Unbounded eigenfunctions in the stability problem
for a three-layer flow in porous media

Gelu Paşa

Simion Stoilow Institute of Mathematics, Calea Grivitei 21, Bucharest, Romania
E-mail: gelu.pasa@imar.ro and pasa.gelu@gmail.com

We study the linear stability of the displacement of three Stokes fluids with constant viscosity in a porous medium when the middle fluid is contained in a bounded region. We use the Hele-Shaw model. The eigenfunctions of the stability system are the amplitudes of the linear perturbations. These amplitudes must be small. We get unbounded eigenfunctions. So the stability problem has no physical sense.

AMS 2010 Subject Classification: 34B09; 34D20; 35C09; 35J20; 76S05.
Keywords: Hele-Shaw displacements; Constant viscosity fluids; Hydrodynamic stability.

1. Introduction

We consider the linear stability problem for the displacement of three Stokes fluids in a Hele-Shaw cell, parallel with the horizontal fix plane $x_1Oy$ - see [1], [21]. The three constant viscosities $\mu_L, \mu, \mu_R$ are (in fact) divided by the constant permeability of the equivalent porous medium. The fluid $\mu_L$ is displacing the middle fluid $\mu$, which in turn is pushing a fluid with viscosity $\mu_R$, in the positive direction $Ox_1$. The middle liquid $\mu$ is contained in a bounded region. We suppose

$$\mu_L < \mu < \mu_R.$$  \hfill (1)

A large number of successive intermediate liquids can be considered between the initial displacing fluids with viscosities $\mu_L, \mu_R$. On this way some authors try to minimize the Saffman-Taylor instability (see [5] - [12]), by using the seminal paper [4]. The obtained results have some weak points, explained in some works of the present author (see [15] -[20]). We give here only a selective list of related papers. Some have been published recently and contain more detailed references.

The linear stability system was obtained in [4] and also in [19]. An existence condition for the non-zero solution of the stability system was used in [4] to obtain a quadratic equation for the growth rates; some numerical results were given. A compatibility condition was derived in [19]; we proved that this condition is not fulfilled. Thus the stability problem has no solution. However, some (natural) additional hypothesis were used in [19].

In this paper we use the above existence condition for obtain the behavior of the eigenfunctions of the stability system in the range of very large wavenumbers. The new element of this paper is following. We get unbounded eigenfunctions for increasing wavenumbers. Therefore the perturbations are also unbounded for large wavenumbers. Thus the stability problem has no physical sense. This time, we not use any additional hypothesis.

2. The stability problem

The displacement is given by the velocity $(U, 0)$ of the fluid $\mu_L$ far upstream. We use the moving reference $x = x_1 - Ut$, where $t$ is the time. The intermediate liquid is contained in the segment $(a, b), b \leq 0$. The relevant (basic) flow equations are quite similar with Darcy’s law in porous media. A stationary solution exists, with two straight interfaces $x = a, x = b$ between the fluid layers - see [13]. The Laplace-Young law acts on both interfaces, where two positive surface tensions $T(a), T(b)$ exist.
We perform a linearized stability analysis by normal modes. We use the Fourier decomposition of the velocity perturbation in the $Ox$ direction:

$$u' = \epsilon f(x) \exp(iky + \sigma t).$$  \hspace{1cm} (2)

Here $k, \sigma, f$ are the wave numbers, growth rates and amplitudes. $\epsilon$ is a small positive number. The function $f$ must be bounded. Moreover, $f$ must decay to zero in the far field and is continuous. The derivative of $f$ is not continuous in $a, b$. In [21] are imposed the conditions

$$f(x) = f(a)e^{k(x-a)}, \ \forall x \leq a, \ k \geq 0;$$
$$f(x) = f(b)e^{-k(x-b)}, \ \forall x \geq b, \ k \geq 0. \hspace{1cm} (3)$$

We insert (2) in the linearized disturbance equations obtained from the Darcy’s law. We follow the procedure given in [3], [4] and we obtain the problem (2a), (2b), (3) of [4]:

$$f_{xx}(x, k) - k^2 f(x, k) = 0, \ x \in (a, b), \ b \leq 0, \ \forall k \geq 0; \hspace{1cm} (4)$$
$$f^+_x(a, k) = [-kE_a(k)/\sigma(k) + \mu_Lk/\mu]f(a, k); \hspace{1cm} (5)$$
$$f^{-}_x(b, k) = [kE_b(k)/\sigma(k) - \mu_Rk/\mu]f(b, k); \hspace{1cm} (6)$$
$$E_a(k) := (\mu - \mu_L)Uk - T(a)k^3; \hspace{1cm} (7)$$
$$E_b(k) := (\mu_R - \mu)Uk - T(b)k^3;$$

$f_x$ is the derivative; $f^+_x, f^-_x$ are the lateral derivatives; $f$ is depending also on $k$.

3. The non-existence result

The general solution of (4) is

$$f(x, k) = A(k)e^{kx} + B(k)e^{-kx}. \hspace{1cm} (8)$$

The coefficients $A, B$ were considered constant in [4]. We use the relations (5)-(6) and obtain a stability system for $A(k), B(k)$, with variable coefficients $c(k), d(k), g(k), h(k)$ (see formulas (2b), (3), (5), pag. 112101-5 in [4] and the system (12) of [19]):

$$A(k)e^{ka}c(k) + B(k)e^{-ka}d(k) = 0, \ A(k)e^{kb}g(k) + B(k)e^{-kb}h(k) = 0; \hspace{1cm} (9)$$
$$c(k) = [\mu_L - \mu - E_a(k)/\sigma(k)], \ d(k) = [\mu_L + \mu - E_a(k)/\sigma(k)]; \hspace{1cm} (10)$$
$$g(k) = [\mu_R + \mu - E_b(k)/\sigma(k)], \ h(k) = [\mu_R - \mu - E_b(k)/\sigma(k)]. \hspace{1cm} (11)$$

A nontrivial solution $(A(k), B(k))$ exists if the following condition is verified

$$e^{2k(a-b)}c(k)h(k) - g(k)d(k) = 0, \ \forall k \geq 0. \hspace{1cm} (12)$$

Our main goal is to find $\lim_{k \to \infty} A(k)$ and $\lim_{k \to \infty} B(k)$ by using (9) - (12).

Remark 1. We have

$$d(k), c(k), g(k), h(k) \neq 0, \forall k > 0. \hspace{1cm} (13)$$

Indeed, suppose $\exists m$ s.t. $d(m) = 0, c(m) = -2\mu$. From (9) and (8) we get

$$A(m) = 0, \ f(x, m) = B(m)e^{-mx}, \ f_x(x, m) = -mB(m)e^{-mx}.$$
We emphasize that \( B(m) \neq 0 \), otherwise \( f(x, m) \equiv 0 \); the eigenfunctions cannot be identically zero. We simplify with \( B(m) \), then \([5]-[9]\) in \( k = m \) lead us to

\[
\mu_L e^{-ma} + \mu e^{-ma} = \frac{E_a(m)}{\sigma(m)} e^{-ma}, \quad \mu_R e^{-mb} + \mu e^{-mb} = \frac{E_b(m)}{\sigma(m)} e^{-mb}.
\] (14)

In both relation we have the same \( \sigma(m) \) then we get \( E_a(m)/(\mu_L + \mu) = E_b(m)/(\mu_R + \mu) \). This is an unexpected restriction on \( \mu \), not considered as a hypothesis.

We get \( c(k), g(k), h(k) \neq 0, \forall k > 0 \) by using the same arguments. For example, suppose \( \exists n \) s.t. \( c(n) = 0 \). Then \([5], [9]\) and \([5]-[6]\) in \( k = n \) give us \( B(n) = 0 \), \( f(x, n) = A(n)e^{nx}, \ A(n) \neq 0 \), \( f_x(x, n) = nA(n)e^{nx} \), thus \( E_a(n)/(\mu_L - \mu) = E_b(n)/(\mu_R + \mu) \).

Remark 2. The following properties hold:

\[
\lim_{k \to \infty} d(k) = 0 \Rightarrow \lim_{k \to \infty} g(k), \lim_{k \to \infty} c(k), \lim_{k \to \infty} h(k) \quad \text{are real bounded (not zero) quantities.} \quad (15)
\]

\[
\lim_{k \to \infty} g(k) = 0 \Rightarrow \lim_{k \to \infty} d(k), \lim_{k \to \infty} c(k), \lim_{k \to \infty} h(k) \quad \text{are real bounded (not zero) quantities.} \quad (16)
\]

Indeed, suppose \( \lim_{k \to \infty} d(k) = 0 \). Thus \( \lim_{k \to \infty} E_a/\sigma = (\mu_L + \mu) \) and we have

\[
\lim_{k \to \infty} g(k) = \mu_R + \mu - (T_b/T_a)(\mu_L + \mu),
\]

\[
\lim_{k \to \infty} h(k) = \mu_R - \mu - (T_b/T_a)(\mu_L + \mu), \quad \lim_{k \to \infty} c(k) = -2\mu.
\]

We use the same arguments and get \([10]\).

For a function \( H(k) \), we use also the notation \( H(k) \to \alpha \) when \( \lim_{k \to \infty} H(k) = \alpha \).

From Remark 2 it follows that \( d(k) \to 0 \) is a possible “solution” of \([12]\) when \( k \to \infty \).

Indeed, the limits of \( c(k), h(k), g(k) \) are bounded, thus \( e^{2k(a-b)}c(k)h(k) \to 0 \) and \( d(k)g(k) \to 0 \). Next we prove that \( d(k) \to 0 \) or \( g(k) \to 0 \) are necessary conditions.

Proposition 1. We have the following two possibilities

\[
\text{or } d(k) \to 0 \quad \text{or } g(k) \to 0. \quad (17)
\]

Proof. We recall \([12]\). If \( c(k) \to \infty \) or \( h(k) \to \infty \), then \( \lim_{k \to \infty} e^{2k(a-b)}c(k)h(k) \) it might not exist. First, we prove here that both \( c(k), h(k) \) have bounded limits limits to infinity.

Indeed, suppose \( c(k) \to \infty \). That means, in fact, \( E_a/\sigma(k) \to \infty \), because only this term of \( c(k) \) is depending on \( k \). We use the notation \( I(k) := E_a(k)/\sigma \) and get

\[
e^{2k(a-b)}I^2(k)[\frac{\mu_L - \mu}{I(k)} - 1][\frac{\mu_R - \mu}{I(k)} - \frac{E_b(k)}{E_a(k)}] - I^2(k)[\frac{\mu_L + \mu}{I(k)} - 1][\frac{\mu_R + \mu}{I(k)} - \frac{E_b(k)}{E_a(k)}] = 0. \quad (18)
\]

We simplify with \( I^2(k) \). As \( E_b(k)/E_a(k) \to T_b/T_a \) (which is a bounded quantity), when \( k \to \infty \) we get \( 0 - T_a/T_b = 0 \). Therefore the equation \([12]\) is not verified when \( c(k) \to \infty \).

Suppose \( c(k), h(k) \to \infty \). Thus \( I(k) := E_a(k)/\sigma(k) \to \infty \), \( J(k) := E_b(k)/\sigma(k) \to \infty \) and from \([12]\) we get

\[
e^{2k(a-b)}I(k)J(k)[\frac{\mu_L - \mu}{I(k)} - 1][\frac{\mu_R - \mu}{J(k)} - 1] - I(k)J(k)[\frac{\mu_L + \mu}{I(k)} - 1][\frac{\mu_R + \mu}{J(k)} - 1] = 0. \quad (19)
\]
When $k \to \infty$ we get $0 - 1 = 0$. Thus (12) is not verified when $c(k), h(k) \to \infty$.

The second step is following. Both $c(k), h(k)$ have bounded limits when $k \to \infty$, therefore $e^{2k(a-b)c(k)}h(k) \to 0$. From (12) it follows $d(k)g(k) \to 0$ and we obtain the result (17).

**Remark 3.** The above proposition gives us two possible eigenvalues, only for large $k$:

$$k \to \infty \Rightarrow \sigma_1(k) \approx \frac{E_{a}(k)}{(\mu_{L} + \mu)}, \quad \sigma_2(k) \approx \frac{E_{b}(k)}{(\mu_{R} + \mu)}. \quad (20)$$

This was also obtained in [19] by direct calculations from (12), with the natural hypothesis

$$e^{-2k^{3}} \approx 0, \quad e^{-2k} \ll k^{3}; \quad e^{-2k^{6}} \approx 0, \quad e^{-2k^{6}} \ll k^{6} \quad \text{for large enough} \quad k.

**Proposition 2.** There exists unbounded eigenfunctions $f(x, k)$ with $A(k), B(k)$ given by (9).

**Proof.** Consider $d(k) \to 0$ and $f(a, k)$ bounded. Then from (9) we obtain

$$(-2\mu)A(k)e^{ka} \to 0. \quad (21)$$

Let $Y(k) = -g(k)/h(k)$. From [13], [15], [16] it follows that $Y(k)$ is bounded for large $k$. We use (22) and obtain

$$B(k)e^{-ka} = Y(k)A(k)e^{k(2b-a)}. \quad (22)$$

Here is the main point of our paper. We use (22) to get the limit of $B(k)e^{-ka}$. There exist a lot of possibilities, related with $A(k)$ and the magnitude of the interval $(a, b)$. For example:

i) if $A(k) = D = constant$ and $a = -10, b = -1$, then $2b - a = 8 > 0$ and $|B(k)e^{-ka}| \to \infty$. This contradicts the hypothesis $f(a, k)$ bounded;

ii) if $A = D$ and $a = -2, b = -1.5$ then $2b - a = -1 < 0$ and both terms of $f(a, k)$ are convergent to zero for increasing $k$. This is a “good” case.

iii) if $A(k)$ is a polynomial, $a = -10, b = -1$, we still have $A(k)e^{ka} \to 0$ and $|B(k)e^{-ka}| \to \infty$. This contradicts again the hypothesis $f(a, k)$ bounded.

Moreover, the formulas (21)-(22) give us

$$\lim_{k \to \infty} B(k)e^{-ka} = \lim_{k \to \infty} \{Y(k)[A(k)e^{ka}]e^{2k(b-a)}\}. \quad (23)$$

Thus $\lim_{k \to \infty} f(a, k)$ might even not exist.

**Remark 4.** Consider the case iii) with $b = 0$ and formula (23) with $A(k) = e^{-ka}$. We get a very strange conclusion. The eigenfunctions (so the perturbations amplitudes) could be exponentially increasing, for fixed $k$, as functions of $(b - a)$. This is in total contradiction with the results of [2], [3] and [13]. In these three papers, a variable profile was considered in the intermediate region - denoted by (I.R). On the water-liquid interface, it was considered a continuous viscosity. The main conclusion was: a large enough (I.R.) (and a suitable viscosity profile) can almost suppress the instability Saffman-Taylor. This effect was confirmed by experimental results. On the contrary, there are no experimental results for the models studied in [4] and [5] - [12].
4. Conclusions

In the seminal paper [4] were given the following results. From the relations (5)-(6) was obtained the system (9) with \(A, B\) constant. The condition (12) was used to get a quadratic equation for \(\sigma\). Numerical values and some estimates of \(\sigma\) were also given. All these results were used later in [5] - [12]. Some weak points exist in these cited papers, which were explained in [15] - [19].

In this paper we consider the general case when \(A, B\) are depending on \(k\). The eigenfunctions \(f(x, k)\) are not bounded (in general) as functions of \(k\), so the amplitudes of perturbations could be not small. Thus the solution of the stability system has no physical meaning, even if \(A = A(k), B = B(k)\). The equation (12) is still true. But we can say that the corresponding eigenvalues also do not make physical sense.

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