Hardy-Littlewood-Sobolev Inequality on Product Spaces

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Abstract

We prove Hardy-Littlewood-Sobolev inequality on product spaces by using strong maximal functions.

1 Introduction

In 1928, Hardy and Littlewood first established an $L^p$-regularity theorem of fractional integrals in one dimensional space, as was shown in [1]. Ten years later, Sobolev extended the result into higher dimensions in [2]. This has been known as Hardy-Littlewood-Sobolev inequality. It was later revealed by Hedberg in [4], such that the inequality can be obtained as a consequence of Hardy-Littlewood-Wiener theorem of maximal functions. A well illustrative proof is given in the book [3]. In this paper, in the same spirit of using maximal functions, we extend Hardy-Littlewood-Sobolev inequality into product spaces.

Let $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. For $0 < \alpha < m$ and $0 < \beta < n$, we define

$$\Omega(x, y) = \left( \frac{1}{|x|} \right)^{m-\alpha} \left( \frac{1}{|y|} \right)^{n-\beta}.$$  \hspace{1cm} (1. 1)

Our main result is the following:

Theorem Let $1 < p < q < \infty$ and $f \in L^p(\mathbb{R}^m \times \mathbb{R}^n)$. We have

$$\|f * \Omega\|_{L^q(\mathbb{R}^m \times \mathbb{R}^n)} \leq A_{p,q} \|f\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)}$$  \hspace{1cm} (1. 2)

if and only if

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m} = \frac{\beta}{n}.$$  \hspace{1cm} (1. 3)

We prove hereby a 2-parameters product theorem. The result of general multi-parameters can be obtained by developing the estimation in the same spirit.
2 Some Remarks

Let $B_\delta \times B_\lambda \subset \mathbb{R}^m \times \mathbb{R}^n$ be the product of balls with radius $\delta$ and $\lambda$, centering on $(x, y) = (0, 0)$. A strong maximal function is defined by

$$\begin{align*}
(Mf)(x, y) &= \sup_{\delta, \lambda > 0} \frac{1}{|B_\delta||B_\lambda|} \int_{B_\delta \times B_\lambda} |f(x - u, y - v)| \, du \, dv. 
\end{align*}$$

(2.1)

It is not hard to observe that

$$\begin{align*}
(Mf)(x, y) \lesssim (M_1(M_2f))(x, y) 
\end{align*}$$

(2.2)

where $M_i$, $i = 1, 2$ are standard maximal functions respectively on the coordinate subspace.

We define simultaneously the $L^p$-norms as functions by

$$\begin{align*}
\|M_1f \cdot \|_{L^p(\mathbb{R}^n)} &= \left( \int_{\mathbb{R}^n} (M_1f)^p(x, v) \, dv \right)^{\frac{1}{p}}, 
\end{align*}$$

(2.3)

$$\begin{align*}
\|M_2f \cdot \|_{L^p(\mathbb{R}^m)} &= \left( \int_{\mathbb{R}^m} (M_2f)^p(u, y) \, du \right)^{\frac{1}{p}}.
\end{align*}$$

(2.4)

Let

$$\begin{align*}
(Gf)(x, y) &= \|M_1f \cdot \|_{L^p(\mathbb{R}^n)} \|M_2f \cdot \|_{L^p(\mathbb{R}^m)}.
\end{align*}$$

(2.5)

Since $M_1$ and $M_2$ are $L^p$-bounded respectively on the coordinate subspaces, we have

$$\begin{align*}
\left( \int_{\mathbb{R}^m \times \mathbb{R}^n} (Gf)^p(x, y) \, dx \, dy \right)^{\frac{1}{p}} 
\lesssim \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} \left( \int_{\mathbb{R}^n} (M_1f)^p(x, v) \, dv \right) \left( \int_{\mathbb{R}^m} (M_2f)^p(u, y) \, du \right) \, dx \, dy \right)^{\frac{1}{p}} 
\end{align*}$$

(2.6)

$$\begin{align*}
= \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} (M_1f)^p(x, v) \, dx \, dv \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} (M_2f)^p(u, y) \, du \, dy \right)^{\frac{1}{p}} 
\lesssim \|f\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)}^2.
\end{align*}$$

Lastly, the necessity of (1.3) follows by changing the dilatations with respect to each variable $x$ and $y$. It is obviously that (1.3) implies

$$\begin{align*}
\frac{1}{q} &= \frac{1}{p} - \frac{\alpha + \beta}{m + n},
\end{align*}$$

(2.7)
3 Proof of Theorem

By (1.1), we write

\[
|(f * \Omega)(x, y)| \leq \iint_{\mathbb{R}^m \times \mathbb{R}^n} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} \, dudv
\]

(3.1)

\[
= \iint_{|u| \leq R_1, |v| \leq R_2} + \iint_{|u| > R_1, |v| \leq R_2} + \iint_{|u| \leq R_1, |v| > R_2} + \iint_{|u| > R_1, |v| > R_2}
\]

where \( R_1 > 0 \) and \( R_2 > 0 \). The estimation will be carried out in several steps.

1. We first estimate

\[
\iint_{|u| \leq R_1, |v| \leq R_2} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} \, dudv.
\]

(3.2)

Observe that both \(|u|^{-m+\alpha}\) and \(|v|^{-n+\beta}\) are both radial and radially decreasing, provide that 
\(0 < \alpha < m\) and \(0 < \beta < n\). They can be approximated in the sense of that

\[
|u|^{-m+\alpha} \sim \sum_j a_j \chi_{B_j}, \quad \sum_j a_j |B_j| \sim \int_{|u| \leq R_1} |u|^{-m+\alpha} \, du
\]

(3.3)

and

\[
|v|^{-n+\beta} \sim \sum_k b_k \chi_{B_k}, \quad \sum_k b_k |B_k| \sim \int_{|v| \leq R_2} |v|^{-n+\beta} \, dv
\]

(3.4)

where \( a_j, b_k \) are positive real numbers, and \( \chi_{B_j}, \chi_{B_k} \) are characteristic functions of \( B_j \subset \mathbb{R}^m \), \( B_k \subset \mathbb{R}^n \) for every \( j, k = 1, 2, \ldots \) respectively.

We have

\[
\iint_{|u| \leq R_1, |v| \leq R_2} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} \, dudv
\]

\[
\sim \left( \sum_j a_j |B_j| \right) \left( \sum_k b_k |B_k| \right) \left( \frac{1}{|B_j||B_k|} \right) \iint_{B_j \times B_k} |f(x - u, y - v)| \, dudv
\]

(3.5)

\[
\leq (Mf)(x, y) \iint_{|u| \leq R_1, |v| \leq R_2} \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} \, dudv
\]

\[
= (Mf)(x, y) R_1^\alpha R_2^\beta.
\]

2. Next, we turn to

\[
\iint_{|u| > R_1, |v| > R_2} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} \, dudv.
\]

(3.6)
By Hölder inequality,
\[
\int_{|u| > R_1, |v| > R_2} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} \, du \, dv
\leq \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} |f(u, v)|^p \, du \, dv \right)^{\frac{1}{p}} \left( \int_{|u| > R_1, |v| > R_2} \left( \frac{1}{|u|} \right)^{(m-\alpha)p} \left( \frac{1}{|v|} \right)^{(n-\beta)p} \, du \, dv \right)^{\frac{1}{p-1}}.
\]
(3.7)

We have \((-m + \alpha) p/(p - 1) < m\) provided that
\[
\frac{m}{p - 1} - \frac{\alpha p}{p - 1} = \frac{m}{q} \left( \frac{p}{p - 1} \right) > 0
\]
and similarly \((-n + \beta) p/(p - 1) < n\) provided that
\[
\frac{n}{p - 1} - \frac{\beta p}{p - 1} = \frac{n}{q} \left( \frac{p}{p - 1} \right) > 0.
\]

Therefore, (3.7) is further bounded by
\[
\|f\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)} R_1^{\frac{\alpha}{m/p - \alpha}} R_2^{\frac{\beta}{n/p - \beta}}.
\]
(3.10)

3. What remains to be estimated are
\[
\int_{|u| \leq R_1, |v| > R_2} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} \, du \, dv
\]
(3.11)
and
\[
\int_{|u| > R_1, |v| \leq R_2} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} \, du \, dv.
\]
(3.12)

By symmetry, it is suffice to estimate one of them.

Observe that
\[
\int_{|u| \leq R_1, |v| > R_2} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} \, du \, dv
\leq \int_{|v| > R_2} \left( \frac{1}{|v|} \right)^{\beta} \left\{ \int_{|u| \leq R_1} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \, du \right\} \, dv
\]
(3.13)
\[
\leq R_1^{\alpha} \int_{|v| > R_2} (M_1 f)(x, y - v) \left( \frac{1}{|v|} \right)^{n-\beta} \, dv
\]
whereas the second inequality above is carried out in analogue to step 1.
On the other hand, by Hölder inequality,

\[ R_1^a \int_{|v| > R_2} (M_1 f)(x, y - v) \left( \frac{1}{|v|} \right)^{n-\beta} dv \]

\[ \leq R_1^a \left( \int_{R^n} (M_1 f)'(x, v) dv \right)^{\frac{1}{\beta}} \left( \int_{|v| > R_2} \left( \frac{1}{|v|} \right)^{\frac{(n-\beta)p}{m}} dv \right)^{\frac{1}{1-\beta}} \]  \hspace{1cm} (3.14)

\[ \leq \left\| (M_1 f)(x, \cdot) \right\|_{L^p(R^n)} R_1^a R_2^{\beta-n/p} \]

as showed in step 2.

By carrying out the same estimation as above for (3.12), with \( u \) and \( v \) switched in role, we obtain

\[ \int \int_{|u| > R_1, |v| \leq R_2} |f(x - u, y - v)| \left( \frac{1}{|u|} \right)^{m-\alpha} \left( \frac{1}{|v|} \right)^{n-\beta} dudv \]

\[ \leq \left\| (M_2 f)(\cdot, y) \right\|_{L^p(R^n)} R_1^{a-m/p} R_2^\beta . \]  \hspace{1cm} (3.15)

4. Suppose that

\[ (G f)(x, y) \leq (M f)(x, y) \left\| f \right\|_{L^p(R^n \times R^n)} . \]  \hspace{1cm} (3.16)

In summary of the estimates obtained in step 1-3, we choose \( R_1 \) and \( R_2 \) such that

\[ (M f)(x, y) R_1^a R_2^\beta = \left\| f \right\|_{L^p(R^n \times R^n)} R_1^{a-m/p} R_2^{\beta-n/p} \]  \hspace{1cm} (3.17)

and

\[ \left\| (M_1 f)(x, \cdot) \right\|_{L^p(R^n)} R_1^a R_2^{\beta-n/p} = \left\| (M_2 f)(\cdot, y) \right\|_{L^p(R^n)} R_1^{a-m/p} R_2^\beta . \]  \hspace{1cm} (3.18)

As a result, we have

\[ \frac{(M f)(x, y)}{\left\| f \right\|_{L^p(R^n \times R^n)}} = R_1^{-m/p} R_2^{-n/p} \]  \hspace{1cm} (3.19)

and

\[ \frac{\left\| (M_1 f)(x, \cdot) \right\|_{L^p(R^n)}}{\left\| (M_2 f)(\cdot, y) \right\|_{L^p(R^n)}} = R_1^{-m/p} R_2^{-n/p} . \]  \hspace{1cm} (3.20)

By solving equations (3.19)-(3.20), we have

\[ R_1^{-m/p} = \left( \frac{(M f)(x, y)}{\left\| f \right\|_{L^p(R^n \times R^n)}} \left\| (M_1 f)(x, \cdot) \right\|_{L^p(R^n)} \right)^{\frac{1}{2}} \]  \hspace{1cm} (3.21)

and

\[ R_2^{-n/p} = \left( \frac{(M f)(x, y)}{\left\| f \right\|_{L^p(R^n \times R^n)}} \left\| (M_2 f)(\cdot, y) \right\|_{L^p(R^n)} \right)^{\frac{1}{2}} . \]  \hspace{1cm} (3.22)
5. By inserting estimates (3.21)-(3.22) into (3.17), we have

\[
\left( \mathbf{M} f \right)(x, y) R_1^{a} R_2^{\beta} = \left\| f \right\|_{L^p(R^n \times R^n)} R_1^{a-m/p} R_2^{\beta-n/p} \tag{3.23}
\]

\[
= \left( \mathbf{M} f(x, y) \right)^\frac{p}{n} \left( \left\| f \right\|_{L^p(R^n \times R^n)} \right)^{\left(1 - \frac{p}{n}\right)}.
\]

On the other hand, by inserting estimates (3.21)-(3.22) into (3.18), we have

\[
\left\| (\mathbf{M}_1 f)(x, \cdot) \right\|_{L^p(R^n)} R_1^{a} R_2^{\beta-n/p} = \left\| (\mathbf{M}_2 f)(\cdot, y) \right\|_{L^p(R^n)} R_1^{a-m/p} R_2^{\beta} \tag{3.24}
\]

\[
= \left( \mathbf{M} f(x, y) \right)^\frac{2}{n} \left( \frac{\left( \mathbf{G} f \right)(x, y)}{\left\| f \right\|_{L^p(R^n \times R^n)}} \right) \left\| f \right\|_{L^p(R^n \times R^n)} \left(1 - \frac{p}{n}\right).
\]

6. Suppose that

\[
\left( \mathbf{G} f \right)(x, y) > \left( \mathbf{M} f \right)(x, y) \left\| f \right\|_{L^p(R^n \times R^n)} . \tag{3.25}
\]

We replace \((\mathbf{M} f)(x, y)\) with \((\mathbf{G} f)(x, y) / \left\| f \right\|_{L^p(R^n \times R^n)}\) in (3.5). The resulting estimates in step 4 will be changed accordingly. Namely, we choose \(R_1\) and \(R_2\) such that

\[
\left( \mathbf{G} f \right)(x, y) R_1^{a} R_2^{\beta} = \left\| f \right\|_{L^p(R^n \times R^n)} R_1^{a-m/p} R_2^{\beta-n/p} \tag{3.26}
\]

and

\[
\left\| (\mathbf{M}_1 f)(x, \cdot) \right\|_{L^p(R^n)} R_1^{a} R_2^{\beta-n/p} = \left\| (\mathbf{M}_2 f)(\cdot, y) \right\|_{L^p(R^n)} R_1^{a-m/p} R_2^{\beta}. \tag{3.27}
\]

As a result, we have

\[
\left( \mathbf{G} f \right)(x, y) = \frac{\left\| f \right\|_{L^p(R^n \times R^n)}}{R_1^{a-m/p} R_2^{\beta-n/p}} \tag{3.28}
\]

and

\[
\left\| (\mathbf{M}_1 f)(x, \cdot) \right\|_{L^p(R^n)} = \frac{R_1^{a-m/p}}{R_2^{\beta-n/p}}. \tag{3.29}
\]

By solving equations (3.28)-(3.29), we have

\[
R_1^{a-m/p} = \left( \frac{\left( \mathbf{G} f \right)(x, y) \left\| (\mathbf{M}_1 f)(x, \cdot) \right\|_{L^p(R^n)}}{\left\| f \right\|_{L^p(R^n \times R^n)}} \right)^{\frac{1}{2}} \tag{3.30}
\]

and

\[
R_2^{\beta-n/p} = \left( \frac{\left( \mathbf{G} f \right)(x, y) \left\| (\mathbf{M}_2 f)(\cdot, y) \right\|_{L^p(R^n)}}{\left\| f \right\|_{L^p(R^n \times R^n)}} \right)^{\frac{1}{2}}. \tag{3.31}
\]
7. Therefore, by inserting estimates (3. 30)-(3. 31) into (3. 26), we have

\[
(Gf)(x, y)R_1^a R_2^b = \|f\|_{L^p(R^n \times R^n)}^2 R_1^a R_2^b
\]

\[
= (Gf(x, y))^p \left( \|f\|_{L^p(R^n \times R^n)} \right)^{(1 - \frac{2}{q})} \tag{3. 32}
\]

On the other hand, by inserting estimates (3. 30)-(3. 31) into (3. 27), we have

\[
\left\| (M_1 f)(x, \cdot) \right\|_{L^p(R^n)} R_1^a R_2^b = \left\| (M_2 f)(\cdot, y) \right\|_{L^p(R^n)} R_1^a R_2^b
\]

\[
= \left( \frac{(Gf)(x, y)}{\|f\|_{L^p(R^n \times R^n)}} \right)^p \|f\|_{L^p(R^n \times R^n)} \tag{3. 33}
\]

\[
= (Gf)^p (x, y) \left( \|f\|_{L^p(R^n \times R^n)} \right)^{(1 - \frac{2}{q})}.
\]

References

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