BEILINSON-TATE CYCLES ON SEMIABELIAN VARIETIES

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In [AK], it was shown that the Beilinson-Hodge conjecture holds for a product of smooth curves and semiabelian varieties. In this companion paper, we establish the “Tate” version that given such a variety over (say) a number field, the Galois invariant cycles of highest weight on étale cohomology come from motivic cohomology. We also give criteria for similar statements to hold for lower weight cycles in both the Hodge and Tate cases. More precisely, given a smooth not necessarily proper variety $U$ over a field $k$, we have the so called regulator or cycle maps

- $CH^i(U, j) \otimes \mathbb{Q} \to \text{Hom}_{\mathbb{MHS}}(\mathbb{Q}(0), H^{2i-j}(U, \mathbb{Q}(i)))$ ($k = \mathbb{C}$, Hodge-version)
- $CH^i(U, j) \otimes \mathbb{Q}_l \to H^{2i-j}_{\text{et}}(U \times_k \bar{k}, \mathbb{Q}_l(i))^G_k$ ($k$ finitely generated over $\mathbb{Q}$, Tate-version)

from Bloch’s higher Chow groups [B2]. Here $G_k$ is the absolute Galois group $\text{Gal} (\bar{k}/k)$. Following Asakura and Saito [AS], we refer to the surjectivity of the first map when $i = j$ as the Beilinson-Hodge conjecture, and surjectivity of the second when $i = j$ as Beilinson-Tate. One may consider, as Beilinson originally had, the surjectivity question in either case for all pairs $(i, j)$, but then things becomes a bit more subtle. Surjectivity is known to fail for some pairs $i \neq j$ when the field of definition is transcendental, but surjectivity is expected over number fields, c.f. [J]. We will refer to this form as the “strong” Beilinson’s Hodge and Tate conjecture.

Our goal here is to extend the results of [AK] to include strong version of Beilinson’s Hodge conjecture. In proposition 3.5 and 3.7 it is shown that the strong version of Beilinson Hodge conjecture holds for product of curves and semiabelian varieties if the classical Hodge conjecture for their smooth compactification is true. The conjecture is shown to hold even for varieties dominated by product of curves (corollary 4.2).

We also show the Beilinson-Tate conjecture for products of curves, semiabelian varieties and more generally for varieties dominated by products of curves. We recall that a semiabelian variety over a field is an extension of abelian variety by a (possibly nonsplit) torus. Also the strong version of Beilinson-Tate conjecture holds in these cases if the Tate conjecture holds for their smooth compactification. The proof of the main result in [AK] was based on analysing invariants under the Mumford-Tate group. A similar method is used for Beilinson-Tate conjectures, but the Galois group plays the role of the Mumford-Tate group in this case.

1. Cycle map on higher Chow groups

The regulator map was originally described using Chern classes in higher $K$-theory [Be1, Be2, G, So]. Bloch [B1, B2] later recast this as a more explicit cycle map on his higher Chow group. This group is a fundamental object. It can

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be interpreted as motivic cohomology [L] [MVW], and rationally it coincides with the associated graded pieces of $K$-theory with respect to the $\gamma$-filtration. From the point of view of motives, the regulator can be understood as coming from an appropriate realization functor [Hu].

We felt it useful to describe Bloch’s construction in some detail, in order to make certain properties clear. Let $U$ be a smooth variety defined over a field $k$. In Bloch’s original definition, he viewed $\mathbb{A}^n_k$ as a simplex and defined $CH^q(U, n)$ as the homology of the complex $z^q(U, n) \subseteq Z^q(U \times \mathbb{A}^n)$ of codimension $q$ cycles meeting the simplicial faces properly. Here it is more convenient to use the cubical approach [To], where we view $\mathbb{A}^n_k$ as a cube with faces obtained by setting some of the coordinates to 0 or 1. The codimension one faces can be described as images of the face maps $\partial^i : \mathbb{A}^{n-1} \to \mathbb{A}^n$, $\epsilon \in \{0, 1\}$ given by

$$(t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_{i-1}, \epsilon, t_i, \ldots)$$

Let $C^q(U, n) \subset Z^q(U \times \mathbb{A}^n)$ denote the group generated by codimension $q$ cycles meeting faces properly. Then we have a complex

$$\ldots \to C^q(U, n) \to C^q(U, n-1) \to \ldots$$

with differential

$$\partial = \sum_i (-1)^i(\partial^0_i - \partial^1_i)$$

So as to use cohomology indexing, put $C^q(U, n)$ in degree $-n$. In order to get the right cohomology, we have to divide out by the subcomplex of degenerate cycles $D^q(U, n)$ spanned by pullbacks of cycles from $U \times \mathbb{A}^{n-1}$ along coordinate projections $p_i$. Note that the condition is vacuous for $n = 0$, so $D^q(U, 0) = 0$. Put $\tilde{C}^q(U, n) = C^q(U, n)/D^q(U, n)$. Then $CH^q(U, n) = H^{-n}(\tilde{C}^q(U, \bullet))$.

Suppose now that the ground field $k = \mathbb{C}$. Let $S^\bullet(U)$ be the singular cochain complex. Given Zariski closed $Z \subseteq U$, let $S^\bullet_Z(U) = \ker[S^\bullet(U) \to S^\bullet(U - Z)]$. This computes the relative cohomology $H^*_Z(U) := H^*(U, U - Z) = H^*(S^*_Z(U))$. For a cycle $Z$, we take $S^*_Z = S^{supp}_Z$. We define a double complex

$$S^{ab} = S^{ab}(U) = \lim_{Z \in CH(U, -a)} S^b_Z(U \times \mathbb{A}^{-a})$$

in the second quadrant with vertical differential given by simplicial coboundary and horizontal differential $\partial$ given by $[U]$. To avoid convergence issues, we should truncate this below a certain $a \ll 0$ as in [BlH]. However, to avoid excessive notation we leave this step implicit. We obtain a spectral sequence

$$E_1^{-a,b} = H^b(\lim_{Z} S^a_Z(U \times \mathbb{A}^{-a})) \cong \lim_{Z} H^b_Z(U \times \mathbb{A}^{-a}) \Rightarrow H^{b-a}(Tot(S^{\bullet\bullet}))$$

As above, we have a subcomplex of degenerate cohomology

$$D E_1^{-a,b} = \sum p^*_i H^b_Z(U \times \mathbb{A}^{-a-1})$$

and we set $E_1^{ab} = E_1^{ab}/D E_1^{ab}$ to the quotient spectral sequence. It’s not quite clear what $E_1^{ab}$ converges to, but we don’t really care. If we drop the support condition by setting

$$S^{ab} = S^b(U \times \mathbb{A}^{-a})$$
and define \( \mathcal{E}_{1}^{ab}, \mathcal{E}_{1}^{ab}, \mathcal{E}_{1}^{ab} \) exactly as before using \( S_{ab} \), then

\[
\mathcal{E}_{1}^{ab} = \begin{cases} 
H^{b}(U) & \text{if } a = 0 \\
0 & \text{otherwise}
\end{cases}
\]

Since we have a map of spectral sequences \( \mathcal{E}_{1}^{ab} \rightarrow \mathcal{E}_{1}^{ab} \), we conclude the abutment of the first spectral sequence \( E^{a} \) maps to the abutment of the second which is just \( H^{a}(U) \).

We have semipurity that says \( H^{2i}(U) = 0 \) for \( i < 2\text{codim}(Z) \). Therefore \( \mathcal{E}_{1}^{-a,b} \) vanishes below the line \( b = 2q \). Thus \( \mathcal{E}_{2}^{-a,2q} \) maps onto \( \mathcal{E}_{\infty}^{-a,2q} \subseteq \mathcal{E}_{\infty}^{2q-a} \) which in turn maps to \( H^{2q-a}(U) \). This map can be described more explicitly using the following diagram

\[
\begin{array}{ccc}
\lim_{q} H_{2q}(U \times \mathbb{A}^{a}, U \times \partial \mathbb{A}^{a}) & \longrightarrow & \ker[d_{1}: \mathcal{E}_{1}^{-a,2q} \rightarrow \mathcal{E}_{1}^{-a,1,2q}]
\\
& \downarrow & \\
H^{2q}(U \times \mathbb{A}^{a}, U \times \partial \mathbb{A}^{a}) & \sim & H^{2q-a}(U)
\end{array}
\]

where \( \partial \mathbb{A}^{a} \) is the union of codimension one faces. The top line comes from the long exact sequence for the pair \((U \times \mathbb{A}^{a}, U \times \partial \mathbb{A}^{a})\). The map labelled \( \sim \) is an isomorphism by the Künneth formula. This description shows that the map is compatible with mixed Hodge structures, provided we use \( H^{2q-a}(U)(q) \) at the last step.

We have a map of complexes

\[
c: C^{a}(U, \bullet) \rightarrow E_{1}^{+2q}
\]

given by sending a cycle to its fundamental class, which induces

\[
c: \mathcal{C}^{a}(U, \bullet) \rightarrow \mathcal{E}_{1}^{+2q}
\]

This induces a map

\[
\text{reg}: CH^{q}(U, n) \rightarrow \mathcal{E}_{2}^{-n,2q} \rightarrow H^{2q-n}(U)
\]

By the previous remark, the image lies in the group of cycles of type \((0,0)\) in \( H^{2q-n}(U, \mathbb{Z}(q)) \).

1.1. Properties. We now show that \( \text{reg} \) is compatible with pushforwards, pullbacks, and products. Also that

\[
\text{reg}: CH^{1}(U, 1) \cong \mathcal{O}(U)^{\ast} \rightarrow H^{1}(U, \mathbb{Z}(1))
\]

is the connecting map associated to the exponential sequence.

Given a proper map \( f: U \rightarrow V \), and cycle \( \alpha \in C^{q}(U \times \mathbb{A}^{n}) \) we can push it forward in the usual way [Fu] to get a cycle in \( C^{q+c}(U \times \mathbb{A}^{n}) \) where \( c = \text{dim } V - \text{dim } U \). This can be seen to define map of complexes \( f_{\ast}: \mathcal{C}^{q}(U, \bullet) \rightarrow \mathcal{C}^{q+c}(V, \bullet) \) [132]. Note we also have a pushforward on the spectral sequences

\[
f_{\ast}: \mathcal{E}_{1}^{ab}(U) \rightarrow \mathcal{E}_{1}^{a,b+2c}(V),
\]

which can be checked to be compatible with the cycle map \( c \) defined above (cf [Fu] §19.1). Thus \( \text{reg} \) is compatible with pushforward.

Bloch originally described the groups \( CH^{q}(U, n) \) as the cohomology of the complex \( z^{q}(U, \bullet) \subset Z^{q}(U \times \mathbb{A}^{\bullet}) \). In this setting it is no longer necessary to divide out the degenerate cycles. There is a spectral sequence \( E_{1}^{ab}(U) \) analogous to \( E_{1}^{ab}(U) \),
and $\text{reg}$ can be described in terms of a map $z^q(U, \bullet) \rightarrow \mathcal{E}_1^{1,2q}(U)$ as above [Bl1]. If $V$ is affine then Levine [L, part I chap II, §3.5] showed that $z^q(V, \bullet)$ is quasiisomorphic to a subcomplex $z^q(V, \bullet)_f$ consisting of cycles whose scheme theoretic pullback defines a cycle in $z^q(U, \bullet)$. One can see that one has a commutative diagram

$$
\begin{array}{c}
z^q(U, \bullet)_f \rightarrow \mathcal{E}_1^{1,2q}(U) \\
\downarrow \\
z^q(V, \bullet) \rightarrow \mathcal{E}_1^{1,2q}(V)
\end{array}
$$

This shows compatibility with $\text{reg}$ and pullbacks when $V$ is affine. The general case can be reduced to this using the Mayer-Vietoris for $H^*$ and $CH^*$ [L, part I, chap II] and the 5-lemma.

Returning to the cubic viewpoint, the external product

$$CH^q(U, n) \times CH^p(V, m) \rightarrow CH^{q+p}(U \times V, n + m)$$

is induced by the map of complexes

$$C^q(U, \bullet) \otimes C^p(V, \bullet) \rightarrow C^{q+p}(U \times V, \bullet)$$

given by $Z \otimes W \mapsto Z \times W$. It is fairly clear that $c(Z \otimes W) = c(Z) \cup c(W)$, and therefore $\text{reg}$ preserves external products. The cup product which is the external product followed by the pullback under the diagonal map $\Delta : U \rightarrow U \times U$ must therefore also be preserved.

By Bloch [Bl2], $CH^1(U, 1) = \mathcal{O}(U)^*$. 

**Proposition 1.1.** $\text{reg} : CH^1(U, 1) \rightarrow H^1(U, \mathbb{Z}(1))$ coincides with the connecting map associated with the exponential sequence, at least up to sign.

Here is a sketch. Set $\Delta = \mathbb{A}^1$. Let $\mathcal{I}$ denote the ideal sheaf of $U \times \partial \Delta = U \times \{0, 1\}$ in $U \times \Delta$. Let $j : \Delta - \{0, 1\} \rightarrow \Delta$ be the inclusion. Consider the diagram

$$
\begin{array}{c}
0 \rightarrow j^* \mathbb{Z}(1) \rightarrow \mathbb{Z}_{U \times \Delta}(1) \rightarrow \mathbb{Z}_{U \times \{0, 1\}}(1) \rightarrow 0 \\
\downarrow \\
0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{U \times \Delta} \rightarrow \mathcal{O}_{U \times \{0, 1\}} \rightarrow 0 \\
\exp \rightarrow \exp \rightarrow \exp \\
1 \rightarrow (1 + \mathcal{I})^* \rightarrow \mathcal{O}_{U \times \Delta}^* \rightarrow \mathcal{O}_{U \times \{0, 1\}}^* \rightarrow 1
\end{array}
$$
where \((1 + \mathcal{I})^*\) is defined as the kernel of \(r\). From this we obtain a diagram

\[
\begin{array}{c}
\mathcal{O}(U \times \{0, 1\})^* \xrightarrow{c_1} H^1((1 + \mathcal{I})^*) \\
H^1(U \times \Delta) \xrightarrow{} H^1(U \times \{0, 1\}, \mathbb{Z}) \xrightarrow{} H^2(U \times \Delta, U \times \{0, 1\}) \\
\end{array}
\]

The cohomology group \(H^1(U, (1 + \mathcal{I})^*)\) is a relative Picard group, which can be described in terms of certain divisor classes on \(U \times \Delta\) [Bl2, §6]. In particular, a divisor \(Z\) in \(\ker[C^1(U, 1) \to C^1(U, 0)]\), locally defined by \(f_i = 0\), gives an element \(\mathcal{O}(Z)_{rel} = \{f_i/f_j\}\) of \(H^1((1 + \mathcal{I})^*)\). By refining the usual arguments, one can see that the fundamental class of \(Z\) in \(H^2(U \times \Delta, U \times \{0, 1\})\) coincides with \(\pm c_1(\mathcal{O}(Z)_{rel})\).

1.2. Étale version. For a scheme \(U\) over a field \(k\) let \(\bar{U}\) denote \(U \times_k \bar{k}\). Choose a prime \(l \neq \text{char}(k)\). We can then define a double complex

\[
S^n(U) = \lim_{\rightarrow} \Gamma_Z(U \times \mathbb{A}^{-a}, I_n^{-a}(q))
\]

as above, where \(I_n^{-a}(q)\) is an injective resolution of the étale sheaf \(\mu_l^{-a}(q)\) on \(\bar{U} \times \mathbb{A}^n\).

In this way, we obtain spectral sequences \(E_1^{ab}, \bar{E}_1^{ab}\) and \(\bar{E}_2^{ab}\) as before, and a cycle map

\[\bar{C}_n(U, n) \to \bar{E}_1^{-n, 2q}\]

This induces the regulator map to étale cohomology

\[\text{reg} : CH^n(U, n) \to \bar{E}_2^{-n, 2q} \to H^{2q-n}_{et}(\bar{U}, \mu_l^n(q))\]

Moreover, the image lies in the \(G_k\)-invariant part. Passing to the limit yields a map

\[\text{reg} : CH^n(U, n) \to H^{2q-n}_{et}(\bar{U}, \mathbb{Z}_l(q))^{G_k}\]

Functorial properties of this regulator map follows in the same way as the singular case. Also as in singular case, this regulator map on \(CH^1(U, 1)\) can be described explicitly. From the Kummer sequence

\[1 \to \mu_l \to \mathcal{O}_U^* \to \mathcal{O}_U^*/l^N \to 1\]

the connecting homomorphism induces the map

\[\mathcal{O}^*(U)/l^N \mathcal{O}^*(U) \to H^1_{et}(\bar{U}, \mu_l^N)\]

Taking the inverse limit over \(N\), we get the map

\[\lim_{\rightarrow N} \mathcal{O}^*(U)/l^N \mathcal{O}^*(U) \to H^1_{et}(\bar{U}, \mathbb{Z}_l(1))\]

This is the same as the regulator map. The proof is similar to the argument with the exponential sequence in the singular case.
2. Invariant theory

This section refines some results from our earlier paper [AK]. We start by reviewing some notation from that paper. The category of rational mixed Hodge structures forms a neutral Tannakian category over $\mathbb{Q}$. Let $\langle H \rangle$ denote the Tannakian category generated by $H$. This is the full subcategory consisting of all subquotients of tensor powers $T^{m,n}H = H^\otimes m \otimes (H^*)^\otimes n$. This construction extends to any set of Hodge structures. The Mumford-Tate group $MT(H)$ is the group of tensor automorphisms of the forgetful functor from $\langle H \rangle$ to $\mathbb{Q}$-vector spaces.

By Tannaka duality $\langle H \rangle$ is equivalent to the category of representations of this group. When $H$ is a pure Hodge structure, $MT(H)$ can be defined in a more elementary fashion as the smallest $\mathbb{Q}$-algebraic group whose real points contains the image of the torus defining the Hodge structure. We define two auxiliary groups. The extended Mumford-Tate group $EMT(H) = MT(\langle H, \mathbb{Q}(1) \rangle)$ (some authors consider this to be the Mumford-Tate group). The special Mumford-Tate group $SMT(H) = \ker \left( EMT(H) \to \mathbb{G}_m \right)$ with respect to the map that is induced by the inclusion $\langle \mathbb{Q}(1) \rangle \subset \langle H, \mathbb{Q}(1) \rangle$.

Let $H$ be the first cohomology of a smooth quasi projective variety. We have a description of $SMT(H)$ given by Lemma 2.1.

**Lemma 2.1.** As a subgroup of $GL(H) = GL(V \oplus W)$

$$SMT(U) = \{ \begin{pmatrix} I & 0 & 0 \\ f & S & \end{pmatrix} \mid S \in SMT(W) \text{ and } f \in \Phi \}.$$

**Proof.** See [AK].

We want to refine this slightly. We define three subspaces $V_i \subset H$. Let $V_3 = W_1 H$, and let $V_1 \subseteq H^{SMT(H)}$ be a complement to $V_3$ in $W_1 H + H^{SMT(H)}$, and finally choose $V_2$ to be a complement to $V_1 + V_3$ in $H$. Thus we have a decomposition

$$H = V_1 \oplus V_2 \oplus V_3$$

into vector spaces. Under this decomposition, $W_1 H$ maps to $V_3$. For the arguments below, it is convenient to assign weights to elements of a mixed Hodge structure. Say that $x$ has weight $k$ if $x \in W_k$ and $x \notin W_{k-1}$. With this convention, we see that elements of $V_1$ and $V_2$ are of weight 2 and elements of $V_3$ are of weight 1.

Let $G_c = SMT(V_3)$. With respect to (2), $SMT(H)$ is a subgroup of the following matrix group:

$$\left\{ \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & f & S \end{pmatrix} \mid S \in G_c \text{ and } f \in Hom(V_2, V_3) \right\}.$$

The unipotent radical $U(SMT(H))$ lies in the subgroup

$$\left\{ \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & f & I \end{pmatrix} \mid f \in Hom(V_2, V_3) \right\}.$$

**Lemma 2.2.** For any nonzero $u \in V_2$, we can find a $g \in U(SMT(H))$ such that $gu \neq u$, or equivalently such that $f(u) \neq 0$ with respect to the matrix (3).

**Proof.** See [AK].
Let $BH^{n,s}(H)$ denote the space of Beilinson-Hodge cycles of weight $2s$ in $H^\otimes n$. More precisely,

$$BH^{n,s}(H) = \text{Hom}_{\text{MHS}}(\mathbb{Q}(-s), H^\otimes n) \subseteq \text{Hom}_{\text{MHS}}(\mathbb{Q}(-s), (H^{\text{split}})^\otimes n)$$

Note that if $n > 2s$ then $BH^{n,s}(H) = 0$ hence the results stated below are trivially true. So we shall also assume $n \leq 2s$. To simplify book keeping, we will usually write tuples $(j_1, \ldots, j_n)$ as strings $j_1 \cdots j_n$. In particular, juxtaposition is used to denote concatenation of strings, with exponents used for repetition. For example, $1^2 2^3 0 = 112$. Note that

$$H^\otimes n = \bigoplus_{j_1, \ldots, j_n} V(j_1 \cdots j_n),$$

where

$$V(j_1 \cdots j_n) = V_{j_1} \otimes \cdots \otimes V_{j_n}.$$

Define $|j_1 j_2 \ldots j_n|_3 = \#\{i : j_i = 3\}$.

**Lemma 2.3.** The weight of a nonzero element of $V(j_1 \cdots j_n)$ is $2n - |j_1 \cdots j_n|_3$.

**Proof.** This follows trivially from the fact that elements of $V_1$ and $V_2$ are of weight 2 and elements of $V_3$ are of weight 1. \hfill \Box

Define the space of elementary Hodge cycles $EH^{n,s}(H) \subseteq H^\otimes n$ as the subspace generated by products of elements of $V_1$ and $V_3^{G_c}$ of weight $2s$. Thus from lemma 2.3,

$$EH^{n,s}(H) = \bigoplus_{j_1, \ldots, j_n \in \{1,3\}, |j_1 \cdots j_n|_3 = 2n-2s} V(j_1 \cdots j_n)^{G_c}$$

where the action of $G_c$ on $V_1$ is trivial. Note that $G_c$ can be viewed as a quotient of $SMT(H)$ since $W$ is sub-Hodge structure of $H$.

**Theorem 2.4.** $BH^{n,s}(H) = EH^{n,s}(H)$

**Proof.** Since $SMT(H)$ acts trivially on $V_1$ and the action of $SMT(H)$ on $V_3$ factors through $G_c$, the action of $SMT(H)$ on $EH^{n,s}(H)$ is trivial. Also elements of $EH^{n,s}(H)$ are of weight $2s$, hence they are Beilinson-Hodge cycles of weight $2s$ in $H^\otimes n$. This proves $EH^{n,s}(H) \subseteq BH^{n,s}(H)$.

Let $\tau \in BH^{n,s}(H)$, our goal is to show that $\tau \in EH^{n,s}(H)$. Let us decompose

$$\tau = \sum \tau_{j_1 \cdots j_n}$$

with respect to $\mathbf{4}$. Let $\tau^{EH}$ be the sum of those terms $\tau_{j_1 \cdots j_n}$ which are in $EH^{n,s}(H)$. We replace $\tau$ by $\tau - \tau^{EH}$. Then it is enough to show that $\tau$ equals 0. Suppose that it is nonzero. Since $\tau \in W_{2s} H^\otimes n$, by lemma 2.3 we have that $|k_1 \cdots k_n|_3 \geq 2n - 2s$ for every nonzero term $\tau_{k_1 \cdots k_n}$ of $\mathbf{4}$. Also since $\tau$ is nonzero of weight $2s$, $\tau$ must have a nonzero term $\tau' = \tau_{j_1 \cdots j_n}$ so that $|j_1 \cdots j_n|_3 = 2n - 2s$. Without loss of generality, suppose that $j_1 = j_2 = \ldots = j_{2n-2s} = 3$ and the remainder of the $j_i$’s are either 1 or 2. When at least one of these is 2, then we will derive a contradiction. We have $j_1 \ldots j_n = 3^{2n-2s-1}1^{n_1}2^{n_2}1^{n_3} \ldots$ with $n_2 > 0$. So using $\mathbf{3}$ and the lemma 2.2, we can obtain a $g \in U(SMT(H))$ so that $g\tau' - \tau'$ has a nonzero component in $V(3^{2n-2s-1}1^{n_1}2^{n_2-1}1^{n_3} \ldots)$. Also if $\tau''$ is any other term in $\tau$ then its image under $g - I$ does not lie in $V(3^{2n-2s-1}1^{n_1}2^{n_2-1}1^{n_3} \ldots)$. This
corollary 2.6. In other words, Φ is a subspace of Hom
In particular, Its Zariski closure with respect to the decomposition H
is a subgroup of Q
m
lemma 2.5. proof. This is evident from the above description.

2.1. Beilinson-Tate case. Let k be a finitely generated field over Q. In analogy with the constructions of the Mumford-Tate group above, we define MG(H) to be the image of the map Gk → GL(H) for a Gk-module H. Let EMG(H) = MG(H ⊕ Qi(1)). This group acts on both factors H and Qi(1). The second action determines a homomorphism EMG(H) → Gm. Let SMG(H) = ker[EMG(H) → Gm].

For a smooth variety U over k, let MG(U) and SMG(U) denote MG(H1 et( ¯U, Qi)) and SMG(H1 et( ¯U, Qi)) respectively. Recall that H1 et( ¯U, Qi) carries an increasing filtration Wj called the weight filtration [D1]. The space Wj is defined as the sum of the (generalized) eigenspaces of a geometric Frobenius Fm, at an unramified place m, with eigenvalues having norms (Normk/Q(m))j/2 for some j′ ≤ j. When i = 1, this can be described explicitly as follows. Choose a smooth compactification X of U (which exists by [H]). Then

0 = W0 ⊆ W1 = im[H1 et( ¯X, Qi) → H1 et( ¯U, Qi)] ⊆ W2 = H1 et( ¯U, Qi)

Lemma 2.5. MG(U) preserves the weight filtration on H1 et( ¯U, Qi).

Proof. This is evident from the above description.

Let H = H1 et( ¯U, Qi) and let W = W1H = H1 et( ¯X, Qi). Choose a (not necessarily Gk-invariant) complementary subspace V to W in H. SMG(H) being a subgroup of MG(H), by lemma 2.5, it preserves the weight filtration on H1(U). Therefore, with respect to the decomposition H = V ⊕ W, we can identify

SMG(H) ⊆ \{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \}.

In particular,

Φ = ker[SMG(U) → SMG(H/W ⊕ W)]

is a subgroup of

\{ \begin{pmatrix} I & 0 \\ f & I \end{pmatrix} | f ∈ HomQi(V, W) \}.

In other words, Φ is a subspace of HomQi(V, W).

Corollary 2.6. The group

SMG(H) = \{ \begin{pmatrix} I & 0 \\ f & S \end{pmatrix} | S ∈ SMG(W) and f ∈ Φ \}.

Its Zariski closure

\overline{SMG}(H) = \{ \begin{pmatrix} I & 0 \\ f & S \end{pmatrix} | S ∈ \overline{SMG}(W) and f ∈ Φ \}.

The group Φ is the unipotent radical of \overline{SMG}(H).
Proof. The only thing to observe, for the first statement, is that $V \cong H/W \cong \mathbb{Q}_l(-1)^N$, so $SMG(H)$ acts trivially on it. The second follows from this. Finally, by a theorem of Faltings [F, Satz 3] [FW, p 211] the action of $G_k$ on $W = H^1_{et}(\bar{X}, \mathbb{Q}_l)$ is semisimple. Therefore the Zariski closure of its image $MG(W)$ is reductive. $SMG(W)$ is also reductive, because up to isogeny it is a direct factor of $MG(W)$.

Since $SMT(H)$ and $SMG(H)$ are similar in structure, the results on invariants proved above for $SMT(H)$ holds for $SMG(H)$ as well. In particular, we can decompose $H = V_1 \oplus V_2 \oplus V_3$ so that $V_3 = W$, $SMG(H)$ acts trivially on $V_1$ and for every nonzero $u \in V_2$, there exist an element $g \in \Phi$ such that $gu \neq u$.

As in Hodge case, let $BT_{n,s}(H) = (H^n \otimes \mathbb{Q}_l(s))^G_k \otimes \mathbb{Q}_l(-s)$

$$BT_{n,s}(H) = \text{Hom}_{MG(H)}(\mathbb{Q}_l(s), H^n)$$

Let $G_c = SMG(V_3)$. Define the space of elementary Tate-cycles $ET_{n,s}(H) \subseteq H^{\otimes n}$ as the subspace

$$ET_{n,s}(H) = \bigoplus_{j_1, \ldots, j_n \in \{1, 3\}, \sum_{j_1, \ldots, j_n} = 2n-2s} V(j_1 \ldots j_n)^{G_c}$$

where the action of $G_c$ on $V_1$ is trivial. Again $G_c$ can be viewed as a quotient of $SMG(H)$ since $W$ is $G_k$-submodule of $H$.

**Theorem 2.7.** $BT_{n,s}(H) = ET_{n,s}(H)$

The proof is same as the Hodge case.

3. Beilinson’s Hodge and Tate cycles on product of curves and semiabelian varieties

For a variety $U$ over a field $k$, let $\bar{U} = U \times_k \bar{k}$. From the first section, we get maps

$$\begin{cases} CH^1(U, j) \to \text{Hom}_{MHS}(\mathbb{Z}(0), H^{2i-j}(U, \mathbb{Z}(i))) & \text{if } U \text{ is over } \mathbb{C} \\ CH^1(U, j) \otimes \mathbb{Z}_l \to H^{2i-j}_{et}(\bar{U}, \mathbb{Z}_l(i))^{G_k} & \text{in general} \end{cases}$$

The first map is easily seen to be surjective for $i = j = 1$ (see [J] Thm 5.13) or [AK] Thm 1.1), and therefore the Beilinson-Hodge conjecture holds integrally at this level. For a finitely generated field $k$, the same is true for Beilinson-Tate:

**Theorem 3.1.** ([Jannsen, [J] Thm 5.15]) For any smooth variety $U$ over a finitely generated field $k$, the map

$$\text{reg} : CH^1(U, 1) \otimes \mathbb{Z}_l \to H^1(U, \mathbb{Z}_l(1))^{G_k}$$

is surjective.

In [AK], we defined

$$BH^q(U) = \text{Hom}_{MHS}(\mathbb{Q}(0), H^q(U, \mathbb{Q}(q)))$$

Similarly, we define the space of Beilinson-Tate cycles

$$BT^q(U) = H^q_{et}(\bar{U}, \mathbb{Q}_l(q))^{G_k}$$
More generally, let
\[ BH^{n,s}(U) = \text{Hom}_{\text{MHS}}(\mathbb{Q}(0), H^n(U, \mathbb{Q}(s))) \]
\[ BT^{n,s}(U) = H^n_{et}(\overline{U}, \mathbb{Q}_l(s))^G_k \]

The Beilinson-Hodge (respectively Beilinson-Tate) conjecture asserts that the regulator maps from \( CH^q(U, q) \) (respectively \( CH^q(U, q) \otimes \mathbb{Q}_l \)) surjects onto \( BH^q(U) \) (respectively \( BT^q(U) \)). As in the Beilinson-Hodge case, note that the conjecture is only interesting for open varieties, because it is vacuously true if the variety is projective. In case of smooth projective varieties \( BT^q(U) = 0 \) because \( H^q_{et}(\overline{U}, \mathbb{Q}_l) \) is pure of weight 2q.

**Lemma 3.2.** If the products \( BT^1(U) \times \ldots \times BT^1(U) \rightarrow BT^q(U) \) are surjective for all q, then the Beilinson-Hodge conjecture holds for U.

**Proof.** This follows from the following commutative diagram and theorem 3.1

\[
\begin{array}{ccc}
CH^1(U,1) \times \ldots \times CH^1(U,1) & \longrightarrow & CH^q(U,q) \\
\downarrow & & \downarrow \\
BT^1(U) \times \ldots \times BT^1(U) & \longrightarrow & BT^q(U)
\end{array}
\]

\[ \square \]

**Corollary 3.3.** The Beilinson-Tate conjecture holds for a product of smooth curves over a finitely generated field of characteristic 0.

**Proof.** Let \( U = \prod U_i \), where \( U_i \) are smooth curves. Let \( H = H^1_{et}(U) \) and note that by K"unneth’s formula and the theorem 2.7
\[ BT^q(U) = BT^{q,q}(U) = BT^{q,q}(H) \otimes \mathbb{Q}_l(q) = ET^{q,q}(H) \otimes \mathbb{Q}_l(q) = BT^1(U)^{\otimes q}. \]
So the hypothesis of lemma 3.2 holds. \[ \square \]

**Corollary 3.4.** The Beilinson-Tate conjecture holds for a semiabelian variety over a finitely generated field of characteristic 0.

**Proof.** Let \( U \) be a semiabelian variety. Let \( H = H^1_{et}(U) \). By the theorem 2.7 we have that \( BT^n(H) = BT^1(H)^{\otimes n} \). Now observe that \( H^*(U) = \wedge^* H \) which is a direct summand of the tensor algebra. So the Beilinson-Tate cycles on \( H^n(U) \) are given by products of Beilinson-Tate cycles on \( H \). \[ \square \]

### 3.1. Strong Beilinson’s Hodge and Tate conjecture.

\( k \) will stand for either \( \mathbb{C} \) or a finitely generated field of characteristic 0.

**Proposition 3.5.** Let \( C_1, C_2, \ldots, C_n \) be smooth curves defined over \( k \). Let \( U = C_1 \times C_2 \ldots C_n \) and let \( X \times \overline{X} = \overline{C}_1 \times \overline{C}_2 \ldots \overline{C}_n \) where \( \overline{C}_i \) denote the smooth compactification of \( C_i \).

1. If \( k = \mathbb{C} \) and the Hodge conjecture is true for \( X \) then \( BH^{n,s}(U) \) is generated by algebraic cycles on \( U \).
2. If \( k \) is finitely generated and the Tate conjecture is true for \( X \) then \( BT^{n,s}(U) \) is generated by algebraic cycles on \( U \).
Proof. Let $H = H^1(U, \mathbb{Q})$. By the Künneth formula, $H^*(U)$ can be written as a sum of tensor products of powers of $H$ with spaces generated by cycles. By theorem \(2.4\), it is enough to prove that every cycle in $EH^s(H)$ is algebraic. Moreover, it suffices to show that every summand in \(\mathbb{A}\) is algebraic. Without loss of generality, we shall show that $V(3^{2n-2s}1^{2s-n})G_e$ is algebraic.

The Hodge conjecture for $X$ implies the surjectivity of the map $CH^{n-s}(X) \rightarrow BH^{2n-2s,n-s}(X)$. We know that $H^{2n-2s}(X)$ surjects onto $W_{2n-2s}H^{2n-2s}(U)$ \(\mathbb{D}2\), and this induces the surjection $BH^{2n-2s,n-s}(X) \rightarrow BH^{2n-2s,n-s}(U)$. Hence we get the surjection $CH^{n-s}(U) \rightarrow BH^{2n-2s,n-s}(U) = V(3^{2n-2s}G_e \otimes \mathbb{Q}(n-s))$.

By \(\mathbb{A}3\), we have the surjection

$$
CH^{2s-n}(U, 2s-n) \rightarrow BH^{2s-n,2s-n}(U) = V(1^{2s-n}) \otimes \mathbb{Q}(2s-n).
$$

Since regulators preserve products, we have the surjection from

$$
CH^s(U, 2s-n) \rightarrow V(3^{2n-2s}G_e \otimes V(1^{2s-n}) \otimes \mathbb{Q}(s)) = V(3^{2n-2s}1^{2s-n})G_e \otimes \mathbb{Q}(s)
$$

In view of theorem \(2.7\), the proof of (2) is similar. \(\square\)

The above proposition can be reduced to verifying the Hodge and the Tate conjecture for a certain abelian variety.

**Corollary 3.6.** The conclusion of the proposition is true if the Hodge (respectively Tate) conjecture holds for the product of Jacobians $J = J(\bar{C}_1) \times \ldots \times J(\bar{C}_n)$.

**Proof.** Let $X = \bar{C}_1 \times \bar{C}_2 \times \ldots \bar{C}_n$. The Abel-Jacobi map induces a surjection $H^*(J) \rightarrow H^*(X)$. A Hodge (resp. Tate) cycle $\gamma$ on $X$ can be pulled back to a Hodge (resp. Tate) cycle on $J$, and then the corresponding algebraic cycle can be pushed back down to $X$ to prove algebraicity of $\gamma$. \(\square\)

**Proposition 3.7.** Let $U$ be a semiabelian variety and $X$ be a smooth compactification of $U$ both defined over $k$.

1. If $k = \mathbb{C}$ and the Hodge conjecture is true for $X$ then $BH^{n,s}(U)$ is generated by algebraic cycles on $U$.
2. If $k$ is finitely generated and the Tate conjecture is true for $X$ then $BT^{n,s}(U)$ is generated by algebraic cycles on $U$.

**Proof.** Let $H = H^1(U, \mathbb{Q})$. Again note that $H^s(U, \mathbb{Q})$ is a direct summand of $H^s$. So $BH^{n,s}(U)$ is a direct summand of $BH^{n,s}(H) \otimes \mathbb{Q}(s)$. Theorem \(2.4\) yields that $BH^{n,s}(U)$ is a direct summand of $EH^{n,s}(H) \otimes \mathbb{Q}(s)$. So without loss of generality, it is enough to show that $BH^{n,s}(U) \cap V(3^{2n-2s}1^{2s-n})G_e \otimes \mathbb{Q}(s)$ is algebraic. Again because of the splitting from wedge-products to tensor products, we know that

$$
BH^{n,s}(U) \cap V(3^{2n-2s}1^{2s-n})G_e \otimes \mathbb{Q}(s)) = BH^{2n-2s,n-s}(U) \otimes BH^{2s-n,2s-n}(U).
$$

The rest of the proof is same as for product of curves. \(\square\)

**Corollary 3.8.** Let $A$ be an abelian variety, $T$ be a torus and $U$ be a semiabelian variety given by an extension of $A$ by $T$. The conclusion of the above proposition is true if Hodge (resp. Tate) conjecture holds for $A$.

**Proof.** After replacing $k$ by a finite extension (which is harmless), we can assume that $T$ is split. Then $U$ can be compactified by the projective space bundle $X$ over $A$. Then $H^*(X) = H^*(A) \otimes H^*(\mathbb{P}^r)$ where $r$ is the rank of torus $T$. So Hodge (resp. Tate) conjecture for $X$ is true if and only if Hodge (resp. Tate) conjecture for $A$ is true. \(\square\)
There are a number of well known criteria for the Hodge conjecture for abelian varieties (cf. [G]). Many of them are obtained by analyzing the Mumford-Tate group of the first cohomology of abelian varieties. For a smooth projective curve \(C\) of nonzero genus, let \(\text{Lef}(C)\) be the centralizer of \(\text{End}(J(C)) \otimes \mathbb{Q}\) in the symplectic group \(\text{Sp}(H^1(C))\). The following is one of the basic criteria to decide Hodge conjecture for \(J(C)^n\).

**Proposition 3.9 (Murty).** Let \(C\) be a smooth projective curve such that \(\text{End}(J(C)) \otimes \mathbb{Q}\) is field and \(\text{Lef}(C) = \text{SMT}(C)\) then Hodge conjecture holds for \(J(C)^n\) for all \(n\).

**Proof.** This follows from [G, Theorem 6.2]. \(\square\)

In fact for most smooth projective curves \(C\) the Hodge conjecture holds for \(C^n\) as shown by the following proposition in [A, Prop 6.5].

**Proposition 3.10.** There exists a countable union \(S\) of proper Zariski closed sets in the moduli space \(M_g(\mathbb{C})\) of curves of genus \(g \geq 2\), such that if \(C \in M_g(\mathbb{C}) \setminus S\) then the generalized Hodge conjecture holds for all powers of its Jacobian \(J(C)\).

Any such \(C\) is called a very general curve. We combine some of these results to obtain the following corollary.

**Corollary 3.11.** Let \(C\) be an open subset of a smooth projective complex curve \(\bar{C}\). The strong version of the Beilinson-Hodge conjecture holds for \(C^n\) if \(C\) is one of the following

1. a curve of genus 1, 2 or 3.
2. a curve of prime genus such that its Jacobian is simple.
3. a Fermat curve \(x^m + y^m + z^m = 0\) with \(m\) prime or less than 21.
4. a curve admitting a surjection from a modular curve \(X_1(N)\).
5. a very general curve.

**Proof.** In view of corollary 3.6, it is enough to show that Hodge conjecture holds for \(J(C)^n\) in all the cases:

1. When \(C\) is an elliptic curves, Hodge conjecture for \(C^n\) was first proved by Tate but never published his proof. In [M] Murasaki showed that Hodge cycles on \(C^n\) are generated from divisors. For curves of genus 2 and 3 it was worked out by Mumford in an unpublished work but a proof can be found in [MZ].
2. For these curves the result was shown by Tankeev and Ribet ([R, p. 525]).
3. Shioda ([S], Theorem IV]) treated the Fermat curves.
4. This is given by work of Hazama and Murty (see [H]).
5. This follows from the above proposition. \(\square\)

**Corollary 3.12.** Let \(U\) be a semiabelian variety obtained by an extension of the abelian variety \(A\) by a torus. Strong versions of Beilinson-Hodge and Beilinson-Tate conjectures hold for \(U\) if \(A\) is defined over a number field and is one of the following type:

1. \(\dim A = 2\) or an odd number and \(\text{End}^0(A \times_k \bar{k}) = \mathbb{Q}\).
2. \(\dim A = 2d\) where \(d\) is an odd number and \(\text{End}^0(A \times_k \bar{k})\) is a real quadratic field or an indefinite quaternion algebra over \(\mathbb{Q}\).
dim $A = 4d$ where $d$ is an odd number and $\text{End}^0(A \times k \bar{k})$ is an indefinite quaternion algebra over a real quadratic field.

Proof. In view of corollary 3.8, it suffices to check that Hodge and Tate conjectures are true for $A$ in the three cases. The first case was proved by Serre in an unpublished notes. Serre’s methods were extended by W. Chi to show Mumford-Tate conjecture, Hodge conjecture and Tate conjecture in all the three cases ([C, Theorem 8.5, 8.6, 8.8]).

Tankeev ([Ta]) has extended these results of Serre and Chi to some other classes of abelian varieties. So corollary 3.8 can be applied in these cases as well.

4. CASE OF SMOOTH VARIETIES DOMINATED BY PRODUCTS OF CURVES

In this section we will deduce Beilinson-Hodge and Beilinson-Tate conjectures for smooth varieties $U$ which are dominated by product of curves by a proper surjective morphism.

Lemma 4.1. Suppose that the Beilinson-Hodge (respectively Beilinson-Tate) conjecture holds for a smooth variety $Y$ and there is a proper surjective map $p : Y \to X$ with $X$ smooth. Then the Beilinson-Hodge (respectively Beilinson-Tate) conjecture holds for $X$.

Proof. We treat the case of Beilinson-Hodge only, since the Beilinson-Tate case is identical. Let $r = \dim(Y) - \dim(X)$ be the relative dimension. The properties of the regulator map in section I shows the commutativity of the following diagram.

$$
\begin{array}{ccc}
CH^i(Y, j) \otimes \mathbb{Q} & \overset{\text{reg}_Y}{\longrightarrow} & BH^{i, 2i-j}(Y) \\
\downarrow p_* & & \downarrow p_* \\
CH^{i-r}(X, j) \otimes \mathbb{Q} & \overset{\text{reg}_X}{\longrightarrow} & BH^{i-r, 2i-r-2j}(X)
\end{array}
$$

By assumption, $\text{reg}_Y$ is surjective. Let $f : Z \to Y$ be a subvariety so that the restriction $g : Z \to X$ is generically finite and surjective. Then $g_* : H^*(Z) \to H^*(X)$ is surjective by the projection formula. Since $g = p \circ f$, $p_*$ on the right in the above commutative diagram is surjective. Therefore $\text{reg}_X$ must be surjective. □

Corollary 4.2. Let $U$ be a product of smooth curves over $k$, $V$ be a smooth variety over $k$ and $p : U \to V$ be a proper surjective morphism. Let $X$ be a smooth compactification of $U$.

1. If $k = \mathbb{C}$ then the Beilinson-Hodge conjecture holds for $V$ and if Hodge conjecture holds for $X$ then strong version of Beilinson Hodge conjecture also holds for $V$.

2. If $k$ is a number field then the Beilinson-Tate conjecture holds for $V$ and if Tate conjecture holds for $X$ then the strong version of Beilinson Tate conjecture also holds for $V$.

Proposition 4.3. Every semiabelian variety $U$ is dominated by product of curves.

Proof. Let $C_1, C_2, \ldots, C_n$ be (possibly affine) curves in $U$ such that they generate $U$ and $X_i$ be a one-point-compactification of $C_i$. By [Sc Chapter V] there exist commutative group schemes called generalized Jacobians $J_i = J(X_i)$ and morphisms $\phi_i : C_i \to J_i$ which are universal for commutative group schemes. This
defines a surjective morphism $\phi : J_1 \times J_2 \times \ldots \times J_n \to U$ given by $\phi(x_1, \ldots, x_n) = \phi_1(x_1) \cdot \phi_2(x_2) \cdot \ldots \cdot \phi_n(x_n)$. There is also a surjective morphism from $C^n_\mathbb{Q} \to J_i$ where $\pi_i$ is the arithmetic genus of $X_i$. Hence $U$ is dominated by product of curves. \hfill \Box

We can recover corollary 3.3 from the above proposition. We give another example where corollary 3.2 applies. Let $C$ be a smooth projective curve of genus $g$ and $P$ be a $k$-rational point on it. Let $J = J(C)$ be the Jacobian of $C$. Viewing points on $J$ as degree zero divisors on $C$, let $f_n : C^n \to J$ be the Abel-Jacobi morphism given by $(P_1, \ldots, P_n) \mapsto P_1 + \ldots + P_n - nP$. We can identify $C$ with its image $f_1(C)$. By Jacobi inversion, $f_g$ is a generically finite surjective morphism. The image of $f_{g-1}$ is the $\Theta$-divisor on $J$ ($\Theta$).

**Corollary 4.4.** Given rational points $Q_1, \ldots, Q_m \in C(k)$, the Beilinson-Hodge and Beilinson-Tate conjectures holds for $J \setminus \cup_i (\Theta + Q_i)$. Moreover if Hodge (resp. Tate) conjecture holds for $J$ then the strong Beilinson-Hodge (resp. Beilinson-Tate) conjecture holds for $J \setminus \cup_i (\Theta + Q_i)$.

**Proof.** The map $(C \setminus \{Q_1, \ldots, Q_m\})^g \to J \setminus \cup_i (\Theta + Q_i)$ given by the restriction of $f_g$ is a proper surjective morphism. \hfill \Box

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