The Ring of Invariants for Smooth Completions of Kac-Moody Lie Algebras

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Abstract. It is proved that the ring of invariants of the standard smooth completion of a Kac-Moody Lie algebra is functionally generated by two elements: the coefficient of the center and the Killing form.

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§1. Introduction

In this paper we address a very specific question:

What are the smooth $\text{ad}^*$-invariant functions for the smooth completion $\hat{\mathfrak{g}}$ of the full affine Lie algebra based on a finite dimensional Lie algebra $\mathfrak{g}$?

Given is a finite dimensional complex orthogonal Lie algebra $\mathfrak{g}$: that is to say $\mathfrak{g}$ admits an invariant symmetric non-degenerate bilinear form. It gives rise in the usual fashion to the smooth completion $\hat{\mathfrak{g}}$ of the affine Lie algebra: $\hat{\mathfrak{g}}$ is the semidirect product of the complex line $\mathbb{C}$ with the standard central extension $\hat{\mathfrak{g}}$ of the loop algebra $\tilde{\mathfrak{g}} = C^\infty(S^1, \mathfrak{g})$.

We will show that the ring of coadjoint-invariant smooth functions on the dual $\hat{\mathfrak{g}}^*$ of $\hat{\mathfrak{g}}$ is functionally generated by two elements: the quadratic form defined by the naturally induced symmetric bilinear form on $\hat{\mathfrak{g}}$ and the map associating to an element of $\hat{\mathfrak{g}}^*$ the coefficient of its center. It is worth emphasising that when we say generated, we mean generated over $\mathbb{C}$, which means that for any $\mathfrak{g}$ there are essentially only two complex–valued coadjoint invariants on $\hat{\mathfrak{g}}$.

A similar result was obtained from an algebraic point of view by Chari and Ilango-\-van [1984]. They show that exactly the same two elements polynomially generate the center of a carefully chosen formal completion of the universal enveloping algebra of a contragredient Lie algebra defined by a given Cartan matrix. The present result can be viewed therefore as an extension of their algebraic theorem to the category of smooth completions of affine Lie algebras. Other important references are Kac [1984] and Kac and Peterson [1983].

Our proof is very different to that of Chari and Ilango-\-van, being geometric instead of algebraic. It is based on the idea introduced by Weinstein [1983], of a dual pair of Poisson maps defined on a symplectic manifold. From this geometric point of view, the present paper can be interpreted as a result on dual pairs for the cotangent bundle of a loop group. The natural temptation, based on experience gained from the finite dimensional case, is to mould the proof in such a way as to be able to derive the result from standard facts about dual pairs. Unfortunately, as is mostly the case when working with infinite dimensional objects, such a proof remains a formal exercise, since the theorems one needs to quote do not in most cases have infinite dimensional generalizations, and even if they do, their proofs require the use of technical machinery from the theory of Fréchet manifolds. We have chosen a different way to deal with these problems: every technical fact for the case at hand is proved directly, without any appeal to infinite dimensional symplectic or Poisson geometry. It is important to mention the idea of dual pairs as this was crucial for leading us to the result. The interested reader might like to try and reconstruct our result from such a point of view, comparing with Marshall [1994], always bearing in mind that a demand for total rigour will have to be laid on one side to do this.

An important motivation for constructing coadjoint invariants on $\hat{\mathfrak{g}}^*$ comes from the field of integrable systems. Invariant functions on the dual of any Lie algebra can be used to construct commuting flows. From this point of view, the result of the present work is disappointing for it rules out the search for interesting integrable systems based on $\hat{\mathfrak{g}}$ (at least by the standard approach). However, on both $\mathfrak{g}^*$ and $\hat{\mathfrak{g}}^*$ (these are the duals respectively to $\mathfrak{g}$ and to the central extension of the loop algebra based on $\mathfrak{g}$),
there are a lot of invariant functions which have been put to good use in the works on
integrable systems.

§2. Statement of the theorem

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra endowed with a nondegenerate $\text{ad}$–invariant inner product $(\cdot, \cdot): \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, that is,

$$(X, [Y, Z]) = ([X, Y], Z) \quad \forall X, Y, Z \in \mathfrak{g}. \quad (2.1)$$

The full affine Lie algebra $\bar{\mathfrak{g}}$ based on $\mathfrak{g}$ is obtained by the following construction (see Kac [1985], Pressley and Segal [1986]). Consider the central extension $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}$ of the loop algebra $\tilde{\mathfrak{g}} = \mathbb{C} \infty(S^1, \mathfrak{g})$ by $\mathbb{C}$, having the Lie bracket

$$[(X, a), (Y, b)](x) = \left( [X(x), Y(x)], \int_0^{2\pi} (X(y), Y'(y)) dy \right), \quad (2.2)$$

where the prime “$'$” denotes derivation. Let the abelian Lie algebra $\mathbb{C}$ act on $\hat{\mathfrak{g}}$ by

$$z \cdot (X, a) = (zX', 0). \quad (2.3)$$

and form the semidirect product $\bar{\mathfrak{g}} = \mathbb{C} \oplus \hat{\mathfrak{g}}$ with Lie bracket

$$[(z_1, X, a), (z_2, Y, b)] = \left( 0, [X, Y] + z_1 Y' - z_2 X', \int_0^{2\pi} (X(y), Y'(y)) dy \right). \quad (2.4)$$

If we identify $\bar{\mathfrak{g}}^*$ with $\bar{\mathfrak{g}}$ by

$$\langle (\alpha, \xi, e), (z, X, a) \rangle = \text{Re} \left( az + \int_0^{2\pi} (\xi(x), X(x)) dx + ea \right) \quad (2.5)$$

then we obtain

$$\text{ad}^*_\circ(z, X, a)(\alpha, \xi, e) = \left( \int_0^{2\pi} (\xi(x), X'(x)) dx, [X, \xi] + z\xi' + eX', 0 \right), \quad (2.6)$$

where $\text{ad}^*$ denotes minus the dual map of $\text{ad}$. We will denote by the same symbol $\langle, \rangle$ the pairing identifying $\bar{\mathfrak{g}}^*$ with $\bar{\mathfrak{g}}$, i.e., take in (2.5) $\alpha = e = z = a = 0$.

If $F \in C^\infty(\bar{\mathfrak{g}}^*)$, we will denote by $\nabla_{(\alpha, \xi, e)} F \in \bar{\mathfrak{g}}$ its functional derivative, defined to be the vector representative of the Fréchet derivative of $F$ at $(\alpha, \xi, e)$ relative to the pairing (2.5). That is

$$f((\alpha, \xi, e) + t(\beta, \eta, e')) = f(\alpha, \xi, e) + t(\beta, \eta, e'), \nabla_{(\alpha, \xi, e)} f + O(t^2) \quad \text{as} \quad t \rightarrow 0, \quad t \in \mathbb{R} \quad \forall (\alpha, \xi, e), (\beta, \eta, e') \in \bar{\mathfrak{g}}^*. \quad (2.7)$$

Our goal is to find the set $I(\bar{\mathfrak{g}}^*)$ ($I$ stands for invariant) of all functions $F \in C^\infty(\bar{\mathfrak{g}}^*)$ which are $\text{ad}^*$–invariant, that is, satisfying

$$\langle (\alpha, \xi, e), [(z, X, a), \nabla_{(\alpha, \xi, e)} F] \rangle = 0 \quad \forall (z, X, a) \in \bar{\mathfrak{g}}, \forall (\alpha, \xi, e) \in \bar{\mathfrak{g}}^*. \quad (2.8)$$
We will prove the following theorem:

**Theorem** Any $\text{ad}^*-$invariant function on $\mathfrak{g}^*$ is of the form $F(\kappa, \pi)$, where $\kappa : \mathfrak{g}^* \rightarrow \mathbb{C}$ is given by

$$\kappa(\alpha, \xi, e) = e^{\alpha} - \frac{1}{2} \int_0^{2\pi} (\xi(x), \xi(x)) dx, \quad (2.9)$$

$\pi : \mathfrak{g}^* \rightarrow \mathbb{C}$ is the projection onto the center, and $F \in C^\infty(\mathbb{C}^2)$.

Harnad and Kupershmidt [1991] is also relevant as it contains many of the formulae for the momentum maps in section 3.

§3. **Proof of the theorem**

Suppose that $G$ is a connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}$, regarded as a real Lie algebra. Let $\tilde{G} = C^\infty(S^1, G)$ be the loop group of $G$ and let $\tilde{\mathfrak{g}} = C^\infty(S^1, \mathfrak{g})$ be the corresponding Lie algebra. The loop group $\tilde{G}$ is connected. On the cotangent bundle $T^*\tilde{G}$, which we will henceforth identify with $\tilde{G} \times \tilde{\mathfrak{g}}^*$ by left translations we define the family of non–canonical symplectic structures by

$$\omega_k = \omega_{\text{CAN}} + \pi^* \Omega_k, \quad k \in \mathbb{C}. \quad (3.1)$$

Here $\omega_{\text{CAN}}$ is the canonical 2-form and $\pi : T^*\tilde{G} \rightarrow \tilde{G}$ is the canonical projection to the base of the bundle. $\pi^*$ denotes the pull-back defined by $\pi$. $\Omega_k$ is the closed 2-form on $\tilde{G}$ given by

$$\Omega_k(g)(V, W) = \text{Re} \int_0^{2\pi} (X(x), kY'(x))(x) dx = \langle X, kY' \rangle \quad (3.2)$$

where $V, W \in T_g \tilde{G}$ are vectors tangent to the curves

$$t \mapsto \text{ge}^{tX}, \quad t \mapsto \text{ge}^{tY} \quad (3.3)$$

respectively. We will refer to the symplectic manifold $(T^*\tilde{G}, \omega_k)$ as $T^*\tilde{G}_k$.

The left and right actions of $\tilde{G}$ on itself lift to the following symplectic actions on $T^*\tilde{G}_k$:

$$\tilde{G} \times T^*\tilde{G}_k \rightarrow T^*\tilde{G}_k, \quad (h, (g, \mu)) \mapsto L_h(g, \mu) = (hg, \mu) \quad (3.4)$$

and

$$\tilde{G} \times T^*\tilde{G}_k \rightarrow T^*\tilde{G}_k, \quad (h, (g, \mu)) \mapsto R_{h^{-1}}(g, \mu) = (gh^{-1}, h\mu h^{-1} + kh'h^{-1}). \quad (3.5)$$

The corresponding momentum maps are given respectively by

$$J^L(g, \mu) = g\mu g^{-1} + kg'g^{-1} \quad (3.6)$$

and

$$J^R(g, \mu) = -\mu. \quad (3.7)$$
See Section 4. In the above formulae \( h\mu h^{-1} \) means \( \text{Ad}^*_h\mu \) and similarly for \( g\mu g^{-1} \), where \( \text{Ad}^* \) is the coadjoint action of \( G \) on \( \mathfrak{g}^* \) (that is \( \text{Ad}^* \) is the dual of the inverse of \( \text{Ad} \)). Below, we prefer to reserve the notations \( \text{Ad}^* \) and \( \text{ad}^* \) for actions on \( \tilde{\mathfrak{g}}^* \). In order not to clutter the flow of the proof, we will prove (3.6) and (3.7) separately in Section 4.

Let us now observe that \( J^L \) and \( J^R \) are independent on \( T^*\tilde{G}_k \). To see this it is sufficient to check that for \( \xi, \eta \in \tilde{\mathfrak{g}} \) the function \( \Phi = \langle \xi, J^L \rangle + \langle J^R, \eta \rangle \) cannot have zero exterior derivative in any open set unless \( \xi = 0 = \eta \). If \( V \in T_{(g,\mu)}T^*\tilde{G}_k \) is tangent to the curve \( t \mapsto (ge^{tX}, \mu + t\nu) \), we get

\[
(V \llcorner d\Phi)(g\mu) \bigg|_{t=0} = \Phi(ge^{tX}, \mu + t\nu) = \langle g^{-1}\xi g, \nu + [X, \mu] + kX' \rangle - \langle \eta, \nu \rangle,
\]

from where the assertion follows.

Let us also compute the Poisson brackets. For any \( X, Y \in \tilde{\mathfrak{g}} \) we have

\[
\{ \langle J^L, X \rangle, \langle J^L, Y \rangle \} = \langle J^L, [X, Y] \rangle + \langle X, kY' \rangle,
\]

\[
\{ \langle J^R, X \rangle, \langle J^R, Y \rangle \} = \langle J^R, [X, Y] \rangle - \langle X, kY' \rangle,
\]

\[
\{ \langle J^L, X \rangle, \langle J^R, Y \rangle \} = 0.
\]

It follows that all Poisson brackets of the form \( \{J^L^*\phi, J^R^*\psi\} \) vanish, where \( J^L^* \) and \( J^R^* \) denote the respective pull-back mappings. Note also that it follows from (3.9a) and from (3.9b) that both \( J^L \) and \( J^R \) are not infinitesimally equivariant. See the last paragraph of section 4.

We will denote by \( \ell(X) \) and \( r(X) \) the infinitesimal generators of the left and right actions (3.4) and (3.5); they are vector fields on \( T^*\tilde{G}_k \). Thus, by definition of the momentum map,

\[
\ell(X) = \mathcal{X}_{<J^L, X>} \quad \text{and} \quad r(X) = \mathcal{X}_{<J^R, X>},
\]

where \( \mathcal{X}_H \) denotes the Hamiltonian vector field corresponding to the function \( H \).

Suppose now that \( \Phi \in C^\infty(T^*\tilde{G}_k) \) and that \( \Phi \) can be factored through \( J^L \times J^R \). If \( \Phi \) is invariant under the left action \( L \) given by (3.4), then \( \Phi \) is a function of \( J^R \) only. That is, if \( \Phi = f(J^L, J^R) \) then

\[
\Phi \circ L_h = \Phi \quad \forall h \in \tilde{G} \quad \Leftrightarrow \quad \frac{\partial f}{\partial a}(a, b) = 0 \quad \forall a, b \in \tilde{\mathfrak{g}}^*.
\]

We define the infinitesimal action \( v \) of \( \mathbb{C} \), considered to be a real abelian Lie algebra, on \( T^*\tilde{G}_k \):

\[
v : \mathbb{C} \to \text{vect}(T^*\tilde{G}_k),
\]

by declaring \( v(z)(g, \mu) \in T_{(g,\mu)}T^*\tilde{G}_k \) to be the vector tangent at \( t = 0 \) to the curve

\[
t \mapsto (ge^{tzg^{-1}}g', \mu + t\mu').
\]
Clearly $v$ defines an infinitesimal action, i.e. $[v(z_1), v(z_2)] = 0$ for any $z_1, z_2 \in \mathbb{C}$. It is also an infinitesimally symplectic action, since for any $F, H \in C^\infty(T^*\tilde{G}_k)$ we have

$$\{v(z) \, \lrcorner \, dF, H\} + \{F, v(z) \, \lrcorner \, dH\} = v(z) \, \lrcorner \, \{F, H\}, \quad (3.14)$$

where $\lrcorner$ denotes the interior derivative operation (contraction on the first index) of a vector field on a form. The momentum map for the action given by (3.13) is computed in Section 4 and has the expression

$$J(g, \mu) = \int_0^{2\pi} \left( g^{-1}g', \mu \right) + \frac{1}{2} \int_0^{2\pi} \left( kg^{-1}g', g^{-1}g' \right). \quad (3.15)$$

The map $J$ is clearly infinitesimally equivariant. If $k \neq 0$, (3.15) can be rewritten in the following way:

$$J = \frac{1}{2k} \int_0^{2\pi} \left( (J^L, J^L) - (J^R, J^R) \right). \quad (3.16)$$

Note that the algebraic dual to $\mathbb{C}$ has been identified with $\mathbb{C}$ itself via the real pairing that associates to $(z_1, z_2)$ the real part of the product $z_1 z_2$.

The $\mathbb{C}$ action $v$ and the left action $L$ of $\tilde{G}$ on $T^*\tilde{G}_k$ are compatible in the sense that

$$[v(z), \ell(X)] = -\ell(z \cdot X) \quad (3.17)$$

where $z \cdot X = z X'$. Therefore the semi–direct product $\mathbb{C} \ltimes \tilde{g}$ acts infinitesimally symplectically on $T^*\tilde{G}_k$ and this action has the momentum map

$$J \oplus J^L : T^*\tilde{G}_k \to \mathbb{C} \times \tilde{g}^*. \quad (3.18)$$

This is a non-equivariant momentum map from $T^*\tilde{G}_k$ to the dual of the semi–direct product $\mathbb{C} \ltimes \tilde{g}$. As recalled in Section 4, the cocycle term needed to define the canonical equivariant extension of this momentum map is found by the following computation:

$$\{< (J, J^L), (z, X)>, < (J, J^L), (\zeta, Y) > \}$$
$$= < J^L, [X, Y] + zY' - \zeta X' > + < X, kY' >$$
$$= < (J, J^L), [(z, X), (\zeta, Y)] > + < X, kY' >$$
$$= < (J, J^L, k), [(z, X, a), (\zeta, Y, b)] >. \quad (3.19)$$

where the Lie bracket in the final expression is that given by (2.4) and $a$ and $b$ are any complex numbers. In other words, the map

$$\mathcal{S} = (J, J^L, k) : T^*\tilde{G}_k \to \tilde{g}^* \quad (3.20)$$

is infinitesimally equivariant, i.e for any $\varphi \in C^\infty(\tilde{g}^*)$, and for any $(z, X, \sigma) \in \tilde{g}$,

$$(v(z) + \ell(X)) \, \lrcorner \, d(\varphi \circ \mathcal{S}) = < \mathcal{S}, [\nabla\mathcal{S}\varphi, (z, X, \sigma)] >. \quad (3.21)$$
Suppose now that \( F \in I(\mathfrak{g}^*) \). By (2.8) we have
\[
\langle (\alpha, \xi, e), \nabla_{(\alpha, \xi, e)} F \rangle \geq 0 \quad \forall (z, X, \sigma) \in \bar{\mathfrak{g}}, \; \forall (\alpha, \xi, e) \in \mathfrak{g}^*.
\tag{3.22}
\]
Letting \( \mathcal{G}(g, \mu) = (\alpha, \xi, e) \) and combining relations (3.21) and (3.22) for \( z = 0 \), we get
\[
\ell(X) \cdot d(F \circ \mathcal{G}) = 0 \quad \forall X \in \bar{\mathfrak{g}}.
\tag{3.23}
\]
Thus we have shown that \( F \circ \mathcal{G} \in C^\infty(T^*\tilde{G}_k) \) is invariant with respect to the infinitesimal left action of \( \mathfrak{g} \) on \( T^*\tilde{G}_k \). Moreover, from (3.16), \( F \circ \mathcal{G} \) factors through \( J^L \times J^R \). By (3.11) it follows then that \( F \circ \mathcal{G} \) must be of the form
\[
F \circ \mathcal{G} = \psi \circ J^R
\tag{3.24}
\]
for some \( \psi \in C^\infty(\mathfrak{g}^*) \), i.e. we have
\[
\psi \circ J^R = F(J^L, J^R, k)
= F\left(\frac{1}{2k} \int_0^{2\pi} ((J^L, J^L)) - (J^R, J^R)), J^L, k)\right).
\tag{3.25}
\]
i.e. \( F \) is a function of \( \int_0^{2\pi} (J^R, J^R) \). Denoting \( J^L(g, \mu) = \xi, \mathcal{J}(g, \mu) = \alpha \) and observing that
\[
-\frac{1}{2} \int_0^{2\pi} (J^R, J^R)(g, \mu) = k\alpha - \frac{1}{2} \int_0^{2\pi} (\xi, \xi),
\tag{3.26}
\]
it follows that for each fixed \( k \), \( F \) has the form
\[
F = \mathcal{G} \circ \kappa
\tag{3.27}
\]
for some \( \mathcal{G} \in C^\infty(\mathbb{C}) \), where
\[
\kappa(\alpha, \xi, k) = k\alpha - \frac{1}{2} \int_0^{2\pi} (\xi(x), \xi(x)) dx.
\tag{3.28}
\]
Functions depending only on \( k \) are clearly invariant since \( k \) is the coordinate of the center. Therefore, the above result holds for each \( k \) and we conclude that
\[
F = \mathcal{F}(\kappa, \pi),
\tag{3.29}
\]
where \( \pi : \mathfrak{g}^* \to \mathbb{C} \) is the projection onto the center, \( \kappa \) is given by (3.28) and \( \mathcal{F} \in C^\infty(\mathbb{C}^2) \).

§4. The momentum maps \( J^L, J^R, \mathcal{J} \)

For an introduction to symplectic manifolds the reader is referred to one of the many standard books on the subject, for example Abraham and Marsden [1978] or Arnold [1978]. Let us begin with a closer look at the symplectic manifold \( T^*\tilde{G}_k \).
Let $\varphi, \psi \in C^\infty(\tilde{G} \times \tilde{g}^\ast)$ and denote by $\varphi^\mu, \psi^\mu \in C^\infty(\tilde{G}), \varphi^g, \psi^g \in C^\infty(\tilde{g})$ the partial functions $\varphi^\mu(g) = \varphi^g(\mu) = \varphi(g, \mu), \psi^\mu(g) = \psi^g(\mu) = \psi(g, \mu)$. The Poisson bracket is given by

$$\{\varphi, \psi\}(g, \mu) = \left. \frac{d}{dt} \right|_{t=0} \left( \varphi^\mu(ge^t\nabla \psi^g(\mu)) - \psi^\mu(ge^t\nabla \varphi^g(\mu)) \right)$$

$$- <\mu, [\nabla \varphi^g(\mu), \nabla \psi^g(\mu)] > - <k\nabla \varphi^g(\mu), (\nabla \psi^g(\mu))' > \quad (4.1)$$

where $\nabla \varphi^g(\mu), \nabla \psi^g(\mu) \in \tilde{g}$ are the gradients of $\varphi^g$ and $\psi^g$ relative to the pairing $<, >$. (The reader might find it helpful to note that the rule for computing the gradient of a function relative to a pairing was given explicitly in (2.7) for the pairing given by (2.5).)

It is straightforward to verify that $L_h$ and $R_{h^{-1}}$ define symplectic actions by checking,

$$\{\varphi \circ L_h, \psi \circ L_h\} = \{\varphi, \psi\} \circ L_h \quad \forall \varphi, \psi \in C^\infty(\tilde{G} \times \tilde{g}^\ast), \text{ for any } h \in \tilde{G} \quad (4.2)$$

and similarly for $R_{h^{-1}}$.

The values at $(g, \mu)$ of the infinitesimal generators of these two actions, $\ell(X)$ and $r(X)$, are the derivatives with respect to $t$ at $t = 0$ to the curves in $\tilde{G} \times \tilde{g}^\ast$ given by

$$t \mapsto (e^{tX}g, \mu) \quad \text{and} \quad t \mapsto (ge^{-tX}, e^{tX} \mu e^{-tX} + tkX'). \quad (4.3)$$

By the definition of the momentum map, $J^L$ and $J^R$ are given as the solutions to the equations

$$X_{<J^L, X>} = \ell(X) \quad X_{<J^R, X>} = r(X), \quad (4.4)$$

where $X_H \in vect(\tilde{G} \times \tilde{g}^\ast)$ is the Hamiltonian vector field corresponding to the function $H$, i.e.

$$X_H \lrcorner dK = \{K, H\}, \quad \forall K \in C^\infty(\tilde{G} \times \tilde{g}^\ast). \quad (4.5)$$

Let us verify (4.4) for the map $J^L$. If $\varphi \in C^\infty(\tilde{G} \times \tilde{g}^\ast)$, we have

$$(\ell(X) \lrcorner d\varphi)(g, \mu) = \left. \frac{d}{dt} \right|_{t=0} \varphi(e^{tX}g, \mu) = \{\varphi, <J^L, X>\}(g, \mu). \quad (4.6)$$

The last equality in (4.6) is obtained by using (4.1) and the formulae

$$\nabla <J^L, X>^g(\mu) = g^{-1}Xg,$$

$$\left. \frac{d}{dt} \right|_{t=0} <J^L, X>^\mu(ge^{tY}) = <[Y, \mu] + kY', g^{-1}Xg> \quad \text{for any } Y \in \tilde{g}. \quad (4.7)$$

One similarly verifies formula (4.4) for the momentum map $J^R$ by using (4.1) and

$$\nabla <J^R, X>^g(\mu) = -X,$$

$$\left. \frac{d}{dt} \right|_{t=0} <J^R, X>^\mu(ge^{tY}) = 0 \quad \text{for any } Y \in \tilde{g}. \quad (4.8)$$
Next, we turn to the computation of the momentum map for the \( C \) action \( v \). For \( \varphi \in C^\infty(\tilde{G} \times \tilde{g}^*) \), we have

\[
(v(z)_*d\varphi)(g, \mu) = \left. \frac{d}{dt} \right|_{t=0} \varphi(ge^{tzg^{-1}g'}, \mu + tz\mu') = \{\varphi, <J, z>\}(g, \mu),
\]

which again follows from (4.1) and the formulae

\[
\nabla <J, z>^g(\mu) = zg^{-1}g', \tag{4.10a}
\]

\[
\left. \frac{d}{dt} \right|_{t=0} <J, z>^\mu(g e^{tY}) = [g^{-1}g', Y] + Y', z\mu + <[g^{-1}g', Y] + Y', zk g^{-1}g'> \quad \text{for any } Y \in \tilde{g}. \tag{4.10b}
\]

In general, if \( P : M \to a^* \) is a momentum map for the infinitesimal symplectic action of a Lie algebra \( a \) on a connected symplectic manifold \( M \), we have

\[
[X_{<P,A>}, X_{<P,B>}] = X_{<P,[AB]>} \quad \forall A, B \in a, \tag{4.11}
\]

but the left hand side of (4.11) is \( X_{\{<P,A>, <P,B>\}} \) and it follows that

\[
\{<P,A>, <P,B>\} = <P, [A, B]> + \Sigma(A, B), \tag{4.12}
\]

where \( \Sigma(A, B) \) is a constant function on \( M \). It can be checked that the constant term in (4.12) corresponds to a coadjoint one-cocycle on \( a \) and hence although \( P \) may not be equivariant (equivariance would be the case when the constant is zero), an equivariant map \( \hat{P} \) can be constructed from \( P \), having its image in the central extension \( \hat{a} = a \oplus \mathbb{R} \) of \( a \). This is the reason for the computation in (3.19) and for the remark following the formulae in (3.9).

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