In this paper, we study the $\sigma_k$ curvature flow of noncompact spacelike hypersurfaces in Minkowski space. We prove that if the initial hypersurface satisfies certain conditions, then the flow exists for all time. Moreover, we show that after rescaling, the flow converges to a self-expander.

1. INTRODUCTION

Let $\mathbb{R}^{n,1}$ be the Minkowski space with the Lorentzian metric

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$ 

In this paper, we study the $\sigma_k$ curvature flows of noncompact spacelike hypersurfaces in Minkowski space. Spacelike hypersurfaces $M \subset \mathbb{R}^{n,1}$ have an everywhere timelike normal field, which we assume to be future directed and to satisfy the condition $\langle \nu, \nu \rangle = -1$. Such hypersurfaces can be locally expressed as the graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ satisfying $|Du(x)| < 1$ for all $x \in \mathbb{R}^n$.

Given an entire spacelike hypersurface $M_0$ embedded in $\mathbb{R}^{n,1}$, we let

$$X_0 : \mathbb{R}^n \to \mathbb{R}^{n,1}$$

be an embedding with $X_0(\mathbb{R}^n) = M_0$. For a given $\alpha \geq 1$, we say a family of spacelike embeddings is a solution of the $\sigma_k^\alpha/k$ curvature flow, if for each $t > 0$, $X(\mathbb{R}^n, t) = M_t$ is an entire spacelike hypersurface embedded in $\mathbb{R}^{n,1}$, and $X(\cdot, t)$ satisfies

$$\frac{\partial X(p, t)}{\partial t} = F^\alpha(\kappa[M_t](p, t))\nu$$

$$X(\cdot, 0) = M_0,$$

where $F^\alpha(\kappa[M_t](p, t)) = \sigma_k^\alpha/k(\kappa[M_t](p, t))$ is the $\sigma_k^\alpha/k$ curvature of $M_t$ at $X(p, t)$, $\nu$ is the future directed unit normal vector of $M_t$ at $X(p, t)$, and $\sigma_k$ is the $k$-th elementary symmetric polynomial, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$
Since the embeddings $X(\cdot, t)$ are spacelike, the position vector of $X(\cdot, t)$ can be written as $\mathcal{M}_t = \{ (x, u(x,t)) \mid x \in \mathbb{R}^n \}$. In particular, we assume $\mathcal{M}_0 = \{ (x, u_0(x)) \mid x \in \mathbb{R}^n \}$. After reparametrization, we can rewrite (1.1) as following equation

$$
\begin{align*}
\frac{\partial u}{\partial t} &= F^\alpha \left( \frac{1}{w} \gamma^{ik} u_{kl} \gamma^{lj} \right) w, \\
u(x, 0) &= u_0(x),
\end{align*}
$$

where $w = \sqrt{1 - |Du|^2}$, $\gamma^{ik} = \delta_{ik} + \frac{u_i u_k}{w(1+w)}$, and $u_{kl} = D^2_{x_k x_l} u$ is the ordinary Hessian of $u$.

The curvature flow problem in Euclidean space has been extensively studied in the literature. For the mean curvature flow, Huisken [9] proved that if the initial hypersurface $\mathcal{M}_0$ is smooth, closed, and strictly convex, then the mean flow exists on a finite time interval $0 \leq t \leq T$, and the $\mathcal{M}_t$ converges to a point as $t \to T$. Moreover, by a suitable rescaling, it is shown that the normalized hypersurfaces converge to a sphere. A similar result has been obtained by Chow [7] for the $n$-th root of the Gauss curvature flow. In [3], Andrews generalized Huisken’s result via the Gauss map to a large family of curvature flows including the $k$-th root of the $\sigma_k$ curvature flow.

Recall that the hyperboloid in Minkowski space is the analogue of the sphere in Euclidean space. Natural questions to ask are:

a). Let $\mathcal{M}_0$ be an entire, spacelike, strictly convex hypersurface in $\mathbb{R}^{n,1}$, does there exists a solution to (1.1)?

b). If the answer to part a) is “yes”, then after rescaling, does $\mathcal{M}_t$ converge to a hyperboloid?

These questions were investigated by Andrews-Chen-Fang-McCoy in [2]. Under the assumption that the initial hypersurface $\mathcal{M}_0$ is spacelike, co-compact, and strictly convex, Andrews-Chen-Fang-McCoy have given affirmative answers to questions a) and b). Note that we say a hypersurface is co-compact, if it is invariant under a discrete group of ambient isometries and the quotient space with respect to this group is compact. Such hypersurfaces only constitute a small collection of spacelike and strictly convex hypersurfaces. Moreover, with this assumption, the standard maximum principle can be applied directly to (1.1) without worrying about the infinity (i.e., when establish a priori estimates, $\mathcal{M}_t$ can be treated as compact hypersurfaces without boundary). In [17], for $k = n$ and $\alpha > 0$, the authors were able to give affirmative answers to quations a) and b) with much weaker assumptions on $\mathcal{M}_0$ (for details see Section 1 of [17]).

In this paper, we assume the initial hypersurface $\mathcal{M}_0$ satisfying **Condition A:**

1. spacelike,
2. strictly convex,
3. $u_0(x) - |x| \to \varphi(x/|x|) > 0$ as $|x| \to \infty$,
4. there exists constant $c_0, C > 0$ such that

\[ c_0 < \sigma_k(\kappa[\mathcal{M}_0])^{\alpha/k} < -C \langle X, \nu \rangle. \]
By theorem 1 in [14] we know there is a large collection of hypersurfaces satisfying Condition A. Therefore, Condition A is not a strong assumption.

We consider the long time existence and convergence of equation (1.1) with initial hypersurfaces \( \mathcal{M}_0 \) satisfying Condition A. Unlike curvature flows in Euclidean space (see [9, 7, 3] and references therein), we show that in general the rescaling of \( \mathcal{M}_t \) does not converge to the hyperboloid.

We want to explain Condition A a little bit more. It is easy to see that for any strictly convex spacelike hypersurface \( \mathcal{M} = \{(x, u(x)) \mid x \in \mathbb{R}^n\} \) with Gauss image equals unit ball \( \bar{B}_1 \), we can move \( \mathcal{M} \) vertically such that (3) is satisfied. Condition (4) can be viewed as a growth condition on the \( \sigma_k \) curvature of \( \mathcal{M}_0 \), i.e., \( \sigma_k(\kappa[\mathcal{M}_0]) \) cannot grow too fast as \( |x| \to \infty \). We want to point out that hypersurfaces with \( \sigma_k \) curvature bounded from above and below satisfy condition (4). Assumptions on the curvature of the initial hypersurface are often needed in proving the long-time existence and convergence of non-compact curvature flows (see [1, 2, 5, 8] for example).

Before we state our main result, we need the definition of self-expander.

**Definition 1.** A \( k \)-convex hypersurface \( \mathcal{M}_u \) is called a self-expander of the flow (1.1), if it satisfies the equation

\[
\sigma_k^\frac{\dot{X}}{X} (\kappa[\mathcal{M}_u]) = - \langle X, \nu \rangle.
\]

Here \( X = (x, u(x)) \) is the position vector of \( \mathcal{M}_u \) and \( \nu \) is the future time like unit normal of \( \mathcal{M}_u \).

It is clear, the hyperboloid is a self-expander. Moreover, in [18], the authors have proved that for an arbitrary \( \varphi \in C^2(\mathbb{S}^{n-1}), \varphi > 0 \), there exists a unique self-expander \( \mathcal{M}_u \) such that

\[
u(x) - |x| \to \varphi \left( \frac{x}{|x|} \right), \text{ as } |x| \to \infty.
\]

This work is concerned with the long time existence and convergence of (1.1) for initial data \( \mathcal{M}_0 \) satisfying Conditions A. In particular, we prove the following theorem.

**Theorem 2.** Suppose \( \varphi \in C^2(\mathbb{S}^{n-1}), \varphi > 0 \), and the initial spacelike hypersurface \( \mathcal{M}_{u_0} \) satisfying Conditions A. Then, the curvature flow (1.1) admits a solution \( \mathcal{M}_{u(x,t)} \) for all \( t > 0 \). Moreover, the rescaled flow

\[
\tilde{X} = \left( A(t)x, \frac{1}{A(t)} u(A(t)x,t) \right), A(t) = \left[ (1 + \alpha)t + 1 \right]^{\frac{1}{1+\alpha}},
\]

converges to a self-expander \( \mathcal{M}_{u,\infty} \) with the asymptotic behavior

\[
u^\infty(x) - |x| \to \varphi \left( \frac{x}{|x|} \right), \text{ as } |x| \to \infty.
\]

**Remark 3.** In this remark, we will explain that \( \varphi > 0 \) is only needed for technical reasons. In other words, without this assumption we can still prove the existence and convergence of the flow (1.1). When \( u_0(x) - |x| \to \varphi \) as \( |x| \to \infty \) and \( \varphi \) is not positive on \( \mathbb{S}^{n-1} \), we can define

\[
u_0^c := u_0 + c, \text{ where } c > 0 \text{ such that } \varphi^c := \varphi + c > 0 \text{ on } \mathbb{S}^{n-1}.
\]
Applying Theorem 2 we know there exists \( u^c(x,t) \) satisfies
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= F^{\alpha} \left( \frac{1}{w} \gamma^{ik} u_{kl} \gamma^{lj} \right) w, \\
\end{aligned}
\]
\( u(x,0) = u_0^c(x) \).

Moreover, the rescaled flow \( \left( A(t)x, \frac{1}{A(t)} u^c(A(t)x,t) \right) \) converges to a self-expander \( u^c_\infty(x) \) as \( t \to \infty \), and
\[
\begin{aligned}
u^\infty(x) - |x| &\to \varphi^c \left( \frac{x}{|x|} \right).
\end{aligned}
\]

Let \( u(x,t) := u^c(x,t) - c \), then it is clear that \( u(x,t) \) satisfies (1.2) and \( \left( A(t)x, \frac{1}{A(t)} \left( u(A(t)x,t) + c \right) \right) \) converges to a self-expander \( u^c_\infty(x) \) as \( t \to \infty \). In other words, when \( \varphi \) is not positive, the curvature flow (1.1) still admits a solution \( M_{u(x,t)} \) for all \( t > 0 \). Moreover, after moving vertically, the rescaled flow still converges to a self-expander.

The organization of this paper is as follows. In Section 2 we prove the long time existence of the approximate problem (2.3). Local \( C^1 \) and \( C^2 \) estimates are established in Section 3. Combining Section 2 with Section 3, we prove the existence of solution to (1.1) for all time \( t > 0 \). In Section 4 we show that after rescaling the solution of (1.1) converges to a self-expander as \( t \to \infty \).

## 2. Solvability of the Approximate Problem

In the rest of this paper, the constant \( C \) in (♣) is assumed to be 1, namely we assume
\[
(2.1)
\]
\[
c_0 < \sigma^u_\infty (\kappa [M_{u_0}]) < - \langle X_{u_0}, \nu_{u_0} \rangle.
\]
The equivalence of (♣) and (2.1) will be explained in the Appendix. We will also always denote \( A(t) = [(1 + \alpha) t + 1]^{\frac{1}{1+\alpha}} \).

Applying Theorem 1 and 3 in [18], we know there exists a locally strictly convex function \( \underline{u}(x) \) such that
\[
\sigma^u_\infty (\kappa [M_{\underline{u}}]) = - \langle X_{\underline{u}}, \nu_{\underline{u}} \rangle \text{ and } \underline{u}(x) - |x| \to \varphi \left( \frac{x}{|x|} \right) \text{ as } |x| \to \infty.
\]
A straightforward calculation yields \( A(t)\underline{u} \left( \frac{x}{A(t)} \right) \) satisfying the flow equation (1.2). Moreover, \( A(t)\underline{u} \left( \frac{x}{A(t)} \right) - |x| \to A(t) \varphi \left( \frac{x}{|x|} \right) \) as \( |x| \to \infty \). This inspires us to look for a locally strictly convex solution \( u(x,t) \) of the initial value problem (1.1) satisfying \( u(x,t) - |x| \to A(t) \varphi \left( \frac{x}{|x|} \right) \) as \( |x| \to \infty \). If such solution \( u(x,t) \) exists, then by [6] we obtain that its Legendre transform \( u^* \) is defined in the unit ball \( \bar{B}_1 \subset \mathbb{R}^n \). Furthermore, by Lemma 14 of [15] we know that \( u^* \) satisfies \( u^*(\xi,t) = - A(t) \varphi(\xi) \) for \( \xi \in S^{n-1} \).
In the following, we will always use $u^*$ to denote the Legendre transform of $u$ and we will also denote $F_* = \left( \frac{\sigma_n}{\sigma_n - k} \right)^{\frac{1}{k}}$. In this paper, we will consider the solvability of the following problem:

\[
\begin{cases}
  u^*_t = -F_*^{-\alpha}(w^* \gamma^*_i \gamma^*_j w^*) & \text{in } B_1 \times (0, \infty), \\
  u^*(\cdot, t) = [(1 + \alpha)\bar{t}]^{1+\alpha} \varphi^* & \text{on } \partial B_1 \times [0, \infty), \\
  u^*(\cdot, 0) = u^*_0 & \text{on } B_1 \times \{0\},
\end{cases}
\]

(2.2)

where $u^*_0$ is the Legendre transform of $u_0$, $\varphi^*(\xi) = -\varphi(\xi)$ for any $\xi \in \partial B_1$, $\bar{t} = t + (1 + \alpha)^{-1}$, $w^* = \sqrt{1 - |\xi|^2}$, $\gamma^*_i = \delta_{ik} - \frac{\xi_i \xi_k}{1 + |\xi|^2}$, and $u^*_k = \frac{\partial^2 u}{\partial \xi_k \partial \xi_l}$. It is easy to see that if $u^*(\xi, t)$ solves (2.2), then its Legendre transform $u(x, t)$ is the unique solution of the initial value problem (1.1).

Due to the degeneracy of equation (2.2), in this section, we study the solvability of the approximate problem

\[
\begin{cases}
  (u^*_0)_t = -F_*^{-\alpha}(w^* \gamma^*_i \gamma^*_j w^*) & \text{in } B_r \times (0, T), \\
  u^*_t (\cdot, t) = [(1 + \alpha)\bar{t}]^{1+\alpha} u^*_0 & \text{on } \partial B_r \times [0, T), \\
  u^*_t (\cdot, 0) = u^*_0 & \text{on } B_r \times \{0\}.
\end{cases}
\]

(2.3)

2.1. $C^0$ estimates for $u^*_r$. We first construct the supersolution of (1.2). Let $\bar{u}^*$ is the Legendre transform of $u_0$. By the assumption (2.1), we can see

\[
F_*^{-\alpha}(w^* \gamma^*_i \bar{u}^*_k \gamma^*_j) = \frac{1}{k} \left( \kappa[M_{u_0}] \left( \frac{x}{A(t)} \right) \right) < \frac{-\bar{u}^*}{w^*}.
\]

Consider $\tilde{u}^* = [(1 + \alpha)\bar{t}]^{\frac{1}{1+\alpha}} \bar{u}^*$, we want to point out that $\tilde{u}^*$ is the Legendre transform of $\bar{A(t) u_0} \left( \frac{x}{A(t)} \right)$. Then we get

\[
\begin{align*}
\bar{u}^* &= [(1 + \alpha)\bar{t}]^{-\frac{1}{1+\alpha}} \tilde{u}^* \\
&< -[(1 + \alpha)\bar{t}]^{-\frac{1}{1+\alpha}} \bar{w}^* F_*^{-\alpha}(w^* \gamma^*_i \bar{u}^*_k \gamma^*_j) \\
&= -F_*^{-\alpha}(w^* \gamma^*_i \bar{u}^*_k \gamma^*_j) w^*.
\end{align*}
\]

(2.4)

Therefore, by the standard maximum principle we obtain that $\bar{u}^* \leq u^*$.

Similar we can show $\bar{u}^* = [(1 + \alpha)\bar{t}]^{\frac{1}{1+\alpha}} \bar{u}^*$ is a supersolution to (2.3) and $u^*_r < \bar{u}^*$. Here $\bar{M}_u = \{ (x, u(x)) \mid x \in \mathbb{R}^n \}$ is a strictly convex spacelike hypersurface satisfying

\[
\sigma_k \left( \kappa[M_u] \left( \frac{x}{A(t)} \right) \right) = -\left< X_u, \nu_u \right>
\]

(2.5)

with $u(x) \to \varphi \left( \frac{x}{|x|} \right)$ as $|x| \to \infty$, and $u^*$ is the Legendre transform of $u$. Moreover, $\tilde{u}^*$ is the Legendre transform of $A(t) u \left( \frac{x}{A(t)} \right)$.

We conclude

**Lemma 4.** Let $u^*_r$ be a solution of (2.3), then $u^*_r$ satisfies

$$
\bar{u}^* \leq u^*_r < \tilde{u}^*,
$$
which implies
\[-C_1[(1 + \alpha)\hat{t}]^{\frac{1}{1+\alpha}} < u_+^* < C_0[(1 + \alpha)\hat{t}]^{\frac{1}{1+\alpha}},\]
where $C_0, C_1$ are positive constants depending on $\varphi^*$ and $M_0$.

2.2. The bounds for $F_*$. In this subsection we will show that along the flow, $F_*$ is bounded from above and below.

We take the hyperplane $P := \{X = (x_1, \cdots, x_n, x_{n+1}) \mid x_{n+1} = 1\}$ and consider the projection of $\mathbb{H}^n(-1)$ from the origin into $P$. Then $\mathbb{H}^n(-1)$ is mapped in a one-to-one fashion onto an open unit ball $B_1 := \{\xi \in \mathbb{R}^n \mid \sum \xi_k^2 < 1\}$. The map $P$ is given by
\[
P : \mathbb{H}^n(-1) \rightarrow B_1; \quad (x_1, \cdots, x_{n+1}) \mapsto (\xi_1, \cdots, \xi_n),
\]
where $x_{n+1} = \sqrt{1 + x_1^2 + \cdots + x_n^2}$, $\xi_i = \frac{x_i}{x_{n+1}}$. When $u_+^*$ satisfies (2.3), let $v_r = \frac{u_+^*}{\sqrt{r}}$, then a straightforward calculation yields $v_r$ satisfies
\[
\begin{cases}
(v_r)_t = -\hat{F}^{-1}(\Lambda_{ij}) := \tilde{G}(\Lambda_{ij}) & \text{in } P^{-1}(B_r) \times (0, T) := U_r \times (0, T) \\
v_r(\cdot, t) = [(1 + \alpha)\hat{t}]^{\frac{1}{1+\alpha}} \frac{u_0^*}{\sqrt{1-r^2}} & \text{on } \partial U_r \times [0, T] \\
v_r(\cdot, 0) = u_0^* \xi_{n+1} & \text{on } U_r \times \{0\},
\end{cases}
\]
(2.6)
where $\hat{F} = F_*^\alpha$, $\Lambda_{ij} = \tilde{\nabla}_{ij} v_r - v_r \delta_{ij}$, and $\tilde{\nabla}$ is the Levi-Civita Connection of the hyperbolic space.

**Lemma 5.** Assume $v$ is a solution of (2.6). Then we have
\[
\frac{[(1 + \alpha)T + 1]^{\frac{\alpha}{1+\alpha}}}{C_3} > \hat{F} > \frac{1}{C_2 x_{n+1}} \quad \text{on } \bar{U}_r \times (0, T).
\]
Here, $C_2 = C_2(|u_0^*|_{C^0})$ and $C_3 = C_3(c_0, |u_0^*|, r)$, where $c_0 > 0$ is the lower bound of $\sigma_k^\alpha(\kappa[M_0])$ in (2.1).

**Proof.** Since
\[
\tilde{G}_{ij} = \tilde{G}^{ij}(v_t)_{ij} - v_t \delta_{ij} = \tilde{G}^{ij}(\tilde{\nabla}_{ij} \tilde{G} - \tilde{G} \delta_{ij}),
\]
we have $L\tilde{G} = -\tilde{G} \sum_i \tilde{G}_{ii}$, where $L := \frac{\partial}{\partial t} - \tilde{G}^{ij} \tilde{\nabla}_{ij}$. It is clear that $L x_{n+1} = -x_{n+1} \sum_i \tilde{G}_{ii}$. Therefore, we get
\[
L \frac{-\tilde{G}}{x_{n+1}} = \frac{L(-\tilde{G})}{x_{n+1}} + \frac{\tilde{G} L(x_{n+1})}{x_{n+1}^2} + 2\tilde{G}_{ij}^i \left(\frac{x_{n+1}}{x_{n+1}^2}\right)_{ij} \left(\frac{-\tilde{G}}{x_{n+1}}\right)_{ij} = 2\tilde{G}^{ij} \left(\frac{x_{n+1}}{x_{n+1}^2}\right)_{ij} \left(\frac{-\tilde{G}}{x_{n+1}}\right)_{ij}.
\]
Applying the maximum principal, we get \( \tilde{F}_{x_{n+1}} \) achieves its maximum and minimum at the parabolic boundary. Now denote \( \tilde{v}_r = v_r - [(1 + \alpha)\tilde{t}]^{\frac{\alpha}{1 + \alpha}} \frac{u_0^*}{\sqrt{1 - r^2}} \), then \( \tilde{v}_r \) satisfies

\[
\begin{cases}
(\tilde{v}_r)_t = -\tilde{F}^{-1} - [(1 + \alpha)\tilde{t}]^{\frac{\alpha}{1 + \alpha}} \frac{u_0^*}{\sqrt{1 - r^2}} & \text{in } U_r \times (0, T], \\
v_r(\cdot, t) = 0 & \text{on } \partial U_r \times [0, T], \\
v_r(\cdot, 0) = u_0^* x_{n+1} - \frac{u_0^*}{\sqrt{1 - r^2}} & \text{on } U_r \times \{0\}.
\end{cases}
\]

In view of the short time existence theorem, we obtain on \( \partial U_r \times (0, T] \)

\[
\tilde{F} = \frac{[(1 + \alpha)\tilde{t}]^{\frac{\alpha}{1 + \alpha}} u_0^*}{\sqrt{1 - r^2}}.
\]

Therefore, we get, on \( \partial U_r \times (0, T] \),

\[
\tilde{F}_{x_{n+1}} = \frac{[(1 + \alpha)\tilde{t}]^{\frac{\alpha}{1 + \alpha}}}{-u_0^*};
\]

on \( U_r \times \{0\} \), by (2.1), we have

\[
c_0 x_{n+1}^r < \tilde{F}^{-1} x_{n+1}^r < |u_0^*|.
\]

This completes the proof of Lemma 5. \( \square \)

2.3. \( C^1 \) estimates for \( u_r^* \).

**Lemma 6.** Let \( u_r^* \) be a solution of (2.3), then \( |Du_r^*| \leq C \) \( := C(\mathcal{M}_0, r, t, T) \) in \( \tilde{B}_r \times [0, T] \).

**Proof.** By Lemma 5 we know that, for any \( t \in (0, T] \) we have

\[
C_1 \leq \frac{\sigma_n}{\sigma_{n-k}} (w^* \gamma_{ik}^* (u_r^*)_{kl} \gamma_{lj}^*) \leq C_2
\]

for some \( C_1, C_2 \) depending on \( \mathcal{M}_0, r, T, \) Applying Section 5 of [15], we get, for any fixed \( t \in (0, T] \) there exists \( \bar{u}^* \) and \( \underline{u}^* \) such that

\[
\begin{cases}
\frac{\sigma_n}{\sigma_{n-k}} (w^* \gamma_{ik}^* (\bar{u}^*)_{kl} \gamma_{lj}^*) = C_1 \quad & \text{in } B_r, \\
\bar{u}^* = [(1 + \alpha)\tilde{t}]^{\frac{1}{1 + \alpha}} u_0^* \quad & \text{on } \partial B_r,
\end{cases}
\]

and

\[
\begin{cases}
\frac{\sigma_n}{\sigma_{n-k}} (w^* \gamma_{ik}^* (\underline{u}^*)_{kl} \gamma_{lj}^*) = C_2 \quad & \text{in } B_r, \\
\underline{u}^* = [(1 + \alpha)\tilde{t}]^{\frac{1}{1 + \alpha}} u_0^* \quad & \text{on } \partial B_r,
\end{cases}
\]

where \( \tilde{t} = t + (1 + \alpha)^{-1} \). By the maximum principle we derive

\[
\underline{u}^* \leq u^*(\cdot, t) \leq \bar{u}^* \text{ in } B_r.
\]
Therefore, we obtain

$$|Du^*_r| \leq \max_{\xi \in \partial B_r} \{|Du^*_t|, |Du^*|\},$$

here we have used the convexity of $u^*_r$. $\square$

2.4. $C^2$ boundary estimates for $u^*_r$. In this subsection, we will show $D^2u^*_r$ is bounded on $\partial B_r \times [0, T]$. From now on, for our convenience, we will consider the solution of equation (2.6). We suppose $\{\tau_1, \cdots, \tau_n\}$ is the orthonormal frame of the boundary $\partial U_r$. Moreover, $\tau_1, \cdots, \tau_{n-1}$ are the tangential vectors and $\tau_n$ is the unit interior normal vector.

**Lemma 7.** Let $v$ be the solution of (2.6), then the second order tangential derivatives on the boundary satisfy

$$|\bar{\nabla}_{\alpha\beta}v| \leq C$$

on $\partial U_r \times (0, T]$ for $\alpha, \beta < n$. Here $C$ depends on $M_0, r, T$.

**Proof.** Recall the calculation in Subsection 2.1 we know that $\bar{u}^*$ is a subsolution of (2.6). Let

$$v = \frac{\bar{u}^*}{\sqrt{1 - |\xi|^2}}.$$  

Since $v - u \equiv 0$ on $\partial U_r \times [0, T]$, we have

$$\bar{\nabla}_{\alpha\beta}(v - u) = \bar{\nabla}_n(v - u)II(\tau_\alpha, \tau_\beta)$$

on $\partial U_r \times (0, T]$. Here $II$ is the second fundamental form of $\partial U_r$. Applying Lemma 6 we complete the proof of this Lemma. $\square$

Now, we will show that $|\nabla_{\alpha n}v|$ is bounded. In the following we will denote

$$L\phi := \phi_t - \bar{F}_v^{-2}\tilde{F}^{ij}_{\bar{v}}\tilde{\nabla}^{ij}\phi + \phi\tilde{F}_v^{-2}\sum_i \tilde{F}^{ii}_{\bar{v}}$$

for any smooth function $\phi$. Here $\tilde{F}_v(\Lambda_{ij}) = F^*_s(\Lambda_{ij}), \tilde{F}^{ij}_{\bar{v}} = \frac{\partial \tilde{F}_v}{\partial \Lambda_{ij}},$ and $\Lambda_{ij} = \tilde{\nabla}_{ij}v - v\delta_{ij}$. 

**Lemma 8.** Let $v$ be a solution of (2.6), $v$ be the subsolution of (2.6) which is defined by (2.7), and $h = (v - u) + B \left(\frac{1}{\sqrt{1 - r^2}} - x_{n+1}\right)$, where $B > 0$ is a constant. Then for any given constant $B_1 > 0$, there exists a sufficiently large $B$ depending on $M_0, r, T$, such that $Lh > \frac{B_1}{\tilde{F}_v} \sum \tilde{F}^{ii}_{\bar{v}}$.

**Proof.** A direct calculation gives

$$L \left(\frac{1}{\sqrt{1 - r^2}} - x_{n+1}\right)$$

$$= -\tilde{F}_v^{-2}\tilde{F}^{ij}_{\bar{v}}\tilde{\nabla}^{ij}(-x_{n+1}) + \left(\frac{1}{\sqrt{1 - r^2}} - x_{n+1}\right)\tilde{F}_v^{-2}\sum \tilde{F}^{ii}_{\bar{v}}$$

$$= \frac{1}{\sqrt{1 - r^2}} \tilde{F}_v^{-2}\sum \tilde{F}^{ii}_{\bar{v}},$$
here we have used $\nabla_{ij} x_{n+1} = x_{n+1} \delta_{ij}$. It is easy to see that to prove Lemma 8 we only need to show there exists some $B_2 > 0$ such that

\begin{equation}
\mathcal{L}(v - \bar{v}) > - \left( \bar{v} - \tilde{F}^{-1}(\bar{\Delta}_j) + \frac{B_2}{F^2_v} \sum_i \tilde{F}^\prime_{ii} \right),
\end{equation}

where $\bar{\Delta}_j = \nabla_{ij} \bar{v} - \bar{v} \delta_{ij}$.

In the rest of this proof, the value of $B_2$ may vary from line to line. Notice that inequality (2.9) is equivalent to

\begin{equation}
(v - \bar{v})_t - \tilde{F}^{-2} \tilde{F}^\prime_{ij} \nabla_{ij} (v - \bar{v}) + (v - \bar{v}) \tilde{F}^{-2} \sum_i \tilde{F}^\prime_{ii}
\end{equation}

\begin{equation}
> - \bar{v} - \tilde{F}^{-1}(\bar{\Delta}_j) - \frac{B_2}{F^2_v} \sum_i \tilde{F}^\prime_{ii},
\end{equation}

which can be simplified as

\begin{equation}
- \frac{(1 + \alpha)}{F_v} + \tilde{F}^{-2} \tilde{F}^\prime_{ii} \bar{\Delta}_i + \frac{B_2}{F^2_v} \sum_i \tilde{F}^\prime_{ii} > - \tilde{F}^{-1}(\bar{\Delta}_j).
\end{equation}

Since $F^{\frac{1}{\alpha}}$ is concave we have

\begin{equation}
\frac{1}{\alpha} F^{\frac{1}{\alpha}} \tilde{F}^\prime_{ij} \bar{\Delta}_j \geq \tilde{F}^{\frac{1}{\alpha}}(\bar{\Delta}_j) \quad \text{and} \quad \frac{1}{\alpha} F^{\frac{1}{\alpha}} - 1 \sum_i \tilde{F}^\prime_{ii} \geq \tilde{F}^{\frac{1}{\alpha}}(1, \ldots, 1).
\end{equation}

This implies

\begin{equation}
\text{l.h.s. of (2.10)} \geq - \frac{1 + \alpha}{F_v} + \tilde{F}^{-2} \left[ \frac{\alpha \tilde{F}^{\frac{1}{\alpha}}(\bar{\Delta}_j)}{F^{\frac{1}{\alpha}} - 1} \right] + \frac{\alpha B_2}{\tilde{F}^{\frac{1}{\alpha}} + 1},
\end{equation}

where $c = (C_n^k)^{1/k}$ is the combination number. One can see that (2.10) can be derived from the following inequality:

\begin{equation}
\alpha \tilde{F}^{\frac{1}{\alpha}}(\bar{\Delta}_j) + \frac{B_2 \alpha}{c} + \frac{\tilde{F}^{1 + \frac{1}{\alpha}}}{\tilde{F}(\bar{\Delta}_j)} \geq (1 + \alpha) \tilde{F}^{\frac{1}{\alpha}}.
\end{equation}

Recall (2.7), we have

\begin{equation}
\tilde{F}(\bar{\Delta}_j) = \frac{1}{\alpha} \tilde{F}^{\frac{1}{\alpha}}(w^* \sigma^* \tilde{u}_k \tilde{\gamma}_l) < [(1 + \alpha) \tilde{T}]^{\frac{1}{1 + \alpha} c_0^{-1}},
\end{equation}

where $c_0$ is given by assumption (2.1). When $\tilde{F}_v > [(1 + \alpha) \tilde{T}]^{\frac{1}{1 + \alpha} c_0^{-1}}(1 + \alpha)$, we have $\tilde{F}_v^{1 + \frac{1}{\alpha}} \geq (1 + \alpha) \tilde{F}_v^{\frac{1}{\alpha}}$ and (2.11) follows. When $\tilde{F}_v \leq [(1 + \alpha) \tilde{T}]^{\frac{1}{1 + \alpha} c_0^{-1}}(1 + \alpha)$, choose $B_2 \geq c(1 + \alpha)^{\frac{1}{1 + \alpha} [(1 + \alpha) \tilde{T}]^{\frac{1}{1 + \alpha} c_0^{-1}}}$, then (2.11) follows. Here $\tilde{T} = T + (1 + \alpha)^{-1}$. Therefore, we conclude that when $B_2 := B_2(n, k, c_0, T)$ is sufficiently large, (2.11) always holds. This completes the proof of this Lemma.

**Lemma 9.** Let $v$ be a solution of (2.6) and suppose $\tau_n$ is the interior unit normal vector field of $\partial U_r$. We have $|\nabla_{\tau_n} v| \leq C$ on $\partial U_r \times (0, T)$. Here $C$ depends on $M_0, r, T$. 


Proof. Denote the angular derivative $\mathcal{T} := \xi_\alpha \partial_n - \xi_n \partial_\alpha$, where $\partial_i := \frac{\partial}{\partial x_i}$ and $\alpha < n$. Let $\phi(\cdot, t) = u^*_\alpha(\cdot, t) - \tilde{u}^*(\cdot, t)$, then on $(\partial B_r \times [0, T]) \cup (B_r \times \{0\})$ we have $\phi \equiv 0$. Therefore, we have

$$\mathcal{T} \phi(x, t) = 0 \text{ for } (x, t) \in \partial B_r \times [0, T] \text{ and } \mathcal{T} \phi(x, 0) = 0 \text{ for } x \in B_r.$$ 

In the following we denote $\tilde{G}(w^*\gamma^*_ik(\alpha^*_r)_{kl}\gamma^*_lj) = -F_{-\alpha} \kappa^*(w^*\gamma^*_ik(\alpha^*_r)_{kl}\gamma^*_lj))$, then $u^*_\alpha$ satisfies

$$\frac{\partial u^*_\alpha}{\partial t} - \tilde{G}(w^*\gamma^*_ik(\alpha^*_r)_{kl}\gamma^*_lj)w^* = 0. \tag{2.12}$$

Notice that $\mathcal{T}$ is an angular derivative vector with respect to the origin of $B_r$. By Lemma 29 of [13], we get

$$\frac{\partial (\mathcal{T} u^*_\alpha)}{\partial t} - \tilde{G}^{ij}(w^*\gamma^*_ik(\mathcal{T} u^*_\alpha)_{kl}\gamma^*_lj)w^* = 0, \tag{2.13}$$

where $\tilde{G}^{ij} = \frac{\partial \tilde{G}}{\partial a_{ij}}$ for $a_{ij} = w^*\gamma^*_ik(\alpha^*_r)_{kl}\gamma^*_lj$.

Recall that $\tilde{u}^* = [(1 + \alpha)\tilde{t}]^{1+\alpha} u^*$ and $\tilde{u}^*$ is the Legendre transform of $u_0$. It is clear that $\mathcal{T} \tilde{u}^* = [(1 + \alpha)\tilde{t}]^{1+\alpha} \partial U^* \tilde{u}^*$. Applying Lemma 15 of [15], we know

$$\nabla_{ij} \left( w^* \right) - \frac{w^*}{w^*} \delta_{ij} = w^* \gamma^*_ik \alpha^*_r \gamma^*_lj.$$ 

Combining with (2.13) we obtain

$$(\mathcal{T} v)_t - \tilde{G}^{ij} \nabla_{ij}(\mathcal{T} v) + (\mathcal{T} v) \sum_i \tilde{G}^{ii} = 0,$$

where $\mathcal{T} v = \mathcal{T} \left( \frac{u^*}{w^*} \right) = \frac{\mathcal{T} u^*}{w^*}$, and $\tilde{G}^{ij} = F_{-\alpha}^{ij} 2 \tilde{F}_{ij}$. A straightforward calculation yields

$$[\mathcal{L} \mathcal{T} v] = \left( [(1 + \alpha)\tilde{t}]^{1+\alpha} \frac{\partial}{\partial \tilde{t}} \frac{\tilde{u}^*}{w^*} - \tilde{G}^{ij} \nabla_{ij} \left( \frac{\tilde{t}^*}{w^*} \right) \cdot [(1 + \alpha)\tilde{t}]^{1+\alpha} + [(1 + \alpha)\tilde{t}]^{1+\alpha} \frac{\tilde{u}^*}{w^*} \sum_i \tilde{G}^{ii} \right)$$

$$\leq C_4 \sum_i \tilde{G}^{ii},$$

where $C_4$ is some constant depending on $\tilde{u}^*, \mathcal{T}, r$.

Let $\tilde{\psi}(x, t) = \mathcal{T} v(x, t) - \mathcal{T} v(x, t)$, then we have

$$\tilde{\psi}(x, 0) = 0 \text{ for any } x \in U_r, \quad \tilde{\psi}(x, t) = 0 \text{ for any } (x, t) \in \partial U_r \times (0, T).$$

Moreover, we have $|\mathcal{L} \tilde{\psi}| \leq C_4 \sum_i \tilde{G}^{ii}$. Applying Lemma [8], we have, when $B > 0$ very large

$$\mathcal{L}(\tilde{\psi} - h) \leq 0 \text{ in } U_r \times (0, T).$$

It’s easy to see that $\tilde{\psi} \leq h$ on the parabolic boundary $(\partial U_r \times (0, T)) \cup (U_r \times \{0\})$. By the maximum principle, we get $h \geq \tilde{\psi}$ in $U_r \times (0, T)$. Therefore, $h_n > \tilde{\psi}_n$ on $\partial U_r \times (0, T)$, which yields $\nabla_{\alpha n} v \leq C_5$. Similarly, by considering $\mathcal{T} v - \mathcal{T} u$, we obtain $\nabla_{\alpha n} v \geq C_6$. \qed
Lemma 10. Let \( v \) be a solution of (2.6) and suppose \( \tau_n \) is the interior unit normal vector filed of \( \partial U_r \). Then we have \( |\nabla_{nn}v| \leq C \) on \( \partial U_r \times (0, T) \), where \( C \) is a positive constant depending on \( M_0, r, T \).

Proof. Recall that in the proof of Lemma 5 we have shown, on the boundary \( \partial U_r \times (0, T) \),

\[
F^\alpha = \left[ \frac{((1 + \alpha)\ell)^{\frac{k}{1 + \alpha}}}{u_0^*} \right],
\]

where \( F^\alpha = \left( \frac{\sigma_{n-k}}{\sigma_{n-k}} \right)^{\frac{1}{k}} \). This is equivalent to, on \( \partial U_r \times (0, T) \),

\[
\frac{\sigma_n}{\sigma_{n-k}} = \left[ (1 + \alpha)^{\frac{1}{1 + \alpha}} \left( \frac{\sqrt{1 - r^2}}{-u_0^*} \right)^{\frac{k}{\alpha}} \right].
\]

We will adapt the idea of [12] to prove this Lemma. Recall the formula (2.5) in [12], we have

\[
\sigma_k(\kappa[\Lambda_{ij}]) = \sigma_{k-1}(\kappa[\Lambda_{\alpha\beta}])\Lambda_{nn} + \sigma_k(\kappa[\Lambda_{\alpha\beta}]) - \sum_{\gamma=1}^{n-1} \sigma_{k-2}(\kappa[\Lambda_{\alpha\beta}\Lambda_{\gamma\gamma}])\Lambda_{\gamma\gamma}^2,
\]

where \( 1 \leq i, j \leq n \) and \( 1 \leq \alpha, \beta, \gamma \leq n - 1 \). We define

\[
A_{k-1} = \sigma_{k-1}(\kappa[\Lambda_{\alpha\beta}]) \quad \text{and} \quad B_k = \sigma_k(\kappa[\Lambda_{\alpha\beta}]) - \sum_{\gamma=1}^{n-1} \sigma_{k-2}(\kappa[\Lambda_{\alpha\beta}\Lambda_{\gamma\gamma}])\Lambda_{\gamma\gamma}^2.
\]

Therefore, we obtain on \( \partial U_r \times (0, T) \)

\[
(2.14) \quad \frac{\sigma_n}{\sigma_{n-k}} = \frac{A_{n-1}A_{nn} + B_n}{A_{n-k-1}A_{nn} + B_n} = \left[ (1 + \alpha)^{\frac{1}{1 + \alpha}} \left( \frac{\sqrt{1 - r^2}}{-u_0^*} \right)^{\frac{k}{\alpha}} \right].
\]

Now, we prove

\[
(2.15) \quad \frac{A_{n-1}}{A_{n-k-1}} \geq \frac{\sigma_n}{\sigma_{n-k}} = \frac{A_{n-1}A_{nn} + B_n}{A_{n-k-1}A_{nn} + B_n},
\]

which is equivalent to the following inequality

\[
\sigma_{n-1}(\kappa') \left[ \sigma_{n-k}(\kappa') - \sum_{\gamma=1}^{n-1} \sigma_{n-k-2}(\kappa'\gamma)\Lambda_{\gamma\gamma}^2 \right] \geq \sigma_{n-k-1}(\kappa') \left[ -\sum_{\gamma=1}^{n-1} \sigma_{n-2}(\kappa'\gamma)\Lambda_{\gamma\gamma}^2 \right],
\]

where \( \kappa' = \kappa'[\Lambda_{\alpha\beta}] \) are eigenvalues of \( (\Lambda_{\alpha\beta}) \). For any fixed \( \gamma \), we have

\[
\sigma_{n-k-1}(\kappa') \sigma_{n-2}(\kappa'\gamma) = [\sigma_{n-k-1}(\kappa'\gamma) + \sigma_{n-k-2}(\kappa'\gamma)\kappa]\sigma_{n-2}(\kappa'\gamma) > \sigma_{n-1}(\kappa') \sigma_{n-k-2}(\kappa'\gamma),
\]

which gives (2.15) directly.
In the rest of this proof, we will denote \( f = \left( \frac{\sigma_{n-1}(\kappa')}{\sigma_{n-k-1}(\kappa')} \right)^{\frac{1}{k'}} \). In view of (2.15), we get, on \( \partial U_r \times (0, T) \)
\[
f \geq [(1 + \alpha)\tilde{t}]^{\frac{1}{1+\alpha}} \left( \frac{\sqrt{1 - r^2}}{-u_0^*} \right)^{\frac{1}{\alpha}}.
\]
Now consider the following function defined on \( \partial U_r \times [0, T] \),
\[
d(x, t) := f - [(1 + \alpha)\tilde{t}]^{\frac{1}{1+\alpha}} \left( \frac{\sqrt{1 - r^2}}{-u_0^*} \right)^{\frac{1}{\alpha}}.
\]
By our assumption (2.1) on \( u_0 \), we derive
\[
\frac{\sigma_n}{\sigma_{n-k}}(w^*\gamma_{ik}(u_0^*(\xi))\kappa^*\gamma_{lj}) > \left( \frac{\sqrt{1 - r^2}}{-u_0^*(\xi)} \right)^{\frac{1}{\alpha}} \quad \text{for } \xi \in \partial B_r,
\]
which implies \( d(x, 0) \geq c > 0 \) on \( \partial U_r \times \{0\} \). Therefore, we may assume \( d(x, t) \) achieves its minimum at the point \( (y, t_0) \in \partial U_r \times (0, T) \). If not, we would have \( d \geq c \) on \( \partial U_r \times [0, T] \). Then in view of (2.13), we would derive that \( \Lambda_{nn} \) has an uniform upper bound on \( \partial U_r \times [0, T] \). The Lemma would follow easily.

Note that at any \( (x, t) \in \partial U_r \times (0, T) \), for \( \alpha, \beta < n \), we have
\[
\nabla_{\alpha\beta}(v - \underline{v}) = -\nabla_{\alpha}(v - \underline{v})\rho_{\alpha\beta},
\]
where \( \rho_{\alpha\beta} \) is the second fundamental form of \( \partial U_r \) with respect the \( \tau_n \). For a symmetric matrix \( r = (r_{\alpha\beta}) \) with eigenvalues \( \lambda_1, \cdots, \lambda_{n-1} \), let’s define \( G(r) = f(\lambda_1, \cdots, \lambda_{n-1}) = \left( \frac{\sigma_{n-1}(\lambda)}{\sigma_{n-k-1}(\lambda)} \right)^{\frac{1}{k'}} \),
\[
G^\alpha = \frac{\partial G}{\partial r_{\alpha\beta}}, \quad \text{and } G^\alpha_{\alpha\beta} = G^{\alpha\beta}\big|_{\Lambda_{\alpha\beta}(y, t_0)}.
\]
By the concavity of \( f \), we have
\[
G^\alpha_{\alpha\beta} \Lambda_{\alpha\beta}(x, t) \geq G(\Lambda_{\alpha\beta}(x, t)), \quad \text{and } G^\alpha_{\alpha\beta} \Delta_{\alpha\beta}(x, t) \geq G(\Delta_{\alpha\beta}(x, t)).
\]
Therefore, for all \( (x, t) \in \partial U_r \times [0, T] \) we have
\[
G^\alpha_{\alpha\beta} \Lambda_{\alpha\beta}(x, t) - [(1 + \alpha)\tilde{t}]^{\frac{1}{1+\alpha}} \left( \frac{\sqrt{1 - r^2}}{-u_0^*} \right)^{\frac{1}{\alpha}} \geq d(x, t) \geq d(y, t_0).
\]
When \( (x, t) \in \partial U_r \times [0, T] \), using the above inequity we get,
\[
G^\alpha_{\alpha\beta} \nabla_n (v - \underline{v})\rho_{\alpha\beta} = G^\alpha_{\alpha\beta} \nabla_{\alpha\beta} [w(x, t) - v(x, t)]
\]
\[
= G^\alpha_{\alpha\beta} \left[ \nabla_{\alpha\beta}w(x, t) - w(x, t)\delta_{\alpha\beta} \right] - G^\alpha_{\alpha\beta} [\nabla_{\alpha\beta}v(x, t) - v(x, t)\delta_{\alpha\beta}]
\]
\[
\leq G^\alpha_{\alpha\beta} \Delta_{\alpha\beta}(x, t) - d(y, t_0) - [(1 + \alpha)\tilde{t}]^{\frac{1}{1+\alpha}} \left( \frac{\sqrt{1 - r^2}}{-u_0^*} \right)^{\frac{1}{\alpha}}.
\]
Suppose \( U_{r\delta} := \{ x \in U_r \mid \frac{1}{\sqrt{1-r^2}} - x_{n+1} < \delta \} \) and \( \Gamma_{\theta} := \{ x \in U_r \mid \frac{1}{\sqrt{1-r^2}} - x_{n+1} = \theta \} \) for \( 0 \leq \theta \leq \delta \). Now we extend \( \tau_1, \tau_2, \cdots, \tau_n \) to \( U_{r\delta} \) such that it is still an orthonormal frame and the first \( n-1 \) vectors are the tangential vectors of \( \Gamma_{\theta} \). Therefore, we can extend \( \rho_{\alpha\beta} \) to be the second fundamental form of \( \Gamma_{\theta} \). Thus, it is clear that there exists \( c_1 > 0 \) such that

\[
G^a_0 \rho_{\alpha\beta}(x) \geq c_1 \text{ in } \bar{U}_{r\delta} \times [0, T].
\]

By (2.17), we get, on \( \partial U_r \times [0, T] \),

\[
\nabla_n (v-v) \leq \frac{G^a_0 \Delta_{\alpha\beta}(x,t) - d(y,t_0) - \lfloor (1+\alpha) \rfloor \tau^4 \left( \frac{\sqrt{1-r^2}}{u_0} \right)^{\frac{\alpha}{\beta}}}{G^a_0 \rho_{\alpha\beta}(x)}.
\]

We denote the right hand side of the above inequality by \( \psi(x,t) \). Then we define

\[
\phi := \nabla_n (v-v) - \psi - C \left( \frac{1}{\sqrt{1-r^2}} - x_{n+1} \right) \text{ on } \bar{U}_{r\delta} \times [0, T].
\]

It is obvious that \( \phi(y, t_0) = 0 \), and \( \phi \leq 0 \) on \( \partial U_r \times [0, T] \). We use the coordinate \( x_1, x_2, \cdots, x_n \) for the hyperboloid. Thus, \( U_r \) is the \( n-1 \)-dimensional ball \( B_{n-1} \subset \mathbb{H}^n \) defined by \( |x| \leq 1/\sqrt{1-r^2} \). For any \( x \in B_{n-1} \), we suppose \( p \) is the point on \( \partial B_{n-1} \) such that \( \text{dist}(x, \partial B_{n-1}) = \text{dist}(x, p) \). Here, \( \text{dist}(\cdot, \cdot) \) is the Euclidean distance function. We further let

\[
\tilde{\psi} = \psi - \nabla_n (v - v).
\]

Thus, we get

\[
-\tilde{\psi}(x, 0) \leq \tilde{\psi}(p, 0) - \tilde{\psi}(x, 0) \leq |D\tilde{\psi}(\cdot, 0)||x-p| \leq C_0 (r_0 - |x|),
\]

where \( r_0 = \frac{r}{\sqrt{1-r^2}} \) and \( C_0 > 0 \) is a constant depending on \( U_r, M_0 \). On the other hand, we have

\[
\frac{1}{\sqrt{1-r^2}} - x_{n+1} = \frac{(r_0 - |x|)(r_0 + |x|)}{\sqrt{1-r^2} + x_{n+1}} \geq \frac{r}{2}(r_0 - |x|).
\]

Therefore, we can choose a sufficiently large \( C > 0 \) such that on \( U_r \times \{0\}, \phi \leq 0 \); and on \( \Gamma_{\delta} \times [0, T] \), \( \phi < 0 \).

Differentiating the flow equation \( v_t + \tilde{\xi}^{-1}(v_{ij} - v\delta_{ij}) = 0 \) with respect to \( \tau_n \), we get

\[
v_{nt} - \tilde{\xi}^{-2} F^i_v (v_{ni} - v_n \delta_{ij}) = 0.
\]

This yields \( |\xi| \leq C_7 \tilde{\xi}^{-2} \sum_i \tilde{F}^i_v \), where \( C_7 \) depends on \( r, M_0, \) and \( T \). Therefore, if we choose sufficiently large \( B \) in the definition of \( h \), we conclude

\[
\phi \leq h \text{ in } \bar{U}_r \times [0, T],
\]

which implies \( \nabla_{nn} v(y, t_0) < C_8 \). Thus, there exists a constant \( C_9 > 0 \) such that \( d(x, t) \geq C_9 \) on \( \partial U_r \times [0, T] \), which in turn gives \( \nabla_{nn} v(x, t) < C_1 0 \) on \( \partial U_r \times [0, T] \). Thus we prove that \( \nabla_{nn} v \) is
bounded from above on the parabolic boundary. The lower bound for \( \nabla_{nn} v \) comes from the matrix \((\Lambda_{ij})\) is positive definite. This completes the proof of this Lemma. \( \square \)

The global \( C^2 \) estimates for \( u^*_r \)- solution of (2.3) follows from Lemma 20 in [17] directly. Therefore, we have proved the solvability of the approximate problem (2.3).

3. LOCAL ESTIMATES FOR \( u_r \)

In this section, we will establish local estimates for \( u_r \), the Legendre transform of \( u^*_r \)-solution of (2.3).

3.1. Local \( C^0 \) estimates for \( u^*_r \). By Lemma 4, we have

\[
\tilde{u}^* < u^*_r < \bar{u}^*, \quad \text{where} \quad \tilde{u}^* = [(1 + \alpha \tilde{t})^{\frac{1}{1+\alpha}} u^*], \quad \text{and} \quad \bar{u}^* = [(1 + \alpha \tilde{t})^{\frac{1}{1+\alpha}} u^*].
\]

By Lemma 13 in [15], we get

\[
\frac{x}{[(1 + \alpha \tilde{t})^{\frac{1}{1+\alpha}} u]} < u_r(x, t) < \frac{x}{[(1 + \alpha \tilde{t})^{\frac{1}{1+\alpha}} \bar{u}]},
\]

which is a local \( C^0 \) estimate for \( u_r \).

3.2. Local \( C^1 \) estimates. We introduce a new subsolution \( \tilde{u}_1 \) satisfying

\[
\sigma^k(\nabla \mathcal{M}_{\tilde{u}_1}(x)) = -10 \langle X_{\tilde{u}_1}, \nu_{\tilde{u}_1} \rangle
\]

and as \( |x| \to \infty \)

\[
\tilde{u}_1 \to |x| + \varphi \left( \frac{x}{|x|} \right).
\]

By the strong maximum principle we have, when \( x \in \mathbb{R}^n \)

\[
\tilde{u}_1(x) < \tilde{u}(x).
\]

We let

\[
\tilde{u}_1(x, t) = A(t) \tilde{u}_1 \left( \frac{x}{A(t)} \right), \quad \bar{u}(x, t) = A(t) \bar{u} \left( \frac{x}{A(t)} \right), \quad \text{and} \quad \tilde{u}(x, t) = A(t) \tilde{u}_0 \left( \frac{x}{A(t)} \right),
\]

where \( A(t) = [(1 + \alpha \tilde{t})^{\frac{1}{1+\alpha}}] \geq 1 \). Moreover, for any compact convex domain \( K \) and positive constant \( T \), let

\[
2\delta = \min_{K \times [0, T]} (\tilde{u} - \bar{u}_1).
\]

We define a spacelike function \( \Psi = \tilde{u}_1 + \delta \). Denote \( \Omega = \{(x, t) \in \mathbb{R}^n \times [0, T]; \Psi \leq \tilde{u}\} \). It is clear that \( K \times [0, T] \subset \Omega \). Since as \( |x| \to \infty \), \( \tilde{u}_1 - \tilde{u} \to 0 \), we know that \( \Omega \) is a compact set only depending on \( K \) and \( T \). Applying Lemma 5.1 of [5], if \( \Omega \subset \bigcup_{t \in [0, T]} \Omega_r(t) \), then we have the gradient estimate:

\[
\sup_{K \times [0, T]} \frac{1}{\sqrt{1 - |Du_r|^2}} \leq \frac{1}{\delta} \sup_{\Omega} \frac{\tilde{u} - \Psi}{\sqrt{1 - |D\Psi|^2}}.
\]
3.3. Local estimates for $F$. Due to the complication of the local $C^2$ estimates, in this subsection we need to establish the local estimates for $F$. More precisely, we want to bound $F$ from above and below in terms of $u = -\langle X, e_{n+1} \rangle$ and $v = -\langle \nu, e_{n+1} \rangle$. By Subsection 3.2, we know, this is equivalent to bounding $F$ from above and below by the height function $u$. For our convenience, we will denote $\Phi = F^{\alpha}$, where $F = \sigma_k^{\frac{1}{k}}$. A straightforward calculation yields the following Lemma.

**Lemma 11.** Under the flow (1.1) we have

\begin{align*}
(3.1) & \quad \mathfrak{L} u = (1 - \alpha) \Phi v, \\
(3.2) & \quad \mathfrak{L} v = - \sum_i \Phi^{ii} \kappa_i^2 v, \\
(3.3) & \quad \mathfrak{L} \Phi = - \sum_i \Phi^{ii} \kappa_i^2 \Phi, \\
\end{align*}

and

\begin{align*}
(3.4) & \quad \mathfrak{L} h_i^j = (\alpha - 1) \Phi \sum_k h_i^k h_k^j - h_i^j \sum_{k,l,s} \Phi^{kl} h_k^s h_l^i + \sum_{p,q,r,s} \Phi^{pq,rs} \nabla_i h_{pq} \nabla_j h_{rs}.
\end{align*}

Here, $\mathfrak{L} = \partial_t - \Phi^{ij} \nabla_i \nabla_j$.

Recall Lemma 5, since $\widetilde{F} = \Phi^{-1}$ we have $\Phi < C_2 v$. This leads to a local upper bound of $F$ directly.

In order to obtain the local lower bound of $F$, we consider the following test function,

$$\varphi = \gamma \log(c - t - u) - \log \Phi + Av,$$

where $\gamma, A$ are two positive constants to be determined later and $c > 0$ is an arbitrary positive constant. At the maximal value point $(x_0, t_0)$ of $\varphi$, we have

\begin{align*}
(3.5) & \quad 0 = \varphi_i = \gamma \frac{-u_i}{c - t - u} - \frac{\Phi_i}{\Phi} + Av_i, \\
\end{align*}

and

$$0 \geq \varphi_{ii} = \gamma \frac{-u_{ii}}{c - t - u} - \gamma \frac{u_i^2}{(c - t - u)^2} - \frac{\Phi_{ii}}{\Phi} + \frac{\Phi^2}{\Phi^2} + Av_{ii}.$$ 

Therefore, we get

\begin{align*}
(3.6) & \quad \mathfrak{L}(\varphi) = \gamma \left( \frac{-1 - \mathfrak{L} u}{c - t - u} + \frac{\Phi^{ii} u_i^2}{(c - t - u)^2} - \frac{\mathfrak{L} \Phi}{\Phi} - \Phi^{ii} \frac{\Phi^2}{\Phi^2} + Alv \right) \\
& \quad = \frac{-1 + (\alpha - 1) \Phi v}{c - t - u} + \frac{\Phi^{ii} u_i^2}{(c - t - u)^2} - \frac{\Phi^{ii} \Phi^2}{\Phi^2} - (Av - 1) \Phi^{ii} \kappa_i^2.
\end{align*}

By (3.5), we have

$$\gamma \frac{\Phi^{ii} u_i^2}{(c - t - u)^2} \leq \frac{2 \Phi^{ii} \Phi^2}{\gamma \Phi^2} + \frac{2A^2}{\gamma} \Phi^{ii} \kappa_i^2 u_i^2 \leq \frac{2 \Phi^{ii} \Phi^2}{\gamma \Phi^2} + \frac{2A^2 v^2}{\gamma} \Phi^{ii} \kappa_i^2.$$
where we have used $\sum_i u_i^2 = v^2 - 1 < v^2$. Combining with (3.6), we obtain at $(x_0, t_0)$

$$0 \leq \gamma \frac{-1 + (\alpha - 1) \Phi v}{c - t - u} - \left( Av - 1 - \frac{2A^2v^2}{\gamma} \right) \Phi^{ii} \kappa_i^2 - \left( 1 - \frac{2}{\gamma} \right) \frac{\Phi^{ii} \Phi^2}{\Phi^2}.$$

When $A = 2$ and $\gamma$ is chosen so large that

$$\gamma \geq 4 + 2A^2v^2,$$

we get

$$(\alpha - 1) \Phi v \geq 1.$$ This leads to the desired estimate. We conclude

**Lemma 12.** Let $u_r^*$ be the solution of (2.3) and $u_r$ be the Legendre transform of $u_r^*$. For any $c > 0$, denote $K := \{(x, t) \mid u_r(x, t) + t \leq c\}$ and $V_0 := \max_{(x, t) \in K} v$. Then we have

$$\left( \frac{c - t - u}{c} \right)^{\gamma} e^{2(v - V_0)} \frac{1}{(\alpha - 1)V_0} \leq \Phi < C_2 V_0,$$

where $C_2 = C_2(u_0^*)$ is determined by Lemma 5 and $\gamma = 4 + 8V_0^2$. We note that here $c$ is always chosen such that $K \subset \bigcup_{t \in [0, \infty)} (Du_r^*(B_r, t) \times \{t\})$.

### 3.4. Local $C^2$ estimates

In this subsection, we will establish the local $C^2$ estimate for $u_r$.

**Lemma 13.** Let $u_r^*$ be the solution of (2.3) and $u_r$ be the Legendre transform of $u_r^*$. Denote $\Omega_r(t) := Du_r^*(B_r, t)$. For any given $c > 0$, let $r_c \in (0, 1)$ such that when $r > r_c$, $u_r(\cdot, t) |_{\partial \Omega_r(t)} > c$ for all $t \in [0, \infty)$. Then for $r > r_c$ we have

$$(c - u_r)^m \log \kappa_{\max}(x, t) \leq C,$$

where $\kappa_{\max}(x, t)$ is the largest principal curvature of $\mathcal{M}_{u_r}$ at $(x, t)$, $m$ is a large constant only depending on $k$, and $C := C(\Phi, c) > 0$ is independent of $r$.

**Proof.** In this proof, we will drop the subscript $r$ and denote $u_r$ by $u$. Consider

$$\varphi = \frac{(c - u)^m \log P_m}{1 - \frac{m \Phi}{M}},$$

where $m$ is some positive integer to be determined later, $M \geq 2m \sup_{u \leq c} \Phi$, and

$$P_m = \sum_i \kappa_i^m.$$ Then we get

$$\log \varphi = m \log(c - u) + \log \log P_m - \log \left( 1 - \frac{m \Phi}{M} \right).$$
We suppose \( \varphi \) achieves its maximum value at \((x_0, t_0)\). We may choose a local orthonormal frame \( \{\tau_1, \cdots, \tau_n\} \) such that at \((x_0, t_0)\), \( h_{ij} = \kappa_k \delta_{ij} \) and \( \kappa_1 \geq \cdots \geq \kappa_n \). At the maximal value point \((x_0, t_0)\), differentiating \( \log \varphi \) twice we get

\[
0 = \frac{\varphi_i}{m \varphi} = -\frac{u_i}{c - u} + \frac{1}{\log P_m} \frac{\sum_j \kappa_j^{m-1} h_{jji}}{P_m} + \frac{\Phi_i}{M - m \Phi},
\]

and

\[
0 \geq \frac{\varphi_{ii}}{m \varphi} = -\frac{u_{ii}}{c - u} - \frac{u_i^2}{(c - u)^2} - \frac{m}{(\log P_m)^2} \left( \frac{\sum_j \kappa_j^{m-1} h_{jji}}{P_m} \right)^2 + \frac{1}{P_m \log P_m} \left[ \sum_j \kappa_j^{m-1} h_{jii} + (m - 1) \sum_j \kappa_j^{m-2} h_{jji} + \sum_{p \neq q} \kappa_p^{m-1} - \kappa_q^{m-1} \right] \frac{h_{pqi}^2}{P_m}
\]

\[
+ \frac{1}{P_m \log P_m} \left[ \sum_j \kappa_j^{m-1} h_{jji} \right]^2 + \frac{\Phi_{ii}}{M - m \Phi} + \frac{m(\Phi_i)^2}{(M - m \Phi)^2}.
\]

Therefore, at \((x_0, t_0)\) we obtain

\[
0 \leq \frac{\Delta \varphi}{m \varphi} = -\frac{\Delta u}{c - u} + \frac{1}{\log P_m} \frac{\sum_j \kappa_j^{m-1} \mathcal{L} \kappa_j}{P_m}
\]

\[
+ \frac{\Delta \Phi}{M - m \Phi} + \frac{\Phi_{ii} u_i^2}{(c - u)^2} + \frac{m \Phi_{ii}}{(\log P_m)^2} \left( \frac{\sum_j \kappa_j^{m-1} h_{jji}}{P_m} \right)^2
\]

\[
- \frac{\Phi_{ii}}{(\log P_m) P_m} \left[ \sum_j \kappa_j^{m-1} h_{jii} + (m - 1) \sum_j \kappa_j^{m-2} h_{jji} + \sum_{p \neq q} \kappa_p^{m-1} - \kappa_q^{m-1} \right] \frac{h_{pqi}^2}{P_m}
\]

\[
+ \frac{m \Phi_{ii}}{\log P_m} \left( \frac{\sum_j \kappa_j^{m-1} h_{jji}}{P_m} \right)^2 - \frac{m \Phi_{ii} (\Phi_i)^2}{(M - m \Phi)^2}.
\]

For our convenience, in below, we will denote \( \beta = \frac{\alpha_k}{\kappa} \), then \( \Phi = \alpha_k^\beta \). Plugging

\[
\Phi_{pq} = \beta \sigma_k^{\beta - 1} \sigma_k^{pq} \quad \text{and} \quad \Phi_{pq,rs} = \beta \sigma_k^{\beta - 1} \sigma_k^{pq,rs} + \beta(\beta - 1) \sigma_k^{\beta - 2} \sigma_k^{pq} \sigma_k^{rs}
\]
into (3.9), we get

$$0 \leq \frac{(\beta k - 1)\Phi v}{c - u} + \frac{1}{(\log P_m)P_m} \sum_j \kappa_j^{m-1} \left[ (\beta k - 1)\Phi \kappa_j^2 - \kappa_j \Phi ii \kappa_i^2 \right]$$

$$+ \beta \sigma_k^{\beta - 1} \sigma_k \Phi_{pq} \sigma_k^{s} \Phi_{pq} \sigma_k^{r} + \beta (\beta - 1)\sigma_k^{\beta - 2} (\sigma_k^2)$$

$$- \frac{\Phi \Phi ii \kappa_i^2}{M - m \Phi} + \frac{\beta \sigma_k^{\beta - 1} \sigma_k^{ii} u^2}{(c - u)^2} + \frac{m \beta \sigma_k^{\beta - 1} \sigma_k^{ii}}{(\log P_m)^2} \left( \frac{\sum_j \kappa_j^{m-1} h_{jjj}^2}{P_m} \right)^2 \right)$$

$$- \frac{\Phi \Phi ii \kappa_i^2}{M - m \Phi} + \frac{\beta \sigma_k^{\beta - 1} \sigma_k^{ii} u^2}{(c - u)^2} + \frac{m \beta \sigma_k^{\beta - 1} \sigma_k^{ii}}{(\log P_m)^2} \left( \frac{\sum_j \kappa_j^{m-1} h_{jjj}^2}{P_m} \right)^2$$

$$+ \frac{m \beta \sigma_k^{\beta - 1} \sigma_k^{ii} (\Phi_i)^2}{(M - m \Phi)^2}$$

For any fixed index $1 \leq i \leq n$, we denote

$$A_i = \frac{\kappa_i^{m-1}}{P_m} \left[ K(\sigma_k)_i^2 - \sum_{p,q} \sigma_k^{pp} \Phi_{pp} \sigma_k^{qq} \Phi_{qq} \right], \quad B_i = \frac{2}{P_m} \sum_j \kappa_j^{m-1} \sigma_k^{ij} \Phi_{jj} h_{jjj}^2,$$

$$C_i = \frac{m - 1}{P_m} \sigma_k^{ii} \sum_j \kappa_j^{m-2} h_{jjj}^2, \quad D_i = \frac{2 \sigma_k^{ij}}{P_m} \sum_{j \neq i} \kappa_j^{m-1} \sigma_k^{ij} \Phi_{jj} h_{jjj}^2,$$

and

$$E_i = \frac{m}{P_m} \sigma_k^{ii} \left( \sum_j \kappa_j^{m-1} h_{jjj}^2 \right)^2,$$

where $K > 0$ depends on $\min_{u \leq c} \sigma_k$. Then (3.10) becomes

$$0 \leq \frac{(\beta k - 1)\Phi v}{c - u} + \frac{n(\beta k - 1)\Phi K_1}{\log P_m} - \frac{\beta \sigma_k^{\beta - 1} \sigma_k^{ii} \kappa_i^2}{\log P_m}$$

$$- \frac{\beta \sigma_k^{\beta - 1}}{\log P_m} \sum_i \left[ A_i + B_i + C_i + D_i - \left( 1 + \frac{1}{\log P_m} \right) E_i \right]$$

$$+ \frac{\beta \sigma_k^{\beta - 1}}{\log P_m} \sum_i \frac{\kappa_i^{m-1}}{P_m} K(\sigma_k)_i^2 + \frac{\beta (\beta - 1)\sigma_k^{\beta - 2}}{(\log P_m)P_m} \sum_j \kappa_j^{m-1} (\sigma_k^2)_j$$

$$- \frac{\Phi \Phi ii \kappa_i^2}{M - m \Phi} + \frac{\beta \sigma_k^{\beta - 1} \sigma_k^{ii} u^2}{(c - u)^2} - \frac{m \beta \sigma_k^{\beta - 1} \sigma_k^{ii} (\Phi_i)^2}{(M - m \Phi)^2}.$$
Moreover, it is easy to see that \( \sigma_k^{11,1} \geq \eta_0 \sigma_k \) for some \( \eta_0 = \eta_0(n,k) \), which implies \( \sum \sigma_k^{ii} \geq \eta_0 \sigma_k \). If \( \log \kappa_1 > c_1 M^2 \) for \( c_1 = c_1(\sigma_k, \eta_0, m, K, \beta) \), then we have

\[
(3.12) \quad \frac{m}{2M^2} \sigma_k^{ii} \left[ \alpha \sigma_k^{\beta-1}(\sigma_k) \right]^2 \geq \frac{\kappa_i^{m-1}}{P_m \log P_m} K(\sigma_k)^2 + \frac{|\beta - 1| \sigma_k^{m-1}(\sigma_k)^2}{(\log P_m) P_m}. 
\]

Note by \( (3.7) \) we obtain

\[
(3.13) \quad \left( \frac{u_i}{c - u} \right) = \left( \frac{1}{\log P_m} \frac{\sum \kappa_j^{m-1} h_{jj} i}{P_m} + \frac{\Phi_i}{M - m \Phi} \right)^2 \leq \frac{2}{(\log P_m)^2} \left( \frac{\sum \kappa_j^{m-1} h_{jj} i}{P_m} \right)^2 + \frac{2(\Phi_i)^2}{(M - m \Phi)^2}. 
\]

Combining \( (3.12) \) and \( (3.13) \) with \( (3.11) \), we get

\[
(3.14) \quad 0 \leq \frac{(\alpha - 1) \Phi \nu}{c - u} + \frac{n(\alpha - 1) \Phi \kappa_1}{\log P_m} - \frac{\beta \sigma_k^{\beta-1} \sigma_k^{ii} \kappa_i^2}{\log P_m} 
- \frac{\beta \sigma_k^{\beta-1}}{\log P_m} \sum_i \left[ A_i + B_i + C_i + D_i - \left( 1 + \frac{1}{\log P_m} + \frac{2}{m \log P_m} \right) E_i \right] 
- \frac{\Phi \Phi^{ii} \kappa_i^2}{M - m \Phi} - \frac{m}{2 - 2 \beta} \frac{\sigma_k^{\beta-1} \sigma_k^{ii} (\Phi_i)^2}{(M - m \Phi)^2}.
\]

If we assume \( m, K \) and \( \kappa_1 \) are all sufficiently large, the Lemma 8 and Lemma 9 of \([10]\) gives

\[
\sum_i \left[ A_i + B_i + C_i + D_i - \left( 1 + \frac{1}{\log P_m} + \frac{2}{m \log P_m} \right) E_i \right] \geq 0.
\]

It’s clear that when \( \log \kappa_1 > CM \) for some \( C = C(n, k, \sigma_k, m, \beta) \), we have

\[
\frac{\Phi \Phi^{ii} \kappa_i^2}{2M^2} > \frac{n(\alpha - 1) \Phi \kappa_1}{\log P_m}.
\]

Thus, \( (3.14) \) yields

\[
0 \leq \frac{(\alpha - 1) \Phi \nu}{c - u} - \frac{\eta_0 \beta \Phi \kappa_1}{2(M - m \Phi)}
\]

which gives the desired estimate. \( \square \)

4. Convergence

In Section 2 we have shown there exists a solution to the initial value problem \( (2.3) \). Now, denote

\[
(4.1) \quad \tilde{u}_r(x, t) = \frac{u_r^*(x, t)}{A(t)}, \text{ where } A(t) = \left[ (1 + \alpha) \bar{t} \right]^{\frac{1}{1 + \alpha}} = \left[ (1 + \alpha) t + 1 \right]^{\frac{1}{1 + \alpha}}.
\]
Let \( \tau = \int_0^t [(1 + \alpha)s + 1]^{-1} ds \), then \( \tilde{u}_r^* \) satisfies
\[
\begin{aligned}
(\tilde{u}_r^*)_t &= -F_*^{-\alpha}(w^* \gamma_{ik}^*(\tilde{u}_r^*)_{kl} \gamma_{lj}^*)w^* - \tilde{u}_r^* & \text{in } B_r \times (0, T], \\
\tilde{u}_r^*(\cdot, t) &= u_0^* & \text{on } \partial B_r \times [0, T], \\
\tilde{u}_r^*(\cdot, 0) &= u_0^* & \text{on } B_r \times \{0\}.
\end{aligned}
\]
(4.2)

Notice that if \( X_r = (x, u_r(x, t)) \) for \((x, t) \in Du_r^*(B_r, t) \times \{t\}\), is the position vector for the graph \( u_r \) which is the Legendre transform of \( u_r^* \), then \( \tilde{X}_r = \frac{X_r}{A(t)} = \left(A(t)x, \frac{1}{A(t)} u_r(A(t)x, t)\right) \), where \((x, t) \in \frac{1}{A(t)} Du_r^*(B_r, t) \times \{t\}\), is the position vector for the graph of the Legendre transform of \( \tilde{u}_r^* \).

In the following we will prove two Lemmas.

**Lemma 14.** Let \( \tilde{u}_r^* \) be defined as in (4.1), then we have \( \tilde{u}_r^*(\cdot, t) \rightarrow u_r^{\infty*}(\cdot) \) uniformly in \( B_r \) as \( r \rightarrow \infty \). Here \( u_r^{\infty*} \) satisfies
\[
\begin{aligned}
F_*^{-\alpha}(w^* \gamma_{ik}^*(u_r^{\infty*})_{kl} \gamma_{lj}^*)w^* &= -u_r^{\infty*} & \text{in } B_r, \\
u_r^{\infty*} &= u_0^* & \text{on } \partial B_r.
\end{aligned}
\]
(4.3)

The proof of this Lemma will be given in the Subsection 4.1.

**Lemma 15.** Let \( u_r \) be the Legendre transform of \( u_r^* \). Then \( \frac{1}{A(t)} u_r(A(t)x, t) \rightarrow \frac{1}{A(t)} u(A(t)x, t) \) as \( r \rightarrow 1 \) uniformly in any compact subset of \( \mathbb{R}^n \times [0, \infty) \).

The proof of this Lemma will be given in the Subsection 4.1.

In view of Section 3 of [18] we can see that as \( r \rightarrow 1 \), \( u_r^{\infty*} \) which is the Legendre transform of \( u_r^{\infty*} \) converges to \( u^\infty \) uniformly on any compact set \( K \subset \mathbb{R}^n \), and \( u^\infty \) satisfies
\[
\begin{aligned}
\sigma^\infty_k(\kappa [M_{u^\infty}]) &= \langle X_{u^\infty}, \nu_{u^\infty} \rangle \\
u^\infty - |x| &\rightarrow \varphi \left( \frac{x}{|x|} \right).
\end{aligned}
\]

Combining this fact with Lemma 14 and Lemma 15 we conclude

**Corollary 16.** Let \( u^* \) be the solution of the initial value problem (2.2) and \( u \) be the Legendre transform of \( u^* \). Then for any sequence \( \{t_j\} \rightarrow \infty \) there exists a subsequence \( \{t_{jk}\} \rightarrow \infty \) such that
\[
\frac{1}{A(t_{jk})} u(A(t_{jk})x, t_{jk}) \rightarrow u^\infty(x)
\]
uniformly in any compact set \( K \subset \mathbb{R}^n \). Moreover, \( u^\infty \) satisfies
\[
\begin{aligned}
\sigma^\infty_k(\kappa [M_{u^\infty}]) &= \langle X_{u^\infty}, \nu_{u^\infty} \rangle \\
u^\infty - |x| &\rightarrow \varphi \left( \frac{x}{|x|} \right).
\end{aligned}
\]
(4.4)

4.1. **Proof of Lemma 14 and Lemma 15**
4.1.1. $C^0$ estimates for $\tilde{u}_r^*$. By our assumption (♣) on the initial hypersurface we have

$$\sigma_k (\kappa [\mathcal{M}_{u_0}]) < -\frac{u_0^*}{w^*} = - \langle X_{u_0}, \nu_{u_0} \rangle,$$

which yields

$$F_*^{-\alpha} (w^* \gamma_{ik}^* (u_0^*)_{kl} \gamma_{lj}^*) w^* + u_0^* < 0.$$

Therefore, $u_0^*$ is a subsolution of (4.2). It’s clear that $u^*$ constructed in Subsection 2.1 is a supersolution of (4.2). We conclude that

$$u_0^* \leq \tilde{u}_r^* \leq u_*^*.$$

Here, the $C^0$ estimate of $\tilde{u}_r^*$ is independent of $r$ and $t$.

4.1.2. $C^1$ estimate for $\tilde{u}_r^*$. Consider

$$F_* (w^* \gamma_{ik}^* \tilde{u}_r^* \gamma_{lj}^*) = \left( \frac{-u_*^*}{w^*} \right)^{\frac{1}{\alpha}} \text{ in } B_r,$$

$$u_* = u_0^* \text{ on } \partial B_r.$$

By Section 3 of [18], we know there exist a solution $\hat{u}_r^*$ of (4.5). In view of the standard maximum principle we have

$$u_0^* \leq \tilde{u}_r^* \leq \hat{u}_r^* = u_0^* \text{ on } \partial B_r \times [0, \infty).$$

This gives $|D\tilde{u}_r^*| \leq C$, for some $C > 0$ independent of $t$.

4.1.3. Bounds for $F_* (w^* \gamma_{ik}^* (\tilde{u}_r^*)_{kl} \gamma_{lj}^*) := \tilde{F}_*$. Let’s denote $\tilde{H} := -\tilde{F}_*^{-\alpha} w^* - \tilde{u}_r^*$, a straightforward calculation yields

$$L \tilde{H} := -\tilde{H},$$

where $L := \frac{\partial}{\partial \tau} - \alpha (w^*)^2 \tilde{F}_*^{-\alpha - 1} \tilde{F}_*^{ij} \gamma_{ik}^* \gamma_{lj}^* \partial_{kl}$. By the assumption (♣) on the initial surface we know at $\tau = 0$, $\tilde{H} \geq 0$. Moreover, applying the standard short time existence Theorem, we get on $\partial B_r \times (0, \infty)$, $\tilde{H} = 0$. Therefore, we conclude that $\tilde{H} \geq 0$ in $\tilde{B}_r \times [0, \infty)$, which yields

$$\tilde{F}_* \geq \left( \frac{-u_*^*}{w^*} \right)^{\frac{1}{\alpha}} \geq C_1,$$

where $C_1$ is independent of $t$.

On the other hand, recall Lemma [5] we know

$$F_*^\alpha < \frac{A(T)^\alpha}{C_3} \text{ on } \tilde{U}_r \times [0, T].$$

In particular, for any $T > 0$ we get

$$F_*^\alpha (w^* \gamma_{ik}^* (\tilde{u}_r^*(\cdot, T))_{kl} \gamma_{lj}^*) < \frac{A(T)^\alpha}{C_3},$$

where $C_3$ depends on $\mathcal{M}_0$ and $r$. This gives

$$F_*^\alpha (w^* \gamma_{ik}^* (\tilde{u}_r^*)_{kl} \gamma_{lj}^*) A(t)^\alpha < \frac{A(t)^\alpha}{C_3}.$$


which is equivalent to
\[ \tilde{F}^\alpha < \frac{1}{C_3}. \]
Here, note that \( C_3 \) is independent of \( t \).

4.1.4. \( C^2 \) estimates for \( \tilde{u}_r^\ast \). Now let \( \tilde{v}_r = \frac{\tilde{u}_r^\ast}{w^\ast} \) then \( \tilde{v}_r \) satisfies
\[
\begin{cases}
(\tilde{v}_r)_\tau = -\tilde{F}^{-\alpha}_\ast \left( \tilde{\Lambda}_{ij} \right) - \tilde{v}_r & \text{in } U_r \times (0, \infty), \\
\tilde{v}_r = v^\ast_0 = \frac{u^\ast_0}{w^\ast} & \text{on } \partial U_r \times [0, \infty), \\
\tilde{v}_r = v^\ast_0 & \text{on } U_r \times \{0\},
\end{cases}
\]
where \( \tilde{\Lambda}_{ij} = \tilde{\nabla}_{ij} \tilde{v}_r - \tilde{v}_r \delta_{ij} \). We will denote \( \tilde{F}^{-1} = F^{-\alpha}_\ast(\tilde{\Lambda}_{ij}) \). For any smooth function \( \phi \), we define
\[ L\phi := \phi_\tau - \tilde{F}^{-2}_\ast \tilde{F}^{ij}_\ast \nabla_{ij} \phi + \left( \tilde{F}_\ast^{-2} \sum_i \tilde{F}^{ii}_\ast + 1 \right) \phi. \]
Notice that we have proved the \( C^0, C^1 \) estimates of \( \tilde{u}_r^\ast \) are independent of \( t \). This implies the \( C^0, C^1 \) estimates of \( \tilde{v}_r \) are independent of \( t \). Moreover, we also know the upper and lower bounds of \( \tilde{F}^{-1} \) are independent of \( t \). By a small modification of the argument in Subsection 2.4, we obtain a \( C^2 \) boundary estimate of \( \tilde{v}_r \) that is independent of \( t \). The global \( C^2 \) estimate for \( \tilde{v}_r \) follows from a small modification of the proof of Lemma 20 in [17], and it is not hard to see that this estimate is also independent of \( t \).

4.1.5. Proof of Lemma 14. By these uniform estimates of \( \tilde{u}_r^\ast \), we conclude
\[ \lim_{\tau \to \infty} \tilde{u}_r^\ast(\xi, \tau) = u^\infty_r(\xi) \]
uniformly in \( B_r \). It’s clear that
\[ \tilde{u}_r^\ast(x, \tau) - \tilde{u}_r^\ast(x, 0) = \int_0^\tau \tilde{H} ds. \]
Then the uniform \( C^0 \) bound for \( \tilde{u}_r^\ast \) implies \( \int_0^\infty \tilde{H} ds < \infty \). This yields as \( \tau \to \infty, \tilde{H} \to 0 \). Therefore, \( u^\infty_r \) satisfies
\[
\begin{cases}
F^{-\alpha}_\ast (w^\ast \gamma^\ast_{ik}(u_r^\infty)_{kl} \gamma^\ast_{lj}) w^\ast = -u^\infty_r & \text{in } B_r, \\
u^\infty_r = u^\ast_0 & \text{on } \partial B_r.
\end{cases}
\]
\]
Choosing $\beta > 0$ strictly convex, and therefore, we can see that $\hat{u}$ when $r > r_K$.

Proof of Lemma 15.

4.1.6. We want to show that for any $K \subset \mathbb{R}^n$, there exists $r_K > 0$, such that when $r > r_K$, \( \frac{1}{A(t)} u_r(A(t)x,t) \) is defined in $K$ for any $t > 0$. We denote $\hat{\Omega}(r,t) = D\hat{u}_r^*(B_r,t)$ and $\hat{\Omega}(r) = D\hat{u}_r^*(B_r)$, where $\hat{u}_r^*$ is the solution of (4.5). In the following, we only need to show if $r > r_K$ then $K \subset \hat{\Omega}(r,t)$ for any $t > 0$. It’s clear that

$$\hat{u}_r|_{\partial\hat{\Omega}(r,t)} = (\xi \cdot D\hat{u}_r^* - \hat{u}_r)\big|_{\partial B_r} \geq (\xi \cdot D\hat{u}_r^* - \hat{u}_r)\big|_{\partial B_r} \geq \hat{u}_r|_{\partial\hat{\Omega}(r)},$$

where $\hat{u}_r$ is the Legendre transform of $\hat{u}_r^*$. Recall [13] we know when $r \to 1$, $\hat{u}_r|_{\partial\hat{\Omega}(r)} \to +\infty$, which yields as $r \to 1$, $\hat{u}_r|_{\partial\hat{\Omega}(r)} \to +\infty$. By virtue of Subsubsection 4.1.1 we know $\hat{u}_r > c$ for some constant $c$ independent of $r$ and $t$. Since $|D\hat{u}_r| < 1$, we get $\hat{u}_r$ is uniformly bounded from above in $K$. We conclude that for any compact set $K \subset \mathbb{R}^n$, there exists $r_K > 0$ such that when $r > r_K$, $K \subset \hat{\Omega}(r,t)$ for any $t > 0$.

Combining the discussion above with estimates obtained in Section 3, it is easy to see that $u_r$ convergence to $u$ on any compact subset $K \times [a,b] \subset \mathbb{R}^n \times [0,\infty)$, $0 \leq a < b$. Therefore, we prove Lemma [15].

APPENDIX

Proof of the equivalence of (4.6) and (2.1). When $C > 1$, we consider $\hat{u}_0(x) = \beta u_0 \left( \frac{x}{\beta} \right)$, where $\beta$ is some undetermined positive constant. A straightforward calculation yields

$$F^\alpha(\kappa[\mathcal{M}\hat{u}_0]) = \beta^{-\alpha} F^\alpha(\kappa[\mathcal{M}u_0]),$$

and

$$-\langle X_{\hat{u}_0}, \nu_{\hat{u}_0} \rangle = \frac{-x \cdot Du_0 \left( \frac{x}{\beta} \right) + \beta u_0 \left( \frac{x}{\beta} \right)}{\sqrt{1 - |Du_0 \left( \frac{x}{\beta} \right)|^2}} = -\beta \langle X_{u_0}, \nu_{u_0} \rangle.$$ 

Therefore, we can see that $\hat{u}_0$ satisfies

$$F^\alpha(\kappa[\mathcal{M}\hat{u}_0]) < C \beta^{-\alpha-1}(-\langle X_{u_0}, \nu_{u_0} \rangle).$$

Choosing $\beta > 0$ sufficiently large, then $\hat{u}_0$ satisfies (2.1). It is easy to see that $\hat{u}_0$ is also spacelike, strictly convex, and

$$\hat{u}_0(x) - |x| \to \beta \varphi \left( \frac{x}{|x|} \right), \text{ as } |x| \to \infty.$$ 

Applying Theorem [2] we know there exists $\hat{u}(x,t)$ such that

$$\begin{cases} \hat{u}_t = F^\alpha w \text{ in } \mathbb{R}^n \times (0,\infty), \\ \hat{u}(x,0) = \hat{u}_0(x) \text{ in } \mathbb{R}^n. \end{cases}$$
Moreover, the rescaled flow \( (A(t,x), \frac{\hat{u}(A(t)x,t)}{A(t)} ) \) converges to a self-expander \( \mathcal{M}_{\hat{u}^\infty} := \{ (x, \hat{u}^\infty(x)) \mid x \in \mathbb{R}^n \} \) with
\[
\hat{u}^\infty(x) - |x| \to \beta \varphi \left( \frac{x}{|x|} \right), \text{ as } |x| \to \infty.
\]

Now, let \( u(x,t) = \frac{1}{\beta} \hat{u}(\beta x, \beta^{\alpha+1} t) \). One can verify that \( u \) satisfies
\[
\begin{cases}
  u_t = F^\alpha u & \text{in } \mathbb{R}^n \times (0, \infty), \\
  u(x,0) = u_0(x) & \text{in } \mathbb{R}^n.
\end{cases}
\]

Moreover, the rescaled flow \( (A(t)x, \frac{u(A(t)x,t)}{A(t)} ) \) converges to the self-expander \( \mathcal{M}_{u^\infty} := \{ (x, u^\infty(x)) \mid x \in \mathbb{R}^n \} \) with
\[
u^\infty(x) - |x| \to \varphi \left( \frac{x}{|x|} \right), \text{ as } |x| \to \infty.
\]
Here \( u^\infty(x) = \frac{1}{\beta} \hat{u}^\infty(\beta x) \). \( \square \)

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