Parameterized Complexity of Minimum Membership Dominating Set

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Received: 25 May 2022 / Accepted: 23 May 2023 / Published online: 13 June 2023
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Abstract
Given a graph $G = (V, E)$ and an integer $k$, the Minimum Membership Dominating Set (MMDS) problem seeks to find a dominating set $S \subseteq V$ of $G$ such that for each $v \in V$, $|N[v] \cap S|$ is at most $k$. We investigate the parameterized complexity of the problem and obtain the following results for the MMDS problem. First, we show that the MMDS problem is NP-hard even on planar bipartite graphs. Next, we show that the MMDS problem is W[1]-hard for the parameter pathwidth (and thus, for treewidth) of the input graph. Then, for split graphs, we show that the MMDS problem is W[2]-hard for the parameter $k$. Further, we complement the pathwidth lower bound by an FPT algorithm when parameterized by the vertex cover number of input graph. In particular, we design a $2^{O(vc)}|V|^{O(1)}$ time algorithm for the MMDS problem where $vc$ is the vertex cover number of the input graph. Finally, we show that the running time lower bound based on ETH is tight for the vertex cover parameter.

Keywords Dominating set · Pathwidth · Vertex cover number · FPT · Split graphs · Planar bipartite graphs
1 Introduction

For a graph $G = (V, E)$, a set $S \subseteq V$ is a dominating set for $G$, if for each $v \in V$, either $v \in S$, or a neighbor of $v$ in $G$ is in $S$. The DOMINATING SET problem takes as input a graph $G = (V, E)$ and an integer $k$, and the objective is to test if there is a dominating set of size at most $k$ in $G$. The DOMINATING SET problem is a classical NP-hard problem [1], which together with its variants, is a well-studied problem in Computer Science. We refer the reader to the book by Haynes, Hedetniemi and Slater [2] for more information about the minimum dominating set problem and its applications.

A variant of the DOMINATING SET problem that is of particular interest to us in this paper, is the one where we have an additional constraint that the number of closed neighbors that a vertex has in a dominating set is bounded by a given integer as input. We call this version (to be formally defined shortly) of the DOMINATING SET problem as the MINIMUM MEMBERSHIP DOMINATING SET (MMDS) problem. For a graph $G = (V, E)$, a vertex $u \in V$ and a set $S \subseteq V$, the membership of $u$ in $S$ is $M(u, S) = |N[u] \cap S|$. Next, we formally define the MMDS problem.

| MINIMUM MEMBERSHIP DOMINATING SET (MMDS) |
|-----------------------------------------|
| **Input:** A graph $G = (V, E)$ and a positive integer $k$. |
| **Parameter:** $k$. |
| **Question:** Does there exist a dominating set $S$ of $G$ such that $\max_{u \in V} M(u, S) \leq k$? |

We refer to a solution of the MMDS problem as a $k$-membership dominating set ($k$-mds). Unless, otherwise specified, for the MMDS problem, by $k$ we mean the membership. The term “membership” is borrowed from a similar version of the SET COVER problem by Kuhn et al. [3], that was introduced to model reduction in interference among transmitting base stations in cellular networks.

In this paper we give complete proofs for all theorems and improve the result on split graphs, as compared to our WALCOM 2022 paper [4].

**Our Results** Most of the variants of the DOMINATING SET problems are primarily constrained by the solution size. In particular, these problems seek to find a minimum sized solution. However, we do not have such restriction in the MMDS problem, that is, the solution size is not a constraint for us. Therefore, a fundamental question is “what is the complexity of the MMDS problem?”. We prove that the MMDS problem is NP-complete and study the problem in the realm of parameterized complexity.

**Theorem 1** The MMDS problem is NP-complete even on planar bipartite graphs for $k = 1$.

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1 For a vertex $v$ in a graph $G = (V, E)$, the closed neighborhood of $v$ in $G$, $N_G[v]$, is the set $\{u \in V \mid uv \in E\} \cup \{v\}$. 
Further, our reduction also shows that the MMDS problem restricted to planar bipartite graphs does not have a \((2 - \epsilon)\) approximation for any \(\epsilon > 0\). Parameterized complexity is an approach to deal with problems that are \(\text{NP}\)-hard. The variants of the DOMINATING SET problems consider the solution size as the natural parameter to explore the parameterized complexity of the problems. As we mentioned earlier, the solution size is not a constraint in the MMDS problem. Therefore, the natural parameter would be the membership value \((k)\). The membership parameter is not so helpful since the problem is \(\text{NP}\)-hard even for \(k = 1\). To complement the \(\text{NP}\)-hardness, we study the structural parameterizations of the MMDS problem. The DOMINATING SET problem and its variants are well-studied for the parameter treewidth. In particular, the DOMINATING SET problem parameterized by the treewidth admits an FPT algorithm that runs in time \(3^{\text{tw}}n^{O(1)}\) [5]. In contrast to the above, we show that such an algorithm is unlikely to exist for the MMDS problem.

**Theorem 2** The MMDS problem is \(\text{W}[1]\)-hard when parameterized by the pathwidth of the input graph.

It is known that the pathwidth of a graph is at least as large as its treewidth, and thus the above theorem implies that the MMDS problem parameterized by the treewidth does not admit any FPT algorithm. We prove Theorem 2 by demonstrating an appropriate parameterized reduction from the well-known \(\text{W}[1]\)-hard MULTI-COLORED CLIQUE problem (see [6] for its \(\text{W}[1]\)-hardness). The above theorem shows a difference in complexities of the DOMINATING SET problem and the MMDS problem when parameterized by the treewidth. It is known that the DOMINATING SET problem is known to be \(\text{W}[2]\)-complete for split graphs [7] for the parameter solution size. So naturally, we study the MMDS problem on split graphs, and prove the following theorem.

**Theorem 3** The MMDS problem is \(\text{W}[2]\)-hard on split graphs when parameterized by \(k\).

We had proved the \(\text{W}[1]\)-hardness of the MMDS problem on split graphs for parameter \(k\) in [4] by a reduction from MULTI-COLORED INDEPENDENT SET problem. In this paper, we improve the result to show that the MMDS problem is in fact \(\text{W}[2]\)-hard in split graphs for the parameter \(k\). We prove the above theorem by a parameterized reduction from the MULTI-COLORED SET COVER problem which is shown to be \(\text{W}[2]\)-hard for the parameter solution size by [8].

As it is proved that the MMDS problem is \(\text{W}[1]\)-hard for the parameter pathwidth, we consider a larger parameter vertex cover number of the input graph. Therefore, we study the MMDS problem parameterized by the vertex cover number \((\text{vc})\) of the input graph and prove the following theorem.

**Theorem 4** The MMDS problem admits an algorithm running in time \(2^{O(\text{vc})}n^{O(1)}\), where \(\text{vc}\) is the size of the minimum vertex cover of the input graph.

We remark that computing the minimum vertex cover \(X \subseteq V\) is itself in FPT when parameterized by \(|X|\) [9], and so we do not need \(X\) on the input. Further, we obtain a matching conditional lower bound as follows.

**Theorem 5** Assuming ETH, the MMDS problem does not admit an algorithm running in time \(2^{o(\text{vc})}n^{O(1)}\), where \(\text{vc}\) is the size of a minimum-sized vertex cover of the input graph.
Related works Kuhn et al. [3] introduced the “membership” variant, in a spirit similar to what we have, for the Set Cover problem, called Minimum Membership Set Cover (MMSC, for short). For the above problem, they obtained several results, including \( \text{NP} \)-completeness, an \( \mathcal{O}(\ln n) \) approximation algorithm, and a matching approximation hardness result. A special case of the MMSC problem is studied in [10] where the collection of sets has consecutive ones property. In such a set system, the problem is shown to be polynomial-time solvable. The dual of the MMSC problem which is the Minimum Membership Hitting Set (MMHS) is studied in [11] and [12] in various geometric settings. Narayanaswamy et al. [11] showed that the MMHS problem does not admit a \( 2 - \varepsilon \) approximation even in geometric settings where vertical line segments are to be hit by horizontal line segments. A polynomial time algorithm was given for the case when the horizontal segments are intersected by vertical lines. Recently, Mitchell and Pandit [12] studied both the MMSC problem and the MMHS problem on various types of geometric objects in the plane, including axis-parallel line segments, axis-parallel strips, rectangles that are anchored on a horizontal line from one side, rectangles that are stabbed by a horizontal line, and rectangles that are anchored on one of two horizontal lines. For each of these problems, they either proved the \( \text{NP} \)-hardness or gave a polynomial time algorithm.

The problem Perfect Code is a variant of Dominating Set where (in addition to the size constraint) we require the membership of each vertex in the dominating set to be exactly one. Perfect Code is another well-studied variant of Dominating Set, see for instance [13–20]. Telle [21, 22] studied a variant of Dominating Set where two vectors \( \sigma, \rho \) are additionally given as input, and the membership of vertices in the dominating set and outside this set needs to be determined by \( \sigma \) and \( \rho \), respectively. They obtained several results with respect to parameterized complexity of the above variant of Dominating Set. Also, Chapelle [23] studied the above variant with respect to treewidth as the parameter and gave an algorithm running in time \( k^{tw}n^{O(1)} \), where \( tw \) is the treewidth of the input graph. The MMDS problem with membership constraint \( k \), is the same as \([\sigma, \rho]\)-Dominating Set, when \( \sigma = [0, k - 1] \) and \( \rho = [1, k] \), except for the latter aims to minimize solution size. Thus the problem also admits such an algorithm.

Chellali et al. [24] introduced a version called \([j, \ell]\)-Dominating Set, where we seek a dominating set where the membership of each vertex is at least \( j \) and at most \( \ell \). They studied the above problem for the viewpoint of combinatorial bounds on special graph classes like claw-free graphs, \( P_4 \)-free graphs, and caterpillars, for restricted values of \( j \) and \( \ell \). Recently Meybodi et al. [25] studied the problems \([1, j]\)-Dominating Set and \([1, j]\)-Total Dominating Set in the realm of parameterized complexity. Though these problems involve constrained membership, unlike the MMDS problem, they require a membership constraint only on the open neighborhood of vertices.

2 Preliminaries

We recall in this section some notations and definitions used throughout this article. For any two positive integers \( x \) and \( y \), by \([x, y]\) we mean the set \( \{x, x + 1, \ldots, y\} \), and by \([x]\) we mean \([1, x]\). We assume that all our graphs are simple and undirected. Given a
graph $G = (V, E)$, $n$ represents the number of vertices, and $m$ represents the number of edges. We denote an edge between any two vertices $u$ and $v$ by $uv$. For a subset $S \subseteq V$, by $G[S]$ we mean the subgraph of $G$ induced by $S$, and by $G - S$ we mean $G[V \setminus S]$. For every vertex $u \in V$, by $N(u)$ we mean open neighborhood of $u$, and by $N[u]$ we mean closed neighborhood of $u$. Similarly, for any set $S \subseteq V$, $N(S) = \bigcup_{u \in S} N(u) \setminus S$ and $N[S] = \bigcup_{u \in S} N[u]$. Other than this, we follow the standard graph-theoretic notations based on Diestel [26]. We refer to the recent books of Cygan et al. [27] and Downey and Fellows [28] for detailed introductions to parameterized complexity.

**Treewidth and pathwidth** For an undirected graph $G = (V, E)$, a tree decomposition of $G$ is a pair $(T, X)$, where $T$ is a tree and $X = \{X_i \subseteq V \mid i \in V(T)\}$ such that

- $\bigcup_{i \in V(T)} X_i = V$,
- for each edge $uv \in E$, there exists a node $i \in V(T)$ such that $u, v \in X_i$, and
- for each $u \in V$, the set of nodes $\{i \in V(T) \mid u \in X_i\}$ induces a connected subtree in $T$.

The *width* of a tree decomposition $(T, X)$ is $\max_{i \in V(T)}(|X_i| - 1)$. The *treewidth* of $G$ is the minimum width over all possible tree decompositions of $G$. A tree decomposition $(T, X)$ is said to be a *path decomposition* if $T$ is a path. The *pathwidth* of a graph $G$ is the minimum width over all possible path decompositions of $G$. Let $\text{pw}(G)$ and $\text{tw}(G)$ denote the pathwidth and treewidth of the graph $G$, respectively. The pathwidth of a graph $G$ is one less than the minimum clique number of an interval supergraph $H$ which contains $G$ as an induced subgraph. It is well-known that the maximal cliques of an interval graph can be linearly ordered such that for each vertex, the maximal cliques containing it occur consecutively in the linear order. This gives a path decomposition of the interval graph. A path decomposition of the graph $G$ is the path decomposition of the interval supergraph $H$ which contains $G$ as an induced subgraph. In our proofs we start with the path decomposition of an interval graph and then reason about the path decomposition of graphs that are constructed from it.

### 3 The MMDS Problem on Planar Bipartite Graphs is $\text{NP}$-Complete

We show that the MMDS problem is $\text{NP}$-hard for $k = 1$ even when restricted to planar bipartite graphs. The $\text{NP}$-hardness is proved by a reduction from planar positive 1-in-3 SAT (PP1in3SAT) as follows. Let $\phi$ be a 3-CNF boolean formula with no negative literals on $n$ variables $X = \{x_1, x_2, \ldots, x_n\}$ having $m$ clauses $C = \{C_1, C_2, \ldots, C_m\}$. Further we consider the restricted case when the graph encoding the variable-clause incidence is planar. The variable-clause incidence graph is defined as a planar bipartite graph $G_{\phi} = (C \cup X, E)$ where $X = \{x_1, x_2, \ldots, x_n\}$, $C = \{C_1, C_2, \ldots, C_m\}$ and $E = \{x_i C_j \mid \text{variable } x_i \text{ appears in the clause } C_j\}$. A planar representation of an instance of the PP1in3SAT problem is illustrated in Fig. 1.
Fig. 1 A planar representation of an input instance of the PP1in3SAT problem (left) and equivalent graph of the reduced instance to the MMDS problem (right). The red-colored vertices denote the clauses and blue color vertices denote the literals. The vertices with blue-shade denote the variables which are set to 1 in a feasible assignment.

**Planar Positive 1-in-3 SAT (PP1in3SAT)**

**Input:** A 3-CNF boolean formula \( \phi(X) \) without negative literals and \( G_{\phi} \) is planar

**Question:** Does there exist an assignment of values \( a_1, a_2, \ldots, a_n \) to the variables \( x_1, x_2, \ldots, x_n \) such that exactly one variable in each clause is set to true?

It is known that PP1in3SAT is \( \text{NP} \)-complete [29]. A reduction from PP1in3SAT to the MMDS problem is shown to prove that the MMDS problem is \( \text{NP} \)-hard even on planar bipartite graphs.

**Proof of Theorem 1** Let \( \phi \) be a positive 3-CNF formula such that \( G_{\phi} \) is planar. We construct a planar bipartite graph \( \hat{G}_{\phi} \) as follows. The graph \( \hat{G}_{\phi} \) is a super graph of \( G_{\phi} \). Therefore, in graph \( \hat{G}_{\phi} \) we will have all vertices and edges in \( G_{\phi} \). Further, we add \( n \) more vertices \( \hat{X} = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\} \) to \( \hat{G}_{\phi} \). For each vertex \( x_i \in G_{\phi} \), we add an edge \( x_i, \hat{x}_i \) to \( E(\hat{G}_{\phi}) \). The resulting graph \( \hat{G}_{\phi} = (((X \cup \hat{X}) \cup C), E(G_{\phi}) \cup \{x_i, \hat{x}_i | 1 \leq i \leq n\}) \) is also a planar graph. Observe that \( \hat{G}_{\phi} \) is planar and bipartite. An instance of the PP1in3SAT and the corresponding reduced instance of the MMDS problem are illustrated in Fig. 1. Let, \( \langle \hat{G}_{\phi}, k = 1 \rangle \) be the reduced instance of the MMDS problem.

Next we prove the correctness of the reduction. We show that \( \phi \) is satisfiable if and only if \( \hat{G}_{\phi} \) has a dominating set which hits the closed neighborhood of each vertex exactly once. Let \( A = \{a_1, a_2, \ldots, a_n\} \) be a satisfying assignment for \( \phi \) such that exactly one variable is set to true in each clause. For the graph \( \hat{G}_{\phi} \), we define a set \( S \subseteq V(\hat{G}_{\phi}) \) as follows:

\[
S = \{x_i | a_i = 1, \text{ for } 1 \leq i \leq n\} \cup \{\hat{x}_i | a_i = 0, \text{ for } 1 \leq i \leq n\}
\]
Clearly, the set $S$ is a dominating set for $\hat{G}_\phi$. Consider a clause vertex $c \in C$. Let the three variable vertices adjacent to $c$ be $x$, $y$ and $z$, and out of which exactly one will be assigned value 1 by the satisfying assignment $A$. Without loss of generality, let $y$ be the variable which is assigned value 1 in $A$. The vertex $y$ will dominate $c$ and $\hat{y}$, and vertices $x$ and $z$ will be dominated by $\hat{x}$ and $\hat{z}$, respectively. Therefore, the set $S$ is a feasible solution for the MMDS problem on instance $\langle \hat{G}_\phi, 1 \rangle$. Thus, $\langle \hat{G}_\phi, 1 \rangle$ is a YES-instance of the MMDS problem.

Now we show that if $\langle \hat{G}_\phi, 1 \rangle$ is a YES-instance of the MMDS problem then $\langle \phi \rangle$ is a YES-instance of the PPI1in3SAT. More precisely, we show that there exists a satisfying assignment for $\phi$ such that exactly one literal in each clause is set to 1. Let $S$ be a dominating set for $\hat{G}_\phi$ such that the closed neighborhood of each vertex is intersected exactly once by $S$. For each $i \in [n]$, $\hat{x}_i$ or $x_i$ must be in solution in order to dominate $\hat{x}_i$ since $\hat{x}_i$ is a pendent vertex. This enforces that no other neighbors of $x_i$ for any $i \in [n]$ can be in any feasible solution in order to satisfy the membership constraint. Therefore, $C \cap S = \emptyset$. Further, for each $1 \leq i \leq n$, either $x_i \in S$ or $\hat{x}_i \in S$ but not both. Consider the assignment $A = \{a_1, \ldots, a_n\}$ for $\phi$ as follows: For each $x_i \in S$, assign $a_i = 1$, and for each $x_i \notin S$, assign $a_i = 0$. Since $S$ is a dominating set of $\hat{G}_\phi$, $A$ is a satisfying assignment of $\phi$. Further, for each clause $c \in C$, there exists a unique variable with value 1 in $A$ since membership of $c$ is one in $\hat{G}_\phi$. Therefore, $A$ is a feasible solution for $\phi$ and, thus $\langle \phi \rangle$ is a YES-instance of the PPI1in3SAT problem.

Next, we show that the MMDS problem is in $\mathsf{NP}$. Given a set $S$, we can check the feasibility of the set $S$ to the instance $\langle G, k \rangle$ of the MMDS problem in polynomial time. Therefore, the MMDS problem is in $\mathsf{NP}$. Hence, the MMDS problem is $\mathsf{NP}$-complete for $k = 1$ even on planar bipartite graphs.\hfill $\square$

**Remark** The reduction also shows that the MMDS problem does not have a polynomial time $(2 - \epsilon)$ approximation algorithm unless $\mathsf{P} = \mathsf{NP}$. This is because such an algorithm can solve the MMDS problem for $k = 1$. The complexity dichotomy of the MMDS problem is still unsettled. Our hardness reduction shows that the MMDS problem is $\mathsf{NP}$-hard when $k = 1$. It is trivially known that the MMDS problem is in $\mathsf{P}$ when $k = \Delta$ where $\Delta$ is the maximum degree of the input graph. Therefore, it is a natural question that for what values of $k$, the MMDS problem switches from the easy complexity class to the hard complexity class.

**4 $\mathsf{W}[1]$-Hardness with Respect to Pathwidth**

We prove Theorem 2 by a reduction from the **MULTI-COLORED CLIQUE** problem to the MMDS problem. It is well-known that the **MULTI-COLORED CLIQUE** problem is $\mathsf{W}[1]$-hard for the parameter solution size [30].
Multi-Colored Clique

Input: A positive integer $k$ and a $k$-colored graph $G$

Parameter: $k$

Question: Does there exist a clique of size $k$ with one vertex from each color class?

Let $(G = (V, E), k)$ be an instance of the Multi-Colored Clique problem. We assume that the colors are $1, \ldots, k$ and denote by $V_i$ the set of vertices of color $i$. We assume, without loss of generality, $|V_i| = n$ for each $i \in [k]$. We usually use $n$ to denote number of vertices in the input graph. However, we use $n$ here to denote the number of vertices in each color class. For each $1 \leq i \leq k$, let $V_i = \{u_i, \ell \mid 1 \leq \ell \leq n\}$.

4.1 Gadget Based Reduction from Multi-Colored Clique

For an input instance $(G, k)$ of the Multi-Colored Clique problem, the reduction outputs an instance $(H, k')$ of the MMDS problem where $k' = n + 1$. The graph $H$ is constructed using two types of gadgets, $D$ and $I$ (illustrated in Fig. 2). The gadget $I$ is the primary gadget and the gadget $D$ is secondary gadget that is used to construct the gadget $I$.

Gadget of Type $D$. For two vertices $u$ and $v$, the gadget $D_{u,v}$ is an interval graph consisting of vertices $u$, $v$ and $n + 2$ additional vertices that form an independent set. The vertices $u$ and $v$ are adjacent, and both $u$ and $v$ are adjacent to every other vertex. We refer to the vertices $u$ and $v$ as heads of the gadget $D_{u,v}$. Intuitively, for any feasible solution $S$, and for any gadget $D_{u,v}$, either $u$ or $v$ should be in $S$. Otherwise, the remaining $n + 2$ vertices must be in $S$ which contradicts the optimality of $S$ because membership for both $u$ and $v$ is at least $n + 2$.

Observation 6 The pathwidth of the gadget $D$ is two. Indeed, it is an interval graph with maximum clique of size three and thus, by definition, has pathwidth 2.

Fig. 2 To the left is the type-$I$ gadget for $n = 4$ and to the right is the type-$D$ gadget. The zigzag edges between vertices $u$ and $v$ represent the gadget $D_{u,v}$.
Fig. 3 Illustration of a vertex block $H_i$ for some $i \in [k]$. An edge block $H_{i,j}$ for some $1 \leq i < j \leq k$ will have $|E_{i,j}|$-many internal gadgets

Gadget of type I. Let $n \geq 1$ be an integer. The gadget has two vertices $h_1$ and $h_2$, and two disjoint sets: $A = \{a_1, \ldots, a_n\}$ and $D = \{d_1, \ldots, d_n\}$. For each $i \in [n]$, vertices $a_i$ and $d_i$ are connected by the gadget $D_{a_i,d_i}$. Let $h_2$ and $h_1$ be two additional vertices which are adjacent. The vertices in the sets $A$ and $D$ are adjacent to $h_2$ and $h_1$, respectively. For each $1 \leq i \leq n$, $a_i$ and $h_1$ are connected by the gadget $D_{a_i,h_1}$, and $d_i$ and $h_2$ are connected by the gadget $D_{d_i,h_2}$. In the reduction a gadget of type $I$ is denoted by the symbol $I$ and an appropriate subscript.

Claim 7 The pathwidth of a gadget type $I$ is at most four.

Proof We observe that the removal of the vertices $h_1$ and $h_2$ results in a graph in which for each $i \in [n]$, there is a connected component consisting $a_i$ and $d_i$ which are the heads of a gadget of type $D$ and they are both adjacent to $n + 2$ vertices of degree 1. Each component is an interval graph with a triangle as the maximum clique and from Observation 6 is of pathwidth 2.

Let $(T', X')$ be the path decomposition of $I - \{h_1, h_2\}$ with width two. Thus adding $h_1$ and $h_2$ into all the bags of the path decomposition $(T', X')$ gives a path decomposition for the gadget $I$, and thus the pathwidth of the gadget $I$ is at most 4. $\square$

In the following parts, when we refer to a gadget, we mean the primary gadget $I$ unless the gadget $D$ is specified. For each vertex and edge in the given graph, our reduction has a corresponding gadget in the instance output by the reduction.

Description of the reduction For $1 \leq i < j \leq k$, let $E_{i,j}$ denote the set of edges with one end point in $V_i$ and the other in $V_j$, that is $E_{i,j} = \{xy \mid x \in V_i, y \in V_j\}$.

For each vertex and edge in $G$, the reduction uses a gadget of type $I$. For each $1 \leq i < j \leq k$, the graph $H$ has an induced subgraph $H_i$ corresponding to $V_i$, and has an induced subgraph $H_{i,j}$ for the edge set $E_{i,j}$. We refer to $H_i$ as a vertex-partition block and $H_{i,j}$ as an edge-partition block. Inside block $H_i$, there is a gadget of type $I$ for each vertex in $V_i$, and in the block $H_{i,j}$ is a gadget for each edge in $E_{i,j}$. For a vertex $u_{i,x}, I_{x}$ denotes the gadget corresponding to $u_{i,x}$ in the partition $V_i$, and for an edge $e$, $I_e$ denotes the gadget corresponding to $e$. Finally, the blocks are connected by the connector vertices which we describe below. We next define the structure of a block which we denote by $B$. The definition of the block applies to both the vertex-partition block and the edge-partition block.

A block $B$ consists of the following gadgets, additional vertices, and edges (Illustrated in Fig. 3).
The block $B$ corresponding to the vertex-partition block $H_i$ for any $i \in [k]$ is as follows: for each $\ell \in [n]$, add a gadget $I_\ell$ to the vertex-partition block $H_i$, to represent the vertex $u_{i,\ell} \in V_i$.

The block $B$ corresponding to the edge-partition gadget $H_{i,j}$ for any $1 \leq i < j \leq k$ is as follows: for each $e \in E_{i,j}$, add a gadget $I_e$ in the edge-partition block $H_{i,j}$, to represent the edge $e$.

In addition to the gadgets, we add $(n+1)(n+3)+2$ vertices to the block $B$ as follows: Let $C(B)$ denote the set $\{f, f', c_1, c_2, \ldots, c_{n+1}, b_1, b_2, \ldots, b_{(n+1)(n+2)}\}$, which is the set of additional vertices that are added to the block $B$. Let $C'(B)$ denote the subset $\{c_1, c_2, \ldots, c_{n+1}\}$. For each gadget $I$ in $B$, and for each $t \in [n]$, $a_t$ in $I$ is adjacent to $f$, and the vertex $f$ is adjacent to $f'$. Further, the vertex $f'$ is adjacent to each vertex $c_{p}$ for $p \in [n+1]$. Finally, for each $p \in [n+1]$ and $(p-1)(n+2) < q \leq p(n+2)$, $c_p$ is adjacent to $b_q$.

Next, we introduce the connector vertices to connect the edge-partition blocks and vertex-partition blocks. Let $R = \{r_{i,j}^t, s_{i,j}^t, s_{i,j}^j, r_{i,j}^1 \mid 1 \leq i < j \leq k\}$ be the connector vertices. The blocks are connected based on the following exclusive and exhaustive cases, and is illustrated in Fig. 4:

For each $i \in [k]$, each $i < j \leq k$ and each $\ell \in [n]$, the edges are described below.

- for each $1 \leq t \leq \ell$, the vertex $a_t$ in the gadget $I_\ell$ of $H_i$ is adjacent to the vertex $s_{i,j}^t$.
- for each $\ell \leq t \leq n$, the vertex $a_t$ in the gadget $I_\ell$ of $H_i$ is adjacent to the vertex $r_{i,j}^t$.

For each $i \in [k]$, each $1 \leq j < i$ and each $\ell \in [n]$,

- for each $1 \leq t \leq \ell$, the vertex $a_t$ in the gadget $I_\ell$ of $H_i$ is adjacent to the vertex $s_{j,i}^t$.
- for each $\ell \leq t \leq n$, the vertex $a_t$ in the gadget $I_\ell$ of $H_i$ is adjacent to the vertex $r_{j,i}^t$.

For each $1 \leq i < j \leq k$, and for each $e = u_{i,x}u_{j,y} \in E_{i,j}$,
• for each \(1 \leq t \leq x\), the vertex \(a_i\) in the gadget \(I_e\) of \(H_{i,j}\) is adjacent to the vertex \(r^i_{i,j}\)
• for each \(x \leq t \leq n\), the vertex \(a_i\) in the gadget \(I_e\) of \(H_{i,j}\) is adjacent to the vertex \(s^i_{i,j}\)
• for each \(1 \leq t \leq y\), the vertex \(a_i\) in the gadget \(I_e\) of \(H_{i,j}\) is adjacent to the vertex \(r^j_{i,j}\)
• for each \(y \leq t \leq n\), the vertex \(a_i\) in the gadget \(I_e\) of \(H_{i,j}\) is adjacent to the vertex \(s^j_{i,j}\)

This completes construction of the graph \(H\) with \(O(mn^2)\) vertices and \(O(mn^3)\) edges. We next bound the pathwidth of the graph \(H\) as a polynomial function of \(k\).

**Claim 8** The pathwidth of a block \(B\) is at most five.

**Proof** If we remove the vertex \(f\) from the block \(B\), then the resulting graph is a disjoint collection of gadgets and a tree of height two. We know that the pathwidth of a gadget is four from Claim 7, and the pathwidth of a tree of height two is two. Let \((T', X')\) be a path decomposition of \(B - \{f\}\) with pathwidth four. Thus adding \(f\) into all bags of \((T', X')\) gives a path decomposition for the block \(B\), and thus the pathwidth of the block is at most five.

\(\square\)

**Lemma 9** The pathwidth of the graph \(H\) is at most \(4\binom{k}{2} + 5\).

**Proof** Removal of the connector vertices from \(H\) results in a collection of disjoint blocks. From Claim 8, the pathwidth of a block is five. Let \((T', X')\) be a path decomposition of \(H - R\) with pathwidth five. Therefore, adding all connector vertices to the path decomposition \((T', X')\) gives a path decomposition for the graph \(H\) with pathwidth at most \(4\binom{k}{2} + 5\).

\(\square\)

### 4.1.1 Properties of a Feasible Solution for the Instance \((H, k')\) of the MMDS Problem

Let \(S\) be a feasible solution for the MMDS instance \((H, k')\). We state the following properties of the set \(S\). In all the arguments below, we crucially use the property that for each \(u \in V(H)\), \(M(u, S) \leq n + 1\).

**Claim 10** For each block \(B\) in the graph \(H\), \(C'(B) \subseteq S\).

**Proof** By construction of graph \(H\), for each \(1 \leq p \leq n + 1\), the vertex \(c_p\) must be in the set \(S\) since it has \(n + 2\) vertices of degree one as neighbors. Otherwise, its membership will be at least \(n + 2\), contradicting that \(S\) is a feasible solution for \((H, k')\). Hence the claim.

\(\square\)

**Claim 11** For each block \(B\) in \(H\), the vertices \(f\) and \(f'\) in \(B\) are not in the set \(S\).

**Proof** We know that \(f\) is made adjacent to \(f'\), and \(f'\) is adjacent to each vertex in \(C'(B)\). From Claim 10, we know that \(C'(B)\) is a subset of \(S\). Thus, \(n + 1\) neighbors of \(f'\) is in \(S\). If either \(f\) or \(f'\) is in the set \(S\), then \(M(f', S)\) is \(n + 2\). This contradicts the feasibility of the set \(S\). Hence the claim.

\(\square\)
Claim 12 For each gadget of type I in each block B in the graph H, either \( A \cap S = A \) or \( A \cap S = \emptyset \).

Proof We prove this by contradiction. Assume that \( \emptyset \subsetneq A \cap S \subsetneq A \). Let \( J = \{ j \in [n] \mid a_j \in S \} \), that is \( J \) is the index of the elements in \( A \cap S \). Note that by our premise \( J \) is non-empty and it is not all of \([n]\). Since \( J \) is a strict subset of \([n]\), we observe that the vertex \( h_1 \) is in \( S \). This is because, for each \( i \in [n]\setminus J \), \( h_1 \) and \( a_i \) is connected by the gadget of type \( D \). If both \( a_i \) and \( h_1 \) are not in \( S \), then the \( n+2 \) neighbours in the gadget of type \( D \) containing the edge \( \{a_i, h_1\} \) will be in \( S \), and thus \( M(a_i, S) \) and \( M(h_1, S) \) are both at least \( n+2 \). This violates the hypothesis that for each \( u \in V(H) \), \( M(u, S) \leq n+1 \). We now consider two cases, one in which the vertex \( h_2 \) is in \( S \) and the other in which \( h_2 \) is not in \( S \).

First, we consider \( h_2 \in S \). For each \( i \in [n] \), by using the same argument which we used for \( a_i \) and \( h_1 \), it follows that at least one of the \( a_i \) or \( d_i \) is in the set \( S \) since \( a_i \) and \( d_i \) are both in a gadget of type \( D \). Therefore, for each \( i \in [n]\setminus J \), the vertex \( d_i \) is in \( S \). That is \( |D \cap S| \geq n - |J| \). Consequently, using the fact that \( h_1 \in S \) and the premise that \( h_2 \in S \), it follows that the membership of \( h_1 \) is

\[
M(h_1, S) \geq |A \cap S| + |D \cap S| + 2 \geq |J| + n - |J| + 2 \geq n + 2.
\]

This contradicts the feasibility of \( S \).

Next we consider the case that \( h_2 \) is not in \( S \). For each \( i \in [n] \), \( d_i \) is in \( S \) since \( d_i \) and \( h_2 \) are in a gadget of type \( D \). Then, the \( N[h_1] = (A \cap S) \cup D \cup \{h_1\} \). Further, we know that \( J \) is a non-empty set, and thus, the membership of \( h_1 \) is

\[
M(h_1, S) \geq |A \cap S| + |D| + 1 \geq |J| + n + 1 \geq n + 2.
\]

Therefore, our assumption that that \( A \subsetneq S \) and \( A \cap S \neq \emptyset \) is wrong. Therefore, either the set \( A \) is completely included in the set \( S \) or completely excluded from the set \( S \). □

Claim 13 For each block \( B \) in the graph \( H \), there exists a unique gadget of type I in the block \( B \) such that the set \( A \) in the gadget is in \( S \).

Proof The vertices \( f \) and \( f' \) in \( B \) are not in the solution \( S \) due to Claim 11. The Claim 12 states that either the set \( A \) in any gadget is completely included in the set \( S \) or completely excluded in the set \( S \). If for each gadget in \( B \), the set \( A \) is not in \( S \) then the vertex \( f \) is not dominated by \( S \). This contradicts the feasibility of \( S \). If the set \( A \) of at least two gadgets in the block \( B \) are in the set \( S \), then the membership of \( f \) will be \( 2n > n + 1 \). This contradicts the feasibility of \( S \). Thus, there exists an unique gadget \( I \) in each block such that the set \( A \) in \( I \) is in \( S \). □

Using these properties in the following two lemmas, we prove the correctness of the reduction.

Lemma 14 If \((G, k)\) is a YES-instance of the **Multi-Colored Clique** problem, then \((H, k')\) is a YES-instance of the **MMDS** problem.
Proof Let $K = \{u_i, x_i \mid i \in [k]\}$ be a $k$-clique in $G$. That is, for each $i \in [k]$, $x_i$-th vertex of the partition $V_i$ is in the clique. Now we construct a feasible solution $S$ for the instance $(H, k')$ of the MMDS problem. The set $S$ consists of the following vertices. For each $i \in [k]$,

- for each $\ell \in [n]$ with $\ell \neq x_i$, add $D \cup \{h_1\}$ in the gadget $I_\ell$ in the vertex-partition block $H_i$ to $S$, and
- in the gadget $I_{x_i}$ of the vertex-partition block $H_i$, add $A \cup \{h_2\}$ to $S$, and
- add $C'(H_i)$ to $S$.

For each $1 \leq i < j \leq k$,

- for each edge $e \in E_{i, j}$ with $e \neq u_i, x_i, u_j, x_j$, add $D \cup \{h_1\}$ in the gadget $I_e$ the edge-partition block $H_{i, j}$ to $S$, and
- for the edge $e = u_i, x_i, u_j, x_j$, add $A \cup \{h_2\}$ in the gadget $I_e$ the edge-partition block $H_{i, j}$ to $S$, and
- add $C'(H_{i, j})$ to $S$.

We show that $S$ is a feasible solution to the MMDS problem in $H$ for membership value $k' = n + 1$.

First we show that the set $S$ is a dominating set in $H$. In each gadget in each block, we have added either $D \cup \{h_1\}$ or $A \cup \{h_2\}$ into $S$. Therefore, in every gadget of type $D$ at least one head is in $S$. That is, $S$ dominates every vertex which is part of some gadget of type $D$. Since every vertex in a gadget of type $I$ is part of some gadget of type $D$, the gadget of type $I$ is dominated by $S$. Thus, every gadget of type $I$ is dominated by $S$.

Then we consider the vertices outside any gadget of type $I$. In any block $B$, this is the set $C(B)$. In each block $B$ in $H$, from Claim 13, vertices in the set $A$ of exactly one gadget of type $I$ is in $S$. Each of these vertices dominate $f$. All other vertices in the block which are outside the gadgets are dominated by $C'(B)$ which is a subset of $S$ by definition. For $1 \leq i < j \leq k$, consider a connector vertex pair $(s_{i, j}, r_{i, j})$ that connects the blocks $H_i$ and $H_{i, j}$. Both vertices were made adjacent to the vertices in the set $A$ of each gadget in the block $H_i$. Since $S$ contains the set $A$ in the gadget $I_{x_i}$ of $H_i$, both connector vertices are dominated. Thus all the connector vertices are dominated by $S$. Therefore, $S$ is a dominating set of $H$.

Next we show that the membership of any vertex $u \in V(H)$ in $S$ is $k'$, that is, we show that $M(u, S) = n + 1$. Observe that the vertices in any gadget of type $I$ are solely dominated by the vertices of $S$ which are inside the gadget. The maximum membership of $n + 1$ is achieved by the vertices $h_1$ and $h_2$ for any gadget $I$. Therefore, the membership of any vertex in a gadget of type $I$ is $n + 1$. In each block $B$, among the vertices $C(B)$, the maximum membership of value $n + 1$ is achieved by the vertices $f$ and $f'$.

We next show crucially that the membership of the connector vertices is at most $n + 1$. For each $1 \leq i < j \leq k$, consider the edge $e = u_i, x_i, u_j, x_j \in E_{i, j}$. By construction of the set $S$, we picked the set $A$ only from the gadgets $I_e$ in $H_{i, j}$, $I_{x_i}$ from $H_i$, and $I_{x_j}$ from $H_j$. From the reduction and the definition of $S$, it is clear that for all the other gadgets in a block, the vertices in $S$ are not adjacent to the connector vertices. Therefore, the $M(s_{i, j}, S)$ is $x_i + (n - x_i + 1) = n + 1$, and
$M(r_{i,j}^i, S)$ is $(n - x_i + 1) + x_i = n + 1$. Next, we consider the membership of $s_{i,j}^j = x_j + (n - x_j + 1) = n + 1$ and the membership of $r_{i,j}^j$ is $x_j + (n - x_j + 1) = n + 1$. These membership values can be seen clearly from Fig. 4. Hence, the membership of any vertex in $V(H)$ is $n + 1$. Thus, the instance $(H, k')$ is a YES-instance of the MMDS problem. □

**Lemma 15** If $(H, k')$ is a YES-instance of the MMDS problem, then $(G, k)$ is a YES-instance of the Multi-Colored Clique problem.

**Proof** Let $S$ be a feasible solution to the instance $(H, k')$ of the MMDS problem. For each $i \in [k]$, let $I_{x_i}$ be the unique gadget for some $x_i \in [n]$, where the set $A$ of $I_{x_i}$ is in $S$. For each $1 \leq i < j \leq k$, let $I_e$ be the unique gadget for some $e = u_i, x_i' u_j, x_j' \in E_{i,j}$, where the set $A$ of $I_e$ is in $S$. The existence of such gadgets is ensured by Claim 13. Let $K = \{u_i, x_i | i \in [k]\}$. We show that the set $K$ is a clique in $G$ as follows. Observe that we picked one vertex from each partition $V_i$ for $i \in [k]$. Next we show that for each $1 \leq i < j \leq k$, there is an edge $u_i, x_i u_j, x_j \in E(G)$. Let $i, j \in [k]$ such that $i < j$. The vertex $s_{i,j}^i$ is adjacent to $x_j$ vertices in $I_{x_j}$ from $H_i$, and $n - x_j' + 1$ vertices in $I_e$ from $H_i, j$. The vertex $r_{i,j}^i$ is adjacent to $n - x_i + 1$ vertices in $I_{x_i}$ from $H_i$, and $x_j'$ vertices in $I_e$ from $H_i, j$. Then, the membership of the connector vertices $r_{i,j}^i$ and $s_{i,j}^i$ in $S$ are

$$M(r_{i,j}^i, S) \geq (n - x_i + 1) + (x_j') \geq n + x_j' - x_i + 1,$$

$$M(s_{i,j}^i, S) \geq x_i + (n - x_j' + 1) \geq n + x_i - x_j' + 1.$$ 

Further, the membership of the vertices is at least one and at most $n + 1$, that is $1 \leq M(r_{i,j}^i, S), M(s_{i,j}^i, S) \leq n + 1$. Therefore, $n + 1 \geq n + x_j' - x_i + 1 \Rightarrow x_i \geq x_j'$ and $n + 1 \geq n + x_i - x'_j + 1 \Rightarrow x'_j \geq x_i$. Thus, we have $x_i = x_j'$. Similarly, we will get $x_j = x_j'$. Therefore, by construction of the graph $H$, there is an edge $u_i, x_i u_j, x_j \in E(G)$. Thus, the set $K$ is a feasible solution for the instance $(G, k)$ of the Multi-Colored Clique problem. □

Thus, we conclude the section with the proof of Theorem 2.

**Proof of Theorem 2** On an instance $(G, k)$ of Multi-Colored Clique the reduction constructs $(H, k' = n + 1)$ in polynomial time. From Lemma 9 we know that the pathwidth of $H$ is a quadratic function of $k$. Finally, from Lemma 14 and Lemma 15 it follows that the MMDS instance $(H, k')$ output by the reduction is equivalent to the Multi-Colored Clique instance $(G, k)$ that was input to the reduction. Since Multi-Colored Clique is known to be W[1]-hard for the parameter $k$, it follows that the MMDS problem is W[1]-hard with respect to the parameter pathwidth of the input graph.
5 \textit{W[2]}-Hardness Parameterized by Membership on Split Graphs

In this section, we prove that the MMDS problem is \textit{W[2]}-hard on split graphs when parameterized by the membership parameter $k$ by a reduction from the \textsc{Multi-Colored Set Cover} (MCSC) problem. The MCSC problem is defined as follows:

**\textsc{Multi-Colored Set Cover} (MCSC)**

\begin{itemize}
  \item **Input:** A set of elements $U$ and $k$ families $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_k)$ of subsets of $U$.
  \item **Parameter:** $k$
  \item **Question:** Does there exist $k$ sets $S_1, \ldots, S_k$ such that $\bigcup_{i=1}^{k} S_i = U$ and for $i \in [k]$, $S_i \in \mathcal{F}_i$?
\end{itemize}

The proof is by demonstrating a parameterized reduction from the MCSC problem to \textsc{Minimum Membership Dominating Set}. \textsc{Multi-Colored Set Cover} problem is shown to be \textit{W[2]}-hard for the parameter solution size in [8].

Given an instance $\langle U, \mathcal{F}, k \rangle$ of the MCSC problem, we define an equivalent instance $\langle H, k \rangle$ of the MMDS problem where $H$ is a split graph with a clique $C$ and an independent set $I$. We construct the split graph $H$ as follows.

- For $i \in [k]$, we add $|\mathcal{F}_i|$ new vertices, denoted by $V_i = \{c_F | F \in \mathcal{F}_i\}$. Let $C = V_1 \cup \cdots \cup V_k$ be the clique in $H$.
- For $i \in [k]$, we add $k + 1$ new vertices, denoted by $I_i$. We add $n$ new vertices, denoted by $D = \{x_i | i \in [n]\}$. Let $I = I_1 \cup \cdots \cup I_k \cup D$ be the independent set in $H$.
- Since $C$ is a clique, we add edges between every pair of vertices in $C$. For any two vertices $u \in C$ and $v \in I$, there is an edge $uv \in E$ if one of the following is true:
  \begin{itemize}
    \item $u = c_F$ for some $F \in \mathcal{F}_i$ and $v \in I_i$, or
    \item $v = c_F$ for some $F \in \mathcal{F}_i$, $v = x_j \in D$ for some $j \in [n]$, and $a_j \in F$
  \end{itemize}

The above construction is depicted in Fig. 5.

**Lemma 16** If $\langle U, \mathcal{F}, k \rangle$ is a YES-instance of the MCSC problem, then $\langle H, k \rangle$ is a YES-instance of the MMDS problem.

**Proof** Let $S = \{\mathcal{F}_1, \ldots, \mathcal{F}_k\}$ be a feasible solution to the input instance of the MCSC problem. We define a set $S' \subseteq V$ as follows:

$$S' = \{c_{F_i} | F_i \in S\}$$

We show that the set $S'$ is a feasible solution for the instance $\langle H, k \rangle$ of the MMDS problem. First we show that the set $S'$ is a dominating set. Since $C \cap S' \neq \emptyset$, $S'$ dominates $C$. For each $i \in [k]$, $c_{F_i} \in S'$ dominates $I_i$. Since $S$ is a set cover, for each $j \in [n]$, there exists an $i \in [k]$ such that $x_j \in F_i$. Similarly, for each $j \in [n]$, there exists an $i \in [k]$ such that $x_j$ is dominated by $C_{F_i}$. Therefore, $S'$ dominates $V$. 
Observe that \( \max_{u \in V} M(u, S') \leq k \) since \( S' = k \). Hence, \( S' \) is a feasible solution for the instance \( \langle H, k \rangle \) of the MMDS problem, and thus the instance is a YES-instance.

**Lemma 17** If \( \langle H, k \rangle \) is a YES-instance of the MMDS problem then \( \langle U, F, k \rangle \) is a YES-instance of the MCSC problem.

**Proof** Let \( S \) be a feasible solution for the MMDS problem on the graph \( H \) with membership value \( k \). Since \( S \) is a \( k \) membership dominating set of \( H \), \( S \) exhibits the following properties.

1. \( |S \cap V_i| = 1, i \in [1, k] \). At least one vertex from each of the set \( V_i \) must belong to \( S \). Otherwise in order to dominate \( I_i \), all \( k + 1 \) vertices from \( I_i \) need to be included in \( S \), thus violating the membership constraint for the vertices in \( V_i \). Observe that \( S \) contains exactly one vertex from each set \( V_i, i \in [k] \). If there exists an \( i \in [k] \) such that more than one vertex from a \( V_i \) is included in the solution, then \( |C \cap S| > k \) and the membership constraint of vertices in \( C \) will be violated. Therefore, exactly one vertex from each \( V_i \) is included in \( S \).

2. \( |I_i \cap S| = 0, i \in [1, k] \). Every vertex in \( I_i, i \in [1, k] \) is already dominated by a vertex in the corresponding \( V_i \), and has membership value \( 1 \). If a vertex from \( I_i \) for some \( i \in [1, k] \) is included in \( S \), all vertices in \( V_i \) will have membership \( k + 1 \) leading to a violation of the membership constraint.

3. \( |S \cap D| = 0 \). For each \( j \in [n] \), the vertex \( x_j \) is adjacent to some vertex in \( C \). As \( C \cap S = k \), adding any vertex \( x_j \) to \( S \) will violate the membership constraint of vertex adjacent to \( x_j \) in \( C \).

Therefore, the set \( S \) is a subset of \( C \). We define a set \( S' \) as follows:

\[
S' = \{ F \mid c_F \in S \}
\]

We claim that the set \( S' \) is a feasible solution for the instance \( \langle U, F, k \rangle \) of the MCSC problem. Observe that for each \( i \in [k] \), \( |S' \cap F_i| = 1 \) since \( |S \cap V_i| = 1 \). Next, we
show that $S'$ is a set cover. For each $j \in [n]$, there exists an $i \in [k]$ such that $x_j$ is dominated by $c_{F_i}$. Similarly, the element $a_j$ will be covered by the set $F_i$. Therefore, the set $S$ is a feasible solution for the instance $\langle U, F, k \rangle$ of the MCSC problem, and thus the instance is a YES-instance.

Proof of Theorem 3 From Lemma 16 and Lemma 17, it follows that the MULTI-COLORED SET COVER instance $\langle U, F, k \rangle$ is equivalent to the MMDS instance $\langle H, k \rangle$. Since MULTI-COLORED SET COVER is known to be $\text{W}[2]$-hard for parameter $k$, it follows that the MMDS problem is $\text{W}[2]$-hard when parameterized by the membership.

6 Vertex Cover Parameterization

First, we show that the MMDS problem is FPT parameterized by vertex cover number $\text{vc}$. We then show that the MMDS problem does not have a subexponential algorithm in the size of vertex cover conditioned on the famous Exponential Time Hypothesis (ETH).

6.1 The MMDS Problem is FPT Parameterized by Vertex Cover

In order to design an FPT algorithm parameterized by the size of a vertex cover of the input graph, we construct an FPT-time Turing reduction from the MMDS problem to Integer Linear Programming. In the reduced instance the number of constraints is at most twice the size of a minimum vertex cover. We then use the recent result by Dvořák et al. [31] which proves that ILP is FPT parameterized by the number of constraints. The following theorem directly follows from Corollary 9 of [31].

**Theorem 18** [Corollary 9, [31]] ILP is FPT in the number of constraints and the maximum number of bits for one entry.

**FPT time Turing reduction from the MMDS problem to ILP:** Let $(G, k)$ be the input instance of the MMDS problem. Compute a minimum vertex cover of $G$, denoted by $C$, in time FPT in $|C|$ [27]. Let $I$ denote the maximum independent set $V \setminus C$. The following lemma is crucial to the correctness of the reduction.

**Lemma 19** Let $D$ be a $k$ membership dominating set of $G$. Let $C_1 = D \cap C$, $I_1 = I \setminus (N(C_1) \cap I)$, and $R = N(C_1) \cap I \cap D$. Then, $I_1 \subseteq D$, and $C \setminus (N[C_1] \cup N(I_1))$ is dominated by $R$. 

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**Proof** The outline is that $I_1$ cannot be dominated by any other vertex other than by itself. Further, $R \subseteq D$ is the only vertices which can dominate $C \setminus (N[C_1] \cup N(I_1))$. Hence the lemma.

As a consequence of this lemma, it is clear that the choice of $C_1$ immediately fixes $I_1$. Thus, to compute the set $D$, the task is to compute $R$. We pose this problem as the *constrained* MMDS problem. A CMMDS problem instance is a 4-tuple $(G, k, C, C_1)$ where $C$ is a vertex cover and $C_1$ is a subset of $C$. The decision question is whether there is a $k$ membership dominating set $D$ of $G$ such that $D \cap C = C_1$. From Lemma 19, given an instance of $(G, k, C, C_1)$, we know that $C_1$ immediately fixes $I_1 \subseteq I = V \setminus C$. Thus, to compute $D$, we need to compute $R$ as defined in Lemma 19. We now describe the ILP formulation to compute $R$ once $C_1$ (and thus $I_1$) is fixed. Since $R$ is a subset of $I \setminus I_1$, it follows that the variables correspond to vertices in $I \setminus I_1$ which do not already have $k$ neighbors in $C_1$; we use $I_e$ to denote this set. It can be immediately checked if $C_1 \cup I_1$ can be part of a feasible solution—we check that for no vertex is the intersection of its closed neighborhood greater than $k$. We now assume that this is the case, and specify the linear constraints. The linear constraints in the ILP are associated with the vertices in $C$. For each vertex in $C$ there are at most two constraints—if $v$ is in $C \setminus (N[C_1] \cup N(I_1))$, then at least one neighbor and at most $k$ neighbors from $I_e$ must be chosen into $R$. On the other hand, for $v \in (N[C_1] \cap C) \cup N(I_1)$, we have the constraint that at most $k$ neighbors must be in $C_1 \cup I_1 \cup R$. The choice of variables in $I_e$ does not affect any other vertex in $I$, and thus there are no constraints among the vertices in $I$. To avoid notation, we assume that an instance of CMMDS $(G, k, C, C_1)$, also denotes the ILP.

**Lemma 20** The CMMDS problem on an instance $(G, k, C, C_1)$ can be solved in time which is FPT in the size of the vertex cover.

**Proof** Since the instance $(G, k, C, C_1)$ uniquely specifies the ILP for the choice of $R$, it follows that this ILP has $O(|C|)$ constraints. From Theorem 18, we know that the ILP can be specified in FPT time with $|C|$ as the parameter, and this proves the Lemma.

**Proof of Theorem 4** Given an input instance $(G, k)$ of the MMDS problem, we first compute a minimum vertex cover $C$ in FPT time (in size of the cover as the parameter) using any of the well-known methods (see the book by Cygan et al. [27], for example). Let $I = V \setminus C$ be the independent set. Now we iterate through each subset $C_1$ of $C$, and check if it can be extended to a $k$ membership dominating set $D$ such that $D \cap C = C_1$. For each such $C_1$, we know from Lemma 19 that $I_1 = I \setminus (N(C_1) \cap I)$ must be added to the solution, if one exists. For each subset $C_1 \subseteq C$, we assume that $C_1 \cup I_1$ goes into the solution set. Then, we solve CMMDS on the instance $(G, k, C, C_1)$ to check if there is an $R \subseteq I_e$ such that $C_1 \cup I_1 \cup R$ in a $k$ membership dominating set. From Lemma 20, we know that this check can be solved in FPT time. It thus follows that the MMDS problem is FPT when parameterized by the size of vertex cover.
6.2 Lower Bound for the Vertex Cover Parameterization

We show that there is no sub-exponential-time parameterized algorithm for the MMDS problem when the parameter is the vertex cover number, using a reduction from 3-SAT. By the ETH, we know that 3-SAT does not have a sub-exponential-time algorithm, and thus the reduction proves the lower bound for the MMDS problem when parameterized by vertex cover number.

Reduction

Let \( \phi \) be a 3-CNF boolean formula on \( n \) variables \( X = \{x_1, x_2, \ldots, x_n\} \) having \( m \) clauses \( C = \{C_1, C_2, \ldots, C_m\} \). Given an instance \( \langle \phi \rangle \) of the 3-SAT problem, we define an instance \( \langle G, k \rangle \) of the MMDS problem such that the minimum vertex cover of \( G \) has size \( O(n) \).

To construct the graph \( G \), first, we define the variable gadgets. Let \( k \geq 3 \) be a constant. For \( i \in [n] \), we construct a variable gadget \( H_i \) as follows. We add two vertices \( v_{x_i} \) and \( v_{\neg x_i} \) to represent the corresponding literals, and connect them by an edge in \( H_i \). Then, we add \( k - 1 \) vertices \( b_1^i, \ldots, b_{k-1}^i \), and each vertex is connected to \( k + 1 \) pendent vertices. Further, for \( j \in [k-1] \), \( b_j^i \) is connected to both \( v_{x_i} \) and \( v_{\neg x_i} \). Finally, we add \( k + 1 \) vertices \( a_1^i, \ldots, a_{k+1}^i \) and each vertex is connected to both \( v_{x_i} \) and \( v_{\neg x_i} \). An illustration of a variable gadget is shown in Fig. 6.

The graph \( G \) is constructed as follows: We add \( n \) variable gadgets \( H_1, \ldots, H_n \) in \( G \). We add \( m \) vertices \( v_{C_1}, \ldots, v_{C_m} \) to represent the clauses in \( \phi \). For each clause \( C \) in \( \phi \) and for each literal \( x \) in \( C \), we connect the vertices \( v_C \) and \( v_x \) by an edge. Then, we add a vertex \( Y \), and it is connected to \( v_C \) for each clause \( C \) in \( \phi \). Finally, we add \( k \) vertices \( u_1, \ldots, u_k \), and each vertex is connected to \( k + 1 \) pendent vertices. For \( i \in [k] \), \( u_i \) is connected to the vertex \( Y \). This completes the construction of the graph \( G \). The construction of graph \( G \) is illustrated in Fig. 7.

Claim 21 The vertex cover number of the graph \( G \) is \( (n + 1)(k + 1) \).
Fig. 7 Construction of the graph $G$ from a 3-SAT instance

**Proof** Consider the vertex subset

$$T = \{ Y \} \cup \bigcup_{i=1}^{n} \{ v_{x_i}, v_{\overline{x_i}}, b_1^i, \ldots, b_{k-1}^i \} \cup \bigcup_{i=1}^{k} \{ u_i \}.$$  

Note that $|T| = 1 + n(k + 1) + k = (n + 1)(k + 1)$. Observe that $G - T$ is an empty graph (edge-less graph). That is, $V \setminus T$ is an independent set in $G$. Therefore, the minimum vertex cover size should be at most the size of the set $T$. Thus, $\text{vc}(G) \leq |T| = (n + 1)(k + 1)$.  

**Lemma 22** If $\phi$ has a satisfying assignment then $G$ has a dominating set with membership value $k$.

**Proof** Let $A : X \to \{0, 1\}$ be a satisfying assignment for $\phi$. Now, we construct a feasible solution $S$ for the MMDS problem as follows.

- For each $i \in [n],$
  - add $v_{x_i}$ if $A(x_i) = 1$ or $v_{\overline{x_i}}$ if $A(x_i) = 0$ to $S$, and
  - for $j \in [k-1]$, add $b_j^i$ to $S$.
- For $i \in [k]$, add $u_i$ to $S$.

We claim that $S$ is a dominating set for $G$ and has membership at most $k$. First, we show that $S$ is a dominating set. For each $i \in [n]$, either $v_{x_i}$ or $v_{\overline{x_i}}$ is in $S$, and $b_1^i, \ldots, b_{k-1}^i$ are in $S$. Therefore, $S$ dominates all the vertices in the variable gadget $H_i$. The vertices $u_1, \ldots, u_k$ in $S$ dominates all its pendent vertices and the vertex $Y$. Since $A$ is a satisfying assignment, for each $i \in [m]$, the clause $C_i$ has a satisfying literal $x$ where $x$ is either $x_i$ with $A(x_i) = 1$ or $\overline{x_i}$ with $A(x_i) = 0$ for some $i \in [n]$.  

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Therefore, the vertex \( v_{C_i} \) is dominated by the vertex \( v_x \) in \( S \). Thus, the set \( S \) is a dominating set in \( S \).

Next, we show that the membership of any vertex in \( G \) is at most \( k \). In each variable gadget, maximum membership of \( k \) is attained by the vertices \( v_{x_i} \) and \( v_{\overline{x_i}} \). Each clause vertex has membership at most \( 3 \leq k \). The vertex \( Y \) has the maximum membership of \( k \) due to \( u_1, \ldots, u_k \) in \( S \). Therefore, \( S \) is a feasible solution for the instance \( \langle G, k \rangle \) of the MMDS problem.

**Lemma 23** If \( G \) has a dominating set with membership value \( k \), then \( \phi \) has a satisfying assignment.

**Proof** Let \( S \) be feasible solution for MMDS in graph \( G \). Then \( S \) has the following properties:

- If a vertex has more than \( k \) pendent neighbors, then the vertex should be in \( S \). Otherwise, the inclusion of its pendent neighbors violates the membership of the vertex. This ensures that

  \[
  \bigcup_{i=1}^{n} \{b_1^i, \ldots, b_k^i\} \cup \bigcup_{i=1}^{k} \{u_i\} \subseteq S.
  \]

- For each \( i \in [n] \), either \( v_{x_i} \) or \( v_{\overline{x_i}} \) must be in \( S \) to dominate \( a_1^i, \ldots, a_{k+1}^i \). Note that both \( v_{x_i} \) and \( v_{\overline{x_i}} \) together cannot be there in \( S \) since it violates the membership property of both vertices.

- For \( i \in [m], \{v_{C_i} \notin S \) since inclusion of \( v_{C_i} \) makes \( M(Y, S) = k \).

- For each \( i \in [m] \), there exists a literal \( x \) in \( C_i \) such that \( v_x \) is in \( S \). Otherwise, \( C_i \) left to be not dominated.

Now we define a satisfying assignment \( A \) as follows: for each \( i \in [n] \),

\[
A(x_i) = \begin{cases} 
1 & \text{if } v_{x_i} \in S \\
0 & \text{otherwise.}
\end{cases}
\]

It follows from the above properties that \( A \) is a satisfying assignment for \( \phi \). Therefore, \( \phi \) has a satisfying assignment. \( \square \)

**Proof of Theorem 5** From Lemma 22 and Lemma 23, it follows that the 3-SAT can be reduced to the MMDS problem parameterized by vertex cover number. Therefore, a \( 2^{o(\text{vc}(G))} n^{O(1)} \)-time algorithm for the MMDS problem implies a \( 2^{o(n)} \)-time algorithm for the 3-SAT which violates the ETH. Hence, it is proved that there is no sub-exponential-time algorithm for the MMDS problem when parameterized by vertex cover number.

**7 Conclusion**

In this paper, we studied the parameterized complexity of the **minimum membership dominating set** problem, which requires finding a dominating set such that each
vertex in the graph is dominated minimum possible times. We started our analysis by showing that in spite of having no constraints on the size of the solution, unlike DOMINATING SET, MMDS turns out to be \( \mathcal{W}[1] \)-hard when parameterized by path-width (and hence treewidth). We further showed that the problem is \( \mathcal{W}[2] \)-hard for split graphs when the parameter is the size of the membership. For general graphs we prove that MMDS is FPT when parameterized by the size of vertex cover. Finally, we showed that assuming ETH, the problem does not admit a sub-exponential algorithm when parameterized by the size of vertex cover, thus showing our FPT algorithm to be optimal. There are many related open problems that are yet to be explored. One such problem is analyzing the complexity of the MMDS problem in chordal graphs. Other directions involve structural parameterizations of the MMDS problem with respect to other parameters such as maximum degree, distance to bounded degree graphs, bounded genus and maximum number of leaves in a spanning tree.

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