial_\mu \eta^\nu)) \\
& \quad m_\mu n_\nu (m \times n) (m_\rho + n_\rho) S^\rho (m + n) \\
\psi^W_{13} &= S^{(\kappa\lambda\rho\sigma\tau)} (3\epsilon_{\mu\kappa} \epsilon_{\nu\lambda} \epsilon^{\alpha\beta} \partial_\alpha \partial_\beta \xi^\mu \partial_\lambda \eta^\nu - 4\epsilon_{\mu\nu} \partial_\kappa \partial_\lambda \xi^\mu \partial_\sigma \partial_\tau \eta^\nu) \\
& \quad - \xi \leftrightarrow \eta \\
& \quad (3(m \times n) \epsilon_{\mu\kappa} \epsilon_{\nu\lambda} m_\rho m_\sigma n_\tau - 4\epsilon_{\mu\nu} m_\kappa m_\rho m_\sigma n_\lambda n_\tau \cdot S^{(\kappa\lambda\rho\sigma\tau)} (m + n) - m \leftrightarrow n \\
\psi^W_{14} &= S^{(\alpha\beta\kappa\lambda\rho\sigma\tau)} (\epsilon_{\mu\alpha} \epsilon_{\nu\beta} \partial_\kappa \partial_\lambda \xi^\mu \partial_\sigma \partial_\tau \eta^\nu) - \xi \leftrightarrow \eta \\
& \quad \epsilon_{\mu\alpha} \epsilon_{\nu\beta} m_\kappa m_\rho m_\sigma n_\lambda n_\tau S^{(\alpha\beta\kappa\lambda\rho\sigma\tau)} - m \leftrightarrow n
\end{align*}
\]

As in subsection [2.3], \( \epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = 1 \) and \( m \times n = m_1 n_2 - m_2 n_1 = \epsilon^{\mu\nu} m_\mu n_\nu \).

The Jacobi identities, in the Fourier basis, were verified numerically on a computer. To this end, it was useful to write the cocycles as

\[
\begin{align*}
c(\xi, \eta) &= R(\xi, \xi, \eta, \eta) - R(\eta, \eta, \eta, \xi, \xi) \\
\epsilon_{\mu\nu}(m, n) &= R_{\mu\nu}(m, m, m, n|m+n) - R_{\nu\mu}(n, n, n, m|m+n),
\end{align*}
\]

where e.g.

\[
R(m, r, s, n, t|u) = m_\nu R_{\lambda\rho} s_\rho n_\sigma t_\tau R^{(\kappa\lambda\rho)(\sigma\tau)}(u),
\]

and this operator carries conformal weight 1. The Jacobi identities now lead to the conditions

\[
\begin{align*}
n_\mu R_{\nu\sigma}(m, m, n, s, s|m+n+s) + s_\nu R_{\mu\sigma}(n, n, s, m, m|m+n+s) \\
+ m_\sigma R_{\mu\nu}(s, s, m, n|m+n+s) - s_\mu R_{\sigma\nu}(m, s, n, n|m+n+s) \\
- m_\nu R_{\mu\sigma}(n, n, s, s|m+n+s) - n_\sigma R_{\nu\mu}(s, s, n, m|m+n+s) &= 0.
\end{align*}
\]

Note how the epsilons conspire to yield the correct assignments of conformal weights: \( \epsilon^{\mu\nu} \) and \( \epsilon_{\mu\nu} \) carry weight +1 and -1, respectively. In particular, \( S^\rho \) has weight zero and is therefore not a one-chain, so it is not possible to write \( S^\rho (\partial_\mu F) = \Phi(F) \) for \( \Phi \) a tensor density.
In one dimension vectors have only one component, so we can use the simplified notation $L_m = L_1(m)$. A density with weight $\lambda$ (often called a primary field) transforms as $[L_m, A_n] = (n + (1 - \lambda)m)A_{m+n}$. Dzhumadil’daev describes the cocycles as follows.

$$
\begin{align*}
\lambda & \quad \text{Cocycle} \\
-1 & \quad L_0 \wedge L_2 \mapsto 1 \\
-4 & \quad L_2 \wedge L_3 \mapsto 1 \\
-6 & \quad L_2 \wedge L_5 \mapsto 1, L_3 \wedge L_4 \mapsto -3
\end{align*}
$$

(3.22)

It is not easy to guess the explicit forms of the cocycles from this description, but fortunately they were given in [4]. This list was later rediscovered by Ovsienko and Roger [22], and the super generalization has recently been worked out by Marcel [12]. I follow his naming scheme for the cocycles.

$$
\begin{align*}
\lambda & \quad c(m, n) \\
\gamma_1 & = \psi_1^W \quad 1 \quad (m-n)A_{m+n} \\
\gamma_2 & = \psi_1^W \quad 0 \quad (m^2n-2mn^2)A_{m+n} \\
\gamma_3 & = \psi_1^W \quad 0 \quad (m^2-n^2)A_{m+n} \\
\gamma_4 & = \psi_1^W \quad -1 \quad (m^3n-3mn^3)A_{m+n} \\
\gamma_5 & = \psi_1^W \quad -1 \quad (m^3-n^3)A_{m+n} \\
\gamma_6 & = \psi_1^W \quad -4 \quad (m^3n^4-n^3m^4)A_{m+n} \\
\gamma_7 & = \psi_1^W \quad -6 \quad (2m^3n^6-9m^4n^5+9n^4m^5-2n^3m^6)A_{m+n}
\end{align*}
$$

(3.23)

For some reason, $\gamma_1$, $\gamma_3$ and $\gamma_5$ are not included in Dzhumadil’daev’s 1996 classification, although they are present in his 1992 paper. Moreover, we have the Virasoro cocycle with values in the trivial module.

Generalize primary fields to translated primary fields:

$$
[L_m, A_n] = (n + r + (1 - \lambda)m)A_{m+n}.
$$

(3.24)

Now consider the redefinition

$$
L_m \mapsto L'_m = L_m + am^pA_m,
$$

(3.25)

where $a$ is a parameter. This redefinition gives rise to a trivial cocycle, except when $\lambda = \lambda_0$ and $r = 0$, where $\lambda_0$ is the weight in the table above. In this critical case, (3.23) gives rise to no cocycle at all. Now set $a = 1/(\lambda - \lambda_0)$, $r = 0$, and take the limit $\lambda \rightarrow \lambda_0$. This limiting procedure yields the cocycle
$c_\lambda(m, n)$. Or set $a = 1/r$, $\lambda = \lambda_0$, and take the limit $r \to 0$, giving cocycle $c_r(m, n)$. The result is

| $p$ | $\lambda_0$ | $c_\lambda(m, n)$ | $(m, n)$ |
|-----|-------------|-------------------|----------|
| 0   | 1           | $\gamma_1$       | $-$      |
| 1   | any         | $-$               | $\gamma_1$|
| 2   | 0           | $\gamma_2$       | $\gamma_3$|
| 3   | $-1$        | $\gamma_4$       | $\gamma_5$|

In this way, the cocycles $\gamma_1 - \gamma_5$ arise as limits of trivial cocycles. To “explain” $\gamma_6$, assume that in the $p = 3$ case,

$$[A_m, A_n] = (m - n)B_{m+n},$$

(“locality”) implying that $B$ transforms with $\lambda = -4$. The same assumption can be made also when $p = 0, 1, 2$, but this gives nothing new.

4 Dzhumadil’daev’s classification: subalgebra cocycles

4.1 Divergence-free vector fields

The cocycles are obtained by restriction from $\text{diff}(N)$. Since $\partial_\mu \xi^\mu = 0$, cocycles $\psi^W_1$, $\psi^W_2$, $\psi^W_3$, $\psi^W_4$, $\psi^W_7$ vanish, and the remaining extensions for $N \geq 3$ are denoted in $[5]$, Table 2, by

$$\text{svect}(N) \quad \psi^S_1 \quad \psi^S_2 \quad \psi^S_3 \quad \psi^S_4 \quad \psi^S_5$$

$$\text{diff}(N) \quad \psi^W_4 \quad \psi^W_5 \quad \psi^W_8 \quad \psi^W_9 \quad \psi^W_{10}$$

The treatment of the special two-dimensional cocycles is deferred to the next subsection, because $\text{svect}(2) \sim \text{Ham}(2)$
4.2 Hamiltonian vector fields

$Ham(N)$ has nontrivial cocycles in the following modules:

| Name     | $N$ | HW | Tensor                                   |
|----------|-----|----|------------------------------------------|
| $\psi_1^H$ (Moyal) | $N \geq 2$ | 0 | $\Phi$                                   |
| $\psi_2^H$ | $N \geq 4$ | $\pi_2$ | $A^{[\mu \nu]}$ |
| $\psi_3^H$ | $N \geq 4$ | $2\pi_2$ | $B^{[(\mu \rho)(\nu \sigma)]} \equiv B^{[\mu \nu][\rho \sigma]}$ |
| $\psi_4^H$ | $N \geq 4$ | $3\pi_2$ | $C^{[(\mu \rho \lambda)(\rho \sigma \tau)]} = \tilde{C}^{[\mu \nu \rho \sigma \tau]}$ |
| $\psi_5^H$ | $N \geq 4$ | $4\pi_2 + \pi_2$ | $D^{[\rho \lambda \sigma \tau][\mu \nu]}$ |
| $\psi_6^H$ | $N \geq 2$ | $\pi_1$ | $S^\rho$ |
| $\psi_7^H$ | $N = 2$ | $7\pi_1$ | $S^{(\mu \nu \kappa \lambda \rho \sigma \tau)}$ |
| $\psi_8^H$ | $N = 2$ | $2\pi_1$ | $S^{(\mu \nu)}$ |

The tensors are demanded to be totally traceless, in the following sense.

$$\epsilon_{\mu \nu} A^{[\mu \nu]} = \epsilon_{\mu \nu} B^{[(\mu \rho)(\nu \sigma)]} = \epsilon_{\mu \nu} C^{[(\mu \rho \lambda)(\nu \sigma \tau)]} = \epsilon_{\mu \nu} D^{[\rho \lambda \sigma \tau][\mu \nu]} = 0. \quad (4.1)$$
Explicitly, the cocycles are given by (4.1, Table 4)

\[
\begin{align*}
\text{Cocycle} & \quad c_H(f, g) \\
N & \quad c_H(m, n) \\
\psi^H_1 & \quad \Phi(\epsilon^{\mu\nu}\epsilon^{\sigma\tau}\epsilon_{\rho\sigma}f\partial_{\rho}f\partial_{\sigma}g) \\
N & \quad (m \times n)^3 \Phi(m + n) \\
\psi^H_2 & \quad A^{[\mu\nu]}(\epsilon^{\sigma\tau}\epsilon_{\rho\sigma}f\partial_{\rho}f\partial_{\sigma}g) \\
N & \quad (m \times n)^2 m_n A^{[\mu\nu]}(m + n) \\
\psi^H_3 & \quad B^{(\mu\rho)(\nu\sigma)}(\epsilon^{\lambda\tau}\epsilon_{\rho\sigma}f\partial_{\rho}f\partial_{\sigma}g) \\
N & \quad (m \times n) m_\rho m_\sigma B^{(\mu\rho)(\nu\sigma)}(m + n) \\
\psi^H_4 & \quad C^{(\mu\rho\lambda)(\nu\sigma\tau)}(\partial_{\rho}\partial_{\sigma}\partial_{\tau}g) \\
N & \quad m_\mu m_\rho m_\sigma m_\tau C^{(\mu\rho\lambda)(\nu\sigma\tau)}(m + n) \\
\psi^H_5 & \quad D^{(\rho\lambda\sigma\tau)[\mu\nu]}(\partial_{\rho}\partial_{\lambda}\partial_{\sigma}\partial_{\tau}g) \\
N & \quad m_\mu m_\rho m_\lambda m_\sigma m_\tau D^{(\rho\lambda\sigma\tau)[\mu\nu]}(m + n) \\
\psi^H_6 & \quad S^\rho(\epsilon^{\mu\nu}\epsilon^{\kappa\sigma}\epsilon^{\lambda\tau}\partial_{\rho}(\partial_{\mu}\partial_{\kappa}\partial_{\sigma}\partial_{\lambda}\partial_{\tau}g)) \\
N & \quad (m \times n)^3 (m_\rho + n_\rho) S^\rho(m + n) \\
\psi^H_7 & \quad S^{\mu\nu\kappa\lambda\rho\sigma}\partial_{\rho}\partial_{\lambda}\partial_{\rho}\partial_{\kappa}\partial_{\nu}\partial_{\sigma}\partial_{\tau}g - f \leftrightarrow g \\
N & \quad (m \times n)^2 m_\mu m_\kappa m_\lambda m_\sigma m_\sigma m_\nu m_\tau S^{\mu\nu\kappa\lambda\rho\sigma}(m + n) - m \leftrightarrow n, \\
\psi^H_8 & \quad \epsilon^{\kappa\lambda\tau}S^{\mu\nu}(7\partial_{\mu}\partial_{\nu}\partial_{\kappa}\partial_{\lambda}\partial_{\rho}\partial_{\sigma}\partial_{\tau}g + 3(\partial_{\mu}\partial_{\nu}\partial_{\rho}\partial_{\sigma}\partial_{\tau}g + \partial_{\nu}\partial_{\rho}\partial_{\sigma}\partial_{\tau}g + \partial_{\rho}\partial_{\sigma}\partial_{\tau}g) \partial_{\mu}\partial_{\rho}\partial_{\lambda}\partial_{\kappa}\partial_{\nu}\partial_{\tau}g)) \\
N & \quad (7m_\mu m_\nu + 3(m_\mu m_\nu + n_\mu m_\nu))(m \times n)^3 S^{\mu\nu}(m + n)
\end{align*}
\]

\(\psi^H_1 \rightarrow \psi^H_2\) arise from restriction of the traceless cocycles of \(\psi^W_3 \rightarrow \psi^W_10\), i.e. \(\psi^S_5 \rightarrow \psi^S_3\). The modules appear different because we can use the symplectic form to eliminate all lower indices. Explicitly, we have

\[
R^{(\lambda\rho)(\sigma\tau)} = \epsilon_{\alpha\beta} \epsilon_{\lambda\rho} R^{(\lambda\rho)(\beta\sigma)} \quad \text{and} \quad R^{(\sigma\tau)(\alpha\lambda)} = -R^{(\alpha\lambda)(\beta\sigma)}.
\]

(4.2)

In this way, each traceless module in \(\psi^W_3 \rightarrow \psi^W_10\) can be expanded as a direct sum of the modules in \(\psi^H_1 \rightarrow \psi^H_2\). E.g., the field in \(\psi^W_4\) can be written as \(F^{[\mu\nu]} = \epsilon^{\mu\nu}\Phi + A^{[\mu\nu]}\) with \(\epsilon^{\mu\nu}A^{[\mu\nu]} = 0\), and thus we obtain \(\psi^H_1\) and \(\psi^H_2\).
The conformal weight is irrelevant for Hamiltonian vector fields, wherefore we can consistently substitute $\Phi(\phi) \equiv S^0(\partial_\mu \phi)$ in $\psi^H_1$; this gives $\psi^H_6$. $\psi^H_6$ also follows by restriction from $\psi^W_{11}$ in two dimensions, but exists in all dimensions. For the special two-dimensional cocycles, $\psi^H_7$ is the restriction of $\psi^W_{14}$, whereas $\psi^W_{12}$ is a divergence which restricts to zero. Finally, it was checked numerically that $\psi^H_8$ is a cocycle. It may be related to $\psi^W_{13}$.

As is well known, the Moyal cocycle $\psi^W_1$ can be integrated to a full-fledged deformation of $Ham(N)$. Consider the Moyal (or sine) algebra, which is the Lie algebra with brackets (in Fourier basis)

$$[H(m), H(n)] = \frac{1}{\hbar} \sin(h(m \times n))H(m + n). \quad (4.3)$$

The Moyal cocycle appears at the lowest non-trivial order in $\hbar$.

$$[H(m), H(n)] = (m \times n)H(m + n) + (m \times n)^3 \Phi(m + n),
$$

$$[H(m), \Phi(n)] = (m \times n)\Phi(m + n)$$

$$[\Phi(m), \Phi(n)] = 0,$$

where $\Phi(m) = (h^2/6)H(m)$.

### 4.3 Contact vector fields

Dzhumadil'daev lists the $K(N)$ cocycles in his Table 5. His results for the relevant modules and thee cocycles are

| Cocycle | HW | $N$ | $c_K$ | $\psi^W_N$ |
|----------|----|-----|-------|------------|
| $\psi^K_1$ | 0  | $N \geq 3$ | $\psi^H_1$ | $\psi^W_1$ |
| $\psi^K_2$ | $\pi_2$ | $N \geq 5$ | $\psi^H_2$ | $\psi^W_2$ |
| $\psi^K_3$ | $\pi_2$ | $N \geq 5$ | $\psi^W_4$ | $\psi^H_4$ |
| $\psi^K_4$ | $2\pi_2$ | $N \geq 5$ | $\psi^H_5$ | $\psi^W_5$ |
| $\psi^K_5$ | $3\pi_2$ | $N \geq 5$ | $\psi^W_4$ | $\psi^H_4$ |
| $\psi^K_6$ | $4\pi_1 + \pi_2$ | $N \geq 5$ | $\psi^W_5$ | $\psi^H_5$ |
| $\psi^K_7$ | $2\pi_1 + \pi_2$ | $N \geq 5$ | $\psi^W_5$ | $\psi^H_5$ |
| $\psi^K_8$ | $4\pi_1$ | $N \geq 3$ | $\psi^W_5$ | $\psi^H_5$ |
| $\psi^K_9$ | $\pi_1$ | $N = 3$ | $\psi^W_6 = \psi^W_{11}$ | $\psi^H_6$ |
| $\psi^K_{10}$ | $\pi_1$ | $N = 3$ | $\psi^W_{12}$ | $\psi^H_6$ |
| $\psi^K_{11}$ | $3\pi_1$ | $N = 3$ | $\psi^W_{13}$ | $\psi^H_6$ |
| $\psi^K_{12}$ | $5\pi_1$ | $N = 3$ | $\psi^W_{13}$ | $\psi^H_6$ |
| $\psi^K_{13}$ | $7\pi_1$ | $N = 3$ | $\psi^W_{14}$ | $\psi^H_6$ |

20
\(\psi^K_7\) and \(\psi^K_8\) arise by restriction from \(\psi^W_6 - \psi^W_7\). An important point is that contact vector fields have non-zero divergence, \(\text{div}K_f \propto \partial_0 f\). Here \(\psi^H_n\) means that the restriction to the Hamiltonian subalgebra is \(\psi^H_n\), and \(\psi^W_n\) that the cocycle is obtained by restriction from to unrestricted diffeomorphism algebra. \(\psi^K_11\) is a special cocycle which seems to be unique to the contact algebra; I have not verified its existence.

However, I fail to understand Dzhumadil’daev’s results on two points.

1. In view of the results in the previous subsection, the eight cocycles \(\psi^K_1 - \psi^K_8\) arise by restriction from the eight MF cocycles \(\psi^W_3 - \psi^W_10\). However, moving all indices upstairs as in (4.2) requires the existence of an invariant and invertible two-form \(\epsilon_{\mu\nu}\). This exists for the Hamiltonian algebra but not, as far as I understand, for the contact algebra.

2. \(K(N)\) contains a \textit{diff}(1) subalgebra obtained by requiring that the function \(f(x^0)\) depends on \(x^0\) only. In this case, (2.18) becomes

\[
K_f = 2f(x^0)\partial_0 + f'(x^0)x^i\partial_i. \tag{4.5}
\]

These operators generate \textit{diff}(1) and \(K_f\) is recognized as the expression for a primary field. Upon the restriction \textit{diff}(N) → \(K(N)\) → \textit{diff}(1), the cocycle \(\psi^K_1\) for \textit{diff}(N) becomes \(\psi^K_1\) for \textit{diff}(1). Hence there must exist a nontrivial \(K(N)\) cocycle with coefficients in \(\pi_1\), also for \(N \geq 5\).

Nevertheless, almost all cocycles follow by restriction from \textit{diff}(N).

5 Beyond tensor modules

5.1 Higher-dimensional Virasoro algebras

We start from the tensor extensions \(\psi^W_3\) and \(\psi^W_4\), which involve the skew tensor field \(F^{\rho\sigma}\) of type (2,0;1), i.e. a two-chain. However, the extensions have the form

\[
\begin{align*}
c(\xi, \eta) & \quad c(\xi, \eta) \\
F^{\rho\tau}(\partial_{\rho} \xi^\mu \partial_\mu \eta^\nu), & \quad c_{\mu\nu}(m, n) \\
F^{\rho\tau}(\partial_{\tau} \xi^\mu \partial_\mu \eta^\nu), & \quad m_\mu n_\rho n_\nu (m_\tau + n_\tau) F^{\rho\tau}(m + n), 
\end{align*}
\]

respectively. By (2.13), we can now rewrite the extensions as

\[
\begin{align*}
c(\xi, \eta) & \quad c(\xi, \eta) \\
S^\rho(\partial_\rho \xi^\mu \partial_\mu \eta^\nu), & \quad m_\mu n_\rho n_\nu S^\rho(m + n), \\
S^\rho(\partial_{\tau} \xi^\mu \partial_\mu \eta^\nu), & \quad m_\nu n_\rho n_\mu S^\rho(m + n), 
\end{align*}
\]

(5.2)
where $S^\rho$ is the exact one-chain defined by

$$S^\rho(\phi_\rho) = F^{\rho\tau}(\partial_\tau \phi_\rho), \quad S^\rho(m) = m_\tau F^{\rho\tau}(m).$$

(5.3)

In particular, exact one-chains are also closed, and it turns out that this is enough to satisfy the cocycle condition. Thus, (5.2) defines cocycles, to be denoted by $\bar{\psi}_3^W$ and $\bar{\psi}_4^W$, provided that

$$S^\rho(\partial_\rho \phi) \equiv 0, \quad m_\rho S^\rho(m) \equiv 0$$

holds identically. $\bar{\psi}_4^W$ is the Eswara Rao-Moody cocycle [7], and $\bar{\psi}_3^W$ was first described by myself [16]. Contrary to $\psi_3^W$ and $\psi_4^W$, these cocycles are defined for all $N$ including $N = 1$, and in one dimension both reduce to the Virasoro cocycle.

There was some confusion in [17] regarding these cocycles. The reason that they are not included in Dzhumadil’dyev’s list is that they do not involve tensor modules, but rather submodules thereof. $\bar{\psi}_3^W$ and $\bar{\psi}_4^W$ arise naturally in toroidal Lie algebras.

### 5.2 Mickelsson-Faddeev

Let $d^{\rho\tau\beta}_{\kappa\nu\gamma}$ be totally symmetric under interchange of the pairs $(\rho,\kappa)$, $(\tau,\nu)$, $(\beta,\gamma)$. Such structure constants can be defined in terms of Kronecker deltas, but the interesting case in $N \geq 3$ dimensions is

$$d^{\rho\tau\beta}_{\kappa\nu\gamma} = \epsilon^{\rho\tau\beta\mu_1..\mu_{N-3}} \epsilon_{\kappa\nu\gamma\mu_1..\mu_{N-3}}$$

(5.5)

Then we can add an inhomogeneous term to the transformation law for the $(4,2;1)$-type tensor field in (3.10).

$$[L_\mu(m), R^{(\lambda\rho)(\sigma\tau)}_{\kappa\nu}(n)] = (4,2;1) + d^{\rho\tau\sigma}_{\kappa\nu\mu} m_\alpha m_\beta S^\lambda_{\alpha\beta}(m + n) + \text{symm}(\lambda\rho,\sigma\tau),$$

(5.6)

where $\text{symm}(\lambda\rho,\sigma\tau)$ stands for the three extra terms needed to give the rhs the appropriate symmetries. This follows immediately by specializing (2.24) to $g = gl(N)$ and (2.7). In three dimensions,

$$R^{(\lambda\rho)(\sigma\tau)}_{\kappa\nu}(n) = \epsilon^{\lambda\sigma\alpha} d^{\rho\tau\beta}_{\kappa\nu\gamma} \Gamma_{\alpha\beta}^\gamma(n) + \text{symm}(\lambda\rho,\sigma\tau),$$

(5.7)

where $\Gamma_{\alpha\beta}^\gamma$ is the connection (2.14). The additional term in (5.6) is new. It can not be embedded in a larger algebra using (3.17), because that would violate the Jacobi identities [20].
5.3 Dzhumadil’daev-Ovsienko-Roger cocycles in higher dimensions

In this subsection I describe higher-dimensional generalizations of the cocycles $\gamma_1-\gamma_3$ of (3.23). Since such generalizations contain non-trivial extensions of $\text{diff}(1)$, these new cocycles are also non-trivial. First tensor modules (2.5) must be generalized to translated tensor modules with dead indices. This concept is best illustrated by an example. If

$$[L_\mu(m), \Phi^{\nu\rho}_{\sigma\tau}(n)] = (n_\mu + \lambda m_\mu + r_\mu)\Phi^{\nu\rho}_{\sigma\tau}(m + n) + \delta_\mu^\nu m_\lambda \Phi^{\lambda\rho}_{\sigma\tau}(m + n) - m_\rho \Phi^{\nu\rho}_{\mu\tau}(m + n),$$

(5.8)

we say that $\Phi^{\nu\rho}_{\sigma\tau}(m - r)$ is of type $(1, 1; 1)$ with one dead upper index ($\rho$) and one dead lower index ($\tau$). The remaining indices are, of course, alive.

1. Consider the redefinition

$$L_\mu(m) \mapsto L_\mu(m) + aA_\mu(m),

[L_\mu(m), A_\nu(n)] = (n_\mu + (1 - \lambda)m_\mu + r_\mu)A_\nu(m + n).$$

(5.9)

Thus, $A_\nu(m - r)$ is of type $(0, 0; \lambda)$ with a dead lower index. The limit $\lambda \to 1$, $r_\mu = 0$, $a(1 - \lambda) = 1$, gives rise to the cocycle

$$c(\xi, \eta) = A_\rho(\partial_\mu \xi^\mu \eta^\rho - \partial_\nu \eta^\nu \xi^\rho),

c_{\mu\nu}(m, n) = m_\mu A_\nu(m + n) - n_\nu A_\mu(m + n),$$

(5.10)

which is a higher-dimensional generalization of $\gamma_1$. The limit $r_\mu \to 0$, $\lambda = 1$, $ar_\mu = e_\mu$, yields

$$c(\xi, \eta) = A_\rho(e_\mu \xi^\mu \eta^\rho - e_\nu \eta^\nu \xi^\rho),

c_{\mu\nu}(m, n) = e_\mu A_\nu(m + n) - e_\nu A_\mu(m + n),$$

(5.11)

This cocycle vanishes when $N = 1$.

2. Consider the redefinition $L_\mu(m) \mapsto L_\mu(m) + amB_\nu^\mu(m)$, where either $B_\mu^\nu(m - r)$ is of type $(1, 0; \lambda)$ with a dead lower index, or it is of type $(0, 1; \lambda)$ with a dead upper index. The limit $\lambda \to 1$, $r_\mu = 0$, $a(1 - \lambda) = 1$, gives rise to the cocycle

$$c(\xi, \eta) = B_\sigma^\rho(\partial_\mu \xi^\mu \partial_\rho \eta^\sigma - \partial_\nu \eta^\nu \partial_\rho \xi^\sigma),

c_{\mu\nu}(m, n) = m_\mu n_\rho B_\nu^\mu(m + n) - n_\nu m_\rho B_\mu^\rho(m + n).$$

(5.12)

This cocycle vanishes when $N = 1$. The limit $r_\mu \to 0$, $\lambda = 1$, $ar_\mu = e_\mu$, yields a trivial cocycle.
3. Consider the redefinition $L_\mu(m) \mapsto L_\mu(m) + am_\rho m_\sigma K_\mu^{\rho\sigma}(m)$, where $K_\mu^{\rho\sigma}(m-r)$ is of type $(2, 1; \lambda)$, symmetric and $\rho$ and $\sigma$, and all indices are alive. As described above, the limit $\lambda \to 1$, $r_\mu = 0$, $a(1 - \lambda) = 1$, gives rise to the cocycles $\psi_1^W$ and $\psi_2^W$, for the trace and traceless parts, respectively. These are higher-dimensional generalizations of $\gamma_2$. The limit $r_\mu \to 0$, $\lambda = 1$, $a_{r_\mu} = e_\mu$, yields

$$c(\xi, \eta) = K_\mu^{\rho\tau}(\partial_\sigma \partial_\tau \xi^\mu e_\nu \eta^\nu - \partial_\sigma \partial_\nu \eta^\nu e_\mu \xi^\mu),$$
$$c_{\mu\nu}(m, n) = \partial_\nu m_\sigma m_\tau K_\mu^{\sigma\tau}(m+n) - e_\mu n_\sigma n_\tau K_\mu^{\sigma\tau}(m+n),$$

(5.13)

which is a higher-dimensional generalization of $\gamma_3$. With $K_\mu^{\sigma\tau} = \delta_\rho^{(\tau} S_\sigma^{\sigma)}$, we obtain

$$c(\xi, \eta) = S_\sigma(\partial_\sigma \partial_\mu \xi^\mu e_\nu \eta^\nu - \partial_\sigma \partial_\nu \eta^\nu e_\mu \xi^\mu),$$
$$c_{\mu\nu}(m, n) = (e_\nu m_\sigma m_\tau - e_\mu n_\sigma n_\tau) S_\sigma(m+n),$$

(5.14)

where $S_\rho$ is of type $(1, 0; 1)$.

4. The natural way to generalize $\gamma_4$ and $\gamma_5$ would be to redefine $L_\mu(m) \mapsto L_\mu(m) + am_\rho m_\sigma m_\tau D_\mu^{\rho\sigma\tau}(m)$, where $D_\mu^{\rho\sigma\tau}(m-r)$ is of type $(3, 1; \lambda)$ and totally symmetric. However, as noted in (3.15), this gives rise to a trivial cocycle even when $\lambda = 1$ and $r_\mu = 0$, except in one dimension. Hence I suspect that $\gamma_4$ and $\gamma_5$ have no $N > 1$ counterparts.

5.4 Anisotropic extensions

In [17] I constructed two complicated cocycles satisfied by the representations introduced by Eswara-Rao and Moody [7]. It turned out [18, 19] that they could be obtained from the DRO algebra defined below (section 9), by imposing the second-class constraint $L_f \approx 0$, $Q_0(t) \approx t$. The former conditions can be viewed as a first class constraint and the latter as a gauge condition. Other cocycles can be found by replacing the gauge condition, as long as the constraints together are second class, i.e. the Poisson bracket matrix is invertible.

6 Extensions of $\text{diff}(N) \times \text{map}(N, \text{diff}(d))$

Replace $N$ by $N + d$ everywhere in the previous sections. The total space $\mathbb{R}^{N+d}$ have coordinates $z^A = (x^\mu, y^i)$, where greek indices $\mu, \nu, \rho, \sigma, \tau = 1, \ldots, N$ label horizontal (base space) directions, latin indices $i, j, k, \ell = 1, \ldots, d$ label vertical (target space) directions, and capitals $A = (\mu, i)$, etc.
label directions in total space. This induces splits \( \partial_A \equiv \partial/\partial z^A = (\partial_{\mu}, \partial_1) \equiv (\partial/\partial x^\mu, \partial/\partial y^i) \), \( \Xi^A(z) = (\xi^\mu(x), X^i(x,y)) \), \( \mathcal{L}_\xi = (\mathcal{L}_\xi, \mathcal{J}_X) \), etc. What makes this split a proper embedding is that the horizontal components of the vector fields \( \xi^\mu(x) \) are taken to be independent of the vertical coordinates \( y^i \), so \( \partial_i \xi^\mu = 0 \).

An extension of \( \text{diff}(N) \times \text{map}(N, \text{diff}(d)) \) has the form

\[
\begin{align*}
[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}[\xi,\eta] + c(\xi,\eta), \\
[\mathcal{L}_\xi, \mathcal{J}_X] &= \mathcal{J}_\xi^\nu \partial_\nu X + c(\xi,X), \\
[\mathcal{J}_X, \mathcal{J}_Y] &= \mathcal{J}_{[X,Y]} + c(X,Y).
\end{align*}
\]

Tensor densities are described by \( (2.5), \mathcal{L}_\xi = \Xi^A \partial_A + \partial_B \Xi^A T^B_A \), where \( T^A_B \) satisfy \( gl(N + d) \). Hence

\[
\begin{align*}
\mathcal{L}_\xi &= \xi^\mu \partial_\mu + \partial_\nu \xi^\mu T^\nu_\mu, \\
\mathcal{J}_X &= X^i \partial_i + \partial_j X^i T^j_i + \partial_\mu X^i T^j_i.
\end{align*}
\]

Since the \( T^j_i \) component never enters any formulas, its value is unimportant and may be set to zero. We can then perform a similarity transformation \( T^A_B \rightarrow T'^A_B = \tilde{S}^A_C T^C_D S^D_B \), with

\[
S^A_B = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 0 & \varepsilon \delta^j_i \end{pmatrix}, \quad \tilde{S}^A_B = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 0 & \varepsilon^{-1} \delta^j_i \end{pmatrix}, \quad T'^A_B = \begin{pmatrix} T^\mu_\nu & \varepsilon T^\mu_j \\ 0 & T^j_i \end{pmatrix}.
\]

This amounts to multiplying the last term in \( (6.2) \) by \( \varepsilon \). For convenience, the transformation laws for a tensor field of type (1,1;1) in base space and (1,1) in target space is given explicitly; the general case follows readily.

\[
\begin{align*}
[\mathcal{L}_\xi, \Phi^\sigma_k(\phi^\tau_i)] &= \Phi^\sigma_k(\xi^\mu \partial_\mu \phi^\tau_k + \partial_\sigma \xi^\mu \phi^\tau_k - \partial_\nu \xi^\sigma \phi^\nu_k), \\
[\mathcal{J}_X, \Phi^\sigma_k(\phi^\tau_i)] &= \Phi^\sigma_k(X^i \partial_k \phi^\tau_i + \partial_k X^i \phi^\tau_k - \partial_j X^i \phi^\tau_k + \varepsilon \partial_\nu X^i \phi^\nu_k),
\end{align*}
\]

where \( \phi^\tau_i(x,y) \) is an arbitrary function on total space. Of course, a similarity transformation does not bring anything essentially new, but we can set \( \varepsilon = 0 \) in \( (6.3) \), corresponding to a singular matrix \( S^A_B \). Base space and target space indices then decouple which makes the transformation laws particularly simple.

The similarity transformation amounts to a rescaling of \( \partial_\sigma X^i \) by \( \varepsilon \) without affecting other components of \( \partial_B \Xi^A \). This is equivalent to rescaling \( X^i \) by \( \varepsilon \) and \( \partial_j \) by \( \varepsilon^{-1} \). \( \partial_j \xi^\mu \) would also rescale by \( \varepsilon^{-1} \), but this is no problem
since it vanishes anyway. What is a problem is that \( \partial_j \partial_k X^i \) also rescales by \( \varepsilon^{-1} \). Since all cocycles contain such terms, we can in fact not put \( \varepsilon = 0 \), but it will become possible in the next section. For the remainder of this section, we set \( \varepsilon = 1 \).

The restrictions of the generic extensions are as follows.

\[ \overline{\psi}_3^W : \]
\[
\begin{align*}
  c(\xi, X) &= S^\mu (\partial_\mu \xi^\mu \partial_i X^i), \\
  c(X, Y) &= S^\rho (\partial_\rho \partial_i X^i \partial_j Y^j) + S^k (\partial_k \partial_i X^i \partial_j Y^j). 
\end{align*}
\]

\[ \overline{\psi}_4^W : \]
\[
\begin{align*}
  c(\xi, X) &= 0, \\
  c(X, Y) &= S^\rho (\partial_\rho \partial_i X^i \partial_j Y^j) + S^k (\partial_k \partial_i X^i \partial_j Y^j). 
\end{align*}
\]

where \( S^C(\phi_C) = S^\rho (\phi_\rho) + S^k (\phi_k) \) is a tensor density in total space of type \( (1, 0; 1) \), satisfying the auxiliary condition \( S^C(\partial_C \phi) \equiv 0 \). Explicitly, the transformation laws are given by

\[
\begin{align*}
  [\mathcal{L}_\xi, S^\rho (\phi_\rho)] &= S^\rho (\xi^\mu \partial_\mu \phi_\rho + \partial_\rho \xi^\mu \phi_\mu), \\
  [\mathcal{L}_\xi, S^k (\phi_k)] &= S^k (\xi^\mu \partial_\mu \phi_k), \\
  [\mathcal{J}_X, S^\rho (\phi_\rho)] &= S^\rho (X^i \partial_i \phi_\rho), \\
  [\mathcal{J}_X, S^k (\phi_k)] &= S^k (X^i \partial_i \phi_k + \partial_k X^i \phi_i) + S^\rho (\partial_\rho X^i \phi_i), 
\end{align*}
\]

where \( S^\rho (\partial_\rho \phi) + S^k (\partial_k \phi) \equiv 0 \).

\[ \overline{\psi}_2^W : \]
\[
\begin{align*}
  c(\xi, X) &= S^\rho (\partial_\rho \partial_i \xi^\mu \partial_j X^j) - \partial_\rho \xi^\mu \partial_i \partial_j X^j - S^k (\partial_\rho \xi^\mu \partial_k \partial_i X^j), \\
  c(X, Y) &= S^\rho (\partial_\rho \partial_i X^i \partial_j Y^j) - \partial_i X^i \partial_\rho \partial_j Y^j \\
  &\quad + S^k (\partial_k \partial_i X^i \partial_j Y^j - \partial_i X^i \partial_k \partial_j Y^j). 
\end{align*}
\]

where \( S^C \) is as above but the closedness condition is no longer necessary.

\[ \overline{\psi}_1^W : \]
\[
\begin{align*}
  c(\xi, X) &= K^{(AB)}_i (\partial_\rho \xi^\mu \partial_A \partial_B X^i) - K^{(\sigma \tau)}_i (\partial_\sigma \partial_\tau \xi^\mu \partial_i X^i), \\
  c(X, Y) &= K^{(AB)}_j (\partial_i X^i \partial_A \partial_B Y^j) - K^{(\mathcal{A}B)}_i (\partial_A \partial_B X^i \partial_j Y^j), 
\end{align*}
\]

where \( K^{(AB)}_C \) is a tensor field of type \( (2, 1; 1) \).
ψ^W - ψ^W:
\[
c(\xi, X) = R^{(AB)(CD)}_{\mu\lambda\rho\sigma} (\partial_\lambda \partial_\rho \xi^\mu \partial_C \partial_D X^i),
\]
\[
c(X, Y) = R^{(CD)}_{ij} (\partial_A \partial_B X^i \partial_C \partial_D Y^j),
\]
where \( R^{(AB)(CD)}_{EF} \) is a tensor field of type (4, 2; 1).

7 Extensions of \( \text{diff}(N) \ltimes \text{map}(N, \text{gl}(d)) \)

Now consider the subalgebra \( \text{gl}(d) \subset \text{diff}(d) \), with vertical vector fields \( X = X^i(x, y) \partial_i = X_j^i(x) y^j \partial_i \). In the previous section, we substitute \( \partial_j X^i = X^i_j \), \( \partial_j \partial_k X^i = 0 \). The algebra formally takes the same form (6.1), but now \( [X, Y] = (X^i_j Y^k_j - X^i_k Y^k_j) y^j \partial_i \). Tensor fields are decomposed into components which are homogeneous in \( y^i \), e.g.,

\[
\Phi^{\sigma k}_{\tau \ell} (\phi^{\tau \ell}_{\sigma k}) = \sum_{n=0}^{\infty} \Phi^{\sigma k|_{m_1..m_n}}_{\tau \ell} (\phi^{\tau \ell}_{\sigma k|_{m_1..m_n}}),
\]

where \( \phi^{\tau \ell}_{\sigma k|_{m_1..m_n}}(x) \) is a function independent of the vertical coordinate \( y^i \) and

\[
\Phi^{\sigma k|_{m_1..m_n}} (\cdot) \equiv \Phi^{\sigma k}_{\tau \ell} (y^{m_1} \ldots y^{m_n}).
\]

The base space transformation law (6.4) is unchanged, whereas (6.5) is replaced by

\[
[X, \Phi^{\sigma k|_{m_1..m_n}}_{\tau \ell} (\phi^{\tau \ell}_{\sigma k})] = \Phi^{\sigma k|_{m_1..m_n}}_{\tau \ell} (X^i_j \phi^{\tau \ell}_{\sigma k} + \sum_{r=1}^{n} \partial_j X^i_{m_r} \phi^{\tau \ell}_{\sigma k|_{m_1..m_n}}) - \sum_{r=1}^{n} \partial_j X^i_{m_r} \phi^{\tau \ell}_{\sigma k|_{m_1..m_n}} - \varepsilon \Phi^{\sigma k|_{m_1..m_n}}_{\tau \ell} (\partial_j X^i_{\phi^{\tau \ell}_{\sigma k}}).
\]

In this section we can set \( \varepsilon = 0 \), since the dangerous term \( \partial_j \partial_k X^i = 0 \) anyway. Because \( X^i_j = \partial_j X^i \), this amounts to rescalings of \( y^i \) by \( \varepsilon \) and of \( \partial_j \) by \( \varepsilon^{-1} \), and hence \( \Phi^{\sigma k|_{m_1..m_n}} (\cdot) \) must be multiplied by \( \varepsilon^n \).

With any value of \( \varepsilon \), transformation laws are readily read off from the index structure. In particular, target space indices transform in the same way independent of if they appear to the left or to the right of a vertical bar.

27
\[ c(\xi, X) = S^\rho(\partial_\rho \partial_\mu \xi^\mu X_i^j), \]
\[ c(X, Y) = S^\rho(\partial_\rho X_i^j Y^j), \]  
(7.4)

\[ c(\xi, X) = 0, \]
\[ c(X, Y) = S^\rho(\partial_\rho X_i^j Y^j). \]  
(7.5)

Note that only the horizontal component of the one-form \( S^C(\phi_C) \) appears, and that its argument is independent of \( y^i \). Therefore, we can limit our attention to \( S^\rho(\phi_\rho) \), where \( \phi_\rho(x) \) is independent of the vertical coordinates and \( \phi_i = 0 \). The transformation laws read
\[ [\mathcal{L}_\xi, S^\rho(\phi_\rho)] = S^\rho(\xi^\mu \partial_\mu \phi_\rho + \partial_\rho \xi^\mu \phi_\mu), \]
\[ [\mathcal{J}_X, S^\rho(\phi_\rho)] = 0, \]
(7.6)

where \( S^\rho(\partial_\rho \phi) \equiv 0 \).

\[ c(\xi, X) = \varepsilon K_i^{(\sigma\tau)}|j (\partial_\mu \xi^\mu \partial_\sigma X_j^i) + 2 K_i^{(\sigma j)}(\partial_\mu \xi^\mu \partial_\sigma X_j^i) - K_i^{(\sigma\tau)}(\partial_\rho \partial_\tau \xi^\mu X_i^j), \]
(7.8)

where \( K_i^{(\sigma\tau)}(\cdot) = K_i^{(\sigma\tau)}(y^j \cdot) \). The two cocycles that survive when \( \varepsilon = 0 \) are independent.

\[ c(\xi, X) = \varepsilon R_{\mu i}^{(\lambda \rho)(\sigma \tau)} |j (\partial_\lambda \partial_\rho \xi^\mu \partial_\sigma \partial_\tau X_j^i) + 2 R_{\mu i}^{(\lambda \rho)(\sigma \tau)}(\partial_\lambda \partial_\rho \xi^\mu \partial_\sigma X_j^i), \]
(7.9)

\[ c(X, Y) = \varepsilon^2 R_{ij}^{(\lambda \rho)(\sigma \tau)}|k (\partial_\mu \partial_\rho X_k^i \partial_\sigma \partial_\tau Y_j^j) + 2 \varepsilon (R_{ij}^{(k \rho)(\sigma \tau)}|k (\partial_\mu \partial_\rho X_k^i \partial_\sigma \partial_\tau Y_j^j) + R_{ij}^{(\lambda \rho)(\ell \tau)}|k (\partial_\mu \partial_\rho X_k^i \partial_\sigma \partial_\tau Y_j^j)) + 4 R_{ij}^{(k \rho)(\ell \tau)}(\partial_\mu X_k^i \partial_\sigma Y_j^j), \]
where

\[
R^{(\lambda \rho)(\sigma \tau)}_{\mu i}(\cdot) = R^{(\lambda \rho)(\sigma \tau)}_{\mu i}(y^j \cdot),
\]

\[
R^{(\lambda \rho)(\sigma \tau)k\ell}_{ij}(\cdot) = R^{(\lambda \rho)(\sigma \tau)k\ell}_{ij}(y^k \cdot y^\ell \cdot),
\]

\[
R^{(k\rho)(\sigma \tau)ij}_{k\ell}(\cdot) = R^{(k\rho)(\sigma \tau)ij}_{k\ell}(y^\ell \cdot).
\]

(7.10)

The two cocycles that survive when \(\varepsilon = 0\) are independent, and the last term in (7.9) is recognized as the MF cocycle (2.24) for \(\text{map}(N, \mathfrak{gl}(d))\).

8 Extensions of \(\text{diff}(N) \ltimes \text{map}(N, \mathfrak{g})\)

Assume that the finite-dimensional Lie algebra \(\mathfrak{g}\) has a \(d\)-dimensional representation with matrices \(\sigma^a = (\sigma^a_j)\). In the previous section, we substitute \(X_j = X_a \sigma^a_j\). Set \(\text{tr} \sigma^a = \sigma^a_i = z_M \delta^a_i\), where either \(\delta^c_{abc} = 0\) or \(z_M = 0\), and \(\text{tr} \sigma^a \sigma^b = \sigma^a_i \sigma^b_i = y_M \delta^{ab}\). Now \([X, Y]_c = if^a_{
\text{bc}} X_a Y_b\), \(X_i = z_M \delta^a X_a\) and \(X_i Y^j_i = y_M \delta^{ab} X_a Y_b\). Tensor fields are given by

\[
L_\xi = \xi^\mu \partial_\mu + \partial_\nu \xi^\mu T^\nu_
\]

\[
J_X = X_a \sigma^a_j y^j \partial_i + X_a \sigma^a_j T^j_i + \varepsilon \partial_\mu X_a \sigma^a_i T^\mu_i.
\]

(8.1)

\(\psi^W_3:\)

\[
c(\xi, X) = z_M \delta^a S^\rho (\partial_\rho \partial_\mu \xi^\mu X_a),
\]

\[
c(X, Y) = z_M^2 \delta^{ab} S^\rho (\partial_\rho X_a Y_b).
\]

(8.2)

\(\psi^W_4:\)

\[
c(\xi, X) = 0,
\]

\[
c(X, Y) = y_M \delta^{ab} S^\rho (\partial_\rho X_a Y_b).
\]

(8.3)

In particular, in one dimension we get

\[
[L_m, J^a_n] = nJ^a_m + z_M \delta^a m^2 \delta_{m+n},
\]

\[
[J^a_m, J^b_n] = if^a_{
\text{bc}} J^c_{m+n} + z_M \delta^{ab} m \delta_{m+n} + y_M \delta^{ab} m \delta_{m+n}.
\]

(8.4)

The last term is recognized as the Kac-Moody cocycle. The other two are not so well known, because they vanish for \(\mathfrak{g}\) semisimple. However, all three cocycles are non-trivial.
\(\psi^W_1:\)
\[
c(\xi, X) = z_M \delta^a S^p (\partial_\mu \partial_\nu \xi^\mu X_a - \partial_\nu \xi^\mu \partial_\mu X_a),
\]
\[
c(X, Y) = z_M^2 \delta^a \delta^b S^p (\partial_\mu X_a Y_b - X_a \partial_\mu Y_b).
\]
(8.5)

\(\psi^W_2:\)
\[
c(\xi, X) = \epsilon \sigma_i a^j R_i^{(\sigma j)} (\partial_\lambda \partial_\rho \xi^\mu \partial_\tau X_a) + 2 \sigma_i a^j R_i^{(\sigma j)} (\partial_\mu \partial_\rho \xi^\mu \partial_\tau X_a) - z_M \delta^a R_i^{(\sigma j)} (\partial_\mu \partial_\sigma \xi^\mu X_a),
\]
\[
c(X, Y) = 2 \epsilon \sigma_i a^k R_i^{(\sigma j)} (X_a \partial_\rho \partial_\tau Y_b) + 2 \sigma_i a^k R_i^{(\sigma j)} (X_a \partial_\rho \partial_\tau Y_b).
\]
(8.6)

The two cocycles that survive when \(\epsilon = 0\) are independent.

\(\psi^W_3 - \psi^W_{10}:\)
\[
c(\xi, X) = \epsilon \sigma_i a^j R_i^{(\lambda \rho)(\sigma j)} (\partial_\lambda \partial_\mu \partial_\sigma \partial_\tau X_a) + 2 \sigma_i a^j R_i^{(\lambda \rho)(\sigma j)} (\partial_\mu \partial_\nu \partial_\sigma \partial_\tau X_a) - 2 \epsilon \sigma_i a^k R_i^{(\lambda \rho)(\sigma j)} (X_a \partial_\rho \partial_\sigma \partial_\tau Y_b) + 2 \sigma_i a^k R_i^{(\lambda \rho)(\sigma j)} (X_a \partial_\rho \partial_\sigma \partial_\tau Y_b) - z_M \delta^a R_i^{(\lambda \rho)(\sigma j)} (\partial_\nu \partial_\rho \partial_\tau X_a),
\]
\[
c(X, Y) = \epsilon \sigma_i a^k R_i^{(\lambda \rho)(\sigma j)} (X_a \partial_\rho \partial_\tau Y_b) - X \leftrightarrow Y.
\]
(8.7)

The two cocycles that survive when \(\epsilon = 0\) are independent, and the latter is recognized as the MF cocycle (2.24) for map\((N, g)\).

9 Extensions of \(diff(N) \oplus diff(1): DRO\) algebra

The DRO (Diffeomorphism, Reparametrization, Observer) algebra \(DRO(N)\) was introduced in [18, 19] as an extension of \(diff(N) \oplus diff(1)\) by the observer’s trajectory \(q^\mu(t)\). The reason for giving this algebra a special name is its importance for Fock representations of \(diff(N)\). Expand all fields in a Taylor series around \(q^\mu(t)\), where \(t \in S^1\). The Taylor coefficients, or \(jets\), are
\[
\Phi, m(t) = \partial_m \Phi(q(t)) \equiv \partial_1^{m_1} \ldots \partial_N^{m_N} \Phi(q(t)).
\]
(9.1)

where \(m = (m_1, \ldots, m_N)\) is a multi-index. Note that the \(jets\) depend on \(t\) although the field \(\Phi(x)\) does not, since this dependence enters through
the expansion point. In [18] I took the space of $p$-jets $\Phi_m(t)$, with $|m| = \sum_{\mu=1}^N m_\mu \leq p$, as the starting point for the Fock construction. This leads to consistent results because the jet space consists of finitely many functions of a single variable $t$. The full DRO algebra acts naturally on the jets; the additional $\text{diff}(1)$ factor describes reparametrizations of the observer’s trajectory.

Any extension of $\text{diff}(N) \oplus \text{diff}(1)$ has the form

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]} + c(\xi, \eta),$$
$$[\mathcal{L}_\xi, \mathcal{L}_f] = \mathcal{L}_{[\xi, f]} + c(\xi, f),$$
$$[\mathcal{L}_f, \mathcal{L}_g] = \mathcal{L}_{[f, g]} + c(f, g),$$

(9.2) where $f = f(t) d/dt$ is a vector field on the circle and $[f, g] = (f \dot{g} - g \dot{f}) d/dt$.

We embed $\text{diff}(N) \oplus \text{diff}(1) \subset \text{diff}(N+1)$ in the natural way: set $z^A \equiv (z^\mu, z^0) = (x^\mu, t)$, partials $\partial_A = (\partial_\mu, d/dt)$, $\Xi^A(z) = (\xi^\mu(x), f(t))$, $\mathcal{L}_\Xi = (\mathcal{L}_\xi, \mathcal{L}_f)$.

Tensor densities restrict to

$$\mathcal{L}_\xi = \xi^\mu \partial_\mu + \partial_\nu \xi^\mu T^\nu_\mu,$$
$$\mathcal{L}_f = f \frac{d}{dt} + \dot{f} T^0_0,$$

(9.3) where $T^0_0$ was called the causal weight in [17]–[19]. Thus, both the $T^\mu_\mu$ and $T^0_0$ components of the $\text{gl}(N + 1)$ generator $T^A_B$ decouple.

$\tilde{\psi}^W_3$:

$$c(\xi, f) = S^\rho(\partial_\mu \partial_\rho \xi^\mu \dot{f}),$$
$$c(f, g) = S^0(\dot{f} \dot{g}).$$

(9.4)

$\tilde{\psi}^W_4$:

$$c(\xi, f) = 0,$$
$$c(f, g) = S^0(\dot{f} \dot{g}).$$

(9.5)

With $\phi(x)$ independent of $t$ and $f(t)$ independent of $x^\mu$, closedness implies

$$S^\rho(\partial_\rho \phi f) + S^0(\dot{\phi} \dot{f}) \equiv 0.$$  

(9.6)

In particular, $S^0(\dot{f}) \equiv 0$, so $S^0(f) \propto \int dt \ f(t)$ and $c(f, g)$ is the Virasoro cocycle in both cases.

Thus $DRO(N)$ has four independent Virasoro-like cocycles, namely the terms proportional to $c_1$, $c_2$, $c_3$ and $c_4$ in the notation of [18]. In the notation
of the present paper, \( c_1 = \psi^W_1 \), \( c_2 = \psi^W_2 \), \( c_3 = c(\xi,f) \) from (9.4), and \( c_4 = c(f,g) \) from (9.3) or (9.4). As described in subsection 5.4, we can eliminate reparametrizations by a second class constraint, trading the last two cocycles for anisotropic cocycles of \( \text{diff}(N) \).

\( \psi^W_1 \):

\[
\begin{align*}
c(\xi,f) &= S^0(\partial_\mu \xi^\mu \dot{f}) - S^0(\partial_\mu \xi^\mu \ddot{f}), \\
c(f,g) &= S^0(\ddot{f} \dot{g} - \dot{f} \ddot{g}).
\end{align*}
\]

where (9.6) no longer holds. The second formula is the Virasoro generalization (3.6).

\( \psi^W_2 \):

\[
\begin{align*}
c(\xi,f) &= K^{00}_0(\partial_\mu \xi^\mu \dot{f}) - K^{(\sigma\tau)}(\partial_\sigma \partial_\tau \xi^\mu \dot{f}), \\
c(f,g) &= K^{00}_0(\ddot{f} \dot{g} - \dot{f} \ddot{g}).
\end{align*}
\]

where (9.6) no longer holds. The second formula is the Virasoro generalization (3.6).

\( \psi^W_3 - \psi^W_{10} \):

\[
\begin{align*}
c(\xi,f) &= R^{(\lambda\rho)}_\mu(\partial_\lambda \partial_\rho \xi^\mu \ddot{f}), \\
c(f,g) &= 0.
\end{align*}
\]

where \( R^{(\lambda\rho)}_\mu \) is a tensor field of type (3,1;1).

10 Conclusion

In this paper I have reviewed Dzhumadil’daev’s exhaustive classification of tensor extensions of \( \text{diff}(N) \) and subalgebras, extended it beyond tensor modules, and studied the chain of restrictions down to \( \text{map}(N,g) \). The method proves existence for the cocycles of the subalgebras, but it neither proves non-triviality nor exhaustion. However, since the extensions obtained in the last step are in fact recognized as non-trivial (Kac-Moody, MF, etc.), the entire chain is non-trivial. Moreover, I tautologically exhaust the class of subalgebra cocycles with values in tensor modules, which can be lifted to the diffeomorphism algebra in total space.

The construction of projective Fock modules of \( \text{diff}(N) \) was initiated in [6] and further developed in [17, 18]. By restriction, this gives Fock modules of subalgebras, of the type described in [6, 21] and in the papers just cited. Berman and Billig [8] constructed another type of module, postulating the two cocycles \( \bar{\psi}^W_3 \) and \( \bar{\psi}^W_4 \) from the outset. It seems likely that
a deep generalization of their modules exists, if one starts from the four inequivalent Virasoro-like extensions of $DRO(N)$ instead. The $diff(1)$ factor should then provide the necessary extra-grading. Finally, Fock modules for extensions of current algebras that are similar to, but different from, the Mickelsson-Faddeev algebra have recently been constructed [21].

This work can be extended in several directions. One can consider subalgebras of $diff(N)$ such as algebras of divergence-free, Hamiltonian or contact vector fields, or superize by letting some coordinates become fermionic. I expect no essential difficulties here, except that I am not aware of any classification of extensions of superdiffeomorphism algebras.

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