On Sequences with Non-Learnable Subsequences

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Abstract. The remarkable results of Foster and Vohra was a starting point for a series of papers which show that any sequence of outcomes can be learned (with no prior knowledge) using some universal randomized forecasting algorithm and forecast-dependent checking rules. We show that for the class of all computationally efficient outcome-forecast-based checking rules, this property is violated. Moreover, we present a probabilistic algorithm generating with probability close to one a sequence with a subsequence which simultaneously miscalibrates all partially weakly computable randomized forecasting algorithms.

According to the Dawid’s prequential framework we consider partial recursive randomized algorithms.

1 Introduction

Let a binary sequence \( \omega_1, \omega_2, \ldots, \omega_{n-1} \) of outcomes is observed by a forecaster whose task is to give a probability \( p_n \) of a future event \( \omega_n = 1 \). The evaluation of probability forecasts is based on a method called calibration: informally, following Dawid [1] forecaster is said to be well-calibrated if for any \( p^* \) the event \( \omega_n = 1 \) holds in \( 100p^*\% \) of moments of time as he choose \( p_n \approx p^* \). (see also [2]).

Let us give some notations. Let \( \Omega \) be the set of all infinite binary sequences, \( \Xi \) be the set of all finite binary sequences and \( \lambda \) be the empty sequence. For any finite or an infinite sequence \( \omega = \omega_1 \ldots \omega_n \ldots \), we write \( \omega^n = \omega_1 \ldots \omega_n \) (we put \( \omega^0 = \lambda \)). Also, \( l(\omega^n) = n \) denotes the length of the sequence \( \omega^n \). If \( x \) is a finite sequence and \( \omega \) is a finite or infinite sequence then \( x \omega \) denotes the concatenation of these sequences, \( x \sqsubseteq \omega \) means that \( x = \omega^n \) for some \( n \).

In the measure-theoretic framework we expect that the forecaster has a method for assigning probabilities \( p_n \) of a future event \( \omega_n = 1 \) for all possible finite sequences \( \omega_1, \omega_2, \ldots, \omega_{n-1} \). In other words, all conditional probabilities

\[
p_n = P(\omega_n = 1|\omega_1, \omega_2, \ldots, \omega_{n-1})
\]

must be specified and the overall probability distribution in the space \( \Omega \) of all infinite binary sequences will be defined. But in reality, we should recognize that we have only individual sequence \( \omega_1, \omega_2, \ldots, \omega_{n-1} \) of events and that the corresponding forecasts \( p_n \) whose testing is considered may fall short of defining a full probability distribution in the whole space \( \Omega \). This is the point of the prequential principle proposed by Dawid [1]. This principle says that the evaluation of a
probability forecaster should depend only on his actual probability forecasts and the corresponding outcomes. The additional information contained in a probability measure that has these probability forecasts as conditional probabilities should not enter in the evaluation. According to Dawid’s prequential framework we do not consider numbers $p_n$ as conditional probabilities generated by some overall probability distribution defined for all possible events. In such a way, a deterministic forecasting system is a partial recursive function $f : \Xi \to [0, 1]$. We suppose that a valid forecasting system $f$ is defined on all finite initial fragments $\omega_1, \ldots, \omega_{n-1}$ of an analyzed individual sequence of outcomes.

First examples of individual sequences for which well-calibrated deterministic forecasting is impossible (non-calibrable sequences) were presented by Oakes [6] (see also Shervish [9]). Unfortunately, the methods used in these papers, and in Dawid [1], [2], do not comply with prequential principle; they depend on some mild assumptions about the measure from which probability forecasts are derived as conditional probabilities. The method of generation the non-calibrable sequences with probability arbitrary close to one presented in V’yugin [11] also is based on the same assumptions. In this paper we modify construction from [11] for the case of partial deterministic and randomized forecasting systems do not corresponding to any overall probability distributions.

Oakes [6] showed that any everywhere defined forecasting system $f$ is not calibrated for a sequence $\omega = \omega_1\omega_2 \ldots$ defined

$$\omega_i = \begin{cases} 1 & \text{if } p_i < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

and $p_i = f(\omega_1 \ldots \omega_{i-1})$, $i = 1, 2, \ldots$

Foster and Vohra [3] showed that the well-calibrated forecasts are possible if these forecasts are randomized. By a randomized forecasting system they mean a random variable $f(\alpha; x)$ defined on some probability space $\Omega_x$ supplied by some probability distribution $P_{\omega}x$, where $x \in \Xi$ is a parameter. As usual, we omit the argument $\alpha$. For any infinite $\omega$, these probability distributions $P_{\omega_{i-1}}$ generate the overall probability distribution $P_{\omega}$ on the direct product of probability spaces $\Omega_{\omega_{i-1}}$, $i = 1, 2, \ldots$.

It was shown in [3], [4] that any sequence can be learned: for any $\Delta > 0$, a universal randomized forecasting system $f$ was constructed such that for any sequence $\omega = \omega_1\omega_2 \ldots$ the overall probability $P_{\omega}$ of the event

$$\left| \frac{1}{n} \sum_{i=1}^{n} I(p_i)(\omega_i - \tilde{p}_i) \right| \leq \Delta$$

(1)

tends to one as $n \to \infty$, where $\tilde{p}_i = f(\omega^{n-1})$ is the random variable, $I(p)$ is the characteristic function of an arbitrary subinterval of $[0, 1]$; we call this function a forecast-based checking rule.

Lehrer [5] and Sandrony et al. [8] extended the class of checking rules to combination of forecast- and outcome-based checking rules: a checking rule is a function $c(\omega^{i-1}, p) = \delta(\omega^{i-1})I(p)$, where $\delta : \Xi \to \{0, 1\}$ is an outcome-based
checking rule, and \( I(p) \) is a characteristic function of a subinterval of \([0, 1]\). They also considered a more general class of randomized forecasting systems - random variables \( \tilde{p}_i = f(\alpha; \omega_{i-1}, p_{i-1}) \), where \( p_{i-1} = p_1, \ldots, p_{i-1} \) is the sequence of past realized forecasts.

For \( k = 1, 2, \ldots \), let \( \{\delta_k\} \) be any sequence of outcome-based checking rules and \( \{I_k\} \) be any sequence of characteristic functions of subintervals of \([0, 1]\). Sandrony et al. [8] defined a randomized universal forecasting system which calibrates all checking rules \( \{\delta_kI_k\} \), \( k = 1, 2, \ldots, \), i.e., such that for any \( \Delta > 0 \) and for any sequence \( \omega = \omega_1\omega_2\ldots \), the overall probability of the event (1) tends to one as \( n \to \infty \), where \( \tilde{p}_i = f(\omega^{n-1}, p_{i-1}) \) and \( I(\tilde{p}_i) \) is replaced on \( \delta_k(\omega^{i-1})I_k(\tilde{p}_i) \) for all \( k = 1, 2, \ldots \).

In this paper we consider the class of all computable (partial recursive) outcome-based checking rules \( \{\delta_k\} \) and a slightly different class of randomized forecasting systems: our forecasting systems are random variables \( \tilde{p}_i = f(\alpha; \omega_{i-1}) \) not depending on past realized forecasts (this take a place for the universal forecasting systems defined in [3] and [10] \(^1\)). Concurrently, such a function can be undefined outside \( \omega \), it requires that any well defined forecasting system must be defined on all initial fragments of an analyzed sequence of outcomes. This peculiarity is important, since we consider forecasting systems possessing some computational properties: there is an algorithm computing the probability distribution function of such forecasting system. This algorithm when fed to some input can never finish its work, and so, is undefined on this input.

In this case, a universal randomized forecasting algorithm which calibrates all computationally efficient outcome-forecast-based checking rules does not exist. Moreover, we construct a probabilistic generator (or probabilistic algorithm) of non-learnable (in this way) sequences. This generator outputs with probability close to one an infinite sequence such that for each randomized forecasting system \( \tilde{p}_i = f(\alpha; \omega^{i-1}) \) some computable outcome-based checking rule \( \delta \) selects an infinite subsequence of \( \omega \) on which the property (1) fails for some characteristic function \( I \) with the overall probability one, where the overall probability is associated with the forecasting system \( f \).

2 Miscalibrating the forecasts

We use standard notions of the theory of algorithms. This theory is systematically treated in, for example, Rogers [7]. We fix some effective one-to-one enumeration of all pairs (triples, and so on) of nonnegative integer numbers. We identify any pair \((t, s)\) and its number \(\langle t, s \rangle\); let \( p(\langle t, s \rangle) = t \).

A function \( \phi: A \to \mathbb{R} \) is called (lower) semicomputable if \( \{(r, x) : r < \phi(x)\} \) (\( r \) is a rational number) is a recursively enumerable set. A function \( \phi \) is upper

\(^1\) Note that the algorithm from [8] can be modified in a fashion of [3], i.e., such that at any step of the construction past forecasts can be replaced on measures with finite supports defined on previous steps. Since these measures are defined recursively in the process of the construction, they can be eliminated from the condition of the universal forecasting algorithm.
semicomputable if $-\phi$ is lower semicomputable. Standard argument based on the recursion theory shows that there exist the lower and upper semicomputable real functions $\phi^-(j, x)$ and $\phi^+(k, x)$ universal for all lower semicomputable and upper semicomputable functions from $x \in \Xi$; in particular every computable real function $\phi(x)$ can be represented as $\phi(x) = \phi^-(j, x) = \phi^+(k, x)$ for all $x$, for some $j$ and $k$. Let $\phi^s_n(j, x)$ be equal to the maximal rational number $r$ such that the triple $(r, j, x)$ is enumerated in $s$ steps in the process of enumerating of the set $\{ (r, j, x) : r < \phi(j, x), r$ is rational $\}$ and equals $-\infty$, otherwise. Any such function $\phi^s_n(j, x)$ takes only finite number of rational values distinct from $-\infty$.

By definition, $\phi^s_n(j, x) \leq \phi^s_{n+1}(j, x)$ for all $j, s, x$, and $\phi^-(j, x) = \lim_{s \to \infty} \phi^s_n(j, x)$.

An analogous non-increasing sequence of functions $\phi^s_n(k, x)$ exists for any upper semicomputable function.

Let $i = (t, k)$. We say that a real function $\phi_i(x)$ is defined on $x$ if given any degree of precision - positive rational number $\kappa > 0$, it holds $|\phi^+_i(t, x) - \phi^-_i(k, x)| \leq \kappa$ for some $s$; $\phi_i(x)$ undefined, otherwise. If any such $s$ exists then for minimal such $s$, $\phi_{i,s}(x) = \phi^+_i(k, x)$ is called the rational approximation (from below) of $\phi_i(x)$ up to $\kappa$; $\phi_{i,s}(x)$ undefined, otherwise.

To define a measure $P$ on $\Omega$, we define values $P(z) = P(\Gamma_z)$ for all intervals $\Gamma_z = \{ \omega \in \Omega : z \subseteq \omega \}$, where $z \in \Xi$, and extend this function on all Borel subsets of $\Omega$ in a standard way.

We use also a concept of computable operation on $\Xi \cup \Omega$ (see [12]). Let $\hat{F}$ be a recursively enumerable set of ordered pairs of finite sequences satisfying the following properties: (i) $(x, \lambda) \in \hat{F}$ for each $x$; (ii) if $(x, y) \in \hat{F}$, $(x', y') \in \hat{F}$ and $x \subseteq x'$ then $y \subseteq y'$ or $y' \subseteq y$ for all finite binary sequences $x, x', y, y'$. A computable operation $F$ is defined as follows

$$F(\omega) = \sup\{ y \mid x \subseteq \omega \text{ and } (x, y) \in \hat{F} \text{ for some } x\},$$

where $\omega \in \Omega \cup \Xi$ and sup is in the sense of the partial order $\subseteq$ on $\Xi$.

A probabilistic algorithm is a pair $(L, F)$, where $L(x) = L(\Gamma_x) = 2^{-l(x)}$ is the uniform measure on $\Omega$ and $F$ is a computable operation. For any probabilistic algorithm $(L, F)$ and a set $A \subseteq \Omega$, we consider the probability $L(\{ \omega : F(\omega) \in A \})$ of generating by means of $F$ a sequence from $A$ given a uniformly distributed sequence $\omega$.

A partial randomized forecasting system $f$ is weakly computable if its weak probability distribution function $\varphi_n(\omega^{n-1}) = Pr_n\{ f(\omega^{n-1}) < \frac{1}{2} \}$ is a partial recursive function from $\omega^{n-1}$.

Any function $\delta : \Xi \to \{0, 1\}$ is called an outcome-based selection (or checking) rule. For any sequence $\omega = \omega_1\omega_2\ldots$, the selection rule $\delta$ selects a sequence of indices $n_i$ such that $\delta(\omega^{n_i-1}) = 1$, $i = 1, 2, \ldots$, and the corresponding subsequence $\omega_{n_1}\omega_{n_2}\ldots$ of $\omega$.

The following theorem is the main result of this paper. In particular, it shows that the construction of the universal forecasting algorithm from Sandrony et al. [8] is computationally non-efficient in a case when the class of all partial recursive outcome-based checking rules $\{ \delta_k \}$ is used.
Theorem 1. For any $\epsilon > 0$ a probabilistic algorithm $(L, F)$ can be constructed, which with probability $\geq 1 - \epsilon$ outputs an infinite binary sequence $\omega = \omega_1 \omega_2 \ldots$ such that for every partial weakly computable randomized forecasting system $f$ defined on all initial fragments of the sequence $\omega$ there exists a computable selection rule $\delta$ defined on all these fragments and such that for $\nu = 0$ or $\nu = 1$ the overall probability of the event

$$\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{i=1}^{n} \delta(\omega_{i-1}) I_{\nu}(\tilde{p}_i)(\omega_i - \tilde{p}_i) \right| \geq \frac{1}{16}$$

equals one, where $I_0$ and $I_1$ are the characteristic functions of the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$, $\tilde{p}_i = f(\omega_{i-1})$ is a random variable, $i = 1, 2, \ldots$, and the overall probability distribution is associated with $f$.

Proof. For any probabilistic algorithm $(L, F)$, we consider the function

$$Q(x) = L\{\omega : x \sqsubseteq F(\omega)\}. \quad (3)$$

It is easy to verify that this function is lower semicomputable and satisfies: $Q(\lambda) \leq 1$; $Q(x0) + Q(x1) \leq Q(x)$ for all $x$. Any function satisfying these properties is called semicomputable semimeasure. For any semicomputable semimeasure $Q$ a probabilistic algorithm $(L, F)$ exists such that (3) holds. Though the semimeasure $Q$ is not a measure, we consider the corresponding measure on the set $\Omega$

$$\bar{Q}(I_x) = \inf_{n} \sum_{l(y) = n, x \sqsubseteq y} Q(y).$$

We will construct a semicomputable semimeasure $Q$ as a some sort of network flow. We define an infinite network on the base of the infinite binary tree. Any $x \in \Xi$ defines two edges $(x, x0)$ and $(x, x1)$ of length one. In the construction below we will mount to the network extra edges $(x, y)$ of length $> 1$, where $x, y \in \Xi$, $x \sqsubseteq y$ and $y \neq x0, x1$. By the length of the edge $(x, y)$ we mean the number $l(y) - l(x)$. For any edge $\sigma = (x, y)$ we denote by $\sigma_1 = x$ its starting vertex and by $\sigma_2 = y$ its terminal vertex. A computable function $q(\sigma)$ defined on all edges of length one and on all extra edges and taking rational values is called a network if for all $x \in \Xi$

$$\sum_{\sigma : \sigma_1 = x} q(\sigma) \leq 1.$$

Let $G$ be the set of all extra edges of the network $q$ (it is a part of the domain of $q$). By $q$-flow we mean the minimal semimeasure $P$ such that $P \geq R$, where the function $R$ is defined by the following recursive equations $R(\lambda) = 1$ and

$$R(y) = \sum_{\sigma : \sigma_2 = y} q(\sigma)R(\sigma_1) \quad (4)$$

for $y \neq \lambda$. A network $q$ is called elementary if the set of extra edges is finite and $q(\sigma) = 1/2$ for almost all edges of unit length. For any network $q$, we define the
network flow delay function \((q\text{-delay function})\)

\[d(x) = 1 - q(x, x0) - q(x, x1).\]

The construction below works with all computable real functions \(\phi_t(x), x \in \Xi, t = 1, 2, \ldots\) We suppose that for any computable function \(\varphi\) there exist infinitely many programs \(t\) such that \(\phi_t = \varphi\). \(^2\) Any pair \(i = \langle t, s \rangle\) is considered as a program for computing the rational approximation \(\phi_{t,s,i}(\omega^{n-1})\) of \(\phi_t\) from below up to \(\kappa_s = 1/s\).

By the construction below we visit any function \(\phi_t\) on infinitely many steps \(n\). To do this, we use the function \(p(n)\): for any positive integer number \(i\) we have \(p(n) = i\) for infinitely many \(n\).

Let \(\beta\) be a finite sequence and \(1 \leq k < l(\beta)\). A bit \(\beta_k\) of the sequence \(\beta\) is called hardly predictable by a program \(i = \langle t, s \rangle\) if \(\phi_{t,s,i}(\beta^{k-1})\) is defined and

\[
\beta_k = \begin{cases} 
0 & \text{if } \phi_{t,s,i}(\beta^{k-1}) \geq \frac{1}{2} \\
1 & \text{otherwise}
\end{cases}
\]

**Lemma 1.** Let \(i = \langle t, s \rangle\) be a program and \(\mu\) be an arbitrary sufficiently small positive real number. Then for any binary sequence \(x\) of length \(n\) the portion of all sequences \(\gamma\) of length \(K = \left\lfloor (2 + \mu) i \right\rfloor n\) (in the set of all finite sequences of length \(K\)) such that

1) \(\phi_{t,s,i}(x\gamma^k)\) is defined for all \(0 \leq k < K\),

2) the number of hardly predictable bits of \(\gamma\) by the forecasting program \(i\) is less than \(in\),

is \(\leq 2^{-2\mu^2 in + O(log(in))}\) for all sufficiently large \(n\).

**Proof.** Any function \(\sigma(x)\), where \(x \in \Xi\) and \(\sigma(x) \in \{A, B\}\), is called labelling if \(\sigma(x0) \neq \sigma(x1)\) for all \(x \in \Xi\). For any \(\gamma\) of length \(K\) and for any \(k\) such that \(1 \leq k < K\), define \(\sigma(\gamma^{k+1}) = A\) and \(\sigma(\gamma^k\gamma_{k+1}) = B\) if the bit \(\gamma_{k+1}\) of the sequence \(x\gamma^k\) is hardly predictable, where we denote \(\theta = 1 - \theta\) for any binary bit \(\theta\). Since \(\phi_{t,s,i}(x\gamma^k)\) is defined for all \(0 \leq k < K\), then \(\sigma(\gamma^{k+1})\) is also defined for all these \(k\). This partial labelling \(\sigma\) can be easily extended on the set of all binary sequences of length \(K\) in many different ways. We fix some such extension. Then the total number of all \(\gamma\) satisfying 1)-2) does not exceed the total number of all binary sequences of length \(K\) with \(\leq in\) labels \(A\). Therefore, for all sufficiently large \(n\), the portion of these \(\gamma\) does not exceed

\[
\sum_{i \leq in} \binom{K}{i} 2^{-K} \leq 2^{-(1-H(1/2-\mu))2in+O(log(in))} \leq 2^{-2\mu^2 in + O(log(in))},
\]

where \(H(r) = -r \log r - (1 - r) \log(1 - r)\). \(\square\)

In the following we put \(\mu = 1/\log(i + 1)\).

We define an auxiliary relation \(B(i, q^{n-1}, \sigma, n)\) and a function \(\beta(x, q^{n-1}, n)\). Let \(x, \beta \in \Xi\). The value of \(B(i, q^{n-1}, (x, \beta), n)\) is true if the following conditions hold:

\(^2\) To obtain this property, we can replace the sequence \(\phi_t(x)\) on a sequence \(\phi'_{t,s}(x) = \phi_t(x)\) for all \(s\).
- \( n \geq (1 + [(2 + \log^{-1}(i + 1))i])l(x); \)
- \( l(\beta) = n \) and \( x \subseteq \beta; \)
- \( d^{n-1}(\beta^j) < 1 \) for all \( j \) such that \( 1 \leq j < n; \)
- for all \( j, l(x) < j \leq (1 + [(2 + \log^{-1}(i + 1))i])l(x) \), the value \( \phi_{t,x}(\beta^{j-1}) \) is computed in \( \leq n \) steps, and for at least \( il(x) \) of these \( j \) the bit \( \beta_j \) is hardly predictable by the program \( i = \langle t, s \rangle. \)

The value of \( B(i, q^{n-1}, (x, \beta), n) \) is false, otherwise. Define

\[ \beta(x, q^{n-1}, n) = \min \{ y : p(l(y)) = p(l(x)), B(p(l(x)), q^{n-1}, (x, y), n) \}. \]

Here \( \min \) is considered for lexicographical ordering of strings; we suppose that \( \min \emptyset \) is undefined.

**Construction.** Let \( \rho(n) = (n + n_0)^2 \) for some sufficiently large \( n_0 \) (the value \( n_0 \) will be specified below in the proof of Lemma 5).

Using the mathematical induction by \( n \), we define a sequence \( q^n \) of elementary networks. Put \( q^0(\sigma) = 1/2 \) for all edges \( \sigma \) of length one.

Let \( n > 0 \) and a network \( q^{n-1} \) is defined. Let \( d^{n-1} \) be the \( q^{n-1} \)-delay function and let \( G^{n-1} \) be the set of all extra edges. We suppose also that \( l(\sigma_2) < n \) for all \( \sigma \in G^{n-1}. \)

Let us define a network \( q^n \). At first, we define a network flow delay function \( d^n \) and a set \( G^n \). The construction can be split up into two cases.

Let \( w(i, q^{n-1}) = \text{be equal to the minimal} \ m \ \text{such that} \ p(m) = i \ \text{and} \ m > l(\sigma_2) \ \text{for each extra edge} \ \sigma \in G^{n-1} \ \text{such that} \ p(l(\sigma_1)) < i. \)

The inequality \( w(i, q^m) \neq w(i, q^{m-1}) \) can be induced by some task \( j < i \) that mounts an extra edge \( \sigma = (x, y) \) such that \( l(x) > w(i, q^{m-1}) \) and \( p(l(x)) = p(l(y)) = j. \) Lemma 2 (below) will show that this can happen only at finitely many steps of the construction.

**Case 1.** \( w(p(n), q^{n-1}) = n \) (the goal of this part is to start a new task \( i = p(n) \) or to restart the existing task \( i = p(n) \) if it was destroyed by some task \( j < i \) at some preceding step).

Put \( d^n(y) = 1/p(n) \) for \( l(y) = n \) and define \( d^n(y) = d^{n-1}(y) \) for all other \( y. \)

Put also \( G^n = G^{n-1}. \)

**Case 2.** \( w(p(n), q^{n-1}) < n \) (the goal of this part is to process the task \( i = p(n) \)). Let \( C_n \) be the set of all \( x \) such that \( w(i, q^{n-1}) \leq l(x) < n, 0 < d^{n-1}(x) < 1, \) the function \( \beta(x, q^{n-1}, n) \) is defined 3 and there is no extra edge \( \sigma \in G^{n-1} \) such that \( \sigma_1 = x. \)

In this case for each \( x \in C_n \) define \( d^n(\beta(x, q^{n-1}, n)) = 0, \) and for all other \( y \) of length \( n \) such that \( x \subseteq y \) define

\[ d^n(y) = \frac{d^{n-1}(x)}{1 - d^{n-1}(x)}. \]

Define \( d^n(y) = d^{n-1}(y) \) for all other \( y. \) We add an extra edge to \( G^{n-1}, \) namely, define

\[ G^n = G^{n-1} \cup \{(x, \beta(x, q^{n-1}, n)) : x \in C_n\}. \]

3 In particular, \( p(l(x)) = i \) and \( l(\beta(x, q^{n-1}, n)) = n. \)
We say that the task \( i = p(n) \) mounts the extra edge \((x, \beta(x, q^{n-1}, n))\) to the network and that all existing tasks \( j > i \) are destroyed by the task \( i \).

After Case 1 and Case 2, define for any edge \( \sigma \) of unit length

\[
q^n(\sigma) = \frac{1}{2}(1 - d^n(\sigma_1))
\]

and \( q^n(\sigma) = d^n(\sigma_1) \) for each extra edge \( \sigma \in G^n \).

**Case 3.** Cases 1 and 2 do not hold. Define \( d^n = d^{n-1}, q^n = q^{n-1}, G^n = G^{n-1} \).

As the result of the construction we define the network \( q = \lim_{n \to \infty} q^n \), the network flow delay function \( d = \lim_{n \to \infty} d^n \) and the set of extra edges \( G = \bigcup_n G^n \).

The functions \( q \) and \( d \) are computable and the set \( G \) is recursive by their definitions. Let \( Q \) denotes the \( q \)-flow.

The following lemma shows that any task can mount new extra edges only at finite number of steps. Let \( G(i) \) be the set of all extra edges mounted by the task \( i, w(i, q) = \lim_{n \to \infty} w(i, q^n) \).

**Lemma 2.** The set \( G(i) \) is finite, \( w(i, q) \) exists and \( w(i, q) < \infty \) for all \( i \).

**Proof.** Note that if \( G(j) \) is finite for all \( j < i \), then \( w(i, q) < \infty \). Hence, we must prove that the set \( G(i) \) is finite for any \( i \). Suppose that the opposite assertion holds. Let \( i \) be the minimal such that \( G(i) \) is infinite. By choice of \( i \) the sets \( G(j) \) for all \( j < i \) are finite. Then \( w(i, q) < \infty \).

For any \( x \) such that \( l(x) \geq w(i, q) \), consider the maximal \( m \) such that for some initial fragment \( x^m \subseteq x \) there exists an extra edge \( \sigma = (x^m, y) \in G(i) \). If no such extra edge exists define \( m = w(i, q) \). By definition, if \( d(x^m) \neq 0 \) then \( 1/d(x^m) \) is an integer number. Define

\[
u(x) = \begin{cases} 
1/d(x^m) & \text{if } d(x^m) \neq 0, l(x) \geq w(i, q) \\
p(w(i, q)) & \text{if } l(x) < w(i, q) \\
0 & \text{otherwise}
\end{cases}
\]

By construction the integer valued function \( u(x) \) has the property: \( u(x) \geq u(y) \) if \( x \subseteq y \). Besides, if \( u(x) > u(y) \) then \( u(x) > u(z) \) for all \( z \) such that \( x \subseteq z \) and \( l(z) = l(y) \). Then the function

\[
\hat{u}(\omega) = \min\{ n : u(\omega^i) = u(\omega^n) \text{ for all } i \geq n \}
\]

is defined for all \( \omega \in \Omega \). It is easy to see that this function is continuous. Since \( \Omega \) is compact space in the topology generated by intervals \( \Gamma_x \), this function is bounded by some number \( m \). Then \( u(x) = u(x^m) \) for all \( l(x) \geq m \). By the construction, if any extra edge of \( i \)th type was mounted to \( G(i) \) at some step then \( u(y) < u(x) \) holds for some new pair \( (x, y) \) such that \( x \subseteq y \). This is contradiction with the existence of the number \( m \). ☐

An infinite sequence \( \alpha \in \Omega \) is called an \( i \)-extension of a finite sequence \( x \) if \( x \subseteq \alpha \) and \( B(i, q^{n-1}, x, \alpha^n, n) \) is true for almost all \( n \).

A sequence \( \alpha \in \Omega \) is called \( i \)-closed if \( d(\alpha^n) = 1 \) for some \( n \) such that \( p(n) = i \), where \( d \) is the \( q \)-delay function. Note that if \( \sigma \in G(i) \) is some extra edge (i.e. an edge of \( i \)th type) then \( B(i, q^{n-1}, \sigma, n) \) is true, where \( n = l(\sigma_2) \).
Lemma 3. Let for any initial fragment $\omega^n$ of an infinite sequence $\omega$ some $i$-extension exists. Then either the sequence $\omega$ will be $i$-closed in the process of the construction or $\omega$ contains an extra edge of $i$th type (i.e. $\sigma_2 \subseteq \omega$ for some $\sigma \in \mathcal{G}(i)$).

Proof. Let a sequence $\omega$ is not $i$-closed. By Lemma 2 the maximal $m$ exists such that $p(m) = i$ and $d(\omega^m) > 0$. Since the sequence $\omega^m$ has an $i$-extension and $d(\omega^m) < 1$, by Case 2 of the construction a new extra edge $(\omega^m, y)$ of $i$th type must be mounted to the binary tree. By the construction $d(y) = 0$ and $d(z) \neq 0$ for all $z$ such that $\omega^m \subseteq z$, $l(z) = l(y)$, and $z \neq y$. By the choice of $m$ we have $y \subseteq \omega$. \(\square\)

Lemma 4. It holds $Q(y) = 0$ if and only if $q(\sigma) = 0$ for some edge $\sigma$ of unit length located on $y$ (this edge satisfies $\sigma_2 \subseteq y$).

Proof. The necessary condition is obvious. To prove that this condition is sufficient, let us suppose that $q(y^n, y^{n+1}) = 0$ for some $n < l(y)$ but $Q(y) \neq 0$. Then by definition $d(\omega^n) = 1$. Since $Q(y) \neq 0$ an extra edge $(x, z) \in G$ exists such that $x \subseteq y^n$ and $y^{n+1} \subseteq z$. But, by the construction, this extra edge can not be mounted to the network $q_i^{(z)} - 1$ since $d(z^n) = 1$. This contradiction proves the lemma. \(\square\)

For any semimeasure $P$ define $E_P = \{ \omega \in \Omega : \forall n (P(\omega^n) \neq 0) \}$ - the support set of $P$. It is easy to see that $\mathcal{P}(E_P) = \mathcal{P}(\Omega)$. By Lemma 4 $E_Q = \Omega \setminus \cup_{d(x) = 1} \Gamma_x$.

Lemma 5. It holds $Q(E_Q) > 1 - \frac{1}{2} \epsilon$.

Proof. We bound $Q(\Omega)$ from below. Let $R$ be defined by (4). By definition of the network flow delay function, we have

$$\sum_{u : l(u) = n+1} R(u) = \sum_{u : l(u) = n} (1 - d(u)) R(u) + \sum_{\sigma, \sigma \in G, l(\sigma_2) = n+1} q(\sigma) R(\sigma_1). \quad (5)$$

Define an auxiliary sequence $S_n = \sum_{u : l(u) = n} R(u) - \sum_{\sigma, \sigma \in G, l(\sigma_2) = n} q(\sigma) R(\sigma_1)$. At first, we consider the case $w(p(n), q^{n-1}) < n$. If there is no edge $\sigma \in G$ such that $l(\sigma_2) = n$ then $S_{n+1} \geq S_n$. Suppose that some such edge exists. Define

$$P(u, \sigma) \iff l(u) = l(\sigma_2) \& \sigma_1 \subseteq u \& u \neq \sigma_2 \& \sigma \in G.$$  

By definition of the network flow delay function, we have

$$\sum_{u : l(u) = n} d(u) R(u) = \sum_{\sigma, \sigma \in G, l(\sigma_2) = n} d(\sigma_2) \sum_{u : P(u, \sigma)} R(u) = \sum_{\sigma, \sigma \in G, l(\sigma_2) = n} \frac{d(\sigma_1)}{1 - d(\sigma_1)} \sum_{u : P(u, \sigma)} R(u) \leq \sum_{\sigma, \sigma \in G, l(\sigma_2) = n} d(\sigma_1) R(\sigma_1) = \sum_{\sigma, \sigma \in G, l(\sigma_2) = n} q(\sigma) R(\sigma_1). \quad (6)$$
Since Lemma is proved. Here we used the inequality \( \sum_{u:P(u,\sigma)} R(u) \leq R(\sigma_1) - d(\sigma_1)R(\sigma_1) \) for all \( \sigma \in G \) such that \( l(\sigma_2) = n \). Combining this bound with (5) we obtain \( S_{n+1} \geq S_n \).

Let us consider the case \( w(p(n), q^{n-1}) = n \). Then \( \sum_{u:l(u)=n} d(u)R(u) \leq \rho(n) = (n + n_0)^{-2} \). Combining (5) and (6) we obtain \( S_{n+1} \geq S_n - (n + n_0)^{-2} \) for all \( n \).

Since \( S_0 = 1 \), this implies \( S_n \geq 1 - \sum_{i=1}^{\infty} (i + n_0)^{-2} \geq 1 - \frac{1}{2} \epsilon \) for some sufficiently large constant \( n_0 \). Since \( Q \geq R \), it holds

\[
\tilde{Q}(\Omega) = \inf_n \sum_{l(u)=n} Q(u) \geq \inf_n S_n \geq 1 - \frac{1}{2} \epsilon.
\]

Lemma is proved. □

**Lemma 6.** There exists a set \( U \) of infinite binary sequences such that \( \tilde{Q}(U) \leq \epsilon/2 \) and for any sequence \( \omega \in E_Q \setminus U \) for each partial computable forecasting system the condition (2) holds.

**Proof.** Let \( \omega \) be an infinite sequence and let \( f \) be a partial computable forecasting system such that the corresponding \( \phi_i(\omega^{n-1}) \) is defined for all \( n \). Let \( i = (t, s) \) be a program for computing the rational approximation \( \phi_i, \kappa_s \) from below up to \( \kappa_s = 1/s \).

If \( d(\omega^m) = 1 \) for some \( m \) such that \( p(m) = i \) then for every \( \beta \) of length \( (1 + [(2 + \log^{-1}(i + 1)]i)m \) such that \( \omega^m \subseteq \beta \) there are \( \langle im \rangle \) bits hardly predictable by the forecasting program \( i \).

We show that \( Q \)-measure of all intervals generated by such \( \beta \) becomes arbitrary small for all sufficiently large \( i \). Since there are no extra edges \( \sigma \) such that \( \omega^m \subseteq \sigma_1 \), the measure \( Q \) when restricted on interval \( I_{\omega, m} \) is proportional to the uniform measure. Then by Lemma 1, where \( \mu = \log^{-1}(i + 1) \), \( Q \)-measure of all such \( \beta \) decreases exponentially by \( im \). Therefore, for each \( j \) there exists a number \( m_j \) such that \( Q(U_j) \leq 2^{-j+1} \), where \( U_j \) is the union of all intervals \( I_{\beta} \) defined by all \( \beta \) of length \( (1 + [(2 + \log^{-1}(i + 1)]i)m \) for \( m \geq m_j \) containing \( \langle im \rangle \) bits hardly predictable by the forecasting program \( i = p(m) \). Define \( U = \cup_{j \in U_j} \).

Define a selection rule \( \gamma \) as follows:

- define \( \gamma(\omega^{j-1}) = 1 \) if \( \sigma_1 \subseteq \omega^{j-1} \subseteq \sigma_2 \) for some \( \sigma \in G(i) \) and the \( j \)th bit of \( \sigma_2 \) is hardly predictable by the forecasting program \( i \);
- define \( \gamma(\omega^{j-1}) = 0 \) otherwise.

We also define two selection rules \( J_\nu \), where \( \nu = 0, 1 \),

\[
J_\nu(\omega^{j-1}) = \begin{cases} 1 - \nu & \text{if } \phi_i, \kappa_s(\omega^{j-1}) < \frac{1}{2} \\ \nu & \text{if } \phi_i, \kappa_s(\omega^{j-1}) \geq \frac{1}{2} \end{cases}
\]

Suppose that \( \omega \notin U \) and \( \phi_i(\omega^n) \) is defined for all \( n \). Then \( \omega \) is an \( i \)-extension of \( \omega^n \) for each \( n \). Since for each \( n \) the sequence \( \omega^n \) is not \( i \)-closed, by Lemma 3
there exists an extra edge $\sigma \in G(i)$ such that $\sigma_2 \subseteq \omega$. In the following, let $m = l(\sigma_1), n = (1 + \lceil (2 + \log^{-1}(i + 1))i \rceil)m$.

Then by the construction the selection rule $\delta_\nu(\omega^{i-1}) = \gamma(\omega^{i-1})J_\nu(\omega^{i-1})$, for $\nu = 0$ or for $\nu = 1$, selects from a fragment of $\omega$ of length $n$ a subsequence $\omega_{i_1}, \ldots, \omega_{i_l}$ of length $l \geq im/2$. Since by definition these bits are hardly predictable, we have $\omega_{i_j} = 1$ for all $j$ such that $1 \leq j \leq l$ if $\nu = 0$, and $\omega_{i_j} = 0$ for all these $j$ if $\nu = 1$.

Let $\tilde{p}_j = f(\omega^{j-1})$, $j = 1, 2, \ldots$, be an arbitrary computable randomizing forecasting system (it is a random variable) defined on all initial fragments of $\omega = \omega_1\omega_2\ldots$ Then $\phi(\omega^{j-1}) = Pr\{\tilde{p}_j \geq \frac{1}{2}\}$ is a computable real function. By definition $\phi = \phi_t$ for infinitely many $t$ and

$$\phi_{t,\kappa,\nu}(\omega^{j-1}) \leq \phi_t(\omega^{j-1}) \leq \phi_{t,\kappa,\nu}(\omega^{j-1}) + \kappa.$$

for all $s$ and $j$. Consider two random variables, for $\nu = 0$ and for $\nu = 1$,

$$\vartheta_{n,\nu} = \sum_{j=1}^{n} \delta_\nu(\omega^{j-1})I_\nu(\tilde{p}_j)(\omega_j - \tilde{p}_j).$$

Suppose that $l \geq im/2$ holds for $\nu = 0$. Then using (7) we obtain

$$E(\vartheta_{n,0}) \geq \sum_{j=m+1}^{n} \delta_0(\omega^{j-1})Pr\{\tilde{p}_j < \frac{1}{2}\} \geq m \geq \frac{im}{4}(\frac{1}{2} - \kappa_s) - m$$

Since $n = (1 + \lceil (2 + \log^{-1}(i + 1))i \rceil)m$, $i$ can be arbitrary large and we visit any pair $i = (t, s)$ infinitely often, we obtain from (8)

$$\limsup_{n \to \infty} \frac{1}{n}E(\vartheta_{n,0}) \geq 1/16.$$  \hspace{1cm} (9)

Analogously, if $\nu = 1$ we obtain

$$\liminf_{n \to \infty} \frac{1}{n}E(\vartheta_{n,1}) \leq -1/16.$$  \hspace{1cm} (10)

The martingale strong law of large numbers says that for $\nu = 0, 1$ with $Pr$-probability one

$$\frac{1}{n} \sum_{j=1}^{n} \delta_\nu(\omega^{j-1})I_\nu(\tilde{p}_j)(\omega_j - \tilde{p}_j) - \frac{1}{n}E(\vartheta_{n,\nu}) \to 0$$

as $n \to \infty$. Combining (9), (10) and (11) we obtain (2).

Lemma 6 and Theorem 1 are proved. \sq

The following theorem is a generalization of the result from V'yugin [11] for partial defined computable deterministic forecasting systems.
Theorem 2. For any $\epsilon > 0$ a probabilistic algorithm $(L, F)$ can be constructed, which with probability $\geq 1 - \epsilon$ outputs an infinite binary sequence $\omega = \omega_1\omega_2\ldots$ such that for every partial deterministic forecasting algorithm $f$ defined on all initial fragments of the sequence $\omega$ a computable outcome-based selection rule $\delta$ exists defined on all these fragments such that

$$
\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{i=1}^{n} \delta(\omega_{i-1})(\omega_i - f(\omega_{i-1})) \right| \geq \frac{1}{8}.
$$

The proof of this theorem is based on the same construction.

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References

1. Dawid, A.P.: The Well-Calibrated Bayesian [with discussion], J. Am. Statist. Assoc. 77, 605-613 (1982)
2. Dawid, A.P.: Calibration-Based Empirical Probability [with discussion], Ann. Statist. 13, 1251-1285 (1985)
3. Foster, D.P., Vohra, R.: Asymptotic Calibration, Biometrika 85, 379-390 (1998)
4. Kakade, S.M., Foster, D.P.: Deterministic Calibration and Nash Equilibrium, In John Shawe Taylor and Yoram Singer, editors Proceedings of the Seventeenth Annual Conference on Learning Theory Volume 3120 of Lecture Notes in Computer Science, 33-48, Heidelberg, Springer (2004)
5. Lehrer, E.: Any Inspection Rule is Manipulable, Econometrica, 69-5, 1333-1347 (2001)
6. Oakes, D.: Self-Calibrating Priors Do not Exists [with discussion], J. Am. Statist. Assoc. 80, 339-342 (1985)
7. Rogers, H.: Theory of Recursive Functions and Effective Computability, New York, McGraw Hill (1967)
8. Sandroni, A., Smorodinsky R., and Vohra, R.: Calibration with Many Checking Rules, Mathematics of Operations Research 28-1, 141-153 (2003)
9. Schervish, V.: Comment [to Oakes, 1985], J. Am. Statist. Assoc. 80, 341-342 (1985)
10. Vovk, V.: Defensive forecasting for optimal prediction with expert advice, arXiv:0708.1503v1 (2007)
11. V’yugin, V.V.: Non-Stochastic Infinite and Finite Sequences, Theor. Comp. Science. 207, 363-382 (1998)
12. Zvonkin, A.K., Levin, L.A.: The Complexity of Finite Objects and the Algorithmic Concepts of Information and Randomness, Russ. Math. Surv. 25, 83-124 (1970)