On the Bertini theorem in arbitrary characteristic

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Abstract We give a simple direct proof of the Kleiman Bertini theorem in arbitrary characteristic. We also give a simple proof of Serre splitting theorem.

Keywords Bertini theorem · Serre splitting theorem

Mathematics Subject Classification (1991) 14 A

1 Introduction

Kleiman in the paper [4] proved Bertini theorem in arbitrary characteristic (Corollary 12 in [4]). Kleiman deduced this theorem from his general transversality theorem: if $X$ is $G$-homogenous algebraic variety and $Z$, $W$ are smooth varieties over $X$, then the fiber product of $Z$ and a general translate $gY$ over $X$ is smooth. This theorem is valid in characteristic zero, however Kleiman shows that his result still works in positive characteristic, if we additionally assume that the action of the group $G$ is sufficiently good (this assumption is satisfied for the action of the group $PGL(n)$ on the projective space $\mathbb{P}^n$) and that $Z$, $W$ are unramified over $X$.

The aim of this note is to give a simple, direct proof of the Bertini Theorem in arbitrary characteristic. We also partially generalize our method to locally free sheafs of higher ranks (mainly for char $k = 0$). In particular we give a simple proof of the
fact that if a variety $X$ is smooth and $\mathcal{F}$ is a locally free sheaf on $X$, with sufficiently many sections, then a generic section of $\mathcal{F}$ is transversal to $X$. As a Corollary we give a proof of the Atiyah–Serre Splitting Theorem (compare with [6], [1] and [5]).

2 Notations and definitions

We assume for simplicity, that the base field $k$ is algebraically closed. Let $X$ be algebraic variety and $\mathcal{F}$ be a locally free sheaf on $X$ of rank $r$. Take a section $s \in \Gamma(X, \mathcal{F})$. We describe a scheme of zeroes of $s^{-1}(0)$ in the following way: locally we can assume that $X = U$ is an affine variety and $\mathcal{F} = \mathcal{O}_U$ is trivial. Hence $s = (s_1, \ldots, s_r)$ where $s_i \in \Gamma(U, \mathcal{O}_U) = k[U]$. Then $s^{-1}(0)$ is given by the ideal $(s_1, \ldots, s_r) \subset k[U]$. We say that the section $s$ is transversal to $X$ if either $s^{-1}(0)$ is smooth and it has dimension $\dim X - r$ or $s^{-1}(0) = \emptyset$.

Let $X, Y$ be smooth varieties and $f : X \to Y$ be a morphism. We say that $f$ is unramified, if $f$ separates infinitely near points of $X$, i.e., the mapping $d_x f : T_x X \to T_{f(x)} Y$ is a monomorphism for every closed point $x \in X$ (see [3], p. 15).

If $X$ is an affine variety we will denote by $k[X]$ the ring $\Gamma(X, \mathcal{O}_X)$. If $M$ is a $k[X]$ module then by $M^\sim$ we denote the sheafification of $M$—see [2], Definition on the page 110.

3 Main result

Theorem 3.1 (Bertini theorem in arbitrary characteristic) Let $X$ be a smooth algebraic variety of dimension $d$ and let $\mathcal{F}$ be an invertible sheaf on $X$. Assume that $\mathcal{F}$ is generated by global sections $s_1, \ldots, s_r \in \Gamma(X, \mathcal{F})$. Assume that the morphism $\Phi$ from $X$ to the projective space $\mathbb{P}^{r-1}$ given by $s_1, \ldots, s_r$ is unramified. Then there is a Zariski open non-empty subset $U \subset k^r$ such that for every $c = (c_1, \ldots, c_r) \in U$ the zero set of the section $s = \sum_{i=1}^r c_i s_i$ is smooth.

Proof Since $X$ is quasi-compact we can assume that $X$ is affine and the sheaf $\mathcal{F}$ is trivial. Hence we can identify $\mathcal{F}$ with $\mathcal{O}_X$ and now $s_i : X \to k$ are regular functions. Additionally we can assume that $s_r \equiv 1$. Indeed, since $s_1, \ldots, s_r$ generates $\mathcal{F}$ we have that open (affine!) subsets $U_i := X \setminus \{s_i = 0\}$ cover $X$. Consequently we can assume that $s_r \neq 0$ in $X$. Take $s'_i = s_i/s_r$, $i = 1, \ldots, r$. If we prove our theorem for $s'_i$, then we automatically prove it also for originals $s_i$. Indeed, note that for a fixed $c = (c_1, \ldots, c_r) \in k^r$ the ideals $(\sum_{i=1}^r c_i s_i)$ and $(\sum_{i=1}^r c_i s'_i)$ are equal in $k[X]$. Moreover, we can assume that on $X$ we have global local coordinates $x_1, \ldots, x_d$, i.e., the mapping $(x_1, \ldots, x_d) : X \to k^d$ is etale.

Let

$$V = \left\{(c, x) \in k^r \times X : c_r + \sum_{i=1}^{r-1} c_i s_i(x) = 0\right\}.$$

The variety $V$ is smooth. Indeed, let $(g_1, \ldots, g_r)$ be the set of generators of the ideal $I(V) \subset k[X][c_1, \ldots, c_r]$ and let $J(V)(c, x) = [\frac{\partial g_i}{\partial c_j}(c, x)]$, where $z =$

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(c_1, \ldots, c_r, x_1, \ldots, x_d) is a set of global local coordinates on k^r \times X. Let us note that a polynomial h = \sum_{i=1}^r c_i s_i(x) does belong to the ideal I(V). Now we see that rank J(V) \geq 1, because partial derivatives of h with respect to c_i form a matrix [s_i(x)]_{1 \leq i \leq r}, which has a rank 1. Hence dim T_{(c,x)} V \leq d + r - 1.

On the other hand dim V = d + r - 1 and consequently we have the equality dim T_{(c,x)} V = d + r - n = dim V.

Consider the projection:

\[ q : V \ni (c, x) \mapsto c \in k^r. \]

We show that for generic c \in k^r the mapping q is smooth on the set q^{-1}(U_c), where U_c is suitable neighborhood of c.

Indeed, let us compute the tangent space at (c, x). It is given by the equation

\[ dc_r + \sum_{i=1}^{r-1} s_i(x) dc_i + \sum_j \left( \sum_{i=1}^{r-1} c_i \frac{\partial s_i(x)}{\partial x_j} \right) dx_j. \]

Let us note that rank \left[ \frac{\partial s_i(x)}{\partial x_j} \right]_{1 \leq i \leq r-1, 1 \leq j \leq d} = d. Indeed the mapping \Phi : X \ni x \mapsto (s_1(x), \ldots, s_{r-1}(x)) \in k^{r-1} is unramified.

Put \( L_c := \{ x \in X : \sum_{i=1}^{r-1} c_i \frac{\partial s_i(x)}{\partial x_j} = 0, j = 1, \ldots, d \}. \) Let us note that the mapping q is not a submersion in a neighborhood of q^{-1}(c) a exactly if \( L_c \neq \emptyset. \) We show that the set \( S := \{ c \in k^r : L_c \neq \emptyset \} \) is a constructible subset of k^r of dimension less than r.

Indeed, let \( W = \{ (c', x) \in k^{r-1} \times X : \sum_{i=1}^{r-1} c_i \frac{\partial s_i(x)}{\partial x_j} = 0, j = 1, \ldots, d \}. \) Let \( \pi : W \ni (c', x) \mapsto x \in X. \) Let us note that the fiber of \( \pi \) is a linear subspace of k^{r-1} of dimension r - 1 - d (as a kernel of a suitable linear mapping). Hence dim W \leq r - 1.

Let \( W' = \{ (c, x) \in V : \sum_{i=1}^{r-1} c_i \frac{\partial s_i(x)}{\partial x_j} = 0, j = 1, \ldots, d \}. \) We have a surjective mapping

\[ s : W \ni (c', x) \mapsto \left( \left( c', -\sum_{i=1}^{r-1} c_i s_i(x) \right), x \right) \in W'. \]

Since \( S = \rho(W'), \) where \( \rho : W' \ni (c', x) \mapsto c \in k^r \) and dim W = dim s(W) = dim W' we have dim cl(S) \leq r - 1.

This means that the projection \( q : V \ni x \mapsto k^r \) is a submersion outside a proper algebraic subset \( \overline{S} \subset k^r. \) In particular the zero set of a generic section \( s = \sum_{i=1}^r c_i s_i \) is smooth.

Using a similar method we can generalize this result to higher dimension. As a Corollary we obtain Atiyah-Serre Spliting Theorem.

**Theorem 3.2** Let \( X \) be an algebraic variety of dimension d and let \( \mathcal{F} \) be a locally-free sheaf on X of rank n. Assume that \( \mathcal{F} \) is generated by global sections \( s_1, \ldots, s_r \in \Gamma(X, \mathcal{F}). \) Then there is a Zariski open non-empty subset \( U \subset k^r \) such that for every
c = (c_1, \ldots, c_r) \in U the zero set of the section \( s = \sum_{i=1}^{r} c_i s_i \) is either empty or it has dimension \( d - n \). Moreover, if \( \text{char } k = 0 \) and \( X \) is smooth, then a generic section \( s = \sum_{i=1}^{r} c_i s_i \) is transversal to \( X \).

**Proof** Since \( X \) is quasi-compact we can assume that \( X \) is affine and the sheaf \( \mathcal{F} \) is trivial, i.e., we can identify a global local coordinates \( x_1, \ldots, x_d \). Hence \( s_i = (s_{i1}, \ldots, s_{in}) : X \to k^n \) are regular mappings. Let

\[
V = \left\{ (c, x) \in k^r \times X : \sum_{i=1}^{r} c_i s_i(x) = 0 \right\}.
\]

Let us note that \( \dim V = r + d - n \), where \( d = \dim X \). Indeed, we have a surjection \( \pi : V \ni (c, x) \to x \in X \). Any fiber of \( \pi \) is a linear subspace of \( k^r \) of dimension \( r - n \); it is a kernel of surjective linear mapping \( F_x : k^r \ni c \to \sum_{i=1}^{r} c_i s_i(x) \in E^n_x \), where \( E^n = X \times k^n \) is a trivial vector bundle of rank \( n \). Consequently \( \dim V = \dim X + r - n = d + r - n \). Now consider the second projection:

\[
q : V \ni (c, x) \to c \in k^r.
\]

If it is dominated then the generic fiber has dimension \( \dim V - \dim k^r = d + r - n - r = d - n \), otherwise the generic fiber is empty.

Moreover, if \( X \) is smooth over \( k \), then \( V \) is smooth. Indeed, let \( (g_1, \ldots, g_r) \) be the set of generators of the ideal \( I(V) \subset k[X][c_1, \ldots, c_r] \) and let \( J(V)(c, x) = [\frac{\partial g_i}{\partial x_j}(c, x)] \), where \( z = (c_1, \ldots, c_r, x_1, \ldots, x_d) \) is a set of global local coordinates on \( k^r \times X \). Let us note that polynomials \( h_j = \sum_{i=1}^{r} c_i s_{ij}(x) \) does belong to the ideal \( I(V) \). Now we see that rank \( J(V) \geq n \), because partial derivatives of \( h_j \) with respect to \( c_i \) form a matrix \( [s_{ij}(x)]_{1 \leq i \leq r, 1 \leq j \leq n} \), which has a rank \( n \). Hence \( \dim T_{(c, x)} V \leq d - r - n \). On the other hand \( \dim V = d + r - n \) and consequently we have the equality \( \dim T_{(c, x)} V = d + r - n = \dim V \). If additionally \( \text{char } k = 0 \), then the generic fiber is also smooth (generic smoothness—see [2], Corollary 10.7, p. 272).

**Corollary 3.3** (Atiyah–Serre Splitting Theorem) *Let \( X \) be an algebraic variety of dimension \( d \) and let \( \mathcal{F} \) be a locally free sheaf on \( X \) of rank \( n \). Assume that \( \mathcal{F} \) is generated by global sections \( s_1, \ldots, s_r \in \Gamma(X, \mathcal{F}) \). If \( n > d \) then \( \mathcal{F} \) contains a trivial subsheaf \( \mathcal{A} \subset \mathcal{F} \) of rank \( n - d \).*

**Proof** Indeed, by induction we can find \( n - d \) linearly independent sections \( t_j = \sum_{i=1}^{r} c_{ji} s_i \) (note that the quotient sheaf of \( \mathcal{F} \) is generated by the same sections (or rather their classes) as \( \mathcal{F} \)).

In particular if we assume that \( X \) is affine we have the Serre result in a classical form:

**Corollary 3.4** (Serre splitting theorem) *Let \( X \) be an affine algebraic variety of dimension \( d \) and let \( \mathcal{F} \) be a locally free sheaf on \( X \) of rank \( n \). If \( n > d \) then \( \mathcal{F} \) contains a locally free subsheaf \( \mathcal{F}' \) of rank \( d \) such that

\[
\mathcal{F} = \mathcal{F}' \oplus \mathcal{O}_X^{n-d}.
\]
Proof. From the previous statement we know that $O_X^{n-d} \subset \mathcal{F}$. Take $\mathcal{F}' = \mathcal{F}/O_X^{n-d}$. It is easy to see that $\mathcal{F}'$ is also locally free sheaf of rank $d$. Moreover we have a short exact sequence

$$0 \to O_X^{n-d} \to \mathcal{F} \to \mathcal{F}' \to 0.$$ 

This gives the following exact sequence

$$0 \to \Gamma(X, O_X^{n-d}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}') \to H^1(X, O_X^{n-d}) = 0.$$ 

Since $k[X]$ modules $\Gamma(X, O_X^{n-d})$, $\Gamma(X, \mathcal{F})$, $\Gamma(X, \mathcal{F}')$ are projective (it is true locally hence also globally!) we have

$$\Gamma(X, \mathcal{F}) = \Gamma(X, \mathcal{F}') \oplus \Gamma(X, O_X^{n-d}).$$ 

Finally

$$\Gamma(X, \mathcal{F})^\sim = \Gamma(X, \mathcal{F}')^\sim \oplus \Gamma(X, O_X^{n-d})^\sim,$$

and we have

$$\mathcal{F} = \mathcal{F}' \oplus O_X^{n-d}$$ 

(see [2], Prop. 5.1, p. 110). □

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