We consider an arbitrary qubit channel depending on a single parameter, which is to be estimated by a physical process. Using the quantum Fisher information per channel invocation to quantify the estimation accuracy, we consider various estimation protocols when the available initial states are mixed with very low purity. We compare a protocol using a single channel invocation on one out of $n$ qubits prepared in a particular correlated input state to the optimal protocol using uncorrelated input states, with the same initial state purity. We show that, to lowest order in initial-state purity, for a unital channel this correlated state protocol enhances the estimation accuracy by a factor between $n−1$ and $n$. We also show that to lowest order in initial-state purity, a broad class of non-unital channels yields no gain regardless of the input state.

## I. INTRODUCTION

Quantum parameter estimation, or metrology, considers the use of physical quantum systems as measuring devices. Typically a system is prepared in a known initial state and is then subjected to an evolution of a known type but which depends on an unknown parameter that to be estimated. The parameter must subsequently be inferred from measurements on the system. Classical statistics and quantum physics constrain the success of such procedures; combining these has led to a quantum estimation framework [1–9].

This has been applied to various situations, including estimation of parameters in phase-shifts [5], depolarizing channels [10–12] Pauli channels [13, 14] and amplitude damping channels [15]. A key issue is whether using states only available to quantum systems (such as entangled states) enhances the estimation accuracy compared to that for “classical” repeat and average strategies using uncorrelated states. Sometimes this is true.

Most studies focus on the absolute optimal situations, which require pure initial states. However, in some situations such as solution-state nuclear magnetic resonance (NMR), pure states are unavailable and the issue becomes whether advantages arise when correlating mixed or noisy states that would otherwise be used in an uncorrelated estimation protocol. This has been addressed for the phase-shift [16–19], phase-flip [20] and depolarizing channels [21]. These studies focused on the enhancement of estimation accuracy in terms of the quantum Fisher information. So far there is no general result for all situations where the available initial states are very noisy. Section IV applies this to a single qubit protocol, which serves as a baseline for comparison with a multiple qubit correlated state protocol that is described in Sec. V. This contains the main results of this article.

## II. PARAMETER DEPENDENT QUBIT CHANNELS

We consider general single qubit quantum channels. Prior to evolution, the channel input state for a single qubit can be represented as

$$\hat{\rho}_i = \frac{1}{2} \left( \hat{I} + r \hat{r}_i \cdot \hat{\sigma} \right),$$  

(1)

where $\hat{I}$ is the identity operator, $\hat{r}_i$ is the input state Bloch-sphere direction, a three dimensional real unit vector and $\hat{r}_i \cdot \hat{\sigma} = r_{i_x} \hat{\sigma}_x + r_{i_y} \hat{\sigma}_y + r_{i_z} \hat{\sigma}_z$. Here $r$, which satisfies $0 \leq r \leq 1$, is called the purity of the state and quantifies the mixedness or noisiness of the state. Under the channel, $\hat{\rho}_i \mapsto \hat{\rho}_i$ where, again generically, $\hat{\rho}_i = \left( \hat{I} + r_f \cdot \hat{\sigma} \right)/2$ where $r_f$ is the final state Bloch-
sphere direction with $|r_t| \leq 1$. For any channel \[22\],

$$r_t = M(r_i) + d = rM \hat{r}_i + d$$

where $M$ is a $3 \times 3$ real Bloch-sphere matrix, $d$ and a
real Bloch-sphere shift vector. Note that $|r_t| \leq 1$ for all possible inputs and for $r = 0$ this implies $|d| \leq 1$. We argue that if $M \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $|d| \neq 1$. To do so, $|r_t|^2 = r^2 r_i^T M^T r_i + |d|^2 + r r_i^T M^T d + r d^T r_i$, where $\top$ indicates the transpose. Assuming that $|d| = 1$, taking the case where $r = 1$ and noting that the first term is positive this implies that for all unit vectors $\hat{r}_i$, the product $d^T M \hat{r}_i$ must be negative. However, invoking the singular value decomposition for $M$ implies that, regardless of $d$, there will always be some choices of $\hat{r}_i$ such that this is positive. Thus, $|d| = 1$ is only possible when $M = 0$.

Qubit channels for which $d = 0$ are called unital; this includes unitary channels, Pauli channels and the depolarizing channel. Non-unital channels, for which $d \neq 0$, include the amplitude damping channel.

Note that by the linearity of any channel action, the channel maps $I \mapsto I + d \cdot \sigma$ and also $\hat{r}_i \cdot \sigma \mapsto M \hat{r}_i \cdot \sigma$.

We will consider channels for which the mapping depends on a single parameter, $\lambda$, which we assume is independent of the channel input state. This implies that $M$ and $d$ might depend on $\lambda$ but are independent of $r$ and $\hat{r}_i$. The task will be to estimate the parameter by a physical process in which one or more qubits, prepared in known initial states, undergo evolution via one or more identical copies of the channel. The channel actions are followed by measurements, whose outcomes are used to infer the parameter. The goal will be to choose input states, measurements and statistical inference processes that minimize fluctuations in the estimates they generate; we assume that the key cost of such procedures is the number of channel invocations.

### III. ENTANGLEMENT-ASSISTED METROLOGY WITH NOISY INITIAL STATES

A standard formalism for assessing physical quantum estimation assumes that the parameter to be estimated is encoded into a physical state of a system, described by a (possibly multiple qubit) density operator, $\rho_t(\lambda)$. The estimate, $\lambda_{\text{est}}$, is inferred from measurement outcomes via a known estimator function. We will require that this estimator is unbiased, i.e. the mean of the estimates equals the true parameter value. The accuracy of the process can be quantified in terms of the variance in the estimate, $\text{var}(\lambda_{\text{est}}) := \left\langle \left( \lambda_{\text{est}} - \langle \lambda \rangle \right)^2 \right\rangle$ with the angle brackets indicating the mean over all possible measurement outcomes. Although this will depend on the choice of estimator, an ultimate limit is given by the classical Cramér-Rao bound (CRB), $\text{var}(\lambda_{\text{est}}) \geq 1/F(\lambda)$, for any choice of estimator \[23\]. Here the classical Fisher in-

formation,

$$F(\lambda) := \int \left[ \frac{\partial \ln p(x_1, x_2, \ldots | \lambda)}{\partial \lambda} \right]^2 dx_1 dx_2 \ldots,$$

is determined from the probability distribution for the possible measurement outcomes, denoted $x_1, x_2, \ldots$ for the process. There is always an estimator which asymptotically attains the lower bound \[23\].

A key additional feature of quantum estimation is that the choice of measurement affects the probability distribution used to compute the classical Fisher information. However, a further constraint is given by the quantum Cramér-Rao bound (QCRB), $\lambda_{\text{est}} \leq H(\lambda)$, where the quantum Fisher information (QFI) is

$$H(\lambda) = \text{Tr} \left[ \hat{\rho}(\lambda) \hat{L}^2(\lambda) \right],$$

and $\hat{L}(\lambda)$ is the symmetric logarithmic derivative (SLD) defined via

$$\frac{\partial \hat{\rho}(\lambda)}{\partial \lambda} = \frac{1}{2} \left[ \hat{L}(\lambda) \hat{\rho}(\lambda) + \hat{\rho}(\lambda) \hat{L}(\lambda) \right].$$

The SLD and the QFI only depend on the pre-measurement system state and thus the quantum Cramér-Rao bound together with the classical Cramér-Rao bound constrain the variance, $\text{var}(\lambda_{\text{est}}) \geq 1/H(\lambda)$. This bound is independent of both the choice of measurement and estimator \[3, 6, 24-26\].

The SLD can always be computed from a diagonal decomposition, $\hat{\rho}(\lambda) = \sum_j p_j(\lambda) |\phi_j(\lambda)\rangle \langle \phi_j(\lambda)|$, according to \[6, 9\],

$$\hat{L}(\lambda) = 2 \sum_{j,k} \frac{\phi_j |\hat{\rho}(\lambda)\phi_k\rangle}{p_j + p_k} |\phi_j\rangle \langle \phi_k|$$

where the dot indicates differentiation with respect to the parameter. In some cases there are simpler algebraic methods for computing the SLD \[20\]. Also, simple matrix algebra allows for rearrangement of the SLD to give

$$H(\lambda) = \text{Tr} \left[ \frac{\partial \hat{\rho}(\lambda)}{\partial \lambda} \hat{L}(\lambda) \right].$$

It is always possible to saturate this bound by choosing a projective measurement in the eigenbasis of the SLD but is cannot be assured that this choice is independent of the unknown parameter \[6, 25, 27\]. In such cases, there exist various other measurement schemes that asymptotically saturate the quantum Cramér-Rao bound \[28\].

Thus the QFI quantifies the accuracy of possible physical measurement procedures. Generally the task in any quantum parameter estimation study has been to engineer a final system state $\hat{\rho}_t$ that maximizes the QFI, subject to various system constraints (i.e. number of channel invocations, number of available systems,
types of initial states available, . . . ) [3–6, 8–10, 12–15, 17, 18, 20, 21, 24, 29–33]. We adopt this approach.

Within this context, using multiple copies of the channel in any single estimation protocol might enhance the QFI. Indeed, a protocol which invokes the channel once on each of $m$ systems each in the same state. An entanglement-assisted protocol using $n$ systems with the channel invoked once on each of $m$ of these. The lower $n - m$ systems serve as ancillas.

In contrast, entanglement-assisted metrology considers protocols, illustrated in Fig 1b), where a subset of the available quantum systems are subjected to the channel while the remaining ancilla systems function as spectators in a noiseless environment. The entire system is prepared in an entangled or otherwise correlated state prior to channel invocation and the issue is to determine whether this can enhance the QFI per channel invocation over uncorrelated or independent protocols. Sometimes such entanglement assistance has been shown to be advantageous for estimation [9, 33–35]; entanglement assistance has even been investigated experimentally [36].

We consider situations where a fixed finite number of qubits, $n$ is available and each is initially in the same state $\rho_0$ so that the initial state of the entire system is $\rho_0^n$. In the entanglement-assisted protocol the entire system is subjected to a preparatory unitary $U_{\text{prep}}$, yielding a channel input state $\hat{\rho}_i := U_{\text{prep}}\rho_0^n U_{\text{prep}}^\dagger$. This input state will be subjected to channel invocations on a subset of the systems, yielding a final, pre-measurement state.

Additionally we consider situations where the purity of the initial qubit states is very small, i.e. $r \ll 1$, and we ask, for a given channel, whether to lowest order in the purity, an entanglement assisted protocol can enhance the QFI per channel invocation over an independent channel invocation protocol that uses inputs with the same purity. It is crucial to note that we do not compare protocols with different initial-state purities. However various initial-state Bloch sphere directions can be accommodated via parameter independent single qubit rotations, which could be incorporated into the preparatory unitary. Solution-state NMR, for which $r \approx 10^{-4}$ offers one example of this situation [22].

Exact expressions for the QFI for all purities have so far been attained only in particular circumstances and even here their complicated nature renders interpretation difficult [18, 20, 21]. More generally there are currently no known techniques for calculating the SLD algebraically; this inhibits further analysis. We therefore aim for approximate expressions for the SLD and QFI that will be true to lowest order in the purity and we propose the following power series approach.

Using $\hat{\rho}_0 = (I + r\hat{r}_0 \cdot \hat{\sigma})/2$, the state prior to the preparatory unitary can be expressed as

$$\hat{\rho}_0^{\otimes n} = \sum_{j=0}^n r^j \hat{\rho}_0^{(j)},$$

where $\hat{\rho}_0^{(j)}$ is an operator independent of the purity, $r$.

The preparatory unitary maps this to the input state

$$\hat{\rho}_i = \sum_{j=0}^n r^j \hat{\rho}_i^{(j)}$$

where $\hat{\rho}_i^{(j)} = U_{\text{prep}}^\dagger \hat{\rho}_0^{(j)} U_{\text{prep}}$ is again independent of $r$.

Similarly the final state can be expressed as

$$\hat{\rho}_f(\lambda) = \sum_{j=0}^n r^j \hat{\rho}_f^{(j)}(\lambda)$$

where $\hat{\rho}_f^{(j)}(\lambda)$ is completely determined by evaluating the channel actions on $\hat{\rho}_i^{(j)}$.

Importantly, the two lowest order terms in the initial-state series are

$$\hat{\rho}_0^{(0)} = \frac{1}{N} \hat{f}^{\otimes n}$$

and

$$\hat{\rho}_0^{(1)} = \frac{1}{N} \left[ \hat{r}_0 \cdot \hat{\sigma} \otimes \hat{f}^{\otimes (n-1)} + \hat{I} \otimes \hat{r}_0 \cdot \hat{\sigma} \otimes \hat{f}^{\otimes (n-2)} + \cdots + \hat{f}^{\otimes (n-1)} \otimes \hat{r}_0 \cdot \hat{\sigma} \right]$$

where $N = 2^n$. For any preparatory unitary

$$\hat{\rho}_i^{(0)} = \frac{1}{N} \hat{f}^{\otimes n}$$

since $U_{\text{prep}}^\dagger \hat{U}_{\text{prep}} = \hat{I}$. The remaining terms in $\hat{\rho}_i$ depend on the nature of the preparatory unitary. There are no simple general expressions for the lowest order terms in the final state as certain channels, such as the amplitude damping channel map the identity in a non-trivial way.

Similarly the SLD and QFI can be expressed as power series, possibly with infinitely many terms, in $r$. Thus

$$\hat{L}(\lambda) = \sum_{j=0}^\infty r^j \hat{L}^{(j)}(\lambda),$$
where $\hat{L}^{(j)}(\lambda)$ is an operator independent of $r$, and

$$H = \sum_{j=0}^{\infty} r^j H^{(j)}, \quad (14)$$

where $H^{(j)}$ is independent of $r$. The operators $\hat{L}^{(j)}(\lambda)$ can be evaluated by substituting from Eqs. (10) and (13) into Eq. (5). The result must be true for all values of $r$ and comparing terms order by order gives

$$\frac{\partial \hat{p}^{(k)}_{\lambda}}{\partial \lambda} = \frac{1}{2} \sum_{j=0}^{k} \left( \Delta (k-j) \hat{p}^{(j)}_{\lambda} + \hat{p}^{(j)}_{\lambda} \hat{L}^{(k-j)} \right). \quad (15)$$

Similarly substituting from Eqs. (10) and (13) into Eq (7) gives

$$H^{(j)} = \sum_{k=0}^{\infty} \text{Tr} \left[ \frac{\partial \hat{p}^{(j-k)}_{\lambda}}{\partial \lambda} \hat{L}^{(k)} \right]. \quad (16)$$

This allows for an iterative calculation of the QFI in increasing orders of the purity parameter; for sufficiently low purities, the QFI can be approximated by truncation.

It is also useful to determine series expressions for the eigenstates of the SLD in order to assess possible measurements that saturate the quantum CRB. Denote the normalized eigenstate of the SLD by $|\phi\rangle$ and the associated eigenvalue by $\mu$. Again these can be expanded as power series in $r$, giving

$$|\phi\rangle = \sum_{j=0}^{\infty} r^j |\phi^{(j)}\rangle \quad (17)$$

and

$$\mu = \sum_{j=0}^{\infty} r^j \mu^{(j)}. \quad (18)$$

The normalization condition $\langle \phi | \phi \rangle = 1$ must hold for all $r$ and implies that $\langle \phi^{(0)} | \phi^{(0)} \rangle = 1$. Then order-by-order comparison of terms in $\hat{L} |\phi\rangle = \mu |\phi\rangle$ gives for each $k = 0, 1, \ldots,$

$$\sum_{j=0}^{k} \hat{L}^{(k-j)} |\phi^{(j)}\rangle = \sum_{j=0}^{k} \mu^{(k-j)} |\phi^{(j)}\rangle. \quad (19)$$

This yields an iterative scheme for determining the eigenstates of the SLD and hence one possible projective measurement which saturates the quantum CRB.

### A. Unital channels

A unital channel maps $\hat{1} \rightarrow \hat{1}$ and here, $\hat{p}^{(0)}_{\lambda} = \hat{1} \otimes n / N$. Repeatedly using Eq. (15) gives a convenient iterative procedure, resulting in

$$\hat{L}^{(0)} = 0, \quad (20a)$$

$$\hat{L}^{(1)} = N \frac{\partial \hat{p}^{(1)}_{\lambda}}{\partial \lambda}, \quad (20b)$$

$$\hat{L}^{(2)} = N \frac{\partial \hat{p}^{(2)}_{\lambda}}{\partial \lambda} - \frac{N^2}{2} \left\{ \frac{\partial \hat{p}^{(1)}_{\lambda}}{\partial \lambda}, \hat{p}^{(1)}_{\lambda} \right\}. \quad (20c)$$

where $\{ \},$ indicates the anti-commutator. Thus, for unital channels, Eq. (16) yields

$$H^{(0)} = 0, \quad (21a)$$

$$H^{(1)} = 0, \quad (21b)$$

$$H^{(2)} = N \text{Tr} \left[ \left( \frac{\partial \hat{p}^{(1)}_{\lambda}}{\partial \lambda} \right)^2 \right]. \quad (21c)$$

This immediately establishes the result, found earlier [18, 20, 21] for the qubit phase-shift, phase-flip and depolarizing channels, that the lowest order terms for the QFI are second order in the purity, provided that the preparation step consists of unitary operations only.

Additionally Eq. (19) allows for computation of the eigenstates of the SLD via

$$\hat{L}^{(0)} |\phi^{(0)}\rangle = \mu^{(0)} |\phi^{(0)}\rangle = \mu^{(0)} |\phi^{(0)}\rangle + \mu^{(1)} |\phi^{(0)}\rangle \quad (22a)$$

$$\hat{L}^{(1)} |\phi^{(0)}\rangle + \hat{L}^{(0)} |\phi^{(1)}\rangle = \mu^{(0)} |\phi^{(1)}\rangle + \mu^{(1)} |\phi^{(0)}\rangle \quad (22b)$$

Since $|\phi^{(0)}\rangle \neq 0$, but $\hat{L}^{(0)} = 0$, the first gives $\mu^{(0)}$, leaving

$$\hat{L}^{(1)} |\phi^{(0)}\rangle = \mu^{(1)} |\phi^{(0)}\rangle = \frac{\partial \hat{p}^{(1)}_{\lambda}}{\partial \lambda} |\phi^{(0)}\rangle. \quad (23)$$

Thus, to lowest order in the purity, a measurement that suffices to saturate the quantum CRB is one which is done in the eigenbasis of $\frac{\partial \hat{p}^{(1)}_{\lambda}}{\partial \lambda}$.

### IV. SINGLE-QUBIT, SINGLE-CHANNEL PROTOCOLS

A baseline against which to compare any metrology protocol is one where the channel is invoked once on a single qubit. This is called the single-qubit, single-channel (SQSC) protocol and is illustrated in Fig. 2a).

\[\begin{array}{c}
\begin{array}{c}
\hat{\rho}_0 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hat{\Gamma} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hat{\rho}_0 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hat{U}_{\text{prep}} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hat{f} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hat{\rho}_0 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\end{array}\]

**FIG. 2.** Single channel qubit metrology protocols. a) The SQSC protocol with a single channel invocation on a single qubit. b) An entanglement-assisted protocol using $n$ qubits with the channel invoked once on one of them.

The main results of this article will be described in terms of the Bloch-sphere mapping of Eq. (2). The analysis depends on whether the channel is unital nor not.
A. SQSC protocols for unital channels

For a single qubit unital channel \( \hat{\rho}_i^{(1)} := \tilde{\rho}_0 \cdot \hat{\sigma}/2 \), giving \( \hat{\rho}_i^{(1)} = M \tilde{\rho}_0 \cdot \hat{\sigma}/2 \). This and Eqs. (20) and (21) imply that
\[
H = r^2 \tilde{\rho}_0^T M^T M \tilde{\rho}_0 + O(r^3)
\]  
(24)
where the dot indicates the derivative with respect to the parameter. Thus to lowest order in purity, \( H = r^2 \tilde{\rho}_0^T M^T M \tilde{\rho}_0 \). This forms a general result for SQSC protocols for unital channels.

Further analysis, all to lowest order only in the QFI, uses the singular value decomposition for real matrices. Here \( M = ASB \) where \( A \) and \( B \) are each orthogonal \( 3 \times 3 \) matrices and \( S = s_1 P_1 + s_2 P_2 + s_3 P_3 \) is a diagonal matrix with positive entries arranged so that \( s_1 \geq s_2 \geq s_3 \); here \( \{ P_i \} \) are projectors onto each of the three orthogonal directions associated with unit vectors \( \{ \hat{e}_i \} \), which are a permutation of \( \hat{x}, \hat{y} \) and \( \hat{z} \). The orthogonality of \( A \) and projective nature of \( P_i \) implies that
\[
H = r^2 \sum_{i=1}^3 s_i^2 \tilde{\rho}_0^T B^T P_i B \tilde{\rho}_0.
\]
(25)
Now \( \tilde{\rho}_0^T B^T P_i B \tilde{\rho}_0 \geq 0 \) and \( \sum_{i=1}^3 \tilde{\rho}_0^T B^T P_i B \tilde{\rho}_0 = 1 \) implies that the \textit{optimal lowest order} SQSC protocol QFI is
\[
H_{s \text{ opt}} = r^2 s_1^2.
\]
(26)
This is attained with \( \tilde{\rho}_0 = B^T \hat{e}_1 \) where \( \hat{e}_1 \) is the unit vector associated with the maximum singular value in \( S \). Note that, depending on the singular value decomposition this might depend on the parameter to be estimated.

One measurement which can saturate the quantum CRB bound is a projective measurement onto the eigenbasis of \( \frac{\partial \hat{\rho}_i^{(1)}}{\partial \lambda} \). Here, for the optimal choice of input state,
\[
\frac{\partial \hat{\rho}_i^{(1)}}{\partial \lambda} = \frac{1}{2} M B^T \hat{e}_1 \cdot \hat{\sigma} = \frac{1}{2} AS \hat{e}_1 \cdot \hat{\sigma}
\]
(27)
and the resulting projective measurement operators are
\[
\hat{P}_\pm := \frac{1}{2} \left[ \hat{I} \mp A \hat{e}_1 \cdot \hat{\sigma} \right].
\]
(28)
Whenever the direction of \( A \hat{e}_1 \) depends on the parameter, these projectors will also depend on the parameter to be estimated and adaptive measurement schemes [28] must be invoked to attain the QFI. But if the direction of \( A \hat{e}_1 \) is independent of the parameter, then the method described here will yield a parameter-independent saturating measurement.

To summarize, with a unital channel subject to the SQSC protocol, the optimal QFI to lowest order in the purity is determined by finding the Bloch-sphere matrix \( M \) that represents the channel action and determining the singular value decomposition, \( M = ASB \). The optimal QFI depends only on the maximal singular value \( s_1 \) and the protocol which attains this is to prepare the input state along the Bloch sphere direction \( B^T \hat{e}_1 \), where \( \hat{e}_1 \) is the direction associated with the maximal singular value in \( S \), and then subject the qubit to the channel. One measurement that saturates the QCRB is a projection onto the Bloch-sphere direction \( A \hat{e}_1 \).

\textit{Example: Unitary phase shift.} The unitary phase shift about the \( z \) axis through angle \( \lambda \), is represented by \( \hat{\rho}_i \mapsto \hat{\rho}_i = U^\dagger \hat{\rho}_i U \) where \( U = e^{-i\lambda \hat{\sigma}_z/2} \). The Bloch-sphere matrix is
\[
M = \begin{pmatrix}
\cos \lambda & -\sin \lambda & 0 \\
\sin \lambda & \cos \lambda & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
(29)
Then
\[
\dot{M} = \begin{pmatrix}
-\sin \lambda & -\cos \lambda & 0 \\
\cos \lambda & -\sin \lambda & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
(30)
which gives \( S = \text{diag}(1,1,0) \) with various possibilities for \( A \) and \( B \). The vector associated with the maximal singular value is any unit vector in the the \( xy \) plane. This gives an optimal lowest order QFI of \( H_{s \text{ opt}} = r^2 \). The optimal QFI is attained using a state with Bloch-sphere input direction in the \( xy \) plane, for example \( B^T \hat{\hat{x}} \).

The saturating measurement of Eq (28) is a projective measurement along the direction \( A \hat{e}_1 \). It is not possible that both the choice of initial Bloch-sphere direction and measurement direction can both be independent of the parameter; this is consistent with exact calculations.

\textit{Example: Phase-flip channel.} The phase-flip channel maps \( \hat{\rho}_i \mapsto \hat{\rho}_i = (1-\lambda) \hat{\rho}_i + \lambda \hat{\sigma}_z \hat{\rho}_i \hat{\sigma}_z \). Here \( M = \text{diag}(1-2\lambda, 1-2\lambda, 1) \) and \( M = \text{diag}(-2, -2, 0) \) so that \( S = \text{diag}(2, 2, 0) \) with \( A = \text{diag}(-1, -1, 0) \) and \( B = \hat{I} \) as one possibility. This gives an optimal lowest order QFI is \( H_{s \text{ opt}} = 4r^2 \), attained when the initial-state Bloch-sphere direction is in the \( xy \) plane. This is consistent with approximations from the exact QFI [20]. A saturating measurement of Eq (28) is a projection along the direction \( \tilde{\rho}_0 \) and is parameter independent.

\textit{Example: Depolarizing channel.} The depolarizing channel maps \( \hat{\rho}_i \mapsto \hat{\rho}_i = (1-\lambda) \text{Tr} [\hat{\rho}_i] \hat{I} + \lambda \hat{\rho}_i \) and \( M = \lambda \hat{I} \) with \( \dot{M} = \lambda \hat{I} \). This indicates that the optimal lowest order QFI is \( H_{s \text{ opt}} = r^2 \) and this is attained regardless of the choice of initial-state vector. This is consistent with approximations for the exact QFI [21]. Again a saturating measurement from the SLD is parameter independent.

B. SQSC protocols for non-unital channels

For the more general non-unital channel acting on a single qubit \( I \xrightarrow{\gamma} I + d \cdot \hat{\sigma} \) and thus
\[
\hat{\rho}_i^{(0)} = \frac{1}{2} \left( I + d \cdot \hat{\sigma} \right).
\]
(31)
The resulting analysis, again all to lowest order in the purity, depends on whether \( \mathbf{d} \) is parameter dependent. In either case, the lowest order version of Eq. (15) gives
\[
\tilde{\mathcal{L}}^{(0)} = \frac{1}{2} \partial \ln (1 - d^2) \hat{I} + \left[ \mathbf{d} - \frac{1}{2} \partial \ln (1 - d^2) \right] \cdot \hat{\mathbf{\sigma}}.
\]
(32)
where \( d := |\mathbf{d}| \neq 1 \) and
\[
\tilde{\mathcal{L}}^{(0)} = \mathbf{d} \cdot \hat{\mathbf{\sigma}}
\]
(33)
if \( d = 1 \).

Direct substitution into the lowest order version of Eq. (16) yields,
\[
H_{s \text{ opt}} = \begin{cases} \mathbf{d} \cdot \mathbf{d} + \frac{1}{4(1 - d^2)} \left[ \partial d^2 \right] \mathbf{d} \cdot \hat{\mathbf{\sigma}}^2 & \text{if } d \neq 1, \\ \mathbf{d} \cdot \mathbf{d} & \text{if } d = 1. \end{cases}
\]
(34)
If \( \mathbf{d} \) is parameter dependent then these are the optimal lowest order QFIs for non-unital SQSC protocols. A key feature of such channels is that to lowest order in the purity, the QFI is independent of \( r \) and this could be attained by an input state with zero purity. A sufficient measurement that would attain this is a projective measurement onto the eigenbasis of the lowest order score operator, \( \tilde{\mathcal{L}}^{(0)} \), and is thus a measurement along the Bloch sphere direction determined by \( \mathbf{d} \) (if \( d = 1 \) or \( \mathbf{d} + \frac{\partial \ln (1 - d^2)}{\partial \lambda} \mathbf{d} \) (if \( d \neq 1 \)).

If \( \mathbf{d} \) is parameter independent then \( \tilde{\mathcal{L}}^{(0)} = 0 \) and \( \tilde{\mathcal{L}}^{(0)} = 0 \) and, as before, \( H^{(0)} = H^{(1)} = 0 \). The next order term is attain via Eq. (15), which with Eq. (31) gives
\[
\tilde{\mathcal{L}}^{(1)} = \frac{1}{2} \left[ \tilde{\mathcal{L}}^{(1)} \hat{\mathcal{L}}^{(0)} + \hat{\mathcal{L}}^{(0)} \tilde{\mathcal{L}}^{(1)} \right] = \frac{1}{2} \tilde{\mathcal{L}}^{(1)} + \frac{1}{4} \left[ \hat{\mathcal{L}}^{(1)} \mathbf{d} \cdot \hat{\mathbf{\sigma}} + \mathbf{d} \cdot \hat{\sigma} \tilde{\mathcal{L}}^{(1)} \right]
\]
(35)
If \( d \neq 1 \), then the solution to this, which can be verified by direct substitution, is
\[
\tilde{\mathcal{L}}^{(1)} = \frac{2 - d^2}{1 - d^2} \tilde{\mathcal{L}}^{(1)} - \frac{1}{1 - d^2} \left[ \tilde{\mathcal{L}}^{(1)} \mathbf{d} \cdot \hat{\mathbf{\sigma}} + \mathbf{d} \cdot \hat{\sigma} \tilde{\mathcal{L}}^{(1)} \right] + \frac{1}{1 - d^2} \left[ \mathbf{d} \cdot \hat{\sigma} \tilde{\mathcal{L}}^{(1)} \mathbf{d} \cdot \hat{\mathbf{\sigma}} \right].
\]
(36)
The remaining case where \( \mathbf{d} \) is a parameter-independent unit vector requires \( M = 0 \) and this leaves no parameter dependence. We can ignore this. Eqs. (14) and (36) give that if \( \mathbf{d} \) is parameter independent, then to lowest order in the purity,
\[
H = \frac{2 - d^2}{1 - d^2} \text{Tr} \left[ \left( \tilde{\mathcal{L}}^{(1)} \right)^2 \right] - \frac{2}{1 - d^2} \text{Tr} \left[ \left( \tilde{\mathcal{L}}^{(1)} \right)^2 \mathbf{d} \cdot \hat{\mathbf{\sigma}} \right] + \frac{1}{1 - d^2} \text{Tr} \left[ \mathbf{d} \cdot \hat{\mathbf{\sigma}} \tilde{\mathcal{L}}^{(1)} \mathbf{d} \cdot \hat{\mathbf{\sigma}} \right].
\]
(37)
Then \( \hat{\rho}_t^{(1)} = M \hat{r}_0 \cdot \hat{\mathbf{\sigma}}/2 \) gives
\[
H = \hat{r}_0^\dagger M^\dagger M \hat{r}_0 + \frac{1}{1 - d^2} \left( \mathbf{d}^\dagger M \hat{r}_0 \right)^2.
\]
(38)
Then
\[
H = \hat{r}_0^\dagger \left[ M^\dagger M + \frac{d^2}{1 - d^2} M^\dagger P_d M \right] \hat{r}_0,
\]
(39)
where \( P_d \) is the projector onto the direction \( \hat{\mathbf{d}} \). The entire operator within braces is positive and a singular value decomposition of this will eventually yield the optimal lowest order QFI and initial Bloch-sphere direction. Note that this indicates that channels with nonzero constant \( \mathbf{d} \) will typically enhance the estimation accuracy by effectively increasing the purity of the state.

Example: Generalized amplitude damping The generalized amplitude damping channel maps \( \hat{\rho}_t \mapsto \hat{\rho}_t = \sum_{i=1}^4 E_i^\dagger \hat{\rho}_t E_i \) where
\[
E_1 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \lambda} \end{pmatrix}, \quad E_2 = \sqrt{p} \begin{pmatrix} 0 & \sqrt{\lambda} \\ 0 & 0 \end{pmatrix}, \quad E_3 = \sqrt{1 - p} \begin{pmatrix} \sqrt{1 - \lambda} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_4 = \sqrt{1 - p} \begin{pmatrix} 0 & 0 \\ \sqrt{\lambda} & 0 \end{pmatrix}
\]
(40)
with \( 0 \leq p \leq 1 \).

Then [22], \( M = \text{diag} (\sqrt{1 - \lambda}, \sqrt{1 - \lambda}, 1 - \lambda) \) and \( \mathbf{d} = \lambda(2p - 1) \hat{\mathbf{d}} \). This yields \( H_{s \text{ opt}} = 1/[1 - \lambda^2(2p - 1)^2] \). The optimal measurement that saturates the quantum CRB bound is a projective measurement along \( \hat{\mathbf{d}} \).

Compiling these results gives in a complete characterization of the lowest order QFI terms for SQSC protocols for all channels: Eq. (26) for unital channels, Eq. (34) for non-unital channels with a parameter-dependent Bloch-sphere shift vector and Eq (39) for non-unital channels with a parameter-independent shift.

V. SYMMETRIC PAIRWISE CORRELATED PROTOCOLS

The central question is whether there is an entanglement-assisted protocol which can yield a larger QFI per channel invocation. Previous results for parameter estimation for the phase-shift, phase-flip and depolarizing channel showed that this is possible for a particular correlating preparatory unitary [18, 20, 21]. We consider a generalization of this for any qubit channel. Here we consider the case where the channel is only invoked once.

The previous correlated state parameter estimation protocol assumed subjected all qubits, each initially in the same state, to a succession of controlled-Z gates acting on each distinct pair of qubits followed by a
the input state to the channels are $\hat{\sigma}$, and in many physical settings these basic two qubit gates scale quadratically in the total number of qubits and in such physical settings these gates are relatively easily constructed. For example, in solution-state NMR implementations of quantum information processing they have been implemented experimentally since the outset of that field [37–39].

Under this preparatory unitary, the two lowest order terms in the input state to the channels are $\rho_0^{(0)} = I^\otimes n/N$ and $\rho_1^{(1)} = U_{\text{prep}} \rho_0^{(0)} U_{\text{prep}}^\dagger$; these will be sufficient for determining the lowest order terms in the QFI. The crucial result for determining $\rho_1^{(1)}$ is

\begin{equation}
U_{\hat{c}}\left(\hat{r}_0 \cdot \hat{\sigma} \otimes \hat{I}\right) U_{\hat{c}} = \hat{r}_0 \cdot \hat{\sigma} \otimes \hat{\sigma} + \hat{r}_0 \cdot \hat{c} \left(\hat{\sigma} \otimes \hat{I} - \hat{\sigma} \otimes \hat{\sigma}\right). \tag{42}
\end{equation}

To demonstrate this, decompose the initial-state vector as $\hat{r}_0 = r_0 c \hat{c} + r_0^\perp$ where $r_0 c = \hat{r}_0 \cdot \hat{c}$ is the component of $\hat{r}_0$ along $\hat{c}$ and $r_0^\perp = \hat{r}_0 - (\hat{r}_0 \cdot \hat{c}) \hat{c}$ is the component perpendicular to $\hat{r}_0$. Then

\begin{align*}
U_{\hat{c}}\left(\hat{r}_0 \cdot \hat{\sigma} \otimes \hat{I}\right) U_{\hat{c}} &= r_0 c U_{\hat{c}}\left(\hat{\sigma} \otimes \hat{I}\right) U_{\hat{c}} + \\
&\quad + U_{\hat{c}}\left(r_0^\perp \cdot \hat{\sigma} \otimes \hat{I}\right) U_{\hat{c}} \\
&= r_0 c \hat{\sigma} \otimes \hat{I} + \\
&\quad + U_{\hat{c}}\left(r_0^\perp \cdot \hat{\sigma} \otimes \hat{I}\right) U_{\hat{c}} \tag{43}
\end{align*}

where we have used the facts that $U_{\hat{c}}$ and $\hat{\sigma} \otimes \hat{I}$ commute and that $U_{\hat{c}}$ is unitary. Now

\begin{align*}
U_{\hat{c}}\left(r_0^\perp \cdot \hat{\sigma} \otimes \hat{I}\right) U_{\hat{c}} &= \frac{1}{2} U_{\hat{c}}\left(r_0^\perp \cdot \hat{\sigma} \otimes \hat{I}\right) \left(\hat{I} \otimes \hat{I} + \\
&\quad + \hat{I} \otimes \hat{\sigma} + \hat{\sigma} \otimes \hat{I} - \hat{\sigma} \otimes \hat{\sigma}\right) \\
&= \frac{1}{2} U_{\hat{c}}\left(\hat{I} \otimes \hat{I} + \hat{I} \otimes \hat{\sigma} - \hat{\sigma} \otimes \hat{I} + \hat{\sigma} \otimes \hat{\sigma}\right) \\
&= \frac{1}{2} U_{\hat{c}}\left(\hat{I} \otimes \hat{I} + \hat{I} \otimes \hat{\sigma} - \hat{\sigma} \otimes \hat{I} + \hat{\sigma} \otimes \hat{\sigma}\right) \\
&= r_0 c \hat{\sigma} \otimes \hat{\sigma} + r_0^\perp \cdot \hat{\sigma} \otimes \hat{\sigma} \tag{44}
\end{align*}

since $\hat{r}_0$ and $r_0^\perp$ are perpendicular and thus $r_0^\perp \cdot \hat{\sigma}$ and $\hat{\sigma} \otimes \hat{\sigma}$ anticommute. Then

\begin{align*}
U_{\hat{c}}\left(\hat{I} \otimes \hat{I} + \hat{I} \otimes \hat{\sigma} - \hat{\sigma} \otimes \hat{I} + \hat{\sigma} \otimes \hat{\sigma}\right) &= \hat{I} \otimes \hat{\sigma}, \\
\text{yields } U_{\hat{c}}\left(\hat{r}_0 \cdot \hat{\sigma} \otimes \hat{I}\right) &= r_0 c \hat{\sigma} \otimes \hat{I} + r_0^\perp \cdot \hat{\sigma} \otimes \hat{\sigma} \\
\text{and finally substitution using } r_0 c = \hat{r}_0 \cdot \hat{c} \text{ and } U_{\hat{c}} \text{ acting on the leftmost and center qubits produces}
\end{align*}

\begin{align*}
\hat{r}_0 \cdot \hat{\sigma} \otimes \hat{I} \\ + (\hat{r}_0 \cdot \hat{c}) \hat{\sigma} \otimes \hat{I} \\
- (\hat{r}_0 \cdot \hat{c}) \hat{\sigma} \otimes \hat{\sigma} \otimes \hat{I}. \tag{45}
\end{align*}

Then $U_{\hat{c}}$ acting on the leftmost and rightmost qubits leaves the second and third terms in Eq. (45) unaltered since it commutes with this. It acts on the first term to produce

\begin{align*}
\hat{r}_0 \cdot \hat{\sigma} \otimes \hat{I} + (\hat{r}_0 \cdot \hat{c}) \hat{\sigma} \otimes \hat{I} \\
- (\hat{r}_0 \cdot \hat{c}) \hat{\sigma} \otimes \hat{\sigma} \otimes \hat{I} + (\hat{r}_0 \cdot \hat{c}) \hat{\sigma} \otimes \hat{\sigma} \otimes \hat{I}. \tag{46}
\end{align*}

Substituting gives that, under the preparatory unitary,

\begin{align*}
\hat{r}_0 \cdot \hat{\sigma} \otimes \hat{I} + (\hat{r}_0 \cdot \hat{c}) \hat{\sigma} \otimes \hat{I} \\
- (\hat{r}_0 \cdot \hat{c}) \hat{\sigma} \otimes \hat{\sigma} \otimes \hat{I}. \tag{47}
\end{align*}

FIG. 3. The preparatory unitary for a general symmetric pairwise correlated scheme considered in this article. The symbols within the blue dashed frame represent a single iteration of $U_{\hat{c}}$; the boxes indicate the two qubits on which the gate acts.
Extending this to arbitrary numbers of qubits yields
\[ \hat{U}_{\text{prep}} (\hat{r}_0 \cdot \hat{\sigma} \otimes \hat{I}^{\otimes (n-1)}) \hat{U}_{\text{prep}} = \hat{r}_0 \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes n} \]
\[ + (\hat{r}_0 \cdot \hat{\epsilon}) \hat{c}_e \otimes \hat{I}^{\otimes (n-1)} \]
\[ - (\hat{r}_0 \cdot \hat{\epsilon}) \hat{c}_e^{\otimes n}. \] (48)

Applied to each term in $\hat{\rho}_1^{(1)}$ this gives
\[ \hat{\rho}_1^{(1)} = \frac{1}{N} \left( \hat{r}_0 \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes (n-1)} + \cdots + \hat{c}_e^{\otimes (n-1)} \otimes \hat{r}_0 \cdot \hat{\sigma} \right) \]
\[ + \frac{\hat{r}_0 \cdot \hat{\epsilon}}{N} \left( \hat{c}_e \otimes \hat{I}^{\otimes (n-1)} + \cdots + \hat{I}^{\otimes (n-1)} \otimes \hat{c}_e \right) \]
\[ - \frac{n \hat{r}_0 \cdot \hat{\epsilon}}{N} \hat{c}_e^{\otimes n}. \] (49)

Note that within the first parentheses, there are $n$ distinct terms, each containing a single factor of $\hat{r}_0 \cdot \hat{\sigma}$. Similarly within the second parentheses there are also $n$ terms, each containing a single factor of $\hat{c}_e$.

The effects of invoking the channel on multiple qubits prepared via the symmetric pairwise correlated protocol could be assessed by using the Bloch-sphere mappings of Eqs. (2). It is known that for the particular cases of the phase-flip [20] and depolarizing channels [21] such correlated state protocols do not yield advantages for all parameter values when there are more than two channel invocations. On the other hand, when there is only one channel invocation, these protocols definitely are advantageous over all parameter values for the phase-flip channel [20] and are probably so for the depolarizing channel [21]. Thus, we only consider the situation where the channel is invoked once and assess whether the remaining ancilla qubits assist in the parameter estimation.

Furthermore the analysis varies depending on whether the channel is unital or not and we now consider these cases separately.

### A. Symmetric pairwise correlated protocol for unital channels

Assume that a unital channel acts once on the leftmost qubit in the tensor product representation. Then the terms in the first order term in the input state of Eq. (49) are mapped by the channel as
\[ \hat{r}_0 \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes (n-1)} \rightarrow (M \hat{r}_0) \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes (n-1)} \]
\[ \hat{c}_e \otimes \hat{r}_0 \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes (n-2)} \rightarrow (M \hat{c}_e) \cdot \hat{\sigma} \otimes \hat{r}_0 \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes (n-2)} \]
\[ \hat{c}_e \otimes \hat{I}^{\otimes (n-1)} \rightarrow (M \hat{c}_e) \cdot \hat{\sigma} \otimes \hat{I}^{\otimes (n-1)} \]
\[ \hat{I} \otimes \hat{c}_e \otimes \hat{I}^{\otimes (n-2)} \rightarrow \hat{I} \otimes \hat{c}_e \otimes \hat{I}^{\otimes (n-2)} \]
\[ \hat{c}_e \otimes \hat{c}_e^{\otimes (n-1)} \rightarrow (M \hat{c}_e) \otimes \hat{c}_e^{\otimes (n-1)} \] (50)

where $M$ is the Bloch-sphere matrix for the channel. Computing the lowest order term in the QFI requires
\[ \hat{\rho}_1^{(1)} = \frac{1}{N} \left[ (M \hat{r}_0) \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes (n-1)} \right. \]
\[ + (M \hat{c}_e) \cdot \hat{\sigma} \otimes \hat{r}_0 \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes (n-2)} \]
\[ + \cdots + (M \hat{c}_e) \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes (n-2)} \otimes \hat{r}_0 \cdot \hat{\sigma} \]
\[ + \frac{\hat{r}_0 \cdot \hat{\epsilon}}{N} \left( M \hat{c}_e \cdot \hat{\sigma} \otimes \hat{I}^{\otimes (n-1)} \right. \]
\[ - \frac{n \hat{r}_0 \cdot \hat{\epsilon}}{N} (M \hat{c}_e \cdot \hat{\sigma} \otimes \hat{c}_e^{\otimes (n-1)}). \] (51)

This yields our main result (see Appendix A for a proof) for unital channels: if the channel is invoked once on a single qubit when the symmetric pairwise correlated protocol is used then to the lowest order in the QFI (remaining analysis is all to lowest order)
\[ H = r^2 \hat{r}_0^\top \left[ (I - P_e) M^\top M (I - P_e) \right. \]
\[ + (2 - n) P_e M^\top M P_e \]
\[ + (n - 1) \hat{c}_e^\top M^\top M \hat{c}_e \] (52)

where $I$ is the $3 \times 3$ identity matrix and $P_e$ is the projector onto the control direction vector $\hat{c}_e$.

For a given channel and Bloch-sphere matrix, $M$, there remains the task of choosing the control direction vector $\hat{c}_e$ and initial-state Bloch-sphere vector $\hat{r}_0$ so as to maximize the QFI of Eq. (52). Again, this can be analyzed in terms of the singular value decomposition $M = A S B$, with $S = \sum_i s_i P_i$ arranged so that $s_1 \geq s_2 \geq s_3 \geq 0$ and where $P_i$ is a projector onto the unit vector $\hat{c}_i$ and each of these is one of $\hat{x}, \hat{y}$ and $\hat{z}$. Then
\[ H = \sum_i s_i^2 \left\{ \hat{r}_0^\top (I - P_e) P_i (I - P_e) \hat{r}_0 \right. \]
\[ + (n - 1) \hat{c}_i^\top P_i \hat{c}_i - (n - 2) \hat{r}_0^\top P_e P_i P_e \hat{r}_0 \} \] (53)

We show that it is possible to choose $\hat{r}_0$ and $\hat{c}_e$ so that
\[ n s_1^2 - s_2^2 \left( 1 - \frac{s_2^2}{s_1^2} \right) \leq H \leq n s_1^2. \] (54)

The lower bound can be established by the particular choice of $\hat{c}_e = B^\top \hat{e}_2$ and $\hat{r}_0 = B^\top \hat{e}_2$. The upper bound can be established by noting that, since $s_1^2 \geq s_2^2 \geq s_3^2 \geq 0$,
\[ H \leq s_1^2 \sum_i \left\{ \hat{r}_0^\top (I - P_e) P_i (I - P_e) \hat{r}_0 \right. \]
\[ + (n - 1) \hat{c}_e^\top P_i \hat{c}_e - (n - 2) \hat{r}_0^\top P_e P_i P_e \hat{r}_0 \} \] (55)

Then the facts that $\sum_i P_i = I$, and $\hat{c}, \hat{r}_0$ are unit vectors and $P_e, I - P_e$ are projectors give
\[ H \leq s_1^2 \left[ n - (n - 1) \hat{r}_0^\top P_e \hat{r}_0 \right]. \] (56)
The left side attains a maximum of \( n s_1^2 \) when \( \hat{c} \) and \( \hat{r}_0 \) are perpendicular. This proves the result for the upper bound. It also implies that for the upper bound to be saturated the initial state Bloch-sphere vector direction and control direction must be perpendicular. But it does not guarantee that the upper bound can be saturated and, if not, it makes no statement about the directions of these vectors in order to attain the maximum QFI.

Equations (26) and (54) allow for comparison of the symmetric pairwise correlated protocol against the SQSC protocol to lowest order for unital channels. Here

\[
\left[ n - \left( 1 - \frac{s_2^2}{s_1^2} \right) \right] H_s \lesssim H_{\text{corr opt}} \lesssim n H_s \quad (57)
\]

where \( H_{\text{corr opt}} \) is the optimal QFI for the correlated protocol over all choices of \( \hat{c} \) and \( \hat{r}_0 \). Thus

\[
n - \left( 1 - \frac{s_2^2}{s_1^2} \right) \lesssim \frac{H_{\text{corr opt}}}{H_s} \lesssim n. \quad (58)
\]

Since \( s_2 \leq s_1 \) this means that for large \( n \) and to lowest order in the purity, the symmetric pairwise correlated protocol roughly gives an \( n \)-fold gain over the SQSC protocol for any unital channel.

Sometimes a precise statement about the optimal QFI for this correlated state protocol can be made. If \( s_1 = s_2 \), as is true for several common considered channels, the the two bounds of Eq. (54) are identical and \( H_{\text{corr opt}} = n H_s \), which is attained when \( \hat{c} = B^\dagger \hat{e}_1 \) and \( \hat{r}_0 = B^\dagger \hat{e}_2 \). As another example, if \( s_2 = s_3 = 0 \), the analysis of Appendix B shows that \( H_{\text{corr opt}} = (n - 1) H_s \) and this is attained when \( \hat{\epsilon}, \hat{r}_0 \) and \( \hat{e}_1 \) are all perpendicular.

The remaining issue with this optimal protocol is to find a QRB saturating measurement. A projective measure in the eigenbasis of \( \rho_{\text{opt}}^{(1)} \) suffices. For the optimal symmetric pairwise correlated protocol,

\[
\rho_{\text{opt}}^{(1)} = \frac{1}{N} \left[ (\hat{M} \hat{r}_0) \cdot \hat{\sigma} \otimes \hat{\sigma}_c^{\otimes(n - 1)} + (\hat{M} \hat{c}) \cdot \hat{\sigma} \otimes \hat{r}_0 \cdot \hat{\sigma}_c^{\otimes(n - 2)} + \cdots + (\hat{M} \hat{c}) \cdot \hat{\sigma} \otimes \hat{r}_0 \cdot \hat{\sigma}_c^{\otimes(n - 2)} \otimes \hat{r}_0 \cdot \hat{\sigma} \right]. \quad (59)
\]

While it is always possible to find the useful eigenbasis, sometimes this will depend on the parameter value, thus suggesting a measurement that would require knowledge of the parameter; this is a general issue which has been addressed elsewhere [27, 28]. Although the matrix \( M \) may depend on the parameter, this does not necessarily imply that it the resulting eigenstates do. We assess this for various important channels.

\textbf{Example: Phase shift} For the phase shift channel about the \( z \) axis, \( s_1 = s_2 = 1, s_3 = 0 \) and by the bounds of Eq. (54), this gives an optimal QFI, \( H_{\text{corr opt}} = n H_s \). Taking \( B = I \) in the singular value decomposition, the choices \( \hat{c} = \hat{e}_1 \) and \( \hat{r}_0 = \hat{e}_2 \) attain this. Then \( \hat{M} \hat{c} = -\sin \lambda \hat{e}_1 + \cos \lambda \hat{e}_2 \) and \( \hat{M} \hat{r}_0 = -\cos \lambda \hat{e}_1 - \sin \lambda \hat{e}_2 \). Thus the eigenstates will depend on the parameter and the condition for a saturating measurement based on the SLD does not yield a parameter-independent measurement.

\textbf{Example: Phase flip} For the phase flip \( s_1 = s_2 = -2, s_3 = 0 \) and this gives an optimal QFI, \( H_{\text{corr opt}} = n H_s \). This is attained when \( \hat{c} = \hat{e}_1 \) and \( \hat{r}_0 = \hat{e}_2 \). Here \( \hat{M} \hat{c} = -2 \hat{e}_1 \) and \( \hat{M} \hat{r}_0 = -2 \hat{e}_2 \). Thus the SLD measurement that saturates the bound is independent of the parameter. These results agree with lowest order approximations from exact expressions for all purities [20].

\textbf{Example: Depolarizing channel} For the phase flip \( s_1 = s_2 = 1, s_3 = 0 \) and this gives an optimal QFI, \( H_{\text{corr opt}} = n H_s \), which is attained when \( \hat{c} = \hat{e}_1 \) and \( \hat{r}_0 = \hat{e}_2 \). Again the SLD measurement that saturates the bound is independent of the parameter. Again these agree with lowest order approximations from exact expressions for all purities [21].

\section*{B. Symmetric pairwise correlated protocol for non-unital channels}

For non-unital channels where \( d \) depends on the parameter, the lowest non-zero term in the QFI is the zero order term. The zero order term in the density operator in the symmetric pairwise correlated state protocol will be the same as that in the SQSC protocol; this is evident from setting \( r = 0 \) in the formalism. Thus for such channels, the lowest order term in QFI in the symmetric pairwise correlated protocol is the same as that for the SQSC protocol. To lowest order there is nothing to gain from the correlated state protocol for such cases.

\section*{VI. CONCLUSION}

We have compared quantum parameter estimation protocols for qubit channels when the available states are mixed with very low purity and where the channel is invoked once. We have shown that for any unital channel, to lowest order in the purity, the particular \( n \) qubit correlated input state generated by the symmetric pairwise correlated protocol provides a roughly \( n \)-fold increase in estimation accuracy over protocols that use uncorrelated states. These results agree with approximations from exact results for the known special cases of the phase-shift [18], phase-flip [20] and depolarizing channels [21]. There still remains the issue of finding generic measurement choices that are independent of the parameter and that yield the optimal classical Fisher information. Also, the matter of whether the classical Fisher information can be attained by uncorrelated measurements immediately after the channel invocation awaits an answer.

For non-unital channels with a parameter-dependent shift, to lowest order in the purity, there is no improvement in estimation accuracy using these particular parameters; this provides a first glimpse into amplitude-damping channel parameter with non-pure initial states.
The formalism used here could be extended to situations where the channel is invoked on more than one of the qubits. However, it is already known for certain unital channels [20, 21] that these protocols only offer advantages over a subset of possible parameter values and thus the universal results will not apply. Studies into restricted regions of the parameter space, where these protocols could offer advantages would still be warranted.

The symmetric pairwise correlated state protocol considered here generalizes those used previously and for unital channels. This is some improvement over previous studies [18, 20, 21] in estimation with noisy states, but even here the scheme awaits optimization over choices of initial-state Bloch-sphere and control vector directions. It is also possible that another type of preparation scheme might be optimal. The structure of the lowest order terms in the density operator in this formalism gives some insight into the origin of the increase in the QFI. After the preparatory unitary, every qubit provides a term in the expression for the system state such that the channel has a non-trivial action on this term. This might be able to yield some insights into the origins of the accuracy enhancement and possible ways to improve it.

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Appendix A: Lowest order symmetric pairwise correlated protocol QFI

To prove the result of Eq. (52) note that it emerges from $H = r^2 H^{(2)} + O(r^3)$ and the trace operation of Eq. (21), which requires computing the trace of the square of the entire right-hand side of Eq. (51). This results in a sum of the trace of each term squared together with the traces of all “cross terms”; we evaluate and list these separately. In both cases a useful tool is that $\text{Tr}[a \cdot \hat{\sigma}(b \cdot \hat{\sigma})] = 2a^\top b.$ for any vectors $a$ and $b$. Also note that the result will contain terms of the form $(\hat{r}_0 \cdot \hat{e}) \hat{M} \hat{e}$ and this can be expressed as $\hat{M}P_\hat{e} \hat{r}_0$ where $P_\hat{e}$ is the projection operator onto $\hat{e}$.

Then the squared terms of Eq. (51) are listed in Table I. Similarly consider the “cross terms” of Eq. (51) are listed in Table II.

Adding these gives

$$H^{(2)} = N \left[ \left( \frac{\partial \hat{H}^{(1)}}{\partial \lambda} \right)^2 \right]$$

$$= \hat{r}_0 \left[ \hat{M}^\top \hat{M} + (3 - n) P_\hat{e} \hat{M}^\top M P_\hat{e} - 2 P_\hat{e} \hat{M}^\top \hat{M} \right] \hat{r}_0 + (n - 1) \hat{e}^\top \hat{M}^\top \hat{M} \hat{e}.$$  \hspace{1cm} (A1)

Note that $\hat{r}_0^\top P_\hat{e} M^\top \hat{M} \hat{r}_0 = \hat{r}_0^\top M^\top P_\hat{e} \hat{r}_0$ and thus a symmetric expression is

$$H^{(2)} = \hat{r}_0^\top \left[ M^\top \hat{M} + (3 - n) P_\hat{e} M^\top P_\hat{e} \right] \hat{r}_0$$

$$- P_\hat{e} \hat{M}^\top \hat{M} - (n - 2) \hat{e}^\top P_\hat{e} \hat{e} - (n - 1) \hat{e}^\top \hat{M}^\top \hat{M} \hat{e}.$$  \hspace{1cm} (A2)

Algebra then gives the stated result.

Appendix B: Optimal QFI for $s_2 = s_3 = 0$.

To prove the result that the optimal choice of control and initial-state directions is one where they are perpendicular, consider any fixed choice of $\hat{e}$. Then within Eq. (53) there appears the term $W := \hat{r}_0^\top (I - P_\hat{e}) P_1 (I - P_\hat{e}) \hat{r}_0 + (n - 1) \hat{e}^\top P_1 \hat{e} - (n - 2) \hat{r}_0^\top P_\hat{e} P_1 P_\hat{e} \hat{r}_0$ and we will show that this is maximized when $\hat{r}_0$ and $\hat{e}$ are perpendicular. Note that $\hat{e}$ and $\hat{e}_1$ span a plane. Then $\hat{r}_0$ can be decomposed into a vector perpendicular to the plane $\hat{r}_0^\parallel$ and a vector parallel to the plane $\hat{r}_0^\perp$. Neither of these necessarily has unit norm. Also let $\Phi$ be the angle from $\hat{e}_1$ to $\hat{e}$ and $\theta$ be the angle from $\hat{e}$ to $\hat{r}_0^\parallel$. As these three vectors lie in the same plane the angles can be chosen so that the from $\hat{e}_1$ to $\hat{r}_0^\parallel$ is $\theta + \phi$. Then vector algebra gives

$$P_1 P_\hat{e} \hat{r}_0 = r_0^\parallel \cos \theta \cos \phi \hat{e}_1 \hat{e}_1$$  \hspace{1cm} (B1a)

$$P_1 (I - P_\hat{e}) \hat{r}_0 = -r_0^\perp \sin \theta \sin \phi$$  \hspace{1cm} (B1b)
where $r_0^\parallel$ is the magnitude of $r_0^\parallel$.

Then

$$
\hat{r}_0^\top P_c P_1 P_e \hat{r}_0 = \left( r_0^\parallel \right)^2 \cos^2 \theta \cos^2 \phi \quad \text{and}
$$

$$
\hat{r}_0^\top (I - P_c) P_1 (I - P_e) \hat{r}_0 = \left( r_0^\parallel \right)^2 \sin^2 \theta \sin^2 \phi \quad \text{(B2a)}
$$

Separately $\hat{c}^\top P_1 \hat{c} = \cos^2 \phi$ Thus

$$
W = \left( r_0^\parallel \right)^2 \left[ \sin^2 \theta \sin^2 \phi + (2 - n) \cos^2 \theta \cos^2 \phi \right] + (n - 1) \cos^2 \phi
$$

$$
= \left( r_0^\parallel \right)^2 \left\{ \cos^2 \theta \left[ (3 - n) \cos^2 \phi - 1 \right] - \cos^2 \phi \right\} + (n - 1) \cos^2 \phi \quad \text{(B3)}
$$

For a given channel and choice of control direction $\hat{c}$, the variable $\phi$ is fixed and this must be optimized with respect to $\hat{r}_0$, i.e. with respect to $\theta$ and $r_0^\parallel$. Here, noting that for $n \geq 2$, the term $(3-n)\cos^2 \phi - 1 \leq 0$. This implies that the factor multiplying $(r_0^\parallel)^2$ is never positive. So the maximum for $W$ is attained when $r_0^\parallel = 0$. This gives $W = (n-1)\cos^2 \phi$ and this attains a maximum of $n-1$ when $\hat{c}$ is perpendicular to $\hat{e}_1$.

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