Formation of Quantum Turbulence from Dark Solitons in Atomic Bose-Einstein Condensates

Takuya Kusumura\(^1\), Makoto Tsubota\(^{1,2}\), Hiromitsu Takeuchi\(^3\)

\(^1\)Department of Physics, Osaka City University, Sumiyoshi-Ku, Osaka 558-8585, Japan
\(^2\)The OCU Advanced Research Institute for Natural Science and Technology (OCARINA), Osaka City University, Sumiyoshi-Ku, Osaka 558-8585, Japan
\(^3\)Graduate school of Integrated Arts and Sciences, Hiroshima University, Kagamiyama 1-7-1, Higashi-Hiroshima, 739-8511 Japan

E-mail: kusumura@sci.osaka-cu.ac.jp

Abstract. We theoretically propose a new method of making quantum turbulence from many dark solitons in atomic Bose-Einstein condensates. We solve numerically the two-dimensional Gross-Pitaevskii equation. We set initially many solitons so that they can form a square grid. A dark soliton is known to be stable in one-dimensional systems, but unstable in two- or three-dimensional systems and decay to vortices. Our simulation shows that these solitons decay to a lot of vortices which move around in the system and eventually lead to two-dimensional quantum turbulence. The probability distribution function of the superfluid velocity obeys a Gaussian distribution in the low-velocity region and a power-law distribution in the high-velocity region. The decay of the total number of vortices obeys a power-law for a relatively long period. This scenario may be experimentally realized through interference of Bose-Einstein condensates in a trap potential.

1. Introduction
Quantum turbulence (QT) of atomic Bose-Einstein condensates (BECs) is one of the most challenging problems in low temperature physics \[1\]. The circulation of vortices in classical turbulence (CT) has an arbitrary value and the radius of an eddy is not clear due to viscous diffusion. On the other hand, the circulation of vortices in QT takes a discrete value \(\kappa\) and a quantized vortex is well defined as a topological defect. The Kolmogorov law, which is the most important statistical law in CT, is numerically and experimentally confirmed in QT of superfluid helium. These results show some analogies between CT and QT in spite of the difference of the nature of vortices.

The previous works numerically confirm that QT in BECs is consistent with the Kolmogorov law with the Gross-Pitaevskii (GP) model \[2, 3, 4\]. However, QT in BECs has not been adequately studied in experiments. How to make experimentally QT in BECs? Recently, QT is realizable also in an atomic BEC \[5\]. In actual experiments, it would be important to create QT in BECs and control many physical parameters for better understanding of quantum fluid dynamics. We expect that QT should be the more experimentally realizable through interference of BECs in a trap potential \[7, 8, 9\]. Solitons are produced by interference of segmented BECs. These solitons decay to a number of vortices and anti-vortices \[6\]. The greatest advantage of
making QT by interference of BECs is that one can make effectively many quantized vortices in actual BEC experiments.

In this paper, we make QT from many dark solitons in a uniform system in order to simulate the dynamics after the solitons are made by the interference of BECs. Then we discuss the universal statistics such as the probability distribution function (PDF) of superfluid velocity and the time dependence of the total number of vortices. It is important to investigate statistical laws because turbulent flow momentarily changes but its statistical properties will be invariant. This study should propose a new method of effectively making QT.

Figure 1. Time evolution of the condensate density $|\psi(r,t)|^2$. Time $t$ is normalized by $\tau = \hbar/g n^0$ with $\tilde{t} = t/\tau$. (a) $\tilde{t} = 0$, many dark solitons are parallel and perpendicular to each other. (b) $\tilde{t} = 50$, the solitons become unstable exciting some waves. (c) $\tilde{t} = 56.25$, dark solitons decay to vortices by the snake instability. (d) $\tilde{t} = 112.5$, the quasistationary turbulence.

Figure 2. Time evolution of the PDFs $Pr(v_x)$. Figure 3. Decay of the total number of vortices for $\tilde{t} > 56.25$.

2. Quantum turbulence from dark solitons
We consider a dilute atomic gas BEC described by the condensate wave function $\psi = \sqrt{n} e^{i\theta}$ at $T = 0$ K. We solve the two-dimensional Gross-Pitaevskii equation (GPE) in a uniform system

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + g|\psi|^2 \psi,$$

(1)
where \( n(r, t) = |\psi(r, t)|^2 \) is the atomic density of the condensate, \( m \) is a particle mass and the coupling constant is \( g = 4\pi\hbar^2a_s/m \) with the s-wave scattering length \( a_s \).

Figure 1 shows the time evolution of the condensate density \( n(r, t) \). We start the simulation with the initial stationary state obtained by the imaginary time step of the GPE [Fig.1(a)]. The initial condition is that many solitons are set as a square grid of the interval 20\( \xi \) with the healing length \( \xi = \hbar/\sqrt{2m\hbar g} \) and the bulk density is \( n_0 \). We inject small random seeds for \( \psi \) in order to trigger the instability of solitons. The unstable waves are excited along the solitons in Fig. 1(b) [6]. Then these solitons decay to a lot of vortices through the snake instability in Fig. 1(c) and lead to two-dimensional QT with equal number of quantized vortices and anti-vortices which move around in the system in Fig. 1(d). When a vortex and an anti-vortex make a bound state, they are called a vortex pair. The total number of vortices decreases due to annihilation of the pairs when the distance between a vortex and an anti-vortex becomes smaller than the healing length.

3. The distribution function of superfluid velocity

We show that the PDF of velocity in the QT obeys a \( v^{-3} \) power-law. These features differ from the near-Gaussian statistics of homogenous and isotropic CT observed in experiments and simulations of the Navier-Stokes equation [10].

The superfluid velocity is defined as \( v = \hbar(\psi^* \nabla \psi - \psi \nabla \psi^*)/(2mi|\psi|^2) \). Figure 2 shows the time evolution of the PDF’s \( \Pr(v_x) \) of the \( v_x \) component. The PDFs \( \Pr(v_x) \) at \( \tilde{t} = 50, \tilde{t} = 56.25 \) and \( \tilde{t} = 112.5 \) respectively correspond to Fig. 1 (b), (c), (d). The PDF at \( \tilde{t} = 112.5 \) is unique in that it exhibits dominantly a Gaussian distribution in the low-velocity region and a \( v^{-3} \) power-law distribution in the high-velocity region (Fig. 2). This property of PDF was investigated in the previous works [11, 12, 13]. Both the probability density of position \( \Pr(r) \) and the probability density of velocity \( \Pr(v(r)) \) is normalized by unit respectively as \( \int_0^{\infty}\Pr(v)dv = \int_0^{\infty}\Pr(r)dr = 1 \). For the case of a single quantized vortex, the probability of separation occurring between \( r \) and \( r + dr \) is \( 2\pi r dr \), and the velocity \( v = \kappa/2\pi r \), which leads to

\[
\Pr(v) \sim \Pr(r(v)) \frac{dr}{dv} \sim v^{-3}, \tag{2}
\]

where \( \Pr(v)dv \) is the probability of observing a velocity between \( v \) and \( v + dv \).

The PDF changes from a Gaussian distribution in the low-velocity region to a \( v^{-3} \) power-law in the high-velocity region. A Gaussian distribution of the PDF comes from the random configuration of the vortices. It is possible to roughly estimate the transition velocity between two kinds of behavior. Since the system size is \( L_x = L_y = 160\xi \), the mean distance \( l \) between the vortices is estimated to be \( l \sim (L_xL_y/N_v)^{1/2} \) with the total number \( N_v \) of vortices. For the case of a single quantized vortex, the midpoint between vortices is located at \( l/2 \), thus the transition velocity is roughly \( v_l = \kappa/\pi l \). At \( \tilde{t} = 112.5 \), \( N_v = 200 \) and \( l = 11.3\xi \), so that the transition is velocity \( \tilde{v}_l \sim 0.35 \) which is normalized by the healing length \( \xi \) and the time \( \tau = \hbar/gn_0^0 \). In Fig. 2, the transition velocity agrees approximately with the estimated transition velocity \( v_l \). This result is consistent with the previous works of three-dimensional QT.

4. Decay of the number of vortices

Figure 3 shows the time dependence of the total vortex number after \( \tilde{t} = 56.25 \) corresponding to Fig. 1(c). This simulation shows that \( N_v(t) \) obeys the \( t^{-1} \) power-law decay for \( 60 < \tilde{t} < 250 \) and the \( t^{-2/3} \) power-law decay for \( 250 < \tilde{t} < 700 \) and deviate from it after that to become stationary eventually.

The decay of \( N_v \) can be understood as the following. For \( 60 < \tilde{t} < 250 \), our simulation shows that the vortices do not tend to behave as free isolated vortices. A vortex and an anti-vortex
rather makes a vortex pair a distance \( d \) apart, which move with velocity \( V_{\text{pair}} = \kappa/2\pi d \). The distance \( d \) is about same for all pairs, being a few times larger than \( \xi \). We suppose that positions of these pairs are totally random and any pair is annihilated whenever two pairs encounter each other. A mean free path of a pair should be \( \lambda \approx 2/dn_v \) by following the kinetic theory of gas, where \( n_v = N_v/L_xL_y \) is the number density of vortices. If we assume the distance \( d \) is constant for \( 60 < \tilde{t} < 250 \), the number of collision of pairs per unit time is \( V_{\text{pair}}/\lambda \), so the decay of \( N_v \) is represented by

\[
\frac{dN_v(t)}{dt} = -A\frac{V_{\text{pair}}}{\lambda}N_v(t),
\]

where \( A \) is a constant of the order unit. The solution of Eq. (3) gives \( N_v(t) \propto t^{-1} \). The decay of \( N_v \) until \( \tilde{t} \approx 250 \) would be attributable to this mechanism.

On the other hand, for \( 250 < \tilde{t} < 700 \), our simulation shows that the vortices tend to be free isolated vortices. If we assume that vortices do not move with the velocity \( V_{\text{pair}} \) but the velocity \( V_{\text{vort}} = \kappa/2\pi l \) with the mean distance \( l \) between the vortices as a separate vortex, then the solution of Eq. (3) gives \( N_v(t) \propto t^{-2/3} \) in Fig. 3. However, after \( \tilde{t} \approx 700 \) in Fig. 3, this picture is broken. Probably the above assumptions may be wrong since the vortices are dilute.

5. Conclusion
By numerically solving the GPE, we studied the formation of two-dimensional decaying QT from many dark solitons which are initially set as a square grid in the GP model. We investigated the statistical properties of this QT, such as the PDF of superfluid velocity and the time dependence of the total number of vortices. The PDF is consistent with those of the three-dimensional turbulence. The decay of \( N_v \) obeys the \( t^{-1} \) power-law and the \( t^{-2/3} \) power-law for a relatively long period and we proposed a simple model for understanding the decay.

We expect the similar dynamics as interference of BECs in two- or - three dimensional trapped systems. This would be another method making of QT easily. The detail will be reported elsewhere.

References
[1] Halperin W P and Tsubota M 2008 Progress in Low Temperature Physics vol 16 (Amsterdam: Elsevier)
[2] Kobayashi M and Tsubota M 2005 Phys. Rev. Lett 94 065302
[3] Kobayashi M and Tsubota M 2005 J. Phys. Soc. Jpn. 74 3248
[4] Kobayashi M and Tsubota M 2007 Phys. Rev. A 76 045603
[5] Henn E A L, Seman J A, Roati G, Magalhães K M F, and Bagnato V S 2007 Phys. Rev. Lett 103 045301
[6] Manjun Ma, Carretero-Gonzalez R, Kevrekidis P G, Frantzeskakis D J, and Malomed B A 2010 Phys. Rev. A 82 02362
[7] Carretero-Gonzalez R, Whitaker N, Kevrekidis P G, and Frantzeskakis D J 2007 Phys. Rev. A 77 023605
[8] Carretero-Gonzalez R, Anderson B P, Kevrekidis P G, Frantzeskakis D J, and Weiler C N 2008 Phys. Rev. A 77 033625
[9] Ruben G, Paganin D M, and Morgan M J 2008 Phys. Rev. A 78 013631
[10] Vincent Aand Meneguzzi M 1991 J. Fluid Mech. 225 1
[11] Adachi H and Tsubota M 2011 Phys. Rev. B 83 132503
[12] Paoletti M S, Michael E. Fisher, Sreenivasan K R, and Lathrop D P 2008 Phys. Rev. Lett 101 154501
[13] White A C, Bareghni C F, Proukakis N P, Yould A J, and Wacks D H 2010 Phys. Rev.Lett 104 075301