Edge colorings of graphs without monochromatic stars

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Abstract
In this note, we improve on results of Hoppen, Kohayakawa and Lefmann about the maximum number of edge colorings without monochromatic copies of a star of a fixed size that a graph on \(n\) vertices may admit. Our results rely on an improved application of an entropy inequality of Shearer.

1 Introduction
Let \(r\) be a positive integer and \(G\) and \(H\) be (simple) graphs. We define \(c_{r,H}(G)\) as the number of \(r\)-edge-colorings of \(G\) (i.e., functions \(c : E(G) \to [r] = \{1, \ldots, r\}\)) without a monochromatic copy of \(H\) as a subgraph. For instance, when \(H\) is the path on 3 vertices (we denote it by \(P_3\)), \(c_{r,H}(G)\) is simply the number of proper \(r\)-edge-colorings of \(G\). Furthermore, let \(c_{r,H}(n)\) be the maximum value of \(c_{r,H}(G)\) as \(G\) runs through all graphs on \(n\) vertices. A graph \(G\) is called \((r,H)\)-extremal if \(c_{r,H}(G) = c_{r,H}(|V(G)|)\).

For every \(r\), \(n\) and \(H\), we have the following general bounds:
\[
   r^{\text{ex}(n,H)} \leq c_{r,H}(n) \leq r^{n-\text{ex}(n,H)},
\]
where \(r^{\text{ex}(n,H)} \leq c_{r,H}(n) \leq r^{n-\text{ex}(n,H)}\).

The lower bound is obtained by taking \(G\) as an \(H\)-free graph on \(n\) vertices and \(\text{ex}(n,H)\) edges (i.e., an \(H\)-free extremal graph); the upper bound follows from
the fact that in any \( r \)-coloring of a graph on \( n \) vertices and at least \( r \cdot \text{ex}(n, H) + 1 \) edges there is a monochromatic subgraph on at least \( \text{ex}(n, H) + 1 \) edges, by the Pigeonhole Principle, and hence a monochromatic \( H \).

This problem traces back to a question of Erdős and Rothschild (\cite{4}) that corresponds to \( r = 2 \) and \( H = K_3 \) in the setup above. More precisely, they conjectured that \( c_{2, K_3}(n) \) matches the lower bound in (1) for all \( n \) large enough, which was proved by Yuster:

**Theorem 1.** \cite{8} \( c_{2, K_3}(n) = 2^{\lfloor n^2/4 \rfloor} \) for all \( n \geq 6 \).

He conjectured further that the same result holds for \( H = K_t \) and proved an asymptotic version of the conjecture, which was settled later by Alon and others for 2 and 3 colors:

**Theorem 2.** \cite{1} For every fixed \( t \), there is \( n_0 \) such that \( c_{2, K_t}(n) = 2^{|\text{ex}(n, K_t)|} \) and \( c_{3, K_t}(n) = 3^{|\text{ex}(n, K_t)|} \) hold for \( n > n_0 \).

They also dealt with the case \( r > 3 \), showing that the lower bound in (1) is not the correct value of \( c_{r, K_t}(n) \) in this case, and their proofs can be extended to any non-bipartite graph \( H \). We refer to their paper (\cite{1}) for the detailed results.

Considering the disjoint union of two \((r, H)\)-extremal graphs on \( n \) and \( m \) vertices, it is easy to see, assuming \( H \) is a connected graph, that \( c_{r, H}(n + m) \geq c_{r, H}(n) \cdot c_{r, H}(m) \) holds for all positive integers \( m \) and \( n \) (i.e., the function \( c_{r, H}(n) \) is supermultiplicative). A lemma of Fekete (\cite{5}) implies, then, that the limit \( b_{r, H} = \lim_{n \to \infty} c_{r, H}(n)^{1/n} \in \mathbb{R} \cup \{\infty\} \) exists.

Hoppen, Kohayakawa and Leffman addressed the problem for some graphs \( H \) with linear Turán number (i.e., \( \text{ex}(n, H) = O(n) \)). By (1), these are exactly the graphs for which \( b_{r, H} \) is finite. They settled the question when \( H \) is a matching of fixed size (\cite{6}), and studied it for other classes of bipartite graphs, including paths and stars (\cite{7}). Surprisingly, only very few exact values of \( b_{r, H} \) are known in these cases. In this note, we will improve some of the current upper bounds when the forbidden graph is a star. We now state the best known upper and lower bounds followed by our corresponding improvements to the upper bounds in each case.

First, we consider small forbidden stars \((S_3 \text{ and } S_4)\) and 2-colorings. For \( S_3 \), Hoppen, Kohayakawa and Leffman had the following bounds:

**Theorem 3.** \cite{7} \( b_{2, S_3} \leq \sqrt{6} \approx 2.45 \). On the other hand, the graph consisting of \( n/6 \) disjoint copies of the complete bipartite graph \( K_{3,3} \) gives \( b_{2, S_3} \geq \sqrt[4]{102} \approx 2.16 \).

We improve the upper bound above to:

**Theorem 4.** There is a constant \( c \) such that \( c_{2, S_3}(n) \leq c \cdot 18^{5n/10} \). In particular, \( b_{2, S_3} \leq 18^{3/10} \approx 2.38 \).

Their result for \( S_4 \) is:
Theorem 5. $b_{2,S_4} \leq \sqrt{20} \approx 4.47$. On the other hand, the graph consisting on the union of $n/10$ disjoint bipartite graphs $K_{5,5}$ gives $b_{2,S_4} \geq 3.61$.

Our improved upper bound in this case is:

Theorem 6. $b_{2,S_4} \leq 200^{5/18} \approx 4.36$.

Next, we consider 2-colorings that forbid monochromatic big stars. Hoppen, Kohayakawa and Lefmann, in the same paper, proved the following:

Theorem 7. For every $t$, $b_{2,S_t} \leq \left(\frac{2^t-2}{t-1}\right)^{1/2}$. Furthermore, a certain complete bipartite graph gives $b_{2,S_t} \geq 3.61$.

We improve the upper bound for large $t$ as follows:

Theorem 8. For large values of $t$, we have:

$$b_{2,S_t} \leq \left(\frac{\sqrt{2}}{2} \cdot \left(\frac{2^t-2}{t-1}\right)^{1/2} \right)^{2^{t-3}} \approx \left(1 - \frac{\log t}{10} + O(\frac{1}{t})\right) \cdot \left(\frac{2^t-2}{t-1}\right)^{1/2}.$$  

Finally, we fix the forbidden star to be $S_3$ and consider $r$-colorings. The bounds in Hoppen, Kohayakawa and Lefmann’s paper are:

Theorem 9. For every $r$, $b_{r,S_3} \leq \left(\frac{(2^r)!}{2^r}\right)^{1/2}$. On the other hand, some complete bipartite graph shows that $b_{r,S_3} \geq r^{-3\sqrt{\log(3)/4+\Theta(1))}r} \cdot \left(\frac{(2^r)!}{2^r}\right)^{1/2}$.

The new upper bound for this quantity that we prove here is:

Theorem 10. Let $r \geq 2$. Then

$$b_{r,S_3} \leq \left(\frac{r(2^r-1)!}{2^{2^r-2}}\right)^{\frac{1}{2^{r-1}}} \approx \sqrt[2^{r-1}]{\frac{\sqrt{2}}{\sqrt{e}}} \cdot \left(\frac{(2^r)!}{2^r}\right)^{1/2} \approx 0.85 \cdot \left(\frac{(2^r)!}{2^r}\right)^{1/2}.$$  

2 Notation and preliminary lemma

Given a graph $G$, we call an edge $e = uv \in E(G)$ an $ab$-edge ($a \leq b$) if $\{d(u), d(v)\} = \{a, b\}$. Furthermore, we denote by $m_{ab}$ the number of $ab$-edges (sometimes we will write $m_a$ instead of $m_{aa}$ for short) and by $v_a$ the number of vertices of degree $a$ in $G$.

We now state and prove a simple lemma that will be used throughout the proofs of this paper.

Lemma 1. For every $r \geq 2$, $t \geq 3$ and $n$, there is an $(r, S_t)$-extremal graph $G$ on $n$ vertices and a constant $c(r,t)$ with the following properties: $\Delta(G) \leq r(t-1)-1$, and $d(v) \geq \left[\frac{t}{2}\right] \cdot (t-1)$ holds for all but at most $c(r,t)$ vertices $v \in V(G)$.  

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Proof. Let $G$ be a graph on $n$ vertices. If $G$ has a vertex of degree at least $r(t-1)+1$, all of its $r$-edge colorings contains a monochromatic $S_t$, by Pigeonhole Principle, so $c_{r,S_t}(G) = 0$. Furthermore, if there is a vertex $v$ of degree exactly $r(t-1)$, then for an edge $e$ incident to $v$, the graph $G' = G - e$ has at least as many colorings as $G$. Indeed, every coloring of $G$ induces a coloring of $G'$ in an injective way, since the color of the other $(r-1)(t-1) - 1$ edges incident in $v$ define the color of the edge $e$ uniquely.

On the other hand, if $G$ has two vertices $u, v$ of degree less than $\lceil \frac{r}{2} \rceil \cdot (t-1)$ not joined by an edge, the graph $G' = G + uv$ has at least as many good colorings as $G$, since in every partial coloring of $G'$ that comes from a coloring of $G$, there is at least one free color for the edge $uv$. Therefore, we may assume that all such vertices induce a clique, which implies that there is at most a constant number of them.

\[ \Box \]

3 Applying an entropy lemma

In this section, we will outline the general framework on which our proofs will rely. We start by stating a crucial lemma from [3]:

**Lemma 2.** Let $F$ be a family of vectors in $F_1 \times \cdots \times F_m$. Let $G = \{G_1, \ldots, G_n\}$ be a collection of subsets of $M = \{1, \ldots, m\}$, and suppose that each element $i \in M$ belongs to at least $k$ members of $G$. For $j = 1, \ldots, n$ let $F_j$ be the set of all projections of the members of $F$ on $G_j$. Then

\[ |F|^k \leq \prod_{j=1}^n |F_j|. \tag{2} \]

In our proofs, we will take $F$ to be the set of $r$-edge-colorings of a graph $G$ without monochromatic copies of $S_t$. It is a family of vectors in $[r]^{E(G)}$, where an edge-coloring $c : E(G) \to [r]$ is identified with the vector indexed by the edges of $G$ whose value in entry $e \in E(G)$ is $c(e)$.

For each $ab$-edge $e_i$ of $G$, we will take a set $G_{ij}$ to be the set of indexes of $e_i$ and the edges incident to it, and we take $2r(t-1) - 2 - (a+b)$ identical unit sets $G_{ij}^1, \ldots, G_{ij}^{2r(t-1) - 2 - (a+b)}$ containing the index of $e_i$. This choice guarantees that each edge is counted $2r(t-1) - 3$ times among the sets in $G$, so we may apply inequality (2) with $k = 2r(t-1) - 3$.

Let us estimate now the size of the $F_j$. It is the number of restrictions of $r$-edge-colorings of $G$ without monochromatic $S_t$ to the subgraph spanned by the edges in the set $G_j$. The number of $r$-edge-colorings without monochromatic $S_t$ of this subgraph is an upper bound for $|F_j|$.

For the unit sets $G_{ij}^j$, it is clear that $|F_{ij}^j| \leq r$. Otherwise, let us denote by $f(x)$ the number of $r$-edge-colorings without monochromatic $S_t$ of a star on $x$ edges
in which the color of exactly one edge is fixed. If we color an \(ab\)-edge \(e_i\) and then the stars hanging on its endpoints, we get \(|F_i| \leq r f(a) f(b)\).

Taking into account both types of set, an \(ab\)-edge contributes to the right-hand side of (2) with a factor of \(g(a, b) = r^{2r(t-1)-1-(a+b)} f(a) f(b)\).

Plugging this bound on (2), we get an optimization problem in terms of the number of \(ab\)-edges of \(G\). This problem would be significantly simplified if we could assume that almost all edges of \(G\) are \(aa\)-edges.

This is indeed the case, since whenever we have a pair of independent \(ab\)-edges \((a \neq b)\ e = uv\ and \ f = xy\), say, \(d(u) = d(x) = a\) and \(d(v) = d(y) = b\), such that \(ux\ and vy\ are not edges, we may consider the graph \(G'\ formed by \(G\) by deleting \(uv\ and xy\ and adding \(ux\ and vy\). Note that \(G'\ has two less \(ab\)-edges, one more \(aa\)-edge and one more \(bb\)-edge than \(G\). On the other hand, the upper bounds on the number of colorings of \(G\) and \(G'\ given by (2) are the same, since \(g(a, b)^2 = g(a, a) \cdot g(b, b)\), and the degree of the endpoints of all other edges remain unchanged. Therefore, repeating this procedure as long as we can, we may assume that \(G\ has at most a constant number of \(ab\)-edges with \(a \neq b\). In particular, we may rewrite (2) as

\[
|F|^{2r(t-1)-3} \leq c \cdot \prod_{a=\lceil \frac{t}{2} \rceil}^{r(t-1)-1} (r^{2r(t-1)-1-2a} f(a)^2)^{m_a} \prod_{a=\lceil \frac{t}{2} \rceil}^{r(t-1)-1} (r^{2r(t-1)-1-2a} f(a)^2)^{a v_a / 2},
\]

where the range of \(a\) in the product comes from Lemma 1.

By taking logarithms, it is clear that we are maximizing a linear function of the \(v_i\). This means that the maximum is attained when all but one of the \(v_i\ are zero, and the exceptional \(v_i\ corresponds to the value that maximizes the function \(g(a) = (r^{2r(t-1)-1-2a} f(a)^2)^a\).

### 4 Forbidding small stars in 2-edge-colorings

In this section, we prove Theorems 4 and 6. Following the setup in the previous section, the proofs are quite straightforward:

**Proof of Theorem 4**: By (3), we have the following bound:

\[
|F|^5 \leq c \cdot \prod_{a=2}^{3} (2^{7-2a} f(a)^2)^{a v_a / 2} = c' \cdot 32^{v_2} \cdot 18^{3v_3 / 2},
\]
since \( f(2) = 2 \) and \( f(3) = 3 \) in this case. The fact that \( 32 < 18^{3/2} \approx 76 \) concludes the proof.

\[ \square \]

Proof of Theorem 6. In this case, simple computations show that \( f(3) = 4 \), \( f(4) = 7 \) and \( f(5) = 10 \). Therefore, the bound reads as

\[
\left| F \right| \leq c \cdot 512^{m_3} \cdot 392^{m_4} \cdot 200^{m_5} = c' \cdot 512^{3v_3/2} \cdot 392^{4v_4/2} \cdot 200^{5v_5/2}.
\]

As \( 512^{3/2} \approx 11585 \), \( 392^{4/2} = 153664 \) and \( 200^{5/2} \approx 565685 \), the maximum is achieved when \( v_3 = v_4 = 0 \) and \( v_5 = n \), and the proof is complete.

\[ \square \]

5 Forbidding large monochromatic stars in two-edge-colorings

In this section, we prove Theorem 8.

Proof of Theorem 8. In this case, \( f(x) = \sum_{k=x-t}^{t-2} \binom{x-1}{k} \), since given a star on \( x \) edges with one edge colored with color \( c \), we may choose at least \( a-t \) and at most \( t-2 \) of the remaining \( x-1 \) edges to assign \( c \) without having a monochromatic \( S_t \) in any of the colors.

We are done, then, if we find the maximum of 

\[
g(a) = \left(2^{2t-5-2a} \left( \sum_{k=a-t}^{t-2} \binom{a-1}{k} \right)^2 \right)^a,
\]

for \( t-1 \leq a \leq 2t-3 \). We claim that, for \( t \) large enough, the maximum value of \( g \) is attained for \( a = 2t-3 \).

To prove this claim, we will use the following well-known bounds for large \( a \) and \( t \):

\[
\binom{2t-3}{t-2} \geq 0.9 \cdot \frac{2^{2t-3}}{\sqrt{\pi t}} \tag{6}
\]

and

\[
\binom{a-1}{\frac{a-1}{2}} \leq 1.01 \cdot \frac{2^{a-1}}{\sqrt{\pi a}} \tag{7}
\]

The first one implies that

\[
g(2t-3) = \left(2^{2t-3} \right)^{2t-3} > \left(0.9^2 \cdot 2^{4t-7} \right)^{2t-3} \approx 2^{8t^2 - 2t \log_2 t - 25.92t} + O(\log(t)).
\]
Also, we have \( f(a) \leq 2^{a-1} \), since \( f(a) \) is a sum of binomial coefficients in the \((a-1)\)-st row of Pascal’s triangle. Hence,

\[
g(a) \leq (2^{4t-5-2a}(2^{a-1})^2)^a = 2^{(4t-7)a}.
\]

Suppose first that \( a \leq 2t - \log_2 t \). Then the last inequality implies that

\[
g(a) \leq 2^{(4t-7)(2t-\log_2 t)} = 2^{8t^2 - 4t \log_2 t + O(t)} \leq g(2t - 3)
\]

for large \( t \).

On the other hand, if \( 2t - \log_2 t \leq a \leq 2t - 4 \), notice that, as the central binomial coefficient is the maximum in its row, we have

\[
f(a) = \sum_{k=a-t}^{t-a} \binom{a-1}{k} \leq (2t-a+1) \binom{a-1}{\lfloor \frac{a-1}{2} \rfloor} \leq 1.01(2t-a+1) \frac{2^{a-1}}{\sqrt{\pi a}},
\]

by (7).

The latter estimate implies that

\[
g(a) \leq 2^{(4t-5-2a)(1.01(2t-a+1) \cdot 2^{a-1}/\sqrt{\pi a})^2}^a
\]

\[
= 2^{a(4t-7+2 \log_2 (2t-a+1)+\log_2 (1.01^2/\pi)-\log_2 a)}.
\]

By taking the derivative (for fixed \( t \), with respect to \( a \)) of the function in the exponent, it is easy to see that this bound on \( g \) is increasing for \( 2t - \log_2 t \leq a \leq 2t - 4 \) and large \( t \). Therefore, the maximum of the bound in this range is attained for \( a = 2t - 4 \), which gives, for large \( t \),

\[
g(a) \leq 2^{(2t-4)(4t-7+2 \log_2 (5)+\log_2 (1.01^2/\pi)-\log_2 2t-4)}
\]

\[
< 2^{8t^2 - 2 \log_2 t - 25.95t + O(\log(t))}
\]

\[
< g(2t - 3),
\]

and concludes the proof.

\[\square\]

### 6 More colors

Finally, we prove Theorem 10.

**Proof of Theorem 10.** The bound in (8) can be written as

\[
|F|^{4r-3} \leq c \prod_{a=r}^{2r-1} (r^{4r-2a-1} f(a)^2)^{m_a} = c' \prod_{a=r}^{2r-1} (r^{4r-2a-1} f(a)^2)^{m_a}/2.
\]
Again, all it is left to do is to prove that the maximum of $g(a) = (r^{4r-2a-1} f(a)^2)^a$ is obtained for $a = 2r - 1$. With this result, our theorem follows by plugging $v_i = 0$ for $i < 2r - 1$ and $v_{2r-1} = n$ in (3) and by the fact that $f(2r-1) = \frac{(2r-1)!}{2r-2}$. We have

$$g(2r-1) = \left( \frac{r(2r-1)!^2}{2r-2} \right)^{2r-1} = r^{8r^2-2(4-\log(4)) \log(r)} + o(r^{\log(r)}).$$

We are going to bound $f(a)$ in two different ways and use each of the bounds for a different range of the value of $a$.

First, notice that $f(a) \leq r^{a-1}$, since this is the total number of $r$-colorings of a star with $a-1$ edges. This bound is enough if $a \leq 2r - 2r/\log(r)$. Indeed, in this case,

$$g(a) \leq (r^{4r-2a-1} \cdot r^{2a-2})^a$$
$$< r^{(4r-3)(2r-2) \log(r)}$$
$$= r^{8r^2 - 8 \log(r)} + O(r)$$
$$< g(2r-1),$$

for large $r$.

Suppose now that that $a \geq 2r - 2r/\log(r)$. Let us divide the colorings according to the number of times each color appears on it. There are exactly $\frac{(a-1)!}{\prod_{i=1}^{a} c_i!}$ colorings where the color $i$ appears exactly $c_i$ times (without loss of generality: $c_1 \leq 1; c_i \leq 2$, for $i \geq 2; \sum_{i=1}^{a} c_i = a - 1$). Notice that, in any valid coloring with $a \leq r + 1$, at least $a - 1 - r$ colors appear twice, so $\prod_{i=1}^{a} c_i! \leq \frac{(a-1)!}{2^{a-1-r}}$ holds for any choice of the $c_i$. The number of choices for the $c_i$ satisfying the condition above is bounded from above by $\binom{3r-a-2}{r-1}$ (see, for instance, Corollary 2.4 in [2]). Hence we have the following estimate for $g$:

$$g(a) \leq \left( r^{4r-2a-1} \left( 3r - a - 1 \right)^{2} \cdot (a-1)!^2 \right)^a \frac{2^{2a-2-2r}}{r-1}. \tag{9}$$

We will prove that the upper bound for $g(a)$ in (9) is increasing with $a$ in this range, and that for $a = 2r - 2$ it gives a value smaller than $g(2r-1)$.

Plugging $a = 2r - 2$ in (9), we get

$$g(2r-2) \leq \left( \frac{r^5(r+1)^2(2r-3)!}{24^{r-4}} \right)^{2r-2}.$$

So
\[ \frac{g(2r - 1)}{g(2r - 2)} \geq \left( \frac{r(2r - 1)!^2}{2^{2r-2}} \right)^{2r-1} \cdot \left( \frac{r^5(r + 1)^2(2r - 3)!}{2^{2r-4}} \right)^{-(2r-2)} \]
\[ \approx \frac{\pi}{4e^{10}} \cdot \frac{2^6r^4}{e^{2r}} > 1, \]

for large \( r \), since \( 2^6 > e^4 \).

To prove that the bound in (9) is increasing in this range, we first rewrite it as
\[ g(a) \leq (r^{4r-1}2^{2r+2})^a \cdot \left( \frac{(a - 1)!}{(2r)^a} \left( \frac{3r - a - 1}{r - 1} \right) \right)^{2a} \]
The first term in the right-hand side of the inequality above is clearly increasing with \( a \). Let us show that the second term, call it \( h(a) \), grows with \( a \) as well. It is enough to prove \( \frac{h(a + 1)}{h(a)} \geq 1 \). The following calculation shows that this is indeed the case:
\[ \frac{h(a + 1)}{h(a)} = \frac{a! \cdot a^a}{(2r)^{2a+1}} \cdot \left( \frac{3r - a - 2}{r - 1} \right) \cdot \left( \frac{2r - a}{3r - a - 1} \right)^a \]
\[ \geq \frac{(2r - r/ \log(r))!(2r - r/ \log(r))^{(2r-r/ \log(r))}}{(2r)^{2r+1}} \cdot \left( \frac{r + r/ \log(r)}{r + r/ \log(r) - 1} \right)^{2r} \]
\[ \to \infty, \]
as \( r \to \infty \), where we used the fact that \( r - r/ \log(r) \leq a \leq 2r - 2 \) and that \( \left( \frac{3r - a - 2}{r - 1} \right) \geq 1 \), together with the monotonicity of the functions involved.

\[ \square \]

7 Final remarks and open problems

Our argument could be generalized by taking the sets \( G_j \) to include bigger neighborhoods of the edge \( e_j \). However, in this case, new technical problems arise when we try to estimate the \( |F_i| \). Somewhat better results could be achieved, but we do not believe that they get substantially closer to the lower bounds.

We conjecture that \( b_{2,S_3} = \sqrt[6]{102} \), i.e., the union of disjoint \( K_{3,3} \)'s is the graph with the biggest number of 2-edge-colorings without monochromatic \( S_3 \). In general, for 2-colorings forbidding monochromatic stars of a fixed size, we think that the extremal configuration is given by a collection of copies of a fixed (possibly complete bipartite) graph of constant size.
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