State space realisation and model reduction of potential-flow aerodynamics for HAWT applications

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Abstract. This paper presents a general state-space realisation of the unsteady vortex-lattice method and combines it with a novel model-order reduction strategy. The aim is to provide a computationally efficient aerodynamic description suitable for integration in horizontal axis wind turbine aerelasticity. The consistent linearisation is in the 3D components of the vortex-lattice geometry. The resulting linear-time invariant system can, therefore, resolve all the component of forces, hence being suitable for linearisation around arbitrary wake shapes and blade geometries/deflections. The wake modelling captures unsteady aerodynamic effects but it results in large state-space models. Projection on low-dimensional degrees-of-freedom and balanced residualisation are, therefore, employed to reduce the model dimensionality. An iterative balancing algorithm based on Smith’s method is also developed so as to contain the computational cost of the process. The paper also presents an initial numerical investigation on aerofoils linearised around non-zero reference conditions, showing that this approach can reduce the size of the problem by several orders of magnitude and at a lower computational cost than standard direct methods for system balancing.

1. Introduction
The boosting demand for green energy has lead the wind industry to gradually increase the size of horizontal axis wind turbines (HAWT). This is especially relevant for offshore wind turbines, whose rotor dimension is subjected to no restrictions other than technological. As HAWT grow larger, however, lighter and, consequently, more flexible blades will be required to contain structural weight. From a modelling perspective, this poses the challenge of characterising the kinematics of large blade deflections and how this impacts the aerodynamics of the rotor. Other factors, such as the interaction between rotors, tower and, for floating turbines, the platform support motion, can further complicate the aeroelastic response of very flexible HAWT. Finally, as critical loading conditions are typically associated to unsteady phenomena, these effects will need to be characterised through dynamical analyses.

In order to cover wide envelopes of operative conditions, tools for HAWT aeroelastic analysis have historically been based on relatively simple, yet computationally cheap, aerodynamics models, commonly relying on quasi-steady blade element momentum (BEM) theory and tip loss corrections [1]. Overall, these methods are adequate for relatively stiff blades but can not characterise relevant aspects of very flexible blades dynamics. From a structural perspective, novel aeroelastic frameworks account for the blades arbitrary kinematics through geometrically-exact beam models [2, 3, 4]. While BEM could be coupled with these descriptions [2], it heavily relies on empiric corrections and has been shown to over-predict critical loading as compared

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to potential flow methods [5]. This becomes especially true when wake dynamics effects are important, as in the presence of flexible/rigid body motion [6]. Generalised dynamic wake models are also less accurate than potential flow solutions at low-speed operative conditions [6].

Amongst potential flow methods, the unsteady vortex-lattice method (UVLM) results particularly suitable for dynamic analyses under arbitrary kinematics, as it naturally arises in a time domain description and captures unsteady aerodynamics effects by modelling the wake explicitly [7, 8, 9]. For aeroelastic integration, furthermore, linearisation in the degrees-of-freedom of a beam finite-element model can provide a more general approach in which 3D forces, arbitrary motions and the effects of steady-state aerodynamic loading are accounted for [8]. Coupling this with a linearised geometrically-exact beam model allowed Ng et al. [4] to derive a state-space model of a very flexible HAWT suitable for control synthesis and design.

Vortex methods, however, require a large number of degrees-of-freedom to discretize the wake in a converged model [7, 8]. Upon linearisation, this can be addressed through model order reduction. Hall [10] first proposed projecting the UVLM equations onto a reduced set of aerodynamic eigenstates, but better convergence properties have been achieved using internal balancing [11, 12]. This approach has been subsequently extended to unsteady aerodynamic problems around non-zero reference conditions [13]. Internal balancing projects the state-space model onto a basis which is optimal in terms on input/output mapping [12]. As a result, the achievable states reduction is primarily bounded by the amount of input and output of the state-space model itself [13]. Standard implementations of balancing, however, still need optimising for UVLM applications. Direct methods based on Shur decomposition [14, 15], in fact, unnecessarily compute all the projected modes and do not exploit neither the sparsity of the model [11] or the low-rank nature of the system observability and controllability Gramians [16].

Recently Simpson et al. [17] have shown that Joukowski’s theorem becomes necessary to compute all three components of the aerodynamic force for problems with arbitrary kinematics. This has been further explored by Palacios et al. [18], who presented a consistent linearisation of the UVLM with no kinematics assumption and resolving all components of the unsteady aerodynamic force. The description also scales with the dynamic pressure, hence eliminating the need to re-evaluate aerodynamic coefficients at different operative conditions. For these features, consistent linearisation results particularly suitable to describe the unsteady aerodynamics of very flexible HAWT in the context of aeroelastic integration. Nonetheless, the computational efficiency of the description, which results into large state-space models, needs to be improved.

With the aim of achieving an efficient computational framework for the aeroelasticity of very flexible blades, therefore, this work proposes a three-stages strategy for model reduction of UVLM-based state-space models. In first place, projection over low degree-of-freedom input/output and balanced residualisation [19] will provide minimal-size reduced order models (ROMs). Secondarily, the convergence properties of Palacios et al. formulation [18] is improved by increasing, from first to second order, the time resolution of the added mass forcing term. This second strategy aims to facilitate model-order reduction by scaling-down the size of the full-states model. Finally, a low-rank method for balancing based on Smith’s iteration [16, 20] is proposed to further reduce the computational cost of ROM generation. The process is finally demonstrated for non-zero reference linearisations of resolved two-dimensional aerofoil models with up to $10^5$ states.

2. Methodology

The first part of this section describes the state-space realisation of the UVLM equations as introduced by Palacios et al. [18]. A higher order formulation, based on second-order accuracy evaluation of the added mass effects is here proposed. Finally, the application of Smith’s iterative method to balancing is discussed [16, 20].
2.1. Nondimensionalised discrete-time UVLM

This section outlines the discrete-time state-space description of the unsteady vortex-lattice-based aerodynamics and its linearization around arbitrary reference conditions. When required, dimensional quantities are indicated by the \( \hat{\cdot} \) symbol. The aerodynamic surfaces, and their wake, are discretised in vortex rings of vertex coordinates \( \hat{\zeta} \) and \( \hat{\zeta}_w \). A scalar circulation strength, \( \hat{\Gamma} \) and \( \hat{\Gamma}_w \), is associated to each bound and wake vortex ring. A background fluid velocity field, which includes the presence of a mean flow \( U \) and atmospheric turbulence, \( \nu \), is defined at both the surface and the wake. The aerodynamic forces, \( \hat{f} \), are also defined at the bound lattice nodes. Variables are normalised as

\[
\hat{\Gamma} = bU_\infty \Gamma, \quad \hat{\zeta} = b\zeta, \quad \hat{U} = U_\infty U, \quad \hat{\nu} = U_\infty \nu, \quad \hat{f} = 2q_\infty b^2 f, \quad \hat{t} = \frac{b}{U_\infty} t, \tag{1}
\]

where \( q_\infty \) is the dynamic pressure and the reference time \( b/U_\infty \) is set by a typical semi-chord, \( b \), and a reference wind speed \( U_\infty \). Finally, time derivatives with respect to the normalised time \( t \) are indicated by the \( (\cdot)' \) symbol.

2.1.1. Geometrically-exact formulation: \quad Induced velocities at the mid-point of each vortex ring are calculated through the Biot-Savart law [21]. The non-penetration condition at the lifting surfaces lattice allows expressing the bound circulation at time \( n+1 \) as

\[
\mathcal{A}(\zeta^{n+1}) \Gamma^{n+1} + \mathcal{A}_w(\zeta^{n+1}, \zeta_w^{n+1}) \Gamma_w^{n+1} + W(\zeta^{n+1})(U^{n+1} + \nu^{n+1} - \zeta^{n+1}) = 0, \tag{2}
\]

where \( \mathcal{A}(\zeta) \) and \( \mathcal{A}_w(\zeta, \zeta_w) \) are, respectively, the wing to wing and wake to wing aerodynamic influence coefficient matrices, while \( W(\zeta) \) projects the kinematic and background fluid velocities along the normal vectors at the collocation points. Unsteady effects are included through convection of the wake coordinates \( \zeta_w \) and circulation \( \Gamma_w \) in the fixed reduced time-step \( \Delta t \) [18]. The convection of circulation strengths can be written as

\[
\Gamma_{w}^{n+1} = C_T \Gamma^{n} + C_{\Gamma_w} \Gamma_{w}^{n}, \tag{3}
\]

in which the sparse matrices \( C_T \) and \( C_{\Gamma_w} \) shift the circulation downstream. The wake lattice shape, \( \zeta_w \), can also be obtained as part of the solution [8].

Unsteady aerodynamic forces at the lattice nodes, \( f \), are expressed through the Joukowski method [17]. A quasi-steady contribution, accounting for induced-drag and leading-edge suction, is expressed at the mid-point of the \( n \)-th bound lattice segment, \( l_n \), as

\[
f_{sn} = \Gamma_n (u_n \times l_n), \tag{4}
\]

where the local induced velocity \( u_n \) is also provided by the Biot-Savart law. Unsteady forces, accounting for the added mass effect, are expressed at the centroid of the \( c \)-th bound panel of normal \( n_c \) and area \( A_c \) by [17]:

\[
f_{uc} = \Gamma'_c A_c n_c. \tag{5}
\]

All quantities in Equation 4 and 5 are normalised as per (1). Being the formulation in discrete-time, Equation 5 requires determining a model for the time-derivative \( \Gamma' \). The following two schemes, respectively of first (6) and second (7) order accuracy, are implemented in this work:

\[
\Gamma^{n+1} \Delta t = \Gamma^{n+1} - \Gamma^{n} \tag{6}
\]

\[
2 \Gamma^{n+1} \Delta t = 3 \Gamma^{n+1} - 4 \Gamma^{n} + \Gamma^{n-1} \tag{7}
\]

If Equation 6 is used, the overall UVLM scheme is first order accurate in time, which is the most commonly found implementation [21, 8]. When (7) is employed, the overall time accuracy is improved, yet not second order due to (3); hence this solution is simply referred to as high order.
2.1.2. Consistent linearisation: Equation 2 to 5 can be linearised around arbitrary geometry/in-flow conditions [18] with the only assumption of frozen wake geometry, $\zeta_{w0}$. Contrarily to other potential-flow methods (e.g. DLM), however, the wake can still take any arbitrary shape, either prescribed (e.g. elicoidal) or obtained from nonlinear time-marching of the aerodynamic equations [18]. Perturbation of the variables in Equation 2 to 7 as

$$\Gamma = \Gamma_0 + \delta \Gamma,$$
$$\zeta = \zeta_0 + \delta \zeta,$$
$$\zeta' = \zeta'_0 + \delta \zeta',$$
$$\nu = \nu_0 + \delta \nu,$$
$$f = f_0 + \delta f,$$

allows forming the discrete-time state-space system [18]

$$x^{n+1} = A x^n + B u^n, \quad y = C x + D u,$$  \hspace{1cm} (9)$$

where the input and output vectors are $u = (\delta \zeta^T, \delta \zeta'^T, \delta \nu^T)^T$ and $y = \delta f$, respectively. The state vector is, instead, defined as $x^n = (\delta \Gamma_0^T, \delta \Gamma_0^n T, \Delta t \delta \Gamma_0^m T)^T$ if a first order scheme (6) is used to compute $\delta \Gamma^n T$; otherwise $x^n = (\delta \Gamma_0^m T, \delta \Gamma_0^{m-1} T, \delta \Gamma_0^n T, \Delta t \delta \Gamma_0^m T)^T$. In both cases, using $\Delta t \delta \Gamma_0^m$ as state variable instead of $\delta \Gamma_0^m$ eliminates the dependency of $A$ on the reduced time-step $\Delta t$.

The state-space model (9) is referred in Figure 1 through the discrete-time transfer function $G_0$. For multi-disciplinary integration, and with the aim of reducing the input/output dimensionality, $\zeta$ and $\zeta'$ can be projected in generalised coordinates, $\eta$, through the gain $K_\eta$. A discrete-time model ($G_{\eta'}$) based on (6) or (7) of their time-derivatives, $\eta'$, is also required to produce $\zeta'$. Similarly, a gust model ($G_w$) can be obtained from a generalised set of input, $w$, while the gain $K_f$ retrieves the total aerodynamic forces, $F$. Upon projection onto $\eta, w$ and $F$, a reduced input/output degrees-of-freedom state-space model, $G_{tot}$, can be obtained.

Overall, the approach is very general: discrete-time state-space models include the effects of a nonzero reference condition in all the states and inputs of the model, and can predict linear induced drag effects. Additionally, the description is independent of any structural model and, as forces scale with the free-stream dynamic pressure, there is no need to re-evaluate the aerodynamic coefficients at different operative conditions as long as the geometrical features at the linearisation point, $\zeta_0$ and $\zeta_{w0}$, are unchanged. While computing the added mass through a second order accuracy scheme (7) improves its convergence properties, UVLM models still require thousands of states for modelling the wake unsteady aerodynamics. This makes the approach impractical when a large number of analyses are required, e.g. for fatigue loads calculations. This issue is here tackled through internal balancing.

2.2. Internal Balancing

Internal balancing projects the state-space model $G_{tot}$ onto a basis which optimises the input/output mapping [12]. More specifically, balanced states are equally controllable/observable and ranked according to the importance of their contribution to the system response [11]. This
makes the method appealing for state-space models having a low-dimensional set of degrees-of-freedom — here \( \eta, u \) and \( F \) in Figure 1. In these cases, balancing has already been shown to achieve UVLM ROMs of order comparable to the input/output dimensionality [22].

Balancing the discrete-time system \( G_{\text{tot}} \) (Figure 1) requires solving the discrete-time Lyapunov equations,

\[
AW_cA^T - W_c = -BB^T \quad \text{and} \quad A^T W_o A - W_o = -C^T C ,
\]

for the observability and controllability Gramians, \( W_o \) and \( W_c \). If the system is controllable and observable, such a solution always exists and the Gramians can be shown to be positive-definite [23]. Under these conditions, a transformation \( x_b = T^{-1} x \) such that the Gramians of the balanced system, \( W_{\text{c-bal}} \) and \( W_{\text{o-bal}} \), are equal to a positive-definite diagonal matrix, \( \Sigma = \text{diag}(\sigma) \), i.e.

\[
W_{\text{c-bal}} = T^{-1} W_c T^{-T} = \Sigma = T^T W_o T = W_{\text{o-bal}} ,
\]

also exists [11, 16]. The elements of \( \sigma \) are referred to as Hankel singular values (HSV). Starting from the Cholevski factorisation of the system Gramians,

\[
W_c = Q_cQ_c^T \quad \text{and} \quad W_o = Q_oQ_o^T ,
\]

an expression for \( T \) is given by

\[
T = Q_cV\Sigma^{-1/2} \quad \text{and} \quad T^{-1} = \Sigma^{-1/2}U^TQ_o^T ,
\]

where \( U \), \( V^T \) and \( \Sigma \) are the factors of the singular value decomposition (SVD) of \( Q_c^T Q_c \), i.e.: \( Q_o^T Q_o = U \Sigma V^T \).

Once the transformation \( T \) has been identified, the balanced state vector is partitioned as \( x_b = (x_{b1}^T, x_{b2}^T)^T \), where \( x_{b1} \) contains the states corresponding to the largest \( N_b \) HSVs. In balanced truncation, ROMs are obtained discarding the less observable/controllable states, hence setting \( x_{b2} = 0 \). A better low frequency approximation of \( G_{\text{tot}} \) is, however, obtained by residualisation [19], i.e. setting \( x_{b2}^{n+1} = x_{b2}^n \), which is the approach used in this work. Importantly, both methods share the same error bounding criterion:

\[
\|G_b(k) - G_{\text{tot}}(k)\|_{\infty} \leq \sum_{i=N_b+1}^{N_x} \sigma_i ,
\]

where \( G_b \) is the transfer function of an \( N_b \) order balanced truncated/residualised ROM of \( G_{\text{tot}} \).

2.2.1. Direct methods: Standard direct methods employ either the Bartels-Stewart [14] or the Hammarling algorithms [15] to solve the Lyapunov equations in (10). The latter approach, in particular, provides the Cholevski factors, \( Q_c \) and \( Q_o \), directly. In both cases, however, a Schur decomposition is applied to the \( A \) matrix [16] and a SVD is required for building the \( U \), \( V^T \) and \( \Sigma \) matrices in Equation 13 [24]. Consequently, direct methods result in algorithms of computational complexity \( O(N_x^3) \), where \( N_x \) is the number of system states, and are inefficient for converged UVLM models with \( N_x \geq 10^4 \). Furthermore, in the context of larger-scale UVLM models, Schur decomposition does not allow exploiting the \( A \) sparsity structure.

2.2.2. A low-rank Smith method: In the context of low degrees-of-freedom input/output systems, the Gramians \( W_c \) in Equation 10 are expected to have a low numerical rank [25]. Hence, approximate solutions of the form \( W_c \approx Z \Sigma Z^T \), with rank(\( Z \)) \( \ll N_x \), can be sought. To this aim, the Lyapunov equations are here solved through the squared-Smith iteration [16]. In its
factorised form, this leads to \([20]\):

\[
Z_{c0} = B, \quad \tilde{A}_0 = A^T \\
Z_{o0} = C^T, \quad \tilde{A}_0 = A
\]

and

\[
Z_{*k+1} = \left[ Z_{*k}, (\tilde{A}_k^2)^T Z_{*k} \right]. \tag{15}
\]

While Equation 15 converges quadratically, \(Z_{*k}\) doubles its number of columns at each iterations. The exponential growth of \(Z_{*k}\) can, however, be controlled through SVD truncation \([20]\). Namely, defining \(\sigma_1\) and \(\sigma_2\) as the singular-values of \(Z_{*k}\) respectively above and below a tolerance \(\sigma_{svd}\), one can re-define \(Z_{*k}\) as:

\[
Z_{*k} = [U_1, U_2]\text{diag}([\sigma_1, \sigma_2])V \Rightarrow W_* = Z_{*k}Z_{*k}^T \approx U_1\text{diag}(\sigma_1^2)U_1^T \Rightarrow Z_{*k} \approx U_1\text{diag}(\sigma_1). \tag{16}
\]

The balancing transformation \(T\) is finally provided by Equation 13, where \(U, V^T\) and \(\Sigma\) are now obtained through SVD of \(Z_{*k}^2Z_c\) \([16]\).

Overall, the computational cost of the algorithm is determined by the number of iterations required in Equation 15 and the rank of \(Z_c\) and \(Z_o\) \([16]\). These, in turn, depend of the input/output dimensionality of \(G_{tot}\), as well as the radius of \(A\) \([20]\), which for UVLM models is inconveniently close to 1. With this respect, acceleration techniques based on alternating direction implicit iteration can boost the converge speed of Smith’s method \([25, 20]\). Furthermore, the \(O(N^3_d)\) computationally expensive evaluation of \((\tilde{A}_k^2)^T Z_{*k}\) in Equation 15 can be parallelised \([16]\) or speeded-up using a block-Krylov method \([20]\). However, only matrix products parallelisation is included in the current implementation, which aims primarily to explore the impact of the low-rank process.

3. Results

This section starts with an assessment of the convergence properties of the high order UVLM scheme (subsection 3.1) and the state-space realisation around non-zero reference, subsection 3.2. In subsection 3.3 the effect of producing ROMs starting from a first and a high order UVLM scheme (subsection 3.1) and the state-space realisation around non-zero reference, subsection 3.2.

All studies will consider two-dimensional aerofoils. With reference to \(G_{tot}\) in Figure 1, these account for camber deflections, pitch (about a point at 0.25 chords behind the leading-edge), plunge and planar motion \((\eta, \eta')\), while \(w\) and \(G_w\) model a sinusoidal gust. Aerodynamic lift, drag and moment coefficients define the model output, \(F\).

3.1. Convergence of high order UVLM

The better convergence properties of the high order UVLM scheme are shown for a flat plate of infinite span at zero incidence. The output lift and moment coefficients, \(C_L\) and \(C_M\), responses to pitch, plunge and gust input are compared against Theodorsen’s and Sears’s analytical solutions. For each output/input combination \(ij\), the \(H_\infty\) norm of the error over the reduced frequencies range \(k \in [0, 1]\)

\[
\mathcal{E}_\infty[ij] = \frac{1}{2\pi} \int_0^1 |G_{tot}(k) - G_{ref}(k)|_{ij} \, dk \tag{17}
\]

is shown in Figure 2. In both schemes the \(C_M\) response generally converges slower, hence setting a bound for the spacial/temporal discretisation. Expectedly, the high order scheme delivers higher accuracy. For example, \(\mathcal{E}_\infty[ij] < 10^{-2}\) for all output/input combinations \(ij\) is achieved with just 80\% of the states required by a first order UVLM (3200 vs 4050 states).
3.2. State-space realisation around non-zero reference

Realisation around non-zero reference is demonstrated in Figure 3, where the response of a family of cambered plates at zero incidence is shown. Aerofoils camberlines follow the 4-digits NACA convention, with the point of maximum camber at 40% of the chord, c. A maximum camber of 0.08c has been considered. Due to the small curvature of these profiles, the lift response does not depart considerably from Theodorsen’s analytical solution for flat plates. This is shown for the case of pitch input Figure 3a and 3b: only for the largest camber of 0.08c and high reduced frequencies (k > 0.5) the added mass, which acts perpendicular to the aerofoil surface, produces a noticeable effect.

The response magnitude of the $C_M$ about 0.25c, shown in Figure 3c for pitch input, also follows closely the linear trend of Theodorsen’s solution. This is expected as, at small angles of attack, moments are mainly due to lifting forces. However, consistent linearisation can capture the steady changes ($k = 0$) of the $C_M$ with the pitch angle, linked to the fact that cambered aerofoil aerodynamic center in not located at a quarter-chord from the leading edge. While small-amplitude, these drastically impact the phase response with respect to Theodorsen’s solution in the $k < 0.3$ range (Figure 3d). As $k$ increases, added mass effects become dominant, and the phase response smoothly transits from 0 deg to Theodorsen’s solution.

Finally, it is worth noticing that, when the total aerodynamic force is non-zero at the linearisation point, the realisation can also capture first-order induced drag effects. This is shown by the drag coefficient, $C_D$, response to plunge in Figure 3e and Figure 3f. For flat plates, the induced drag effects in plunge motion are only second-order [26], hence they are not detected by the linear assumption. However, when the aerofoil has camber, i.e. a non-zero lift at the realisation point, consistent linearisation captures the tilting of the aerodynamic force due to the changes in effective angle of attack resulting from plunging.

3.3. Balanced realisation: first vs. high order UVLM

As seen in Figure 2, the high order UVLM model provides better convergence properties. In the next study, therefore, we verify whether this translates into any advantage for model reduction. To this aim, balanced residualisation is applied to first and high order UVLM models of a zero incidence flat plate. Different mesh refinements are considered. For each output $F_i$, the number of states retained in the ROM is chosen so as to bound the gap against Theodorsen’s and Sears’s analytical solutions as

$$\varepsilon_{\infty}[F_i] = \max_j \varepsilon_{\infty,j}[F_i] < 1.5 \cdot 10^{-2}. \quad (18)$$
As shown in Figure 4a, there is a minimum full-states model size above which very low-order ROMs, reducing the number of states of up to three orders of magnitude, can be obtained. Importantly, converged ROMs obtained with a first and high order scheme have a comparable number of states, being balancing driven by the input/output dimensionality. However, due to its better convergence properties, balancing over the high order UVLM generally allows manipulating smaller systems. ROMs of order $N_x \approx 10$ are obtained from full-states high-order UVLM models of about 3500 states or more, against a minimum of 5000 states required by a first order UVLM. When considering that direct methods for balancing have computational complexity $O(N_x^3)$, this translates into a 65% computationally cheaper model reduction process.

Figure 4b also shows the HSVs associated to a first and a high accuracy UVLM model of comparable size ($\approx 3900$ states). For models of order below 10, the HSVs of the first order accuracy UVLM have lower values that those linked to the high-accuracy UVLM. The same applies to the error $\mathcal{E}_\infty(C_L)$, consistently with Equation 14. However, as the order is increased, ROMs derived from the higher-accuracy UVLM scheme achieve lower $\mathcal{E}_\infty(C_L)$ values, hence explaining the trends observed in Figure 4a.

### 3.4. Balancing through Smith’s method

Balancing based on the Smith’s method is verified for UVLM models of a cambered plate at a 2° angle of attack. The aerofoil profile is defined according to the 4-digit NACA convention, with maximum camber of 0.04 c at 0.4 c from the leading edge. High order UVLMs are used and convergence is measured against a highly resolved (7809 states) solution. Smith’s algorithm is run for different values of $\sigma_{svd}$ and the deriving ROMs are compared to those obtained via direct balanced reduction in Figure 5. Converged ROMs require $\sigma_{svd} \leq 10^{-5}$ and compare perfectly against those derived through direct balancing. For instance, the number of states
needed to reach a convergence level $\mathcal{E}_\infty[F_j] < 0.02$ is in good agreement (Figure 5a). Also the HSVs and $\mathcal{E}_\infty[C_L]$ compare well when varying the ROM size (Figure 5b). As a result of the low tolerance levels, $\sigma_{\text{end}} < 10^{-5}$, required to converge the Smith iteration, however, the factors $Z_c$ and $Z_\sigma$ resulting from SVD truncation (Equation 16) are almost full-rank, $\text{rank}(Z_c) \approx 0.9 \, N_x$. Despite this, the computational time of the process is considerably reduced (Figure 5c): for the largest system of 7809 states, iterative balancing is approximatively 40\% less computationally expensive.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure4a}
\includegraphics[width=0.45\textwidth]{figure4b}
\caption{(a) ROMs with $\mathcal{E}_\infty[F_j] < 1.5 \cdot 10^{-2}$ vs. full-model size. (b) HSV and $\mathcal{E}_\infty[C_L]$ for ROMs derived from full-states model of 3900 states.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure5a}
\includegraphics[width=0.45\textwidth]{figure5b}
\caption{(a) ROMs with $\mathcal{E}_\infty[F_j] < 0.02$ vs. full-size model. (b) HSV and $\mathcal{E}_\infty[C_L]$ for ROMs derived from a 7809 states model.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure5c}
\caption{(c) Computational time of iterative Smith algorithm expressed as a fraction of the time required for balancing using a direct method.}
\end{figure}

\section{4. Conclusions}
This work presented a general state-space realisation of the unsteady-vortex lattice method and a three-fold strategy to reduce the dimensionality of the resulting models. Numerical studies focused on two-dimensional aerofoil models. Consistent linearisation has been shown
to resolve all force components and to be suitable for linearisation around arbitrary geometries. Normalisation of the equations also scales the description with the dynamic pressure.

Having the problems of interest very few inputs and outputs, balanced residualisation allowed reducing the models order of several orders of magnitude. The size of converged ROMs, in particular, has been seen to depend only on the state-space model input/output dimensionality. Second-order modelling of added-mass effects improved the convergence properties of the linearised UVLM description, hence allowing to derive converged ROMs from smaller-size full-states models. An iterative Smith algorithm has been seen to further reduced the computational cost of internal balancing. Numerical studies suggest that the high-order UVLM/low-rank Smith combination can produce highly converged ROMs in about 20% of the time required when using a first-order UVLM scheme and a direct method balancing.

Overall, the UVLM state-space realisation presented, combined with an efficient internal balancing strategy, can lead to an accurate, yet computationally cheap, aerodynamic description which includes unsteady aerodynamics effects and accounts for arbitrary geometries/deformations. For these reasons, the model proposed offers an appealing alternative to BMT-based tools in the context of very flexible HAWT aeroelasticity. Numerical studies, however, have been restricted to two-dimensional models, hence the portability of the approach to larger-scale 3D application needs further verifications.

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