GROUP DIVISIBLE DESIGNS WITH BLOCK SIZE FIVE

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ABSTRACT. We report some group divisible designs with block size five, including types $6^{15}$ and $10^{15}$. As a consequence we are able to extend significantly the known spectrum for 5-GDDs of type $g^u$.

1. Introduction

For the purpose of this paper, a group divisible design, $K$-GDD, of type $g_1^{u_1}g_2^{u_2} \ldots g_r^{u_r}$ is an ordered triple $(V, G, B)$ such that:
(i) $V$ is a base set of cardinality $u_1g_1 + u_2g_2 + \cdots + u_rg_r$;
(ii) $G$ is a partition of $V$ into $u_i$ subsets of cardinality $g_i$, $i = 1, 2, \ldots, r$, called groups;
(iii) $B$ is a non-empty collection of subsets of $V$ with cardinalities $k \in K$, called blocks; and
(iv) each pair of elements from distinct groups occurs in precisely one block but no pair of elements from the same group occurs in any block.

We abbreviate $\{k\}$-GDD to $k$-GDD, and a $k$-GDD of type $q^k$ is also called a transversal design, TD($k, q$). A pairwise balanced design, $(v, K, 1)$-PBD, is a $K$-GDD of type $1^v$.

A parallel class in a group divisible design is a subset of the block set that partitions the base set. A $k$-GDD is called resolvable, and is denoted by $k$-RGDD, if the entire set of blocks can be partitioned into parallel classes. If there exist $k$ mutually orthogonal Latin squares (MOLS) of side $q$, then there exists a $(k+2)$-GDD of type $q^{k+2}$ and a $(k+1)$-RGDD of type $q^{k+1}$, [4, Theorem III.3.18]. Furthermore, as is well known, there exist $q - 1$ MOLS of side $q$ whenever $q$ is a prime power.

Because of their widespread use in design theory, especially in the construction of infinite classes of combinatorial designs by means of the technique known as Wilson’s Fundamental Construction, [17, 13, Theorem IV.2.5], group divisible designs are useful and important structures. The existence spectrum problem for group divisible designs with constant block sizes, $k$-GDDs, $k \geq 3$, appears to be a long way from being completely

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solved. Nevertheless, for $k \in \{3, 4, 5\}$ where all the groups have the same size, considerable progress has been made.

The necessary conditions for the existence of $k$-GDDs of type $g^u$, namely
\begin{align*}
u & \geq k, \\
g(u - 1) & \equiv 0 \pmod{k - 1}, \\
g^2u(u - 1) & \equiv 0 \pmod{k(k - 1)},
\end{align*}
are known to be sufficient for $k = 3$, [14], [9, Theorem IV.4.1], and for $k = 4$ except for types $24$ and $64$, [7], [9, Theorem IV.4.6]. For 5-GDDs of type $g^u$, a partial solution to the design spectrum problem has been achieved, [1, 2], [5, 6, 9, 10, 11, 14, 15, 16, 18], and for future reference, we quote the main result concerning 5-GDDs in the important paper of Wei and Ge, [16], which represents a considerable advance on [9, Theorem IV.4.16] in the Colbourn–Dinitz Handbook.

**Theorem 1.1** (Wei, Ge, 2014). The necessary conditions (1) for the existence of a 5-GDD of type $g^u$ are sufficient except for types $2^5, 2^{11}, 3^5, 6^5$, and except possibly for:

- $g = 3$ and $u \in \{45, 65\}$;
- $g = 2$ and $u \in \{15, 35, 71, 75, 95, 111, 115, 195, 215\}$;
- $g = 6$ and $u \in \{15, 35, 75, 95\}$;
- $g \in \{14, 18, 22, 26\}$ and $u \in \{11, 15, 71, 111, 115\}$;
- $g \in \{34, 46, 62\}$ and $u \in \{11, 15\}$;
- $g \in \{38, 58\}$ and $u \in \{11, 15, 71, 111\}$;
- $g = 2\alpha$, $\gcd(\alpha, 30) = 1$, $33 \leq \alpha \leq 2443$, and $u = 15$;
- $g = 10$ and $u \in \{5, 7, 15, 23, 27, 33, 35, 39, 47\}$;
- $g = 30$ and $u = 15$;
- $g = 50$ and $u \in \{15, 23, 27\}$;
- $g = 90$ and $u = 23$;
- $g = 10\alpha$, $\alpha \in \{7, 11, 13, 17, 35, 55, 77, 85, 91, 119, 143, 187, 221\}$ and $u = 23$.

**Proof.** This is Theorem 2.25 of [16].

The objective of this paper is to prove Theorem 1.2 below, which improves Theorem 1.1 by eliminating many possible exceptions.

**Theorem 1.2.** The necessary conditions (7) for the existence of a 5-GDD of type $g^u$ are sufficient except for types $2^5$, $2^{11}$, $3^5$, $6^5$, and except possibly for:

- $g = 3$ and $u = 65$;
- $g = 2$ and $u \in \{15, 75, 95, 115\}$;
- $g = 6$ and $u \in \{35, 95\}$;
- $g \in \{14, 18, 22, 26, 38, 58\}$ and $u \in \{11, 15\}$;
- $g \in \{74, 82, 86, 94\}$ and $u = 15$;
- $g = 10$ and $u \in \{5, 7, 27, 39, 47\}$;
- $g = 50$ and $u = 27$. 

2. GDDs with block size 5 and type $g^u$

We begin with some directly constructed group divisible designs.

**Theorem 2.1.** There exist 5-GDDs of types $2^{35}$, $2^{71}$, $2^{111}$, $3^{45}$, $6^{15}$, $10^{15}$, $10^{23}$ and $10^{23}$.

**Proof.** For $2^{35}$, $2^{71}$, and $10^{23}$ see [S] Lemma 4.1.

### $2^{111}$

With the point set $\{0, 1, \ldots, 221\}$ partitioned into residue classes modulo 111 for $\{0, 1, \ldots, 221\}$, the design is generated from

- $\{137, 73, 211, 182, 50\}$, $\{138, 74, 212, 183, 51\}$, $\{148, 201, 185, 107, 206\}$, $\{149, 202, 186, 108, 207\}$, $\{202, 148, 11, 152, 191\}$, $\{203, 149, 12, 153, 192\}$, $\{119, 166, 168, 153, 212\}$, $\{120, 167, 169, 154, 213\}$, $\{123, 106, 46, 71, 188\}$, $\{124, 107, 47, 72, 189\}$, $\{84, 132, 77, 65, 156\}$, $\{0, 3, 12, 122, 136\}$, $\{0, 8, 38, 126, 154\}$, $\{0, 7, 83, 156, 219\}$, $\{0, 10, 32, 101, 102\}$, $\{0, 27, 55, 75, 182\}$, $\{0, 33, 51, 57, 108\}$, $\{0, 1, 107, 121, 204\}$, $\{0, 79, 119, 151, 189\}$, $\{1, 9, 31, 97, 123\}$, $\{0, 6, 26, 62, 159\}$, $\{0, 9, 71, 127, 195\}$

by the mapping: $x \mapsto x + 2j$ (mod 222), $0 \leq j < 111$.

### $3^{45}$

With the point set $\{0, 1, \ldots, 134\}$ partitioned into residue classes modulo 44 for $\{0, 1, \ldots, 131\}$, and $\{132, 133, 134\}$, the design is generated from

- $\{121, 84, 8, 48, 108\}$, $\{82, 9, 79, 86, 124\}$, $\{133, 30, 56, 57, 35\}$, $\{131, 80, 60, 9, 37\}$, $\{95, 70, 122, 60, 91\}$, $\{0, 2, 8, 30, 49\}$, $\{0, 3, 18, 85, 115\}$, $\{0, 12, 75, 77, 86\}$, $\{0, 14, 53, 78, 93\}$, $\{0, 16, 45, 50, 119\}$, $\{0, 9, 43, 84, 95\}$, $\{0, 7, 23, 83, 131\}$, $\{1, 7, 19, 33, 97\}$, $\{0, 33, 66, 99, 134\}$

by the mapping: $x \mapsto x + 2j$ (mod 132) for $x < 132$, $x \mapsto (x + j$ (mod 2)) + 132 for $132 \leq x < 134$, $134 \mapsto 134$, $0 \leq j < 66$ for the first 13 blocks, $0 \leq j < 33$ for the last block.

### $6^{15}$

With the point set $\{0, 1, \ldots, 89\}$ partitioned into residue classes modulo 15 for $\{0, 1, \ldots, 89\}$, the design is generated from

- $\{80, 41, 45, 18, 25\}$, $\{0, 1, 41, 67, 88\}$, $\{0, 21, 29, 63, 73\}$, $\{0, 14, 39, 40, 71\}$, $\{0, 5, 34, 81, 83\}$, $\{0, 4, 13, 16, 84\}$, $\{0, 8, 32, 52, 85\}$, $\{0, 11, 17, 42, 79\}$, $\{0, 18, 36, 54, 72\}$, $\{1, 19, 37, 55, 73\}$

by the mapping: $x \mapsto x + 2j$ (mod 90), $0 \leq j < 45$ for the first eight blocks, $0 \leq j < 9$ for the last two blocks.

### $10^{15}$

With the point set $\{0, 1, \ldots, 149\}$ partitioned into residue classes modulo 15 for $\{0, 1, \ldots, 149\}$, the design is generated from

- $\{101, 21, 43, 132, 59\}$, $\{12, 85, 61, 88, 129\}$, $\{29, 9, 85, 93, 147\}$, $\{141, 39, 26, 48, 88\}$, $\{7, 76, 86, 25, 110\}$, $\{0, 1, 12, 108, 137\}$, $\{0, 14, 32, 111, 145\}$, $\{0, 17, 57, 63, 67\}$, $\{0, 16, 84, 107, 143\}$, $\{0, 2, 21, 102, 146\}$, $\{0, 8, 86, 95, 112\}$, $\{0, 7, 35, 36, 130\}$, $\{0, 11, 37, 58, 109\}$, $\{0, 3, 5, 70, 122\}$

by the mapping: $x \mapsto x + 2j$ (mod 150), $0 \leq j < 75$. 

With the point set \(\{0, 1, \ldots, 329\}\) partitioned into residue classes modulo 33 for \(\{0, 1, \ldots, 329\}\), the design is generated from
\[
\{102, 84, 56, 8, 268\}, \{145, 251, 217, 214, 137\}, \{57, 303, 73, 97, 184\},
\{304, 149, 216, 134, 104\}, \{203, 229, 88, 107, 278\}, \{170, 150, 53, 139, 229\},
\{300, 246, 79, 41, 278\}, \{108, 129, 65, 133, 48\}, \{0, 13, 120, 193, 222\},
\{0, 7, 42, 65, 214\}, \{0, 1, 148, 153, 162\}, \{0, 27, 63, 110, 201\},
\{0, 10, 62, 136, 197\}, \{0, 2, 55, 105, 144\}, \{0, 6, 57, 98, 202\},
\{0, 12, 56, 151, 229\}\]
by the mapping: \(x \mapsto x + j \pmod{330}\), \(0 \leq j < 330\). \(\square\)

For our proof of Theorem 1.2, we require some definitions and constructions.

A **double group divisible design**, \(k\)-DGDD, is an ordered quadruple \((V, G, H, B)\) such that:

(i) \(V\) is a base set of points;
(ii) \(G\) is a partition of \(V\), the **groups**;
(iii) \(H\) is another partition of \(V\), the **holes**;
(iv) \(B\) is a non-empty collection of subsets of \(V\) of cardinality \(k\), the **blocks**;
(v) for each block \(B \in B\), each group \(G \in G\) and each hole \(H \in H\), we have \(|B \cap G| \leq 1\) and \(|B \cap H| \leq 1\);
(vi) each pair of elements of \(V\) not in the same group and not in the same hole occurs in precisely one block.

A \(k\)-DGDD of type
\[
(g_1, h_1^u)w_1 (g_2, h_2^w)w_2 \ldots (g_r, h_r^w)w_r, \quad g_i = wh_i, \quad i = 1, 2, \ldots, r,
\]
is a double group divisible design where:

(i) there are \(u_i\) groups of size \(g_i\), \(i = 1, 2, \ldots, r\);
(ii) there are \(w\) holes;
(iii) for \(i = 1, 2, \ldots, r\), each group of size \(g_i\) intersects each hole in \(h_i\) points.

A **modified group divisible design**, \(k\)-MGDD, of type \(g^w\) is a \(k\)-DGDD of type \((g, 1^w)\). By interchanging groups and holes we see that a \(k\)-MGDD of type \(g^w\) exists if and only if a \(k\)-MGDD of type \(w^g\) exists. See [1] for an extensive treatment of \(5\)-MGDDs.

**Lemma 2.1.** Suppose there exists a 5-GDD of type \(g_1^w g_2^w \ldots g_n^w\). Then for positive integer \(h \notin \{2, 3, 6, 10\}\), there exists a 5-GDD of type \((g_1h)^w (g_2h)^w \ldots (g_nh)^w\).

**Proof.** Inflate each point of the 5-GDD by a factor of \(h\) and replace the blocks with 5-GDDs of type \(h^5\). By Theorem 1.1 there exists a 5-GDD of type \(h^5\) for \(h \geq 1\), \(h \notin \{2, 3, 6, 10\}\). \(\square\)

**Lemma 2.2.** Suppose there exists a \(K\)-GDD of type \(g_1^w g_2^w \ldots g_r^w\), and let \(w\) be a positive integer. Suppose also that for each \(k \in K\), there exists a
5-MGDD of type $w^k$, and for $i = 1, 2, \ldots, r$, there exists a 5-GDD of type $g_i^w$. Then there exists a 5-GDD of type $(u_1 g_1 + u_2 g_2 + \cdots + u_r g_r)^w$.

**Proof.** This is a combination of Constructions 2.19 and 2.20 in [16], and it also appears (for block size 4) as Constructions 1.8 and 1.10 in [12].

Take the $K$-GDD and inflate each point by a factor of $w$. Replace each inflated block by a 5-MGDD of type $w^k$, $k \in K$ to obtain a 5-DGDD of type $(wg_1, g_1^w)(wg_2, g_2^w)w \cdots (wg_r, g_r^w)w_r$. Then overlay the holes of this 5-DGDD with 5-GDDs of types $g_i^w$, $i = 1, 2, \ldots, r$.

We can now prove our main result.

**Proof of Theorem 1.2**

For types $2^{35}$, $2^{11}$, $2^{111}$, $3^{45}$, $6^{15}$, $10^{15}$, $10^{23}$ and $10^{33}$, see Theorem 2.1.

For types $2^{195}$ and $2^{215}$, take a 5-GDD of type $68^848^1$ or $68^88^5$, [15] (alternatively, see [5] Theorem 2.1 or [9] Theorem IV.4.17), and adjoin 2 extra points. Overlay each group together with the new points with a 5-GDD of type $2^{25}$ or $2^{35}$ or $2^{45}$, as appropriate.

For type $6^{75}$, take a 5-GDD of type $90^5$ and overlay the groups with 5-GDDs of type $6^{15}$.

For type $g^t$, $g \in \{14, 18, 22, 26, 38, 58\}$, $t \in \{71, 111\}$, use Lemma 2.1 with type $2^{71}$ or $2^{111}$ and $h = g/2$.

For type $g^{115}$, $g \in \{14, 18, 22, 26\}$, construct a 5-GDD of type $(5g)^{23}$ using Lemma 2.1 with a 5-GDD of type $10^{23}$ and $h = g/2$; then replace each group with a 5-GDD of type $g^5$.

For types $10^{35}$, $30^{15}$ and $50^{15}$, use Lemma 2.1 with a 5-GDD of type $2^{35}$ or $6^{15}$ or $10^{15}$, as appropriate, and $h = 5$.

For type $(10\alpha)^{23}$, odd $\alpha \geq 5$, use Lemma 2.1 with a 5-GDD of type $10^{23}$ and $h = \alpha$.

Let

$$G = \{34, 46, 62\} \cup \left\{\text{even } g \geq 66 : \gcd \left(\frac{g}{2}, 30\right) = 1\right\}$$

$$\setminus \{74, 82, 86, 94, 98, 106, 118, 178\}.$$  

For $g \in G$, there exists a $(g+1, \{5, 7, 9\}, 1)$-PBD, [3] Table IV.3.23. Take this PBD, remove a point and the blocks containing it to get a $\{5, 7, 9\}$-GDD of type $4^a6^b8^c$ for some non-negative integers $a$, $b$, $c$ satisfying $4a + 6b + 8c = g$. Now use Lemma 2.2 with this $\{5, 7, 9\}$-GDD and $w = 11$ or $15$ to obtain 5-GDDs of types $g^{11}$ and $g^{15}$ for every $g \in G$. For the existence of 5-MGDDs of types $w^5$, $w^7$ and $w^9$, see [1]. For the existence of 5-GDDs of types $4^w$, $6^w$ and $8^w$, see Theorems 1.1 and 2.1.

For type $98^{15}$, take a TD$(9, 11)$, fill in the groups with blocks of size 11 and remove a point together with the blocks containing it to get a $\{9, 11\}$-GDD of type $8^{11}10^1$. Now use Lemma 2.2 with this $\{9, 11\}$-GDD and $w = 15$ to obtain a 5-GDD of type $98^{15}$. For the existence of 5-MGDDs of types $15^9$
and $15^{11}$, see [1]. For the existence of 5-GDDs of types $8^{15}$ and $10^{15}$, see Theorems 1.1 and 2.1.

For types $106^{15}$, $118^{15}$ and $178^{15}$, we refer the reader to Lemma 3.16 of [11], which proves that there exists a 5-GDD of type $h^{11}$ for $h \equiv 2 \pmod{4}$, $h \geq 66$. By [11] Theorem 1.3, there exists a 4-frame of type $6^{15}$, i.e. a 4-GDD $(V, \mathcal{G}, \mathcal{B})$ of type $6^{15}$ in which the block set can be partitioned into 30 partial parallel classes of size 21 each of which partitions $V \setminus G$ for some $G \in \mathcal{G}$. Also we have the 5-GDD of type $6^{15}$ from Theorem 2.1 as well as 5-GDDs of type $h^{15}$ for $h \equiv 0 \pmod{4}$ from Theorem 1.1. Then, by a straightforward adaptation of the proof of [11] Lemma 3.16, we obtain 5-GDDs of type $g^{15}$ for $g \in \{6n, 6n + 4, \ldots, 8n - 2\}$ whenever there exists a TD(15, $n$) with odd $n$. This interval contains 106 and 118 when $n = 17$, and 178 when $n = 23$. 

$\square$

3. GDDs with block size 5 and type $g^u m^1$

Assuming they might be of some use for future research, we collect together an assortment of directly constructed 5-GDDs of type $g^u m^1$ that we have found during our investigations.

**Theorem 3.1.** There exist 5-GDDs of types $2^{38} 10^1$, $4^{12} 8^1$, $4^{21} 20^1$, $4^{22} 8^1$, $4^{24} 24^1$, $4^{25} 12^1$, $4^{26} 20^1$, $4^{27} 8^1$, $4^{30} 12^1$, $4^{31} 20^1$, $4^{32} 8^1$, $4^{37} 8^1$, $6^{12} 21^1$, $7^{20} 19^1$, $8^{10} 4^1$, $8^{12} 16^1$, $8^{33} 12^1$, $8^{18} 12^1$, $8^{20} 4^1$, $8^{20} 24^1$, $16^8 24^1$ and $24^7 8^1$.

**Proof.** $2^{38} 10^1$ With the point set $\{0, 1, \ldots, 81\}$ partitioned into residue classes modulo 36 for $\{0, 1, \ldots, 71\}$, and $\{72, 73, \ldots, 81\}$, the design is generated from

- $\{21, 76, 30, 35, 0\}$,
- $\{38, 9, 33, 7, 30\}$, $\{65, 23, 8, 15, 4\}$,
- $\{32, 79, 55, 30, 61\}$, $\{72, 63, 9, 64, 54\}$, $\{1, 35, 80, 60, 34\}$,
- $\{9, 61, 28, 21, 65\}$, $\{6, 12, 28, 40, 60\}$, $\{0, 14, 59, 69, 73\}$

by the mapping: $x \mapsto x + 2j \pmod{72}$ for $x < 72$, $x \mapsto (x - 72 + 5j \pmod{10}) + 72$ for $x \geq 72$, $0 \leq j < 36$.

$4^{12} 8^1$ With the point set $\{0, 1, \ldots, 55\}$ partitioned into residue classes modulo 12 for $\{0, 1, \ldots, 47\}$, and $\{48, 49, \ldots, 55\}$, the design is generated from

- $\{0, 1, 3, 11, 32\}$, $\{0, 4, 9, 34, 48\}$, $\{0, 6, 26, 41, 51\}$

by the mapping: $x \mapsto x + j \pmod{48}$ for $x < 48$, $x \mapsto (x + j \pmod{8}) + 48$ for $x \geq 48$, $0 \leq j < 48$.

$4^{21} 20^1$ With the point set $\{0, 1, \ldots, 103\}$ partitioned into residue classes modulo 21 for $\{0, 1, \ldots, 83\}$, and $\{84, 85, \ldots, 103\}$, the design is generated from

- $\{84, 57, 27, 22, 28\}$, $\{13, 86, 79, 82, 32\}$, $\{6, 65, 52, 63, 92\}$,
- $\{98, 1, 10, 24, 63\}$, $\{77, 44, 3, 70, 85\}$, $\{0, 4, 16, 56, 64\}$

by the mapping: $x \mapsto x + j \pmod{84}$ for $x < 84$, $x \mapsto (x - 84 + 5j \pmod{20}) + 84$ for $x \geq 84$, $0 \leq j < 84$. 

\[\]
With the point set \( \{0, 1, \ldots, 95\} \) partitioned into residue classes modulo 22 for \( \{0, 1, \ldots, 87\} \), and \( \{88, 89, \ldots, 95\} \), the design is generated from
\[
\{70, 92, 1, 27, 15\}, \{89, 73, 66, 69, 39\}, \{53, 55, 5, 76, 37\}, \{43, 53, 12, 18, 54\}, \{0, 5, 13, 64, 73\}
\]
by the mapping: \( x \mapsto x + j \mod 88 \) for \( x < 88 \), \( x \mapsto (x + j \mod 8) + 88 \) for \( x \geq 88 \), \( 0 \leq j < 88 \).

With the point set \( \{0, 1, \ldots, 119\} \) partitioned into residue classes modulo 24 for \( \{0, 1, \ldots, 95\} \), and \( \{96, 97, \ldots, 119\} \), the design is generated from
\[
\{72, 99, 61, 64, 49\}, \{13, 65, 45, 43, 101\}, \{85, 72, 109, 44, 54\}, \{38, 106, 84, 77, 63\}, \{26, 53, 102, 87, 93\}, \{112, 82, 46, 65, 91\}, \{0, 1, 5, 38, 54\}
\]
by the mapping: \( x \mapsto x + j \mod 96 \) for \( x < 96 \), \( x \mapsto (x + j \mod 24) + 96 \) for \( x \geq 96 \), \( 0 \leq j < 96 \).

With the point set \( \{0, 1, \ldots, 111\} \) partitioned into residue classes modulo 25 for \( \{0, 1, \ldots, 99\} \), and \( \{100, 101, \ldots, 111\} \), the design is generated from
\[
\{30, 84, 46, 16, 72\}, \{0, 110, 99, 94, 9\}, \{58, 105, 77, 60, 95\}, \{1, 44, 67, 109, 22\}, \{8, 4, 75, 28, 35\}, \{0, 3, 11, 39, 52\}
\]
by the mapping: \( x \mapsto x + j \mod 100 \) for \( x < 100 \), \( x \mapsto (x - 100 + 3j \mod 12) + 100 \) for \( x \geq 100 \), \( 0 \leq j < 100 \).

With the point set \( \{0, 1, \ldots, 123\} \) partitioned into residue classes modulo 26 for \( \{0, 1, \ldots, 103\} \), and \( \{104, 105, \ldots, 123\} \), the design is generated from
\[
\{23, 9, 54, 110, 8\}, \{61, 123, 58, 88, 67\}, \{104, 2, 9, 95, 72\}, \{17, 78, 27, 122, 56\}, \{76, 106, 59, 22, 57\}, \{13, 68, 21, 26, 92\}, \{0, 4, 16, 36, 64\}
\]
by the mapping: \( x \mapsto x + j \mod 104 \) for \( x < 104 \), \( x \mapsto (x - 104 + 5j \mod 20) + 104 \) for \( x \geq 104 \), \( 0 \leq j < 104 \).

With the point set \( \{0, 1, \ldots, 115\} \) partitioned into residue classes modulo 27 for \( \{0, 1, \ldots, 107\} \), and \( \{108, 109, \ldots, 115\} \), the design is generated from
\[
\{39, 18, 67, 92, 87\}, \{114, 43, 30, 72, 65\}, \{1, 63, 77, 73, 79\}, \{44, 113, 1, 62, 59\}, \{69, 29, 5, 38, 46\}, \{0, 1, 12, 38, 57\}
\]
by the mapping: \( x \mapsto x + j \mod 108 \) for \( x < 108 \), \( x \mapsto (x - 108 + 2j \mod 8) + 108 \) for \( x \geq 108 \), \( 0 \leq j < 108 \).

With the point set \( \{0, 1, \ldots, 131\} \) partitioned into residue classes modulo 30 for \( \{0, 1, \ldots, 119\} \), and \( \{120, 121, \ldots, 131\} \), the design is generated from
\[
\{72, 2, 10, 117, 78\}, \{108, 2, 104, 128, 35\}, \{30, 116, 7, 27, 68\}, \{8, 102, 75, 104, 92\}, \{130, 83, 90, 115, 41\}, \{27, 83, 6, 28, 43\}, \{0, 9, 66, 101, 127\}
\]
by the mapping: \( x \mapsto x + j \pmod{120} \) for \( x < 120 \), \( x \mapsto (x + j \pmod{12}) + 120 \) for \( x \geq 120 \), \( 0 \leq j < 120 \).

**43120**

With the point set \{0, 1, \ldots, 143\} partitioned into residue classes modulo 31 for \{0, 1, \ldots, 123\}, and \{124, 125, \ldots, 143\}, the design is generated from

\[
\{106, 98, 1, 50, 64\}, \{76, 59, 53, 127, 6\}, \{92, 140, 19, 9, 54\}, \\
\{23, 101, 44, 133, 110\}, \{65, 67, 80, 141, 62\}, \{124, 112, 18, 17, 43\}, \\
\{100, 93, 8, 19, 41\}, \{0, 4, 16, 44, 104\}
\]

by the mapping: \( x \mapsto x + j \pmod{124} \) for \( x < 124 \), \( x \mapsto (x - 124 + 5j \pmod{20}) + 124 \) for \( x \geq 124 \), \( 0 \leq j < 124 \).

**4328**

With the point set \{0, 1, \ldots, 135\} partitioned into residue classes modulo 32 for \{0, 1, \ldots, 127\}, and \{128, 129, \ldots, 135\}, the design is generated from

\[
\{95, 104, 64, 21, 74\}, \{113, 53, 79, 56, 12\}, \{20, 22, 4, 70, 42\}, \\
\{22, 113, 125, 126, 121\}, \{42, 57, 130, 6, 115\}, \{72, 133, 3, 55, 66\}, \\
\{0, 7, 42, 56, 89\}
\]

by the mapping: \( x \mapsto x + j \pmod{128} \) for \( x < 128 \), \( x \mapsto (x + j \pmod{8}) + 128 \) for \( x \geq 128 \), \( 0 \leq j < 128 \).

**4378**

With the point set \{0, 1, \ldots, 155\} partitioned into residue classes modulo 37 for \{0, 1, \ldots, 147\}, and \{148, 149, \ldots, 155\}, the design is generated from

\[
\{86, 63, 119, 43, 147\}, \{21, 22, 53, 15, 13\}, \{50, 34, 120, 37, 130\}, \\
\{149, 122, 88, 53, 147\}, \{14, 133, 112, 150, 3\}, \{60, 13, 9, 108, 135\}, \\
\{32, 113, 47, 77, 89\}, \{0, 5, 19, 90, 107\}
\]

by the mapping: \( x \mapsto x + j \pmod{148} \) for \( x < 148 \), \( x \mapsto (x - 148 + 2j \pmod{8}) + 148 \) for \( x \geq 148 \), \( 0 \leq j < 148 \).

**6122**

With the point set \{0, 1, \ldots, 73\} partitioned into residue classes modulo 12 for \{0, 1, \ldots, 71\}, and \{72, 73\}, the design is generated from

\[
\{32, 70, 25, 41, 21\}, \{14, 31, 46, 56, 0\}, \{9, 11, 48, 39, 70\}, \\
\{64, 58, 60, 41, 63\}, \{26, 55, 21, 34, 54\}, \{57, 72, 32, 47, 50\}, \\
\{0, 19, 37, 45, 51\}
\]

by the mapping: \( x \mapsto x + 2j \pmod{72} \) for \( x < 72 \), \( x \mapsto (x + j \pmod{2}) + 72 \) for \( x \geq 72 \), \( 0 \leq j < 36 \).

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With the point set \{0, 1, \ldots, 158\} partitioned into residue classes modulo 19 for \{0, 1, \ldots, 132\}, \{133, 134, \ldots, 139\}, and \{140, 141, \ldots, 158\}, the design is generated from

\[
\{64, 48, 14, 54, 115\}, \{39, 2, 156, 51, 94\}, \{39, 101, 24, 128, 21\}, \\
\{0, 4, 91, 98, 145\}, \{0, 1, 14, 22, 147\}, \{0, 2, 25, 30, 88\}, \\
\{0, 17, 48, 81, 133\}, \{0, 9, 68, 122, 140\}, \{0, 24, 97, 138, 158\}
\]

by the mapping: \( x \mapsto x + j \pmod{133} \) for \( x < 133 \), \( x \mapsto (x + j \pmod{7}) + 133 \) for \( 133 \leq x < 140 \), \( x \mapsto (x - 140 + j \pmod{19}) + 140 \) for \( x \geq 140 \), \( 0 \leq j < 133 \).
$8^{10}4^1$ With the point set $\{0, 1, \ldots, 83\}$ partitioned into residue classes modulo 10 for $\{0, 1, \ldots, 79\}$, and $\{80, 81, 82, 83\}$, the design is generated from
\[\{56, 2, 24, 70, 3\}, \{80, 42, 19, 60, 57\}, \{14, 49, 6, 30, 77\}\]
by the mapping: $x \mapsto x + j \pmod{80}$ for $x < 80$, $x \mapsto (x + j \pmod{4}) + 80$ for $x \geq 80, 0 \leq j < 80$.

$8^{12}16^1$ With the point set $\{0, 1, \ldots, 111\}$ partitioned into residue classes modulo 12 for $\{0, 1, \ldots, 95\}$, and $\{96, 97, \ldots, 111\}$, the design is generated from
\[\{34, 42, 100, 36, 59\}, \{92, 89, 55, 85, 36\}, \{88, 3, 12, 66, 103\}, \{111, 28, 66, 56, 1\}, \{43, 4, 22, 48, 108\}, \{0, 1, 14, 46, 81\}\]
by the mapping: $x \mapsto x + j \pmod{96}$ for $x < 96$, $x \mapsto (x + j \pmod{16}) + 96$ for $x \geq 96, 0 \leq j < 96$.

$8^{13}12^1$ With the point set $\{0, 1, \ldots, 115\}$ partitioned into residue classes modulo 13 for $\{0, 1, \ldots, 103\}$, and $\{104, 105, \ldots, 115\}$, the design is generated from
\[\{52, 16, 14, 24, 64\}, \{38, 99, 70, 95, 79\}, \{90, 5, 0, 109, 87\}, \{41, 103, 10, 113, 68\}, \{35, 2, 17, 72, 105\}, \{0, 1, 7, 60, 81\}\]
by the mapping: $x \mapsto x + j \pmod{104}$ for $x < 104$, $x \mapsto (x - 104 + 3j \pmod{12}) + 104$ for $x \geq 104, 0 \leq j < 104$.

$8^{18}12^1$ With the point set $\{0, 1, \ldots, 155\}$ partitioned into residue classes modulo 18 for $\{0, 1, \ldots, 143\}$, and $\{144, 145, \ldots, 155\}$, the design is generated from
\[\{49, 57, 14, 17, 15\}, \{137, 122, 77, 61, 55\}, \{52, 21, 14, 65, 150\}, \{56, 79, 60, 23, 32\}, \{6, 84, 32, 11, 59\}, \{53, 12, 92, 152, 142\}, \{2, 71, 13, 83, 100\}, \{0, 10, 30, 95, 149\}\]
by the mapping: $x \mapsto x + j \pmod{144}$ for $x < 144$, $x \mapsto (x + j \pmod{12}) + 144$ for $x \geq 144, 0 \leq j < 144$.

$8^{20}4^1$ With the point set $\{0, 1, \ldots, 163\}$ partitioned into residue classes modulo 20 for $\{0, 1, \ldots, 159\}$, and $\{160, 161, 162, 163\}$, the design is generated from
\[\{70, 95, 117, 58, 51\}, \{9, 133, 124, 148, 61\}, \{88, 99, 57, 3, 89\}, \{67, 144, 10, 136, 14\}, \{13, 117, 94, 123, 156\}, \{15, 66, 80, 64, 148\}, \{56, 99, 10, 38, 51\}, \{0, 3, 58, 93, 160\}\]
by the mapping: $x \mapsto x + j \pmod{160}$ for $x < 160$, $x \mapsto (x + j \pmod{4}) + 160$ for $x \geq 160, 0 \leq j < 160$.

$8^{20}24^1$ With the point set $\{0, 1, \ldots, 183\}$ partitioned into residue classes modulo 20 for $\{0, 1, \ldots, 159\}$, and $\{160, 161, \ldots, 183\}$, the design is generated from
\[\{142, 54, 150, 133, 40\}, \{172, 8, 137, 115, 2\}, \{112, 17, 6, 69, 153\}, \{72, 114, 39, 181, 129\}, \{78, 137, 183, 114, 116\}, \{46, 19, 145, 176, 108\}, \{89, 40, 179, 43, 134\}, \{125, 52, 120, 42, 174\}, \{35, 54, 6, 36, 140\},

\{0, 4, 16, 125, 132\} by the mapping: \(x \mapsto x + j \pmod{160}\) for \(x < 160\), \(x \mapsto (x - 160 + 3j \pmod{24}) + 160\) for \(x \geq 160\), \(0 \leq j < 160\).

16\(^8\)24\(^1\) With the point set \{0,1,…,151\} partitioned into residue classes modulo 8 for \{0,1,…,127\}, and \{128,129,…,151\}, the design is generated from
\[
\{62,141,95,9,19\}, \{94,11,93,55,146\}, \{18,115,0,15,148\},
\{30,77,9,23,96\}, \{137,31,22,81,101\}, \{3,80,106,102,135\},
\{40,70,3,5,97\}, \{0,5,11,116,145\}
\]
by the mapping: \(x \mapsto x + j \pmod{128}\) for \(x < 128\), \(x \mapsto (x - 128 + 3j \pmod{24}) + 128\) for \(x \geq 128\), \(0 \leq j < 128\).

24\(^7\)8\(^1\) With the point set \{0,1,…,175\} partitioned into residue classes modulo 7 for \{0,1,…,167\}, and \{168,169,…,175\}, the design is generated from
\[
\{135,1,159,70,81\}, \{13,63,15,54,32\}, \{159,28,29,3,114\},
\{107,162,91,87,55\}, \{127,17,12,173,104\}, \{115,161,55,88,155\},
\{90,16,24,120,133\}, \{0,3,18,47,170\}
\]
by the mapping: \(x \mapsto x + j \pmod{168}\) for \(x < 168\), \(x \mapsto (x + j \pmod{8}) + 168\) for \(x \geq 168\), \(0 \leq j < 168\).

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