Natural lacunae method and Schatten-von Neumann classes of the convergence exponent

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Abstract

The first our aim is to clarify the results obtained by Lidskii devoted to the decomposition on the root vector system of the non-selfadjoint operator. We use a technique of the entire function theory and introduce a so-called Schatten-von Neumann class of the convergence exponent. Considering strictly accretive operators satisfying special conditions formulated in terms of the norm, we construct a sequence of contours of the power type on the contrary to the results by Lidskii, where a sequence of contours of the exponential type was used.

Keywords: Strictly accretive operator; Abel-Lidskii basis property; Schatten-von Neumann class; convergence exponent; counting function.

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1 Introduction

Generally the concept origins from the well-known fact that the eigenvalues of the compact selfadjoint operator form a basis in the closure of its range. The question what happens in the case when the operator is non-selfadjoint is rather complicated and deserves to be considered as a separate part of the spectral theory. Basically, the aim of the mentioned part of the spectral theory are propositions on the convergence of the root vector series in one or another sense to an element belonging to the closure of the operator range. Here, we should note when we say a sense we mean Bari, Riesz, Abel (Abel-Lidskii) senses of the series convergence [2], [13]. A reasonable question that appears is about minimal conditions that guaranty the desired result, for instance in the mentioned papers there considered a domain of the parabolic type containing the spectrum of the operator. In the paper [2], non-selfadjoint operators with the special condition imposed on the numerical range of values are considered. The main advantage of this result is a comparatively weak condition imposed upon the numerical range of values comparatively with the sectorial condition (see definition of the sectorial operator). Thus, the convergence in the Abel-Lidskii sense was established for an operator class wider when the class of sectorial operators. Here, we make comparison between results devoted to operators with the discrete spectra and operators with the compact resolvent for they can be easily reformulated from one to the other field.
The central idea of this paper is to formulate sufficient conditions of the Abel-Lidskii basis property of the root functions system for a sectorial non-selfadjoint operator of the special type. Considering such an operator class we strengthen a little the condition regarding the semi-angle of the sector, but weaken a great deal conditions regarding the involved parameters. Moreover, the central aim generates some prerequisites to consider technical peculiarities such as a newly constructed sequence of contours of the power type on the contrary to the Lidskii results [26], where a sequence of the contours of the exponential type was considered. Thus, we clarify results [26] devoted to the decomposition on the root vector system of the non-selfadjoint operator. We use a technique of the entire function theory and introduce a so-called Schatten-von Neumann class of the convergence exponent. Considering strictly accretive operators satisfying special conditions formulated in terms of the norm, using a sequence of contours of the power type, we invent a peculiar method how to calculate a contour integral involved in the problem in its general statement. Finally, we produce applications to differential equations in the abstract Hilbert space. In particular, the existence and uniqueness theorems for evolution equations with the right-hand side – a differential operator with a fractional derivative in final terms are covered by the invented abstract method. In this regard such operators as a Riemann-Liouville fractional differential operator, Kipriyanov operator, Riesz potential, difference operator are involved. Note that analysis of the required conditions imposed upon the right-hand side of the evolution equations that are in the scope leads us to relevance of the central idea of the paper. In this regard we should note a well-known fact [44], [20] that a particular interest appears in the case when a senior term of the operator at least is not selfadjoint for in the contrary case there is a plenty of results devoted to the topic wherein the following papers are well-known [16], [24], [31], [32], [44]. The fact is that most of them deal with a decomposition of the operator to a sum where the senior term must be either a selfadjoint or normal operator. In other cases the methods of the papers [21], [20] become relevant and allow us to study spectral properties of operators whether we have the mentioned above representation or not. Here, we should remark that the results of the papers [2], [31] applicable to study non-selfadjoin operators are based on the sufficiently strong assumption regarding the numerical range of values of the operator, whereas the methods [20] can be used in the natural way, if we deal with more abstract constructions formulated in terms of the semigroup theory [22]. The central challenge of the latter paper is how to create a model representing a composition of fractional differential operators in terms of the semigroup theory. Here we should note that motivation arouse in connection with the fact that a second order differential operator can be presented as a some kind of a transform of the infinitesimal generator of a shift semigroup and stress that the eigenvalue problem for the operator was previously studied by methods of theory of functions [36], [7]. Having been inspired by novelty of the idea we generalize a differential operator with a fractional integro-differential composition in the final terms to some transform of the corresponding infinitesimal generator of the shift semigroup. By virtue of the methods obtained in the paper [20] we managed to study spectral properties of the infinitesimal generator transform and obtained an outstanding result – asymptotic equivalence between the real component of the resolvent and the resolvent of the real component of the operator. The relevance is based on the fact that the asymptotic formula for the operator real component can be established in most cases due to well-known asymptotic relations for the regular differential operators as well as for the singular ones [11]. It is remarkable that the results establishing spectral properties of non-selfadjoint operators allow us to implement a novel approach to the problem of the basis property of the root vectors of non-selfadjoint operators. In its own turn the application of results connected with the basis property covers many problems in
the framework of the theory of evolution equations. However, as a main advantage, we establish an abstract formula for the solution. Moreover, the norm-convergence of the series representing the solution allows us to apply the methods of the approximation theory. Thus, we can claim that the offered approach is undoubtedly novel and relevant. As an application we produce the artificially constructed normal operator. This example indicates the relevance and significance of a variant of the natural lacunae method allowing us to formulate the optimal conditions in comparison with the Lidskii results [26].

2 Preliminaries

Let \( C, C_i, i \in \mathbb{N}_0 \) be real constants. We assume that a value of \( C \) is positive and can be different in various formulas but values of \( C_i \) are certain. Denote by \( \text{int} M, \text{Fr} M \) the interior and the set of boundary points of the set \( M \) respectively. Everywhere further, if the contrary is not stated, we consider linear densely defined operators acting on a separable complex Hilbert space \( \mathcal{H} \). Denote by \( \mathcal{B}(\mathcal{H}) \) the set of linear bounded operators on \( \mathcal{H} \). Denote by \( \tilde{L} \) the closure of an operator \( L \). We establish the following agreement on using symbols \( \tilde{L} \).

Denote by \( \text{D}(\mathcal{H}) \) the domain of definition, the range, and the kernel or null space of an operator \( L \) respectively. The deficiency (codimension) of \( \text{R}(L) \), dimension of \( \text{N}(L) \) are denoted by \( \text{def} T, \text{mul} T \) respectively. Assume that \( L \) is a closed operator acting on \( \mathcal{H} \), \( \text{N}(L) = 0 \), let us define a Hilbert space \( \mathcal{H}_L := \{ f, g \in \text{D}(L), (f, g)_{\mathcal{H}L} = (Lf, Lg)_{\mathcal{H}} \} \). Consider a pair of complex Hilbert spaces \( \mathcal{H}, \mathcal{H}_L \), the notation \( \mathcal{H}_+ \subset \mathcal{H} \) means that \( \mathcal{H}_+ \) is dense in \( \mathcal{H} \) as a set of elements and we have a bounded embedding provided by the inequality

\[
\| f \| \leq C_0 \| f \|_{\mathcal{H}_+}, \quad C_0 > 0, \quad f \in \mathcal{H}_+,
\]

moreover any bounded set with respect to the norm \( \mathcal{H}_+ \) is compact with respect to the norm \( \mathcal{H} \). Let \( L \) be a closed operator, for any closable operator \( \tilde{S} = L \), its domain \( \text{D}(S) \) will be called a core of \( L \). Denote by \( \text{D}_0(L) \) a core of a closable operator \( L \). Let \( \text{P}(L) \) be the resolvent set of an operator \( L \) and \( R_L(\zeta), \zeta \in \text{P}(L), [R_L := R_L(0)] \) denotes the resolvent of an operator \( L \). Denote by \( \lambda_i(L), i \in \mathbb{N} \) the eigenvalues of an operator \( L \). Suppose \( L \) is a compact operator and \( N := (L^*L)^{1/2}, r(N) := \dim \text{R}(N) \); then the eigenvalues of the operator \( N \) are called the singular numbers (s-numbers) of the operator \( L \) and are denoted by \( s_i(L), i = 1, 2, ..., r(N) \). If \( r(N) < \infty \), then we put by definition \( s_i = 0, i = r(N) + 1, 2, ..., \). According to the terminology of the monograph [13] the dimension of the root vectors subspace corresponding to a certain eigenvalue \( \lambda_k \) is called the algebraic multiplicity of the eigenvalue \( \lambda_k \). Let \( \nu(L) \) denotes the sum of all algebraic multiplicities of an operator \( L \). Denote by \( n(r) \) a function equals to the quantity of the elements of the sequence \( \{ a_n \}_{n=1}^\infty, [a_n] \uparrow \infty \) within the circle \( |z| < r \). Let \( A \) be a compact operator, denote by \( n_A(r) \) counting function a function \( n(r) \) corresponding to the sequence \( \{ s_i^{-1}(A) \}_{i=1}^\infty \). Let \( \mathcal{S}_p(\mathcal{H}) \), \( 0 < p < \infty \) be a Schatten-von Neumann class and \( \mathcal{S}_\infty(\mathcal{H}) \) be the set of compact operators. Suppose \( L \) is an operator with a compact resolvent and \( s_n(R_L) \leq C n^{-\nu}, n \in \mathbb{N}, 0 \leq \mu < \infty \); then we denote by \( \mu(L) \) order of the operator \( L \) in accordance with the definition given in the paper [14]. Denote by \( \text{Re} L := (L^* + L)/2, \text{Im} L := (L^* - L)/2i \) the real and imaginary components of an operator \( L \) respectively. In accordance with the terminology of the monograph [15] the set \( \Theta(L) := \{ z \in \mathbb{C} : z = (Lf, f)_{\mathcal{H}}, f \in \text{D}(L), \| f \|_{\mathcal{H}} = 1 \} \) is called the numerical range of an operator \( L \). An operator \( L \) is called sectorial if its numerical range belongs to a closed sector \( \Sigma_\nu(\theta) := \{ \zeta : |\arg(\zeta - i)| \leq \theta < \pi/2 \} \), where \( i \) is the vertex and \( \theta \) is the semi-angle of the sector \( \Sigma_\nu(\theta) \). If
we want to stress the correspondence between $\iota$ and $\theta$, then we will write $\theta_i$. An operator $L$ is called bounded from below if the following relation holds $\text{Re}(Lf, f)_0 \geq \gamma_L \|f\|_0^2$, $f \in D(L)$, $\gamma_L \in \mathbb{R}$, where $\gamma_L$ is called a lower bound of $L$. An operator $L$ is called accretive if $\gamma_L = 0$. An operator $L$ is called strictly accretive if $\gamma_L > 0$. An operator $L$ is called $m$-accretive if the next relation holds $(A + \zeta)^{-1} \in B(\mathcal{H})$, $\|(A + \zeta)^{-1}\| \leq (\text{Re}\zeta)^{-1}$, $\text{Re}\zeta > 0$. An operator $L$ is called $m$-sectorial if $L$ is sectorial and $L + \beta$ is $m$-accretive for some constant $\beta$. An operator $L$ is called symmetric if one is densely defined and the following equality holds $(Lf, g)_0 = (f, Lg)_0$, $f, g \in D(L)$.

Consider a sesquilinear form $t[\cdot, \cdot]$ (see [15]) defined on a linear manifold of the Hilbert space $\mathcal{H}$. Denote by $\tilde{t}$ the quadratic form corresponding to the sesquilinear form $t[\cdot, \cdot]$. Let $\mathfrak{h} = (t + t^*)/2$, $\mathfrak{t} = (t - t^*)/2i$ be a real and imaginary component of the form $t$ respectively, where $t[u, v] = t[v, u]$, $D(t^*) = D(t)$. According to these definitions, we have $\mathfrak{h}[\cdot] = \text{Re}t[\cdot]$, $\mathfrak{t}[\cdot] = \text{Im}t[\cdot]$. Denote by $\tilde{t}$ the closure of a form $t$. The range of a quadratic form $t[f]$, $f \in D(t)$, $\|f\|_0 = 1$ is called range of the sesquilinear form $t$ and is denoted by $\Theta(t)$. A form $t$ is called sectorial if its range belongs to a sector having a vertex $\iota$ situated at the real axis and a semi-angle $0 \leq \theta_i < \pi/2$. Suppose $t$ is a closed sectorial form; then a linear manifold $D_0(t) \subset D(t)$ is called core of $t$, if the restriction of $t$ to $D_0(t)$ has the closure $t$ (see [15], p.166)). Due to Theorem 2.7 [15], p.323] there exist unique $m$-sectorial operators $T_\iota$, $T_\mathfrak{h}$ associated with the closed sectorial forms $t$, $\mathfrak{h}$ respectively. The operator $T_\iota$ is called a real part of the operator $T_\mathfrak{h}$ and is denoted by $\text{Re}T_\mathfrak{h}$. Suppose $L$ is a sectorial densely defined operator and $t[u, v] := (Lu, v)_0$, $D(t) = D(L)$; then due to Theorem 1.27 [15], p.318] the corresponding form $t$ is closable, due to Theorem 2.7 [15], p.323] there exists a unique $m$-sectorial operator $T_\mathfrak{t}$ associated with the form $\tilde{t}$. In accordance with the definition [15], p.325] the operator $T_\mathfrak{t}$ is called a Friedrichs extension of the operator $L$. Everywhere further, unless otherwise stated, we use notations of the papers [13], [15], [17], [18], [43].

**Some properties of non-selfadjoint operators**

In this section we explore a special operator class for which a number of spectral theory theorems can be applied. As an application of the obtained abstract results we study a basis property of the root vectors of the operator in terms of the order of the operator real part. By virtue of such an approach we express a convergence exponent of s-numbers through the order of the operator real part. The theorem given below gives us a description of spectral properties of some class of non-selfadjoint operators.

**Theorem 1.** Assume that $L$ is a non-selfadjoint operator acting in $\mathcal{H}$, the following conditions hold

(H1) There exists a Hilbert space $\mathcal{H}_+ \subset \mathcal{H}$ and a linear manifold $\mathfrak{M}$ that is dense in $\mathcal{H}_+$. The operator $L$ is defined on $\mathfrak{M}$.

(H2) $|(Lf, g)_0| \leq C_1 \|f\|_{\mathcal{H}_+} \|g\|_{\mathcal{H}_+}$, $\text{Re}(Lf, f)_0 \geq C_2 \|f\|_{\mathcal{H}_+}^2$, $f, g \in \mathfrak{M}$, $C_1, C_2 > 0$.

Let $W$ be a restriction of the operator $L$ on the set $\mathfrak{M}$. Then the following propositions are true.

(A) We have the following classification

$$R_{W} \in \mathfrak{S}_p, p = \begin{cases} l, & l > 2/\mu, \mu \leq 1, \\ 1, & \mu > 1. \end{cases}$$
here and further $\mu$ is the order of $H := \text{Re} \hat{W}$. Moreover under the assumptions $\lambda_n(R_H) \geq C n^{-\mu}$, $n \in \mathbb{N}$, we have the following implication

$$\{R_{\hat{W}} \in \mathcal{S}_p, 1 \leq p < \infty\} \Rightarrow \mu p > 1.$$ 

\textbf{(B)} The following relation holds

$$\sum_{i=1}^{n} |\lambda_i(R_{\hat{W}})|^p \leq C \sum_{i=1}^{n} \lambda_i^p(R_H), 1 \leq p < \infty, \ (n = 1, 2, ..., \nu(R_{\hat{W}})),$$

moreover if $\nu(R_{\hat{W}}) = \infty$ and $\mu \neq 0$, then the following asymptotic formula holds

$$|\lambda_i(R_{\hat{W}})| = o \left(i^{-\tau}\right), \ i \to \infty, \ 0 < \tau < \mu.$$

\textbf{(C)} Assume that $\theta < \pi \mu/2$, where $\theta$ is the semi-angle of the sector $\Sigma_0(\theta) \supset \Theta(\hat{W})$. Then the system of root vectors of $R_{\hat{W}}$ is complete in $\mathcal{H}$.

Throughout the paper we formulate results in terms of the restriction $W$ on the set $\mathfrak{M}$ of the operator $L$ satisfying the Theorem \textbf{I} conditions. We also use the short-hand notations $A := R_{\hat{W}}, \mu := \mu(H)$, where $H := \text{Re} \hat{W}$.

\textbf{Some facts of the entire functions theory}

Here we introduce some notions and facts of the entire functions theory, we follow the monograph [25]. In this subsection, we use the following notations

$$G(z, p) := \left(1 - z\right) e^{z + \frac{z^2}{2} + ... + \frac{z^p}{p}}, \ G(z, 0) := (1 - z).$$

Consider such an entire function that its zeros satisfy the following relation for some $\lambda > 0$

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^\lambda} < \infty. \tag{1}$$

In this case we denote by $p$ the smallest integer number for which the following condition holds

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty. \tag{2}$$

It is clear that $0 \leq p < \lambda$. It is proved that under the assumption \textbf{(I)} the infinite product

$$\prod_{n=1}^{\infty} (z) := \prod_{n=1}^{\infty} G \left(\frac{z}{a_n}, p\right) \tag{3}$$

is uniformly convergent, we will call it a canonical product and call $p$ the genus of the canonical product. By the \textit{convergence exponent} $\rho$ of the sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}, \ a_n \neq 0, \ a_n \to \infty$ we mean
the greatest lower bound for numbers λ for which series (1) converges. Note that if λ equals to a convergence exponent then series (1) may or may not be convergent. For instance, the sequences \(a_n = 1/n^\lambda\) and \(1/(n\ln^2 n)^\lambda\) have the same convergence exponent \(\lambda = 1\), but in the first case the series (1) is divergent when \(\lambda = 1\) while in the second one it is convergent. In this paper, we have a special interest regarding the first case. Consider the following obvious relation between the convergence exponent \(\rho\) and the genus \(p\) of the corresponding canonical product:

\[\rho \leq p \leq p + 1.\]

It is clear that if \(\rho = p\), then the series (1) diverges for \(\lambda = \rho\), while \(\rho = p + 1\) means that the series converges (in accordance with the definition of \(p\)). In the monograph [25], it is considered a more precise characteristic of the density of the sequence \(\{a_n\}_{1}^{\infty}\) than the convergence exponent. Thus, there is defined a so-called counting function \(n(r)\) equals to a number of points of the sequence in the circle \(|z| < r\).

By upper density of the sequence, we call a number

\[\Delta = \lim \frac{n(t)}{t^\rho}.\]

If a limit exists in the ordinary sense (not in the sense of the upper limit), then \(\Delta\) is called the density. Note that it is proved in Lemma 1 [25] that

\[\lim \frac{n(t)}{t^{\rho + \varepsilon}} \to 0, \varepsilon > 0.\]

We need the following fact (see [25] Lemma 3).

**Lemma 1.** If the series (2) converges, then the corresponding infinite product (3) satisfies the following inequality in the entire complex plane

\[\ln \left| \prod_{n=1}^{\infty} (z) \right| \leq Cr^p \left( \int_{0}^{r} \frac{n(t)}{t^{p+1}} dt + r \int_{r}^{\infty} \frac{n(t)}{t^{p+2}} dt \right), \quad r := |z|.\]

Using this result, it is not hard to prove a relevant fact mentioned in the monograph [25]. Since it has a principal role in the further narrative, then we formulate it as a lemma in terms of the density.

**Lemma 2.** Assume that the following series is convergent for some values \(\lambda > 0\) i.e.

\[\sum_{n=1}^{\infty} \frac{1}{|a_n|^\lambda} < \infty.\]

Then the following relation holds

\[\left| \prod_{n=1}^{\infty} (z) \right| \leq e^{\beta(r)r^{p_1}}, \quad \beta(r) = r^{\rho - p_1} \left( \int_{0}^{r} \frac{n(t)}{t^{p+1}} dt + r \int_{r}^{\infty} \frac{n(t)}{t^{p+2}} dt \right), \quad (4)\]

In the case \(p_1 = \rho\), we have \(\beta(r) \to 0\), if at list one of the following conditions holds: the convergence exponent \(\rho < \lambda\) is non-integer and the density equals zero, the convergence exponent \(\rho = \lambda\) is arbitrary. In addition, the equality \(\rho = \lambda\) guaranties that the density equals zero. In the case \(p_1 > \rho\), we claim that \(\beta(r) \to 0\), without any additional conditions.
Proof. Applying Lemma 1 we establish relation (4). Consider a case when \( \rho < \lambda \) is non-integer. Taking into account the fact that the density equals zero, using L’Hôpital’s rule, we easily obtain

\[
r^{p-\rho} \int_0^r \frac{n(t)}{t^{p+1}} dt \to 0; \quad r^{p+1-\rho} \int_0^\infty \frac{n(t)}{t^{p+2}} dt \to 0,
\]

(here we should remark that if \( \rho \) is integer, then \( p = \rho \)). Therefore \( \beta(r) \to 0 \). Consider the case when \( \rho = \lambda \), then let us rewrite the series (1) in the form of the Stilites integral

\[
\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}} = \int_0^{\infty} \frac{dn(t)}{t^\rho}.
\]

Using integration by parts formulae, we get

\[
\int_0^r \frac{dn(t)}{t^\rho} = \frac{n(r)}{r^\rho} - \frac{n(\gamma)}{C^\rho} + \rho \int_0^r \frac{n(t)}{t^{p+1}} dt.
\]

Here, we should note that there exists a neighborhood of the point zero in which \( n(t) = 0 \). The latter representation shows us that the following integral converges i.e.

\[
\int_0^{\infty} \frac{n(t)}{t^{p+1}} dt < \infty.
\]

In its own turn, it follows that

\[
\frac{n(r)}{r^\rho} = n(r)\rho \int_0^r \frac{1}{t^{p+1}} dt < \rho \int_0^r \frac{n(t)}{t^{p+1}} dt \to 0, \quad r \to \infty.
\]

Using this fact, analogously to the above applying L’Hôpital’s rule, we conclude that (5) holds if \( \rho = \lambda \) is non-integer. If \( \rho = \lambda \) is integer then it is clear that we have \( \rho = p + 1 \), here we should remind that it is not possible to assume \( \rho = p \) due to the definition of \( p \). In the case \( \rho = p + 1 \), using the above reasonings, we get

\[
r^{-1} \int_0^r \frac{n(t)}{t^{p+1}} dt \to 0; \quad \int_r^{\infty} \frac{n(t)}{t^{p+2}} dt \to 0,
\]

from what follows the fact that \( \beta(r) \to 0 \). The reasonings related to the case \( \rho_1 > \rho \) is absolutely analogous, we left the proof to the reader. The proof is complete.

Lemma 3. We claim that the following implication holds

\[
\ln r \frac{n(r)}{r^{\rho_1}} \to 0, \quad \implies \beta(r) \ln r \to 0,
\]

where

\[
\beta(r) = r^{p-\rho_1} \left( \int_0^r \frac{n(t)}{t^{p+1}} dt + r \int_r^{\infty} \frac{n(t)}{t^{p+2}} dt \right), \quad \rho_1 \neq p, \quad \rho_1 \neq p + 1.
\]
Proof. To prove the fact $\beta(r) \ln r \to 0$, we should consider representation (4), we have
\[
\beta(r) \ln r = r^{p-\rho} \left( \int_0^r \frac{n(t)}{t^{p+1}} dt + r \int_r^\infty \frac{n(t)}{t^{p+2}} dt \right) \ln r.
\]
Let us define the following auxiliary functions
\[
u_1(r) := \ln r \int_0^r \frac{n(t)}{t^{p+1}} dt, \quad u_2(r) := r^{p-\rho} \quad \text{and} \quad v_1(r) := \ln r \int_r^\infty \frac{n(t)}{t^{p+2}} dt, \quad v_2(r) := r^{p-1}.
\]
It is clear that
\[
u_1(r) = \frac{1}{r} \int_0^r \frac{n(t)}{t^{p+1}} dt + \ln r \int_0^r \frac{n(r)}{t^{p+1}} dt; \quad \text{and} \quad v_1(r) = \frac{1}{r} \int_r^\infty \frac{n(t)}{t^{p+2}} dt + \ln r \int_r^\infty \frac{n(r)}{t^{p+2}} dt.
\]
Therefore
\[
\frac{u_1'(r)}{u_2'(r)} = C r^{p-\rho} \int_0^r \frac{n(t)}{t^{p+1}} dt + C \ln r \frac{n(r)}{r^{p+1}}; \quad \frac{v_1'(r)}{v_2'(r)} = C r^{p+1-\rho} \int_r^\infty \frac{n(t)}{t^{p+2}} dt + C \ln r \frac{n(r)}{r^\rho}.
\]
Notice that $\beta(r) \ln r = u_1(r)/u_2(r) + v_1(r)/v_2(r)$, applying L'Hôpital’s rule, we have
\[
\beta(r) \ln r \sim \frac{u_1'(r)}{u_2'(r)} + \frac{v_1'(r)}{v_2'(r)}, \quad r \to \infty.
\]
In an analogous way, we obtain the following implication
\[
\frac{n(r)}{r^\rho} \to 0, \quad \Rightarrow \quad \left\{ r^{p-\rho} \int_0^r \frac{n(t)}{t^{p+1}} dt \to 0; \quad r^{p+1-\rho} \int_r^\infty \frac{n(t)}{t^{p+2}} dt \to 0 \right\}.
\]
Thus, taking into account the premise $\ln r \cdot n(r)/r^\rho \to 0$, combining (6), (7), (8), we obtain the desired result.

Regarding Lemma 3, we can produce the following example that indicates the relevance of the issue itself.

Example 1. There exists a sequence $\{a_n\}^\infty_1$ such that density equals zero, moreover
\[
\beta(r) \ln r \to 0, \quad \sum_{n=1}^\infty \frac{1}{|a_n|^\rho} = \infty.
\]
We can construct the required sequence supposing $n(r) \sim r^\rho (\ln r \cdot \ln \ln r)^{-1}$, $\rho > 0$. It follows from the latter relation directly that the density equals zero. It is clear that we can represent partial sums of series (11) due to the Stieltjes integral
\[
\sum_{n=1}^k \frac{1}{|a_n|^\lambda} = \int_0^r \frac{dn(t)}{t^\lambda}, \quad \lambda \geq \rho.
\]
Thus the sequence \( \{a_n\}_1^\infty \) is defined by the function \( n(r) \). Applying the integration by parts formula, we get

\[
\int_0^r \frac{dn(t)}{t^\lambda} = \frac{n(r)}{r^\lambda} + \lambda \int_0^r \frac{n(t)}{t^{\lambda+1}} dt.
\]

Using the latter relation, we can easily establish the fact that the density equals zero while the last integral is divergent when \( \lambda = \rho, r \to \infty \), we have

\[
\int_0^r \frac{n(t)}{t^{\rho+1}} dt \geq C \int_0^r \frac{dt}{t \ln t \cdot \ln \ln t} = \ln \ln r - C.
\]

On the other hand, we have

\[
\int_0^\infty \frac{n(t)}{t^{\lambda+1}} dt = \int_0^\infty \frac{dt}{t^{1+\lambda-\rho} \ln t \cdot \ln \ln t} < \infty, \lambda > \rho.
\]

Therefore, the series \( \{1\} \) is divergent if \( \lambda = \rho \) and convergent if \( \lambda = \rho + \varepsilon, \varepsilon > 0 \). Thus, the denotation \( \rho \) is justified. We should explain that in these reasonings \( \rho \) has two meanings: a power and a convergence exponent, we have established the identity of them. Let us prove the fact \( \beta(r) \ln r \to 0 \), for this purpose in accordance with Lemma \( 3 \), it suffices to show that

\[
\ln r \frac{n(r)}{r^\rho} \to 0, r \to \infty.
\]

Substituting \( r^\rho (\ln r \cdot \ln \ln r)^{-1} \) instead of \( n(r) \), we get

\[
\ln r \frac{n(r)}{r^\rho} = \frac{1}{\ln \ln r} \to 0, r \to \infty.
\]

Thus, we have established the fulfilment of the made claims.

Schatten-von Neumann class and the particular case corresponding to the normal operator

Let \( \mathcal{G}_q(\mathcal{H}) \), \( 0 < q < \infty \) be a Schatten-von Neumann class and \( \mathcal{G}_\infty(\mathcal{H}) \) be the set of compact operators. By definition, put

\[
\mathcal{G}_q(\mathcal{H}) := \left\{ T : \mathcal{H} \to \mathcal{H}, \sum_{i=1}^\infty s_i^q(L) < \infty, 0 < q < \infty \right\}.
\]

Denote by \( \tilde{\mathcal{G}}_\rho(\mathcal{H}) \) the class of the operators such that

\[
\tilde{\mathcal{G}}_\rho(\mathcal{H}) := \{ T \in \mathcal{G}_{\rho+\varepsilon}, T \in \mathcal{G}_{\rho-\varepsilon}, \forall \varepsilon > 0 \}.
\]

This operator class we will call Schatten-von Neumann class of the convergence exponent. Note that there exists a one to one correspondence between selfadjoint compact operators and monotonically decreasing sequences. If we consider example \( \{1\} \) then we see that the made definition becomes relevant in this regard.
Lemma 4. Assume that
\[(\ln^{\kappa+1} x)^{\prime}_{\lambda_i(H)} = o(i^{-\kappa}), \ \kappa \in (0, 1].\]

Then in the general case, we get
\[A \in \tilde{\mathcal{S}}_\rho, \ \rho \in [0, 2/\kappa], \ n_A(r) = o(r^{2/\kappa}/\ln r).\]

In the particular case, when \(\tilde{W}\) is normal, we get
\[A \in \tilde{\mathcal{S}}_\rho, \ \rho \in [0, 1/\kappa], \ n_A(r) = o(r^{1/\kappa}/\ln r).\]

Moreover, the additional assumption \(\lambda_i(H) = O(i^{\kappa+\varepsilon}), \ \forall \varepsilon > 0\) gives us the estimate \(\rho \geq 1/\kappa\) in both cases, thus in the case when \(\tilde{W}\) is normal, we get
\[A \in \tilde{\mathcal{S}}_{1/\kappa}.\]

Proof. Note that the fact \(A \in \tilde{\mathcal{S}}_\rho, \ \rho \in [0, 2/\kappa]\), follows directly from the Theorem 1 claim (A). In accordance with relation (54) \([20]\), we have \(|A|^2 f, f)_B = \|Af\|_B^2 \leq C \cdot \text{Re}(A_f, f)_B = C(Vf, f)_B\), where \(V := (A + A^*)/2\). In accordance with the Theorem 5 \([20]\), we have \(\lambda_i(V) \approx \lambda_i(R_H)\), thus we have \(s_i(A) \leq C\lambda_i^{1/2}(R_H); \ s_i^{-1}(A) \geq C\lambda_i^{1/2}(H)\), the detailed proof of the latter fact see in the Theorem 7 \([20]\). Using the monotonous property of the functions, we have
\[
\frac{\ln^{\kappa} s_i^{-1}(A)}{s_i^{-2}(A)} \leq \frac{\ln^{\kappa} \lambda_i(H)}{\lambda_i(H)} \leq C \cdot \frac{\alpha_i}{i^{1/\kappa}},
\]
where \(\alpha_i \to 0\). Hence
\[
\frac{i \ln s_i^{-1}(A)}{s_i^{-2}(A)} \leq C \cdot \frac{\alpha_i^{1/\kappa}}{i^{1/\kappa}}.
\]
Taking into account the facts \(n(s_i^{-1}) = i; \ n(r) = n(s_i^{-1}), \ s_i^{-1} < r < s_{i+1}^{-1}\), using the monotonous property of the functions, we get
\[
\frac{n(r) \ln r}{r^{2/\kappa}} < C \cdot \alpha_i, \ s_i^{-1} < r < s_{i+1}^{-1}.
\]

The proof corresponding to the general case is complete. Assume that the operator \(\tilde{W}\) is normal, then it is not hard to prove that \(A\) is normal also. Let us show that the operator \(V := (A + A^*)/2\) has a complete orthonormal system of the eigenvectors. Using formula (53) \([20]\), we get
\[V^{-1} = 2H^{\frac{1}{2}}(I + B^2)H^{\frac{1}{2}}.\]

Note that in accordance with relation (67) \([20]\), we have
\[(V^{-1} f, f)_B = 2(SH^{\frac{1}{2}} f, H^{\frac{1}{2}} f)_B \geq 2\|H^{\frac{1}{2}} f\|_B^2 = 2(H f, f)_B, \ f \in D(V^{-1}), \]
where \(S = I + B^2\). Since \(V\) is selfadjoint, then due to Theorem 3 \([1]\, \text{p.136}\) the operator \(V^{-1}\) is selfadjoint also. Combining \([9]\) with Lemma 3 \([20]\), we get that \(V^{-1}\) is strictly accretive. Using these facts we can write
\[\|f\|_{V^{-1}} \geq C\|f\|_H, \ f \in \mathcal{D}_{V^{-1}},\]
where the above norms are understood as the norms of the energetic spaces generated by the operators $V^{-1}$ and $H$ respectively. Since the operator $H$ has a discrete spectrum (see Theorem 5.3 [19]), then any set bounded with respect to the norm $\mathcal{H}$ is a compact set with respect to the norm $\mathcal{H}$ (see Theorem 4 [35, p.220]). Combining this fact with (9), Theorem 3 [35, p.216], we get that the operator $V^{-1}$ has a discrete spectrum, i.e. it has the infinite set of the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \to \infty$, $i \to \infty$ and the complete orthonormal system of the eigenvectors. Now note that the operators $V, V^{-1}$ have the same eigenvectors. Therefore the operator $V$ has the complete orthonormal system of the eigenvectors. Recall that any complete orthonormal system forms a basis in the separable Hilbert space. Hence, the complete orthonormal system of the eigenvectors of the operator $V$ is a basis in the space $\mathcal{H}$. Since the operator $A$ is compact and normal, then in accordance with the well-known theorem we have a fact that there exists an orthonormal system of the eigenvectors $\{\psi_i\}_{i=1}^{\infty}$ of the operator $A$. The system is complete in $\mathcal{R}(A)$ in the following sense

$$f = \sum_{i=1}^{\infty} \psi_i(f, \psi_i)_{\mathcal{H}}, \ f \in \mathcal{R}(A).$$

The corresponding system of eigenvalues is such that

$$A\psi_i = \lambda_i(A)\psi_i, \ A^*\psi_i = \overline{\lambda_i(A)}\psi_i, \ i \in \mathbb{N}.$$

The latter facts give us $A^*A\psi_i = |\lambda_i(A)|^2\psi_i$. Since the operator $A^*A$ is selfadjoint and compact, then it is not hard to prove that $s_i(A) = |\lambda_i(A)|$ (see Lemma 3.3 Chapter II [13]). Thus, we get

$$s_i(A) = |(A\psi_i, \psi_i)_{\mathcal{H}}| = (1 + \tan^2 \theta_i) |\text{Re}(A\psi_i, \psi_i)_{\mathcal{H}}| = (1 + \tan^2 \theta_i) |(V\psi_i, \psi_i)_{\mathcal{H}}| = (1 + \tan^2 \theta_i) \lambda_i(V),$$

where the sequence $\{\tan^2 \theta_i\}_{i=1}^{\infty}$ is bounded by virtue of the sectorial property of the operator. Note that the fact $\mathcal{R}(A) = \mathcal{H}$ indicates that $\{\psi_i\}_{i=1}^{\infty}$ is complete in $\mathcal{H}$. It follows that the operators $V$ and $A$ have the same eigenvectors (since the complete system of the eigenvectors of the operator $V$ is minimal and at the same time it contains all eigenvectors of the operator $A$). Therefore, we can claim that all eigenvalues of the operator $V$ are involved in the right-hand side of relation (10). Taking into account the fact $\lambda_i(V) \asymp \lambda_i(R_H)$, we obtain the following relation

$$\sum_{i=1}^{\infty} |s_i(A)|^p \leq C_2 \sum_{i=1}^{\infty} |\lambda_i(R_H)|^p, \ p > 0.$$ 

Using the theorem condition, we have $A \in \mathcal{S}_p, p > 1/\kappa$. Hence $A \in \hat{\mathcal{S}}_{\rho}, \rho \leq 1/\kappa$. At the same time applying the above reasonings, we get

$$\frac{\ln^\kappa s_i^{-1}(A)}{s_i^{-1}(A)} \leq C \cdot \frac{\ln^\kappa \lambda_i(H)}{\lambda_i(H)} \leq C \cdot \frac{\alpha_i}{i^\kappa},$$

Using this fact, we obtain the following relation

$$\frac{n(r) \ln r}{r^{1/\kappa}} \to 0, \ r \to \infty.$$
Consider the additional condition \( \lambda_i(H) = O(i^{\kappa+\varepsilon}), \forall \varepsilon > 0 \) and let \( \{ \psi_i \}_1^\infty \) still be the complete orthonormal system of the eigenvectors of the operator \( V \). Suppose \( A \in \mathfrak{S}_p, p \geq 1 \), then by virtue of inequalities (7.9) Chapter III [13], the fact \( \lambda_i(V) \asymp \lambda_i(\mathcal{R}_H) \) (see Theorem 5 [19]), we get

\[
\sum_{i=1}^{\infty} |s_i(A)|^p \geq \sum_{i=1}^{\infty} |(A \varphi_i, \varphi_i)_H|^p \geq \sum_{i=1}^{\infty} |\text{Re}(A \varphi_i, \varphi_i)_H|^p = \\
= \sum_{i=1}^{\infty} |(V \varphi_i, \varphi_i)_H|^p = \sum_{i=1}^{\infty} |\lambda_i(V)|^p \geq C \sum_{i=1}^{\infty} i^{-(\kappa+\varepsilon)p}.
\]

Therefore \( A \in \mathfrak{S}_\rho, \rho \geq 1/\kappa \) since in the contrary case the relation \( p(\kappa + \varepsilon) > 1 \) does not hold. The proof is complete.

Consider the following example.

**Example 2.** Here we would like to produce an example of the sequence \( \{\lambda_i\}_1^\infty \) that satisfies the condition

\[
(\ln^{\kappa+1} x)_{\lambda_i} = o(i^{-\kappa}), \kappa \in (0, 1],
\]

\[
\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{1/\kappa}} = \infty.
\]

Consider a sequence \( \lambda_i = i^\kappa \ln^\kappa (i+q) \cdot \ln^\kappa \ln (i+q), q > e^\varepsilon - 1, i = 1, 2, \ldots \). Using the integral test for convergence, we can easily see that the previous series is divergent. At the same time substituting, we get

\[
\frac{\ln^\kappa \lambda_i}{\lambda_i} \leq \frac{C \ln^\kappa (i+q)}{i^\kappa \ln^\kappa (i+q) \cdot \ln^\kappa \ln (i+q)} = \frac{C}{i^\kappa \cdot \ln^\kappa \ln (i+q)},
\]

what gives us the fulfilment of the condition.

Bellow, we produce an auxiliary technique to study the central problem of the paper. The estimates for the Fredholm Determinant were studied by Lidskii in the paper [26] and gave a main tool in questions related to the estimation of the contour integrals. We have slightly improved results by Lidskii having involved the auxiliary function \( \beta \) and obtaining in this way more accurate results.

**Estimates for the Fredholm Determinant**

In this section we produce an adopted version of the propositions given in the paper [26], we consider a case when a compact operator belongs to the class \( \tilde{\mathfrak{S}}_\rho \). Having taken into account the facts considered in the previous subsection, we can reformulate Lemma 2 [26] in the refined form.

**Lemma 5.** Assume that a compact operator \( B \) satisfies the condition \( B \in \tilde{\mathfrak{S}}_\rho \), then for arbitrary numbers \( R, \delta \) such that \( R > 0, 0 < \delta < 1 \), there exists a circle \( |\lambda| = \tilde{R}, (1-\delta)R < \tilde{R} < R \), so that the following estimate holds

\[
\|(I - \lambda B)^{-1}\|_\rho \leq e^{\gamma(|\lambda|)|\lambda|^\rho} |\lambda|^m, |\lambda| = \tilde{R}, m = [\varrho], \varrho \geq \rho,
\]
Lemma 2, we get
\[ B \]
where \( \Delta \) with Theorem 11 [25, p.33], we have
\[ \xi \]
Therefore, applying Lemma 2, we get
\[ R,\delta \]
following estimate
\[ | \]

We can easily see that to obtain the desired result it suffices to estimate the term
\[ \| (I - \lambda^{m+1} B^{m+1})^{-1} \|_\delta \]
Using the obtained estimates, we have
\[ \| (I - \lambda^{m+1} B^{m+1})^{-1} \|_\delta \cdot \| (I + \lambda B + \lambda^2 B^2 + ... + \lambda^m B^m) \|_\delta \leq \]
\[ \leq \| (I - \lambda^{m+1} B^{m+1})^{-1} \|_\delta \cdot \frac{|\lambda|^{m+1} \| B \|^{m+1} - 1}{|\lambda| \cdot \| B \|^m - 1} \].

We can easily see that to obtain the desired result it suffices to estimate the term
\[ \| (I - \lambda^{m+1} B^{m+1})^{-1} \|_\delta \]
Using the obtained estimates, we have
\[ \| (I - \lambda^{m+1} B^{m+1})^{-1} \|_\delta \leq e^{\gamma(|\lambda|)\| B \|^m} \] where \( \gamma(|\lambda|) = \beta(|\lambda|^{m+1} + (2 + \ln\{12e/\delta\})\beta_m(2eλ|\lambda|^{m+1}(2e)^{m+1}) \). Thus, we obtain the desired result.

Proof. We consider the case \( \varrho = \rho \), the reasonings corresponding to the case \( \varrho > \rho \) can be fulfilled in accordance with the same scheme but much simpler and left to the reader. Using the definition, we have
\[ B \in \mathcal{G}_{\rho + \varepsilon}, \varepsilon > 0 \]. By direct calculation we get
\[ (I - \lambda^{m+1} B^{m+1})^{-1}(I + \lambda B + \lambda^2 B^2 + ... + \lambda^m B^m) = (I - \lambda B)^{-1}. \] (11)

In accordance with Lemma 3 [26], for sufficiently small \( \varepsilon > 0 \), we have
\[ \sum_{i=1}^{\infty} \lambda_i^{m+1}(\tilde{B}) \leq \sum_{i=1}^{\infty} \lambda_i^{m+1}(B) < \infty, \]
where \( \tilde{B} := (B^{m+1}B^{m+1})^{1/2} \). Applying inequality (1.27) [26, p.10] (since \( \rho/(m + 1) < 1 \), using Lemma 2 we get
\[ \| \Delta_{B^{m+1}}(\lambda^{m+1})^{-1}(I - \lambda^{m+1} B^{m+1})^{-1} \|_\delta \leq C \prod_{i=1}^{\infty} (1 + |\lambda_i^{m+1} s_i(B^{m+1})|) \leq C e^{\beta(r^{m+1})r^\rho}, \]
where \( \Delta_{B^{m+1}}(\lambda^{m+1}) \) is a Fredholm determinant of the operator \( B^{m+1} \) (see [26, p.8]). In accordance with Theorem 11 [25, p.33], we have
\[ \Delta_{B^{m+1}}(\lambda^{m+1}) \geq e^{-(2 + (\ln(12e/\delta)) \ln \xi_m) \xi_m} \]
where \( R,\delta \) arbitrary numbers such that \( R > 0, 0 < \delta < 1 \), the values of \( \lambda \) belong to the circle \( |\lambda| = \tilde{R} \), which radius is defined by \( R,\delta \) and satisfy the condition \( (1 - \delta)R < \tilde{R} < R \). Note that in accordance with the estimate (1.21) [26, p.10], we have
\[ \Delta_{B^{m+1}}(\lambda) \leq C \prod_{i=1}^{\infty} (1 + |\lambda s_i(B^{m+1})|). \]
Therefore, applying Lemma 2 we get \( \xi_m \leq e^{\beta(2e\tilde{R}^{(m+1)}(2e)^\rho)} \). Consider relation (11), we have the following estimate
\[ \| (I - \lambda B)^{-1} \|_\delta \leq \| (I - \lambda^{m+1} B^{m+1})^{-1} \|_\delta \cdot \| (I + \lambda B + \lambda^2 B^2 + ... + \lambda^m B^m) \| \leq \]
\[ \leq \| (I - \lambda^{m+1} B^{m+1})^{-1} \|_\delta \cdot \frac{|\lambda|^{m+1} \| B \|^{m+1} - 1}{|\lambda| \cdot \| B \|^m - 1}. \]

We can easily see that to obtain the desired result it suffices to estimate the term
\[ \| (I - \lambda^{m+1} B^{m+1})^{-1} \|_\delta. \]
Abel-Lidsky summarizing the series

In this subsection, we reformulate results obtained by Lidskii [26] in a more convenient form applicable to the reasonings of this paper. However, let us begin our narrative. In accordance with the Hilbert theorem (see [42], [13, p.32]) the spectrum of an arbitrary compact operator $B$ consists of the so called normal eigenvalues it gives us the opportunity to consider a decomposition

$$H = H_q + \mathfrak{M}_q,$$  \hspace{1cm} (12)

where both summands are invariant subspaces regarding the operator $B$, the first one is a finite dimensional root subspace corresponding to the eigenvalue $\mu_q$ and the second one is a subspace wherein the operator $B - \mu_q I$ is invertible. Let $n_q$ is a dimension of $H_q$ and let $B_q$ is the operator induced in $H_q$. We can choose a basis (Jordan basis) in $H_q$ that consists of Jordan chains of eigenvectors and root vectors of the operator $B_q$. Each chain $e_{q_1}, e_{q_1+1}, ..., e_{q_k+1}$, where $e_{q_i}, \xi \in \mathbb{N}$ are the eigenvectors corresponding to the eigenvalue $\mu_q$ and other terms are root vectors, can be transformed by the operator $A$ according with the following formulas

$$Be_{q_1} = \mu_q e_{q_1}, \quad Be_{q_1+1} = \mu_q e_{q_1+1} + e_{q_1}, ..., \quad Be_{q_k+1} = \mu_q e_{q_k+1} + e_{q_k+1}.$$  \hspace{1cm} (13)

Considering the sequence $\{\mu_i\}_{1}^{\infty}$ of the eigenvalues of the operator $B$ and choosing a Jordan basis in each corresponding space $H_i$ we can arrange a system of vectors $\{e_k\}_{1}^{\infty}$ which we will call a system of the root vectors or following Lidskii a system of the major vectors of the operator $A$. Let $e_1, e_2, ..., e_n$ be the Jordan basis in the subspace $H_i$, then in accordance with Lidskii [26] there exists a corresponding biorthogonal basis $g_1, g_2, ..., g_n$ in the space $M_i^\perp$ (see [26, p.14]), note that in accordance with our clarification $M_i^\perp = H_i$. Moreover the set $\{g_k\}_{1}^{n_i}$ consists of the Jordan chains of the operator $B^\ast$ which correspond to the Jordan chains (13) due to the following formula

$$B^* g_{q_k} = \mu_q g_{q_k}, \quad B^* g_{q_k+1} = \mu_q g_{q_k+1} + g_{q_k}, ..., \quad B^* g_{q_k} = \mu_q g_{q_k} + g_{q_k+1}.$$  \hspace{1cm} (14)

Let us show that $H_i \subset M_j$, $i \neq j$ for this purpose note that in accordance with the representation $P_{\mu_i} H = H_i$ and the property $P_{\mu_i} P_{\mu_j} = 0$, $i \neq j$, where $P_{\mu_i}$ is a Riesz projector (integral) corresponding to the eigenvalue $\mu_i$ (see [13] Chapter I §1.3), we have an orthogonal decomposition $H = H_i + H_j$ + $M_ij$, where $M_ij = (I - P_{q_k+\xi}) H_i$. On the other hand in accordance with [13] Chapter I §2.1 we can claim that the following orthogonal decomposition is unique

$$H = H_j + M_j,$$

hence we have an orthogonal sum $M_j = H_i + M_{ij}$, what proves the desired result. Taking into account relation (14), we conclude that the set $g_1, g_2, ..., g_n$, $i \neq j$ is orthogonal to the set $e_1, e_2, ..., e_n$. Gathering the sets $g_1, g_2, ..., g_n$, $i = 1, 2, ...,$, we can obviously create a biorthogonal system $\{g_i\}_{1}^{\infty}$ with respect to the system of the major vectors of the operator $B$. It is rather reasonable to call it as a system of the major vectors of the operator $B^\ast$. Note that if an element $f \in H$ allows a decomposition in the strong sense

$$f = \sum_{n=1}^{\infty} e_n c_n, \quad c_n \in \mathbb{C},$$

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then by virtue of the biorthogonal system existing, we can claim that such a representation is unique. Further, let us come to the previously made agreement that the vectors in each Jourdan chain are arranged in the same order as in (13) i.e. at the first place there stands an eigenvector. It is clear that under such an assumption we have

\[ c_{q\xi + i} = \frac{(f, g_{q\xi + k - i})}{(e_{q\xi + i}; g_{q\xi + k - i})}, \quad 0 \leq i \leq k(q\xi), \]

where \( k(q\xi) + 1 \) is a number of elements in the \( q\xi \)-th Jourdan chain. In particular, if the vector \( e_{q\xi} \) is included to the major system solo, there does not exist a root vector corresponding to the same eigenvalue, then

\[ c_{q\xi} = \frac{(f; g_{q\xi})}{(e_{q\xi}; g_{q\xi})}. \]

Note that in accordance with the property of the biorthogonal sequences, we can expect that the denominators equal to one in the previous two relations. Consider a formal series corresponding to a decomposition on the major vectors of the operator \( B \)

\[ f \sim \sum_{n=1}^{\infty} e_n c_n, \]

where each number \( n \) corresponds to a number \( q\xi + i \) (thus, the coefficients \( c_n \) are defined in accordance with the above and numerated in a simplest way). Consider a set of the polynomials with respect to a real parameter \( t \)

\[ P_m^\alpha(\zeta^{-1}, t) = \frac{e^{\zeta^{-\alpha}}}{m!} \frac{d^m}{d\zeta^m} e^{-\zeta^{-\alpha}}, \quad \alpha > 0, \quad m = 1, 2, \ldots. \]

Consider a series

\[ \sum_{n=1}^{\infty} c_n(t)e_n, \quad (14) \]

where the coefficients \( c_n(t) \) are defined in accordance with the correspondence between the indexes \( n \) and \( q\xi + i \) in the following way

\[ c_{q\xi + i}(t) = e^{-\lambda_q t} \sum_{m=0}^{k-i} P_m^\alpha(\lambda_q, t) c_{q\xi+i+m}, \quad i = 0, 1, 2, \ldots, k, \quad (15) \]

here \( \lambda_q = 1/\mu_q \) is a characteristic number corresponding to \( e_{q\xi} \). It is clear that in any case, we have \( c_n(t) \to c_n, \quad t \to 0 \) (it can be established by direct calculations). In accordance with the definition given in [26, p.17] we will say that series (14) converges to the element \( f \) in the sense \((B, \lambda, \alpha)\), if there exists a sequence of the natural numbers \( \{N_j\}^\infty_1 \) such that

\[ f = \lim_{t \to +0} \lim_{j \to \infty} \sum_{n=1}^{N_j} c_n(t)e_n. \]

Note that sums of the latter relation forms a subsequence of the partial sums of the series (14).
To establish the main results we need the following lemmas by Lidskii. Note that in spite of the fact that we have rewritten the lemmas in the refined form the proof has not been changed and can be found in the paper [26]. Further, considering an arbitrary compact operator $B : H \to H$ such that $\Theta(B) \subset L_0(\theta)$, $-\pi < \theta < \pi$, we put the following contour in correspondence to the operator

$$
\gamma(B) := \{ \lambda : |\lambda| = r > 0, |\arg \lambda| \leq \theta + \varepsilon \} \cup \{ \lambda : |\lambda| > r, |\arg \lambda| = \theta + \varepsilon \},
$$

(16)

where $\varepsilon > 0$ is an arbitrary small number, the number $r$ is chosen so that the operator $(I - \lambda B)^{-1}$ is regular within the corresponding closed circle. Here we should note that the compactness property of $B$ gives us the fact $(I - \lambda B)^{-1} \in B(H)$, $\lambda \in \mathbb{C} \setminus \text{int} \gamma(B)$. It can be proved easily if we note that in accordance with the Corollary 3.3 [15, p.268], we have $P(B) \subset \mathbb{C} \setminus \Theta(B)$.

**Lemma 6.** Assume that $B$ is a compact operator, $\Theta(B) \subset L_0(\theta)$, $-\pi < \theta < \pi$, then on each ray $\zeta$ containing the point zero and not belonging to the sector $L_0(\theta)$ as well as the real axis, we have

$$
\| (I - \lambda B)^{-1} \| \leq \frac{1}{\sin \varphi}, \lambda \in \zeta,
$$

where $\varphi = \min \{|\arg \zeta - \theta|, |\arg \zeta + \theta|\}$.

**Lemma 7.** Assume that the operator $B$ satisfies conditions of Lemma 6, $f \in R(B)$, then

$$
\lim_{t \to 0^+} \int_{\gamma(B)} e^{-\lambda \alpha t} B(I - \lambda B)^{-1} f d\lambda = f, \alpha > 0.
$$

**Lemma 8.** Assume that $B$ is a compact operator, then in the pole $\lambda^q$ of the operator $(I - \lambda B)^{-1}$, the residue of the vector function $e^{-\lambda \alpha t} B(I - \lambda B)^{-1} f$, $(f \in H)$, $\alpha > 0$ equals to

$$
- \sum_{q=1}^{m(q)} \sum_{i=0}^{k(q)} c_{q \xi+i}(t),
$$

where $m(q)$ is a geometrical multiplicity of the $q$-th eigenvalue, $k(q) + 1$ is a number of elements in the $q \xi$-th Jourdan chain, the coefficients $c_{q \xi+i}(t)$ are defined in accordance with formula [15].

## 3 Main results

In this section, considering the class $\tilde{S}_\alpha$ under additional assumptions, we improve results obtained by Lidskii [26]. As an application we consider differential equations in the Hilbert space. We should stress that a significant refinement takes place in comparison with the reasonings [26]. We consider the operator classes under the point of view made in the latter section. Firstly, we consider a general statement with the made refinement related to the involved notion of the convergence exponent. Secondly, having formulated conditions in terms of the operator order, we produce an example establishing the fact in accordance with which the contours may be chosen in a concrete way, under the assumption $\rho = \alpha$, what provides a peculiar validity of the statement. Finally, we consider applications to the differential equations in the Hilbert space. For convenience, we will use auxiliary denotations

$$
I = \sum_{\nu=0}^{\infty} I_{\nu}; \quad J^+ = \sum_{\nu=0}^{\infty} J^+_{\nu}; \quad J^- = \sum_{\nu=0}^{\infty} J^-_{\nu}.
$$

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The structure of the proof of the following theorem completely belongs to Lidskii. However, we produce the proof since we make a refinement corresponding to consideration of the case when a convergence exponent does not equals the index of the Schatten-von Neumann class.

**Theorem 2.** Assume that $B$ is a compact operator, $\Theta(B) \subset \mathfrak{L}_0(\theta)$, $\theta < \min\{\pi/2\alpha, \pi\}$, $B \in \mathfrak{S}_\rho$, $0 < \rho \leq \alpha$. Moreover in the case $B \in \mathfrak{S}_\rho \setminus \mathfrak{S}_\rho$ the additional condition holds

$$\frac{n_{B^m+1}(r^{m+1})}{r^\rho} \to 0, \ m = [\rho].$$

Then a sequence of natural numbers $\{N_\nu\}_0^\infty$ can be chosen so that

$$\frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \sum_{\nu=0}^\infty \sum_{q=N_\nu+1}^{m(q)k(q)} \sum_{\xi=1}^\infty \sum_{i=0}^{\infty} e_{q+i\xi} c_{q+i\xi}(t),$$

moreover

$$\sum_{\nu=0}^\infty \left| \sum_{q=N_\nu+1}^{m(q)k(q)} \sum_{\xi=1}^\infty \sum_{i=0}^{\infty} e_{q+i\xi} c_{q+i\xi}(t) \right| < \infty.$$  \hspace{1cm} (18)

**Proof.** Consider a contour $\gamma(B)$. Having fixed $R > 0$, $0 < \delta < 1$, so that $R(1 - \delta) = r$, consider a monotonically increasing sequence $\{R_\nu\}_0^\infty$, $R_\nu = R(1 - \delta)^{-\nu+1}$. Using Lemma 5, we get

$$\| (I - \lambda B)^{-1} \|_\delta \leq e^{\gamma(|\lambda|)|\lambda|^{m}}, \ m = [\rho], \ |\lambda| = \tilde{R}_\nu, \ R_\nu < \tilde{R}_\nu < R_{\nu+1},$$

where the function $\gamma(r)$ is defined in Lemma 5.

$$\beta(r) = r^{-\frac{\rho}{\pi+\rho}} \left( \int_0^r \frac{n_{B^m+1}(t)}{t} dt + r \int_r^\infty \frac{n_{B^m+1}(t)}{t^2} dt \right).$$

Denote by $\nu_\nu$ a bound of the intersection of the ring $\tilde{R}_\nu < |\lambda| < R_{\nu+1}$ with the interior of the contour $\gamma(B)$, denote by $N_\nu$ a number of poles being contained in the set int $\gamma(B) \cap \{\lambda : r < |\lambda| < \tilde{R}_\nu\}$. In accordance with Lemma 8, we get

$$\frac{1}{2\pi i} \int_{\nu_\nu} e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \sum_{q=N_\nu+1}^{m(q)k(q)} \sum_{\xi=1}^\infty \sum_{i=0}^{\infty} e_{q+i\xi} c_{q+i\xi}(t).$$

Let us estimate the above integral, for this purpose split the contour $\nu_\nu$ on terms $\tilde{\nu}_\nu := \{\lambda : |\lambda| = \tilde{R}_\nu, |\arg\lambda| < \theta + \varepsilon\}, \ \nu_\nu^+ := \{\lambda : \tilde{R}_\nu < |\lambda| < R_{\nu+1}, \arg\lambda = \theta + \varepsilon\}, \ \nu_\nu^- := \{\lambda : \tilde{R}_\nu < |\lambda| < R_{\nu+1}, \arg\lambda = -\theta - \varepsilon\}$. In accordance with Lemma 5, we have

$$I_\nu := \left\| \int_{\tilde{\nu}_\nu} e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda \right\|_{\delta} \leq \int_{\tilde{\nu}_\nu} e^{-\lambda t} \| B(I - \lambda B)^{-1} f \|_{\delta} |d\lambda| \leq$$

$$\leq e^{\gamma(|\lambda|)|\lambda|^{m+1}} \int_{-\theta-\varepsilon}^{\theta+\varepsilon} e^{-\text{Re}\lambda t} |d\lambda|, \ |\lambda| = \tilde{R}_\nu.$$
Using the theorem conditions, we get \(|\arg \lambda| < \pi/2\alpha, \lambda \in \tilde{\gamma}_\nu, \nu = 0, 1, 2, \ldots\) It follows that
\[
\Re \lambda^\alpha \geq |\lambda|^\alpha \cos [(\pi/2\alpha - \delta)\alpha] = |\lambda|^\alpha \sin \alpha \delta,
\]
where \(\delta\) is a sufficiently small number. Thus, we get
\[
I_\nu \leq C e^{\gamma(|\lambda|)}|\lambda|^{\rho - t}|\lambda|^\alpha \sin \alpha \delta |\lambda|^{m+1} = e^{\gamma(|\lambda|)}|\lambda|^{\alpha - \rho} \sin \alpha \delta |\lambda|^{m+1}, m = |\rho|, |\lambda| = \tilde{R}_\nu.
\]
Let us show that for a fixed \(t\) and a sufficiently large \(|\lambda|\), we have \(\gamma(|\lambda|) - t|\lambda|^{\alpha - \rho} \sin \alpha \delta < 0\). It follows directly from Lemma 2 in the case when \(B \in \tilde{\mathcal{S}}_\rho\) as well as in the case \(B \in \tilde{\mathcal{S}}_\rho \setminus \tilde{\mathcal{S}}_p\) but here we should involve the additional condition \((17)\). Therefore, the series \(I\) converges. Using the analogous estimates, applying Lemma 3 we get
\[
J^+_{\nu} := \left\| \int_{\tilde{\gamma}_\nu} e^{-\lambda^\alpha t} B(I - \lambda B)^{-1} f d\lambda \right\|_{\tilde{\mathcal{S}}_\nu} \leq C \|f\|_{\tilde{\mathcal{S}}_\nu} \int_{R_{\nu}} e^{-\lambda^\alpha t} |d\lambda| \leq C e^{-tR^\nu_\alpha \sin \alpha \varepsilon} \int_{R_{\nu}} |d\lambda| =
\]
\[
= C e^{-tR^\nu_\alpha \sin \alpha \varepsilon} \{R_{\nu+1} - R_\nu\};
\]
\[
J^-_{\nu} := \left\| \int_{\tilde{\gamma}_\nu} e^{-\lambda^\alpha t} B(I - \lambda B)^{-1} f d\lambda \right\|_{\tilde{\mathcal{S}}_\nu} \leq C e^{-tR^\nu_\alpha \sin \alpha \varepsilon} \{R_{\nu+1} - R_\nu\}.
\]
Therefore, the series \(J^+, J^-\) are convergent. Thus, we obtain relation \((18)\), from what follows the rest part of the theorem claim. \(\square\)

**Sequence of power type contours**

It is remarkable that we can choose a sequence of contours in various ways. For instance, a sequence of contours of the exponential type was considered in the paper \([24]\). In this paragraph, we produce an application of the previous section results, we study a concrete operator class for which it is possible to choose a sequence of contours of the power type. At the same time having involved an additional condition we can spread the principal result of the paragraph on a wider operator class. Note that using condition H2, it is not hard to prove that \(\Re(\tilde{W} f, f)_{\theta} - k|\Im(\tilde{W} f, f)_{\theta}| \geq (C_2 - kC_1)\|f\|^2_{H_2}, k > 0\), from what follows a fact \(\Theta(A) \subset \mathcal{L}_0(\theta), \theta = \arctan(C_1/C_2)\). In general, the last relation gives us a range of the semi-angle \(\pi/4 < \theta < \pi/2\), thus the conditions H1, H2 are not sufficient to guaranty a value of the semi-angle less than \(\pi/4\). However, we should remark that some relevant results can be obtained in the very case of sufficiently small values of the semi-angle, this gives us a motivation to consider a more specific additional assumption
\[
(H3) \quad |\Im(\tilde{f}, g)_{\theta}| \leq C_3\|f\|_{H_2} \|g\|_{H_2}, f, g \in \mathfrak{M}, C_3 > 0.
\]
In this case, we have \(\Re(\tilde{W} f, f)_{\theta} - k|\Im(\tilde{W} f, f)_{\theta}| \geq C_2\|f\|_{H_2} - kC_3 \left\{\varepsilon\|f\|^2_{H_2} + \|f\|^2_{H_2} / 2\varepsilon\right\} \geq (C_2 - k\varepsilon C_3)\|f\|^2_{H_2} / 2 + (C_2/C_0 - kC_3/\varepsilon)\|f\|^2_{H_2} / 2, k > 0\). Thus, choosing \(\varepsilon = C_2/kC_3\), we get \(\Theta(\tilde{W}) \subset \mathcal{L}_*(\theta_\varepsilon)\), where \(\theta_\varepsilon = C_2/2C_0 - (kC_3)^2/2C_2\), \(\theta_\varepsilon = \arctan(1/k)\). This relation guarantees
that we can choose a sufficiently small value of the semi-angle $\theta$. We put the following contour in correspondence to an operator $L$ satisfying the additional condition $H3$

$$\Gamma(A) := \text{Fr}\{ (\mathcal{L}_0(\theta_0+\varepsilon) \cap \mathcal{L}_i(\theta_i+\varepsilon)) \setminus \mathcal{C}_r \}, \ i < 0, \ \mathcal{C}_r := \{ \lambda : |\lambda| < r, |\arg\lambda| \leq \theta_0 \},$$

where $r$ is chosen so that the operator $(I - \lambda A)^{-1}$ is regular within the corresponding closed circle, $\varepsilon > 0$ is sufficiently small.

**Lemma 9.** Assume that condition $H3$ holds, then

$$\| (I - \lambda A)^{-1} \|_B \leq C, \ \lambda \in \text{Fr}\{ \mathcal{L}_0(\theta_0+\varepsilon) \cap \mathcal{L}_i(\theta_i+\varepsilon) \}, \ i < 0,$$

where $i = C_2/2C_0 - (kC_3)^2/2C_2$, $\theta_i = \arctan(1/k)$, $\varepsilon > 0$ is an arbitrary small number.

**Proof.** Firstly, we should note that in accordance with condition $H3$, for an arbitrary large value $k$, we have $\Theta(W) \subset \mathcal{L}_i(\theta_i)$, where $i = C_2/2C_0 - (kC_3)^2/2C_2$, $\theta_i = \arctan(1/k)$. Hence $\Theta(W) \subset \mathcal{L}_0(\theta_0) \cap \mathcal{L}_i(\theta_i)$, where $i$ is arbitrary negative. Therefore $\Theta(A) \subset \mathcal{L}_0(\theta_0) \cap \mathcal{L}_i(\theta_i)$, it can be verified directly due to the geometrical methods. Note that by virtue of the Lemma 4 [26], we have $\| (I - \lambda A)^{-1} \|_B \leq C, \ \lambda \in \text{Fr}\{ \mathcal{L}_0(\theta_0+\varepsilon) \}$. Thus to obtain the desired result it suffices to prove that $\| (I - \lambda A)^{-1} \|_B \leq C, \ \lambda \in \text{Fr}\{ \mathcal{L}_0(\theta_0+\varepsilon) \}, \ \text{Re} \lambda \geq 0$. Note that in this case $\lambda \in P(W)$ and we have a chain of reasonings $\forall f \in \mathcal{H}: \ (W - \lambda I)^{-1} f = h \in D(W); \ (W - \lambda I)h = f; \ (f, h)_B = (Wh, h)_B - \|h\|^2_B$. Using the latter relation, we get $\| (f, h)_B \|_B = \| (Wh, h)_B \|_B - \lambda \geq |\lambda - i| \sin \varepsilon$. Therefore, using the Cauchy-Schwartz inequality, we get

$$\| (W - \lambda I)^{-1} f \|_B \leq \frac{1}{|\lambda - i| \sin \varepsilon} \cdot \| f \|_B, \ f \in \mathcal{H}.$$

Taking into account the fact $(W - \lambda I)^{-1} = (I - \lambda A)^{-1} A = \{(I - \lambda A)^{-1} - I\}/\lambda$, we get $\| (I - \lambda A)^{-1} \|_B - 1 \leq \| (I - \lambda A)^{-1} - I \|_B \leq |\lambda|/|\lambda - i| \sin \varepsilon$, from what follows the desired result. \[\square\]

**Lemma 10.** Assume that the condition $H3$ holds, $f \in R(A)$, then

$$\lim_{t \to 0} \int_{\Gamma(A)} e^{-\lambda^\alpha t} A(I - \lambda A)^{-1} f d\lambda = f, \ \alpha > 0.$$

**Proof.** The proof is analogous to the proof of the Lemma 5 [26] and the only difference is in the following, we should use Lemma 9 instead of Lemma 6. \[\square\]

The theorems given below are formulated under the assumption that either the condition $\Theta(A) \subset \mathcal{L}_0(\theta)$, $\theta < \pi/2\alpha$ or condition $H3$ holds. In accordance with such an alternative, we put in correspondence $\omega := \gamma(A)$ or $\omega := \Gamma(A)$ respectively. The following theorem is similar to the result [26], but formulated in terms of the operator order. Although the principal clarification $\alpha = \rho$ has not been obtained it can be interesting by virtue of the different way of choosing a sequence of contours.

**Theorem 3.** Assume that the operator $\tilde{W}$ satisfies the condition $\alpha > 2/\mu, \mu \in (0, 1]$ and $\alpha > 1, \mu \in (1, \infty)$. Then a sequence of natural numbers $\{N_\nu\}^\infty_0$ can be chosen so that

$$\frac{1}{2\pi i} \int_\omega e^{-\lambda^\alpha t} A(I - \lambda A)^{-1} f d\lambda = \sum_{\nu=0}^\infty \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)k(q)} \sum_{i=0}^{e_q+i} e_q+i+c_{q+i}(t),$$
where

$$
\sum_{\nu=0}^{\infty} \left\| \sum_{q=N_{\nu}+1}^{m(q)} \sum_{k(q)} e_{q} e_{k+1}(t) \right\| < \infty,
$$

(19)

the following relation holds for the eigenvalues

$$
|\lambda_{N_{\nu}+k} - |\lambda_{N_{\nu}+k-1}| \leq C|\lambda_{N_{\nu}+k}|^{1-1/\tau}, \ k = 2, 3, \ldots, N_{\nu}+1 - N_{\nu}, \ 0 < \tau < \mu.
$$

Proof. In accordance with Theorem 1 we have

$$
|\lambda_{i}^{-1}| = o\left(i^{-\tau}\right), \ i \to \infty, \ 0 < \tau < \mu,
$$

Thus using the fact $\lambda_{i}/i^{\tau} \geq C$, we can prove that there exists a subsequence $\{\lambda_{N_{\nu}}\}_{\nu=0}^{\infty}$, such that

$$
|\lambda_{N_{\nu}+1} - |\lambda_{N_{\nu}}| \geq K|\lambda_{N_{\nu}+1}|^{1-1/\tau}, \ K > 0,
$$

for this purpose it suffices to establish the following implication

$$
\lim_{n \to \infty} (\lambda_{n+1} - \lambda_{n})/\lambda_{n}^{(p-1)/p} = 0, \implies \lim_{n \to \infty} \lambda_{n}/n^{p} = 0, \ p > 0
$$

(see proof of Lemma 2 [2]). Now, consider

$$
|\lambda_{N_{\nu}+1} - |\lambda_{N_{\nu}}| \geq K|\lambda_{N_{\nu}}|^{q}, \ q := 1 - 1/\tau,
$$

and let us find $\delta_{\nu}$ from the condition $R_{\nu} = K|\lambda_{N_{\nu}}|^{q} + |\lambda_{N_{\nu}}|, \ R_{\nu}(1 - \delta_{\nu}) = |\lambda_{N_{\nu}}|$, then $\delta_{\nu}^{-1} = 1 + K^{-1}|\lambda_{N_{\nu}}|^{1-q}$. Further, we restrict our reasonings considering the case $\mu \in (0,1]$, since the reasonings corresponding to the case $\alpha > 1, \mu \in (1, \infty)$ is absolutely analogous. Note that in accordance with Lemma 3 [26] the following relation holds

$$
\sum_{i=1}^{\infty} \lambda_{i}^{(m+1)}(\hat{A}) \leq \sum_{i=1}^{\infty} \lambda_{i}^{(m+1)}(A) < \infty,
$$

where $\rho$ is chosen so that $2/\mu < \rho < \alpha, \ m = [\rho], \ \hat{A} := (A^{m+1}A^{m+1})^{1/2}$. It is clear that $\hat{A} \in \mathcal{S}_{\rho/(m+1)}$. Consider a function

$$
\beta(r) = r^{-\rho/(m+1)} \left( \int_{0}^{r} n_{A^{m+1}}(t) \frac{dt}{t} + r \int_{r}^{\infty} n_{A^{m+1}}(t) \frac{dt}{t^{2}} \right).
$$

Here, we produce the variant of the proof corresponding to the case H1. The variant of the proof corresponding to the case $\Theta(A) \subset \mathcal{S}_{\rho}(\theta), \ \theta < \pi/2\alpha$ is analogous and left to the reader. Consider a contour $\Gamma(A)$, absolutely analogously to the reasonings of Theorem 2 applying Lemma 5 we claim that there exists an arch $\hat{\gamma}_{\nu} := \{\lambda : |\lambda| = \hat{R}_{\nu}, |\arg \lambda| < \theta_{\nu} + \varepsilon\}$ in the ring $(1 - \delta_{\nu})R_{\nu} < |\lambda| < R_{\nu}$, on which the following estimate holds for a sufficiently small value $\delta > 0$

$$
I_{\nu} = \left\| \int_{\hat{\gamma}_{\nu}} e^{-\lambda^{\nu}t} A(I - \lambda A)^{-1} f d\lambda \right\|_{\delta} \leq C e^{(1-\delta_{\nu}) \frac{m}{2} |\theta_{\nu}|} \leq C e^{\rho/2}, \ |\lambda| = \hat{R}_{\nu},
$$

$\Theta_{\nu} = \{\lambda : |\lambda| = \hat{R}_{\nu}, |\arg \lambda| < \theta_{\nu} + \varepsilon\}$.
where \( \gamma(|\lambda|) = \beta(|\lambda|^{m+1}) + (2 + \ln\{12e/\delta_\nu\})\beta(|2e\lambda|^{m+1})(2e)^{\nu}. \) It is clear that within the contour \( \Gamma(A) \) between the arches \( \tilde{\gamma}_\nu, \tilde{\gamma}_{\nu+1} \) (we denote the boundary of this domain by \( \gamma_\nu \)) there lie the eigenvalues only for which the following relation holds

\[
|\lambda_{\nu+k}| - |\lambda_{\nu+k-1}| \leq C|\lambda_{\nu+k}|^\eta, \ k = 2, 3, ..., N_{\nu+1} - N_\nu.
\]

Using Lemma 8 we obtain a relation

\[
\frac{1}{2\pi i} \int_{\gamma_\nu} e^{-\lambda^\alpha} A(I - \lambda A)^{-1} f d\lambda = \sum_{q=N_{\nu+1}}^{N_{\nu+1}} \sum_{m(q)=1}^{m(q)} \sum_{i=0}^{k(q)} e^{-\lambda_{\nu+i}} e_{q+i} f(t).
\]

Hence, to prove the main claim of the theorem, we should show that the series composed of the above terms converges. Here, we want to realize the idea of splitting \( \gamma_\nu \) on terms. Let us prove that the series \( I \) converges. Substituting \( \delta_\nu^{-1} \), we have

\[
\ln\{12e/\delta_\nu\} = \ln\{12e + 12eK^{-1}|\lambda_{\nu}|^{-1-\eta}\} \leq C\ln\{|\lambda_\nu|^{1-\eta}\}. \]

It is clear that to obtain the desired result we should prove that

\[
\ln |\lambda_\nu|^{-1-\eta} \beta(|\lambda_\nu|^{m+1}) \to 0, \ \nu \to \infty.
\]

Using simple reasonings based on the fact \( \ln|\lambda_\nu|/|\lambda_\nu| \to 0, \ \nu \to \infty \), applying Lemma 2 we obtain the desired result. Finally, we should consider the integrals along the contours \( \gamma_\nu^+: = \{\lambda : (1 - \delta_\nu)R_\nu < |\lambda| < R_\nu, \ \arg\lambda = \theta_s + \epsilon\}, \ \gamma_\nu^-: = \{\lambda : (1 - \delta_\nu)R_\nu < |\lambda| < R_\nu, \ \arg\lambda = -\theta_s - \epsilon\}, \)

where \( s = 0, \ i. \) Analogously to Theorem 2 applying Lemma 9 we have

\[
J_\nu^+ := \left\| \int_{\gamma_\nu^+} e^{-\lambda^\alpha} A(I - \lambda A)^{-1} f d\lambda \right\|_B \leq C\|f\|_B \int_{(1-\delta_\nu)R_\nu}^{R_\nu} |e^{-\lambda^\alpha}| |d\lambda| \leq C e^{-t(1 - \delta_\nu)^{\alpha}} \sin\alpha \delta_\nu R_\nu;
\]

\[
J_\nu^- := \left\| \int_{\gamma_\nu^-} e^{-\lambda^\alpha} A(I - \lambda A)^{-1} f d\lambda \right\|_B \leq C e^{-t(1 - \delta_\nu)^{\alpha}} \sin\alpha \delta_\nu R_\nu.
\]

Therefore the series \( J^+, J^- \) are convergent. Thus, we obtain (19), from what follows the rest part of the theorem claim.

Remain that a sequence of contours of the exponential type was considered in the paper 26, under the imposed condition \( \alpha > \rho \). We improve this result in the following sense, we produce a sequence of contours of the power type, what gives us a solution of the problem in the case \( A \in \hat{S}_\rho, \ \alpha = \rho \).

**Theorem 4.** Assume that a normal operator \( \tilde{W} \) satisfies the condition \( (\ln^{1+1/\alpha} x)', H) = o(i^{-1/\alpha}), \ \alpha > 1. \) Then a sequence of the natural numbers \( \{N_\nu\}_{\nu} \) can be chosen so that

\[
\frac{1}{2\pi i} \int_{\infty}^{\infty} e^{-\lambda^\alpha} A(I - \lambda A)^{-1} f d\lambda = \sum_{\nu=0}^{\infty} \sum_{q=N_{\nu+1}}^{N_{\nu+1}} m(q) k(q) \sum_{i=1}^{k(q)} e^{\lambda_{\nu+i}} e_{q+i} f(t),
\]
Consider a contour $\Gamma(\delta)$ for a sufficiently small $\delta > 0$. Here, we produce the variant of the proof corresponding to the case $H1$. Furthermore, moreover

$$
\sum_{v=0}^{\infty} \left\| \sum_{q=N_{v+1}}^{m(q)k(q_{v})} \sum_{\xi=1}^{\gamma} e_{q\xi+i} c_{q\xi+i}(t) \right\| < \infty, \tag{20}
$$

the following relation holds for the corresponding eigenvalues

$$
|\lambda_{N_v+k} - |\lambda_{N_v+k-1}| \leq C|\lambda_{N_v+k}|^{1-1/\tau}, \ k = 2, 3, ..., N_{v+1} - N_v, \ 0 < \tau < 1/\alpha.
$$

**Proof.** Applying Theorem 1, we get

$$|\lambda_{v}^{-1}| = o \left( i^{-\tau} \right), \ i \to \infty, \ 0 < \tau < 1/\alpha,$$

Thus, using the fact $\lambda_1/i^\tau \geq C$, we can prove that there exists a subsequence $\{\lambda_{N_v}\}_{v=0}^\infty$, such that

$$|\lambda_{N_v+1} - |\lambda_N| \geq K|\lambda_{N_v+1}|^{1-1/\tau}, \ K > 0,$$

for this purpose it suffices to establish the following implication

$$\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n)/\lambda_{n}^{(p-1)/p} = 0, \ \Rightarrow \ \lim_{n\to\infty} \lambda_n/n^p = 0, \ p > 0$$

(see proof of Lemma 2 [2]). Now, consider

$$|\lambda_{N_v+1} - |\lambda_N| \geq K|\lambda_N|^{q}, \ q := 1 - 1/\tau,$$

and let us find $\delta_v$ from the condition $R_v = K|\lambda_N|^{q} + |\lambda_N|$, $R_v(1 - \delta_v) = |\lambda_N|$, then $\delta_v^{-1} = 1 + K^{-1}|\lambda_N|^{1-q}$. In accordance with Lemma 3, we have $A \in S_\rho, \ \rho \in [0, \alpha]$, $n_A(r) = o(r^\alpha/\ln r)$. Here, we produce the variant of the proof corresponding to the case $H1$. The variant of the proof corresponding to the case $\Theta(A) \subset S_0(\theta), \ \theta < \pi/2\alpha$ is absolutely analogous and left to the reader.

Consider a contour $\Gamma(A)$, applying Lemma 4 analogously to the reasonings of Theorem 2, we claim that for a sufficiently small $\delta > 0$, there exists an arch $\tilde{\gamma}_v := \{\lambda : \ |\lambda| = \tilde{R}_v, \ \arg \lambda < \theta + \varepsilon\}$ in the ring $(1 - \delta_v)R_v < |\lambda| < R_v$, on which the following estimate holds

$$I_v \left( \gamma_v \right) = \left\| \int_{\tilde{\gamma}_v} e^{-\lambda^{\alpha}t} A(I - \lambda A)^{-1} f d\lambda \right\| \leq e^{\|\lambda\|_{\gamma(\lambda)|t\sin |\alpha}}} |\lambda|^{m+1}, \ m = [\alpha], \ |\lambda| = \tilde{R}_v, \tag{21}
$$

where $\gamma(|\lambda|) = \beta(|\lambda|^{m+1}) + (2 + \ln\{12e/\delta_v\})\beta(|2e\lambda|^{m+1})(2e)^\alpha$,

$$\beta(r) = r^{-\frac{\alpha}{m+1}} \left( \int_{0}^{r} \frac{n_{A}^{m+1}(t)}{t} dt + \int_{r}^{\infty} \frac{n_{A}^{m+1}(t)}{t^2} dt \right).$$

It is clear that within the contour $\Gamma(A)$, between the arches $\tilde{\gamma}_v, \tilde{\gamma}_{v+1}$ (we denote the boundary of this domain by $\gamma_v$) there lie the eigenvalues only for which the following relation holds

$$|\lambda_{N_v+k} - |\lambda_{N_v+k-1}| \leq C|\lambda_{N_v+k}|^{q}, \ k = 2, 3, ..., N_{v+1} - N_v.$$

Using Lemma 5, we obtain a relation

$$\frac{1}{2\pi i} \int_{\gamma_v} e^{-\lambda^{\alpha}t} A(I - \lambda A)^{-1} f d\lambda = \sum_{q=N_v+1}^{m(q)k(q_v)} \sum_{\xi=1}^{\gamma} e_{q\xi+i} c_{q\xi+i}(t).$$
It is clear that to obtain the desired result, we should prove that the series composed of the above terms converges. However, we can prove that the series $I$ converges, what is more stronger condition. Here, we want to realize the idea of splitting $\gamma_\nu$ on terms. Consider the right-hand side of formula (21). Substituting $\delta^{-1}$, we have

$$\ln\{12e/\delta\} = \ln\{12e + 12eK^{-1}|\lambda_N|^{1-q}\} \leq C\ln\{|\lambda_N|^{1-q}\}.$$ 

Hence, to obtain the desired result we should prove that $\ln|\lambda_N|^{1-q}\beta(|\lambda_N|^{m+1}) \to 0$, $\nu \to \infty$. In its own turn, using Lemma 3, we can prove the latter relation, if we show that

$$\ln r^{\frac{N_{m+1}(r^{m+1})}{r^\alpha}} \to 0, \ r \to \infty. \quad (22)$$

We need establish some facts. Notice that the following operators have the same eigenfunctions i.e.

$$\left(A^*A\right)^{1/2}f_n = \mu_n f_n \iff \left(A^{*s}A^s\right)^{1/2}f_n = \mu_n^s f_n, \ s = m + 1. \quad (23)$$

To prove this fact, firstly let us show that $A^*A = A^{*s}$, it follows easily from the inclusion $A^{*s} \subset A^*$ and the fact $D(A^*) = \Omega$. Thus, for a normal operator we have $(A^*A)^s = A^{*s}A^s$. Let us involve the spectral theorem for the selfadjoint non-negative operator, in accordance with a definition (see [23] Chapter 3), we have

$$(A^*A)^\vartheta = \int_0^{\|A^*A\|} \lambda^\vartheta dP_\lambda, \ \vartheta > 0,$$

where the latter integral is understood in the Riemann sense as a limit of the partial sums

$$\sum_{i=0}^n \xi_i^\vartheta P_{\Delta \lambda_i} \to \int_0^{\|A^*A\|} \lambda^\vartheta dP_\lambda, \ \omega \to 0,$$

where $(0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \|A^*A\|)$ is an arbitrary splitting of the segment $[0, \|A^*A\|]$, $\omega := \max(\lambda_{i+1} - \lambda_i)$, $\xi_i$ is an arbitrary point belonging to $[\lambda_i, \lambda_{i+1}]$, the operators $P_{\Delta \lambda_i}$ are projectors corresponding to the selfadjoint operator. It follows easily from the well-known facts that if in additional $A^*A$ is a compact operator, then the above formula reduces to

$$(A^*A)^\vartheta f = \sum_{n=1}^\infty \lambda_n^\vartheta (f, \varphi_n)\varphi_n,$$

where $\{\varphi_n\}_1^\infty, \{\lambda_n\}_1^\infty$ are sets of the eigenvectors and the eigenvalues of the operator $A^*A$ respectively. Taking into account the latter representation, an obvious fact that $(A^*A)^\vartheta$ is selfadjoint, it is not hard to prove

$$(A^*A)^{\frac{1}{2}\cdot s} = \left(A^{*s}A^s\right)^{\frac{1}{2}}.$$

Thus, using the property $A^*A = AA^*$, we get

$$(A^*A)^{\frac{1}{2}\cdot s} = \left(A^{*s}A^s\right)^{\frac{1}{2}},$$
from what follows the implication from the left-hand side of formula (23). To obtain the contrary implication we should establish the fact that the operator and its positive powers have the same eigenvectors. For this purpose, let us notice that

\[ T^g \varphi_i = \lambda_i^g \varphi_i, \quad i \in \mathbb{N}, \]

where \( T := A^*A \). It follows that

\[ T^g f = \sum_{n=1}^{\infty} \lambda_n^g (f, \varphi_n)_{\mathcal{H}} \varphi_n = \sum_{n=1}^{\infty} (T^g \varphi_n)_{\mathcal{H}} \varphi_n = \sum_{n=1}^{\infty} (T^g f, \varphi_n)_{\mathcal{H}} \varphi_n. \]

Hence, we have a fact

\[ g = \sum_{n=1}^{\infty} (g, \varphi_n)_{\mathcal{H}} \varphi_n, \quad g \in R(T^g). \]

Let us assume that there exists an eigenfunction \( h \) of the operator \( T^g \) that differs from \( \varphi_i, \quad i \in \mathbb{N} \). Using the proved above fact, we get

\[ T^g h = \sum_{n=1}^{\infty} \lambda_n^g (h, \varphi_n)_{\mathcal{H}} \varphi_n = \zeta \sum_{n=1}^{\infty} (h, \varphi_n)_{\mathcal{H}} \varphi_n, \]

where \( \zeta \) is a corresponding eigenvalue. Multiplying (in the sense of the inner product) both sides of the latter relation on \( \varphi_k, \varphi_{k+1} \) we get \( \lambda_k^g = \zeta = \lambda_{k+1}^g \), this contradiction proves the desired result. Thus, we complete the proof of formula (23). To complete the proof of relation (22) we need mention the fact \( n_A(\lambda) = n_{A^m}(\lambda^m) \) that follows easily from relation (23). Thus, making a substitution and using the theorem condition, we claim that relation (22) holds, hence the series \( I \) is convergent. To complete the proof, we should note that the integrals along the following contours converges uniformly \( \gamma_{\nu} := \{ \lambda: (1-\delta_\nu)R_\nu < |\lambda| < R_\nu, \quad \arg \lambda = \theta_\nu + \epsilon \}, \gamma_{\nu} := \{ \lambda: (1-\delta_\nu)R_\nu < |\lambda| < R_\nu, \quad \arg \lambda = -\theta_\nu - \epsilon \} \), where \( s = 0, \epsilon \). Analogously to the reasonings of Theorem 2 applying Lemma 9 we have

\[ J^+_\nu := \left\| \int_{\gamma_{\nu}} e^{-\lambda^{\alpha t}} A(I - \lambda A)^{-1} f d\lambda \right\|_{\mathcal{H}} \leq C \| f \|_{\mathcal{H}} \int_{(1-\delta_\nu)R_\nu}^{R_\nu} |e^{-\lambda^{\alpha t}}| |d\lambda| \leq Ce^{-t(1-\delta_\nu)^{\alpha} R_\nu^{\alpha} \sin \delta_\nu R_\nu}; \]

\[ J^-_\nu := \left\| \int_{\gamma_{\nu}} e^{-\lambda^{\alpha t}} A(I - \lambda A)^{-1} f d\lambda \right\|_{\mathcal{H}} \leq Ce^{-t(1-\delta_\nu)^{\alpha} R_\nu^{\alpha} \sin \delta_\nu R_\nu}. \]

Therefore, the series \( J^+, J^- \) converge. Thus, we obtain relation (20), from what follows the rest part of the proof.

**Corollary 1.** Under the Theorem 2 assumptions, we get

\[ f = \lim_{t \to +0} \sum_{\nu_0}^{\infty} \sum_{\nu=q}^{\nu_0} \sum_{q=0}^{q_{\nu+1}} \sum_{\xi=1}^{q_{\nu+1}} e_{q_{\nu}+i} c_{q_{\nu}+i}(t), \quad f \in D(\tilde{W}). \]

This fact follows immediately from Lemmas 7,10 respectively.
Differential equations in the Hilbert space

Further, we will consider a Hilbert space $\mathcal{H}$ consists of element-functions $u : \mathbb{R}_+ \to \mathcal{H}$, $u := u(t)$, $t \geq 0$ and we will assume that if $u$ belongs to $\mathcal{H}$ then the fact holds for all values of the variable $t$. Notice that under such an assumption all standard topological properties as completeness, compactness e.t.c. remain correctly defined. We understand such operations as differentiation and integration in the generalized sense that is caused by the topology of the Hilbert space $\mathcal{H}$. The derivative is understood as the following limit

$$
\frac{u(t + \Delta t) - u(t)}{\Delta t} \xrightarrow{\Delta t \to 0} \frac{du}{dt}, \quad \Delta t \to 0.
$$

Let $t \in I := [a, b]$, $0 < a < b < \infty$. The following integral is understood in the Riemann sense as a limit of partial sums

$$
\sum_{i=0}^{n} u(\xi_i)\Delta t_i \xrightarrow{\lambda \to 0} \int_{I} u(t)dt, \quad \lambda \to 0,
$$

where $(a = t_0 < t_1 < \ldots < t_n = b)$ is an arbitrary splitting of the segment $I$, $\lambda := \max_{i}(t_{i+1} - t_i)$, $\xi_i$ is an arbitrary point belonging to $[t_i, t_{i+1}]$. The sufficient condition of the last integral existence is a continuous property (see [23, p.248]) i.e. $u(t) \xrightarrow{t \to t_0} u(t_0)$, $\forall t_0 \in I$. The improper integral is understood as a limit

$$
\int_{a}^{b} u(t)dt \xrightarrow{b \to c} \int_{a}^{c} u(t)dt, \quad b \to c, \quad c \in [0, \infty].
$$

Combining these operations we can consider a fractional differential operator in the Riemann-Liouville sense (see [43]) i.e. in the formal form, we have

$$
D_{-\alpha}^{1/\alpha} f(t) := -\frac{1}{\Gamma(1-1/\alpha)} \frac{d}{dt} \int_{0}^{\infty} f(t + x)x^{-1/\alpha}dx, \quad \alpha > 1.
$$

Let us study a Cauchy problem

$$
D_{-\alpha}^{1/\alpha} u = \tilde{W}u, \quad u(0) = h \in D(\tilde{W}),
$$

(24)

in the case when the operator composition $D^{1-1/\alpha}\tilde{W}$ is accretive we assume that $h \in \mathcal{H}$.

**Theorem 5.** Assume that the Theorem 4 conditions hold, then there exists a solution of the Cauchy problem (24) in the form

$$
u(t) = \frac{1}{2\pi i} \int_{\mathbb{C}} e^{-\lambda \xi t} A(I - \lambda A)^{-1}hd\lambda = \sum_{\nu=0}^{\infty} \sum_{q=N_{\nu+1}}^{m(q)} \sum_{k(q_{\xi})}^{m(q)} e_{q_{\xi}+i}c_{q_{\xi}+i}(t),
$$

(25)

where

$$
\sum_{\nu=0}^{\infty} \left| \sum_{q=N_{\nu+1}}^{m(q)} \sum_{k(q_{\xi})}^{m(q)} e_{q_{\xi}+i}c_{q_{\xi}+i}(t) \right| < \infty,
$$

a sequence of natural numbers $\{N_{\nu}\}_{0}^{\infty}$ can be chosen in accordance with the claim of Theorem 4. Moreover, the existing solution is unique if the operator composition $D_{-\alpha}^{1-1/\alpha}\tilde{W}$ is accretive.
Proof. Let us find a solution in the form \((25)\) satisfying the initial condition \((24)\). Below, we produce the variant of the proof corresponding to the case \(\Theta(A) \subset \mathcal{L}_0(\theta)\), \(\theta < \pi/2\alpha\). The variant of the proof corresponding to the case \(H1\) is absolutely analogous and left to the reader. Consider a contour \(\gamma(A)\). Using Lemma 6, it is not hard to prove that the following integral exists i.e.

\[
\frac{1}{2\pi i} \int_{\gamma(A)} e^{-\lambda^\prime t}(E - \lambda A)^{-1}h d\lambda \in \mathcal{H}; \quad \frac{du}{dt} = \frac{1}{2\pi i} \int_{\gamma(A)} e^{-\lambda^\prime t}\lambda A (E - \lambda A)^{-1}h d\lambda \in \mathcal{H}.
\]

Note that the first relation gives us the fact \(u(t) \in D(\hat{W})\). Using Lemmas 5,6 analogously to the methods of the ordinary calculus, we can establish the following formulas

\[
\int_0^\infty x^{-1/\alpha} dx \int_{\gamma(A)} e^{-\lambda^\prime(t+x)} A (E - \lambda A)^{-1}h d\lambda = \int_{\gamma(A)} \lambda^{1-\alpha} e^{-\lambda^\prime t} A (E - \lambda A)^{-1}h d\lambda = \int_{\gamma(A)} \lambda e^{-\lambda^\prime t} A (E - \lambda A)^{-1}h d\lambda.
\]

Therefore, combining these formulas, taking into account a relation

\[
\int_0^\infty x^{-1/\alpha} e^{-\lambda^\prime x} dx = \Gamma(1 - 1/\alpha)\lambda^{1-\alpha},
\]

we get

\[
\mathcal{D}_-^{1/\alpha} u = \frac{1}{2\pi i} \int_{\gamma(A)} e^{-\lambda^\prime t}\lambda A (E - \lambda A)^{-1}h d\lambda.
\]

Making a substitution using the formula \(\lambda A (E - \lambda A)^{-1} = (E - \lambda A)^{-1} - E\), we obtain

\[
\mathcal{D}_-^{1/\alpha} u = \frac{1}{2\pi i} \int_{\gamma(A)} e^{-\lambda^\prime t}(E - \lambda A)^{-1}h d\lambda - \frac{1}{2\pi i} \int_{\gamma(A)} e^{-\lambda^\prime t}h d\lambda = I_1 + I_2.
\]

The second integral equals zero by virtue of the fact that the function under the integral is analytical inside the intersection of the domain int \(\gamma(A)\) with the circle of an arbitrary radius \(R\) and it decreases sufficiently fast on the arch of the radius \(R\), when \(R \to \infty\). Now, if we consider the expression for \(u\), we obtain the fact that \(u\) is a solution of the equation i.e. \(\mathcal{D}_-^{1/\alpha} u = \hat{W} u\). The decomposition on the series of the root vectors \((25)\) is obtained due to Theorem 4. Let us show that the initial condition holds in the sense \(u(t) \xrightarrow{\gamma} h, t \to +0\). It becomes clear in the case \(h \in D(\hat{W})\), it suffices to apply Lemma 7 what gives us the desired result. Consider a case when \(h\) is an arbitrary element of the Hilbert space \(\mathcal{H}\). Let us involve the accretive property of the operator composition \(\mathcal{D}_-^{1-1/\alpha} \hat{W}\). It follows from Lemma 6 that for a fixed \(t\) the following operator is bounded

\[
S_t h = \frac{1}{2\pi i} \int_{\gamma(A)} e^{-\lambda^\prime t} A (E - \lambda A)^{-1}h d\lambda.
\]

Let us show that \(\|S_t\| \leq 1, t > 0\). Firstly, assume that \(h \in D(\hat{W})\), then in accordance with the above, we get \(u(t) \xrightarrow{\mathcal{D}} h, t \to +0\). Thus, we can claim the fact that \(u(t)\) is continuous at
the right-hand side of the point zero. Let us apply the operator $\mathcal{D}_{-1/\alpha}^{-1}$ to the both sides of relation (24). Taking into account a relation $\mathcal{D}_{-1/\alpha}^{-1/\alpha}u = -u^\prime$, we get $u^\prime + \mathcal{D}_{-1/\alpha}^{-1/\alpha}\hat{W}u = 0$. Let us multiply the both sides of the latter relation on $u$ in the sense of the inner product, we get $(u^\prime, u)_\beta + (\mathcal{D}_{-1/\alpha}^{-1/\alpha}\hat{W}u, u)_\beta = 0$. Consider a real part of the latter relation, we have $\Re(u^\prime, u)_\beta + \Re(\mathcal{D}_{-1/\alpha}^{-1/\alpha}\hat{W}u, u)_\beta = (u^\prime, u)_\beta /2 + (u, u^\prime)_\beta /2 + \Re(\mathcal{D}_{-1/\alpha}^{-1/\alpha}\hat{W}u, u)_\beta$. Therefore $((u(t))_\beta^2)' = -2\Re(\mathcal{D}_{-1/\alpha}^{-1/\alpha}\hat{W}u, u)_\beta \leq 0$. Integrating both sides, we get

$$
\|u(t)\|_\beta^2 - \|u(0)\|_\beta^2 = \int_0^\tau \frac{d}{dt}\|u(t)\|_\beta^2 dt \leq 0.
$$

The last relation can be rewritten in the form $\|S_t h\|_\beta \leq \|h\|_\beta$, $h \in D(\hat{W})$. Since $D(\hat{W})$ is a dense set in $\mathfrak{F}$, then we obviously obtain the desired result i.e. $\|S_t\| \leq 1$. Now consider the following reasonings, having assumed that $h_n \overset{n}{\to} h$, $n \to \infty$, $\{h_n\} \subset D(\hat{W})$, $h \in \mathfrak{F}$, we have $\|u(t) - h\|_\beta = \|S_t h - h\|_\beta = \|S_t h - S_t h_n + S_t h_n - h + h - h_n\|_\beta \leq \|S_t\| \cdot \|h - h_n\|_\beta + \|S_t h_n - h\|_\beta + \|h - h_n\|_\beta$. It is clear that if we chose $n$ so that $\|S_t h - h_n\|_\beta < \varepsilon/3$ and after that chose $t$ so that $\|S_t h_n - h\|_\beta < \varepsilon$, then we obtain $\forall \varepsilon > 0, \exists \delta(\varepsilon)$ : $\|u(t) - h\|_\beta < \varepsilon$, $t < \delta$. Thus the initial condition holds. The uniqueness follows easily from the fact that $\mathcal{D}_{-1/\alpha}^{-1/\alpha}\hat{W}$ is accretive. In this case, repeating the previous reasonings we come to

$$
\|g(\tau)\|_\beta^2 - \|g(0)\|_\beta^2 = \int_0^\tau \frac{d}{dt}\|g(t)\|_\beta^2 dt \leq 0,
$$

where $g$ is a sum of two solutions $u_1$ and $u_2$. Notice that by virtue of the initial conditions, we have $g(0) = 0$, thus relation (26) can hold only if $g = 0$. The proof is complete. \hfill \Box

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