1. Introduction

Let $H$ be the upper half-plane with the Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$. A cofinite group is a discrete group $\Gamma \subset \text{PSL}(2, \mathbb{R})$ with noncompact fundamental domain $F$ whose area $|F|$ with respect to the invariant measure $d\mu = y^{-2}dx\,dy$ is finite. In what follows, we only deal with cofinite groups $\Gamma$.

The Laplace operator $\Delta = y^2(\partial_x^2 + \partial_y^2)$ extends to be a self-adjoint operator on $L^2(F, d\mu)$ with continuous spectrum covering the interval $[1/4, \infty)$ and with discrete spectrum $\{\lambda_n\} \ (\Delta \varphi_n + \lambda_n \varphi_n = 0, 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots, \varphi_n \in L^2(F, d\mu))$. Little is known about the structure of the discrete spectrum. In particular, it is not known for what groups $\Gamma$ this spectrum is infinite.

Selberg posed the following question: what cofinite groups $\Gamma$ satisfy the Weyl formula

$$N_\Gamma \left(T^2 + \frac{1}{4}\right) = \# \left\{ n \mid \lambda_n \leq T^2 + \frac{1}{4} \right\} \sim \frac{|F|}{4\pi} T^2 \quad (T \to \infty)?$$

Nowadays, such groups are said to be essentially cuspidal. Formula (1.1) has been proved for a number of groups (see [1–4]), in particular, for the congruence subgroups of $\text{SL}(2, \mathbb{Z})$. However, all these groups correspond to nongeneric points in the Teichmüller space. Roelke (e.g., see [1]) conjectured that $N_\Gamma(T^2 + 1/4) \to \infty$ as $T \to \infty$.

The interest in these questions arose in connection with the papers [5,6]. These papers, as well as [7,8], provide a number of sufficient conditions for the Weyl law (1.1) to be violated. Based on these results, Sarnak [2] conjectured that neither the Weyl law nor even the Roelke conjecture holds for generic cofinite groups $\Gamma$.

One main approach to studying the spectrum $\{\lambda_n\}$ is based on the Selberg formula, and all the results of the present paper are corollaries of this formula.
The Selberg formula for cofinite groups is given in Sec. 2. Symbolically, it can be written as
\begin{equation}
\sum_{n \geq 0} h(r_n) = \Phi_{\Gamma}[h\{\lambda_n\}, \{N(P)\}, \varphi] \quad \forall h \in \{h\}, \quad \lambda_n = r_n^2 + \frac{1}{4},
\end{equation}
where $\Phi_{\Gamma}$ is a functional on the space $\{h\}$ (see Sec. 2). The functional $\Phi_{\Gamma}$ depends on the spectrum $\{\lambda_n\}$, the set $\{N(P)\}$ of norms of hyperbolic conjugacy classes, the function $\varphi$ defined by $\varphi(s) = \det \Phi(s)$ (where $\Phi(s)$ is the scattering matrix), and finitely many parameters such as $|F|$, the number of elliptic and parabolic classes, $\text{tr} \Phi(1/2)$, etc. The Selberg formula for a cocompact group $\Gamma$ reads
\begin{equation}
\sum_{n \geq 0} h(r_n) = \Phi_{\Gamma}[\lambda_n, \{N(P)\}].
\end{equation}

It was shown in [10] that, for strictly hyperbolic groups, the Selberg formula (supplemented by an additional condition on the function $h$) implies that the spectrum $\{\lambda_n\}$ satisfies equations of the form
\begin{equation}
\sum_{n \geq 0} h(r_n) = \tilde{\Phi}_{\Gamma}[\lambda_n].
\end{equation}

One aim of the present paper is to generalize this result to arbitrary cofinite groups. We show (Theorem 1) that the following analog of formula (1.4) holds:
\begin{equation}
\sum_{n \geq 0} h(r_n) = \tilde{\Phi}_{\Gamma}[\lambda_n, \{s_\alpha\}].
\end{equation}

Here $\{s_\alpha\}$ is the set of poles $s_\alpha = \beta_\alpha + i\gamma_\alpha$ of $\varphi$ such that $\beta_\alpha < 1/2$ and $\gamma_\alpha \neq 0$. This set will be called the resonance spectrum.

We take various functions $h$ and obtain relations that should be satisfied by the discrete spectrum and the resonance spectrum. In Sec. 6 we compute the asymptotics of $\tilde{\Phi}_{\Gamma}[h\{\lambda_n\}, \{s_\alpha\}]$ as $t \to 0$ for the case of $h(r) = e^{-tr^2(r^2 + p^2)^{-1}}$ and use this asymptotics to prove the Roelke conjecture (Theorem 2).

Note that if $\Gamma = \text{SL}(2, \mathbb{Z})$, then $s_\alpha = \rho_\alpha/2$, where the $\rho_\alpha$ are the nontrivial zeros of the Riemann zeta function, and formulas (1.5) specify relations between the sets $\{\lambda_n\}$ and $\{\rho_\alpha\}$.

A preliminary version of Theorem 1 containing a number of inaccuracies, was published in [11]. The second theorem in [11], which gave some sufficient conditions for the Roelke conjecture to be true, was based on the assumption that the series $D_\Gamma = \sum s_\alpha |s_\alpha|^{-2}$ converges for any group $\Gamma$. I am grateful to the referee of [11] for pointing out that this assumptions has never been proved and is most likely false.

2. Preliminaries

This section provides some insight into the Selberg formula for cofinite groups and introduces notation to be used in the paper. All the information given here can be found in [13][12][17].

Throughout the paper, we assume that the functions $h(\cdot)$ belong to the class $\{h\}$, that is, satisfy the following conditions:
1. $h(r) = h(-r)$.
2. The function $h$ is holomorphic in the strip $\{|\text{Im} r| \leq 1/2 + \varepsilon, \varepsilon > 0\}$.
3. In this strip, one has $|h(r)| = O(1 + |r|^2)^{-1-\varepsilon}$ ($|r| \to \infty$).
By $g$ we denote the Fourier transform of $h$,

\begin{equation}
(2.1) \quad g(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)e^{-iry} \, dr, \quad h(r) = \int_{-\infty}^{\infty} e^{iry}g(y) \, dy.
\end{equation}

The numbers $r_n$, $s_n$, and $\pi_n$ with a Latin subscript are defined by the formulas

\begin{equation}
(2.2) \quad \lambda_n = s_n\pi_n, \quad \pi_n = 1 - s_n, \quad s_0 = 1, \quad s_n = \frac{1}{2} + ir_n, \quad \lambda_n = \frac{1}{4} + r_n^2.
\end{equation}

The eigenvalues $\lambda_n$ in the interval $0 \leq \lambda_n < 1/4$ are said to be exceptional; the number $M$ of such eigenvalues is finite. For the exceptional eigenvalues, one has

\begin{equation}
(2.3) \quad r_n = -i\left(\frac{1}{4} - \lambda_n\right)^{1/2}, \quad r_0 = -\frac{i}{2}.
\end{equation}

The eigenvalues $\lambda_n \geq 1/4$ will be numbered by a subscript $j$, so that $\lambda_j = 1/4 + r_j^2$, $r_j \geq 0$.

When considering the Selberg formula, we restrict ourselves to the case of the trivial one-dimensional representation $\chi: \Gamma \to \mathbb{C}$, $\chi(\gamma) = 1$, $\forall \gamma \in \Gamma$. Then the Selberg formula reads

\begin{equation}
(2.4) \quad \sum_{n \geq 0} h(r_n) = H[h] + S_R[h] + S_P[g] + \mathcal{P}[h|\varphi]
\end{equation}

for every $h \in \{h\}_\Gamma$ and determines the form of the functional $\Phi_\Gamma$ (1.2).

Let us give the definitions of the objects occurring on the right-hand side in (2.3).

First,

\begin{equation}
(2.5) \quad H[h] = \frac{|F|}{4\pi} \int_{-\infty}^{\infty} r \tanh \pi rh(r) \, dr.
\end{equation}

Second, $S_R[h]$ is the contribution of the conjugacy classes (in $\Gamma$) of elliptic elements (the contribution of elliptic conjugacy classes), and

\begin{equation}
(2.6) \quad S_R[h] = \sum_{\{R\}} \sum_{k=1}^{p-1} \frac{1}{p\sin \pi k/p} \int_{-\infty}^{\infty} h(r) \frac{e^{-2\pi kr/p}}{1 + e^{-2\pi r}} \, dr,
\end{equation}

where the sum is over the set $\{R\}$ of primitive elliptic conjugacy classes and $p = p(R)$ is the order of a class $R$. The number $\{|\{R\}|$ of elliptic conjugacy classes and their maximum order are finite.

The third term on the right-hand side in (2.3) is the contribution of hyperbolic conjugacy classes; it is given by the formula

\begin{equation}
(2.7) \quad B_0 = \min_{\{P\}} N(P), \quad B_0 > 1, \quad b_0 = \ln B_0.
\end{equation}
The last term $\mathcal{P}[h|\varphi]$ on the right-hand side in (2.3) is the contribution of parabolic conjugacy classes (the contribution of the continuous spectrum), and

$$
\mathcal{P}[h|\varphi] = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi'(1/2 + ir)}{\varphi(1/2 + ir)} dr - \frac{n}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1 + ir)}{\Gamma(1 + ir)} dr 
$$

$$
- \frac{h(0)}{4} \left( n - \text{tr} \Phi \left( \frac{1}{2} \right) \right) - ng(0) \ln 2.
$$

(2.8)

Here $\Phi(s)$ is the $n \times n$ matrix of free terms in the Eisenstein series (the scattering matrix),

$$
\varphi(s) = \det \Phi(s),
$$

(2.9)

$n$ is the number of primitive parabolic conjugacy classes (the number of pairwise nonequivalent parabolic points in $F$), and $\Gamma(\cdot)$ is the gamma function.

The properties of $\varphi$ are described in [1,3,12,17]. This is a meromorphic function satisfying the functional equations

$$
\varphi(s)\varphi(1-s) = 1, \quad \varphi(s) = \overline{\varphi(\bar{s})},
$$

(2.10)

where the tilde stands for complex conjugation. The function $\varphi$ is holomorphic in the half-plane $\Re s > 1/2$ except for finitely many poles on the interval $(1/2, 1]$. The poles $s_\alpha$ of $\varphi$ with $\Re s_\alpha < 1/2$ lie in the strip $1 - \mu_0 < \Re s < 1/2$ symmetrically with respect to the real axis, and

$$
\sum_\alpha \left( \frac{1}{2} - \beta_\alpha \right) |s_\alpha|^2 = C_\Gamma < \infty, \quad \sum_{0<\gamma_\alpha \leq x} 1 \leq A_\Gamma x^2.
$$

(2.11)

Throughout the following, Greek subscripts are used to number the poles of $\varphi$.

The Selberg zeta function $Z(\cdot)$ for cofinite groups is defined in the same way as for cocompact groups; for $\Re s > 1$, one has

$$
Z(s) = \prod_{\{P_0\}} \prod_{k=1}^\infty [1 - N(P_0)^{-k-s}].
$$

This function has an analytic continuation into the entire plane of the variable $s = \sigma + it$ as a meromorphic function and satisfies a functional equation of the form $Z(1-s) = A(s)Z(s)$. An explicit expression for the factor $A(s)$ can be found in the papers [1,12], which give a complete description of all the zeros and poles of $Z$, their multiplicities being indicated. Let us present this description and simultaneously introduce a numbering to be used for the nontrivial zeros of the Selberg zeta function.

The nontrivial zeros of $Z(\cdot)$ are

1. The zeros $s_j$ on the critical line $\Re s = 1/2$. They are arranged symmetrically with respect to the real axis, and one has the corresponding eigenvalues

$$
\lambda_j = s_j (1 - s_j), \quad s_j = 1/2 + r_j \quad (j \geq 0).
$$

2. The zeros $s_m \in (0, 1)$, $m = 1, \ldots, M_1$. They are arranged symmetrically with respect to the point $s = 1/2$, and one has the corresponding eigenvalues

$$
\lambda_m = s_m (1 - s_m), \quad s_m = \sigma_m.
$$

3. The zeros

$$
\sigma_\alpha = \beta_\alpha + i\gamma_\alpha, \quad 1 - \mu_0 < \beta_\alpha < 1/2,
$$
at the poles of the function \( \varphi \). These zeros are arranged symmetrically with respect to the real axis.

4. The zeros \( s_\nu = \sigma_\nu, 1/2 < \sigma_\nu \leq 1, \nu = 0, 1, \ldots, M_2 - 1, \) at the poles of \( \varphi \). One has the corresponding eigenvalues \( \lambda_\nu = \sigma_\nu(1 - \sigma_\nu) (\nu \neq 0) \) and \( \lambda_0 = 0 (\sigma_0 = 1) \). The poles of \( Z(\cdot) \) lie at the points \( s = -l + 1/2, l = 0, 1, \ldots, \) and the trivial zeros of \( Z(\cdot) \) lie at the points \( s = -l, l = 0, 1, \ldots. \)

The numbers \( \lambda_m, \lambda_\nu, \) and \( \lambda_j \) exhaust the whole discrete spectrum, and so \( \{\lambda_n\} = \{\lambda_m\} \cup \{\lambda_\nu\} \cup \{\lambda_j\} \).

### 3. Explicit formula for \( S_P[g] \)

In analytic number theory, the term *explicit formulas* refers to formulas representing the object of study by a series over zeros and poles of the corresponding analytic function. The main example is given by the explicit formula representing the Chebyshev function \( \Psi(\cdot) \) by a series over the nontrivial zeros of the Riemann zeta function. In our setting, the function (see \cite{14, 15})

\[
\Lambda^\Gamma(P) = \frac{\ln N(P_0)}{1 - N(P)^{-1}}
\]

is an analog of the Mangoldt function \( \Lambda(\cdot) \), and

\[
\Psi^\Gamma(x) = \sum_{B_0 \leq N(P) \leq x} \Lambda^\Gamma(P)
\]

is the corresponding analog of the Chebyshev function. Based on the results in \cite{15, 10}, let us present a definitive version of an explicit formula for the function

\[
\Psi^\Gamma_1(x) = \int_{B_0}^{x} \Psi^\Gamma(\xi) d\xi.
\]

This formula reads

\[
\Psi^\Gamma_1(x) = \Sigma_{R,\Delta}(x) + \Sigma_{R,\varphi}(x) + \Psi^\Gamma_{1,0} + \Delta_R(x).
\]

The functions \( \Sigma_{R,\Delta}(x) \) and \( \Sigma_{R,\varphi}(x) \) on the right-hand side in (3.4) are given by the formulas

\[
\Sigma_{R,\Delta}(x) = \sum_{0 \leq \nu \leq R} \frac{x^{1+s_j}}{s_j(1+s_j)} + \frac{x^{1+s_j}}{s_j(1+s_j)} + \sum_{1/2 \leq s_m < 1} \left( \frac{x^{1+s_m}}{s_m(1+s_m)} + \frac{x^{1+s_m}}{s_m(1+s_m)} \right) + \sum_{\nu=0}^{M_2-1} \frac{x^{1+s_\nu}}{s_\nu(1+s_\nu)},
\]

\[
\Sigma_{R,\varphi}(x) = \sum_{(\alpha,R)} \frac{x^{1+s_\alpha}}{s_\alpha(1+s_\alpha)} + \frac{x^{1+s_\alpha}}{s_\alpha(1+s_\alpha)} (s_\alpha = \beta_\alpha + i\gamma_\alpha).
\]

Just as above, the tilde stands for complex conjugation, \( \bar{s} = 1 - s \), and the summation in \( \Sigma_{(\alpha,R)} \) is over all the poles \( s_\alpha \) of \( \varphi \) such that

\[
\beta_\alpha < 1/2, \quad \gamma_\alpha > 0, \quad 0 < \gamma_\alpha < R.
\]

Throughout the following, we assume that

\[
B_0 \geq x_0.
\]
Note that $B_0 = (1/4)(3 + \sqrt{5})^2 \approx 6.8541$ if $\Gamma = \text{SL}(2, \mathbb{Z})$.

Formula (3.4) can be proved by standard methods of analytic number theory based on the integral representation [15]

$$\Psi_{\Gamma}^1(x) = \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{x^{s+1}}{s(s+1)} \frac{Z'(s)}{Z(s)} ds, \quad \sigma_1 > 1.$$ 

An analog of formula (3.4) for cocompact groups was proved in [15]; in this case, one can take $x_0 = 2$. Note that it is sufficient for us to have the coarse estimate

$$|\Delta_R(x)| = O\left(\frac{x^k \ln x}{v(R)}\right), \quad v(R) \to \infty \quad (R \to \infty).$$

The first two terms on the right-hand side in (3.4) are the sum of residues of the integrand in the rectangular domain with vertices $\sigma_1 \pm iR$ and $-A \pm iR$ ($A \to \infty$). The residues at the points $s = 0, -1$ must be considered separately (see [15]), and their contribution is included in $\Psi_{\Gamma,0}^1(x)$. This function includes the contributions of the poles and trivial zeros of $Z(\cdot)$, and

$$\Psi_{\Gamma,0}^1(x) = -\frac{2}{3} x^{3/2} (n - \text{tr} \Phi(1/2)) + \tilde{\Psi}_{\Gamma,0}^1(x).$$

The first term on the right-hand side in this formula is the contribution of the pole at $s = 1/2$. The function $\Psi_{\Gamma,0}^1(x)$ can be explicitly evaluated based on the results in [112], and $\tilde{\Psi}_{\Gamma,0}^1(\cdot)$ is a function differentiable for $x \geq 2$ and such that

$$\tilde{\Psi}_{\Gamma,0}^1(x) = C_1^\Gamma \ln x + O(x), \quad \frac{d\tilde{\Psi}_{\Gamma,0}^1(x)}{dx} = C_1^\Gamma \ln x + C_2^\Gamma + O(x^{-1}).$$

For example, if $\Gamma = \text{SL}(2, \mathbb{Z})$, then

$$\tilde{\Psi}_{\Gamma,0}^1(x) = C_1 x \ln x + C_2 x + C_3 + C_4 x \ln(1 - x^{-1}) + C_5 \ln(1 - x^{-1})$$
$$+ C_6 x^{-1/2} + C_7 (1 + x^{-1}) \ln \frac{1 + x^{-1/2}}{1 - x^{-1/2}},$$

where the $C_i$ are some constants. Let us introduce the function

$$F(x) = \frac{d}{dx} \frac{g(\ln x)}{\sqrt{x}}.$$

**Lemma 1** (en explicit formula for $S_P[g]$). Let $B_0 \geq x_0$ [38], and let $h \in \{h\}_S$ be a function such that

$$\int_{b_0}^{\infty} e^{y/2} |g(y)| + |g^{(1)}(y)| + |g^{(2)}(y)| dy = C_T[g] < \infty, \quad g^{(3)} \in L^1(b_0, \infty).$$

Then

$$S_P[g] = S_P^\infty[g] + S_{\text{ex}}[g] + S_0[g]$$

The first two terms on the right-hand side in (3.4) are the sum of residues of the integrand in the rectangular domain with vertices $\sigma_1 \pm iR$ and $-A \pm iR$ ($A \to \infty$). The residues at the points $s = 0, -1$ must be considered separately (see [15]), and their contribution is included in $\Psi_{\Gamma,0}^1(x)$. This function includes the contributions of the poles and trivial zeros of $Z(\cdot)$, and

$$\Psi_{\Gamma,0}^1(x) = -\frac{2}{3} x^{3/2} (n - \text{tr} \Phi(1/2)) + \tilde{\Psi}_{\Gamma,0}^1(x).$$

The first term on the right-hand side in this formula is the contribution of the pole at $s = 1/2$. The function $\Psi_{\Gamma,0}^1(x)$ can be explicitly evaluated based on the results in [112], and $\tilde{\Psi}_{\Gamma,0}^1(\cdot)$ is a function differentiable for $x \geq 2$ and such that

$$\tilde{\Psi}_{\Gamma,0}^1(x) = C_1^\Gamma \ln x + O(x), \quad \frac{d\tilde{\Psi}_{\Gamma,0}^1(x)}{dx} = C_1^\Gamma \ln x + C_2^\Gamma + O(x^{-1}).$$

For example, if $\Gamma = \text{SL}(2, \mathbb{Z})$, then

$$\tilde{\Psi}_{\Gamma,0}^1(x) = C_1 x \ln x + C_2 x + C_3 + C_4 x \ln(1 - x^{-1}) + C_5 \ln(1 - x^{-1})$$
$$+ C_6 x^{-1/2} + C_7 (1 + x^{-1}) \ln \frac{1 + x^{-1/2}}{1 - x^{-1/2}},$$

where the $C_i$ are some constants. Let us introduce the function

$$F(x) = \frac{d}{dx} \frac{g(\ln x)}{\sqrt{x}}.$$
In view of the estimates (see [15, 16]) we rewrite (2.6) in the form

\[ \Psi(I)F(x)dx, \]

(3.15) \[ S_0[g] = - \int_B^\infty \frac{dF(x)}{dx} \Psi_I,0(x)dx - \int_{B_0}^B \Psi_I(x)F(x)dx, \]

(3.16) \[ S_{\text{ex}}[g] = - \sum_{1/2<s_m<1} \int_B^\infty \left( \frac{x^s_m + x^{s_m}}{s_m} \right) F(x)dx - \sum_{\nu=0}^{M_2-1} \int_B^\infty \frac{x^{s_{\nu}}}{s_{\nu}} F(x)dx, \]

(3.17) \[ S_P^\infty[g] = - \sum_{(\alpha)} \int_B^\infty \left( \frac{x^{s_{\alpha}}}{s_{\alpha}} + \frac{x^{s_{\infty}}}{s_{\infty}} \right) F(x)dx - \sum_{r_j>0} \int_B^\infty \left( \frac{x^{s_j}}{s_j} + \frac{x^{\infty}}{s_j} \right) F(x)dx, \]

and the symbol \( \sum_{(\alpha)} \) stands for a sum over the poles \( s_{\alpha} = \beta_{\alpha} + i\gamma_{\alpha} \) of \( \varphi \) such that \( \beta_{\alpha} < 1/2 \) and \( \gamma_{\alpha} > 0 \) (see [4, 11]). Here and in what follows, \( \sum_{r_j>0} + \sum_{(\alpha)} = \lim_{R \to \infty} \left( \sum_{r_j \leq R} + \sum_{|\gamma_{\alpha}| \leq R} \right). \)

**Proof.** An analog of Lemma [1] for strictly hyperbolic groups was proved in [10]. Let us rewrite (2.6) in the form

\[ S_P[g] = \sum_{\{P\}} \Lambda^I(P)f(N(P)), \quad f(x) = \frac{g(\ln x)}{\sqrt{x}}. \]

By Abel’s partial summation formula,

\[ \sum_{B_0 \leq N(P) \leq x} \Lambda^I(P)f(N(P)) = \Psi_I(x)f(x) - \int_{B_0}^x \Psi_I(\xi)F(\xi)d\xi. \]

In view of the estimates (see [15, 16])

\[ \Psi_I(x) = O(x), \quad \Psi_I,1(x) = O(x^2), \quad x \to \infty, \]

and condition (3.13), we obtain

\[ S_P[g] = - \int_{B_0}^\infty \Psi_I(x)F(x)dx = - \int_{B}^\infty \frac{d\Psi_I}{dx}(x)F(x)dx - \int_{B_0}^B \Psi_I(x)F(x)dx. \]

Further, we integrate by parts and find that

(3.18) \[ S_P[g] = \Psi_I,1(B)F(B) - \int_{B_0}^B \Psi_I(x)F(x)dx + \int_{B}^\infty \Psi_I(x)\frac{dF}{dx}(x)dx. \]

Let us use formula (3.4). We obtain

\[ S_P[g] = \int_{B}^\infty \left( \Sigma_{R,\Delta}(x) + \Sigma_{R,\varphi}(x) + \Psi_I,0(x) \right) \frac{dF}{dx}(x)dx \]

\[ + \Psi_I,1(B)F(B) - \int_{B_0}^B \Psi_I(x)F(x)dx + \int_{B}^\infty \frac{dF}{dx}(x)\Delta_R(x)dx, \]

whence it follows that

\[ S_P[g] = - \Psi_I,1(B)F(B) + \Delta_R(B)F(B) \]

\[ - \int_{B}^\infty \frac{d}{dx}[\Sigma_{R,\Delta}(x) + \Sigma_{R,\varphi}(x) + \Psi_I,0(x)]F(x)dx \]

\[ + \Psi_I,1(B)F(B) + \int_{B_0}^B \Psi_I(x)F(x)dx + \int_{B}^\infty \frac{dF}{dx}(x)\Delta_R(x)dx. \]
We differentiate, change the order of summation and integration, and then pass to
the limit as $R \to \infty$, thus obtaining the desired result. The proof of Lemma 1 is complete.

Formally, one can obtain Eq. (3.14) by substituting an explicit function for $\Psi(x)$
(see [16]) into (3.18) with subsequent integration by parts and changing the order
of integration and summation.

Note that if we use the estimate (3.9), then the entire derivation remains valid
except that $e^{y/2}$ must be replaced with $e^{(k-3/2)y}$ in condition (3.13).

Lemma 2. Let the assumptions of Lemma 1 be satisfied. Then

$$
\sum_{n \geq 0} h(r_n) = H[h] + S_R[h] + S_P^\infty[g] + S_{ex}[g] + S_0[g] + \mathcal{P}[h|\phi].
$$

Here

$$
S_P^\infty[g] = -\sum_{r_j \neq 0} \frac{1}{r_j^2 + 1/4} \int_b^\infty (\cos r_j y + 2r_j \sin r_j y) f(y) dy
$$

$$
- \sum_{(\alpha)} \int_B f(x) \left( \frac{x^{\tilde{s}_\alpha}}{s_\alpha} + \frac{x^{\bar{s}_\alpha}}{\bar{s}_\alpha} \right) F(x) dx,
$$

$$
f(y) = -\frac{1}{2} g(y) + g^{(1)}(y), \quad b = \ln B,
$$
and the remaining terms on the right-hand side in (3.20) have been defined above.

Proof. The proof amounts to the substitution of the expression (3.14) into the
Selberg formula (2.3) with regard to the remark that

$$
\left( \frac{x^{s_j}}{s_j} + \frac{x^{\tilde{s}_j}}{\tilde{s}_j} \right) F(x) dx = \frac{1}{1/4 + r_j} (\cos r_j y + 2r_j \sin r_j y) f(y) dy \quad (\tilde{s}_j = \bar{s}_j)
$$
for $y = \ln x$. The proof of Lemma 2 is complete.

4. Explicit formula for $\mathcal{P}[h|\phi]$

We introduce the notation

$$
J[h|\phi] = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \varphi' \left( \frac{1}{2} + ir \right) dr,
$$

$$
\Delta \mathcal{P}[h|\phi] = -\frac{n}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma} (1 + ir) dr + \frac{h(0)}{4} (n - \text{tr } \Phi(1/2)) - n g(0) \ln 2
$$

and represent $\mathcal{P}[h|\phi]$ (2.8) in the form

$$
\mathcal{P}[h|\phi] = J[h|\phi] + \Delta \mathcal{P}[h|\phi].
$$

As was already noted, the main properties of the function $\varphi$ (2.9) are indicated
in [1, 3, 12]. According to these papers,

$$
\varphi(s) = \sqrt{\pi} \left( \frac{\Gamma(s - 1/2)}{\Gamma(s)} \right) \sum_{n=1}^{\infty} \frac{a_n}{b_{2s}} e^{1/2}, \quad a_1 \neq 0, \quad 0 < b_1 < b_2 < \cdots,
$$
It was proved in [1, 12] that the series

\[ \sum_{\beta_\alpha < 1/2} \left( \frac{1}{s - 1 + s_\mu} - \frac{1}{s - s_\mu} \right) + \sum_{\beta_\alpha < 1/2} \left( \frac{1}{s - 1 + \tilde{s}_\alpha} - \frac{1}{s - s_\alpha} \right) - 2\ln b_1. \]

The summation in \( \sum_\mu \) is over all poles of \( \varphi \) such that \( 1/2 < s_\mu \leq 1 \) \( (s_\mu = \sigma_\mu) \). If \( s = 1/2 + ir \), then \( (r \in \mathbb{R}) \)

\[ \frac{1}{s - 1 + s_\mu} - \frac{1}{s - s_\mu} = -(1 - 2s_\mu) \frac{1}{r^2 + (s_\mu - 1/2)^2} > 0 \quad (1/2 < s_\mu \leq 1), \]

\[ \frac{1}{s - 1 + \tilde{s}_\alpha} - \frac{1}{s - s_\alpha} = \frac{1}{ir - a_\alpha^{(1)}} - \frac{1}{ir - a_\alpha^{(2)}} = -\frac{1 - 2\beta_\alpha}{(r - \gamma_\alpha)^2 + (\beta_\alpha - 1/2)^2} < 0. \]

In the last formula,

\[ a_\alpha^{(1)} = s/2 - \tilde{s}_\alpha, \quad a_\alpha^{(2)} = s_\alpha - 1/2. \]

It was proved in [1,12] that the series

\[ \sum_{\beta_\alpha < 1/2} \left( \frac{1 - 2\beta_\alpha}{(s - \gamma_\alpha)^2 + (\beta_\alpha - 1/2)^2} \right) \]

converges uniformly on compact sets. We substitute the expressions (4.5) and (4.6) into (4.1), change the order of integration and summation, and obtain

\[ J[h|\varphi] = J_0[h|\varphi] + J_1[h|\varphi], \]

where

\[ J_0[h|\varphi] = -\sum_\mu (1 - 2s_\mu) \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(r)}{r^2 + (s_\mu - 1/2)^2} dr - 2g(0) \ln b_1, \]

\[ J_1[h|\varphi] = \sum_{\beta_\alpha < 1/2} (I(a_\alpha^{(1)}) - I(a_\alpha^{(2)})), \]

and \( I(a) \) is the integral

\[ I(a) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(r)}{r - a} dr. \]

By Parseval’s identity,

\[ I(a) = \frac{1}{2} \int_{-\infty}^{\infty} g(y) f(-y) dy, \quad f(-y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{iy}}{r + ia} dr. \]

The last integral can be evaluated using residue theory.

As a result, we obtain

\[ I(a_\alpha^{(1)}) - I(a_\alpha^{(2)}) = -\frac{1}{2} \int_{0}^{\infty} g(y) [e^{-y/2 + \tilde{s}_\alpha y} + e^{-y/2 + s_\alpha y}] dy. \]

We substitute this expression into (4.11), take into account our definition of \( \sum_\alpha \)

\( (\sum_\alpha = \sum_{\beta_\alpha < 1/2, \ |\gamma_\alpha| > 0}) \), and arrive at the relation

\[ J_1[h|\varphi] = -\sum_\alpha \int_{0}^{\infty} g(y) [e^{-y/2 + \tilde{s}_\alpha y} + e^{-y/2 + s_\alpha y}] dy. \]
Lemma 3 ((an explicit formula for $\mathcal{S}[h|\varphi]$)). For every cofinite group $\Gamma$, one has
\begin{equation}
\mathcal{S}[h|\varphi] = J_1[h|\varphi] + \Delta \mathcal{S}[h|\varphi] + J_0[h|\varphi].
\end{equation}
Here $J_1[h|\varphi]$ is defined in (4.12), $\Delta \mathcal{S}[h|\varphi]$ is defined in (4.12), and $J_0[h|\varphi]$ is defined in (4.9).

Proof. The proof amounts to the substitution of the right-hand sides of (4.8) and (4.12) into (4.3). The series $J_1[h|\varphi]$ converges in view of (4.13). □

5. MAIN THEOREM

Theorem 1 proved in the section specifies the explicit form of the functional $\tilde{\Phi}_\Gamma$ in formula (1.10). Let us preliminarily transform the expression (3.21) for $S^\gamma_\Gamma[g]$.

Lemma 4. Let the assumptions of Lemma 1 be satisfied, and let $\lambda_n \neq 1/4$. Then
\begin{equation}
S^\gamma_\Gamma[g] = W[g] + S^1_\Gamma[g|\Delta] + S^2_\Gamma[g|\varphi] + S^3_\Gamma[g|\varphi] - J_1[h|\varphi].
\end{equation}

Here $J_1[h|\varphi]$ is defined in (4.12), and
\begin{equation}
W[g] = -2f(b)\left[\sum_{r_j \geq 0} \frac{\cos r_j b}{r_j^2 + 1/4} + \sum_{(\alpha)} \frac{\cos b \gamma_\alpha}{\gamma_\alpha^2} e^{(\beta_\alpha - 1/2)b}\right].
\end{equation}
\begin{equation}
S^1_\Gamma[g|\Delta] = \sum_{r_j > 0} \frac{1}{r_j^2 + 1/4} r_j \left\{ \sin r_j b \left[ -\frac{1}{2} g(b) + 2g^{(2)}(b) \right] + \int_0^\infty \sin r_j y \left[ -\frac{1}{2} g^{(1)}(y) + 2g^{(3)}(y) \right] dy, \right. \right.
\end{equation}
\begin{equation}
S^2_\Gamma[g|\varphi] = 2 \sum_{(\alpha)} \left[ g(b) e^{(\beta_\alpha - 1/2)b} \sin \gamma_\alpha b \left( \frac{\gamma_\alpha}{\beta_\alpha^2 + \gamma_\alpha^2} - \frac{1}{\gamma_\alpha} \right) + \frac{(\beta_\alpha - 1/2)}{\gamma_\alpha^2} g(0) - g(b) \frac{\beta_\alpha^3}{\gamma_\alpha^2 (\beta_\alpha^2 + \gamma_\alpha^2)} e^{(\beta_\alpha - 1/2)b} \right],
\end{equation}
\begin{equation}
S^3_\Gamma[g|\varphi] = 2 \sum_{(\alpha)} \gamma_\alpha^2 \int_0^b \frac{d^2}{dy^2} (e^{(\beta_\alpha - 1/2)y} g(y)) \cos \gamma_\alpha y dy.
\end{equation}
If $\lambda_n = 1/4$, then one must add $-4k \int_0^b f(y) dy$ to the right-hand side of (5.1), where $k$ is the multiplicity of the eigenvalue $\lambda_n = 1/4$.

Proof. Consider the integral
\begin{equation}
A_\alpha = \int_B \left( \frac{x^{s_\alpha}}{s_\alpha} + \frac{x^{\bar{s}_\alpha}}{\bar{s}_\alpha} \right) F(x) \, dx
\end{equation}
oncurring on the right-hand side in (3.21). By definition (3.12) of the function $F$,
\begin{equation}
A_\alpha = \frac{2g(b)}{s_\alpha \bar{s}_\alpha} e^{(\beta_\alpha - 1/2) b} (\beta_\alpha \cos \gamma_\alpha + \gamma_\alpha \sin \gamma_\alpha) - 2 \int_0^\infty e^{(\beta_\alpha - 1/2)y} \cos(\gamma_\alpha y) g(y) \, dy.
\end{equation}

One derives Eq. (5.1) from (3.20) by twice integrating by parts. The absolute convergence of the series $S^3_\Gamma[g|\varphi]$ and $S^2_\Gamma[g|\varphi]$ follows from (2.11) and the convergence of the series $\sum_{(\alpha)} \gamma_\alpha^2$ and $\sum_{r_j > 0} r_j^3$. To prove the convergence of the series $S^1_\Gamma[g|\varphi]$, it suffices to integrate by parts. Now the convergence of the series $W[g]$ follows from formula (5.1).

The proof of Lemma 4 is complete. □
Corollary. For any \( b > b_0 \) and any cofinite group \( \Gamma \) such that \( B_0 \geq x_0 \), the series in the definition of \( W[g] \) converges; i.e.,

\[
\sum_{r_j \geq 0} \frac{\cos r_j b}{r_j^2 + 1/4} + \sum_{(\alpha)} \frac{\cos b \gamma_{\alpha}}{\gamma_{\alpha}^2} e^{(\beta_{\alpha} - 1/2)b} = C_\Gamma(b) < \infty.
\]

Now we are in a position to prove Theorem 1.

Theorem 1. Let a function \( h \in \{h\}_S \) satisfy condition (3.10). Then

\[
(5.6) \quad \sum_{n \geq 0} h(r_n) = H[h] + G[h] + S_1[g|\Delta] + S_2[g|\varphi] + S_3[g|\varphi] + M[g]
\]

for every cofinite group \( \Gamma \) such that \( B_0 \geq x_0 \).

Here

\[
(5.7) \quad G[h] = -\frac{n}{2\pi} \int_{-\infty}^{+\infty} h(r) \psi(1 + ir) \, dr \quad (\psi(x) = \frac{\Gamma'}{\Gamma}(x)),
\]

\[
M[g] = W[g] + S_{\infty}[g] + S_R[h] + S_0[g] - \sum_{\mu} \frac{1}{4} \int_{-\infty}^{+\infty} \frac{h(r)dr}{r^2 + (s_\mu - 1/2)^2} +
\]

\[
(5.8) \quad + \left( n - \text{tr} \left( \frac{1}{2} \right) \right) h(0) - g(0) (n \ln 2 + 2b),
\]

where the summation in \( \sum_{\mu} \) is over the poles of \( \varphi \) such that \( 1/2 < s_\mu \leq 1 \) \( (s_\mu = \sigma_\mu) \).

Proof. The proof amounts to the substitution of (5.1) and (5.3) into (3.20). The proof of Theorem 1 is complete. \( \square \)

Note that it follows from (5.3), (5.4), and (5.5) that the contribution of the discrete spectrum \( \{\lambda_n\} \) to \( \sum_{n \geq 0} h(r_n) \) is determined by the behavior of \( g(y) \) for \( y > b \), while the contribution of the resonance spectrum \( \{s_\alpha\} \) is determined by the behavior of \( g(y) \) for \( y \leq b \).

6. Proof of the Roelke Conjecture

The proof of the Roelke conjecture is based on an analysis of the asymptotics as \( t \to 0 \) of the expressions on the right-hand side in (5.6) for the case in which

\[
(6.1) \quad h(r) = h(r, p) = \frac{e^{-tr^2}}{r^2 + p^2} \quad (p > 1/2).
\]

Throughout the following, \( h(r) = h(r, p) \) and \( g(y) = g(y, p) \) in (2.1). The dependence of the objects in question on \( p \) will be indicated explicitly. By \( C_i \) we denote various constants independent of \( p \). We write \( A(t, p) = O_p(t^k) \) if \( A(t, p) = C(p)O(t^k) \) \( (t \to 0) \) and \( A(t, p) = R(t, p) \) if \( A(t, p) = C_0(p) + C_1(p)t + C_2(p)t^2 + \cdots \) \( (t \to 0) \). By \( F_i(t) \) we denote functions independent of \( p \) such that \( F_i(t) \leq C_i \). The constants \( C_i \) and \( C_i(p) \), as well as the functions \( F_i(t) \), may be different in different formulas.
Lemma 5. For every cofinite group $\Gamma$ such that $B_0 \geq x_0$, one has the following asymptotic formula as $t \to 0$:

$$\sum_{n \geq 1} \frac{e^{-tr_n^2}}{r_n^2 + p^2} + \sum_{\alpha} \gamma_{\alpha}^{-2} e^{-t\gamma_{\alpha}^2} = \frac{|F|}{4\pi} e^{tp^2} \ln \frac{1}{t} - \frac{n}{\pi} I(t, p) + p^2 e^{tp^2} S(t) + B(t, p),$$

where

$$B(t, p) = e^{tp^2} \left[ C_0(p) + C_1(p) \sqrt{t} F_1(t) + C_2(p) t F_2(t) + C_3(p) t^{3/2} F_3(t) + C_4(p) p^2 F_4(t) + O(t^3) \right],$$

and hence

$$\sum_{n \geq 1} \frac{e^{-tr_n^2}}{r_n^2 + p^2} + \sum_{\alpha} \gamma_{\alpha}^{-2} e^{-t\gamma_{\alpha}^2} = \frac{|F|}{4\pi} e^{tp^2} \ln \frac{1}{t} - \frac{n}{\pi} I(t, p) + p^2 e^{tp^2} S(t) + B(t, p),$$

where

$$I(t, p) = \int_0^\infty \frac{e^{-tr^2}}{r^2 + p^2} \ln r \, dr, \quad S(t) = \sum_{\alpha} \gamma_{\alpha}^{-4} e^{-t\gamma_{\alpha}^2}.$$ 

Proof. We use the standard notation \[20\]

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi, \quad \text{erfc}(x) = 1 - \text{erf}(x)$$

and the following properties of the probability integral erf $x$:

$$\text{erf}(x) = -\text{erf}(-x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{n!(2n+1)},$$

$$\frac{e^{-x^2}}{2 x + \sqrt{x^2 + 4/\pi}} \leq \text{erf}(x) \leq \frac{e^{-x^2}}{2 x + \sqrt{x^2 + 4/\pi}} (x \geq 0)$$

$$\text{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi} z} \left( 1 + \sum_{m=1}^\infty \frac{(-1)^m 1 \cdot 3 \cdots (2m-1)}{(2z^2)^m} \right) \quad \left( |z| \to \infty, \ |\arg(z)| \leq \frac{\pi}{2} \right).$$

The function

$$g(y, p) = \frac{1}{\pi} \int_0^\infty \frac{e^{-tr^2} \cos ry}{r^2 + p^2} \, dr$$

and its derivatives can be expressed via the probability integral \[19\][20],

$$g(y) = \frac{1}{4p} e^{tp^2} \left[ 2e^{-yp} - e^{-yp} \text{erfc}\left( \frac{y}{2\sqrt{t}} - p\sqrt{t} \right) + e^{yp} \text{erfc}\left( \frac{y}{2\sqrt{t}} + p\sqrt{t} \right) \right].$$

Thus,

$$g(y) = \frac{1}{2p} e^{tp^2 - yp} + \Delta g(y), \quad |\Delta g(y)| \leq Ce^{-y^2/(4t)}.$$

Likewise,

$$g^{(1)}(y) = -\frac{1}{\pi} \int_0^\infty \frac{re^{-tr^2}}{r^2 + p^2} \sin ry \, dr$$

$$= \frac{1}{4} e^{tp^2} \left[ -2e^{-yp} + e^{-yp} \text{erfc}\left( \frac{y}{2\sqrt{t}} - p\sqrt{t} \right) + e^{yp} \text{erfc}\left( \frac{y}{2\sqrt{t}} + p\sqrt{t} \right) \right]$$

and hence

$$g^{(1)}(y) = -\frac{1}{2} e^{tp^2 - yp} + \Delta g^{(1)}(y), \quad |\Delta g^{(1)}(y)| \leq Ce^{-y^2/(4t)}.$$
Finally,

\[(6.9) \quad g^{(2)}(y) = \frac{1}{\pi} \int_0^\infty \frac{r^2 \cos ry}{p^2 + r^2} e^{-\frac{1}{\sqrt{4\pi t}}} e^{-r^2/(4t)} \, dr = p^2 g(y) - \frac{1}{\pi} e^{-y^2/(4t)}.\]

These results mean that the assumptions under which Theorem 1 applies hold for \( p > 1/2 \) and \( B_0 > x_0 \).

According to definition (2.4),

\[\begin{align*}
H[h] &= H_0[h] + H_1[h], \\
H_0[h] &= \frac{|F|}{2\pi} \int_0^\infty \frac{r e^{-tr^2}}{p^2 + r^2} \, dr, \\
H_1[h] &= -\frac{|F|}{\pi} \int_0^\infty \frac{r e^{-tr^2}}{p^2 + r^2} \frac{e^{-2\pi r}}{1 + e^{-2\pi r}} \, dr,
\end{align*}\]

and hence

\[H_1[h] = R_1(t, p).\]

On the other hand (see [19]),

\[\begin{align*}
H_0[h] &= \frac{|F|}{4\pi} e^{tp^2} (-\text{Ei}(-a)), \quad a = p^2 t, \\
-\text{Ei}(-a) &= \int_a^\infty e^{-t} t^{-1} \, dt,
\end{align*}\]

and we can use the relation

\[\text{Ei}(-a) = \gamma + \ln a + \sum_{n=1}^\infty \frac{(-a)^n}{n \cdot n!}\]

to obtain

\[(6.10) \quad H[h] = \frac{|F|}{4\pi} e^{tp^2} \ln \frac{1}{t} + R_2(t, p).\]

The next term on the right-hand side in (5.6) is

\[(6.11) \quad G[h] \equiv G(t, p) = -\frac{n}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-tr^2} \psi(1 + ir) \, dr}{r^2 + p^2} \psi(z) = \left( \frac{\Gamma'}{\Gamma} \right)(z).
\]

Since the function \( \text{Re} \psi(1 + ir) \) is even and \( \text{Im} \psi(1 + ir) \) is odd [20], we have

\[(6.12) \quad G(t, p) = G_1(t, p) + G_2(t, p),\]

\[(6.13) \quad G_1(t, p) = -\frac{n}{\pi} \int_0^1 \frac{e^{-tr^2}}{r^2 + p^2} \text{Re} \psi(1 + ir) \, dr,\]

\[(6.14) \quad G_2(t, p) = -\frac{n}{\pi} \int_1^\infty \frac{e^{-tr^2}}{r^2 + p^2} \text{Re} \psi(1 + ir) \, dr.\]

We use the relation

\[\text{Re} \psi(1 + ir) = -\gamma + r^2 \sum_{n=1}^\infty \frac{1}{n(n^2 + r^2)}\]

to obtain

\[G_1(t, p) = R_3(t, p).\]

To analyze \( G_2(t, p) \), we use the asymptotic relation

\[\text{Re} \psi(1 + ir) = \ln r + \frac{1}{12r^2} + \frac{1}{120r^4} + \frac{1}{252r^6} + \cdots \quad (r \to \infty).\]
This relation implies that

\[ G_2(t, p) = C_0(p) + C_1(p)t + C_2(p)t^{3/2} + C_3(p)t^2 + O_p(t^{5/2}) \]

and hence

(6.15) \[ G(t, p) = C_0(p) + C_1(p)t - \frac{n}{\pi} I(t, p) + C_2(p)t^{3/2} + C_3(p)t^2 + O_p(t^{5/2}), \]

where the integral \( I(t, p) \) is defined (6.3). Using the above-mentioned properties of \( g(y) \) and \( g^{(1)}(y) \) (see (6.6), (6.8) and (5.3), (5.4)), we have

(6.16) \[ S_1^p[g|\Delta] = R(t, p), \]
\[ S_2^p[g|\varphi] = R(t, p) + Cg(0). \]

Since

(6.17) \[ g(0) = \frac{1}{2p} e^{tp^2} (1 - \text{erf}(p\sqrt{t})) = \]
\[ = \frac{1}{2p} + C_0\sqrt{t} + C_1p^2t^{3/2} + O_p(t^{5/2}), \]

(see (6.5)), it follows that

(6.18) \[ S_2^p[g|\varphi] = C(p) + C_0(\sqrt{t} + C_1(p)t + C_2(p)t^{3/2} + C_3(p)t^2 + O_p(t^{5/2}) \quad (t \to 0). \]

We represent \( S_3^p[g|\varphi] \) (5.5) in the form

(6.19) \[ S_3^p[g|\varphi] = 2 \sum_{(\alpha)} \frac{(\beta_\alpha - 1/2)^2}{\gamma_\alpha^2} I^{(0)}_{\alpha}(t, p) + \sum_{(\alpha)} \frac{(\beta_\alpha - 1/2)}{\gamma_\alpha^2} I^{(1)}_{\alpha}(t, p) + \sum_{(\alpha)} \frac{1}{\gamma_\alpha^2} I^{(2)}_{\alpha}(t, p), \]

where

(6.20) \[ I^{(i)}_{\alpha}(t, p) = \int_0^b e^{(\beta_\alpha - 1/2)y} \cos(\gamma_\alpha y) g^{(i)}(y, p) dy. \]

Consider the integrals \( I^{(0)}_{\alpha}(t, p) \). To this end, we represent \( g(y, p) \) (6.3) in the form

(6.21) \[ g(y, p) = \frac{1}{2p} e^{tp^2 - y^2} + \frac{e^{tp^2}}{4} f_0(y, p), \]

(6.22) \[ f_0(y, p) = \frac{1}{p} \left[ e^{yp} \text{erfc} \left( \frac{y}{2\sqrt{t}} + p\sqrt{t} \right) - e^{-yp} \text{erfc} \left( \frac{y}{2\sqrt{t}} - p\sqrt{t} \right) \right]. \]

Note that

\[ f_0(y, -p) = f_0(y, p). \]

We introduce the notation

\[ \xi = \frac{y}{2\sqrt{t}}, \quad (\text{erfc} \xi)^{(n)} = -\Phi^{(n)}(\xi) \quad (n \geq 1). \]

Then

\[ |\Phi^{(n)}(\xi)| = \frac{2}{\sqrt{\pi}} e^{-\xi^2} |H_{n-1}(\xi)|, \]

where \( H_n(\xi) \) is the Hermite polynomial.
Let us expand \( f_0(y, p) \) in a Taylor series in powers of \( p \sqrt{t} \). This expansion has the form

\[
f_0(y, p) = \text{erfc} \frac{2 \sinh yp}{p} - \frac{\Phi^{(1)}(\xi)}{1!} p \sqrt{t} \frac{2 \cosh yp}{p} + \frac{\Phi^{(2)}(\xi)}{2!} p^2 t \frac{2 \sinh yp}{p} \]

\[
- \frac{\Phi^{(3)}(\xi)}{3!} p^3 t^{3/2} \frac{2 \cosh yp}{p} + \frac{\Phi^{(4)}(\xi)}{4!} p^4 t^2 \frac{2 \sinh yp}{p} + \cdots.
\]

Further, we expand \( \sinh yp \) and \( \cosh yp \) in Taylor series and obtain

\[ (6.23) \quad f_0(y, p) = 4 \xi \text{erfc}(\xi) \sqrt{t} + \frac{4}{\sqrt{\pi}} e^{-\xi^2} \sqrt{t} + \]

\[ + \text{erfc}(\xi) \left[ c_1 t^{3/2} \xi^{3/2} p^2 + c_2 t^{5/2} \xi^{5/2} p^4 + \ldots \right] \sum_{k=1}^{\infty} t^{k+1/2} p^{2k} e^{-\xi^2} Q_{2k+1}(\xi) \quad (t \to 0), \]

where \( Q_{2k+1}(\xi) \) is a polynomial of degree \( 2k + 1 \).

We substitute (6.23) into (6.21), take into account definition (6.20), and find the desired expansion of \( I_0^{(0)}(t, p) \),

\[ (6.24) \quad I_0^{(0)}(t, p) = e^{tp^2} [C_0^{(0)}(p) + t F_1^{(0)}(t) + t^2 p^2 F_2^{(0)}(t) + t^3 p^3 F_3^{(0)}(t) + \cdots] \quad (t \to 0), \]

where

\[ (6.25) \quad |F_{k, \alpha}^{(0)}(t)| \leq C_k. \]

Since

\[ (6.26) \quad g^{(1)}(y) = \frac{1}{2} e^{-tp^2 - yp} + \frac{1}{4} e^{tp^2} f_1(y, p), \]

\[ f_1(y, p) = 2 \text{erfc} \xi + \text{erfc} \xi (C_1 t^2 \xi^2 p^2 + C_2 t^2 \xi^4 p^4 + \ldots) + \]

\[ + e^{-\xi^2} (C_3 t^2 \xi p^2 + C_4 t^2 Q_2(\xi) p^4 + C_5 t^3 Q_3(\xi) p^6 + \ldots), \]

similarly to (6.24), we obtain

\[ (6.27) \quad I_0^{(1)}(t, p) = e^{tp^2} [C_0^{(1)}(p) + \sqrt{t} F_1^{(1)}(t) + t^{3/2} p^2 F_2^{(1)}(t) + t^{5/2} p^4 F_3^{(1)}(t) + \cdots], \]

where

\[ (6.28) \quad |F_{k, \alpha}^{(1)}(t)| \leq C_k^{(1)}. \]

Since the series

\[ \sum_{(\alpha)} \frac{(\beta_{\alpha} - 1/2)^2}{\gamma_{\alpha}^2}, \quad \sum_{(\alpha)} \frac{(\beta_{\alpha} - 1/2)}{\gamma_{\alpha}^2} \]

converge and the constants \( C_k^{(i)} \) are independent of \( \alpha \) and \( p \), we find from (6.19) that

\[
S_p[g, \varphi] = e^{tp^2} [C_0(p) + \sqrt{t} F_1(t) + t F_2(t) + p^2 t^{3/2} F_3(t) + \]

\[ + p^2 t^2 F_4(t) + p^4 t^{5/2} F_5(t) + p^4 t^3 F_6(t) + \cdots] + A(t, p). \]

In this formula, \( A(t, p) \) is the last term on the right-hand side in (6.19); i.e.,

\[ A(t, p) = 2 \sum_{(\alpha)} \frac{1}{\gamma_{\alpha}^2} t^{(2)}(t, p), \]
where the integral $I_{\alpha}^{(2)}(t, p)$ is defined in (6.20). Using (6.30), we obtain

\begin{equation}
A(t, p) = A_1(t) + A_2(t, p),
\end{equation}

\begin{equation}
A_1(t) = -\frac{1}{\sqrt{\pi t}} \sum_{(\alpha)} \gamma_{\alpha}^{-2} \int_0^b e^{(\beta_{\alpha}-1/2)y} \cos(\gamma_{\alpha}y) e^{-y^2/4t} dy,
\end{equation}

\begin{equation}
A_2(t, p) = 2p^2 \sum_{(\alpha)} \gamma_{\alpha}^{-2} \int_0^b e^{(\beta_{\alpha}-1/2)y} \cos(\gamma_{\alpha}y) g(y, p) dy.
\end{equation}

We point out that $A_1(t)$ is independent of $p$. Consider $A_2(t, p)$. We expand $e^{(\beta_{\alpha}-1/2)y}$ in a Taylor series and write

\begin{align*}
A_2(t, p) &= 2p^2 \sum_{(\alpha)} \frac{1}{\gamma_{\alpha}^2} \left[ g(b) \sin(\gamma_{\alpha}b) - \int_0^b g^{(1)}(y, p) \sin(\gamma_{\alpha}y) dy \right] \\
&\quad + 2p^2 \sum_{(\alpha)} \left[ \frac{(\beta_{\alpha} - 1/2)}{\gamma_{\alpha}^2} \int_0^b y \cos(\gamma_{\alpha}y) g(y) dy \right] \\
&\quad + 2p^2 \sum_{(\alpha)} \frac{(\beta_{\alpha} - 1/2)^2}{\gamma_{\alpha}^2} \int_0^b \frac{y^2}{2!} \cos(\gamma_{\alpha}y) g(y) dy + \cdots \right].
\end{align*}

The series on the right-hand side in this relation converge absolutely, and the integrals can be estimated by the same scheme as $I_{\alpha}^{(0)}$ and $I_{\alpha}^{(1)}$. As a result, we obtain the expansion

\begin{equation}
A_2(t, p) = e^{p^2/2} [C_0(p) + p^2 S(t) + C_1(p) \sqrt{t} F_1(t) + C_2(p) t F_2(t) + C_3(p) t^{3/2} F_3(t) + C_4(p) t^2 F_4(t) + C_5 t^{5/2} F_5(t) + O_t^3 + \cdots]
\end{equation}

Now consider $A_1(t)$ (6.31). We have

\begin{equation}
A_1(t) = -\frac{1}{\sqrt{\pi t}} \sum_{(\alpha)} \int_0^b \cos(\gamma_{\alpha}y) e^{-y^2/4t} \left[ 1 + \frac{\beta_{\alpha} - 1/2}{1!} y + \frac{(\beta_{\alpha} - 1/2)^2}{2!} y^2 + \cdots \right].
\end{equation}

It follows that

\begin{equation}
A_1(t) = -\frac{1}{\sqrt{\pi t}} \int_0^\infty \cos(\gamma_{\alpha}y) e^{-y^2} dy + \Delta A_1(t)
\end{equation}

\begin{equation}
|\Delta A_1(t)| \leq Ct^{1/2}.
\end{equation}

Since

\[ \int_0^\infty e^{-\beta x^2} \cos bx \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} e^{-\frac{b^2}{4\beta}} \quad (\text{Re } \beta > 0) \]

(see [13]), we obtain

\begin{equation}
A_1(t) = -\sum_{(\alpha)} \gamma_{\alpha}^{-2} e^{-t\gamma_{\alpha}^2} + \Delta A_1(t).
\end{equation}
Now the desired estimate for $S^3_P[g|\varphi]$ follows from (6.29), (6.36), and (6.33); it has the form
\begin{equation}
S^3_P[g|\varphi] = e^{t p^2} \left[ C_0(p) + p^2 S(t) + C_1(p) \sqrt{t} F_1(t) + C_2(p) t F_2(t) + C_3(p) t^{3/2} F_3(t) + C_4(p) t^2 F_4(t) + C_5(p) t^{5/2} F_5(t) + O_p(t^3) + \cdots \right] - \sum_{(\alpha)} \gamma_\alpha^2 e^{-t r_\alpha^2} + \Delta A_1(t).
\end{equation}

where $\Delta A_1(t)$ satisfies the estimate (6.35). The sum
\begin{equation}
B(t) = \sum_{(\alpha)} \gamma_\alpha^2 e^{-t r_\alpha^2}
\end{equation}
in this expression can be estimated using (2.11). The Abel transform (summation by parts formulas) gives the estimate
\begin{equation}
|B(t)| \leq C A \ln \frac{1}{t}.
\end{equation}

It remains to consider the expressions on the right-hand side in (5.8). In view of the properties of $g(\cdot, p)$, we have
\begin{equation}
W[g] = R(t, p), \quad S_R[h] = R(t, p), \quad S_0[g] = R(t, p), \quad n - \text{tr } \Phi \left( \frac{1}{2} \right) h(0) = R(t, p), \quad S_{ex}[g] = R(t, p).
\end{equation}

Since $g(0)$ has already been considered (6.17), it remains to find the asymptotics of the finite sum
\begin{equation}
S_{ex}^1[h] = - \sum \frac{1 - 2 s_\mu}{4\pi} \int_{-\infty}^{+\infty} \frac{h(r, p)}{r^2 + (s_\mu - 1/2)^2} \, dr.
\end{equation}

occurring in the expression (5.8) for $M[g]$. This sum can be represented in the form
\begin{equation}
S_{ex}^1[g] = \sum \mu C_\mu J_\mu(t, p),
\end{equation}
\begin{equation}
J_\mu(t, p) = \int_0^\infty \frac{e^{-tr^2}}{(r^2 + a_\mu^2)(r^2 + p^2)} \, dr \quad \left( |a_\mu| < \frac{1}{2} \right).
\end{equation}

We use the formula [19]
\begin{equation}
\int_0^\infty \frac{e^{-\lambda^2 x^2}}{x^2 + \beta^2} \, dx = (1 - \text{erf}(\beta \lambda)) \frac{\pi}{2\beta} e^{\beta^2 \lambda^2}
\end{equation}
and obtain
\begin{equation}
S_{ex}^1[g] = e^{t p^2} \sqrt{t} \left[ C_0 + C_1 p^2 t + C_2 p^4 t^2 + \cdots \right] + F(t),
\end{equation}
\begin{equation}
|F(t)| < C \sqrt{t}.
\end{equation}

Now we gather all the estimates and arrive at (6.2). The proof of Lemma 5 is complete. \qed
Let us proceed to the proof of the Roelke conjecture. V. A. Bykovskii noticed to me that, given Eq. (6.2), to prove the Roelke conjecture it suffices to consider the difference

\[(6.41) \quad Q(t, p_1, p_2) = \sum_{n \geq 0} h(r, p_1) - \sum_{n \geq 0} h(r, p_2) \quad (p_2 > p_1 > \frac{1}{2}).\]

**Theorem 2** (proof of the Roelke conjecture). For every cofinite group $\Gamma$ such that $B_0 \geq x_0$, one has

\[(6.42) \quad N_{\Gamma}(T^2 + \frac{1}{4}) = \# \{n | r_n \leq T \} \to \infty, \quad T \to \infty.\]

**Proof.** Consider the function $Q(t, p_1, p_2)$. The relation (6.2) implies that the equality

\[(6.43) \quad \frac{p_2^2 - p_1^2}{p_1^2 e^{tr_1} - p_2^2 e^{tr_2}} \sum_{m \geq 0} \frac{e^{-tr_m^2}}{(r_m^2 + p_1^2)(r_m^2 + p_2^2)} =\]

\[= S(t) + \frac{e^{tr_1^2} - e^{tr_2^2}}{4 \pi} \ln \frac{1}{t} - \frac{n}{\pi} \frac{I(t, p_1) - I(t, p_2)}{p_1^2 e^{tr_1^2} - p_2^2 e^{tr_2^2}} + \frac{B(t, p_1) - B(t, p_2)}{p_1^2 e^{tr_1^2} - p_2^2 e^{tr_2^2}}\]

holds for any $p_1, p_2$ ($p_2 > p_1$). M.A. Korolev noticed that the results from [21] imply the following asymptotic expansion for the integrand $I(t, p)$ defined by (6.3)

\[(6.44) \quad I(t, p) = \frac{\pi}{2} \ln p \sum_{n=0}^{\infty} \frac{p^{2n-1}}{n!} t^n + \sum_{n=0}^{\infty} p^{2n} t^{n+1/2} d_n + \ln \frac{1}{t} \sum_{n=0}^{\infty} p^{2n} t^{n+1/2} d_n.\]

Suppose that the spectrum $\{r_m\}$ is finite. Then we have

\[(6.45) \quad \sum_{m \geq 0} \frac{e^{-tr_m^2}}{(r_m^2 + p_1^2)(r_m^2 + p_2^2)} = \sum_{m=0}^{M} \frac{e^{-tr_m^2}}{(r_m^2 + p_1^2)(r_m^2 + p_2^2)} = \sum_{k=0}^{\infty} a_k (p_1, p_2) t^k.\]

In this case, the left hand side of (6.43) becomes an analytic function of $t$. At the same time, both the formula (6.2) for $B(t, p)$ and (6.44) imply that the asymptotic expansion of the right hand side of (6.43) contains the terms

\[- \frac{n}{\pi} t^{5/2} \ln \frac{1}{t} (p_1^2 + p_2^2)(d_2 - d_1).\]

The explicit expressions for the coefficients $d_n$ in (6.44) can be derived, and it follows that $d_2 > d_1$. Thus, the hypothesis about the finiteness of the spectrum leads to the contradiction. Theorem 2 is proved.

If we consider the case

\[h(r) = e^{-tr^2}, \quad g(y) = \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t},\]

then (5.6) implies the asymptotic relation

\[(6.46) \quad \sum_{n \geq 0} e^{-tr_n^2} + \sum_{(\alpha)} e^{-t\gamma_\alpha^2} = \frac{|F|}{4\pi} \frac{1}{t} + \frac{n \ln t}{4\sqrt{\pi t}} + \frac{C}{\sqrt{t}} + C_0 + C_1 \sqrt{t} + O(t).\]
We omit its proof. We only note that it approves with the result of [17, Theorem 11.1].

Remark. We point out that the left-hand side of (6.43) is the sum \( \sum_{n \geq 0} h(r_n) + \sum_{(\alpha)} h(\gamma_\alpha) \). For \( r_n \gg p \), the same sum is on the left-hand side in (6.2) (see also (6.43) when \( r_m \to +\infty \)). Such a symmetry \((\{r_n\} \leftrightarrow \{\gamma_\alpha\})\) apparently holds for a fairly broad class of functions \( h \in \{h\}_S \). This symmetry is related to the fact that the sum \( S_p^V[g|\varphi] \) (5.5) contains a term of the form
\[
2\gamma_\alpha^{-2} \int_0^b \cos \gamma_\alpha y g^{(2)}(y) \, dy \simeq h(\gamma_\alpha),
\]
provided that \( g^{(2)}(y) \) decays sufficiently rapidly as \( |y| \to \infty \).

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