GORENSTEIN SEMIGROUP ALGEBRAS OF WEIGHTED TREES

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Abstract. We classify exactly when the toric algebras \( \mathbb{C}[S_T(r)] \) are Gorenstein. These algebras arise as toric deformations of algebras of invariants of the Cox-Nagata ring of the blow-up of \( n-1 \) points on \( \mathbb{P}^{n-3} \), or equivalently algebras of the ring of global sections for the Plücker embedding of weight varieties of the Grassmanian \( Gr_2(\mathbb{C}^n) \), and algebras of global sections for embeddings of moduli of weighted points on \( \mathbb{P}^1 \). As a corollary, we find exactly when these families of rings are Gorenstein as well.

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1. Introduction

Let \( \mathfrak{k} \) be a field, then a \( \mathbb{Z} \)-graded \( \mathfrak{k} \)-algebra \( R \) is said to be Gorenstein if the Matlis Dual

\[ H_m^{\dim(R)}(R)^* = \text{Hom}_m(H_m^{\dim(R)}(R), \mathfrak{k}), \]

is isomorphic to grade-shifted copy \( R(-a) \) of \( R \). Here \( m \) is the maximal ideal generated by elements in \( R \) of positive degree, and \( \text{Hom}_m(\cdot, \cdot) \) is the functor of graded \( \mathfrak{k} \)-morphisms. The number \(-a\) is called the \( a \)-invariant of the graded Gorenstein algebra \( R \). We refer the reader to the book by Bruns and Herzog, [BH] for this and all other relevant definitions. We study this property for a specific family of normal semigroup algebras \( \mathbb{C}[S_T(r)] \). Here \( T \) is a trivalent tree with \( n \) ordered leaves, and \( r \) is an \( n \)-tuple of positive integers which sums to an even number. We let \( E(T) \) denote the set of edges of \( T \), \( L(T) \) denote the set of edges incident to a leaf of \( T \), \( I(T) \) denote the set \( E(T) \setminus L(T) \), and \( V(T) \) denote the set of vertices of \( T \). For each trinode \( \tau \in V(T) \) we let \((\tau, i)\) be the \( i \)-th edge incident on \( \tau \). When we are dealing with a cone or polytope \( P \) we denote the lattice points in the interior by \( \text{int}(P) \). We will work over \( \mathbb{C} \), but our analysis works for an arbitrary field \( \mathfrak{k} \).

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Definition 1.1. The k-th piece of $S_T(r)$ is the set of weightings $\omega : E(T) \to \mathbb{Z}_+$ defined by the following conditions.

1. For all $\tau \in V(T)$ the numbers $\omega(\tau, i)$ satisfy $|\omega(\tau, 1) - \omega(\tau, 2)| \leq \omega(\tau, 3) \leq |\omega(\tau, 1) + \omega(\tau, 2)|$.
2. $\sum_{i=1}^3 \omega(\tau, i)$ is even.
3. For all $v_m \in L(T)$, $\omega(v_m) = kr_m$.

The expressions in item 1 above are called the triangle inequalities, item 2 is referred to as the parity condition. These semigroups are all embedded in the semigroup of lattice points in the cone defined by the triangle inequalities $P_T$, with the lattice $L_2$ defined by the parity condition. From now on, when three numbers $A$, $B$ and $C$ satisfy both the parity condition and the triangle inequalities, we write $\Delta_2(A, B, C)$.

Presentations of these semigroups and their associated algebras were constructed by the author in [M]. In [HMSV], Howard, Millson, Snowden, and Vakil independently constructed presentations of $\mathbb{C}[S_T(r)]$ in order to find presentations of a projective coordinate ring of the moduli space of $r$-weighted points on $\mathbb{P}^1$, denoted $M_r$. The embedding they studied comes from the homeomorphism of projective varieties, $M_r \cong Gr_2(\mathbb{C}^n) / rT$, where the right hand side is the $r$-weight variety of $Gr_2(\mathbb{C}^n)$. They constructed a toric deformation of this algebra, $\mathbb{C}[M_r]$, to $\mathbb{C}[S_T(r)]$ for each tree $T$ by means of the Speyer-Sturmfels [SpSt] toric deformations of $Gr_2(\mathbb{C}^n)$, which deform the projective coordinate ring given by the Plücker embedding to $\mathbb{C}[P_T]$, the algebra of lattice points for the cone $P_T$, see [HMSV] and [SpSt] for details. Also, Sturmfels and Xu [StXu] have shown the Cox ring $R_{n-1,n-3}$ of the blow-up of $\mathbb{P}^{n-3}$ at $n-1$ points to be isomorphic (as a multigraded algebra) to the Plücker algebra of $Gr_2(\mathbb{C}^n)$. The multigrading, given by the Picard group of this blow-up, is equivalent to the multigrading on $\mathbb{C}[P_T]$ given by the weights on $L(T)$. This then implies that the subrings of $R_{n-1,n-3}$ given by taking invariants with respect to a character of the corresponding “Picard torus” are isomorphic to $\mathbb{C}[M_r]$. The relevance of all the above work to this project comes from the following theorem.

Theorem 1.2. The graded algebra $\mathbb{C}[M_r]$ is Gorenstein if and only if $\mathbb{C}[S_T(r)]$ is Gorenstein, for any tree $T$.

This follows from a theorem of Stanley, appearing as corollary 4.3.8 in [BH], which characterizes the Gorenstein properties of Cohen-Macaulay domains by details of their Hilbert function. We get the above theorem because both $\mathbb{C}[M_r]$ and $\mathbb{C}[S_T(r)]$ are Cohen-Macaulay domains, and have the same Hilbert function by a standard theorem of deformation theory. We are therefore able to study the Gorenstein property for projective coordinate rings of weighted points on $\mathbb{P}^1$, weight varieties of $Gr_2(\mathbb{C}^n)$, and the invariant subrings of $R_{n-1,n-3}$ by learning about the family $\mathbb{C}[S_T(r)]$ of normal semigroup algebras. We also get the following corollary.

Corollary 1.3. $\mathbb{C}[S_T(r)]$ is Gorenstein if and only if $\mathbb{C}[S_{T'}(r)]$ is Gorenstein for any trees $T$, $T'$ with the same number of leaves.

Since we are dealing with algebras generated by the lattice points of convex cones we are allowed the full range of available literature on the commutative algebra
of toric rings, in particular the following proposition, which is a consequence of corollary 6.3.8 in [BH].

**Proposition 1.4.** Let $S_P$ be the semigroup given by the lattice points in a convex cone $P$. Then the algebra $\mathbb{C}[S_P]$ is Gorenstein if and only if there is a lattice point $\omega \in \text{int}(P)$ with $\text{int}(P) = \omega + P$. Furthermore, in the presence of a grading, we have $a(\mathbb{C}[S_P]) = -\deg(\omega)$.

This proposition follows from the fact that the ideal $(\text{int}(P))\mathbb{C}[S_P]$ can be identified with the canonical module of the algebra $\mathbb{C}[S_P]$ (resp. the *-canonical module in the presence of a grading), see [BH] for details. We wish to prove this property for the cone over $P_T(r) \times \{1\}$ in $\mathbb{R}^{I(T)} \times \mathbb{R}$ with the product lattice $L_2 \times \mathbb{Z}$. In order to do this, we break the problem into two parts. First, we analyze when some minkowski sum of $P_T(r)$ contains a unique interior lattice point, a necessary but not sufficient condition for the Gorenstein property.

**Theorem 1.5.** $P_T(r)$ has a unique interior point if and only if $r = \hat{2} + \hat{R}$ where $\hat{R}$ is of one of the following types.

1. $R_i = \sum_{j \neq i} R_j$ for some $i$.
2. $\Delta_2(R_i, R_j, R_k)$ holds for some $i, j, k$ and $R_\ell = 0$ for all $\ell \neq i, j, k$.

This will be proved in section 2. Next, we find when every other interior point of the cone has the unique interior point of the appropriate minkowski sum of $P_T(r)$ as a summand. Theorem 1.5 allows us to significantly narrow our search for $P_T(r)$ which satisfy this condition. In order to carry this out, we bring in an alternative description of the weightings $\omega \in P_T(r)$, which can be found in [HMM], [HMSV], and [SpSt]. We start by considering a weighting $\omega$ on a single trinode, $\tau$. Since $\Delta_2(\omega(\tau, 1), \omega(\tau, 2), \omega(\tau, 3))$ always holds, we may apply the 1-1 transformation of cones

$$T : P_3 \rightarrow \mathbb{R}_+^3$$

given by $T(\omega)(x_{ij}) = \frac{1}{2}(\omega(\tau, i) + \omega(\tau, j) - \omega(\tau, k))$. This is an isomorphism of semigroups when $\mathbb{R}^3$ is given the standard lattice. We represent the image of $\omega$ via the piping model on the leaves of the tree, where the number of pipes going from $i$ to $j$ is $T(\omega)(x_{ij})$, an example is picture below.
The map $T$ has an inverse $S$ given by $S(\eta)(\tau, i) = \eta(x_{ij}) + \eta(x_{ik})$. In this way we may go from weightings on the trinode $\tau$ to planar graphs on the set $\{(\tau, i)\}$. The transformation $T$ is useful as it clarifies divisibility issues in the semigroup of lattice points in $P_3$: $\omega$ divides $\omega'$ if and only if $T(\omega)(x_{ij}) \leq T(\omega')(x_{ij})$ for all $i, j$. For a general tree $T$, something similar happens for graphs on the set $L(T)$. Given a graph $G$ on the set $E(T)$, we may construct a simultaneous weighting of the edges of each trinode $\tau \in V(T)$ as follows. For each "pipe" $e \in G$ we consider the unique path $\gamma$ in $T$ joining the end points of $e$, each trinode edge $(\tau, i)$ traversed by this path gets weight $+1$. We call this map $S_T$. There is a section to this map, $T_T$, which is defined by the following algorithm. Given a weighting $\omega$ of $T$, consider the simultaneous weighting of trinodes given by restricting $\omega$ to each $\tau \in V(T)$. Apply $T$ to each of these weightings, and join up the ends of the resulting pipes in the unique way such that the resulting graph is planar. We leave it to the reader to show that this is well-defined (that is, there is a unique planar graph for such a weighting, this can also be found in [HMM]). An example is illustrated in figure 2.

![Figure 2. Piping model for a general $T$.](image)

The map $T_T$ cannot be an inverse as there are many (non-planar) graphs that give the same weighting under $S_T$, this redundancy accounts for the elements in a certain basis of $\mathbb{C}[M_r]$, linearly related by the Plücker equations, which degenerate to the same element under the flat degeneration defined by $T$. For more on this see [HMM]. The reader will also note that a weighting $\omega$ divides another weighting $\omega'$ if and only if $\omega|_T$ divides $\omega'|_T$ for all $\tau \in V(T)$, so the map $T$ is still very useful for questions of divisibility in the case of general $T$. Note that the piping model only makes sense for a tree $T'$ which has a specified embedding into $\mathbb{R}^2$, because of our need to deal with planar graphs. For this reason we restrict our attention to trees $T$ with such an embedding assumed. This has no effect on our results because the isomorphism class of $P_T$ as a multigraded algebra is determined by the topological type of $T$, and therefore any $\mathbb{C}[S_T(r)]$ is isomorphic to some $\mathbb{C}[S_{T'}(r')]$ with $T'$ planar and $r'$, a permutation of the entries of $r$.

We let $N_{ij}(\omega)$ be the multiplicity of edges between $i$ and $j$ on the graph $T_T(\omega)$. We are now ready to state our main theorem. From now on we denote the unique internal weighting for the cone on $P_T(r)$ by $\omega_T(T)$, if it exists. Also, we let $2r_T$ be the weighting which assigns 2 to each edge of $T$, an example is illustrated in figure 2 above. In general, $T_T(2r_T)$ is always a complete planar cycle on the set $L(T)$. 
Theorem 1.6. \( \mathbb{C}[S_T(\mathbf{r})] \) is Gorenstein if and only if

1. \( \mathbf{r} \) is as in theorem \( \ref{thm:gorenstein} \), for some \( a \).
2. In this degree, \( N_{ij}(\omega_T(T) - 2_T) \geq n - 4 \) when it is nonzero.

This will be proved in section \( \ref{sec:proof} \). We finish with a theorem which restricts the \( a \)-invariant of \( \mathbb{C}[S_T(\mathbf{r})] \).

Theorem 1.7.

\[
a(\mathbb{C}[S_T(\mathbf{r})]) \mid 2(n - 2)
\]

This is proved in section \( \ref{sec:proof} \). We mention here that this theorem also restricts the \( a \)-invariant for \( \mathbb{C}[M_\mathbf{r}] \) as well, because the \( a \)-invariant may be read off the Hilbert function of the algebra.

2. Proof of theorem \( \ref{thm:proof} \)

In this section we will classify the polytopes \( P_T(\mathbf{r}) \) that have a unique interior point. This is the first step to understanding when the ring \( \mathbb{C}[S_T(\mathbf{r})] \) is Gorenstein. Let \( P_T \) denote the cone of weightings \( \omega \) on the tree \( T \) such that

\[
\Delta_2(\omega(\tau, 1), \omega(\tau, 2), \omega(\tau, 3))
\]

holds for each internal vertex \( \tau \in V(T) \). A weighting \( \omega \) is on a face of this cone if and only if one of these triangle inequalities is an equality for some vertex \( \tau \). Let \( D_n \) be the cone of side lengths for \( n \)-sided polygons (so \( D_3 = P_3 \)), there is a map of cones \( \pi : P_T \rightarrow D_n \) given by forgetting the weights on internal edges of \( T \). We have the obvious identification,

\[
P_T(\mathbf{r}) = \pi^{-1}(\mathbf{r}).
\]

Let us study the dimension of these fibers. For \( n = 3 \) there is nothing to say, so suppose \( n > 3 \). If for every side length \( r_i < \sum_{j \neq i} r_j \) then we may form a convex planar \( n \)-gon \( \mathcal{P} \) with sidelengths \( \mathbf{r} \) with non-zero area, this in turn implies we may find such an \( n \)-gon where all triangles in \( \mathcal{P} \) formed by the diagonals and sides have non-zero area. Let \( \omega \) be the (not necessarily integer) weighting of \( T \) formed by the diagonal lengths and sidelengths of \( \mathcal{P} \). Now, observe that for any diagonal \( d \) of \( \mathcal{P} \) specified by \( T \) which borders two triangles with non-zero area, we may stretch and contract the length of \( d \) within some neighborhood \( \epsilon \) without changing the lengths of any other sides and specified diagonals non-zero, see figure \( \ref{fig:proof} \). This implies that there is a small neighborhood of dimension \( |I(T)| \) in \( P_T(\mathbf{r}) \) which contains \( \omega \). On the other hand, if some entry \( r_j \) of \( \mathbf{r} \) has \( r_j = \sum_{i \neq j} r_i \) then \( P_T(\mathbf{r}) \) can be nothing but a single point. Thus we conclude that the dimension of \( P_T(\mathbf{r}) \) can be either \( |I(T)| \) or 1. This discussion has some bearing on the following proposition.

Proposition 2.1. A weighting \( \omega \in P_T(\mathbf{r}) = \pi^{-1}(\mathbf{r}) \) is on a facet if and only if it is on a facet of \( P_T \).

Proof. First note that if \( \mathbf{r} \) has some entry equal to the sum of the other entries, then \( P_T(\mathbf{r}) \) is a fiber over a point in a facet of \( D_n \), therefore the unique point of \( P_T(\mathbf{r}) \) is on a facet of \( P_T \). If \( \mathbf{r} \) does not satisfy this property, by definition the triangle inequalities give facets of \( P_T(\mathbf{r}) \), hence if \( \omega \) is on a facet of \( P_T \) and has sidelengths \( \mathbf{r} \), it is on a facet of \( P_T(\mathbf{r}) \). If \( \omega \) is not on a facet of \( P_T \), then it has all triangle inequalities strict, and it represents a polygon \( \mathcal{P} \) with all interior triangles
having non-zero area. From the discussion above we know that \( \omega \) must then lie in the interior of \( P_T(r) \) because it has a neighborhood of full-dimension. \( \square \)

\[\text{Figure 3. Creating a neighborhood of } \omega. \text{ The case for a general polygon can be reduced to the case of a quadrilateral by excising all but the 4 edges incident on the diagonal in question.}\]

The content of this proposition can also be found in [KM]. The following gives an algebraic characterization of interior lattice points.

**Proposition 2.2.** A weighting \( \omega \in P_T \) is in the interior if and only if \( \omega = \eta + 2T \) for some \( \eta \in P_T \).

**Proof.** If \( \omega \) is in the interior of \( P_T \) then all inequalities defined by the condition \( \Delta_2 \) are strict. After converting to the piping model, we must have \( T_T(\omega)(x_{ij}(\tau)) \geq 1 \), for each trinode \( \tau \in V(T) \). This implies that \( \omega \) has \( 2T \) as a factor. Running this argument in reverse gives the converse. \( \square \)

**Corollary 2.3.** If \( \omega \in P_T(r) \) is an interior point, then \( \omega = 2T + \eta \) for some \( \eta \in P_T(r - \vec{2}) \)

**Corollary 2.4.** The toric algebra \( \mathbb{C}[P_T] \) is Gorenstein.

This second corollary also follows from the same theory that gave us theorem 1.2 and the fact that the algebra of the Plücker embedding of \( Gr_2(\mathbb{C}^n) \) is Gorenstein. As a consequence of proposition 2.2 and corollary 2.3 we get the following proposition.

**Proposition 2.5.** \( P_T(r) \) has a unique interior point if and only if \( r = \vec{2} + \vec{R} \) such that \( P_T(\vec{R}) \) is a single point.

The following proposition classifies all \( \vec{R} \) which have this property.

**Proposition 2.6.** \( P_T(\vec{R}) \) is exactly one point if and only if \( \vec{R} \) is of one of the following forms

1. \( R_i = \Sigma_{i \neq j} R_j \) for some \( i \).
2. \( \Delta_2(R_i, R_j, R_k) \) holds for some \( i, j, k \) and \( R_\ell = 0 \) for all \( \ell \neq i, j, k \).

**Proof.** For sufficiency, note that there is exactly one polygon fitting the description given by both cases above. For necessity, we consider the piping model \( T_T(\omega) \) of the tree weighting. We suppose \( P_T(r) \) is a single point and classify the pipe arrangements allowed for weightings of \( T \). Suppose we were allowed an arrangement of pipes where two edges do not share any common vertex. Then we may swap
pipes while maintaining the edge weights. This implies that any pair of pipes must share a common vertex. There are exactly two ways for this to happen, the reader can verify that $S_T \circ T_T(\omega) = \omega$ must satisfy the edge weight conditions in the statement of the theorem.

\[ \square \]

**Figure 4.** Creating another weighting.

**Corollary 2.7.** $P_T(\mathbf{v})$ has a unique interior lattice point if and only if $P_{T'}(\mathbf{v})$ has a unique interior lattice point, for all $T'$.

When we convert the above weighting conditions to their graphical representation on the set $L(T)$, we get the possibilities represented below in Figure 5. One possibility is a graph where every pipe shares a common incident vertex, the second possibility has exactly three vertices with incident pipes. Propositions 2.5 and 2.6 then prove theorem 1.5.

**Figure 5.** Associated graphs for $\vec{R}$.

### 3. Proof of theorem 1.6

Theorem 1.5 gives a necessary condition for $\mathbb{C}[S_T(\mathbf{v})]$ to be Gorenstein. Now we see what must be added in order to ensure that all interior lattice points carry the unique interior lattice point $\omega_r(T)$ as a summand. We will make use of the piping model for most of this section. For the cases presented in the statement of theorem 1.5, the first case has $N_{ij}(\omega_r(T) - 2\tau) = R_j$ and $N_{kj}(\omega_r(T) - 2\tau) = 0$ for all $k, j \neq i$, and the second case has $N_{ij}(\omega_r(T) - 2\tau) = \frac{1}{2}(R_i + R_j - R_k)$ and $N_{im}(\omega_r(T) - 2\tau) = 0$ for $l$ or $m \neq i, j, k$.

**Proposition 3.1.** $\mathbb{C}[S_T(\mathbf{v})]$ is Gorenstein if and only if there is no interior weighting $\omega$ in degree $k \geq a$ such that $N_{ij}(\omega - 2\tau) < N_{ij}(\omega_r(T) - 2\tau)$.

**Proof.** After converting $\omega$ to the piping model and removing the complete cycle on $L(T)$ corresponding to $2\tau$ we get the graph of $\omega - 2\tau$. It is clear that if $N_{ij}(\omega - 2\tau) \geq N_{ij}(\omega_r(T) - 2\tau)$ for all $i, j$ then $\omega_r(T)$ is a summand of $\omega$. For the converse, we find a weighting $\omega'$ on a new tree $T'$ which has $i$ and $j$ connected to a common
trinode \( \tau \), with the number of pipes between \( i \) and \( j \) in the trinode equal to \( N_{ij}(\omega) \).

To do this, simply exchange members of \( L(T) \) with a permutation \( P_k \) so that \( i \) and \( j \) are next to each other, and choose a \( T' \) such that these leaves are now incident on a common internal vertex \( \tau \). This new graph may have crossings, but this does not matter, we consider the weighting \( S_{T'} \circ P_k \circ T(\omega) = \omega' \). By corollary \( 2.7 \) there exists a unique internal lattice point \( \omega(P_k(T)) \) in the polytope \( P_{T'}(P_k(T)) \) with \( N_{kj}(\omega(P_k(T))) = N_{ij}(\omega_r(T)) \). By construction we have \( N_{kj}(\omega_r(T) - 2T') < N_{ij}(\omega(P_k(T)) \circ T) \), implying that \( \omega(P_k(T)) \circ T(\omega) \) cannot divide \( \omega'(\tau) \). This implies that \( \omega(P_k(T)) \circ T(\omega) \) cannot divide \( \omega' \), and that \( \mathbb{C}[S_{T'}(P_k(T))] \) is not Gorenstein. The permutation group \( S_n \) acts on the algebra of global sections of \( Gr_2(\mathbb{C}^n) \) given by the Plücker embedding by permuting the entries of the multigrading, so we get \( \mathbb{C}[M_r] \cong \mathbb{C}[M_{P_k(r)}] \). Now by theorem \( 1.2 \) and corollary \( 1.3 \) \( \mathbb{C}[S_{T'}(r)] \) cannot be Gorenstein either. 

\[ \begin{align*}
\text{Figure 6. Proof of theorem 1.6}
\end{align*} \]

Now we are ready to prove theorem \( 1.6 \) this is accomplished with the next proposition.

**Proposition 3.2.** For any \( \mathbb{C}[S_{T'}(r)] \) such that some multiple of \( r \) satisfies the criteria of theorem \( 1.5 \) there is a weighting \( \omega \) which has \( N_{ij}(\omega - 2T) < N_{ij}(\omega_r(T) - 2T) \) if and only if \( N_{ij}(\omega_r(T) - 2T) \) is less than \( n - 4 \) when it is nonzero.

**Proof.** Using the piping model, this is a simple counting argument. The inequality \( N_{ij}(\omega - 2T) < N_{ij}(\omega_r(T) - 2T) \) holds if and only if we may allocate the ends of the remaining pipes connected to \( i \) and \( j \) to vertices in the rest of the graph. In order to keep faithful to the edgeweight condition, it is necessary and sufficient to have

\[ \Sigma_{\ell \neq i,j} \left( \frac{k}{a}(R_{\ell} + 2) - 2 \right) - \left( \frac{k}{a}(R_i + 2) - 2 \right) - \left( \frac{k}{a}(R_j + 2) - 2 \right) + 2N_{ij}(\omega_r(T) - 2T) > 0, \]
where \( a \) is the degree of \( \omega_r(T) \) and \( k \) is the degree of \( \omega \). In both cases this reduces to \( N_{ij}(\omega_r(T) - 2r) < n - 4 \).

4. The \( a \)-invariant

Since the polytopes \( P_T(r) \) are the fibers of \( \pi \), a morphism of convex cones induced by ambient linear map, we get \( P_T(kr) = kP_T(r) \). This allows us to prove theorem 1.7 classifying the \( a \)-invariant of our family of Gorenstein rings. This theorem is implied by the following proposition.

**Proposition 4.1.** If \( P_T(kr) \) has a unique internal lattice point then \( k \) must divide \( 2(n - 2) \)

**Proof.** If \( n \leq 3 \) then our ring is isomorphic to \( \mathbb{C}[x] \). Furthermore, if any \( kr_i = 2 \) then \( k = 1 \) or \( 2 \). This takes care of all cases except when \( R_i = \sum_{i \neq j} R_j \) and all \( R_j > 0 \). Note that \( k \) must divide

\[
\sum_{i \neq j}(R_j + 2) - (R_i + 2) = R_i + 2(n - 1) - R_i - 2 = 2(n - 2)
\]

□

**Example 4.2** (Gorenstein property first shown by B. Howard and M. Herring, [HH]). Consider the case \( r = (1, \ldots, 1) = \vec{1} \), this case satisfies all the conditions of theorem 1.7 with the unique interior point occuring in the polytope \( P_T(\vec{1}) \), the lattice points of which give the degree 2 part of the algebra. Therefore \( \mathbb{C}[ST(\vec{1})] \) and \( \mathbb{C}[M_{\vec{1}}] \) are Gorenstein, with \( a \)-invariant equal to \(-2\). The latter algebra is of particular importance in [HMSV].

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REFERENCES

[BH] W. Bruns and J. Herzog, *Cohen Macaulay Rings*. Cambridge Studies in Advanced Mathematics 39 (1993).

[HH] B. J. Howard, M. Herring, private communication.

[HMM] B. J. Howard, C. A. Manon, J. J. Millson, *The toric geometry of triangulated polygons in Euclidean space*, in preparation.

[HMSV] B. J. Howard, J. J. Millson, A. Snowden, and R. Vakil, *The projective invariants of ordered points on the line*, [http://lanl.arXiv.org/math.AG/0505096](http://lanl.arXiv.org/math.AG/0505096).

[KM] M. Kapovich, J. J. Millson, *The symplectic geometry of polygons in Euclidean space*, J. Differential Geom. **44** (1996), no. 3, 479-513.

[M] C. A. Manon, *Presentations of Semigroup Algebras of Weighted Trees*, [http://arxiv.org/abs/0808.1320](http://arxiv.org/abs/0808.1320).

[SpSt] D. Speyer and B. Sturmfels, *The tropical Grassmannian*, [http://lanl.arXiv.org/math.AG/0304218](http://lanl.arXiv.org/math.AG/0304218).

[StXu] B. Sturmfels and Z. Xu, *Sagbi Bases of Cox-Nagata Rings*, [arXiv:0803.0892v2 [math.AG]](http://arxiv.org/abs/0803.0892v2).