On the scattering Aharonov–Bohm effect

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Abstract
In this paper we review some aspects of the scattering Aharonov–Bohm effect and Berry’s phase. Specifically, the problem of scattering of free 2d electrons on the system of an arbitrary number of parallel, infinitely thin and infinitely long coils (non-interacting, point magnetic fluxes) is modeled as free semi-classical particle propagation on a topologically non-trivial background. We show that in this case it is possible to obtain Berry’s phase by integration. First the case of a single coil is analyzed, upon which a particular solution for the case of scattering on an arbitrary number of coils is found. Considering the solution in the momentum representation, an analytic continuation of the angle coordinate in the momentum space is introduced, in accord with the non-trivial geometry of the configuration space. Finally, some simple experiments with possible applications for quantum computing are proposed.

Keywords: Aharonov–Bohm effect, Berry phase, multiple solenoid system

(Some figures may appear in colour only in the online journal)

1. Introduction
In their seminal paper, Aharonov and Bohm [1] solved the problem of scattering of free electrons on infinitely thin and infinitely long coil carrying constant current, producing a field-free vector potential outside the coil. The solution wave-function in coordinate space in terms of Bessel functions of generally non-integer (and non-half-integer) order suffers from non-single-valuedness at the branch line.

There exist theoretical explanations for this effect, most notably in terms of Berry’s phase (holonomies) [2, 3], anyonic nature of such systems [4], as well as through gauge and other symmetries [5]. Notably, the physical meaning of the phase of a wave-function was discussed already by Born and Fock [6] and Dirac [7] at the very beginnings of the (new) quantum theory. It is also interesting that meanwhile the analyses of geometrical phases found application in a variety of very different physical contexts, from anomalies in field theory to Hall’s effect in solid state physics [5, 8, 9], with high expectations for application in future quantum computing [10].

However, the relations between different approaches, and different contexts in which the effect appears are not completely understood (see review in [8]), the behavior of phase for non-cyclic evolutions is barely discussed, and the case of $n \geq 2$ parallel coils is not yet solved [11, 12]. We also note that the analysis of the problem in the momentum representation, apart from few exceptions [11, 13] is almost absent as opposed to coordinate representation.

Here we study the interaction of electrons with field-free vector potential as a propagation of free particles on the topologically non-trivial background configuration space, for which cotangent spaces at different points can not be identified. We show that motion on this background produces a phase that is single valued and integrable, the familiar discontinuity emerging upon the projection of the solution on real Euclidean space. Furthermore, we show that instead of variation of an external parameter of the Hamiltonian, one can also arrive at Berry’s phase by considering variation of an internal parameter, that is the cotangent space of the configurational space, as the particles propagate over it (compare to [14]).

In section 2 we lay down the procedure for obtaining Berry’s phase on example of plane-waves scattering on a single coil, reproducing also the general solution of Aharonov and Bohm [1], and analyzing the case of multiple coils. Section 3 studies the solution in the momentum representation, showing the phase does not depend on the...
representation. In section 4 some simple experimental proposals to test our findings, as well as their possible practical applications in quantum computing, are shortly discussed.

2. Configuration space

Consider the action of two infinitesimal displacements operators \( \nabla \) and \( \nabla_\alpha \), with eigenvalues \( P \) and \( p \)

\[
\nabla \psi(r; P) = P \psi(r; P),
\]

(1)

\[
\nabla_\alpha \psi(r; p) = p \psi(r; p).
\]

(2)

In particular, consider the case where the spectra of the squares of the two operators are the same

\[
-\nabla^2 \psi(r; P) = 2mE \psi(r; P),
\]

(3)

\[
-\nabla^2_\alpha \psi(r; p) = 2mE \psi(r; p).
\]

(4)

It is obvious that the operators \( \nabla \) and \( \nabla_\alpha \) can not act on the same vector space, otherwise their eigenvalues would be identical. We take operator \( \nabla \) to be the standard infinitesimal displacement operator acting on vectors on Euclidean space, \( \psi(r; P) \in L^2(\mathbb{R}^3) \). The operator \( \nabla_\alpha \), a covariant momentum, is a global infinitesimal displacement operator acting on vectors defined on non-trivial background \( \psi(r; p) \in L^2(\mathbb{R}^3) \) for a single coil, and belonging to the cotangent space at the point \( \nabla \) where the wave-function identically vanishes. On this background, operator \( \nabla \) is defined as the element of the position dependent cotangent space, and is given as a combination of displacement and multiplication operators

\[
\nabla(r) = \nabla_\alpha - ieA(r),
\]

(5)

where \( A(r) \) is the introduced vector potential whose rotation vanishes, in this paper considered to be

\[
eA(r) = \sum_i \frac{\alpha_i}{|r - a_i|} \hat{\mathbf{z}} \times (r - a_i),
\]

(6)

with \( \alpha_i \) real constants equal to the strengths of point magnetic fluxes at positions \( r = a_i \). It is the displacement operator for vectors defined on flat space constructed around some point \( r \), or on the projection of non-trivial background onto Euclidean space, i.e.

\[
(-i\nabla + eA(r))^2 \psi^{pr}(p; r) = 2mE\psi^{pr}(p; r)
\]

(7)

where \( \psi^{pr}(r; p) \in L^2(\mathbb{R}^3 \setminus \text{cuts}) \).

Non-standard geometry of the configuration space is also seen in the algebra of the operators. While the Heisenberg algebra for the operators \( \nabla \) and \( r \) is a standard one, i.e.

\[
[r_i, r_j] = 0, \quad [r_i, P_j] = i\delta_{ij}, \quad [P_i, P_j] = 0,
\]

(8)

the algebra of operators \( \nabla_\alpha \) and \( r \) gets deformed

\[
[r_i, r_j] = 0, \quad [r_i, P_j] = i\delta_{ij}, \quad [P_i, P_j] = 0.
\]

(9)

Noncommutativity of infinitesimal displacements in different directions locally corresponds to a global effect of the non-vanishing phase for cyclic evolutions.

Some elements of the geometry of \( \psi(r; p) \in L^2(\mathbb{R}^3) \) are given in [12]. Most importantly, the metric is unchanged, meaning that the phase can be simply integrated

\[
\psi(r; p) = \psi(r; P)\exp\left(\int_{p = p + eA(r)}^{p = p + eA(r')}(\mathbf{p} - \mathbf{P})d\mathbf{r}\right).
\]

(10)

This follows from the fact that the spectra of the squares of the two operators are the same, meaning their eigenvectors, when projected onto each other, can differ only by an unit phase. It also requires the existence of a part of the domain where the two sets of eigenvectors are identical. This is achieved by extending the domain with a so called ideal point [12], where \( P = p \), which we take to be \( (r = \infty, \varphi = \varphi_0) \) but also discuss other possibilities.

One can also arrive at (10) by simply introducing Ansatz \( \psi^{pr}(r; p) = \psi(r; P)\exp\left(\int_{p = p + eA(r)}^{p = p + eA(r')}(\mathbf{p} - \mathbf{P})d\mathbf{r}\right) \) into (7), and evaluate the unknown function \( G(r) \) to be as given in (10) upon inverse projection (extension).

2.1. Single coil

In this case

\[
A(r) = \frac{\alpha}{r} \hat{\varphi},
\]

(11)

with magnetic flux of strength \( \alpha \) at the origin.

A particular solution of (3) that will be considered in practical application is a simple plane wave, \( \psi(r; P) = e^{iPr} \), so we start with it. From (10) is seen that the radial component of the difference of the displacements vanishes, so that the radial motion does not acquire a phase. Only the angular motion matters, giving

\[
\psi(r; p) = e^{iPr}e^{-i\varphi(\varphi - \varphi_0)},
\]

(12)

which is a particular solution of (4). The phase factor is Berry’s phase, an angle between vectors as they are parallel transported from point at infinity (point where interaction vanishes) to some point \( r \), via operators \( \nabla \) and \( \nabla_\alpha \). The solution is single valued, since it is defined on the domain \( L^2(\mathbb{R}^3) \).

Irrelevance of the radial motion for the phase means that the ideal point could be taken anywhere on the line \( \varphi = \varphi_0 \). In particular one may consider defining an ideal point by extending the domain with the point \( (r = 0, \varphi = \varphi_0) \) [12]. This allows for an interpretation of the phase as coming from parallel transport of cotangent space of a point flux to the position of an electron. Note that our considerations are directly applicable to localized systems, whose background domain is an open subset of \( \mathbb{R}^3 \). In this case ideal point is taken to be on the boundary of the domain, and is identified with Berry’s external parameter \( R \) [2].

Of some interest might be to reproduce the full solution of Aharonov and Bohm [1], in terms of standing waves (Bessel functions) that vanish at infinity. Remembering that the usual derivation of Bessel’s standing waves comes from separation of the variables, \( \psi(r; P) = R(r; P)\Phi(\varphi; P) \), after
which
\[ -\frac{\partial^2}{\partial \varphi^2} \Phi(\varphi; \mathbf{P}) = M^2 \Phi(\varphi; \mathbf{P}), \]  
(13)

\[ \left( r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + 2meE^2 \right) R(r; \mathbf{P}) = M^2 R(r; \mathbf{P}). \]  
(14)

The single-valuedness of the angular part fixes \( M \) to be a non-negative integer, upon which follows the general solution
\[ \psi(\mathbf{r}; \mathbf{P}) = \sum a_M J_M(\sqrt{2meE} r) e^{iM\varphi}. \]  
(15)

When vector potential is introduced, equation (13) changes
\[ -\left( \frac{\partial}{\partial \varphi} + i \alpha \right)^2 \Phi(\varphi; \mathbf{P}) = M^2 \Phi(\varphi; \mathbf{P}), \]  
(16)

implying that the change \( \mathbf{P} \to \mathbf{p} + \frac{\alpha}{r} \hat{\mathbf{r}} \) corresponds to changing \( M \to |m + \alpha| \). This can also be seen from the change in the angular momentum operator
\[ \mathbf{J} = \mathbf{r} \times \mathbf{P} = \frac{\partial}{\partial \Phi} \mathbf{z} \to \mathbf{r} \times \mathbf{P} + \alpha \mathbf{z} = \left( \frac{\partial}{\partial \varphi} + \alpha \right), \]
(17)

where operators \( \mathbf{J} \) and \( \mathbf{j} \) have positive integers \( M \) and \( m \) as eigenvalues and act on vectors on the background of \( \mathbb{R}^2 \) and \( \mathbb{R}_+ \times \mathbb{R} \), respectively.

From (10) then follows the general solution of Aharonov and Bohm:
\[ \psi(\mathbf{r}; \mathbf{p}) = \psi(\mathbf{r}; \mathbf{P})|_{M=|m+\alpha|} \]

\[ \exp\left( i \int_{M=|m+\alpha|}^{M=|m+\alpha|} (|m-M|) d\varphi' \right) \]

\[ = e^{i\alpha r_0} \sum_m a_m J_{m+\alpha}(\sqrt{2meE} r) e^{im\varphi}. \]  
(19)

Projecting it on \( \mathbb{R}^2 \), \( \psi^{pr}(\mathbf{r}; \mathbf{p}) \) has a branch line (cut) at \( \varphi = \varphi_0 \).

2.2. Multiple coils

The rotational symmetry that was deformed by the introduction of a single coil disappears when other coils are added, as in this case angular and radial part of the equation do not separate. This makes the analyses of the full solution difficult. But a particular solution of plane waves, which are not eigenstates of the angular momentum operator, is directly obtainable from (10)
\[ \psi(\mathbf{p}; \mathbf{r}) = e^{i(\mathbf{p}+\mathbf{A}(\mathbf{r})\mathbf{r})} \prod_{i=1}^{n} e^{-i\alpha_i(\varphi - \varphi_0_i)}. \]  
(20)

Here \( (\varphi - \varphi_0_i) \) is an angle that the ray connecting \( i \)th point flux and \( \mathbf{r} \) makes with the \( i \)th point flux and the ideal point. When projected onto \( \mathbb{R}^2 \) background, the solution \( \psi^{pr}(\mathbf{r}; \mathbf{p}) \) has \( n \) cuts, going from positions of the \( n \) fluxes to the ideal point at infinity \( (r = \infty, \varphi = \varphi_0) \). When considering a localized system, the phase it gets in a cyclic evolution is equal to the total encircled flux times \( 2\pi \).

In this case the background geometry is more complicated then in the case of a single coil. Separability of phases due to different point fluxes
\[ \psi(\mathbf{p}; \mathbf{r}) = e^{i\alpha P} \prod_{i=1}^{n} e^{iA_i(r_i)} e^{-i\alpha_i(\varphi - \varphi_0_i)}, \]  
(21)

where \( A_i(\mathbf{r}) \) is the strength of the vector potential at point \( \mathbf{r} \) from \( i \)th point flux only, allows for a highly formal expression for the domain
\[ \psi(\mathbf{r}; \mathbf{p}) = \in \mathbb{L}_2 \left( \sum_{i=1}^{n} (\mathbb{R}_+ \times \mathbb{R}_+), \right) \]  
(22)

where \( (\mathbb{R}_+ \times \mathbb{R}_+) \) is \( \mathbb{R}_+ \times \mathbb{R} \) space around the \( i \)th flux, and with the understanding that the full phase is a sum of individual phases.

3. Momentum representation

Identifying Berry’s phase with the angle that a vector from cotangent space makes as it is parallel transported from the point at infinity (point of no interaction) to some point should be independent of the representation. We proceed to show that on an example of plane waves.

In the absence of flux we have
\[ \mathbf{P} \ddot{\psi}(\mathbf{P}) = 2me\ddot{\psi}(\mathbf{P}), \]  
(23)

with the particular solution
\[ \ddot{\psi}(\mathbf{P}) = \frac{1}{P} \delta(P - \sqrt{2meE}) \delta(\Theta - \Theta_0), \]  
(24)

where \( \Theta_0 \) is the angle incoming electrons make with \( x \)-axes. When flux is introduced, equation becomes
\[ \left( \mathbf{p} + c \int d\mathbf{p}' A(\mathbf{p} - \mathbf{p}') \right)^2 \ddot{\psi}(\mathbf{p}) = 2me\ddot{\psi}(\mathbf{p}), \]  
(25)

where \( A(\mathbf{p} - \mathbf{p}') \) is the Fourier transform of the vector potential
\[ eA(\mathbf{p}) = \frac{\alpha}{P} \dot{\varphi}. \]  
(26)

In coordinate representation differential operator was replaced by a differential operator plus multiplication; in the momentum representation, multiplication operator is replaced by multiplication plus integral operator
\[ \mathbf{P} = \mathbf{p} + \int d\mathbf{p}' \frac{\alpha}{|P - P'|} \dot{\varphi}, \]  
(27)

which makes the analyses more difficult. Note that the above highly formal notation hides the fact that integral operator is not local (it acts on \( \ddot{\psi}(\mathbf{p}') \)), and also that the square of the vector potential vanishes upon Fourier transformation. Following (10), we write
\[ \ddot{\psi}(\mathbf{p}) = \ddot{\psi}(\mathbf{P})|_{\mathbf{P} \to \mathbf{p}} \int d\mathbf{p}' \frac{\alpha}{|P - P'|} \dot{\varphi} \]

\[ \exp\left( i \int_{P \to P}(r = \infty) (\mathbf{P} - \mathbf{p}) d\mathbf{f}' \right). \]  
(28)
It is particularly difficult to implement the condition (27). The radial component (square) of the momentum does not change, so that \( p \) is everywhere to be replaced simply by \( p \). The angular component does not change either; we argue heuristically that the only change is the extension of the domain of the angular component of the momentum, due to non-trivial geometry of the configurational space.

That is

\[
\Psi(p) = \frac{1}{p} \delta(p - \sqrt{2mE}) \delta(\theta - \theta_0) e^{-i\varphi - \varphi_0}, \tag{29}
\]

and we proceed to treat cases of integer/non-integer \( \alpha \) separately.

### 3.1. Integer \( \alpha \)

When \( \alpha \) is an integer, the background geometry is equivalent to \( \mathbb{R}^2 \) and we use the standard inverse Fourier transform in two dimensions

\[
\mathcal{F}\{\Psi(p)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \int_0^\infty p dpe^{-ipr\sin(\theta + \phi)} \Psi(p) \equiv \Psi(r; \mathbf{p})
\]

where \( \phi = \pi/2 - \varphi \) is introduced for convenience. Inserting the solution (29), doing the radial integral and representing the Dirac’s delta in the angular part as a series

\[
\delta(\alpha) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{i\alpha m}, \tag{31}
\]

gives

\[
\psi(r; \mathbf{p}) = \sum_{m=-\infty}^{\infty} (-i)^m \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-irk\sin(\theta + \phi)} e^{i(\theta - \theta_0)} e^{i(\phi - \varphi_0)} d\theta,
\]

\[
\hspace{1cm} = \sum_{m=-\infty}^{\infty} (-i)^m e^{-im\phi} e^{i(m+\alpha)\theta_0} e^{i\phi} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\alpha r\sin(\theta + \phi)} e^{i(m+\alpha)(\theta + \phi)} d(\theta + \phi),
\]

\[
\hspace{1cm} \equiv \sum_{m=-\infty}^{\infty} \psi_m(r; \mathbf{p}), \tag{34}
\]

where in the first equality a factor \( e^{i\alpha(\theta - \theta_0)} \), equal to unity upon integration is inserted, and \( k \equiv \sqrt{2mE} \) is introduced for short. Due to periodicity (single-valuedness) of the phase, this becomes

\[
\psi(r; \mathbf{p}) = \sum_{m=-\infty}^{\infty} (-i)^m e^{-im\phi} \int_{-\pi}^{\pi} e^{-i\alpha r\sin(\theta + \phi)} e^{i(m+\alpha)\theta_0} d\theta, \tag{35}
\]

and with a familiar integral representation of Bessel’s function of non-negative integer order (e.g. [15], p 1399),

\[
J_{\alpha}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iz\sin(\theta + \varphi)} d\theta, \tag{36}
\]

gives

\[
\psi_m(r; \mathbf{p}) = (-i)^m k e^{-im\phi} J_{m+\alpha}(kr) \tag{37}
\]

for \( m + \alpha \) positive, and

\[
\psi_m(r; \mathbf{p}) = \exp(i\alpha \varphi) \sum_{m=-\infty}^{\infty} (-i)^m k e^{-im\phi} J_{m+\alpha}(kr) \tag{38}
\]

where we introduced \( \theta_R \) and \( i\theta_I \) to distinguish the angular momentum variable on it is standard and extended domains, one immediately sees that the first contribution is the same as in the integer magnetic flux case, while the second contribution only adds in case of non-integer \( \alpha \). Interpreting first term as incident, and second term as scattered wave, the scattering cross section proportionality with \( \sin^2(\alpha \pi) \) is reproduced. Thus the imaginary part of the domain of the momentum angle, which does not have a classical analog, turns out responsible for the quantum scattering effect.

We note that while the above argumentation is heuristic and rather indirect, the imaginary part of the domain appearing for non-integer \( \alpha \) only when plane waves are...
We have seen the appearance of Berry’s phase on the wave-function of free-flowing electrons. With respect to the original Berry’s proposal, there is a possible practical benefit in that it is not necessary to localize electrons and move them around coil, but rather one can manipulate free-flowing electrons like in figure 2.

The effect has already been proposed as an idea for quantum computing (check e.g. a pedagogical review in [10]). Interestingly, figure 2 may also represent a scheme of a prototype of a basic building block for the architecture of the quantum computer. Consider cases \( \alpha = 1/2, 1/3, 1/4 \) etc as an example. When \( \alpha = 1/2 \) figure one is bimodal and corresponds to a classical computer—the flows can be either in phase or anti-phase, depending on the evenness/oddness of the winding number. But when \( \alpha \) is an inverse of an integer greater than two, the scheme in figure 2 is a multimodal logic. If it would be possible to incorporate stable current flows and coherent sources of free-flowing electrons into a design with controllable back-reaction and dissipation effects, the scheme of figure 2 could perhaps indeed offer (one) basic solution for quantum computer. In addition, with the ability to variate the current flows in coils the logical structure of the machine becomes dynamic, allowing for locally switching between different logical modalities (like from quantum to classical computing) by locally manipulating current in a coil.

5. Conclusion and outlook

Formulation of the problem on a non-trivial background geometry makes it possible to obtain Berry’s phase by simple integration, as shown in this paper. It also allowed us to obtain a particular solution for the scattering on multiple coils. In the latter case, we were not able to define the background geometry, which will be the topic of further investigations.

Analyzing the single-coil solution in the momentum representation, we found that the phase does not depend on the realization. We also found that the domain of the momentum angular component gets extended, in correspondence with the same extension in the configuration space.

Finally, from practical point of view, this formulation treats motion of free electrons on the same footing as global motions of the system. The wave-function of the electron only records the total motion it made on the configuration space, regardless of the cause. We offer a scheme for experimental verification of this fact.

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