Supercongruences and Complex Multiplication

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Abstract

We study congruences involving truncated hypergeometric series of the form

\[ 3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda \right)_{\left(\frac{mp^s-1}{2}\right)} := \sum_{k=0}^{\left(\frac{mp^s-1}{2}\right)} \frac{(1/2)_{k!}}{k!} (1/2) \lambda^k \]

where \( p \) is a prime and \( m, s \) are positive integers. These truncated hypergeometric series are related to the arithmetic of a family of K3 surfaces. For special values of \( \lambda \), with \( s = 1 \), our congruences are stronger than those predicted by the theory of formal groups, because of the presence of elliptic curves with complex multiplications. They generalize a conjecture made by Stienstra and Beukers for the \( \lambda = 1 \) case and confirm some other supercongruence conjectures at special values of \( \lambda \).

Keywords: Complex multiplication, truncated hypergeometric series, supercongruences, Atkin and Swinnerton-Dyer congruences

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1. Introduction

The hypergeometric series \( _rF_{r-1} \) is defined as

\[ _rF_{r-1}\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_{r-1} \end{array}; \lambda \right) := \sum_{k=0}^{\infty} \frac{(a_1)_{k!}(a_2)_{k!} \cdots (a_r)_{k!}}{(b_1)_{k!}(b_2)_{k!} \cdots (b_{r-1})_{k!}} \lambda^k \]

where \((a)_{k!} := a(a+1) \cdots (a+k-1)\) and where none of the \( b_i \) is 0 or a negative integer [5]. The truncated hypergeometric series \( _rF_{r-1}\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_{r-1} \end{array}; \lambda \right)_{n} \) is the degree \( n \) polynomial in \( \lambda \) obtained by truncating the hypergeometric series to the sum over \( k \) from 0 to \( n \).

In this paper, we study the arithmetic of \( _3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array}; \lambda \right)_{n} \); these values are related to a family of K3 surfaces

\[ S_\lambda : W^2 = X_1X_2X_3(X_1 - X_2)(X_2 - X_3)(X_3 - \lambda X_1) \]

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with generic Picard number 19, that has been studied in [4, 17]. The variation of the complex structure of this family is depicted by its Picard-Fuchs differential equation, which is an order-3 ordinary differential equation. Up to multiplication by a scalar, its unique holomorphic solution near 0 is \( 3F_2 \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3} \right) \). Moreover, the Picard-Fuchs equation of the family \( S_\lambda \) is projectively equivalent to the symmetric square of the Picard-Fuchs equation of

\[ E_\lambda : y^2 = (x - 1) \left( x^2 - \frac{1}{1 - \lambda} \right); \]

see [17]. In terms of arithmetic, if we let \( A_p(\lambda) = \#(S_\lambda/\mathbb{F}_p) - p^2 - 1 \) and \( a_p(\lambda) = p + 1 - \#(E_\lambda/\mathbb{F}_p) \), then \( A_p(\lambda) = \left( \frac{1 - \lambda}{p} \right) \left( a_p(\lambda)^2 - p \right) \) [4].

Deuring’s argument [10, pp. 255] shows that for any \( \lambda \in \mathbb{F}_p \),

\[ A_p(\lambda) \equiv 3F_2(\lambda)_{p-1} \equiv 3F_2(\lambda)_{p-1} \quad (\text{mod } p). \]

More generally Dwork showed in [11] that for any \( \lambda \in \mathbb{Z}_p \), there is a \( p \)-adic number \( \gamma(\lambda) \) such that

\[ 3F_2(\lambda)_{mp^s-1} \equiv \gamma(\lambda) \cdot 3F_2(\lambda^p)_{mp^s-1} \quad (\text{mod } p^s) \quad (1) \]

for all integers \( m, s \geq 1 \).

It can be shown that these congruences come from a formal group structure attached to \( S_\lambda \), as constructed by Stienstra [24]. In particular, when \( 3F_2(\lambda)_{p-1} \neq 0 \) (mod \( p \)) (i.e. \( p \) ordinary for \( S_\lambda \)), one can use the so-called Shioda-Inose structure of the K3 surfaces \( S_\lambda \) to show that \( \gamma(\lambda) = \left( \frac{1 - \lambda}{p} \right) \alpha_p(\lambda)^2 \), with \( \alpha_p(\lambda) \) being the unit root of \( X^2 - a_p(\lambda)X + p = 0 \).

At special values of \( \lambda \), stronger congruences have been observed for the truncations \( 3F_2(\lambda)_n \). Such congruences, that are stronger than what can be predicted from the formal group structure, are known as supercongruences. For example, Stienstra and Beukers conjectured the following supercongruences involving truncated hypergeometric series in [25], corresponding to our \( \lambda = 1 \) case: for odd primes \( p \),

\[ 3F_2 \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 1 \right)_{p-1} = \sum_{k=0}^{p-1} \left( \frac{1}{3} \right) \left( \frac{k}{p-1} \right) \equiv b_p \quad (\text{mod } p^2) \]

where \( b_p \) is the \( p \)-th coefficient of the weight 3 cusp form \( \eta(4z)^6 \), where \( \eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) with \( q = e^{2\pi iz} \), is the eta function. This conjecture was proved by Van Hamme in

\[ \text{[1] The surfaces } X_3 \text{ with affine model } X_3 : s^2 = xy(x+1)(y+1)(x+\lambda y) \text{ studied in [4] are isomorphic to } S_\lambda \text{ via } \lambda = -\bar{\lambda}, X_1 = 1, X_2 = -1/y, X_3 = x/y, \text{ and } W = s/y^3. \text{ Note that our } \lambda \text{ is the negative of the } \bar{\lambda} \text{ in [4].} \]
[30], with subsequent proofs by Ishikawa [14] and Ahlgren [1]. More recently, using a different technique, it is shown in [19] that for any prime $p \equiv 1 \pmod{4}$,

$$3F_2\left(\frac{1}{2}, 1, 1; 1; 1\right)_{p-1} = -\Gamma_p \left(\frac{1}{4}\right)^4 \pmod{p^3}$$

where $\Gamma_p(\cdot)$ denotes the $p$-adic Gamma function; there is a similar expression for primes which are congruent to 3 modulo 4.

Similarly, Z.-W. Sun conjectured (see remark 1.4 in [27]) a congruence for the $\lambda = 64$ case:

$$3F_2\left(\frac{1}{2}, 1, 1; 1; 1; 64\right)_{p-1} = \sum_{k=0}^{p-1} \left(\frac{1}{2}k\right)^3 (64)^k \equiv a_p \pmod{p^2}$$

where $a_p = 0$ if $p \equiv 3, 5, 6 \pmod{7}$ and $a_p = 4x^2 - 2p$ where $p = x^2 + 7y^2, x, y \in \mathbb{Z}$, if $p \equiv 1, 2, 4 \pmod{7}$. In fact, this $a_p$ is just the $p$th coefficient of $\eta(z)^3 \eta(7z)^3$.

We show that such supercongruences occur for $3F_2(\lambda)_n$ whenever the elliptic curve $E_{\lambda}$ has complex multiplications (CM):

**Theorem 1.** Let $\lambda$ be an algebraic number such that $E_{\lambda}$ has complex multiplications. Let $p$ be a prime and $E_{\lambda}$ have a model defined over $\mathbb{Z}_p$ with good reduction modulo $p\mathbb{Z}_p$. Then

$$3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda\right)_{p-1} = \sum_{k=0}^{p-1} \left(\frac{1}{2}k\right)^3 \lambda^k \equiv \left(\frac{1 - \lambda}{p}\right) \alpha_p(\lambda)^2 \pmod{p^2}$$

where $\alpha_p(\lambda) \in \mathbb{Z}_p$ is the unit root of $X^2 - [p+1-\#(E_{\lambda}/\mathbb{F}_p)]X + p = 0$ if $E_{\lambda}$ is ordinary at $p$; and $\alpha_p(\lambda) = 0$ if $E_{\lambda}$ is supersingular at $p$.

This result confirms the Conjecture of Sun mentioned above. The rational values of $\lambda$ such that $E_{\lambda}$ has CM are $\lambda = -1, 4, -8, 64, \frac{1}{4}, \frac{1}{8}$, and $\frac{1}{64}$ [4]; but note that this theorem applies also to algebraic CM values of $\lambda$ and primes $p$ such that $\lambda$ can be embedded in $\mathbb{Z}_p$. For example, $E_{\lambda}$ is CM when $\lambda = \frac{7}{8} + \frac{5\sqrt{2}}{8}$, and Theorem 1 applies to both embeddings of $\lambda$ in $\mathbb{Z}_p$ for $p \equiv \pm 1 \pmod{8}$.

At each CM value $\lambda$ with $|\lambda| < 1$ in each embedding, there is a Ramanujan-type formula of the form $\sum_{k=0}^{\infty} (ak + 1) \left(\frac{1}{2}k\right)^3 \lambda^k = b \pi$ where $a, b$ are algebraic numbers depending on $\lambda$. Corresponding supercongruences for $\sum_{k=0}^{p-1} (ak + 1) \left(\frac{1}{2}k\right)^3 \lambda^k$ have been obtained in [7].

We derive the following corollary to Theorem 1 in section 4:
Corollary 2. Let $H_k$ be the harmonic sum $\sum_{j=1}^{k} \frac{1}{j}$. If $E_{\lambda}$ is a CM elliptic curve, then for almost all primes $p$ such that $\lambda$ embeds in $\mathbb{Z}_p$,

$$\sum_{i=0}^{p-1} \binom{2i}{i}^3 \left( \frac{\lambda}{64} \right)^i \left( 6(H_{2i} - H_i) + \left( \frac{\lambda}{p} \right)^{p-1} - 1 \right) \equiv 0 \pmod{p}.$$  

Below is one simple, special case of these congruences for $\lambda = 64$.

Corollary 3. For all primes $p > 3$, we have

$$\sum_{i=1}^{p-1} \binom{2i}{i}^3 \sum_{j=1}^{i} \frac{1}{i+j} \equiv 0 \pmod{p}. \quad (2)$$

In general, such congruences are difficult to prove. For similar work, see [1, 2] and remark 1 of [18].

Here are some other well-known examples of supercongruences. Beukers conjectured that for all odd primes $p$

$$4F3 \left( \frac{\frac{1}{2}, \frac{1}{p}, \frac{1}{2} + \frac{1}{p}, \frac{1}{2} + \frac{1}{p}}{1, 1, 1, 1} ; 1 \right) \equiv c_p \pmod{p^2}$$

where the left hand side\(^2\) is the $(p-1)/2$th Apéry number $\sum_{k=0}^{(p-1)/2} \binom{p-1/2}{k}^2 \binom{p-1/2+k}{k}^2$ and $c_p$ is the $p$th coefficient of the weight-4 modular form $\eta(2z)^4 \eta(4z)^4$.

In [22], Rodriguez-Villegas made many supercongruence conjectures, including that for all odd primes $p$

$$4F3 \left( \frac{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}}{1, 1, 1, 1} ; 1 \right)_{(p-1)/2} \equiv c_p \pmod{p^3}.$$  

Ahlgren and Ono proved the modulo $p^2$ conjecture of Beukers using Gaussian hypergeometric functions (see [2] and [21, Chapter 11]). Kilbourn applied these methods to prove the modulo $p^3$ conjecture of Rodriguez-Villegas [15], and McCarthy proved another of Rodriguez-Villegas’s modulo $p^3$ conjectures using a $p$-adic analogue of Gaussian hypergeometric functions [20].

We end our introduction with another motivation for supercongruences. It is known that the coefficients of weight-$k$ noncongruence modular forms satisfy the so-called Atkin and Swinnerton-Dyer congruences [6, 23]. These congruences are supercongruences if $k > 2$ [23] and have played an important role in understanding the characterizations of genuine noncongruence modular forms [16].

\(\text{Note that this hypergeometric series terminates after the } \frac{p-1}{2} \text{th term, because of the negative integer argument } \frac{1-p}{2}, \text{ while Rodriguez-Villegas’s conjecture that follows is a genuine truncation.}
The paper is organized as follows. We present some background in Section 2. Section 3 discusses supercongruences and uses a theorem of Coster and van Hamme to show that the function \(3F_2(\lambda)_n\) exhibits supercongruences whenever \(E_\lambda\) has CM. In section 4, we relate these supercongruences to some interesting combinatorial congruences.

2. Preliminaries

2.1. Legendre polynomials

Let \(P_n(x)\) denote the \(n\)th Legendre polynomial, which can be defined by

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad [5, 9, 26].
\]

Equivalently, the degree \(n\) Legendre polynomial can be defined as

\[
P_n(x) := 2F_1 \left( -n, n+1 ; 1 \middle| \frac{1-x}{2} \right) = \sum_{k=0}^{n} \binom{n}{k} \binom{-n-1}{k} \left( \frac{1-x}{2} \right)^k.
\]

The first few Legendre Polynomials are \(P_0(x) = 1\), \(P_1(x) = x\), and \(P_2(x) = \frac{3}{2}(3x^2 - 1)\). These polynomials form an important class of orthogonal polynomials and have several nice properties; one relevant to our application is that they have generating function

\[
(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n.
\]

Because of this, special values of \(P_n(x)\) show up in certain expansions of differential forms on elliptic curves.

2.2. The Atkin and Swinnerton-Dyer congruences

For elliptic curves of the form \(E : y^2 = x(x^2 + Ax + B)\) defined over \(\mathbb{Z}_p\) with \(t = x/y\) as a local parameter at the point at infinity (where \(t\) has a simple zero), Coster and van Hamme showed that the coefficients of the \(t\)-expansion of the invariant differential form \(-\frac{dx}{2y}\) of \(E\) come from special values of Legendre Polynomials (see formula (1) of [9]). Explicitly,

\[
-\frac{dx}{2y} = \sum_{k=0}^{\infty} a_k t^k \frac{dt}{t} = \sum_{k=0}^{\infty} P_k \left( \frac{A}{\sqrt{A^2 - 4B}} \right) (\sqrt{A^2 - 4B})^k t^{2k+1} \frac{dt}{t}.
\]

where \(a_{2k+1} = P_k \left( \frac{A}{\sqrt{A^2 - 4B}} \right) (\sqrt{A^2 - 4B})^k\) and \(a_{2k} = 0\).

The Atkin and Swinnerton-Dyer congruences (ASD) for elliptic curves (Theorem 4 of [6]) imply that if \(E\) has good reduction modulo \(p\), then for all positive integers \(m\) and for \(s \geq 0\),

\[
a_{mp^{s+1}} - A_p a_{mp^s} + p a_{mp^{s-1}} \equiv 0 \pmod{p^{s+1}}
\]

where \(A_p = p + 1 - \#(E/\mathbb{F}_p)\). We define \(a_k\) to be 0 if \(k\) is not integral, as may happen for the final term if \(s = 0\).

Essentially, the ASD congruences say that for fixed \(p\) and \(m\), terms of the sequence \(\{a_{mp^s}\}\) satisfy a three-term congruence with increasing \(p\)-adic precision as \(s\) increases.
The ASD congruences generalize the Hecke recursion: Fourier coefficients $b_n$ of weight $k = 2$, normalized Hecke newforms with trivial nebentypus satisfy the three-term recursion, for all positive integers $m$, for $s \geq 0$, and for all $p$,

$$b_{mp^s + 1} - b_p b_{mp^s} + pb_{mp^s - 1} = 0. \quad (6)$$

In the ASD congruences for an elliptic curve $E$, we distinguish two cases. If the middle coefficient $A_p$ is divisible by $p$, we say that $E$ is \textit{supersingular} at $p$ or simply that $p$ is supersingular. Otherwise, we say $E$ is \textit{ordinary} at $p$ or that $p$ is ordinary. Dwork’s congruences, in which consecutive ratios of certain terms in a sequence converge to a $p$-adic limit, are related to ASD congruences at ordinary primes. If $p$ is ordinary and is unramified in $K_p := \mathbb{Q}_p(\sqrt{A^2 - 4B})$, let $\beta_p$ be the $p$-adic unit root of $T^2 - [p + 1 - \#(E/\mathbb{F}_p)]T + p$. Then the ASD congruences imply that $a_{mp^s} \equiv \beta_p \cdot a_{mp^s - 1} \pmod{p^s}$.

Using the relation between $a_{2k+1}$ and the Legendre polynomial, we have for any good odd ordinary prime $p$ for $E$ unramified at $K_p$

$$P_{\frac{mp^s - 1}{2}} \left( \frac{A}{\sqrt{A^2 - 4B}} \right) \equiv \chi_p^{mp^s - 1} \cdot \beta_p \cdot P_{\frac{mp^s - 1}{2} - 1} \left( \frac{A}{\sqrt{A^2 - 4B}} \right) \pmod{p^s}, \quad (7)$$

where $\chi_p \in K_p$ is the (not necessarily primitive) order-4 root of unity satisfying $\chi_p \equiv \left( \frac{1}{\sqrt{A^2 - 4B}} \right)^{\frac{p-1}{2}} \pmod{p}$.

2.3. Clausen formula

It follows from the well-known Clausen formula for hypergeometric series and a Pfaff transformation that

$$2F_1 \left( -a, a + 1, 1 \pm \sqrt{1-x} \right) \equiv 3F_2 \left( 1, -a, a + 1 \right. \left. \frac{1}{2}; 1, 1, 1; x \right). \quad (8)$$

See equation (3.3) of [7] for a derivation of this formula.

Thus,

$$P_n(\sqrt{1-\lambda})^2 \equiv 3F_2 \left( \frac{1}{2}, -n, n + 1 \right. \left. \frac{1}{2}; 1, 1, 1; \lambda \right). \quad (9)$$

The following congruence of degree $(p - 1)/2$ polynomials holds coefficient-wise.

**Lemma 4.** Let $p$ be any odd prime. Then

$$3F_2 \left( \frac{1}{2}, \frac{1-p}{2}, \frac{1+p}{2} \right. \left. 1, 1, \frac{1}{2}; x \right) \equiv 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right. \left. 1, 1, \frac{1}{2}; x \right) \pmod{p^2 \mathbb{Z}_p[x]}.$$
Proof. We use Zudilin’s observation about rising factorials (see Lemma 1 in [8], [18]),
\[
\left(\frac{1}{2} + \epsilon\right)_k = \left(\frac{1}{2} + \epsilon\right)\left(\frac{1}{2} + \epsilon + 1\right) \cdots \left(\frac{1}{2} + \epsilon + k - 1\right)
= \left(\frac{1}{2}\right)_k \left(1 + 2\epsilon \sum_{j=1}^k \frac{1}{2j-1} + O(\epsilon^2)\right),
\]
to expand \((\frac{1+\sqrt{p}}{2})_k\) in terms of \((\frac{1}{2})_k\). When we take the product \((\frac{1-\sqrt{p}}{2})_k(\frac{1+\sqrt{p}}{2})_k\), the coefficients of \(p^1\) cancel; and so the product is congruent to \((\frac{1}{2})_k^2\) modulo \(p^2\), which establishes the coefficient-wise congruence. \qed

3. Supercongruences

To prove our main theorem, we use the following theorem of Coster and van Hamme.

**Theorem 5** (Coster and van Hamme, [9]). Let \(p\) be an odd prime. Let \(d\) be a square-free positive integer such that \(\left(\frac{-d}{p}\right) = 1\). Let \(K\) be an algebraic number field such that \(\sqrt{-d} \in K\) and \(K \subset \mathbb{Q}_p\). Consider the elliptic curve
\[
\mathcal{E} : Y^2 = X(X^2 + AX + B)
\]
with \(A, B \in K\), where \(A\) and \(\Delta = A^2 - 4B\) are \(p\)-adic units. Let \(\omega\) and \(\omega'\) be a basis of periods of \(\mathcal{E}\) and suppose that \(\tau = \omega'/\omega \in \mathbb{Q}(\sqrt{-d})\) (which implies that the curve has complex multiplication), \(\tau\) has positive imaginary part, and \(A = 3\wp(\frac{1}{2}\omega)\), \(\wp(\sqrt{\Delta}) = \wp(\frac{1}{2}\omega') - \wp(\frac{1}{2}\omega)\), where \(\wp\) is the Weierstrass \(\wp\)-function. Let \(\pi, \bar{\pi} \in \mathbb{Q}(\sqrt{-d})\) such that \(\pi\bar{\pi} = p\), with \(\bar{\pi}\) a \(p\)-adic unit, \(\pi = u_1 + v_1\tau\), and \(\pi\tau = u_2 + v_2\tau\) with \(u_1, v_1, u_2, v_2\) integers and \(v_1\) even. Then we have
\[
P_{mp^{r-1}}\left(\frac{A}{\sqrt{\Delta}}\right) \equiv \varepsilon^{mp^{r-1}} \cdot \bar{\pi} \cdot P_{mp^{r-1}-1}\left(\frac{A}{\sqrt{\Delta}}\right) \pmod{p^{2r}},
\]
where \(m\) and \(r\) are positive integers, with \(m\) odd, and \(\varepsilon = (\sqrt{-1})^{-u_2v_2 + v_2 + p - 2}\), where \(P_n(x)\) is the \(n\)th Legendre polynomial.

The main point of the theorem is the existence of supercongruences arising from an elliptic curve \(\mathcal{E}\) with complex multiplication. While Coster and van Hamme interpreted the congruence as inclusion in an ideal of the ring of integers of \(K\), we interpret all of our congruences \(p\)-adically. Since the number field \(K\) embeds into \(\mathbb{Q}_p\) and \(\pi\) is just \(p\) times a \(p\)-adic unit under this embedding, we may simply replace \(\pmod{p^{2r}}\) with \(\pmod{p^{2r}}\) when we view the congruence \(p\)-adically. These congruences are twice as strong as formal group theory predicts.

Note that \(\varepsilon\) and \(\bar{\pi}\) in the theorem above must correspond to \(\pm \chi_p\) and \(\pm \beta_p\) (with the same sign) in our notation. The conditions that \(\tau\) has positive imaginary part,
that $A = 3\varphi(\frac{1}{2}\omega)$, and that $\sqrt{\Delta} = \varphi(\frac{1}{2}\omega' + \frac{1}{2}\omega) - \varphi(\frac{1}{2}\omega')$, can always be satisfied by a suitable choice of the basis of periods; and we can additionally ensure that $\varepsilon = \chi_p$ and $\overline{\pi} = \beta_p$.

**Proposition 6.** For CM values $\lambda$ of the family $E_\lambda : y^2 = (x - 1)(x^2 - \frac{1}{1-\lambda})$, such that $\lambda \in \mathbb{Z}_p$ and $p$ is ordinary, for all positive integers $m$ and $s$ with $m$ odd,

$$3F_2\left(\frac{1}{2}, \frac{1-mp^s}{2}, \frac{1+mp^s}{2}; \lambda\right) \equiv \left(\frac{1-\lambda}{p}\right) \cdot \alpha_p(\lambda)^2 \cdot 3F_2\left(\frac{1}{2}, \frac{1-mp^{s-1}}{2}, \frac{1+mp^{s-1}}{2}; \lambda\right) \quad \text{(mod } p^{2s})$$

where $\alpha_p(\lambda)$ is the unit root of $X^2 - [p + 1 - \#(E_\lambda/\mathbb{F}_p)]X + p = 0$.

**Proof.** Letting $X = x - 1$, $Y = y$, the elliptic curve $E_\lambda$ can be rewritten as $Y^2 = X(X^2 + 2X + \frac{\lambda}{\lambda-1})$. Then we have $A = 2, B = \frac{\lambda}{\lambda-1}, \Delta = A^2 - 4B = \frac{4}{1-\lambda}$, and $A\sqrt{AB-4B} = \sqrt{1-\lambda}$.

By our assumptions that $E_\lambda$ has complex multiplication, that $\lambda \in \mathbb{Z}_p$, and that $p$ is ordinary, we satisfy the conditions of Theorem 5: $K = \mathbb{Q}(\lambda)$ embeds into $\mathbb{Q}_p$ and $\lambda \not\equiv 1 \pmod{p}$, so $\Delta$ is a $p$-adic unit. Combining (9) and (10), we have

$$3F_2\left(\frac{1}{2}, \frac{1-mp^s}{2}, \frac{1+mp^s}{2}; \lambda\right) = P_{mp^{s-1}}\left(\sqrt{1-\lambda}\right)^2$$

$$\equiv (\varepsilon^{mp^{s-1}}\overline{\pi})^2 \cdot P_{mp^{s-1}-1}\left(\sqrt{1-\lambda}\right)^2 \quad \text{(mod } p^{2s})$$

$$= (\chi_p^{mp^{s-1}}\beta_p)^2 \cdot 3F_2\left(\frac{1}{2}, \frac{1-mp^{s-1}}{2}, \frac{1+mp^{s-1}}{2}; \lambda\right)$$

$$= \left(\frac{1-\lambda}{p}\right) \cdot \alpha_p(\lambda)^2 \cdot 3F_2\left(\frac{1}{2}, \frac{1-mp^{s-1}}{2}, \frac{1+mp^{s-1}}{2}; \lambda\right).$$

For the final equality, note that we have chosen $A$ and $B$ so that $\beta_p = \alpha_p(\lambda)$ and so that $\chi_p^2$ is the Legendre symbol $\left(\frac{1-\lambda}{p}\right)$, which does not change when we raise it to the odd power $mp^{s-1}$. \hfill \Box

**Proof of Theorem 1.** Note that for all primes $p$ that are ordinary for $E_\lambda$, Theorem 1 follows from Proposition 6 and Lemma 4. For primes $p$ that are supersingular for $E_\lambda$, we can conclude from the ASD congruence that $P_{mp^{s-1}}\left(\sqrt{1-\lambda}\right) \equiv 0 \pmod{p}$. Hence

$$3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda\right)_{p^{s-1}} \equiv 3F_2\left(\frac{1}{2}, \frac{1-p}{2}, \frac{1+p}{2}; \lambda\right) = P_{mp^{s-1}}\left(\sqrt{1-\lambda}\right)^2 \equiv 0 \pmod{p^2},$$

which concludes the proof. \hfill \Box

We note that this establishes, modulo $p^2$, all cases of Conjecture 5.2 of [27] by Z.-W. Sun. These conjectures can be written as

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \left(\frac{\lambda}{64}\right)^k \equiv \begin{cases} \left(\frac{p}{2}\right) (4a^2 - 2p) \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = 1 \text{ where } a^2 + Db^2 = p, \\ 0 \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = -1. \end{cases}$$

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with appropriate choices of $D \in \mathbb{Z}_+$ and character $\left( \frac{3}{p} \right)$. Note that $\sum_{k=0}^{p-1} \left( \frac{2k}{p} \right)^3 \left( \frac{\lambda}{64} \right)^k = 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda \right)_{\frac{1}{2}} - \frac{1}{p}$, via the identity $\frac{(1/2)x^3}{k!} = \left( \frac{2k}{p} \right)^3 \frac{1}{64^k}$. These conjectures address the $\lambda$-values $\lambda = -8, 1, -\frac{1}{4}, 4, \frac{1}{4}, 64, \frac{1}{64}, -1$, which are all of the CM values for $E_\lambda$ over $\mathbb{Q}$, as verified in [4], with the exception of the degenerate case $\lambda = 1$, for which $E_\lambda$ is not an elliptic curve. The supercongruence for $\lambda = 1$ was proved by Van Hamme in [29] and by Ono in [21].

If $E_\lambda$ has CM over $K = \mathbb{Q}(\sqrt{-D})$, then the attached 2-dimensional representation $\rho$ decomposes into 2 Grossencharacters when $\rho$ is restricted to $\text{Gal}(\overline{\mathbb{Q}}/K)$. At splitting primes $p$, which are precisely the ordinary primes of $E_\lambda$, the trace of the Frobenius is $\alpha_p(\lambda) + \beta_p(\lambda)$, where both $\alpha_p(\lambda)$ and $\beta_p(\lambda)$ are in the ring of integers of the quadratic field $K$ and have absolute value $\sqrt{p}$. In the case that $K$ has class number 1 (all Sun $\lambda$ values correspond to class number 1 cases), then ideals $(\alpha_p(\lambda))$ and $(\beta_p(\lambda))$ are the two distinct prime ideals above $p$. That is, $\alpha_p(\lambda) = a + b\sqrt{-D}$ and $\beta_p(\lambda) = a - b\sqrt{-D}$, where $a$ and $b$ are integers or half integers depending on $D \equiv 1$ or 3 (mod 4), such that $a^2 + b^2D = p$. In the ordinary case, our congruences involve $\alpha_p(\lambda)^2$, which is just $a^2 - Db^2 + 2ab\sqrt{-D}$. Using $\beta_p(\lambda)^2 = a^2 - Db^2 - 2ab\sqrt{-D} \equiv 0$ (mod $p^2$) and $a^2 + b^2D = p$, we have $\alpha_p(\lambda)^2 \equiv 4a^2 - 2p$ (mod $p^2$), which, along with the character $\left( \frac{1-\lambda}{p} \right)$, is the target of Z.-W. Sun’s congruences in the case that $\left( \frac{\lambda}{D} \right) = 1$. In the case that $\left( \frac{\lambda}{D} \right) = -1$, $p$ is a supersingular prime of $E_\lambda$ and so $3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda \right)_{\frac{1}{2}} - \frac{1}{p} \equiv 0$ (mod $p^2$), establishing the other half of Z.-W. Sun’s congruences.

Alternatively, we note that Ono has explicitly identified the values $\alpha_p(\lambda)$, for all CM curves $E_\lambda$ with $\lambda \in \mathbb{Z}$, in Theorem 6 of [21]. These values $\alpha_p(\lambda)$ determine the formal group structure and the ASD congruences (i.e., that $\alpha_p(\lambda) \equiv \left( \frac{1-\lambda}{p} \right) \alpha_p(\lambda)^2$ (mod $p$)); combining this with Coster and Van Hamme’s supercongruences gives another proof of Sun’s conjectures, that $\alpha_p(\lambda) \equiv \left( \frac{1-\lambda}{p} \right) \alpha_p(\lambda)^2$ (mod $p^2$).

Theorem 1, and the following Conjecture 7, apply not only to the cases considered by Ono and Z.-W. Sun, which correspond to CM values of $\lambda$ over $\mathbb{Q}$, but also to infinitely many other algebraic CM values of $\lambda$, for those primes $p$ such that $\lambda$ embeds in $\mathbb{Z}_p$ with $\lambda \not\equiv 0, 1$ (mod $p$). This is satisfied by almost all primes $p$ such that there is a prime ideal $p$ above $p$ in $\mathbb{Q}(\lambda)$ with inertia degree 1.

Based on numeric evidence, we have

**Conjecture 7.** For CM values $\lambda$ of the family $E_\lambda : y^2 = (x - 1)(x^2 - \frac{1}{1-\lambda})$, such that $\lambda \in \mathbb{Z}_p$ and $p$ is ordinary, for all positive integers $m$ and $s$ with $m$ odd,

$$3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda \right)_{mp^s-1} \equiv \left( \frac{1 - \lambda}{p} \right) \alpha_p(\lambda)^2 \quad 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda \right)_{mp^s-1} \equiv \alpha_p(\lambda)^2 \quad (\text{mod } p^{3s})$$

where $\alpha_p(\lambda)$ is the unit root of $X^2 - [p + 1 - \#(E_\lambda/\mathbb{F}_p)]X + p = 0$. 

4. Corollaries

An idea of Gessel for dealing with the supercongruences of the Apéry numbers

\[ c_n = 4F_3\left(\begin{array}{c}-n, -n, 1 + n, 1 + n; \\ 1, 1, 1 \end{array}; 1\right) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n + k}{k}^2 \]

is as follows [12]. He identified the auxiliary sequence \( d_n = 2 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 (H_{n+k} - H_{n-k}) \), where \( H_k \) is the harmonic sum \( \sum_{j=1}^{k} \frac{1}{j} \), and showed that \( c_{k+pn} \equiv (c_k + pd_k)c_n \) (mod \( p^2 \)) where \( 0 \leq k < p \). Using the idea of Ishikawa [13], we take \( k = n = \frac{p-1}{2} \).

It follows that when \( c_{(p-1)/2} \not\equiv 0 \pmod{p} \), we have the supercongruence \( c_{(p^2-1)/2} \equiv c_{(p-1)/2}^2 \pmod{p^2} \) precisely when \( d_{(p-1)/2} \equiv 0 \pmod{p} \), which follows from the \( p \)-adic properties of Harmonic sums. In [2], Ahlgren and Ono also need an entity similar to \( d_{(p-1)/2} \) to be zero modulo \( p \), which they established using a binomial coefficient identity proved by the WZ method [3].

In the above examples, supercongruences of a sequence \( c_n \) were shown to be equivalent to congruences of an auxiliary sequence \( d_n \); and the congruences for \( d_n \) were proved using whatever method applied in each case. Similarly, the supercongruence in Theorem 1 for the sequence \( a_n = \sum_{i=0}^{n} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \) is equivalent to the auxiliary congruence in Corollary 2 for the sequence \( d_n = \sum_{i=0}^{n} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i (6(H_{2i} - H_i) + \frac{(\lambda/64)^{p-1}-1}{p}) \). However, we proved our supercongruence using the theorem of Coster and Van Hamme, and thus obtain our auxiliary congruence. We know of no direct proof of Corollary 2; we expect a proof for each fixed individual \( \lambda \) might require some combinatorial identity and additional intelligent guesses of WZ pairs to prove the identity, see [1, 2].

**Lemma 8.** For the sequence \( a_n = \sum_{i=0}^{n} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \), we introduce the auxiliary sequence \( d_n = \sum_{i=0}^{n} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i (6(H_{2i} - H_i) + \frac{(\lambda/64)^{p-1}-1}{p}) \). Then for any prime \( p \), any \( k \) with \( \frac{p-1}{2} \leq k < p \), and any \( n \),

\[ a_{k+pn} \equiv a_{k}a_{n} + pd_k \sum_{i=0}^{n} i \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \pmod{p^2}. \]

**Proof.** Notice we can write \( a_{k+pn} - a_{k}a_{n} \) as the telescoping sum \( \sum_{i=1}^{n} T_{k,i} \), where

\[ T_{k,n} = (a_{k+pn} - a_{k}a_{n}) - (a_{k+p(n-1)} - a_{k}a_{n-1}) \]
\[ = (a_{k+pn} - a_{k+p(n-1)}) - a_k(a_{n} - a_{n-1}) \]
\[ = \sum_{i=-p+k+1}^{k} \binom{2i + 2pn}{i + pn}^3 \left(\frac{\lambda}{64}\right)^i - \left(\sum_{i=0}^{k} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \right) \binom{2n}{n} \binom{\lambda}{64}^n \]
Using the condition that \( \frac{p-1}{2} \leq k < p \), we notice that \( \binom{2i+2pm}{i+pn} \equiv 0 \pmod{p} \) if \(-p+k+1 < i < 0\). Simplifying modulo \( p^2 \), these terms disappear and we can factor.

\[
T_{k,n} \equiv \sum_{i=0}^{k} \left( \binom{2i+2pn}{i+pn} \right)^3 \left( \frac{\lambda}{64} \right)^{pn} - \left( \binom{2n}{n} \right)^3 \left( \frac{\lambda}{64} \right)^n \left( \frac{2i}{i} \right)^3 \left( \frac{\lambda}{64} \right)^i \quad \text{(mod } p^2\text{)}
\]

The factor \( \binom{2i+2pn}{i+pn}^3 \) may be rewritten as \( \frac{-\Gamma_p(1+2i+2pn)^3}{\Gamma_p(1+i+pn)^6} \binom{2n}{n}^3 \), where \( \Gamma_p \) is the \( p \)-adic gamma function (see Chapter 11 [21]). Let \( T_{k,n} \equiv \left( \frac{\lambda}{64} \right)^n \left( \frac{2n}{n} \right)^3 U_{k,n} \pmod{p^2} \), where

\[
U_{k,n} = \sum_{i=0}^{k} \left( \frac{-\Gamma_p(1+2i+2pn)^3}{\Gamma_p(1+i+pn)^6} \right) \left( \frac{\lambda}{64} \right)^{(p-1)n} \left( \frac{2i}{i} \right)^3 \left( \frac{\lambda}{64} \right)^i .
\]

To simplify the \( p \)-adic gamma function modulo \( p^2 \), we expand \( \Gamma_p \) in terms of factorials and harmonic sums \( H_n = \sum_{i=1}^{n} \frac{1}{i} \). (By convention, \( H_0 = 0 \).) We also use the congruence, for \( p > 3 \), that \( H_{p-1} \equiv 0 \pmod{p} \). (Wolstenholme has shown this congruence holds modulo \( p^2 \), though we only need modulo \( p \) [31].)

\[
\Gamma_p(1+i+pn)^r \equiv (-1)^{(1+i+pn)r} i^r (1+pnrH_i) \prod_{j=0}^{n-1} (p-1)!^r (1+pjrH_{p-1}) \pmod{p^2}
\]

\[
\equiv (-1)^{(1+i+pn)r} i^r (1+pnrH_i)(-1)^nr \pmod{p^2}
\]

\[
\equiv (-1)^{(1+i)r} i^r (1+pnrH_i) \pmod{p^2}
\]

Plugging this into \( U_{k,n} \), we have

\[
U_{k,n} \equiv \sum_{i=0}^{k} \left( \frac{(2i)!^6(1+6pnH_{2i})}{(i)!^6(1+6pnH_i)} \right) \left( \frac{\lambda}{64} \right)^{(p-1)n} \left( \frac{2i}{i} \right)^3 \left( \frac{\lambda}{64} \right)^i \pmod{p^2}
\]

\[
\equiv \sum_{i=0}^{k} \left( \frac{2i}{i} \right)^3 \left( \frac{\lambda}{64} \right)^i \left( 1+6pn(H_{2i}-H_i) \right) \left( \frac{\lambda}{64} \right)^{(p-1)n} - 1 \pmod{p^2}
\]

Using \( \left( \frac{\lambda}{64} \right)^{(p-1)n} = \left( 1+p \left( \frac{\lambda}{64} \right)^{p-1} \right)^n \equiv 1+pn \left( \frac{\lambda}{64} \right)^{p-1} \pmod{p^2}, \)

\[
U_{k,n} \equiv \sum_{i=0}^{k} \left( \frac{2i}{i} \right)^3 \left( \frac{\lambda}{64} \right)^i \left( 1+6pn(H_{2i}-H_i) \right) \left( 1+pn \left( \frac{\lambda}{64} \right)^{p-1} \right) - 1 \pmod{p^2}
\]

\[
\equiv pn \sum_{i=0}^{k} \left( \frac{2i}{i} \right)^3 \left( \frac{\lambda}{64} \right)^i \left( 6(H_{2i}-H_i) + \left( \frac{\lambda}{64} \right)^{p-1} \right) \pmod{p^2}
\]
So \( T_{k,n} \equiv p n (2n)^3 (\frac{\lambda}{64})^n d_k \pmod{p^2} \). Combining this congruence with the telescoping sum \( a_{k+pn} - a_{k}a_n = \sum_{i=1}^{n} T_{k,i} \) completes the proof of the lemma.

Using this lemma, we show the equivalence of Theorem 1 and Corollary 2.

**Proof of Corollary 2.** We consider \( T_{k,n} \) with \( k = \frac{p-1}{2} \) and \( n = 1 \). By definition, \( T_{\frac{p-1}{2},1} = a_{3 \frac{p-1}{2}} - a_{\frac{p-1}{2}} a_{3-1}; \) we can rewrite this, modulo \( p^2 \), as \( P_{3 \frac{p-1}{2}} (\sqrt{1-\lambda})^2 - P_{\frac{p-1}{2}} (\sqrt{1-\lambda})^2 P_{3-1} (\sqrt{1-\lambda})^2 \). Since the sequence \( P_{n-1} (\sqrt{1-\lambda}) \) satisfies ASD congruences, we know that \( T_{\frac{p-1}{2},1} \equiv 0 \pmod{p} \). However, Theorem 1 is precisely the information we need to conclude that \( T_{\frac{p-1}{2},1} \equiv 0 \pmod{p^2} \) whenever \( \lambda \) is a CM value of \( E_\lambda \) that embeds in \( \mathbb{Z}_p \).

Thus, since

\[
T_{\frac{p-1}{2},1} \equiv \frac{p\lambda}{8} \sum_{i=0}^{(p-1)/2} \left( 2i \right)^3 \left( \frac{\lambda}{64} \right)^i \left( 6(H_{2i} - H_i) + \left( \frac{\lambda}{64} \frac{p-1}{p} - 1 \right) \right) \pmod{p^2},
\]

we have the desired congruence \( d_{\frac{p-1}{2}} \equiv 0 \pmod{p} \) whenever we have supercongruences for \( a_{\frac{p-1}{2}} \).

**\( \square \)**

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