ON OCCUPATION TIMES IN THE RED OF LÉVY RISK MODELS

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Abstract. In this paper, we complement the existing literature on the occupation time in the red (below level 0) of a spectrally negative Lévy process, and later extend the analysis to the refracted spectrally negative Lévy process. For both classes of processes, we derive an explicit expression for the distribution of such occupation time up to an independent exponential time. As an application, we consider the inverse occupation time (also known as the time of cumulative Parisian ruin in [14]), where ruin is deemed to occur at the earliest time the risk process cumulatively stays below a critical level over a pre-determined time-threshold. Some particular examples of spectrally negative Lévy processes are also examined in more detail.

1. Introduction

The study of occupation times is a long-standing research topic in applied probability. In finance, occupation-time-related derivatives were studied under various dynamics for the underlying asset (e.g., Cai et al. [9], Cai and Kou [10] and Linetsky [27]). In actuarial mathematics, the occupation time of an insurer’s surplus process below a given threshold level (often chosen to be the "symbolic" level 0) is particularly critical in the assessment of an insurer’s solvency risk (e.g., Landriault et al. [21] and Guérin and Renaud [14]). With this application in mind, we define the occupation time in the red of a risk process \(X\) in the time interval \((0,t)\) as

\[
O_t^X = \int_0^t 1_{(-\infty,0)}(X_s) \, ds,
\]

where

\[
1_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise}. \end{cases}
\]

Its infinite-time counterpart \(O_X^\infty\) is defined as \(O_X^\infty = \int_0^\infty 1_{(-\infty,0)}(X_s) \, ds\).

There exists a number of results on the occupation time \(O_t^X\) in the literature. For the standard Brownian motion, the distribution of \(O_t^X\) appeared in Lévy’s [22] famous arc-sine law. This formula was generalized by Akahori [11] and Takács [37] to a Brownian motion with drift. For the classical compound Poisson process, Dos Reis [12] studied many quantities involving the duration of negative surplus using a martingale approach. Zhang and Wu [39] further considered the generalization to a compound Poisson process perturbed by an independent Brownian motion. The reader is also referred to Li and Zhou [23] for occupation time related work for the class of time-homogeneous diffusion processes.

More specifically, for spectrally negative Lévy processes, many quantities of interest involving the duration of the negative surplus are already known. For instance, Landriault et

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al. [20] derived an expression for the Laplace transform of $O_X^\infty$ in terms of the process’ scale function. Loeffen et al. [33] generalized the results in [20] by characterizing the joint Laplace transform of $(\tau_b^+, O_X^\tau_b^+)$ where $\tau_b^+ = \inf\{t > 0: X_t > b\}$. Li and Palmowski [24] considered a further extension by studying weighted occupation times. Also, potential measures involving occupation times were also analyzed by Guérin and Renaud [13] and Li et al. [26].

As alluded to earlier, in actuarial mathematics, occupation times can naturally be used as a measure of the risk inherent to an insurance portfolio. However, the existing analysis on the occupation time distribution fails to provide a solvency early warning mechanism (in the form of a stopping time or others) that the insurer can act on in periods of financial distress. This motivates us to consider the inverse occupation time, that is the first time the accumulated duration of all periods of financial distress (periods in which the risk process is below the solvency threshold level) exceeds a deterministic grace period. In actuarial ruin terminology, the inverse occupation time is also known as the cumulative Parisian ruin time (see Guérin and Renaud [14]). In fact, occupation times problems are known to be closely related to Parisian ruin models (e.g., [4, 20, 28, 29, 31, 32]). Under these models, insurers are granted a grace period to re-emerge above the threshold level before ruin is deemed to occur. More precisely, Parisian ruin occurs if the time spent below a pre-determined critical level exceeds a stochastic or deterministic time threshold. Guérin and Renaud [14] studied the inverse occupation time for the Cramér–Lundberg risk model with exponential claims by giving an explicit representation for the distribution of $O_t^X$. Lkabous and Renaud [30] studied a VaR-type risk measure defined as the (minimum) amount of initial capital required to bound the finite-time probability of cumulative Parisian ruin, generalizing the work of Trufin et al. [38]. Recently, Bladt et al. [8] studied the inverse occupation time for a class of dependent risk-reserve processes.

In this paper, we study the distribution of the occupation time below zero of a spectrally negative Lévy process up to an (independent) exponential time horizon. As an application, we study the inverse occupation time, generalizing results derived by Guérin and Renaud [14]. We later extend the analysis to the class of refracted spectrally negative Lévy process introduced by Kyprianou and Loeffen [35]. This class of processes is of interest in a number of insurance applications. This includes dividend payouts under a threshold strategy (e.g., Hernández-Hernández [15] and Czarna et al. [11]) and variable annuities with a state-dependent fee structure (e.g., Bernard et al. [6]). Results on occupation times on the refracted spectrally negative Lévy process can be found in Kyprianou et al. [34], Renaud [35] and Li and Zhou [25]. Note that in these papers, the aim was to identify the Laplace transform of some occupation times while our analysis is more distribution focused.

The rest of the paper is organized as follows. In Section 2, we first present the necessary background material on spectrally negative Lévy processes and scale functions. We then derive the main results of this paper and consider some relevant applications. We conclude this section by examining two examples of Lévy risk processes. Finally, in Section 3 we extend our study to the class of refracted spectrally negative Lévy process.

2. Occupation Times of Spectrally Negative Lévy Processes

2.1. Preliminaries. First, we present the necessary background material on spectrally negative Lévy processes. A Lévy insurance risk process $X$ is a process with stationary and independent increments and no positive jumps. To avoid trivialities, we exclude the case where $X$ has monotone paths. As the Lévy process $X$ has no positive jumps, its Laplace
transform exists: for all $\lambda, t \geq 0$,

$$
\mathbb{E}[e^{\lambda X_t}] = e^{\psi(\lambda)},
$$

where

$$
\psi(\lambda) = \gamma \lambda + \frac{1}{2} \sigma^2 \lambda^2 + \int_0^\infty \left( e^{-\lambda z} - 1 + \lambda z 1_{(0,1]}(z) \right) \Pi(dz),
$$

for $\gamma \in \mathbb{R}$ and $\sigma \geq 0$, and where $\Pi$ is a $\sigma$-finite measure on $(0, \infty)$ called the Lévy measure of $X$ such that

$$
\int_0^\infty (1 \wedge z^2) \Pi(dz) < \infty.
$$

Throughout, we will use the standard Markovian notation: the law of $X$ when starting from $X_0 = x$ is denoted by $P_x$ and the corresponding expectation by $E_x$. We write $P$ and $E$ when $x = 0$.

We recall the definitions of standard first-passage stopping times: for $b \in \mathbb{R}$,

$$
\tau_b^- = \inf\{ t > 0 : X_t < b \} \quad \text{and} \quad \tau_b^+ = \inf\{ t > 0 : X_t > b \},
$$

with the convention $\inf\emptyset = \infty$.

We now present the definition of the scale functions $W_q$ and $Z_q$ of $X$. First, recall that there exists a function $\Phi: [0, \infty) \to [0, \infty)$ defined by $\Phi_q = \sup\{ \lambda \geq 0 \mid \psi(\lambda) = q \}$ (the right-inverse of $\psi$) such that

$$
\psi(\Phi_q) = q, \quad q \geq 0.
$$

Now, for $q \geq 0$, the $q$-scale function of the process $X$ is defined as the continuous function on $[0, \infty)$ with Laplace transform

$$
\int_0^\infty e^{-\lambda y} W_q(y)dy = \frac{1}{\psi_q(\lambda)}, \quad \text{for } \lambda > \Phi_q,
$$

where $\psi_q(\lambda) = \psi(\lambda) - q$. This function is unique, positive and strictly increasing for $x \geq 0$ and is further continuous for $q \geq 0$. We extend $W_q$ to the whole real line by setting $W_q(x) = 0$ for $x < 0$. We write $W = W_0$ when $q = 0$.

We also define another scale function $Z_q(x, \theta)$ by

$$
Z_q(x, \theta) = e^{\theta x} \left( 1 - \psi_q(\theta) \int_0^x e^{-\theta y} W_q(y)dy \right), \quad x \geq 0,
$$

and $Z_q(x, \theta) = e^{\theta x}$ for $x < 0$. For $\theta = 0$,

$$
Z_q(x, 0) = Z_q(x) = 1 + q \int_0^x W_q(y)dy, \quad x \in \mathbb{R}.
$$

For $\theta \geq \Phi_q$, using (1), the scale function $Z_q(x, \theta)$ can be rewritten as

$$
Z_q(x, \theta) = \psi_q(\theta) \int_0^\infty e^{-\theta y} W_q(x + y)dy, \quad x \geq 0.
$$

We recall the delayed $q$-scale function of $X$ introduced by Loeffen et al. [32] defined as

$$
\Lambda^{(q)}(x, r) = \int_0^\infty W_q(x + z) \frac{z}{r} \mathbb{P}(X_r \in dz),
$$

and we write $\Lambda = \Lambda^{(0)}$ when $q = 0$. Note that we can show

$$
\Lambda^{(q)}(0, r) = e^{qr}.
$$
We also denote the partial derivative of \( \Lambda^{(q)} \) with respect to \( x \) by

\[
\Lambda^{(q)'}(x, r) = \frac{\partial \Lambda^{(q)}}{\partial x}(x, r) = \int_0^\infty W'_q(x + z) \frac{z}{r} \mathbb{P}(X_r \in dz),
\]

For \( x \leq 0 \), we also have

\[
\Lambda'(x, r) = \Lambda^{(q)'}(x, r) - q \int_0^r \Lambda^{(q)'}(x, s) ds - qW_q(x).
\]

Also, we recall the following identity stated in [33] and related to scale functions: for \( p, q, x \geq 0 \),

\[
(s - p) \int_0^x W_p(x - y)W_q(y)dy = W_s(x) - W_p(x).
\]

Finally, we recall Kendall’s identity that provides the distribution of the first upward crossing of a specific level (see [7, Corollary VII.3]): on \((0, \infty) \times (0, \infty)\), we have

\[
\mathbb{P}(\tau_+^r \in dz) = z\mathbb{P}(X_r \in dz)dr.
\]

We refer the reader to [17] for more details on spectrally negative Lévy processes and fluctuation identities. More examples and numerical computations related to scale functions can be found in e.g., [16] and [36].

2.2. Main results.

2.2.1. Distribution of occupation times. We are now ready to state our main results. First, we provide an explicit expression for the distribution of \( \mathcal{O}^X_{e_\lambda} \), where, throughout this paper, \( e_\lambda \) is an exponential random variable with rate \( \lambda > 0 \) that is independent of the process \( X \).

**Theorem 1.** For \( \lambda > 0 \), \( x \in \mathbb{R} \) and \( y \geq 0 \),

\[
\mathbb{P}_x \left( \mathcal{O}^X_{e_\lambda} \in dy \right) = \left( 1 - \left( \frac{Z_\lambda(x)}{\Phi_\lambda} - \lambda \right) \right) \delta_0 \left( dy \right) + \lambda e^{-\lambda y} \left( \mathcal{B}^{(\lambda)}(x, y) - \lambda \int_0^y \mathcal{B}^{(\lambda)}(x, s) ds \right) dy + \lambda e^{-\lambda y} \left( Z_\lambda(x) - \frac{\lambda}{\Phi_\lambda} W_\lambda(x) \right) dy,
\]

where

\[
\mathcal{B}^{(\lambda)}(x, s) = \frac{\Lambda^{(\lambda)'}(x, s)}{\Phi_\lambda} - \Lambda^{(\lambda)}(x, s),
\]

and \( \delta_0(\cdot) \) is the Dirac mass at 0.

**Proof.** First, we observe that

\[
\mathbb{E}_x \left[ e^{-q \mathcal{O}^X_{e_\lambda}} \right] = \mathbb{P}_x \left( \mathcal{O}^X_{e_\lambda} < e_q \right) = \mathbb{P}_x \left( \kappa^q > e_\lambda \right) = 1 - \mathbb{E}_x \left[ e^{-\lambda \kappa^q} \right],
\]

where the Laplace transform of \( \kappa^q \) can be extracted from [4] (see also [2]) and it is given by

\[
\mathbb{E}_x \left[ e^{-\lambda \kappa^q} \right] = \frac{q Z_\lambda(x)}{q + \lambda} - \lambda \left( \Phi_{q+\lambda} - \Phi_\lambda \right) Z_\lambda(x, \Phi_{q+\lambda}).
\]

Thus, we obtain

\[
\mathbb{E}_x \left[ e^{-q \mathcal{O}^X_{e_\lambda}} \right] = \lambda \left( \frac{\Phi_{q+\lambda} - \Phi_\lambda}{\Phi_\lambda} \right) Z_\lambda(x, \Phi_{q+\lambda}) - \frac{q Z_\lambda(x)}{q + \lambda} + 1.
\]
Given that
\[ \mathbb{P}_x (\mathcal{O}^X_{e\lambda} = 0) = \mathbb{P}_x (\tau^-_0 > e\lambda) = 1 - Z_\lambda (x) + \frac{\lambda}{\Phi_\lambda} W_\lambda (x), \]
which is 0 if \( x < 0 \), we first rewrite (13) as
\[
\mathbb{E}_x \left[ e^{-q \mathcal{O}^X_{e\lambda}} \right] = \left\{ 1 - Z_\lambda (x) + \frac{\lambda}{\Phi_\lambda} W_\lambda (x) \right\} + \frac{\lambda}{\lambda + q} \frac{\Phi_{\lambda+q} - \Phi_\lambda}{\Phi_\lambda} Z_\lambda (x, \Phi_{\lambda+q}) + \frac{\lambda}{\lambda + q} Z_\lambda (x) - \frac{\lambda}{\Phi_\lambda} W_\lambda (x).
\]
By simple manipulations, the above expression can also be rewritten as
\[
\mathbb{E}_x \left[ e^{-q \mathcal{O}^X_{e\lambda}} \right] = 1 - Z_\lambda (x) + \frac{\lambda}{\Phi_\lambda} W_\lambda (x)
\]
\[
+ \frac{\lambda}{\lambda + q} \left( \frac{\Phi_{\lambda+q} - \Phi_\lambda}{\Phi_\lambda} Z_\lambda (x, \Phi_{\lambda+q}) + Z_\lambda (x) - \frac{\lambda + q}{\Phi_\lambda} W_\lambda (x) \right)
\]
\[
= 1 - Z_\lambda (x) + \frac{\lambda}{\Phi_\lambda} W_\lambda (x)
\]
\[
+ \frac{\lambda}{\lambda + q} \left( \Phi_{\lambda+q} Z_\lambda (x, \Phi_{\lambda+q}) - q W_\lambda (x) - Z_\lambda (x, \Phi_{\lambda+q}) + Z_\lambda (x) - \frac{\lambda}{\Phi_\lambda} W_\lambda (x) \right)
\]
\[
= 1 - Z_\lambda (x) + \frac{\lambda}{\Phi_\lambda} W_\lambda (x) + \frac{\lambda}{\lambda + q} \left( Z_\lambda (x) - \frac{\lambda}{\Phi_\lambda} W_\lambda (x) \right)
\]
\[
+ \lambda \left( 1 - \frac{\lambda}{\lambda + q} \right) \left( \Phi_{\lambda+q} Z_\lambda (x, \Phi_{\lambda+q}) - q W_\lambda (x) - Z_\lambda (x, \Phi_{\lambda+q}) \right). \quad (14)
\]
Using the following identities
\[
\frac{Z_\lambda (x, \Phi_{\lambda+q})}{q} = \int_0^\infty e^{-qy} (e^{-\lambda y} \Lambda (\lambda) (x, y)) \, dy, \quad (15)
\]
and
\[
\frac{\Phi_{\lambda+q} Z_\lambda (x, \Phi_{\lambda+q}) - q W_\lambda (x)}{q} = \int_0^\infty e^{-qy} (e^{-\lambda y} \Lambda (\lambda)' (x, y)) \, dy, \quad (16)
\]
which can be proved using Kendall’s identity [13] and Tonelli’s Theorem, leads to
\[
\mathbb{E}_x \left[ e^{-q \mathcal{O}^X_{e\lambda}} \right] = \left\{ 1 - Z_\lambda (x) + \frac{\lambda}{\Phi_\lambda} W_\lambda (x) \right\} + \frac{\lambda}{\lambda + q} \left( Z_\lambda (x) - \frac{\lambda}{\Phi_\lambda} W_\lambda (x) \right)
\]
\[
+ \lambda \left( 1 - \frac{\lambda}{\lambda + q} \right) \int_0^\infty e^{-qy} \left\{ e^{-\lambda y} \mathcal{B} (\lambda) (x, y) \right\} \, dy.
\]
Hence, by Laplace inversion, we obtain
\[
\mathbb{P}_x (\mathcal{O}^X_{e\lambda} \in dy) = \left\{ 1 - Z_\lambda (x) + \frac{\lambda}{\Phi_\lambda} W_\lambda (x) \right\} \delta_0 (dy)
\]
\[
+ \int_0^\infty e^{-qy} \left\{ \lambda e^{-\lambda y} \left( Z_\lambda (x) - \frac{\lambda}{\Phi_\lambda} W_\lambda (x) \right) \right\} \, dy
\]
\[
+ \int_0^\infty e^{-qy} \left\{ \lambda e^{-\lambda y} \left( \mathcal{B} (\lambda) (x, y) - \lambda \int_0^y \mathcal{B} (\lambda) (x, s) \, ds \right) \right\} \, dy.
\]
This ends the proof. ■
Note that when \( x = 0 \), using (6) and (7), Equation (10) reduces to
\[
\mathbb{P} \left( \mathcal{O}_t^X \in dy \right) = \frac{\lambda}{\Phi_\lambda} \left( W_\lambda (0) \delta_0 (dy) + e^{-\lambda r} \Lambda' (0, y) dy \right). \tag{17}
\]
Furthermore, letting \( \lambda \to 0 \) in (10), we obtain the following expression for the distribution of \( \mathcal{O}_\infty^X \).

**Corollary 2.** For \( x \in \mathbb{R}, \ y \geq 0 \) and \( \mathbb{E}[X_1] > 0 \),
\[
\mathbb{P}_x \left( \mathcal{O}_\infty^X \in dy \right) = \mathbb{E}[X_1] \left( W (x) \delta_0 (dy) + \Lambda'(x, y) dy \right). \tag{18}
\]

We point out that the terms involving the Dirac mass \( \delta_0 \) in (10) and (18) are
\[
\mathbb{P}_x \left( \tau_0^- > e_\lambda \right) = 1 - \left( Z_\lambda (x) - \frac{\lambda}{\Phi_\lambda} W_\lambda (x) \right),
\]
and
\[
\mathbb{P}_x \left( \tau_0^- = \infty \right) = \mathbb{E}[X_1] W (x),
\]
respectively.

**Remark 3.** Many drawdown quantities have drawn considerable interest in insurance contexts (see, e.g., Avram et al. [3] and Landriault et al. [19]). Recently, Baurdoux et al. [5] introduced the future drawdown extreme defined as
\[
\bar{D}_{s,t} = \sup_{0 \leq u \leq s} \inf_{u \leq w \leq t+s} (X_w - X_u),
\]
where \( s, t > 0 \). The infinite-horizon version is denoted by
\[
\bar{D}_s = \lim_{t \to \infty} \bar{D}_{s,t}.
\]
From [5, Corollary (ii)]}, we have
\[
\mathbb{P} \left( -\bar{D}_{s,t} < x \right) = \mathbb{E}_x \left[ e^{-q \mathcal{O}_s^X} \right] = \mathbb{E}[X_1] \frac{\Phi_q}{q} Z (x, \Phi_q).
\]

By inversion, using (18), we conclude that
\[
\mathbb{P} \left( -\bar{D}_s < x \right) = \mathbb{E}[X_1] \left( W (x) \delta_0 (ds) + \Lambda'(x, s) ds \right).
\]

Next, we consider two applications of Theorem 1.

2.3. **Applications.**

2.3.1. **Inverse occupation time.** The inverse occupation time with delay \( r > 0 \) is defined as
\[
\sigma_r = \inf \left\{ t > 0 : \mathcal{O}_t^X > r \right\}.
\]
The stopping time \( \sigma_r \) is deemed to occur at the first time the process \( X \) cumulatively stays below level 0 in excess of \( r \). Here, the parameter \( r \) can represent the insurer’s tolerance level for the surplus process to cumulatively stay below threshold 0. The finite-time probability of inverse occupation time is given by
\[
\mathbb{P}_x (\sigma_r \leq t) = \mathbb{P}_x \left( \mathcal{O}_t^X > r \right), \tag{19}
\]
while in the infinite-time horizon case
\[
\mathbb{P}_x (\sigma_r < \infty) = \mathbb{P}_x \left( \mathcal{O}_\infty^X > r \right). \tag{20}
\]
The Laplace transform of $\sigma_r$ is given by

$$E_x \left[ e^{-\lambda \sigma_r} \mathbf{1}_{\{\sigma_r < \infty\}} \right] = \mathbb{P}_x (\sigma_r < e\lambda) = \mathbb{P}_x \left( O_{\sigma_r}^X > r \right),$$

which can be readily obtained from Theorem 1. This result is stated without proof.

\textbf{Theorem 4.} For $r, \lambda > 0$ and $x \in \mathbb{R}$,

$$E_x \left[ e^{-\lambda \sigma_r} \mathbf{1}_{\{\sigma_r < \infty\}} \right] = e^{-\lambda r} \left( 1 - Z_{\lambda}(x) + \frac{\lambda}{\Phi_{\lambda}} W_{\lambda}(x) \right) - \lambda \int_0^r e^{-\lambda u} \left( \mathcal{B}^{(\lambda)}(x, u) - \lambda \int_u^\infty \mathcal{B}^{(\lambda)}(x, s) ds \right) du. \quad (22)$$

Using (18) (or letting $\lambda \to 0$ in (22)), we obtain the following expression for the probability of inverse occupation time.

\textbf{Corollary 5.} For $r > 0$, $x \in \mathbb{R}$ and $\mathbb{E}[X_1] > 0$,

$$\mathbb{P}_x (\sigma_r < \infty) = 1 - \mathbb{E}[X_1] \left( W(x) + \int_0^r \Lambda'(x, s) ds \right). \quad (23)$$

Finally, we point out that (23) reduces to $\mathbb{P} (\tau^\tau_0 < \infty) = 1 - \mathbb{E}[X_1] W(x)$ when $r \to 0$ and using also the fact that the stopping time $\sigma_r$ converges $\mathbb{P}_x$-a.s. to the time of classical ruin $\tau_0^\tau$ (see Proposition 3.3 in [14]).

2.3.2. \textit{Parisian ruin with exponential delays.} Another type of actuarial ruin with strong ties to the distribution of $O_X^\infty$ is the Parisian ruin with exponential delays defined as

$$\kappa^q = \inf \{ t > 0 \mid t - g_t > e_{q_t} \},$$

where $g_t = \sup \{ 0 \leq s \leq t : X_s \geq 0 \}$ and $e_{q_t}$ is an exponentially distributed random variable with rate $q > 0$. An expression for the probability of Parisian ruin with exponential delays was first given in [20] through the relation between the occupation time $O_X^\infty$ and $\kappa^q$, that is, for $q > 0$ and $x \in \mathbb{R}$,

$$\mathbb{P}_x (\kappa^q < \infty) = 1 - \mathbb{E}_x \left[ e^{-q \mathcal{O}_x^\infty} \right] = 1 - \left( \mathbb{E}[X_1] \right)_+ \frac{\Phi_{\lambda}}{q} Z(x, \Phi_{\lambda}), \quad (24)$$

where $(x)_+ = \max(x, 0)$. Given that from Proposition 3.4 in [14], it is known that $\kappa^q$ and $\sigma_{e_q}$ have the same distribution, one can readily obtain Equation (24) from (23) by replacing the delay $r$ by an exponential random time $e_q$. Indeed, we have

$$\mathbb{P}_x (\sigma_{e_q} < \infty) = \mathbb{P}_x (\kappa^q < \infty) = 1 - \mathbb{E}[X_1] \int_0^\infty q e^{-qr} \left( W(x) + \int_r^\infty \Lambda'(x, s) ds \right) dr = 1 - \mathbb{E}[X_1] \left( W(x) + \int_0^\infty e^{-qs} \Lambda'(x, s) ds \right) = 1 - \mathbb{E}[X_1] \frac{\Phi_{\lambda}}{q} Z(x, \Phi_{\lambda}),$$

where the last equality follows from identity (16).
2.4. Examples. In this subsection, we utilize Equation (10) to derive in a rather straightforward manner the (known) distribution of $O^X_t$ for two special cases of spectrally negative Lévy processes, namely the Brownian motion model and the Cramér-Lundberg process with exponential claims. For instance, the probability of inverse occupation time in (19) can then be computed for these two special cases.

2.4.1. Brownian risk process. Let $X_t = x + \mu t + \sigma B_t$, where $\mu > 0$, $\sigma > 0$ and $B = \{B_t, t \geq 0\}$ is a standard Brownian motion. For this process, the scale function and the right-inverse of the Laplace exponent are given by

$$W(x) = \frac{1}{\mu} \left(1 - e^{-2\mu x/\sigma^2}\right), \quad x \geq 0,$$

and

$$\Phi_\lambda = \left(\sqrt{\mu^2 + 2\lambda \sigma^2} - \mu\right)\sigma^{-2}, \quad \lambda > 0.$$

respectively. Also, since $X_t$ has a normal distribution with mean $\mu s$ and variance $s\sigma^2$,

$$\Lambda(x, s) = \left(\sigma e^{-\frac{\mu^2}{2\sigma^2} s} + N\left(\frac{\mu\sqrt{s}}{\sigma}\right)\right) \left(1 - e^{-\frac{2\mu}{\sigma^2} x}\right) + e^{-\frac{2\mu}{\sigma^2} x},$$

and consequently,

$$\Lambda'(x, s) = \frac{2}{\sigma^2} e^{-\frac{2\mu}{\sigma^2} x} \left(\sigma e^{-\frac{\mu^2}{2\sigma^2} s} \sqrt{2\pi s} - \mu \tilde{N}\left(\frac{\mu\sqrt{s}}{\sigma}\right)\right), \quad (25)$$

where $N = 1 - \tilde{N}$ is the cumulative distribution function of the standard normal distribution. One can easily check that

$$\frac{e^{-\lambda s}}{\Phi_\lambda} = \int_0^\infty e^{-\lambda t} \left(\mu + \frac{\sigma e^{-\left(\frac{\mu^2}{2\sigma^2} t\right) - s}}{\sqrt{2\pi (t - s)}} - \mu \tilde{N}\left(\frac{\mu\sqrt{t - s}}{\sigma}\right)\right) dt. \quad (26)$$

Since $X$ has paths of unbounded variation (i.e., $W_\lambda(0) = 0$) and from (25) at $x = 0$, we have

$$\lambda^{-1} P \left(O^X_{\nu_\lambda} \in ds\right) = \frac{e^{-\lambda s}}{\Phi_\lambda} \Lambda'(0, s) = \frac{e^{-\lambda s}}{\Phi_\lambda} \frac{2}{\sigma^2} \left(\sigma e^{-\left(\frac{\mu^2}{2\sigma^2} s\right)} \sqrt{2\pi s} - \mu \tilde{N}\left(\frac{\mu\sqrt{s}}{\sigma}\right)\right) ds.$$

Using Laplace inversion, we finally obtain

$$P \left(O^X_t \in ds\right) = \frac{2}{\sigma^2} \left\{\sigma e^{-\left(\frac{\mu^2}{2\sigma^2} s\right)} \sqrt{2\pi s} - \mu \tilde{N}\left(\frac{\mu\sqrt{s}}{\sigma}\right)\right\} \times \left\{\mu + \frac{\sigma e^{-\left(\frac{\mu^2}{2\sigma^2} (t-s)\right)}}{\sqrt{2\pi (t-s)}} - \mu \tilde{N}\left(\frac{\mu\sqrt{t - s}}{\sigma}\right)\right\} ds,$$

which corresponds to the well-known result of Akahori [1]. For $\sigma = 1$ and $\mu = 0$ and integrating the last expression over $[r, \infty)$, one obtains Paul Lévy’s arcsine law, that is,

$$P \left(O^X_t > r\right) = 1 - \frac{2}{\pi} \arcsin\left(\sqrt{\frac{r}{t}}\right),$$

with $0 < r < t$. 

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2.4.2. Cramér-Lundberg process with exponential claims. Let $X$ be a Cramér-Lundberg risk process with exponentially distributed claims, i.e.

$$X_t = x + ct - \sum_{i=1}^{N_t} C_i,$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity $\eta > 0$, and $\{C_1, C_2, \ldots\}$ are independent and exponentially distributed random variables with parameter $\alpha$, also independent of $N$. The scale function of $X$ is known to be

$$W(x) = \frac{1}{c - \eta/\alpha} \left(1 - \frac{\eta}{\alpha} e^{(2 - \alpha)x}\right),$$

and the right-inverse has the closed-form expression

$$\Phi \lambda = \frac{1}{2c} \left(\lambda - \eta + \alpha \sqrt{\lambda^2 + 4\alpha c \lambda}\right),$$

As noted in [31], we have

$$\mathbb{P}\left(\sum_{i=1}^{N_s} C_i \in dy\right) = e^{-\eta s} \left(\delta_0(dy) + e^{-\alpha y} \sum_{m=0}^{\infty} \frac{(\alpha \eta s)^{m+1}}{m!(m+1)!} y^m dy\right),$$

and consequently

$$\int_0^\infty z \mathbb{P}(X_s \in dz) = \int_0^cs e^{-\eta s} \left(\delta_0(cs - dz) + e^{-\alpha (cs - z)} \sum_{m=0}^{\infty} \frac{(\alpha \eta s)^{m+1}}{m!(m+1)!} (cs - z)^m dz\right)$$

$$= e^{-\eta s} \left(cs + \sum_{m=0}^{\infty} \frac{(\eta s)^{m+1}}{m!(m+1)!} cs \Gamma(m + 1, cs \alpha) - \frac{1}{\alpha} \Gamma(m + 2, cs \alpha)\right),$$

where $\Gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$ is the incomplete gamma function, and

$$\frac{\eta}{\alpha} \int_0^\infty e^{(2 - \alpha)z} z \mathbb{P}(X_s \in dz) = \int_0^\infty z \mathbb{P}(X_s \in dz) - (c - \eta/\alpha)s.$$

Then,

$$\Lambda'(0, s) = \frac{\alpha}{c} e^{-\eta s} \left(c + \sum_{i=0}^{\infty} \frac{\eta^{m+1} \alpha^m}{m!(m+1)!} \left(cs \Gamma(m + 1, cs \alpha) - \frac{1}{\alpha} \Gamma(m + 2, cs \alpha)\right)\right),$$

and

$$\frac{1}{\Phi \lambda c} = \frac{1}{\sqrt{\lambda^2 + 4\alpha c \lambda - (c \alpha - \lambda - \eta)}}.$$

Since $X$ is of bounded variation paths (i.e., $W_\lambda(0) > 0$), we have

$$\lambda^{-1} \mathbb{P}(O_{\lambda}^X \in ds) = \frac{1}{\Phi \lambda} W_\lambda(0) \delta_0(ds) + \frac{e^{-\lambda s}}{\Phi \lambda} \Lambda'(0, s) ds$$

$$= \frac{1}{\Phi \lambda c} \delta_0(ds) + \frac{e^{-\lambda s}}{c \Phi \lambda} e^{-\eta s}$$

$$+ \frac{\alpha e^{-\lambda s}}{c \Phi \lambda} e^{-\eta s} \sum_{i=0}^{\infty} \frac{\eta^{m+1} \alpha^m}{m!(m+1)!} \left(cs \Gamma(m + 1, cs \alpha) - \frac{1}{\alpha} \Gamma(m + 2, cs \alpha)\right).$$
As shown in [14], we have
\[ \frac{1}{\Phi_{\lambda c}} = \int_0^\infty e^{-\lambda t} a_t dt, \]
where
\[ a_t = \left( 1 - \frac{\lambda}{c} \right) + \frac{2\lambda}{\pi} e^{-(\lambda+c) t} \int_{-1}^1 \frac{1}{\sqrt{1-u^2} e^{-2\sqrt{c\lambda t}}} du. \]
Thus, we obtain the following expression for the distribution of the occupation time \( O_t^X \) which is consistent with [14]: for \( t > 0 \),
\[ \mathbb{P} (O_t^X \in ds) = a_t \delta_0 (ds) + a_{t-s} (\lambda - c\alpha (1-a_s)) 1_{(0,t)} (s) ds. \]

3. OCCUPATION TIMES OF THE REFRAC TED LÉVY PROCESS

We now extend our results to a refracted spectrally negative Lévy process \( U = \{U_t, t \geq 0\} \) at level 0 defined as
\[ U_t = X_t - \delta \int_0^t 1_{\{U_s > 0\}} ds, \quad t \geq 0, \]
where \( \delta \geq 0 \) is the refraction parameter. As discussed in Kyprianou and Loeffen [18], such process exists and it is a skip-free upward strong Markov process. Above 0, the surplus process \( U \) evolves as \( Y = \{Y_t = X_t - \delta t, t \geq 0\} \) for which the Laplace exponent is given by
\[ \lambda \mapsto \psi(\lambda) - \delta \lambda, \]
with right-inverse \( \varphi_q = \sup \{ \lambda \geq 0 \mid \psi(\lambda) - \delta \lambda = \varphi \} \). Then, for each \( q \geq 0 \), we define the scale functions of \( Y \), namely \( \mathbb{W}_q \) and \( \mathbb{Z}_q \), by
\[ \int_0^\infty e^{-\lambda y} \mathbb{W}_q (y) dy = \frac{1}{\psi_q (\lambda) - \delta \lambda}, \quad \lambda > \varphi_q, \]
and
\[ \mathbb{Z}_{\delta, q} (x, \theta) = e^{\theta x} \left( 1 - (\psi_q (\theta) - \delta \theta) \int_0^x e^{-\theta z} \mathbb{W}_q (z) dz \right). \]
We also have
\[ \mathbb{Z}_q (x) = \mathbb{Z}_{\delta, q} (x, 0) = 1 + q \int_0^x \mathbb{W}_q (y) dy. \]
We denote the delayed \( q \)-scale function of \( Y \) by
\[ \Lambda_q (x, s) = \int_0^\infty \mathbb{W}_q (x + z) \frac{z}{s} \mathbb{P} (X_s \in dz). \]
In [18] and [35], many fluctuation identities for the refracted process are expressed in terms of the scale function of \( U \), that is, for \( q \geq 0 \) and for \( x, z \in \mathbb{R} \), set
\[ w(q)(x; z) = W_q (x - z) + \delta 1_{\{x \geq 0\}} \int_0^x \mathbb{W}_q (x - y) W_q (y - z) dy. \tag{27} \]
Note that when \( x < 0 \), we have
\[ w(q)(x; z) = W_q (x - z), \]
and when \( q = 0 \), we will write \( w(0)(x; z) = w(x; z) \). First, for \( a \in \mathbb{R} \), we define the following first-passage stopping times:
\[ \nu^-_a = \inf \{ t > 0 : Y_t < a \} \quad \text{and} \quad \nu^+_a = \inf \{ t > 0 : Y_t \geq a \} \]
\[ \kappa^-_a = \inf \{ t > 0 : U_t < a \} \quad \text{and} \quad \kappa^+_a = \inf \{ t > 0 : U_t \geq a \}. \]
Since $Y$ is also a spectrally negative Lévy process, the identities for $X$ also hold for $Y$. For example, for $x \in \mathbb{R}$,

$$
E_x \left[ e^{-\lambda \kappa_0^-} r^{Y_0^-} \mathbf{1}_{\{\kappa_0^- < \infty\}} \right] = Z_{\delta,\lambda} (x, r) - \left( \frac{\psi_\lambda (r) - \delta r}{r - \varphi_\lambda} \right) W_\lambda (x). \tag{28}
$$

We denote by $\kappa_{U}^q$, the time of Parisian ruin with exponential delays for the refracted Lévy process $U$

$$
\kappa_{U}^q = \inf \left\{ t > 0 \mid t - g^U_t > e^{0U}_q \right\}.
$$

We have the following new results for the Laplace transforms of $\kappa_{U}^q$ and $\mathcal{O}_{e\lambda}^U$.

**Lemma 6.** For $q, \lambda > 0$ and $x \in \mathbb{R}$,

$$
E_x \left[ e^{-\lambda \kappa_{U}^q} \mathbf{1}_{\{\kappa_{U}^q < \infty\}} \right] = \frac{q}{\lambda + q} \left( Z_{\lambda} (x) - \frac{\lambda (\Phi_{q + \lambda} - \varphi_\lambda)}{(q - \delta \Phi_{q + \lambda}) \varphi_\lambda} Z_{\delta,\lambda} (x, \Phi_{q + \lambda}) \right), \tag{29}
$$

and consequently,

$$
E_x \left[ e^{-q \mathcal{O}_{e\lambda}^U} \right] = \frac{q \lambda (\Phi_{q + \lambda} - \varphi_\lambda)}{(\lambda + q) (q - \delta \Phi_{q + \lambda}) \varphi_\lambda} Z_{\delta,\lambda} (x, \Phi_{q + \lambda}) - \frac{q Z_{\lambda} (x)}{q + \lambda} + 1. \tag{30}
$$

**Proof.** For $x < 0$, using the strong Markov property of $U$ and the fact that $U_{\kappa_0^+} = 0$ on $\{\kappa_0^- < \infty\}$, we have

$$
E_x \left[ e^{-\lambda \kappa_{U}^q} \mathbf{1}_{\{\kappa_{U}^q < \infty\}} \right] = E_x \left[ e^{-\lambda \kappa_0^-} \mathbf{1}_{\{\kappa_0^- < \infty\}} \right] + E_x \left[ e^{-(q + \lambda) \kappa_0^+} \right] E \left[ e^{-\lambda \kappa_{U}^q} \mathbf{1}_{\{\kappa_{U}^q < \infty\}} \right].
$$

Since $\{X_t, t < \tau_0^+\}$ and $\{U_t, t < \kappa_0^+\}$ have the same distribution with respect to $\mathbb{P}_x$ when $x < 0$, we further have

$$
E_x \left[ e^{-\lambda \kappa_{U}^q} \mathbf{1}_{\{\kappa_{U}^q < \infty\}} \right] = E_x \left[ e^{-\lambda \kappa_0^-} \mathbf{1}_{\{\tau_0^- > e_0\}} \right] + E_x \left[ e^{-(q + \lambda) \tau_0^+} \right] E \left[ e^{-\lambda \kappa_{U}^q} \mathbf{1}_{\{\kappa_{U}^q < \infty\}} \right].
$$

For $x \geq 0$, using the strong Markov property of $U$, we get

$$
E_x \left[ e^{-\lambda \kappa_{U}^q} \mathbf{1}_{\{\kappa_{U}^q < \infty\}} \right] = E_x \left[ e^{-\lambda \kappa_0^-} \mathbf{1}_{\kappa_0^- < \infty} \right] e^{-(q + \lambda) \kappa_0^+} \mathbf{1}_{\kappa_0^- < \infty} \right] + E_x \left[ e^{-\lambda \kappa_0^-} \mathbf{1}_{\kappa_0^- < \infty} \right] e^{-(q + \lambda) \tau_0^+} \mathbf{1}_{\kappa_0^- < \infty} \right] \right] \right] 
$$

$$
= \frac{q}{\lambda + \lambda} \left( E_x \left[ e^{-\lambda \kappa_0^-} \mathbf{1}_{\kappa_0^- < \infty} \right] - E_x \left[ e^{-\lambda \kappa_0^- + \Phi_{q + \lambda} Y_0^-} \mathbf{1}_{\kappa_0^- < \infty} \right] \right) 
$$

$$
- E_x \left[ e^{-\lambda \kappa_0^- + \Phi_{q + \lambda} Y_0^-} \mathbf{1}_{\kappa_0^- < \infty} \right] \right] \right] 
$$

$$
= \frac{q}{\lambda + \lambda} \left( E_x \left[ e^{-\lambda \kappa_0^-} \mathbf{1}_{\kappa_0^- < \infty} \right] - E_x \left[ e^{-\lambda \kappa_0^- + \Phi_{q + \lambda} Y_0^-} \mathbf{1}_{\kappa_0^- < \infty} \right] \right) 
$$

where in the last equality we used the fact that $\{Y_t, t < \nu_0^-\}$ and $\{U_t, t < \kappa_0^+\}$ have the same distribution under $\mathbb{P}_x$. Note that the above expression holds for all $x \in \mathbb{R}$. 

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Now, we assume $X$ and $Y$ have paths of bounded variation. Solving for $\mathbb{E} \left[ e^{-\lambda \kappa_0^y} \mathbf{1}_{\{\kappa_0^y < \infty\}} \right]$ and using (28), we get

\[
\mathbb{E} \left[ e^{-\lambda \kappa_0^y} \mathbf{1}_{\{\kappa_0^y < \infty\}} \right] = \frac{q}{q + \lambda} \left( \mathbb{E} \left[ e^{-\lambda \kappa_0^x} \mathbf{1}_{\{\kappa_0^x < \infty\}} \right] - \mathbb{E} \left[ e^{-\lambda \kappa_0^x + \Phi_{\lambda + q} Y_{\kappa_0^x}^y} \mathbf{1}_{\{\kappa_0^x < \infty\}} \right] \right) \frac{1}{1 - \mathbb{E} \left[ e^{-\lambda \kappa_0^x + \Phi_{\lambda + q} Y_{\kappa_0^x}^y} \mathbf{1}_{\{\kappa_0^x < \infty\}} \right]} \]

Substituting (28) and (32) into (31), we have

\[
\mathbb{E} \left[ e^{-\lambda \kappa_0^y} \mathbf{1}_{\{\kappa_0^y < \infty\}} \right] = \frac{q}{q + \lambda} \left( \frac{\lambda}{(\lambda + q) \varphi_\lambda (q - \delta \Phi_{q + \lambda})} \mathbb{E}_x \left[ \mathbf{1}_{\{\kappa_0^x < \infty\}} \right] \right)
\]

The case where $X$ has paths of unbounded variation follows using the same approximating procedure as in (18) (see also (13)).

Finally, Equation (30) is immediate using the identity

\[
\mathbb{E}_x \left[ e^{-\lambda \kappa_0^y} \right] = 1 - \mathbb{E}_x \left[ e^{-\lambda \kappa_0^y} \mathbf{1}_{\{\kappa_0^y < \infty\}} \right].
\]

Using similar techniques as in the proof of Theorem 11 we obtain the following expression for the distribution of $\mathcal{O}_{\delta,\lambda}^U$. The result is stated without proof.

**Theorem 7.** For $\lambda > 0$, $x \in \mathbb{R}$ and $y \geq 0$,

\[
\mathbb{P}_x \left( \mathcal{O}_{\delta,\lambda}^U \in dy \right) = \left( 1 - \left( Z_\lambda (x) - \frac{\lambda}{\varphi_\lambda} Z_\lambda (x) \right) \right) \delta_0 (dy) + \lambda e^{-\lambda y} \left( B_{\delta}^{\lambda} (x, y) - \lambda \int_0^y B_{\delta}^{(\lambda)} (x, s) ds \right) dy + \lambda e^{-\lambda y} \left( Z_\lambda (x) - \frac{\lambda}{\varphi_\lambda} Z_\lambda (x) \right) dy,
\]

(33)

(34)
where
\[
B^{(\lambda)}_\delta(x, s) = \frac{\Lambda^{(\lambda)\prime}(x, s)}{\varphi_\lambda} - \Lambda^{(\lambda)}_\delta(x, s).
\]

We denote the inverse occupation time of the refracted process \(U\) by
\[
\sigma^U_r = \inf \{ t > 0 : O^U_t > r \},
\]
and for which we obtain the following Laplace transform.

\textbf{Theorem 8.} For \(r, \lambda > 0\) and \(x \in \mathbb{R}\),
\[
\mathbb{E}_x \left[ e^{-\lambda \sigma^U_r} 1_{\{\sigma^U_r < \infty\}} \right] = e^{-\lambda x} \left( 1 - Z_\lambda(x) + \frac{\lambda}{\varphi_\lambda} \mathbb{W}_\lambda(x) \right) - \lambda \int_0^r e^{-\lambda u} \left( B^{(\lambda)}_\delta(x, u) - \lambda \int_0^u B^{(\lambda)}_\delta(x, s) \, ds \right) \, du. \tag{35}
\]

We also obtain the following expression of the probability of inverse occupation time for the refracted process \(U\).

\textbf{Corollary 9.} For \(r > 0\), \(x \in \mathbb{R}\) and \(E[X_1] > \delta\),
\[
\mathbb{P}_x(\sigma^U_r < \infty) = 1 - (E[X_1] - \delta) \left( \mathbb{W}(x) + \int_0^r \Lambda^\prime_\delta(x, s) \, ds \right). \tag{36}
\]

It is a trivial exercise to show that when \(\delta = 0\), the results reduced to those given in Section 2.

\textbf{Remark 10.} The above expression can also be expressed as follows,
\[
\mathbb{P}_x(\sigma^U_r < \infty) = 1 - (E[X_1] - \delta) \left( \frac{w(x; 0)}{1 - \delta W(0)} + \int_0^r \Lambda^\prime_\delta(x, s) \, ds \right), \tag{37}
\]
which is due to the following useful identity relating different scale functions and taken from [35], that is, for \(p, q \geq 0\) and \(x \in \mathbb{R}\),
\[
(q - p) \int_0^x \mathbb{W}_p(x - y) W_q(y) \, dy = W_q(x) - \mathbb{W}_p(x) + \delta \left( W_q(0) \mathbb{W}_p(x) + \int_0^x \mathbb{W}_p(x - y) W_q'(y) \, dy \right),
\]
and for which we consider \(p = q = 0\). Note that when \(\delta = 0\), we recover the spectrally negative analogue in [36]. Letting \(r \rightarrow 0\) in (37), we recover the classical probability of ruin of \(U\)
\[
\mathbb{P}_x(\kappa^-_0 < \infty) = 1 - \frac{(E[X_1] - \delta)}{1 - \delta W(0)} w(x; 0).
\]

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