Reducing Linear Programs into Min-max Problems

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Abstract

We show how to reduce a general, strictly-feasible LP problem, into a min-max problem, which can be solved by the algorithm from the third section of [1].

1 Reduction

Problem 1. Let us consider a linear program in the following form,

\[
\text{maximize}_{x \in \mathbb{R}^d} \quad (0,0,\ldots,1)^T x
\]

subject to \(Ax \leq b\)

and let us assume that the problem is strictly feasible; that is, there exists a point \(x\) for which \(Ax < b\). Further assume that the origin \((0,0,\ldots,0)\) is a strictly feasible point.

Any strictly feasible linear program can be rotated such that the objective function is \((0,0,\ldots,1) \cdot x\), and translated such that the origin is a strictly feasible point. The translation is discussed in Strict Feasibility of the Origin, below, while the rotation is explained further below, in Rotation.

We now show how to solve Problem 1 using the algorithm described in the third section of [1].

Definition 2 (z-axis). Let \(z\) denote the last coordinate of the space of our problem. \(z\)-intersect of a hyperplane refers to its intersection with the \(z\)-axis, while the last coordinate of a point is its \(z\) value. For example, in a 5-dimensional space, the \(z\)-coordinate denotes the fifth coordinate.

Definition 3 (Planes). For terseness, we denote the plane \(z \cdot p = \sigma\) as \((\pi,\sigma)\).

Definition 4 (Projective Duality). Let \(p \in \mathbb{R}^d\) be a point. Then its dual, \(p^*\), is the plane \((p, -1)\). Conversely, let \((\pi,\sigma)\) be a plane with \(\sigma \neq 0\). Then its dual, \((\pi,\sigma)^*\), is the point \(-\frac{\pi}{\sigma}\).

It is straightforward to confirm that the projective duality is self-dual and incidence preserving. For future use, we note that the \(z\)-intersect of a dual \(p^*\) to a point \(p\) is \(-\frac{1}{\sigma}\).

Definition 5 (Constraints and their Duals). The set of constrains in Problem 1, \(Ax \leq b\), can be described by a set of planes. Let us denote these planes as the set \(\Pi = (A_i, b_i)\), and their duals as \(\Pi^* = -A_i/b_i\).

Note that we exclude the definition of duality for planes which intersect the origin; however, since the origin is strictly feasible in Problem 1, no constraint plane intersects it.

Claim 6. Let \(p\) be a point and \((\pi,\sigma)\) a plane. Then \(p\) and the origin are on the same side of \((\pi,\sigma)\), if and only if the point \((\pi,\sigma)^*\) and the origin are on the same side of the plane \(p^*\).

Proof. \(p\) and the origin are on the same side of \((\pi,\sigma)\) iff,

\[
\text{sign} \left( (p \cdot (\pi - \sigma) \cdot (\pi \cdot 0 - \sigma)) \right) = \text{sign} \left( \sigma^2 \cdot \left( -\frac{1}{\sigma} \cdot \pi \cdot p + 1 \right) \right)
\]

\[
= \text{sign} \left( -\frac{1}{\sigma} \cdot \pi \cdot p + 1 \right)
\]

\[
= 1
\]

Similarly, \((\pi,\sigma)^*\) (which equals \(-\pi/\sigma\)) and the origin are on the same side of the plane \(p^*\) (which equals \((p, -1)\)) iff,

\[
\text{sign} \left( (p \cdot \left( -\frac{\pi}{\sigma} \right) + 1) \cdot (p \cdot 0 + 1) \right) = \text{sign} \left( -\frac{\pi}{\sigma} \cdot p + 1 \right) = 1.
\]

Claim 7. Assume the origin is a feasible point. Then, a point \(p\) is feasible iff the set of points \(\Pi^*\) representing the problem constraints, and the origin, are on the same side of the point’s dual plane, \(p^*\).
Proof. Since the origin is feasible, any other feasible point must share with it the same side of all the constraint planes $\Pi$. By Claim 6, this implies all duals to these planes, $\Pi^*$, and the origin, must be on the same side of $p^*$.

Claim 7 is illustrated in Figure 1. The point $F$ is a feasible point and is on the same side as the origin relative to all of the constraint planes (left figure). Its dual, $F^*$ has all the constraint points $\Pi^*$ and the origin on its same side (right figure).

**Definition 8** (Feasible Dual Plane). A plane $(\pi, \sigma)$ is feasible if its dual point, $\pi/\sigma$ is a feasible solution to Problem 1. Applying Claim 7, this implies that all dual constraint point $\Pi^*$, and the origin, are on the same side of $(\pi, \sigma)$.

Since the origin is a strictly feasible point, an optimal solution $p$ to Problem 1 must have a positive $z$ value. As a result, its dual must have a negative $z$–intersect. Moreover, since $p$ has a largest $z$ value amongst all feasible points, its dual must have the largest (negative) $z$–intersect amongst all feasible dual planes. In the case that the dual plane can be made to have an arbitrarily small negative $z$–intersect, the problem is unbounded.

It follows, then, that a plane which supports the set of points $\Pi^*$ from below and has a maximal (negative) $z$–intersect, is a solution to Problem 1, and this is exactly the problem which the algorithm from the third section of [1] solves.

**1.1 Strict Feasiblity of the Origin**

Given a strictly feasible solution $p_0$ to Problem 1, set $v \triangleq Ap_0$, and replace $b$ by $b' = b - v$. Because $Ap \leq b$ if and only if $A(p - p_0) \leq b'$, the feasible set of the new problem equals the feasible set of the original problem, translated by $p_0$. In addition, because $v < b$, it holds that $b' > 0$, which means that $A \cdot 0 < b$. That is, the origin is a strictly feasible point.

Finding a strictly feasible solution to Problem 1 can be performed by solving the following LP problem,

$$\begin{align*}
\min_{s \in \mathbb{R}, p \in \mathbb{R}^d} & \quad s \\
\text{s.t.} & \quad A \cdot p - b \leq s
\end{align*}$$

for which $p = 0$ and $s = -\min(b) + 1$ are a feasible solution. If the optimal solution $s^*$ is negative, $p^*$ is a strictly feasible point.

Alternatively, the equivalent min-max problem can be solved in the way described in the third section of [1],

$$\min_{p \in \mathbb{R}^d} \max (Ap - b);$$

if the solution is negative, $p^*$ is a strictly feasible point.

**1.2 Rotation**

Let $c \in \mathbb{R}^d$ be a general vector, and $u = c - (0, 0, \ldots, 1)^T$. Define the following matrix:

$$R' = I - \frac{uu^T}{\|u\|^2},$$

and set $R$ to be $R'$ with its first row negated. It is straightforward to verify that $R$ is a rotation matrix, and that $Rc = (0, 0, \ldots, 1)^T$.

Applying $R$ to a general LP program,

$$\begin{align*}
\max_{x \in \mathbb{R}^d} & \quad (Rc)^T x \\
\text{subject to} & \quad (AR^T) x \leq b,
\end{align*}$$

results in the form of Problem 1. The solutions of the two problems are related by rotation with $R$.

The computational cost of this rotation is bounded by $O(dn)$.

**References**

[1] Carmi Grushko. *Continuous Symmetries of Non-rigid Shapes*. MSc thesis, Technion - Israel Institute of Technology, 2012.