On a generalization of Kelly’s combinatorial lemma

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Abstract: Kelly’s combinatorial lemma is a basic tool in the study of Ulam’s reconstruction conjecture. A generalization in terms of a family of $t$-elements subsets of a $v$-element set was given by Pouzet. We consider a version of this generalization modulo a prime $p$. We give illustrations to graphs and tournaments.

Key words: Set, matrix, graph, tournament, isomorphism

1 Introduction

Kelly’s combinatorial lemma is the assertion that the number $s(F, G)$ of induced subgraphs of a given graph $G$, isomorphic to $F$, is determined by the deck of $G$, provided that $|V(F)| < |V(G)|$, namely $s(F, G) = \frac{1}{|V(G)| - |V(F)|} \sum_{x \in V(G)} s(F, G - x)$ (where $G - x$ is the graph induced by $G$ on $V(G) \setminus \{x\}$).

In terms of a family $\mathcal{F}$ of $t$-elements subsets of a $v$-element set, it simply says that $|\mathcal{F}| = \frac{1}{v - t} \sum_{x \in V(G)} |\mathcal{F}_{-x}|$ where $\mathcal{F}_{-x} := \mathcal{F} \cap [E \setminus \{x\}]^t$.

Pouzet [23, 24] gave the following extension of this result.

Lemma 1.1 (M. Pouzet [23]) Let $t$ and $r$ be integers, $V$ be a set of size $v \geq t + r$ elements, $U$ and $U'$ be sets of subsets $T$ of $t$ elements of $V$. If for every subset $K$ of $k = t + r$ elements of $V$, the number of elements of $U$ which are contained in $K$ is equal to the number of elements of $U'$ which are contained in $K$, then for every finite subsets $T'$ and $K'$ of $V$, such that $T'$ is contained in $K'$ and $K' \setminus T'$ has at least $t + r$ elements, the number of elements of $U$ which contain $T'$ and are contained in $K'$ is equal to the number of elements of $U'$ which contain $T'$ and are contained in $K'$.

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In particular if $|V| \geq 2t + r = t + k$, we have this particular version of the combinatorial lemma of Pouzet:

**Lemma 1.2 (M. Pouzet [23])** Let $v, t$ and $k$ be integers, $V$ be a set of $v$ elements with $t \leq \min(k, v-k)$, $U$ and $U'$ be sets of subsets $T$ of $t$ elements of $V$. If for every subset $K$ of $k$ elements of $V$, the number of elements of $U$ which are contained in $K$ is equal to the number of elements of $U'$ which are contained in $K$, then $U = U'$.

We denote by $n(U, K)$ the number of elements of $U$ which are contained in $K$, thus Lemma 1.2 says that if $n(U, K) = n(U', K)$ for every subset $K$ of $k$ elements of $V$ then $U = U'$. Here we consider the case where $n(U, K) \equiv n(U', K)$ modulo a prime $p$ for every subset $K$ of $k$ elements of $V$; our main result, Theorem 1.3 is then a version, modulo a prime $p$, of the particular version of the combinatorial lemma of Pouzet.

Kelly’s combinatorial lemma is a basic tool in the study of Ulam’s reconstruction conjecture. Pouzet’s combinatorial lemma has been used several times in reconstruction problems (see for example [1, 11, 15, 16, 18, 19]). Pouzet gave a proof of his lemma via a counting argument [24] and latter by using linear algebra (related to incidence matrices) [23] (the paper was published earlier).

Let $n, p$ be positive integers, the decomposition of $n = \sum_{i=0}^{n(p)} n_i p^i$ in the basis $p$ is also denoted $[n_0, n_1, \ldots, n_{n(p)}]_p$ where $n_{n(p)} \neq 0$ if and only if $n \neq 0$.

**Theorem 1.3** Let $p$ be a prime number. Let $v, t$ and $k$ be non-negative integers, $k = [k_0, k_1, \ldots, k_{k(p)}]_p$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$. Let $V$ be a set of $v$ elements with $t \leq \min(k, v-k)$, $U$ and $U'$ be sets of subsets $T$ of $t$ elements of $V$. We assume that for every subset $K$ of $k$ elements of $V$, the number of elements of $U$ which are contained in $K$ is equal (mod $p$) to the number of elements of $U'$ which are contained in $K$.

1) If $k_i = t_i$ for all $i \neq t(p)$ and $k_{t(p)} > t_{t(p)}$, then $U = U'$.
2) If $t = t_{t(p)} p^{t(p)}$ and $k = \sum_{i=t(p)+1}^{k(p)} k_i p^i$, we have $U = U'$, or one of the sets $U, U'$ is the set of all $t$ element-subsets of $V$ and the other is empty, or (whenever $p = 2$) for all $t$-element subsets $T$ of $V$, $T \in U$ if and only if $T \notin U'$.

Our proof of Theorem 1.3 is an application of properties of incidence matrices due to D.H. Gottlieb [16], W. Kantor [17] and R.M. Wilson [27], we use Wilson’s Theorem (Theorem 2.2).

In a reconstruction problem of graphs up to complementation [10], Wilson’s Theorem yielded the following result:

**Theorem 1.4 ([10])** Let $k$ be an integer, $2 \leq k \leq v - 2$, $k \equiv 0$ (mod $4$). Let $G$ and $G'$ be two graphs on the same set $V$ of $v$ vertices (possibly infinite). We assume that $e(G_{G(K)}$ has the same parity as $e(G'_{G(K)})$ for all $k$-element subsets $K$ of $V$. Then $G' = G$ or $G' = \overline{G}$.

Here we look for similar results whenever $e(G_{G(K)}) \equiv e(G'_{G(K)})$ modulo a prime $p$. As an illustration of Theorem 1.3 we obtain the following result.
Theorem 1.5 Let \( p \) be a prime number and \( k \) be an integer, \( 2 \leq k \leq v-2 \). Let \( G \) and \( G' \) be two graphs on the same set \( V \) of \( v \) vertices (possibly infinite). We assume that for all \( k \)-element subsets \( K \) of \( V \), \( e(G_{\mid K}) \equiv e(G'_{\mid K}) \pmod{p} \).

1) If \( p \geq 3 \), \( k \not\equiv 0 \pmod{p} \), then \( G' = G \).

2) If \( p \geq 3 \), \( k \equiv 0 \pmod{p} \), then \( G' = G \), or one of the graphs \( G, G' \) is the complete graph and the other is the empty graph.

3) If \( p = 2 \), \( k \equiv 2 \pmod{4} \), then \( G' = G \).

We give another illustration of Theorem 1.3 to graphs in section 4, and to tournaments in section 5.

2 Incidence matrices

We consider the matrix \( W_{t,k} \) defined as follows: Let \( V \) be a finite set, with \( v \) elements. Given non-negative integers \( t, k \), let \( W_{t,k} \) be the \( \binom{v}{t} \times \binom{v}{k} \) matrix of 0's and 1's, the rows of which are indexed by the \( t \)-element subsets \( T \) of \( V \), the columns are indexed by the \( k \)-element subsets \( K \) of \( V \), and where the entry \( W_{t,k}(T,K) \) is 1 if \( T \subseteq K \) and is 0 otherwise. The matrix transpose of \( W_{t,k} \) is denoted \( W_{t,k}^T \).

We say that a matrix \( D \) is a diagonal form for a matrix \( M \) when \( D \) is diagonal and there exist unimodular matrices (square integral matrices which have integral inverses) \( E \) and \( F \) such that \( D = EMF \). We do not require that \( M \) and \( D \) are square; here ”diagonal” just means that the \((i,j)\) entry of \( D \) is 0 if \( i \neq j \). A fundamental result, due to R.M. Wilson [27], is the following.

Theorem 2.1 (R.M. Wilson [27]) For \( t \leq \min(k, v-k) \), \( W_{t,k} \) has as a diagonal form the \( \binom{v}{t} \times \binom{v}{k} \) diagonal matrix with diagonal entries

\[
\binom{k-i}{t-i} \text{ with multiplicity } \binom{v}{i} - \binom{v}{i-1}, \quad i = 0, 1, \ldots, t.
\]

Clearly from Theorem 2.1 rank \( W_{t,k} \) over the field \( \mathbb{Q} \) is \( \binom{v}{t} \), that is Theorem 2.3 due to Gottlieb [16]. On the other hand, from Theorem 2.1 follows rank \( W_{t,k} \) over the field \( \mathbb{Z}/p\mathbb{Z} \), as given by Theorem 2.2.

Theorem 2.2 (R.M. Wilson [27]) For \( t \leq \min(k, v-k) \), the rank of \( W_{t,k} \) modulo a prime \( p \) is

\[
\sum \binom{v}{i} - \binom{v}{i-1}
\]

where the sum is extended over those indices \( i \), \( 0 \leq i \leq t \), such that \( p \) does not divide the binomial coefficient \( \binom{k-i}{t-i} \).

In the statement of the theorem, \( \binom{v}{-1} \) should be interpreted as zero.

A fundamental result, due to D.H. Gottlieb [16], and independently W. Kantor [17], is this:
Theorem 2.3 (D.H. Gottlieb [16], W. Kantor [17]) For $t \leq \min(k, v - k)$, $W_{t,k}$ has full row rank over the field $\mathbb{Q}$ of rational numbers.

It is clear that $t \leq \min(k, v - k)$ implies \( \binom{t}{i} \leq \binom{v}{k} \) then, from Theorem 2.3 we have the following result:

Corollary 2.4 For $t \leq \min(k, v - k)$, the rank of $W_{t,k}$ over the field $\mathbb{Q}$ of rational numbers is $\binom{v}{k}$ and thus $\ker(W_{t,k}) = \{0\}$.

If $k := v - t$ then, up to a relabelling, $W_{t,k}$ is the adjacency matrix $A_{t,v}$ of the Kneser graph $KG(t,v)$ [15], graph whose vertices are the $t$-element subsets of $V$, two subsets forming an edge if they are disjoint. The eigenvalues of Kneser graphs are computed in [15] (Theorem 9.4.3), and thus an equivalent form of Theorem 2.3 is:

Theorem 2.5 $A_{t,v}$ is non-singular for $t \leq \frac{v}{2}$.

We characterize values of $t$ and $k$ so that $\dim \ker(W_{t,k}) \in \{0, 1\}$ and give a basis of $\ker(W_{t,k})$, that appears in the following result.

Theorem 2.6 Let $p$ be a prime number. Let $v, t$ and $k$ be non-negative integers, $k = [k_0, k_1, \ldots, k_{k(p)}]$, $t = [t_0, t_1, \ldots, t_{t(p)}]$, $t \leq \min(k, v - k)$. We have:

1) $k_j = t_j$ for all $j < t(p)$ and $k_{t(p)} = t_{t(p)}$ if and only if $\ker(W_{t,k}) = \{0\}$ (mod $p$).

2) $t = t_{t(p)}p^{t(p)}$ and $k = \sum_{i=t(p)+1}^{k(p)} k_ip^i$ if and only if $\dim \ker(W_{t,k}) = 1$ (mod $p$) and \( \{1, 1, \ldots, 1\} \) is a basis of $\ker(W_{t,k})$.

The proof of Theorem 2.6 uses Lucas’s Theorem. The notation $a \mid b$ (resp. $a \nmid b$) means $a$ divide $b$ (resp. $a$ not divide $b$).

Theorem 2.7 (Lucas’s Theorem [12]) Let $p$ be a prime number, $t, k$ be positive integers, $t \leq k$, $t = [t_0, t_1, \ldots, t_{t(p)}]$ and $k = [k_0, k_1, \ldots, k_{k(p)}]$. Then

\[
\binom{k}{t} = \prod_{i=0}^{t(p)} \binom{k_i}{t_i} \pmod{p}, \quad \text{where} \quad \binom{k_i}{t_i} = 0 \text{ if } t_i > k_i.
\]

As a consequence of Theorem 2.7, we have the following result which is very useful in this paper.

Corollary 2.8 Let $p$ be a prime number, $t, k$ be positive integers, $t \leq k$, $t = [t_0, t_1, \ldots, t_{t(p)}]$, and $k = [k_0, k_1, \ldots, k_{k(p)}]$. Then

\[ p \mid \binom{k}{t} \] if and only if there is $i \in \{0, 1, \ldots, t(p)\}$ such that $t_i > k_i$. 
Proof of Theorem 2.6. 1) We begin by the direct implication. We will prove \( p \nmid \binom{k-i}{t-i} \)
for all \( i = [i_0, i_1, \ldots, i_{t(p)}] \in \{0, \ldots, t\} \) with \( i_{t(p)} \leq t_{t(p)} \). Since \( k_j = t_j \) for all \( j < t(p) \), then \( (t-i)_j = (k-i)_j \) for all \( j < t(p) \). As \( k_{t(p)} \geq t_{t(p)} \geq t_{t(p)} \) then \( (k-i)_{t(p)} \geq (t-i)_{t(p)} \), thus, by Corollary 2.8 \( p \nmid \binom{k-i}{t-i} \) for all \( i \in \{0,1,\ldots,t\} \). Now from Theorem 2.2 rank \( W_{tk} = \sum_{i=0}^{t} \binom{i}{t} - \binom{i}{t-i} = \binom{i}{t} \). Then the kernel of \( W_{tk} \) (mod \( p \)) is \{0\}.

Now we prove the converse implication. From Theorem 2.1 \( \text{Ker} (W_{tk}) = \{0\} \) implies \( p \nmid \binom{k-i}{t-i} \) for all \( i \in \{0,1,\ldots,t\} \), in particular \( p \nmid \binom{i}{t} \). Then by Corollary 2.8 \( k_j \geq t_j \) for all \( j \leq t(p) \). We will prove that \( k_j = t_j \) for all \( j \leq t(p) \). By contradiction, let \( s \) be the least integer in \( \{0,1,\ldots,t(p)-1\} \), such that \( k_s > t_s \). We have \( (t-(t_s+1)p^s)_s = p-1 \), \( (k-(t_s+1)p^s)_s = k_s-t_s-1 \) and \( p-1 > k_s-t_s-1 \). From Corollary 2.8 \( p \mid \binom{k-(t_s+1)p^s}{t-(t_s+1)p^s} \), that is impossible.

2) Set \( n := t(p) \). We begin by the direct implication. Since \( 0 = k_n < t_n \) then, by Corollary 2.8 \( p \mid \binom{k}{t} \). We will prove \( p \nmid \binom{k-i}{t-i} \) for all \( i = [i_0, i_1, \ldots, i_n] \in \{1,2,\ldots,t\} \).

Since \( k_j = t_j = 0 \) for all \( j < n \), then \( (t-i)_j = (k-i)_j \) for all \( j < n \). From \( t_n \geq i_n \), we have \( (t-i)_n \in \{t_n-i_n, t_n-i_n-1\} \). Note that \( (k-i)_n \in \{p-i_n-1, p-i_n\} \) and \( p-i_n-1 \geq n-i_n \); thus \( (k-i)_n \geq (t-i)_n \). So for all \( j \leq n \), \( (k-i)_j \geq (t-i)_j \). Then, by Corollary 2.8 \( p \nmid \binom{k-i}{t-i} \) for all \( i \in \{1,2,\ldots,t\} \). Now from Theorem 2.2 rank \( W_{tk} = \sum_{i=1}^{t} \binom{i}{t} - \binom{i}{t-i} = \binom{i}{t} - 1 \), and thus dim \( \text{Ker} (W_{tk}) = 1 \). Now \( (1,1,\ldots,1)W_{tk} = \left( \binom{1}{t}, \binom{1}{t}, \ldots, \binom{1}{t} \right) \).

Since \( p \mid \binom{1}{t} \), then \((1,1,\ldots,1)W_{tk} \equiv 0 \) (mod \( p \)). Then \( \{(1,1,\ldots,1)\} \) is a basis of the kernel of \( W_{tk} \) (mod \( p \)).

Now we prove the converse implication. Since \( \{(1,1,\ldots,1)\} \) is a basis of the kernel of \( W_{tk} \) (mod \( p \)) and \((1,1,\ldots,1)W_{tk} = \left( \binom{1}{t}, \binom{1}{t}, \ldots, \binom{1}{t} \right) \), then \( p \mid \binom{1}{t} \). Since \( \dim \text{Ker} (W_{tk}) = 1 \), then from Theorem 2.2 \( p \nmid \binom{k-i}{t-i} \) for all \( i \in \{1,2,\ldots,t\} \).

First, let us prove that \( t = t_n p^s \). Note that \( t_n \neq 0 \) since \( t \neq 0 \). Since \( p \mid \binom{1}{t} \) then, from Corollary 2.8 there is an integer \( j \in \{0,1,\ldots,n\} \) such that \( t_j > k_j \). Let \( A := \{j < n : t_j \neq 0\} \). By contradiction, assume \( A \neq \emptyset \).

Case 1. There is \( j \in A \) such that \( t_j > k_j \). We have \( (t-p^n)_j = t_j, (k-p^n)_j = k_j \). Then from Corollary 2.8 we have \( p \mid \binom{k-p^n}{t-p^n} \), that is impossible.

Case 2. For all \( j \in A, t_j \leq k_j \). Then \( t_n > k_n \). We have \( (t-p^n)_n = t_n, (k-p^n)_n = k_n \). Then, from Corollary 2.8 we have \( p \mid \binom{k-p^n}{t-p^n} \), that is impossible.

From the above two cases, we deduce \( t = t_n p^s \).

Secondly, since \( p \mid \binom{1}{t} \), then by Corollary 2.8 \( t_n > k_n \). Let us show that \( k_n = 0 \). By contradiction, if \( k_n \neq 0 \) then \( (t-p^n)_n = t_n - 1 > k_n - 1 = (k-p^n)_n \). From Corollary 2.8 \( p \mid \binom{k-p^n}{t-p^n} \), that is impossible. Let \( s \in \{0,1,\ldots,n-1\} \), let us show that \( k_s = 0 \). By contradiction, if \( k_s \neq 0 \) then \( (t-p^n)_s = p-1, (k-p^n)_s = k_s-1 \), thus \( (t-p^n)_s > (k-p^n)_s \), so, from Corollary 2.8 \( p \mid \binom{k-p^n}{t-p^n} \), that is impossible.
3 Proof of Theorem 1.3.

Let $T_1, T_2, \ldots, T_{(v)}$ be an enumeration of the $t$-element subsets of $V$, let $K_1, K_2, \ldots, K_{(v)}$ be an enumeration of the $k$-element subsets of $V$ and $W_{t,k}$ be the matrix of the $t$-element subsets versus the $k$-element subsets.

Let $w_U$ be the row matrix $(u_1, u_2, \ldots, u_{(v)})$ where $u_i = 1$ if $T_i \in U$, 0 otherwise. We have

$$w_U W_{t,k} = (|\{T_i \in U : T_i \subseteq K_1\}|, \ldots, |\{T_i \in U : T_i \subseteq K_{(v)}\}|).$$

$$w' U W_{t,k} = (|\{T_i \in U' : T_i \subseteq K_1\}|, \ldots, |\{T_i \in U' : T_i \subseteq K_{(v)}\}|).$$

Since for all $j \in \{1, \ldots, (\binom{v}{k})\}$, the number of elements of $U$ which are contained in $K_j$ is equal (mod $p$) to the number of elements of $U'$ which are contained in $K_j$, then $(w_U - w_{U'}) W_{t,k} = 0$ (mod $p$), so $w_U - w_{U'} \in \text{Ker}(W_{t,k})$.

1) Assume $k_i = t_i$ for all $i < t(p)$ and $k_{t(p)} \geq t_{t(p)}$. From 1) of Theorem 2.6 $w_U - w_{U'} = 0$, that gives $U = U'$.

2) Assume $t = t_{t(p)}p^{t(p)}$ and $k = \sum_{i=(t(p))}^{k_{t(p)}} k_ip^i$. From 2) of Theorem 2.6 there is an integer $\lambda \in [0, p-1]$ such that $w_U - w_{U'} = \lambda(1, 1, \ldots, 1)$. It is clear that $\lambda \in \{0, 1, -1\}$. If $\lambda = 0$ then $U = U'$. If $\lambda = 1$ and $p \geq 3$ then $U = \{T_1, T_2, \ldots, T_{(v)}\}$, $U' = \emptyset$. If $\lambda = -1$ and $p = 2$ then $U' = \{T_1, T_2, \ldots, T_{(v)}\}$, $U = \emptyset$, or $T \in U$ if and only if $T \not\in U'$. If $\lambda = -1$ and $p \geq 3$ then $U' = \{T_1, T_2, \ldots, T_{(v)}\}$, $U = \emptyset$, or $T \in U$ if and only if $T \not\in U'$. \hfill \square

4 Illustrations to graphs

Our notations and terminology follow [2]. A digraph $G = (V, E)$ or $G = (V(G), E(G))$, is formed by a finite set $V$ of vertices and a set $E$ of pairs of distinct vertices, called arcs of $G$. The order (or cardinal) of $G$ is the number of its vertices. If $K$ is a subset of $V$, the restriction of $G$ to $K$, also called the induced subdigraph of $G$ on $K$ is the digraph $G|_K := (K, K^2 \cap E)$. If $K = V \setminus \{x\}$, we denote this digraph by $G_{-x}$. Let $G = (V, E)$ and $G' = (V', E')$ be two digraphs. A one-to-one correspondence $f$ from $V$ onto $V'$ is an isomorphism from $G$ onto $G'$ provided that for $x, y \in V$, $(x, y) \in E$ if and only if $(f(x), f(y)) \in E'$. The digraphs $G$ and $G'$ are then said to be isomorphic, which is denoted by $G \simeq G'$. A subset $I$ of $V$ is an interval [13, 26] (or a clan [11], or an homogenous subset [14]) of $G$ provided that for all $a, b \in I$ and $x \in V \setminus I$, $(a, x) \in E(G)$ if and only if $(b, x) \in E(G)$, and the same for $(x, a)$ and $(x, b)$. For example, $\emptyset$, $\{x\}$ where $x \in V$, and $V$ are intervals of $G$, called trivial intervals. A digraph is then said to be indecomposable [26] (or primitive [11]) if all its intervals are trivial, otherwise it is said to be decomposable. We say that $G$ is a graph (resp. tournament) when for every distinct vertices $x, y$ of $V$, $(x, y) \in E$ if and only if $(y, x) \in E$ (resp $(x, y) \in E$ if and only if $(y, x) \not\in E$); we say that $\{x, y\}$ is an edge of the graph $G$ if $(x, y) \in E$, thus $E$ is identified with a subset of
Let $G = (V, E)$ be a graph, the complement of $G$ is the graph $\overline{G} := (V, [V]^2 \setminus E)$. We denote by $e(G) := |E(G)|$ the number of edges of $G$. The degree of a vertex $x$ of $G$, denoted $d_G(x)$, is the number of edges which contain $x$. A 3-element subset $T$ of $V$ such that all pairs belong to $E(G)$ is a triangle of $G$. Let $T(G)$ be the set of triangles of $G$ and let $t(G) := |T(G)|$. A 3-element subset of $V$ which is a triangle of $G$ or of $\overline{G}$ is a 3-homogeneous subset of $G$. We set $H^{(3)}(G) := T(G) \cup T(\overline{G})$, the set of 3-homogeneous subsets of $G$, and $h^{(3)}(G) := |H^{(3)}(G)|$.

Another proof of Theorem 1.4 using Theorem 1.3. Here $p = 2$, $t = 2 = [0, 1]_p$ and $k = [0, 0, k_2, \ldots]_p$. From 2) of Theorem 1.3 $U = U'$, or one of the sets $U, U'$ is the set of all 2-element-subsets of $V$ and the other is empty, or for all 2-element subsets $T$ of $V$, $T \in U$ if and only if $T \not\in U'$. Thus $G' = G$ or $G' = \overline{G}$. \hfill \qed

Proof of Theorem 1.5. We set $U := E(G), U' := E(G')$. For all $K \subseteq V$ with $|K| = k$, we have: $\{|\{(x, y) \subseteq K : (x, y) \in U\}| = E(G_{1|K})\}$ and $\{|\{(x, y) \subseteq K : (x, y) \in U'\}| = E(G'_{1|K})\}$. Since $e(G_{1|K}) \equiv e(G'_{1|K}) (\mod p), then |\{|\{(x, y) \subseteq K : (x, y) \in U\}| = |\{|\{(x, y) \subseteq K : (x, y) \in U'\}| (\mod p)\}$.

1) $p \geq 3, t = 2 = [2]_p$ and $k_0 \geq 2$. From 1) of Theorem 1.3 $U = U'$, thus $G = G'$.
2) $p \geq 3, t = 2 = [2]_p$ and $k_0 = 0$. From 2) of Theorem 1.3 we have $U = U'$ or one of $U, U'$ is the set of all 2-element subsets of $V$ and the other is empty. Then $G = G'$ or one of the graphs $G, G'$ is the complete graph and the other is the empty graph.
3) $p = 2, t = 2 = [0, 1]_p$ and $k = [0, 1, k_2, \ldots]_p$. From 1) of Theorem 1.3 we have $U = U'$, thus $G = G'$. \hfill \qed

The following result concerns graphs $G$ and $G'$ such that $h^{(3)}(G_{1|K}) \equiv h^{(3)}(G'_{1|K})$ modulo a prime $p$, for all $k$-element subsets $K$ of $V$.

Theorem 4.1 Let $G$ and $G'$ be two graphs on the same set $V$ of $v$ vertices. Let $p$ be a prime number and $k$ be an integer, $3 \leq k \leq v - 3$.

1) If $h^{(3)}(G_{1|K}) = h^{(3)}(G'_{1|K})$ for all $k$-element subsets $K$ of $V$ then $G$ and $G'$ have the same 3-element homogeneous sets.

2) Assume $p \geq 5$. If $k \neq 1, 2 (\mod p)$ and $h^{(3)}(G_{1|K}) \equiv h^{(3)}(G'_{1|K}) (\mod p)$ for all $k$-element subsets $K$ of $V$, then $G$ and $G'$ have the same 3-element homogeneous sets.

3) If $(p = 2$ and $k \equiv 3 (\mod 4))$ or $(p = 3$ and $3 \mid k), and h^{(3)}(G_{1|K}) \equiv h^{(3)}(G'_{1|K}) (\mod p)$ for all $k$-element subsets $K$ of $V$, then $G$ and $G'$ have the same 3-element homogeneous sets.

Proof. $H^{(3)}(G) = \{\{a, b, c\} : G_{1[a,b,c]}$ is a 3-element homogeneous set$\}$. We set $U := H^{(3)}(G)$ and $U' := H^{(3)}(G')$. For all $K \subseteq V$ with $|K| = k$, we have: $\{|\{T \subseteq K : T \in U\}| = H^{(3)}_{G_{1|K}}$ and $\{|\{T \subseteq K : T \in U'\}| = H^{(3)}_{G'_{1|K}}$. Set $t := |T| = 3$.

1) Since $h^{(3)}(G_{1|K}) = h^{(3)}(G'_{1|K})$ for all $k$-element subsets $K$ of $V$ then $|\{T \subseteq K : T \in U\}| = |\{T \subseteq K : T \in U'\}$. From Lemma 1.2 it follows that $U = U'$, then $G$ and $G'$
have the same 3-element homogeneous sets.

2) Since \( h^{(3)}(G_{|K}) \equiv h^{(3)}(G'_{|K}) \) (mod \( p \)) for all \( k \)-element subsets \( K \) of \( V \) then \(|\{T \subseteq K : T \in U\}| \equiv |\{T \subseteq K : T \in U'\}| \) (mod \( p \)).

Case 1. \( p \geq 5, t = 3 = [3]_p, k = [k_0, \ldots]_p \) and \( t_0 = 3 \leq k_0 \). From 1) of Theorem 1.3 we have \( U = U' \), thus \( G \) and \( G' \) have the same 3-element homogeneous sets.

Case 2. \( p \geq 5, t = 3 = [3]_p, k = [0, k_1, \ldots]_p \). By Ramsey’s Theorem [25], every graph with at least 6 vertices contains a 3-element homogeneous set. Then \( U \) and \( U' \) are nonempty, so from 2) of Theorem 1.3 \( U = U' \), thus \( G \) and \( G' \) have the same 3-element homogeneous sets.

3) Since \( h^{(3)}(G_{|K}) \equiv h^{(3)}(G'_{|K}) \) (mod \( p \)) for all \( k \)-element subsets \( K \) of \( V \) then \(|\{T \subseteq K : T \in U\}| \equiv |\{T \subseteq K : T \in U'\}| \) (mod \( p \)).

Case 1. \( p = 2, t = 3 = [1, 1]_p \) and \( k \equiv 3 \) (mod \( 4 \)). In this case, \( k = [1, 1, k_2, \ldots]_p \), then from 1) of Theorem 1.3 we have \( U = U' \), thus \( G \) and \( G' \) have the same 3-element homogeneous sets.

Case 2. \( p = 3, t = 3 = [0, 1]_p \) and \( k = [0, k_1, \ldots, k_{k(p)}]_p \).

Case 2.1. \( k_1 \in \{1, 2\} \), then from 1) of Theorem 1.3 we have \( U = U' \), thus \( G \) and \( G' \) have the same 3-element homogeneous sets.

Case 2.2. \( k_1 = 0 \). By Ramsey’s Theorem [25], every graph with at least 6 vertices contains a 3-element homogeneous set. Then \( U \) and \( U' \) are nonempty, so from 2) of Theorem 1.3 \( U = U' \), thus \( G \) and \( G' \) have the same 3-element homogeneous sets.

Let \( G = (V, E) \) be a graph. From [26], every indecomposable graph of size 4 is isomorphic to \( P_4 = (\{0, 1, 2, 3\}, \{\{0, 1\}, \{1, 2\}, \{2, 3\}\}) \). Let \( P^{(4)}(G) \) be the set of indecomposable induced subgraphs of \( G \) of size 4, we set \( p^{(4)}(G) := |P^{(4)}(G)| \). The following result concerns graphs \( G \) and \( G' \) such that \( p^{(4)}(G_{|K}) \equiv p^{(4)}(G'_{|K}) \) modulo a prime \( p \), for all \( k \)-element subsets \( K \) of \( V \).

**Theorem 4.2** Let \( G \) and \( G' \) be two graphs on the same set \( V \) of \( v \) vertices. Let \( p \) be a prime number and \( k \) be an integer, \( 4 \leq k \leq v - 4 \).

1) If \( p^{(4)}(G_{|K}) = p^{(4)}(G'_{|K}) \) for all \( k \)-element subsets \( K \) of \( V \) then \( G \) and \( G' \) have the same indecomposable sets of size 4.

2) Assume \( p^{(4)}(G_{|K}) \equiv p^{(4)}(G'_{|K}) \) (mod \( p \)) for all \( k \)-element subsets \( K \) of \( V \).

a) If \( p \geq 5 \) and \( k \not\equiv 1, 2, 3 \) (mod \( p \)), then \( G \) and \( G' \) have the same indecomposable sets of size 4.

b) If \( (p = 2, 4 \mid k \) and \( 8 \nmid k) \) or \( (p = 3, 3 \mid k - 1 \) and \( 9 \nmid k - 1) \), then \( G \) and \( G' \) have the same indecomposable sets of size 4.

c) If \( p = 2 \) and \( 8 \mid k \), then \( G \) and \( G' \) have the same indecomposable sets of size 4, or for all 4-element subsets \( T \) of \( V \), \( G_{|T} \) is indecomposable if and only if \( G'_{|T} \) is decomposable.

**Proof.** Let \( U := \{T \subseteq V : |T| = 4, G_{|T} \simeq P_4\} = P^{(4)}(G), U' := \{T \subseteq V : |T| = 4, G'_{|T} \simeq P_4\} = P^{(4)}(G') \). For all \( K \subseteq V \), we have \( \{T \subseteq K : T \in U\} = P_4(G_{|K}) \) and \( \{T \subseteq K : T \in U'\} = P_4(G'_{|K}) \). Set \( t := |T| = 4 \).

1) Since \( p^{(4)}(G_{|K}) = p^{(4)}(G'_{|K}) \) then \(|\{T \subseteq K : T \in U\}| = |\{T \subseteq K : T \in U'\}| \). From
Lemma 1.2: $U = U'$, then $G$ and $G'$ have the same indecomposable sets of size 4.

2) We have $p^4(G|K) \equiv p^4(G'|K)$ (mod p) for all $k$-element subsets $K$ of $V$, then $|\{T \subseteq K : T \in U\}| \equiv |\{T \subseteq K : T \in U'\}|$ (mod p).

a) Case 1. $p \geq 5$, $t = 4 = [4]_p$, $k = [k_0, \ldots]_p$ and $t_0 = 4 \leq k_0$. From 1) of Theorem 1.3 we have $U = U'$, thus $G$ and $G'$ have the same indecomposable sets of size 4.

Case 2. $p = 5$, $t = 4 = [4]_p$, $k = [0, k_1, \ldots]_p$. Since in every graph of order 5, there is a restriction of size 4 not isomorphic to $P_4$ then, from 2) of Theorem 1.3 $U = U'$, thus $G$ and $G'$ have the same indecomposable sets of size 4.

b) Case 1. $p = 2$, $t = 4 = [0, 0, 1]_p$ and $k = [0, 0, 1, k_3, \ldots, k_{k(p)}]_p$. From 1) of Theorem 1.3 we have $U = U'$, thus $G$ and $G'$ have the same indecomposable sets of size 4.

c) Case 2. $p = 2$, $t = 4 = [1, 1]_p$, $k = [1, k_1, \ldots, k_{k(p)}]_p$ and $t_1 = 1 \leq k_1$. From 1) of Theorem 1.3 $U = U'$, or for all 4-element subsets $T$ of $V$, $T \in U$ if and only if $T \notin U'$. Thus $G$ and $G'$ have the same indecomposable sets of size 4, or for all 4-element subsets $T$ of $V$, $G|T$ is indecomposable if and only if $G'|T$ is decomposable. □

In a reconstruction problem of graphs up to complementation [10], Wilson’s Theorem yielded the following result:

**Theorem 4.3 (10)** Let $G$ and $G'$ be two graphs on the same set $V$ of $v$ vertices (possibly infinite). Let $k$ be an integer, $5 \leq k \leq v - 2$, $k \equiv 1$ (mod 4). Then the following properties are equivalent:

(i) $e(G|K)$ has the same parity as $e(G'|K)$ for all $k$-element subsets $K$ of $V$; and $G|K$, $G'|K$ have the same $3$-homogeneous subsets;

(ii) $G' = G$ or $G' = \overline{G}$.

Here, we just want to point out that we can obtain a similar result for $k \equiv 3$ (mod 4), namely Theorem 4.4 using the same proof as that of Theorem 1.3.

The boolean sum $G \oplus G'$ of two graphs $G = (V, E)$ and $G' = (V, E')$ is the graph $U$ on $V$ whose edges are pairs $e$ of vertices such that $e \in E$ if and only if $e \notin E'$.

**Theorem 4.4** Let $G$ and $G'$ be two graphs on the same set $V$ of $v$ vertices (possibly infinite). Let $k$ be an integer, $3 \leq k \leq v - 2$, $k \equiv 3$ (mod 4). Then the following properties are equivalent:

(i) $e(G|K)$ has the same parity as $e(G'|K)$ for all $k$-element subsets $K$ of $V$; and $G|K$, $G'|K$ have the same $3$-homogeneous subsets;

(ii) $G' = G$.

**Proof.** It is exactly the same as that of Theorem 1.3 (see [10]). The implication $(ii) \Rightarrow (i)$ is trivial. We prove $(i) \Rightarrow (ii)$. We suppose $V$ finite, we set $U := G \oplus G'$, let $T_1, T_2, \cdots, T_{\binom{k}{2}}$ be an enumeration of the 2-element subsets of $V$, let $K_1, K_2, \cdots, K_{\binom{k}{2}}$ be
an enumeration of the \( k \)-element subsets of \( V \). Let \( w_U \) be the row matrix \( (u_1, u_2, \cdots, u_{\binom{v}{2}}) \) where \( u_i = 1 \) if \( T_i \) is an edge of \( U \), 0 otherwise.

We have \( w_U W_{2k} = (e(U_{1K_1}), e(U_{1K_2}), \cdots, e(U_{1K_{\binom{v}{2}}})) \). From the facts that \( e(G_{1K}) \) has the same parity as \( e(G'_{1K}) \) and \( e(U_{1K}) = e(G_{1K}) + e(G'_{1K}) - 2e(G_{1K} \cap G'_{1K}) \) for all \( k \)-element subsets \( K \), \( w_U \) belongs to the kernel of \( W_{2k} \) over the 2-element field. According to Theorem 2.2, the rank of \( W_{2k} \) (mod 2) is \( \binom{v}{2} - v + 1 \). Hence \( \text{dim} \text{Ker}(W_{2k}) = v - 1 \).

We give a similar claim as Claim 2.8 of [10], the proof is identical.

Claim 4.5 Let \( k \) be an integer such that \( 3 \leq k \leq v - 2 \), \( k \equiv 3 \) (mod 4), then the kernel of \( W_{2k} \) consists of complete bipartite graphs (including the empty graph).

Proof. Let us recall that a star-graph of \( v \) vertices consists of a vertex linked to all other vertices, those \( v - 1 \) vertices forming an independent set. First we prove that each star-graph \( S \) belongs to \( \mathbb{K} \), the kernel of \( W_{2k} \). Let \( w_S \) be the row matrix \( (s_1, s_2, \cdots, s_{\binom{v}{2}}) \) where \( s_i = 1 \) if \( T_i \) is an edge of \( S \), 0 otherwise. We have \( w_S W_{2k} = (e(S_{1K_1}), e(S_{1K_2}), \cdots, e(S_{1K_{\binom{v}{2}}})) \). For all \( i \in \{1, \ldots, \binom{v}{2}\} \), \( e(S_{1K_i}) = k - 1 \) if \( 1 \in K_i \), 0 otherwise. Since \( k \) is odd, each star-graph \( S \) belongs to \( \mathbb{K} \). The vector space (over the 2-element field) generated by the star-graphs on \( V \) consists of all complete bipartite graphs; since \( v \geq 3 \), these are distinct from the complete graph (but include the empty graph). Moreover, its dimension is \( v - 1 \) (a basis being made of star-graphs). Since \( \text{dim} \text{Ker}(W_{2k}) = v - 1 \), then \( \mathbb{K} \) consists of complete bipartite graphs as claimed.

\( \square \)

A claw is a star-graph of four vertices, that is a graph made of a vertex joined to three other vertices, with no edges between these three vertices. A graph is claw-free if no induced subgraph is a claw.

Claim 4.6 ([10]) Let \( G \) and \( G' \) be two graphs on the same set and having the same 3-homogeneous subsets, then the boolean sum \( U := G \oplus G' \) is claw-free.

From Claim 4.5, \( U \) is a complete bipartite graph and, from Claim 4.6, \( U \) is claw-free. Since \( v \geq 5 \), it follows that \( U \) is the empty graph. Hence \( G' = G \) as claimed.

\( \square \)

5 Illustrations to tournaments

Let \( T = (V, E) \) be a tournament. For two distinct vertices \( x \) and \( y \) of \( T \), \( x \rightarrow_T y \) (or simply \( x \rightarrow y \)) means that \( (x, y) \in E \) and \( (y, x) \notin E \). For \( A \subseteq V \) and \( y \in V \), \( A \rightarrow y \) means \( x \rightarrow y \) for all \( x \in A \). The degree of a vertex \( x \) of \( T \) is \( d_T(x) := |\{y \in V : x \rightarrow y\}| \). We denote by \( T^* \) the dual of \( T \) that is \( T^* = (V, E^*) \) with \( (x, y) \in E^* \) if and only if \( (y, x) \in E \). A transitive tournament or a total order or \( k \)-chain (denoted \( O_k \)) is a tournament of cardinality \( k \), such that for \( x, y, z \in V \), if \( x \rightarrow y \) and \( y \rightarrow z \), then \( x \rightarrow z \). If \( x \) and \( y \) are two distinct vertices of a total order, the notation \( x < y \) means that \( x \rightarrow y \). The tournament \( C_3 := \{\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\}\} \)
Let $T = (V, E)$ and $T' = (V, E')$ be two tournaments. Let $p$ be a prime number and $k$ be an integer, $2 \leq k \leq v - 2$. Let $G := T + T'$. We assume that for all $k$-element subsets $K$ of $V$, $e(G_{|K}) \equiv 0 \pmod{p}$.

1) If $p \geq 3$, $k \not\equiv 0, 1 \pmod{p}$, then $T' = T$.
2) If $p \geq 3$, $k \equiv 0 \pmod{p}$, then $T' = T$ or $T' = T^*$.
3) If $p = 2$, $k \equiv 2 \pmod{4}$, then $T' = T$.
4) If $p = 2$, $k \equiv 0 \pmod{4}$, then $T' = T$ or $T' = T^*$.

**Proof.** We set $G' :=$ The empty graph. Then $e(G_{|K}) \equiv e(G'_{|K}) \pmod{p}$.

1) From 1) of Theorem 1.5, $G$ is the empty graph, then $T' = T$.
2) From 2) of Theorem 1.5, $G$ is empty or the complete graph, then $T' = T$ or $T' = T^*$.
3) From 3) of Theorem 1.5, $G$ is the empty graph, then $T' = T$.
4) From Theorem 1.4, $G$ is the empty graph or the complete graph, then $T' = T$ or $T' = T^*$.

Let $T$ be a tournament, we set $C^{(3)}(T) := \{\{a, b, c\} : T_{|\{a,b,c\}} \text{ is a 3-cycle}\}$, and $c^{(3)}(T) := |C^{(3)}(T)|$. Let $T = (V, E)$ and $T' = (V, E')$ be two tournaments, let $k$ be a non-negative integer, $T$ and $T'$ are $k$-hypomorphic [7, 21] (resp. $k$-hypomorphic up to duality) if for every $k$-element subset $K$ of $V$, the induced subtournaments $T'_{|K}$ and $T_{|K}$ are isomorphic (resp. $T'_{|K}$ is isomorphic to $T_{|K}$ or to $T^*_{|K}$). We say that $T$ and $T'$ are $(\leq k)$-hypomorphic if $T$ and $T'$ are $h$-hypomorphic for every $h \leq k$. Similarly, we say that $T$ and $T'$ are $(\leq k)$-hypomorphic up to duality if $T$ and $T'$ are $h$-hypomorphic up to duality for every $h \leq k$.

**Theorem 5.2** Let $T$ and $T'$ be two tournaments on the same set $V$ of $v$ vertices. Let $p$ be a prime number and $k$ be an integer, $3 \leq k \leq v - 3$.

1) If $c^{(3)}(T_{|K}) = c^{(3)}(T'_{|K})$ for all $k$-element subsets $K$ of $V$ then $T$ and $T'$ are $(\leq 3)$-hypomorphic.
2) Assume $p \geq 5$. If $k \not\equiv 1, 2 \pmod{p}$, and $c^{(3)}(T_{|K}) \equiv c^{(3)}(T'_{|K}) \pmod{p}$ for all $k$-element subsets $K$ of $V$, then $T$ and $T'$ are $(\leq 3)$-hypomorphic.

3) If $(p = 2$ and $k \not\equiv 3 \pmod{4})$ or $(p = 3$ and $3 \mid k)$, and $c^{(3)}(G_{|K}) \equiv c^{(3)}(G'_{|K}) \pmod{p}$ for all $k$-element subsets $K$ of $V$, then $T$ and $T'$ are $(\leq 3)$-hypomorphic.

**Proof.** Since every tournament, of cardinality $\geq 4$, has at least a restriction of cardinality 3 which is not a 3-cycle, then the proof is similar to that of Theorem 4.1.

Let $T$ be a tournament, we set $D^+_4(T) := \{\{a, b, c, d\} : T_{\{a,b,c,d\}} \simeq \delta^+\}$, $D^-_4(T) := \{\{a, b, c, d\} : T_{\{a,b,c,d\}} \simeq \delta^-\}$, $d^+_4(T) := |D^+_4(T)|$ and $d^-_4(T) := |D^-_4(T)|$.

It is well-known that every subtournament of order 4 of a tournament is either a diamond, a 4-chain, or a 4-cycle subtournament. We have $c^{(3)}(O_4) = 0$, $c^{(3)}(\delta^+) = c^{(3)}(\delta^-) = 1$, $c^{(3)}(C_4) = 2$ and $C_4 \simeq C^*_4$.

**Theorem 5.3** Let $T$ and $T'$ be two $(\leq 3)$-hypomorphic tournaments on the same set $V$ of $v$ vertices. Let $p$ be a prime number and $k$ be an integer, $4 \leq k \leq v - 4$.

1) If $d^+_4(T_{|K}) = d^+_4(T'_{|K})$ for all $k$-element subsets $K$ of $V$ then $T'$ and $T$ are $(\leq 5)$-hypomorphic.

2) Assume $d^+_4(T_{|K}) \equiv d^+_4(T'_{|K}) \pmod{p}$ for all $k$-element subsets $K$ of $V$.

a) If $p \geq 5$ and, $k \not\equiv 1, 2, 3 \pmod{p}$, then $T'$ and $T$ are $(\leq 5)$-hypomorphic.

b) If $(p = 3, 3 \mid k - 1$ and $9 \nmid k - 1)$ or $(p = 2, 4 \mid k$ and $8 \nmid k)$, then $T'$ and $T$ are $(\leq 5)$-hypomorphic.

c) If $p = 2$ and $8 \mid k$, then $T'$ and $T$ are $(\leq 5)$-hypomorphic or for all 4-element subset $S$ of $V$, $T_{|S}$ is isomorphic to $\delta^+$ if and only if $T'_{|S}$ is isomorphic to $\delta^-$.

**Proof.** To prove that $T'$ and $T$ are $(\leq 5)$-hypomorphic, the following lemma shows that it is sufficient to prove that $T'$ and $T$ are $(\leq 4)$-hypomorphic.

**Lemma 5.4** Let $T$ and $T'$ be two $(\leq 4)$-hypomorphic tournaments on at least 5 vertices. Then, $T$ and $T'$ are $(\leq 5)$-hypomorphic.

Now, let $U^+ := \{S \subseteq V : T_{|S} \simeq \delta^+\} = D^+_4(T)$, $U'^+ := D^+_4(T')$, $U^- := D^-_4(T)$ and $U'^- := D^-_4(T')$.

**Claim 5.5** If $T$ and $T'$ are $(\leq 3)$-hypomorphic and $U^+ = U'^+$, then $U^- = U'^-$; $T$ and $T'$ are $(\leq 5)$-hypomorphic.

**Proof.** Let $S \in U^-$, $T_{|S} \simeq \delta^-$. Since $T$ and $T'$ are $(\leq 3)$-hypomorphic, then $T'_{|S} \simeq \delta^+$ or $T'_{|S} \simeq \delta^-$. We have $\{S \subseteq V : T'_{|S} \simeq \delta^+\} = \{S \subseteq V : T_{|S} \simeq \delta^+\}$, then $T'_{|S} \simeq \delta^-$, $S \in U'^-$ and $U^- = U'^-$. So, for $X \subseteq V$, if $T_{|X}$ is a diamond then $T'_{|X}$ is a diamond.

Now we prove that $T$ and $T'$ are $4$-hypomorphic. Let $X \subseteq V$ such that $|X| = 4$. If $T_{|X} \simeq C_4$, then $c^{(3)}(T_{|X}) = 2$. Since $T$ and $T'$ are $(\leq 3)$-hypomorphic then $c^{(3)}(T'_{|X}) = 2$, thus $T'_{|X} \simeq T_{|X} \simeq C_4$. The same, if $T_{|X} \simeq O_4$, then $T'_{|X} \simeq T_{|X} \simeq O_4$. So, $T'$ and $T$ are $(\leq 4)$-hypomorphic. Then, From Lemma 5.4, $T'$ and $T$ are $(\leq 5)$-hypomorphic. From Claim 5.5, it is sufficient to prove that $U^+ = U'^+$.

For all $K \subseteq V$ with $|K| = k$, we have $\{S \subseteq K : S \in U^+\} = D^+_4(T_{|K})$ and
\{S \subseteq K : S \in U^+\} = D_1^+(T_K').
1) Since \(d_1^+(T_K) = d_1^+(T_K')\) then \(|\{S \subseteq K : S \in U^+\}| = |\{S \subseteq K : S \in U'^+\}|\). From Lemma 1.2, we have \(U^+ = U'^+\).

2) We have \(d_1^+(T_K) \equiv d_1^+(T_K') \pmod{p}\) for all \(k\)-element subsets \(K\) of \(V\), then \(|\{S \subseteq K : S \in U^+\}| \equiv |\{S \subseteq K : S \in U'^+\}| \pmod{p}\).

a) Case 1. \(p \geq 5\), \(t = 4 = [4]_p\), \(k = [k_0, \ldots ]_p\) and \(t_0 = 4 \leq k_0\). From 1) of Theorem 1.3, we have \(U^+ = U'^+\).

Case 2. \(p \geq 5\), \(t = 4 = [4]_p\), \(k = [0, k_1, \ldots ]_p\). Since every tournament of cardinality \(\geq 5\) has at least a restriction of cardinality 4 which is not a diamond, then from 2) of Theorem 1.3, \(U^+ = U'^+\).

b) Case 1. \(p = 3\), \(t = 4 = [1, 1]_p\), \(k = [1, k_1, \ldots , k_{k(p)}]_p\) and \(t_1 = 1 \leq k_1\). From 1) of Theorem 1.3, we have \(U^+ = U'^+\).

Case 2. \(p = 2\), \(t = 4 = [0, 0, 1]_p\) and \(k = [0, 0, 1, k_3, \ldots , k_{k(p)}]_p\). From 1) of Theorem 1.3, we have \(U^+ = U'^+\).

c) We have \(p = 2\), \(t = 4 = [0, 0, 1]_p\), \(k = [0, 0, 0, k_3, \ldots , k_{k(p)}]_p\). Since every tournament of cardinality \(\geq 5\) has at least a restriction of cardinality 4 which is not a diamond, and the fact that \(T\) and \(T'\) are 3-hypomorphic, then from 2) of Theorem 1.3, \(U^+ = U'^+\), thus \(T'\) and \(T\) are \((\leq 5)\)-hypomorphic, or for all 4-element subsets \(S\) of \(V\), \(T_{\mid S}\) is isomorphic to \(\delta^+\) if and only if \(T'_{\mid S}\) is isomorphic to \(\delta^-\).

\(\square\)

Given a digraph \(S = (\{0, 1, \ldots , m - 1\}, A)\), where \(m \geq 1\) is an integer, for \(i \in \{0, 1, \ldots , m - 1\}\) we associate a digraph \(G_i = (V_i, A_i)\), with \(|V_i| \geq 1\), such that the \(V_i\)'s are mutually disjoint. The lexicographic sum of \(S\) by the digraphs \(G_i\) or simply the \(S\)-sum of the \(G_i\)'s, is the digraph denoted by \(S(G_0, G_1, \ldots , G_{m-1})\) and defined on the union of the \(V_i\)'s as follows: given \(x \in V_i\) and \(y \in V_j\), where \(i, j \in \{0, 1, \ldots , m - 1\}\), \((x, y)\) is an arc of \(S(G_0, G_1, \ldots , G_{m-1})\) if either \(i = j\) and \((x, y) \in A_i\) or \(i \neq j\) and \((i, j) \in A\): this digraph replaces each vertex \(i\) of \(S\) by \(G_i\). We say that the vertex \(i\) of \(S\) is diluted by \(G_i\).

Let \(h\) be a non-negative integer. The integers below are considered modulo \(2h + 1\).

The circular tournament \(T_{2h+1}\) (see Figure 2) is defined on \(\{0, 1, \ldots , 2h\}\) by:

\(T_{2h+1, \{0, 1, \ldots , h\}}\) is the usual total order on \(\{0, 1, \ldots , h\}\), \(T_{2h+1, \{h+1, \ldots , 2h\}}\) is also the usual order on \(\{h+1, h+2, \ldots , 2h\}\), however \(\{i+1, i+2, \ldots , i+h\}\) \(\rightarrow\) \(T_{2h+1, \{0, 1, \ldots , i\}}\) for every \(i \in \{0, 1, \ldots , h-1\}\). A tournament \(T\) is said to be an element of \(D(T_{2h+1})\) if \(T\) is obtained by dilating each vertex of \(T_{2h+1}\) by a finite chain \(p_i\), then \(T = T_{2h+1}(p_0, p_1, \ldots , p_{2h})\). We recall that \(T_{2h+1}\) is indecomposable and \(D(T_{2h+1})\) is the class of finite tournaments without diamond [21].

We define the tournament \(\beta_6^+ := T_3(p_0, p_1, p_2)\) with \(p_0 = (0 < 1 < 2)\), \(p_1 = (3 < 4)\) and \(|p_2| = 1\) (see Figure 3). We set \(\beta_6^- := (\beta_6^+)^*\). For a tournament \(T = (V, E)\), we set \(B_6^+(T) := \{S \subseteq V : T_{\mid S} \simeq \beta_6^+\}\), \(B_6^-(T) := \{S \subseteq V : T_{\mid S} \simeq \beta_6^-\}\), \(b_6^+(T) := |B_6^+(T)|\) and \(b_6^-(T) := |B_6^-(T)|\).
Two tournaments $T$ and $T'$ on the same vertex set $V$ are hereditarily isomorphic if for all $X \subseteq V$, $T|_X$ and $T'|_X$ are isomorphic [3].

Figure 2: Circular tournament $T_{2h+1}$

Figure 3: $\beta_6^+$. Let $G = (V, E)$ and $G' = (V, E')$ be two ($\leq 2$)-hypomorphic digraphs. Denote $D_{G,G'}$ the binary relation on $V$ such that: for $x \in V$, $xD_{G,G'}x$; and for $x \neq y \in V$, $xD_{G,G'}y$ if there exists a sequence $x_0 = x, ..., x_n = y$ of elements of $V$ satisfying $(x_i, x_{i+1}) \in E$ if and only if $(x_i, x_{i+1}) \notin E'$, for all $i$, $0 \leq i \leq n - 1$. The relation $D_{G,G'}$ is an equivalence relation called the difference relation, its classes are called difference classes.

Using difference classes, G. Lopez [19, 20] showed that if $T$ and $T'$ are ($\leq 6$)-hypomorphic then $T$ and $T'$ are isomorphic. One may deduce the next corollary.

Corollary 5.6 ([19, 20]) Let $T$ and $T'$ be two tournaments. We have the following properties:
1) If $T$ and $T'$ are ($\leq 6$)-hypomorphic then $T$ and $T'$ are hereditarily isomorphic.
2) If for each equivalence class $C$ of $D_{T,T'}$, $C$ is an interval of $T$ and $T'$, and $T'|_C$, $T|_C$ are ($\leq 6$)-hypomorphic, then $T$ and $T'$ are hereditarily isomorphic.

Lemma 5.7 [22] Given two ($\leq 4$)-hypomorphic tournaments $T$ and $T'$, and $C$ an equivalence class of $D_{T,T'}$, then:
1) $C$ is an interval of $T'$ and $T$.
2) Every 3-cycle in $T|_C$ is reversed in $T'|_C$.
3) There exists an integer $h \geq 0$ such that $T|_C = T_{2h+1}(p_0, p_1, ..., p_{2h})$ and $T'|_C = T_{2h+1}^*(p'_0, p'_1, ..., p'_{2h})$ with $p_i$, $p'_i$ are chains on the same basis, for all $i \in \{0, 1, ..., 2h\}$. 
Theorem 5.8 Let $T$ and $T'$ be two $(\leq 4)$-hompomorphic tournaments on the same set $V$ of $v$ vertices. Let $p$ be a prime number and $k = [k_0, k_1, \ldots, k_{k(p)}]_p$ be an integer, $6 \leq k \leq v - 6$.

1) If $b_6^0(T|_K) = b_6^0(T'|_K)$ for all $k$-element subsets $K$ of $V$ then $T'$ and $T$ are hereditarily isomorphic.

2) Assume $b_6^0(T|_K) \equiv b_6^0(T'|_K) \pmod{p}$ for all $k$-element subsets $K$ of $V$.

a) If $p \geq 7$, and $k_0 \geq 6$ or $k_0 = 0$, then $T'$ and $T$ are hereditarily isomorphic.

b) If $p = 5$, $k_0 = 1$ and $k_1 \neq 0$ or $(p = 3$, $k_0 = 0$ and $k_1 = 2)$ or $(p = 3$ and $k_0 = 1 = k_2 = 1$), then $T'$ and $T$ are hereditarily isomorphic.

Proof. Let $U^+ := \{S \subseteq V, T|_S \simeq \beta_6^+\} = B_6^+(T)$, $U'^+ := B_6^+(T')$, $U^- := \{S \subseteq V, T|_S \simeq \beta_6^-\} = B_6^-(T)$, $U'^- := B_6^-(T')$.

Every tournament of cardinality $\geq 7$ has at least a restriction of cardinality 6 which is not isomorphic to $\beta_6^+$ and $\beta_6^-$. Then for all cases, similarly to the proof of Theorem 5.3, we have $U^+ = U'^+$. Let $C$ be an equivalence class of $D_{T,T'}$, $S \in U^-$, $T|_S \simeq \beta_6^-$. Since $T$ and $T'$ are $(\leq 3)$-hompomorphic, then $T|_S \simeq \beta_6^+$ or $T'|_S \simeq \beta_6^-$. We have $\{S \subseteq V, T|_S \simeq \beta_6^+\} = \{S \subseteq V, T|_S \simeq \beta_6^-\}$, then $T'|_S \simeq \beta_6^-$, $S \in U^-$ and $U = U'^-$. Let $X \subseteq C$ such that $|X| = 6$; if $T_X \simeq \beta_6^-$ then, from 2) of Lemma 5.7, $T_X \simeq \beta_6^-$, that is impossible, so $T_C$ and $T'_C$ has not a restriction of cardinality 6 isomorphic to $\beta_6^+$ and $\beta_6^-$. Now we will prove that $T|_C$ and $T'|_C$ are $(\leq 6)$-hompomorphic.

From 3) of Lemma 5.7 there exists an integer $h \geq 0$ such that $T|_C = T_{2h+1}(p_0, p_1, \ldots, p_{2h})$, with $p_i$ is a chain and $a_i \in p_i$ for all $i \in \{0, 1, \ldots, 2h\}$. Since $T|_C$ hasn’t a tournament isomorphic to $\beta_6^+$, then $h \leq 3$. Indeed, if $h \geq 4$, then $T|_{\{a_0,a_1,a_2,a_3,a_4,a_{3+h}\}} \simeq \beta_6^+$, and $\{a_0,a_1,a_2\}, \{a_3,a_4\}$ are two intervals of $T|_{\{a_0,a_1,a_2,a_3,a_4,a_{3+h}\}}$, that is impossible.

a) If $h = 3$, then $T|_C = T_7$. Indeed, if $a_0,b_0 \in V(p_0)$ then $T|_{\{a_0,b_0,a_1,a_2,a_3,a_5\}} \simeq \beta_6^+$, and $\{a_0,b_0,a_1\}, \{a_2,a_3\}$ are two intervals of $T|_{\{a_0,b_0,a_1,a_2,a_3,a_5\}}$, that is impossible.

b) If $h = 2$, then $T|_C = T_5$, or $T|_C$ is obtained by dilating one vertex of $T_5$ by a chain of cardinality 2. Indeed:

Case 1. $a_0,b_0,c_0 \in V(p_0)$, then $T|_{\{a_0,b_0,c_0,a_1,a_2,a_3\}} \simeq \beta_6^+$ and $\{a_0,b_0,c_0\}, \{a_1,a_2\}$ are two intervals of $T|_{\{a_0,b_0,c_0,a_1,a_2,a_3\}}$, that is impossible.

Case 2. If $a_i,b_i \in V(p_i)$ for all $i \in \{0,1\}$, then $T|_{\{a_0,b_0,a_1,a_3,a_4\}} \simeq \beta_6^+$ and $\{a_0,b_0,a_4\}, \{a_1,b_1\}$ are two intervals of $T|_{\{a_0,b_0,a_1,a_3,a_4\}}$, that is impossible.

Case 3. If $a_i,b_i \in V(p_i)$ for all $i \in \{0,1,2\}$, then $T|_{\{a_0,b_0,a_1,a_2,a_4\}} \simeq \beta_6^+$ and $\{a_0,b_0,a_1\}, \{a_2,b_2\}$ are two intervals of $T|_{\{a_0,b_0,a_1,a_2,a_4\}}$, that is impossible.

Case 3. If $h = 1$, then $T|_C$ is obtained by dilating one vertex of $C_3$ by a chain or by dilating two or three vertices of $C_3$ by a chain of cardinality 2.

d) If $h = 0$, then $T|_C$ is a chain.

In all cases, $T|_C$ and $T'|_C$ are $(\leq 6)$-hompomorphic. From 1) of Lemma 5.7 $C$ is an interval of $T'$ and $T$. Then, from 2) of Corollary 5.6 $T$ and $T'$ are hereditarily isomorphic. □
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