The Parallel-Repeated Magic Square Game is Rigid

Matthew Coudron*    Anand Natarajan †

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Abstract

We show that the \( n \)-round parallel repetition of the Magic Square game of Mermin and Peres is rigid, in the sense that for any entangled strategy succeeding with probability \( 1 - \epsilon \), the players’ shared state is \( O(\text{poly}(n\epsilon)) \)-close to \( 2n \) EPR pairs under a local isometry. Furthermore, we show that, under local isometry, the players’ measurements in said entangled strategy must be \( O(\text{poly}(n\epsilon)) \)-close to the “ideal” strategy when acting on the shared state.

1 Introduction

Nonlocal games have long been a fundamental topic in quantum information, starting from Bell’s pioneering work in the 1960s. In the language of games, Bell [Bel64] showed that for a certain two-player nonlocal game, two players sharing a single EPR pair between them can win with substantially higher probability than they could by following the best classical strategy. In Bell’s original game, the messages between the players and the referee were real numbers, but soon afterward, Clauser, Horne, Shimony, and Holt [CHSH69] discovered a game (called the CHSH game) with similar properties, but with messages consisting of just one bit. The CHSH game can be viewed as a test for the “quantumness” of a system, with good soundness: that is, the probability of a non-quantum system fooling the test is at most \( 3/4 \). However, the test lacks the property of so-called perfect completeness: as shown by Tsirelson [Cir80], even the optimal quantum strategy succeeds with probability at most \( (2 + \sqrt{2})/4 \approx 0.854 \). To remedy this drawback, Mermin [Mer90] and independently Peres [Per90] independently introduced the Magic Square game: a two-player game with two-bit inputs and outputs, and for which the best classical strategy succeeds with probability \( 8/9 \), but there exists a quantum strategy using only two shared EPR pairs succeeding with probability \( 1 \).

Later, Mayers and Yao [MY98] realized that the CHSH game could be used not only to test for “quantumness,” but to test for a specific quantum state: namely, the EPR pair. Such a test is often called a “self-test.” Mayers and Yao showed that in any optimal quantum strategy for CHSH, the players’ shared state is equivalent under a local isometry\( ^{1} \) to an EPR pair. This result was not robust in that required the CHSH correlations to hold exactly: however, the subsequent work of McKague, Yang, and Scarani [MYS12] was able to achieve a robust self-test based on CHSH

\(^{1}\text{Massachusetts Institute of Technology. Email m.coudron@mit.edu.}\)

\(^{†}\text{Center for Theoretical Physics, MIT. anandn@mit.edu}\)

\(^{1}\text{Since either player could apply a local unitary to their half of the state and their measurements, without affecting their winning probability, equivalence under local isometry is the best one could hope for.}\)
for a single EPR pair. That is, they showed that for any strategy that wins CHSH with probability $\geq p_{\text{max}} - \epsilon$, there exists an isometry $V$ mapping the players’ state $|\psi\rangle$ to a state $|\phi\rangle$ which is $O(\sqrt{\epsilon})$-close to the EPR pair state in 2-norm. Moreover, they showed that the measurements applied by the players must also be close to the measurements used in the ideal strategy, as measured in a state-dependent distance: for instance, if $X$ is the operator applied by player 1 when asked to measure a Pauli $X$, then under the same isometry $V$, $\| V(X|\psi\rangle) - \sigma_X|\phi\rangle \| \leq O(\sqrt{\epsilon})$, where $\sigma_X$ is the Pauli $X$-matrix. Such a result is called a rigidity result, because it shows that any strategy that is close to optimal must have the same structure as the ideal strategy. We refer to the bound that appears in the right-hand side of the norm inequalities (here $\sqrt{\epsilon}$) as the robustness of the test. More recently, Wu et al. [WBMS16] showed rigidity for Mermin and Peres’s Magic Square game, demonstrating that it serves as robust self-test for a single EPR pair.

In recent years, self-testing has found applications to quantum cryptography (QKD, device independent QKD, and randomness expansion), as well as to multiprover quantum interactive proof systems (the complexity class MIP*) [RUV13]. However, these applications all rely on testing multi-qubit states, whereas known robust self-testing results are directly applicable only to states of a few qubits. A natural strategy to obtain a multi-qubit test is to repeat the single-qubit tests, either in series (i.e. over many rounds) or in parallel (i.e. in one round)—for instance, the work of Reichardt, Unger, and Vazirani [RUV13] uses a serially repeated CHSH test, and McKague [McK15] gives a parallel self-test based on CHSH. The lack of perfect completeness considerably complicates the analysis of these tests, since one cannot demand that the players win every repetition of the test—rather, one has to check whether the fraction of successful repetitions is above a certain threshold.

In this paper, we circumvent these issues by studying the $n$-round parallel repetition of the Magic Square game. We achieve a proof of rigidity, showing that if the players win with probability $1 - \epsilon$, their state is $O(\text{poly}(n\epsilon))$-close to $2n$ EPR pairs, under a local isometry. This is an exponential improvement in error dependence over the strictly parallel self-testing result of [McK15], which has error dependence $O(\exp(n)\text{poly}(\epsilon))$ and is the previous best known result for rigidity of strictly parallel repeated non-local games (McKague’s result is stated for the parallel repeated CHSH game with a threshold test, rather than the parallel repeated Magic Square game). We note that McKague’s result has $O(\log(n))$-bit questions, whereas our game has $O(n)$-bit questions and answers, but additionally robustly certifies all $n$-qubit measurement operators. This means that our result is a strictly parallel test, that can be used to “force” untrusted provers to apply all $n$-qubit Pauli operators faithfully (in expectation), which is a new feature that we believe will be valuable in the context of complexity applications.

As a fundamental building block for our result, we make use of the rigidity of a single round of the Magic Square game, which was established in [WBMS16]. A key observation of our work is that, by leveraging a “global consistency check” which occurs naturally within the parallel repeated Magic Square game, we can establish approximate commutation between the different copies (or “rounds”) of the game in the parallel repeated test. This then allows us to extend the single round analysis of [WBMS16] to a full $n$-round set of approximate anti-commutation relations for the provers measurements, which is expressed in Theorem 8. A second important technical tool in our proof is a theorem (Theorem 9) which, given operators on the players’ state that approx-

\[ \text{Note that, by repeating the test in section 4 of [McK15] a polynomial number of times, one can achieve a self-test for } n \text{ EPR pairs with polynomial error dependence. However, the test given in section 4 is not a strictly parallel test, and does not robustly certify } n\text{-qubit measurement operators, as our result does.} \]
imately satisfy the algebraic relations of single-qubit Pauli matrices, constructs an isometry that maps the players’ “approximate Paulis” close to exact Pauli operators acting on a $2^n$-qubit space. The proof of Theorem 9 relies on an isometry inspired by the works of McKague [McK10, McK16a], but is designed to take the guarantees produced by Theorem 8 and conclude closeness of the players “approximate Paulis” to exact Pauli operators in expectation, where all $2^n$-qubit Pauli operators are handled simultaneously, with polynomial error dependence.

Very recently, we became aware of two independent works achieving related results in this area. The first is an unpublished paper of Chao, Reichardt, Sutherland, and Vidick [CRSV16], which proves a theorem similar to our Theorem 9. The second is a paper by Coladangelo [Col16], which proves a self-testing result for the parallel repeated Magic Square game that is similar to our own, albeit with slightly different polynomial factors. Furthermore, the robustness analysis of the results in [Col16] makes use of the same key theorem of [CRSV16], which is, in turn, similar to our own Theorem 9. The theorem of [CRSV16] (and consequently the robustness result of Coladangelo) achieve a robustness of $n^{3/2} \sqrt{\varepsilon}$ for for all single-qubit operators (i.e., to achieve constant robustness, $\varepsilon$ must scale as $1/n^3$). On the other hand, our Theorem 9 achieves a robustness of $n \varepsilon^{1/4}$ (i.e. $\varepsilon \sim 1/n^4$), but for operators acting on all $2^n$ qubits simultaneously. It is natural to ask whether one can prove a single result which combines the strengths of these two different error dependencies. We expect that this is possible, but leave it for future work.

2 Preliminaries

We use the standard quantum formalism of states and measurements. An observable is a Hermitian operator whose eigenvalues are $\pm 1$, and encodes a two-outcome projective measurement (the POVM elements of the two outcomes are the projections on to the $+1$ and $-1$ eigenspaces). Throughout this paper, we make use of the Pauli matrices. These are $2 \times 2$ Hermitian matrices defined by

\[
\sigma_X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

They satisfy the anticommutation relation

\[XZ = -ZX.\]

3 The Magic Square game

In this section we introduce the nonlocal game analyzed in this work: the $n$-round parallel repeated Magic Square game. We also introduce notation to describe entangled strategies for the game and state some simple properties they satisfy.

The parallel repeated Magic Square game is played between players (which we will refer to as Alice and Bob), and a verifier. First, let us define the single-round Magic Square game, originally introduced by Mermin [Mer90] and Peres [Per90]. The rules of the game are described in Fig. 1.
The magic square game is a one-round, two-player game, played as follows.

1. The verifier sends Alice a question \( r \in \{0, 1, 2\} \) and Bob a question \( c \in \{0, 1, 2\} \).
2. Alice sends the verifier a response \( (a_0, a_1) \in \{0, 1\}^2 \), and Bob sends a response \( (b_0, b_1) \in \{0, 1\}^2 \).
3. Let \( a_2 := a_0 \oplus a_1 \) and \( b_2 := 1 \oplus b_0 \oplus b_1 \). Then Alice and Bob win the game if \( a_c = b_r \) and lose otherwise.

Any entangled strategy for this game is described by a shared quantum state \( |\psi\rangle_{AB} \) and projectors \( P_{a_0,a_1} \) for Alice and \( Q_{b_0,b_1} \) for Bob. It can be seen that the game can be won with certainty for the following strategy:

\[
|\psi\rangle = \frac{1}{2} \sum_{ij \in \{0,1\}} |ij\rangle_A \otimes |ij\rangle_B
\]

\[
P_{0,a_1} = \frac{1}{4} (I + (-1)^{a_0}Z)_{A1} \otimes (I + (-1)^{a_1}Z)_{A2} \otimes I_B
\]

\[
P_{1,a_1} = \frac{1}{4} (I + (-1)^{a_1}X)_{A1} \otimes (I + (-1)^{a_0}X)_{A2} \otimes I_B
\]

\[
Q_{0,b_1} = \frac{1}{4} I_A \otimes (I + (-1)^{b_0}Z)_{B1} \otimes (I + (-1)^{b_1}X)_{B2}
\]

\[
Q_{1,b_1} = \frac{1}{4} I_A \otimes (I + (-1)^{b_1}X)_{B1} \otimes (I + (-1)^{b_0}Z)_{B2}
\]

This strategy is represented pictorially in Fig. 2, where each row contains a set of simultaneously-measurable observables that give Alice's answers, and likewise each column for Bob.

![Figure 2: The ideal strategy for a single round of magic square. Alice and Bob share the state \( |EPR\rangle \otimes 2 \).](image)

The game we study in this paper is the \( n \)-fold parallel repetition of the above game.

**Definition 1.** The \( n \)-fold parallel repeated Magic Square game is a game with two players, Alice and Bob, and one verifier. The player sends Alice a vector \( r \in \{0, 1, 2\}^n \) and Bob a vector \( c \in \{0, 1, 2\}^n \), where each coordinate of \( r \) and \( c \) is chosen uniformly at random. Alice responds with two \( n \)-bit strings \( a_0, a_1 \), and Bob with two \( n \)-bit strings \( b_0, b_1 \). The players win if for every \( k \in [n] \), the \( k \)th components of Alice and Bob's answers \( a_{0,k}, a_{1,k}, b_{0,k}, b_{1,k} \) satisfy the win conditions of the Magic Square game with input \( r_k \) and \( c_k \).

Throughout this paper we will refer to the non-local entangled strategy applied by the players according to the following definitions:
Remark 5. By definition, it follows that $A_{r,p}^{c} = A_{r,p}^{c'}$. Note that, by Remark 5, these two definitions are equivalent because $p \equiv 0 \pmod{2}$.

Definition 3. Define $a_2 \equiv a_0 + a_1 \pmod{2}$ and $b_2 \equiv b_0 + b_1 + 1 \pmod{2}$.

Definition 4. Define the column-$c$ output observables for Alice as $A_{r,p}^{c} \equiv \sum_{a_0,a_1} (-1)^{a_0} a_0 \cdot P_{r}^{b_0,a_1}$.

Where $a_0$ is defined to be the $n$ dimensional vector whose $i$th component is defined by $(a_0)_i \equiv (a_0)_i$.

Similarly, define the row-$r$ observables for Bob as $B_{r,q}^{b} \equiv \sum_{b_0,b_1} (-1)^{b_0} b_0 \cdot Q_{c}^{b_0,b_1}$.

Remark 5. By definition, it follows that $A_{r,p}^{c} = A_{r,p}^{c'}$ if $c$ and $c'$ differ only on rounds where the coordinate of $p$ is 0, and likewise for $B$ and $r$.

The win conditions for magic square:

Fact 6. Suppose Alice and Bob win the magic square game with probability $\geq 1 - \varepsilon$. Then it holds that

$$\forall p, \quad E_{r,c} \langle \psi | A_{r,p}^{c} B_{r,p}^{c} | \psi \rangle \geq 1 - \varepsilon. \quad (1)$$

In Remark 5 we noted that we can freely change the output column for Alice (resp. row for Bob) on the “ignored” rounds. In the following lemma, we show that we can also change the input row (resp. column), up to an $O(\varepsilon)$ error, provided that the strategy is $\varepsilon$ close to optimal.

Lemma 7. Suppose Alice and Bob have an $\varepsilon$-optimal strategy. Then, $\forall i, r, c$,

$$\left| 1 - E_{r,r',r''=r_i=1} \langle \psi | A_{r,e}^{c} \cdot A_{r',e_i}^{c'} | \psi \rangle \right| \leq 36\varepsilon$$

Proof. To start we define an extended state $|s\rangle \equiv |\psi\rangle \otimes \frac{1}{\sqrt{3^{n-1}}} \sum_{r_{-i}} | r_{-i} \rangle \otimes \frac{1}{\sqrt{3^{n-1}}} \sum_{r_{-i}} | r_{-i} \rangle \otimes \sum_{s_{-i}} | s_{-i} \rangle$ as well as extended operators:

$$T \equiv \sum_{r_{-i}} A_{r,e_i}^{c} \otimes | r_{-i} \rangle \otimes I \otimes I = \sum_{r_{-i}} \sum_{s_{-i}} A_{r,e_i}^{c} \otimes | r_{-i} \rangle \otimes I \otimes | s_{-i} \rangle \langle s_{-i} |$$

Note that, by Remark 5 these two definitions are equivalent because $A_{r,e_i}^{c}$ is identically equal to $A_{r,e_i}^{c} \otimes s_{-i}$ by definition, regardless of the value of $s_{-i}$. Further define

$$T' \equiv \sum_{r_{-i}} A_{r,e_i}^{c} \otimes I \otimes | r_{-i} \rangle \otimes I = \sum_{r_{-i}} \sum_{s_{-i}} A_{r,e_i}^{c} \otimes I \otimes | s_{-i} \rangle \langle s_{-i} |$$

and

$$S \equiv \sum_{r_{-i}} \sum_{s_{-i}} B_{r,e_i}^{c} \otimes | r_{-i} \rangle \otimes I \otimes | s_{-i} \rangle \langle s_{-i} | = \sum_{r_{-i}} \sum_{s_{-i}} B_{r,e_i}^{c} \otimes | r_{-i} \rangle \otimes \sum_{r_{-i}} | r_{-i} \rangle \langle r_{-i} | \otimes | s_{-i} \rangle \langle s_{-i} |$$

$$= \sum_{r_{-i}} \sum_{s_{-i}} B_{r,e_i}^{c} \otimes \sum_{r_{-i}} \langle s_{-i} | \langle s_{-i} | \langle s_{-i} |$$

and
Where, to conclude equivalence of the different versions of the last definition, we are using Remark 5 as well as the fact that \( r_i = r'_i = r \), some fixed value.

Now, note that:

\[
\langle \sigma | T \cdot S | \sigma \rangle = \left( \langle \psi | \otimes \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{r_{-i}} \langle r_{-i} | \otimes \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{r'_{-i}} \langle r'_{-i} | \otimes \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{s_{-i}} \langle s_{-i} | \right) \\
\times \left( \sum_{r_{-i}, s_{-i}} A^c_{r_{-i}, s_{-i}} \otimes | r_{-i} \rangle \langle r_{-i} | \otimes I \otimes | s_{-i} \rangle \langle s_{-i} | \right) \left( \sum_{r'_{-i}, s'_{-i}} B^r_{r'_{-i}, s'_{-i}} \otimes | r'_{-i} \rangle \langle r'_{-i} | \otimes I \otimes | s'_{-i} \rangle \langle s'_{-i} | \right) \\
= \frac{1}{3^{-(n-1)}} \sum_{r_{-i}, s_{-i}} \langle \psi | A^c_{r_{-i}, s_{-i}} B^r_{r_{-i}, s_{-i}} | \psi \rangle \cdot \left( \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{r'_{-i}} \langle r'_{-i} | \right) \left( \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{s'_{-i}} \langle s'_{-i} | \right) \\
= \frac{1}{3^{-(n-1)}} \sum_{r_{-i}, s_{-i}} \langle \psi | A^c_{r_{-i}, s_{-i}} B^r_{r_{-i}, s_{-i}} | \psi \rangle = E_{r_{-i}, s_{-i}} \langle \psi | A^c_{r_{-i}, s_{-i}} B^r_{r_{-i}, s_{-i}} | \psi \rangle \geq 1 - 9\epsilon
\]

Where the last line follows by Fact 6. Similarly,

\[
\langle \sigma | T' \cdot S | \sigma \rangle = \left( \langle \psi | \otimes \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{r'_{-i}} \langle r'_{-i} | \otimes \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{r''_{-i}} \langle r''_{-i} | \otimes \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{s_{-i}} \langle s_{-i} | \right) \\
\times \left( \sum_{r'_{-i}, s_{-i}} A^c_{r'_{-i}, s_{-i}} \otimes I \otimes | r''_{-i} \rangle \langle r''_{-i} | \otimes | s_{-i} \rangle \langle s_{-i} | \right) \left( \sum_{r''_{-i}, s'_{-i}} B^{r''}_{r''_{-i}, s'_{-i}} \otimes I \otimes | r''_{-i} \rangle \langle r''_{-i} | \otimes | s'_{-i} \rangle \langle s'_{-i} | \right) \\
= \frac{1}{3^{-(n-1)}} \sum_{r'_{-i}, s_{-i}} \langle \psi | A^c_{r'_{-i}, s_{-i}} B^{r''}_{r'_{-i}, s_{-i}} | \psi \rangle \cdot \left( \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{r''_{-i}} \langle r''_{-i} | \right) \left( \frac{1}{\sqrt{3^{-(n-1)}}} \sum_{s'_{-i}} \langle s'_{-i} | \right) \\
= \frac{1}{3^{-(n-1)}} \sum_{r'_{-i}, s_{-i}} \langle \psi | A^c_{r'_{-i}, s_{-i}} B^{r''}_{r'_{-i}, s_{-i}} | \psi \rangle = E_{r'_{-i}, s_{-i}} \langle \psi | A^c_{r'_{-i}, s_{-i}} B^{r''}_{r'_{-i}, s_{-i}} | \psi \rangle \geq 1 - 9\epsilon
\]

Where the last line again follows by Fact 6. It follows by Lemma 30 that

\[
\langle \sigma | T' \cdot T | \sigma \rangle \geq 1 - 36\epsilon
\]

Noting that
Theorem 8. Suppose that two players Alice and Bob have an entangled strategy for the \(n\)-round parallel repeated Magic Square game, which wins with probability at least \(1 - \varepsilon\). Then, if we adjoin an ancilla register to Alice’s space in the appropriate state \(|\text{ancilla}\rangle_A\) and similarly for Bob in the appropriate state \(|\text{ancilla}\rangle_B\), there exist observables \(\hat{A}_{r,k}^c\) indexed by \(r, c \in \{0, 1, 2\}\) and \(k \in [n]\) acting on Alice’s space such that

\[
\forall k, r, c, r', c', \quad d_\psi(\hat{A}_{r,k}^c \hat{A}_{r',k}^{c'} (-1)^{f(r,c,c')} \hat{A}_{r,k}^{c} \hat{A}_{r,k}^{c'}) \leq O(\sqrt{\varepsilon})
\]

\[
\forall k \neq k', r, c, r', c', \quad d_\psi(\hat{A}_{r,k}^c \hat{A}_{r',k}^{c'} \hat{A}_{r,k}^{c} \hat{A}_{r,k}^{c'}) \leq O(\sqrt{\varepsilon}).
\]

where \(|\psi\rangle = |\text{ancilla}\rangle_A \otimes |\text{ancilla}\rangle_B\) denotes the state together with the ancilla registers, and \(f(r,r',c,c') = 1\) if \(r \neq r'\) and \(c \neq c'\), and 0 otherwise.

Likewise, there exist observables \(\hat{B}_{c,k}^r\) on Bob’s space such that

\[
\forall k, r, c, r', c', \quad d_\psi(\hat{B}_{c,k}^r \hat{B}_{c',k}^{r'} (-1)^{f(r,c,c')} \hat{B}_{c,k}^{r} \hat{B}_{c,k}^{r'}) \leq O(\sqrt{\varepsilon})
\]

\[
\forall k \neq k', r, c, r', c', \quad d_\psi(\hat{B}_{c,k}^r \hat{B}_{c',k}^{r'} \hat{B}_{c,k}^{r} \hat{B}_{c,k}^{r'}) \leq O(\sqrt{\varepsilon}).
\]

So, we have,

\[
|1 - E_{r,r':r''=r''} \langle \psi | \hat{A}_{r,k}^c \hat{A}_{r',k}^{c'} | \psi \rangle| = |1 - \langle \sigma | T \cdot T' | \sigma \rangle| \leq 36\varepsilon
\]

\[\square\]

4 Results

In this section, we state and prove our technical results on the structure of strategies for the \(n\)-round parallel repeated Magic Square game. We first give an overview of the proof and then fill in the technical details.

4.1 Overview

Our result has two main technical components. The first is a theorem that, given a near-optimal strategy, shows how to construct observables on each players’ Hilbert space that approximately satisfy a set of pairwise commutation and anticommutation relations.

Theorem 8. Suppose that two players Alice and Bob have an entangled strategy for the \(n\)-round parallel repeated Magic Square game, which wins with probability at least \(1 - \varepsilon\). Then, if we adjoin an ancilla register to Alice’s space in the appropriate state \(|\text{ancilla}\rangle_A\) (and similarly for Bob in the appropriate state \(|\text{ancilla}\rangle_B\), there exist observables \(\hat{A}_{r,k}^c\) indexed by \(r, c \in \{0, 1, 2\}\) and \(k \in [n]\) acting on Alice’s space such that

\[
\forall k, r, c, r', c', \quad d_\psi(\hat{A}_{r,k}^c \hat{A}_{r',k}^{c'} (-1)^{f(r,c,c')} \hat{A}_{r,k}^{c} \hat{A}_{r,k}^{c'}) \leq O(\sqrt{\varepsilon})
\]

\[
\forall k \neq k', r, c, r', c', \quad d_\psi(\hat{A}_{r,k}^c \hat{A}_{r',k}^{c'} \hat{A}_{r,k}^{c} \hat{A}_{r,k}^{c'}) \leq O(\sqrt{\varepsilon}).
\]

where \(|\psi\rangle = |\text{ancilla}\rangle_A \otimes |\text{ancilla}\rangle_B\) denotes the state together with the ancilla registers, and \(f(r,r',c,c') = 1\) if \(r \neq r'\) and \(c \neq c'\), and 0 otherwise.

Likewise, there exist observables \(\hat{B}_{c,k}^r\) on Bob’s space such that

\[
\forall k, r, c, r', c', \quad d_\psi(\hat{B}_{c,k}^r \hat{B}_{c',k}^{r'} (-1)^{f(r,c,c')} \hat{B}_{c,k}^{r} \hat{B}_{c,k}^{r'}) \leq O(\sqrt{\varepsilon})
\]

\[
\forall k \neq k', r, c, r', c', \quad d_\psi(\hat{B}_{c,k}^r \hat{B}_{c',k}^{r'} \hat{B}_{c,k}^{r} \hat{B}_{c,k}^{r'}) \leq O(\sqrt{\varepsilon}).
\]
Moreover, the following consistency relations hold in expectation:

\[ \forall c, p, \quad E_r d_\psi(A^c_{r,p} \otimes I_{\text{ancilla}} \prod_{k=1}^n (\tilde{A}^c_{r,k} p_k)^2 \leq O(n\sqrt{\varepsilon}) \]  

(4)

\[ \forall r, p, \quad E_c d_\psi(B^r_{c,p} \otimes I_{\text{ancilla}} \prod_{k=1}^n (\tilde{B}^r_{c,k} p_k)^2 \leq O(n\sqrt{\varepsilon}) \]  

(5)

**Proof of Theorem 8**. The single-round phase relations in Equations (2) and (3) follow from Lemma 13. The commutation relations between rounds follow from Lemma 14. The consistency relations (Equations (4) and (5)) follow from Lemma 18. \[ \square \]

Having constructed these observables, we use them to build an isometry that “extracts” a 2n-qubit state out of the shared state of Alice and Bob. This isometry is *local*: it does not create any entanglement between Alice and Bob. Moreover, it maps the measurements in the players’ strategy to 2n-qubit measurements that are close to the ideal strategy.

**Theorem 9.** Suppose that two players share an entangled state in a Hilbert space \( \mathcal{H} \) and operators \( \tilde{A}_{r,k}^c \tilde{B}_{c,k}^r \) satisfying Equations (2) and (3). Then there exists an isometry \( V : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \), and for every \( s, t \in \{0, 1\}^{2n} \), there exists an operator \( W^A_{s,t} \) on Alice’s space, and for every \( u, v \in \{0, 1\}^{2n} \), there exists an operator \( W^B_{u,v} \) on Bob’s space, such that

\[ \forall a, b, c, d, \quad \left| \langle \phi | \sigma_X^A(s) \sigma_Z^A(t) \sigma_X^B(u) \sigma_Z^B(v) | \phi \rangle - \langle \psi | W^A_{s,t} W^B_{u,v} | \psi \rangle \right| \leq O(n^2 \sqrt{\varepsilon}), \]  

(6)

where \( | \phi \rangle = V(| \psi \rangle) \), \( \sigma_X^A, \sigma_Z^A \) are Pauli operators acting on the second output register of \( V \), and \( \sigma_X^B, \sigma_Z^B \) are Pauli operators acting on the fourth output register of \( V \).

The proof of this theorem is deferred to Section 4.3. As a corollary, we show that the output state of the isometry has high overlap with the state \( | \text{EPR} \rangle^{\otimes 2n} \) consisting of 2n EPR pairs shared between Alice and Bob.

**Corollary 10.** Suppose that two players have an entangled strategy for the n-round parallel repeated Magic Square game, which wins with probability at least \( 1 - \varepsilon \). Then, letting \( | \phi \rangle = V(| \psi \rangle) \) as in Theorem 9

\[ \langle \phi | \text{EPR} \langle \text{EPR} |^{\otimes 2n} \otimes I_{\text{junk}} | \phi \rangle \geq 1 - O(n^2 \sqrt{\varepsilon}), \]

where the identity operator \( I_{\text{junk}} \) acts on the first, third, and fifth register of the isometry output.

**Proof.** This follows from Lemma 25 and Lemma 22. \[ \square \]

### 4.2 Single-round observables

**Definition 11.** Let \( k \in [n] \) be the index of a round, and denote the single round observables associated with that round by \( A^c_{r,k} := A^c_{r,e_k} \) and \( B^r_{c,k} := B^r_{c,e_k} \), where \( c \) and \( r \) are any vectors whose kth coordinates are \( r \) and \( c \) respectively, and \( e_k \) is the vector with a 1 in the kth position and 0s elsewhere.
Definition 12. For each round $k$, define the state $|\text{ancilla}_k\rangle := \frac{1}{\sqrt{3n-1}} \sum_{r,s \in \{0,1,2\}^{n-1}} |r-k\rangle$. Define the dilated state

$$|\psi'\rangle := |\psi\rangle \otimes |\text{ancilla}_1\rangle^A_1 \otimes \ldots \otimes |\text{ancilla}_n\rangle^A_1 \otimes |\text{ancilla}_1\rangle^B_2 \otimes \ldots \otimes |\text{ancilla}_n\rangle^B_2$$

and define dilated observables on Alice’s side

$$\tilde{A}^c_{r,k} := \sum_{a_k} \sum_{a_1} (-1)^{(a_k)} P_{a_k,a_1}^r \otimes I_1 \otimes \ldots \otimes I_{k-1} \otimes |r-k\rangle \langle r-k| \otimes I_{k+1} \otimes \ldots \otimes I_n$$

$$= \sum_{c_k} \tilde{A}^c_{k,c_k} \otimes I_1 \otimes \ldots \otimes I_{k-1} \otimes |r-k\rangle \langle r-k| \otimes I_{k+1} \otimes \ldots \otimes I_n$$

Where $c$ in the last line can be any $c$ satisfying $c_k = c$, and wherever we write a sum over $r-k$ it is implicit that $r_k$ is fixed to be $r_k = r$.

Observe that the operators $\tilde{A}^c_{r,k}$ are true observables, i.e. they are Hermitian and square to $I$. Moreover, $\tilde{A}^c_{r,k}$ simulates the two-outcome POVM whose elements are given by $M^a_c := E_{r_k} P_{r_k}^a$.

Similarly, define dilated observables on Bob’s side

$$\tilde{B}^c_{r,k} := \sum_{c_k} \sum_{b_k} (-1)^{(b_k)} Q_{c_k}^r P_{b_k,b_1}^r \otimes I_1 \otimes \ldots \otimes I_{k-1} \otimes |r-k\rangle \langle r-k| \otimes I_{k+1} \otimes \ldots \otimes I_n$$

$$= \sum_{c_k} \tilde{B}^c_{k,c_k} \otimes I_1 \otimes \ldots \otimes I_{k-1} \otimes |r-k\rangle \langle r-k| \otimes I_{k+1} \otimes \ldots \otimes I_n$$

Where $r$ in the last line can be any $r$ satisfying $r_k = r$, and wherever we write a sum over $c-k$ it is implicit that $c_k$ is fixed to be $c_k = c$.

Observe that the operators $\tilde{B}^c_{r,k}$ are true observables, i.e. they are Hermitian and square to $I$. Moreover, $\tilde{B}^c_{r,k}$ simulates the two-outcome POVM whose elements are given by $M^b_c := E_{c_k} P_{c_k}^b$.

Lemma 13. For all $k, r, r', c, c'$, it holds that

$$\| (\tilde{A}^c_{r,k} \tilde{A}^{c'}_{r',k} - (-1)^{f(r,r',c,c')} \tilde{A}^{c'}_{r',k} \tilde{A}^c_{r,k}) |\psi'\rangle \| \leq O(\sqrt{\varepsilon}).$$

The analogous statement also holds for Bob operators.

Proof. Follows from single round analysis. See Appendix [B]. Replacing the operators $A^c_{r,k}$ in that analysis with $\tilde{A}^c_{r,k}$, and replacing $B^c_{r,k}$ in that analysis with $\tilde{B}^c_{r,k}$, one may observe that the analysis in Appendix [B] still holds. \qed

Lemma 14. For all $k \neq k', r, r', c, c'$, it holds that

$$\| (\tilde{A}^c_{r,k} \tilde{A}^{c'}_{r',k'} - \tilde{A}^{c'}_{r',k'} \tilde{A}^c_{r,k}) |\psi'\rangle \| \leq O(\sqrt{\varepsilon}).$$

The analogous statement also holds for Bob operators.

Proof. Let $c$ be any choice of columns such that $c_k = c, c_{k'} = c'$. Recall that by equation (1) we have that

$$\forall p, E_{r,c} \langle p | A^c_{r,p} B^r_{c,p} |\psi\rangle \geq 1 - \varepsilon.$$ (7)

Setting $p = e_k$ gives that, for all fixed values of $r_k$ and $c_k$,
\begin{align}
\forall k, \quad & \mathbb{E}_{r_k,c_k} (\psi | A_{r_k,c_k}^c B_{c_k,e_k}^r | \psi) \geq 1 - 9\epsilon. \quad (8) \\
\text{So,} \\
\forall k, \quad & \mathbb{E}_{r_k,c_k} d_\psi (A_{r_k,c_k}^c B_{c_k,e_k}^r)^2 \leq 18\epsilon \\
\end{align}

Further, recall that \( A_{r_k,c_k}^c = A_{r_k,c_k}^c \) as long as the \( k \)th coordinate of \( c \) and \( c' \) agree. Denote by \( \mathbb{E}_{c|k,k'} \) the uniform distribution over choices of column vector \( c \) such that \( c_k = c \) and \( c_{k'} = c' \). Then

\[
d_\psi (\tilde{A}_{r_k,c_k}^c \tilde{A}_{r_{k'},c_{k'}}^c, \tilde{A}_{r_k,c_k}^c) = \mathbb{E}_{c|k,k'} d_\psi \left( \sum_{r_k,r'_{k'}} A_{r_k,c_k}^c A_{r_{k'},c_{k'}}^c \otimes |r_{-k},r'_{-k}| \langle r_{-k},r'_{-k} | \right)
\]

\[
\sum_{r_k,r'_{k'}} A_{r_k,c_k}^c A_{r_{k'},c_{k'}}^c \otimes |r_{-k},r'_{-k}| \langle r_{-k},r'_{-k} |)
\]

Note that the column vector \( c \) is common to both \( A \) operators. Also, as a convention, wherever there is a sum or expectation over \( r_{-k} \) or \( r'_{-k} \) in this proof, it is implicit that the values of \( r_k \) and \( r'_{k'} \) are fixed to be \( r_k = r \) and \( r'_{k'} = r' \). Now, we apply Lemma 27 to move the leftmost \( A \) operator to Bob.

\[
\leq \mathbb{E}_{c|k,k'} \left[ d_\psi \left( \sum_{r_k,r'_{k'}} A_{r_k,c_k}^c B_{c_k,e_k}^r \otimes |r_{-k},r'_{-k}| \langle r_{-k},r'_{-k} | \right) \right] + \\
\sum_{r_k,r'_{k'}} A_{r_k,c_k}^c A_{r_{k'},c_{k'}}^c \otimes |r_{-k},r'_{-k}| \langle r_{-k},r'_{-k} |)
\]

\[
\sum_{r_k,r'_{k'}} A_{r_k,c_k}^c \otimes |r_{-k}| \langle r_{-k} | I_k \otimes |r'_{-k}| \langle r'_{-k} |)
\]

\[
\sum_{r_k,r'_{k'}} A_{r_k,c_k}^c \otimes |r_{-k}| \langle r_{-k} | I_k \otimes |r'_{-k}| \langle r'_{-k} |)
\]

Note that \( \| \sum_{r_k} A_{r_k,e_k}^c \otimes |r_{-k}| \langle r_{-k} | I_k \| \leq 1 \). Hence, applying Lemma 28 and Lemma 29, we get

\[
\leq \mathbb{E}_{c|k,k'} \left[ d_\psi \left( \sum_{r_k,r'_{k'}} A_{r_k,c_k}^c B_{c_k,e_k}^r \otimes |r_{-k},r'_{-k}| \langle r_{-k},r'_{-k} | \right) \right] + \\
\sum_{r_k,r'_{k'}} A_{r_k,c_k}^c A_{r_{k'},c_{k'}}^c \otimes |r_{-k},r'_{-k}| \langle r_{-k},r'_{-k} |)
\]

By performing the same steps on the other \( A \) operator, we obtain

\[
\leq \mathbb{E}_{c|k,k'} \left[ d_\psi \left( \sum_{r_k,r'_{k'}} B_{c_k,e_k}^r \otimes |r_{-k},r'_{-k}| \langle r_{-k},r'_{-k} | \right) \right] + \\
\sum_{r_k,r'_{k'}} A_{r_k,c_k}^c A_{r_{k'},c_{k'}}^c \otimes |r_{-k},r'_{-k}| \langle r_{-k},r'_{-k} |)
\]

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Now the $B$ operators can be commuted exactly since they share the same input $c$.

\[
\leq \mathbf{E}_{c,k,k'} \left[ d_{\psi'}(\sum_{\mathbf{r}, \mathbf{r}' \neq \mathbf{k}} B_{\mathbf{c},\mathbf{r}} B_{\mathbf{c},\mathbf{r}'}) \otimes |\mathbf{r}_{-k}, \mathbf{r}'_{-k}\rangle \langle \mathbf{r}_{-k}, \mathbf{r}'_{-k}|) + \sum_{\mathbf{r}, \mathbf{r}' \neq \mathbf{k}} A_{\mathbf{c},\mathbf{r}} A_{\mathbf{c},\mathbf{r}'} \otimes |\mathbf{r}_{-k}, \mathbf{r}'_{-k}\rangle \langle \mathbf{r}_{-k}, \mathbf{r}'_{-k}|) + 2 \mathbf{E}_{\mathbf{r}, \mathbf{k}} d_{\psi}(A_{\mathbf{r}, \mathbf{c}} B_{\mathbf{c}, \mathbf{r}}) + 2 \mathbf{E}_{\mathbf{r}', \mathbf{k}} d_{\psi}(A_{\mathbf{r}', \mathbf{c}} B_{\mathbf{c}, \mathbf{r}'}) \right]
\]

We move the $B$s back to Alice by reversing the previous steps, again using Lemmas 27, 28, and 29.

Finally, we bound this by Equation (9). Note that Equation (9) is stated with $E$ on $E'$ implies the same statement with $E$ on $E'$. So, continuing our computation:

\[
\leq 4 \cdot 3 \cdot 3 \sqrt{2\varepsilon} = 36 \sqrt{2\varepsilon}.
\]

\[\]

**Lemma 15.**

\[\forall \mathbf{r}, \mathbf{c}, \mathbf{k}, \quad d_{\psi'}(\tilde{A}_{\mathbf{r}, \mathbf{k}} \tilde{B}_{\mathbf{c}, \mathbf{k}}) \leq O(\sqrt{\varepsilon})\]

**Proof.** In the argument below, let $\mathbf{r}$ be the row vectors agreeing with $\mathbf{r}$ on index $k$ and $\mathbf{r}_{-k}$ on the remaining indices; likewise for $\mathbf{c}$ (note that $\mathbf{r}_{-k}$ is stored in Alice’s register and and $\mathbf{c}_{-k}$ in Bob’s). The main trick is to use the freedom of choice of $\mathbf{c}$ on Alice’s operators to pick $\mathbf{c}$ agreeing with Bob’s ancilla register $\mathbf{c}_{-k}$.

\[
d_{\psi'}(\tilde{A}_{\mathbf{r}, \mathbf{k}} \tilde{B}_{\mathbf{c}, \mathbf{k}})^2 = \frac{1}{3n-1} \sum_{\mathbf{r}, \mathbf{c}, \mathbf{k}} A_{\mathbf{r}, \mathbf{k}} |\psi\rangle \langle \mathbf{A} |_{AB} \otimes |\mathbf{r}_{-k}\rangle^A \otimes |\mathbf{c}_{-k}\rangle^B -
\]

\[
\frac{1}{3n-1} \sum_{\mathbf{r}, \mathbf{c}, \mathbf{k}} B_{\mathbf{c}, \mathbf{k}} |\psi\rangle \langle \mathbf{A} |_{AB} \otimes |\mathbf{r}_{-k}\rangle^A \otimes |\mathbf{c}_{-k}\rangle^B)^2.
\]

By Lemma 29 with $i = (\mathbf{r}_{-k}, \mathbf{c}_{-k})$,

\[
= \mathbf{E}_{\mathbf{r}_{-k}, \mathbf{c}_{-k}} d_{\psi'}(\tilde{A}_{\mathbf{r}, \mathbf{k}} \tilde{B}_{\mathbf{c}, \mathbf{k}})^2
\]

This is bounded by the probability that round $k$ of the test succeeds with inputs $r$ and $c$

\[
\leq O(\varepsilon).
\]
Lemma 16. \( \forall r, c, p \) and \( \forall i \in [n] \)

\[
\left| \langle \psi' | \prod_{k=n}^{i+1} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i} \tilde{A}_{r,k}^c ) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i-1} \tilde{A}_{r,k}^c ) | \psi' \rangle \right| \leq O(\sqrt{\varepsilon})
\]

Proof. Fixing \( r, c, p \), and fixing \( i \in [n] \) we have

\[
\left| \langle \psi' | (\prod_{k=n}^{i+1} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i} \tilde{A}_{r,k}^c ) | \psi' \rangle - \langle \psi' | \prod_{k=n}^{i} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i-1} \tilde{A}_{r,k}^c ) | \psi' \rangle \right|
\]

\[
= \left| \langle \psi' | (\prod_{k=n}^{i} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i-1} \tilde{A}_{r,k}^c ) (\tilde{B}_{c_i,i}^r) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i-1} \tilde{A}_{r,k}^c ) (\tilde{A}_{c_i,i}^r - \tilde{B}_{c_i,i}^r) | \psi' \rangle \right|
\]

\[
\leq \left| \langle \psi' | (\prod_{k=n}^{i+1} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i} \tilde{A}_{r,k}^c ) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i-1} \tilde{A}_{r,k}^c ) | \psi' \rangle \right|
\]

\[
+ \left| \langle \psi' | (\prod_{k=n}^{i+1} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i} \tilde{A}_{r,k}^c ) \left( \tilde{A}_{c_i,i}^r - \tilde{B}_{c_i,i}^r \right) | \psi' \rangle \right|
\]

\[
\leq \left| \langle \psi' | (\prod_{k=n}^{i+1} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i} \tilde{A}_{r,k}^c ) \left( \tilde{A}_{c_i,i}^r - \tilde{B}_{c_i,i}^r \right) | \psi' \rangle \right|
\]

\[
\leq 0 + O(\sqrt{\varepsilon}) = O(\sqrt{\varepsilon}).
\]

Here the last inequality uses Lemma 15 and the second to last inequality uses that \( \tilde{B}_{c_i,i}^r \) commutes with all Alice operators, and that \( (\prod_{k=n}^{i} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i-1} \tilde{A}_{r,k}^c ) \) is a unitary, so that

\[
\left| \langle \psi' | (\prod_{k=n}^{i+1} (\tilde{B}_{c_i,k}^r)^p ) A_{r,p}^c (\prod_{k=1}^{i} \tilde{A}_{r,k}^c ) \left( \tilde{A}_{c_i,i}^r - \tilde{B}_{c_i,i}^r \right) | \psi' \rangle \right| \leq \| (\tilde{A}_{c_i,i}^r - \tilde{B}_{c_i,i}^r) | \psi' \| = d_{\psi'}(\tilde{A}_{c_i,i}^r, \tilde{B}_{c_i,i}^r).
\]

\[\square\]

Lemma 17.

\[
E_r \left( \langle \psi' | (\prod_{k=n}^{i+1} (\tilde{B}_{c_i,k}^r)^p ) (\prod_{k=1}^{i} A_{r,p_k}^e_k \otimes I) (\tilde{A}_{c_i+1}^{r_i+1}) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i} (\tilde{B}_{c_i,k}^r)^p ) (\prod_{k=1}^{i} A_{r,p_k}^e_k \otimes I) (\tilde{A}_{c_i}^{r_i}) | \psi' \rangle \right) \leq O(\sqrt{\varepsilon})
\]

and

\[
E_r \left( 1 - \langle \psi' | A_{r,p_k}^e_k (\tilde{A}_{c_n}^{r_n}) | \psi' \rangle \right) \leq O(\sqrt{\varepsilon})
\]
Proof.

\[
\begin{align*}
E_r \left( \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) (\tilde{A}_{c_{i+1},i+1}^{c}) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i+1} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i} A_{r,p,k}^{c} e_k \otimes I) (\tilde{A}_{c_i}^{c}) | \psi' \rangle \right) \\
= E_r \left( \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) (\tilde{A}_{c_{i+1},i+1}^{c}) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) (\tilde{A}_{c_{i+1},i+1}^{c}) | \psi' \rangle \\
+ \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) (\tilde{B}_{c_{i+1},i+1}^{r_k}) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i} A_{r,p,k}^{c} e_k \otimes I) (\tilde{A}_{c_{i+1},i+1}^{c}) | \psi' \rangle \right) \\
\leq E_r \left( \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) (\tilde{A}_{c_{i+1},i+1}^{c}) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) (\tilde{B}_{c_{i+1},i+1}^{r_k}) | \psi' \rangle \right) \\
+ E_r \left( \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) (\tilde{B}_{c_{i+1},i+1}^{r_k}) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i} A_{r,p,k}^{c} e_k \otimes I) (\tilde{A}_{c_{i+1},i+1}^{c}) | \psi' \rangle \right) \\
\leq E_r \left( d_{\psi'}(\tilde{A}_{c_{i+1},i+1}^{c+1}, \tilde{B}_{c_{i+1},i+1}^{r_k}) \right) \\
+ E_r \left( \langle \psi' | (\prod_{k=n}^{i+1} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i+1} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i} A_{r,p,k}^{c} e_k \otimes I) \cdot A_{r,p,i}^{c} e_i \otimes I \cdot (\tilde{A}_{c_{i+1},i+1}^{c}) | \psi' \rangle \right) \\
\leq E_r \left( d_{\psi'}(\tilde{A}_{c_{i+1},i+1}^{c+1}, \tilde{B}_{c_{i+1},i+1}^{r_k}) \right) + E_r \left( d_{\psi'}(I, A_{r,p,i}^{c} e_i \otimes I) \cdot \tilde{A}_{c_{i+1},i+1}^{c} \right)
\end{align*}
\]

Where the second to last inequality uses the fact that \(\| \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) \| = 1\), and the third inequality uses the fact that \(\| \langle \psi' | (\prod_{k=n}^{i+1} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) \| = 1\). Now, applying Lemma 15 we have

\[
\begin{align*}
\left| E_r \left( \langle \psi' | (\prod_{k=n}^{i+2} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i+1} A_{r,p,k}^{c} e_k \otimes I) (\tilde{A}_{c_{i+1},i+1}^{c}) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{i+1} \tilde{B}_{c,k}^{r_k}) (\prod_{k=n}^{i} A_{r,p,k}^{c} e_k \otimes I) (\tilde{A}_{c_{i+1},i+1}^{c}) | \psi' \rangle \right) \right| \\
\leq E_r \left( d_{\psi'}(\tilde{A}_{c_{i+1},i+1}^{c+1}, \tilde{B}_{c_{i+1},i+1}^{r_k}) \right) + E_r \left( d_{\psi'}(I, A_{r,p,i}^{c} e_i \otimes I) \cdot \tilde{A}_{c_{i+1},i+1}^{c} \right) \\
\leq O(\sqrt{\epsilon}) + E_r \left( d_{\psi'}(I, A_{r,p,i}^{c} e_i \otimes I) \cdot \tilde{A}_{c_{i+1},i+1}^{c} \right)
\end{align*}
\]

(12)

And we note that

(13)
\[
\mathbb{E}_r \left( d_{\psi'}(I, (A_{r,p_i}^e \otimes I) \cdot \tilde{A}_{r,i}^c)^2 \right) \\
= \mathbb{E}_r \left( \|\psi'\| - (A_{r,p_i}^e \otimes I) \cdot \tilde{A}_{r,i}^c \|\psi'\| \right) \\
= \mathbb{E}_r \left( 2 - 2\langle \psi'|(A_{r,p_i}^e \otimes I) \cdot \tilde{A}_{r,i}^c|\psi'\rangle \right) \\
= \mathbb{E}_r \left( 2 - 2 \left( \langle \psi \otimes \frac{1}{\sqrt{3^{n-1}}} \sum_{r_i \in \{0,1,2\}^{n-1}} |r_i\rangle \right) \left( \sum_{r',r'_i=r_i} A_{r',p_i}^e \otimes |r'\rangle \langle r' - i| \right) \times \ldots \times \left( \langle \psi \otimes \frac{1}{\sqrt{3^{n-1}}} \sum_{r_i \in \{0,1,2\}^{n-1}} |r_i\rangle \right) \right) \\
= \mathbb{E}_r \left( 2 - 2 \cdot \frac{1}{3^{n-1}} \sum_{r',r'_i=r_i} \langle \psi|A_{r',p_i}^e \cdot A_{r'_i}^c|\psi\rangle \cdot \langle \psi'\| \langle \psi'\| \right) \\
= \mathbb{E}_r \left( 2 - 2 \cdot \mathbb{E}_{r',r'_i=r_i} \langle \psi|A_{r',p_i}^e \cdot A_{r'_i}^c|\psi\rangle \right) = 2 \left( 1 - \mathbb{E}_{r',r'_i=r_i} \langle \psi|A_{r',p_i}^e \cdot A_{r'_i}^c|\psi\rangle \right) \\
\leq 2 \cdot 3 \cdot 36\varepsilon 
\tag{14}
\]

Where the last inequality follows from Lemma[7] Furthermore, by Jensen’s inequality it follows that:

\[
\mathbb{E}_r \left( d_{\psi'}(I, (A_{r,p_i}^e \otimes I) \cdot \tilde{A}_{r,i}^c)^2 \right) \leq \sqrt{\mathbb{E}_r \left( d_{\psi'}(I, (A_{r,p_i}^e \otimes I) \cdot \tilde{A}_{r,i}^c)^2 \right)} \leq O(\sqrt{\varepsilon})
\]

Now, resuming the calculation in equation (12), we have that

\[
\left| \mathbb{E}_r \left( \langle \psi'\| (\prod_{k=n}^{i+1} \tilde{B}_{c,k}^i) (\prod_{k=n}^{i+1} A_{r,p_k}^c \otimes I) (\tilde{A}_{r,i}^{c_{i+1}}) |\psi'\rangle - (\prod_{k=n}^{i+1} \tilde{B}_{c,k}^i) (\prod_{k=n}^{i+1} A_{r,p_k}^c \otimes I) (\tilde{A}_{r,i}^{c_i}) |\psi'\rangle \right) \right| \\
\leq O(\sqrt{\varepsilon}) + \mathbb{E}_r \left( d_{\psi'}(I, (A_{r,p_i}^e \otimes I) \cdot \tilde{A}_{r,i}^c) \right) \leq O(\sqrt{\varepsilon})
\]

Finally, note that, since Equation[14]is valid for every \( i \), it follows by the same calculation, with \( i = n \), that:

\[
\left| \mathbb{E}_r \left( 1 - \langle \psi'\| A_{r,p_n}^c \cdot (\tilde{A}_{r,n}^{c_n}) |\psi'\rangle \right) \right| \leq O(\varepsilon) \leq O(\sqrt{\varepsilon})
\]

\[\square\]

**Lemma 18.**

\[\forall c, p, \quad \mathbb{E}_r d_{\psi'}^2 \left( A_{r,p}^c \otimes I, \prod_{k=1}^n (\tilde{A}_{r,k}^c)_{p_k}^c \right) \leq O(n \sqrt{\varepsilon})
\]

The analogous statement also holds for Bob operators

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Proof. For simplicity of notation, throughout this proof, we will denote $A^c_r \otimes I$ simply by $A^c_r$. Start by noting that we have the following exact property:

$$A^c_{r,p} A^c_{r,p'} = A^c_{r,p+p'}.$$ 

As a consequence, we may decompose each observable $A^c_{r,p}$ into a product of single-round observables

$$A^c_{r,p} = A^c_{r,p_1} \cdots A^c_{r,p_k}.$$ 

So, fixing any value of $c$, and $p$, we have

$$\mathbb{E}_r d\psi' \left( A^c_{r,p} \prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k} \right)^2$$

$$= \mathbb{E}_r \left( \langle \psi' | A_{r,p}^c A_{r,p}^c | \psi' \rangle + \langle \psi' | (\prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k})^+ (\prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle - \langle \psi' | A_{r,p}^c (\prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle - \langle \psi' | (\prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k})^+ A_{r,p}^c | \psi' \rangle \right)$$

$$= 2 \mathbb{E}_r \left( 1 - \langle \psi' | A_{r,p}^c (\prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle \right)$$

Where, in the second equality we are using the fact that $A_{r,p}^c$ is Hermitian to get that

$$\langle \psi' | A_{r,p}^c (\prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle = \langle \psi' | A_{r,p}^c (\prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle = \langle \psi' | (\prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k})^+ A_{r,p}^c | \psi' \rangle.$$

Continuing, we have

$$\mathbb{E}_r d\psi' \left( A^c_{r,p} \prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k} \right)^2$$

$$= 2 \mathbb{E}_r \left( 1 - \langle \psi' | A_{r,p}^c (\prod_{k=1}^n (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle \right)$$

$$= 2 \mathbb{E}_r \left( 1 - \langle \psi' | (\prod_{k=n}^{n+i} (\tilde{B}_r^{c_k})^{p_k}) A_{r,p}^c (\tilde{A}_r^{c_1}) | \psi' \rangle \right)$$

$$- \sum_{i=n}^{i+1} \left( \langle \psi' | (\prod_{k=n}^{n+i} (\tilde{B}_r^{c_k})^{p_k}) A_{r,p}^c (\prod_{k=1}^i (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{n+i} (\tilde{B}_r^{c_k})^{p_k}) A_{r,p}^c (\prod_{k=1}^i (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle \right)$$

$$\leq 2 \mathbb{E}_r \left( 1 - \langle \psi' | (\prod_{k=n}^{n+i} (\tilde{B}_r^{c_k})^{p_k}) A_{r,p}^c (\tilde{A}_r^{c_1}) | \psi' \rangle \right)$$

$$+ \sum_{i=n}^{i+1} 2 \mathbb{E}_r \left( \langle \psi' | (\prod_{k=n}^{n+i} (\tilde{B}_r^{c_k})^{p_k}) A_{r,p}^c (\prod_{k=1}^i (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle - \langle \psi' | (\prod_{k=n}^{n+i} (\tilde{B}_r^{c_k})^{p_k}) A_{r,p}^c (\prod_{k=1}^i (\tilde{A}_r^{c_k})^{p_k}) | \psi' \rangle \right).$$
We now apply Lemma 16 inside the expectation:

\[ E[1 - \langle \psi' | (\prod_{k=n}^{n+2} B_{c,k}^r) | \psi' \rangle] = E[1 - \langle \psi' | (\prod_{k=n}^{n+2} B_{c,k}^r) | \psi' \rangle] \]

\[ + \sum_{i=1}^{n} 2 \cdot O(\sqrt{\varepsilon}) \]

\[ = 2 E[r \left[ 1 - \langle \psi' | (\prod_{k=n}^{n+2} B_{c,k}^r) | \psi' \rangle \right] + \sum_{i=1}^{n} 2 \cdot O(\sqrt{\varepsilon}) \]

\[ = 2 E[r \left[ 1 - \langle \psi' | (\prod_{k=n}^{n+2} B_{c,k}^r) | \psi' \rangle \right] + \sum_{i=1}^{n} 2 \cdot O(\sqrt{\varepsilon}) \]

\[ \leq 2 \left| E[r \left[ 1 - \langle \psi' | A_{r,p_k}^c | \psi' \rangle \right] + 2 \sum_{i=n-1}^{n} E[r \left[ \langle \psi' | (\prod_{k=n}^{n+2} B_{c,k}^r) | \psi' \rangle \right] + O(n \sqrt{\varepsilon}) \]

\[ \leq 2 \cdot O(\sqrt{\varepsilon}) + 2(n-1)O(\sqrt{\varepsilon}) + O(n \sqrt{\varepsilon}) = O(n \sqrt{\varepsilon}) \]

Where the last inequality follows by Lemma 17.

\[ \square \]

4.3 The Isometry

**Definition 19.** Define the single round “approximate Pauli” operators on Alice’s space by:

\[ X_{2k-1} = \bar{A}_{1,k}^1 \]

\[ X_{2k} = \bar{A}_{1,k}^0 \]

\[ Z_{2k-1} = \bar{A}_{0,k}^0 \]

\[ Z_{2k} = \bar{A}_{0,k}^1 \]

Likewise define the single round approximate Pauli operators on Bob’s space by

\[ X_{2k-1}^B = \bar{B}_{1,k}^1 \]

\[ X_{2k}^B = \bar{B}_{1,k}^0 \]

\[ Z_{2k-1}^B = \bar{B}_{0,k}^0 \]

\[ Z_{2k}^B = \bar{B}_{0,k}^1 \]

**Lemma 20** (Approximate single-round Pauli relations). Suppose Alice and Bob share an entangled strategy that wins with probability 1 − ε. Then the single-round Pauli operators as defined above satisfy the following relations:

\[ \forall i, \quad d_\phi(X_i, X_i^B) \leq \sqrt{\varepsilon} \]

\[ \forall i, \quad d_\phi(Z_i, Z_i^B) \leq \sqrt{\varepsilon} \]

\[ \forall i, \quad d_\phi(X_iZ_i, Z_iX_i) \leq \sqrt{\varepsilon} \]  \hspace{1cm} (15)

\[ \forall i \neq j, \quad d_\phi(X_iX_j, X_jX_i) \leq \sqrt{\varepsilon} \]

\[ \forall i \neq j, \quad d_\phi(Z_iZ_j, Z_jZ_i) \leq \sqrt{\varepsilon} \]
Proof. The consistency relations follow from Lemma 15. The other relations come from Theorem 8.

We will now build up multi-round Paulis from products of these.

Lemma 21 (Approximate Pauli relations). Suppose $X_i$, $Z_i$ are observables on Alice and $X_i^b, Z_i^b$ are observables on Bob indexed by $i \in [n]$ satisfying Equation (15). Let $X^a := \prod_{i=1}^{n} X_i^a$ and $Z^b := \prod_{i=1}^{n} Z_i^b$, and likewise let $(X^a)^b := \prod_{i=1}^{n} (X_i^b)^a$ and $(Z^b)^a := \prod_{i=1}^{n} (Z_i^a)^b$. Then

$$\forall a, b, a', b', \quad d_\psi((X^a Z^b)(X^a' Z^b'), (-1)^{a-b} X^{a+a'} Z^{b+b'}) \leq O(n^2 \sqrt{\varepsilon})$$

(16)

$$\forall a, b, \quad d_\psi((X^a Z^b), (Z^b)^a) \leq O(n \sqrt{\varepsilon}).$$

(17)

Proof. Equation (17) is an immediate consequence of Lemma 31. We obtain Equation (16) in two steps. First, by Equation (18) of Lemma 34, we have that

$$d_\psi(X^a Z^b, (-1)^{a-b} Z^b X^a) \leq O(n^2 \sqrt{\varepsilon}).$$

Further, by Equation (19) of Lemma 34 we have that

$$d_\psi(Z^b Z^b, Z^{b+b'}) \leq O(n^2 \sqrt{\varepsilon}).$$

Hence,

$$d_\psi((X^a Z^b X^a' Z^b'), (-1)^{a-b} X^{a+a'} Z^{b+b'}) \leq d_\psi((X^a Z^b X^a' Z^b', (Z^b)^a X^a Z^b X^a')$$

$$+ d_\psi((Z^b)^b X^a Z^b X^a', (-1)^{a-b} (Z^b)^b X^a X^a' Z^b)$$

$$+ d_\psi((-1)^{a-b} (Z^b)^b' X^a X^a' Z^b, (-1)^{a-b} (Z^b)^b' (Z^b)^b X^a X^a')$$

$$+ d_\psi((-1)^{a-b} (Z^b)^b' (Z^b)^b X^a X^a', (-1)^{a-b} (Z^b)^b' (Z^b)^b X^a X^a' Z^b')$$

$$+ d_\psi((-1)^{a-b} X^a X^a' Z^b Z^b', (Z^b)^b X^a X^a' Z^b Z^b')$$

$$\leq O(n^2 \sqrt{\varepsilon}).$$

Proof of Theorem 22. Let $W^{A_{a,b}} := X^a Z^b$ and $W^{B_{a,b}} := (X^b)^a (Z^b)^b$, and let $H$ be the provers’ Hilbert space, together with the ancillas adjoined in Section 4.2. Then we define the isometry $V : H \rightarrow H \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$ by

$$V(|\psi\rangle) = \frac{1}{2^{3n}} \sum_{a,b,c,d,e,f} (-1)^{b(a+c)} (-1)^{e(d+f)} W^{A_{a,b}} \otimes W^{B_{d,e}} |\psi\rangle \otimes |a+c, c) \otimes |d+f, f\rangle.$$
alternate description in terms of a circuit that “swaps” the input into the output register, which is initialized to be maximally entangled with the junk register.

We now show the expectation value of any multi-qubit Pauli operator on the output of the isometry is close to the corresponding expectation value of approximate Paulis in the isometry input. In the equations below, \( |\phi\rangle = V(|\psi\rangle) \), the Paulis \( \sigma_X^A, \sigma_Y^A \) act on output register 2, and \( \sigma_Z^B, \sigma_Z^E \) on output register 4.

\[
P = \langle \phi | \sigma_X^A(s) \sigma_Y^A(t) \sigma_Z^B(u) \sigma_Z^E(v) | \phi \rangle
\]

\[
= \frac{1}{2^{6n}} \sum_{a,b,c',d,e,f} \sum_{b',c,d,e,f} \left( \langle \psi | \otimes |a' + c', c'| \otimes (d' + f', f') |W_{A,t}^{a,b'} \otimes W_{B,t}^{d',e'}(-1)^{b'-(a'+c') + e'-(d'+f')} \right.

\times \langle \sigma_X^A(s) \sigma_Y^A(t) \sigma_Z^B(u) \sigma_Z^E(v)(-1)^{b-(a+c)+e-(d+f)} |W_{A,ab}^t \otimes W_{B,de}^t | \psi \rangle \otimes |a + c, c \rangle \otimes |d + f, f \rangle \right)

\]

\[
= \frac{1}{2^{6n}} \sum_{a,b,c,d,e,f} \sum_{b',c,d,e,f} \left( \langle \psi | \otimes |a' + c', c'| \otimes (d' + f', f') |W_{A,t}^{a,b'} \otimes W_{B,t}^{d',e'}(-1)^{b'-(a'+c')}(1-e'-(d'+f')} \right.

\times (-1)^{(b+t)(a+c)}(-1)^{(e+v)(d+f)} |W_{A,ab}^t \otimes W_{B,de}^t | \psi \rangle \otimes |a + c + s, c \rangle |d + f + u, f \rangle \right)

\]

\[
= \frac{1}{2^{6n}} \sum_{a,b,c,d,e,f} \left( \langle \psi | \sigma_X^A \sigma_Y^A \sigma_Z^B \sigma_Z^E |W_{A,t}^{a,s} \otimes W_{B,t}^{d,u}(-1)^{b'-s(a+c)}(-1)^{e'-(d+u+f)} \right.

\times (-1)^{(b+t)(a+c)}(-1)^{(e+v)(d+f)} |W_{A,ab}^t \otimes W_{B,de}^t | \psi \rangle \right).
\]

Now we do the sum over \( c \) and \( f \) to force \( b' = b + t \) and \( e' = e + v \):

\[
= \frac{1}{2^{4n}} \sum_{a,b,d,e} \left( (-1)^{(b+t)(s(-1)^{(e+v)}u)} |W_{A,t}^{a,s} \otimes W_{B,t}^{d,u} \otimes | \psi \rangle \right).
\]

Finally, we apply Lemma 22 to merge the \( W^A \) and \( W^B \) operators, picking up an error of \( O(n^2 \sqrt{\varepsilon}) \) in the process.

\[
\approx_{O(n^2 \sqrt{\varepsilon})} \langle \psi | W^A_s W^B_{u,v} | \psi \rangle.
\]

\]

\[
\square
\]

Lemma 22. Let \( M_n \) be the 4n-qubit operator defined by

\[
M_n = \left( \frac{1}{2} III + \frac{1}{18} (IXIX + XIXI + XXXX + ZIZI + IIZI + ZZZZ + XZZZ + ZXZX + YYYY) \right)^{\otimes n}.
\]

Then if a density matrix \( \rho \) satisfies \( \text{Tr}[M_n \rho] \geq 1 - \delta, \langle \text{EPR} | |\text{EPR}\rangle^{\otimes 2n} \rangle \geq 1 - \frac{\delta}{4} \).

Proof. Observe that the highest eigenvalue of \( M_1 \) is 1, with unique eigenvector \( |\text{EPR}\rangle^{\otimes 2} \). Moreover all other eigenvalues of \( M_1 \) have absolute value at most 5/9. Hence, the highest eigenvalue of \( M_n \) is also 1 with the unique eigenvector is \( |\text{EPR}\rangle^{\otimes 2n} \), and all other eigenvalues have absolute value at most 5/9. Hence

\[
M_n \leq |\text{EPR}\rangle \langle \text{EPR}|^{\otimes 2n} + \frac{5}{9} (I - |\text{EPR}\rangle \langle \text{EPR}|^{\otimes 2n}).
\]

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So

\[
1 - \delta \leq \text{Tr}[M_n\rho]
\]

\[
\leq \frac{4}{9}\text{Tr}[\rho|\text{EPR}\rangle\langle\text{EPR}|^{\otimes 2n}] + \frac{5}{9}
\]

\[
\frac{4}{9} - \delta \leq \frac{4}{9}\text{Tr}[\rho|\text{EPR}\rangle\langle\text{EPR}|^{\otimes 2n}]
\]

\[
1 - \frac{9}{4}\delta \leq \text{Tr}[\rho|\text{EPR}\rangle\langle\text{EPR}|^{\otimes 2n}].
\]

Lemma 23. For every single round operator $\tilde{A}^r_c$, let $X^aZ^b$ be approximate Pauli operator formed by taking the row-$r$, column-$c$ entry in the Magic Square (Figure 2), and converting $X$ and $Z$ on the first and second qubits to the approximate Paulis on qubits $2k - 1$ and $2k$, respectively. Then

\[
d_\psi(\tilde{A}^r_c X^aZ^b) \leq O(\sqrt{\epsilon}).
\]

Likewise, for Bob,

\[
d_\psi(\tilde{B}^r_c (X^a)^b(Z^b)^b) \leq O(\sqrt{\epsilon}).
\]

Proof. First consider Alice. Then the conclusion follows by definition of the approximate Paulis for $r \in \{0, 1\}$. When $r = 2$, use the fact that $d_\psi(\tilde{A}^c_0, \tilde{B}^c_0) \leq O(\sqrt{\epsilon})$. By definition, $\tilde{B}^c_0 = -\tilde{B}^c_1$. Each of these two operators can be switched back to Alice, to yield

\[
d_\psi(\tilde{A}^r_{2k}, -\tilde{A}^c_0, \tilde{A}^c_1) \leq O(\sqrt{\epsilon}).
\]

This establishes the result for single round operators. For the Bob, we follow the same argument, interchanging the role of the row and column indices. □

Lemma 24. For every product of single-round operators $\prod_{k=1}^n (\tilde{A}^c_{r,k})^{p_k}$, let $X^aZ^b$ be the approximate Pauli operator formed by applying the procedure of Lemma 23 to each single-round operator. Then

\[
d_\psi(\prod_{k=1}^n (\tilde{A}^c_{r,k})^{p_k}, X^aZ^b) \leq O(n\sqrt{\epsilon}).
\]

The analogous statement holds for $B$.

Proof. This is a consequence of Lemma 23 and Lemma 32. □

Lemma 25. Suppose Alice and Bob win the test with probability $1 - \epsilon$. Then for the operator $M_n$ defined in Lemma 22 $\langle \phi|M_n|\phi \rangle \geq 1 - O(n^2\sqrt{\epsilon})$, where $|\phi\rangle = V(|\psi\rangle)$ is the output of the isometry in Theorem 9 applied to Alice and Bob’s shared state $|\psi\rangle$.

Proof. Recall from Fact 6, we know that

\[
\forall p, E_{r,c} \langle \psi|A^c_{r,p}B^c_{c,p}|\psi \rangle \geq 1 - \epsilon.
\]
By applying the consistency relations Equation (4) and Equation (5) guaranteed by Theorem 8 we obtain that
\[ \forall p, \quad E_{r,c}(\psi) \prod_{k=1}^{n} (\hat{A}_{r_{k}, k}^{c_{k}})^{p_{k}} \prod_{k=1}^{n} (\hat{B}_{r_{k}, k}^{c_{k}})^{p_{k}} |\psi\rangle \geq 1 - O(n \sqrt{\epsilon}). \]

Now, by Lemma 24 we can switch the \( \hat{A} \) and \( \hat{B} \) operators to approximate Paulis:
\[ \forall p, \quad E_{r,c}(\psi|(X^a Z^b) ((X^{B^c} Z^{B^d}) |\psi\rangle \geq 1 - O(n \sqrt{\epsilon}). \]

Applying Theorem 9 we obtain that
\[ \forall p, \quad \langle \phi | E_{r,c}(\sigma_{X^A}^{A}(a) \sigma_{Z^B}^{B}(b) \sigma_{X^C}^{C}(c) \sigma_{Z^D}^{D}(d)) |\phi\rangle \geq 1 - O(n^2 \sqrt{\epsilon}). \]

In particular, taking an expectation over uniformly random choices of \( p \), we obtain that
\[ \langle \phi | E_{r,c,p}(\sigma_{X^A}^{A}(a) \sigma_{Z^B}^{B}(b) \sigma_{X^C}^{C}(c) \sigma_{Z^D}^{D}(d)) |\phi\rangle \geq 1 - O(n^2 \sqrt{\epsilon}). \]

It is not hard to see that \( E_{r,c,p}(\sigma_{X^A}^{A}(a) \sigma_{Z^B}^{B}(b) \sigma_{X^C}^{C}(c) \sigma_{Z^D}^{D}(d)) \) is precisely the operator \( M_n \), corresponding to the magic square test performed on an unknown state \( |\phi\rangle \) using the measurement operators of the ideal strategy.

5 Discussion and open questions

The reader familiar with previous self-testing results may notice that our Theorem 9 gives a robustness bound on the expectation value of operators without explicitly characterizing the state, whereas previous works often state a bound on the 2-norm \( \| V(|\psi\rangle) - |\psi\rangle \otimes |\text{junk}\| \), where \( |\psi\rangle \) is a fixed target state. While it is possible to translate from one to the other by means of the techniques in Lemma 25, we think the guarantee on expectation values is more natural in applications where one does not want to test closeness to a fixed target state, but rather to test whether the state satisfies a certain property described by a measurement operator.

Self-testing and rigidity have been very active areas of research in recent years, and we believe that many more interesting questions remain to be answered. One open question of interest is to reduce the question and answer length of the test without sacrificing the error scaling. This is especially interesting from the perspective of computational complexity, where self-testing results have been used to show computational hardness for estimating the value of non-local games [115, 116]. Rigidity has also been applied to secure delegated computation and quantum key distribution: in particular, the work of Reichardt, Unger, and Vazirani [117] achieves these applications using a serial (many-round) version of the CHSH test; it would be interesting to see if their results could be improved using the Magic Square test.

A further way to generalize our result would be to adapt it to test states made up of qudits, with local dimension \( d \neq 2 \). As our techniques relied heavily on the algebraic structure of the qubit Pauli group, this may require significant technical advances. In fact, a variant of the Magic Square game for which the ideal strategy consists of “generalized Paulis” (i.e. the mod \( d \) shift- and clock-matrices) was recently proposed by McKague [MCK16b], and it would be interesting to see if our analysis could extend to the parallel repetition of this game. Likewise, it would be interesting to extend our analysis to states other than the EPR state—for instance, could we do something like McKague’s self-test for \( n \)-qubit graph states [MCK16a], but with only two provers instead of \( n \)?
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A Properties of the State-Dependent Distance

Definition 26. Given a state $|\psi\rangle$ and two operators $A, B$, the state-dependent distance $d_\psi(A, B)$ between $A$ and $B$ is defined to be

$$d_\psi(A, B) := ||A|\psi\rangle - B|\psi\rangle||.$$

Lemma 27. The state-dependent distance satisfies the triangle inequality

$$\forall A, B, C, \quad d_\psi(A, C) \leq d_\psi(A, B) + d_\psi(B, C).$$

Lemma 28. Let $A, B, C, D$ be bounded operators. Then

$$d_\psi(DA, DC) \leq d_\psi(DA, DB) + \|D\|d_\psi(B, C).$$

Proof. By Lemma 27

$$d_\psi(DA, DC) \leq d_\psi(DA, DB) + d_\psi(DB, DC).$$

Expand the second term:

$$d_\psi(DB, DC) = \|D(B|\psi\rangle - C|\psi\rangle)\|_2$$

$$\leq \|D\| \cdot \|B|\psi\rangle - C|\psi\rangle\|_2$$

$$= \|D\|d_\psi(B, C).$$

The following lemma tells us that guarantees on the state-dependent distance on average can be made “coherent.”

Lemma 29. Let $\{A_i\}$ and $\{B_i\}$ be two sets of operators indexed by $i \in [N]$, and suppose that

$$E_i d_\psi(A_i, B_i)^2 = \delta.$$

Define the extended state $|\psi'\rangle = \frac{1}{\sqrt{N}} \sum_{i \in [N]} |\psi\rangle \otimes |i\rangle$, and the extended operators $\tilde{A} = \sum_i A_i \otimes |i\rangle\langle i|$ and $\tilde{B} = \sum_i B_i \otimes |j\rangle\langle j|$. Then

$$d_{\psi'}(\tilde{A}, \tilde{B})^2 = \delta.$$

Proof.

$$d_{\psi'}(\tilde{A}, \tilde{B}) = ||\tilde{A}|\psi'\rangle - \tilde{B}|\psi'\rangle||^2$$

$$= ||\frac{1}{\sqrt{N}} \sum_i A_i|\psi\rangle \otimes |i\rangle - \frac{1}{\sqrt{N}} \sum_i B_i|\psi\rangle \otimes |i\rangle||^2$$

$$= \frac{1}{N} \sum_i \langle \psi |(A_i^+A_i + B_i^+B_i - A_i^+B_i - B_i^+A_i)|\psi\rangle$$

$$= E_i d_\psi(A_i, B_i)^2$$

$$= \delta.$$
Lemma 30. Given three Hermitian, unitary operators $T, T', S$, and a unit vector $|\sigma\rangle$, if: $\langle \sigma | T \cdot S | \sigma \rangle \geq 1 - \delta$ and $\langle \sigma | T' \cdot S | \sigma \rangle \geq 1 - \delta$, then $\langle \sigma | T \cdot T' | \sigma \rangle \geq 1 - 4\delta$.

Proof. Note that

$$\| (T - S) | \sigma \rangle \| ^2 = 2 - 2 \langle \sigma | T \cdot S | \sigma \rangle \leq 2\delta$$

and, similarly,

$$\| (T' - S) | \sigma \rangle \| ^2 = 2 - 2 \langle \sigma | T' \cdot S | \sigma \rangle \leq 2\delta.$$

So, by the Cauchy-Schwarz inequality,

$$\left| \langle \sigma | (T - S)(T' - S) | \sigma \rangle \right| \leq \| (T - S) | \sigma \rangle \| \cdot \| (T' - S) | \sigma \rangle \| \leq \sqrt{2\delta} \cdot \sqrt{2\delta} = 2\delta.$$

Expanding out the Left Hand Side, now gives

$$2\delta \geq \left| \langle \sigma | (T - S)(T' - S) | \sigma \rangle \right| = \left| \langle \sigma | T \cdot T' | \sigma \rangle - \langle \sigma | T \cdot S | \sigma \rangle - \langle \sigma | S \cdot T' | \sigma \rangle + \langle \sigma | S \cdot S | \sigma \rangle \right|$$

$$= \left| \langle \sigma | T \cdot T' | \sigma \rangle - \langle \sigma | T \cdot S | \sigma \rangle - \langle \sigma | S \cdot T' | \sigma \rangle + 1 \right|$$

So,

$$- 2\delta \leq \langle \sigma | T \cdot T' | \sigma \rangle - \langle \sigma | T \cdot S | \sigma \rangle - \langle \sigma | S \cdot T' | \sigma \rangle + 1$$

and

$$\langle \sigma | T \cdot T' | \sigma \rangle \geq \langle \sigma | T \cdot S | \sigma \rangle + \langle \sigma | S \cdot T' | \sigma \rangle - 1 - 2\delta \geq (1 - \delta) + (1 - \delta) - 1 - 2\delta = 1 - 4\delta,$$

where the last inequality again uses the assumption of this lemma.

We now state and prove some “utility” lemmas, about what happens when we commute words of operators past each other.

Lemma 31. Let $A_1, \ldots, A_k$ be Hermitian operators on Alice’s space, and $B_1, \ldots, B_k$ be Hermitian operators on Bob’s space, such that

$$\forall i, \quad d_\phi(A_i, B_i) \leq \epsilon_i.$$

Then

$$d_\phi\left( \prod_{i=1}^{k} A_i, \prod_{i=k}^{1} B_i \right) \leq \sum_{i=1}^{k} \epsilon_i.$$

Proof.

$$d_\phi\left( \prod_{i=1}^{k} A_i, \prod_{i=k}^{1} B_i \right) \leq d_\phi(A_1 \ldots A_k, B_k A_1 \ldots A_{k-1}) + d_\phi(B_k A_1 \ldots A_{k-1}, B_k B_{k-1} A_1 \ldots A_{k-2})$$

$$+ \cdots + d_\phi(B_k \ldots B_2 A_1, B_k \ldots B_1)$$

$$\leq d_\phi(A_k, B_k) + d_\phi(A_{k-1}, B_{k-1}) + \cdots + d_\phi(A_1, B_1)$$

$$= \sum_{i} \epsilon_i.$$

\[\square\]
Lemma 32. Let $A_1, \ldots A_k$ and $A'_1, \ldots A'_k$ be operators on Alice, and $B_1, \ldots B_k$ be operators on Bob, such that
\[ \forall i, \quad d_{\psi}(A_i, B_i) \leq \varepsilon_1 \]
\[ \forall i, \quad d_{\psi}(A'_i, B_i) \leq \varepsilon_2. \]
Then
\[ d_{\psi}(A_1 \ldots A_k, A'_1 \ldots A'_k) \leq n(\varepsilon_1 + \varepsilon_2). \]

Proof. This is a straightforward application of the Lemma 31
\[ d_{\psi}(A_1 \ldots A_k, A'_1 \ldots A'_k) \leq d_{\psi}(A_1 \ldots A_k, B_k \ldots B_1) + d_{\psi}(B_k \ldots B_1, A'_1 \ldots A'_k) \]
\[ \leq n\varepsilon_1 + n\varepsilon_2. \]
\[ \square \]

Lemma 33. Let $A_1, \ldots A_k$ be Hermitian operators on Alice’s space, and $B_1, \ldots, B_k$ be Hermitian operators on Bob’s space. Suppose that
\[ \forall i, \quad d_{\psi}(A_i, B_i) \leq \varepsilon_1 \]
and
\[ \forall i, j \in \{1, \ldots, k-1\}, j \in \{k\}, \quad d_{\psi}(A_i, A_j, \alpha_{ij}A_jA_i) \leq \varepsilon_2 \]
where $\alpha_{ij} \in \{\pm 1\}$ for each choice of $i, j$. Then
\[ d_{\psi}(A_1 \ldots A_k, a_{1k}a_{2k} \ldots a_{k-1,k}A_kA_1A_2 \ldots A_{k-1}) \leq 2(k-2)\varepsilon_1 + (k-1)\varepsilon_2. \]

Proof.
\[ d_{\psi}(A_1 \ldots A_k, \prod_{i=1}^{k-1} a_{ik})A_kA_1 \ldots A_{k-1} \]
\[ \leq d_{\psi}(A_1 \ldots A_k, a_{k-1,k}A_1 \ldots A_{k-2}A_kA_{k-1}) \]
\[ + d_{\psi}(a_{k-1,k}A_1 \ldots A_{k-2}A_kA_{k-1}, a_{k-1,k}B_{k-1}A_1 \ldots A_{k-2}A_k) \]
\[ + d_{\psi}(a_{k-1,k}B_{k-1}A_1 \ldots A_{k-2}A_k, a_{k-1,k}a_{k-2,k}B_{k-1}A_1 \ldots A_{k-3}A_kA_{k-2}) \]
\[ + d_{\psi}(a_{k-1,k}a_{k-2,k}B_{k-1}A_1 \ldots A_{k-3}A_kA_{k-2}, a_{k-1,k}a_{k-2,k}B_{k-1}B_{k-2}A_1 \ldots A_{k-3}A_k) \]
\[ + \ldots \]
\[ + d_{\psi}(\prod_{i=2}^{k} a_{ik}B_{k-1} \ldots B_2A_1A_k, \prod_{i=1}^{k-1} a_{ik}B_{k-1} \ldots B_2A_kA_1) \]
\[ + d_{\psi}(\prod_{i=1}^{k} a_{ik}B_{k-1} \ldots B_2A_kA_1, \prod_{i=1}^{k-1} a_{ik}A_kA_1 \ldots A_{k-1}) \]
\[ \leq d_{\psi}(A_{k-1}A_k, ak-1, ka_{k}A_{k-1}) + d_{\psi}(A_{k-1}, B_{k-1}) + \ldots + d_{\psi}(A_2A_k, a2kA_kA_2) + d_{\psi}(A_2, B_2) \]
\[ + d_{\psi}(A_1A_k, a1kA_kA_1) + d_{\psi}(B_2, A_2) + \ldots + d_{\psi}(B_k, A_k) \]
\[ \leq 2(k-2)\varepsilon_1 + (k-1)\varepsilon_2 \]
\[ \square \]
As a consequence of the preceding lemma

**Lemma 34.** Let $S_1, \ldots, S_k, T_1, \ldots, T_k$ be Hermitian operators on Alice’s space and let $S_1^B, \ldots, S_k^B, T_1^B \ldots T_k^B$ be Hermitian operators on Bob’s space, satisfying

\[
\forall i, \quad d_\varphi(S_i, S_i^B) \leq \varepsilon_1 \\
\forall i, \quad d_\varphi(T_i, T_i^B) \leq \varepsilon_2 \\
\forall i, j, \quad d_\varphi(S_i T_j, \alpha_{ij} T_i S_i) \leq \varepsilon_3.
\]

Then

\[
d_\varphi(S_1 \ldots S_k T_1 \ldots T_k, \prod_{i,j=1}^k \alpha_{ij} T_1 \ldots T_k S_1 \ldots S_k) \leq 2(k-1)\varepsilon_2 + k(2(k-1)\varepsilon_1 + k\varepsilon_3). \tag{18}
\]

Likewise,

\[
d_\varphi(S_1 \ldots S_k T_1 \ldots T_k, \prod_{i=2}^{k-1} \prod_{j=1}^i \alpha_{ij} S_1 T_1 S_2 T_2 \ldots S_k T_k) \leq 2(k-1)\varepsilon_2 + \sum_{j=2}^k (2(j-2)\varepsilon_2 + (j-1)\varepsilon_3) \tag{19}
\]

**Proof.** We first prove Equation (18).

\[
d_\varphi(S_1 \ldots S_k T_1 \ldots T_k, \prod_{i,j=1}^k \alpha_{ij} T_1 \ldots T_k S_1 \ldots S_k) \\
\leq d_\varphi(S_1 \ldots S_k T_1 \ldots T_k, T_1^B \ldots T_k^B S_1 \ldots S_k T_1) \\
+ d_\varphi(T_k^B \ldots T_2^B S_1 \ldots S_k T_1, \prod_{i=1}^k \alpha_{i1} T_k^B \ldots T_2^B T_1 S_1 \ldots S_k) \\
+ d_\varphi(\prod_{i=1}^k \alpha_{i1} T_k^B \ldots T_2^B T_1 S_1 \ldots S_k, \prod_{i=1}^k \alpha_{i1} T_k^B \ldots T_3^B T_1 S_1 \ldots S_k T_2) \\
+ d_\varphi(\prod_{i=1}^k \alpha_{i1} T_k^B \ldots T_3^B T_1 S_2 \ldots S_k T_2, \prod_{i=1}^k \alpha_{i1} \alpha_{i2} T_k^B \ldots T_3^B T_2 S_1 \ldots S_k) \\
+ \ldots \\
+ d_\varphi(\prod_{i=1}^k \prod_{j=1}^{k-1} \alpha_{ij} T_k^B T_1 \ldots T_{k-1} S_1 \ldots S_k, \prod_{i=1}^k \prod_{j=1}^{k-1} \alpha_{ij} T_1 \ldots T_{k-1} S_1 \ldots S_k T_k) \\
+ d_\varphi(\prod_{i=1}^k \prod_{j=1}^{k-1} \alpha_{ij} T_1 \ldots T_{k-1} S_1 \ldots S_k T_k, \prod_{i,j=1}^k \alpha_{ij} T_1 \ldots T_k S_1 \ldots S_k) \\
\leq 2(k-1)\varepsilon_2 + k(2(k-1)\varepsilon_1 + k\varepsilon_3).
\]

The derivation of Equation (19) is very similar. The only difference is that the number of commutations of $S$ with $T$ is different. \qed
B The Single Round Case

In this section, we review the self-testing result of [WBMS16] on the single-round magic square game, and write out the measurement definitions concretely for use in our setting. The rules of the game are described in Fig. 1. Any entangled strategy for this game is described by a shared quantum state $|\psi\rangle_{AB}$ and projectors $P_{a_0,a_1}$ for Alice and $Q_{b_0,b_1}$ for Bob. It can be seen that the game can be won with certainty for the following strategy:

$$|\psi\rangle = \frac{1}{2} \sum_{i,j \in \{0,1\}} |ij\rangle_A \otimes |ij\rangle_B$$

$$P_{a_0,a_1} = \frac{1}{4} (I + (-1)^{a_0} Z_{A1} \otimes (I + (-1)^{a_1} Z_{A2} \otimes I_B$$

$$Q_{b_0,b_1} = \frac{1}{4} I_{A1} \otimes (I + (-1)^{b_0} Z_{B1} \otimes (I + (-1)^{b_1} X_{B2})$$

This strategy is represented pictorially in Fig. 2, where each row contains a set of simultaneously-measurable observables that give Alice’s answers, and likewise each column for Bob.

Inspired by this ideal strategy, for any strategy we can define the following induced observables on Alice’s system:

$$X_1 = \sum_{a_0,a_1} (-1)^{a_1} P_{a_0,a_1}^A = A_1^1$$

$$X_2 = \sum_{a_0,a_1} (-1)^{a_0} P_{a_0,a_1}^A = A_1^0$$

$$Z_1 = \sum_{a_0,a_1} (-1)^{a_0} P_{a_0,a_1}^A = A_0^1$$

$$Z_2 = \sum_{a_0,a_1} (-1)^{a_1} P_{a_0,a_1}^A = A_0^0$$

and on Bob’s system:

$$X_3 = \sum_{b_0,b_1} (-1)^{b_1} Q_{b_0,b_1}^B = B_1^1$$

$$X_4 = \sum_{b_0,b_1} (-1)^{b_1} Q_{b_0,b_1}^B = B_1^0$$

$$Z_3 = \sum_{b_0,b_1} (-1)^{b_0} Q_{b_0,b_1}^B = B_0^1$$

$$Z_4 = \sum_{b_0,b_1} (-1)^{b_0} Q_{b_0,b_1}^B = B_0^0$$

The X and Z observables correspond to the first two rows and columns of the square. From the
third row and third column, we obtain four more observables; two for Alice:

\[ W_1 = \sum_{a_0,a_1} (-1)^{a_0} P_2^{a_0,a_1} = A_2^0 \]

\[ W_2 = \sum_{a_0,a_1} (-1)^{a_1} P_2^{a_0,a_1} = A_2^1 \]

and two for Bob:

\[ W_3 = \sum_{b_0,b_1} (-1)^{b_0} Q_2^{b_0,b_1} = B_2^0 \]

\[ W_4 = \sum_{b_0,b_1} (-1)^{b_1} Q_2^{b_0,b_1} = B_2^1. \]

There are nine consistency conditions implied by winning the game with probability \(1 - \varepsilon\):

\[
\langle \psi | Z_1 Z_3 | \psi \rangle \geq 1 - 9\varepsilon \quad (20)
\]

\[
\langle \psi | Z_2 Z_4 | \psi \rangle \geq 1 - 9\varepsilon \quad (21)
\]

\[
\langle \psi | Z_1 Z_2 W_3 | \psi \rangle \geq 1 - 9\varepsilon \quad (22)
\]

\[
\langle \psi | X_2 X_4 | \psi \rangle \geq 1 - 9\varepsilon \quad (23)
\]

\[
\langle \psi | X_1 X_3 | \psi \rangle \geq 1 - 9\varepsilon \quad (24)
\]

\[
\langle \psi | X_1 X_2 W_4 | \psi \rangle \geq 1 - 9\varepsilon \quad (25)
\]

\[
-\langle \psi | W_1 Z_3 X_4 | \psi \rangle \geq 1 - 9\varepsilon \quad (26)
\]

\[
-\langle \psi | W_2 Z_4 X_3 | \psi \rangle \geq 1 - 9\varepsilon \quad (27)
\]

\[
-\langle \psi | W_1 W_2 W_3 W_4 | \psi \rangle \geq 1 - 9\varepsilon. \quad (28)
\]

From this we obtain anticommutation conditions

\[ X_1 Z_1 \approx X_1 Z_2 W_3 \quad \text{(by (22))} \]

\[ = W_3 X_1 Z_2 \]

\[ \approx W_3 X_1 Z_4 \quad \text{(by (21))} \]

\[ \approx W_3 Z_4 X_3 \quad \text{(by (24))} \]

\[ \approx - W_3 W_2 \quad \text{(by (27))} \]

\[ \approx W_3 W_1 W_3 W_4 \quad \text{(by (28))} \]

\[ = W_1 W_4 \]

\[ \approx - W_4 Z_3 X_4 \quad \text{(by (26))} \]

\[ \approx - Z_1 W_4 X_4 \quad \text{(by (20))} \]

\[ \approx - Z_1 X_2 W_4 \quad \text{(by (23))} \]

\[ \approx - Z_1 X_2 X_2 X_1 \quad \text{(by (25))} \]

\[ = - Z_1 X_1. \]
We can also get commutation relations on different qubits:

\[ X_1 Z_2 \approx X_1 Z_4 \quad \text{(by (21))} \]
\[ \approx Z_4 X_3 \quad \text{(by (24))} \]
\[ = X_3 Z_4 \quad \text{(by construction)} \]
\[ \approx X_3 Z_2 \quad \text{(by (21))} \]
\[ \approx Z_2 X_1 \quad \text{(by (24))}. \]

The other cases follow similarly. See [WBMS16] for further details.