The transformation of Hermite-Gauss beams with embedded optical vortex by lens system

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Abstract. In this paper the transformation of Hermite-Gauss beams with embedded vortex phase by lens system is investigated theoretically and numerically. A particular attention is devoted to the formation of vortex phase singularities in focal area. It is shown, that under appointed relations between HG mode indices and the number of embedded optical vortex the vortex phase singularity in a focal plane centre may disappear.

1. Introduction
Vortex beams are laser beams with the vortex phase singularity which is characterized by zero intensity in a point of phase uncertainty [1-7]. Well known vortex beams are Laguerre-Gauss modes [8-12] and Bessel modes [13-18]. Hypergeometric modes [19-22] and Zernike functions [23-25] contain angular harmonic components also. All said distributions are separable in the polar coordinates system – they are representable as a product of the function depending on radius only, and angular harmonic component \( \exp(\text{imag}) \), \( m \) is optical vortex order.

Beams which are separable in Cartesian coordinates system, such as Hermite-Gauss modes (HG) [26-28]. Airy beams [29-34] are not vortex ones, but their certain superpositions are used for transformation to vortex beams [35, 36]. However, Airy beams with embedded vortex phase are examined lately [37-39]. Besides, in paper [40] vortex HG beams depending on complex argument were examined. In paper [41] the optical element matched with HG modes was used as a low-frequency grating for a multiplication of first-order phase singularity.

In this paper we investigate HG beams with embedded optical vortex of an arbitrary order. A particular attention is devoted to a picture in the focal plane and to the formation of vortex phase singularities depending on relation of HG mode indices and order of embedded optical vortex. We obtain an analytical condition of the disappearance of vortex phase singularity in the focal plane centre. Also we have examined a propagation of such beam through paraxial lens system on base of usage of fractional Fourier transformation.

2. Theoretical foundations.
Let us consider the Hermite-Gauss beams with embedded vortex phase (embedded vortex Hermite-Gaussian, EVHG):

\[
\Psi_{\text{evh}}(x, y) = \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)H_n\left(\frac{x}{\sigma}\right)H_m\left(\frac{y}{\sigma}\right)(x + iy)^m,
\]

where \( H_n(x) \) is Hermite polynomial.

Hermite polynomials have a particular property, namely all terms have equal power parity. However, now we shall not take this in account, and, for commonness, we shall examine a case of the arbitrary polynomial of two variables. An individual term looks like \( P_{pq}(x, y) = A(r) x^p y^q (x + iy)^m \), where \( A(r) \) is an arbitrary axial-symmetric function depending on radius only. For a convenience of following analysis \( P_{pq}(x, y) \) is written in polar coordinates:
\[ P_{pq}(r, \varphi) = A(r)(r \cos \varphi)^p (r \sin \varphi)^q r' \exp(i\varphi). \]  

(2)

Let us consider the Fourier transformation of function (2):

\[ F_{pq}(p, 0) = \frac{2\pi}{\lambda f} \int_0^\infty \int_0^{2\pi} P_{pq}(r, \varphi) \exp\left[ -\frac{i2\pi}{\lambda f} pr \cos(\varphi - 0) \right] r \, dr \, d\varphi, \]  

(3)

where \( \lambda \) is the wavelength of an illuminating light, \( f \) is focal length of a parabolic lens implementing the Fourier transformation.

It is well known that vortex beams retain the vortex phase singularity and the zero intensity on optical axis under propagation in homogeneous medium and under passing through parabolic lens [1-7]. However in anisotropic media [42, 43] or at an astigmatic transformation [35, 36, 44] such beams lose the axial symmetry of intensity distribution, in particular, non-zero value may appear on the axis. Since the beam (1) has not axial symmetry, then under its propagation or passing through lens it may undergo changes similar to anisotropic or astigmatic transformation.

Let us clarify whether zero value will be in the focal plane centre (on axis). For this purpose we may use results from Ref. [45]. Zero amplitude on axis will be under conditions:

\[
(s > p + q) \text{ or } (p + q + s \text{ is odd}) \quad s > 0 \\
p \text{ or } q \text{ is odd} \quad s = 0
\]  

(4)

These conditions are sufficient, but the condition \((p + q + s \text{ is odd})\) may not be necessary (at its violation amplitude on axis may be zero).

3. The numerical simulation

In this section we present simulation results for EVHG beams prescribed by expression (1).

3.1 Field distribution in the focal plane.

Table 1 contains distribution in the focal plane obtained on base of expression (3) for different values \( n, m \) and \( s \).

| \( n \) \(=0, m=1 \) | \( n=1, m=1 \) | \( n=1, m=2 \) | \( n=2, m=2 \) | \( n=3, m=3 \) |
|----------------|----------------|----------------|----------------|----------------|
| \( s=0 \)     | ![Image](image1) | ![Image](image2) | ![Image](image3) | ![Image](image4) |
| \( s=1 \)     | ![Image](image5) | ![Image](image6) | ![Image](image7) | ![Image](image8) |
| \( s=2 \)     | ![Image](image9) | ![Image](image10) | ![Image](image11) | ![Image](image12) |
The first row of Table 1 contains distributions for Fourier-images of conventional HG modes \((s=0)\). In this case distributions are similar to initial functions and they have a binary phase. According to formula (4) value in centre is zero if at least one index is odd.

The second row of Table 1 contains distributions of spatially-spectral picture for HG beams with embedded optical vortex of order one \((s=1)\). In this case we have set of separated optical vortex of order one instead of singular lines (linear phase jumps) Number of vortices is defined by formula:

\[ N_{v1} = (n+1)(m+1) + nm. \] (5)

For determining what value will be in centre (zero or nonzero) one may use formula (4).

The third row of Table 1 contains focal pictures for HG beams with embedded optical vortex of order two \((s=2)\). As we see, in this case the distinction between sufficient and necessary condition takes place at \(n=2, m=2, s=2\).

Thus, theoretical discussions listed above are conformed to calculation results completely.

An interesting result is obtained at embedding of the optical vortex into HG modes with one index equal zero (columns 1 and 2). In this case the number of zero intensity points which are easily found out by sight, is connected with the vortex order unambiguously:

\[ N_s = m + s. \] (6)

Formula (6) is true for vortex orders \(s<m+2\). Consequently, Hermite-Gauss modes with one zero and other large enough index may be used for determination of the optical vortex order.

### 3.2 The beam propagation through lens system.

In conclusion let us consider the propagation of beam (1) through paraxial lens system. This process may be described with usage of fractional Fourier transformation [46, 47]. In polar coordinates its expression looks like [48]:

\[
E(r, \varphi, z) = \frac{ik}{2\pi \sin(\alpha z)} \exp\left(ikz\right) \exp\left(\frac{ikr^2}{2f \tan(\alpha z)}\right) \times
\]

\[
\times \int_{0}^{2\pi} \int_{0}^{\pi} E_0(\rho, \theta) \exp\left(\frac{ik\rho^2}{2f \tan(\alpha z)}\right) \exp\left(-\frac{ik}{f \sin(\alpha z)} \rho \cos(\theta - \varphi)\right) \rho \ d\rho \ d\theta,
\] (7)

where \(\alpha = \pi/(2f)\); \(k = 2\pi/\lambda\) is the wavenumber, \(\lambda\) is the wavelength of an illuminating light.

**Table 2.** Longitudinal distributions (diagonal sections as shown by red line in Table 1) for different values of \(n, m\) and \(s\) (amplitudes are shown).
The operator (7) allows to obtain a beam shape (taking into account a scale) in any paraxial domain – both in Fresnel diffraction domain and in far domain. Table 2 shows passing of field (1) from input plane \((z=0)\) through focal plane \((z=f)\) to output plane \((z=2f)\).

In output plane \((z=2f)\) in common case the distribution which is centrally symmetrical to initial one is being formed, but the amplitude of beam (1) is symmetrical initially, therefore the distribution coincides with initial \((z=0)\). This explains a symmetry of longitudinal pictures in Table 2 relative to the plane \(z=f\), so as examination of changes from \(z=0\) to \(z=f\) is quite enough.

An analysis of images in Table 2 allows us to arrive at the following conclusions. At vortex absence the distribution is similar to initial one along the whole length from \(z=0\) to \(z=f\).

At vortex presence there is passing between the domain with distribution similar to initial one, and the domain with distribution similar to one in focal plane. The fact itself is obvious, but the longitudinal distribution shows that domains with other structure are absent.

4. Conclusion

In this paper we have examined HG modes with embedded vortex phase. As against axially symmetrical distributions the vortex phase (and zero intensity value on optical axis) for such beams is not remained under propagation or under passing through parabolic lens.

Theoretical and numerical analysis of Fourier transformation for HG modes with embedded vortex phase has been implemented. We have verified theoretical conditions for zero intensity value in the centre of focal plane by the numerical simulation.

With aid of the fractional Fourier transformation the modeling of HG beams passing through paraxial lens system is implemented. Results show that even at the vortex presence, there is a continuous transformation from initial distribution to focal one.

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6. References

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