MONODROMY AND BIRATIONAL GEOMETRY OF O’GRADY’S SIXFOLDS

GIOVANNI MONGARDI AND ANTONIO RAPAGNETTA

Abstract. We prove that the bimeromorphic class of an hyperkähler manifolds deformation equivalent to O’Grady’s six dimensional is determined by the Hodge structure of its Beauville-Bogomolov lattice by showing that the monodromy group is maximal. As applications, we give the structure for the kähler and the birational Kähler cones in this deformation class and we prove that the existence of a square zero divisor implies the existence a rational lagrangian fibration with fixed fibre types.

1. Introduction

This paper deals with a deformation class of hyperkähler manifolds, which was first discovered by O’Grady [29]. These manifolds are sixfolds with second Betti number 8, and are usually called manifolds of $OG6$ type. Manifolds in this family are obtained in two ways. The first construction, is obtained by taking a generic abelian surface and a Mukai vector $w$ of square 2. The moduli space of Gieseker semistable sheaves with Mukai vector $2w$ is a singular tenfold with rational singularities, whose Albanese fibre admits a crepant resolution that is a hyperkähler manifold in the family we are dealing with. This was proven by O’Grady [29] for a special Mukai vector. Later M. Lehn and Sorger [15] showed that, under our assumption on $w$, the blow up of the Albanese fibre of the moduli space along its singular locus always gives a crepant resolution and Perego and the second named author proved in [34] that these crepant resolutions are deformation equivalent, along smooth projective deformations, to the original O’Grady’s example. A second construction was obtained in [24], by considering a principally polarized abelian surface $A$ and its Kummer K3 surface $S$. On a moduli space of sheaves on $S$, the authors construct a non regular involution, whose quotient is birational to a manifold of $OG6$ type. This last construction was used to compute the Hodge numbers of manifolds of $OG6$ type, and the present paper is a continuation of [24] aiming at a greater understanding of their geometry.

For every hyperkähler manifold $X$, the second cohomology group $H^2(X, \mathbb{Z})$ has a natural lattice structure induced by the Beauville-Bogomolov form $B_X$, compatible with the weight two Hodge structure and this datum is a fundamental invariant for $X$: by Verbitsky’s Global Torelli Theorem it determines, up to a finite indeterminacy, the bimeromorphic class of $X$ among hyperkähler manifolds in the same hyperkähler deformation equivalence class of $X$.

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In this paper we show that in the case of $OG6$ some basic geometric are completely determined by the weight two Hodge structure of the Beauville-Bogomolov lattice. The main result is Theorem 5.4(2) stating that the Classical Bimeromorphic Global Torelli Theorem, as conjectured in Speculation 10.1 of [9], holds for manifolds of $OG6$ type:

**Theorem.** Let $X,Y$ be two hyperkähler manifold of $OG6$ type and equip $H^2(X,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$ by the lattice structures induced by the Beauville-Bogomolov forms of $X$ and $Y$. Then $X$ and $Y$ are bimeromorphic if and only if there exists an isometric isomorphism of integral Hodge structures $H^2(X,\mathbb{Z}) \cong H^2(Y,\mathbb{Z})$.

We remark that the Classical Bimeromorphic Global Torelli Theorem rather rarely happens to hold for deformation equivalence classes of known hyperkähler manifolds: it only holds (among known hyperkähler manifolds) for $K3$ surfaces and their Hilbert schemes of $n$ points if $n−1$ is a prime power. The theorem fails for Hilbert schemes on $K3$ surfaces if $n$ is not a prime power (cf. [16] Section 9) and fails for O’Grady’s ten dimensional manifolds thanks to a counter example contained in [23]. Finally the Classical Bimeromorphic Global Torelli Theorem always fails for generalized Kummer manifolds, due to a classical counterexample by Namikawa [24], as replacing the abelian surface used to construct the generalized Kummer manifold with its dual does not change the second Hodge structure but the two Kummer manifolds are not birational. Due to this counterexample and due to the role of an abelian surface in the construction of O’Grady’s six dimensional manifolds, one could expect a similar failure of the Global Torelli Theorem for O’Grady’s sixfolds. However, this is not the case and an intuitive explanation of this fact could be that the relevant manifolds for O’Grady’s construction are not only an abelian surface $A$, but rather $A \times A^\vee$ and the Kummer K3 surface $\widetilde{A}/\pm 1$ (cf. [24]).

We present two main applications. The first one is Theorem 6.9 and gives the description of the Kähler cone $K(X) \subset H^{1,1}(X,\mathbb{R})$ and of the closure $\overline{BK}(X) \subset H^{1,1}(X,\mathbb{R})$ of the birational Kähler cone of a hyperkähler manifold of $OG6$ type $X$, in terms of its weight two Hodge structure in a purely lattice theoretic way.

**Theorem.** Let $X$ be on hyperkähler manifold of $OG6$ type and let the positive cone $C(X)$ of $X$ be the connected component of the cone

$$\{\alpha \in H^{1,1}(X,\mathbb{R}) : B_X(\alpha,\alpha) > 0\}$$

containing a Kähler class. Then

(1) The closure of the birational Kähler cone $\overline{BK}(X)$ of $X$ is the closure in $C(X)$ of the connected component of

$$C(X) \setminus \bigcup_{\alpha \in H^{1,1}(X,\mathbb{Z}), B_X(\alpha,\alpha) = -2 \text{ or } -4, \text{ div}(\alpha) = 2} \alpha^\perp \oplus \beta_X$$

containing a Kähler class.
(2) The Kähler cone $K(X)$ is the connected component of
$$C(X) \setminus \bigcup_{\alpha \in H^{1,1}(X,\mathbb{Z}), \ B_X(\alpha,\alpha)=-2 \text{ or } B_X(\alpha,\alpha)=-4 \text{ and } \text{div}(\alpha)=2} \alpha^\perp_{B_X}$$
containing a Kähler class.

In this statement, for $\alpha \in H^2(X,\mathbb{Z})$ the subspace $\alpha^\perp_{B_X} \subseteq H^2(X,\mathbb{R})$ is the perpendicular to $\alpha$ with respect to the real extension of $B_X$ and $\text{div}(\alpha)$ is the divisibility of $\alpha$ in $H^2(X,\mathbb{Z})$, i.e. the minimum strictly positive integer that can be obtained as $B_X(\alpha,\beta)$ for $\beta \in H^2(X,\mathbb{Z})$ (see Definition 2.1). We remark that, when $X$ is projective, the ample cone is the intersection of the Kähler cone with $H^{1,1}(X,\mathbb{Z}) \otimes \mathbb{R}$ and the movable cone is $BK(X) \cap (H^{1,1}(X,\mathbb{Z}) \otimes \mathbb{R})$.

The second main application concerns the existence of lagrangian fibrations on hyperkähler manifolds of $OG_6$ type. A famous conjecture due to Hassett-Tschinkel, Huybrechts and Sawon predicts the existence of a lagrangian fibration on every hyperkähler manifold admitting a divisor whose class is isotropic with respect to the Beauville-Bogomolov form. Theorem 7.2 settles this conjecture for hyperkähler manifolds of $OG_6$ type.

**Theorem.** Let $X$ be a manifold of $OG_6$ type with a square zero divisor. Then $X$ has a bimeromorphic model which has a dominant map to $\mathbb{P}^3$ whose general fiber is a $(1,2,2)$-polarized abelian threefold.

As a corollary of this theorem we also prove Beauville’s weak splitting property for hyperkähler manifolds of $OG_6$ type with a square zero divisor (see Corollary 7). Our proof of the Classical Bimeromorphic Global Torelli for $OG_6$ type rests on Markman’s Hodge theoretic version [16, Thm 1.3] of Verbitsky’s global Torelli theorem [44] stating that if $X$ and $Y$ are two hyperkähler manifolds in the same deformation equivalence class, they are bimeromorphic if and only if there exists an isometric, with respect to the Beauville-Bogomolov forms, isomorphism $H^2(X,\mathbb{Z}) \cong H^2(Y,\mathbb{Z})$ coming from a parallel transport. This theorem reduces the problem of the validity of the Classical Bimeromorphic Global Torelli Theorem to the computation of the monodromy group, which is the group of all transformations of the second cohomology which can be obtained by taking parallel transport along loops in families of smooth Kähler deformations of our hyperkähler manifold in the chosen deformation equivalence class. The result of this calculation (see Theorem 5.4 (1)) can be considered the core of this paper.

**Theorem.** Let $X$ be an hyperkähler manifold of $OG_6$ type, Then the Monodromy group $\text{Mon}^2(X)$ is the subgroup $O^+(H^2(X,\mathbb{Z}))$ of the Z linear automorphism preserving the Beauville-Bogomolov form and the orientation of the positive cone $C(H^2(X,\mathbb{Z}))$ of $H^2(X,\mathbb{Z})$.

The explanation of the definitions of $O^+$ and $C(H^2(Y,\mathbb{Z}))$ is given in Subsection 2.1 and Remark 2.2. Since every isometry of $H^2(X,\mathbb{Z})$ is contained, up to sign, in $O^+(H^2(Y,\mathbb{Z}))$, every Hodge isometry $H^2(X,\mathbb{Z}) \cong H^2(Y,\mathbb{Z})$ comes, up to sign, by a parallel transport and the Classical Bimeromorphic Global Torelli Theorem holds for $OG_6$ type manifolds.

To contextualize our result on the monodromy group we recall that monodromy groups of hyperkähler manifolds of the deformation types of Beauville’s examples
were already known: the monodromy group of manifolds of $K3^{[n]}$ type was computed by Markman [16, Section 9] and that of generalized Kummers by the first named author [23] using fundamental results of Markman [19]. On the other hand, the monodromy group of manifolds of $OG_{10}$ type (the deformation type of the ten dimensional O’Grady example) is still unknown although very important progress has been recently made by Onorati [31].

Our two main applications follow directly from the computation of the monodromy group.

By the work of the first named author [22] and Markman [16, Section 6], the Kähler and the Birational Kähler cones are cut out by wall divisors and Stably prime exceptional divisors respectively. Using that these are two classes of monodromy invariant divisors, we determine them by means of explicit geometric constructions in specific examples of hyperkähler manifolds of $OG_6$ type in Proposition 6.8.

They key observation in the proof of the existence of a lagrangian fibration on a hyperkähler manifold $X$ of $OG_6$ type with a divisor whose Beauville-Bogomolov square is 0 is that primitive isotropic elements in $H^2(X,\mathbb{Z})$ are in the same monodromy orbit. By a fundamental results due to Markman ([17, Section 5.3 and Lemma 5.17(ii)]) and Matsushita ([21, Theorem 1.2]) this reduces the proof of the Hassett-Tschinkel, Hybrechts and Sawon conjecture to the existence of a lagrangian fibration in a single case.

The paper is structured as follows: in Section 2, we introduce some basic facts about lattices and monodromy which will be used in the proofs. In Section 3, we prove that parallel transport of complex tori of dimension two can all be obtained by considering families of projective tori. We believe this result is known to experts, but we could not find it in the literature. In Section 4, we study the monodromy group of the singular moduli spaces of semistable sheaves, using parallel transport along projective families and Fourier-Mukai transformations. The key result of this section is that this group is already maximal, see Proposition 4.12 for details. In Section 5, we use the previous result to finish the computation of the Monodromy group. Finally, the last two sections are dedicated to applications: in Section 6, we give the structure of the Kähler and the birational Kähler cones for manifolds of $OG_6$ type and in Section 7 we prove that the existence of divisors of square zero implies the existence of a birational lagrangian fibration with fibres of polarization type $(1,2,2)$ and we prove Beauville’s weak splitting property for these manifolds.

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2. Preliminaries

In this section, we will gather notation, definitions and some results concerning Lattices and Monodromy groups, which we will use in the following.

2.1. Lattices: Notation and basic results. In this subsection, we will fix some notation concerning lattices and will recall the results which we will use in the following.

Definition 2.1. An even lattice $L$ is a finitely generated free $\mathbb{Z}$ module equipped with a non-degenerate bilinear symmetric form $(\cdot, \cdot)$, with values in $\mathbb{Z}$, such that the associated quadratic form takes only even values.

The discriminant group of $L$ is the finite abelian group $A_L := L^\vee/L$ and the discriminant form $q_{A_L} : A_L \to \mathbb{Q}/2\mathbb{Z}$ is the quadratic form induced by the bilinear form $(\cdot, \cdot)$ on $L$.

The divisibility $\text{div}_L(v)$ of an element $v \in L$ is the positive generator of the ideal $(v, L)$, if no confusion can arise we simply denote it by $\text{div}(v)$.

We are interested in certain subgroups of the group of isometries $O(L)$ of the even lattice $L$, when it is not negative definite. To introduce them, we recall that the Grassmannian $Gr^+(L)$ parametrizing maximal positive definite subspaces of $L \otimes \mathbb{R}$ is contractible and the Grassmannian $Gr^{+\text{or}}(L)$ parametrizing oriented maximal positive definite subspaces of $L \otimes \mathbb{R}$ has two connected components.

The subgroups of $O(L)$ we are interested in, are the following:

- $SO(L)$, the group of isometries of determinant one.
- $O^+(L)$, the group of isometries acting trivially on the set of connected components of $Gr^{+\text{or}}(L)$.
- $SO^+(L) := SO(L) \cap O^+(L)$.
- $\tilde{O}(L)$ The group of isometries whose induced action on $A_L$ is trivial.
- $S\tilde{O}^+(L) = SO^+(L) \cap \tilde{O}(L)$.

Remark 2.2. For every even lattice $L$ that is not negative definite the positive cone of $L$ is

$$C(L) := \{ v \in L \otimes \mathbb{R} | (v, v) > 0 \} \subseteq L \otimes \mathbb{R}.$$  

As shown in [16, §4], for every maximal positive subspace $W \subseteq L \otimes \mathbb{R}$, the complement of the origin $W \setminus \{0\}$ is a deformation retract of $C(L)$: hence $C(L)$ has the Homotopy type of a sphere. The subgroup $O^+(L)$ is equivalently defined, in [16, §4], as the subgroup of $O(L)$ acting trivially on the homology of $C(L)$, i.e. the subgroup of $O(L)$ preserving the orientation of the positive cone $C(L)$.

All results of this subsection are well known to experts, and we use standard references for them.

Lemma 2.4 is a folklore result, which allows to extend isometries of a sublattice to the ambient lattice. We will use it several times. The following Lemma 2.3 is used in the proof of Lemma 2.4.

Lemma 2.3. [27, Prop. 1.4.1] Let $L, M$ be two even lattices and $H_N \subset A_L \oplus A_M$ an isotropic subgroup. Then there is a unique even lattice $N \subset L^\vee \oplus M^\vee$ such that $H_N = N/(L \oplus M)$. Conversely, every finite extension $L \oplus M \subset N$ determines an isotropic subgroup $H_N \subset A_L \oplus A_M$. Moreover, $A_N = H_N^\vee / H_N$. 
Proof. Let us sketch how $H_N$ is determined by $N$, as it will be the only part of interest for us. An element $n \in N$ determines an element of $(L \oplus M)^\vee$ by taking the map $v \to (n, v)$. If $n \notin L \oplus M$, this gives us a non zero class $[n]$ inside $A_{L \oplus M} = A_L \oplus A_M$. The value of the discriminant form on $[n]$ is the value of the quadratic form on $n$ modulo $2\mathbb{Z}$, that is $0$ as $N$ is even. The non zero elements of $H_N$ are then given by all elements of $N$ which are not in $L \oplus M$. \hfill \Box

In the special and relevant for us case where $N/M$ and $N/L$ have no torsion, the group $H_N$ intersects $A_L$ (and $A_M$) only in the zero element.

**Lemma 2.4.** Let $L$ and $M$ be two lattices and let $M \oplus L \subset N$ be a finite order extension by an even lattice. Suppose moreover that $N/L$ and $N/M$ have no torsion. Let $H_N \subset A_L \oplus A_M$ be the finite order isotropic subgroup given by $N$. Let $f \in \overline{O}(M)$ be an isometry. Then, there exists an isometry $\overline{f}$ of $N$ such that $\overline{f}|_M = f$ and $\overline{f}|_L = 1d_L$.

**Proof.** As in the proof of Lemma 2.3, elements of $H_N$ are represented by elements $n$ of $N$ not in $L \oplus M$. For all of them, there is an integer $r$ such that $rn = l + m$, with $l \in L$ and $m \in M$. Let us define $\overline{f}(n) = \frac{1}{2}(m + l)$ for $n \notin L \oplus M$. The image of $\overline{f}$ is again contained in $N$, therefore it is a well defined isometry and we are done. \hfill \Box

**Remark 2.5.** The above lemma will be often applied to the case where $f \in \overline{O}(M)$ is an isometry of a sublattice $M$ of $U^3$ and $N$ has rank one: if $N$ is generated by an element of square 2 and $M$ is its perpendicular, the discriminant group $A_M$ has order two and $\overline{f}$ always exists.

The following is a famous result of Eichler, which allows to determine when two elements of a lattice are in the same orbit of the isometry group.

**Lemma 2.6.** [7, Lemma 3.5] Let $L'$ be an even lattice and let $L = U^2 \oplus L'$. Let $v, w \in L$ be two primitive elements such that the following holds:

- $v^2 = w^2$.
- $[v/\text{div}(v)] = [w/\text{div}(w)]$ in $A_L$.

Then there exists an isometry $g \in SO^+(L)$ such that $g(v) = g(w)$.

In the reference above, the result is stated for $g \in \overline{O}^+(L)$, but the proof uses a class of isometries, called Eichler’s transvections, which have determinant one. Indeed, let $L$ be any indefinite lattice, let $e \in L$ be isotropic and let $a \in e^\perp$, the Eichler’s transvection $t(e, a)$ with respect to $e$ and $a$ is defined as follows:

$$t(e, a)(v) = v - (a, v)e + (e, v)a - \frac{1}{2}(a, a)(e, v)e.$$  

In particular, we have the following three properties:

- $t(e, a)^{-1} = t(e, -a)$
- $t(e, a) \circ t(e, b) = t(e, a + b)$.
- $g^{-1} \circ t(e, a) \circ g = t(g(e), g(a))$.

where $g$ is any isometry. If $L = U \oplus L_1$, we will denote with $E_U(L_1)$ the group of all Eichler’s transvections $t(e, a)$ with $e \in U$ and $a \in L_1$.

Two useful results giving a finite number of generators of isometry groups and concerning Eichler’s transvections are the following:
Lemma 2.7. [6] Lemma 3.2 Let $U^2$ be two copies of the hyperbolic lattice with standard basis $\{e_1, f_1, e_2, f_2\}$. Then $SO^+(U^2)$ is generated by $t(e_2, e_1), t(e_2, f_1), t(f_2, e_1)$ and $t(f_2, f_1)$.

Lemma 2.8. [6] Prop. 3.3 (iii) Let $L := U \oplus L_1$ be an even lattice. Then $O^+(L)$ is generated by $O^+(L_1)$ and $E_U(L_1)$.

2.2. Monodromy: definitions and facts. In this subsection we recall the notions of parallel transport operator and monodromy groups.

Definition 2.9. A proper morphism $f : \mathcal{X} \to T$ between complex analytic spaces, is a proper analytically locally trivial family if every $x \in \mathcal{X}$ has an analytic neighborhood isomorphic, over $T$, to a product of a neighborhood of $U_x$ of $x$ in its fibre $f^{-1}(f(x))$ and a neighborhood $V_{f_x}$ of $f_x \in T$.

The main example of a proper analytically locally trivial family is given by a standard basis $\{e_1, f_1, e_2, f_2\}$ of $H^n(X, \mathbb{Z})$ and $\gamma$ is the loop $\gamma : [0, 1] \to T$, the sheaf \(\gamma^*(R^n f_*(\mathbb{Z}))\) is constant.

Definition 2.10. (1) Set $X := f^{-1}(\gamma(0))$ and $X' := f^{-1}(\gamma(1))$, the parallel transport operator on $H^n$ associated with $f$ and $\gamma$ is the isomorphism
\[
t^n_{\gamma,f} : H^n(X, \mathbb{Z}) \to H^n(X', \mathbb{Z})
\]
induced between the stalks at 0 and 1 by the trivialization of $\gamma^*(R^n f_*(\mathbb{Z}))$.

(2) A monodromy operator on $H^n(X, \mathbb{Z})$ induced by $f$ is an isomorphism of the form
\[
t^n_{\gamma,f} : H^n(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})
\]
where $\gamma$ is a loop $\gamma : [0, 1] \to T$.

(3) The group of monodromy operator on $H^n(X, \mathbb{Z})$ induced by $f$ is
\[
Mon^n_f(X) := \{t^n_{\gamma,f} \mid \gamma(0) = \gamma(1)\}.
\]

By construction the parallel transport operator $t^n_{\gamma,f}$ only depends on the fixed endpoints homotopy class of $\gamma$. The notions of parallel transport operator and monodromy operator allow to introduce the monodromy groups that we are interested in.

Definition 2.11. (1) If $X$ is a compact Kähler manifold, the monodromy group $Mon^n(X)$ is the subgroup of $\text{Aut}_{\mathbb{Z}-\text{mod}}(H^n(X, \mathbb{Z}))$ generated by the subgroups of the form $Mon^n_f(X)$ where $f : \mathcal{X} \to T$ is a proper smooth morphism having $X$ as fibre and the normalization of $\mathcal{X}$ is Kähler.

(2) If $X$ is a projective manifold, the projective monodromy group $Mon^n(X)^\text{pr}$ is the subgroup of $\text{Aut}_{\mathbb{Z}-\text{mod}}(H^n(X, \mathbb{Z}))$ generated by the subgroups of the form $Mon^n_f(X)$ where $f : \mathcal{X} \to T$ is a projective smooth morphism between algebraic varieties having $X$ as fibre.

(3) If $X$ is a singular projective variety, the locally trivial projective monodromy group $Mon^n(X)^\text{pr}$ is the subgroup of $\text{Aut}_{\mathbb{Z}-\text{mod}}(H^n(X, \mathbb{Z}))$ generated by the subgroups of the form $Mon^n_f(X)$ where $f : \mathcal{X} \to T$ is a projective proper analytically locally trivial family having $X$ as fibre.
We will be interested in the special case of the group \( \text{Mon}^2(X) \), where either \( X \) is a hyperkähler manifold or \( X \) is a projective variety admitting a resolution by an hyperkähler manifold: in the latter, \( X \) is a singular symplectic variety such that \( H^1(\mathcal{O}_X) = 0 \) admitting a unique, up to scalar, holomorphic two form on the smooth locus, i.e. \( X \) is a projective primitive symplectic variety according to [2, definition 3.1] or a projective Namikawa symplectic variety according to [33, Definition 2.18]. In both cases, \( H^2(X, \mathbb{Z}) \) has a pure Hodge structure with a compatible deformation invariant quadratic form \( q_X \), the Beauville-Bogomolov form in the smooth case and the Beauville-Bogomolov-Namikawa form in the singular case (see [26]). The deformation invariance of \( q_X \) implies that, \( \text{Mon}^2(X) \) in the smooth case or in \( \text{Mon}^2(X)^{pr} \) in the singular case, actually lies in \( O(H^2(X, \mathbb{Z})) \), where the lattice structure is given by \( q_X \).

For a hyperkähler manifold \( X \) or a projective primitive symplectic variety, the group \( O^+(H^2(X, \mathbb{Z})) \) coincides with the group of isometries preserving the two components of the cone of the classes in \( H^{1,1}(X, \mathbb{R}) \) having strictly positive Beauville-Bogomolov(-Namikawa) square (see [16, §4]). As a Kähler class in the smooth case, or an ample class in the singular projective case, gives a preferred component of this cone, we have the following fundamental constraint on the monodromy groups that we are going to study:

**Lemma 2.12.**

1. \( \text{Mon}^2(X) \subseteq O^+(H^2(X, \mathbb{Z})) \) for every hyperkähler manifold \( X \).
2. \( \text{Mon}^2(X)^{pr} \subseteq O^+(H^2(X, \mathbb{Z})) \) for every projective primitive symplectic variety \( X \).

3. **Abelian monodromy**

In this section we study the monodromy group \( \text{Mon}^2(A) \) on the second cohomology for an abelian surface \( A \). The elements of \( \text{Mon}^2(A) \) will be used in the next section to induce monodromy operators on the Albanese fibres of moduli spaces of sheaves on abelian surfaces. It is well known, essentially already contained in [41], that

\[
\text{Mon}^2(A) = SO^+(H^2(A, \mathbb{Z})),
\]

where the lattice structure on \( H^2(A, \mathbb{Z}) \) is given by the intersection form. We recall that the intersection form on \( H^2(A, \mathbb{Z}) \) is unimodular and even hence, by classification, there exists an isometry \( H^2(A, \mathbb{Z}) \cong U^3 \) and we have an isomorphism

\[
\text{Mon}^2(A) \cong SO^+(U^3).
\]

Unfortunately we cannot use directly this result in the next section to induce monodromy operators on Albanese fibres of moduli spaces of sheaves because of the absence of a satisfactory theory of moduli spaces of sheaves on non necessarily projective surfaces.

We need to prove a more precise result stating that the whole \( \text{Mon}^2(A) \) comes from compositions of monodromy operators induced by projective families.

In the following Proposition 3.3 we analyze monodromy operators coming from families of polarized abelian surfaces. Before stating it we recall the notion of a polarization on an abelian surface.
Definition 3.1. A polarization on an abelian surface $A$ is a primitive class $\alpha \in H^2(A, \mathbb{Z}) \cap H^{1,1}(A)$ such that, $\forall \beta \in H^{1,0}(A, \mathbb{C})$, the real number

$$i \int_A \alpha \wedge \beta \wedge \beta$$

is positive and

$$\int_A \alpha \wedge \alpha = 2d.$$

Remark 3.2. Starting from Section 4 we follow the usual convention according to which a polarization on a projective variety is an ample divisor or an ample line bundle.

Proposition 3.3. For every $d \in \mathbb{N} \setminus \{0\}$ there exists a smooth quasi-projective variety $T_{2d}$, a smooth family of abelian surfaces $f_{2d} : A_{2d} \to T_{2d}$ and a relatively very ample line bundle $\mathcal{L}$ on $A_{2d}$ such that:

1. the class $[\mathcal{L}] \in H^2(A, \mathbb{Z})$ of the restriction $\mathcal{L}_t$ of $\mathcal{L}$ to $A_t := f^{-1}(t)$ is the triple of a polarization $\alpha_t$ of type $(1, d)$ on $A_t$;
2. every polarized abelian surface of type $(1, d)$ is isomorphic, as a polarized abelian surface, to some fibre of $f$;
3. the group $\text{Mon}^\alpha_t(A_t)$ of monodromy operators on $H^2(A_t, \mathbb{Z})$ associated with the family $f$ is

$$SO^+_n(H^2(A_t, \mathbb{Z})): = \{ \varphi \in SO^+(H^2(A_t, \mathbb{Z})) | \varphi(\alpha_t) = \alpha_t \}.$$

Proof. We construct $f_{2d} : A_{2d} \to T_{2d}$ as the restriction of the universal family of a suitable Hilbert scheme to an open subset.

Since for every abelian variety the triple of an ample divisor is always very ample [4, Theorem 4.5.1] and since the dimension of the space of sections of every ample line bundle on a polarized abelian variety only depends on the type of the polarization, the polarized abelian surface of type $(1, d)$ can all be embedded in a fixed projective space $\mathbb{P}^m$ ($m = 9d - 1$) using a line bundle representing the triple of a polarization. It follows that there exists an open subset $T$ of the Hilbert scheme parametrizing subschemes of $\mathbb{P}^m$ with Hilbert polynomial $P(x) = 9dx^2$ such that every $t \in T$ parametrizes an abelian surface $A_t$ of type $(1, d)$ and every such abelian surface is parametrized by some $t \in T$. Since the moduli space of polarized abelian surfaces of a fixed type is connected and the same holds for $\text{PGL}(m + 1)$, we may also assume that $T$ is connected.

Moreover $T$ is a smooth manifold of dimension $(m + 1)^2 + 2$. In fact, since the moduli space of polarized abelian surfaces has dimension 3 and $\text{PGL}(m + 1)$ acts on $T$ with finite stabilizers, for every $t \in T$ the dimension of $T$ at $t$ is at least $(m + 1)^2 + 2$. On the other hand, the tangent space of $T$ at $t$ is the space of sections of the normal bundle $N_t$ to $A_t$ in $\mathbb{P}^m$ and, letting $T_{A_t}$ be the tangent bundle of $A_t$ and $T_{\mathbb{P}^m|A_t}$ be the restriction on $A_t$ of the tangent bundle of $\mathbb{P}^m$, we have the exact sequence

$$0 \to H^0(T_{A_t}) \to H^0(T_{\mathbb{P}^m|A_t}) \to H^0(N) \to H^1(T_{A_t}).$$

Since the image of the last map is contained in the 3 dimensional subspace of projective first order deformations of $A_t$, we deduce the inequality $h^0(N) \leq h^0(T_{\mathbb{P}^m|A_t}) + 1$. The dimension $h^0(T_{\mathbb{P}^m|A_t})$ may be estimated by restricting the Euler exact sequence of $\mathbb{P}^m$ to $A_t$. The associated long exact sequence in cohomology is

$$0 \to H^0(\mathcal{O}_{A_t}) \to H^0((\mathcal{O}_{\mathbb{P}^m}(1)|_{A_t})^{m+1}) \to H^0(T_{\mathbb{P}^m|A_t}) \to$$
\[ H^1(\mathcal{O}_{A_t}) \rightarrow H^1((\mathcal{O}_{p^m}(1)|_{A_t})^{m+1}). \]

Since \( A_t \subset \mathbb{P}^m \) is linearly normal and \( H^1(\mathcal{O}_{p^m}(1)|_{A_t}) = 0 \) by Kodaira vanishing, we conclude that \( h^0(T_{p^m}|_{A_t}) \leq (m+1)^2 + 1 \). It follows that, for every \( t \in T \), the tangent space of \( T \) at \( t \) has dimension \( \leq (m+1)^2 + 2 \) and, since \( T \) has dimension at least \( (m+1)^2 + 2 \) at \( t \), it is smooth of dimension \( (m+1)^2 + 2 \).

We set \( T_{2d} := T \) and let \( f_{2d} : A_{2d} \rightarrow T_{2d} \) be the restriction of the universal family of the given Hilbert scheme. Let \( \mathcal{L} \) be the line bundle induced by the tautological line bundle \( \mathcal{O}(1) \) on \( \mathbb{P}^m \). In this way, (1) and (2) are true by construction.

We prove (3) for a fixed \( 0 \in T_{2d} \): we set \( A := A_0 \) and \( \alpha := \alpha_0 \). We first analyze the group \( \text{Mon}_{f_{2d}}(A) \) of monodromy operators on \( H^1(A, \mathbb{Z}) \) associated with the family \( f_{2d} \). The class \( \alpha \in H^2(A, \mathbb{Z}) \) induces an integral bilinear alternating form \( F_\alpha : H^1(A, \mathbb{Z}) \times H^1(A, \mathbb{Z}) \rightarrow \mathbb{Z} \) defined by

\[ F_\alpha(\beta_1, \beta_2) = \int_A \alpha \wedge \beta_1 \wedge \beta_2 \]

for \( \beta_i \in H^1(A, \mathbb{Z}) \). Since \( \alpha \) is a polarization of type \( (1, d) \) the form \( \varphi_\alpha \) is symplectic and there exists a basis \( B \) of \( H^1(A, \mathbb{Z}) \) such that the matrix of \( F_\alpha \) with respect to \( B \) is

\[ M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & -d & 0 \end{pmatrix}, \]

equivalently, there exists an isomorphism \( \iota : \mathbb{Z}^4 \rightarrow H^1(A, \mathbb{Z}) \) such that \( \Lambda^2(\varphi) \) sends the integral alternating form \( F \) on \( \mathbb{Z}^4 \), whose associated matrix is \( M \), to \( F_\alpha \).

First, we claim that \( \text{Mon}_{f_{2d}}(A) \) is the subgroup \( SL_{F_\alpha}(H^1(A, \mathbb{Z})) \) of the automorphism group of \( H^1(A, \mathbb{Z}) \), preserving the form \( F_\alpha \), i.e.

\[ \text{Mon}_{f_{2d}}(A) = SL_{F_\alpha}(H^1(A, \mathbb{Z})): = \{ \psi \in SL(H^1(A, \mathbb{Z})) : F_\alpha(\psi(\beta_1), \psi(\beta_2)) = F_\alpha(\beta_1, \beta_2), \forall \beta_i \in H^1(A, \mathbb{Z}) \}. \]

The fundamental group \( \pi_1(T, 0) \) acts as \( \text{Mon}_{f_{2d}}(A) \) on \( SL_{F_\alpha}(H^1(A, \mathbb{Z})) \) by composition on the left. Let \( h : T' \rightarrow T_{2d} \) be the étale cover of \( T_{2d} \) induced by this right action of \( \pi_1(T, 0) \) on \( SL_{F_\alpha}(H^1(A, \mathbb{Z})) \). More explicitly, the manifold \( T' \) is in natural bijective correspondence with the set of pairs \( (t, \zeta) \) where \( t \in T_{2d} \) and \( \zeta : H^1(A, \mathbb{Z}) \rightarrow H^1(A_t, \mathbb{Z}) \) is an isomorphism sending \( F_\alpha \) to the alternating bilinear form \( F_{\alpha_t} \), induced by \( \alpha_t \) on \( H^1(A_t, \mathbb{Z}) \). By construction, our claim holds if and only if the complex manifold \( T' \) is connected.

Connectedness of \( T' \) will be shown by analyzing the relevant period map for the corresponding family of abelian surfaces.

Let \( \text{Gr}^F(2, 4) \) be the isotropic Grassmannian parametrizing 2-dimensional complex vector spaces in \( \mathbb{C}^4 \) that are isotropic with respect to the \( \mathbb{C} \)-linear extension \( F_{\mathbb{C}} \) of \( F \) and set

\[ D := \{ V \in \text{Gr}^F(2, 4) : V \cap \overline{V} = 0 \& iF_{\mathbb{C}}(v, \overline{v}) > 0 \forall v \in V \}. \]

Let \( P : T' \rightarrow D \) be the holomorphic period map defined by

\[ P(t, \zeta) := (\zeta \circ \iota)^{-1}(H^{1,0}(A_t)). \]
Since for every $V \in D$ the quotient $A_V := \mathbb{C}^2/\mathbb{Z}^2$ is a complex torus of dimension 2 equipped with an identification $H^1(A_V, \mathbb{Z}) \simeq \mathbb{Z}^4$ sending $H^{1,0}$ to $V$, the holomorphic map $P$ is surjective.

Moreover the fibres of $P$ are irreducible of dimension $(m+1)^2 - 1$. More precisely the action of the connected group $PGL(m+1)$ on the manifold $T_{2d}$ lifts to an action on the manifold $T'$: the lifted action is defined by setting, $g(t, \zeta) := (gt, g_* \circ \zeta)$ where $g_* : H^1(A_t, \mathbb{Z}) \to H^1(A_{gt}, \mathbb{Z})$ is the isomorphism induced by the restriction of $g \in PGL(m+1)$ to a morphism from $A_t$ to $A_{gt}$.

Since $g_*$ is a morphism of Hodge structures, points of $T'$ in the same $PGL(m+1)$ orbit have the same image in $D$. Viceversa, $P(t_1, \psi_1) = P(t_2, \psi_2)$ implies that $\psi_2 \circ \psi_1^{-1} : H^1(A_{t_1}, \mathbb{Z}) \to H^1(A_{t_2}, \mathbb{Z})$ is an isomorphism of integral polarized Hodge structures: hence it comes from an isomorphism $A_{t_1} \simeq A_{t_2}$ and, using translations, we may assume that this isomorphism sends $\mathcal{E}_{t_1}$ to $\mathcal{E}_{t_2}$, i.e it is the restriction of an element of $PGL(m+1)$. We conclude that the fibres of $P$ are irreducible because they are the orbits of the $PGL(m+1)$-action on $T'$. The dimension of every fibre is $(m + 1)^2 - 1$, since the group of automorphisms of an abelian surface fixing an ample line bundle is finite, hence $PGL(m+1)$ acts with finite stabilizers on $T_{2d}$.

Since $T'$ is smooth of pure dimension $(m + 1)^2 + 2$ and $P : T' \to D$ is surjective with connected equidimensional fibres, $T'$ is connected if $D$ is connected. Connectedness of $D$ can be proved directly by observing that

(1) $V \in D$ if and only if there exist 1-dimensional vector subspaces $U$ and $W$ of $\mathbb{C}^4$ such that $iF_C(u, \overline{w}) > 0$, $F_C(w, u) = F_C(w, \overline{w}) = 0$ and $iF_C(w, \overline{w}) > 0$ for every $0 \neq u \in U$ and $0 \neq w \in W$;

(2) For every real vector space $Z$ equipped with a symplectic form $F$, the set of one dimensional vector spaces of $Z \otimes_{\mathbb{R}} \mathbb{C}$ of the form $Cv$ with $iF_C(v, \overline{w}) > 0$ is connected by a direct computation.

We conclude that $T'$ is connected and our claim holds.

Since $H^2(A, \mathbb{Z}) = \Lambda^2(H^1(A, \mathbb{Z}))$, it remains to show that the image of the group morphism $\Lambda : SL_F(H^1(A, \mathbb{Z})) \to O(H^2(A, \mathbb{Z}))$ sending $\psi$ to $\Lambda^2(\psi)$ is exactly $SO^+_m(H^2(A, \mathbb{Z}))$. We may replace $H^1(A, \mathbb{Z})$ and $H^2(A, \mathbb{Z})$ by their abstract models $\mathbb{Z}^4$ and $\Lambda^2(\mathbb{Z}^4) \simeq U^3$ and it suffices to prove that the image of $SL_F(U^3)$ under the group morphism $\Lambda_R : SL(4, \mathbb{R}) \to O(U^3 \otimes \mathbb{R})$, sending $\psi$ to $\Lambda^2(\psi)$, is

$$SO^+_m(U^3) = \{ \varphi \in SO^+(U^3) | \varphi(a) = a \},$$

where $a \in \Lambda^2(U^3)$ induces, by wedge product with pairs of elements of $\mathbb{Z}^4$, the alternating form $F$ on $\mathbb{Z}^4$.

We first show that

$$\Lambda_R(SL(\mathbb{Z}^4)) = SO^+(U^3).$$

The real Lie group $O(U^3 \otimes \mathbb{R})$ has four connected components [11 Lemma 4.4(b)] and the one containing the identity is the subgroup $SO^+(U^3 \otimes \mathbb{R})$ of isometries preserving the orientation of the positive cone of $U^3$ (see Remark 2.2) and having determinant 1. Since the kernel of $\Lambda_R$ is generated by $-1$, the Lie group $SL(4, \mathbb{R})$ is connected and $SL(4, \mathbb{R})$ and $O(U^3 \otimes \mathbb{R})$ have both dimension 15, the morphism $\Lambda_R$ is surjective, proper and étale.

Equation (5) follows if we prove that $\Lambda_R^{-1}(SO^+(U^3)) = SL(\mathbb{Z}^4)$. Since $\Lambda_R$ is proper and $SO^+(U^3)$ is discrete , the group $\Lambda_R^{-1}(SO^+(U^3))$ is a discrete subgroup of $SL(4, \mathbb{R})$ containing $SL(4, \mathbb{Z})$. By [5] Theorem 7, the subgroup $SL(n, \mathbb{Z})$ is
discrete maximal in $SL(n, \mathbb{R})$: therefore $\lambda^{-1}_{\mathbb{R}}(SO^+(U^3)) = SL(4, \mathbb{Z})$ and equation (5) holds.

Finally (3) holds because $\lambda(\varphi)$ preserves $\alpha \in \Lambda^2(\mathbb{Z}^4)$ if and only if it preserves the associated alternating form $F$ on $\mathbb{Z}^4$. □

Remark 3.4. The Plucker embedding identifies $D$ with one of the two components of the period domain $\Delta_\alpha := \{[l] \in \mathbb{P}(U^3 \otimes \mathbb{C}) : l \cdot \alpha = 0, l \cdot \mathbb{I} = 0 \& l \cdot \mathbb{I} > 0\}$ for the Hodge structure on $H^2(A, \mathbb{Z})$.

Remark 3.5. Using an isometry $H^2(A_t, \mathbb{Z}) \cong U^3$ and letting $a \in U^3$ be the image of $\alpha$, item (3) of 3.3 can be restated as

$$\text{Mon}^2_{f,\alpha}(A_t) = SO^+_e(U^3) := \{\varphi \in SO^+_e(U^3) | \varphi(a) = a\}.$$ 

The following, purely lattice theoretic lemma shows that a few subgroups of the form $SO^+_e(U^3)$ are sufficient to generate the whole $SO^+(U^3)$.

Lemma 3.6. Let $\{e_1, f_1\}$ be a standard basis for a copy $U_1$ of $U$ in $U^3$. For every $d \in \mathbb{N} \setminus \{0, 1\}$, the union

$$(6) \quad SO^+_{e_1+df_1}(U^3) \cup SO^+_{d-1+f_1}(U^3) \cup SO^+_{e_1+(d+1)f_1}(U^3) \cup SO^+_{e_1+(d+1)f_1}(U^3)$$

in $SO^+(U^3)$ generates the whole group.

Proof. Let $g$ be any isometry of $SO^+(U^3)$. Let $L_1 \cong U^2$ be the orthogonal of $U_1$ inside $U^3$ and let $\{e_2, f_2, e_3, f_3\}$ be the standard basis for $L_1$. By Lemma 2.8, elements of $O^+(U^3)$ are a composition of elements in $O^+(L_1)$ and transvections $t(e_1, a)$ or $t(f_1, a)$ with $a \in L_1$. As $g$ is in $SO^+(U^3)$, we can write it as a composition of elements in $SO^+(L_1)$ and transvections as above. Therefore, to prove our claim it is enough to prove that all these factors can be obtained by compositions of elements contained in the four subgroups of equation (6).

Since $SO^+(L_1)$ acts trivially on $U_1$, it is contained in each of the four subgroups. We will now prove that $t(e_1, a)$ and $t(f_1, a)$ are contained in the subgroup generated by the union (6) at once. By the previous step of the proof, by Lemma 2.6 applied to the lattice $L_1$ and by equation 4, we can suppose that $a$ lies in the second copy of $U$ spanned by $e_2, f_2$. In particular, the two isometries $t(e_1, a)$ and $t(f_1, a)$ act trivially on the third copy of $U$ and can therefore be considered as an isometry of $\langle e_1, f_1, e_2, f_2 \rangle$. By Lemma 2.7, $t(e_1, a)$ and $t(f_1, a)$ can be written as a composition of $t(e_2, e_1), t(e_2, f_1), t(f_2, e_1)$ and $t(f_2, f_1)$. For every positive integer $d$, the transvections $t(e_2, e_1 - df_1)$ and $t(f_2, e_1 - df_1)$ fix $e_1 + df_1$: therefore they are elements of $SO^+_{e_1+df_1}(U^3)$. Analogously, the transvections $t(e_2, e_1 - (d+1)f_1)$ and $t(f_2, e_1 - (d+1)f_1)$ fix $e_1 + (d+1)f_1$, therefore they are elements of $SO^+_{e_1+(d+1)f_1}(U^3)$.

A direct computation using (2) and (3) shows

$$t(e_2, e_1 - df_1) \circ t(e_2, e_1 - (d+1)f_1)^{-1} = t(e_2, f_1)$$

$$t(f_2, e_1 - df_1) \circ t(f_2, e_1 - (d+1)f_1)^{-1} = t(f_2, f_1)$$

Analogously, we obtain $t(e_2, e_1)$ and $t(f_2, e_1)$ by fixing the polarizations $de_1 + f_1$ and $(d+1)e_1 + f_1$, hence our claim. □
As a consequence of Proposition 3.3 and Lemma 3.6 we obtain that the monodromy group $Mon^2(A)$ of an abelian surface $A$ is generated by monodromy operators appearing in a finite set of smooth projective families containing $A$ as a fibre.

In the following corollary, we keep the notation as in 3.3 and let $f_{2d} : A_{2d} → T_{2d}$ be the family of polarized abelian surfaces of type $(1, d)$ constructed therein. In particular, by (2) of Proposition 3.3 every polarized abelian surface $(A, α)$ of type $(1, d)$ is isomorphic, as a polarized abelian surface, to $f^1_{2d}(t)$ for some $t ∈ T_{2d}$ and this isomorphism allows to consider the group $Mon^2 f_{2d}(f^1_{2d}(t))$ of the monodromy operators associated with the family $f_{2d}$ as a subgroup of the monodromy group $Mon^2(A) ⊆ SO^+(H^2(A, Z))$ of $A$.

**Corollary 3.7.** Let $A = E × E'$ be an abelian surface with $E, E'$ very general elliptic curves, so that $NS(A) ≅ U$. Let $e_1, f_1$ be the classes of the two elliptic curves inside $U$. Let $t_1, t_2 ∈ T_{2d}$ and $s_1, s_2 ∈ T_{2d+2}$ be such that there are isomorphism of polarized abelian surfaces

$$(A, e + df) ≃ f^1_{2d}(t_1), \ (A, de + f) ≃ f^1_{2d}(t_2),$$

$$(A, e + (d + 1)f) ≃ f^1_{2d+2}(s_1), \ (A, (d + 1)e + f) ≃ f^1_{2d+2}(s_2).$$

The isometry group $SO^+(H^2(A, Z))$ is generated by the union

$$\bigcup_{i=1}^2 Mon^2 f_{2d}(f^1_{2d}(t_i)) \cup \bigcup_{j=1}^2 Mon^2 f_{2d+2}(f^1_{2d+2}(s_j)).$$

**Proof.** Using an isometry $H^2(A, Z) ≃ U^3$ that sends $e$ and $f$ to the elements $e_1$ and $f_1$ of the standard basis of the first copy $U_1$ of $U$ in $U^3$, we get an isomorphism $SO^+(H^2(A, Z)) ≃ SO^+(U^3)$. By Proposition 3.3 (see also Remark 3.5), there are isomorphisms

$$Mon^2 f_{2d}(f^1_{2d}(t_1)) ≃ SO^+_{e_1 + df_1}(U^3), \ Mon^2 f_{2d+2}(f^1_{2d+2}(s_1)) ≃ SO^+_{e_1 + (d+1)f_1}(U^3),$$

$$Mon^2 f_{2d}(f^1_{2d}(t_2)) ≃ SO^+_{de_1 + f_1}(U^3), \ Mon^2 f_{2d+2}(f^1_{2d+2}(s_2)) ≃ SO^+_{(d+1)e_1 + f_1}(U^3).$$

Hence, the statement follows from Lemma 3.6. □

**Remark 3.8.** The previous corollary implies that for every complex torus $A$ of dimension two the index-2 subgroup $SO^+(H^2(A, Z)) ⊆ O^+(H^2(A, Z))$ is contained in $Mon^2(A)$. On the other hand every monodromy operator in $φ ∈ Mon^2(A)$ has to be the second wedge power of an automorphism of $H^1(A, Z)$ preserving the orientation given on $H^1(A, Z)$ by the complex structure: this implies that $φ$ has to preserve the induced orientation on $H^2(A, Z) = A^2H^1(A, Z)$, hence its determinant has to be 1. By Corollary 3.7 we reprove that $SO^+(H^2(A, Z)) = Mon^2(A)$.

**Remark 3.9.** The most natural example of an isometry in $O^+(H^2(A, Z))$ that does not belong to $SO^+(H^2(A, Z))$ is provided by the Poincaré duality morphism.

Let $\hat{A} := H^1(A, Z) / H^1(A, Z)$ be the dual complex torus. There are identifications

$$H^*(\hat{A}, Z) = H^*(A, Z)^\vee, \ H^{p,q}(\hat{A}) = H^{q,p}(A)^\vee$$

and the Poincaré duality morphism

$$PD : H^2(A, Z) → H^2(\hat{A}, Z) = H^2(A, Z)^\vee$$
defined by
\[ PD(\alpha) := \int_A \alpha \wedge (\cdot) \]
is a isomorphism and a Hodge isometry. By a direct computation \( PD \) is incompatible with the orientations on \( H^2(A, \mathbb{Z}) \) and \( H^2(\tilde{A}, \mathbb{Z}) \) and is not the second wedge power of an isomorphism between \( H^1(A, \mathbb{Z}) \) and \( H^1(\tilde{A}, \mathbb{Z}) \) (see [20, Lemma 4.5]).

If \( h \in H^2(A, \mathbb{Z}) \) is a polarization, \( \tilde{h} := PD(h) \in H^2(\tilde{A}, \mathbb{Z}) \) is called the dual polarization and \((\tilde{A}, \tilde{h})\) is called the dual abelian surface of \((A, h)\). If \( h \) is of type \((1, 1)\) there exists an isomorphism of polarized abelian surfaces \( g : A \to \tilde{A} \) and \( g^* \circ PD : H^2(A, \mathbb{Z}) \to H^2(\tilde{A}, \mathbb{Z}) \) is an element of \( O^+(H^2(A, \mathbb{Z})) \setminus SO^+(H^2(A, \mathbb{Z})) \)

4. Singular Monodromy

In this section we study locally trivial monodromy of singular symplectic varieties arising as Albanese fibres of moduli spaces of sheaves on general Abelian surfaces, whose desingularization are manifolds of \( OG_6 \) type.

In order to properly state the results of this section, we need to fix our setting for this and the next section.

Setting 4.1. Let \( A \) be an abelian surface and let
\[ w = (w_0, w_2, w_4) \in H^0(A, \mathbb{Z}) \oplus NS(A) \oplus H^4(A, \mathbb{Z}) \]
be a Mukai vector such that \( w_0 > 0 \) or \( w_0 = 0 \), \( w_2 \) is effective and \( w_4 \neq 0 \) or \( w_0 = 0 \), \( w_2 = 0 \) and \( w_4 > 0 \). Assume that the Mukai square of \( w \) is \( w^2 = 2 \) and set \( v = (v_0, v_2, v_4) := (2w_0, 2w_2, 2w_4) = 2w \).

Fix a \( v \)-generic polarization \( H \) on \( A \) (see [36, Definition 2.1]), let \( h \in H^2(A, \mathbb{Z}) \) be its class.

Let \( M_v(A) \) be the Gieseker moduli space of \( H \)-semistable sheaves on \( A \) with Mukai vector \( v \).

Let \( K_v(A) \subset M_v(A) \) be a fibre of the (isotrivial) Albanese fibration of \( M_v(A) \) and let \( \tilde{K}_v(A) \) be the the blow up of \( K_v(A) \) along its singular locus with reduced structure.

By Theorem 1.6 of [34], the projective variety \( \tilde{K}_v(A) \) is a Hyperkähler manifold in the deformation class of \( OG_6 \).

In this section we determine the group \( \text{Mon}^2(K_v(A))_{lt}^{pr} \) of monodromy operators on \( H^2(K_v(A), \mathbb{Z}) \) that are compositions of parallel transport operators along projective families which are analytically locally trivial deformations at every point of the domain. The cohomology \( H^2(K_v(A), \mathbb{Z}) \) has a lattice structure given by the Beauville-Bogomolov-Namikawa pairing, i.e. the restriction of the Beauville-Bogomolov pairing on the resolution of \( K_v(A) \). This lattice structure is invariant under deformations which are analytically locally trivial deformations at every point of the domain (see [2] Lemma 5.5]), therefore \( \text{Mon}^2(K_v(A))_{lt}^{pr} \) is contained in \( O^+(H^2(K_v(A), \mathbb{Z})) \). In this section, we prove that
\[ \text{Mon}^2(K_v(A))_{lt}^{pr} = O^+(H^2(K_v(A), \mathbb{Z})). \]

We remark that we will only use deformations of moduli spaces of sheaves and their Albanese fibres coming from deformations of the underlying abelian surfaces and
suitable isomorphisms of moduli spaces obtained by Yoshioka using Fourier-Mukai transforms.

**Remark 4.2.** We are mainly interested in the a priori bigger monodromy group \( \text{Mon}^2(K_v(A))_{lt} \) of monodromy operators on \( H^2(K_v(A),\mathbb{Z}) \) that are compositions of parallel transport operators along proper families over smooth bases which are analytically locally trivial deformations at every point of the domain and such that the blow up of every fibre along its singular locus is a hyperkähler manifold of \( OG6 \) type. The last condition is always true if we impose that our families are also projective.

Since \( \text{Mon}^2(K_v(A))_{lt} \subseteq \text{Mon}^2(\tilde{K}_v(A)) \), by Lemma 2.12 every monodromy operator of \( \text{Mon}^2(K_v(A))_{lt} \) preserves the orientation of the positive cone of \( H^2(K_v(A),\mathbb{Z}) \) (see Remark 2.2). We conclude that

\[
\text{Mon}^2(K_v(A))_{lt}^{pr} = \text{Mon}^2(\tilde{K}_v(A))_{lt} = O^+(H^2(K_v(A),\mathbb{Z})).
\]

### 4.1. Monodromy from the underlying abelian surface.

In this subsection we use Proposition 3.3 and Corollary 3.7 to describe the monodromy operators in \( \text{Mon}^2(K_v(A))_{lt}^{pr} \) induced by monodromy operators of \( \text{Mon}^2(A) \). In order to relate \( \text{Mon}^2(K_v(A))_{lt}^{pr} \) and \( \text{Mon}^2(A) \), we need to recall the relation provided by the Mukai-Donaldson-Le Potier morphism (see [34, §3.2]) between the cohomology of \( A \) and \( H^2(K_v(A),\mathbb{Z}) \). For every

\[
v = (v_0, v_2, v_4) = (2w_0, 2w_2, 2w_4) = 2w
\]

as in Setting 3.3 let \( v^\perp = w^\perp \) be the perpendicular lattice to \( v \) in the Mukai lattice of \( A \) and let \( H^2(K_v(A),\mathbb{Z}) \) be endowed with the lattice structure given by the Beauville-Bogomolov-Namikawa form.

By [34, Theorem 1.7], the Mukai-Donaldson-Le Potier morphism

\[
\nu_v : w^\perp \to H^2(K_v(A),\mathbb{Z})
\]

is an isomorphism of abelian groups and an isometry of lattices respecting the natural weight two Hodge structures on \( v^\perp \) and \( H^2(K_v(A),\mathbb{Z}) \).

The Mukai-Donaldson-Le Potier morphism induces an identification

\[
O^+(w^\perp) = O^+(H^2(K_v(A),\mathbb{Z})�).
\]

The subgroup \( \text{SO}^+_{w_2}(H^2(A,\mathbb{Z})) \subseteq \text{SO}^+(H^2(A,\mathbb{Z})) \) fixing \( w_2 \) is naturally a subgroup of \( O^+(w^\perp) \): the injection is given by extending every \( \gamma \in \text{SO}^+_{w_2}(H^2(A,\mathbb{Z})) \) to the isometry of the Mukai lattice of \( A \) acting as the identity on \( H^0(A,\mathbb{Z}) \oplus H^4(A,\mathbb{Z}) \) and then restricting to \( w^\perp \). In particular, in the important special case where \( w_2 = 0 \), i.e. \( v = (2,0,-2) \), the group \( \text{SO}^+(H^2(A,\mathbb{Z})) \) is naturally a subgroup of \( O^+(w^\perp) \).

In the following proposition we compare \( \text{SO}^+_{w_2}(H^2(A,\mathbb{Z})) \) and \( \text{Mon}^2(K_v(A))_{lt}^{pr} \) as subgroups of \( O^+(w^\perp) = O^+(H^2(K_v(A),\mathbb{Z})) \) in relevant cases.

**Proposition 4.3.** Using the identification (8) we have:

1. If \( w_2 \in \text{NS}(A) \) is proportional to the class \( h \) of the \( v \)-generic polarization \( H \), there is an inclusion

\[
\text{SO}^+_{w_2}(H^2(A,\mathbb{Z})) \subseteq \text{Mon}^2(K_v(A))_{lt}^{pr}.
\]

2. If \( v = (2,0,-2) \), there is an inclusion

\[
\text{SO}^+(H^2(A,\mathbb{Z})) \subseteq \text{Mon}^2(K_v(A))_{lt}^{pr}.
\]

**Proof.** (1) Let \((1, d)\) be the type of the polarized abelian surface \((A, H)\) and let \( f_{2d} : \mathcal{A}_{2d} \to T_{2d} \) be the family of polarized abelian surfaces constructed in Proposition 3.3. By item (2) of Proposition 3.3 there exists \( 0 \in T_{2d} \) such that \((A, H)\) and
$A_0 := f_{2d}^{-1}(0)$ are isomorphic as polarized abelian surfaces. In order to study $Mon^2(K_v(A))^p_{it}$ we need to introduce the relevant local systems.

Let $\mathcal{H}^{cv} := \oplus_{i=0}^2 R^2 f_{2d} \mathcal{K}(Z)$ be the local system on $T_{2d}$ whose stalk $\mathcal{H}^{cv}_t$ at $t$ is the Mukai lattice of $A_t := f_{2d}^{-1}(t)$. Since $\nu_2$ is proportional to the class $h$ which is constant, it comes from a global section of $\mathcal{H}^{cv}$ and evaluating this section at $t$ we get a Mukai vector $v_t = 2w_t$ on $A_t$ sharing the same numerical properties of $v = 2w$. Let $\mathcal{W}^\perp \subset \mathcal{H}^{cv}$ be the sub local system on $T_{2d}$ whose stalk at every $t \in T_{2d}$ is $w_t = v_t^\perp$. The monodromy action centered at 0 of the local system $\mathcal{H}^{cv}$ is trivial on $H^0(A, Z) \oplus H^4(A, Z)$ and, by Proposition $3.3$, is given by $SO^+_2(H^2(A, Z))$ on $H^2(A, Z)$. Since $\mathcal{W}^\perp \subset \mathcal{H}^{cv}$ we conclude that $SO^+_2(H^2(A, Z))$ is the monodromy centered at 0 of the local system $\mathcal{W}^\perp$.

It remains to prove that group of monodromy operators of the local system $\mathcal{W}^\perp$ comes from a projective proper analytically locally trivial (on the domain) family having $K_v(A)$ as a fibre.

Let $p : \mathcal{M}_v \rightarrow T_{2d}$ be the relative moduli space of semistable sheaves on the fibres of $f_{2d}$ with Mukai vector specializing to $v$ on $A_0$ (see [10, Theorem 4.3.7]). By [33, Corollary 4.2 and Lemma 4.4], the polarization of $A_t$ is $v_t$-generic in the complement $U$ of a locally finite union of complex analytic subvarieties of $T_{2d}$. By [33, Proposition 2.16] it follows that, over $U$, the relative moduli space $\mathcal{M}_v$ is a proper analytically locally trivial family at every point of the domain.

Recall from [33, §3.2] that there exists a Mukai-Donaldson-Le Potier morphism $\lambda_v : w^\perp \rightarrow M_v(A)$ such that $\nu_v = f_v^* \circ \lambda_v$ where $j : K_v(A) \rightarrow M_v(A)$ is the inclusion of a fibre of the Albanese morphism and the construction of this Mukai-Donaldson-Le Potier morphism can be done in families using relative semiversal families. It follows that $\lambda_v$ induces a morphism of local systems over $U$

$$\Lambda : \mathcal{W}^\perp \rightarrow R^2 p_* \mathbb{Z}. $$

If we could construct a family $q : \mathcal{K} \rightarrow U$ with an inclusion $j : \mathcal{K} \rightarrow \mathcal{M}_v$ such that $j(q^{-1}(t))$ is an Albanese fibre of $p^{-1}(t)$ for every $t \in U$, the relative Mukai-Donaldson-Le Potier morphism

$$\mathcal{V} = j^* \circ \Lambda : \mathcal{W}^\perp \rightarrow R^2 q_* \mathbb{Z} $$

would be an isomorphism and the monodromy operator of $\mathcal{W}^\perp$ would come as a monodromy operator associated with the family $q : \mathcal{K} \rightarrow U$.

We can bypass the non existence (in general) of the family $\mathcal{K}$ by using the relative Albanese morphism (see [33, §3]). By [33, Theorem 3.3.iii], on some non empty Zariski open subset $U' \subset T_{2d}$, there exist a relative Albanese variety $s : \mathcal{Y} \rightarrow U'$, a relative Albanese morphism $alb : p^{-1}(U') \rightarrow \mathcal{Y}$ and a factorization $p_{U'} = s \circ alb$, where $p_{U'} : p^{-1}(U') \rightarrow U'$ is induced by $p$ by restriction. Since the fundamental group of $\mathcal{Y}$ surjects on the fundamental group of $U'$, there exists a smooth curve $C \subset \mathcal{Y}$, containing a point $o \in s^{-1}(0)$, such that $s_o$ induces a surjection $\pi_1(\mathcal{Y}, o) \rightarrow \pi_1(U', 0)$ of fundamental groups.

Moreover, $s$ is smooth and its fibres are four dimensional abelian varieties. Set $\mathcal{K}_C := alb^{-1}(C)$ and let $q_C : \mathcal{K}_C \rightarrow C$ be the restriction of $alb$. The morphism $q_c$ is a deformation of $\mathcal{K}_{C,o} := q_C^1(o) \simeq K_v(A)$ analytically locally trivial at any point of $\mathcal{K}_C$ and there exists an embedding $j_C : \mathcal{K}_C \rightarrow \mathcal{M}_v \times T_{2d} C$.

Letting $s_C : C \rightarrow U'$ be the restriction of $s$, we have a morphism of local systems

$$\mathcal{V}_C := j_C^* \circ s_C^* (\Lambda) : s_C^* \mathcal{W}^\perp \rightarrow R^2 q_{C,*} \mathbb{Z}. $$
that is an isomorphism since its restriction at $o$ is $v$: hence the monodromy of the family $K_C$ at $o$ equals the monodromy of the local system $s_C^*\mathcal{W}^\perp$ at $o$. Since $s_C$ induces a surjection on fundamental groups, the group of the monodromy operators of the local system $s_C^*\mathcal{W}^\perp$ at $o$ equals the group of the monodromy operators of $\mathcal{W}^\perp$ at 0 which is $SO^+_b(H^2(A,\mathbb{Z}))$. We conclude that, using the identification $[5]$, the inclusion $SO^+_b(H^2(A,\mathbb{Z})) \subseteq Mon^2(K_v(A))_{lt}^{pr}$ holds.

Let us now prove item (2). In this proof the chosen $v$-generic polarization will play an important role, so we include it in the notation. We denote by $M_v(A, H)$ the moduli space of $H$-semistable sheaves on $A$ with Mukai vector $v$, by $K_v(A, H)$ the Albanese fibre and by $\nu_H : v^+ \to H^2(K_v(A, H), \mathbb{Z})$ the associated Mukai-Donaldson-Le Potier morphism. We first prove the statement for a set of polarizations in the case where $A := E_1 \times E_2$ is the product of a very general pair of elliptic curves and $NS(A) = \mathbb{Z}c \oplus \mathbb{Z}f$ is generated by the classes $e$ and $f$ of $E_1$ and $E_2$.

A polarization on $A$ is a class of the form $ae + bf$ for $a, b > 0$. A straightforward computation[6] shows that for a non-stable polystable sheaf $F$ of $M_v(A, aE_1 + bE_2)$ there are only two possibilities:

a) $F \simeq I_{x_i} \otimes L_1 \oplus I_{x_2} \otimes L_2$, where $I_{x_i}$ is the ideal of the point $x_i \in A$ and $L_i \in Pic^0(A)$;

b) $F \simeq M_1 \oplus M_2$, where the $M_i$ are line bundles, the class of $M_1$ is $e - f$ and the class of $M_2$ is $f - e$.

Case a) always happens while case b) only happens for $a = b$. Moreover, if $F$ is $aE_1 + bE_2$-semistable and $a'E_1 + b'E_2$-unstable, there exists a polarization $a_0E_1 + b_0E_2$ in the segment joining the given polarizations such that $F$ is $a_0E_1 + b_0E_2$-semistable. This implies $a_0 = b_0$ and the point of $M_v(A, E_1 + E_2)$ corresponding to $F$ is represented by a polystable sheaf as in b).

It follows that $M_v(A, aE_1 + bE_2) = M_v(A, a'E_1 + b'E_2)$ and $K_v(A, aE_1 + bE_2) = K_v(A, a'E_1 + b'E_2)$ if $a - b$ and $a' - b'$ have the same sign. Let $a < b$ and $a' > b'$ and set $H := aE_1 + bE_2$ and $H' := a'E_1 + b'E_2$. The same analysis shows that a $H$-semistable sheaf $F$ with Mukai vector $v$ being $H'$-unstable is actually $H$-stable and fits in a non trivial extension of the form

$$0 \to M_2 \to F \to M_1 \to 0,$$

with $M_1$ and $M_2$ as in b). The locus of $M_v(H)$ parametrizing unstable sheaves is a $\mathbb{P}^1$-bundle over $A \times A^\vee$, contained in the smooth locus of $M_v(A, H)$. By passing to the Albanese fibres, we have that $K_v(A, H')$ is the Mukai flop of $K_v(A, H)$ along the restriction of this $\mathbb{P}^1$-bundle over $A \times A^\vee$ (which could have several components, all of them isomorphic to $\mathbb{P}^1$). Therefore, we obtain that $K_v(A, H)$ and $K_v(A, H')$ are birational and the birational map $\varphi : K_v(A, H) \dashrightarrow K_v(A, H')$ becomes an isomorphism after removing codimension-3 subvarieties contained in the smooth loci of domain and codomain: hence it induces an isomorphism

$$\varphi_* : H^2(K_v(A, H), \mathbb{Z}) \to H^2(K_v(A, H'), \mathbb{Z}).$$

Since $K_v(A, H)$ and $K_v(A, H')$ are 2-factorial ([35 Theorem 1.2]), By Theorem 1.1 of [14] (see also [2 Theorem 6.17]), there exist one parameter projective locally (on the domain) trivial deformations $K \to D$ and $K' \to D$ over a smooth curve $D$, having $K_v(A, H)$ and $K_v(A, H')$ as fibres over $0 \in D$ such that there exists a closed

---

1 The first Chern $e$ class of a direct summand of a polystable sheaf with Mukai vector $(2, 0, -2)$ should be perpendicular to $ae + bf$ and should satisfy $e^2 \geq -1$. 
subvariety $\Gamma \subset K \times K'$ restricting to the closure of the graph of $\varphi$ over 0 and to the graph of an isomorphism over every $t \in D \setminus \{0\}$. As a consequence, since $\varphi$ is regular on the singular locus, the morphism $\varphi_*$ is a parallel transport operator. We are going to use $\varphi_*$ to induce monodromy operators on $K_v(A,H)$ from monodromy operators on $K_v(A,H')$.

The group of induced monodromy operators is easily computed since

$$\nu_{v,H'} = \varphi_* \circ \nu_{v,H}.$$  

Let $i : U \to K_v(A,H)$ be the open subset where $\varphi$ is regular and let $i' : U' \to K_v(A,H)$ be the open embedding of its image. Since $K_v(A,H) \setminus U$ and $K_v(A,H') \setminus U'$ are codimension-3 subvarieties contained in the smooth loci of $K_v(A,H)$ and $K_v(A,H')$, the morphisms $i$ and $i'$ induce isomorphisms on $H^2$ and (9) holds if

$$i^* \circ \nu_{v,H'} = i'^* \circ \varphi_* \circ \nu_{v,H}.$$  

The last equality holds since $U$ and $U'$ represents the same set of sheaves on which $\varphi$ gives the identity and, by definition, the composition of Mukai-Donaldson-Le Potier morphism with restriction to a subvariety, only depends on semiuniversal sheaf on that subvariety.

Fix now $a = 1$ and $b = d > 1$. By item (1) we have

$$SO^+_{\ell}((H^2(A,Z)) \subseteq Mon^2(K_v(A,E_1 + dE_2))_{II}.$$  

Since $K_v(A,E_1 + (d + 1)E_2) = K_v(A,E_1 + dE_2)$ and $\nu_{E_1 + (d + 1)E_2} = \nu_{E_1 + dE_2}$ we also have

$$SO^+_{\ell + (d + 1)f}((H^2(A,Z)) \subseteq Mon^2(K_v(A,E_1 + dE_2))_{II}.$$  

Item (1) applied to the polarization $E_1 + dE_2$ gives the injective morphism of groups

$$SO^+_{\ell + f}(H^2(A,Z)) \subseteq Mon^2(K_v(A,dE_1 + E_2))_{II},$$  

and by the factorization 9 we get

$$SO^+_{\ell + f}(H^2(A,Z)) \subseteq Mon^2(K_v(A,E_1 + dE_2))_{II}.$$  

Similarly,

$$SO^+_{\ell + f}(H^2(A,Z)) \subseteq Mon^2(K_v(A,E_1 + dE_2))_{II}.$$  

By Lemma 3.4, it follows that the subgroup $Mon^2(K_v(A,E_1 + dE_2))_{II}$ of the group

$$O^+(H^2(K_v(A,h),Z)) \to H^2(K_v(E_1 \times E_2,e + df),Z)$$  

extending in the natural way a parallel transport operator $t : H^2(A,Z) \to H^2(E_1 \times E_2,Z)$. It follows that

$$Mon^2(K_v(A,H))_{II} \geq t^{-1} \circ SO^+(H^2(E_1 \times E_2,Z)) \circ t = SO^+(H^2(A,Z)).$$  

If the $\nu$-generic polarization $h$ is principal, i.e. of type $(1,1)$, and the rank of $NS(A)$ is at least 2, by openness of chambers, there exists an ample divisor $H'$ whose class $h'$ is a primitive, non principal and $\nu$-generic polarization in the same chamber of $h$: hence $K_v(A,H) = K_v(A,H')$ and

$$Mon^2(K_v(A,H))_{II} = Mon^2(K_v(A,H))_{II} \geq SO^+(H^2(A,Z)).$$  

Finally, if $h$ is principal and $NS(A) \simeq Z$, by density of the Noether Lefschetz locus, there exists a projective small deformation $A'$ of $A$ such that $NS(A')$ has rank at
least 2. As in the previous case, using a parallel transport operator induced by a parallel transport operator between $A$ and $A'$ we get the result.

Remark 4.4. In the proof of (2) of Proposition 4.3 we analyzed the dependence of $K_{(2,0,-2)}(A,H)$ on the polarization $H$ in the case where $A$ is the product of two elliptic curves $E_1$ and $E_2$ and $NS(A)$ is generated by the classes $e$ and $f$ of $E_1$ and $E_2$.

We saw that if the class of $H$ is $ae + bf$ for $a \neq b$, the variety only depends on the sign of $a - b$: if $H'$ is a polarization whose class is $be + af$, we saw that the non $H'$-semistable locus of $K_{(2,0,-2)}(A,H)$ consists of a finite union of copies of $\mathbb{P}^3$, each parametrizing extensions of two fixed line bundles with classes $e - f$ and $f - e$. In particular the non $H'$-semistable locus of $K_{(2,0,-2)}(A,H)$ parametrizes locally free sheaves and does not intersect the singular locus of $K_{(2,0,-2)}(A,H)$.

Here we want to discuss the case of the non $(2,0,-2)$-generic polarization $H_0$ whose class is $e + f$. The argument in the proof of (2) of Proposition 4.3 shows that every $H$-semistable coherent sheaf with Mukai vector $(2,0,-2)$ is also $H_0$-semistable and a stable sheaf in $K_{(2,0,-2)}(A,H)$ becomes strictly $H_0$-semistable if and only if it belongs to the non $H'$-semistable locus of $K_{(2,0,-2)}(A,H)$.

It follows that there is a contraction $c : K_{(2,0,-2)}(A,H) \to K_{(2,0,-2)}(A,H_0)$ where $K_{(2,0,-2)}(A,H_0)$ is the Albanese fibre of the moduli space of $H_0$-semistable sheaves with Mukai vector $(2,0,-2)$.

Moreover, the morphism $c$ contracts a finite union of copies of $\mathbb{P}^3$ disjoint from the singular locus of $K_{(2,0,-2)}(A,H)$, hence the singular locus of $K_{(2,0,-2)}(A,H_0)$ is the disjoint union of the isomorphic image of the singular locus of $K_{(2,0,-2)}(A,H)$ and a finite set of isolated points.

4.2. Yoshioka’s isomorphisms of moduli spaces. In this subsection we recall, and adapt to our context where necessary, two results on isomorphisms between moduli spaces exhibited by Yoshioka in §3 of [12].

The induced morphisms in cohomology are conveniently described by using, for every $v = 2w$ as in our assumptions, the Mukai-Donaldson-Le Potier morphism

$$\nu_v : v^\perp \to H^2(K_v(A),\mathbb{Z})$$

introduced in the previous subsection.

The isomorphism $\psi$ of the following proposition will allow to construct monodromy operators in $Mon^2(K_v(A,H))_{pr}^\perp \setminus O^+(H^2(A,\mathbb{Z}))$, i.e. monodromy operators that cannot be obtained by deforming the underlying abelian surface.

Proposition 4.5. [12] Thm. 3.15] Let $A = E_1 \times E_2$ be an abelian surface that is the product of a very general pair $(E_1, E_2)$ of elliptic curves and let $e$ and $f \in H^2(A,\mathbb{Z})$ be the classes of the factors. Choosing a polarization of the form $e + kf$ for $k \gg 0$, the following hold:

i) There exists an isomorphism $\psi : K_{(2,0,-2)}(A) \to K_{(0,2e+2f,2)}(A)$.

ii) There exists a commutative diagram

$$\begin{array}{ccc}
(1,0,-1)^\perp & \xrightarrow{\varphi} & (0,e+f,1)^\perp \\
\nu_{(2,0,-2)} & \downarrow & \nu_{(0,2e+2f,2)} \\
H^2(K_{(2,0,-2)}(A)) & \xrightarrow{\psi_*} & H^2(K_{(0,2e+2f,2)}(A))
\end{array}$$

\[ (10) \]
where \( \varphi : (1,0,-1) \rightarrow (0,e+f,1) \) is the isometry given by

\[
\varphi((r,ae+bf+a,s)) = (-a, re - (s + a)f + a, r + b)
\]

for \( \alpha \in H^2(A, \mathbb{Z}) \) perpendicular to both \( e \) and \( f \).

**Proof.** Let \( \mathcal{P} \) be the normalized Poincaré line bundle on \( E_2 \times E_2 \) and let \( N \) be a line bundle of degree one on \( E_1 \). Set \( \mathcal{Q} := p_{2,3}^*\mathcal{P} \otimes p_1^*N \) where the morphisms \( p_{2,3} : E_1 \times E_2 \times E_2 \rightarrow E_2 \times E_2 \) and \( p_1 : E_1 \times E_2 \times E_2 \rightarrow E_1 \) are the projections. Since \( E_1 \times E_2 \times E_2 \) is naturally identified with the fibre product \( A \times E_1 \subset A \times A \), the sheaf \( \mathcal{Q} \) can be regarded as a coherent sheaf on \( A \times A \).

By [42, Thm. 3.15] the sheaf \( \mathcal{Q} \) is the kernel of a Fourier-Mukai transform satisfying the weak index theorem with index one on every semistable sheaf in \( M_{(2,0,-2)}(A) \) and inducing the isomorphism \( \psi : K_{(2,0,-2)}(A) \rightarrow K_{(0,2e+2f,2)}(A) \).

In order to prove commutativity of diagram (10), we need first to compute the map \( \psi^H : H^{ev}(A, \mathbb{Z}) \rightarrow H^{ev}(A, \mathbb{Z}) \) induced on the even cohomology of \( A \) by the kernel \( \mathcal{Q} \), i.e. the map given by

\[
\psi^H((r,ae+bf+a,s)) := p_{1,3*}(ch(\mathcal{Q})p_{1,2}^*((r,ae+bf+a,s)));
\]

where \( \eta \in H^1(A, \mathbb{Z}) \) is the Poincaré dual of a point and \( p_{1,j} : E_1 \times E_2 \times E_2 \rightarrow E_1 \times E_2 \) is the projection on the product of the first and the \( j \)-th factors.

By definition of \( \mathcal{Q} \), we have

\[
\psi^H((r,ae+bf+a,s)) = p_{1,3*}(ch(p_{2,3}^*(\mathcal{P}) \otimes p_1^*N)p_{1,2}^*((r,ae+bf+a,s)\eta)) =
\]

\[
p_{1,3*}(p_{2,3}^*(ch(\mathcal{P}))p_{1,2}^*((r,ae+bf+a,s)\eta)(1,f,0))) =
\]

\[
p_{1,3*}(p_{2,3}^*(ch(\mathcal{P}))p_{1,2}^*((r,ae+bf+\alpha + (b + r)f + \alpha, s + a)\eta))
\]

and, using the Künneth decomposition of the classes \( 1, e, f, \alpha, \eta \) for \( A = E_1 \times E_2 \) and letting \( \mathcal{P} \) act on the factors coming from the cohomology of \( E_2 \), we get

\[
p_{1,3*}(p_{2,3}^*(ch(\mathcal{P}))p_{1,2}^*((r,ae+bf+\alpha + (b + r)f + \alpha, s + a)\eta)) =
\]

\[
(a, -re + (s + a)f - r, -b + r)\eta.
\]

It follows that

\[
\psi^H((r,ae+bf+a,s)) = (a, -re + (s + a)f + \alpha, -(b + r)\eta).
\]

To deduce the commutativity of diagram (10), from formula (12) we notice that \( \varphi \) equals the opposite of the restriction \( \psi^H_{(1,0,-1)} : (1,0,-1) \rightarrow (0,e+f,1) \) of \( \psi^H \). We recall that, by [34, Lemma 3.7], the inclusions of the stable loci \( i_{(2,0,-2)} : K^s_{(2,0,-2)}(A) \rightarrow K_{(2,0,-2)}(A) \) and \( i_{(0,2e+2f,2)} : K^s_{(0,2e+2f,2)}(A) \subset K_{(0,2e+2f,2)}(A) \) induce injective maps on 2-cohomology groups: hence it suffices to show the commutativity of the following diagram

\[
\begin{array}{ccc}
(1,0,-1) & \xrightarrow{-\psi^H_{(1,0,-1)}} & (0,e+f,1) \\
\xrightarrow{i_{(2,0,-2)}^*} & & \xrightarrow{i_{(0,2e+2f,2)}^*} \\
H^2(K^s_{(2,0,-2)}(A)) & \xrightarrow{\psi^s} & H^2(K^s_{(0,2e+2f,2)}(A))
\end{array}
\]

where \( \psi^s : K^s_{(2,0,-2)}(A) \rightarrow K^s_{(0,2e+2f,2)}(A) \) is the restriction of \( \psi \) to the stable locus. \footnote{Recall that \( \mathcal{P} \) sends the class of a point to the fundamental class of \( E_2 \) and acts as \(-1 \) on \( H^1 \).}
The commutativity of diagram \[13\] follows by copying the proof of \[12\] Proposition 2.4. This Proposition is stated only for primitive Mukai vectors, but its proof works in our case since, on the stable loci, the vertical arrows can be computed by the same formula defining $\nu_\psi$ for primitive $\nu'$ (see §3.2 of \[34\]). Finally, as in \[12\] Proposition 2.4, we have the minus in diagram \[13\] because the sheaves parametrized by $M'_{(2,0,-2)}(A)$ satisfy the weak index theorem with odd index. □

Using the identification \[8\], we see $SO^+_{e+f}(H^2(E_1 \times E_2, \mathbb{Z}))$ as a subgroup of $O^+((0, e+f, 1)^\perp)$ and keeping notation as in the previous proposition set

$$
\varphi^{-1} \circ SO^+_{e+f}(H^2(E_1 \times E_2, \mathbb{Z})) \circ \varphi := \left\{ \varphi^{-1} \circ g \circ \varphi \mid g \in SO^+_{e+f}(H^2(E_1 \times E_2, \mathbb{Z})) \subset O^+((1,0,-1)^\perp) \right\}.
$$

As a consequence of Proposition 1.5 we get the following corollary

**Corollary 4.6.** Under the assumption of Proposition 4.5, the union of subgroups

$$
SO^+(H^2(E_1 \times E_2, \mathbb{Z})) \cup \varphi^{-1} \circ SO^+_{e+f}(H^2(E_1 \times E_2, \mathbb{Z})) \circ \varphi
$$

is contained in

$$
Mon^2(K_{2,0,-2}(E_1 \times E_2))_{lt}^{pr} \subseteq O^+((1,0,-1)^\perp).
$$

**Proof.** The inclusion $SO^+(H^2(E_1 \times E_2, \mathbb{Z})) \subseteq Mon^2(K_{2,0,-2}(E_1 \times E_2))_{lt}^{pr}$ follows from (2) of Proposition 4.3. Commutativity of diagram \[10\] implies that

$$
\varphi^{-1} \circ g \circ \varphi \in Mon^2(K_{2,0,-2}(E_1 \times E_2))_{lt}^{pr} \subseteq O^+((1,0,-1)^\perp)
$$

for every $g \in Mon^2(K_{(0,2e+2f,2)}(E_1 \times E_2))_{lt}^{pr} \subseteq O^+((0,e+f,1)^\perp)$. It remains to show that

$$
SO^+_{e+f}(H^2(E_1 \times E_2, \mathbb{Z})) \subseteq Mon^2(K_{(0,2e+2f,2)}(E_1 \times E_2))_{lt}^{pr}.
$$

This does not follows directly from Proposition 4.5(1) since, under the assumptions of Proposition 4.5, the intermediate component $2e + 2f$ of the Mukai vector $(0, 2e+2f, 2)$ is not a multiple of the $(0, 2e+2f, 2)$-generic polarization $e+kf$ and, moreover, $e+f$ is not a $(0, 2e+2f, 2)$-generic polarization.

To deal with this problem we include the polarization in the notation and argue as follows. By density of the Noether-Lefschetz locus there exists a small deformation $A'$ of $E_1 \times E_2$ where both $e + f$ and $e+kf$ remain algebraic (i.e. $e$ and $f$ remain algebraic) along the deformation and NS($A'$) has rank at least 3. Let $e', f' \in H^2(E_1 \times E_2, \mathbb{Z})$ and $v' = (0, 2e' + 2f', 2) \in H^{pr}(E_1 \times E_2, \mathbb{Z})$ be the parallel transport images of $e, f$ and $v = (0, 2e + 2f, 2)$ and let $H'$ be an ample divisor whose cohomology class is $e' + kf'$. By \[34\] Corollary 4.2 and Lemma 4.4 $H'$ is $\nu'$-generic. Since the construction of the Mukai-Donaldson-Le Potier morphism works in families where the Mukai vector $\nu$ stays algebraic and the polarization stays algebraic and $\nu$ generic \[34\] §3.2, it suffices to prove that

$$
SO^+_{e+f}(H^2(A', \mathbb{Z})) \subseteq Mon^2(K_{(0,2e'+2f',2)}(A', H'))_{lt}^{pr}.
$$

By boundedness of the Hilbert scheme the set $S$ of the classes in $H^2(A', \mathbb{Z})$ that can be represented by subcurves of curves representing $2e' + 2f'$ is finite. Hence, in every open subset of the ample cone of $A'$, there exists a polarization $b'$ such that the saturation $L$ of the lattice generated by $e' + f'$ and $b'$ intersects $S$ only in $e' + f'$ and $2e' + 2f'$. If we take $b'$ in the same open chamber of $e' + f'$ and denote by $H'$ an
ample divisor representing $h'$, we have $K_{(0, 2e' + 2f', 2)}(A', H') = K_{(0, 2e' + 2f', 2)}(A', H')$ and we need to prove

$$SO_{e' + f'}^+(H^2(A', Z)) \subseteq \text{Mon}^2(K_{(0, 2e' + 2f', 2)}(A', H'))_{it}^{pr}.$$  

Let $A''$ be a small deformation of $A'$ such that $L$ remains algebraic along the deformation and $NS(A'')$ has rank 2 (i.e. $L \cong NS(A'')$). Let $\gamma, h'' \in H^2(A'', Z)$ and $v'' = (0, 2e'' + 2f'', 2) \in H^{e''}(A'', Z)$ be the parallel transport images of $e' + f', h'$ and $v' = (0, 2e' + 2f', 2)$ and let $H''$ be an ample divisor whose cohomology class is $h''$. 

By properness of the relative Hilbert scheme and by construction, the only cohomology classes of curves that are contained in curves whose class is $2\gamma$, are $\gamma$ and $2\gamma$. By definition of $\nu$-genericity for Mukai vectors of dimension 1 sheaves (see [36, Definition 2.1]), every polarization on $A''$ is $\nu''$-generic: hence $\gamma$ and $h''$ are in the same chamber and $K_{(0, 2\gamma, 2)}(A'', H'') = K_{(0, 2\gamma, 2)}(A'', \Gamma)$, where $\Gamma$ is an ample divisor representing $\gamma$.

As a consequence it remains to show

$$SO_{\gamma}^+(H^2(A'', Z)) \subseteq \text{Mon}^2(K_{(0, 2\gamma, 2)}(A', \Gamma))_{it}^{pr}.$$  

and here we can apply (2) of Proposition 4.3.

\[ \square \]

Using the dual abelian surface and the Poincaré duality morphism, introduced in Remark 3.3, the following proposition shows the existence of an isomorphism $\rho$ that will allow to construct elements in $\text{Mon}^2(K_c(A, H))_{it}^{pr}$ with determinant $-1$.

**Proposition 4.7.** ([22] Prop. 3.2) Let $(A, h)$ be a polarized abelian surface with polarization of type $(1, 2)$, i.e. $h^2 = 4$, and such that $NS(A) = Zh$. Let $(\hat{A}, \hat{h})$ be its dual abelian surface. The following hold:

i) There exists an isomorphism $\rho : K_{(2, 2h, 2)}(A) \to K_{(2, 2\hat{h}, 2)}(\hat{A})$.

ii) There exists a commutative diagram

\[
\begin{array}{ccc}
(1, h, 1)^- & \xrightarrow{\varphi} & (1, \hat{h}, 1)^- \\
\nu_{(2, 2h, 2)} & \downarrow & \nu_{(2, 2\hat{h}, 2)} \\
H^2(K_{(2, 2h, 2)}(A)) & \xrightarrow{\rho_*} & H^2(K_{(2, 2\hat{h}, 2)}(\hat{A}))
\end{array}
\]

where $\varphi : (1, h, 1)^- \to (1, \hat{h}, 1)^-$ is the isometry given by

\[
\varphi((r, \alpha, s)) = -(s, PD(\alpha), r).
\]

**Proof.** Let $P$ be the Poincaré line bundle on $A \times \hat{A}$ and let $p : A \times \hat{A} \to A$ and $q : A \times \hat{A} \to \hat{A}$ be the projections. Let $G_P$ be the contravariant equivalence of the derived categories $D(A)$ and $D(\hat{A})$ of $A$ and $\hat{A}$ defined by

$$G_P(\cdot) := R\text{Hom}_q(p^*(\cdot) \otimes P, \mathcal{O}) = R(q_* \circ \text{Hom})(p^*(\cdot) \otimes P, \mathcal{O}).$$

We are going to show that for every sheaf $E \in M_{(2, 2h, 2)}(A)$ the complex $G_P(E)$ has non zero cohomology only in degree 2, i.e. the weak index theorem with index 2 (WIT(2)) holds for $E$, and $H^2(G_P(E))$ is a semistable sheaf in $M_{(2, 2\hat{h}, 2)}(\hat{A})$.

Since $G_P$ is an equivalence, this implies that sending a sheaf $E$ to $H^2(G_P(E))$ gives
an isomorphism between $M_{(2,2h,2)}(A)$ and $M_{(2,2\hat{h},2)}(\hat{A})$ and by restriction to the Albanese fibres we get the desired isomorphism

$$p : K_{(2,2h,2)}(A) \rightarrow K_{(2,2\hat{h},2)}(\hat{A}).$$

By [42, Prop. 3.2] this result holds for the Mukai vector $(2,2h,2)$ replaced by $(1,h,1)$ and Mukai vector $(2,2\hat{h},2)$ replaced by $(1,\hat{h},1)$. Since the Mukai vector of $G_P(E)$ only depends on the Mukai vector of $E$ it has to be $(2,2\hat{h},2)$ for every $E \in M_{(2,2h,2)}(A)$ if WIT(2) holds for $E$. By definition $H^1(G_P(E))$ is the relative extension $\mathcal{E}xt_1^1(p^*(E) \otimes \mathcal{P}, \mathcal{O})$ (see [13]) and we need to prove that

1. $\mathcal{E}xt_1^i(p^*(E) \otimes \mathcal{P}, \mathcal{O}) = 0$ for $i \neq 2$
2. $\mathcal{E}xt_2^2(p^*(E) \otimes \mathcal{P}, \mathcal{O})$ is a semistable sheaf

for $E \in M_{(2,2h,2)}(A)$.

We distinguish 3 cases

a) $E$ is strictly semistable
b) $\mathcal{E}xt^i(E,L) = 0$ for $i \neq 2$ and $L \in \hat{A}$
c) $\mathcal{E}xt^1(E,L) \neq 0$ for some $L \in \hat{A}$.

In case a) the sheaf $E$ fits in an exact sequence of the form

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

with $E_1$ and $E_2$ in $M_{(2,2h,2)}(A)$: 1) and 2) follows from [42, Prop. 3.2].

In case b) we study the sheaves $\mathcal{E}xt_1^i(p^*(E) \otimes \mathcal{P}, \mathcal{O})$ by using the base change theorem for relative Ext-sheaves [13, Thm 1.4]. By stability $\text{Hom}(E \otimes M, \mathcal{O}) = 0$ for $M \in \hat{A}$ and, since $\mathcal{E}xt^1(E \otimes M, \mathcal{O}) = \mathcal{E}xt^1(E, M') = 0$, we deduce that the dimension of $\mathcal{E}xt_2^2(E \otimes M, \mathcal{O})$ is $\chi(M) = 2$. Theorem 1.4. of [13] implies (1) and that $\mathcal{E}xt_2^2(p^*(E) \otimes \mathcal{P}, \mathcal{O})$ is a rank two vector bundle with Mukai vector $(2,2\hat{h},2)$.

In this case the sheaf $\mathcal{E}xt_1^2(p^*(E) \otimes \mathcal{P}, \mathcal{O})$ is actually stable. A destabilizing quotient should be properly contained in a line bundle whose first Chern class is $\hat{h}$: hence, if $\mathcal{E}xt_2^2(p^*(E) \otimes \mathcal{P}, \mathcal{O})$ is unstable, it admits a non zero morphism to a line bundle $\bar{G}$ with Mukai vector $(1,\hat{h},1)$. By [42, Prop. 3.2], by applying the inverse of $G_P$, there would exist a line bundle $G$ on $A$, with Mukai vector $(1,h,1)$, such that $G_P(G) = \bar{G}$ and, since $G_P$ is an equivalence, $G$ would have a non trivial morphism to $E$ making $E$ unstable.

In case c) let us first assume that $E$ is a vector bundle. Under this assumption there exists $L \in \hat{A}$ and a non trivial extension

$$0 \rightarrow L \rightarrow F \rightarrow E \rightarrow 0$$

such that $F$ is stable.

In fact, as $F$ has rank 3, its stability may be checked by considering only rank 1 subsheaves and rank 1 quotients. The saturation of every rank one subsheaf of $F$ is either $L$ or a line bundle that injects into $E$: its first Chern class cannot be strictly positive by stability of $E$. Every rank 1 quotient of $F$ has to have strictly positive first Chern class by stability of $E$ and because the extension is not trivial. It follows that $F \in M_{(2,2\hat{h},3)}$.

By [13, Thm 3.7] every stable sheaf with Mukai vector $(2,2\hat{h},3)$ satisfies the WIT(2) with respect to $G_P$. We deduce that $\bar{F} := G_P(F) \in M_{(2,2\hat{h},3)}$. By applying
\( \mathcal{G}_P \) to the exact sequence (10), since WIT(2) holds for \( L \) too, we get (1) and the exact sequence
\[ 0 \to \mathcal{E}xt^2_p(p^*(E) \otimes \mathcal{P}, \mathcal{O}) \to \hat{F} \to C_L \to 0 : \]
in particular \( \mathcal{E}xt^2_p(p^*(E) \otimes \mathcal{P}, \mathcal{O}) \) is torsion free. This sheaf is also stable: otherwise it would admit a non trivial map to a sheaf in \( \mathcal{M}_{(1, \hat{h}, 1)} \) and, since \( \mathcal{G}_P \) is an equivalence inducing an isomorphism between \( \mathcal{M}_{(1, \hat{h}, 1)} \) and \( \mathcal{M}_{(1, \hat{h}, 1)} \), this would imply the existence of a non trivial map from a sheaf in \( \mathcal{M}_{(1, \hat{h}, 1)} \) to \( E \), contradicting stability of \( E \).

In the remaining case where \( E \) is stable and not locally free, torsion freeness of \( \mathcal{E}xt^2_p(p^*(E) \otimes \mathcal{P}, \mathcal{O}) \) requires a different argument. The double dual \( E^{\vee \vee} \) of the stable sheaf \( E \) is a \( \mu \)-stable vector bundle with Mukai vector \((2, 2h, 2 + i)\) for \( i > 0 \). Starting from observing that
\[ \dim(M_{(2, \hat{h}, 2+i)}(A)) = (2, 2h, 2 + i)^2 = 16 - 4(2 + i) + 2 = 10 - 4i \]
is non negative only for \( i \leq 2 \), one can check that and \( E^{\vee \vee} \) fits in an exact sequence of the form
\[ 0 \to L_1 \otimes \mathcal{O}(H) \to E^{\vee \vee} \to L_2 \otimes \mathcal{O}(H) \to 0 \]
where \( L_i \in \hat{\mathcal{A}} \) and \( H \) is a divisor whose class is \( h \).

As a consequence, using stability of \( E \), the cokernel \( E/E^{\vee \vee} \) is the structure sheaf of a length two subscheme \( Z \in A \) and we have a non trivial extension
\[ 0 \to L_1 \otimes \mathcal{O}(H) \otimes I_Z \to E \to L_2 \otimes \mathcal{O}(H) \to 0, \]
where \( I_Z \) is the sheaf of ideals of \( Z \) in \( A \).

By [43, Proposition 3.11] and [43, Thm 3.7] \( L_1 \otimes \mathcal{O}(H) \otimes I_Z \) and \( L_2 \) satisfy the weak index theorem with index two: hence the same holds for \( E \) and there is a non trivial extension
\[ 0 \to A \to \mathcal{E}xt^2_p(p^*(E) \otimes \mathcal{P}, \mathcal{O}) \to B \to 0 \]
where \( A \in \mathcal{M}_{(2, \hat{h}, 1)} \) and \( B \in \mathcal{M}_{(0, \hat{h}, 1)} \). Since \( \text{Hom}(B, A) = 0 \) the extension induces a non trivial section of the pure one dimensional sheaf \( \mathcal{E}xt^1(B, A) \): as a consequence \( \mathcal{E}xt^2_p(p^*(E) \otimes \mathcal{P}, \mathcal{O}) \) is a vector bundle outside a zero dimensional subset.

The sheaf \( \mathcal{E}xt^2_p(p^*(E) \otimes \mathcal{P}, \mathcal{O}) \) is actually torsion free: otherwise it would contain the structure sheaf of a point as a subsheaf and this is impossible since \( \mathcal{G}_P \) induces an isomorphism \( \text{Hom}(\mathcal{C}_p, \mathcal{E}xt^2_p(p^*(E) \otimes \mathcal{P}, \mathcal{O})) \cong \text{Hom}(E, L_p) \) and \( \text{Hom}(E, L_p) = 0 \) by stability.

Stability of \( \mathcal{E}xt^2_p(p^*(E) \otimes \mathcal{P}, \mathcal{O}) \) follows as in the locally free case.

The commutativity of the diagram (14) is formally identical to the commutativity of the diagram (10) in Proposition 1.3. In this case the morphism
\[ \rho^H : H^{ev}(A, \mathbb{Z}) \to H^{ev}(\hat{A}, \mathbb{Z}) \]
induced in Cohomology by \( \mathcal{G}_P \) satisfies \( \rho^H(r, \alpha, s) = (s, PD(\alpha), r) \) (see [12, Lemma 3.1]) and using [12, Proposition 2.5] we get that the restriction
\[ \varrho : (1, h, 1)^\perp \to (1, \hat{h}, 1)^\perp \]
of the opposite of \( \rho^H \) to \( (1, h, 1)^\perp \) makes the diagram (14) commutative. \( \square \)

As a consequence, we get the the existence of a monodromy operator whose determinant is \(-1\)
Corollary 4.8. Under the assumptions of Proposition 4.7, there exists 
\[ m \in Mon^2(K\{2,2n,2\}(A,H)) pt \setminus SO^+(1,h,1) \].

Proof. By connectedness of moduli spaces of polarized abelian surfaces there exists a parallel transport operator \( t : H^1(\hat{A}, \mathbb{Z}) \rightarrow H^1(A, \mathbb{Z}) \) coming from a projective family. Let \( t^{ev} : H^{ev}(\hat{A}, \mathbb{Z}) \rightarrow H^{ev}(A, \mathbb{Z}) \) be the induced isomorphism on the even cohomologies and \( t^{pr} : (1, \hat{h}, 1)^{\perp} \rightarrow (1, h, 1)^{\perp} \) be its restriction.

Set \( m := t^{pr} \circ \varphi \) and let \( \tilde{m} : H^{ev}(A, \mathbb{Z}) \rightarrow H^{ev}(\hat{A}, \mathbb{Z}) \) be the extension of \( \varphi \) defined by the same formula (15), so that \( \tilde{m} : H^{ev}(A, \mathbb{Z}) \rightarrow H^{ev}(A, \mathbb{Z}) \) extends \( m \). Since \(-PD : H^2(A, \mathbb{Z}) \rightarrow H^2(\hat{A}, \mathbb{Z}) \) is not the second wedge power of an isomorphism compatible with complex orientations (see Remark 4.9), the composition \( \Lambda^2(t) \circ -PD \) cannot be obtained as the second wedge power of an element of \( SL(H^1(A, \mathbb{Z})) \). By formula (5) the determinant of \( \Lambda^2(t) \circ -PD \) is \(-1 \). As a consequence, the determinant of \( \tilde{m} \) is 1 and since \( \tilde{m}((1,h,1)) = -(1,h,1) \), its restriction \( m \) to \((1,h,1)^{\perp} \) has determinant \(-1 \). \( \square \)

4.3. Monodromy of the singular models. In this subsection, for every \( A,v = 2w \) and \( H \) as in Setting 4.1 we prove that the group \( Mon^2(K_v(A)) pt \), of monodromy operators on \( H^2(K_v(A), \mathbb{Z}) \) is the whole group \( O^+(H^2(K_v(A), \mathbb{Z})) \) of isometries of \( H^2(K_v(A), \mathbb{Z}) \) preserving the orientation of the positive cone of the lattice \( H^2(K_v(A), \mathbb{Z}) \) (see Remark 2.2).

The proof of this result contains a computational part. In the following remark we collect elementary facts on isometries that we will use.

Remark 4.9. For every \( w \in H^{ev}(A, \mathbb{Z}) \) such that \( w^2 = 2 \) and every \( \gamma \in O(w^{\perp}) \) there exists a unique \( \tilde{\gamma} \in O(H^{ev}(A, \mathbb{Z})) \) extending \( \gamma \) and such that \( \tilde{\gamma}(w) = w \) (see Remark 2.5). If \( w = (1,0,-1) \) we necessarily have
\[ \tilde{\gamma}(1,0,1) = (2m + 1,2\alpha,2m + 1) \]
with \( \alpha^2 = 2m(m+1) \) and \( \tilde{\gamma}(1,0,0) = (m+1,\alpha,m) \), \( \tilde{\gamma}(0,0,1) = (m,\alpha,m+1) \).

Indeed, we can write \( \tilde{\gamma}(1,0,1) = (l,\beta,l) \) as it must be orthogonal to the element \( \tilde{\gamma}(1,0,-1) = (1,0,-1) \). Now by linearity we have:
\[ \tilde{\gamma}(2,0,0) = (l+1,\beta,l-1). \]
Thus, \( \beta = 2\alpha \) and \( l-1 = 2m \), therefore we also have
\[ \tilde{\gamma}(0,0,2) = (2m,2\alpha,2m+2). \]
And our claim follows by computing the square of these elements. Notice moreover that \( \alpha \) and \( 2m+1 \) are coprime as \((1,0,-1)\) is indivisible.

We first show that \( Mon^2(K_v(A)) pt \) contains all isometries preserving the orientation of the positive cone of \( H^2(K_v(A), \mathbb{Z}) \) (see Remark 2.2) and having determinant 1

Proposition 4.10. Using the identification
\[ SO^+(w^{\perp}) = SO^+(H^2(K_v(A), \mathbb{Z})) \subseteq Mon^2(K_v(A)) pt. \]

Proof. By [34, Theorem 1.6, Proposition 2.16] or [36, Theorem 1.17, Remark 1.18] every two varieties of the form \( K_v(A) \) appear as fibres of a projective family, over a connected base, which is an analytically locally trivial deformation at every point of the domain.
Hence it suffices to prove the statement in the special case where $A = E_1 \times E_2$, the Neron Severi group $NS(A)$ is generated by the classes $e$ and $f$ of the curves $E_1$ and $E_2$, the Mukai vector $v$ is $(2, 0, -2)$ and the polarization is $e + kf$ for a big $k$ (as in Proposition 4.4).

Let $G \subseteq O^+((1, 0, -1)\perp)$ be the subgroup generated by $SO^+(H^2(E_1 \times E_2, \mathbb{Z}))$ and $\varphi^{-1} \circ SO^+_{\epsilon + f}(H^2(E_1 \times E_2, \mathbb{Z})) \circ \varphi$, by Corollary 4.9 the statement follows from the inclusion

$$SO^+(w\perp) \subseteq G$$

(the opposite inclusion is trivial).

We are going to show that for every $\gamma \in SO^+((1, 0, -1)\perp)$ there exists $\delta$ belonging to $G$ such that $\gamma((1, 0, 1)) = \delta((1, 0, 1))$.

If this is the case, letting $\tilde{\gamma}$ and $\tilde{\delta}$ be the extensions of $\gamma$ and $\delta$ as in Remark 4.9, the composition $\tilde{\delta}^{-1} \circ \tilde{\gamma} \in O(H^{ev}(E_1 \times E_2, \mathbb{Z}))$ has determinant 1 and is the identity on $H^0(E_1 \times E_2, \mathbb{Z}) \oplus H^4(E_1 \times E_2, \mathbb{Z})$: this implies that $\tilde{\delta}^{-1} \circ \tilde{\gamma}$ acts on $H^2(E_1 \times E_2, \mathbb{Z})$ with determinant 1. Moreover $\tilde{\delta}^{-1} \circ \tilde{\gamma}$ preserves the orientation of the positive cone of the lattice $(1, 0, -1)\perp$, hence the same holds for its restriction to $H^2(E_1 \times E_2, \mathbb{Z})$. It follows that $\delta^{-1} \circ \gamma \in SO^+(H^2(E_1 \times E_2, \mathbb{Z}))$ and $\gamma \in G$.

Given $\gamma \in SO^+((1, 0, -1)\perp)$, in order to find $\delta$ we notice, from the definition of $\varphi$ in equation (11), that

$$\varphi(l, \chi + bf, l) = (0, l(e - f) + \chi, l + b)$$

for every $l, b \in \mathbb{Z}$ and $\chi \in H^2(E_1 \times E_2, \mathbb{Z})$ perpendicular to both $e$ and $f$.

By Remark 4.9 we know $\gamma((1, 0, 1)) = (2m + 1, 2\alpha, 2m + 1)$.

Let us first suppose that $\alpha$ is primitive. By equation (17) we have

$$\varphi((1, 0, 1)) = (0, e - f, 1)$$

and, by Eichler’s criterion 2.6 and Remark 2.5, there exists an isometry $g$ belonging to $SO^+(H^2(E_1 \times E_2, \mathbb{Z}))$ such that $g(\alpha + f) = \alpha + f$ and $g(\alpha - f) = (2m + 1)(f - e) + 2\rho$, with $\rho$ primitive and orthogonal to $e$ and $f$.

We have $(g \circ \varphi)((1, 0, 1)) = g(0, f - e, 1) = (0, (2m + 1)(e - f) + 2\rho, 1)$ and, by equation (17),

$$\varphi^{-1} \circ g \circ \varphi((1, 0, 1)) = (2m + 1, 2(mf + \rho), 2m + 1).$$

Notice that $\rho' := \rho + mf$ is primitive as $\rho$ is and $f$ is orthogonal to it. We again apply Eichler’s criterion 2.6 to find an isometry $g' \in SO^+(H^2(E_1 \times E_2, \mathbb{Z}))$ such that $g'(\rho') = \alpha$. It follows that $\beta := g' \circ \varphi^{-1} \circ g \circ \varphi \in G$ satisfies $\beta((1, 0, 1)) = \gamma((1, 0, 1))$.

Let us now suppose that $\alpha = n\beta$, with $n \neq \pm 1$. By Eichler’s criterion there exists an element $g'' \in SO^+(H^2(E_1 \times E_2, \mathbb{Z}))$ such that $g''(\beta') = \beta' \in \langle e, f \rangle^\perp$. Therefore $\varphi(2m + 1, 2n\beta', 2m + 1) = (0, (2m + 1)(e - f) + 2n\beta', 2m + 1)$. As $\beta'$ is orthogonal to $f - e$, by applying Eichler’s criterion again, there exists an element $g''' \in SO^+_{\epsilon + f}(H^2(E_1 \times E_2, \mathbb{Z}))$ fixing the element $e + f$ and such that $g'''((2m + 1)(e - f) + 2n\beta') = (2m + 1)(e - f) + 2\beta''$, with $\beta''$ primitive and perpendicular to $e$ and $f$. Let now $h := \varphi^{-1} \circ g''' \circ \varphi \circ g'' \in G$. We have

$$h \circ \gamma((1, 0, 1) = h((2m + 1, 2n\beta', 2m + 1)) = (2m + 1, 2\beta'', 2m + 1)$$

and by the first part of the proof, the equality $\delta((1, 0, 1)) = (2m + 1, 2\beta', 2m + 1)$ holds for some $\delta \in G$. We conclude that $\gamma((1, 0, -1)) = (h^{-1} \circ \delta)((1, 0, -1))$ and since $h^{-1} \circ \delta \in G$, this finishes the proof. \qed
Remark 4.11. The argument of the above proof could be applied to construct monodromy operators also for generalized Kummer manifolds $K_n(A)$ of dimension $2n$ for $n > 1$. In this case

$$SO^+((1,0,−1−n)^+) \subseteq Mon^2(K_n(A)),$$

and the above arguments only show that $Mon^2(K_n(A))$ contains the subgroup $S\tilde{O}^+((1,0,−1−n)^+)$ consisting of isometries that can be extended to the whole Mukai lattice with trivial action $(1,0,−1−n)$. This difference has a pure lattice theoretic explanation: every isometry of $(1,0,−1−n)^+$ extends to an isometry of the Mukai lattice fixing $(1,0,−1−n)$ if and only if $n = 0$. Using the argument of the proof of the previous Proposition and the analogue of Corollary 4.8 based on [34, Proposition 3.2] one can prove that $Mon^2(K_n(A))$ contains an extension of $S\tilde{O}^+((1,0,−1−n)^+)$ of index two. However, to prove that this extension is the monodromy group of generalized Kummer manifold, one would still need some argument to show that a monodromy operator has extend to an isometry of the Mukai lattice like (see [19, 23, 30]).

Combining Proposition 4.10 and Corollary 4.8 we determine the monodromy group $Mon^2(K_v(A))_{lt}$.  

Proposition 4.12. For every $A,v = 2w,H$ and $K_v(A)$ as in Setting 4.1,

$$Mon^2(K_v(A))_{lt} = O^+(H^2(K_v(A),\mathbb{Z})) = O^+(w^+).$$

Proof. Since $SO^+(w^+)$ has index two in $O^+(w^+)$ and $SO^+(w^+) \subseteq Mon^2(K_v(A))_{lt}$ by Proposition 4.10 it suffices to find a monodromy operator in $Mon^2(K_v(A))_{lt}$ with determinant $−1$. As in the proof of Proposition 4.10 by [34, Theorem 1.6, Proposition 2.16] or [36, Theorem 1.17, Remark 1.18], it is enough to find this element in a particular case. This has been done in Corollary 4.8.

As an immediate consequence we determine $Mon^2(K_v(A))_{lt} \supseteq Mon^2(K_v(A))_{pr}$.

Corollary 4.13. 

$$Mon^2(K_v(A))_{lt} = O^+(H^2(K_v(A),\mathbb{Z})) = O^+(w^+).$$

5. Total Monodromy and the Classical Bimeromorphic Global Torelli

In this section we prove that the monodromy group of a hyperkähler manifold of $O'G6$ type is maximal and that Classical Bimeromorphic Global Torelli Theorem holds for this class of manifolds.

In order to show these results we will only use monodromy operators coming from the singular models studied in the previous subsection and one more monodromy operator induced by a specific prime exceptional divisor or a divisorial contraction.

Let $A,v = 2w,h$ and $K_v(A)$ be as in Setting 4.1 and let $\pi_v : \tilde{K}_v(A) \to K_v(A)$ be the blow up of $K_v(A)$ along its singular locus with reduced structure. By [34, Theorem 1.6], the variety $\tilde{K}_v(A)$ is a hyperkähler manifold of $O'G6$ type. Moreover, by [35, Theorem 2.4, Remark 3.3],

$$\pi_v^* : H^2(K_v(A),\mathbb{Z}) \to H^2(\tilde{K}_v(A),\mathbb{Z})$$

is a Hodge isometric embedding and

$$H^2(\tilde{K}_v(A),\mathbb{Z}) = \pi_v^*(H^2(K_v(A),\mathbb{Z})) \oplus_{\perp} \mathbb{Z}c_v$$
where $\epsilon_v$, a half the class of the (irreducible) exceptional divisor and $\epsilon_v^2 = -2$. Using the Mukai-Donaldson-Le Potier isomorphism $w^\perp \simeq H^2(K_v(A), \mathbb{Z})$, we have an isomorphism
\begin{equation}
H^2(\tilde{K}_v(A), \mathbb{Z}) \simeq w^\perp \oplus_\perp \mathbb{Z} \epsilon_v
\end{equation}
that allows the identification
\begin{equation}
O^+(H^2(\tilde{K}_v(A), \mathbb{Z})) = O^+(w^\perp \oplus_\perp \mathbb{Z} \epsilon_v).
\end{equation}
Hence we may see the group $\text{Mon}^2(\tilde{K}_v(A))$ of monodromy operators on $H^2(\tilde{K}_v(A), \mathbb{Z})$ for families of Kähler manifolds as a subgroup of $O^+(w^\perp \oplus_\perp \mathbb{Z} \epsilon_v)$. The following Proposition describes the contribution of $\text{Mon}^2(\tilde{K}_v(A))_{\text{lt}}$ to $\text{Mon}^2(\tilde{K}_v(A))$.

**Proposition 5.1.** Using the identification (19),
\[ O^+(w^\perp) \subset \text{Mon}^2(\tilde{K}_v(A)) \subset O^+(w^\perp \oplus_\perp \mathbb{Z} \epsilon_v). \]

**Proof.** By Proposition (11) $O^+(w^\perp) = \text{Mon}^2(K_v(A))_{\text{lt}}$. By functoriality of the blow up and injectivity of $\pi_v^*$ we have $\text{Mon}^2(K_v(A))_{\text{pr}} \subset \text{Mon}^2(\tilde{K}_v(A))$ and, since the exceptional divisor of $\pi_v$ is irreducible, $\text{Mon}^2(K_v(A))_{\text{pr}}$ acts trivially on $\epsilon_v$, as desired. \[ \square \]

Monodromy operators in $\text{Mon}^2(\tilde{K}_v(A))$ of a different nature can be obtained by considering prime exceptional divisors on particular projective hyperkähler manifolds of $OG6$ type. A divisor $D$ on a projective hyperkähler manifold $M$ is prime exceptional if it is irreducible and the Beauville-Bogomolov square of the class $[D] \in H^2(M, \mathbb{Z})$ is negative. By [10, Prop. 6.2] for every prime exceptional divisor $D$ on $M$ and every class $\alpha \in H^2(M, \mathbb{Z})$, the value $2\langle[D], \alpha \rangle$ is integral and the associated reflection
\[ R_D : H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}) \]
defined by the formula
\[ R_D(\alpha) = \alpha - 2\langle[D], \alpha \rangle \frac{[D]}{|[D]|^2} \]
is a monodromy operator in $\text{Mon}^2(M)$.

The projective model of $OG6$ that we need is the original O’Grady example and the modular image of the corresponding prime exceptional divisor parametrizes non locally free sheaves.

**Proposition 5.2.** Let $A$ be the Jacobian of a generic curve of genus 2 and let $v = (2, 0, -2) \in H^{ev}(A, \mathbb{Z})$. Let $B \subset \tilde{K}_{(2,0,-2)}(A)$ be the strict transform in $\tilde{K}_{(2,0,-2)}(A)$ of the locus of non locally free sheaves in $K_{(2,0,-2)}(A)$.

1. The subvariety $B \subset \tilde{K}_{(2,0,-2)}(A)$ is a prime exceptional divisor of Beauville-Bogomolov square $-4$.
2. The reflection $R_B : H^2(\tilde{K}_{(2,0,-2)}(A), \mathbb{Z}) \rightarrow \tilde{K}_{(2,0,-2)}(A), \mathbb{Z})$ belongs to the monodromy group $\text{Mon}^2(\tilde{K}_{(2,0,-2)}(A))$.

**Proof.** Irreducibility of $B$ follows from [23, Lemma 4.3.3] and $|B|^2 = -4$ is proven in [33, Theorem 3.5.1] or in [32, Theorem 9.1]: this proves (1). (2) follows from (1) and [10, Prop. 6.2]. \[ \square \]

**Remark 5.3.** Nagai proved furthermore that $B$ is the contracted locus of a divisorial contraction.
Our main result follows from Proposition 5.1 and Proposition 5.2.

**Theorem 5.4.** (1) The monodromy group of a hyperkähler manifold $X$ of $OG6$ type is maximal, i.e.

$$\text{Mon}^2(Y) = O^+(H^2(Y, \mathbb{Z})).$$

(2) The Classical Bimeromorphic Global Torelli Theorem holds for hyperkähler manifolds of $OG6$ type, i.e. two hyperkähler manifolds $X'$ and $X''$ of $OG6$ type are bimeromorphic if and only if there exists an isometric isomorphism of Hodge structures between $H^2(X', \mathbb{Z})$ and $H^2(X'', \mathbb{Z})$.

**Proof.** (1) By definition of deformation equivalence type of hyperkähler manifolds it suffices to prove the statement for the original O’Grady example $\tilde{K}_{(2,0,-2)}(A)$ where $A$ is the Jacobian of a generic curve of genus 2. In this case we have

$$H^2(\tilde{K}_{(2,0,-2)}(A)) = (1, 0, -1)^+ \oplus_\perp \mathbb{Z} \epsilon =$$

$$H^2(A, \mathbb{Z}) \oplus_\perp \mathbb{Z} \epsilon \oplus_\perp \mathbb{Z} \zeta,$$

where $\epsilon := \epsilon_{(2,0,-2)}$ and $\zeta^2 = \epsilon^2 = -2$. By [35, Theorem 3.5.1] the class $[B]$ of the prime exceptional divisor is in $\mathbb{Z} \zeta \oplus_\perp \mathbb{Z} \epsilon$ and since $[B]^2 = -4$, up to changing the signs of $\zeta$ and $\epsilon_{(2,0,-2)}$, we may suppose that $[B] = \zeta + \epsilon$.

By Proposition 5.1 we know that $O^+((1, 0, -1)^+) \subseteq \text{Mon}^2(\tilde{K}_{(2,0,-2)}(A))$ and by Proposition 5.2 we know that $R_B := R_{\zeta + \epsilon} \in \text{Mon}^2(\tilde{K}_{(2,0,-2)}(A))$: hence the statement follows if we prove that the the subgroup generated by $R_{\zeta + \epsilon}$ and

$$O^+((1, 0, -1)^+) = O^+(H^2(A, \mathbb{Z}) \oplus_\perp \mathbb{Z} \zeta)$$

is the whole

$$O^+(H^2(A, \mathbb{Z}) \oplus_\perp \mathbb{Z} \epsilon \oplus_\perp \mathbb{Z} \zeta).$$

We have to show that every $f \in O^+(H^2(A, \mathbb{Z}) \oplus_\perp \mathbb{Z} \epsilon \oplus_\perp \mathbb{Z} \zeta)$ can be obtained by composing $R_{\zeta + \epsilon}$ and elements of $O^+((1, 0, -1)^+) = O^+(H^2(A, \mathbb{Z}) \oplus_\perp \mathbb{Z} \zeta)$.

First, notice that $R_{\zeta + \epsilon}$ sends $\zeta$ in $-\epsilon$, $\epsilon$ to $-\zeta$ and leaves their orthogonal invariant. Since the divisibility of $\epsilon$ is 2, its image has to be of the form

$$f(\epsilon) = 2u + a\zeta + b\epsilon$$

for $a, b \in \mathbb{Z}$ and $u \in H^2(A, \mathbb{Z}) \simeq U^3$.

We split the proof in four steps:

- **b = 0** This means that $2u + a\zeta$ is primitive and has divisibility 2 inside the lattice $H^2(A, \mathbb{Z}) \oplus_\perp \mathbb{Z} \epsilon$. By Eichler’s criterion (cf. Lem. 2.6), there is an isometry $g \in O^+(H^2(A, \mathbb{Z}) \oplus_\perp \mathbb{Z} \epsilon) = O^+((1, 0, -1)^+)$ such that $g(2u + a\zeta) = -\zeta$, therefore the isometry $h := R_{\zeta + \epsilon} \circ g \circ f$ sends $\epsilon$ into itself, hence $h \in O^+(H^2(A, \mathbb{Z}) \oplus_\perp \mathbb{Z} \zeta)$. Since $g^{-1} \circ R_{\zeta + \epsilon} \circ h = f$, our claim holds.

- **a = 0** This case reduces to the first one after composing $f$ with $R_{\zeta + \epsilon}$.

- **a, b ≠ 0 and 2 | u** Since $(f(\epsilon))^2 = \epsilon^2 = -2$, either $a$ or $b$ is even and, since we can compose with $R_{\zeta + \epsilon}$, we can suppose that $a = 2c$. As $u$ is not divisible by 2, a primitive sub multiple of the element $u + c\zeta$ has divisibility 1. Therefore, by Eichler’s criterion (Cf. Lem. 2.6), there exists $g \in O^+(H^2(A, \mathbb{Z}) \oplus_\perp \mathbb{Z} \epsilon)$ such that $g(u + c\zeta) = \bar{u}$ for an element $\bar{u} \in H^2(A, \mathbb{Z})$. Thus, the isometry $gf$ falls in the previous case and our claim holds.
a, b \neq 0 \text{ and } 2|b \text{. By the same argument of the previous case we may suppose that } b = 2c. \text{ In this case } a \text{ has to be odd. A primitive submultiple of } 2u + a\zeta \text{ is of the form } 2u' + a'\zeta \text{ with } a' \text{ odd. By Eichler’s criterion (cf. Lem. 2.6), there exists } g \in O^+(H^2(A, \mathbb{Z}) \oplus \perp \mathbb{Z}) \text{ such that } g(2u' + a'\zeta) = 2u'' + a'\zeta \text{ with } u'' \text{ not divisible by } 2. \text{ It follows that } gf \text{ falls in the previous case, hence our claim holds for } f.\

(2) This is a standard consequence of (1) and Markman’s Hodge theoretic version of Verbitsky’s Global Torelli. If } X' \text{ and } X'' \text{ are bimeromorphic there exists a Hodge isometry } H^2(X', \mathbb{Z}) \simeq H^2(X'', \mathbb{Z}) \text{ by } [28] \text{ Proposition 1.6.2]. Conversely, given a Hodge isometry } \varphi : H^2(X', \mathbb{Z}) \to H^2(X'', \mathbb{Z}) \text{ and a parallel transport operator } t : H^2(X'', \mathbb{Z}) \to H^2(X', \mathbb{Z}), \text{ since } Mon^2(X') = O^+(X') \text{ and } -1 \text{ reverses the orientation of the positive cone of the lattice } H^2(X', \mathbb{Z}) \text{ (see Remark 2.2], either } t\varphi \in Mon^2(X') \text{ or } -(t\varphi) \in Mon^2(X'). \text{ Hence either } \varphi \text{ or } -\varphi \text{ is a Hodge isometry and a parallel transport operator: by Markman’s Hodge theoretic Torelli theorem } [16] \text{ Theorem 1.3] the hyperkähler manifolds } X' \text{ and } X'' \text{ are birational.} \]

\text{ Remark 5.5. We have actually proven that } Mon^2(\widetilde{K}_{(2,0,-2)}(A)) \text{ consist of monodromy operators for families of projective manifolds. This statement holds for } \widetilde{K}_v(A) \text{ for every } v, A \text{ and a } v\text{-generic } H \text{ and follows since the proof of } [34] \text{ Theorem 1.7] shows that the hyperkähler manifold } \widetilde{K}_v(A) \text{ may be deformed to } \widetilde{K}_{(2,0,-2)}(A) \text{ using only families of projective manifolds.}

6. The Kähler cone and the birational Kähler cone

The aim of this section is to compute the Kähler and Birational Kähler cones for manifolds of } OG6 \text{ type. In general, for a hyperkähler manifold } X \text{ these cones are subcones of the positive cone } C(X) \subset H^{1,1}(X, \mathbb{R}) \text{ that is defined as the connected component of the set of the classes with positive Beauville-Bogomolov square containing a Kähler class. The Birational Kähler cone is the union } \bigcup_f f^*(K(X')) \text{ of Kähler cones of } X', \text{ where } f : X \to X' \text{ runs on all birational maps between } X \text{ and other hyperkähler manifolds } X'. \text{ The Kähler and the birational Kähler cone are dual in } C(X) \text{ to wall divisors and stably prime exceptional divisors respectively ([16] Section 6 and [22] Proposition 1.5]). The main result of this section says that the Kähler and birational Kähler cones of manifolds of } OG6 \text{ type can be explicitly determined only with lattice theory.}

Before discussing hyperkähler manifolds of } OG6 \text{ type, we recall definitions and basic properties of prime exceptional divisors and wall divisors.}

\text{Definition 6.1. A prime exceptional divisor of an hyperkähler manifold } X \text{ is a reduced and irreducible divisor of negative Beauville-Bogomolov square. A stably prime exceptional divisor } D \in \text{Div}(X) \text{ is a divisor which is prime exceptional in a general deformation of the pair } (X, O(D)).

Prime exceptional divisors are stably prime exceptional divisors (see [16] Proposition 6.6(1))], but the converse does not hold in general. The easiest example of a stably prime exceptional divisor which is not prime exceptional is given by a reducible } -2 \text{ curve on a K3 surface.}

\text{Remark 6.6(1)]}, but the converse does not hold in general. The easiest example of a stably prime exceptional divisor which is not prime exceptional is given by a reducible } -2 \text{ curve on a K3 surface.}
By removing the orthogonal hyperplanes to stably prime exceptional divisors, the positive cone $C(X)$ is cut in a wall and chamber decomposition. One such chamber is the closure of the Birational Kähler cone (see [16, Section 5.2]) and its algebraic part is the movable cone, i.e. the cone generated (over $\mathbb{R}$) by all divisors which do not have a divisorial base locus (see [16, Theorem 5.8]). We remind the reader that the Birational Kähler cone is the union of the Kähler cones of the hyperkähler bimeromorphic models of $X$ and is not connected in general.

**Definition 6.2.** A wall divisor on an hyperkähler manifold $X$ is a primitive divisor $D$ such that for every monodromy operator $g : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ that is a Hodge isometry, the perpendicular to the class $g([D])$ does not intersect the Birational Kähler cone.

By using the natural lattice embedding $H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{Q})$, a wall divisor is precisely a multiple of an extremal rational curve, up to the action of monodromy Hodge isometries, see [12, Proposition 2.3]. Also orthogonals to wall divisors give a wall and chamber decomposition of the positive cone (whence their name), and one of the open chambers is the Kähler cone. In particular, if we restrict this wall and chamber decomposition to the birational Kähler cone, we obtain the Kähler cones of all hyperkähler birational models of $X$.

Every stably prime exceptional divisor is a wall divisor (or more precisely, its primitive multiple is), but the converse does not hold and non-stably prime exceptional wall divisors are those responsible for the non connectedness of the birational Kähler cone.

**Remark 6.3.** A very important property of these classes of divisors is their invariance under parallel transport: if $X, Y$ are hyperkähler and $D \in \text{Div}(X)$ is a wall divisor (resp. stably prime exceptional), and $\varphi$ is a parallel transport operator between $X$ and $Y$ such that $\varphi([D]) \in \text{Pic}(Y)$, then $\varphi([D])$ is the class of a wall divisor (resp. stably prime exceptional), see [22, Theorem 1.3] (resp. [16, Proposition 6.6]). Therefore it suffices to determine the classes of stably prime exceptional and wall divisors up to parallel transport.

To determine classes in Picard group of stably prime exceptional divisors and wall divisors in the case of hyperkähler manifolds of $OG6$ type, we will use two tools. The first is explicit birational geometry of of O’Grady six dimensional manifolds, to prove that some divisors are either stably prime exceptional or wall divisors. The second is the construction of ample divisors on Albanese fibres of moduli space of sheaves to prove that some divisors are not wall divisors. Let us start with the first approach:

**Lemma 6.4.** Let $X$ be a manifold of $OG6$ type. Let $D \in \text{Div}(X)$ , let $[D] \in H^2(X, \mathbb{Z})$ be its class and let $\text{div}(D)$ be the divisibility of $[D]$ in $H^2(X, \mathbb{Z})$. Then $[D]$ is the class of a (multiple of a) stably prime exceptional divisor if one of the following holds:

- $[D]^2 = -4$ and $\text{div}(D) = 2$,
- $[D]^2 = -2$ and $\text{div}(D) = 2$.

**Proof.** Let $A$ be an abelian surface and let $X$ be the crepant resolution of the Albanese fibre of the moduli space of stable sheaves on $A$ with Mukai vector $v := (2, 0, -2)$ and a $v$-generic stability condition. Then, $\text{Pic}(X) \cong NS(A) \oplus -2^2$. We have two effective divisors $\tilde{\Sigma}$ and $\tilde{B}$ on $X$, which are respectively the exceptional
divisor of the resolution $\pi : X := \widetilde{K}_v(A) \to K_v(A)$ and the strict transform of the locus on non locally free sheaves. By [38] Prop. 3.3.2], there exists a divisor $E$ such that $2[E] = [\Sigma]$. By [32] and [39], the classes $[E]$ and $[\Sigma]$ have divisibility 2 and square respectively $-2$ and $-4$. These are therefore classes of prime exceptional divisors, hence also stably prime exceptional, and every element in their parallel transport orbit is. By Lemma 6.6 and Theorem 5.4 the square and divisibility determine the orbits of these classes.

To construct wall divisors that are not stably prime exceptional we need some birational geometry of a specific hyperkähler manifold of $OG6$ type whose singular model has already been investigated in the proof of (2) of Proposition 4.3. We discuss the needed birational geometry in the following example.

Example 6.5. Let $A = E_1 \times E_2$ for two very general distinct elliptic curves $E_1, E_2$. Let $e, f$ be their classes inside $NS(A) \cong U$ and let $H, H_0$ be two ample divisors whose image in $NS(A)$ are respectively $e + df, e + f$ with $b > a$.

By Remark 6.4 there is a small contraction $c : K(2,0,-2)(A, H) \to K(2,0,-2)(A, H_0)$ where $K(2,0,-2)(A, H_0)$ is the Albanese fibre of the moduli space of $H_0$-semistable sheaves on $A$ with Mukai vector $(2,0,-2)$ and the contracted locus of $c$ is a finite union of copies of $\mathbb{P}^3$ parametrizing locally free sheaves and disjoint from the singular locus of $K(2,0,-2)(A, H)$.

Set $X := \widetilde{K}(2,0,-2)(A, H)$ and let $X_0 = \widetilde{K}(2,0,-2)(A, H_0)$, be the blow up of $K(2,0,-2)(A, H_0)$ along the isomorphic image of the singular locus of $K(2,0,-2)(A, H)$. We have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\tilde{c}} & X_0 \\
\downarrow & & \downarrow \\
K(2,0,-2)(A, H) & \xrightarrow{c} & K(2,0,-2)(A, H_0)
\end{array}
$$

By the Hodge isometry $H^2(X, \mathbb{Z}) \cong (2,0,-2)^\perp \oplus \mathbb{Z} \epsilon$ (see equation (18)), the rank of the Picard group of $X$ is four. Since the exceptional divisor of the blow up $X \to K(2,0,-2)(A, H)$ and the strict transform $B$ in $X$ of the locus parametrizing non locally free sheaves do not intersect the contracted locus of $\tilde{c}$, their classes descend to $X_0$ and, since $X_0$ is projective, its Picard rank is three. It follows that $\tilde{c}$ is a relative Picard rank one contraction.

To determine the class $[D]$ of the extremal curve contracted by $\tilde{c}$, we notice that it has to be a Hodge class perpendicular to $\epsilon$ and $[B]$ since the contracted locus of $\tilde{c}$ is contained in the locally free locus. By [38] Theorem 3.5.1 or [32] Theorem 9.1 the classes $\epsilon$ and $[B]$ generate the perpendicular to the image of $H^2(A, \mathbb{Z})$ in $H^2(X, \mathbb{Z}) \cong (2,0,-2)^\perp \oplus \mathbb{Z} \epsilon$: hence $[D] \in H^2(A, \mathbb{Z})$.

By naturality of the restriction of the Mukai-Donaldson-Le Potier morphism to the algebraic part of $(1,0,-1)^\perp$ (see [10] Theorem 8.1.5), the class $(0, e + f, 0) \in (1,0,-1)^\perp \subseteq H^2(X, \mathbb{Z})$ descends to a class in $H^2(X_0, \mathbb{Z})$: hence $[D]$ is also perpendicular to $(0, e + f, 0)$. This implies that, up to scalars, $[D] = \epsilon - f$ and the saturation of the sublattice of $Pic(X)$ of line bundles descending to $X_0$ is generated by $(0, e + f, 0), \epsilon$ and $\zeta$.

Lemma 6.6. Let $X$ be a manifold of $OG6$ type. Let $D \in Div(X)$, let $[D] \in H^2(X, \mathbb{Z})$ be its class and let $div(D)$ be the divisibility of $[D]$ in $H^2(X, \mathbb{Z})$. Then
[D] is the class of a wall divisor but not the class of a multiple of a stably prime exceptional divisor if \([D]^2 = -2\) and \(\text{div}(D) = 1\).

**Proof.** By Remark 6.3 since \(\text{Mon}^2(X) \simeq O^+(U^3 \oplus (-2)^2)\) by Theorem 5.2 using Lemma 2.6 it suffices to show the existence of a specific hyperkähler manifold \(X\) of \(\text{OG}6\) type and a wall divisor \(D \in \text{Div}(X)\) such that \(D\) is not a stably prime exceptional divisor and \([D]^2 = -2\) and \(\text{div}(D) = 1\). We consider the case where \(X\) is as in Example 6.5 and \([D] = (0, e - f, 0) \in H^2(X, \mathbb{Z})\); clearly \([D]^2 = -2\) and \(\text{div}(D) = 1\). As shown in Example 6.5 the class \([D]\) is, up to scalars, the image in \(H^2(X, \mathbb{Q})\) of the class of an extremal curve giving a small contraction; by [12] Proposition 2.3 this implies that \(D\) is a wall divisor and no multiple of \([D]\) is the class of a stably prime exceptional divisor.

Let us move to the second part of the proof. We exhibit ample line bundles on hyperkähler manifolds of \(\text{OG}6\) type by using a classical construction that produces ample line bundles on the the loci of moduli spaces of sheaves where the determinant is fixed.

To produce the ample line bundles we consider again the case of Example 6.5. Before stating the result we recall that the Albanese fibre of the moduli space of \(H\)-semistable sheaves with Mukai vector \((2, 0, -2)\) does not depend on the choice of \(a\) an \(b\) such that \(0 < a < b\) (see Remark 4.4); as usual we simply denote it by \(K_{(2,0,-2)}(A)\) and let \(\widetilde{K}_{(2,0,-2)}(A)\) be the hyperkähler manifold of \(\text{OG}6\) type obtained by blowing up the singular locus, with reduced structure, of \(K_{(2,0,-2)}(A)\).

In this setting, we have the following:

**Lemma 6.7.**

1. Using the identification provided by the Mukai-Donaldson-Le Potier morphism [7], for \(k \gg 0\), the class

   \((-1, kh, -1) \in (1, 0, -1)^\perp = H^2(K_{(2,0,-2)}(A), \mathbb{Z})\)

   is the class of a very ample line bundle on \(K_{(2,0,-2)}(A)\).

2. Using the identification [13], for \(k \gg c \gg d > 0\), the class

   \((-c + c, -c) \perp -de \in (1, 0, -1)^\perp \perp \mathbb{Z}\epsilon = H^2(\tilde{K}_{(2,0,-2)}(A), \mathbb{Z})\)

   is the class of an ample line bundle on \(\tilde{K}_{(2,0,-2)}(A)\).

3. Using the identification [13], there exists integral numbers \(a, c, d > 0\) such that the class

   \((-c, ae + (a + 1)f, -c) \perp -de \in (1, 0, -1)^\perp \perp \mathbb{Z}\epsilon = H^2(\tilde{K}_{(2,0,-2)}(A), \mathbb{Z})\)

   is the class of an ample line bundle on \(\tilde{K}_{(2,0,-2)}(A)\).

**Proof.**

(1) By [13] Theorem 8.1.11] the class \((-1, 0, -1) + k^2 h^2 / 2\) is the class of a very ample divisor on the Albanese fibre \(K_{(2,2kh, -2 + k^2h^2)}(A)\) of the moduli space of \(H\)-semistable sheaves on \(A\) with Mukai vector \((2, 2kh, -2 + k^2h^2)\). Since tensoring by a multiple of the polarization preserves stability and semistability and is compatible with the Mukai-Donaldson-Le Potier morphism, tensoring back by \(O(-kh)\) we get that \((-1, kh, -1) \in (1, 0, -1)^\perp\) is the class of a very ample line bundle on \(K_{(2,0,-2)}(A)\).

(2) Since \(K_{(2,0,-2)}(A)\) is a divisorial contraction of an extremal curve such that the class of the contracted divisor is a positive multiple of \(\epsilon\), item (1) implies (2).
(3) By (2), if \( b \geq a \geq 0 \), the class \( (0, ae + bf, 0) \in H^2(\widetilde{K}_{(2, 0, -2)}(A), \mathbb{Z}) \) is limit of classes of ample line bundles; hence it is the class of a nef line bundle. By Example 6.5, the class of a line bundle on \( \widetilde{K}_{(2, 0, -2)}(A) \) descending to an ample line bundle on \( X_0 \) is of the form \((-c, a(e + f), -c) \oplus_1 -de\) moreover, since \((0, e + f, 0)\) is nef and positive multiples of \((1, 0, 1)\) and \(e\) are effective with negative Beauville-Bogomolov square (see [32, Theorem 9.1]), we get \(a, c, d > 0\). Since \(X_0\) is the contraction of an extremal curve of class \((0, e - f, 0)\) and \((0, f, 0)\) is nef with degree non zero on that curve we obtain that \((-c, a(e + f) + f, -c) \oplus_1 -de\) is the class of an ample divisor on \( \widetilde{K}_{(2, 0, -2)}(A) \). \(\square\)

With the above, we are now ready to determine wall divisors and stably prime exceptional divisors on hyperkähler manifolds of OG6 type:

**Proposition 6.8.** Let \( X \) be a manifold of OG6 type. Let \( D \in \text{Div}(X) \), let \([D] \in H^2(X, \mathbb{Z})\) be its class and let \( \text{div}(D) \) be the divisibility of \([D]\) in \( H^2(X, \mathbb{Z})\). Then \( D \) is a wall divisor if and only if one of the following holds:

i) \([D]^2 = -4\) and \( \text{div}(\langle [D] \rangle) = 2\),

ii) \([D]^2 = -2\) and \( \text{div}(\langle [D] \rangle) = 2\),

iii) \([D]^2 = -2\) and \( \text{div}(\langle [D] \rangle) = 1\).

In cases i) and ii) a multiple of \([D]\) is the class of a stably prime exceptional and in case iii) no multiple of \([D]\) is represented by an effective divisor.

**Proof.** As wall divisors are invariant under parallel transport which preserves their Hodge type by [22, Theorem 1.3], we can prove the statement on a specific hyperkähler manifold of OG6 type: consider the case \( X := \widetilde{K}_{(2, 0, -2)}(A) \) as in Lemma 6.7. Then, \( \text{Pic}(X) \cong U \oplus_1 (-2)^2 \) and by Lemma 2.6 and Theorem 5.4, any class in \( H^2(X, \mathbb{Z}) \) can be moved with the monodromy group inside \( \text{Pic}(X) \), therefore every wall divisor shows up in this case.

By Lemmas 6.4 and 6.6, elements of square \(-2\) or \(-4\) and divisibility \(2\) are indeed classes of wall divisors with an effective multiple (that is, stably prime exceptional divisors) and elements of square \(-2\) and divisibility \(1\) are classes of wall divisors with no effective multiples (that is, wall divisors which are not stably prime exceptional). This proves the "if" part of the statement.

By Lemma 2.6, beyond the cases listed in i), ii) and iii) we have four standard forms for classes of primitive divisors \([D]\) of strictly negative squares:

A) \([D] = (0, e - bf, 0)\) with \(b > 1\),

B) \([D] = (-1, 2(e - bf), -1) - \epsilon\) with \(b \geq 1\),

C) \([D] = (-1, 2(e - bf), -1)\) with \(b \geq 1\),

D) \([D] = (0, 2(e - bf), 0) - \epsilon\) with \(b \geq 1\),

where \(\epsilon\) is a half the class of the exceptional divisor of the blow up \( X = \widetilde{K}_{(2, 0, -2)}(A) \rightarrow K_v(A) \). By Theorem 5.4 (1), every primitive class, in the Picard group of an hyperkähler manifold of OG6 type, of negative square and not satisfying i),ii) or iii), can be moved by a parallel transport operator to a class \([D] \in \text{Pic}(X)\) as in A), B), C) or D). Since the image under a parallel transport operator of the class of a wall divisor is a class of a wall divisors if it is of Hodge type (see Remark 6.5), we have to prove that every class as in A), B), C) or D) is not the class of a wall divisor. Since cases C) and D) are monodromy equivalent, it suffices to show that every class as in A), B), or C) is not the class of a wall divisor. We will prove this claim for each form separately and similarly.
In case A), let $b > 1$ and let $h = e + bf \in NS(A)$ be a primitive ample class. Let $e - bf$ be a generator of $h^1$ in $NS(A)$. By Lemma 6.7(2), $(-c, k h, -c) \oplus_{\perp} -d e$ is ample on $K_c(A)$ for $k \gg c \gg d > 0$. This ample divisor is orthogonal to $(0, e - bf, 0)$: therefore $(0, e - bf, 0)$ is not the class of a wall divisor.

To deal with case B), we use again Lemma 6.7. By Lemma 6.7(3) there exists an ample divisor on $X$ whose class is $(-c, ae + (a + 1)f, -c) \oplus_{\perp} -d e$ for strictly positive $a, c, d \in \mathbb{Z}$. By Lemma 6.7(1), the class $f$ is limit of classes of ample divisors, hence it is the class of a nef line bundle. In particular, since $ab - a + c + d - 1 \geq 0$, the class

$$
\Gamma := (-c, ae + (a + 1)f, -c) \oplus_{\perp} -d e + (0, (ab - a + c + d - 1)f, 0) =
$$

$$
= (-c, ae + (ab + c + d)f, -c) \oplus_{\perp} -d e = (c, a(e + bf) + (c + d)f, -c) \oplus_{\perp} -d e.
$$
is the sum of classes of an ample and a nef divisor, hence it is the class of an ample divisor on $X$. Since $(-1, 2(e - bf), -1) \oplus_{\perp} -e$ is perpendicular to $\Gamma$ we conclude that it is not the class a wall divisor.

In case C) one can argue as in case B) and show that the class

$$
\Gamma := (-c, ae + (a + 1)f, -c) \oplus_{\perp} -d e + (0, (ab - a + c - 1)f, 0) =
$$

$$
= (-c, ae + (ab + c)f, -c) \oplus_{\perp} -d e = (-c, a(e + bf) + cf, -c) \oplus_{\perp} -d e
$$
is the class of a ample divisor on $X$ perpendicular to $(-1, 2(e - bf), -1)$. \hfill \Box

As a consequence of Proposition 6.8, using of [16 Section 6] and [22], we get the main result of this section. In the statement, following the standard notation, for every hyperkähler manifold $X$ and for every $\alpha \in H^2(X, \mathbb{Z})$ we denote by $\alpha^\perp_{BX}$ the perpendicular to $\alpha \in H^2(X, \mathbb{R})$ with respect to the Beauville-Bogomolov form and we denote by $\text{div}(\alpha)$ the divisibility of $\alpha$ in the lattice $H^2(X, \mathbb{R})$ (see Definition 2.1).

**Theorem 6.9.** Let $X$ be on hyperkähler manifold of OG6 type and let the positive cone $C(X)$ of $X$ be the connected component of the cone

$$
\{ \alpha \in H^{1,1}(X, \mathbb{R}) : B_X(\alpha, \alpha) > 0 \}
$$
containing a Kähler class. Then

1. The birational Kähler cone $\overline{BK}(X)$ of $X$ is the closure in $C(X)$ of the connected component of

$$
C(X) \setminus \bigcup_{\substack{\alpha \in H^{1,1}(X, \mathbb{Z}), \\
B_X(\alpha, \alpha) = -2 \text{ or } -4, \\
\text{div}(\alpha) = 2}} \alpha^\perp_{BX}
$$
containing a Kähler class.

2. The Kähler cone $K(X)$ is the connected component of

$$
C(X) \setminus \bigcup_{\substack{\alpha \in H^{1,1}(X, \mathbb{Z}), \\
B_X(\alpha, \alpha) = -2 \text{ or } \text{div}(\alpha) = 2}} \alpha^\perp_{BX}
$$
containing a Kähler class.
Proof. (1) Let $S$ be the set of stably prime exceptional divisors on $X$. By Proposition 6.10], the closure of the birational Kähler cone is the closure of the component of $C(X) \setminus \bigcup_{D \in S} \alpha_{D(X)^-}$ containing a Kähler class. By Proposition 6.8 stably prime exceptional divisors are (up to multiples) those of divisibility 2 and squares $-2$ or $-4$, so the claim follows.

(2) Let $W$ be the set of wall divisors on $X$. Analogously, by Proposition 1.5, the Kähler cone is the connected component of $C(X) \setminus \bigcup_{D \in W} \alpha_{D(X)^-}$ containing a Kähler class. By Proposition 6.8 wall divisors are those of square $-4$ and divisibility 2 or of square $-2$ and any divisibility, therefore the claim follows. \hfill \square

Remark 6.10. Since for every hyperkähler manifold $X$ the ample cone and the movable cone can be obtained by intersecting the Kähler cone and the closure of the birational Kähler cone with $H^{1,1}(X, \mathbb{Q}) \otimes \mathbb{R}$, Theorem 6.9 also determine the ample and the movable cone of every hyperkähler manifold of $OG6$ type.

7. Lagrangian fibrations and applications

The aim of this section is to prove that, whenever a manifold of $OG6$ type has a square zero divisor, it has a rational lagrangian fibration. First, we establish the number of monodromy orbits of a square zero divisor:

Lemma 7.1. Let $l \in L := U^3 \oplus (-2)^2$ be a primitive element of square zero. Then $\text{div}(d) = 1$ and there is a single orbit for the action of $O^+(L)$.

Proof. As the discriminant group of $L$ is of two torsion, the divisibility can be either one or two. Any primitive element of divisibility 2 can be written as $2w + at + bs$ for some $w \in U^3$ and $t, s$ such that $(t, s) = (-2)^2$ and $(t, s)^\perp = U^3$, moreover if $a$ is odd $b$ is even. This means that, modulo 8, the square of such an element is congruent to either $-2$ or $-4$, which is not the case for an element of square zero. Therefore $d$ has divisibility one and, by lemma 2.6 the action of $O^+(L)$ has a single orbit. \hfill \square

Recall that, for any hyperkähler manifold $X$, if $p : X \to \mathbb{P}^n$ is a lagrangian fibration, the divisor $p^*(O(1))$ is primitive, nef and isotropic. In particular, if a divisor is induced by a lagrangian fibration on a different birational model of $X$, it will be isotropic and in the boundary of the Birational Kähler cone. The following is a converse for manifolds of $OG6$ type:

Theorem 7.2. Let $X$ be an hyperkähler manifold of $OG6$ type and let $O(D) \in \text{Pic}(X)$ be a non-trivial line bundle whose Beauville-Bogomolov square is 0. Assume that the class $[D]$ of $O(D)$ belongs to the boundary of the birational Kähler cone of $X$.

Then, there exists a smooth hyperkähler manifold $Y$ and a bimeromorphic map $\psi : Y \to X$ such that $O(D)$ induces a lagrangian fibration $p : Y \to \mathbb{P}^3$. Moreover, smooth fibres of $p$ are $(1, 2, 2)$-polarized abelian threefolds.

Proof. By the work of Matsushita Theorem 1.2, the locus inside the base of universal deformations of the pair $(X, O(D))$ where the parallel transport of $[D]$ defines a birational lagrangian fibration is either the locus where the parallel transport of $[D]$ belongs to the boundary of the birational kähler cone or empty. It follows that the statement holds for $(X, O(D))$ if it holds for a deformation $(X', O(D'))$ of the pair $(X, O(D))$: therefore the proof of the first part of our claim
only requires one example in every connected component of the space of pairs $(X, \mathcal{O}(D))$. By [17, Section 5.3 and Lemma 5.17(ii)], the number of connected components of the space of pairs $(X, \mathcal{O}(D))$ with $\mathcal{O}(D)$ in the boundary of the positive cone and $[D]$ primitive and isotropic, corresponds to twice the cardinality of $\text{Im}(f) \subset \Sigma$ for a faithful monodromy invariant $f : I(X) \to \Sigma$, where $I(X)$ is the set of all primitive isotropic elements of $H^2(X, \mathbb{Z})$. Trivially, the quotient map $I(X) \to I(X)/\text{Mon}^2(X) := \Sigma$ is one such faithful monodromy invariant. Therefore, by Lemma 7.1, $\Sigma$ is a singleton and the moduli space we are interested in is connected. By [14, Theorem 1.1], the polarization type of a general fibre is constant in every component, so to prove our claim it suffices to find one example of a lagrangian fibration where the general fibre is $(1, 2, 2)$ polarized. But such an example is well known, it suffices to take a principally polarized Abelian surface $(A, h)$, a Mukai vector $(0, 2h, 2a)$ and the Lagrangian fibration associated to it is the one induced by the fitting morphism, sending a sheaf into its support. These fibres have étale double covers which are Jacobians of genus three curves (see [37] and [24, Remark 5.1]), and have the required polarization type by [4, Corollary 12.1.5].

Remark 7.3. In particular, if we have a primitive isotropic divisor $[D] \in \text{Pic}(X)$ with $X$ of $\text{OG}_6$ type, then a birational model of $X$ has a lagrangian fibration. Indeed, by the above proposition it suffices to show that there exists another isotropic divisor on $X$ which is in the boundary of the birational Kähler cone. This follows from [10, Section 6], where Markman proves that $\text{Mon}^2(X) \cap \text{Hdg}(H^2(X))$ (the group of Hodge isometries which are monodromy operators) acts transitively on the set of exceptional chambers of the positive cone, one of which contains the Birational Kähler cone and has its same closure (moreover, every element of the closure of the positive cone is in the closure of one such exceptional chamber). Thus, either $[D]$ or $-[D]$ is in the closure of one such exceptional chamber, and a monodromy Hodge isometry can move it to the boundary of the birational Kähler cone.

As an immediate consequence of the above proposition by using [40, Theorem 4.2], we obtain that the Weak Splitting property conjectured by Beauville [3] holds when the manifold has a square zero divisor.

**Corollary 7.4.** Let $X$ be a projective hyperkähler manifold of $\text{OG}_6$ type and let $D$ be a square zero divisor on it. Let $\text{DCH}(X) \subset \text{CH}_Q(X)$ be the subalgebra generated by divisor classes. Then the restriction of the cycle class map $\text{cl}|_{\text{DCH}(X)} : \text{DCH}(X) \to H^*(X, \mathbb{Q})$ is injective.

**Proof.** The proof is straightforward: [40, Theorem 4.2] proves that the Weak Splitting property holds for all manifolds $X$ such that one of their birational model has a lagrangian fibration, and Proposition 7.2 and Remark 7.3 prove that this lagrangian fibration exists for any manifold of $\text{OG}_6$ type with a square zero divisor.

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