Model-independent $\tan \beta$ bounds in the MSSM

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**Abstract**

We demonstrate, through the study of the one-loop effective potential in the MSSM, the existence of fully model-independent lower and upper theoretical bounds on $\tan \beta$. We give their general analytic form and illustrate some of their implications.
Phenomenological scenarios for the electroweak symmetry breaking in supersymmetric theories have attracted much interest over the past [], and even more so in the very recent years, motivated by encouraging hints from accelerator data, like the discovery of a heavy top quark [], the possible unification of the three coupling constants [], to mention just two. Several tantalizing theoretical approaches exist of how this breaking would occur. They are, however, generically tributary of assumptions about physics at the GUT or Planck scales and thus suffer from some related theoretical uncertainties. Whatever the correct theory turns out to be at these scales though, the low-energy physics is thought to be described by an effective Lagrangian where a linearly realized global supersymmetry is softly broken and the electroweak symmetry broken at the electroweak scale.

In a typical such model like the Minimal Supersymmetric Standard Model (MSSM) [], the very many free parameters can be theoretically correlated through specific model assumptions together with the requirement of radiative electroweak symmetry breaking (EWSB), leading to phenomenological predictions for the full mass spectrum []. In this context, it is commonly assumed that some theoretical constraints on the effective parameters are obtainable only in the above mentioned model-dependent context. The aim of the present letter is to investigate a model-independent alternative which questions this assumption.

It is indeed important, given the theoretical uncertainties, to disentangle the constraints which are a direct reflection of specific model assumptions from those which proceed from general physical requirements. Clearly, the first general physical requirement is that the effective potential (of the MSSM) should allow for a stable EWSB minimum. This requirement should of course be comitant with that of an unstable minimum. This requirement should of course be comitant with that of an unstable minimum. The conditions for the existence of an electroweak symmetry breaking minimum at the electroweak scale are usually written as

\[
\frac{1}{2} M_Z^2 = \frac{m_1^2 - m_2^2}{\tan^2 \beta - 1} \sin 2\beta = -\frac{2m_2^2}{m_1^2 + m_2^2}
\]  

in the MSSM, where \(\tan \beta \equiv \frac{v_{H_2}}{v_{H_1}}\) is the ratio of the vev’s of the two Higgs doublets, and the \(m_i\)’s involve the soft susy breaking masses and are determined from the effective potential (EP). As they stand, Eqs.[] are only first order derivatives, and as such should not be, generally speaking, expected to define fully a (local) EWSB minimum. It so happens, however, that they do so in the MSSM but only in the lowest order of the effective potential, namely at tree-level or renormalization-group-improved tree-level (RGITL) approximations. In these approximations the vanishing of the first order derivatives implies the positivity of the second order ones. [The reader is referred to [] for detailed proofs of this and all subsequent results presented in this letter.] Therefore, it should come as no surprise that, going beyond these approximations, one would have to resort to extra conditions from the positivity of the second order derivatives and check whether these are still automatically satisfied. We find that they are generically not, and imposing them on top of Eqs.[] leads to model-independent bounds on \(\tan \beta\). A characteristic of the tree-level or RGITL approximations is that the \(m_i^2\)’s in Eqs.[] have no dependence on \(\tan \beta\). If we assume for illustration that \(m_1^2 > 0\), then one has the following model-independent constraints:

- \(i)\) if \(m_2^2 < 0\) then \(|\tan \beta| > 1\)
- \(ii)\) if \(m_2^2 > 0\) then \(1 < |\tan \beta| \leq \frac{|m_1|}{|m_2|}\) (resp. \(\frac{|m_1|}{|m_2|} < |\tan \beta| < 1\)) for \(m_1^2 > m_2^2\) (resp. \(m_1^2 < m_2^2\))

Let us now go one step beyond the above approximation by considering the finite (non-logarithmic) one-loop corrections to the effective potential.

The 1-loop EP [] in the \(\overline{MS}\) scheme reads

\[
V = V_{\text{tree}} + \frac{\hbar}{64\pi^2} \text{Str}[M^4 (\log \frac{M^2}{\mu_R^2}) - 3/2]
\]  

where \(V_{\text{tree}}\) is the tree-level MSSM potential [], and \(M^2\) the field dependent squared mass matrix of the scalar or vector or fermion fields. We will assume for definiteness, hereafter, that all mass scales in \(M^2\) are of comparable magnitudes. This rough approximation allows a simultaneous resummation of all the logs by an appropriate choice of the renormalization scale \(\mu_R = \mu_{\text{Str}}^R\), leading to

\[
V = V_{\text{tree}}(\mu_R^2) + \frac{\hbar}{64\pi^2} (-3/2) \text{Str} M^4
\]  

where now \(V_{\text{tree}}(\mu_R^2)\) is obtained from \(V_{\text{tree}}\) by replacing all the tree-level quantities by their running counterparts, and \(\text{Str} [...] \equiv \sum_{\text{spin}} (-1)^{2s}(2s + 1)(...)\) sums over all gauge boson, fermion and scalar contributions. It should be clear that in the approximation leading to Eq.[] we bypass the problem of log resummations in the presence of multi-mass scales []. We consider this as the \(0^\text{th}\) order approximation in which we state our results, being understood that mass scale disparities should be ultimately considered. Specifying to the Higgs fields directions we find,
The effective potential Eq.(6) has the same functional dependence as the tree-level, i.e. all loop corrections are absorbed in the definitions of $\tilde{\beta}, X$ and the $X^2_m, \alpha$'s [we ignore throughout the implicit scale dependence on the fields], except for the $\tilde{\alpha}$ term which is a genuine one-loop effect, Eq.(6). Although quantitatively small, we will see that this new term changes the qualitative features which prevailed at the tree-level and RGITL approximations.

Let us now determine the conditions for the existence of a (local) minimum which breaks the electroweak symmetry. On one hand, the eight conditions for a stationary point with respect to the eight real-valued Higgs fields boil down, in the natural direction

$$< H_1 >= \frac{1}{\sqrt{2}} \left( \begin{array}{c} v_1 \\ 0 \end{array} \right) \quad \quad < H_2 >= \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v_2 \end{array} \right)$$

$(v_1, v_2$ real valued), to the two equations:

$$X^2_m (\tilde{\alpha} - X)t^4 + \tilde{\alpha}X^2_{m_1} - X (X^2_{m_1} + X^2_{m_2})t^3 + (\tilde{\alpha}X^2_{m_2} + X (X^2_{m_1} + X^2_{m_2}))t + X^2_{m_3} (\tilde{\alpha} + X) = 0$$

$$u = \frac{1}{\tilde{\alpha}(t^2 - 1)}(X^2_{m_3}(t^2 + 1) + (X^2_{m_1} + X^2_{m_2}))t$$

where $t \equiv \tan \beta$ and $u \equiv v_1 v_2$. On the other hand, the stability conditions at the points satisfying Eqs.(11 - 12) give:

$$-(v_1^2 + v_2^2)X^2_m \frac{v_{m_3}}{v_1 v_2} \geq 0$$

$$2\tilde{\alpha}(v_1^2 - v_2^2) + (v_1^2 + v_2^2)(2X - \frac{X^2_{m_3}}{v_1 v_2}) \geq 0$$

$$-4\tilde{\alpha}v_1^2 v_2^2 + 2(v_2^2 - v_1^2)(v_1^2 + v_2^2)\tilde{\alpha}$$

$$(\frac{-X^2_{m_3}}{v_1 v_2} + \frac{\tilde{\alpha}}{2})(v_1^2 + v_2^2) \geq 0 \quad \text{(twice)}$$

plus three zeros corresponding to the three goldstone degrees of freedom. The latter inequalities boil down in turn, due to the perturbative positivity of $\tilde{\alpha}$ and $X \pm \tilde{\alpha}$, to the two conditions

$$\frac{X^2_m}{v_1 v_2} \leq 0$$

$$\tan^2 \beta \leq t_+ \quad \text{or} \quad \geq t_+$$

where $t_\pm$ are defined in Eq.(23) below.

Eqs.(11 - 12) supplemented with the requirement $M_Z^2 = g^2u(t + 1/t)/4$ are a special form of Eqs.(6), except that now the dependence on $\tan \beta$ in the $m^2$s is made explicit. They remain fully analytically solvable in our approximation, but we do not dwell further on this aspect here. Eqs.(17 - 18) are equivalent to the requirement that the one-loop corrected squared Higgs masses be positive, in the approximation of Eq.(6). It is straightforward then to see from Eqs.(13 - 15) that at tree-level (or RGITL for that matter), conditions (17, 18) reduce to

$$\frac{m^2_{\tilde{\alpha}}}{v_1 v_2} \leq 0$$

Furthermore, the latter inequality is a direct consequence of Eqs.(6) in this limit, and thus the only necessary bounds that one can obtain on $\tan \beta$ in a model-independent way are in this case given by i) and ii). At the one-loop level the situation becomes more involved. The requirement that $u$ and $t$ have the same sign (i.e. $v_1$ and $v_2$ are real-valued) together with Eqs.(12, 17, 18) allow us to determine new analytic bounds on $\tan \beta$. 
Here the sign of $\tilde{\alpha}$ plays an important role. Furthermore, given its dependence on the Yukawa couplings, it will concomitantly lead to further consistency bounds involving $m_t/m_b$, the ratio of the top to down quark masses. A detailed analysis \cite{9} leads to the following bounds

a) $\tilde{\alpha} \leq 0$

if $\tan \beta > 1$ then $\max[\sqrt{T_+}, T_+] \leq \tan \beta \leq \frac{m_t}{m_b}$ \hspace{1cm} (19)

if $\tan \beta < 1$ then $T_- \leq \tan \beta \leq \sqrt{t_-}$ \hspace{1cm} (20)

b) $\tilde{\alpha} \geq 0$

$$\max[\frac{m_t}{m_b}, \sqrt{T_+}] \leq \tan \beta \leq T_+$$ \hspace{1cm} (21)

where

$$T_\pm = \frac{-X_{m_1}^2 - X_{m_2}^2 \mp \sqrt{(X_{m_1}^2 + X_{m_2}^2)^2 - 4X_{m_3}^4}}{2X_{m_3}^2}$$ \hspace{1cm} (22)

and

$$t_\pm = \frac{\tilde{\alpha}^2 \frac{v_1 v_2}{X_{m_3}} - X \mp \sqrt{X - \tilde{\alpha}^2 \frac{v_1 v_2}{X_{m_3}}^2 + \tilde{\alpha}^2 - X^2}}{\tilde{\alpha} - X}$$ \hspace{1cm} (23)

A couple of remarks are in order. We stress first that the above bounds are fully model-independent. No unification or universality assumptions are needed or assumed. $T_\pm$ and $t_\pm$ are generally calculable in terms of the full-fledged MSSM parameters. When writing the bounds a) and b) we only considered, without loss of generality, positive $\tan \beta$ values. This is of course just a convention for the relative phase of the fields $H_1, H_2$. It implies, however, that $X_{m_3}^2 > 0$ is forbidden in view of Eq.(17). \cite{9} [An equivalent discussion can be carried out in the opposite convention.] From this constraint on the sign of $X_{m_3}^2$ and the requirement of boundedness from below of the potential \cite{3} the $|H_1| = |H_2|$ direction, namely $X_{m_1}^2 + X_{m_2}^2 \pm 2X_{m_3}^2 \geq 0$, it readily follows that $T_\pm$ are always real-valued and positive. Similarly, $t_\pm$ are always real-valued as can be seen from Eq.(23) upon use of Eq.(17), and positive since $X \pm \tilde{\alpha} \geq 0$ is always perturbatively satisfied. Furthermore, one shows that $t_- < 1 \leq t_+$ and $T_- < 1 \leq T_+$. The inequalities in a) and b) are thus always consistent; In particular, one sees that a band around $\tan \beta = 1$ is always excluded. More generally, and depending on the chosen values of the MSSM parameters, the relative magnitudes of $t_- T_-$ or $t_+, T_+$ will exclude some domains for $\tan \beta$.

It is also worth noting that the appearance of $m_t/m_b$ in the bounds is rather peripheral in the sense that it follows from the tree-level Yukawa masses and the form of $\tilde{\alpha}$, not from the effective potential itself. For instance, had we not neglected the $\tau$ quark Yukawa coupling in $\tilde{\alpha}$, $m_t/m_b$ would have been replaced by $(m_t/m_b)(1 - m_b^2/6m_t^2)$ in a) and b). Thus the main information extracted from \cite{3} are the model-independent bounds $t_\pm$ and $T_\pm$. It is instructive to compare Eq.(19) with the SUGRA-GUT qualitative constraint $1 \lesssim \tan \beta \lesssim m_t/m_b$ \cite{12}, which relies on universality assumptions and the trend of the running of $m_1^2$ and $m_2^2$. What we learn here is that the mere consideration of the finite (non-logarithmic) contribution to the effective potential improves quantitatively the lower bound even if the model assumptions are loosened. More generally, conditions a) and b) distinguish naturally between small and large $\tan \beta$ and allow to tell when the respective windows are closed or not, due to the general requirement of EWSB. For instance in case a), $\tan \beta < 1$ would be excluded if $\sqrt{T_-} < T_-$ while $1 \lesssim \tan \beta \lesssim m_t/m_b$ would be forbidden if $m_t/m_b < \max[\sqrt{T_+}, T_+]$. Similarly case b) would be forbidden in the region where $T_+ < \max[m_t/m_b, \sqrt{T_+}]$. An exhaustive study of the above situations lies out of the scope of the present letter as it necessitates a scan over a wide range of the parameter space. Here we aim at a simple illustration of how conditions a) and b) can be used. For this we chose to correlate the tree-level parts of the $X_{m_i}^2$’s, i.e. the $m_{m_i}^2$’s, by requiring that they satisfy Eqs.\cite{9}. This eliminates all the Higgs soft masses (appearing also in the one-loop contributions) in terms of $\tan \beta^0$ and $m_{A^0}$ (the CP-odd Higgs mass) which are consistent with EWSB at the tree-level\cite{11}. We then study the behaviour of $T_+$ as a function of $\tan \beta^0$ and $m_{A^0}$ after having assigned some values to the remaining free parameters. Fig.1 illustrates the case a) with $\tan \beta > 1$ and $t_+ < T_+$.

\footnote{It should be clear that such a correlation assumption is just for the sake of illustration and is by no means mandatory. One could choose the free parameters differently and compute the bounds from them.}
While the usual bounds [12] would have just told us that tan β = 5, 20 are allowed and tan β = 40 forbidden (if we commit ourselves to a given model), we see from Fig.1 that for instance the tree-level guess tan β = 5 cannot be made consistent with EWSB, the good candidate values being well above it, unless m_A0 ≥ 100 GeV or so. At the other extreme, tan β = 40 (a value qualitatively inconsistent with our input for Y_t, Y_b) leads to the situation where only the region m_A0 ≤ 170 GeV is allowed and corresponds to a tan β well below 40. The intermediate guess values lying between T^0_+ ≈ 11 and m_t/m_h can be made a priori consistent with EWSB for any m_A0, as seen for instance for tan β = 20. One should keep in mind, however, that we did not require here the symmetry breaking to be consistent with the Z and top masses yet, nor did we implement the full information from Eqs.([11], [12]). These would of course constrain further the allowed domains for tan β. It is worth noting that, when α << 1, T_+ is a good estimate for tan β satisfying Eq.([11]). Thus even when the guess value of tan β = 0 is not excluded by the lower bound it remains true, as can be seen for tan β = 20 in Fig.1, that the correct tan β would appreciably differ from it except for very heavy m_A0.

To conclude, we believe we have shown that the general form of the MSSM 1-loop EP in the MS scheme contains more information than what was a priori expected from model-independent phenomenological analyses. This information can be easily implemented as analytical constraints on tan β in the MSSM. Finally, a further treatment of the Logs beyond the naive RGITT will certainly improve our approximation, keeping though the above conclusion qualitatively unchanged.

[1] L.E. Ibañez and G.G. Ross, Phys. Lett. B110 (1982) 215; K.Inoue, A. Kakuto, H. Komatsu and S. Takeshita, Prog. Theor. Phys. 68 (1982) 927; 71 ( 1984) 413; L. Alvarez-Gaumé, M. Claudson and M.B. Wise, Nucl. Phys. B207 (1982) 96; J. Ellis, D.V. Nanopoulos and K. Tamvakis, Phys. Lett B121 (1983) 123; L.E. Ibanez, Nucl. Phys. B218 (1983) 514; L. Alvarez-Gaumé, J. Polchinski and M.B. Wise, Nucl. Phys. B221 (1983) 495; J. Ellis, J.S. Hagelin, D.V. Nanopoulos and K. Tamvakis, Phys. Lett. B125 (1983) 275; L.E. Ibañez and C. Lopez, Phys. Lett. B126 (1983) 54; Nucl. Phys. B236 (1984) 438; [2] D0 Collaboration (Krzysztof Genser for the collaboration). FERMILAB-CONF-97-233-E,Presented at La Thuile, Italy, 2-8 Mar 199; CDF Collaboration (Sandra Leone for the collaboration). FERMILAB-CONF-97-319-E, to be published in the proceedings of QCD 97 Montpellier, France, 3-9 Jul 1997; [3] J. Ellis , S. Kelley and D.V. Nanopoulos, Phys. Lett. B 260 (1991) 131; U. Amaldi, W. de Boer and H. Fürstenau, Phys. Lett. B 260 (1991) 447; P. Langacker and M. Luo, Phys. Rev. D 44 (1991) 817; C. Giunti, C.W. Kim and U.W. Lee, Mod. Phys. Lett. A6 (1991) 1745; [4] For a review of the MSSM and related topics see H. P. Nilles, Phys. Rep. 110 (1984) 1; H. E. Haber and G. L. Kane, Phys. Rep. 117 (1985) 75; A. B. Lahanas and D. V. Nanopoulos, Phys. Rep. 145 (1987) 1; [5] D.J. Castaño, E.J. Piard and P. Ramond, Phys.Rev.D49 (1994) 4882; W. de Boer, R. Ehret and D.I. Kazakov, Z. Phys. C67 (1994) 647; V. Barger, M.S. Berger and P. Ohmann, Phys.Rev.D49 (1994) 4908; [6] see for instance J.A. Casas, A. Lledya, C. Muñoz, Nucl. Phys. B471 (1995) 3, and references therein; [7] A.H. Chamseddine, R.Arnowitt and P.Nath, Phys.Rev.Lett. 49 (1982) 970; R.Bariberg, S.Ferrara and C.A Savoy, Phys.Lett. B119 (1982) 343; L.Hall, J.Lykken and S.Weinberg, Phys.Rev.D27 (1983) 2359; [8] M.Dine and A.E.Nelson, Phys.Rev.D48 (1993) 1277; [9] C. Le Mouël, G. Moulatak, hep-ph/9711356; [10] S. Coleman, E. Weinberg, Phys. Rev. D 7 (1973) 1888; [11] C. Ford, D.R.T. Jones, P.W. Stephenson, M.B. Einhorn Nucl. Phys. B 395 (1993) 17; M. Bando, T. Kugo, N. Maekawa, H. Nakano, Phys. Lett. B 301 (1993) 83 and Prog.Theor.Phys.90 (1993) 405; [12] G.F. Giudice, G. Ridolfi, Z. Phys. C 41 (1988) 447; see also F. Zwirner in “Physics and Experiments in Linear Colliders”, Saariselkä, Finland Sept.91, Eds. R. Orava, P. Eerola, M. Nordberg;