Integral inequalities for $s$-convex functions via generalized conformable fractional integral operators

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Abstract

We introduce new operators, the so-called left and right generalized conformable fractional integral operators. By using these operators we establish new Hermite–Hadamard inequalities for $s$-convex functions and products of two $s$-convex functions in the second sense. Also, we obtain two interesting identities for a differentiable function involving a generalized conformable fractional integral operator. By applying these identities we give Hermite–Hadamard and midpoint-type integral inequalities for $s$-convex functions. Different special cases have been identified and some known results are recovered from our general results. These results may motivate further research in different areas of pure and applied sciences.

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1 Introduction

The theory of inequalities is known to play an important role in almost all areas of pure and applied sciences. Richard Bellman stated succinctly, at the Second International Conference on Mathematical Inequalities, Oberwolfach, Germany, July 30–August 5, 1978, that “there are three reasons for the study of inequalities: practical, theoretical, and aesthetic”. In the last few decades the theory of inequalities has attracted the attention of great number of researchers.

We use $I$ to denote an interval in the real line $\mathbb{R} = (-\infty, +\infty)$, and $L[\lambda_1, \lambda_2]$, where $\lambda_1 < \lambda_2$, the set of integrable functions on $[\lambda_1, \lambda_2]$.

Definition 1 A function $h : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if

$$h(x\lambda_1 + (1 - x)\lambda_2) \leq xh(\lambda_1) + (1 - x)h(\lambda_2)$$

for all $\lambda_1, \lambda_2 \in I$ and $x \in [0, 1]$.
The interesting mean type inequality, known as the Hermite–Hadamard inequality for convex functions, is given by the following theorem.

**Theorem 1** Let \( h : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function, and let \( \lambda_1, \lambda_2 \in I \) with \( \lambda_1 < \lambda_2 \). Then

\[
h \left( \frac{\lambda_1 + \lambda_2}{2} \right) \leq \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} h(x) \, dx \leq \frac{h(\lambda_1) + h(\lambda_2)}{2}.
\]

Inequality (2) is also acknowledged as the trapezium inequality.

The trapezium inequality has an extraordinary interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (2) in the premises of newly invented definitions due to the motivation of convex functions. The interested readers can see the references [2–7, 9, 12–15, 18, 21, 22, 24–28, 30–32, 34–37].

**Definition 2** ([13]) A function \( h : [0, +\infty) \rightarrow \mathbb{R} \) is said to be \( s \)-convex \( (s \in (0, 1]) \) in the second sense if

\[
h(a\lambda_1 + (1-a)\lambda_2) \leq a^s h(\lambda_1) + (1-a)^s h(\lambda_2)
\]

for all \( \lambda_1, \lambda_2 \geq 0 \) and \( a \in [0, 1] \).

The \( s \)-convex functions in the second sense are presented in [13]. Also, researchers started to study conformable fractional integrals; see [1, 8, 12, 18, 19, 30, 32]. Khalil et al. [17] defined the fractional integral only of order \( 0 < \alpha \leq 1 \), whereas Abdeljawad [1] generalized the definition of left and right conformable fractional integrals to any order \( \alpha > 0 \). In 2017, Khan et al. implemented this definition by providing a class of Hermite-type inequalities.

**Definition 3** Let \( f : [\lambda_1, +\infty) \rightarrow \mathbb{R}, \xi \in (n, n+1] \), set \( \zeta = \xi - n \). Then the left conformable fractional integral starting at \( \lambda_1 \) is defined by

\[
\left( I_{\lambda_1}^\zeta f \right)(x) = \frac{1}{n!} \int_{\lambda_1}^{x} (x-\theta)^n (\theta - \lambda_1)^{\zeta-1} f(\theta) \, d\theta, \quad x > \lambda_1.
\]

Analogously, if \( f : (-\infty, \lambda_2) \rightarrow \mathbb{R}, \xi \in (n, n+1] \), and \( \zeta = \xi - n \), then the right conformable fractional integral at \( \lambda_2 \) is defined by

\[
\left( I_{\lambda_2}^\zeta f \right)(x) = \frac{1}{n!} \int_{x}^{\lambda_2} (\theta-x)^n (\lambda_2 - \theta)^{\zeta-1} f(\theta) \, d\theta, \quad x < \lambda_2.
\]

Note that if \( \xi = n + 1 \), then \( \zeta = \xi - n = n + 1 - n = 1 \), where \( n = 0, 1, 2, \ldots \).

Set et al. [30] obtained a new generalized class of Hermite–Hadamard-type inequalities for \( s \)-convex functions by applying conformable fractional integrals.

**Theorem 2** ([30]) Let \( f : [\lambda_1, \lambda_2] \rightarrow \mathbb{R} \) be a function with \( 0 \leq \lambda_1 < \lambda_2, s \in (0, 1] \), and \( f \in L[\lambda_1, \lambda_2] \). If \( f \) is an \( s \)-convex function on \( [\lambda_1, \lambda_2] \), then we have the following inequalities for
conformable fractional integrals:

\[
\frac{\Gamma' (\xi - n)}{\Gamma (\xi + 1)} f \left( \frac{\lambda_1 + \lambda_2}{2} \right) \\
\leq \frac{1}{2^\xi (\lambda_2 - \lambda_1)^\xi} \left[ \left( I_{\lambda_1}^{\lambda_2} f \right) (\lambda_2) + \left( I_{\lambda_2}^{\lambda_1} f \right) (\lambda_1) \right] \\
\leq \left[ \frac{\xi (n + s + 1, \xi - n) + \xi (n + 1, \xi - n + s)}{n!} \right] \left[ f(\lambda_1) + f(\lambda_2) \right] \\
\leq \left[ \Phi(\xi) \right] \left[ \xi + s + 1, \xi - n \right] + \left[ \Phi(\xi) \right] \left[ \xi + 1, \xi - n + s \right] \\
\frac{n!}{2^\xi} \left[ f(\lambda_1) + f(\lambda_2) \right]
\]

with \( \xi \in (n, n + 1), n = 0, 1, 2, \ldots \), where \( \Gamma \) is the Euler gamma function.

Sarikaya et al. [28] defined a function \( \Phi : (0, +\infty) \rightarrow [0, +\infty) \) satisfying the following conditions:

\[
\int_0^1 \frac{\Phi(\theta)}{\theta} \, d\theta < +\infty, \quad (4)
\]

\[
\frac{1}{\mathcal{A}} \leq \frac{\Phi(\theta_1)}{\Phi(\theta_2)} \leq \frac{\Phi(\theta_2)}{\Phi(\theta_1)} \text{ for } 1 \leq \frac{\theta_1}{\theta_2} \leq 2, \quad (5)
\]

\[
\Phi(\theta_2) \leq \frac{\Phi(\theta_1)}{\theta_2^\xi} \text{ for } \theta_1 \leq \theta_2, \quad (6)
\]

\[
\left| \frac{\Phi(\theta_2)}{\theta_2^\xi} - \frac{\Phi(\theta_1)}{\theta_1^\xi} \right| \leq \mathcal{C} |\theta_2 - \theta_1| \left( \frac{\Phi(\theta_2)}{\theta_2^\xi} \right) \text{ for } 1 \leq \frac{\theta_1}{\theta_2} \leq 2, \quad (7)
\]

where \( \mathcal{A}, \mathcal{B}, \mathcal{C} > 0 \) are independent of \( \theta_2, \theta_1 > 0 \). If \( \Phi(\theta_1) \theta_2^\xi \) is increasing for some \( \xi \geq 0 \) and \( \frac{\Phi(\theta_2)}{\theta_2^\xi} \) is decreasing for some \( \xi \geq 0 \), then \( \Phi \) satisfies (4)–(7); see [29]. Therefore the left–sided and right–sided generalized integral operators are defined as follows:

\[
\lambda_1 I_{\phi} f(x) = \int_{\lambda_1}^x \Phi(x - \theta) \frac{f(\theta)}{x - \theta} \, d\theta, \quad x > \lambda_1, \quad (8)
\]

\[
\lambda_2 I_{\phi} f(x) = \int_x^{\lambda_2} \Phi(\theta - x) \frac{f(\theta)}{\theta - x} \, d\theta, \quad x < \lambda_2. \quad (9)
\]

The most significant element of generalized integrals is that they produce Riemann–Liouville fractional integrals, \( k \)-Riemann–Liouville fractional integrals, Katugampola fractional integrals, and so on; see [10, 11, 16, 20, 23, 28, 29, 31, 33].

Now we are in position to introduce the following definitions of left and right generalized conformable fractional integral operators of any order \( \xi > 0 \), where \( \Phi : [0, +\infty) \rightarrow [0, +\infty) \) satisfies conditions (4)–(7).

**Definition 4** Let \( \xi \in (n, n + 1] \) and \( \xi = n \), where \( n = 0, 1, 2, 3, \ldots \). The left generalized conformable fractional integral operator starting at \( \lambda_1 \) of order \( \xi > 0 \) is defined by

\[
\lambda_1 T_{\phi} f(x) = \int_{\lambda_1}^x \Phi(x - \theta) \left( \theta - \lambda_1 \right)^{\xi - 1} f(\theta) \, d\theta, \quad x > \lambda_1. \quad (10)
\]
Analogously, the right generalized conformable fractional integral operator of order $\xi > 0$ is defined by
\[
\lambda_2 \mathcal{T}_\sigma f(x) = \int_x^{\lambda_2} \frac{\Phi(\theta - x)(\lambda_2 - \theta)^{\xi-1}f(\theta)}{\theta - x} \, d\theta, \quad x < \lambda_2.
\]

**Remark 1** Taking $\zeta = 1$ in Definition 4, we obtain the generalized fractional integral operators given from (8) and (9). Also, choosing $\Phi(t) = \frac{t^{n+1}}{n!}$ in Definition 4, we get Definition 3.

Motivated by the literature cited, our paper is organized as follows: In Sect. 2, using the new operators, we establish the so-called left and right generalized conformable fractional integral operators, new Hermite–Hadamard inequalities for $s$-convex functions and products of two $s$-convex functions in the second sense. In Sect. 3, we obtain two interesting identities for differentiable function involving generalized conformable fractional integral operator. By applying these identities we give Hermite–Hadamard and midpoint-type integral inequalities for $s$-convex functions. Various particular cases will be identified, and some known results will be recaptured from our general results. In Sect. 4, we give a brief conclusion.

### 2 Hermite–Hadamard inequalities

Throughout this study, let $\xi \in (n, n+1]$ and $\xi = \xi - n$, where $n = 0, 1, 2, 3, \ldots$. Also, for all $\theta \in [0, 1]$, we define
\[
\Omega^\xi_{\phi}(\theta) = \int_0^{\theta} \frac{\Phi(x(\lambda_2 - \lambda_1))}{x} (1 - x)^{\xi-1} \, dx < +\infty, \quad \lambda_1 < \lambda_2,
\]
\[
\Sigma^\xi_{\phi,1}(x, \theta) = \int_0^{\theta} \frac{\Phi(u(x - \lambda_1))}{u} (1 - u)^{\xi-1} \, du < +\infty, \quad x > \lambda_1,
\]
\[
\Sigma^\xi_{\phi,2}(x, \theta) = \int_0^{\theta} \frac{\Phi(u(\lambda_2 - x))}{u} (1 - u)^{\xi-1} \, du < +\infty, \quad x < \lambda_2,
\]
and
\[
\Psi^\xi_{\phi} = \int_{\lambda_1}^{\lambda_2} \frac{\Phi(x - \lambda_1)(\lambda_2 - x)^{\xi-1}}{x - \lambda_1} \, dx = \int_{\lambda_1}^{\lambda_2} \frac{\Phi(\lambda_2 - x)}{\lambda_2 - x} (x - \lambda_1)^{\xi-1} \, dx.
\]

Let represent Hermite–Hadamard inequalities for $s$-convex functions in the second sense via general conformable fractional integral operators as follows.

**Theorem 3** Let $f : [\lambda_1, \lambda_2] \to \mathbb{R}$ be a positive function with $0 \leq \lambda_1 < \lambda_2$ such that $f \in L[\lambda_1, \lambda_2]$. If $f$ is $s$-convex in the second sense on $[\lambda_1, \lambda_2]$, then for any $\xi > 0$ and $s \in (0, 1]$, we have the following inequalities for generalized conformable fractional integral operators:
\[
f\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq \frac{1}{2s\Psi^\xi_{\phi}} \left[\lambda_1 \mathcal{T}_\sigma f(\lambda_2) + \lambda_2 \mathcal{T}_\sigma f(\lambda_1)\right] \leq \left[\frac{f(\lambda_1) + f(\lambda_2)}{2s}\right] \Delta^\xi_{\phi}.
\]
where
\[
\Delta_{\Phi}^{s,c} = \frac{1}{(\lambda_2 - \lambda_1)^{s+1}} \int_{\lambda_1}^{\lambda_2} \frac{\Phi(x - \lambda_1)}{x - \lambda_1} (\lambda_2 - x)^{s-1} \left[ (x - \lambda_1)^\nu + (\lambda_2 - x)^\nu \right] dx,
\]  
(17)
and \( \Psi_{\Phi}^{c} \) is defined by (15).

**Proof** Let \( x, y \in [\lambda_1, \lambda_2] \). Since \( f \) is \( s \)-convex in the second sense on \( [\lambda_1, \lambda_2] \), we have
\[
f\left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2^s}.
\]
Taking \( x = \theta \lambda_1 + (1 - \theta) \lambda_2 \) and \( y = (1 - \theta) \lambda_1 + \theta \lambda_2 \), we get
\[
2^sf\left( \frac{\lambda_1 + \lambda_2}{2} \right) \leq f(\theta \lambda_1 + (1 - \theta) \lambda_2) + f((1 - \theta) \lambda_1 + \theta \lambda_2). \tag{18}
\]
Multiplying both sides of inequality (18) by \( \Phi(\theta(\lambda_2 - \lambda_1)) (1 - \theta)^{\zeta - 1} \) and integrating the resulting inequality with respect to \( \theta \) over \([0, 1]\), we obtain
\[
2^s \Psi_{\Phi}^{c} f\left( \frac{\lambda_1 + \lambda_2}{2} \right) \leq f(\theta \lambda_1 + (1 - \theta) \lambda_2) + f((1 - \theta) \lambda_1 + \theta \lambda_2). \tag{19}
\]
So, we have
\[
2^s \Psi_{\Phi}^{c} f\left( \frac{\lambda_1 + \lambda_2}{2} \right) \leq \left[ \lambda_2^T \Phi f(\lambda_2) + \lambda_1^T \Phi f(\lambda_1) \right], \tag{19}
\]
which means that the left side of (16) is proved. To prove the right side of (16), since \( f \) is \( s \)-convex in the second sense on \( [\lambda_1, \lambda_2] \), we have the inequalities
\[
f\left( \theta \lambda_1 + (1 - \theta) \lambda_2 \right) \leq \theta^s f(\lambda_1) + (1 - \theta)^s f(\lambda_2) \tag{20}
\]
and
\[
f\left( (1 - \theta) \lambda_1 + \theta \lambda_2 \right) \leq (1 - \theta)^s f(\lambda_1) + \theta^s f(\lambda_2). \tag{21}
\]
Adding (20) and (21), we get
\[
f(\theta \lambda_1 + (1 - \theta) \lambda_2) + f((1 - \theta) \lambda_1 + \theta \lambda_2) \leq \left[ \theta^s + (1 - \theta)^s \right] \left[ f(\lambda_1) + f(\lambda_2) \right]. \tag{22}
\]
Multiplying both sides of inequality (22) by \( \Phi(\theta(\lambda_2 - \lambda_1)) (1 - \theta)^{\zeta - 1} \) and integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain
\[
\int_{0}^{1} \frac{\Phi(\theta(\lambda_2 - \lambda_1))}{\theta} (1 - \theta)^{\zeta - 1} f(\theta \lambda_1 + (1 - \theta) \lambda_2) d\theta
\]
\[ + \int_0^1 \frac{\Phi(\theta(\lambda_2 - \lambda_1))}{\theta} (1 - \theta)^{\xi - 1} f((1 - \theta)\lambda_1 + \theta \lambda_2) \, d\theta \]
\[ \leq \left[ f(\lambda_1) + f(\lambda_2) \right] \int_0^1 \frac{\Phi(\theta(\lambda_2 - \lambda_1))}{\theta} (1 - \theta)^{\xi - 1} [\theta^\xi + (1 - \theta)^\xi] \, d\theta. \]

So, we have
\[ \left[ \int_1^2 T_\phi f(\lambda_2) \, d\lambda_2 \right] \left[ \int_1^1 T_\phi f(\lambda_1) \, d\lambda_1 \right] \leq \left[ f(\lambda_1) + f(\lambda_2) \right] \int_0^1 \frac{\Phi(\theta(\lambda_2 - \lambda_1))}{\theta} (1 - \theta)^{\xi - 1} \left[ \theta^{\xi} + (1 - \theta)^{\xi} \right] \, d\theta, \]
which means that the right side of (16) is proved. The proof of Theorem 3 is completed. □

**Corollary 1** Taking \( s = 1 \) in Theorem 3, we get the following inequalities for convex functions via generalized conformable fractional integral operators:
\[ f\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq \frac{1}{2\Psi_\phi}\left[ \int_1^2 T_\phi f(\lambda_2) \, d\lambda_2 \right] \leq \frac{f(\lambda_1) + f(\lambda_2)}{2}. \] (24)

**Corollary 2** Taking \( \Phi(\theta) = \theta^{n+1}/n! \) in Theorem 3, we get Theorem 2.

**Remark 2** Taking \( \xi = s = 1 \) in Corollary 2, we get the well–known Hermite–Hadamard inequality (2).

Let us represent now Hermite–Hadamard inequalities for the product of two \( s \)-convex functions in the second sense via general conformable fractional integral operators.

**Theorem 4** Let \( f, g : [\lambda_1, \lambda_2] \to \mathbb{R} \) be two positive functions with \( 0 \leq \lambda_1 < \lambda_2 \) and \( f \in L[\lambda_1, \lambda_2] \). If \( f \) and \( g \) are \( s \)-convex in the second sense on \([\lambda_1, \lambda_2], \) then for any \( \xi > 0 \) and \( s \in (0, 1], \) we have the following inequalities for generalized conformable fractional integral operators:
\[ f\left(\frac{\lambda_1 + \lambda_2}{2}\right) g\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq \frac{1}{4\Psi_\phi}\left[ \int_1^2 T_\phi f(\lambda_2) g(\lambda_2) + \int_1^2 T_\phi f(\lambda_1) g(\lambda_1) \right] \]
\[ \leq \frac{1}{4\Psi_\phi}\left[ \int_1^2 T_\phi f(\lambda_2) g(\lambda_1) + \int_1^2 T_\phi f(\lambda_1) g(\lambda_2) \right] \]
\[ \leq \frac{1}{4\Psi_\phi}\left[ 2M(\lambda_1, \lambda_2) \Theta_\phi^{\xi} + N(\lambda_1, \lambda_2) \Delta_\phi^{2\xi} \right], \]
(25)
where
\[ M(\lambda_1, \lambda_2) = f(\lambda_1) g(\lambda_1) + f(\lambda_2) g(\lambda_2), \] (26)
\[ N(\lambda_1, \lambda_2) = f(\lambda_1) g(\lambda_2) + f(\lambda_2) g(\lambda_1), \] (27)
\[ \Theta_\phi^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{2\xi}} \int_{\lambda_1}^{\lambda_2} \frac{\Phi(x - \lambda_1)}{x - \lambda_1} (x - \lambda_1)^{\xi - 1} (x - \lambda_1)^\xi (x - \lambda_1)\Phi(\theta) \, dx, \]
(28)
and \( \Psi_\phi^{\xi} \) and \( \Delta_\phi^{2\xi} \) are defined by (15) and (17).
Proof Let \( x, y \in [\lambda_1, \lambda_2] \). Since \( f \) and \( g \) are \( s \)-convex in the second sense on \([\lambda_1, \lambda_2]\), we have
\[
\frac{f\left(\frac{x + y}{2}\right)}{2^s} \leq f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right), \quad \frac{g\left(\frac{x + y}{2}\right)}{2^s} \leq g\left(\frac{x}{2}\right) + g\left(\frac{y}{2}\right).
\]
Taking \( x = \theta \lambda_1 + (1 - \theta) \lambda_2 \) and \( y = (1 - \theta) \lambda_1 + \theta \lambda_2 \), we get
\[
2^s f\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq f(\theta \lambda_1 + (1 - \theta) \lambda_2) + f((1 - \theta) \lambda_1 + \theta \lambda_2) \tag{29}
\]
and
\[
2^s g\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq g(\theta \lambda_1 + (1 - \theta) \lambda_2) + g((1 - \theta) \lambda_1 + \theta \lambda_2). \tag{30}
\]
Multiplying both sides of inequalities (29) and (30), we obtain
\[
4^s f\left(\frac{\lambda_1 + \lambda_2}{2}\right) g\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq f(\theta \lambda_1 + (1 - \theta) \lambda_2) g(\theta \lambda_1 + (1 - \theta) \lambda_2) + f((1 - \theta) \lambda_1 + \theta \lambda_2) g((1 - \theta) \lambda_1 + \theta \lambda_2) + f((1 - \theta) \lambda_1 + \theta \lambda_2) g((1 - \theta) \lambda_1 + \theta \lambda_2). \tag{31}
\]
Multiplying both sides of inequality (31) by \( \frac{\Phi(\theta \lambda_2 - \lambda_1)}{\theta} (1 - \theta)^{\frac{r}{\gamma}} \) and integrating the resulting inequality with respect to \( \theta \) over \([0, 1]\), we have
\[
4^s f\left(\frac{\lambda_1 + \lambda_2}{2}\right) g\left(\frac{\lambda_1 + \lambda_2}{2}\right) \int_0^1 \frac{\Phi(\theta \lambda_2 - \lambda_1)}{\theta} (1 - \theta)^{\frac{r}{\gamma} - 1} d\theta \leq f(\theta \lambda_1 + (1 - \theta) \lambda_2) g(\theta \lambda_1 + (1 - \theta) \lambda_2) \int_0^1 \frac{\Phi(\theta (\lambda_2 - \lambda_1))}{\theta} (1 - \theta)^{\frac{r}{\gamma} - 1} d\theta
+ f((1 - \theta) \lambda_1 + \theta \lambda_2) g((1 - \theta) \lambda_1 + \theta \lambda_2) \int_0^1 \frac{\Phi(\theta (\lambda_2 - \lambda_1))}{\theta} (1 - \theta)^{\frac{r}{\gamma} - 1} d\theta
+ f((1 - \theta) \lambda_1 + \theta \lambda_2) g((1 - \theta) \lambda_1 + \theta \lambda_2) \int_0^1 \frac{\Phi(\theta (\lambda_2 - \lambda_1))}{\theta} (1 - \theta)^{\frac{r}{\gamma} - 1} d\theta.
\]
So, we get
\[
4^s \Psi f\left(\frac{\lambda_1 + \lambda_2}{2}\right) g\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq \left[\int_{\lambda_1}^{\lambda_2} T_{\Phi} f(\lambda_2) g(\lambda_1) + \int_{\lambda_2}^{\lambda_1} T_{\Phi} f(\lambda_1) g(\lambda_2)\right]
+ \left[\int_{\lambda_1}^{\lambda_2} T_{\Phi} f(\lambda_2) g(\lambda_2) + \int_{\lambda_2}^{\lambda_1} T_{\Phi} f(\lambda_1) g(\lambda_1)\right]. \tag{32}
\]
which means that the left side of (25) is proved. To prove the right side of (25), since \( f \) and \( g \) are \( s \)-convex in the second sense on \([\lambda_1, \lambda_2]\), we have the inequalities
\[
f(\theta \lambda_1 + (1 - \theta) \lambda_2) \leq \theta f(\lambda_1) + (1 - \theta) f(\lambda_2), \tag{33}
\]
Applying inequalities (33) to (36), we have

\[
f(1 - \theta)\lambda_1 + \theta\lambda_2 \leq (1 - \theta)f(\lambda_1) + \theta f(\lambda_2),
\]
\[
g(\theta\lambda_1 + (1 - \theta)\lambda_2) \leq \theta^r g(\lambda_1) + (1 - \theta)^r g(\lambda_2)
\]

and

\[
g((1 - \theta)\lambda_1 + \lambda_2) \leq (1 - \theta)^r g(\lambda_1) + \theta^r g(\lambda_2).
\]

Applying inequalities (33) to (36), we have

\[
f(\theta\lambda_1 + (1 - \theta)\lambda_2)g(\theta\lambda_1 + (1 - \theta)\lambda_2) + f((1 - \theta)\lambda_1 + \theta\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2)
\]
\[
+ f(\theta\lambda_1 + (1 - \theta)\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2) + f((1 - \theta)\lambda_1 + \theta\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2)
\]
\[
\leq f(\theta\lambda_1 + (1 - \theta)\lambda_2)g(\theta\lambda_1 + (1 - \theta)\lambda_2) + f((1 - \theta)\lambda_1 + \theta\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2)
\]
\[
+ [\theta^r f(\lambda_1) + (1 - \theta)^r f(\lambda_2)] \cdot [(1 - \theta)^r g(\lambda_1) + \theta^r g(\lambda_2)]
\]
\[
+ [(1 - \theta)^r f(\lambda_1) + \theta^r f(\lambda_2)] \cdot [\theta^r g(\lambda_1) + (1 - \theta)^r g(\lambda_2)]
\]
\[
= f(\theta\lambda_1 + (1 - \theta)\lambda_2)g(\theta\lambda_1 + (1 - \theta)\lambda_2) + f((1 - \theta)\lambda_1 + \theta\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2)
\]
\[
+ 2\theta^r (1 - \theta)^r M(\lambda_1, \lambda_2) + [\theta z_2 + (1 - \theta)^z_2] N(\lambda_1, \lambda_2).
\]

Multiplying both sides of inequality (37) by \(\Phi(\theta)\) and integrating the resulting inequality with respect to \(\theta\) over \([0, 1]\), we obtain

\[
\int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} f(\theta\lambda_1 + (1 - \theta)\lambda_2)g(\theta\lambda_1 + (1 - \theta)\lambda_2) d\theta
\]
\[
+ \int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} f((1 - \theta)\lambda_1 + \theta\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2) d\theta
\]
\[
+ \int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} f(\theta\lambda_1 + (1 - \theta)\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2) d\theta
\]
\[
+ \int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} f((1 - \theta)\lambda_1 + \theta\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2) d\theta
\]
\[
\leq \int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} f(\theta\lambda_1 + (1 - \theta)\lambda_2)g(\theta\lambda_1 + (1 - \theta)\lambda_2) d\theta
\]
\[
+ \int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} f((1 - \theta)\lambda_1 + \theta\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2) d\theta
\]
\[
+ \int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} f(\theta\lambda_1 + (1 - \theta)\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2) d\theta
\]
\[
+ \int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} f((1 - \theta)\lambda_1 + \theta\lambda_2)g((1 - \theta)\lambda_1 + \theta\lambda_2) d\theta
\]
\[
+ 2M(\lambda_1, \lambda_2) \int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} \theta^r (1 - \theta)^s d\theta
\]
\[
+ N(\lambda_1, \lambda_2) \int_0^1 \frac{\Phi(\theta)}{\theta} (1 - \theta)^{\xi - 1} [\theta^z_2 + (1 - \theta)^z_2] d\theta.
\]

So, we get

\[
\left[\lambda_1 T^f(\lambda_2)g(\lambda_2) + \lambda_2 T^f(\lambda_2)g(\lambda_1)\right] + \left[\lambda_1 T^f(\lambda_2)g(\lambda_1) + \lambda_2 T^f(\lambda_1)g(\lambda_2)\right]
\]
\[
\leq \left[\lambda_1 T^f(\lambda_2)g(\lambda_2) + \lambda_2 T^f(\lambda_2)g(\lambda_1)\right] + 2M(\lambda_1, \lambda_2)\theta^r + N(\lambda_1, \lambda_2)\psi_0 \lambda_2^{z_2} + N(\lambda_1, \lambda_2)\psi_1 \lambda_2^{z_1},
\]

(38)
which means that the right side of (25) is proved. The proof of Theorem 4 is completed. □

Corollary 3 Taking \(s = 1\) in Theorem 4, we get the following inequalities for products of two convex functions via generalized conformable fractional integral operators:

\[
\begin{align*}
&f\left(\frac{\lambda_1 + \lambda_2}{2}\right)g\left(\frac{\lambda_1 + \lambda_2}{2}\right) - \frac{1}{4\psi_\phi^\zeta} \left[ \lambda_1 T_\phi f(\lambda_2)g(\lambda_1) + \lambda_2 T_\phi f(\lambda_1)g(\lambda_2) \right] \\
&\leq \frac{1}{4\psi_\phi^\zeta} \left[ \lambda_1 T_\phi f(\lambda_1)g(\lambda_1) + \lambda_2 T_\phi f(\lambda_1)g(\lambda_2) \right] \\
&\leq \frac{1}{4} \left[ 2M(\lambda_1, \lambda_2)\Theta_\phi^\zeta + N(\lambda_1, \lambda_2)\Delta_\phi^{2\zeta} \right], \quad (39)
\end{align*}
\]

where

\[
\Theta_\phi^\zeta = \frac{1}{(\lambda_2 - \lambda_1)^2}\int_{\lambda_1}^{\lambda_2} \Phi(x - \lambda_1)(\xi_2 - x)^\zeta dx
\]

and

\[
\Delta_\phi^{2\zeta} = \frac{1}{(\lambda_2 - \lambda_1)^2}\int_{\lambda_1}^{\lambda_2} \Phi(x - \lambda_1)(\xi_2 - x)^{\zeta-1} [(x - \lambda_1)^2 + (\xi_2 - x)^2] dx.
\]

Corollary 4 Taking \(f = g\) in Theorem 4, we get

\[
\begin{align*}
&f^2\left(\frac{\lambda_1 + \lambda_2}{2}\right) - \frac{1}{4\psi_\phi^\zeta} \left[ \lambda_1 T_\phi f^2(\lambda_2) + \lambda_2 T_\phi f^2(\lambda_1) \right] \\
&\leq \frac{1}{4\psi_\phi^\zeta} \left[ \lambda_1 T_\phi f^2(\lambda_1) + \lambda_2 T_\phi f^2(\lambda_1) \right] \\
&\leq \frac{1}{4} \left[ 2P(\lambda_1, \lambda_2)\Theta_\phi^{2\zeta} + Q(\lambda_1, \lambda_2)\Delta_\phi^{2\zeta} \right], \quad (40)
\end{align*}
\]

where

\[
P(\lambda_1, \lambda_2) = f^2(\lambda_1) + f^2(\lambda_2), \quad Q(\lambda_1, \lambda_2) = 2f(\lambda_1)f(\lambda_2).
\]

3 Some other results

To establish the results of this section regarding general conformable fractional integral operators, we first prove the following two lemmas.

Lemma 3.1 Let \(f : [\lambda_1, \lambda_2] \to \mathbb{R}\) be a differentiable function on \((\lambda_1, \lambda_2)\). If \(f' \in L[\lambda_1, \lambda_2]\), then we have the following identity for generalized conformable fractional integrals:

\[
\left[ f(\lambda_1) + f(\lambda_2) \right] \frac{1}{2} \psi_\phi^{\zeta} - \frac{1}{2} \left[ \lambda_1^\zeta T_\phi f(\lambda_2) + \lambda_2^\zeta T_\phi f(\lambda_1) \right] \\
= \frac{(\lambda_2 - \lambda_1)^{\zeta}}{2} \int_{\theta = 0}^{1} \left[ \Omega_\phi^{\zeta}(1 - \theta) - \Omega_\phi^{\zeta}(\theta) \right] f'(\theta \lambda_1 + (1 - \theta)\lambda_2) d\theta,
\]

where \(\Omega_\phi^{\zeta}(\theta)\) is defined by (12).
Proof We denote
\[ I_{f,Ω^{λ}_{φ}}(λ_{1},λ_{2}) = \frac{1}{2} [ f^{(1)}_{f,Ω^{λ}_{φ}}(λ_{1},λ_{2}) - f^{(2)}_{f,Ω^{λ}_{φ}}(λ_{1},λ_{2}) ], \] (43)
where
\[ f^{(1)}_{f,Ω^{λ}_{φ}}(λ_{1},λ_{2}) = \int_{0}^{1} Ω^{λ}_{φ}(1-θ) f'(θλ_{1} + (1-θ)λ_{2}) dθ \] (44)
and
\[ f^{(2)}_{f,Ω^{λ}_{φ}}(λ_{1},λ_{2}) = \int_{0}^{1} Ω^{λ}_{φ}(θ) f'(θλ_{1} + (1-θ)λ_{2}) dθ. \] (45)
Integrating by parts (44) and changing the variable of integration
\[ x = θλ_{1} + (1-θ)λ_{2}, \]
we get
\[ I^{(1)}_{f,Ω^{λ}_{φ}}(λ_{1},λ_{2}) = Ω^{λ}_{φ}(1-θ)f'(θλ_{1} + (1-θ)λ_{2}) \bigg|_{0}^{1} \]
\[ + \frac{1}{λ_{1} - λ_{2}} \int_{0}^{1} Φ((1-θ)(λ_{2} - λ_{1})) (1 - (1-θ)^{λ_{1} - 1}) f'(θλ_{1} + (1-θ)λ_{2}) dθ \]
\[ = Ψ^{λ}_{φ} f(λ_{2}) - \frac{1}{(λ_{2} - λ_{1})^{λ_{1}}} × λ_{1} T_{Ω^{λ}_{φ}}(λ_{1}). \] (46)
Similarly, using (45), we obtain
\[ I^{(2)}_{f,Ω^{λ}_{φ}}(λ_{1},λ_{2}) = -Ψ^{λ}_{φ} f(λ_{1}) + \frac{1}{(λ_{2} - λ_{1})^{λ_{2}}} × λ_{2} T_{Ω^{λ}_{φ}}(λ_{2}). \] (48)
Substituting (48) and (49) into (43), we obtain the desired equality (41). \(\square\)

Remark 3 Taking \( Φ(θ) = \frac{θ^{λ_{1}+1}}{n!} \) in Lemma 3.1, we get ([30], Lemma 3.1).

Lemma 3.2 Let \( f : [λ_{1},λ_{2}] \rightarrow \mathbb{R} \) be a differentiable function on \( (λ_{1},λ_{2}) \). If \( f' \in L[λ_{1},λ_{2}] \), then we have the following identity for generalized conformable fractional integrals:
\[ f(x)Ψ^{λ}_{φ} - \frac{1}{2} [ x \cdot T_{Ω^{λ}_{φ}}(λ_{2}) + x \cdot T_{Ω^{λ}_{φ}}(λ_{1}) ] \]
\[ = \frac{(x - λ_{1})^{λ_{1}}}{2} \int_{0}^{1} Σ^{λ}_{φ,1}(x,θ) f'(θx + (1-θ)λ_{1}) dθ \]
\[ - \frac{(λ_{2} - x)^{λ_{1}}}{2} \int_{0}^{1} Σ^{λ}_{φ,2}(x,θ) f'(θx + (1-θ)λ_{2}) dθ. \] (50)
where $\Sigma^\xi_{\Phi,1}(x,\theta)$ and $\Sigma^\xi_{\Phi,2}(x,\theta)$ are defined by (13) and (14), respectively. We denote

\begin{align}
I_{f,\Sigma^\xi_{\Phi,1},\Sigma^\xi_{\Phi,2}}(x;\lambda_1,\lambda_2)
&= \frac{(x-\lambda_1)^{\xi}}{2} \int_0^1 \Sigma^\xi_{\Phi,1}(x,\theta) f'(\theta x + (1-\theta)\lambda_1) \, d\theta \\
&- \frac{(\lambda_2-x)^{\xi}}{2} \int_0^1 \Sigma^\xi_{\Phi,2}(x,\theta) f'(\theta x + (1-\theta)\lambda_2) \, d\theta.
\end{align}

(51)

**Proof** See the proof of Lemma 3.1. \qed

**Theorem 5** Let $f : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be a differentiable function on $(\lambda_1, \lambda_2)$. If $f' \in L[\lambda_1, \lambda_2]$ and $|f'|^q$ is $s$-convex in the second sense with $s \in (0,1]$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have the following inequality for generalized conformable fractional integrals:

\begin{equation}
|I_{f,\Omega^\xi_{\Phi}}(\lambda_1, \lambda_2)| \leq \frac{(\lambda_2 - \lambda_1)^{\xi}}{2} \sqrt{\Pi^\xi_{\Phi}(p)} \sqrt{\frac{|f'(\lambda_1)|^q + |f'(\lambda_2)|^q}{s+1}},
\end{equation}

(52)

where

\begin{equation}
\Pi^\xi_{\Phi}(p) = \int_0^{\frac{1}{2}} \left( \int_0^{1-t} \Phi(x(\lambda_2 - \lambda_1)/\lambda_1)(1-x)^{\xi-1} \, dx \right)^p \, d\theta.
\end{equation}

**Proof** By Lemma 3.1, the $s$-convexity in the second sense of $|f'|^q$, the Hölder inequality, and properties of the modulus we have

\begin{align}
|I_{f,\Omega^\xi_{\Phi}}(\lambda_1, \lambda_2)|
&\leq \frac{(\lambda_2 - \lambda_1)^{\xi}}{2} \int_0^1 |\Omega^\xi_{\Phi}(1-\theta) - \Omega^\xi_{\Phi}(\theta)| \left| f'(\theta \lambda_1 + (1-\theta)\lambda_2) \right| \, d\theta \\
&\leq \frac{(\lambda_2 - \lambda_1)^{\xi}}{2} \left( \int_0^1 |\Omega^\xi_{\Phi}(1-\theta) - \Omega^\xi_{\Phi}(\theta)|^p \, d\theta \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(\theta \lambda_1 + (1-\theta)\lambda_2) \right|^q \, d\theta \right)^{\frac{1}{q}} \\
&\leq \frac{(\lambda_2 - \lambda_1)^{\xi}}{2} \sqrt{\Pi^\xi_{\Phi}(p)} \left( \int_0^1 \left[ \theta^q |f'(\lambda_1)|^q + (1-\theta)^q |f'(\lambda_2)|^q \right] \, d\theta \right)^{\frac{1}{q}} \\
&= \frac{(\lambda_2 - \lambda_1)^{\xi}}{2} \sqrt{\Pi^\xi_{\Phi}(p)} \sqrt{\frac{|f'(\lambda_1)|^q + |f'(\lambda_2)|^q}{s+1}}.
\end{align}

The proof of Theorem 5 is completed. \qed

We point out some particular cases of Theorem 5.

**Corollary 5** Taking $\Phi(\theta) = \frac{\theta^{s+1}}{m}$ in Theorem 5, we get ([30], Theorem 3.2).

**Corollary 6** Taking $s = 1$ in Theorem 5, we have the following inequality for convex function via generalized conformable fractional integral operators:

\begin{equation}
|I_{f,\Omega^\xi_{\Phi}}(\lambda_1, \lambda_2)| \leq \frac{(\lambda_2 - \lambda_1)^{\xi}}{2} \sqrt{\Pi^\xi_{\Phi}(p)} \sqrt{\frac{|f'(\lambda_1)|^q + |f'(\lambda_2)|^q}{2}}.
\end{equation}

(53)
Corollary 7 Taking $|f'| \leq K$ in Theorem 5, we obtain

\[
|f_{\Omega_{\phi}}(\lambda_1, \lambda_2)| \leq K \frac{(\lambda_2 - \lambda_1)^{\xi}}{2 \sqrt{s + 1}} \sqrt{s} \frac{1}{\Pi_s(p)}. \tag{54}
\]

Theorem 6 Let $f : [\lambda_1, \lambda_2] \to \mathbb{R}$ be a differentiable function on $(\lambda_1, \lambda_2)$. If $f' \in L[\lambda_1, \lambda_2]$ and $|f'|^q$ is $s$-convex in the second sense with $s \in (0, 1]$, then for $q \geq 1$, we have the following inequality for generalized conformable fractional integrals:

\[
|f_{\Omega_{\phi}}(\lambda_1, \lambda_2)| \leq \frac{(\lambda_2 - \lambda_1)^{\xi}}{2 \sqrt{s + 1}} \left( \left[ L_{\phi, 1}^{\xi} \right]^{1 - \frac{q}{s}} \left[ L_{\phi, 2}^{\xi} + L_{\phi, 3}^{\xi} \right] |f'(\lambda_1)|^q + \left( \Psi_{\phi}^{\xi} - L_{\phi, 4}^{\xi} + L_{\phi, 5}^{\xi} \right) |f'(\lambda_2)|^q \right), \tag{55}
\]

where

\[
L_{\phi, 1}^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \left[ \int_{\lambda_1}^{\lambda_2} \frac{\Phi(\theta - \lambda_1)}{\theta - \lambda_1} \frac{\Phi(\lambda_2 - \theta)\theta^{\xi - 1} d\theta}{(\lambda_2 - \theta)^{\xi - 1}} \right],
\]

\[
L_{\phi, 2}^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \int_{\lambda_1}^{\lambda_2} \frac{\Phi(\theta - \lambda_1)}{\theta - \lambda_1} (\lambda_2 - \theta)^{\xi - 1} d\theta,
\]

\[
L_{\phi, 3}^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \int_{\lambda_1}^{\lambda_2} \Phi(\lambda_2 - \theta) (\theta - \lambda_1)^{\xi - 1} (\lambda_2 - \theta)^{\xi} d\theta,
\]

\[
L_{\phi, 4}^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \int_{\lambda_1}^{\lambda_2} \frac{\Phi(\lambda_2 - \theta)}{\lambda_2 - \theta} (\theta - \lambda_1)^{\xi - 1} d\theta,
\]

\[
L_{\phi, 5}^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \int_{\lambda_1}^{\lambda_2} \Phi(\lambda_2 - \theta) (\theta - \lambda_1)^{\xi - 1} (\lambda_2 - \theta)^{\xi} d\theta,
\]

\[
L_{\phi, 6}^{\xi} = \Psi_{\phi}^{\xi} - \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \left[ \int_{\lambda_1}^{\lambda_2} \frac{\Phi(\theta - \lambda_1)}{\theta - \lambda_1} (\lambda_2 - \theta)^{\xi} d\theta \right.
\]

\[
\left. + \int_{\lambda_1}^{\lambda_2} \frac{\Phi(\lambda_2 - \theta)}{\lambda_2 - \theta} (\theta - \lambda_1)^{\xi - 1} d\theta \right],
\]

\[
L_{\phi, 7}^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \int_{\lambda_1}^{\lambda_2} \Phi(\lambda_2 - \theta) (\theta - \lambda_1)^{\xi - 1} (\lambda_2 - \theta)^{\xi} d\theta,
\]

\[
L_{\phi, 8}^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \int_{\lambda_1}^{\lambda_2} \Phi(\theta - \lambda_1) (\lambda_2 - \theta)^{\xi - 1} (\lambda_2 - \theta)^{\xi} d\theta,
\]

\[
L_{\phi, 9}^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \int_{\lambda_1}^{\lambda_2} \frac{\Phi(\lambda_2 - \theta)}{\lambda_2 - \theta} (\theta - \lambda_1)^{\xi - 1} d\theta,
\]

\[
L_{\phi, 10}^{\xi} = \frac{1}{(\lambda_2 - \lambda_1)^{\xi}} \int_{\lambda_1}^{\lambda_2} \Phi(\theta - \lambda_1) (\lambda_2 - \theta)^{\xi - 1} (\theta - \lambda_1)^{\xi} d\theta,
\]

and $\Psi_{\phi}^{\xi}$ is defined by (15).
Proof By Lemma 3.1, the s-convexity in the second sense of $|f'|^q$, the well-known power mean inequality, and properties of the modulus we have

$$
|L_{\phi}^\xi (\lambda_1, \lambda_2)|
\leq \frac{(\lambda_2 - \lambda_1)^\xi}{2} \int_0^1 |\Omega^\xi_{\phi}(1 - \theta) - \Omega^\xi_{\phi}(\theta)||f'(\theta \lambda_1 + (1 - \theta) \lambda_2)| \, d\theta
$$

where

$$
\leq \frac{(\lambda_2 - \lambda_1)^\xi}{2} \left( \int_0^1 |\Omega^\xi_{\phi}(1 - \theta) - \Omega^\xi_{\phi}(\theta)| \, d\theta \right)^{1 - \frac{1}{q}}
$$

\times \left( \int_0^1 |\Omega^\xi_{\phi}(1 - \theta) - \Omega^\xi_{\phi}(\theta)||f'(\theta \lambda_1 + (1 - \theta) \lambda_2)|^q \, d\theta \right)^{\frac{1}{q}}

\leq \frac{(\lambda_2 - \lambda_1)^\xi}{2} \left( \int_0^1 |f'(\lambda_1) + (1 - \theta)^q f'(\lambda_2)|^q \, d\theta \right)^{\frac{1}{q}}

\times \left( \int_0^1 |\Omega^\xi_{\phi}(\theta)||f'(\lambda_1) + (1 - \theta)^q f'(\lambda_2)|^q \, d\theta \right)^{\frac{1}{q}}

\times \left( \int_0^1 |\Omega^\xi_{\phi}(1 - \theta) - \Omega^\xi_{\phi}(\theta)||f'(\lambda_1) + (1 - \theta)^q f'(\lambda_2)|^q \, d\theta \right)^{\frac{1}{q}}

= \frac{(\lambda_2 - \lambda_1)^\xi}{2 \sqrt{2} \sqrt{3}} \left\{ \left[ L_{\phi,1} \right]^{1 - \frac{1}{q}} \left[ (L_{\phi,2}^\xi + L_{\phi,3}^\xi) \right]^{\frac{1}{q}} \left( \psi^\xi_{\phi} - L_{\phi,4}^\xi - L_{\phi,5}^\xi \right)^{\frac{1}{q}}

+ \left[ L_{\phi,6}^\xi \right]^{1 - \frac{1}{q}} \left[ (\psi^\xi_{\phi} - L_{\phi,7}^\xi - L_{\phi,8}^\xi) \right]^{\frac{1}{q}} \left( L_{\phi,9}^\xi + L_{\phi,10}^\xi \right)^{\frac{1}{q}} \right\}. \tag{56}

The proof of Theorem 6 is completed. \qed

We point out some particular cases of Theorem 6.

Corollary 8 Taking $\Phi(\theta) = \frac{\mu_{n+1}}{m!}$ and $q = 1$ in Theorem 6, we get ([30], Theorem 3.1).

Corollary 9 Taking $s = 1$ in Theorem 6, we get the following inequality for convex function via generalized conformable fractional integral operators:

$$
|L_{\phi}^\xi (\lambda_1, \lambda_2)|
\leq \frac{(\lambda_2 - \lambda_1)^\xi}{2 \sqrt{2} \sqrt{3}} \left\{ \left[ L_{\phi,1} \right]^{1 - \frac{1}{q}} \left[ (L_{\phi,2}^\xi + L_{\phi,3}^\xi) \right]^{\frac{1}{q}} \left( \psi^\xi_{\phi} - L_{\phi,4}^\xi - L_{\phi,5}^\xi \right)^{\frac{1}{q}}

+ \left[ L_{\phi,6}^\xi \right]^{1 - \frac{1}{q}} \left[ (\psi^\xi_{\phi} - L_{\phi,7}^\xi - L_{\phi,8}^\xi) \right]^{\frac{1}{q}} \left( L_{\phi,9}^\xi + L_{\phi,10}^\xi \right)^{\frac{1}{q}} \right\}, \tag{56}

where

$$
L_{\phi,2}^\xi = \frac{1}{(\lambda_2 - \lambda_1)^{\xi+1}} \int_{\theta_1 \xi + \frac{1}{2}}^{\theta_2 \xi + \frac{1}{2}} \Phi(\theta) (\theta - \lambda_1)^\xi \, d\theta,
$$

$$
L_{\phi,3}^\xi = \frac{1}{(\lambda_2 - \lambda_1)^{\xi+1}} \int_{\theta_1 \xi + \frac{1}{2}}^{\theta_2 \xi + \frac{1}{2}} \Phi(\theta) (\theta - \lambda_1)^\xi \, d\theta,
$$

$$
L_{\phi,6}^\xi = \frac{1}{(\lambda_2 - \lambda_1)^{\xi+1}} \int_{\theta_1 \xi + \frac{1}{2}}^{\theta_2 \xi + \frac{1}{2}} \Phi(\theta) (\theta - \lambda_1)^\xi \, d\theta.
$$
The following inequality for generalized conformable fractional integrals

\[ L_{\phi,4}^\xi = \frac{1}{(\lambda_2 - \lambda_1)^{s+1}} \int_{\lambda_1+\frac{\lambda_2}{2}}^{\lambda_2} \Phi(\theta - \lambda_1)(\lambda_2 - \theta)^{s+1}(\theta - \lambda_1) \, d\theta, \]

\[ L_{\phi,5}^\xi = \frac{1}{(\lambda_2 - \lambda_1)^{s+1}} \int_{\lambda_1+\frac{\lambda_2}{2}}^{\lambda_2} \frac{\Phi(\lambda_2 - \theta)}{\lambda_2 - \theta} (\theta - \lambda_1)^{s+1} \, d\theta, \]

\[ L_{\phi,7}^\xi = \frac{1}{(\lambda_2 - \lambda_1)^{s+1}} \int_{\lambda_1}^{\lambda_1+\frac{\lambda_2}{2}} \Phi(\lambda_2 - \theta)(\theta - \lambda_1)^{s+1}(\lambda_2 - \theta) \, d\theta, \]

\[ L_{\phi,8}^\xi = \frac{1}{(\lambda_2 - \lambda_1)^{s+1}} \int_{\lambda_1}^{\lambda_1+\frac{\lambda_2}{2}} \frac{\Phi(\lambda_2 - \theta)}{\theta - \lambda_1} (\theta - \lambda_1)^{s+1} \, d\theta, \]

\[ L_{\phi,9}^\xi = \frac{1}{(\lambda_2 - \lambda_1)^{s+1}} \int_{\lambda_1}^{\lambda_1+\frac{\lambda_2}{2}} \frac{\Phi(\lambda_2 - \theta)}{\lambda_2 - \theta} (\theta - \lambda_1)^{s+1} \, d\theta, \]

and

\[ L_{\phi,10}^\xi = \frac{1}{(\lambda_2 - \lambda_1)^{s+1}} \int_{\lambda_1}^{\lambda_1+\frac{\lambda_2}{2}} \Phi(\theta - \lambda_1)(\lambda_2 - \theta)^{s+1}(\theta - \lambda_1) \, d\theta. \]

**Corollary 10** Taking \(|f'| \leq K\) in Theorem 6, we obtain

\[ |f_{t,\Omega_\phi}(\lambda_1, \lambda_2)| \leq K \frac{(\lambda_2 - \lambda_1)^s}{2 \sqrt[3]{s+1}} \left[ (L_{\phi,1}^\xi)^{1-\frac{s}{2}} (L_{\phi,2}^{\xi} + L_{\phi,3}^{\xi} + (L_{\phi,4}^{\xi} - L_{\phi,5}^{\xi})) + (L_{\phi,6}^{\xi})^{1-\frac{s}{2}} \right] \]

\[ + (L_{\phi,7}^{\xi} + L_{\phi,8}^{\xi} + (L_{\phi,9}^{\xi} + L_{\phi,10}^{\xi})). \]  

(57)

**Theorem 7** Let \( f: [\lambda_1, \lambda_2] \rightarrow \mathbb{R} \) be a differentiable function on \((\lambda_1, \lambda_2)\). If \( f' \in L[\lambda_1, \lambda_2] \) and \(|f'|^q\) is \(s\)-convex in the second sense with \( s \in (0,1) \), then for \( q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), we have the following inequality for generalized conformable fractional integrals:

\[ |f_{t,\Sigma_\phi}(x; \lambda_1, \lambda_2)| \leq \frac{(x - \lambda_1)^s}{2} \sqrt[3]{\Sigma_{\phi,1}(x, p)} \sqrt[3]{\left( |f'(x)|^q + |f'(\lambda_1)|^q \right)^{s+1}} \]

\[ + \frac{(\lambda_2 - x)^s}{2} \sqrt[3]{\Sigma_{\phi,2}(x, p)} \sqrt[3]{\left( |f'(x)|^q + |f'(\lambda_2)|^q \right)^{s+1}}, \]  

(58)

where

\[ \Sigma_{\phi,1}(x, p) = \int_0^1 \left[ \Sigma_{t,1}(x, \theta) \right]^p \, d\theta, \quad \Sigma_{\phi,2}(x, p) = \int_0^1 \left[ \Sigma_{t,2}(x, \theta) \right]^p \, d\theta. \]

**Proof** By Lemma 3.2, the \(s\)-convexity in the second sense of \(|f'|^q\), the H"{o}lder inequality, and properties of the modulus we have

\[ |f_{t,\Sigma_\phi}(x; \lambda_1, \lambda_2)| \leq \frac{(x - \lambda_1)^s}{2} \int_0^1 \left[ \Sigma_{t,1}(x, \theta) |f'(\theta x + (1 - \theta) \lambda_1)| \right] \, d\theta \]
The proof of Theorem 7 is completed. \(\square\)

We point out some particular cases of Theorem 7.

**Corollary 11** Taking \(x = \frac{1 + \xi x}{2}\) in Theorem 7, we get the following midpoint inequality via generalized conformable fractional integral operators:

\[
\begin{align*}
&\left|I_{\mu,\Sigma_1^\mu,\Sigma_2^\mu}(\frac{\lambda_1 + \lambda_2}{2};\lambda_1,\lambda_2)\right| \\
\leq &\frac{(\lambda_2 - \lambda_1)^\xi}{2^{\xi+1} \sqrt{s+1}} \sqrt{\Xi_\phi(p)} \\
&\times \left\{ f'(x)^q + f'(\frac{\lambda_1 + \lambda_2}{2})^q + \sqrt{ \frac{\lambda_1 + \lambda_2}{2} } \right\}, \quad (59)
\end{align*}
\]

where

\[
\Xi_\phi(p) = \int_0^1 \left( \int_0^\theta \frac{\Phi(u(\frac{\lambda_1 + \lambda_2}{2}))}{\mu} (1 - u)^\mu - 1 \right)^p du d\theta.
\]

**Corollary 12** Taking \(s = 1\) in Theorem 7, we have the following inequality for convex function via generalized conformable fractional integral operators:

\[
\begin{align*}
&\left|I_{\mu,\Sigma_1^\mu,\Sigma_2^\mu}(x;\lambda_1,\lambda_2)\right| \\
\leq &\frac{(x - \lambda_1)^\xi}{2} \sqrt{\Xi_\phi_1(x,p)} \sqrt{ f'(x)^q + f'(\lambda_1)^q } \\
&+ \frac{(\lambda_2 - x)^\xi}{2} \sqrt{\Xi_\phi_2(x,p)} \sqrt{ f'(x)^q + f'(\lambda_2)^q }. \quad (60)
\end{align*}
\]
Corollary 13 Taking $|f| \leq K$ in Theorem 7, we obtain

$$
|f, \Sigma_{\phi,1}^{\xi}, \Sigma_{\phi,2}^{\xi} (x; \lambda_1, \lambda_2)| \leq \frac{K}{2} \sqrt[\frac{2}{s+1}]{(x - \lambda_1)^{\xi} \sqrt{\frac{2}{s+1}} \Sigma_{\phi,1}^{\xi} (x, p) + (\lambda_2 - x)^{\xi} \sqrt{\frac{2}{s+1}} \Sigma_{\phi,2}^{\xi} (x, p)}. 
$$

(61)

Theorem 8 Let $f : [\lambda_1, \lambda_2] \to \mathbb{R}$ be a differentiable function on $(\lambda_1, \lambda_2)$. If $f' \in L[\lambda_1, \lambda_2]$ and $|f'|^{q \xi}$ is $s$-convex in the second sense with $s \in (0, 1]$, then for $q \geq 1$, we have following inequality for generalized conformable fractional integrals:

$$
|f, \Sigma_{\phi,1}^{\xi}, \Sigma_{\phi,2}^{\xi} (x; \lambda_1, \lambda_2)| \\
\leq \frac{(x - \lambda_1)^{\xi}}{2} \left[ M_{\phi,1}^{\xi} (x) \right]^{1 - \frac{1}{q \xi}} \sqrt{\frac{2}{s+1}} M_{\phi,2}^{\xi} (x) |f'(x)|^{q} + M_{\phi,1}^{\xi} (x) \left| f'(\lambda_1) \right|^{q} \\
+ \frac{(\lambda_2 - x)^{\xi}}{2} \left[ M_{\phi,2}^{\xi} (x) \right]^{1 - \frac{1}{q \xi}} \sqrt{\frac{2}{s+1}} M_{\phi,2}^{\xi} (x) |f'(x)|^{q} + M_{\phi,2}^{\xi} (x) \left| f'(\lambda_2) \right|^{q}, 
$$

(62)

where

$$
M_{\phi,1}^{\xi} (x) = \frac{\Psi_{\phi}^{\xi}}{(x - \lambda_1)^{\xi - 1}} - \frac{1}{(x - \lambda_1)^{\xi}} \int_{\lambda_1}^{x} \Phi (\theta - \lambda_1) (x - \theta)^{\xi - 1} d\theta,
$$

$$
M_{\phi,2}^{\xi} (x) = \frac{\Psi_{\phi}^{\xi}}{(s + 1)(x - \lambda_1)^{\xi - 1}} - \frac{1}{(s + 1)(x - \lambda_1)^{\xi}} \int_{\lambda_1}^{x} \Phi (\theta - \lambda_1) (\theta - \lambda_1)^{\xi - 1} d\theta,
$$

$$
M_{\phi,3}^{\xi} (x) = \frac{1}{(s + 1)(x - \lambda_1)^{\xi}} \int_{\lambda_1}^{x} \Phi (\theta - \lambda_1) (x - \theta)^{\xi - 1} d\theta,
$$

$$
M_{\phi,4}^{\xi} (x) = \frac{\Psi_{\phi}^{\xi}}{(\lambda_2 - x)^{\xi - 1}} - \frac{1}{(\lambda_2 - x)^{\xi}} \int_{x}^{\lambda_2} \Phi (\lambda_2 - \theta) (\theta - x)^{\xi - 1} d\theta,
$$

$$
M_{\phi,5}^{\xi} (x) = \frac{\Psi_{\phi}^{\xi}}{(s + 1)(\lambda_2 - x)^{\xi - 1}} - \frac{1}{(s + 1)(\lambda_2 - x)^{\xi}} \int_{x}^{\lambda_2} \Phi (\theta - x) (\theta - x)^{\xi - 1} d\theta,
$$

$$
M_{\phi,6}^{\xi} (x) = \frac{1}{(s + 1)(\lambda_2 - x)^{\xi}} \int_{x}^{\lambda_2} \Phi (\theta - x) (\lambda_2 - \theta)^{\xi - 1} d\theta,
$$

and $\Psi_{\phi}^{\xi}$ is defined from (15).

Proof By Lemma 3.2, the $s$-convexity in the second sense of $|f'|^{q \xi}$, the well–known power mean inequality, and properties of the modulus we have

$$
|f, \Sigma_{\phi,1}^{\xi}, \Sigma_{\phi,2}^{\xi} (x; \lambda_1, \lambda_2)| \\
\leq \frac{(x - \lambda_1)^{\xi}}{2} \left( \int_{0}^{1} \Sigma_{\phi,1}^{\xi} (x, \theta) d\theta \right)^{1 - \frac{1}{q \xi}} \left( \int_{0}^{1} \Sigma_{\phi,2}^{\xi} (x, \theta) |f'(\theta x + (1 - \theta) \lambda_1)|^{q} d\theta \right)^{\frac{1}{q}} \\
+ \frac{(\lambda_2 - x)^{\xi}}{2} \left( \int_{0}^{1} \Sigma_{\phi,2}^{\xi} (x, \theta) d\theta \right)^{1 - \frac{1}{q \xi}} \left( \int_{0}^{1} \Sigma_{\phi,2}^{\xi} (x, \theta) |f'(\theta x + (1 - \theta) \lambda_2)|^{q} d\theta \right)^{\frac{1}{q}}.
$$
The proof of Theorem 8 is completed. \(\Box\)

We point out some particular cases of Theorem 8.

**Corollary 14** Taking \(x = \frac{\lambda_1 + \lambda_2}{2}\) in Theorem 8, we get the following midpoint inequality via generalized conformable fractional integral operators:

\[
\left| I_{f, \Sigma^{c,2}_{\phi_1}, \Sigma^{c,2}_{\phi_2}} \left( \frac{\lambda_1 + \lambda_2}{2} ; \lambda_1, \lambda_2 \right) \right| \\
\leq \frac{(\lambda_2 - \lambda_1)^c}{2^{c+1}} \\
\times \left\{ \left[ M_{\phi,1}^{c} \left( \frac{\lambda_1 + \lambda_2}{2} \right) \right]^{1 - \frac{1}{q}} \\
\times \sqrt[\frac{1}{q}]{M_{\phi,2}^{c} \left( \frac{\lambda_1 + \lambda_2}{2} \right) \frac{f \left( \frac{\lambda_1 + \lambda_2}{2} \right)^q}{M_{\phi,2}^{c} \left( \frac{\lambda_1 + \lambda_2}{2} \right) \frac{f \left( \lambda_1 \right)^q}{M_{\phi,2}^{c} \left( \frac{\lambda_1 + \lambda_2}{2} \right) \frac{f \left( \lambda_2 \right)^q}{M_{\phi,2}^{c} \left( \frac{\lambda_1 + \lambda_2}{2} \right)}}} + M_{\phi,3}^{c} \left( \frac{\lambda_1 + \lambda_2}{2} \right) \frac{f \left( \lambda_1 \right)^q}{M_{\phi,3}^{c} \left( \frac{\lambda_1 + \lambda_2}{2} \right) \frac{f \left( \lambda_2 \right)^q}{M_{\phi,3}^{c} \left( \frac{\lambda_1 + \lambda_2}{2} \right)}} \right) \right\}.
\]

**Corollary 15** Taking \(s = 1\) in Theorem 8, we have the following inequality for convex functions via generalized conformable fractional integral operators:

\[
\left| I_{f, \Sigma^{c,2}_{\phi_1}, \Sigma^{c,2}_{\phi_2}} (x ; \lambda_1, \lambda_2) \right| \\
\leq \frac{(x - \lambda_1)^c}{2} \left[ M_{\phi,1}^{c} (x) \right]^{1 - \frac{1}{q}} \sqrt[\frac{1}{q}]{M_{\phi,2}^{c} (x) \frac{f (x)^q}{M_{\phi,2}^{c} (x) \frac{f (\lambda_1)^q}{M_{\phi,2}^{c} (x) \frac{f (\lambda_2)^q}{M_{\phi,2}^{c} (x)}}} + M_{\phi,3}^{c} (x) \frac{f (\lambda_1)^q}{M_{\phi,3}^{c} (x) \frac{f (\lambda_2)^q}{M_{\phi,3}^{c} (x)}}},
\]

where

\[
M_{\phi,1}^{c} (x) = \frac{\Psi_{\phi}^{c}}{2(x - \lambda_1)^{c-1}} - \frac{1}{2(x - \lambda_1)^{c+1}} \int_{\lambda_1}^{x} \Phi(\theta - \lambda_1)(\theta - \lambda_1)(x - \theta)^{c-1} d\theta, \\
M_{\phi,2}^{c} (x) = \frac{1}{2(x - \lambda_1)^{c+1}} \int_{\lambda_1}^{x} \Phi(\theta - \lambda_1)(x - \theta)^{c+1} d\theta,
\]
\[
M_{\phi,3}(x) = \frac{\Phi^\xi}{2(\lambda_2 - x)^\xi - 1} - \frac{1}{2(\lambda_2 - x)^\xi + 1} \int_x^{\lambda_2} \Phi(\theta - x)(\theta - x)(\lambda_2 - \theta)^\xi - 1 d\theta,
\]

and
\[
M_{\phi,4}(x) = \frac{1}{2(\lambda_2 - x)^\xi + 1} \int_x^{\lambda_2} \Phi(\theta - x) \frac{\theta - x}{(\lambda_2 - \theta)^\xi + 1} d\theta.
\]

**Corollary 16** Taking \(|f'| \leq K\) in Theorem 8, we obtain
\[
|\int_{F_{\phi,1}^{\xi} \cdot F_{\phi,2}^{\xi}} (x; \lambda_1, \lambda_2) |
\leq \frac{K}{2 \xi} \left[ (x - \lambda_1)^{1 - \frac{1}{\xi}} \sqrt{M_{\phi,1}(x) + M_{\phi,3}(x)} 
+ (\lambda_2 - x)^{1 - \frac{1}{\xi}} \sqrt{M_{\phi,2}(x) + M_{\phi,4}(x)} \right]. \quad (65)
\]

**Remark 4** Applying our Theorems 5, 6, 7, and 8 to suitable functions \(\Phi(\theta) = \theta, \theta \xi \Gamma(\xi)，\theta \xi \exp(-A\theta), \) where \(A = \frac{1 - \xi}{\xi}\) for \(\xi \in (0, 1),\) we can construct some new generalized conformable fractional integral inequalities. Also, we can obtain some new integral inequalities using special means (arithmetic, geometric, logarithmic, etc.). Finally, some new bounds for the midpoint and trapezium quadrature formula using our results can be provided as well. We omit their proofs, and the details are left to the interested readers.

**4 Conclusion**

Trapezium-type integral inequalities for functions of diverse natures are useful in numerical computations. Using the generalized conformable fractional integral operators defined in our paper, the interested reader can obtain in a similar way new results for different operators, such as \(k\)-Riemann–Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, \((p, q)\)-quantum calculus, Hadamard fractional integrals, and so on. These results can be applied in convex analysis, optimization, probability, and also different areas of pure and applied sciences. The ideas and techniques of this paper may stimulate further research in the fascinating field of integral inequalities.

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**Authors’ contributions**

The authors have worked equally when writing this paper. All authors read and approved the final manuscript.

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