NONCOMMUTATIVE IMAGES OF COMMUTATIVE SPECTRA

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Abstract. We initiate a unified, axiomatic study of noncommutative algebras $R$ whose prime spectra are, in a natural way, finite unions of commutative noetherian spectra. Our results illustrate how these commutative spectra can be functorially “sewn together” to form $\text{Spec } R$. In particular, we construct a bimodule-determined functor $\text{Mod } Z \to \text{Mod } R$, for a suitable commutative noetherian ring $Z$, from which there follows a finite-to-one, continuous surjection $\text{Spec } Z \to \text{Spec } R$. Algebras satisfying the given axiomatic framework include PI algebras finitely generated over fields, noetherian PI algebras, enveloping algebras of complex finite dimensional solvable Lie algebras, standard generic quantum semisimple Lie groups, quantum affine spaces, quantized Weyl algebras, and standard generic quantizations of the coordinate ring of $n \times n$ matrices. In all of these examples (except for the non-finitely-generated noetherian PI algebras), $Z$ is finitely generated over a field, and the constructed map of spectra restricts to a surjection $\text{Max } Z \to \text{Prim } R$.

1. Introduction

1.1. It is well known that the prime ideal theories of the following noncommutative algebras have many of the basic properties found in commutative settings:
   (a) noetherian PI algebras and PI algebras finitely generated over a field (see §3 below),
   (b) enveloping algebras of finite dimensional completely solvable Lie algebras (§4),
   (c) standard generic quantized coordinate rings of semisimple Lie groups (§5),
   (d) certain other generic quantized coordinate rings, including quantized Weyl algebras and the standard quantized coordinate rings of $n \times n$ matrices (§6).

Moreover, a longstanding (essentially achieved) goal for all of these algebras has been the parameterization of their prime and primitive ideals using algebraic geometric data.

Our aim in this paper is to present, and begin the study of, an abstract axiomatic setting that includes all of these examples. In our main result, given in (2.18), we show that the prime spectra occurring within this framework are continuous, functorial, finite-to-one images of spectra of commutative noetherian rings.

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1.2. Our analysis throughout is based on the deep structure theories available for the classes of algebras listed in (a), (b), (c), and (d). These theories encompass decades of work, by legions of mathematicians. The reader unfamiliar with any of the relevant material is directed, for example, to the texts [2; 4; 9; 13; 16; 20].

1.3. The specific algebras $R$ listed in (1.1) all have the following in common: Their prime spectra are unions of finitely many locally closed subsets (in the Jacobson/Zariski topology), each homeomorphic to the spectrum of some commutative noetherian ring, and each naturally homeomorphic to a localization of a factor of $R$. Hypotheses A, B, and C (and stronger variants), presented in §2, specify these properties in detail. A further hypothesis D, under which $R$ satisfies the Nullstellensatz, allows applications to primitive ideals.

1.4. To more precisely but still briefly summarize our main results, stated in (2.18), suppose that $R$ satisfies the hypotheses A, B, and C (referred to in the preceding paragraph and exactly specified in §2). Then there is a homomorphism from $R$ into an algebra $T$, whose ideals are completely determined by their intersection with its noetherian center $Z$, such that

$$
\theta : \text{Spec } Z \xrightarrow{q \mapsto \text{ann } \left( (Z/q) \otimes^Z T \right)_R} \text{Spec } R
$$

is a continuous, finite-to-one, surjection. In cases (c) and (d) of (1.1), $\theta$ is bijective. In all of the cases (a), (b), (c), and (d), except for non-finitely-generated noetherian PI algebras, we can assume both that $Z$ is finitely generated over the ground field and that $\theta$ restricts to a surjection from $\text{Max } Z$ onto $\text{Prim } R$.

Note that $\theta$ is a consequence of the functor

$$
\text{Mod } Z \xrightarrow{- \otimes^Z T} \text{Mod } R.
$$

In the approach to noncommutative algebraic geometry developed in [19, 22] (for example), $\text{Mod } R$ is viewed as a category of quasi-coherent sheaves on a not-explicitly-defined noncommutative scheme, and the preceding functor is an affine map from $\text{Mod } Z$ into $\text{Mod } R$. Recall, using a theorem of Serre, that $\text{Mod } Z$ can be identified with the category of quasi-coherent sheaves on the commutative scheme $\text{Spec } Z$.

Applications of the above results to the specific classes of algebras listed in (1.1) are given in sections 3, 4, 5, and 6.

1.5. Let $V$ be a topological space that is a (not necessarily disjoint) union of subspaces $V_1, \ldots, V_t$, each homeomorphic to the prime spectrum of a commutative noetherian ring. The coproduct $V_1 \sqcup \cdots \sqcup V_t$ (i.e., the disjoint union equipped with the topology induced from the individual spaces $V_\ell$) is homeomorphic to the prime spectrum of a commutative noetherian ring (i.e., the direct product of the rings corresponding to the $V_\ell$). Also, there is a continuous surjection from $V_1 \sqcup \cdots \sqcup V_t$ onto $V$. So, the fact that the spectra we are considering in this paper are continuous images of commutative noetherian spectra is, in essence, only an elementary afterthought. What is new in our results are the explicitly described ring homomorphisms $R \to T$ and functors $\text{Mod } Z \to \text{Mod } R$ that produce these continuous surjections.
1.6. The continuous maps presented in (2.18) are not, in general, topological quotient maps; see (4.3).

1.7. If $R$ is the quantized coordinate ring of affine $n$-space, over an algebraically closed field $k$, and $-1$ is not included in the multiplicative group generated by the quantizing parameters, then $\text{Spec } R$ is a topological quotient of $\text{Spec } k[x_1, \ldots, x_n]$; see [6]. For more general quantizations $\mathcal{O}_q(V)$ of coordinate rings $\mathcal{O}(V)$, Goodearl has conjectured that $\text{Spec } \mathcal{O}_q(V)$ is a topological quotient of $\text{Spec } \mathcal{O}(V)$; see [5]. If $R$ is the enveloping algebra of a finite dimensional complex solvable Lie algebra $\mathfrak{g}$, then there is a topological quotient map from $\mathfrak{g}^*$ onto $\text{Prim } R$, given by the Dixmier map [4; 15].

1.8. In (2.16) we note that our hypotheses A, B, C, and D imply the Dixmier-Moeglin equivalence, following arguments similar to those found in [7].

1.9. In [3] an axiomatic framework is developed for certain quantum algebras. Our approach here can be viewed as a further generalization of this framework.

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2. The General Theory

Throughout this section, $R$ will denote an associative ring with identity.

2.1 Preliminaries. Let $\Lambda$ be a ring, and let $I$ be an ideal of $\Lambda$.

(i) The center of $\Lambda$ will be denoted $Z(\Lambda)$. The set of elements of $\Lambda$ regular modulo $I$ will be denoted $C_\Lambda(I)$.

(ii) Recall that $n \in \Lambda$ is normal modulo $I$ exactly when $\Lambda.n = n.\Lambda + I$, and that $c \in \Lambda$ is central modulo $I$ exactly when $a.c = c.a + I$ for all $a \in \Lambda$. When $I = 0$, elements normal modulo $I$ are normal (and elements central modulo $I$ are, of course, central). Also recall that if $n$ is a normal element of $\Lambda$, and if $Q$ is a prime ideal of $\Lambda$, then $n$ is regular modulo $Q$ if and only if $n \notin Q$. In particular, a nonzero normal element of a prime ring is regular.

(iii) (Cf., e.g., [9, Chapter 10].) A multiplicatively closed subset $D$ of $\Lambda$ is a right denominator set [9, p. 168–169] provided $D$ is a right Ore set [9, p. 82] and the canonical right-Ore-localization map $\Lambda \to \Lambda D^{-1}$ is injective [9, p. 167]. If $n$ is a normal and regular element of $\Lambda$, then it is not hard to show that $\{1, n, n^2, \ldots \}$ is a right (and left) denominator set of $\Lambda$. Localizations at right denominator sets of regular elements will be referred to as classical right quotient rings.

(iv) Let $\varphi: \Lambda \to \Lambda'$ be a ring homomorphism. For $a \in \Lambda$ and $b \in \Lambda'$, we will use $a.b$ to denote $\varphi(a).b$, and we will use $b.a$ to denote $b.\varphi(a)$. Adapting standard terminology to our uses below, we will say that the homomorphism is normalizing if $\Lambda'$ is generated as a left $\Lambda$-module by elements $b$ such that $\Lambda.b = b.\Lambda$, and we will say that the homomorphism is centralizing if $\Lambda'$ is generated as a left $\Lambda$-module by elements $c$ such that $a.c = c.a$ for all $a \in \Lambda$.

(v) We will use $\text{Spec } \Lambda$, $\text{Prim } \Lambda$, and $\text{Max } \Lambda$ to denote, respectively, the sets of prime, right primitive, and maximal ideals of $\Lambda$; throughout this note, each of these sets will be...
equipped with the Zariski/Jacobson topology. In particular, the closed subsets of \( \text{Spec} \Lambda \) will have the form
\[
V(E) = V_\Lambda(E) = \{ P \in \text{Spec} \Lambda \mid P \supseteq E \},
\]
for subsets \( E \) of \( \Lambda \). Analogous notation will be used for closed subsets of \( \text{Prim} \Lambda \) and \( \text{Max} \Lambda \). We will use \textit{primitive} to mean “right primitive.”

(vi) For an ideal \( I \) of \( \Lambda \), we will let \( \sqrt{I} \) denote the prime radical of \( I \) (i.e., the intersection of all of the prime ideals containing \( I \)).

(vii) We will let \( k \) denote an arbitrary field.

2.2 Setup and Notation \((I_\ell, J_\ell, X_\ell, S_\ell, S, Y_\ell, Y, \mu)\). We now begin to describe our general setup. The notation developed here will remain in effect for the rest of this section.

(i) Choose ideals \( I_1, J_1, \ldots, I_t, J_t \) of \( R \) such that
\[
\text{Spec} R = X_1 \cup \cdots \cup X_t,
\]
where
\[
X_\ell = V(I_\ell) \setminus V(J_\ell),
\]
for \( \ell = 1, \ldots, t \). Note that we are not requiring \( X_1, \ldots, X_t \) to be (pairwise) disjoint. Equip each \( X_\ell \) with the subspace topology.

(ii) For \( 1 \leq \ell \leq t \), let
\[
S_\ell = R/I_\ell,
\]
and set
\[
S = S_1 \times \cdots \times S_t.
\]
Let \( e_1, \ldots, e_t \) denote the canonical central idempotents corresponding to this ring product. We obtain a ring homomorphism
\[
\mu: R \rightarrow S.
\]
The standard theory of finite centralizing extensions (cf., e.g., [16, Chapter 10]) ensures that the function
\[
\mu^*: \text{Spec} S \rightarrow \mu^{-1}(Q) \rightarrow \text{Spec} R
\]
is continuous, with image \( \text{Spec} R \), since \( S = R.e_1 + \cdots + R.e_t \), and since \( \ker \mu \) is contained within the prime radical of \( R \). Continuity and image can also be directly derived from the given setup.

(iii) Identify \( \text{Spec} S \), homeomorphically, with the disjoint union
\[
\text{Spec} S_1 \sqcup \cdots \sqcup \text{Spec} S_t,
\]
equipped with the topology induced from the spaces \( \text{Spec} S_\ell \). Note that each \( X_\ell \) is naturally homeomorphic to the complement in \( \text{Spec} S_\ell \) of
\[
V_{S_\ell}(J_\ell + I_\ell/I_\ell).
\]
We will use \( Y_\ell \) to denote this complement. Setting \( Y = Y_1 \sqcup \cdots \sqcup Y_t \), and equipping \( Y \) with the topology induced from the spaces \( Y_1, \ldots, Y_t \), we may view \( Y \) as a subspace of \( \text{Spec} S \).

(Note that \( Y \) need not be homeomorphic to \( \text{Spec} R \), even when \( X_1, \ldots, X_t \) are disjoint, because inclusions among prime ideals in \( \text{Spec} R \) may not lift to inclusions among prime ideals in \( Y \).)

(iv) Let \( P \) be a prime ideal of \( R \). Then the preimage of \( P \) under \( \mu^* \) is exactly the finite (and nonempty) set of kernels in \( S \) of the maps,

\[
S \xrightarrow{\text{projection}} S_\ell = R/I_\ell \xrightarrow{\text{natural surjection}} R/P,
\]

for all \( \ell \) such that \( I_\ell \subseteq P \). Furthermore, it is not hard to see that the preimage of \( P \) under \( \mu^* \) intersects \( Y \) nontrivially.

(v) Now consider the restriction

\[
\mu^*: Y \to \text{Spec} R.
\]

This function is continuous, by (ii), and surjective, by (iv).

(vi) Note that the restriction of \( \mu^* \) in (v) is bijective if \( X_1, \ldots, X_t \) are disjoint.

2.3. For a prime ideal \( Q \) of \( S \), it is easy to see that \( \mu^*(Q) \) is primitive if and only if \( Q \) is primitive, and that \( \mu^*(Q) \) is maximal if and only if \( Q \) is maximal. (These conclusions also follow from the standard theory of finite centralizing extensions.) Therefore, all of the statements in (2.2) remain valid if the word “prime” is replaced with the word “primitive” and if “Spec” is replaced with “Prim.” We can make a similar substitution with maximal ideals and maximal spectra.

2.4 Additional Setup and Notation \((D_\ell, T_\ell, T, \nu, \eta, Z_\ell, Z)\). The following additional notation – to be used in conjunction with later hypotheses – will be retained for the remainder of this section: Let \( D_\ell \) be a right denominator set of \( S_\ell \), for each \( \ell \), and let \( T_\ell \) be the right Ore localization of \( S_\ell \) at \( D_\ell \). Set \( T = T_1 \times \cdots \times T_t \). (Note that \( T \) is itself an Ore localization of \( S \).) Let \( \nu: S \to T \) be the natural product homomorphism, and let \( \eta: R \to T \) be the composition of \( \mu \) with \( \nu \).

For each \( \ell \), let \( Z_\ell \) denote the center of \( T_\ell \), and identify \( Z_1 \times \cdots \times Z_t \) with the center \( Z \) of \( T \).

2.5. We now begin to introduce the hypotheses that will guide our analysis. We will always explicitly indicate when they are in effect.

**Hypothesis A.** \( T_\ell \) is right noetherian for all \( \ell \). Consequently, \( T \) is right noetherian.

**Hypothesis B.** \( Y_\ell = \{ Q \in \text{Spec} S_\ell \mid D_\ell \subseteq C_{S_\ell}(Q) \} \), for all \( \ell \).

2.6. Assume hypotheses A and B. (i) It follows (e.g.) from [9, 10.17, 10.18] that there is a lattice isomorphism

\[
\{ \text{semiprime ideals of } T_\ell \} \to \{ \text{semiprime ideals } N \text{ of } S_\ell \text{ such that } D_\ell \subseteq C_{S_\ell}(N) \},
\]
mapping each semiprime ideal of $T_\ell$ to its natural preimage in $S_\ell$. When we restrict the preceding function to prime ideals, we obtain a homeomorphism

$$\text{Spec } T_\ell \longrightarrow Y_\ell.$$ 

Now let $Q$ be a prime ideal of $S_\ell$. Under the preceding map, the preimage of $Q$ is $Q.T_\ell$ (by, e.g., [9, 10.18]). Next, let $\overline{D_\ell}$ denote the image modulo $Q$ of $D_\ell$. It follows from hypothesis B that $\overline{D_\ell}$ is a right Ore set, in $S_\ell/Q$, of regular elements. Hence,

$$T_\ell/Q.T_\ell \cong (S_\ell/Q) \otimes_{S_\ell} T_\ell \cong (S_\ell/Q)\overline{D_\ell}^{-1}.$$ 

In particular, $T_\ell/Q.T_\ell$ is a classical right quotient ring for $S_\ell/Q$.

(ii) Now consider the homomorphism $\nu: S \rightarrow T$ described in (2.4). We see that

$$\nu^*: \text{Spec } T \longrightarrow \nu^{-1}(p) \longrightarrow \text{Spec } S$$

is homeomorphic onto its image, $Y$. Let $Q$ be a prime ideal of $S$. It is not hard to prove, given (i), that the preimage of $Q$ under $\nu^*$ is $Q.T$ and that $T/Q.T$ is a classical right quotient ring of $S/Q$.

(iii) The necessity in (i) and (ii) of right (or left) noetherianity, as assumed in hypothesis A, is discussed in [9, p. 179].

2.7 Lemma. Assume hypotheses A and B. Consider the function

$$\eta^*: \text{Spec } T \longrightarrow \eta^{-1}(p) \longrightarrow \text{Spec } R.$$ 

(i) $\eta^* = \mu^* \circ \nu^*$.

(ii) $\eta^*$ is a continuous, finite-to-one, surjective function.

(iii) When $X_1, \ldots, X_\ell$ are disjoint, $\eta^*$ is injective.

Proof. First, part (i) follows from (2.2) and (2.6ii).

Next, in view of (i), the continuity and surjectivity of $\eta^*$ follow from (2.6ii) and (2.2v). That $\eta^*$ is finite-to-one follows from (i) and (2.2iv). Part (ii) follows.

Part (iii) follows from (i) and (2.2vi). □

2.8. As noted in (2.1ii), a normal element $n$ of a ring $\Lambda$ is regular modulo a prime ideal $Q$ of $\Lambda$ if and only if $n \notin Q$. Also, as noted in (2.1iii), the non-negative powers of a regular normal element form a right denominator set. It follows from these two facts that the following implies hypothesis B:

Hypothesis B’. For each $\ell$, the ideal $I_\ell + J_\ell$ is generated by a single element $b_\ell$ normal and regular modulo $I_\ell$, and $D_\ell = \{1 + I_\ell, b_\ell + I_\ell, b_\ell^2 + I_\ell, \ldots\} \subset S_\ell$.

2.9 Remarks. (i) When B’ holds and $R$ is a finitely generated $k$-algebra, then $T$ is also a finitely generated $k$-algebra.

(ii) When B’ holds, the homomorphism $\eta: R \rightarrow T$ is normalizing. Moreover, if each of the $b_\ell$ is central modulo $I_\ell$ then the homomorphism is centralizing.
2.10. Let \( \Lambda \) be a ring, and suppose there is a product preserving, one-to-one correspondence between the sets of ideals of \( \Lambda \) and \( Z(\Lambda) \), given by mutually inverse assignments
\[
a \mapsto a \cap Z(\Lambda), \quad \text{and} \quad b \mapsto b\Lambda,
\]
for arbitrary ideals \( a \) of \( \Lambda \) and \( b \) of \( Z(\Lambda) \). We will say in this situation that \( \Lambda \) is ideally commutative. There is then a homeomorphism between \( \text{Spec} \, \Lambda \) and \( \text{Spec} \, Z(\Lambda) \), given by
\[
a \mapsto a \cap Z(\Lambda), \quad \text{and} \quad b \mapsto b\Lambda,
\]
for prime ideals \( a \) of \( \Lambda \) and \( b \) of \( Z(\Lambda) \). This homeomorphism restricts to a homeomorphism between \( \text{Max} \, \Lambda \) and \( \text{Max} \, Z(\Lambda) \). If, in addition, \( Z(\Lambda) \) is a finitely generated \( k \)-algebra, we will say that \( \Lambda \) is ideally \( k \)-affine commutative.

We now present the third hypothesis, and a stronger variant:

**Hypothesis C.** For all \( \ell \), \( T_\ell \) is ideally commutative.

**Hypothesis C’.** For all \( \ell \), \( T_\ell \) is ideally \( k \)-affine commutative.

2.11. Assume hypothesis C.

(i) It immediately follows that \( T \) is ideally commutative.

(ii) If hypothesis A also holds then \( Z \) is noetherian, by (i), since \( T \) is right noetherian. If hypothesis C’ holds then \( Z \) is a finitely generated \( k \)-algebra.

(iii) As in (2.10), extension and contraction of ideals provide homeomorphisms between \( \text{Spec} \, T \) and \( \text{Spec} \, Z \), and between \( \text{Max} \, T \) and \( \text{Max} \, Z \).

2.12. In the next subsections, before presenting our main result, we collect information on primitive and related ideals. Our approach generalizes, and is directly derived from, [7, §2]. We first record the following preparatory lemma:

**Lemma.** Assume hypotheses A and B. (i) Let \( \mathfrak{p} \) be a prime ideal of \( T \), and set \( P = \eta^*(\mathfrak{p}) \). Then \( T/\mathfrak{p} \) is a classical right quotient ring of \( R/P \). (ii) Every prime factor of \( R \) is right Goldie.

**Proof.** (i) Set \( Q = \nu^*(\mathfrak{p}) \). By (2.7i), \( \mu^*(Q) = P \), and by (2.6ii), \( P = Q.T \). Next, it follows from (2.2ii) that \( R/P \) is isomorphic to \( S/Q \). As noted in (2.6ii), \( T/Q.T = T/\mathfrak{p} \) is a classical right quotient ring of \( S/Q \). In particular, part (i) follows.

(ii) First recall that a prime ring is right Goldie if and only if it has a simple artinian classical right quotient ring; see, for example, [9, Chapter 6], and in particular [9, 6.18], for details. By hypothesis A, \( T/Q.T \) is right noetherian and so possesses a simple artinian classical right quotient ring. However, by (2.6ii), \( T/Q.T \) is a classical right quotient ring for \( S/Q \). Therefore, \( S/Q \) possesses a simple artinian classical right quotient ring, and (ii) follows. \( \square \)

2.13. We now briefly review the Nullstellensatz, for noncommutative algebras, and the Dixmier-Moeglin equivalence; the reader is referred (e.g) to [16, Chapter 9; 18; 20, §8.4; 2, II.8] for background information.
(i) Using the notation of [16, 9.1.4], a \( k \)-algebra \( \Lambda \) satisfies the Nullstellensatz (over \( k \)) if \( \Lambda \) satisfies the endomorphism property (i.e., \( \text{End}_M \Lambda \) is algebraic over \( k \) for every simple right \( \Lambda \)-module \( M \)), and if \( \Lambda \) satisfies the radical property (i.e., the Jacobson radical of every factor ring of \( \Lambda \) is nil). When \( k \) is uncountable, every countably generated \( k \)-algebra satisfies the Nullstellensatz (cf., e.g., [16, 9.1.8]). When every prime factor of \( \Lambda \) is right or left Goldie, \( \Lambda \) satisfies the radical property if and only if \( \Lambda \) is a Jacobson ring (i.e., every prime ideal is an intersection of primitive ideals).

(ii) Recall that a prime ideal \( P \) of a \( k \)-algebra \( \Lambda \) is rational (over \( k \)) provided \( \Lambda /P \) is right (or left) Goldie, and provided the center of the Goldie right (or left) quotient ring of \( R/P \) is algebraic over \( k \).

(iii) Also recall that a prime ideal of a ring \( \Lambda \) is locally closed if it is a locally closed point of \( \text{Spec} \Lambda \). In other words, \( P \in \text{Spec} \Lambda \) is locally closed exactly when

\[
Q \subseteq \bigcap\{ Q' \in \text{Spec} \Lambda \mid Q' \supseteq Q \}.
\]

(iv) Following [18], a \( k \)-algebra \( \Lambda \) satisfies the Dixmier-Moeglin equivalence (over \( k \)) provided a prime ideal \( P \) of \( \Lambda \) is primitive if and only if \( P \) is rational, if and only if \( P \) is locally closed.

(v) If \( \Lambda \) is a Jacobson ring, it follows immediately that every locally closed prime ideal of \( \Lambda \) is primitive. It follows from (i) and (2.12), when \( R \) satisfies the Nullstellensatz, that every locally closed prime ideal of \( R \) is primitive.

(vi) Suppose that \( \Lambda \) is a \( k \)-algebra satisfying the Nullstellensatz, that \( P \) is a primitive ideal of \( \Lambda \), and that \( \Lambda /P \) is right or left Goldie. A proof that \( P \) must then be rational can be found (e.g.) in [4, 4.1.6].

2.14 Lemma. Assume hypotheses A and B, and let \( p \) be a prime ideal of \( T \). Then \( p \) is locally closed if and only if \( \eta^*(p) \) is a locally closed prime ideal of \( R \).

Proof. It is easy to see that a prime ideal \( Q \) of \( S \) is locally closed if and only if \( \mu^*(Q) \) is a locally closed prime ideal of \( R \). Next, let \( \nu_\ell \) be the canonical map \( S_\ell \rightarrow T_\ell \). Let \( p \) be a prime ideal of \( T_\ell \), and set \( Q = \nu_\ell^{-1}(p) \). Note that \( p = Q.T_\ell \), as noted in (2.6i).

To prove the lemma it only remains to check that \( p \) is locally closed in \( \text{Spec} T_\ell \) if and only if \( Q \) is locally closed in \( \text{Spec} S_\ell \). To start, when \( Q \) is locally closed it follows immediately from (2.6i) that \( p \) must also be locally closed.

So now suppose that \( p \) is locally closed. Let \( \Sigma \) be the set of prime ideals of \( S_\ell \) properly containing \( Q \), let \( \Delta \) be the set of prime ideals of \( T_\ell \) properly containing \( p \), and let \( \Gamma \) be the set of prime ideals \( \nu_\ell^{-1}(p') \) for \( p' \in \Delta \). Since the intersection of the prime ideals in \( \Delta \) must strictly contain \( p \), it follows from (2.6i) that the intersection \( K_\ell \) of the prime ideals in \( \Gamma \) must strictly contain \( Q \).

Next, let \( L_\ell \) denote the natural image of \( J_\ell \) in \( S_\ell \). By (2.6i), \( L_\ell \) is not contained in \( Q \). It also follows from (2.6i) that if \( Q' \in \Sigma \setminus \Gamma \) then \( Q' \) contains \( L_\ell \). We now see that

\[
M_\ell := \bigcap_{Q' \in \Sigma} Q' = \left( \bigcap_{Q' \supseteq K_\ell} Q' \right) \bigcap \left( \bigcap_{Q' \supseteq L_\ell} Q' \right).
\]
However, since $Q$ is prime, $K_{q} \cap L_{q} \not\subseteq Q$, and so $M_{q} \not\subseteq Q$. Hence $M_{q}$ properly contains $Q$, and $Q$ is locally closed. The lemma follows. □

2.15. We now introduce the last hypothesis:

**Hypothesis D.** $R$ is a $\mathbb{k}$-algebra satisfying the Nullstellensatz.

2.16 Theorem. Assume that $R$ satisfies hypotheses A, B, C, and D. Then $R$ satisfies the Dixmier-Moeglin equivalence.

Proof. Let $P$ be a rational ideal of $R$ – recall from (2.12) that every prime factor of $R$ is right Goldie. By (2.13v, vi), to prove the theorem it suffices to prove that $P$ is locally closed. Now, by (2.7), we can choose a prime ideal $p$ of $T$ such that $\eta^{*}(p) = P$. By (2.12i), $T/p$ is a classical right quotient ring for $R/P$; hence the rationality of $p$ ensures the rationality of $P$. However, since $p$ is rational, $p \cap Z$ is a maximal ideal of $Z$. Therefore, by (2.10) and (2.11i), $p$ is a maximal ideal of $T$, and in particular, $p$ is locally closed. It now follows from (2.14) that $P$ is locally closed. □

2.17. Choose $q \in \text{Spec } Z$, and set

$$\theta(q) = \text{ann}(T/qT)_{R} = \text{ann}((Z/q) \otimes_{Z} T)_{R}.$$  

By (2.11iii), $qT$ is a prime ideal of $T$. By (2.7), $\text{ann}(T/qT)_{R} = \eta^{*}(qT)$ is a prime ideal of $R$. Consequently, $\theta(q)$ is a prime ideal of $R$.

We now present the main result of this section.

2.18 Theorem. Assume hypotheses A, B, and C.

(i) The function $\theta: \text{Spec } Z \to \text{Spec } R$, described in (2.17), is continuous, finite-to-one, and surjective. If $X_{1}, \ldots, X_{t}$ are disjoint then $\theta$ is bijective.

(ii) For all $q \in \text{Spec } Z$, $\theta(q)$ is locally closed if and only if $q$ is locally closed.

(iii) Assume that $R$ satisfies hypothesis D. Then for all $q \in \text{Spec } Z$, $\theta(q)$ is primitive if and only if $q$ is maximal. Consequently, $\theta: \text{Max } Z \to \text{Prim } R$ is continuous, finite-to-one, and surjective.

Proof. Part (i) follows from (2.7) and (2.11iii). Part (ii) follows from (2.11iii) and (2.14).

To prove part (iii), let $q$ be a prime ideal of $Z$ and let $P = \theta(q)$. Set $p = qT$; by (2.11iii), $p$ is a prime ideal of $T$, and following (2.17), $P = \theta(q) = \eta^{*}(p)$.

Now suppose that $P$ is primitive. By (2.16), $P$ is a rational ideal of $R$. Therefore, by (2.12i), $p$ is a rational ideal of $T$, and so $p \cap Z$ must be a maximal ideal of $Z$.

Conversely, suppose that $q$ is maximal. Then, by (2.11iii), $p$ is a maximal ideal of $T$. Therefore, $\theta(q) = \eta^{*}(p)$ is a locally closed prime ideal of $R$, by (2.14). Thus $\theta(q)$ is a primitive ideal of $R$, by (2.16). Part (iii) follows. □

2.19. Let $\Lambda$ and $\Lambda'$ be rings, and let $U$ be an $\Lambda$-$\Lambda'$-bimodule. In the approach to noncommutative algebraic geometry developed in [19, 22], the categories Mod$\Lambda$ and Mod$\Lambda'$ are viewed as categories of quasi-coherent sheaves on not-explicitly-defined noncommutative affine schemes, and the functor

$$\text{Mod } \Lambda \xrightarrow{- \otimes_{\Lambda} U} \text{Mod } \Lambda'$$
is an affine map.

Now assume hypotheses A, B, and C. We obtain the affine map

$$\operatorname{Mod} \mathbb{Z} \xrightarrow{- \otimes_{\mathbb{Z}} T} \operatorname{Mod} R.$$ 

Here, of course, $\operatorname{Mod} \mathbb{Z}$ is naturally equivalent to the category of quasi-coherent sheaves on the affine scheme $\operatorname{Spec} \mathbb{Z}$, which is noetherian by (2.11ii). We see that the function $\theta$ of (2.17) is derived from this affine map.

2.20. We conclude this section with useful sufficient conditions for hypotheses A, B', C, and C'. To start, say that a prime ring $\Lambda$ is generically ideally commutative provided the right Ore localization of $\Lambda$ at the non-negative powers of some nonzero normal (and so regular) element is – in the notation of (2.10) – ideally commutative. Say that a prime ring $\Lambda$ is generically ideally $k$-affine commutative provided the right Ore localization of $\Lambda$ at the non-negative powers of some nonzero normal element is $k$-affine ideally commutative.

We will find the following criteria useful:

**Proposition.** Assume that the semiprime ideals of $R$ satisfy the ascending chain condition.

(i) If every prime factor of $R$ is generically ideally commutative, then hypotheses A, B', and C are satisfied, for suitable choices of $I_1, \ldots, I_t$, $J_1, \ldots, J_t$, and $D_1, \ldots, D_t$.

(ii) If every prime factor of $R$ is generically ideally $k$-affine commutative, then hypotheses A, B', and C' are satisfied, for suitable choices of $I_1, \ldots, I_t$, $J_1, \ldots, J_t$, and $D_1, \ldots, D_t$.

**Proof.** (i) Assume every prime factor of $R$ is generically ideally commutative. Since $\operatorname{Spec} R = V(N)$, where $N$ is the prime radical of $R$, we may assume without loss of generality that $R$ is semiprime. Moreover, by noetherian induction, we may further assume that (i) holds for all proper semiprime factors of $R$.

Suppose that $R$ is not prime. Then there exist nonzero semiprime ideals $I$ and $J$ of $R$ such that $I \cap J = 0$. Since (i) holds for $R/I$ and $R/J$, and since $\operatorname{Spec} R = V(I) \cup V(J)$, we see that (i) holds for $R$.

So now suppose that $R$ is prime. Since $R$ is generically ideally commutative, we can choose a regular normal element $b_1$ of $R$ such that the right Ore localization of $R$ at the non-negative powers of $b_1$ is ideally commutative. It follows that the conditions in hypotheses A, B', and C hold for $I_1 = 0$, $J_1 = \langle b_1 \rangle$, $X_1 = V(I_1) \setminus V(J_1)$, $S_1 = R$, $D_1 = \{1, b_1, b_1^2, \ldots\}$, and $T_1 = S_1 D_1^{-1}$.

Now take $I_2 = \sqrt{J_1}$. We know from the induction hypothesis that (i) holds for $R/I_2$. Since $\operatorname{Spec} R = X_1 \cup V(I_2)$, we see that (i) holds for $R$.

(ii) This follows similarly to (i). □

3. PI Algebras

In this section, assume that $R$ is a ring satisfying a polynomial identity; see, for example, [16, Chapter 13] or [21] for necessary background information.
3.1. (i) Let $\Lambda$ be a (prime) Azumaya algebra. In the notation of (2.10), $\Lambda$ is ideally commutative. Also, if $\Lambda$ is a finitely generated $k$-algebra, then $Z(\Lambda)$ is finitely generated as a $k$-algebra.

(ii) Also in the notation of (2.10), it follows from (i) and the Artin-Procesi theorem that every prime factor of $R$ is generically ideally commutative. Moreover, if $R$ is a finitely generated $k$-algebra then every prime factor of $R$ is generically ideally $k$-affine commutative.

3.2 Finitely generated PI $k$-algebras. Suppose that $R$ is a finitely generated $k$-algebra. Then the semiprime ideals of $R$ satisfy the ascending chain condition (see, e.g., [21, 4.5.7]). In particular, by (3.1ii) and (2.20ii), $R$ satisfies hypotheses A, B', and C'. Also, $R$ satisfies hypothesis D (see, e.g., [16, 13.10.4]). Consequently, for a suitably chosen, finitely generated (by (2.11ii)), commutative $k$-algebra $Z$, (2.18) provides a continuous, finite-to-one, surjection from $\text{Spec } Z$ onto $\text{Spec } R$, restricting to a continuous surjection from $\text{Max } Z$ onto $\text{Prim } R = \text{Max } R$.

The fact that $\text{Prim } R$ is a disjoint union of finitely many locally closed subsets, each homeomorphic to an open subset of a $k$-affine algebraic variety, can already be found in [1; 17].

3.3 Noetherian PI algebras. Suppose that $R$ is right noetherian (but not necessarily finitely generated as an algebra). Then by (3.1ii) and (2.20i), $R$ satisfies hypotheses A, B', and C. Therefore, (2.18) provides a continuous, finite-to-one, surjection $\theta : \text{Spec } Z \to \text{Spec } R$, for a suitably chosen commutative ring $Z$, noetherian by (2.11ii). If $R$ and $Z$ are Jacobson rings, then, by (2.18ii), $\theta$ restricts to a surjection from $\text{Max } Z$ onto $\text{Prim } R = \text{Max } R$.

4. Solvable Lie algebras

Assume in this section that $k$ has characteristic zero and that $R$ is the enveloping algebra of a finite dimensional completely solvable $k$-Lie algebra $g$.

4.1. By [16, 14.9.19], every prime factor of $R$ is generically ideally $k$-affine commutative. Also, $R$ is noetherian by the Poincaré-Birkhoff-Witt theorem. Therefore, by (2.20ii), $R$ satisfies hypotheses A, B', and C'. Furthermore, $R$ satisfies hypothesis D; see, for example, [4, 2.6.4, 3.1.15]. Hence, (2.18) provides a continuous, finite-to-one, surjection from $\text{Spec } Z$ onto $\text{Spec } R$, restricting to a continuous surjection from $\text{Max } Z$ onto $\text{Prim } R$, for a suitable finitely generated (by (2.11ii)) commutative $k$-algebra $Z$.

4.2. When $k$ is algebraically closed, the Dixmier map (see, e.g., [4]) provides a Zariski-continuous map from $g^*$ onto $\text{Prim } R$; bicontinuity of the factorized map is given in [15].

4.3. An example: Let $g$ be the nonabelian 2-dimensional solvable $k$-Lie algebra, and let $R$ be the enveloping algebra of $g$. Then $R$ is a domain and contains a unique minimal nonzero prime ideal $P$. Set $I_1 = 0$, $J_1 = P$, $I_2 = P$, and $J_2 = R$. Following the constructions in (2.2), and in the rest of §2, we see that $Y_1 \cong X_1 = V(0) \setminus V(P) = \{0\}$, and $Y_2 \cong X_2 = V(P)$. Since 0 is a generic point of $\text{Spec } R$, and since $Y_1$ is closed, we see that $\theta$ in this case is not a topological quotient map.
5. Quantum Semisimple Groups

5.1. Let $G$ be a connected, semisimple Lie group, and let $R$ denote one of the following algebras: $R_q[G]$ as defined in [13], $C_{q,p}[G]$ as defined in [12], or $O_q(G)$ as defined in [2, I.7]. For each of the possible choices of $R$ we let $k$ denote the appropriate ground field (over which $R$ is finitely generated), and we assume that $k$ is algebraically closed. These choices for $R$ are “non-root-of-unity” or “generic” quantizations.

Throughout, we equip $R$ with the rational $H$-action by automorphisms described, for example, in [2, II.1.18], where $H$ is a torus over $k$ corresponding to a maximal torus of $G$.

Note, when a $k$-algebra is equipped with an $H$-action by automorphisms, that there is a naturally induced $H$-action on the prime and primitive spectra.

5.2. The following information is well known, and can be extracted, for example, from [2, II.4; 3; 12, §4; 13, Chapter 10]. The original results describing the prime and primitive ideals in this way can mostly be found in [10; 11; 12; 14].

(i) To start, $R$ is a noetherian domain. Also, $R$ satisfies the Nullstellensatz; see (e.g.) [2, II.7.20].

(ii) Let $W$ be the Weyl group associated to $G$. For each $w \in W \times W$ there exists a prime ideal $I_w$ of $R$ and a finite set $E_w$ of elements of $R$ normal modulo $I_w$ such that

$$\text{Spec } R = \coprod_{w \in W \times W} \text{Spec}_w R, \quad \text{and} \quad \text{Prim } R = \coprod_{w \in W \times W} \text{Prim}_w R,$$

where

$$\text{Spec}_w R = \{ P \in \text{Spec } R : P \supseteq I_w \text{ and } P \cap E_w = \emptyset \},$$

and

$$\text{Prim}_w = (\text{Spec}_w R) \cap (\text{Prim } R).$$

Moreover, $\text{Spec}_w R$ and $\text{Prim}_w R$ are nonempty. (That $E_w$ can be chosen to be finite is explicitly justified in [3, 5.6].)

Let $b_w$ denote the product of the elements in $E_w$, and set $J_w = \langle b_w \rangle$. It is easy to see that $b_w$ is normal modulo $I_w$ and that

$$\text{Spec}_w R = V(I_w) \setminus V(J_w).$$

(iii) Set $S_w = R/I_w$, and let $D_w$ denote the image in $S_w$ of the non-negative powers of $b_w$. Let $T_w$ denote the localization of $S_w$ at $D_w$, and let $Z_w$ be the center of $T_w$. Each $T_w$ is ideally $k$-affine commutative, following (e.g.) [12, 4.15].

(iv) Each $Z_w$ is isomorphic as a $k$-algebra to a Laurent polynomial ring $k[y_1^{\pm}, \ldots, y_m^{\pm}]$, for some non-negative integer $m_w$ (see, e.g., [2, II.4.14]). The value of $m_w$ can be explicitly determined in one-parameter quantizations and for some multiparameter cases; see [12, 4.17].

(v) Under the given action of $H$ on $R$ by automorphisms, each $I_w$ is $H$-stable and the elements of $E_w$ can be chosen to be $H$-eigenvectors. There is then an induced $H$-action on $T_w$, and $Z_w$ is generated by $H$-eigenvectors. The inclusion of $Z_w$ in $T_w$ is $H$-equivariant, each $\text{Spec}_w R$ is stable under the induced $H$-action, and each $\text{Prim}_w R$ is a single $H$-orbit.
5.3. For the given \( I_w, J_w, b_w, \) and \( T_w \) of (5.2), hypotheses A follows from (5.2i), hypothesis \( B' \) follows from (5.2ii), hypothesis \( C' \) follows from (5.2iii), and hypothesis D follows from (5.2i). Moreover, setting \( X_w = V(I_w) \setminus V(J_w) = \text{Spec}_w R \), we see that the \( X_w \) are disjoint, as noted in (5.2ii).

Using (2.18) and (5.2v), we obtain an \( H \)-equivariant, continuous, bijection \( \theta: \text{Spec} Z \to \text{Spec} R \), restricting to an \( H \)-equivariant continuous bijection from \( \text{Max} Z \) onto \( \text{Prim} R \), for \( Z \) equal to the ring direct product of the \( Z_w \).

6. Other Quantized Coordinate Algebras

In this section, \( k \) denotes an infinite field, \( H \) denotes a \( k \)-algebraic \( n \)-torus, and \( R \) denotes a finitely generated noetherian \( k \)-algebra equipped with a rational \( H \)-action by automorphisms. Assume further that there are only finitely many fixed points in the naturally induced \( H \)-action on \( \text{Spec} R \). In [2; 7; 8] this and similar settings are used as unified frameworks to study both the algebras considered in §4 and other quantum function algebras.

6.1. That the algebras \( R_q[G], C_{q,p}[G], \) and \( O_q(G) \) considered in §4 satisfy the current assumptions follows from [12; 14]; see (5.2). Other examples satisfying the current assumptions include quantized coordinate rings of affine \( n \)-space, quantized Weyl algebras (at non-roots of unity), and quantized coordinate rings (also at non-roots-of-unity) of \( n \times n \) matrices, \( GL_n \), symplectic \( n \)-space, and euclidean \( n \)-space; see [2, II; 7] for details. All of these examples satisfy hypothesis D; see (e.g.) [2, II.7.17, II.7.20].

6.2. We now verify hypotheses A, B, and \( C' \), for suitable choices of \( I_\ell, J_\ell, \) and \( D_\ell \). To start, let \( I_1, \ldots, I_t \) be the \( H \)-stable prime ideals of \( R \), and for each \( 1 \leq \ell \leq t \), let \( J_\ell \) be the intersection of the \( H \)-stable prime ideals of \( R \) not contained in \( I_\ell \).

(i) Following [2, II.2; 7, §2; 8],

\[
\text{Spec} R = \coprod_{\ell=1}^{t} X_\ell,
\]

where

\[
X_\ell = V(I_\ell) \setminus V(J_\ell),
\]

for all \( 1 \leq \ell \leq t \).

(ii) For each \( \ell \), set \( S_\ell = R/I_\ell \). It follows from [8] (or [2, II.2.13]) that there exist right denominator sets \( D_\ell \) of \( S_\ell \), for all \( \ell \), such that hypothesis B holds true. Now set \( T_\ell = S_\ell D^{-1}_\ell \). Since \( R \) is noetherian, each \( T_\ell \) is noetherian, and so hypothesis A is satisfied. Let \( Z_\ell \) denote the center of \( T_\ell \), for each \( \ell \). Hypothesis \( C' \) holds in this setting, by [8] (or [2, II.2.13]).

6.3. Retain the notation of (6.2).

(i) Since each \( I_\ell \) is \( H \)-stable, it follows that each \( S_\ell \) inherits the \( H \)-action on \( R \). By [8] (or [2, II.2.13]), we can assume that \( D_\ell \) consists entirely of \( H \)-eigenvectors. Hence each \( T_\ell \)
is naturally equipped with a rational $H$-action by automorphisms. Furthermore, it follows from [8] (or see [2, II.2.13]) that each $Z_{\ell}$ is generated by $H$-eigenvectors.

(ii) Set $T = T_1 \times \cdots \times T_t$, and identify the center $Z$ of $T$ with $Z_1 \times \cdots \times Z_t$. Then $T$ and $Z$ inherit rational $H$-actions by automorphisms, from the $H$-actions on each $T_{\ell}$ and $Z_{\ell}$.

The embedding of $Z$ in $T$ is $H$-equivariant, as is the $k$-algebra homomorphism $\eta: R \to T$ of (2.4). Following (2.11ii), $Z$ is finitely generated as a $k$-algebra.

6.4. By (6.2), hypotheses A, B, and C’ hold for the given choices of $I_{\ell}$, $J_{\ell}$, and $D_{\ell}$; moreover, the sets $X_{\ell}$ are (pairwise) disjoint. In view of (6.3) and (2.18), we obtain a continuous, $H$-equivariant bijection $\theta: \text{Spec } Z \to \text{Spec } R$.

All of the algebras mentioned in (6.1) satisfy the Nullstellensatz over $k$, by [2, II.7.18, II.7.20]; in this case, by (2.18), the preceding function $\theta$ restricts to a continuous, $H$-equivariant bijection from Max $Z$ onto Prim $R$.

6.5. Suppose that $R$ is the quantized coordinate ring of $k$-affine $n$-space, and assume further that $-1$ is not contained in the multiplicative group generated by the quantizing parameters. It is proved in [6] that Spec $R$ is a topological quotient of Spec $k[x_1, \ldots, x_n]$ and that Prim $R$ is a topological quotient of Max $k[x_1, \ldots, x_n]$. Goodearl has conjectured in [5] that the prime spectra of more general quantum function algebras should be topological quotients of commutative prime spectra.

References
1. M. Artin, On Azumaya algebras and finite dimensional representations of rings, J. Algebra 11 (1969), 532–563.
2. K. A. Brown and K. R. Goodearl, Lectures on Algebraic Quantum Groups, Advanced Courses in Mathematics – CRM Barcelona, Birkhäuser Verlag, Basel, 2002.
3. ______, Prime spectra of quantum semisimple groups, Trans. Amer. Math. Soc. 348 (1996), 2465–2502.
4. Jacques Dixmier, Enveloping algebras, Graduate Studies in Mathematics, vol. 11, American Mathematical Society, Providence, 1996.
5. K. R. Goodearl, Prime spectra of quantized coordinate rings, Interactions between Ring Theory and Representations of Algebras (Murcia, 1998) (F. Van Oystaeyen and M. Saorín, eds.), Dekker, New York, 2000, pp. 205–237.
6. K. R. Goodearl and E. S. Letzter, Quantum $n$-space as a quotient of classical $n$-space, Trans. Amer. Math. Soc. 352 (2000), 5855–5876.
7. ______, The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras, Trans. Amer. Math. Soc. 352 (2000), 1381–1403.
8. K. R. Goodearl and J. T. Stafford, The graded version of Goldie’s theorem, Algebra and its applications (Athens, OH, 1999), Contemp. Math. 259, Amer. Math. Soc., Providence, 2000, pp. 237–240.
9. K. R. Goodearl and R. B. Warfield, Jr., An Introduction to Noncommutative Noetherian Rings, Second Edition, London Mathematical Society Student Texts 61, Cambridge University, Cambridge, 2004.
10. T. J. Hodges and T. Levasseur, Primitive ideals of $C_q[SL(n)]$, J. Algebra 168, 455–468.
11. ______, Primitive ideals of $C_q[SL(3)]$, Commun. Math. Phys. 156 (1993), 581-605.
12. T. J. Hodges, T. Levasseur, and M. Toro, Algebraic structure of multiparameter quantum groups, Adv. Math. 126 (1997), 52–92.
13. A. Joseph, Quantum Groups and Their Primitive Ideals, Ergebnisse der Mathematik und ihrer Grenzberei- giete 3. Folge · Band 29, Springer-Verlag, Berlin, 1995.
14. A. Joseph, On the prime and primitive spectra of the algebra of functions on a quantum group, J. Algebra, 169 (1994), 441–511.
15. Olivier Mathieu, *Bicontinuity of the Dixmier map*, J. Amer. Math. Soc. 4 (1991), 837–863.
16. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics 30, American Mathematical Society, Providence, 2000.
17. C. Procesi, *Finite dimensional representations of algebras*, Israel J. Math. 19 (1974), 169–182.
18. R. Rentschler, *Primitive ideals in enveloping algebras (general case)*, in Noetherian Rings and their Applications (Oberwolfach, 1983) (L. W. Small, ed.), Math. Surveys Monographs 24, Amer. Math. Soc., Providence, 1987, pp. 37–57.
19. A. L. Rosenberg, *Noncommutative algebraic geometry and representations of quantized algebras*, Mathematics and its applications, vol. 330, Kluwer, Dordecht, 1995.
20. L. H. Rowen, *Ring Theory, Volumes I and II*, Pure and Applied Mathematics 127, 128, Academic Press, Boston, 1988.
21. ____, *Polynomial Identities in Ring Theory*, Pure and Applied Mathematics 84, Academic Press, London, 1980.
22. M. Van den Bergh, *Blowing up of non-commutative smooth surfaces*, Mem. Amer. Math. Soc. 154 (2001).

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