Algebraic Bethe Ansatz for $U(1)$ Invariant Integrable Models: Compact and non-Compact Applications

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Abstract

We apply the algebraic Bethe ansatz developed in our previous paper [1] to three different families of $U(1)$ integrable vertex models with arbitrary $N$ bond states. These statistical mechanics systems are based on the higher spin representations of the quantum group $U_q[SU(2)]$ for both generic and non-generic values of $q$ as well as on the non-compact discrete representation of the $SL(2,R)$ algebra. We present for all these models the explicit expressions for both the on-shell and the off-shell properties associated to the respective transfer matrices eigenvalue problems. The amplitudes governing the vectors not parallel to the Bethe states are shown to factorize in terms of elementary building blocks functions. The results for the non-compact $SL(2,R)$ model are argued to be derived from those obtained for the compact systems by taking suitable $N \rightarrow \infty$ limits. This permits us to study the properties of the non-compact $SL(2,R)$ model starting from systems with finite degrees of freedom.

PACS numbers: 05.50+q, 02.30.IK

Keywords: Algebraic Bethe Ansatz, Lattice Integrable Models

March 2009
1 Introduction

This article is a continuation of the paper \[1\] in which we have developed the algebraic Bethe ansatz for $U(1)$ invariant vertex models. The central object in this method turns out to be the monodromy matrix $T_{\mathcal{A}}(\lambda)$ \[2, 3\] depending on the spectral parameter $\lambda$. For a $N$-dimensional auxiliary space $\mathcal{A}$ we view $T_{\mathcal{A}}(\lambda)$ as,

\[ T_{\mathcal{A}}(\lambda) = \begin{pmatrix} T_{1,1}(\lambda) & T_{1,2}(\lambda) & \cdots & T_{1,N}(\lambda) \\ T_{2,1}(\lambda) & T_{2,2}(\lambda) & \cdots & T_{2,N}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ T_{N,1}(\lambda) & T_{N,2}(\lambda) & \cdots & T_{N,N}(\lambda) \end{pmatrix}_{N \times N}, \tag{1} \]

where the elements $T_{a,b}(\lambda)$ are operators acting on the system quantum space.

The physical quantities such as partition function can be expressed in terms of the trace of the monodromy matrix. This operator is the generating function of the conserved currents and is called the transfer matrix $T(\lambda)$,

\[ T(\lambda) = \text{Tr}_{\mathcal{A}} [T_{\mathcal{A}}(\lambda)] = \sum_{i=1}^{N} T_{a,a}(\lambda). \tag{2} \]

A necessary condition to diagonalize $T(\lambda)$ by the algebraic Bethe ansatz is the existence of a vector $|0\rangle$ such that the action of the monodromy on it results in a triangular matrix for arbitrary values of the spectral parameter. For instance, if $T_{\mathcal{A}}(\lambda) |0\rangle$ is annihilated by its lower left elements we have,

\[ T_{\mathcal{A}}(\lambda) |0\rangle = \begin{pmatrix} \omega_1(\lambda) |0\rangle & \# & \cdots & \# \\ 0 & \omega_2(\lambda) |0\rangle & \cdots & \# \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_N(\lambda) |0\rangle \end{pmatrix}_{N \times N}, \tag{3} \]

where the symbol $\#$ denotes non-null states and $\omega_i(\lambda)$ for $i = 1, \cdots, N$ are complex valued functions.

After having property (3) fulfilled one can in principle use the algebraic Bethe ansatz method to propose an ansatz for other eigenvectors $|\Phi_n\rangle$ of $T(\lambda)$. In general, the states $|\Phi_n\rangle$ are searched
as linear combination of certain product of creation fields $T_{a,b}(\lambda)$ with $a < b$ acting on the reference state $|0\rangle$. The next step in this framework is to compute the action of the diagonal operators $T_{a,a}(\lambda)$ on the ansatz state $|\Phi_n\rangle$. This operation generates not only the state $|\Phi_n\rangle$ but also a number of vectors that are not parallel to such proposed eigenvectors of $T(\lambda)$ which are often called unwanted terms. This analysis is then able to produce informations on the on-shell Bethe ansatz properties which turns out to be the eigenvalues of $T(\lambda)$ and the Bethe ansatz equations needed to cancel out the unwanted terms. In addition, it also gives us the off-shell properties which consist on both the determination of the pattern of the vectors not parallel to $|\Phi_n\rangle$ and the functional form of the functions proportional to these states. We shall refer to these functions as the off-shell amplitudes of a given algebraic Bethe ansatz analysis. We remark that the knowledge of the off-shell Bethe ansatz data is of relevance since its semi-classical limit can provide solutions of integrable long-range systems such as the Gaudin models [4] as well as representations for the solutions of equations of Knizhnik-Zamolodchikov type [5].

In our previous paper [1] we have developed the above discussed algebraic framework to solve arbitrary integrable vertex models that are invariant by one $U(1)$ symmetry. Here we will use the general results of [1] to compute the explicit expressions for both the on-shell and off-shell parts of the algebraic Bethe ansatz solution of three distinct classes of vertex models. We recall that the off-shell properties have been given in terms of recurrence relations whose solution for a specific model requires additional analysis. At first we consider the vertex model derived from the higher spin representation of the $U_q[SU(2)]$ algebra for generic values of the deformation parameter $q$. This gives origin to the celebrated integrable spin-$s$ extension of the Heisenberg XXZ chain [6,7,8,9]. The on-shell properties of this system have been obtained long ago by using the fusion hierarchy procedure [10,11]. For a recent discussion on the algebraic Bethe ansatz of the XXZ-$s$ in the context of the super-integrable chiral Potts model see [12]. However, to the best of our knowledge the off-shell Bethe ansatz structure of the XXZ-$s$ is still unknown. The second family of models we shall consider are those directly associated to the colored solutions of the Yang-Baxter equation [13,14,15]. These vertex models are intimately connected to the representations of the $U_q[SU(2)]$ algebra when $q$ is a root of unity [16,17]. The third system is
based on the discrete $D_{s^-}$ representation of the $SL(2,R)$ symmetry leading us to a vertex model with an infinite number of states per bond.

We have organized this paper as follows. For sake of completeness we review the main results of our previous work [1] in next section. This helps us to elaborate on the on-shell and off-shell Bethe ansatz results discussed in [1] as well as to present them in a self-consistent way. In section 3 we consider the classical analogue of the solvable spins-$s$ XXZ model. Its $R$-matrix is expressed in the Weyl basis in order to allow us to compute the respective off-shell properties in closed forms. In section 4 we discuss the vertex models derived from the braid group representations associated to the $U_q[SU(2)]$ quantum algebra when $q$ is a root of unity. The main feature of this system is that its Boltzmann weights may depend on three distinct continuous variables and this freedom is used to define vertex models with additive and non-additive $R$-matrices. The corresponding on-shell and off-shell Bethe ansatz properties are then exhibited. In section 5 we consider a non-compact vertex model associated to the discrete $D_{s^-}$ representation of the $SL(2,R)$ symmetry. We present its algebraic Bethe ansatz properties and argue that they can be viewed as an appropriate limit of that derived for the vertex model defined in section 4.1. Our conclusions are presented in section 6. In Appendices A, B, C and D we summarize technical details that are useful for the comprehension of the main text.

2 Definitions and Results

We shall here review our previous general results for the algebraic Bethe ansatz solution for the $U(1)$ invariant vertex models [1]. These vertex models are statistical systems defined on a square lattice of $L$ rows and $L$ columns whose intersections are denominated vertices. The statistical configurations of these models are characterized by assigning to each lattice edge a variable that takes value on a set of integer numbers $\{1, 2, \cdots, N\}$. The Boltzmann weight at the $i$-th vertex is generally represented by $R(\lambda, \mu_i)_{a,b}^{c,d}$ where $a,b,c,d = 1, \cdots, N$ and the parameters $\mu_i$ play the role of horizontal inhomogeneities. The underlying $U(1)$ symmetry implies that the vertex
weights satisfy the following ice rule constraint,

- \( R(\lambda, \mu)^{c,d}_{a,b} \neq 0 \), for \( a + b = c + d \).
- \( R(\lambda, \mu)^{c,d}_{a,b} = 0 \), for \( a + b \neq c + d \).  \( \text{(4)} \)

As usual the integrability of the vertex model is guaranteed by imposing the Yang-Baxter equation for a \( N^2 \times N^2 \) \( R \)-matrix \( R(\lambda, \mu) \) which in the notation of [1] is given by,

\[
R(\lambda, \mu) = \sum_{a,b,c,d=1}^{N} R(\lambda, \mu_i)^{c,d}_{a,b} e_{a,c} \otimes e_{b,d}, \quad \text{(5)}
\]

where \( e_{a,b} \) denote \( N \times N \) Weyl matrices.

The monodromy matrix associated to vertex models is build up by considering the product of weights on the horizontal line which formally can be written as,

\[
\mathcal{T}_A(\lambda) = \mathcal{L}_{AL}(\lambda, \mu_L) \mathcal{L}_{AL-1}(\lambda, \mu_{L-1}) \ldots \mathcal{L}_{A1}(\lambda, \mu_1). \quad \text{(6)}
\]

The expression for the operators \( \mathcal{L}_{Ai}(\lambda, \mu_i) \) in terms of the statistical weights is,

\[
\mathcal{L}_{Ai}(\lambda, \mu_i) = \sum_{a,b,c,d=1}^{N} R(\lambda, \mu_i)^{c,d}_{a,b} e_{a,c} \otimes e_{b,d}^{(i)}, \quad \text{(7)}
\]

where \( e_{b,d}^{(i)} \) are \( N \times N \) Weyl matrices acting on the quantum space \( \bigotimes_{i=1}^{L} \mathcal{L}_i \) of a one-dimensional chain of length \( L \).

It turns out that the ice rule (4) permits us to construct a reference state in terms of the tensor product of standard local ferromagnetic vectors \( |s\rangle_i \) with spin \( s = (N - 1)/2 \) for the monodromy matrix (6,7). This pseudo-vacuum state is,

\[
|0\rangle = \prod_{i=1}^{L} |s\rangle_i, \quad |s\rangle_i = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_N, \quad \text{(8)}
\]
where functions $\omega_a(\lambda)$ introduced in Eq. (3) are,

$$w_a(\lambda) = \prod_{i=1}^{L} R(\lambda, \mu_i)^{a,1}_{a,1}. \quad (9)$$

The other eigenstates of $T(\lambda)$ are constructed in terms of a linear combination of product of creation operators defined by the first row of the monodromy matrix. Due to the $U(1)$ symmetry these states are viewed as $n$-particle states parameterized by the number of rapidities $\lambda_1, \cdots, \lambda_n$ which can be written as,

$$|\Phi_n\rangle = \phi_n(\lambda_1, \cdots, \lambda_n) |0\rangle. \quad (10)$$

The states $|\Phi_n\rangle$ can be interpreted as excitations of spin $(N-1)/2 - n$ over the reference state $|0\rangle$. The mathematical structure of the vector $\phi_n(\lambda_1, \cdots, \lambda_n)$ is given by the following $(N-1)$-step recurrence relation,

$$\phi_n(\lambda_1, \cdots, \lambda_n) = \sum_{e=1}^{m(n,N-1)} T_{1,1+e}(\lambda_1) \sum_{2\leq j_2<\cdots<j_e\leq n}^{*} \phi_{n-e}(\lambda_{j_{e+1}}, \cdots, \lambda_{j_n}) e^{-1} \mathcal{F}^{(2)}_{e-1}(\lambda_1, \lambda_{j_2}, \cdots, \lambda_{j_e}) \prod_{k_1=2}^{e} T_{1,1}(\lambda_{j_{k_1}}) \prod_{k_2=e+1}^{n} R(\lambda_{j_{k_2}}, \lambda_{j_{k_1}})^{1,1}_{1,1} \times R(\lambda_{j_{k_2}}, \lambda_{j_{k_1}})^{2,1}_{2,1} \theta_{<}(\lambda_{j_{k_2}}, \lambda_{j_{k_1}}), \quad (11)$$

where the symbol $*$ means that terms with $j_k = j_l$ are excluded in the sum and $m(x,y)$ denotes the minimum integer of the pair $\{x,y\}$.

The functions entering Eq. (11) are well defined in terms of the $R$-matrix elements apart from the overall normalization $\mathcal{F}^{(2)}_0(\lambda)$ which can be set to unity. In particular, $\theta_{<}(\lambda_i, \lambda_j)$ is defined by,

$$\theta_{<}(\lambda_i, \lambda_j) = \begin{cases} \theta(\lambda_i, \lambda_j), & \text{for } i < j \\ 1, & \text{for } i \geq j. \end{cases} \quad (12)$$

where the expression for $\theta(\lambda, \mu)$ is,

$$\theta(\lambda, \mu) = \begin{vmatrix} R(\lambda, \mu)^{2,2}_{2,2} & R(\lambda, \mu)^{2,2}_{3,1} \\ R(\lambda, \mu)^{3,1}_{2,2} & R(\lambda, \mu)^{3,1}_{3,1} \end{vmatrix} \quad (13)$$
The auxiliary function \( b \mathcal{F}^{(2)}_b(\lambda_1, \lambda_2, \ldots, \lambda_{b+1}) \) is a special class of more general functions that are companion of the unwanted vectors generated by the action of the monodromy matrix elements \( T_{a,a}(\lambda) \) on the \( b \)-particle state \( |\Phi_b\rangle \). We shall denote these generalized off-shell amplitudes by \( c \mathcal{F}^{(a)}_b(\lambda_1, \lambda_2, \ldots, \lambda_{b+1}) \) where the range of the extra indices are \( c = 0, \ldots, b \) and \( a = 1, \ldots, N-b \). Here we remark that the index \( c \) accounts for the number of weights \( \omega_1(\lambda_i) \) that is present in the respective undesirable term proportional to the operators \( \prod_{i=1}^{c} \omega_1(\lambda_i) T_{a,a+b}(\lambda) \). It turns out that they satisfy a set of recurrence relations whose initial conditions are,

\[
0 \mathcal{F}^{(a)}_1(\lambda, \mu) = -1 \mathcal{F}^{(a)}_1(\lambda, \mu) = \frac{R(\lambda, \mu) a_{a+1}}{R(\lambda, \mu) a_{a+1}} \text{ for } a = 1, \ldots, N - 1. \tag{14}
\]

The structure of the off-shell amplitudes for \( c \neq 0 \) and \( c \neq b \) has a direct factorized form of the following type,

\[
\begin{align*}
c \mathcal{F}^{(a)}_b(\lambda_1, \ldots, \lambda_b) &= \mathcal{F}^{(a)}_{b-c}(\lambda, \lambda_c+1, \ldots, \lambda_b) c \mathcal{F}^{(a+b-c)}_c(\lambda, \lambda_1, \ldots, \lambda_c) \prod_{i=c+1}^{b} \prod_{j=1}^{c} R(\lambda_i, \lambda_j)_{1,1}^{1,1} \prod_{i=c+1}^{b} R(\lambda_i, \lambda_j)_{2,1}^{2,1}, \\
&\quad \text{for } b = 2, \ldots, N - 1; \quad a = 1, \ldots, N - b; \quad c = 1, \ldots, b - 1. \tag{15}
\end{align*}
\]

However, in order to iterate Eq. \((15)\) we still need to know the off-shell amplitudes for the extremum values \( c = 0 \) and \( c = b \). These functions satisfy more complicated recurrence relations involving the sum of products of many distinct terms, namely

\[
0 \mathcal{F}^{(a)}_b(\lambda_1, \ldots, \lambda_b) = \sum_{e=1}^{b} \frac{R(\lambda, \lambda_1)_{a+1,1}^{a+1,1} \sum^*} {R(\lambda, \lambda_1)_{a+1,1}^{a+1,1}} \prod_{i=e+1}^{b} \prod_{j=1}^{c} R(\lambda_{j_1}, \lambda_{j_2})_{1,1}^{1,1} \prod_{i=e+1}^{b} R(\lambda_{j_1}, \lambda_{j_2})_{2,1}^{2,1} \theta_\lambda(\lambda_{j_1}, \lambda_{j_2}), \\
\times (-1)^{b-1} \mathcal{F}^{(2)}_{b-e}(\lambda_1, \lambda_{j(b-e+1)}, \ldots, \lambda_{j(b-e)}) \prod_{l_1=1}^{b-e} \prod_{l_2=b-e}^{b-1} R(\lambda_{j_1}, \lambda_{j_2})_{1,1}^{1,1} \prod_{l_1=1}^{b-e} R(\lambda_{j_1}, \lambda_{j_2})_{2,1}^{2,1}, \tag{16}
\]

and

\[
\begin{align*}
b \mathcal{F}^{(a)}_b(\lambda_1, \ldots, \lambda_b) &= -\sum_{f=0}^{b-1} \sum_{1 \leq l_1 < l_2 < \cdots < l_{b-f} \leq b} \mathcal{F}^{(a)}_b(\lambda, \lambda_{i=1}^{b-f} \lambda_{l}, \lambda_1, \ldots, \lambda_l) \\
&\times \prod_{s=1}^{b-f} \prod_{i \neq l_1, \ldots, l_{b-f}} \theta_\lambda(\lambda_{l}, \lambda_{l}) \frac{R(\lambda_i, \lambda_{t})_{1,1}^{1,1} R(\lambda_i, \lambda_{t})_{2,1}^{2,1}}{R(\lambda_i, \lambda_{t})_{2,1}^{2,1} R(\lambda_i, \lambda_{t})_{1,1}^{1,1}}, \\
&\quad \text{for } b = 1, \ldots, N - 1; \quad a = 1, \ldots, N - b. \tag{17}
\end{align*}
\]
where \( \{ \lambda_i \}_{i=1}^{b} \) means that out of the possible variables \( \lambda_1, \ldots, \lambda_b \) those indexed by \( \lambda_{i_1}, \ldots, \lambda_{i_p} \) are absent in the set.

At this point we emphasize that relations (14-17) provide a self-consistent way to determine all the off-shell amplitudes entering in a given \( n \)-particle state. For example, in order to generate the two-particle off-shell amplitudes one has to substitute the one-particle initial condition (14) in expressions (15-17). These data together are then able to provide us the three-particle off-shell amplitudes and this procedure can then be iterated until we reach the final step with the total number of \( (N-1) \) particles. We observe that in practice, for the case of a specific model, it is sufficient to find closed expressions for the amplitudes \( aF^{(a)}_b(\lambda, \lambda_1, \ldots, \lambda_b) \) and \( bF^{(a)}_b(\lambda, \lambda_1, \ldots, \lambda_b) \) since the remaining ones are automatically fixed by Eq. (15).

We now turn our attention to the action of the transfer matrix \( T(\lambda) \) on the multi-particle state \( |\Phi_n \rangle \). Following [1] we find that \( T(\lambda) |\Phi_n \rangle \) can be rewritten as,

\[
T(\lambda) |\Phi_n \rangle = \sum_{a=1}^{N} \omega_a(\lambda) \prod_{i=1}^{n} P_a(\lambda, \lambda_i) |\Phi_n \rangle - \sum_{t=1}^{N-t} \sum_{a=1}^{N} T_{a,a+t}(\lambda) \sum_{p=0}^{n} \sum_{t \leq j_1 < \cdots < j_p < n} w_{p,t}^{a}(\lambda, \lambda_1, \ldots, \lambda_t) |\Phi_n \rangle,
\]

where functions \( P_a(\lambda, \mu) \) proportional to the states \( |\Phi_n \rangle \) are,

\[
P_a(\lambda, \mu) = \left\{ \begin{array}{ll}
R(\mu, \lambda)^{1,1}_{1,1}, & \text{for } a = 1 \\
R(\mu, \lambda)^{2,1}_{2,1}, & \\
R(\lambda, \mu)^{a,2}_{a,2} R(\lambda, \mu)^{a,2}_{a,1,1}, & \text{for } 2 \leq a \leq N-1 \\
R(\lambda, \mu)^{a+1,1}_{a,2} R(\lambda, \mu)^{a+1,1}_{a+1,1}, & \\
R(\mu, \lambda)^{a,1}_{a,1} R(\lambda, \mu)^{a+1,1}_{a+1,1}, & \\
R(\lambda, \mu)^{N,2}_{N,2}, & \text{for } a = N.
\end{array} \right.
\]

The expression (18) tells us that the unwanted terms, i.e the vectors not parallel to the state \( |\Phi_n \rangle \), have a universal pattern governed by the following type of operators \( T_{a,a+t}(\lambda) \phi_{n-t}(\{ \lambda_i \}_{i=1}^{n}) |0 \rangle \) for \( t = 1, \ldots, m(n, N-1) \). We also see that the respective amplitudes are product of two distinct classes of functions \( aF^{(a)}_b(\lambda, \lambda_1, \ldots, \lambda_t) \) and \( bF^{(a)}_b(\lambda, \lambda_1, \ldots, \lambda_t) \). The first part carries a
dependence on both the spectral parameter $\lambda$ and the Bethe rapidities $\lambda_1, \ldots, \lambda_n$ and satisfies the non-trivial recurrence relations (14-17). By contrast, the second one depends only on the Bethe variables and has a rather simple factorized form,

$$ p_\mathcal{H}_t(n)(\lambda_{j_1}, \ldots, \lambda_{j_t}) = \prod_{s=1}^{p} w_1(\lambda_{j_s}) \prod_{i=1}^{n} \left( \prod_{i \neq j_1, \ldots, j_t} R(\lambda_i, \lambda_{j_s})^{1,1}_{1,1} \theta_{<}(\lambda_i, \lambda_{j_s}) \prod_{r=p+1}^{t} \prod_{s=1}^{p} \theta_{<}(\lambda_{j_r}, \lambda_{j_s}) \right) $$

$$ \times \prod_{i=1}^{n} \theta_{<}(\lambda_i, \lambda_{j_r}) \left[ \prod_{r=p+1}^{t} \prod_{i \neq j_1, \ldots, j_t} w_2(\lambda_{j_r}) \prod_{i=1}^{n} \left( \prod_{i \neq j_1, \ldots, j_t} R(\lambda_i, \lambda_{j_r})^{1,1}_{1,1} \theta(\lambda_{j_r}, \lambda_i) \prod_{s=1}^{p} \theta(\lambda_{j_r}, \lambda_{j_s}) \right) \right] , $$

which is easily computed from the knowledge of few statistical weights. In fact it only depends on the amplitudes $R(\lambda, \mu)^{a,1}_{a,1}$ and function $\theta(\lambda, \mu)$.

In order to enforce that $|\Phi_n\rangle$ is an eigenstate of $T(\lambda)$ we need to find variables $\lambda_1, \ldots, \lambda_n$ such that all the unwanted terms vanish for arbitrary values of the spectral parameter. Considering the functional form of functions proportional to the undesirable terms we conclude that this is achieved by imposing that $p_\mathcal{H}_t(n)(\lambda_{j_1}, \ldots, \lambda_{j_t}) = 0$. From Eq. (20) it is not difficult to see that this leads us to the following Bethe equations for the rapidities $\lambda_1, \ldots, \lambda_n$,

$$ \frac{w_1(\lambda_j)}{w_2(\lambda_j)} = \prod_{i=1}^{n} \theta(\lambda_j, \lambda_i) R(\lambda_j, \lambda_i)^{1,1}_{1,1} \frac{R(\lambda_i, \lambda_{j})^{2,1}_{2,1}}{R(\lambda_{j}, \lambda_{j})^{2,1}_{2,1}} \frac{R(\lambda_{j}, \lambda_{j})^{1,1}_{1,1}}{R(\lambda_i, \lambda_j)^{1,1}_{1,1}} \text{ for } j = 1, \ldots, n. $$

As a consequence of that we conclude that the $n$-particle transfer matrix eigenvalue associated to the on-shell Bethe states,

$$ T(\lambda) |\Phi_n\rangle = \Lambda_n(\lambda) |\Phi_n\rangle , $$

is determined by the expression,

$$ \Lambda_n(\lambda) = \sum_{a=1}^{N} w_a(\lambda) \prod_{i=1}^{n} P_a(\lambda, \lambda_i). $$

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In next sections we shall present the explicit expressions for the non-trivial off-shell amplitudes (14-17), the Bethe ansatz equations (21) and the eigenvalues (23) in the case of three different types of vertex models.

3 The XXZ-s model

The study of integrable higher spin Heisenberg XXZ-s chains started with the search of $N$-state vertex model solutions of Yang-Baxter equation for $N = 3, 4$ [6, 7]. The respective $R$-matrix for arbitrary values of $s = (N - 1)/2$ was first proposed within the fusion procedure in the special case of the isotropic $SU(2)$ XXX-s chain [8]. Other progress on higher spin descendants of the Heisenberg model has been made with the notion of universal $R$-matrix [9], the connection with higher spin representations of the $U_q[SU(2)]$ algebra for generic values of $q$ [18] and the relationship with new link polynomials and generalized braid monoid algebras [19]. The $R$-matrix associated to the XXZ-s chain is conveniently written with the help of an auxiliary matrix $\tilde{R}(\lambda)$,

$$R(\lambda, \mu) = P \tilde{R}(\lambda, \mu), \quad (24)$$

where $P = \sum_{a,b=1}^{N} e_{a,b} \otimes e_{b,a}$ is the $C^N \otimes C^N$ permutator.

The matrix $\tilde{R}(\lambda, \mu)$ can be expressed in a closed form by using the projectors $\tilde{P}_j(q)$ on the tensor product of two irreducible representations of $U_q[SU(2)]$ with spin $s = (N - 1)/2$ [18],

$$\tilde{R}(\lambda) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} \frac{\sinh[i k \gamma + \lambda - \mu]}{\sinh[i k \gamma - \lambda + \mu]} \tilde{P}_j(q), \quad (25)$$

where the deformation parameter $q$ and the the anisotropy $\gamma$ are related by $q = \exp[-2\gamma t]$.

The expressions for the operators $\tilde{P}_j(q)$ can in principle be given by means of an interpolation among the roots of the $U_q[SU(2)]$ Casimir operator [20]. Though $\tilde{P}_j(q)$ can be expressed in terms of such operator in a simple way its explicit expression on the Weyl basis requires an extra amount of work. This step is essential to provide us the weights $R_{a,b}^{c,d}(\lambda, \mu)$ which are the main ingredient to establish a statistical mechanics interpretation and to compute the respective Bethe
ansatz properties as described in section 2. We found that these projectors can be written as,

\[
P_j(q) = \sum_{k=0}^{N-1} \frac{\hat{S}(q) - (-1)^k q^{(k+1)/2}}{(-1)^k q^{(j+1)/2} - (-1)^k q^{(k+1)/2}}.
\] (26)

The operator \( \hat{S}(q) \) in Eq. (26) play the role of a braid satisfying a form of the Yang-Baxter equation without spectral parameter, namely

\[
[\hat{S}(q) \otimes I_N][I_N \otimes \hat{S}(q)][\hat{S}(q) \otimes I_N] = [I_N \otimes \hat{S}(q)][\hat{S}(q) \otimes I_N][I_N \otimes \hat{S}(q)],
\] (27)

where \( I_N \) is the \( N \times N \) identity matrix.

In Weyl basis the expression for the braid \( \hat{S}(q) \) is,

\[
\hat{S}(q) = \sum_{\substack{a,b,c,d \in \mathbb{Z} \cap [0,N-1] \atop a \geq d \atop c \geq b}} S_{c,d}^{a,b}(q) e_{b,d} \otimes e_{a,c},
\] (28)

where the weights \( S_{c,d}^{a,b}(q) \) are given by

\[
S_{c,d}^{a,b}(q) = -(-1)^N q^\frac{(a-1)(b-1)q^{N-1} + (d-1)(b-1)q^{N-1}}{2} \frac{1}{W_0(a-d)W_0(c-b)} \delta_{a+b,c+d},
\] (29)

while function \( W_\epsilon(n) \) for \( \epsilon = 0,1 \) is defined by the following product,

\[
W_\epsilon(n) = \prod_{k=1}^{n} (1 - q^{k-\epsilon N}).
\] (30)

The relevant feature of Eqs. (24-30) is that they can be easily implemented to compute the \( R \)-matrix for relatively large values of \( N \). By substituting the \( R \)-matrix amplitudes in Eqs. (21,23) we are able to obtain the respective on-shell Bethe ansatz properties. After some manipulations we find that the transfer matrix eigenvalues are,

\[
\Lambda_n(\lambda) = \sum_{a=1}^{N} \prod_{l=1}^{L} \prod_{k=1}^{a-1} \frac{\sinh[\lambda - \mu_l - i(k-1)\gamma]}{\sinh[\lambda - \mu_l + i(N-k)\gamma]} \times \prod_{i=1}^{n} \frac{\sinh[\lambda - \lambda_i - i(N-1)\gamma]}{\sinh[\lambda - \lambda_i - i(a-1)\gamma]} \frac{\sinh[\lambda - \lambda_i + i\gamma]}{\sinh[\lambda - \lambda_i - i(a-2)\gamma]},
\] (31)
while the Bethe ansatz equations for the variables $\lambda_j$ are
\[
\prod_{i=1}^L \frac{\sinh[\lambda_j - \mu_i + i(N - 1)\gamma]}{\sinh[\lambda_j - \mu]} = \prod_{i=1}^n \frac{\sinh[\lambda_j - \lambda_i + i\gamma]}{\sinh[\lambda_j - \lambda_i - i\gamma]}, \quad j = 1, \ldots, n. \tag{32}
\]

We remark that the Bethe ansatz results (31,32) have been obtained before by the mechanism of fusion \cite{10,11}. This method allows us to write the transfer matrix of higher spin systems by means of traces taken on the smaller two-dimensional $s = 1/2$ auxiliary space and the respective eigenvalues are determined recursively. The Bethe equations are then proposed by requiring the analyticity of the eigenvalues rather than by explicit cancellation of the unwanted terms preventing us information on the off-shell data. By contrast, the results described in section 2 provide us the means to determine the off-shell Bethe ansatz structure by working out the recurrence relations (14-17). This step, however, requires a considerable amount of additional work specially when we are interested on the results for arbitrary values of $N$. In appendix A we summarize the technical details that we have devised to perform this computation for all the models considered in this paper. By implementing the analysis of Appendix A for the XXZ-$s$ model we find that the expressions for the basic amplitudes $0F_b^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b)$ and $bF_b^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b)$ factorize in terms of elementary trigonometric functions,
\[
0F_1^{(a)}(\lambda, \mu) = -F_1^{(a)}(\lambda, \mu) = \exp[\mu - \lambda]\frac{\sqrt{\sinh[i(N - 1)\gamma] \sinh[i(N - a)\gamma] \sinh[ia\gamma]}}{\sqrt{\sinh[i\gamma] \sinh[i(a - 1)\gamma - \lambda + \mu]}}, \tag{33}
\]
\[
0F_b^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b) = G_0^{(a, b)}(\gamma) \prod_{i,j=1}^b \frac{\sinh[\lambda_i - \lambda_j - i(N - 1)\gamma]}{\sinh[\lambda_i - \lambda_j - i\gamma]} \prod_{i=1}^b \frac{\sinh[\lambda_i - \lambda_j - i(a\gamma)]}{\sinh[\lambda_i - \lambda_j - i\gamma]} \prod_{i=1}^b \frac{\sinh[i(a - 1)\gamma - \lambda + \mu]}{\sinh[i(a - 1)\gamma - \lambda + \mu]}. \tag{34}
\]
\[
bF_b^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b) = G_0^{(N+1-a-b, b)}(\gamma) \prod_{i,j=1}^b \frac{\sinh[\lambda_i - \lambda_j - i(N - 1)\gamma]}{\sinh[\lambda_i - \lambda_j - i\gamma]} \prod_{i=1}^b \frac{\sinh[i(a + b - 1 - l)\gamma]}{\sinh[i(a + b - 1)\gamma]} \frac{\sinh[i(N + 1 - a - l)\gamma]}{\sinh[i(N + 1 - a - b)\gamma]}. \tag{35}
\]

The constant $G_0^{(a, b)}(\gamma)$ does not depend either on the spectral parameter $\lambda$ or on the variables $\lambda_1, \ldots, \lambda_n$ and it is given by,
\[
G_0^{(a, b)}(\gamma) = \prod_{l=1}^{b-1} \sqrt{\frac{\sinh[i(a + b - 1 - l)\gamma] \sinh[i(N + 1 - a - l)\gamma]}{\sinh[i(a + b - 1)\gamma] \sinh[i(N + 1 - a - b)\gamma]}}, \tag{36}
\]
\footnote{The Bethe ansatz equations (32) can be made more symmetrical by performing the shift $\lambda_j \rightarrow \lambda_j - \frac{i(N-1)\gamma}{2}$.}
To complete the off-shell data we are only left with the computation of function $p\mathcal{H}^{(n)}(\lambda_{j_1}, \ldots, \lambda_{j_t})$ which depends on the weights $R(\lambda, \mu)_{a,1}$ and the Bethe ansatz function $\theta(\lambda, \mu)$. For the XXZ-s model they are given by,

$$R(\lambda, \mu)_{a,1}^{a,1} = \prod_{k=1}^{a-1} \frac{\sinh[\lambda - \mu - i(k-1)\gamma]}{\sinh[\lambda - \mu + i(N - k)\gamma]},$$

and

$$\theta(\lambda, \mu) = \frac{\sinh[\lambda - \mu - i(N - 1)\gamma] \sinh[\lambda - \mu + i\gamma]}{\sinh[\lambda - \mu + i(N - 1)\gamma] \sinh[\lambda - \mu - i\gamma]}.$$  

We close this section remarking that the off-shell part produced by a complete algebraic Bethe ansatz analysis contains much more information than we would think at first sight [5]. Indeed, the semi-classical limit of the off-shell terms play an important role in the solution of Gaudin like models and Knizhnik-Zamolodchikov equations [5]. Due to the factorizability of the off-shell data in terms of simple “two-body” functions their semi-classical study should not be complicated. It is expected that such analysis will give us a representation for the solution of the trigonometric Knizhnik-Zamolodchikov equation corresponding to higher spin representations of $U_q[\text{SU}(2)]$. Recall that this study has so far been pursued in the particular case of the six ($s = 1/2$) and nineteen ($s = 1$) vertex models [21, 22].

4 Colored Vertex Models

The aim of this section is to discuss the transfer matrix diagonalization of solvable $U(1)$ vertex models directly connected to non-generic braid group representations first discovered by Couture, Lee and Schmeing [23]. Such braid solution was then generalized to include color variables on the braid strings leading to the proposition of new $N$-state vertex models [13, 14, 15]. These models have been also viewed as Yang-Baxter solutions associated with the finite dimensional representation of $U_q[\text{SU}(2)]$ when $q$ is a root of unity [16, 17].
4.1 Additive $R$-matrices

From colored braid matrices it is possible to construct $R$-matrices that are additive with respect to the spectral parameters. This happens when we consider that the color variable attached to the $i$-th string is the same for all $i$-th indices playing the role of an extra continuous parameter denoted here by $\bar{\gamma}$. This variable characterizes the additional freedom of non-cyclic irreducible representation of quantum group at roots of unity \[16\]. To avoid confusion with the deformation parameter of the previous section we shall denote the roots of unity by the variable $\omega$,

$$\omega = \exp\left[\frac{2\pi ik}{N}\right] \text{ for } k \text{ and } N \text{ coprime.}$$  \hspace{1cm} (39)

This means that out of the possible values $k = 1, \cdots, N - 1$ the only admissible ones are those that are prime with $N$. The underlying $U(1)$ symmetry guarantees that the corresponding braid $\hat{S}(\bar{\gamma}, \omega)$ can once again be represented as,

$$\hat{S}(\bar{\gamma}, \omega) = \sum_{\substack{a,b,c,d=1 \atop a\geq d, c\geq b}}^{N} S_{c,d}^{a,b}(\bar{\gamma}, \omega) e_{b,d} \otimes e_{a,c}. \hspace{1cm} (40)$$

Considering the results of \[13\] we find that the amplitudes $S_{c,d}^{a,b}(\bar{\gamma}, \omega)$ can be written as,

$$S_{c,d}^{a,b}(\bar{\gamma}, \omega) = \frac{\omega^{(b-1)(d-1)} \exp[\bar{\gamma}(b+d-2)]}{H(\omega, a-d)} \sqrt{\frac{H(\omega, a-1)H(\omega, c-1)}{H(\omega, d-1)H(\omega, b-1)}} \times \sqrt{\frac{H(\exp(2\bar{\gamma}), a-1)H(\exp(2\bar{\gamma}), c-1)}{H(\exp(2\bar{\gamma}), d-1)H(\exp(2\bar{\gamma}), b-1)}} \delta_{a+b,c+d}, \hspace{1cm} (41)$$

where function $H(x, n)$ denotes the following factorial product,

$$H(x, n) = \prod_{k=0}^{n-1} (1 - xw^k). \hspace{1cm} (42)$$

The construction of spectral parameter dependent $R$-matrices for models based on the colored braids was first discussed for general $N$ by Deguchi and Akutsu \[14, 15\] within the quantum group framework. Here we shall present an alternative manner to generate a solution of the Yang-Baxter equation from the braid representation \[40, 42\]. This procedure is usually called Baxterization.
and it is able to produce additive $R$-matrices directly on the Weyl basis for arbitrary values of $N$. This analysis offers us a practical computational way to determine the Boltzmann weights and to our knowledge it is original in the literature. The first step in this method is to examine the eigenvalue structure of the braid $[25]$. The diagonalization of the braid $[40,41]$ reveals us that it satisfies the following polynomial relation,

$$
\prod_{i=1}^{N} [\hat{S}(\bar{\gamma}, \omega) - \xi_i I_N \otimes I_N] = 0,
$$

(43)

where the $N$ distinct eigenvalues $\xi_i$ are,

$$
\xi_i = (-1)^{i+1} w^{\frac{(i-2)(i-1)}{2}} \exp[2\bar{\gamma}(i - 1)].
$$

(44)

The knowledge of the eigenvalues of the braid permits us to formally decompose it as $\hat{S}(\bar{\gamma}, \omega) = \sum_{i=1}^{N} \xi_i \hat{P}_i(\bar{\gamma}, \omega)$ where $\hat{P}_i(\bar{\gamma}, \omega)$ is the projector on the subspace $\xi_i$,

$$
\hat{P}_i(\bar{\gamma}, \omega) = \prod_{\substack{k=1\atop k \neq i}}^{N} \frac{[\hat{S}(\bar{\gamma}, \omega) - \xi_k]}{\xi_i - \xi_k}.
$$

(45)

The form of the $R$-matrices derived in the context of the quantum group framework suggests us to consider the following ansatz for $\hat{R}(\lambda)$,

$$
\hat{R}(\lambda, \mu) = \sum_{i=1}^{N} \rho_i(\lambda - \mu) \hat{P}_i(\bar{\gamma}, \omega).
$$

(46)

The scalar functions $\rho_i(\lambda)$ can be fixed by means of the unitarity property $\hat{R}(\lambda, \mu)\hat{R}(\mu, \lambda) = I_N \otimes I_N$ as well as by imposing that the original braid should be recovered when one takes the spectral parameter to infinity. The simplest functional form for $\rho_i(\lambda)$ fulfilling such properties is,

$$
\rho_i(\lambda) = \prod_{k=i}^{N-1} \frac{1 + \exp[2\lambda] \frac{\xi_{k+1}}{\xi_k}}{\left(\exp[2\lambda] + \frac{\xi_{k+1}}{\xi_k}\right)}.
$$

(47)

Now the remaining freedom we have at hand to fix the underlying $R$-matrix is only concerned with the permutation of the $N$ eigenvalues $\xi_i$. The suitable ordering of $\xi_i$ is selected out by
imposing that the ansatz (46,47) for the $R$-matrix indeed satisfy the Yang-Baxter equation.

Putting the above considerations altogether we find that the corresponding $R$-matrix is,

$$
\tilde{R}(\lambda, \mu) = \sum_{l=1}^{N-1} \prod_{j=l}^{N-1} \frac{\sinh \frac{i\pi k(j-1)}{N} + \gamma + \lambda - \mu}{\sinh \frac{i\pi k(j-1)}{N} + \gamma - \lambda + \mu} P_l(\gamma, \omega). \tag{48}
$$

As before the $R$-matrix representation (48) provides us the means to compute the respective weights $R_{a,b}^{c,d}(\lambda, \mu)$ for moderate large values of $N$. This is the basic ingredient to determine the Bethe ansatz properties of this type of vertex model. Once again by using Eqs.(21,23) we find that the corresponding eigenvalue is,

$$
\Lambda_n(\lambda) = \sum_{a=1}^{N} \prod_{l=1}^{L} \frac{\sinh[\lambda - \mu_l + \frac{i\pi k(j-1)}{N}]}{\sinh[\lambda - \mu_l + \gamma + \frac{i\pi k(j-1)}{N}]} \times \prod_{i=1}^{n} \frac{\sinh[\lambda - \lambda_i - \frac{i\pi k}{N}(1-a)] \sinh[\lambda - \lambda_i + \frac{i\pi k}{N}(2-a)]}{\sinh[\lambda - \lambda_i - \frac{i\pi k}{N}] \sinh[\lambda - \lambda_i + \frac{i\pi k}{N}]} \sinh[\lambda - \lambda_i - \frac{i\pi k}{N}], \tag{49}
$$

provided that the rapidities $\lambda_j$ satisfy the following Bethe ansatz equations,

$$
\prod_{l=1}^{L} \frac{\sinh[\lambda_j - \mu_l + \gamma]}{\sinh[\lambda_j - \mu_l]} = \prod_{i=1}^{n} \frac{\sinh[\lambda_j - \lambda_i - \frac{i\pi k}{N}]}{\sinh[\lambda_j - \lambda_i + \frac{i\pi k}{N}]} \sinh[\lambda_j - \lambda_i - \frac{i\pi k}{N}], \quad j = 1, \ldots, n. \tag{50}
$$

Here we remark that the on-shell Bethe ansatz results (49,50) have been previously discussed in the literature [26]. The derivations are sketched according to the lines used to solve the XXZ-s chain through a partial algebraic Bethe ansatz analysis. The Bethe ansatz equations are obtained by using the hypothesis of analyticity of the proposed eigenvalues and the off-shell structure is not presented. However, we point out that even the on-shell Bethe ansatz results proposed in [26] are not complete. The main branch $k = 1$ for $N$ odd is not predicted as well as the many other possible choices of $k$ for a given $N$ have been overlooked. There exists also an unnecessary distinction whether the dimension of the representation is even or odd. In this sense our findings (49,50) provide a non-trivial complement to those proposed in [26] for the on-shell properties.

We shall now discuss the results for the corresponding off-shell amplitudes. This is again done along the lines described in Appendix A. However, the computations of the respective constants are more cumbersome than that of the XXZ-s model due to the many possible branches $k$ for
The final results are,

$$0\mathcal{F}_1^{(a)}(\lambda, \mu) = -1\mathcal{F}_1^{(a)}(\lambda, \mu) = \exp[\mu - \lambda] \frac{\sqrt{\sinh[\gamma]\sinh[\frac{i\pi k}{N}a]\sinh[\frac{i\pi k}{N}(a - 1)]}}{\sqrt{\sinh[\frac{i\pi k}{N}]\sinh[\mu - \lambda - \frac{i\pi k}{N}(a - 1)]}},$$  \hspace{1cm} (51)$$

$$0\mathcal{F}_b^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b) = G_1^{(a,b)}(\bar{\gamma}) \prod_{i,j=1}^{b} \frac{\sinh[\lambda_i - \lambda_j - \bar{\gamma}]}{\sinh[\lambda_i - \lambda_j + \frac{i\pi k}{N}]} \prod_{i=1}^{b} 0\mathcal{F}_1^{(a+b-1)}(\lambda, \lambda_i),$$  \hspace{1cm} (52)$$

$$b\mathcal{F}_b^{(a)}(\lambda, \lambda_1, \ldots, \lambda_b) = G_2^{(a,b)}(\bar{\gamma}) \prod_{i,j=1}^{b} \frac{\sinh[\lambda_i - \lambda_j - \bar{\gamma}]}{\sinh[\lambda_i - \lambda_j + \frac{i\pi k}{N}]} \prod_{i=1}^{b} b\mathcal{F}_1^{(a)}(\lambda, \lambda_i).$$  \hspace{1cm} (53)$$

where the rapidity independent constants $G_1^{(a,b)}(\bar{\gamma})$ and $G_2^{(a,b)}(\bar{\gamma})$ are given by

$$G_1^{(a,b)}(\bar{\gamma}) = \prod_{l=1}^{b-1} \sqrt{\frac{\sinh[\bar{\gamma} + \frac{i\pi k}{N}(a + l - 2)]\sinh[\frac{i\pi k}{N}(a + b - 1 - l)]}{\sinh[\bar{\gamma} + \frac{i\pi k}{N}(a + b - 2)]\sinh[\frac{i\pi k}{N}(a + b - 1)]}},$$  \hspace{1cm} (54)$$

$$G_2^{(a,b)}(\bar{\gamma}) = \prod_{l=1}^{b-1} \sqrt{\frac{\sinh[\bar{\gamma} + \frac{i\pi k}{N}(a + l - 1)]\sinh[\frac{i\pi k}{N}(a + b - l)]}{\sinh[\bar{\gamma} + \frac{i\pi k}{N}(a - 1)]\sinh[\frac{i\pi k}{N}a]}}. \hspace{1cm} (55)$$

The main ingredients to calculate the off-shell properties are completed by presenting the explicit expressions for functions $R_0^{a}^{a,1}(\lambda, \mu)$ and $\theta(\lambda, \mu)$. For this model they are given by,

$$R(\lambda, \mu)_{a,1}^{a,1} = \prod_{j=1}^{a-1} \frac{\sinh[\lambda - \mu + \frac{i\pi k}{N}(j - 1)]}{\sinh[\lambda - \mu + \bar{\gamma} + \frac{i\pi k}{N}(j - 1)]},$$  \hspace{1cm} (56)$$

and

$$\theta(\lambda, \mu) = \frac{\sinh[\lambda - \mu - \frac{i\pi k}{N}]\sinh[\lambda - \mu - \bar{\gamma}]}{\sinh[\lambda - \mu + \frac{i\pi k}{N}]\sinh[\lambda - \mu + \bar{\gamma}]}. \hspace{1cm} (57)$$

We would like to conclude this section with the following remarks. Direct inspection of Eqs. (49-57) reveals us that for the special point $\bar{\gamma} = -(N - 1)\frac{i\pi k}{N}$ we are able to recover the corresponding results (31-38) concerning the solution of the XXZ-$s$ model with anisotropy $\gamma = -\frac{i\pi k}{N}$. We next note that the braid structure of the model at roots of unity is richer than that of the braid associated to the XXZ-$s$ model. In fact, by substituting $\omega = q$ and $\bar{\gamma} = i(N - 1)\gamma$ in Eq.(41) we can reproduce the braid (29) of the XXZ-$s$ model up to a multiplicative normalization. This means that at least formally one can use such prescription in the Bethe ansatz results of this section to obtain the corresponding ones of the XXZ-$s$ model.
4.2 Non-additive $R$-matrices

Colored braid matrices can in general be thought as $R$-matrices depending on two independent spectral parameters. In this case the braid carries two color variables $\lambda$ and $\mu$ attached to neighbor strings which play the role of continuous variables. This two-parameter braid $\hat{S}(\lambda, \mu)$ satisfies the following generalized braid relation,

$$[I_N \otimes \hat{S}(\lambda_1, \lambda_2)][\hat{S}(\lambda_1, \lambda_3) \otimes I_N][I_N \otimes \hat{S}(\lambda_2, \lambda_3)] = [\hat{S}(\lambda_2, \lambda_3) \otimes I_N][I_N \otimes \hat{S}(\lambda_1, \lambda_3)][\hat{S}(\lambda_1, \lambda_2) \otimes I_N],$$

(58)

where $\lambda_i$ are the color variables on the strings.

It is immediate to see that solutions of the colored braid relation (58) provide us integrable vertex models with non-additive $R$-matrices $R(\lambda, \mu)$ upon the identification,

$$R(\lambda, \mu) = PS(\lambda, \mu).$$

(59)

It turns out that the class of braid discussed in section (4.1) admits such color extension [13, 14, 15]. The color variables distinguish different representations of $U_q[SU(2)]$ with dimension $N$ when $q$ is a root of unity [16, 17]. The simplest case $N = 2$ is directly related to the Felderhof parameterization of the free-fermion models [27]. Its $R$-matrix is that of a six-vertex model whose weights satisfy the free-fermion condition,

$$R(\lambda, \mu) = (1 - \lambda \mu)(e_{1,1} \otimes e_{1,1} + e_{2,2} \otimes e_{2,2}) + (\lambda - \mu)(e_{1,1} \otimes e_{2,2} - e_{2,2} \otimes e_{1,1})$$

$$+ \sqrt{(1 - \lambda^2)(1 - \mu^2)}(e_{1,2} \otimes e_{2,1} + e_{2,1} \otimes e_{1,2}).$$

(60)

For $N \geq 3$ new vertex models start to emerge. The three-state case $N = 3$ turns out to be
an interesting nineteen-vertex model whose $R$-matrix in our notation is,

$$R(\lambda, \mu) = R(\lambda, \mu)_{1,1}^{1,1}(e_{1,1} \otimes e_{1,1} + e_{3,3} \otimes e_{3,3}) + R(\lambda, \mu)_{1,2}^{1,2}(e_{1,1} \otimes e_{2,2} - e_{2,2} \otimes e_{1,1})$$

$$+ R(\lambda, \mu)_{1,3}^{2,1}(e_{1,2} \otimes e_{2,1} + e_{2,1} \otimes e_{1,2}) + R(\lambda, \mu)_{1,3}^{1,3}e_{1,1} \otimes e_{3,3}$$

$$+ R(\lambda, \mu)_{2,1}^{2,2}(e_{1,2} \otimes e_{3,2} + e_{2,1} \otimes e_{2,3}) + R(\lambda, \mu)_{1,3}^{3,1}(e_{1,3} \otimes e_{3,1} + e_{3,1} \otimes e_{1,3})$$

$$+ R(\lambda, \mu)_{2,2}^{2,2}e_{2,2} \otimes e_{2,2} + R(\lambda, \mu)_{2,3}^{3,1}(e_{2,3} \otimes e_{2,1} + e_{3,2} \otimes e_{1,2})$$

$$+ R(\lambda, \mu)_{2,3}^{2,3}(e_{2,2} \otimes e_{3,3} - e_{3,3} \otimes e_{2,2}) + R(\lambda, \mu)_{2,3}^{3,2}(e_{2,3} \otimes e_{3,2} + e_{3,2} \otimes e_{2,3})$$

$$+ R(\lambda, \mu)_{3,1}^{3,1}e_{3,3} \otimes e_{1,1},$$

(61)

where the Boltzmann weights amplitudes $R(\lambda, \mu)_{a,b}^{c,d}$ are given by [13]

$$R(\lambda, \mu)_{1,1}^{1,1} = (1 - \mu \lambda)(1 - \mu \lambda w),$$

(62)

$$R(\lambda, \mu)_{1,2}^{1,2} = (\lambda - \mu)(1 - \mu \lambda w),$$

(63)

$$R(\lambda, \mu)_{1,2}^{2,1} = (1 - \mu \lambda w)\sqrt{(1 - \mu^2)(1 - \lambda^2)},$$

(64)

$$R(\lambda, \mu)_{1,3}^{1,3} = (\lambda - \mu)(\lambda - \mu w),$$

(65)

$$R(\lambda, \mu)_{1,3}^{2,2} = (\lambda - \mu)\sqrt{(1 - \lambda^2)(1 - \mu^2 w)(1 + w)},$$

(66)

$$R(\lambda, \mu)_{1,3}^{3,1} = \sqrt{(1 - \lambda^2)(1 - \lambda^2 w)(1 - \mu^2)(1 - \mu^2 w)},$$

(67)

$$R(\lambda, \mu)_{2,2}^{2,2} = (1 - \lambda^2)(1 - \mu^2 w) - (\mu - \lambda)(\mu - \lambda w),$$

(68)

$$R(\lambda, \mu)_{2,2}^{3,1} = (\mu - \lambda)\sqrt{(1 - \mu^2)(1 - \lambda^2 w)(1 + w)},$$

(69)

$$R(\lambda, \mu)_{2,3}^{2,3} = (1 + w)(\lambda - \mu)(1 - \mu \lambda),$$

(70)

$$R(\lambda, \mu)_{2,3}^{3,2} = (1 - \mu \lambda)\sqrt{(1 - \mu^2 w)(1 - \lambda^2 w)},$$

(71)

$$R(\lambda, \mu)_{3,1}^{3,1} = (\mu - \lambda)(\mu - \lambda w),$$

(72)

such that $\omega = \exp(\frac{2\pi i}{3})$ or $\omega = -\exp(\frac{2\pi i}{3}).$

For this family of vertex models the expression for the respective $R$-matrix are rather involving for arbitrary $N$ [13, 14, 15, 17]. In the case $N = 4$, we recall that the explicit formulae for all the weights $R_{a,b}^{c,d}(\lambda, \mu)$ are available in [13] and for sake of completeness we include them in Appendix
B. Fortunately, by substituting the \( R \)-matrix amplitudes for \( N = 2, 3, 4 \) in Eqs.\((21,23)\), we noticed that such on-shell Bethe ansatz results have a uniform dependence on \( N \). This makes possible to propose the transfer matrix eigenvalues for general \( N \),

\[
\Lambda_n(\lambda) = \sum_{a=1}^{N} \prod_{j=1}^{L} \left[ \prod_{i=a}^{N-1} (1 - \lambda \mu_j w^{i-1}) \prod_{i=1}^{a-1} (\mu_j - \lambda w^{i-1}) \right] \prod_{j=1}^{n} \frac{(1 - \lambda \lambda_j)(\lambda - \lambda_j w)}{\lambda w^{a-1} - \lambda_j \lambda w^{a-2} - \lambda_j}, \tag{73}
\]

while the Bethe ansatz equations for the variables \( \lambda_j \) are,

\[
\prod_{j=1}^{L} \frac{1 - \lambda \mu_j}{\mu_j - \lambda_l} = \prod_{j=1}^{n} \frac{\lambda_l - \lambda_j w}{\lambda l w - \lambda_j}, \quad l = 1, \ldots, n. \tag{74}
\]

The same observation made above also works for the off-shell properties. A case by case analysis of Eqs.\((14-17)\) up to \( N = 4 \), following the strategy of Appendix A, is able to reveal us the general pattern for functions \( 0F^{(a)}_1(\lambda, \mu) \) and \( bF^{(a)}_1(\lambda, \lambda_1, \ldots, \lambda_b) \), namely

\[
0F^{(a)}_1(\lambda, \mu) = -F^{(a)}_1(\lambda, \mu) = \frac{\sqrt{(1 - w^a)(1 - w^{a-1}\lambda^2)(1 - \mu^2)}}{\sqrt{1 - w(\mu - w^{a-1}\lambda)}}, \tag{75}
\]

\[
0F^{(a)}_b(\lambda, \lambda_1, \ldots, \lambda_b) = w^{-\frac{b(b-1)}{2}} \prod_{i,j=1}^{b} \frac{(1 - \lambda_i \lambda_j)}{(w \lambda_i - \lambda_j)} \prod_{i=1}^{b-1} \sqrt{\frac{1 - w^{a+b-1-i}}{1 - w^{a+b-1}}} \left( \frac{1 - \lambda^2 w^{a+i-2}}{1 - \lambda^2 w^{a+b-2}} \right)
\times \prod_{i=1}^{b} 0F^{(a+b-1)}_1(\lambda, \lambda_i), \tag{76}
\]

\[
bF^{(a)}_b(\lambda, \lambda_1, \ldots, \lambda_b) = \prod_{i,j=1}^{b} \frac{(1 - \lambda_i \lambda_j)}{(w \lambda_i - \lambda_j)} \prod_{i=1}^{b-1} \sqrt{\frac{1 - w^{a+b-1-i}}{1 - w^{a}}} \left( \frac{1 - \lambda^2 w^{a+i-1}}{1 - \lambda^2 w^{a-1}} \right)
\times \prod_{i=1}^{b} F^{(a)}_1(\lambda, \lambda_i). \tag{77}
\]

As before the off-shell amplitudes are completely determined by presenting the respective functions \( R^{a,1}_{a,1}(\lambda, \mu) \) and \( \theta(\lambda, \mu) \). For this model they are,

\[
R^{a,1}_{a,1}(\lambda, \mu) = \prod_{i=1}^{a-1} (\mu - \lambda w^{i-1}) \prod_{i=a}^{N-1} (1 - \mu \lambda w^{i-1}), \tag{78}
\]
and
\[ \theta(\lambda, \mu) = -\frac{\lambda - \mu w}{\lambda w - \mu}. \] (79)

We close this section by mentioning that a colored vertex model with an infinite number of edge states has also been proposed by Deguchi and Akutsu [14, 15]. This system is invariant by the $U(1)$ symmetry and therefore can in principle be solved within the algebraic Bethe ansatz approach developed by the authors of this paper [1]. We hope to address the problem of presenting the on-shell and off-shell Bethe ansatz properties of this model in a future publication.

5 Non-Compact $SL(2, \mathcal{R})$ model

In this section we present the on-shell and off-shell algebraic Bethe ansatz properties of the vertex model based on the discrete $D_s^{-}$ representation of the $SL(2, \mathcal{R})$ algebra. The interest in the study of such non-compact vertex models emerged from the discovery of integrable structures in high energy QCD scattering [28, 29, 30]. It has been argued that the scale dependence of certain scattering amplitudes is governed by the eigenspectrum of exactly solvable Hamiltonians with $SL(2, \mathcal{R})$ symmetry [31]. In recent years similar connection was found in the context of the duality between $\mathcal{N} = 4$ Yang-Mills gauge theory and the world-sheet theory of strings on $AdS^5 \times S^5$ [32, 33]. In particular, the simplest non-compact subsector of the one-loop dilatation operator of the $\mathcal{N} = 4$ gauge theory is directly related to the $s = -\frac{1}{2}$ integrable $SL(2, \mathcal{R})$ spin magnet [34].

We start by recalling that the $SL(2, \mathcal{R})$ algebra is generated by three operators obeying the following commutation rules,
\[ [S^z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z. \] (80)

Here we will consider integrable models associated to the discrete highest weight $D_s^{-}$ representation of $SL(2, \mathcal{R})$. This representation is labeled by the generalized spin variables such that $s \in \mathcal{R}^-$ for the universal covering group $SL(2, \mathcal{R})$ [35]. The respective states can be represented in terms of the following infinite dimensional angular momenta basis,
\[ |s, m + s\rangle, \quad m = 0, 1, \ldots, \infty, \] (81)
where $S^z|s,m+s⟩ = (m+s)|s,m+s⟩$ and $S^+|s,s⟩ = 0$. The state $|s,s⟩$ plays therefore the role of a highest weight vector.

The simplest $R$-matrix associated to the $SL(2,R)$ algebra has a two dimensional auxiliary space. It can be viewed as a $2 \times 2$ matrix given by \[ R_{1/2, s}(\lambda, \mu) = \begin{pmatrix} (\lambda - \mu)I_s + iS^z & iS^- \\ iS^+ & (\lambda - \mu)I_s - iS^z \end{pmatrix}, \] where $I_s$ denotes an infinite dimensional unity matrix.

However, to construct non-compact spin magnets described by next-neighbor Hamiltonians we have to consider a $R$-matrix whose auxiliary space belongs to the infinite dimensional representation $D_s^-$. Such $R$-matrix turns out to be a non-trivial generalization of (82) and has the following form [8],

\[ R_{s,s}(\lambda, \mu) = \sum_{j=0}^{\infty} \prod_{k=1}^{j} \left( \frac{k_l + \lambda - \mu}{k_l - \lambda + \mu} \right) \hat{P}_j(s), \]

where $\hat{P}_j(s)$ are operators projecting the tensor product space $D_s^- \otimes D_s^-$ on the representation with total spin $j$.

The corresponding Hamiltonian can be derived by taking the logarithmic derivative of the transfer matrix \[ T_{a,b}(\lambda, \mu) \] whose respective weights $R(\lambda, \mu)_{a,b}^{c,d}$ are obtained from the $R$-matrix (83). We recall that this derivative is computed at the regular point $\lambda = 0$ and also by setting the inhomogeneities $\mu_l$ to zero. The Hamiltonian is then given by the standard expression,

\[ H = \sum_{l=1}^{L-1} H_{i,i+1}(s) + H_{L,1}(s), \]

where $H_{1,2}(s) = \frac{d}{d\lambda} ln R_{s,s}(\lambda,0)|_{\lambda=0}$.

As far as we know the explicit expression for density Hamiltonian $H_{1,2}(s)$ has only been discussed in the particular case $s = -\frac{1}{2}$ [34]. In what follows we shall complement this result by presenting the action of $H_{1,2}(s)$ on the tensor product of states (81) for arbitrary $s \in \mathcal{R}^-$. We emphasize that this knowledge is essential for practical applications of such non-compact spin magnets.
chains. The computation is somehow involving but the final result is rather simple.

\[
H_{1,2}(s) |m_1, m_2\rangle = \left[ \sum_{k=1}^{m_1} h_1(k) + \sum_{k=1}^{m_2} h_1(k) \right] |m_1, m_2\rangle \\
+ \sum_{k=1}^{m_1} h_2(k, m_2, m_1) |m_1 - k, m_2 + k\rangle \\
+ \sum_{k=1}^{m_2} h_2(k, m_1, m_2) |m_1 + k, m_2 - k\rangle ,
\]

(85)

where \( |m_1, m_2\rangle \) denotes a sort notation for the tensor product \( |s, s + m_1\rangle \otimes |s, s + m_2\rangle \) state. In addition, functions \( h_1(k) \) and \( h_2(k, m_1, m_2) \) are given by

\[
h_1(k) = \frac{2s}{2s + 1 - k},
\]

(86)

\[
h_2(k, m_1, m_2) = \frac{2s}{k} \prod_{i=1}^{k} \sqrt{\frac{(m_1 + i)}{(m_1 + i - 2s - 1)}} \frac{(m_2 + 1 - i)}{(m_2 - 2s - i)}.
\]

(87)

At the present the algebraic Bethe ansatz analysis of the vertex models has been restricted to the diagonalization of the transfer matrix whose weights are the elements of the simplest \( R \)-matrix \( R_{1/2}^{s,s} (\lambda, \mu) \) \cite{29, 36}. In this case the auxiliary space is two-dimensional and the algebraic Bethe ansatz is fairly parallel to that developed for six-vertex models \cite{2, 3}. The same problem for the transfer matrix based on the \( R \)-matrix \( R_{s,s} (\lambda, \mu) \) could in principle be handled by extending the approach used to solve the \( SU(2) \) higher spin XXX-s chain \cite{10} to the situation of an infinite-dimensional auxiliary space. This method, however, is not capable to keep track of all the unwanted terms and to find the Bethe equations from the condition of their equality to zero. This drawback prevents us to find a complete algebraic Bethe ansatz solution and consequently to obtain information on the off-shell properties.

By way of contrast we certainly can use the results of section 2 to tackle the problem for the transfer matrix constructed from the \( R \)-matrix \( R_{s,s} (\lambda, \mu) \). The suitable reference state for this

\[\text{We have set the overall normalization such that } H_{1,2}(s) |0, 0\rangle = 0.\]

\[\text{Note that for } s = -\frac{1}{2} \text{ functions } h_1(k) \text{ and } h_2(k, m_1, m_2) \text{ drastically simplify to the form } \pm \frac{1}{k} \text{ of harmonic numbers.}\]
model is constructed by considering the tensor product of highest vector $|s, s\rangle$,

$$|0\rangle = \prod_{i=1}^{L} \otimes |s, s\rangle_i.$$  \(88\)

Our next task is to compute the $R$-matrix elements of $R_{s,s}(\lambda, \mu)$ that are necessary to calculate the corresponding on-shell and off-shell properties. In general this task is rather complicated since both the auxiliary and quantum spaces of $|\lambda, \mu\rangle$ are infinite dimensional. This problem can be circumvented by exploring the $U(1)$ invariance and expressing the $R$-matrix in terms of the sectors $n$ labeling the eigenvalues of the $U(1)$ operator. More precisely, we can always decompose $R_{s,s}(\lambda, \mu)$ as,

$$R_{s,s}(\lambda, \mu) = \sum_{n=0}^{\infty} \sum_{a,c=1}^{n+1} R(\lambda, \mu)^{c,n+2-c}_{a,n+2-a} e_{a,c} \otimes e_{n+2-a,n+2-c}.$$  \(89\)

We now project out the operators $\hat{P}_j(s)$ on a given sector $n$ and by comparing Eqs.$(83,89)$ we are able to calculate the weights $R(\lambda, \mu)^{c,n+2-c}_{a,n+2-a}$. As concrete examples of this approach we have summarized in Appendix C such amplitudes up to sector $n = 4$ for arbitrary value of $s \in \mathcal{R}$. By substituting such elements in Eq.$(21,23)$ and carrying on some algebraic simplifications one observes that each transfer matrix eigenvalue term has a very simple dependence on sector $n$. This enables us to propose the exact expression for the eigenvalues,

$$\Lambda_n(\lambda) = \sum_{a=1}^{\infty} \prod_{l=k=1}^{L} \frac{[\lambda - \mu_l - (k-1)\delta]}{[\lambda - \mu_l + (2s + 1 - k)\delta]} \times \prod_{i=1}^{n} \frac{[\lambda - \lambda_i - 2s\delta][\lambda - \lambda_i + \delta]}{[\lambda - \lambda_i - (a-1)\delta][\lambda - \lambda_i - (a-2)\delta]}.$$  \(90\)

The corresponding Bethe ansatz equations are fixed from the knowledge of the $R$-matrix elements up to the sector $n = 2$. We find that the rapidities $\lambda_j$ satisfy the following equation,

$$\prod_{l=1}^{L} \frac{(\lambda_j - \mu_l + 2s\delta)}{(\lambda_j - \mu_l)} = \prod_{i=1}^{n} \frac{(\lambda_j - \lambda_i + \delta)}{(\lambda_j - \lambda_i - \delta)} \quad j = 1, \ldots, n.$$  \(91\)

Before proceeding we remark that Eq.$(91)$ can be reproduced within the coordinate Bethe ansatz formulation for the non-compact Hamiltonian $|\lambda, \mu\rangle$. This has been done in the very

\footnote{The Bethe ansatz equations $|\lambda, \mu\rangle$ can be symmetrized through the shift $\lambda_j \rightarrow \lambda_j - s\delta$.}
special case of the spin \( s = -\frac{1}{2} \) up to the two particle excitation sector \([37]\). For sake of completeness we have presented an extension of this analysis for arbitrary values of \( s \) in Appendix D. In the case of integrable theories this is enough to supply us the main form of the Bethe equations for the rapidities but not sufficient to provide us the general structure of the eigenvectors. The situation is even more complicated for non-compact systems due to the possibility of an infinite number of particle excitations per site.

However, an algebraic representation for the eigenvectors of the non-compact vertex model can formally be obtained by taking the limit \( N \to \infty \) in Eq. (11). Therefore, to benefit from the knowledge of the eigenvectors we have to compute the off-shell functions \( bF_b^{(2)}(\lambda, \lambda_1, \cdots, \lambda_b) \) where now the indices \( a \) and \( b \) are unlimited. Fortunately, this computation can be implemented by using the same strategy explained above for the on-shell data. It turns out that the final results for the off-shell properties are,

\[
oF_1^{(a)}(\lambda, \mu) = -1F_1^{(a)}(\lambda, \mu) = -\sqrt[2]{2s}(2s + 1 - a) \frac{2s + 1 - a}{\mu - \lambda + (a - 1)i} \tag{92}
\]

\[
oF_b^{(a)}(\lambda, \lambda_1, \cdots, \lambda_b) = \prod_{k=1}^{b-1} \sqrt{\frac{(2s + 2 - a - k)(a + b - 1 - k)}{(2s + 2 - a - b)(a + b - 1)}} \prod_{i, j=1}^{b} \frac{(\lambda_i - \lambda_j - 2si)}{(\lambda_i - \lambda_j - i)} \times \prod_{i=1}^{b} oF_1^{(a+b-1)}(\lambda, \lambda_i), \tag{93}
\]

\[
bF_b^{(a)}(\lambda, \lambda_1, \cdots, \lambda_b) = \prod_{k=1}^{b-1} \sqrt{\frac{(2s + 1 - a - k)(a + b - k)}{(2s + 1 - a)}} \prod_{i, j=1}^{b} \frac{(\lambda_i - \lambda_j - 2si)}{(\lambda_i - \lambda_j - i)} \times \prod_{i=1}^{b} |iF_1^{(a)}(\lambda, \lambda_i), \tag{94}
\]

where \( a, b = 1, \ldots, \infty \).

The off-shell data is completed by exhibiting functions \( R_{a,1}^{a,1}(\lambda, \mu) \) and \( \theta(\lambda, \mu) \). For such non-compact vertex model they are,

\[
R(\lambda, \mu)_{a,1}^{a,1} = \prod_{k=1}^{a-1} \frac{[\lambda - \mu - (k - 1)i]}{[\lambda - \mu + (2s + 1 - k)i]}, \tag{95}
\]

24
and

\[ \theta(\lambda, \mu) = \frac{(\lambda - \mu - 2s \bar{\gamma}) (\lambda - \mu + \bar{\gamma})}{(\lambda - \mu + 2s \bar{\gamma}) (\lambda - \mu - \bar{\gamma})}. \]  

(96)

At this point we note that the Bethe ansatz properties of the non-compact \( SL(2, \mathcal{R}) \) model can be viewed as an analytic continuation of those associated to the XXX-s model. We just have to extend the spin variable to take values on \( s \in \mathcal{R}^- \) as well as to consider the total number of degrees of freedom per site infinite. This feature can be seen by considering the isotropic limit \( \gamma \to 0 \) in the results for the XXZ-s and afterwards comparing them with the structure of Eqs.(90-96). This property, to what concerns the form of the Bethe ansatz equations, was expected from the algebraic Bethe ansatz for the \( R \)-matrix \( R_{\frac{1}{2}s}(\lambda, \mu) \) \[29, 36\]. The fact that it extends to the off-shell amplitudes is however a novelty which strengthens the relationship between representation theory and Bethe ansatz properties.

Other remarkable feature of the on-shell and off-shell results for the non-compact model is as follows. It turns out that they can also be obtained from a particular limit of those derived in section 4.1 for the vertex model based on the quantum algebra at roots of unity. This is achieved by choosing the free parameter \( \bar{\gamma} = -2s \frac{\pi k}{N} \) and also by re-scaling all the spectral variables as follows \( \lambda \to -\frac{\pi k}{N} \lambda, \lambda_j \to -\frac{\pi k}{N} \lambda_j \) and \( \mu_l \to -\frac{\pi k}{N} \mu_l \). By performing these operations in Eqs.(49-57) and afterwards taking the \( N \to \infty \) limit we indeed obtain the results (90-96) associated to the non-compact model. This fact offers us the possibility to study the properties of such non-compact model considering a well defined limit of a system with finite number of degrees of freedom. This truncated approach can be seen as one way to infer on the physical behavior of the non-compact model avoiding the complications of dealing with infinite dimensional Hilbert space. We conclude by mentioning that such mechanism also works if we use as a compact system the XXZ-s model. In this case we have to tune the spin and anisotropy by choosing for example \( \gamma = \frac{2\pi}{N} \). We then take the limit \( N \to \infty \) and as a result we are able to recover the on-shell and off-shell behavior associated to the non-compact \( s = -\frac{1}{2} \) chain. From the higher spin Heisenberg model, however, the possible values we can reach for the non-compact spin variable \( s \) are restricted.
6 Conclusions

We have presented the algebraic Bethe ansatz solution of three distinct classes of integrable vertex models that are invariant by one $U(1)$ charge symmetry. The on-shell and off-shell properties associated to the respective transfer matrix eigenvalue problems are exhibited. In particular, all the off-shell data can be presented in term of products of elementary building block functions. This fact could be of relevance to compute properties that require the knowledge of the exact form of the eigenvectors such wave-function norms and correlation functions of $U(1)$ higher spin chains \[^{38,39}\].

The first two families of vertex models are derived from the braid group representations of the quantum $U_q(SU(2))$ algebra either for generic values of the deformation parameter producing the XXZ-s chain or when it takes values on the roots of unity leading us to colored models. It is noted that the latter braid representation is richer than the one associated to generic values of $q$. Formally, this property allows one to obtain the Bethe ansatz results for the XXZ-s chain by adapting those derived for the colored vertex model. Here we remark that recently these solvable models have been discussed in the context of the calculation of their partition function on the presence of certain domain wall boundary condition \[^{40}\]. Considering that this kind of partition functions can in principle be calculated by the algebraic Bethe ansatz method \[^{41}\] it seems interesting to investigate whether the results of \[^{40}\] can be reproduced or even extended to other possible fix boundary conditions by using the algebraic Bethe ansatz framework described in this paper.

The third class of model considered is that based on the discrete $D^-_s$ representation of the $SL(2, \mathbb{R})$ algebra. We have derived the expression for the corresponding Hamiltonian acting on the standard angular momenta basis. This provides us the means to derive the coherent state representation of this integrable spin chain and to find in the continuum limit the respective two-dimensional quantum field theory. Both on-shell and off-shell Bethe ansatz results for such non-compact model can be seen as a kind of analytic continuation of those associated to the XXZ-s. In addition, we argued that these Bethe ansatz properties can be derived from the
diagonalization of the transfer matrix of the $U_q[SU(2)]$ vertex model at roots of unity. This observation offers us the possibility to unveil the physical behavior of the non-compact $SL(2, \mathcal{R})$ model from a bona fide $N \to \infty$ limit of the properties of a compact vertex model. This avoids us to deal with infinite dimensional Hilbert space specially to what concerns the study of the anti-ferromagnetic behavior of the $SL(2, \mathcal{R})$ chain.

It is reasonable to believe that the above remarks are not restricted to the $q$-deformation of the classical $SU(2)$ symmetry. In fact, the existence of colored vertex models associated to quantum groups other than $U_q[SU(2)]$ has already been outlined in the literature [14]. This then could supply us with a general method to extract information about an integrable non-compact model based on a given algebra $\mathcal{G}$ from the respective compact model associated to the $U_q[\mathcal{G}]$ deformation at roots of unity. It would be rather interesting to investigate this fact in the case of superalgebras since an extra continuum variable besides the deformation parameter is allowed [42]. In particular, if this approach could bring any new insight to exactly solved models based on the non-compact $PSU(2,2|4)$ algebra due to their apparent relevance in the understanding of the integrable properties found for planar $N = 4$ super Yang-Mills theory [32, 33, 34].

**Acknowledgments**

The authors thank the Brazilian Research Agencies FAPESP and CNPq for financial support.

**Appendix A : Off-shell Amplitudes**

In this appendix we present the technical details concerning the computation of the off-shell amplitudes $a \mathcal{F}_b^a(\lambda, \lambda_1, \cdots, \lambda_b)$ and $b \mathcal{F}_b^a(\lambda, \lambda_1, \cdots, \lambda_b)$. We first note that for $b = 1$ such amplitudes are directly computed from the knowlodge of the Boltzmann weights by using Eq.(14). For $b \geq 2$ we have to iterate the recurrence relations (15)-(17) for all the possible values of the
index $c = 0, \cdots, b$. In the simplest case $b = 2$ such equations lead us to the expressions,

$$
1_{F}^{(a)}(\lambda, \lambda_1, \lambda_2) = 0_{F}^{(a)}(\lambda, \lambda_2) \frac{R(\lambda_2, \lambda_1)_{1,1}}{R(\lambda_2, \lambda_1)_{2,1}} \ (A.1)
$$

$$
0_{F}^{(a)}(\lambda, \lambda_1, \lambda_2) = \frac{R(\lambda_1)_{a+1,1}^2}{R(\lambda)_{a+1,1}} 0_{F}^{(a+1)}(\lambda, \lambda_2) + \frac{R(\lambda_1)_{a+2,1}^3}{R(\lambda)_{a+2,1}} 1_{F}^{(2)}(\lambda, \lambda_2) \ (A.2)
$$

$$
2_{F}^{(a)}(\lambda, \lambda_1, \lambda_2) = -0_{F}^{(a)}(\lambda, \lambda_1, \lambda_2) - 1_{F}^{(a)}(\lambda, \lambda_2, \lambda_1) \frac{R(\lambda_1, \lambda_2)_{2,1}^{2,1}}{R(\lambda_1, \lambda_2)_{1,1}^{1,1}} \frac{R(\lambda_2, \lambda_1)_{2,1}^{1,1}}{R(\lambda_2, \lambda_1)_{1,1}^{1,1}} \theta(\lambda_1, \lambda_2). \ (A.3)
$$

From Eq.(A.1) we see that $1_{F}^{(a)}(\lambda, \lambda_1, \lambda_2)$ is already given in terms of product of the one-particle $b = 1$ off-shell amplitudes and the ratio of elementary weights. Therefore, we only have to carry on simplifications on the amplitudes $0_{F}^{(a)}(\lambda, \lambda_1, \lambda_2)$ and $2_{F}^{(a)}(\lambda, \lambda_1, \lambda_2)$ associated to the extremum values of the index $c = 0, 2$. This step is done by substituting in Eqs.(A.2 A.3) the previous results for the $b = 1$ off shell amplitudes as well as the expression for $1_{F}^{(a)}(\lambda, \lambda_1, \lambda_2)$. It turns out that for all the models considered in this paper we find that such amplitudes can be presented in the following factorized form,

$$
0_{F}^{(a)}(\lambda, \lambda_1, \lambda_2) = \mathcal{A}_{0}^{(a,2)} Q(\lambda_1, \lambda_2) \prod_{i=1}^{2} 0_{F}^{(a+1)}(\lambda, \lambda_i) \ (A.4)
$$

and

$$
2_{F}^{(a)}(\lambda, \lambda_1, \lambda_2) = \mathcal{A}_{1}^{(a,2)} Q(\lambda_1, \lambda_2) \prod_{i=1}^{2} 1_{F}^{(a)}(\lambda, \lambda_i) \ (A.5)
$$

The parameters $\mathcal{A}_{0}^{(a,2)}$ and $\mathcal{A}_{1}^{(a,2)}$ are constants which depend of the model we are considering but they are independent of the variables $\lambda, \lambda_1$ and $\lambda_2$. In addition, function $Q(\lambda, \mu)$ depends on the corresponding weights by an expression that is model independent, namely

$$
Q(\lambda, \mu) = \theta(\lambda, \mu) \frac{R(\lambda, \mu)_{1,1}^{1,1}}{R(\lambda, \mu)_{2,1}^{2,1}} + \frac{R(\mu, \lambda)_{1,1}^{1,1}}{R(\mu, \lambda)_{2,1}^{2,1}} \ (A.6)
$$

where $\theta(\lambda, \mu)$ is determined Eq.(13).

The factorized form of all off-shell amplitudes associated to the two-particle sector can now be used to determine the structure of the off-shell amplitudes for the next sector $b = 3$. This is
once again done by iterating the recurrence relations \((15-17)\). By performing this procedure up to the four-particle sector we conclude that the structure of functions 

\[ 0 F_b^a(\lambda, \lambda_1, \ldots, \lambda_b) \]

and

\[ b F_b^a(\lambda, \lambda_1, \ldots, \lambda_b) \]

are given by the expressions,

\[
0 F_b^a(\lambda, \lambda_1, \ldots, \lambda_b) = A^{(a,b)}_0 \prod_{i,j=1 \atop i<j}^b Q(\lambda_i, \lambda_j) \prod_{i=1}^b 0 F_{b-1}^{(a+b-1)}(\lambda, \lambda_i), \quad (A.7)
\]

and

\[
b F_b^a(\lambda, \lambda_1, \ldots, \lambda_b) = A^{(a,b)}_1 \prod_{i,j=1 \atop i<j}^b Q(\lambda_i, \lambda_j) \prod_{i=1}^b 1 F_{b-1}^{(a)}(\lambda, \lambda_i), \quad (A.8)
\]

where \(A^{(a,b)}_0\) and \(A^{(a,b)}_1\) are model dependent constants.

Having found the functional dependence of the off-shell functions on the rapidities the next task is to determine the constants \(A^{(a,b)}_0\) and \(A^{(a,b)}_1\). This is done by direct comparison of explicit calculations for functions 

\[ 0 F_b^a(\lambda, \lambda_1, \ldots, \lambda_b) \]

and

\[ b F_b^a(\lambda, \lambda_1, \ldots, \lambda_b) \]

with the expression given by Eqs.\((A.7,A.8)\). The explicit computation of such constants are more cumbersome for the colored vertex model due to the many possible branches.
Appendix B : Non-additive $N = 4$ $R$-matrix

Here we rewrite the colored $R$-matrix for $N = 4$ given in [13] considering the notation used in this paper. The result is,

\[
\begin{align*}
R_{12}(\lambda, \mu) &= R(\lambda, \mu)^{11}_{1,1}(e_{1,1} \otimes e_{1,1} + e_{4,4} \otimes e_{4,4}) + R(\lambda, \mu)^{12}_{1,2}(e_{1,1} \otimes e_{2,2} - e_{2,2} \otimes e_{1,1}) \\
&+ R(\lambda, \mu)^{21}_{1,2}(e_{1,2} \otimes e_{1,2} + e_{2,1} \otimes e_{1,2}) + R(\lambda, \mu)^{13}_{1,3}e_{1,1} \otimes e_{3,3} \\
&+ R(\lambda, \mu)^{22}_{1,3}(e_{1,2} \otimes e_{3,2} + e_{2,1} \otimes e_{2,3}) + R(\lambda, \mu)^{31}_{1,3}(e_{1,3} \otimes e_{3,1} + e_{3,1} \otimes e_{1,3}) \\
&+ R(\lambda, \mu)^{41}_{1,4}(e_{1,4} \otimes e_{4,1} + e_{4,1} \otimes e_{4,1}) + R(\lambda, \mu)^{32}_{1,4}(e_{1,3} \otimes e_{4,2} + e_{3,1} \otimes e_{2,4}) \\
&+ R(\lambda, \mu)^{33}_{1,4}(e_{1,3} \otimes e_{4,3} + e_{2,1} \otimes e_{3,4}) + R(\lambda, \mu)^{14}_{1,4}(e_{1,1} \otimes e_{1,4} + e_{4,1} \otimes e_{2,1}) \\
&+ R(\lambda, \mu)^{24}_{1,4}(e_{2,3} \otimes e_{1,1} + e_{3,2} \otimes e_{1,2}) + R(\lambda, \mu)^{22}_{2,2}(e_{2,2} \otimes e_{2,2}) \\
&+ R(\lambda, \mu)^{23}_{2,2}(e_{2,2} \otimes e_{3,2} + e_{3,2} \otimes e_{2,3}) + R(\lambda, \mu)^{23}_{3,2}(e_{3,2} \otimes e_{2,3} + e_{2,3} \otimes e_{3,2}) \\
&+ R(\lambda, \mu)^{31}_{3,1}e_{3,3} \otimes e_{1,1} + R(\lambda, \mu)^{33}_{3,3}(e_{3,3} \otimes e_{3,3} + R(\lambda, \mu)^{41}_{4,1}(e_{4,1} \otimes e_{1,1} + R(\lambda, \mu)^{42}_{4,2}(e_{4,4} \otimes e_{2,2}) \\
&+ R(\lambda, \mu)^{24}_{2,4}(e_{2,2} \otimes e_{4,4} + R(\lambda, \mu)^{44}_{2,4}(e_{2,4} \otimes e_{3,1} + e_{4,2} \otimes e_{1,3}) \\
&+ R(\lambda, \mu)^{33}_{2,4}(e_{2,3} \otimes e_{4,3} + e_{3,2} \otimes e_{3,4}) + R(\lambda, \mu)^{42}_{2,4}(e_{2,4} \otimes e_{4,2} + e_{4,2} \otimes e_{2,4}) \\
&+ R(\lambda, \mu)^{41}_{3,2}(e_{3,4} \otimes e_{2,1} + e_{4,3} \otimes e_{1,2}) + R(\lambda, \mu)^{43}_{3,2}(e_{3,4} \otimes e_{3,2} + e_{4,3} \otimes e_{2,3}) \\
&+ R(\lambda, \mu)^{43}_{3,4}(e_{3,4} \otimes e_{4,3} + e_{4,3} \otimes e_{3,4}) + R(\lambda, \mu)^{34}_{3,4}(e_{3,3} \otimes e_{4,4} - e_{4,4} \otimes e_{3,3}). \quad (B.1)
\end{align*}
\]

The corresponding amplitudes $R(\lambda, \mu)^{cd}_{ab}$ are given by,

\[
\begin{align*}
R(\lambda, \mu)^{11}_{1,1} &= (1 - \mu \lambda)(1 - \mu \lambda w)(1 - \mu \lambda w^2), \quad (B.2) \\
R(\lambda, \mu)^{12}_{1,2} &= (\lambda - \mu)(1 - \mu \lambda w)(1 - \mu \lambda w^2), \quad (B.3) \\
R(\lambda, \mu)^{21}_{1,2} &= \sqrt{(1 - \mu^2)(1 - \lambda^2)}(1 - \mu \lambda w)(1 - \mu \lambda w^2), \quad (B.4) \\
R(\lambda, \mu)^{13}_{1,3} &= (\lambda - \mu)(\lambda - \mu w)(1 - \mu \lambda w^2), \quad (B.5) \\
R(\lambda, \mu)^{22}_{1,3} &= \sqrt{(1 - \lambda^2)(1 - \mu^2 w)(1 + w)(\lambda - \mu)(1 - \mu \lambda w)}, \quad (B.6) \\
R(\lambda, \mu)^{31}_{1,3} &= \sqrt{(1 - \lambda^2)(1 - \lambda^2 w)(1 - \mu^2)(1 - \mu^2 w)(1 - \mu \lambda w^2)}, \quad (B.7)
\end{align*}
\]
\[
R(\lambda, \mu)_{2,2}^{2,2} = (1 - \lambda^2)(1 - \mu^2 w) - (\mu - \lambda)(\mu - \lambda w)(1 - \mu \lambda w^2), \quad (B.8)
\]
\[
R(\lambda, \mu)_{2,2}^{3,1} = (\mu - \lambda)\sqrt{(1 - \mu^2)(1 - \lambda^2 w)(1 + w)(1 - \mu \lambda w^2)}, \quad (B.9)
\]
\[
R(\lambda, \mu)_{2,3}^{2,3} = (\mu - \lambda)((1 - \mu^2 \lambda^2)(1 - w^3) - w(\lambda - \mu w)(\lambda - \mu)), \quad (B.10)
\]
\[
R(\lambda, \mu)_{2,3}^{3,2} = \sqrt{(1 - \mu^2 w)(1 - \lambda^2 w)((1 - \mu^2)(1 - \lambda^2 w^2) - (1 + w)(\lambda - \mu w)(\lambda - \mu))(B.11)
\]
\[
R(\lambda, \mu)_{3,1}^{3,1} = (\mu - \lambda)(\mu - \lambda w)(1 - \lambda \mu w^2), \quad (B.12)
\]
\[
R(\lambda, \mu)_{1,4}^{1,4} = \sqrt{(1 - \lambda^2)(1 - \lambda^2 w)(1 - \lambda^2 w^2)\sqrt{(1 - \mu^2)(1 - \mu^2 w)(1 - \mu^2 w^2)}}, \quad (B.13)
\]
\[
R(\lambda, \mu)_{1,4}^{2,2} = \sqrt{(1 - \lambda^2)(1 - \lambda^2 w)(1 - \lambda^2 w^2) \sqrt{(1 + w + w^2)(\lambda - \mu)}}, \quad (B.14)
\]
\[
R(\lambda, \mu)_{1,4}^{2,3} = \sqrt{(1 - \lambda^2)(1 - \mu^2 w^2)\sqrt{(1 + w + w^2)\lambda - \mu w}}, \quad (B.15)
\]
\[
R(\lambda, \mu)_{1,4}^{2,4} = (1 + w + w^2)(1 - \lambda \mu)(\lambda - \mu)(\lambda - \mu w), \quad (B.16)
\]
\[
R(\lambda, \mu)_{2,3}^{3,2} = ((1 - \lambda^2 \mu^2)(1 + w) - w(\mu - \lambda)(\mu - \lambda w))((\mu - \lambda)), \quad (B.17)
\]
\[
R(\lambda, \mu)_{2,3}^{3,2} = ((1 - \lambda^2)(1 - \mu^2 w^2) - (1 + w)(\mu - \lambda)(\mu - \lambda w))\sqrt{(1 - \lambda^2 w)(1 - \mu^2 w)}(B.22)
\]
\[
R(\lambda, \mu)_{3,2}^{4,1} = \sqrt{(1 - \lambda^2 w^2)(1 - \mu^2)(1 - w^2)(\mu - \lambda)(\mu - \lambda w)}, \quad (B.23)
\]
\[
R(\lambda, \mu)_{3,3}^{4,2} = \sqrt{(1 - \lambda^2 w^2)(1 - \mu^2 w^2)\sqrt{(1 + w)(1 + w + w^2)(1 - \lambda \mu)(\mu - \lambda)}, \quad (B.24)
\]
\[
R(\lambda, \mu)_{3,3}^{3,3} = ((1 - \lambda^2 w)(1 - \mu^2 w^2) - (1 + w + w^2)(\mu - \lambda)(\mu - \lambda w))(1 - \lambda \mu), \quad (B.25)
\]
\[
R(\lambda, \mu)_{3,4}^{3,3} = \sqrt{(1 - \lambda^2 w)(1 - \mu^2 w^2)(1 - \lambda \mu)(1 - \lambda \mu w)}, \quad (B.26)
\]
\[
R(\lambda, \mu)_{3,4}^{3,4} = (1 + w + w^2)(1 - \lambda \mu)(1 - \lambda \mu w)(\lambda - \mu), \quad (B.27)
\]
\[
R(\lambda, \mu)_{4,1}^{4,1} = (\mu - \lambda)(\mu - \lambda w)(\mu - \lambda w^2), \quad (B.28)
\]
\[
R(\lambda, \mu)_{4,2}^{4,2} = (1 + w + w^2)(1 - \lambda \mu)(\mu - \lambda)(\mu - \lambda w), \quad (B.29)
\]

where \( \omega = \pm i \).
Appendix C : \( SL(2, \mathcal{R}) \) R-matrix elements

Here we present the amplitudes \( R(\lambda, \mu)^{c,d}_{a,b} \) corresponding to the submatrices of the R-matrix for \( SL(2, \mathcal{R}) \). The results are given up to the particle sector \( n = 4 \),

\( n = 0 \)

\[ R(\lambda, \mu)^{1,1}_{1,1} = 1. \] (C.1)

\( n = 1 \)

\[ R(\lambda, \mu)^{1,2}_{1,2} = R(\lambda, \mu)^{2,1}_{2,1} = \frac{i(\mu - \lambda)}{p_2(\lambda, \mu)}, \] (C.2)

\[ R(\lambda, \mu)^{2,1}_{1,2} = R(\lambda, \mu)^{1,2}_{2,1} = \frac{2s}{p_2(\lambda, \mu)}. \] (C.3)

\( n = 2 \)

\[ R(\lambda, \mu)^{1,3}_{1,3} = R(\lambda, \mu)^{3,1}_{3,1} = \frac{(\mu - \lambda)(i + \mu - \lambda)}{p_3(\lambda, \mu)}, \] (C.4)

\[ R(\lambda, \mu)^{2,2}_{1,3} = R(\lambda, \mu)^{1,3}_{2,2} = R(\lambda, \mu)^{3,2}_{2,2} = R(\lambda, \mu)^{2,2}_{3,1} = \frac{2\sqrt{i}s\sqrt{i(2s - 1)}(\mu - \lambda)}{p_3(\lambda, \mu)}, \] (C.5)

\[ R(\lambda, \mu)^{3,1}_{1,3} = R(\lambda, \mu)^{1,3}_{3,1} = \frac{2s(2s - 1)}{p_3(\lambda, \mu)}, \] (C.6)

\[ R(\lambda, \mu)^{2,2}_{2,2} = \frac{2s(2s - 1) - (\mu - \lambda)(-i + \mu - \lambda)}{p_3(\lambda, \mu)}. \] (C.7)

\( n = 3 \)

\[ R(\lambda, \mu)^{1,4}_{1,4} = R(\lambda, \mu)^{4,1}_{4,1} = \frac{(2 - i\mu + i\lambda)(\mu - \lambda)(i + \mu - \lambda)}{p_4(\lambda, \mu)}, \] (C.8)

\[ R(\lambda, \mu)^{2,3}_{1,4} = R(\lambda, \mu)^{1,4}_{2,3} = R(\lambda, \mu)^{4,1}_{3,2} = R(\lambda, \mu)^{3,2}_{4,1} = \frac{2i\sqrt{3}\sqrt{i(-1 + s)}\sqrt{i}s(\mu - \lambda)(i + \mu - \lambda)}{p_4(\lambda, \mu)}, \] (C.9)

\[ R(\lambda, \mu)^{3,2}_{1,4} = R(\lambda, \mu)^{4,1}_{2,3} = R(\lambda, \mu)^{1,4}_{3,2} = R(\lambda, \mu)^{2,3}_{4,1} = \frac{2\sqrt{3}i\sqrt{i(1 - s)}\sqrt{i}s(-1 + 2s)(\mu - \lambda)}{p_4(\lambda, \mu)}, \] (C.10)
\[ R(\lambda, \mu)_{1,4}^{1,1} = R(\lambda, \mu)_{4,1}^{1,4} = \frac{4(-1+s)(1+2s)}{p_4(\lambda, \mu)}, \quad (C.11) \]
\[ R(\lambda, \mu)_{2,3}^{2,3} = R(\lambda, \mu)_{3,2}^{3,2} = \frac{-\iota(\mu - \lambda) [-8(-1+s)s + (\mu - \lambda)(-\iota + \mu - \lambda)]}{p_4(\lambda, \mu)}, \quad (C.12) \]
\[ R(\lambda, \mu)_{2,3}^{3,2} = R(\lambda, \mu)_{3,2}^{2,3} = \frac{2(-1+2s)[2(-1+s)s - (\mu - \lambda)(-\iota + \mu - \lambda)]}{p_4(\lambda, \mu)}. \quad (C.13) \]

- \( n = 4 \)

\[ R(\lambda, \mu)_{1,5}^{1,5} = R(\lambda, \mu)_{5,1}^{5,1} = \frac{(\mu - \lambda)(-1+s)(2+\mu - \lambda)(3+\mu - \lambda)}{p_5(\lambda, \mu)}, \quad (C.14) \]
\[ R(\lambda, \mu)_{1,5}^{2,4} = R(\lambda, \mu)_{2,4}^{1,5} = R(\lambda, \mu)_{4,2}^{5,1} = R(\lambda, \mu)_{5,1}^{4,2}, \quad (C.15) \]
\[ R(\lambda, \mu)_{1,5}^{3,3} = R(\lambda, \mu)_{3,3}^{1,5} = R(\lambda, \mu)_{3,3}^{5,1} = R(\lambda, \mu)_{5,1}^{3,3}, \quad (C.16) \]
\[ R(\lambda, \mu)_{1,5}^{4,2} = R(\lambda, \mu)_{2,4}^{5,1} = R(\lambda, \mu)_{5,1}^{4,2}, \quad (C.17) \]
\[ R(\lambda, \mu)_{1,5}^{5,1} = R(\lambda, \mu)_{5,1}^{1,5} = \frac{4(-1+s)(1+2s)(-1+2s)}{p_5(\lambda, \mu)}, \quad (C.18) \]
\[ R(\lambda, \mu)_{2,4}^{2,4} = R(\lambda, \mu)_{4,2}^{2,4} = \frac{(\mu - \lambda)(-1+s)(3-2s)s + (\mu - \lambda)(-\iota + \mu - \lambda)}{p_5(\lambda, \mu)}, \quad (C.19) \]
\[ R(\lambda, \mu)_{2,4}^{3,3} = R(\lambda, \mu)_{3,3}^{2,4} = R(\lambda, \mu)_{3,3}^{4,2} \]
\[ = \frac{2\sqrt{3}(\mu - \lambda)(-1+s)(3-2s)s + (\mu - \lambda)(-\iota + \mu - \lambda)}{p_5(\lambda, \mu)}, \quad (C.20) \]
\[ R(\lambda, \mu)_{2,4}^{4,2} = R(\lambda, \mu)_{4,2}^{4,2} = \frac{2(-1+s)(-1+2s)(3-2s)s - (\mu - \lambda)(-\iota + \mu - \lambda)}{p_5(\lambda, \mu)}, \quad (C.21) \]
\[ R(\lambda, \mu)_{3,3}^{3,3} = \frac{4(-1+s)(3-2s)(-1+2s) + 2s[3 + 4s(-3+2s)](\mu - \lambda)}{p_5(\lambda, \mu)} \]
\[ + \frac{-7 + 8s(-3+2s)](\mu - \lambda)^2 - 2s(\mu - \lambda)^3 + (\mu - \lambda)^4}{p_5(\lambda, \mu)}. \quad (C.22) \]
The auxiliary function \( p_i(\lambda, \mu) \) entering in the above expressions is defined by,

\[
p_i(\lambda, \mu) = \prod_{j=1}^{i-1} [2s + 1 + i(\mu - \lambda) - j].
\]  

(C.23)

Appendix D : The coordinate Bethe ansatz

In this appendix we present the coordinate Bethe ansatz diagonalization of the non-compact Hamiltonian,

\[
H(s)\psi_n = E_n\psi_n
\]  

(D.1)

up to the two-particle sector \( n = 2 \).

Considering Eqs.(85,87) the action of the Hamiltonian on the subspace of states up to the sector \( n = 2 \) are,

\[
\begin{align*}
H_{12}(s) |0, 0\rangle &= 0 \\
H_{12}(s) |0, 1\rangle &= |0, 1\rangle - |1, 0\rangle \\
H_{12}(s) |1, 0\rangle &= |1, 0\rangle - |0, 1\rangle \\
H_{12}(s) |1, 1\rangle &= 2 |1, 1\rangle + d(s)(|0, 2\rangle + |2, 0\rangle) \\
H_{12}(s) |2, 0\rangle &= e(s) |0, 2\rangle + d(s) |1, 1\rangle + c(s) |2, 0\rangle \\
H_{12}(s) |0, 2\rangle &= c(s) |0, 2\rangle + d(s) |1, 1\rangle + e(s) |2, 0\rangle
\end{align*}
\]  

(D.2-7)

where the parameters \( c(s), d(s) \) and \( e(s) \) are given by,

\[
c(s) = 2 + \frac{1}{2s - 1}, \quad d(s) = -\frac{2\sqrt{s}}{\sqrt{2s - 1}}, \quad e(s) = \frac{1}{2s - 1}
\]  

(D.8)

The sector \( n = 0 \) only contains the reference state and the wave-function is \( \psi = |0...0\rangle \) with \( E_0 = 0 \). The wave-function \( \psi_1 \) for the \( n = 1 \) sector is the linear combination of states \( |0...\frac{1}{x}...0\rangle \) made by inserting one excitation of type 1 on a \( x \)-th site,

\[
\psi_1 = \sum_{x=1}^{L} \phi(x)|0...\frac{1}{x}...0\rangle
\]  

(D.9)
The action of $H(s)$ on $\psi_1$ leads us to the following difference equation for $\phi(x)$

$$E_1 \phi(x) = 2\phi(x) - \phi(x + 1) - \phi(x - 1)$$  \hspace{1cm} (D.10)

The standard plane-wave assumption $\phi(x) = \exp(ikx)$ solves Eq.\text{[D.10]} provided the eigenvalue $E_1$ is

$$E_1 = 2 [1 - \cos(k)]$$  \hspace{1cm} (D.11)

while the one-particle momenta is fixed by periodic boundary condition $\phi(x + L) = \phi(x)$,

$$\exp(i k L) = 1$$  \hspace{1cm} (D.12)

The subspace of states for $n = 2$ is constituted of $L(L-1)$ states of two excitation of type 1

$$\big| 0 \cdots \frac{1}{x_1} \cdots \frac{1}{x_2} \cdots 0 \big>$$

and $L$ states of one excitation of type 2

$$\big| 0 \cdots \frac{2}{x} \cdots 0 \big>.$$

Therefore, the ansatz for the two-particle wave-function $\psi_2$ is,

$$\psi_2 = \sum_{1 \leq x_1 < x_2 \leq L} \phi(x_1, x_2) \big| 0 \cdots \frac{1}{x_1} \cdots \frac{1}{x_2} \cdots 0 \big> + \sum_{x=1}^{L} \varphi(x) \big| 0 \cdots \frac{2}{x} \cdots 0 \big>.$$  \hspace{1cm} (D.13)

By solving the eigenvalue equation $\text{[D.11]}$ for $\psi_2$ we derive the following set of difference equations,

- $(E - 4)\phi(x_1, x_2) = -\phi(x_1 + 1, x_2) - \phi(x_1 - 1, x_2) - \phi(x_1, x_2 + 1) - \phi(x_1, x_2 - 1), \ x_2 > x_1 + 1$  \hspace{1cm} (D.14)
- $(E - 4)\phi(x_1, x_2) = -\phi(x_1, x_2 + 1) - \phi(x_1 - 1, x_2) + d(s) [\varphi(x_1) + \varphi(x_2)], \ x_2 = x_1 + 1$  \hspace{1cm} (D.15)
- $(E - 2c)\varphi(x) = d(s) [\phi(x - 1, x) + \phi(x, x + 1)] + e(s) [\varphi(x - 1) + \varphi(x + 1)],$  \hspace{1cm} (D.16)

To solve Eqs.\text{[D.14]}\text{[D.16]} we have to consider three basic steps. First, guided by the $n = 1$ case and the structure of \text{[D.14]}, we proposed for $\phi(x_1, x_2)$ the typical Bethe ansatz form,

$$\phi(x_1, x_2) = \exp(ik_1 x_1 + ik_2 x_2) + S(k_1, k_2) \exp(ik_2 x_1 + ik_1 x_2)$$  \hspace{1cm} (D.17)

where $S(k_1, k_2)$ is the two-particle scattering matrix. This solves Eq.\text{[D.14]} provided the eigenvalue is

$$E_2 = 2 [1 - \cos(k_1)] + 2 [1 - \cos(k_2)].$$  \hspace{1cm} (D.18)
Next we consider the compatibility between Eq. (D.15) and Eq. (D.16) at the point $x_2 = x_1 + 1$. By subtracting Eq. (D.15) from Eq. (D.14) for $x_1 = x$ and $x_2 = x + 1$ we obtain the following matching condition,

$$d(s)\phi(x) = -\phi(x, x).$$

(D.19)

Now by substituting the ansatz Eq. (D.17) in Eq. (D.16) after considering condition (D.19) we are able to fix the scattering matrix as,

$$S(k_1, k_2) = -\frac{1 + \exp[i(k_1 + k_2)] + (2s - 1)\exp[ik_1] - (2s + 1)\exp[ik_2]}{1 + \exp[i(k_1 + k_2)] + (2s - 1)\exp[ik_2] - (2s + 1)\exp[ik_1]}.$$ 

(D.20)

The final step is to impose the periodic boundary conditions $\phi(x_1, x_2 + L) = \phi(x_2, x_1)$. As a result we obtain the two-particle Bethe ansatz constraint,

$$\exp(ik_i L)\prod_{j=1, j\neq i}^2 S(k_i, k_j) = 1,$$

(D.21)

where we have used the unitarity property $S(k_1, k_2)S(k_2, k_1) = 1$.

In order to reproduce the Bethe ansatz equations (91) we have to parameterize the momenta $k_j$ as,

$$\exp(ik_j) = \frac{\lambda_j + 2s i}{\lambda_j}.$$ 

(D.22)

References

[1] C.S. Melo and M.J. Martins, *Nucl.Phys.B* 806 (2009) 567

[2] E.K. Sklyanin, L.A. Takhtadzhan and L.D. Faddeev, *Theor.Math.Fiz. 40* (1979) 688; L.A. Takhtadzhan and L.D. Faddeev, *Russ.Math.Sur. 34* (1979) 11

[3] V.E. Korepin, G. Izergin and N.M. Bogoliubov, “Quantum Inverse Scattering Method and Correlation Functions, Cambridge University Press, Cambridge, 1993

[4] M. Gaudin, *J. Physique* 37 (1976) 1087
[5] H.M. Babujian, *J.Phys.A: Math.Gen.* 26 (1993) 6981; H.M. Babujian and R. Flume, *Mod.Phys.Lett. A* 9 (1994) 2029

[6] A.B. Zamolodchikov and V.A. Fateev, *Sov.J.Nucl.Phys.* 32 (1980) 298

[7] K. Sogo, Y. Akutsu and T. Abe, *Progr. Theor. Phys.* 70 (1983) 730.

[8] P.P. Kulish and N.Yu. Reshetikhin and E.K. Sklyanin, *Lett.Math.Phys.* 5 (1981) 393

[9] L.D. Faddeev, V.O. Tarasov and L.A. Takhtajan, *Theor.Math.Phys.* 57 (1983) 1059

[10] H.M. Babujian, *Nucl.Phys.B* 215 (1983) 317; L.A. Takhtajan, *Phys.Lett. A* 87 (1982) 479

[11] H.M. Babujian and A.M. Tsvelick, *Nucl.Phys.B* 265 (1986) 24; A.N. Kirillov and N.Y. Reshetikhin, *J.Phys. A: Math. Gen.* 20 (1987) 1565

[12] A. Nishino and T. Deguchi, *J.Stat.Phys.* 133 (2008) 587

[13] T. Deguchi and Y. Akutsu, *J.Phys.Soc.Jpn.* 60 (1991) 4051; *Phys.Rev.Lett.* 67 (1991) 777

[14] T. Deguchi and Y. Akutsu, *Mod.Phys.Lett.A* 7 (1992) 767.

[15] T. Deguchi and Y. Akutsu, *J.Phys.Soc.Jpn.* 62 (1993) 19

[16] M. Couture, *J.Phys.A: Math. Gen.* 24 (1991) L103

[17] C. Gomez, M. Ruiz-Altaba and G. Sierra, *Phys.Lett.B* 265 (1991) 95; C. Gomez and G. Sierra, *Nucl.Phys.B* 373 (1992) 761

[18] M. Jimbo, *Lett.Math.Phys.* 10 (1985) 63

[19] M. Wadati, T. Deguchi and Y. Akutsu, *Phys.Rep.* 180 (1989) 247

[20] A.G. Bytsko, *J. Math. Phys.* 44 (2003) 3698

[21] H.M. Babujian, *J.Phys.A: Math.Gen.* 27 (1994) 7753

37
[22] A. Lima-Santos and W. Utiel, *Nucl.Phys.B* 600 (2001) 512; V. Kurak and A. Lima-Santos, *Nucl.Phys.B* 701 (2004) 497

[23] M. Couture, H.C. Lee and N.C. Schmeing, *in Proceedings of “Physics, Geometry and Topology”, NATO ASI V 238 (1989)* 573

[24] V.F.R. Jones, *Commun.Math.Phys.* 125 (1989) 459; *Int.J.Mod.Phys. A* 6 (1991) 2035

[25] M.L. Ge, Y.S. Wu and K. Xue, *Int.J.Mod.Phys. A* 6 (1991) 3735

[26] A. Berkovich, C. Gomez and G. Sierra, *J.Phys.A:Math.Gen.* 26 (1993) L45, *Int. J. Mod. Phys. B* 6 (1992) 1939

[27] B.U. Felderhof, *Physica* 66 (1973) 279

[28] L.N. Lipatov, *JETP Lett.* 59 (1994) 596

[29] F.D. Faddeev and G.P. Korchemsky, *Phys.Lett.B* 342 (1995) 311

[30] H.J. de Vega and L.N. Lipatov, *Phys.Rev.D* 64 (2001) 114019

[31] V.M. Braun, S.E. Derkachov and A.N. Manashov, *Phys.Rev.Lett.* 81 (1998) 2020

[32] J.A. Minahan and K. Zarembo, *JHEP* 0303 (2003) 013

[33] V.A Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *JHEP* 0405 (2004) 024

[34] N. Beisert, *Nucl.Phys.B* 676 (2004) 3; N. Beisert and M. Staudacher, *Nucl.Phys.B* 670 (2003) 439

[35] A.O. Barut and C. Fronsdal, *Proc.Roy.Soc. A* 287 (1965) 532

[36] V.M. Braun, S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, *Nucl.Phys.B* 553 (1999) 355; A.V. Belisky, *Phys.Lett.B* 453 (1999) 59; S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, *Nucl.Phys.B* 566 (2000) 203.

[37] M. Staudacher, *JHEP* 0505 (2005) 054

38
[38] N. Kitanine, *J.Phys.A:Math.Gen.* 34 (2001) 8151; O.A. Castro-Alvaredo and J.M. Maillet, *J.Phys.A:Math.Teor.* 40 (2007) 7451

[39] T. Deguchi and C. Matsui, *Nucl.Phys.B* 814 (2009) 405

[40] A. Caradoc, O. Foda and N. Kitanine, *J.Stat.Mech.* (2006) P03012; O. Foda, M. Wheeler and M. Zuparic, *J.Stat.Mech.* (2007) P10016; A. Caradoc, O. Foda, M. Wheeler and M. Zuparic, *J.Stat.Mech.* (2007) P03010

[41] V.E. Korepin, *Commun.Math.Phys.* 86 (1982) 391, A.G. Izergin, *Sov.Phys.Dokl.* 32 (1987) 878, N.M. Bogoliubov, A.G. Pronko and M.B. Zvonarev, *J.Phys.A:Math.Gen.* 35 (2002) 5525

[42] A.J. Braken, M.D. Gould, Y-Z. Zhang and G.W. Delius, *J.Phys.A:Math.Gen* 27 (1994) 6551; Z. Maassarani, *J.Phys.A:Math.Gen.* 28 (1995) 1305