A Lichnerowicz-type estimate for Steklov eigenvalues on graphs and its rigidity

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Abstract
In this paper, we obtain a Lichnerowicz-type estimate for the first positive Steklov eigenvalues on graphs and discuss its rigidity.

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1 Introduction

One of the main themes in the research of differential geometry is to explore the relations of local invariants and global invariants. The local invariants mostly come from the curvature tensor while there are various of global invariants depending on the structure we are interested in. For example, the Gauss-Bonnet-Chern theorem [4] and Chern characteristic classes [5] reveal the relation of curvature and topology; the Bonnet-Myers theorem (see [8]) reveals the relation of Ricci curvature and diameter; the positive mass theorem of Schoen-Yau [28] reveals the relation of scalar curvature and ADM mass and the Lichnerowicz estimate [18] reveals the relation of Ricci curvature and spectrum.

This theme of research was transplanted into the study of graphs. Two notions of Ricci curvature lower bounds for graphs, the Bakry-Émery curvature from the diffusion process point of view and the Ollivier curvature from the optimal transport point of view, were introduced and developed in [1, 7, 17, 21, 22, 24]. Classical results in differential geometry revealing the relations of local invariants and global structures such as Lichnerowicz estimate ([3, 16, 20, 21, 24]), Bonnet-Myer diameter estimate ([6, 19–21]), Li-Yau gradient estimate ([2]) etc. all have its corresponding versions in the discrete setting.
In this paper, we further study Lichnerowicz-type estimates on graphs according to this theme of research. Let’s recall some classical Lichnerowicz-type estimates first. Let \((M^n, g)\) be a closed Riemannian manifold with Ricci curvature not less than a positive constant \(K\). Then, Lichnerowicz’s estimate \([18]\) says that its first positive eigenvalue for the Laplacian operator must be no less than \(\frac{nK}{n-1}\). The rigidity part of Lichnerowicz’s estimate was later obtained by Obata \([23]\) which says that the equality of Lichnerowicz’s estimate holds if and only if \((M, g)\) is isometric to a round sphere. This result was later extended to the first Dirichlet eigenvalue and first positive Neumann eigenvalue for compact Riemannian manifolds with boundary by Reilly \([27]\) via his famous integral formula which is now called the Reilly formula. For Steklov eigenvalues on compact Riemannian manifolds with boundary, there is a conjecture by Escobar \([9]\) in a similar spirit which was partially solved by Xia-Xiong \([32]\) recently.

Motivated by all these works, we consider Lichnerowicz-type estimates for Steklov eigenvalues on graphs in this paper. Let’s recall some preliminaries before stating the main results of the paper.

Let \(G\) be a graph with the set of vertices denoted by \(V(G)\) and the set of edges denoted by \(E(G)\). We will also write \(V(G)\) and \(E(G)\) as \(V\) and \(E\) for simplicity if no confusion was made. Without further indications, throughout this paper, the graph \(G\) is assumed to be finite, simple and connected. For \(A, B \subset V\), we will use \(E(A, B)\) to denote the set of edges in \(G\) joining a vertex in \(A\) to a vertex in \(B\).

A triple \((G, m, w)\) is called a weighted graph where \(m\) and \(w\) are positive functions on \(V\) and \(E\) which are called the vertex measure and edge weight respectively. For simplicity, the edge weight \(w\) is also viewed as a symmetric function on \(V \times V\) such that for any \(x, y \in V\), \(w(x, y)\) is the weight of the edge \(\{x, y\}\) when \(\{x, y\} \in E\), otherwise \(w(x, y) = 0\). We will also write \(m(x)\) as \(m_x\) and \(w(x, y)\) as \(w_{xy}\) for simplicity. For each \(x \in V\), we call

\[
\text{Deg}(x) := \frac{1}{m_x} \sum_{y \in V} w_{xy}
\]

(1.1)

the weighted degree of \(x\). Moreover, for any \(S \subset V\), we call

\[
V_S := \sum_{x \in S} m_x
\]

(1.2)

the volume of \(S\).

Some special weights are attracting special interests when studying the structure of graphs. For example, the unit weight with \(m \equiv 1\) and \(w \equiv 1\) is usually considered when considering combinatorial structure of graphs. The normalized weight with \(\text{Deg}(x) = 1\) for any \(x \in V\) is usually considered when considering random walks on graphs.

For \(u \in \mathbb{R}^V\), the Laplacian of \(u\) is defined as

\[
\Delta u(x) = \frac{1}{m_x} \sum_{y \in V} (u(y) - u(x))w_{xy}
\]

(1.3)

for any \(x \in V\). The differential of \(u\) is a skew-symmetric function on \(V \times V\) defined as:

\[
du(x, y) = \begin{cases} 
  u(y) - u(x) & \{x, y\} \in E \\
  0 & \text{otherwise}
\end{cases}
\]

(1.4)

In fact, as mentioned in \([29]\), skew-symmetric functions \(\alpha\) on \(V \times V\) such that \(\alpha(x, y) = 0\) whenever \(x \sim y\) are called 1-forms of \(G\). The space of 1-forms on \(G\) is denoted as \(A^1(G)\).
A natural inner product on $A_1^1(G)$ is defined as:

$$\langle \alpha, \beta \rangle = \sum_{\{x,y\} \in E} \alpha(x,y)\beta(x,y)w_{xy} = \frac{1}{2} \sum_{x,y \in V} \alpha(x,y)\beta(x,y)w_{xy}$$

(1.5)

for any $\alpha, \beta \in A_1^1(G)$. Note that functions on $V$ can be viewed as 0-forms. So the space $A_0^0(G)$ of 0-forms on $G$ is just $\mathbb{R}^V$. The natural inner product on $A_0^0(G)$ is:

$$\langle u, v \rangle = \sum_{x \in V} u(x)v(x)m_x$$

(1.6)

for any $u, v \in A_0^0(G)$. Let $d^* : A_1^1(G) \to A_0^0(G)$ be the adjoint operator of $d : A_0^0(G) \to A_1^1(G)$ with respect to the two natural inner products defined on $A_0^0(G)$ and $A_1^1(G)$ respectively. Then, by direct computation, it is not hard to see that

$$\Delta = -d^*d.$$  

(1.7)

This is the same as the smooth case. From this point of view, it is clear that

$$\langle \Delta u, v \rangle = -\langle du, dv \rangle$$

(1.8)

for any $u, v \in \mathbb{R}^V$. This implies that $-\Delta$ is a nonnegative self-adjoint operator on $\mathbb{R}^V$. Let

$$0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_{|V|}$$

(1.9)

be the eigenvalues of $-\Delta$ on $(G, m, w)$. It is clear that $\mu_1 = 0$ because constant functions are the corresponding eigenfunctions. Moreover, $\mu_2 > 0$ because we always assume that $G$ is connected.

We next recall the notion of Bakry-Émery curvature on graphs. A weighted graph $(G, m, w)$ is said to satisfy the Bakry-Émery curvature-dimension condition $CD(K, n)$ if for any $f \in \mathbb{R}^V$ and $x \in V$,

$$\Gamma_2(f, f)(x) \geq \frac{1}{n} (\Delta f)^2(x) + K \Gamma(f, f)(x).$$

(1.10)

Here $K$ is a real number and $n$ is positive and can be taken to be $\infty$. Moreover,

$$\Gamma(u, v) := \frac{1}{2} (\Delta(uv) - v\Delta u - u\Delta v)$$

(1.11)

and

$$\Gamma_2(u, v) := \frac{1}{2} (\Delta \Gamma(u, v) - \Gamma(\Delta u, v) - \Gamma(u, \Delta v))$$

(1.12)

for any $u, v \in \mathbb{R}^V$. By direct computation,

$$\Gamma(u, v)(x) = \frac{1}{2m_x} \sum_{y \in V} (u(x) - u(y))(v(x) - v(y))w_{xy} = \frac{1}{2m_x} \langle du, dv \rangle_{E_x}$$

(1.13)

Here $E_x$ means the set of edges in $G$ adjacent to $x$ and for a set $S$ of edges in $G$,

$$\langle \alpha, \beta \rangle_S := \sum_{\{x,y\} \in S} \alpha(x,y)\beta(x,y)w_{xy}$$

(1.14)

for any $\alpha, \beta \in A_1^1(G)$. So,

$$\langle \Gamma(u, v), 1 \rangle = \sum_{x \in V} \Gamma(u, v)(x)m_x = \langle du, dv \rangle.$$  

(1.15)
The following Lichnerowicz-type estimate on graphs was shown independently by several authors [3, 16].

**Theorem 1.1** Let \((G, m, w)\) be a connected weighted finite graph satisfying the Bakry-Émery curvature-dimension condition \(CD(K, n)\) with \(K > 0\) and \(n > 1\). Then

\[
\mu_2(G) \geq \frac{nK}{n-1}. \tag{1.16}
\]

Some discussions on rigidity of (1.16) in the case \(n = \infty\) can be found in [20].

Finally, we need to recall some notions on Steklov eigenvalues for graphs that was introduced in [14] and [25] recently.

First recall the notion of graphs with boundary. A pair \((G, B)\) is called a graph with boundary if \(G\) is a simple graph and \(B \subseteq V(G)\) such that (i) any two vertices in \(B\) are not adjacent, (ii) any vertex in \(B\) is adjacent to some vertex in \(\Omega_1 := V \setminus B\). The set \(B\) is called the vertex-boundary of \((G, B)\) and the set \(\Omega_1\) is called the vertex-interior of \((G, B)\).

An edge joining a boundary vertex and an interior vertex is called a boundary edge. Note that the notion of graphs with boundary was naturally introduced by Friedman in [10] when considering nodal domains of graphs.

Let \((G, m, w, B)\) be a weighted graph with boundary. For each \(x \in \Omega_1\), we call

\[
\text{Deg}_b(x) := \frac{1}{m_x} \sum_{y \in B} w_{xy} \tag{1.17}
\]

the weighted boundary degree at \(x\). For any \(u \in \mathbb{R}^V\) and \(x \in B\), define the normal derivative of \(u\) at \(x\) as:

\[
\frac{\partial u}{\partial n}(x) := \frac{1}{m_x} \sum_{y \in V} (u(x) - u(y)) w_{xy} = -\Delta u(x). \tag{1.18}
\]

The reason to define this is that one has the following Green’s formula which is a straightforward consequence of (1.8):

\[
\langle \Delta u, v \rangle_{\Omega_1} = -\langle du, dv \rangle + \left\langle \frac{\partial u}{\partial n}, v \right\rangle_B. \tag{1.19}
\]

Here, for any set \(S \subseteq V\),

\[
\langle u, v \rangle_S := \sum_{x \in S} u(x) v(x) m_x. \tag{1.20}
\]

Because of (1.19), similar machinery in defining Dirichlet-to-Neumann maps and Steklov eigenvalues in the smooth case can be also run in the discrete case. For each \(f \in \mathbb{R}^B\), let \(u_f\) be the harmonic extension of \(f\) into \(\Omega\):

\[
\begin{align*}
\Delta u_f(x) &= 0 \quad x \in \Omega \\
u_f(x) &= f(x) \quad x \in B.
\end{align*} \tag{1.21}
\]

Define the Dirichlet-to-Neumann map \(\Lambda : \mathbb{R}^B \to \mathbb{R}^B\) as

\[
\Lambda(f) = \frac{\partial u_f}{\partial n}. \tag{1.22}
\]

By (1.19), it is clear that

\[
\langle \Lambda(f), g \rangle_B = \langle du_f, du_g \rangle \tag{1.23}
\]
for any \( f, g \in \mathbb{R}^B \). This implies that \( \Lambda \) is a nonnegative self-adjoint operator on \( \mathbb{R}^B \). Let

\[
0 = \sigma_1 < \sigma_2 \leq \cdots \leq \sigma_{|B|}
\]  

(1.24)

be the eigenvalues of \( \Lambda \). It is clear that \( \sigma_1 = 0 \) because constant functions are the corresponding eigenfunctions. Moreover, \( \sigma_2 > 0 \) because we always assume that \( G \) is connected.

We are now ready to state the first main result of this paper, a Lichnerowicz-type estimate for Steklov eigenvalues on graphs.

**Theorem 1.2** Let \((G, m, w, B)\) be a connected weighted finite graph with boundary. Suppose that \((G, m, w)\) satisfy the Bakry-Émery curvature-dimension condition \(CD(K, n)\) with \(K > 0\) and \(n > 1\). Then

\[
\sigma_2 \geq \frac{nK}{n - 1}.
\]

(1.25)

The Lichnerowicz-type estimate for Steklov eigenvalues looks quite different with that of the smooth case (see [32]). One reason for this may be that the Bakry-Émery curvature-dimension condition \(CD(K, n)\) on \((G, m, w)\) is a curvature condition on the whole graph including curvature restrictions on the boundary. Another reason may be that, although Dirichlet-to-Neumann maps on graphs are defined by the same machinery with the smooth case, the normal derivative

\[
\frac{\partial u}{\partial n} = -(\Delta u)|_B
\]

(1.26)

is different with the smooth case where the Dirichlet-to-Neumann map is somehow like the square root of the Laplacian operator.

In this paper, we will present a direct proof to Theorem 1.2. We would like to mention that there is another simple proof by observing that

\[
\sigma_i \geq \mu_i
\]

(1.27)

for \(i = 1, 2, \cdots, |B|\) and combining Theorem 1.1 (see [30] for details). In fact, the eigenvalue comparison (1.27) was mentioned in [15, Corollary 1.6] for normalized weighted graphs.

The other main results of this paper are considering the rigidity of (1.25). For graphs with unit weight, we have the following conclusion for rigidity of (1.25).

**Theorem 1.3** Let \((G, B)\) be a connected finite graph with boundary, equipped with the unit weight and satisfying the Bakry-Émery curvature-dimension condition \(CD(K, n)\) with \(K > 0\) and \(n > 1\). Moreover, suppose that \(\sigma_2 = \frac{nK}{n - 1}\). Then, \(|V(G)| = 3\) or 4. When \(|V(G)| = 3\), one has \(K = \frac{1}{2}\) and \(n = 2\). Moreover \((G, B)\) is a path with 3 vertices and with the two end points the boundary vertices. When \(|V(G)| = 4\), one has \(n = \infty\) and \(K = 2\). Moreover \((G, B)\) is one of the following graphs:

1. \(V(G) = \{1, 2, 3, 4\}, E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\} \) and \(B = \{1, 3\}\);
2. \(V(G) = \{1, 2, 3, 4\}, E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{2, 4\}\} \) and \(B = \{1, 3\}\).

For general weighted graphs, by trying to squeeze the curvature-dimension condition into the interior of the graph, we have the following rigidity of (1.25).

**Theorem 1.4** Let \((G, m, w, B)\) be a connected weighted finite graph with boundary. Suppose that \((G, m, w)\) satisfy the Bakry-Émery dimension-curvature condition \(CD(K, n)\) with \(K > 0\) and \(n > 1\). Then \(\sigma_2 = \frac{nK}{n - 1}\) if and only if all following statements are true:
(1) \(|B| = 2\) and every interior vertex is adjacent to both of the two boundary vertices.
(2) let \(B = \{1, 2\}\). Then \(m_1 = m_2 := m\) and for \(x \in \Omega\), \(w_{1x} = w_{2x} := w_x\);
(3) \(\text{Deg}(1) = \text{Deg}(2) = \sum_{x \in \Omega} w_x = \frac{nk}{n-1}\);
(4) For any \(x \in \Omega\), \(\text{Deg}_b(x) = \frac{2w_x}{m_x} = \frac{(n+2)K}{n-1}\);
(5) Either \(n = 2\) and \(|\Omega| = 1\), or \(n > 2\) and for any \(x \in \Omega\) and \(f \in \mathbb{R}^\Omega\) with \(f(x) = 0\),
\[
\Gamma^\Omega_2(f, f)(x) - \frac{1}{n-2}(\Delta_\Omega f)^2(x) + \frac{3K}{n-1} \Gamma^\Omega(f, f)(x) + \frac{(n+2)K^2}{8m(n-1)}(f, f)_\Omega
\]
\[
- \frac{(n+2)K}{(n-1)(n-2)m}(f, 1)\Omega \Delta_\Omega f(x) - \frac{n(n+2)K^2}{8(n-2)(n-1)^2m^2}(f, 1)\Omega^2 \geq 0.
\]
In particular, when \(n = \infty\), the last inequality becomes to
\[
\Gamma^\Omega_2(f, f)(x) + \frac{K^2}{8m}(f, f)_\Omega - \frac{K^2}{8m^2}(f, 1)\Omega^2 \geq 0
\]for any \(f \in \mathbb{R}^\Omega\) with \(f(x) = 0\).
Here, \(\Gamma^\Omega_2\), \(\Gamma^\Omega\) and \(\Delta_\Omega\) are the corresponding \(\Gamma_2\), \(\Gamma\) and \(\Delta\) operators for the induced subgraph on \(\Omega\).

By Theorem 1.4, one can construct many weighted graphs satisfying the Bakry-Émery curvature-dimension condition \(\text{CD}(K, n)\) with \(K > 0\) and \(n > 2\), on which the equality of (1.25) holds. For example, first fix a complete graph \(\Omega\) induced subgraph on \(\Omega\) with \(f(x) = 0\),
\[
\Gamma^\Omega_2(f, f)(x) + \frac{K^2}{8m}(f, f)_\Omega - \frac{K^2}{8m^2}(f, 1)\Omega^2 \geq 0
\]
for any \(f \in \mathbb{R}^\Omega\) with \(f(x) = 0\).

Because (1.28) looks complicated, we first consider rigidity of (1.25) for the case that the induced subgraph on \(\Omega\) is trivial. In this case,
\[
\Gamma^\Omega_2(f, f) = \Gamma^\Omega(f, f) = \Delta_\Omega f = 0.
\]
So, (1.28) becomes very simple and we are able to get a classification in this case.

**Theorem 1.5** Let \((G, m, w, B)\) be a connected weighted finite graph with boundary and \(E(\Omega, \Omega) = \emptyset\). Suppose that \((G, m, w)\) satisfy the Bakry-Émery curvature-dimension condition \(\text{CD}(K, n)\) with \(K > 0\) and \(n > 1\), and \(\sigma_2 = \frac{nK}{n-1}\). Then, \(|V(G)| = 3\) or \(4\). When \(|V(G)| = 3\), one has \(n \geq 2\) and \((G, B)\) is a path on three vertices with the two end points as the boundary vertices. Moreover, if denote the two boundary vertices as 1 and 2, and the interior vertex as \(x\), then \(m_1 = m_2 := m\), \(w_{1x} = w_{2x} = \frac{mk}{n-1}\), and \(m_x = \frac{2nm}{n+2}\). When \(|V(G)| = 4\), one has \(n = \infty\) and \((G, B)\) is a square with two diagonal vertices as the boundary vertices. Moreover, if denote the two boundary vertices as 1 and 2, and the interior vertices as \(x\) and \(y\), then \(m_1 = m_2 = m_x = m_y := m\) and \(w_{1x} = w_{2x} = w_{1y} = w_{2y} = \frac{mK}{2}\).
By applying the last theorem, one has the following rigidity of (1.25) for graphs with normalized weight.

**Theorem 1.6** Let \((G, m, w, B)\) be a connected finite graph with boundary equipped with a normalized weight. Suppose that \((G, m, w)\) satisfy the curvature-dimension condition \(\text{CD}(K, n)\) with \(K > 0\) and \(n > 1\) and \(\sigma_2 = \frac{nK}{n-1}\). Then, \(n = \infty\), \(K = 1\), and \(V(G) = 3\) or 4. Moreover, \((G, m, w, B)\) is the graphs listed in Theorem 1.5 with \(n = \infty\) and \(K = 1\).

For general weighted graphs, we have the following information about the structures of graphs satisfying the Bakry-Émery curvature-dimension condition \(\text{CD}(K, n)\) with \(K > 0\) and \(n > 1\) and that the equality of (1.25) holds derived from (1.28) and (1.29).

**Theorem 1.7** Let \((G, m, w, B)\) be a connected weighted finite graph with boundary. Suppose that \((G, m, w)\) satisfy the Bakry-Émery dimension-curvature condition \(\text{CD}(K, n)\) with \(K > 0\) and \(n > 1\) and \(\sigma_2 = \frac{nK}{n-1}\). Then,

1. When \(2 < n < \infty\), the induced subgraph on \(\Omega\) can not contain two disjoint balls of radius 2. In particular, the induced subgraph on \(\Omega\) must be connected and with diameter not greater than 4.
2. When \(n = \infty\), the induced subgraph on \(\Omega\) can not contain two disjoint balls of radius 2 unless \((G, B)\) is a square with two diagonal vertices the boundary vertices (i.e. the case with \(|V(G)| = 4\) in Theorem 1.5.). In particular, if \(G\) is not a square, then the induced subgraph on \(\Omega\) must be connected and with diameter not greater than 4.

At the end of this section, we would like to mention that, although Dirichlet-to-Neumann maps and Steklov eigenvalues on graphs were introduced very recently by Hua-Huang-Wang [14] and Hassannezhad-Miclo [12] independently, there are already a number of works on the topic. See for example [11, 13, 15, 25, 26, 29, 30]. In [31], we obtained Lichnerowicz-type estimates for Dirichlet and Neumann eigenvalues on graphs.

The rest of the paper is organized as follows. In Sect. 2, we prove Theorem 1.2 and obtain crucial properties for graphs on which equality of (1.25) holds. Moreover, as a simple application of the properties we obtained, we prove Theorem 1.3 which gives a classification of graphs with unit weight such that equality of (1.25) holds. In Sect. 3, we deal with the rigidity of (1.25) for general weighted graphs.

## 2 Lichnerowicz estimate and rigidity for unit weighted

In this section, we prove Theorems 1.2 and 1.3 and obtain crucial properties for graphs on which equality of (1.25) holds. Throughout this section, \(u \in \mathbb{R}^V\) is used to denote the harmonic extension of an eigenfunction of \(\sigma_2\) for \((G, m, w, B)\). More precisely, \(u\) is a function on \(V\) satisfying:

\[
\begin{aligned}
\Delta u(x) &= 0 \quad x \in \Omega \\
\frac{\partial u}{\partial n}(x) &= \sigma_2 u(x) \quad x \in B \\
\langle u, u \rangle_B &= 1.
\end{aligned}
\]  

(2.1)

We first prove Theorem 1.2. The argument is rather classical by just integrating (1.10).
**Proof of Theorem 1.2** By (1.8), (1.15), (1.26) and (2.1),

\[
\sum_{x \in V(G)} \Gamma_2(u, u)(x)m_x = - \sum_{x \in V} \Gamma(\Delta u, u)(x)m_x
\]

\[
= - (d\Delta u, du)
\]

\[
= \langle \Delta u, \Delta u \rangle
\]

\[
= \left( \frac{\partial u}{\partial n}, \frac{\partial u}{\partial n} \right)_B
\]

\[
= \sigma_2^2 (u, u)_B
\]

\[
= \sigma_2^2.
\]

Here, in the first equality, we have used the fact that

\[
\sum_{x \in V} \Delta f(x)m_x = \langle \Delta f, 1 \rangle = - \langle df, d1 \rangle = 0
\]

for any \( f \in \mathbb{R}^V \). Furthermore, by (1.15), (1.26) and (2.1),

\[
\sum_{x \in V} \left( \frac{1}{n} (\Delta u(x))^2 + K\Gamma(u, u)(x) \right)m_x
\]

\[
= \frac{\sigma_2^2}{n} (u, u)_B + K \langle du, du \rangle
\]

\[
= \frac{\sigma_2^2}{n} - K \langle \Delta u, u \rangle
\]

\[
= \frac{\sigma_2^2}{n} + K \left( \frac{\partial u}{\partial n} \right)_B
\]

\[
= \frac{\sigma_2^2}{n} + K \sigma_2.
\]

Finally, because \((G, m, \mu)\) satisfies the Bakry-Émery curvature-dimension condition \(\text{CD}(K, n)\), one has

\[
\sigma_2^2 \geq \frac{\sigma_2^2}{n} + K \sigma_2
\]

which gives us the conclusion since \(\sigma_2 > 0\).

\[\Box\]

We next discuss the rigidity of (1.25). We first have the following lemma mainly claiming that \(u\) is also an eigenfunction for \(\mu_2\) when the equality in (1.25) holds.

**Lemma 2.1** Let the notations and assumptions be the same as in Theorem 1.2. Moreover, suppose that \(\sigma_2 = \frac{nK}{n-1}\). Then, \(\mu_2(G) = \frac{nK}{n-1}\) and \(u\) is also an eigenfunction for \(\mu_2(G)\). Moreover, \(u(x) = 0\) for \(x \in \Omega\).

**Proof** Let \(v = u + c\) be such that \(\langle v, 1 \rangle = 0\) where \(c\) is some constant. Then,

\[
\langle du, du \rangle = \langle dv, dv \rangle.
\]
Moreover, note that $\langle u, 1 \rangle_B = 0$, so
\[
\langle v, v \rangle_B = \langle u + c, u + c \rangle_B = \langle u, u \rangle_B + \langle c, c \rangle_B \geq \langle u, u \rangle_B.
\] (2.6)
Then, by the above inequality and Theorem 1.1,
\[
\frac{nK}{n - 1} = \sigma_2 = \frac{\langle du, du \rangle}{\langle u, u \rangle_B} \geq \frac{\langle dv, dv \rangle}{\langle v, v \rangle_B} \geq \frac{\langle dv, dv \rangle}{\langle v, v \rangle_B} \geq \mu_2 \geq \frac{nK}{n - 1}.
\] (2.7)
So, the inequalities in the expression above must be all equalities which gives us the conclusions of the lemma. $\square$

The following lemma is a key ingredient in the study of rigidity for (1.25).

**Lemma 2.2** Let the notations and assumptions be the same as in Theorem 1.2. Moreover, suppose that $\sigma_2 = \frac{nK}{n - 1}$. Then,

1. for any $x, z \in B$ with $d(x, z) = 2$, one has $u(x) = -u(z)$;
2. for any $x \in B$, $u(x) \neq 0$;
3. for any $x \in B$ and $y \in \Omega$, $x \sim y$;
4. $|B| = 2$;
5. let $B = \{1, 2\}$. Then $m_1 = m_2$ and for any $x \in \Omega$, $w_{1x} = w_{2x}$;
6. for each $x \in B$, $\text{Deg}(x) = \frac{nK}{n - 1}$;
7. for each $x \in \Omega$, $\text{Deg}_b(x) = \frac{(n+2)K}{n - 1}$.

**Proof** By the proof of Theorem 1.2, because the equality of (1.25) holds,
\[
\Gamma_2(u, u)(x) = \frac{1}{n} (\Delta u)^2(x) + K \Gamma(u, u)(x)
\] (2.8)
for any $x \in V$. By the same argument as in the proof of Theorem 3.4 (b) in [20, P. 15], one has
\[
\frac{u(z) + u(x)}{2} = \frac{\sum_{y \in V} u(y)w_{xy}w_{yz}/m_y}{\sum_{y \in V} w_{xy}w_{yz}/m_y}.
\] (2.9)
for any $x, z \in V$ with $d(x, z) = 2$. We now come to the proofs of each conclusions.

1. By Lemma 2.1, for any $y \in \Omega$, $u(y) = 0$. So, by (2.9),
\[
u(z) + u(x) = 0.
\]
2. Let $N = \{x \in B \mid u(x) \neq 0\}$. We first claim that for any $x \in \Omega$, $d(x, N) = 1$. Otherwise, let $x \in \Omega$ with $d(x, N) = l \geq 2$. Let
\[
N \ni x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_l = x
\]
be a shortest path joining $x$ to $N$. It is clear that $x_1 \in \Omega$. If $x_2 \in B$, then by (1), $x_2 \in N$ because $d(x_0, x_2) = 2$. However, this is impossible since the path above is the shortest path joining $x$ to $N$. So, $x_2 \in \Omega$ and moreover $d(x_0, x_2) = 2$. This is also because the path we chosen above is the shortest path joining $x$ to $N$. Then, substituting the two vertices $x_0$ and $x_2$ into (2.9), we find that the L.H.S. of (2.9) is nonzero while the R.H.S. is zero which is a contradiction.
The claim above implies that
\[
\Gamma(u, u)(y) > 0
\]
for any \( y \in \Omega \). Now, if there is a \( x \in B \) with \( u(x) = 0 \), then
\[
\Delta u(x) = 0
\]
and
\[
\Gamma(u, u)(x) = 0,
\]
since \( u|_{\Omega} = 0 \) by Lemma 2.1. Moreover,
\[
\Delta \Gamma(u, u)(x) \frac{1}{m_{xy}} \sum_{y \in \Omega} (\Gamma(u, u)(y) - \Gamma(u, u)(x)) \mu_{xy} > 0.
\]
However, by (2.8),
\[
\frac{1}{2} \Delta \Gamma(u, u)(x) - \Gamma(\Delta u, u)(x) = \Gamma(\Delta u, u)(x) = 0
\]
which implies that
\[
\Delta \Gamma(u, u)(x) = 0
\]
since
\[
\Gamma(\Delta u, u)(x) = \mu_2 \Gamma(u, u)(x) = 0.
\]
This is a contradiction. So, \( B = N \).

(3) By the proof of the claim in (2), we know that \( d(y, B) = 1 \) for any \( y \in \Omega \). Then, for any \( y \in \Omega \), let \( x_1, x_2, \ldots, x_k \in B \) be all the neighbors of \( y \) in \( B \). By that \( u|_{\Omega} = (\Delta u)|_{\Omega} = 0 \),
\[
0 = \Delta u(y) \frac{1}{m_{xy}} \sum_{i=1}^{k} u(x_i) w_{x_i y}.
\]
Combining this with (2), we know that \( k \geq 2 \). Moreover, by noting that \( d(x_i, x_j) = 2 \) for \( 1 \leq i < j \leq k \) and (1), we know that \( k = 2 \). So each interior vertex will be adjacent exactly to two boundary vertices.

Now, if there are \( x \in B \) and \( y \in \Omega \) such that \( d(x, y) = l \geq 2 \). Let
\[
x = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_l = y.
\]
be a shortest path joining \( x \) and \( y \). It is clear that \( x_1 \in \Omega \). If \( x_2 \in \Omega \), then \( d(x_0, x_2) = 2 \) because the path we chosen is shortest. Now, similar as in the proof the claim in (2), we get a contradiction by substituting \( x_0 \) and \( x_2 \) into (2.9). Therefore \( x_2 \in B \) and \( x_3 \in \Omega \). Now \( d(x_1, x_3) = 2 \) and they have only one common adjacent boundary vertex \( x_2 \). Substituting \( x_1 \) and \( x_3 \) into (2.9), one can see that the L.H.S of (2.9) is zero while the R.H.S. is nonzero. This is a contradiction.

(4) If there are more than three boundary vertices, then two of them must of the same sign. This violates (1) because of (2) and (3).

(5) Because
\[
u(1) = -u(2) \neq 0
\]
and
\[
u(1)m_1 + u(2)m_2 = \langle u, 1 \rangle_{B} = 0,
\]
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we know that \( m_1 = m_2 \). Moreover, because \( u(x) = \Delta u(x) = 0 \) for \( x \in \Omega \) by Lemma 2.1,

\[
0 = \Delta u(x) = \frac{1}{m_x} (u(1)w_{1x} + u(2)w_{2x}).
\]

(2.11)

So, \( w_{x1} = w_{x2} \).

(6) Note that, for any \( x \in B \),

\[
\frac{nK}{n-1} u(x) = \frac{\partial u}{\partial n}(x) = \frac{1}{m_x} \sum_{y \in \Omega} (u(x) - u(y))w_{xy} = \text{Deg}(x)u(x)
\]

since \( u(y) = 0 \) for \( y \in \Omega \). So, we get the conclusion because \( u(x) \neq 0 \).

(7) Let \( B = \{1, 2\}, m_1 = m_2 = m \) and \( w_{1x} = w_{2x} = w_{x} \) for any \( x \in \Omega \). Then, by direct computation,

\[
\Gamma(u, u)(x) = \begin{cases} 
\frac{\text{Deg}_b(x)}{2} u(1)^2 & x \in \Omega \\
\frac{\text{Deg}(1)}{2} u(1)^2 & x = 1, 2.
\end{cases}
\]

(2.12)

Moreover, by (2.8) and Lemma 2.1,

\[
\Delta \Gamma(u, u)(x) = -\frac{2K}{n-1} \Gamma(u, u)(x).
\]

(2.13)

for any \( x \in \Omega \). Then, for any \( x \in \Omega \),

\[
\Delta_\Omega \Gamma(u, u)(x)
= \frac{1}{m_x} \sum_{y \in \Omega} (\Gamma(u, u)(y) - \Gamma(u, u)(x))w_{xy}
= \frac{1}{m_x} \sum_{y \in V(G)} (\Gamma(u, u)(y) - \Gamma(u, u)(x))w_{xy} - \frac{1}{m_x} \sum_{y \in B} (\Gamma(u, u)(y) - \Gamma(u, u)(x))w_{xy}
= \Delta \Gamma(u, u)(x) - (\text{Deg}(1) - \text{Deg}_b(x)) \Gamma(u, u)(x)
= \left( \text{Deg}_b(x) - \frac{n+2}{n-1} K \right) \Gamma(u, u)(x)
\]

(2.14)

where we have used (2.12), (2.13) and (6).

Let \( x_0 \) be a maximum point of \( \Gamma(u, u)(x) \) in \( \Omega \) which is equivalent to say that \( x_0 \) is a maximum point of \( \text{Deg}_b(x) \) in \( \Omega \) by (2.12). Then,

\[
\Delta_\Omega \Gamma(u, u)(x_0) \leq 0
\]

which implies that

\[
\text{Deg}_b(x) \leq \text{Deg}_b(x_0) \leq \frac{n+2}{n-1} K
\]

(2.15)

for any \( x \in \Omega \), by (2.14). Similarly, by using a minimum point of \( \Gamma(u, u)(x) \) in \( \Omega \), one can see that

\[
\text{Deg}_b(x) \geq \frac{n+2}{n-1} K
\]

(2.16)

for any \( x \in \Omega \). These give us the conclusion. \( \square \)
We are now ready to prove Theorem 1.3, the rigidity of (1.25) for graphs with unit weight.

**Proof of Theorem 1.3** By Lemma 2.2, we know that, when the graph is of unit weight,  
\[ 2 = |B| = \text{Deg}_b(x) = \frac{(n + 2)K}{n - 1} \]  
(2.17)  
for any \( x \in \Omega \), and  
\[ |\Omega| = \text{Deg}(x) = \frac{nK}{n - 1} \]  
(2.18)  
for any \( x \in B \). So,  
\[ |\Omega| = \frac{2n}{n + 2} \leq 2. \]  
(2.19)  
Therefore, \( |\Omega| = 1 \) or 2. When \( |\Omega| = 1 \), by (2.17) and (2.19), \( n = 2 \) and \( K = \frac{1}{2} \). When \( |\Omega| = 2 \), by (2.17) and (2.19) again, \( n = \infty \) and \( K = 2 \). The other conclusions of the theorem are not hard to be checked. \( \square \)

### 3 Rigidity for general weighted graphs

In this section, we discuss the rigidity of (1.25) for general weighted graphs. By Lemma 2.2, we know that for a connected weighted finite graph \((G, m, w, B)\) with boundary satisfying the Bakry-Émery curvature-dimension condition \(\text{CD}(K, n)\) with \(K > 0\) and \(n > 1\), such that the equality of (1.25) holds, the statements (1),(2),(3),(4) in Theorem 1.4 must be true. So, without further indication, we assume the graph \((G, m, w, B)\) satisfy (1),(2),(3),(4) in Theorem 1.4 throughout this section. More precisely, we assume that

- (A1) \( B = \{1, 2\} \) and \( m_1 = m_2 = m \);
- (A2) for any \( x \in \Omega \), \( w_{1x} = w_{2x} = w_x \);
- (A3) \( \text{Deg}(1) = \text{Deg}(2) = \frac{\sum_{x \in \Omega} w_x}{m} = \frac{nK}{n - 1} := \text{Deg} \);
- (A4) for any \( x \in \Omega \), \( \text{Deg}_b(x) = \frac{(n + 2)K}{n - 1} := \text{Deg}_b \).

By (A3) and (A4), one has  
\[ \text{Deg}_b = \frac{2 \sum_{x \in \Omega} w_x}{\sum_{x \in \Omega} m_x} = \frac{\sum_{x \in \Omega} w_x}{m} \cdot \frac{2m}{V_\Omega} = \frac{2m \text{Deg}}{V_\Omega}. \]  
(3.1)  
So,  
\[ V_\Omega = \frac{2m \text{Deg}}{\text{Deg}_b} = \frac{2mn}{n + 2} \]  
(3.2)  
We now compute the expressions of some related quantities in terms of some quantities in the interior.

**Lemma 3.1** For any \( f, g \in \mathbb{R}^V \),  
\[ \Gamma(f, g)(x) = \Gamma(\Omega)(f, g)(x) + \frac{\text{Deg}_b}{2} f(x)g(x) - \frac{\text{Deg}_b}{4} (f(1) + f(2)) g(x) - \frac{\text{Deg}_b}{4} (g(1) + g(2)) f(x) \]  
\[ + \frac{\text{Deg}_b}{4} (f(1)g(1) + f(2)g(2)) \]  
(3.3)
for any $x \in \Omega$,
\[
\Gamma(f, g)(1) = \frac{\text{Deg}_b}{4m} \langle f, g \rangle_{\Omega} - \frac{\text{Deg}_b}{4m} f(1) \langle g, 1 \rangle_{\Omega} - \frac{\text{Deg}_b}{4m} g(1) \langle f, 1 \rangle_{\Omega} + \frac{\text{Deg}}{2} f(1) g(1)
\]  
(3.4)

and similarly,
\[
\Gamma(f, g)(2) = \frac{\text{Deg}_b}{4m} \langle f, g \rangle_{\Omega} - \frac{\text{Deg}_b}{4m} f(2) \langle g, 1 \rangle_{\Omega} - \frac{\text{Deg}_b}{4m} g(2) \langle f, 1 \rangle_{\Omega} + \frac{\text{Deg}}{2} f(2) g(2).
\]  
(3.5)

Proof The proof is just by direct computation using the assumptions (A1),(A2),(A3),(A4). First, we have
\[
\Gamma(f, g)(x) = \frac{1}{2m_x} \sum_{z \in V} (f(z) - f(x))(g(z) - g(x))w_{xz}
\]
\[
= \frac{1}{2m_x} \sum_{z \in \Omega} (f(z) - f(x))(g(z) - g(x))w_{xz} + \frac{1}{2m_x} \sum_{z \in B} (f(z) - f(x))(g(z) - g(x))w_{xz}
\]
\[
= \Gamma^\Omega(f, g)(x) + \frac{w_x}{2m_x} ((f(1) - f(x))(g(1) - g(x)) + (f(2) - f(x))(g(2) - g(x)))
\]
\[
= \Gamma^\Omega(f, g)(x) + \frac{\text{Deg}_b}{4m} ((f(1) - f(x))(g(1) - g(x)) + (f(2) - f(x))(g(2) - g(x)))
\]
\[
= \Gamma^\Omega(f, g)(x) + \frac{\text{Deg}_b}{2} f(x)g(x) - \frac{\text{Deg}_b}{4} (f(1) + f(2)) g(x) - \frac{\text{Deg}_b}{4} (g(1) + g(2)) f(x)
\]
\[
+ \frac{\text{Deg}_b}{4} (f(1) g(1) + f(2) g(2)).
\]
(3.6)

Moreover,
\[
\Gamma(f, g)(1) = \frac{1}{2m} \sum_{x \in \Omega} (f(x) - f(1))(g(x) - g(1))w_x
\]
\[
= \frac{1}{2m} \sum_{x \in \Omega} f(x)g(x)w_x - \frac{1}{2m} f(1) \sum_{x \in \Omega} g(x)w_x - \frac{1}{2m} g(1) \sum_{x \in \Omega} f(x)w_x + \frac{1}{2m} f(1) g(1) \sum_{x \in \Omega} w_x
\]
\[
= \frac{1}{2m} \sum_{x \in \Omega} f(x)g(x)m_x \frac{w_x}{m_x} - \frac{1}{2m} f(1) \sum_{x \in \Omega} g(x)m_x \frac{w_x}{m_x} - \frac{1}{2m} g(1) \sum_{x \in \Omega} f(x)m_x \frac{w_x}{m_x} + \frac{\text{Deg}}{2} f(1) g(1)
\]
\[
= \frac{\text{Deg}_b}{4m} \langle f, g \rangle_{\Omega} - \frac{\text{Deg}_b}{4m} f(1) \langle g, 1 \rangle_{\Omega} - \frac{\text{Deg}_b}{4m} g(1) \langle f, 1 \rangle_{\Omega} + \frac{\text{Deg}}{2} f(1) g(1)
\]
(3.7)

By similar computation as in the proof of the last lemma, we have the following expressions for $\Delta f$.

Lemma 3.2 Let $f \in \mathbb{R}^V$. Then,
\[
\Delta f(x) = \Delta^\Omega f(x) - \text{Deg}_b f(x) + \frac{\text{Deg}_b}{2} (f(1) + f(2))
\]
(3.8)
for any $x \in \Omega$,

\[
\Delta f (1) = \frac{\text{Deg}_b (f, 1)}{2m} \langle f, 1 \rangle_\Omega - \text{Deg} f (1)
\]

and similarly,

\[
\Delta f (2) = \frac{\text{Deg}_b (f, 1)}{2m} \langle f, 1 \rangle_\Omega - \text{Deg} f (2)
\]

Proof The proof is similar as before by direct computation, we omit it for simplicity.

Combining the two lemmas above, we have the following expression for $\Gamma_2 (f, f)$.

**Lemma 3.3** (1) Let $x \in \Omega$, for any $f \in \mathbb{R}^V$ with $f (x) = 0$,

\[
\Gamma_2 (f, f) (x)
\]

\[
= \Gamma_2^\Omega (f, f) (x) + \text{Deg}_b \Gamma_2 (f, f) (x) + \frac{\text{Deg}_b^2 (f, f)}{8m} \langle f, f \rangle_\Omega
\]

\[
+ \frac{\text{Deg}_b}{8} \left(3 \text{Deg} f^2 (1) + 2 \text{Deg}_b f (1) f (2) + 3 \text{Deg} f^2 (2)\right) - \frac{\text{Deg}_b^2}{4m} \langle f, 1 \rangle_\Omega (f (1) + f (2)).
\]

(3.11)

(2) For any $f \in \mathbb{R}^V$ with $f (1) = 0$,

\[
\Gamma_2 (f, f) (1)
\]

\[
= \frac{\text{Deg}_b}{2m} \langle df, df \rangle_\Omega + \frac{\text{Deg}_b (3 \text{Deg}_b - \text{Deg})}{8m} \langle f, f \rangle_\Omega + \frac{\text{Deg}_b^2}{8m^2} \langle f, 1 \rangle_\Omega^2
\]

\[
+ \frac{\text{Deg}_b \text{Deg}}{8} f^2 (2) - \frac{\text{Deg}_b^2}{4m} \langle f, 1 \rangle_\Omega f (2)
\]

and similarly, for any $f \in \mathbb{R}^V$ with $f (2) = 0$,

\[
\Gamma_2 (f, f) (2)
\]

\[
= \frac{\text{Deg}_b}{2m} \langle df, df \rangle_\Omega + \frac{\text{Deg}_b (3 \text{Deg}_b - \text{Deg})}{8m} \langle f, f \rangle_\Omega + \frac{\text{Deg}_b^2}{8m^2} \langle f, 1 \rangle_\Omega^2
\]

\[
+ \frac{\text{Deg}_b \text{Deg}}{8} f^2 (1) - \frac{\text{Deg}_b^2}{4m} \langle f, 1 \rangle_\Omega f (1)
\]
Proof (1) By Lemmas 3.1 and 3.2,

\[
\Delta \Gamma(f, f)(x) \\
= \Delta \Omega \Gamma(f, f)(x) - \text{Deg}_b \Gamma(f, f)(x) + \frac{\text{Deg}_b}{2} (\Gamma(f, f)(1) + \Gamma(f, f)(2)) \\
= \Delta \Omega \Gamma^\Omega(f, f)(x) + \frac{\text{Deg}_b}{2} \Delta \Omega f^2(x) - \frac{\text{Deg}_b}{2} (f(1) + f(2)) \Delta \Omega f(x) \\
- \text{Deg}_b \left( \Gamma^\Omega(f, f)(x) + \frac{\text{Deg}_b}{4} (f^2(1) + f^2(2)) \right) \\
+ \frac{\text{Deg}_b}{2} \left( \frac{\text{Deg}_b}{2m} (f, f)_{\Omega} - \frac{\text{Deg}_b}{2m} (f(1) + f(2))(f, 1)_{\Omega} + \frac{\text{Deg}_b}{2} (f^2(1) + f^2(2)) \right) \\
= \Delta \Omega \Gamma^\Omega(f, f)(x) + \frac{\text{Deg}_b^2}{4m} (f, f)_{\Omega} - \frac{1}{4} (\text{Deg}_b - \text{Deg}) \text{Deg}_b (f^2(1) + f^2(2)) \\
- \left( \frac{\text{Deg}_b}{2} \Delta \Omega f(x) + \frac{\text{Deg}_b^2}{4m} (f, 1)_{\Omega} \right) (f(1) + f(2)) \\
(3.14)
\]

by noting that

\[
\Gamma^\Omega(f, f)(x) = \frac{1}{2} \Delta \Omega f^2(x) - f(x) \Delta \Omega f(x) = \frac{1}{2} \Delta \Omega f^2(x)
\]
since we have assumed that \( f(x) = 0 \).

Moreover, by Lemmas 3.1 and 3.2 again,

\[
\Gamma(\Delta f, f)(x) \\
= \Gamma^\Omega(\Delta f, f)(x) - \frac{\text{Deg}_b}{4} (f(1) + f(2)) \Delta f(x) + \frac{\text{Deg}_b}{4} (f(1) \Delta f(1) + f(2) \Delta f(2)) \\
= \Gamma^\Omega(\Delta \Omega f, f)(x) - \text{Deg}_b \Gamma^\Omega(f, f)(x) - \frac{\text{Deg}_b}{4} (f(1) \\
+ f(2)) \left( \Delta \Omega f(x) + \frac{\text{Deg}_b}{2} (f(1) + f(2)) \right) \\
+ \frac{\text{Deg}_b}{4} \left( \frac{\text{Deg}_b}{2m} (f, 1)_{\Omega} (f(1) + f(2)) - \text{Deg}(f^2(1) + f^2(2)) \right) \\
(3.15)
\]

Hence, by (3.14) and (3.15),

\[
\Gamma_2(f, f)(x) \\
= \frac{1}{2} \Delta \Gamma(f, f)(x) - \Gamma(\Delta f, f)(x)
\]
\[
\begin{align*}
\Gamma_2^{\Omega}(f, f)(x) + \text{Deg}_b \Gamma^{\Omega}(f, f)(x) + \frac{\text{Deg}_b^2}{8m} (f, f)_{\Omega}
+ \frac{\text{Deg}_b}{8} (3\text{Deg} f^2(1) + 2\text{Deg}_b f(1) f(2) + 3\text{Deg} f^2(2)) - \frac{\text{Deg}_b^2}{4m} (f, 1)_{\Omega} f(1) + f(2)).
\end{align*}
\]

(2) By Lemmas 3.1 and 3.2 and noting that \( f(1) = 0 \) by assumption,
\[
\Delta \Gamma(f, f)(1)
= \frac{\text{Deg}_b}{2m} \Gamma(f, f)(1) - \frac{\text{Deg}_b}{2m} \Gamma(f, f)(1)
- \frac{\text{Deg}_b (3\text{Deg} - \text{Deg})}{4m} (f, f)_{\Omega} + \frac{\text{Deg}_b^2 V_{\Omega}}{8m} f^2(2) - \frac{\text{Deg}_b^2 (f, 1)_{\Omega}}{4m} f(2),
\]
where \( \langle df, df \rangle_{\Omega} := \langle df, du \rangle_{E(\Omega, \Omega)} \). Moreover,
\[
\Gamma(\Delta f, f)(1)
= \frac{\text{Deg}_b}{4m} \langle \Delta f, f \rangle_{\Omega} - \frac{\text{Deg}_b}{4m} \Delta f(1)_{\Omega}(f, 1)_{\Omega}
= \frac{\text{Deg}_b}{4m} \langle \Delta f, f \rangle_{\Omega} - \frac{\text{Deg}_b^2}{4m} (f, f)_{\Omega} - \frac{\text{Deg}_b^2}{8m^2} (f, 1)_{\Omega}^2 + \frac{\text{Deg}_b^2}{8m} (f, 1)_{\Omega} f(2)
= - \frac{\text{Deg}_b}{4m} \langle df, df \rangle_{\Omega} - \frac{\text{Deg}_b}{4m} (f, f)_{\Omega} - \frac{\text{Deg}_b^2}{8m^2} (f, 1)_{\Omega}^2 + \frac{\text{Deg}_b^2}{8m} (f, 1)_{\Omega} f(2).
\]

Combining (3.17) and (3.18),
\[
\Gamma_2(f, f)(1)
= \frac{1}{2} \Delta \Gamma(f, f)(1) - \Gamma(\Delta f, f)(1)
= \frac{\text{Deg}_b}{2m} \langle df, df \rangle_{\Omega} + \frac{\text{Deg}_b (3\text{Deg} - \text{Deg})}{8m} (f, f)_{\Omega} + \frac{\text{Deg}_b^2}{8m^2} (f, 1)_{\Omega}^2 + \frac{\text{Deg}_b^2}{8m} (f, 1)_{\Omega} f(2)
\]
\[
\frac{\text{Deg}_b}{8} f^2(2) - \frac{\text{Deg}_b^2}{4m} (f, 1)_{\Omega} f(2)
\]
where we have used (3.2).  \( \square \)

We are now ready to compute the curvature of a graph satisfying the assumptions (A1), (A2), (A3), (A4).

**Theorem 3.1** Let \((G, m, w, B)\) be a connected weighted finite graph with boundary satisfying the assumptions (A1), (A2), (A3) and (A4) with \( K > 0 \) and \( n > 1 \). Then,

(1) \((G, m, w)\) satisfies the Bakry-Émery curvature-dimension condition \( \text{CD}(K, n) \) at the boundary vertices.

\( \square \) Springer
(2) $(G, m, w)$ satisfies the Bakry-Émery curvature-dimension condition $\text{CD}(K, n)$ at $x \in \Omega$ if and only if either $n = 2$ and $|\Omega| = 1$, or $n > 2$ and

$$
\Gamma_2^\Omega(f, f)(x) = \frac{1}{n-2}(\Delta f)^2(x) + \frac{3K}{n-1} \Gamma^\Omega(f, f)(x) + \frac{(n+2)^2K^2}{8m(n-1)^2} (f, f)_\Omega
$$

$$
- \frac{(n+2)K}{(n-1)(n-2)m} f_\Omega \Delta f(x) - \frac{n(n+2)K^2}{8(n-2)(n-1)^2m^2} (f, f)_\Omega \geq 0
$$

for any $f \in \mathbb{R}^\Omega$ with $f(x) = 0$. In particular, $(G, m, w)$ satisfies the Bakry-Émery curvature-dimension condition $\text{CD}(K, \infty)$ at $x \in \Omega$ if and only if

$$
\Gamma_2^\Omega(f, f)(x) + \frac{K^2}{8m} (f, f)_\Omega - \frac{K^2}{8m^2} (f, 1)_\Omega^2 \geq 0
$$

for any $f \in \mathbb{R}^\Omega$ with $f(x) = 0$.

**Proof** (1) Note that $(G, m, w)$ satisfies the Bakry-Émery curvature-dimension condition $\text{CD}(K, n)$ at 1 if and only if

$$
\Gamma_2(f, f)(1) - \frac{1}{n} (\Delta f)^2(1) - K \Gamma(f, f)(1) \geq 0
$$

for any $f \in \mathbb{R}^V$ with $f(1) = 0$. By Lemmas 3.1, 3.2 and 3.3,

$$
\Gamma_2(f, f)(1) - \frac{1}{n} (\Delta f)^2(1) - K \Gamma(f, f)(1)
$$

$$
= \frac{\text{Deg}_b}{2m} \langle df, df \rangle_\Omega + \frac{\text{Deg}_b}{4m} \left( \frac{3\text{Deg}_b - \text{Deg}}{2} - K \right) \langle f, f \rangle_\Omega + \frac{\text{Deg}_b^2}{4m^2} \left( \frac{1}{2} - \frac{1}{n} \right) (f, 1)_\Omega^2
$$

$$
+ \frac{\text{Deg}_b \text{Deg}}{8} f^2(2) - \frac{\text{Deg}_b^2}{4m} \langle f, 1 \rangle_\Omega f(2)
$$

$$
= \frac{\text{Deg}_b}{2m} \langle df, df \rangle_\Omega + \frac{\text{Deg}_b}{4m} \left( \frac{3\text{Deg}_b - \text{Deg}}{2} - K \right) \langle f, f \rangle_\Omega + \frac{\text{Deg}_b^2}{4m^2} \left( \frac{1}{2} - \frac{1}{n} \right) (f, 1)_\Omega^2
$$

$$
+ \frac{\text{Deg}_b \text{Deg}}{8} \left( f(2) - \frac{\text{Deg}_b}{m \text{Deg}} \langle f, 1 \rangle_\Omega \right)^2 - \frac{\text{Deg}_b^3}{8m^2 \text{Deg}} \langle f, 1 \rangle_\Omega^2
$$

$$
\geq \frac{\text{Deg}_b}{4m} \left( \frac{3\text{Deg}_b - \text{Deg}}{2} - K \right) \langle f, f \rangle_\Omega + \frac{\text{Deg}_b^2}{4m^2} \left( \frac{1}{2} - \frac{1}{n} - \frac{\text{Deg}_b}{2 \text{Deg}} \right) (f, 1)_\Omega^2
$$

$$
= \frac{\text{Deg}_b}{4m} \left( \frac{4K}{n-1} (f, f)_\Omega - \frac{2 \text{Deg}_b}{nm} (f, 1)_\Omega^2 \right)
$$

$$
\geq \frac{\text{Deg}_b}{4m} \left( \frac{4K}{n-1} (f, f)_\Omega - \frac{2 \text{Deg}_b \text{Vol}}{nm} (f, f)_\Omega \right)
$$

$$
= \frac{\text{Deg}_b}{4m} \left( \frac{4K}{n-1} (f, f)_\Omega - \frac{4 \text{Deg}_b}{n} (f, f)_\Omega \right)
$$

$$
= 0
$$

(3.23)

where we have used the Cauchy-Schwartz inequality and (3.2). Similarly, the Bakry-Émery curvature-dimension condition $\text{CD}(K, n)$ is also satisfied at 2.
(2) Note that \((G, m, w)\) satisfies the Bakry-Émery curvature-dimension condition \(\text{CD}(K, n)\) at \(x \in \Omega\) if and only if
\[
\Gamma_2(f, f)(x) - \frac{1}{n} (\Delta f)^2(x) - K \Gamma(f, f)(x) \geq 0
\]
(3.24)
for any \(f \in \mathbb{R}^V\) with \(f(x) = 0\). By Lemmas 3.1, 3.2 and 3.3,
\[
\begin{align*}
\Gamma_2(f, f)(x) &= \frac{1}{n} (\Delta f)^2(x) - K \Gamma(f, f)(x) \\
&= \Gamma_2^\Omega(f, f)(x) - \frac{1}{n} (\Delta f)^2(x) + (\text{Deg}_b - K) \Gamma^\Omega(f, f)(x) + \frac{\text{Deg}_b^2}{8m} (f, f)_{\Omega} \\
&\quad + \frac{\text{Deg}_b}{8} \left( 3\text{Deg} - \frac{2\text{Deg}_b}{n} - 2K \right) f^2(1) + \left( 2 - \frac{4}{n} \right) \text{Deg}_b (1) f(2) \\
&\quad + \left( 3\text{Deg} - \frac{2\text{Deg}_b}{n} - 2K \right) f^2(2) - \left( \frac{\text{Deg}_b^2}{4m} (f, 1)_{\Omega} + \frac{\text{Deg}_b}{n} \Delta f(x) \right) (f(1) + f(2)) \\
&= \Gamma_2^\Omega(f, f)(x) - \frac{1}{n} (\Delta f)^2(x) + \frac{3K}{n-1} \Gamma^\Omega(f, f)(x) + \frac{\text{Deg}_b^2}{8m} (f, f)_{\Omega} \\
&\quad + \frac{1}{8} \left( 1 - \frac{2}{n} \right) \text{Deg}_b^2 (f(1) + f(2))^2 - \left( \frac{\text{Deg}_b}{4m} (f, 1)_{\Omega} + \frac{1}{n} \Delta f(x) \right) \text{Deg}_b (f(1) + f(2)).
\end{align*}
\]
(3.25)

When \(n \in (1, 2)\), the expression (3.25) will be negative if we choose \(f \in \mathbb{R}^V\) such that \(f|_{\Omega} \equiv 0\) and \(f(1) + f(2) \neq 0\). So, in this case, the graph will not satisfy the Bakry-Émery curvature-dimension \(\text{CD}(K, n)\) at \(x\). Moreover, when \(n = 2\) and \(|\Omega| \geq 2\), let \(f(y) = 1\) for any \(y \in \Omega \setminus \{x\}\), so that
\[
\frac{\text{Deg}_b}{4m} (f, 1)_{\Omega} + \frac{1}{n} \Delta f(x) > 0.
\]
(3.26)

Then, by letting \(f(1) + f(2)\) be large enough, the expression (3.25) will be negative. So, in this case, the graph will not satisfy the Bakry-Émery curvature-dimension \(\text{CD}(K, n)\) at \(x\). Furthermore, it is clear that when \(n = 2\) and \(|\Omega| = 1\), the expression (3.25) is zero. So, in this case, the graph will satisfy the Bakry-Émery curvature-dimension \(\text{CD}(K, n)\) at \(x\).

In the following, we will assume that \(n > 2\). We can first fix suitable \(f(1)\) and \(f(2)\) so that
\[
\frac{1}{8} \left( 1 - \frac{2}{n} \right) \text{Deg}_b^2 (f(1) + f(2))^2 - \left( \frac{\text{Deg}_b}{4m} (f, 1)_{\Omega} + \frac{1}{n} \Delta f(x) \right) \text{Deg}_b (f(1) + f(2))
\]
achieves its minimum
\[
- \frac{2n}{n-2} \left( \frac{\text{Deg}_b}{4m} (f, 1)_{\Omega} + \frac{1}{n} \Delta f(x) \right)^2.
\]
(3.27)

So, in this case, the graph satisfies the Bakry-Émery curvature-dimension condition \(\text{CD}(K, n)\) at \(x\) if and only if
\[
\begin{align*}
\Gamma_2^\Omega(f, f)(x) - \frac{1}{n} (\Delta f)^2(x) + \frac{3K}{n-1} \Gamma^\Omega(f, f)(x) + \frac{\text{Deg}_b^2}{8m} (f, f)_{\Omega} \\
&\quad - \frac{2n}{n-2} \left( \frac{\text{Deg}_b}{4m} (f, 1)_{\Omega} + \frac{1}{n} \Delta f(x) \right)^2 \geq 0.
\end{align*}
\]
(3.28)
for any \( f \in \mathbb{R}^\Omega \) with \( f(x) = 0 \). Simplifying the last expression, we arrive at

\[
\Gamma^\Omega_2(f, f)(x) = -\frac{1}{n - 2}(\Delta f)^2(x) + \frac{3K}{n - 1} \Gamma^\Omega(f, f)(x) + \frac{(n + 2)^2 K^2}{8m(n - 1)^2} (f, f)_\Omega
\]

(3.29)

for any \( f \in \mathbb{R}^\Omega \) with \( f(x) = 0 \). This completes the proof of the Theorem. \( \square \)

We are now ready to prove Theorem 1.4. In fact, it is a direct consequence of Lemma 2.2 and Theorem 3.1.

**Proof of Theorem 1.4** Assume that \( (G, m, w, B) \) satisfies the Bakry-Émery curvature-dimension condition \( \text{CD}(K, n) \) with \( K > 0 \) and \( n > 1 \), and that (1.25) holds with equality. Then, by Lemma 2.2, we know that (1),(2),(3),(4) must be true. Moreover, by Theorem 3.1, we know that (5) must be true.

Conversely, by Theorem 3.1, a graph satisfying (1),(2),(3),(4),(5) must satisfy the Bakry-Émery curvature condition \( \text{CD}(K, n) \). So

\[
\sigma_2 \geq \frac{nK}{n - 1}
\]

by Theorem 1.2. Moreover, let \( u \in \mathbb{R}^\Omega \) be such that \( u(1) = -u(2) = 1 \) and \( u|\Omega = 0 \). Then, for any \( x \in \Omega \),

\[
\Delta u(x) = \frac{1}{m_x} [(u(1) - u(x))w_x + (u(2) - u(x))w_x] = \frac{w_x}{m_x} (u(1) + u(2)) = 0,
\]

and

\[
\frac{\partial u}{\partial n}(i) = \frac{1}{m_i} \sum_{x \in \Omega} (u(i) - u(x))w_x = \text{Deg}(i)u(i) = \frac{nK}{n - 1} u(i)
\]

for \( i = 1, 2 \). So \( \frac{nK}{n - 1} \) is a Steklov eigenvalue of the graph. Thus

\[
\sigma_2 = \frac{nK}{n - 1}.
\]

\( \square \)

We next come to prove Theorem 1.5, a rigidity of (1.25) for general weighted graphs with trivial induced subgraph on the interior.

**Proof of Theorem 1.5** When \( n = 2 \), there is nothing to prove by Theorem 1.4. So, in the following, we assume that \( n > 2 \). If \( |\Omega| = 1 \), then there is also nothing to prove by Theorem 1.4. So, we further assume that \( |\Omega| \geq 2 \). By Theorem 1.4, we know that equality of (1.25) holds if and only if assumptions (A1),(A2),(A3),(A4) are satisfied and

\[
\frac{(n + 2)^2 K^2}{8m(n - 1)^2} (f, f)_\Omega - \frac{n(n + 2)^2 K^2}{8(n - 2)(n - 1)^2m^2} (f, 1)^2_\Omega \geq 0
\]

(3.30)

for any \( f \in \mathbb{R}^\Omega \) which is vanished at some vertex in \( \Omega \). For each \( x \in \Omega \), let \( f = \chi_{\Omega \setminus \{x\}} \) in (3.30). We get

\[
V_{\Omega \setminus \{x\}} \leq \frac{(n - 2)m}{n}.
\]

(3.31)
Then,
\[ m_x = V_\Omega - V_{\Omega \setminus \{x\}} \geq \frac{(n^2 + 4)m}{n(n+2)} \geq \frac{nm}{n+2} = \frac{V_\Omega}{2} \]  
(3.32)
by using (3.2). The equality holds if and only if \( n = \infty \). This implies that when \( n < \infty \), it is impossible for \( |\Omega| \geq 2 \). Moreover, when \( n = \infty \), then \( |\Omega| = 2 \) and \( m_x = m \) for any \( x \in \Omega \). This completes the proof of the theorem.

We next come to prove Theorem 1.6, the rigidity of (1.25) for graphs with normalized weights.

**Proof of Theorem 1.6** Because the graph is equipped with a normalized weight, by Lemma 2.2,
\[ 1 = \text{Deg}(x) = \frac{nK}{n-1} \]  
(3.33)
for any \( x \in B \), and moreover,
\[ 1 \geq \text{Deg}_b(x) = \frac{(n+2)K}{n-1} \geq \frac{nK}{n-1} = 1 \]  
(3.34)
for any \( x \in \Omega \). This means that \( n = \infty \), \( K = 1 \) and \( \text{Deg}_b(x) = 1 \) for any \( x \in \Omega \). Because \( \text{Deg}(x) = 1 \) for any \( x \in V \), we know that \( E(\Omega, \Omega) = \emptyset \). Then, by Theorem 1.5, we complete the proof of the theorem.

Finally, we come to prove theorem 1.7.

**Proof of Theorem 1.7** (1) Let \( B_\chi^\Omega(2) \) and \( B_\chi^\Omega(2) \) be two disjoint balls of radius 2 in the induced subgraph on \( \Omega \). Let \( f = \chi_{\Omega \setminus B_\chi^\Omega(2)} \). Then, it is clear that
\[ \Gamma_2^\Omega(f, f)(x) = \Delta_\Omega f(x) = \Gamma^\Omega(f, f)(x) = 0. \]  
(3.35)
Substituting this into (1.28), we get
\[ V_{\Omega \setminus B_\chi^\Omega(2)} \leq \frac{(n-2)m}{n} \]  
(3.36)
which implies that
\[ V_{B_\chi^\Omega(2)} = V_\Omega - V_{\Omega \setminus B_\chi^\Omega(2)} = \frac{(n^2 + 4)m}{n(n+2)} > \frac{V_\Omega}{2}. \]  
(3.37)
Similarly,
\[ V_{B_\Omega^\chi(2)} > \frac{V_\Omega}{2}. \]  
(3.38)
However, this impossible since the two balls are disjoint.

(2) Let \( B_\chi^\Omega(2) \) and \( B_\chi^\Omega(2) \) be two disjoint balls of radius 2 in the induced subgraph on \( \Omega \). By the same argument in (1), we know that
\[ V_{B_\chi^\Omega(2)} = V_{B_\chi^\Omega(2)} = \frac{V_\Omega}{2} = m. \]  
(3.39)
So $\Omega = B^\Omega_x(2) \cup B^\Omega_y(2)$. We are only left to show that $B^\Omega_x(2) = \{x\}$ and $B^\Omega_y(2) = \{y\}$. Otherwise, let $z$ be another vertex in $B^\Omega_x(2)$. Let $f \in \mathbb{R}^\Omega$ be such that

$$
 f(\xi) = \begin{cases} 
 0 & \xi \in B^\Omega_x(2) \text{ and } \xi \neq z \\
 1 & \xi = z \\
 c & \xi \in B^\Omega_y(2).
\end{cases}
$$

(3.40)

Substituting this into (1.29), we have

$$
\Gamma_2^\Omega(f, f)(x) + \frac{K^2}{8} \left( \frac{m_z}{m} - \frac{m_z^2}{m^2} \right) - \frac{K^2m_z}{4m} \cdot c \geq 0.
$$

(3.41)

However, this is impossible when $c$ is large enough since $\Gamma_2^\Omega(f, f)$ depends only on the values of $f$ on $B^\Omega_x(2)$. This completes the proof of the conclusion. \hfill \Box

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