Abstract

Neural networks have become standard tools in the analysis of data, but they lack comprehensive mathematical theories. For example, there are very few statistical guarantees for learning neural networks from data, especially for classes of estimators that are used in practice or at least similar to such. In this paper, we develop a general statistical guarantee for estimators that consist of a least-squares term and a regularizer. We then exemplify this guarantee with $\ell_1$-regularization, showing that the corresponding prediction error increases at most sub-linearly in the number of layers and at most logarithmically in the total number of parameters. Our results establish a mathematical basis for regularized estimation of neural networks, and they deepen our mathematical understanding of neural networks and deep learning more generally.

1 Introduction

Neural networks have proved extremely useful across a variety of applications, including speech recognition [Hinton et al., 2012, Graves et al., 2013, Chorowski et al., 2015], natural language processing [Jozeowicz et al., 2016], object categorization [Girshick et al., 2014, Szegedy et al., 2015], and image segmentation [Long et al., 2015, Badrinarayanan et al., 2017]. But our mathematical understanding of neural networks and deep learning has not developed at the same speed.

A central objective is to equip methods for learning neural networks with statistical guarantees. Some guarantees are available for unconstrained estimators [Anthony and Bartlett, 2009], but these bounds are linear in the number of parameters, which conflicts with the large sizes of typical networks. The focus has thus shifted to estimators that involve constraints or regularizers. Recently surged in popularity have estimators with $\ell_1$-regularizers [Bartlett, 1998, Bartlett and Mendelson, 2002, Anthony and Bartlett, 2009, Barron and Klusowski, 2018, 2019, Liu and Ye, 2019], motivated by the success of this type of regularization in linear regression [Tibshirani, 1996], compressed sensing [Candès et al., 2006, Donoho, 2006], and many other parts of data science. A key feature of $\ell_1$-regularization is that it is easy to include into optimization schemes and, at the same time, induces sparsity, which has a number of favorable effects in deep learning [Glorot et al., 2011]. There has been some progress on guarantees for least-squares with constraints based on the sparsity of the networks [Schmidt-Hieber, 2017] or group-type norms on the weights [Neyshabur et al., 2015b]. These developments have provided valuable intuition, for example, about the role of network widths and depths, but important problems remain: for example, the combinatorial constraints in the first paper render the corresponding estimators infeasible in practice, the exponential dependence of the bounds in the second paper are contrary to the trend toward very deep networks, and practitioners usually prefer regularized rather than constraint formulations of the estimators. More generally,
we first establish an alternative parametrization of neural networks and use this parameterization
The organization of the paper is as follows: In Section 2, we introduce our regularization scheme
1 which decreases essentially as \( \frac{L \log(nP)}{n} \log(n) \),
which ensures accurate prediction even of very wide and deep networks.
The organization of the paper is as follows: In Section 2, we introduce our regularization scheme
and establish a general prediction bound. In Section 3, we specify this bound to \( \ell_1 \)-regularization.
In Section 4, we compare to related literature. In Section 5, we establish further mathematical
properties of neural networks. Section 6, we state all proofs. In Section 7, we conclude our paper.

2 Scale regularization for neural networks

We first establish an alternative parametrization of neural networks and use this parameterization
to define our regularization strategy. We then provide a prediction guarantee for the corresponding
estimators.

2.1 Alternative Parametrization

Consider data \((x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^d \times \mathbb{R} \) that follow a regression model
\[ y_i = g_\star(x_i) + u_i \quad \text{for } i \in \{1, \ldots, n\} \]
for some function \( g_\star : \mathbb{R}^d \to \mathbb{R} \). We are interested in estimating an approximation of \( g_\star \) based on
neural networks. Following first standard approaches, we consider feed-forward neural networks of
the form
\[ g_\Theta : \mathbb{R}^d \to \mathbb{R} \]
\[ x \mapsto g_\Theta(x) := W^L f^L(\ldots W^1 f^1(W^0 x)), \]
indexed by the network parameter \( \Theta = (W^L, \ldots, W^0) \) that summarizes the weight matrices
\( W^l \in \mathbb{R}^{p_{l+1} \times p_l} \). The \( x_i \) and \( y_i \) are the network’s inputs and outputs, respectively, and the \( u_i \) are the noise
variables. For ease of notation, we assume that the \( x_i \) are fixed; generalizations to random \( x_i \) are
straightforward. The network’s architecture is specified by the number of hidden layers or depth
\( L \in \{1, 2, \ldots\} \) and by the the number of neurons in each layer or width
\( p = (p_{L+1}, p_L, \ldots, p_1, p_0) \in \{1, 2, \ldots\}^{L+2} \). The 0th layer is the input layer with
\( p_0 = d \), and the \((L + 1)\)th layer is the output layer with \( p_{L+1} = 1 \). The functions
\( f^l : \mathbb{R}^{p_l} \to \mathbb{R}^{p_l} \) are called activation functions. We omit shifts in the activation functions
for notational simplicity, but such can often be incorporated as additional
neurons [Barron and Klusowski, 2018].
The parameter space in the above formulation is
\[ \mathcal{A} := \{ \Theta = (W^L, \ldots, W^0) : W^l \in \mathbb{R}^{p_{l+1} \times p_l} \}. \]
In the following, however, we propose an alternative parametrization. We say that a function
\( q : \mathbb{R}^s \to \mathbb{R} \) is non-negative homogeneous of degree \( k \in (0, \infty) \) if
\[ q(az) = a^k q(z) \quad \text{for all } a \in [0, \infty) \text{ and } z \in \mathbb{R}^s, \]
many questions about the statistical properties of constraint and regularized estimation of neural
networks remain open.
In this paper, we introduce a general class of regularized least-squares estimators. Our strategy
is to disentangle the parameters into a “scale” and a “direction”—similarly to introducing polar
coordinates—which allows us to focus the regularization on a one-dimensional parameter. We call
our approach scale regularization. We then equip the scale regularized least-squares estimators with
a general statistical guarantee for prediction. A main feature of this guarantee is that it connects
neural networks to standard empirical process theory through a quantity that we call the effective
noise. This connection facilitates the specification of the bound to different types of regularization.
In a second step, we exemplify the general bound for \( \ell_1 \)-regularization. We find a guarantee for the
squared prediction error of the order of
\[ \sqrt{\frac{L \log(nP)}{n} \log(n)}, \]
which decreases essentially as \( 1/\sqrt{n} \) in the number of samples \( n \), is sub-linear in the number
of hidden layers \( L \), and is logarithmic in the total number of parameters \( P \). This result suggests that
\( \ell_1 \)-regularization can ensure accurate prediction even of very wide and deep networks.

In Section 2, we introduce our regularization scheme and establish a general prediction bound. In Section 3, we specify this bound to \( \ell_1 \)-regularization. In Section 4, we compare to related literature. In Section 5, we establish further mathematical
properties of neural networks. Section 6, we state all proofs. In Section 7, we conclude our paper.
and we say that a function \( q : \mathbb{R}^r \to [0, \infty) \) is positive definite if
\[
q(z) = 0 \iff z = 0_n.
\]
The corresponding properties for functions on \( A \) are defined accordingly. For example, every norm on \( \mathbb{R}^n \) or \( A \) is non-negative homogeneous of degree 1 and positive definite. We then find the following:

**Proposition 1** (Equivalence between neural networks). Assume that the activation functions \( f^1, \ldots, f^L \) are non-negative homogeneous of degree 1. Consider a function \( h : A \to [0, \infty) \) that is non-negative homogeneous of degree \( k \in (0, \infty) \) and positive definite, and denote the corresponding unit ball by
\[
A_h := \{ \Theta \in A : h(\Theta) \leq 1 \}.
\]
Then, for every \( \Theta \in A \), there exists a pair of \( \kappa \in [0, \infty) \) and \( \Omega \in A_h \) such that
\[
g_{\Theta}(x) = \kappa g_{\Omega}(x) \quad \text{for all } x \in \mathbb{R}^d,
\]
and vice versa, for every pair of \( \kappa \in [0, \infty) \) and \( \Omega \in A_h \), there exists a \( \Theta \in A \) such that the above equality holds.

The proposition shows that the standard parametrization of neural networks over the set \( A \) can be replaced by a parametrization over \( [0, \infty) \times A_h \). The equivalence requires the activation functions to be non-negative homogeneous (ReLU activations are popular examples), but it motivates a different parametrization of neural networks more generally. We thus change the parameter space for estimating the true data generating function \( g_* \) to \( [0, \infty) \times A_h \) and the corresponding space of networks to \( \{ \kappa g_{\Omega} : \kappa \in [0, \infty), \Omega \in A_h \} \) (which, of course, equals the original space \( \{ g_{\Theta} : \Theta \in A \} \) under the conditions of the proposition). In other words, we study neural networks
\[
\kappa g_{\Omega} : \mathbb{R}^d \to \mathbb{R}
\]
\[
x \mapsto \kappa g_{\Omega}(x) := \kappa U^L f^L (\ldots U^1 f^1(U^0 x)),
\]
indexed by the network parameters \( \kappa \in [0, \infty) \) and \( \Omega = (U^L, \ldots, U^0) \in A_h \). We can interpret \( \kappa \) as the network’s “scale” and \( \Omega \) as the network’s “orientation.”

### 2.2 Estimation

The most basic approach to fit the model parameters of the network (2) to the model (1) is the least-squares estimator
\[
\hat{\Theta}_{\text{LS}} \in \operatorname{arg\,min}_{\Theta \in A} \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - g_{\Theta}(x_i))^2 \right\}.
\]
But to account for the high-dimensionality of the parameter space \( A \), the least-squares estimator is often complemented with a regularizer \( h : A \to [0, \infty) \); popular choices for \( h \) are the \( \ell_1 \)-norm [Zhang et al., 2016] or group versions of it [Scardapane et al., 2017]. A straightforward way to incorporate such regularizers is
\[
\hat{\Theta}_{\text{reg}, h} \in \operatorname{arg\,min}_{\Theta \in A} \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - g_{\Theta}(x_i))^2 + \lambda h(\Theta) \right\},
\]
where \( \lambda \in [0, \infty) \) is a tuning parameter. But in neural network frameworks, it turns out difficult to analyze such estimators statistically.

We introduce, therefore, a different way to incorporate regularizers. The approach is based on our new parametrization. The equivalent of the above least-squares estimator in the framework (3) is
\[
\left( \hat{\kappa}_{\text{LS}}, \hat{\Omega}_{\text{LS}} \right) \in \operatorname{arg\,min}_{\kappa \in [0, \infty), \Omega \in A_h \} \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - \kappa g_{\Omega}(x_i))^2 \right\}.
\]
It holds that \( g_{\Theta_{\text{LS}}} = \kappa_{\text{LS}} g_{\Omega_{\text{LS}}} \) under the conditions of Proposition 1, but we can take this estimator as a starting point more generally. This allows us to focus the regularization on the scale-parameter \( \kappa \); in other words, we propose the estimators
\[
\left( \hat{\kappa}_h, \hat{\Omega}_h \right) \in \operatorname{arg\,min}_{\kappa \in [0, \infty), \Omega \in A_h \} \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - \kappa g_{\Omega}(x_i))^2 + \lambda \kappa \right\},
\]
where $\lambda \in [0, \infty)$ is a tuning parameter. The fixed constraint $\Omega \in \mathcal{A}_h$ captures the type of regularization (such as $\ell_1$), while the actual regularization concerns only on the scale $\kappa \in [0, \infty)$. We thus call our approach scale regularization.

The concentration of the regularization on a one-dimensional parameter greatly facilitates the statistical analysis. Specifically, it will allow us to focus our attention on

$$
z_h := \sup_{\omega \in \mathcal{A}_h} \left| \frac{1}{n} \sum_{i=1}^{n} g_{\Omega}(\mathbf{x}_i) u_i \right|,
$$

which can be thought of as the neural network equivalent of what high-dimensional linear regression refers to as the effective noise [Lederer and Vogt, 2020]. We need to ensure—just as in high-dimensional linear regression—that the effective noise is controlled by the tuning parameter with high probability. In this spirit, we define quantities $\lambda_{h,t}$ of the effective noise for given level $t \in [0, 1]$ through

$$
\lambda_{h,t} \in \min\{\delta \in [0, \infty) : P(z_h \leq \delta) \geq 1 - t\}.
$$

In other words, $\lambda_{h,t}$ is the smallest tuning parameter that controls the effective noise $z_h$ at level $1 - t$.

To measure the accuracy of the regularized estimators, we consider the in-sample prediction error with respect to the data generating function $g_s$:

$$
\text{err}(\kappa g_{\Omega}) := \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\kappa g_{\Omega}(\mathbf{x}_i) - g_s(\mathbf{x}_i))^2} \quad \text{for } \kappa \in [0, \infty), \Omega \in \mathcal{A}_h.
$$

This is a standard measure for how well the data generating function is learned. We find the following guarantee in this measure:

**Theorem 1** (Prediction guarantee). Assume that $\lambda \geq \lambda_{h,t}$ for a $t \in [0, 1]$. Then,

$$
\text{err}^2(\hat{\lambda}_h g_{\hat{\Omega}_h}) \leq \inf_{\kappa \in [0, \infty)} \left\{ \text{err}^2(\kappa g_{\Omega}) + 2\lambda \kappa \right\}
$$

with probability at least $1 - t$.

The bound is an analog of what has been called sparsity-bound in high-dimensional linear regression [Lederer et al., 2019]. For neural networks, however, it is the first such bound. It states that the squared prediction error of the regularized estimator is governed by an approximation error or squared bias $\text{err}^2(\kappa g_{\Omega})$ and an excess error or variance $2\lambda \kappa$. In other words, the estimator is guaranteed to have a small prediction error if (i) the quantile $\lambda_{h,t}$ is small and (ii) the data generating function can be represented well by a neural network with reasonably small $\kappa$. A typical example for (i) is provided in the following section; recent results on approximation theory support (ii) especially for wide and deep networks [Yarotsky, 2017].

Since $z_h$ is a supremum over an empirical process, it allows us to connect our statistical theorems with theories on empirical processes. Deviation inequalities that bound quantities such as $\lambda_{h,t}$ have been established even for noise $u_i$ that has very heavy tails [Lederer and van de Geer, 2014]. In Section 3, we derive an explicit bound for $\lambda_{h,t}$ for $\ell_1$-regularization and sub-Gaussian noise. Crucial in this derivation, and in controlling $z_h$ in general, is that the index set of the empirical process is the constraint parameter space $\mathcal{A}_h$, rather than the entire parameter space $\mathcal{A}$. This key feature of $z_h$ is due to our novel way of regularizing.

The standard parametrization $\Theta \in \mathcal{A}$ of neural networks is ambiguous: there are typically uncountably many parameters $\Theta \in \mathcal{A}$ that yield the same network $g_\Theta$. This ambiguity remains in our new framework with $(\kappa, \Omega) \in [0, \infty) \times \mathcal{A}_h$. But importantly, our guarantees hold for every solution $(\hat{\kappa}_h, \hat{\Omega}_h)$.

### 3 An example: $\ell_1$-regularization

Theorem 1 can be specified readily to different types of regularization. Indeed, concrete bounds for the prediction error follow directly from concrete bounds for the quantiles $\lambda_{h,t}$. We highlight this
feature in the case of $\ell_1$-regularization, that is, we define $h$ as

$$h(\Omega) := \|\Omega\|_1 := \sum_{l=0}^{L} \sum_{k=1}^{p_l} \sum_{j=1}^{p_{l+1}} |U_{kj}|.$$ 

To fix ideas, we impose two assumptions on the activation functions and the noise: First, we assume that the activation functions satisfy $f'(0_{p_l}) = 0_{p_l}$ and are $a_{\text{Lip}}$-Lipschitz continuous for a constant $a_{\text{Lip}} \in [0, \infty)$ and with respect to the Euclidean norms on their input and output spaces:

$$|f'(z) - f'(z')|_2 \leq a_{\text{Lip}} \|z - z'\|_2 \quad \text{for all } z, z' \in \mathbb{R}^{p_l}.$$ 

This assumption is satisfied by many popular activation functions: for example, the coordinates $x$ versions of ReLU with probability at least $1 - \epsilon$; $h$-versions of ReLU $x \mapsto 0 \lor x$ [Nair and Hinton, 2010], “ leaky” versions of ReLU $x \mapsto (0 \lor x) + (0 \land c x)$ for $c \in (0, 1)$, ELU functions $x \mapsto x \lor 0 + c(e^{x/0} - 1)$ for $c \in (0, 1)$ [Clevert et al., 2015], hyperbolic tangent functions $x \mapsto (e^{2x} - 1)/(e^{2x} + 1)$, or SiL/Swish functions $x \mapsto x/(1 + e^{-x})$ [Ramachandran et al., 2017, Elfwing et al., 2018] (throughout, we use the shorthands $r \lor s := \max\{r, s\}$ and $r \land s := \min\{r, s\}$ for $r, s \in \mathbb{R}$). Feasible Lipschitz constants for these examples are $a_{\text{Lip}} = 1.1$ for SiL/Swish and $a_{\text{Lip}} = 1$ for all other functions.

Second, we assume that the noise variables $u_i$ are independent, centered, and uniformly sub-Gaussian for constants $K, \gamma \in (0, \infty)$ [van de Geer, 2000, Page 126; Vershynin, 2018, Section 2.5]:

$$\max_{i \in \{1, \ldots, n\}} K^2 (E\|u_i\|_2^2 - 1) \leq \gamma^2.$$ 

Using the shorthands $A_1 := \{\Theta \in A : \|\Theta\|_1 \leq 1\}$, $|x|_n = \sqrt{\sum_{i=1}^{n} |x_i|_2^2}/n$, and $P := \sum_{l=0}^{L} p_{l+1} p_l$, we then find the following prediction guarantee for the estimator in (4):

**Theorem 2 (Prediction guarantee for $\ell_1$-regularization).** Assume that $\lambda \geq a(a_{\text{Lip}})^L \|x\|_n \sqrt{L \log(2nP)/n \log(2n)}$, where $a \in (0, \infty)$ is a constant that depends only on the sub-Gaussian parameters $K$ and $\gamma$ of the noise. Then, for $n$ large enough,

$$\text{err}^2(\hat{k}_h g_{\Omega h}) \leq \inf_{\kappa \in [0, \infty)} \left\{ \text{err}^2(\kappa \Omega_h) + 2\lambda \kappa \right\}$$

with probability at least $1 - 1/n$.

The bound establishes essentially a $1/\sqrt{n}$-dependence on the sample size $n$, a sub-linear dependence on the number of hidden layers $L$ (note that $a_{\text{Lip}} \leq 1$ for typical activation functions), and a logarithmic dependence on the number of parameters $P$. The dependencies on the sample size $n$ and the number of parameters $P$ match those of standard bounds in $\ell_1$-regularized linear regression [Hebiri and Lederer, 2013]. But one can argue that the logarithmic dependence on the number of parameters is even more crucial for neural networks: already a small network with $L = 10, p_0 = 100$, and $p_1, \ldots, p_L = 50$ involves $P = 27550$ parameters, which highlights that neural networks typically involve very large $P$.

As an illustration, we can simplify Theorem 2 further in a parametric setting:

**Corollary 1 (Parametric setting).** Assume that $\lambda = a(a_{\text{Lip}})^L \|x\|_n \sqrt{L \log(2nP)/n \log(2n)}$ and that there exist parameters $(\kappa, \Omega_h) \in [0, \infty) \times A_1$ such that $\kappa \ Omega_h (x_i) = g_a(x_i)$ for all $i \in \{1, \ldots, n\}$. Then, for $n$ large enough,

$$\text{err}^2(\hat{k}_h g_{\Omega_h}) \leq 2a \kappa (a_{\text{Lip}})^L \|x\|_n \sqrt{L \log(2nP)/n \log(2n)}$$

with probability at least $1 - 1/n$.

The above choice of $h$ is not the only way to formulate $\ell_1$-constraints. Another way is, for example, $h(\Omega) := \max_{k \in \{0, \ldots, L\}} \sum_{j=1}^{p_{l+1}} |U_{kj}|$. The proofs and results remain virtually the same, and one may choose in practice whatever regularizer is more appropriate or easier to compute. And more broadly, our theories provide a general scheme for deriving prediction guarantees that could account for different regularizers (such as grouped versions of $\ell_1$), activation functions (such as non-Lipschitz functions), and noise (such as heavy-tailed noise) through corresponding bounds for $z_h$. 


4 Related literature

In this section, we relate our results to literature. An immediate difference of most papers mentioned below to ours is that their estimators are regularized through constraints, while we add regularizers to the objective functions. Adding regularization terms is more common in practice than adding constraints. And more importantly, while the constraints always involve model parameters that are unknown in practice (such as good bounds on the sparsity level or the Frobenius norms of the weights), we detail how suitable tuning parameters for the regularization term relate to the known model parameters (such as the sample size or the number of parameters)—see, for example, Theorem 2.

Another difference is that most papers bound misclassification probabilities (probability that a new input vector is mislabeled) or generalization errors (square-root of the expected squared-difference between the estimator and the true data generating function evaluated on a new input vector), while we bound in-sample prediction errors (square-root of the averaged squared-differences between the estimator and the true data generating function—not the outcome—evaluated on the available data).

The papers Bartlett [1998], Bartlett and Mendelson [2002], Ledent et al. [2019], Neyshabur et al. [2015b] derive bounds by using fat-shattering dimensions [Kearns and Schapire, 1994; Anthony and Bartlett, 2009, Section 11.3] or Rademacher/Gaussian complexities [Shalev-Shwartz and Ben-David, 2014, Chapter 26] of sparse or \( \ell_1 \)-related classes of neural networks. Such bounds translate into misclassification bounds or risk bounds for empirical risk minimizers over those classes—see, for example, [Bartlett, 1998, Section 2] and [Bartlett and Mendelson, 2002, Theorem 8], respectively.

A bound for the misclassification probabilities for empirical risk minimization over \( \ell_1 \)-balls is Bartlett [1998, Theorem 28]. The measure of complexity used for deriving these bounds is fat-shattering. A common denominator of their theories and our Section 3 is the \( \ell_1 \)-regularization. But besides considering classification rather than prediction, their bounds differ in their dependence on the network architecture: for example, they allow for infinite widths, have a slightly different dependence on the input (their \( \max_{i \in \{1, \ldots, n\}} |x_i|_\infty \) versus our \( \|x\|_n \)), and have an exponential rather than sub-linear dependence on the depths \( L \).

Other bounds for the misclassification probabilities and prediction errors of risk minimizers can be derived from Bartlett and Mendelson [2002] and Ledent et al. [2019]. For example, Bartlett and Mendelson [2002, Theorem 18] entails bounds for \( \ell_1 \)-regularized empirical risk minimization over two-layer neural networks; the bounds are similar to the ones in our Corollary 1 when \( L = 1 \). Ledent et al. [2019] derive guarantees that cater to classification with many classes.

Bounds for Rademacher complexities of network classes with general group-norm constraints are provided in [Neyshabur et al., 2015b]. A specification to \( \ell_1 \)-constraints is their [Neyshabur et al., 2015b, Corollary 2]. A feature of their bounds is that they allow for infinite widths. But besides having a slightly different dependence on the input than our bounds in Theorem 2 and Corollary 2 (their \( \max_{i \in \{1, \ldots, n\}} |x_i|_\infty \) versus our \( \|x\|_n \)), their bounds are restricted to ReLU activation and have an exponential dependence on the number of hidden layers \( L \).

An approach for deriving guarantees that is different to the fat-shattering/Rademacher approaches relates to nonparametric statistics [Schmidt-Hieber, 2017]. Their paper provides upper and lower bounds for empirical risk minimization over classes of sparse networks. A feature of these bounds is that they apply beyond the empirical risk minimizer; in particular, they provide insights into how inaccuracies in computing empirical risk estimators affect the estimators’ risks. A connection to our results is that \( \ell_1 \)-regularization typically induces sparsity. But the estimators, bounds, proofs, and the framework more generally in Schmidt-Hieber [2017] differ largely from ours; for example: 1. Empirical risk minimization over sparse networks requires prior knowledge about the necessary sparsity level. 2. In contrast to our setup, theirs assumes that the parameters are bounded by one, that the network functions are bounded, and that the noise is standard normal. 3. The assumptions on the setup and the restriction to sparse networks ensures small covering numbers (while we leverage our novel parametrization to bound covering numbers) and changes the range of potential proof techniques more generally. 4. The results in Schmidt-Hieber [2017] are limited to ReLU activation, while our results hold for arbitrary (Section 2.2) or a wide variety of activations (Section 3).
Empirical risk minimizers over sparse networks such as in Schmidt-Hieber [2017] are very different from practical estimators, especially because they are based on (i) non-convex objective functions and (ii) computationally intractable given the combinatorial constraints. Our objective functions are still non-convex, but in view of our general and practical regularization strategy, we make considerably forward in closing this gap. In particular, since $\ell_1$-regularization is well-known to induce sparsity, one could see our $\ell_1$-approach in Section 3 as a more practical version of the $\ell_0$-constraint in Schmidt-Hieber [2017].

Although from very different angles, both Schmidt-Hieber [2017] and our paper highlight that estimating neural network can benefit from sparsity. Kohler and Langer [2019], in contrast, argue that sparsity might not be necessary for classes of extremely deep networks (in particular, they consider $L$ polynomially increasing in $n$), but such architectures are currently not used in practice.

The rates of the bounds in our Section 3 as well as in most other mentioned papers are essentially $1/\sqrt{n}$ in the number of samples. In contrast, the results in Schmidt-Hieber [2017] indicate the possibility of rates as fast as $1/n$. But one can verify that these faster rates follow only under very restrictive assumptions, and we believe that the $1/\sqrt{n}$-rate cannot be improved in general: while a formal proof still needs to be established, a corresponding statement has already been proved for $\ell_1$-regularized linear regression [Dalalyan et al., 2017, Proposition 4]. In this sense, we might claim some optimality of our results.

Feng and Simon [2017] provide theoretical and practical insights into the effect of group regularization at the input level.

Recent insights into the roles of widths and depths in Rademacher analyses of neural networks with Frobenius norm constraints are provided in Golowich et al. [2018]. In particular, the bounds in Golowich et al. [2018, Theorem 1] share our Theorem 2’s sub-linear dependence on the number of layers. Hence, both Golowich et al. [2018] and our paper highlight that even very deep networks can be learned effectively.

5 Further technical results

We now establish Lipschitz and complexity properties of neural networks. These results are used in our proofs but might also be of interest by themselves. To start, we define operator norms of the parameters and the weight matrices by

$$\|\Theta\|_2 := \sqrt{\sum_{l=0}^{L} |W^l|^2_2} \quad \text{and} \quad |W^l|_2 := \sigma_{\max}(W^l),$$

respectively, where $\sigma_{\max}(W^l)$ is the largest singular value of $W^l$. We also define Frobenius norms of the parameter and weight matrices by

$$\|\Theta\|_F := \sqrt{\sum_{l=0}^{L} |W^l|^2_F} \quad \text{and} \quad |W^l|_F := \sqrt{\sum_{k=1}^{p_{l+1}} \sum_{j=1}^{p_l} (W^l_{kj})^2}.$$

We then define the Euclidean norm of vectors by $\|v\|_2 := \sqrt{\sum_{i=1}^{d} (v_i)^2}$ for $v \in \mathbb{R}^d$. And finally, the prediction distance of any two networks $g_{\Theta}$ and $g_{\Gamma}$ with $\Theta, \Gamma \in \mathcal{A}$ is

$$\|g_{\Theta} - g_{\Gamma}\|_n := \frac{1}{n} \sum_{i=1}^{n} (g_{\Theta}(x_i) - g_{\Gamma}(x_i))^2,$$

and similarly,

$$\|g_{\Theta}\|_n := \frac{1}{n} \sum_{i=1}^{n} (g_{\Theta}(x_i))^2.$$

The Lipschitz property of neural networks is then as follows.
Proposition 2 (Lipschitz property of NNs). Assume that the activation functions $f^l : \mathbb{R}^p \to \mathbb{R}^p$ are $a_{Lip}$-Lipschitz with respect to the Euclidean norms on their input and output spaces. Then, it holds for every $x \in \mathbb{R}^d$ and $\Theta = (W^L, \ldots, W^0), \Gamma = (V^L, \ldots, V^0) \in A$ that

$$\|g_\Theta(x) - g_\Gamma(x)\| \leq c_{Lip}(x)\|\Theta - \Gamma\|_F$$

with $c_{Lip}(x) := 2(a_{Lip})^L \sqrt{L} \|x\|_2 \max_{l \in \{0, \ldots, L\}} \prod_{j \in \{0, \ldots, L\}, j \neq l} (|W^j|_2 \vee |V^j|_2)$.

And similarly, it holds that

$$\|g_\Theta - g_\Gamma\|_n \leq \tau_{Lip}\|\Theta - \Gamma\|_F$$

with $\tau_{Lip} := 2(a_{Lip})^L \sqrt{L} \|x\|_n \max_{l \in \{0, \ldots, L\}} \prod_{j \in \{0, \ldots, L\}, j \neq l} (|W^j|_2 \vee |V^j|_2)$.

This property is extremely helpful in bounding the quantiles of the empirical processes. The above inequalities do not guarantee that the networks are Lipschitz in the parameter in general (because $c_{Lip}(x)$ and $\tau_{Lip}$ depend on the parameters), but they guarantee that the networks are Lipschitz and bounded over typical sets that originate from our regularization scheme: For every $\Omega = (U^L, \ldots, U^0) \in A_1$, it holds that $\|U^{j}\|_{2} \leq \|U^{j}\|_{F} \leq \|\Omega\|_{F} \leq \|\Omega\|_{1} \leq 1$. Moreover, for $\Gamma = (V^L, \ldots, V^0) = (0_{p_{l+1} \times p_{l}}, \ldots, 0_{p_{1} \times p_{0}})$, we get that $g_\Gamma(x) = 0$. Hence, Proposition 2 entails the following:

Corollary 2 (Lipschitz and boundedness on $A_1$). Under the conditions of Proposition 2, it holds for every $\Omega, \Gamma \in A_1$ that

$$\|g_\Omega - g_\Gamma\|_n \leq c_{Lip} \|\Omega - \Gamma\|_F$$

and that

$$\|g_\Omega\|_n \leq c_{Lip}$$

with $c_{Lip} := 2(a_{Lip})^L \sqrt{L} \|x\|_n$.

To derive the complexity properties, we denote covering and entropy numbers by $N(r, \mathcal{T}, \| \cdot \|)$ and $H(r, \mathcal{T}, \| \cdot \|) := \log N(r, \mathcal{T}, \| \cdot \|)$, respectively, where $r \in (0, \infty)$, $\mathcal{T}$ is a set, and $\| \cdot \|$ is a (pseudo-)norm on an ambient space of $\mathcal{T}$ [van der Vaart and Wellner, 1996, Page 98]. We use these numbers to define a complexity measure for a collection of networks $\mathcal{G}_h := \{g_\Omega : \Omega \in A_h\}$ by

$$J(\delta, \sigma, A_h) := \int_{\delta/(8\sigma)}^\infty H^{1/2}(r; \mathcal{G}_h, \| \cdot \|_n) dr$$

(8)

for $\delta, \sigma \in (0, \infty)$ [van de Geer, 2000, Section 3.3]. Almost in line with standard terminology, we call this complexity measure the Dudley integral [Vershynin, 2018, Section 8.1]. We can bound the complexity of the class of neural networks $A_1 := \{g_\Omega : \Omega \in A_1\}$ that have parameters in the constraint set $A_1$ as follows:

Proposition 3 (Complexity properties of NNs). Assume that the activation functions $f^l : \mathbb{R}^p \to \mathbb{R}^p$ are $a_{Lip}$-Lipschitz continuous with respect to the Euclidean norms on their input and output spaces. Then, it holds for every $r \in (0, \infty)$ and $\delta, \sigma \in (0, \infty)$ that satisfy $\delta \leq 8\sigma c_{Lip}$ that

$$H(r, A_1, \| \cdot \|_n) \leq \frac{15(c_{Lip})^2}{r^2} \log \left( \frac{c_{Lip} P}{r} \vee eP \right)$$

and

$$J(\delta, \sigma, A_1) \leq 4c_{Lip} \sqrt{\log \left( \frac{8\sigma c_{Lip} P}{\delta} \vee eP \right) \log \left( \frac{8\sigma c_{Lip}}{\delta} \right)},$$

where we recall that $c_{Lip} := 2(a_{Lip})^L \sqrt{L} \|x\|_n$.

6 Additional materials and proofs

We now state some auxiliary results and then prove our claims.
6.1 Additional materials

We first provide three auxiliary results that we use in our proofs. We start with a slightly adapted version of van de Geer [2000, Corollary 8.3]

**Lemma 1** (Suprema over Gaussian processes). Consider a set \( \mathcal{A} \subset \mathcal{A} \) and a constant \( R \in [0, \infty) \) such that \( \sup_{\Theta \in \mathcal{A}} |g_\Theta|_{\mathcal{A}} \leq R \). Assume that the noise random variables \( u_1, \ldots, u_n \) are independent, centered, and uniformly sub-Gaussian as specified on Page 5. Then, there is a constant \( a_{sub} \in (0, \infty) \) that depends only on \( K \) and \( \gamma \) such that for all \( \delta, \sigma \in (0, \infty) \) that satisfy \( \delta < \sigma R \)

\[
\sqrt{n}\delta \geq a_{sub}(J(\delta, \sigma, \mathcal{A}) \lor R),
\]

it holds that

\[
P\left\{ \sup_{\Theta \in \mathcal{A}} \left| \frac{1}{n} \sum_{i=1}^{n} g_\Theta(x_i)u_i \right| \geq \delta \right\} \leq \frac{1}{\delta^2} \left\{ \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 \leq \sigma^2 \right\} \leq a_{sub}e^{-\frac{a_{sub}^2}{(\sigma R)^2}}.
\]

This result is used to bound \( \lambda_{\ell_1,t} \).

We then turn to a Lipschitz property of metric entropy:

**Lemma 2** (Entropy transformation for Lipschitz functions). Consider sets \( \mathcal{A} \subset \mathcal{A} \) and \( \mathcal{G} := \{g_\Theta : \Theta \in \mathcal{A} \} \) and a metric \( \rho : \mathcal{A} \times \mathcal{A} \to \mathbb{R} \). Assume that \( |g_\Theta(x) - g_\Gamma(x)| \leq k_{Lip}(\Theta, \Gamma) \) for every \( \Theta, \Gamma \in \mathcal{A} \) and \( x \in \mathbb{R}^d \) and a fixed function \( k_{Lip} : \mathbb{R}^d \to [0, \infty) \). Then,

\[
H(r, \mathcal{G}, \| \cdot \|_n) \leq H\left(\frac{r}{k_{Lip}}\right) \leq k_{Lip} \quad \text{for all } r \in (0, \infty),
\]

where \( k_{Lip} = \sqrt{\sum_{i=1}^{n} (k_{Lip}(x_i))^2/n} \).

We use the convention \( a/0 = \infty \) for \( a \in (0, \infty) \). The result allows us to bound entropies on the parameter spaces instead of the network spaces. We prove the lemma in the following section.

We continue with a standard bound on entropies [van der Vaart and Wellner, 1996, Page 94; van de Geer, 2000, Page 20]:

**Lemma 3** (Entropy of a ball). Let \( B_d(v) \) be a ball in \( d \)-dimensional Euclidean space with radius \( v \in (0, \infty) \). Then,

\[
H\left( r, B_d(v), \| \cdot \|_2 \right) \leq d \log\left( \frac{4r + v}{r} \right) \leq d \log\left( \frac{5v}{r} \lor 5 \right) \quad \text{for all } r \in (0, \infty).
\]

This result is used for bounding the entropies over the parameter spaces.

We conclude with a deviation inequality for the noise.

**Lemma 4** (Deviation of sub-Gaussian noise). Assume that the noise variables \( u_1, \ldots, u_n \) are independent, centered, and uniformly sub-Gaussian as stipulated on Page 5. Then,

\[
P\left( \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 \geq v \right) \leq e^{-\frac{nv}{12\gamma^2}} \quad \text{for all } v \in (2\gamma^2, \infty).
\]

This deviation inequality is tailored to our needs in the proof of Theorem 2.

6.2 Proofs

We provide here the proofs of our claims.

6.2.1 Proof of Proposition 1

**Proof.** We prove the two directions in order.

**Direction 1:** Fix a \( \Theta = (W^L, \ldots, W^0) \in \mathcal{A} \). Assume first that \( W^l = 0_{p_{l+1} \times p_l} \) for an \( l \in \{0, \ldots, L \} \). In view of the definition of neural networks in (2) and the assumed non-negative homogeneity of the activation functions, it then holds that

\[
g_\Theta(x) = W^L \cdot f^L(0_{p_{L+1} \times p_L} \cdot f^L(\ldots W^1 \cdot f^1(W^0 x)))
\]

\[
= W^L \cdot f^L(0 \cdot 0_{p_{L+1} \times p_L} \cdot f^L(\ldots W^1 \cdot f^1(W^0 x))) = 0
\]


for all $x \in \mathbb{R}^d$. Therefore, $\kappa := 0$ and all $\Omega \in \mathcal{A}_h$ satisfy $\kappa g_\Omega = g_\Theta$, as desired.

Assume now that $W^l \neq 0_{p_{l+1} \times p_l}$ for all $l \in \{0, \ldots, L\}$. Define $\kappa := (h(\Theta))^{(L+1)/k}$ and $\Omega := \Theta/\kappa^{1/(L+1)} = (W^L/\kappa^{1/(L+1)}, \ldots, W^0/\kappa^{1/(L+1)})$ if $\kappa^{1/(L+1)} \neq 0$. We need to show that 1. $\kappa \in (0, \infty)$ and $\Omega \in \mathcal{A}_h$ and 2. $g_\Theta = \kappa g_\Omega$.

Since $h$ is assumed positive definite, it holds that $h(\Theta) \in (0, \infty)$ and, therefore, $\kappa \in (0, \infty)$. The fact that $\kappa > 0$ also ensures that the parameter $\Omega$ is well-defined, and we can invoke the assumed non-negative homogeneity of degree $k$ of $h$ to derive

$$h(\Omega) = h(\Theta/\kappa^{1/(L+1)}) = (\kappa^{-1/(L+1)})^k h(\Theta) = ((h(\Theta))^{(L+1)/k})^{-k/(L+1)} h(\Theta) = 1.$$

This verifies 1.

We can then invoke the assumed non-negative homogeneity of degree 1 of the activation functions to derive for all $x \in \mathbb{R}^d$ that

$$\kappa g_\Omega(x) = \frac{W^L}{\kappa^{1/(L+1)}} f^L \left( \ldots \frac{W^1}{\kappa^{1/(L+1)}} f^1 \left( \frac{W^0}{\kappa^{1/(L+1)}} x \right) \right)$$

$$= \frac{W^L}{\kappa^{1/(L+1)}} f^L \left( \ldots \frac{W^1}{\kappa^{1/(L+1)}} f^1 (W^0 x) \right)$$

$$= \ldots$$

$$= \kappa \frac{W^L f^L (\ldots W^1 f^1 (W^0 x))}{(\kappa^{1/(L+1)})^{L+1}}$$

$$= W^L f^L (\ldots W^1 f^1 (W^0 x))$$

$$= g_\Theta(x).$$

This verifies 2.

**Direction 2:** Fix a $\kappa \in [0, \infty)$ and a $\Omega = (U^L, \ldots, U^0) \in \mathcal{A}_h$, and define $\Theta := \kappa^{1/(L+1)} \Omega = (\kappa^{1/(L+1)} U^L, \ldots, \kappa^{1/(L+1)} U^0)$. We then invoke the assumed non-negative homogeneity of degree 1 of the activation functions to derive for all $x \in \mathbb{R}^d$ that

$$g_\Theta(x) = \kappa^{1/(L+1)} U^L f^L \left( \ldots \kappa^{1/(L+1)} U^1 f^1 (\kappa^{1/(L+1)} U^0 x) \right)$$

$$= \kappa^{1/(L+1)} U^L f^L \left( \ldots (\kappa^{1/(L+1)})^2 U^1 f^1 (U^0 x) \right)$$

$$= \ldots$$

$$= \kappa U^L f^L (\ldots U^1 f^1 (U^0 x))$$

$$= \kappa g_\Omega(x),$$

as desired. \(\square\)

**6.2.2 Proof of Theorem 1**

**Proof.** Since $(\hat{\kappa}_h, \hat{\Omega}_h)$ is a minimizer of the objective function in (4), we find for every $\kappa \in [0, \infty)$ and $\Omega \in \mathcal{A}_h$ that

$$\frac{1}{n} \sum_{i=1}^n (y_i - \hat{\kappa}_h g_{\hat{\Omega}_h}(x_i))^2 + \lambda \hat{\kappa}_h \leq \frac{1}{n} \sum_{i=1}^n (y_i - \kappa g_{\Omega}(x_i))^2 + \lambda \kappa.$$

Replacing the $y_i$'s via the model in (1) then yields

$$\frac{1}{n} \sum_{i=1}^n (g_\star(x_i) + u_i - \hat{\kappa}_h g_{\hat{\Omega}_h}(x_i))^2 + \lambda \hat{\kappa}_h \leq \frac{1}{n} \sum_{i=1}^n (g_\star(x_i) + u_i - \kappa g_{\Omega}(x_i))^2 + \lambda \kappa.$$
Expanding the squared-terms and rearranging terms, we get
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \hat{\kappa}_h g_{\Omega_h}(x_i) - g_\ast(x_i) \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} (\kappa g_{\Omega}(x_i) - g_\ast(x_i))^2 \\
+ \frac{2}{n} \sum_{i=1}^{n} \hat{\kappa}_h g_{\Omega_h}(x_i) u_i - \frac{2}{n} \sum_{i=1}^{n} \kappa g_{\Omega}(x_i) u_i + \lambda \kappa - \lambda \hat{\kappa}_h.
\]

We can then bound the sums on the second line to obtain
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \hat{\kappa}_h g_{\Omega_h}(x_i) - g_\ast(x_i) \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} (\kappa g_{\Omega}(x_i) - g_\ast(x_i))^2 \\
+ \hat{\kappa}_h \sup_{\Gamma \in A_h} \left| \frac{2}{n} \sum_{i=1}^{n} g_{\Gamma}(x_i) u_i \right| + \kappa \sup_{\Gamma \in A_h} \left| \frac{2}{n} \sum_{i=1}^{n} g_{\Gamma}(x_i) u_i \right| + \lambda \kappa - \lambda \hat{\kappa}_h.
\]

The second line can then be consolidated by virtue of the assumption on $\lambda$: with probability at least $1 - t$, it holds that
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \hat{\kappa}_h g_{\Omega_h}(x_i) - g_\ast(x_i) \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} (\kappa g_{\Omega}(x_i) - g_\ast(x_i))^2 + 2\lambda \kappa.
\]

Taking the infimum over $\kappa \in [0, \infty)$ and $\Omega \in A_h$ and invoking the definition of $\text{err}^2(\cdot)$ on Page 4 gives the desired result. \(\square\)

### 6.2.3 Proof of Proposition 2

**Proof.** The proof peels the networks into inner and outer subnetworks. The inner subnetworks of a network $g_\Theta \in \mathcal{G} := \{g_\Theta : \Theta \in \mathcal{A}\}$ are vector-valued functions defined by
\[
S_{0}g_\Theta : \mathbb{R}^d \rightarrow \mathbb{R}^d \\
x \mapsto S_{0}g_\Theta(x) := x
\]
and
\[
S_{l}g_\Theta : \mathbb{R}^d \rightarrow \mathbb{R}^{p_l} \\
x \mapsto S_{l}g_\Theta(x) := f^l(W^{l-1}f^{l-1}(\ldots W^1f^1(W^0x)))
\]
for $l \in \{1, \ldots, L\}$. Similarly, the outer subnetworks of $g_\Theta$ are real-valued functions defined by
\[
S_{l}^1g_\Theta : \mathbb{R}^{p_{l-1}} \rightarrow \mathbb{R} \\
z \mapsto S_{l}^1g_\Theta(z) := W^{L}f^L(\ldots W^lf^l(W^{l-1}z))
\]
for $l \in \{1, \ldots, L\}$ and
\[
S_{L+1}^Lg_\Theta : \mathbb{R}^L \rightarrow \mathbb{R} \\
z \mapsto S_{L+1}^Lg_\Theta(z) := W^{L}z.
\]

The initial network can be split into an inner and an outer network along every layer $l \in \{1, \ldots, L + 1\}$:
\[
g_\Theta(x) = S_{l-1}g_\Theta(S_{l}g_\Theta(x)).
\]

This observation is the basis for the following derivations.

We now show a contraction property for the inner subnetworks and a Lipschitz property for the outer subnetworks. Using the assumption that $z \mapsto f^{l-1}(z)$ is $a_{L_{lip}}$-Lipschitz, we get for every $\Theta = (W^L, \ldots, W^0)$ and $x \in \mathbb{R}^d$ that
\[
\|S_{l-1}g_\Theta(x)\|_2 = \|f^{l-1}(W^{l-2}S_{l-2}g_\Theta(x))\|_2 \\
\leq a_{L_{lip}}\|W^{l-2}S_{l-2}g_\Theta(x)\|_2
\]
We now use these contraction and Lipschitz properties for the subnetworks to derive a Lipschitz property for the entire network. We consider two networks \( f \) and \( g \) with parameters \( \Theta \) and \( \Gamma \) respectively, with respect to the Euclidean norms on the input space \( \mathbb{R}^d \) and output spaces \( \mathbb{R}^{p_i} \) and \( \mathbb{R}^p \), respectively.

By similar arguments, we get for every \( z_1, z_2 \in \mathbb{R}^{p_i} \) that
\[
|S^{l+1}g_{\Theta}(z_1) - S^{l+1}g_{\Theta}(z_2)| \\
= |W^L f^L(\ldots W^{l+1} f^{l+1}(W^l z_1)) - W^L f^L(\ldots W^{l+1} f^{l+1}(W^l z_2))| \\
\leq \|W^L\|_2 \|W^{l+1} f^{l+1} - f^l(\ldots W^{l+1} f^{l+1}(W^l z_2))\|_2 \\
\leq a_{\text{Lip}} \|W^L\|_2 \|W^{l+1} f^{l+1} - f^l(\ldots W^{l+1} f^{l+1}(W^l z_2))\|_2 \\
\leq \ldots \\
\leq (a_{\text{Lip}})^{L-l} \|z_1 - z_2\|_2 \prod_{j=l}^{L} \|W^j\|_2
\]
for \( l \in \{0, \ldots, L\} \). In other words, \( z \mapsto S^{l+1}g_{\Theta}(z) \) is Lipschitz with constant \( (a_{\text{Lip}})^{L-l} \prod_{j=l}^{L} \|W^j\|_2 \) with respect to the Euclidean norms on the input space \( \mathbb{R}^{p_i} \) and output space \( \mathbb{R}^p \).

We now use these contraction and Lipschitz properties for the subnetworks to derive a Lipschitz property for the entire network. We consider two networks \( g_{\Theta} \) and \( g_{\Gamma} \) with parameters \( \Theta = (W^L, \ldots, W^0) \in \mathcal{A} \) and \( \Gamma = (V^L, \ldots, V^0) \in \mathcal{A} \), respectively. Our above splitting of the networks applied to \( l = 1 \) and \( l = L + 1 \) and the fact that \( S_0 g_{\Theta}(x) = S_0 g_{\Gamma}(x) = x \) yield
\[
|g_\Theta(x) - g_\Gamma(x)| = |S^1 g_{\Theta}(S_0 g_{\Theta}(x)) - S^{L+1} g_{\Gamma}(S_L g_{\Gamma}(x))| \\
= |S^1 g_{\Theta}(S_0 g_{\Gamma}(x)) - S^{L+1} g_{\Gamma}(S_L g_{\Gamma}(x))|
\]
Elementary algebra and the fact that \( S^{l+1} g_{\Theta}(S_L g_{\Gamma}(x)) = S^{l+1} g_{\Theta}(f^l(V^{l-1} S_{l-1} g_{\Gamma}(x))) \) then allow us to derive
\[
|g_\Theta(x) - g_\Gamma(x)| \\
= \left| S^1 g_{\Theta}(S_0 g_{\Gamma}(x)) - \sum_{l=1}^{L-1} \left( S^{l+1} g_{\Theta}(S_l g_{\Gamma}(x)) - S^{l+1} g_{\Theta}(S_l g_{\Gamma}(x)) \right) \right| \\
- \left( S^{L+1} g_{\Theta}(S_{L+1} g_{\Gamma}(x)) - S^{L+1} g_{\Theta}(S_{L+1} g_{\Gamma}(x)) \right) \\
= \left| S^2 g_{\Theta}(f^L(W^0 S_0 g_{\Gamma}(x))) - \sum_{l=1}^{L-1} \left( S^{l+1} g_{\Theta}(f^l(V^{l-1} S_{l-1} g_{\Gamma}(x))) - S^{l+1} g_{\Theta}(f^l(V^{l-1} S_{l-1} g_{\Gamma}(x))) \right) \right| \\
- \left( S^{L+1} g_{\Theta}(f^L(V^L S_{L-1} g_{\Gamma}(x))) + S^{L+1} g_{\Theta}(S_L g_{\Gamma}(x)) - S^{L+1} g_{\Theta}(S_L g_{\Gamma}(x)) \right) \\
= \sum_{l=1}^{L} \left( S^{l+1} g_{\Theta}(f^l(W^{l-1} S_{l-1} g_{\Gamma}(x))) - S^{l+1} g_{\Theta}(f^l(V^{l-1} S_{l-1} g_{\Gamma}(x))) \right) \\
+ \left( S^{L+1} g_{\Theta}(S_L g_{\Gamma}(x)) - S^{L+1} g_{\Theta}(S_L g_{\Gamma}(x)) \right) \\
= \sum_{l=1}^{L} \left( S^{l+1} g_{\Theta}(f^l(W^{l-1} S_{l-1} g_{\Gamma}(x))) - S^{l+1} g_{\Theta}(f^l(V^{l-1} S_{l-1} g_{\Gamma}(x))) \right) + W^L S_L g_{\Gamma}(x) - V^L S_L g_{\Gamma}(x)
\]
We now study the last sum in that bound: First, we observe that
\[ S^{l+1} g_\Theta \left( f^l (W^{l-1} S_{l-1} g_{l'} (x)) \right) - S^{l+1} g_\Theta \left( f^l (V^{l-1} S_{l-1} g_{l'} (x)) \right) + \|(W^L - V^L) S_L g_{l'} (x)\|.
\]

We bound this further by using 1. the above-derived Lipschitz property of \(S^{l+1} g_\Theta\), 2. the assumption that the \(f^l\) are \(a_{Lip}\)-Lipschitz, 3. the properties of the \(\ell_2\)-norm, and 4. the above-derived contraction property of \(S_{l-1} g_{l'}\):
\[
|g_\Theta (x) - g_{l'} (x)| \\
\leq \sum_{l=1}^{L} \left[ (a_{Lip})^{L-1} \prod_{j=l}^{L} \|W^j\|_2 \right] \|f^l (W^{l-1} S_{l-1} g_{l'} (x)) - f^l (V^{l-1} S_{l-1} g_{l'} (x))\|_2 + \|(W^L - V^L) S_L g_{l'} (x)\| \\
\leq \sum_{l=1}^{L} \left[ (a_{Lip})^{L-l+1} \prod_{j=l}^{L} \|W^j\|_2 \right] \|W^{l-1} S_{l-1} g_{l'} (x) - V^{l-1} S_{l-1} g_{l'} (x)\|_2 + \|(W^L - V^L) S_L g_{l'} (x)\|_2 \\
\leq \sum_{l=1}^{L} \left[ (a_{Lip})^{L-l+1} \prod_{j=l}^{L} \|W^j\|_2 \right] \|W^{l-1} - V^{l-1}\|_2 \|S_{l-1} g_{l'} (x)\|_2 + \|(W^L - V^L) S_L g_{l'} (x)\|_2 \\
\leq \sum_{l=1}^{L} \left[ (a_{Lip})^{L-l+1} \prod_{j=l}^{L} \|W^j\|_2 \right] \|W^{l-1} - V^{l-1}\|_2 \left( a_{Lip} \prod_{j=0}^{L-2} \|V^j\|_2 \right) \|x\|_2 + \|(W^L - V^L) \left( a_{Lip} \prod_{j=0}^{L-1} \|V^j\|_2 \right) \|x\|_2,
\]

where we set \(\prod_{j=0}^{L-1} \|V^j\|_2 := 1\). Consolidating and rearranging then yields
\[
|g_\Theta (x) - g_{l'} (x)| \\
\leq (a_{Lip})^L \left( \sum_{l=1}^{L} \left[ \prod_{j \in \{0, \ldots, L\}} \|W^j\|_2 \lor \|V^j\|_2 \right] \|W^{l-1} - V^{l-1}\|_2 \|x\|_2 + \prod_{j=0}^{L-1} \left( \|W^j\|_2 \lor \|V^j\|_2 \right) \|W^L - V^L\|_2 \|x\|_2 \right) \\
= (a_{Lip})^L \left( \sum_{l=1}^{L+1} \left[ \prod_{j \in \{0, \ldots, L\}} \|W^j\|_2 \lor \|V^j\|_2 \right] \|W^{l-1} - V^{l-1}\|_2 \|x\|_2 \right) \\
\leq (a_{Lip})^L \|x\|_2 \max_{l \in \{1, \ldots, L+1\}} \left\{ \prod_{j \in \{0, \ldots, L\}} \left( \|W^j\|_2 \lor \|V^j\|_2 \right) \right\} \sum_{m=1}^{L+1} \|W^{m-1} - V^{m-1}\|_2 \\
= (a_{Lip})^L \|x\|_2 \max_{l \in \{1, \ldots, L\}} \left\{ \prod_{j \in \{0, \ldots, L\}} \left( \|W^j\|_2 \lor \|V^j\|_2 \right) \right\} \sum_{m=0}^{L} \|W^{m} - V^{m}\|_2.
\]

We now study the last sum in that bound: First, we observe that
\[
\sum_{m=0}^{L} \|W^{m} - V^{m}\|_2 = \sqrt{\left( \sum_{m=0}^{L} \|W^{m} - V^{m}\|_2^2 \right)} \\
\leq \sqrt{(L + 1) \sum_{m=0}^{L} \|W^{m} - V^{m}\|_2^2} \\
= \sqrt{L + 1} \sqrt{\sum_{m=0}^{L} \|W^{m} - V^{m}\|_2^2}.
\]
where we use \((\sum_{m=0}^{L} a_m)^2 \leq (L + 1) \sum_{m=0}^{L} (a_m)^2\) with \(a_m := \|W^m - V^m\|_2\). We then bound the last line further to obtain
\[
\sum_{m=0}^{L} \|W^m - V^m\|_2 = \sqrt{L + 1} \|\Theta - \Gamma\|_2 \\
\leq 2\sqrt{L} \|\Theta - \Gamma\|_2 \\
\leq 2\sqrt{L} \|\Theta - \Gamma\|_F,
\]
where we use 1. the definition of the operator norm on Page 7, 2. where we use Page 9 and the definition of the entropy on Page 8. We thus assume as desired.

Using the definition of the prediction distance on Page 7 and the Lipschitz property stipulated in the lemma, we find that
\[
|g_\Theta(x) - g_\Gamma(x)| \leq 2(a_{\text{Lip}})^L \sqrt{L} \|x\|_2 \max_{l \in \{0, \ldots, L\}} \left\{ \prod_{j \in \{0, \ldots, L\} \setminus \{l\}} (\|W^j\|_2 \lor \|V^j\|_2) \right\} \|\Theta - \Gamma\|_F \\
= c_{\text{Lip}}(x) \|\Theta - \Gamma\|_F,
\]
as desired.

The second claim then follows readily:
\[
\|g_\Theta - g_\Gamma\|_n = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (g_\Theta(x_i) - g_\Gamma(x_i))^2} \\
\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} (c_{\text{Lip}}(x_i) \|\Theta - \Gamma\|_F)^2} \\
= \sqrt{\frac{1}{n} \sum_{i=0}^{n} \left(2(a_{\text{Lip}})^L \sqrt{L} \|x_i\|_2 \max_{l \in \{0, \ldots, L\}} \left\{ \prod_{j \in \{0, \ldots, L\} \setminus \{l\}} (\|W^j\|_2 \lor \|V^j\|_2) \right\} \|\Theta - \Gamma\|_F \right)^2} \\
= 2(a_{\text{Lip}})^L \sqrt{\frac{1}{n} \sum_{i=0}^{n} \|x_i\|_2^2 \max_{l \in \{0, \ldots, L\}} \left\{ \prod_{j \in \{0, \ldots, L\} \setminus \{l\}} (\|W^j\|_2 \lor \|V^j\|_2) \right\} \|\Theta - \Gamma\|_F} \\
= \tau_{\text{Lip}} \|\Theta - \Gamma\|_F,
\]
as desired.

\[\square\]

### 6.2.4 Proof of Lemma 2

**Proof.** The case \(\|k_{\text{Lip}}\|_n = 0\) follows directly from our convention \(a/0 = \infty\) for \(a \in (0, \infty)\) on Page 9 and the definition of the entropy on Page 8. We thus assume \(\|k_{\text{Lip}}\|_n > 0\) in the following.

Using the definition of the prediction distance on Page 7 and the Lipschitz property stipulated in the lemma, we find that
\[
|g_\Theta - g_\Gamma| = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (g_\Theta(x_i) - g_\Gamma(x_i))^2} \\
\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} (k_{\text{Lip}}(x_i))^2 (\rho(\Theta, \Gamma))^2} \\
= \|k_{\text{Lip}}\|_n \rho(\Theta, \Gamma).
\]

Now, let \(A'\) be an \(r/\|k_{\text{Lip}}\|_n\)-covering of \(A'\) with respect to the metric \(\rho\). This means that for every \(\Theta \in A'\), there is a \(\Theta_r \in A'\) such that \(\rho(\Theta, \Theta_r) \leq r/\|k_{\text{Lip}}\|_n\). This insight together with the
We then bound the right-hand side of this equality similarly as in the proof

We now think of entropy bound

Claim 1: 

We prove the two claims in order.

Proof. We prove the two claims in order.

6.2.5 Proof of Proposition 3

Proof. We prove the two claims in order.

Claim 1: entropy bound

Our strategy is to move from \(H(r, G_1, \| \cdot \|_n)\) to \(H(r/c_{\text{lip1}}, A_1, \| \cdot \|_F)\) via Proposition 2 and Lemma 2 and then bound the latter covering number by using counting arguments.

Corollary 2 ensures that the function \(\Omega \mapsto g_0\) restricted to the parameter space \(A_1\) is \(c_{\text{lip1}}\)-Lipschitz with respect to the prediction distance \(\| \cdot \|_n\) on the network space and Frobenius norm \(\| \cdot \|_F\) on the parameter space with \(c_{\text{lip1}} = 2(a_{\text{lip}}^2L\sqrt{L|\mathbf{x}|}_n)\). If \(c_{\text{lip1}} = 0\), then \(N(r, G_1, \| \cdot \|_n) = 1\) for all \(r \in (0, \infty)\) and, therefore, \(H(r, G_1, \| \cdot \|_n) = 0\) for all \(r \in (0, \infty)\), which commensurates with the alleged bound. We can thus assume \(c_{\text{lip1}} > 0\) in the following.

We then apply Lemma 2 with \(A' := A_1, G' := G_1, k_{\text{lip}}|n := c_{\text{lip1}}, \) and \(\rho := \| \cdot \|_F\) to obtain

\[
H\left(r, G_1, \| \cdot \|_n\right) \leq H\left(\frac{r}{c_{\text{lip1}}}, A_1, \| \cdot \|_F\right).
\]

We now think of \(A_1\) as a set in \(\mathbb{R}^P\). Defining \(A'' := \{\omega = (\omega_1, \ldots, \omega_P)\in \mathbb{R}^P : \sum_{j=1}^P |\omega_j| \leq 1\}\), we find for every \(r \in (0, \infty)\) and \(\epsilon := r/(\sqrt{2}c_{\text{lip1}}) \in (0, \infty)\),

\[
H\left(\frac{r}{c_{\text{lip1}}}, A_1, \| \cdot \|_F\right) = H(\sqrt{2}\epsilon, A'', \| \cdot \|_2).
\]

We then bound the right-hand side of this equality similarly as in the proof of Loubès and van de Geer [2000, Lemma 4.3]. Let \(\omega_\epsilon = ((\omega_\epsilon)_1, \ldots, (\omega_\epsilon)_P) \in A''\) be a sparse \(\epsilon\)-approximation of a given \(\omega \in A''\) in the sense that

\[
\sum_{j \in I_\epsilon} |\omega_j - (\omega_\epsilon)_j|^2 \leq \epsilon^2
\]

with \(I_\epsilon := \{j \in \{1, \ldots, P\} : |\omega_j| > \epsilon^2\}\) and in the sense that \((\omega_\epsilon)_j = 0\) for all \(j \in I_\epsilon^c := \{j \in \{1, \ldots, P\} : |\omega_j| \leq \epsilon^2\}\). (Such a vector always exists: a trivial solution is \((\omega_\epsilon)_j = \omega_j\) for \(j \in I_\epsilon\)}
and \((\omega_j) = 0\) for \(j \in I^c\)). The \(\ell_2\)-distance between \(\omega\) and \(\omega_\varepsilon\) is bounded by \(\sqrt{2}\varepsilon\):

\[
\|\omega - \omega_\varepsilon\|_2 = \sqrt{\sum_{j=1}^{P} |\omega_j - (\omega_\varepsilon)_j|^2} = \sqrt{\sum_{j \in I_\varepsilon} |\omega_j|^2 + \sum_{j \in I^c_\varepsilon} |\omega_j - (\omega_\varepsilon)_j|^2} \\
\leq \sqrt{\varepsilon^2 + \sum_{j \in I^c_\varepsilon} |\omega_j|^2} \\
\leq \sqrt{\varepsilon^2 + \varepsilon^2 \sum_{j \in I^c_\varepsilon} |\omega_j|^2} \\
\leq \sqrt{\varepsilon^2 + \varepsilon^2 \sum_{j=1}^{P} |\omega_j|^2} \\
\leq \sqrt{2}\varepsilon.
\]

Now define

\[
N_\omega(\varepsilon) := |I_\varepsilon| \quad \text{and} \quad N(\varepsilon) := \left\lfloor \frac{1}{\varepsilon^2} \right\rfloor,
\]

where \(|I|\) denotes the cardinality of a set \(I\) and \(\left\lfloor 1/\varepsilon^2 \right\rfloor\) the integer part of \(1/\varepsilon^2\). Observe that

\[
N_\omega(\varepsilon)\varepsilon^2 \leq \sum_{j \in I_\varepsilon} |\omega_j| \leq \sum_{j=1}^{P} |\omega_j| \leq 1,
\]

that is, \(N_\omega(\varepsilon) \leq 1/\varepsilon^2\). Since \(N_\omega(\varepsilon)\) is an integer, this observation entails

\[
N_\omega(\varepsilon) \leq \left\lfloor \frac{1}{\varepsilon^2} \right\rfloor = N(\varepsilon).
\]

We conclude that the \(\sqrt{2}\varepsilon\)-covering number of \(A''\) with respect to \(\| \cdot \|_2\) is bounded by the product of (i) the number of possible sparsity patterns of \(P\)-dimensional vectors with \(N(\varepsilon)\) non-zero entries and (ii) the \(\varepsilon\)-covering number of an \(\ell_1\)-ball with radius 1 in the \(N(\varepsilon)\)-dimensional Euclidean space.

The quantity in (i) is bounded by \(\binom{P}{N(\varepsilon)} \leq P^{N(\varepsilon)}\) by counting arguments; since \(\ell_1\)-balls are contained their \(\ell_2\)-counterparts, the quantity in (ii) is bounded by \(\exp \left( N(\varepsilon) \log (5/\varepsilon \vee 5) \right)\) by Lemma 3. Plugging these observations into the earlier display yields

\[
H(\sqrt{2}\varepsilon, A'', | \cdot |_2) = \log N(\sqrt{2}\varepsilon, A'', \| \cdot \|_2) \\
\leq \log (P^{N(\varepsilon)} \cdot \exp (N(\varepsilon) \log (5/\varepsilon \vee 5))) \\
= N(\varepsilon) \log (P(5/\varepsilon \vee 5)).
\]

Collecting the pieces and recalling that \(\varepsilon = r/(\sqrt{2}c_{\text{lip}})\) then yields

\[
H(r, G_1, \| \cdot \|_n) \leq H(\sqrt{2}\varepsilon, A'', \| \cdot \|_2) \\
\leq N(\varepsilon) \log (P(5/\varepsilon \vee 5)) \\
\leq \frac{1}{\varepsilon^2} \log (P(5/\varepsilon \vee 5)) \\
= \frac{2(c_{\text{lip}})^2}{r^2} \log \left( \frac{5\sqrt{2}c_{\text{lip}} \sqrt{P}}{r} \vee 5P \right) \\
\leq \frac{15(c_{\text{lip}})^2}{r^2} \log \left( \frac{c_{\text{lip}} P}{r} \vee eP \right),
\]

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as desired.

Claim 2: Dudley bound

Our strategy is to use Claim 1 to prove that

$$J(\delta, \sigma, A_1) \leq 4c_{\text{Lip1}} \sqrt{\log \left( \frac{8\sigma c_{\text{Lip1}} P}{\delta} \vee eP \right) \log \left( \frac{8\sigma R}{\delta} \right)}$$

and then to use Corollary 2 to formulate the bound in the desired way.

We first split the Dudley integral into two parts according to

$$J(\delta, \sigma, A_1) = \int_{\delta/(8\sigma)}^R H^{1/2}(r, \mathcal{G}_1, \| \cdot \|_n) dr + \int_{r > R} H^{1/2}(r, \mathcal{G}_1, \| \cdot \|_n) dr$$

for $R := \sup_{\Gamma \in A_1} \| g_{\Gamma} \|_n$. Since $\| g_{\Gamma} - g_{\Gamma'} \|_n = \| g_{\Gamma'} \|_n \leq \sup_{\Gamma \in A_1} \| g_{\Gamma} \|_n = R$ for all $\Gamma \in A_1$ and $0_A := (0_{p_L \times p_1}, \ldots, 0_{p_1 \times p_0})$, it holds that $N(r, \mathcal{G}_1, \| \cdot \|_n) = 1$ for all $r > R$ and, consequently, $H^{1/2}(r, \mathcal{G}_1, \| \cdot \|_n) = 0$ for all $r > R$. Therefore, the Dudley integral simplifies to

$$J(\delta, \sigma, A_1) = \int_{\delta/(8\sigma)}^R H^{1/2}(r, \mathcal{G}_1, \| \cdot \|_n) dr.$$

Using this equality together with the bound from Claim 1, we obtain that

$$J(\delta, \sigma, A_1) = \int_{\delta/(8\sigma)}^R H^{1/2}(r, \mathcal{G}_1, \| \cdot \|_n) dr$$

$$= \int_{\delta/(8\sigma)}^R \left( \frac{15(c_{\text{Lip1}})^2}{r^2} \log \left( \frac{c_{\text{Lip1}} P}{r} \vee eP \right) \right)^{1/2} dr$$

$$\leq 4c_{\text{Lip1}} \sqrt{\log \left( \frac{8\sigma c_{\text{Lip1}} P}{\delta} \vee eP \right) \log \left( \frac{8\sigma R}{\delta} \right)}.$$

Since $R \leq c_{\text{Lip1}}$ by Corollary 2, we can bound the last term further to get

$$J(\delta, \sigma, A_1) \leq 4c_{\text{Lip1}} \sqrt{\log \left( \frac{8\sigma c_{\text{Lip1}} P}{\delta} \vee eP \right) \log \left( \frac{8\sigma c_{\text{Lip1}}}{\delta} \right)},$$

as desired.

\[\square\]

6.2.6 Proof of Lemma 4

Proof. There are several ways to derive such a deviation inequality. We choose an approach based on a version of Bernstein’s inequality.

A Taylor expansion of the sub-Gaussian assumption on Page 5 gives

$$\max_{i \in \{1, \ldots, n\}} K^2 \left( \mathbb{E}[|u_i|^2/K^2 + (|u_i|^2/K^2)^2/2! + (|u_i|^2/K^2)^3/3! + \ldots] \right) \leq \gamma^2.$$

Hence, the individual terms of the expansion satisfy the moment inequality

$$\max_{i \in \{1, \ldots, n\}} K^2 \mathbb{E} \left[ (|u_i|^2/K^2)^m/m! \right] \leq \gamma^2 \quad \text{for all } m \in \{1, 2, \ldots\}.$$

By exchanging the maximum for an average, we then find

$$\frac{1}{n} \sum_{i=1}^n K^2 \mathbb{E} \left[ (|u_i|^2/K^2)^m/m! \right] \leq \gamma^2 \quad \text{for all } m \in \{1, 2, \ldots\},$$

which can be reformulated as

$$\sum_{i=1}^n \mathbb{E} \left[ (|u_i|^2)^m \right] \leq \frac{m!}{2} (2n \gamma^2 K^2)^{m-2} \quad \text{for all } m \in \{1, 2, \ldots\}.$$
We can thus apply a Bernstein-type deviation inequality such as Boucheron et al. [2013, Corollary 2.11] to derive
\[
\mathbb{P}\left( \sum_{i=1}^{n} \left( (u_i)^2 - \mathbb{E}[(u_i)^2] \right) \geq \frac{nv}{2} \right) \leq e^{-\frac{n^2 \gamma^2}{2(2n^2 R^2 + K \gamma n \sigma)}},
\]
which can be reformulated as
\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \left( (u_i)^2 - \mathbb{E}[(u_i)^2] \right) \geq \frac{v}{2} \right) \leq e^{-\frac{n^2 \gamma^2}{2(2n^2 R^2 + K \gamma n \sigma)}},
\]
Using that \( v \geq 2 \gamma^2 \) by assumption, we then find further
\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \left( (u_i)^2 - \mathbb{E}[(u_i)^2] \right) \geq \frac{v}{2} \right) \leq e^{-\frac{8n^2 \gamma^2}{12n^2 R^2}} = e^{-\frac{n \gamma^2}{12R^2}}.
\]
By 1. adding a zero-valued term, 2. invoking the above-derived property on the \((u_i)^2\) (set \( m = 1 \)), 3. using again that \( v \geq 2 \gamma^2 \), and 4. invoking the above display, we conclude that
\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 \geq v \right) = \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 - \mathbb{E}[(u_i)^2] \geq v - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(u_i)^2] \right)
\leq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 - \mathbb{E}[(u_i)^2] \geq v - \gamma^2 \right)
\leq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 - \mathbb{E}[(u_i)^2] \geq \frac{v}{2} \right)
\leq e^{-\frac{n \gamma^2}{12R^2}},
\]
as desired. \( \square \)

6.2.7 Proof of Theorem 2

Proof. The idea of the proof is to bound the effective noise and then apply Theorem 1.

If \( c_{Lip1} = 0 \), then \( g_1(x_i) = 0 \) for all \( \Omega \in \mathcal{A}_1 \) and \( i \in \{1, \ldots, n\} \) in view of Corollary 2. Hence,
\[
\mathbb{P}\left( \sup_{\Omega \in \mathcal{A}_1} \left| \frac{1}{n} \sum_{i=1}^{n} g_1(x_i) u_i \right| \leq \delta \right) = \mathbb{P}(0 \leq \delta) = 1 \quad \text{for all } \delta \in (0, \infty),
\]
which makes a proof straightforward. We can thus assume \( c_{Lip1} > 0 \) in the following.

Our first step is to apply Lemma 1 about suprema of empirical processes with \( \mathcal{A}':= \mathcal{A}_1 \). For this, we need to find 1. a constant \( R \in (0, \infty) \) that satisfies \( \sup_{\Omega \in \mathcal{A}_1} \| g_1 \|_n \leq R \) and 2. suitable \( \delta, \sigma \in (0, \infty) \) that satisfy \( \delta < \sigma R \) and
\[
\sqrt{n} \geq \frac{d_{sub}}{\delta} \left( J(\delta, \sigma, \mathcal{A}_1) \vee R \right).
\]

Condition 1 is verified by \( R := c_{Lip1} \) according to Corollary 2.

For Condition 2, we define \( \delta = \delta(n, R, a_{sub}, c_{Lip1}) := 96 a_{sub} c_{Lip1} \sqrt{\log(2nP)/n \log(2n)} \in (0, \infty) \) and \( \sigma := (2\delta/c_{Lip1}) \vee (\sqrt{2} \gamma) \). Then, \( \delta < \sigma R \) by the definitions of \( \delta, \sigma, R \). Moreover, Proposition 3 (note that \( \delta \leq 8c_{Lip1} \)), the definitions of \( \delta \) and \( R \), and
\[
\frac{8c_{Lip1}}{\delta} = 16 \vee \frac{8\sqrt{2} \gamma c_{Lip1}}{96 a_{sub} c_{Lip1} \sqrt{\log(2nP)/n \log(2n)}} \leq 16 \vee \frac{\gamma n}{4a_{sub}} \leq 16n
\]
we can thus apply Lemma 1 with the above-specified parameters to obtain that
\[
\frac{a_{\text{sub}}}{\delta}(J(\delta, \sigma, A_1) \cup \text{Lip}_1)
\]
\[
\leq \frac{a_{\text{sub}}}{\delta} \left( 4\text{Lip}_1 \log\left(\frac{8\sigma\text{Lip}_1^P}{\delta} \lor eP\right) \log\left(\frac{8\sigma\text{Lip}_1}{\delta} \lor \text{Lip}_1\right) \right)
\]
\[
\leq \sqrt{\frac{n a_{\text{sub}}}{96\text{Lip}_1^P \log(2n^2P) \log(2n)}} \left( 4\text{Lip}_1 \log(16nP \lor eP) \log(16n) \lor \text{Lip}_1 \right)
\]
\[
\leq \sqrt{\frac{n a_{\text{sub}}}{96\text{Lip}_1^P \log(2n^2P) \log(2n)}} \left( 96\text{Lip}_1 \log(2n^2P) \log(2n) \lor \text{Lip}_1 \right)
\]
\[
= \sqrt{n},
\]
which verifies Condition 2.

We can thus apply Lemma 1 with the above-specified parameters to obtain that
\[
\mathbb{P}\left( \left\{ \sup_{\Omega \in A_1} \left| \frac{1}{n} \sum_{i=1}^{n} g_\Omega(x_i) u_i \right| \geq \delta \right\} \cap \left\{ \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 \leq \sigma^2 \right\} \right) \leq a_{\text{sub}} e^{-\left(\frac{a_{\text{sub}}^2}{(\text{Lip}_1)^2}\right)}.
\]

We use that \(\mathbb{P}(C \cap D) \leq \alpha\) implies \(\mathbb{P}(C^c \cup D^c) \geq 1 - \alpha\) and that \(\mathbb{P}(C^c) \geq \mathbb{P}(C^c \cup D^c) - \mathbb{P}(D^c)\) to rewrite this inequality as
\[
\mathbb{P}\left( \left\{ \sup_{\Omega \in A_1} \left| \frac{1}{n} \sum_{i=1}^{n} g_\Omega(x_i) u_i \right| \leq \delta \right\} \right) \geq \mathbb{P}\left( \sup_{\Omega \in A_1} \left| \frac{1}{n} \sum_{i=1}^{n} g_\Omega(x_i) u_i \right| < \delta \right)
\]
\[
\geq 1 - a_{\text{sub}} e^{-\left(\frac{a_{\text{sub}}^2}{(\text{Lip}_1)^2}\right)} - \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 > \sigma^2 \right).
\]

Since \(\sigma^2 \geq 2\gamma^2\) by the definition of \(\sigma\), Lemma 4 with \(v := \sigma^2\) allows us to bound the last term according to
\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 > \sigma^2 \right) \leq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} (u_i)^2 \geq \sigma^2 \right) \leq e^{-\frac{n a_{\text{sub}}^2}{4K^2}}.
\]

Combining this inequality with the previous one yields
\[
\mathbb{P}\left( \sup_{\Omega \in A_1} \left| \frac{1}{n} \sum_{i=1}^{n} g_\Omega(x_i) u_i \right| \leq \delta \right) \geq 1 - a_{\text{sub}} e^{-\left(\frac{a_{\text{sub}}^2}{(\text{Lip}_1)^2}\right)} - e^{-\frac{n a_{\text{sub}}^2}{4K^2}}.
\]

By the definitions of \(\delta\) and \(\sigma\), and assuming that \(n\) is large enough (depending on \(\gamma, K\), we find that
\[
a_{\text{sub}} e^{-\left(\frac{a_{\text{sub}}^2}{(\text{Lip}_1)^2}\right)} + e^{-\frac{n a_{\text{sub}}^2}{4K^2}}
\]
\[
\leq a_{\text{sub}} e^{-96\text{Lip}_1^P \log(2n^2P) \log(2n)} + e^{-\frac{496\text{Lip}_1^2 \log(2n^2P) \log(2n)^2}{12K^2}} + e^{-\frac{n a_{\text{sub}}^2}{4K^2}}
\]
\[
\leq e^{-\log(4n)} + e^{-\log(4n)} + e^{-\frac{n a_{\text{sub}}^2}{6K^2}}
\]
\[
\leq \frac{1}{n}
\]
that is,\[
\mathbb{P}\left( \sup_{\Omega \in A_1} \left| \frac{1}{n} \sum_{i=1}^{n} g_\Omega(x_i) u_i \right| \leq \delta \right) \geq 1 - \frac{1}{n}.
\]

In other words, \(\lambda_{h,t} \leq 2\delta\) for \(t = 1/n\).

The claim then follows directly from Theorem 1 with \(\lambda \geq 2\delta = 192a_{\text{sub}} \text{Lip}_1 \sqrt{\log(2nP) / n \log(2n)}\), \(\text{Lip}_1 = 2(a_{\text{Lip}})^L \sqrt{\lambda} |x_i|_n\) (see Corollary 2), and \(a := 384a_{\text{sub}}\).
7 Discussion

Our theories in Section 3 show that $\ell_1$-regularization can guarantee accurate prediction even when the neural networks are very wide (see the logarithmic dependence on the number of parameters) and deep (see the sub-linear dependence on the number of layers). More generally, our theories in Section 2 facilitate the derivation of concrete guarantees by connecting regularization with the rich literature on suprema of empirical processes.

Our paper shares two limitations with other theory papers on neural networks and deep learning: 1. It accounts only for explicit regularization, while some features of deep learning might also be due to implicit regularization [Neyshabur, 2017]. 2. It accounts only for global optima, while current algorithms often provide local optima or stationary points [Murty and Kabadi, 1987, Agarwal et al., 2017, Neyshabur et al., 2015a]. Hence, our paper is still not an exact representation of the pipelines that are currently used in practice, but it reduces the gap between statistical theory and practice considerably, especially through the new, practical regularization scheme and the new guarantees that cater to the deep networks that have become popular in applications.

In summary, our paper highlights the effectiveness of regularisation in deep learning, and it furthers the mathematical understanding of neural networks more broadly. As practical advice, our results suggest the use of large networks (to minimize approximation errors) together with regularization (to avoid overfitting).

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