Extraction of Scattering Lengths from Production Reactions

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Abstract. We review a method based on dispersion theory, that allows one to extract the scattering length of a hadronic two-body system from corresponding final-state interactions in production reactions. Possible theoretical uncertainties of the method are discussed. A particular case of the hyperon-nucleon final-state interaction is analyzed. A generalization of the method to the case of strangeness \( S = -2 \) and \( S = -3 \) systems is considered. The possibility to disentangle spin-triplet and spin-singlet scattering lengths by means of various polarization measurements is demonstrated by the examples of several production reactions in \( K^-d \) and \( \gamma d \) scattering.

1. Introduction

Only a few hadronic systems can be studied directly via scattering experiments. For the others it is difficult or impossible to prepare a corresponding beam or target. Therefore, one is forced to utilize more sophisticated methods to investigate interactions among such particles. One possible way to study such systems is to look at the interaction of the produced hadrons in the final state. A method for extracting hadronic scattering lengths from production reactions was proposed in [1, 2, 3]. The presentation of the method in those publications was done with special emphasis on its application to the hyperon-nucleon \((YN)\) interaction. In particular, the reactions \( NN \rightarrow KYN \) and \( \gamma d \rightarrow KYN \) were analyzed, and possible uncertainties of the method were established. Polarization observables needed to disentangle different spin states of the final \( YN \) system were identified. However, the method can be applied just as well to the baryon-baryon interaction in the strangeness \( S = -2 \) sector [4]. Here in particular the strength of the \( \Lambda \Lambda \) interaction is of specific interest, first for a better understanding of the role played by SU(3) flavor symmetry but also in the quest for double-\( \Lambda \) hypernuclei [5]. Actually, there have been theoretical investigations of the \( S = -2 \) sector not only within the framework of chiral effective field theory [6] but also within lattice QCD [7] in recent times. The purpose of this talk is to review the method proposed in [1, 2, 3] and to discuss its possible application for various systems, including \( S = -2 \) and \( S = -3 \) baryon-baryon states, and for various production reactions.

2. Review of the method

The basic idea of the method is to exploit the scale separation between a short range production operator and long range final state interaction (FSI). In this case the production operator can
be regarded as point-like, and the FSI can be factored out. These conditions restrict the class of reactions and kinematic regimes, that one can consider. Namely, one can only apply the method to reactions with large momentum transfer \( q_t \). On the other hand the scattering length in the system under consideration must be large (\( a \gg 1/q_t \)). Sufficiently large scattering lengths are expected in the baryon-baryon sector. In particular it is interesting to study hyperon-nucleon and hyperon-hyperon interaction with different strangeness content of hyperons. An elegant way to utilize the condition of scale separation is a dispersion-relation approach. Imposing unitarity and analyticity constraints on the amplitude one arrives at the following expression for the reaction amplitude \( A_S \) \[8, 9, 1\]

\[ A_S(s, t, m^2) = \exp \left[ \frac{1}{\pi} \int_{m_0^2}^{m_{\text{max}}^2} \frac{\delta_S(m^2)}{m^2 - m^2 - i0} dm^2 \right] \Phi(s, t, m^2), \]

where \( m \) is the invariant mass of the produced baryon-baryon system with its threshold value \( m_0 \), \( s \) is the total center-of-mass (CM) energy squared, and \( t \) represents all the remaining kinematic variables the amplitude depends upon. The function \( \Phi(s, t, m^2) \) slowly varies with \( m^2 \), which is a consequence of large momentum transfer. The cut off \( m_{\text{max}} \) in the integral is determined by the region, where FSI effects are expected to be important. For the hyperon-nucleon interaction (and probably for other baryon-baryon sectors) a typical cut off from the scale arguments is equal to \( \epsilon_{\text{max}} = m_{\text{max}} - m_0 \approx 40 \text{ MeV} \) \[1\]. The baryon-baryon scattering is assumed to be elastic in this region (there are no other open channels) and dominated by the \( s \)-wave amplitude parametrized by the phase shift \( \delta_S(m^2) \). Note that formula (1) can only be applied to amplitudes for a specific baryon-baryon spin state \( S \). Therefore, one has to be able to separate spin-singlet and spin-triplet states experimentally.

It was shown in \[1\] how one can invert Eq. (1) to express the scattering length via the reaction amplitude squared (or the differential cross section \( \frac{d^2 \sigma_s}{dm^2 dt} \) corresponding to the production of the baryon-baryon system with spin “S”)

\[ a_S = \lim_{m^2 \to m_0^2} \frac{1}{2\pi} \frac{m_1 + m_2}{\sqrt{m_1 m_2}} \frac{1}{\sqrt{m^2 - m_{\text{min}}^2}} \times \frac{1}{\sqrt{m^2 - m^2}} \left[ \frac{1}{\sqrt{m^2 - m_{\text{max}}^2}} \right] \left( \frac{d^2 \sigma_s}{dm^2 dt} \right)^{-1}, \]

where \( m_1, m_2 \) are the masses of the two baryons, and \( p \) is the CM momentum in the baryon-baryon system. An analogous equation can be derived for the effective range.

Possible theoretical uncertainties of the method originate from the following sources: (i) energy dependence of the production operator, (ii) influence of scattering at the higher energy (\( m > m_{\text{max}} \)), (iii) contributions from inelastic channel (e.g. \( \Sigma N \to \Lambda N \) transition) and (iv) final state interaction among other pairs of particles. For the hyperon-nucleon FSI the theoretical uncertainty in determination of the scattering length was estimated not to exceed 0.3 fm \[1\]. This estimate was confirmed by model calculations of production amplitudes using several different models for the hyperon-nucleon interactions with triplet and singlet scattering lengths varying from \(-0.7 \) to \(-2.5 \text{ fm} \). Fig. 1 shows the dependence of the difference of the exact scattering length of the model and the one extracted via Eq. (1) as a function of the cut off \( \epsilon_{\text{max}} = m_{\text{max}} - m_0 \). This difference lies inside the estimated error interval for reasonable values of the cut off.

The general form of Eq. (1) admits approximations under certain conditions. One of those approximative treatments follows from the assumption that the phase shifts are given by the first two terms in the effective range expansion,

\[ p \cot(\delta(m^2)) = -\frac{1}{a} + \frac{r_e}{2} \frac{p^2}{a^2}, \]
Figure 1. Dependence of the extracted scattering lengths on the value of the upper limit of integration, $\epsilon_{\text{max}} = m_{\text{max}} - m_0$. Shown is the difference from the exact results for the spin singlet scattering length $a_s$ (left panel) and the spin triplet scattering length $a_t$ (right panel). The solid and the dot–dashed line correspond respectively to model NSC97a and NSC97f of Ref. [10] and the dashed one corresponds to the $YN$ model of Ref. [11]. The shaded area indicates the estimated error of the proposed method and the arrows indicate the value for $\epsilon_{\text{max}}$ as estimated based on scale arguments.

usually called the effective range approximation (ERA), over the whole energy range. Here $p$ is the relative momentum of the final-state particles under consideration in their CM system, corresponding to the invariant mass $m^2$. In this case the relevant integrals (1) can be evaluated in closed form as [12]

$$ A(m^2) \propto \frac{(p^2 + \alpha^2)r_e / 2}{-1/a + (r_e/2)p^2 - ip}, \quad (3) $$

where $\alpha = 1/\sqrt{1 - 2r_e/a}$. Because of its simplicity Eq. (3) is often used for the treatment of the final-state interaction (FSI).

A further simplification can be made if one assumes that $a \gg r_e$. This situation is practically realized in the $^1S_0$ partial wave of the $NN$ system. Then the energy dependence of the quantity in Eq. (3) is given by the energy dependence of the elastic amplitude

$$ A(m^2) \propto \frac{1}{-1/a + (r_e/2)p^2 - ip}, \quad (4) $$

as long as $p \ll 1/r_e$. Therefore one expects that, at least for small kinetic energies, $NN$ elastic scattering and meson production in $NN$ collisions with a $NN$ final state exhibit the same energy dependence [12, 13, 14], which indeed was experimentally confirmed. This treatment of FSI effects is often referred to as Migdal-Watson (MW) approach [13, 14]. The reliability of such
approximations as compared to the formula (1) can be seen from Figs. 2, 3, where in analogy with Fig. 4 the differences between the model and extracted scattering lengths and effective ranges are shown. In general the method based on Eq. (1) works systematically better than the approximations and gives scattering lengths within 0.3 fm accuracy even for the rather large proton-nucleon value. The uncertainty in the extraction of the effective range is typically larger (see Fig. 3).

![Figure 2](image_url)

**Figure 2.** Comparison of different extraction methods for the scattering length $a$. Shown are the differences between results predicted by various $YN$ and $NN$ models and corresponding values extracted via the dispersion integral method (circles), the Jost-ERA approach (3) (squares) and the Migdal-Watson prescription (4) (triangles). The lines are drawn to guide the eye.

The method was also applied to the existing experimental data on unpolarized $pp \rightarrow K^+X$ mass spectrum [15] to see a possible influence of the experimental errors. By fitting the data and applying Eq. (1) one obtains the scattering length $a = -1.5 \pm 0.15$ fm. One could consider this as a spin-averaged results for the scattering length. Note, however, that there is no theoretically founded argument that the value above should lie indeed between the spin-singlet and spin-triplet scattering lengths. That is why a high precision polarization measurements are required to really extract spin dependent scattering lengths.

### 3. Strangeness $S = -2$ and $S = -3$ sectors

The technique utilized for the hyperon-nucleon interactions in [1, 2, 3] is applicable also in the case of strangeness $S = -2$ and $S = -3$ systems. The necessary condition that baryon-baryon scattering should be elastic up to some $m = m_{\text{max}}$ is satisfied for the following $S = -2$ channels: $\Lambda\Lambda$, $\Sigma^+\Sigma^+$, $\Sigma^-\Sigma^-$, $\Xi^0p$, $\Xi^-n$, and for the $S = -3$ channels: $\Xi^-\Lambda$, $\Xi^0\Lambda$, $\Xi^-\Sigma^-$, $\Xi^0\Sigma^+$. In what follows we are going to consider as examples $\gamma d$ and $K^-d$ scattering in complete analogy with the hyperon-nucleon production reactions considered in [3, 16]. The following three types of reactions can be used to produce baryon-baryon $S = -2$ and $S = -3$ states: $K^-d \rightarrow KB_1B_2$ ($K^-d \rightarrow K^0\Lambda\Lambda$, $K^-d \rightarrow K^+\Xi^-$), $\gamma d \rightarrow K_1K_2B_1B_2$ ($\gamma d \rightarrow K^0K^0\Xi^0p$, $\gamma d \rightarrow K^+K^0\Lambda\Lambda$,
the following set of polarization observables $\vec{k}$ necessary and the momentum of the emitted kaon is denoted by $K$. One to separate different spin states. For the reaction $\gamma d \rightarrow K^+ K^+ \Xi^- n$, $K^- d \rightarrow K_1 K_2 B_1 B_2$ ($K^- d \rightarrow K^0 K^0 \Xi^0 \Lambda$, $K^- d \rightarrow K^+ K^0 \Xi^- \Lambda$, $K^- d \rightarrow K^+ K^+ \Xi^- \Sigma^-$).

Now we come to the question of separating the spin-triplet and spin-singlet states in the baryon-baryon system. As in the case of the hyperon-nucleon interaction it is sufficient to consider reactions with polarized initial particles. We start from the general form for the reaction amplitude in the CM system for the three processes

\begin{align*}
\mathcal{M}_{K^- d \rightarrow K B_1 B_2} &= a_1^a (\vec{e}_d \times \hat{p}) \cdot \vec{k} + a_1^b (\vec{e}_d \cdot \hat{S}) (\vec{S} \cdot \vec{k}) + a_1^c (\vec{e}_d \cdot \hat{S}) (\vec{S} \cdot \vec{p}) ,
\mathcal{M}_{K^- d \rightarrow K_1 K_2 B_1 B_2} &= b_1^a (\vec{e}_d \times \hat{S}) \cdot \vec{p} + b_1^b (\vec{e}_d \cdot \hat{S}) (\vec{S} \cdot \vec{p}) + b_1^c (\vec{e}_d \cdot \vec{S}) (\vec{S} \cdot \vec{p}) ,
\mathcal{M}_{K^- d \rightarrow K_1 K_2 B_1 B_2} &= c_1^a (\vec{e}_d \cdot \hat{p}) + c_1^b (\vec{e}_d \times \vec{S}) \cdot \hat{p} ,
\end{align*}

where $a$, $b$, $c$ are some functions of $s$, $t$, and $m$, and $\vec{e}_d$, $\vec{e}_d$ are polarization vectors of the photon and deuteron, respectively, and their upper indices indicate whether they correspond to spin-singlet ($s$) or spin-triplet ($t$) amplitude. The spin vector $\vec{S}$ is used for the spin-triplet final state. For the last two reactions we assume the momenta of the final kaons to be either aligned or anti-aligned with the initial CM momentum $\hat{p}$. This leads to a significant simplification allowing one to separate different spin states. For the reaction $K^- d \rightarrow KB_1 B_2$ such a restriction is not necessary and the momentum of the emitted kaon is denoted by $\vec{k}$. It is convenient to introduce the following set of polarization observables

\begin{align*}
O_1 &= (1 - \sqrt{2} T_{20}^0) \frac{d\sigma_0}{dm^2 dt} , O_2 = (2 + \sqrt{2} T_{20}^0) \frac{d\sigma_0}{dm^2 dt} , O_3 = T_{10}^0 \frac{d\sigma_0}{dm^2 dt} , \\
O_4 &= \left(2 + \sqrt{2} T_{20}^0 + \sqrt{3} (T_{22}^0 + T_{22}^0)\right) \frac{d\sigma_0}{dm^2 dt} = \sqrt{3} (-\sqrt{2} T_{10}^c + (T_{22}^1 + T_{22}^1)) \frac{d\sigma_0}{dm^2 dt} ,
\end{align*}
where the various $T$’s for $\gamma d$ initial state are defined in [3], and $\frac{d\sigma}{d\Omega^2 dt}$ is the unpolarized differential cross section. $T^{0}_{20}$ and $T^{0}_{10}$ have the same definition also for the $K^- d$ initial state, the only difference being the absence of the summation over the photon polarizations. The observable $O_4$ selects the amplitudes with longitudinal target polarization $\vec{\epsilon}_d \parallel \hat{p}$, whereas $O_2$, $O_3$, $O_4$ select the amplitudes with $\vec{\epsilon}_d \perp \hat{p}$. In addition $O_4$ contain only that part of the amplitude which is antisymmetric with respect to an interchange of $\vec{\epsilon}_d$ and $\vec{\epsilon}_\gamma$.

Now inspecting the structure of the reaction amplitudes (5) we can identify the observables that allows one to separate a particular spin state.

For the reaction $K^- d \rightarrow KB_1B_2$ the triplet final state can be singled out by the observable $O_4$ for any direction of the emitted kaon, or one can measure the unpolarized differential cross section for $\vec{k} \parallel \hat{p}$. The spin singlet state cannot be separated. An exception is the $\Lambda\Lambda$ system, which can only be in the spin-singlet state in the s-wave due to the Pauli principle.

For the reaction $\gamma d \rightarrow K_1K_2B_1B_2$ the triplet final state can be separated by measuring $O_4$. The observable $O_4$ provides access to the spin-singlet amplitude.

For the reaction $K^- d \rightarrow K_1K_2B_1B_2$ the observable $O_4$ separates spin-singlet contribution, whereas $O_2$ and $O_3$ separate spin-triplet state.

4. Summary

We reviewed a method that allows one to extract hadronic scattering lengths from production reactions by studying final state interactions. Its generalization to the case of the baryon-baryon interaction in the strangeness $S = -2$ and $S = -3$ sectors was presented. We emphasized the importance of separating different spin states of the interacting particles. Considering as examples the reactions $K^- d \rightarrow KB_1B_2$, $\gamma d \rightarrow K_1K_2B_1B_2$ and $K^- d \rightarrow K_1K_2B_1B_2$, it was demonstrated that it is possible to construct polarization observables that provide access to spin-singlet and spin-triplet scattering lengths.
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