A novel SUSY energy bound-states treatment of the Klein-Gordon equation with PT-symmetric and q-deformed parameter Hulthén potential

M. Aktas

Energy Systems Engineering Department, Engineering and Natural Sciences Faculty, Ankara Yıldırım Beyazıt University - 06030, Ankara, Turkey

received 15 September 2017; accepted in final form 20 February 2018
published online 9 March 2018

PACS 03.65.Ca – Formalism
PACS 03.65.Ge – Solutions of wave equations: bound states
PACS 03.65.Pm – Relativistic wave equations

Abstract – In this study, we focus on investigating the exact relativistic bound-state spectra for supersymmetric, PT-supersymmetric and non-Hermitian versions of the q-deformed parameter Hulthén potential. The Hamiltonian hierarchy mechanism, namely the factorization method, is adopted within the framework of SUSYQM. This algebraic approach is used in solving the Klein-Gordon equation with the potential cases. The results obtained analytically by executing the straightforward calculations are in consistent forms for certain values of q. Achieving the results may have a particular interest for such applications. That is, they can be involved in determining the quantum structural properties of molecules for ro-vibrational states, and optical spectra characteristics of semiconductor devices with regard to the lattice dynamics. They are also employed to construct the broken or unbroken case of the supersymmetric particle model concerning the interaction between the elementary particles.

Copyright © EPLA, 2018

Introduction. – The dynamical, spontaneous and meta-stable supersymmetry breaking mechanisms [1–8] arising from the concept of supersymmetry [9–14] are of great importance for quantum mechanical systems involved in a symmetry viewpoint. Recent studies and interests on quantum mechanical systems with supersymmetry as well as on PT-supersymmetric Hamiltonian systems [15–21] have just been in progress. In addition, their significant impacts have just kept going for a long time [22–29]. These type of systems which are Hermitian require the real and discrete energy eigenvalues. The concept of PT-symmetry (i.e., space-time reflectional symmetry) referring to the \((P \rightarrow \text{parity})\) and the \((T \rightarrow \text{time reversal})\) allows us to understand the characteristics of complex potential systems. It is suggested that the PT-symmetric but non-Hermitian Hamiltonians can be characterized by real spectra for spontaneously broken PT-symmetry [15]. The other crucial concept for a class of these types of Hamiltonians is the pseudo-Hermiticity [30–33]. The pseudo-Hermitian Hamiltonians imply the real or the complex eigenvalue spectra.

A variety of techniques such as the Nikiforov-Uvarov method was performed in solving the Klein-Gordon (KG) equation within the PT-symmetric case [34–36], variational, semi-classical estimates, group theoretical approach, relativistic shifted \(l\)-expansion and large-\(N\) expansion [37–50]. Moreover, several applications in the solutions of the KG equation with scalar and vector-type potentials (e.g., harmonic oscillator, ring-shaped, exponential types, etc.) were introduced by the authors of [51–61]. Recent engineering applications based on the concepts of supersymmetry and PT-symmetry for optical system devices have also been implemented [62–65].

In this work, we investigate the exact relativistic bound states of the Klein-Gordon equation for PT-supersymmetric and non-Hermitian Hulthén potential cases within the formalism of SUSYQM [9]. The SUSYQM approach provides an alternative tool to investigate the potential problems algebraically in relativistic and non-relativistic quantum mechanics. Several interesting features of SUSYQM are suggested in this context. It is mainly concerned with both the shape invariance requirement and the Hamiltonian hierarchy mechanism.

This paper is organized as follows: in the following section, the hierarchy of a Hamiltonian procedure, known as the factorization method is introduced. In the third
section, we deal with the energy eigenvalue problem of the Klein-Gordon equation for the \( q \)-deformed Hulthén potential and its \( PT \)-symmetric, \( PT \)-supersymmetric and non-Hermitian versions. The last section deals with the discussion of the results.

**Hierarchy of a Hamiltonian: factorization method.** – The main concern of this approach is the reduction of second-order ordinary differential equations (ODEs), e.g., the Schrödinger equation for a given potential to the first-order nonlinear Riccati equation involving the superpotential \( W(x) \). This method was first proposed by Darboux and later was provided in the derivation of exact spectra of the hydrogen atom problem by Schrödinger. The SUSYQM algebraic approach allows us to write Hamiltonians by setting (\( \text{Schrödinger} \). The SUSYQM algebraic approach allows us to get the general Riccati equations as

\[
W_n^2 - W_n' = W_2(x) - E_2^{(0)}.
\]

Therefore, the Riccati equation becomes

\[
W_n^2 + W_n' = V_2(x) - E_2^{(0)}.
\]

By iterating the Hamiltonians \( n \)-th times similarly, one can get the general Riccati equations as

\[
W_n^2 - W_n' = \left( \hat{\Omega}_n^+ \hat{\Omega}_n^{-} \right) + E_n^{(0)},
\]

and

\[
\hat{H}_n(x) = -\frac{d^2}{dx^2} + V_n(x) = \hat{\Omega}_n^- \hat{\Omega}_n^+ + E_n^{(0)},
\]

\[
\hat{\Omega}_n^+ = \frac{d}{dx} + \int d\psi_0(x), \quad \hat{\Omega}_n^- = \frac{d}{dx} - \int d\psi_0(x).
\]

For the unbroken SUSY case, the partner Hamiltonians satisfy the energy eigenvalue and eigenfunction expressions as \([11]\)

\[
E_{n+1}^{(0)} = E_n^{(1)}, \quad E_0^{(0)} = 0, \quad n = 0, 1, 2, \ldots
\]

and

\[
\psi_n = \frac{1}{\sqrt{E_n^{(0)}}} \left[ \hat{\Omega}_n^- \psi_n^{(0)} \right], \quad \psi_{n+1} = \frac{1}{\sqrt{E_n^{(0)}}} \left[ \hat{\Omega}_n^+ \psi_n^{(0)} \right],
\]

with the same eigenvalue.

**Klein-Gordon equation and Hulthén potential.** – The Klein-Gordon (KG) equation is a relativistic version of the Schrödinger equation. In relativistic quantum mechanics, it describes the elementary particles with zero-spin (spin-0) quantum number (e.g., Higgs bosons). Its solution implies a quantum scalar or pseudoscalar field. There are various schemes on this equation by using different techniques. The s-wave KG equation is studied for the vector and scalar type of the Hulthén potential \([37,38]\). The same problems are discussed for scattering state solutions with the regular and irregular boundary conditions \([39]\). The Green functions of these potentials are obtained by means of the path integral approach \([40]\).
Moreover, many attempts have been made to develop approximation techniques such as (shifted)large-$N$, 1/$N$ expansion etc. for the KG equation with the Coulomb-like and Coulomb plus linear and Coulomb plus Aharonov-Bohm potential [41–44]. Furthermore, the s-wave bound-state solutions of the KG equation with the generalized Hulthén potential are examined by using the Nikiforov-Uvarov (NU) method [34–36], the alternative SUSYQM approach [45] as well.

The 1-D Klein-Gordon equation for the Lorentz vector and scalar potentials can be defined as

$$\left\{ \frac{d^2}{dx^2} + \left[ E - V(x) \right]^2 - [m + S(x)]^2 \right\} \Psi_0(x) = 0, \quad \text{(16)}$$

where $\Psi(x) = \frac{1}{2} \Psi_0(x)$, $V(x)$ and $S(x)$ are Lorentz vector and scalar forms of the Hulthén potential.

The generalized Hulthén potential. The $q$-deformed Hulthén potential is

$$V^H_q(x) = -V_0 \frac{e^{-\lambda x}}{1 - qe^{-\lambda x}}. \quad \text{(17)}$$

The Lorentz vector and scalar forms of the Hulthén potential can be written as

$$V(x) = -V_0 \frac{e^{-\lambda x}}{1 - qe^{-\lambda x}}, \quad S(x) = -S_0 \frac{e^{-\lambda x}}{1 - qe^{-\lambda x}}. \quad \text{(18)}$$

Therefore, the effective Hulthén potential becomes

$$V_{\text{eff}}(x) = \left[ S^2(x) - V^2(x) \right] + 2[mS(x) + EV(x)], \quad \text{(19)}$$

$$\epsilon = E^2 - m^2 < 0. \quad \text{We can write the final form of the effective potential as}$$

$$V_{\text{eff}}(x) = \Gamma_1 \frac{e^{-2\lambda x}}{(1 - qe^{-\lambda x})^2} - \Gamma_2 \frac{e^{-\lambda x}}{(1 - qe^{-\lambda x})}, \quad \text{(20)}$$

where $\Gamma_1 = (S_0^2 - V_0^2) > 0$ and $\Gamma_2 = 2(mS_0 + EV_0) > 0$.

Supersymmetric energy bound spectra of the KG equation with Hulthén potential. Now, we want to construct the successive superpotentials of the $q$-deformed Hulthén potential, by proposing an ansatz superpotential as

$$W_1(x) = -\nu_1 \frac{e^{-\lambda x}}{1 - qe^{-\lambda x}} + \mu_1. \quad \text{(21)}$$

The Riccati equation is defined in terms of $W_1(x)$

$$V_{\text{eff}}'(x) = W_1^2(x) - W_1' = V_{\text{eff}}(x) - \epsilon_0^{(3)}. \quad \text{(22)}$$

Substituting the square of ansatz (21) and differential form of it into eq. (22) yields

$$V_{\text{eff}}'(x) = \nu_1(\nu_1 - q\lambda) \frac{e^{-2\lambda x}}{(1 - qe^{-\lambda x})^2} - \nu_1(2\mu_1 + \lambda) \frac{e^{-\lambda x}}{(1 - qe^{-\lambda x})} + \mu_1^2, \quad \text{(23)}$$

when we compare the right sides of these equalities in eq. (23) term-by-term,

$$\mu_1 = \frac{(\Gamma_1 + q\Gamma_2 - \nu_1^2)}{2\nu_1} \quad \text{(24)}$$

is obtained with $-\epsilon_0^{(1)} = \mu_1$. Now, we should determine $V_{\text{eff}}'(x)$ for the case

$$V_{\text{eff}}'(x) = W_1^2(x) + W_1' = V_{\text{eff}}(x) + \mu_1. \quad \text{(25)}$$

Hence, one can write

$$V_{\text{eff}}'(x) = \nu_1(\nu_1 + q\lambda) \frac{e^{-2\lambda x}}{(1 - qe^{-\lambda x})^2} - \nu_1(2\mu_1 - \lambda) \frac{e^{-\lambda x}}{(1 - qe^{-\lambda x})} + \mu_1^2. \quad \text{(26)}$$

Let us propose the second ansatz superpotential for $W_2$ as

$$W_2(x) = -\nu_2 \frac{e^{-\lambda x}}{(1 - qe^{-\lambda x})} + \mu_2. \quad \text{(27)}$$

Therefore, the Riccati equation will become

$$V_{\text{eff}}''(x) = W_2^2(x) + W_2' = V_{\text{eff}}(x) - \epsilon_0^{(2)}. \quad \text{(28)}$$

By inserting the ansatz equation (27) into eq. (28), we obtain

$$V_{\text{eff}}''(x) = \nu_2(\nu_2 + q\lambda) \frac{e^{-2\lambda x}}{(1 - qe^{-\lambda x})^2} - \nu_2(2\mu_2 \pm \lambda) \frac{e^{-\lambda x}}{(1 - qe^{-\lambda x})} + \mu_2^2,$$

$$V_{\text{eff}}''(x) = \Gamma_1 \frac{e^{-2\lambda x}}{(1 - qe^{-\lambda x})^2} - \Gamma_2 \frac{e^{-\lambda x}}{(1 - qe^{-\lambda x})} - \epsilon_0^{(2)}. \quad \text{(29)}$$

From eqs. (23), (26) and (29), one gets

$$\nu_2 = \nu_1 + q\lambda \quad \text{and} \quad \mu_2 = \frac{(\Gamma_1 + q\Gamma_2 - \nu_2^2)}{2\nu_2}. \quad \text{(30)}$$

By using the relation $-\epsilon_0^{(2)} = \mu_2^2$, we get

$$\epsilon_0^{(2)} = \left[ \frac{(\nu_2 - (\Gamma_1 + q\Gamma_2))^2}{2q\nu_2} \right]. \quad \text{(31)}$$

can be written.

Consequently, if the process is repeated $n$-times cautiously, we can get the following results:

$$V_{\text{eff}}''(x) = \nu_n(\nu_n + q\lambda) \frac{e^{-2\lambda x}}{(1 - qe^{-\lambda x})^2} - \nu_n(2\mu_n + \lambda) \frac{e^{-\lambda x}}{(1 - qe^{-\lambda x})} + \mu_n^2,$$

$$W_n(x) = -\nu_n \frac{e^{-\lambda x}}{(1 - qe^{-\lambda x})} + \mu_n,$$

$$\epsilon_0^{(n)} = -\left[ \frac{(\nu_n + q\lambda)^2 - (\Gamma_1 + q\Gamma_2)^2}{2q(\nu_n + q\lambda)} \right]; \quad n = 1, 2, \ldots \quad \text{(32)}$$

10005-p3
The use of notation $\epsilon_0^{(n)} = E_n^2 - m^2$ gives the energy eigenvalues of the general $q$-deformed Hulthén potential for the KG equation as
\[
E_n = \pm \frac{i}{2q} \sqrt{\left[\nu_n + inq\lambda - \left(\frac{\Gamma_1 + q\Gamma_2}{\nu_n + inq\lambda}\right)^2\right]} - (4q^2m^2).
\] (33)
Also, the $n$-th ground-state wave function can be defined from eq. (4) as
\[
\Psi_0^{(n)} = (1 - qe^{i\lambda x})^{\nu_n/i\lambda} e^{-\mu_n x}.
\] (34)

**Bound spectra of PT-supersymmetric Hulthén potential.**

Let us now consider the PT-symmetric form of the Hulthén potential by taking $\lambda \rightarrow i\lambda$ in eq. (17). The effective potential can be written as follows:
\[
V_{\text{eff}}(x) = \Gamma_1\left(\frac{e^{-2i\lambda x}}{1 - qe^{-i\lambda x}}\right)^2 - \Gamma_2\left(\frac{e^{-i\lambda x}}{1 - qe^{-i\lambda x}}\right)^2.
\] (35)

The ansatz superpotential of this potential will be
\[
W_1(x) = -\nu_1\left(\frac{e^{-i\lambda x}}{1 - qe^{-i\lambda x}}\right) + \mu_1.
\] (36)

Putting the ansatz equation (36) into eq. (22), we can obtain
\[
V_{\text{eff}}(x) = \nu_1(\nu_1 - iq\lambda)\left(\frac{e^{-2i\lambda x}}{1 - qe^{-i\lambda x}}\right)^2
- \nu_1(2\mu_1 + iq\lambda)\left(\frac{e^{-i\lambda x}}{1 - qe^{-i\lambda x}}\right) + \mu_1^2,
\]
\[
\tilde{V}_{\text{eff}}(x) = \Gamma_1\left(\frac{e^{-2i\lambda x}}{1 - qe^{-i\lambda x}}\right)^2
- \Gamma_2\left(\frac{e^{-i\lambda x}}{1 - qe^{-i\lambda x}}\right)^2 - \epsilon_0^{(1)}.
\] (37)

The same procedure followed in the above section leads to the energy eigenvalues of KG equation for the PT-symmetric potential as
\[
E_n = \pm \frac{i}{2q} \sqrt{\left[\nu_n + inq\lambda - \left(\frac{\Gamma_1 + q\Gamma_2}{\nu_n + inq\lambda}\right)^2\right]} - (4q^2m^2).
\] (38)

By iterating the same procedure as in the previous subsection, we can obtain the energy eigenvalues of the KG equation for the potential
\[
E_n = \pm \frac{i}{2q} \sqrt{\left[\nu_n + inq\lambda - \left(\frac{\Gamma_1 + q\Gamma_2}{\nu_n + inq\lambda}\right)^2\right]} - (4q^2m^2).
\] (41)

with $\epsilon_0^{(n)} < 0$. Hence, the ground-state wave function takes the form as
\[
\Psi_0^{(n)} = (1 - qe^{i\lambda x})^{\nu_n/i\lambda} e^{-\mu_n x}.
\] (42)

**Concluding remarks.** – In this work, relativistic bound-state spectra and the corresponding wave functions of the Klein-Gordon (KG) equation with the generalized, PT-symmetric, PT-supersymmetric and non-Hermitian $q$-deformed Hulthén potential case are carried out by implementing the Hamiltonian hierarchy procedure within the SUSYQM formalism. Various forms of the potential are also treated within the solution of the KG equation. We have proved that the bound-state spectra of PT-invariant complex-valued non-Hermitian potentials may be real or complex depending on the actual potential parameters ($q, \lambda, V_0, S_0$) in eqs. (17) and (18), and the superpotential parameters ($\nu_1, \mu_1$) in eq. (21). For instance, in cases of $q = 0$ referring to the exponential potential, $q = 1$ referring to the Hulthén potential, $q = -1$ referring to the Woods-Saxon potential as well as with the condition of the Lorentz scalar and vector potentials $S_0 = V_0$, the bound-state results depend on the parameter $\Gamma_2 = 2(m + E)V_0$. Likewise, when $q = 0$, the negative and positive bound energy results in eqs. (33), (38) and (41) will tend to go infinity automatically. Therefore, no explicit forms are existed in that case. However, in cases of $q = \pm 1$, there exist limitations on the bound-state spectra. Furthermore, the corresponding ground-state wave functions for some values of the potential parameters or only $x \rightarrow 0$ should explicitly require to be finite. As a final remark, achieving the results in the study is likely to leading new perspectives both in spontaneous and in dynamical supersymmetry breaking case studies as well as determining the optical spectra characteristics of particular semiconductor devices [62,65].

**REFERENCES**

[1] Witten E., *Nucl. Phys. B*, 188 (1981) 513.
[2] Ralchenko Y. V. and Semenov V. V., *J. Phys. A: Math. Gen.*, 24 (1991) L1305.
[3] Dutt R., Gangopadhyaya A., Khare A., Pagnamenta A. and Sukhatme U. P., *Phys. Rev. A*, 48 (1993) 1845.
[4] Gangopadhyaya A., Mallow J. V. and Sukhatme U. P., *Phys. Lett. A*, 283 (2001) 279.
[5] Intriligator K. and Seiberg N., *Class. Quantum Grav.*, 24 (2007) S741 (arXiv:hep-th/0702069v3).
[6] Martin S. P., arXiv:hep-ph/9709356v7.
[7] Stergiou A., *New Ways of Supersymmetry Breaking*, M. Sc. Thesis (Faculteit Der Natuurwetenschappen Wiskunde En Informatica, Amsterdam) 2007.
A novel SUSY energy bound-states treatment of the Klein-Gordon equation etc.