Optical singular and dark solitons to the nonlinear Schrödinger equation in magneto-optic waveguides with anti-cubic nonlinearity

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Abstract
The present paper aims to investigate the coupled nonlinear Schrödinger equation in magneto-optic waveguides having anti-cubic (AC) law nonlinearity. The solitons secured to magneto-optic waveguides with AC law nonlinearity are extremely useful to fiber-optic transmission technology. Three constructive techniques, namely, the $(G'/G)$-expansion method, the modified simple equation method, and the extended tanh method are utilized to find the exact soliton solutions of this model. Consequently, dark, singular, combined dark-singular and periodic soliton solutions are obtained. The behaviours of soliton solutions are presented by 3D and 2D plots.

Keywords Nonlinear Schrödinger equation · Magneto-optic · Optical soliton solutions · $(G'/G)$-expansion · extended tanh method · Modified simple equation method

1 Introduction

Many nonlinear physical phenomena arise in various fields of science and engineering such as quantum mechanics, fluid dynamics, molecular biology, nuclear dynamics plasma physics, solid state physics and optical fibres. To express these complex physical phenomena
in science and engineering, nonlinear partial differential equations (PDE) are widely used. Besides, it’s one of the trending topics for scientists and engineers to obtain solutions for nonlinear PDE. For solving them, there are many analytical and numerical methods developed, such as the differential transform method (Akinyemi 2020), the variational iteration method (Das 2009; He 1998), the first integral method (Mathanaranjan 2021), the q-homotopy analysis method (Akinyemi 2019; Akinyemi et al. 2020), the Bernoulli collocation method (Adel and Sabir 2020), the residual power series method (Senol et al. 2019; Senol 2020), the Euler matrix method (El-Gamel and Adel 2018), the iterative shehu transform method (Akinyemi and Iyiola 2020), the Decomposition Method (Mathanaranjan and Himalini 2019; Mathanaranjan and Vijayakumar 2021), the sub-equation method (Akinyemi et al. 2021), the improved Sardar sub-equation method (Akinyemi et al. 2021), the \((G'/G)\)-expansion method (Wang et al. 2008; Zayed and Gepreel 2009; Bilal et al. 2021b), the modified simple equation method (Zahran and Khater 2014; Mathanaranjan 2020), the tanh method (Wazwaz 2007; Abdou 2007), the solitary wave Ansatz Method (Mathanaranjan 2021; Sassaman and Biswas 2010), the direct algebraic method (Bilal et al. 2021a, c), the extended sinh–Gordon equation expansion (Bilal et al. 2021d, e), the Lie symmetry analysis (Inc et al. 2018; Baleanu et al. 2017, 2018), and so on Darvishi et al. (2018, 2021a, b), Javeed et al. (2020), Arshed and Raza (2020), Raza et al. (2020), Afzal et al. (2019), Ghanbari (2021a, b), Ghanbari et al. (2019, 2020), Hosseini et al. (2021a, b, c), Akbar et al. (2021).

The propagation of the optical solitons is usually governed by the nonlinear Schrödinger equation (NLSE) which is one of the most important models in modern nonlinear science. Optical solitons have promising potential to become information carriers in telecommunication due to their capability of propagating long distance without attenuation and changing their shapes. In recent years, considerable attentions have been paid to analyze the NLSE and to calculate their optical soliton solutions (Younas et al. 2021; Inc et al. 2016; Tchier et al. 2016a; Tchier and Sommezoglu 2016b; Kilic and Inc 2017; Aslan and Inc 2017; Aslan et al. 2017; Tchier et al. 2017).

Recently, the coupled system of NLSE are investigated to obtain the optical soliton by many scientists. The optical soliton solutions of the coupled NLSE in magneto-optic waveguides with several perturbation terms are obtained in Vega-Guzman et al. (2014). Conservation laws for soliton propagating in birefringent optical fibers and magneto-optic waveguides are studied in Kara et al. (2016). A \((2 + 1)\)-dimensional coupled NLSE with spatially modulated nonlinearity and transverse modulation is investigated in Dai et al. (2017). Bright, singular and combo optical solitons in magneto-optic waveguides with quadratic-cubic nonlinearity of coupled NLSE are obtained in Asma et al. (2020). Solitons in magneto-optic waveguides with parabolic law nonlinearity by Jacobi’s elliptic function expansion method are developed in Zayed et al. (2020). Solitons in magneto-optic waveguides with quadratic-cubic nonlinearity by Riccati equation expansion method are derived in Zayed et al. (2020). Optical solitons in magneto-optic waveguides with two types of nonlinearities namely quadratic-cubic and dual-power laws are studied by implementing the mathematical technique called extended Fan-sub equation method (Younas and Ren 2021). Finally, solitons in magneto-optic waveguides with anti-cubic (AC) nonlinearity by Jacobi’s elliptic function are extracted in Zayed et al. (2020). In this article, our goal is to obtain the optical soliton to the NLSE in magneto-optic waveguides with AC nonlinearity, which was studied in Zayed et al. (2020), implementing the \((G'/G)\)-expansion method, the extended tanh method, and the modified simple equation method.

The paper’s structure is proposed as follows. The mathematical analysis of the governing model is presented in Sect. 2. The implementations of the above proposed three
methods are devoted to Sect. 3. The Physical explanation of the obtained solutions are provided in Sect. 4. Finally, the last Section is reserved for the concluding remarks.

2 Mathematical analysis of the model

Consider the coupled NLSE in magneto-optic waveguides having AC law nonlinearity of the form (Zayed et al. 2020; Biswas et al. 2021)

\[ iu_t + a_1 u_{xx} + \left( \frac{b_{11}}{|u|^4} + b_{12} |u|^2 + b_{13} |u|^4 + \frac{c_{11}}{|v|^4} + c_{12} |v|^2 + c_{13} |v|^4 \right) u = d_1 v + i \left[ \alpha_1 u_x + \lambda_1 (|u|^2)_x + v_1 (|u|^2)_x u + \tau_1 |u|^2 u_x \right], \]  

(1)

\[ iv_t + a_2 v_{xx} + \left( \frac{b_{21}}{|v|^4} + b_{22} |v|^2 + b_{23} |v|^4 + \frac{c_{21}}{|u|^4} + c_{22} |u|^2 + c_{23} |u|^4 \right) v = d_2 u + i \left[ \alpha_2 v_x + \lambda_2 (|v|^2)_x + v_2 (|v|^2)_x v + \tau_2 |v|^2 v_x \right], \]  

(2)

where \( a_j, b_{jk}, c_{jk}, d_j, \alpha_j, \lambda_j, v_j, \) and \( \tau_j \) are constants. Since \( u(x, t) \) and \( v(x, t) \) are complex functions, so we assume that

\[ u(x, t) = \phi_1(\xi) e^{i\chi(x,t)}, \]  

(3)

and

\[ v(x, t) = \phi_2(\xi) e^{i\chi(x,t)}, \]  

(4)

where

\[ \xi = x - vt, \quad \chi(x, t) = -kx + \omega t, \]  

(5)

and \( v, k, \) and \( \omega \) are respectively the speed of wave, wave number and frequency of the soliton. By inserting Eqs. (4) into (1) and (2), we get the real parts

\[ c_{11} \phi_1(\xi)^4 + b_{11} \phi_2(\xi)^4 + (b_{12} - k(\tau_1 + \lambda_1)) \phi_1(\xi)^6 \phi_2(\xi)^4 + b_{13} \phi_1(\xi)^3 \phi_2(\xi)^5 - (\omega + k(\alpha_1 + \lambda_1)) \phi_1(\xi)^4 \phi_2(\xi)^3 = 0, \]  

(6)

\[ b_{21} \phi_1(\xi)^4 + c_{21} \phi_2(\xi)^4 + (b_{22} - k(\tau_2 + \lambda_2)) \phi_1(\xi)^6 \phi_2(\xi)^6 + b_{23} \phi_1(\xi)^3 \phi_2(\xi)^5 - (\omega + k(\alpha_2 + \lambda_2)) \phi_1(\xi)^4 \phi_2(\xi)^3 = 0, \]  

(7)

while the imaginary parts are as follows:

\[ \nu_1 + 2k\alpha_1 + \alpha_1 + (\tau_1 + 3\lambda_1 + 2v_1) \phi_1(\xi)^2 = 0, \]  

(8)

\[ \nu_2 + 2k\alpha_2 + \alpha_2 + (\tau_2 + 3\lambda_2 + 2v_2) \phi_2(\xi)^2 = 0. \]  

(9)

From Eqs. (8) and (9) we get:
\[
\nu_1 = \frac{1}{2}(-\tau_1 - 3\lambda_1), \quad \nu_2 = \frac{1}{2}(-\tau_2 - 3\lambda_2), \quad k = \frac{-\alpha_1 + \alpha_2}{2(a_1 - a_2)}, \quad \text{with } a_1 \neq a_2. \quad (10)
\]

Now, to solve the above coupled pair of Eqs. (6) and (7), we introduce the ansatz:

\[\phi_2(\xi) = K\phi_1(\xi),\]

where \(K \neq 1\), is a nonzero constant. Then, Eqs. (6) and (7) transform to

\[
(c_{11} + K^4b_{11}) - K^4(\omega + Kd_1 + k(ka_1 + \alpha_1))\phi_1(\xi)^4 + K^4(b_{12} + K^2c_{12})
- k(\tau_1 + \lambda_1)\phi_1(\xi)^6 + K^4(b_{13} + K^4c_{13})\phi_1(\xi)^8 + K^4a_1\phi_1(\xi)^3\phi''(\xi) = 0,
\]

\[
(b_{21} + K^4c_{21}) - K^3(K\omega + d_2 + kK(ka_2 + \alpha_2))\phi_1(\xi)^4 + K^4(c_{22} + K^2b_{22})
- k(\tau_2 + \lambda_2)\phi_1(\xi)^6 + K^4(K^4b_{23} + c_{23})\phi_1(\xi)^8 + K^4a_2\phi_1(\xi)^3\phi''(\xi) = 0. \quad (13)
\]

Now the necessary and sufficient condition for a nontrivial solution of the function \(\phi_1(\xi)\) satisfying both Eqs. (12) and (13) is that; the coefficients of Eqs. (12) and (13) satisfying the proportional ratios are as follows:

\[
\frac{b_{21} + K^4c_{21}}{c_{11} + K^4b_{11}} = \frac{K\omega + d_2 + kK(ka_2 + \alpha_2)}{K(\omega + Kd_1 + k(ka_1 + \alpha_1))} = \frac{c_{22} + K^2(b_{22} - k(\tau_2 + \lambda_2))}{b_{12} + K^2c_{12} - k(\tau_1 + \lambda_1)} = \frac{K^4b_{23} + c_{23}}{b_{13} + K^4c_{13}} = \frac{a_2}{K\alpha_1}.
\]

For the sake of simplicity, we set

\[
b_0 = K^3a_2, \\
b_1 = -K^3(K\omega + d_2 + kK(ka_2 + \alpha_2)), \\
b_2 = K^4(c_{22} + K^2(b_{22} - k(\tau_2 + \lambda_2))), \\
b_3 = K^4(K^4b_{23} + c_{23}), \\
b_4 = b_{21} + K^4c_{21},
\]

then the Eq. (13) can be written as

\[
b_0\phi_1(\xi)^3\phi''_1(\xi) + b_1\phi_1(\xi)^4 + b_2\phi_1(\xi)^6 + b_3\phi_1(\xi)^8 + b_4 = 0. \quad (16)
\]

Balancing \(\phi_1^3\phi''_1\) and \(\phi_1^6\) in Eq. (16), yields \(N = \frac{1}{2}\). Since \(N\) is not an integer, we introduce the transformation

\[
\phi_1(\xi) = \left[\psi(\xi)\right]^{\frac{1}{2}},
\]

where \(\psi\) is a known function of \(\xi\). Inserting Eq. (17) into Eq. (16), we obtain the new equation

\[
2b_0\psi(\xi)\psi''(\xi) - b_0\psi'(\xi)^2 + 4b_1\psi(\xi)^2 + 4b_2\psi(\xi)^2 + 4b_3\psi(\xi)^4 + 4b_4 = 0. \quad (18)
\]

Now, our task is to solve Eq. (18) by above mentioned three different analytical methods.
3 Solitons to the NLSE in magneto-optic waveguides with anti-cubic nonlinearity

In this section, we apply the proposed three methods to find the soliton solutions of NLSE in magneto-optic waveguides having AC nonlinearity.

3.1 Applying the \((G'/G)\)-expansion method

From the homogeneous balance of \(\psi(\xi)\psi''(\xi)\) and \(\psi(\xi)^4\) in Eq. (18), we find \(N = 1\). According to the proposed method (Wang et al. 2008; Zayed and Gepreel 2009), the solution of Eq. (18) is given as:

\[
\psi(\xi) = \alpha_0 + \alpha_1 (G'/G),
\]

where the constants \(\alpha_0\), and \(\alpha_1\) are to be determined later and \(G(\xi)\) satisfies the following ordinary differential equation:

\[
G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0,
\]

(20)

Here, \(\lambda\) and \(\mu\) are constants. Substituting Eq. (19) together with Eq. (20) into Eq. (18), and making the coefficients of \((G'/G)^N\), \((N = 0, 1, 2, 3, 4)\) equal to zero, yield the system of algebraic equations as follows:

\[
\begin{align*}
[G'/G]^0 & : 4b_4 + 4a_0^2(b_1 + a_0(b_2 + b_3a_0)) + 2\lambda\mu b_0a_0a_1 - \mu^2b_0a_1^2 = 0, \\
[G'/G]^1 & : 2a_0((\lambda^2 + 2\mu)b_0 + 4b_1 + 6b_2a_0 + 8b_3a_0^2)a_1 = 0, \\
[G'/G]^2 & : a_1(4(b_1 + 3a_0(b_2 + 2b_3a_0))a_1 + b_0(6\lambda a_0 + (\lambda^2 + 2\mu)a_1)) = 0, \\
[G'/G]^3 & : 4a_1((b_2 + 4b_3a_0)a_1^2 + b_0(a_0 + \lambda a_1)) = 0, \\
[G'/G]^4 & : 3b_0a_1^2 + 4b_3a_1^4 = 0.
\end{align*}
\]

(21)

The following outcome is found by solving the above equations:

\[
\begin{align*}
\alpha_0 &= 0, \quad \alpha_1 = \pm \frac{1}{2} \sqrt{-\frac{3b_0}{b_3}}, \quad \lambda = \mp \frac{b_2}{2} \sqrt{-\frac{3}{b_0b_3}}, \quad \mu = \frac{3b_2^2 - 16b_1b_3}{8b_0b_3}, \\
b_4 &= -\frac{3(3b_2^2 - 16b_1b_3)^2}{1024b_3^2}.
\end{align*}
\]

(22)

Provided that \(b_0b_3 < 0\). By inserting Eq. (22) into Eq. (19), we obtain

\[
\psi(\xi) = \pm \frac{1}{2} \sqrt{-\frac{3b_0}{b_3}} \left( \frac{G'}{G} \right).
\]

(23)

Inserting general solutions of Eq. (20) into Eq. (23), we obtain three types of travelling wave solutions to the coupled pair of Eqs. (1) and (2) as follows:

**Type 1.** When \(\lambda^2 - 4\mu = \frac{-9b_2^2 + 32b_1b_3}{4b_0b_3} > 0\), we obtain the hyperbolic function solutions of Eqs. (1) and (2) as follows:
$$u(x,t) = \left\{ \begin{array}{ll}
\frac{-3b_2}{8b_3} \pm \frac{\sqrt{3(9b_2^2 - 32b_1b_3)}}{8b_3} \\
c_1 \sin \left[ \frac{1}{4} \xi \sqrt{\frac{-9b_2^2 + 32b_1b_3}{b_0b_3}} \right] + c_2 \cos \left[ \frac{1}{4} \xi \sqrt{\frac{-9b_2^2 + 32b_1b_3}{b_0b_3}} \right] \\
\frac{1}{2}
\end{array} \right. e^{i(-kx+\omega t)}$$

and $$v(x,t) = Ku(x,t)$$. In particular, if we choose $$c_1 \neq 0$$ and $$c_2 = 0$$ in Eq. (24), then the dark soliton solution is revealed as follows:

$$u(x,t) = \left\{ \begin{array}{ll}
\frac{-3b_2}{8b_3} \pm \frac{\sqrt{3(9b_2^2 - 32b_1b_3)}}{8b_3} \tanh \left[ \frac{1}{4} \xi \sqrt{\frac{-9b_2^2 + 32b_1b_3}{b_0b_3}} \right] \\
\frac{1}{2}
\end{array} \right. e^{i(-kx+\omega t)}$$

and $$v(x,t) = Ku(x,t)$$.

If we choose $$c_1 = 0$$ and $$c_2 \neq 0$$ in Eq. (24), then the singular soliton solution falls out:

$$u(x,t) = \left\{ \begin{array}{ll}
\frac{-3b_2}{8b_3} \pm \frac{\sqrt{3(9b_2^2 - 32b_1b_3)}}{8b_3} \coth \left[ \frac{1}{4} \xi \sqrt{\frac{-9b_2^2 + 32b_1b_3}{b_0b_3}} \right] \\
\frac{1}{2}
\end{array} \right. e^{i(-kx+\omega t)}$$

and $$v(x,t) = Ku(x,t)$$.

**Type 2.** When $$\lambda^2 - 4\mu = \frac{-9b_2^2 + 32b_1b_3}{4b_0b_3} < 0$$, we obtain the trigonometric function solutions of Eqs. (1) and (2) as follows:

$$u(x,t) = \left\{ \begin{array}{ll}
\frac{-3b_2}{8b_3} \pm \frac{\sqrt{3(-9b_2^2 + 32b_1b_3)}}{8b_3} \\
c_1 \sin \left[ \frac{1}{4} \xi \sqrt{\frac{9b_2^2 - 32b_1b_3}{b_0b_3}} \right] + c_2 \cos \left[ \frac{1}{4} \xi \sqrt{\frac{9b_2^2 - 32b_1b_3}{b_0b_3}} \right] \\
\frac{1}{2}
\end{array} \right. e^{i(-kx+\omega t)}$$

and $$v(x,t) = Ku(x,t)$$.

In particular, if we choose $$c_1 \neq 0$$ and $$c_2 = 0$$ in Eq. (27), then the periodic solitary wave solutions is reveled as:
and \( v(x, t) = ku(x, t) \).

If we choose \( c_1 = 0 \) and \( c_2 \neq 0 \) in Eq. (27), then the periodic solitary wave solutions falls out:

\[
\begin{aligned}
  u(x, t) &= -\frac{3b_2}{8b_3} \pm \frac{\sqrt{3}}{2} \frac{\sqrt{9b_2^2 + 32b_1b_3}}{8b_3} \cot \left[ \frac{1}{4} \xi \sqrt{\frac{9b_2^2 - 32b_1b_3}{b_0b_3}} \right] e^{i(\xi x + \omega t)}, \\
  v(x, t) &= ku(x, t)
\end{aligned}
\]  

and \( v(x, t) = ku(x, t) \).

**Type 3.** When \( \lambda^2 - 4\mu = -\frac{9b_2^2 + 32b_1b_3}{4b_0b_3} = 0 \), we obtain the rational function solutions as follows:

\[
\begin{aligned}
  u(x, t) &= -\frac{3b_2}{8b_3} \pm 2 \frac{3b_0}{b_3} \frac{c_2}{c_1 + c_2^2} \frac{1}{2} e^{i(\xi x + \omega t)}, \\
  v(x, t) &= ku(x, t),
\end{aligned}
\]

and \( v(x, t) = ku(x, t) \), where \( c_1 \) and \( c_2 \) are constants.

### 3.2 Applying the modified simple equation method

Considering the homogeneous balance among \( \psi(\xi)\psi''(\xi) \) and \( \psi(\xi)^4 \) in Eq. (18), one get \( N = 1 \). Based on the MSEM (Zahran and Khater 2014; Mathanaranjan 2020), we suppose that:

\[
\psi(\xi) = A_0 + A_1 \frac{\psi''(\xi)}{\psi(\xi)},
\]

where \( A_0 \) and \( A_1 \neq 0 \) are constants to be determined later. Inserting Eq. (31) into Eq. (18), and then setting the coefficients \( \Psi(\xi)^{-j}, (j = 0, 1, 2, 3, 4) \) equal to zero, we find a set of algebraic equations as follows:

\[
\begin{aligned}
  \Psi(\xi)^0 : & \ 4(b_4 + a_0^2(b_1 + a_0(b_2 + b_3a_0))) = 0, \\
  \Psi(\xi)^{-1} : & \ 22a_0a_1(2(2b_1 + a_0(3b_2 + 4b_3a_0))\psi''(\xi) + b_0\psi'(\xi)(\xi)) = 0, \\
  \Psi(\xi)^{-2} : & \ a_1(4(b_1 + 3a_0(b_2 + 2b_3a_0))a_1\psi'(\xi)^2 - b_0a_1\psi''(\xi) \\
  & \ - 6b_0a_0\psi'(\xi)\psi''(\xi) + 2b_0a_1\psi'(\xi)(\xi)\psi''(\xi)(\xi)) = 0, \\
  \Psi(\xi)^{-3} : & \ 4a_1\psi'(\xi)^2((b_0a_0 + (b_2 + 4b_3a_0)a_1^2)\psi'(\xi) - b_0a_1\psi''(\xi)) = 0, \\
  \Psi(\xi)^{-4} : & \ a_1^2(3b_0 + 4b_3a_1^2)\psi'(\xi)^4 = 0.
\end{aligned}
\]  

Solving above algebraic equations in Eq. (32), we obtain:
\[ \alpha_1 = \pm \frac{1}{2} \sqrt{\frac{3b_0}{b_3}}, \quad \alpha_0 = -\frac{-3b_2 \mp \sqrt{3}}{8b_3} \sqrt{9b_2^2 - 32b_1b_3}, \quad b_4 = -\frac{3(3b_2^2 - 16b_1b_3)^2}{1024b_3^3}, \]  

and

\[ 9b_2^2\Psi'(\xi) + 4b_3(-8b_1\Psi'(\xi) + b_0\Psi^{(3)}(\xi)) = 0, \]  

provided that \( b_0b_3 < 0 \) and \( (9b_2^2 - 32b_1b_3) > 0 \). Consequently, the Eq. (34) reduces to

\[ 2e^{-\frac{3}{2} \xi} \sqrt{-\frac{9b_2^2 + 32b_1b_3}{b_3}} \left[ -c_1 + c_2 \xi \sqrt{-\frac{-9b_2^2 + 32b_1b_3}{b_3}} \right] \sqrt{b_0b_3}, \]  

where \( c_1, c_2 \) and \( c_3 \) are constants. Substituting Eqs. (35) into (4) with (17), we find the exact solution to coupled pair of Eqs. (1) and (2) as follows:

\[ u(x, t) = \left\{ \frac{-3b_2}{8b_3} \pm \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{8b_3} \right\} \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{4b_3} \left( c_1 + c_2 \frac{\xi}{\sqrt{b_0b_3}} \right)^{\frac{1}{2}} e^{i(-kx+\omega t)}, \]  

and \( v(x, t) = Ku(x, t) \). As a particular selection, if we take \( c_1 = 0 \) and \( \frac{c_1}{c_2} = \sqrt{\frac{4b_0b_3}{-9b_2^2 + 32b_1b_3}} \) in Eq. (36), the dark soliton solution can be obtained as:

\[ u(x, t) = \left\{ \frac{-3b_2}{8b_3} \pm \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{8b_3} \right\} \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{8b_3} \tanh \left[ \frac{1}{4} \xi \sqrt{-\frac{-9b_2^2 + 32b_1b_3}{b_0b_3}} \right] \frac{1}{2} e^{i(-kx+\omega t)}, \]  

while, if we set \( c_1 = 0 \) and \( \frac{c_1}{c_2} = -\sqrt{\frac{4b_0b_3}{-9b_2^2 + 32b_1b_3}} \) in Eq. (36), singular soliton solution can be obtained as:

\[ u(x, t) = \left\{ \frac{-3b_2}{8b_3} \pm \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{8b_3} \right\} \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{8b_3} \coth \left[ \frac{1}{4} \xi \sqrt{-\frac{-9b_2^2 + 32b_1b_3}{b_0b_3}} \right] \frac{1}{2} e^{i(-kx+\omega t)}, \]  

and \( v(x, t) = Ku(x, t) \).

It should be noted that the solitary wave solutions Eqs. (37) and (38) are similar to the previous solutions Eqs. (25) and (26) respectively.
### 3.3 Applying the extended tanh method

According to the proposed method (Wazwaz 2007; Abdou 2007), we assume that

\[ Y = \tanh(\mu \xi), \]  

which gives to the change of variables

\[ \frac{d\psi}{d\xi} = \mu (1 - Y^2) \frac{d\psi}{dY}, \]  

\[ \frac{d^2\psi}{d\xi^2} = -2 \mu^2 Y(1 - Y^2) \frac{d\psi}{dY} + \mu^2 (1 - Y^2) \frac{d^2\psi}{dY^2}. \]

The solution of above equation is considered as

\[ \psi(\xi) = \sum_{k=-N}^{N} a_k Y^k, \]

where \( N \) is balancing number. Considering the homogeneous balance among \( \psi(\xi)\psi''(\xi) \) and \( \psi(\xi)^4 \) in Eq. (18), provides \( N = 1 \). Then, from Eq. (42) we get:

\[ \psi(\xi) = a_0 + a_1 Y + \frac{a_{-1}}{Y}. \]

Substituting Eq. (43) into Eq. (18), and making all the coefficients of powers \( Y \) equal to zero, we get a set of equations as follows:

\[ \begin{align*}
Y^{-4} & : 3 \mu^2 b_0 a_{-1}^2 + 4 b_3 a_{-1}^4 = 0, \\
Y^{-3} & : 4(b_2 a_{-1}^2 + \alpha_{-1}(\mu^2 b_0 + 4 b_3 a_{-1})) = 0, \\
Y^{-2} & : 2 \alpha_{-1}(-\mu^2 b_0(\alpha_{-1} - 3 a_1) + 2 \alpha_{-1}(b_1 + 3 a_0(b_2 + 2 b_3 a_0) + 4 b_3 a_{-1} a_1)) = 0, \\
Y^{-1} & : 4 \alpha_{-1}(a_0(\mu^2 b_0 + 2 b_1 + a_0(3 b_2 + 4 b_3 a_0)) + 3 \alpha_{-1}(b_2 + 4 b_3 a_0 a_1)) = 0, \\
Y^0 & : 4 b_4 + 4 a_0^2(b_1 + a_0(b_2 + b_3 a_0)) + 8 \alpha_{-1} a_1(b_1 + 3 a_0(b_2 + 2 b_3 a_0)) + 3 b_3 a_{-1} a_1 = 0, \\
Y^1 & : 4 \alpha_1(a_0(\mu^2 b_0 + 2 b_1 + a_0(3 b_2 + 4 b_3 a_0)) + 3 \alpha_{-1}(b_2 + 4 b_3 a_0 a_1)) = 0, \\
Y^2 & : 2 \alpha_1(\mu^2 b_0(3 \alpha_{-1} - a_1) + 2 \alpha_1(b_1 + 3 a_0(b_2 + 2 b_3 a_0) + 4 b_3 a_{-1} a_1)) = 0, \\
Y^3 & : 4(\mu^2 b_0 a_0 a_1 + (b_2 + 4 b_3 a_0 a_1)) = 0, \\
Y^4 & : 3 \mu^2 b_0 a_1^2 + 4 b_3 a_1^4 = 0.
\end{align*} \]

Solving the above system, we find the following three sets of solutions:

1. The first set:
2. The second set:

\[ a_0 = -\frac{3b_2}{8b_3}, \quad \alpha_{-1} = 0, \quad \alpha_1 = \pm \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{8b_3}, \]
\[ \mu = \pm \frac{1}{4} \sqrt{-\frac{9b_2^2 + 32b_1b_3}{b_0b_3}}, \quad b_4 = -\frac{3(3b_2^2 - 16b_1b_3)^2}{1024b_3^2}. \] (45)

3. The third set:

\[ a_0 = -\frac{3b_2}{8b_3}, \quad \alpha_{-1} = \pm \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{16b_3}, \]
\[ \alpha_1 = 0, \quad \mu = \pm \frac{1}{8} \sqrt{-\frac{9b_2^2 + 32b_1b_3}{b_0b_3}}, \quad b_4 = -\frac{3(3b_2^2 - 16b_1b_3)^2}{1024b_3^2}. \] (46)

Provided that \( b_0b_3 < 0 \) and \( (9b_2^2 - 32b_1b_3) > 0 \). From Eqs. (45)–(47), the dark, singular, and combined dark-singular soliton solutions to the coupled pair of Eqs. (1) and (2) read as follows:

\[ u(x, t) = \begin{cases} 
-\frac{3b_2}{8b_3} \pm \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{8b_3} \tanh \left[ \frac{1}{4} \sqrt{-\frac{9b_2^2 + 32b_1b_3}{b_0b_3}} \right], \\
\exp(i(-kx + \omega t))
\end{cases} \] (48)

\[ u(x, t) = \begin{cases} 
-\frac{3b_2}{8b_3} \pm \frac{\sqrt{3} \sqrt{9b_2^2 - 32b_1b_3}}{8b_3} \coth \left[ \frac{1}{4} \sqrt{-\frac{9b_2^2 + 32b_1b_3}{b_0b_3}} \right], \\
\exp(i(-kx + \omega t))
\end{cases} \] (49)
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\[ u(x,t) = \begin{cases} 
- \frac{3b_2}{8b_3} \pm \frac{\sqrt{3}}{4} \frac{\sqrt{9b_2^2 - 32b_1b_3}}{16b_3} \\
\tanh \left[ \frac{1}{8} \sqrt{\frac{-9b_2^2 + 32b_1b_3}{b_0b_3}} \right] + \coth \left[ \frac{1}{8} \sqrt{\frac{-9b_2^2 + 32b_1b_3}{b_0b_3}} \right] \end{cases} \]

(50)

and \( v(x,t) = Ku(x,t) \).

**Remark 1** Note that the first two solutions are equivalent to the previous solutions Eqs. (25) and (26) respectively.

### 4 Physical explanation of the solutions

In this section, we present the physical interpretation of the obtained exact soliton solutions of the coupled NLSE in magneto-optic waveguides having AC law nonlinearity. The graphical illustrations of 3D and 2D plots of some solutions are given in Figs. 1, 2, 3, 4 and 5 for special values of the free parameters. From the above figures, one can see that the obtained solutions possess the dark, singular, combined dark-singular and periodic soliton solutions. Figure 1 of the solution Eq. (25) shows the shape of the dark soliton with \( K = \omega = k = a_2 = a_2 = \theta_2 = \lambda_2 = b_2 = b_22 = b_23 = 1, b_21 = c_22 = 2, c_21 = -1, c_23 = -2, \) and \( d_2 = -4 \). On the other hand, Fig. 2 of the solution Eq. (26) represent singular soliton with \( K = \omega = k = a_2 = a_2 = \theta_2 = \lambda_2 = b_2 = b_22 = b_23 = 1, b_21 = c_22 = 2, c_21 = -1, c_23 = -2, \) and \( d_2 = -4 \). The solutions in Figs. 3 and 4 shows the shape of the periodic soliton solution with \( K = \omega = k = a_2 = a_2 = \theta_2 = \lambda_2 = b_2 = b_22 = b_23 = 1, b_21 = c_22 = c_23 = 2, \) and \( d_2 = -4 \). Indeed, Fig. 5 of the solution Eq. (50) shows the shape of the combined dark-singular soliton with \( K = \omega = k = a_2 = a_2 = \theta_2 = \lambda_2 = b_2 = b_22 = b_23 = 1, b_21 = c_22 = 2, c_21 = -1, c_23 = -2, \) and \( d_2 = -4 \). For convenience, other figures are omitted as they exhibit the same behaviour of the above solutions. The exact solutions and figures obtained

![Fig. 1 a 3D graph of Eq. (25) with \( K = \omega = k = a_2 = a_2 = \theta_2 = \lambda_2 = b_2 = b_22 = b_23 = 1, b_21 = c_22 = 2, c_21 = -1, c_23 = -2, d_2 = -4 \), b corresponding 2D graph for \( t = 1 \)](image-url)
in this paper gives us a different physical interpretation for the coupled NLSE in magneto-optic waveguides having AC law nonlinearity.
5 Conclusions

This study investigates the optical solitons solution of the nonlinear Schrödinger equation in magneto-optic waveguides with AC nonlinearity. The solitons secured to magneto-optic waveguides with AC law nonlinearity will be extremely advantageous in fiber-optic transmission technology. Therefore, based on three effective methods, namely the \((G'/G)\)-expansion method, the modified simple equation method and the extended tanh method, we have successfully obtained the dark, singular and combined dark-singular soliton solutions of the above model. To our best knowledge, the application of proposed methods to the model, and the received combined soliton solutions are new, which have not been reported earlier. The obtained solutions are illustrated by 3D and 2D graphs to express the dynamical behaviour of solutions.

References

Abdou, M.A.: The extended tanh method and its applications for solving nonlinear physical models. Appl. Math. and Comput. 190, 988–996 (2007)

Adel, W., Sabir, Z.: Solving a new design of nonlinear second-order Lane–Emden pantograph delay differential model via Bernoulli collocation method. Eur. Phys. J. Plus 135(6), 427 (2020)

Afzal, U., Raza, N., Murtaza, I.G.: On soliton solutions of time fractional form of Sawada–Kotera equation. Nonlinear Dyn. 95(1), 391–405 (2019)

Akbar, M.A., Akinyemi, L., Yao, S.W., Jhangeneh, A., Rezazadeh, H., Khater, M.M., Ahmad, H., Inc, M.: Soliton solutions to the Boussinesq equation through sine–Gordon method and Kudryashov method. Res. Phys. 25, 104228 (2021)

Akinyemi, L.: q-Homotopy analysis method for solving the seventh-order time-fractional Lax’s Korteweg–de Vries and Sawada-Kotera equations. Comput. Appl. Math. 38(4), 1–22 (2019)

Akinyemi, L.: A fractional analysis of Noyes–Field model for the nonlinear Belousov–Zhabotinsky reaction. Comput. Appl. Math. 39, 1–34 (2020)

Akinyemi, L., Iyiola, O.S.: Exact and approximate solutions of time-fractional models arising from physics via Shehu transform. Math. Meth. Appl. Sci. 43(12), 7442–7464 (2020)

Akinyemi, L., Iyiola, O.S., Akpan, U.: Iterative methods for solving fourth- and sixth order time-fractional Cahn–Hilliard equation. Math. Meth. Appl. Sci. 43(7), 4050–4074 (2020)

Akinyemi, L., Senol, M., Iyiola, O.S.: Exact solutions of the generalized multidimensional mathematical physics models via sub-equation method. Math. Comput. Simul. 182, 211–233 (2021)
Hosseini, K., Salahshour, S., Mirzazadeh, M., Ahmadian, A., Baleanu, D., Khoshrang, A.: The (2 + 1) -dimensional Heisenberg ferromagnetic spin chain equation: its solitons and Jacobi elliptic function solutions. Eur. Phys. J. Plus 136(2), 1–9 (2021c)
Inc, M., Ates, E., Tchier, F.: Optical solitons of the coupled nonlinear Schrödinger equations with spatiotemporal dispersion. Nonlinear Dyn. 85, 1319–1329 (2016)
Inc, M., Yusuf, A., Aliyu, A.I., Baleanu, D.: Time-fractional Cahn–Allen and time-fractional Klein–Gordon equations: lie symmetry analysis, explicit solutions and convergence analysis. Physica A 493, 94–106 (2018)
Javeed, S., Saleem Alimeer, K., Nawaz, S., Waheed, A., Suleman, M., Baleanu, D., Atif, M.: Soliton solutions of mathematical physics models using the exponential function technique. Symmetry 12(1), 176 (2020)
Kara, A.H., Biswas, A., Belic, M.: Conservation laws for optical solitons in birefringent fibers and magneto-optic waveguides. Optik 127(24), 11662–11673 (2016)
Kilic, B., Inc, M.: Optical solitons for the Schrödinger–Hirota equation with power law nonlinearity by the Backlund transformation. Optik 138, 6467 (2017)
Mathanaranjan, T.: Solitary wave solutions of the Camassa–Holm–Nonlinear Schrödinger Equation. Res. Phys. 19, 103549 (2020)
Mathanaranjan, T.: Soliton solutions of deformed nonlinear Schrödinger equations using ansatz method. Int. J. Appl. Comput. Math. 7, 159 (2021)
Mathanaranjan, T.: Exact and explicit traveling wave solutions to the generalized Gardner and BBMB equations with dual high-order nonlinear terms. Partial Differ. Equ. Appl. Math. 4, 100120 (2021)
Mathanaranjan, T., Himalini, K.: Analytical solutions of the time-fractional non-linear Schrödinger equation with zero and non zero trapping potential through the Sumudu decomposition method. J. Sci. Univ. Kelaniya 21, 12–33 (2019)
Mathanaranjan, T., Vijayakumar, D.: Laplace decomposition method for time-fractional Fornberg–Whitham type equations. J. Appl. Math. Phys. 9, 260–271 (2021)
Raza, N., Osman, M.S., Abdel-Aty, A.H., Abdel-Khalek, S., Besbes, H.R.: Optical solitons of space-time fractional Fokas–Lenells equation with two versatile integration architectures. Adv. Differ. Equ. 2020(1), 1–15 (2020)
Sassaman, R., Biswas, A.: Topological and non-topological solitons of the Klein–Gordon equations in (1 + 2) dimensions. Nonlinear Dyn. 61(1–2), 23–28 (2010)
Senol, M.: Analytical and approximate solutions of (2 + 1)-dimensional time-fractional Burgers–Kadomtsev–Petviashvili equation. Commun. Theor. Phys. 72(5), 1–11 (2020)
Senol, M., Iyiola, O.S., Daei Kasmaei, H., Akinfenwa, L.: Efficient analytical techniques for solving time-fractional nonlinear coupled Jaulet–Miodek system with energy-dependent Schrödinger potential. Adv. Differ. Equ. 2019, 1–21 (2019)
Tchier, F., Sonmezoglu, A.: Optical solitons with resonant NLSE using three integration scheme. J. Optoelectron. Adv. Metar. 18, 950–973 (2016b)
Tchier, F., Aslan, E.C., Inc, M.: Optical solitons in parabolic law medium: Jacobi elliptic function solution. Nonlinear Dyn. 85, 2577–2582 (2016a)
Tchier, F., Aslan, E.C., Inc, M.: Nanoscale waveguides in optical metamaterials: Jacobi elliptic function solutions. J. Nanelectron. Optoelectron. 12, 526–531 (2017)
Vega-Guzman, J., Alshaery, A.A., Hilal, E.M., Bhrawy, A.H., Mahmood, M.F., Moraru, L., Biswas, A.: Optical soliton perturbation in magneto-optic waveguides with spatio-temporal dispersion. J. Optoelectron. Adv. Mater. 16, 1063–1070 (2014)
Wang, M., Li, X., Zhang, J.: The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys. Lett. A 372(4), 417–423 (2008)
Wazwaz, A.M.: The extended tanh method for abundant solitary wave solutions of nonlinear wave equations. Appl. Math. Comput. 187, 1131–1142 (2007)
Younas, U., Ren, J.: Investigation of exact soliton solutions in magneto-optic waveguides and its stability analysis. Res. Phys. 21, 103816 (2021)
Younas, U., Bilal, M., Ren, J.: Propagation of the pure-cubic optical solitons and stability analysis in the absence of chromatic dispersion. Opt. Quant. Electron. 53, 490 (2021)
Zahran, E.H., Khater, M.M.: The modified simple equation method and its applications for solving some nonlinear evolutions equations in mathematical physics. Jokull J. 64(5), 297–312 (2014)
Zayed, E.M.E., Gepreel, K.A.: The (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics. J. Math. Phys. 50(1), 013502 (2009)
Zayed, E.M., Alngar, M.E., El-Horbaty, M.M., Biswas, A., Guggilla, P., Ekici, M., Belic, M.R.: Solitons in magneto-optic waveguides with parabolic law nonlinearity. Optik 222, 165314 (2020)
Zayed, E.M., Shohib, R.M., El-Horbaty, M.M., Biswas, A., Asma, M., Ekici, M., Belic, M.R.: Solitons in magneto-optic waveguides with quadratic-cubic nonlinearity. Phys. Lett. A 384(25), 126456 (2020)

Zayed, E.M., Alngar, M.E., Shohib, R.M., Biswas, A., Ekici, M., Alzahrani, A.K., Belic, M.R.: Solitons in magneto-optic waveguides with anti-cubic nonlinearity. Optik 222, 165313 (2020)

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