The Bishop-Phelps-Bollobás and approximate hyperplane series properties

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ABSTRACT. We study the Bishop-Phelps-Bollobás property for operators between Banach spaces. Sufficient conditions are given for generalized direct sums of Banach spaces with respect to a uniformly monotone Banach sequence lattice to have the approximate hyperplane series property. This result implies that Bishop-Phelps-Bollobás theorem holds for operators from $\ell_1$ into such direct sums of Banach spaces. We also show that the direct sum of two spaces with the approximate hyperplane series property has such property whenever the norm of the direct sum is absolute.

1. Introduction

The motivation for this paper comes from recent intensive study of the famous Bishop-Phelps Theorem [10], which states that every Banach space is subreflexive, i.e., the set of norm attaining (continuous and linear) functionals on a Banach space is dense in its topological dual.

The first who initiated the study of the denseness of norm-attaining operators between two Banach spaces was Lindenstrauss [22]. Later a lot of attention was devoted to extend Bishop-Phelps result in the setting of operators on Banach spaces (see, e.g., [2, 13]).

In 1970, Bollobás showed the following “quantitative version” which is now called Bishop-Phelps-Bollobás Theorem [11]. To state this result we mention that for a normed space $X$, we denote by $B_X$ and $S_X$ the closed unit ball and the unit sphere of $X$, respectively. As usual, $X^*$ denotes the dual Banach space of $X$.

The mentioned above version of the Bishop-Phelps-Bollobás Theorem from [12, Theorem 16.1] states that if $X$ is a Banach space and $0 < \varepsilon < 1$, then given $x \in B_X$ and...
$x^* \in S_{X^*}$ with $|1 - x^*(x)| < \varepsilon^2/4$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

For a refinement of the above result see [15, Corollary 2.4(a)]. In 2008 Acosta, Aron, García and Maestre initiated the study of parallel versions of this result for operators [3]. For two normed spaces $X$ and $Y$ over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $\mathcal{L}(X,Y)$ denotes the space of (bounded and linear) operators from $X$ into $Y$, endowed with the usual operator norm.

We recall the following definition from [3].

**Definition 1.1.** Let $X$ and $Y$ be both either real or complex Banach spaces. It is said that the pair $(X,Y)$ has the Bishop-Phelps-Bollobás property for operators (BPBp), if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for any $T \in S_{\mathcal{L}(X,Y)}$, if $x \in S_X$ is such that $\|Tx\| > 1 - \eta(\varepsilon)$, then there exist an element $u$ in $S_X$ and an operator $S$ in $S_{\mathcal{L}(X,Y)}$ satisfying the following conditions

$$\|Su\| = 1, \quad \|u - x\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

During the last years there are a number of interesting results where it is shown versions of Bishop-Phelps-Bollobás Theorem for operators (see for instance [7, 14 and 20]). It is known that the pair $(X,Y)$ has the BPBp whenever $X$ and $Y$ are finite dimensional spaces (see [3] Proposition 2.4]). If a Banach space $Y$ has the property $\beta$ of Lindenstrauss, then $(X,Y)$ has the BPBp for every Banach space $X$ (see [3] Theorem 2.2]). In the case when $X = \ell_1$ a characterization of the Banach spaces $Y$ such that the pair $(\ell_1,Y)$ has the BPBp was given in [3] Theorem 4.1].

It should be pointed out that very little is known about the stability under direct sums of the property that a pair of Banach spaces $(X,Y)$ has the Bishop-Phelps-Bollobás property for operators. In order to state some results of this kind we recall the following notion used in [4]. Given two Banach spaces $X$ and $Y$ (both real or complex), we say that $Y$ has property $P_X$ if the pair $(X,Y)$ has the BPBp for operators.

It was shown in [8] that the pairs $(X, (\oplus \sum_{n=1}^{\infty} Y_n)_{c_0})$ and $(X, (\oplus \sum_{n=1}^{\infty} Y_n)_{\ell_\infty})$ satisfy the Bishop-Phelps-Bollobás property for operators whenever all pairs $(X,Y_n)$ have the Bishop-Phelps-Bollobás property for operators “uniformly”. In general the analogous stability result does not hold for every Banach sequence lattice $E$ instead of $c_0$. For instance, the subset of norm attaining operators from any Banach space $X$ into $\ell_p$ $(1 \leq p < \infty)$ is not dense in the space of operators from $X$ into $\ell_p$ ([18, 11]) for every Banach space $X$. Indeed it is a longstanding open question if for every (real) Banach space $X$,
the subset of norm attaining operators from \(X\) into the euclidean space \(\mathbb{R}^2\) is dense in the corresponding space of operators. However, it is also known that \(\mathcal{P}_{\ell_1}\) is stable under finite \(\ell_p\)-sums for \(1 \leq p \leq \infty\) (see [4, Corollary 2.8]).

In this paper we provide two nontrivial extensions of the above stability results. On one hand we prove that the property \(\mathcal{P}_{\ell_1}\) is stable under absolute summands (Theorem 2.6). This extends the above mentioned result for finite \(\ell_p\)-sums. We also prove under mild additional assumptions, that the property \(\mathcal{P}_{\ell_1}\) is stable under \(E\)-sums, being \(E\) a uniformly monotone Banach sequence lattice (Theorem 2.10). As a consequence we deduce, for instance, that if \(\{X_k : k \in \mathbb{N}\}\) is a sequence of spaces such that \(X_k\) is either some \(C(K)\) or \(L_1(\mu)\) or a Hilbert space, then the pair \((\ell_1, (\sum_{k=1}^{\infty} X_k)_{\ell_p})\) has the BPBp for operators (Corollary 2.11).

On the other hand, in case that the range is a Hilbert space, we also prove some optimal stability result of BPBp under \(\ell_1\)-sums on the domain (Proposition 2.3). This result extends [21, Proposition 9], where the authors show the above result for the \(\ell_1\)-sum of copies of the same space.

As we already mentioned there is a characterization of the Banach spaces \(Y\) such that the pair \((\ell_1, Y)\) has the Bishop-Phelps-Bollobás property for operators [3]. The property on \(Y\) equivalent to the previous fact was called the AHSp (Approximate hyperplane series property).

We will use the following definition, where in what follows by a convex series we mean a series \(\sum \alpha_n\), where \(0 \leq \alpha_n \leq 1\) for each \(n \in \mathbb{N}\) and \(\sum_{n=1}^{\infty} \alpha_n = 1\).

**Definition 1.2.** A Banach space \(X\) has the approximate hyperplane series property (AHSp) if and only if for every \(0 < \varepsilon < 1\) there exists \(0 < \eta < \varepsilon\) such that for every sequence \(\{x_n\}\) in \(S_X\) and every convex series \(\sum \alpha_n\) with

\[
\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| > 1 - \eta,
\]

there exist a subset \(A \subset \mathbb{N}\) and a subset \(\{z_k : k \in A\} \subset S_X\) satisfying

1. \(\sum_{k \in A} \alpha_k > 1 - \varepsilon\),
2. \(\|z_k - x_k\| < \varepsilon\) for all \(k \in A\) and
3. there is \(x^* \in S_{X^*}\) such that \(x^*(z_k) = 1\) for every \(k \in A\).

We will use the following characterization of the AHSp (see [4, Proposition 1.2].)

**Proposition 1.3.** Let \(X\) be a Banach space. The following conditions are equivalent:
(a) $X$ has the AHSp.
(b) For every $0 < \varepsilon < 1$ there exist $\gamma_X(\varepsilon) > 0$ and $\eta_X(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0} \gamma_X(\varepsilon) = 0$ such that for every sequence $\{x_n\}$ in $B_X$ and every convex series $\sum_n \alpha_n$ with $\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta_X(\varepsilon)$, there are a subset $A \subseteq \mathbb{N}$ with $\sum_{k \in A} \alpha_k > 1 - \gamma_X(\varepsilon)$, an element $x^* \in S_{X^*}$, and $\{z_k : k \in A\} \subseteq (x^*)^{-1}(1) \cap B_X$ such that $\|z_k - x_k\| < \varepsilon$ for all $k \in A$.
(c) For every $0 < \varepsilon < 1$ there exists $0 < \eta < \varepsilon$ such that for any sequence $\{x_n\}$ in $B_X$ and every convex series $\sum_n \alpha_n$ with $\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta$, there are a subset $A \subseteq \mathbb{N}$ with $\sum_{k \in A} \alpha_k > 1 - \varepsilon$, an element $x^* \in S_{X^*}$, and $\{z_k : k \in A\} \subseteq (x^*)^{-1}(1) \cap B_X$ such that $\|z_k - x_k\| < \varepsilon$ for all $k \in A$.
(d) The same statement holds as in (c) but for every sequence $\{x_n\}$ in $S_X$.

2. The main results

In the section we study the Bishop-Phelps-Bollobás property for operators between special types of Banach spaces. In particular we are interested in stability of this property when the domain is an $\ell_1$ sum of Banach spaces. Throughout the paper we consider either real or complex Banach spaces.

We will need the following lemma (see [3] Lemma 3.3).

**Lemma 2.1.** Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leq 1$ for each $n$ and let $\eta > 0$ be such that there is some sequence $\{\alpha_n\}$ of nonnegative real numbers satisfying $\sum_{n=1}^{\infty} \alpha_n \leq 1$ and $\Re \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \Re c_i > r\}$, satisfies the estimate

$$\sum_{i \in A} \alpha_i > 1 - \frac{\eta}{1 - r}.$$ 

We also need the following technical lemma. For the sake of completeness we include a proof.

**Lemma 2.2.** Let $H$ be a real or complex Hilbert space and assume that $u, v \in S_H$. Then there is a surjective linear isometry $\Phi$ on $H$ such that $\Phi(u) = v$ and $\|\Phi - I\| = \|u - v\|$.

**Proof.** The result is obvious in the case $\dim H = 1$. Assume that $\dim H \geq 2$. Thus there is an element $v^\perp \in S_H$ orthogonal to $v$ and such that $[u, v] \subset [v, v^\perp]$, where $[x, y]$ is the linear span of the vectors $x$ and $y$ in $H$. Let $u_1, u_2 \in \mathbb{K}$ such that $u = u_1 v + u_2 v^\perp$ and...
write $u^\perp = -\overline{w_2}v + \overline{v_1}w_1$. It is clearly satisfied that
\[
1 = \|u\|^2 = |u_1|^2 + |u_2|^2 \quad \text{and} \quad \langle u, u^\perp \rangle = 0.
\]

Let $M$ be a subspace of $H$ orthogonal to $[v, v^\perp] = [u, u^\perp]$ and such that $H = [u, u^\perp] \oplus M$. Define the mapping $\Phi : H \to H$ given by
\[
\Phi(zu + wu^\perp + m) = vz + wv^\perp + m, \quad \forall (z, w) \in \mathbb{K}^2, m \in M,
\]
which is a surjective linear isometry on $H$. It clearly satisfies $\Phi(u) = v$ and $\Phi(u^\perp) = v^\perp$.

Clearly $(\Phi - I)(u) = v - u, (\Phi - I)(u^\perp) = v^\perp - u^\perp$ and $\|v - u\| = \|v^\perp - u^\perp\|$. Also we have that
\[
\langle v - u, v^\perp - u^\perp \rangle = -\left(\langle v, u^\perp \rangle + \langle u, v^\perp \rangle\right) = 0.
\]
Hence $\Phi - I$ restricted to $[u, u^\perp]$ is a multiple of a linear isometry from this subspace into itself. As a consequence $\|\Phi - I\| = \|v - u\|$. \hfill $\square$

The next result uses the argument outlined in \cite{MR1820208} Proposition 9 in the case that the domain is the $\ell_1$-sum of one space.

**Proposition 2.3.** Assume that $\{X_i : i \in I\}$ is a family of Banach spaces, $H$ is a Hilbert space such that the pair $(X_i, H)$ has the BPBp for operators for every $i \in I$ and with the same function $\eta$. Then the pair $\left((\bigoplus_{i \in I} X_i), H\right)$ has the BPBp.

**Proof.** We write $Z = \left((\bigoplus_{i \in I} X_i), H\right)$. Given $0 < \varepsilon < 1$, we choose positive real numbers $r, s$ and $t$ such that
\[
(2.1) \quad r < \frac{\varepsilon}{4}, \quad s < \min\left\{\frac{\varepsilon}{4}, \frac{\delta_H(r)}{3}\right\} \quad \text{and} \quad t < \min\left\{\frac{\varepsilon}{4}, \eta(s), \frac{\delta_H(r)}{3}\right\},
\]
where $\delta_H$ is the modulus of convexity of $H$.

Assume that $z_0 = \{z_0(i)\} \in S_Z$ and $T \in S_{\mathcal{L}(Z, H)}$ satisfies $\|T(z_0)\| > 1 - t^2$. For every $i \in I$, we denote by $T_i$ the restriction of $T$ to $X_i$, that is embedded in $Z$ in a natural way. Assume that $y^* \in S_{H^*}$ satisfies that $\text{Re} y^*(T(z_0)) = \|T(z_0)\| > 1 - t^2$.

Denote by $B = \{i \in I : \text{Re} y^*(T_i(z_0(i))) > (1 - t)\|z_0(i)\|\}$. We clearly have that
\[
1 - t^2 < \text{Re} y^*(T(z_0)) = \sum_{i \in I} \text{Re} y^*(T_i(z_0(i)))
\]
\[
= \sum_{i \in B} \text{Re} y^*(T_i(z_0(i))) + \sum_{i \in I \setminus B} \text{Re} y^*(T_i(z_0(i)))
\]
\[
\leq \sum_{i \in B} \|z_0(i)\| + \sum_{i \in I \setminus B} (1 - t)\|z_0(i)\|
\]
\[
= 1 - t \sum_{i \in I \setminus B} \|z_0(i)\|.
\]
Hence

\[(2.2) \quad \sum_{i \in I \setminus B} \|z_0(i)\| \leq t.\]

By assumption, for every \(i \in B\) there is an operator \(S_i \in \mathcal{L}(X_i,H)\) and an element \(x_i \in S_{X_i}\) such that

\[(2.3) \quad \left\| S_i - \frac{T_i}{\|T_i\|} \right\| < s, \quad \left\| x_i - \frac{z_0(i)}{\|z_0(i)\|} \right\| < s \quad \text{and} \quad \|S_i(x_i)\| = 1, \quad \forall i \in B.\]

It follows by (2.3) that for every \(i, j \in B\) we have that

\[
\begin{align*}
\|S_i(x_i) + S_j(x_j)\| &\geq \left\| \frac{S_i(z_0(i))}{\|z_0(i)\|} + \frac{S_j(z_0(j))}{\|z_0(j)\|} \right\| - 2s \\
&\geq \left\| \frac{T_i(z_0(i))}{\|T_i\|\|z_0(i)\|} + \frac{T_j(z_0(j))}{\|T_j\|\|z_0(j)\|} \right\| - 4s \\
&\geq 2(1 - t) - 4s \\
&> 2(1 - \delta_H(r)).
\end{align*}
\]

As a consequence \(\|S_i(x_i) - S_j(x_j)\| \leq r\) for each \(i, j \in B\).

Since \(B \neq \emptyset\), we choose some element \(i_0 \in B\) and define \(y_0 = S_{i_0}(x_{i_0})\). By Lemma 2.2, for every \(i \in B\), there is a linear surjective isometry \(\Phi_i : H \to H\) such that \(\Phi_i(S_i(x_i)) = y_0\) and \(\|\Phi_i - I\| = \|S_i(x_i) - y_0\| \leq r\).

We define an operator \(R = \{R_i\}_{i \in I} \in \mathcal{L}(Z,H)\) by

\[R_i = \Phi_i \circ S_i, \quad \forall i \in B \quad \text{and} \quad R_i = T_i, \quad \forall i \in I \setminus B.\]

Clearly that \(R\) is in the unit ball of \(\mathcal{L}(Z,H)\) and it satisfies

\[
\|R - T\| = \sup\{\|R_i - T_i\| : i \in B\}
\leq \sup\{\|\Phi_i - I\| : i \in B\} + \sup\{\|S_i - T_i\| : i \in B\}
\leq r + \sup\left\{ \left\| S_i - \frac{T_i}{\|T_i\|} \right\| : i \in B \right\} + \sup\left\{ \left\| T_i \right\| - T_i : i \in B \right\}
\leq r + s + \sup\left\{ \left| 1 - \frac{T_i}{\|T_i\|} \right| : i \in B \right\}
\leq r + s + t < \varepsilon.
\]

Let \(P_B\) be the natural projection on the subspace of elements in \(Z\) whose support is contained in \(B\).

Now observe that \(x_0\) given by

\[
x_0(i) = \begin{cases} 
\frac{\|z_0(i)\| x_i}{\|P_B(z_0)\|}, & \text{if } i \in B \\
0 & \text{if } i \in I \setminus B
\end{cases}
\]
belongs to $S_Z$ and also satisfies
\[
\|x_0 - z_0\| \leq \|x_0 - P_B(z_0)\|_x + \|P_B(z_0)\|_x_0 - z_0 + \|z_0 \chi_{I \setminus B}\|
\]
\[
\leq \|1 - P_B(z_0)\| + \sum_{i \in B} \|z_0(i)\|_x_i - z_0 + \|z_0 \chi_{I \setminus B}\|
\]
\[
\leq 2\|z_0 \chi_{I \setminus B}\| + s \sum_{i \in B} \|z_0(i)\| \quad \text{(by (2.3))}
\]
\[
\leq 2t + s \quad \text{(by (2.2))}
\]
\[
< \varepsilon.
\]
It remains to check that $R$ attains its norm at $x_0$. Indeed,
\[
\|R(x_0)\| = \frac{1}{\|P_B(z_0)\|} \left\| \sum_{i \in B} z_0(i)\|R_i(x_i)\| \right\|
\]
\[
= \frac{1}{\|P_B(z_0)\|} \left\| \sum_{i \in B} z_0(i)\|\Phi_i(S_i(x_i))\| \right\|
\]
\[
= \frac{1}{\|P_B(z_0)\|} \left\| \sum_{i \in B} z_0(i)\|y_0\| \right\| = 1.
\]
Hence $R \in S_{L(Z,H)}$ and $\|R(x_0)\| = 1$. This completes the proof that the pair $(Z,H)$ has the BPBp.

Let us note that it follows from [8, Theorem 2.1] that $(X_i,H)$ has the BPBp for every $i \in I$ with the same function $\eta$ provided that $((\oplus \sum_{i \in I} X_i)_{\ell_1},H)$ has the BPBp. This shows that the assumption in Proposition 2.3 is a necessary condition.

Now we prove stability results of the Bishop-Phelps-Bollobás property for operators when the domain is $\ell_1$.

As we already mentioned it was proved that the pair $(\ell_1,Y)$ has the BPBp for operators if, and only if, $Y$ has the approximate hyperplane series property (see [3, Theorem 4.1]). Since the AHSp is an isometric property, if a space is the (topological) direct sum of two subspaces with the AHSp, in general it does not have the AHSp. However, we will prove that this property is stable under sums involving an absolute (or monotone) norm. First we recall this notion.

**Definition 2.4.** Let $X$ and $Y$ be Banach spaces, and $Z = X \oplus Y$, a norm $\| \cdot \|_f$ in $Z$ is said to be absolute if there is a function $f : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \rightarrow \mathbb{R}^+_0$ such that
\[
(2.4) \quad \|x + y\|_f = f(\|x\|,\|y\|), \quad \forall x \in X, \ y \in Y.
\]
The absolute norm is normalized if $f(1,0) = 1 = f(0,1)$.
It is immediate to check that in case that the equality (2.4) gives a norm in $Z$, the function $f$ can be extended to a norm $|\cdot|$ on $\mathbb{R}^2$ satisfying $|(r, s)| = f(|r|, |s|)$ for every pair of real numbers $(r, s)$.

We also recall that the norm $|\cdot|$ is absolute on $\mathbb{R}^2$ if, and only if, it satisfies

$$|r| \leq |s|, \ |t| \leq |u| \ \Rightarrow \ f(r, t) \leq f(s, u)$$

(see for instance [12, Lemma 21.2]).

Clearly the usual $\ell_p$-norm of the sum of two Banach spaces is an absolute norm for every $1 \leq p \leq \infty$.

Next result is a far reaching extension of Proposition 2.1, Theorems 2.3 and 2.6 in [4], where the $\ell_p$-norm on $\mathbb{R}^2$ for $1 \leq p < \infty$ is considered. Part of the essential idea of the argument we will use is contained there, however our proof is simpler.

The following technical lemma will be useful in the proof of the main result.

**Lemma 2.5.** Let $|\cdot|$ be an absolute and normalized norm on $\mathbb{R}^2$. For every $\varepsilon > 0$ there is $\delta > 0$ satisfying the following conditions:

$$(r, s) \in \mathbb{R}^2, \quad |(r, s)| = 1, \ s > 1 - \delta \ \Rightarrow \ \exists t \in \mathbb{R} : |(t, 1)| = 1 \ \text{and} \ |t - r| < \varepsilon$$

and

$$(r, s) \in \mathbb{R}^2, \quad |(r, s)| = 1, \ r > 1 - \delta \ \Rightarrow \ \exists t \in \mathbb{R} : |(1, t)| = 1 \ \text{and} \ |t - s| < \varepsilon.$$

**Proof.** Of course it suffices to check only the first assertion. Assume that it is not true. Hence there is some $\varepsilon_0 > 0$ such that

$$\forall \delta > 0 \ \exists (r_\delta, s_\delta) \in S_{(\mathbb{R}^2, |\cdot|)}, \ s_\delta > 1 - \delta \ \text{and} \ t \in \mathbb{R} \ \text{with} \ |(t, 1)| = 1 \ \Rightarrow \ |t - r_\delta| \geq \varepsilon_0.$$

We choose any sequence $\{\delta_n\}$ of positive real numbers converging to 0. By assumption there is a sequence $\{(r_n, s_n)\}$ in $S_{(\mathbb{R}^2, |\cdot|)}$ satisfying for each $n \in \mathbb{N}$ that

$$(2.5) \quad s_n > 1 - \delta_n \ \text{and} \ |t - r_n| \geq \varepsilon_0 \ \forall t \in \mathbb{R} \ \text{with} \ |(t, 1)| = 1.$$  

By passing to a subsequence, we may assume that $(r_n, s_n) \to (r, s)$. Since $|(0, 1)| = 1$ and the norm is absolute on $\mathbb{R}^2$ it is satisfied

$$s = |(0, s)| \leq |(r, s)| = 1.$$  

Since $s_n > 1 - \delta_n$ for each $n$ we also have $s \geq 1$. So $s = 1$. So $|(r, 1)| = 1$. We also know that $r_n \to r$, hence $(r_n, s_n) \to (r, 1)$ and this contradicts condition (2.5). \qed
Theorem 2.6. Assume that $| \cdot |$ is an absolute and normalized norm on $\mathbb{R}^2$. Let $X$ be a (real or complex) Banach space that can be decomposed as $X = M \oplus N$ for certain subspaces $M$ and $N$ and such that

$$
\|(m,n)\| = |(\|m\|, \|n\|)|, \quad \forall m \in M, \ n \in N.
$$

Then $X$ has the AHSp if, and only if, both $M$ and $N$ has the AHSp. In such case, both subspaces satisfy Definition 1.2 with the same function $\eta$.

Proof. We can clearly assume that both $M$ and $N$ are non-trivial. Let $P$ and $Q$ be the natural projections from $X$ onto $M$ and $N$, respectively.

First we check the necessary condition. So assume that $X$ has the AHSp and we show that $M$ also has the AHSp. Let us fix $0 < \varepsilon < 1$ and let $\eta_0$ be the positive number satisfying Definition 1.2 for the space $X$ and $\varepsilon/2$.

Assume that $\sum_{k=1}^{\infty} \alpha_k m_k$ is a convex series with $\{m_k : k \in A\} \subset S_M$ satisfying

$$
\left\| \sum_{k=1}^{\infty} \alpha_k m_k \right\| > 1 - \eta_0.
$$

By the assumption there are $A \subset \mathbb{N}$ and $\{x_k : k \in \mathbb{N}\} \subset S_X$ such that

$$
\sum_{k \in A} \alpha_k > 1 - \frac{\varepsilon}{2} > 0, \quad \|x_k - m_k\| < \frac{\varepsilon}{2}, \quad \forall k \in A \quad \text{and} \quad \text{co}\{x_k : k \in A\} \subset S_X.
$$

So $A \neq \emptyset$.

Since the norm $| \cdot |$ on $\mathbb{R}^2$ is an absolute norm it is satisfied

\begin{equation}
(2.6) \quad \|P(x_k) - m_k\| = \|P(x_k - m_k)\| \leq \|x_k - m_k\| < \frac{\varepsilon}{2},
\end{equation}

and

$$
\|Q(x_k)\| \leq \|x_k - m_k\| < \frac{\varepsilon}{2}.
$$

Hence we have that

\begin{equation}
(2.7) \quad \|P(x_k)\| > 1 - \frac{\varepsilon}{2} \quad \text{and} \quad \|Q(x_k)\| < \frac{\varepsilon}{2}, \quad \forall k \in A.
\end{equation}

On the other hand, since $\text{co}\{x_k : k \in A\} \subset S_X$ there is $x^* \in S_{X^*}$ that can be decomposed as $x^* = m^* + n^*$, for some $m^* \in M^*$ and $n^* \in N^*$ and such that for each $k \in A$ it is
satisfied

\[ 1 = \text{Re} \, x^*(-x_k) \]
\[ = \text{Re} \, m^*(P(x_k)) + \text{Re} \, n^*(Q(x_k)) \]
\[ \leq \|m^*\| \|P(x_k)\| + \|n^*\| \|Q(x_k)\| \]
\[ = (\|m^*\|, \|n^*\|) (\|P(x_k)\|, \|Q(x_k)\|) \]
\[ \leq \|x^*\| \|x_k\| = 1. \]

As a consequence, we obtain that

\[ m^*(P(x_k)) = \|m^*\| \|P(x_k)\|, \quad \forall k \in A. \]

Let us fix \( k \in A \). If \( m^* = 0 \), in view of (2.8) we obtain that \( \|Q(x_k)\| = 1 \), which contradicts (2.7). By using again (2.7) we also know that \( P(x_k) \neq 0 \), so we can write \( u_k = \frac{P(x_k)}{\|P(x_k)\|} \). By (2.9) we obtain that

\[ \frac{m^*}{\|m^*\|}(u_k) = 1 \quad \forall k \in A \]

and clearly \( \frac{m^*}{\|m^*\|} \in S_{M^*} \subset S_{X^*} \).

For \( k \in A \) we also have

\[ \|u_k - m_k\| \leq \frac{P(x_k)}{\|P(x_k)\|} - P(x_k) + \|P(x_k) - m_k\| \]
\[ \leq 1 - \|P(x_k)\| + \|P(x_k) - m_k\| \]
\[ < \varepsilon \quad \text{(by (2.7) and (2.6))}. \]

We checked that \( M \) has the AHSp.

Conversely, assume that \( M \) and \( N \) have the AHSp. We will prove that \( X \) also has the AHSp. Let \( \varepsilon \) be a real number with \( 0 < \varepsilon < 1 \). In view of Lemma 2.5 there is \( 0 < \delta < 1 \) satisfying the following conditions

\[ (a, b) \in S_{(\mathbb{R}^2, |\cdot|)}, \quad b > 1 - \delta \implies \exists c \in \mathbb{R} : \|(c, 1)\| = 1 \quad \text{and} \quad |a - c| < \frac{\varepsilon}{5} \]

and

\[ (a, b) \in S_{(\mathbb{R}^2, |\cdot|)}, \quad a > 1 - \delta \implies \exists c \in \mathbb{R} : \|(1, c)\| = 1 \quad \text{and} \quad |b - c| < \frac{\varepsilon}{5}. \]

Let us choose \( 0 < \varepsilon_1 < \frac{\varepsilon}{8} \). Assume that the pair \((\varepsilon_1, \eta_1)\) satisfy condition (c) in Proposition 1.3 for both \( M \) and \( N \). We also fix real numbers \( r, s \) and \( \varepsilon_0 \) such that

\[ 0 < s < \min \left\{ \frac{\delta}{2}, \frac{\eta_1}{2} \right\}, \quad 0 < r < \min \left\{ \frac{\delta}{2}, s^2 \eta_1 \right\} \quad \text{and} \quad 0 < \varepsilon_0 < \frac{r \varepsilon}{8}. \]
By [3] Proposition 3.5] finite-dimensional spaces have the AHSp. So for every \( \varepsilon_0 > 0 \) there is \( 0 < \eta_0 < \varepsilon_0 \) satisfying condition (d) in Proposition [13] for \( \mathbb{R}^2 \) endowed with the norm \( | \cdot | \).

Let \( \{ x_k \} \) be a sequence in \( S_X \) and \( \sum \alpha_k \) be a convex series such that
\[
\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta_0.
\]
Hence we have
\[
1 - \eta_0 < \left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| = \left\| \sum_{k=1}^{\infty} \alpha_k (P(x_k) + Q(x_k)) \right\|
\leq \left( \left\| \sum_{k=1}^{\infty} \alpha_k P(x_k) \right\| + \left\| \sum_{k=1}^{\infty} \alpha_k Q(x_k) \right\| \right).
\]

Since \( (\mathbb{R}^2, | \cdot |) \) has the AHSp, it follows that for the convex series \( \sum_{k=1}^{\infty} \alpha_k (\| P(x_k) \|, \| Q(x_k) \|) \), there are a subset \( A \subset \mathbb{N} \), \( \{(r_k, s_k) : k \in A\} \subset S_{\mathbb{R}^2} \) and \( (\alpha, \beta) \in S(\mathbb{R}^2) \), satisfying
\[
\sum_{k \in A} \alpha_k > 1 - \varepsilon_0, \quad r_k, s_k \geq 0, \quad \alpha r_k + \beta s_k = 1, \quad \forall k \in A,
\]
and
\[
\| P(x_k) \| - r_k < \varepsilon_0, \quad \| Q(x_k) \| - s_k < \varepsilon_0, \quad \forall k \in A.
\]

It is clearly satisfied that
\[
\left\| \sum_{k \in A} \alpha_k x_k \right\| \geq \left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| - \left\| \sum_{k \in \mathbb{N} \setminus A} \alpha_k x_k \right\|
\]
\[
\geq \left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| - \sum_{k \in \mathbb{N} \setminus A} \alpha_k
\]
\[
> 1 - \eta_0 - \varepsilon_0 \quad (\text{by (2.13)})
\]
\[
> 1 - 2\varepsilon_0.
\]

Now fix arbitrary elements \( m_0 \in S_M \) and \( n_0 \in S_N \) and define the following elements:
\[
m_k := \begin{cases} 
\frac{r_k P(x_k)}{\| P(x_k) \|} & \text{if } k \in A \text{ and } P(x_k) \neq 0 \\
\frac{r_k m_0}{r_k m_0} & \text{if } k \in A \text{ and } P(x_k) = 0
\end{cases}
\]
and

\[ n_k := \begin{cases} \frac{s_k Q(x_k)}{\|Q(x_k)\|} & \text{if } k \in A \text{ and } Q(x_k) \neq 0 \\ \frac{s_k n_0}{|s_k|} & \text{if } k \in A \text{ and } Q(x_k) = 0. \end{cases} \]

Next we write \( y_k := m_k + n_k \) for all \( k \in A \). Since \( |(r_k, s_k)| = 1 \) for every \( k \in A \), it is clear that \( \{y_k : k \in A\} \subset S_X \) and in view of (2.14) we obtain

\[ \|y_k - x_k\| \leq |r_k - \|P(x_k)\|| + |s_k - \|Q(x_k)\|| < 2\varepsilon_0, \quad \forall k \in A. \]

By the previous inequality and bearing in mind (2.15) we have

\[ \left\| \sum_{k \in A} \alpha_k y_k \right\| > \left\| \sum_{k \in A} \alpha_k x_k \right\| - 2\varepsilon_0 > 1 - 4\varepsilon_0. \]

In view of Hahn-Banach theorem there is a functional \( x^* \in S_X^* \) such that

\[ \text{Re} x^* \left( \sum_{k \in A} \alpha_k y_k \right) = \left\| \sum_{k \in A} \alpha_k y_k \right\| > 1 - 4\varepsilon_0. \]

Now we define \( B = \{ k \in A : \text{Re} x^*(y_k) > 1 - r \} \). In view of Lemma 2.1 we have that

\[ \sum_{k \in B} \alpha_k > 1 - \frac{4\varepsilon_0}{r} > 0. \]

If we decompose \( x^* = m^* + n^* \), for each \( k \in B \) we have that

\[ 1 - r < \text{Re} x^*(y_k) = \text{Re}(m^*(m_k) + n^*(n_k)) \]

\[ \leq \|m^*\| \|m_k\| + \text{Re} n^*(n_k) \]

\[ \leq \|m^*\| \|m_k\| + \|n^*\| \|n_k\| \leq 1. \]

As a consequence of (2.18), for each \( k \in B \), we also have that

\[ \|m^*\| r_k = \|m^*\| \|m_k\| \leq \text{Re} m^*(m_k) + r \]

and

\[ \|n^*\| s_k = \|n^*\| \|n_k\| \leq \text{Re} n^*(n_k) + r. \]

In order to show the result we will consider three cases:

Case 1) Assume that \( \|m^*\| \leq s \).

Since \( \|n^*\| \leq \|x^*\| = 1 \), in view of (2.18) we know that

\[ s_k \geq \|n^*\| s_k \geq 1 - r - s > 1 - \delta, \quad \forall k \in B. \]

By using also (2.20) we obtain that

\[ \text{Re} n^*(n_k) \geq 1 - 2r - s > 1 - \eta_1, \quad \forall k \in B. \]
Since $N$ has the AHSp there are $C \subset B$, \( \{v_k : k \in C\} \subset S_N \) and \( n_1^* \in S_{N^*} \) such that
\[
(2.22) \quad \sum_{k \in C} \alpha_k > (1 - \varepsilon_1) \sum_{k \in B} \alpha_k, \quad n_1^*(v_k) = 1 \quad \text{and} \quad \|v_k - n_k\| < \varepsilon_1, \quad \forall k \in C.
\]

By $(2.21)$ we can use $(2.10)$, and so for every $k \in C$ there is $a_k \in \mathbb{R}$ such that
\[
(2.23) \quad \langle (a_k, 1) \rangle = 1, \quad |a_k - r_k| < \frac{\varepsilon}{5}.
\]
So we define the subset \( \{z_k : k \in C\} \subset X \) by
\[
z_k = \begin{cases} \frac{m_k}{\|m_k\|} + v_k & \text{if } m_k \neq 0, \\ a_k m_0 + v_k & \text{if } m_k = 0, \end{cases} \quad \forall k \in C.
\]
Clearly we have that
\[
\|z_k\| = \langle (a_k, 1) \rangle = 1, \quad \forall k \in C.
\]

By $(2.16)$, $(2.23)$ and $(2.22)$ we obtain that
\[
\|z_k - x_k\| \leq \|z_k - y_k\| + \|y_k - x_k\| \\
\leq |a_k - r_k| + \|v_k - n_k\| + 2\varepsilon_0 \\
\leq \frac{\varepsilon}{5} + \varepsilon_1 + 2\varepsilon_0 \\
< \varepsilon.
\]

We also have that
\[
n_1^*(z_k) = n_1^*(v_k) = 1, \quad \forall k \in C.
\]

Finally from $(2.22)$ and $(2.17)$ we also know that
\[
\sum_{k \in C} \alpha_k > (1 - \varepsilon_1) \sum_{k \in B} \alpha_k > (1 - \varepsilon_1) \left(1 - \frac{4\varepsilon_0}{r}\right) > 1 - \varepsilon_1 - \frac{4\varepsilon_0}{r} > 1 - \varepsilon.
\]
So the proof is finished in this case.

Case 2) Assume that $\|n^*\| \leq s$.

We can proceed in the same way that in Case 1, but by using that $M$ has the AHSp.

Case 3) Assume that $\|m^*\|, \|n^*\| > s$.

We define the set $B_1$ given by
\[
B_1 = \{k \in B : r_k \geq s\}.
\]

For each element $k \in B_1$, in view of $(2.19)$ we have that
\[
\frac{\text{Re} m^*(m_k)}{\|m^*\| r_k} \geq 1 - \frac{r}{\|m^*\| r_k} \geq 1 - \frac{r}{s^2} > 1 - \eta_1.
\]

Since $M$ has the AHSp there is a set $D_1 \subset B_1$, \( \{u_k : k \in D_1\} \subset S_M \) and \( m^*_1 \in S_{M^*} \) such that
\[
(2.24) \quad \sum_{k \in D_1} \alpha_k > (1 - \varepsilon_1) \sum_{k \in B_1} \alpha_k \geq \sum_{k \in B_1} \alpha_k - \varepsilon_1
\]
and

\[(2.25) \quad \left\| u_k - \frac{m_k}{r_k} \right\| < \varepsilon_1, \quad m_1^*(u_k) = 1, \quad \forall k \in D_1. \]

In an analogous way, we can proceed by defining the set \( C_1 = \{ k \in B : s_k \geq s \} \) and by using that \( N \) has the AHSp we obtain that there is a set \( F_1 \subset C_1, \{ v_k : k \in F_1 \} \subset S_N \) and \( n_1^* \in S_{N*} \) such that

\[(2.26) \quad \sum_{k \in F_1} \alpha_k \geq (1 - \varepsilon_1) \sum_{k \in C_1} \alpha_k \geq \sum_{k \in C_1} \alpha_k - \varepsilon_1 \]

and

\[(2.27) \quad \left\| v_k - \frac{n_k}{s_k} \right\| < \varepsilon_1, \quad n_1^*(v_k) = 1, \quad \forall k \in F_1. \]

Let us notice that for \( k \in B \setminus B_1 \) we have that \( r_k \leq s \) and since \( 1 = |(r_k, s_k)| \leq s + s_k < \frac{1}{2} + s_k \) then \( s_k > \frac{1}{2} > s \). Hence \( k \in C_1 \). Hence we checked that

\[(2.28) \quad B \setminus B_1 \subset C_1 \quad \text{and so} \quad B \setminus C_1 \subset B_1. \]

Clearly we have that

\[(2.29) \quad \sum_{k \in B \setminus B_1} \alpha_k \leq \sum_{k \in D_1 \cap F_1} \alpha_k + \sum_{k \in B_1 \setminus D_1} \alpha_k + \sum_{k \in C_1 \setminus F_1} \alpha_k \]

\[\leq \sum_{k \in D_1 \cap F_1} \alpha_k + 2\varepsilon_1 \quad \text{(by (2.24) and (2.26)).} \]

We also obtain

\[(2.30) \quad \sum_{k \in B \setminus B_1} \alpha_k = \sum_{k \in (B \setminus B_1) \cap F_1} \alpha_k + \sum_{k \in (B \setminus B_1) \setminus F_1} \alpha_k \]

\[\leq \sum_{k \in (B \setminus B_1) \cap F_1} \alpha_k + \sum_{k \in C \setminus F_1} \alpha_k \quad \text{(by (2.28))} \]

\[\leq \sum_{k \in (B \setminus B_1) \cap F_1} \alpha_k + \varepsilon_1 \quad \text{(by (2.26)).} \]

By arguing as above we get

\[(2.31) \quad \sum_{k \in B \setminus C_1} \alpha_k \leq \sum_{k \in (B \setminus C_1) \cap D_1} \alpha_k + \sum_{k \in B \setminus (C_1 \setminus D_1)} \alpha_k \]

\[\leq \sum_{k \in (B \setminus C_1) \cap D_1} \alpha_k + \sum_{k \in B_1 \setminus D_1} \alpha_k \quad \text{(by (2.28))} \]

\[\leq \sum_{k \in (B \setminus C_1) \cap D_1} \alpha_k + \varepsilon_1 \quad \text{(by (2.24)).} \]

Now we take the set \( C \) given by \( C = (D_1 \cap F_1) \cup ((B \setminus B_1) \cap F_1) \cup ((B \setminus C_1) \cap D_1) \). Let us notice that in view of (2.28) the three subsets whose union is \( C \) are pairwise disjoint.
We deduce that

\[
\sum_{k\in C} \alpha_k = \sum_{k\in D_1 \cap F_1} \alpha_k + \sum_{k\in (B\setminus B_1) \cap F_1} \alpha_k + \sum_{k\in (B\setminus C_1) \cap D_1} \alpha_k \\
\geq \sum_{k\in B_1 \cap C_1} \alpha_k + \sum_{k\in B\setminus B_1} \alpha_k + \sum_{k\in B\setminus C_1} \alpha_k - 4\varepsilon_1 \quad \text{(by (2.29), (2.30) and (2.31))} \\
= \sum_{k\in B} \alpha_k - 4\varepsilon_1 \\
> 1 - \frac{4\varepsilon_0}{r} - 4\varepsilon_1 \quad \text{(by (2.17))} \\
> 1 - \varepsilon.
\]

If \(D_1 = \emptyset\), then \(C = (B\setminus B_1) \cap F_1\). In this case we choose any elements \(u_0 \in S_M\) and \(m^*_1(u_0) = 1\). Analogously, in case that \(F_1 = \emptyset\), we have \(C = (B\setminus C_1) \cap D_1\) and we choose \(v_0 \in S_N\) and \(n^*_1(v_0) = 1\). Otherwise \(D_1 \neq \emptyset\) and \(F_1 \neq \emptyset\) and so the elements \(m^*_1\) and \(n^*_1\) satisfying (2.25) and (2.27) attain their norms; so in this case we can choose \(u_0 \in S_M\) and \(v_0 \in S_N\) with \(m^*_1(u_0) = 1\) and \(n^*_1(v_0) = 1\).

For each \(k \in C\) we define

\[
z_k = \begin{cases} 
    r_k u_k + s_k v_k & \text{if } k \in D_1 \cap F_1 \\
    r_k u_0 + s_k v_k & \text{if } k \in (B\setminus B_1) \cap F_1 \\
    r_k u_k + s_k v_0 & \text{if } k \in (B\setminus C_1) \cap D_1.
\end{cases}
\]

We claim that \(\|z_k - x_k\| < \varepsilon\) for each \(k \in C\). To see this observe that for \(k \in D_1 \cap F_1\) we have

\[
\|z_k - x_k\| \leq \|z_k - y_k\| + \|y_k - x_k\| \\
\leq \left( r_k \left\| u_k - \frac{m_k}{r_k} \right\|, s_k \left\| v_k - \frac{n_k}{s_k} \right\| \right) + 2\varepsilon_0 \quad \text{(by (2.16))} \\
\leq \left( r_k \varepsilon_1, s_k \varepsilon_1 \right) + 2\varepsilon_0 \quad \text{(by (2.25) and (2.27))} \\
\leq \varepsilon_1 + 2\varepsilon_0 < \varepsilon.
\]

For \(k \in (B\setminus B_1) \cap F_1\) we have that

\[
\|z_k - x_k\| \leq \|z_k - y_k\| + \|y_k - x_k\| \\
\leq 2r_k + s_k \left\| v_k - \frac{n_k}{s_k} \right\| + 2\varepsilon_0 \quad \text{(by (2.16))} \\
\leq 2s + \varepsilon_1 + 2\varepsilon_0 \quad \text{(by (2.27))} \\
< \varepsilon.
\]
In case when $k \in (B \setminus C_1) \cap D_1$,
\[
\|z_k - x_k\| \leq \|z_k - y_k\| + \|y_k - x_k\|
\leq r_k \left\| u_k - \frac{m_k}{r_k} \right\| + 2s_k + 2\varepsilon_0 \quad \text{(by (2.16))}
\leq \varepsilon_1 + 2s + 2\varepsilon_0 \quad \text{(by (2.25))}
\leq \varepsilon
\]
and this proves the claim.

Now we observe that $\alpha m_1^* + \beta n_1^* \in X^*$ and $\|\alpha m_1^* + \beta n_1^*\| = |(\alpha, \beta)|^* = 1$. In view of (2.25), (2.27) and the choice of $u_0$ and $v_0$, for each $k \in C$ one clearly has
\[
(\alpha m_1^* + \beta n_1^*)(z_k) = \alpha m_1^*(P(z_k)) + \beta n_1^*(Q(z_k)) = \alpha r_k + \beta s_k = 1.
\]

Let us remark that we have been informed by the referee about the paper by F.J. García-Pacheco [17], where the easier part of the above result was independently obtained.

Before we state and prove a stability result of AHSp for some infinite sums of Banach spaces that includes infinite $\ell_p$-sums, we recall the following notion that was introduced in [16, Definition 2.1].

**Definition 2.7.** A Banach space $X$ has the *approximate hyperplane property* (AH$p$) if there exists a function $\delta : (0, 1) \rightarrow (0, 1)$ and a 1-norming subset $C$ of $S_{X^*}$ satisfying the following property.

Given $\varepsilon > 0$ there is a function $\Upsilon_{X,\varepsilon} : C \rightarrow S_{X^*}$ with the following condition
\[
x^* \in C, \ x \in S_X, \quad \text{Re} \ x^*(x) > 1 - \delta(\varepsilon) \quad \Rightarrow \quad \text{dist}(x, F(\Upsilon_{X,\varepsilon}(x^*))) < \varepsilon,
\]
where $F(y^*) = \{y \in S_X : \text{Re} y^*(y) = 1\}$ for any $y^* \in S_{X^*}$.

A family of Banach spaces $\{X_i : i \in I\}$ has AH$p$ uniformly if every space $X_i$ has property AH$p$ with the same function $\delta$.

Clearly we can assume that the 1-norming subset $C$ in the previous definition satisfies $TC \subset C$, where $T$ is the unit sphere of the scalar field.

Let us notice that a similar property to AH$p$ was implicitly used to prove that several classes of spaces have AHSp (see [3]).

It is known that property AH$p$ implies AHSp (see for instance [16, Proposition 2.2]). Examples of spaces having AH$p$ are finite-dimensional spaces, uniformly convex spaces,
$L_1(\mu)$ for every measure $\mu$ and also $C(K)$ for every compact Hausdorff topological space $K$ (see [3] Propositions 3.5, 3.8, 3.6 and 3.7 and also [16] Corollary 2.12). In what follows we will use the standard notation from the theory of Banach lattices as presented for example in [23]. We denote by $\omega$ the space of all real sequences. As usual, the order $|x| := (|x_n|) \leq |y|$ for $x = (x_n), y = (y_n) \in \omega$ means that $|x_n| \leq |y_n|$ for each $n \in \mathbb{N}$.

A (real) Banach space $E \subset \omega$ is **solid** whenever $x \in w$, $y \in E$ and $|x| \leq |y|$ then $x \in E$ and $\|x\| \leq \|y\|_E$. $E$ is said to be a **Banach sequence lattice** (or Banach sequence space) if $E \subset \omega$, $E$ is solid and there exists $u \in E$ with $u > 0$. A Banach sequence lattice $E$ is said to be **order continuous** if for every $0 \leq f_n \downarrow 0$, it follows that $\|f_n\| \to 0$. If $E$ is an order-continuous Banach sequence lattice, then $E^*$ can be identified in a natural way with the Köthe dual space $(E', \| \cdot \|_{E'})$ of all $x = (x_k) \in \omega$ equipped with the norm

$$\|x\|_{E'} := \sup_{(y_k) \in B_E} \sum_{k=1}^{\infty} |x_k y_k|.$$ 

Let $E$ be a Banach sequence lattice. For a given sequence $(X_k, \| \cdot \|_{X_k})_{k=1}^{\infty}$ of Banach spaces the vector space of sequences $x = (x_k)_{k=1}^{\infty}$, with $x_k \in X_k$ for each $k \in \mathbb{N}$ and with $(\|x_k\|) \in E$, becomes a Banach space when equipped with the norm

$$\|(x_k)\| = \left\|(\|x_k\|_{X_k})\right\|_E;$$

this space will be denoted by $(\oplus_{k=1}^{\infty} X_k)_E$.

Finally we recall that a Banach lattice $E$ is **uniformly monotone** (UM) if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $x \in S_E$, $y \in E$ and $x, y \geq 0$ the condition $\|x + y\| \leq 1 + \delta$ implies that $\|y\| \leq \varepsilon$. It is known that every UM Banach lattice is order continuous (see [9] Theorem 22).

We will use the following duality result which is well known in the case $E = \ell_p$ with $1 \leq p < \infty$ or $E = c_0$ (see, e.g., [6] Theorem 12.6)). Since the proof of the general case is similar we omit it.

**Theorem 2.8.** Let $E$ be an order continuous Banach sequence lattice and let $(X_n)$ be a sequence of Banach spaces. Then the mapping $(\oplus_{n=1}^{\infty} X_n^*)_E \ni x^* = (x_n^*) \mapsto \phi_{x^*}$ defined by

$$\phi_{x^*}(x_n) = \sum_{n=1}^{\infty} x_n^*(x_n), \quad (x_n) \in \left(\oplus_{n=1}^{\infty} X_n\right)_E,$$

is an isometrical isomorphism from $(\oplus_{n=1}^{\infty} X_n^*)_E$ onto $(\oplus_{n=1}^{\infty} X_n)_E^*$.
Lemma 2.9. Let $E$ be a Banach sequence lattice which is order continuous and $\{X_k : k \in \mathbb{N}\}$ be a sequence of (nontrivial) Banach spaces. For each natural number $k$ assume that $C_k \subset S_{X_k^*}$ is a 1-norming set for $X_k$. Then the set $C$ given by

$$C = \{(e_k^* \lambda_k x_k^*) : e^* \in S_{E'}, e^* \geq 0, \lambda_k \in \mathbb{K}, |\lambda_k| = 1, x_k^* \in C_k, \forall k \in \mathbb{N}\}$$

is a subset of $S_{Z^*}$, a 1-norming set for $Z$, where $\mathbb{K}$ is the scalar field and $Z = \left( \bigoplus_{k=1}^{\infty} X_k \right)_E$.

**Proof.** By Theorem 2.8 the set $C$ is contained in $S_{Z^*}$. Let $z = (z_k) \in Z$ and $\varepsilon > 0$. By assumption we know that $(\|z_k\|) \in E$. In view of Theorem 2.8, $E^*$ coincides with $E'$, so there is a nonnegative element $e^* \in S_{E'}$ such that $e^*(\|z_k\|) = \|(z_k)\|_E = \|z\|$. For each $k \in \mathbb{N}$, $C_k$ is a 1-norming set for $X_k$ and so there exists $z_k^* \in C_k$ and a scalar $\lambda_k$ with $|\lambda_k| = 1$ such that $\text{Re} \lambda_k z_k^*(z_k) > \|z_k\| - \frac{\varepsilon}{(e_k^* + 1)^2}$. The element $z^* = (e_k^* \lambda_k z_k^*) \in C$ and

$$\text{Re} \ z^*(z) = \sum_{k=1}^{\infty} \text{Re} \ e_k^* \lambda_k z_k^*(z_k) > \sum_{k=1}^{\infty} e_k^* \left( \|z_k\| - \frac{\varepsilon}{(e_k^* + 1)^2} \right) \geq \|z\| - \varepsilon.$$

We proved that $C$ is a 1-norming set for $Z$. \hfill $\square$

Now we are ready to prove the stability of the AHSp.

Theorem 2.10. Let $E$ be a Banach sequence lattice with the AHSp and such that it is uniformly monotone. Assume that $\{X_k : k \in \mathbb{N}\}$ has property AH$p$ uniformly. Then the space $\left( \bigoplus_{k=1}^{\infty} X_k \right)_E$ has the AHSp.

**Proof.** We take $M = \{k \in \mathbb{N} : X_k \neq 0\}$. If $M$ is infinite, there is no loss of generality in assuming that $M = \mathbb{N}$. Otherwise the proof of the statement is essentially the same but easier.

So we assume that $X_k \neq \{0\}$ for each $k$. We put $Z := \left( \bigoplus_{k=1}^{\infty} X_k \right)_E$.

Let us fix $0 < \varepsilon < 1$. By assumption, $\{X_k : k \in \mathbb{N}\}$ has AH$p$ uniformly, so there is $\delta : (0,1) \rightarrow (0,1)$ satisfying Definition 2.7 for each $k \in \mathbb{N}$. We choose $0 < \eta < \min\{\frac{\varepsilon}{4}, \delta(\frac{\varepsilon}{4})\}$. Since $E$ is uniformly monotone, we can use condition ii) in [19, Theorem 6], so there is $0 < \alpha < \varepsilon/4 < 1$ satisfying that

$$(2.32) \quad e \in S_E, \ e \geq 0, \ A \subset \mathbb{N}, \ |\|e\chi_A\|_E > \frac{\varepsilon}{4} \Rightarrow |\|e\chi_{\mathbb{N}\setminus A}\|_E < 1 - \alpha.$$

For $r = (1 + 2\eta - \alpha\eta)/(1 + 2\eta)$, we choose $0 < \varepsilon' < (1 - r)\varepsilon/3$. Then by our assumption, it follows that there is $0 < \eta' < \varepsilon'$ such that $E$ satisfies the statement (d) in Proposition 1.3 for $(\varepsilon', \eta')$.

In order to prove that $Z$ satisfies the AHSp we will show that condition (d) in Proposition 1.3 is satisfied for $(\varepsilon, \eta')$. 

Assume that \((z_n)\) is a sequence in \(S_Z\) and \(\sum \alpha_n\) is a convex series such that 
\[ \left\| \sum_{n=1}^{\infty} \alpha_n z_n \right\| > 1 - \eta'. \]

Then 
\[ 1 - \eta' < \left\| \sum_{n=1}^{\infty} \alpha_n z_n \right\| \]
\[ = \left\| \left( \left\| \sum_{n=1}^{\infty} \alpha_n z_n(k) \right\| \right) \right\| _E \]
\[ \leq \left\| \left( \sum_{n=1}^{\infty} \alpha_n \| z_n(k) \| \right) \right\| _E \]
\[ = \left\| \sum_{n=1}^{\infty} \alpha_n \left( \| z_n(k) \| \right) \right\| _E. \]

Combining our hypothesis that \(E\) has the AHSp with \((\| z_n(k) \|) \in SE\) for each positive integer \(n\), we conclude that there is a finite subset \(A \subset \mathbb{N}\) and \(\{r_n : n \in A\} \subset SE\) such that 
\[ \sum_{n \in A} \alpha_n > 1 - \varepsilon' \]
and also 
\[ r_n \geq 0, \|r_n - (\| z_n(k) \|)\|_E < \varepsilon' \]
and there is \(r^* \in SE'\) with \(r^*(r_n) = 1\), for all \(n \in A\). Hence from (2.33) and (2.34) we obtain that 
\[ 1 - \eta' - \varepsilon' < \left\| \sum_{n \in A} \alpha_n z_n \right\|. \]

For each \(k \in \mathbb{N}\) we choose an element \(x_k \in S_{X_k}\) and define for every \(n \in A\) the element \(u_n\) in \(Z\) given by 
\[ u_n(k) = \begin{cases} 
  r_n(k) \frac{z_n(k)}{\| z_n(k) \|} & \text{if } z_n(k) \neq 0 \\
  r_n(k)x_k & \text{otherwise.}
\end{cases} \]

By (2.35) it is clearly satisfied that 
\[ \|u_n - z_n\| = \|r_n - (\| z_n(k) \|)\|_E < \varepsilon', \quad \forall n \in A. \]

So in view of (2.36) we obtain that 
\[ 1 - \eta' - 2\varepsilon' < \left\| \sum_{n \in A} \alpha_n u_n \right\|. \]

By assumption, \(\{X_k : k \in \mathbb{N}\}\) has AHp uniformly. For each \(k \in \mathbb{N}\) let \(G_k \subset S_{X_k^*}\) be the 1-norming set for \(X_k\) satisfying Definition 2.7. We can also assume that \(G_k = \{\lambda r^* : \lambda \in \mathbb{R}\} \).
\( |\lambda| = 1, x^* \in G_k \) for each \( k \in \mathbb{N} \). By Lemma 2.9, there is \( z^* \in S_{Z^*} \) that can be written as \( z^* = (z^*_k) = (e_k^* x_k^*) \) where \( e_k^* \in S_{E^*}, e^* \geq 0 \) and \( x_k^* \in G_k \) for each \( k \in \mathbb{N} \) satisfying that

\[
1 - \eta' - 2\varepsilon' < \Re z^* \left( \sum_{n \in A} \alpha_n u_n \right).
\]

Now we define the set \( C \) by

\[ C = \left\{ n \in A : \Re z^*(u_n) > r \right\} \]

By Lemma 2.1, we obtain that

\[
(2.39) \quad \sum_{n \in C} \alpha_n > 1 - \frac{\eta' + 2\varepsilon'}{1 - r} > 1 - \varepsilon > 0.
\]

For each element \( n \in C \) we have that

\[
r < \Re z^*(u_n) = \sum_{k=1}^{\infty} \Re z_k^*(u_n(k))
\]

(2.40) \[
\leq \sum_{k=1}^{\infty} \left| z_k^*(u_n(k)) \right| \\
\leq \sum_{k=1}^{\infty} \left\| z_k^* \right\| \left\| u_n(k) \right\| \\
\leq \left\| \left( \left\| z_k^* \right\| \right)_{E^*} \left( \left\| u_n(k) \right\| \right) \right\|_E \\
= 1.
\]

For each \( n \in C \) and \( k \in \mathbb{N} \) we put

\[ d_n(k) = \| z_k^* \| \left\| u_n(k) \right\| - \Re z_k^*(u_n(k)). \]

The chain of inequalities (2.40) implies that

(2.41) \[
\sum_{k=1}^{\infty} d_n(k) \leq 1 - r, \quad \forall n \in C.
\]

We now fix a positive integer \( k \). If \( z_k^* = 0 \), then \( d_n(k) = 0 \) for every \( n \in C \). If \( n \in C \) and \( u_n(k) = 0 \) for some \( k \in \mathbb{N} \) then \( d_n(k) = 0 \). Otherwise it is satisfied that

\[
(2.42) \quad \Re \frac{z_k^*}{\| z_k^* \|} \left( \frac{u_n(k)}{\| u_n(k) \|} \right) = 1 - \frac{d_n(k)}{\| z_k^* \| \left\| u_n(k) \right\|}.
\]

In what follows, for each \( n \in C \), we consider the following subset

\[ B_n = \left\{ k \in \mathbb{N} : d_n(k) < \eta \| z_k^* \| \left\| u_n(k) \right\| \right\}. \]
By (2.40) we know that

\[ r < \sum_{k=1}^{\infty} \| z_k^* \| \| u_n(k) \| \]

\[ = \sum_{k \in B_n} \| z_k^* \| \| u_n(k) \| + \sum_{k \in \mathbb{N} \setminus B_n} \| z_k^* \| \| u_n(k) \| \]

\[ \leq \sum_{k \in B_n} \| z_k^* \| \| u_n(k) \| + \frac{1}{\eta} \sum_{k \in \mathbb{N} \setminus B_n} d_n(k) \]

\[ \leq \sum_{k \in B_n} \| z_k^* \| \| u_n(k) \| + \frac{1}{\eta} (1 - r) \quad (\text{by } (2.41)). \]

As a consequence,

\[ (2.43) \quad \sum_{k \in B_n} \| z_k^* \| \| u_n(k) \| > r - \frac{1-r}{\eta} > 0 \]

and in view of (2.40) we deduce that

\[ (2.44) \quad \sum_{k \in \mathbb{N} \setminus B_n} \| z_k^* \| \| u_n(k) \| < 1 - r + \frac{1-r}{\eta}, \quad \forall n \in C. \]

In view of (2.42), for every \( n \in C \) and \( k \in B_n \) it is satisfied that

\[ \text{Re} \ x_k^* \left( \frac{u_n(k)}{\| u_n(k) \|} \right) = \text{Re} \ z_k^* \left( \frac{u_n(k)}{\| u_n(k) \|} \right) = 1 - \frac{d_n(k)}{\| z_k^* \| \| u_n(k) \|} > 1 - \eta. \]

Now we will use that for each \( k \) the space \( X_k \) has the property AHp for the function \( \delta, \eta < \delta \left( \frac{\xi}{2} \right) \) and \( x_k^* \in G_k \). Hence for each \( k \in \bigcup_{l \in C} B_l \), there is \( y_k^* \in S_{X_k}^* \) such that if \( n \in C \) and \( k \in B_n \) there is \( m_n(k) \in S_{X_k} \) with

\[ (2.45) \quad \left\| m_n(k) - \frac{u_n(k)}{\| u_n(k) \|} \right\| < \frac{\varepsilon}{4}, \quad \text{and} \quad \text{Re} \ y_k^*(m_n(k)) = 1, \quad \forall n \in C, \quad \forall k \in B_n. \]

Let \( D = \mathbb{N} \setminus \bigcup_{l \in C} B_l \). For each \( k \in D \), we choose any element \( y_k^* \in S_{X_k}^* \) such that \( y_k^*(x_k) = 1 \).

For each \( n \in C \), we write \( C_n = \bigcup_{l \in C} B_l \setminus B_n \) and define \( v_n \in Z \) by

\[ v_n(k) = \begin{cases} r_n(k)m_n(k) & \text{if } k \in B_n \\ r_n(k)m_p(k) & \text{if } k \in C_n \\ r_n(k)x_k & \text{if } k \in D, \end{cases} \]

where \( p(k) = \min\{ s \in C : k \in B_s \} \) if \( k \in \bigcup_{l \in C} B_l \). It is clear that \( \| v_n \| = \| r_n \|_E = 1 \) for each \( n \in C \).
We clearly have that
\[
\|r_n\chi_{B_n}\|_E = \|u_n\chi_{B_n}\|
\]
\[
\geq \text{Re} z^*(u_n\chi_{B_n})
\]
\[
= \text{Re} \sum_{k \in B_n} z^*_k(u_n(k))
\]
\[
= \sum_{k=1}^{\infty} \text{Re} z^*_k(u_n(k)) - \sum_{k \in \mathbb{N}\setminus B_n} \text{Re} z^*_k(u_n(k))
\]
\[
> r - \sum_{k \in \mathbb{N}\setminus B_n} \text{Re} z^*_k(u_n(k)) \quad \text{(by (2.40))}
\]
\[
\geq r - \sum_{k \in \mathbb{N}\setminus B_n} \|z^*_k\|\|u_n(k)\|
\]
\[
> r - \left(1 - r + \frac{1-r}{\eta}\right) \quad \text{(by (2.41))}
\]
\[
= 2r - 1 - \frac{1-r}{\eta} = 1 - \alpha.
\]

Since $0 \leq r_n$ for each $n \in C$ and \{\(r_n : n \in C\)\} $\subset S_E$, from (2.46) and (2.32) it follows that
\[
\|r_n\chi_{\mathbb{N}\setminus B_n}\|_E \leq \frac{\varepsilon}{4}. \tag{2.47}
\]

For every $n \in C$ and $k \in B_n$, in view of (2.45) we have that
\[
\|v_n(k) - u_n(k)\| = \|r_n(k)m_n(k) - u_n(k)\|
\]
\[
\leq \frac{\varepsilon}{4} r_n(k). \tag{2.48}
\]

Hence from (2.48), for every $n \in C$ we have that
\[
\|v_n - u_n\| \leq \|(v_n - u_n)\chi_{B_n}\| + \|v_n\chi_{\mathbb{N}\setminus B_n}\| + \|u_n\chi_{\mathbb{N}\setminus B_n}\|
\]
\[
\leq \frac{\varepsilon}{4}\|r_n\|_E + 2\|r_n\chi_{\mathbb{N}\setminus B_n}\|_E \quad \text{(by (2.48))}
\]
\[
\leq \frac{3\varepsilon}{4} \quad \text{(by (2.47)).}
\]

Combining with (2.37), we conclude that for each $n \in C$,
\[
\|v_n - z_n\| \leq \|v_n - u_n\| + \|u_n - z_n\|
\]
\[
\leq \frac{3\varepsilon}{4} + \varepsilon'
\]
\[
< \varepsilon.
\]
Let \( v^* \) be the element in \( Z^* \) given by \( v^* = \{ r_k^* y_k^* \} \). By Theorem 2.8 it is satisfied that \( \| v^* \| = \| r^* \|_{E'} = 1 \). For each \( n \in C \) we clearly have that
\[
v^*(v_n) = \sum_{k=1}^{\infty} r_k^* y_k^*(v_n(k))
\]
\[
= \sum_{k \in B_n} r_k^* r_n(k) y_k^*(m_n(k)) + \sum_{k \in C_n} r_k^* r_n(k) y_k^*(m_{pl}(k)(k)) + \sum_{k \in D} r_k^* r_n(k) y_k^*(x_k)
\]
\[
= \sum_{k=1}^{\infty} r_k^* r_n(k) \quad \text{(by (2.35))}
\]
\[
= r^*(r_n) = 1 \quad \text{(by (2.35)).}
\]
From (2.39) we also know that \( \sum_{n \in C} \alpha_n > 1 - \varepsilon \), so the proof is finished. \( \square \)

As we mentioned above uniformly convex spaces have AHp. Indeed in this case the modulus of convexity plays the role of the function \( \delta \) satisfying Definition 2.7 and the identity function on the unit sphere of the dual plays the role of the function \( \Upsilon_{\delta} \) [5] Lemma 2.1]. So a family \( \{ X_i : i \in I \} \) of uniformly convex Banach spaces has the AHp uniformly in case that \( \inf\{ \delta_i(\varepsilon) : i \in I \} > 0 \), for any \( \varepsilon > 0 \), being \( \delta_i \) the modulus of convexity of \( X_i \). Also \( C(K) \) spaces and \( L_1(\mu) \) have AHp uniformly for any compact Hausdorff space \( K \) and any measure \( \mu \) [16] Corollary 2.8]. As a consequence of Theorem 2.10 and [3] Theorem 4.1] we deduce, for instance, the following result.

**Corollary 2.11.** Let \( \{ X_k : k \in \mathbb{N} \} \) be a sequence of (nontrivial) Banach spaces such that any of them is either a uniformly convex space or \( C(K) \) (some compact \( K \)) or \( L_1(\mu) \) (some measure \( \mu \)). Let \( A = \{ k \in \mathbb{N} : X_k \text{ is a uniformly convex space} \} \) and assume that \( \inf\{ \delta_k(\varepsilon) : k \in A \} > 0 \) for every \( \varepsilon > 0 \), being \( \delta_k \) the modulus of convexity of \( X_k \). Then the pair \( (\ell_1, (\ominus \sum_{k=1}^{\infty} X_k)_{\ell_p}) \) satisfies the BPBp for every \( 1 \leq p < \infty \).

Let us remark that in general AHSp is not stable under infinite \( \ell_1 \)-sums (see [8] Corollary 4.6]). So in order to have the stability result in Theorem 2.10 some additional restriction is needed. Now we show the following partial converse of Theorem 2.10 that extends to some infinite sums the necessary condition obtained in Theorem 2.6.

**Proposition 2.12.** Let \( \{ X_k : k \in \mathbb{N} \} \) be a sequence of (nontrivial) Banach spaces and \( E \) be an order continuous Banach sequence lattice. Assume that the space \( Z = (\ominus \sum_{k=1}^{\infty} X_k)_{E} \) has the approximate hyperplane series property. Then there is a function \( \tilde{\eta} : (0, 1) \rightarrow (0, 1) \) such that \( X_k \) satisfies the approximate hyperplane series property with the function \( \tilde{\eta} \) for every \( k \in \mathbb{N} \). More precisely, one can take the function given by \( \tilde{\eta}(\varepsilon) = \eta(\varepsilon^{1/2}) \), where \( \eta \) is the function satisfying Definition 1.2 for \( Z \).
Proof. It suffices to prove that \( X_1 \) has the property AHSp for \( \tilde{\eta} \). Consider the subspace \( Z_1 \) of \( Z \) given by

\[
Z_1 = \{ z \in Z : z(k) = 0, \forall k \geq 2 \}.
\]

Notice that the mapping from \( Z_1 \) into \( X_1 \) given by \( z \mapsto z(1)\|e_1\|_E \) is a linear isometry, where \( e_1 \) is the sequence given by \( e_1(k) = \delta_k^1 \) for each natural number \( k \). Since AHSp is clearly preserved by linear isometries (and the function \( \eta \) satisfying AHSp also) then it suffices to prove that \( Z_1 \) satisfies AHSp with the function \( \tilde{\eta} \).

So let us fix \( 0 < \varepsilon < 1 \). Assume that \( \alpha_n \geq 0, u_n \in S_{Z_1} \) for every \( n \), \( \sum_{n=1}^{\infty} \alpha_n = 1 \) and it is also satisfied that

\[
\left\| \sum_{n=1}^{\infty} \alpha_n u_n \right\| > 1 - \eta\left(\frac{\varepsilon}{2}\right).
\]

By assumption \( Z \) has the AHSp, so there is a subset \( A \subset \mathbb{N} \) such that \( \sum_{n \in A} \alpha_n > 1 - \frac{\varepsilon}{2} > 1 - \varepsilon \), \( z^* \in S_{Z^*} \) and \( \{ z_n : n \in A \} \subset S_Z \) such that

\[
(2.49) \quad \| z_n - u_n \| < \frac{\varepsilon}{2} \quad \text{and} \quad z^*(z_n) = 1, \quad \forall n \in A.
\]

For every \( n \in A \) we define the element \( y_n \in Z_1 \) given by

\[
y_n(1) = z_n(1), \quad y_n(k) = 0, \quad \forall k \geq 2.
\]

Let us fix \( n \in A \). We clearly have that

\[
(2.50) \quad \| y_n - u_n \| = \| (\| y_n(k) - u_n(k) \|) \|_E \leq \| (\| z_n(k) - u_n(k) \|) \|_E = \| z_n - u_n \| < \frac{\varepsilon}{2}.
\]

Since we know that

\[
\| y_n \| \leq \| z_n \| = 1, \quad \forall n \in A,
\]

in view of (2.50) we deduce that

\[
(2.51) \quad 1 - \frac{\varepsilon}{2} \leq \| y_n \| \leq 1, \quad \forall n \in A.
\]

As a consequence of Theorem 2.38 we know that \( z^* \in \left( \bigoplus_{k=1}^{\infty} X_k^* \right)_E \) and we also have

\[
(2.52) \quad z^*(1)(y_n(1)) = z^*(1)(z_n(1)) = \| z^*(1) \| \| z_n(1) \| = \| z^*(1) \| \| y_n(1) \|, \quad \forall n \in A.
\]
On the other hand, it is satisfied that
\[
|z^*(1)(yn(1))| = |z^*(yn)| \\
\geq |z^*(zn)| - |z^*(yn - zn)| \\
\geq 1 - \|zn - yn\| \\
\geq 1 - \|zn - un\| - \|un - yn\| \\
\geq 1 - 2\|zn - un\| \quad \text{(by (2.50))} \\
> 1 - \varepsilon > 0 \quad \text{(by (2.49))}.
\]

(2.53)

We denote by \(w^*\) the element in \(Z^*\) given by
\[
w^*(1) = z^*(1), \quad w^*(k) = 0, \quad \text{if} \quad k \geq 2.
\]

Notice that \(\|e_1\|_{E'}\|e_1\|_E = 1\). So it is clearly satisfied
\[
\text{Re } w^*(yn) = \text{Re } z^*(yn) \\
= \|z^*(1)\| \|yn(1)\| \quad \text{(by (2.52))} \\
= \|w^*\| \|yn\| \\
= \|e_1\|_{E'} \|e_1\|_E \\
= \|w^*\| \|yn\|,
\]
and bearing in mind (2.53) we deduce that \(w^*(yn) \neq 0\).

Since for each \(n \in A\) we have also that
\[
\left\|un - \frac{yn}{\|yn\|}\right\| \leq \left\|un - yn\right\| + \left\|yn - \frac{yn}{\|yn\|}\right\| \\
< \frac{\varepsilon}{2} + 1 - \|yn\| \leq \varepsilon \quad \text{(by (2.50) and (2.51))},
\]
we checked that \(Z_1\) has the AHSp for the function \(\tilde{\eta}\) as we wanted to show. \(\square\)

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