Gauge theory and $G_2$-geometry on Calabi–Yau links

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Abstract. The 7-dimensional link $K$ of a weighted homogeneous hypersurface on the round 9-sphere in $\mathbb{C}^5$ has a nontrivial null Sasakian structure which is contact Calabi–Yau, in many cases. It admits a canonical co-calibrated $G_2$-structure $\varphi$ induced by the Calabi–Yau 3-orbifold basic geometry. We distinguish these pairs $(K, \varphi)$ by the Crowley–Nordström $\mathbb{Z}_{48}$-valued $\nu$ invariant, for which we prove odd parity and provide an algorithmic formula.

We describe moreover a natural Yang–Mills theory on such spaces, with many important features of the torsion-free case, such as a Chern–Simons formalism and topological energy bounds. In fact, compatible $G_2$-instantons on holomorphic Sasakian bundles over $K$ are exactly the transversely Hermitian Yang–Mills connections. As a proof of principle, we obtain $G_2$-instantons over the Fermat quintic link from stable bundles over the smooth projective Fermat quintic, thus relating in a concrete example the Donaldson–Thomas theory of the quintic threefold with a conjectural $G_2$-instanton count.

1. Introduction

We propose a contemporary angle on Milnor’s celebrated study of singular hypersurface links [32], from the perspective of special metrics and higher-dimensional gauge theory. Our starting point is the observation that several topological properties of the Milnor fibre and its boundary, the Milnor link (see Section 2), resemble those of the $G_2$-invariant $\nu$ recently introduced by Crowley and Nordström [16], suggesting to optimists that Milnor’s construction might be related to $G_2$-geometry.

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1.1. G₂-metrics on Calabi–Yau links

Let \( V \subseteq \mathbb{C}^{n+1} \) be a complex analytic variety with an isolated singularity at the origin. Milnor proved that \( V \) intersects transversally every sufficiently small sphere \( S^{2n+1} := \partial B_\varepsilon(0) \), and the link

\[
K := V \cap S^{2n+1}
\]

is a \((n-2)\)-connected smooth manifold with \( \dim_{\mathbb{R}} K = 2 \dim_{\mathbb{C}} V - 1 \). The topologies of \( V \) and of its embedding in \( \mathbb{C}^{n+1} \) are completely determined by the embedding \( K \hookrightarrow S^{2n+1} \).

Suppose henceforth that \( V = (f) \) is an affine hypersurface defined by a homogeneous polynomial \( f \in \mathbb{C}[z_0, \ldots, z_\ell] \), with \( f(0) = 0 \) and \( \text{Crit}(f) \cap B_\varepsilon(0) = \{0\} \). The Hopf fibration \( \pi: S^{2n+1} \to \mathbb{P}^n \) characterises the corresponding link \( K_f \) in a natural way as the total space of a \( S^1 \)-bundle over the smooth projective hypersurface \( V \) defined by \( f \):

\[
\pi: K_f \xrightarrow{S^1} V \subset \mathbb{P}^n.
\]

As a circle bundle, \( K_f \) carries a global angular form \( \theta \in \Omega^1(K) \), whose restriction to each fibre \( \pi^{-1}(x) \) generates the cohomology \( H^1(\pi^{-1}(x), \mathbb{R}) \). Its exterior derivative \( d\theta = -\pi^*e \in \Omega^2(K) \) is the pullback of the Euler class on the base (compare with Lemma 2.12,below).

If the link has degree \( n+1 \), then the projective variety \( V \) is a Calabi–Yau \((n-1)\)-fold. Fixing \( n = 4 \), a quintic (possibly weighted) link \( K_f \) is a smooth Sasakian 7-manifold fibering by circles over the smooth Calabi–Yau 3-fold \( V \), and it is the simplest example of a Calabi–Yau (CY) link (see Definition 2.15). Now, it is well known that the Riemannian product of a Calabi–Yau 3-fold and a circle carries a torsion-free \( G_2 \)–structure, so we define naturally (see also Theorem 2.5 in [24]) the following \( G_2 \)-structure on \( K_f \):

\[
\varphi := \theta \wedge \omega + \text{Im} \, \varepsilon, \\
\psi := \frac{1}{2} \omega \wedge \omega + \theta \wedge \text{Re} \, \varepsilon = *\varphi,
\]

where \( \omega \) and \( \varepsilon \) are respectively the Kähler and holomorphic volume forms defining the Calabi–Yau structure on \( V \), and we denote identically differential forms and their pullbacks under \( \pi \). Although in the nontrivial fibration case this structure has torsion, it is still cocalibrated (see Section 2.2), in the sense that \( d\psi = 0 \):

**Theorem 1.1.** Every quintic link \( K_f \) is a 2-connected, compact, smooth real 7-manifold admitting the natural cocalibrated \( G_2 \)-structure (1.1).

It should be stressed that Theorem 1.1 has recently been found and subsumed, independently, by Habib and Vezzoni [26], §6.2, in the context of contact Calabi–Yau (cCY) geometry. Their theory extends the above discussion to weighted homogeneous links and therefore yields many more examples of CY links, fibering over CY 3-orbifolds in weighted \( \mathbb{P}^4(w) \) (see Section 2.4). This is very fortunate, because otherwise the Fermat quintic would be the only strictly homogeneous quintic with an isolated singularity at the origin.
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In the light of substantial recent progress in the classification of 2-connected 7-manifolds with $G_2$-structures [15], [14], [16], it is a natural task to sort such CY links $(K_f, \varphi)$. The important $\mathbb{Z}_{48}$-valued invariant $\nu(\varphi)$ introduced by Crowley and Nordström [16] allows us to distinguish such pairs, up to diffeomorphism of $K_f$ and homotopy of $\varphi$, but its definition is non-constructive and it requires an ad hoc spin coboundary 8-manifold $W$ such that $K_f = \partial W$. In Section 3, we show that this coboundary can be essentially taken to be a typical Milnor fibre, and we find an explicit formula for $\nu(\varphi)$ in terms of topological data:

**Theorem 1.2.** Let $K_f \xrightarrow{S^1} V \subset P^4(w)$ be a Calabi–Yau link of degree $d$ and weight $w = (w_0, \ldots, w_4)$; then the Crowley–Nordström $\nu$ invariant of any $S^1$-invariant $G_2$-structure $\varphi$ on $K_f$ is an odd integer given by

$$\nu(\varphi) = \left(\frac{d}{w_0} - 1\right) \cdots \left(\frac{d}{w_4} - 1\right) - 3(\mu_+ - \mu_-) + 1,$$

where $(\mu_-, \mu_+)$ is the signature of the intersection form on $H^4(\tilde{V}, \mathbb{R})$, for $\tilde{V} = f^{-1}(1) \subset \mathbb{C}^5$.

Using a method by Steenbrink to calculate the signature, we obtain an effective algorithm to determine $\nu(\varphi)$ for any CY link, with straightforward computational methods (see Appendix A). We observe that several values of $\nu$ are realised in this manner, and conjecture that indeed all possible 24 values can be realised by the ‘natural’ cocalibrated $G_2$-structure (1.1) of a weighted CY link. In particular, for the homogeneous case we find:

**Corollary 1.3.** The Crowley–Nordström $\nu$ invariant of the Fermat quintic link with $G_2$-structure (1.1) is $\nu(\varphi) = 5$.

To the best of our knowledge, this large class of 7-manifolds with $G_2$-structure of the form $(K_f, \varphi)$ is the first instance besides the original reference [16] in which the $\nu$ invariant has been computed explicitly.

1.2. Gauge theory on contact Calabi–Yau manifolds

In Section 4, we turn to the second axis of interest in $G_2$-geometry, as a model for 7-dimensional gauge theory. Since that concept appeared in the Physics literature [11], physicists pursue an analogous definition of Witten’s topological quantum field theory [49] on spaces with $G_2$-metrics [1]. Moreover, it was noticed in [29] that the superpotential for M–theory compactifications on $G_2$-manifolds ‘counts’ associative 3-manifolds (i.e., submanifolds calibrated by $\varphi$) in the same way as the prepotential of type II strings counts holomorphic curves in CY 3-folds. Mathematicians, on the other hand, following the seminal viewpoint of [20], expect the theory to culminate in a topological count of instantons, yielding an invariant for 7-manifolds with a $G_2$-structure, in the same vein as the Casson invariant for flat connections over 3-manifolds [19]. At the current stage, however, major compactification issues remain and a more thorough analytical understanding might have to be postponed in favour of exploring a good number of examples [37], [47], [9], [38], [39], [40].
We propose a consistent formulation of elementary Yang–Mills theory on 7-dimensional CY manifolds. In Section 4.1, we define a connection $A$ on a complex vector bundle $E \to K$ to be a $G_2$-instanton if $F_A \wedge \psi = 0$, where $\psi$ is the $G_2$-structure 4-form (cf. [20], [45]), which characterises $A$ at first as a critical point of the Chern–Simons functional. In Section 4.2 we endow $E$ with a suitable holomorphic Sasakian vector bundle structure, following the framework of Biswas and Schumacher [5], to obtain a notion of Chern connection, compatible at once with the holomorphic structure and some Hermitian bundle metric (Proposition 4.3). In Section 4.3, we verify that the $G_2$-instanton condition is exactly equivalent to a natural transverse Hermitian Yang–Mills condition (Lemma 4.4 and Corollary 4.5), somewhat similarly to the classical identification of selfdual and HYM connections on compact Kähler surfaces [18, §2]. Furthermore, going over into Section 4.4, the cCY $G_2$-geometry allows us to move past some generally expected difficulties in the presence of torsion: since in these cases $d\varphi = \omega \wedge \omega$, integrable $G_2$-instantons are indeed Yang–Mills solutions, i.e., $d^*_A F_A = 0$, and we define a secondary characteristic class leading to topological energy bounds, thus proving:

**Theorem 1.4.** Let $E$ be a holomorphic Sasakian bundle over a 7-dimensional closed contact Calabi–Yau manifold $M$ endowed with its natural cocalibrated $G_2$-structure (cf. Proposition 2.14). Then,

1. integrable $G_2$-instantons (cf. (4.2)) on $E$ are critical points of the Yang–Mills functional $S_{YM}$;

2. if the absolute topological minimum of $S_{YM}$ is attained among integrable connections, then the minima are exactly the $G_2$-instantons, i.e., the critical points of the Chern–Simons functional $S_{CS}$.

To conclude with an example, in Section 4.5 we focus on the simplest case in which $E$ is a pullback from the basic CY$^3$, and we derive the explicit local equations of $G_2$-instantons in that setting:

**Theorem 1.5.** Suppose $\pi: K \to V$ is a 7-dimensional CY link, and let $E := \pi^* E_0 \to V$. Then,

(i) if an integrable connection $A = A + \sigma \theta$ on $E$ is a $G_2$-instanton, then $A$ defines locally a family $\{A_t\}_{t \in S^1}$ of Hermitian Yang–Mills connections on $E_0$, satisfying

$$\left( \frac{\partial A_t}{\partial t} - d_A^* \sigma \right) \wedge \theta = 0;$$

(ii) if $E$ is indecomposable, there is a one-to-one correspondence between $S^1$-invariant $G_2$-instantons on $E$ and Hermitian Yang–Mills connections on $E_0$.

In particular, Theorem 1.5 implies that $S^1$-invariant $G_2$-instantons on $E$ are ‘counted’ by the Donaldson–Thomas invariants of $E_0$, and this count should remain constant at least for any $S^1$-invariant deformations of the $G_2$-structure (1.1). Finally, we underscore that the homogeneous case is offered as proof of principle, since our narrative seems to readily extend to crepant resolutions of weighted projective Calabi–Yau 3-orbifolds.
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Readers interested in a more detailed account of instanton theory on $G_2$-manifolds are kindly referred to the introductory sections of [39], [40] and citations therein. A more thorough study of the moduli spaces of $G_2$-instantons, Sasakian HYM connections and also contact instantons on contact Calabi–Yau 7-manifolds will appear shortly as part of Luis Portilla’s PhD thesis [35].

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2. Geometric structures on links

We address the possibilities of $G_2$-geometry on Calabi–Yau links, starting from the motivational fact that a 7-manifold admits a $G_2$-structure if and only if it is orientable and spin [24], as is the case of links weighted homogeneous hypersurface singularities in $\mathbb{C}^5$ [6], Theorem 9.3.2. Such links have a very rich tautological geometry, including a null Sasakian structure with a compatible non-degenerate 3-form which is ‘transversely’ holomorphic, fitting in the category of contact Calabi–Yau manifolds proposed by Tomassini and Vezzoni [46]. In this section we compile relevant definitions and known properties of weighted homogeneous links, and derive some straightforward consequences.

2.1. Hypersurface links of isolated singularities

We begin by reviewing more carefully Milnor’s fibration theorem, following the original reference [32], §5-7. We denote by $B_\varepsilon$ (respectively $\overline{B}_\varepsilon$) the open (resp. closed) ball of radius $\varepsilon$ centered at the origin $0 \in \mathbb{C}^{n+1}$, and by $S^{2n+1}_\varepsilon = \partial B_\varepsilon$ its boundary; the explicit radius $\varepsilon$ will be omitted whenever one may assume, for simplicity, $\varepsilon = 1$. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a complex analytic map with $f(0) = 0$ and denote $\mathcal{V} := f^{-1}(0)$ and $K_f := \mathcal{V} \cap S^{2n+1}_\varepsilon$ (Figure 1).

![Figure 1. Link $K$ of a hypersurface $\mathcal{V}$ in $\mathbb{C}^{n+1}$.](image-url)
Theorem 2.1. Let $\varepsilon > 0$ be sufficiently small. Then the map
\[ \phi : \mathbb{S}_n^{2n+1} \setminus K_f \to S^1, \quad \phi = \frac{f(x)}{|f(x)|}, \]
is a locally trivial fibration, each fibre $F = \phi^{-1}(a)$ is smooth parallelisable and has the homotopy type of a finite CW-complex of dimension $n$. Furthermore, if $f$ has an isolated singularity at 0, then each fibre $F$ has the homotopy type of a bouquet $S^n \vee \ldots \vee S^n$ of spheres, and it is homotopy-equivalent to its closure $\overline{F}$ which is a compact manifold with boundary, with common boundary $\partial F = K_f$. Likewise, $K_f$ is a smooth $(n-2)$-connected real manifold of dimension $2n-1$.

The number $\mu$ of spheres $S^n$ in the bouquet described in Theorem 2.1 is called the Milnor number. It is an extremely important topological invariant of the link.

Theorem 2.2. The Milnor number $\mu$ has the following interpretations:

(i) $\mu$ is the complex dimension of the vector space obtained by taking the quotient of the local ring $O_0(\mathbb{C}^{n+1})$ of holomorphic functions at $0 \in \mathbb{C}^{n+1}$ by the Jacobian ideal $J_f = (\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$ of $f$:
\[ \mu = \dim \mathbb{C} O_0(\mathbb{C}^{n+1}) / J_f, \]

(ii) $\mu$ is the rank of the (free Abelian) middle homology group $H_n(F, \mathbb{Z})$,

(iii) $\mu$ is determined by the Euler characteristic of $F$:
\[ \chi(F) = 1 + (-1)^n \mu. \]

The following result [32], Theorem 5.11, gives a useful alternative description of the Milnor fibre.

Theorem 2.3. If a complex number $c \neq 0$ is sufficiently close to zero, then the complex hypersurface $f^{-1}(c)$ intersects the open ball $B_\varepsilon$ along a smooth manifold which is diffeomorphic to the fibre $F$.

Now we focus on the particular case in which $f$ is a weighted homogeneous polynomial with an isolated singularity at 0 in $\mathbb{C}^{n+1}$. This case is special because $V := f^{-1}(0)$ intersects transversally every sphere $S^{2n+1}_\varepsilon$ around the origin. Recall that a weight vector $w = (w_0, \ldots, w_n) \in \mathbb{Q}^{n+1}$ defines a weighted $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$, denoted by
\[ \mathbb{C}^*(w) : (z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n), \quad \lambda \in \mathbb{C}^*. \]

Definition 2.4. A polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ is said to be weighted homogeneous of degree $d$ and weight $w = (w_0, \ldots, w_n)$ (or simply $w$-homogeneous) if $f$ is homogeneous of order $d$ with respect to $\mathbb{C}^*(w)$, i.e.,
\[ f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n). \quad \forall \lambda \in \mathbb{C}^*. \]

NB.: a homogeneous polynomial is weighted homogeneous of weights $(1, \ldots, 1)$. 
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Proposition 2.5 ([33], Theorem 1). Let $f \in \mathbb{C}[z_0, \ldots, z_n]$ be $w$-homogeneous of degree $d$, having an isolated singularity at $0 \in \mathbb{C}^{n+1}$. Then the cohomology $H_n(F, \mathbb{Z})$ is free Abelian of rank $\mu = (d/w_0 - 1) \cdots (d/w_n - 1)$.

The Milnor fibration associated to a weighted homogeneous polynomial can appear under a different dressing, as the following lemma shows (Lemma 9.4 in [32]; see also [17], Chapter 3, Exercises 1.11 and 1.13).

Lemma 2.6. Let $f \in \mathbb{C}[z_0, \ldots, z_n]$ be weighted homogeneous. Then its restriction $f : \mathbb{C}^{n+1} \setminus \mathcal{V} \to \mathbb{C}^*$ is a locally trivial fibration. Moreover, the preimage of the unit circle $S^1$ under that fibration is fibre-diffeomorphic to the Milnor fibration $\phi$ of Theorem 2.1 associated to $f$. In particular, the Milnor fibre is diffeomorphic to the non-singular affine hypersurface $\mathcal{V} := f^{-1}(1) \subset \mathbb{C}^{n+1}$.

Weighted homogeneous polynomials give rise in a natural way to links, fibering by circles over weighted projective hypersurfaces:

Definition 2.7. Let $f \in \mathbb{C}[z_0, \ldots, z_n]$ be $w$-homogeneous with an isolated critical point at $0$, so that each sphere $S^{2n+1} = \partial B_\varepsilon$ intersects $\mathcal{V} := f^{-1}(0) \subset \mathbb{C}^{n+1}$ transversally. Then $K_f := \mathcal{V} \cap S^{2n+1}$ is called a weighted link of degree $\deg f$ and weight $w$.

We have the commutative diagram

$$
\begin{array}{ccc}
K_f & \longrightarrow & S^9 \\
\downarrow & & \downarrow \\
\mathcal{V} & \longrightarrow & \mathbb{P}^1(w),
\end{array}
$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal $S^1$-orbibundles and orbifold Riemannian submersions. As a complex orbifold, the hypersurface $V \subset \mathbb{P}^1(w)$ can be represented as the quotient $(\mathcal{V} \setminus \{0\}) / \mathbb{C}^*(w)$, with $\mathcal{V} = f^{-1}(0)$.

2.2. $G_2$-geometry

We now address the context of Theorem 1.1, concerning the natural cocalibrated $G_2$-structure (1.1). This section serves the double purpose of recalling notions of $G_2$-geometry and setting the scene for the gauge theoretical investigation in Section 4.

Let $Y$ be an oriented smooth 7-manifold. A $G_2$-structure is a smooth tensor $\varphi \in \Omega^3(Y)$ identified, at every $p \in Y$, by some frame $f_p : T_p Y \to \mathbb{R}^7$, with the model

$$
\varphi_0 = e^{567} + \omega_1 \wedge e^5 + \omega_2 \wedge e^6 + \omega_3 \wedge e^7 
$$

(2.3)
(same convention as in [41]), in the sense that \( \varphi_p = \Gamma^p_0(\varphi_0) \), where
\[
\omega_1 = e^{12} - e^{34}, \quad \omega_2 = e^{13} - e^{42}, \quad \text{and} \quad \omega_3 = e^{14} - e^{23}
\]
are the canonical generators of selfdual 2-forms in \( \Lambda^2_+ (\mathbb{R}^4)^* \). The pointwise inner-product
\[
\langle u, v \rangle_{e_1 \ldots e_7} := \frac{1}{6} (u \lrcorner \varphi_0) \wedge (v \lrcorner \varphi_0) \wedge \varphi_0
\]
determines a Riemannian metric \( g_{\varphi} \) on \( Y \), under which \( *_{\varphi} \varphi \) is given pointwise by
\[
*_{\varphi_0} = e^{1234} - \omega_1 \wedge e^{67} - \omega_2 \wedge e^{75} - \omega_3 \wedge e^{56}.
\]
In the language of calibrated geometry [28], a 7-manifold with \( G_2 \)-structure \((Y, \varphi)\) is said to be calibrated if \( d\varphi = 0 \) and cocalibrated if \( d*_{\varphi} \varphi = 0 \); moreover it is common to omit \( Y \) and refer simply to \( \varphi \) in those terms. Cocalibrated \( G_2 \)-structures appear in the Fernández–Gray classification [21] of \( G_2 \)-structures by their intrinsic torsion. An \( AG_2 \)-structure \( \varphi \) is both calibrated and cocalibrated if and only if
\[
\nabla^g \varphi = 0,
\]
in which case \( \text{Hol}(g_{\varphi}) \subseteq G_2 \) and it is said to be torsion-free [41], Lemma 11.5.

Let us consider the following familiar example found, for example, in Proposition 11.1.2 of [30].

**Example 2.8.** Let \((Z, \omega, \varepsilon)\) be a Calabi–Yau 3-fold. Then the product manifold \( Z \times S^1 \) has a natural torsion-free \( G_2 \)-structure defined by
\[
\varphi := dt \wedge \omega + \text{Im} \varepsilon,
\]
where \( t \) is the variable in \( S^1 \) and tensors are denoted identically to their pullbacks under projection onto the \( Z \) factor. The Hodge dual of \( \varphi \) is
\[
\psi := *\varphi = \frac{1}{2} \omega \wedge \omega + dt \wedge \text{Re} \varepsilon,
\]
and the induced metric \( g_{\varphi} = g_Z + dt \otimes dt \) is the Riemannian product metric on \( Z \times S^1 \), with holonomy \( \text{Hol}(g_{\varphi}) = \text{SU}(3) \subseteq G_2 \).

In the case of a CY link, we only deviate from the product model of Example 2.8 in the sense that \( K_f \) is necessarily nontrivial as a circle bundle over the CY\(^3\) base \( V \), since \( \pi_1(K) = \{1\} \) by Theorem 2.1, so it is fair to ask whether \( K_f \) also inherits a ‘globally twisted’ \( G_2 \)-structure from the Calabi–Yau structure of \( V \).

**Remark 2.9.** If a Lie group \( G \) induces a \( G \)-structure on a manifold \( M \), then every bundle of tensors splits into summands corresponding to irreducible representations of \( G \). The link \( K_f \) carries a \( G_2 \)-structure so, in particular, 2-forms split as
\[
\Omega^2(K) = \Omega^2_7(K) \oplus \Omega^2_{14}(K),
\]
where \( \Omega^2_7(K) \) and \( \Omega^2_{14}(K) \) are vector subbundles of \( \Omega^2(K) \) with fibres isomorphic to the irreducible 7 and 14 representations of \( G_2 \), respectively. It is a well-known fact about manifolds with a \( G_2 \)-structure [7], [39] that \( \Omega^2_7 \) (respectively \( \Omega^2_{14} \)) is the \((-2)\)-eigenspace (resp. \((+1)\)-eigenspace) of the \( G_2 \)-equivariant linear map
\[
T_\varphi : \Omega^2 \to \Omega^2, \quad \eta \mapsto T_\varphi \eta := * (\eta \wedge \varphi).
\]
2.3. Links as Sasakian 7–manifolds

A contact manifold \((M, \theta)\) is given by a smooth \((2n+1)\)-manifold \(M\) and a contact structure \(\theta \in \Omega^1(M)\) such that \(\theta \wedge (d\theta)^n \neq 0\), everywhere on \(M\). On a contact manifold there exists a unique Reeb vector field \(\xi \in \Gamma(TM)\), such that \(\xi \wedge \theta = 1\) and \(\xi \wedge d\theta = 0\). The Reeb vector field is nowhere-vanishing, so it uniquely determines a 1-dimensional foliation \(\mathcal{N}_\xi\) called the characteristic foliation. It is customary to think of contact manifolds as odd-dimensional analogues of symplectic manifolds, with the 2-form \(d\theta\) being ‘transversely symplectic’ with respect to the characteristic foliation. From that perspective, Sasakian geometry encodes the notion of ‘transversely Kähler structure’:

**Definition 2.10.** A Sasakian structure on \(M\) is a quadruple \((M, \theta, g, \Phi)\) such that \((M, g)\) is a Riemannian manifold, \((M, \theta)\) is a contact manifold with Reeb vector field \(\xi, \Phi\) is a global section of \(\text{End}(\Theta M)\), and the following relations hold:

\[
\begin{align*}
g(\xi, \xi) &= 1, \quad \Phi \circ \Phi = -\text{Id}_{\Theta M} + \theta \otimes \xi, \quad g(\Phi X, \Phi Y) = g(X, Y) - \theta(X) \theta(Y), \\
\nabla^g_\xi \xi &= -\Phi X, \quad (\nabla^g_\Phi)(Y) = g(X, Y) \xi - \theta(Y) X,
\end{align*}
\]

where \(X, Y\) are vector fields on \(M\) and \(\nabla^g\) is the Levi-Civita connection corresponding to \(g\). If \((M, \theta, g, \Phi)\) satisfies these conditions, we say \(M\) is a Sasakian manifold.

If the orbits of \(\xi\) are all closed, hence circles, then \(\xi\) integrates to an isometric \(\text{U}(1)\) action on \(M\), in particular this action is locally free. If the action is in fact free, then the Sasakian structure is said to be regular, otherwise, it is said to be quasi-regular. The leaf space \(Z := M/\mathcal{N}_\xi = M/\text{U}(1)\) has the structure of a manifold or orbifold, in the regular or quasi-regular case respectively.

The sphere \(S^{2n+1}\) has a natural contact structure given by the Hopf fibration \(S^{2n+1} \xrightarrow{\pi} \mathbb{P}^n\):

\[
\theta_c = -\frac{1}{2} \sum_{j=0}^{n} (z_j d\bar{z}_j - \bar{z}_j dz_j) = \sum_{j=0}^{n} (y_j dx_j - x_j dy_j)
\]

(in the real coordinates \(z_j = x_j + iy_j\)). It carries moreover \([42]\) a regular Sasakian structure \((S^{2n+1}, \theta_c, g_c, \Phi_c)\) in which the Reeb vector field is

\[
\xi_c = \sum_{i=0}^{n} \left( y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right) = -\sum_{j=0}^{n} \left( z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right),
\]

the metric \(g_c\) is given by the inclusion \(S^{2n+1} \subset \mathbb{R}^{2n+2}\), and

\[
\Phi_c = \sum_{i,j} \left\{ (x_i x_j - \delta_{ij}) \partial_{x_i} + (x_j y_i) \partial_{y_i} \right\} dy_j - \left\{ (y_i y_j - \delta_{ij}) \partial_{y_i} + x_i y_j \partial_{x_j} \right\} dx_j.
\]

The links of isolated hypersurface singularities admit Sasakian structures in a natural way.

**Proposition 2.11** (Proposition 9.2.2 in [6]). Let \(K_f\) be the link of a hypersurface singularity. Then the Sasakian structure \(S_c := (\theta_c, g_c, \Phi_c)\) on \(S^{2n+1}\) defined above induces by restriction a Sasakian structure, also denoted by \(S_c\), on the link \(K_f\).
2.4. Contact Calabi–Yau structures on links

Contact Calabi–Yau manifolds were introduced by Tomassini and Vezzoni in [46] and thoroughly studied by Habib and Vezzoni in [26], as a development of Reinhardt’s general theory of Riemannian foliations [36]. This concept describes Sasakian manifolds endowed with a closed basic complex volume form, which is ‘transversely holomorphic’ in a certain sense (see Definition 2.13). Most importantly for us, it allows for a vast generalisation of the $G_2$-geometry on homogeneous links discussed in Section 2.2.

Let $(M,\theta)$ be a contact manifold with contact 1-form $\theta$ and denote $B := \ker \theta$ its contact distribution of rank $2n$, i.e., $TM = B \oplus N_\xi$. Let $X$ denote an arbitrary vector field tangent to the characteristic foliation $N_\xi$. A differential form $\beta \in \Omega^k(M)$ is said to be transversal if $X \lrcorner \beta = 0$ and $L_X \beta = 0$ for every such $X$.

If $(M,\theta,\Phi)$ is a Sasakian manifold, and $x \in M$, it follows from Definition 2.10 that $(\Phi|_B)_x^2 = -\text{Id}_B_x$. Then we can decompose the complexification $B_x \otimes \mathbb{C}$ into the eigenspaces of the complexified automorphism $\Phi|_B_x \otimes \mathbb{C}$:

$$B_x \otimes \mathbb{C} = B_x^{1,0} \oplus B_x^{0,1},$$

where $B_x^{1,0}$ and $B_x^{0,1}$ correspond to the eigenvalues $i := \sqrt{-1}$ and $-i$ respectively.

This induces a splitting of the exterior differential algebra over $B_C := B \otimes \mathbb{C}$:

$$\Omega^k(B_C) = \bigoplus_{p+q=k} \Omega^{p,q}(M),$$

where $\Omega^{p,q}(M) := \Gamma((B^{1,0})^p \otimes (B^{0,1})^q)$ and $p, q \geq 0$. Then we have an obvious decomposition of exterior forms on $M$ given by

$$\Omega^j(M) = \bigoplus_{p+q=j} \Omega^{p,q}(M) \oplus \bigoplus_{p+q=j-1} \Omega^{p,q}(M) \wedge \theta.$$  

If $\beta \in \Omega^k(M)$ is a transversal differential form, we will say that $\beta$ is of type $(p,q)$ if it belongs to $\Omega^{p,q}(M)$. The following lemma (Corollary 3.1 in [5]) will be crucial for our applications in gauge theory, so we sketch it for convenience.

**Lemma 2.12.** Let $(M,\theta,\Phi)$ be a Sasakian manifold. Then $d\theta \in \Omega^{1,1}(M)$.

**Proof.** That $d\theta$ is transversal is clear from Definition 2.10. It is easy to prove that $d\theta(X,Y) = -\theta(\Phi X, Y)$ for all $X, Y \in B$, then $d\theta$ is of type $(1,1)$. \hfill \Box

**Definition 2.13.** A contact Calabi–Yau manifold (cCY) is a quadruple $(M,\theta,\Phi,\varepsilon)$ such that:

- $(M,\theta,\Phi)$ is a $2n+1$-dimensional Sasakian manifold;
- $\varepsilon$ is a nowhere vanishing transversal form on $B = \ker(\theta)$ of type $(n,0)$:

$$\varepsilon \wedge \bar{\varepsilon} = c_n \omega^n, \quad d\varepsilon = 0,$$

where $c_n = (-1)^{n(n+1)/2} \frac{2^n}{n!}$ and $\omega := d\theta$. We denote accordingly

$$\text{Re}\, \varepsilon := \frac{\varepsilon + \bar{\varepsilon}}{2} \quad \text{and} \quad \text{Im}\, \varepsilon := \frac{\varepsilon - \bar{\varepsilon}}{2i}. $$
Gauge theory and $G_2$-geometry on Calabi–Yau links

Our interest in cCY structures for $G_2$-geometry derives from the following fundamental result.

**Proposition 2.14** ([26], Subsection 6.2.1). Let $(M, \theta, \Phi, \varepsilon)$ be 7-dimensional contact Calabi–Yau manifold. Then $M$ carries a cocalibrated $G_2$-structure defined by

$$\varphi := \theta \wedge \omega + \text{Im} \varepsilon$$

with torsion $d\varphi = \omega \wedge \omega$ (cf. Definition 2.13) and corresponding dual 4-form

$$\psi = *\varphi = \frac{1}{2} \omega \wedge \omega + \theta \wedge \text{Re} \varepsilon.$$

The existence of cCY structures on links is equivalent to a simple numerical criterion on the weighted homogeneous data, which we adopt as a definition.

**Definition 2.15.** A weighted link $K_f$ (cf. Definition 2.7) of degree $d$ and weight $w = (w_0, \ldots, w_n)$ is said to be a Calabi–Yau (CY) link if

$$d = \sum_{i=0}^{n} w_i.$$

The condition $d - \sum_{i=0}^{n} w_i = 0$ means precisely that the Sasakian structure $(K, \theta_c, \Phi_c)$ on $K_f$ induced from the canonical Sasakian structure of the sphere $S^{2n+1}$ is null Sasakian, i.e., the (basic) first Chern class of $(K, \theta_c, \Phi_c)$ vanishes. Recall also this vanishing is exactly the requirement for the weighted projective $V$ to be a Calabi–Yau orbifold [8], thus CY links are nontrivial circle fibrations over Calabi–Yau 3-orbifolds. Furthermore, the Reeb vector field the unit tangent to the $S^1(w)$-action and the 3-form $\varepsilon$ is transversal, so the $G_2$-structure (2.6) is $S^1$-invariant. In the terms of Definition 2.15, Habib and Vezzoni’s existence result can be restated as:

**Proposition 2.16** ([26], Proposition 6.7). Every Calabi–Yau link admits a $S^1$-invariant contact Calabi–Yau structure.

The proof of Proposition 2.16 relies on a Sasakian version of the El Kacimi theorem to prove that any null Sasakian structure on a compact simply-connected manifold can be deformed into a contact Calabi–Yau one. Combining the previous two propositions:

**Corollary 2.17** ([26], Corollary 6.8). Every Calabi–Yau link has a cocalibrated $S^1$-invariant $G_2$-structure of the form (2.6).

3. The $\nu$ invariant of Calabi–Yau links

For an arbitrary closed 7-manifold with $G_2$-structure $(Y^7, \varphi)$, Crowley and Nordström define a pair of homotopy invariants $(\nu(\varphi), \xi(\varphi))$ which completely classifies the data, up to diffeomorphism and homotopy, if $Y$ is 2-connected ([16], Theorem 1.17). Subsequently this has been refined as an analytic invariant of manifolds with $G_2$-metrics [14], and similar ideas also intervene in the authors’ topological classification of spin 2-connected 7-manifolds [15].
We will be interested in the first invariant $\nu(\varphi)$, which is a $\mathbb{Z}_{48}$-valued combination of topological data from a compact coboundary 8-manifold with a Spin(7)-structure $(W^8, \Psi)$ filling $(Y, \varphi)$, in the sense that $Y = \partial W$ and $\Psi|_Y = \varphi$:

\begin{equation}
\nu(\varphi) := \chi(W) - 3\sigma(W) \mod 48
\end{equation}

($\chi$ and $\sigma$ denote the real Euler characteristic and the signature, respectively). This quantity is preserved under diffeomorphisms of $Y$ and homotopies of the $G_2$-structure $\varphi$ [16], Theorem 1.3. Moreover, $\nu(\varphi)$ is independent of the particular choice of coboundary $W$ [16], Corollary 3.2, thus it is interpreted as an “$\hat{A}$-defect” from certain integral characteristic classes of principal Spin(8)-bundles evaluated on $TW$ and the half-spinor bundles $S^\pm W$.

A central aspect is the fact that such a filling $W$ always exists (see [16], Lemma 3.4(ii)). The argument relies on the fact that the bordism group $\Omega_{\text{spin}}^7$ is trivial, hence there always exists some (connected) coboundary $(W, \Psi)$ inducing a reference $G_2$-structure on $Y$, but it is totally non-constructive. For example, the authors must resort to an elaborate construction of an explicit coboundary $W$ to calculate $\nu = 24$ (Theorem 1.7 in [16]) for the important class of manifolds with holonomy $G_2$ obtained as twisted connected sums [13]. This allows one to distinguish, for instance, whether a given $G_2$-structure is not a gluing of asymptotically cylindrical Calabi–Yau 3-folds [12].

3.1. Construction of a spin coboundary

Let $K_f$ be a weighted link (see Definition 2.7) of degree $d$ and weight $w = (w_0, \ldots, w_4)$. In order to calculate the $\nu$ invariant for our $G_2$-structure (2.6) on $K_f$, we must find an ad hoc compact Spin(7)-coboundary $(W, \Psi)$ such that

$K = \partial W$ and $\Psi|_{K_f} = \varphi$.

The ambient 4-form

$$
\Psi := \frac{1}{f} \sum_{i=0}^4 z_i dz_0 \wedge \cdots \wedge dz_4 \in \Lambda^{4,0}(\mathbb{C}^5)
$$

is $S^1 \subset \mathbb{C}^*$-invariant under the action (2.1) if, and only if, $d - \sum_{i=0}^4 w_i = 0$, i.e., exactly when the link $K_f$ is Calabi–Yau (Definition 2.15). Let $f_\varepsilon$ be a smoothing of $f$, e.g. $f_\varepsilon := f - \varepsilon$, and let

$$X_\varepsilon := f_\varepsilon^{-1}(0) \cap \overline{B} \subset \mathbb{C}^5$$

be the 8-manifold inside the (compact component of the complement of the) sphere $S^9$ with boundary $K_f = \partial X_\varepsilon$ (Figure 2). The restriction $\Psi|_{X_\varepsilon}$ induces an SU(4)-structure, hence a Spin(7)-structure on $X_\varepsilon$, which is $S^1$-invariant by construction.

Restricting to the boundary we get an $S^1$-invariant $G_2$-structure on $K_f$, which corresponds exactly to an SU(3)-structure $\varphi'$ on the basic Calabi–Yau $V$. Now, all
SU(3)-structures on a 6-manifold are homotopic, as sections of a bundle of rank 8, so $\varphi'$ is homotopic to our $\varphi$ and we can take $W = X_\varepsilon$:

\begin{equation}
\nu(\varphi) = \nu(\varphi') = \chi(X_\varepsilon) - 3\sigma(X_\varepsilon) \mod 48.
\end{equation}

Now all we need is to calculate the topology of the smoothing of an affine hypersurface.

**Proposition 3.1.** Let $f \in \mathbb{C}[z_0, \ldots, z_n]$ be weighted homogeneous with typical Milnor fibre $F$ (cf. Theorem 2.1), and consider the model affine variety

$$\tilde{V} := f^{-1}(1) \subset \mathbb{C}^{n+1}.$$ 

Given $\varepsilon > 0$ sufficiently small, the smoothing $f_\varepsilon := f - \varepsilon$ defines in $\mathbb{C}^{n+1}$ a compact manifold with boundary $X_\varepsilon := f_\varepsilon^{-1}(0) \cap \overline{B_{2^{n+2}}}$, and the following are diffeomorphic:

\begin{equation}
X_\varepsilon \cong F, \quad X_\varepsilon \setminus \partial X_\varepsilon \cong F \cong \tilde{V}.
\end{equation}

**Proof.** Taking $\varepsilon = c = 1$ in Theorem 2.3, we identify diffeomorphically the smoothing $X_\varepsilon$ with the closure of the Milnor fibre $F$. Then the second identification is immediate from Lemma 2.6. \hfill $\square$

**3.2. Explicit formula for $\nu$ on Calabi-Yau links**

In view of Proposition 3.1, we will obtain the $\nu$ invariant from the following:

\begin{equation}
\chi(X_\varepsilon) = \chi(F) \quad \text{and} \quad \sigma(X_\varepsilon) = \sigma(\tilde{V}).
\end{equation}

We begin with Steenbrink’s method [43] for the signature of $\tilde{V}$. Let

$$\{z^\alpha \in \mathbb{C}[z_0, \ldots, z_n] : \alpha = (\alpha_0, \ldots, \alpha_n) \in I \subset \mathbb{N}^{n+1}\}$$
be a set of monomials representing a basis over \( \mathbb{C} \) for \( \frac{C[[z_0, \ldots, z_n]]}{(\partial f/\partial z_0, \ldots, \partial f/\partial z_n)} \) (cf. (i) of Theorem 2.2). For each \( \alpha \in I \), define

\[
(3.5) \quad l(\alpha) := \sum_{i=0}^{n} (\alpha_i + 1) \frac{w_i}{d}
\]

Assume that \( n \) is even (in our case, indeed \( n = 4 \)), and denote by \((\mu_-, \mu_0, \mu_+)\) the signature of the intersection form on \( H^n(\tilde{V}, \mathbb{R}) \) i.e., \( \mu_- \), \( \mu_0 \) and \( \mu_+ \) denote the numbers of negative, zero and positive entries, respectively, on the diagonal of the intersection matrix. Then

\[\sigma(\tilde{V}) = \mu_+ - \mu_- .\]

Note that the sum \( \mu_+ + \mu_- + \mu_0 \) equals the Milnor number \( \mu \) by (ii) of Theorem 2.2.

On the other hand, by (iii) of Theorem 2.2, the Euler characteristic of the Milnor fiber is determined by the Milnor number, which is given by Proposition 2.5 for weighted homogenous links. By Theorem 2.1, \( F \) is homotopy-equivalent to \( \overline{F} \), so for \( n = 4 \):

\[\chi(\overline{F}) = \chi(F) = 1 + \left( \frac{d}{w_0} - 1 \right) \cdots \left( \frac{d}{w_4} - 1 \right) .\]

Finally, replacing (3.4) in (3.2), we establish the formula of Theorem 1.2:

\[
(3.6) \quad \nu(\varphi) = \left( \frac{d}{w_0} - 1 \right) \cdots \left( \frac{d}{w_4} - 1 \right) - 3(\mu_+ - \mu_-) + 1 .
\]

Steenbrink proved (Theorem 2 in [43]) that the signature \((\mu_-, \mu_0, \mu_+)\) can be computed as follows:

\[
\mu_+ = |\{ \beta \in I : l(\beta) \notin \mathbb{Z}, [l(\beta)] \in 2\mathbb{Z} \}| ,
\mu_- = |\{ \beta \in I : l(\beta) \notin \mathbb{Z}, [l(\beta)] \notin 2\mathbb{Z} \}| ,
\mu_0 = |\{ \beta \in I : l(\beta) \in \mathbb{Z} \}| ,
\]

where \( [x] \) denotes the integer part of \( x \in \mathbb{Q} \), hence the above process can be easily implemented. We offer the code for a working algorithm in a combination of SINGULAR and MATHEMATICA, but surely readers will be able to formulate leaner alternatives. We display in Table 1 the invariants given by (3.6) for some examples from Candelas’ list of weighted Calabi–Yau threefolds.

Inspection of a few examples suggests a parity constraint for the \( \nu \) invariant, and this is indeed the case:

**Proposition 3.2.** The Crowley–Nordström \( \nu \) invariant of a weighted link is odd in \( \mathbb{Z}_{48} \).

**Proof.** We know from Theorem 1.3 in [16] that \( \nu(\varphi) \equiv \chi_\mathbb{Q}(K) \mod 2 \), where

\[
\chi_\mathbb{Q}(K) := \sum_{i=0}^{n-1} b_i(K)
\]
Table 1. The $\nu$ invariant for certain Calabi–Yau links.

| degree | weights | polynomial | $\nu$ |
|--------|---------|------------|-------|
| 75     | (10, 12, 13, 15, 25) | $z_0^7 z_4 + z_1^7 z_3 + z_2^3 z_5 + z_3^3 z_7 + z_4^3$ | 1 |
| 135    | (1, 18, 32, 39, 45) | $z_0^{135} + z_1^9 z_4 + z_2^9 z_3 + z_3^9 z_1 + z_4^9$ | 3 |
| 36     | (18, 12, 4, 1, 1) | $z_0 + z_1^9 + z_2^9 + z_3^{30} + z_4^9$ | 5 |
| 81     | (3, 7, 18, 26, 27) | $z_0^7 + z_1^7 z_2 + z_2^7 z_4 + z_3^3 z_7 + z_4$ | 7 |
| 45     | (3, 5, 8, 14, 15) | $z_0^{15} + z_1^9 z_2 + z_2^9 z_3 + z_3^9 z_5 + z_4^9$ | 9 |
| 45     | (4, 7, 9, 10, 15) | $z_0^5 z_2 + z_1^5 z_3 + z_2^5 z_4 + z_3^5 z_7 + z_4^5$ | 11 |
| 75     | (5, 8, 12, 15, 35) | $z_0^{15} + z_1^7 z_4 + z_2^7 z_3 + z_3^7 z_1 + z_4^7$ | 13 |
| 180    | (90, 60, 20, 9, 1) | $z_0 + z_1^7 + z_2^7 + z_3^7 + z_4^{80}$ | 15 |
| 45     | (15, 15, 5, 9, 1) | $z_0^7 + z_1^7 + z_2^7 + z_3^7 + z_4^{45}$ | 17 |
| 16     | (4, 8, 2, 1, 1) | $z_0^7 z_1 + z_1^7 z_2 + z_2^7 z_3 + z_3^{16} + z_4^{10} + z_5^{6}$ | 19 |
| 81     | (2, 9, 19, 24, 27) | $z_0^{27} z_4 + z_1^3 z_3 + z_2^3 z_1 + z_3^9 + z_4^{3}$ | 21 |
| 24     | (12, 8, 2, 1, 1) | $z_0 + z_1^3 + z_2^3 + z_3^{24} + z_4^{24}$ | 23 |
| 1806   | (42, 258, 903, 602, 1) | $z_0^{13} + z_1^7 + z_2^7 + z_3^7 + z_4^{806}$ | 25 |
| 51     | (2, 6, 9, 17, 17) | $z_0^{14} z_4 + z_1^7 z_2 + z_2^7 z_3 + z_3^7 z_0 + z_4^{6}$ | 29 |
| 93     | (3, 8, 21, 30, 31) | $z_0^{14} + z_1^7 z_2 + z_2^7 z_3 + z_3^7 z_0 + z_4^{4}$ | 31 |
| 63     | (3, 4, 14, 21, 21) | $z_0^{24} + z_1^{14} z_0 + z_2^4 z_3 + z_3^4 + z_4^3$ | 33 |
| 103    | (1, 16, 23, 29, 34) | $z_0^{103} + z_1^7 z_2 + z_2^7 z_3 + z_3^7 z_1 + z_4^{3}$ | 37 |
| 135    | (5, 6, 14, 45, 65) | $z_0^{37} + z_1^7 z_0 + z_1^{10} z_2 + z_2^7 z_4$ | 39 |
| 60     | (30, 20, 5, 4, 1) | $z_0^{20} + z_1^{15} + z_2^{15} + z_3^{15} + z_4^{60}$ | 41 |
| 55     | (4, 4, 11, 17, 19) | $z_0^{11} z_2 + z_1^{10} z_4 + z_2^9 z_3 + z_3^9 z_1 + z_4^{4}$ | 43 |
| 135    | (1, 21, 30, 38, 45) | $z_0^{135} + z_1^7 z_2 + z_2^7 z_4 + z_3^7 z_1 + z_4^7$ | 45 |
| 45     | (5, 5, 9, 11, 12) | $z_0^5 z_2 + z_1^5 z_3 + z_2^5 z_4 + z_3^5 z_2 + z_4^5$ | 47 |

is the rational semi-characteristic of $K_f$. On the other hand, $b_1 = b_2 = 0$ because $K_f$ is 2-connected (cf. Theorem 2.1), and we know from Theorem 9.3.2 in [6] that the Betti number $b_{n-1}$ is even, if $n$ is even. Therefore $b_3$ is even when $n = 4$, thus $\chi_{\mathbb{Q}}(K)$ is odd. 

Together with formula (3.6), this completes the proof of Theorem 1.2.

**Example 3.3.** Let us calculate the $\nu$ invariant for our G$_2$-structure (1.1) on the Fermat quintic

$$f(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5.$$ 

In this case the Milnor algebra is just $\mathbb{C}[z_0, \ldots, z_4]/(z_0^5, \ldots, z_4^5)$, the Milnor number is $\mu = 1024$ and we can take, as a basis of the Milnor algebra, all monomials of the form $z_0^{\alpha_0} \ldots z_4^{\alpha_4}$, with $0 \leq \alpha_i \leq 3$, $\forall i$. A simple computation gives

$$(\mu_-, \mu_0, \mu_+) = (240, 204, 580),$$

therefore $\sigma(\hat{V}) = 340$. On the other hand, Theorem 2.2 gives $\chi(X_\varphi) = 1 + \mu = 1025$. Hence $\nu(\varphi) = 5$ as claimed in Corollary 1.3.
4. Gauge theory on contact Calabi–Yau 7-manifolds

Let \((M, \phi)\) be a closed contact Calabi–Yau manifold and consider a \(G\)-bundle \(E \to M\) with \(G\) a compact semi-simple Lie group, denote by \(G := \Gamma(\text{Aut} E)\) its gauge group with \(g := \text{Lie}(G)\) the associated adjoint bundle and by \(\mathcal{A}(E)\) its space of connections. We address the classical problem of describing the absolute minima of the Yang–Mills functional

\[
\mathcal{S}_{\text{YM}} : \mathcal{A}(E) \to \mathbb{R}^+
\]

\[
\mathcal{S}_{\text{YM}}(A) := \|F_A\|^2_{L^2(M)} = \int_M \langle F_A \wedge *F_A \rangle_g
\]

i.e., solutions of the Yang–Mills equation:

\[
(4.1) \quad d_A^* F_A = 0.
\]

4.1. Yang–Mills connections, \(G_2\)-instantons and the Chern–Simons action

The paradigmatic PDE for gauge theory in the presence of a \(G_2\)-structure is the \(G_2\)-instanton equation \([20], [45]\), which can be formulated equivalently in terms of \(\phi\) or \(\psi := *\phi\):

\[
F_A \wedge \psi = 0 \iff *F_A = F_A \wedge \phi.
\]

This is the natural Euler–Lagrange equation for the Chern–Simons action, defined relatively to a fixed reference connection \(A_0 \in \mathcal{A}(E)\) by

\[
\mathcal{S}_{\text{CS}} : \mathcal{A}(E) \simeq A_0 + \Omega^1(g) \to \mathbb{R}
\]

\[
\mathcal{S}_{\text{CS}}(A_0 + a) := \frac{1}{2} \int_M \text{Tr} \left( d_A a \wedge a + \frac{2}{3} a \wedge a \wedge a \right) \wedge \psi
\]

with \(\mathcal{S}_{\text{CS}}(A_0) = 0\). Supposing the cocalibrated condition \(d\psi = 0\), the action is well-defined and \(G_2\)-instantons are manifestly critical points. Its gradient is the Chern–Simons 1-form \(\rho = d\mathcal{S}_{\text{CS}}\), defined on vector fields \(b : \mathcal{A}(E) \to \Omega^1(g)\) by

\[
(4.3) \quad \rho(b)_A = \int_M \text{Tr} \left( F_A \wedge b_A \right) \wedge \psi.
\]

and indeed the solutions of \((4.2)\) are precisely its zeroes (for a more detailed exposition, see [38].)

Now, if the \(G_2\)-structure was closed, then by the Bianchi identity every solution of \((4.2)\) would automatically solve \((4.1)\). In other words, \(G_2\)-instantons would manifestly be critical points of the Yang–Mills functional, somewhat in analogy to (anti-)selfdual connections in dimension 4 \([18]\). This indeed was the starting point of our predecessors in proposing gauge theory on \(G_2\)-manifolds. Since then, such Yang–Mills \(G_2\)-instantons have been constructed on Joyce manifolds \([47]\),
Bryant–Salamon manifolds [9], associative fibrations [38], asymptotically cylindrical $G_2$-manifolds [37], [39] and their twisted connected sums [40], [48]. However, the implication $(4.2) \Rightarrow (4.1)$, that $G_2$-instantons are Yang–Mills minima, does not hold for generic cocalibrated $G_2$-structures. In particular, a naive attempt to apply standard Chern–Weil arguments would depend on a certain characteristic class in de Rham cohomology which generically does not exist\(^1\). This is unfortunate, since every oriented spin 7-manifold admits such a structure [15], of which indeed many explicit examples are now known (see e.g. [10], [2], [31], [23] and references therein). As we will show in Section 4.4, a suitable version of the argument does hold for the natural cocalibrated $G_2$-structure on contact Calabi–Yau manifolds and in particular for Calabi–Yau links. Explicitly, integrable $G_2$-instantons on holomorphic Sasakian vector bundles are solutions of the Yang–Mills equation (4.1), even though the $G_2$-structure is not closed.

4.2. Sasakian vector bundles

Interpreting Sasakian structures as a setup for ‘transversely Kähler’ geometry, any compatible formulation of gauge theory requires a good notion of transversely compatible structures on vector bundles. We adopt the lexicon proposed by Biswas and Schumacher [5], §3.3.

Let $E \to M$ be a $C^\infty$ complex vector bundle over a smooth manifold and let
\begin{equation}
L \subset TM_C := TM \otimes \mathbb{C}
\end{equation}
be an integrable subbundle, i.e., closed under the Lie bracket. A partial connection along $L$ is a $C^\infty$ differential operator
\[ D_L : E \to L^* \otimes E \]
satisfying the Leibniz condition for $f \in C^\infty(M)$ and $s \in \Gamma(E)$:
\[ D_L(fs) = fD_L(s) + q_L(df) \]
relative to the dual $q_L : T^*M_C \to L^*$ of the inclusion (4.4). Since $L$ is integrable, $q_L$ induces a natural exterior derivative $\hat{d} : L^* \to \bigwedge^2 L^*$, hence an extension of $D_L$ to $E$-valued sections of $L^*$:
\[ D_L : L^* \otimes E \to (\bigwedge^2 L^*) \otimes E \]
\[ D_L(t \otimes s) := \hat{d}(t) \otimes s - t \otimes D_L(s). \]

Denoting by $\mathfrak{g}$ the adjoint bundle of $E$, the curvature of $D_L$ is the $C^\infty(M)$-linear section
\[ F_{D_L} := D_L^2 \in \Gamma((\bigwedge^2 L^*) \otimes \mathfrak{g}) \]
and $D_L$ is said to be flat if $F_{D_L} = 0$.

\(^1\)The implication may still hold, in special circumstances, despite the absence of the de Rham class, see for instance the scalar torsion case in [27].
Definition 4.1. A Sasakian (vector) bundle \( E \rightarrow M \) over a Sasakian manifold \( (M,\theta,g,\Phi) \) with Reeb field \( \xi \) is a pair \( (E,D_\xi) \) given by a \( C^\infty \) complex vector bundle \( E \) over \( M \) and a partial connection \( D_\xi : E \rightarrow \theta \otimes E \) along \( \xi \).

To be completely precise, we are applying the previous discussion to the line subbundle \( L = \xi_C := N_\xi \otimes_R \mathbb{C} \subset TM_C \) spanned by \( \xi \) over \( \mathbb{C} \) (cf. Section 2.4); it is clear that any such \( D_\xi \) is flat. Moreover, the natural partial connection induced by \( D_\xi \) on \( E^* \) gives natural definitions of \( E^* \) and \( \text{End}(E) \) as Sasakian bundles, and we define a Hermitian structure on \( E \) as a smooth Hermitian structure \( h \) on \( E \) preserved by \( D_\xi \), in the sense that

\[
d(h(s_1, s_2))|_{\xi_C} = h(D_\xi(s_1), s_2) + h(s_1, D_\xi(s_2)).
\]

Clearly a Hermitian structure on \( E \) induces a Hermitian structure on \( E^* \) and on \( \text{End}(E) \). A unitary connection on \( (E,h) \) is a connection \( A \) on \( E \) such that \( d_A \) preserves \( h \) in the usual sense.

Finally, in the notation of (2.5), we obtain a natural notion of transversely holomorphic structures over a Sasakian manifold \( M^{2n+1} \) relative to the integrable ‘extended anti-holomorphic’ \((n+1)\)-dimensional foliation

\[
(4.5) \quad \tilde{B}^{0,1} := B^{0,1} \oplus \xi_C \subset TM_C.
\]

Definition 4.2. A holomorphic (Sasakian) bundle \( \mathcal{E} \rightarrow M \) over a Sasakian manifold \( M \) with Reeb field \( \xi \) is a pair \( (E,\bar{\partial}) \) given by a Sasakian bundle \( E = (E,D_\xi) \) (cf. Definition 4.1) and a flat partial connection \( \bar{\partial} = D_{\tilde{B}^{0,1}} \) such that \( \bar{\partial}|_{\xi_C} = D_\xi \).

An integrable connection on \( \mathcal{E} = (E,\bar{\partial}) \) is connection \( A \) on \( E \) such that its induced partial connection along \( \tilde{B}^{0,1} \) given by \( D_{\tilde{B}^{0,1}} := d_A|_{\tilde{B}^{0,1}} \) coincides with \( \bar{\partial} \). We denote by \( \mathcal{A}(\mathcal{E}) \) the subset of integrable connections inside \( \mathcal{A}(E) \). We are now in position to extend the well-known concept of a Chern connection, mutually compatible with the holomorphic structure and the Hermitian metric \([18]\), Proposition 2.1.56:

Proposition 4.3 ([5], p. 552). Let \( (\mathcal{E},h) \) be a holomorphic Sasakian bundle with Hermitian structure; then there exists a unique unitary and integrable Chern connection \( A_h \) on \( \mathcal{E} \), and

\[
F_{A_h} \in \Omega^{1,1}(\mathfrak{g}).
\]

Moreover, the expression

\[
(4.6) \quad \det \left( \text{id}_E + \frac{i}{2\pi} F_{A_h} \right) =: \sum_{j=0}^n c_j(\mathcal{E},h)
\]

defines closed Chern forms \( c_j(\mathcal{E},h) \in \Omega^{0,j}(M) \).

In a local holomorphic trivialization \( \tau \), i.e., such that \( \mathcal{E} \) is locally spanned by sections in \( \text{ker} \bar{\partial} \), the Chern connection of \( h \) is represented by the matrix of \((1,0)\)-forms \( A^*_h = h^{-1}\bar{\partial}h \) and its curvature has the form \( F^*_h = \bar{\partial}(h^{-1}\bar{\partial}h) \). Moreover, it is clear from (4.5) and Definition 4.2 that any other Hermitian structure \( h' \) induces a Chern connection on \( \mathcal{E} \) satisfying \( A_{h'} - A_h \in \Omega^{1,0}(\mathfrak{g}) \).
4.3. $G_2$-instantons and the Hermitian Yang–Mills condition

A connection $A$ on a complex vector bundle over a Kähler manifold is *Hermitian Yang–Mills (HYM)* if

$$\hat{F}_A := (F_A, \omega) = 0 \quad \text{and} \quad F_A^{0,2} = 0.$$

This notion extends literally to Sasakian bundles $E \to M$, taking $\omega = d\theta \in \Omega^{1,1}(M)$ as the transverse Kähler form. Fixing a holomorphic structure on $E$, it is easy to check that compatible HYM connections are exactly $G_2$-instantons:

**Lemma 4.4.** Let $E$ be a holomorphic Sasakian bundle over a 7-dimensional closed contact Calabi–Yau manifold $M$ endowed with its natural $G_2$-structure (2.6). Then a Chern connection $A$ on $E$ is HYM if, and only if, it is a $G_2$-instanton.

**Proof.** A Chern connection $A$ satisfies $F_A \in \Omega^{1,1}(M)$ (Proposition 4.3), so taking account of the bidegree of the transverse holomorphic volume form (cf. Definition 2.13) we have $F_A \wedge \varepsilon = F_A \wedge \bar{\varepsilon} = 0$. Therefore

$$F_A \wedge \text{Im} \varepsilon = \frac{1}{2i} F_A \wedge (\varepsilon - \bar{\varepsilon}) = 0.$$

Now, taking the product with the 4-form we have

$$F_A \wedge \psi = \frac{1}{2} F_A \wedge \omega \wedge \omega = (\text{est.}) \hat{F}_A(*\theta),$$

hence $A$ is a solution of (4.2) if, and only if, $\hat{F}_A = 0$. \hfill $\Box$

This result generalises the well-known fact that HYM connections compatible with a fixed holomorphic structure over a smooth Calabi–Yau 3-fold pull back bijectively to $S^1$-invariant $G_2$-instantons over the product $\text{CY}^3 \times S^1$ [39], Proposition 8. Indeed, it is easy to deduce the corresponding claim for arbitrary circle fibrations:

**Corollary 4.5.** Let $X$ be a Calabi–Yau threefold, let $\pi: Y \to X$ be a Sasakian circle fibration endowed with the natural $G_2$-structure (1.1), and let $E := \pi^* E_0 \to Y$ be the pullback from a holomorphic vector bundle $E_0 \to X$. Then $E$ is a holomorphic Sasakian bundle, and a Chern connection $A$ on $E_0$ is HYM if, and only if, $\pi^* A$ is a $G_2$-instanton on $E$.

**Proof.** The contact Calabi–Yau structure is trivially given by the global angular form $\theta \in \Omega^1(Y)$ and the pullbacks of the Calabi–Yau data from $X$, under $\pi$, with natural Reeb field determined by $\theta(\xi) = 1$, tangent to the $S^1$-action. Then the underlying complex vector bundle $\pi^* E_0$ is trivial along $\xi$ and we can adopt $D_\xi = d_\xi$ the trivial vertical connection, which is manifestly flat. This defines a Sasakian bundle structure (cf. Definition 4.1). Moreover, the 6-dimensional distribution $B := \ker \theta \subset TY$ maps under $\pi_*$ isomorphically to $TX$, which induces a natural bi-degree decomposition $B = \bigoplus B^{i,j}$. It is immediate to check that the holomorphic structure $\partial_0$ on $E_0$ pulls back to a holomorphic structure $\partial := \pi^* \partial_0$ on $E$. \hfill $\Box$
This gives a correspondence
\[
\{ \text{S}^1\text{-invariant unitary connections on } \mathcal{E} = \pi^* \mathcal{E}_0 \} \overset{1-1}{\longleftrightarrow} \{ \text{HYM Chern connections on } \mathcal{E}_0 \}
\]
which proves part (ii) of Theorem 1.5. Notice that the right-hand side is bijectively parametrised by stable holomorphic structures on the underlying complex vector bundle \( E_0 \to X \), by the Hitchin–Kobayashi correspondence. At this point it is sharply relevant to ask whether a Sasakian version of that correspondence may be obtained, via a suitable notion of transverse stability. While Biswas and Schumacher do outline some progress in that direction [5], §3.4, in the course of submission of the present article some fundamental progress has been achieved by Baraglia and Hekmati, who established a Hitchin–Kobayashi correspondence for transversely Kähler foliated geometries [4].

**Remark 4.6.** The coincidence of \( G_2 \)-instantons and HYM connections in the holomorphic Sasakian context of Lemma 4.4 has as striking consequence on contact Calabi–Yau manifolds (cf. Proposition 2.14). Using the Bianchi identity, the Yang–Mills equation (4.1) for any cocalibrated \( G_2 \)-instanton \( A \) (cf. (4.2)) is equivalently rephrased as follows:
\[
0 = d^*_A F_A = d_A(F_A \wedge \varphi) = F_A \wedge d\varphi \iff F_A \in \ker d\varphi \subset (\Omega^\bullet(M), \wedge).
\]
But in the cCY case, the natural \( G_2 \)-structure (2.6) satisfies \( d\varphi = \omega^2 \), and \( F_A \in \ker \omega^2 \) is precisely the HYM condition. Therefore integrable \( G_2 \)-instantons are actually Yang–Mills critical points, even though the \( G_2 \)-structure is not closed.

### 4.4. Characteristic classes and topological energy bounds

We will show that 7-dimensional contact Calabi–Yau manifolds admit a naturally defined secondary characteristic class representing topological charge, which is another peculiar feature among \( G_2 \)-structures with torsion. From the perspective of gauge theory, this means that critical points of the Chern–Simons functional indeed saturate the Yang–Mills energy, just like in classical 4-dimensional theory or more familiar torsion-free higher dimensional models.

**Definition 4.7.** Let \( E \to M \) be a Sasakian bundle (cf. Definition 4.2) over a 7-dimensional closed contact Calabi–Yau manifold (cf. Definition 2.13) with \( G_2 \)-structure (2.6) given by Proposition 2.14. We define the *charge* of a connection \( A \in \mathcal{A}(E) \) by
\[
(4.7) \quad \kappa(A) := \int_M \text{Tr} F_A^2 \wedge \varphi
\]

**Lemma 4.8.** In the context of Proposition 4.3 and Definition 4.7, the quantity (4.7) assessed among Chern connections in \( \mathcal{A}(\mathcal{E}) \) is independent of the Hermitian structure and it defines a topological charge \( \kappa(\mathcal{E}) \).
Proof. Fix a Hermitian metric $h$ on a holomorphic Sasakian bundle $E$ with Chern connection $A = A_h \in \mathcal{A}(E)$; then any other Chern connection on $E$ has the form $A' = A + b$, for some $b \in \Omega^{1,0}(g)$. We know from standard Chern–Weil theory that

$$\text{Tr} F_{A+b}^2 - \text{Tr} F_{A}^2 = d(\text{Tr} \eta)$$

for some

$$\eta = \eta(A, b) := F_A \wedge b + \frac{1}{2} d_A b \wedge b + \frac{1}{3} b \wedge b \wedge b \in \Omega^3(g).$$

Since by assumption $M$ is a closed manifold, the quantity $\kappa$ in (4.7) is defined up to a term given by Stokes’ theorem after integration by parts:

$$(4.8) \quad \int_M \text{Tr} \eta \wedge d\varphi = \int_M \text{Tr}(F_A \wedge b + \frac{1}{2} d_A b \wedge b + \frac{1}{3} b \wedge b \wedge b) \wedge d\varphi.$$

Since $A$ is a Chern connection, Proposition 4.3 specifies the bi-degree of $F_A \wedge b \in \Omega^{2,1}(g)$. Moreover, since $b$ is transversal to the characteristic foliation and $A$ is locally of type $(1,0)$, the term $d_A b$ has basic type $(2,0) + (1,1)$, so

$$d_A b \wedge b \in (\Omega^{3,0} \oplus \Omega^{2,1})(g).$$

Finally, one obviously has $b \wedge b \wedge b \in \Omega^3(g)$.

Recall from Lemma 2.12 and Proposition 2.14 that $d\varphi = (d\theta)^2 \in \Omega^{2,2}(M)$, hence all three terms on the right-hand side of (4.8) vanish by excess in bi-degree.

Now, following a classical argument, on one hand we have the orthogonal decomposition of the Yang–Mills functional:

$$(4.9) \quad \mathcal{S}_{YM}(A) = \|F_A\|^2 = \|F_7\|^2 + \|F_{14}\|^2.$$

On the other hand, applying the $G_2$-equivariant eigenspace decomposition from Remark 2.9 to integrable connections as in (ii) of Lemma 4.8, a straightforward calculation relates the topological charge to these components:

$$\kappa(E) = -2 \|F_7\|^2 + \|F_{14}\|^2.$$

Combining with (4.9), we can isolate the topological charge as a lower bound of the Yang–Mills energy among integrable connections:

$$(4.10) \quad \mathcal{S}_{YM}[\mathcal{A}(E)](A) = -\frac{1}{2} \kappa(E) + \frac{3}{2} \|F_{14}\|^2 = \kappa(E) + 3 \|F_7\|^2.$$

Hence, if $\mathcal{S}_{YM}$ attains on $\mathcal{A}(E)$ its absolute topological minimum, this occurs at a connection whose curvature lies either in $\Omega_7^0$ or in $\Omega_{14}^0$. Moreover, since $\mathcal{S}_{YM} \geq 0$, the sign of $\kappa(E)$ obstructs the existence of one type or the other, so we fix $\kappa(E) \geq 0$, compatibly with the existence of our $G_2$-instantons (4.2) with $F_7 = 0$, i.e., such that $\mathcal{S}_{YM}(A) = \kappa(E)$. Together with Remark 4.6, this proves Theorem 1.4.

In summary, under the natural cocalibrated $G_2$-structure of a contact Calabi–Yau 7-manifold, $G_2$-instantons (4.2) over are Yang–Mills critical points and indeed absolute minima among compatible connections.
4.5. Example: $G_2$-instantons on pullback bundles

Motivated by Corollary 4.5 in the previous section, let us explore the simplest model case for gauge theory on a 7-dimensional CY link. Let $\pi: Y = K_f \to V$ be a CY link, which fibres nontrivially by circles over the smooth 3-fold $V = (f) \subset \mathbb{P}^4$, and consider the holomorphic Sasakian bundle given by pullback $E := \pi^*E_0 \to K_f$ of a holomorphic bundle over $V$. We would like to describe the explicit local form of the constraint imposed on a Chern connection $A \in \mathcal{A}(E)$ by the $G_2$-instanton equation (4.2).

Over a trivialising neighbourhood of $K_f$ as a circle fibration, i.e., an open set $U \subset V$ such that $K_f \supset \pi^{-1}(U) \simeq S^1 \times U$, given points $y \in \pi^{-1}(U)$ and $x = \pi(y) \in U$, an arbitrary integrable connection $A$ on $E_0$ can be written as

$$A(y) \equiv \pi^*A_t(x) + \sigma(x,t) \theta$$

where $\{A_t\}_{t \in S^1}$ is a family of integrable connections on $E_0$ and $\sigma \in \Omega^0(K, g)$, where $g := \pi^*g_{E_0}$ is the corresponding adjoint bundle of $E$. Let us denote this fact informally by

$$A = A_t + \sigma \theta.$$ 

The curvature of $A$ is the gauge-covariant global 2-form

$$F_A = F_{A_t} \oplus \left( d_{A_t} \sigma - \frac{\partial A_t}{\partial t} \right) \wedge \theta \in \Omega^2(K, g).$$

and, replacing that expression in the $G_2$-instanton equation (4.2), one obtains in particular

$$\hat{F}_{A_t}(\ast \theta) = F_{A_t} \wedge \omega^2 = 0.$$ 

This is exactly the HYM condition on each $A_t$. On the other hand, by Theorem 1.4, if $A$ is an integrable $G_2$-instanton, then it minimises the Yang–Mills functional (4.9). This implies locally

$$\left( d_{A_t} \sigma - \frac{\partial A_t}{\partial t} \right) \wedge \theta = 0,$$

since otherwise the pullback component $A_t$ alone would violate the minimum topological energy (4.10):

$$S_{\text{YM}}(A_t) = \|F_{A_t}\|^2 \geq \|F_A\|^2 = S_{\text{YM}}(A) = \kappa(E).$$

Moreover, if the family $A_t \equiv A_{t_0}$ is constant, i.e., $S^1$-invariant, then $d_{A_{t_0}} \sigma = 0$ implies $\sigma \equiv 0$, since by assumption $E$ is indecomposable and therefore does not admit nonzero parallel sections, and so $A$ is indeed a pullback. If the moduli space $\mathcal{M}$ of HYM connections on the base $V$ is discrete, then by continuity the family $\{A_t\}$ is contained in a gauge orbit. This concludes the proof of Theorem 1.5.

Remark 4.9. Let $\pi: Y = K_f \to V$ be the Fermat quintic link (cf. Example 3.3). Since the moduli space of stable holomorphic bundles on a Fermat quintic Calabi–Yau 3-fold $V$ is known to be discrete, we infer that $S^1$-invariant $G_2$-instantons
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should be counted in some sense by the Donaldson–Thomas invariant of $V$, which is deformation-invariant because $h^{0,2}(V) = 0$ [44], Definition 3.34. Thus we envisage a ‘conservation of number’ property for $S^1$-invariant $G_2$-instantons over such Fermat quintic links, to be made precise in upcoming work.

Afterword: Atiyah’s conjecture and singular $G_2$-metrics

Atiyah predicted that the Casson invariant $\lambda(\Sigma)$ of a homology sphere which is the link of a normal complete intersection singularity equals $\frac{1}{2}\sigma(F)$, where $F$ is the Milnor fibre. This was verified for Brieskorn spheres by Fintushel and Stern [22], and Neumann and Wahl [34] inductively use that fact to confirm the conjecture for weighted homogeneous surface singularities and for links of hypersurfaces of the form $f(x, y) + z^n = 0$, among others. Their theorem suggests a general relation between the Floer homology (or at least the Casson invariant) of a link in $\mathbb{C}^3$ and the signature of $F$. Arnold and Floer [3] suggested higher-dimensional analogues, which would require extra structure on the links (e.g. CR or contact structure) and Milnor fibre (e.g. symplectic structure).

In our context, Chern–Simons theory (4.3) suggests thinking of $G_2$-instantons as 7-dimensional analogues of flat connections. Applying the above intuition to the holomorphic Casson invariant of R. Thomas over a CY 3-fold base [44], we wonder whether a version of Atiyah’s conjecture may hold for CY links.

Finally, from the perspective of M-theory, examples of compact $G_2$-metrics with prescribed singularities might be within reach, starting from some suitably singular CY link and taking adiabatic limits on the circle fibres near an orbifold singularity. As we have shown, meaningful Yang–Mills theory results may be established on such spaces, even though the $G_2$-structure has some torsion.

A. Algorithm for Steenbrink’s signature theorem

As discussed in Section 3.2, Steenbrink’s method for the signature of a compactified affine variety depends solely on computing the nonzero signature $(\mu_+, \mu_-)$. This requires an explicit basis of the Milnor algebra, which several computational tools provide. We use the following code in SINGULAR [25].

We first compute the numbers (3.5), for a given polynomial $f$ of degree $d$ and a list of weights $W$, and arrange them into list $L$:

```plaintext
proc Signature(poly f, list W, int d) {
    ring A = 0, (a,b,c,d,e), lp;
    list L;
    int s;
    ideal J = jacob(f);
    J = groebner(J);
    ideal K = kbase(J);
```
s=size(K);
for ( int j=1; j <= s; j++ )
 { L[j]=(1+leadexp(K[j])[1])*(W[1]/d)
 + (1+leadexp(K[j])[2])*(W[2]/d)+ (1+leadexp(K[j])[3])*(W[3]/d)
 + (1+leadexp(K[j])[4])*(W[4]/d)+(1+leadexp(K[j])[5])*(W[5]/d) ; }
return(L);
write("list.txt", L);

Then we use Mathematica code to compute the $\nu$ invariant from the list $L$:
$\nu= \text{Mod}[\text{Length}[L]+1-3*\text{Length}[\text{Select}[\text{Select}[L, \# \notin \text{Integers } \&], \text{Mod}[\text{IntegerPart}[\#], 2] == 0 \&]] - \text{Length}[\text{Select}[\text{Select}[L, \# \notin \text{Integers } \&], \text{Mod}[\text{IntegerPart}[\#], 2] == 1 \&]], 48$}

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