OPTIMAL CONTROL OF THE DISCRETE-TIME FRACTIONAL-ORDER CUCKER–SMALE MODEL

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Abstract. We obtain necessary optimality conditions for the discrete-time fractional-order Cucker–Smale optimal control problem. By using fractional order differences on the left side of nonlinear system we introduce memory effects to the considered problem.

1. Introduction and problem formulation. Models describing emerging collective behaviors and self-organization in groups of interacting agents have received significant research attention in biology and ecology [2, 9], robotics and control theory [8, 18], as well in sociology and economics [7, 15]. In the seminal works [10, 11] Cucker and Smale proposed the following discrete-time model:

\[
\begin{align*}
    x_i(t+h) - x_i(t) &= hv_i(t), \\
    v_i(t+h) - v_i(t) &= h \sum_{j=1}^{N} a_{ij} (v_j(t) - v_i(t)), \quad i = 1, \ldots, N,
\end{align*}
\]

(1)

that describes the emergent behavior of \( N \) autonomous agents. Each agent is represented by coordinates \((x_i(t), v_i(t)) \in \mathbb{R}^d\), where \( x_i(t) \in \mathbb{R}^d \) and \( v_i(t) \in \mathbb{R}^d \) are the time-dependent state and consensus parameter, respectively. The weights

\[
a_{ij} = \frac{c}{(1 + \|x_i - x_j\|^2)^{\beta}},
\]

for fixed \( c > 0 \) and \( \beta \geq 0 \), quantify the way that the agents influence each other, and \( h > 0 \) is the magnitude of the step size. In the mentioned papers the continuous-time version of model (1), given by the system of differential equations

\[
\begin{align*}
    x'_i(t) &= v_i(t) \\
    v'_i(t) &= \sum_{j=1}^{N} a_{ij} (v_j(t) - v_i(t)), \quad i = 1, \ldots, N,
\end{align*}
\]

was also considered. In both models the communication rate between agents is ruling by the parameter \( \beta \). Cucker and Smale proved that if \( \beta < 1/2 \), then for every initial conditions \( v_i(t) \) converge, when \( t \to \infty \), to a common limit \( \bar{v} \in \mathbb{R}^d \) (consensus

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value). The same happens if $\beta \geq 1/2$ provided the initial conditions satisfy a given, explicite, relation [10]. For the continuous-time model in the situation when the system does not converge to a consensus pattern it was proposed, by Caponigro et al. [6], to use external control strategies in order to facilitate the formation of consensus. One such strategy is the finite time optimal control problem of determining a trajectory solution of

$$
\begin{cases}
x'_i(t) = v_i(t) \\
v'_i(t) = \sum_{j=1}^{N} a_{ij}(v_j(t) - v_i(t)) + u_i(t), \quad i = 1, \ldots, N,
\end{cases}
$$

starting at $(x_0, v_0)$, and minimizing a cost functional

$$
\int_0^T \left( \sum_{i=1}^{N} \|v_i(t) - \frac{1}{N} \sum_{j=1}^{N} v_j(t)\|^2 + \gamma \sum_{i=1}^{N} \|u_i(t)\| \right) dt,
$$

under the control constraint $\sum_{i=1}^{N} \|u_i(t)\| \leq K$, where $\|\cdot\|$ denotes the $l_2^d$-Euclidean norm on $\mathbb{R}^d$. The use of a mixed $l_1^N - l_2^d$-norm is justified by the fact that $l_1^N$-minimization leads to sparse controls (in other words to the minimal amount of intervention of an external policy maker). For a deeper discussion of this issue we refer the reader to [6] and references given there.

In this paper we apply the mentioned external control strategy to the Discrete-time Fractional-order Cucker–Smale (DFCS) model. By using fractional order differences we introduce memory effects to the system, that is the present state and consensus parameter depend on all past states and consensus parameters. In other words DFCS model has the memory parameter which is the order of the fractional difference. One can ask here about conditions under which this fractional system converges to a consensus. This question, to our knowledge, still remains open. However, in the present work we are rather interested in optimal control of DFCS model.

Let $T = \{t_k\}_{k=0, \ldots, M} = \{kh\}_{k=0, \ldots, M}, \ M \geq 2, \ h = \frac{T}{M}$ be the usual regular partition of the interval $[0, T]$. By $N \in \mathbb{N}$ we denote the number of interacting agents and by $(x_i, v_i)$ the state of each agent. For the simplicity of the presentation, without loss of generality, we suppose that all agents are in one-dimensional space. It means that the main state of $N$ agents is described by $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and the consensus parameter by $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$. Assume that $\alpha > 0, \beta \geq 0, \ \gamma > 0, \ c, c \in \mathbb{R}, \ (x, v) \in C(T; \mathbb{R}^N \times \mathbb{R}^N), \ \text{and} \ u \in C(T^\alpha; \mathbb{R}^N)$. The Discrete-time Fractional-order Cucker–Smale (DFCS) optimal control problem consists in finding a solution to the system

$$
\begin{align}
\Delta_{0+}^\alpha [x_i](t_{k+1}) &= v_i(t_k) \\
\Delta_{0+}^\alpha [v_i](t_{k+1}) &= \frac{1}{N} \sum_{j=1}^{N} \frac{c(v_j(t_k) - v_i(t_k))}{(1 + (x_i(t_k) - x_j(t_k))^2)^\alpha} + u_i(t_k),
\end{align}
$$

(2)

$i = 1, \ldots, N, \ k = 0, \ldots, M - 1$, where $\Delta_{0+}^\alpha$ denotes the Grünwald–Letnikov fractional difference, and initialized at

$$
(x(0), v(0)) = (x_0, v_0) \in \mathbb{R}^N \times \mathbb{R}^N,
$$

(3)
that minimizes the functional
\[
J(x,v,u) = h \sum_{k=0}^{M-1} \left[ \sum_{i=1}^{N} \left( v_i(t_k) - \frac{1}{N} \sum_{j=1}^{N} v_j(t_k) \right)^2 + \gamma \sum_{i=1}^{N} |u_i(t_k)| \right]
\]
(4)
under the control constraint
\[
\sum_{i=1}^{N} \|u_i\|_1 := \sum_{i=1}^{N} \sum_{k=0}^{M-1} |u_i(t_k)| \leq K.
\]
(5)
In other words, we want to apply an external control to the system with memory in the way that minimizes a combination of the discrepancy of the consensus variables to the mean of those variables with a \(l^1\)-norm term of the control. The memory effect is introduced to the system by the use of the Grünwald–Letnikov fractional difference. It should be noted that computing the value of \(\Delta^\alpha_{a+}[x](t_{k+1})\) (or \(\Delta^\alpha_{b-}[v](t_{k+1})\)) we must take into account all past points from \(t_0\) to the current point due to the summation operator (see Definition 2.1). For a deeper discussion of the discrete-time fractional calculus and its applications we refer the reader to books [16, 19, 24, 25].

The paper is organized as follows. Section 2 is a preparatory section where we introduce the notation of fractional differences and recall the extremum principle for the smooth-convex problems. Section 3 is devoted to the main result of the paper in which we prove necessary optimality conditions for problem (2)–(5). Our Theorem 3.2 is a version of the Pontryagin Maximum Principle for DFCS optimal control problem. The important point to note here is that till now only the Weak Pontryagin Maximum Principle for discrete-time fractional optimal control problems is available in the literature [5] (see also [14]). We end with Section 4 of conclusions.

2. Preliminaries. A variety of definitions of the fractional differences have been presented in the recent years, we can mention here those introduced by Diaz and Osler [13], Miller and Ross [23], Atici and Eloe [3] or the Caputo difference [1]. In this paper, we use the notion of Grünwald–Letnikov [19, 25].

Let \(T = \{t_k\}_{k=0}^{M} = \{a + kh\}_{k=0}^{M}\) be the usual regular partition of the interval \([a,b]\) with \(M \geq 2\) and \(h = \frac{(b-a)}{M}\). By \(C(T, \mathbb{R}^n)\) we denote the set of all functions defined on \(T\) with values in \(\mathbb{R}^n\), since all functions acting from \(T\) to \(\mathbb{R}^n\) are continuous. Moreover, we set
\[
\alpha_i := \begin{cases} 
1, & \text{if } i = 0 \\
(-1)^i \alpha(\alpha-1)\cdots(\alpha-i+1), & \text{if } i = 1, 2, \ldots .
\end{cases}
\]

Definition 2.1. Let \(y \in C(T, \mathbb{R}^n)\). The left Grünwald–Letnikov fractional difference of order \(\alpha > 0\) of function \(y\) is defined by
\[
\forall k = 1, \ldots , M \quad \Delta^\alpha_{a+}[y](t_k) := \frac{1}{h^\alpha} \sum_{r=0}^{k} \alpha_r y(t_{k-r}),
\]
while the right Grünwald–Letnikov fractional difference of order \(\alpha > 0\) of function \(y\) is defined by
\[
\forall k = 0, \ldots , M - 1 \quad \Delta^\alpha_{b-}[y](t_k) := \frac{1}{h^\alpha} \sum_{r=0}^{M-k} \alpha_{r-k} y(t_{k+r}).
\]
Remark 1. Note that $\Delta_0^\alpha : C(T; \mathbb{R}^n) \to C(T_\alpha; \mathbb{R}^n)$ (resp. $\Delta_0^{-\alpha} : C(T; \mathbb{R}^n) \to C(T^n; \mathbb{R}^n)$), where $T_\alpha = T \setminus \{a\}$ (resp. $T^n = T \setminus \{b\}$).

For the convenience of the reader we recall the extremum principle for the smooth-convex problems [17, 20] that will be used in the next section. Let $Y$, $Z$ be Banach spaces, $U$ be an arbitrary set, and $f_0$ be a function on $Y \times U$ and $F : Y \times U \to Z$. We consider the following problem:

$$f_0(y, u) \to \inf$$

subject to

$$F(y, u) = 0 \quad \text{for any pair } (y, u) \in V \times U \text{ satisfying constraints (7)–(8)}.$$  

Definition 2.2. Pair $(y_*, u_*)$ satisfying constraints (7)–(8) is a local minimum of problem (6)–(8) if there exists a neighborhood $V$ of $y_*$ such that $f_0(y_*, u_*) \leq f_0(y, u)$ for any pair $(y, u) \in V \times U$ satisfying constraints (7)–(8).

The Lagrange function for (6)–(8) is defined by

$$\mathcal{L}(y, u, p^*) = \lambda_0 f_0(y, u) + \langle p^*, F(y, u) \rangle,$$

where $\lambda_0$ is real number and $p^* \in Z^*$ ($Z^*$ the dual space of $Z$).

Theorem 2.3. [Theorem 3, p. 71 in [17]] Assume that the point $(y_*, u_*)$ satisfies the conditions (7)–(8) and suppose that there exists a neighborhood $V \subset Y$ of $y_*$ such that

A) for every $u \in U$, the mapping $y \mapsto F(y, u)$ and the function $y \mapsto f_0(y, u)$ belong to the class $C^1$ at the point $y_*$;

B) for every $y \in V$, the mapping $u \mapsto F(y, u)$ and the function $u \mapsto f_0(y, u)$ satisfy the following convexity condition: for every $u_1, u_2 \in U$ and $\mu \in [0, 1]$, there exists $u \in U$ such that

$$F(y, u) = \mu F(y, u_1) + (1 - \mu) F(y, u_2),$$

$$f_0(y, u) \leq \mu f_0(y, u_1) + (1 - \mu) f_0(y, u_2);$$

C) the range $\text{Im} F_y(y_*, u_*)$ of the linear operator $F_y(y_*, u_*) : Y \to Z$ is closed and has finite codimension in $Y$.

Then, if $(y_*, u_*)$ is a local minimum point of problem (7)–(8), there exist Lagrange multipliers $\lambda_0, p^* \in Z^*$, not all zero, such that

$$\mathcal{L}_y(y_*, u_*, p^*) = \lambda_0 f_0(y_*, u_*) + F_y(y_*, u_*) p^* = 0\quad \text{and}$$

$$\mathcal{L}(y_*, u_*, p^*) = \min_{u \in U} \mathcal{L}(y_*, u, p^*).$$

In addition, if

D) the image of the set $Y \times U$ under the mapping

$$(y, u) \mapsto F_y(y_*, u_*) y + F(y_*, u)$$

contains a neighborhood of the origin of $Z$, and if there exists a point $(y, u)$ such that

$$F_y(y_*, u_*) y + F(y_*, u) = 0,$$

then $\lambda_0 \neq 0$ and, for simplicity, one can take $\lambda_0 = 1$. 

3. Necessary optimality conditions for DFCS optimal control problem. First, let us observe that system (2)–(3) has the unique solution for every \( u^* \in C(T;\mathbb{R}^N) \). As a consequence, the set of feasible solutions in problem (2)–(3) is nonempty. We shall define what we mean by the solution to this problem.

**Definition 3.1.** Let \((x^*,v^*) \in C(T;\mathbb{R}^N \times \mathbb{R}^N)\) be a solution to (2)–(3) associated with a control \( u^* \in C(T;\mathbb{R}^N) \) satisfying (5). We say, that \((x^*,v^*,u^*)\) is a local minimum of problem (2)–(5) if there exists \( \delta > 0 \) such that

\[
J(x^*,v^*,u^*) \leq J(x,v,u) \quad \text{for any } (x,v,u) \in C(T;\mathbb{R}^N \times \mathbb{R}^N) \times C(T;\mathbb{R}^N) \quad \text{with } \|(x(t_k),v(t_k)) - (x^*(t_k),v^*(t_k))\| < \delta \quad \text{for all } t_k \in T,
\]

and satisfying conditions (2)–(3) and (5).

The main idea of obtaining necessary optimality conditions is to rewrite DFCS optimal control problem as a finite dimensional optimization problem and then to apply the extremum principle given by Theorem 2.3. For this reason let us define:

\[
\begin{align*}
x^*_k &:= x_i(t_k), v^*_k := v_i(t_k), \quad k = 0, \ldots, M, \quad i = 1, \ldots, N, \\
u^*_k &:= u_i(t_k), \quad k = 0, \ldots, M - 1, \quad i = 1, \ldots, N;
\end{align*}
\]

set \( U := \{ u \in \mathbb{R}^{NM} : \sum_{i=1}^{N} \sum_{k=0}^{M-1} |u^k_i| \leq K \}; \)

\[
\begin{align*}
v &:= (v_1^1, \ldots, v_1^M, \ldots, v_N^1, \ldots, v_N^M), \\
x &:= (x_1^1, \ldots, x_1^M, \ldots, x_N^1, \ldots, x_N^M), \\
u &:= (u_0^1, \ldots, u_0^{M-1}, \ldots, u_N^0, \ldots, u_N^{M-1});
\end{align*}
\]

map

\[
F := \begin{bmatrix} F^v \\ F^x \\ \vdots \\ F^x \end{bmatrix} : \mathbb{R}^{NM} \times \mathbb{R}^{NM} \times U \rightarrow \mathbb{R}^{2NM}
\]

\[
F^v_i := \begin{bmatrix} F^v_{i,0} \\ \vdots \\ F^v_{i,M-1} \end{bmatrix} : \mathbb{R}^{NM} \times \mathbb{R}^{NM} \times U \rightarrow \mathbb{R}^{2M},
\]

\[
F^x_i := \begin{bmatrix} F^x_{i,0} \\ \vdots \\ F^x_{i,M-1} \end{bmatrix} : \mathbb{R}^{NM} \times \mathbb{R}^{NM} \times U \rightarrow \mathbb{R}^{2M},
\]

where

\[
F^v_{i,k} := -\frac{1}{h^\alpha} \left( v_i^{k+1} + \sum_{r=1}^{k+1} \alpha_r v_i^{k+1-r} \right) + \frac{1}{N} \sum_{j=1}^{N} \frac{c(v_j^k - v_i^k)}{(1 + (x_j^k - x_i^k)^2)^\beta} + u_i^k,
\]

\[
F^x_i := -\frac{1}{h^\alpha} \left( x_i^{k+1} + \sum_{r=1}^{k+1} \alpha_r x_i^{k+1-r} \right) + v_i^k,
\]
for $i = 1, \ldots, N$ and $k = 0, \ldots, M - 1$; and function

$$f_0 : \mathbb{R}^{NM} \times \mathbb{R}^{NM} \times U \rightarrow \mathbb{R},$$

$$f_0(v, x, u) := \sum_{k=0}^{M-1} \left( \sum_{i=1}^{N} \left( v_i^k - \frac{1}{N} \sum_{j=1}^{N} v_j^k \right)^2 + \gamma \sum_{i=1}^{N} |u_i|^2 \right).$$

Now we shall consider the following problem:

$$f_0(v, x, u) \rightarrow \min$$

subject to

$$F(v, x, u) = 0,$$

$$u \in U.$$

It is clear that for every $u \in U$, the mapping $(v, x) \mapsto F(v, x, u)$ and the function $(v, x) \mapsto f_0(v, x, u)$ belong to the class $C^1$. Moreover, for every $(v, x) \in \mathbb{R}^{NM} \times \mathbb{R}^{NM}$, the mapping $u \mapsto F(v, x, u)$ and the function $u \mapsto f_0(v, x, u)$ satisfy the convexity condition B) of Theorem 2.3. Note that $F(v, x) \in \mathcal{M}_{2MN \times 2MN}$ and, by Theorem 3.2 in [26],

$$\det F_{v,x}(v, x, u) = \det \left[ \frac{\partial F^v}{\partial v} \quad \frac{\partial F^v}{\partial x} \right]$$

$$= \det \left( \frac{\partial F^v}{\partial v} - \frac{\partial F^v}{\partial x} \left( \frac{\partial F^x}{\partial x} \right)^{-1} \frac{\partial F^x}{\partial v} \right) \cdot \det \frac{\partial F^x}{\partial x} = 1,$$

where

$$\frac{\partial F^x}{\partial x} = \begin{bmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ \vdots & \ldots & \ddots & \vdots \\ 0 & \ldots & 0 & A \end{bmatrix}, \quad A = \frac{-1}{h^\alpha} \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ \alpha_1 & 1 & 0 & \ldots & 0 \\ \alpha_2 & \alpha_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \alpha_{M-1} & \alpha_{M-2} & \alpha_{M-3} & \ldots & 1 \end{bmatrix};$$

$$\frac{\partial F^x}{\partial v} = \begin{bmatrix} B & 0 & \ldots & 0 \\ 0 & B & \ldots & 0 \\ \vdots & \ldots & \ddots & \vdots \\ 0 & \ldots & 0 & B \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & -1 \end{bmatrix};$$

$$\frac{\partial F^v}{\partial x} = \left[ \frac{\partial F^v}{\partial x_l} \right]_{l=1}^{N}, \quad \frac{\partial F^v}{\partial x_l} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ \frac{\partial F^{v,1}}{\partial x_l} & 0 & 0 & \ldots & 0 \\ 0 & \frac{\partial F^{v,2}}{\partial x_l} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \frac{\partial F^{v,M-1}}{\partial x_{M-1,l}} \end{bmatrix}.$$
\[
\frac{\partial F^v}{\partial v_i} = \begin{cases}
-\frac{1}{N} \sum_{j \neq i} \frac{2\beta c(x^k_j - x^k_i)(v^k_j - v^k_i)}{(1 + (x^k_j - x^k_i)^2)^{\beta + 1}}, & i \neq l, \\
\frac{1}{N} \sum_{j \neq i} \frac{2\beta c(x^k_j - x^k_i)(v^k_j - v^k_i)}{(1 + (x^k_j - x^k_i)^2)^{\beta + 1}}, & i = l, \quad k = 1, \ldots, M - 1;
\end{cases}
\]

\[
\frac{\partial F^v}{\partial v} = \left[ \frac{\partial F^v_i}{\partial v_j} \right]_{i,j=1,\ldots,N},
\]

\[
\frac{\partial F^v_i}{\partial v_l} = A + \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{N} \frac{c}{(1 + (x^k_l - x^k_i)^2)^{\beta}} & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{N} \frac{c}{(1 + (x^k_l - x^k_i)^2)^{\beta}} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{1}{N} \frac{c}{(1 + (x^k_l - x^k_i)^2)^{\beta}}
\end{bmatrix},
\]

for \(i \neq l\) and

\[
\frac{\partial F^v_i}{\partial v_l} = A + \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
\frac{-1}{N} \frac{\sum_{j \neq i} c}{(1 + (x^k_l - x^k_i)^2)^{\beta}} & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{-1}{N} \frac{\sum_{j \neq i} c}{(1 + (x^k_l - x^k_i)^2)^{\beta}} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{-1}{N} \frac{\sum_{j \neq i} c}{(1 + (x^k_l - x^k_i)^2)^{\beta}}
\end{bmatrix},
\]

for \(i = l\). Since \(\det F_{v,x}(v, x, u) = 1 \neq 0\), the range \(\text{Im} F_{v,x}(v, x, u)\) of the linear operator \(F_{v,x}(v, x, u) : \mathbb{R}^{2NM} \rightarrow \mathbb{R}^{2NM}\) is the whole space \(\mathbb{R}^{2NM}\). It means that assumptions C) and D) of Theorem 2.3 are satisfied. Therefore, if \((v, x, u) \in \mathbb{R}^{2NM} \times \mathbb{R}^{2NM} \times U\) is a local minimum point of problem (9)–(11) then, by Theorem 2.3, there exists \(p^* \in \mathbb{R}^{2NM}\), such that

\[
L_{(v,x)}(v, x, u, p^*) = 0, \\
L(v, x, u, p^*) = \min_{u \in U} L(v, x, u, p^*),
\]

where the Lagrangian \(L\) is given by

\[
L(v, x, u, p) := f_0(v, x, u) + \langle p, F(v, x, u) \rangle \\
= \sum_{k=0}^{M-1} \left[ \sum_{i=1}^{N} \left( v^k_i - \frac{1}{N} \sum_{j=1}^{N} v^k_j \right)^2 + \gamma \sum_{i=1}^{N} |u^k_i| \right] \\
+ \sum_{k=0}^{M-1} \sum_{i=1}^{N} \left( p^k_{i,v} F^v_i + p^k_{i,x} F^x_i \right),
\]
with \( p = \begin{bmatrix} p_v \\ p_x \end{bmatrix}, \ p_v = \begin{bmatrix} p_{1,v} \\ \vdots \\ p_{N,v} \end{bmatrix}, \ p_x = \begin{bmatrix} p_{1,x} \\ \vdots \\ p_{N,x} \end{bmatrix} \) and \( p_{i,v} = \begin{bmatrix} p_{1,v}^i \\ \vdots \\ p_{M,v}^i \end{bmatrix}, \ p_{i,x} = \begin{bmatrix} p_{1,x}^i \\ \vdots \\ p_{M,x}^i \end{bmatrix}, \)

\( i = 1, \ldots, N. \) Set the Hamiltonian function \( H^k \) as follows

\[
H^k(v^k, x^k, u^k, p^{k+1}) := f_0(v^k, x^k, u^k) + \sum_{i=1}^N \left( p_{i,v}^{k+1} f^v_i + p_{i,x}^{k+1} f^x_i \right),
\]

where \( v^k = (v_1^k, \ldots, v_N^k), \ x^k = (x_1^k, \ldots, x_N^k), \ u^k = (u_1^k, \ldots, u_N^k), \ p^{k+1} = (p_{1,v}^{k+1}, \ldots, p_{N,v}^{k+1}, p_{1,x}^{k+1}, \ldots, p_{N,x}^{k+1}), \) and

\[
f^v_i := \frac{1}{N} \sum_{j=1}^N \frac{c(v_j^k - v_i^k)}{(1 + (x_j^k - x_i^k)^2)^{\alpha}} + u_i^k,
\]

\[
f^x_i := u_i^k,
\]

for \( i = 1, \ldots, N, \ k = 0, \ldots, M - 1. \) Hence

\[
L(v, x, u, p) = \sum_{k=0}^{M-1} H^k(v^k, x^k, u^k, p^{k+1}) - \frac{1}{\hbar^\alpha} \sum_{k=0}^{M-1} \sum_{i=1}^N \left( p_{i,v}^{k+1} \left( v_i^{k+1} + \sum_{r=1}^{k+1} \alpha_r v_i^{k+1-r} \right) + p_{i,x}^{k+1} \left( x_i^{k+1} + \sum_{r=1}^{k+1} \alpha_r x_i^{k+1-r} \right) \right).
\]

Since there exists \( p^* \in \mathbb{R}^{2NM}, \) such that \( L(v, x, u, p^*) = 0, \) we have:

\[
\frac{\partial H^k}{\partial v_i^k}(v^*_u, x^*_u, u^*_k, p^{*,k+1}) = \frac{1}{\hbar^\alpha} \sum_{r=0}^{M-k} \alpha_r p^*_{i,v}^{*,k+r},
\]

\[
\frac{\partial H^k}{\partial x_i^k}(v^*_u, x^*_u, u^*_k, p^{*,k+1}) = \frac{1}{\hbar^\alpha} \sum_{r=0}^{M-k} \alpha_r p^*_{i,x}^{*,k+r},
\]

\[
p^*_{i,v} = 0, \quad p^*_{i,x} = 0,
\]

for \( i = 1, \ldots, N, k = 1, \ldots, M - 1. \) As \( L(v, x, u, p^*) = \min_{u \in U} L(v, x, u, p^*), \) we get

\[
\sum_{k=0}^{M-1} H^k(v^k, x^k, u^k, p^{*,k+1}) = \min_{u \in U} \sum_{k=0}^{M-1} H^k(v^k, x^k, u^k, p^{*,k+1}).
\]

Additionally we have

\[
\frac{\partial H^k}{\partial p_{i,v}^{k+1}}(v^k, x^k, u^k, p^{k+1}) = f_i^v,k,
\]

\[
\frac{\partial H^k}{\partial p_{i,x}^{k+1}}(v^k, x^k, u^k, p^{k+1}) = f_i^x,k
\]

for \( i = 1, \ldots, N, k = 1, \ldots, M - 1. \)

Concluding, we have proved the following theorem.
Theorem 3.2. If the pair \((x_*, v_*) \in C \left( \mathbb{T}; \mathbb{R}^N \times \mathbb{R}^N \right) \), associated with a control \(u_* \in C \left( \mathbb{T}^n; \mathbb{R}^N \right) \), is a local minimum of problem (2)-(5), then there exists a mapping \(p^* \in C \left( \mathbb{T}; \mathbb{R}^N \right) \), such that \(p^*(T) = 0 \) and

\[
\begin{align*}
\frac{\partial H}{\partial v}(v_*(t_k), x_*(t_k), u_*(t_k), p^*(t_{k+1})) &= \Delta_0^- [p^*]_i(t_k), \\
\frac{\partial H}{\partial x}(v_*(t_k), x_*(t_k), u_*(t_k), p^*(t_{k+1})) &= \Delta_0^- [p^*]_i(t_k), \\
\frac{\partial H}{\partial p_i}(v_*(t_k), x_*(t_k), u_*(t_k), p^*(t_{k+1})) &= \Delta_0^+ [v]_i(t_{k+1}), \\
\frac{\partial H}{\partial p_i}(v_*(t_k), x_*(t_k), u_*(t_k), p^*(t_{k+1})) &= \Delta_0^+ [x]_i(t_{k+1}),
\end{align*}
\]

for \(i = 1, \ldots, N \). Moreover, it holds

\[
\sum_{k=0}^{M-1} H(v_*(t_k), x_*(t_k), u_*(t_k), p^*(t_{k+1})) = \min_{u \in U} \sum_{k=0}^{M-1} H(v_*(t_k), x_*(t_k), u_k, p^*(t_{k+1})),
\]

where the Hamiltonian \(H : \mathbb{R}^{2N} \times \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R} \) is given by

\[
H(v, x, u, p) = \sum_{i=1}^{N} \left( v_i - \frac{1}{N} \sum_{j=1}^{N} v_j \right)^2 + \gamma \sum_{i=1}^{N} |u_i| + \sum_{i=1}^{N} \left( \frac{1}{N} \sum_{j=1}^{N} \frac{c(v_j - v_i)}{1 + (x_j - x_i)^2} + u_i \right) + p_i x_i,
\]

and \(U := \{ u \in \mathbb{R}^{NM} : \sum_{i=1}^{N} \sum_{k=0}^{M-1} |u_{i}^k| \leq K \} \).

3.1. Example. Let us consider problem (2)-(5) with \(h = 1, \alpha = \frac{1}{2}, M = 11, N = 2, K = 10, c = 1, \beta = 1 \) and the initial conditions

\[
(x_1(0), x_2(0), v_1(0), v_2(0)) = (1, 5, 5, 0) \in \mathbb{R}^2 \times \mathbb{R}^2.
\]

Figure 1. Consensus parameters with using control.
Figure 2. Consensus parameters without control.

In Figure 1, one can see the graph of consensus parameters that are a solution to the considered DFCS optimal control problem, while in Figure 2 the graph of consensus parameters that are the solution to the DFCS system:

\[
\begin{align*}
\Delta_0^\frac{1}{2} [x_i](t_{k+1}) &= v_i(t_k) \\
\Delta_0^\frac{1}{2} [v_i](t_{k+1}) &= \frac{1}{2} \sum_{j=1}^2 c(v_j(t_k) - v_i(t_k)) \\
&\quad \cdot \frac{1}{1 + (x_j(t_k) - x_i(t_k))^2},
\end{align*}
\]

\(i = 1, 2, k = 0, \ldots, 10, (x_1(0), x_2(0), v_1(0), v_2(0)) = (1, 5, 5, 0) \in \mathbb{R}^2 \times \mathbb{R}^2\). Obviously, values of \(|v_1(t_k) - v_2(t_k)|\) for \(k = 1, \ldots, M\) are significantly smaller in the case of a solution to the DFCS optimal control problem.

4. Conclusions. In the recent years, fractional-order systems and fractional-order controls systems have attracted increasing interest due to their many applications in various field of science and engineering [4, 19, 21, 22, 24]. It was shown that many phenomena in nature can be better explained using fractional-order systems. Moreover, those systems are excellent tool for the description of memory. This was our motivation for introducing the study of the DFCS model. In this paper a version of the Pontryagin Maximum Principle for the DFCS optimal control problem was proved. Another interesting and open question that can be addressed is whether the considered optimization problem, or more generally a discrete-time fractional optimal control problem, has an optimal solution or not. We believe that the technique of proof used in this work can be applied for answering the mentioned question, as well as for proving the Pontryagin Maximum Principle for a general form of discrete-time fractional optimal control problem, and this will be considered in a forthcoming paper.

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