A general method for stability controllability in the theory of fractional-order differential systems

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Abstract. The main purpose of this paper is to present a general method for the controllability of the stability of a system of fractional-order differential equations around its equilibrium states. This method is applied to analyze and control the fractional stability of the fractional 2-dimensional fractional Toda lattice with one linear control.

1 Introduction

The theory of fractional differential equations (i.e. fractional calculus) and its applications are based on non-integer order of derivatives and integrals [31, 27, 3, 37]. The fractional calculus has deep and natural connections with many fields of science and engineering.

In the last three decades, one increasing attention has been paid to the study of the dynamic behaviors (in particular, the chaotic behavior) of some classical differential systems, as well as some fractional-order differential systems. For example, the fractional models played an important role in applied mathematics [13, 11, 23], mathematical physics [10, 33, 19, 28, 22], applied physics [11, 37, 21], study of biological systems [2, 32, 26], chaos synchronization, secure communications [38, 4, 17, 30] and so on.

The Lie groups, Lie algebroids and Leibniz algebroids have proven to be powerful tools for geometric formulation of the Hamiltonian mechanics [7, 14, 15]. Also, they have been used in the investigation of many fractional dynamical systems [10, 16, 25].

This paper is structured as follows. The Section 2 is devoted to the exposition of the controllability method of fractional stability at an equilibrium state of a given fractional system. This method consists in associating a given fractional-order system with a new system of fractional-order differential equations, called the controlled fractional system around of an equilibrium point. In Section 3 we investigate the fractional differential systems associated to 2-dimensional Toda lattice with one linear control (3.4) in terms of fractional Caputo derivatives. For this fractional-order model we investigate the existence and uniqueness of solution of initial value problem and asymptotic stability of its equilibrium states. In Section 4, for the asymptotic stabilization of fractional model (3.4), we associate the controlled fractional-order system with controls \( c_1, c_2 \) at the equilibrium point \( x_e \), denoted by (4.4). In Proposition (4.1) are established sufficient conditions on parameters \( k, c_1, c_2 \) to control the chaos in the fractional-order system (4.4).

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2 Stability analysis and controllability of a fractional-order differential system

We recall the Caputo definition of fractional derivatives, which is often used in concrete applications. Let \( f \in C^\infty(\mathbb{R}) \) and \( q \in \mathbb{R}, q > 0 \). The \( q \)-order Caputo differential operator \( \mathbb{S} \), is described by \( D^q_0 f(t) = \Gamma(q) \int_0^t (t - s)^{q-1} f(s)ds, \ q > 0 \), where \( f^{(m)}(t) \) represents the \( m \)-order derivative of the function \( f, \ m \in \mathbb{N}^* \) is an integer such that \( m - 1 \leq q \leq m \) and \( \Gamma \) is the \( q \)-order Riemann-Liouville integral operator, which is expressed by \( D^q_0 f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s)ds, \ q > 0 \), where \( \Gamma \) is the Euler Gamma function. If \( q = 1 \), then \( D^1_0 f(t) = df/dt \).

In this paper we suppose that \( q \in (0, 1] \).

We consider the following system of fractional-order differential equations on \( \mathbb{R}^n \):

\[
D^q_0 x^i(t) = f_i(x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)), \quad i = 1, n, \tag{2.1}
\]

where \( q \in (0, 1), f_i \in C^\infty(\mathbb{R}^n, \mathbb{R}) \), \( D^q_0 x^i(t) \) is the Caputo fractional derivative of order \( q \) for \( i = 1, n \) and \( t \in [0, \tau) \) is the time.

The fractional dynamical system (2.1) can be written as follows:

\[
D^q_0 x(t) = f(x(t)), \tag{2.2}
\]

where \( f(x(t)) = (f_1(x^{(1)}(t), \ldots, x^{(n)}(t)), f_2(x^{(1)}(t), \ldots, x^{(n)}(t)), \ldots, f_n(x^{(1)}(t), \ldots, x^{(n)}(t)))^T \)

and \( D^q_0 x(t) = (D^q_0 x^{(1)}(t), \ldots, D^q_0 x^{(n)}(t))^T \).

A point \( x_e = (x_1^e, x_2^e, \ldots, x_n^e) \in \mathbb{R}^n \) is said to be equilibrium state of the fractional differential system (2.2), if \( D^q_0 x^i(t) = 0 \) for \( i = 1, n \). Its equilibrium states are determined by solving the set of equations: \( f_i(x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)) = 0, \ i = 1, n. \)

The Jacobian matrix associated to (2.2) is \( J(x) = (\frac{\partial f_i}{\partial x^j}), \quad i, j = 1, n. \)

The stability of the fractional system (2.2) has been studied by Matignon in [29], where necessary and sufficient conditions have been established.

**Proposition 2.1** ([29]) Let \( x_e \) be an equilibrium state of fractional differential system (2.2) and \( J(x_e) \) be the Jacobian matrix \( J(x) \) evaluated at \( x_e \).

(i) \( x_e \) is locally asymptotically stable, if and only if all eigenvalues \( \lambda(J(x_e)) \) of \( J(x_e) \) satisfy:

\[
|arg(\lambda(J(x_e)))| > \frac{q\pi}{2}. \tag{2.3}
\]

(ii) \( x_e \) is locally stable, if and only if either it is asymptotically stable, or the critical eigenvalues satisfying \( |arg(\lambda(J(x_e)))| = \frac{q\pi}{2} \) have geometric multiplicity one. \( \square \)

Using the notation: \( \bar{q} := \frac{2}{\pi}|arg(\lambda(J(x_e)))| \) and applying Proposition (2.1) one obtains the following corollary.

**Corollary 2.1** ([30]) (i) The equilibrium state \( x_e \) of the fractional model (2.2) is asymptotically stable if and only if the difference \( q - \bar{q} \) is strictly negative. More precisely, \( x_e \) is asymptotically stable for all \( q \in (0, \bar{q}) \).
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(ii) If $q - \tilde{q} > 0$, then $x_e$ is unstable and the fractional model (2.2) may exhibit chaotic behavior. More precisely, $x_e$ is unstable $(\forall)q \in (\tilde{q}, 1)$. □

Corollary 2.2 Let $x_e$ be an equilibrium state of the fractional model (2.2) and $\lambda_i, i = 1, n$ the eigenvalues of $J(x_e)$.

(i) If one of the eigenvalues $\lambda_i, i = 1, n$ is equal to zero or it is positive, then $x_e$ is unstable for all $q \in (0, 1)$.

(ii) If $\lambda_i < 0$, for all $i = 1, n$, then $x_e$ is asymptotically stable $(\forall) q \in (0, 1)$.

Proof. (i) We suppose $\lambda_1 \leq 0$. Then $\tilde{q} = 0$, since $\arg(\lambda_1) = 0$. We have $q - \tilde{q} = q > 0$ and applying Corollary 2.1(ii), it follows that $x_e$ is unstable $(\forall) q \in (0, 1)$.

(ii) Let $\lambda_i < 0, i = 1, n$. Then $|\arg(\lambda(J(x_e)))| = \pi$ and $\tilde{q} = 2$. We have $q - \tilde{q} < 0$. By Corollary 2.1(i), it follows that $x_e$ is asymptotically stable $(\forall) q \in (0, 1)$. □

In the case when $x_e$ is a unstable equilibrium state of the fractional differential system (2.2), we propose a simple method for to control the stability of fractional model (2.2) around its equilibrium point $x_e$. This method consists in associating the fractional-order system (2.2) with a new sistem of fractional-order differential equations, determined by (2.2) and a control function $u(x(t)) \in C^\infty(\mathbb{R}^n, \mathbb{R})$. To apply this method, the following four steps must be performed:

Step 1. We associate to (2.2) a new fractional-order differential system defined by

$$D_t^q x(t) = f(x(t)) + u(x(t)), \quad (2.4)$$

where $u(x(t)) = (u_1(x^1(t), \ldots, x^n(t)), \ldots, x^n(t))^T$ is a control function.

Step 2. We choose the function control $u(x(t))$ such that $u(x_e) = 0$.

Step 3. A good option for choosing the control function $u(t)$ that satisfies the relation $u(x_e) = 0$ is the following

$$u_i(t) = c_i(x^i(t) - x^i_e), \quad i = 1, n \quad (2.5)$$

where $c_i \in \mathbb{R}$ are control parameters.

With the control function $u(t)$ given by (2.6), the fractional system (2.4) becomes

$$D_t^q x^i(t) = f_i(x(t)) + c_i(x^i(t) - x^i_e), \quad i = 1, n, \quad q \in (0, 1) \quad (2.6)$$

where $c_i \in \mathbb{R}, i = 1, n$.

The fractional model (2.5) is called the controlled fractional system associated to fractional system (2.2) at equilibrium point $x_e$.

Step 4. For to analyse the fractional stability of the controlled fractional system (2.6) we apply the Matignon’s test.

This above method will be called the controllability method of fractional stability at an equilibrium state of a given fractional system.

It is easy to see that (2.2) and (2.6) have $x_e$ a common equilibrium state.

Remark 2.1 (i) This method was applied to analyse and control the fractional stability type of steady states for the fractional differential equations 3D Maxwell-Bloch type in [12].
(ii) Other fractional modeling of classical dynamical systems in which this method is applied to control their fractional stability have been discussed in [19, 20, 23]. □

If one selects the appropriate parameters \( c_i, i = 1, \ldots, n \) which then make the eigenvalues of the linearized equation of (2.6) satisfy one of the conditions from Proposition 2.1, then the trajectories of (2.6) asymptotically approaches the unstable equilibrium state \( x_e \) in the sense that \( \lim_{t \to \infty} \| x(t) - x_e \| = 0 \), where \( \| \cdot \| \) is the Euclidean norm. In the case when the equilibrium state \( x_e \) is unstable, then fractional model (2.6) may exhibit chaotic behavior.

3 Stability analysis of the 2–dimensional fractional-order Toda lattice with one linear control

The Toda-type systems [5] are described by the following equations on \( \mathbb{R}^{2n-1} \):

\[
\dot{x}^i(t) = x^i(t)(y^{i+1}(t) - y^i(t)), \quad \dot{y}^j(t) = 2[(x^j)^2(t) - (x^{j-1})^2(t)],
\]

where \( x^0(t) = x^n(t) = 0, \ x^i, i = 1, \ldots, n-1, \ y^j, j = 1, \ldots, n \) are state variables, \( \dot{x}^i(t) = dx^i(t)/dt, \ \dot{y}^j(t) = dy^j(t)/dt \) and \( t \) is the time. The system (3.1) is called the \( n \)-dimensional Toda lattice.

The \( n \)-dimensional fractional-order Toda lattice associated to dynamics (3.1) is defined by the following set of fractional differential equations:

\[
\begin{align*}
D_q^t x^i(t) & = x^i(t)(y^{i+1}(t) - y^i(t)), \quad i = 1, \ldots, n-1 \\
D_q^t y^j(t) & = 2[(x^j)^2(t) - (x^{j-1})^2(t)], \quad j = 1, \ldots, n \quad q \in (0,1), \\
x^0(t) & = 0, \quad x^n(t) = 0.
\end{align*}
\]

In this section we investigate the \( n \)-dimensional fractional-order Toda lattice for \( n = 2 \) with one linear control about \( Oy^2 \)-axis. This fractional model is described by:

\[
D_q^t x^1 = x^1(-y^1 + y^2), \quad D_q^t y^1 = 2(x^1)^2, \quad D_q^t y^2 = -2(x^1)^2 - ky^2,
\]

where \( k \in \mathbb{R}^* \) is a control parameter.

Using the transformations \( x^1 = x^1, \ y^1 = x^2, \ y^2 = x^3 \), the system (3.3) becomes:

\[
\begin{align*}
D_q^t x^1(t) & = x^1(-x^2(t) + x^3(t)), \\
D_q^t x^2(t) & = 2(x^1(t))^2, \quad q \in (0,1), \\
D_q^t x^3(t) & = -2(x^1(t))^2 - kx^3(t).
\end{align*}
\]

The initial value problem of the fractional system (3.4) can be represented in the following matrix form:

\[
D_q^t x(t) = x^1(t)Ax(t) + x^3(t)Bx(t), \quad x(0) = x_0,
\]

where \( 0 < q < 1, \ x(t) = (x^1(t), x^2(t), x^3(t))^T, \quad t \in (0, \tau) \) and

\[
A = \begin{pmatrix}
0 & -1 & 1 \\
2 & 0 & 0 \\
-2 & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -k
\end{pmatrix}.
\]
**Proposition 3.1** The initial value problem of the 2-dimensional fractional-order Toda lattice with one control (3.4) has a unique solution.

**Proof.** Let \( f(x(t)) = x^1(t)Ax(t) + x^3(t)Bx(t) \). It is obviously continuous and bounded on \( D = \{ x \in \mathbb{R}^3 \mid x^i \in [x^i_0 - \delta, x^i_0 + \delta], i = 1, 3 \} \) for any \( \delta > 0 \). We have \( f(x(t)) - f(y(t)) = x^1(t)Ax(t) - y^1(t)Ay(t) + x^3(t)Bx(t) - y^3(t)By(t) = g(t) + h(t) \), where \( g(t) = x^1(t)Ax(t) - y^1(t)Ay(t) \) and \( h(t) = x^3(t)Bx(t) - y^3(t)By(t) \). Then

(a) \( |f(x(t)) - f(y(t))| \leq |g(t)| + |h(t)| \).

Using reasoning analogous to that in the proof of the Proposition 2.1 in [19], we can show that:

(b) \( |g(t)| \leq (\|A\| + |y^1(t)|) \cdot |x(t) - y(t)| \) and \( |h(t)| \leq (\|B\| + |y^3(t)|) \cdot |x(t) - y(t)| \).

According to (b) the relation (a) becomes

(c) \( |f(x(t)) - f(y(t))| \leq (\|A\| + \|B\| + |y^1(t)| + |y^3(t)|) \cdot |x(t) - y(t)| \).

Replacing \( \|A\| = \sqrt{10}, \|B\| = |k| \) and using the inequalities \( |y^1(t)| \leq |x_0| + \delta, i = 1, 3 \) from the relation (c), we deduce that

(d) \( |f(x(t)) - f(y(t))| \leq L \cdot |x(t) - y(t)| \), where \( L = \sqrt{10} + |k| + 2(|x_0| + \delta) > 0 \).

The inequality (d) shows that \( f(x(t)) \) satisfies a Lipschitz condition. Based on the results of Theorems 1 and 2 in [9], we can conclude that the initial value problem of the system (3.3) has a unique solution. \( \square \)

For the fractional system (3.4) we introduce the following notations:

\[
\begin{align*}
f_1(x) &= -x^1x^2 + x^3^3, \\
f_2(x) &= 2(x^1)^2, \\
f_3(x) &= -2(x^1)^2 - kx^3.
\end{align*}
\]  

**Proposition 3.2** The equilibrium states of the 2-dimensional fractional-order Toda lattice (3.4) are given as the following family:

\[
E := \{ e_m = (0, m, 0) \in \mathbb{R}^3 \mid m \in \mathbb{R} \}.
\]

**Proof.** The equilibrium states are solutions of the equations \( f_i(x) = 0, i = 1, 3 \) where \( f_i, i = 1, 3 \) are given by (3.6).

Let us we present the study of asymptotic stability of equilibrium states for the fractional system (3.4). Finally, we will discuss how to stabilize the unstable equilibrium states of the system (3.4) via fractional order derivative. For this study we apply the Matignon’s test.

The Jacobian matrix associated to system (3.4) is:

\[
J(x, k) = \begin{pmatrix}
-x^2 + x^3 & -x^1 & x^1 \\
4x^3 & 0 & 0 \\
-4x^1 & 0 & -k
\end{pmatrix}.
\]

**Proposition 3.3** The equilibrium states \( e_m \in E \) are unstable \((\forall)q \in (0, 1)\).
Proof. The characteristic polynomial of the matrix $J(e_m, k) = \begin{pmatrix} -m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -k \end{pmatrix}$ is

$$p_{J(e_m, k)}(\lambda) = \det(J(e_m, k) - \lambda I) = -\lambda(\lambda + m)(\lambda + k).$$

The equation $p_{J(e_m, k)}(\lambda) = 0$ has the root $\lambda_1 = 0$. By Corollary 2.2(i), follows that $e_m, m \in \mathbb{R}$ are unstable for all $q \in (0, 1)$.

4 Controllability of chaotic behavior of the fractional model (3.4)

For the controllability of chaotic behavior of the 2-dimensional Toda lattice with one linear control, we apply the controllability method of fractional stability of the fractional model (3.4) at an equilibrium point.

Let $x_e$ be an unstable equilibrium state. We associate to (3.4) a new fractional-order system with (external) controls and given by:

$$\begin{cases} D_t^q x^1(t) = x^1(t)(-x^2(t) + x^3(t)) + u_1(t), \\ D_t^q x^2(t) = 2(x^1(t))^2 + u_2(t), \\ D_t^q x^3(t) = -2(x^1(t))^2 - kx^3(t) + u_3(t), \end{cases} \quad q \in (0, 1), \quad (4.1)$$

where $u_i(t), i = 1, 3$ are control functions.

In this section we take the control functions $u_i(t), i = 1, 3$, given by:

$$u_1(t) = c_1(x^1(t) - x_e^1), \quad u_2(t) = c_2(x^2(t) - x_e^2), \quad u_3(t) = 0, \quad c_1, c_2 \in \mathbb{R}^*. \quad (4.2)$$

With the control functions (4.2), the system (4.1) becomes:

$$\begin{cases} D_t^q x^1(t) = x^1(t)(-x^2(t) + x^3(t)) + c_1(x^1(t) - x_e^1), \\ D_t^q x^2(t) = 2(x^1(t))^2 + c_2(x^2(t) - x_e^2), \\ D_t^q x^3(t) = -2(x^1(t))^2 - kx^3(t), \end{cases} \quad q \in (0, 1), \quad (4.3)$$

where $k, c_1, c_2 \in \mathbb{R}^*$ are control parameters.

The fractional system (4.3) is called the controlled fractional-order system associated to (3.4) at $x_e$.

The controlled fractional-order system associated to (3.4) at $x_e = e_m$, is written:

$$\begin{cases} D_t^q x^1(t) = x^1(t)(-x^2(t) + x^3(t)) + c_1 x^1(t), \\ D_t^q x^2(t) = 2(x^1(t))^2 + c_2(x^2(t) - m), \\ D_t^q x^3(t) = -2(x^1(t))^2 - kx^3(t), \end{cases} \quad q \in (0, 1), \quad (4.4)$$

where $k, c_1, c_2 \in \mathbb{R}^*$ are control parameters and $m \in \mathbb{R}$.

The Jacobian matrix of the fractional model (4.4) is

$$J(x, k, c_1, c_2) = \begin{pmatrix} -x^2 + x^3 + c_1 & -x^1 & x^1 \\ 4x^1 & c_2 & 0 \\ -4x^1 & 0 & -k \end{pmatrix}.$$
**Proposition 4.1** Let be the fractional system (4.4) and \( e_m = (0, m, 0) \in E. 

1. If \( k > 0 \).
   (i) If \( c_2 < 0 \), then \( e_m \) is asymptotically stable for all \( m \in (c_1, \infty) \) and \( q \in (0, 1) \).
   (ii) If \( c_2 < 0 \), then \( e_m \) is unstable for all \( m \in (-\infty, c_1] \) and \( q \in (0, 1) \).
   (iii) If \( c_2 > 0 \), then \( e_m \) is unstable \( (\forall)m \in \mathbb{R} \) and \( q \in (0, 1) \).

2. If \( k < 0 \). If \( c_1, c_2 \in \mathbb{R}^+ \), then \( e_m \) is unstable for all \( m \in \mathbb{R} \) and \( q \in (0, 1) \).

**Proof.** The characteristic polynomial of matrix \( J(e_m, k, c_1, c_2) = \begin{pmatrix} -m + c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & -k \end{pmatrix} \) is \( p_{J(e_m, k, c_1, c_2)}(\lambda) = \det(J(e_m, k, c_1, c_2) - \lambda I) = -(\lambda + m - c_1)(\lambda - c_2)(\lambda + k) \). The roots of equation \( p_{J(e_m, k, c_1, c_2)}(\lambda) = 0 \) are \( \lambda_1 = c_1 - m, \lambda_2 = c_2, \lambda_3 = -k. \)

1. Case \( k > 0 \) and \( q \in (0, 1) \). Then \( \lambda_3 < 0 \).
   (i) We have \( \lambda_1 < 0 \) and \( \lambda_2 < 0 \) if and only if \( c_2 < 0 \) and \( m \in (c_1, \infty) \). It follows that \( \lambda_i < 0, i = 1, 3 \) and according to Corollary 2.2(ii), the equilibrium state \( e_m \) is asymptotically stable.
   (ii) We suppose \( c_2 < 0 \). Then \( \lambda_2 < 0 \). We have \( \lambda_1 > 0 \) if and only if \( m \in (-\infty, c_1] \). In this case, \( J(e_m, k, c_1, c_2) \) has a positive eigenvalue and by Corollary 2.2(i), it follows that \( e_m \) is unstable.
   (iii) We suppose \( c_2 > 0 \). In this case, one from the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) is positive. Since \( J(e_m, k, c_1, c_2) \) has at least a positive eigenvalue, it follows that \( e_m \) is unstable for all \( m \in \mathbb{R} \).

2. Case \( k < 0 \) and \( q \in (0, 1) \). Then \( \lambda_3 > 0 \). Since \( J(e_m, k, c_1, c_2) \) has at least a positive eigenvalue, it follows that \( e_m \) is unstable \( (\forall)m \in \mathbb{R} \). Hence, the assertions (1) and (2) hold.

**Example 4.1** Let be the 2-dimensional fractional-order Toda lattice with controls \( k, c_1, c_2 \) described by (4.4).

1. We select \( k = 0.4, c_1 = -0.02 \) and \( c_2 = -0.3 \). According to Corollary 2.2(ii), it follows that \( e_0 = (0, 0, 0) \) is asymptotically stable for all \( q \in (0, 1) \).
2. We consider \( k = 1, c_1 = 0.02 \) and \( c_2 = -0.3 \). Applying Corollary 2.2(i), it follows that \( e_0 = (0, 0, 0) \) is unstable for all \( q \in (0, 1) \). In other words, the fractional model (3.3) with \( k = 1, c_1 = 0.02 \) and \( c_2 = -0.3 \), behaves chaotically around the equilibrium point \( e_0 \).
3. We consider \( k = -0.25, c_1 = -0.02 \) and \( c_2 = -0.3 \). Applying Corollary 2.2(i), it follows that \( e_0 = (0, 0, 0) \) is unstable for all \( q \in (0, 1) \).

**Remark 4.1** Toda-type dynamic systems have been studied from various research directions by many authors. More specifically, from the point of view of Poisson geometry, Toda lattices were discussed in the papers \([5, 31, 35, 36] \), and as fractional-order differential systems they were investigated in \([24, 18] \).

**Conclusions.** This paper presents the 2-dimensional fractional-order Toda lattice with one linear control, denoted by (3.4). The asymptotic stability of equilibrium states of (3.4) was investigated. Sufficient conditions on the parameters \( k, c_1, c_2 \) in the fractional...
model (4.4) so that its equilibrium states are asymptotically stable have proved. By choosing the right parameter $k, c_1, c_2$ in (4.4), this work offers a series of chaotic fractional differential systems.

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