ON IWASAWA INVARIANTS OF MODULAR FORMS WITH REDUCIBLE AND NON-$p$-DISTINGUISHED RESIDUAL GALOIS REPRESENTATIONS

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Abstract. In the present paper, we study the $p$-adic $L$-functions and the (strict) Selmer groups over $\mathbb{Q}_\infty$, the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, of the $p$-adic weight one cusp forms $f$, obtained via the $p$-stabilization of weight one Eisenstein series, under the assumption that a certain Eisenstein component of the $p$-ordinary universal cuspidal Hecke algebra is Gorenstein. As an application, we compute the Iwasawa invariants of ordinary modular forms of weight $k \geq 2$ with the same residual Galois representations as the one of $f$, which in our setting, is reducible and non-$p$-distinguished. Combining this with a result of Kato [Kat04, Theorem 17.4.2], we prove the Iwasawa main conjecture for these forms. Also, we give numerical examples that satisfy the Gorenstein hypothesis.

A crucial point for the $p$-adic $L$-functions of $f$ is that under the Gorenstein hypothesis, we are able to define them, following Greenberg–Vatsal, as elements in the one-dimensional Iwasawa algebra by using Mazur–Kitagawa two-variable $p$-adic $L$-functions and then, to compute them explicitly via local explicit reciprocity law. For the (strict) Selmer groups of $f$, we compute them via the knowledge of the Galois representations of $f$ studied in [BDP22].

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1. Introduction

Let $p$ be a prime number, and let $N$ be a positive integer relatively prime to $p$. Let $g = \sum_{n \geq 1} a_n(g)q^n$ be a $p$-ordinary normalized cuspidal eigenform of level $\Gamma_1(Np)$, weight $k \in \mathbb{Z}_{\geq 1}$, and character $\varphi$ of conductor $N$ or $Np$. If $k \geq 2$, the associated $p$-adic $L$-function
\(L(g)\), constructed by Mazur–Tate–Teitelbaum [MTTS0], is an element in the one-dimensional Iwasawa algebra and interpolates special values of the complex \(L\)-function of \(g\). When \(k = 1\) and \(g\) is classical whose associated Galois representation is absolutely irreducible, if there exists a unique cuspidal Hida family passing through \(g\), then its \(p\)-adic \(L\)-function, also denote by \(L(g)\), was defined by Greenberg–Vatsal [GV00] and is again an element in the one-dimensional Iwasawa algebra. Recently, many works on the trivial zero conjecture, the Bloch–Kato conjecture, etc. rely on the study of \(p\)-adic \(L\)-functions and the Iwasawa main conjecture for modular forms.

1.1. Iwasawa main conjecture for modular forms. For any field \(L\), we denote by \(G_L := \text{Gal}(\bar{L}/L)\) the absolute Galois group of \(L\). Let \(\omega_p : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times\) denote the Teichmüller character, \(\kappa_p : \text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \to \mathbb{Z}_p^\times\) denote the \(p\)-adic cyclotomic character, and \(\mathbb{Q}_\infty \subset \mathbb{Q}(\zeta_{p^\infty})\) denote the cyclotomic \(\mathbb{Z}_p\)-extension of \(\mathbb{Q}\). Let \(K\) be a finite field extension of \(\mathbb{Q}_p\) containing \(a_n(g)\) for all \(n \in \mathbb{Z}_{\geq 1}\), and let \(\mathcal{O}\) denote the ring of integers of \(K\).

It is known that for any cusp form \(g\) as above, there exists a continuous irreducible ordinary Galois representation \(\rho_g : G_{\mathbb{Q}} \to \text{Aut}_K(V_g)\), where \(V_g\) is a \(2\)-dimensional vector space over \(K\), such that \(\rho_g\) is unramified outside \(NP\), \(\text{det}(\rho_g) = \varphi n_p^{k-1}\) and \(\text{Tr}(\text{Frob}_l) = a_l(g)\) for all \(l \not= NP\). Let \(T_g \subset V_g\) be a \(\mathbb{Q}_p\)-stable \(\mathcal{O}\)-lattice, and set \(A_{g,j} := (V_g/T_g) \otimes \omega_j^p\) for integers \(0 \leq j \leq p - 1\). It was proved by Greenberg [Gre89 Proposition 6] that the Pontryagin dual \(\text{Sel}(\mathbb{Q}_\infty, A_{g,j})^\vee\) of the Greenberg’s Selmer group \(\text{Sel}(\mathbb{Q}_\infty, A_{g,j})\) of \(g\) over \(\mathbb{Q}_\infty\) (see Section 3.1 for the definition) is a finitely generated \(\Lambda_{\mathcal{O}} := \mathcal{O}[X] \cong \mathcal{O}[1+p\mathbb{Z}_p]\)-module. If it is torsion, then we let \(\text{Char}_{\Lambda_{\mathcal{O}}}\text{Sel}(\mathbb{Q}_\infty, A_{g,j})^\vee \in \Lambda_{\mathcal{O}}\) denote the characteristic polynomial of \(\text{Sel}(\mathbb{Q}_\infty, A_{g,j})^\vee\) over \(\Lambda_{\mathcal{O}}\) (see (3.3) for the definition), and let \(L(g, \omega_j^p) \in \Lambda_{\mathcal{O}}\) denote the \(\omega_j^p\)-branch of the \(p\)-adic \(L\)-function \(L(g) \in \mathcal{O}[\mathbb{Z}_p]\) that interpolates the special values \(L(g, \omega_j^p, r)\) of the \(L\)-function of \(g\) for \(r = 1, \ldots, k - 1\) if \(k \geq 2\); see (2.1) and (2.2). Following [GV00 Section 3] (for \(k \geq 2\) and [GV20] (for \(k = 1\)), the Iwasawa main conjecture for \((g, j)\) is as follows.

**Conjecture 1.1** (Main conjecture). Let the notation be as above. Then, the \(\Lambda_{\mathcal{O}}\)-module \(\text{Sel}(\mathbb{Q}_\infty, A_{g,j})^\vee\) is torsion, and we have

\[
L(g, \omega_j^p) \cdot \Lambda_{\mathcal{O}}[1/p] = \text{Char}_{\Lambda_{\mathcal{O}}[1/p]}\text{Sel}(\mathbb{Q}_\infty, A_{g,j})^\vee.
\]

Since we do not assume that the residual representation \(\bar{\rho}_g\) of \(\rho_g\) is irreducible, the lattice \(T_g\) is not unique in general and hence, we have to invert \(p\) in the statement of Conjecture 1.1 as different choices of lattices can only change the characteristic polynomial by a power of \(p\) [Gre91 p. 102] (also see [BP19 Section 4.3]).

Conjecture 1.1 is known for many cases when \(k \geq 2\). In his seminal work [Kat04 Theorem 17.4], Kato proved that \(\text{Sel}(\mathbb{Q}_\infty, A_{g,j})^\vee\) is torsion and the divisibility

\[
\text{Char}_{\Lambda_{\mathcal{O}}[1/p]}\text{Sel}(\mathbb{Q}_\infty, A_{g,j})^\vee | L(g, \omega_j^p) \cdot \Lambda_{\mathcal{O}}[1/p].
\]
If we further assume that the residual representation $\overline{\rho}_g$ is irreducible and $p$-distinguished, the other divisibility of (1.2) was proved by Skinner–Urban [SU14, Theorem 1] under some technical assumptions. When the residual representation $\overline{\rho}_g$ is reducible and $p$-distinguished, some results were proved by Greenberg–Vatsal [GV00, (16) and Theorem 3.12] and Bellaïche–Pollack [BP19, Corollary 5.13]. In the case of $k = 1$ and $\overline{\rho}_g$ being absolutely irreducible, it was proved by Greenberg–Vatsal [GV20] that $\text{Sel}(\mathbb{Q}_\infty, A_{g,j})^\vee$ is torsion and proved by Maksoud [Mak18, Theorem C] that the divisibility (1.2) holds if one further assumes that $\overline{\rho}_g$ is $p$-distinguished and $p$ does not divide the order of the image of $\rho_g$. However, the equality (1.1) is still open. Also, it is not clear a priori, but a consequence of (1.1), that the $p$-adic $L$-functions $L(g, \omega_j^p)$ are non-trivial; see [GV20, Remark 5.5].

For modular forms of weight $k \geq 1$ whose residual Galois representations are reducible and non-$p$-distinguished, the conjectural equality (1.1) is still open.

1.2. Main results. A main goal of this paper is to study the $p$-adic $L$-functions and the (strict) Selmer groups over $\mathbb{Q}_\infty$ of the weight one $p$-adic cusp forms $f$ obtained by the $p$-stabilization of classical weight one Eisenstein series, under a Gorenstein hypothesis on a certain Eisenstein component of the ordinary universal cuspidal Hecke algebra. Another aim is to use results of the first goal to compute the Iwasawa invariants of ordinary modular forms of weight $k \geq 2$ whose residual Galois representations are the same as $\overline{\rho}_f$ that is reducible and non-$p$-distinguished. As an application, we prove Conjecture 1.1 for these forms.

To be more precise, we now assume $p \geq 3$, $Np \geq 5$, and $p \nmid \phi(N)$, where $\phi$ is the Euler’s phi function. Let $\chi$ be an odd primitive Dirichlet character of conductor $N$ with $\chi(p) = 1$, set $\theta := \chi \omega_p$, and set $\mathcal{O} := \mathbb{Z}_p[\theta]$ the $\mathbb{Z}_p$-algebra generated by the values of $\theta$. In this setting, the $p$-stabilization of the weight one Eisenstein series

$$E_1(\chi, \frac{1}{2}) := 2^{-1}L(0, \chi) + \sum_{n=1}^{\infty} \left( \sum_{d|n} \chi(d) \right) q^n$$

is unique, denote by $f$. In their work [BDP22, Proposition 4.7], Betina–Dimitrov–Pozzi showed that the weight one form $f$ is a $p$-adic cusp form (i.e. the constant terms of $f$ at varies $p$-ordinary cusps are zero) and studied the geometry of the eigencurve at the weight one point arise from $f$. As a result, they proved that there exists a unique $p$-ordinary cuspidal Hida family passing through $f$; see Theorem A(i) of op. cit.

Let $\mathcal{E}(\theta, \frac{1}{2})$ be the $\Lambda_{\mathcal{O}}$-adic Eisenstein series with nonzero constant term. We refer the reader to [SW21, p. 9] for the definition of $\Lambda_{\mathcal{O}}$-adic modular forms and the definition of $\mathcal{E}(\theta, \frac{1}{2})$. By definition, the weight one specialization of $\mathcal{E}(\theta, \frac{1}{2})$ is $f$. Let $h_{\theta,m}$ denote the localization, at the maximal ideal $m$ containing the Eisenstein ideal $I_\theta$ of $\mathcal{E}(\theta, \frac{1}{2})$ defined by (2.5), of the universal ordinary cuspidal Hecke algebra $\mathcal{H}_\theta$, generated over $\Lambda_{\mathcal{O}}$ by the adjoining Hecke operators $T_\ell^*$ for all $\ell \nmid Np$ and $U_q^*$ for all $q\mid Np$. The subindex “$\theta$” indicates that the action of the diamond operators $(l)$ is given by $\theta(l)$ for all $l \nmid Np$. 


From now on, we assume that the cuspidal Hecke algebra $h_{\theta, m}$ is Gorenstein; see Section 5 for examples satisfying this assumption. We can then define the $p$-adic $L$-function $L(f, \omega_p^j)$ of $f$ following Greenberg–Vatsal; see (2.6). From the discussion in Section 1.1, one can ask many interesting questions in the study of the Iwasawa theory for modular forms in this setting. For instance,

(Q1) can one compute the $p$-adic $L$-function $L(f, \omega_p^j)$ of $f$ is without using Conjecture 1.1?

(Q2) which lattice $T_g$ of $V_g$ can make Conjecture 1.1 hold with the same algebraic and analytic $\mu$-invariants?

A novelty of our work is to assert that the answer of the question (Q1) is affirmative by computing $L(f, \omega_p^j)$ explicitly via explicit local reciprocity law (Theorem 1.2 (1)). This is the first example of the $p$-adic $L$-functions of weight one forms that can be computed explicitly.

Another novelty of our work is that we find a way to compute the (residual) Selmer groups when the (residual) Galois representations are semisimple and non-$p$-distinguished (1.4 and 1.8). As an application, we prove some cases of Conjecture 1.1 and answer the question (Q2) for which the choice of $T_g$ will be addressed in Section 3.2.

Let $X^{(\chi^{-1} \omega_p^j)}_{Np^\infty}$ (resp. $X^{(\omega_p^j)}_{p^\infty}$) denote $\chi^{-1} \omega_p^j$-component (resp. $\omega_p^j$) of the unramified everywhere (resp. unramified outside $p$) Iwasawa module defined via (2.13). It was proved by Iwasawa [Iwa73] that they are finitely generated torsion $\Lambda_{O}$-modules if $j \in \mathbb{Z} \geq 0$ is even. In addition, let $u \in 1 + p\mathbb{Z}_p$ be a fixed topological generator. The following is our main result for the weight one form $f$.

**Theorem 1.2.** Let the notation be as above, and let $0 \leq j \leq p-1$ be any even integer. Assume that the cuspidal Hecke algebra $h_{\theta, m}$ is Gorenstein.

1. One has

$$L(f, \omega_p^j) \sim_{O} G_{\omega_p^j}(X) G_{X^{\chi^{-1}}_{Np^\infty}}(u(1+X)^{-1} - 1),$$

where $\sim_{O}$ means equal up to a unit in $O$ and for an even Dirichlet character $\eta$, the formal power series $G_{\eta}(X)$ is the formal power series expression of the Kubota–Leopoldt $p$-adic $L$-function $L_p(s, \eta)$; see (2.8).

2. The $\Lambda_O$-module $\text{Sel}(\mathbb{Q}_\infty, A_{f,j})^\vee$ is torsion and one has

$$\text{Char}_{\Lambda_O} \text{Sel}(\mathbb{Q}_\infty, A_{f,j})^\vee = \begin{cases} \text{Char}_{\Lambda_O} X^{(\chi^{-1})}_{Np^\infty} & \text{if } j = 0 \\ \text{Char}_{\Lambda_O} X^{(\omega_p^j)}_{p^\infty} \cdot \text{Char}_{\Lambda_O} X^{(\chi^{-1} \omega_p^j)}_{Np^\infty} & \text{if } j \neq 0. \end{cases}$$

**Remark 1.3.** It was proved in [BDP22, Theorem A(i)] that the localization of $h_{\theta, m}$ at the height one prime $p_f$ corresponding to $f$ is Gorenstein. Therefore, for even integers $j$, we know $L(f, \omega_p^j)$ is non-zero without any assumption by Theorem 1.2 (1).

A direct consequence of Theorem 1.2 is that when $\chi$ is a quadratic character, the weight one Eisenstein series $E_1(\chi, 1)$ coincides with the CM theta series $\theta_1$ attached to the trivial
character. Then, \([1.3]\) reads that the \(p\)-adic \(L\)-function \(L(f, \omega^0)\) coincides, up to a unit in \(\mathcal{O}\), with the trivial character branch of the cyclotomic Katz \(p\)-adic \(L\)-function (Corollary \([2.3]\)).

Let \(\mu^{an}(g,j)\), \(\lambda^{an}(g,j)\), \(\mu^{alg}(A_{g,j})\), and \(\lambda^{alg}(A_{g,j})\) be analytic and algebraic Iwasawa invariants defined via (2.11) and (3.5), respectively. The following is our main result for weight \(k \geq 2\) cusp forms \(g\).

**Theorem 1.4.** Let \(g\) be an ordinary cusp form of weight \(k \geq 2\) passing through by a cuspidal family corresponding to a minimal prime ideal of \(\mathfrak{h}_{\theta,m}\), and let \(T_{g,j}\) be the lattice defined in Section \([3.2]\). Also, let other notation and the assumption be as in Theorem \([1.2]\). Then, one has

\[
\lambda^{an}(g,j) = \lambda(G_{\omega_p^j}(X)) + \lambda(G_{\chi^{-1}\omega_p^j}(u(1 + X)^{-1} - 1)).
\]

Moreover, one has \(\mu^{alg}(A_{g,j}) = 0\) and

\[
\lambda^{alg}(A_{g,j}) = \begin{cases} 
\lambda(\text{Char}_{\Lambda_{\mathcal{O}}} X_{Np^\infty}^{(\chi^{-1})}) - 1 & \text{if } j = 0 \\
\lambda(\text{Char}_{\Lambda_{\mathcal{O}}} X_{p^\infty}^{(\omega_p^j)}) + \lambda(\text{Char}_{\Lambda_{\mathcal{O}}} X_{Np^\infty}^{(\chi^{-1}\omega_p^j)}) & \text{if } j \neq 0.
\end{cases}
\]

Here, the \(\lambda\)-invariant of a formal power series in \(\Lambda_{\mathcal{O}}\) is defined by \((2.10)\).

Recall that the Iwasawa main conjecture for abelian extensions of \(\mathbb{Q}\), proved by Mazur–Wiles \([MWS]\), asserts that

\[
\text{Char}_{\Lambda_{\mathcal{O}}} X_{Np^\infty}^{(\chi^{-1}\omega_p^j)} = (G_{\chi^{-1}\omega_p^j}(u(1 + X)^{-1} - 1)) \text{ and } \text{Char}_{\Lambda_{\mathcal{O}}} X_{p^\infty}^{(\omega_p^j)} = (G_{\chi^{-1}}(X)).
\]

Thus, by \((1.2)\), Theorems \([1.2]\) and \([1.4]\) would yield the following corollary.

**Corollary 1.5.** Let the notation and the assumption be as in Theorem \([1.4]\). Let \(0 \leq j \leq p - 1\) be any even integer. Then, Conjecture \([1.4]\) holds for \((g,j)\). Also, Conjecture \([1.4]\) holds for \((f,j)\) if \(j \neq 0\).

To end this subsection, we make a remark when \(j\) is an odd integer.

**Remark 1.6.** Let \(j \leq p - 1\) be a positive odd integer. We show in Proposition \([2.2]\) that \(L(f, \omega_p^j) = 0\), and hence, one has \(\mu^{an}(g,j) \neq 0\); see Corollary \([2.3]\). In addition, by the same argument of the proof of \((1.6)\), one can show that \(\mu^{alg}(A_{g,j}) \neq 0\); see Remark \([1.5]\). However, we can not say anything about the \(\lambda\)-invariants \(\lambda^{an}(g,j)\) and \(\lambda^{alg}(A_{g,j})\).

1.3. **Idea of the proof.** We now explain the idea of the proof of Theorems \([1.2]\). We first explain the proof of \((1.3)\) presented in Section \([2]\). By the works of Sharifi \([Sha11]\) and Fukaya–Kato \([FK12]\), we know that the plus part \(L^+_{\theta,m}\) of the Mazur–Kitagawa two-variable \(p\)-adic \(L\)-function modulo the Eisenstein ideal \(I_{\theta}\) coincides with the cup product of cyclotomic units \((2.14)\). We then show in Lemma \((2.5)\) that if one further modulo the height one prime \(p_f\) of \(\mathfrak{h}_{\theta,m}\) corresponding to \(f\), the cup product \((2.14)\) is indeed in the local Galois cohomology and
hence, can be computed explicitly via local explicit law. The computation will be presented in Section 2.4.

We next explain the proof of (1.4) presented in Section 4. By [SW21, Theorem 1.5], the action of $G_{\mathbb{Q}}$ on the lattice $T_f$ chosen in Section 3.2 is semisimple. We can then compute $\text{Sel}(\mathbb{Q}_\infty, A_{f,j})$ by writing elements as $(x, y) \in H^1(G_{\mathbb{Q}_\infty, \Sigma_p, K/\text{slash}O}), H^1(G_{\mathbb{Q}_\infty, \Sigma_{Np}, K/\text{slash}O}(\omega_j^p))$ satisfying some local conditions. When $j = 0$, the computation is straightforward, since $x$ is always zero by Iwasawa theory. When $j \neq 0$, the computation is more complicated as both $x$ and $y$ could be nonzero and hence, we have to study the Selmer conditions on $x$ and $y$.

A crucial point is that one knows the basis of the $G_{\mathbb{Q}_p}$-stable line in the ordinary filtration (3.2) (Proposition 3.1). By using this, we see the condition on $x$ and $y$ (4.9) and then further prove (1.4).

Finally, we explain the idea of the proof of Theorem 1.4. The assertion on the analytic invariants follows from (1.3) (Corollary 2.4). Our idea to prove (1.6) is inspired by the work of Emerton–Pollack–Weston [EPW06] generalizing the work of Greenberg–Vatsal [GV00]. The argument will reduce to compute the residual Selmer groups $\text{Sel}(\mathbb{Q}_\infty, A_{g,j})[\pi]$ defined in Section 3.4. Here $\pi$ is a uniformizer of the Hecke ring of $g$. To this end, there are three steps.

1. Show that the Iwasawa module $\text{Sel}(\mathbb{Q}_\infty, A_{g,j})^\vee$ does not have any finite submodules (Proposition 3.5). As a consequence (Corollary 3.6), we know that $\mu^\text{alg}(A_{g,j}) = 0$ if and only if $\text{Sel}(\mathbb{Q}_\infty, A_{g,j})[\pi]$ is finite, and if this is the case, $\lambda^\text{alg}(g, j)$ coincides with $\dim_{\mathbb{F}} \text{Sel}(\mathbb{Q}_\infty, A_{g,j})[\pi]$. Here $\mathbb{F}$ is the residue field of the Hecke ring of $g$.

2. Find the explicit relationship between $\text{Sel}(\mathbb{Q}_\infty, A_{g,j})[\pi]$ and $\text{Sel}(\mathbb{Q}_\infty, A_{g,j})[\pi]]$ (Proposition 3.7).

3. Compute the residue Selmer group $\text{Sel}(\mathbb{Q}_\infty, A_{g,j})[\pi]$ explicitly (Section 4).

The result in step (3) is computed via a similar computation of $\text{Sel}(\mathbb{Q}_\infty, A_{f,j})$ and can be stated as follows

$$\dim_{\mathbb{F}} \text{Sel}(\mathbb{Q}_\infty, A_{g,j})[\pi] = \begin{cases} \dim_{\mathbb{F}} \text{Hom}(X_{Np}^{(\chi^{-1})}, \mathbb{F}) & \text{if } j = 0 \\ \dim_{\mathbb{F}} \text{Hom}(X_{p}^{(\omega_j^p)}, \mathbb{F}) + \dim_{\mathbb{F}} \text{Hom}(X_{Np}^{(%1-\omega_j^p)}, \mathbb{F}) & \text{if } j \neq 0. \end{cases}$$

1.4. Outline. Section 2 focuses on the $p$-adic $L$-functions of modular forms. We will briefly review the interpolation property of the $p$-adic $L$-functions of ordinary modular forms of weight $k \geq 2$ and define the $p$-adic $L$-function of the weight one form $f$ under the Gorenstein hypothesis. Main results will state in Section 2.3 and the proof will be given in Section 2.4. In Section 3, we review the definition of Greenberg’s Selmer group and collect its properties used for the reduction explained in Section 1.3. Also, we explain the choice of lattice $T_g$ and study the associated residual representation. Section 4 is devoted to computing (residual)
Selmer groups. In Section 5, we give examples of \((N, p)\) for which the cuspidal Hecke algebra \(\mathfrak{h}_{\theta,m}\) is Gorenstein when \(\chi\) is a quadratic character.

Acknowledgments. The first authors (S.S.) would like to thank A. Betina, M. Dimitrov, A. Maksoud, and C. Wang-Erickson for stimulating discussions and helpful comments on the earlier version of the manuscript.

Shih was supported by Austrian Science Fund (FWF) under Grant No. START-Prize Y966. Wang was partially supported by Morningside Center of Mathematics in his postdoctoral studies.

2. Analytic Iwasawa invariants

Let the notation be as in the Introduction. Throughout this paper, we fix, once and for all, embeddings \(t_p : \mathbb{Q} \to \overline{\mathbb{Q}}_p\) and \(t_\infty : \mathbb{Q} \to \mathbb{C}\). We can then consider modular forms over \(\overline{\mathbb{Q}}_p\) via \(t_p\).

The aim of this section is to prove Theorems 1.2.1 and 1.6.1. We will first review in Section 2.1 the interpolation properties of the \(p\)-adic \(L\)-functions of ordinary modular forms of weight \(k \geq 2\) following [FK12, Section 4.5.1]. We then explain in Section 2.2 how one can recover these \(p\)-adic \(L\)-functions and how to define the \(p\)-adic \(L\)-function of \(f\) from Mazur-Kitagawa two-variable \(p\)-adic \(L\)-functions under the Gorenstein hypothesis. The main result is stated in Section 2.3 and the proof is presented in Section 2.4.

2.1. \(p\)-adic \(L\)-functions for ordinary cusp forms of weight \(k \geq 2\). Let \(g = \sum_{n \geq 1} a_n(g)q^n\) be a \(p\)-ordinary normalized eigenform of weight \(k \geq 2\) in \(S_k(\Gamma_1(Np), \phi, \overline{\mathbb{Q}}_p)\), where \(\phi\) is a primitive character of conductor \(N\) or \(Np\). In case of the conductor of \(\phi\) being \(N\), \(g\) is the ordinary \(p\)-stabilization of a modular form of level \(N\) and character \(\phi\). Let \(K_g := \mathbb{Q}_p(a_n(g) \mid n \in \mathbb{Z}_{\geq 1})\) be a finite field extension of \(\mathbb{Q}_p\), and let \(\mathcal{O}_g\) denote the ring of integers of \(K_g\).

Let \(\Omega_g^*\) be a period of \(g\). The \(p\)-adic \(L\)-function of \(g\) with respect to the period \(\Omega_g^k\), denoted by \(\mathcal{L}(g)\), is the unique element \(\in \mathcal{O}_g[\mathbb{Z}^*_p]\) satisfying the following relation. For all \(r = 1, \ldots, k-1\) and for a primitive Dirichlet character \(\psi\) such that \(\psi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p\), let \(\mathcal{L}(g, \psi, r) \in \overline{\mathbb{Q}}_p\) be the image of \(\mathcal{L}(g)\) under the map \(\psi \cdot \sigma^{-1} : \mathcal{O}_g[\mathbb{Z}^*_p] \to \overline{\mathbb{Q}}_p\) sending the group element \([a]\) for \(a \in \mathbb{Z}^*_p\) to \(a^{-r}\psi(a)\). If \(\psi\) is non-trivial, one has

\[
\mathcal{L}(g, \psi, r) = (r-1)!p^n(r-1)a_p(g)^{-n}\tau(\psi)(-2\pi i)^{1-r}\mathcal{L}(f, \psi, r) \in \overline{\mathbb{Q}}_p.
\]

Here, \(\pm = (-1)^{r-1}\psi(-1)\) and \(\tau(\psi)\) is the Gauss sum of \(\psi\). If \(\psi\) is a trivial character, then one has

\[
\mathcal{L}(f, 1, r) = (r-1)!(1-\mu(p)p^{-1}a_p(g)^{-1})(1-\mu(p)p^{-1}a_p(g)^{-1})(-2\pi i)^{1-r}\mathcal{L}(f, r) \in \overline{\mathbb{Q}}_p.
\]

Here \(\mathcal{L}(f, \psi, r)\) and \(\mathcal{L}(f, r)\) are the complex \(L\)-functions. For \(j = 0, \ldots, p-1\), denote by \(\mathcal{L}(g, \omega_p^j) \in \Lambda\mathcal{O}_g\) the \(\omega_p^j\)-branch of \(\mathcal{L}(g)\).
2.2. Mazur–Kitagawa two-variable $p$-adic $L$-functions. Let the notation be as in the previous subsection. Set $H := \lim_{\to} H^1_{\et}(X_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\text{ord}}$, where the superscript “ord” means the ordinary part for the dual Hecke operator $U_p^*$. Let $\mathfrak{h} := \mathbb{Z}_p[T_n \mid n \geq 1]$ be the universal ordinary Hecke algebra acting on $H$. One can identify $H^1_{\et}(X_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)$ with a certain homology group by Poincaré duality and the comparison between Betti (co)homology groups and étale (co)homology groups. Namely, one has an isomorphism of $\mathbb{Z}_p[\mathbb{G}_a]$-modules

\begin{equation}
H^1_{\et}(X_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)(1) \cong H_1(X_1(Np^r)(\mathbb{C}), \mathbb{Z}_p).
\end{equation}

We will view modular symbols $\{\alpha, \beta\}_{r \geq 1} \in \lim_{\to} H_1(X_1(Np^r)(\mathbb{C}), \mathbb{Z}_p)$ as elements of $H$ via the isomorphism (2.3) without any further notice.

In [Kit94], Kitagawa constructed Mazur–Kitagawa two-variable $p$-adic $L$-functions by varying the $p$-adic $L$-functions $\mathcal{L}(g)$ $p$-adically analytically in $k$. Following [FK12 §4.4.2], the Mazur–Kitagawa two-variable $p$-adic $L$-function $\mathcal{L}$ is defined as

\[ \mathcal{L} := \left( \sum_{\alpha \in (\mathbb{Z}/p^r \mathbb{Z})^*} U^*(p)^{-r}(\infty, a/p^r) \otimes [a] \right)_{r \geq 1} = \left( \sum_{\alpha \in (\mathbb{Z}/p^r \mathbb{Z})^*} U^*((p)^{-r}[Na : 1_r] \otimes [a]) \right)_{r \geq 1}. \]

By [Sha11, Lemma 3.1], it is an element in $H[\mathbb{Z}_p^\times]$ since $([Na : 1_r]_{r \geq 1}$ is in $H$. For a Dirichlet character $\eta$ of conductor $N$ or $Np$, we denote by $H_\eta$ the $\eta$-component of $H$ on which diamond operators act via the character $\eta$, and denote by $\mathcal{L}_\eta$ the image of $\mathcal{L}$ in $H_\eta$. Also, for a maximal ideal $\mathfrak{m}$ of the cuspidal algebra $\mathfrak{h}_\eta$ acting on $H_\eta$, we denote by $H_{\eta, \mathfrak{m}}$ the localization of $H_\eta$ at $\mathfrak{m}$, and denote by $\mathcal{L}_{\eta, \mathfrak{m}}$ the image of $\mathcal{L}_\eta$ in $H_{\eta, \mathfrak{m}}$. Moreover, we denote by $\mathcal{L}_{\eta, \mathfrak{m}}(\omega_p^\mathfrak{m}) \in H_{\eta, \mathfrak{m}}[1 + p\mathbb{Z}_p] \cong H_{\eta, \mathfrak{m}}[X]$ the $\omega_p^\mathfrak{m}$-branch of $\mathcal{L}_{\eta, \mathfrak{m}}$.

If the cuspidal Hecke algebra $\mathfrak{h}_{\eta, \mathfrak{m}}$ is Gorenstein, the cohomology $H_{\eta, \mathfrak{m}}$ is free $\mathfrak{h}_{\eta, \mathfrak{m}}$-module of rank two and hence, one can view $\mathcal{L}_{\eta, \mathfrak{m}}$ as an element in $\mathfrak{h}_{\eta, \mathfrak{m}}[\mathbb{Z}_p^\times]$ via an isomorphism

\begin{equation}
H_{\eta, \mathfrak{m}} \cong \mathfrak{h}_{\eta, \mathfrak{m}}^2.
\end{equation}

If this is the case, we fix, once and for all, such an isomorphism which allows us to take the period $\Omega^\mathfrak{m}_\theta$ to be the canonical period defined in [Vat99 (5)] (or see [EPW06 Section 3.3]) for all weight $k \geq 2$ modular forms $g$ in passing through by a cuspidal family corresponding to a minimal prime of $\mathfrak{h}_{\eta, \mathfrak{m}}$. The canonical period is well-defined up to a $p$-adic unit. One can then recover $\mathcal{L}(g, \omega_p^\mathfrak{m})$ by

\[ \mathcal{L}(g, \omega_p^\mathfrak{m}) = \mathcal{L}_{\eta, \mathfrak{m}}(\omega_p^\mathfrak{m})(\text{mod} p_g) \in \mathfrak{h}_{\eta, \mathfrak{m}}/p_g[X] \cong \mathcal{O}_g[X], \]

where $p_g$ is the kernel of the algebraic homomorphism $\mathfrak{h}_{\eta, \mathfrak{m}} \to \mathcal{O}_g$ corresponding to $g$.

We now take $\eta = \theta := \chi \omega_p$, where $\chi$ is an odd primitive Dirichlet character of conductor $N$ with $\chi(p) = 1$. Recall that $f$ denotes the unique $p$-stabilization of the weight one Eisenstein series $E_1(\chi, 1)$. For simplicity, we write $\mathcal{O} := \mathcal{O}_f = \mathbb{Z}_p[\chi]$ and write $\Lambda := \mathcal{O}[X]$. Let $\pi$ denote the uniformizer of $\mathcal{O}$, and let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{h}_\theta$ containing the Eisenstein ideal $I_\theta$.

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of the Eisenstein family $E(\theta, 1)$ generated by
\begin{equation}
U_q^* - 1 \text{ and } T_q^* - \theta^{-1}(\ell) - 1 \text{ for all } q \mid Np \text{ and } \ell \not\mid Np.
\end{equation}
As the weight one specialization of $E(\theta, 1)$ is $E_1(\chi, 1)$, one sees that $p_f$ contains $I_\theta$.

From now on, we assume that the cuspidal Hecke algebra $\mathfrak{H}_\eta$ is Gorenstein. Following [GV20], the $\omega_p^j$-branch $L(f, \omega_p^j)$ of the $p$-adic $L$-function of $f$ is defined by
\begin{equation}
L(f, \omega_p^j) := L_{\theta, m}(\omega_p^j)(\text{mod}p_f) \in \Lambda.
\end{equation}
This is well-defined since it was shown in [BDP22, Theorem A(i)] that there exists a unique ordinary cuspidal Hida family passing through $f$.

2.3. The $p$-adic $L$-function of weight one form $f$. The aim of this subsection is to describes the $p$-adic $L$-function $L(f, \omega_p^j)$ explicitly.

Recall that for an even Dirichlet character $\eta$, the Kubota–Leopoldt $p$-adic $L$-function $L_p(s, \eta)$ admits the interpolation property
\begin{equation}
L_p(1 - n, \eta) = L(1 - n, \eta \omega_p^{-n})(1 - \eta \omega_p^{-n}(p)p^{n-1})
\end{equation}
for all $n \in \mathbb{Z}_{\geq 1}$ and admits a power series expression
\begin{equation}
L_p(1 - s, \eta) = G_\eta(u^s - 1),
\end{equation}
if $\eta$ of type $S$ (i.e. $F_\eta \cap \mathbb{Q}_\infty = \mathbb{Q}$, where $F_\eta$ is the field extension of $\mathbb{Q}$ cut out by $\eta$). Moreover, one has
\begin{equation}
G_{\eta \psi}(T) = G_\eta(\psi(u)(1 + T) - 1)
\end{equation}
for all characters $\psi$ of type $W$ (i.e. $F_\psi \subset \mathbb{Q}_\infty$). By Weierstrass Preparation Theorem [Was97, Theorem 7.3], every element $F(X)$ in $\Lambda$ can be written uniquely as $\pi^n f(X)u(X)$ for some $n \in \mathbb{Z}_{\geq 0}$, $u(X) \in \Lambda^*$, and distinguished polynomial $f(X)$. The $\lambda$- and $\mu$-invariants of $F(X)$, denote respectively by $\lambda(F(X))$ and $\mu(F(X))$, are defined as
\begin{equation}
\lambda(F(X)) = \deg(f(X)) \text{ and } \mu(F(X)) = n.
\end{equation}
To state the result of $L(f, \omega_p^j)$, we set
\begin{equation}
\lambda_{an}(f, j) := \lambda(L(f, \omega^j)) \text{ and } \mu_{an}(f, j) := \mu(L(f, \omega^j)).
\end{equation}

**Theorem 2.1.** Assume that the cuspidal Hecke algebra $\mathfrak{H}_{\eta, m}$ is Gorenstein. Then, one has
\begin{equation}
L(f, \omega_p^j) \sim \mathcal{O} G_{\omega_p^j}(X)G_{\chi_{p^{-1}}}(u(X + 1)^{-1} - 1)
\end{equation}
for all even integers $j = 0, \ldots, p - 1$. In particular, one has $\mu_{an}(f, j) = 0$ and $\lambda_{an}(f, j) = \lambda(G_{\omega_p^j}(X)) + \lambda(G_{\chi_{p^{-1}}}(u(X + 1)^{-1} - 1))$. Recall from the Introduction that $\sim \mathcal{O}$ means equal up to a unit in $\mathcal{O}$. 

Proof. Let \( j \leq p-1 \) be an even positive integer. Applying Corollary 2.7 below to any finite order characters \( \psi \) of \( \mathbb{Z}_p^* \) with \( \omega_p^j = \psi|_{(\mathbb{Z}/p\mathbb{Z})^\times} \) and \( \psi' := \psi|_{1+p\mathbb{Z}_p} \), we see that \( L(f, \omega_p^j) \) is, up to a unit in \( \mathcal{O} \), a product of two elements \( F(X) \) and \( H(X) \) in \( \Lambda \) such that

\[
F(\psi'(u) - 1)H(\psi'(u) - 1) = L_p(1, \omega_p^j \psi')L_p(0, \chi \omega_p^{1-j} \psi')
\]

for all finite order characters \( \psi' \) of \( 1+p\mathbb{Z}_p \). Then, the assertion \( (2.12) \) follows from \( (2.8) \) and \( (2.9) \). Moreover, the vanishing of the \( \mu \)-invariant follows from a work of Ferrero–Washington [FW79] and \( (1.7) \) that \( \mu(G_{\omega_p}(X)) = 0 = \mu(G_{\chi \omega_p^{1-j}}(X)) \).

We next study \( L_{\theta,m}(f, \omega^j) \) for odd integers \( j \). Let \( L_{\theta,m}^\infty(f) \) denote the support of \( L_{\theta,m}(f) \) on all odd character branches of \( L_{\theta,m}(f) \).

Proposition 2.2. Let the notation be as above. Then one has \( L_{\theta,m}^\infty(f) = 0 \).

Proof. Set \( 0L \in H[\mathbb{Z}_p^*] \) to be the image of \( L \in H[\mathbb{Z}_p^*] \) under the isomorphism

\[
H[\mathbb{Z}_p^*] \simeq H[\mathbb{Z}_p^*]; \quad x \otimes [a] \mapsto ax \otimes [a],
\]

and let \( 0L_{\theta,m}^\infty(f, \omega_p^j) \in \Lambda \) denote the image of \( 0L_{\theta,m}^\infty(\omega_p^j) \) in \( H_{\theta,m}^\infty /[f]X \) under the canonical isomorphism

\[
H_{\theta,m}^\infty /[f]X \simeq (H_{\theta,m}/[I_0])/(\langle f \rangle/I_0) \simeq \gamma(I_{\theta,m}/[I_0])/(\langle f \rangle/I_0) \simeq \mathcal{O}.
\]

By definition, for any odd integer \( 1 \leq j \leq p-1 \), \( L(f, \omega_p^j) \) coincides with \( 0L_{\theta,m}^\infty(\omega_p^j) \)(mod \( f \)) in \( \Lambda \). Therefore, to prove the assertion, it suffices to show that \( 0L_{\theta,m}^\infty(\text{mod } f) \) is trivial.

Following the notation in [FK12, Theorem 8.1.2], we let \( -1 \xi_{Np^\infty} \in \mathbb{Z}_p[(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^*] \) denote the \( p \)-adic \( L \)-function whose image under any character \( \psi \) of \( (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^* \) is the complex \( L \)-value of the partial \( L \)-function \( L^{(Np)}(s, \psi^{-1}) \) at \( s = -1 \). By Theorem 2.2.1 of \textit{op. cit.}, one has \( 0L_{\theta}^\infty \mod I_0 = 2\sigma_{-1} \cdot A_0 \xi \) for some \( A_0 \in Q(\mathbb{Z}_p[\mathbb{Z}_p^*]) \) and \( \xi \in \Lambda[\mathbb{Z}_p^*] \) obtained by the image of the \( p \)-adic \( L \)-function \( -1 \xi_{Np^\infty} \) under the morphism

\[
\mathbb{Z}_p[(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^*] \rightarrow \mathbb{Z}_p[(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^*][\mathbb{Z}_p^*] \rightarrow \Lambda[\mathbb{Z}_p^*].
\]

Here the first map sends \( [a] \) to \( [a]a \) and the second map is the natural projection. Note that the image \( -1 \xi_{Np^\infty} \) of \( -1 \xi_{Np^\infty} \) under the natural projection \( \mathbb{Z}_p[(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^*] \rightarrow \Lambda \) is the \( p \)-adic \( L \)-function whose image under any character \( \psi \) of \( 1+p\mathbb{Z}_p \) is the complex \( L \)-value \( L^{(Np)}(-1, \chi^{-1} \omega_p^{1} \psi^{-1}) \). It is clear that the \( p \)-adic \( L \)-function \( -1 \xi_{Np^\infty} \) admits a trivial zero and hence, the coefficients of \( \xi \) in \( \Lambda \) generates an ideal divisible by \( p_1 \), the ideal of \( \Lambda \) corresponding to the weight one specialization. As \( \langle f \rangle \cap \Lambda \) coincides with \( p_1 \), one obtains from the above discussion that \( 0L_{\theta,m}^\infty(\text{mod } f) \) is trivial. Thus, the assertion follows.

Note that if the character \( \chi \) is quadratic, then the weight one Eisenstein series \( E_1(\chi, 1) \) coincides with the theta series \( \theta_1 \) attached to the trivial character \( 1 \), defined in [Hid93, p. 235, (1)]. The following consequence of Theorem 2.1 asserts that the \( p \)-adic \( L \)-function
of \( f \) coincides, up to a unit in \( \mathcal{O} \), with the trivial character branch of the cyclotomic Katz \( p \)-adic \( L \)-function \( L^{K\alpha}_{cyc}(1) \) constructed in [Kat78].

**Corollary 2.3.** Let the assumption be as in Theorem 2.1. If \( \chi \) is quadratic, then one has

\[
L(f, \omega^0) \sim_\mathcal{O} L^{K\alpha}_{cyc}(1).
\]

**Proof.** The assertion follows from Theorem 2.1 and the Gross’s factorization [Gro80] for the trivial character.

\( \square \)

Define \( \lambda^{an}(g, j) \) and \( \mu^{an}(g, j) \) in the same manner as in (2.11). The following is a consequence of Theorem 2.1 and Proposition 2.2.

**Corollary 2.4.** Let \( g \) be a classical weight \( k \geq 2 \) ordinary cusp form in a Hida family corresponding to a minimal prime ideal of \( \mathfrak{h}_{\theta, m} \). If \( j \) is even, then one has

\[
\mu^{an}(g, j) = 0 \quad \text{and} \quad \lambda^{an}(g, j) = \lambda(G_{\omega_j}^0(X)) + \lambda(G_{\chi_\omega_{j-1}^0}(u(X + 1)^{-1} - 1))
\]

and if \( j \) is odd, then one has \( \mu^{an}(g, j) \neq 0 \).

**Proof.** The assertion essentially follows from the discussion in [EPW06, Section 3.7]. For the sake of convenience, we give a proof here. Let \( I(L_{\theta, m}(\omega_j^0)) \) be the ideal of \( \mathfrak{h}_{\theta, m} \) generated by the coefficient of \( L_{\theta, m}(\omega_j^0) \). It follows from Theorem 2.1 that the ideal \( I(L_{\theta, m}(\omega_j^0)) \) modulo the height one prime \( \mathfrak{p}_f \) is \( \mathcal{O} \) if \( j \) is even and is divisible by \( \mathfrak{p}_g \) if \( j \) is odd. It then follows from Proposition 3.7.3 of loc. cit. that \( L_{\theta, m} I(\mathfrak{h}_{\theta, m}) \) modulo \( \mathfrak{p}_g \) is \( \mathcal{O}_g \) in the former case, which is equivalent to the vanishing of \( \mu^{an}(g, j) \), and divisible by \( p \) in the latter case, which is equivalent to the non-vanishing of \( \mu^{an}(g, j) \). When \( j \) is even, one further has \( \lambda(g, j) = \lambda(f, j) \) by a remark after Definition 3.7.6 of loc. cit. that if \( \mu \)-invariants vanish, then \( \lambda \)-invariants do not change under local homomorphism of complete local rings.

\( \square \)

### 2.4. Computation of local cup products.

For \( N \in \mathbb{Z}_{\geq 1} \) prime to \( p \), let \( L_{\infty} \) (resp. \( M_{\infty} \)) be the maximal unramified (resp. unramified outside \( p \)) abelian pro-\( p \) extension of \( \mathbb{Q}(\zeta_{Np^\infty}) := \bigcup_{n=1}^{\infty} \mathbb{Q}(\zeta_{Np^n}) \) for \( N \in \mathbb{Z}_{\geq 1} \). Set

\[
X_{Np^\infty} := \text{Gal}(L_{\infty}/\mathbb{Q}(\zeta_{Np^\infty})) \quad \text{and} \quad \mathfrak{x}_{Np^\infty} := \text{Gal}(M_{\infty}/\mathbb{Q}(\zeta_{Np^\infty})).
\]

In addition, let \( L_{\infty, p} \subset L_{\infty} \) denote the field extension of \( \mathbb{Q}(\zeta_{Np^\infty}) \) in which \( p \) splits completely, and set \( X_{Np^\infty, p} := \text{Gal}(L_{\infty, p}/\mathbb{Q}(\zeta_{Np^\infty})) \). These groups have a natural action of \( \text{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}) \cong (\mathbb{Z}/Np\mathbb{Z})^* \times \mathbb{Z}_p \) via conjugation. For any Dirichlet character \( \eta \) of \( (\mathbb{Z}/Np\mathbb{Z})^* \), we denote by \( X_{Np^\infty, \eta} \) the submodule of \( X_{Np^\infty} \) on which \( (\mathbb{Z}/Np\mathbb{Z})^* \) acts via \( \eta \). Also, we denote by \( \mathfrak{x}_{Np^\infty, \eta} \) and \( X_{Np^\infty, p, \eta} \) in the same manner.

The map

\[
\varpi_\theta : H^1_\theta(1) \to \lim_{\leftarrow} H^2(\mathbb{Z}[\zeta_{Np^\infty}, \tfrac{1}{p}], \mathbb{Z}_p(2))^*_\theta =: S_\theta.
\]
defined by Sharifi [Sha11, Proposition 5.7] sends the Manin symbols \([u : v]_{r \geq 1}^+\) to the cup product \((1 - \zeta_{Np^r}^u, 1 - \zeta_{Np^r}^v)_{r \geq 1, \theta}\). Moreover, it is shown by Fukaya–Kato [FK12] Theorem 5.2.3] that the homomorphism \(\varpi_{\theta}\) factors through the quotient of \(I_{\theta}H_{\theta}^-(1)\).

Set \(p_1 := p_f \cap \Lambda\) which corresponds to the exceptional zero factor of \(G_{\chi_{\omega_p}}(X)\) in \(\Lambda\) and also corresponds to the kernel of the weight 1 specialization. We have seen in the previous subsection that \(p_f\) contains the Eisenstein ideal \(I_{\theta}\), so the map \(\varpi_{\theta}\) induces the following map

\[
\varpi_{\theta} : (H_{\theta,m}^-(1)/p_f H_{\theta,m}^-(1))[\mathbb{Z}_p^\times] \to (S_\theta/p_1 S_\theta)[\mathbb{Z}_p^\times].
\]

Recall from [SW21, Section 3.1] that we have a short exact sequence of \(\Lambda\)-modules

\[
0 \to X_{Np^r, p}^{(1)} \to S_\theta \to S_\theta' \to 0,
\]

where \(S_\chi' = \mathbb{Z}_p[\chi] is the \(\chi\)-component of the Brauer groups. By Lemma 3.1 of loc. cit., one sees that \(X_{Np^r, p}^{(1)}/p_1\) is finite since its characteristic ideal is prime to \(p_1\) and that \(S_\theta'/p_1 S_\theta' = S_\theta'\) since it characteristic ideal is \(p_1\).

**Lemma 2.5.** Let the notation be as above. If the cuspidal Hecke algebra \(b_{\theta,m}\) is Gorenstein, then the homomorphism \(\varpi_{\theta}\) is injective and its image is \(S_\chi'(1)\). Consequently, the image of \(L_{\theta,m}(\mod p_f)\) under the map \(\varpi_{\theta}\) is in \(S_{\chi'(1)}[\mathbb{Z}_p^\times]\).

**Proof.** Recall that \(\mathcal{O} := b_{\theta,m}/p_f = \mathbb{Z}_p[\chi]\). Suppose that \(\varpi_{\theta}\) is not injective. Then the image of \(\varpi_{\theta}\) is isomorphic to a quotient of \(\mathbb{Z}_p[\chi]\) and hence is finite, which is impossible since \(\varpi_{\theta}\) is an isomorphism after localizing at \(p_1\) ([SW21, Theorem 1.2]). Therefore, \(\varpi_{\theta}\) is injective and hence, its image is \(S_\chi'(1)\) as \(\varpi_{\theta}\) is a \(\mathbb{Z}_p[\chi]\)-module homomorphism and \(S_\chi'(1)\) is isomorphic to \(\mathbb{Z}_p[\chi]\) as a \(\mathbb{Z}_p[\chi]\)-module by [SW21, Proposition 3.2].

Denote by \(L_\theta^+\) the plus part of \(L_\theta\) defined in terms of \([Na : 1]_{r, \theta}^+\). By [FK12, Proposition 5.2.12], the map \(\varpi_{\theta}\) sends \(L_\theta^+ \in H_{\theta}^-[\mathbb{Z}_p^\times]\) to

\[
(2.14) \quad C_\theta := \sum_{a \in (\mathbb{Z}/p^r \mathbb{Z})^\times} (1 - \zeta_{p^r}^u, 1 - \zeta_{p^r}^v)_\theta[a] \in S_\theta[\mathbb{Z}_p^\times].
\]

Since the map \(\varpi_{\theta}\) factors through the quotient by the Eisenstein ideal \(I_{\theta}\), \(L_\theta^+ \mod I_{\theta}\) coincides with \(C_\theta\) in \(S_\theta[\mathbb{Z}_p^\times]\). For any finite order character \(\psi\) of \(\mathbb{Z}_p^\times\), we have

\[
(2.15) \quad \psi(C_\theta) = \sum_{a \in (\mathbb{Z}/p^r \mathbb{Z})^\times} \tilde{\psi}(a)(1 - \zeta_{p^r}^u, 1 - \zeta_{p^r}^v)_{r \geq 1, \theta} \in S_\theta,
\]

and we denote by \(\psi(C_\theta)'\) the image of \(\psi(C_\theta)\) in \(S_{\chi'(1)}\). Note that on the right hand side of (2.15), we have \(\tilde{\psi}(a)\) rather than \(\psi(a)\) because of the interpolation property (2.1). When \(\psi\) is the trivial character, it is known (for example, see [SW21, Corollary 3.4]) that \(\psi(L_\theta) = (1 - U^+(p^{-1})\{0, \infty\}_{r \geq 1, \theta}\) sends to the cup product \((p, 1 - \zeta_{p^r})_{r \geq 1, \theta}\) via \(\varpi_{\theta}\) and its image in \(S_{\chi'(1)}\) has been computed in [SW21, Theorem 1.3]. We now compute \(\psi(C_\theta)'\) for non-trivial characters \(\psi\). To this end, we first prove the following proposition.
Proposition 2.6. Let the notation be as above. Then for any even non-trivial character $\psi$ of $\mathbb{Z}_p^\times$ of conductor $p^r$ for some $r \in \mathbb{Z}_{>1}$, we have

$$\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \bar{\psi}(a)(1 - \zeta_p^a, 1 - \zeta_p^{pr}, \zeta_p^{pr})_{r,\theta} = -\frac{\tau(\chi)}{\phi(N)} L_p(1, \psi)L_p(0, \chi \omega_p \bar{\psi}) \in (\mathbb{O}/p^r\mathbb{O})(1)$$

Proof. We follow the proof of [SW21, Theorem 4.2]. The action of $(\mathbb{Z}/p\mathbb{Z})^\times$ on the group of $p^r$th roots of unity is given by $\alpha \cdot \zeta_p^\alpha = \zeta_p^{\omega(\alpha)}$ for all $\alpha \in (\mathbb{Z}/p\mathbb{Z})^\times$. Then, one has

$$\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \bar{\psi}(a)(1 - \zeta_p^a, 1 - \zeta_p^{pr}, \zeta_p^{pr})_{r,\theta}$$

$$= \phi(Np)^{-1} \sum_{g \in \mathbb{G}_N \cdot G_p} \chi(g^{-1})\omega_p(g^{-1}) \sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \bar{\psi}(a)(1 - \zeta_p^{\omega_p(g)}, 1 - \zeta_p^{pr} \zeta_p^{\omega_p(g)})_r$$

$$= \phi(Np)^{-1} \sum_{g \in \mathbb{G}_N \cdot G_p} \chi(g^{-1})\omega_p(g^{-1}) \sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \bar{\psi}(a)(1 - \zeta_p^a, 1 - \zeta_p^{pr} \zeta_p^{\omega_p(g)})_r.$$ 

Here $g_N$ and $g_p$ run through all elements in $(\mathbb{Z}/N\mathbb{Z})^\times$ and $(\mathbb{Z}/p\mathbb{Z})^\times$, respectively. By changing of variable and using the assumption that $\chi(p) = 1$, one can rewrite the above summation as

$$(2.16) \quad \phi(Np)^{-1} \sum_{g \in \mathbb{G}_N \cdot G_p} \chi(g^{-1})\omega_p(g^{-1}) \psi(\omega_p(g)) \sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \bar{\psi}(a)(1 - \zeta_p^a, 1 - \zeta_p^{pr} \zeta_p^{\omega_p(g)})_r.$$ 

Since $\prod_a \bar{\psi}(a)(1 - \zeta_p^a)$ is a $p$-adic unit, by the local explicit reciprocity law [Iwa65, Theorem 8.18] or [DSS7, Theorem I.4.2] (taking $\lambda(X) = 1 + X$) one can write (2.16) as

$$\frac{1}{p^r \phi(Np)} \sum_{G_N \cdot G_p} \chi(G_N^{-1})\omega_p(g_p^{-1}) \psi(\omega_p(g_p)) \log \left( \prod_{a} \bar{\psi}(a)(1 - \zeta_p^a) \cdot \omega_p(g_p) \zeta_N^{G_p} \zeta_N^{G_p} \right),$$

where $\text{Tr}$ is the trace $\text{Tr}_{\mathbb{Q}(\mu_p^\infty)/\mathbb{Q}_p}$. By computing the trace, we can simply it as

$$(2.17) \quad \frac{(p - 1)}{p^r \phi(Np)} \left( \sum_{a} \bar{\psi}(a) \log (1 - \zeta_p^a) \right) \sum_{G_N \cdot G_p} \chi(G_N^{-1})\psi(G_p) \frac{\zeta_N^{G_p} \zeta_N^{G_p}}{\zeta_N^{G_p} \zeta_N^{G_p} - 1},$$

where $G_N$ and $G_p$ run through elements in $(\mathbb{Z}/N\mathbb{Z})^\times$ and $(\mathbb{Z}/p\mathbb{Z})^\times$, respectively. Recall that [SW21, Lemma 4.1] asserts

$$\frac{\zeta_N^{G_p} \zeta_N^{G_p}}{\zeta_N^{G_p} \zeta_N^{G_p} - 1} = \sum_{a \in \mathbb{Z}/Np^r\mathbb{Z}} \zeta_a(Np^r)(0)(\zeta_N^{G_p} \zeta_N^{G_p})^a,$$

where $\zeta_a(Np^r)(0)$ is the special value of the partial Riemann zeta function at $s = 0$. A direction computation shows

$$(2.18) \quad \sum_{G_N \cdot G_p} \chi(G_N^{-1})\psi(G_p) \zeta_a(Np^r)(0)(\zeta_N^{G_p} \zeta_N^{G_p})^a = \begin{cases} \tau(\chi^{-1})\tau(\psi)\chi \bar{\psi}(a) \zeta_a(Np^r)(0) & \text{if } (a, Np) = 1 \\ 0 & \text{otherwise.} \end{cases}$$
Hence, we see that (2.17) equals to
\[
\frac{\tau(y^{-1}) \tau(y)}{p^r \phi(N)} \left( \sum_{\alpha} \tilde{\psi}(A) \log_p (1 - \zeta_{p^r}^\alpha) \right) \sum_{\alpha \in \mathbb{Z}/p^r \mathbb{Z}} \chi \tilde{\psi}(\alpha) \zeta_{\alpha(Np^r)}(0).
\]
Then, the assertion follows from the facts
\[
L_p(1, \psi) = -(1 - \frac{\psi(p)}{p}) \tau(\psi)p^{-r} \sum_{\alpha \in \mathbb{Z}/p^r \mathbb{Z}} \tilde{\psi}(\alpha) \log_p (1 - \zeta_{p^r}^\alpha) \quad (\text{[Wast97, Theorem 5.18]})
\]
and
\[
\sum_{\alpha \in \mathbb{Z}/p^r \mathbb{Z}} \chi \tilde{\psi}(\alpha) \zeta_{\alpha(Np^r)}(0) = L(0, \chi \tilde{\psi}) = L_p(0, \chi \omega_p \tilde{\psi}).
\]

The following corollary is a direct consequence of Proposition 2.6 by writing \( \zeta_{NP^r} \) in terms of \( \zeta_N^{NP^r} \).

**Corollary 2.7.** For any non-trivial even character \( \psi = \omega_p \psi_p \) of \( \mathbb{Z}_p \) with conductor \( p^r \), we have
\[
\psi(C)^j = \frac{\tau(y^{-1}) \omega_p(N^{-1})}{\phi(N)} L_p(1, \psi)L_p(0, \chi \omega_p \tilde{\psi}).
\]

**Proof.** The argument is the same as the proof of [SW21, Corollary 4.3].

3. Selmer groups

The goal of this section is to review the definition and to discuss the properties of Greenberg’s (strict) Selmer groups. There are three ingredients play important role in the proof of (1.6). First is the choice of the lattice \( T_g \) in the definition of Selmer groups. This will be addressed in Section 3.2. Second is Proposition 3.5 asserting that the Pontryagin dual of Selmer groups do not have submodule of finite index, which allows us to reduce the proof of (1.6) to the computation of residual Selmer groups. In Section 3.3 we study the difference between Selmer groups and residual Selmer groups.

In this section, we let \( g = \sum_{n \geq 1} a_n(g) q^n \in S_k(\Gamma_1(Np), \varphi, \overline{Q}_p) \) be an ordinary normalized eigenform with \( k \geq 1 \), where \( \varphi \) is a primitive character of conductor \( N \) or \( Np \). Recall that we denote by \( K_g \) the Hecke field of \( g \) over \( \overline{Q}_p \), by \( \mathcal{O}_g \) the ring of integers of \( K_g \), by \( \pi_g \) a uniformizer of \( \mathcal{O}_g \), by \( \mathbb{F} := \mathcal{O}_g/\pi_g \) the residue field of \( \mathcal{O}_g \). Also recall that we denote by \( \rho_g : G/Q \to Aut(V_g) \) the corresponding ordinary Galois representation, where \( V_g \) is a 2-dimensional vector space over \( K_g \).

3.1. Definition of Selmer groups. Let \( T_g \subset V_g \) be a \( G/Q \)-stable \( \mathcal{O}_g \)-lattice. Since \( g \) is ordinary, for \( 0 \leq j \leq p - 1 \), we have an exact sequence of \( \mathcal{O}_g[G_{Q^p}] \)-modules [Hid12, Theorem 4.2.7(2)]

\[
0 \to \mathcal{O}_g(\eta^{-1} \varphi \kappa^{-1} \omega_p^j) \to T_{g,j} := T_g \otimes \omega_p^j \to \mathcal{O}_g(\eta \omega_p^j) \to 0,
\]

(3.1)
where \( \eta : G_{\mathbb{Q}_p} \to O_g^\times \) is an unramified character sending \( \text{Frob}_p \) to \( a_p(g) \). Set \( A_{g,j} := V_{g,j}/T_{g,j} \cong (K_g/O_g)^2 \) endowed with an action of \( G_{\mathbb{Q}_p} \), where \( V_{g,j} := V_g \otimes \omega_p^{j} \). Also, set \( A'_{g,j} := (K_p/O_p)(\eta^{-1} \cdot \kappa_p^{-1} \cdot \omega_p^{j}) \) and \( A''_{g,j} := (K_p/O_p)(\eta \cdot \omega_p^{j}) \). Then, the exact sequence (3.1) yields an exact sequence of \((K_g/O_g)(\mathbb{Q}_p)\)-modules

\[
0 \to A'_{g,j} \to A_{g,j} \to A''_{g,j} \to 0.
\]

For a finite place \( v \) of \( \mathbb{Q}_\infty \), we denote by \( \mathbb{Q}_\infty,v \) the completion of \( \mathbb{Q}_\infty \) at \( v \) and denote by \( I_{\infty,v} \) the inertia subgroup of \( G_{\mathbb{Q}_\infty} \). Following Greenberg [Gre89], the Selmer group of \( g \) twisted by \( \omega_p^{j} \) over \( \mathbb{Q}_\infty \) is defined as

\[
\text{Sel}(\mathbb{Q}_\infty,A_{g,j}) := \ker(H^1(\mathbb{Q}_\infty,A_{g,j})) \to \prod_{v \not= p} H^1(\mathbb{Q}_\infty,v,A_{g,j}) \times H_p(A_{g,j})
\]

(3.3)

\[
\cong \ker(H^1(G_{\mathbb{Q}_\infty,\Sigma_{Np}},A_{g,j})) \to \prod_{v \in \Sigma_{Np},v \not= p} H^1(\mathbb{Q}_\infty,v,A_{g,j}) \times H_p(A_{g,j}).
\]

where \( \Sigma_{Np} \) is the set of finite places of \( \mathbb{Q}_\infty \) above \( Np \), and \( H_p(A_{g,j}) \) is defined as

\[
H_p(A_{g,j}) := \text{im}(H^1(\mathbb{Q}_\infty,w,A_{g,j}) \to H^1(\mathbb{Q}_\infty,w,A_{g,j}')) \cong H^1(\mathbb{Q}_\infty,w,A_{g,j})/L_w,
\]

where \( w \) is the unique prime of \( \mathbb{Q}_\infty \) above \( p \) and

\[
L_w = \ker(H^1(\mathbb{Q}_\infty,w,A_{g,j}) \to H^1(\mathbb{Q}_\infty,w,A_{g,j}')).
\]

Notice that the set \( \Sigma_{Np} \) is not required to contain infinity places as the prime \( p \) is odd. If we replace \( I_{\infty,w} \) by \( \mathbb{Q}_\infty,w \) in the definition of \( H_p(A_{g,j}) \), this would define possible smaller Selmer group, called strict Selmer group and denote by \( \text{Sel}^\text{str}(\mathbb{Q}_\infty,A_{g,j}) \).

The strict Selmer group is a discrete \( O_g \)-module with natural \( \Gamma := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \) action and hence, a \( \Lambda_{O_g} \)-module, where \( \Lambda_{O_g} = O_g[G] \cong O_g[X] \). As in the Introduction, let \( \text{Sel}(\mathbb{Q}_\infty,A_{g,j})^\vee \) denote the Pontryagin dual of \( \text{Sel}(\mathbb{Q}_\infty,A_{g,j}) \), which is a finitely generated \( \Lambda_{O_g} \)-module. Recall that the structure theory of finitely generated Iwasawa modules [Was97] Theorem 13.12] asserts that if \( \text{Sel}(\mathbb{Q}_\infty,A_{g,j})^\vee \) is torsion, then it is pseudo-isomorphic to \( \bigoplus_j \Lambda_{O_g}/(\pi_g^n)^j \oplus \bigoplus_i \Lambda_{O_g}/(f_i(X)) \) for some \( n_j \in \mathbb{Z}_{\geq 0} \) and for some distinguished polynomials \( f_i(X) \in \Lambda_{O_g} \). The characteristic polynomial of \( \text{Sel}(\mathbb{Q}_\infty,A_{g,j})^\vee \) over \( \Lambda_{O_g} \), denote by \( \text{Char}_{\Lambda_{O_g}} \text{Sel}(\mathbb{Q}_\infty,A_{g,j})^\vee \), is defined as

\[
\text{Char}_{\Lambda_{O_g}} \text{Sel}(\mathbb{Q}_\infty,A_{g,j})^\vee := \prod_j \pi_g^n \times \prod_i f_i(X).
\]

Following (2.10), we define algebraic \( \mu \)- and \( \lambda \)- invariants of \( g \) by

\[
\mu^\text{alg}(A_{g,j}) := \mu(\text{Char}_{\Lambda_{O_g}} \text{Sel}(\mathbb{Q}_\infty,A_{g,j})^\vee) \quad \text{and} \quad \lambda^\text{alg}(A_{g,j}) := \lambda(\text{Char}_{\Lambda_{O_g}} \text{Sel}(\mathbb{Q}_\infty,A_{g,j})^\vee).
\]
3.2. Choice of lattices. In this subsection, we fix our choice $T_g$ and show that the obtained residual representation is semisimple. This is crucial for us to compute the residual Selmer groups in Section 4.

We first briefly review the Galois representation associated with cohomology of modular curves following [ FK12, Section 1.7.14] and refer the reader to loc. cit. for more details. Recall in Section 2.2 that $H$ denotes the project limits of the ordinary component of the cohomology of modular curves. It admits an exact sequence of $G_{Q_p}$-modules

$$0 \to H_{\text{sub}} \to H \to H_{\text{quo}} \to 0,$$

and the action of $G_{Q}$ on $H \otimes_b Q(h)$ yields a big Galois representation $\rho_{Q(b)}$ satisfying the following properties

(G1) $\text{Tr}(\rho_{Q(b)}(\text{Frob}_\ell)) = T^*(\ell)$ and $\det(\rho_{Q(b)}(\text{Frob}_\ell)) = \ell(\ell)^{-1}$ for all $l \not\equiv Np$. Here $\text{Frob}_\ell$ denotes the arithmetic Frobenius at $\ell$ and $(\ell)$ is the image of $\ell$ under the natural embedding $(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times \to \mathcal{O}[(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times]$.

(G2) For all $\sigma \in G_{Q}$, $\det(\rho_{Q(b)}(\sigma)) = \kappa_p(\sigma)^{-1}(\sigma)^{-1}$.

(G3) The action of $G_{Q_p}$ on $H_{\text{quo}}(1)$ is unramified and $\text{Frob}_p$ acts as $U^*(p)$.

(G4) The action of $G_{Q_p}$ on $H_{\text{sub}}(1)$ is given by $\kappa_p(\sigma)(\sigma)^{-1}$.

(G5) The action of $G_{Q}$ on $H^-(1)/IH^-(1)$ is trivial and on $H^+(1)/IH^+(1)$ is via $\kappa_p(\sigma)(\sigma)^{-1}$ ([FK12, Proposition 6.3.2]).

Recall that the universal cuspidal Hecke algebra $h_{\theta,m}$ is assumed to be Gorenstein and that $h_{\theta,m}$ is a $\Lambda := \mathbb{Z}_p[\theta][X]$-module. Then the Galois representation $\rho_{Q(h_{\theta,m})}$ induces a Galois representation

$$\rho_{h_{\theta,m}} : G_{Q} \to GL(H_{\theta,m}) \cong GL_{2}(h_{\theta,m}).$$

Thus, for any cusp form $g$ of weight $k \geq 1$ in a Hida family corresponding to a minimal prime ideal of $h_{\theta,m}$, we take $T_{g,j}$ to be $H_{\theta,m}(1)/p\varphi \otimes \omega_{g}^{j}$ and the ordinary filtration (3.2) is taken to be

$$0 \to (H_{\text{sub},\theta,m}(1)/p\varphi) \otimes \omega_{g}^{j} \to T_{g,j} \to (H_{\text{quo},\theta,m}(1)/p\varphi) \otimes \omega_{g}^{j} \to 0.$$ (3.6)

From the above discussion, it makes sense to consider $\rho_{h_{\theta,m}} \bmod m$ in this situation.

Proposition 3.1. Suppose that the Hecke algebra $h_{\theta,m}$ is Gorenstein. Let $\{e_-, e_+\}$ be a fixed basis of $(H_{\theta,m}(1)^{-} \otimes H_{\theta,m}(1)^{+})/m$ over $h_{\theta,m}/m$. Then under the basis $\{e_-, e_+\}$, the action of $G_{Q}$ on $A_{g,j}[\pi_g]$ is given by $\binom{\omega_{g}^{j} \ 0}{0 \ \chi^{-1}\omega_{g}^{j}}$ and the $G_{Q_p}$-module $A_{g,j}[\pi_g]$ is generated by $e_- + e_+$.

Proof. It follows from [SW21, Theorem 1.5] that the action of $G_{Q}$ on the lattice $T_f := (H_{\theta,m}(1)/p_f) = (H_{\tilde{\theta},m}(1) \oplus H_{\tilde{\theta},m}(1))/p_f$ is semisimple, and hence, so is the action of $G_{Q}$ on $(H_{\theta,m}(1) \oplus H_{\tilde{\theta},m}(1))/m$. The action of $G_{Q}$ under the basis $\{e_- , e_+\}$ can be examined from action on $(H_{\theta,m}(1) \oplus H_{\tilde{\theta},m}(1))/m$ described in (G5).
To see the basis of $A'_{g,j}[\pi]$, from the discussion in [DPV] Section 3.4.1, we know that if the Galois representation of $f$ is semisimple under a complex conjugation basis $\{e'_-, e'_+\}$, then the ordinary $G_{Q_p}$-stable is generated by $e'_- + e'_+$. Moreover, from the computation in the proof of Lemma 3.9 of op. cit. and the fact that the localization and completion of $\mathfrak{h}_{\theta,m}$ at $p_f$ is $\tilde{\mathcal{O}}_p[X]$ ([BDP22 Theorem A(i)]) we know that the Galois representation $G_Q \xrightarrow{\rho_{\mathfrak{h}_{\theta,m}}} \text{GL}_2(\mathbb{Q}_p[X])/(X^2)$ is of the form $(1 + \begin{pmatrix} a & b \\ c & d \end{pmatrix} X) \begin{pmatrix} 1 & 0 \\ \chi & -1 \end{pmatrix}$, where $a, b, c, d$ not nonzero function on $G_Q$. Hence, $G_Q$-stable line $H_{sub,\theta,m}(1)/p_f$ is generated by $e_- + e_+$ and hence, so is $H_{sub,\theta,m}(1)/\mathfrak{m}$.

**Remark 3.2.** The assumption of $\mathfrak{h}_{\theta,m}$ being Gorenstein is essential. Without this assumption, we only have a GMA $\left( \begin{array}{cc} B & \mathfrak{h}_{\theta,m} \\ C & \mathfrak{h}_{\theta,m} \end{array} \right)$, where $B$ and $C$ are fractional ideals in $Q(\mathfrak{h}_{\theta,m})$ respectively generated by values of $b(g)$ and $c(g)$ for all $g \in G_Q$.

To end this subsection, we show the following lemma which will be used in the proof of Proposition 3.5.

**Lemma 3.3.** Suppose that the Hecke algebra $\mathfrak{h}_{\theta,m}$ is Gorenstein. Set $A^*_{g,j} := \text{Hom}(A_{g,j}, \mu_{p^\infty})$.

1. The group $H^0(Q_\infty, A^*_{g,j}) = 0$ if and only if $j \neq 1$
2. Then, one has $\dim_{F} H^0(Q_\infty, A^*_{g,j}[\pi_\gamma]) = 1$ if $j = 0$ and $0$ if $j \neq 0$.

**Proof.** From the above discussion, we see that the action of $G_Q$ on $A^*_{g,j}/\pi_\gamma$ under the basis $\{e_-, e_+\}$ is

\[
\begin{pmatrix} \kappa_p \omega_p^{-j} \\ \chi \omega_p^{-j} \kappa_p \end{pmatrix} \mod \pi_\gamma = \begin{pmatrix} \omega_p^{1-j} \\ \chi \omega_p^{1-j} \end{pmatrix}.
\]

The above equality is obtained by the definition of $p$-adic cyclotomic character that $\kappa_p \equiv \omega_p \mod p$. Hence, $H^0(Q, A^*_{g,j}/\pi) = 0$ if and only if $j \neq 1$. The assertion (1) then follows. The assertion (2) follows from Proposition 3.1 immediately.

By [GV00 Proposition 2.1], the following corollary follows from Lemma 3.3 immediately.

**Corollary 3.4.** Suppose that the Hecke algebra $\mathfrak{h}_{\theta,m}$ is Gorenstein. The map $\gamma$ in (3.3) is surjective if $j \neq 1$.

3.3. **Properties of Selmer groups.** Let the notation be as in previous subsections. The following proposition will be used in Section 4.

**Proposition 3.5.** The group $\text{Sel}(Q_\infty, A_{g,j})^\gamma$ has no finite $\Lambda$-submodules.

**Proof.** One can adapted the proof in [Gre99 Section 4] to show the desired assertion. For the sake of convenience, we sketch a proof and left the detail to the reader.
When \( j \neq 0 \), it follows from Proposition 3.1 that \( H^0(\mathbb{Q}, A_{g,j}[[\pi]]) = 0 \) and hence, one has \( H^0(\mathbb{Q}, A_{g,j}) = 0 = H^0(\mathbb{Q}_\infty, A_{g,j}) \) since \( \Gamma \) is pro-\( p \). The assertion then follows from a similar proof of [Gre99, Proposition 4.14].

When \( j = 0 \), the morphism \( \gamma \) in (3.3) is surjective by Corollary 3.4. Then, the \( \Lambda \)-module \( H_p(A_g, 0) \) is \( \Lambda \)-cofree by [GV00, Proposition 2.5] since \( A_g'' \) is unramified. Let \( \text{Sel}'(\mathbb{Q}_\infty, A_g, 0) \) be the non-primitive Selmer group defined by omitting the condition \( H_p(A_g, 0) \) in (3.3). Since \( \gamma \) is surjective, we have

\[
\text{Sel}'(\mathbb{Q}_\infty, A_g, 0)/\text{Sel}(\mathbb{Q}_\infty, A_g, 0) \simeq H_p(A_g, 0).
\]

By Lemma 2.6 of op. cit., to show the assertion, it then suffices to show that \( \text{Sel}'(\mathbb{Q}_\infty, A_g, 0) \) has no submodules of finite index, which follows from a similar argument of [Gre99, Proposition 4.15]. \( \square \)

The following is a consequence of Proposition 3.5.

**Corollary 3.6.** We have

\[
\mu^{alg}(g, \omega_p^j) = 0 \iff \text{Sel}(\mathbb{Q}_\infty, A_g, j)[\pi_g] \text{ is finite.}
\]

If this is the case, then \( \text{Sel}(\mathbb{Q}_\infty, A_g, j) \) is \( \mathcal{O}_g \)-divisible and

\[
\lambda^{alg}(g, \omega_p^j) = \dim_{F}(\text{Sel}(\mathbb{Q}_\infty, A_g, j)[\pi_g]).
\]

**Proof.** For a proof, see the proof of [EPW06, Theorem 4.1.1]. \( \square \)

### 3.4. Residual Selmer groups

We now define the residual Selmer groups following [GV00, §2] and then discuss its relation to the usual Selmer groups.

Recall that \( F \) denotes the residual field of \( \mathcal{O}_g \). Consider the exact sequence of \( F[G_{Q_p}] \)-modules

\[
0 \to A_g'[\pi_g] \to A_g[\pi_g] \to A_g''[\pi_g] \to 0.
\]

The residual Selmer group is defined as

\[
\text{Sel}(\mathbb{Q}_\infty, A_g, j)[\pi_g]) := \ker(H^1(G_{Q_\infty}, \Sigma_{N_p}, A_{g,j}[\pi_g]) \to \prod_{v \notin \Sigma_{N_p}, v \nmid p} H^1(I_{\infty,v}, A_{g,j}[\pi_g]) \times \mathcal{H}_p(\mathbb{Q}_\infty, A_{g,j}[\pi_g])),
\]

where \( \mathcal{H}_p(\mathbb{Q}_\infty, A_g, j)[\pi_g]) := \text{Im}(H^1(\mathbb{Q}_\infty, \Sigma_{N_p}, A[\pi_g]) \to H^1(I_{\infty,w}, A''_{g,j}[\pi_g])).
\]

**Lemma 3.7.** For any \( v | N \), the map

\[
H^1(I_{\infty,v}, A_{g,j}[\pi_g]) \to H^1(I_{\infty,v}, A_{g,j})
\]

is injective. Also, the map

\[
H^1(I_{\infty,w}, A''_{g,j}[\pi_g]) \to H^1(I_{\infty,w}, A''_{g,j})
\]

is injective, where \( w \) is the unique prime of \( \mathbb{Q}_\infty \) above \( p \).
Proof. The kernel of these two maps are $A^1_{\infty,w}/\pi_g$ and $(A''_{g,j})^1_{\infty,w}/\pi_g$, respectively. They are zero since both $A^1_{\infty,w}$ and $(A''_{g,j})^1_{\infty,w} = A''_{g,j}$ are divisible. Notice that $A^1_{\infty,w}$ is divisible follows from the proof of [EPW06 Lemma 4.1.2] as the character $\chi$ is primitive of conductor $N$. 

**Proposition 3.8.** We have a short exact sequence

\[(3.7) \quad 0 \to H^0(\mathbb{Q}_\infty, A_{g,j})/\pi_g \to \text{Sel}(\mathbb{Q}_\infty, A_{g,j}[\pi_g]) \to H^0(\mathbb{Q}_\infty, A_{g,j})[\pi_g] \to 0\]

and

\[(3.8) \quad \dim_F H^0(\mathbb{Q}_\infty, A_{g,j})/\pi_g = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases}\]

**Proof.** The exact sequence \((3.7)\) is known [BP19 Lemma 4.3]. For the sake of convenience, we give a proof here. Since $A_{g,j}$ is $O_g$-divisible, multiplication by $\pi_g$ yields a short exact sequence

\[0 \to A_{g,j}[\pi_g] \to A_{g,j} \xrightarrow{x_{\pi_g}} A_{g,j} \to 0.\]

Hence, we have the following exact sequence of $O_g$-modules

\[0 \to H^0(\mathbb{Q}_\infty, A_{g,j})/\pi_g \to H^1(\mathbb{Q}_\infty, A_{g,j}[\pi_g]) \to H^1(\mathbb{Q}_\infty, A_{g,j})[\pi_g] \to 0\]

and the following isomorphism of $F$-vector spaces

\[(3.9) \quad H^0(\mathbb{Q}_\infty, A_{g,j}[\pi_g]) \simeq H^0(\mathbb{Q}_\infty, A_{g,j})[\pi_g].\]

One can verify the surjectivity of $\text{Sel}(\mathbb{Q}_\infty, A_{g,j}[\pi_g]) \to \text{Sel}(\mathbb{Q}_\infty, A_{g,j})[\pi_g]$ by Lemma \[3.7\] and by diagram chasing.

To see the injectivity of $H^0(\mathbb{Q}_\infty, A_{g,j})/\pi_g \to \text{Sel}(\mathbb{Q}_\infty, A_{g,j}[\pi_g])$, Lemma \[3.3(2)\] yields the vanishing of $H^0(\mathbb{Q}_\infty, A_{g,j})$ if $j \neq 0$. Thus, there is nothing to show in this case. We now assume $j = 0$. It suffices to show that $H^0(\mathbb{Q}_\infty, A_{g,0})/\pi_g$ is contained in $\text{Sel}(\mathbb{Q}_\infty, A_{g,0}[\pi_g])$, i.e. it satisfies the Selmer condition at all places of $\mathbb{Q}_\infty$ above $Np$. It satisfies the Selmer conditions at the place above $p$, since the composition

\[H^0(\mathbb{Q}_\infty, A_{g,0})/\pi_g \to H^1(\mathbb{Q}_\infty, A_{g,0}[\pi_g]) \to H^1(1_{\mathbb{Q}_\infty}, A''_{g,0}[\pi_g])\]

coincides with the composition

\[H^0(\mathbb{Q}_\infty, A_{g,0})/\pi_g \to H^0(1_{\mathbb{Q}_\infty}, A''_{g,0})/\pi_g \to H^1(1_{\mathbb{Q}_\infty}, A''_{g,0}[\pi_g]),\]

and $H^0(1_{\mathbb{Q}_\infty}, A''_{g,0})/\pi_g = A''_{g,0}/\pi_g = 0$ as $A''_{g,0}$ is divisible. Similarly, it satisfies the Selmer conditions at all the places $v$ above $N$, since we have seen in the proof of Lemma \[3.7\] that $H^0(1_{\mathbb{Q}_\infty,v}, A_{g,0})/\pi_g = 0$.

Finally, we show \[3.8\] when $j = 0$. Note that $H^0(\mathbb{Q}_\infty, A_{g,j})$ has divisible part and torsion part. One can adapt the proof of [Rib81 Theorem 3] to show that the divisible part is trivial. Then, \[3.9\] and Lemma \[3.3(2)\] can yield

\[\dim_F H^0(\mathbb{Q}_\infty, A_{g,0})/\pi_g = \dim_F H^0(\mathbb{Q}_\infty, A_{g,0})[\pi_g]/\pi_g = 1.\]
4. Algebraic Iwasawa invariants

Throughout this section, $g$ denotes ordinary cusp form of weight $k \geq 2$ passing through by a cuspidal family corresponding to a minimal prime of $\mathfrak{p}_{\theta,m}$. Recall that we denote by $f$ the $p$-stabilization of $E_{1}(\chi, 1)$ and that the Iwasawa modules $X_{NP}^{\infty}$ and $X_{NP}^{\infty}$ was defined in Section 2.4. This section is devoted to computing the (residual) Selmer groups that completes the proof of the algebraic counter parts of Theorems 1.2 and 1.4.

4.1. Trivial twist. The goal of this subsection is to present the proof for the case $j = 0$. We first compute the (strict) Selmer group of $f$.

**Proposition 4.1.** One has $\text{Sel}(\mathbb{Q}_{\infty}, A_{f,0}) \simeq X_{NP}^{(\chi^{-1})}$ and $\text{Sel}^\dagger(\mathbb{Q}_{\infty}, A_{f,0}) \simeq X_{NP}^{(\chi^{-1})}$. In particular, the $\Lambda$-module $\text{Sel}(\mathbb{Q}_{\infty}, A_{f,0})$ is torsion, $\mu_{\text{alg}}(A_{f,0}) = 0$, and (1.4) holds for $j = 0$.

**Proof.** Recall from Section 3.2 that the action of $G_{\mathbb{Q}}$ on $A_{f,0}$ is given by the split sequence

$$0 \to K_{f}/O_{f} \to A_{f,0} \to K_{f}/O_{f}(\chi^{-1}) \to 0. \tag{4.1}$$

As $\chi(p) = 1$, the ordinary filtration (3.2) is given by the $G_{\mathbb{Q}_{p}}$-split sequence

$$0 \to K_{f}/O_{f} \to A_{f,0} \to K_{f}/O_{f} \to 0. \tag{4.2}$$

The split sequence (4.1) yields the following split exact sequence of cohomology groups

$$0 \to H^{1}(G_{Q_{\infty}, \Sigma_{NP}}, K_{f}/O_{f}) \to H^{1}(G_{Q_{\infty}, \Sigma_{NP}}, A_{f,0}) \to H^{1}(G_{Q_{\infty}, \Sigma_{NP}}, K_{f}/O_{f}(\chi^{-1})) \to 0 \tag{4.3}$$

From diagram chasing and the Selmer conditions, one can verify that the Selmer group $\text{Sel}(\mathbb{Q}_{\infty}, A_{f,0})$ is isomorphic to the product of

$$\ker(H^{1}(G_{Q_{\infty}, \Sigma_{NP}}, K_{f}/O_{f})) \to \prod_{v|N} H^{1}(I_{\infty,v}, K_{f}/O_{f}(\chi^{-1})) \times H^{1}(I_{\infty,w}, K_{f}/O_{f})) \tag{4.4}$$

and

$$\ker(H^{1}(Q_{\infty}, \Sigma_{NP}, K_{f}/O_{f}(\chi^{-1}))) \to \prod_{v|N} H^{1}(I_{\infty,v}, K_{f}/O_{f}(\chi^{-1})) \times H^{1}(I_{\infty,w}, K_{f}/O_{f})). \tag{4.5}$$

This would yield the assertion for $\text{Sel}(\mathbb{Q}_{\infty}, A_{f,0})$ since the Pontryagin dual of (4.4) is contained in $X_{NP}^{\Lambda} = 0$, where $\Delta = \text{Gal}(\mathbb{Q}(\zeta_{p})/\mathbb{Q})$, and the Pontryagin dual of (4.5) is $X_{NP}^{(\chi^{-1})}$. By replacing $I_{\infty,w}$ by $\mathbb{Q}_{\infty,w}$ in the above discussion, the assertion for $\text{Sel}^\dagger(\mathbb{Q}_{\infty}, A_{f,0})$ follows from the same argument. \hfill \blacksquare

Next, we compute the residual Selmer groups for weight $k \geq 2$ cusp forms $g$.

**Proposition 4.2.** One has $\dim F \text{Sel}(\mathbb{Q}_{\infty}, A_{g,0}[\pi]) = \dim F \text{Hom}(X_{NP}^{(\chi^{-1})}, F)$. Moreover, one has $\mu_{\text{alg}}(A_{g,0}) = 0$ and (1.6) for $j = 0$ holds.
Proposition 4.3. Let \( \Omega \) be \( \mathbb{F} \) or \( K_f/\mathcal{O}_f \). Suppose that \( j \leq p-1 \) is a positive even integer. For any \( x \in \text{Hom}(X^{(\chi_{\omega_j}^{p-2})}_{N_{p^\infty}}, \Omega) \), there is a unique \( \bar{y} \) in \( \text{Hom}(X^{(\chi_{\omega_j}^{p-2})}_{N_{p^\infty}}, \Omega)/\text{Hom}(X^{(\chi_{\omega_j}^{p-2})}_{N_{p^\infty}}, \Omega) \) such that \( x-y = 0 \) on \( \mathbb{I}_p \), where \( y \) is any lifting of \( \bar{y} \) in \( \text{Hom}(X^{(\chi_{\omega_j}^{p-2})}_{N_{p^\infty}}, \Omega) \).
Proof. If $x|_{l_p} = 0$, then $y|_{l_p}$ must be 0 as well and hence, one has $y \in \text{Hom}(X_{N_{l_p}^{\infty}}^{(x_1,\omega_p^j)}, \Omega)$. Now assume $x|_{l_p} \neq 0$. Then, $x \neq 0 \in \text{Hom}(\text{Gal}(K_{ab}/\mathbb{Q}_p(\mu_{l_p^n}))^{(\omega_p^j)}, \Omega)$, where $K_{ab}$ be the maximal abelian pro-$p$ extension of $\mathbb{Q}_p(\mu_{l_p^n})$. We claim that one has a surjective homomorphism of abelian groups

$$\text{Gal}(K_{ab}/\mathbb{Q}_p(\mu_{l_p^n}))^{(\omega_p^j)} \twoheadrightarrow \text{Gal}(M_{\infty}/L_{\infty})^{(x_1,\omega_p^j)}$$

that is an isomorphism if $j$ is even. The short exact sequence

$$0 \twoheadrightarrow \text{Gal}(M_{\infty}/L_{\infty}) \rightarrow X_{N_{l_p}^{\infty}} \rightarrow X_{N_{l_p}^{\infty}} \twoheadrightarrow 0$$

induces a surjective homomorphism

$$(4.10) \quad \text{Hom}(X_{N_{l_p}^{\infty}}^{(x_1,\omega_p^j)}, \Omega) / \text{Hom}(X_{N_{l_p}^{\infty}}^{(x_1,\omega_p^j)}, \Omega) \twoheadrightarrow \text{Hom}(\text{Gal}(M_{\infty}/L_{\infty})^{(x_1,\omega_p^j)}, \Omega),$$

which is an isomorphism if $j$ is even since in this case both $X_{N_{l_p}^{\infty}}^{(x_1,\omega_p^j)}$ and $X_{N_{l_p}^{\infty}}^{(x_1,\omega_p^j)}$ are free $\mathbb{Z}_p$-modules (their $\mu$-invariants are zeros [FW79] and they do not have any finite submodules [Gre01 Proposition 2.5]). Therefore, the claim can yield the desired assertions.

To prove the claim, let $\mathbb{Q}_\chi$ denote the field extension of $\mathbb{Q}$ cut out by the character $\chi$, and let $F_n = \mathbb{Q}_\chi(\mu_{l_p^n})$ for all $n \in \mathbb{Z}_{\geq 1}$. There are $d := [F_0 : \mathbb{Q}]$-distinct primes of $F_0$ above $p$. For each prime $p$ of $F_0$ above $p$ and for each $n \in \mathbb{Z}_{\geq 1}$, denote by $p_n$ the unique prime of $F_n$ above $p$. Let $U_{F_n,p_n}$ denote the group of principal units in the completion $F_{n,p_n}$ of $F_n$ at $p_n$. Put $\mathcal{U} := \lim_{\leftarrow} \prod_{p_n} U_{F_n,p_n}$. By [Was97 Corollary 13.6], we have

$$\text{Gal}(M_{\infty}/L_{\infty})^{(x_1,\omega_p^j)} = (\mathcal{U}/\mathcal{E})^{(x_1,\omega_p^j)},$$

which is non-trivial as $\chi(p) = 1$. Here $\mathcal{E}$ is the closure of the global units $E$ in $\mathcal{U}$. Finally, to prove the claim, one has isomorphisms

$$\text{Gal}(K_{ab}/\mathbb{Q}_p(\mu_{l_p^n}))^{(\omega_p^j)} \simeq \mathcal{U}^{(x_1,\omega_p^j)} \twoheadrightarrow (\mathcal{U}/\mathcal{E})^{(x_1,\omega_p^j)},$$

where the first isomorphism is obtained by Iwasawa theory of local fields [Iwa86 Proposition 4.12]. Moreover, the second map is an isomorphism if $j$ is even since $\chi\omega_p^j$ is an odd character. Thus, the claim follows.

**Corollary 4.4.** Suppose $j \neq 0$ is an even integer. Then the $\Lambda$-module $\text{Sel}(\mathbb{Q}_\infty, A_{f,j})^\vee$ is torsion with $\mu^{\text{alg}}(A_{f,j}) = 0$ and (1.3) holds. Moreover, one has $\mu^{\text{alg}}(A_{f,j}) = 0$ and (1.6) holds.

**Proof.** The assertion for $A_{f,j}$ follows from Proposition 4.3 directly by taking $\Omega = K_f/\mathcal{O}_f$. Also, Proposition 4.3 for $\Omega = \mathbb{F}$ yields

$$\dim_{\mathbb{F}} \text{Sel}(\mathbb{Q}_\infty, A_{f,j}[\pi]) = \dim_{\mathbb{F}}(\text{Hom}(X_{N_{l_p}^{\infty}}^{(x_1,\omega_p^j)}, \mathbb{F})) + \dim_{\mathbb{F}}(\text{Hom}(X_{N_{l_p}^{\infty}}^{(x_1,\omega_p^j)}, \mathbb{F})) < \infty.$$ 

Then, the assertion for $A_{f,j}$ follows from the same proof of Proposition 4.2. 

To end this section, we make a remark when $j$ is an odd integer.
Remark 4.5. Suppose \( j \leq p-1 \) is a positive odd integer. By the same proof of Theorem 4.3, one can show a similar result but \( y \) is not unique since the surjective homomorphism (4.10) is not necessary injective. Note that in this case, Greenberg’s conjecture asserts that \( X^\infty_{NP=\pm 1} \) is finite. In addition, it is known that \( X_{NP=\pm 1}^{(\omega_j)} \) has \( \Lambda \)-free part and hence, its \( \mathbb{F} \)-dual is not finite. As a consequence, one sees that the residual Selmer group \( \text{Sel}(\mathbb{Q}_{\infty}, A_{g,i}[\pi]) \) is infinity and hence, \( \mu_{\text{alg}}(A_{g,j}) \) is nonzero by Corollary 3.6 and Proposition 3.8.

5. Examples

We verify that for \((N,p) = (3,7)\) and \((11,3)\), the universal \( \mathfrak{p} \)-ordinary cuspidal Hecke algebra \( \mathfrak{h}_{\chi=\omega_p,m} \) is free of rank one as \( \mathbb{Z}_p \langle X \rangle \)-module and hence is Gorenstein, where the character \( \chi \) is the unique quadratic character corresponding to the field extension \( \mathbb{Q}(\sqrt{-N}) \) of \( \mathbb{Q} \). Note that for these examples, \( p \) splits completely in \( \mathbb{Q}(\sqrt{-N}) \) and hence, one has \( \chi(p) = 1 \).

Our method is as follows. Let \( S(Np, \chi=\omega_p; \Lambda)_{\text{ord}} \) denote the localization at \( \mathfrak{m} \) of the space of \( \mathfrak{p} \)-ordinary cuspidal families of tame level \( N \) and character \( \chi=\omega_p \). Our convention is that the weight \( k \) specialization of \( S(Np, \chi=\omega_p; \Lambda)_{\text{ord}} \) is the space of weight \( k \) ordinary cusp forms of level \( Np \) and character \( \chi=\omega_p^{k-1} \). Since the character \( \chi \) cuts out the imaginary quadratic field \( \mathbb{Q}(\sqrt{-N}) \), the weight one Eisenstein series \( E_1(\mathbb{1}, \chi) \) coincides with the weight one theta series \( \theta_1 \) attached to the trivial character \( \mathbb{1} \). As before, we denote by \( f \) the \( \mathfrak{p} \)-stabilization of \( E_1(\mathbb{1}, \chi) \). By [BDP22] Theorem A(i), the CM theta family \( \Theta_1 \) defined in [Hid93] p. 235, (2a)] is the unique cuspidal family passing through \( f \). By Hida’s control theory [Hid93] p. 215, Theorem 3], the weight \( k := p \) specialization of \( S(Np, \chi=\omega_p; \Lambda)_{\text{ord}} \) is isomorphic to the space \( S_k(\Gamma_1(N)\cap\Gamma_0(p), \chi)_{\text{ord}} \) of \( \mathfrak{p} \)-ordinary cusp form of level \( \Gamma_1(N)\cap\Gamma_0(p) \), weight \( k \), and character \( \chi \) and hence, isomorphic, induced by the ordinary \( \mathfrak{p} \)-stabilization, to the space \( S_k(\Gamma_1(N), \chi) \) of cusp forms of level \( \Gamma_1(N) \), weight \( k \), and character \( \chi \) by [Miy06] Theorem 4.6.17] (or see [Hid85] Section 3). Moreover, we verify that the Eisenstein component of the former space is free of rank one as \( \mathbb{Z}_p \)-module by checking the database LMFDB [LMF22], that the latter space is free of rank one as \( \mathbb{Z}_p \)-module consisting of exactly one CM form. Hence, the space \( S(Np, \chi=\omega_p; \Lambda)_{\text{ord}} \) is free of rank one as \( \Lambda \)-module and therefore, so is \( \mathfrak{h}_{\chi=\omega_p,m} \) by using the Hecke duality.

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