Continuum Approach to Non-equilibrium Quantum Functional Integral

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Abstract

Standard derivations of the functional integral in non-equilibrium quantum field theory are based on the discrete time representation [1, 2]. In this work we derive the non-equilibrium functional integral for non-interacting bosons and fermions using a continuum time approach by accounting for the statistical distribution through the boundary conditions and using them to evaluate the Green’s function.

Non-equilibrium quantum field theory has its foundations on the closed time contour. The contour, introduced by Schwinger [3, 4], is used to evaluate expectation values and replaces the real time axis of the equilibrium theory. This contour was developed in statistical quantum mechanics by Kadanoff and Baym [5] and its formulation was made simpler and more transparent by Keldysh [6]. Keldysh’s formulation is even more transparent in its functional integral analogue, developed by Kamenev [7]. This formulation, used also in Refs. [1, 2], accounts for the statistical distribution as an off-diagonal component of the inverse Green’s function in the discrete time representation. The Green’s function is calculated by matrix inversion and the continuum limit is eventually taken. Instead, in this work we adopt a continuum approach: we derive the functional integral by accounting for the statistical distribution through the boundary conditions and we evaluate the Green’s function as a solution of a differential equation with boundary conditions derived from those of the functional integral.

The central object of non-equilibrium quantum functional integral is the partition function along the closed time contour,

$$Z = \frac{\text{Tr} \left[ \hat{U} (t_i, t_f) \hat{U} (t_f, t_i) \hat{\rho} \right]}{\text{Tr} [\hat{\rho}]}, \quad (1)$$

where $\hat{\rho}$ is the initial distribution, $\hat{U}(t_f, t_i) = T e^{-i \int_{t_i}^{t_f} dt \hat{H}(t)}$ ($\hbar = 1$) is the time-evolution operators from the initial time $t_i$ to the final time $t_f$ and $T$ is the time-order operator [11]. The operators $\hat{U} (t_f, t_i)$ and $\hat{U} (t_i, t_f)$ are referred to as forward and backward time-evolution operators.
This partition function has meaning only if there are different external sources on forward and backward time-evolution operators. These sources are used to generate correlation functions and are usually set to zero at the end of a calculation [1]. When the external sources are the same on forward and backward time-evolution operators, $\hat{U}(t_i,t_f)\hat{U}(t_f,t_i) = 1$ and the partition function is identically $Z = 1$. However, since the following derivation is independent of sources, we omit them, keeping in mind that they should be present.

We start by constructing the functional integral representation of $Z$ for a non-interacting single-level system of energy $\varepsilon$ populated by bosons or fermions and we will extend it to many-level systems at the end. The Hamiltonian is $\hat{H}(t) = \varepsilon \hat{a}^\dagger \hat{a}$, where $\hat{a}$ and $\hat{a}^\dagger$ are bosonic or fermionic creation and annihilation operators, satisfying the canonical commutation relation $[\hat{a},\hat{a}^\dagger]_+ = 1$ for bosons or anti-commutation relation $[\hat{a},\hat{a}^\dagger]_- = 1$ for fermions. To derive the functional integral representation of the partition function, we consider the coherent states of bosons and fermions, $\phi$ and $\tilde{\phi}$, respectively associated to $\hat{a}$ and $\hat{a}^\dagger$ through the relations $\hat{a}|\phi\rangle = \phi |\phi\rangle$ and $\langle \phi | \hat{a}^\dagger = \tilde{\phi} \langle \phi |$, where $\phi$ and $\tilde{\phi}$ are complex numbers for bosons or Grassmann number for fermions. Inserting the resolution of identity of coherent states between the three operators in the numerator of Eq. (1) and using the coherent state representation of the trace, the partition function becomes,

$$Z = \frac{1}{\text{Tr}[\hat{\rho}]} \int d[\tilde{\phi}_i,\phi_i^\dagger] d[\tilde{\phi}_f,\phi_f] e^{-\tilde{\phi}_f \phi_f^\dagger} \langle \phi_f^\dagger | \hat{U}(t_i,t_f) | \phi_f \rangle e^{-\tilde{\phi}_f \phi_f^\dagger} \langle \phi_f \hat{U}(t_f,t_i) | \phi_f^\dagger \rangle,$$

(2)

where $d[\tilde{\phi},\phi] = d(\text{Re} \phi) d(\text{Im} \phi) / \pi$, $\zeta = +1$ for bosons and $d[\tilde{\phi},\phi] = d\tilde{\phi} d\phi$, $\zeta = -1$ for fermions. We used the superscript + and − for initial states acting respectively on forward and backward time-evolution operators. As opposed to the initial states, $|\phi_f^\dagger \rangle$ and $|\phi_f \rangle$, the final state, $|\phi_f \rangle$, acts on both forward and backward time-evolution operators.

The partition function assumes a simpler form for initial states defined by the distribution,

$$\hat{\rho} = \tilde{\rho}^{\dagger} \hat{\rho}, \quad \rho = \frac{\bar{n}}{1 + \zeta \bar{n}},$$

(3)

where $\bar{n}$ is the average occupation of the level,

$$\bar{n} = \langle \hat{a}^\dagger \hat{a} \rangle_\rho = \frac{\text{Tr}[\hat{a}^\dagger \hat{a} \hat{\rho}]}{\text{Tr}[\hat{\rho}]}.$$  

(4)

Here, $\bar{n} \geq 0$ for bosons and $0 \leq \bar{n} \leq 1$ for fermions. An example of a state described by distribution (3) is the thermal state of a non-interacting system, where $\bar{n} = (e^{\frac{T \mu}{\varepsilon} - \zeta} - 1)^{-1}$, $\rho = e^{-\frac{T \mu}{\varepsilon}}$ is the Boltzmann factor, $T$ is the temperature and $\mu$ is the chemical potential. Using distribution (3) and the identity $\langle \phi | \tilde{\rho} \phi' \rangle = e^{\tilde{\phi} \phi'}$, the integrand in the first line of Eq. (2) becomes $e^{-\tilde{\phi}_f \phi_f^\dagger} \langle \phi_f^\dagger | \hat{U}(t_i,t_f) | \phi_f \rangle e^{-\tilde{\phi}_f \phi_f^\dagger} \langle \phi_f \hat{U}(t_f,t_i) | \phi_f^\dagger \rangle$ and integration over $\tilde{\phi}_f$ enforces the constraint $\phi_f^\dagger = \zeta \phi_f$. Integrating over $\phi_f^\dagger$ the partition function (2) becomes,

$$Z = (1 - \zeta \rho)^\zeta \int d[\tilde{\phi}_i,\phi_i] d[\tilde{\phi}_f,\phi_f] e^{-\tilde{\phi}_f \phi_f^\dagger} \langle \phi_i^\dagger \hat{U}(t_i,t_f) | \phi_f \rangle e^{-\tilde{\phi}_f \phi_f^\dagger} \langle \phi_f \hat{U}(t_f,t_i) | \zeta \phi_i \rangle,$$

(5)
where we remove the label \( \phi \) from the initial state. The pre-factor of the integral is the normalization of the partition function, \( \text{Tr} [\hat{\rho}] = (1 - \zeta)\zeta \). We express the expectation value of the time-evolution operator in the functional integral form,

\[
e^{-\hat{\rho} \phi_f \langle \phi_f | \hat{U} (t_f, t_i) | \phi_i \rangle} = \int_{\phi(t_i) = \phi_i}^{\phi(t_f) = \phi_f} D[\varphi, \phi] e^{i \int_{t_i}^{t_f} \varphi(t) (i \partial_t - \epsilon) \phi(t)} .
\]  

(6)

Here, the functional integral measure and the action in the exponent are defined by splitting the time interval \( t_f - t_i \) in \( N \) smaller intervals \( \delta t = (t_f - t_i)/N \) and taking the symbolical limit,

\[
D[\varphi, \phi] = \left. \lim_{N \to \infty} \prod_{n=1}^{N-1} d[\varphi_n, \phi_n] \right|_{t_i}^{t_f}
\]

(7)

\[
\int_{t_i}^{t_f} dt \phi(t) (i \partial_t - \epsilon) \phi(t) = \lim_{N \to \infty} \sum_{n=1}^{N} \delta t \left[ \varphi_n \frac{\phi_n - \phi_{n-1}}{\delta t} - \epsilon \varphi_n \phi_{n-1} \right]
\]

where \( \phi_0 = \phi_i \) and \( \phi_N = \phi_f \). Integrating over initial and final fields, we arrive at,

\[
Z = (1 - \zeta)\zeta \int_{\phi^+(t_i) = \zeta \rho \phi^-(t_i)}^{\phi^-(t_f)} D[\varphi^+, \phi^+] D[\varphi^-, \phi^-] e^{i \int_{t_i}^{t_f} [\varphi^+(t) (i \partial_t - \epsilon) \phi^+(t) - \varphi^-(t) (i \partial_t - \epsilon) \phi^-(t)]} .
\]

(8)

where \( + \) and \( - \) fields are associated to forward and backward time-evolutions and the minus sign in front of the second term in the exponent comes from the inversion of the integration limits. The integration measure has been redefined to contain the additional integration over the initial and final variables, \( \varphi_0, \varphi_\infty \) and \( \phi^+_N, \phi^-_N \), which are \( \phi^-(t_i), \phi^-(t_i) \) and \( \phi^+(t_f), \phi^+(t_f) \) in continuum notation. This partition function is the sum over all the possible configurations of \( + \) and \( - \) fields, related only through the boundary conditions in distant past and future, with a weight that depends on the difference of their actions. The stationary configurations satisfy \( \phi^+(t) = \phi^-(t) \) and \( \tilde{\phi}^+(t) = \tilde{\phi}^-(t) \). These configurations are the most important as, among the individual configurations, they carry most of the weight of the partition function.\(^1\)

As a consequence, the quantity \( \phi^+(t) - \phi^-(t) \) measures how far away configurations are from the most important ones.

**Bosons**

The goal of the remaining derivation is the evaluation of the Green’s function and the procedure is different for bosons and fermions. We start from bosons. In analogy to equilibrium quantum field theory \([2]\), we associate stationary configurations, \( \phi^+(t) = \phi^-(t) \) and \( \tilde{\phi}^+(t) = \tilde{\phi}^-(t) \), to classical ones. Motivated by this identification and following Ref. \([1]\), we define a classical field, \( \phi^c(t) \), and a quantum field, \( \phi^q(t) \), as

\[
\begin{align*}
\phi^c(t) &= \frac{1}{\sqrt{2}} (\phi^+(t) + \phi^-(t)) , \\
\phi^q(t) &= \frac{1}{\sqrt{2}} (\phi^+(t) - \phi^-(t)) .
\end{align*}
\]

(9)

\(^1\)We note that configurations satisfying \( \phi^+ = -\phi^- \) and \( \tilde{\phi}^+ = -\tilde{\phi}^- \) are stationary, as well. However, contrary to the previous ones, they may not be stationary for an interacting system \([1]\).
and the same for the complex conjugate. This transformation is known as Keldysh rotation [6]. In terms of these new fields, classical configurations have $φ^0(t) = 0$ and, as we will see later, $φ^c(t)$ satisfies the classical equations of motion. Moreover, the quantum field $φ^q(t)$ measures how far away configurations are from classical ones. In these new fields, the partition function becomes,

$$Z = \frac{1}{1 + n} \int_{φ^0(t_f) = 0}^{φ^c(t_i) = -φ^c(t_f)} D[φ^c, φ^q] D[φ^0, φ^q] e^{i \int_{t_i}^{t_f} dt φ^0(t_i) [G_n^{-1}]^{αβ}(t) φ^β(t)},$$  \hspace{1cm} (10)

where $α, β = cl, q$ and we use the conventional sum over repeated indices. The inverse Green’s function is,

$$[G_n^{-1}]^{αβ}(t) = \left( \begin{array}{cc} 0 & i ∂_t - ε \\ i ∂_t - ε & 0 \end{array} \right).$$  \hspace{1cm} (11)

The inverse Green’s function depends on the initial average occupation, $n$, because, by construction, it is an operator whose domain is the set of fields, $φ^α(t)$, that satisfy the boundary conditions of Eq. (10) and the initial boundary condition depends on $n$. Integration over the quantum fields, $φ^α(t)$ and $φ^q(t)$, enforces the classical equations of motion, $(i ∂_t - ε) φ^c(t) = 0$ and $(-i ∂_t - ε) φ^cl(t) = 0$. This fact originally motivated the name “classical field” [1].

It is worth to pause for a moment and discuss the initial boundary condition in the partition function (10). This condition relates the values of initial classical and quantum variables, $φ^c(t_i)$ and $φ^q(t_i)$, and initial occupation number, $n$. Taking the modulus, the equation reads,

$$|φ^α(t_i)| = \frac{1}{1 + 2n} |φ^c(t_i)|.$$  \hspace{1cm} (12)

Since for bosons $n ≥ 0$, for a fixed $φ^c(t_i)$ the quantum variable is initially bounded, $0 ≤ |φ^q(t_i)| ≤ |φ^c(t_i)|$ (A more restrictive condition applies in the final state, $φ^α(t_f) = 0$). When $n ≪ 1$, the level occupation is microscopic and $φ^q(t_i) ∼ φ^c(t_i)$. Instead, when $n ≫ 1$, the level occupation becomes macroscopic and $|φ^q(t_i)| ≪ |φ^c(t_i)|$. This, as expected [2], means that the larger the initial occupation of the level, the closer the initial state of the system is to a classical one. These results are summarized in Fig. [1].

We proceed and calculate the Green’s function, $G^{αβ}(t, t') = -i ⟨φ^α(t) φ^β(t')⟩$, which satisfies the differential equation,

$$\left( \begin{array}{cc} 0 & i ∂_t - ε \\ i ∂_t - ε & 0 \end{array} \right) \left( \begin{array}{cc} G^{ccl}(t, t') & G^{ccl}(t, t') \\ G^{qcl}(t, t') & G^{qcl}(t, t') \end{array} \right) = \left( \begin{array}{cc} δ(t - t') & 0 \\ 0 & δ(t - t') \end{array} \right),$$  \hspace{1cm} (13)

where we used the matrix representation (11) of the inverse Green’s function. The general solution of this equation is,

$$G^{αβ}(t, t') = -ie^{−iε(t−t')} \left( \begin{array}{cc} a(b + 1) & bθ(t − t') + bθ(t′ − t) \\ (c + 1) θ(t − t') + cθ(t′ − t) & d \end{array} \right).$$  \hspace{1cm} (14)

The constants $a, b, c$ and $d$ are fixed by the boundary conditions in (10). For this, we multiply each boundary condition of the partition function (10) by $φ^α(t')$, take the average and use the
Figure 1: Example of a configuration of the modulus of quantum and classical fields, \(|\phi^{cl}(t)|\) and \(|\phi^{q}(t)|\). The smaller \(|\phi^{q}(t)|\) the closer is the configuration to a classical one. At the final time, \(t = t_f\), \(|\phi^{q}(t_f)| = 0\). Instead, at \(t = t_i\), the initial value of the quantum field, \(|\phi^{q}(t_i)|\), is related to the initial value of the classical field, \(|\phi^{cl}(t_i)|\), through Eq. (12). For a fixed value of \(|\phi^{cl}(t_i)|\), \(|\phi^{q}(t_i)| \rightarrow |\phi^{cl}(t_i)|\) for \(\tilde{n} \rightarrow 0\) and \(|\phi^{q}(t_i)| \rightarrow 0\) for \(\tilde{n} \rightarrow \infty\).

The Green’s function, \(G^{\alpha\beta}(t,t') = -i\langle \phi^{\alpha}(t)\bar{\phi}^{\beta}(t') \rangle\), to obtain the boundary conditions of the Green’s function,

\[
\begin{align*}
G^{q\alpha}(t_f,t') & = 0, \\
G^{clo}(t_i,t') & = -(1 + 2\tilde{n})G^{q\alpha}(t_i,t').
\end{align*}
\]  

(15)

The first boundary condition fixes the constants of \(G^{q\alpha}(t,t')\) to \(c = -1\) for \(\alpha = cl\) and \(d = 0\) for \(\alpha = q\), where we use the fact that the first and second theta functions in \(G^{q\alpha}(t_f,t')\) are respectively equal to 1 and 0 as \(t_i < t' < t_f\). Using \(G^{q\alpha}(t_i,t')\) in the second boundary condition, the constants of \(G^{clo}(t,t')\) are fixed to \(a = 1 + 2\tilde{n}\) for \(\alpha = cl\) and \(b = 0\) for \(\alpha = q\). Substituting these values in Eq. (14), the Green’s function becomes,

\[
G^{\alpha\beta}(t,t') = \begin{pmatrix} G^K(t,t') & G^R(t,t') \\ G^A(t,t') & 0 \end{pmatrix},
\]

(16)

where,

\[
\begin{align*}
G^R(t,t') & = -i\theta(t-t') e^{-i\varepsilon(t-t')} , \\
G^A(t,t') & = i\theta(t'-t) e^{-i\varepsilon(t-t')} , \\
G^K(t,t') & = -i(1+2\tilde{n}) e^{-i\varepsilon(t-t')} .
\end{align*}
\]  

(17)

The Green’s functions (17) are retarded, advanced and Keldysh Green’s functions, in agreement with Ref. [1] for a thermal occupation of the level, \(\tilde{n} = (e^{-T\mu} - 1)^{-1}\). Retarded

\footnote{To fix the constants we did not consider \(t' = t_i, t_f\) because this would lead to equal time Green’s functions and the theta functions in \(G^{rel}(t,t')\) and \(G^{clq}(t,t')\) have not been regularized at equal times, \(t = t'\), yet.}
and advanced Green’s functions are still undefined at equal times because the theta function, $\theta(t)$, needs to be regularized at $t = 0$. First, using the Fourier transform, $G^{R,A}(\omega) = (\omega - \omega_0 + i0^\pm)^{-1}$, we have,

$$G^R(t, t) - G^A(t, t) = \int \frac{d\omega}{2\pi} [G^R(\omega) - G^A(\omega)] = -i.$$  \hspace{1cm} (18)

Second, $\langle \phi(t) \bar{\phi}(t) \rangle = -\langle \phi(t) \bar{\phi}(t) \rangle^\dagger$ which leads to $G^R(t, t) = [G^A(t, t)]^\dagger$, where $\dagger$ denotes complex conjugation and time transposition. It follows that the regularization is $\theta(t = 0) = 1/2$.

**Fermions**

The derivation is different for fermions, since they do not have a classical analogue. Instead of classical and quantum, we use the labels 1 and 2 through the transformation [1],

$$\begin{align*}
\begin{cases}
\phi_1(t) = \frac{1}{\sqrt{2}} (\phi^+(t) + \phi^-(t)) \\
\phi_2(t) = \frac{1}{\sqrt{2}} (\phi^+(t) - \phi^-(t))
\end{cases},
\begin{cases}
\bar{\phi}_1(t) = \frac{1}{\sqrt{2}} (\bar{\phi}^+(t) - \bar{\phi}^-(t)) \\
\bar{\phi}_2(t) = \frac{1}{\sqrt{2}} (\bar{\phi}^+(t) + \bar{\phi}^-(t))
\end{cases}.
\end{align*} \hspace{1cm} (19)$$

In these new variables, the partition function reads,

$$Z = \frac{1}{1 + \bar{n}} \int_{\phi_2(t_f) = 0} D[\bar{\phi}_1, \phi_1] D[\bar{\phi}_2, \phi_2] e^{\int_{t_i}^{t_f} dt \bar{\phi}_2(t)[G_{n}^{-1}]^{ab}(t)\phi_b(t)},$$  \hspace{1cm} (20)

where $a, b = 1, 2$ and the inverse Green’s function is,

$$[G_{n}^{-1}]^{ab}(t) = \begin{pmatrix} i\partial_t - \varepsilon & 0 \\ 0 & i\partial_t - \varepsilon \end{pmatrix}_{\bar{n}}.$$  \hspace{1cm} (21)

The Green’s function, $G^{ab}(t, t') = -i\langle \phi^a(t) \bar{\phi}^b(t') \rangle$, satisfies the differential equation,

$$\begin{pmatrix} i\partial_t - \varepsilon & 0 \\ 0 & i\partial_t - \varepsilon \end{pmatrix}_{\bar{n}} \begin{pmatrix} G^{11}(t, t') & G^{12}(t, t') \\ G^{21}(t, t') & G^{22}(t, t') \end{pmatrix} = \begin{pmatrix} \delta(t - t') & 0 \\ 0 & \delta(t - t') \end{pmatrix},$$  \hspace{1cm} (22)

for which the general solution is,

$$G^{ab}(t, t') = -ie^{-i\varepsilon(t-t')} \begin{pmatrix} (a + 1) \theta(t - t') + a\theta(t' - t) \\ c \end{pmatrix} \begin{pmatrix} b \\ (d + 1) \theta(t - t') + d\theta(t' - t) \end{pmatrix},$$  \hspace{1cm} (23)

We fix the constants $a, b, c$ and $d$ using the boundary conditions of the Green’s function,

$$\begin{align*}
G^{2a}(t_f, t_f) = 0, \\
G^{1a}(t_i, t_i) = -(1 - 2\bar{n})G^{2a}(t_i, t_i).
\end{align*} \hspace{1cm} (24)$$
obtained from the partition function (20) in the same way we did for bosons. The boundary conditions at \( t = t_f \) fix \( c = 0 \) and \( d = -1 \) and the ones at \( t = t_i \) fix \( a = 0 \) and \( b = 1 - 2\mathcal{n} \). Substituting these values in Eq. (23), the Green’s function becomes,

\[
G_{ab}(t, t') = \begin{pmatrix} G^R(t, t') & G^K(t, t') \\ 0 & G^A(t, t') \end{pmatrix},
\]

where,

\[
G^R(t, t') = -i\theta(t - t')e^{-i\varepsilon(t-t')},
\]

\[
G^A(t, t') = i\theta(t' - t)e^{-i\varepsilon(t-t')},
\]

\[
G^K(t, t') = -i(1 - 2\mathcal{n})e^{-i\varepsilon(t-t')}.
\]

(26)

As in the bosonic case, we have retarded, advanced and Keldysh Green’s function, in agreement with Ref. [1] for a thermal occupation of the level, \( \mathcal{n} = (e^{-\frac{\varepsilon}{kT}} + 1)^{-1} \). The regularization of the theta functions are the same as for bosons. The only difference between Eqs. (17) and (26) is the minus sign in front of \( \mathcal{n} \) in the Keldysh Green’s function.

**Many levels**

In the previous pages we derived the Green’s function for systems with a single level. In this section, we extend the results to systems with many levels. In a system with many levels, for every level \( i \), we consider the annihilation and creation operators \( \hat{a}_i \) and \( \hat{a}_i^\dagger \), satisfying the commutation or anti-commutation relations \([\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}\). The single-level non-interacting Hamiltonian \( H = \varepsilon\hat{a}_i^\dagger\hat{a}_i \), is replaced by \( H = \hat{a}_i^\dagger\varepsilon_{ij}\hat{a}_j \) where, \( \varepsilon_{ij} \), is a Hermitian matrix and the average particle number, \( \mathcal{n} \), is replaced by the one-body density matrix \( \mathcal{n}_{ij} = \langle \hat{a}_i^\dagger\hat{a}_j \rangle \), a Hermitian matrix, as well. For clarity, we employ the bold notation for vectors, \( \hat{a} = (\ldots, \hat{a}_{i-1}, \hat{a}_i, \hat{a}_{i+1}, \ldots) \), and matrices, \( \varepsilon \) and \( \mathcal{n} \). To proceed, we diagonalize the density matrix through a unitary transformation, so that the initial distribution \( \hat{\rho} \) is the product of single-level distributions (3), derive the partition function and transform back to the original density matrix. Associating the coherent states vectors \( \phi \) and \( \bar{\phi} \) to the operators \( \hat{a} \) and \( \hat{a}_i^\dagger \), the partition function of bosons and fermions are respectively (10) and (20) with bold notation and boundary conditions given by,

\[
\begin{align*}
\phi^0(t_f) &= 0, \\
\phi^0(t_i) &= -(I + 2\mathcal{n}^T)^{-1}\phi^d(t_i),
\end{align*}
\]

(27)

and

\[
\begin{align*}
\phi_2(t_f) &= 0, \\
\phi_1(t_i) &= -(I - 2\mathcal{n}^T)\phi_2(t_i),
\end{align*}
\]

(28)

where \( I \) is the identity matrix over the many level indices and \( T \) denotes transposition. The procedure to calculate the Green’s function is the same as before with the difference that the columns on which we apply the boundary conditions are over the many-level indices, as well.
The Green’s function are identical in form to (16) for bosons and (25) for fermions, but now each component is a matrix over the many-level indices given by,

\[
G^R(t, t') = -i\theta(t - t') e^{-i\varepsilon(t-t')},
\]

\[
G^A(t, t') = i\theta(t' - t) e^{-i\varepsilon(t-t')},
\]

\[
G^K(t, t') = -ie^{-i\varepsilon(t-t_i)} \left( I + 2\zeta \bar{n}^T \right) e^{i\varepsilon(t'-t_i)}.
\]

(29)

The transposition of the many-body density matrix is due to the fact that the order of creation and annihilation operator in the definitions of Green’s function and one-body density matrix are exchanged.

Conclusions

We saw how to derive the non-equilibrium functional integral for bosons and fermions by accounting for the initial distribution through the boundary conditions of the functional integral and without resorting to a discrete representation to find the Green’s function. Using this approach, not only is the derivation shorter and simpler than the discrete approach but highlights the properties of the classical configurations. Moreover, it is easy to extend the procedure to many-level systems with non-diagonal density matrices.

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