Intermittent boundary layers and torque maxima in Taylor-Couette flow

Hannes J. Brauckmann and Bruno Eckhardt
Fachbereich Physik, Philipps-Universität Marburg, D-35032 Marburg, Germany and
J.M. Burgerscentrum, TU Delft, Delft, The Netherlands

(Dated: May 22, 2014)

Turbulent Taylor-Couette flow between counter-rotating cylinders develops intermittently fluctuating boundary layers for sufficient counter-rotation. We demonstrate the phenomenon in direct numerical simulations for radius ratios $\eta = 0.5$ and 0.71 and propose a theoretical model for the critical value in the rotation ratio. Numerical results as well as experiments show that the onset of this intermittency coincides with the maximum in torque. The variations in torque correlate with the variations in mean Taylor vortex flow which is first enhanced for weak counter-rotation, and then reduced as intermittency sets in. To support the model, we compare to numerical results, experiments at higher Reynolds numbers, and to Wendt’s data.

PACS numbers: 47.20.Qr; 47.27.N-

I. INTRODUCTION

The flow between concentric cylinders has served as a paradigm for the transition to turbulence since Taylor's 1923 characterization of the bifurcation from laminar to vortical flows [1]. Many transitions between spatially and temporally simple flow states like vortices, modulated vortices and traveling waves, are accessible by standard bifurcation theory and have been described and studied in considerable detail [2, 3]. The turbulent states that are reached after several of these bifurcations are not always homogeneous but can show patchiness in the form of turbulent spots or turbulent spirals [4–7]. In addition to such azimuthal and axial modulations, Coughlin and Marcus [8] have described a radial inhomogeneity for counter-rotating cylinders, which consists of turbulent bursts that have been seen in experiments by Colovas and Andereck [9]. Measurements in the Twente Taylor-Couette facility have confirmed the presence of this inhomogeneity up to Reynolds numbers of $10^6$ [10]. Moreover, the onset of this instability has been linked to the maximum in torque that appears for moderate counter-rotation [10, 11].

Coughlin and Marcus [8] already suggested that this radial inhomogeneity should be linked to the presence of a neutral surface of vanishing angular velocity in the laminar Couette profile for counter-rotating cylinders. The region between the inner cylinder and the neutral surface is inviscidly unstable by the Rayleigh criterion, whereas the region between the neutral surface and the outer cylinder is Rayleigh stable. These considerations do not provide an immediate prediction for the onset of inhomogeneity since the neutral surface appears with infinitesimal amounts of counter-rotation already. We here discuss the extensions needed in order to derive a predictive theory for the onset of this intermittency, compare it with observations for different radius ratios, and describe the link to the torque maxima.

The outline of the paper is as follows. In section II we introduce the numerical simulations and describe the phenomenon. We also identify the onset of intermittency and summarize numerical and experimental results for torque maxima, including a reanalysis of the data of Wendt [12] which is documented in appendix A. In section III we present the argument for the boundary layer intermittency, and in section IV we describe the link to the torque maximum. We conclude with a few remarks in section V.

II. BOUNDARY LAYER INTERMITTENCY

In order to introduce the phenomenon we present results from direct numerical simulations for different radius ratios. We solve the incompressible Navier-Stokes equation with the spectral scheme explained in [13]. For the dimensionless units we measure all lengths and times in units of the gap width $d = r_o - r_i$, where $r_o$ and $r_i$ are the radii of the outer and inner cylinders, respectively, and the viscous time $d^2/\nu$. To avoid the end-effects caused by top and bottom lids in experiments, an additional periodicity in axial direction of length $L_z$ is introduced resulting in an aspect ratio $\Gamma = L_z/d$. Here $\Gamma = 2$, which allows for one Taylor vortex pair when the outer cylinder is at rest. Fourier modes and Chebyshev polynomials are employed for expansions in the two periodic and the wall-normal direction, respectively. We simulate a domain of reduced azimuthal length, i.e. one third for $\eta = r_i/r_o = 0.5$ and one ninth for $\eta = 0.71$ of the full azimuthal length. Consequently the flow field repeats three (nine) times to fill the entire circumference. We tested that the shorter azimuthal period does not influence the computed torques. The criteria used to test and verify the code are detailed in [14]. Specifically, the spatial resolution, characterized by the number of modes ($N_z, N_\phi, N_r$) in each direction, is chosen so that three convergence criteria are satisfied: torque computed at the inner and outer cylinder has to agree within a relative deviation of $5 \cdot 10^{-3}$, the expansion coefficients in each direction have to cover a range of at least $10^4$ and the energy dissipation estimated from the torque has to agree with the volume energy dissipation rate to within $10^{-2}$. 


All these requirements are met in all simulations shown here.

We consider radius ratios $\eta = 0.5$ and $\eta = 0.71$. In order to assess the influence of the mean system rotation on turbulent characteristics, the simulations are performed at a fixed shear between the cylinder walls, defined by Dubrulle et al. [15] as

$$Re_S = \frac{2}{1 + \eta} |\eta Re_i - Re_o|,$$  \hspace{1cm} (1)

with $Re_i = (r_o - r_i) r_i \omega_i/\nu$ and $Re_o = (r_o - r_i) r_o \omega_o/\nu$. Here, $\omega_i$ and $\omega_o$ denote the angular velocity of the inner and outer cylinder and $\nu$ the kinematic viscosity. For both radius ratios we realize various mean rotations characterized by the rotation ratio $\mu = \omega_o/\omega_i$ for the same shear $Re_S = 2 \cdot 10^4$. According to Lathrop et al. [16] this value is high enough that the vortices noticeable for lower Re have disappeared for $\mu = 0$. The dimensionless torque $G$ exerted on the inner and outer cylinder is obtained from

$$G(t) = \nu^{-2} J^\omega = \nu^{-2} r^3 \left( \langle u_r \omega \rangle_{A(r)} - \nu \partial_r \langle \omega \rangle_{A(r)} \right),$$  \hspace{1cm} (2)

where $u_r$ and $\omega = u_{\varphi}/r$ denote the radial and angular velocity, respectively, and $\langle \cdot \cdot \cdot \rangle_{A(r)}$ stands for an area average over the surface of a concentric cylinder [17]. The long-time mean value is obtained from an additional average over time.

For co-rotation, i.e. $\mu \geq 0$, torque values at the inner and outer cylinders agree on average and in their fluctuations [14]. However, the situation drastically changes for strong counter-rotation as illustrated by the torque time-series for $\mu = -0.5$ in Fig. 1(a). Near the inner cylinder, fluctuations are small and without long time variation. At the outer cylinder the torque exhibits fluctuations of relatively strong amplitude and slow dynamics that qualitatively differ from the ones at the inner cylinder. These slow fluctuations reflect an intermittent turbulent activity in the vicinity of the outer cylinder as demonstrated by the cross-flow energy in Fig. 1(b). The cross-flow energy,

$$E_{cf}(r, t) = \langle u_r^2 + u_{\varphi}^2 \rangle_{A(r)},$$  \hspace{1cm} (3)

measures the energy content in the transverse velocity components at a radial distance $r$ and at an instant of time. Small values indicate a flow close to laminar, large values a turbulent state. One notes that near the inner cylinder the flow is more or less homogeneously turbulent, whereas towards the outer cylinder one observes an increased activity synchronized with the increase in mean torque.

For this geometry, the Rayleigh-stable region of the laminar profile extends over the interval $(r - r_i)/d \in [0.414, 1]$, with the lower bound marked by a dashed line in Fig. 1(b). The presence of the bursts shows that this stability is not maintained, but the position of the laminar line still marks the transition between small radial variations near the inner cylinder and larger ones towards the outer cylinder.

For $Re_S \sim 2.3 \cdot 10^3$ Conghin and Marcus [8] described the bursts as instability of spatially ordered 'interpenetrating spiral’ flow. We here observe turbulent flow in the Rayleigh unstable inner region at $Re_S = 2.0 \cdot 10^4$ that is accompanied by intermittent turbulence in the outer one. The connections to spirals cannot be followed up because of the limited radial and azimuthal domain size that can be computed. The experiments of [10] extend this observation to even larger Reynolds numbers $Re_S \sim 10^6$ where the presence of turbulent bursts in the outer region is deduced from a bimodal distributions of the angular velocity.

The onset of the intermittent behavior is accompanied by an increase in the torque fluctuations. Therefore, we study the standard deviation $\sigma_G$ of the torque relative to the mean for the outer cylinder, i.e. the ratio $\sigma_G/G$, as an indicator and deduce a critical value $\mu_c(\eta)$ for the onset from the requirement that $\sigma_G/G$ exceeds the base level for $\mu = 0$, cf. Fig. 2. The choice of $\sigma_G/G$ as measure for the transition is supported by the observation that
it remains relatively unaffected by variations of \( \mu \) at the inner cylinder. For the numerical simulations we find the critical values

\[
\begin{align*}
\mu_c(0.5) &= -0.207 \pm 0.014 \\
\mu_c(0.71) &= -0.351 \pm 0.050,
\end{align*}
\]

marked by vertical lines in Fig. 2. The uncertainties are estimated as half the gap in \( \mu \) between the two data points next to the maximum. The critical point varies with \( \eta \), and we will return to this feature in section 11. The onset of intermittent bursts was determined experimentally at \( \mu_c(0.716) \approx -0.368 \) for \( Re_S \sim 10^6 \). The differences between this value and our observation \( \mu_c \) may be due to our uncertainty in \( \mu_c \) or the difference in Reynolds number, but they are not significant.

The rotation ratio \( \mu_{\text{max}}(\eta) \) of maximal torque was identified independently in two experiments, \([10, 11, 12, 13]\). For a constant shear, they report a torque maximization for \( \mu_{\text{max}}(0.716) = -0.33 \pm 0.04 \) \([10]\) and for \( \mu_{\text{max}}(0.7245) = -0.333 \) \([11]\). Numerical simulations show that the torque maximum for counter-rotation only occurs after a shift of the maximizing \( \mu \)-value with increasing \( Re_S \) \([14]\).

Figure 3 shows the computed torques for \( Re_S = 2.0 \cdot 10^4 \) just at the beginning of the asymptotic regime. We determine the rotation ratio of optimal transport as the maximum of a quadratic fit, \( G/G_{\text{lam}} = c_2 \mu^2 + c_1 \mu + c_0 \), to five data points and find

\[
\begin{align*}
\mu_{\text{max}}(0.5) &= -0.195 \pm 0.019 \\
\mu_{\text{max}}(0.71) &= -0.357 \pm 0.060.
\end{align*}
\]

The uncertainties are deduced from the relative confidence interval \( \Delta G/G \) which results from temporal torque fluctuations. This uncertainty in the torque values transforms into an uncertainty in the maximum location of the quadratic fit, i.e.

\[
\Delta \mu_{\text{max}} = \sqrt{-\frac{\Delta G}{c_2 G(G/G_{\text{lam}})_{\text{max}}}}
\]

with the fit coefficient \( c_2 \) and the maximal rescaled torque \( (G/G_{\text{lam}})_{\text{max}} \).

For both radius ratios, the maximizing global rotation \( \mu_{\text{max}} \) compares well with the transition to the radial inhomogeneity at \( \mu_c \), as it was previously observed experimentally by van Gils et al. \([10]\) for \( \eta = 0.716 \). On the other hand, the critical values for \( \eta = 0.71 \) are more uncertain, which complicates the identification of a correspondence between them. We note that the torque maximization at \( \mu_{\text{max}}(0.71) = -0.357 \) falls in line with the experimental observations \( \mu_{\text{max}}(0.716) = -0.33 \pm 0.04 \) and \( \mu_{\text{max}}(0.7245) = -0.333 \) \([10, 11]\).

A final data point is provided by Wendt’s data \([12]\), which are reanalyzed in the way used in the recent experiments in appendix A. Wendt worked with the radius ratio of \( \eta = 0.680 \), and the maximum in his torque data lies near

\[
\mu_{\text{max}}(0.680) = -0.295 \pm 0.113.
\]
III. ONSET OF THE RADIAL INHOMOGENEITY

In order to connect the onset of fluctuations with the rotation ratio for maximal torque and its dependence on the radius ratio, we ask the following questions: (i) What is the physical mechanism that determines the onset of fluctuations and can one derive a prediction for the onset from it? (ii) Why should the rotation ratio of torque maximization coincide with the onset of fluctuations?

A first answer to these questions is provided by van Gils et al. [10]: They suggest that the rotation ratio for maximal torque is determined by locations in parameter space \((Re_i, Re_o)\) that are equally distant from the Rayleigh stability lines \(\mu = \eta^2\) and \(\mu = -\infty\). This condition results in the so called “angle bisector”,

\[
\mu_{\text{bis}}(\eta) = \frac{-\eta}{\tan \left[ \frac{\eta}{2} - \frac{\pi}{2} \arctan(\eta^{-1}) \right]}, \tag{8}
\]

for the location of the torque maximum [10]. Secondly, they argue that the onset of fluctuations has to coincide with \([5]\) since the intermittent behavior in the outer layer reduces radial transport of momentum the torque starts to drop. While the angle bisector agrees well with their measured optimum \(\mu_{\text{max}}(0.716) = -0.33\), it disagrees with our simulation result for \(\eta = 0.5\), since \(\mu_{\text{max}}(0.5) = -0.195\) whereas Eq. \([8]\) gives \(\mu_{\text{bis}}(0.5) = -0.309\).

We here propose an explanation for the onset of fluctuations that is not based on the stability of laminar flow but on properties of turbulent flows in general and turbulent Taylor vortices in particular. The key idea is that the turbulent flow detaches because the inner part that is driven by the Rayleigh-unstable region is not sufficiently strong to maintain persistent turbulence across the Rayleigh-stable region to the outer cylinder. But the outer region cannot return to laminar for all times, because the turbulent transport and the friction has to be the same, independent of the radial distribution [17]. The radial range over which the inner unstable region may maintain a turbulence is somewhat larger than the inner Rayleigh-unstable region. The simple inviscid stability calculations for counter-rotation alluded to before gives a neutral surface at radius

\[
r_n(\mu) = r_i \sqrt{\frac{1 - \mu}{\eta^2 - \mu}}, \tag{9}
\]

that separates stable from unstable flow [19] and implies a detachment of the unstable flow for any \(\mu < 0\). However, experiments and viscous calculations show that Taylor vortices extend beyond this neutral surface when counter-rotation sets in, see for example [11]. Esser and Grossmann [20] deduced from their stability calculation that flow structures protrude the neutral surface by a factor \(a(\eta)\), i.e. the effective extension of secondary flow is

\[
r_{\text{EG}}(\mu) = r_i + a(\eta) (r_n - r_i) \tag{10}
\]

that takes values between 1.4 and 1.6. The rotation ratio \(\mu_{\text{pred}}\) where the unstable flow detaches from the outer cylinder wall then follows from the condition \(r_{\text{EG}}(\mu_{\text{pred}}) = r_o\) and reads

\[
\mu_{\text{pred}}(\eta) = -\eta^2 \left( \frac{a^2 - 2a + 1}{2(a - 1)} + \frac{1}{\eta(2a - 1)} \right). \tag{12}
\]

For \(\mu < \mu_{\text{pred}}\) the turbulence can no longer fill the whole cylinder gap, i.e. \(r_{\text{EG}} < r_o\), and intermittency has to set in. The extended range of the inner unstable region as captured by the factor \(a(\eta)\) then mandates a minimal counter-rotation for the neutral surface to fall inside the cylinders. The evaluation of \([12]\) yields predictions \(\mu_{\text{pred}}(0.5) = -0.191\) and \(\mu_{\text{pred}}(0.71) = -0.344\) that compare well with the empirically found onsets of intermittency [11], see also Tab. II.

| \(\eta\) | \(\mu_c\) | \(\mu_{\text{max}}\) | \(\mu_{\text{pred}}\) | \(\mu_{\text{bis}}\) | \(\mu_{\text{LSC}}\) |
|-------|--------|----------------|----------------|--------|----------------|
| 0.5   | -0.207 | -0.195         | -0.191         | -0.309 | -0.223         |
| 0.68  | -0.295 | -0.321         | -0.360         |         |                |
| 0.71  | -0.351 | -0.357         | -0.344         | -0.367 | -0.357         |
| 0.716 | -0.368 | -0.33          | -0.349         | -0.368 |                |
| 0.7245| -0.333 | -0.356         | -0.370         |         |                |

TABLE I: Rotation ratio of the onset of intermittency \(\mu_c\) and of the torque maximum \(\mu_{\text{max}}\) together with the new prediction \(\mu_{\text{pred}}\) \([12]\) and the angle bisector \(\mu_{\text{bis}}\) \([5]\) for various radius ratios. The rotation ratio \(\mu_{\text{LSC}}\) of the maximal mean-flow contribution to the torque is also given for the numerical simulations in the last column.

IV. ENHANCED LARGE SCALE CIRCULATION

We now turn to the question of why the torque maximum coincides with the onset of the radial intermittency. Van Gils et al. [10] argue that the torque decreases when turbulent bursting sets in because of the reduced radial transport. However, to obtain a torque maximum at the bursting onset \(\mu_c\), additionally, the torque has to increase with \(\mu\) decreasing from zero to \(\mu_c < 0\). We argue that this increase is caused by a strengthening of the mean Taylor vortex flow. Such a large-scale circulation (LSC) is able to effectively transport momentum and, thus, to increase the torque in addition to turbulent fluctuations. The strengthening of the LSC is due to a change in the effective outer boundary condition: While for \(\mu > \mu_{\text{pred}}\) the LSC seeks to extend beyond the outer cylinder and is restricted by the rigid wall, i.e. \(r_{\text{EG}} > r_o\), the rigid boundary conditions become replaced
The mixed terms complete torque $G / G \text{lam}$ in (14) to measure the mean weight of each with the radius. Therefore, we introduced an additional radial average in [14] to measure the mean weight of each contribution. Moreover to accurately capture the mean Taylor vortex motion, we axially shift the instantaneous flow fields during the temporal average of $\bar{u}$ and in [14] so that Taylor vortices always stay at a fixed height.

For $\mu \gtrsim 0$, the torque is mainly caused by turbulent fluctuations and the mean-flow contribution nearly drops to the laminar level, as shown in Fig. 4. Turbulent fluctuations also dominate the torque for strong counter-rotation, i.e. $\mu = 0.5$ for $\eta = 0.5$ and $\mu \lesssim 0.71$ for $\eta = 0.71$. For intermediate rotation ratios, mean-flow vortices (LSC) contribute the major share to the torque. Note that the onset of mean vortical flow for $\mu > -\eta$, with $\mu = -\eta$ corresponding to perfect counter-rotation $Re_o = -Re_i$, was previously observed by Ravelet et al. [21]. Additionally, the LSC contribution grows with $\mu$ decreasing from zero which is consistent with our picture of a change in the outer boundary condition from no-slip to a less restrictive free-surface condition. The mean Taylor vortices are strongest (as measured by their contribution to the torque) at the rotation ratio $\mu_{LSC}(\eta)$ where $G$ is maximized. Using a quadratic fit with an uncertainty estimation analogue to (6) we find

$$
\mu_{LSC}(0.5) = -0.223 \pm 0.018
$$

$$
\mu_{LSC}(0.71) = -0.357 \pm 0.075.
$$

Consequently, the rotation ratio of optimal momentum transport by the mean flow coincides with the empirically found onset of intermittency, Eq. (4), within the given uncertainties. Furthermore, the mean-flow contribution is responsible for the maximum in the total torques, cf. Fig. 4 and Eq. (5), thereby establishing the connection between the onset of intermittency and the torque maximum within the framework depicted above.

This connection implies that the prediction for the intermittency onset also acts as a prediction for torque maxima. We thus compare the predictions from the boundary layer argument $\mu_{pred}(\eta)$ from Eq. (12) and the bisection argument $\mu_{bis}(\eta)$ from Eq. (8) [10] with experimental and numerical results for torque maxima in Fig. 5 (and Tab. 1). The rotation ratio $\mu = -2.797$ of the turbulent bursts found by Coughlin and Marcus [8] lies below both predictions and, thus, clearly in the intermittent range. Our simulation result for $\eta = 0.71$ and Wendt’s experimental result for $\eta = 0.680$ are consistent with both the angle bisector and the current prediction within the error bars. Moreover, we note that the torque maximum $\mu_{max}(0.7245) = -0.333$ measured by Paoletti and Lathrop [11] as well as $\mu_{max}(0.716) = -0.33 \pm 0.04$ measured by van Gils et al. [10] tend towards our prediction, Eq. (12). However considering the usual error bars, these values are also consistent with the angle bisector line. The simulation results for $\eta = 0.5$ provide a better test of both predictions, since the values obtained from (8) and (12) differ. The numerical data are in better agreement with the boundary layer estimate (12).
\[ \eta = 0 \] the current theory predicts a maximum for a rotation ratio of \(-0.6\), whereas the angle bisection gives a value close to \(-0.4\). However, this limit is delicate because the linear instability disappears and both theories will most likely have to be refined or replaced. Evidence for this is provided for by the change from a continuous to a dashed line for \(\eta > 0.9\).

Furthermore, numerical simulations revealed that at lower \(Re_s \leq 4 \cdot 10^3\) the torque is maximized at \(\mu = 0\) for \(\eta = 0.71\) [14]. This larger rotation ratio is not covered with the current theory, so that further refinements are needed for lower Reynolds numbers and mildly turbulent flows.

**Acknowledgments**

We are grateful to Marc Avila for developing and providing the code used for our simulations. This work was supported in part by Deutsche Forschungsgemeinschaft. Most computations were done at the LOEWE-CSC in Frankfurt.

**Appendix A: Re-analysis of Wendts data**

Recent experimental studies analyze the dependence of torque on the shear rate and on the mean system rotation independently. This decomposition is advantageous since torques can be compensated either by dividing by the effective scaling with the shear [18] or by talking the ratio to \(G(\mu = 0)\) [11] to study the rotation dependence. The resulting torque amplitudes are
FIG. 7: Compensated torques by Wendt for $\eta = 0.680$ averaged in the range $7.8 \cdot 10^4 < \text{Re}_S < 1.3 \cdot 10^5$ for each rotation ratio independently. The blue line indicates a quadratic least-square fit to the four largest values. Its maximum $\mu_{\text{max}} = -0.295$ is marked by the dashed line.

Based on numerous measurements for each rotation ratio which improves statistical significance. In contrast, Wendt presented the torque dependence on the rotation for some selected shear Reynolds numbers in figure 10 of Ref. [12]. Since this evaluation is based on single measurements, uncertainties may play a major role.

Therefore, we here apply the current analysis method to Wendt’s torque measurements for $\eta = 0.680$ digitized from figure 9 in [12]. Figure 6 shows the torques for various rotation ratios compensated with $\text{Re}_S^{0.7}$ which Wendt found as effective scaling for $10^4 \lesssim \text{Re}_S \lesssim 10^5$. One easily sees that the torque depends on the mean rotation with the largest values for high $\text{Re}_S$ at $\mu = -0.25$. We closely follow the analysis in [10, 11, 18] and average the compensated torques in the range $7.8 \cdot 10^4 < \text{Re}_S < 1.3 \cdot 10^5$ to find amplitudes depending on the rotation only, see Fig. 7. We chose a Reynolds number range that starts after the shift of the torque maximum [14] and includes the highest data points for $-0.50 \leq \mu \leq -0.17$ (cf. Fig. 6). One observes a maximum in the statistical more significant mean amplitudes for moderate counter-rotation which was also found in recent studies [10, 11, 18] and in current simulations. Based on a quadratic fit to the largest amplitudes we find

$$\mu_{\text{max}}(0.680) = -0.295 \pm 0.113,$$

with the uncertainty calculated in analogy to Eq. [6]. Its relative high level is due to the broad maximum in Fig. 7 and due to the few rotation ratios investigated by Wendt. In spite of the high uncertainty, the torque maximization for counter-rotation, i.e. $\mu_{\text{max}} < 0$, is clear without ambiguity. Moreover, the new maximum, $\mu_{\text{max}}(0.680) = -0.295$, lies consistently between the maxima identified here, cf. Eq. [5].

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