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Optimality conditions for the minimal time problem for Complementarity Systems.

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Abstract: In this paper, we tackle the minimal time problem for systems with complementarity constraints. A special focus is then made on LCS, and we investigate a bang-bang property. Finally, for sake of completeness, the results are completed by an HJB equation, giving necessary and sufficient conditions of optimality.

Keywords: Linear optimal control, Minimum-time control, Complementarity problems.

1. INTRODUCTION

This paper focuses on finding optimality conditions for the minimal time problem:

\[ T^* = \min_{T(x,u)} T(x,u) \]

\[ \begin{align*}
    \dot{x}(t) &= \phi(x(t),u(t)), \\
    g(x(t),u(t)) &\leq 0, \\
    h(x(t),u(t)) &= 0, \text{ a.e. on } [0,T(x,u)] \\
    0 &\leq G(x(t),u(t)) \perp H(x(t),u(t)) \geq 0, \\
    u(t) &\in \mathcal{U} \\
    (x(0),x(T(x,u))) &= (x_0,x_f),
\end{align*} \]

with \( \phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l \), \( h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q \), \( G,H : [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \), \( \mathcal{U} \subset \mathbb{R}^m \), \( x_0,x_f \in \mathbb{R}^n \) given. We suppose that \( F \) and \( \phi \) are \( \mathcal{L} \times \mathcal{B} \)-measurable, where \( \mathcal{L} \times \mathcal{B} \) denotes the \( \sigma\)-algebra of subsets of appropriate spaces generated by product sets \( M \times N \), where \( M \) is a Lebesgue (\( \mathcal{L} \)) measurable subset in \( \mathbb{R} \), and \( N \) is a Borel (\( \mathcal{B} \)) measurable subset in \( \mathbb{R}^n \times \mathbb{R}^m \). We denote a solution of this problem by \( (T^*,x^*,u^*) \). If we don’t bound \( u \) with the constraint \( \mathcal{U} \), then the solution will most probably be \( T^* = 0 \) (i.e. an impulsive control, provided (2) is given a mathematical meaning). The condition

\[ 0 \leq G \perp H \geq 0 \]

means that \( G,H \geq 0 \) and \( \langle G,H \rangle = 0 \) for almost all \( t \in [0,T] \). Systems like (2), despite their simple look, gives rise to several challenging questions, mainly because conditions (3) introduce non-differentiability at switching points and non-convexity of the set of constraints. It provides a modeling paradigm for many problems, as Nash equilibrium games, hybrid engineering systems (Brogliato (2003)), contact mechanics or electrical circuits (Acary et al. (2011)). Several problems have already been tackled, let us mention observer-based control (Çamlibel et al. (2006), Heemels et al. (2011)) and Zeno behavior (Çamlibel and Schumacher (2001), Pang and Shen (2007), Shen (2014)).

Another difficulty comes from the fact that the constraints involve both the control and the state. These mixed constraints make the analysis even more challenging. For instance, deriving a maximum principle with wide applicability involves the use of non-smooth analysis, even in the case of smooth and/or convex constraints (see e.g. Clarke and De Pinho (2010)). Some first order conditions were given in Guo and Ye (2016) for systems with complementarity constraints, but they do not tackle this problem, since therein, the final time \( T^* \) is fixed beforehand. However, slight changes in the proof made for (Vinter, 2010, Theorem 8.7.1) allow us to derive first order conditions for (1)(2). This paper is organized as follows: first, the necessary conditions for (1)(2) will be derived. Then, we will show how these results are adapted to the problem of minimal time control for Linear Complementarity Systems (LCS), and some cases where the hypothesis are met. The focus will be made on proving a bang-bang property of the minimal-time control, and a Hamilton-Jacobi-Bellman characterization of the minimal time function. The results will be illustrated for a certain class of one dimensional LCS, deriving the analytical solution.

2. NECESSARY CONDITIONS

Since we have to compare different trajectories that are defined on different time-intervals, it should be understood that for \( T > T^* \), a function \( w \) defined on \( [0,T^*] \) is extended to \( [0,T] \) by assuming constant extension: \( w(t) = w(T^*) \) for all \( t \in [T^*,T] \).

Definition 1. We refer to any absolutely continuous function as an arc, and to any measurable function on \( [0,T^*] \) as a control. An admissible pair for (1)(2) is a pair of functions \( (x,u) \) on \( [t_0,t_1] \) for which \( u \) is a control and \( x \) is an arc, that satisfy all the constraints in (2). The complementarity cone is defined by \( C^c = \{(v,w) \in \mathbb{R}^m \mid 0 \leq v \perp w \geq 0 \} \). We define the set constraint by:
For every given \( t \in [t_0, t_1] \) and a positive constants \( R \) and \( \epsilon \), we define a neighbourhood of the point \((x^*(t), u^*(t))\) as:
\[
S_{r} = \{(x, u) \in S : \|x - x^*(t)\| \leq \epsilon, \|u - u^*(t)\| \leq R\}.
\]

For every such \( (x, u) \), denote:
\[
\text{dist}_{S}(x, u) \leq \tau(\max\{0, g(x, u)\}) + \|h(x, u)\| + \text{dist}_{C}(G(x, u), H(x, u)).
\]

For every given \( t \in [t_0, t_1] \) and a positive constants \( R \) and \( \epsilon \), we define a neighbourhood of the point \((x^{*}(t), u^{*}(t))\) as:
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\]

For every such \( (x, u) \), denote:
\[
\text{dist}_{S}(x, u) \leq \tau(\max\{0, g(x, u)\}) + \|h(x, u)\| + \text{dist}_{C}(G(x, u), H(x, u)).
\]
Proposition 3.7). We give some cases for which this condition holds when the underlying system is an LCS:

\[
\|u^*(t) - u(t)\| \leq \|u^*(t) - b(t)\| + \|u(t) - b(t)\| \\
\leq \frac{R}{2} + \|\tilde{u}(\sigma) - b(\tau(\sigma))\| \text{ where } t = \tau(\sigma)
\]

and:

\[
\|x - x^*\|_{w^{1,1}} = \int_0^T \|\dot{x}(t) - \dot{x}^*(t)\| dt \\
\leq \int_0^T \|\dot{x}(t) - \dot{u}(t)\| dt + \int_0^T \|\dot{x}^*(t) - \dot{u}(t)\| dt \\
\leq \int_0^T \|\phi(x(t), u(t)) - \dot{u}(t)\| dt + \varepsilon \\
\leq \int_0^T \|\phi(\dot{u}(\sigma), \dot{u}(\sigma)) - \dot{u}(\tau(\sigma))\| dt + \varepsilon \\
\leq \int_0^T \|\phi(\dot{u}(\sigma), \dot{u}(\sigma)) - \dot{u}(\tau(\sigma))\| dt + \frac{\varepsilon}{2} \\
\leq \varepsilon.
\]

Therefore, since they are in the same neighbourhood, the two trajectories can be compared. Since \((x^*, u^*)\) is supposed to be a local minimizer for (1)(2), we should have \(T^* \leq T\). This is a contradiction, so the claim that (11) is the minimizer was right.

Remark that since we supposed that Assumption 2 is verified, the same assumptions adapted for problem (9)(10) are also valid. Therefore, the results of (Guo and Ye, 2016, Theorem 3.2) for (9)(10) state that there exists an arc \(p : [0, T^*] \rightarrow \mathbb{R}^n\), a scalar \(\lambda_0 \in [0,1]\) and multipliers \(\lambda^g : [0, T^*] \rightarrow \mathbb{R}^g\), \(\lambda^h : [0, T^*] \rightarrow \mathbb{R}^p\), \(\lambda^G, \lambda^H : [0, T^*] \rightarrow \mathbb{R}^m\) such that (8a)-(8c) hold, along with the Weierstrass condition (8g).

Notice that since \(z(T^*) - z(0) < \varepsilon\) and \(\|u^*(t) - b(t)\| \leq \frac{R}{2}\)

Theorem 3.1 Sufficient condition for the bounded slope condition

These results still rely on assumptions, among which the bounded slope condition is a stringent, non-intuitive, and hard to verify condition. A sufficient condition for the bounded slope condition to hold is given by (Guo and Ye, 2016, Proposition 3.7). We give some cases for which this condition holds when the underlying system is an LCS:

\[
T^* = \min\ T(x, u, v)
\]

s.t. \[
\dot{x}(t) = Ax(t) + Bu(t) + Fu(t), \quad 0 \leq v(t) \leq Cx(t) + Cv(t) + Ku(t) + Eu(t) \geq 0, \quad u(t) \in U, \text{ a.e. on } [0, T^*]
\]

where \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{m \times n}\), \(C \in \mathbb{R}^{m \times n}\), \(F \in \mathbb{R}^{m \times m}\), \(U \subseteq \mathbb{R}^m\). However, a direct application of (Guo and Ye, 2016, Theorem 3.5(b)) proves the following proposition.

Proposition 4. Suppose Assumption 2 for the problem (12)(13) holds. Suppose also that \(U\) is a union of finitely many polyhedral sets. Let \((x^*, u^*, v^*)\) be a local minimizer for (12)(13). Then, \((x^*, u^*, v^*)\) is M-stationary, meaning it is W-stationary with arc \(p\), and moreover, there exist measurable functions \(\eta^g, \eta^H : [0, T^*] \rightarrow \mathbb{R}^m\) such that:

\[
\begin{align*}
0 &= B^Tp + D^T\eta^H + \eta^g, \\
0 &= -F^Tp + E^T\eta^H + \lambda^G(u^*(t)), \\
\eta^g(t) &= 0, \quad \forall \ t \in I^0(x^*, u^*, v^*), \\
\eta^H(t) &= 0, \quad \forall \ t \in I^0(x^*, u^*, v^*), \\
\eta^g > 0 &\text{ or } \eta^H > 0, \quad \forall \ t \in I^0(x^*, u^*, v^*)
\end{align*}
\]

Since the system is linear, (Guo and Ye, 2016, Proposition 2.3) asserts that the local error bound condition holds at every admissible point. There is one case for which one can check that the bounded slope condition hold when \(U = \mathbb{R}^m\), as proved in Vieira et al. (2018). However, this case of unbounded \(U\) is rather unrealistic, since it could lead to \(T^* = 0\) (in the sense that the target \(x_f\) can be reached from \(x_0\) given any positive time \(T^* > 0\); see for instance Lohéac et al. (2018) for an example). When one attempts to add a constraint \(U\) to the previous proof, (Guo and Ye, 2016, Theorem 3.7) adds a normal cone that prevents checking the inequality, unless one supposes that the optimal trajectory is inside an \(R\)-neighbourhood which lies in the interior of \(U\). Nonetheless, there are two cases when (13) verifies the bounded slope condition, even with constraints on \(u\).

Proposition 5. Suppose either \(C = 0\), or \(D\) is a diagonal matrix with positive entries. Then the bounded slope condition for (13) holds.

Proof. The case when \(C = 0\) is obvious, when one applies directly Proposition (Guo and Ye, 2016, Proposition 3.7). Assume \(D = \text{diag}(d_1, ..., d_m)\), where the \(d_i > 0\), \(i \in \mathbb{N}\), are the diagonal entries of \(D\), and diag means that \(D\) is a diagonal matrix built with these entries. First of all, remark that:

\[
\forall \lambda^H \in \mathbb{R}^m, \quad\|C^T\lambda^H\| \leq \|C^T D^{-1}\| \|D\lambda^H\|. \\
\]

Now, for \(t \in [0, T^*] \) and \((x, u) \in S^R_{+}\), take \(\lambda^g\) and \(\lambda^H\) in \(\mathbb{R}^m\) such that:

\[
\lambda_i^g = 0, \quad \forall i \in I^0(x, u), \quad \lambda_i^H = 0, \quad \forall i \in I^0(x, u), \\
\lambda_i^g > 0, \quad \lambda_i^H > 0, \quad \forall i \in I^0(x, u).
\]

It yields:

\[
\|D\lambda^H\|^2 = \sum_{i \in I^0(x, u)} (D_i\lambda_i^H)^2 + \sum_{i \in \mathbb{N}^*} (D_i\lambda_i^H)^2 \\
+ \sum_{i \in \mathbb{N}^*} (D_i\lambda_i^H)^2
\]
One can easily see that:
\[
\sum_{i \in I^n_{\theta}(x,u)} (D_i \lambda^H) \leq \sum_{i \in I^n_{\theta}(x,u)} (D_i \lambda^H + \lambda_G^2),
\]
which is proved using Aumann's theorem (see for instance (Guo and Ye, 2016, Proposition 3.7)), we see that
\[
0 = \sum_{i \in I^n_{\theta}(x,u)} (D_i \lambda^H)^2 \leq \sum_{i \in I^n_{\theta}(x,u)} (D_i \lambda^H + \lambda_G^2)^2,
\]
and, therefore,
\[
\|D\lambda^H\|_2 \leq \sum_{i=1}^m (D_i \lambda^H + \lambda_G^2)^2 = \|D\lambda^H + \lambda_G^2\|_2.
\]
One finally proves: \(\forall \zeta \in \mathcal{N}u(u),\)
\[
\|C^T \lambda^H\| \leq \|C^T D^{-1} \| \|D\lambda^H + \lambda_G^2\|_2 \\
\leq \|C^T D^{-1} \| \|D\lambda^H + \lambda_G^2\|_2.
\]
Using (Guo and Ye, 2016, Proposition 3.7), we see that (13) complies with the bounded slope condition.

3.2 A bang-bang property

Reachable set for linear systems

We turn ourselves to the reachability set of linear systems in order to state a result that will be useful in order to prove a bang-bang property for LCS. Consider the following system:
\[
\begin{align*}
\dot{x}(t) &= Mx(t) + Nu(t), \\
u(t) &\in \mathcal{V},
\end{align*}
\]
for some matrices \(M \in \mathbb{R}^{n \times n}\) and \(N \in \mathbb{R}^{n \times m}\). We define the reachable (or accessible) set from \(x_0 \in \mathbb{R}^n\) at time \(t \geq 0\), with controls taking values in \(\mathcal{V}\), denoted by \(\mathcal{A}ccy(x_0, t)\), the set of points \((x, t)\), where \(x : [0, t] \to \mathbb{R}^n\) is a solution of (15), with \(u(s) \in \mathcal{V}\) for almost all \(s \in [0, t]\) and \(x(0) = x_0\). As stated in (Trélat, 2005, Corollary 2.1.2), which is proved using Aumann’s theorem (see for instance Clarke (1981)), the following Proposition shows that the set of constraints \(\mathcal{V}\) can be embedded in its convex hull:

**Proposition 6.** (Trélat, 2005, Corollary 2.1.2) Suppose that \(\mathcal{V}\) is compact. Then:
\[
\mathcal{A}ccy(x_0, t) = \mathcal{A}cc_{\text{conv}}(\mathcal{V})(x_0, t)
\]
where \(\mathcal{A}cc_{\text{conv}}(\mathcal{V})\) denotes the convex hull of \(\mathcal{V}\).

Thanks to Krein-Milman’s Theorem (see Appendix A), this justifies that minimal-time optimal controls can be searched as bang-bang controls (meaning, \(u\) only takes values that are extremal points of \(\mathcal{V}\) if one supposes in addition that \(\mathcal{V}\) is convex).

**Extremal points for LCS** For this section, let us state the following Assumption:

**Assumption 7.** In (13), \(C = 0\), \(D\) is a \(P\)-matrix, and \(\mathcal{U}\) is a finite union of polyhedral compact convex sets.

As it can be expected for a minimal time problem with linear dynamics, a bang-bang property can be proved, where the bang-bang controls have to be properly defined. Let us define first some notions. Denote by \(\Omega\) the constraints on the controls \((u, v)\) in (13), meaning:
\[
\Omega = \{(u, v) \in \mathcal{U} \times \mathbb{R}^m | 0 \leq v \perp Du + Eu \geq 0\}.
\]
The set \(\mathcal{A}cc_{\mathcal{F}}(x_0, t)\) denotes the reachable set from \(x_0 \in \mathbb{R}^n\) at time \(t \geq 0\) with controls with values in \(\Omega\). For a convex set \(\mathcal{C}\), a point \(c \in \mathcal{C}\) is called an extreme point if \(\mathcal{C} \{c\}\) is still convex. The set of extreme points of \(\mathcal{C}\) will be denoted \(\mathcal{E}(\mathcal{C})\). Suppose \(\Omega\) is compact (which is not necessarily the case: take for instance \(D = 0\) with \(0 \in \mathcal{U}\)). Applying Proposition 6, one proves that:
\[
\mathcal{A}cc_{\mathcal{F}}(x_0, t) = \mathcal{A}cc_{\text{conv}}(\Omega)(x_0, t).
\]
The set \(\Omega\) is not convex and has empty interior; finding its boundary or extreme points is not possible in this case. However, Krein-Milman’s Theorem (see Appendix A) proves that \(\mathcal{E}(\Omega)\) can be generated by its extreme points. In what follows, we will prove that the extreme points of \(\mathcal{E}(\Omega)\) are actually points of \(\Omega\) that can be easily identified from the set \(\mathcal{U}\). For an index set \(\alpha \subseteq m\), denote by \(\mathbb{R}^m_\alpha\) the set of points \(q \in \mathbb{R}^m\) such that \(q_\alpha \geq 0\), \(\sum_{\alpha} q_\alpha \leq 0\), and define \(E^- \mathbb{R}^m_\alpha = \{u \in \mathbb{R}^m | Eu \in \mathbb{R}^m_\alpha\}\) (\(E\) is not necessarily invertible).

**Lemma 8.** Suppose Assumption 7 holds true. For a certain \(\alpha \subseteq m\), denote by \(\mathcal{P}_\alpha\) the set:
\[
\mathcal{P}_\alpha = \{(u, v) \in (\mathcal{U} \cap E^- \mathbb{R}^m_\alpha) \times \mathbb{R}^m | v_\alpha = 0, D\pi_\alpha + E\pi_\alpha = 0, v \geq 0, Du + Eu \geq 0\},
\]
and by \(\mathcal{E}_\alpha\) the set:
\[
\mathcal{E}_\alpha = \{(u, v) \in \mathcal{E}(\mathcal{U} \cap E^- \mathbb{R}^m_\alpha) \times \mathbb{R}^m | v_\alpha = 0, D\pi_\alpha + E\pi_\alpha = 0, v \geq 0, Du + Eu \geq 0\}.
\]
Then \(\mathcal{E}(/\mathcal{P}_\alpha) = \mathcal{E}_\alpha\).

**Proof.** If \(\mathcal{U} \cap E^- \mathbb{R}^m_\alpha\) is empty, then the equality is obvious. Choose \(\alpha\) such that \(\mathcal{U} \cap E^- \mathbb{R}^m_\alpha\) is not empty.

- \(\mathcal{E}_\alpha \subseteq \mathcal{E}(\mathcal{P}_\alpha)\): Let \((u, v) \in \mathcal{E}_\alpha\). Suppose that \((u, v) \notin \mathcal{E}(\mathcal{P}_\alpha)\). Thus, there exist \((u^1, v^1)\) and \((u^2, v^2)\) in \(\mathcal{P}_\alpha\), both different than \((u, v)\), such that \((u, v) = \frac{1}{2}[(u^1, v^1) + (u^2, v^2)]\). But this implies that \(u = \frac{1}{2}(u^1 + u^2)\), and since \(u \in \mathcal{E}(\mathcal{U} \cap E^- \mathbb{R}^m_\alpha)\), \(u = u_1 = u_2\). Therefore, since \(D\) is a \(P\)-matrix, \(v = SOL(D, Eu) = SOL(D, Eu^i) = v^i\) for \(i \in \{1, 2\}\). Therefore, \((u, v) = (u^1, v^1) = (u^2, v^2)\), and \((u, v)\) is an extremal point of \(\mathcal{P}_\alpha\). This is a contradiction.

- \(\mathcal{E}(\mathcal{P}_\alpha) \subseteq \mathcal{E}_\alpha\): Let \((u, v) \in \mathcal{E}(\mathcal{P}_\alpha)\). Suppose that \((u, v) \notin \mathcal{E}(\mathcal{U} \cap E^- \mathbb{R}^m_\alpha)\). Therefore, there exists \(u^i\) and \(u^j\) in \(\mathcal{U} \cap E^- \mathbb{R}^m_\alpha\), different than \(u\), such that \(u = \frac{1}{2}(u^i + u^j)\). Define for \(i \in \{1, 2\}\) \(v^i = SOL(D, Eu^i)\). Since \(Eu^i\) and \(Eu^2\) are membe of \(\mathbb{R}^m_\alpha\), for \(i \in \{1, 2\}\):
\[
v^i = - (D\pi_{\alpha i})^{-1}(Eu^i)\pi_\alpha, \quad v_{\alpha i} = 0.
\]
So:
\[
v^\pi = - (D\pi_{\alpha})^{-1}(Eu\pi)\pi_\alpha = \frac{1}{2}(v^1 + v^2),
\]
and \(v_\alpha = \frac{1}{2}(v_{1\alpha} + v_{2\alpha})\).

So \((u, v) = \frac{1}{2}[(u^1, v^1) + (u^2, v^2)]\) with \((u^i, v^i) \in \mathcal{P}_\alpha\), \(i \in \{1, 2\}\). But since \((u, v) \in \mathcal{E}(\mathcal{P}_\alpha)\), \(u = u^i = u^2\).

This is a contradiction.
The interest of Proposition 10 is twofold: first, the com-

Proof. The function SOL(D, ·) : q \mapsto v = \text{SOL}(D, q)

where a, b, d, f, e are scalars, and we suppose \( d > 0 \) and \( e \neq 0 \). We suppose also that there exist at least one trajectory stirring \( x_0 \) to \( x_f \).

In this case, there are two index sets \( \alpha \) as described in Lemma 8: \( \emptyset \) or \( \{1\} \). Therefore, we should have a look at the extreme points of \( \mathcal{U} \cap \mathbb{R}_+^m = \mathcal{U} \cap \mathbb{R}_-^m = [-1, 0] \) and of \( \mathcal{U} \cap \mathbb{R}_+^m = \mathcal{U} \cap [-1, 0] \). Thus, it is sufficient to look at input functions \( u(t) \) with values in \( \{-1, 0, 1\} \). Suppose that the constants in (18) (with \( u(t) \) supposed unconstrained for the moment) is completely controllable, which means:

**If** \( e > 0 \) : if \( f > 0 \), then \( b > 0 \) or \( b - \frac{Ld}{e} < 0 \). if \( f > 0 \), then \( b < 0 \) or \( b > \frac{Ld}{e} > 0 \). If \( e < 0 \) : the same cases as with \( e > 0 \) hold by inverting the sign of \( f \).

All other cases (like \( f = 0 \) or \( e = 0 \)) are discarded. Let us now deduce from Theorem 3 the only stationary solution. First of all, the equation (8b) tells us that the adjoint state satisfies with the ODE \( \dot{p} = -ap \). Therefore there exists \( p_0 \) such that, for all \( t \in [0, T^*] \), \( p(t) = p_0 e^{-at} \). Could we have \( p_0 = 0 \)? It would imply that \( p \equiv 0 \) and then, \( \lambda_0 = (p(t), \alpha x(t) + bu(t) + fu(t)) = 0 \), so \( (p(t), \lambda_0) = 0 \) for almost all \( t \in [0, T^*] \). This is not allowed, so \( p_0 \neq 0 \). Moreover, there exist multipliers \( \lambda^G \) and \( \lambda^H \) such that, for almost all \( t \in [0, T^*] \):

\[
\lambda^G(t) = -bp(t) - \lambda^H(t),
\]

\[
\lambda^G(t) = 0 \text{ if } v(t) > 0 = dv(t) + eu(t),
\]

\[
\lambda^H(t) = 0 \text{ if } v(t) < 0 < dv(t) + eu(t),
\]

\[
\lambda^G(t) + e\lambda^H(t) \in N_{[-1, 1]}(u(t)) = \begin{cases} \{0\} & \text{if } |u(t)| \neq 1, \\ \mathbb{R}^+ & \text{if } u(t) = 1, \\ -\mathbb{R}^+ & \text{if } u(t) = -1, \\ \end{cases}
\]

(22)

(23)

(24)

Let us now discuss all possible cases for \( u \). If \( u(t) = 0 \), then \( v(t) = 0 = dv(t) + eu(t) \). We use stationarity conditions for (23) since the MPEC Linear Condition holds, there exists multipliers \( \eta^H \) and \( \eta^G \) such that \( \eta^H = -\frac{Ld}{e}p(t) \), \( \eta^G = \left(\frac{d}{e} - b\right)p(t) \), and \( \eta^G \eta^H = 0 \) or \( \eta^G > 0, \eta^H > 0 \) (one has M-stationarity, see Proposition 4). However, \( \eta^H \neq 0 \) and \( \eta^G \neq 0 \). Furthermore, \( \eta^H \) has the same sign as \( -f \) and \( \eta^G \) has the same sign as \( f \). Therefore, the two have opposite signs, and \( u(t) = 0 \) can not be an M-stationary solution. Therefore, it proves that necessarily, the optimal control \( u^* \) complies with \( |\eta^H(t)| = 1 \) for almost all \( t \in [0, T^*] \). If \( u(t) = 1 \), then \( v(t) < 0 < dv(t) + eu(t) \), and by (21), \( \lambda^H(t) = 0 \). Then by (22), \( f \) needs to be \( \geq 0 \). If \( u(t) = -1 \), then \( v(t) > 0 = dv(t) + eu(t) \), and by (19)(20), \( \lambda^H(t) = -\frac{b}{e}p(t) \). Then by (22), \( \left(1 - \frac{d}{e} \right) \) \( \neq 0 \). It is impossible to have
It is clear that there exists no solution to the LCP\((−u)\) appearing in (27) for \(P\) but \(D\) however, some examples show that even when \(x = \Omega\) if \(x_0\) is a trajectory and \(T^*\) is a state, then the optimal time \(T^*\) is positive. Therefore, one can infer that:

\[
x^*(t) = \begin{cases} 
  \frac{f t}{d} & \text{if } u^*(t) = 1, \\
  x_0 + \frac{be - fd}{ad} & \text{if } u^*(t) = -1.
\end{cases}
\]

One must then find the solution that complies with \(x(T^*) = x_f\). One can isolate the optimal time \(T^*\):

\[
T^* = \begin{cases} 
  1 - \ln \left( \frac{ax_0 + f}{ax_0 + f} \right) & \text{if } ax_0 + f \neq 0 \text{ and } \frac{ax_0 + f}{ax_0 + f} > 0, \\
  1 - \ln \left( \frac{adx_0 + be - fd}{adx_0 + be - fd} \right) & \text{if } adx_0 + be - fd \neq 0 \text{ and } adx_0 + be - fd > 0.
\end{cases}
\]

Since we supposed that there exists at least one trajectory stirring \(x_0\) to \(x_f\), one of these two expressions of \(T^*\) must be positive. Therefore, one can infer that:

\[
u^* \equiv \begin{cases} 
  1 & \text{if } ax_0 + f \neq 0 \text{ and } \frac{ax_0 + f}{ax_0 + f} > 0, \\
  1 & \text{if } ax_0 + f \neq 0 \text{ and } \frac{ax_0 + f}{ax_0 + f} > 0.
\end{cases}
\]

If \(a = 0\):

\[
x^*(t) = \begin{cases} 
  \frac{ft}{d} & \text{if } u^*(t) = 1, \\
  \frac{be - fd}{d} & \text{if } u^*(t) = -1.
\end{cases}
\]

With the same calculations made in the case \(a \neq 0\), one proves that:

\[
u^* \equiv \begin{cases} 
  1 & \text{if } f \neq 0 \text{ and } \frac{x_f}{f} > 0, \\
  1 & \text{if } f \neq 0 \text{ and } \frac{x_f}{f} > 0.
\end{cases}
\]

The proof of Proposition 10 relies on the fact that when \(D\) is a P-matrix, \(Ω\) is the union of compact convex polyhedra. However, some examples show that even when \(D\) is not a P-matrix, then this property may hold.

**Example 12.**

\[
T^* = \min_{T(x,u,v)} T(x,u,v) = \begin{cases} 
  \frac{f t}{d} & \text{if } u^*(t) = 1, \\
  x_0 + \frac{be - fd}{ad} & \text{if } u^*(t) = -1.
\end{cases}
\]

It is clear that there exists no solution to the LCP\((-u)\) appearing in (27) for \(u \in [-1,0]\), so \(U\) can actually be restricted to \([0,1]\). A graphic showing the shape of \(Ω\) and its convex hull is shown in Figure 1.

It clearly appears that \(\text{conv}(Ω)\) is generated by three extreme points: \((u,v) \in Ε = \{(0,0),(1,0),(1,1)\}\). Exactly the same way as in the proof of Proposition 10, one can simply show that for all \(t \geq 0\) and for all \(x_0 \in \mathbb{R}\), \(\text{Acc}_{Ω}(x_0,t) = \text{Acc}_{E}(x_0,t)\). Therefore, the optimal trajectory can be searched with controls \((u,v)\) with values in \(Ε\). It is also interesting to note that this bang-bang property can be guessed from the condition of maximisation of the Hamiltonian in (8g) and from Figure 1. Indeed, (8g) state that at almost all time \(t\), the linear function \(Λ : (u,v) \mapsto (p(t),Bv + Fu)\) must be maximized with variables \((u,v)\) in \(Ω\). When one tries to maximize \(Λ\) over \(\text{conv}(Ω)\) it becomes a Linear Program (LP) over a simplex. It is well known that linear functions reach their optimum over simplexes at extreme points; in this case, the extreme points are the points of \(Ε\).

### 3.3 Characterisation through HJB equation

An other way to solve the minimal time optimal control problem is through the Dynamic Programming Principle and the Hamilton-Jacobi-Bellman (HJB) equation. The theory needs pure control constraints, but not convexity of the set of constraints. Therefore, the Assumption that \(C = 0\) in (13) still holds. However, one doesn’t need \(D\) to be a P-matrix anymore. The only necessary Assumption needed is an assumption of compactness.

**Assumption 13.** In (13), \(C = 0\), and the set \(Ω\) defined in (16) is a compact subset of \(\mathbb{R}^{m_u} × \mathbb{R}^m\).

The HJB equation is a non-linear PDE that the objective cost must comply with. In this framework, the minimal time \(T^*\) is seen as a function of the target \(x_f\). However, the equation will not be directly met by \(T^*(x_f)\), but by a discounted version of it, called the Kružkov transform, and defined by:

\[
z^*(x_f) = \begin{cases} 
  1 - e^{-T^*(x_f)} & \text{if } T^*(x_f) < +\infty, \\
  1 & \text{if } T^*(x_f) = +\infty.
\end{cases}
\]

This transformation comes immediately when one tries to solve this optimal control problem with the running cost \(C(t(x_f)) = \int_{x(t)}^{x_f} e^{-t} dt = 1 - e^{-t(x_f)}\) where \(t(x_f)\) is a free variable. Minimizing \(C(t(x_f))\) amounts to minimizing \(T^*\).
Once one finds the optimal solution \( z(x_f) \), it is easy to recover \( T^* \), since \( T^*(x_f) = -\ln(1 - z(x_f)) \).

The concept of solution for the HJB equation needs the concept of viscosity solution. A reminder of the definitions of sub- and supersolutions appears in Appendix B. But the most useful definitions are recalled here. First of all, one needs the notion of lower semicontinuous envelope.

**Definition 14.** Denote \( z : X \to [-\infty, +\infty], \; X \subseteq \mathbb{R}^n \).

We call lower semicontinuous envelope of \( z \) the function \( \tilde{z} \) defined pointwise by:

\[
\tilde{z}(x) = \liminf_{y \to x} z(y) = \liminf_{r \to 0^+} \{ z(y) : y \in X, \; |y - x| \leq r \}
\]

One can see easily that \( \tilde{z} = z \) at every point where \( z \) is (lower semi-)continuous. Secondly, one needs the definition of an envelope solution.

**Definition 15.** Consider the Dirichlet problem

\[
\begin{cases}
F(x, z(x), \nabla z(x)) = 0 & x \in \kappa \\
z(x) = g(x) & x \in \partial \kappa 
\end{cases}
\tag{29}
\]

with \( \kappa \subseteq \mathbb{R}^n \) open, \( F : \kappa \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) continuous, and \( g : \partial \kappa \to \mathbb{R} \). Denote \( \mathcal{S} = \{ \text{subolutions of } (29) \} \) and \( \overline{\mathcal{S}} = \{ \text{supersolutions of } (29) \} \). Let \( z : \pi \to \mathbb{R} \) be locally bounded.

(1) \( z \) is an envelope viscosity subolution of (29) if there exists \( \mathcal{S}(z) \subseteq \mathcal{S}, \; \mathcal{S}(z) \neq \emptyset \), such that:

\[
z(x) = \sup_{w \in \mathcal{S}(z)} w(x), \; x \in \pi
\]

(2) \( z \) is an envelope viscosity supersolution of (29) if there exists \( \overline{\mathcal{S}}(z) \subseteq \overline{\mathcal{S}}, \; \overline{\mathcal{S}}(z) \neq \emptyset \), such that:

\[
z(x) = \inf_{w \in \overline{\mathcal{S}}(z)} w(x), \; x \in \pi
\]

(3) \( z \) is an envelope viscosity solution of (29) if it is an envelope viscosity sub- and supersolution.

With these definitions, one can formulate the next Theorem, stating the HJB equation for \( z^* \):

**Theorem 16.** \( z^* \) is the envelope viscosity solution of the Dirichlet problem:

\[
\begin{cases}
z + H(x, \nabla z) = 1 & x \in \mathbb{R}^n \setminus \{ x_f \}, \\
z = 0 & x \in \{ x_f \}
\end{cases}
\tag{30}
\]

where \( H(x, p) = \sup_{(u, v) \in \tilde{\Omega}} (-p, Ax + Bv + Fu) \).

**Remark 17.** The target \( \{ x_f \} \) could be changed to any closed nonempty set \( \mathcal{T} \) with compact boundary.

**Example 18.** Example 11 revisited.

Let us check that the Krzukov transform of \( T^* \) found in (25) complies with (30). The verification will be carried in the case when \( \frac{ax_1 + f}{ax_0 + f} > 0 \), the other cases being treated with the same calculations. In this case, the Krzukov transform of \( T^* \) defined in (28) amounts to:

\[
z^*(x_f) = 1 - \left( \frac{ax_1 + f}{ax_0 + f} \right)^{-\frac{1}{2} - 1}
\]

Therefore, one must check that

\[
1 - z^*(x_f) = - \left( \frac{ax_1 + f}{ax_0 + f} \right)^{-\frac{1}{2}}
= \sup_{(u,v) \in \tilde{\Omega}} \left\{ (ax_f + bv + Fu) \frac{dz^*}{dx}(x_f) \right\}
\tag{32}
\]

where \( \Omega \) is defined as \( \tilde{\Omega} = \{(u,v) \in [-1,0,1] \times \mathbb{R} \mid 0 \leq v \leq dx + eu \geq 0 \} \).

As it has been shown in Example 11, the sup in (32) is attained at \( u = 1, \; v = 0 \). Therefore:

\[
\sup_{(u,v) \in \tilde{\Omega}} \left\{ (ax_f + bv + Fu) \frac{dz^*}{dx}(x_f) \right\} = (ax_f + f) \left( -1 + \frac{ax_1 + f}{ax_0 + f} \right)^{-\frac{1}{2} - 1}
= \frac{ax_1 + f}{ax_0 + f} - 1 = 1 - z^*
\]

Therefore, using the same definition of \( H \) made in (31), it is proven that \( z^* \) complies with the equation \( z^* + H(x_f, \frac{dz^*}{dx}) = 1 \), which is the HJB Equation (30).

4. CONCLUSION

The necessary conditions for optimality exposed in Guo and Ye (2016) were extended to the case of minimal time problem. These results were precised for LCS, and some special properties that the optimum possess the case of LCS, were also shown. As future work, one could extend the class of LCS complying for the Bounded Slope Condition, and also prove the bang-bang property for a broader class of LCS, as Example 12 suggests.

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Brogliato, B. (2003). Some perspectives on the analysis and control of complementarity systems. IEEE Transactions on Automatic Control, 48(6), 918–935.
Appendix A. KREIN-MILMAN THEOREM

Since the Krein-Milman Theorem is used in this paper, it is worth recalling its statement. Let us start with a definition.

Definition 19. Let $C$ be a convex compact subset of a Hausdorff locally convex set. Let $c \in C$. The point $c$ is called an extremal point of $C$ if $C \setminus \{c\}$ is still convex. Equivalently, $c$ is an extreme point of $C$ if the following implication holds: $c_1, c_2 \in C, c = \frac{1}{2}(c_1 + c_2) \implies c = c_1 = c_2$. The set of extreme points of $C$ is denoted by $\text{Ext}(C)$.

Theorem 20. (Krein-Milman). Let $C$ be a convex compact subset of a Hausdorff locally convex set. Then

$$C = \text{cl conv} \{\text{Ext}(C)\}$$

Appendix B. VISCOSITY SOLUTIONS

In order to understand some results concerning the HJB equation, one needs to know some definitions related to the concept of viscosity solutions. The definitions given here, extracted from Bardi and Capuzzo-Dolcetta (2008), are only the ones useful for this manuscript. In particular, the definitions given here are the ones useful to handle the concept of discontinuous viscosity solutions. The interested reader can find broader results in Bardi and Capuzzo-Dolcetta (2008) and the references therein. Let us first define the notion of subsolution and supersolution of a first order equation

$$F(x, u, \nabla u) = 0 \text{ in } \Omega, \quad (B.1)$$

with $\Omega \subseteq \mathbb{R}^n$ and $F : \Omega \times \mathbb{R}\times \mathbb{R}^n \to \mathbb{R}$ continuous. For this, let us fix some notations: for $E \subseteq \mathbb{R}^n$, denote $\text{USC}(E) = \{u : E \to \mathbb{R} \text{ upper semicontinuous}\}$, $\text{LSC}(E) = \{u : E \to \mathbb{R} \text{ lower semicontinuous}\}$.

Definition 21. A function $u \in \text{USC}(\Omega)$ (resp. $\text{LSC}(\Omega)$) is a viscosity subsolution (resp. supersolution) of (B.1) if, for any $\phi \in C^1(\Omega)$ and $x \in \Omega$ such that $u - \phi$ has a local maximum (resp. minimum) at $x$,

$$F(x, u(x), \nabla \phi(x)) \leq 0 \text{ (resp. } \geq 0).$$