Hessian–information geometric formulation of Hamiltonian systems and generalized Toda’s dual transform

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Abstract
In this paper a class of classical Hamiltonian systems is geometrically formulated. This class is such that a Hamiltonian can be written as the sum of a kinetic energy function and a potential energy function. In addition, these energy functions are assumed strictly convex. For this class of Hamiltonian systems Hessian and information geometric formulation is given. With this formulation, a generalized Toda’s dual transform is proposed, where his original transform was used in deriving his integrable lattice system. Then a relation between the generalized Toda’s dual transform and the Legendre transform of a class of potential energy functions is shown. As an extension of this formulation, dissipation-less electric circuit models are also discussed in the geometric viewpoint above.

Keywords: Hamilton mechanics, Legendre transform, information geometry, Toda lattice, Hessian geometry

1. Introduction

Ideas of duality shed light on various aspects in mathematics and physics. One of such is the Legendre duality by the Legendre transform associated with a convex function, and this transform connects different viewpoints.

In mathematics, the Legendre transform plays a role in Hessian geometry [1]. This geometry is deeply connected to information geometry, where information geometry is a geometrization of mathematical statistics [2, 3]. Applications of information geometry include statistical...
interference, thermodynamics, and so on. Nowadays, some of Hessian geometry and that of information geometry are amalgamated [4, 5]. Then, it is expected that the development of Hessian and information geometries influences various mathematical sciences. The Legendre transform also appears in contact geometry, where contact geometry is known to be an odd-dimensional cousin of symplectic geometry [6, 7]. As is well-known, symplectic geometry is a geometrization of analytical mechanics, and contact and symplectic geometries have influenced various branches of pure and applied mathematics. Such examples include geometric formulations of electric circuit models [8, 9] and a geometric characterization for dynamical systems discussed in information geometry [10].

In physics, the subjects where the use of the Legendre transform is emphasized are analytical mechanics and thermodynamics [11, 12]. Since analytical mechanics is placed at the center of theoretical physics, its development influences various physical sciences, such as condensed matter physics, high energy physics, and so on. Thermodynamics is related to black hole physics, information geometry and so on [13–15]. Although the use of the Legendre transform has been stressed in analytical mechanics [16], the emphasis is not placed on recent development of Hessian geometry and information one. Also, the application of the Legendre transform has mainly been to kinetic energy functions. We then feel that the link between geometry consisting of Hessian and information geometries and analytical mechanics or symplectic geometry should be explored more [17]. In addition, in Toda’s paper of 1967 [18], the way to find his integrable lattice model is to apply a transform, called the dual transform. His dual transform makes nonlinear force terms linear ones for Hamilton’s equations of motion. On the other hand the applications of the Legendre transform to a class of potential functions yield the linearized forces as well. Thus we should explore a relation between these transforms. If such a relation exists, it is then expected that this relation can be introduced in the theory of dissipation-less electric circuit models since model equations are similar to Hamilton’s equations.

In this paper the term Hessian–information geometry stands for the geometry consisting of Hessian geometry and information one, and it is shown how the Legendre transform is applied to a class of classical Hamiltonian systems with emphasis on Hessian–information geometry. Here such a Hamiltonian is the sum of a strictly convex kinetic energy functions and strictly convex potential energy function, so that the Legendre transform is invertible. In this Hessian–information geometric formulation, it is shown that the $\alpha$-connection invented in information geometry also appears in canonical equations of motion. Then Toda’s dual transform and a class of lattice system are interpreted from the viewpoint of the Legendre transform and that of Hessian–information geometry. Also, some dual lattice systems are constructed explicitly. Finally the Hessian–information geometric formulation for dissipation-less electric circuit theory is discussed. In appendix, a brief explanation of information geometry and Hessian geometry is given.

### 2. Natural Hamiltonian systems with strictly convex energies

In this paper a class of Hamiltonian systems is considered. In this class a Hamiltonian is the sum of a kinetic energy function and a potential energy function.

Let $\mathcal{M}$ be a $2n$-dimensional manifold, $H : \mathcal{M} \rightarrow \mathbb{R}$ a Hamiltonian, $(p, q)$ a set of canonical coordinates such that a symplectic 2-form is $\omega = \sum_{a=1}^{n} dp_a \wedge dq^a = dp_a \wedge dq^a$, $K : \mathcal{M} \rightarrow \mathbb{R}$ a kinetic energy function depending on $p$ only, and $U : \mathcal{M} \rightarrow \mathbb{R}$ a potential energy function depending on $q$ only. Here and in what follows the Einstein convention is used and every object is differentiable. Thus, $H$ can be written as
The canonical equations of motion are then
\[
\frac{d}{dt}q^a = \frac{\partial K}{\partial p_a}, \quad \text{and} \quad \frac{d}{dt}p_a = -\frac{\partial U}{\partial q_a}, \quad a \in \{1, \ldots, n\},
\]
where \( t \in \mathbb{R} \) denotes time. The equation (2) also referred to as Hamilton’s equations of motion. These can be derived from
\[
\iota_{X_H} \omega = -dH,
\]
where \( X_H \in TM \) is a Hamiltonian vector field, and \( \iota_Y \) the interior product operator with \( Y \in TM \), with \( TM \) being the tangent bundle. In this geometric context the triplet \((M, \omega, H)\) is referred to as a (classical) Hamiltonian system.

If a Hamiltonian can be written as (1), then \( H \) is referred to as a natural Hamiltonian and its system is referred to as a natural Hamiltonian system. Throughout this section it is assumed that

- the manifold \( M \) can be written as \( M = M_K \times M_U \) with some \( n \)-dimensional manifolds \( M_K \) and \( M_U \). Local coordinates of \( M_K \) and \( M_U \) are denoted as \( p \) and \( q \), respectively.

To specify \( H \) given in (1) further, strictly convex function is introduced. Let \( N \) be an \( n \)-dimensional manifold, \( \{x^a\} \) a set of coordinates, and \( f : N \rightarrow \mathbb{R} \) a function. If \( f \) satisfies
\[
\frac{\partial^2 f}{\partial x^a \partial x^b} \succ 0,
\]
in some convex domain \( D \subset N \), then \( f \) is referred to as a strictly convex function in \( D \). Here \( A \succ 0 \) denotes that a matrix \( A \) is positive definite.

Then, strictly convex energy functions are introduced.

**Definition 2.1 (Strictly convex energy functions).** If \( K \) and \( U \) satisfy
\[
\frac{\partial^2 K}{\partial p_a \partial p_b} \succ 0, \quad \text{and} \quad \frac{\partial^2 U}{\partial q^a \partial q^b} \succ 0, \quad a, b \in \{1, \ldots, n\}
\]
in some convex domains, then \( K \) is referred to as a strictly convex kinetic energy function and \( U \) a strictly convex potential energy function.

If one considers a natural Hamiltonian system whose Hamiltonian is the sum of strictly convex energy functions, then one can apply Hessian geometry to the system. Since a part of Hessian geometry has been applied to information geometry, one can also apply known facts found in information geometry to Hamiltonian systems with strictly convex energy functions.

### 2.1. Non-vanishing potential systems

In this subsection it is assumed that

- a system is a natural Hamiltonian system whose Hamiltonian is the sum of strictly convex energy functions, \( H = K + U \) with \( K \) being a function of \( p = \{p_a\} \), and \( U \) being a function of \( q = \{q^a\} \).

From this assumption, the conditions \( (\partial^2 K/\partial p_a \partial p_b) \succ 0 \) and \( (\partial^2 U/\partial q^a \partial q^b) \succ 0 \), are satisfied.

From convex analysis the following coordinates play various roles...
Definition 2.2 (Dual coordinates). The coordinates defined by
\[ p_a^* = \frac{\partial K}{\partial p_a}, \quad \text{and} \quad q_a^* = \frac{\partial U}{\partial q_a^*}, \quad a \in \{1, \ldots, n\} \]
are referred to as dual coordinates. In particular, \( p_a^* \) is referred to as being dual to \( p_a \), and \( q_a^* \) is referred to as being dual to \( q_a \).

Remark 2.1. Since \( K \) is strictly convex, one has that the correspondence between \( p_a \) and \( p_a^* \) is one-to-one. Similarly, the correspondence between \( q_a \) and \( q_a^* \) is also one-to-one.

Due to strict convexity of \( K \) and \( U \), one has the Riemannian metric tensor fields
\[ h_K^{ab} = \frac{\partial^2 K}{\partial p_a \partial p_b}, \quad \text{and} \quad h_U^{ab} = \frac{\partial^2 U}{\partial q_a^* \partial q_b^*}, \quad a, b \in \{1, \ldots, n\}. \] (3)

Definition 2.3 (Riemannian metric tensor fields associated with convex energy functions). The \( h_K^{ab} \) in (3) with (4) is referred to as the Riemann metric tensor field associated with \( K \), and \( h_U^{ab} \) in (3) with (4) is referred to as that associated with \( U \), respectively.

There exist the inverse matrices of \((h_K^{ab})\) and \((h_U^{ab})\). Such inverse matrices \((h_K^{ab})\) and \((h_U^{ab})\) can be written as
\[ h_{K}^{ab} = \frac{\partial^2 K^*}{\partial p_a \partial p_b^*}, \quad \text{and} \quad h_{U}^{ab} = \frac{\partial^2 U^*}{\partial q_a^* \partial q_b^*}, \quad a, b \in \{1, \ldots, n\}, \] (4)
where \( K^* \) and \( U^* \) are the Legendre transforms of \( K \) and \( U \):
\[ K^*(p_a) = \sup_p \{ p_a p_a^* - K(p) \}, \quad \text{and} \quad U^*(q_a^*) = \sup_q \{ q_a^* q_a^* - U(q) \}. \] (5)

It can be shown that [2]
\[ p_a = \frac{\partial K^*}{\partial p_a}, \quad \text{and} \quad q_a^* = \frac{\partial U^*}{\partial q_a^*}. \] (6)

The following inequalities are consequences of the strict convexity of \( K \) and \( U \).

Proposition 2.1. Let \( z_K \) and \( z_K' \) be two points of \( M_K \), \( p = \{ p_a \} \) and \( p' = \{ p'_a \} \) coordinates of \( z_K \) and \( z_K' \), \( p_a = \{ p_a^* \} \) and \( p'_a = \{ p'_a^* \} \) dual coordinates of \( z_K \) and \( z_K' \), and \( D_K : M_K \times M_K \rightarrow \mathbb{R} \) a function such that
\[ D_K (z_K \parallel z_K') = K(p) + K^*(p'_a) - p_a p'_a. \]

Then, it follows that
\[ D_K (z_K \parallel z_K') \geq 0. \]

In addition, the equality holds when \( z_K = z_K' \).

Proof. See [1] for example. \( \square \)
Similar to this, one has the following.

**Proposition 2.2.** Let \( z_U \) and \( z'_U \) be two points of \( M_U \), \( q = \{ q^a \} \) and \( q' = \{ q'^a \} \) coordinates of \( z_U \) and \( z'_U \), \( q^* = \{ q^*_a \} \) and \( q'^* = \{ q'^*_a \} \) dual coordinates of \( z_U \) and \( z'_U \), and \( D_U : M_U \times M_U \rightarrow \mathbb{R} \) a function such that 

\[
D_U(z_U \parallel z'_U) = U(q) + U^*(q'^*_a) - q_a q'^*_a.
\]

Then, it follows that 

\[
D_U(z_U \parallel z'_U) \geq 0.
\]

In addition, the equality holds when \( z_U = z'_U \).

**Proof.** See [1] for example.

In information geometry the functions similar to \( D_K \) and \( D_U \) in propositions 2.1 and 2.2 are often used. Such functions are known as the canonical divergences, and they are used in various applications [2, 3, 19]. As shown in these propositions, it should be emphasized that the canonical divergences can be introduced in the present class of Hamilton’s equations, and that the existence of these functions enables one to discuss information geometric aspects of Hamiltonian systems.

Applying Hessian-information geometry to natural Hamiltonian systems, one can write canonical equations of motion in terms of geometric objects developed in such geometry. To this end, introducing some connections on Riemannian manifolds, one has Hessian manifolds. To discuss Hessian geometry of canonical equations of motion one defines the following connections.

**Definition 2.4 (Flat connections associated with energy functions).** The connections \( \nabla^K \) and \( \nabla^U \) such that

\[
h^K = \nabla^K dK, \quad \text{and} \quad h^U = \nabla^U dU,
\]

are referred to as the connection associated with \( K \), and referred to as that associated with \( U \), respectively. Also, the connections \( \nabla^{K^*} \) and \( \nabla^{U^*} \) such that

\[
h^K = \nabla^{K^*} dK^*, \quad \text{and} \quad h^U = \nabla^{U^*} dU^*.
\]

are referred to as the connection associated with \( K^* \), and referred to as that associated with \( U^* \), respectively.

Let \((N, g)\) be a Riemannian manifold, and \( \nabla \) a connection such that there exists a coordinate system so that connection components vanish. Such coordinate system is referred to as a \( \nabla \)-affine coordinate system. If there exists a function \( \psi \) on \( N \) such that \( g = \nabla d\psi \), then \((N, \nabla, g)\) is referred to as a Hessian manifold.

By definition, \( \nabla^K \) and \( \nabla^U \) are flat connections, where \( \nabla^K \)-affine coordinates are \( \{ p^a \} \), and \( \nabla^U \)-affine ones are \( \{ q^a \} \). Then the triplets \((M_K, \nabla^K, h^K)\) and \((M_U, \nabla^U, h^U)\) are Hessian manifolds. Similarly, \( \nabla^{K^*} \) and \( \nabla^{U^*} \) are flat connections, where \( \nabla^{K^*} \)-affine coordinates are \( \{ p^*_a \} \), and \( \nabla^{U^*} \)-affine ones are \( \{ q^*_a \} \). Then the triplets \((M_K, \nabla^{K^*}, h^K)\) and \((M_U, \nabla^{U^*}, h^U)\) are Hessian manifolds.

There is some overlap between information geometry and Hessian geometry, and cubic forms are defined in information geometry. Such cubic forms also appear in rewriting Hamilton’s equations.
Definition 2.5 (Cubic form). The following \((0,3)\)-tensor fields

\[ C^K = \nabla^k h^K, \quad \text{and} \quad C^U = \nabla^U h^U, \]

are referred to as the cubic form associated with \(K\) and referred to as that with \(U\), respectively.

Similarly,

\[ C^K^* = \nabla^{K^*} h^K, \quad \text{and} \quad C^U^* = \nabla^{U^*} h^U, \]

are referred to as the cubic form associated with \(K^*\) and referred to as that with \(U^*\), respectively.

Note that cubic forms are not 3-forms. The components of cubic form are given as follows.

Lemma 2.1. In terms of \(\nabla^K\)-affine coordinates \(\{ p_a \}\) and \(\nabla^U\)-affine coordinates \(\{ q^a \}\), the components of the cubic forms

\[ C^K = C^K_{abc} dp_a \otimes dp_b \otimes dp_c, \]

and

\[ C^U = C^U_{abc} dq^a \otimes dq^b \otimes dq^c, \]

are written as

\[ C^K_{abc} = \frac{\partial^3 K}{\partial p_a \partial p_b \partial p_c}, \quad \text{and} \quad C^U_{abc} = \frac{\partial^3 U}{\partial q^a \partial q^b \partial q^c}. \]

Similarly, in terms of \(\nabla^{K^*}\)-affine coordinates \(\{ p^*_a \}\) and \(\nabla^{U^*}\)-affine coordinates \(\{ q^*_a \}\), the components of the cubic forms

\[ C^{K^*} = C^{K^*}_{abc} dp^*_a \otimes dp^*_b \otimes dp^*_c, \]

and

\[ C^{U^*} = C^{U^*}_{abc} dq^*_a \otimes dq^*_b \otimes dq^*_c, \]

are written as

\[ C^{K^*}_{abc} = \frac{\partial^3 K^*}{\partial p^*_a \partial p^*_b \partial p^*_c}, \quad \text{and} \quad C^{U^*}_{abc} = \frac{\partial^3 U^*}{\partial q^*_a \partial q^*_b \partial q^*_c}. \]

Proof. Let \(X, Y, Z\) be vector fields whose basis is \(\{ \partial / \partial p_a \}\). Then one has

\[ C^K(X, Y, Z) = (\nabla^K h^K)(Y, Z) = X(h^K(Y, Z) - h^K(\nabla^K Y, Z) - h^K(Y, \nabla^K Z)). \]

Since \(\{ p_a \}\) is a set of \(\nabla^K\)-affine coordinates, one has \(\nabla^K X = 0\) and \(\nabla^K Y = 0\). Combining these and (4), one arrives at

\[ C^K_{abc} = C^K \left( \frac{\partial}{\partial p_a}, \frac{\partial}{\partial p_b}, \frac{\partial}{\partial p_c} \right) = \frac{\partial h^K_{bc}}{\partial p_a} = \frac{\partial^3 K}{\partial p_a \partial p_b \partial p_c}. \]

Similarly one can calculate the components \(C^U_{abc}, C^{K^*}_{abc}\) and \(C^{U^*}_{abc}\). □
As shown below, the components of the cubic forms $C^K$ and $C^U$ are related to the connection components of the Levi-Civita connections associated with $h^K$ and $h^U$.

**Lemma 2.2.** Let $\nabla^{K(0)}$ and $\nabla^{U(0)}$ be the Levi-Civita connections associated with $h^K$ and $h^U$, $\Gamma^{ab}_{K(0)}$ connection coefficients for $\nabla^{K(0)}$ such that $\nabla^{K(0)}_b \partial^c = \Gamma^{ab}_{K(0)} \partial^c$, $(\partial^a := \partial/\partial q^a)$, $\Gamma^{ab}_{U(0)}$ connection coefficients for $\nabla^{U(0)}$ such that $\nabla^{U(0)}_b \partial^c = \Gamma^{ab}_{U(0)} \partial^c$. Then $\Gamma^{ab}_{K(0)} := h^K_{ij} \Gamma^{ab}_{K(0)}$ and $\Gamma^{ab}_{U(0)} := h^K_{ij} \Gamma^{ab}_{U(0)}$ are given by

$$\Gamma^{abc}_{K(0)} = \frac{1}{2} \frac{\partial^3 K_+}{\partial q^a \partial q^b \partial q^c} = \frac{1}{2} c^{abc}_K,$$

and $\Gamma^{abc}_{U(0)} = \frac{1}{2} \frac{\partial^3 U_+}{\partial q^a \partial q^b \partial q^c} = \frac{1}{2} c^{abc}_U$.

**Proof.** A proof for $\Gamma^{U(0)}_{ab}$ is given as follows. Substituting $h^{U}_{ab} = \partial^2 U/\partial q^a \partial q^b$ into

$$\Gamma^{U(0)}_{abc} = \frac{1}{2} \frac{\partial^3 U_+}{\partial q^a \partial q^b \partial q^c},$$

one has

$$\Gamma^{U(0)}_{abc} = \frac{1}{2} \frac{\partial^3 U_+}{\partial q^a \partial q^b \partial q^c}.$$ 

Combining this with lemma 2.1, one has $\Gamma^{U(0)}_{abc} = C^{U}_{abc}/2$. Similarly proofs for $\Gamma^{K(0)}_{ab}$, $\Gamma^{K(0)}_{abc}$ and $\Gamma^{U(0)}_{abc}$ can be given.

The canonical equations of motion (2) can then be written as

$$\frac{d}{dt} \left( \frac{\partial U_+}{\partial q^a} \right) = p^a_+, \quad \frac{d}{dt} \left( \frac{\partial K_+}{\partial q^a} \right) = -q^a_+, \quad a \in \{1, \ldots, n\}. \quad (9)$$

It should be noted that the force term $-\partial U/\partial q$ in the original coordinate system is linear $-q_+$ in the dual coordinate system. This linearization scheme for force term may be seen as an extension or a variant of Toda’s dual transform [18] (see section 2.2).

From the viewpoint above, one arrives at the following set of transformed equations, and this is summarized as the main theorem in this paper.

**Theorem 2.1 (Generalized Toda’s dual transformed equations).** Consider the natural Hamiltonian system (2). If $\{h^K_{ab}\}$ are constant, then the canonical equations are written as

$$\frac{d}{dt} \left( \frac{\partial U_+}{\partial q^a} \right) = h^K_{ab} p_b + h^K_{a(0)}, \quad a \in \{1, \ldots, n\}, \quad (10)$$

where $p_a := \partial p_a/\partial t$ and $\{h^K_{a(0)}\}$ are constant.
Proof. One has second order equations of motion in the transformed coordinates as follows. It follows from (2) that

\[ \frac{d^2 q^a}{dt^2} = - \frac{\partial^2 K}{\partial p_a \partial p_b} \frac{\partial U}{\partial q^b}, \quad a \in \{1, \ldots, n\} \]

from which

\[ \frac{d^2}{dt^2} \left( \frac{\partial U^*}{\partial q^a} \right) = - h_{ab}^b q^a, \quad a \in \{1, \ldots, n\}. \quad (11) \]

Substituting \( \dot{p}_a = - q^a \) coming from the second equation of (9) into (11), one has

\[ \frac{d^2}{dt^2} \left( \frac{\partial U^*}{\partial q^a} \right|_{q^* = -\dot{p}_a} = h_{ab}^b \frac{dp_b}{dt}. \]

Since \( \{ h_{ab}^b \} \) are constant, one can integrate the equations above with respect to \( t \). These calculations yield (10).

In this paper the set of equation (11) is referred to as the generalized Toda’s dual transformed equations, and it will be shown how these transformed equations are related to the original Toda’s equations in section 2.2.

One can integrate (11) once more by changing variables. Choice of such new variables depends on the given system. However the following change of variables is a generalization for the case of the Toda lattice system. First, the abbreviation

\[ U^* a (q^*) := \frac{\partial U^*}{\partial q^*_a}, \quad a \in \{1, \ldots, n\}, \]

is introduced, then one introduces the variables \( \tau^U = \{ \tau^U_a \} \) that depend on \( t \) as

\[ p_a (\tau^U) = \delta_{ab} \frac{d}{dt} U^* b (\tau^U). \]

It follows that

\[ p_a (\tau^U (t)) = \delta_{ab} h_{bc}^c \frac{dr_c}{dt}, \quad \text{and} \quad \frac{d}{dt} p_a (\tau (t)) = \delta_{ab} \frac{d^2 U^* b}{dt^2} = \delta_{ab} \left( \frac{\partial h_{bc}^c}{\partial \tau^U} \frac{dr^U_c}{dt} \frac{dr^U_c}{dt} + h_{bc}^c \frac{d^2 \tau^U_c}{dt^2} \right), \]

where \( \delta_{ab} \) is the Kronecker delta giving unity for \( a = b \), and zero otherwise. In these coordinates, one has from (10) that

\[ U^* a \left( - \delta_{ab} \frac{d^2 U^* b}{dt^2} \right) = h_{bc}^b \delta_{bc} U^* c (\tau^U) + h_{bc}^b \delta_{bc} U^* c (\tau^U) + h_{bc}^b U^* c (\tau^U) + h_{bc}^b \delta_{bc} U^* c (\tau^U), \]

where \( \{ h_{bc}^b \} \) are constant. For the case of the Toda lattice system, \( U^* \) is a logarithm function, and the functions \( \tau^U \) is the so-called \( \tau \)-functions (see section 2.2).

The following states how the canonical equations of motion are written in terms of Hessian–information geometry.

**Theorem 2.2 (Canonical equations of motion written in terms of Hessian geometry).** The canonical equations of motion in terms of the dual coordinates (11) can be written in the forms

\[ C_{\dot{U} a}^{a b} \dot{q}_b \dot{q}_c + h_{\dot{U} a}^{a b} \dot{q}_b = - h_{\dot{U} a}^{a b} q_b, \quad a \in \{1, \ldots, n\}, \quad (12) \]
or equivalently
\[ 2\Gamma_{U^*}^{abc}(0) \dot{q}_a^* \dot{q}_b^* + h_{U^*}^{ab} q_b^* = -h_K^{ab} q_b^*, \quad a \in \{1, \ldots, n\}, \]
where \( \dot{q}_a^* := dq_a^*/dt \) and \( \ddot{q}_a^* := d^2 q_a^*/dt^2 \).

**Proof.** First, (11) can be written as
\[ \frac{\partial}{\partial q_a^*} \dot{q}_b^* \frac{\partial}{\partial q_c^*} \dot{q}_b^* + \frac{\partial}{\partial q_a^*} \frac{\partial}{\partial q_c^*} \dot{q}_b^* = -h_K^{ab} q_b^*, \quad a \in \{1, \ldots, n\}. \]
Then substituting the explicit forms of \( C_{U^*}^{abc}, h_{U^*}^{ab}, h_K^{ab} \), and \( \Gamma_{U^*}^{abc}(0) \) obtained in lemmas 2.1 and 2.2 into the equations above, one completes the proof.

In addition to this theorem, one can see how the bases \( \{\partial/\partial q^a\}, \{\partial/\partial q_a\}, \{\partial/\partial p_a\}, \) and \( \{\partial/\partial p_a^*\} \) are oriented. From discussions in information geometry [8, 10], they are oriented. From discussions in information geometry [8, 10], they are oriented. From discussions in information geometry [8, 10], they are oriented.

In addition, the solution to the canonical equations of motion satisfy \( q_a^* = -\dot{p}_a \). This yields that the vector \( \partial/\partial q^a \) is anti-parallel to \( \partial/\partial p_a \).

Extending \( \Gamma_{U^*}^{abc}(0) \), one can have the one-parameter family of connection coefficients as
\[ \Gamma_{U^*}^{abc}(\alpha) := 1 - \frac{\alpha}{2} \frac{\partial}{\partial q_a^*} \frac{\partial}{\partial q_b^*} \frac{\partial}{\partial q_c^*}, \quad \alpha \in \mathbb{R}. \]

It follows from (13) that \( \Gamma_{U^*}^{abc}(\alpha) \) and \( \Gamma_{U^*}^{abc}(-\alpha) \) satisfy
\[ \Gamma_{U^*}^{abc}(\alpha) + \Gamma_{U^*}^{abc}(-\alpha) = \frac{\partial}{\partial q_a^*} h_{U^*}^{bc}. \]

In the context of information geometry, the pair of connection coefficients \( \Gamma_{U^*}^{abc}(\alpha) \) and \( \Gamma_{U^*}^{abc}(-\alpha) \) are referred to as the components of dual connections with respect to \( h_{U^*} \) [2]. It should be noted that the \( \alpha \)-connection plays a role in information geometry. As shown below, this family of connections can also appear in this geometric formulation of classical Hamiltonian systems.

**Proposition 2.3.** The canonical equations of motion are written in terms of the \( \alpha \)-connection with \( \alpha = -1 \) as
\[ \frac{d^2 q_i^*}{dt^2} + h_{U^*}^{ij} \Gamma_{U^*}^{jbc} \frac{dq_j^*}{dt} \frac{dq_c^*}{dt} = -h_{U^*}^{ik} h_K^{jk} q_b^*, \quad i \in \{1, \ldots, n\}. \]

**Proof.** The canonical equations of motion (11) can be written with \( \Gamma_{U^*}^{abc}(\alpha) \) as follows. Combining lemma 2.2 and (13), one has
\[ C_{U^*}^{abc} = 2\Gamma_{U^*}^{abc}(0) = \Gamma_{U^*}^{abc}(-1). \]
Substituting (15) into (12), one has (14).

One can also write (14) as
\[ \frac{d^2 q_i^*}{dt^2} + h_{U^*}^{ij} h_K^{jk} q_b^* = -h_{U^*}^{ik} \Gamma_{U^*}^{jbc} \frac{dq_j^*}{dt} \frac{dq_c^*}{dt}, \quad i \in \{1, \ldots, n\}. \]
This form of the equations is similar to a form of a perturbed harmonic oscillator if \( \Gamma_{ij}^{ab} U_{i}(-1) \) are small enough. A variety of applications of perturbed harmonic oscillators are found in physics, and a solution to the unperturbed system is the basis of discussion in general. Thus the case where \( \Gamma_{ij}^{ab} U_{i}(-1) \) vanish is of interest.

As shown below, the following duality holds in the case where \( U \) is quadratic.

**Proposition 2.4.** Consider the case where \( U \) is quadratic. Then a solution to the canonical equations of motion is also a solution to the equations obtained by replacing the \((-1)\)-connection in (14).

**Proof.** Throughout this proof, the set of variables \( \{ \hat{q}_a \} \) is introduced in order to emphasize that a quadratic potential is focused, and \( \{ q_a \} \) is distinguished from \( \{ \hat{q}_a \} \). Similarly \( \{ \hat{q}_a \} \) is introduced. Since \( U \) is quadratic, one has from (4) that

\[
U(\hat{q}) = \frac{1}{2} h_{ab}^{ij} U(\hat{q}) \hat{q}_a \hat{q}_b .
\]

Then the Legendre transform of \( U \) is calculated from (5) as

\[
U^*(\hat{q}) = \sup_q \{ \hat{q}^a \hat{q}_a - U(\hat{q}) \} = [\hat{q}^a \hat{q}_a - U(\hat{q})]_{\hat{q}^a = h_{ab}^{ij} U(\hat{q}) \hat{q}_b} = \frac{1}{2} h_{ab}^{ij} U(\hat{q}) \hat{q}_a \hat{q}_b .
\]

For \( U^* \), it follows from (13) that \( \Gamma_{ij}^{abc} U^*(-1) \equiv 0 \). With this, \( h_{ij}^U = h_{ij}^U(0) \), and (16), one has

\[
\frac{d^2 \hat{q}_i^{(1)}}{dt^2} + h_{ij}^U(0) h_{ij}^{ab} \hat{q}_b^{(1)} = 0, \quad i \in \{1, \ldots, n\}.
\tag{17}
\]

Consider the equations obtained by replacing \( \Gamma_{ij}^{abc} U^*(-1) \) with \( \Gamma_{ij}^{abc} U^*(-1) \) in (16):

\[
\frac{d^2 \hat{q}_i^{(1)}}{dt^2} + h_{ij}^U(0) h_{ij}^{ab} \hat{q}_b^{(1)} = -h_{ij}^U(0) \Gamma_{ij}^{abc} \frac{d \hat{q}_b^{(1)}}{dt} \frac{d \hat{q}_a^{(1)}}{dt}, \quad i \in \{1, \ldots, n\},
\tag{18}
\]

where the set of variables \( \{ \hat{q}_a^{(1)} \} \) has been introduced to emphasize that the \( \alpha \)-connection with \( \alpha = 1 \) has been focused. It follows from (13) that \( \Gamma_{ij}^{abc} U^*(-1) \equiv 0 \). Then with \( h_{ij}^U = h_{ij}^U(0) \) the equations (18) reduce to

\[
\frac{d^2 \hat{q}_i^{(1)}}{dt^2} + h_{ij}^U(0) h_{ij}^{ab} \hat{q}_b^{(1)} = 0, \quad i \in \{1, \ldots, n\}.
\tag{19}
\]

Thus, a solution of (19) is that of (17). \( \square \)

**Remark 2.2.** If \( K \) is also quadratic, then (17) is a set of linear equations.

Before closing this subsection, the present geometric formulation is compared with the one proposed by Teruel [16].

Define the function \( J \) that was originally introduced in [16] as the total Legendre transform of \( H \)

\[
J(\hat{q}, \hat{p}) := \inf_{q,p} \{ \hat{p}_a q^a - \hat{q}_a p_a + H(q, p) \} ,
\]

from which

\[
J(\hat{q}, \hat{p}) = - \sup_{p,q} \{ \hat{q}_a p_a - \hat{p}_a q^a - U(q) - K(p) \} .
\]

\[
J(\hat{q}, \hat{p}) = - \sup_{p,q} \{ \hat{q}_a p_a - \hat{p}_a q^a - U(q) - K(p) \} .
\]
In the dual coordinates introduced in definition 2.2

\[ p_*^a = \frac{\partial K}{\partial p_a} = \dot{q}^a \quad \text{and} \quad q_*^a = \frac{\partial U}{\partial q^a} = -\dot{p}_a, \]

where (2) has been used, one can express \( J \) as

\[ J(p_*, -q^*) = -\sup_{q^*} \left[ p_*^a p_a + q_*^a q^a - U(q) - K(p) \right]. \]

With this and (5), one has

\[ J(p_*, -q^*) = -\left[ K^*(p_*) + U^*(q^*) \right]. \tag{20} \]

This states how \( J \) in [16] is related to the present formulation. Hamilton’s equations can also be written with \( J \) as follows. Applying (6) to differentiation of (20), one has

\[ \frac{\partial J}{\partial p_a} = -\frac{\partial K^*}{\partial p_a} = -p_a, \quad \text{and} \quad \frac{\partial J}{\partial q^a} = -\frac{\partial U^*}{\partial q^a} = -q^a, \]

from which

\[ \frac{d}{dt} \left( \frac{\partial J}{\partial p_a} \right) = -\dot{p}_a = q_*^a, \quad \text{and} \quad \frac{d}{dt} \left( \frac{\partial J}{\partial q^a} \right) = -\dot{q}^a = -p_*^a, \quad a \in \{1, \ldots, N\}. \]

These derived equations correspond to (9).

2.2. Relation to Toda’s original dual transform

In this subsection it is shown how Toda’s original dual transform is related to the geometric formulation introduced in section 2.1. To this end, Toda’s original idea of the dual transform and the dual lattice system are briefly reviewed below first.

Toda considered in [18] Hamilton’s equations of motion for the \( a \)th particle (\( a \in \{0, \ldots, N\} \)) in a chain

\[ m \frac{d^2 u^a}{dt^2} = -\phi'(u^a - u^{a-1}) + \phi'(u^{a+1} - u^a), \tag{21} \]

where \( m > 0 \) stands for the mass of the particles, \( u^a \) the position of the \( a \)th particle, \( \phi \) the interaction potential energy function depending on the distance between adjacent particles, and \( \phi' \) its derivative with respect to the argument. In this paper the system (21) is referred to as the original lattice system. Then the potential energy function \( U \) is the sum of \( \phi \). Introducing the variables

\[ q_a := u^{a+1} - u^a, \quad \text{and} \quad f_a := -\phi'(q_a), \quad a \in \{1, \ldots, N\} \]

one can derive

\[ \ddot{q}^a = -\frac{1}{m} \left( f_{a+1} + f_{a-1} - 2f_a \right), \]

or equivalently with \( p_a \) satisfying \( \dot{p}_a = f_a, \ (a \in \{1, \ldots, N\}) \),

\[ \dot{q}^a = -\frac{1}{m} \left( p_{a+1} + p_{a-1} - 2p_a \right), \quad \text{and} \quad \dot{p}_a = -\frac{d \phi(q^a)}{dq^a}. \tag{22} \]

If one finds a function \( \chi \) by solving the second equation of (22) for \( q^a \) such that
for all $a \in \{0, \ldots, N\}$, then one has
\[
\frac{d}{dt} \chi(p_a) = p_{a+1} + p_{a-1} - 2p_a.
\]

Toda proposed the nonlinear form of $\phi$,
\[
\phi(q) = \frac{A}{B} e^{-Bq} + Aq,
\] with $A$ and $B$ being positive constant. This interaction potential energy function $\phi$ is referred to as the Toda potential function. In this case the function $\chi$ is obtained as
\[
\chi(p_a) = m \frac{B}{A} \ln \left( 1 + \frac{p_a}{A} \right),
\] and one has the Toda lattice system:
\[
m \frac{d}{dt} \left[ \frac{1}{B} \ln \left( 1 + \frac{p_a}{A} \right) \right] = p_{a+1} + p_{a-1} - 2p_a.
\] The lattice system (24) is referred to as the dual lattice system with respect to the original lattice system (21). Note that the meaning of dual in Toda’s original theory was not directly related to that in Hessian–information geometry.

In his derivation of the Toda lattice system reviewed above, one observes the following.

• The interaction potential energy function $\phi$ is strictly convex, since
\[
\frac{d^2 \phi}{dq^2} = AB e^{-Bq} > 0.
\]
• The existence of the function $\chi$ in (23) is a key to find the second order equations of the dual lattice system.

These observations lead to the following

• The potential energy function is strictly convex, since the sum of strictly convex functions is also a strictly convex function.
• If an explicit form of the Legendre transform of the potential energy function is found, then this is a key to find the dual lattice system.

The following is the main theorem in this subsection and it states how Toda’s dual lattice system can be written in terms of the Legendre transform of the interaction potential energy function.

**Theorem 2.3 (The Legendre transform of interaction potential for dual lattice).** Consider (22), where $\phi$ is strictly convex. Assume that the Legendre transform $\phi^*$ of $\phi$, is explicitly written. Then the function $\chi$ in (24) is written in terms of $p_a$ and $\phi^*$ as
\[
\chi(p_a) = -m \left. \frac{d\phi^*}{dq^*_a} \right|_{q^*_a = -p_a}, \quad a \in \{1, \ldots, N\}.
\] **Proof.** Since $\phi$ is strictly convex, one can define the dual coordinates $\{q^*_a\}$, and they are related to $\{\dot{p}_a\}$ such that
\begin{equation}
q_a^* = \frac{d\phi(q^*)}{dq^a} = -\dot{p}_a.
\end{equation}

Due to the assumption that the Legendre transform \(\phi^*\) of \(\phi\) can explicitly be written, one can write
\begin{equation}
q^a = \frac{d\phi^*}{dq^a}.
\end{equation}

The relation between \(q^a\) and \(\dot{p}_a\) is obtained by combining (28) and (29) as
\begin{equation}
q^a = \left. \frac{d\phi^*}{dq^a} \right|_{q^a = -\dot{p}_a}.
\end{equation}

Substituting this into \(\chi(\dot{p}_a) = -mq^a(\dot{p}_a)\) coming from (23), one obtains (27).

\begin{remark}
With this theorem, the system (24) is written as
\begin{equation}
-m \frac{d}{dt} \left( \left. \frac{d\phi^*}{dq^a} \right|_{q^a = -\dot{p}_a} \right) = p_{a+1} + p_{a-1} - 2p_a,
\end{equation}
which is a generalization of (26). Also, the equation (30) are obtained from theorem 2.1 with
\begin{align*}
U &= \sum_i \phi(q^i), \\
K &= \sum_i (p_{i+1} - p_i)^2/(2m), \text{ and } \{ h^n_i \} = 0.
\end{align*}

From this theorem it turns out that one significance of the proposed Hessian–information geometric formulation of Hamiltonian systems is to give how to systematically obtain an explicit form of Toda’s dual transformed equations from a given lattice system. Thus, roughly speaking, dual in the sense of Toda is equivalent to that in the sense of Legendre.

In analyzing an integrable system, a set of \(\tau\)-functions may be focused, and these functions are used for constructing soliton solutions [20, 21]. For the Toda lattice system (26) with \(A = B = m = 1\), a set of \(\tau\)-functions \(\{\tau_a\}\) is such that
\begin{equation}
p_a = \frac{d}{dt} \ln \tau_a, \quad a \in \{1, \ldots, N\}.
\end{equation}

From (31) and (28), one has the relation between \(q_a^*\) and \(\tau_a\):
\begin{equation}
q_a^* = \frac{-d^2}{dt^2} \ln \tau_a.
\end{equation}

Then, the equations written in terms of a set of \(\tau\)-functions \(\{\tau_a\}\) are obtained as follows. One can integrate (24) with respect to \(t\) by introducing \(\{\tau_a\}\) defined in (31) as
\begin{equation}
-m \chi(\dot{p}_a)_{\dot{p}_a = \dot{p}_a(\tau_a)} = \ln \left( \frac{\tau_{a+1}\tau_{a-1}}{\tau_a^2} \right) + \text{Const.},
\end{equation}
where
\begin{equation}
\dot{p}_a(\tau_a) = \frac{\tau_a\dot{\tau}_a - (\tau_a)^2}{\tau_a^2}.
\end{equation}

The left hand sides of the equations above are written as
\begin{equation}
-m \chi(\dot{p}_a)_{\dot{p}_a = \dot{p}_a(\tau_a)} = -m \left. \frac{d\phi^*}{dq^a} \right|_{q^a = -\dot{p}_a(\tau_a)} = \frac{m}{B} \ln \left( 1 + \frac{\dot{p}_a(\tau_a)}{A} \right).
\end{equation}

For the case where \(m = A = B = 1\) and \(\text{Const.} = 0\), one derives the set of equations for \(\{\tau_a\}\):
\[ \tau_a \tau_a = (\tau_a)^2 = \tau_{a+1} \tau_{a-1} - (\tau_a)^2. \]

To state the applicability of Hessian–information geometry for (22), one considers a system (22) with φ being strictly convex under some wider boundary conditions. The system is rewritten as the Hamiltonian system

\[ \dot{q}^a = \frac{\partial K}{\partial p_a}, \quad \text{and} \quad \dot{p}_a = -\frac{\partial U}{\partial q^a}, \]

where

\[ K(p) = \frac{1}{2m} \sum_i (p_{i+1} - p_i)^2, \quad \text{and} \quad U(q) = \sum_i \phi(q^i), \quad (32) \]

under some boundary conditions. Note that U is strictly convex. Then it follows that

\[ h_{ab} = \frac{\partial^2 K}{\partial p_a \partial p_b} = -\frac{1}{m} \left( \delta^{b,a+1} + \delta^{b,a-1} - 2 \delta^{b,a} \right). \quad (33) \]

Thus, to apply the general theory developed in section 2.1 to this system, one needs to verify that the condition \((\partial^2 K / \partial p_a \partial p_b) > 0\) holds under boundary conditions. For example if \(p_{i+1} = p_i\) for any \(i\), then

\[ h_{ab} = \frac{\partial^2 K}{\partial p_a \partial p_b} = \frac{1}{m} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \]

The eigenvalues of the matrix \((h_{ab})\) are obtained as \((2 \pm \sqrt{2})/m\) and \(2/m\), and they are positive due to \(m > 0\). It is known that if all the eigenvalues of a matrix \(M\) are positive, then \(M\) is positive definite, \(M > 0\). Applying this, one has that \((h_{ab}) > 0\). From this, \(K\) is strictly convex. Thus, Hessian–information geometric formulation can be applied to this periodic system.

Consider the case where the Hessian–information geometric formulation can be applied to a system whose Hamiltonian is \(H = K + U\) with \(K\) and \(U\) given by (32). Also choose \(\phi\) to be the Toda potential, (32). In this case the set of equations of motion (11) becomes (26). Then the quantities used in (12) are shown below. The coordinates being dual to \(\{q_a\}\) and the components \(h_{ab}^U\) are

\[ q^*_a = \frac{d}{dq^a} = -Ae^{-Bq^*} + A, \quad \text{and} \quad h_{ab}^U = AB e^{-Bq^*} \delta_{ab} = B(A - q^*_a) \delta_{ab}, \quad (\text{no sum}). \]

The Legendre transform of \(\phi\) is

\[ \phi^*(q^*_a) = \sup_q \left[ q_a q^*_a - \phi(q) \right] = \frac{A - q^*_a}{B} \left[ \ln \left( 1 - \frac{q^*_a}{A} \right) - 1 \right], \quad (\text{no sum}) \]

from which

\[ q^a = \frac{d}{dq^*_a} q^*_a = -\frac{1}{B} \ln \left( 1 - \frac{q^*_a}{A} \right), \quad h_{ab} = \frac{1}{B} \delta_{ab}, \quad (\text{no sum}), \]

and

\[ C_{abc}^{U*} = \begin{cases} \frac{1}{B(A - q^*_a)} & \text{for } a = b = c \\ 0 & \text{otherwise} \end{cases}. \]
The components of the Riemannian metric tensor field $h_k$ have been calculated as (33).

It has not been known how to systematically obtain the dual lattice systems from given lattice systems, and only a few examples have been known, where explicit expressions of dual lattice systems are obtained. They include linear and the Toda lattice systems. In what follows, some other examples of nonlinear lattice systems are shown, where dual lattice systems are explicitly expressed.

**Example 2.1.** Consider a class of lattice systems of the form (21) with the interaction potential energy function $\phi_\beta$ given by

$$
\phi_\beta(q) = \frac{(q^2)^\beta}{2\beta},
$$

where $\beta$ is fixed. Observe that $\phi_\beta(q) \in \mathbb{R}$ for any $q \in \mathbb{R}$.

The second order equations of the dual lattice system can be obtained by applying theorem 2.3 to this system. To this end, one calculates the Legendre transform

$$
\phi_\beta^*(q^*) = \sup_q [qq^* - \phi_\beta(q)] = \frac{(q^*)^2}{2\beta^*},
$$

where $\beta^*$ satisfies

$$
\frac{1}{2\beta} + \frac{1}{2\beta^*} = 1.
$$

With (34) and (30), the explicit form of the dual lattice system is obtained as

$$
-m \frac{d}{dt} \left( (-\dot{p}_a)^2 \right)^{\beta^*-1/2} = p_{a+1} + p_{a-1} - 2p_a.
$$

Note that the dual lattice system obtained above is linear ($2\beta^* = 2$) if the original lattice system is linear, $2\beta = 2$.

The Hamiltonian written in terms of $\{q^a\}$ and $\{p_a\}$ is

$$
H(p,q) = K(p) + U(q), \quad \text{where} \quad K(p) = \frac{1}{2m} \sum_i (p_{i+1} - p_i)^2, \quad \text{and} \quad U(q) = \sum_i \phi_\beta(q^i), \quad 2\beta > 1.
$$

Consider the case where the Hessian–information geometric formulation can be applied to this system. Then the quantities used in (12) are shown below. The coordinates being dual to $\{q^a\}$ and the components $h_{ab}^U$ are

$$
q^a = \frac{d\phi_\beta}{dq^a} = \left( (q^a)^2 \right)^{\beta^*-1/2}, \quad \text{and} \quad h_{ab}^U = (2\beta^* - 1) \left( (q^a)^2 \right)^{\beta^*-1} \delta_{ab}, \quad (\text{no sum}).
$$

The Legendre transform of $\phi$ has been obtained as (34), from which

$$
q^a = \frac{d\phi_\beta^*}{dq^a} = \left( (q^a)^2 \right)^{\beta^*-1/2}, \quad \text{and} \quad h_{ab}^U = (2\beta^* - 1) \left( (q^a)^2 \right)^{\beta^*-1} \delta_{ab}, \quad (\text{no sum}),
$$

and

$$
C_{\hat{U}^a} = \begin{cases} 
(2\beta^* - 1)(2\beta^* - 2) \left( (q^a)^2 \right)^{\beta^*-3/2} & \text{for } a = b = c \\
0 & \text{otherwise}
\end{cases}.
$$
The components of the Riemannian metric tensor field $h_{K}$ have been calculated as (33).

**Example 2.2.** In the example below, general interaction potential function is focused, where such potential functions have been discussed in the development of information geometry.

To this end, one introduces a positive monotonically increasing function and considers a generalized logarithmic and exponential functions [22, 23]. Let $\varphi$ be a function of $\zeta \in \mathbb{R}$ such that

$$\varphi(\zeta) > 0, \quad \text{and} \quad \frac{d \varphi}{d \zeta} > 0, \quad \text{for} \quad \zeta > 0.$$  

Then define the generalized logarithmic function associated with $\varphi$

$$\ln_\varphi(\zeta) = \int_{1}^{\zeta} \frac{1}{\varphi(\zeta')} d\zeta',$$

and the generalized exponential function as the inverse function of $\ln_\varphi(\zeta)$, so that

$$\exp_\varphi(\ln_\varphi(\zeta)) = \zeta, \quad \text{and} \quad \ln_\varphi(\exp_\varphi(\zeta)) = \zeta.$$

It follows from

$$\frac{d}{d \zeta} \exp_\varphi(\zeta) = \varphi(\exp_\varphi(\zeta)) > 0, \quad \text{and} \quad \frac{d^2}{d \zeta^2} \exp_\varphi(\zeta) = \varphi(\exp_\varphi(\zeta)) \frac{d \varphi}{d \zeta} \bigg|_{\zeta' = \exp_\varphi \zeta} > 0$$

that the generalized exponential function is strictly convex.

Choose $\varphi$ and the interaction potential energy function

$$\phi_{\varphi}(q) = \int_{0}^{q} \exp_\varphi(q') dq'.$$

The Legendre transform of this is obtained as

$$\phi_{\varphi}^*(q^*) = \int_{1}^{q^*} \ln_\varphi(q^*) dq^*,$$

from which

$$\frac{d}{dq^*} \phi_{\varphi}^*(q^*) = \ln_\varphi(q^*), \quad \text{and} \quad \frac{d^2}{dq^*^2} \phi_{\varphi}^*(q^*) = \frac{1}{\varphi(q^*)} > 0.$$  

(36)

With (36) and (30), the explicit form of the dual lattice system is obtained as

$$-m \frac{d}{dt} \ln_\varphi(-\dot{p}_a) = p_{a+1} + p_{a-1} - 2p_a,$$

or equivalently,

$$\frac{m}{\varphi(-\dot{p}_a)} \frac{d^2 p_a}{dt^2} = p_{a+1} + p_{a-1} - 2p_a, \quad \text{(no sum)}.$$
Consider the case where the Hessian–information geometric formulation can be applied to this system. Then the quantities used in (12) are shown below. The coordinates being dual to \( \{ q^a \} \) and the components \( h^U_{ab} \) are

\[
q^*_a = \frac{d\phi_\varphi}{dq^a} = \exp \varphi(q^a), \quad \text{and} \quad h^U_{ab} = \varphi(\exp \varphi q^a) \delta_{ab}, \quad (\text{no sum}).
\]

The Legendre transform of \( \phi_\varphi \) has been obtained as (35), from which

\[
q^a = \frac{d\phi^*}{dq^*_a} = \ln \varphi(q^*_a), \quad h^U_{ab} = \frac{1}{\varphi(q^*_a)} \delta_{ab}, \quad (\text{no sum}),
\]

and

\[
C^{abc}_U = \begin{cases} 
- [\varphi(q^*_a)]^{-2} \left( \frac{d\varphi}{dq^*_a} \right) & \text{for } a = b = c \\
0 & \text{otherwise}
\end{cases}.
\]

The components of the Riemannian metric tensor field \( h_K \) have been calculated as (33).

### 2.3. Vanishing potential systems

In this subsection it is assumed that

- the dimension of the manifold \( \mathcal{M} \) is \( n \), and its local coordinates are \( p = \{ p_a \} \),
- a system is a natural Hamiltonian system whose Hamiltonian is a strictly convex kinetic energy function, \( H = K \) with \( K \) being a function of \( p \).

From these assumptions, the condition \( (\partial^2 K/\partial p_a \partial p_b) > 0 \) is satisfied.

The canonical equations of motion (2) are

\[
\frac{d}{dt} q^a = \frac{\partial K}{\partial p_a}, \quad \text{and} \quad \frac{d}{dt} p_a = 0,
\]

from which \( \ddot{q}^a = 0 \). In these coordinates, one has

\[
q^a(t) = p^a_+ t + q^a(0),
\]

where \( \{ p^+_a \} \) has been defined as the set of the dual coordinates (see definition 2.2),

\[
p^+_a = \frac{\partial K}{\partial p_a}.
\]

One can write the canonical equations as

\[
\frac{d}{dt} p^+_a = 0, \quad \text{from which} \quad p^+_a(t) = p^+_a(0).
\]

Similar to discussions in section 2.1, one has that the triplet \( (\mathcal{M}, \nabla^K, h^K) \) is a Hessian manifold, where \( \nabla^K \)-affine coordinates are \( \{ p_a \} \), \( h^K \) and \( \nabla^K \) have been defined in (3) and (7), respectively. Also, the triplet \( (\mathcal{M}, \nabla^K^*, h^K) \) is a Hessian manifold, where \( \nabla^K^* \)-affine coordinates are \( \{ p^*_a \} \), and \( \nabla^K^* \) has been defined in (8).

One immediately has the following.

**Proposition 2.5.** In addition to that the set of coordinates \( \{ p_a \} \) is constant, the set of dual coordinates \( \{ p^*_a \} \) is also constant.
3. Extension to LC circuit models

Since there are some similarities between non-dissipative electric circuit models and classical Hamiltonian systems, one is interested in how to extend the above geometric formulation of Hamiltonian systems to circuit theory. In this section it is shown how this is given.

Consider the following series LC circuit model

$$\frac{d}{dt} Q = I, \quad \frac{d}{dt} \Phi = -V,$$

where $t \in \mathbb{R}$ is time, $Q \in \mathbb{R}$ the electric charge stored in a capacitor with electric capacitance $C$, $I \in \mathbb{R}$ the current, $\Phi \in \mathbb{R}$ the magnetic flux due to $I$ whose flux is stored in an inductor with electric inductance $L$, $V \in \mathbb{R}$ the capacitor voltage. The electromagnetic energy for this circuit model is expressed as

$$E^*(V,I) = E^*_C(V) + E^*_L(I), \quad E^*_C(V) = \int Q(V) dV, \quad E^*_L(I) = \int \Phi(I) dI,$$

where $E^* : \mathbb{R}^2 \to \mathbb{R}$ is referred to as a co-energy function, and the total Legendre transform of $E^*$ is the energy function denoted by $\mathcal{E}$. The value $\mathcal{E}_C^*(V)$ is interpreted as the energy due to the capacitor, and $\mathcal{E}_L^*(I)$ the energy due to the inductor. The constitutive relations are

$$Q(V) = \frac{\partial E^*}{\partial V} = \frac{dE^*_C}{dV}, \quad \Phi(I) = \frac{\partial E^*}{\partial I} = \frac{dE^*_L}{dI}.$$  

In what follows it is assumed that

- $\mathcal{E}_C^*$ and $\mathcal{E}_L^*$ are strictly convex:
  $$\frac{d^2 \mathcal{E}_C^*}{dV^2} > 0, \quad \text{and} \quad \frac{d^2 \mathcal{E}_L^*}{dI^2} > 0,$$

  in some domains.

Similar to definition 2.2, one defines the following.

**Definition 3.1 (Dual coordinates).** The coordinates $Q$ and $\Phi$ defined by (38) are referred to as dual coordinates. In particular, $Q$ is referred to as being dual to $V$, and $\Phi$ is referred to as being dual to $I$.

From the assumption it can be shown that

$$V = \frac{\partial \mathcal{E}}{\partial Q} = \frac{d\mathcal{E}_C}{dQ}, \quad \text{and} \quad I = \frac{\partial \mathcal{E}}{\partial \Phi} = \frac{d\mathcal{E}_L}{d\Phi},$$

where $\mathcal{E}_C$ is the Legendre transform of $\mathcal{E}_C^*$, and $\mathcal{E}_L$ the Legendre transform of $\mathcal{E}_L^*$.

The series LC circuit model (37) can then be written as

$$\frac{d^2 Q}{dt^2} = -\frac{d^2 \mathcal{E}_L}{d\Phi^2} V,$$

from which

$$\frac{d^2}{dt^2} \left( \frac{d\mathcal{E}_C^*}{dV} \right) = -\frac{d^2 \mathcal{E}_L}{d\Phi^2} V.$$  

(39)
This equation is an analogue of (11).

The following is analogous to theorem 2.1.

**Theorem 3.1 (Generalized dual transformed equations for LC circuits).** Assume that \( d^2 \mathcal{E}_L / d\Phi^2 \) is constant. Then the series LC circuit model is written of the form

\[
\frac{d}{dt} \left( \frac{d\mathcal{E}_C^*}{dV} \bigg|_{V=-\Phi} \right) = \frac{d^2 \mathcal{E}_L}{d\Phi^2} \Phi + \text{Const.}.
\]

**Proof.** A way to prove this is analogous to that of theorem 2.1. \( \square \)

One has the following theorem, and this is analogous to theorem 2.2.

**Theorem 3.2 (LC circuits written in terms of Hessian geometry).** The series LC circuit model in terms of the dual coordinates can be written in the form

\[
\mathcal{C}^* (\dot{V})^2 + h_c \mathcal{V} = -h_L V, \quad \mathcal{C}^* := \frac{d^3 \mathcal{E}_C}{dV^3}, \quad h_c := \frac{d^2 \mathcal{E}_C}{dV^2}, \quad h_L := \frac{d^2 \mathcal{E}_L}{d\Phi^2}.
\]

**Proof.** A way to prove this is analogous to that of theorem 2.2. \( \square \)

**Remark 3.1.** It is straightforward to see that \( \mathcal{C}^* \) is the component of a cubic form.

Note that constitutive relations are obtained by differentiating energy or co-energy functions as in (38). By contrast, if constitutive relations are expressed as monotonically increasing functions, then integration of such constitutive relations gives strictly convex energy or co-energy functions.

The following is an example. Unlike the general discussion above, constitutive relations are given first. Second, co-energy and energy functions are calculated. Then the second order equation written in terms of introduced geometric objects is shown.

**Example 3.1.** Choose the following constitutive relations

\[
\Phi(I) = LI, \quad \text{and} \quad Q(V) = Q_0 \ln \left( 1 + \frac{V}{V_0} \right),
\]

where \( L > 0 \), and \( Q_0, V_0 > 0 \) are constants. Note that

\[
Q_0 \neq \lim_{V \to 0} Q(V), \quad \text{and} \quad \lim_{V \to 0} \frac{dQ}{dV} = \frac{Q_0}{V_0}.
\]

Then the co-energy function is

\[
\mathcal{E}^* (V, I) = \mathcal{E}^*_L (I) + \mathcal{E}^*_C (V),
\]

where

\[
\mathcal{E}^*_L (I) = L \frac{I^2}{2}, \quad \text{and} \quad \mathcal{E}^*_C (V) = Q_0 V_0 \left[ \left( 1 + \frac{V}{V_0} \right) \ln \left( 1 + \frac{V}{V_0} \right) - \frac{V}{V_0} \right].
\]

They are strictly convex for \( \{(V, I) \mid V + V_0 > 0\} \), since

\[
\frac{d^2 \mathcal{E}^*_L (I)}{dI^2} = L > 0, \quad \text{and} \quad \frac{d^2 \mathcal{E}^*_C (V)}{dV^2} = \frac{Q_0}{V + V_0} > 0.
\]

The total Legendre transforms of \( \mathcal{E}^*_L \) and \( \mathcal{E}^*_C \),
\[ E_L(\Phi) = \sup_I \left[ \Phi I - E^*_L(I) \right], \quad \text{and} \quad E_C(Q) = \sup_V \left[ QV - E^*_C(V) \right], \]

are obtained from the relation \(1 + \frac{V}{V_0} = \exp(Q/Q_0)\) as

\[ E_L(\Phi) = \frac{\Phi^2}{2L}, \quad \text{and} \quad E_C(Q) = V_0 \left( Q_0 e^{Q/Q_0} - Q_0 \right). \]

From \(V(Q) = V_0(\exp(Q/Q_0) - 1)\), one verifies that

\[ V_0 \neq \lim_{Q \to 0} V(Q), \quad \text{and} \quad \lim_{Q \to 0} \frac{dV}{dQ} = \frac{V_0}{Q_0}. \]

The well-known quadratic co-energy and energy functions are obtained with the Taylor expansion of \(E^*_C(V)\) and that of \(E_C(Q)\) as

\[ E^*_C(V) = \frac{C_0 V^2}{2} + (\text{higher order terms}), \quad \text{and} \quad E_C(Q) = \frac{Q^2}{2C_0} + (\text{higher order terms}), \]

where \(C_0 := Q_0/V_0\).

The second order circuit equation in terms of the dual coordinate \(V\) is obtained from (39) as

\[ \frac{d^2}{dt^2} \left[ Q_0 \ln \left( 1 + \frac{V}{V_0} \right) \right] = -\frac{1}{L} V. \]

The generalized dual transformed equation is obtained from theorem 3.1 as

\[ \frac{d}{dt} \left[ Q_0 \ln \left( 1 - \frac{1}{V_0} \frac{d\Phi}{dt} \right) \right] = \frac{1}{L} \Phi + \text{Const..} \]

The components of geometric objects in theorem 3.2 are calculated as

\[ h^*_C = \frac{Q_0}{V + V_0}, \quad \dot{c}^* = \frac{dh^*_C}{dV} = -\frac{Q_0}{(V + V_0)^2}, \quad \text{and} \quad h_L = \frac{1}{L}. \]

4. Conclusions

This paper has offered a formulation of a class of classical Hamiltonian systems in terms of Hessian–information geometry. It has been shown that the \(\alpha\)-connection, cubic forms and so on, invented in information geometry, appear in canonical equations of motion. From the theorems in this paper, one significance of the proposed geometric formulation of Hamiltonian systems is to give an explicit form of Toda’s dual transformed equations from a given lattice system. In this sense, dual in the sense of Toda is equivalent to that in the sense of Legendre. With this formulation, some explicit forms of dual lattice systems have been obtained. Also, this paper has offered a similar formulation of non-dissipative electric circuit models.

There are some potential future works that follow from this paper. One is to apply the present approach to dissipative systems written in terms of contact geometry [24, 25]. Since the present study has been restricted to conservative Hamiltonian systems, it is interesting to see if this approach can be extended for dissipative systems. With these future works based on this
study, it is expected that the proposed geometric formulation yields a step to importing various theorems in Hessian–information geometry to Hamiltonian mechanics and electric circuit theory, and vice versa.

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Appendix. A brief explanation of information and Hessian geometries

This appendix provides a brief explanation of information geometry and Hessian geometry. This is intended to give a rough sketch of these geometries for the readers who are not familiar with them but familiar with Riemannian geometry. For more comprehensive discussions for these geometries see [2] and [3] from a mathematical viewpoint, and [26] from a physical viewpoint.

Information geometry is a geometrization of mathematical statistics. In particular, most of cases parametric distribution functions are focused. In such a case, a finite set of parameters associated with a given probability distribution function is identified with a set of coordinates of a manifold. Although a metric tensor field called the Fisher metric tensor field is often introduced, information geometry is not exactly classified as a Riemannian geometry. Instead, a manifold, a metric tensor field, and some two connections play roles in information geometry. The following is a list of geometrical objects used in information geometry.

- Coordinates: parameters for expressing a distribution function
- Components of a metric tensor field: the Fisher information matrix
- Two connections: a connection $\nabla$ and its dual connection $\nabla^*$, which are torsion-free and need not to be the Levi-Civita connection. Here dual connection is defined through the Fisher metric tensor field $g$ so that $X[g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$ is satisfied for any vector fields $X$, $Y$ and $Z$.

The importance of these objects, or a structure, is summarized as the Chentsov theorem. A manifold equipped with this structure is referred to as a statistical manifold.

Dually flat spaces are particularly important in information geometry. If a connection $\nabla$ and dual connection $\nabla^*$ are flat, then such a statistical manifold is referred to as a dually flat space, and is denoted by $(\mathcal{M}, g, \nabla, \nabla^*)$, where $\mathcal{M}$ is a manifold expressing probability transform and dual coordinates. In a dually flat space, by definition, $\nabla$-affine coordinates and $\nabla^*$-affine coordinates exist. Let $\theta = \{\theta^a\}$ and $\eta = \{\eta_a\}$ be such coordinates. It turns out that strictly convex functions are guaranteed to exist. With these functions, denoted by $\psi$ and $\phi$, the coordinate transforms between $\theta$ and $\eta$ can be written as $\eta_a = \partial \psi / \partial \theta^a$ and $\theta^a = \partial \phi / \partial \eta_a$.

In addition, the Legendre transform of $\psi$ is $\phi$, and that of $\phi$ is $\psi$. These relations above are employed in the main text. In addition, since $\nabla$ and $\nabla^*$ are flat, geodesic curves are immediately obtained as follows. Geodesic curves with respect to $\nabla$ reduce to $\theta^a(t) = \dot{\theta}^a(0)t + \theta^a(0)$ with $t \in \mathbb{R}$ parameterizing a set of points on a geodesic curve. Similarly, geodesic curves with respect to $\nabla^*$ reduce to $\eta_a(t) = \dot{\eta}_a(0)t + \eta_a(0)$.
Hessian geometry is linked to information geometry as mentioned in Introduction of this paper. In what follows this is explained. As discussed in the main text, Hessian geometry is a geometry where a Hessian structure $(\tilde{\nabla}, \tilde{g})$ is provided, and strictly convex functions play fundamental roles. Here $\tilde{\nabla}$ is a connection, and $\tilde{g}$ a (pseudo) Riemannian metric tensor field. It can be shown that a strictly convex function induces a Hessian manifold. To show a link between information geometry and Hessian geometry, consider the cumulant generating function of a distribution function belonging to the exponential family. Here, the cumulant generating function of the exponential family is convex, where the exponential family is a class of parametric probability distribution functions. It can be shown that a strictly convex function induces a dually flat space. Thus, by restricting a domain for the cumulant generating function so that it is strictly convex, one has a Hessian manifold and a dually flat space. This shows how information geometry is related to Hessian geometry.

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