Surface holonomy and gauge 2-group

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Just as point objects are parallel transported along curves, giving holonomies, string-like objects are parallel transported along surfaces, giving surface holonomies. Composition of these surfaces correspond to products in a category theoretic generalization of the gauge group, called a 2-group. I consider two different ways of constructing surface holonomies, one by using a pair of one and two form connections, and another by using a pair of one-form connections. Both procedures result in the structure of a 2-group.

I. INTRODUCTION

The notion of parallel transport plays an important role in physics. Elementary particles and the fields whose point-like excitations describe them, may be thought of as carrying representations of Lie algebras. To describe the dynamics, vector fields at nearby points must be compared, and parallel transport provides a self-consistent way of doing so. When the particles are charged, parallel transport requires the existence of a connection on a fiber bundle constructed on space-time. This connection is called a gauge field, and all interactions of particles can be described by these gauge fields. The dynamics of gauge fields are described by nonlinear (Yang-Mills) theories, and it is expected that in the strong coupling regime there are extended objects, such as strings, or flux tubes in the theory. So it is worthwhile to consider parallel transport of string-like objects.

There is an immediate obstacle to parallel transporting ‘charged’ flux tubes. For particles, parallel transport involves carrying a vector (in some Lie algebra) along a specified curve in space-time. Consider an infinitesimal curve of length $\epsilon$ and tangent $\tau^\mu$. Then parallel transport is a group action, where the vector is acted upon by a group element of the form $g(\epsilon, \tau, A) \sim 1 + \epsilon \tau^\mu A_\mu$, where $A_\mu$ is the connection. Now, curves can be joined end to end, infinitesimal curves can be joined to produce a finite curve. Joining curves is the same as composition of the corresponding group elements, and thus it becomes possible to define the parallel transport of a vector along any finite curve in terms of group elements. So given a starting point and a connection, a curve can be uniquely identified with a group element by taking the path ordered exponential of the gauge field $A$ along that curve. This identification is called the holonomy of the curve.

This procedure cannot be generalized to extended objects except trivially. That is, suppose I could associate an element of some group to the infinitesimal parallel transport of a string, a sort of ‘surface holonomy’. For simplicity, let me take the string to be infinitesimal as well. Now let me try to construct the parallel transport of a finite string between two finitely separated configurations, say with the same end points. I should break up a surface bounded by these two configurations into infinitesimal surface elements, bounded by infinitesimal portions of ‘intermediate’ configurations of the string. Since I know the surface holonomy of each little area element, can I compose them to get the holonomy for the entire surface? The answer is No, for the simple reason that there is no canonical way of ordering surfaces. So the infinitesimal areas may be composed in any order one likes, and each different ordering will give a different holonomy for the whole surface. Unless, that is, the holonomy for infinitesimal surfaces is either trivial (identity) or Abelian.

One way of defining a surface holonomy is to equate it with the holonomy of a closed loop around the surface, i.e. by taking the path ordered exponential of a gauge field along the boundary of the surface element. If the gauge field is non-Abelian, it follows that surface holonomy defined this way is well defined only if the gauge field is flat, i.e. has vanishing curvature. The question that remains is whether it is possible to give an alternative definition of the surface...
holonomy such that it can vanish without forcing the gauge field to be flat. However there is still the problem that
two infinitesimal squares can be composed in two ways, along a common edge, and at a common vertex. So it seems
clear that holonomy for surfaces cannot be thought of as elements of a group. It turns out that categorical Lie groups,
or Lie 2-groups, which naturally have two types of group composition rules, provide an appropriate description of
surface holonomy. In this paper I discuss and relate different approaches to defining a surface holonomy in terms of
Lie 2-groups.

In [11] the definition of a Lie 2-Group, or a categorical Lie group, is given following [2]. Two dimensional parallel
transport will be described by these structures. One-dimensional flux tubes are transported along two-dimensional
surfaces, whose geometric composition corresponds to composition of elements in a Lie 2-group.

In [111] the construction of surface holonomy is briefly described following [3]. Separate group elements, or
holonomies, are associated to the face and edges of an infinitesimal surface element. The edge holonomy requires
a gauge field, a one-form $A$ valued in a Lie algebra, while the face holonomy requires a two-form $B$ valued in another
Lie algebra. The face and edge holonomies are thus elements of two different groups, and combine according to
the composition laws of the 2-group when the surfaces are composed. The total holonomy of the surface vanishes,
$F + B = 0$.

In [11V] a different approach is employed. Instead of thinking in terms of parallel transporting infinitesimal strings,
fields are transported along strings as well as along paths between nearby string configurations. In addition to
the connection field which transports along a string configuration, a second one-form connection $\bar{A}$ is introduced,
rather than a two-form, for transporting between nearby configurations. Demanding that such parallel transport be
unambiguous leads to an integrability condition involving both $A$ and $\bar{A}$. Allowing $\bar{A}$ to take any value subject to
this constraint produces an effective field theory of two-forms.

The paper ends with some discussions in [11V] about the meaning of surface holonomy and related constructions,
given that category theory does not magically create a canonical ordering for surfaces.

II. LIE 2-GROUP

Parallel transports of fields defined at a point are described by groups. It is natural to expect that parallel transport
of objects defined on curves or 'strings' will require two groups, one for comparing the values of a field at nearby
points of the string, and another group for comparisons between nearby string configurations. It turns out that the
two groups combine to form a categorical group, or a Lie 2-group.

A category consists of a set $O$ of objects and a set $\{M_{BA}\}$ of morphisms from the object $A$ to the object $B$ for
all $A, B \in O$, and a composition rule $\circ$ on the set $\{M_{BA}\}$ such that $\mu_{CB} \circ \mu_{BA} \in M_{CA}$ for all $\mu_{BA} \in M_{BA}$ and
$\mu_{CB} \in M_{CB}$, and the following two conditions hold.

- Composition is associative, i.e.,
  \[
  \mu_{DC} \circ (\mu_{CB} \circ \mu_{BA}) = (\mu_{DC} \circ \mu_{CB}) \circ \mu_{BA}.
  \]  \tag{2.1}

- Identities exist, i.e., for each object $A$ there is a morphism $\iota_A$ from $A$ to $A$ such that
  \[
  \mu_{BA} \circ \iota_A = \mu_{BA}, \quad \iota_A \circ \mu_{AC} = \mu_{AC}
  \]  \tag{2.2}

Clearly, a category is a generalization of the concept of a group, rather a monoid, with objects replacing elements and
morphisms replacing maps.

A Lie 2-group is an example of a category in which the set of objects and the set of morphisms are both Lie groups,
and the composition of morphisms is a homomorphism. A trivial example of a Lie 2-group is a Lie group $G$, whose
elements are now the objects, so that $O = G$, and each morphism takes one element to another $\tilde{3}$,
\[
\mu_{21} = g_2(g_1)^{-1}, \quad \mu_{21}(g_1) = g_2.
\]  \tag{2.3}

It is easy to see that in general the Lie 2-group has two types of composition rules. Let the objects be elements of
$G$ and let the morphisms be elements of some other Lie group $\widehat{G}$. Then each morphism is some $\hat{g}(g_2, g_1)$ which takes
$g_1$ to $g_2 \in G$. Since this is a category, a morphism which takes $g_1$ to $g_2$ can be composed with another which takes
$g_2$ to $g_3$, and the resulting morphism takes $g_1$ to $g_3$,
\[
\hat{g}(g_3, g_2) \circ \hat{g}(g_2, g_1) = \hat{g}(g_3, g_1).
\]  \tag{2.4}
This composition rule makes no reference to the fact that $O = G$ is a group.

On the other hand, consider morphisms $\tilde{g}(g_4, g_3)$ and $\tilde{g}(g_2, g_1)$ between different elements of $G$. Now the idea is that the composition $\tilde{g}(g_4, g_3) \cdot \tilde{g}(g_2, g_1)$ in $\tilde{G}$ should be a morphism which takes $g_1g_1$ to $g_4g_2$, and this morphism should be a function of $g_1, g_2, g_3, g_4$. This idea can be easily implemented if $\tilde{G}$ is the semi-direct product of $G$ with some other group $H$, i.e., the morphisms are elements of $H \rtimes G$, and the action of the morphism on elements of $G$ is a homomorphism from $H$ to $G$.

Recall that $H \rtimes G$ consists of groups $G$ and $H$, along with an action of $G$ on $H$ given by $\alpha(g)(h)$, and the composition rule is

$$(h, g) \cdot (h', g') = (h\alpha(g)(h'), gg').$$

(2.5)

Note that $\alpha$ is a homomorphism from $G$ to $\text{Aut}(H)$, the group of automorphisms of $H$, and therefore

$$\alpha(g)(\alpha(g')(h)) = \alpha(gg')(h),$$

$$\alpha(g)(h) \alpha(g')(h') = \alpha(g)(hh').$$

(2.6)

Then the Lie 2-group thus defined consists of a pair of Lie groups $G$ and $H$, a homomorphism $\alpha : G \to \text{Aut}(H)$, and two types of composition rules: $\cdot$, which is the composition in the semi-direct product $H \rtimes G$ as given in Eq. (2.5); and $\circ$, which is the obvious ‘composition of morphisms’

$$(h, g) \circ (\tilde{h}, \tilde{g}) = (hh, g).$$

(2.7)

The order of the objects on the left hand side is a matter of convention. Often the homomorphism $t : H \to G$ is made explicit so that the result of applying the morphism $(h, g)$ on $g$ is written as

$$(h, g) : g \mapsto t(h)g.$$ 

(2.8)

I will usually keep the homomorphism implicit and write $t(h)g$ simply as $hg$. Note also that the composition of morphisms in Eq. (2.5) is defined only if the two elements $(h, g)$ and $(\tilde{h}, \tilde{g})$ are composable, meaning $\tilde{g} = hg$, so that the second morphism can act on the result of the first one. An important property of a 2-group, which can be checked directly, is that an exchange law is satisfied,

$$(f_1 \cdot f_2) \circ (f_3 \cdot f_4) = (f_1 \circ f_3) \cdot (f_2 \circ f_4),$$

(2.9)

where $f_1 = (h_1, g_1)$, etc.

Many of the interesting examples of Lie 2-groups are for the case where the automorphism $\alpha(g)(h)$ can be written as $ghg^{-1}$. For convenience of calculation, I will consider only these cases below, and write the composition of the semi-direct product accordingly as

$$(h, g) \cdot (h', g') = (hgh'g^{-1}, gg').$$

(2.10)

Finally, a Lie 2-group is equivalent to what is called a Lie crossed module [2], and in what follows the two phrases can be used interchangeably.

### III. SURFACE HOLOMONY

Usual gauge theories are theories of point particles, which are described by fields valued in the Lie algebra of $G$. The connection one-form or gauge field $A$ parallel transports a field along infinitesimal paths. This means that the field changes by the action of a group element equal to the path ordered exponential of $A$ along a continuous curve. For parallel transport of a string the corresponding object should be an element of a Lie 2-group. Such an object can be defned directly [2][3][4], which I now proceed to describe.

Let me start with an infinitesimal string. Consider parallel transporting this string infinitesimally, keeping the end points fixed. This results in the pair of configurations schematically drawn in Fig. 1(a), which will be called a bi-gon. This object can be associated with an element $(h, g)$ of a Lie 2-group by first thinking of a string configuration in terms of its associated holonomy, i.e., as an element $g \in G$. Then parallel transporting the string can be thought of as a morphism, so that an element $h \in H$ needs to be associated with the surface element bounded by the two configurations.

For concreteness, let me think of the top edge as the ‘initial’ configuration, which is then parallel transported (morphed) to the bottom or final configuration, using an element $h \in H$. Similarly, when constructing holonomies
$g \in G$ for the edges, I will think of the edges as being directed from left to right. The morphism from the top edge to the bottom edge is a homomorphism $t : H \to G$, so that its action on the holonomy of the top edge can be written as $t(h)g$, or as $hg$ as mentioned earlier. Such a bi-gon will be termed as ‘carrying’ $(h, g)$. In general, the holonomy along the bottom edge of a bi-gon need not be the same as the parallel transported holonomy of the top edge, but two bi-gons can be composed along a common edge only if these two things are in fact equal, as explained below.

There are two types of compositions for bi-gons, the horizontal and vertical compositions, shown in Fig. 1(b) and Fig. 1(c), respectively. For horizontal composition in Fig. 1(b), going along the top edges I should find a composition of the corresponding edge holonomies in $G$. Similarly going along the bottom edges gives a composition of the morphed holonomies. There should be a corresponding morphism, which takes the top product to the bottom product, and this should be made of the two individual morphisms. For the vertical composition, there is a crucial condition. The holonomy of the bottom edge of the upper bi-gon in Fig. 1(c), which results from morphing the top edge holonomy, must be the same as the top edge holonomy of the lower bi-gon. That is, two bi-gons carrying $(h, g)$ and $(\bar{h}, \bar{g})$ can be composed as in Fig. 1(c) only if $\bar{g} = t(h)g$. Otherwise the composition of the two bi-gons cannot make sense.

It should be now quite obvious how to relate the bi-gons to Lie 2-groups. A bi-gon carrying $(h, g)$ is to be identified with the element $(h, g)$ of a Lie 2-group, in the same sense a curve can be identified with its holonomy which lives in some group $G$. Horizontal composition, as in Fig. 1(b), is to be thought of as the product of morphisms given by the composition rule of the semi-direct product of $H \times G$ as in Eq. (2.10). Vertical composition of bi-gons as in Fig. 1(c), whenever composable, is to be thought of as a composition of morphisms, i.e. as given in Eq. (2.11). The exchange law ensures that (composable) bi-gons may be composed in any order with the same result. Thus the holonomy for a finite surface, between two configurations of a finite string with the same end points, may be computed by breaking up the surface into infinitesimal bi-gons and composing their surface holonomies as elements of a Lie 2-group.

This is an obvious generalization to categories of composing holonomies along infinitesimal line elements to get the holonomy of a finite curve. In the latter case the holonomy is the path ordered exponential of one-form connection or gauge field $A$, valued in the Lie algebra of some group. The infinitesimal holonomy along a curve of length $\epsilon$ and tangent $\tau^\mu$ is $1 + \epsilon \tau^\mu A_\mu$. To generalize this to bi-gons, two objects are needed, one for the holonomy along an edge, another for the morphism between edges. So let me introduce a one-form gauge field $A$ valued in the Lie algebra of $G$ in order to compute the holonomy along an edge. In addition, let me also introduce a two-form field $B$ valued in the Lie algebra of $H$.

Then the holonomy along an infinitesimal edge is of the form $g \sim 1 + \epsilon \tau^\mu A_\mu \sim 1 + \int A$ as before. I have written an integral because it does not make sense to represent the edges of bi-gons by tangent vectors. In fact, calculations become easier if a bi-gon is replaced by a triangle with the base being identified as the lower edge. Then the ‘integral’ is the sum of $\epsilon \tau^\mu A_\mu$ terms. Now there is also a contribution from the infinitesimal surface, of the form $h \sim 1 + a^2 \sigma^{\mu \nu} B_{\mu \nu}$. Here $\sigma^{\mu \nu}$ is the tensor characterizing the surface, and $a^2 \sim O(\epsilon^2)$ is its area. Let me write this holonomy as $1 + \int B$ in analogy with the one-form. If $g$ belongs to the top edge of the bi-gon, the morphism to the bottom edge takes $g$ to

$$h g \sim (1 + \int B)(1 + \int A) \sim 1 + \int B + \int A. \quad (3.1)$$

Since each ‘integral’ is actually an infinitesimal itself, their product can be ignored to the order of the area. Further, $A$ and $B$ do not belong to the same Lie algebra, but the homomorphism $t$ induces an obvious map so that $B$ can be brought to the same space as $A$ and added. Just as curves are identified with group elements via a gauge field, this completes the identification of bi-gons with elements of a Lie 2-group, via a pair of ‘connection’ fields $(A, B)$.
IV. TWO CONNECTIONS FOR 2-GROUP

One can take an alternative approach to constructing a surface holonomy, perhaps somewhat closer in philosophy to quantum field theories. In this approach, briefly described earlier, one introduces two one-form gauge fields, valued in the Lie algebras of two groups $G$ and $H$, rather than a one-form and a two-form. Then instead of surface holonomy, one considers the holonomy between identified points on nearby string configurations. Any surface can be decomposed into a sum of infinitesimal squares, and thus the result of parallel transporting a field along an arbitrary path on any surface is unambiguous if and only if a certain integrability condition holds. Further, it is possible to think of a field theoretic action on which this condition is imposed as a constraint. The corresponding Lagrange multiplier field is a two-form in four dimensions (a $(D-2)$-form in $D$ dimensions), leading to the usual gauge theories of a non-Abelian two-form field. The composition of parallel transports around squares again follow the structure of a Lie 2-group.

In this section parallel transport will always mean that of some field along a curve. Consider an infinitesimal piece of a string, or flux tube, and another one infinitesimally close to the first one. These are the string configurations. The pieces are directed, and there is a notion of going continuously from one to the other. This produces the picture of a square, as in Fig. 2(a). In this square, the top and the bottom edges belong to different string configurations.

![Diagram](image)

**FIG. 2:** (a) A square with two connections, (b) horizontal composition, (c) vertical composition.

For smooth string configurations, it is possible to unambiguously define vectors along the string and normal to the string, so squares as these can always be drawn in such cases.

Consider fields living on the string. Parallel transport is always integrable in one dimension, so knowledge of the field at any point on the string determines the field at any other point. Thus given a connection on the string, any field is well defined at every point of the string, and can be calculated in terms of its value at some ‘zero point’ on the string. Now consider an infinitesimally close string configuration. Again parallel transport along the string determines the field at any point of the string, in terms of its value at the transported zero point. But now both the field and the connection (to be used along the string) must have been parallel transported to the new configuration from the previous one. This parallel transport between configurations could have been done by any connection, not necessarily the one transporting fields along the string.

So let $A$ transport fields along the string, and let $\bar{A}$ transport normal to the string. Also let $A$ and $\bar{A}$ belong to the Lie algebras of $G$ and $H$, respectively. Note that $\bar{A}$ can live in a subalgebra of the Lie algebra to which $A$ belongs, but not the other way around, for fields living on the string must remain in the same algebra as $A$ for all configurations.

Let me assume for the moment that the two Lie algebras are in fact isomorphic, so that I can write $\bar{A} = A + V$ , treating the isomorphism as an equality. Then in Fig. 2(a) a field can be parallel transported form vertex 1 to vertex 4 either along the bottom and right edges, or along the left and top edges. The results of parallel transporting a field using these two routes around the infinitesimal square will be the same if

$$F_{\mu\nu} + \frac{1}{2} \left( \partial_{[\mu} V_{\nu]} + [A_{[\mu}, V_{\nu]} \right) = 0. \quad (4.1)$$

If I used the same connection $A$ for all sides of the square, I would have found a condition of vanishing curvature, $F = 0$. Conditions of this type are referred to as integrability conditions in the literature. When a connection
satisfies it, the parallel transport of a field between two points can be ‘integrated’ along any path connecting the two points, leading to a unique definition of the field at each point.

The integrability condition of Eq. (4.41) should be interpreted in a similar fashion. Any field is completely determined at all points of the string by parallel transport. If the string is moved to a nearby configuration, the field can be calculated at every point in the new configuration, without regard to how intermediate configurations were traversed, provided the integrability condition holds. Then transporting fields around squares is unambiguous. And any surface can be broken up into infinitesimal squares, and thus a field can be unambiguously transported along paths on finite surfaces as well.

The composition of the squares now follows the rules of composition in a Lie 2-group, as is easy to see. The integrability condition allows me to choose any route around a square, so let me choose one that is the most convenient for the purpose of comparison with the bi-gon picture. Let me bring a field from vertex 4 to vertex 1 by first dragging it left along the top edge then down along the left edge. Suppose the top edge has a holonomy \( h \) for this route. Then the ‘total’ holonomy along this route is \( hg \), where again I have kept the homomorphism \( t : H \rightarrow G \) implicit.

Suppose I now compose two squares by joining them at a corner as in Fig. 2(b). The total holonomy for bringing an object from the top right corner to the bottom left corner, along the top and left edges of the squares, is then \( h_1g_1h_2g_2 \). This is obviously the same as a square with \( g_1g_2 \) on the top edge and \( h_1g_1h_2^{-1}g_1^{-1} \) on the left edge. Clearly this can be identified with the product of morphisms as in Eq. (2.10). On the other hand, if I compose two squares along an edge as in Fig. 2(c), the holonomy from the top right corner to the bottom left corner is \( h_2h_1g_1 \), same as that for a rectangle with \( g_1 \) on the top edge and \( h_2h_1 \) on the left edge. This can be identified with the composition of morphisms in the Lie 2-group, as in Eq. (2.10)\( ^\cdot \). Also quite obviously, these compositions of squares are exactly the same as the horizontal and vertical compositions of bi-gons.

The integrability condition is the only one which restricts the choice of connection \( \tilde{A} \) for transporting fields between strings. Then using a principle typical to quantum theory, I can sum over all possible choices of the second connection. In other words, suppose I start with the free action of the gauge field \( V \).

The path integral becomes

\[
Z = \int D\tilde{A} D\tilde{V} \delta \left[ F + \frac{1}{2} d_A V \right] \exp(-i \int \frac{1}{2} F \wedge * F). \tag{4.2}
\]

The Lie algebra indices have been summed over as usual. The \( \delta \)-functional which enforces the constraint on the theory, can be rewritten by introducing a Lagrange multiplier field \( B \). Then the path integral becomes

\[
Z = \int D\tilde{A} D\tilde{V} \exp \left[ -i \int \left( \frac{1}{2} F \wedge * F - B \wedge (F + \frac{1}{2} d_A V) \right) \right], \tag{4.3}
\]

It is easy to integrate out \( V \) from this path integral, and the result is a constraint \( d_A B = 0 \), which is imposed on a theory with action \( I = \int \left( \frac{1}{2} F \wedge * F + B \wedge F \right) \).

Alternatively I can choose to take a Gaussian average over \( V \). This is the same as saying that the second connection \( \tilde{A} \) is peaked around \( A \), or that \( V \) is peaked around zero. Then the path integral includes a term proportional to \( V^2 \) in the exponent, and can be written as

\[
Z = \int D\tilde{A} D\tilde{V} \exp \left[ -i \int \left( \frac{1}{2} F \wedge * F + \frac{1}{2} m^2 V^2 - m B \wedge (F + \frac{1}{2} d_A V) \right) \right]. \tag{4.4}
\]

Here \( m \) is a constant of mass dimension one, introduced so that the dimensions of all terms agree, and I have also rescaled \( B \rightarrow mB \) so that \( B \) has the same dimensionality as the gauge field \( A \). If \( V \) is now integrated over, the result is the path integral

\[
Z = \int D\tilde{A} D\tilde{B} \exp(iI_{\text{eff}}), \tag{4.5}
\]

with

\[
I_{\text{eff}} = \int \left( -\frac{1}{2} F \wedge * F - \frac{1}{2} H \wedge * H + mB \wedge F \right), \tag{4.6}
\]

where \( H = d_A B \) is the field strength of \( B \). This action has several interesting physical consequences, including the appearance of a pole in the propagator of the gauge field without a residual Higgs field \( \Pi \).
V. DISCUSSION

There is something seemingly very odd about the relation between surfaces and Lie 2-groups. Recall that the lack of a canonical ordering for surfaces implied that surface holonomy was not well defined unless it was trivial or Abelian. Does the bi-gon construction in terms of Lie 2-groups now allow non-trivial non-Abelian surface holonomy? There are two ways of answering this question. Suppose I forget about the category structure of Lie 2-groups and naively associate a one-form $A$ to edges and a two-form $B$ to faces, valued in the Lie algebras of two groups $G$ and $H$, respectively. The holonomy around an infinitesimal closed loop can be written in terms of the surface it encloses, as $g \sim 1 + a^2 \sigma^{\mu\nu} F_{\mu\nu}$, where $F$ is the curvature or field strength of $A$. The total surface holonomy is then $1 + a^2 \sigma^{\mu\nu}(F_{\mu\nu} + B_{\mu\nu})$, where as before $B$ is brought to the same space as $A$ before addition. This is then the object associated with an infinitesimal surface, and a product of these objects must be taken when composing infinitesimal surfaces. Since there is no canonical ordering for surfaces, the infinitesimal surfaces may be composed in any order one likes, and the product of the corresponding holonomies must give a unique result irrespective of the order. Clearly, this can happen only if the total holonomy is trivial or Abelian, i.e. either $F + B = 0$ or the sum lives in an Abelian algebra.

But surely the bi-gon construction showed that the ordering of the infinitesimal surfaces did not matter when composing the elements of the Lie 2-group? After all, the two-form $B$ was introduced just for this purpose! It is true that any surface can be decomposed into infinitesimal bi-gons — simply flatten the bottom edge of a bi-gon, bend the top edge sharply rather than smoothly, and the bi-gon becomes a triangle, and any surface can be broken up in triangles. It is also true that given such a decomposition, the corresponding surface holonomies will compose as in a Lie 2-group, and the exchange law of Eq. (2.9) ensures that I can compose the bi-gons in any order I like, leading to the same final result. However, there is no contradiction, because not any arbitrary decomposition is allowed. Only a decomposition in which adjacent bi-gons are composable, is acceptable. The ‘zero curvature’ condition $F + B = 0$, is a direct consequence of the condition of composability, $\tilde{g} = t(h)g$, where $g$ and $\tilde{g}$ are the holonomies of the top and bottom edges and $t(h)$ is the contribution from $B$ to the surface holonomy $\tilde{g}$.

If the two-form $B$ had not been introduced, the infinitesimal surface holonomy would be simply $1 + a^2 \sigma^{\mu\nu} F_{\mu\nu}$, and the ordering independence of surface composition would imply that either $F = 0$ or $A$ is Abelian. Similarly, if I tried to define a surface holonomy by only a two-form $B$ and ignored the possibility of composing holonomies along the edges, I would find that either $B = 0$ or $B$ is Abelian. This result is the original one due to Teitelboim $\mathbb{R}$.

The same sort of argument holds, even more transparently, in the construction of Lie 2-groups based on two connections. In this case the important object is the holonomy around surfaces. Any surface can be decomposed in terms of infinitesimal squares, but the result of transporting a field along the boundary of the surface is uniquely defined if and only if the integrability condition holds. Either way, there is no non-trivial surface holonomy which belongs to some non-Abelian group. The real issue is of course whether it is possible to define a (trivial or Abelian) surface holonomy involving a non-Abelian gauge field $A$ which is not flat, i.e. for which $F \neq 0$. This is obviously true for both the procedures I have considered, and this is what distinguishes the 2-group construction from the usual results for the integrability of parallel transport $\mathbb{R}$.

Let me end with a comment about the relationship of surface holonomy with field theory. For the construction with two connections, a field theory of non-Abelian two forms appeared almost naturally by imposing the integrability condition Eq. (1.1) as a constraint on usual Yang-Mills theory. For the bi-gon construction, a two-form field is already present. So it is tempting to try to derive the flatness condition $F + B = 0$ as an equation of motion in some field theory. Unfortunately the simplest such theory is rather trivial, with action

$$\int (B \wedge F + \frac{1}{2} B \wedge B) \simeq - \int \frac{1}{2} F \wedge F.$$  \hspace{1cm} (5.1)

Of course it is possible to write down other actions using $A$ and $B$, including the actions found in $\mathbb{W}$. But it is only in four dimensions that the Lagrange multiplier field $B$ of one construction has the same structure as the surface gauge connection $B$ of the other one.

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