A VISCOITY APPROACH TO THE DIRICHLET PROBLEM
FOR COMPLEX MONGE-AMPÈRE EQUATIONS

YU WANG

Abstract. The Dirichlet problem for complex Monge-Ampère equations with
continuous data is considered. In particular, a notion of viscosity solutions is
introduced; a comparison principle and a solvability theorem are proved; the
equivalence between viscosity and pluripotential solutions is established; and an
ABP-type of $L^\infty$-estimate is achieved.

1. Introduction

Viscosity methods provide a powerful tool for the study of non-linear partial dif-
ferential equations (see e.g. [CIL92, IL90, CC95] and references therein). However,
viscosity methods have been developed largely so far for real equations, where no
complex structure plays a particular role. It is only relatively recently that the
incorporation of complex structures, such as plurisubharmonicity in the case of
the complex Monge-Ampère equation, has been considered by Harvey and Lawson
[HL09] and particularly by Eyssidieux, Guedj, and Zeriahi [EGZ10].

In the complex case, even the notion of viscosity solutions, especially viscosity
supersolutions, presents some subtleties. A detailed discussion of this can be found
in [EGZ10], who solved this problem in the case of the complex Monge-Ampère
equation $(\omega_0 + \frac{i}{2} \partial \bar{\partial} u)^n = e^{u \phi(z)} \omega^n$, on a compact Kähler manifold $(X, \omega)$ with
$\phi > 0$ and $\omega_0 \geq 0$. Their approach is partly motivated by the desire to apply
existing results on generalized solutions [IL90, CIL92, BEGZ], and by the specific
objective of establishing the continuity of the potentials for the singular Kähler-
Einstein metrics which they constructed earlier in [EGZ09]. In [HL09], Monge-
Ampère equations with constant right hand sides were considered. But to the best
of our knowledge, a general and systematic viscosity treatment of general complex
Monge-Ampère equations is not yet available in the literature.

The main goal of this paper is to develop such a treatment. Specifically we
consider the Dirichlet problem

$$
\begin{cases}
M_C(u) := \det(2u_{ki}) = f(z, u) & \text{in } \Omega \\
u = g & \text{on } \partial \Omega \\
u \in PSH(\Omega) \cap C(\overline{\Omega})
\end{cases}
$$

(1.1)

where $\Omega$ is a strictly pseudoconvex bounded domain in $\mathbb{C}^n$, $PSH(\Omega)$ denotes the
space of plurisubharmonic functions on $\Omega$, and $f, g$ are given continuous functions
with $f \geq 0$, $f(z,u)$ non-decreasing in $u$. We note that $f$ is not required to be strictly increasing, so our discussion includes automatically the case $f(z,u) = \phi(z)$.

We formulate a notion of viscosity solutions for such equations (see Definition 3.2 below). A new feature in our viscosity formulation is the decomposition of the space of testing functions. This is important for the invariance properties of the equations, and does not seem to have been considered before. Our definition is somewhat similar to that of [Gut01] and [IL90] for the real Monge-Ampère equation. However, the treatment of real Monge-Ampère equations in [IL90] makes use of the local Lipschitz property of convex functions, and hence cannot be directly carried over.

A key ingredient for our approach is the following comparison theorem, which allows a right-hand side $f(z,u)$ depending on $u$:

**Theorem 1.1.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. Let $f \in C(\Omega \times \mathbb{R})$ be non-negative and for any fixed $z \in \Omega$, $f(z,\cdot)$ is non-decreasing; Assume that $u,v \in C(\Omega) \cap PSH(\Omega)$ are viscosity subsolution and supersolution of the equation $M_C(\cdot) = f(z,\cdot)$ respectively. Then

$$u \leq v \text{ on } \partial \Omega \Rightarrow u \leq v \text{ in } \Omega.$$

The main idea in the proof of Theorem 1.1 originates from the work of Caffarelli and Cabre [CC95], and we make use of convex envelope and contact set techniques. Although they are known in the study of many non-divergence equations, including the real Monge-Ampère equations [Caf89], [Caf90], these techniques do not appear to have been applied to the complex Monge-Ampère equations. An immediate advantage of these techniques is that we can give a unified approach to comparison principles without treating separately the cases where $f(z,u)$ has zeros or $f(z,u)$ does not increase in $u$ strictly. We also exploit a simple but useful inequality between real and complex Monge-Ampère equations. The use of this inequality goes back to Cheng and Yau (unpublished), and subsequent works of Bedford [Bel88], Cegrell and Persson [CL92], and Blocki [Blo05]. Combining with viscosity techniques, we can give a more transparent version of the proof by Cheng and Yau (see p.75 [Bel88]) and obtain the following ABP-type of $L^\infty$-estimate, which improves the $L^2$-stability in [CL92]:

**Theorem 1.2.** Let $\Omega$ be a domain such that $\Omega \subset B_r \subset B_{2r}$ and $f \in C(\Omega)$ non-negative. Let $u \in C(\overline{\Omega})$ be a viscosity supersolution of the equation $M_C(u) = f$ in $\Omega$ and $u \geq 0$ on $\partial \Omega$. Then

$$\sup_\Omega u^- \leq C(n)r||f \cdot \chi_{\{u = \Gamma_u\}}||_{L^2(\Omega)}^{1/n},$$

where $C$ only depends on $n$.

The comparison principle (Theorem 1.1) implies readily the following existence theorem for viscosity solutions:
Theorem 1.3. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $g \in C(\partial \Omega)$. Assume that:

i) $f \in C(\overline{\Omega} \times \mathbb{R})$ be non-negative and for any fixed $z \in \Omega$, $f(z, \cdot)$ is non-decreasing;

ii) there exists a harmonic function $\pi$ with boundary value $g$;

iii) there exists a viscosity subsolution $u \in C(\overline{\Omega})$ of (1.1) such that $u = g$ on $\partial \Omega$. Then there exists a unique viscosity solution of the Dirichlet problem (1.1).

We note that the interest in Theorem 1.3 resides in the fact that the data is only continuous. For smooth data, the existence of smooth solutions has been established directly by Guan [Gua00], using the method of continuity and building on the works of Caffarelli, Kohn, Nirenberg, and Spruck [CKNS85].

Our proof of Theorem 1.3 also differs from standard reference ([CIL92], [IL90]). We learned this argument in Prof. Ovidiu Savin’s lectures at Columbia University. Besides proving the existence of the solution, this argument also shows that the modulus of continuity of solution is controlled by the given subsolution and given datum (See. Corollary 5.3).

Finally, we relate viscosity solutions in our sense to generalized solutions in the sense of pluripotential theory:

Theorem 1.4. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ and $u \in C(\overline{\Omega}) \cap \text{PSH}(\Omega)$. Then $M_C(u) = f(z)$ in viscosity sense if and only if $M_C(u) = f(z)$ in pluripotential sense.

An immediate consequence of Theorems 1.3 and 1.4 is the following Corollary, which can be viewed as a generalization of the classic result of Bedford and Taylor for the case when $f(z, u)$ does not depend on $u$:

Corollary 1.5. Let $\Omega$ be a strictly pseudoconvex domain and $g \in C(\partial \Omega)$. Assume that $f \in C(\overline{\Omega} \times \mathbb{R})$ is non-negative and for any fixed $z \in \Omega$, $f(z, \cdot)$ is non-decreasing. Then there exists a unique pluripotential (equivalent, viscosity) solution of the Dirichlet problem (1.1).

The paper is organized as follows. In §1, we summarize some basic linear algebra which is not standard. In §2, we introduce our notion of viscosity solutions and establish some of their basic properties. In §3, we adapt some standard tools from viscosity theory. §4 is devoted to the proof of the comparison principle and the study of the Dirichlet problem for the complex Monge-Amplere equation in strictly pseudoconvex domain. The equivalence between pluripotential solutions and viscosity solutions is established in §5. We prove Theorem 1.2 in §6.

Acknowledgments: I would like to express my great gratitude to my advisor Prof. Duong Hong Phong, for his penetrating remarks, various suggestions and many encouragement. Also I am heartily grateful to Prof. Ovidiu Savin, for his valuable and inspirational discussions. I would also like to thank Prof. Zbigniew Blocki, for providing me with some important references.

This paper is part of my forthcoming thesis at Columbia University.
2. Linear Algebra Preliminaries

Let $\mathbb{R}^{2n}$ be equipped with canonical coordinates ordered as $x^1, \ldots, x^n, y^1, \ldots, y^n$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by the relation $x^i + \sqrt{-1} y^i = z^i$, and we let $J$ be the canonical complex structure:

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where $I$ is the $n \times n$ identity matrix. We introduce the following terminology and notations:

- $D^2$ is the Hessian operator computed with respect to the ordered coordinates of $\mathbb{R}^{2n}$.
- $dd^c$ is the operator sending a smooth function $\varphi$ to the Hermitian matrices $2\varphi_{ji}$, computed with respect to the coordinate $z^i$.
- $T(n)$ is the space of polynomials of the form $A_{ji}z^i\bar{z}^j$, where $A_{ji}$ is a Hermitian matrices. $T^+(n)$ is the non-negative cone in $T(n)$ consisting of $p = A_{ji}z^i\bar{z}^j$ with $A$ being non-negative.
- $H(n) := \mathbb{C}[n](2) \oplus \mathbb{C}[n](1)$, where $\mathbb{C}[n](d)$ denotes space of homogeneous polynomials over $\mathbb{C}$ of degree $d$ with respect to $z^1, \ldots, z^n$.
- Herm$(n)$ and Sym$(2n)$ are the spaces of Hermitian matrices over $\mathbb{C}$ and symmetric matrices over $\mathbb{R}$ respectively. Recall that each $M \in$ Sym$(2n)$ that commutes with $J$ can be canonically identified as an element in Herm$(n)$.
- $\det_\mathbb{C}$ and $\det_\mathbb{R}$ are the determinant functions on Herm$(n)$ and Sym$(2n)$ respectively. For any smooth function $\varphi$, we define

$$M_\mathbb{C}(\varphi) := \det_\mathbb{C}(dd^c \varphi), \quad M_\mathbb{R}(\varphi) := \det_\mathbb{R}(D^2 \varphi).$$

The following lemmas regarding quadratic polynomials and matrices are very elementary; we shall omit the proof.

**Lemma 2.1.** Let $\varphi$ be a quadratic polynomial in $\mathbb{R}[x^1, \ldots, x^n, y^1, \ldots, y^n]$. Then under the identification $x^i + \sqrt{-1} y^i = z^i$, $\varphi$ can be uniquely written as:

$$\varphi = p + \Re(h), \quad p \in T(n), \quad h \in H(n).$$

**Lemma 2.2.** Let $\varphi$ be a quadratic polynomial (or a smooth function), then $\frac{1}{2} (D^2 \varphi + J^t D^2 \varphi J)$ can be canonically identified with $dd^c \varphi$, and

$$\det_\mathbb{R}(\frac{1}{2} (D^2 \varphi + J^t D^2 \varphi J)) = (\det_\mathbb{C}(dd^c \varphi))^2.$$

Moreover if $\varphi$ is convex, then

$$M_\mathbb{C}^{1/2n}(\varphi) \geq M_\mathbb{R}^{1/2n}(\varphi).$$
3. Viscosity Solutions

We consider the following equation:

\[(3.1) \quad M_C(u)(z) = f(z, u(z)) \quad \forall z \in \Omega \]

Henceforth, we shall always assume the right-hand side \( f \) to be a function from \( \Omega \times \mathbb{R} \) to \([0, \infty)\) (not necessarily continuous) and non-decreasing with respect to the second variable.

We recall the following terminology:

**Definition 3.1.** Let \( u, \varphi \) be two continuous functions.

(a) \( \varphi \) is said to touch \( u \) from above at \( z \) in an open neighborhood \( V \) if \( \varphi \geq u \) in \( V \) and \( \varphi(z) = u(z) \).

(b) \( \varphi \) is said to touch \( u \) from above at \( z \), if there exists some open neighborhood \( V \) of \( z \) such that \( \varphi \) touches \( u \) from above in \( V \).

The notions of \( \varphi \) touching \( u \) from below at \( x \) are defined in an analogous way.

We can now define the notion of viscosity solutions of the equation \( M_C(\cdot) = f(z, \cdot) \) as follows:

**Definition 3.2.** Let \( u \) be a continuous plurisubharmonic (“psh”) function on a domain \( \Omega \subset \mathbb{C}^n \). The function \( u \) is said to be a viscosity subsolution (resp. supersolution) of \( M_C(\cdot) = f(z, \cdot) \), if the following condition holds:

\[(\star) \quad \text{For any } p \in T^+(n), \text{ if there exists } h \in H(n) \text{ such that } p + \Re(h) \text{ touches } u \text{ from above (resp. below) at some } z_0 \in \Omega, \text{ then } M_C(p) \geq f(z_0, u(z_0)) \text{ (resp. } M_C(p) \leq f(z_0, u(z_0))) \]

We say that \( M_C(u) \geq f(z, u) \) (resp. \( M_C(u) \geq f(z, u) \ )) in viscosity sense if \( u \) is a viscosity subsolution (resp. supersolution). We say that \( u \) is a viscosity solution of \( M_C(\cdot) = f(z, \cdot) \) if it is both a supersolution and a subsolution.

**Remark 3.3.** i) In the Definition 3.2, \( \Omega \) is not required to be strictly pseudoconvex or bounded.

ii) One can similarly define \( M_C^{1/n} \), and \( M_C^{1/n}(u) = f^{1/n}(z, u) \) (resp. \( \geq, \leq \)) in viscosity sense if and only if \( M_C(u) = f(z, u) \) (resp. \( \geq, \leq \)) in viscosity sense.

The following proposition is useful in exploring the generality of our definition:

**Proposition 3.4.** (a) \( u \) is a viscosity subsolution (resp. supersolution) of \( M_C(\cdot) = f(z, \cdot) \) according to Definition 3.2 if and only if the following condition holds:

For any \( \varphi \in C^2(\Omega) \cap \text{PSH}(\Omega) \), if \( u - \varphi \) takes a local maximum (resp. minimum) at \( z_0 \in \Omega \), then \( M\varphi(z_0) \geq f(z_0, \varphi(z_0)) \) (resp. \( M\varphi(z_0) \leq f(z_0, \varphi(z_0)) \))

(b) Let \( u \in \text{PSH}(\Omega) \cap C(\Omega) \). Then \( u \) is a subsolution if and only if \( u \) satisfies the condition in (a) with \( \varphi \in C^2(\Omega) \cap \text{PSH}(\Omega) \) replaced by \( \varphi \in C^2(\Omega) \).
Proof. Note that if $p \in T^+$ and $M_C(p) > 0$, then there exists some $\epsilon_0$ such that for all $\epsilon < \epsilon_0$, $dd^c(p - \frac{\epsilon}{2}|z|^2)$ is strictly positive. If $M_C(p) = 0$, then $M_C(p) < f(z, u(z))$ for all $z$ as $f$ is non-negative.

With the above observation, (a) can be proved by mimicking the argument for Proposition 2.4 in [CC95].

To see (b), one just needs to notice that we assume $u$ to be a psh. In this case, if $\varphi \in C^2(\Omega)$ and $u - \varphi$ takes a local maximum at $z_0 \in \Omega$, $dd^c \varphi$ has to be non-negative. □

We make the following definition for convenience:

Definition 3.5. $u \in C(\Omega)$ is said to be $T_2$ at a point $z_0 \in \Omega$ if there exists a quadratic polynomial $\varphi$ such that:

$$u(z) = \varphi(z) + o(|z-z_0|^2) \quad \text{as } z \to z_0.$$ 

In this case, we define $D^2u$ to be $D^2\varphi$ and $dd^c u$ to be $dd^c \varphi$.

It is clear that such a $\varphi$ is unique if exists, and hence $D^2u, dd^c u$ are well-defined.

Proposition 3.6. Let $u$ be a continuous in $\Omega$.

(a) if $u$ is a viscosity subsolution (resp. supersolution) of $M_C(\cdot) = f(z, \cdot)$, then for any holomorphic function $h$ on $\Omega$, $M_C(u + \Re(h)) \geq f(z,u(z))$ (resp. $M_C(u + \Re(h)) \leq f(z,u(z))$)) in the viscosity sense in $\Omega$.

In particular, if $u$ is a viscosity subsolution (resp. supersolution) of $M_C(\cdot) = f(z, \cdot)$, then so is $u - c$ (resp. $u + c$) for any positive constant $c$.

(b) Let $u$ be a subsolution (resp. supersolution) for the equation $M_C(\cdot) = f(z, \cdot)$ and $w$ be a continuous psh function that touches $u$ from above (resp. below) at some $z_0 \in \Omega$. Suppose that $w$ is $T_2$ at $z_0$. Then $M_C(w)(z_0) \geq f(z_0, u(z_0))$ (resp. $M_C(w)(z_0) \leq f(z_0, u(z_0))$).

In particular, if $u$ is viscosity subsolution (resp. supersolution) of $M_C(u) = f(z, u)$ and $u$ is $T_2$ at a point $z_0 \in \Omega$, then $M_C(u)(z_0) \geq f(z_0, u(z_0))$ (resp. $M_C(u)(z_0) \leq f(z_0, u(z_0))$).

(c) If $u \in C^2(\Omega) \cap PSH(\Omega)$, then $u$ is a viscosity subsolution (resp. supersolution) of $M_C(u) = f(z, u)$ if and only if $M_C(u) \geq f(z, u)$ (resp. $M_C(u) \leq f(z, u)$) in classical sense.

Proof. With the observations mentioned in the proof of Proposition 3.4 to prove the above proposition, we just need to carry out the arguments for Lemma 2.5 and Corollary 2.6 in [CC95]. □

We end this section with the following convergence Proposition We refer to Proposition 2.9 in [CC95] for a proof.

Proposition 3.7. If $u_k$ be a sequence of viscosity subsolution (resp. supersolution) of $M_C(\cdot) = f(z, \cdot)$ converging uniformly in compact subset of $\Omega$ to $u$. Then $M_C(u) \geq f(z, u)$ (resp. $M_C(u) \leq f(z, u)$) in $\Omega$ in viscosity sense.
4. Convex Envelopes and Jensen’s Approximation

We recall some results for convex envelopes. Our main reference is Ch.3 of [CC95].

**Definition 4.1.** Let \( w \) be a continuous function in an open bounded domain \( \Omega \) such that \( \Omega \subset B_r \subset B_{2r} \) and \( w \geq 0 \) on \( \partial \Omega \). Let \( w^- := -\inf \{ w, 0 \} \) and extend it to \( B_{2r} \) by zero. Define:

\[
\Gamma_w(x) := \sup \{ v(x) : v \text{ is convex in } B_{2r}, \ v \leq -w^- \text{ in } \Omega \}.
\]

We call the set \( \{ w = \Gamma_w \} \) the contact set of \( w \) in \( \Omega \);

**Remark 4.2.** Unless \( w^- \) is identically zero, \( \Gamma_w \) is strictly negative in the interior of \( \Omega \).

We recall some terminology in order to state an essential estimate – the Alexandrov-Bakelman-Pucci (ABP) estimate.

A function \( P \) is called a paraboloid of opening \( K \) if it is of the form

\[
P(x) = \frac{K}{2} |x|^2 + l
\]

where \( l \) is an affine function.

**Definition 4.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and let \( w \in C(\Omega) \). A function \( w \) is said to be \( K \)-semi-concave in \( \Omega \), if for any point \( x \in \Omega \), there exists a paraboloid of opening \( K \) touch \( w \) from above in \( \Omega \).

Similarly, one define \( K \)-semi-convexity.

\( w \) is semi-concave (resp. semi-convex) if there exists a finite \( K > 0 \) such that \( w \) is \( K \)-semi-concave (resp. semi-convex).

**Remark 4.4.** By Alexandrov theorem on second order differentiability (see §1.1 of [CC]), a semi-convex (semi-concave) functions are almost every \( T_2 \).

Now we shall state the ABP estimate

**Theorem 4.5.** Let \( w \) be continuous in \( \Omega \subset B_r \subset B_{2r} \subset \mathbb{R}^n \) and \( w \geq 0 \) on \( \partial \Omega \). Assume that \( w \) is semi-concave. Then

i) \( \Gamma_w \in C^{1,1}(\overline{\Omega}) \);

ii) \( \{ w = \Gamma_w \} \subset \Omega \) unless \( w^- \) is identically zero.

iii) The image of the set \( \{ w = \Gamma_w \} \) under the normal mapping of \( \Gamma_u \) contains a ball of radius \( \sup w^- / 2r \), that is,

\[
B_{\sup w^- / 2r} \subset \nabla \Gamma_w(\{ w = \Gamma_w \})
\]

**Remark 4.6.** ABP estimates holds under weaker assumptions. For details, one may refer to §3.1 of [CC95].

Following immediate consequence of \( \text{Theorem 4.5} \) will play key role in our proof of Theorem \( \text{Theorem 1.1} \).
Corollary 4.7. Let \( w \in C(\overline{\Omega}) \) be semi-concave and \( E \subset \Omega \) be a set such that \(|\Omega \setminus E| = 0\). Suppose \( w \geq 0 \) on \( \partial \Omega \) and \( \min_{\Omega} w = -a, a > 0 \). Then for any \( \delta \in (0, a/2d), d = \text{diam}(\Omega) \), there exists a point \( x_0 \in E \) such that:

i) \( w(x_0) = \Gamma_w(x_0) < 0 \).

ii) \( \Gamma_w \) is \( T_2 \) at \( x_0 \) and

\[
|\nabla w|(x_0) = |\nabla \Gamma_w|(x_0) < \delta, \quad (\det D^2 \Gamma_w(x_0))^{1/n} \geq \frac{\delta}{d}
\]

Proof. We may assume \( 0 \in \Omega \subset B_d \). Let \( H \) be the set consisting \( T_2 \)-points of \( \Gamma_u \) in \( \Omega \). i) of Theorem 4.5 implies

\[
|\Omega \setminus H| = 0.
\]

Fix \( \delta \in (0, a/2r) \), then by ii) of Theorem 4.5, there exists a non-empty subset \( A \) of the contact set \( \{ w = \Gamma_w \} \) such that

\[
\nabla \Gamma_w(A) = B_\delta
\]

and any point \( x \in A \cap H \) satisfies:

\[
w(x) = \Gamma_w(x) < 0, \quad |\nabla w|(x) = |\nabla \Gamma_w(x)| < \delta
\]

Moreover, the \( C^{1,1} \)-regularity along with the area formula implies.

\[
|B_\delta| \leq |\nabla \Gamma_w(A)| = \int_{A \cap E \cap H} \det D^2 \Gamma_w \, dx.
\]

Here we used the fact \(|A \setminus (E \cup H)| = 0\).

Since \( A \subset \Omega \subset B_d \), (4.1) implies \( A \setminus (E \cup H) \) has positive measure and there exists \( x_0 \in A \cap E \cap H \) such that

\[
\det D^2 \Gamma_w(x_0) \geq \frac{\delta^n}{d^n}.
\]

This completes the proof of the corollary. \( \square \)

Remark 4.8. For the definition and properties of normal mapping of a convex function, one may refer to [Gut01].

We now adapt standard Jensen’s approximation theorem to our setting. Our main reference is Ch.5 of [CC95].

Definition 4.9. Let \( u \in C(\overline{\Omega}) \cap PSH(\Omega) \) and \( M_\mathcal{C}(u) \geq f(x, u) \) in viscosity sense in \( \Omega \). For \( \epsilon > 0 \), define

\[
u^\epsilon(z_0) := \sup_{z \in \overline{\Omega}} \{ u(z) - \frac{1}{\epsilon} |z - z_0|^2 \}, \quad z_0 \in \Omega;
\]

\[
f^\epsilon(z_0, t) := \inf \{ f(z, t) : z \in B_\tau(x_0) \cap \overline{\Omega} \}, \quad \tau = (\epsilon \text{osc} u)^{1/2}
\]

Similarly, one define \( v^\epsilon, f^\epsilon \) for a supersolution.
Proposition 4.10. Assume that \( u \in C^1(\Omega) \cap PSH(\Omega) \) and \( M_C(u) \geq f(x,u) \). Then:

(a) \( u^e \in C^{0,1}(\Omega) \cap PSH(\Omega) \) with Lipschitz constant smaller than \( \frac{4}{e} \text{diam} (\Omega) \).

(b) \( u^e \) decreases uniformly to \( u \) in \( \Omega \).

(c) \( u^e \) is semi-convex in \( \Omega \).

(d) \( f^e \) increases uniformly to \( f \).

(e) For any compact domain \( U \) of \( \Omega \), there exists a \( \epsilon_0 \), depending on \( \text{osc}_{\Omega} u \) and \( \text{dist}(U, \partial\Omega) \), such that, for any \( \epsilon < \epsilon_0 \)

\[
M_C(u^e) \geq f^e(x,u^e), \quad \text{in } U
\]

in viscosity sense.

Corresponding statements holds for \( v^e, f^e \) of a supersolution.

Proof. To prove (b), (c) and (e), one can carry out the argument on page 44 of [CC95] with obvious modification. (d) is clear. In (a), Lipschitz part is also proved in [CC95], the PSH part follows from the change of variable \( y = z - z_0 \) and Choquet Lemma.

5. Dirichlet Problem

First, we establish the comparison principle in Theorem 1.1. The key idea of our proof originates from Sec.5 of [CC95].

Proof of Theorem 1.1. Without lose of generality, we may assume \( 0 \in \Omega \). Let \( d = \text{diam}(\Omega) \). Moreover, by replacing \( v \) by \( v + \delta \), we may assume \( v > u \) on \( \partial\Omega \). Theorem 1.1 will follow from this case by taking \( \delta \) to be zero.

Argue by contradiction. Write \( w = v - u \). Assume there exists \( x_0 \in \Omega \) such that

\[
w(x_0) = -a < 0.
\]

Regularizing \( v, u \) via Jensen’s approximation, and write \( w_\epsilon = v_\epsilon - u^e \). Since \( w > 0 \) on \( \partial\Omega \) and \( w_\epsilon \) increases uniformly to \( w \), we may fix a compact subdomain \( U \) of \( \Omega \) such that \( w_\epsilon \geq 0 \) in \( \Omega \setminus U \) for all \( \epsilon \) sufficiently small,

Fix any \( \epsilon > 0 \) small, denote \( E_\epsilon \subset U \) the set consists of points on which \( w_\epsilon, v_\epsilon, -u^e \) are \( T_2 \). By (b) of Proposition 4.10 \( |U \setminus E_\epsilon| = 0 \) and \( w_\epsilon \) is semi-concave. Apply Corollary 4.7 we may choose \( x_\epsilon \in E_\epsilon \) such that

\[
w_\epsilon(x_\epsilon) = \Gamma_w(x_\epsilon) < 0, \quad \det^{1/2n}(D^2 \Gamma_{w_\epsilon})(x_\epsilon) \geq \frac{a}{3d^2}.
\]

By applying (e) of Proposition 4.10 with respect \( U \), we obtain

\[
M_C(v^e)(x_\epsilon) \leq f^e(x_\epsilon, v_\epsilon(x_\epsilon)).
\]

and

\[
M_C(u^e)(x_\epsilon) \geq f_\epsilon(x_\epsilon, u^e(x_\epsilon)).
\]

for all sufficiently small \( \epsilon \). Here the expression are computed in the sense of (b) of Proposition 3.6.
Combine Eq. (5.1, 5.2, 5.3) and use the Minkowski inequality of determinant and Lemma 2.2 along with the fact \( \Gamma_{u_{\epsilon}} + u \) touches \( v_{\epsilon} \) from below at \( x_{\epsilon} \). we obtain:

\[
f_{\epsilon}(x_{\epsilon}, v_{\epsilon}(x_{\epsilon})) \geq f_{\epsilon}(x_{\epsilon}, u_{\epsilon}(x_{\epsilon})) + \frac{a}{3d^2}
\geq f_{\epsilon}(x_{\epsilon}, v_{\epsilon}(x_{\epsilon})) + \frac{a}{3d^2}
\]

(5.4)

In the last inequality, we have used the fact that \( f(x, t) \) is non-decreasing in \( t \).

Since \( a/3d^2 \) is independent of \( \epsilon \) and \( f_{\epsilon}, f_{\epsilon} \) converges uniform to \( f \), Eq. 5.4 leads to a contradiction when \( \epsilon \) is taken sufficiently small. \( \Box \)

Remark 5.1. i) The continuity of \( u, v \) is not essential to the proof. Semi-continuity with corresponding boundary condition is sufficient.

ii) This proof can be applied to more general operator with suitable structure conditions.

Now, we shall apply Perron Method to prove Theorem 1.3. Our argument, learned from Prof. Ovidiu Savin, differs from standard argument in [IL90].

The key part is the following lemma:

**Lemma 5.2.** Under assumption of Theorem 1.3. For each subsolution \( u \) with \( u|_{\partial \Omega} \leq g \), there exists \( \tilde{u} \in C(\Omega) \cap PSH(\Omega) \) such that

i) \( \tilde{u} \) is a viscosity subsolution of \( M(\cdot) = f(x, \cdot) \) in \( \Omega \).

ii) \( \tilde{u} \geq u \) in \( \Omega \) and \( \tilde{u}|_{\partial \Omega} = g \).

iii) The modulus of continuity \( \omega_{\tilde{u}} \) of \( \tilde{u} \) satisfies: for all \( z_1, z_2 \in \Omega \)

\[
\omega_{\tilde{u}}(|z_1 - z_2|) \leq (3d \text{osc } f) |z_1 - z_2| + d^2 \omega_f(|z_1 - z_2| + \omega(|z_1 - z_2|)).
\]

where \( \omega = \max\{\omega_u, \omega_f\} \) and \( \omega_f \) is the modulus of continuity of \( f^{1/n} \) in the region

\[
V = \overline{\Omega} \times [-M, M], \quad M = \max\{|g|_\infty + d^2, |u|_\infty + d^2\}.
\]

**Proof.** The proof is similar to that of Jensen’s approximation (see §5.1 [CC]). However this lemma seems not standard and well-known, we shall include details for readers’ convenience.

Without lose of generality, we shall assume \( 0 \in \Omega \) and \( d = \text{diam}(\Omega) \). By taking \( \sup\{u, \tilde{u}\} \), we may assume \( u|_{\partial \Omega} = g \) and \( u \geq \tilde{u} \) in \( \partial \Omega \).

Define \( \tilde{u} \) as following:

\[
\tilde{u}(z_0) := \sup_{y \in \Omega} \left\{ \max[u(y) - \omega(|y - z_0|) + \omega_f(|y - z_0|)|z_0|^2 - d^2], u(z_0) \right\}.
\]

Write \( z_0^* \) for the point where the supreme occur. We shall show \( \tilde{u} \) satisfies desired properties.
We prove iii) first and $u \in PSH(\Omega)$ follows from Choquet lemma. The key point is that (optimal) modulus of continuity function is sub-additive. Fix $z_1, z_2 \in \overline{\Omega}$.

By additivity,
$$|\omega(|z_1 - y|) - \omega(|z_2 - y|)| \leq \omega(|z_2 - z_1|), \quad \forall y \in \overline{\Omega}$$

Similarly holds for $\omega_f$

Therefore,
$$\omega_f(|z_1 - y|)(|z_1|^2 - d^2) - \omega_f(|z_2 - y|)(|z_2|^2 - d^2)$$
$$= \omega_f(|z_1 - y|)(|z_1|^2 - |z_2|^2) + (\omega_f(|z_1 - y|) - \omega_f(|z_2 - y|))(|z_2|^2 - d^2)$$
$$\geq -3d(\text{osc } f)(|z_1 - z_2| - d^2\omega_f(|z_1 - z_2|))$$

Combine all these, we obtain
$$\hat{u}(z_1) - \hat{u}(z_2) \geq -(3d \text{osc } f)|z_1 - z_2| - d^2\omega(|z_1 - z_2|) - \omega(|z_1 - z_2|)$$

Since $z_1, z_2$ are chosen arbitrarily
$$\omega_\delta(|z_1 - z_2|) \leq (3d \text{osc } f)|z_1 - z_2| + d^2\omega_f(z_1 - z_2) + \omega(|z_1 - z_2|)$$

This proves iii).

To prove i). Let $p \in T^+(n), h \in H(n)$ and $P = p + \Re(h)$ touches $\hat{u}$ from above at $x_0 \in \Omega$. If $\hat{u}(x_0) = \underline{u}(x_0)$, then
$$M_C(P) \geq f(x_0, \underline{u}(x_0)),$$

because $\underline{u}$ is a subsolution.

If $\hat{u}(x_0) > \underline{u}(x_0)$, then the quadratic polynomial
$$Q(z) := P(z + x_0 - x_0^*) + \omega(|x_0 - x_0^*|) - \frac{\omega_f(|x_0 - x_0^*|)}{2}(|z + x_0 - x_0^*|^2 - d^2)$$
touches $u$ at $x_0^*$.

Claim $x_0^* \in \Omega$. Suppose otherwise $x_0^* \in \partial \Omega$, then
$$\hat{u}(x_0) = \underline{u}(x_0) - \omega(|x_0^* - x_0|) + \frac{\omega_f(|x_0^* - x_0|)}{2}(|x_0|^2 - d^2)$$
$$\leq \overline{u}(x_0^*) - \omega(|x_0^* - x_0|) \leq \overline{u}(x_0^*) = \underline{u}(x_0^*).$$
Here we used the fact that $u$ is subharmonic and $u = \overline{u}$ on $\partial \Omega$. But this contradicts to the assumption that $\hat{u}(x_0) > \underline{u}(x_0)$.

Now, recall $u$ is a subsolution, thus
$$M_C(Q) \geq f(x_0^*, \underline{u}(x_0^*)) = f(x_0^*, \underline{u}(x_0^*))$$

Apply Minkowski inequality of determinant, we obtain
$$M_C^{1/n}(P) \geq f^{1/n}(x_0^*, \underline{u}(x_0^*)) + \omega(|x_0^* - x_0|)$$
$$\geq f^{1/n}(x_0^*, \hat{u}(x_0)) + \omega(|x_0^* - x_0|) \geq f^{1/n}(x_0, \hat{u}(x_0))$$
Here we have used the fact
\[ \tilde{u}(x_0) = u(x_0) - \omega(|x_0^* - x_0|) + \frac{\omega(|x_0^* - x_0|)}{2}(|x_0|^2 - d^2) \leq u(x_0). \]
Thus, we have shown \( \tilde{u} \) is a subsolution.

To see ii), same as in the previous step, if \( z_0 \in \partial \Omega \), then for any \( y \in \Omega \)
\[ u(y) - \omega(|y - z_0|) + \frac{\omega(|y - z_0|)}{2}(|z_0|^2 - d^2) \leq u(y) - \omega(|z_0 - y|) \leq u(z_0). \]
Hence
\[ \tilde{u}|_{\partial \Omega} = u|_{\partial \Omega} = g. \]
That \( \tilde{u} \geq u \) in \( \Omega \) is obvious.

This completes the proof. \( \Box \)

Proof of Theorem 1.3 Define
\[ \mathcal{S} := \{ v : M_C(v) \geq f(x, v) \text{ in viscosity sense in } \Omega, \ v|_{\partial \Omega} \leq g \}. \]
and
\[ u = \sup \{ v : v \in \mathcal{S} \}. \]
Consider the following sub-family of \( \mathcal{S} \)
\[ \tilde{\mathcal{S}} := \{ \tilde{v} : v \in \mathcal{S} \}. \]
By the Lemma 5.2, \( \tilde{\mathcal{S}} \) is a equi-continuous subset of \( \mathcal{S} \) and
\[ u = \sup \{ \tilde{v} : \tilde{v} \in \tilde{\mathcal{S}} \}. \]
By Arzel–Ascoli, \( u \) is uniform limit of a sequence of subsolution. Hence, by recalling Proposition 3.7, \( u \in C(\Omega) \) is again a subsolution.

To finish the proof, we only left to show that \( u \) is also a supersolution. Argue by contradiction, there exists a \( p \in T^+(n), h \in H(n) \) such that \( p + \Re(h) \) touches \( u \) from below at \( z_0 \in \Omega \), but \( M_C(p) > f(z, u(z_0)) \). Then, taking a small ball \( B_r \) of \( z_0 \), and define
\[ \psi := p + \Re(h) - \frac{\epsilon}{2}(|z - z_0^2| - r^2). \]
For \( r, \epsilon \) small enough, we have \( \psi \) is Psh,
\[ M_C(\psi) > f(z, u(z)), \quad \forall z \in B_r(z_0). \]
and
\[ \psi(z_0) > u(z_0), \quad \psi|_{\partial B_r} \leq u|_{\partial B_r}. \]
Then define
\[ \tilde{u} := \begin{cases} \max\{ \psi, u \} & x \in \overline{B_r}(z_0) \\ u & x \in \Omega \setminus B_r(z_0). \end{cases} \]
It is easy to check that \( \hat{u} \) is again a subsolution but \( \hat{u}(z_0) > u(z_0) \). This contradicts the maximality of \( u \).

The proof is then complete. \( \square \)

Besides solvability, Lemma 5.2 and the proof of Theorem 1.3 yields Corollary 5.3.

\[ \text{Let } \Omega \text{ be a strictly pseudoconvex domain and } \phi \in C(\Omega) \text{ and } \psi(u) \in C(\mathbb{R}) \text{ non-decreasing. Suppose } M_{\mathbb{C}}(\cdot) = \phi(x)\psi(u) \text{ admits an subsolution } \underline{u} \text{ with modulus of continuity } \omega_{\underline{u}}. \text{ Then the modulus of continuity } \omega_u \text{ unique viscosity solution } u \text{ satisfies} \]

\[ \omega_u \leq C(\max\{\omega_\phi, \omega_{\underline{u}}\}) \]

where \( C \) depends on \( n, \text{diam}(\Omega), ||\phi||_{L^∞(\Omega)} \) and the \( L^∞ \) norm of \( \psi \) on the interval \([\min_\Omega u, \max_\Omega u]\).

In particular, if \( u \) and \( \phi \) are \( \alpha \)-Hölder in \( \overline{\Omega} \), then \( u \) is \( \alpha \)-Hölder with

\[ ||u||_{\alpha, \overline{\Omega}} \leq C \max\{||u||_{\alpha, \overline{\Omega}}, ||\phi||_{\alpha, \overline{\Omega}}\} \]

**Proof.** Since \( \Omega \) is strictly pseudoconvex, in particular of \( C^2 \)-boundary. By standard harmonic analysis, there exists a harmonic function \( h \) such that \( h|_{\partial \Omega} = u \) and \( \omega_h \leq C \omega_{\underline{u}} \)

where \( C \) only depends on \( n, \text{diam}(\Omega) \).

Apply Lemma 5.2 to the viscosity solution \( u \), then \( \hat{u} \) \( \geq u \). By tracking the proof of Lemma 5.2, the specific form \( f(x, t) = \phi(x)\psi(u) \) allows one to conclude that \( \omega_h \) has required modulus of continuity. But \( u \geq \hat{u} \) by comparison principle, hence \( u = \hat{u} \). And the corollary follows. \( \square \)

### 6. Relations with Pluripotential Solutions

In this section, we assume that \( \Omega \) is a strictly bounded pseudoconvex domain. We use the following normalization of Lebesgue measure:

\[ d\lambda = \frac{1}{n!} \left( \sum_{i} \sqrt{-1} dz^i \wedge d\bar{z}^i \right). \]

Hence

\[ (dd^c u) = \det(2u_{\bar{k}j}) \, d\lambda. \]

**Proposition 6.1.** Let \( f \in C(\Omega) \) be non-negative and \( u \in C(\Omega) \). Then \( (dd^c u) = f(z) \, d\lambda \) in pluripotential-potential sense implies that \( M_{\mathbb{C}}(u) = f(z) \) in viscosity sense.

**Proof.** Let \( u \) be a pluripotential solution of \( M_{\mathbb{C}}(u) = f(z) \). To see that \( u \) is a viscosity subsolution, one may carry out argument of proposition [EGZ10] words by words. Now, it suffices to show that \( u \) is a viscosity supersolution.

Let \( p \in T^+(n), h \in H(n) \) and \( p + \Re(h) \) touches \( u \) from below at some \( z_0 \in \Omega \). We need to verify that \( M_{\mathbb{C}}(p) \leq f(z_0) \). Without lose of generality, we assume \( z_0 = 0 \).
Suppose on the contrary, $M_C(p) > f(0) \geq 0$. By continuity of $f$, there exists some $r_0$ and $\epsilon_0$ such that

$$dd^c(p - \frac{\epsilon_0}{2}|z|^2) > 0; \quad M_C(p - \frac{\epsilon_0}{2}|z|^2)(z) > f(z), \quad \forall z \in \overline{B_{r_0}}$$

Choose $0 < \delta < \frac{\epsilon_0}{2}r_0^2$, then

$$p + \Re(h) - \frac{\epsilon}{2}|r_0|^2 + \delta < \epsilon_0, \text{ on } \partial B_{r_0}$$

$$p(0) + \Re(h)(0) - \frac{\epsilon}{2}|z|^2|_{z=0} + \delta > u(0)$$

But this contradicts the pluripotential comparison principle. Thus a pluripotential solution $u$ is also a viscosity supersolution.

Theorem 1.4 then follows easily from Proposition 6.1.

Proof of Theorem 1.4: Let $u \in C(\overline{\Omega})$ satisfy $M_C(\cdot) = f$ in viscosity sense. Solve the Dirichlet problem (1.1) with data $f, g = u|_{\partial \Omega}$ in pluripotential sense. Denote the unique solution by $\tilde{u}$. By Proposition 6.1, $\tilde{u}$ is also a viscosity solution of the Dirichlet problem. The viscosity uniqueness forces $\tilde{u} = u$.

Finally, we prove the corollary 1.5.

Proof of Corollary 1.5: By Theorem 1.4, it suffices to solve the Dirichlet problem (1.1) in the viscosity sense.

First consider the case when $g \in C^\infty(\overline{\Omega})$. Let $\rho \in C^2(\overline{\Omega})$ be the exhaustion function of $\Omega$, then $A\rho + g$ is a subsolution with boundary value $g$ for $A$ sufficiently large. And the $C^2$-boundary allows the existence of harmonic functions for arbitrary given continuous boundary data. Therefore following Theorem (1.1), we obtain the existence and uniqueness.

The general case when $g$ is continuous follows from the above special case by applying a standard approximation procedure based on Theorem 1.4 and Proposition 3.7.

Remark 6.2. We use here the fact that strictly pseudoconvex domains have $C^2$ exhaustion functions.

7. An ABP-type of $L^\infty$-estimate

Let $\Omega$ be a bounded pseudoconvex domain and $u$ solves

$$\begin{cases}
(dd^c u)^n = f \in C(\overline{\Omega}) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

It was originally established by Cheng and Yau (see [7], p. 75) that

$$||u||_{L^\infty(\Omega)} \leq c_n \text{diam}(\Omega)||f||_{L^2(\Omega)}^{1/n}.$$
The Cheng-Yau argument was made precise by Cegrell and Persson [CL92]. Moreover, it is pointed out in Bedford [Bef88] that
\[ ||u||_{L^\infty(\Omega)} \leq c_n \text{diam}(\Omega) \left( ||f\chi_{\{u=\Gamma_u\}}||_{L^2}^{1/n} \right). \]
when $\Omega$ is convex. Using the viscosity techniques developed in this paper, we can make the proof in [Bef88] more transparent:

**Lemma 7.1.** Under the hypotheses of Theorem 1.2, $\Gamma_u$ is a viscosity supersolution of the real Monge-Ampère equation
\[ M_C(\cdot) = f^2 \chi_{u=\Gamma_u}, \]
and $\Gamma_u = 0$ on $\partial B_{2r}$. (see [Gut01] for the definition of viscosity solutions for real Monge-Ampère equations)

**Proof.** The fact that $\Gamma_u = 0$ on $\partial B_{2r}$ is trivial.

Let $z_0 \in B_{2r} \setminus \{u = \Gamma_u\}$. It is shown in p. 27 of [CC95] that there exists an open line segment $L$ through $z_0$ on which $\Gamma_u$ is affine. Hence, if $P$ is a convex polynomial touching $\Gamma_u$ from below at $z_0$, then $P$ is affine on $L$, hence $\det D^2 P = 0$.

Let $z_0 \in B_{2r} \in \{u = \Gamma_u\}$ and $P$ be a convex polynomial touching $\Gamma_u$ from below at $z_0$. Then $P$ touches $u$ from below at $z_0$. Since $u$ is a viscosity supersolution of $M_C(\cdot) = f$ and we can apply Lemma 2.2, we obtain
\[ M_{1/2n}^R(P) \leq M_1^C(P) \leq f^{1/n}(z_0). \]

**Proof of Theorem 1.2** By the standard theory of real Monge-Ampère equation (see Proposition 1.7.1 of [Gut01] for example), Lemma 7.1 implies that
\[ |\nabla \Gamma_u(U)| \leq \int_U f^2 \, dx \]
for all open sets $U \subset B_{2r}$. Thus, by Alexandrov’s maximum principle (Theorem 1.4.1 of [Gut01]), we can conclude that
\[ (7.1) \sup_{B_{2r}} -\Gamma_u \leq C(n) r \left( ||f\chi_{\{u=\Gamma_u\}}||_{L^2(B_{2r})}^{1/n} \right). \]
Since $u^- \leq -\Gamma_u$ and $\{u = \Gamma_u\} \subset \Omega$ unless $u^- = 0$ (see Theorem 3.6 in [CC95]), the desired estimate follows. Theorem 1.2 is proved. □

**Remark 7.2.** I) Kolodziej [Kol98] has shown that for any $p > 1$, $(dd^c u)^n = f \in L^p(\Omega)$ and $u \geq 0$ on $\partial \Omega$ implies
\[ \sup_{\Omega} u^- \leq C(p, n, \text{diam}(\Omega)) ||f||_{L^p(\Omega)}^{1/n}. \]

It is not clear whether $||f\chi_{\{u=\Gamma_u\}}||_{L^2}$ can be controlled by $||f||_{L^p}$ with $1 < p < 2$, or vice versa.

ii) If $\Omega$ is convex, then Lemma 7.1 holds for the convex envelope of $u$ in $\Omega$. 
Along the same line of proof as in [CL92], Theorem 1.2 implies the following stability theorem:

**Corollary 7.3.** Let \( f_1, f_2 \in C(\Omega) \) be non-negative and let \( u_1, u_2 \in C(\Omega) \) be pluripotential-potential solutions of \( M_C(\cdot) = f_1 \) (resp. \( M_C(\cdot) = f_2 \)). Then

\[
||u_1 - u_2||_{L^\infty(\Omega)} \leq ||u_1 - u_2||_{L^\infty(\partial\Omega)} + C(n)\text{diam}(\Omega)|| (f_1 - f_2) \cdot \chi_{\{u=\Gamma_u\}}||_{L^2(\Omega)}^{1/n}.
\]

**References**

[Bef88] E. Bedford. Survey of pluripotential theory. In J.E. Fornæss, editor, *Several Complex Variables*, Mittag-Leffler Institute, 1987/1988. Princeton University Press.

[BEGZ] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. Monge ampère equations in big cohomology classes. *Acta Math.*

[Bl05] Z. Block. On uniform estimate in calabi-yau theorem. In *Proceedings of SCV2004*, volume 48 of *Science in China Series A*, pages 244–247, Beijing, 2005. Science in China Press.

[Caf89] L.A. Caffarelli. Interior a priori estimates for solutions of fully nonlinear. *Ann. of Math.*, 131(1):189–213, 1989.

[Caf90] L.A. Caffarelli. Interior \( w^{2,p} \) estimates for solutions of the monge-ampère. *Ann. of Math.*, 131(1):135 –150, 1990.

[CC95] L.A. Caffarelli and Cabre. *Fully nonlinear elliptic equations*, volume 43 of *Colloquium Publications*. 1995.

[CL92] M. Crandall, H Ishii, and P.L. Lions. User’s guide to viscosity solutions of second order partial. *Bull. Amer. Math. Soc.*, 27:1–67, 1992.

[CKNS85] L.A. Caffarelli, J.J. Kohn, L. Nirenberg, and J. Spruck. The dirichlet problem for nonlinear second-order elliptic equations. *Comm. of P. and A. Math.*, 38:209–252, 1985.

[CL92] U. Cegrell and P. Lersson. The dirichlet problem for the complex monge-ampère operator: stability in \( P^2 \). *Michigan Math. J.*, 39:145–151, 1992.

[EGZ09] P. Eyssidieux, V. Guedj, and A. Zeriahi. Singular kähler-einstein metrics. *J. Amer. Math. Soc.*, 22(3):607–639, 2009.

[EGZ10] P. Eyssidieux, V. Guedj, and A. Zeriahi. Viscosity solutions to degenerate complex monge-ampère equations. *arXiv:1007.0076v1 (July)*, 2010.

[Gua00] B. Guan. The dirichlet problem for complex monge-ampère equations and. *Comm. in Anal. Geom.*, 8:213–218, 2000.

[Gut01] Cristian.E. Gutiérrez. *The Monge-Ampère Equation*, volume 44 of *Progress in nonlinear differential equations and their applications*. Birkhäuser Boston, 2001.

[HL09] F.R. Harvey and H.B. Lawson. Dirichlet duality and the nonlinear dirichlet problem. *Comm. of P. and A. Math.*, 62:396–443, 2009.

[IL90] H Ishii and P.L. Lions. Viscosity solutions of fully nonlinear second order elliptic partial. *Journ. Diff. Equations*, 83:26 –78, 1990.

[Kol98] S. Kolodziej. The complex monge-ampère equation. *Acta Math.*, 180:69–117, 1998.