Parity-based formalism for high spin matter fields.

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Using the recent parity-based construction of a covariant basis for operators acting on the \((j,0)\oplus(0,j)\) representation of the Homogeneous Lorentz Group, we propose a formalism for the description of high spin matter fields, based on the projection over subspaces of well-defined parity. We identify two possibilities for the projection (on-shell and off-shell projection) which in general yield equivalent free theories but different interacting theories. For all \(j\) except for \(j = 1/2\), we find that the projection does not completely fix the properties of the interacting theory. This freedom is related to the fact that the covariant form of parity can be written in terms of one of the symmetric traceless tensors in the covariant basis and in general allows for a free magnetic dipole term in the lagrangian. We gauge the theory and construct the charge conjugation operator showing that it commutes with parity for bosons, and anticommute in the case of fermions as expected. In the case of bosons, the parity invariant subspaces are also invariant under charge conjugation and time reversal and the formulation of a quantum field theory can be done using only these subspaces.

As a first exhaustive example we work out the electrodynamics for \(j = 1\) matter bosons, rewrite the theory in terms of an antisymmetric tensor field and compare our results with existing formalisms in the literature, either in tensor or “spinor” language. We find that there are three essentially different formalisms: i) formalisms equivalent to the on-shell parity projection, ii) formalisms equivalent to the off-shell parity projection and iii) the Poincaré projector formalism which describes a degenerate parity doublet. In particular, the tensor formalism used in chiral perturbation theory with resonances \((R\chi PT)\) is the same as a theory proposed by Shay and Good in “spinor” language and corresponds to our theory based on the off-shell parity projection. Also, a theory proposed by Joos and Weinberg in “spinor” basis is the same as the “antisymmetric tensor matter field” formalism used by Chizhov (in the massless case) and corresponds to our on-shell parity projection.

Naive power counting admits anomalous magnetic-dipole terms and self-interactions at tree level. We perform a chiral decomposition of these theories and show that chiral symmetry can be realized linearly only for the theory based on the on-shell projection. Chiral symmetry forbids mass and anomalous magnetic dipole terms and in general admits six self-interaction terms. We conclude that this is the appropriate framework to attempt the incorporation of spin 1 matter bosons in chiral theories like the standard model.

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I. INTRODUCTION.

The largely awaited first results of the Large Hadron Collider (LHC) for the structure of the Higgs sector have been released, confirming the existence of a scalar particle with a mass of 126 GeV, as required by precision data of the standard model [1, 2]. These results put serious constraints on the existence of supersymmetric particles, large extra dimensions and other proposals for physics beyond the standard model [3].

In this context, it is reasonable to explore different possibilities for the existence of new particles. It is remarkable that in the standard model there is a clear correspondence of types of fields with representations of the Homogeneous Lorentz Group (HLG). Matter fields transform in the \((\frac{1}{2},0)\) and \((0,\frac{1}{2})\) representation, gauge fields in the \((\frac{3}{2},\frac{3}{2})\) and the Higgs, the particle that endows the matter (and some of the gauge) fields with mass, transforms in the \((0,0)\) representation; only particles of spin 0, \(\frac{1}{2}\), and 1 are present. The most appealing extensions of the standard model consider the enlargement of the gauge group with the incorporation of the only space-time symmetry beyond the Poincaré symmetries, namely, supersymmetry. Although in the supersymmetric extensions of the standard model particles with spin \(\frac{3}{2}\) appear, they enter the formalism as gauge fields transforming in the \((1,\frac{1}{2})\oplus(\frac{3}{2},1)\) representation of the HLG, not as matter fields.

The tension between the existing proposals for physics beyond the standard model and the recent results of the LHC, together with the aforementioned correspondence of the standard model building blocks with specific representations of the Lorentz group, encourages us to study the incorporation of high spin matter fields in the formulation of theories beyond the standard model. Guided by this correspondence, we posit that high spin matter elementary particles, if realized in nature, should be described by fields transforming in the \((j,0)\oplus(0,j)\) representation of the HLG. The first step towards the incorporation of these particles in extensions of the standard model is to have a consistent description of such fields.
The problems in the description of high spin particles have been addressed by many authors, leading to a variety of formulations for high spin matter, which have eventually been found unsatisfactory in general. The interest on this topic decreased upon the completion of the standard model at the beginning of the seventies, when the focus of the community was put on the precision phenomenology of the standard model, in the uncovering of the nature of the non-perturbative regime of QCD, and in the formulation of the extension of the ideas behind the standard model to new scenarios which did not require to address the problems of high spin particles.

In this work we propose a formalism for the description of high spin matter fields transforming in the $(j,0) \oplus (0,j)$ family of representations. Our approach is motivated by an alternative view of the Dirac theory as a manifestation of the discrete symmetries satisfied by the interactions of the particle in the representation in which the field transforms. This view has been put forth before, and is in the spirit of the quantum theory of fields developed by Weinberg. To be specific, we take the position that the free theory for spin $j$ and well defined parity is dictated by the properties of the representation chosen and by the projection over its parity subspaces. In order to properly realize charge conjugation and time reversal we use as the equation of motion the projection condition, and we gauge this equation using electromagneticism as a simple example. Then we construct charge conjugation showing that, as expected, this operator commutes with parity for bosons and anticommutes for fermions.

In this construction we actually find two possibilities for the realization of the parity projection, associated to the on-shell projection (projection operation valid only in the case when the on-shell relation $p^2 = m^2$ holds) and off-shell projection (projection operation valid for arbitrary values of $p^2$). We find that the on-shell formulation reproduces Dirac theory in the case $j = 1/2$ and this is the appropriate framework for the description of high spin fermion fields. As for bosonic fields both approaches are feasible yielding equivalent free field theories but different interacting theories. We also find that parity projection fixes completely the properties of the interacting theory only in the case $j = 1/2$. For $j \geq 1$ the parity projection used as a dynamical principle yields ambiguities related to the fact that the covariant form of parity operator involves only a symmetric space-time tensor. Solving this ambiguity requires to introduce an additional anti-symmetric term to which the free theory is insensitive, introducing a free parameter in the theory.

As a first exhaustive example, we work out our formalism for the electrodynamics of spin 1 matter bosons. We consider both possibilities (on-shell and off-shell projection) for the interacting theory, and performing a Gordon-like decomposition of the corresponding electromagnetic currents we show that the free parameters are actually “anomalous” magnetic moments related to the existence of a Pauli term in the corresponding Lagrangian. This term is gauge invariant and due to mass dimension one of the field, turns out to have mass dimension four thus it must be included in the naively renormalizable lagrangian anyway.

We translate out theory to the antisymmetric tensor language and compare our formalism with existing formulations in the literature for the $(1,0) \oplus (0,1)$ representation, either in terms of tensor or spinor fields.

Concerning formulations in spinor language, the on-shell formulation in the case of positive parity and for a vanishing “anomalous” magnetic moment, coincides with a proposal by Joos and Weinberg. The off-shell formulation, with a vanishing “anomalous” magnetic moment, corresponds to a theory considered previously in spinor language. This theory with a nonvanishing “anomalous” magnetic moment was analyzed in, where it is shown that the classical theory is causal only for a total gyromagnetic factor $g = 1$.

As to tensor language, the most general second order lagrangian written down in terms of tensor fields was considered in. There, appropriate constraints were imposed in order to have a theory with a single pole in the propagator, and the formalism was used to formulate effective theories for light hadrons in a non-linear implementation of chiral symmetry. We find that this formalism is the tensor language version of the theory proposed in for either positive or negative parity. We show that there is another possible theory arising from the most general lagrangian in tensor language written in and not considered there. This theory corresponds with the Poincaré projector formalism for spin 1 matter bosons discussed in, which has been recently shown to describe a parity doublet. In this last reference the parity projector in $(1,0) \oplus (0,1)$ is also introduced in tensor language and it is extracted from an explicit construction of the states induced as derivatives of the states in the $(1/2,1/2)$ representation. This formalism corresponds to an “anomalous” gyromagnetic factor of $1/2$ (total gyromagnetic factor $g = 1$) of our theory based on the off-shell parity projection. In it has been shown that at the classical level this theory is causal in agreement with results in.

We analyze the chiral properties of our formalism finding that the formulation based on the on-shell projection is the only one admitting a linear realization of chiral symmetry in the massless case. Therefore, in the massless limit this formalism is the only one suitable for the description of chiral matter spin 1 bosons.

Our paper is organized as follows. In the next section we describe the structure of the $(j,0) \oplus (0,j)$ representation. In section III we describe the general structure for the covariant basis for operators acting on this representation. In section IV we construct the projectors over subspaces of definite parity and give the Lagrangian for the free field theory. Interactions are introduced in section V, where a comprehensive discussion of the discrete symmetries of the theory is given. We work out the electrodynamics of spin 1 matter bosons in section VI. In section VII we rewrite our
theory in the tensor basis for the \((1, 0) \oplus (0, 1)\) representation and make a detailed comparison with different proposals in the literature both in spinor and tensor language. The chiral structure of the two theories is discussed in section VIII and our conclusions are given in section IX.

II. STRUCTURE OF THE \((j, 0) \oplus (0, j)\) REPRESENTATION.

It is well known that the HLG is locally isomorphic to \(SU(2)_A \otimes SU(2)_B\). There is a well known isomorphism between classical algebras \(so(4) \simeq su(2) \otimes su(2)\), which allow us to relate the representations of the HLG with those of \(SU(2)_A \otimes SU(2)_B\). The generators of the latter group are related to the rotations, \(J\), and boosts, \(K\), generators as

\[
A = \frac{1}{2}(J-iK), \quad B = \frac{1}{2}(J+iK).
\]

The generators of the HLG, \(\{J, K\}\), transform as the components of a second rank antisymmetric tensor \(M_{\mu\nu}\) according to

\[
M^{\mu i} = K_i, \quad M^{ij} = \epsilon_{ijk}J_k \quad (2)
\]

and for the simplest representations \((j, 0)\) and \((0, j)\), here denoted by “right” and “left” respectively, this tensor has the following components

\[
M^{\mu i}_R = (K_R)_i, \quad M^{ij}_R = \epsilon_{ijk}(J_R)_k \quad (3)
\]

\[
M^{\mu i}_L = (K_L)_i, \quad M^{ij}_L = \epsilon_{ijk}(J_L)_k \quad (4)
\]

which satisfy

\[
K_R = iJ_R, \quad K_L = -iJ_L \quad (5)
\]

where \(J_R = J_L = \tau\) are the conventional \((2j + 1) \times (2j + 1)\) rotation matrices for spin \(j\) in the \(\{|j, m\}\) basis. Due to these relations each of the second rank antisymmetric tensors \(M_{\mu\nu}^{R,L}\), has only 3 independent components. This lack of independence can be covariantly written as

\[
\widetilde{M}_{\mu\nu}^R = -iM_{\mu\nu}^R, \quad \widetilde{M}_{\mu\nu}^L = iM_{\mu\nu}^L \quad (6)
\]

with the dual tensor defined by

\[
\widetilde{M}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} M_{\alpha\beta}, \quad (7)
\]

where we use the convention \(\varepsilon^{0123} = 1\).

Now, the generators for the \((j, 0) \oplus (0, j)\) representation in this basis (denoted as chiral basis in the following) are

\[
M_{\mu\nu} = \begin{pmatrix} M_{\mu\nu}^R & 0 \\ 0 & M_{\mu\nu}^L \end{pmatrix} \quad (8)
\]

and the dual to this tensor is

\[
\widetilde{M}^{\mu\nu} = \begin{pmatrix} \widetilde{M}_{\mu\nu}^R & 0 \\ 0 & \widetilde{M}_{\mu\nu}^L \end{pmatrix} = -i \begin{pmatrix} M_{\mu\nu}^R & 0 \\ 0 & -M_{\mu\nu}^L \end{pmatrix} = -i\chi M_{\mu\nu}, \quad (9)
\]

where \(\chi\) is the chirality operator

\[
\chi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (10)
\]

In vector language, for \((j, 0) \oplus (0, j)\) we have

\[
J = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}, \quad K = \begin{pmatrix} i\tau & 0 \\ 0 & -i\tau \end{pmatrix} \quad (11)
\]
These operators are related as
\[ K = i\chi J. \] (12)

Parity exchanges \((j, 0) \leftrightarrow (0, j)\). Hence, modulo a global phase, in the rest frame the parity operator in the chiral basis reads
\[ \Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (13)

The generators satisfy the conventional Lie algebra
\[ [M^{\mu\nu}, M^{\alpha\beta}] = -i \left( g^{\mu\alpha} M^{\nu\beta} - g^{\mu\beta} M^{\nu\alpha} - g^{\nu\alpha} M^{\mu\beta} + g^{\nu\beta} M^{\mu\alpha} \right). \] (14)

It is interesting that for the \((j, 0) \oplus (0, j)\) representations the chirality operator can be written as
\[ \chi = \frac{i}{4j(j+1)} \tilde{M}^{\mu\nu} M_{\mu\nu}, \] (15)
i.e. it is proportional to the second Casimir operator of the Lorentz group. As such, it commutes with the generators,
\[ [\chi, M^{\mu\nu}] = 0, \] (16)
and a straightforward calculation yields
\[ \{\chi, \Pi\} = 0. \] (17)

Notice that Eqs. (15) yields an explicit matrix representation for the boost generators of the \((j, 0)\) and \((0, j)\) representations, which allows us to explicitly construct the corresponding boost operators. Acting with this operator on the rest frame states we construct the explicit form of the corresponding states in an arbitrary frame. By extension of the \(j = 1/2\) case, we will call also “spinors” the matrix representation of the states for arbitrary \(j\) (chiral spinors in this case, see below) even though they do not furnish a representation of a Clifford algebra. Instead, as shown in [26], they possess of a more complicated algebraic structure.

The \((j, 0) \oplus (0, j)\) representation has \(2(2j + 1)\) complex degrees of freedom, twice the needed to describe a spin \(j\) particle, but in contrast with \((j, 0)\) and \((0, j)\), it admits a representation for discrete symmetries. In other words, these are the irreducible representations for the complete HLG, not just the connected part.

The construction of spinors transforming in the \((j, 0) \oplus (0, j)\) representation can be done following the same procedure but now using the second of Eqs. (11). In this case the boost operator is the same for all states in the \((j, 0) \oplus (0, j)\) and the transformation properties under discrete symmetries of the corresponding spinors are dictated by the specific choice of the rest frame spinors. Since parity historically played an important role in the description of electromagnetic and strong interactions, states with well defined parity have been considered in the past and the corresponding states in momentum space have been constructed using this procedure without reference to a Lagrangian or to an equation of motion for \(j = 1/2, 1, 3/2, 2\) [24, 27]. It is worth to remark, however, that spinors with different transformation properties under discrete symmetries can also be constructed this way, and specially it is possible to obtain explicit representations for Weyl spinors - representations of the Weyl states embedded in this larger space- or Majorana spinors if we have an appropriate construction of charge conjugation in the \((j, 0) \oplus (0, j)\) representation space.

In a quantum field theory, the states are the one-particle amplitudes of the corresponding field. This amplitude - the wave function- satisfies some equation of motion. We take in this paper the position that this equation of motion is related to the chosen properties of the particle in the rest frame, and therefore these properties dictate the structure of the corresponding quantum field theory when a suitable Lagrangian is constructed and the gauge principle is used. We study here the case of particles with well defined parity, and thus we impose the parity projection condition on the rest frame spinors. In an arbitrary frame this constraint eliminates the redundant degrees of freedom, leaving only the \(2j + 1\) degrees of freedom necessary to describe a spin \(j\) and mass \(m\) particle, with the additional property of well defined parity. The proper-covariant- implementation of the parity projection in an arbitrary frame requires to work out the transformation properties under Lorentz transformations of operators acting on the \((j, 0) \oplus (0, j)\) representation, specially of parity, and to find a basis with definite transformation properties under the Lorentz group. This calculation is nontrivial and the details are given in a separate publication [26]. In that work, an algorithm is given for the calculation of the parity-based covariant basis for arbitrary \(j\) and the \(j = 1/2, 1, 3/2\) have been worked out explicitly. We refer the reader to Ref. [26] for the details, but in order to make the paper as self-contained as possible and settle down our notation for the parity-based formalism in the case \(j = 1\) to be worked out in detail below, we briefly review the main results in the next section where we re-derive the explicit form of one of the the symmetric traceless tensor entering this basis for \(j = 1\) in a way convenient for the purposes of this paper.
III. COVARIANT BASIS FOR OPERATORS IN \((j,0) \oplus (0,j)\) CONSTRUCTED FROM PARITY.

The construction of a quantum field theory for particles with well-defined parity transforming in the \((j,0) \oplus (0,j)\) representation requires that we elucidate the covariant properties of parity and find a basis with well-defined properties under Lorentz transformations for this space. In the following we will refer to this basis as the covariant basis. The \((j,0) \oplus (0,j)\) space has dimension \(2(2j+1)\) and there are \(4(2j+1)^2\) independent matrices acting in this space. Any operator \(O\) acting on this space can be written as a sum of the external product of the states

\[
O = \sum_n c_n |n\rangle \langle n| \tag{18}
\]

where \(\{ |n\rangle \}\) is a basis of the representation space \((j,0) \oplus (0,j)\). Using the chiral basis, \(\{|j,m\rangle_R, |j,m\rangle_L\}\), this external product space (space of squared matrices of dimension \(2(2j+1)^2\)) has the symbolic decomposition

\[
[(j,0) \oplus (0,j)] \vee [(j,0) \oplus (0,j)] = \bigoplus_{i=0}^{2j} [(i,0) \oplus (0,i)] \oplus 2(j,j) \tag{19}
\]

where the right hand side enumerates the Lorentz representations under which the operators transform. For every \(j\), it is possible to construct a set of operators transforming in these Lorentz representations which form a basis of the operators acting on \((j,0) \oplus (0,j)\). In general this set contains two Lorentz scalar operators, i.e. operators commuting with the Lorentz generators \(M_{\mu\nu}\), which are easily identified as the unit matrix of dimension \(2(2j+1)\) and the chirality operator \(\chi\) in Eq. (10). The decomposition in Eq. (19) also contains six operators transforming in the \((1,0) \oplus (0,1)\) forming a rank-2 anti-symmetric tensor which we identify with the generators of the HLG, \(M_{\mu\nu}\). We have also a pair of symmetric traceless tensors transforming in the \((j,j)\). It was shown in [26] that the rest frame parity operator is the time component of one of these symmetric traceless tensors denoted \(S^{\mu_1 \mu_2 ... \mu_{2j}}\), and that the second tensor is simply given by \(\chi S^{\mu_1 \mu_2 ... \mu_{2j}}\). The number of elements in the basis increases with \(j\), the remaining operators must be explicitly constructed for every \(j\) and an algorithm for this construction is given in [26].

The explicit matrix form of the operators in the covariant basis for \(j = 1/2, 1, 3/2\) was given in [26]. In particular, the symmetric traceless tensor to which parity belongs, \(S^{\mu_1 \mu_2 ... \mu_{2j}}\), of relevance here, was obtained for these values of \(j\) by analyzing the transformation properties of the rest frame parity operator under Lorentz transformations in an inductive process. However, once we know that parity is the totally temporal part of the symmetric \(S\) tensor and we have constructed the boost operator, it is easier to obtain its explicit form just boosting the rest frame parity operator since

\[
B(p)\Pi B^{-1}(p) = \frac{S^{\mu_1 \mu_2 ... \mu_{2j}} p_{\mu_1} p_{\mu_2} ... p_{\mu_{2j}}}{m^{2j}} = \frac{S_j(p)}{m^j}. \tag{20}
\]

In the case \(j = 1/2\) we have two scalar operators, 1 and \(\chi\), an antisymmetric tensor, \(M_{\mu\nu}\), and and two vector operators (the “symmetric” operators of rank \(2j = 1\)).

\[
\{1, \chi, S^\mu, \chi S^\mu, M_{\mu\nu}\}, \tag{21}
\]

where

\[
S^\mu = \Pi \left(g^{0\mu} - 2iM^{0\mu}\right). \tag{22}
\]

This is the conventional set used in the literature up to a 1/2 factor in \(M_{\mu\nu}\). Boosting the rest frame parity operator we get

\[
B(p)\Pi B^{-1}(p) = \frac{S^\mu p_\mu}{m}. \tag{23}
\]

In the case \(j = 1\) there are 36 independent matrices, two scalars, an antisymmetric tensor, two symmetric traceless tensors and a tensor transforming in the \((2,0) \oplus (0,2)\) representation. In this case it can be shown that

\[
B(p)\Pi B^{-1}(p) = \frac{S^{\mu\nu} p_\mu p_\nu}{m^2}, \tag{24}
\]

where \(S^{\mu\nu}\) is a symmetric tensor given by

\[
S^{\mu\nu} = \Pi \left(g^{\mu\nu} - i(g^{0\mu} M^{0\nu} + g^{0\nu} M^{0\mu}) - \{M^{0\mu}, M^{0\nu}\}\right). \tag{25}
\]
This tensor is traceless in the Lorentz indices
\[ S_{\mu}^{\mu} = 0, \]  
and has 9 independent components. The second symmetric traceless tensor is given by \( \chi S^{\mu\nu} \).

The tensor transforming in the \((2,0) + (0,2)\) representation is given by
\[
C^{\mu\nu\alpha\beta} = 4\{M^{\mu\nu}, M^{\alpha\beta}\} + 2\{M^{\mu\alpha}, M^{\nu\beta}\} - 2\{M^{\mu\beta}, M^{\nu\alpha}\} - 8(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}).
\]  
(27)

It has the symmetries of the Weyl tensor of general relativity, namely

\[
C_{\mu\nu\alpha\beta} = -C_{\nu\mu\alpha\beta} = -C_{\mu\alpha\nu\beta}, \quad C_{\mu\nu\alpha\beta} = C_{\alpha\beta\mu\nu},
\]
(28)

the contraction of any pair of indices vanishes and it satisfies the algebraic Bianchi identity

\[
C_{\mu\nu\alpha\beta} + C_{\mu\alpha\beta\nu} + C_{\mu\beta\nu\alpha} = 0.
\]
(29)

These symmetries leave only 10 independent components out of the 256 components of a rank-4 tensor.

In summary, in the case of spin 1 the basis of matrices with well defined Lorentz transformation properties is

\[
\{1, \chi, S^{\mu\nu}, \chi S^{\mu\nu}, M^{\mu\nu}, C^{\mu\nu\alpha\beta}\}.
\]
(30)

Important relation for our formalism can be obtained from Eqs. (16, 17, 25, 27)

\[
\{\chi, S^{\mu\nu}\} = 0, \quad [\chi, C^{\mu\nu\alpha\beta}] = 0.
\]
(31)

In general, for arbitrary \(j\) the covariant basis is given by

\[
\{1, \chi, M^{\mu\nu}, S_{\mu_1\mu_2...\mu_{2j}}, \chi S_{\mu_1\mu_2...\mu_{2j}}, C^{\mu\nu\alpha\beta}, ...\}
\]
(32)

where ... stands for additional tensor operators transforming in the \((3,0) \oplus (0,3) \oplus ... \oplus (2j,0) \oplus (0,2j)\), whose explicit form can be obtained following the algorithm developed in Ref. [26].

IV. PARITY PROJECTORS AND FREE THEORY.

The condition for a state transforming in \((j,0) \oplus (0,j)\) to have well defined parity can be written in the rest frame as

\[
P_{\pm}(0)u_{\pm}(0) = u_{\pm}(0),
\]
(33)

where \(P_{\pm}\) stands for the projector over the subspaces of well defined parity in the rest frame given as

\[
P_{\pm}(0) = \frac{1}{2} \left( 1 \pm \Pi \right).
\]
(34)

with the \(+(-)\) sign corresponding to the positive (negative) parity case. For an arbitrary frame, the condition for the state to have well defined parity is obtained just boosting this equation. The parity operator can be written in terms of the symmetric traceless tensor operator transforming in the \((j,j)\) representation according to Eq. (20), thus the projectors in an arbitrary frame are given by

\[
P_{\pm}(p) = \frac{1}{2} \left( 1 \pm S_j(p) \right).
\]
(35)

The operator \(S_j\) satisfies the following relation

\[
(S_j(p))^2 = (p^2)^2j,
\]
(36)

which can be used to show that, on-shell, the following relations hold

\[
(P_{\pm}(p))^2 = P_{\pm}(p), \quad P_+(p)P_-(p) = P_-(p)P_+(p) = 0.
\]
(37)
Considering the projector in Eq. (35) the parity projection in an arbitrary frame yields the condition
\[(\pm S_j(p) - m^2) u_\pm(p, \lambda) = 0.\]  
(38)

Writing the corresponding wave function as
\[\psi_\pm(x) = u_\pm(p, \lambda)e^{-ip\cdot x},\]
(39)

it is easy to show that it obeys the following equation of motion
\[(\pm S_j(i\partial) - m^2) \psi_\pm(x) = 0.\]
(40)

In particular, for \(j = 1/2\) and positive parity this is the conventional Dirac equation
\[[iS^\mu\partial_\mu - m] \psi = 0.\]
(41)

In general, Eqs. (40) can be derived from the following Lagrangians
\[L_\pm = \overline{\psi_\pm(x)} \left(\pm S_j(i\partial) - m^2\right) \psi_\pm(x).\]
(42)

Here, the adjoint spinor is defined as
\[\overline{\psi} = \psi^\dagger \Pi,\]
(43)

where \(\Pi\) is the rest-frame parity operator in Eq. (13). The \(\Pi\) factor here is necessary to make a scalar product in the case
\[\pm = 1\] and will show that this subtlety is at the root of the differences between two of the different formalisms existing in the literature for \(j = 1\) in the \((1, 0) \oplus (0, 1)\) representation.
V. INTERACTING THEORY AND DISCRETE SYMMETRIES.

In this section we study in detail the discrete symmetries of our formalism, specially charge conjugation. In the following we consider in detail positive parity in Eq. (40) but our results in this section are also valid for negative parity or the equation derived using the mass and parity projector in Eq. (46).

Using the gauge principle we obtain the equation of motion satisfied by $\psi$ interacting with an external electromagnetic field as

$$\left[S_j (i\partial - qA) - m^2j\right] \psi = 0,$$

(48)

where $q$ is the charge of the particle. Taking the complex conjugate of Eq. (40) and multiplying on the left by $\eta_c \Gamma$ where $\eta_c$ is a phase and $\Gamma$ is a matrix in the $(j,0) \oplus (0,j)$ space, we obtain

$$\left[(-1)^{2j} S_j^* \Gamma^{-1} (i\partial + qA) - m^2j\right] \psi^c = 0.$$  

(49)

where the conjugate field is given by

$$\psi^c = \eta_c \Gamma \psi^*.$$  

(50)

Requiring $\psi^c$ to satisfy the same equation as $\psi$ but with the opposite charge we obtain

$$\Gamma (S_{\mu_1 \mu_2 \cdots \mu_{2j}})^* \Gamma^{-1} = (-1)^{2j} S_{\mu_1 \mu_2 \cdots \mu_{2j}}.$$  

(51)

The construction of the matrix $\Gamma$ satisfying Eq. (51) can be done for general $j$ in terms of time reversal. This is an antilinear operator and it can always be written as

$$\Theta = U \mathcal{K},$$  

(52)

with $\mathcal{K}$ the complex conjugation operator and $U$ a unitary operator. In general, it is easy to show that the action of time reversal on the generators of rotations is given by

$$\Theta J \Theta^{-1} = U J^* U^{-1} = -J,$$

(53)

and that states with well defined angular momentum transform as

$$\Theta |j, m\rangle = \eta(j) i^{2m} |j, -m\rangle,$$

(54)

with $\eta(j)$ a phase depending only on $j$ (not on $m$). Furthermore, the squared time reversal operator satisfies

$$\Theta^2 = (-1)^{2j}.$$  

(55)

We can make a matrix representation for $U$ from Eq. (54). Alternatively, if we take the conventional representation of the angular momentum generators where $J_2$ is purely imaginary and $J_{1,3}$ are real matrices, it is equivalent and easier to make the following representation

$$U = \xi(j) \exp(-i\pi J_2),$$

(56)

where $\xi(j)$ is a phase depending only on $j$.

Concerning boosts, it can be shown that the generators transform under time reversal as

$$\Theta(K) \Theta^{-1} = K,$$

(57)

thus the generators in Eq. (1) have the following transformation properties

$$\Theta(A) \Theta^{-1} = \frac{1}{2} \Theta(J - iK) \Theta^{-1} = -\frac{1}{2} (J - iK) = -A$$  

(58)

$$\Theta(B) \Theta^{-1} = \frac{1}{2} \Theta(J + iK) \Theta^{-1} = -\frac{1}{2} (J + iK) = -B$$

i.e $A$ and $B$ have the same transformation properties as $J$.  

Using these relations for the specific \((j,0)\) and \((0,j)\) representations, we can see that time reversal interchanges these irreps of the HLG. Indeed,

\[
\begin{align*}
\Theta \Lambda_R \Theta^{-1} &= \Theta (e^{-iJ \cdot (\theta + i\varphi)} ) \Theta^{-1} = e^{i \Theta J \Theta^{-1} \cdot (\theta - i\varphi)} = e^{-i J \cdot (\theta + i\varphi)} = \Lambda_L, \\
\Theta \Lambda_L \Theta^{-1} &= \Theta (e^{-iJ \cdot (\theta - i\varphi)} ) \Theta^{-1} = e^{i \Theta J \Theta^{-1} \cdot (\theta + i\varphi)} = e^{-i J \cdot (\theta - i\varphi)} = \Lambda_R.
\end{align*}
\]

(59)

Now, for the \((j,0) \oplus (0,j)\) representation, the construction of charge conjugation can be done in terms of \(\Theta\) as follows

\[
C = \left( \begin{array}{cc} 0 & \Theta \\ \Theta^{-1} & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & U \\ U^{-1} & 0 \end{array} \right) K \equiv C K.
\]

(60)

where we defined the unitary matrix

\[
\Gamma = \left( \begin{array}{cc} 0 & U \\ U^{-1} & 0 \end{array} \right)
\]

with

\[
U = e^{-i \pi J_2}.
\]

(61)

The matrix \(U\) satisfies

\[
U^2 = e^{-i 2 \pi J_2} = (-1)^{2j} = \begin{cases} +1 & \text{for bosons} \\
-1 & \text{for fermions} \end{cases},
\]

(62)

thus

\[
U^{-1} = \begin{cases} +U & \text{for bosons} \\
-U & \text{for fermions} \end{cases},
\]

(63)

and in both cases

\[
\Gamma^2 = 1.
\]

(64)

Furthermore, since \(J_2\) is purely imaginary (in the Condon-Shortley phase convention for the \(|j, m\rangle\) states and in the \(|j, m\rangle\) basis for \((j,0)\) and \((0,j)\)), then \(U\) is a real matrix. We can check that

\[
\Gamma J^\star \Gamma^{-1} = \left( \begin{array}{cc} 0 & U \\ U^{-1} & 0 \end{array} \right) \left( \begin{array}{cc} J_R^\star & 0 \\ 0 & J_L^\star \end{array} \right) \left( \begin{array}{cc} 0 & U \\ U^{-1} & 0 \end{array} \right) = \left( \begin{array}{cc} U J_L^\star U^{-1} & 0 \\ 0 & U^{-1} J_R^\star U \end{array} \right) = -J
\]

(65)

The transformation properties of \(K\) are easily obtained from

\[
K = i \chi J,
\]

(66)

and taking into account that

\[
\Gamma \chi^\star \Gamma^{-1} = -\chi
\]

(67)

which yields

\[
\{C, \chi\} = 0
\]

(68)

and

\[
\Gamma K^\star \Gamma^{-1} = \Gamma (i \chi J)^\star \Gamma^{-1} = -i \Gamma (\chi)^\star \Gamma^{-1} (-J) = -i \chi J = -K.
\]

(69)

(70)

Summarizing

\[
\Gamma M_{\mu \nu}^\star \Gamma^{-1} = -M_{\mu \nu}.
\]

(71)

Now, using Eqs. \ref{20} it is easy to show that

\[
\Gamma \left( \frac{S_{j}(p)}{m^2} \right)^\star \Gamma^{-1} = B(p) |\Pi \Gamma^{-1} B^{-1}(p) |.
\]

(72)
A straightforward calculation yields

\[ \Gamma \Pi^\ast \Gamma^{-1} = (-1)^{2j} \Pi, \quad (73) \]

thus the key transformation property in Eq. (71) is satisfied and Eq. (11) indeed describes the motion of a particle with the same mass and opposite electric charge. Interestingly, Eq. (73) can be rewritten as

\[
\begin{align*}
\{C, \Pi\} &= 0 \quad \text{for bosons}, \\
\{C, \Pi\} &= 0 \quad \text{for fermions}. 
\end{align*}
\]

Thus, for states of well defined parity, the anti-particle (the particle described by the charge conjugated field) has the same parity as the particle in the case of bosons and the opposite parity in the case of fermions. In order to work out explicitly this result in a generalization of the well known properties of spin 1/2 spinors, let us make a transformation to the basis of well defined parity states. As well known, this is done transforming all observables and states with the matrix

\[ M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (75) \]

In this basis the states are given as

\[ \psi_\pi = \left( \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_R + \phi_L \\ \phi_R - \phi_L \end{pmatrix} \right) = \begin{pmatrix} \phi_u \\ \phi_v \end{pmatrix}. \quad (76) \]

The Lorentz transformations in this basis are (we use in the following the suffix \( \chi \) for the chiral representation above)

\[ \Lambda_\pi = M \Lambda_\chi M^{-1} = \frac{1}{2} \begin{pmatrix} \Lambda_R + \Lambda_L & \Lambda_R - \Lambda_L \\ \Lambda_R - \Lambda_L & \Lambda_R + \Lambda_L \end{pmatrix}. \quad (77) \]

It is easy to extract the generators which in this basis read

\[ J_\pi = MJ_\chi M^{-1} = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}, \quad K_\pi = MK_\chi M^{-1} = \begin{pmatrix} 0 & i \tau \\ i \tau & 0 \end{pmatrix}, \quad (78) \]

and similarly for parity and chirality operators

\[ \Pi_\pi = M \Pi_\chi M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \chi_\pi = M \chi_\chi M^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (79) \]

Now, the charge conjugation matrix \( \Gamma \) in this basis reads

\[ \Gamma_\pi = M \Gamma_\chi M^{-1} = \frac{1}{2} \begin{pmatrix} U + U^{-1} & -U + U^{-1} \\ U - U^{-1} & -U - U^{-1} \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & -U \end{pmatrix} \quad \text{for bosons} \]

\[ \begin{pmatrix} U & 0 \\ 0 & -U \end{pmatrix} \quad \text{for fermions} \]

The charge conjugated states in this basis are

\[ \psi^c_\pi = \Gamma_\pi \left( \begin{pmatrix} \phi_u^* \\ \phi_v^* \end{pmatrix} \right) = \begin{pmatrix} U \phi_u^* \\ -U \phi_v^* \\ -U \phi_u^* \\ U \phi_v^* \end{pmatrix} \quad \text{for bosons} \]

\[ \begin{pmatrix} U \phi_u^* \\ -U \phi_v^* \\ -U \phi_u^* \\ U \phi_v^* \end{pmatrix} \quad \text{for fermions}. \quad (81) \]

We remark that since charge conjugation is an anti-linear operator at this level, care must be taken in the physical interpretation of results when changing the basis. In our case, however, the transformation matrix in Eq. (75) is real and we can proceed as with conventional linear operators. Clearly, in the case of fermions, \( u \)-type spinors (positive parity states) are mapped by charge conjugation onto \( v \)-type spinors (negative parity states) and vice versa as is usual in the case of spin 1/2. However, for bosons, \( u \)-type spinors are mapped by charge conjugation onto \( u \)-type spinors and \( v \)-type spinors are mapped onto \( v \)-type spinors. This is just an explicit confirmation that in the case of bosons charge conjugation does not change the parity of the states. Now the question is if the conjugated fields are independent
degrees of freedom in the case of bosons. In order to check this point let us work out explicitly the spin 1 case. The charge conjugation operator in the parity basis for $(1,0) \oplus (0,1)$ is given by

$$\psi^c = C \psi \equiv \Gamma \mathcal{K} \psi,$$

with

$$\Gamma_\pi = \begin{pmatrix} U & 0 \\ 0 & -U \end{pmatrix}$$

and

$$U = e^{-i \pi J_2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (82)

The conjugate state vectors to $u(0, \lambda)$ are

$$u(0, +)^c = \left( U \phi^*_u(0, +) \right) = \left( -\phi^*_u(0, -) \right) = -u^*(0, -)$$

$$u(0, 0)^c = \left( U \phi^*_u(0, 0) \right) = \left( \phi^*_u(0, 0) \right) = u^*(0, 0)$$

$$u(0, -)^c = \left( U \phi^*_u(0, -) \right) = \left( -\phi^*_u(0, +) \right) = -u^*(0, +).$$  \hspace{1cm} (85)

The conclusion is that if the fields are real then the conjugated fields are not independent. Only for truly complex fields, the conjugate fields can be independent and couple with the opposite charge to the electromagnetic field. This is a similar situation to the scalar field where the coupling to the electromagnetic field requires to have a complex field and the electric charge is the charge of the corresponding $U(1)$ symmetry. In the fermion case the electric charge is also associated with this symmetry but in this case the parity subspaces are not invariant under charge conjugation ($\{C, P\} = 0$) and we require the whole $(j, 0) \oplus (0, j)$ to make a Poincaré $P, C$ and $T$ invariant formalism. In the boson case, parity subspaces are charge conjugation subspaces ($\{C, P\} = 0$) and according to Eq. (59) these subspaces are also invariant under time reversal, thus in the boson case it is possible to make a quantum field theory using only the subspaces of well defined parity in $(j, 0) \oplus (0, j)$. In general, in the diagonal parity representation of operators (Dirac representation of gamma matrices in the case of spin 1/2), rest-frame spinors of positive (negative) parity are of the $u$-type ($v$-type) i.e. have vanishing components in the lower (upper) part. In an arbitrary frame, these spinors develop components in the lower (upper) part, but these components are not dynamical and dictated by the kinematics. The distinctive property of fermions and bosons is that charge conjugation associate a state of different parity in the former case and of the same parity in the latter case. The fermion case is transparent in the case $j = 1/2$ described by the Dirac theory. The explicit case of spin 1 bosons will be worked out in the following.

**VI. ELECTRODYNAMICS OF SPIN 1 BOSONS**

In this section we consider in detail the case of spin 1. From our previous discussion on discrete symmetries it is clear that we will be able to formulate independent theories for the different parities and on the other side we must take care of the subtleties mentioned in Section II.

**A. Theory based on the on-shell parity projector: Theory I.**

We start with the on-shell projectors in Eq. (83) but use the freedom to add an antisymmetric part to the symmetric traceless tensor. The general form of the antisymmetric tensor can be obtained as a linear combination of the elements of the covariant basis in Eq. (30) but the only possibility is a multiple of the Lorentz generators tensor thus the equations of motion for particles with positive (+) or negative (−) parity read

$$\left( \mp \Sigma_{\mu\nu} \partial_\mu \partial_\nu - m^2 \right) \psi(x) = 0,$$

where

$$\Sigma_{\mu\nu} = S_{\mu\nu} - i \kappa M_{\mu\nu}$$  \hspace{1cm} (86)
with \(\kappa\) a free parameter. These equations can be derived from the following Lagrangians

\[
L_\pm = \overline{\psi}(x) \left( \mp \Sigma^{\mu \nu} \partial_\mu \partial_\nu - m^2 \right) \psi(x).
\]

These Lagrangians can be recast as

\[
L_\pm = \mp \partial_\mu \left[ \overline{\psi}(x) \Sigma^{\mu \nu} \partial_\nu \psi(x) \right] \pm m^2 \overline{\psi}(x) \psi(x).
\]

In the following we skip the surface term and use the lagrangians

\[
L_\pm^0 = \pm \partial_\mu \overline{\psi}(x) \Sigma^{\mu \nu} \partial_\nu \psi(x) - m^2 \overline{\psi}(x) \psi(x),
\]

which are Hermitian since the \(\Sigma^{\mu \nu}\) tensor satisfy

\[
\Sigma^{\mu \nu} = \Pi (\Sigma^{\mu \nu})^\dagger \Pi = \Sigma^{\nu \mu}.
\]

The Lagrangians for interacting fields of positive (+) or negative (−) parity are obtained using the \(U(1)_{em}\) gauge principle as

\[
L_\pm = \pm \overline{D_\mu \psi} \Sigma^{\mu \nu} D_\nu \psi - m^2 \overline{\psi} \psi = L_\pm^0 + L_{\text{int}},
\]

where \(D^\mu = \partial^\mu - ieA^\mu\) and \(-e\) is the charge of the particle. A straightforward calculation yields the following interacting Lagrangian

\[
L_{\pm \text{int}} = \pm im \overline{\psi} \Sigma^{\mu \nu} \partial_\mu \psi - (\partial_\nu \psi) \Sigma^{\mu \nu} \partial_\nu \psi \pm ie \overline{\psi} \Sigma^{\mu \nu} \psi \partial_\nu A_\mu A_\nu.
\]

In the case of positive parity, the electromagnetic current is

\[
J_\mu^I(x) = \overline{\psi} [\Sigma_{\mu \nu} \partial^\nu \psi - (\partial^\nu \psi) \Sigma^\nu_{\mu \nu}] \psi.
\]

In momentum space this current reads

\[
J_\mu^I(p,p') = \overline{u}(p', \lambda') \left[ S^{\nu \mu} (p' + p)^\nu + i \kappa M_{\mu \nu} (p' - p)^\nu \right] u(p, \lambda).
\]

The electromagnetic current admits a Gordon-like decomposition. Indeed, the external spinors satisfy

\[
\frac{S(p)}{m^2} u(p, \lambda) = u(p, \lambda), \quad \overline{u}(p', \lambda') \frac{S(p')}{m^2} = \overline{u}(p', \lambda'),
\]

thus

\[
J_{\mu}^I(p,p') = \overline{u}(p', \lambda') \left[ S^{\nu \mu} p'^\nu + S_{\mu \nu} p^\nu + S(p') \frac{S(p)}{m^2} + i \kappa M_{\mu \nu} (p' - p)^\nu \right] u(p, \lambda).
\]

The symmetric traceless tensor \(S^{\mu \nu}\) in Eq. (25) satisfies the following commutation rules

\[
[S^{\mu \nu}, S^{\alpha \beta}] = -i \left( g^{\mu \alpha} M^{\beta \nu} + g^{\nu \alpha} M^{\beta \mu} + g^{\nu \beta} M^{\mu \alpha} + g^{\mu \beta} M^{\nu \alpha} \right),
\]

\[
\{ S^{\mu \nu}, S^{\alpha \beta} \} = \frac{4}{3} \left( g^{\mu \alpha} g^{\nu \beta} + g^{\nu \alpha} g^{\mu \beta} - \frac{1}{2} g^{\mu \nu} g^{\alpha \beta} \right) - \frac{1}{6} \left( C^{\mu \alpha \nu} + C^{\nu \beta \alpha} \right).
\]

which can be used to show that

\[
S(p') S^{\mu \nu} p_\nu = p'^2 (g^\mu \nu + i M^{\mu \nu} p_\nu),
\]

\[
S^{\mu \nu} S(p) p_\nu = p^2 (g^\mu \nu - i M^{\mu \nu} p_\nu),
\]

and using the on-shell condition \(p'^2 = p^2 = m^2\) we get

\[
J_{\mu}^I(p,p') = \overline{u}(p', \lambda') \left[ (p' + p)_\mu + i (1 + \kappa) M_{\mu \nu} (p' - p)^\nu \right] u(p, \lambda).
\]

This is the Gordon-like decomposition of this theory from where it is clear that the particle has a gyromagnetic factor

\[
g = 1 + \kappa,
\]

and reveals the free parameter \(\kappa\) as an “anomalous” contribution to the magnetic moment. A similar calculation to the one presented in Ref. [24] shows that the multipoles of this particle are given by

\[
q = e, \quad \mu = \frac{eg}{m} = \frac{e(1 + \kappa)}{m} \quad Q = \frac{e(g - 1)}{m^2} = \frac{e\kappa}{m^2}.
\]
B. Theory based on the mass, spin and parity off-shell projector: Theory II.

Now we turn to the theories based on the mass and parity off-shell projector in Eq. 46. Similarly to the previous case the equations of motion for particles with positive (+) or negative (−) parity read

\[
\left(-\Sigma_{\pm}^{\mu\nu} \partial_{\mu} \partial_{\nu} - m^2 \right) \psi(x) = 0,
\]

and the Lagrangian can be written as

\[
\mathcal{L}_{\pm}^0 = \partial_{\mu} \bar{\psi}(x) \Sigma_{\pm}^{\mu\nu} \partial_{\nu} \psi(x) - m^2 \bar{\psi}(x) \psi(x),
\]

where now the tensor is given by

\[
\Sigma_{\pm}^{\mu\nu} = \frac{1}{2} \left( g^{\mu\nu} \pm S_{\mu\nu} - i2\rho M_{\mu\nu} \right)
\]

with \( \rho \) a free parameter. Gauging the theory we get the following interacting Lagrangian

\[
\mathcal{L}_{\pm \text{int}} = i e (\bar{\psi} \Sigma_{\pm}^{\mu\nu} \partial_{\nu} \psi - (\partial_{\nu} \bar{\psi}) \Sigma_{\pm}^{\mu\nu} \psi) A_{\mu} + e^2 (\bar{\psi} \Sigma_{\pm}^{\mu\nu} \psi A_{\mu} A_{\nu}.
\]

The electromagnetic current in momentum space for this theory reads

\[
J_{\mu}^\gamma(p, p') = \frac{1}{2} \overline{\sigma}(p', \lambda') \left( (p' + p)^\rho \pm S^{\rho\nu} (p' + p)_\nu + i2\rho M_{\mu\nu} (p' - p)_\nu \right) u(p, \lambda).
\]

A similar calculation to the previous theory yields the following Gordon-like decomposition

\[
J_{\mu}^\gamma = \overline{\sigma}(p', \lambda') \left[ (p' + p)^\rho + i \left( \frac{1}{2} + \rho \right) M^{\rho\nu} (p' - p)_\nu \right] u(p, \lambda).
\]

Particles in this theory have a gyromagnetic factor

\[
g = \frac{1}{2} + \rho
\]

thus the free parameter in this case also corresponds to an “anomalous” magnetic moment and the gyromagnetic factor inherent to the mass and parity projector is \( g = 1/2 \). The electromagnetic multipole moments of this particle are given by

\[
q = e, \quad \mu = \frac{e(\frac{1}{2} + \rho)}{m}, \quad Q = \frac{e(-\frac{1}{2} + \rho)}{m^2}.
\]

VII. MAPPING TO THE ANTISYMMETRIC FIELD AND COMPARISON WITH EXISTING FORMALISMS.

A. Parity-based formalism in tensor basis.

Since the description of massive spin 1 particles has a long history a side by side comparison with existing formalisms for the description of massive spin 1 particles transforming in the \((1, 0) \oplus (0, 1)\) is mandatory. With this aim we translate our formalism for spin 1 matter fields to tensor language. We denote the corresponding field as \( F^{\alpha\beta}(x) \) where \( \alpha, \beta \) are Lorentz indices associated to the \((1, 0) \oplus (0, 1)\) representation. In the tensor basis, instead of a pair of state vector indices, every operator has two pairs of internal Lorentz indices. The explicit form of the operators in the covariant, for the description of massive spin 1 particles transforming in the \((1, 0) \oplus (0, 1)\) is mandatory. With this aim we translate our formalism for spin 1 matter fields to tensor language. We denote the corresponding field as \( F^{\alpha\beta}(x) \) where \( \alpha, \beta \) are Lorentz indices associated to the \((1, 0) \oplus (0, 1)\) representation. In the tensor basis, instead of a pair of state vector indices, every operator has two pairs of internal Lorentz indices. The explicit form of the operators in the covariant basis is given by

\[
1_{\alpha\beta\gamma\delta} = \frac{1}{2} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}),
\]

\[
\chi_{\alpha\beta\gamma\delta} = \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta},
\]

\[
(M_{\mu\nu})_{\alpha\beta\gamma\delta} = -i (g_{\mu\gamma} \chi_{\alpha\beta\delta\gamma} + g_{\mu\delta} \chi_{\alpha\beta\gamma\delta} - g_{\nu\gamma} \chi_{\alpha\beta\delta\gamma} - g_{\nu\delta} \chi_{\alpha\beta\gamma\delta})
\]

\[
(S_{\mu\nu})_{\alpha\beta\gamma\delta} = g_{\mu\gamma} \chi_{\alpha\beta\delta\gamma} - g_{\mu\delta} \chi_{\alpha\beta\gamma\delta} - g_{\nu\gamma} \chi_{\alpha\beta\delta\gamma} - g_{\nu\delta} \chi_{\alpha\beta\gamma\delta}
\]

\[
(C_{\mu\nu\rho})_{\alpha\beta\gamma\delta} = 321_{\alpha\tau\mu\nu} 1_{\beta\delta\rho\sigma} - 321_{\alpha\tau\rho\sigma} 1_{\beta\delta\mu\nu} + 6g_{\alpha\tau} X_{\beta\delta\rho\sigma\mu\nu} + 6g_{\beta\delta} X_{\alpha\tau\rho\sigma\mu\nu} + 81_{\alpha\beta\delta\tau} 1_{\rho\sigma\mu\nu}
\]

\[
+ 161_{\alpha\beta\rho\sigma} 1_{\delta\tau\mu\nu} + 161_{\alpha\delta\mu\nu} 1_{\beta\tau\rho\sigma} - 161_{\alpha\delta\rho\sigma} 1_{\beta\tau\mu\nu} - 161_{\alpha\delta\mu\nu} 1_{\beta\tau\rho\sigma}
\]

\[
+ 161_{\alpha\beta\rho\sigma} 1_{\delta\tau\mu\nu} + 161_{\alpha\delta\mu\nu} 1_{\beta\tau\rho\sigma} - 161_{\alpha\delta\rho\sigma} 1_{\beta\tau\mu\nu} - 161_{\alpha\delta\mu\nu} 1_{\beta\tau\rho\sigma}
\]

\[
+ 161_{\alpha\beta\rho\sigma} 1_{\delta\tau\mu\nu} + 161_{\alpha\delta\mu\nu} 1_{\beta\tau\rho\sigma} - 161_{\alpha\delta\rho\sigma} 1_{\beta\tau\mu\nu} - 161_{\alpha\delta\mu\nu} 1_{\beta\tau\rho\sigma}
\]
where

\[ X_{\alpha \tau \rho \mu \nu} = 2g_{\alpha \rho}T_{\mu \nu \tau \rho} - 2g_{\alpha \sigma}T_{\mu \nu \tau \rho} - 2g_{\alpha \mu}1_{\nu \tau \rho \sigma} + 2g_{\alpha \sigma}1_{\mu \tau \rho \sigma}. \]  

(113)

all the structure previously described can be translated into tensor language using the operators in Eq. (112). The Lorentz generators in tensor basis are purely imaginary whereas the symmetric traceless tensor turns out to be real. It is easy to show that charge conjugation in this case is simply the complex conjugation operation, thus real fields are self-conjugated (Majorana fields) and the coupling to an electromagnetic field requires to complexify this representation, similarly to the case of scalars. Also, the commutation of charge conjugation with parity is apparent from the real nature of \( S^{\mu \nu} \) in tensor basis.

Another simplification in tensor basis concerns the action of parity and the “bar” notation. The rest-frame parity operator in tensor basis is given by

\[ \Pi_{\alpha \beta \gamma \delta} = (S_{00})_{\alpha \beta \gamma \delta} = 1_{\alpha \beta \gamma \delta} - 2g_{0 \gamma}1_{\alpha \beta 0 \delta} - 2g_{0 \delta}1_{\alpha \beta \gamma 0}. \]  

(114)

When acting on the field tensor this operator selects the components of well defined parity

\[ F_{\pi}^{\alpha \beta} \equiv \Pi_{\gamma \delta} F_{\gamma \delta} = F_{\alpha \beta} - 2 \left( g_{\alpha 0} F_{0 \beta} - g_{\beta 0} F_{0 \alpha} \right), \]  

(115)

thus

\[ F_{\pi}^{0i} = -F_{0i}, \quad F_{\pi}^{ij} = F^{ij}. \]  

(116)

Since our operator projects onto well defined parity states and antiparticles have the same parity there is no need to carry the “bar” operation thorough and we can use directly the sign in the Lagrangian as done usually, e.g. the Lagrangian Eq. (92) in the tensor basis reads

\[ \mathcal{L}_{\pm} = \left( D_{\mu} F^{\alpha \beta} \right)^{\dagger} \left( \Sigma^{\mu \nu} \right)_{\alpha \beta \gamma \delta} D_{\nu} F_{\gamma \delta} \mp m^{2} \left( F^{\alpha \beta} \right)^{\dagger} F_{\alpha \beta}. \]  

(117)

Special attention deserves the chirality operator in Eqs. (112). In tensor basis this transformation is related to the concept of a dual tensor as

\[ (\chi F)^{\alpha \beta} = i\tilde{F}^{\alpha \beta}. \]  

(118)

Chirality operator is purely imaginary in tensor basis making transparent that Eq. (69) is satisfied. Also, an explicit calculation shows that Eq. (31) holds.

**B. Existing formalisms in the light of the parity-based formalism.**

The \((1,0) \oplus (0,1)\) field has been considered in the literature either in the spinor language \([7],[8],[11],[12],[15]\), or in the form of an antisymmetric tensor field \([22]\). Concerning the spinor formalisms, in Refs. \([7],[8]\) the following equation of motion was put forth

\[ [S^{\mu \nu} \partial_{\mu} \partial_{\nu} + m^{2}] \psi_{+(x)} = 0, \]  

(119)

where \( \psi_{+(x)} = u(p, \lambda)e^{-ip \cdot x} \) is the \((1,0) \oplus (0,1)\) wave function. The corresponding state vector \( u(p, \lambda) \) satisfies the condition

\[ [S^{\mu \nu} p_{\mu} p_{\nu} - m^{2}] u(p, \lambda) = 0. \]  

(120)

From the perspective of our construction in Eq. (85), this is the projection over subspaces of positive parity in \((1,0) \oplus (0,1)\), and corresponds to the particular value \( \kappa = 0 \) of the positive parity case of Theory I discussed above (see Eq. (85)). The corresponding equation for negative parity states differ from this equation just by a relative sign between the mass term and the \( S(p) \) operator,

\[ [-S^{\mu \nu} \partial_{\mu} \partial_{\nu} + m^{2}] \psi_{-(x)} = 0. \]  

(121)

It is worth to remark that the construction of discrete symmetries above, specially charge conjugation, apply to this theory and the charge conjugated fields have the same parity as the field itself. There are claims in the literature on
the possibility that the charge conjugated vector bosons in the \((1, 0) \oplus (0, 1)\) have the opposite parity to the boson itself \cite{28}. At odds with our construction, the charge conjugation operation used in \cite{28} is obtained from the interacting theory imposing that the field satisfy Eq. \(119\) and the conjugate field satisfy a different equation, the one satisfied by the field of opposite parity in Eq. \(121\).

The electrodynamics of spin 1 bosons in the Joos-Weinberg formalism has been discussed in the literature \cite{16,17}, being considered as a phenomenological approach due to the “unphysical” solutions inherent to this formalism \cite{12}. The problem seems to be that the propagator of the theory, in addition to the conventional pole at \(p^2 = m^2\), has an unphysical pole at \(p^2 = -m^2\). However, these conclusions are based on a naive calculation of the two-point Green function. The proper calculation of Green functions requires to work out the corresponding quantum field theory and in particular the calculation of the physical causal propagator needs an appropriate handling of the poles for which the conventional \(m^2 \rightarrow m^2 - \varepsilon\) prescription is not enough. This calculation is beyond the scope of this paper and will be published elsewhere. Below we discuss the naive calculation from the perspective of the most general lagrangian in the tensor formalism.

In order to avoid the “problems” detected for the massive Joos-Weinberg formalism, new proposals with the mass shell condition as an auxiliary condition were studied in \cite{29} which eventually lead to Shay an Good \cite{11} to propose the following modified equation

\[
[\partial^2 + S^{\mu\nu} \partial_\mu \partial_\nu + 2m^2] \psi(x) = 0. \tag{122}
\]

Performing a non-relativistic expansion of the gauged theory they conclude that it describes particles with a gyromagnetic factor \(g_{SG} = \frac{1}{2}\). The same equation was considered in \cite{13} where the electrodynamics of a spin boson is developed in this context. In that work, this equation is the first of a set of equations for high spin. A naive calculation of the propagator for these theories yields an unphysical pole at \(p^2 = 0\) for all \(j\) except for \(j = 1\) which corresponds to the Shay-Good equation. From the perspective of our parity-based construction, the Shay-Good equation corresponds to the particular value \(\rho = 0\) of the positive parity case of Theory II (see Eq. \(114\)). The classical causality properties of the Shay-Good equation when an “anomalous” magnetic-dipole term is included was studied in \cite{21} which conclude that only in the case when the gyromagnetic factor take the value \(g = 1\), i.e. when \(\rho = 1/2\), the theory is causal.

Concerning formalisms in the tensor language, aiming to construct an effective field theory description of the interactions of spin 1 hadrons, in Ref. \cite{22} a tensor formalism for spin 1 was developed. This formalism is grounded on general arguments and it is worth to make a careful comparison with our results and the existing literature. First, they write the most general second order Lagrangian for the antisymmetric tensor field as

\[
\mathcal{L} = a \partial^\mu F_{\mu\beta} \partial^\nu F^{\nu\beta} + b \partial^\mu F_{a\beta} \partial^\mu F^a_{\alpha\beta} + c F^{\alpha\beta} F_{\alpha\beta}. \tag{123}
\]

This is indeed the most general second order Lagrangian and can be put in a form resembling our Lagrangian using the symmetries of the field

\[
\mathcal{L} = \partial^\mu F^{\alpha\beta} \left[ (S_{\mu\nu}^G)_{\alpha\beta\gamma\delta} \right] \partial^\nu F_{\gamma\delta} + F^{\alpha\beta} [c1_{\alpha\beta\gamma\delta}] F_{\gamma\delta}. \tag{124}
\]

where the most general symmetric tensor allowed by Lorentz covariance is

\[
(S_{\mu\nu}^G)_{\alpha\beta\gamma\delta} = bg_{\mu\nu} 1_{\alpha\beta\gamma\delta} + \frac{a}{4} (g_{\mu\gamma} 1_{\alpha\beta\nu\delta} + g_{\nu\delta} 1_{\alpha\gamma\nu\delta} + g_{\nu\gamma} 1_{\alpha\beta\mu\delta} + g_{\nu\delta} 1_{\alpha\gamma\mu\delta}). \tag{125}
\]

In terms of the symmetric traceless tensor in Eq. \(112\) and skipping the representation indices we get

\[
S_{\mu\nu}^G = \left( b + \frac{a}{4} \right) g_{\mu\nu} - \frac{a}{4} S_{\mu\nu}. \tag{126}
\]

The operator of the most general Lagrangian in Eq. \(123\) in momentum space is

\[
O^G = \left( b + \frac{a}{4} \right) p^2 - \frac{a}{4} S(p) + c \equiv Ap^2 + BS(p) + C. \tag{127}
\]

In a naive calculation, the propagator is simply the inverse of this operator and can be found as follows. Defining the operator

\[
\tilde{O}^G = \alpha p^2 + \beta S(p) + \kappa \tag{128}
\]

we obtain the product

\[
\tilde{O}^G O^G = f(p^2) + [(\beta A + \alpha B) p^2 + (\beta C + \kappa B)] S(p), \tag{129}
\]
with
\[ f(p^2) = (\alpha A + \beta B) p^4 + (\alpha C + \kappa A) p^2 + \kappa C. \] (130)

Imposing to have an operator proportional to 1 on the r.h.s. of Eq. (129) requires the coefficient of \( S(p) \) in Eq. (129) to vanish for all \( p^2 \) thus
\[ \beta A + \alpha B = 0, \quad \beta C + \kappa B = 0. \] (131)

Under this condition we get the inverse of \( O^G \) as
\[ (O^G)^{-1} = \frac{\tilde{O}^G}{f(p^2)}. \] (132)

The poles of the propagator would be the zeros of \( f(p^2) \) which are given by
\[ f(p^2) = (\alpha A + \beta B) p^4 + (\alpha C + \kappa A) p^2 + \kappa C = 0. \] (133)

Multiplying by \( A/\alpha \) and using Eqs. (131) we get the condition
\[ (A^2 - B^2) p^4 + 2ACp^2 + C^2 = 0, \] (134)

which yields the poles at
\[ p^2_1 = -\frac{C}{A - B} = -\frac{2c}{a + 2b} \equiv M^2_1, \quad p^2_2 = -\frac{C}{A + B} = -\frac{c}{b} \equiv M^2_2. \] (135)

These are the results for the naive poles quoted in Ref. 22 and in general we have two of them. There are only three possibilities to avoid this naive double pole structure: i) \( A = -B \quad (b = 0) \), ii) \( A = B \quad (a = -2b) \), iii) \( B = 0 \quad (a = 0) \). The first two cases were considered in 22. In any of these cases \( f(p^2) \) is linear in \( p^2 \) and there is only one simple pole located at
\[ p^2 = -\frac{C}{2A} = M^2. \] (136)

In both of these cases the naive calculation outlined here yields the correct two-point Green function. For \( A = B \) and removing a global factor \( 2A \) (taking \( A = 1/2 \)) yields the following operator
\[ O^G_+ = \frac{1}{2} \left( p^2 + S(p) \right) - M^2 \] (137)
i.e. we obtain the positive parity projection in Eq. (34). The inverse operator in this case is given by
\[ (O^G_+)^{-1} = \frac{\Delta_+(p)}{p^2 - M^2}, \] (138)

with
\[ \Delta_+(p) = -\frac{1}{2} (p^2 - S(p)) + M^2 \] (139)

In the case \( A = -B \), removing a global factor \( 2A \) (taking \( A = -1/2 \)) the operator is
\[ O^G_- = \frac{1}{2} \left( p^2 - S(p) \right) - M^2, \] (140)

which is the positive parity projection in our Eq. (34). The inverse operator in this case is given by
\[ (O^G_-)^{-1} = \frac{\Delta_-(p)}{p^2 - M^2}. \] (141)

with
\[ \Delta_-(p) = -\frac{1}{2} (p^2 + S(p)) + M^2. \] (142)
These are precisely the propagators in our Theory II for positive and negative parity respectively. The corresponding Lagrangians are

\[ \mathcal{L}_A = b \left( \partial^\mu F_{\alpha\beta} \partial_\mu F^{\alpha\beta} - 2\partial^\mu F_{\mu\beta} \partial_\nu F^{\nu\beta} - M_A^2 F^{\alpha\beta} F_{\alpha\beta} \right) = b \left[ \partial_\mu F^{a\beta} \left( \Sigma_+^{\mu\nu} \right)_{\alpha\beta\gamma\delta} \partial^\nu F^{\gamma\delta} - M_A^2 F^{a\beta} F_{\alpha\beta} \right], \]  

(143)

for the axial case and in the vector case it reads

\[ \mathcal{L}_V = a \left( \partial^\mu F_{\mu\beta} \partial_\nu F^{\nu\beta} - \frac{1}{2} M_V^2 F^{\alpha\beta} F_{\alpha\beta} \right) = a \left[ \partial_\mu F^{a\beta} \left( \Sigma_-^{\mu\nu} \right)_{\alpha\beta\gamma\delta} \partial^\nu F^{\gamma\delta} - M_V^2 F^{a\beta} F_{\alpha\beta} \right]. \]  

(144)

These Lagrangians coincide with the tensor version of Theory II for the specific value \( \rho = 0 \). In [22] it is shown that in the first case \( (A = B) \) the degrees of freedom of the vector part \( (F^0) \) of the tensor are frozen (they are not dynamical) and we are left only with the vector part \( (F^0) \). Similarly, in the second case \( (A = B) \), the axial components are frozen and we are left with the vector ones. In our formalism this is simply a consequence of the projection over subspaces of well defined parity as can be seen from Eq. (116).

The third possibility to avoid the naive double pole structure is \( B = 0 \) \( (a = 0) \). In this case there are no further constrictions and the corresponding theory describes a degenerate parity-doublet. This possibility corresponds to the case \( q = 0 \) of the Poincaré projector formalism [23][24] for spin 1 in the \((1,0) \oplus (0,1)\) developed in Ref. [22]. In that work also the possibility of a theory based on the projection over parity eigensubspaces was considered. The parity projector is constructed directly from the states which in turn are derived from states in the \((1/2,1/2)\) representation. In the perspective of the present work, the tensor in the so obtained projector is of the form of Eq. (106) with \( \rho = 1/2 \). The classical causality properties of this theory are also analyzed in [23] concluding that it is causal. This result is consistent with the conclusions of Ref. [21].

Before ending this section we would like to remark that beyond the analyzed values, the naive calculation of the propagator in a second order formalism in general yields two poles. The only possibility for getting two symmetric poles at \( p^2 = M^2 \) and \( p^2 = -M^2 \) from Eqs. (135) is to take \( A = 0 \). This correspond to \( b = -\frac{a}{\lambda} \) in whose case \( SO^\pi \) reduces to \( S \) modulo a global factor. This case corresponds to \( \kappa = 0 \) in our Theory I (Joos-Weinberg formalism in the case of positive parity). Interestingly the massless limit \((c = 0)\) of this lagrangian has been discussed in the literature in tensor language for \( a = -1 \) (see [30] for a review). Here, the corresponding action has been shown to be conformally invariant and the fields have been named “antisymmetric tensor matter fields”. A proper choice of the annihilation and creation operators allows for a correct canonical quantization in the massless case and interesting applications to hadron physics and Yukawa interactions in physics beyond the standard model and further references can be found in Ref. [30]. The possibility of an action built of an antisymmetric second rank tensor field acting as the “potential” of a gauge invariant third rank completely antisymmetric third rank tensor field discussed in this work corresponds to \( a = -1/2, b = 1/4 \), i.e is the same as the Shay-Good theory or the formalism used in \( R\chi PT \). We remark that a proper calculation of the propagator in the massive case requires to work out the corresponding quantum field theory and to find an appropriate prescription for the proper handling of the naive poles. Otherwise we would have the tachyon propagation of the second pole which causes the classical causality problems that have been discussed in the literature. The conformal quantization of the massless case of the theory based on the on-shell projection (same as Joos-Weinberg or Chizhov theory) done in [30] are interesting guidelines in this concern.

Finally, it is clear that the freedom in the choice of the antisymmetric part of the space-time tensor appearing when boosting the rest-frame parity operator is relevant in the sense that it defines crucial properties of the theory such as the multipole moments. Indeed, our construction of the covariant basis shows that these terms can contain only the Lorentz generators tensor \( M_{\mu\nu} \) which defines not only the value of the gyromagnetic factor but also all multipole moments. The corresponding term in the lagrangian is an “anomalous” magnetic-dipole interaction which is gauge invariant by itself. Furthermore, the matter fields have mass dimension one and this term is dimension four, thus naively renormalizable, and must be included in the tree level lagrangian. The lesson is that we can either consider the theory with the tensors in Eqs. (87-100), or take \( \kappa = 0 \) and \( \rho = 0 \) and include an anomalous magnetic term in the lagrangian. Even more, the mass dimension one of the matter field allows also for dimension four self-interactions, which are also naively renormalizable and must be included in the Lagrangian. In the following section we analyze the chiral structure of the theories and the reliability of these terms in a chiral theory as the standard model.

VIII. CHIRAL DECOMPOSITION.

The Lagrangian for positive parity in the case of Theory I including all terms of dimension four reads

\[ \mathcal{L}_I = \partial^\mu \bar{\psi} \left( S_{\mu\nu} - i\kappa M_{\mu\nu} \right) \partial_\nu \psi - m^2 \bar{\psi} \psi + \mathcal{L}_{\text{self}}, \]  

(145)
where $L_{\text{self}}$ stands for all self-interaction terms which must be constructed from the following bilinears

$$
\bar{\psi}\psi, \quad \bar{\psi}\chi\psi, \quad \bar{\psi}S_{\mu\nu}\psi, \quad \bar{\psi}\chi S_{\mu\nu}\psi, \quad \bar{\psi}M_{\mu\nu}\psi, \quad \bar{\psi}C_{\mu\nu\alpha\beta}\psi, \quad \bar{\psi}\chi M_{\mu\nu}\psi, \quad \bar{\psi}\chi C_{\mu\nu\alpha\beta}\psi. \quad (146)
$$

The last two bilinears arises from the contractions of the previous two with the Levi-Civita tensor (contractions with the metric tensor vanish) which can be rewritten in terms of the chirality operators using the relations

$$
\tilde{M}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} M_{\rho\sigma}, \quad \tilde{C}_{\mu\nu\alpha\beta} \equiv \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma\alpha\beta} = -i\chi C_{\mu\nu\alpha\beta}. \quad (147)
$$

There are ten independent non-vanishing terms that can be built from the products of these bilinears, thus

$$
L_{\text{self}} = c_1 (\bar{\psi}\psi)^2 + c_2 (\bar{\psi}\chi\psi)^2 + c_3 (\bar{\psi}S_{\mu\nu}\psi)^2 + c_4 (\bar{\psi}\chi S_{\mu\nu}\psi)^2
$$

$$
+ c_5 (\bar{\psi}M_{\mu\nu}\psi)^2 + c_6 (\bar{\psi}C_{\mu\nu\alpha\beta}\psi)^2 + c_7 (\bar{\psi}\chi\psi) (\bar{\psi}\chi\psi) + c_8 (\bar{\psi}S_{\mu\nu}\psi) (\bar{\psi}\chi S_{\mu\nu}\psi)
$$

$$
+ c_9 (\bar{\psi}M_{\mu\nu}\psi) (\bar{\psi}\chi M_{\mu\nu}\psi) + c_{10} (\bar{\psi}C_{\mu\nu\alpha\beta}\psi) (\bar{\psi}\chi C_{\mu\nu\alpha\beta}\psi). \quad (148)
$$

Now, the chirality operator commutes with all the covariant basis elements except for $S^{\mu\nu}$ and $\chi S^{\mu\nu}$, for which it anti-commutes.

Chiral fields are naturally defined in terms of the projectors onto subspaces of well defined chirality

$$
\psi_R = P_R \psi \quad \text{and} \quad \psi_L = P_L \psi, \quad (149)
$$

where

$$
P_R = \frac{1}{2} (1 + \chi), \quad P_L = \frac{1}{2} (1 - \chi). \quad (150)
$$

These operators have the following properties

$$
P_R + P_L = 1, \quad P_R P_L = 0, \quad P_R^2 = P_R, \quad P_L^2 = P_L, \quad (151)
$$

and the commutation relations in Eqs. (1631) yield

$$
M^{\mu\nu} P_{R,L} = P_{R,L} M^{\mu\nu}, \quad S^{\mu\nu} P_R = P_L S^{\mu\nu}. \quad (152)
$$

A chiral transformation of the field is defined by

$$
\psi' = U \psi \equiv \exp (i\theta \chi) \psi. \quad (153)
$$

The chiral transformation of the bilinears, in terms of the chiral fields read

$$
\begin{align*}
(\bar{\psi}\psi) & \rightarrow (\bar{\psi}',\psi') = e^{2i\theta} (\bar{\psi}_L \psi_R) + e^{-2i\theta} (\bar{\psi}_R \psi_L), \\
(\bar{\psi}\chi\psi) & \rightarrow (\bar{\psi}'\chi\psi') = e^{2i\theta} (\bar{\psi}_L \chi \psi_R) - e^{-2i\theta} (\bar{\psi}_R \chi \psi_L), \\
(\bar{\psi}S_{\mu\nu}\psi) & \rightarrow (\bar{\psi}'S_{\mu\nu}\psi') = (\bar{\psi}_R S_{\mu\nu} \psi_R) + (\bar{\psi}_L S_{\mu\nu} \psi_L), \\
(\bar{\psi}\chi S_{\mu\nu}\psi) & \rightarrow (\bar{\psi}'\chi S_{\mu\nu}\psi') = (\bar{\psi}_R S_{\mu\nu} \psi_R) - (\bar{\psi}_L S_{\mu\nu} \psi_L), \\
(\bar{\psi}M_{\mu\nu}\psi) & \rightarrow (\bar{\psi}'M_{\mu\nu} U^2 \psi') = e^{2i\theta} (\bar{\psi}_L M_{\mu\nu} \psi_R) + e^{-2i\theta} (\bar{\psi}_R M_{\mu\nu} \psi_L), \\
(\bar{\psi}C_{\mu\nu\alpha\beta}\psi) & \rightarrow (\bar{\psi}'C_{\mu\nu\alpha\beta} \psi') = e^{2i\theta} (\bar{\psi}_L C_{\mu\nu\alpha\beta} \psi_R) + e^{-2i\theta} (\bar{\psi}_R C_{\mu\nu\alpha\beta} \psi_L), \\
(\bar{\psi}\chi M_{\mu\nu}\psi) & \rightarrow (\bar{\psi}'\chi M_{\mu\nu} U^2 \psi') = e^{2i\theta} (\bar{\psi}_L M_{\mu\nu} \psi_R) - e^{-2i\theta} (\bar{\psi}_R M_{\mu\nu} \psi_L), \\
(\bar{\psi}\chi C_{\mu\nu\alpha\beta}\psi) & \rightarrow (\bar{\psi}'\chi C_{\mu\nu\alpha\beta} \psi') = e^{2i\theta} (\bar{\psi}_L C_{\mu\nu\alpha\beta} \psi_R) - e^{-2i\theta} (\bar{\psi}_R C_{\mu\nu\alpha\beta} \psi_L). 
\end{align*} \quad (154)
$$

From these transformation properties, we see that most of the terms in our lagrangian are not chirally invariant. In particular, mass terms and anomalous magnetic moments are forbidden by chiral symmetry. It is straightforward to show that the chirally invariant lagrangian is

$$
\mathcal{L}_\chi = \partial^\mu \bar{\psi} \left( S_{\mu\nu} \right) \partial_\nu \psi + \mathcal{L}_\chi^\times + \mathcal{L}_{\text{self}}^\times, \quad (155)
$$

with

$$
\begin{align*}
\mathcal{L}_\chi^\times = & a_1 (\bar{\psi}\psi)^2 - (\bar{\psi}\chi\psi)^2 + a_2 (\bar{\psi}S_{\mu\nu}\psi)^2 + a_3 (\bar{\psi}\chi S_{\mu\nu}\psi)^2 + a_4 (\bar{\psi}M_{\mu\nu}\psi)^2 - (\bar{\psi}\chi M_{\mu\nu}\psi)^2
\end{align*}
$$

$$
+ a_5 (\bar{\psi}C_{\mu\nu\alpha\beta}\psi)^2 - (\bar{\psi}\chi C_{\mu\nu\alpha\beta}\psi)^2 + a_6 (\bar{\psi}S_{\mu\nu}\psi) (\bar{\psi}\chi S_{\mu\nu}\psi) \quad (156)
$$
The decomposition of the chiral Lagrangian in terms of the chiral field reads

$$\mathcal{L}_\chi = \partial^\mu \bar{\psi}_R S_{\mu \nu} \partial^\nu \psi_R + \partial^\mu \bar{\psi}_L S_{\mu \nu} \partial^\nu \psi_L + \mathcal{L}_{\text{self}}^\chi,$$

(157)

with

$$\mathcal{L}_{\text{self}}^\chi = b_1 \left( \bar{\psi}_R \psi_L \right) \left( \bar{\psi}_L \psi_R \right) + b_2 \left( \bar{\psi}_L S_{\mu \nu} \psi_L \right)^2 + b_3 \left( \bar{\psi}_R S_{\mu \nu} \psi_R \right)^2 + b_4 \left( \bar{\psi}_R M_{\mu \nu} \psi_L \right) \left( \bar{\psi}_L M_{\mu \nu} \psi_R \right) + b_5 \left( \bar{\psi}_R C_{\mu \nu \alpha \beta} \psi_L \right) \left( \bar{\psi}_L C_{\mu \nu \alpha \beta} \psi_R \right) + b_6 \left( \bar{\psi}_L S_{\mu \nu} \psi_L \right) \left( \bar{\psi}_R S_{\mu \nu} \psi_R \right).$$

(158)

In this form, it is clear that this lagrangian is invariant under the following independent transformations of the chiral fields

$$\psi_R' = \exp (i \alpha R) \psi_R \quad \psi_L' = \exp (i \alpha L) \psi_L.$$

(159)

For this Lagrangian, the decoupling of left and right fields allows them to have different interactions with gauge fields thus this is an appropriate formalism to attempt the inclusion of spin 1 matter fields in chiral theories like extensions of the standard model. In addition the self-interaction terms produce a richer structure than in the spin one-half case. Similar results are obtained for the negative parity case.

As for theory II, the self-interacting lagrangian is the same but the decomposition of the Lagrangian reads

$$\mathcal{L}_{II} = \frac{1}{2} \left( \partial^\mu \bar{\psi}_R g_{\mu \nu} \partial^\nu \psi_L + \partial^\mu \bar{\psi}_R S_{\mu \nu} \partial^\nu \psi_R + \partial^\mu \bar{\psi}_L S_{\mu \nu} \partial^\nu \psi_L \right) - i e \kappa \partial^\mu \bar{\psi}_R M_{\mu \nu} \partial^\nu \psi_L - m^2 \bar{\psi}_R \psi_L + R \leftrightarrow L.$$

(160)

The first term in the kinetic piece couples always the left to the right field thus it is not possible to realize chiral symmetry linearly. This formalism is therefore appropriate for (and in the tensor basis has been used in the formulation of ) theories with a chiral symmetry realized non-linearly. If confirmed that the scalar boson found at the LHC is the Higgs boson, chiral symmetry must be linearly realized and this formalism (Theory II) is not appropriate to be used in possible extensions of the standard model by spin 1 matter fields.

IX. CONCLUSIONS.

In this work we use our recent parity-based construction of a covariant basis for operators acting on the $(j, 0) \oplus (0, j)$ representation of the HLG to propose a formalism for the description of particles of spin $j$ and well defined parity, transforming in this representation which by extension of the notation of spin 1/2 in the standard model we call matter fields. We show that for all $j$, except for $j = 1/2$, there is a freedom in the writing of the covariant form of parity operator which is irrelevant for free theories but which becomes important in the interacting case. Using our covariant basis we show that this freedom is related to a magnetic-dipole term in the lagrangian. In addition to this freedom, we show that we have the choice of using on-shell or off-shell projectors which yield different nonequivalent interacting theories.

We construct the operators implementing the discrete symmetries of the theory, specially charge conjugation, and show that it commutes with parity in the case of bosons and anti-commutes in the case of fermions as expected. As an explicit example of an interacting theory we work out the electrodynamics of spin 1 matter bosons finding two nonequivalent interacting theories for a given parity. Using the properties of the covariant basis we perform a Gordon-like decomposition of the corresponding electromagnetic current and show that in the formulation based on the on-shell projectors the spin 1 matter boson has a gyromagnetic factor $g = 1 + \kappa$ while in the formalism based on the off-shell projectors it has $g = \frac{1}{2} + \rho$, where $\kappa$ and $\rho$ are parameters associated to the above mentioned freedom and turn out to be “anomalous” magnetic moments. We rewrite our formalisms for spin one matter bosons in the form an antisymmetric tensor field and make a comparison with existing formalisms in the literature, either in spinor or tensor language.

Concerning the spinor formalism, in the case $\kappa = 0$ and for positive parity, our theory based on the on-shell projectors reproduces an equation proposed by Joos [7] and Weinberg [8]. In the case $\rho = 0$ and for positive parity our theory based on the off-shell projectors reproduces an equation proposed by Shay and Good in [11]. For arbitrary $\rho$ this theory reproduces the Shay-Good equation with a magnetic-dipole term which has been shown in [21] to be classically causal only for $\rho = 1/2$. As for the tensor formalism, we show that the conformally invariant action used by Chizhov in Ref. [30] corresponds to $\kappa = 0$, $m = 0$ and positive parity of our theory based on the on-shell projectors, thus it the same as the massless limit of the Joos-Weinberg formalism. In the case $\rho = 0$ our theory based on the off-shell projectors recover the tensor formalism used in chiral perturbation theory with resonances $(R_\chi PT)$ [22].
which is the same as the Shay-Good theory just written in tensor language. This theory in tensor language and for the specific case $\rho = 1$ coincides with a recent proposal based on parity projection where the projector is extracted from an explicit construction of the states, in turn induced from states in $(1/2, 1/2)$ and which has been proved to propagate causally in an electromagnetic background \[25\], in agreement with results in \[21\].

Naive power counting admits anomalous magnetic-dipole terms and self-interactions at tree level. We perform a chiral decomposition of these theories and using the properties of the covariant basis we show that chiral symmetry can be realized linearly only for the theory based on the on-shell projectors. Chiral symmetry forbids not only mass terms but also anomalous magnetic dipole terms and some of the self-interaction terms, leaving only six of them. We conclude that this is the appropriate framework to attempt the incorporation of spin 1 matter bosons in chiral theories like the standard model.

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