A NEW CLASS OF GLOBAL FRACTIONAL-ORDER PROJECTIVE DYNAMICAL SYSTEM WITH AN APPLICATION

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Abstract. In this article, some existence and uniqueness of solutions for a new class of global fractional-order projective dynamical system with delay and perturbation are proved by employing the Krasnoselskii fixed point theorem and the Banach fixed point theorem. Moreover, an approximating algorithm is also provided to find a solution of the global fractional-order projective dynamical system. Finally, an application to the idealized traveler information systems for day-to-day adjustments processes and a numerical example are given.

1. Introduction. It is well known that the delay fractional differential equation is a type of fractional differential equation in which the fractional derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Introduction of delay in the fractional differential equation (system) enriches its dynamics and allows a precise description of the real life phenomena. Since the (delay) fractional differential equations have been proved to be valuable tools to model many phenomena in various fields of physics, physiology, ecology, viscoelasticity, electrochemistry, control, electromagnetic, engineering and economics, there are many authors to study various theoretical results, numerical algorithms and applications concerned with many kinds of (delay) fractional differential equations in the literature (see, for example, [2–4, 9, 20–23, 41, 44] and the references therein).

On the other hand, it was showed by Dupuis and Nagurney [5] in 1993 that the stationary points of a projective dynamical systems are solutions to some associated variational inequalities. Since the variational inequality is a very powerful tool in investigating various network equilibrium problems arising in economic, management, and engineering, much effort has been made in the projective dynamical
systems in recent decades. Some related results in these directions can be found in [11–13, 16, 32, 33, 37–40, 42, 45–47].

Motivated by the work in connection with fractional differential equations and projective dynamical systems, Wu and Zou [34] introduced and studied the following fractional-order projective dynamical systems

\[ \begin{align*}
\frac{D_0^\alpha}{\alpha} x(t) &= P_K [x(t) - \rho M x(t) - \rho b] - x(t), \quad t \geq 0, \ 0 < \alpha < 1, \\
x_i(0) &= x_{i0}, \quad i = 1, 2, \ldots, n,
\end{align*} \tag{1} \]

where \( M \) is a real matrix. They proved the existence and uniqueness of the solution for this system and showed the stability for the equilibrium point. They also provided a predictor-corrector algorithm to approximating a solution to fractional-order projective dynamical system. Very recently, based on a network tatonnement model, Wu et al. [35] introduced a system of fractional-order interval projection neural networks as follows

\[ \begin{align*}
\frac{D_0^\alpha}{\alpha} x(t) &= P_{K_1} [x(t) - \rho (Ax(t) + A^* y(t)) - \rho a] - x(t), \quad t \geq 0, \\
x(0) &= x_0, \\
\frac{D_0^\alpha}{\alpha} y(t) &= P_{K_2} [y(t) - \lambda (B y(t) + B^* x(t)) - \lambda b] - y(t), \quad t \geq 0, \\
y(0) &= y_0,
\end{align*} \tag{2} \]

where \( 0 < \alpha \leq 1 \) and

\[ \begin{align*}
A &\in A_I = \left\{ (a_{ij})_{n \times n} : A \leq \bar{A}, \quad i.e., \ a_{ij} \leq \bar{a}_{ij} \leq \bar{a}_{ij} \right\}, \\
A^* &\in A_I^* = \left\{ (a^*_{ij})_{n \times m} : A^* \leq \bar{A}^*, \quad i.e., \ a^*_{ij} \leq \bar{a}^*_{ij} \leq \bar{a}^*_{ij} \right\}, \\
B &\in B_I = \left\{ (b_{ij})_{m \times m} : B \leq \bar{B}, \quad i.e., \ b_{ij} \leq \bar{b}_{ij} \leq \bar{b}_{ij} \right\}, \\
B^* &\in B_I^* = \left\{ (b^*_{ij})_{m \times n} : B^* \leq \bar{B}^*, \quad i.e., \ b^*_{ij} \leq \bar{b}^*_{ij} \leq \bar{b}^*_{ij} \right\},
\end{align*} \]

and showed the existence and uniqueness of the equilibrium point for the fractional-order interval projection neural networks under mild conditions. Moreover, some results concerned with the existence and stability of solutions for the following global fractional-order projective dynamical systems involving set-valued perturbations

\[ \begin{align*}
\frac{D_0^\alpha}{\alpha} x(t) &\in P_{K_1} [x(t) - \rho M (x(t), y(t)) - \rho a] - x(t) + G_1(x(t)), \quad \text{for a.e. } t \in [0, h], x(0) = x_0, \\
\frac{D_0^\alpha}{\alpha} y(t) &\in P_{K_2} [y(t) - \lambda N (y(t), x(t)) - \lambda b] - y(t) + G_2(y(t)), \quad \text{for a.e. } t \in [0, h], y(0) = y_0,
\end{align*} \tag{3} \]

were presented by Wu et al. [36]. Although many contributions for the fractional differential equations and the (fractional-order) projective dynamical systems, to our best knowledge, there is no researchers to study the fractional-order projective dynamical systems with delay appeared as IVP (4). This fact is the motivation of the present work.

In this paper, we consider an initial value problem (for short, IVP) of fractional-order projective dynamical system with delay and perturbation of the following form

\[ \begin{align*}
\frac{D_t^\alpha}{\alpha} x(t) &= P_K [g(x(t)) - \rho M(x_t)] - g(x(t)) + N(t, x_t), \quad t \in [t_0, t_0 + T], \ t_0 \geq 0, \\
x_{t_0} &= \phi,
\end{align*} \tag{4} \]

where \( \frac{D_t^\alpha}{\alpha} x(t) \) is Caputo’s fractional derivative of order \( 0 < \alpha \leq 1 \), \( P_K \) is the projection operator, \( \rho \) is a positive constant, \( g : R^n \rightarrow R^n \), \( M : C([-r, 0], R^n) \rightarrow R^n \), and showed the existence and uniqueness of the equilibrium point for this system and showed the stability for the equilibrium point. They also provided a predictor-corrector algorithm to approximating a solution to fractional-order projective dynamical system. Very recently, based on a network tatonnement model, Wu et al. [35] introduced a system of fractional-order interval projection neural networks as follows

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x(0) &= x_0, \\
\frac{D_0^\alpha}{\alpha} y(t) &= P_{K_2} [y(t) - \lambda (B y(t) + B^* x(t)) - \lambda b] - y(t), \quad t \geq 0, \\
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and showed the existence and uniqueness of the equilibrium point for the fractional-order interval projection neural networks under mild conditions. Moreover, some results concerned with the existence and stability of solutions for the following global fractional-order projective dynamical systems involving set-valued perturbations

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\frac{D_0^\alpha}{\alpha} x(t) &\in P_{K_1} [x(t) - \rho M (x(t), y(t)) - \rho a] - x(t) + G_1(x(t)), \quad \text{for a.e. } t \in [0, h], x(0) = x_0, \\
\frac{D_0^\alpha}{\alpha} y(t) &\in P_{K_2} [y(t) - \lambda N (y(t), x(t)) - \lambda b] - y(t) + G_2(y(t)), \quad \text{for a.e. } t \in [0, h], y(0) = y_0,
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were presented by Wu et al. [36]. Although many contributions for the fractional differential equations and the (fractional-order) projective dynamical systems, to our best knowledge, there is no researchers to study the fractional-order projective dynamical systems with delay appeared as IVP (4). This fact is the motivation of the present work.

In this paper, we consider an initial value problem (for short, IVP) of fractional-order projective dynamical system with delay and perturbation of the following form

\[ \begin{align*}
\frac{D_t^\alpha}{\alpha} x(t) &= P_K [g(x(t)) - \rho M(x_t)] - g(x(t)) + N(t, x_t), \quad t \in [t_0, t_0 + T], \ t_0 \geq 0, \\
x_{t_0} &= \phi,
\end{align*} \tag{4} \]
and $N : [t_0, t_0 + T] \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ are given maps, $K$ is a nonempty closed convex subset of $\mathbb{R}^n$, $x(t) \in \mathbb{R}^n$, $\phi \in C([-r, 0], \mathbb{R}^n)$, $x_t$ stands for the history of state function up to the time $t$, i.e., $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

We note that IVP (4) is presented in the most abstract form and also covers many important problems in projective dynamical system, variational inequality and fractional differential equation as special cases [11–13, 16, 32–37, 39, 40, 42, 45–47]. If $M$ is a linear map, $g(x(t)) = x(t)$, $r = 0$ and $N = 0$, then IVP (4) reduces to the general form of (1). Furthermore, if $\alpha = 1$, then IVP (4) reduces to the global projective dynamical system which has been studied by many authors during the past decades (see, for example, [12, 13, 33, 39, 40, 46]). It is worth mentioning that model (4) captures the desired features of both the projective dynamical system and the fractional differential equation with delay and perturbation within the same framework. To the best of our knowledge, no author has studied the fractional-order projective dynamical system with delay and perturbation. Therefore, the study concerned with the fractional-order projective dynamical system with delay and perturbation is important and interesting in theory and practice. The main purpose of this paper is to give some new results in connection with solutions of IVP (4) under some suitable assumptions.

The rest of this paper is organized as follows. Section 2 introduces some definitions and preliminary facts. Section 3 establishes some sufficient conditions for the existence and uniqueness of the solution of IVP (4). Section 4 provides a numerical algorithm for approximating the solution of IVP (4). Finally, we present an application to the idealized traveler information systems for day-to-day adjustments processes and a numerical example in Section 5.

2. Preliminaries. In this section, we introduce some basic definitions, notations and preliminary facts. Let $I \subset \mathbb{R}$ and $C(I, \mathbb{R}^n)$ be the Banach space of all continuous functions $x(t)$ from $I$ into $\mathbb{R}^n$ with the norm

$$
\|x\|_{C(I, \mathbb{R}^n)} = \sup_{t \in I} \|x(t)\|
$$

where $\| \cdot \|$ denotes a suitable complete norm on $\mathbb{R}^n$.

**Definition 2.1.** If $K$ is a closed convex subset of $\mathbb{R}^n$, then $P_K : \mathbb{R}^n \rightarrow K$ is defined by

$$
P_K[x] = \arg \min_{v \in K} \|x - v\|.
$$

**Definition 2.2.** [15, 26] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $x(t)$ is defined as follows:

$$
I_{t_0}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} x(s)ds \quad (t > t_0).
$$

**Definition 2.3.** [15, 26] The Caputo fractional derivative of order $\alpha$ for a function $x(t) \in \mathbb{C}^n([t_0, +\infty), \mathbb{R})$ is defined as follows:

$$
C_{t_0}^t D_\alpha^x(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^{t} (t - s)^{n-\alpha-1} x^{(n)}(s)ds,
$$

where $t > t_0$ and $n$ is a positive integer such that $n - 1 < \alpha < n$.

**Remark 1.** If $\alpha = n$, then the Caputo fractional derivative of $\alpha$ for a function $x(t)$ is usual derivative $x^{(n)}(t)$.
Lemma 2.4. [16] The projection operator $P_K : R^n \rightarrow K$ is non-expansive, i.e.
\[
\|P_K[x] - P_K[y]\| \leq \|x - y\|, \quad \forall x, y \in R^n.
\]

A function $G : [a, b] \times B_D \rightarrow R^n$ is said to satisfy the $C_\gamma$-condition, if $G(t, y)$ is continuous with respect to $t$ in $[a, b]$ for each given $\gamma$ times continuously differentiable function $y : [a - r, b] \rightarrow D$, where $B_D$ stands for state space $C^\gamma([-r, 0], D)$ and $D$ is a subset of $R^n$.

Lemma 2.5. [20] Let $\alpha > 0$, $m = -[\alpha]$ and $\gamma = m - 1$. If $G : [a, b] \times B_D \rightarrow R^n$ satisfies the $C_\gamma$-condition, then a $\gamma$ times continuously differentiable function $y : [a - r, b] \rightarrow D$ is a solution of the equations
\[
\begin{cases}
C_{\alpha}D^\alpha_t y(t) = G(t, y), & t \in [a, b], \\
x_0 = \phi, & \phi \in B_D
\end{cases}
\]
if and only if
\[
y(t) = \begin{cases}
m - 1 \sum_{j=0}^{m-1} \frac{\phi^{(j)}(0)}{j!} (t - a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} G(s, y_\gamma) ds, & a \leq t \leq b, \\
\phi(t - a), & a - r \leq t \leq a.
\end{cases}
\]

Lemma 2.6. (Krasnoselskii Fixed Point Theorem) [17] Let $X$ be a Banach space and $E$ be a bounded closed convex subset of $X$. Assume that $U$ and $V$ are two mappings of $E$ into $X$ such that
(i) $Ux + Vy \in E$ whenever $x, y \in E$;
(ii) $U$ is a contraction mapping;
(iii) $V$ is a completely continuous.

Then there exists a point $x \in E$ such that $x = Ux + Vx$.

3. Existence and uniqueness. Let
\[
A_\delta = \left\{ x \in C([t_0 - r, t_0 + T], R^n) \mid x_{t_0} = \phi, \quad \sup_{t_0 \leq t \leq t_0 + T} \|x(t) - \phi(0)\| \leq \delta \right\},
\]
where $\delta$ is positive constant.

In what follows, we introduce the following hypotheses.

(H1) $g$ is Lipschitz continuous, i.e., for any $x^1, x^2 \in A_\delta$,
\[
\|g(x^1) - g(x^2)\| \leq l_1 \|x^1 - x^2\|_{C([t_0, t_0 + T], R^n)}, \quad l_1 > 0;
\]

(H2) $M$ is Lipschitz continuous, i.e., for any $x^1, x^2 \in A_\delta$, $t \in [t_0, t_0 + T],
\[
\|M(x^1_t) - M(x^2_t)\| \leq l_2 \|x^1 - x^2\|_{C([t_0, t_0 + T], R^n)}, \quad l_2 > 0;
\]

(H3) $N(t, x_t)$ is measurable with respect to $t$ on $[t_0, t_0 + T]$;

(H4) $N(t, \cdot)$ is continuous with respect to $\varphi$ on $C([t_0, t_0 + T], R^n)$;

(H5) There exist $\beta \in (0, \alpha)$ and a real-valued function $f(t) \in L^\frac{\beta}{\alpha}([t_0, t_0 + T], R)$ such that, for any $x \in A_\delta$,
\[
\|N(t, x_t)\| \leq f(t), \quad \forall t \in [t_0, t_0 + T];
\]

(H6) The following inequality holds
\[
L^* := \frac{2l_1 + \rho l_2}{\Gamma(\alpha + 1)} T^\alpha < 1.
\]
Now, we are ready to state the existence of solution which is based on the Krasnoselskii’s fixed point theorem.

**Theorem 3.1.** If \((H_1)-(H_6)\) hold, then IVP (4) has a least one solution on \([t_0, t_0 + T]\).

**Proof.** According to Lemma 2.5, IVP (4) is equivalent to the following equation

\[
x(t) = \begin{cases} 
\phi(0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} \left\{ P_K [g(x(s)) - \rho M(x_s)] \right\} ds, & t \in [t_0, t_0 + T], \\
\phi(t - t_0), & t \in [t_0 - r, t_0].
\end{cases}
\]

(5)

Let \(\tilde{\phi} \in A_{\delta}\) be define as \(\tilde{\phi}_{t_0} = \phi\) and \(\tilde{\phi}(t_0 + t) = \phi(0)\) for all \(t \in [0, T]\). If \(x\) is a solution of the IVP (4), then let \(x(t_0 + t) = \tilde{\phi}(t_0 + t) + y(t)\) for \(t \in [-r, T]\) and so we know that \(x_{t_0 + t} = \tilde{\phi}(t_0 + t) + y(t)\) for all \(t \in [0, T]\). Obviously, \(y\) satisfies the following equation

\[
y(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_{0}^{t} (s - t)^{\alpha - 1} \left\{ P_K [g(y(s) + \phi(0)) - \rho M(y_s + \tilde{\phi}_{t_0 + s})] \right\} ds, & t \in [0, T], \\
0, & t \in [-r, 0].
\end{cases}
\]

(6)

Since \(K\) is nonempty, we can choose a point \(x_0 \in K\). By the definition of the projection operator, we know that \(\|P_K[0]\| \leq \|x_0\|\). Let \(\delta\) be a constant satisfying the following condition

\[
\delta \geq \frac{\|x_0\| + \|y(0)\| + \rho M(0) + 2\|\phi(0)\| + \rho L_2 \|\phi\| T^\alpha + \frac{1}{\Gamma(\alpha)} \left( \frac{1 - \beta}{\alpha - \beta} \right) T^{\alpha - \beta} F}{1 - L^+},
\]

where \(F = \|f\|_{L^1([t_0, t_0 + T], R)}\). Let

\[E_{\delta} = \{ y \in C([-r, T], R^n) \mid y(t) = 0, \forall t \in [-r, 0], \|y\|_{C([0, T], R^n)} \leq \delta \}.\]

Then it is easy to check that \(E_{\delta}\) is a closed convex bounded subset of \(C([-r, T], R^n)\). Define two mappings \(U\) and \(V\) on \(E_{\delta}\) by

\[
Uy(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} P_K [g(y(s) + \phi(0))] ds, & t \in [0, T], \\
0, & t \in [-r, 0]
\end{cases}
\]

and

\[
Vy(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} \{ -g(y(s) + \phi(0)) \} ds, & t \in [0, T], \\
0, & t \in [-r, 0].
\end{cases}
\]

(7)

Clearly, the operator equation

\[
y = Uy + Vy
\]

has a solution \(y \in E_{\delta}\) if and only if \(y\) is a solution of equation (6). Thus, if \(y \in E_{\delta}\) is a solution of operator equation (7), then \(x(t_0 + t) = y(t) + \tilde{\phi}(t_0 + t)\) is a solution.
of equation (5) on $[0,T]$. Therefore, IVP (4) has a solution is equivalent to that $U + V$ has a fixed point in $E_{\delta}$.

Next, we prove that $U + V$ has a fixed point in $E_{\delta}$. The proof is divided into three steps.

**Step 1.** We show that $U y^1 + V y^2 \in E_{\delta}$, whenever $y^1, y^2 \in E_{\delta}$.

In fact, for every pair $y^1, y^2 \in E_{\delta}$, we know that $U y^1 + V y^2 \in C([0,T], \mathbb{R}^n)$. Also, it is easy to see that $(U y^1 + V y^2)(t) = 0$ for all $t \in [-r,0]$ and $y^1_s + \phi t_{0+s}, y^2_s + \tilde{\phi} t_{0+s} \in A_{\delta}$ for all $s \in [0,T]$. Moreover, for $t \in [0,T]$, it follows from assumption $(H_1)$-$(H_5)$ and Lemma 2.4 that

$$
\| U y^1(t) \| \\
= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ P_K \left[ g \left( y^1(s) + \phi(0) \right) - \rho M \left( y^1_s + \tilde{\phi} t_{0+s} \right) \right] \right. \\
- P_K[0] + P_K[0] \right\} ds \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ P_K \left[ g \left( y^1(s) + \phi(0) \right) - \rho M \left( y^1_s + \tilde{\phi} t_{0+s} \right) \right] \right. \\
- P_K[0] + P_K[0] \right\} ds \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ \| x_0 \| + \| g(0) \| + \rho \| M(0) \| \int_0^t (t-s)^{\alpha-1} ds \right\} ds \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ \| x_0 \| + \| g(0) \| + \rho \| M(0) \| + |l_1(\delta + \| \phi(0) \|) + \rho l_2(\delta + \| \phi \|) | \int_0^t (t-s)^{\alpha-1} ds \right\} \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ \| x_0 \| + \| g(0) \| + \rho \| M(0) \| + |l_1(\delta + \| \phi(0) \|) + \rho l_2(\delta + \| \phi \|) | \int_0^t (t-s)^{\alpha-1} ds \right\} (8)
$$

and

$$
\| V y^2(t) \| \\
= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ - g \left( y^2(s) + \phi(0) \right) + g(0) \right. \\
\left. + N \left( t_0 + s, y^2_s + \tilde{\phi} t_{0+s} \right) \right\} ds \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ \| g \left( y^2(s) + \phi(0) \right) - g(0) \| + \| g(0) \| \right. \\
\left. + N \left( t_0 + s, y^2_s + \tilde{\phi} t_{0+s} \right) \right\} ds \\
\leq \frac{l_1(\delta + \| \phi(0) \|) + \| g(0) \| \int_0^t (t-s)^{\alpha-1} ds}
\[
\begin{align*}
\frac{1}{\Gamma(\alpha)} \left( \int_0^t (t-s)^{-\beta} ds \right)^{1-\beta} 
& \left( \int_{t_0}^{t_0+t} (f(s))^\beta ds \right) \leq \frac{l_1(\delta + \|\phi(0)\| + \|g(0)\|)}{\Gamma(\alpha + 1)} T^\alpha \\
& + \frac{1}{\Gamma(\alpha)} \left( \frac{1 - \beta}{\alpha - \beta} \right)^{1-\beta} T^{\alpha - \beta} F.
\end{align*}
\] (9)

In light of (8), (9) and assumption (H$_6$), we have

\[
\begin{align*}
\|Uy^1(t) + V y^2(t)\| 
& \leq \frac{\|x_0\| + \|g(0)\| + \rho\|M(0)\| + l_1(\delta + \|\phi(0)\|) + \rho l_2(\|\phi\|)}{\Gamma(\alpha + 1)} T^\alpha \\
& + \frac{l_1(\delta + \|\phi(0)\|) + \|g(0)\|}{\Gamma(\alpha + 1)} T^\alpha + \frac{1}{\Gamma(\alpha)} \left( \frac{1 - \beta}{\alpha - \beta} \right)^{1-\beta} T^{\alpha - \beta} F \\
& = \frac{\|x_0\| + 2\|g(0)\| + \rho\|M(0)\| + 2l_1\|\phi(0)\| + \rho l_2\|\phi\|}{\Gamma(\alpha + 1)} T^\alpha \\
& + \frac{1}{\Gamma(\alpha)} \left( \frac{1 - \beta}{\alpha - \beta} \right)^{1-\beta} T^{\alpha - \beta} F + \frac{2l_1 + \rho l_2}{\Gamma(\alpha + 1)} T^\alpha \delta
\end{align*}
\]

and so

\[
\|Uy^1 + V y^2\|_{C([0,T],R^n)} = \sup_{t \in [0,T]} \|Uy^1(t) + V y^2(t)\| \leq \delta,
\]

which implies that $Uy^1 + V y^2 \in E_\delta$, whenever $y^1, y^2 \in E_\delta$.

**Step 2.** We prove that $U$ is a contraction mapping on $E_\delta$.

For any $y^1, y^2 \in E_\delta$, similar to the proof in Step 1, we can show that $y^1_s + \tilde{\phi}_{t,s}, y^2_s + \tilde{\phi}_{t,s} \in A_\delta$ for all $s \in [0, T]$. By Lemma 2.4 and assumption (H$_1$)-(H$_2$), we obtain

\[
\begin{align*}
\|Uy^1(t) - Uy^2(t)\| 
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| P_K \left[ g \left( y^1(s) + \phi(0) \right) - \rho M \left( y^1_s + \tilde{\phi}_{t,s} \right) \right] \right\| ds \\
& - P_K \left[ g \left( y^2(s) + \phi(0) \right) - \rho M \left( y^2_s + \tilde{\phi}_{t,s} \right) \right] \right\| ds \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| (g \left( y^1(s) + \phi(0) \right) - g \left( y^2(s) + \phi(0) \right) ) \right\| ds \\
& - \rho \left( M \left( y^1_s + \tilde{\phi}_{t,s} \right) - M \left( y^2_s + \tilde{\phi}_{t,s} \right) \right) \right\| ds \\
& \leq \frac{l_1 + \rho l_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \|y^1 - y^2\|_{C([0,T],R^n)} \\
& \leq \frac{l_1 + \rho l_2}{\Gamma(\alpha + 1)} T^\alpha \|y^1 - y^2\|_{C([0,T],R^n)}.
\end{align*}
\] (10)

In view of $\frac{l_1 + \rho l_2}{\Gamma(\alpha + 1)} T^\alpha < L^* < 1$, we know that $U$ is a contraction mapping on $E_\delta$.

**Step 3.** $V$ is a completely continuous mapping.

**Claim 1.** The set $\{Vy : y \in E_\delta\}$ is uniformly bounded.
For any \( t \in [0, T] \) and \( y \in E_\delta \), one has

\[
\|Vy(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ \|g(y(s) + \phi(0)) - g(0)\| + \|g(0)\| \right\} ds
+ \frac{l_1(\delta + \|\phi(0)\| + \|g(0)\|)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds
+ \frac{1}{\Gamma(\alpha)} \left( \int_0^t (t-s)^{\frac{2\alpha-1}{\alpha}} ds \right)^{1-\beta} \left( \int_{t_0}^{t_0+t} (f(s))^{\frac{1}{\beta}} ds \right) \beta
\]

\[
\leq \frac{l_1(\delta + \|\phi(0)\| + \|g(0)\|)}{\Gamma(\alpha)} T^\alpha + \frac{1}{\Gamma(\alpha)} \left( \frac{1-\beta}{\alpha-\beta} \right) T^{\alpha-\beta} F,
\]

which means that \( \{Vy : y \in E_\delta\} \) is uniformly bounded.

**Claim 2.** The set \( \{Vy : y \in E_\delta\} \) is equicontinuous.

For any \( 0 \leq t_1 < t_2 \leq T \) and \( y \in E_\delta \), we have

\[
\|Vy(t_2) - Vy(t_1)\| \leq \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_1} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) g(y(s) + \phi(0)) ds \\
+ \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} g(y(s) + \phi(0)) ds \right\}
+ \frac{l_1(\delta + \|\phi(0)\| + \|g(0)\|)}{\Gamma(\alpha)} \int_0^{t_1} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) N \left( t_0 + s, y_s + \tilde{\phi}_{t_0+s} \right) ds
+ \frac{l_1(\delta + \|\phi(0)\| + \|g(0)\|)}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) N \left( t_0 + s, y_s + \tilde{\phi}_{t_0+s} \right) ds \right\}
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_1} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) g(y(s) + \phi(0)) ds \\
+ \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} g(y(s) + \phi(0)) ds \right\}
+ \frac{l_1(\delta + \|\phi(0)\| + \|g(0)\|)}{\Gamma(\alpha)} \int_0^{t_1} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) N \left( t_0 + s, y_s + \tilde{\phi}_{t_0+s} \right) ds
+ \frac{l_1(\delta + \|\phi(0)\| + \|g(0)\|)}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) N \left( t_0 + s, y_s + \tilde{\phi}_{t_0+s} \right) ds \right\}.
\]  

(11)
It follows from the assumption \( (H_1) \) that
\[
\left\| \int_0^{t_1} [(t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}] g(y(s) + \phi(0)) \, ds \right\|
\]
\[
= \left\| \int_0^{t_1} [(t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}] (g(y(s) + \phi(0)) - g(0) + g(0)) \, ds \right\|
\]
\[
\leq \int_0^{t_1} [(t_1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1}] (\|g(y(s) + \phi(0)) - g(0)\| + \|g(0)\|) \, ds
\]
\[
\leq (\delta + \|\phi(0)\| + \|g(0)\|) \int_0^{t_1} [(t_1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1}] \, ds
\]
\[
\leq \frac{\delta + \|\phi(0)\| + \|g(0)\|}{\alpha} (t_2 - t_1)^\alpha
\] (12)

and
\[
\left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} g(y(s) + \phi(0)) \, ds \right\|
\]
\[
= \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} (g(y(s) + \phi(0)) - g(0) + g(0)) \, ds \right\|
\]
\[
\leq \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} (\|g(y(s) + \phi(0)) - g(0)\| + \|g(0)\|) \, ds
\]
\[
\leq (\delta + \|\phi(0)\| + \|g(0)\|) \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} \, ds
\]
\[
\leq \frac{\delta + \|\phi(0)\| + \|g(0)\|}{\alpha} (t_2 - t_1)^\alpha
\] (13)

Using the fact that \((a - b)^\eta \leq a^\eta - b^\eta\) for all \(a > b > 0\) with \(\eta > 1\) and assumption \((H_3)-(H_5)\), we obtain
\[
\left\| \int_0^{t_1} [(t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}] N \left( t_0 + s, y_s + \tilde{\phi}_{t_0 + s} \right) \, ds \right\|
\]
\[
\leq \left( \int_0^{t_1} [(t_1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1}] \left\| \left( \int_{t_0}^{t_0 + t} (f(s))^\frac{1}{\beta} \, ds \right)^\beta \right\| \, ds \right)^{1 - \beta}
\]
\[
\leq \left( \int_0^{t_1} [(t_1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1}] \, ds \right)^{1 - \beta} F
\]
\[
\leq \left( \frac{1 - \beta}{\alpha - \beta} \right)^{1 - \beta} \left( (t_2 - t_1)^{\frac{2 - \beta}{\alpha - \beta}} + t_1^{\frac{2 - \beta}{\alpha - \beta}} - t_2^{\frac{2 - \beta}{\alpha - \beta}} \right) \left( t_2 - t_1 \right)^{\alpha - \beta} F
\]
\[
\leq \left( \frac{1 - \beta}{\alpha - \beta} \right)^{1 - \beta} (t_2 - t_1)^{\alpha - \beta} F
\] (14)

and
\[
\left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} N \left( t_0 + s, y_s + \tilde{\phi}_{t_0 + s} \right) \, ds \right\|
\]
\[
\leq \left( \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} \, ds \right)^{1 - \beta} \left( \int_{t_0}^{t_0 + t} (f(s))^\frac{1}{\beta} \, ds \right)^\beta
\]
Combining (11), (12), (13), (14) and (15), one has
\[
\|V_y(t_2) - V_y(t_1)\| \leq \frac{2(\delta + \|\phi(0)\| + \|g(0)\|)}{\alpha} (t_2 - t_1)^\alpha + 2 \left( \frac{1 - \beta}{\alpha - \beta} \right)^{1 - \beta} (t_2 - t_1)^{\alpha - \beta} F.
\]
which implies that \( \{V_y : y \in E_\delta\} \) is equicontinuous.

Claim 3. The operator \( V \) is continuous on \( E_\delta \).

Let
\[
V_1 y(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} (-g(y(s) + \phi(0))) \, ds, & t \in [0, T], \\
0, & t \in [-r, 0],
\end{cases}
\]
and
\[
V_2 y(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} N(t_0 + s, y_s + \tilde{\phi}_{t_0+s}) \, ds, & t \in [0, T], \\
0, & t \in [-r, 0],
\end{cases}
\]
Clearly, \( V = V_1 + V_2 \). Let \( \{y^m\} \) be a sequence such that \( y^m \to y \) in \( E_\delta \) as \( m \to \infty \). Then for each \( t \in [0, T] \), we have
\[
\|V_1 y^m(t) - V_1 y(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \|g(y^m(s) + \phi(0)) - g(y(s) + \phi(0))\| \, ds
\]
\[
\leq \frac{l_1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} ds \|y^m - y\|_{C([0,T], R^n)}
\]
\[
\leq \frac{l_1 T^\alpha}{\Gamma(\alpha + 1)} \|y^m - y\|_{C([0,T], R^n)} \to 0, \quad \text{as} \quad m \to \infty,
\]
which means that \( V_1 \) is continuous.

On the other hand, by the assumption (H4), for any given number \( \epsilon > 0 \), we can choose \( \delta' > 0 \) such that, for \( \varphi_1, \varphi_2 \in C([-r, 0], R^n) \), when \( \|\varphi_1 - \varphi_2\|_{C([-r, 0], R^n)} < \delta' \), one has
\[
\|N(t, \varphi_1) - N(t, \varphi_2)\| < \frac{\Gamma(\alpha + 1)}{T^\alpha} \epsilon.
\]
Choose \( m_0 \in Z_+ \) such that \( \|y^m - y\|_{C([0,T], R^n)} < \delta' \) for all \( m > m_0 \). Then we know that
\[
y^m_{t_0} + \tilde{\phi}_{t_0+t}, \ y_t + \tilde{\phi}_{t_0+t} \in C([-r, 0], R^n)
\]
and, for any \( t \in [0, T] \),
\[
\left\|\left(y^m_{t_0} + \tilde{\phi}_{t_0+t}\right) - \left(y_t + \tilde{\phi}_{t_0+t}\right)\right\|_{C([-r, 0], R^n)} = \|y^m_t - y_t\|_{C([-r, 0], R^n)} < \delta'.
\]
Thus, one has
\[
\|V_2 y^m(t) - V_2 y(t)\| \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \left\| N \left( t_0 + s, y^m_s + \tilde{\varphi}_{t_0+s} \right) - N \left( t_0 + s, y_s + \tilde{\varphi}_{t_0+s} \right) \right\| ds \\
\leq \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} \frac{\epsilon}{T^\alpha} \int_0^t (t - s)^{\alpha - 1} ds \\
\leq \epsilon
\]
and so
\[
\|V_2 y^m - V_2 y\|_{C([0,T],\mathbb{R}^n)} < \epsilon, \text{ when } m > m_0,
\]
which implies that \( V_2 \) is continuous on \( E_\delta \). This shows that \( V \) is continuous on \( E_\delta \) and so we know that \( V \) is a completely continuous operator.

Therefore, we show that mappings \( U \) and \( V \) satisfy all the conditions of Krasnoselskii fixed point theorem. Thus, Lemma 2.6 implies that \( U + V \) has a fixed point on \( E_\delta \) and so the IVP (4) has a solution
\[
x(t) = \phi(0) + y(t - t_0), \quad t \in [t_0, t_0 + T].
\]
This completes the proof. \( \square \)

Next, we will show the uniqueness result concerned with the solution of IVP (4) by employing the Banach fixed point theorem. To this end, we need the following hypotheses:

\( (H_7) \) For any \( x^1, x^2 \in A_\delta, t \in [t_0, t_0 + T], \)
\[
\left\| N(t, x^1_t) - N(t, x^2_t) \right\| \leq l_3 \left\| x^1 - x^2 \right\|_{C([t_0, t_0 + T], \mathbb{R}^n)}, \quad l_3 > 0;
\]

\( (H_8) \) The following inequality holds
\[
L^\ast := \frac{2l_1 + \rho l_2 + l_3 T^\alpha}{\Gamma(\alpha + 1)} < 1.
\]

**Theorem 3.2.** Assume that all the hypotheses \((H_1)-(H_5)\) and \((H_7)-(H_8)\) are satisfied. Then IVP (4) has a unique solution on \([t_0, t_0 + T]\).

**Proof.** According to the argument of Theorem 3.1, it suffices to prove that \( U + V \) has a unique fixed point on \( E_\delta \). Obviously, \( U + V \) is a mapping from \( E_\delta \) into itself. Similar to the proof of Theorem 3.1, for any \( y^1, y^2 \in E_\delta \), one has
\[
\left\| U y^1(t) - U y^2(t) \right\| \leq \frac{l_1 + \rho l_2}{\Gamma(\alpha + 1)} T^\alpha \left\| y^1 - y^2 \right\|_{C([0,T], \mathbb{R}^n)}
\]
and
\[
\left\| V y^1(t) - V y^2(t) \right\| \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \left\{ \left\| g \left( y^1(s) + \phi(0) \right) - g \left( y^2(s) + \phi(0) \right) \right\| \\
+ \left\| N \left( t_0 + s, y^1_s + \tilde{\varphi}_{t_0+s} \right) - N \left( t_0 + s, y^2_s + \tilde{\varphi}_{t_0+s} \right) \right\| \right\} ds \\
\leq \frac{l_1 + l_3}{\Gamma(\alpha)} \| y^1 - y^2 \|_{C([0,T], \mathbb{R}^n)} \int_0^t (t - s)^{\alpha - 1} ds \\
\leq \frac{l_1 + l_3}{\Gamma(\alpha + 1)} \| y^1 - y^2 \|_{C([0,T], \mathbb{R}^n)}
\]
It follows that
\[
\|(U + V)y^1(t) - (U + V)y^2(t)\| \leq \frac{2l_1 + \rho l_2 + l_3 T^\alpha}{\Gamma(\alpha + 1)} \|y^1 - y^2\|_{C([0, T], \mathbb{R}^n)}
\]
\[
= L^{**} \|y^1 - y^2\|_{C([0, T], \mathbb{R}^n)}
\]
and so
\[
\|(U + V)y^1 - (U + V)y^2\|_{C([0, T], \mathbb{R}^n)} \leq L^{**} \|y^1 - y^2\|_{C([0, T], \mathbb{R}^n)},
\]
where \(L^{**} < 1\). By the Banach fixed point theorem, we know that \(U + V\) has a unique fixed point on \(E_3\). This completes the proof.

4. **An approximating algorithm.** Based on the predictor-corrector scheme, Diethelm et al. [6–8] introduced Adams-Bashforth-Moulton algorithms for the numerical solution of differential equations of fractional order. They also provided detailed error analysis for the Adams-Bashforth-Moulton algorithms. Later, Bhailekar et al. [4] and Wang [31] modified this numerical method to solve delayed fractional-order differential equations, respectively. Now, we take the advantage of the modified Adams-Bashforth-Moulton algorithm for solving IVP (4). To explain the method for constructing the approximating algorithm, we first state a brief introduction.

Consider the following delayed fractional-order differential equation
\[
\begin{cases}
\frac{d^m}{dt^m} y(t) = G_1(t, y(t), y(t - r)), & t \in [0, T], \\
y(t) = \phi(t), & t \in [-r, 0],
\end{cases}
\]
where \(m - 1 < \alpha \leq m\) with \(m \geq 1\) and \(m \in \mathbb{Z}_+\). Let \(h = T/L, L \in \mathbb{Z}_+, t_n = nh, n = 0, 1, \cdots, L, (l - \tau)h = r\) and \(0 \leq \tau < 1\). It is known that the Adams-Bashforth-Moulton algorithm for solving (16) can be discretized as follows:
\[
\begin{aligned}
y_{h}(t_{n+1}) &= \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!}\phi(k)(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} G_1(t_{n+1}, y_{h}^n(t_{n+1}), v_{n+1}) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} G_1(t_j, y_h(t_j), v_j), \\
y_{h}^n(t_{n+1}) &= \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!}\phi(k)(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} G_1(t_j, y_h(t_j), v_j), \\
v_{n+1} &= \{\tau y_{h}(t_{n-l+2}) + (1 - \tau)y_{h}(t_{n-l+1}), \quad \text{if} \quad l > 1, \\
&\quad \tau y_{h}^n(t_{n+1}) + (1 - \tau)y_{h}(t_{n}), \quad \text{if} \quad l = 1,
\end{aligned}
\]
where
\[
a_{j,n+1} = \begin{cases}
\frac{n^{\alpha+1} - (n - \alpha)(n + 1)^\alpha}{1}, & \text{if} \quad j = 0, \\
\frac{(n - j + 2)^{\alpha+1} + (n - j)^{\alpha+1} - 2(n - j + 1)^{\alpha+1}}{1}, & \text{if} \quad 1 \leq j \leq n,
\end{cases}
\]
and
\[
b_{j,n+1} = \frac{h^\alpha}{\alpha} \frac{(n + 1 - j)^\alpha - (n - j)^\alpha}{1}.
\]
Following the Adams-Bashforth-Moulton algorithm for solving (16) mentioned above, the numerical scheme for solving IVP (4) can be depicted as the following form:
where a

An application and a numerical example. First, we will illustrate how this type of system can be put into practice. In 1994, wardropian user equilibrium tatonnement model was constructed to consider idealized traveler information systems for day-to-day adjustments of flows and costs basis in the presence of information, as explained in detail by Friesz et al. [12], who showed that the network tatonnement model was constructed to consider idealized traveler information systems. Hence, the perturbation item is nothing but a form the fractional-order projective dynamical systems with delay and perturbation.

Thus, it is reasonable to replace the integer order derivative by the fractional order derivative in model (19). Therefore, by taking into account all the facts mentioned above, it is important and interesting to extend model (19) as the following form:

\[
\begin{align*}
\frac{C^\alpha_0 D^\alpha_0 x(t)}{dt} &= \Lambda \{ P_K [ x(t) - bM(x(t))] - x(t) \}, \quad \forall t \in [0, T], \\
x(0) &= x_0 = (h_0, u_0)^\top,
\end{align*}
\]

where \( \alpha \in (0, 1] \)

5. An application and a numerical example. First, we will illustrate how this type of system can be put into practice. In 1994, wardropian user equilibrium tatonnement model was constructed to consider idealized traveler information systems for day-to-day adjustments of flows and costs basis in the presence of information, as explained in detail by Friesz et al. [12], who showed that the network tatonnement model can be formulated as the following projective dynamical systems:

\[
\begin{align*}
N(t_{n+1}, v_{n+1}) &= \sum_{j=0}^{m} \left\{ \sum_{j=0}^{m} b_{j,n+1} \{ P_K [ g(x_h(t_j)) - \rho M(v_j)] \} \right\}, \\
N(t_{n+1}, v_{n+1}) &= \sum_{j=0}^{m} b_{j,n+1} \{ P_K [ g(x_h(t_j)) - \rho M(v_j)] \} - g(x_h(t_j)), \\
x_h(t_{n+1}) &= \phi(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{m} b_{j,n+1} \{ P_K [ g(x_h(t_j)) - \rho M(v_j)] \} - g(x_h(t_j)), \\
v_{n+1} &= \left\{ \begin{array}{ll}
\tau x_h(t_{n+1}) + (1 - \tau) x_h(t_{n-1}), & \text{if } l > 1, \\
\tau x_h(t_{n+1}) + (1 - \tau) x_h(t_{n+1}), & \text{if } l = 1,
\end{array} \right.
\]

where \( a_{j,n+1} \) and \( b_{j,n+1} \) are defined by (17) and (18), respectively.

In particular, we take \( \Lambda = I \), where \( I \) is an identity matrix. Then model (20) is nothing but a form the fractional-order projective dynamical systems with delay and perturbation.
Next, we will give a numerical example to illustrate validity of the main results presented in Section 3.

**Example 5.1.** Suppose that \( \alpha = 0.92, t_0 = 0, T = 0.4, \rho = 0.2, r = 0.05, \)
\[
g(x(t)) = (g_1(x(t)), g_2(x(t)))^\top = (x_1(t), x_2(t))^\top,
\]
\[
M(x_t) = M \cdot x_t = \begin{pmatrix}
0.4 & 0.1 \\
0.25 & -0.25
\end{pmatrix}
\begin{pmatrix}
x_1(t-0.05) \\
x_2(t-0.05)
\end{pmatrix}
= \begin{pmatrix}
0.4x_1(t-0.05) + 0.1x_2(t-0.05) \\
0.25x_1(t-0.05) - 0.25x_2(t-0.05)
\end{pmatrix},
\]
and
\[
N(t, x_t) = (0.06 \cos(x_1(t-0.05)), 0.08 \sin(x_2(t-0.05)))^\top.
\]
This means that we should consider the following fractional-order projective dynamical systems with delay \( r = 0.05 \) and perturbation \( N(t, x_t) \):
\[
\begin{aligned}
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t} D^\alpha \right) x_1(t) = x_1(t) - \rho \left( 0.4x_1(t-0.05) + 0.1x_2(t-0.05) \\
\quad - 0.25x_1(t-0.05) - 0.25x_2(t-0.05) \right) \\
\quad + 0.06 \cos(x_1(t-0.05)) \\
\quad + 0.08 \sin(x_2(t-0.05)), \\
\end{array} \right.
\end{aligned}
\tag{21}
\]
\[
x(t) = \phi(t) = (\phi_1(t), \phi_2(t))^\top = (-0.15, 0.24)^\top, \quad t \in [-0.05, 0],
\]
where \( x(t) = (x_1(t), x_2(t))^\top \) and \( K = \{x \mid 0 \leq x_1 \leq 0.2, -0.15 \leq x_2 \leq 0\} \).

By the assumptions, it is easy to check that
\[
\|g(x^1) - g(x^2)\|_1 = 1 \times \|x^1 - x^2\|_{C([0,0.4], \mathbb{R}^2)} = l_1 \|x^1 - x^2\|_{C([0,0.4], \mathbb{R}^2)},
\]
\[
\|M(x^1_t) - M(x^2_t)\| \leq 0.4752 \times \|x^1 - x^2\|_{C([0,0.4], \mathbb{R}^2)} = l_2 \|x^1 - x^2\|_{C([0,0.4], \mathbb{R}^2)},
\]
\[
\|N(t, x_t)\|_1 \leq 0.14 = f(t),
\]
\[
L^* = \frac{2l_1 + \rho l_2 T^\alpha}{\Gamma(\alpha + 1)} = 0.9308 < 1,
\]
where \( l_1 = 1 \) is the Lipschitz constant of \( g \), \( l_2 = 0.5398 \) is the largest singular value of matrix \( M \), \( \| \cdot \|_1 \) denotes 1-norm and the superscript \( ^\top \) denotes transpose operation.

Obviously, we know that (H1)-(H4) hold and so IVP (21) has a least one solution on \( [0, 0.4] \). Furthermore, one has
\[
\|N(t, x^1_t) - N(t, x^2_t)\|_1 \leq 0.08 \|x^1 - x^2\|_{C([0,0.4], \mathbb{R}^2)} = l_3 \|x^1 - x^2\|_{C([0,0.4], \mathbb{R}^2)}
\]
and
\[
L^{**} = \frac{2l_1 + \rho l_2 + l_3 T^\alpha}{\Gamma(\alpha + 1)} = 0.9664 < 1.
\]
Thus, all the conditions of Theorem 3.2 are satisfied and it follows that IVP (21) has a unique solution on \( [0, 0.4] \). Figure 1 shows the trajectories of the system (21) on \( [0, 0.4] \), and the trajectories reveal the adjustment processes of state variables \( x_1 \) and \( x_2 \) with the change of time \( t \), here \( h = 0.001, \tau = 0, L = 400, l = 50 \).
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