GRASSMANNIAN ESTIMATION

CLAUDE AUDERSET, CHRISTIAN MAZZA AND ERNST A. RUH

ABSTRACT. This paper discusses the family of distributions on the Grassmannian $G(m, r)$ of the linear span of $r$ central normal vectors in $\mathbb{R}^m$ or $\mathbb{C}^m$, parametrized by the covariance matrix (up to a positive factor). Our main result is an existence and uniqueness criterion for the maximum likelihood estimate of a sample in $G(m, r)$, based on convexity and asymptotic properties of the log-likelihood. By coupling methods of algebraic geometry and linear programming, we show that almost all samples of size $n > m^2/r(m - r)$ in $G(m, r)$ have a unique MLE.

In the real case, a new, unexpected phenomenon takes place for some values $1 < r < m$, which does not occur in the angular Gaussian case $r = 1$. Random samples of some critical size in $G(m, r)$ may have a unique estimate or not, with a positive probability in either case.

1. Introduction

As stated in [20], the current data deluge inundating science is remarkable for the rapid proliferation in new data type. Typical examples are directions in $\mathbb{R}^n$ or elements of the Grassmann manifold $G(m, r)$ of all vector subspaces of dimension $r$ of $\mathbb{R}^m$ ($0 < r < m$), as introduced in [4]. Being of increasing importance in practical situations (see e.g. [7], [14], [15], [18], [19], [20] or [21]), there is a strong need for studying various classical inference problems, like for example maximum likelihood estimation. To deal with these problems, one can in most cases reparametrize the manifold and recast the inference problem in some Euclidean space. However, this can have the effect of hiding intrinsic geometric properties of the statistical relevant objects (see below).

A typical example is obtained when dealing with $G(m, r)$ when $r = 1$, the set of axes or directions in $\mathbb{R}^m$, see e.g. [12] and [21]. In [13], the manifold is endowed with the angular Gaussian distribution, that is of the law of the random direction obtained by retaining only the axis of a multivariate centered gaussian random vector in $\mathbb{R}^m$ of covariance matrix $\Sigma$. Kent and Tyler [13] derived sufficient conditions for the existence of the maximum likelihood estimator (MLE) based on an i.i.d. sample by working on $\mathbb{R}^{m-1}$; the angular Gaussian distribution is then equivalent to the Cauchy law. The mathematical analysis can then be performed in $\mathbb{R}^{m-1}$, at the cost of loosing nice properties of the problem. In [2], the whole picture was obtained using mainly convexity. The parameter space $\text{Pos}(m)$ consists of positive definite self adjoint matrices of determinant 1, which is considered as a Riemanian manifold with a natural metric. The results derived in [2] make strong use of this manifold structure, of the particular form of the log likelihood function and of the geometric link between the parameter space $\text{Pos}(m)$ and the sample space $G(m, r)$, $r = 1$. Interestingly, the estimated scatter matrix plays a fundamental rôle for...
multivariate nonparametric tests, where it is known as the Tyler’s transformation matrix, see e.g. [17], or in finance where the maximum likelihood estimator is used to fit financial data, see [3].

When \( r \) is arbitrary, we obtain random subspaces by retaining only the linear span \( U = \langle x_1, \ldots, x_r \rangle \) of an i.i.d. sample of \( r \) multivariate centered gaussian random vectors of covariance matrix \( \Sigma \in \text{Pos}(m) \). The law of this random subspace has been considered previously in the literature and has been termed as the matrix angular Gaussian distribution (see e.g. [4], [5] or [6]); however, basic questions like the existence of the MLE remain unexplored.

We will show that a new phenomenon emerges: In most statistical settings, the MLE based on some sample \( u_1, \ldots, u_n \) exists with probability one when the size \( n \) is larger than a critical value \( n_c \) and does not exist with probability one when \( n \leq n_c \), like for example in the angular Gaussian case with \( r = 1 \) (see e.g. [2]). In the Grassmannian setting, we show in Example 2 of Section 3 that there are sizes \( n \) such that the MLE exists with positive probability and does not exist with positive probability (see e.g. [1] where a similar phenomenon occurs in logistic regression).

Section 2 introduces the Grassmannian statistical model and the related likelihood function. Section 3 considers the problem of existence and uniqueness of the Grassmannian maximum likelihood estimate (GE). Our main results, Theorems 1 and 2 give necessary and sufficient conditions for the existence of a unique GE. The geometrical setting is illustrated in Examples 1 and 2. Section 4 provides fundamental properties of the likelihood function like its convexity when restricted to the geodesics of \( \text{Pos}(m) \). This nice property is then used to prove Theorems 1 and 2. Theorem 4 of Section 5 shows finally that the GE of almost all samples of size \( n \) is unique when

\[
\frac{m^2}{r(m - r)}. 
\]
GRASSMANNIAN ESTIMATION

\( U \in G(m, r) \) is

\[
A G_\Sigma = G_{A^* \Sigma A^*}.
\]

In fact, the Grassmannian statistical model \((G_\Sigma)_{\Sigma \in \text{Pos}(m)}\) is the unique family of Borel probability measures on \(G(m, r)\) indexed by \(\text{Pos}(m)\) enjoying the equivariance property \(\ref{eq:1}\) for all matrices \(A \in \mathbb{F}^{m \times m}\) of determinant 1. To see this, observe that condition \(\ref{eq:1}\) implies the invariance of \(G_\Sigma\) under the group of invertible matrices \(A\) of determinant 1 such that \(A \Sigma A^* = \Sigma\). As this group is compact and acts continuously and transitively on \(G(m, r)\), there is a unique Borel probability measure on \(G(m, r)\) which is invariant under it, namely \(G_\Sigma\).

Let us represent a point \(U \in G(m, r)\) as the linear span \(U = \langle x_1, \ldots, x_r \rangle\) of linearly independent vectors \(x_1, \ldots, x_r\) of \(U\) or, equivalently, as the range \(U = \langle X \rangle\) of the matrix \(X = (x_1, \ldots, x_r)\) of rank \(r\). Then, a computation shows that the density, or Radon-Nikodym derivative, of the Grassmannian distribution \(G_\Sigma\) \((\Sigma \in \text{Pos}(m))\) with respect to the uniform distribution \(G_I\) on \(G(m, r)\) \((I = \text{identity matrix})\) is given by

\[
\frac{dG_\Sigma}{dG_I}(X) = \left(\frac{\det(X^*X)}{\det(X^*X^{-1}X)}\right)^{i\gamma m/2},
\]

where \(i\gamma = \dim_{\mathbb{F}}(\mathbb{F})\) (see \(\ref{eq:2}\) for the real case). The meaning of this formula is perhaps more apparent in the form

\[
\frac{dG_\Sigma}{dG_I}(U) = \left(\frac{\text{vol}(E_I \cap U)}{\text{vol}(E_\Sigma \cap U)}\right)^m \quad (U \in G(m, r)),
\]

where \(E_\Sigma = \{x \in \mathbb{F}^m \mid x^*\Sigma^{-1}x \leq 1\}\) denotes the ellipsoid associated to \(\Sigma\) \((E_I = \text{unit ball})\), and \(\text{vol}\) the Lebesgue measure on \(U\).

When \(r = 1\), the Grassmannian distribution \(G_\Sigma\) is known as the (real or complex) angular Gaussian distribution of parameter \(\Sigma \in \text{Pos}(m)\) on the projective space \(\mathbb{F}^{m-1} = G(m, 1)\) (see \(\ref{eq:3}\)). For any \(0 < r < m\), the Grassmann manifold \(G(m, r)\) can be viewed as the space of projective subspaces of dimension \(r - 1\) of \(\mathbb{F}^{m-1}\) by identifying a vector \(r\)-subspace \(U\) of \(\mathbb{F}^m\) with the projective subspace \(\{y \in \mathbb{F}^{m-1} \mid y \subseteq U\}\). In this projective interpretation, the Grassmannian distribution \(G_\Sigma\) on \(G(m, r)\) is the law of the projective span of i.i.d. random points \(y_1, \ldots, y_r\) of \(\mathbb{F}^{m-1}\) with angular Gaussian distribution of parameter \(\Sigma\).

2.2. Grassmannian maximum likelihood estimates. Let \(P\) be a Borel probability measure on \(G(m, r)\). Typically, we think of \(P\) as being the empirical measure \((\delta_{U_1} + \cdots + \delta_{U_n})/n\) of a sample \(U_1, \ldots, U_n\) in \(G(m, r)\), but other cases are of interest too. A parameter \(\Sigma \in \text{Pos}(m)\) is called a Grassmannian (maximum likelihood) estimate — abbreviated GE in the sequel — of \(P\) if maximizes the log-likelihood \(\int_{G(m, r)} \log(dG_{\Sigma}/dP) dP\). It is called a GE of a sample \(U_1, \ldots, U_n \in G(m, r)\) when \(P\) is the empirical measure \((\delta_{U_1} + \cdots + \delta_{U_n})/n\).

For convenience, we shall rather work with the following negative version of the log-likelihood

\[
\ell_P(\Sigma) = \frac{-1}{i\gamma m} \int_{G(m, r)} \log(dG_{\Sigma}/dG_I) dP = \int_{G(m, r)} \ell_U(\Sigma) dP(U),
\]

where the (negative) log-density \(\ell_U\) is defined by
having a unique GE. What is needed is a bound for $k$ so that $n$ can choose any number of the sample all of which are met by some line $V$ has a unique GE if and only if $n > k$ in $P$.

In the case of an empirical measure $P = (\delta_{U_1} + \cdots + \delta_{U_n})/n$, we find that the sample contained in a nontrivial vector subspace $V$ of $\mathbb{F}^m$ for all nontrivial linear subspaces $V$ of $\mathbb{F}^m$ be less than $n \dim(V)/m$ (see [2] for a more precise result).

Now, almost all samples in $\mathbb{P}^{m-1}$ are in general position, i.e., any nontrivial vector subspace $V$ of $\mathbb{F}^m$ contains at most $\dim(V)$ points of the sample. Thus almost all samples of size $n > m$ in $\mathbb{P}^{m-1}$ have a unique angular Gaussian maximum likelihood estimate. This result goes back to [23]. On the other hand, no samples of size $n \leq m$ in $\mathbb{P}^{m-1}$ have a unique angular Gaussian maximum likelihood estimate since any point $U \in \mathbb{P}^{m-1}$ of a sample is, of course, contained in the one-dimensional subspace $V = U$ of $\mathbb{F}^m$, so that the condition for the number of points of the sample contained in $V$ to be less than $n \dim(V)/m$ is not satisfied when $n \leq m$.

For a Grassmannian $G(m, r)$ which is not a projective space, the situation is more involved, even in the simplest case $m = 4, r = 2$.

Example 2. Let $U_1, \ldots, U_n$ be a sample in the Grassmann manifold $G(4, 2)$, viewed as the space of lines in the projective space $\mathbb{P}^3$. Suppose that the lines $U_1, \ldots, U_n$ are pairwise skew, i.e., $U_i \cap U_j = 0$ for $i \neq j$. Examining case by case all of the possible values of $\dim(V)$ and $\dim(U_i \cap V)$ in Corollary [1] we find that the sample has a unique GE if and only if $n > k$, where $k$ is the maximum number of lines of the sample all of which are met by some line $V \in G(4, 2)$. Now, given a line $V$, we can choose any number $n$ of pairwise skew lines $U_1, \ldots, U_n \in G(4, 2)$ meeting $V$, so that $n = k$. Hence, there are arbitrary large samples of pairwise skew lines not having a unique GE. What is needed is a bound for $k$. 

### 3. Existence and uniqueness of the Grassmannian estimate

**Theorem 1.** A Borel probability measure $P$ on the real or complex Grassmannian $G(m, r)$ has a unique GE if and only if

$$
\int_{G(m, r)} \dim(U \cap V) dP(U) < \frac{r}{m} \dim(V)
$$

for all nontrivial linear subspaces $V$ of $\mathbb{F}^m$ ($0 \neq V \neq \mathbb{F}^m$).

In the case of an empirical measure $P = (\delta_{U_1} + \cdots + \delta_{U_n})/n$, we find that the sample contained in a nontrivial vector subspace $V$ of $\mathbb{F}^m$ be less than $n \dim(V)/m$ (see [2] for a more precise result).

**Corollary 1.** A sample $U_1, \ldots, U_n$ in the real or complex Grassmannian $G(m, r)$ has a unique GE if and only if

$$
\frac{1}{n} \sum_{i=1}^{n} \dim(U_i \cap V) < \frac{r}{m} \dim(V)
$$

for all nontrivial linear subspaces $V$ of $\mathbb{F}^m$ ($0 \neq V \neq \mathbb{F}^m$).
Recall that the lines meeting each of three pairwise skew lines $U_1$, $U_2$ and $U_3$ form a one-dimensional family $\mathcal{F}_1$ of lines on a quadric surface $Q \subset \mathbb{P}^3$, whereas the other family $\mathcal{F}_2$ of lines on $Q$ consists of the lines meeting every line of $\mathcal{F}_1$. A point of intersection $x$ of a further line $U_4 \in G(4,2)$ with the quadric $Q$ determines a line meeting each of the four lines $U_1$, $U_2$, $U_3$ and $U_4$, namely the line $V \in \mathcal{F}_1$ through $x$, and vice versa (see Fig. 1).

The number of lines meeting four pairwise skew lines $U_1$, $U_2$, $U_3$ and $U_4$ is thus

- $2$ if $U_4$ meets $Q$ transversally,
- $0$ if $U_4$ does not meet $Q$, which can occur only when $F = \mathbb{R}$,
- $1$ if $U_4$ is tangent to $Q$,
- $\infty$ if $U_4$ lies on $Q$, in which case $U_4 \in \mathcal{F}_2$ so that every line meeting $U_1$, $U_2$ and $U_3$ necessarily meets $U_4$ too.

In both the real and the complex case, there are at most two lines $V \in G(4,2)$ meeting each of four pairwise skew lines, except when the four lines belong to the same family of lines on a smooth quadric. So, almost all samples of size $n$ in $G(4,2)$ consist of pairwise skew lines of which at most four are intersected by a line $V \in G(4,2)$. We conclude from the criterion above that almost all samples of size $n > 4$ in the real or complex Grassmann manifold $G(4,2)$ have a unique GE.

In the complex case, there is a line meeting each of any four pairwise skew lines $U_1$, $U_2$, $U_3$ and $U_4$ since $U_4$ always meets the quadric $Q$. The same holds if some of the four lines meet together or even coincide. Thus, by Corollary 1 no samples of size $n \leq 4$ in the complex Grassmann manifold $G(4,2)$ have a unique GE.

The situation is different in the real case since $U_4$ need not meet the quadric $Q$. If we choose four lines at random, there may be a line meeting each of them or not, with a positive probability in both cases. Therefore, the probability that a random sample of size $n = 4$ in the real Grassmann manifold $G(4,2)$ has a unique GE is positive and $< 1$.

On the other hand, by Corollary 1 no samples of size $n < 4$ in the real Grassmann manifold $G(4,2)$ have a unique GE since any $n < 4$ lines are intersected by some line (in fact, by infinitely many lines).

4. Likelihood equation

We first introduce notions from linear algebra which are necessary to settle the likelihood equation on the symmetric space Pos(m). Consider the scalar product

\[ (x|y)_\Sigma = x^* \Sigma^{-1} y \quad (x, y \in \mathbb{R}^m) \]
associated to a parameter $\Sigma \in \text{Pos}(m)$. We denote by $\pi_U(\Sigma)$ the $\Sigma$-orthogonal projector onto a vector subspace $U$ of $\mathbb{F}^m$. It is the linear map $\pi_U(\Sigma) : \mathbb{F}^m \to \mathbb{F}^m$ defined by $\pi_U(\Sigma)u = u$ if $u \in U$, and $\pi_U(\Sigma)v = 0$ if $v \in \mathbb{F}^m$ is $\Sigma$-orthogonal to $U$, i.e., $(v|u)_\Sigma = 0$ for all $u \in U$. In matrix notation,

$$
\pi_U(\Sigma) = X(X^*\Sigma^{-1}X)^{-1}X^*\Sigma^{-1},
$$

where $U = \langle X \rangle$ is the range of $X \in \mathbb{F}^{m \times r}$. We call a matrix $A \in \mathbb{F}^{m \times m}$ self-$\Sigma$-adjoint if $(Ax|y)_\Sigma = (x|Ay)_\Sigma$ for all $x, y \in \mathbb{F}^m$ or, equivalently, if it coincides with its $\Sigma$-adjoint $\Sigma A^*\Sigma^{-1}$.

The parameter space $\text{Pos}(m)$ is a Riemannian manifold, in fact a symmetric space. Its tangent space $T_\Sigma$ at $\Sigma \in \text{Pos}(m)$ consists of the self-$\Sigma$-adjoint matrices $v \in \mathbb{F}^{m \times m}$ of trace zero, and the Riemannian metric is defined by the scalar products

$$
\langle v_1, v_2 \rangle = \text{tr}(v_1v_2) \quad (v_1, v_2 \in T_\Sigma)
$$

on the tangent spaces $T_\Sigma$, where $\text{tr}(A)$ denotes the trace of a matrix $A$. The geodesic $\gamma : \mathbb{R} \to \text{Pos}(m)$ of velocity $\ell \in T_\Sigma$ issuing from $\Sigma \in \text{Pos}(m)$ is

$$
\gamma(t) = e^{tv}\Sigma(e^{tv})^* = e^{2tv}\Sigma \quad (t \in \mathbb{R}),
$$

where $e^{tv}$ denotes the matrix exponential.

Deriving the expression (4) along a geodesic (10) and using the matrix form (8) of the $\Sigma$-orthogonal projector $\pi_U(\Sigma)$ onto $U$, we find the gradient (with respect to the Riemannian metric (9) defined above) of the log-density

$$
\text{grad} \ell_U(\Sigma) = \frac{\nu}{m} I - \pi_U(\Sigma) \quad (U \in G(m,r), \Sigma \in \text{Pos}(m)),
$$

and the covariant derivative of $\text{grad} \ell_U$ in the direction of $v \in T_\Sigma$

$$
\nabla_v \text{grad} \ell_U(\Sigma) = \pi_U(\Sigma)v(I - \pi_U(\Sigma)) + (I - \pi_U(\Sigma))v\pi_U(\Sigma).
$$

By integrating these formulas with respect to $P$, and interchanging integration and derivation by means of the Lebesgue dominated convergence theorem, we get the gradient of the log-likelihood (3)

$$
\text{grad} \ell_P(\Sigma) = \frac{\nu}{m} I - \int_{G(m,r)} \pi_U(\Sigma) dP(U) \quad (\Sigma \in \text{Pos}(m)),
$$

and its covariant derivative

$$
\nabla_v \text{grad} \ell_P(\Sigma) = \int_{G(m,r)} [\pi_U(\Sigma)v(I - \pi_U(\Sigma)) + (I - \pi_U(\Sigma))v\pi_U(\Sigma)]dP(U).
$$

A function $f$ on $\text{Pos}(m)$ is called convex if its restriction $f(\gamma(t))$ $(t \in \mathbb{R})$ to any geodesic $\gamma$ is convex in the usual sense. This amounts to saying that the Hessian $\nabla^2 f$ is positive semi-definite, i.e., $\nabla^2 f(\Sigma) = \langle \nabla_v \text{grad} f(\Sigma), v \rangle \geq 0$ for all $\Sigma \in \text{Pos}(m)$ and $v \in T_\Sigma$, since

$$
\frac{d^2}{dt^2} f(\gamma(t)) = \langle \nabla^2 f(\gamma(t)), v \rangle = \langle \nabla_v \text{grad} f(\gamma(t)), v \rangle,
$$

where $v$ is the velocity of the geodesic $\gamma$.

**Proposition 1.** The log-likelihood function $\ell_P$ is convex. More precisely, its restriction $\ell_P(\gamma(t))$ $(t \in \mathbb{R})$ to a geodesic $\gamma$ is either strictly convex or affine linear. The latter case occurs if and only if $\ell(U) \subset U$ for $P$-almost all $U \in G(m,r)$, where $v$ is the velocity of the geodesic.
Proof. The convexity can be obtained directly by proceeding as in [2]. On the other hand, one can use the fact that the log-likelihood function is a Busemann function for the symmetric space Pos(m) (see e.g. [9]), and convexity follows. □

As the log-likelihood function $\ell_P$ is convex, its minima are exactly the zeroes of its gradient hence, by formula (13),

Theorem 2. A parameter $\Sigma \in \text{Pos}(m)$ is a GE of a Borel probability measure $P$ on $G(m, r)$ if and only if it satisfies the maximum likelihood equation

$$\int_{G(m, r)} \pi_U(\Sigma) \, dP(U) = \frac{r}{m} I.$$  

Proof of Theorem 2

One can either proceed as in [2], or use the fact that the log-likelihood functions is a Busemann function of the symmetric space Pos(m), see e.g. [9]. The maximum likelihood estimator is then the barycenter of the related probability measure on the Grassman manifold, viewed as an orbit in the Tits boundary. Theorem 2 then follows from Proposition 6.2 of [11].

5. The linear programming bound

In order to apply the criterion of Corollary 1 for the existence and uniqueness of the GE of a sample, we must first answer the following question.

Given vector subspaces $U_1, \ldots, U_n \in G(m, r)$ of dimension $r$ of $\mathbb{F}^m$ and integers $d_1, \ldots, d_n \geq 0$, on what conditions is there a vector subspace $V \in G(m, s)$ of dimension $s$ of $\mathbb{F}^m$ such that $\dim(U_k \cap V) = d_k$ for $k = 1, \ldots, n$? A necessary condition, using methods of algebraic geometry, is given by Proposition 2 below.

In a second step, we look for all possibilities with $0 < s = \dim V < m$ and

$$\frac{1}{n} \sum_{k=1}^{n} d_k < \frac{rs}{m}$$

using methods of linear programming. This leads to the following.

Theorem 3. Almost all samples of size

$$n > \frac{m^2}{r(m-r)}$$

in the real or complex Grassmann manifold $G(m, r)$ have a unique GE.

Our main tool is the Schubert calculus on the Grassmannian $G(m, s)$. In general, the Schubert variety ([10], [8]) associated to a Young diagram or partition

$$\lambda = (\lambda_1, \ldots, \lambda_{m-s}) \quad (s \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-s} \geq 0)$$

with at most $s$ rows and $m - s$ columns and a complete flag

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = \mathbb{F}^m$$

of vector subspaces of $\mathbb{F}^m$ is defined as

$$\Omega_\lambda = \{ V \in G(m, s) \mid \dim(F_{m-s+i-\lambda_i} \cap V) \geq i, 1 \leq i \leq s \}.$$  

It is an irreducible algebraic subvariety of codimension $|\lambda| = \lambda_1 + \cdots + \lambda_{m-s}$ of the Grassmannian $G(m, s)$ of dimension $s(m-s)$.
In particular, given \( U \in G(m, r) \) and an integer \( d \) such that
\[
\max\{0, r + s - m\} \leq d_k \leq \min\{r, s\},
\]
the set
\[
S_d(U) = \{ V \in G(m, s) \mid \dim(U \cap V) \geq d \}
\]
is the Schubert variety \( \Omega_{\lambda} \) associated to the rectangular Young diagram \( \lambda = d^k \) with \( k = m + d - r - s \) rows and \( d \) columns if we choose the flag in such a way that \( F_r = U \). So,
\[
\text{codim } S_d(U) = d(m + d - r - s).
\]

**Proposition 2.** The following property holds for almost all samples \( (U_1, \ldots, U_n) \) in the real or complex Grassmann manifold \( G(m, r) \). For any vector subspace \( V \) of dimension \( s \) of \( \mathbb{F}^m \),
\[
\begin{align*}
(17) & \quad \max\{0, r + s - m\} \leq d_k \leq \min\{r, s\} \text{ for } k = 1, \ldots, n, \text{ and} \\
(18) & \quad \sum_{k=1}^n d_k(m + d_k - r - s) \leq s(m - s),
\end{align*}
\]
where \( d_k = \dim(U_k \cap V) \).

**Remark.** The conditions (17) and (18) are necessary for the existence of a vector subspace \( V \) such that \( d_k = \dim(U_k \cap V) \) for \( k = 1, \ldots, n \). But they are not sufficient, as shown by the example \( m = 6, r = 3, s = 3, n = 2, d_1 = d_2 = 2 \). In this case, the inequalities (17) and (18) are satisfied, although there is in general no \( V \in G(6, 3) \) meeting \( U_1 \) and \( U_2 \) in subspaces of dimension 2.

To get necessary and sufficient conditions, we need the Schubert calculus. But computations in the Schubert calculus (Littlewood-Richardson coefficients) are algorithmically hard [16] so we must content ourselves with Proposition 2.

**Proof of Proposition 2** The inequalities (17) for \( d_k = \dim(U_k \cap V) \) follow from the dimension formula
\[
\dim(U_k \cap V) + \dim(U_k + V) = \dim(U_k) + \dim(V).
\]
The proof of the rest of the proposition uses standard methods of algebraic geometry.

Let \( d_1, \ldots, d_n \) be arbitrary integers satisfying the inequalities (17) and consider the algebraic correspondence
\[
C = \{ ((U_1, \ldots, U_n), V) \in G(m, r)^n \times G(m, s) \mid \dim(U_k \cap V) \geq d_k, k = 1, \ldots, n \}.
\]
The range of \( C \) is the whole of \( G(m, r) \), and its domain \( A_{d_1, \ldots, d_n} \) consists of the samples \( (U_1, \ldots, U_n) \in G(m, r)^n \) for which there is some \( V \in G(m, s) \) with \( \dim(U_k \cap V) \geq d_k \) for \( k = 1, \ldots, n \). Let \( ((U_1, \ldots, U_n), V) \) be a generic point of \( C \). Observe that
\[
C^{-1}(V) = \{ (U_1, \ldots, U_n) \in G(m, r)^n \mid ((U_1, \ldots, U_n), V) \in C \}
= S_{d_1}(V) \times \cdots \times S_{d_n}(U_n),
\]
where \( S_{d_k}(V) = \{ U \in G(m, r) \mid \dim(U \cap V) \geq d_k \} \) is a Schubert variety with
\[
\text{codim } S_{d_k}(V) = d_k(m + d_k - r - s)
\]
as explained above for \( S_d(U) \).
According to the principle of counting constants [10],

$$\dim A_{d_1, \ldots, d_n} + \dim C(U_1, \ldots, U_n) = \dim G(m, s) + \dim C^{-1}(V),$$

where $C(U_1, \ldots, U_n)$ consists of all $V \in G(m, s)$ such that $\dim (U_k \cap V) \geq d_k$ for $k = 1, \ldots, n$, hence

$$\dim A_{d_1, \ldots, d_n} \leq \dim G(m, s) + \dim C^{-1}(V)$$

$$= s(m - s) + \sum_{k=1}^{n} (r(m - r) - d_k(m + d_k - r - s))$$

$$= \dim G(m, r)^n + s(m - s) - \sum_{k=1}^{n} d_k(m + d_k - r - s).$$

This shows that $A_{d_1, \ldots, d_n}$ is a proper algebraic subset of $G(m, r)^n$ if the inequality (18) is not satisfied. Let $N_s$ be the union of $A_{d_1, \ldots, d_n}$ where $(d_1, \ldots, d_n)$ runs over all those lists of integers satisfying the inequalities (17) but not the inequality (18), and let $N$ be the union of $N_s$ for $s = 0, \ldots, m$. As a finite union of proper algebraic subsets, $N$ is also a proper algebraic subset by the irreducibility of $G(m, r)^n$, hence negligible. Now, take a sample $(U_1, \ldots, U_n) \in G(m, r)^n$ not belonging to $N$, and any vector subspace $V$ of any dimension $s$ of $\mathbb{F}^m$. Set $d_k = \dim(U_k \cap V)$ for $k = 1, \ldots, n$, so that $(U_1, \ldots, U_n) \in A_{d_1, \ldots, d_n}$. Then $(d_1, \ldots, d_n)$ must satisfy the inequality (18), otherwise $(U_1, \ldots, U_n)$ would belong to $N$ by the very definition of $N$. This proves the Proposition. □

Consider next the set $B(m, r, s)$ of those positive integers $n$ for which there are integers $d_1, \ldots, d_n$ satisfying the inequalities

$$\max\{0, r + s - m\} \leq d_k \leq \min\{r, s\} \quad \text{for } k = 1, \ldots, n,$$

$$\sum_{k=1}^{n} d_k(m + d_k - r - s) \leq s(m - s),$$

and set $B(m, r) = \bigcup_{s=1}^{m-1} B(m, r, s)$.

**Lemma 1.** Almost all samples of size $n \notin B(m, r)$ in the real or complex Grassmann manifold $G(m, r)$ have a unique GE.

**Proof.** Suppose that $n \notin B(m, r)$. According to Proposition [2], the following holds for almost all $(U_1, \ldots, U_n) \in G(m, r)^n$. For any proper vector subspace $V$ of dimension $s$ of $\mathbb{F}^m$, the integers $d_k = \dim(U_k \cap V)$ satisfy the inequalities (17) and (18). But they do not satisfy the inequality (19) since $n \notin B(m, r)$ hence $n \notin B(m, r, s)$. Thus $m \sum_{k=1}^{n} d_k < nrs$, which is precisely the condition (19) of Corollary [1] for the sample $U_1, \ldots, U_n$ to have a unique GE. □

**Lemma 2.** For any integers $m, r$ with $0 < r < m$, the set $B(m, r)$ is bounded above by $m^2/r(m - r)$.

**Proof.** As $B(m, r) = \bigcup_{s=1}^{m-1} B(m, r, s)$, we first look for an upper bound of $B(m, r, s)$. To this end, we replace the unknowns $d_1, \ldots, d_n$ in the definition of $B(m, r, s)$ by the number

$$n_i = \#\{k \in \{1, \ldots, n\} \mid d_k = i\}$$
of occurrences among \(d_1, \ldots, d_n\) of each integer \(i\) between \(i_0\) and \(i_1\), where
\[
i_0 = \max\{0, r + s - m\} \quad \text{and} \quad i_1 = \min\{r, s\}.
\]
With these new unknowns \(n_{i_0}, \ldots, n_{i_1}\), the inequations \((17–19)\) translate into the system of linear inequations
\[
\begin{align*}
n_i &\geq 0 \quad \text{for } i_0 \leq i \leq i_1, \\
\sum_{i=i_0}^{i_1} i(m + i - r - s)n_i &\leq s(m - s), \\
\sum_{i=i_0}^{i_1} (rs - mi)n_i &\leq 0,
\end{align*}
\]
with \(n = \sum_{i=i_0}^{i_1} n_i\). So, \(B(m, r, s)\) consists of those integers \(n\) which decompose into a sum \(n = \sum_{i=i_0}^{i_1} n_i\) of integers \(n_i\) satisfying the inequalities \((20–22)\). The maximum of \(B(m, r, s)\) (if any) is the solution of the integer linear program
\[
\begin{align*}
&\text{maximize } \sum_{i=i_0}^{i_1} n_i \\
&\text{subject to the constraints } (20–22).
\end{align*}
\]
Relaxing the integrality condition on \(n_i\) yields a usual linear program with real \(n_{i_0}, \ldots, n_{i_1}\), whose solution is an upper bound of \(B(m, r, s)\). Standard methods of linear programming \cite{22} show that the constraints \((17–19)\) define a bounded polytope whose vertices are of one of the following two types.
\[
\begin{aligned}
&\diamond \quad n_i = \frac{s(m - s)}{i(m + i - r - s)} \text{ for some } i \text{ and } n_k = 0 \text{ for } k \neq i. \\
&\diamond \quad n_i \text{ and } n_j \text{ are the solutions of the system of equations}
\end{aligned}
\]
\[
\begin{align*}
i(m + i - r - s)n_i + j(m + j - r - s)n_j &= s(m - s), \\
(rs - mi)n_i + (rs - m_j)n_j &= 0.
\end{align*}
\]
and \(n_k = 0\) for \(k \neq i, j\).
Now, routine computations show that the sum \(n = \sum_{i=i_0}^{i_1} n_i\) reaches its maximum on vertices of the first type when \(i = \lceil rs/m \rceil\), and on vertices of the second type when \(i + 1 = j = \lceil rs/m \rceil\). It can then be checked that these maxima are bounded above by the quantity \(m^2/r(m - r)\). \(\square\)

Theorem 3 immediately follows from Lemma 1 and 2.

6. Numerical Algorithms

Let \(P\) be a probability measure admitting a unique maximum likelihood estimator. We propose here two algorithms to locate this estimator, using the geometry of the problem (see Section 4). The first one is a gradient-descent dynamics. The second one is a faster method which avoids the time consuming steps of the first one.

We look for the solution \(\hat{\Sigma}\) to the equation \((16)\). The Exponential map \(\text{Exp}_\Sigma\) from \(T\Sigma\) to \(\text{Pos}(m)\) is given explicitly by \(\text{Exp}_\Sigma(v) = e^v\Sigma\). Given some \(\Sigma_k\) and \(\Sigma' = \text{Exp}_{\Sigma_k}(v)\), the idea is to approximate the gradient \(\ell_P(\Sigma')\) using the parallel
transport of \( \nabla \ell_P(\Sigma_k) + \nabla v_k + \nabla \ell_P(\Sigma_k) \). One then computes the solution \( v_{k+1} \in T\Sigma_k \) to the linear system
\[
(23) \quad \nabla \ell_P(\Sigma_k) + \nabla v_{k+1} + \nabla \ell_P(\Sigma_k) = 0.
\]
The loop is closed by setting \( \Sigma_{k+1} = \text{Exp}_{\Sigma_k}(v_{k+1}) \).

The step which consists in solving (23) is time consuming, so that we propose a faster dynamics: Given \( \Sigma_k \), we use the geodesic \( \gamma_k(t) = e^{2t \nabla \ell_P(\Sigma_k) \Sigma_k} \), and set
\[
(24) \quad \Sigma_{k+1} = \gamma_k(1) = e^{2 \nabla \ell_P(\Sigma_k) \Sigma_k}.
\]

Our simulations indicate that the sequence \( (\Sigma_k)_{k \geq 0} \) converges toward the maximum likelihood estimator \( \hat{\Sigma}_n \). We have performed a simulation study using \( n = 50, 500, 5000 \) i.i.d. random samples \( \langle X^1 \rangle, \ldots, \langle X^n \rangle, \langle X^n \rangle \in G(4,2) \), distributed according to the Grassmannian distribution of parameter \( \Sigma_0 \) given by

\[
\begin{align*}
1.23943 & \quad 0.53234 & \quad 0.21763 & \quad 0.33038 \\
0.53234 & \quad 1.12502 & \quad 0.76236 & \quad 0.20842 \\
0.21763 & \quad 0.7626 & \quad 1.52821 & \quad 0.82655 \\
0.33038 & \quad 0.20842 & \quad 0.82655 & \quad 1.52298
\end{align*}
\]

The probability measure \( P \) is then the empirical distribution on \( G(4,2) \) associated with the random sample. Our simulations indicate that the maximum likelihood is consistent. For a random sample of size \( n = 500 \), we found that the difference between \( \Sigma_0 \) and the estimate \( \hat{\Sigma}_n \) is given by

\[
\begin{align*}
0.0282495 & \quad 0.0095817 & \quad 0.0791341 & \quad -0.0819841 \\
0.0269432 & \quad 0.1031291 & \quad -0.0447055 & \quad 0.1444463 \\
0.33038 & \quad 0.20842 & \quad 0.82655 & \quad 1.52298
\end{align*}
\]

For \( n = 5000 \), this difference was given by

\[
\begin{align*}
0.01223629 & \quad -0.0100086 & \quad 0.0110916 & \quad -0.0221974 \\
-0.0209799 & \quad -0.0366614 & \quad -0.0114825 & \quad -0.0571491 \\
-0.0571491 & \quad 0.0010570 & \quad 0.0380609
\end{align*}
\]

REFERENCES

[1] A. Albert and J.A. Anderson. On the existence of maximum likelihood estimates in logistic regression models. *Biometrika*, 71 1-10 (1984).
[2] C. Auderset, C. Mazza, and E. Ruh. Angular Gaussian and Cauchy Estimation, *Journal of Multivariate Analysis*, 93 180-197 (2005).
[3] J. Bouchaud, M. Potters Theory of Financial Risk and Derivative Pricing, From Statistical Physics to Risk Management. Cambridge University Press, Cambridge, 2003.
[4] Y. Chikuse The Matrix Angular Central Gaussian Distribution *Journal of Multivariate Analysis*, 33 265-274 (1990).
[5] Y. Chikuse Statistics on special manifolds Lecture Notes in Statistics 174 Springer (2000).
[6] Y. Chikuse State Space Models on Special Manifolds *Journal of Multivariate Analysis*, 97 1284-1294 Springer (2006).
[7] I.L. Dryden, K.V. Mardia Statistical Shape Analysis Wiley, 1998.
[8] W. Fulton. Young Tableaux, Cambridge University Press (1999).
[9] R. Fluege, E. Ruh Barycenter and maximum likelihood. Diff. Geom. Appl. 24 (2006), 660-669.
[10] W.V.D. Hodge W.V.D., B.A. Pedoe. Methods of Algebraic Geometry Vol. II, Cambridge University Press (1952).
[11] M. Kapovich, B. Leeb, J. Millson Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity. arXiv: math/0301118v, (2005).
[12] D.G. Kendall, Shape manifolds, Procrustean metrics, and complex projective spaces, Bull. London Math. Soc. 16 (1984) 81-121.
[13] J.T. Kent, D.E. Tyler, Redescending M estimators of multivariate location and scatter, Ann. Statist. 19 (1991) 2102-2119.
[14] K.V. Mardia Statistics of Directional Data. Academic Press, New-York, 1972.
[15] K.V. Mardia, Patrangenaru, V. and Sugathadasa, S. Protein gels matching. In Quantitative Biology, Shape Analysis, and Wavelets. (S. Barber, P.D. Baxter, K.V. Mardia and R.E. Walls(Eds.).) 163-165, 2005. Leeds, Leeds University Press.
[16] N. Narayanan On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients, Journal of Algebraic Combinatorics 24 347 - 354 (2006)
[17] H. Oja, R. Randles. Multivariate Nonparametric Tests. Statistical Sciences 19 598-605 (2004)
[18] V. Patrangenaru, K.V. Mardia Affine shape analysis and image analysis. In Proceedings of the LASR Workshop, 57-62, (2003).
[19] V. Patrangenaru,S. Sugathadasa A Covariance Formula for Shape Statistics on Grassmani-ans, In Proceedings of the ICIA05 Conference, Colombo, Sri Lanka, 15-20, (2005).
[20] I. Rahman, I. Drori, V. Stodden, D. Donoho, P. Schroeder Multiscale Representations for Manifold-Valued Data Multiscale Model Simul. 4 1201-1232 (2005).
[21] A. Srivastava, E. Klassen Bayesian and geometric subspace tracking, Adv. in Appl. Probab 36 43-56, (2004).
[22] A. Schrijver, Theory of Linear and Integer Programming. Wiley (1986).
[23] D.E. Tyler, Statistical analysis for the angular central Gaussian distributions on the sphere, Biometrika 74 (1987) 579-589
[24] G.S. Watson, Statistics on Spheres, Wiley (1983).