A Paradigm for Instantaneously-Wideband Impedance Matching by Temporal Switching of Transmission Line Parameters

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Introduction.—Transmission-lines (TL) are used in any high-frequency electronic device or acoustic guiding network to connect signal sources to various loads. The lossless TL is characterized by a real-valued capacitance $C$ and inductance $L$ per unit length, or equivalently, by impedance, $Z_c = \sqrt{L/C}$, and phase velocity, $v = 1/\sqrt{LC}$, that are independent of frequency if the line operates in its fundamental mode, e.g., Transverse Electromagnetic mode in electronics [1], or plane-wave mode in acoustics [2]. It is well known that in order to avoid any back-reflections in the line, and to guarantee maximal power transfer from the source to the load, ideally, the internal source impedance, $R_g$, and the load impedance, $R_L$, should be equal to the characteristic impedance of the line, $Z_c$. In this case, no reflection occurs at the load, and, therefore, the energy transfer from the source to the load becomes optimal with efficiency $\eta_{\text{opt}} = R_g/(R_g + Z_c) = 0.5$ [1]. It is important to note that in this case, since the TL is practically dispersionless, the optimal matching is instantaneously-wideband, namely, with linear-phase response of transmission coefficient, or equivalently, with frequency independent group velocity, thus enabling the transmission of short temporal pulses with no signal distortion [3]. Unfortunately, in most practical cases, $R_L \neq R_g$, leading necessarily to reflections and degradation in performance that deteriorates further when the contrast is high, namely, when $R_g/R_L \ll 1$ or vice versa. This is often the case, e.g., in the contexts of matching small antennas for communication and imaging [4], for cloaking [5], as well as at the acoustic interface between liquid and gas [6], and for ultrasonic absorbers [8].

Impedance matching, a fundamental practice in wave engineering, is used to tackle these issues. However, it is limited by the Bode-Fano criterion [7,11] which implies a stringent tradeoff between the matching efficiency and the matching bandwidth [12]. Due to the Bode-Fano bound, while it is possible to obtain perfect matching (no reflections) at a discrete set of frequencies, it is impossible to achieve that on an entire finite frequency band. Thus, typically, the matching efficiency, and particularly the transmission group delay, are highly dispersive. Yielding that an instantaneously-wideband signal, namely, a short pulse in time, that propagates in such a system will be severely distorted. Thus introducing challenges on modern-days technologies that are required to incorporate with the ever-increasing demand for fast, ultra-wideband, communication networks and medical imaging [4], as well as on broadband cloaking [11], and high contract acoustics [6,8]. To demonstrate this issue we consider a linear and passive wideband matching network that is based on 3-stages Chebyshev transformer, shown in Fig. 1(a), and is used to match a generator with $R_g = 50\Omega$ through a $Z_0 = 50\Omega$ line to a $R_L = 1\Omega$ load. Due to the highly dispersive power transmission and group delay (see Fig. 1(b)), the signal excited by the generator is drastically distorted before reaching the load, as seen in Figs. 1(c) and 1(d).

How can we overcome the Bode-Fano bound? The idea is to revoke its underlying assumptions [2,9,10]. Namely, linearity, passivity, and time-invariance. Indeed, in the past years active matching networks that are implemented by non-Foster components have been successfully explored as a means to overcome the matching bound. These have been suggested to increase also the
FIG. 1. (a) A passive matching network based on 3-stages Chebyshev transformer. Here, the matching is performed between a generator with $R_g = 50\Omega$, connected to a load $R_L = 1\Omega$ through a line with $Z_0 = 50\Omega$, the required matching fractional bandwidth is $\Delta f/f_0 = 1$, and center frequency $f_0$. The transformer parameters are $(Z_1, Z_2, Z_3) = (19.6, 7.1, 2.6)\Omega$. $t_1 = t_2 = t_3 = 0.25c/f_0$ where $c$ is the phase velocity in the line and $f_0$ the center frequency of the pulse. This linear time-invariant matching network exhibits severe frequency dispersion of the power transmission and group delay, as shown in (b). Thus, through this matching network, the generator signal that is shown in (c) experiences significant distortion when reaching the load as shown in (d).

bandwidth of small antennas [13] and cloaking devices [14, 15]. However, unfortunately, this approach comes with the price of an unavoidable increase in the internal noise of the network [10], as well as with inherent potential for instability [16, 17]. Moreover, while non-Foster matching can in principle compensate reactive loads, it cannot be applied to match small resistive load to a TL.

In this letter we propose to mitigate the Bode-Fano bound by violating the time-invariance assumption through temporal switching of the TL parameters. Time-varying media have been explored in the past for various purposes such as for signal parametric amplification, magnetoelastic delaylines, and pumped non-linear media [13, 21]. Recently, temporal modulation has regained vast interest in the context of amplification [22], energy accumulation [24], inverse prism functionality [25], but particularly, in the context of magnet-less non-reciprocity [26, 31], and implementation of synthetic magnetic field [32].

As opposed to pervious attempts, here, we explore a paradigm to achieve a linear and passive impedance matching for short temporal pulses. The proposed approach uses an abrupt temporal switching of the TL parameters. It therefore bypasses the Bode-Fano assumptions, and moreover, increases the number of degrees of freedom for the matching process. Using the proposed approach we enable matching capabilities beyond today’s limitations. Thus, opening a unique venue to resolve a fundamental issue in wave-engineering with significant potential in various wave based applications [4, 6, 11].

Pulse dynamics in abruptly switched TL.—We consider the network shown in Fig. 2(a). A source with internal impedance $R_g$ is connected to a TL that is terminated by a load $R_L$. The TL is characterized by capacitance and inductance per unit length. It is assumed that these characteristics can be abruptly switched, at some time $t_s$, between two states #1 and #2, with $(L_i, C_i)$ or equivalently with impedance and phase velocity $(Z_{c_i}, v_i) = (\sqrt{L_i/C_i}, 1/\sqrt{L_iC_i})$, for state #i. See Fig. 2(b) for schematic implementation of such a metamaterial TL. Assume that the TL is initially at state #1. The voltage on the source, $V_g(t)$, excites a forward propagating pulse with temporal width $T_1$ that propagates along the line toward the load,

$$V_{1+}^+(t, z) = Z_{c1}V_g(t - z/v_1)/(Z_{c1} + R_g), \quad (1)$$

Then, at some time $t = t_s$, that is sufficiently larger than the pulse width $T_1$, the TL characteristics are abruptly switched to state #2, see Fig. 2(a). During the switching moment, the electric charge and the magnetic flux along the line remain continuous [33, Sec. 1]. In order to satisfy this continuity, the pulse $V_{1+}^+$ splits into a forward and backward propagating pulses, $V_{2+}^-$ and $V_{2-}^-$, respectively. They are related to the original pulse $V_{1+}^+(z, t)$ via [33]

$$V_{2+}^-(z, t) = TV_{1+}^+((v_2/v_1)\tau - \zeta/v_1), \quad (2a)$$

$$V_{2-}^+(z, t) = \Gamma V_{1+}^+(-(v_2/v_1)\tau - \zeta/v_1), \quad (2b)$$

with $\tau = t - t_s$, $\zeta = z - z_s$ where $z_s = v_1t_s$, and the transmission and reflection coefficients are given by

$$\mathcal{T} = \frac{1}{2} \left( \frac{v_2}{v_1} \right) \left[ \frac{Z_{c2}}{Z_{c1}} + 1 \right], \quad \Gamma = \frac{1}{2} \left( \frac{v_2}{v_1} \right) \left[ \frac{Z_{c2}}{Z_{c1}} - 1 \right]. \quad (3)$$

Assuming that the pulse width of $V_{1+}^+$ is $T_1$, then, it follows from Eq. (2) that the pulse width of $V_{2+}^-$ is $T_2 = (v_1/v_2)T_1$. Thus, the TL switching from state #1 to #2 results in only a temporal up/down pulse compression that can be readily restored, as opposed to the complex signal distortion that is obtained by a conventional matching scheme such as in Fig. 1. Furthermore, denoting $\mathcal{E}_1 = ||V_{1+}^+||^2/Z_{c1}$ and $\mathcal{E}_2^{\pm} = ||V_{2+}^\pm||^2/Z_{c2}$, where $||$ is the $L_2$ norm, as the energy delivered by the corresponding pulses, the energy balance $\Delta \mathcal{E} = \mathcal{E}_2^- + \mathcal{E}_2^+ - \mathcal{E}_1$ reads,

$$\Delta \mathcal{E} = \left\{ \frac{1}{2} \left( \frac{v_2}{v_1} \right) \left[ \frac{Z_{c2}}{Z_{c1}} + 1 \right] \right\} \mathcal{E}_1. \quad (4)$$

Using Eq. (4) we identify three switching regimes: (i) $\Delta \mathcal{E} < 0$ where energy is absorbed (dissipated) from the wave-system; (ii) $\Delta \mathcal{E} = 0$ where no energy change occurred, and (iii) $\Delta \mathcal{E} > 0$ where energy is pumped into the wave-system (See [33, Sec. 2] and Fig. S1 there). While the first two are considered ‘passive’, the last one involves
parametric amplification and is therefore considered ‘active’. Each of these switching cases constitutes a different wave dynamic in terms of compression, velocity, energy conversion efficiency between source and load.

Formulation of impedance matching as a constrained optimization problem.—The energy delivered to the load \( R_L \), by the first pulse reaching the load at \( t > t_s \), reads, \[ \eta = \frac{\mathcal{E}_L}{\mathcal{E}_L + \Delta \mathcal{E} \mathcal{H}(\Delta \mathcal{E})}. \] (6)

Note that in Eq. (6) we used the Heaviside function \( \mathcal{H}(x) = 1 \) for \( x > 0 \) and \( \mathcal{H}(x) = 0 \) for \( x < 0 \), to imply that in the ‘active’ case the matching efficiency accounts also for the additional power by the switching. It should also be noted that as a quality measure an ‘efficiency’ is not a natural parameter of a system but can be defined in various ways as to capturing (and emphasize) different and desired characteristics, see e.g. [1]. Thus, if required, the proposed approach can be easily augmented to these definitions as well. Inspection of Eq. (6) with Eq. (5) suggests that for a given \( R_g \) and \( R_L \), \( \eta \) is a function of three normalized free parameters \( Z_{c1}/R_g \), \( Z_{c2}/R_L \) and \( v_2/v_1 \) that are constrained by a given \( \Delta \mathcal{E} \) of Eq. (4) (see, e.g., Fig. S1 [33]). Our goal is to maximize the matching efficiency \( \eta \) with respect to these free parameters. Thus, the matching problem has been replaced by a three-dimensional, generally non-convex, optimization problem. Note that the switching between the two states of the TL enables us to increase the number of degrees of freedom we have for the matching, from one to three, and therefore make it possible to reach optimal points that could not be reached otherwise. We stress that the two state switching is only an example for this idea, and one may in principle use switching between larger number of states to relax the optimization process and improve its performance.

Passive switching \( [\Delta \mathcal{E} = 0] \).—By using Eq. (6) subject to Eq. (4) with \( \Delta \mathcal{E} = 0 \), and setting \( \rho = R_L/R_g \), \( x = Z_{c1}/R_g \), \( y = Z_{c1}/R_L \) and \( \bar{v} = v_2/v_1 \), the efficiency optimization problem is set as, \[ \max_{x,y,\bar{v}} \left\{ \frac{2 \bar{v}}{\rho} \frac{x^2}{(1 + x)^2} \frac{1}{(1 + y)^2} \left[ \frac{y}{x} + 1 \right]^2 \right\} \] (7a)

subject to \( \left( \frac{\rho y}{x} \right) + \left( \frac{\rho y}{x} \right)^{-1} = \frac{2}{\bar{v}} \). (7b)

In this case the optimization proposed in Eq. (7) can be transformed to the solution of a single transcendental equation that is solved numerically [33]. Figure 2(c) depicts the required ratio of the TL characteristic parameters, before and after the switching, \( C_1/C_2 \) and \( L_1/L_2 \), as function of the load-generator contrast \( \rho \), to achieve the optimal efficiency. The optimal efficiency attained by the proposed switching scheme is shown in the continuous curve in (2d). The dashed curve in (2d) depicts, for comparison, the best efficiency for non switchable TL, i.e., TL with a fixed characteristic impedance \( Z_c = \sqrt{R_g R_l} \). The results in the current and following sections have been verified by a circuit simulation of the metamaterial TL that its unit cell is shown in Fig. 2(b), for several cases, and with very good agreement. It is clearly noted that switching of the TL parameters provides a significant increase in the energy delivered to the load (actual efficiency) in comparison with the non-switching case. Remarkably, the increase is more dominant for the challenging high contract cases where \( R_g \gg R_L \) and \( R_g \ll R_L \), and moreover, in this case becomes nearly independent on the contrast.

FIG. 2. (a) Physical layout of the switched TL system. (b) A possible realization of the switched TL as a discrete periodic structure of parallel varactor diodes and banks of switched series inductors. (c) The ratio between the TL parameters required for the optimal realization: \( L_1/L_2 \) in blue line and \( C_1/C_2 \) in red line. (d) Continuous line - the optimal efficiency (with switching), and dashed line - the efficiency for the “no switching” case.

Active switching \( [\Delta \mathcal{E} > 0] \).—In the previous section signal amplification by the switching scheme was not allowed, this is due to the constraint \( \Delta \mathcal{E} = 0 \). for this reason we called this scheme ‘passive’. Here, as opposed to that, we allow injection of energy into the system
by the switching, i.e., $\Delta \mathcal{E} > 0$. With that in mind, and setting $\Delta \mathcal{E} = \delta \mathcal{E}_a$ with $\delta > 0$, Eq. (6) becomes $\eta = \mathcal{E}_L/[(1 + \delta)\mathcal{E}_a]$ (where $\mathcal{E}_L$ also depends on $\delta$ via the characteristic impedances, see Eq. (5)). For a given energy balance between the source energy and the switching, i.e., $\delta$. The optimization problem is set as, (6)

$$\max_{x, y, \bar{v}} \left\{ \frac{2}{1 + \delta} \frac{x^2}{(1+x)^2} \left( \frac{y}{x} + 1 \right)^2 \right\}$$

subject to

$$\left( \frac{y}{x} + \frac{1}{\rho} \right) = \left[ 2 + \frac{(1+x)^2}{x} \right] \frac{1}{\bar{v}}$$

(8)

where $x$, $y$, $\bar{v}$ and $\rho$ are defined as in Eq. (7). Note that Eq. (8) is a generalization of Eq. (7) for the case $\delta > 0$. The constraint structure in (8a) is more complicated than that in Eq. (7b), rendering an involved direct maximization of $\eta$ (in Eq. (8a)). To this end, standard optimization tools are used [34]. The optimization results are shown in Fig. 3 as function of the contrast $\rho = R_L/R_g$ and the additional energy to the wave system by the switching $\delta = \Delta \mathcal{E}/\mathcal{E}_a$. In panels (a) and (b) we show $x = Z_{c1}/R_g$ and $\eta$, respectively. Similar figures for $y = Z_{c2}/R_L$ and $\bar{v}$ are shown in Fig. S2 [33]. We note that in the ‘passive’ matching case, namely, for $\Delta \mathcal{E} = 0$ ($\delta = 0$), the optimization procedure leads to a single optimum point, which is, therefore, a global maximum. As opposed to that, for $\delta > 0$, there are several local optima. Nevertheless, the global optimum behaves similar to the ‘passive’, $\delta = 0$, solution. However, as soon as $\delta$ increases above the 0.5 threshold, namely, as soon as the power that enters into the wave-system by the switching mechanism exceeds the maximal power that can be delivered to a matched load without switching (recall that for $R_g = R_L = Z_0$, $\mathcal{E}_L = 0.5\mathcal{E}_a$), the optimization, and therefore, also the matching dynamics significantly alter for the high contrast cases, see e.g., Fig. 3(a) for $\rho \lesssim 0.1$. In this case, interestingly, the global optimum exhibits a remarkably different matching mechanism compare to the case of $0 < \delta < 0.5$, as evident by the ‘jump’ in Fig. 3(a). Here, the global optimum is obtained when the line impedance at state #1 is extremely low, i.e., $Z_{c1}/R_g \ll 1$. Thus, the line merely sense the generator signal. Then, at the switching moment to state #2, the sampled signal is amplified by the power that enters through the switching process. Hence, in these cases, the actual role of $V_1^+$ is to provide a sampling of the source’s waveform that would be amplified later by the switching to provide the energy for the matching. In this regime, therefore, the switched matching system can be described as a cascaded system of a sampler and an ideal amplifier that buffers between the generator and the load, as schematically shown in Fig. 3(b). In light of the nature of this, so-called, ‘active’ regime, here, the matching efficiency, as defined in Eq. (7) can exceed values up to $\eta = 0.75$ for $\delta = 1$ as shown in Fig. 3(c), and up to $\eta = 1$ for $\delta \gg 1$ (Fig. Sec. 4).

**FIG. 3.** (a) The optimal value of $x = Z_{c1}/R_g$ as a function of the contrast, $\rho$, and the amount of energy balance $\delta$. (b) Equivalent circuit model for $0.5 < \delta < 1$ and high generator-load contrast $\rho$. (c) The optimal value the efficiency, $\eta$ as a function of $\rho$ and $\delta$.

**Conclusions and Discussion.**—In this letter we have discussed a new paradigm for short pulse impedance matching in TL networks by switching the TL’s parameters between two states. Generally, this matching system maybe passive or active depending on the two switching states. The latter are determined by solving a nonlinear constrained optimization for the matching efficiency. We have demonstrated that the proposed approach can overcome severe challenges that exist in other matching techniques, particularly for cases of high contrast between the source and load impedances. Our approach, therefore, introduces possibilities to cope with today’s ever-ending demand for high-speed ultra-wideband communication,
for small antennas matching, it may be applied to achieve wideband cloaking, and can be used for wideband matching of acoustic waves at the interface between liquids and gases, and so on. Finally, we note that a byproduct of the suggested switching scheme is the ability to obtain significant delay of signals along the line. Thus, an alternative design goal can be taken by defining a different quality measure where the switched TL can be considered as a ‘controlled delay line’ to be used in, e.g., true time delay pulsed systems [35, 36].

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Nevertheless, the case of $R_0 = R_g$ ($\rho = 1$) can be treated heuristically as discussed in [33] to give $x = y = 1$, $\bar{v} = 1 + 2\delta$ and $\eta = (1 + 2\delta)/(2 + 2\delta)$ to give $\eta = 0.75$ at $\delta = 1$ as can also be noted in Fig. [3].

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