Some eigenstates for a model associated with solutions of tetrahedron equation

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Abstract

Here we present some eigenstates for a 2+1-dimensional model associated with a solution of the tetrahedron equation. The eigenstates include those "particle-like" (namely one-particle and two-particle ones), constructed in analogy with the usual 1 + 1-dimensional Bethe ansatz, and some simple "string-like" ones.

Introduction

The problem of extending the Bethe ansatz onto the 2 + 1-dimensional models, or maybe finding some other method for constructing their eigenvectors, is surely of great importance. In this paper, we are going to present some results obtained in this direction.

Our model will be a model on the cubic lattice: the lattice vertices are points whose all three coordinates are integers. The lattice will be assumed infinite in all directions, unless the contrary is stated explicitly. Thus, the calculations in this paper will be in part formally-algebraic. In each vertex there is an “S-operator” acting in the tensor product of three linear spaces attached to the links, as in paper [2]. To be exact, it will be convenient for us to assume that there is a transposed matrix $S^T$ in each vertex. This is because we will need some objects called “vacuum vectors”, and the “vacuum covectors” of $S$ studied in [2] are “vacuum vectors” of $S^T$.

The transfer matrix we will deal with will be a “diagonal” one: it is cut out of the lattice by two planes perpendicular to the vector $(1, 1, 1)$ in such a way that it consist of separate, not linked to each other, “anti-tank hedgehogs” (vertices). In each of those planes, the intersection with the cubic lattice yields a kagome lattice
consisting, as known, of triangles and hexagons. We can group all vertices of the kagome lattice in triples—vertices of triangles—in such a way that the transfer matrix acts on each triangle separately, turning it inside out and making a linear transformation in the tensor product of three corresponding subspaces.

The mentioned product of three subspaces is comprised of the 0-, 1-, 2- and 3-particle sectors. According to papers [1, 2], the sectors with even and odd particle numbers do not mix together and, moreover, in the even sector the S-operator acts as an identical unity. The 1- and 3-particle sectors do mix together, but it turns out that there are two eigenvectors of the S-operator in the one-particle sector, with eigenvalues 1 and −1. Their explicit form can be extracted out of the end of p. 94 and the beginning of p. 95 of [1]. Namely, denote as \((x, y, z)^T\) a one-particle state describing the situation when the “amplitudes” for a particle to be in the 1st, 2nd and 3rd spaces are \(x\), \(y\) and \(z\). According to [2], and taking into account the fact that we are considering the vectors without a 3-particle component, we can take any “isotropic” vector, i.e. such that \(x^2 - y^2 + z^2 = 0\), as the eigenvector corresponding to the eigenvalue 1, and any other “isotropic” vector similarly for the eigenvalue \(-1\). Let us write those vectors as

\[
\begin{align*}
\text{(1)} & \quad (\sin \lambda_+, 1, \cos \lambda_+)^T \\
\text{(2)} & \quad (\sin \lambda_-, 1, \cos \lambda_-)^T,
\end{align*}
\]

respectively.

A vector \((x, y, z)^T\) is a linear combination of the vectors (1) and (2) iff

\[
y = ax + bz,
\]

where

\[
a = \frac{\cos \lambda_- - \cos \lambda_+}{\sin(\lambda_+ - \lambda_-)}, \quad b = \frac{\sin \lambda_+ - \sin \lambda_-}{\sin(\lambda_+ - \lambda_-)}.
\]

1 One-particle states

Consider now the whole kagome lattice. For it, the one-particle space is the direct sum of one-particle spaces over all its vertices multiplied by vacuums in other places. To indicate a one-particle vector \(\varphi\) means to attach a number—amplitude \(\varphi_A\)—to each vertex \(A\) of the kagome lattice. It is evident that in order to ensure that the vector never comes out of the one-particle space on applying to it any number of transfer matrices, we must take it to be an eigenvector of the composition of a transfer matrix and a proper lattice shift. The latter arises from the fact that a
transfer matrix, when acting on a “slice vector”, turns inside out half of the triangles of the kagome lattice to which that vector belongs, thus moving the lines.

So, let us write down the conditions for a vector in the one-particle space to be an eigenvector. Consider the following picture (Fig. 1) representing a fragment of the kagome lattice. Here the triangle $DCE$ is going to be turned inside out, while the triangle $BCA$ has been obtained by turning inside out a triangle on the previous step. So, the two conditions arise:

$$\varphi_C = a\varphi_D + b\varphi_E$$  \hspace{1cm} (5)

and

$$\varphi_C = a\varphi_B + b\varphi_A$$  \hspace{1cm} (6)

(compare (3)).

When the triangle $DCE$ is turned inside out, it yields a triangle $D'C'E'$ (Fig. 2), and the new “field” variables are expressed through the old ones as

$$\begin{pmatrix} \varphi_{D'} \\ \varphi_{E'} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \varphi_D \\ \varphi_E \end{pmatrix},$$  \hspace{1cm} (7)

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{\sin(\lambda_+ - \lambda_-)} \begin{pmatrix} \sin(\lambda_+ + \lambda_-) & -2\sin\lambda_+\sin\lambda_- \\ 2\cos\lambda_+\cos\lambda_- & \sin(\lambda_+ + \lambda_-) \end{pmatrix}.$$

Note that

$$\alpha = -\delta, \quad \alpha^2 + \beta\gamma = 1.$$  \hspace{1cm} (8)
The points $E'$ and $D'$ of the “new” lattice are analogs of the points $A$ and $B$ correspondingly belonging to the “old” lattice. Thus, in order to obtain on the new lattice a vector proportional to the vector on the old lattice, we must require that

$$\frac{\varphi_{E'}}{\varphi_{D'}} = \frac{\varphi_A}{\varphi_B}. \tag{9}$$

If this condition holds, one can extend both the old vector $\varphi$ and the new “primed” vector periodically onto the whole lattice and in such way that the new one will be proportional to the (shifted) old one.

The condition (9) together with (5, 6, 7) is enough to obtain $\varphi_A$ and $\varphi_B$ (as well as $\varphi_C$, $\varphi_D'$ and $\varphi_{E'}$) out of given $\varphi_D$ and $\varphi_E$. Thus, only one essential free parameter, e.g. $\varphi_D/\varphi_E$, remains for our construction of one-particle eigenvectors.

## 2 Two-particle states

How can the superposition of two one-particle states of Section 1 look like? The experience of studying the 2 + 1-dimensional classical integrable models hints that probably the “scattering” of two particles on one another must be trivial, i.e. it makes sense to assume for the “amplitude of the event that two particles are in two different points $F$ and $G$ of the lattice” the form

$$\Phi_{FG} = \varphi_F \psi_G + \varphi_G \psi_F, \tag{10}$$

where $\varphi_\ldots$ and $\psi_\ldots$ are one-particle amplitudes like those constructed in Section 1.

To see how $\Phi_{FG}$ transforms under the action of transfer matrix, let us decompose $\varphi_\ldots$ and $\psi_\ldots$, considered as functions of $F$, in sums over triangles of the type $DCE$ in Fig. 1, i.e. represent $\varphi_F$ and $\psi_F$ as sums of summands each of which equals zero if $F$ lies beyond the corresponding triangle. In this way, $\Phi_{FG}$ naturally decomposes in a sum over (non-ordered) pairs of such triangles, including pairs of two coinciding...
triangles. We want that $\Phi_{FG}$ be transformed by the transfer matrix in an expression of the same form (10), with $\varphi_\ldots$ and $\psi_\ldots$ changed to their images with respect to this action.

It is easy to see that this holds automatically if $F$ and $G$ belong to different triangles. So, it remains to consider the case where $F$ and $G$ belong to the same triangle, say triangle $DCE$ in Fig. 1. When this triangle is transformed by the transfer matrix in the triangle $D'C'E'$ of Fig. 2, the one-particle amplitudes are transformed according to (7):

\[
\begin{pmatrix}
\varphi_{D'} \\
\varphi_{E'}
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \begin{pmatrix}
\varphi_D \\
\varphi_E
\end{pmatrix}, \quad \begin{pmatrix}
\psi_{D'} \\
\psi_{E'}
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \begin{pmatrix}
\psi_D \\
\psi_E
\end{pmatrix},
\]

(11)

where one must add the conditions of type (8):

\[
\begin{align*}
\varphi_C &= a\varphi_D + b\varphi_E, \\
\psi_C &= a\psi_D + b\psi_E, \\
\varphi_{C'} &= a\varphi_{D'} + b\varphi_{E'}, \\
\psi_{C'} &= a\psi_{D'} + b\psi_{E'}.
\end{align*}
\]

(12)

On the other hand, the $S$-matrix of the work [2] acts trivially, i.e. as a unity matrix, in the 2-particle sector. Thus, it must be

\[
\Phi_{C'D'} = \Phi_{CD}, \quad \Phi_{C'E'} = \Phi_{CE}, \quad \Phi_{D'E'} = \Phi_{DE}.
\]

(13)

Together, the formulae (12) and (13) lead to the following conditions on the one-particle amplitudes $\varphi_\ldots$ and $\psi_\ldots$:

\[
\begin{align*}
\varphi_{E'}\psi_{D'} + \varphi_{D'}\psi_{E'} &= \varphi_{E}\psi_{D} + \varphi_{D}\psi_{E}, \\
\varphi_{D'}\psi_{D'} &= \varphi_{D}\psi_{D}, \\
\varphi_{E'}\psi_{E'} &= \varphi_{E}\psi_{E}
\end{align*}
\]

(14, 15, 16)

(with making no use of the explicit form (4) of coefficients $a$ and $b$).

The three conditions (14–16) together with (11) and (8) give, remarkably, just one condition

\[
-\gamma \varphi D \psi D + \alpha (\varphi D \psi E + \varphi E \psi D) + \beta \varphi E \psi E = 0
\]

(17)

on $\varphi_\ldots$ and $\psi_\ldots$. Recall that, according to Section 4, each of the vectors $\varphi$ and $\psi$ is parametrized by one parameter (besides a trivial scalar factor). Together they are parametrized by two parameters, but condition (17) subtracts one parameter. Thus, the two-particle eigenstates constructed in this section depend on one significant parameter.
3 The simplest string-like states

The matrix $S^T$, the transposed to the matrix $S$ of the work [4], has two families of vacuum vectors (which are vacuum covectors for $S$, see [2], formulae (2.13–2.15), and also (2.21, 2.22) and the text in a neighborhood of these latter ones). Here we will restrict ourselves to considering the first family, i.e. the vacuum vectors transformed by $S^T$ into themselves:

$$S^T(X^T(\zeta) \otimes Y^T(\zeta) \otimes Z^T(\zeta)) = X^T(\zeta) \otimes Y^T(\zeta) \otimes Z^T(\zeta),$$

(18)

$\zeta$ being a parameter taking values in an elliptic curve (compare with formula (1.12) from [2]). What we are going to do in this section can be done with the same success for the second family as well. Instead of doing that, let us note that the eigenstates from Sections 1 and 2 are built using both families of vacuum vectors, and in that sense the considerations in this section are (still) more trivial.

The simplest eigenvectors $\Omega(\zeta)$ of the transfer matrix, with the eigenvalue 1, are built as follows: fix $\zeta$ and put in correspondence to each point of type $D$ (Fig. 1) of the kagome lattice the vector $X(\zeta)^T$, to each point of type $D$—the vector $Y(\zeta)^T$, and of type $E$—the vector $Z(\zeta)^T$. Then take the tensor product of all those vectors. The formula (18) shows at once that this is indeed an eigenvector with eigenvalue 1.

A little bit more intricate eigenvectors, for which the eigenvalues in case of a finite lattice are roots of unity, can be constructed as follows. It is seen from formulae (2.13–2.15) of paper [2], where enter the values $x$, $y$ and $z$—ratios of two coordinates of vectors $X$, $Y$ and $Z$ respectively,—that the triple $X,Y,Z$ will remain vacuum if one makes one of the following changes:

$$\begin{align*}
(x, y, z) & \rightarrow (x, \frac{1}{y}, \frac{1}{z}), \\
(x, y, z) & \rightarrow (\frac{1}{x}, y, -\frac{1}{z}), \\
(x, y, z) & \rightarrow (-\frac{1}{x}, -\frac{1}{y}, z).
\end{align*}$$

(19) \quad (20) \quad (21)

It can be said that the changes (19), (20) and (21) affect respectively the sides $CE$, $DE$ and $DC$ of the triangle $DCE$ in Fig. 1. Obviously, two such changes, if applied successively, commute with one another.

To construct a vector whose transformation under the action of transfer matrix is easy to trace, let us act like this: first, select arbitrarily some strait lines—strings—going along the edges of the kagome lattice. Then, take the vector $\Omega(\zeta)$ and change it as follows: make in each triangle of the type $DCE$ the transformation(s) of type (19–21) if its corresponding side lies in a selected line.
The obtained vector—let us call it Θ—goes under the action of transfer matrix $T$ into a vector of a similar form, but with the properly shifted lines (the latter, let us recall, result from the intersection of the cubic lattice faces with a plane perpendicular to the vector $(1, 1, 1)$, and move in that plane when the plane itself moves). An eigenvector of $T$ can be now built in the form

$$
\cdots + \omega^{-1}T^{-1}\Theta + \Theta + \omega T\Theta + \omega^2T^2\Theta + \cdots,
$$

(22)

where in the case of a finite lattice the sum must be finite, and the number $\omega$—a root of unity of a proper degree, determined by the sizes of the lattice.

4 Discussion

In this paper, we are not trying to discuss the “physical” consequences of the eigenvectors constructed. Its only modest aim is to show that the classical Bethe ansatz could be relevant for 2+1-dimensional models and that, probably, it makes sense to search for eigenvectors of another, “string-like”, form as well (certainly, it would be of great interest to find a “string-like” eigenvector of a less trivial type than here).

The model discussed here was discovered by the author in 1989 [1]. Later on, a similar but different model was discovered by J. Hietarinta [3], and then it was shown by S.M. Sergeev, V.V. Mangazeev, Yu.G. Stroganov [4] that both those models are particular cases of one model, parallel in some sense to the Zamolodchikov model. This allows one to hope that the reach mathematical structures already discovered in connection with our model will be extended some time onto the Zamolodchikov model as well.

References

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