Abstract. This paper treats the stationary Stokes problem in exterior domain of \( \mathbb{R}^3 \) with Navier slip boundary condition. The behavior at infinity of the data and the solution are determined by setting the problem in \( L^p \)-spaces, for \( p > 2 \), with weights. The main results are the existence and uniqueness of strong solutions of the corresponding system.

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1. Introduction

We consider a simply-connected bounded domain $\Omega' \subset \mathbb{R}^3$, of class $C^{2,1}$ with boundary $\Gamma$. Let $\Omega$ be an exterior domain given by $\Omega := \mathbb{R}^3 \setminus \Omega'$. The motion of viscous incompressible fluid in the exterior domain $\Omega$ around the obstacle $\Omega'$ is described by the Navier-Stokes equations, which are non-linear. The Stokes systems is a linear approximation of this model. Precisely, the velocity $u$ of fluid and the pressure $\pi$ satisfy the following stationary Stokes problem:

$$
-\nu \Delta u + \nabla \pi = f \quad \text{in} \quad \Omega,
$$

$$
\text{div} \, u = 0 \quad \text{in} \quad \Omega,
$$

(1.1)

where $f$ the external forces acting on the fluid, there are several possibilities of boundary conditions. Under the hypothesis of impermeability of the boundary, the velocity field $u$ satisfies:

$$
u \Delta u + \nabla \pi = f \quad \text{in} \quad \Omega,
$$

$$\text{div} \, u = 0 \quad \text{in} \quad \Omega,
$$

where $n$ stands for the outer normal vector. According to the idea that the fluid cannot slip on the wall due to its viscosity, we get the no-slip condition:

$$
u \Delta u + \nabla \pi = f \quad \text{in} \quad \Omega,
$$

$$\text{div} \, u = 0 \quad \text{in} \quad \Omega,
$$

(1.2)

$$u \cdot n = 0 \quad \text{in} \quad \Gamma,$n

where $u \cdot n$ denotes, as usual, the tangential component of $u$. The problem (1.1) in an exterior domain with the Dirichlet boundary condition, is the combination of (1.2)–(1.3), has been studied by many authors. We can mention [6, 26, 27, 28, 40, 41] and references therein. However, there are many other kinds of boundary conditions which also match in the reality. In the physical applications, we are often encountering situations where the no-slip boundary conditions does not quite feasible. In this case, it is really important to introduce another boundary condition to describe the behavior of fluid on the wall. For example, hurricanes and tornadoes do slip along the ground and lose energy as they slip [13]. For the skin of sharks [22, 23] or golf balls in the case that the obstacle may have rough boundaries. Another application of interest can also be found, for instance, in aerodynamics in drag control of aircraft wings, in order to reduce the drag, small injection jets are introduced over the wings of the plane [4]. In
1827, C. Navier \[37\] was the first mathematician who considered the slip phenomena and proposed the following boundary conditions, called Navier-slip boundary conditions:

\[
\begin{align*}
\{ & u \cdot n = 0 \\
& 2[D(u)n]_\tau + \alpha(x)u_\tau = 0
\end{align*}
\quad \text{on } \Gamma,
\]

where \(D(u) = \frac{1}{2} (\nabla u + \nabla^T u)\), \(n\) and \(\tau\) are unit outer normal and tangential vector of the boundary \(\Gamma\). In (1.4), \(\alpha(x)\) is a physical parameter, which can be a positive constant or a function in \(L^\infty(\Gamma)\). Here, we consider the case that \(\alpha(x)\) is constant which called the slip coefficient. The first condition in (1.4) is the no-penetration condition and the second condition expresses the fact that the tangential velocity, instead of being zero as in the slip condition (1.3), is proportional to the tangential stress. The Navier slip conditions have been extensively studied, see for instance [2, 3, 14, 17, 25, 31, 32, 39] and references therein. For the case of Navier boundary conditions without friction \((\alpha = 0)\), let us mention [10, 34, 20], where the following boundary conditions were used:

\[
\begin{align*}
\{ & u \cdot n = 0 \\
& \text{curl } u \times n = 0
\end{align*}
\quad \text{on } \Gamma,
\]

where \(\text{curl } u\) is the vorticity field. These conditions coincide with (1.4) on flat boundaries when \(\alpha = 0\). They were also used in [16] for the study of the non stationary Navier-Stokes equations in half-spaces of \(\mathbb{R}^3\). We finally refer to [35, 36] for the study of the non stationary problem of Navier-Stokes with mixed boundary conditions that include (1.4) without friction.

The problem (1.1)–(1.4) set in bounded domains has been well studied by various authors (see for instance [11, 15] or [12, 39] for the case \(\alpha = 0\) and references therein). Although in the exterior domain, to the best of our knowledge, we can just mention [38] where (1.4) was used for the stationary Navier-Stokes equations in exterior domains with also the assumption that the velocity tends to a non zero constant vector at infinity. But in the case that the velocity tends to a zero at infinity we can mention [19, 21], where the authors studied the Stationary Stokes problem with Navier slip boundary condition in an exterior domain, they posed...
the problem in weighted spaces in order to control the infinite behavior of the solutions. They obtained in the Hilbertian framework, existence results, uniqueness of variational and strong solutions and another class of solutions called very weak solution for less regular data.

Since the domain $\Omega$ is unbounded, we set the problem in weighted Sobolev spaces, the weight functions are polynomials and enable to describe the growth or the decay of functions at infinity which allows to look for solutions of the problem with various behaviors at infinity this is one of the main advantages of the weighted Sobolev spaces. In this work, we study the problem (1.1)–(1.4) in $L^p$-theory where $p > 2$, we look for the strong solutions that have a different behavior at infinity. To that end, we combine results on the Stokes problem set in the whole space to catch the behavior at infinity and results in bounded domains to take into account the boundary conditions.

The paper is organized as follows. In Section 2, we introduce the notations, the functional framework based on weighted Sobolev space. We recall some basic results concerning the Stokes in the whole space, we give a result concerning Laplace problem with Neumann boundary conditions. We end this section by solving a mixed Stokes problem with Navier and Dirichlet boundary conditions. Finally, in Section 3, we prove the existence and uniqueness of Strong solutions for the exterior Stokes problem (1.1)–(1.4).

2. Notations and Preliminary Results

2.1. Notations.
In this section, we recall the main notation which we shall use later. In what follows, $p$ is a real number in the interval $[1, \infty[$. The dual exponent of $p$ denoted by $p'$ is given by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. We will use bold characters for vector and matrix fields. A point in $\mathbb{R}^3$ is denoted by $\mathbf{x} = (x_1, x_2, x_3)$ and its distance to the origin by

$$r = |\mathbf{x}| = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}.$$
For any multi-index $\lambda \in \mathbb{N}^3$, we denote by $\partial^{\lambda}$ the differential operator of order $\lambda$,

$$D^{\lambda} = \frac{\partial^{|\lambda|}}{\partial_1^{\lambda_1} \partial_2^{\lambda_2} \partial_3^{\lambda_3}}, \quad |\lambda| = \lambda_1 + \lambda_2 + \lambda_3.$$ 

We denote by $[s]$ the integer part of $s$. For any $k \in \mathbb{Z}$, $\mathcal{P}_k$ stands for the space of polynomials of degree less than or equal to $k$ and $\mathcal{P}_k^\Delta$ the harmonic polynomials of $\mathcal{P}_k$. If $k$ is a negative integer, we set by convention $\mathcal{P}_k = \{0\}$. Let $\Omega' \subset \mathbb{R}^3$ is a simply connected bounded domain that has a boundary $\partial \Omega' = \Gamma$ of class $\mathcal{C}^{2,1}$ and let $\Omega$ be the complement of its closure in $\mathbb{R}^3$. We denote by $\mathcal{D}(\Omega)$ the space of $\mathcal{C}^\infty$ functions with compact support in $\Omega$, $\mathcal{D}(\Omega)$ the restriction to $\Omega$ of functions belonging to $\mathcal{D}(\mathbb{R}^3)$. We recall that $\mathcal{D}'(\Omega)$ is the well-known space of distributions defined on $\Omega$. We recall that $L^p(\Omega)$ is the well-known Lebesgue real space and for $m \geq 1$, we recall that $W^{m,p}(\Omega)$ is the well-known classical Sobolev space. We shall write $u \in W^{m,p}_{loc}(\Omega)$ to mean that $u \in W^{m,p}(\mathcal{O})$, for any bounded domain $\mathcal{O}$, with $\overline{\mathcal{O}} \subset \Omega$. In this work, we shall also denote by $B_R$ the open ball of radius $R > 0$ centred at the origin with boundary $\partial B_R$. In particular, since $\Omega'$ is bounded, we can find some $R_0$, such that $\Omega' \subset B_{R_0}$ and we introduce, for any $R \geq R_0$, the set $\Omega_R = \Omega \cap B_R$. If $X$ is a Banach space, with dual space $X'$, and $Y$ is a closed subspace of $X$, we denote by $X' \perp Y$ the subspace of $X'$ orthogonal to $Y$, i.e.

$$X' \perp XY = \{ f \in X', \forall v \in Y, \langle f, v \rangle = 0 \} = (X/Y)'.$$

Given $A$ and $B$ two matrices fields, such that $A = (a_{ij})_{1 \leq i, j \leq 3}$ and $B = (b_{ij})_{1 \leq i, j \leq 3}$, then we define $A : B = (a_{ij} b_{ij})_{1 \leq i, j \leq 3}$. Finally, as usual, $C > 0$ denotes a generic constant the value of which may change from line to line and even at the same line.

### 2.2. Weighted Sobolev spaces.

In order to control the behavior at infinity of our functions and distributions we use for basic weights the quantity $\rho(x) = (1 + r^2)^{1/2}$ which is equivalent to $r$ at infinity, and to one on any bounded subset of $\mathbb{R}^3$.

For $k \in \mathbb{Z}$, we introduce

$$W^0_k(\Omega) = \left\{ u \in \mathcal{D}'(\Omega), \rho^k u \in L^p(\Omega) \right\},$$
which is a Banach space equipped with the norm:

$$\|u\|_{W^{m,p}_k(\Omega)} = \|\rho^k u\|_{L^p(\Omega)}.$$ 

For any non-negative integers $m$, real numbers $p > 1$ and $k \in \mathbb{Z}$. We define the weighted Sobolev space for $3/p + k \notin \{1, \cdots, m\}$:

$$W^{m,p}_k(\Omega) = \left\{ u \in \mathcal{D}'(\Omega); \forall \lambda \in \mathbb{N}^3 : 0 \leq |\lambda| \leq m, \rho^{k-m+|\lambda|} D^\lambda u \in L^p(\Omega) \right\}.$$

It is a reflexive Banach space equipped with the norm:

$$\|u\|_{W^{m,p}_k(\Omega)} = \left( \sum_{0 \leq |\lambda| \leq m} \|\rho^{k-m+|\lambda|} D^\lambda u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

We define the semi-norm

$$|u|_{W^{m,p}_k(\Omega)} = \left( \sum_{|\lambda|=m} \|\rho^k D^\lambda u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

Let us give some examples of such space that will be often used in the remaining of this work.

1) For $m = 1$, we have

$$W^{1,p}_k(\Omega) := \{u \in \mathcal{D}'(\Omega); \rho^{k-1} u \in L^p(\Omega), \rho^k \nabla u \in L^p(\Omega)\}.$$

2) For $m = 2$, we have

$$W^{2,p}_{k+1}(\Omega) := \{u \in W^{1,p}_k(\Omega), \rho^{k+1} \nabla^2 u \in L^p(\Omega)\}.$$

Now, we present some basic properties on weighted Sobolev spaces. For more details, the reader can refer to [9, 8, 29].

**Properties 2.1.**

- The space $\mathcal{D}(\overline{\Omega})$ is dense in $W^{m,p}_k(\Omega)$.

- For any $m \in \mathbb{N}^*$ and $3/p + k \neq 1$, we have the following continuous embedding:

$$W^{m,p}_k(\Omega) \hookrightarrow W^{m-1,p}_{k-1}(\Omega).$$
• For any $k, m \in \mathbb{Z}$ and for any $\lambda \in \mathbb{N}^3$, the mapping
\begin{equation}
    u \in W_k^{m,p}(\Omega) \longrightarrow \partial^\lambda u \in W_k^{m-|\lambda|,p}(\Omega)
\end{equation}
is continuous.

• If \(3/p + k \notin \{1, \cdots, m\}, 3/p + k - \mu \notin \{1, \cdots, m\}\) and \(m \in \mathbb{Z}\) the mapping
\[ u \in W_k^{m,p}(\Omega) \longrightarrow \rho^\mu u \in W_{k-\mu}^{m,p}(\Omega) \]
is an isomorphism.

The space \(W_k^{m,p}(\Omega)\) sometimes contains some polynomial functions. Let \(j\) be defined as follow:
\begin{equation}
    j = \begin{cases} 
        [m - (3/p + k)] & \text{if } 3/p + k \notin \mathbb{Z}, \\
        m - 3/p - k - 1 & \text{otherwise.} 
    \end{cases}
\end{equation}
Then \(P_j\) is the space of all polynomials included in \(W_k^{m,p}(\Omega)\).

The norm of the quotient space \(W_k^{m,p}(\Omega)/P_j\) is given by:
\[ ||u||_{W_k^{m,p}(\Omega)/P_j} = \inf_{\mu \in P_j} ||u + \mu||_{W_k^{m,p}(\Omega)}. \]
All the local properties of \(W_k^{m,p}(\Omega)\) coincide with those of the corresponding classical Sobolev spaces \(W^{m,p}(\Omega)\). Hence, it also satisfies the usual trace theorems on the boundary \(\Gamma\). Therefore, we can define the space
\[ \tilde{W}_k^{m,p}(\Omega) = \{ u \in W_k^{m,p}(\Omega), \gamma_0 u = 0, \gamma_1 u = 0, \cdots, \gamma_{m-1} u = 0 \}. \]
Note that when \(\Omega = \mathbb{R}^3\), we have \(\tilde{W}_k^{m,p}(\mathbb{R}^3) = W_k^{m,p}(\mathbb{R}^3)\). The space \(D(\Omega)\) is dense in \(\tilde{W}_k^{m,p}(\Omega)\). Therefore, the dual space of \(\tilde{W}_k^{m,p}(\Omega)\), denoting by \(W_{-k}^{-m,p'}(\Omega)\), is a space of distributions with the norm
\[ ||u||_{W_{-k}^{-m,p'}(\Omega)} = \sup \frac{\langle u, v \rangle_{W_{-k}^{-m,p'}(\Omega) \times \tilde{W}_k^{m,p}(\Omega)}}{||v||_{\tilde{W}_k^{m,p}(\Omega)}}. \]
The proof of the following theorem can be found in [33, Proposition 2.1].
Theorem 2.2.
Let $k, l$ be real numbers. Let $\lambda$ be a polynomial that belongs to $W^{1,p}_k(\Omega) + W^{1,q}_l(\Omega)$. Then $\lambda$ belongs to $P_\gamma$ where

$$\gamma = \max \left(\left[1 - \frac{3}{p} - k\right], \left[1 - \frac{3}{q} - l\right]\right).$$

We note that the vector-valued Laplace operator of a vector field $v = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta v = 2 \text{div} \mathbf{D}(v) - \text{grad}(v)$$

This leads to the following definition.

Definition 2.3. For all integers $k \in \mathbb{Z}$ and $1 < p < \infty$. The space $H^p_k(\text{div}, \Omega)$ is defined by

$$H^p_k(\text{div}, \Omega) = \{ v \in W^{0,p}_k(\Omega); \text{div } v \in W^{0,p}_{k+1}(\Omega) \},$$

and is provided with the norm

$$\| v \|_{H^p_k(\text{div}, \Omega)} = \left( \| v \|^p_{W^{0,p}_k(\Omega)} + \| \text{div } v \|^p_{W^{0,p}_{k+1}(\Omega)} \right)^{\frac{1}{p}}.$$

This definition will be also used with $\Omega$ replaced by $\mathbb{R}^3$.

The argument used by Hanouzet (see [30]) to prove the density of $\mathcal{D}(\Omega)$ in $W^{m,p}_k(\Omega)$ can be adapted to establish that $\mathcal{D}(\Omega)$ is dense in $H^p_k(\text{div}, \Omega)$. Therefore, denoting by $n$ the exterior unit normal to the boundary $\Gamma$, the normal trace $v \cdot n$ can be defined in $W^{-1/p',p}(\Gamma)$ for the functions of $H^p_k(\text{div}, \Omega)$, where $W^{-1/p,p}(\Gamma)$ denotes the dual space of $W^{1/p,p'}(\Gamma)$. They satisfy the trace theorems; i.e, there exists a constant $C$ such that

$$\forall v \in H^p_k(\text{div}, \Omega), \quad \| v \cdot n \|_{W^{-1/p,p}(\Gamma)} \leq C \| v \|_{H^p_k(\text{div}, \Omega)}.$$  

In addition, the following Green’s formulas holds: For any $v \in H^p_k(\text{div}, \Omega)$ and $\varphi \in W^{-1/p',p}(\Omega)$

$$\int_{\Omega} v \cdot \nabla \varphi \, dx + \int_{\Omega} \varphi \, \text{div } v \, dx = \langle v \cdot n, \varphi \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}.$$  

The closures of $\mathcal{D}(\Omega)$ in $H^p_k(\text{div}, \Omega)$ is denoted by $\hat{H}^p_k(\text{div}, \Omega)$ and can be charac-
characterized by
\[
\tilde{H}^p_k(\text{div}, \Omega) = \{ v \in H^p_k(\text{div}, \Omega) : v \cdot n = 0 \text{ on } \Gamma \}.
\]
The proof of the following result can be found in [18]:

**Proposition 2.4.** A distribution \( f \) belongs to \( \tilde{H}^p_k(\text{div}, \Omega)' \) if and only if there exist \( \psi \in \mathcal{W}^{0,p}_{-k}(\Omega) \) and \( \chi \in \mathcal{W}^{0,p}_{-k-1}(\Omega) \), such that \( f = \psi + \nabla \chi \). Moreover
\[
\| \psi \|_{\mathcal{W}^{0,p}_{-k}(\Omega)} + \| \chi \|_{\mathcal{W}^{0,p}_{-k-1}(\Omega)} \leq C \| f \|_{\tilde{H}^p_k(\text{div}, \Omega)'},
\]

2.3. **The Stokes problem in the whole space \( \mathbb{R}^3 \).**

We recall here some basic results concerning the Stokes problem in \( \mathbb{R}^3 \):

\[
(2.7) \quad -\Delta u + \nabla \pi = f \quad \text{and} \quad \text{div} \, u = \chi \quad \text{in} \quad \mathbb{R}^3.
\]

These results can be found in [5]. Let us first introduce the kernel of the Stokes operator

\[
\mathcal{N}^p_k(\mathbb{R}^3) = \left\{ (u, \pi) \in \mathcal{W}^{1,p}_k(\mathbb{R}^3) \times \mathcal{W}^{0,p}_k(\mathbb{R}^3), \quad -\Delta u + \nabla \pi = 0 \quad \text{and} \quad \text{div} \, u = 0 \quad \text{in} \quad \mathbb{R}^3 \right\}
\]

and the space of polynomials

\[
N_k = \left\{ (\lambda, \mu) \in \mathcal{P}_k \times \mathcal{P}^{\Delta}_{k-1}, \quad -\Delta \lambda + \nabla \mu = 0 \quad \text{and} \quad \text{div} \, \lambda = 0 \right\}.
\]

Recall that by agreement on the notation \( \mathcal{P}_k \), the space \( N_k = \{(0, 0)\} \) when \( k < 0 \) and \( N_0 = \mathcal{P}_0 \times \{0\} \).

The next proposition characterizes the kernel of (2.7).

**Proposition 2.5.** Let \( 1 < p < \infty \) and \( k \) be integers. Then \( \mathcal{N}^p_k(\mathbb{R}^3) = N_{[1-k-3/p]} \). In particular, \( \mathcal{N}^p_k(\mathbb{R}^3) = \{(0, 0)\} \) if \( k > 1 - 3/p \).

The next theorem states an existence, uniqueness and regularity result for problem (2.7).
Theorem 2.6. Assume that \( k \in \mathbb{Z} \) and \( 1 < p < \infty \). If \( (f, \chi) \in W^{-1,p}_k(\mathbb{R}^3) \times W^{0,p}_k(\mathbb{R}^3) \) satisfies the compatibility condition:

\[
\forall (\lambda, \mu) \in N_{[1-3/p'+k]}, \quad \langle f, \lambda \rangle_{W^{-1,p}_k(\mathbb{R}^3) \times W^{0,p'}_k(\mathbb{R}^3)} - \langle \chi, \mu \rangle_{W^{0,p}_k(\mathbb{R}^3) \times W^{0,p'}_k(\mathbb{R}^3)} = 0,
\]

then problem (2.7) has a solution \((u, \pi) \in W^{1,p}_k(\mathbb{R}^3) \times W^{0,p}_k(\mathbb{R}^3)\) unique up to an element of \( N_{[1-3/p-k]} \) and we have the estimate

\[
\inf_{(\lambda, \mu) \in N_{[1-3/p-k]}} \left( \|u + \lambda\|_{W^{1,p}_k(\mathbb{R}^3)} + \|\pi + \mu\|_{W^{0,p}_k(\mathbb{R}^3)} \right) \leq C \left( \|f\|_{W^{-1,p}_k(\mathbb{R}^3)} + \|\chi\|_{W^{0,p}_k(\mathbb{R}^3)} \right).
\]

Furthermore, if \((f, \chi) \in W^{0,p}_{k+1}(\mathbb{R}^3) \times W^{1,p}_{k+1}(\mathbb{R}^3)\), then \((u, \pi) \in W^{2,p}_{k+1}(\mathbb{R}^3) \times W^{1,p}_{k+1}(\mathbb{R}^3)\).

The proofs of Proposition 2.5 and Theorem 2.6 can be found in [5].

2.4. Generalized Neumann problem.

In this section, we are interested into the following Neumann problem:

\[
-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma.
\]

Our first proposition is established also in [33], it characterizes the kernel of the Laplace operator with Neumann boundary condition. For any integer \( k \in \mathbb{Z} \) and \( 1 < p < \infty \),

\[
N_{\Delta}^{\Delta} = \left\{ w \in W^{1,p}_k(\Omega); \quad \Delta w = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma \right\}.
\]

Proposition 2.7. For any integer \( k \geq 1 \), \( N_{\Delta}^{\Delta} \) the subspace of all functions in \( W^{1,p}_k(\Omega) \) of the form \( w(p) - p \), where \( p \) runs over all polynomials of \( P_{[1-3/p-k]} \) and \( w(p) \) is the unique solution in \( W^{1,2}_0(\Omega) \cap W^{1,p}_k(\Omega) \) of the Neumann problem

\[
\Delta w(p) = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial w(p)}{\partial n} = \frac{\partial p}{\partial n} \quad \text{on } \Gamma.
\]

Here also, we set \( N_{\Delta}^{\Delta} = \{0\} \) when \( k \leq 0 \); \( N_{\Delta}^{\Delta} \) is a finite-dimentional space of the same dimension as \( P_{[1-3/p-k]} \) and in particular, \( N_{\Delta}^{\Delta} = \mathbb{R} \).

The next theorem states an existence, uniqueness and regularity result for problem (2.9), for \( k = 0 \) and \( 1 < p < \infty \).
Theorem 2.8. For any $f$ in $L^p(\Omega)$ and $g$ in $W^{-1/p,p}(\Gamma)$. Then, the problem (2.9) has a solution $u \in W^{-1/p,p}(\Omega)$ unique up to element of $N_{-1,p}(\Omega)$ and we have the following estimate:

$$\|u\|_{W^{-1/p,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)}).$$

If in addition, $g$ in $W^{1/p',p}(\Gamma)$, the solution $u$ of problem (2.9) belongs to $W^2_{0,p}(\Omega)$ and satisfies

$$\|u\|_{W^2_{0,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{W^{1/p',p}(\Gamma)}).$$

Proof. Let us extend $f$ by zero in $\Omega'$ and let $\tilde{f}$ denote the extended function. Then $\tilde{f}$ belongs to $L^p(\mathbb{R}^3)$. Applying [8], there exists a unique function $\tilde{v} \in W^2_{0,p}(\mathbb{R}^3)/P_{2-3/p}$ such that

$$-\Delta \tilde{v} = \tilde{f} \text{ in } \mathbb{R}^3.$$ (2.13)

Then $\nabla \tilde{v} \cdot n$ belongs to $W^{1-1/p,p}(\Gamma) \hookrightarrow W^{-1/p,p}(\Gamma)$. It follows from [33, Theorem 3.12], that the following problem:

$$\Delta w = 0 \text{ in } \Omega \quad \text{and} \quad \nabla w \cdot n = g - \nabla \tilde{v} \cdot n \text{ on } \Gamma,$$ (2.14)

has a solution $w \in W^{-1/p,p}(\Omega)$ unique up to element of $N_{-1,p}(\Omega)$. Thus $u = \tilde{v}|_\Omega + w \in W^{-1/p,p}(\Omega)$ is the required solution of (2.9). The uniqueness follows immediately from Proposition 2.7.

Now, suppose that $g$ belongs to $W^{1/p',p}(\Gamma)$. The aim is to prove that $u$ belongs to $W^2_{0,p}(\Omega)$. To that end, let us introduce the following partition of unity:

$$\varphi, \psi \in C^\infty(\mathbb{R}^3), \quad 0 \leq \varphi, \psi \leq 1, \quad \varphi + \psi = 1 \text{ in } \mathbb{R}^3,$$ (2.15)

$$\varphi = 1 \text{ in } B_R, \quad \text{supp } \varphi \subset B_{R+1}.$$ Let $P$ be a continuous linear mapping from $W^{-1/p,p}(\Omega)$ to $W^{-1/p,p}(\mathbb{R}^3)$, such that $Pu = \tilde{u}$. Then $\tilde{u}$ belongs to $W^{1,p}(\mathbb{R}^3)$ and can be written as:

$$\tilde{u} = \varphi \tilde{u} + \psi \tilde{u}.$$ Next, one can easily observe that $\tilde{u}$ satisfies the following problem:

$$-\Delta \psi \tilde{u} = f_1 \text{ in } \mathbb{R}^3,$$ (2.16)
with 

\[ f_1 = \tilde{f}\psi - (2\nabla \tilde{u}\nabla \psi + \tilde{u}\Delta \psi). \]

Owing to the support of \( \psi \), \( f_1 \) has the same regularity as \( f \) and so belongs to \( L^p(\mathbb{R}^3) \). It follows from [3], that there exists \( z \) in \( W^{2,p}_0(\mathbb{R}^3) \) such that \( -\Delta z = f_1 \) in \( \mathbb{R}^3 \). This implies that \( \psi \tilde{u} - z \) is a harmonic tempered distribution and therefore a harmonic polynomial that belongs to \( \mathcal{P}_{[2-3/p]} \). The fact that \( \mathcal{P}_{[2-3/p]} \subset W^{2,p}_0(\mathbb{R}^3) \) yields that \( \psi \tilde{u} \) belongs to \( W^{2,p}_0(\mathbb{R}^3) \). In particular, we have \( \psi \tilde{u} = u \) outside \( B_{R+1} \), so the restriction of \( u \) to \( \partial B_{R+1} \) belongs to \( W^{2-1/p,p}(\partial B_{R+1}) \). Therefore, \( \varphi \tilde{u} \in W^{1,p}(\Omega_{R+1}) \) satisfies the following problem:

\[
\begin{aligned}
-\Delta \varphi u &= f_2 \quad \text{in } \Omega_{R+1}, \\
\nabla \varphi u \cdot n &= g \quad \text{on } \Gamma, \\
\varphi u &= \psi \tilde{u} \quad \text{on } \partial B_{R+1},
\end{aligned}
\]

(2.17)

where \( f_2 \in L^p(\Omega_{R+1}) \) have similar expression as \( f_1 \) with \( \psi \) remplaced by \( \varphi \). According [Remark 3.2, [9]], \( \varphi u \in W^{2,p}(\Omega_{R+1}) \), which in turn shows that \( \varphi \tilde{u} \) also belongs to \( W^{2,p}(\Omega_{R+1}) \). This implies that \( u \in W^{2,p}(\Omega) \).

\[
\square
\]

2.5. A mixed Stokes problem.

Let \( R > 0 \) be a real number large enough so that \( \overline{\Omega'} \subset B_R \). We recall that \( \Omega_R = \Omega \cap B_R \) and \( \partial B_R = \Gamma_R \). Now, we study the following mixed boundary value problem: Given \( f, \chi, g, h \) and \( a \). We look for \((u, \pi)\) satisfying

\[
\begin{aligned}
-\Delta u + \nabla \pi &= f \quad \text{and} \quad \text{div } u = \chi \quad \text{in } \Omega_R, \\
\nabla u \cdot n &= g \quad \text{and} \quad 2[D(u)n]_\tau + \alpha u_\tau = h \quad \text{on } \Gamma, \\
u &= a \quad \text{on } \Gamma_R.
\end{aligned}
\]

(2.18)

**Theorem 2.9.** Assume that \( 1 < p < \infty \). Let the pair \((u, \pi) \in H^2(\Omega_R) \times H^1(\Omega_R)\) be a solution to the problem (2.18) with data \( f \in L^p(\Omega_R), \chi \in W^{1,p}(\Omega_R), g \in W^{1+1/p',p}(\Gamma), h \in W^{1-1/p,p}(\Gamma), a \in W^{1+1/p',p}(\partial B_R) \) such that \( h \cdot n = 0 \) on \( \Gamma \). Then, we also have \( u \in W^{2,p}(\Omega_R) \) and \( \pi \in W^{1,p}(\Omega_R) \).
Proof. The proof of the theorem is made of two steps.

Step 1. The case $1 < p \leq 2$.

Since $(u, \pi) \in H^2(\Omega_R) \times H^1(\Omega_R)$ then it is clear that $(u, \pi) \in W^{2,p}(\Omega_R) \times W^{1,p}(\Omega_R)$ for any $1 < p \leq 2$.

Step 2. The case $2 < p < \infty$.

Let us introduce the following partition of unity:

$$\theta_1, \theta_2 \in C^\infty(\Omega_R), \quad 0 \leq \theta_1, \theta_2 \leq 1, \quad \theta_1 + \theta_2 = 1 \text{ in } \Omega_R,$$

$$\theta_1 = 1 \text{ in } B_{R/3}, \quad \text{supp } \theta_1 \subset B_{2R/3}.$$

Then we can write $u = \theta_1 u + \theta_2 u = u_1 + u_2$ and $\pi = \theta_1 \pi + \theta_2 \pi = \pi_1 + \pi_2$.

Now it is clear that

$$-\Delta u_1 + \nabla \pi_1 = f_1 \quad \text{and} \quad \text{div } u_1 = \chi_1 \text{ in } \Omega_R,$$

where $f_1 = \theta_1 f - u \Delta \theta_1 - 2\nabla u \cdot \nabla \theta_1 + \pi \nabla \theta_1 \in L^p(\Omega_R)$ and $\chi_1 = \theta_1 \chi + u \nabla \theta_1 \in W^{1,p}(\Omega_R)$ for $2 \leq p \leq 6$.

It is also clear that on the boundaries, we have

$$\begin{cases} u_1 \cdot n = g & \text{on } \Gamma, \\ u_1 \cdot n = 0 & \text{on } \partial B_R, \end{cases}$$

where $h \in W^{1-1/p,p}(\Gamma)$ and $g \in W^{1+1/p',p}(\Gamma)$, we deduce that $(u_1, \pi_1)$ that satisfies (2.19)-(2.20) belongs to $W^{2,p}(\Omega_R) \times W^{1,p}(\Omega_R)$ for $2 < p \leq 6$ see [1, Theorem 2.1].

Similar arguments show $(u_2, \pi_2)$ satisfies the following Stokes problem with the Dirichlet boundary

$$\begin{cases} -\Delta u_2 + \nabla \pi_2 = f_2 & \text{and} \quad \text{div } u_2 = \chi_2 \text{ in } \Omega_R, \\ u_2 = 0 & \text{on } \Gamma \text{ and } u_2 = a \text{ on } \partial B_R. \end{cases}$$

Where $f_2 \in L^p(\Omega_R)$ and $\chi_2 \in W^{1,p}(\Omega_R)$ have similar expression as $f_1$ and $\chi_1$ with $\theta_1$ replaced by $\theta_2$. Since $a \in W^{1+1/p',p}(\partial B_R)$, the problem (2.21) has a solution $(u_2, \pi_2)$ belongs to $W^{2,p}(\Omega_R) \times W^{1,p}(\Omega_R)$ (see for instance [7] or [24]).

Now, suppose that $p > 6$. The above argument shows that $(u, \pi)$ belongs to $W^{2,6}(\Omega_R) \times W^{1,6}(\Omega_R)$ and we can repeat the same argument with $p = 6$.
instead \( p = 2 \). We have \( W^{1,6}(\mathcal{O}) \hookrightarrow L^q(\mathcal{O}) \) for any real number \( q > 6 \). We know that the embedding \( W^{2,p}(\Omega') \hookrightarrow W^{1,q}(\Omega') \), for any \( q \in [1, \infty) \) if \( p > 3 \). Then we have \( u|_\Gamma \in W^{2-1/6,6}(\Gamma) \hookrightarrow W^{1-1/p}(\Gamma) \) for all \( p > 6 \). Consequently, by the same reasoning we deduce that the solution \((u, \pi)\) belongs to \( W^{2,p}(\Omega_R) \times W^{1,p}(\Omega_R) \).

\[
\square
\]

3. Strong solutions for the exterior Stokes problem

In this section, we are interested in the following problem:

\[
\begin{aligned}
(S_T) \quad \begin{cases}
-\Delta u + \nabla \pi = f & \text{and} \quad \text{div } u = \chi & \text{in } \Omega, \\
u \cdot n = g & \text{and} \quad 2[D(u)n]_r + \alpha u_r = h & \text{on } \Gamma.
\end{cases}
\end{aligned}
\]

In this part, we investigate the well-posedness of strong solutions in \( W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega) \) with \( k \in \mathbb{Z} \). In order to deal with the uniqueness issues, we first need to characterize the kernel of problem \((S_T)\). We define the kernel of problem \((S_T)\).

For \( k \in \mathbb{Z} \) and \( p \geq 2 \), we introduce:

\[
\mathcal{N}^p_{k+1}(\Omega) = \left\{ (u, \pi) \in W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega); \right. \\
-\Delta u + \nabla \pi = 0, \text{div } u = 0 \text{ in } \Omega \text{ and } u \cdot n = 0, 2[D(u)n]_r + \alpha u_r = 0 \text{ on } \Gamma \bigg\}.
\]

The characterization of the kernel \( \mathcal{N}^p_{k+1}(\Omega) \) is given by the following proposition:

**Proposition 3.1.** Consider an exterior domain \( \Omega \) with boundary \( \Gamma \) of class \( C^{2,1} \).

We have

\[
\mathcal{N}^p_{k+1}(\Omega) = \left\{ (v - \lambda, \theta - \mu); \quad (\lambda, \mu) \in \mathcal{N}_{[1-k-3/p]} \right\},
\]

where \( (v, \theta) \in W^{2,p}_{k+1}(\Omega) \cap W^{1,2}_{1}(\Omega) \times W^{1,p}_{k+1}(\Omega) \cap W^{1,2}_{1}(\Omega) \) is the unique solution of the following problem:

\[
\begin{aligned}
(3.1) \quad \begin{cases}
-\Delta v + \nabla \theta = 0 & \text{and} \quad \text{div } v = 0 & \text{in } \Omega, \\
v \cdot n = \lambda \cdot n & \text{and} \quad 2[D(v)n]_r + \alpha v_r = 2[D(\lambda)n]_r + \alpha \lambda_r & \text{on } \Gamma.
\end{cases}
\end{aligned}
\]
In particular, \( N_{k+1}^p(\Omega) = \{(0,0)\} \) if \( k > 1 - 3/p \).

Proof. Let us assume that \((u, \pi)\) belongs to \( N_{k+1}^p(\Omega) \). The pair \((u, \pi)\) has an extension \((\tilde{u}, \tilde{\pi})\) that belongs to \( W_{k+1}^{2,p}(\mathbb{R}^3) \times W_{k+1}^{1,p}(\mathbb{R}^3) \). Set now

\[
F = -\Delta \tilde{u} + \nabla \tilde{\pi} \quad \text{and} \quad e = \text{div} \tilde{u}.
\]

Then the pair \((F, e)\) belongs to \( W_{k+1}^{0,p}(\mathbb{R}^3) \times W_{k+1}^{1,p}(\mathbb{R}^3) \) and has a compact support. Therefore \((F, e)\) also belongs to \( W_{1}^{0,2}(\mathbb{R}^3) \times W_{1}^{1,2}(\mathbb{R}^3) \). It follows from Theorem 3.2, that there exists a unique solution \((v, \theta)\) in \( W_{1}^{2,2}(\mathbb{R}^3) \times W_{1}^{1,2}(\mathbb{R}^3) \) such that

\[
-\Delta v + \nabla \theta = -\Delta \tilde{u} + \nabla \tilde{\pi} \quad \text{and} \quad \text{div} v = \text{div} \tilde{u} \quad \text{in} \quad \mathbb{R}^3.
\]

It follows that \((v - \tilde{u}, \theta - \tilde{\pi})\) belongs to \( (W_1^{2,2}(\mathbb{R}^3) + W_1^{2,p}(\mathbb{R}^3)) \times (W_1^{1,2}(\mathbb{R}^3) + W_1^{1,p}(\mathbb{R}^3)) \).

Hence, there exits \((\lambda, \mu) \in N_{k-3/p}^{1} \) such that \((v - \tilde{u}, \theta - \tilde{\pi}) = (\lambda, \mu)\), Thus, \((v, \theta)\) belongs to \( (W_1^{2,2}(\mathbb{R}^3) \cap W_1^{2,p}(\mathbb{R}^3)) \times (W_1^{1,2}(\mathbb{R}^3) \cap W_1^{1,p}(\mathbb{R}^3)) \), its restriction to \( \Omega \) belongs to \( (W_1^{2,2}(\Omega) \cap W_1^{2,p}(\Omega)) \times (W_1^{1,2}(\Omega) \cap W_1^{1,p}(\Omega)) \) and satisfies (3.1).

The next Theorem solves the problem \((\mathcal{S}_T)\) when \( p \geq 2 \), our study is based on strong solutions in a Hilbertian framework (see [19]), that’s why we will take the data \( f \) and \( \chi \) have a compact support.

**Theorem 3.2.** Assume that \( p \geq 2 \). Let \( f \in W_{k+1}^{0,p}(\Omega), \chi \in W_{k+1}^{1,p}(\Omega), \) \( g \in W_{k+1}^{1+p',p}(\Gamma) \) and \( h \in W_{k+1}^{1-p,p}(\Gamma) \) such that \( f \) and \( \chi \) have a compact support, \( h \cdot \mathbf{n} = 0 \) on \( \Gamma \) and the following compatibility condition is satisfied

\[
(3.2)
\forall (\xi, \eta) \in N_{k+1}^p(\Omega), \quad \int_{\Omega} f \cdot \xi dx - \int_{\Omega} \chi \eta dx = \int_{\Gamma} g \left( 2[D(\xi)\mathbf{n}] \cdot \mathbf{n} - \eta \right) d\sigma - \int_{\Gamma} h \cdot \xi d\sigma.
\]

Then, the Stokes problem \((\mathcal{S}_T)\) has a solution \((u, \pi) \in W_{k+1}^{2,p}(\Omega) \times W_{k+1}^{1,p}(\Omega) \) unique up to an element of \( N_{k+1}^p(\Omega) \). In addition, we have the following estimate:
\begin{align*}
\inf_{(\lambda, \mu) \in \mathcal{N}_{k+1}^p(\Omega)} \left( \| u + \lambda \|_{W_{k+1}^{2,p}(\Omega)} + \| \pi + \mu \|_{W_{k+1}^{1,p}(\Omega)} \right) \\
\leq C \left( \| f \|_{W_{k+1}^{0,p}(\Omega)} + \| \chi \|_{W_{k+1}^{1,p}(\Omega)} + \| h \|_{W^{1-1/p,p}(\Gamma)} + \| g \|_{W^{1+1/p',p}(\Gamma)} \right).
\end{align*}

\textbf{Proof.} Observe first that the uniqueness is a straightforward consequence of Proposition 3.1. We now divide the proof of the theorem into several parts.

- \textbf{Compatibility condition.} In this part, we prove that (3.2) is a necessary condition.

Let \((\xi, \eta)\) be in \(\mathcal{N}_{k+1}^{p'}(\Omega)\). For any \((\varphi, \psi)\) \(\in D(\overline{\Omega}) \times D(\overline{\Omega})\). Using the same calculation as in the proof of \([19, \text{Theorem 3.7}]\), we have

\[
\int_{\Omega} \left[ \left( -\Delta \varphi + \nabla \psi \right) \cdot \xi - \eta \div \varphi \right] \, dx \\
= \int_{\Gamma} \left( \varphi \cdot n \right) \left( 2[D(\xi) n] \cdot n - \eta \right) \, d\sigma - \int_{\Gamma} \xi \tau \cdot \left( 2[D(\varphi) n] \tau + \alpha \varphi \tau \right) \, d\sigma.
\]

Then, the last Green’s formula holds for any pair \((\varphi, \psi)\) \(\in W_{k+1}^{2,p}(\Omega) \times W_{k+1}^{1,p}(\Omega)\) by density. In particular, if \((u, \pi)\) \(\in W_{k+1}^{2,p}(\Omega) \times W_{k+1}^{1,p}(\Omega)\) is a solution of (\(S_T\)), then (3.2) holds.

- \textbf{Existence.} Here we prove that problem \((S_T)\) has a solution \((u, \pi)\) that belongs to \(W_{k+1}^{1,p}(\Omega) \times W_{k+1}^{0,p}(\Omega)\). We start with the case \(2 \leq p \leq 6\), the proof is made of two steps.

Step 1. The case \(g = 0\).

Since \(p \geq 2\) and \((f, \chi)\) have a support compact, then we have \((f, \chi)\) belongs to \(W_{1}^{0,2}(\Omega) \times W_{1}^{1,2}(\Omega)\). In addition, its clear that \(h\) belongs to \(H^{1/2}(\Gamma)\). Thanks to \([19, \text{Theorem 3.6}]\), problem \((S_T)\) has a solution \((u, \pi)\) \(\in W_{1}^{2,2}(\Omega) \times W_{1}^{1,2}(\Omega)\). It remains now to prove that \((u, \pi)\) belongs to \(W_{k+1}^{1,p}(\Omega) \times W_{k+1}^{0,p}(\Omega)\). To that end, we shall use again properties of the Stokes problem in the whole space \(\mathbb{R}^3\). Now, we
first need appropriate extensions of \( u \) and \( \pi \) defined in \( \mathbb{R}^3 \). So let us consider the following Stokes problem in the bounded domain \( \Omega' \)

\[
\begin{aligned}
-\Delta u' + \nabla \pi' &= 0 & \text{in} & & \Omega', \\
u' &= u & \text{on} & & \Gamma.
\end{aligned}
\]

(3.4)

Since \( u \cdot n = 0 \), problem (3.4) has a solution \( (u', \pi') \in H^2(\Omega') \times H^1(\Omega') \) (see for instance [7] or [24]). Setting

\[
\tilde{u} = \begin{cases}
u & \text{in} & \Omega, \\
u' & \text{in} & \Omega',
\end{cases}
\quad \text{and} \quad
\tilde{\pi} = \begin{cases}\pi & \text{in} & \Omega, \\
\pi' & \text{in} & \Omega'.
\end{cases}
\]

Then clearly, the pair \((\tilde{u}, \tilde{\pi})\) belongs to \( W^{1,2}_k(\mathbb{R}^3) \times W^{0,2}_k(\mathbb{R}^3) \). The goal is now to identify \((\tilde{u}, \tilde{\pi})\) with a solution of the Stokes problem in \( \mathbb{R}^3 \) that belongs to \( W^{1,p}_k(\Omega) \times W^{0,p}_k(\Omega) \). Let us set

\[
F = -\Delta \tilde{u} + \nabla \tilde{\pi} \quad \text{and} \quad e = \text{div} \tilde{u}.
\]

(3.5)

In order to apply Theorem [2.6] we need to show that \((F, e)\) belongs to \( W^{-1,p}_k(\mathbb{R}^3) \times W^{0,p}_k(\mathbb{R}^3) \) and satisfies (2.8). Therefore, denoting by \( \tilde{f} \in W^{0,p}_k(\mathbb{R}^3) \) the extension of \( f \) by zero in \( \Omega' \), we deduce that for any \( \varphi \in D(\mathbb{R}^3) \)

\[
\langle F, \varphi \rangle_{D'(\mathbb{R}^3) \times D(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \tilde{f} \cdot \varphi \, dx + 2 \int_{\Gamma} \varphi \cdot (D(u)n - D(u')n) \, d\sigma
\]

\[
+ \int_{\Gamma} (\varphi \cdot n)(\pi' - \pi) \, d\sigma - \int_{\Gamma} (\varphi \cdot n) \chi \, d\sigma.
\]

(3.6)

This calculations are the same as in the previous proof of [19] Theorem 3.7.

Since \((2D(u)n - 2D(u')n)\mid_{\Gamma}\) and \((\pi' - \pi)\mid_{\Gamma}\) belongs to \( H^{1/2}(\Gamma) \).

\[
\left| \langle F, \varphi \rangle_{D'(\mathbb{R}^3) \times D(\mathbb{R}^3)} \right| \leq \left| \int_{\mathbb{R}^3} \tilde{f} \cdot \varphi \, dx \right| + \left| \langle \varphi \cdot n, \chi \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} \right|
\]

\[
+ \left| 2\langle D(u)n - D(u')n, \varphi \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} \right| + \left| \langle \varphi \cdot n, \pi' - \pi \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} \right|
\]

\[
\leq \| \tilde{f} \|_{W^{-1,p}_k(\mathbb{R}^3)} \| \varphi \|_{W^{0,p'}_k(\mathbb{R}^3)} + \| \chi \|_{k+1}(\Omega) \| \varphi \|_{H^{-1/2}(\Gamma)}
\]

\[
+ \| 2D(u)n - 2D(u')n \|_{H^{1/2}(\Gamma)} \| \varphi \|_{H^{-1/2}(\Gamma)} + \| \varphi \cdot n \|_{H^{-1/2}(\Gamma)} \| \pi' - \pi \|_{H^{1/2}(\Gamma)}
\]

\[
\leq \| \tilde{f} \|_{W^{-1,p}_k(\mathbb{R}^3)} \| \varphi \|_{W^{1,p'}_k(\mathbb{R}^3)} + C \| \varphi \|_{H^{-1/2}(\Gamma)}
\]

As we have

\[
H^{1/2}(\Gamma) \hookrightarrow W^{-1/6,6}(\Gamma) \hookrightarrow W^{-1/p,p}(\Gamma), \text{for any } 2 \leq p \leq 6
\]
We obtain

\[
\|\varphi\|_{H^{-1/2}(\Gamma)} \leq C\|\varphi\|_{W^{1/p',p'}(\Gamma)} \\
\leq C\|\varphi\|_{W_{1-k}^{1,p'}(\Omega)} \\
\leq C\|\varphi\|_{W_{1-k}^{1,p'}(\mathbb{R}^3)}
\]

Then, we have

\[
\left|\langle F, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)}\right| \leq C\|\varphi\|_{W_{1-k}^{1,p'}(\mathbb{R}^3)}.
\]

Since \(\mathcal{D}(\mathbb{R}^3)\) is dense in \(W_{1-k}^{1,p'}(\mathbb{R}^3)\), then (3.6) is still valid for any \(\varphi \in W_{1-k}^{1,p'}(\mathbb{R}^3)\) which implies that \(F\) belongs to \(W_{1-k}^{-1,p}(\mathbb{R}^3)\).

Now for any \(\phi \in \mathcal{D}(\mathbb{R}^3)\), we have

\[
\langle e, \phi \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} = \langle \text{div} \tilde{u}, \phi \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)} = -\int_{\mathbb{R}^3} \tilde{u} \cdot \nabla \phi \, dx
\]

\[
= \int_{\Omega} \text{div} u \, \phi \, dx + \int_{\Omega'} \text{div} u' \, \phi \, dx - \int_{\Gamma} (u \cdot n) \phi \, d\sigma + \int_{\Gamma} (u' \cdot n) \phi \, d\sigma
\]

(3.7) \quad = \int_{\Omega} \chi \phi \, dx.

Since \(\chi \in W_{k+1}^{1,p}(\Omega)\). Then, we have

\[
\left|\langle e, \phi \rangle_{\mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3)}\right| \leq \|\chi\|_{W_{k+1}^{1,p}(\Omega)} \|\phi\|_{W_{1-k}^{0,p'}(\Omega)} \leq C\|\phi\|_{W_{1-k}^{0,p'}(\Omega)}.
\]

Due to the density of \(\mathcal{D}(\mathbb{R}^3)\) in \(W_{1-k}^{0,p'}(\mathbb{R}^3)\), then (3.7) is still valid for any \(\phi \in W_{1-k}^{0,p'}(\mathbb{R}^3)\), which implies that \(e\) belongs to \(W_{1-k}^{0,p}(\mathbb{R}^3)\). As a result, we have proved that \((F, e)\) belongs to \(W_{1-k}^{-1,p}(\mathbb{R}^3) \times W_{1-k}^{0,p}(\mathbb{R}^3)\).

Let us now prove that \(F\) and \(e\) satisfy (2.8), which in view of (3.6) and (3.7), amounts to prove that for any \((\lambda, \mu)\) in \(N_{[1-3/p'+k]}\)

(3.8) \quad \int_{\mathbb{R}^3} \tilde{f} \cdot \lambda \, dx - \int_{\Omega} \chi \mu \, dx + 2 \int_{\Gamma} \lambda \cdot \left( D(u) \, n - D(u') \, n \right) \, d\sigma
\]

\[+ \int_{\Gamma} (\lambda \cdot n) \, (\pi' - \pi) \, d\sigma - \int_{\Gamma} (\lambda \cdot n) \chi \, d\sigma = 0.
\]

So let \((\lambda, \mu) \in N_{[1-3/p'+k]}\) and let \((v(\lambda), \theta(\lambda))\) be in \(W_{1-k+1}^{2,p'}(\Omega) \cap W_{1-k+1}^{1,2}(\Omega) \times W_{1-k+1}^{1,p'}(\Omega) \cap W_{1-k+1}^{1,2}(\Omega)\) such that the pair \((v(\lambda) - \lambda, \theta(\lambda) - \mu)\) belongs to \(N_{1-k+1}^{p'}(\Omega)\).

Now, for any \((u, \theta) \in W_{1-k+1}^{2,p'}(\Omega) \cap W_{1-k+1}^{2,2}(\Omega) \times W_{1-k+1}^{1,p'}(\Omega) \cap W_{1-k+1}^{1,2}(\Omega)\) such that \(\text{div} \, v = 0\), computations in \(\Omega\) yields
\( \int_{\Omega} f \cdot v (\lambda) \, dx - \int_{\Omega} \chi \theta (\lambda) \, dx \)
\[- \int_{\Gamma} (v (\lambda) \cdot n) \chi \, d\sigma - 2 \int_{\Gamma} u \cdot D(v(\lambda)) n \, d\sigma \]
\[+ 2 \int_{\Gamma} v (\lambda) \cdot D(u) n \, d\sigma - \int_{\Gamma} (v (\lambda) \cdot n) \pi \, d\sigma = 0. \]

Now making the difference between (3.9) and (3.2) with \( \xi = v(\lambda) - \lambda \) and \( \eta = \theta(\lambda) - \mu \) and recalling that \( v (\lambda) \cdot n = \lambda \cdot n \) on \( \Gamma \), yields

\( \int_{\Omega} f \cdot \lambda \, dx - \int_{\Omega} \chi \mu \, dx - \int_{\Gamma} (v (\lambda) - \lambda) \cdot h \, d\sigma \)
\[- \int_{\Gamma} (\lambda \cdot n) \chi \, d\sigma - 2 \int_{\Gamma} u \cdot D(v(\lambda)) n \, d\sigma \]
\[+ 2 \int_{\Gamma} v (\lambda) \cdot D(u) n \, d\sigma - \int_{\Gamma} (v (\lambda) \cdot n) \pi \, d\sigma = 0. \]

Computations on \( \Omega' \) yield

\[ \int_{\Omega'} (\Delta u' + \nabla \pi') \cdot \lambda \, dx = 0 \]
\[= \int_{\Omega'} u' \cdot \Delta \lambda \, dx + 2 \int_{\Gamma} \lambda \cdot D(u') n \, d\sigma - 2 \int_{\Gamma} u' \cdot D(\lambda) n \, d\sigma - \int_{\Gamma} (\lambda \cdot n) \pi' \, d\sigma. \]

The fact that \( u = u' \) on \( \Gamma \), implies

\[ \int_{\Omega'} u' \cdot \Delta \lambda \, dx = \int_{\Omega'} u' \cdot \nabla \mu \, dx = - \int_{\Omega'} \mu \, \text{div} \, u' \, dx + \int_{\Gamma} \mu (u' \cdot n) \, d\sigma = 0 \]

and we deduce that

\( \int_{\Omega'} u' \cdot \nabla \mu \, dx = - \int_{\Omega'} \mu \, \text{div} \, u' \, dx + \int_{\Gamma} \mu (u' \cdot n) \, d\sigma = 0 \)

and (3.11) yields

\( 2 \int_{\Gamma} \lambda \cdot D(u') n \, d\sigma - 2 \int_{\Gamma} u' \cdot D(\lambda) n \, d\sigma - \int_{\Gamma} (\lambda \cdot n) \pi' \, d\sigma = 0. \)

Combining (3.10) and (3.11) yields

\[ \int_{\mathbb{R}^3} \tilde{f} \cdot \lambda \, dx - \int_{\Omega} \chi \mu \, dx - \int_{\Gamma} \chi (\lambda \cdot n) \, d\sigma + \int_{\Gamma} (\lambda \cdot n) (\pi' - \pi) \, d\sigma \]

(3.12) \[ - 2 \int_{\Gamma} \lambda \cdot D(u') n \, d\sigma + 2 \int_{\Gamma} u \cdot (D(\lambda) n - D(v(\lambda)) n) \, d\sigma \]
\[+ 2 \int_{\Gamma} v(\lambda) \cdot D(u) n \, d\sigma - \int_{\Gamma} (v(\lambda) - \lambda) h \, d\sigma = 0. \]
Due to the fact that \( g = 0 \) on \( \Gamma \) and using the Navier boundary condition in (3.1), we have
\[
2 \int_{\Gamma} u \cdot \left( D(\lambda)n - D(v(\lambda))n \right) d\sigma = 2 \int_{\Gamma} u_{\tau} \cdot \left( [D(\lambda)n]_{\tau} - [D(v(\lambda))n]_{\tau} \right) d\sigma
\]
\[
= \alpha \int_{\Gamma} u_{\tau} \cdot (v(\lambda)_{\tau} - \lambda_{\tau}) d\sigma.
\]
Next, using the fact that \( v(\lambda) \cdot n = \lambda \cdot n \) on \( \Gamma \),
\[
2 \int_{\Gamma} v(\lambda) \cdot D(u)n d\sigma = 2 \int_{\Gamma} v(\lambda)_{\tau} \cdot [D(u)n]_{\tau} d\sigma + 2 \int_{\Gamma} (\lambda \cdot n)[D(u)n]n d\sigma.
\]
Finally, using that \( h \cdot n = 0 \) on \( \Gamma \), we obtain
\[
\int_{\Gamma} (v(\lambda) - \lambda) h d\sigma = \int_{\Gamma} (v(\lambda)_{\tau} - \lambda_{\tau}) h d\sigma.
\]
Combining these three expressions and after calculation, we obtain
\[
2 \int_{\Gamma} \lambda \cdot D(u)n d\sigma = 2 \int_{\Gamma} u \cdot \left( D(\lambda)n - D(v(\lambda))n \right) d\sigma + 2 \int_{\Gamma} v(\lambda) \cdot D(u)n d\sigma
\]
\[
- \int_{\Gamma} (v(\lambda) - \lambda) h d\sigma.
\]
Plugging this in (3.12) allows to get (3.8).

Therefore it follows from Theorem 2.6 that there exists a solution \((\tilde{z}, \tilde{q}) \in (W^{1,p}_{k}(\mathbb{R}^3) \times W^{0,p}_{k}(\mathbb{R}^3))\) satisfying the following Stokes problem:
\[
-\Delta \tilde{z} + \nabla \tilde{q} = F \text{ and div } \tilde{z} = e \text{ in } \mathbb{R}^3.
\]
Using (3.5), we obtain
\[
-\Delta (\tilde{z} - \tilde{u}) + \nabla (\tilde{q} - \tilde{\pi}) = 0 \text{ and div } (\tilde{z} - \tilde{u}) = 0 \text{ in } \mathbb{R}^3.
\]
It follows that \((\tilde{z} - \tilde{u}, \tilde{q} - \tilde{\pi})\) belongs to \((W^{1,p}_{k}(\mathbb{R}^3) + W^{1,2}_{1}(\mathbb{R}^3)) \times (W^{0,p}_{k}(\mathbb{R}^3) + W^{0,2}_{1}(\mathbb{R}^3))\), then \((\tilde{z} - \tilde{u}, \tilde{q} - \tilde{\pi})\) also belongs to \(N_{[1-3/p-k]}\). We deduce that there exist \((\lambda, \mu) \in N_{[1-3/p-k]}\), then \(\tilde{z} - \tilde{u} = \lambda \text{ and } \tilde{q} - \tilde{\pi} = \mu\) which imply that the solution \((u, \pi)\) belongs indeed to \(W^{1,p}_{k}(\Omega) \times W^{0,p}_{k}(\Omega)\).

**Regularity.** Finally, we prove that the solution \((u, \pi) \in W^{1,p}_{k}(\Omega) \times W^{0,p}_{k}(\Omega)\) of \((\mathcal{S}_T)\) established previously, belongs to \(W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)\). Here we use regularity arguments on the Stokes problem set in bounded domains and in the whole
Step 2. The case 

Let \((\tilde{u}, \tilde{\pi}) \in W_{k}^{1,p}(\mathbb{R}^{3}) \times W_{k}^{0,p}(\mathbb{R}^{3})\) be an extension of \((u, \pi)\) to the whole space \(\mathbb{R}^{3}\). We can write:

\[
\tilde{u} = \varphi \tilde{u} + \psi \tilde{u} \quad \text{and} \quad \tilde{\pi} = \varphi \tilde{\pi} + \psi \tilde{\pi}.
\]

Then it is enough to show that the pairs \((\varphi \tilde{u}, \varphi \tilde{\pi})\) and \((\psi \tilde{u}, \psi \tilde{\pi})\) belong to \(W_{k+1}^{2,p}(\mathbb{R}^{3}) \times W_{k+1}^{1,p}(\mathbb{R}^{3})\). To that end, consider first

\[
-\Delta (\psi \tilde{u}) + \nabla (\psi \tilde{\pi}) = f_{1} \quad \text{and} \quad \text{div} (\psi \tilde{u}) = \chi_{1} \quad \text{in} \quad \mathbb{R}^{3},
\]

where

\[
f_{1} = f \psi - (2\nabla \tilde{u} \nabla \psi + u \Delta \psi) + \tilde{\pi} \nabla \psi \quad \text{and} \quad \chi_{1} = \chi \psi + \tilde{u} \cdot \nabla \psi.
\]

We easily see that \(f_{1}\) and \(\chi_{1}\) have bounded supports and belong to \(L_{loc}^{p}(\mathbb{R}^{3}) \times W_{loc}^{1,p}(\mathbb{R}^{3})\). As a consequence, \((f_{1}, \chi_{1})\) belongs to \(W_{k+1}^{0,p}(\mathbb{R}^{3}) \times W_{k+1}^{1,p}(\mathbb{R}^{3})\). Using the regularity of the Stokes problem (see Theorem 2.9), we deduce that the pair \((\psi \tilde{u}, \psi \tilde{\pi})\) also belongs to \(W_{k+1}^{2,p}(\mathbb{R}^{3}) \times W_{k+1}^{1,p}(\mathbb{R}^{3})\).

Consider now the system

\[
-\Delta (\varphi \tilde{u}) + \nabla (\varphi \tilde{\pi}) = f_{2} \quad \text{and} \quad \text{div} (\varphi \tilde{u}) = \chi_{2},
\]

where \(f_{2}\) and \(\chi_{2}\) have similar expressions as \(f_{1}\) and \(\chi_{1}\) with \(\psi\) remplaced by \(\varphi\). It is easy to check that \((f_{2}, \chi_{2})\) belongs to \(L^{p}(\Omega_{R+1}) \times W^{1,p}(\Omega_{R+1})\). In particular, we have \(\tilde{u} = \psi \tilde{u}\) outside \(B_{R+1}\), so the restriction of \(u\) to \(\partial B_{R+1}\) belongs to \(W^{1+1/p',p}(\partial B_{R+1})\). It’s clear that \((\varphi \tilde{u}, \varphi \tilde{\pi})\) belongs to \(H^{2}(\Omega_{R+1}) \times H^{1}(\Omega_{R+1})\) and satisfies (2.9). Then thanks to Theorem 2.9 we prove that \((\varphi \tilde{u}, \varphi \tilde{\pi}) \in W^{2,p}(\Omega_{R+1}) \times W^{1,p}(\Omega_{R+1})\) solution of problem (2.9), which also implies that \((\varphi \tilde{u}, \varphi \tilde{\pi})\) belongs to \(W_{k+1}^{2,p}(\mathbb{R}^{3}) \times W_{k+1}^{1,p}(\mathbb{R}^{3})\). Consequently, the pair \((u, \pi)\) belongs to \(W_{k+1}^{2,p}(\Omega) \times W_{k+1}^{1,p}(\Omega)\) if \(2 \leq p \leq 6\).

Step 2. The case \(g \neq 0\).

Let \(w\) be in \(W_{k+1}^{3,p}(\Omega)\) such that \(\frac{\partial w}{\partial n} = g\) on \(\Gamma\). According to step 1, the following problem

\[
\begin{align*}
-\Delta z + \nabla \pi &= f + \Delta (\nabla w) \quad \text{and} \quad \text{div} z = \chi - \Delta w \quad \text{in} \quad \Omega, \\
z \cdot n &= 0 \quad \text{and} \quad 2[D(z)n]_{\tau} + \alpha z_{\tau} = K \quad \text{on} \quad \Gamma,
\end{align*}
\]

satisfies (3.13) for the same \(f_{1}\) and \(\chi_{1}\). As a consequence of Theorem 2.9, the pair \((z, \pi)\) also belongs to \(W_{k+1}^{2,p}(\mathbb{R}^{3}) \times W_{k+1}^{1,p}(\mathbb{R}^{3})\). Consequently, the pair \((u, \pi)\) belongs to \(W_{k+1}^{2,p}(\Omega) \times W_{k+1}^{1,p}(\Omega)\) if \(2 \leq p \leq 6\).
has a solution in $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ if and only if, $\forall (\xi, \eta) \in \mathcal{N}^{-p}_{-k+1}(\Omega)$,

\begin{equation}
\int_{\Omega} (f + \Delta (\nabla w)) \cdot \xi \, dx - \int_{\Omega} (\chi - \Delta w) \eta \, dx = - \int_{\Gamma} K \cdot \xi \, d\sigma.
\end{equation}

But if $(\xi, \eta)$ is in $\mathcal{N}^{-p}_{-k+1}(\Omega)$, then since $\nabla w \in W^{2,p}_{k+1}(\Omega)$, we can write (8.3) for the pair $(-\nabla w, 0)$ and we obtain

\begin{equation}
\int_{\Omega} \Delta (\nabla w) \cdot \xi \, dx + \int_{\Omega} \Delta w \, \eta \, dx = - \int_{\Gamma} g\left(2[D(\xi) n] \cdot n - \eta\right) \, d\sigma
\end{equation}

Combining (3.2) and (3.16) allows to obtain (3.15). Thus setting $W^{2,p}_{k+1}(\Omega)$ is a solution of $(S_T)$.

Now, suppose that $p > 6$. The above argument shows that $(u, \pi)$ belongs to $W^{2,6}_{k+1}(\Omega) \times W^{1,6}_{k+1}(\Omega)$ and we can repeat the same argument with $p = 6$ instead of $p = 2$ using the fact if $\mathcal{O}$ is a lipschitzian bounded domain, we have $W^{1,6}(\mathcal{O}) \hookrightarrow L^q(\mathcal{O})$ for any real number $q > 1$. We know that the following embedding holds

$$W^{2,p}(\mathcal{O}') \hookrightarrow W^{1,q}(\mathcal{O}')$$

for any $q \in [1, \infty]$ if $p > 3$. Then we have $u \in W^{2-1/6,6}(\Gamma) \hookrightarrow W^{1-1/p}(\Gamma)$ for all $p > 6$. Thus establishes the existence of solution $(u, \pi)$ in $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ of problem $(S_T)$ when $p > 6$.

We finally close this section by the following theorem.

**Theorem 3.3.** Assume that $p \geq 2$. Let $f \in W^{0,p}_{k+1}(\Omega)$, $\chi \in W^{1,p}_{k+1}(\Omega)$, $g \in W^{1+1/p',p}(\Gamma)$, $h \in W^{1-1/p,p}(\Gamma)$ such that $h \cdot n = 0$ on $\Gamma$ and that the compatibility condition (3.2) is satisfied. Then, the Stokes problem $(S_T)$ has a solution $(u, \pi) \in W^{2,p}_{k+1}(\Omega) \times W^{p}_{k+1}(\Omega)$ unique up to an element of $\mathcal{N}^{-p}_{k+1}(\Omega)$. In addition, we have the following estimate:

$$\inf_{(\lambda, \mu) \in \mathcal{N}^{-p}_{k+1}(\Omega)} \left(\|u + \lambda\|_{W^{2,p}_{k+1}(\Omega)} + \|\pi + \mu\|_{W^{1,p}_{k+1}(\Omega)}\right) \leq C \left(\|f\|_{W^{0,p}_{k+1}(\Omega)} + \|\chi\|_{W^{1,p}_{k+1}(\Omega)} + \|h\|_{W^{1-1/p,p}(\Gamma)} + \|g\|_{W^{1+1/p',p}(\Gamma)}\right)$$.
Proof.

Here we prove that problem \((S_T)\) has a solution \((u, \pi)\) that belongs to \(W^{2p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)\). To that end, we proceed in two steps.

Step 1. The case \(k < 3/p' - 1\).

First, let extend \(f\) by zero in \(\Omega'\) and denote by \(\tilde{f} \in W^{0,p}_{k+1}(\mathbb{R}^3)\) the extended function. Moreover, let \(\tilde{\chi} \in W^{1,p}_{k+1}(\mathbb{R}^3)\) be an extension of \(\chi\). We consider the following problem:

\[
\begin{align*}
\Delta \tilde{w} + \nabla \tilde{\eta} &= \tilde{f} & \text{in } \mathbb{R}^3, \\
\text{div } \tilde{w} &= \tilde{\chi} & \text{in } \mathbb{R}^3.
\end{align*}
\]

Since \(k < 3/p' - 1\), then \(N_{[1-3/p'+k]} = \{(0, 0)\}\) and thus applying Theorem 2.6, we deduce that this problem has a solution \((\tilde{w}, \tilde{\eta}) \in W^{2p}_{k+1}(\mathbb{R}^3) \times W^{1,p}_{k+1}(\mathbb{R}^3)\). Denoting the restriction to \(\Omega\) by \((w, \eta)\) belongs to \(W^{2p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)\). Since \(w \in W^{2p}_{k+1}(\Omega)\), then we have \([D(w)n]_\tau\) belongs to \(W^{1-1/p,p}(\Gamma)\). Consider now the following problem:

\[
\begin{align*}
-D v + \nabla \theta &= 0, & \text{div } v &= 0 & \text{in } \Omega, \\
v \cdot n &= G & \text{and } 2[D(v)n]_\tau + \alpha v_\tau &= H & \text{on } \Gamma,
\end{align*}
\]

Where \(H = -2[D(w)n]_\tau - \alpha w_\tau + h\) and \(G = g - w \cdot n\). It's clear that \(G \in W^{1+1/p',p}(\Gamma)\) and \(H \in W^{1-1/p,p}(\Gamma)\) such that \(H \cdot n = 0\) on \(\Gamma\). According to Theorem 3.2, the problem \((3.17)\) has a solution in \(W^{2p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)\). Hence, the pair \((w + v, \eta + \theta)\) belongs to \(W^{2p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)\) and satisfies problem \((S_T)\).

Step 2. The case \(k \geq 3/p' - 1\). We split this step in two cases:

The case \(g = 0\). Since \(k \geq 3/p' - 1 > 0\), then \((f, \chi)\) belongs to \(W^{0,p}_1(\Omega) \times W^{1,p}_1(\Omega)\). According to Step 1, the problem \((S_T)\) has a solution \((u, \pi)\) belongs to \(W^{2p}_1(\Omega) \times W^{1,p}_1(\Omega)\). It remains now to prove that \((u, \pi)\) belongs to \(W^{2p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)\). To that end, we shall use again properties of the Stokes problem in the whole space \(\mathbb{R}^3\).

Since \(u \cdot n = 0\) on \(\Gamma\), problem \((3.17)\) has a solution \((u', \pi') \in W^{2p}(\Omega') \times W^{1,p}(\Omega')\). Set now

\[
\tilde{u} = \begin{cases} u & \text{in } \Omega, \\
u' & \text{in } \Omega', \end{cases} \quad \text{and} \quad \tilde{\pi} = \begin{cases} \pi & \text{in } \Omega, \\
\pi' & \text{in } \Omega'. \end{cases}
\]
Then clearly, the pair \((\tilde{\mathbf{u}}, \tilde{\pi})\) belongs to \(W_{1}^{1,p}(\mathbb{R}^{3}) \times W_{1}^{0,p}(\mathbb{R}^{3})\). In order to apply Theorem 2.6 with data \(-\Delta \tilde{\mathbf{u}} + \nabla \tilde{\pi}\) and \(\text{div} \tilde{\mathbf{u}}\), we need to show that \((-\Delta \tilde{\mathbf{u}} + \nabla \tilde{\pi}, \text{div} \tilde{\mathbf{u}})\) belongs to \(W_{-k}^{1,p}(\mathbb{R}^{3}) \times W_{-k}^{0,p}(\mathbb{R}^{3})\) and satisfies (2.8). Therefore, denoting by \(\tilde{\mathbf{f}} \in W_{k+1}^{0,p}(\mathbb{R}^{3})\) the extension of \(\mathbf{f}\) by zero in \(\Omega'\). For any \(\varphi \in \mathcal{D}(\mathbb{R}^{3})\), using the same calculation as in the proof of Theorem 3.2, we have:

\[
\langle -\Delta \tilde{\mathbf{u}} + \nabla \tilde{\pi}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^{3}) \times \mathcal{D}(\mathbb{R}^{3})} = \int_{\mathbb{R}^{3}} \tilde{\mathbf{f}} \cdot \varphi \, dx + 2 \int_{\Gamma} \varphi \cdot (\mathbf{D}(\mathbf{u}) \mathbf{n} - \mathbf{D}(\mathbf{u}') \mathbf{n}) \, d\sigma \\
+ \int_{\Gamma} (\varphi \cdot \mathbf{n})(\tilde{\pi}' - \tilde{\pi}) \, d\sigma - \int_{\Gamma} \varphi \cdot \mathbf{n}, \chi \, d\sigma.
\]

(3.19)

Since \(\mathcal{D}(\mathbb{R}^{3})\) is dense in \(W_{k}^{1,p}(\mathbb{R}^{3})\) and \(\tilde{\mathbf{f}}\) belongs to \(W_{k+1}^{0,p}(\mathbb{R}^{3})\), then (3.19) is still valid for any \(\varphi \in W_{-k}^{1,p}(\mathbb{R}^{3})\) which implies that \(-\Delta \tilde{\mathbf{u}} + \nabla \tilde{\pi}\) belongs to \(W_{k}^{-1,p}(\mathbb{R}^{3})\). Now for any \(\phi \in \mathcal{D}(\mathbb{R}^{3})\), we have

\[
\langle \text{div} \tilde{\mathbf{u}}, \phi \rangle_{\mathcal{D}'(\mathbb{R}^{3}) \times \mathcal{D}(\mathbb{R}^{3})} = \int_{\Omega} \chi \phi \, dx.
\]

(3.20)

Since \(\chi\) belongs to \(W_{k+1}^{1,p}(\Omega)\) and due to the density of \(\mathcal{D}(\mathbb{R}^{3})\) in \(W_{-k}^{0,p}(\mathbb{R}^{3})\), then (3.20) is still valid for any \(\phi \in W_{-k}^{0,p}(\mathbb{R}^{3})\), which implies that \(\text{div} \tilde{\mathbf{u}}\) belongs to \(W_{k}^{0,p}(\mathbb{R}^{3})\). As a consequence, we have proved that \((-\Delta \tilde{\mathbf{u}} + \nabla \tilde{\pi}, \text{div} \tilde{\mathbf{u}})\) belongs to \(W_{k}^{-1,p}(\mathbb{R}^{3}) \times W_{k}^{0,p}(\mathbb{R}^{3})\). Using the same calculation as in the proof of Theorem 3.2, we prove that \((-\Delta \tilde{\mathbf{u}} + \nabla \tilde{\pi}, \text{div} \tilde{\mathbf{u}})\) satisfy (2.8). Therefore it follows from Theorem 2.6 that there exists a solution \((\tilde{\mathbf{z}}, \tilde{\mathbf{q}}) \in (W_{k}^{1,p}(\mathbb{R}^{3}) \times W_{k}^{0,p}(\mathbb{R}^{3}))\) satisfying the following Stokes problem:

\[-\Delta (\tilde{\mathbf{z}} - \tilde{\mathbf{u}}) + \nabla (\tilde{\mathbf{q}} - \tilde{\pi}) = \mathbf{0} \quad \text{and} \quad \text{div} (\tilde{\mathbf{z}} - \tilde{\mathbf{u}}) = 0 \quad \text{in} \quad \mathbb{R}^{3}.
\]

It follows that \((\tilde{\mathbf{z}} - \tilde{\mathbf{u}}, \tilde{\mathbf{q}} - \tilde{\pi})\) belongs to \((W_{1}^{1,p}(\mathbb{R}^{3}) + W_{1}^{0,p}(\mathbb{R}^{3})) \times (W_{1}^{0,p}(\mathbb{R}^{3}) + W_{1}^{0,p}(\mathbb{R}^{3}))\), then \((\tilde{\mathbf{z}} - \tilde{\mathbf{u}}, \tilde{\mathbf{q}} - \tilde{\pi})\) also belongs to \(N_{[1-3/p-k]}\). We deduce that there exist \((\lambda, \mu) \in N_{[1-3/p-k]}\), such that \(\tilde{\mathbf{z}} - \tilde{\mathbf{u}} = \lambda\) and \(\tilde{\mathbf{q}} - \tilde{\pi} = \mu\) which imply that the solution \((\mathbf{u}, \pi)\) belongs indeed to \(W_{1}^{1,p}(\Omega) \times W_{1}^{0,p}(\Omega)\).

Finally, to prove that the solution \((\mathbf{u}, \pi) \in W_{k}^{1,p}(\Omega) \times W_{k}^{0,p}(\Omega)\) of \((S_{T})\) established previously, actually belongs to \(W_{k+1}^{2,p}(\Omega) \times W_{k+1}^{1,p}(\Omega)\), we can proceed as in the proof of Theorem 3.2 with the use of the partition of unity (2.15).
For the case $g \neq 0$, we proceed as in the same way as in the proof of Theorem 3.2, we prove that the pair $(u, \pi)$ belongs to $W^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ is a solution of $(S_T)$.

□

References

[1] Paul Acevedo, Chérif Amrouche, Carlos Conca, and Amrita Ghosh. Stokes and Navier-Stokes equations with Navier boundary condition. *C. R. Math. Acad. Sci. Paris*, 357(2):115–119, 2019.

[2] Y. Achdou and O. Pironneau. Domain decomposition and wall laws. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(5):541–547, 1995.

[3] Y. Achdou, O. Pironneau, and F. Valentin. Effective boundary conditions for laminar flows over periodic rough boundaries. *J. Comput. Phys.*, 147(1):187–218, 1998.

[4] Y. Achdou, O. Pironneau, and F. Valentin. Shape control versus boundary control. In *Équations aux dérivées partielles et applications*, pages 1–18. Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998.

[5] F. Alliot and C. Amrouche. The Stokes problem in $\mathbb{R}^n$: an approach in weighted Sobolev spaces. *Math. Models Methods Appl. Sci.*, 9(5):723–754, 1999.

[6] F. Alliot and C. Amrouche. Weak solutions for the exterior Stokes problem in weighted Sobolev spaces. *Math. Methods Appl. Sci.*, 23(6):575–600, 2000.

[7] C. Amrouche and V. Girault. Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. *Czechoslovak Math. J.*, 44(119)(1):109–140, 1994.

[8] C. Amrouche, V. Girault, and J. Giroire. Weighted Sobolev spaces for Laplace’s equation in $\mathbb{R}^n$. *J. Math. Pures Appl. (9)*, 73(6):579–606, 1994.

[9] C. Amrouche, V. Girault, and J. Giroire. Dirichlet and Neumann exterior problems for the $n$-dimensional Laplace operator: an approach in weighted Sobolev spaces. *J. Math. Pures Appl. (9)*, 76(1):55–81, 1997.

[10] C. Amrouche and M. Meslameni. Stokes problem with several types of boundary conditions in an exterior domain. *Electron. J. Differential Equations*, No. 196, 28 pp, 2013.

[11] C. Amrouche and A. Rejaiba. $L^p$-theory for Stokes and Navier-Stokes equations with Navier boundary condition. *J. Differential Equations*, 256(4):1515–1547, 2014.

[12] C. Amrouche and A. Rejaiba. Navier-Stokes equations with Navier boundary condition. *Math. Methods Appl. Sci.*, 39(17):5091–5112, 2016.

[13] S. N. Antontsev and H. B. de Oliveira. Navier-Stokes equations with absorption under slip boundary conditions: existence, uniqueness and extinction in time. In *Kyoto Conference on*
the Navier-Stokes Equations and their Applications, RIMS Kökyûroku Bessatsu, B1, pages 21–41. Res. Inst. Math. Sci. (RIMS), Kyoto, 2007.

14. A. Basson and D. Gérard-Varet. Wall laws for fluid flows at a boundary with random roughness. *Comm. Pure Appl. Math.*, 61(7):941–987, 2008.

15. H. Beirão Da Veiga. Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip-type boundary conditions. *Adv. Differential Equations*, 9(9-10):1079–1114, 2004.

16. H. Beirão da Veiga. Vorticity and regularity for flows under the Navier boundary condition. *Commun. Pure Appl. Anal.*, 5(4):907–918, 2006.

17. J. Casado-Díaz, E. Fernández-Cara, and J. Simon. Why viscous fluids adhere to rugose walls: a mathematical explanation. *J. Differential Equations*, 189(2):526–537, 2003.

18. A. Dhifaoui. Équations de Stokes en domaine extérieur avec des conditions aux limites de type Navier. PhD thesis, 2020. Thèse de doctorat dirigée par Razafison, Ulrich Jerry et Ben Hamed, Bassem Mathématiques Bourgogne Franche-Comté 2020.

19. A. Dhifaoui, M. Meslameni., and U. Razafison. Weighted Hilbert spaces for the stationary exterior Stokes problem with Navier slip boundary conditions. *J. Math. Anal. Appl.*, 472(2):1846–1871, 2019.

20. A. Dhifaoui. $L^p$-theory for the exterior stokes problem with navier’s type slip-without-friction boundary conditions. *arXiv preprint arXiv:2111.05822*, 2021.

21. A. Dhifaoui. Very weak solution for the exterior stationary stokes equations with navier slip boundary condition. *arXiv preprint arXiv:2111.05827*, 2021.

22. E. Friedmann. The optimal shape of riblets in the viscous sublayer. *J. Math. Fluid Mech.*, 12(2):243–265, 2010.

23. E. Friedmann and T. Richter. Optimal microstructures drag reducing mechanism of riblets. *J. Math. Fluid Mech.*, 13(3):429–447, 2011.

24. G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994.

25. D. Gérard-Varet and N. Masmoudi. Relevance of the slip condition for fluid flows near an irregular boundary. *Comm. Math. Phys.*, 295(1):99–137, 2010.

26. V. Girault. The Stokes problem and vector potential operator in three-dimensional exterior domains: an approach in weighted Sobolev spaces. *Differential Integral Equations*, 7(2):535–570, 1994.

27. V. Girault, J. Giroire, and A. Sequeira. A stream-function–vorticity variational formulation for the exterior Stokes problem in weighted Sobolev spaces. *Math. Methods Appl. Sci.*, 15(5):345–363, 1992.

28. V. Girault and A. Sequeira. A well-posed problem for the exterior Stokes equations in two and three dimensions. *Arch. Rational Mech. Anal.*, 114(4):313–333, 1991.
[29] B. Hanouzet. Espaces de Sobolev avec poids application au problème de Dirichlet dans un demi espace. *Rend. Sem. Mat. Univ. Padova*, 46:227–272, 1971.

[30] B. Hanouzet. Espaces de Sobolev avec poids. Application au problème de Dirichlet dans un demi espace. *Rend. Sem. Mat. Univ. Padova*, 46:227–272, 1971.

[31] W. Jäger and A. Mikelić. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations*, 170(1):96–122, 2001.

[32] Daniel D. Joseph and Gordon S. Beavers. Boundary conditions at a naturally permeable wall. *J. Fluid Mech.*, 30:197–207, 1967.

[33] H. Louati, M. Meslameni, and U. Razafison. Weighted $L^p$-theory for vector potential operators in three-dimensional exterior domains. *Math. Methods Appl. Sci.*, 39(8):1990–2010, 2016.

[34] H. Louati, M. Meslameni, and U. Razafison. On the three-dimensional stationary exterior stokes problem with non standard boundary conditions. *ZAMM - Journal of Applied Mathematics and Mechanics*, 2020.

[35] G. Mulone and F. Salemi. On the existence of hydrodynamic motion in a domain with free boundary type conditions. *Meccanica*, 18:136–144, 1983.

[36] G. Mulone and F. Salemi. On the hydrodynamic motion in a domain with mixed boundary conditions: existence, uniqueness, stability and linearization principle. *Ann. Mat. Pura Appl. (4)*, 139:147–174, 1985.

[37] C.L.M.H. Navier. Mémoire sur les Lois du Mouvement des fluides. *Mem. Acad. Sci. Inst. de France (2)*, 6:389–440, 1827.

[38] A. Russo and A. Tartaglione. On the Navier problem for the stationary Navier-Stokes equations. *J. Differential Equations*, 251(9):2387–2408, 2011.

[39] V. A. Solonnikov and V. E. Scadilov. A certain boundary value problem for the stationary system of Navier-Stokes equations. *Trudy Mat. Int. Steklov.*, 125:1515–1547,235, 1973.

[40] M. Specovius-Neugebauer. Exterior Stokes problems and decay at infinity. *Math. Methods Appl. Sci.*, 8(3):351–367, 1986.

[41] M. Specovius-Neugebauer. Weak solutions of the Stokes problem in weighted Sobolev spaces. *Acta Appl. Math.*, 37(1-2):195–203, 1994.

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