More Confining $N = 1$ SUSY Gauge Theories
From Non-Abelian Duality

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ABSTRACT

We expand on an idea of Seiberg that an $N = 1$ supersymmetric gauge theory shows confinement without breaking of chiral symmetry when the gauge symmetry of its magnetic dual is completely broken by the Higgs effect. This has recently been applied to some models involving tensor fields and an appropriate tree-level superpotential. We show how the confining spectrum of a supersymmetric gauge theory can easily be derived when a magnetic dual is known and we determine it explicitly for many models containing fields in second rank tensor representations. We also give the form of the confining superpotential for most of these models.
1 Introduction

There has been much progress in understanding the low-energy limit of $N = 1$ supersymmetric gauge theories during the last years. Due to holomorphy and non-renormalization theorems exact results could be obtained without the need to perform complicated calculations. As a consequence it has become possible to argue that some gauge theories with special matter content confine at low energies. The first example is due to Seiberg \[1\] who found that supersymmetric quantum chromodynamics (SQCD) with gauge group $SU(N_c)$ and $N_f$ quark flavors (i.e. $N_f$ fields in the fundamental representation of the gauge group and the same amount of fields in the antifundamental representation) shows confinement when $N_f = N_c$ or $N_f = N_c + 1$. In the former case the chiral symmetry is spontaneously broken by a quantum deformation of the classical moduli space, whereas in the latter case there exists a point on the quantum moduli space where the full chiral symmetry is unbroken. The low-energy theory is described by mesons and baryons which, for $N_f = N_c + 1$, are coupled by a non-perturbatively generated superpotential.

This has been generalized to more complicated models. All $N = 1$ supersymmetric gauge theories with vanishing tree-level superpotential which confine either with a quantum deformed moduli space and no non-perturbative superpotential or with a smooth confining superpotential and without chiral symmetry breaking could be classified \[2, 3\] because they are constrained by an index argument \[3\]. For the moduli space to be deformed by quantum effects the index\[2\] $\Delta = \mu_{\text{matter}} - \mu_G$ must vanish \[2, 3\], where $\mu_{\text{matter}}$ denotes the sum over the Dynkin indices of all matter fields and $\mu_G$ is the Dynkin index of the adjoint representation. The condition $\Delta = 2$ is necessary for the low-energy theory to be described by a superpotential which is a smooth function of the confined degrees of freedom \[2\] (this has been called $s$-confinement in \[2\]).

When a tree-level superpotential is present the index argument is no longer valid. Because of the lower symmetry the non-perturbative superpotential is less constrained and one expects more confining models to exist. Indeed, Csáki and Murayama \[3\] recently showed that many of the Kutasov-like \[3\] models exhibit confinement (without breaking of chiral symmetry) for special values of the number of quark flavors $N_f$ \[3\]. Fortunately, these theories simplify considerably once an appropriate superpotential for the tensor fields is added. More precisely, for all of the models considered in \[3\] a dual description in terms of magnetic variables is known \[3, 5, 6, 7\] and the authors of \[3\] used the fact that the electric gauge theory confines when its magnetic dual is completely higgsed.

Seiberg already used this idea as an additional consistency check in his original paper establishing electric-magnetic duality for non-Abelian $N = 1$ supersymmetric gauge theories \[11\]. He showed how the confining superpotential of SQCD with $N_f = N_c + 1$ could be obtained by a perturbative calculation in the completely broken magnetic gauge theory. Under duality the fields of the magnetic theory (which are gauge singlets as the gauge symmetry

\[1\] There is another class of confining gauge theories which do not have constraints for the gauge invariant composite fields. They are classified in \[1\].

\[2\] The relevance of this index was noticed by the authors of \[2\]. We take their normalization which assigns $\mu = 1$ to the fundamental of $SU$ and $Sp$ groups and $\mu = 2$ to the vector of $SO$.

\[3\] Some further confining models with non-vanishing tree-level superpotential are discussed in \[5\].
is completely broken) are mapped to the mesons and baryons of the electric theory and the
confining superpotential is easily shown to be the image of the magnetic superpotential un-
der this mapping \[1\]. This a realization in \(N = 1\) supersymmetric gauge theories of an old
idea of ‘t Hooft and Mandelstam \[2\] that confinement is driven by condensation of magnetic
monopoles.

Now, many gauge theory models have been found that possess a dual description in
terms of magnetic variables in the infrared. This allows us to predict many new examples of
confining gauge theories. The idea described in the previous paragraph was first used by the
authors of \[13\] to determine the confining spectrum of the model proposed by Kutasov \[6\]
and has recently been applied by Csáki and Murayama \[5\] to six further models that confine
in the presence of an appropriate superpotential.

In the next section we explain how the duality discovered by Seiberg is used to obtain the
low-energy spectrum and the form of the non-perturbative superpotential of confining gauge
theories. To this end we review the example of SQCD with an additional field in the adjoint
representation discussed in \[13\]. We then generalize the argument of \[2\] determining the most
general non-perturbative superpotential for models with vanishing tree-level superpotential.
We find that when tree-level terms are present the non-perturbative superpotential is still
constrained but no longer uniquely determined by the symmetries alone.

In sections 3 and 4 we present the new confining gauge theories. We consider all gauge
theory models of \[10, 14\] based on simple gauge groups and find that each of them confines
when the gauge group of its magnetic dual gets completely broken by the Higgs effect. Six of
these models have recently been found to show confinement by the authors of \[5\]. For nine of
these theories, however, the confining phase has not yet been discussed. We explain how the
confining spectrum can easily be obtained from the duality mapping and determine the form
of the confining superpotential for most of the models. Constructing the confined low-energy
spectrum for theories with \(SU(N_c)\) or \(SO(N_c)\) gauge groups involves considering generalized
baryons that can be mapped to similar operators of the magnetic theory for any value of \(N_c\).
When \(N_c\) is tuned such that the magnetic gauge group is completely broken these mappings
reduce to a correspondence between the light degrees of freedom of the magnetic theory in
the Higgs phase and the confined degrees of freedom of the electric theory in the confinement
phase. Furthermore, some of the terms of the confining superpotential can be obtained by
applying these mappings to the tree-level superpotential of the magnetic theory.

It is interesting to deform these theories either by giving a large expectation value to one
of the composite operators or by adding a mass term for one quark flavor and integrating
out the massive modes. In the first case one flows to a theory which is again confining with a
smooth non-perturbative superpotential.\[4\] The mass deformation leads to a theory with one
quark flavor less, which in most of the cases of section 3 has no stable vacuum. However, we
find that for \(SU(N_c)\) with an antisymmetric flavor and for \(Sp(2N_c)\) with an antisymmetric
tensor there exists a quantum moduli space with multiple constraints on the composite
operators, one of which is modified quantum mechanically. This is, to our knowledge, the

\[4\]There may be classical flat directions which lead to effective theories with no stable ground state. These
flat directions are removed from the moduli space by quantum effects and therefore do not correspond to
composite operators of the confined low-energy spectrum.\[5\]
Table 1: All s-confining models of \[2\] except those containing spinors. For some of them a magnetic dual is known. The confining phase can in these cases be derived by completely breaking the magnetic gauge group.

| electric gauge group | microscopic spectrum | duality known for . . . | magnetic gauge group | ref. |
|----------------------|----------------------|--------------------------|---------------------|-----|
| $SU(N_c)$            | $(N_c + 1) \Box + \Box$ | $N_f \Box + \Box$        | $SU(N_f - N_c)$     | 11  |
| $SU(N_c)$            | $\Box + \Box + 3 \Box + \Box$ | $\Box + N_f \Box + (N_c + N_f - 4) \Box$ | $SU(N_f - 3) \times Sp(2(N_f - 4))$ | 18  |
| $SU(7)$              | $2 \Box + 3 \Box$     | $-$                      | $-$                 | 2   |
| $SU(6)$              | $\Box + 4 \Box + \Box$ | $\Box + 5 \Box + \Box$   | $Sp(4)$            | 15  |
| $SU(5)$              | $3 \Box + \Box$       | $-$                      | $-$                 | 15  |
| $SU(5)$              | $\Box + 2 \Box + 4 \Box$ | $-$                      | $-$                 | 15  |
| $Sp(2N_c)$           | $2 (N_c + 2) \Box$    | $2N_f \Box$              | $Sp(2(N_f - N_c - 2))$ | 19  |
| $Sp(2N_c)$           | $\Box + 6 \Box$       | $\Box + 8 \Box$          | $Sp(2N_c)$         | 20  |
| $G_2$                | $5 \Box$              | $-$                      | $-$                 | 21  |

first example of theories that possess a quantum modified moduli space in the presence of a tree-level superpotential.

## 2 Confining gauge theories from non-Abelian duality

It is an interesting fact that the s-confining models with vanishing tree-level superpotential found in \[2\] can be derived from the completely higgsed magnetic dual whenever a duality is known for the model that contains one quark flavor more. Let us make two remarks: (i) There are 25 s-confining models with $Spin(N_c)$ gauge groups, $N_c$ ranging from 7 to 14, which we do not consider here. Nine of these models can be derived from a self-dual description of the theory with one more vector as explained in \[15\] and seven further ones can presumably be derived from the self-dual $Spin(N_c)$ models presented in \[14\]. (ii) The confining phase of an $SO(N_c)$ gauge theory with $N_f = N_c - 3$ quarks, as discussed in \[11, 17\], can also be derived from the dual magnetic $SO(N_f - N_c + 4)$ gauge theory. But this model is not s-confining in the sense of \[2\] because its quantum moduli space possesses two distinct branches and confinement is found only on one of these branches.

In table \[1\] we list all s-confining models of \[2\] which do not contain spinors. In the first
two columns we give the gauge group $SU(N_c)$ and the microscopic spectrum of the electric theory that leads to confinement. The third column displays the spectrum of the electric theory for which a dual magnetic description is known. The magnetic gauge group is shown in the fourth column. Finally, for each model the reference is given where the mentioned duality is discussed and where in most cases it is shown how to obtain the s-confining model from the magnetic dual.

Let us explain the idea of using duality to obtain the confining spectrum by briefly reviewing the example worked out in [13]. Consider SQCD with an additional second rank tensor field $X$ transforming in the adjoint representation. Under the gauge and global symmetries the matter fields transform like in the following table. (The charges are chosen such that all these symmetries are non-anomalous.)

| $Q$ | $SU(N_c)$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_B$ | $U(1)_R$ | $Z_{(k+1)N_f}$ |
|-----|-----------|-------------|-------------|----------|----------|-----------------|
| $Q$ | $\Box$    | $\Box$      | 1           | $\frac{1}{N_c}$ | $1 - \frac{2N_c}{(k+1)N_f}$ | $-N_c$        |
| $\bar{Q}$ | $\Box$ | 1 | $\Box$ | $\frac{1}{N_c}$ | $1 - \frac{2N_c}{(k+1)N_f}$ | $-N_c$ |
| $X$ | adj       | 1           | 1           | 0        | $\frac{2}{k+1}$ | $N_f$          |

The low-energy limit of the theory with vanishing tree-level superpotential is not yet understood. However the situation turned out to simplify [6] when a superpotential term

$$W = h \ Tr X^{k+1}$$

is added. Here $k$ denotes a positive integer and $h$ is a dimensionful coupling parameter.

This model has a number of flat directions which can be parametrized by the expectation values of gauge invariant composite operators. These operators are most conveniently written in terms of “dressed” quarks $Q(j) \equiv X^j Q$, $\bar{Q}(j) \equiv X^j \bar{Q}$, where the gauge indices are contracted with a Kronecker delta:

**mesons** $M_j \equiv QQ(j), \ j = 0, \ldots, k - 1,$

**baryons** $B^{(n_0, \ldots, n_{k-1})} \equiv (Q)^{n_0} (Q^{(1)})^{n_1} \cdots (Q^{(k-1)})^{n_{k-1}}, \ \sum_{j=0}^{k-1} n_j = N_c,$

**antibaryons** $\bar{B}^{(\bar{n}_0, \ldots, \bar{n}_{k-1})} \equiv (\bar{Q})^{\bar{n}_0} (\bar{Q}^{(1)})^{\bar{n}_1} \cdots (\bar{Q}^{(k-1)})^{\bar{n}_{k-1}}, \ \sum_{j=0}^{k-1} \bar{n}_j = N_c,$

where the gauge indices are contracted with a Kronecker delta for the mesons and an epsilon tensor for the (anti-)baryons. The classical flat directions corresponding to the operators $Tr X^j, \ j = 2, \ldots k$, in general are lifted by the superpotential (2.1). For vanishing quark expectation values $\langle X \rangle$ must vanish as well if $N_c$ is no multiple of $k$.

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5 In this paper $Sp(2N_c)$ denotes the symplectic group of rank $N_c$.
6 The $G_2$ gauge theory was first treated in [21]. The suggestion of [16] that the theory with six fundamental fields might be self-dual is not well established because the 't Hooft anomalies for the discrete symmetries do not match between the electric and the magnetic theory [22].
7 Anomalies of discrete symmetries are discussed in [22, 24, 25].
The infrared behavior of this theory can equivalently be described by a magnetic $SU(\tilde{N}_c)$ gauge theory, with $\tilde{N}_c = kN_f - N_c$ and matter content $[1]$

| $SU(\tilde{N}_c)$ | $SU(N_j)_L$ | $SU(N_j)_R$ | $U(1)_B$ | $U(1)_R$ | $Z_{(k+1)N_f}$ |
|-------------------|-------------|-------------|----------|----------|----------------|
| $q$               | $\square$  | $\square$  | 1        | $\frac{1}{N_c}$ | $1 - \frac{2N_c}{(k+1)N_f} - \tilde{N}_c$ |
| $\bar{q}$         | $\square$  | 1           | $-\frac{1}{N_c}$ | 1        | $-\frac{2N_c}{(k+1)N_f} - \tilde{N}_c$ |
| $Y$               | $\text{adj}$ | 1           | 1        | $0$      | $N_f$          |
| $M_j$             | 1           | $\square$  | $\square$ | 0        | $2 - \frac{2}{k+1} - \frac{2jN_f}{(k+1)N_f} - 2N_c + jN_f$ |

The magnetic theory contains a tree-level superpotential

$$W_{\text{mag}} = -h \; \text{Tr} \; Y^{k+1} + \frac{h}{\mu^2} \sum_{j=0}^{k-1} M_{k-1-j} qY^j \bar{q}, \quad (2.3)$$

where $\mu$ is a mass scale that has to be introduced to match the magnetic to the electric theory.

Under duality the gauge singlets $M_j$ of the magnetic theory are mapped to the meson operators of (2.2) and are therefore denoted by the same symbols. The correct degrees of freedom $\hat{M}_j$ of the magnetic theory have mass dimension one and are related to the operators $M_j$ by $\mu^{j+1} \hat{M}_j = M_j$. We prefer to express all equations in terms of the $M_j$. One can construct magnetic baryon operators $\tilde{B}^{(m_0,\ldots,m_{k-1})}$, $\tilde{B}^{(m_0,\ldots,m_{k-1})}$ by contracting products of $\tilde{N}_c$ dressed magnetic quarks with an epsilon tensor very much like in the electric theory. The mapping to the electric baryons of (2.2) is given by $[3]

$$B^{(m_0,\ldots,m_{k-1})} \leftrightarrow \tilde{B}^{(m_0,\ldots,m_{k-1})}, \quad \text{with} \; m_j = N_f - n_{k-1-j} \quad (2.4)$$

Now consider the model with $N_f + 1$ quark flavors and choose $N_c = kN_f - 1 \quad [3, 5]$. Adding a mass term for the $(N_f + 1)$th flavor in the electric theory and integrating out the massive modes leads to a complete breaking of the $SU(k+1)$ gauge symmetry on the magnetic side via the Higgs effect (in general the number of magnetic colors is reduced by $k$ when integrating out a flavor from the electric theory). One color component of each quark flavor stays massless after the symmetry breaking. These singlets can be identified with the magnetic baryons $\tilde{B}^{(1,0,\ldots,0)}$, $\tilde{B}^{(1,0,\ldots,0)}$ and via (2.4) are mapped to the electric baryons $B^{(N_f,\ldots,N_f,N_f-1)}$, $\tilde{B}^{(N_f,\ldots,N_f,N_f-1)}$ of (2.2). At low energies the magnetic theory is in the weakly coupled Higgs phase and the electric theory is very strongly coupled and does not flow to a fixed point of the remormalization group $[4]$. Thus, if there still exists a sensible description of the low-energy theory in terms of electric variables it should only consist of the confined degrees of freedom. Indeed, one finds $[3, 5]$ that the electric theory confines with low-energy spectrum given by the mesons $M_j$, $j = 0, \ldots, k-1$, and the baryons $B^{(N_f,\ldots,N_f,N_f-1)}$, $\tilde{B}^{(N_f,\ldots,N_f,N_f-1)}$ of (2.2). The confining superpotential has been derived in $[3]$ and is exactly reproduced by the effective superpotential of the magnetic theory when care is taken of instanton effects in the completely broken magnetic gauge group.

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8 An infrared fixed point is only expected in the range of parameters where both the electric and the magnetic theory are asymptotically free in the ultraviolet $[11, 3]$. 

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Integrating out one quark flavor from the $N_c = kN_f - 1$ theory the authors of [5] found a non-perturbative superpotential of Affleck–Dine–Seiberg [25] type. This was to be expected from the analysis of [6] which showed that the theory has no stable vacuum for $N_f < N_c/k$. However, this behavior is in contrast to the s-confining gauge theories without a tree-level superpotential. Integrating out one flavor from these theories one obtains a quantum modified moduli space with no non-perturbative superpotential.

In the following we want to use this Higgs phase / confinement duality to determine the confining spectrum of many new models. One just has to find the mapping of the magnetic singlets that stay massless after higgsing the gauge group to the gauge invariant composite fields of the electric theory. A straightforward way to find this mapping for models with $SU$ or $SO$ gauge groups is to consider baryonic composite operators of the electric theory and the duality mapping to their magnetic counterparts. When the magnetic gauge group is higgsed the light degrees of freedom are proportional to some generalized baryons and can therefore easily be mapped to the corresponding confined degrees of freedom of the electric theory. Some of the terms of the confining superpotential are obtained by applying this mapping to the magnetic tree-level superpotential others are generated by instanton effects in the completely broken magnetic gauge group. However, in practice it is very difficult to determine the precise form of the superpotential terms that are generated by instantons of the magnetic gauge group. It is therefore important to know the non-perturbative electric superpotential from independent considerations.

For any $N = 1$ supersymmetric model with vanishing tree-level superpotential the form of the most general superpotential that can be generated by non-perturbative effects is completely fixed by the requirement that it be invariant under all symmetries of the considered model [25, 26, 2]. For a theory with gauge group $G$ and chiral matter fields $\phi_l$ in representations $r_l$ of $G$ and dynamically generated scale $\Lambda$ one finds

$$W \propto \left( \prod_l (\phi_l^{(m_l)})^{\mu_l} \right)^\Delta \Lambda^b,$$

(2.5)

where $\mu_l$ is the (quadratic) Dynkin index of the representation $r_l$, $\mu_G$ denotes the index of the adjoint representation, $\Delta = \sum_l \mu_l - \mu_G$ and $b = \frac{1}{2}(3\mu_G - \sum_l \mu_l)$ is the coefficient of the 1-loop $\beta$-function. In general the complete non-perturbative superpotential consists of a sum of terms of the form (2.5) with different possible contractions of all gauge and flavor indices. The relative coefficients of these terms cannot be fixed by symmetry arguments but must be inferred from a different reasoning.

When a tree-level superpotential is present the global symmetries are reduced and hence the non-perturbative corrections less constrained. To see this more explicitly divide the set of matter fields into two subsets $\{\phi_l\} = \{\bar{\phi}_l\} \cup \{\hat{\phi}_l\}$, with $\{\bar{\phi}_l\} \cap \{\hat{\phi}_l\} = \emptyset$, and add one tree-level term for the hatted fields:

$$W_{\text{tree}} = h \prod_l (\hat{\phi}_l)^{n_l},$$

(2.6)

where $h$ is a dimensionful coupling parameter and the $n_l$ are positive integers.
The form of the non-perturbative superpotential can be easily derived by viewing the dynamical scale \( \Lambda \) and the coupling parameter \( h \) as background chiral fields \([27]\). For each field \( \phi_l \) there is classically a \( U(1)_l \) symmetry under which only \( \phi_l \) carries charge 1 and all other fields carry charge 0. The \( U(1)_l \) symmetries are spontaneously broken by the tree-level term (2.6) but can be restored by assigning charge \( -n_l \) under each of them to the background field \( h \). At the quantum level these symmetries are anomalous but this can be cured by assigning charge \( \mu_l \) to \( \Lambda \) \([26, 28]\). Requiring that the non-perturbative superpotential be invariant under all these symmetries and have charge 2 under the non-anomalous R-symmetry we find

\[
W \propto \left( \prod_l (\hat{\phi}_l)^{n_l} / \Lambda^b \right)^\alpha \prod_l (\hat{\phi}_l)^{\beta_l} h^\gamma, \tag{2.7}
\]

where the powers \( \alpha, \beta_l, \gamma \) must verify the following relations:

\[
\gamma = 1 - \frac{1}{2} \alpha \Delta, \\
\beta_l = \mu_l \alpha + \gamma n_l. \tag{2.8}
\]

We conclude that the symmetries do not uniquely fix the form of the non-perturbative superpotential. For a given theory, i.e. fixed \( \Delta \), there exists a superpotential consistent with all symmetries for each value of the power \( \alpha \). However in some cases it is possible to determine the allowed values for \( \alpha \) from a different reasoning, as we shall see later. Note that for \( \alpha = 0 \) we recover the tree-level superpotential. In the following we will be interested in the cases where \( W \) is a smooth function of the \( \phi_l \), i.e. \( \alpha = k \), \( k \) a positive integer. Such a superpotential term can be generated by a \( k \)-instanton effect in the completely broken magnetic gauge group. This is because the dynamically generated scale \( \tilde{\Lambda} \) of the magnetic theory is related to the electric scale \( \Lambda \) by an equation of the form \( \Lambda^b \tilde{\Lambda}^b = f(h, \mu) \), where \( h \) is the coupling of (2.6), \( \mu \) is a mass scale similar to the one introduced in (2.3) and \( f \) is some function of \( h \) and \( \mu \) such that the equation is invariant under all \( U(1)_l \)-symmetries. (For all of the models considered in section 3 this function is given by \( f(h, \mu) = \mu^{N_f + \bar{N}_f} h^{-\Delta} \), where \( N_f \) (\( \bar{N}_f \)) is the number of (anti-)fundamental fields.) Thus the magnetic dual of (2.7) is proportional to \( \tilde{\Lambda}^b \) and can be generated by an \( \alpha \)-instanton effect. In the limit of vanishing tree-level superpotential, which can formally be obtained by setting \( \gamma = 0 \), we find \( \alpha = 2/\Delta \) because of (2.8) and therefore recover the index argument of \([2]\) that \( \Delta \) has to equal 2 when one demands \( W \) to be smooth in the fields.

Next, let us generalize this to the case of two tree-level terms

\[
W_{\text{tree}} = h_1 \prod_{l \in S_1} (\hat{\phi}_l)^{n_l} + h_2 \prod_{l \in S_2} (\hat{\phi}_l)^{m_l}, \tag{2.9}
\]

where \( S_1, S_2 \) are two subsets of \( \{ \hat{l} \} \) such that \( S_1 \cup S_2 = \{ \hat{l} \} \). A similar reasoning leads to a non-perturbative superpotential of the form (2.7) with \( h^\gamma \) replaced by \( h_1^{\gamma_1} h_2^{\gamma_2} \) and (2.8) replaced by

\[
\gamma_1 + \gamma_2 = 1 - \frac{1}{2} \alpha \Delta, \\
\beta_l = \mu_l \alpha + \epsilon_1 \gamma_1 n_l + \epsilon_2 \gamma_2 m_l, \tag{2.10}
\]

where \( \epsilon_{1/2} = \begin{cases} 
0 & \text{if } \hat{\ell} \not\in S_{1/2} \\
1 & \text{if } \hat{\ell} \in S_{1/2} \end{cases} \).
We see that the non-perturbative superpotential is even less constrained. For a given theory the correct form of the superpotential can only be determined if we know $\alpha$ and one of the $\gamma$’s from different arguments. Note that in the limit $S_2 = \emptyset$ and $\gamma_2 = 0$ we recover the case of only one tree-level term (2.7, 2.8). These considerations are only valid if the supersymmetric field strength $W_\alpha$ does not appear in the superpotential. One could however imagine that the superpotential (2.7) is multiplied by $(W_\alpha)^{2\delta}$, where $\delta$ is a positive integer. $W_\alpha$ has R-charge 1 but does not carry charge under any of the $U(1)_I$ symmetries and therefore only the relation for $\gamma$ (first line in (2.8) and (2.10)) is modified:

$$\gamma_1 + \gamma_2 = 1 - \frac{1}{2} \alpha \Delta - \delta.$$  

(2.11)

(The case of only one tree-level term is obtained by setting $\gamma_2 = 0$ and $\gamma = \gamma_1$.)

Additional information on the confining superpotential comes from the fact that the equations of motion derived from this superpotential should reproduce the classical constraints that hold amongst the confined degrees of freedom. In some cases it is easy to determine at least some of these constraints. The superpotential terms necessary to produce them can then be constructed. The authors of [5] found a simple method to determine explicitly the constraints on the gauge invariant composite operators which works for some gauge theories with only adjoint and fundamental matter. In these cases all the confined degrees of freedom can be expressed only in terms of the dressed quarks introduced in the line above eq. (2.2). To derive the constraints the considered theory can therefore be viewed as containing no tensor field but having an enhanced number of quark flavors. In the above example of $SU(N_c)$ with an adjoint tensor this means that we treat it as a theory of $k N_f = N_c + 1$ quark flavors. However, the constraints amongst the meson and baryon operators in SQCD with one more flavor than colors are known [1].

Most of the models containing tensor fields that show the phenomenon of non-Abelian duality in the presence of an appropriate tree-level superpotential have been presented in a systematic way by the authors of [14]. They realized that the superpotentials for the tensor fields in these models resemble $A_k$ or $D_k$ singularity forms, generalized from numbers to matrices. The models containing one tensor (and its conjugate for complex representations) have superpotentials of the form $\text{Tr} X^{k+1}$, corresponding to an $A_k$ singularity under the usual ADE classification [29]. The models containing two tensors (and their conjugates for complex representations) have superpotentials of the form $\text{Tr} X^{k+1} + \text{Tr} X Y^2$, corresponding to a $D_{k+2}$ singularity. We find that all the models of [14] based on simple gauge groups confine when the magnetic gauge group is completely higgsed (for most of the $A_k$ models this has already been established in [5]). Using the methods described above we determine the low-energy spectrum in each case and in most cases also the form of the confining superpotential. An overview of the results is given in table 2. It shows for each model (specified by the gauge group, matter content and tree-level superpotential) the number of colors for which the model confines and the powers $\alpha$, $\beta$ and $\gamma$ of the non-perturbative superpotential (2.7). For some models there are terms with different values of these coefficients. In these cases we display the powers that correspond to the terms with highest $\alpha$ because only these terms are relevant for deriving the classical constraints. For the $D_{k+2}$ models we restrict ourselves to give the value of $\alpha$. The powers $\beta$ and $\gamma$ could not be uniquely fixed. For some superpotential terms they are calculated in section 4.
| $SU(N_c)$ | $Sp(2N_c)$ | $SO(N_c)$ |
|----------------|----------------|----------------|
| tensors | $W_{\text{tree}}$ | $X^{k+1}$ | $X^{2(k+1)}$ | $X^{k+1}$ | $X^{2(k+1)}$ |
| | $\alpha$ | $(2k+1)N_f - 2$ | $(2k+1)N_f + 3$ | $(2k+1)N_f - 2$ | $(2k+1)N_f + 3$ |
| | $\beta$ | $2k(\alpha + \gamma)$ | $3k - N_f$ | $3k - N_f$ | $3k - N_f$ |
| | $\gamma$ | $-(N_c + 1)$ | $3 - N_f$ | $3 - N_f$ | $3 - N_f$ |

Table 2: Gauge theories that confine in the presence of a tree-level superpotential. The coefficients $\alpha$, $\beta$, $\gamma$ refer to the powers in the non-perturbative superpotential (2.7). Some of the terms of this superpotential are generated by an $\alpha$-instanton effect in the dual magnetic gauge theory. For $Sp(2N_c)$ with two tensors we suppose that $\alpha$ has the given values although we were not able to prove this.
3 Models with $A_k$-type superpotentials

3.1 SU($N_c$) with an antisymmetric tensor and its conjugate

Consider SQCD with an additional flavor of antisymmetric tensors $X$, $\bar{X}$ and tree-level superpotential $W_{\text{tree}} = h \text{Tr}(X\bar{X})^{k+1}$. This model was first studied in [10]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

|        | SU($N_c$) | SU($N_f$)$_L$ | SU($N_f$)$_R$ | U(1)$_X$ | U(1)$_B$ | U(1)$_R$ | $\mathbb{Z}_{2(k+1)N_f}$ |
|--------|-----------|--------------|---------------|---------|---------|---------|-----------------|
| $Q$    | $\Box$    | $\Box$       | 1             | 0       | $\frac{1}{N_c}$ | 1 - $\frac{N_c+2k}{(k+1)N_f}$ | $-(N_c-2)$ |
| $\bar{Q}$ | $\Box$    |             |               |         |         |         |                 |
| $X$    | 1         | $\Box$       | 1             | 0       | $\frac{1}{N_c}$ | 1 - $\frac{N_c+2k}{(k+1)N_f}$ | $-(N_c-2)$ |
| $\bar{X}$ | 1         | 1           | 1             | $\frac{2}{N_c}$ | $\frac{1}{k+1}$ | $N_f$          |                 |

We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

**Mesons**

\[ M_j = Q \bar{Q}_{(j)}, \quad P_r = Q \bar{X} Q_{(r)}, \quad \bar{P}_r = \bar{Q} X \bar{Q}_{(r)}, \]

with $Q_{(j)} = (X \bar{X})^j Q$, $\bar{Q}_{(j)} = (\bar{X} X)^j \bar{Q}$, $j = 0, \ldots, k$, $r = 0, \ldots, k - 1$,

**Baryons**

\[ B_n = X^n Q^{N_c-2n}, \quad \bar{B}_n = \bar{X}^n \bar{Q}^{N_c-2n}, \quad n = 0, \ldots, \left\lceil \frac{N_c}{2} \right\rceil, \]

\[ T_i = \text{Tr}(X \bar{X})^i, \quad i = 1, \ldots, k, \]

where the gauge indices of the baryons are contracted with an epsilon tensor.

This theory is dual to an SU($\tilde{N}_c$) gauge theory, with $\tilde{N}_c = (2k + 1)N_f - 4k - N_c$ and matter content [10]

|        | SU($\tilde{N}_c$) | SU($N_f$)$_L$ | SU($N_f$)$_R$ | U(1)$_X$ | U(1)$_B$ | U(1)$_R$ | $\mathbb{Z}_{2(k+1)N_f}$ |
|--------|-------------------|--------------|---------------|---------|---------|---------|-----------------|
| $q$    | $\Box$            | $\Box$       | 1             | $\frac{k(N_f-2)}{N_c}$ | $\frac{1}{N_c}$ | 1 - $\frac{N_c+2k}{(k+1)N_f}$ | $N_c + N_f - 2$ |
| $\bar{q}$ | $\Box$            | 1           | $\Box$       | $-\frac{k(N_f-2)}{N_c}$ | $-\frac{1}{N_c}$ | 1 - $\frac{N_c+2k}{(k+1)N_f}$ | $N_c + N_f - 2$ |
| $Y$    | 1                 | $\Box$       | 1             | $\frac{N_c-\tilde{N}_f}{N_c}$ | $\frac{2}{N_c}$ | $\frac{1}{k+1}$ | $N_f$          |
| $\bar{Y}$ | 1                 | 1           | $\Box$       | $-\frac{N_c-N_f}{N_c}$ | $-\frac{2}{N_c}$ | $\frac{1}{k+1}$ | $N_f$          |

and singlets $M_j$, $P_r$, $\bar{P}_r$ that carry the same quantum numbers as the mesons of the electric theory.

The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

\[ W_{\text{mag}} = -h \text{Tr}(Y \bar{Y})^{k+1} + \frac{h}{\mu^2} \sum_{j=0}^{k} M_{k-j} q \bar{q}_{(j)} + \frac{h}{\mu^2} \sum_{r=0}^{k-1} \left[ P_{k-1-r} q \bar{Y} q_{(r)} + \bar{P}_{k-1-r} \bar{q} Y \bar{q}_{(r)} \right], \quad (3.2) \]
where magnetic dressed quarks have been introduced by \( q_{(j)} = (Y \bar{Y})^j q, \bar{q}_{(j)} = (\bar{Y} Y)^j \bar{q}. \)

The authors of \([10]\) found a mapping of the baryons \( B_n \) of \([3,4]\) to the magnetic baryons \( \tilde{B}_m = Y^n q^{N_c - 2m} \) consistent with all global symmetries:

\[
B_n \leftrightarrow \tilde{B}_m, \quad \text{with} \quad m = k(N_f - 2) - n. \tag{3.3}
\]

For \( N_c = (2k + 1)N_f - 4k - 1 \) the magnetic theory is completely higgsed and the electric theory confines \([3]\) with low-energy spectrum given by the composite fields

\[
M_j, \, P_r, \, \bar{P}_r, \quad j = 0, \ldots, k, \quad r = 0, \ldots, k - 1,
\]

\[
B \equiv B_{k(N_f-2)}, \quad \tilde{B} \equiv \tilde{B}_{k(N_f-2)} \tag{3.4}
\]

of eqs. \((3.1)\). Note that the baryons are of the form \( B = X^{k(N_f-2)}Q^{N_f-1} \) and therefore transform in the antifundamental representation of the \( SU(N_f) \) flavor group. Furthermore they are mapped to the magnetic quark singlets that stay massless after breaking the magnetic gauge group, as can be seen by setting \( \tilde{N}_c = 1 \) in \((3.3)\). The ’t Hooft anomaly matching conditions \([30]\) are trivially satisfied because the only fields that contribute to the global anomalies in the magnetic theory for \( \tilde{N}_c = 1 \) are the \( 2N_f \) quark singlets and the meson singlets which carry the same charges as the electric baryons and mesons respectively.\(^9\) The anomaly matching conditions cannot tell us whether the composite fields \( T_i \) of \((3,1)\) appear as low-energy degrees of freedom or not, because it turns out that their contribution to the global anomalies vanishes. The reason is that the \( T_i \) are only charged under \( U(1)_R \) and \( Z_{2(k+1)N_f} \) and the fermionic charges under these symmetries have equal absolute values but opposite signs for each of the pairs \((T_i, T_{k+1-i})\). For the same reason each of the bilinear terms \( T_iT_{k+1-i} \) has R-charge 2 and is neutral under all other symmetries. They should be present in the effective superpotential, giving mass to all of the fields \( T_i \).\(^9\) Therefore the operators \( T_i \) are removed from the low-energy spectrum. Another way to see this is to realize that the flat directions corresponding to non-vanishing expectation values \( \langle T_i \rangle \) (for vanishing quark VEV’s) are lifted by the tree-level superpotential. Furthermore, the confining superpotential without the fields \( T_i \) shown below reproduces correctly the constraints amongst the classical composite fields by the equations of motion (we checked only the constraints involving the baryons \( B, \tilde{B} \)).

The effective low-energy superpotential of the magnetic theory, deduced from \((3.2)\) for the theory with \( \tilde{N}_c = 2(k + 1) \) and \( (N_f + 1) \) quark flavors by adding a tree-level term \( mM_0 \) and integrating out the massive modes, contains a term \( M_k q \bar{q} \). We thus expect that the confining superpotential of the electric theory has a term proportional to

\[
\tilde{B}M_k B \sim (Q \bar{Q})^{N_f} (X \bar{X})^{k(N_f-1)}. \tag{3.5}
\]

Comparing this to \((2.7)\) we find \( \alpha = 1 \). From \((2.8)\) we then get \( \gamma = 3 - N_f \) and \( \beta_X = \bar{\beta}_X = k(N_f - 1) \). The confining superpotential consequently is of the form

\[
W = \frac{\tilde{B}M_k B + \sum_{(\gamma, j_m), 0 \leq p \leq N_f / 2} \left[ \prod_{l=1}^{p} (P_{r_l} \bar{P}_{r_l}) \prod_{m=1}^{N_f-2p} M_j^m \right]}{h^{N_f-3} \Lambda^{4k(N_f-2)+N_f}}, \tag{3.6}
\]

\(^9\)On more general grounds it has been argued \([31]\) that the matching conditions for the gauge invariant composite operators are satisfied whenever the classical constraints can be derived from a superpotential.\(^9\)

I thank C. Csáki and H. Murayama for a clarifying remark on this point.
by a one-instanton effect in the completely broken magnetic gauge group. The equations of motion of $\mathcal{M}_X$ to transform the tensor $\epsilon_{i_1 \ldots i_{N_f}} X^{\alpha_1 \alpha_2} \ldots X^{\alpha_k(N_f-2)-1\alpha_k(N_f-2)}$ with $\epsilon_{i_1 \ldots i_{N_f}} \epsilon_{\alpha_1 \ldots \alpha_{N_c}} X^{\alpha_1 \alpha_2} \ldots X^{\alpha_k(N_f-2)-1\alpha_k(N_f-2)}$ is such that $((\bar{X}X)^k)_{\alpha}^{\dot{\beta}} \bar{Q}_{\beta}^i = 0$ for $\beta \leq p$. On the other hand

$$\epsilon_{i_1 \ldots i_{N_f}} \epsilon_{\alpha_1 \ldots \alpha_{N_c}} X^{\alpha_1 \alpha_2} \ldots X^{\alpha_k(N_f-2)-1\alpha_k(N_f-2)} \cdot Q^{\alpha_2k(N_f-2)+1i_1} \ldots Q^{\alpha_{N_c}iN_f-1} Q^{\beta_1N_f} = 0 \quad \text{for} \quad \beta > p.$$  

As a consequence $M_k B$ vanishes.

There could be further terms invariant under all symmetries, with $\alpha > 1$ and possibly generated by multi-instanton effects. However only the terms of (3.6) containing the highest power of $1/\Lambda$ are relevant for the derivation of the classical constraints (because the classical limit is reached when the field expectation values are much bigger than $\Lambda$). As the classical constraints seem to follow from the $\alpha = 1$ terms it is not likely that these multi-instanton corrections are present. However, we can not exclude them rigorously.

Integrating out one quark flavor we obtain a theory with $\hat{N}_f = N_f - 1$ flavors and $\hat{N}_c = (2k + 1)\hat{N}_f - 2k$ colors. This theory should possess a stable ground state because the authors of [10] found that there is a stable vacuum if $(2k + 1)\hat{N}_f > \hat{N}_c$. Indeed, adding a mass term $m M_0$ to (3.6) and integrating out the massive modes, we find a quantum moduli space with $k + 1$ constraints amongst the confined degrees of freedom $\hat{M}_f$, $\hat{P}_r$, $\hat{\bar{P}}_r$, $\hat{B}$, $\hat{\bar{B}}$ of the theory with $\hat{N}_f$ quark flavors. One of these constraints is modified quantum mechanically and reads

$$\sum_{\{r_l, j_m\}} \left[ \frac{p}{l=1} \prod_{r_1 \leq r_l \leq \hat{N}_f/2} \frac{\hat{N}_f - 2p}{\hat{N}_f} \prod_{m=1}^{\hat{N}_f - 2p} \hat{M}_{j_m} \right] = \hat{p}^{\hat{N}_f - 2L} L^{4k(\hat{N}_f - 1) + \hat{N}_f + 2}, \quad (3.7)$$

with $\sum_{l=1}^{p} (2r_l + 1) + \sum_{m=1}^{\hat{N}_f - 2p} j_m = k \hat{N}_f$.

\footnote{The hats denote the reduction to $\hat{N}_f$ flavors; the baryons, e.g., are given by $\hat{B} = Q^{\hat{N}_f} X^{k(\hat{N}_f - 1)}$.}
where we again left out all relative coefficients. This is, to our knowledge, the first example of a quantum modified moduli space which does not satisfy the index constraint \( \Delta = 0 \). The argument of \([2, 3]\) to derive this constraint rested on the assumption that it is possible to assign zero R-charge to all fields and then compensate the anomaly by assigning R-charge \( \Delta \) to \( \Lambda^4 \). Of course this reasoning is no longer valid when a tree-level superpotential is present. The constraint (3.7) spontaneously breaks the chiral symmetry of the theory. We checked that the 't Hooft anomaly matchings are satisfied at the point of the moduli space where the \( SU(N_f)_L \times SU(N_f)_R \) flavor symmetry is broken to its diagonal subgroup and all \( U(1) \) symmetries are unbroken.

For the special case \( k = 1 \), \( N_f = 3 \), and consequently \( N_c = 4 \), this can be seen more explicitly. For \( N_c = 4 \) the antisymmetric representation and its conjugate are equivalent. Therefore there is an additional global \( SU(2) \) symmetry which permutes the two tensors \( X_1 \equiv X \) and \( X_2 \equiv \tilde{X} \). The mesons and baryons in this case are defined by \( (M_0)^{ij} = Q_i^r Q_j^s, \) \( (M_1)^{ij} = \frac{1}{4} \epsilon_{rstu} \alpha_{\alpha} \beta_{\beta} Q_i^{st} X_\alpha \beta X_\beta \), \( B_{\alpha i} = \frac{1}{2} \epsilon_{ijm} Q_i^m X_\alpha \), and \( B_{\alpha i} = \frac{1}{2} \epsilon_{ijm} Q_i^m X_\alpha \), with color indices \( r, s, t, \ldots \), flavor indices \( i, j, m, \ldots \) and \( SU(2) \) indices \( \alpha, \beta \). The

![Equation](equation.png)

Adding a mass term for the third quark flavor and integrating out the massive modes, we find a vanishing low-energy superpotential and the two constraints

![Equation](equation.png)

for the low-energy fields of the \( \hat{N}_f = 2 \) theory. In the limit \( \Lambda_L \to 0 \) these reduce to the right constraints amongst the classical composite operators when the equations of motion of the tree-level superpotential are taken into account.

Another interesting special case is \( k = 0 \). The tree-level superpotential \( W_{\text{tree}} = h \ Tr \ X \bar{X} \) gives mass to the antisymmetric tensors. The model reduces to SQCD with \( N_c = N_f - 1 \), which is known to confine with a non-perturbative superpotential that is smooth in the confined degrees of freedom \([4]\). The superpotential (3.6) has the correct form in this limit, \( W_{k=0} = (\hat{B} M_0 B - \det M_0)/\Lambda_L^{2N_f - 3} \), where the scale matching \( h (N_e - 2) \Lambda_L^{3N_e - N_f} = \Lambda_L^{3N_e - N_f} \) was used. The theory with one quark flavor less, i.e. \( \hat{N}_c = \hat{N}_f \), is known to confine with a quantum modified moduli space \([4]\). The equation (3.7) is modified by an additional term \( \hat{B} \hat{B} \) in the limit \( k = 0 \) and reproduces the correct quantum constraint \( \det \hat{M} - \hat{B} \hat{B} \hat{M} = \Lambda_L^{2\hat{N}_f} \), where \( h \hat{N}_f - 2 \Lambda_L^{\hat{N}_f + 2} = \Lambda_L^{2\hat{N}_f} \).

### 3.2 \( SU(N_c) \) with a symmetric tensor and its conjugate

Consider SQCD with an additional flavor of symmetric tensors \( X, \bar{X} \) and tree-level superpotential \( W_{\text{tree}} = h \ Tr(X \bar{X})^{k+1} \). This model was first studied in \([4]\). The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global
symmetries are shown in the following table:

|        | $SU(N_c)$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_X$ | $U(1)_B$ | $U(1)_R$ | $Z_{2(k+1)N_f}$ |
|--------|------------|-------------|-------------|----------|----------|----------|-----------------|
| $Q$    | \(\square\) | \(\square\) | 1 | 0 | \(\frac{1}{N_c}\) | \(\frac{N_c-2k}{N_f}\) | \(-\frac{1}{N_c}(-N_c+2)\) |
| $\bar{Q}$ | \(\square\) | 1 | \(\square\) | 0 | \(-\frac{1}{N_c}\) | \(\frac{N_c-2k}{N_f}\) | \(-\frac{1}{N_c}(-N_c+2)\) |
| $X$    | \(\square\) | 1 | 1 | 1 | \(\frac{2}{N_c}\) | \(\frac{k+1}{1}\) | \(N_f\) |
| $\bar{X}$ | \(\square\) | 1 | 1 | -1 | \(-\frac{2}{N_c}\) | \(\frac{k+1}{1}\) | \(N_f\) |

We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

**mesons** \(M_j = Q\bar{Q}^{(j)}, \quad P_r = Q\bar{X}Q^{(r)}, \quad \bar{P}_r = \bar{Q}X\bar{Q}^{(r)},\)

with \(Q^{(j)} = (XX)^jQ, \quad \bar{Q}^{(j)} = (\bar{X}X)^j\bar{Q}, \quad j = 0, \ldots, k, \quad r = 0, \ldots, k - 1,\)

**baryons** \(B^{(n_0, \ldots, n_{k-1}, n_0, \ldots, n_k)} = (XW_0)^2(X\bar{X}XW_0)^2 \cdots (X\bar{X}XW_0)^{p-1}W_0^2\)

\[\cdot (X\bar{Q})^{n_0} (X\bar{Q}^{(1)})^{\bar{n}_1} \cdots (X\bar{Q}^{(k-1)})^{\bar{n}_{k-1}} Q^{n_1} Q^{n_1} \cdots Q^{n_k},\]

\[B^{(n_0, \ldots, n_{k-1}, n_0, \ldots, n_k)} = (\bar{X}W_0)^2(\bar{X}\bar{X}XW_0)^2 \cdots (\bar{X}\bar{X}XW_0)^{p-1}W_0^2\]

\[\cdot (\bar{X}\bar{Q})^{n_0} (\bar{X}\bar{Q}^{(1)})^{\bar{n}_1} \cdots (\bar{X}\bar{Q}^{(k-1)})^{\bar{n}_{k-1}} \bar{Q}^{n_1} \bar{Q}^{n_1} \cdots \bar{Q}^{n_k},\]

with \(\sum_{j=0}^{k} n_j + \sum_{j=0}^{k-1} \bar{n}_j = N_c - 4p, \quad p = 0, \ldots, \min(k, \left\lceil\frac{N_c}{4}\right\rceil),\)

\[B_n = X^N Q^{N_c-n} \bar{Q}^{N_c-n}, \quad \bar{B}_n = \bar{X}^N \bar{Q}^{N_c-n} \bar{Q}^{N_c-n}, \quad n = 0, \ldots, N_c,\]

\[T_i = \text{Tr}(XX)^i, \quad i = 1, \ldots, k,\]

where the gauge indices are contracted with one epsilon tensor for the baryons \(B^{(-)}, \bar{B}^{(-)}\) and with two epsilon tensors for the baryons \(B_n, \bar{B}_n\).

This theory is dual to an \(SU(\tilde{N}_c)\) gauge theory, with \(\tilde{N}_c = (2k + 1)N_f + 4k - N_c\) and matter content \([10]\)

|        | $SU(\tilde{N}_c)$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_X$ | $U(1)_B$ | $U(1)_R$ | $Z_{2(k+1)N_f}$ |
|--------|-------------------|-------------|-------------|----------|----------|----------|-----------------|
| $q$    | \(\square\) | \(\square\) | 1 | \(\frac{k(N_f+2)}{N_c}\) | \(\frac{1}{N_c}\) | \(1 - \frac{N_c - 2k}{(k+1)N_f}\) | \(N_c + N_f + 2\) |
| $\bar{q}$ | \(\square\) | 1 | \(\square\) | \(\frac{k(N_f+2)}{N_c}\) | \(\frac{1}{N_c}\) | \(1 - \frac{N_c - 2k}{(k+1)N_f}\) | \(N_c + N_f + 2\) |
| $Y$    | \(\square\) | 1 | 1 | \(\frac{N_c - N_f}{N_c}\) | \(\frac{2}{N_c}\) | \(\frac{1}{k+1}\) | \(N_f\) |
| $\bar{Y}$ | \(\square\) | 1 | 1 | \(-\frac{N_c - N_f}{N_c}\) | \(-\frac{2}{N_c}\) | \(\frac{1}{k+1}\) | \(N_f\) |

and singlets \(M_j, P_r, \bar{P}_r\) that carry the same quantum numbers as the mesons of the electric theory.

The following tree-level superpotential of the magnetic theory is invariant under all the
symmetries:

$$W_{\text{mag}} = -h \, \text{Tr}(Y \tilde{Y})^{k+1} + \frac{h}{\mu^2} \sum_{j=0}^{k} M_{k-j} q \bar{q}(j) + \frac{h}{\mu^2} \sum_{r=0}^{k-1} \left[ P_{k-1-r} q \tilde{Y} q(\bar{r}) + \bar{P}_{k-1-r} \bar{q} \tilde{Y} \bar{q}(\bar{r}) \right],$$  \hspace{1cm} (3.11)

where $q_{(j)} = (Y \tilde{Y})^j q$, $\bar{q}_{(j)} = (\tilde{Y} Y)^j \bar{q}$.

The electric baryons of (3.10) can be consistently mapped to similar baryons of the magnetic theory:

$$\mathcal{B}_{p}^{(n_{a},\ldots,n_{k-1},n_{0},\ldots,n_{k})} \leftrightarrow \tilde{\mathcal{B}}_{\bar{q}}^{(\bar{n}_{a},\ldots,\bar{n}_{k-1},\bar{n}_{0},\ldots,\bar{n}_{k})}, \quad \text{with}$$

$$q = k - p, \quad m_{j} = N_{f} - n_{k-j}, \quad \bar{m}_{j} = N_{f} - \bar{n}_{k-1-j},$$

$$B_{n} \leftrightarrow \tilde{B}_{m}, \quad \text{with} \quad m = 2k(N_{f} + 2) - n,$$  \hspace{1cm} (3.12)

where the magnetic baryons $\tilde{\mathcal{B}}_{\bar{q}}^{(\ldots)}$, $\tilde{B}_{m}$ are defined in the same way as the electric baryons of (3.10) replacing all fields by their dual partners and $N_{c}$ by $\tilde{N}_{c}$. The second of these mappings has been found in [10].

For $N_{c} = (2k + 1)N_{f} + 4k - 1$ the magnetic theory is completely higgsed and the electric theory confines [3] with low-energy spectrum given by the composite fields

$$M_{j}, \quad P_{r}, \quad \bar{P}_{r}, \quad j = 0, \ldots, k, \quad r = 0, \ldots, k-1,$$

$$B \equiv \mathcal{B}_{k}^{(N_{f},\ldots,N_{f},N_{f}-1)}, \quad \bar{B} \equiv \tilde{\mathcal{B}}_{k}^{(N_{f},\ldots,N_{f},N_{f}-1)},$$

$$b \equiv B_{2k(N_{f}+2)-1}, \quad \bar{b} \equiv \tilde{B}_{2k(N_{f}+2)-1}$$  \hspace{1cm} (3.13)

of eqs. (3.10). Note that the baryons $b$, $\bar{b}$ are of the form $b = X^{2k(N_{f}+2)-1} Q_{N_{f}} Q^{N_{f}}$ and therefore do not transform under the $SU(N_{f})$ flavor group; the baryons $B$, $\bar{B}$ are in the antifundamental representation of $SU(N_{f})$. Furthermore, from (3.12) we find the mappings $B, \bar{B} \leftrightarrow q, \bar{q}$ and $b, \bar{b} \leftrightarrow Y, \bar{Y}$. One color component of each of the fields $q$, $\bar{q}$, $Y$, $\bar{Y}$ together with the meson singlets are exactly the degrees of freedom that stay massless after breaking the magnetic gauge group. It is easy to see that the 't Hooft anomaly matching conditions are satisfied because the only fields that contribute to the global anomalies in the magnetic theory for $\tilde{N}_{c} = 1$ are the $2N_{f}$ quark singlets, one component of $Y$ and $\bar{Y}$ each and the meson singlets. They carry the same charges as the baryons and mesons of the electric theory and consequently the global anomalies match between the macroscopic and the microscopic description. As in the model of the previous section one could add the fields $T_{i}$ without violating the matching conditions. Their contribution to the global anomalies cancels. But in the presence of the tree-level superpotential their VEV’s do not correspond to flat directions and they are removed from the low-energy spectrum by the mass terms $T_{i} T_{k+1-i}$.

The effective low-energy superpotential of the magnetic theory, deduced from (3.11) for the theory with $\tilde{N}_{c} = 2(k + 1)$ and $(N_{f} + 1)$ quark flavors by adding a tree-level term $m M_{0}$ and integrating out the massive modes, contains the terms

$$\text{Tr}(Y \tilde{Y})^{k+1} + M_{k} q \bar{q} + P_{k-1} q \tilde{Y} q + P_{k-1} \bar{q} \tilde{Y} \bar{q}.$$
We thus expect that the confining superpotential of the electric theory has terms proportional to
\[(bb)^{k+1} \sim (QQ)^{2(k+1)N_f} (X\bar{X})^{(k+1)(2k(N_f+2)−1)},\]
\[BM_kB \sim (W_\alpha)^{4k} (QQ)^{(2k+1)N_f} (X\bar{X})^{k(N_c−2k)},\] \[(3.14)\]
\[BBP_{k-1}\bar{b}, \overline{BB\bar{P}}_{k-1}b \sim (W_\alpha)^{4k} (QQ)^{2(k+1)N_f} (X\bar{X})^{(k+1)(2k(N_f+1)−1)}.\]

Comparing this to (2.7) we find \(\alpha^{(1)} = 2(k+1)\) for the term in the first line of (3.14). From (2.8) we then get \(\gamma^{(1)} = -(N_c+N_f+4)\) and \(\beta^{(1)} = (k+1)(2k(N_f+2)−1)\). In the same way find for the terms in the second and third line \(\alpha^{(2)} = 2k+1, \gamma^{(2)} = −N_c−2(k+1), \beta^{(2)} = k(N_c−2k)\) and \(\alpha^{(3)} = 2(k+1), \gamma^{(3)} = -(N_c + N_f + 4 + 2k), \beta^{(3)} = (k+1)(2k(N_f+1)−1)\), where the formula (2.11) has been used with \(\delta = 2k\). The confining superpotential consequently is of the form
\[W = \frac{BM_kB}{h^{N_c+2(k+1)} \Lambda^{(2k+1)(4k(N_f+2)−4+N_f)}} + \frac{(bb)^{k+1} + h^{-2k}(BBP_{k-1}\bar{b} + \overline{BB\bar{P}}_{k-1}b)}{h^{N_c+N_f+4} \Lambda^{2(k+1)(4k(N_f+2)−4+N_f)}}.\] \[(3.15)\]

Because all mesons have negative R-charge (for \(k > 0\)) no other superpotential terms with \(\alpha \leq 2(k+1)\) are possible. In the semiclassical regime \(\Lambda \rightarrow 0\) (i.e. only the second term of (3.13) is relevant) the equations of motion of this superpotential set the expectation values of the baryons to zero. These are indeed classical constraints because the field strength tensor \(W_\alpha\) vanishes classically.

From the analysis of [10] we know that there is only a stable vacuum if \((2k + 1)N_f \geq N_c - 4k\). Thus, for the theory with one less flavor one expects to find a superpotential of the Affleck-Dine-Seiberg type which destabilizes the ground state. It is not yet clear how this can be obtained by integrating out one quark flavor from (3.15).

In the limit \(k = 0\) the coupling parameter \(h\) represents a mass for the tensors \(X, \bar{X}\). Integrating out the massive modes and using the scale matching relation \(h^{N_c+2}\Lambda^{3N_c−(N_c+2)−N_f} = \Lambda^{3N_c−N_f}\), we find SQCD with \(N_c = N_f − 1\) and the correct confining superpotential \(W_{k=0} = (\mathcal{B}M_0B − \det M_0)/\Lambda^{2N_f−3}\). The term proportional to \(\det M_0\) is only possible for \(k = 0\) and is generated by a one-instanton effect in the magnetic gauge theory. For \(k = 0\) there are no mesons \(P_r, \bar{P}_r\) and \(b, \bar{b}\) acquire a mass from the superpotential (3.13) as anticipated by the authors of [3].

### 3.3 SU\((N_c)\) with an antisymmetric tensor and a conjugate symmetric tensor

Consider an \(SU(N_c)\) gauge theory with \(N_f + 8\) quarks \(Q, N_f\) antiquarks \(\bar{Q}\), an antisymmetric tensor \(X\) and a conjugate symmetric tensor \(\bar{X}\) and tree-level superpotential \(W_{\text{tree}} = h \text{Tr}(X\bar{X})^{2(k+1)}\). This is a chiral theory and was first studied in [10]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global
symmetries are shown in the following table:

|       | $SU(N_c)$ | $SU(N_f+8)_L$ | $SU(N_f)_R$ | $U(1)_X$ | $U(1)_B$ | $U(1)_R$ | $Z_{4(k+1)}(N_f+4)$ |
|-------|-----------|---------------|--------------|----------|----------|----------|---------------------|
| $Q$   | □         | □             | 1            | -2(2k+1) + $\frac{2(4k+3)}{N_f+8}$ | $\frac{1}{N_c}$ | 1 - $\frac{N_c+2(4k+3)}{2(k+1)(N_f+8)}$ | $-N_c$         |
| $\bar{Q}$ | □         | 1             | □            | $2k+1 + \frac{2(4k+3)}{N_f}$ | $-\frac{1}{N_c}$ | 1 - $\frac{N_c-2(4k+3)}{2(k+1)N_f}$ | $-N_c$         |
| $X$   | 1         | 1             | □            | $2 - \frac{2}{N_c}$ | $\frac{1}{2}$ | $\frac{1}{2(k+1)}$ | $N_f + 4$       |
| $\bar{X}$ | 1         | 1             | □            | -1 | $-\frac{2}{N_c}$ | $\frac{1}{2(k+1)}$ | $N_f + 4$       |

We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

**Mesons**

$$M_j = Q \tilde{Q}_{(j)} , \quad P_r = Q \bar{X} Q^{(r)} , \quad \bar{P}_r = \bar{Q} X \bar{Q}^{(r)} ,$$

with $Q_{(j)} = (X \bar{X})^j Q$, $\tilde{Q}_{(j)} = (\bar{X} X)^j \tilde{Q}$, $j = 0, \ldots, 2k + 1$, $r = 0, \ldots, 2k$;

**Baryons**

$$\bar{B}^{(n_0, \ldots, n_{2k}, n_{2k+1})} = (X (X \bar{X})^k W_0)^2(\bar{X} Q)^n (\bar{X} Q^{(1)} )^\bar{n}_1 \cdots (X Q^{(k)})^\bar{n}_{2k} Q^{n_1} \cdots Q^{n_{2k+1}},$$

with $\sum_{j=0}^{2k+1} n_j + \sum_{\bar{j}=0}^{2k} \bar{n}_j = N_c - 4$,

$$B_n = X^n Q^{N_c-2n} , \quad n = 0, \ldots, \left[ \frac{N_c}{2} \right] ,$$

$$\bar{B}_n = \bar{X}^\bar{n} \bar{Q}^{N_c-\bar{n}} \bar{Q}^{N_c-\bar{n}} , \quad \bar{n} = 0, \ldots, N_c ,$$

$$T_i = \text{Tr}(X \bar{X})^i , \quad i = 1, \ldots, 2k + 1 ,$$

where the gauge indices are contracted with one epsilon tensor for the $\bar{B}^{(\cdot)}$, $B_n$ and with two epsilon tensors for the $\bar{B}_n$.

This theory is dual to an $SU(\tilde{N}_c)$ gauge theory, with $\tilde{N}_c = (4k + 3)(N_f + 4) - N_c$ and matter content [10]

|       | $SU(\tilde{N}_c)$ | $SU(N_f+8)_L$ | $SU(N_f)_R$ | $U(1)_X$ | $U(1)_B$ | $U(1)_R$ | $Z_{4(k+1)}(N_f+4)$ |
|-------|-------------------|---------------|--------------|----------|----------|----------|---------------------|
| $q$   | □                 | □             | 1            | $2k+1 - \frac{2(4k+3)}{N_f+8}$ | $\frac{1}{N_c}$ | 1 - $\frac{\tilde{N}_c+2(4k+3)}{2(k+1)(N_f+8)}$ | $-\tilde{N}_c + p$ |
| $\bar{q}$ | □                 | 1             | □            | $-(2k+1) - \frac{2(4k+3)}{N_f}$ | $-\frac{1}{N_c}$ | 1 - $\frac{\tilde{N}_c-2(4k+3)}{2(k+1)N_f}$ | $-\tilde{N}_c - p$ |
| $Y$   | 1                 | 1             | □            | -1 | $\frac{N_c}{2}$ | $\frac{1}{2(k+1)}$ | $N_f + 4 + 2p$ |
| $\bar{Y}$ | 1                 | 1             | □            | 1 | $-\frac{2}{N_c}$ | $\frac{1}{2(k+1)}$ | $N_f + 4 - 2p$ |

and singlets $M_j, P_r, \bar{P}_r$ that carry the same quantum numbers as the mesons of the electric theory. The number $p$ is defined by

$$p = 4 + \frac{(2(k+1)(N_f+4) + 2)(N_f+4)}{\tilde{N}_c} .$$
The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

\[ W_{\text{mag}} = -h \, \text{Tr} (YY)^2 + \frac{h}{\mu^2} \sum_{j=0}^{2k+1} M_{2k+1-j} q q(j) + \frac{h}{\mu^2} \sum_{r=0}^{2k} \left[ P_{2k-r} q \bar{Y} q(r) + \bar{P}_{2k-r} Y q(\bar{q}(r)) \right] \]

where \( q(j) = (Y\bar{Y})^j q, \bar{q}(j) = (\bar{Y}Y)^j \bar{q} \).

Under duality the electric baryons of (3.16) are mapped to the magnetic baryons \( \tilde{B}^{(\tilde{m}_1, \tilde{m}_2)} = (\bar{Y}q)^{\tilde{m}_0} \cdots (\bar{Y}q(2k))^{\tilde{m}_2k}q^{m_0} \cdots q^{m_{2k+1}}, \tilde{B}_m = Y^m q \tilde{N}_c - 2m \) and \( \tilde{B}_{\tilde{m}} = Y^{\tilde{m}} q \tilde{N}_c - \tilde{m} \tilde{q} \tilde{N}_c - \tilde{m} \) according to the following prescription:

\[ \tilde{B}^{(n_0, \ldots, n_{2k}, n_0, \ldots, n_{2k+1})} \leftrightarrow \tilde{B}^{(\tilde{m}_0, \ldots, \tilde{m}_{2k}, \tilde{m}_0, \ldots, \tilde{m}_{2k+1})}, \quad \text{with} \]

\[ m_j = N_f - n_{2k+1-j}, \quad \tilde{m}_j = N_f + 8 - \tilde{n}_{2k-j}, \quad B_n \leftrightarrow \tilde{B}_m, \quad \text{with} \quad m = (2k+1)(N_f+4) - 2 - n, \]

\[ \tilde{B}_{\tilde{n}} \leftrightarrow \tilde{\tilde{B}}_{\tilde{m}}, \quad \text{with} \quad \tilde{m} = 2(2k+1)(N_f+4) + 4 - \tilde{n}. \]

The last two of these mappings have been found in [10].

For \( N_c = (4k+3)(N_f+4) - 1 \) the magnetic theory is completely higgsed and the electric theory confines with low-energy spectrum given by the composite fields

\[ M_j, P_r, \tilde{P}_r, \quad j = 0, \ldots, 2k + 1, \quad r = 0, \ldots, 2k, \]

\[ B \equiv B_{(2k+1)(N_f+4)-2}, \tilde{B} \equiv B^{(N_f+8, \ldots, N_f+8, N_f, \ldots, N_f, N_f-1)}, \]

\[ \bar{b} \equiv \tilde{B}_{2(2k+1)(N_f+4)+3}, \]

of eqs. (3.16).

The baryons \( B, \tilde{B} \) transform in the antifundamental representation of \( SU(N_f+8)_L, SU(N_f)_R \) respectively and \( \bar{b} \) does not transform under the flavor symmetry. The fact that the baryons are very different from the antibaryons is due to the chirality of the theory. The former resemble the baryons of the theory with an antisymmetric flavor whereas the latter are similar to the antibaryons of the theory with a symmetric flavor. From (3.18) we find the magnetic gauge group. The 't Hooft anomaly matching conditions are satisfied because the only fields that contribute to the global anomalies in the magnetic theory for \( \tilde{N}_c = 1 \) are the \( 2N_f \) quark singlets, one component of \( \tilde{Y} \) and the meson singlets. The contribution of the \( T_i \) to the global anomalies again vanishes because they are only charged under \( U(1)_R \) and \( Z_{4(k+1)(N_f+4)} \) and the fermionic charges under these symmetries have equal absolute values but opposite signs for each of the pairs \( (T_i, T_{2(k+1) - i}) \). However, in the presence of the tree-level superpotential their VEV’s do not correspond to flat directions and they are removed from the low-energy spectrum by the mass terms \( T_i T_{2(k+1) - i} \).

As a further consistency check let us consider deformations of the theory along the flat directions corresponding to large expectation values of the baryons \( B, \bar{b} \). A large VEV of
$B$ breaks the gauge symmetry to $Sp(2((2k + 1)(N_f + 4) - 2))$. The low-energy theory contains $2(N_f + 4)$ quarks $Q$, a symmetric tensor $X$ and tree-level superpotential $\text{Tr} \ X^{2(k+1)}$. This model is known to show confinement (cf. section 3.4). A large VEV of $\tilde{b}$ breaks the gauge symmetry to $SO(2(2k + 1)(N_f + 4) + 3)$. The low-energy theory contains $2(N_f + 4)$ quarks $Q$, an antisymmetric tensor $X$ and tree-level superpotential $\text{Tr} \ X^{2(k+1)}$. This model is known to show confinement (cf. section 3.6).

The effective low-energy superpotential of the magnetic theory, deduced from (3.17) for the theory with $\tilde{N}_c = 4(k+1)$, $(N_f + 9)$ quarks and $(N_f + 1)$ antiquarks by adding a tree-level term $mM_0$ and integrating out the massive modes, contains the terms $M_{2k+1}q\bar{q} + P_{2k}q\bar{Y}q$. We thus expect that the confining superpotential of the electric theory has terms proportional to

$$\begin{align*}
\tilde{B}M_{2k+1}B &\sim (W_a)^2 Q^{2(k+1)(N_f+8)} \bar{Q}^{2(k+1)N_f} X^{2(k+1)((2k+1)(N_f+4)-3)} \bar{X}^{2(k+1)((2k+1)(N_f+4)+1)}, \\
BBP_{2k}\tilde{b} &\sim Q^{2(N_f+8)} Q^{2N_f} X^{2((2k+1)(N_f+4)-4+2k)} \bar{X}^{2((2k+1)(N_f+4)+4+2k)}.
\end{align*}$$

\[ (3.20) \]

Comparing this to (2.7) we find $\alpha^{(1)} = 2(k + 1)$ for the term in the first line of (3.20). From (2.8, 2.11) we then get $\gamma^{(1)} = -2(k + 1)(N_f + 4)$ and $\beta_X^{(1)} = 2(k + 1)((2k + 1)(N_f + 4) - 3)$, $\beta_{\bar{X}}^{(1)} = 2(k + 1)((2k + 1)(N_f + 4) + 1)$. In the same way we obtain $\alpha^{(2)} = 2$, $\gamma^{(2)} = 1 - 2(N_f + 4)$, $\beta_X^{(2)} = 2(2k + 1)(N_f + 4) - 4 + 2k$, $\beta_{\bar{X}}^{(2)} = 2(2k + 1)(N_f + 4) + 4 + 2k$ for the term in the second line. The confining superpotential consequently is of the form

$$W = \frac{\tilde{B}M_{2k+1}B}{k^{2(k+1)(N_f+4)}\Lambda^{2((8k+5)(N_f+4)-2)}} + \frac{BBP_{2k}\tilde{b}}{k^{2(N_f+4)-1}\Lambda^{2((8k+5)(N_f+4)-2)}} + \ldots, \quad (3.21)$$

where the dots stand for possible further terms that could be generated by instanton effects in the completely broken magnetic gauge group.

### 3.4 $Sp(2N_c)$ with an adjoint tensor

Consider an $Sp(2N_c)$ gauge theory with $2N_f$ quarks $Q$ in the fundamental representation and a second rank tensor $X$ in the symmetric (=adjoint) representation of the gauge group and tree-level superpotential $W_{\text{tree}} = h \ \text{Tr} \ X^{2(k+1)}$. This model was first studied in [6]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| \( Sp(2N_c) \) | \( SU(2N_f) \) | \( U(1)_R \) | \( Z_{2(k+1)N_f} \) |
|---|---|---|---|
| $Q$ | \( \square \) | \( \square \) | \( 1 - \frac{N_c+1}{(k+1)N_f} \) |
| $X$ | \( \boxdot \) | \( 1 \) | \( -\frac{N_f}{k+1} \) |

There are no baryons in symplectic gauge theories and therefore the only (non-redundant) gauge invariant composite operators that can be built from the elementary fields are:

$$M_j = Q X^j Q, \quad j = 0, \ldots, 2k,$$

$$T_i = \text{Tr} \ X^{2i}, \quad i = 1, \ldots, k.$$

(3.22)
where the gauge indices are contracted with the \(Sp(2N_c)\)-invariant \(J\)-tensor.

This theory is dual to an \(Sp(2\tilde{N}_c)\) gauge theory, with \(\tilde{N}_c = (2k + 1)N_f - 2 - N_c\) and matter content

| \(Sp(2\tilde{N}_c)\) | \(SU(2N_f)\) | \(U(1)_R\) | \(Z_{2(2k+1)N_f}\) |
|-----------------|-----------------|------------|-----------------|
| \(q\)           | \(\Box\)        | \(\Box\)   | \(1 - \frac{\tilde{N}_c+1}{k+1}N_f\) |
| \(Y\)           | \(\Box\)        | \(\Box\)   | \(N_c + N_f + 1\) |

and singlets \(M_j\) that carry the same quantum numbers as the mesons of the electric theory.

The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

\[
W_{\text{mag}} = -h \operatorname{Tr} Y^{2(k+1)} + \frac{h}{\mu^2} \sum_{j=0}^{2k} M_{2k-j}qY^j q. \tag{3.23}
\]

For \(N_c = (2k+1)N_f - 2\) the magnetic theory is completely higgsed and the electric theory confines \(\Box\) with low-energy spectrum given by the composite fields

\[M_j, \quad j = 0, \ldots, 2k, \tag{3.24}\]

of eqs. (3.22). The ’t Hooft anomaly matching conditions are trivially satisfied because the only fields that stay massless after completely breaking the magnetic gauge group are the meson singlets which carry the same charges as the mesons of the electric theory. Again the fields \(T_i\) could be added to the confined spectrum without modifying the anomaly matchings. But for the same reason as in the models considered in the previous sections they are not the right degrees of freedom of the low-energy theory.

The fact that all components of the magnetic quarks get massive when completely breaking the magnetic gauge group makes it more difficult to find the confining superpotential because none of the magnetic tree-level terms survives the symmetry breaking. All the terms of the confining superpotential are generated by instanton effects. However, for the model considered in this section we can use the method developed in \(\Box\) to determine the constraints on the confined degrees of freedom and then try to find a superpotential that reproduces these constraints via the equations of motion. To find the classical constraints we introduce dressed quarks \(Q_{(j)} = X^j Q\) and view the considered model as an \(Sp(2((2k+1)N_f - 2))\) gauge theory with \(2(2k+1)N_f\) quarks \(Q = (Q, Q_{(1)}, \ldots, Q_{(2k)})\) and no tensor. The classical constraints for this reduced theory are known \(\Box\): The mesons \(M = QQ\) must verify

\[\epsilon_{i_1 \cdots i_{2(2k+1)}N_f} M_{i_1 i_2} \cdots M_{i_{2(2k+1)}N_f i\cdot\cdot\cdot} = 0, \quad i_1, i_2 = 1, \ldots, 2(2k+1)N_f. \]

In \(\Box\) the constraints for the special case \(k = 1, N_f = 1\) were explicitly determined in terms of the \(M_j\) and the superpotential that reproduces these constraints was constructed. In general the terms of the confining superpotential should consist of products of \(2(2k+1)N_f\) mesons \(M\). Then one can expect that the equations \(\partial W/\partial M_{ij} = 0\) give the classical constraints. Comparing this to (2.7) we obtain \(\alpha = 2k + 1\) and from (2.8) \(\gamma = -(N_c + 1), \beta = 2k(N_c + 1)\).

The confining superpotential consequently is of the form

\[
W = \frac{\sum_{(ji)} \prod_{i=1}^{2(k+1)N_f} M_{ji}}{h^{N_c+1} \Lambda^{(2(k+1)((4k+1)N_f-2))}}. \tag{3.25}
\]
with \( \sum_{l=1}^{(2k+1)N_f} j_l = 2k((2k+1)N_f - 1) \),

where the flavor indices are contracted with \((2k+1)\) epsilon tensors of rank \(2N_f\).

From the analysis of \([9]\) we know that this model has a stable vacuum if \((2k+1)N_f \geq N_c + 1\). Thus, the theory with \(2\tilde{N}_f = 2(N_f - 1)\) quarks and \(\tilde{N}_c = (2k+1)\tilde{N}_f + 2k - 1\), which is obtained by integrating out two quarks, does not possess a stable ground state.

In the limit \(k = 0\) the tree-level term \(h \text{Tr} X^{2(k+1)}\) gives mass to the adjoint tensor. Integrating it out we find an \(Sp(2\tilde{N}_c)\) gauge theory with \(N_c = N_f - 2\) and the correct \([19]\) confining superpotential \(W_{k=0} = \text{Pf} M_0/\Lambda_L^{2N_f - 3}\), where we used the scale matching relation \(h^N_{c} \Lambda^{2(N_c+1) - N_f} = \Lambda_L^{3(N_c+1) - N_f}\).

### 3.5 \(Sp(2N_c)\) with an antisymmetric tensor

Consider an \(Sp(2N_c)\) gauge theory with \(2N_f\) quarks \(Q\) in the fundamental representation and a second rank tensor \(X\) in the traceless (i.e. \(J_{ab}X^{ba} = 0\)) antisymmetric representation of the gauge group and tree-level superpotential \(W_{\text{tree}} = h \text{Tr} X^{k+1}\). This model was first studied in \([8]\). The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| \(Sp(2N_c)\) | \(SU(2N_f)\) | \(U(1)_R\) | \(Z_{(k+1)N_f}\) |
|----------------|----------------|-------------|----------------|
| \(Q\)          | \(\square\)    | \(\square\) | \(1 - \frac{2(N_c+k)}{(k+1)N_f} - (N_c - 1)\) |
| \(X\)          | \(\overline{\square}\) | \(1\)       | \(N_f\)         |

The (non-redundant) gauge invariant composite operators that can be built from the elementary fields are:

\[
M_j = QX^jQ, \quad j = 0, \ldots, k - 1, \\
T_i = \text{Tr} X^i, \quad i = 2, \ldots, k,
\]

(3.26)

where the gauge indices are contracted with the \(Sp(2N_c)\)-invariant \(J\)-tensor.

This theory is dual to an \(Sp(2\tilde{N}_c)\) gauge theory, with \(\tilde{N}_c = k(N_f - 2) - N_c\) and matter content \([8]\)

| \(Sp(2\tilde{N}_c)\) | \(SU(2N_f)\) | \(U(1)_R\) | \(Z_{(k+1)N_f}\) |
|----------------------|----------------|-------------|----------------|
| \(q\)                | \(\square\)    | \(\square\) | \(1 - \frac{2(N_c+k)}{(k+1)N_f} - N_c + N_f - 1\) |
| \(Y\)                | \(\overline{\square}\) | \(1\)       | \(N_f\)         |

and singlets \(M_j\) that carry the same quantum numbers as the mesons of the electric theory.

The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

\[
W_{\text{mag}} = -h \text{Tr} Y^{k+1} + \frac{h}{\mu^2} \sum_{j=0}^{k-1} M_{k-1-j} q Y^j q. 
\]

(3.27)
For $N_c = k(N_f - 2)$ the magnetic theory is completely higgsed and the electric theory confines with low-energy spectrum given by the composite fields

$$M_j, \quad j = 0, \ldots, k - 1, \quad T_k$$

of eqs. (3.28). It is easy to see that the 't Hooft anomaly matching conditions are satisfied. The contribution of $Y$ to the global anomalies is $(\hat{N}_c(2\hat{N}_c - 1) - 1)$ times its fermionic charge under the considered symmetry. For $\hat{N}_c = 0$ it acts therefore like one field with the negative of the charge of $Y$. Thus, we have to search for a composite field of the electric theory that carries fermionic charge of the same absolute value as $Y$ but of the opposite sign. This condition is satisfied by $T_k$. The only other contribution to the global anomalies in the magnetic theory for $\hat{N}_c = 0$ comes from the meson singlets which carry the same charges as the mesons of the electric theory. The fields $T_i, \quad i = 2, \ldots, k - 1$, do not correspond to classical flat directions. They are removed from the low-energy spectrum by mass terms $T_i T_{k+1-i}$.

Like for the model of the previous section none of the magnetic tree-level terms survives the symmetry breaking. In addition, the method of considering the dressed quarks as the only degrees of freedom to determine the classical constraints does not work for $SP(2N_c)$ with an antisymmetric tensor. For the theory without tree-level superpotential these constraints have been found in [20] for $N_f = 3$ and small values of $N_c$. As the effect of the non-vanishing tree-level superpotential is to remove the $T_2, \ldots, T_{k-1}$ from the moduli space, we expect to find the classical constraints with $W_{\text{tree}} \neq 0$ by setting to zero all the terms in the classical constraints of the model with $W_{\text{tree}} = 0$. All the confining superpotentials determined in [20] are generated by one-instanton effects. Therefore, it is likely that the confining superpotential of the theory with $W_{\text{tree}} \neq 0$ is also due to a one-instanton effect. Furthermore, one can flow from the $SU(N_c)$ gauge theory with an antisymmetric flavor to the model considered in this section by giving a large expectation value to the baryon $B$ of (3.4). The quarks should thus be raised to the same power in the confining superpotentials of both models. For the power $\alpha$ appearing in (2.7) this means $\alpha = 1$. From (2.8) we find $\gamma = 3 - N_f$ and $\beta = (k - 1)(N_f - 1)$. This leads us to the following non-perturbative superpotential which is invariant under all symmetries:

$$W = \frac{\sum_{\{j_l\}, p}(T_k)^p \prod_{l=1}^{N_f} M_{j_l}}{h^{N_f-3} A^{(2k-1)N_f-4(k-1)}},$$

with

$$\sum_{l=1}^{N_f} j_l = (k - 1)(N_f - 1) - pk,$$

where the flavor indices are contracted with an epsilon tensor.

Adding a mass term $mM_0$ for two quarks to the theory with $N_c = k(N_f - 2)$ and integrating out the massive modes we get a low-energy theory with $2\hat{N}_f = 2(N_f - 1)$ quarks and $\hat{N}_c = k(\hat{N}_f - 1)$. From the analysis of [8] we know that this model has a stable vacuum if $k\hat{N}_f > \hat{N}_c$. This condition is clearly satisfied. Indeed, we find a quantum moduli space with $k$ constraints amongst the confined degrees of freedom $\hat{M}_j, \hat{T}_k$ of the theory with $2\hat{N}_f$.
One of these constraints is modified quantum mechanically and reads

$$ \sum_{\{j_l\};p} (T_k)^p \prod_{l=1}^{\hat{N}_f} \hat{M}_{j_l} = h^{\hat{N}_f-2} \Lambda_L^{(2k-1)\hat{N}_f-2(k-2)}, $$

(3.30)

with $\sum_{l=1}^{\hat{N}_f} j_l = (k-1)\hat{N}_f - pk$.

The constraint (3.30) spontaneously breaks the chiral symmetry of the theory. We checked that the 't Hooft anomaly matchings are satisfied at the point of the moduli space where the $SU(2N_f)$ flavor symmetry is broken to its $Sp(2N_f)$ subgroup.

In the limit $k = 1$ the tree-level term $h \text{ Tr } X^{k+1}$ gives mass to the antisymmetric tensor. Integrating it out we find an $Sp(2N_c)$ gauge theory with $N_c = N_f - 2$ and the correct \[^{19}\] confining superpotential $W_{k=1} = \text{ Pf } M_0 / \Lambda_L^{2N_f-3}$, where we used the scale matching relation $h^{N_c-1} \Lambda^{3(N_c+1) - (N_c-1) - N_f} = \Lambda_L^{3(N_c+1) - N_f}$. The theory with two quarks less, i.e. $\hat{N}_c = \hat{N}_f - 1$, is known to confine with a quantum modified moduli space \[^{19}\]. The equation (3.30) reproduces the correct quantum constraint $\text{ Pf } \hat{M} = \hat{\Lambda}_L^{2\hat{N}_f}$, where $h^{\hat{N}_f-2} \Lambda_L^{\hat{N}_f+2} = \hat{\Lambda}_L^{2\hat{N}_f}$.

### 3.6 $SO(N_c)$ with an adjoint tensor

Consider an $SO(N_c)$ gauge theory with $N_f$ quarks $Q$ in the fundamental representation and a second rank tensor $X$ in the antisymmetric (=adjoint) representation of the gauge group and tree-level superpotential $W_{\text{tree}} = h \text{ Tr } X^{2(k+1)}$. This model was first studied in \[^{7}\]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| $SO(N_c)$ | $SU(N_f)$ | $U(1)_R$ | $Z_{2(k+1)N_f}$ | $Z_{2N_f}$ |
|----------|----------|----------|----------------|-----------|
| $Q$      | \(\square\) | \(\square\) | $1 - \frac{N_c-2}{(k+1)N_f}$ | $(N_c-2)$ | $1$ |
| $X$      | \(\square\) | $\sqrt{1}$ | $-\frac{1}{k+1}$ | $N_f$ | $0$ |

We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

- **Mesons** $M_j = Q X^j Q$, $j = 0, \ldots, 2k$,

- **Baryons** $B_n = X^n Q^{N_c-2n}$, $n = 0, \ldots, \left\lfloor \frac{N_c}{2} \right\rfloor$, $T_i = \text{ Tr } X^{2i}$, $i = 1, \ldots, k$,

where the gauge indices are contracted with a Kronecker delta for the mesons and $T_i$ and with an epsilon tensor for the baryons.
This theory is dual to an $SO(\tilde{N}_c)$ gauge theory, with $\tilde{N}_c = (2k + 1)N_f + 4 - N_c$ and matter content \[9\]

|          | $SO(\tilde{N}_c)$ | $SU(N_f)$ | $U(1)_R$ | $Z_{2(k+1)N_f}$ | $Z_{2N_f}$ |
|----------|-------------------|-----------|----------|-----------------|-----------|
| $q$      |                   |           |          |                 |           |
| $Y$      |                   |           |          |                 |           |

and singlets $M_j$ that carry the same quantum numbers as the mesons of the electric theory.

The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

$$W_{mag} = -h \, \text{Tr} \, Y^{2(k+1)} + \frac{h}{\mu^2} \sum_{j=0}^{2k} M_{2k-j} q Y^j q. \quad (3.32)$$

Under duality the electric baryons of (3.31) are mapped to the magnetic baryons $\tilde{B}_m = Y^m q^{N_c - 2m}$ according to

$$B_n \leftrightarrow \tilde{B}_m, \quad \text{with} \ m = kN_f + 2 - n. \quad (3.33)$$

For $N_c = (2k+1)N_f + 3$ the magnetic theory is completely higgsed and the electric theory confines \[5\] with low-energy spectrum given by the composite fields

$$M_j, \quad j = 0, \ldots, 2k,$$

$$B \equiv B_{kN_f+2}$$

of eqs. (3.31). The baryons $B$ are of the form $B = X^{kN_f+2} Q^{N_f-1}$ and transform in the antifundamental representation of $SU(N_f)$. Furthermore from (3.33) we find that they are mapped to the $N_f$ magnetic quark singlets $q$ that stay massless after breaking the magnetic gauge group. It is easy to see that the ’t Hooft anomaly matching conditions are satisfied because the only fields that contribute to the global anomalies in the magnetic theory for $N_c = 1$ are the $N_f$ quark singlets and the meson singlets. They carry the same charges as the baryons and mesons of the electric theory and consequently the global anomalies match between the macroscopic and the microscopic description. As in the models of the previous sections one could add the fields $T_i$ without violating the matching conditions. But they are removed from the low-energy spectrum by the mass terms $T_i T_{k+1-i}$.

The effective low-energy superpotential of the magnetic theory, deduced from (3.32) for the theory with $\tilde{N}_c = 2(k+1)$ and $(N_f + 1)$ quarks by adding a tree-level term $mM_0$ and integrating out the massive modes, contains a term $M_{2k} q q$. We thus expect that the confining superpotential of the electric theory has a term proportional to

$$BM_{2k} B \sim Q^{2N_f} X^{2k(N_f+1)+4}. \quad (3.35)$$

Comparing this to (2.7) we find $\alpha = 1$. From (2.8) we then get $\gamma = 1 - N_f$ and $\beta = 2k(N_f + 1) + 4$. The confining superpotential consequently is of the form

$$W = \frac{BM_{2k} B}{h^{N_f-1} \Lambda^{(4k+1)N_f+2}}. \quad (3.36)$$
Instanton corrections may modify this superpotential by multiplying it with an holomorphic function of the expression
\[
\frac{\left(\prod_{i=1}^{N_f} M_{j_i}\right) B M_{2k} B}{h^{2N_f} \Lambda^{2(4k+1)N_f+4}}, \quad \text{where} \quad \sum_{i=1}^{N_f} j_i = 2k(N_f - 1). \tag{3.37}
\]
The equations of motion of this superpotential give \(M_{2k} B = 0\). This is indeed a classical constraint because from the equations of motion of the tree-level superpotential it follows that \(X^{2k+1} = 0\). Thus, \(X\) can at most have rank \(2k\). As a consequence a totally antisymmetrized product of more than \(2k\) factors of matrix elements of \(X\) must vanish. This means that \(\langle B \rangle = 0\) classically. (For \(N_f = 1\) it vanishes because in this case \(B = \text{Pf } X = \sqrt{\det X} = 0\).)

From the analysis of \([9]\) we know that there is only a stable vacuum if \((2k+1)N_f \geq N_c - 4\). Thus, for the theory with \(\hat{N}_f = N_f - 1\) quarks and \(\hat{N}_c = (2k+1)\hat{N}_f + 2k + 4\) one expects to find a superpotential of the Affleck-Dine-Seiberg type which destabilizes the ground state. We did not check this but suppose that it can be derived by integrating out one quark from (3.36) when the corrections (3.37) are taken into account.

In the limit \(k = 0\) the coupling parameter \(h\) represents a mass for the tensor \(X\). Integrating it out and using the scale matching relation \(h^{N_c - 2} \Lambda^{2(N_c - 2) - N_f} = \Lambda_L^{2(N_c - 2) - N_f}\), we find an \(SO(N_c)\) gauge theory with \(N_c = N_f + 3\) and the correct \([17]\) confining superpotential after replacing the operator \(X^2\) in \(B\) by \((W_0)^2/h\); i.e. setting \(B' \equiv (W_0)^2 Q^{N_f - 1} = hB\), as explained in \([8]\), \(W_{k=0} = B'M_0 B'/\Lambda_L^{2N_f + 3}\).

### 3.7 \(SO(N_c)\) with a symmetric tensor

Consider an \(SO(N_c)\) gauge theory with \(N_f\) quarks \(Q\) in the fundamental representation and a second rank tensor \(X\) in the traceless symmetric representation of the gauge group and tree-level superpotential \(W_{\text{tree}} = h \text{Tr } X^{k+1}\). This model was first studied in \([8]\). The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| \(SO(N_c)\) | \(SU(N_f)\) | \(U(1)_R\) | \(Z_{(k+1)N_f}\) | \(Z_{2N_f}\) |
|----------------|----------------|-------------|-----------------|----------------|
| \(Q\)         | \(\square\)    | \(\square\) | \(1 - \frac{2(N_c - 2k)}{(k+1)N_f}\) | \(-(N_c + 2)\) |
| \(X\)         | \(\Box\)       |             | \(\frac{k+1}{k+1}\) | \(N_f\)         | \(0\)  |

We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

**Mesons** \(M_j = Q Q_{(j)}\), with \(Q_{(j)} = X^j Q\), \(j = 0, \ldots, k - 1\),

**Baryons** \(B_p^{(n_0, \ldots, n_{k-1})} = (W_{a})^2 (X W_{a})^2 \cdots (X^{p-1} W_{a})^2 Q_{n_0} Q_{(1)}^{n_1} \cdots Q_{(k-1)}^{n_{k-1}}\),

\[
\text{with } \sum_{j=0}^{k-1} n_j = N_c - 4p, \quad p = 0, \ldots, \min(k, \left\lceil \frac{N_c}{4} \right\rceil),
\]

\[
T_i = \text{Tr } X^i, \quad i = 2, \ldots, k - 1,
\]

25
where the gauge indices are contracted with a Kronecker delta for the mesons and \( T_i \) and with an epsilon tensor for the baryons.

This theory is dual to an \( SO(\tilde{N}_c) \) gauge theory, with \( \tilde{N}_c = k(N_f + 4) - N_c \) and matter content \[8\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{SO(\tilde{N}_c)} & \text{SU}(N_f) & U(1)_R & Z_{(k+1)N_f} & Z_{2N_f} \\
\hline
q & Y & 1 - \frac{2(\tilde{N}_c - 2k)}{(k+1)N_f} & N_c + N_f + 2 & -1 \\
\hline
Y & 1 & \frac{2k+1}{k+1} & N_f & 0 \\
\hline
\end{array}
\]

and singlets \( M_j \) that carry the same quantum numbers as the mesons of the electric theory.

The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

\[
W_{\text{mag}} = -h \, \text{Tr} \, Y^{k+1} + \frac{h}{\mu^2} \sum_{j=0}^{k-1} M_{k-1-j} \bar{q} Y^j q.
\] (3.39)

The electric baryons of (3.38) can be consistently mapped to similar baryons of the magnetic theory:

\[
\mathcal{B}_p(n_0, \ldots, n_{k-1}) \leftrightarrow \bar{B}_q^{(m_0, \ldots, m_{k-1})}, \quad \text{with } q = k - p, \ m_j = N_f - n_{k-1-j},
\] (3.40)

where the magnetic baryons \( \bar{B}_q^{(\cdot \cdot \cdot)} \) are defined in the same way as the electric baryons of (3.38) replacing all fields by their dual partners and \( N_c \) by \( \tilde{N}_c \). (This mapping is similar to the one that has been found in \[8\].)

For \( N_c = k(N_f + 4) - 1 \) the magnetic theory is completely higgsed and the electric theory confines \[8\] with low-energy spectrum given by the composite fields

\[
M_j, \quad j = 0, \ldots, k-1,
\]

\[
\mathcal{B} \equiv B_k^{(N_f, \ldots, N_f, N_f - 1)}
\] (3.41)

of eqs. (3.38). The baryons \( B \) transform in the antifundamental representation of \( SU(N_f) \). From (3.40) we find that they are mapped to the \( N_f \) magnetic quark singlets \( q \) that stay massless after breaking the magnetic gauge group. It is easy to see that the ’t Hooft anomaly matching conditions are satisfied because the only fields that contribute to the global anomalies in the magnetic theory for \( \tilde{N}_c = 1 \) are the \( N_f \) quark singlets and the meson singlets which carry the same charges as the baryons and mesons of the electric theory. As in the models of the previous sections one could add the fields \( T_i \) without violating the matching conditions. But they are removed from the low-energy spectrum by the mass terms \( T_i T_{k+1-i} \).

The effective low-energy superpotential of the magnetic theory, deduced from (3.39) for the theory with \( \tilde{N}_c = k + 1 \) and \( (N_f + 1) \) quarks by adding a tree-level term \( mM_0 \) and integrating out the massive modes, contains a term \( M_{k-1} \bar{q} q \). We thus expect that the confining superpotential of the electric theory has a term proportional to

\[
BM_{k-1}B \sim (W_\alpha)^{4k} Q^{2kN_f} X^{(k-1)(N_c-2k)}.
\] (3.42)
Comparing this to \( [2.7] \) we find \( \alpha = k \). From \( [2.8, 2.11] \) we then get \( \gamma = -(N_c + 2k) \) and \( \beta = (k - 1)(N_c - 2k) \). The confining superpotential consequently is of the form

\[
W = \frac{BM_{k-1}B}{h^{N_c+2k} \Lambda^{k((2k-1)N_f+8k-10)}}.
\] (3.43)

The equations of motion of this superpotential give \( M_{k-1}B = 0 \). This is the correct result in the classical limit because the field strength tensor \( W_\alpha \) vanishes classically.

From the analysis of \([8]\) we know that there is only a stable vacuum if \( kN_f \geq N_c - 4k \). Thus, for the theory with \( \hat{N}_f = N_f - 1 \) quarks and \( \hat{N}_c = k(\hat{N}_f + 5) - 1 \) one expects to find a superpotential of the Affleck-Dine-Seiberg type which destabilizes the ground state. Possibly this can be obtained by integrating out one quark from \((3.43)\) when instanton corrections are taken into account.

In the limit \( k = 1 \) the coupling parameter \( h \) represents a mass for the tensor \( X \). Integrating out and using the scale matching relation \( h^{N_c+2}\Lambda^{3(N_c-2)-(N_c+2)-N_f} = \Lambda^{3(N_c-2)-N_f} \), we find an \( SO(N_c) \) gauge theory with \( N_c = N_f + 3 \) and the correct \([17]\) confining superpotential \( W_{k=1} = BM_0B/\Lambda_L^{2N_f+3} \).

## 4 Models with \( D_{k+2} \)-type superpotentials

### 4.1 \( SU(N_c) \) with two adjoint tensors

Consider SQCD with two additional second rank tensors \( X, Y \), both transforming in the adjoint representation, and tree-level superpotential \( W_{\text{tree}} = h_1 \text{Tr} X^{k+1} + h_2 \text{Tr} XY^2 \). This model was first studied in \([32]\). The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| \( SU(N_c) \) | \( SU(N_f)_L \) | \( SU(N_f)_R \) | \( U(1)_B \) | \( U(1)_R \) | \( Z_{2(k+1)N_f} \) | \( Z_{2N_f} \) |
|--------------|----------------|----------------|-------------|-------------|-----------------|-------------|
| \( Q \)      | \( \Box \)      | \( \Box \)      | \( 1 \)     | \( \frac{1}{N_c} \) | \( 1 - \frac{N_c}{(k+1)N_f} \) | \( -N_c \) |
| \( \bar{Q} \)| \( \Box \)      | \( 1 \)         | \( -\frac{1}{N_c} \) | \( 1 - \frac{N_c}{(k+1)N_f} \) | \( -N_c \) |
| \( X \)      | \( \text{adj} \) | \( 1 \)         | \( 0 \)     | \( \frac{2k+1}{k+1} \) | \( 2N_f \) |
| \( Y \)      | \( \text{adj} \) | \( 1 \)         | \( 0 \)     | \( \frac{k}{k+1} \) | \( -N_f \) |

We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

\[
\text{mesons} \quad M_{(j,l)} \equiv QQ_{(j,l)}, \quad \text{with } Q_{(j,l)} = X^j Y^l Q, \quad j = 0, \ldots, k - 1, \quad l = 0, 1, 2,
\]

\[
\text{baryons} \quad B_{(n(0,0), \ldots, n(k-1,2))} \equiv Q_{(0,0)}^{n(0,0)} Q_{(1,0)}^{n(1,0)} \cdots Q_{(k-1,2)}^{n(k-1,2)},
\]

\[
\bar{B}_{(n(0,0), \ldots, n(k-1,2))} \equiv \bar{Q}_{(0,0)}^{n(0,0)} \bar{Q}_{(1,0)}^{n(1,0)} \cdots \bar{Q}_{(k-1,2)}^{n(k-1,2)},
\] (4.1)

with \( \sum_{l=0}^{2k-1} n_{(j,l)} = N_c \),

\[27\]
where the gauge indices are contracted with an epsilon tensor for the baryons.

For odd values of $k$ this theory is dual to an $SU(\tilde{N}_c)$ gauge theory, with $\tilde{N}_c = 3kN_f - N_c$ and matter content \[ SU(\tilde{N}_c) \times SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R \times Z_{2(k+1)N_f} \times Z_{2N_f} \]

| $SU(\tilde{N}_c)$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_B$ | $U(1)_R$ | $Z_{2(k+1)N_f}$ | $Z_{2N_f}$ |
|-------------------|-------------|-------------|----------|----------|-----------------|----------|
| $q$               | $\Box$      | $\Box$      | $1$      | $\frac{1}{N_c}$ | $1 - \frac{\tilde{N}_c}{(k+1)N_f}$ | $-\tilde{N}_c$ | $-\tilde{N}_c$ |
| $\bar{q}$         | $\Box$      | $\Box$      | $1$      | $-\frac{1}{N_c}$ | $1 - \frac{\tilde{N}_c}{(k+1)N_f}$ | $-\tilde{N}_c$ | $-\tilde{N}_c$ |
| $\tilde{X}$       | adj         | $1$         | $1$      | $0$      | $\frac{2}{k+1}$ | $2N_f$ | $0$ |
| $\tilde{Y}$       | adj         | $1$         | $1$      | $0$      | $\frac{2}{k+1}$ | $-N_f$ | $N_f$ |

and singlets $M_{(j,l)}$ that carry the same quantum numbers as the mesons of the electric theory.

The magnetic theory contains a tree-level superpotential

$$W_{\text{mag}} = \text{Tr} \tilde{X}^k + \text{Tr} \tilde{X} \tilde{Y}^2 + \sum_{l=0}^{\frac{k-1}{2}} \sum_{j=0}^{\frac{k-1}{2}} M_{(k-1-j,2-l)} q(l,j),$$

where magnetic dressed quarks have been introduced by $q(l,j) = \tilde{X}^j \tilde{Y}^l q$ and we have omitted the dependence of $W_{\text{mag}}$ on $h_1$, $h_2$ and $\mu$.

The electric baryons of (4.1) can be consistently mapped to similar baryons of the magnetic theory \[ B^{(n_{(0,0)},...,n_{(k-1,2)})} \leftrightarrow \tilde{B}^{(m_{(0,0)},...,m_{(k-1,2)})}, \text{ with } m_{(j,l)} = N_f - n_{(k-1-j,2-l)}, \]

where the magnetic baryons $\tilde{B}^{(\cdots)}$ are defined in the same way as the electric baryons of (4.1) replacing all fields by their dual partners and $N_c$ by $\tilde{N}_c$.

For $N_c = 3kN_f - 1$ the magnetic theory is completely higgsed and the electric theory confines with low-energy spectrum given by the composite fields

$$M_{(j,l)}, \quad j = 0, \ldots, k - 1, \quad l = 0, 1, 2,$$

$$B \equiv B^{(N_f,...,N_f,N_f-1)}, \quad \tilde{B} \equiv \tilde{B}^{(N_f,...,N_f,N_f-1)}$$

of eqs. (4.1). From (4.3) we find that the baryons are mapped to the $2N_f$ magnetic quark singlets $q$, $\bar{q}$ that stay massless after breaking the magnetic gauge group. The ’t Hooft anomaly matching conditions are satisfied because the only fields that contribute to the global anomalies in the magnetic theory for $\tilde{N}_c = 1$ are the $2N_f$ quark singlets and the meson singlets which carry the same charges as the baryons and mesons of the electric theory.

The effective low-energy superpotential of the magnetic theory, deduced from (4.2) for the theory with $\tilde{N}_c = 3k + 1$ and $(N_f + 1)$ quark flavors by adding a tree-level term $mM_{(0,0)}$ and integrating out the massive modes, contains a term $M_{(k-1,2)} q(l,j)$. We thus expect that the confining superpotential of the electric theory has a term proportional to

$$\tilde{B} M_{k-1} B \sim (Q\bar{Q})^{3kN_f} X^{(k-1)N_c} Y^{2N_c}. \quad (4.5)$$
Comparing this to (2.7) we find $\alpha = 3k$. From (2.10) we then get $\gamma_1 = -2N_c$ and $\gamma_2 = -(3k - 1)N_c$. The confining superpotential consequently is of the form

$$W = \frac{BM_{(k-1,2)}B + \sum_{\{j_m,l_m\}}(\det M_{(k-1,2)})^{3k-p} \prod_{c=1}^p (M_{(j_m,l_m)} \text{ cof } M_{(k-1,2)})}{h_1^{2Nc} h_2^{(3k-1)N_c} \Lambda^{3k((3k-1)N_f-1)}}$$

(4.6)

with $\sum_{m=1}^p j_m = (k-1)(p-1)$, $\sum_{m=1}^p l_m = 2(p-1)$,

where the cofactor of a matrix $A$ is defined by $(\text{cof } A)^{ij} = \partial \det A / \partial A_{ij}$. To determine the classical constraints we view the $3k$ dressed quarks $Q = (Q, Q_{(1,0)}, \ldots, Q_{(k-1,2)})$ as independent degrees of freedom [5]. We restrict ourselves to the special case where the matrices $X$ and $Y$ commute (in general the constraints are more complicated). The mesons $M = Q \bar{Q}$ and the baryons $B = Q^{N_c}$, $\bar{B} = \bar{Q}^{N_c}$ should then satisfy the constraints that are known for SQCD with one more flavor than colors:

$$MB = \bar{B}M = 0 \quad \text{and} \quad (\text{cof } M)^{ij} = B^i \bar{B}^j.$$  (4.7)

The first condition reduces to $M_{(k-1,2)}B = B M_{(k-1,2)} = 0$ because the baryons $B, \bar{B}$ are given by [4] $B = (0, \ldots, B), \bar{B} = (0, \ldots, \bar{B})$. These constraints are correctly reproduced by the equations of motion of the superpotential (4.6).

In the limit $k = 1$ the coupling parameter $h_1$ represents a mass for the tensor $X$. Integrating out $X$ results in a theory with tree-level superpotential $W_{\text{tree}} = h_1 \text{ Tr } Y^4$, where $h = (h_2)^2 / h_1$, as explained in [32]. This is the model of Kutasov and Schwimmer [4] (reviewed in section 2 of the present paper) for the special case $k = 3$. For $N_c = 3N_f - 1$ it confines [13, 5]. The confining superpotential (4.6) has the correct form $W = BM_{(0,2)}B + \ldots / h_1^{N_c} \Lambda^{3(5N_f-2)}$ in this limit, where we used the scale matching relation $h_1^{N_c} \Lambda^{N_c-N_f} = \Lambda_L^{2N_c-N_f}$.

### 4.2 $SU(N_c)$ with an adjoint tensor, an antisymmetric tensor and its conjugate

Consider an $SU(N_c)$ gauge theory with $N_f$ fundamental flavors $Q, \bar{Q}$, an adjoint tensor $X$, an antisymmetric tensor $Y$ and its conjugate $\bar{Y}$ and tree-level superpotential $W_{\text{tree}} = h_1 \text{ Tr } X^{k+1} + h_2 \text{ Tr } XY \bar{Y}$. This model was first studied in [13]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| $SU(N_c)$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_Y$ | $U(1)_B$ | $U(1)_R$ | $Z_{2(k+1)N_f}$ | $Z_{2N_f}$ |
|-----------|-------------|-------------|--------|--------|--------|----------------|------------|
| $Q$       | $\Box$      | $\Box$      | $1$    | $0$    | $\frac{1}{N_c}$ | $1 - \frac{N_c+2}{(k+1)N_f}$ | $-(N_c+2)$ | $-(N_c-2)$ |
| $\bar{Q}$ | $\Box$      | $1$         | $\Box$ | $0$    | $-\frac{1}{N_c}$ | $1 - \frac{N_c+2}{(k+1)N_f}$ | $-(N_c+2)$ | $-(N_c-2)$ |
| $X$       | $\text{adj}$ | $1$         | $1$    | $0$    | $0$    | $\frac{2}{k+1}$ | $2N_f$ | $0$ |
| $Y$       | $\Box$      | $1$         | $1$    | $1$ | $\frac{2}{N_c}$ | $\frac{k+1}{k}N_f$ | $-N_f$ | $N_f$ |
| $\bar{Y}$ | $\Box$      | $1$         | $1$    | $-1$ | $-\frac{2}{N_c}$ | $\frac{k}{k+1}N_f$ | $-N_f$ | $N_f$ |
We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

\[
\begin{align*}
\text{mesons} & \quad M_j = Q Q(j), \quad N_j = Q Y Y Q(j), \quad P_j = Q Y Q(j), \quad P_j = Q Y Q(j), \\
\text{baryons} & \quad B^{(n_0, \ldots, n_{k-1})}_n = Y^n Q^{n_0} Q^{n_1} \cdots Q^{n_{k-1}}(1), \\
& \quad \bar{B}^{(n_0, \ldots, n_{k-1})}_n = \bar{Y}^n \bar{Q}^{n_0} \bar{Q}^{n_1} \cdots \bar{Q}^{n_{k-1}}(1),
\end{align*}
\]

(4.8)

with \( Q(j) = X^j Q \), \( \bar{Q}(j) = X^j \bar{Q} \), \( j = 0, \ldots, k - 1 \),

where the gauge indices of the baryons are contracted with an epsilon tensor.

For odd values of \( k \) this theory is dual to an \( SU(\tilde{N}_c) \) gauge theory, with \( \tilde{N}_c = 3kN_f - 4 - N_c \) and matter content \( [14] \)

| \( SU(\tilde{N}_c) \) | \( SU(N_f)_L \) | \( SU(N_f)_R \) | \( U(1)_Y \) | \( U(1)_B \) | \( U(1)_R \) | \( Z_{2(k+1)N_f} \) | \( Z_{2N_f} \) |
|-------------------|----------|----------|----------|----------|----------|----------------|----------------|
| \( q \)            | 0        | 1        | \(-\frac{N_c}{N_e} \) | \(-\frac{1}{N_e} \) | 1 - \(-\frac{N_c}{N_e} \) | 0              | 0              |
| \( \bar{q} \)      | 0        | 1        | \(-\frac{N_c}{N_e} \) | \(-\frac{1}{N_e} \) | 1 - \(-\frac{N_c}{N_e} \) | 0              | 0              |
| \( \tilde{X} \)     | 1        | 1        | \( kN_f - N_c \) | \(-\frac{1}{N_e} \) | 1 - \(-\frac{N_c}{N_e} \) | \( \frac{k}{k+1} \) | \( 2N_f \) |
| \( \tilde{Y} \)     | 1        | 1        | \( \frac{k}{k+1} \) | \( \frac{1}{N_e} \) | 1 - \(-\frac{N_c}{N_e} \) | \( -N_f \)    | \( N_f \) |
| \( \bar{\tilde{Y}} \) | 1        | 1        | \(-\frac{N_c}{N_e} \) | \(-\frac{1}{N_e} \) | 1 - \(-\frac{N_c}{N_e} \) | \( \frac{k}{k+1} \) | \( -N_f \) |

and singlets \( M_j, N_j, P_j, \bar{P}_j \) that carry the same quantum numbers as the mesons of the electric theory.

The magnetic theory contains a tree-level superpotential

\[
W_{\text{mag}} = \text{Tr} \tilde{X}^{k+1} + \text{Tr} \tilde{X} \tilde{Y} \tilde{Y} + \sum_{j=0}^{k-1} \left[ M_{k-1-j} q \tilde{Y} \tilde{Y} \tilde{q}(j) + N_{k-1-j} \bar{q} \bar{q}(j) + P_{k-1-j} \bar{q} \bar{q}(j) + \tilde{P}_{k-1-j} \bar{q} \bar{q}(j) \right],
\]

(4.9)

where magnetic dressed quarks have been introduced by \( q(j) = \tilde{X}^j q \).

The authors of \([14]\) found a mapping of the baryons \( B^{(\cdot \cdot \cdot)}_n \) of (4.8) to the magnetic baryons \( \bar{B}^{(\cdot \cdot \cdot)}_m \) consistent with all global symmetries:

\[
B^{(n_0, \ldots, n_{k-1})}_n \leftrightarrow \bar{B}^{(n_0, \ldots, n_{k-1})}_m, \quad \text{with} \quad m = kN_f - 2 - n, \quad m_j = N_f - n_{k-1-j}. \quad (4.10)
\]

For \( N_c = 3kN_f - 5 \) the magnetic theory is completely higgsed and the electric theory confines with low-energy spectrum given by the composite fields

\[
M_j, \quad N_j, \quad P_j, \quad \bar{P}_j, \quad j = 0, \ldots, k - 1, \quad B \equiv B^{(N_f, \ldots, N_f, N_f-1)}_{kN_f-2}, \quad \bar{B} \equiv \bar{B}^{(N_f, \ldots, N_f, N_f-1)}_{kN_f-2} \quad (4.11)
\]
of eqs. (1.8). From (4.10) we find that the baryons are mapped to the \(2N_f\) magnetic quark singlets \(q, \bar{q}\) that stay massless after breaking the magnetic gauge group. The ’t Hooft anomaly matching conditions are satisfied because the only fields that contribute to the global anomalies in the magnetic theory for \(\tilde{N}_c = 1\) are the \(2N_f\) quark singlets and the meson singlets which carry the same charges as the baryons and mesons of the electric theory.

The effective low-energy superpotential of the magnetic theory, deduced from (4.9) for the theory with \(\tilde{N}_c = 3k + 1\) and \((N_f + 1)\) quark flavors by adding a tree-level term \(mM_0\) and integrating out the massive modes, contains a term \(N_{k-1}q\bar{q}\). We thus expect that the confining superpotential of the electric theory has a term proportional to

\[ \bar{B}N_{k-1}B \sim (Q\bar{Q})^{kN_f}X^{(k-1)(kN_f-1)}(Y\bar{Y})^{kN_f-1}. \tag{4.12} \]

Comparing this to (2.7) we find \(\alpha = k\). From (2.10) we then get \(\gamma_1 = -2(kN_f - 1)\) and \(\gamma_2 = -k(3k - 1)N_f + 7k - 1\). The confining superpotential consequently is of the form

\[ W = \bar{B}N_{k-1}B \frac{h_1^{2(kN_f-1)}h_2^{k(3k-1)N_f-7k+1}}{\Lambda^{k((3k-1)N_f-3)}} + \ldots, \tag{4.13} \]

where the dots stand for products of \(M_j, N_j\) and \((P_jP_j)\) that are invariant under the symmetries. These terms may contain different powers of the fields \(X\) and \(Y\).

In the limit \(k = 1\) the coupling parameter \(h_1\) represents a mass for the tensor \(X\). Integrating out \(X\) results in a theory with tree-level superpotential \(W_{\text{tree}} = h \, \text{Tr}(Y\bar{Y})^2\), where \(h = (h_2)^2/h_1\). This is the model discussed in section 3.1 for the special case \(k = 1\). For \(N_c = 3N_f - 5\) it confines \(\mathbb{C}\). The confining superpotential (4.13) has the correct form \(W = (\bar{B}N_0B + \ldots)/h^{N_f-3}N_{\Lambda_L}^{N_f-8}\) in this limit, where we used the scale matching relation \(h_1^{N_c}N_{\Lambda_c+2-N_f} = \Lambda_L^{2N_c+2-N_f}\).

### 4.3 \(SU(N_c)\) with an adjoint tensor, a symmetric tensor and its conjugate

Consider an \(SU(N_c)\) gauge theory with \(N_f\) fundamental flavors \(Q, \bar{Q}\), an adjoint tensor \(X\), a symmetric tensor \(Y\) and its conjugate \(\bar{Y}\) and tree-level superpotential \(W_{\text{tree}} = h_1 \, \text{Tr}X^{k+1} + h_2 \, \text{Tr}XY\bar{Y}\). This model was first studied in \([14]\). The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| \(SU(N_c)\) | \(SU(N_f)\)\(_L\) | \(SU(N_f)\)\(_R\) | \(U(1)_Y\) | \(U(1)_B\) | \(U(1)_R\) | \(Z_{2(k+1)N_f}\) | \(Z_{2N_f}\) |
|-------------|-----------------|-----------------|-------------|-------------|-------------|-----------------|-------------|
| \(Q\)       | \(\Box\)        | \(\Box\)        | \(1\)       | \(0\)       | \(1\)       | \(\frac{1}{N_c}\) | \(-\frac{N_c-2}{(k+1)N_f}\) |
| \(\bar{Q}\)  | \(\Box\)        | \(\Box\)        | \(0\)       | \(-\frac{1}{N_c}\) | \(1\)       | \(\frac{1}{(k+1)N_f}\) | \(-\frac{N_c-2}{N_c+2}\) |
| \(X\)       | \(\text{adj}\)  | \(1\)           | \(1\)       | \(0\)       | \(0\)       | \(\frac{2k+1}{k}\) | \(2N_f\)     |
| \(Y\)       | \(\Box\)        | \(\Box\)        | \(1\)       | \(1\)       | \(\frac{2}{N_c}\) | \(\frac{k+4}{k}\) | \(-N_f\)     |
| \(\bar{Y}\) | \(\Box\)        | \(\Box\)        | \(1\)       | \(-1\)      | \(-\frac{2}{N_c}\) | \(\frac{k+4}{k+2}\) | \(-N_f\)     |
We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

**Mesons** \( M_j = Q \bar{Q}(j), \quad N_j = Q \bar{Y} Y \bar{Q}(j), \quad P_j = Q \bar{Y} Q(j), \quad \bar{P}_j = \bar{Q} Y \bar{Q}(j), \)

with \( Q(j) = X^j Q, \quad \bar{Q}(j) = X^j \bar{Q}, \quad j = 0, \ldots, k - 1, \)

**Baryons** \( \mathcal{B}^{(n_0, \ldots, n_{k-1}, \bar{n}_0, \ldots, \bar{n}_{k-1})} = (Y X^{k-1} W_a)^2 Q^{a_0} Q^{n_1} \cdots Q^{n_{k-1}}, \)

\( \cdot (Y \bar{Q})^{(n_0)} (Y \bar{Q}(1))^{n_1} \cdots (Y \bar{Q}(k-1))^{n_{k-1}} (Y \bar{Y} Q) \bar{n}_0 (Y \bar{Y} Q(1)) \bar{n}_1 \cdots (Y \bar{Y} Q(k-1)) \bar{n}_{k-1}, \)

\( \mathcal{B}^{(n_0, \ldots, n_{k-1}, \bar{n}_0, \ldots, \bar{n}_{k-1})} = (Y X^{k-1} W_a)^2 \bar{Q}^{n_0} \bar{Q}^{n_1} \cdots \bar{Q}^{n_{k-1}}, \)

\( \cdot (Y Q) \bar{n}_0 (Y Q(1)) \bar{n}_1 \cdots (Y Q(k-1)) \bar{n}_{k-1} (Y Y \bar{Q}) (Y Y \bar{Q}(1)) \bar{n}_1 \cdots (Y Y \bar{Q}(k-1)) \bar{n}_{k-1}, \)

with \( \sum_{j=0}^{k-1} (n_j + \bar{n}_j + \bar{n}_j) = N_c - 4, \)

\( B_n^{(n_0, \ldots, n_{k-1})} = Y^n \bar{Q}^{n_0} Q^{n_1} \bar{Q}(1) \cdots Q^{n_{k-1}} \bar{Q}(k-1), \)

\( \bar{B}_n^{(n_0, \ldots, n_{k-1})} = \bar{Y}^n \bar{Q}^{n_0} \bar{Q}^{n_1} \bar{Q}(1) \cdots \bar{Q}^{n_{k-1}} \bar{Q}(k-1), \)

with \( \sum_{j=0}^{k-1} n_j = N_c - n, \)

where the gauge indices of the baryons are contracted with one epsilon tensor for the \( \mathcal{B}^{(\cdots)}, \)

\( \bar{B}^{(\cdots)} \) and with two epsilon tensors for the \( B_n^{(\cdots)}, \bar{B}_n^{(\cdots)} \).

For odd values of \( k \) this theory is dual to an \( SU(\bar{N}_c) \) gauge theory, with \( \bar{N}_c = 3kN_f + 4 - N_c \) and matter content \( [3] \).

|       | SU(\(\bar{N}_c\)) | SU(\(N_f\)) | SU(\(N_f\))_R | U(1)_Y | U(1)_B | U(1)_R | Z_{2(k+1)N_f} | Z_{2N_f} |
|-------|-----------------|-------------|----------------|--------|--------|--------|----------------|---------|
| \( q \) | \( \square \)   | \( \square \) | 1              | \( \frac{kN_f + 2}{N_c} \) | \( \frac{1}{N_c} \) | 1 - \( \frac{N_c - 2}{(k+1)N_f} \) | \( \bar{N}_c - 2 \) | \( \bar{N}_c - 6 \) |
| \( \bar{q} \) | \( \square \) | 1          | \( \square \) | \( \frac{kN_f + 2}{N_c} \) | \( \frac{1}{N_c} \) | 1 - \( \frac{N_c - 2}{(k+1)N_f} \) | \( \bar{N}_c - 2 \) | \( \bar{N}_c - 6 \) |
| \( \bar{X} \) | adj | 1 | 1 | 0 | 0 | \( \frac{2}{k+1} \) | \( 2N_f \) | 0 |
| \( \bar{Y} \) | \( \square \) | 1 | 1 | \( \frac{N_c - kN_f}{N_c} \) | \( \frac{2}{N_c} \) | \( \frac{k}{k+1} \) | \( -N_f \) | \( N_f \) |
| \( \tilde{Y} \) | \( \square \) | 1 | 1 | \( \frac{N_c - kN_f}{N_c} \) | \( \frac{2}{N_c} \) | \( \frac{k}{k+1} \) | \( -N_f \) | \( N_f \) |

and singlets \( M_j, N_j, P_j, \bar{P}_j \) that carry the same quantum numbers as the mesons of the electric theory.

The magnetic theory contains a tree-level superpotential

\[
W_{mag} = \text{Tr} \bar{X}^{k+1} + \text{Tr} \bar{Y} \bar{Y} \bar{Y} + \sum_{j=0}^{k-1} \left[ M_{k-1-j} q Y Y \bar{q}(j) + N_{k-1-j} q \bar{q}(j) + P_{k-1-j} \bar{q} Y q(j) + \bar{P}_{k-1-j} \bar{q} \bar{Y} \bar{q}(j) \right],
\]

where \( q(j) = \bar{X}^j q. \)
The confining superpotential consequently is of the form
\begin{equation}
\gamma_{\alpha} \alpha
\end{equation}
and integrating out the massive modes, contains the terms to each of the fields \(Y\) component of \(B\), \(\bar{Y}\) component of eqs. (4.14). From (4.16) we find the mappings
\begin{equation}
\tilde{m}_j = N_f - \tilde{n}_{k-1-j}, \quad \bar{m}_j = N_f - \bar{n}_{k-1-j}, \quad \bar{m}_j = N_f - n_{k-1-j},
\end{equation}
\begin{equation}
m = 2kN_f + 4 - n, \quad m_j = N_f - n_{k-1-j}.
\end{equation}
The last of these mappings has been found in [14].

For \(N_c = 3kN_f + 3\) the magnetic theory is completely higgsed and the electric theory confines with low-energy spectrum given by the composite fields
\begin{equation}
M_j, \quad N_j, \quad P_j, \quad \bar{P}_j, \quad j = 0, \ldots, k - 1,
\end{equation}
\begin{equation}
B \equiv B^{(N_f, \ldots, N_f, N_f-1)}, \quad \bar{B} \equiv \bar{B}^{(N_f, \ldots, N_f, N_f-1)},
\end{equation}
\begin{equation}
b \equiv B^{(N_f, \ldots, N_f)}_{2kN_f+3}, \quad \bar{b} \equiv \bar{B}^{(N_f, \ldots, N_f)}_{2kN_f+3}
\end{equation}
of eqs. (4.14). From (4.16) we find the mappings \(B, \bar{B} \leftrightarrow q, \bar{q}\) and \(b, \bar{b} \leftrightarrow Y, \bar{Y}\). One color component of each of the fields \(q, \bar{q}, Y, \bar{Y}\) together with the meson singlets are exactly the degrees of freedom that stay massless after breaking the magnetic gauge group. The 't Hooft anomaly matching conditions are satisfied because the only fields that contribute to the global anomalies in the magnetic theory for \(\tilde{N}_c = 1\) are the \(2N_f\) quark singlets, one component of \(Y\) and \(\bar{Y}\) each and the meson singlets.

The effective low-energy superpotential of the magnetic theory, deduced from (4.15) for the theory with \(\tilde{N}_c = 3k N_f + 1\) and \((N_f + 1)\) quark flavors by adding a tree-level term \(mM_0\) and integrating out the massive modes, contains the terms
\begin{equation}
N_{k-1} q \bar{q} + P_{k-1} Y \bar{Y} + \bar{P}_{k-1} \bar{Y} q.
\end{equation}
We thus expect that the confining superpotential of the electric theory has terms proportional to
\begin{equation}
\bar{B} N_{k-1} B \sim (W_0)^4 (Q \bar{Q})^{3kN_f} X^{(k-1)N_f} (Y \bar{Y})^{3kN_f+1},
\end{equation}
\begin{equation}
BBP_{k-1} \bar{b}, \quad \bar{B}B\bar{P}_{k-1} \bar{b} \sim (W_0)^4 (Q \bar{Q})^{4kN_f} X^{(k-1)(N_c+kN_f)} (Y \bar{Y})^{4kN_f+2}.
\end{equation}
Comparing this to (2.7) we find \(\alpha^{(1)} = 3k\) for the term in the first line of (4.18) and from (2.10, 2.11) \(\gamma_1^{(1)} = -6kN_f - 2\) and \(\gamma_2^{(1)} = -3k(3k-1)N_f - 15k + 1\). For the terms in the second line we get \(\alpha^{(2)} = 4k\) and \(\gamma_1^{(2)} = -8kN_f - 3\) and \(\gamma_2^{(2)} = -4k(3k-1)N_f - 20k + 2\). The confining superpotential consequently is of the form
\begin{equation}
W = \frac{\bar{B} N_{k-1} B}{h_1 6kN_f+2 (3k-1)N_f+15k-1 \Lambda^{3k((3k-1)N_f+1)}} + \frac{BBP_{k-1} \bar{b} + \bar{B}B\bar{P}_{k-1} \bar{b}}{h_1 8kN_f+3 (3k-1)N_f+20k-2 \Lambda^{4k((3k-1)N_f+1)}} + \ldots.
\end{equation}
Again there may be further terms generated by instantons of the magnetic gauge group.

In the limit \( k = 1 \) the coupling parameter \( h_1 \) represents a mass for the tensor \( X \). Integrating out \( X \) results in a theory with tree-level superpotential \( W_{\text{tree}} = h \, \text{Tr}(YY) \), where \( h = (h_2)^2/h_1 \). This is the model discussed in section 3.2 for the special case \( k = 1 \). For \( N_c = 3N_f + 3 \) it confines [5]. The confining superpotential (4.19) contains an additional term proportional to \((b\bar{b})^2\) for \( k = 1 \) and has the correct form

\[
W = \frac{\bar{B}N_0B}{h^{3N_f+7}N^{3(5N_f+4)}} + \frac{(b\bar{b})^2 + h^{-2}(BB\bar{P}_0 \bar{b} + \bar{B}\bar{B}\bar{P}_0 b)}{h^{4N_f+7}N^{4(5N_f+4)}}
\]

in this limit, where we used the scale matching relation \( h_1^{N_c-2}N_f = \Lambda_L^{2N_c-2-N_f} \).

### 4.4 \( SU(N_c) \) with an adjoint tensor, an antisymmetric tensor and a conjugate symmetric tensor

Consider an \( SU(N_c) \) gauge theory with \( N_f + 8 \) quarks \( Q \), \( N_f \) antiquarks \( \bar{Q} \), an adjoint tensor \( X \), an antisymmetric tensor \( Y \) and a conjugate symmetric tensor \( \bar{Y} \) and tree-level superpotential \( W_{\text{tree}} = h_1 \, \text{Tr}X^{k+1} + h_2 \, \text{Tr}XY\bar{Y} \). This model was first studied in [14]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| \( SU(N_c) \) | \( SU(N_f+8) \) | \( SU(N_f) \) | \( U(1)_Y \) | \( U(1)_B \) | \( U(1)_R \) | \( Z_{2(k+1)(N_f+4)} \) | \( Z_{2(N_f+4)} \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( Q \)       | \( \square \)  | \( \square \)  | 1              | \( \frac{6}{N_f+8} \) | \( -1 \)      | \( -N_c \)     | \( -N_c \)     |
| \( \bar{Q} \)  | \( \square \)  | \( \square \)  | \( \frac{6}{N_f+1} \) | \( -1 \)      | \( \frac{1}{N_c} \) | \( -N_c \)     | \( -N_c \)     |
| \( X \) adj    | 1              | 1              | 0              | \( \frac{2}{k+1} \) | \( \frac{2}{k+1} \) | \( 2(N_f+4) \) | \( 0 \)        |
| \( Y \)        | 1              | 1              | 1              | \( \frac{2}{N_c} \) | \( \frac{k}{k+1} \) | \( -(N_f+4) \) | \( N_f+4 \)     |
| \( \bar{Y} \)  | 1              | 1              | -1             | \( -\frac{2}{N_c} \) | \( -\frac{k}{k+1} \) | \( -(N_f+4) \) | \( N_f+4 \)     |

We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

**mesons**

\[
M_j = Q\bar{Q}(j), \quad N_j = Q\bar{Y}Y\bar{Q}(j), \quad P_j = Q\bar{Y}Q(j), \quad \bar{P}_j = \bar{Q}Y\bar{Q}(j),
\]

with \( Q(j) = X^jQ, \; \bar{Q}(j) = X^j\bar{Q}, \; j = 0, \ldots, k-1, \)

**baryons**

\[
B_p^{(n_0, \ldots, n_{k-1}, \bar{n}_0, \ldots, \bar{n}_{k-1})} = (YW_\alpha)^2(YXW_\alpha)^2 \cdots (YX^{p-1}W_\alpha)^2
\]

\[
\cdot \bar{Q}^{n_0}Q_{(1)}^{n_1} \cdots \bar{Q}^{n_{k-1}}(\bar{Y}Q)^{\bar{n}_0}Q_{(1)}^{\bar{n}_1} \cdots (\bar{Y}Q_{(k-1)})^{\bar{n}_{k-1}}
\]

\[
\cdot (\bar{Y}Y\bar{Q})^{\bar{n}_0}(\bar{Y}Q_{(1)})^{\bar{n}_1} \cdots (\bar{Y}YQ_{(k-1)})^{\bar{n}_{k-1}},
\]

with \( \sum_{j=0}^{k-1} (n_j + \bar{n}_j + \bar{n}_j) = N_c - 4p, \)

\[
B_p^{(n_0, \ldots, n_{k-1})} = Y^nQ^{n_0}Q_{(1)}^{n_1} \cdots Q_{(k-1)}^{n_{k-1}}, \quad (4.20)
\]

| 34 |
\[ B_{n}^{(\tilde{m}_{0},...,\tilde{m}_{k-1})} = \tilde{Y}^{\tilde{m}_{0}} \tilde{Q}_{\tilde{m}_{0}} \tilde{Q}_{\tilde{m}_{1}} \cdot \cdot \cdot \tilde{Q}_{(k-1)} \tilde{Q}_{(k-1)}, \]

with \( \sum_{j=0}^{k-1} n_j = N_{c} - 2n, \sum_{j=0}^{k-1} \tilde{n}_j = N_{c} - \tilde{n}, \)

where the gauge indices of the baryons are contracted with one epsilon tensor for the \( B_{p}^{(\cdot \cdot \cdot)} \), \( B_{n}^{(\cdot \cdot \cdot)} \) and with two epsilon tensors for the \( \bar{B}_{n}^{(\cdot \cdot \cdot)} \).

This theory is dual to an \( SU(\tilde{N}_{c}) \) gauge theory, with \( \tilde{N}_{c} = 3k(N_{f} + 4) - N_{c} \) and matter content \(^{12}\)

|   | \( SU(\tilde{N}_{c}) \) | \( SU(N_{f}+8) \) | \( SU(N_{f}) \) | \( U(1)_{Y} \) | \( U(1)_{B} \) | \( U(1)_{R} \) | \( Z_{2(k+1)(N_{f}+4)} \) | \( Z_{2(N_{f}+4)} \) |
|---|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| \( q \) | \( \circ \) | \( \circ \) | 1 | \( \frac{6}{N_{f}+8} \) | \( \frac{1}{N_{c}} \) | \( 1 - \frac{N_{c}+6k}{(k+1)(N_{f}+8)} \) | \( -\tilde{N}_{c} - p_{1} \) | \( -\tilde{N}_{c} - p_{2} \) |
| \( \bar{q} \) | \( \circ \) | 1 | \( \circ \) | \( -1 \) | \( \frac{6}{N_{f}} \) | \( -\frac{1}{N_{c}} \) | \( 1 - \frac{N_{c}-6k}{(k+1)N_{f}} \) | \( -\tilde{N}_{c} + p_{1} \) | \( -\tilde{N}_{c} + p_{2} \) |
| \( \bar{X} \) | adj | 1 | 1 | 0 | 0 | \( \frac{2}{k+1} \) | \( 2N_{f}+4 \) | 0 |
| \( \bar{Y} \) | \( \circ \) | 1 | 1 | \( -1 \) | \( \frac{2}{N_{c}} \) | \( \frac{k}{k+1} \) | \( -(N_{f}+4)-2p_{1} \) | \( N_{f}+4-2p_{2} \) |
| \( \bar{\tilde{Y}} \) | \( \circ \) | 1 | 1 | 1 | \( -\frac{2}{N_{c}} \) | \( \frac{k}{k+1} \) | \( -(N_{f}+4)+2p_{1} \) | \( N_{f}+4+2p_{2} \) |

and singlets \( M_{j} \), \( N_{j} \), \( P_{j} \), \( \tilde{P}_{j} \) that carry the same quantum numbers as the mesons of the electric theory. The numbers \( p_{1} \), \( p_{2} \) are defined by

\[ p_{1} = 2 + \frac{2(2k+1)N_{c}}{N_{c}}, \quad p_{2} = \frac{4kN_{c}}{N_{c}}. \]

They are determined by requiring that the baryon mappings below be invariant under the discrete symmetries. We checked that the discrete anomaly matching conditions are satisfied with these charges for the matter fields.

The magnetic theory contains a tree-level superpotential

\[ W_{\text{mag}} = \text{Tr} \bar{X}^{k+1} + \text{Tr} \bar{X} \bar{Y} \bar{Y} + \sum_{j=0}^{k-1} \left[ M_{k-1-j}q_{(j)} + N_{k-1-j}qd_{(j)} + P_{k-1-j}d_{(j)} + \bar{P}_{k-1-j}d_{(j)} \right], \]

where \( q_{(j)} = \bar{X}^{j}q \).

The electric baryons of \(^{420}\) can be consistently mapped to similar baryons of the magnetic theory:

\[ B_{p}^{(n_{0},...,n_{k-1},\tilde{n}_{0},...,\tilde{n}_{k-1})} \leftrightarrow \bar{B}_{q}^{(m_{0},...,m_{k-1},\tilde{m}_{0},...,\tilde{m}_{k-1})}, \quad \text{with} \]

\[ q = k - p, \quad m_{j} = N_{f} + \tilde{n}_{k-1-j}, \]

\[ B_{n}^{(n_{0},...,n_{k-1})} \leftrightarrow \bar{B}_{m}^{(m_{0},...,m_{k-1})}, \quad \text{with} \]

\[ m = k(N_{f}+2) - n, \]

\[ \bar{B}_{n}^{(n_{0},...,n_{k-1})} \leftrightarrow \bar{B}_{m}^{(m_{0},...,m_{k-1})}, \quad \text{with} \]

\[ m = 2k(N_{f}+6) - n, \quad m_{j} = N_{f} - m_{k-1-j}, \]

\(^{12}\)We corrected some misprints in the \( U(1)_{Y} \) charges associated to the matter fields in \(^{14}\).
where the magnetic baryons $\tilde{B}_q(\cdots)$, $\tilde{B}_m(\cdots)$, $\tilde{B}_m(\cdots)$ are defined in the same way as the electric baryons of (4.20) replacing all fields by their dual partners and $N_c$ by $\tilde{N}_c$. The last two of these mappings have been found in [14].

For $N_c = 3k(N_f + 4) - 1$ the magnetic theory is completely higgsed and the electric theory confines with low-energy spectrum given by the composite fields

$$M_j, \ N_j, \ P_j, \ \bar{P}_j, \ j = 0, \ldots, k - 1,$$

$$B \equiv B^{(N_f + 8, \ldots, N_f + 7)}_{k(N_f + 2)}, \ \bar{B} \equiv B^r_{k(N_f + 2)}, \ \tilde{b} \equiv \tilde{B}^{N_f}_{2k(N_f + 6) - 1}$$

of eqs. (2.10).

The effective low-energy superpotential of the magnetic theory, deduced from (4.21) for the theory with $\tilde{N}_c = 3k + 1, (N_f + 9)$ quarks and $(N_f + 1)$ antiquarks by adding a tree-level term $mM_0$ and integrating out the massive modes, contains the terms $N_{k-1}q\bar{q} + P_{k-1}q\bar{q}$. We thus expect that the confining superpotential of the electric theory has terms proportional to

$$\tilde{B}N_{k-1}B \sim (W_0^{2k(N_f + 8)}) \tilde{Q}^{2kN_f} X^{(k-1)(2k(N_f + 4) + k - 1)} Y^{2k(N_f + 1)} Y^{2k(N_f + 5)},$$

$$BBP_{k-1}\bar{b} \sim Q^{2k(N_f + 8)} \bar{Q}^{2kN_f} X^{(k-1)(2k(N_f + 4) - 1)} Y^{2k(N_f + 2)} Y^{2k(N_f + 6)}.$$

Comparing this to (2.7) we find $\alpha^{(1)} = 2k$ for the term in the first line of (4.24) and from (2.10, 2.11) $\gamma^{(1)}_1 = -4k(N_f + 4) + k + 1$ and $\gamma^{(1)}_2 = -2k(3k - 1)(N_f + 4)$. For the term in the second line we get $\alpha^{(2)} = 2k$ and $\gamma^{(2)}_1 = -4k(N_f + 4) + 1$ and $\gamma^{(2)}_2 = -2k(3k - 1)(N_f + 4) + 2k$. The confining superpotential consequently is of the form

$$W = \frac{\tilde{B}N_{k-1}B + h_1^{-k}h_2^{2k}BBP_{k-1}\bar{b}}{h_1^{4k(N_f + 4) - k - 1}h_2^{2k(3k - 1)(N_f + 4) - 1}} \Lambda^{2k(3k - 1)(N_f + 4) - 1} + \ldots,$$

where the dots denote further terms generated by instantons in the completely broken magnetic gauge group.

In the limit $k = 1$ the coupling parameter $h_1$ represents a mass for the tensor $X$. Integrating out $X$ results in a theory with tree-level superpotential $W_{\text{tree}} = h \ Tr(Y\bar{Y})^2$, where $h = (h_2)^2/h_1$. This is the model discussed in section 3.3 for the special case $k = 0$. For $N_c = 3(N_f + 4) - 1$ it confines. The confining superpotential (4.25) has the correct form

$$W = (BN_0B + hBBP_0\bar{b} + \ldots)/h^{2(N_f + 4)} \Lambda_L^{2(N_f + 4) - 2}$$

in this limit, where we used the scale matching relation $h_1^{N_c - N_f} = \Lambda_L^{2N_c - N_f}$.

4.5 $Sp(2N_c)$ with two antisymmetric tensors

Consider an $Sp(2N_c)$ gauge theory with $2N_f$ quarks $Q$ in the fundamental representation and two traceless antisymmetric tensors $X, Y$, and tree-level superpotential $W_{\text{tree}} = h_1 \ Tr X^{k+1} +
This model was first studied in [14]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

|       | $Sp(2N_c)$ | $SU(2N_f)$ | $U(1)_R$ | $Z_{2(k+1)N_f}$ | $Z_{2N_f}$ |
|-------|------------|------------|----------|-----------------|------------|
| $Q$   | $\not\Box$ | $\not\Box$ | $1 - \frac{N_c+2k+1}{(k+1)N_f}$ | $-(N_c-1)$ | $-(N_c-1)$ |
| $X$   | $\Box$     | $1$        | $\frac{2}{k+1}$ | $2N_f$         | $0$        |
| $Y$   | $\Box$     | $1$        | $\frac{k}{k+1}$ | $-N_f$         | $N_f$      |

There are no baryons in symplectic gauge theories and therefore the only (non-redundant) gauge invariant composite operators that can be built from the elementary fields are:

$$M_{(j,l)} = QX^jY^lQ, \quad j = 0, \ldots, k - 1, \quad l = 0, 1, 2,$$

$$T_{(j,l)} = \text{Tr} X^jY^l, \quad (4.26)$$

where the gauge indices are contracted with the $Sp(2N_c)$-invariant $J$-tensor.

For odd $k$, this theory is dual to an $Sp(2\tilde{N}_c)$ gauge theory, with $\tilde{N}_c = 3kN_f - 4k - 2 - N_c$ and matter content [14]

|       | $Sp(2\tilde{N}_c)$ | $SU(2N_f)$ | $U(1)_R$ | $Z_{2(k+1)N_f}$ | $Z_{2N_f}$ |
|-------|---------------------|------------|----------|-----------------|------------|
| $q$   | $\not\Box$         | $\not\Box$ | $1 - \frac{\tilde{N}_c+2k+1}{(k+1)N_f}$ | $N_c - 3kN_f - 1$ | $N_c - 3kN_f - 1$ |
| $\tilde{X}$ | $\Box$     | $1$        | $\frac{2}{k+1}$ | $2N_f$         | $0$        |
| $\tilde{Y}$ | $\Box$     | $1$        | $\frac{k}{k+1}$ | $-N_f$         | $N_f$      |

and singlets $M_{(j,l)}$ that carry the same quantum numbers as the mesons of the electric theory.

The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

$$W_{\text{mag}} = \text{Tr} \tilde{X}^{k+1} + \text{Tr} \tilde{X}Y^2 + \sum_{l=0}^{2} \sum_{j=0}^{k-1} M_{(k-1-j,2-l)} q \tilde{X}^j \tilde{Y}^l q. \quad (4.27)$$

For $N_c = 3kN_f - 4k - 2$ the magnetic theory is completely higgsed and the electric theory confines with low-energy spectrum given by the composite fields

$$M_{(j,l)}, \quad j = 0, \ldots, k - 1, \quad l = 0, 1, 2, \quad T_{(0,2)}, \quad T_{(1,1)} \quad (4.28)$$

of eqs. [1,26]. It is easy to see that the ‘t Hooft anomaly matching conditions are satisfied. The contribution of $X$ and $Y$ to the global anomalies is $(\tilde{N}_c(2\tilde{N}_c-1)-1)$ times their fermionic charge under the considered symmetry. For $\tilde{N}_c = 0$ they act therefore like two fields with the negative of the charge of $X$ and $Y$ respectively. Thus, we have to search for two composite fields of the electric theory that carry fermionic charge of the same absolute value as $X$ and $Y$ but of the opposite sign. This condition is satisfied by $T_{(0,2)}$ and $T_{(1,1)}$ respectively. The only other contribution to the global anomalies in the magnetic theory for $\tilde{N}_c = 0$ comes from the meson singlets which carry the same charges as the mesons of the electric theory.
In the limit \( k = 1 \) the coupling parameter \( h_1 \) represents a mass \( k \) for the tensor \( X \). Integrating out \( X \) results in a theory with tree-level superpotential \( W_{\text{tree}} = h \text{Tr} Y^4 \), where \( h = (h_2)^2/h_1 \). This is the model discussed in section 3.5 for the special case \( k = 3 \). For \( N_c = 3N_f - 6 \) it confines. The confined spectrum (4.28) reduces to the low-energy spectrum (3.28) in this limit: \( M_l = M_{(0,l)} \), \( l = 0, 1, 2, \), \( T_3 = T_{(1,1)} \). The operator \( T_{(0,2)} \) gets massive\(^{13} \) and is removed from the low-energy spectrum. To reproduce the correct superpotential (3.29) \( W = \sum_{\{j,m\}} (T_3)^p \prod_{m=1}^{N_f} M_{jm} / h^{N_f-3} \Lambda^{5N_f-8} \) the confining superpotential of the theory with two antisymmetric tensors should have the expression \( h_1^{2N_f-4} h_2^{2N_f-6} \Lambda^{2N_f-1} \) in the denominator for \( k = 1 \). We deduce \( \gamma_1 + \gamma_2 = -4N_f + 10 \) in this limit and obtain \( \alpha = 1 \) from (2.8).

We suppose that in general \( \alpha = k \) although we were not able to prove this.

### 4.6 \( Sp(2N_c) \) with an antisymmetric and a symmetric tensor

Consider an \( Sp(2N_c) \) gauge theory with \( 2N_f \) quarks \( Q \) in the fundamental representation, a traceless antisymmetric tensor \( X \) and a symmetric tensor \( Y \) and tree-level superpotential \( W_{\text{tree}} = h_{1,1} \text{Tr} X^{k+1} + h_{2,2} \text{Tr} XY^2 \). This model was first studied in [14]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| \( Sp(2N_c) \) | \( SU(2N_f) \) | \( U(1)_R \) | \( Z_{2(k+1)N_f} \) | \( Z_{2N_f} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( Q \)        |    | 1 - \( \frac{N_c+2k-1}{(k+1)N_f} \) | \( N_c - 3 \) | \( -N_c + 1 \) |
| \( X \)        |    | \( \frac{2}{k+1} \) | \( 2N_f \) | 0 |
| \( Y \)        |    | \( \frac{k}{k+1} \) | \( -N_f \) | \( N_f \) |

The (non-redundant) gauge invariant composite operators that can be built from the elementary fields are:

\[
M_{(j,l)} = QX^jY^lQ, \quad j = 0, \ldots, k - 1, \quad l = 0, 1, 2,
\]

\[
T_{(j,l)} = \text{Tr} X^{k+1}Y^l, \quad (4.29)
\]

where the gauge indices are contracted with the \( Sp(2N_c) \)-invariant \( J \)-tensor.

For odd \( k \) this theory is dual to an \( Sp(2\bar{N}_c) \) gauge theory, with \( \bar{N}_c = 3kN_f - 4k + 2 - N_c \) and matter content [14]

| \( Sp(2\bar{N}_c) \) | \( SU(2N_f) \) | \( U(1)_R \) | \( Z_{2(k+1)N_f} \) | \( Z_{2N_f} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( q \)        |    | 1 - \( \frac{\bar{N}_c+2k-1}{(k+1)N_f} \) | \( \bar{N}_c - 3kN_f - 3 \) | \( \bar{N}_c - 3kN_f + 1 \) |
| \( \bar{X} \)   |    | \( \frac{2}{k+1} \) | \( 2N_f \) | 0 |
| \( \bar{Y} \)   |    | \( \frac{k}{k+1} \) | \( -N_f \) | \( N_f \) |

and singlets \( M_{(j,l)} \) that carry the same quantum numbers as the mesons of the electric theory.

\(^{13}\)It has R-charge 1 and is neutral under all other symmetries. Therefore a mass term is possible.
The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

\[ W_{\text{mag}} = \text{Tr} \, \hat{X}^{k+1} + \text{Tr} \, \hat{X} \hat{Y}^2 + \sum_{l=0}^{k-1} \sum_{j=0}^{l} M_{(k-1-j,2-l)} q^{\hat{X}^j \hat{Y}^l} q. \]  

(4.30)

For \( N_c = 3kN_f - 4k + 2 \) the magnetic theory is completely higgeds and the electric theory confines with low-energy spectrum given by the composite fields

\[ M_{(j,l)}, \quad j = 0, \ldots, k - 1, \quad l = 0, 1, 2, \quad T_{(0,2)} \]  

(4.31)
of eqs. (4.29). The argument of the previous section can be repeated to see that the 't Hooft anomaly matching conditions are satisfied.

In the limit \( k = 1 \) the coupling parameter \( h_1 \) represents a mass for the tensor \( X \). Integrating out \( X \) results in a theory with tree-level superpotential \( W_{\text{tree}} = h \text{Tr} \, Y^4 \), where \( h = (h_2)^2/h_1 \). This is the model discussed in section 3.4 for the special case \( k = 1 \). For \( N_c = 3N_f - 2 \) it confines. The confined spectrum (3.31) reduces to the low-energy spectrum (3.24) in this limit: \( M_l = M_{(0,l)}, \quad l = 0, 1, 2 \). The operator \( T_{(0,2)} \) gets massive and is removed from the low-energy spectrum. To reproduce the correct superpotential (3.25) \( W = \sum_{(j,m)} \frac{3N_f}{kN_f} M_{j,m}/h_1 3^{(N_f-3)} \Lambda^{3(5N_f-2)} \) the confining superpotential of the theory with an antisymmetric and a symmetric tensor should have the expression \( h_1^6 h_2^{6N_f-8} h_2^{6N_f-2} \Lambda^{3(2N_f+1)} \) in the denominator for \( k = 1 \). We deduce \( \gamma_1 + \gamma_2 = -12N_f + 10 \) in this limit and obtain \( \alpha = 3 \) from (2.8). We suppose that in general \( \alpha = 3k \) although we were not able to prove this.

### 4.7 \( SO(N_c) \) with two symmetric tensors

Consider an \( SO(N_c) \) gauge theory with \( N_f \) quarks \( Q \) in the fundamental representation and two traceless symmetric tensors \( X, \ Y \) and tree-level superpotential \( W_{\text{tree}} = h_1 \text{Tr} \, X^{k+1} + h_2 \text{Tr} \, X Y^2 \). This model was first studied in [4]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| \( SO(N_c) \) | \( SU(N_f) \) | \( U(1)_R \) | \( \mathbb{Z}_{2(k+1)N_f} \) | \( \mathbb{Z}_{2N_f} \) | \( \mathbb{Z}'_{2N_f} \) |
|---|---|---|---|---|---|
| \( Q \) | \( \square \) | \( \square \) | \( \frac{N_c-4k-2}{(k+1)N_f} \) | \( - (N_c + 2) \) | \( - (N_c + 2) \) | \( 1 \) |
| \( X \) | \( \square \) | \( \square \) | \( \frac{N_c}{k+1} \) | \( 2N_f \) | \( 0 \) | \( 0 \) |
| \( Y \) | \( \square \) | \( \square \) | \( \frac{k-1}{k} \) | \( -N_f \) | \( N_f \) | \( 0 \) |

We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

- **mesons** \( M_{(j,l)} = Q Q_{(j,l)}, \) with \( Q_{(j,l)} = X^j Y^l Q, \quad j = 0, \ldots, k - 1, \quad l = 0, 1, 2, \)

- **baryons** \( B_{p}^{(n_{(0,0)}, \ldots, n_{(k-1,2)})} = (X^{k+1} Y W_\alpha)^2 (Y W_\alpha)^4 (Y X W_\alpha)^4 \cdots (Y X^{p-1} W_\alpha)^4 \cdot Q^{n_{(0,0)}} Q^{n_{(1,0)}} \cdots Q^{n_{(k-1,2)}}. \)  

(4.32)
where the gauge indices are contracted with a Kronecker delta for the mesons and with an epsilon tensor for the baryons and we assumed \( k \) odd.

For odd \( k \) this theory is dual to an \( SO(\bar{N}_c) \) gauge theory, with \( \bar{N}_c = 3kN_f + 8k + 4 - N_c \) and matter content \([13]\)

| \( SO(\bar{N}_c) \) | \( SU(N_f) \) | \( U(1)_R \) | \( Z_2(k+1)N_f \) | \( Z_{2N_f} \) | \( Z'_{2N_f} \) |
|---|---|---|---|---|---|
| \( q \) | | | | | |
| \( \bar{X} \) | | | | | |
| \( \bar{Y} \) | | | | | |

and singlets \( M_{(j,l)} \) that carry the same quantum numbers as the mesons of the electric theory.

The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

\[
W_{\text{mag}} = \text{Tr} \bar{X}^{k+1} + \text{Tr} \bar{X}\bar{Y}^2 + \sum_{l=0}^{2} \sum_{j=0}^{k-1} M_{(k-1-j,2-l)} q \bar{X}^j \bar{Y}^l q. \tag{4.33}
\]

Under duality the electric baryons of \([13,32]\) are mapped to the magnetic baryons \( \bar{B}^{(m_{(j,l)})}_q = (\bar{Y}W_\alpha)^4 \cdots (\bar{Y}X^qW_\alpha)^4 q^{m_{(0,0)}} \cdots q^{m_{(k-1,2)}} \) according to

\[
\bar{B}^{(n_{(0,0)},\ldots,n_{(k-1,2)})}_p \leftrightarrow \bar{B}^{(m_{(0,0)},\ldots,m_{(k-1,2)})}_q, \quad \text{with} \quad q = k - p, \quad m_{(j,l)} = N_f - n_{(k-1-j,2-l)}. \tag{4.34}
\]

For \( N_c = 3kN_f + 8k + 3 \) the magnetic theory is completely higgsed and the electric theory confines with low-energy spectrum given by the composite fields

\[
M_{(j,l)}, \quad j = 0, \ldots, k - 1, \quad l = 0, 1, 2,
\]

\[
B \equiv B^{(N_f,\ldots,N_f,N_f-1)}_k \tag{4.35}
\]

of eqs. (4.32). From (4.34) we find the mapping \( B \leftrightarrow q \). The ’t Hooft anomaly matching conditions are satisfied because the only fields that contribute to the global anomalies in the magnetic theory for \( \bar{N}_c = 1 \) are the \( N_f \) quark singlets staying massless after the symmetry breaking and the meson singlets.

The effective low-energy superpotential of the magnetic theory, deduced from (4.33) for the theory with \( \bar{N}_c = 3k+1 \) and \( (N_f + 1) \) quarks by adding a tree-level term \( m_{(0,0)} \) and integrating out the massive modes, contains the term \( M_{(k-1,2)}qq \). We thus expect that the confining superpotential of the electric theory has a term proportional to

\[
BM_{(k-1,2)}B \sim (W_\alpha)^{8k+4}Q^{6kN_f}X^{(k-1)(3kN_f+4k+1)}Y^{2(3kN_f+4k+1)}. \tag{4.36}
\]

Comparing this to (2.7) we find \( \alpha = 3k \) and from (2.10, 2.11) \( \gamma_1 = -6kN_f - 20k - 2 \) and \( \gamma_2 = -3k(3k - 1)N_f - 24k^2 - 11k + 1 \). The confining superpotential consequently is of the
form
\[ W = \frac{BM_{(k-1,2)}B}{h_1^{6kN_f+20k+2}h_2^{3k(3k-1)N_f+24k^2+11k-1}} \Lambda^{3k((3k-1)N_f+8k-7)} + \ldots \]  
(4.37)

In the limit \( k = 1 \) the coupling parameter \( h_1 \) represents a mass for the tensor \( X \). Integrating out \( X \) results in a theory with tree-level superpotential \( W_{\text{tree}} = h \text{ Tr} Y^4 \), where \( h = (h_2)^2/h_1 \). This is the model discussed in section 3.7 for the special case \( k = 3 \). For \( N_c = 3N_f + 11 \) it confines [3]. The confining superpotential \( (4.37) \) has the correct form \( W = BM_{(0,2)}B/h^{3N_f+17}\Lambda^{3(5N_f+14)} \) in this limit, where we used the scale matching relation \( h_1^{N_c+2}\Lambda^{3(N_c-2)-(N_c+2)-N_f} = \Lambda^{3(N_c-2)-(N_c+2)-N_f} \).

### 4.8 \( SO(N_c) \) with a symmetric and an antisymmetric tensor

Consider an \( SO(N_c) \) gauge theory with \( N_f \) quarks \( Q \) in the fundamental representation, a traceless symmetric tensor \( X \) and an antisymmetric tensor \( Y \) and tree-level superpotential \( W_{\text{tree}} = h_1 \text{ Tr} X^{k+1} + h_2 \text{ Tr} X Y^2 \). This model was first studied in [14]. The transformation properties of the matter fields under the gauge symmetry and the non-anomalous global symmetries are shown in the following table:

| \( SO(N_c) \) | \( SU(N_f) \) | \( U(1)_R \) | \( Z_{(k+1)N_f} \) | \( Z_{2N_f} \) | \( Z'_{2N_f} \) |
|---|---|---|---|---|---|
| \( Q \) | □ | □ | 1 - \( \frac{N_c-4k+2}{(k+1)N_f} \) | \( (N_c + 6) \) | \( (N_c - 2) \) | 1 |
| \( X \) | □ | 1 | \( \frac{2}{k+1} \) | \( 2N_f \) | 0 | 0 |
| \( Y \) | □ | 1 | \( \frac{k+1}{k} \) | \( -N_f \) | \( N_f \) | 0 |

We will be interested in the following gauge invariant composite operators that can be built from the elementary fields:

- **mesons** \( M_{(j,l)} = QY^l Q_{(j)} \), with \( Q_{(j)} = X^j Q \), \( j = 0, \ldots, k-1 \), \( l = 0, 1, 2 \),

- **baryons** \( B_n^{(n_0, \ldots, n_{k-1})} = Y^n Q_{(1)}^{n_0} Q_{(k-1)}^{n_{k-1}} \),

with \( \sum_{j=0}^{k-1} n_j = N_c - 2n, \quad n = 0, \ldots, \left\lfloor \frac{N_c}{2} \right\rfloor \),

where the gauge indices are contracted with a Kronecker delta for the mesons and with an epsilon tensor for the baryons.

For odd \( k \) this theory is dual to an \( SO(\tilde{N}_c) \) gauge theory, with \( \tilde{N}_c = 3kN_f + 8k - 4 - N_c \) and matter content [14]
and singlets \( M_{(j,l)} \) that carry the same quantum numbers as the mesons of the electric theory.

The following tree-level superpotential of the magnetic theory is invariant under all the symmetries:

\[
W_{\text{mag}} = \text{Tr} \, \hat{X}^{k+1} + \text{Tr} \, \hat{X} \hat{Y}^2 + \sum_{l=0}^{2} \sum_{j=0}^{k-1} M_{(k-1-j,2-l)} q \hat{X}^j \hat{Y}^l q. \tag{4.39}
\]

The electric baryons of (4.38) can be consistently mapped to similar baryons of the magnetic theory:

\[
B_{n}^{(q_0,\ldots,q_{k-1})} \leftrightarrow \tilde{B}_{m}^{(m_0,\ldots,m_{k-1})}, \quad \text{with} \quad m = kN_f + 4k - 2 - n, \quad m_j = N_f - n_{k-1-j}, \tag{4.40}
\]

where the magnetic baryons \( \tilde{B}_{m}^{(\ldots)} \) are defined in the same way as the electric baryons of (4.38) replacing all fields by their dual partners and \( N_c \) by \( \tilde{N}_c \).

For \( N_c = 3kN_f + 8k - 5 \) the magnetic theory is completely higgsed and the electric theory confines with low-energy spectrum given by the composite fields

\[
M_{(j,l)}, \quad j = 0, \ldots, k - 1, \quad l = 0, 1, 2,
\]

\[
B \equiv B_{kN_f + 4k - 2}^{(N_f,\ldots,N_f,N_f-1)} \tag{4.41}
\]

of eqs. (4.38). From (4.40) we find the mapping \( B \leftrightarrow q \). The 't Hooft anomaly matching conditions are satisfied because the only fields that contribute to the global anomalies in the magnetic theory for \( \tilde{N}_c = 1 \) are the \( N_f \) quark singlets staying massless after the symmetry breaking and the meson singlets.

The effective low-energy superpotential of the magnetic theory, deduced from (4.39) for the theory with \( \tilde{N}_c = 3k + 1 \) and \( (N_f+1) \) quarks by adding a tree-level term \( mM_{(0,0)} \) and integrating out the massive modes, contains the term \( M_{(k-1,2)}qq \). We thus expect that the confining superpotential of the electric theory has a term proportional to

\[
BM_{(k-1,2)}B \sim Q^{2kN_f} X^{(k-1)(kN_f-1)} Y^{2(kN_f+4k-1)} \tag{4.42}
\]

Comparing this to (2.7) we find \( \alpha = k \) and from (2.10) \( \gamma_1 = -2kN_f - 8k + 2 \) and \( \gamma_2 = -(3k - 1)kN_f - 8k^2 + 11k - 1 \). The confining superpotential consequently is of the form

\[
W = \frac{BM_{(k-1,2)}B}{h_1^{2kN_f+8k-2} h_2^{(3k-1)kN_f+8k^2-11k+1} \Lambda^{k((3k-1)N_f+8k-11)}} + \ldots. \tag{4.43}
\]

In the limit \( k = 1 \) the coupling parameter \( h_1 \) represents a mass for the tensor \( X \). Integrating out \( X \) results in a theory with tree-level superpotential \( W_{\text{tree}} = h \text{Tr} Y^4 \), where \( h = (h_2)^2/h_1 \). This is the model discussed in section 3.6 for the special case \( k = 1 \). For \( N_c = 3N_f + 3 \) it confines \([4]\). The confining superpotential (4.43) has the correct form \( W = BM_{(0,2)}B/h^{N_f-1} \Lambda_L^{5N_f+2} \) in this limit, where we used the scale matching relation \( h_1^{-N_c+2} \Lambda^{2(N_c-2)-(N_c+2)-N_f} = \Lambda_L^{2(N_c-2)-N_f} \).
5 Conclusions

We have used the non-Abelian duality of asymptotically free $N = 1$ supersymmetric gauge theories discovered by Seiberg to find new models that confine in the presence of an appropriate superpotential. This is a very interesting application of the proposed duality because it enables us to obtain non-perturbative results for the electric theory by a perturbative calculation in its magnetic dual. Confinement in the electric theory can be understood from the Higgs phase of the magnetic theory. The confining spectrum can easily be derived from the duality mappings of gauge invariant operators. For $SU$ and $SO$ gauge groups one also obtains the form of the confining superpotential by applying these mappings to the magnetic tree-level superpotential. (To determine the full confining superpotential one has to include instanton corrections in the completely broken gauge group.)

In this paper we have discussed fifteen gauge theory models containing fields in the fundamental representation and in second rank tensor representations which possess a dual description in the infrared when an appropriate tree-level superpotential for the tensor fields is added. All of these models confine without breaking of chiral symmetry when the parameters (number of fundamental flavors $N_f$, number of colors $N_c$ and power of the tensors in the superpotential $k$) are tuned such that the magnetic gauge symmetry is just completely broken, i.e. one formally has $\tilde{N}_c = 1$ for $SU$ and $SO$ gauge groups and $\tilde{N}_c = 0$ for $Sp$ gauge groups, where $\tilde{N}_c$ is the number of colors of the magnetic gauge theory. Reducing $N_f$ further leads to even stronger coupling in the electric theory. In most cases the low-energy theory with one less flavor develops an Affleck-Dine-Seiberg superpotential and has no stable ground state. This is in contrast to the $s$-confining models with vanishing tree-level superpotential which generically lead to confinement with quantum modified moduli when $N_f$ is reduced by one. However, we find two examples for gauge theories that confine with quantum modified moduli in the presence of a tree-level superpotential: $SU(N_c)$ with an antisymmetric flavor and $Sp(2N_c)$ with an antisymmetric tensor. It is an intriguing coincidence that for vanishing tree-level superpotential these two models are the only known examples of gauge theories that contain tensor fields and possess a dual description for each value of $N_c$.

The phase structure of the $A_k$ models is now almost understood. However, a deeper understanding of why just these special models can be analyzed using duality is still missing. As for the $D_{k+2}$ models, more open questions remain to be answered. It is not clear why a dual description could only be found for odd $k$. Furthermore, the stability analysis performed for the $A_k$ models has not been repeated for the $D_{k+2}$ models.

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