GREEN FUNCTIONS WITH SINGULARITIES ALONG
COMPLEX SPACES
Alexander Rashkovskii and Ragnar Sigurdsson
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Abstract
We study properties of a Green function $G_A$ with singularities along a complex subspace $A$ of a complex manifold $X$. It is defined as the largest negative plurisubharmonic function $u$ satisfying locally $u \leq \log |\psi| + C$, where $\psi = (\psi_1, \ldots, \psi_m)$, $\psi_1, \ldots, \psi_m$ are local generators for the ideal sheaf $\mathcal{I}_A$ of $A$, and $C$ is a constant depending on the function $u$ and the generators. A motivation for this study is to estimate global bounded functions from the sheaf $\mathcal{I}_A$ and thus proving a “Schwarz Lemma” for $\mathcal{I}_A$.

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1 Introduction
If $\varphi$ is a bounded holomorphic function on a complex manifold $X$, then it is a natural problem to estimate $|\varphi|$ given some information on the location of the zeros of $\varphi$ and their multiplicities. If $|\varphi| \leq 1$ and the only given information is that $\varphi(a) = 0$ for a single point $a$, then

$$
\log |\varphi| \leq G_{X,a} = G_X(\cdot, a),
$$

where $G_{X,a}$ is the pluricomplex Green function with logarithmic pole at $a$. It is defined as the supremum over the class $\mathcal{F}_{X,a}$ of all negative plurisubharmonic functions $u$ such that $u \leq \log |\zeta| + C$ near $a$, where $\zeta$ are local coordinates near $a$ with $\zeta(a) = 0$ and $C$ is a positive constant depending on $u$ and $\zeta$. The function $G_{X,a}$ was introduced and studied by several authors [16], [24], [9], [19], [4], see also [10], [5].

A generalization is to take $A = (|A|, \{m_a\}_{a \in |A|})$, where $|A|$ is a finite subset of $X$, $m_a$ is a positive real number for every $a \in |A|$, and assume that $\varphi$ has a zero of multiplicity at least $m_a$ at every point $a$ in $|A|$. Then

$$
\log |\varphi| \leq G_A,
$$

where $G_A$ is the Green function with several weighted logarithmic poles. It is defined as the supremum over the class of all negative plurisubharmonic functions $u$ on $X$ satisfying $u \leq m_a \log |\zeta_a| + C$ for every $a$ in $|A|$, where $\zeta_a$ are local coordinates near $a$ with $\zeta_a(a) = 0$ and $C$ is a positive constant depending on $\zeta_a$ and $u$. The function $G_A$ was first introduced by Zaharyuta [24] and independently by Lelong [15].
The notion of multiplicity of a zero of an analytic function has a natural generalization as a Lelong number of a plurisubharmonic function. If \( u \) is plurisubharmonic in some neighbourhood of the origin 0 in \( \mathbb{C}^n \), then the *Lelong number* \( \nu_u(0) \) of \( u \) at 0 can be defined as

\[
\nu_u(0) = \lim_{r \to 0} \sup \left\{ \frac{u(x); |x| \leq r}{\log r} \right\}
\]

and if \( u \) is plurisubharmonic on a manifold \( X \) then the Lelong number \( \nu_u(a) \) of \( u \) at \( a \in X \) is defined as \( \nu_u(a) = \nu_{u \circ \zeta^{-1}}(0) \), where \( \zeta \) are local coordinates near \( a \) with \( \zeta(a) = 0 \). It is clear that this definition is independent of the choice of local coordinates and that \( \nu_u(a) \) equals the multiplicity of \( a \) as a zero of the holomorphic function \( \varphi \) in the case \( u = \log |\varphi| \). Note that the pluricomplex Green function \( G_{X,a} \) with logarithmic pole at \( a \) can be equivalently defined as the upper envelope of all negative plurisubharmonic functions \( u \) on \( X \) satisfying \( \nu_u(a) \geq 1 \) and, similarly, \( \nu_u(a) \geq m_a \) for the Green functions with several weighted logarithmic poles.

For any non-negative function \( \alpha \) on \( X \), Lárusson and Sigurdsson [11], [12] introduced the Green function \( \tilde{G}_\alpha \) as the supremum over the class \( \mathcal{F}_\alpha \) of all negative plurisubharmonic functions with \( \nu_u \geq \alpha \). It is clear that if \( \varphi \) is holomorphic on \( X \), \( |\varphi| \leq 1 \), and every zero \( a \) of \( \varphi \) has multiplicity at least \( \alpha(a) \), then

\[
\log |\varphi| \leq \tilde{G}_\alpha.
\]

In this context it is necessary to note that we assume that the manifold \( X \) is connected, we take the constant function \( -\infty \) as plurisubharmonic, and set \( \nu_{-\infty} = +\infty \). Hence \( -\infty \in \mathcal{F}_\alpha \) for every \( \alpha \). By [11], Prop. 5.1, \( \tilde{G}_\alpha \in \mathcal{F}_\alpha \).

In the special case when \( X \) is the unit disc \( \mathbb{D} \) in \( \mathbb{C} \), we have

\[
\tilde{G}_\alpha(z) = \sum_{w \in \mathbb{D}} \alpha(w) G_{\mathbb{D}}(z, w), \quad z \in \mathbb{D},
\]

where \( G_{\mathbb{D}} \) is the Green function for the unit disc,

\[
G_{\mathbb{D}}(z, w) = \log \left| \frac{z - w}{1 - \overline{w}z} \right|, \quad z, w \in \mathbb{D}.
\]

If \( X \) and \( Y \) are complex manifolds, \( \alpha \) is a non-negative function on \( X \), and \( \Phi : Y \to X \) is a holomorphic map, then the pullback \( \Phi^* u = u \circ \Phi \) satisfies \( \nu_{\Phi^* u} \geq \Phi^* \nu_u \), so \( \Phi^* u \in \mathcal{F}_{\Phi^* \alpha} \) for every \( u \in \mathcal{F}_\alpha \). This implies \( \Phi^* G_\alpha \leq G_{\Phi^* \alpha} \), i.e., \( G_\alpha(x) \leq G_{\Phi^* \alpha}(y) \) if \( x = \Phi(y) \), and in particular

\[
\tilde{G}_\alpha(x) \leq \tilde{G}_{\Phi^* \alpha}(0) = \sum_{w \in \mathbb{D}} f^* \alpha(w) \log |w|, \quad f \in \mathcal{O}(\mathbb{D}, X), \quad f(0) = x.
\]

One of the main results of [11] and [13] is that for every manifold \( X \) and every non-negative function \( \alpha \) we have the formula

\[
\tilde{G}_\alpha(x) = \inf \{ \tilde{G}_{\Phi^* \alpha}(0); f \in \mathcal{O}(\mathbb{D}, X), f(0) = x \}, \quad x \in X.
\] (1.1)

Here \( \mathcal{O}(\mathbb{D}, X) \) is the family of all analytic discs in \( X \) and \( \mathcal{O}(\overline{\mathbb{D}}, X) \) is the subclass of closed analytic discs, i.e., maps from \( \mathbb{D} \) to \( X \) that can be extended to holomorphic maps in some
neighbourhood of the closed disc \( \overline{D} \). Results of this kind originate in Poletsky’s theory of analytic disc functionals, started in \([19]\) and \([20]\).

A natural way of describing the zero set of a holomorphic function \( \varphi \) is to state that its germs \( (\varphi)_x \) are in the stalk \( \mathcal{I}_{A,x} \) of a prescribed coherent ideal sheaf \( \mathcal{I}_A = (\mathcal{I}_{A,x})_{x \in X} \) of a closed complex subspace \( A \) of \( X \). Then, if \( \psi_1, \ldots, \psi_m \) are local generators of \( \mathcal{I}_A \) near the point \( a \), the function \( \varphi \) can be represented as \( \varphi = \varphi_1 \psi_1 + \cdots + \varphi_m \psi_m \) near \( a \), which implies that \( \log |\varphi| \leq \log |\psi| + C \) near \( a \), where \( \psi = (\psi_1, \ldots, \psi_m) \), \( |\cdot| \) is the euclidean norm, and \( C \) is a constant depending on \( \varphi \) and the generators.

We define \( \mathcal{F}_A \) as the class of all negative plurisubharmonic functions \( u \) in \( X \) satisfying \( u \leq \log |\psi| + O(1) \) locally in \( X \), and we define the function \( G_A \), the pluricomplex Green function with singularities along \( A \), as the supremum over the class \( \mathcal{F}_A \).

It follows from the definition of \( G_A \) that if \( A' \) is the restriction of \( A \) to a domain \( X' \) in \( X \), \( \varphi \) is holomorphic function on \( X \), and \( (\varphi)_x \in \mathcal{I}_{A,x} \) for all \( x \in X' \), then

\[
|\varphi| \leq e^{G_{A'}(x)} \sup_{X'} |\varphi|, \quad x \in X',
\]

which is a variant of the Schwarz lemma for the ideal sheaves.

In order to relate \( G_A \) to the Green functions \( G_\alpha \) above, we define the function \( \tilde{\nu}_A \) on \( X \) by \( \tilde{\nu}_A(x) = \nu_{\log |\psi|}(x) \) if \( \psi = (\psi_1, \ldots, \psi_m) \) are local generators for \( \mathcal{I}_A \) in some neighbourhood of \( x \). It is easy to see that \( \tilde{\nu}_A(x) \) is independent of the choice of the generators (actually, it equals the minimal multiplicity of the functions from \( \mathcal{I}_{A,x} \) at \( x \)), so \( \tilde{\nu}_A \) is a well defined function on \( X \) and \( \nu_u \geq \tilde{\nu}_A \) for all \( u \in \mathcal{F}_A \). Hence, with \( \tilde{\nu}_A \) in the role of \( \alpha \) above, we have \( \mathcal{F}_A \subseteq \mathcal{F}_{\tilde{\nu}_A} \) which implies

\[
G_A \leq \tilde{G}_{\tilde{\nu}_A}.
\]

In general, \( G_A \neq \tilde{G}_{\tilde{\nu}_A} \), as seen from the example where \( X = \mathbb{D}^2 \) and \( \mathcal{I}_A \) has the global generators \( \psi = (\psi_1, \psi_2) \) with \( \psi_1(z) = z_1^2 \) and \( \psi_2(z) = z_2 \). Then \( G_A(z) = \max \{2\log |z_1|, \log |z_2|\} \) and \( \tilde{G}_{\tilde{\nu}_A}(z) = \max \{\log |z_1|, \log |z_2|\} \) for \( z = (z_1, z_2) \in \mathbb{D}^2 \). If, on the other hand, \( A \) is an effective divisor generated by the function \( \psi \) in an open subset \( U \) of \( X \), then by \([12]\), Prop. 3.2, the function \( \tilde{G}_{\tilde{\nu}_A} - \log |\psi| \) on \( U \setminus |A| \) can be extended to a plurisubharmonic function on \( U \). This implies that \( G_A = \tilde{G}_{\tilde{\nu}_A} \) for effective divisors \( A \).

Now to the content of the paper. In Section 2 we present the main results, which are proved in later sections. Our first task is to prove that \( G_A \in \mathcal{F}_A \). In Section 2 we show how this follows from the facts that \( \tilde{\nu}_A \in \mathcal{F}_A \), for all \( \alpha : X \to [0, +\infty) \), \( G_A = \tilde{G}_{\tilde{\nu}_A} \) if \( A \) is an effective divisor, and a variant of the Hironaka desingularization theorem. By the same desingularization technique we establish a representation of the Green function as the lower envelope of the analytic disc functional \( f \mapsto G_{f^*A}(0) \). In Section 3 we study decomposition in ideal sheaves as a preparation for Section 4 where we prove that the estimates in the definition of the class \( \mathcal{F}_A \) are locally uniform. This gives a direct proof of the relation \( G_A \in \mathcal{F}_A \) (without referring to desingularization), which in turn implies certain refined maximality properties of the Green function. In Section 5 we get a representation for the current \((dd^cG_A)^p \) in the case when the ideal sheaf \( \mathcal{I}_A \) has global generators, and in Section 6 we study the case when the space is reduced. In Section 7 we prove the product property of Green functions, and finally in Section 8 we give a few explicit examples.
2 Definitions and main results

We shall always let $X$ be a complex manifold and assume that $X$ is connected. We denote by $\text{PSH}(X)$ the class of all plurisubharmonic functions on $X$ and by $\text{PSH}^{-}(X)$ its subclass of all non-positive functions. We take $-\infty \in \text{PSH}(X)$ and set $\nu_{-\infty} = +\infty$. We let $\mathcal{O}_X$ denote the sheaf of germs of locally defined holomorphic functions on $X$. We let $A$ be a closed complex subspace of $X$, $\mathcal{I}_A = (\mathcal{I}_{A,x})_{x \in X}$ be the associated coherent sheaf of ideals in $\mathcal{O}_X$, and $|A|$ be the analytic variety in $X$ defined as the common set of zeros of the locally defined functions on $X$ with germs in $\mathcal{I}_A$. If $U$ is an open subset of $X$, then we let $\mathcal{I}_{A,U}$ denote the space of all holomorphic functions on $U$ with germs in $\mathcal{I}_A$. We let $\mathcal{O}(Y,X)$ denote the set of all holomorphic maps from a complex manifold $Y$ into $X$. A map in $\mathcal{O}(D,X)$ is called an analytic disc, and if it can be extended to a holomorphic map in some neighbourhood of the closed disc $\overline{D}$ then it is said to be closed. The collection of all closed analytic discs is denoted by $\mathcal{O}(\overline{D},X)$.

**Definition 2.1** Given a complex subspace $A$ of a connected complex manifold $X$, the class $\mathcal{F}_A$ consists of all functions $u \in \text{PSH}^{-}(X)$ such that for every point $a \in X$ there exist local generators $\psi_1, \ldots, \psi_m$ for $\mathcal{I}_A$ near $a$ and a constant $C$ depending on $u$ and the generators with $u \leq \log |\psi| + C$ near $a$.

Observe that $-\infty \in \mathcal{F}_A$ for every $A$.

**Definition 2.2** The pluricomplex Green function $G_A$ with singularities along $A$ is the upper envelope of all the functions from the class $\mathcal{F}_A$, i.e.,

$$G_A(x) = \sup \{u(x); u \in \mathcal{F}_A\}, \quad x \in X.$$  

The local estimate $u \leq \log |\psi| + C$ is independent of the choice of generators, i.e., if we have another set of generators $\psi' = (\psi'_1, \ldots, \psi'_k)$, then $u \leq \log |\psi'| + C'$ for some constant $C'$. Furthermore, in the definition of the class $\mathcal{F}_A$, $\psi = (\psi_1, \ldots, \psi_m)$ can be replaced by any holomorphic $\xi = (\xi_1, \ldots, \xi_l)$, defined near $a$ and satisfying

$$\log |\xi| + c_1 \leq \log |\psi| \leq \log |\xi| + c_2,$$

which means precisely that the integral closure of the ideal generated by the germs of the functions $\xi_i$ at $x$ coincides with the integral closure of the ideal $\mathcal{I}_{A,x}$ for all $x$ in some neighbourhood of $a$. (See [6], Ch. VIII, Cor. 10.5.) We occasionally write $\log |\xi| \asymp \log |\psi|$ when inequalities of this kind hold.

Let $X$ and $Y$ be complex manifolds and $\Phi : Y \to X$ be a holomorphic map. If $A$ is a complex subspace of $X$, then we have a natural definition of a pullback $\Phi^*A$ of $A$ as a complex subspace of $Y$. The ideal sheaf $\mathcal{I}_{\Phi^*A}$ is locally generated at a point $b$ by $\Phi^*\psi_1, \ldots, \Phi^*\psi_m$ if $\psi_1, \ldots, \psi_m$ are local generators for $\mathcal{I}_A$ at $\Phi(b)$. It is evident that $\Phi^*u \in \mathcal{F}_{\Phi^*A}$ for all $u \in \mathcal{F}_A$, so

$$\Phi^*G_A \leq G_{\Phi^*A}. \quad (2.1)$$
If $\Phi$ is proper and surjective and $v : Y \to \mathbb{R} \cup \{-\infty\}$ is an upper semi-continuous function, then the push-forward $\Phi_* v$ of $v$ to $X$ is well defined by the formula

$$\Phi_* v(x) = \max_{y \in \Phi^{-1}(x)} v(y), \quad x \in X.$$ 

**Proposition 2.3** Let $X$ and $Y$ be complex manifolds of the same dimension and $\Phi : Y \to X$ be a proper surjective holomorphic map (for example, a finite branched covering). Then $\Phi_* v \in \text{PSH}(X)$ for all $v \in \text{PSH}(Y)$.

**Proof:** In order to show that $\Phi_* v$ is upper semicontinuous, we need to prove the relation $\Phi_* v(a) \geq \limsup_{x \to a} \Phi_* v(x)$ for every $a \in X$. We take a sequence $a_j \to a$ such that $\Phi_* v(a_j) \to \limsup_{x \to a} \Phi_* v(x)$. Since $v$ is upper semicontinuous and $\Phi$ is proper, there exist $b_j \in \Phi^{-1}(a_j)$ such that $v(b_j) = \Phi_* v(a_j)$. By replacing $(b_j)$ by a subsequence we may assume that $b_j \to b \in Y$. Then $\Phi(b) = a$ and

$$\Phi_* v(a) \geq v(b) \geq \limsup_{j \to +\infty} v(b_j) = \lim_{j \to +\infty} \Phi_* v(a_j) = \limsup_{x \to a} \Phi_* v(x).$$

We let $V$ denote the set of all points $y$ in $Y$ for which $d_y \Phi$ is degenerate. Then $V$ is an analytic variety in $Y$ and Remmert’s proper mapping theorem implies that $W = \Phi(V)$ is an analytic variety in $X$. It is sufficient to show that $\Phi_* v$ is plurisubharmonic in a neighbourhood of every point $a \in X \setminus W$, for the upper semicontinuity of $\Phi_* v$ then implies that $\Phi_* v \in \text{PSH}(X)$.

Since $\Phi$ is a local biholomorphism on $Y \setminus \Phi^{-1}(W)$, it follows that the fiber $\Phi^{-1}(a)$ is discrete and compact, thus finite, say that it consists of the points $b_1, \ldots, b_m$. We choose a neighbourhood $U$ of $a$ in $X \setminus W$ and biholomorphic maps $F_j : U \to F_j(U) \subseteq Y \setminus \Phi^{-1}(W)$ with $F_j(a) = b_j$. Then $\Phi_* v(x) = \sup_{1 \leq j \leq m} v \circ F_j(x)$ for all $x \in U$, which shows that $\Phi_* v$ is plurisubharmonic in $U$. \hfill \blacksquare

It is obvious that $u = \Phi_* \Phi^* u$ for all $u \in \text{PSH}(X)$ and $v \leq \Phi^* \Phi_* v$ for all $v \in \text{PSH}(Y)$.

**Proposition 2.4** Let $X$ and $Y$ be complex manifolds of the same dimension, $A$ be a closed complex subspace of $X$, and $\Phi : Y \to X$ be a proper surjective holomorphic map. Then $\Phi_* v \in \mathcal{F}_A$ for all $v \in \mathcal{F}_{\Phi^* A}$ and

$$\Phi^* G_A = G_{\Phi^* A}.$$ 

**Proof:** If $a \in X$ and $\psi_1, \ldots, \psi_m$ are local generators for $\mathcal{I}_A$ near $a$, then $v \leq \Phi^* \log |\psi| + C$ in some neighbourhood of the compact set $\Phi^{-1}(a)$, which implies $\Phi_* v \leq \log |\psi| + C$ near $a$. Hence we conclude from Prop. 2.3 that $\Phi_* v \in \mathcal{F}_A$. Since $\Phi^* G_A \leq G_{\Phi^* A}$, it is sufficient to prove that $v \leq \Phi^* G_A$ for every $v \in \mathcal{F}_{\Phi^* A}$. We have $\Phi_* v \in \mathcal{F}_A$, so $v \leq \Phi^* \Phi_* v \leq \Phi^* G_A$. \hfill \blacksquare

Our first main result is

**Theorem 2.5** If $X$ is a complex manifold and $A$ is a closed complex subspace of $X$, then $G_A \in \mathcal{F}_A$. 

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Observe that in our definition of the class $\mathcal{F}_A$, the constant $C$ in the local estimates $u \leq \log |\psi| + C$ is allowed to depend both on the function $u$ and the local generators. The main work in our proof of Theorem 2.5 in Sections 3 and 4 is to prove that these estimates are indeed locally uniform, i.e., we show that if $U$ is the domain of definition of $\psi$ and $K$ is a compact subset of $U$, then there exists a constant $C_K$, only depending on $K$ and $\psi$, such that $u \leq \log |\psi| + C_K$ on $K$ for all $u \in \mathcal{F}_A$. (See Lemma 3.2)

Let us now show how Theorem 2.5 follows from the facts that $\tilde{G}_a \in \mathcal{F}_a$ for all $a : X \to [0, +\infty)$, $G_A = G_{\partial A}$ if $A$ is an effective divisor, and the following variant of the Hironaka desingularization theorem. (See [1], Theorems 1.10 and 13.4.)

Given a closed complex subspace $A$ on a manifold $X$, there exists a complex manifold $\tilde{X}$ and a proper surjective holomorphic map $\Phi : \tilde{X} \to X$ which is an isomorphism outside $\Phi^{-1}(|A|)$ and such that $\tilde{A} = \Phi^*A$ is a normal-crossing principal ideal sheaf (i.e., generated locally by a monomial in suitable coordinates).

If we let $\Phi$ denote the desingularization map, then

$$G_A = \Phi^*G_{\tilde{A}} = \Phi^*G_{\Phi^*A} = \Phi^*G_A = \Phi^*\tilde{G}_\nu_A.$$ 

Since $\tilde{G}_a \in \mathcal{F}_a$, Proposition 2.4 gives $G_A \in \mathcal{F}_A$ and the theorem is proved.

If $X$ is one-dimensional, i.e., a Riemann surface, then $\mathcal{I}_A$ is a principal ideal sheaf. If $\mathcal{I}_A = 0$, the zero sheaf, then $\tilde{\nu}_A = +\infty$. If $\mathcal{I}_A \neq 0$, then $|A|$ is discrete, for $a \not\in |A|$ we have $\mathcal{I}_{A,a} = \mathcal{O}_{X,a}$ and $\tilde{\nu}_A(a) = 0$, and for $a \in |A|$ the ideal $\mathcal{I}_{A,a}$ is generated by the germ of $z_a^m$ at $a$, where $m = \tilde{\nu}_A(a) > 0$ and $\zeta_a$ is a local generator for $\mathcal{I}_A$ near $a$ with $\zeta_a(a) = 0$. We obviously have

$$G_A \geq \sum_{a \in \mathcal{D}} \tilde{\nu}_A(a)G_X(\cdot, a) \in \mathcal{F}_A,$$

where $G_X(\cdot, a)$ is the Green function on $X$ with single pole at $a$. In the special case $X = \mathbb{D}$, every function $u \in \mathcal{F}_A \setminus \{-\infty\}$ can be represented by the Poisson–Jensen formula

$$u(z) = \frac{1}{2\pi} \int_{\mathbb{D}} G_\mathbb{D}(z, \cdot) \Delta u + \int_{\mathbb{T}} P_\mathbb{D}(z, t) d\lambda_u(t), \quad z \in \mathbb{D},$$

where $P_\mathbb{D}$ is the Poisson kernel for the unit disc $\mathbb{D}$ and $\lambda_u$ is a nonpositive measure on the unit circle $\mathbb{T}$ (the boundary value of $u$). We have $\nu_a(a) = \Delta u(\{a\})/2\pi$, so $\Delta u \geq 2\pi \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a)\delta_a$, where $\delta_a$ is the Dirac measure at the point $a$. Thus the Poisson–Jensen formula implies

$$u(z) \leq \int_{\mathbb{D}} G_\mathbb{D}(z, \cdot) \left( \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a)\delta_a \right) = \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a)G_\mathbb{D}(z, a)$$

and we conclude that

$$G_A(z) = \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a)G_\mathbb{D}(z, a) = \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a) \log \left| \frac{z - a}{1 - \bar{a}z} \right|, \quad z \in \mathbb{D},$$

for every closed complex subspace $A$ of $\mathbb{D}$.

Now we let $X$ be any manifold, $f \in \mathcal{O}(\mathbb{D}, X)$ be an analytic disc, and $a \in \mathbb{D}$. If $\psi_1, \ldots, \psi_m$ are local generators for $A$ at $f(a)$, then $f^*\psi_1, \ldots, f^*\psi_m$ are local generators for $\mathcal{I}_{f^*A}$ near $a$. 6
If all these functions are zero in some neighbourhood of $A$, then $I_{f^\ast A} = 0$ and $\tilde{\nu}_{f^\ast A} = +\infty$. If one of them is not zero at $a$, then $I_{f^\ast A,a} = \mathcal{O}_{X,a}$ and $\tilde{\nu}_{f^\ast A}(a) = 0$, and if they have a common isolated zero at $a$, then $\tilde{\nu}_{f^\ast A}(a)$ is the smallest positive multiplicity of them. Since $f^\ast G_A \leq G_{f^\ast A}$, we get

$$G_A(x) \leq G_{f^\ast A}(0) = \sum_{a \in \mathbb{D}} \tilde{\nu}_{f^\ast A}(a) \log |a|, \quad f \in \mathcal{O}(\mathbb{D}, X), \quad f(0) = x.$$ 

**Theorem 2.6** Let $X$ be a complex manifold and $A$ be a closed complex subspace of $X$. Then

$$G_A(x) = \inf \{ G_{f^\ast A}(0) ; f \in \mathcal{O}(\mathbb{D}, X), x = f(0) \}, \quad x \in X.$$

Let us show how the theorem follows from Hironaka’s desingularization theorem. If we use the disc formula (1.1) for $\tilde{\nu}_{\hat{A}}$ with $\alpha = \hat{\nu}_{\hat{A}}$, the fact that $G_{\hat{A}} = \tilde{G}_\alpha$, and the desingularization map $\Phi$ above with $x = \Phi(\hat{x})$, then

$$G_A(x) = \Phi^\ast G_A(\hat{x}) = G_{\hat{A}}(\hat{x}) = \inf \{ \tilde{G}_{\hat{A},\alpha}(0) ; g \in \mathcal{O}(\hat{\mathbb{D}}, \hat{X}), g(0) = \hat{x} \}$$

$$\geq \inf \{ G_{\hat{A},\alpha}(0) ; g \in \mathcal{O}(\hat{\mathbb{D}}, \hat{X}), g(0) = \hat{x} \} = \inf \{ G_{\Phi^\ast \Phi^\ast A}(0) ; g \in \mathcal{O}(\hat{\mathbb{D}}, \hat{X}), g(0) = \hat{x} \} = \inf \{ G_{\Phi^\ast(\Phi^\ast)\Phi^\ast}(0) ; g \in \mathcal{O}(\hat{\mathbb{D}}, \hat{X}), g(0) = \hat{x} \} \geq \inf \{ G_{f^\ast A}(0) ; f \in \mathcal{O}(\mathbb{D}, X), f(0) = x \}$$

and we have proved Theorem 2.6. We will prove this theorem without reference to desingularization or the disc formula for $G_\alpha$ in a separate paper.

Let $X_1$ and $X_2$ be complex manifolds, $A_1$ and $A_2$ be closed complex subspaces of $X_1$ and $X_2$, respectively, $X = X_1 \times X_2$ be the product manifold of $X_1$ and $X_2$, and $A = A_1 \times A_2$ be the product space of $A_1$ and $A_2$. If $a = (a_1, a_2) \in X$ and $\psi_1, \ldots, \psi_k$ and $\psi_1^2, \ldots, \psi_l^2$ are local generators for $I_{A_1}$ and $I_{A_2}$ near $a_1$ and $a_2$, respectively, then the functions

$$x = (x_1, x_2) \mapsto \psi_1^1(x_1), \ldots, \psi_k^1(x_1), \psi_1^2(x_2), \ldots, \psi_l^2(x_2).$$

are generators for $I_A$ near $a$. This implies that $X \ni x = (x_1, x_2) \mapsto \max \{ u_1(x_1), u_2(x_2) \}$ is in $\mathcal{F}_A$ for all $u_1 \in \mathcal{F}_{A_1}$ and $u_2 \in \mathcal{F}_{A_2}$, so we obviously have

$$G_A(x) \geq \max \{ G_{A_1}(x_1), G_{A_2}(x_2) \}, \quad x = (x_1, x_2) \in X.$$ 

The following is called the product property for Green functions.

**Theorem 2.7** Let $X_1$ and $X_2$ be complex manifolds, $A_1$ and $A_2$ be closed complex subspaces of $X_1$ and $X_2$, respectively, and $A$ be the product of $A_1$ and $A_2$ in $X = X_1 \times X_2$. Then

$$G_A(x) = \max \{ G_{A_1}(x_1), G_{A_2}(x_2) \}, \quad x = (x_1, x_2) \in X.$$ 

We base our proof on Th. 2.6 and give it in Section 7.

It was shown in [12], Prop. 3.2, that if $A$ is given by a single holomorphic function with effective divisor $Z_A$, then $G_A$ satisfies $dd^c G_A \geq Z_A$ and, moreover, it is the largest negative
plurisubharmonic function with this property. (See the last statement of Th. 3.3 in [12]). Here $d = \partial + \bar{\partial}$, $\partial^c = (\partial - \bar{\partial})/2\pi i$.

In the general case, a space $A$ generates holomorphic chains

$$Z^p_A = \sum_i m_{i,p}[A^p_i],$$

where $A^p_i$ are $p$-codimensional components of $|A|$ and $m_{i,p} \in \mathbb{Z}$. Namely, if on a domain $U \subset X$ the space $A$ is given by functions $\psi_1, \ldots, \psi_m$ and $\text{codim} |A| = p$ there, then by the King-Demailly formula ([5], Th. 6.20),

$$(dd^c \log |\psi|)^p = \sum_i m_{i,p}[A^p_i] + R \quad \text{on } U,$$

where $m_{i,p}$ is the generic multiplicity of $\psi$ along $A^p_i$ and $R$ is a positive closed current of bidegree $(p, p)$ on $U$, such that $\chi_{|A|}R = 0$ and $\text{codim} E_c(R) > p$ for every $c > 0$. Here $\chi_S$ is the characteristic function of a set $S$, $E_c(R) = \{x; \nu_R(x) \geq c\}$ and $\nu_R(x)$ is the Lelong number of the current $R$ at $x$. In other words, the holomorphic chain $Z^p_A$ given by (2.2) is the residual Monge-Ampère current of $\log |\psi|$ on $|A| \cap U$.

**Theorem 2.8** Let $A$ have bounded global generators $\psi$ in $X$. Then

(i) $G_A = \log |\psi| + O(1)$ locally near $|A|$.

(ii) If $\text{codim} |A| = p$ on $U \subset X$, then $(dd^c G_A)^p = Z^p_A + Q$ on $U$, where $Q$ is a positive closed current of bidegree $(p, p)$ on $U$, such that $\chi_{|A|}Q = 0$ and $\text{codim} E_c(Q) > p$ for every $c > 0$. If $U \cap |A| \subset J^p$, then $Q$ has zero Lelong numbers; here the set $J^p$ consists of all points $a \in |A|$ such that $p$ is the minimal number of generators of a subideal of $\mathcal{I}_{A,a}$ whose integral closure is equal to the integral closure of $\mathcal{I}_{A,a}$.

A proof is given in Section 3 (and the sets $J^p$ are introduced and studied in Section 3).

### 3 Decomposition in ideal sheaves

In the case of complete intersection, i.e., when for every $a \in |A|$ the local ideal $\mathcal{I}_{A,a}$ is generated by precisely $p = \text{codim}_a |A|$ germs of holomorphic functions, the relation $G_A \in \mathcal{F}_A$ is in fact quite easy to prove without using the desingularization technique. The main result of this section, Prop. 3.5, gives a tool for the reduction of the general situation to the complete intersection case in Section 4. Our approach develops a method from [21].

We recall some basics on complex Grassmannians. (See, e.g., [2], A3.4-5.) The Grassmannian $G(k, m)$ is the set of all $k$-dimensional linear subspaces of $\mathbb{C}^m$ with the following complex structure. Let $S_{1, \ldots, k}$ be the set of all $L \in G(k, m)$ whose projections to the coordinate plane $\mathbb{C}_{1, \ldots, k}$ of the variables $z_1, \ldots, z_k$ are bijective. Choosing a basis $\{(e_j, w_j)\}$ in $L \in S_{1, \ldots, k}$ with $e_j$ the standard basis vectors in $\mathbb{C}^k$ and $w_j$ vectors in $\mathbb{C}^{m-k}$, we get a representation of $L$ as the $k \times m$-matrix $(E, W)$, where $E$ is the unit $k \times k$-matrix and $W$ is a $k \times (m-k)$-matrix. This gives a parametrization of $S_{1, \ldots, k}$ by $k \times (m-k)$-matrices $W$. In a similar way we parametrize all the charts $S_I$, $I = (i_1, \ldots, i_k)$. Since the neighbouring relations are holomorphic, this
determines a complex structure on $G(k, m)$. It is easy to see that $\dim G(k, m) = k(m - k)$. The set $\{(z, L); z \in L\} \subset \mathbb{C}^m \times G(k, m)$ is sometimes called the incidence manifold.

Let $\psi: \Omega \to \mathbb{C}^m$, $m > 1$, be a holomorphic map on a domain $\Omega$ in $\mathbb{C}^n$ and $Z = \{x \in \Omega; \psi(x) = 0\}$. If $U$ is a subdomain of $\Omega$, then the graph $\Gamma_U = \{(x, \psi(x)); x \in U\}$ of $\psi$ over $U$ is an $n$-dimensional complex manifold in $\mathbb{C}^n \times \mathbb{C}^m$. Given $k \leq m - 1$, let $\Gamma^k_U$ be the pullback of $\Gamma_U$ to the incidence variety in $\Gamma_U \times G(k, m)$. Namely, $\Gamma^k_U$ is the closure of the set

$$\{(x, \psi(x), L); x \in U \setminus Z, L \in G(k, m), \psi(x) \in L\}.$$

By $\rho_k$ we denote the projection from $\Gamma^k_U$ to $G(k, m)$, and by $\pi_k$ its projection to $\Omega$.

For $x \in \Omega \setminus Z$, the fiber $\rho_k \circ \pi_k^{-1}(x)$ consists of all $L \in G(k, m)$ passing through $\psi(x) \neq 0$ and thus is isomorphic to $G(k - 1, m - 1)$. Therefore $\dim \Gamma^k_\Omega = n + (k - 1)(m - k)$.

Let $I^k = I^k(\psi)$ be the collection of all points $x$ in $\Omega$ such that $\rho_k(\Gamma^k_U) = G(k, m)$ for every neighbourhood $U$ of $x$, i.e., $\rho_k \circ \pi_k^{-1}(x) = G(k, m)$. Evidently, $I^1 \subseteq I^2 \subseteq \ldots \subseteq I^{m-1} \subseteq Z$.

**Lemma 3.1** $I^k$ is an analytic set of dimension at most $n - m + k - 1$.

**Proof:** We have

$$\pi_k^{-1}(I^k) = I^k \times \{0\} \times G(k, m), \quad (3.1)$$

so $I^k \subset \pi_k \circ \rho_k^{-1}(L)$ for each $L \in G(k, m)$. On the other hand, for every $x \not\in I^k$ there exists $L \in G(k, m)$ such that $x \not\in \pi_k \circ \rho_k^{-1}(L)$. Thus

$$I^k = \bigcap_{L \in G(k, m)} \pi_k \circ \rho_k^{-1}(L).$$

Each $\rho_k^{-1}(L)$ is an analytic set in $\Gamma^k_\Omega$. Since the map $\pi_k$ is proper, Remmert’s theorem implies that $\pi_k \circ \rho_k^{-1}(L)$ is an analytic subset of $\Omega$ for any $L$, and so is $I^k$.

The set $\pi_k^{-1}(I^k)$ is a nowhere dense analytic subset of $\Gamma^k_\Omega$, and thus $\dim \pi_k^{-1}(I^k) < \dim \Gamma^k_\Omega = n + (k - 1)(m - k)$. By (3.1), $\dim \pi_k^{-1}(I^k) = \dim I^k + k(m - k)$. Therefore $\dim I^k < n + (k - 1)(m - k) - k(m - k) = n - m + k$. [\Box]

**Corollary 3.2** If $m > n$, then $I^k = \emptyset$ for all $k \leq m - n$.

**Lemma 3.3** For any $a \in Z \setminus I^k$ there exist a neighbourhood $U$ of $a$ and holomorphic functions $\xi_1, \ldots, \xi_{m-k}$ (linear combinations of $\psi_1, \ldots, \psi_m$) such that $\log |\psi| \propto \log |\xi|$ in $U$.

**Proof:** Given $a \in Z \setminus I^k$, one can find a neighbourhood $U$ of $a$ such that $\rho_k(\Gamma^k_U) \neq G(k, m)$. Since the set $G(k, m) \setminus \rho_k(\Gamma^k_U)$ is open, there exists $L_0$ in the chart $S_{1..k}$ of $G(k, m)$ such that

$$\psi(x) \cap \omega = \emptyset \quad (3.2)$$

for some neighbourhood $\omega \subset S_{1..k}$ of $L_0$ and all $x \in U \setminus Z$.

Let $(E, W_0)$ be the canonical representation of $L_0$. For every $y = (y', y'') \in \mathbb{C}^k \times \mathbb{C}^{m-k}$ with $y' \neq 0$, the map $y \mapsto (y', y'W_0)$ is the projection to the space $L_0$. By elementary linear algebra arguments (see Lemma 3.4 below), relation (3.2) implies existence of $r > 0$ such that

$$|\psi''(x) - \psi'(x)W_0| \geq r|\psi'(x)|, \quad x \in U. \quad (3.3)$$
We define a map $\xi : U \to \mathbb{C}^{m-k}$ by $\xi(x) = \psi''(x) - \psi'(x)W_0$. Then

$$|\xi(x)| \leq C|\psi(x)|, \quad x \in U.$$

Furthermore, inequality (3.3) implies

$$|\psi|^2 \leq |\psi'|^2 + 2|\psi'' - \psi'W_0|^2 + 2|\psi'W_0|^2 \leq C|\psi'' - \psi'W_0|^2 = C|\xi|^2,$$

and the assertion follows.

**Lemma 3.4** Let $W_0$ be a complex $k \times (m - k)$-matrix and a set $S \subset \mathbb{C}^k \times \mathbb{C}^{m-k}$ be such that $|y'' - y'W| > 0$ for all $y = (y', y'') \in S$ and all matrices $W \in \mathbb{C}^{(m-k)}$ with $|W - W_0| < \delta$ (all the norms $|\cdot|$ are the Euclidean norms in the corresponding linear spaces). Then

$$|y'' - y'W| \geq \frac{\delta}{k}|y'|, \quad y \in S, \quad |W - W_0| < \delta.$$

**Proof:** Suppose there exists $y \in S$ and $W$ in the $\delta$-neighbourhood of $W_0$ such that $|y'' - y'W| < \frac{\delta}{k}|y'|$. For the vector $z = (z', z'') := (y', y'' - y'W_0)$ this means $|z''| < \frac{\delta}{k}|z'|$.

We choose $l \in [1, k]$ such that $|z_i| = \max\{|z_i|; 1 \leq i \leq k\}$ and consider the $k \times (m - k)$-matrix $V$ with the entries $V_{ij} = z_{k+i} - z_i$ for $1 \leq j \leq m - k$, and $V_{ij} = 0$ for all $i \neq l$ and $1 \leq j \leq m - k$. Then

$$|V| = \frac{|z''|}{|z_l|} \leq \frac{|z''|}{k|z'|} < \delta$$

and $z'V = z''$. The latter relation is equivalent to $y'' - y'W = 0$ with $W = W_0 + V$. Since $|W - W_0| = |V| < \delta$, this contradicts the hypothesis of the lemma.

We recall that the analytic spread of an ideal $\mathcal{I}$ equals the minimal number of generators of a subideal of $\mathcal{I}$ whose integral closure coincides with the integral closure of $\mathcal{I}$, see [19].

**Proposition 3.5** Let $A$ be a closed complex subspace of a manifold $X$, dim $X = n$. Then the set $\{A\}$ can be decomposed into the disjoint union of local analytic varieties $J^k$, $1 \leq k \leq n$, such that

(i) codim $J^k \geq k$ and

(ii) for each $a \in J^k$, the ideal $\mathcal{I}_{A,a}$ has analytic spread at most $k$.

**Proof:** Let $\psi = (\psi_1, \ldots, \psi_m)$ be generators of $\mathcal{I}_A$ on a domain $\Omega \subset X$. Set $N = \min\{n, m\}$, $Z = |A| \cap \Omega$, $J^1 = Z \setminus I^{m-1}$, $J^k = I^{m-k+1} \setminus I^{m-k}$ for $k = 2, \ldots, N - 1$, and $J^N = I^{m-N+1}$ (some of them can be empty). The sets $J^k$ are pairwise disjoint, dim $J^k \leq n - k$ (Lemma 3.1), and $\cup_k J^k = Z$. On a neighbourhood of each point of $J^k$, the singularity of the function log $|\psi|$ is equivalent to one defined by the function log $|\xi|$ with $\xi = (\xi_1, \ldots, \xi_k)$ (this follows from Lemma 3.3 if $m \leq n$, and Corollary 3.2 in the case $m > n$). This means that the ideal generated by the germs of $\psi$ at $a \in J^k$ has analytic spread at most $k$.

Let $\psi' = (\psi'_1, \ldots, \psi'_m)$ be other generators of $\mathcal{I}_A$ on $\Omega$; by adding some identically zero components to either $\psi$ or $\psi'$ we can assume $m' = m$. For any point $a \in Z \setminus I^k(\psi)$, relation
We recall that a function $\psi'(x) \cap \omega' = \emptyset$ for some neighbourhood $\omega'$ of $L'_0$ and all $x \in U' \setminus Z$, so $a \in Z \setminus I^k(\psi')$. This shows that the sets $J^k$ are independent of the choice of generators of $\mathcal{I}_{A, \Omega}$. Therefore each $J^k$ is well defined as a local (not necessarily closed) analytic variety in $X$ with properties (i) and (ii).

**Example 3.6** Let $A$ be generated by $\psi(x) = (x_1^2, x_2, x_3, x_1x_2x_3)$ in $\mathbb{C}^3$. Then $|A| = \mathbb{C}_{23} \cup \mathbb{C}_1$; here $\mathbb{C}_{23}$ is the coordinate plane of the variables $x_2$ and $x_3$, i.e., $\mathbb{C}_{23} = \{x_1 = 0\}$, and $\mathbb{C}_1 = \{x_2 = x_3 = 0\}$. The variety $|A|$ has the decomposition $|A| = J^1 \cup J^2 \cup J^3$ with $J^1 = \mathbb{C}_{23} \setminus (\mathbb{C}_2 \cup \mathbb{C}_3)$, $J^2 = \mathbb{C}_1^* \cup \mathbb{C}_2^* \cup \mathbb{C}_3^*$, and $J^3 = \{0\}$. Near points of $J^1$ we have $\log |\psi| \asymp \log |x_1|$. As to $J^2$, the relation $\log |\psi| \asymp \log |\xi|$ is satisfied with $\xi = (x_2, x_3)$ near points of $\mathbb{C}_1^*$, and we can take $\xi = (x_1^2, x_1x_3)$ near points of $\mathbb{C}_2^*$ and $\xi = (x_1^2, x_1x_2)$ near points of $\mathbb{C}_3^*$.

4 Upper bounds and maximality

We recall that a function $u \in \text{PSH}(X)$ is called *maximal* in $X$ if for every relatively compact subset $U$ of $X$ and for each upper semicontinuous function $v$ on $\overline{U}$ such that $v \leq u$ on $\partial U$, we have $v \leq u$ in $U$. An equivalent form is that for any $v \in \text{PSH}(X)$ the relation $\{v > u\} \subset X$ implies $v \leq u$ on $X$.

We will use the following variant of the maximum principle for unbounded plurisubharmonic functions.

**Lemma 4.1** Let $D \subset \mathbb{C}^k$ be a bounded domain and $u, v \in \text{PSH}(D)$ such that

(i) $v$ is bounded above,

(ii) the set $S := v^{-1}(\infty)$ is closed in $D$,

(iii) $v$ is locally bounded and maximal on $D \setminus S$,

(iv) for any $\epsilon > 0$ there exists a compact $K_\epsilon \subset D$ such that $u(z) \leq v(z) + \epsilon$ on $D \setminus K_\epsilon$, and

(v) $\limsup_{z \to a, z \in S} (u(z) - v(z)) < \infty$ for each $a \in S$.

Then $u \leq v$ in $D$.

**Proof:** By (i) we may assume that $v$ is negative in $D$. Take any $\epsilon > 0$ and $\delta > 0$. Then it is sufficient to prove that $u_1 = (1 + \delta)(u - \epsilon) \leq v$. By (v) we conclude that each point $a \in S$ has a neighbourhood $U_a \subset D$ where $u_1 \leq v$ and by (iv) that there is a domain $D_1 \subset D$ such that $u_1 \leq v$ on $D \setminus D_1$. By (ii) $S \cap \overline{D}_1$ is compact, so we can take a finite covering of $S \cap \overline{D}_1$ by $U_a$, $1 \leq j \leq N$. Then $D_2 = D_1 \setminus \bigcup_j U_a$ is an open subset of $D$ on which $v$ is bounded and $u_1 \leq v$ holds on $\partial D_2$. By (iii) $v$ is maximal on $D_2$, so $u_1 \leq v$ on $D_2$ and thus on $D$. ■

The next statement is the crucial point in the proof that $G_A \in \mathcal{F}_A$.

**Lemma 4.2** Let $\psi = (\psi_1, \ldots, \psi_m)$ be a holomorphic map on a domain $\Omega \subset \mathbb{C}^n$ and $Z$ be its zero set. Then for every $K \Subset \Omega$ there exists a number $C_K$ such that any function $u \in \text{PSH}^-(\Omega)$ which satisfies $u \leq \log |\psi| + O(1)$ locally near points of $Z$ has the bound $u(x) \leq \log |\psi(x)| + C_K$ for all $x \in K$.
Proof: What we need to prove is that each point \( a \in Z \) has a neighbourhood \( U \) where 
\[ u \leq |\psi| + C \] with \( C \) independent of the function \( u \).

Let \( \text{codim}_a Z = p \). Then, by Prop. 3.5, \( a \in J^k \) for some \( k \in [p, n] \) and thus there exist \( k \) holomorphic functions \( \xi_1, \ldots, \xi_k \) such that \( |\xi| \leq \log |\psi| \) near \( a \). We will argue by induction in \( k \) from \( p \) to \( n \).

Let \( a \in J^p \); this means that there is a neighbourhood \( V \) of \( a \) such that \( Z \cap V \) is a complete intersection given by the functions \( \xi_1, \ldots, \xi_p \). By Thie’s theorem [23], (see also [5], Th. 5.8), there exist local coordinates \( x = (x', x'') \), \( x' = (x_1, \ldots, x_p) \), \( x'' = (x_{p+1}, \ldots, x_n) \), centered at \( a \) and balls \( B' \subset C^p, B'' \subset C^{n-p} \) such that \( B' \times B'' \subset V \), \( Z \cap (B' \times B'') \) is contained in the cone \( \{ |x'| \leq \gamma |x''| \} \) with some constant \( \gamma > 0 \), and the projection of \( Z \cap (B' \times B'') \) onto \( B'' \) is a ramified covering with a finite number of sheets. Let \( r_1 = 2\gamma r_2 \) with a sufficiently small \( r_2 > 0 \) so that \( B_{r_1} \subset B' \) and \( B_{r_2}'' \subset B'' \), then for some \( \delta > 0 \)
\[ |\xi(x)| \geq \delta, \quad x \in \partial B_{r_1}' \times B_{r_2}'' . \]

Given \( x''_0 \in B_{r_2}'' \), denote by \( Z(x''_0) \) and \( \text{Sing} Z(x''_0) \) the intersections of the set \( B_{r_1}' \times \{ x''_0 \} \) with the varieties \( Z \) and \( \text{Sing} Z \), respectively. Since the projection is a ramified covering, \( Z(x''_0) \) is finite for any \( x''_0 \in B_{r_2}'' \), while \( \text{Sing} Z(x''_0) \) is empty for almost all \( x''_0 \in B_{r_2}'' \) because \( \text{dim} \text{Sing} Z \leq n - p - 1 \); we denote the set of all such generic \( x''_0 \) by \( E \).

Fix any \( x''_0 \in E \) and consider the function
\[ v(x') = \log(|\xi(x', x''_0)|/\delta). \]

It is plursubharmonic on \( B_{r_1}' \), nonnegative on \( \partial B_{r_1}' \) and maximal on \( B_{r_1}' \setminus Z(x''_0) \), since the map \( \xi(\cdot, x''_0) : B_{r_1}' \to C^p \) has no zeros outside \( Z(x''_0) \).

For any function \( u \in PSH^-(Y) \) which satisfies \( u \leq \log |\xi| + O(1) \) locally near regular points of \( Z \), we have, by Lemma 4.1, \( u(x', x''_0) < v(x') \) on the whole ball \( B_{r_1}' \).

Since \( x''_0 \in E \) is arbitrary, this gives us \( u \leq \log |\xi| - \log \delta \) on \( B_{r_1}' \times E \). The continuity of the function \( \log |\xi| \) extends this relation to the whole set \( U = B_{r_1}' \times B_{r_2}'' \), which proves the claim for \( k = p \).

Now we make a step from \( k - 1 \) to \( k \). Since \( \text{dim} J^k \leq n - k \), we use Thie’s theorem to get a coordinate system centered at \( a \in J^k \) such that the projection of \( J^k \cap (B' \times B'') \) to \( B'' \subset C^{n-k} \) is a finite map and \( (\partial B' \times \partial B'') \cap J^i = \emptyset \) for all \( i \geq k \). Therefore, by the induction assumption and a compactness argument, \( u \leq \log |\xi| + C \) near \( \partial B' \times \partial B'' \), where the constant \( C \) is independent of \( u \).

Now for any \( x''_0 \in B'' \) we consider the function \( v(x') = \log |\xi(x', x''_0)| + C \). Then Lemma 4.1 gives us \( u(x', x''_0) < v(x') \) on \( B' \) and hence \( u \leq \log |\xi| + C \) on \( B' \times B'' \).

Remark. Note that the uniform bound \( u \leq \log |\psi| + C \) near points \( a \in J^p \), \( \text{codim}_a Z = p \), was deduced from the local bounds only near regular points of \( Z \).

Proof of Theorem 2.5. The relation \( G_A \leq \log |\psi| + O(1) \) follows from Lemma 4.2. This implies that its upper semicontinuous regularization \( G_A^* \) is in \( \mathcal{F}_A \) and thus \( G_A^* = G_A \).

One of the most important properties of the “standard” pluricomplex Green function \( G_{X,a} \) with logarithmic pole at \( a \in X \) is that it satisfies the homogeneous Monge-Ampère equation \( (dd^c G_{X,a})^n = 0 \) outside the point \( a \); in other words, \( G_{X,a} \) is a maximal plurisubharmonic function on \( X \setminus \{ a \} \). In our situation, one can say more.
Theorem 4.3 The function $G_A$ is maximal on $X \setminus |A|$ and locally maximal outside a discrete subset of $|A|$ (actually, the set $J^n$ from Prop. 3.5). If $A$ has $k < n$ global generators on $X$, then $G_A$ is maximal on the whole $X$.

Proof: Take any point $a \notin J^n$. By Proposition 3.5, there exist functions $\xi_1, \ldots, \xi_k \in I_{A,U}$, $k < n$, generating an ideal whose integral closure coincides with the integral closure of $I_{A,U}$, and so $G_A \leq \log |\xi| + C$ on $U$. The function $\log |\xi|$ is maximal on $U$, which follows from the fact that it is the limit of the decreasing sequence of maximal plurisubharmonic functions $u_j = \frac{1}{2} \log(|\xi|^2 + \frac{1}{j})$. (See [21], Example 1.) Take any domain $W \subseteq U$. Given a function $v \in PSH(U)$ with $v \leq G_A$ on $U \setminus W$, we have to show that $v \leq G_A$ on $U$. Consider the function $w$ such that $w = G_A$ on $X \setminus W$ and $w = \max\{G_A, v\}$ on $W$. Since $G_A \leq \log |\xi| + C$ on $U$, we have $w \leq \log |\xi| + C$ on $U \setminus W$, and the maximality of $\log |\xi|$ on $U$ extends this inequality to the domain $W$. Therefore, $w \in F_A$ and thus $w \leq G_A$ on $U$.

When $a \notin |A|$, we can take $U = X \setminus |A|$ and $\xi = 1$, which gives us maximality of $G_A$ on $U = X \setminus |A|$.

Finally, if $A$ has $k < n$ global generators on $X$, then the same arguments with $U = X$ show the maximality of $G_A$ on the whole $X$. ■

Remark. If $J^n = \emptyset$, the Green function is locally maximal on the whole $X$. We don’t know if this implies its maximality on $X$.

5 Complex spaces with bounded global generators

If $A$ has bounded generators $\psi$, which we can choose such that $|\psi| < 1$, then $\log |\psi| \in F_A$. This gives immediately

Proposition 5.1 Let $A$ be a closed complex subspace of a manifold $X$ and assume that $A$ has bounded global generators $\{\psi_i\}$ (for example, $X$ is a relatively compact domain in a Stein manifold $Y$ and $A$ is a restriction to $X$ of a complex space $B$ on $Y$), then

$$G_A = \log |\psi| + O(1)$$

locally near $|A|$.

To describe the boundary behaviour of $G_A$, we recall the notion of strong plurisubharmonic barrier. Let $X$ be a domain in a complex manifold $Y$, and let $p \in \partial X$. A plurisubharmonic function $v$ on $X$ is called a strong plurisubharmonic barrier at $p$ if $v(x) \to 0$ as $x \to p$, while $\sup_{X \setminus V} v < 0$ for every neighbourhood $V$ of $p$ in $Y$. By standard arguments (see, e.g., [12], Proposition 2.4) we get

Proposition 5.2 Let $X$ be a domain in a complex manifold $Y$, and let a closed complex subspace of $X$ have bounded global generators. If $X$ has a strong plurisubharmonic barrier at $p \in \partial X \setminus |A|$, then $G_A(x) \to 0$ as $x \to p$.

A uniqueness theorem for the Green function is similar to that for the divisor case in [12], but the proof is different (since the function $u - \log |\psi|$ need not be plurisubharmonic) and follows from Lemma 4.4 and Proposition 5.1.
**Theorem 5.3** Let a complex space $A$ have bounded global generators $\psi_i$ on $X$, and let a function $u \in PSH^-(X)$ have the properties

(i) $u$ is locally bounded and maximal on $X \setminus |A|$. \\
(ii) For any $\epsilon > 0$ there exists a compact subset $K$ of $X$ such that $u \geq G_A - \epsilon$ on $X \setminus K$; \\
(iii) $u = \log |\psi| + O(1)$ locally near $|A|$. \\

Then $u = G_A$.

Relation (5.1) allows us to derive the properties of the Monge-Ampère current $(dd^c G_A)^p$.

**Proof of Theorem 2.8.** Since $G_A$ is locally bounded on $U \setminus |A|$ and $\text{codim} |A| = p$, the current $(dd^c G_A)^p$ is well defined on $U$. Moreover, Siu’s structural formula for positive closed currents (see also [5], Theorem 6.19) gives us a (unique) representation for the current $(dd^c G_A)^p$ as

$$(dd^c G_A)^p = \sum_j \lambda_j [B_j] + Q,$$

where $B_j$ are some irreducible analytic varieties of codimension $p$, $\lambda_j$ are the generic Lelong numbers of $(dd^c G_A)^p$ along $B_j$, i.e.,

$$\lambda_j = \inf \{ \nu((dd^c G_A)^p, a) : a \in B_j \},$$

and $Q$ is a positive closed current such that $\text{codim} \{ x : \nu(Q, x) \geq c \} > p$ for each $c > 0$.

As $G_A$ has asymptotics near points of the set $|A|$, Demailly’s Comparison Theorem for Lelong numbers ([5], Theorem 5.9) implies

$$\nu((dd^c G_A)^p, a) = \nu((dd^c \log |\psi|)^p, a)$$

at every point $a \in |A| \cap J^p \cap U$. In particular, the generic Lelong number of $(dd^c G_A)^p$ along each variety $A_j^p$ equals the multiplicity of this component in $|A|$. Besides, $\nu((dd^c G_A)^p, a) = 0$ for any $a \notin |A|$. This shows that $\{ B_j \}_j$ are exactly the $p$-codimensional components of the variety $|A|$ in $U$ and $\sum \lambda_j [B_j] = Z_A^p$ on $U$.

Finally, if $U \cap |A| \subset J^p$, then $U \cap |A|$ can be given locally by $p$ holomorphic functions $\xi_i$ with $\log |\xi| \asymp \log |\psi|$. By King’s formula, $(dd^c \log |\xi|)^p = Z_A^p$, which means, in particular, that $(dd^c \log |\xi|)^p$ has zero Lelong numbers outside $\cup_i A_i^p$. Since the currents $(dd^c G_A)^p$ and $(dd^c \log |\xi|)^p$ have the same Lelong numbers, this proves the last statement.

So the Green function satisfies, as in the divisor case, the relation $(dd^c G_A)^p \geq Z_A^p$, but for $p > 1$ it is not the largest negative plurisubharmonic function with this property (even for reduced spaces that are complete intersections). For example, let $X$ be the unit polydisc in $\mathbb{C}^3$ and $A$ be generated by $\psi(z) = (z_1, z_2)$. Then $G_A = \max \{ \log |z_1|, \log |z_2| \}$ and, moreover, $(dd^c G_A)^2 = Z_A = |A|$. But the functions $u_N = \max \{ N \log |z_1|, N^{-1} \log |z_2| \}$, $N > 0$, also satisfy $(dd^c u_N)^2 = |A|$, although they are not dominated by $G_A$. It is easy to see that the upper envelope of all such functions equals 0 outside $|A|$ and $-\infty$ on $|A|$. Therefore, in the case $\text{codim} |A| > 1$ there is no counterpart for the description of the Green function in terms of the current $Z_A$. 

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6 Reduced spaces

Now we return to relations between the functions \( G_A \) and \( \tilde{G}_{\tilde{\nu}_A} \) (see Introduction). As was already mentioned, one has always \( G_A \leq \tilde{G}_{\tilde{\nu}_A} \) and \( G_A < \tilde{G}_{\tilde{\nu}_A} \) for 'generic' spaces \( A \), however \( G_A = \tilde{G}_{\tilde{\nu}_A} \) for effective divisors \( A \). Here we show that the equality holds also in the case of reduced complex spaces.

When \( A \) is a reduced space, it can be identified with the analytic variety \(|A|\). Its generators \( \psi_1, \ldots, \psi_m \) on \( U \) have the property: if a holomorphic function \( \varphi \) vanishes on \( A \cap U \), then \( \varphi = \sum h_i \psi_i \) with \( h_i \in \mathcal{O}(U) \).

Since \( \tilde{\nu}_A = 1 \) at all regular points of \( A \), it is natural to consider the class

\[
\tilde{F}_A^1 = \{ u \in \text{PSH}^-(X); \nu_u(a) \geq 1 \text{ for all } a \in \text{Reg} A \}.
\]

Note that upper semicontinuity of the Lelong numbers implies \( \nu_u \geq 1 \) on the whole \( A \).

We evidently have \( \tilde{F}_A^1 \subseteq \tilde{F}_{\tilde{\nu}_A} \subseteq \tilde{F}_A \).

**Theorem 6.1** If \( A \) is a reduced subspace of \( X \), then \( \tilde{F}_A^1 = \tilde{F}_{\tilde{\nu}_A} = \tilde{F}_A \) and consequently

\[
G_A(x) = \tilde{G}_{\tilde{\nu}_A}(x) = \sup \{ u(x); u \in \tilde{F}_A^1 \}.
\]

**Proof:** It suffices to show that for any function \( u \in \tilde{F}_A^1 \) and every point \( a \in A \) there is a neighbourhood \( U \) of \( a \) and a constant \( C \) such that

\[
u(x) \leq \log |\psi(x)| + C, \quad x \in U.
\]  \hspace{1cm} (6.1)

We will use induction on the dimension of \( X \). The case \( \dim X = 1 \) is evident. Assume it proved for all \( X \) with \( \dim X < n \) and take any \( u \in \tilde{F}_A^1 \). When \( \dim A = 0 \), relation (6.1) follows easily from the fact that \( \log |\psi(x)| = \log |\zeta(x)| + O(1) \) near \( a \in A \), where \( \zeta \) are local coordinates near \( a \) with \( \zeta(a) = 0 \). So we assume \( \dim A > 0 \). We first treat the case when \( a \) is a regular point of \( A \), \( \text{codim}_A A = p < n \). Since the problem is local, we may then assume that \( X \subseteq \mathbb{C}^n \) and contains the unit polydisc \( \mathbb{D}^n \), \( a = 0 \), and the restriction \( A' \) of \( A \) to \( \mathbb{D}^n \) is given by \( \psi(x) = (x_1, \ldots, x_p) \). Then the restriction of \( u \) to \( \mathbb{D}^n \) is dominated by the Green function \( \tilde{G}_{\tilde{\nu}_{A'}} \). By the product property for this type of Green function (12, Theorem 2.5), \( \tilde{G}_{\tilde{\nu}_{A'}}(x) = \max \{ \log |x_j|, 1 \leq j \leq p \} \). This implies (6.1) for \( a \in \text{Reg} A \).

For \( a \in \text{Sing} A \) we will argue similarly to the proof of Lemma 4.2. There is a neighbourhood \( V \) of \( a \) such that \( V \cap \text{Sing} A \subseteq J^p \cup J^{p+1} \cup \ldots J^n \). The proof for \( a \in J^k, p \leq k \leq n \), is then by induction in \( k \).

For \( a \in J^p \cap V \) relation (6.1) follows directly from the remark after Lemma 4.2.

Assuming (6.1) proved for \( a \in J^p \cup \ldots \cup J^k \), we take \( a \in J^{k+1} \). We choose coordinates \( x = (x', x'') \in \mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1} \) such that \( a = 0 \), the projection of \( J^{k+1} \cap \mathbb{D} \) to \( \mathbb{D}'' \) is a finite map and \( \partial \mathbb{B}' \times \mathbb{B}'' \cap J^i = \emptyset \) for all \( i \geq k + 1 \), so the \( k \)-induction assumption gives

\[
u(x) \leq \log |\psi(x)| + C, \quad x \in \partial \mathbb{B}' \times \mathbb{B}''.
\]  \hspace{1cm} (6.2)

Take any \( b = (b', b'') \in \mathbb{B}' \times \mathbb{B}'' \) and consider the \((k+1)\)-dimensional plane \( L = \{ x ; x'' = b'' \} \). Then the restriction \( u_L \) of \( u \) to the plane \( L \) (in the same way we will use the denotation \( \psi_L, \mathbb{B}_L, A_L \), etc.) has Lelong numbers at least 1 at all points of \( A_L \), so \( u_L \in \tilde{F}_{A_L, L}^1 \). Since
dim $\mathcal{B}_L < n$ and the components of $\psi_L$ generate $A_L$, the $n$-induction assumption implies $u_L \in \mathcal{F}_{A_L,\mathcal{B}_L}$. Therefore, $u_L \leq \log |\psi_L| + O(1)$ locally near points of $A_L$.

Since $a \in j^{k+1}$, we can find functions $\xi_1, \ldots, \xi_{k+1}$ such that $\log |\xi| > \log |\psi|$ on $\mathcal{B}$. Therefore $u_L \leq \log |\xi_L| + O(1)$ locally near all points of $A_L$, and, by (6.2), $u_L \leq \log |\psi_L| + C_1$ on a neighbourhood of $\partial\mathcal{B}$ with $C_1$ independent of $L$. The function $\xi_L$ is maximal on $\mathcal{B}_L \setminus A_L$, so by Lemma 4.1 $u_L \leq \log |\xi_L| + C_1$ everywhere on $\mathcal{B}_L$. Since the plane $L$ was chosen arbitrary, this gives us (6.4) for $a \in j^{k+1}$.

This proves the inductive step in the induction in $k$ and, at the same time, in the induction in $n$. 

Theorem 4.3 for reduced spaces has the following form (compare with the remark after the proof of Theorem 4.3).

**Theorem 6.2** The Green function of a reduced space $A$ is maximal on $X \setminus A_0$, where $A_0$ is the collection of 0-dimensional components of $A$.

**Proof:** We need to show that for every domain $U \subset X' := X \setminus A_0$ and a function $u \in PSH(X')$ the condition $u \leq G_A$ on $X' \setminus U$ implies $u \leq G_A$ on $U$.

Consider the set $E_1(u) = \{x \in X : \nu_u(x) \geq 1\}$. Since $u \leq G_A$ on $X' \setminus U$, we have $E_1(u) \setminus U \supset A \setminus U$. By Siu’s theorem, $E_u$ is an analytic variety in $X$, so it must contain the whole $A$. This means that $u \in J_A^1$ and thus is dominated by $G_{\tilde{\nu}_A} = G_A$ on $X$. 

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**7 The product property**

Our proof of Th. 2.7 in this section is based on Th. 2.6. It is a modification of the proof of Th. 2.5 in [12] which in turn generalizes a proof of Edigarian [8] of the product property for the single pole Green function. For the sake of completeness we have repeated some arguments from [8], [12], and [14].

We introduce the following notation: If the function $\varphi$ is holomorphic in some neighbourhood of the point $a$ in $\mathbb{C}$, then we set $m_a(\varphi) = 0$ if $\varphi(a) \neq 0$, $m_a(\varphi) = +\infty$ if $\varphi = 0$ in some neighbourhood of $a$, and let $m_a(\varphi)$ be the multiplicity of $a$ if it is an isolated zero of $\varphi$.

**Lemma 7.1** Let $x \in X$, $\alpha \in (-\infty, 0)$ and assume that $g \in \mathcal{O}(\mathbb{D}, X)$, $g(0) = x$, and $G_{g^*A}(0) < \alpha$. Then there exist $f \in \mathcal{O}(\mathbb{D}, X)$ and finitely many different points $a_1, \ldots, a_k \in \mathbb{D} \setminus \{0\}$ such that $f(0) = x$ and

$$-\infty < \sum_{j=1}^k \tilde{u}_{f^*A}(a_j) \log |a_j| < \alpha. \quad (7.1)$$

**Proof:** We have $G_{g^*A}(0) = \sum_{a \in \mathbb{D}} \tilde{u}_{g^*A}(a) \log |a| < \alpha$, so we can choose finitely many points $a_1, \ldots, a_k \in \mathbb{D} \setminus \{0\}$ such that

$$\sum_{j=1}^k \tilde{u}_{g^*A}(a_j) \log |a_j| < \alpha. \quad (7.2)$$

If the sum in (7.2) is finite, we take $f = g$. If the sum is equal to $-\infty$ and $g(\mathbb{D})$ is not contained in $|A|$, then $a_j = 0$ and $0 < \tilde{u}_{g^*A}(a_j) < +\infty$ for some $j$. We choose $a \in \mathbb{D} \setminus \{0\}$ so
close to 0 that \( \log |a| < \alpha \) and \( g \) is holomorphic in a neighbourhood of the image of \( h : \mathbb{D} \to \mathbb{C} \), \( h(z) = z(z-a) \). If \( \psi_1, \ldots, \psi_n \) are local generators for \( \mathbb{I}_A \) near \( x \), then \( m_a(\psi_j \circ gh) = m_0(\psi_j \circ g) \) for all \( j \), which implies \( \nu^*_g(A) = \nu(g(\cdot)A) \). If we set \( f = g \circ h \), \( k = 1 \), and \( a_1 = a \), then \( f(0) = x \) and \( (7.1) \) holds.

If the sum in (7.2) equals \( -\infty \) and \( g(\mathbb{D}) \) is contained in \( |A| \), then we may replace \( g \) by the constant disc \( z \mapsto x = g(0) \). We choose a neighbourhood \( U \) of \( x \) in \( X \) and a biholomorphic map \( \Phi : U \to \mathbb{D}^n \) such that \( \Phi(x) = 0 \). We take \( v \in \mathbb{C}^n \) with \( |v| < 1 \) such that the disc \( \mathbb{D} \to X \), \( z \mapsto \Phi^{-1}(zv) \) is not contained in \( |A| \) and choose \( a \in \mathbb{D} \setminus \{0\} \) so small that \( \log |a| < \alpha \) and \( z(a-v) \in \mathbb{D}^n \) for all \( z \in \mathbb{D} \). If we take \( k = 1 \), \( a_1 = a \), and let \( f \) be the map \( z \mapsto \Phi^{-1}(z(z-a)v) \), then \( f(0) = f(a) = x \in |A| \), \( 0 < \nu_{f, A}(a) < +\infty \), and \( (7.1) \) holds.

**Proof of Theorem 2.7.** We need to prove that \( G_A(x) \leq \max \{G_{A_1}(x_1), G_{A_2}(x_2)\} \). Take \( \alpha \in (-\infty, 0) \) larger than the right hand side of this inequality. It is then sufficient to show that \( G_A(x) < \alpha \).

By Theorem 2.6 and Lemma 7.1 we have \( f_j \in \mathcal{O}(\overline{\mathbb{D}} \setminus X) \) with \( f_j(0) = x_j \) and \( a_{jk} \in \mathbb{D} \setminus \{0\} \), \( k = 1, \ldots, l_j, j = 1, 2 \), such that

\[
-\infty < \sum_{k=1}^{l_j} \nu_{f_j, A_j}(a_{jk}) \log |a_{jk}| < \alpha, \quad j = 1, 2. \tag{7.3}
\]

We choose \( f_j \) so that \( l_j \) becomes as small as possible. Then \( 0 < \nu_{f_j, A_j}(a_{jk}) < +\infty \) and \( a_{jk} \neq 0 \) for all \( j \) and \( k \). We define the Blaschke products \( B_j \) by

\[
B_j(z) = \prod_{k=1}^{l_j} \left( \frac{a_{jk} - z}{1 - \overline{a_{jk}} z} \right)^{\mu_{jk}}, \quad \text{where } \mu_{jk} = \nu_{f_j, A_j}(a_{jk}).
\]

Then (7.3) implies \( |B_j(0)| < e^\alpha \). We set \( b_j = B_j(0) \) and \( \mu_j = \sum_{k=1}^{l_j} \mu_{jk} \) and we may assume that \( |b_1| > |b_2| \). We have \( B_j'(0) = B_j(0) \sum_{k=1}^{l_j} \mu_{jk}(|a_{jk}|^2 - 1)/a_{jk} \). If \( B_j'(0) = 0 \) we precompose \( f_1 \) with a map \( \mathbb{D} \to \mathbb{D} \) which fixes the origin and makes a slight change of the points \( a_{1k} \) so that \( B_j'(0) \neq 0 \). By Schwarz Lemma this operation increases the value of \( |b_1| \), so we still have \( |b_1| \geq |b_2| \). By precomposing \( f_1 \) by a rotation, we may assume that \( B_1(0) = b_1 \) is not a critical value of \( B_1 \).

If \( c_j \) is one of the points \( a_{jk} \) having largest absolute value, then \( |c_j| e^\beta \leq |b_j| \). For proving this inequality we assume the reverse inequality \( |b_j| < |c_j| e^\beta \) and for simplicity enumerate the points so that \( |a_{j1}| \leq |a_{j2}| \leq \cdots \). Then

\[
\prod_{k=1}^{m_j} \left| \frac{a_{jk}}{c_j} \right|^{\mu_{jk}} < e^\beta
\]

where \( m_j < l_j \) is the smallest natural number with \( |a_{jk}| = |c_j| \) for \( k > m_j \). Hence (7.3) holds with \( f_j \) replaced by \( z \mapsto f_j(c_j z) \), \( a_{jk} \) replaced by \( a_{jk}/c_j \), and \( l_j \) by \( m_j \), which contradicts the fact that \( l_j \) is minimal.

We may assume that \( b_1 = b_2 \). Indeed, if \( |b_1| > |b_2| \), we choose \( t \in (0, 1) \) with \( t^{-\mu_2}|b_2| = |b_1| \). Then \( |a_{2k}| < t \), for

\[
|a_{2k}|^{\mu_2} \leq |c_2|^{\mu_2} \leq |b_2| e^{-\beta} < |b_2/b_1| = t^{\mu_2}.
\]
Replacing $f_2$ by $z \mapsto f_2(tz)$ and $a_{2k}$ by $a_{2k}/t$, we get $|b_1| = |b_2|$. Finally, replacing $f_2$ by $z \mapsto f_2(e^{i\theta}z)$, where $e^{i\theta} = b_2/b_1$ and replacing $a_{2k}$ by $e^{-i\theta}a_{2k}$, we get $b_1 = b_2$.

We let $C$ denote the set of all critical values of $B_1$. We have $B_1(0) = B_2(0)$, so we can take $\varphi_2 : \mathbb{D} \to \mathbb{D} \setminus B_2^{-1}(C)$ as the universal covering map with $\varphi(0) = 0$. A theorem of Frostman, see [14], p. 27, states that an inner function on $\mathbb{D}$ omitting 0 as a non-tangential boundary value is a Blaschke product. It is easy to show, see [14], p. 272, that since $0 \notin B_2^{-1}(C)$, $\varphi_2$ satisfies the assumption in Frostman’s theorem and is thus a Blaschke product. The restriction of $B_1$ to $\mathbb{D} \setminus B_1^{-1}(C)$ is a finite covering over $\mathbb{D} \setminus C$, so by lifting $B_2 \circ \varphi_2$ we conclude that there exists a function $\varphi_1 : \mathbb{D} \to B_1^{-1}(C)$ with $\varphi_1(0) = 0$ and $B_1 \circ \varphi_1 = B_2 \circ \varphi_2$ and Frostman’s theorem implies again that $\varphi_1$ is a Blaschke product. Since $|B_j \circ \varphi_j| = 1$ almost everywhere on $\mathbb{T}$ and $B_j(0) = b_j$, we can choose $r \in (0, 1)$ such that

$$\log |B_j \circ \varphi_j(0)| - \frac{1}{2\pi} \int_0^{2\pi} \log |B_j \circ \varphi_j(re^{i\theta})| d\theta < \alpha.$$  

We set $\sigma(z) = B_1 \circ \varphi_1(rz) = B_2 \circ \varphi_2(rz)$. By the Poisson–Jensen representation formula, the left hand side of this inequality equals $\sum_{i=1}^{n} \nu_i \log |z_i|$, where $z_i$ are the zeros of $\sigma$ in $\mathbb{D}$ with multiplicities $\nu_i$ for $i = 1, \ldots, n$.

We define $g_j \in \mathcal{O}(\overline{\mathbb{D}}, X_j)$ by $g_j(z) = f_j \circ \varphi_j(rz)$ and $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ with $f(0) = (x_1, x_2)$ by $f = (g_1, g_2)$. If $\sigma(z_i) = 0$, then $\varphi_j(rz_i) = a_{jk_j}$ for some $k_j$, and

$$\nu_i = m_{z_i}(\sigma) = \mu_{jk_j} m_{z_i}(a_{jk_j} - \varphi_j(r\cdot)) = \tilde{\nu}_{f_j} A_j(a_{jk_j} m_{z_i}(a_{jk_j} - \varphi_j(r\cdot))) = \tilde{\nu}_{g_j} A_j(z_i)$$

Since the left hand side of this equation is independent of $j$, we get

$$\tilde{\nu}_{f_j} A(z_i) = \min_j \{ \tilde{\nu}_{g_j} A_j(z_i) \} = \nu_i.$$  

Hence

$$G_A(x) \leq G_{f^* A}(0) = \sum_{a \in \mathbb{D}} \tilde{\nu}_{f^* A}(a) \log |a| \leq \sum_{i=1}^{n} \nu_i \log |z_i| = \sum_{i=1}^{n} \nu_i \log |z_i| < \alpha.$$  

\[\Box\]

8 Examples

Example 8.1 Let $X$ be the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, $1 \leq p \leq n$, and let $A$ be generated by $\psi_k(z) = z_k^{\nu_k}$ for $1 \leq k \leq p$ and positive integers $\nu_k$. Then the product property gives

$$G_A(z) = \max_{1 \leq k \leq p} \nu_k \log |z_k|.$$  

Furthermore, we have

$$(dd^c G_A)^p = \nu_1 \cdots \nu_p ||A||.$$  

Example 8.2 Let $X = \mathbb{D}^n$, $n \geq 2$, and let $A$ be generated by $\psi_1(z) = z_1^2$, $\psi_2(z) = z_1 z_2$. Then

$$G_A(z) = \nu(z) := \log |z_1| + \max \{ \log |z_1|, \log |z_2| \}, \quad z = (z_1, z_2, z^\prime) \in \mathbb{D}^n.$$  

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First we take any $z \in \mathbb{D}^n \setminus \{z_1 = 0\}$ with $|z_1| \geq |z_2|$ and consider the disc

$$f(\zeta) = \left( \zeta, \frac{z_2}{z_1}, \zeta'' \right), \quad \zeta \in \mathbb{D}.$$ 

Then for any $u \in \mathcal{F}_A$ we have $f^* u(\zeta) \leq \log |f^* \psi(\zeta)| + C = 2 \log |\zeta| + C$ and so, since $u \leq 0$, $f^* u(\zeta) \leq 2 \log |\zeta| = f^* v(\zeta)$. As $f(z_1) = z_1$, this gives us $u(z) \leq v(z)$.

For $z \in \mathbb{D}^n \setminus \{z_1 = 0\}$ with $|z_1| < |z_2|$, we take the disc

$$g(\zeta) = \left( \frac{z_1}{z_2}, \zeta, \zeta'' \right), \quad \zeta \in \mathbb{D}.$$ 

Then for any $u \in \mathcal{F}_A$ we have again $g^* u(\zeta) \leq 2 \log |\zeta| + C$ near the origin and, since $u(z) \leq \log |z_1|$ (which is the Green function for the polydisc with the poles along the space $z_1 = 0$), $g^* u(\zeta) \leq \log |z_1/z_2|$ near $\partial \mathbb{D}$. Therefore, $g^* u(\zeta) \leq 2 \log |\zeta| + \log |z_1/z_2| = g^* v(\zeta)$ everywhere in $\mathbb{D}$. Since $g(z_2) = z_1$, this shows $u(z) \leq v(z)$ at all such $z$ as well.

Note that $f$ is, up to a Möbius transformation, an extremal disc for the disc functional $f \mapsto G_{f^* A}(0)$, while $g$ is not. Note also that we have $dd^c G_A = [z_1 = 0] + Q$, where the current $Q = dd^c \max \{ \log |z_1|, \log |z_2| \}$ has the property $Q^2 = [z_1 = z_2 = 0]$.

**Example 8.3** Consider the variety $|A| = \{z_1 = z_2 = 0\} \cup \{z_2 = z_3 = 0\} \cup \{z_1 = z_3 = 0\}$ in the unit polydisk $\mathbb{D}^3$ of $\mathbb{C}^3$. It is easy to see that the corresponding reduced complex space $A$ is generated by $\psi(z) = (z_1 z_2, z_2 z_3, z_1 z_3)$ and that $|A|$ has the decomposition (in the sense of Prop. 5.3) $|A| = J^2 \cup J^3$ with $J^3 = \{0\}$. We claim that

$$G_A(z) = v(z) := \max \{ \log |z_1 z_2|, \log |z_2 z_3|, \log |z_1 z_3| \}.$$ 

It suffices to check the relation $u(z) \leq v(z)$ for any function $u \in \mathcal{F}_A$ and each point $z \in \mathbb{D}^3$ with $|z_1| \geq |z_2| \geq |z_3|, |z_2| \neq 0$. We take first any $z$ with $|z_1| = |z_2| \geq |z_3|$ and consider the disc $f(\zeta) = \zeta z_1/z_2, \zeta \in \mathbb{D}$. Then $f^* u \in SH^-(\mathbb{D})$ and, since $u \leq \log |\psi + C_1$ near the origin, $f^* u(\zeta) \leq \log |f^* \psi(\zeta)| + C_1 = 2 \log |\zeta| + C_2$ when $|\zeta| \leq \epsilon$. Therefore, $f^* u(\zeta) \leq 2 \log |\zeta| = f^* v(\zeta)$ and, in particular, $u(z) = f^* u(|z_1|) \leq f^* v(|z_1|) = v(z)$. The disc $f$ is, up to a Möbius transformation, an extremal disc for the disc functional $f \mapsto G_{f^* A}(0)$ at such a point $z$.

Now we can take any $z$ with $|z_1| > |z_2| \geq |z_3|, |z_2| \neq 0$, and consider the analytic disc $g(\zeta) = (z_1, \zeta z_2, \zeta z_3), \zeta \in D_R$ with $R = |z_1|/|z_2| > 1$. We have $|g_1(\zeta)| = |g_2(\zeta)| \geq |g_3(\zeta)|$ when $|\zeta| = R$ and thus $g^* u \leq g^* v$ on $\partial D_R$. Furthermore, $g^* u(\zeta) \leq \log |g^* \psi(\zeta)| + C_3 \leq \log |\zeta| + C_4$ near the origin. Since $g^* v(\zeta) = \log \log |\zeta z_2|$, this shows that $g^* u \leq g^* v$ on $D_R$. Hence we get $u(z) = g^* u(1) \leq f^* v(1) = v(z)$, which proves the claim.

The current $(dd^c G_A)^2$ has Lelong numbers equal 1 at each point $a \in J^2 = |A| \setminus \{0\}$. The point 0 is exceptional: the Lelong number $\nu(dd^c G_A, 0) = 2$, so $\nu((dd^c G_A)^2, 0) \geq 4$, while $\nu(|A|, 0) = 3$.

**Example 8.4** Let $X$ be the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n, 1 \leq p \leq n$, and let $A$ be generated by $\psi_k(z) = z_k$ for all $1 \leq k \leq p$. In the notation $z = (z', z'')$ with $z' \in \mathbb{C}^p$ and $z'' \in \mathbb{C}^{n-p}$, the Green function

$$G_A(z) = \log \frac{|z'|}{\sqrt{1 - |z''|^2}},$$ 

because its restriction to every plane $z'' = c \in \mathbb{B}_{n-p}$ is the pluricomplex Green function for the ball of radius $\sqrt{1 - |c|^2}$ in $\mathbb{C}^p$ with simple pole at the origin.
Example 8.5 The Green function $G_A$ for the unit ball $B_n$ in $\mathbb{C}^n$, $n \geq 2$, with respect to $A$ generated by $(\psi_1, \psi_2) = (z_1^2, z_2)$ is given by

$$G_A(z) = \frac{1}{2} \log \left( \frac{|z_1|^4}{(1-|z''|^2)^2} + \frac{2|z_2|^2}{1-|z''|^2} + \frac{|z_1|^2}{1-|z''|^2} \sqrt{\frac{|z_1|^4}{(1-|z''|^2)^2} + \frac{4|z_2|^2}{1-|z''|^2}} - \frac{1}{2} \log 2, \right)$$

for $z = (z_1, z_2, z'') \in B_n$ (compare with the formula for the pluricomplex Green function with two poles in the ball [3]).

For proving this we let $v$ denote the function defined by the right hand side. Then $v \in \text{PSH}(B_n) \cap C(\partial B_n \setminus |A|)$ and satisfies $v(z) \leq \max\{\log |z_1|^2, \log |z_2|\} + C$ locally near $|A| = \{z_1 = z_2 = 0\}$.

Let us show that its boundary values on $\partial B_n \setminus |A|$ are zero. Take any $z \in \partial B_n \setminus |A|$, then $|z_1|^2 = a$, $|z_2|^2 = b$, $|z''|^2 = 1 - a - b$ with $a, b \geq 0$, $0 < a + b \leq 1$. We get

$$v(z) = \frac{1}{2} \log \left[ \frac{a^2}{(a+b)^2} + \frac{2b}{a+b} + \frac{a}{a+b} \left( 2 - \frac{a}{a+b} \right) \right] - \frac{1}{2} \log 2 = 0.$$  

Finally we show that $v(z) \geq G_A(z)$ for almost all $z \in B_n$ (which implies $v \equiv G_A$). Take any $z \in B_n$ with $z_1 \neq 0$ and consider the analytic curve

$$f(\zeta) = (\zeta, \frac{z_2}{z_1^2}, z'').$$

Note that $f(z_1) = z$. We have $f^*v(\zeta) = 2 \log(|\zeta|/R(z))$, while $f^*G_A$ is a negative subharmonic function in the disc $|\zeta| < R(z)$ with the singularity $2 \log |\zeta|$. So $f^*G_A \leq f^*v$ and, in particular, $G_A(z) = f^*G_A(z_1) \leq f^*v(z_1) = v(z)$.

This shows also that $f$ is, up to a Möbius transformation, an extremal disc for the disc functional $f \mapsto G_{f^*A}(0)$ at $z$ with $z_1 \neq 0$. A corresponding extremal curve for $z = (0, z_2, z'')$ is $f(\zeta) = (0, \zeta, z'')$.

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Alexander Rashkovskii  
Tek/nat, Høgskolen i Stavanger, POB 8002, 4068 Stavanger, Norway  
E-mail: alexander.rashkovskii@his.no

Ragnar Sigurdsson  
Science Institute, University of Iceland, Dunhaga 3, IS-107 Reykjavik, Iceland  
E-mail: ragnar@hi.is