A new way to exclude collisions with order constraints

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Abstract

A new geometric argument is introduced to exclude binary collisions with order constraints. Two applications are given in this paper.

The first application is to show the existence of a new set of periodic orbits in the planar three-body problem with mass $M = [1, m, m]$, where we study the action minimizer under topological constraints in a two-point free boundary value problem. The main difficulty is to exclude possible binary collisions under order constraints, which is solved by our geometric argument.

The second application is to study the set of retrograde orbits in the planar three-body problem with mass $M = [1, m, m]$. We can show the existence for any $m > 0$ and any rotation angle $\theta \in (0, \pi/2)$. Specially, in the case when $\theta = \pi/2$, the action minimizer coincide with either the Schubart orbit or the Broucke-Hénon orbit, which partially answers the open problem proposed by Venturelli.

Key word: Three-body problem, variation method, topological constraint, geometric argument.

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1 Introduction

After the pioneering work of the figure-eight orbit [1], many new orbits have been shown to exist by variational method. One of the main difficulties is to exclude possible collisions in the action minimizer. In the last two decades, two powerful methods have been introduced to study the isolated collisions. One is the local deformation method [3, 7, 9, 10, 13, 16, 17, 18, 21], and the other is the level estimate method [4, 5, 6, 20, 22]. However, when topological constraints are imposed to the N-body problem, it is difficult to eliminate collisions by applying the two methods in general. New progress has been made recently in [17, 18], where the author imposed strong topological constraints and successfully applied
his local deformation method to show the existence of many choreographic and double choreographic solutions.

In this paper, we introduce a geometric argument to exclude collisions and apply it to two sets of periodic orbits in the planar three-body problem. Let \( M = [m_1, m_2, m_3] = [1, m, m] \) with \( m > 0 \). For simplicity, we denote \( Q_i \) as the \( i \)-th (\( i = 1, 2, 3, 4 \)) quadrant in the Cartesian \( xy \) coordinate system and \( \overline{Q}_i \) as its closure. For example, \( Q_1 = \{(x, y) | x > 0, y > 0\} \) and \( \overline{Q}_1 = \{(x, y) | x \geq 0, y \geq 0\} \). Assume the center of mass to be at the origin. That is, \( q \in \mathcal{X} \), where

\[
\mathcal{X} = \left\{ q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_{1x} & q_{1y} \\ q_{2x} & q_{2y} \\ q_{3x} & q_{3y} \end{bmatrix} \in \mathbb{R}^{3 \times 2} \mid \sum_{i=1}^{3} m_i q_i = 0 \right\}. \tag{1.1}
\]

Let

\[
Z_1 = q_2 - q_3, \quad Z_2 = q_1 - \frac{q_2 + q_3}{2} = (1 + \frac{1}{2m}) q_1.
\]

Our geometric result is as follows.

**Theorem 1.1.** Assume the two boundaries \( (Z_1(0), Z_2(0)) \) and \( (Z_1(1), Z_2(1)) \) are fixed and let \( (Z_1, Z_2) \in H^1([0, 1], \mathbb{R}^4) \) be the action minimizer of a fixed boundary value problem

\[
\mathcal{A}(Z_1, Z_2) = \inf_{\mathcal{P}(Z_1, Z_2)} \mathcal{A} = \inf_{\mathcal{P}(Z_1, Z_2)} \int_0^1 (K + U) \, dt,
\]

where \( K \) in \((3.1)\) and \( U \) in \((3.2)\) are the standard kinetic energy and the potential energy respectively, and

\[
\mathcal{P}(Z_1, Z_2) = \{(Z_1, Z_2) \in H^1([0, 1], \mathbb{R}^4) \mid Z_i(0) = Z_i(0), \, Z_i(1) = Z_i(1) \,(i = 1, 2)\}.
\]

Let \( Z_1(0), Z_1(1) \in \overline{Q}_i \) and \( Z_2(0), Z_2(1) \in \overline{Q}_j \), while \( \overline{Q}_i \) and \( \overline{Q}_j \) are two adjacent closed quadrants. Then \( Z_1(t) \) and \( Z_2(t) \) are always in two adjacent closed quadrants for all \( t \in [0, 1] \) and \( (Z_1, Z_2) \) must satisfy one of the following three cases:

(a). \( Z_1(t) \) and \( Z_2(t) \) can not touch the coordinate axes for all \( t \in (0, 1) \);

(b). \( Z_1(t) \) and \( Z_2(t) \) are on the coordinate axes for all \( t \in [0, 1] \);

(c). the motion is a part of the Euler solution with \( Z_2(t) \equiv 0 \) for all \( t \in [0, 1] \).

As its applications, we consider the following two-point free boundary value problems with topological constraints. The first application is as follows. Similar to [22], we set

\[
Q_s = \begin{bmatrix} m(a_1 - a_2) & 0 \\ -(m + 1)a_1 - ma_2 & 0 \\ ma_1 + (m + 1)a_2 & 0 \end{bmatrix}, \quad Q_e = \begin{bmatrix} 2mb_2 & 0 \\ -b_2 & b_1 \\ -b_2 & -b_1 \end{bmatrix} R(\theta), \tag{1.2}
\]

where \( a_1 \geq 0, \, a_2 \geq 0, \, b_1 \in \mathbb{R}, \, b_2 \in \mathbb{R} \), and \( R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \). The two configuration sets are defined as follows:

\[
Q_S = \left\{ Q_s \mid a_1 \geq 0, \, a_2 \geq 0 \right\}, \quad Q_E = \left\{ Q_e \mid b_1 \in \mathbb{R}, \, b_2 \in \mathbb{R} \right\}. \tag{1.3}
\]
Geometrically, the configuration $Q_s$ is on a horizontal line with order constraints $q_{2x}(0) \leq q_{1x}(0) \leq q_{3x}(0)$. The configuration $Q_e$ is an isosceles triangle with $q_1$ as its vertex, and the symmetry axis of each $Q_e$ in (1.3) is a counterclockwise $\theta$ rotation of the $x-$axis. Pictures of the two configurations $Q_s$ and $Q_e$ are shown in Fig. 1 respectively.

Figure 1: The configurations $Q_s$ and $Q_e$ are shown, where blue dots represent $q_1$, red dots represent $q_2$ and black dots represent $q_3$. In $Q_s$, three masses are on the $x-$axis with an order $q_{2x} \leq q_{1x} \leq q_{3x}$. In $Q_e$, three masses form an isosceles triangle with $q_1$ as the vertex, whose symmetry axis is a counterclockwise $\theta$ rotation of the $x-$axis.

For each $\theta \in [0, \pi/2)$, standard results [7, 9, 15, 24] imply that there exists an action minimizer $P_{m, \theta} \in H^1([0,1], \chi)$, such that

$$A(P_{m, \theta}) = \inf_{q \in \mathcal{P}(Q_s, Q_e)} A = \inf_{q \in \mathcal{P}(Q_s, Q_e)} \int_0^1 (K + U) \, dt,$$

where $K = \frac{1}{2} \sum_{i=1}^3 m_i |\dot{q}_i|^2$, $U = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|q_i - q_j|}$ and

$$\mathcal{P}(Q_s, Q_e) = \{ q \in H^1([0,1], \chi) \mid q(0) \in Q_s, q(1) \in Q_e \}.$$

By the celebrated results of Marchal [11] and Chenciner [2], the action minimizer $P_{m, \theta}$ is free of collision in $(0,1)$. We are only left to exclude possible boundary collisions. Our main results are as follows.

**Theorem 1.2.** For each given $\theta \in [0, \pi/2)$ and mass set $M = [m_1, m_2, m_3] = [1, m, \bar{m}]$ with $\bar{m} > 0$, the minimizer $P_{m, \theta}$ is collision-free and it can extended to a periodic or quasi-periodic orbit.

The new idea in the proof is to apply our geometric result (Theorem 1.1) to exclude binary collisions under order constraints. It was first introduced in [23] to study retrograde double-double orbit in the planar equal-mass four-body problem, and it is also used to study the Schubart orbit (Fig. 2(a)) and the Broucke-Hénon orbit (Fig. 2(b)) in the equal mass case [24].

Besides excluding possible collisions, we need to show that $P_{m, \theta}$ is nontrivial, which means that it does not coincide with a relative equilibrium. In the case when $m = 1$, we can show that $P_{m, \theta}$ is nontrivial by introducing test paths for each $\theta$, which extends the result in [22].

**Theorem 1.3.** When $m = 1$ and $\theta \in [0, 0.183 \pi]$, the minimizer $P_{m, \theta}$ is nontrivial.
Figure 2: Motions of the Schubart orbit and the Broucke-Hénon orbit. In each orbit, at $t = 0$, the three masses (in dots) form a collinear configuration. At $t = 1$, they (in crosses) form an Euler configuration or an isosceles configuration.

In [22], we apply Chen’s level estimate method [4, 6] and show that for each $\theta \in [0.084\pi, 0.183\pi]$, $P_{m,\theta}$ is collision-free and can be extended to a nontrivial periodic or quasi-periodic orbit. It is clear that the results (Theorem 1.2 and Theorem 1.3) in this paper are stronger than the results in [22].

Remark 1.4. When $m = 1$ and $\theta = 0$, the orbit is shown in Fig. 3 and it looks very simple. However, to the authors’ knowledge, there is no existence proof for $P_{1,0}$. Actually, it is difficult to apply the local deformation method [9] to $P_{1,0}$, while the level estimate method [4, 6, 22] can only exclude collisions in $P_{m,\theta}$ with $\theta \in [0.084\pi, 0.183\pi]$. In this sense, our geometric argument has its own advantage in eliminating collisions with topological constraints.

Figure 3: At $t = 0$, the three masses (in dots) form a collinear configuration with body 1 in the middle. At $t = 1$, they (in crosses) form an isosceles configuration with body 1 as the vertex. The minimizing path $P_{1,0}$ connects a collinear configuration with order constraints $q_{2x}(0) \leq q_{1x}(0) \leq q_{3x}(0)$ and an isosceles triangle configuration, while its periodic extension has a simple and symmetric shape.
The second application is to show the existence of the retrograde orbits [4, 6]. By introducing the level estimate method, Chen [4, 6] can show the existence of retrograde orbit for most of the rotation angles and most of the masses. Specially, in the case when $M = [m_1, m_2, m_3] = [1, m, m]$, Chen’s result [6] requires two inequalities between the rotation angle $\theta$ and the mass $m > 0$. In fact, when $\theta > 0.4\pi$, the two inequalities in [6] fail for all $m > 0$. Fortunately, our geometric result can be applied successfully to the existence of retrograde orbits for all $\theta \in (0, \pi/2)$ and all $m > 0$. Furthermore, when $\theta = \pi/2$, we can show that the action minimizer coincide with either the Schubart orbit or the Broucke-Hénon orbit, which partially answers the open problem proposed by Venturelli (Problem 6 in [14]).

Before stating our result, we first introduce the variational setting of the retrograde orbits. Let

$$Q_{S_1} = \begin{bmatrix} -2ma_1 - ma_2 & 0 \\ a_1 - ma_2 & 0 \\ a_1 + (m+1)a_2 & 0 \end{bmatrix}, \quad Q_{E_1} = \begin{bmatrix} 2mb_2 & 0 \\ -b_2 & b_1 \\ -b_2 & -b_1 \end{bmatrix} R(\theta),$$

(1.5)

where $M = [1, m, m]$, $a_1 \geq 0$, $a_2 \geq 0$, $b_1 \in \mathbb{R}$, $b_2 \in \mathbb{R}$, and $R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$. The two configuration sets are defined as follows:

$$Q_{S_1} = \left\{ Q_{s_1} \mid a_1 \geq 0, a_2 \geq 0 \right\}, \quad Q_{E_1} = \left\{ Q_{e_1} \mid b_1 \in \mathbb{R}, b_2 \in \mathbb{R} \right\}.$$  

(1.6)

Note that in the two boundary settings $\{Q_S, Q_E\}$ in (1.4) and $\{Q_{S_1}, Q_{E_1}\}$ in (1.6), $Q_{E_1} = Q_E$ at $t = 1$. At $t = 0$, $Q_{S_1}$ in (1.6) is on the $x$-axis with order constraints

$$q_{1x}(0) \leq q_{2x}(0) \leq q_{3x}(0),$$

while $Q_{E_1}$ in (1.6) is on the $x$-axis with different order constraints

$$q_{2x}(0) \leq q_{1x}(0) \leq q_{3x}(0).$$

For each $\theta \in (0, \pi/2]$, standard results [7, 9, 15, 24] imply that there exists an action minimizer $\overline{\mathcal{P}}_{m, \theta} \in H^1([0, 1], \chi)$, such that

$$\mathcal{A}(\overline{\mathcal{P}}_{m, \theta}) = \inf_{q \in P(Q_{S_1}, Q_{E_1})} \mathcal{A} = \inf_{q \in P(Q_{S_1}, Q_{E_1})} \int_0^1 (K + U) \, dt,$$

(1.7)

where

$$P(Q_{S_1}, Q_{E_1}) = \left\{ q \in H^1([0, 1], \chi) \mid q(0) \in Q_{S_1}, q(1) \in Q_{E_1} \right\}.$$

By applying Theorem 1.1, we can show that $\overline{\mathcal{P}}_{m, \theta}$ is collision-free when $\theta \in (0, \pi/2]$, which implies the existence of the retrograde orbits. Furthermore, in the case when $\theta = \pi/2$, $\overline{\mathcal{P}}_{m, \theta}$ is either a part of the Schubart orbit or a part of the Broucke-Hénon orbit.

**Theorem 1.5.** For each given $\theta \in (0, \pi/2]$ and mass set $M = [m_1, m_2, m_3] = [1, m, m]$ with $m > 0$, the action minimizer $\overline{\mathcal{P}}_{m, \theta}$ in (1.7) is collision-free. And it can be extended to a periodic or quasi-periodic orbit.

When $\theta = \pi/2$, the action minimizer $\overline{\mathcal{P}}_{m, \pi/2}$ in (1.7) coincides with either the Schubart orbit or the Broucke-Hénon orbit.
The paper is organized as follows. In Section 2, we show that $\mathcal{P}_{m,\theta}$ is collision free at $t = 1$ and it is also free of total collision. In Section 3 a general geometric result is introduced and it can be applied to $\mathcal{P}_{m,\theta}$ to show that it is collision-free. In Section 4 we show that $\mathcal{P}_{m,\theta}$ is nontrivial when $m = 1$ and $\theta \in [0, 0.183\pi]$. In the last section (Section 5), we apply our geometric result (Theorem 1.1) to the retrograde orbits and the Broucke-Hénon orbit.

2 Exclusion of total collision

In this section, we exclude possible total collisions in the action minimizer $\mathcal{P}_{m,\theta}$ for all $\theta \in [0, \pi/2)$ and $m > 0$. Also, we exclude possible binary collision at $t = 1$ by the standard deformation method in the end.

Lemma 2.1. For any $\theta \in [0, \pi/2)$ and $m > 0$, the action minimizer $\mathcal{P}_{m,\theta}$ has no total collision.

Proof. We first obtain a lower bound of action when $\mathcal{P}_{m,\theta}$ has a total collision in $[0, 1]$. We denote the action by $A_{\text{total}}$ if $\mathcal{P}_{m,\theta}$ has a total collision. By Chen’s level estimate [4],

$$A_{\text{total}} \geq \frac{m^2 + 2m}{1 + 2m} \cdot \frac{3}{2} (1 + 2m)^{2/3} \pi^{2/3} = \frac{3m}{2} \pi^{2/3} \frac{m + 2}{(1 + 2m)^{1/3}}.$$ 

By the setting of $Q_S$ and $Q_E$ in (1.3), we can choose a piece of Euler orbit as the test path. In this Euler orbit, at $t = 0$, it is on the $x$-axis with body 1 at the origin. While at $t = 1$, $Q_e$ degenerates to a straight line. The corresponding action $A_{\text{test}}$ is

$$A_{\text{test}} = \frac{3}{2} (\pi/2 - \theta)^{2/3} \left[ 2m^3 (2 + m/2)^2 \right]^{1/3} \leq \frac{3}{2} \pi^{2/3} \left( \frac{m^3 (2 + m/2)^2}{2} \right)^{1/3} = \frac{3m}{2} \pi^{2/3} \frac{(4 + m)^{2/3}}{2}.$$ 

To show that $\frac{3m}{2} \pi^{2/3} \frac{m + 2}{(1 + 2m)^{1/3}} > \frac{3m}{2} \pi^{2/3} \frac{(4 + m)^{2/3}}{2}$, it is equivalent to show that

$$\frac{m + 2}{(1 + 2m)^{1/3}} > \frac{(4 + m)^{2/3}}{2}.$$ (2.1)

In order to prove (2.1), we consider the difference $g(m) = 8(m + 2)^3 - (4 + m)^2(1 + 2m)$. Then to prove (2.1) is equivalent to show that $g(m) > 0$ for any $m > 0$.

In fact, the derivative $\frac{dg(m)}{dm} = 18m^2 + 62m + 56 > 0$ for any $m > 0$. And $g(0) = 60 > 0$. It follows that $g(m) > 0$ for any $m > 0$. That is, $A_{\text{total}} > A_{\text{test}}$.

Therefore, $\mathcal{P}_{m,\theta}$ has no total collision. The proof is complete.

Note that by the definition of $Q_E$ in (1.3), the only possible binary collision at $t = 1$ is between bodies 2 and 3. Standard local deformation result can imply this fact.

The following blow-up results are needed in proving that $\mathcal{P}_{m,\theta}$ is free of binary collisions at $t = 1$. It is known that the bodies involved in a partial collision or a total collision will approach a set of central configurations. More information can be known if the solution under concern is an action minimizer:
Lemma 2.2 (Theorem 4.1.18 in [13], or Sec. 3.2.1 in [2]). If a minimizer \( q \) of the fixed-ends problem on time interval \([\tau_1, \tau_2]\) has an isolated collision of \( k \leq N \) bodies, then there is a parabolic homothetic collision-ejection solution \( \bar{q} \) of the \( k \)-body problem which is also a minimizer of the fixed-ends problem on \([\tau_1, \tau_2]\).

Lemma 2.3 (Proposition 5 in [7], or Sec. 7 in [9]). Let \( X \) be a proper linear subspace of \( \mathbb{R}^d \). Suppose a local minimizer \( x \) of \( \mathcal{A}_{0, t_1} \) on

\[
B_{0, t_1}(x(t_0), X) := \{ x \in H^1([t_0, t_1], (\mathbb{R}^d)^N) \mid x(t_0) \text{ is fixed, and } x_i(t_1) \in X, i = 1, 2, \ldots, N \}
\]

has an isolated collision of \( k \leq N \) bodies at \( t = t_1 \). Then there is a homothetic parabolic solution \( \bar{y} \) of the \( k \)-body problem with \( \bar{y}(t_1) = 0 \) such that \( \bar{y} \) is a minimizer of \( \mathcal{A}_{\tau, t_1} \) on \( B_{\tau, t_1}(\bar{y}(\tau), X) \) for any \( \tau < t_1 \). Here \( \mathcal{A}_{\tau, t_1} \) denotes the action of this \( k \)-body subsystem.

By applying Lemma 2.2 and Lemma 2.3, the following result holds.

Lemma 2.4. For each given \( \theta \in [0, \pi/2] \) and mass set \( M = [m_1, m_2, m_3] = [1, m, m] \) with \( m > 0 \), there is no binary collision at \( t = 1 \) in \( \mathcal{P}_{m, \theta} \).

Proof. Note that \( Q_E = \left\{ Q_e \mid b_1 \in \mathbb{R}, b_2 \in \mathbb{R} \right\} \) where \( Q_e = \begin{bmatrix} 2b_2/m & 0 \\ -b_2 & b_1 \end{bmatrix} R(\theta) \). It is clear that \( Q_E \) is a two-dimensional vector space and the only possible binary collision in \( Q_E \) is between bodies 2 and 3.

Assume that \( q_2 \) and \( q_3 \) collide at \( t = 1 \) in \( \mathcal{P}_{m, \theta} \). By the analysis of blow up in Lemma 2.2 and Lemma 2.3, there exists a parabolic homothetic solution \( q_i(t) = \xi_i t^\frac{3}{2}, (i = 2, 3) \), which is also a minimizer of the 2-body problem on \([1 - \tau, 1]\) for any \( \tau > 0 \). In fact, \((\xi_2, \xi_3)\) forms a central configuration with \( m\xi_2 = -m\xi_3 \), and the two vectors \( \xi_2, \xi_3 \) satisfy the energy constraint:

\[
\sum_{i=2,3} \frac{1}{2} \frac{(|\xi_i|^2)}{\xi_i} = \frac{1}{\xi_2 - \xi_3} = 0.
\]

For a given \( \varepsilon > 0 \) small enough, we fix \( q_i(\varepsilon) (i = 2, 3) \). Next, we perturb \( q_i \) to \( \bar{q}_i \) (i=2,3) such that \( \bar{q}_i(1 - \varepsilon) = q_i(1 - \varepsilon), (i = 2, 3), \bar{q}_2(1) \neq \bar{q}_3(1) \) and \( \bar{q}_i(1) (i = 2, 3) \) satisfy the boundary condition \( Q_E \). Let

\[
\frac{\bar{q}_3 \bar{q}_2(1)}{\bar{q}_2(1) - \bar{q}_3(1)} = \frac{\bar{q}_2(1) - \bar{q}_3(1)}{|\bar{q}_2(1) - \bar{q}_3(1)|},
\]

where \( \bar{q}_2(1) \) and \( \bar{q}_3(1) \) are the perturbed vectors of \( q_1 \) and \( q_2 \) at \( t = 1 \). Since the boundary set \( Q_E \) is a vector space, it follows that one can always choose the local deformation \( \bar{q}_i \) such that \( \bar{q}_i(1) (i = 2, 3) \) satisfies

\[
\left\langle \bar{q}_3 \bar{q}_2(1), \xi_2 \right\rangle = -1.
\]

By [8, 10, 16], there exist \( \bar{q}_2 \) and \( \bar{q}_3 \), such that the action of \( \bar{q}_2 \) and \( \bar{q}_3 \) in \([1 - \tau, 1]\) is strictly smaller than the action of the parabolic ejection solution: \( q_2 \) and \( q_3 \). Contradiction!

Therefore, there is no binary collision at \( t = 1 \) in \( \mathcal{P}_{m, \theta} \). The proof is complete. \( \square \)
3 Geometric result

In this section, we study the geometric property of the action minimizer connecting two given boundaries. The main result is Theorem 3.1, which can be applied to exclude the possible binary collisions with order constraints.

Let \( M = [m_1, m_2, m_3] = [1, m, m] \). Let
\[
Z_1 = q_2 - q_3, \quad Z_2 = q_1 - \frac{q_2 + q_3}{2} = (1 + \frac{1}{2m})q_1.
\]
The kinetic energy \( K \) and the potential energy \( U \) can be rewritten as
\[
K = \sum_{i=1}^{3} \frac{1}{2} m_i |\dot{q}_i|^2 = \frac{m}{4} |\dot{Z}_1|^2 + \frac{m}{2m+1} |\dot{Z}_2|^2, \tag{3.1}
\]
\[
U = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|q_i - q_j|} = \frac{m^2}{|Z_1|} + \frac{m}{|\frac{1}{2}Z_1 + Z_2|} + \frac{m}{|\frac{1}{2}Z_1 - Z_2|}, \tag{3.2}
\]
while the action functional is
\[
A = \int_0^1 (K + U)dt. \tag{3.3}
\]

For convenience, in the Cartesian \( xy \) coordinate system, the \( i \)-th quadrant is denoted by \( Q_i \) (\( i = 1, 2, 3, 4 \)), while its closure is denoted by \( \overline{Q}_i \). For example, \( Q_1 = \{(x, y) \mid x > 0, y > 0\} \) and \( \overline{Q}_1 = \{(x, y) \mid x \geq 0, y \geq 0\} \). The main result in this section is as follows.

**Theorem 3.1.** Assume the two boundaries \( (Z_1(0), Z_2(0)) \) and \( (Z_1(1), Z_2(1)) \) are fixed and let \( (Z_1, Z_2) \in H^1([0,1], \mathbb{R}^4) \) be the action minimizer of a fixed boundary value problem
\[
A(Z_1, Z_2) = \inf_{p(Z_1, Z_2)} A,
\]
where
\[
P(Z_1, Z_2) = \{(Z_1, Z_2) \in H^1([0,1], \mathbb{R}^4) \mid Z_i(0) = Z_i(0), Z_i(1) = Z_i(1) (i = 1, 2)\}.
\]

Let \( Z_1(0), Z_1(1) \in \overline{Q}_i \) and \( Z_2(0), Z_2(1) \in \overline{Q}_j \), while \( \overline{Q}_i \) and \( \overline{Q}_j \) are two adjacent closed quadrants. Then \( Z_1(t) \) and \( Z_2(t) \) are always in two adjacent closed quadrants for all \( t \in [0, 1] \) and \( (Z_1, Z_2) \) must satisfy one of the following three cases:

(a). \( Z_1(t) \) and \( Z_2(t) \) can not touch the coordinate axes for all \( t \in (0, 1); \)

(b). \( Z_1(t) \) and \( Z_2(t) \) are on the coordinate axes for all \( t \in [0, 1]; \)

(c). the motion is a part of the Euler solution with \( Z_2(t) \equiv 0 \) for all \( t \in [0, 1]. \)
Before proving Theorem 3.1, we introduce several preliminary results. If both \( Z_1(t) \) and \( Z_2(t) \) are nonzero vectors, we define an angle \( \Delta \equiv \Delta(Z_1(t), Z_2(t)) \) by

\[
\Delta \equiv \Delta(Z_1(t), Z_2(t)) = \begin{cases} 
\beta(Z_1(t), Z_2(t)), & \text{if } \beta(Z_1(t), Z_2(t)) \leq \frac{\pi}{2}; \\
\pi - \beta(Z_1(t), Z_2(t)), & \text{if } \beta(Z_1(t), Z_2(t)) > \frac{\pi}{2}, 
\end{cases} \tag{3.4}
\]

where \( \beta(Z_1(t), Z_2(t)) = \arccos \left( \frac{Z_1(t) \cdot Z_2(t)}{|Z_1(t)||Z_2(t)|} \right) \) is the angle between the two nonzero vectors \( Z_1(t) \) and \( Z_2(t) \). By (3.4), it follows that \( \Delta \in [0, \pi/2] \). Intuitively, \( \Delta = \Delta(Z_1, Z_2) \) is the angle between the two straight lines spanned by \( Z_1 \) and \( Z_2 \).

A new formula of \( U \equiv U(Z_1, Z_2) \) can then be derived by the law of cosines:

\[
U(Z_1, Z_2) = \frac{m^2}{|Z_1|} + \frac{m}{|\frac{1}{2}Z_1 + Z_2|} + \frac{m}{|\frac{1}{2}Z_1 - Z_2|} = \frac{m^2}{|Z_1|} + \frac{m}{\sqrt{\frac{1}{4}|Z_1|^2 + |Z_2|^2 + |Z_1||Z_2| \cos(\Delta)}} + \frac{m}{\sqrt{\frac{1}{4}|Z_1|^2 + |Z_2|^2 - |Z_1||Z_2| \cos(\Delta)}}. \tag{3.5}
\]

By (3.5), \( U(Z_1, Z_2) \) is a function of three variables: \( |Z_1| \), \( |Z_2| \) and \( \Delta \) when both \( Z_1 \neq 0 \) and \( Z_2 \neq 0 \) hold. Let \( U(|Z_1|, |Z_2|, \Delta) \equiv U(Z_1, Z_2) \). Indeed, \( U(|Z_1|, |Z_2|, \Delta) \) satisfies the following property.

**Proposition 3.2.** Fix \( |Z_1| \neq 0 \) and \( |Z_2| \neq 0 \) and assume that the potential energy \( U(|Z_1|, |Z_2|, \Delta) = U(Z_1, Z_2) \) in (3.5) is finite. Then \( U(|Z_1|, |Z_2|, \Delta) \) is a strictly decreasing function with respect to \( \Delta \).

**Proof:** Fixing \( |Z_1| \) and \( |Z_2| \) and taking the derivative of \( U(|Z_1|, |Z_2|, \Delta) \) in (3.5) with respect to \( \Delta \), it follows that

\[
\frac{\partial U(|Z_1|, |Z_2|, \Delta)}{\partial \Delta} = \frac{\frac{1}{2}|Z_1||Z_2|\sin(\Delta)}{\left[ \frac{1}{4}|Z_1|^2 + |Z_2|^2 + |Z_1||Z_2| \cos(\Delta) \right]^{3/2}} - \frac{\frac{1}{2}|Z_1||Z_2|\sin(\Delta)}{\left[ \frac{1}{4}|Z_1|^2 + |Z_2|^2 - |Z_1||Z_2| \cos(\Delta) \right]^{3/2}}.
\]

Note that \( \Delta \in [0, \frac{\pi}{2}] \). It implies that

\[
\frac{\partial U(|Z_1|, |Z_2|, \Delta)}{\partial \Delta} \leq 0,
\]

and \( \frac{\partial U(|Z_1|, |Z_2|, \Delta)}{\partial \Delta} = 0 \) if and only if \( \Delta = 0 \) or \( \Delta = \pi/2 \). The proof is complete.

Given a nonzero point \( Z_k = (Z_{kx}, Z_{ky}) \) \((k = 1, 2)\), we consider its four reflection points: \( (\pm |Z_{kx}|, \pm |Z_{ky}|) \). For each \( t \in [0, 1] \), we choose \( Z_{1t}(t) \) to be one of the four reflection points such that \( Z_{1t}(t) \in \overline{Q}_i \). For example, \( Z_{11}(t) = (|Z_{1x}(t)|, |Z_{1y}(t)|) \). Similarly, we can choose \( Z_{2j}(t) \in \overline{Q}_j \) for all \( t \in [0, 1] \). Then the following result holds.
Lemma 3.3. For each \( t \in [0, 1] \) and any \( i, j \in \{1, 2, 3, 4\} \), the potential function \( U(Z_{1i}(t), Z_{2j}(t)) \) must be one of the following two values: \( U_1(t) \) and \( U_2(t) \). It satisfies

\[
U(Z_{1i}(t), Z_{2j}(t)) = \begin{cases} 
U_1(t), & \text{if } Z_{1i}(t), Z_{2j}(t) \text{ are in two adjacent closed quadrants;} \\
U_2(t), & \text{otherwise.}
\end{cases}
\]

(3.6)

Moreover, \( U_1(t) < U_2(t) \).

Proof. Note that if two nonzero vectors \( Z_1(t) \) and \( Z_2(t) \) are in two adjacent closed quadrants, the angle 

\[ \Delta(Z_1(t), Z_2(t)) = \Delta(Z_{1i}(t), Z_{2j}(t)) \]

if their reflection points \( Z_{1i}(t) \) and \( Z_{2j}(t) \) are also in two adjacent closed quadrants.

If \( Z_1(t) \) or \( Z_2(t) \) belongs to the coordinate axes (including the case when \( Z_1(t) = 0 \) or \( Z_2(t) = 0 \) for some \( t \)), it is easy to check that \( Z_{1i}(t) \) and \( Z_{2j}(t) \) are always in two adjacent closed quadrants. Hence, 

\[ U(Z_{1i}(t), Z_{2j}(t)) = U_1(t). \]

Thus we only consider the case when both \( Z_1(t) \) and \( Z_2(t) \) are away from the coordinate axes.

In fact, similar to the definition of \( \Delta \), we define two angles \( \alpha_1 \) and \( \alpha_2 \) as follows

\[
\alpha_1 = \min \left\{ \arccos \frac{\langle Z_1(t), \vec{s}_1 \rangle}{|Z_1(t)|}, \pi - \arccos \frac{\langle Z_1(t), \vec{s}_1 \rangle}{|Z_1(t)|} \right\},
\]

\[
\alpha_2 = \min \left\{ \arccos \frac{\langle Z_2(t), \vec{s}_1 \rangle}{|Z_2(t)|}, \pi - \arccos \frac{\langle Z_2(t), \vec{s}_1 \rangle}{|Z_2(t)|} \right\},
\]

(3.7)

where \( \vec{s}_1 = (1, 0) \). It is clear that \( \alpha_1, \alpha_2 \in (0, \pi/2) \).

If \( Z_{1i} \) and \( Z_{2j} \) are in two adjacent quadrants respectively, then

\[
\Delta_1 \equiv \Delta(Z_{1i}(t), Z_{2j}(t)) = \min \{ \alpha_1 + \alpha_2, \pi - \alpha_1 - \alpha_2 \}.
\]

(3.8)

If \( Z_{1i} \) and \( Z_{2j} \) are not in two adjacent quadrants, we have

\[
\Delta_2 \equiv \Delta(Z_{1i}(t), Z_{2j}(t)) = |\alpha_1 - \alpha_2|.
\]

(3.9)

Note that both \( Z_1(t) \) and \( Z_2(t) \) are away from the coordinate axes, it implies that \( \alpha_1, \alpha_2 \in (0, \pi/2) \).

If \( \alpha_1 + \alpha_2 \leq \frac{\pi}{2} \), then

\[
\min \{ \alpha_1 + \alpha_2, \pi - \alpha_1 - \alpha_2 \} = \alpha_1 + \alpha_2 > |\alpha_1 - \alpha_2|.
\]

If \( \alpha_1 + \alpha_2 > \frac{\pi}{2} \), then

\[
\min \{ \alpha_1 + \alpha_2, \pi - \alpha_1 - \alpha_2 \} = \pi - \alpha_1 - \alpha_2
\]

\[
> \frac{\pi}{2} - \min \{ \alpha_1, \alpha_2 \} > |\alpha_1 - \alpha_2|.
\]

Hence, \( \Delta_1 < \Delta_2 \). Recall that 

\[ U(Z_{1i}(t), Z_{2j}(t)) = U(|Z_1|, |Z_2|, \Delta(Z_{1i}(t), Z_{2j}(t))). \]

Let

\[
U_1(t) = U(|Z_1|, |Z_2|, \Delta_1), \quad U_2(t) = U(|Z_1|, |Z_2|, \Delta_2).
\]
It is clear that for each given \( t \in [0, 1] \), the values of \( U(Z_i(t), Z_j(t)) \) \((i, j = 1, 2, 3, 4)\) can only be either \( U_1(t) \) or \( U_2(t) \). By proposition 3.2, it follows that

\[ U_1(t) < U_2(t). \]

The proof is complete. \( \square \)

**Lemma 3.4.** For the minimizing path \((Z_1, Z_2)\) in Theorem 3.1, if there exists some \( t_0 \in (0, 1) \), such that both \( Z_1(t_0) \) and \( Z_2(t_0) \) are tangent to the axes, then both \( Z_1 \) and \( Z_2 \) must stay on the corresponding axes for all \( t \in [0, 1] \).

**Proof.** The proof basically follows by the uniqueness of solution of the initial value problem of an ODE system. Note that for \( t \in (0, 1) \), \( q_i \) \((i = 1, 2, 3)\) are the solutions of the Newtonian equations. Without loss of generality, we assume \( Z_1(t_0) \) is tangent to the \( x\)-axis with \( 0 < t_0 < 1 \). Note that \( Z_1(t_0) \neq 0 \) and \( Z_1 = q_2 - q_3 \). It follows that

\[ q_{2y}(t_0) = q_{3y}(t_0), \quad \dot{q}_{2y}(t_0) = \dot{q}_{3y}(t_0). \tag{3.10} \]

If \( Z_2(t_0) \) is also on the \( x\)-axis and tangent to it, it implies that

\[ q_{1y}(t_0) = 0, \quad \dot{q}_{1y}(t_0) = 0. \]

Note that the center of mass is fixed at 0, it follows that

\[ q_{1y}(t_0) = q_{2y}(t_0) = q_{3y}(t_0) = 0, \quad \dot{q}_{1y}(t_0) = \dot{q}_{2y}(t_0) = \dot{q}_{3y}(t_0) = 0. \tag{3.11} \]

The Newtonian equations and (3.11) imply that

\[ \dot{q}_{1y}(t_0) = \dot{q}_{2y}(t_0) = \dot{q}_{3y}(t_0) = 0. \tag{3.12} \]

Since the set

\[ \{(q_1, q_2, q_3) \mid q_{1y} = q_{2y} = q_{3y} = 0, \quad \dot{q}_{1y} = \dot{q}_{2y} = \dot{q}_{3y} = 0\} \]

is invariant, it imply that

\[ q_{1y}(t) = q_{2y}(t) = q_{3y}(t) = 0, \quad \forall t \in [0, 1]. \]

It follows that both \( Z_1(t) \) and \( Z_2(t) \) stay on the \( x\)-axis for all \( t \in [0, 1] \).

If \( Z_2(t_0) \) is tangent to the \( y\)-axis, we have

\[ q_{1x}(t_0) = 0, \quad q_{2x}(t_0) = -q_{3x}(t_0), \quad \dot{q}_{1x}(t_0) = 0, \quad \dot{q}_{2x}(t_0) = -\dot{q}_{3x}(t_0). \tag{3.13} \]

Note that in (3.10),

\[ q_{2y}(t_0) = q_{3y}(t_0), \quad \dot{q}_{2y}(t_0) = \dot{q}_{3y}(t_0). \]

By the Newtonian equations, (3.10) and (3.13) imply that

\[ \dot{q}_{1x}(t_0) = 0, \quad \dot{q}_{2x}(t_0) = \dot{q}_{3x}(t_0). \]
Note that the set
\[ \{(q_1, q_2, q_3) \mid q_{1x} = 0, q_{2x} = -q_{3x}, q_{2y} = q_{3y}, q_{1x} = 0, q_{2x} = -q_{3x}, q_{2y} = q_{3y} \} \]
is invariant, it follows that
\[ q_{1x}(t) = 0, \quad q_{2y}(t) = q_{3y}(t), \quad \forall t \in [0, 1]. \]

It implies that \( Z_1 \) stays on the \( x \)-axis and \( Z_2 \) stays on the \( y \)-axis for all \( t \in [0, 1] \). The proof is complete. \( \square \)

**Proof of Theorem 3.1**: The key point in the proof is the observation that for all \( t \in [0, 1] \), \( Z_1(t) \) and \( Z_2(t) \) must belong to two adjacent quadrants.

Without loss of generality, we assume \( Z_1(0), Z_1(1) \in \mathbb{Q}_2 \) and \( Z_2(0), Z_2(1) \in \mathbb{Q}_1 \). We define a path \( (\tilde{Z}_1, \tilde{Z}_2) = (\tilde{Z}_1(t), \tilde{Z}_2(t)) \) by
\[
\tilde{Z}_1(t) = (-|Z_{1x}(t)|, |Z_{1y}(t)|), \quad \tilde{Z}_2(t) = (|Z_{2x}(t)|, |Z_{2y}(t)|), \quad \forall t \in [0, 1]. \tag{3.14}
\]
It is clear that \( \tilde{Z}_1(t) \in \mathbb{Q}_2, \tilde{Z}_2(t) \in \mathbb{Q}_1 \) and \( |\tilde{Z}_i(t)| = |Z_i(t)| \) for all \( t \in [0, 1] \). Also, \( \int_0^1 K(\tilde{Z}_1, \tilde{Z}_2) \, dt = \int_0^1 K(Z_1, Z_2) \, dt \). By lemma 3.6, we have
\[
U(Z_1(t), Z_2(t)) \geq U(\tilde{Z}_1(t), \tilde{Z}_2(t)), \quad \forall t \in [0, 1]. \tag{3.15}
\]
Thus
\[
\mathcal{A}(Z_1, Z_2) \geq \mathcal{A}(\tilde{Z}_1, \tilde{Z}_2).
\]
On the other hand, since \( (Z_1, Z_2) \) is a minimizing path connecting the two fixed boundaries, and \( (\tilde{Z}_1, \tilde{Z}_2) \) connects the same boundaries, it implies that
\[
\mathcal{A}(Z_1, Z_2) \leq \mathcal{A}(\tilde{Z}_1, \tilde{Z}_2).
\]
Hence
\[
\mathcal{A}(Z_1, Z_2) = \mathcal{A}(\tilde{Z}_1, \tilde{Z}_2). \tag{3.16}
\]
By the smoothness of \( U(Z_1(t), Z_2(t)) \) in \((0, 1)\), it follows that
\[
U(Z_1(t), Z_2(t)) = U(\tilde{Z}_1(t), \tilde{Z}_2(t)).
\]
Then by lemma 3.6, \( Z_1(t) \) and \( Z_2(t) \) are in two adjacent closed quadrants for all \( t \in [0, 1] \).

Note that by the celebrated results of Marchal [11] and Chenciner [2], an action minimizer \((Z_1, Z_2)\) is collision-free in \((0, 1)\) and it is a solution of the Newtonian equations in \((0, 1)\).

By (3.16), it implies that both \((Z_1, Z_2)\) and \((\tilde{Z}_1, \tilde{Z}_2)\) are smooth paths in \((0, 1)\). Thus if \( Z_1 \) or \( Z_2 \) touches the coordinate axes in \((0, 1)\), it must be tangent to the coordinate axes.

Note that \( Z_1 = q_2 - q_3 \), it follows that \( Z_1(t) \neq 0 \) for all \( t \in (0, 1) \). If \( Z_2(t_0) = 0 \), by the smoothness of \( Z_2 \) and \( \tilde{Z}_2 \), it follows that \( \dot{Z}_2(t_0) = 0 \). Since \( \{(q_1, q_2, q_3) \mid q_1 = 0, \dot{q}_1 = 0\} \) is an invariant set, it implies...
that $Z_2 \equiv 0$ for all $t \in [0, 1]$. In this case, it is one part of the Euler solution, which is case (c) in Theorem 3.1.

If $Z_2 \neq 0$, by the Newtonian equations, $Z_1$ and $Z_2$ satisfy

$$
\ddot{Z}_1 = \frac{Z_2 - Z_1/2}{|Z_2 - Z_1/2|^3} - \frac{Z_2 + Z_1/2}{|Z_2 + Z_1/2|^3} - \frac{2mZ_1}{|Z_1|^3},
$$

$$
\ddot{Z}_2 = -\frac{1 + 2m}{2} \left[ \frac{Z_2 - Z_1/2}{|Z_2 - Z_1/2|^3} + \frac{Z_2 + Z_1/2}{|Z_2 + Z_1/2|^3} \right].
$$

(3.17)

Next, we show that if there exists some $t_0 \in (0, 1)$, such that $Z_1(t_0)$ or $Z_2(t_0)$ is on the axes, then $Z_1(t)$ and $Z_2(t)$ must stay on the axes for all $t \in [0, 1]$. In fact, we first assume that $Z_1(t_0)$ is on the positive $x$-axis for some $t_0 \in (0, 1)$ and $Z_2(t_0) \in Q_1$. By (3.17), the acceleration in the $y$-direction $\dot{Z}_1y(t_0)$ satisfies

$$
\dot{Z}_1y(t_0) > 0.
$$

The smoothness of $Z_1$ and $\ddot{Z}_1$ implies that the velocity satisfies $\dot{Z}_1y(t_0) = 0$. Consequently, for small enough $\varepsilon > 0$, $Z_1(t_0 + \varepsilon) \in Q_1$, $Z_2(t_0 + \varepsilon) \in Q_1$. Contradiction to the property that $Z_1(t)$ and $Z_2(t)$ belong to two adjacent closed quadrants for all $t \in [0, 1]$! Hence $Z_2(t_0) \notin Q_1$. Similarly, $Z_2(t_0) \notin Q_i$ for $i = 1, 2, 3, 4$, i.e $Z_2(t_0)$ is on the coordinate axes.

We then discuss the case when $Z_1(t_0)$ is on the negative $x$-axis. If $Z_2(t_0) \in Q_4$, it follows that $\dot{Z}_1y(t_0) = 0$ and $\ddot{Z}_1y(t_0) < 0$. Hence, there exists small enough $\varepsilon > 0$, such that $Z_1(t_0 + \varepsilon) \in Q_3$ and $Z_2(t_0 + \varepsilon) \in Q_1$. Contradiction to the result that $Z_1(t)$ and $Z_2(t)$ must belong to two adjacent quadrants! Similarly, $Z_2(t_0) \notin Q_i$ for $i = 1, 2, 3, 4$. Therefore, whenever $Z_1(t_0) \neq 0$ is on the $x$-axis for some $t_0 \in (0, 1)$, $Z_2(t_0)$ must be on the axes. By Lemma 3.4, $Z_1$ and $Z_2$ must stay on the corresponding axes for all $t \in [0, 1]$.

The same argument works for $Z_1(t_0)$ on the $y$-axis or $Z_2(t_0)$ on one of the axes. Therefore, whenever there is some $t_0 \in (0, 1)$ such that $Z_1(t_0)$ or $Z_2(t_0)$ is on the axes, both $Z_1(t)$ and $Z_2(t)$ must stay on the corresponding axes for all $t \in [0, 1]$. That is, case (b) in Theorem 3.1 holds whenever there exists some $t_0 \in (0, 1)$ such that $Z_1(t_0)$ or $Z_2(t_0)$ is on the axes.

In the end, if both case (b) and (c) fail, we have both $Z_1(t)$ and $Z_2(t)$ are away from the axes for all $t \in (0, 1)$. Then case (a) holds. The proof is complete.

Now we can apply Theorem 3.1 to exclude the possible collisions in the action minimizer $\mathcal{P}_{m, \theta}$.

**Theorem 3.5.** For each given $\theta \in [0, \pi/2)$ and mass set $M = [m_1, m_2, m_3] = [1, m, m]$ with $m > 0$, the minimizer $\mathcal{P}_{m, \theta}$ is collision-free.

**Proof.** By the celebrated work of Marchal [11] and Chenciner [2], $\mathcal{P}_{m, \theta}$ is collision-free in $(0, 1)$. By Section 2, $\mathcal{P}_{m, \theta}$ has no total collision. Furthermore, it has no binary collision at $t = 1$.

Recall that at $t = 0$, the three bodies are on the $x$-axis and satisfy the order $q_{2x}(0) \leq q_{1x}(0) \leq q_{3x}(0)$. It implies that he only possible collisions are $q_1(0) = q_2(0)$ and $q_1(0) = q_3(0)$. Note $\mathcal{P}_{m, \theta}$ is free of total collision, it implies that $q_{2x}(0) < 0$ and $q_{3x}(0) > 0$. 

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If $Z_{2x}(0) = (1 + \frac{1}{2m})q_{1x}(0) = 0$, it is an Euler configuration. It is clear that it can not have any binary collision at $t = 0$. If $Z_{2x}(0) > 0$, the only possible binary collision is $q_1(0) = q_3(0)$. Similarly, if $Z_{2x}(0) = (1 + \frac{1}{2m})q_{1x}(0) < 0$, the only possible binary collision is $q_1(0) = q_2(0)$. We only discuss the case when $Z_{2x}(0) = (1 + \frac{1}{2m})q_{1x}(0) > 0$. The other case $Z_{2x}(0) < 0$ is exactly the same.

If $Z_{2x}(0) > 0$, we are left to eliminate the possible binary collision $q_1(0) = q_3(0)$. In what follows, we exclude the binary collision between bodies 1 and 3 by contradiction.

When $\theta \in (0, \pi/2)$, we claim that the collision minimizer $(Z_1, Z_2)$ satisfies

$$Z_1(t) \in \overline{Q_2}, \quad Z_2(t) \in \overline{Q_1}, \quad \forall t \in [0, 1]. \quad (3.18)$$

In fact, by the definition of $Q_S$ and $Q_E$ in (1.3), $Z_1(0) < 0$ is on the negative $x$-axis, while $Z_2(0) > 0$ is on the positive $x$-axis. $Z_1(1) \in Q_2$ or $Q_4$ and $Z_2(1) \in Q_1$ or $Q_3$.

Assume $Z_1(1) \in Q_4$. A new path $(\tilde{Z}_1, \tilde{Z}_2)$ can be defined by (3.14), such that

$$\tilde{Z}_1(t) = (\begin{bmatrix} |Z_{1x}(t)| \end{bmatrix}, \begin{bmatrix} |Z_{1y}(t)| \end{bmatrix}) \in \overline{Q_2}, \quad \tilde{Z}_2(t) = (\begin{bmatrix} |Z_{2x}(t)| \end{bmatrix}, \begin{bmatrix} |Z_{2y}(t)| \end{bmatrix}) \in \overline{Q_1}, \quad \forall t \in [0, 1],$$

and Lemma 3.3 implies that

$$\mathcal{A}(Z_1, Z_2) \geq \mathcal{A}(\tilde{Z}_1, \tilde{Z}_2).$$

Note that $(\tilde{Z}_1, \tilde{Z}_2)$ also satisfies the boundary condition (1.2), it implies that $\mathcal{A}(Z_1, Z_2) \leq \mathcal{A}(\tilde{Z}_1, \tilde{Z}_2)$. Hence,

$$\mathcal{A}(Z_1, Z_2) = \mathcal{A}(\tilde{Z}_1, \tilde{Z}_2).$$

Since $\tilde{Z}_1(1) \in Q_2$ and $\tilde{Z}_2(1) \in \overline{Q_1}$, by Theorem 3.1 we have $\tilde{Z}_1(t) \in Q_2$ for all $t \in (0, 1)$. On the other hand, $Z_1(0)$ is on the negative $x$-axis and $Z_1(1) \in Q_4$. Thus there is some $t_0 \in (0, 1)$ such that $Z_1(t_0)$ is on the $y$-axis. Then $\tilde{Z}_1(t_0)$ is also on the $y$-axis. Contradict to $\tilde{Z}_1(t) \in Q_2$ for all $t \in (0, 1)$! Hence, $Z_1(1) \in Q_2$. A similar argument shows that $Z_2(1) \in \overline{Q_1}$. By Theorem 3.1 (3.18) holds.

When $\theta = 0$, we have $Z_1(1)$ is on the $y$-axis and $Z_2(1)$ is on the $x$-axis. By the same argument as above, we actually have

$$Z_1(t) \in \overline{Q_2}, \quad Z_2(t) \in \overline{Q_1}, \quad \forall t \in [0, 1],$$

or

$$Z_1(t) \in \overline{Q_3}, \quad Z_2(t) \in \overline{Q_4}, \quad \forall t \in [0, 1].$$

Here we only consider the case $Z_1(t) \in \overline{Q_2}, Z_2(t) \in \overline{Q_1}$, while the other case follows similarly.

For $\theta \in [0, \pi/2)$, the minimizer must satisfies that $Z_1(t)$ and $Z_2(t)$ belongs to two adjacent closed quadrants for all $t \in [0, 1]$. This is enough to exclude the binary collision at $t = 0$.

Assume at $t = 0$, $q_1(0) = q_3(0)$. Since $q_{2x} \leq q_{1x} \leq q_{3x}$, it follows that $q_1(0) = q_3(0) > 0$ and $q_2(0) < 0$. In fact, by [8, 10, 19], the following limits hold:

$$\lim_{t \to 0^+} \left| \frac{q_1 - q_3}{q_1 - q_3} \right| = (1, 0), \quad \lim_{t \to 0^+} \frac{d}{dt} \left( \frac{q_1 - q_3}{|q_1 - q_3|} \right) = (0, 0). \quad (3.19)$$

It is known that an isolated binary collision in the three-body problem can be regularized. By (3.19), it implies that for small enough $t > 0$, there exists $\alpha_0 > 2/3$, such that

$$q_1 - q_3 = (at^{2/3} + o(t^{2/3}), bt^{\alpha_0} + o(t^{\alpha_0})).$$
Hence, if \( b \neq 0 \),
\[
\frac{d}{dt} \left( \frac{q_1 - q_3}{q_1 - q_3} \right) = \left( \frac{-ab^2t^2\alpha_0^{-7/3}(3\alpha_0 - 2)}{3(a^2 + b^2t^2\alpha_0^{-4/3})^{3/2}} + o(t^{2\alpha_0 - 7/3}) \right),
\]
By (3.19), it follows that
\[
2\alpha_0 - 7/3 > 0, \quad \alpha_0 - 5/3 > 0.
\]
Therefore,
\[
\alpha_0 > 5/3.
\]
It follows that
\[
\lim_{t \to 0^+} (\dot{q}_{1y} - \dot{q}_{3y}) = 0. \tag{3.20}
\]
If \( b = 0 \), it is clear that (3.20) holds. Hence, at \( t = 0 \), \( q_1 \) and \( q_3 \) has the same \( y \)-velocity. Thus
\[
\dot{q}_{1y}(0) = \dot{q}_{3y}(0) = -\frac{m}{m+1} \dot{q}_{2y}(0). \tag{3.21}
\]
By the above discussion, for \( t \in [0, 1] \) there always holds \( Z_1(t) \in \overline{Q}_2, Z_2(t) \in \overline{Q}_1 \). Together with (3.21), we have
\[
\dot{q}_{2y} \geq 0, \quad \dot{q}_{1y} = \dot{q}_{3y} \leq 0. \tag{3.22}
\]
When \( \dot{q}_{2y} = 0 \), by Lemma 6.2 in [20], the path \((Z_1, Z_2) = (Z_1(t), Z_2(t))(t \in [0, 1]) \) must stay on the \( x \)-axis. However, by the definition of \( Q_E \) in [13], the collision-free isosceles configuration at \( t = 1 \) can never become a collinear configuration on the \( x \)-axis. Contradiction!
When \( \dot{q}_{2y}(0) > 0 \), By (3.21), it follows that \( \dot{q}_{1y}(0) < 0 \). Then for \( t \in (0, \varepsilon) \) with \( \varepsilon > 0 \) small enough we have
\[
\dot{q}_{1y}(t) < 0, \quad \forall t \in (0, \varepsilon).
\]
Hence for \( t \in (0, \varepsilon) \), \( Z_2(t) \notin Q_1 \). But \( Z_1(t) \in Q_2, Z_2(t) \in Q_1 \) holds for all \( t \in (0, 1) \). Contradiction!

The above argument implies that body 1 and 3 can not collide at \( t = 0 \) in \( P_{m, \theta} \) in the case when \( Z_{2x} > 0 \). For the case \( Z_{2x} < 0 \), the argument is similar. Therefore, \( P_{m, \theta} \) is free of collision. The proof is complete.

By Theorem 3.5 for each given \( \theta \in [0, \pi/2] \) and each \( m > 0 \), the action minimizer \( P_{m, \theta} \) is actually a solution of the Newtonian equations. By applying the formulas of first variation as in Section 5 of [22], one can easily show that \( P_{m, \theta} \) can be extended to a periodic or quasi-periodic orbit.

### 4 Compare \( P_{m, \theta} \) and the Euler orbit for \( m = 1 \)

In previous sections, we have shown that \( P_{m, \theta} \) is collision-free. In this section, we show that \( P_{m, \theta} \) is nontrivial when \( m = 1 \) and \( \theta \in [0, 0.183\pi] \).

**Theorem 4.1.** When \( m = 1 \) and \( \theta \in [0, 0.183\pi] \), the minimizer \( P_{m, \theta} \) is nontrivial.
Proof. The case when \( \theta \in [0.084\pi, 0.183\pi] \) has been shown in [22]. In what follows, we only show the case when \( \theta \in [0, 0.084\pi] \).

Let \( \mathcal{A}_{Euler} \) be the action of an Euler orbit connecting \( Q_S \) and \( Q_E \) in (1.3). Then

\[
\mathcal{A}_{Euler} = 3 \left( \frac{5}{4} \right)^{2/3} \left( \frac{\pi}{2} - \theta \right)^{2/3}.
\]

We need to define a test path \( P_{test} \) connecting \( Q_S \) and \( Q_E \), such that its action \( \mathcal{A}_{test} = \mathcal{A}(P_{test}) \) satisfies \( \mathcal{A}_{test} < \mathcal{A}_{Euler} \) for each \( \theta \in [0, 0.084\pi] \). The test path \( P_{test} \) is defined as follows. We choose \( \theta_0 = 0.053\pi \).

Let \( q = \left[ \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \right] \) be the position matrix path of the minimizer \( P_{1,\theta_0} \), and \( \bar{q} = \left[ \begin{array}{c} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \end{array} \right] \) be the position matrix path of \( P_{\text{test}} = P_{\text{test},\theta} \). We can then define a test path \( P_{\text{test},\theta} \) by connecting the following 11 points:

\[
\bar{q} \left( \frac{i}{10} \right) = q \left( \frac{i}{10} \right), \quad (i = 0, 1, \ldots, 9), \quad \bar{q} (1) = q(1)R(-\theta_0)R(\theta).
\]

In fact, \( \bar{q}(t) \) satisfies

\[
\bar{q}(t) = \bar{q} \left( \frac{i}{10} \right) + 10 \left( t - \frac{i}{10} \right) \left[ \bar{q} \left( \frac{i+1}{10} \right) - \bar{q} \left( \frac{i}{10} \right) \right], \quad t \in \left[ \frac{i}{10}, \frac{i+1}{10} \right], \tag{4.1}
\]

where \( i = 0, 1, \ldots, 9 \). It is easy to check that \( \bar{q}(0) \in Q_S \) and \( \bar{q}(1) \in Q_E \), where \( Q_S \) and \( Q_E \) are the boundary sets defined in (1.3). Once the values of \( q \left( \frac{i}{10} \right) (i = 0, 1, \ldots, 10) \) in \( P_{1,\theta_0} \) are given, the action of the test path \( \mathcal{A}_{test} = \mathcal{A}(P_{\text{test},\theta}) \) can be calculated accurately as in Lemma 3.1 of [22]. The data of the test path and the corresponding figures of action values are shown in Table I and Fig. 4. For the given set of 11 interpolation points, it is shown in [22] that the action of the test path \( \mathcal{A}_{test} \) is a smooth function with respect to \( \theta \). In the last step, we compare the value of the two smooth functions: \( \mathcal{A}_{test} \) and \( \mathcal{A}_{Euler} \) when \( \theta \in [0, 0.084\pi] \) in Fig. 4. To do so, we calculate the value of the two functions with a step \( \pi \times 10^{-3} \).

| \( \theta \) = 0.053\pi, \quad \theta \in [0, 0.084\pi] | \begin{array}{c|c|c}
\hline
0 & (0.3067, 0) & (-0.9504, 0) \\
0.1 & (0.34241260, 0.11622575) & (-0.94529039, 0.06856631) \\
0.2 & (0.41572598, 0.18977437) & (-0.93016235, 0.13667709) \\
0.3 & (0.49233787, 0.22736882) & (-0.90538226, 0.20366579) \\
0.4 & (0.56164714, 0.24183944) & (-0.87127416, 0.26870901) \\
0.5 & (0.62118018, 0.24108669) & (-0.82813029, 0.33090644) \\
0.6 & (0.67056091, 0.22979410) & (-0.77626334, 0.38932578) \\
0.7 & (0.70992537, 0.21093951) & (-0.71604331, 0.44303606) \\
0.8 & (0.73947888, 0.18657625) & (-0.64791266, 0.49113901) \\
0.9 & (0.75936088, 0.15823937) & (-0.57237992, 0.53279730) \\
1 & (0.77840337, 0)R(\theta) & (-0.38920168, 0.64061834)R(\theta) \\
\hline
\end{array} 

Table 1: The positions of \( \tilde{q}_{i,j} = \tilde{q}_i(\frac{t}{10}) (i = 1, 2, j = 0, 1, \ldots, 10) \) in \( P_{test} = P_{test,\theta} \) corresponding to \( \theta \in [0, 0.084\pi] \).
The error of the linear interpolation method used to compare the two functions is 
\( \frac{1}{8}(3.14 \times 10^{-3})^2 \Delta \), where \( \Delta \) is the maximum of the second derivative of the corresponding function. For \( \theta \in [0, 0.084\pi] \), it turns out that \( \Delta \leq \frac{40}{\pi} \) for both functions. It implies that the error is bounded by
\[
\frac{1}{8}(3.14 \times 10^{-3})^2 \frac{40}{\pi} \approx 1.57 \times 10^{-5}.
\]
Numerically, for \( \theta \in [0, 0.084\pi] \), the minimum value of \( A_{Euler} - A_{test} \) in Fig. 4 is \( 9.49 \times 10^{-3} > 1.57 \times 10^{-5} \).

Therefore, for each given \( \theta \in [0, 0.084\pi] \), the action of the test path \( A_{test} \) satisfies
\[
A_{test} < A_{Euler}.
\]
It follows that the action minimizer \( P_{1, \theta} \) is nontrivial and collision-free when \( \theta \in [0, 0.084\pi] \). The proof is complete. \( \square \)

5 Application to the retrograde orbit

In this section, we apply our geometric result (Theorem 3.1) to the retrograde orbit. Note that the retrograde orbit can be characterized as an action minimizer connecting two free boundaries. Let

\[
Q_{s1} = \begin{bmatrix} -2ma_1 - ma_2 & 0 \\ a_1 - ma_2 & 0 \\ a_1 + (m+1)a_2 & 0 \end{bmatrix}, \quad Q_{e1} = \begin{bmatrix} 2mb_2 & 0 \\ -b_2 & b_1 \\ -b_2 & -b_1 \end{bmatrix} R(\theta),
\]

where \( M = [1, m, m] \), \( a_1 \geq 0, a_2 \geq 0, b_1 \in \mathbb{R}, b_2 \in \mathbb{R} \), and \( R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \). The two configuration sets are defined as follows:

\[
Q_{S1} = \left\{ Q_{s1} \left| a_1 \geq 0, a_2 \geq 0 \right. \right\}, \quad Q_{E1} = \left\{ Q_{e1} \left| b_1 \in \mathbb{R}, b_2 \in \mathbb{R} \right. \right\},
\]
For each $\theta \in (0, \pi/2]$, standard results [7, 9, 15, 24] imply that there exists an action minimizer $\tilde{P}_{m, \theta} \in H^1([0, 1], \chi)$, such that

$$
A(\tilde{P}_{m, \theta}) = \inf_{q \in P(Q_{S_1}, Q_{E_1})} A = \inf_{q \in P(Q_{S_1}, Q_{E_1})} \int_0^1 (K + U) \, dt,
$$

(5.3)

where

$$
P(Q_{S_1}, Q_{E_1}) = \{ q \in H^1([0, 1], \chi) \mid q(0) \in Q_{S_1}, q(1) \in Q_{E_1} \}.
$$

We first show that the action minimizer $\tilde{P}_{m, \theta}$ is free of total collision by applying the level estimate method [4, 6]. The test path here is a modified version of the test path in [6].

**Lemma 5.1.** $\tilde{P}_{m, \theta}$ is free of total collision for all $m > 0$ and $\theta \in (0, \pi/2]$.

**Proof.** The proof is based on Theorem 2 in [6]. To be consistent, we use the same notation as in [6]. Let the action minimizer $\tilde{P}_{m, \theta}$ is for $t \in [0, T]$ with $T = 1/4$.

We first obtain a lower bound of action when the action minimizer $\tilde{P}_{m, \theta}$ has a total collision. We denote the action by $A_{\text{total}}$ if $\tilde{P}_{m, \theta}$ has a total collision. By Chen’s level estimate [4], if $T = 1/4$, the lower action bound of total collision path is

$$
A_{\text{total}} \geq \frac{m^2 + 2m}{1 + 2m} \cdot \frac{3}{2} (1 + 2m)^{2/3} \pi^{2/3} \left(\frac{1}{4}\right)^{1/2} = \frac{3m}{2} \sqrt[3]{\frac{m + 2}{(1 + 2m)^{1/3}}} \left(\frac{1}{4}\right)^{1/2}.
$$

By the setting of $Q_{S_1}$ and $Q_{E_1}$ in (5.2), we consider an artificial path

$$
x(t) = (x_1(t), x_2(t), x_3(t))
$$
on $[0, 1]$ with $x_i(t) \in \mathbb{C}$.

Let $M = 2m + 1$, $\phi = 4\theta \in (0, 2\pi]$, $\alpha = \left(\frac{M}{2} \right)^{\frac{1}{3}}$ and

$$
J(s) = \int_0^1 \frac{1}{|1 - se^{2\pi i}|} \, dt \quad \text{for} \quad 0 \leq s < 1.
$$

We set

$$
Q(t) = \frac{1}{(M\phi)^{\frac{1}{2}}} e^{\phi \xi}, \quad R(t) = \frac{1}{(2m)^{\frac{1}{2}} (2\pi)^{\frac{1}{2}}} e^{(\phi - 2\pi)\xi}
$$

and consider the path

$$
x(t) = (x_1(t), x_2(t), x_3(t)) = (-2mQ(t), Q(t) - mR(t), Q(t) + mR(t)).
$$

Note that $[0, 1/4]$ is a fundamental domain of this path and $x(0) \in Q_{S_1}, x(1/4) \in Q_{E_1}$. A direct calculation shows that

$$
|\dot{x}_1|^2 = (2m)^2 \left(\frac{\phi}{M^2}\right)^{\frac{1}{2}},
$$

$$
|\dot{x}_2|^2 = \left(\frac{\phi}{M^2}\right)^{\frac{1}{2}} + \frac{m^2(\phi - 2\pi)^2}{(2m)^{\frac{1}{2}} (2\pi)^{\frac{1}{2}}} - \left(\frac{\phi}{M^2}\right)^{\frac{1}{2}} \frac{2m(\phi - 2\pi)}{(2m)^{\frac{1}{2}} (2\pi)^{\frac{1}{2}}} \cos(2\pi t),
$$

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\[ |x_3|^2 = \left( \frac{\phi}{M^2} \right)^\frac{1}{3} + \frac{m^2(\phi - 2\pi)^2}{(2m)^\frac{1}{3}(2\pi)^\frac{1}{3}} + \left( \frac{\phi}{M^2} \right)^\frac{1}{3} \frac{2m(\phi - 2\pi)}{(2m)^\frac{1}{3}(2\pi)^\frac{1}{3}} \cos(2\pi), \]

\[
\int_0^1 K(\dot{x})dt = m \left[ \left( \frac{\phi}{M^2} \right)^\frac{1}{3} + \frac{m^2(\phi - 2\pi)^2}{(2m)^\frac{1}{3}(2\pi)^\frac{1}{3}} \right] + \frac{1}{2} \left( \frac{\phi}{M^2} \right)^\frac{1}{3}. \]

And the potential energy is

\[
\int_0^1 U(x)dt = \int_0^1 \frac{m}{|x_1 - x_2|} + \frac{m}{|x_1 - x_3|} + \frac{m}{|x_2 - x_3|}dt \\
= \int_0^1 \left( \frac{\phi^2}{M} \right)^\frac{1}{3} \left[ \frac{m}{1 - m\alpha e^{-2\pi i}} + \left( \frac{\phi^2}{M} \right)^\frac{1}{3} \frac{m}{1 + m\alpha e^{-2\pi i}} \right] + 2\pi m^2 \pi^2 dt \\
= 2^\frac{1}{3} m^\frac{2}{3} \pi^\frac{2}{3} + 2m \left( \frac{\phi^2}{M} \right)^\frac{1}{3} J(m\alpha). \]

Note that \( m\alpha = m \left( \frac{\phi^2}{M} \right)^\frac{1}{3} \frac{1}{(2m)^\frac{1}{3}} \leq \frac{m}{(2m+1)^\frac{1}{3}(2m)^\frac{1}{3}} < \frac{1}{2} \) and \( J(s) \) is increasing in \([0, 1) \) (4). Therefore the action of the path over the fundamental domain \([0, 1/4]\) is

\[ A_{est} = \frac{1}{4} \int_0^1 K(\dot{x}) + U(x)dt \leq \frac{m}{4} \left( \frac{\pi^2}{M} \right)^\frac{1}{3} \left[ 3 \times 2^{-\frac{2}{3}} m^\frac{2}{3} M^\frac{1}{3} + \left( 2J \left( \frac{1}{2} \right) + 1 \right) \times 2^\frac{2}{3} \right] . \]

Now we are left to prove \( A_{est} < A_{total} \) for all \( m > 0 \).

If \( 0 < m \leq 1 \), we have

\[
\frac{A_{est}}{A_{total}} \leq \frac{3 \times 2^{-\frac{2}{3}} m^\frac{2}{3} M^\frac{1}{3} + \left( 2J \left( \frac{1}{2} \right) + 1 \right) \times 2^\frac{2}{3}}{6 \times \left( \frac{1}{3} \right)^\frac{1}{3} (m + 2)} \\
\leq \frac{1}{2 \times 3^\frac{2}{3}} + \frac{\left[ 2J \left( \frac{1}{2} \right) + 1 \right] \times 2^\frac{2}{3}}{6 \times 2^\frac{1}{3}} \approx 0.9011 < 1. \tag{5.4} \]

If \( m \geq 1 \), let \( f(m) = f_1(m) - f_2(m) \), where

\[
f_1(m) = \frac{m}{4} \left( \frac{\pi^2}{M} \right)^\frac{1}{3} \left[ 3 \times 2^{-\frac{2}{3}} m^\frac{2}{3} M^\frac{1}{3} + \left( 2J \left( \frac{1}{2} \right) + 1 \right) \times 2^\frac{2}{3} \right] \]

and

\[
f_2(m) = \frac{3m}{2} \pi^\frac{2}{3} m^\frac{2}{3} M^\frac{1}{3} (\frac{\phi}{M^2} \beta) \frac{1}{3} . \]

Since \( A_{est} < f_1(m) \) and \( A_{total} \geq f_2(m) \). In order to show \( A_{est} < A_{total} \), it is sufficient to show that \( f(m) = f_1(m) - f_2(m) < 0 \).
In fact,

\[ f(m) := f_1(m) - f_2(m) = m \left( \frac{\pi^2}{4} \right)^{\frac{1}{2}} \left[ 3 \times 2^{-\frac{3}{4}} m^{\frac{3}{4}} + \left( 2J \left( \frac{1}{2} \right) + 1 \right) 2^{\frac{1}{2}} - 3 \times 2^{\frac{3}{4}} (m + 2) \right]. \]

Let \( g(m) = 3 \times 2^{-\frac{3}{4}} m^{\frac{3}{4}} + \left( 2J \left( \frac{1}{2} \right) + 1 \right) 2^{\frac{1}{2}} - 3 \times 2^{\frac{3}{4}} (m + 2) \), then

\[ f(m) = m \left( \frac{\pi^2}{4} \right)^{\frac{1}{2}} g(m). \]

A direct calculation shows that

\[ \frac{dg(m)}{dm} = 2^{\frac{1}{2}} \left[ \left( \frac{2m+1}{m} \right)^{\frac{1}{2}} + \left( \frac{m}{2m+1} \right)^{\frac{1}{2}} \right] - 3 \times 2^{\frac{1}{2}} < 0 \]

for every \( m \geq 1 \). It follows that \( g(m) < g(1) \) for any \( m > 1 \). However, when \( m = 1 \), it is shown that \( f(1) < 0 \). It implies \( g(1) < 0 \) and \( g(m) < 0 \) for all \( m > 1 \). So,

\[ \mathcal{A}_{est} - \mathcal{A}_{total} < f(m) < 0, \quad \forall \ m \geq 1. \tag{5.5} \]

Therefore, by (5.4) and (5.5), the inequality \( \mathcal{A}_{est} < \mathcal{A}_{total} \) holds for any \( m > 0 \). The proof is complete.

By the same argument of Lemma 2.4 in Section 2, the action minimizer \( \tilde{\mathcal{P}}_{m,\theta} = \tilde{\mathcal{P}}_{m,\theta}(\theta, \pi/2) \) in (5.3) is collision-free at \( t = 1 \). We are then left to exclude the possible binary collisions at \( t = 0 \) in \( \tilde{\mathcal{P}}_{m,\theta} \).

Let \( Z_1 = q_2 - q_3 \), and \( Z_2 = q_1 - \frac{q_2 + q_3}{2} = (1 + \frac{1}{2m})q_1 \).

Let \( (\tilde{Z}_1(t), \tilde{Z}_2(t)) \ (t \in [0, 1]) \) be the minimizing path corresponding to the action minimizer \( \tilde{\mathcal{P}}_{m,\theta} \). Next, we apply Theorem 3.1 to exclude possible binary collisions at \( t = 0 \).

**Lemma 5.2.** **For any** \( m > 0 \) **and** \( \theta \in (0, \pi/2), \) \( q_2(0) \neq q_3(0) \) **in** \( \tilde{\mathcal{P}}_{m,\theta} \).

**Proof.** We show it by contradiction! Note that by Lemma 5.1 there is no total collision at \( t = 0 \). If \( q_2(0) = q_3(0) \) in \( \tilde{\mathcal{P}}_{m,\theta} \), it follows that

\[ \dot{\tilde{Z}}_1(0) = 0, \quad \dot{\tilde{Z}}_2(0) < 0, \quad \text{and} \quad \dot{\tilde{Z}}_3(0) = 0, \]

while

\[ \dot{\tilde{Z}}_1(1) \in \mathbb{Q}_2 \text{ or } \mathbb{Q}_4, \quad \dot{\tilde{Z}}_2(1) \in \mathbb{Q}_1 \text{ or } \mathbb{Q}_3. \]

Lemma 5.3 and Theorem 3.1 imply that for any \( m > 0 \) and \( \theta \in (0, \pi/2) \), the minimizing path \( (\tilde{Z}_1(t), \tilde{Z}_2(t)) \ (t \in [0, 1]) \) must satisfy that the two curves \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) stay in two adjacent closed quadrants and they don’t touch the axes in \((0, 1)\). It implies that \( \tilde{Z}_2(t) \in \mathbb{Q}_3 \).
On the other hand, note that by (19), when \( q_2(0) = q_3(0) \) in \( \overline{P}_{m, \theta} \), the motion of the collision pair satisfies

\[
\lim_{t \to 0^+} \frac{q_2 - q_3}{|q_2 - q_3|} = (1, 0), \quad \lim_{t \to 0^+} (\dot{q}_{2y} - \dot{q}_{3y}) = 0.
\]

By Theorem 3.1, \( \dot{Z}_1(t) \in \overline{Q}_4 \). Since \( \dot{Z}_2(t) \in \overline{Q}_3 \), it follows that \( \dot{q}_{1y}(0) \leq 0, \dot{q}_{2y}(0) = \lim_{t \to 0^+} \dot{q}_{2y} \geq 0. \)

When \( \varepsilon > 0 \) small enough, we consider the following identity:

\[
q_{2y}(\varepsilon) - q_{3y}(\varepsilon) = \int_0^\varepsilon \int_0^t [\dot{q}_{2y} - \dot{q}_{3y}] \, ds \, dt. \tag{5.6}
\]

Theorem 3.1 implies that we can choose small enough \( \varepsilon > 0 \), such that \( \dot{Z}_1(\varepsilon) \in Q_4 \). That is \( \dot{Z}_{1y}(\varepsilon) = q_{2y}(\varepsilon) - q_{3y}(\varepsilon) < 0. \)

However, the Newtonian equations imply that

\[
\dot{q}_{2y} - \dot{q}_{3y} = \frac{2(q_{3y} - q_{2y})}{|q_2 - q_3|^3} + \frac{q_{1y} - q_{2y}}{|q_1 - q_2|^3} - \frac{q_{1y} - q_{3y}}{|q_1 - q_3|^3}.
\]

Since \( \dot{Z}_1(t) \in \overline{Q}_4 \) and \( \dot{Z}_2(t) \in \overline{Q}_3 \) for all \( t \in [0, \varepsilon] \) with \( \varepsilon > 0 \) small enough, it follows that for all \( t \in [0, \varepsilon] \),

\[
\frac{2(q_{3y}(t) - q_{2y}(t))}{|q_2(t) - q_3(t)|^3} \geq 0, \quad \frac{q_{1y}(t) - q_{2y}(t)}{|q_1(t) - q_2(t)|^3} - \frac{q_{1y}(t) - q_{3y}(t)}{|q_1(t) - q_3(t)|^3} \geq 0, \quad \forall t \in [0, \varepsilon].
\]

Hence,

\[
\frac{2(q_{1y}(t) - q_{2y}(t))}{|q_1(t) - q_2(t)|^3} \geq 0, \quad \frac{q_{1y}(t) - q_{2y}(t)}{|q_1(t) - q_2(t)|^3} - \frac{q_{1y}(t) - q_{3y}(t)}{|q_1(t) - q_3(t)|^3} \geq 0, \quad \forall t \in [0, \varepsilon].
\]

It implies that

\[
q_{2y}(\varepsilon) - q_{3y}(\varepsilon) = \int_0^\varepsilon \int_0^t [\dot{q}_{2y} - \dot{q}_{3y}] \, ds \, dt \geq 0.
\]

Contradict to \( q_{2y}(\varepsilon) - q_{3y}(\varepsilon) < 0! \)

Therefore, \( \overline{P}_{m, \theta} \) has no binary collision between bodies 2 and 3 at \( t = 0 \). The proof is complete. \( \square \)

**Lemma 5.3.** For any \( m > 0 \) and \( \theta \in (0, \pi/2), q_1(0) \neq q_2(0) \) in \( \overline{P}_{m, \theta} \).

**Proof.** We show it by contradiction! Note that by Lemma 5.1 there is no total collision at \( t = 0 \). If \( q_1(0) = q_2(0) \) in \( \overline{P}_{m, \theta} \), it follows that

\[
\dot{Z}_{1x}(0) < 0, \quad \dot{Z}_{2x}(0) < 0, \quad \text{and} \quad \dot{Z}_{1y}(0) = \dot{Z}_{2y}(0) = 0,
\]

while

\[
\dot{Z}_1(1) \in Q_2 \text{ or } Q_4, \quad \dot{Z}_2(1) \in Q_1 \text{ or } Q_3.
\]

Lemma 3.3 and Theorem 3.1 imply that for any \( m > 0 \) and \( \theta \in (0, \pi/2), \) the minimizing path \( (\dot{Z}_1(t), \dot{Z}_2(t)) \) \( (t \in [0, 1]) \) must satisfy

\[
\dot{Z}_1(t) \in \overline{Q}_2, \quad \dot{Z}_2(t) \in \overline{Q}_3. \tag{5.7}
\]

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By [19], when \( q_1(0) = q_2(0) \) in \( \overline{\mathcal{P}_{m, \theta}} \), the motion of the collision pair satisfies

\[
\lim_{t \to 0^+} \frac{q_1 - q_2}{|q_1 - q_2|} = (1, 0), \quad \lim_{t \to 0^+} (q_{1y} - q_{2y}) = 0.
\]

Since \( \dot{Z}_2(t) \in \overline{\mathcal{Q}_3} \), it follows that \( q_{1y}(0) = \lim_{t \to 0^+} q_{1y} = q_{2y}(0) < 0 \) and \( q_{3y}(0) > 0 \). Then \( \dot{Z}_1(t) = q_2(t) - q_3(t) \) is in \( Q_3 \) for all \( t \in (0, \varepsilon] \). Contradict to \( \dot{Z}_1(t) \in \overline{\mathcal{Q}_2}! \)

Therefore, there is no binary collision between bodies 1 and 2 at \( t = 0 \) in \( \overline{\mathcal{P}_{m, \theta}} \). The proof is complete.

In the case when \( \theta = \pi/2 \), the only difference is that \( Q_{E_3} \) can be degenerated to an Euler configuration on the \( x \)-axis. We show that the action minimizer \( \overline{\mathcal{P}_{m, \theta}} \) is either one part of the Schubart orbit (Fig. 2(a)) or one part of the Broucke-Hénon orbit (Fig. 2(b)).

**Theorem 5.4.** Let \( \theta = \pi/2 \) and \( m > 0 \). The action minimizer \( \overline{\mathcal{P}_{m, \theta}} \) coincide with either the Schubart orbit or the Broucke-Hénon orbit.

**Proof.** By Theorem 3.1, the minimizing path \( (\dot{Z}_1, \dot{Z}_2) \) must satisfy one of the three cases. Similar to the arguments in Lemma 5.2 and Lemma 5.3, case (a) in Theorem 3.1 implies that the action minimizer \( \overline{\mathcal{P}_{m, \theta}} \) is collision-free.

Next, we consider cases (b) and (c) of Theorem 5.1. Note that by the definition of \( Q_{E_1} \) in (1.6), it can be degenerated to an Euler configuration with body 1 at the origin. If case (c) happens, it implies that \( (\dot{Z}_1, \dot{Z}_2) \) coincides with the Euler solution and \( Z_2 \equiv 0 \) for all \( t \in [0, 1] \). However, by the definition of \( Q_{E_1} \) in (1.6), it implies that \( q_1(0) = q_2(0) = q_3(0) = 0 \), which is a total collision. Contradict to Lemma 5.1. Therefore, the minimizing path \( (\dot{Z}_1, \dot{Z}_2) \) must stay on the \( x \)-axis for all \( t \in [0, 1] \).

Next, we show that \( q_2(0) \neq q_3(0) \) in \( \overline{\mathcal{P}_{m, \theta}} \). If not, note that \( \lim_{t \to 0^+} \frac{q_2 - q_3}{|q_2 - q_3|} = (1, 0) \) and \( \overline{\mathcal{P}_{m, \theta}} \) is collision free in \( (0, 1] \), it implies that the three bodies should keep the order at \( t = 1 \):

\[
q_{1x}(1) < q_{3x}(1) < q_{2x}(1), \quad q_{1y}(1) = q_{3y}(1) = q_{2y}(1) = 0. \tag{5.8}
\]

However, by the definition of \( Q_{E_1} \), when it becomes a straight line on the \( x \)-axis, the order of the three bodies can only be

\[
q_{2x}(1) < q_{1x}(1) < q_{3x}(1), \quad \text{or} \quad q_{3x}(1) < q_{1x}(1) < q_{2x}(1).
\]

Contradict to (5.8)! Hence, there is no binary collision between bodies 2 and 3 in \( \overline{\mathcal{P}_{m, \theta}} \).

If at \( t = 0 \), \( q_1(0) = q_2(0) \) in \( \overline{\mathcal{P}_{m, \theta}} \), i.e. bodies 1 and 2 collide at \( t = 0 \). Since \( (\dot{Z}_1, \dot{Z}_2) \) stays on the \( x \)-axis for all \( t \in [0, 1] \), it implies that

\[
q_{1y}(0) = q_{2y}(0) = q_{3y}(0) = 0.
\]

By Lemma 6.2 in [19], the motion must be collinear and it must be part of the Schubart orbit.

Therefore, for any given \( m > 0 \) and \( \theta = \pi/2 \), the action minimizer \( \overline{\mathcal{P}_{m, \theta}} \) coincides with either the Schubart orbit or the Broucke-Hénon orbit. The proof is complete. \( \square \)
References

[1] Chenciner, A., Montgomery, R.: A remarkable periodic solution of the three-body problem in the case of equal masses, *Ann. of Math.* **152** (2000), 881–901.

[2] Chenciner, A.: Action minimizing solutions in the Newtonian n-body problem: from homology to symmetry, *Proceedings of the International Congress of Mathematicians (Beijing, 2002)*, Higher Ed. Press, Beijing, 279–294, 2002.

[3] Chenciner, A., Venturelli, A.: Minima de l’intégrale d’action du problème Newtonien de 4 corps de masses égales dans \( \mathbb{R}^3 \): orbites “Hip-Hop”, *Celest. Mech. Dyn. Astro.* **77** (2000), 139–151.

[4] Chen, K.: Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses, *Ann. of Math.* **167** (2008), 325–348.

[5] Chen, K.: Binary decompositions for planar N-body problems and symmetric periodic solutions, *Arch. Ration. Mech. Anal.* **170** (2003), 247–276.

[6] Chen, K., Lin, Y.: On action-minimizing retrograde and prograde orbits of the three-body problem, *Comm. Math. Phys.* **291** (2009), 403–441.

[7] Chen, K.: Removing collision singularities from action minimizers for the N-body problem with free boundaries, *Arch. Ration. Mech. Anal.* **181** (2006), 311–331.

[8] Chen, K.: A minimizing property of hyperbolic Keplerian orbits, *J. Fixed Point Theory Appl.* **19** (2017) 281–287.

[9] Ferrario, D., Terracini, S.: On the existence of collisionless equivariant minimizers for the classical n-body problem, *Invent. Math.* **155** (2004), 305–362.

[10] Fusco, G., Gronchi, G., Negrini, P.: Platonic polyhedra, topological constraints and periodic solutions of the classical N-body problem, *Invent. Math.* **185** (2011), 283–332.

[11] Marchal, C.: How the method of minimization of action avoids singularities, *Celest. Mech. Dyn. Astro.* **83** (2002), 325–353.

[12] Ouyang, T., Xie, Z.: Star pentagon and many stable choreographic solutions of the Newtonian 4-body problem, *Physica D* **307** (2015), 61–76.

[13] Venturelli, A.: Application de la minimisation de l’action au problème des N corps dans le plan et dans l’espace, *Thesis, Université de Paris 7*, 2002.

[14] Conjectures and open problems concerning Variational Methods in Celestial Mechanics, *https://www.aimath.org/WWN/varcelest/*, 2003.

[15] Shi, B., Liu, R., Yan, D.: Multiple periodic orbits connecting a collinear configuration and a double isosceles configuration in the planar equal-mass four-body problem, *Adv. Nonlinear Stud.* **17** (2017), 819–835.
[16] Yu, G.: Periodic solutions of the planar N-center problem with topological constraints, *Disc. Cont. Dyn. Sys.* 36 (2016), 5131–5162.

[17] Yu, G.: Simple choreography solutions of the Newtonian N-body problem, *Arch. Ration. Mech. Anal.* 225 (2017), 901–935.

[18] Yu, G.: Spatial double choreographies of the Newtonian 2n-body problem, *arXiv*: 1608.07956.

[19] Yu, G.: Shape space figure-8 solution of three body problem with two equal masses, *Nonlinearity* 30 (2017), 2279–2307.

[20] Zhang, S., Zhou, Q.: Nonplanar and noncollision periodic solutions for N-body problems, *Dist. Cont. Dyn. Syst.* 10 (2004), 679–685.

[21] Wang, Z., Zhang, S.: New periodic solutions for Newtonian n-body problems with dihedral group symmetry and topological constraints, *Arch. Ration. Mech. Anal.* 219 (2016), 1185–1206.

[22] Liu, R., Li, J., Yan, D.: New periodic orbits in the planar equal-mass three-body problem, *Dist. Cont. Dyn. Syst.* (2017), to appear.

[23] Kuang, W., Yan, D.: Existence of prograde double-double orbits in the equal-mass four-body problem, *arXiv*: 1710.09960.

[24] Kuang, W., Ouyang, T., Xie, Z., Yan, D.: The Broucke-Hénon orbit and the Schubart orbit in the planar three-body problem with equal masses, *arXiv*: 1607.00580.