Fractional SEIRP model for COVID-19 dynamics incorporating social distancing and environment

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Abstract

We propose a Caputo-based fractional compartmental model for the dynamics of the novel COVID-19 epidemic taking into consideration social distancing and the influence of the environment. Using basic concepts such as continuity and Banach fixed-point theorem, existence and uniqueness of solution to the proposed model was shown. Furthermore, we analyze the stability of the model in the context of Ulam-Hyers and generalized Ulam-Hyers stability criteria. The concept of next-generation matrices was used to compute the basic reproduction number $R_0$, a number which determines the spread or otherwise of the disease into the general population. Numerical simulation of the disease dynamics was carried out using the fractional Adam-Bashforth-Moulton method to validate the obtained theoretical results.

Keywords: Caputo fractional derivative, covid-19, Banach fixed-point, basic reproduction number, Ulam-Hyers stability, epidemics

1 Introduction

Epidemiological modeling of infectious disease using differential equations with integer-order to examine and investigate the transmission dynamics of epidemics is widely studied in literature [1, 2, 3, 4, 5]. Mathematical modeling of epidemics in literature reveals that nonlinear dynamical equations can give some important insight into the transmission dynamics or dynamical behaviors of disease spread. The recent COVID-19 outbreaks around the world have attracted a lot of interest in the mathematical modeling of this highly contagious disease by constructing realistic nonlinear compartmental mathematical models that are driven by data to better understand the transmission dynamics of the epidemics [6, 7, 8, 9, 10, 11, 12, 13, 14].

Fractional order differential equations are a very useful and powerful mathematical modeling tool to study biological and engineering systems. This is due to the fact that differential operators in these types of equations or models are related to systems with memory dynamics which naturally exists in most biological and engineering systems, see [15, 16, 17]. Differential equations that are characterized by Caputo fractional order derivative have been used to study and analysed several infectious diseases transmission dynamics such as HIV/AIDS [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30], TB [31, 32, 33], Malaria [34, 35, 36], Dengue fever [37, 38, 39, 40], Zika [41, 42], Ebola [43, 44, 45] and Hepatitis B [46, 47, 48, 49, 50, 51]. In [52], backward bifurcation dynamics is explored in a basic and realistic vaccination epidemic

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model. Therein, the author considered rigorous qualitative analysis of the formulated mathematical model and also presented numerical simulations to confirm the obtained theoretical results. \[53\] \[53\] extended and studied the cholera epidemic mathematical model formulated by the author in \[54\] to capture Caputo fractional order derivative. The author in \[55\] used Lyapunov functions that are of Volterra–type and investigated uniform asymptotic stability of some basic epidemic models (SIS, SIR, SIRS) and the well-known Ross vector-borne diseases in Caputo sense. A 3-compartmental deterministic modeling structure (normal weight-overweight-obese) is constructed to explore obesity epidemic dynamics in a variable population using Caputo fractional-order nonlinear system \[56\]. The basic SEIR mathematical model driven by Caputo fractional derivative based on the assumption of variable population dynamics is studied in \[57\]. The authors gave a detailed qualitative stability analysis for their new and realistic deterministic model.

Due to usefulness and powerful nature of fractional order differential equation models in fractional calculus \[58, 59, 60, 61, 62\], some recent studies in the literature have considered the mathematical modeling of this deadly COVID-19 pandemic using Caputo based nonlinear equations, see \[63, 64, 65\]. A Caputo fractional-order deterministic epidemic model for COVID-19 infection is developed and studied in \[66\]. They used the well-known Banach contraction mapping principle to established the existence and uniqueness of the solution for the mathematical model. In a new mathematical modeling study \[67\], the authors proposed a SEIQRDP fractional-order deterministic model characterized by Caputo derivative to examine the novel coronavirus epidemic. A SEIPAHRF Caputo fractional-order compartmental model is proposed to analyze COVID-19 transmission dynamics in Wuhan \[68\]. They numerically solved their proposed model and compared it with the initial 67 days reported data on confirmed infected and death cases in Wuhan city. In recent studies, the authors in \[69\] proposed a Caputo-type nonlinear COVID-19 epidemiological model to explore the significance of lockdown dynamics in controlling the spread of the infectious disease. They further presented the uniqueness and existence of solutions for the mathematical model under lockdown using Banach and Schauder fixed theorems. In \[70\], a susceptible-exposed-infected-isolated-recovered compartmental modeling framework with constant total population dynamics is used to construct a new SEIQR Caputo fractional-order COVID-19 mathematical model. Owusu-Mensah and co-authors \[71\], have also studied a new COVID-19 epidemic model using Caputo derivative in fractional calculus. The well-known and powerful generalized form of Adams–Bashforth–Moulton iterative scheme was applied to numerically solve the formulated nonlinear fractional-order differential equation model.

The goal of the present paper is two folds: to establish both the mathematical and biological well-posedness of the integer-order model proposed in \[72\] and employ an approximate analytical technique to obtain long-term dynamics of the disease. Subsequently, we modify the model to a dimensionally consistent Caputo fractional-order model which has been touted in the literature to describe more efficiently, memory effect typically associated with the way human cells respond to diseases. The paper is organized as follows. Following the essential preliminaries on fractional calculus in Section 2, we examine in Section 3 both mathematical and epidemiological well-posedness of the integer-order model in \[72\], and obtain disease dynamics using approximate analytical technique proposed in \[73, 74\]. Section 4 of the paper is devoted to the formulation, well-posedness, local and global stability analysis of the fractional order model. Therein, using basic concepts such as continuity and Banach fixed-point theorem, existence and uniqueness of solution to the proposed model was shown. Furthermore, we discussed stability analysis of the model in the context of Ulam-Hyers and generalized Ulam-Hyers stability criteria. Also, the concept of next-generation matrices was used to compute the basic reproduction number \(R_0\), a number that determines the spread or otherwise of the disease into the general population. We conclude the paper with numerical simulation of the fractional order disease model using
fractional Adam-Bashforth-Moulton; and discussion of results.

2 Preliminaries on fractional calculus

In this section, we introduce well-known definitions and Lemmata in fractional calculus that are relevant for the current article. The interested reader can see the monograph [75] and the article [76] for the proofs and further references.

**Definition 2.1.** [75] The fractional order integral of the function $g \in L^1([0, b], \mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined the sense of Riemann-Liouville as

$$\mathcal{I}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)d\tau, \quad t > 0$$

where $[0, b] \subset \mathbb{R}^+$ and $\Gamma$ is the Euler gamma function defined by $\Gamma(v) = \int_0^\infty t^{v-1}e^{-t}dt$, $v > 0$.

**Definition 2.2.** [75] For a function $g \in C^n([0, b])$, the Caputo fractional-order derivative of order $v$ of $g$ is defined by

$$D^v g(t) = \frac{1}{\Gamma(n-v)} \int_0^t (t-\tau)^{n-v-1} g^{(n)}(\tau)d\tau, \quad t > 0$$

where $n = \lfloor v \rfloor + 1$ and $\lfloor v \rfloor$ denotes the smallest integer that is less or equal to $v$.

**Lemma 2.3.** [75] Let $v \geq 0$ and $n = \lfloor v \rfloor + 1$. Then

$$\mathcal{I}^v(D^v g(t)) = g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} t^k.$$

In particular, if $0 < v \leq 1$, then

$$\mathcal{I}^v(D^v g(t)) = g(t) - g_0.$$  \hspace{1cm} (1)

**Lemma 2.4.** [76] (Generalized Mean Value Theorem) For $0 < v \leq 1$, let $g(t) \in C([a,b])$ and $D^v g(t) \in (a,b]$. Then it holds

$$g(t) = g(a) + \frac{1}{\Gamma(v)} D^v g(\eta)(t-a)^v, \quad 0 \leq \eta \leq t, \quad \forall \ t \in (a,b].$$

3 Model framework: Integer order model

It is well known that the ravaging Covid-19 virus is transmitted through human-to-human interactions and transmission through the environment. When an infected person sneezes or coughs, the virus is released into the immediate environment which, experts believe, stays viable for as long as five days. We, therefore, consider two interacting populations of humans and pathogens denoted by $N(t)$ and $P(t)$ respectively. At any time $t$, the total population $N(t)$ is assumed to comprise the susceptible population $S(t)$, the exposed $E(t)$, the asymptomatic infected $I_A(t)$, the symptomatic infected $I_S(t)$ and the recovered population $R(t)$. Thus, $N(t) = S(t) + I_A(t) + I_S(t) + R(t)$. These interactions are depicted in Figure 1 below as reported in [72].

In the figure, $b$ denotes the rate of at which human population is born into susceptible class $S(t)$. The terms $\beta_1 S N$ and $\beta_2 S(I_A + I_S)$ represent rate at which the susceptible get infected by pathogens and through interactions with infectious asymptomatic $I_A(t)$ and infectious symptomatic $I_S(t)$. The denominators in the above terms factor in the adherence to recent experts advice on social distancing and the use of face masks which minimizes contact with infectious individuals, and transmission through the environment.
Based on the above description, the authors in [72] proposed the following integer-order differential equation to model the dynamics of the transmission of covid-19 taking into account environment and social distancing, namely

\[
\begin{align*}
S'(t) &= b - \frac{\beta_1 SP}{1 + \alpha_1 P} - \frac{\beta_2 S(I_A + I_S)}{1 + \alpha_2 (I_A + I_S)} + \psi E - \mu S, \\
E'(t) &= \frac{\beta_1 SP}{1 + \alpha_1 P} + \frac{\beta_2 S(I_A + I_S)}{1 + \alpha_2 (I_A + I_S)} - \psi E - \mu E - \omega E, \\
I_A'(t) &= (1 - \delta) \omega E - (\mu + \sigma) I_A - \gamma_A I_A, \\
I_S'(t) &= \delta \omega E - (\mu + \sigma) I_S - \gamma_S I_S, \\
R'(t) &= \gamma_S I_S + \gamma_A I_A - \mu R, \\
P'(t) &= \eta_A I_A + \eta_S I_S - \mu_p P
\end{align*}
\]

with initial conditions \(S(0) = S_0 > 0, E(0) = E_0 > 0, I_A(0) = I_{A0} > 0, I_S(0) = I_{S0} > 0, R(0) = R_0 > 0\). Therein, the authors analysed the equilibria and basic reproduction number of the model, and numerical simulation. However, the analysis in article [72] did not examine the essential well-posedness of the proposed model. Here in the present paper, for completeness, we begin our exposition by establishing qualitative properties of the integer order model which ensures that the model is both mathematically and epidemiological well-posed i.e. existence, uniqueness, positivity and boundedness of the solutions to the model. Furthermore, establishing the aforementioned properties allows us the extend the study in [72] to fractional order model, which accommodate the memory effect typically associated with epidemic disease models. In addition, we shall obtain approximate analytical solutions of the model via a multistage method proposed in [73], [74].

### 3.0.1 Well-posedness

For the fact that [2] models human populations, all the model parameters are assumed non-negative, see Table 1 for values and description of the model parameters. It then remains to
Table 1: Model parameters and their descriptions [72]

| Parameter description                                      | Symbol | Value            |
|-----------------------------------------------------------|--------|------------------|
| Birth rate of the human population                        | $b$    | 0.00018 days$^{-1}$ |
| Natural human death rate                                  | $\mu$ | $4.563 \times 10^{-5}$ days$^{-1}$ |
| Human life expectancy                                     | $1/\mu$ | 21915 days or 60years |
| Natural death rate of pathogens in the environment        | $\mu_p$ | 0.1724 days$^{-1}$ |
| Life expectancy of pathogens in the environment           | $1/\mu_p$ | 5.8 days |
| Proportion of interaction with an infectious environment  | $\alpha_1$ | 0.10 |
| Proportion of interaction with an infectious individual    | $\alpha_2$ | 0.10 |
| Rate of transmission from S to E due to contact with P     | $\beta_1$ | 0.00414 |
| Rate of transmission from S to E due to contact with $I_A$ and/or $I_S$ | $\beta_2$ | 0.0115 |
| Proportion of symptomatic infectious people                | $\delta$ | 0.7 |
| Progression rate from E back to S due to robust immune system | $\psi$ | 0.0051 |
| Progression rate from E to either $I_A$ or $I_S$           | $\omega$ | 0.09 |
| Death rate due to the coronavirus                          | $\sigma$ | 0.0018 |
| Rate of recovery of the symptomatic population            | $\gamma_S$ | 0.05 days$^{-1}$ |
| Rate of recovery of the asymptomatic human population     | $\gamma_A$ | 0.0714 days$^{-1}$ |
| Rate of virus spread to environment by symptomatic infectious individuals | $\eta_S$ | 0.1 days$^{-1}$ |
| Rate of virus spread to environment by asymptomatic infectious individuals | $\eta_A$ | 0.1 days$^{-1}$ |

show that unique solution exists for the model and that the state variables remain bounded and non-negative for all time $t > 0$.

**Theorem 3.1.** (Existence and uniqueness of solution) Let $t_f \in \mathbb{R}^+$. The dynamical system \([2]\) admits a unique solution on interval $(0, t_f)$ for initial conditions satisfying $S(0) > 0$, $E(0) > 0$, $I_A(0) > 0$, $I_S(0) > 0$, $R(0) > 0$, $P(0) > 0$.

**Proof.** By defining $y(t) = (S(t), E(t), I_A(t), I_S(t), R(t), P(t))^T$, then \([2]\) can be written as $y'(t) = F(y(t)) = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ where the $f_i$ are functions of $y(t)$. By assumption the initial condition $y(0) = (S(0), E(0), I_A(0), I_S(0), R(0), P(0))^T > 0$. Thus computing and examining the entries of the Jacobian $J(F(y))$, e.g.

$$J_{11} = \frac{\partial f_1}{\partial S} = \frac{\beta_1 P}{1 + \alpha_1 P} - \frac{\beta_2 (I_A + I_S)}{1 + \alpha_2 (I_A + I_S)} - \mu, \quad J_{12} = \frac{\partial f_1}{\partial E} = \psi$$

$$J_{13} = \frac{\partial f_1}{\partial I_A} = -\frac{-\beta_2 S}{\left(1 + \alpha_2 (I_A + I_S)\right)^2} = J_{14} = \frac{\partial f_1}{\partial I_S}, \quad J_{15} = \frac{\partial f_1}{\partial R} = 0$$

$$J_{16} = \frac{\partial f_1}{\partial P} = -\frac{\beta_1 S}{(1 + \alpha_1 P)^2}, \quad \text{etc}$$

reveal that both the right hand side of \([2]\) namely $F$ and its Jacobian are continuous for $t > 0$. Thus, $F$ satisfies a Lipschitz condition on $\mathbb{R}_+^6$. The existence and uniqueness of solution for some time interval $(0, t_f)$ follows from Picard-Lindelof Theorem.

**Theorem 3.2.** (Positivity of solution) The state variables $S(t), E(t), I_A(t), I_S(t), R(t), P(t)$ of \([2]\) with non-negative initial data remains non-negative for all $t > 0$.

**Proof.** Let $S(0) = S_0 > 0$. It follows from the first equation of \([2]\) that

$$\frac{dS}{dt} \geq -\left(\frac{\beta_1 P}{1 + \alpha_1 P} + \frac{\beta_2 (I_A + I_S)}{1 + \alpha_2 (I_A + I_S)} + \mu\right) S$$
and upon integration, one obtains

$$S(t) \geq S_0 e^{-(\mu t + \delta)} > 0$$  \hspace{1cm} (3)$$

for all $t > 0$ where

$$\delta = \int_0^t \left( \frac{\beta_1 P(\tau)}{1 + \alpha_1 P(\tau)} + \frac{\beta_2 (I_A(\tau) + I_S(\tau))}{1 + \alpha_2 (I_A(\tau) + I_S(\tau))} \right) d\tau.$$

Similar arguments yield

$$E(t) \geq E(0) e^{-(\psi + \mu + \omega)t} > 0 \quad I_A(t) \geq I_A(0) e^{-(\mu + \sigma + \gamma_A)t} > 0 \quad I_S(t) \geq I_S(0) e^{-(\psi + \mu + \gamma_S)t} > 0 \quad R(t) \geq R(0) e^{-\mu t} > 0 \quad P(t) \geq P(0) e^{-\mu p t} > 0$$

and as a consequence, non-negativity of the remaining state variables are obtained. \hfill \square

**Theorem 3.3. (Boundedness of solution)** The integer order model (2) has solutions bounded within invariant region given by

$$\Omega = \left\{ (S, E, I_A, I_S, R, P) \in \mathbb{R}^6 : 0 \leq N(t) \leq \frac{b}{\mu}, \quad 0 \leq P(t) \leq \frac{\eta b}{\mu \mu_p} \right\}$$

where $\eta = \eta_A + \eta_S$.

**Proof.** Using the fact that $N(t) = S(t) + E(t) + I_A(t) + I_S(t) + R(t)$, it follows from (2) that

$$N'(t) = \left( b - (\mu + \sigma)(I_A + I_S) - \mu(R + S + E) \right)$$

$$= b - \mu N - \sigma (I_A + I_S)$$

$$\leq b - \mu N.$$ 

Thus considering the initial valued problem $N'(t) = b - \mu N, \quad N(0) = N_0$ and invoking comparison theorem [77], it follows that

$$N(t) \leq N_0 e^{-\mu t} + \frac{b}{\mu} (1 - e^{-\mu t})$$

and consequently

$$\limsup_{t \to \infty} N(t) \leq \frac{b}{\mu}.$$ 

For the pathogen population, using the fact that $I_A + I_S \leq N \leq b/\mu$, the last equation of (2) yields

$$P'(t) = \eta A I_A + \eta S I_S - \mu_p P$$

$$\leq \eta (I_A + I_S) - \mu P, \quad \eta = \eta_A + \eta_S$$

$$\leq \frac{b}{\mu} - \mu_p P$$

and consequently $P(t) \leq P_0 e^{-\mu_p t} + \frac{\eta b}{\mu \mu_p} (1 - e^{-\mu_p t})$. Thus $\limsup_{t \to \infty} P(t) \leq \frac{\eta b}{\mu \mu_p}$. \hfill \square
3.0.2 Approximate analytical solution of the integer order model

Here, we shall adopt the multistage technique proposed in [73], [74] to compute approximate analytical solution to the integer order model (2). For that purpose, let us define $U = \frac{1}{1+\alpha_1 P}$ and $W = \frac{1}{1+\alpha_2(A + I_S)}$ so that (2) is transformed to an equivalent polynomial system

\[
S'(t) = b - \beta_1 S P U - \beta_2 S (I_A + I_S) W + \psi E - \mu S,
\]

\[
E'(t) = \beta_1 S P U + \beta_2 S (I_A + I_S) W - \psi E - \mu E - \omega E,
\]

\[
I_A'(t) = (1 - \delta) \omega E - (\mu + \sigma) I_A - \gamma_1 I_A,
\]

\[
I_S'(t) = \delta \omega E - (\mu + \sigma) I_S - \gamma_1 I_S,
\]

\[
R'(t) = \gamma_1 I_S + \gamma A I_A - \mu R, \tag{4}
\]

\[
P'(t) = \eta A I_A + \eta S I_S - \mu P,
\]

\[
U'(t) = -\alpha_1 U^2 (\eta A I_A + \eta S I_S - \mu P)
\]

\[
W'(t) = -\alpha_2 W^2 \left( \omega E - (\mu + \sigma)(I_A + I_S) - \gamma A I_A - \gamma S I_S \right)
\]

which is now amenable to the proposed technique. By writing $S(t) = \sum_{n=0}^{\infty} S_n t^n$, it follows from the above polynomial system that the coefficients $S_n$ are obtained recursively through

\[
(n + 1) S_n = b - \beta_1 (S P U)_n - \beta_2 \left( S (I_A + I_S) W \right)_n + \psi E_n - \mu S_n, \quad S_0 = S(0).
\]

In a similar manner, approximate analytical solutions of the remaining compartments are computed. Such series solution often have small interval of convergence. Therefore to improve convergence, we choose a safe step length $h$, (see [73] and [74]) and compute a piecewise continuous approximate solution which is convergent in the entire integration interval. The dynamics of the disease progression for each sub-population are displayed in Figures 2a-2d and Figures 3a-3b below, confirming the results in [72].

4 Fractional order model

Fractional derivative are generally believed to model disease epidemic more realistically because of their capability to capture memory effect often associated with human body response to diseases. Here, we propose a fractional order variant of (2) given by

\[
D_t^\alpha S(t) = b^\alpha - \beta_1^\alpha S P \frac{1}{1 + \alpha_1 P} - \beta_2^\alpha S (I_A + I_S) \frac{1}{1 + \alpha_2 (I_A + I_S)} + \psi^\alpha E - \mu^\alpha S, \tag{5}
\]

\[
D_t^\alpha E(t) = \beta_1^\alpha S P \frac{1}{1 + \alpha_1 P} + \beta_2^\alpha S (I_A + I_S) \frac{1}{1 + \alpha_2 (I_A + I_S)} - \psi^\alpha E - \mu^\alpha E - \omega^\alpha E,
\]

\[
D_t^\alpha I_A(t) = (1 - \delta) \omega^\alpha E - (\mu^\alpha + \sigma^\alpha) I_A - \gamma_1^\alpha I_A,
\]

\[
D_t^\alpha I_S(t) = \delta \omega^\alpha E - (\mu^\alpha + \sigma^\alpha) I_S - \gamma_1^\alpha I_S,
\]

\[
D_t^\alpha R(t) = \gamma_1^\alpha I_S + \gamma_1^\alpha I_A - \mu^\alpha R,
\]

\[
D_t^\alpha P(t) = \eta_1^\alpha I_A + \eta_1^\alpha I_S - \mu_1^\alpha P
\]

subject to initial conditions $S(0) = S_0 > 0$, $E(0) = E_0 > 0$, $I_A(0) = I_{A0} > 0$, $I_S(0) = I_{S0} > 0$, $R(0) = R_0 > 0$. In the above, operator $D_t^\alpha$ denotes Caputo fractional derivative of order $0 < \alpha \leq 1$. Noting that all the model parameters except $\alpha_1$, $\alpha_2$ and $\delta$ have dimensions $1/t^\alpha$, we have raised these parameters to power of $\alpha$ for dimensional consistency emphasized by [38].
Figure 2: Disease dynamics of the integer-order model.
4.1 Qualitative properties of solution

In this section, we examine the mathematical and biological well-posedness of the fractional order model. In essence, we prove that solution of the fractional model is bounded and remains positive as long as a positive initial condition is given. Furthermore, we prove the existence and uniqueness of the solution to the modified model.

Let

\begin{align*}
X(t) &= (S, E, I, R, P)^T \\
K(t, X(t)) &= (\phi_i)^T, \quad i = 1, 2, \ldots, 6
\end{align*}

where

\begin{align*}
\phi_1 &= b^\alpha - \frac{\beta_1^a S P}{1 + \alpha_1 P} - \frac{\beta_2^a S (I_A + I_S)}{1 + \alpha_2 (I_A + I_S)} + \psi^a E - \mu^a S \\
\phi_2 &= \frac{\beta_1^a S P}{1 + \alpha_1 P} + \frac{\beta_2^a S (I_A + I_S)}{1 + \alpha_2 (I_A + I_S)} - \psi^a E - \mu^a E - \omega^a E \\
\phi_3 &= (1 - \delta) \omega^a E - (\mu^a + \sigma^a) I_A - \gamma^a A \\
\phi_4 &= \delta \omega^a E - (\mu^a + \sigma^a) I_S - \gamma^a S \\
\phi_5 &= \gamma^a S I_A + \gamma^a A I_A - \mu^a R \\
\phi_6 &= \eta^a A I_A + \eta^a S I_S - \mu^a P
\end{align*}

Then the dynamical system (5) can be written as

\begin{equation}
D_t^\alpha X(t) = K(t, X(t)), \quad X(0) = X_0 \geq 0, \quad t \in [0, b], \quad 0 < \alpha \leq 1.
\end{equation}

In the above, the condition \(X(0) \geq 0\) is to be interpreted component-wise. Problem (6), which is equivalent to fractional differential equation (5), in turn has integral representation

\begin{equation}
X(t) = X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} K(\tau, X(\tau)) d\tau
\end{equation}

Next, we shall analyse model (5) through the integral representation above. For that purpose, let \(E = C([0, b]; \mathbb{R})\) denote the Banach space of all continuous functions from \([0, b]\) to \(\mathbb{R}\) endowed with the norm

\[||X||_E = \sup_{t \in [0, b]} \{|X(t)|\}\]
where \(|X(t)| = |S(t)| + |E(t)| + |I_A(t)| + |I_S(t)| + |R(t)| + |P(t)|\). Note that \(S, E, I_A, I_S, R, P\) all belong to \(C([0, b]; \mathbb{R})\). Furthermore, we define the operator \(P : \mathcal{E} \to \mathcal{E}\) by

\[
(\mathcal{P}X)(t) = X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \mathcal{K}(\tau, X(\tau))d\tau.
\]

(8)

Note that operator \(P\) is well-defined due to obvious continuity of \(\mathcal{K}\).

### 4.1.1 Positivity and boundedness of solution

For the fractional order model to be biologically well-posed, its solution is expected to be positive and bounded at all times. These properties are established in the sequel.

**Theorem 4.1.** Let \(X(t) = (S, E, I_A, I_S, R, P)^T\). Then for \(X(0) > 0\), the solution \(X(t)\) of (5) is bounded, and remains positive for \(t \geq 0\).

**Proof.** We start by establishing positiveness of solution. Following [78], consider the trajectory of solution along the \(S\)-axis where \(E(0) = I_A(0) = I_S(0) = R(0) = P(0)\), and \(S(0) = S_0 > 0\). Then

\[
D_t^\alpha S(t) = b^\alpha - \mu^\alpha S, \quad S(0) = S_0
\]

whose solution is given by

\[
S(t) = S_0 e^{(b^\alpha - \mu^\alpha) t} = S_0 e^{b^\alpha (1 - e^{-\mu^\alpha t})}.
\]

Since \(S(t) > 0\), \(S(0) > 0\), \(R(0) > 0\), and \(P(0) > 0\), similar arguments yield

\[
\begin{align*}
E(t) &= E(0) e^{(-\psi^\alpha + \mu^\alpha + \omega^\alpha) t} > 0 \\
I_A(t) &= I_A(0) e^{(-\mu^\alpha + \sigma^\alpha + \gamma_A^\alpha) t} > 0 \\
I_S(t) &= I_S(0) e^{(-\psi^\alpha + \mu^\alpha + \gamma_S^\alpha) t} > 0 \\
R(t) &= R(0) e^{(-\psi^\alpha + \mu^\alpha + \omega^\alpha) t} > 0 \\
P(t) &= P(0) e^{(-\mu^\alpha + \omega^\alpha) t} > 0
\end{align*}
\]

showing non-negative invariance of the axes.

Now, since the solution to the model (5) is positive in the \(E - I_A - I_S - R - P\) plane, let \(t^* > 0\) such that \(S(t^*) = 0\), \(E(t^*) > 0\), \(I_A(t^*) > 0\), \(I_S(t^*) > 0\), \(R(t^*) > 0\), \(P(t^*) > 0\) and \(S(t) < S(t^*)\). On this plane,

\[
D_t^\alpha S(t) \big|_{t = t^*} = b^\alpha > 0
\]

(9)

By Caputo fractional mean value theorem (see Lemma 2.4), it holds \(S(t) - S(t^*) = \frac{1}{\Gamma(\alpha)} D_t^\alpha (\tau - t^*)^{\alpha - 1} \frac{S(t) - S(t^*)}{t - t^*} = \frac{t^* - \tau}{t^*} S(t)\). Therefore, using (9), we obtain \(S(t) > S(t^*)\), contradicting our earlier assumption for \(t^*\). Thus, any solution \(S(t)\) is non-negative for all \(t \geq 0\). The remaining variables can be treated similarly. Hence, solution \(X(t)\) remains positive for all \(t \geq 0\).

Finally for boundedness, proceeding as in the integer order case (see the proof of Theorem 3.3), one obtains

\[
N(t) \leq N_0 e^{(\beta \omega^\alpha + \frac{\beta \omega^\alpha}{p^\alpha}) (1 - e^{\frac{-\beta \omega^\alpha}{p^\alpha}})}
\]

and consequently \(\lim_{t \to \infty} N(t) \leq \frac{\beta \omega^\alpha}{p^\alpha}\). Similarly, \(\lim_{t \to \infty} P(t) \leq \frac{\beta \omega^\alpha}{p^\alpha}\).

\[\square\]

### 4.1.2 Existence of unique and uniformly stable solution

Here, we establish existence, uniqueness and uniform stability of solutions to (4) through (5). The following preliminary result is in order.

**Lemma 4.2.** Let \(\bar{X} = (\bar{S}, \bar{E}, \bar{I}_A, \bar{I}_S, \bar{R}, \bar{P})^T\). The function \(\mathcal{K} = (\phi_i)^T\) defined above satisfies

\[
||\mathcal{K}(t, X(t)) - \mathcal{K}(t, \bar{X}(t))||_\mathcal{E} \leq L_\mathcal{K} ||X - \bar{X}||_\mathcal{E}
\]

for some \(L_\mathcal{K} > 0\).
Proof. From the first component of \( \mathcal{K} \), we observe that

\[
|\phi_1(t, X(t)) - \phi_1(t, \bar{X}(t))| = |\beta_1 \left( \frac{S(t)P(t)}{1 + \alpha_1P(t)} - \frac{\bar{S}(t)\bar{P}(t)}{1 + \alpha_1P(t)} \right) - \beta_2 \left( \frac{S(t)(I_A(t) + I_S(t))}{1 + \alpha_2(I_A(t) + I_S(t))} - \frac{\bar{S}(t)(\bar{I}_A(t) + \bar{I}_S(t))}{1 + \alpha_2(I_A(t) + I_S(t))} \right) + \psi^\alpha(E(t) - \bar{E}(t)) - \mu^\alpha(S(t) - \bar{S}(t))|
\]

\[
\leq |\beta_1| \left| \frac{S(t)P(t)}{1 + \alpha_1P(t)} - \frac{\bar{S}(t)\bar{P}(t)}{1 + \alpha_1P(t)} \right| + |\beta_2^\alpha| \left| \frac{S(t)(I_A(t) + I_S(t))}{1 + \alpha_2(I_A(t) + I_S(t))} - \frac{\bar{S}(t)(\bar{I}_A(t) + \bar{I}_S(t))}{1 + \alpha_2(I_A(t) + I_S(t))} \right| + |\psi^\alpha|E(t) - \bar{E}(t)| + \mu^\alpha|S(t) - \bar{S}(t)|.
\]

However,

\[
\left| \frac{S(t)P(t)}{1 + \alpha_1P(t)} - \frac{\bar{S}(t)\bar{P}(t)}{1 + \alpha_1P(t)} \right| = \left| \frac{S(t)P(t)(1 + \alpha_1\bar{P}(t)) - \bar{S}(t)\bar{P}(t)(1 + \alpha_1P(t))}{(1 + \alpha_1P(t))(1 + \alpha_1\bar{P}(t))} \right| 
\leq f_1(t)|S(t) - \bar{S}(t)| + f_2(t)|P(t) - \bar{P}(t)|
\]

where

\[ f_1(t) = |P(t)| + \alpha_1|P(t)||\bar{P}(t)|, \quad f_2(t) = |\bar{S}(t)| \]

Similarly we obtain,

\[
\left| \frac{S(t)(I_A(t) + I_S(t))}{1 + \alpha_2(I_A(t) + I_S(t))} - \frac{\bar{S}(t)(\bar{I}_A(t) + \bar{I}_S(t))}{1 + \alpha_2(I_A(t) + \bar{I}_S(t))} \right| \leq g_1(t)|S(t) - \bar{S}(t)| + g_2(t)|I_A(t) - \bar{I}_A(t)| + g_3(t)|I_S(t) - \bar{I}_S(t)|
\]

where

\[ g_1(t) = |I_A(t) + \alpha_2I_A(t)\bar{I}_A(t) + \alpha_2I_A(t)I_S(t)|, \quad g_2(t) = |\bar{S}(t) + \alpha_2S(t)\bar{I}_S(t)|, \quad g_3(t) = |\alpha_2\bar{I}_A(t)S(t)|. \]

Altogether, we have

\[
|\phi_1(t, X(t)) - \phi_1(t, \bar{X}(t))| \leq (\mu^\alpha + \beta_1^\alpha f_1(t) + \beta_2^\alpha g_1(t))|S(t) - \bar{S}(t)| + \psi^\alpha|E(t) - \bar{E}(t)| + \beta_2^\alpha g_2(t)|I_A(t) - \bar{I}_A(t)| + \beta_2^\alpha g_3(t)|I_S(t) - \bar{I}_S(t)| + \beta_1^\alpha f_2(t)|P(t) - \bar{P}(t)| + |P(t) - \bar{P}(t)|
\]

where

\[ L_1 = \mu^\alpha + \psi^\alpha + \max_{t \in [0,b]} \{ \beta_1^\alpha f_1(t) + \beta_1^\alpha f_2(t) + \beta_2^\alpha g_1(t) + \beta_2^\alpha g_2(t) + \beta_1^\alpha g_3(t) \}. \]

In a similar manner, one obtains

\[
|\phi_2(t, X(t)) - \phi_2(t, \bar{X}(t))| \leq L_1 \left( |S(t) - \bar{S}(t)| + |E(t) - \bar{E}(t)| + |I_A(t) - \bar{I}_A(t)| + |I_S(t) - \bar{I}_S(t)| + |P(t) - \bar{P}(t)| \right)
\]

where

\[ L_2 = \mu^\alpha + \psi^\alpha + \omega^\alpha + \max_{t \in [0,b]} \{ \beta_1^\alpha f_1(t) + \beta_2^\alpha g_1(t) + \beta_1^\alpha g_2(t) + \beta_1^\alpha g_3(t) \}. \]
For the remaining components of $\mathcal{K}$, it holds

$$
|\phi_3(t, X(t)) - \phi_3(t, \bar{X}(t))| \leq L_3 \left( |E(t) - \bar{E}(t)| + |I_A(t) - \bar{I}_A(t)| \right)
$$

$$
|\phi_4(t, X(t)) - \phi_4(t, \bar{X}(t))| \leq L_4 \left( |E(t) - \bar{E}(t)| + |I_A(t) - \bar{I}_A(t)| \right)
$$

$$
|\phi_5(t, X(t)) - \phi_5(t, \bar{X}(t))| \leq L_5 \left( |I_A(t) - \bar{I}_A(t)| + |I_S(t) - \bar{I}_S(t)| + |R(t) - \bar{R}(t)| \right)
$$

$$
|\phi_6(t, X(t)) - \phi_6(t, \bar{X}(t))| \leq L_6 \left( |I_A(t) - \bar{I}_A(t)| + |I_S(t) - \bar{I}_S(t)| + |P(t) - \bar{P}(t)| \right)
$$

with $L_3 = |1 - \delta \omega^\alpha + \mu^\alpha + \sigma^\alpha + \gamma_A \omega^\alpha|$, $L_4 = \delta \omega^\alpha + \mu^\alpha + \sigma^\alpha + \gamma_A \omega^\alpha$, $L_5 = \gamma_S \omega^\alpha + \mu^\alpha$, $L_6 = \eta_A \omega^\alpha + \eta_S \omega^\alpha$. Consequently,

$$
||\mathcal{K}(t, X(t)) - \mathcal{K}(t, \bar{X}(t))||_\mathcal{E} = \sup_{t \in [0, b]} \sum_{i=1}^{6} |\phi_i(t, X(t)) - \phi_i(t, \bar{X}(t))|

\leq L_K ||X - \bar{X}||_\mathcal{E}
$$

where $L_K = L_1 + L_2 + L_3 + L_4 + L_5 + L_6$. \hfill \square

**Theorem 4.3.** Let the result of Lemma 4.2 holds and $\Omega = \frac{b^v}{\Gamma(v+1)}$. If $\Omega L_K < 1$, then there exists a unique solution of model (5) on $[0, b]$ which is uniformly Lyapunov stable.

*Proof.* The function $\mathcal{K} : [0, b] \times \mathbb{R}^6 \to \mathbb{R}^6_+$ is clearly continuous on its domain. Thus, existence of solution to (5) follows from [79, Theorem 3.1].

For uniqueness, we shall use Banach contraction mapping principle on operator $P$ defined in (5) above. For that purpose, we show that $P$ is both a self map and a contraction. Firstly, by definition, sup$_{t \in [0, b]} ||\mathcal{K}(t, 0)|| = b^v$. Let us now define $\kappa > \frac{||X_0|| + \Omega b^v}{1 - \Omega L_K}$ and a closed convex set $B_\kappa = \{ X \in \mathcal{E} : ||X||_\mathcal{E} \leq \kappa \}$. Thus for self map property, it suffices to show that $PB_\kappa \subseteq B_\kappa$. So, let $X \in B_\kappa$, then

$$
||PX||_\mathcal{E} = \sup_{t \in [0, b]} \left| \left| X_0 + \frac{1}{\Gamma(v)} \int_0^t (t - \tau)^{v-1} \mathcal{K}(\tau, X(\tau))d\tau \right| \right|

\leq |X_0| + \frac{1}{\Gamma(v)} \sup_{t \in [0, b]} \left\{ \int_0^t (t - \tau)^{v-1} \left( ||\mathcal{K}(\tau, X(\tau)) - \mathcal{K}(\tau, 0)|| + ||\mathcal{K}(\tau, 0)||_\mathcal{E} \right) d\tau \right\}

\leq |X_0| + \frac{1}{\Gamma(v)} \sup_{t \in [0, b]} \left\{ \int_0^t (t - \tau)^{v-1} \left( ||\mathcal{K}(\tau, X(\tau)) - \mathcal{K}(\tau, 0)||_\mathcal{E} + ||\mathcal{K}(\tau, 0)||_\mathcal{E} \right) d\tau \right\}

\leq |X_0| + \frac{L_K ||X||_\mathcal{E} + b^v}{\Gamma(v)} \sup_{t \in [0, b]} \left\{ \int_0^t (t - \tau)^{v-1} d\tau \right\}

\leq |X_0| + \frac{L_K \kappa + b^v}{\Gamma(v) + b^v} \sup_{t \in [0, b]} \left\{ \int_0^t (t - \tau)^{v-1} d\tau \right\}

= |X_0| + \frac{L_K \kappa + b^v}{\Gamma(v) + b^v} \int_0^t (t - \tau)^{v-1} d\tau

\leq \kappa.

\hfill \square

It then follows that operator $PX \subseteq B_\kappa$ and $P$ is indeed a self-map. It remains at this point to prove that $P$ is a contraction. Let $X$ and $\bar{X} \in \mathcal{E}$ satisfy (6). Then again using the result of
Lemma 4.2 we have
\[ ||PX - P\bar{X}||_\varepsilon = \sup_{t \in [0,b]} \left\{ ||(PX)(t) - (P\bar{X})(t)|| \right\} \]
\[ = \frac{1}{\Gamma(v)} \sup_{t \in [0,b]} \left\{ \int_0^t (t - \tau)^{v-1} |\mathcal{K}(\tau, X(\tau)) - \mathcal{K}(\tau, \bar{X}(\tau))|d\tau \right\} \]
\[ \leq \frac{L_K}{\Gamma(v)} \sup_{t \in [0,b]} \left\{ \int_0^t (t - \tau)^{v-1} |X(\tau) - \bar{X}(\tau)|d\tau \right\} \]
\[ \leq \Omega L_K ||X - \bar{X}||_\varepsilon. \]

Thus if \( \Omega L_K < 1 \) then \( P \) is a contraction mapping, and consequently by the Banach contraction mapping principle, \( P \) has a unique fixed point on \([0,b]\) which is solution of (5). Uniformly Lyapunov stability of solution follows from [80, Theorem 3.2].

### 4.2 Equilibria and Basic reproduction number of the fractional order model

By setting the left hand side of (5) to zero, one obtains the equilibria points. Disease-free equilibrium points are those where \( I_A = I_S = 0 \) and \( P = 0 \). These immediately imply \( E = 0 \) and \( R = 0 \). Thus, we obtain \( b^* - \mu^* S^* = 0 \) or \( S^* = b^*/\mu^* \). Consequently, the disease-free equilibrium point of the fractional-order model is \((S^*, E^*, I_A^*, I_S^*, R^*, P^*) = (b^*/\mu^*, 0, 0, 0, 0, 0)\).

The dynamic and stability of a disease model is usually governed by the basic reproduction number \( R_0 \) which simply put, measure the average number of secondary infections resulting from introducing one infected individual into an entirely susceptible population. In other words, it determines the spread or otherwise of the infection into the entire population. We follow the recipes described in [81] to compute the basic reproduction number for the fractional order model. The approach is based on next-generation matrices.

The infected subsystem of (5) comprises the second, third, fourth and sixth equations. Therefore, letting \( x = (E, I_A, I_S, P)^T \), then the infected subsystem can be written in the form
\[ x'(t) = \mathcal{F}(x) - \mathcal{V}(x) \]
where
\[ \mathcal{F}(x) = \begin{pmatrix}
\frac{\beta^* S P}{1 + \alpha_1 P} + \frac{\beta^* S (I_A + I_S)}{1 + \alpha_2 (I_A + I_S)} \\
0 \\
0 \\
\eta_A^* I_A + \eta_S^* I_S
\end{pmatrix}, \quad \mathcal{V}(x) = \begin{pmatrix}
(\psi^* + \mu^* + \omega^*) E \\
(\mu^* + \sigma^* + \gamma_S^*) I_S - \delta^* \omega^* E \\
(\mu^* + \sigma^* + \gamma_A^*) I_A - (1 - \delta) \omega^* E \\
\mu_P^* P
\end{pmatrix}
\]

The matrices \( \mathcal{F} \) and \( \mathcal{V} \) represents the rate of production of new infections and transition of new infections respectively.

Linearizing the infected subsystem about the disease-free equilibrium point, we obtain the corresponding rate of infection and infection transition rate near the equilibrium, respectively, as
\[ F = \begin{pmatrix}
0 & \frac{\beta^* b^*}{\mu^*} & \frac{\beta^* b^*}{\mu^*} & \frac{\beta^* b^*}{\mu^*} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\eta_A^* & \eta_S^* & 0 & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
c_1^0 & 0 & 0 & 0 \\
-\delta \omega^* & 0 & c_2^0 & 0 \\
(\delta - 1) \omega^* & c_3^0 & 0 & 0 \\
0 & 0 & 0 & \mu_P^* 
\end{pmatrix}
\]
where
\[ c_1^0 := \psi^* + \mu^* + \omega^*; \quad c_2^0 := \mu^* + \sigma^* + \gamma_S^*; \quad c_3 := \mu^* + \sigma^* + \gamma_A^*. \]
The basic reproduction number is the spectral radius of matrix \( K = FV^{-1} \), the total production of new infections over the course of outbreak. Using symbolic computation Maple17 software,
Hence, \( \rho \) Consequently, it follows that the disease-free equilibrium point \( R_0 \) of the basic reproduction number \( R_0 \) is locally asymptotically stable (LAS) if \( R_0 < 1 \), and unstable if \( R_0 > 1 \).

The above theorem 4.4 implies that the fractional order epidemic model \( (6) \) is controllable through appropriate control measures (via the parameters \( \alpha_1 \) and \( \alpha_2 \)) such as minimizing contact with infectious individuals as well as shedding of the virus pathogen by infected individuals to the environment.

4.2.1 Global stability

We establish the global stability of the fractional model \( (6) \) in the sense of Ulam-Hyers \([7]\). The Ulam-Hyers stability has been an active research area since it was first introduced by Ulam in 1940 at Winscosing University and a follow-up work by Rassias in the years between 1982 and 1998. Recently the authors in \([66]\) established Ulam-Hyers stability of a nonlinear fractional model of Covid-19 pandemic.

For clarity of the discussion that follows, let us introduce the inequality given by

\[
|D^\alpha_t X(t) - K(t, X(t))| \leq \epsilon, \quad t \in [0, b].
\]

We say a function \( X \in \mathcal{E} \) is a solution of \( (10) \) if and only if there exists \( h \in \mathcal{E} \) satisfying

i. \( |h(t)| \leq \epsilon \)

ii. \( D^\alpha_t X(t) = K(t, X(t)) + h(t), \quad t \in [0, b] \).

It is important to observe that by invoking \( (7) \) and property ii. above, simple simplification yields the fact that any function \( X \in \mathcal{E} \) satisfying \( (10) \) also satisfies the integral inequality

\[
|X(t) - \bar{X}(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} K(\tau, \bar{X}(\tau))| \leq \Omega \epsilon
\]
recalling the expressions (7), (11), we have for \( \epsilon > 0 \) that for every solution \( X \in \mathcal{E} \) satisfying (10), there exists a solution \( \bar{X} \in \mathcal{E} \) of (6) such that

\[
\| \bar{X}(t) - X(t) \|_\mathcal{E} \leq C_K \epsilon, \quad t \in [0, b].
\]

**Definition 4.6.** The fractional order model (6) (and equivalently (5)) is said to be generalized Ulam-Hyers stable if there exists a continuous function \( \phi_K : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \phi_K(0) = 0 \), such that, for each solution \( \bar{X} \in \mathcal{E} \) of (10), there exists a solution \( X \in \mathcal{E} \) of (6) such that

\[
\| \bar{X}(t) - X(t) \|_\mathcal{E} \leq \phi_K \epsilon, \quad t \in [0, b].
\]

We now present our result on stability of the fractional order model.

**Theorem 4.7.** Let the hypothesis and result of Lemma 4.2 hold, \( \Omega = \frac{\kappa}{\Gamma(\alpha + 1)} \) and \( 1 - \Omega L_K > 0 \). Then, the fractional order model (6) (and equivalently (5)) is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.

**Proof.** Let \( X \) be a unique solution of (6) guaranteed by Theorem 4.3. \( \bar{X} \) satisfies (10). Then recalling the expressions (7), (11), we have for \( \epsilon > 0, t \in [0, b] \) that

\[
\| \bar{X} - X \|_\mathcal{E} = \sup_{t\in[0,b]} |\bar{X}(t) - X(t)|
\]

\[
= \sup_{t\in[0,b]} |\bar{X}(t) - X_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \mathcal{K}(\tau, X(\tau)) d\tau|
\]

\[
\leq \sup_{t\in[0,b]} |\bar{X}(t) - X_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \mathcal{K}(\tau, \bar{X}(\tau)) d\tau|
\]

\[
+ \sup_{t\in[0,b]} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |\mathcal{K}(\tau, \bar{X}(\tau)) - \mathcal{K}(\tau, X(\tau))| d\tau
\]

\[
\leq \Omega \epsilon + \frac{L_K}{\Gamma(\alpha)} \sup_{t\in[0,b]} \int_0^t (t-\tau)^{\alpha-1} |\bar{X}(\tau) - X(\tau)| d\tau
\]

\[
\leq \Omega \epsilon + \Omega L_K \| \bar{X} - X \|_\mathcal{E}
\]

from which we obtain \( \| \bar{X} - X \|_\mathcal{E} \leq C_K \epsilon \) where \( C_K = \frac{\Omega}{1 - \Omega L_K} \). \( \square \)

### 5 Numerical simulation and discussion

This section provides some illustrative numerical simulations to explain the dynamical behavior of the Caputo fractional-order deterministic nonlinear COVID-19 mathematical model. The numerical solution of a nonlinear mathematical model using the appropriate iterative scheme is very important in mathematical modeling. For this purpose, we have used the Adams-type predictor-corrector iterative scheme constructed in [83, 84] to solve fractional-order differential equations. The method relies on equivalent integral formulation of our fractional model (6) written in the form (3). Consider a uniform discretization of \([0, b]\) given by \( t_n = nh, \ n = 0, 1, 2, \ldots, N \) where \( 0 < h = b/n \) denote the step size. Now, given any approximation \( X_h(t_i) \approx X(t_i) \), we obtain the next approximation \( X_h(t_{i+1}) \) using the Adams-type predictor-corrector iterative scheme as follows:

**Predictor:**

\[
X_h^p(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} \frac{t_{n+1}^k}{k!} X_0^k + \frac{1}{\Gamma(\alpha)} \sum_{i=0}^n b_{i,n+1} \mathcal{K}(t_i, X_h(t_i))
\]
elucidated the need for compliance with regulations on basic preventive measures such as social
simulations confirmed the established properties of the proposed model, and more importantly
showed the global stability of the model in the sense of Ulam-Hyers stability criteria. Numerical
R
number
point theorem. Using the concept of next-generation matrices for computing basic reproduction
environment. Qualitative analysis of both integer order and fractional order models were carried
on the integer-order model of [72] incorporating social distancing measures and interaction with
We have proposed a fractional order epidemic model for COVID-19 infection dynamics based
6 Conclusion
populations peaked in the former.
α
than through the environment; and
α
there is higher risk of contracting COVID-19 virus through contacts with infectious individuals
α
rise in the exposed population and the infected population within the same time period, figure
α
is expected as more people get exposed through virus-laden environment, explaining the rapid
decline in the susceptible population is noticeable in figure 6a during the first ten days. This
Recall that the constants
α
represent the proportion of interaction with the infectious
α
environment and infectious individuals respectively. Consider a scenario of a contaminated
quarantined, thereby preventing contamination of the environment through virus shedding.
distancing/lockdown regulations, and if infectious and infected individuals are appropriately
α
that for fractional orders considered namely
α
= \{1,0.9,0.8,0.7\}, and with new infection rates
α
= 0.1, the corresponding computed
R
0 \in [0.90850,0.82590] showing that the COVID-19 pandemic is controllable and will effectively
die out as long as there is compliance with social distancing/lockdown regulations, and if infectious and infected individuals are appropriately quarantined, thereby preventing contamination of the environment through virus shedding.
Recall that the constants
α
and
α
represent the proportion of interaction with the infectious
environment and infectious individuals respectively. Consider a scenario of a contaminated
environment
α
= 0.05 where there is higher chances of contracting COVID-19 through the
environment than through infectious individuals
I_A
or
I_S
, \(\alpha_1 = 0.1\). In this situation, a rapid
decline in the susceptible population is noticeable in figure 6a during the first ten days. This
is expected as more people get exposed through virus-laden environment, explaining the rapid
rise in the exposed population and the infected population within the same time period, figure
6b
6c
and figure 6d
In comparison with other scenarios, namely
α
= 0.05,\(\alpha_1 = 0.1\) where there is higher risk of contracting COVID-19 virus through contacts with infectious individuals than through the environment; and \(\alpha_2 = 0.1,\alpha_1 = 0.1\) depicting generally low infection rate from both sources, the number of exposed, asymptomatic infectious and symptomatic infectious populations peaked in the former.

6 Conclusion
We have proposed a fractional order epidemic model for COVID-19 infection dynamics based
on the integer-order model of [72] incorporating social distancing measures and interaction with
environment. Qualitative analysis of both integer order and fractional order models were carried
out. In particular, we established the well-posedness of the fractional model via the Banach fixed
point theorem. Using the concept of next-generation matrices for computing basic reproduction
number
R
0 , local asymptotic stability of the fractional model was proved. Furthermore, we
showed the global stability of the model in the sense of Ulam-Hyers stability criteria. Numerical
simulations confirmed the established properties of the proposed model, and more importantly
elucidated the need for compliance with regulations on basic preventive measures such as social
Figure 4: Population dynamics of the fractional model for $\alpha \in \{1, 0.9, 0.8, 0.7\}$ labelled respectively a-d.
Figure 5: Recovered and Pathogens population dynamics of the fractional model for $\alpha \in \{1, 0.9, 0.8, 0.7\}$ labelled respectively (a), (b) respectively.

distancing, quarantining of infectious and infected individuals, and frequent hand washing to rid the population of the deadly virus.

**Declarations of conflict of interests**

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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