ON HIGHER MONOIDAL ∞-CATEGORIES

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Abstract. In this paper we introduce a notion of $O$-monoidal $\infty$-categories for a finite sequence $O^\otimes$ of $\infty$-operads, which is a generalization of the notion of higher monoidal categories in the setting of $\infty$-categories. We show that the $\infty$-category of coCartesian $O$-monoidal $\infty$-categories and right adjoint lax $O$-monoidal functors is equivalent to the opposite of the $\infty$-category of Cartesian $O_{\text{rev}}$-monoidal $\infty$-categories and left adjoint oplax $O_{\text{rev}}$-monoidal functors, where $O_{\text{rev}}$ is a sequence obtained by reversing the order of $O^\otimes$.

1. Introduction

The purpose of this paper is twofold: Firstly, we generalize the notion of duoidal $\infty$-categories and introduce higher monoidal $\infty$-categories. Secondly, we prove that the $\infty$-category of higher monoidal $\infty$-categories and right adjoint lax monoidal functors is equivalent to the opposite of the $\infty$-category of higher monoidal $\infty$-categories and left adjoint oplax monoidal functors.

A duoidal category is a category equipped with two monoidal structures in which one is (op)lax monoidal with respect to the other. The notion of duoidal category was introduced in [1] and developed further by [3, 4, 5, 6, 12]. In [14] we have introduced duoidal $\infty$-categories which are analogues of duoidal categories in the setting of $\infty$-categories. In this paper we will consider generalizations of duoidal $\infty$-categories to higher monoidal $\infty$-categories.

Higher monoidal categories was also introduced in [1, Chapter 7]. Roughly speaking, $n$-monoidal category is a category equipped with $n$ ordered monoidal structures related by interchange laws. We will define a notion of coCartesian $O$-monoidal $\infty$-categories for a finite sequence $O^\otimes$ of $\infty$-operads over perfect operator categories. This is a generalization of higher monoidal categories and it is appropriate setting to consider lax $O$-monoidal functors. In order to consider oplax monoidal functors, we also introduce a notion of Cartesian $O$-monoidal $\infty$-categories. We will show that the notion of coCartesian $O$-monoidal $\infty$-categories is equivalent to that of Cartesian $O_{\text{rev}}$-monoidal $\infty$-categories (Corollary 3.22), where $O_{\text{rev}}^\otimes$ is a sequence obtained by reversing the order of $O^\otimes$.

Furthermore, by combining these notions, we also introduce mixed $(O, P)$-monoidal $\infty$-categories to discuss bilax $(O, P)$-monoidal functors.

In [13, Proposition 5] we have shown that the left adjoint of a lax monoidal functor between monoidal $\infty$-categories is an oplax monoidal functor. Furthermore, Haugseng [3] and Hebestreit-Linskens-Nuiten [9] have independently shown that the $\infty$-category of $O$-monoidal $\infty$-categories and left adjoint oplax monoidal functors is equivalent to the opposite of the $\infty$-category of $O$-monoidal $\infty$-categories and right adjoint lax monoidal functors for any $\infty$-operad $O^\otimes$.

Our main theorem is a generalization of this equivalence to higher monoidal $\infty$-categories. We denote by $\text{Mon}^{\text{lax}, R}(\text{Cat}_\infty)$ the $\infty$-category of coCartesian $O$-monoidal $\infty$-categories and right adjoint lax $O$-monoidal functors and by $\text{Mon}^{\text{oplax}, L}(\text{Cat}_\infty)$ the $\infty$-category of Cartesian $O_{\text{rev}}$-monoidal $\infty$-categories and left adjoint oplax $O_{\text{rev}}$-monoidal functors.
Theorem 1.1 (Theorem 4.7). There exists an equivalence
\[ \text{Mon}^\text{lax,\,R}(\text{Cat}_{\infty}) \simeq \text{Mon}^\text{oplax,\,L}(\text{Cat}_{\infty})^\text{op} \]
of $\infty$-categories.

The organization of this paper is as follows: In §2 we introduce notions of $S$-(co)Cartesian fibrations and (op)lax $S$-morphisms for a finite sequence $S$ of $\infty$-categories. We also define mixed $(S, T)$-fibrations and bilax $(S, T)$-morphisms by combining $S$-coCartesian fibrations and $T$-Cartesian fibrations. In §3 we define higher monoidal $\infty$-categories and study (op)lax monoidal functors between them. We introduce notions of coCartesian $O$-monoidal $\infty$-categories and lax $O$-monoidal functors, where $O^\circ$ is a finite sequence of $\infty$-operads over perfect operator categories. We also introduce Cartesian $O$-monoidal $\infty$-categories and oplax $O$-monoidal functors. By combining these notions, we also define mixed $(O, P)$-monoidal $\infty$-categories and bilax $(O, P)$-monoidal functors. In §4 we study adjoints of (op)lax monoidal functors between higher monoidal $\infty$-categories, and prove the main theorem (Theorem 4.7).

Notation. We denote by $\text{Cat}_{\infty}$ the $\infty$-category of $\infty$-categories. For an $\infty$-category $C$, we denote by $C^\simeq$ the underlying $\infty$-groupoid. We write $\text{Map}_C(x, y)$ for the mapping space in $C$. For $\infty$-categories $C$ and $D$, we write $\text{Fun}(C, D)$ the $\infty$-categories of functors from $C$ to $D$.

For $\infty$-categories $S$ and $T$, we let $\pi_S : S \times T \to S$ be the projection. For a functor $p : X \to S \times T$ of $\infty$-categories and an object $s \in S$, we set $p_S = \pi_S \circ p$. For an object $s \in S$, we denote by $X_s$ the fiber of $p_S$ at $s$ and $p_s : X_s \to T$ the restriction of $p$ to $X_s$.

Let $S = (S_1, S_2, \ldots, S_n)$ be a finite sequence of $\infty$-categories. We write $l(S)$ for the length of $S$. For an integer $i$, we set $S_{\geq i} = (S_i, S_{i+1}, \ldots, S_n)$ and $S_{\neq i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$, etc. We also set $S^{\text{op}} = (S_1^{\text{op}}, \ldots, S_n^{\text{op}})$ and $S^{\text{rev}} = (S_n, S_{n-1}, \ldots, S_1)$. We denote by $\prod S$ the product $S_1 \times S_2 \times \cdots \times S_n$. For finite sequences $S = (S_1, \ldots, S_n)$ and $T = (T_1, \ldots, T_m)$ of $\infty$-categories, we set $[S, T] = (S_1, S_2, \ldots, S_n, T_1, \ldots, T_m)$.

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2. Iterated (co)Cartesian fibrations and mixed fibrations

In this section we introduce notions of iterated (co)Cartesian fibrations and mixed fibrations in a subcategory $Z$ of $\text{Cat}_{\infty}/Z$ and study their basic properties. In §2.1 we recall the notions of Cartesian fibrations and coCartesian fibrations in $Z$ studied in [14]. In §2.2 we generalize mixed fibrations introduced in [14] §3.2] to mixed fibrations in $Z$. In §2.3 we define $S$-coCartesian fibrations and lax $S$-morphisms, and $T$-Cartesian fibrations and oplax $T$-morphisms for finite sequences $S$ and $T$ of $\infty$-categories by iterating the constructions in §2.4. We also define mixed $(S, T)$-fibrations by combining $S$-coCartesian fibrations and $T$-Cartesian fibrations.

2.1. Cartesian fibrations and coCartesian fibrations in $Z$. In this subsection we recall the notions of Cartesian fibrations and coCartesian fibrations in subcategory of $\text{Cat}_{\infty}/Z$ studied in [14].

Let $S$ and $Z$ be $\infty$-categories. Suppose that $Z$ is a replete subcategory of $\text{Cat}_{\infty}/Z$. In [14] Definition 3.1 we introduced an $\infty$-category $\text{Fun}(S, Z)$. In this paper we call an object of $\text{Fun}(S, Z)$ a coCartesian fibration over $S$ in $Z$. We write $\text{coCart}_{/S}(Z)$ for the $\infty$-category of coCartesian fibrations over $S$ in $Z$ instead of $\text{Fun}(S, Z)$. For the convenience of readers, we will explicitly describe objects and morphisms of $\text{Fun}(S, Z)$.
Definition 2.1. A coCartesian fibration over $S$ in $\mathcal{Z}$ is a functor $p : X \to S \times Z$ of $\infty$-categories that satisfies the following conditions:

(1) The composite $p_S : X \to S$ is a coCartesian fibration and the functor $p$ takes $p_S$-coCartesian morphisms to $\pi_S$-coCartesian morphisms.
(2) For each $s \in S$, the restriction $p_s : X_s \to Z$ is an object of $\mathcal{Z}$.
(3) For each morphism $s \to s'$ in $S$, the induced functor $X_s \to X_{s'}$ over $Z$ is a morphism in $\mathcal{Z}$.

A morphism between coCartesian fibrations $p : X \to S \times Z$ and $q : Y \to S \times Z$ over $S$ in $\mathcal{Z}$ is a functor $f : X \to Y$ over $S \times Z$ that satisfies the following conditions:

(1) The functor $f$ takes $p_S$-coCartesian morphisms to $q_S$-coCartesian morphisms.
(2) For each $s \in S$, the restriction $f_s : X_s \to Y_s$ over $Z$ is a morphism in $\mathcal{Z}$.

We define $\text{coCart}_{/S}(\mathcal{Z})$ to be the subcategory of $\text{Cat}_{\infty/S \times Z}$ with coCartesian fibrations over $S$ in $\mathcal{Z}$ and morphisms between them.

By [14, Proposition 3.2], we have the following lemma.

Lemma 2.2 ([14, Proposition 3.2]). There is an equivalence

$$\text{Fun}(S, \mathcal{Z}) \simeq \text{coCart}_{/S}(\mathcal{Z})$$

of $\infty$-categories.

We also define lax morphisms between coCartesian fibrations over $S$ in $\mathcal{Z}$.

Definition 2.3. A lax morphism between coCartesian fibrations $p : X \to S \times Z$ and $q : Y \to S \times Z$ over $S$ in $\mathcal{Z}$ is a functor $f : X \to Y$ over $S \times Z$ that satisfies the following condition: For each $s \in S$, the restriction $f_s : X_s \to Y_s$ over $Z$ is a morphism in $\mathcal{Z}$.

We define $\text{coCart}_{/S}^{\text{lax}}(\mathcal{Z})$ to be the subcategory of $\text{Cat}_{\infty/S \times Z}$ with coCartesian fibrations over $S$ in $\mathcal{Z}$ and lax morphisms between them.

Since an equivalence in $\text{coCart}_{/S}^{\text{lax}}(\mathcal{Z})$ is a morphism of coCartesian fibrations over $S$ in $\mathcal{Z}$, we obtain the following lemma.

Lemma 2.4. The inclusion functor $\text{coCart}_{/S}^{\text{lax}}(\mathcal{Z}) \to \text{coCart}_{/S}(\mathcal{Z})$ induces an equivalence

$$\text{coCart}_{/S}^{\text{lax}}(\mathcal{Z}) \simeq \text{coCart}_{/S}(\mathcal{Z})$$

of the underlying $\infty$-groupoids.

Dually, we call an object of $\text{Fun}^{\prime}(S, \mathcal{Z})$ in [14, Remark 3.3] a Cartesian fibration over $S$ in $\mathcal{Z}$, and write $\text{Cart}_{/S}(\mathcal{Z})$ for the $\infty$-category of Cartesian fibrations over $S$ in $\mathcal{Z}$ instead of $\text{Fun}^{\prime}(S, \mathcal{Z})$.

Definition 2.5. A Cartesian fibration over $S$ in $\mathcal{Z}$ is a functor $p : X \to S \times Z$ of $\infty$-categories that satisfies the following conditions:

(1) The composite $p_S : X \to S$ is a Cartesian fibration and the functor $p$ takes $p_S$-Cartesian morphisms to $\pi_S$-Cartesian morphisms.
(2) For each $s \in S$, the restriction $p_s : X_s \to Z$ is an object of $\mathcal{Z}$.
(3) For each morphism $s' \to s$ in $S$, the induced functor $X_s \to X_{s'}$ over $Z$ is a morphism in $\mathcal{Z}$.
A morphism between Cartesian fibrations \( p : X \to S \times Z \) and \( q : Y \to S \times Z \) over \( S \) in \( Z \) is a functor \( f : X \to Y \) over \( S \times Z \) that satisfies the following conditions:

1. The functor \( f \) takes \((\pi_S \circ p)\)-Cartesian morphisms to \((\pi_S \circ q)\)-Cartesian morphisms.
2. For each \( s \in S \), the induced map \( f_s : X_s \to Y_s \) over \( Z \) is a morphism in \( Z \).

We define \( \text{Cart}_{/S}(Z) \) to be the subcategory of \( \text{Cat}_{\infty/S \times Z} \) with Cartesian fibrations over \( S \) in \( Z \) and morphisms between them.

**Definition 2.6.** An oplax morphism between Cartesian fibrations \( p : X \to S \times Z \) and \( q : Y \to S \times Z \) over \( S \) in \( Z \) is a functor \( f : X \to Y \) over \( S \times Z \) that satisfies the following condition: For each \( s \in S \), the restriction \( f_s : X_s \to Y_s \) over \( Z \) is a morphism in \( Z \).

We define \( \text{Cart}_{/S}^{\text{oplax}}(Z) \) to be the subcategory of \( \text{Cat}_{\infty/S \times Z} \) with Cartesian fibrations over \( S \) in \( Z \) and oplax morphisms between them.

In the same way as the case of coCartesian fibrations, we obtain the following two lemmas.

**Lemma 2.7** ([14, Remark 3.3]). There is an equivalence
\[
\text{Fun}(S^{\text{op}}, Z) \simeq \text{Cart}_{/S}(Z)
\]
of \( \infty \)-categories.

**Lemma 2.8.** The inclusion functor \( \text{Cart}_{/S}^{\text{oplax}}(Z) \to \text{Cart}_{/S}(Z) \) induces an equivalence
\[
\text{Cart}_{/S}^{\text{oplax}}(Z) \simeq \overset{\simeq}{\text{Cart}_{/S}(Z)}
\]
of the underlying \( \infty \)-groupoids.

**Remark 2.9.** Notice that \( \text{coCart}_{/S}^{\text{ax}}(Z) \) and \( \text{Cart}_{/S}^{\text{oplax}}(Z) \) are replete subcategory of \( \text{Cat}_{\infty/S \times Z} \).
Thus, we can iterate the constructions \( \text{coCart}_{/S}^{\text{ax}}(-) \) and \( \text{Cart}_{/S}^{\text{oplax}}(-) \).

We need the following lemma in §3.2.

**Lemma 2.10.** If the inclusion functor \( \hookrightarrow : \text{Cat}_{\infty/S} \to \text{CoCat}_{\infty/S} \) create finite products, then the inclusion functors \( \text{coCart}_{/S}^{\text{ax}}(Z) \to \text{Cat}_{\infty/S \times Z} \) and \( \text{Cart}_{/S}^{\text{oplax}}(Z) \hookrightarrow \text{Cat}_{\infty/S^{\text{op}} \times Z} \) also create finite products.

**Proof.** We will show that \( \text{coCart}_{/S}^{\text{ax}}(Z) \to \text{Cat}_{\infty/S \times Z} \) creates finite products. The other case can be proved similarly.

It is easy to see that a final object of \( \text{Cat}_{\infty/S} \) is also a final object of \( \text{coCart}_{/S}^{\text{ax}}(Z) \). Let \( p : X \to S \times Z \) and \( q : Y \to S \times Z \) be objects of \( \text{coCart}_{/S}^{\text{ax}}(Z) \). By Lemma 2.2, we regard \( p \) and \( q \) as objects of \( \text{Fun}(S, Z) \). By the assumption on \( Z \), finite products of \( \text{Fun}(S, Z) \) are created in \( \text{Fun}(S, \text{Cat}_{\infty/Z}) \). Thus, \( X \times_{S \times Z} Y \to S \times Z \) is a product of \( p \) and \( q \) in \( \text{coCart}_{/S}(Z) \). We can easily verify that \( X \times_{S \times Z} Y \to S \times Z \) is a product of \( p \) and \( q \) in \( \text{coCart}_{/S}^{\text{ax}}(Z) \). \( \square \)

### 2.2 Mixed fibrations in \( Z \)

In [14] §3.2 we introduced mixed fibrations. In this subsection we generalize them and introduce mixed fibrations in \( Z \).

**Definition 2.11.** Let \( S \) and \( T \) be \( \infty \)-categories. A mixed fibration over \((S, T)\) in \( Z \) is a functor \( p : X \to S \times T \times Z \) of \( \infty \)-categories that satisfies the following conditions:

1. The composite \( p_S : X \to S \) is a coCartesian fibration and the functor \( p \) takes \( p_S \)-coCartesian morphisms to \( \pi_S \)-coCartesian morphisms.
A bilax morphism between mixed fibrations

Definition 2.13. A bilax morphism between mixed fibrations $p : X \to T$ is a Cartesian fibration and the functor $p$ takes $p_T$-Cartesian morphisms to $p_T$-Cartesian morphisms.

Remark 2.12. For each $(s, t) \in S \times T$, the restriction $p_{(s,t)} : X_{(s,t)} \to Z$ is an object of $Z$.

Remark 2.14. Finally, we show that the $\infty$-category of mixed fibrations over $(S, T)$ can be described in terms of the constructions of coCart$^\infty_{/S}(-)$ and Car$^\infty_{/T}(-)$.

We define bilax morphisms between mixed fibrations.

Definition 2.13. A bilax morphism between mixed fibrations $p : X \to S \times T \times Z$ and $q : Y \to S \times T \times Z$ is a functor $f : X \to Y$ over $S \times T \times Z$ that satisfies the following conditions:

1. The functor $f$ takes $p_S$-coCartesian morphisms to $q_S$-coCartesian morphisms.
2. The functor $f$ takes $p_T$-Cartesian morphisms to $q_T$-Cartesian morphisms.
3. For each $(s, t) \in S \times T$, the restriction $f_{(s,t)} : X_{(s,t)} \to Y_{(s,t)}$ over $Z$ is a morphism in $Z$.

We define

$$\text{Mix}_{/(S,T)}(Z)$$

to be the subcategory of $\text{Cat}_{\infty/S \times T \times Z}$ with mixed fibrations over $(S,T)$ in $Z$ and morphisms between them.

Remark 2.12. If $p : X \to S \times T \times Z$ is a mixed fibration over $(S,T)$ in $Z$, then the composite $p_{S \times T} : X \to S \times T$ is a mixed fibration over $(S,T)$ in the sense of [14] Definition 3.15.

We also define bilax morphisms between mixed fibrations.

Definition 2.13. A bilax morphism between mixed fibrations $p : X \to S \times T \times Z$ and $q : Y \to S \times T \times Z$ over $(S,T)$ in $Z$ is a functor $f : X \to Y$ over $S \times T \times Z$ that satisfies the following condition: For each $(s, t) \in S \times T$, the restriction $f_{(s,t)} : X_{(s,t)} \to Y_{(s,t)}$ over $Z$ is a morphism in $Z$.

We define an $\infty$-category

$$\text{Mix}^\text{bilax}_{/(S,T)}(Z)$$

to be the subcategory of $\text{Cat}_{\infty/S \times T \times Z}$ with mixed fibrations over $(S,T)$ in $Z$ and bilax morphisms between them.

Remark 2.14. When $Z = \text{Cat}_{\infty/[0]} \simeq \text{Cat}_{\infty}$, there is an equivalence

$$\text{Mix}^\text{bilax}_{/(S,T)}(Z) \simeq \text{Mfib}_{/(S,T)}$$

of $\infty$-categories, where the right hand side is the $\infty$-category of mixed fibrations over $(S,T)$ defined in [14] Definition 3.15.

We easily obtain the following two lemmas.

Lemma 2.15. There is an equivalence

$$\text{Mix}_{/(S,T)}(Z) \simeq \text{Fun}(S \times T^{\text{op}}, Z)$$

of $\infty$-categories.

Lemma 2.16. The inclusion functor $\text{Mix}^\text{bilax}_{/(S,T)}(Z) \to \text{Mix}_{/(S,T)}(Z)$ induces an equivalence

$$\text{Mix}^\text{bilax}_{/(S,T)}(Z) \xrightarrow{\simeq} \text{Mix}_{/(S,T)}(Z)$$

of the underlying $\infty$-groupoids.
Theorem 2.17. There are equivalences
\[
\text{coCart}^{\text{lax}}_{/S}(\text{Cart}^{\text{oplax}}_{/T}(Z)) \simeq \text{Mix}^{\text{bilax}}_{/(S,T)}(Z) \simeq \text{Cart}^{\text{oplax}}(\text{coCart}^{\text{lax}}_{/S}(Z))
\]
of ∞-categories.

In order to prove Theorem 2.17 we need the following lemmas.

Lemma 2.18. Let \( p : X \to S \times T \times Z \) be a mixed fibration over \((S,T)\) in \(Z\). For each \( s \in S \), the restriction \( p_s : X_s \to T \) is a Cartesian fibration over \( T \) in \( Z \).

Proof. It suffices to show that \( p_s : X_s \to T \) is a Cartesian fibration and \( p_s : X_s \to T \times Z \) preserves Cartesian morphisms for each \( s \in S \). This follows from [14 Remark 3.18]. □

Lemma 2.19. If \( p : X \to S \times T \times Z \) is an object of \( \text{Mix}^{\text{bilax}}_{/(S,T)}(Z) \), then \( p \) is a mixed fibration over \((S,T)\) in \(Z\).

Proof. It suffices to show that \( p_T : X \to T \) is a Cartesian fibration and \( p \) preserves Cartesian morphisms. We can prove this in the same way as the proof of [13 Proposition 3.25]. □

Proof of Theorem 2.17. We shall show that \( \text{Mix}^{\text{bilax}}_{/(S,T)}(Z) \) is equivalent to \( \text{coCart}^{\text{lax}}_{/S}(\text{Cart}^{\text{oplax}}_{/T}(Z)) \).

By the symmetry of the definition of mixed fibrations, we can prove the other equivalence similarly.

We can identify the objects of \( \text{Mix}^{\text{bilax}}_{/(S,T)}(Z) \) with those of \( \text{coCart}^{\text{lax}}_{/S}(\text{Cart}^{\text{oplax}}_{/T})(Z) \) by Lemmas 2.18 and 2.19. Thus, it suffices to show that we can also identify the morphisms in \( \text{Mix}^{\text{bilax}}_{/(S,T)}(Z) \) with those in \( \text{coCart}^{\text{lax}}_{/S}(\text{Cart}^{\text{oplax}}_{/T})(Z) \).

Let \( f : X \to Y \) be a functor over \( S \times T \times Z \), where \( p : X \to S \times T \times Z \) and \( q : Y \to S \times T \times Z \) are mixed fibrations over \((S,T)\) in \(Z\). The functor \( f \) is a morphism of mixed fibrations if and only if \( f_{s,t} : X_{(s,t)} \to Y_{(s,t)} \) over \( Z \) is a morphism in \( Z \) for each \((s,t) \in S \times T \). On the other hand, \( f \) is a morphism in \( \text{coCart}^{\text{lax}}_{/S}(\text{Cart}^{\text{oplax}}_{/T})(Z) \) if and only if \( f_s : X_s \to Y_s \) over \( T \times Z \) is a morphism in \( \text{Cart}^{\text{oplax}}_{/T}(Z) \) for each \( s \in S \). This is equivalent to the condition on \( f \) being a morphism of mixed fibrations. This completes the proof. □

2.3. Iterated (co)Cartesian fibrations in \(Z\). In this subsection we will define \(S\)-coCartesian fibrations and lax \(S\)-morphisms, and \(T\)-Cartesian fibrations and oplax \(T\)-morphisms for finite sequences \(S\) and \(T\) of ∞-categories by iterating the constructions in §2.1. We also define mixed \((S,T)\)-fibrations by combining \(S\)-coCartesian fibrations and \(T\)-Cartesian fibrations.

Definition 2.20. Let \( S = (S_1, S_2, \ldots, S_n) \) be a finite sequence of ∞-categories. We define an ∞-category
\[
\text{coCart}^{\text{lax}}_{/S}(Z)
\]
by induction on \( l(S) \) as follows: When \( l(S) = 0 \), we set \( \text{coCart}^{\text{lax}}_{/S}(Z) = Z \). When \( l(S) > 0 \), we define
\[
\text{coCart}^{\text{lax}}_{/S}(Z) = \text{coCart}^{\text{lax}}_{/S_1}(\text{coCart}^{\text{lax}}_{/S_2}(Z)).
\]
We call \( \text{coCart}^{\text{lax}}_{/S}(Z) \) the ∞-category of \(S\)-coCartesian fibrations and lax \(S\)-morphisms in \(Z\).

Dually, we define an ∞-category of \(T\)-Cartesian fibration and \(T\)-oplax morphisms in \(Z\).

Definition 2.21. Let \( T = (T_1, T_2, \ldots, T_n) \) be a finite sequence of ∞-categories. We define an ∞-category
\[
\text{Cart}^{\text{oplax}}_{/T}(Z)
\]
by induction on \( l(T) \) as follows: When \( l(T) = 0 \), we set \( \operatorname{Cart}^{\oplax}_{/T}(Z) = Z \). When \( l(T) > 0 \), we define

\[
\operatorname{Cart}^{\oplax}_{/T}(Z) = \operatorname{Cart}^{\oplax}_{/T_1}(\operatorname{Cart}^{\oplax}_{/T_2}(Z)).
\]

We call \( \operatorname{Cart}^{\oplax}_{/T}(Z) \) the \( \infty \)-category of \( T \)-Cartesian fibrations and oplax \( T \)-morphisms in \( Z \).

**Remark 2.22.** The notions of \( S \)-coCartesian fibrations and \( T \)-Cartesian fibrations are related to Gray fibrations and Gray tensor products. See \([8, 9]\) of Definition 2.23. We define

\[
\operatorname{Corollary 2.26.} \text{There is an equivalence of } \infty \text{-categories. By Corollary 2.26, we can see that a coCartesian } \infty \text{-category determines a Cartesian } \infty \text{-category, and vice versa. For a coCartesian or Cartesian } S \text{-fibration } p : X \to \prod S, \text{ we denote by } p^\vee : X^\vee \to \prod S^\rev \text{ the corresponding Cartesian or coCartesian } S^\rev \text{-fibration, and we call } p^\vee \text{ the dual fibration to } p.
\]

**Definition 2.23.** We define

\[
\operatorname{Mix}^{\oplax}_{/(S,T)}(Z)
\]

to be the \( \infty \)-category \( \operatorname{coCart}^{\oplax}_{/(S,T)}(\operatorname{Cart}^{\oplax}_{/T}(Z)) \) and call it the \( \infty \)-category of mixed \( (S,T) \)-fibrations and bilax \( (S,T) \)-morphisms.

**Remark 2.24.** By Theorem 2.17 there is an equivalence

\[
\operatorname{Mix}^{\oplax}_{/(S,T)}(Z) \simeq \operatorname{Cart}^{\oplax}_{/T}(\operatorname{coCart}^{\oplax}_{/S}(Z))
\]

of \( \infty \)-categories.

Let \( U \) be an \( \infty \)-category. We describe the mapping space \( \operatorname{Map}_{\operatorname{Cat}_{\infty}}(U, \operatorname{Mix}^{\oplax}_{/(S,T)}(Z)) \) in terms of mixed fibrations in \( Z \).

**Theorem 2.25.** There are natural equivalences

\[
\operatorname{Mix}^{\oplax}_{/([U,S],T)}(Z)^\simeq \operatorname{Map}_{\operatorname{Cat}_{\infty}}(U, \operatorname{Mix}^{\oplax}_{/(S,T)}(Z)) \simeq \operatorname{Mix}^{\oplax}_{/(S,[U,T]^{\op})}(Z)^\simeq
\]

of \( \infty \)-groupoids.

**Proof.** We shall show that \( \operatorname{Map}_{\operatorname{Cat}_{\infty}}(U, \operatorname{Mix}^{\oplax}_{/(S,T)}(Z)) \simeq \operatorname{Mix}^{\oplax}_{/(U,[S],T)}(Z)^\simeq \). The other equivalence can be proved similarly.

By Lemmas 2.22 and 2.3, we have an equivalence

\[
\operatorname{Map}_{\operatorname{Cat}_{\infty}}(U, \operatorname{Mix}^{\oplax}_{/(S,T)}(Z)) \simeq \operatorname{coCart}^{\oplax}_{/U}(\operatorname{Mix}^{\oplax}_{/(S,T)}(Z))^{\simeq}
\]

of \( \infty \)-groupoids. By definition, there is an equivalence

\[
\operatorname{coCart}^{\oplax}_{/U}(\operatorname{Mix}^{\oplax}_{/(S,T)}(\operatorname{Cat}_{\infty})) \simeq \operatorname{Mix}^{\oplax}_{/(U,[S],T)}(\operatorname{Cat}_{\infty})
\]

of \( \infty \)-categories. Combining these equivalences, we obtain the desired equivalence. \( \square \)

**Corollary 2.26.** There is an equivalence

\[
\operatorname{coCart}^{\oplax}_{/S}(Z)^\simeq \operatorname{Cart}^{\oplax}_{/S^\rev}(Z)^\simeq
\]

of the underlying \( \infty \)-groupoids.

**Definition 2.27.** By Corollary 2.26 we can see that a coCartesian \( S \)-fibration canonically determines a Cartesian \( S^\rev \)-fibration, and vice versa. For a coCartesian or Cartesian \( S \)-fibration \( p : X \to \prod S \), we denote by \( p^\vee : X^\vee \to \prod S^\rev \) the corresponding Cartesian or coCartesian \( S^\rev \)-fibration, and we call \( p^\vee \) the dual fibration to \( p \).
3. Higher monoidal ∞-categories

In this section we introduce higher monoidal ∞-categories and study (op)lax monoidal functors between them. In §3.1 we define O-monoidal ∞-categories for an ∞-operad \( O^\otimes \) over a perfect operator category. In §3.2 we generalize the notion of duoidal ∞-categories to that of O-monoidal ∞-categories, where \( O^\otimes \) is a finite sequence of ∞-operads over perfect operator categories. We also define and study (op)lax O-monoidal functors between them.

3.1. O-monoidal ∞-categories over perfect operator categories. In this subsection we define O-monoidal ∞-categories for an ∞-operad \( O^\otimes \) over a perfect operator category, and introduce lax and oplax O-monoidal functors between O-monoidal ∞-categories. We also consider mixed \((O, P)\)-monoidal ∞-categories and mixed \((O, P)\)-monoidal functors.

First, we briefly recall the notion of ∞-operads over a perfect operator category introduced in [2]. Let \( \Phi \) be a prefect operator category in the sense of [2] Definitions 1.2 and 4.6. Associated to \( \Phi \), we have the Leinster category \( \Lambda(\Phi) \) equipped with collections of inert morphisms and active morphisms. According to [2] Definition 7.8, an ∞-operad over \( \Phi \) is a functor \( p : O^\otimes \to \Lambda(\Phi) \) of ∞-categories satisfying the following conditions:

1. For every inert morphism \( \phi : I \to J \) of \( \Lambda(\Phi) \) and every object \( x \in O^\otimes_I \), there is a p-coCartesian morphism \( x \to y \) in \( O^\otimes \) covering \( \phi \).
2. For any objects \( I, J \in \Lambda(\Phi) \), any objects \( x \in O^\otimes_I \) and \( y \in O^\otimes_J \), any morphism \( \phi : I \to J \) of \( \Lambda(\Phi) \), and any p-coCartesian morphisms \( \{ y \to y_j \mid j \in |J| \} \) lying over the inert morphisms \( \{ \rho_j : J \to \{ j \} \mid j \in |J| \} \), the induced map

\[
\text{Map}^\phi_{O^\otimes}(x, y) \to \prod_{j \in |J|} \text{Map}^{\rho_j}_{O^\otimes}(x, y_j)
\]

is an equivalence.
3. For any object \( I \in \Lambda(\Phi) \), the p-coCartesian morphisms lying over the inert morphisms \( \{ I \to \{ i \} \mid i \in |I| \} \) together induce an equivalence

\[
O^\otimes_I \to \prod_{i \in |I|} O^\otimes_{\{ i \}}.
\]

For an ∞-operad \( p : O^\otimes \to \Lambda(\Phi) \) over \( \Phi \), a morphism of \( O^\otimes \) is said to be inert if it is a p-coCartesian morphism over an inert morphism, and active if it covers an active morphism of \( \Lambda(\Phi) \). A morphism of ∞-operads over \( \Phi \) between \( O^\otimes \to \Lambda(\Phi) \) and \( P^\otimes \to \Lambda(\Phi) \) is a functor \( f : O^\otimes \to P^\otimes \) over \( \Lambda(\Phi) \) that preserves inert morphisms.

Example 3.1. Let \( F \) be the category of finite sets. By [2] Example 4.9.3, \( F \) is a perfect operator category. By [2] Example 6.5, \( \Lambda(F) \) is equivalent to the category \( \text{Fin}_* \) of pointed finite sets. By [2] Example 7.9, the notion of ∞-operads over \( F \) coincides with that of Lurie’s ∞-operads in [11] Chapter 2.

Example 3.2. Let \( O \) be the category of ordered finite sets. By [2] Example 4.9.2, \( O \) is a perfect operator category. By [2] Example 6.6, \( \Lambda(O) \) is equivalent to \( \Delta^\text{op} \). The notion of ∞-operads over \( O \) coincides with that of non-symmetric ∞-operads in [7] §3.

For a perfect operator category \( \Phi \), we denote by \( \text{Op}_\Phi \) the ∞-category of ∞-operads over \( \Phi \). By definition, a perfect operator category \( \Phi \) has a final object \( * \) (see [2] Definition 1.2.1 for the definition of operator categories). For an ∞-operad \( O^\otimes \to \Lambda(\Phi) \), we denote by \( O \) the fiber \( O^\otimes_* \) at \( * \in \Lambda(\Phi) \) and say that it is the ∞-category of colors of \( O^\otimes \).

We define a coCartesian \( O \)-monoidal ∞-category for an ∞-operad \( O^\otimes \) over \( \Phi \).
Definition 3.3. Let \( C^\otimes \to O^\otimes \) be a morphism of \( \infty \)-operads over \( \Phi \). We say that \( C \) is a coCartesian \( O \)-monoidal \( \infty \)-category if the functor \( C^\otimes \to O^\otimes \) is a coCartesian fibration. When it is clear from the context, we simply say that it is an \( O \)-monoidal \( \infty \)-category.

Let \( C^\otimes \to O^\otimes \) be a coCartesian \( O \)-monoidal \( \infty \)-category. For each color \( x \in O \), we write \( C_x \) for \( C^\otimes \) and call it the underlying \( \infty \)-category of \( C^\otimes \) over \( x \).

By the same proof of \([11]\) Proposition 2.1.2.12], we have the following lemma.

Lemma 3.4. Let \( p : O^\otimes \to \Lambda(\Phi) \) be an \( \infty \)-operad over \( \Phi \), and let \( f : C^\otimes \to O^\otimes \) be a coCartesian fibration. Then the following conditions are equivalent:

(a) The composite \( q : C^\otimes \xrightarrow{1} O^\otimes \xrightarrow{p} \Lambda(\Phi) \) is an \( \infty \)-operad over \( \Phi \).

(b) For every \( x \in O^\otimes \), the \( p \)-coCartesian morphisms \( \{ x \to x_i \mid i \in |I| \} \) covering the inert morphisms \( \{ I \to \{i\} \mid i \in |I| \} \) induce an equivalence \( C^\otimes_x \to \prod_{i \in |I|} C_{x_i} \) of \( \infty \)-categories.

A coCartesian \( O \)-monoidal \( \infty \)-category \( C \) is equipped with multiplications indexed by active morphisms in \( O^\otimes \) as in \([11]\) Remark 2.1.2.16]. Let \( x, y \in O^\otimes \) with \( p(x) = I \) and \( p(y) = 1 \). We have a unique active morphism \( a : I \to 1 \) in \( \Lambda(\Phi) \). Let \( \{ x \to x_i \mid i \in |I| \} \) be inert morphisms of \( O^\otimes \) covering the inert morphisms \( \{ I \to \{i\} \mid i \in |I| \} \). For every \( \theta \in \Map_{O^\otimes}(x, y) \), we have a multiplication map

\[ \otimes_{\theta} : \prod_{i \in |I|} C_{x_i} \xrightarrow{\sim} C^\otimes_x \xrightarrow{\theta} C_y, \]

where \( \theta \) is a functor induced by \( \theta \) by using the coCartesian fibration \( C^\otimes \to O^\otimes \).

We denote by \( \Op_{\infty/O^\otimes}(\Phi/O^\otimes) \) the \( \infty \)-category of \( \infty \)-operads over \( O^\otimes \).

Definition 3.5. A lax \( O \)-monoidal functor between coCartesian \( O \)-monoidal \( \infty \)-categories is a morphism in \( \Op_{\infty/O^\otimes}(\Phi/O^\otimes) \). Furthermore, if it preserves coCartesian morphisms, then we say that it is a (strong) \( O \)-monoidal functor.

Definition 3.6. We define an \( \infty \)-category

\[ \Mon^\text{lax}_O(Cat_{\infty}) \]

to be the subcategory of \( \text{coCart}^\text{lax}_{\Phi/O^\otimes}(\Cat_{\infty}) \) with coCartesian \( O \)-monoidal \( \infty \)-categories as objects and lax \( O \)-monoidal functors as morphisms.

We write

\[ \Mon^\text{lax}_O(Cat_{\infty}) \]

for the wide subcategory of \( \Mon^\text{lax}_O(Cat_{\infty}) \) with (strong) \( O \)-monoidal functors as morphisms.

Let \( \mathcal{X} \) be an \( \infty \)-category with finite products. Let \( p : O^\otimes \to \Lambda(\Phi) \) be an \( \infty \)-operad over a perfect operator category \( \Phi \). We say that a functor \( F : O^\otimes \to \mathcal{X} \) is an \( O \)-monoid object of \( \mathcal{X} \) if the functor

\[ F(x) \to \prod_{i \in |I|} F(x_i) \]

is an equivalence in \( \mathcal{X} \) for any \( x \in O^\otimes \), where \( p(x) = I \) and \( \{ x \to x_i \mid i \in |I| \} \) are \( p \)-coCartesian morphisms lying over inert morphisms \( \{ I \to \{i\} \mid i \in |I| \} \). We denote by

\[ \Mon_O(\mathcal{X}) \]

the full subcategory of \( \text{Fun}(O^\otimes, \mathcal{X}) \) spanned by \( O \)-monoid objects.

To a coCartesian \( O \)-monoidal \( \infty \)-category \( C \), by the straightening functor of coCartesian fibrations \([11] \) §3.2], we can associate a functor \( C : O^\otimes \to \Cat_{\infty} \), which is an \( O \)-monoid object in \( \Cat_{\infty} \).

Furthermore, by Lemma 3.4 there is an equivalence

\[ \Mon_O(\Cat_{\infty}) \simeq \Mon_\Phi(Cat_{\infty}) \]
of $\infty$-categories.

Let $C : \mathcal{O} \to \text{Cat}_{\infty}$ be an $\mathcal{O}$-monoid object corresponding to a coCartesian $\mathcal{O}$-monoidal $\infty$-category $C^\circ \to \mathcal{O}^\circ$. By the unstraightening functor of Cartesian fibrations [10 §3.2], we obtain a Cartesian fibration $(C^\circ)^{\vee} \to (\mathcal{O}^\circ)^{\text{op}}$. We call it a Cartesian $\mathcal{O}$-monoidal $\infty$-category.

We write $C^{\vee}$ for $(C^\circ)^{\vee}$. For a color $x \in \mathcal{O}$, we write $C^\vee_x$ for $(C^\circ)^{\vee}_x$ and call it the underlying $\infty$-category of $(C^\circ)^{\vee}$ over $x$. Note that there are equivalences

$$C_x \simeq C(x) \simeq C^\vee_x$$

for any $x \in \mathcal{O}$.

We say that a morphism of $(C^\circ)^{\vee}$ is inert if it is a Cartesian morphism over an inert morphism of $(\mathcal{O}^\circ)^{\text{op}}$.

**Definition 3.7.** Let $(C^\circ)^{\vee} \to (\mathcal{O}^\circ)^{\text{op}}$ and $(D^\circ)^{\vee} \to (\mathcal{O}^\circ)^{\text{op}}$ be Cartesian $\mathcal{O}$-monoidal $\infty$-categories. An oplax $\mathcal{O}$-monoidal functor from $C^{\vee}$ to $D^{\vee}$ is a functor $(C^\circ)^{\vee} \to (D^\circ)^{\vee}$ over $(\mathcal{O}^\circ)^{\text{op}}$ which preserves inert morphisms.

We define an $\infty$-category

$$\text{Mon}^\text{oplax}_\mathcal{O}(\text{Cat}_{\infty})$$

to be the subcategory of $\text{Cat}^\text{oplax}_{/\mathcal{O}}(\text{Cat}_{\infty})$ with Cartesian $\mathcal{O}$-monoidal $\infty$-categories as objects and oplax $\mathcal{O}$-monoidal functors as morphisms.

Let $\mathcal{Z}$ be a replete subcategory of $\text{Cat}_{\infty}/Z$. We assume that the inclusion functor $\mathcal{Z} \hookrightarrow \text{Cat}_{\infty}/Z$ creates finite products. As in [13 §3.1], we will introduce $\infty$-categories $\text{Mon}^\text{lax}_\mathcal{O}(\mathcal{Z})$ and $\text{Mon}^\text{oplax}_\mathcal{O}(\mathcal{Z})$ by generalizing $\text{Mon}^\text{lax}_\mathcal{O}(\text{Cat}_{\infty})$ and $\text{Mon}^\text{oplax}_\mathcal{O}(\text{Cat}_{\infty})$.

**Definition 3.8.** We define an $\infty$-category

$$\text{Mon}^\text{lax}_\mathcal{O}(\mathcal{Z})$$

to be a subcategory of $\text{coCart}^\text{lax}_{/\mathcal{O}}(\mathcal{Z})$ as follows. An object of $\text{Mon}^\text{lax}_\mathcal{O}(\mathcal{Z})$ is an object $p : C^\circ \to \mathcal{O}^\circ \times Z$ of $\text{coCart}^\text{lax}_{/\mathcal{O}}(\mathcal{Z})$ that satisfies the following condition: For each $x \in \mathcal{O}^\circ$, the Segal morphism

$$C^\circ_x \xrightarrow{\simeq} \prod_{\mathcal{I}} Z C^\circ_{x,i},$$

is an equivalence in $\mathcal{Z}$, where the right hand side is a product in $\text{Cat}_{\infty}/Z$.

A morphism of $\text{Mon}^\text{lax}_\mathcal{O}(\mathcal{Z})$ between objects $p : C^\circ \to \mathcal{O}^\circ \times Z$ and $q : D^\circ \to \mathcal{O}^\circ \times Z$ is a morphism $f : C^\circ \to D^\circ$ in $\text{coCart}^\text{lax}_{/\mathcal{O}^\circ}(\mathcal{Z})$ that takes $p\mathcal{O}^\circ$-coCartesian morphisms over inert morphisms of $\mathcal{O}^\circ$ to $q\mathcal{O}^\circ$-coCartesian morphisms.

Dually, we define an $\infty$-category

$$\text{Mon}^\text{oplax}_\mathcal{O}(\mathcal{Z})$$

to be the subcategory of $\text{Cart}^\text{oplax}_{/\mathcal{O}}(\mathcal{Z})$ as follows. An object of $\text{Mon}^\text{oplax}_\mathcal{O}(\mathcal{Z})$ is an object $p : C^\circ \to (\mathcal{O}^\circ)^{\text{op}} \times Z$ of $\text{Cart}^\text{oplax}_{/\mathcal{O}}(\mathcal{Z})$ that satisfies the following condition: For each $y \in (\mathcal{O}^\circ)^{\text{op}}$, the Segal morphism

$$C^\circ_y \xrightarrow{\simeq} \prod_{\mathcal{I}} Z C^\circ_{y,i},$$

is an equivalence in $\mathcal{Z}$, where the right hand side is a product in $\text{Cat}_{\infty}/Z$.

A morphism of $\text{Mon}^\text{oplax}_\mathcal{O}(\mathcal{Z})$ between objects $p : C^\circ \to (\mathcal{O}^\circ)^{\text{op}} \times Z$ and $q : D^\circ \to (\mathcal{O}^\circ)^{\text{op}} \times Z$ is a morphism $f : C^\circ \to D^\circ$ in $\text{Cart}^\text{oplax}_{/\mathcal{O}}(\mathcal{Z})$ that takes $p_{\mathcal{O}}\text{op}$-Cartesian morphisms over inert morphisms of $(\mathcal{O}^\circ)^{\text{op}}$ to $q_{\mathcal{O}}\text{op}$-Cartesian morphisms.
We will show that the underlying ∞-groupoids of \( \text{Mon}^{\text{hax}}_\infty(Z) \) and \( \text{Mon}^{\text{op lax}}_\infty(Z) \) are equivalent to that of \( \text{Mon}_\infty(Z) \).

**Lemma 3.9.** The inclusion functors induce equivalences

\[
\text{Mon}^{\text{hax}}_\infty(Z) \cong \text{Mon}_\infty(Z) \cong \text{Mon}^{\text{op lax}}_\infty(Z)
\]

of the underlying ∞-groupoids.

**Proof.** By Lemma 2.10, the inclusion functor \( \text{coCart}^{\text{hax}}_\Theta(Z) \rightarrow \text{coCart}_\Theta(Z) \) induces an equivalence \( \text{coCart}^{\text{hax}}_\Theta(Z) \cong \text{coCart}_\Theta(Z) \) of the underlying ∞-groupoids. This induces an equivalence between \( (\text{Mon}^{\text{hax}}_\infty(Z))^\simeq \) and \( (\text{Mon}_\infty(Z))^\simeq \). The other equivalence can be proved similarly by using Lemma 2.8.

In [13, Definition 4.18] we introduced the ∞-category \( \text{Duo}^{\text{bilax}}_\infty \) of duoidal ∞-categories and bilax monoidal functors. By [13, Theorem 4.20 and Remark 4.21], we showed that there are equivalences

\[
\text{Mon}^{\text{hax}}_\infty(\text{Mon}^{\text{op lax}}_\infty(\text{Cat}_\infty)) \cong \text{Duo}^{\text{bilax}}_\infty \cong \text{Mon}^{\text{op lax}}_\infty(\text{Mon}^{\text{hax}}_\infty(\text{Cat}_\infty))
\]

of ∞-categories. In the following of this subsection we will generalize these equivalences.

First, we prove the following lemma, which guarantees that we can iterate the constructions \( \text{Mon}^{\text{hax}}_\infty(-) \) and \( \text{Mon}^{\text{op lax}}_\infty(-) \).

**Lemma 3.10.** The ∞-category \( \text{Mon}^{\text{hax}}_\infty(Z) \) is a replete subcategory of \( \text{Cat}_\infty/\Theta \), and the inclusion functor \( \text{Mon}^{\text{hax}}_\infty(Z) \hookrightarrow \text{Cat}_\infty/\Theta \) creates finite products. Similarly, \( \text{Mon}^{\text{op lax}}_\infty(Z) \) is a replete subcategory of \( \text{Cat}_\infty/\Theta \), and the inclusion functor \( \text{Mon}^{\text{op lax}}_\infty(Z) \hookrightarrow \text{Cat}_\infty/\Theta \) creates finite products.

**Proof.** By Remark 2.9, we can easily see that \( \text{Mon}^{\text{hax}}_\infty(Z) \) and \( \text{Mon}^{\text{op lax}}_\infty(Z) \) are replete subcategories. Using Lemma 2.10, we can verify that the inclusion functors \( \text{Mon}^{\text{hax}}_\infty(Z) \hookrightarrow \text{Cat}_\infty/\Theta \) and \( \text{Mon}^{\text{op lax}}_\infty(Z) \hookrightarrow \text{Cat}_\infty/\Theta \) create finite products.

Next, we define mixed \((\mathcal{O}, \mathcal{P})\)-monoidal ∞-categories in \( Z \) and bilax \((\mathcal{O}, \mathcal{P})\)-monoidal functors between them.

**Definition 3.11.** Let \( \Theta \rightarrow \Lambda(\Phi) \) and \( \Phi \rightarrow \Lambda(\Psi) \) be ∞-operads over perfect operator categories over \( \Phi \) and \( \Psi \), respectively. A mixed \((\mathcal{O}, \mathcal{P})\)-monoidal ∞-category in \( Z \) is a mixed fibration \( p : \mathcal{C} \rightarrow \mathcal{O} \times (\mathcal{P}^{\text{op}} \times Z) \) over \((\mathcal{O}^{\text{op}}, (\mathcal{P}^{\text{op}})^{\text{op}})\) in \( Z \) that satisfies the following conditions:

1. For each \( x \in \mathcal{O} \), the Segal morphism

\[
\mathcal{C}_x \longrightarrow \prod_{i \in I} (\mathcal{P}^{\text{op}})^{\text{op}} \times Z \mathcal{C}_x^i
\]

is an equivalence in \( \text{Cat}^{\text{op lax}}_\Lambda(\Phi) \), where the right hand side is a product in the ∞-category \( \text{Cat}_\infty/(\mathcal{P}^{\text{op}})^{\text{op}} \).

2. For each \( y \in (\mathcal{P}^{\text{op}})^{\text{op}} \), the Segal morphism

\[
\mathcal{C}^y \longrightarrow \prod_{j \in J} (\mathcal{O} \times Z)^{\text{op}} \mathcal{C}^y_j
\]

is an equivalence in \( \text{coCart}^{\text{hax}}_\Theta(Z) \), where the right hand side is a product in \( \text{Cat}_\infty/(\mathcal{O} \times) \).

A bilax \((\mathcal{O}, \mathcal{P})\)-monoidal functor between mixed \((\mathcal{O}, \mathcal{P})\)-monoidal ∞-categories \( p : \mathcal{C} \rightarrow \mathcal{O} \times (\mathcal{P}^{\text{op}} \times Z) \) and \( q : \mathcal{D} \rightarrow \mathcal{O} \times (\mathcal{P}^{\text{op}})^{\text{op}} \times Z \) in \( Z \) is a bilax \((\mathcal{O}^{\text{op}}, (\mathcal{P}^{\text{op}})^{\text{op}})\)-morphism \( f \) that satisfies the following conditions:
(1) The functor \( f \) takes \( p_{\mathcal{O}^\otimes}\)-coCartesian morphisms over inert morphisms of \( \mathcal{O}^\otimes \) to \( q_{\mathcal{O}^\otimes}\)-coCartesian morphisms.

(2) The functor \( f \) takes \( p_{(\mathcal{P}^\otimes)^{op}}\)-Cartesian morphisms over inert morphisms of \( (\mathcal{P}^\otimes)^{op} \) to \( q_{(\mathcal{P}^\otimes)^{op}}\)-Cartesian morphisms.

We define

\[
\text{Mon}^{\text{bilax}}_{\mathcal{O}, \mathcal{P}}(\mathcal{Z})
\]

to be the subcategory of \( \text{Mix}^{\text{bilax}}_{\mathcal{O}, (\mathcal{P}^\otimes)^{op}}(\mathcal{Z}) \) with mixed \((\mathcal{O}, \mathcal{P})\)-monoidal \( \infty \)-categories and bilax \((\mathcal{O}, \mathcal{P})\)-monoidal functors.

**Theorem 3.12.** There are equivalences

\[
\text{Mon}^{\text{bilax}}_{\mathcal{O}}(\text{Mon}^{\text{plax}}_{\mathcal{P}}(\mathcal{Z})) \simeq \text{Mon}^{\text{bilax}}_{\mathcal{O}, \mathcal{P}}(\mathcal{Z}) \simeq \text{Mon}^{\text{plax}}_{\mathcal{P}}(\text{Mon}^{\text{bilax}}_{\mathcal{O}}(\mathcal{Z}))
\]

of \( \infty \)-categories.

**Proof.** We will prove \( \text{Mon}^{\text{bilax}}_{\mathcal{O}, \mathcal{P}}(\mathcal{Z}) \simeq \text{Mon}^{\text{plax}}_{\mathcal{P}}(\text{Mon}^{\text{bilax}}_{\mathcal{O}}(\mathcal{Z})) \). The other equivalence can be proved similarly.

By Theorem 2.17, we have an equivalence

\[
\text{Mix}^{\text{bilax}}_{\mathcal{O}, (\mathcal{P}^\otimes)^{op}}(\mathcal{Z}) \cong \text{coCart}^{\text{plax}}_{\mathcal{O}^\otimes}(\text{Cart}^{\text{plax}}_{(\mathcal{P}^\otimes)^{op}}(\mathcal{Z})).
\]

Under this equivalence, we can easily verify that an \((\mathcal{O}, \mathcal{P})\)-monoidal \( \infty \)-category in \( \mathcal{Z} \) corresponds to an object of \( \text{Mon}^{\text{plax}}_{\mathcal{P}}(\text{Mon}^{\text{bilax}}_{\mathcal{O}}(\mathcal{Z})) \), and vice versa, by using the same argument in the proof of [14, Proposition 4.13]. Furthermore, we see that a bilax \((\mathcal{O}, \mathcal{P})\)-monoidal functor between \((\mathcal{O}, \mathcal{P})\)-monoidal \( \infty \)-categories corresponds to a morphism of \( \text{Mon}^{\text{plax}}_{\mathcal{P}}(\text{Mon}^{\text{bilax}}_{\mathcal{O}}(\mathcal{Z})) \) by using the argument in the proof of [14, Theorem 4.20]. \( \square \)

### 3.2. Higher monoidal \( \infty \)-categories

In this subsection we generalize the notion of duoidal \( \infty \)-categories to that of \( \mathcal{O} \)-monoidal \( \infty \)-categories, where \( \mathcal{O}^\otimes \) is a finite sequence of \( \infty \)-operads over perfect operator categories.

Let \( \mathcal{O}^\otimes = (\mathcal{O}_1^\otimes, \ldots, \mathcal{O}_n^\otimes) \) be a finite sequence of \( \infty \)-operads over perfect operator categories \( \Phi_1, \ldots, \Phi_n \), respectively. We set \( \mathcal{O} = (\mathcal{O}_1, \ldots, \mathcal{O}_n) \).

First, we define an \( \infty \)-category of coCartesian \( \mathcal{O} \)-monoidal \( \infty \)-categories and lax \( \mathcal{O} \)-monoidal functors by induction on \( l(\mathcal{O}) \).

**Definition 3.13.** When \( l(\mathcal{O}) = 0 \), we set \( \text{Mon}^{\text{lax}}_{\mathcal{O}}(\text{Cat}_\infty) = \text{Cat}_\infty \). When \( l(\mathcal{O}) > 0 \), we define

\[
\text{Mon}^{\text{lax}}_{\mathcal{O}}(\text{Cat}_\infty) = \text{Mon}^{\text{lax}}_{\mathcal{O}_{\geq 2}}(\text{Mon}^{\text{lax}}_{\mathcal{O}_1}(\text{Cat}_\infty)).
\]

For the convenience of readers, we explicitly describe objects and morphisms of \( \text{Mon}^{\text{lax}}_{\mathcal{O}}(\text{Cat}_\infty) \).

**Definition 3.14.** Let \( p : \mathcal{C}^\otimes \to \prod \mathcal{O}^\otimes \) be a functor of \( \infty \)-categories.

When \( l(\mathcal{O}) = 0 \), we say that any functor \( p : \mathcal{C}^\otimes \to [0] \) is a coCartesian \( \mathcal{O} \)-monoidal \( \infty \)-category, where \([0]\) is a final object of \( \text{Cat}_\infty \). A lax \( \mathcal{O} \)-monoidal functor between coCartesian \( \mathcal{O} \)-monoidal \( \infty \)-category is a morphism in \( \text{Cat}_\infty/\{0\} \simeq \text{Cat}_\infty \).

When \( l(\mathcal{O}) > 0 \), we say that \( p \) is a coCartesian \( \mathcal{O} \)-monoidal \( \infty \)-category if it satisfies the following conditions:

1. The composite map \( p_{\mathcal{O}_i^\otimes} : \mathcal{C}^\otimes \to \mathcal{O}_i^\otimes \) is a coCartesian fibration, and \( p \) takes \( p_{\mathcal{O}_i^\otimes} \)-coCartesian morphisms to \( p_{\mathcal{O}_i^\otimes} \)-coCartesian morphisms.
2. For each \( x \in \mathcal{O}_i^\otimes \), the restriction \( \mathcal{C}^\otimes_x \to \prod \mathcal{O}^\otimes_{\geq 2} \) is a coCartesian \( \mathcal{O}_{\geq 2} \)-monoidal \( \infty \)-category.
3. For each morphism \( x \to x' \) in \( \mathcal{O}_i^\otimes \), the induced functor \( \mathcal{C}^\otimes_x \to \mathcal{C}^\otimes_{x'} \) over \( \prod \mathcal{O}^\otimes_{\geq 2} \) is a lax \( \mathcal{O}_{\geq 2} \)-monoidal functor.
(4) For each $x \in O_1^\otimes$, the Segal morphism

$$C^\otimes_x \xrightarrow{\simeq} \prod_{i \in |I|} \prod_{O_2^\otimes} C^\otimes_{x_i},$$

is an equivalence of $O_{\geq 2}$-monoidal $\infty$-categories, where the right hand side is a product in $\text{Cat}_{\infty}/\prod_{O_2^\otimes}$.

A lax $O$-monoidal functor between coCartesian $O$-monoidal $\infty$-categories $p : C^\otimes \to \prod O^\otimes$ and $q : D^\otimes \to \prod O^\otimes$ is a functor $h : C^\otimes \to D^\otimes$ over $\prod O^\otimes$ that satisfies the following conditions:

1. The functor $h$ takes $p_{O_1^\otimes}$-coCartesian morphisms over inert morphisms of $O_1^\otimes$ to $q_{O_1^\otimes}$-coCartesian morphisms.
2. For each $x \in O_1^\otimes$, the induced functor $h_x : C^\otimes_x \to D^\otimes_x$ over $\prod O_{\geq 2}^\otimes$ is a lax $O_{\geq 2}$-monoidal functor.

Unwinding the definition, we obtain the following theorem.

**Theorem 3.15.** The $\infty$-category $\text{Mon}_O^{\text{lax}}(\text{Cat}_{\infty})$ is a subcategory of $\text{Cat}_{\infty}/\prod O^\otimes$ with coCartesian $O$-monoidal $\infty$-categories as objects and lax $O$-monoidal functors as morphisms.

**Example 3.16.** Let $F : C^\otimes \to \Lambda(F)$ be a symmetric monoidal $\infty$-category. We obtain a coCartesian $O$-monoidal $\infty$-category for any finite sequence $O$ of $\infty$-operads over $F$ by taking pullback of $p$ along the map $\prod O^\otimes \to \prod \Lambda(F) \to \Lambda(F)$, where the second map is a smash product functor in $\prod$ Notation 2.2.5.9.

**Example 3.17.** Let $O^\otimes = (O_1^\otimes, \ldots, O_n^\otimes)$ be a finite sequence of $\infty$-operads over $F$. We denote by $\otimes O^\otimes$ the Boardman-Vogt tensor product $O_1^\otimes \otimes \cdots \otimes O_n^\otimes$. Let $p : C^\otimes \to \otimes O^\otimes$ be an $\otimes O^\otimes$-monoidal $\infty$-category. We obtain a coCartesian $O$-monoidal $\infty$-category by taking pullback of $p$ along the map $\prod O^\otimes \to \otimes O^\otimes$.

**Example 3.18.** Let $E_k^\otimes$ be the little $k$-cubes operad. Suppose that $C^\otimes \to E_k^\otimes_{m+n}$ is a presentable $E_{m+n}$-monoidal $\infty$-category and that $A$ is an $E_{m+n}$-algebra object of $C$. Then there exists a coCartesian $(E_m, E_n)$-monoidal $\infty$-category $D^\otimes \to E_k^\otimes_m \times E_k^\otimes_n$ such that the underlying $\infty$-category $D^\otimes_{(1,1)}$ is equivalent to the $\infty$-category $\text{Mod}_{A}^{E_k}(C)$ of $E_k-A$-modules. See [15] for more details.

Dually, we define an $\infty$-category of Cartesian $O$-monoidal $\infty$-categories and oplax $O$-monoidal functors.

**Definition 3.19.** When $l(O) = 0$, we set $\text{Mon}_O^{\text{oplax}}(\text{Cat}_{\infty}) = \text{Cat}_{\infty}$. When $l(O) > 0$, we define

$$\text{Mon}_O^{\text{oplax}}(\text{Cat}_{\infty}) = \text{Mon}_O^{\text{oplax}}(\text{Mon}_O^{\text{oplax}}(\text{Cat}_{\infty})).$$

We call it the $\infty$-category of Cartesian $O$-monoidal $\infty$-categories and oplax $O$-monoidal functors.

For finite sequences $O^\otimes$ and $P^\otimes$ of $\infty$-operads over perfect operator categories, we also define an $\infty$-category of $(O, P)$-monoidal $\infty$-categories and bilax $(O, P)$-monoidal functors.

**Definition 3.20.** We define

$$\text{Mon}_{(O, P)}^{\text{bilax}}(\text{Cat}_{\infty}) = \text{Mon}_O^{\text{lax}}(\text{Mon}_P^{\text{oplax}}(\text{Cat}_{\infty})).$$

Note that there is an equivalence $\text{Mon}_{(O, P)}^{\text{bilax}}(\text{Cat}_{\infty}) \simeq \text{Mon}_P^{\text{oplax}}(\text{Mon}_O^{\text{oplax}}(\text{Cat}_{\infty}))$ by Theorem 3.12.

Now, we show that mixed higher monoidal $\infty$-categories can canonically be identified with coCartesian higher monoidal $\infty$-categories or Cartesian higher monoidal $\infty$-categories.
Theorem 3.21. There are natural equivalences
\[ \text{Mon}^\text{lax}_{P_{\text{rev}},O}(\text{Cat}_\infty)^\simeq \simeq \text{Mon}^\text{bilax}_{(O,P)}(\text{Cat}_\infty)^\simeq \simeq \text{Mon}^\text{oplax}_{O_{\text{rev}},P}(\text{Cat}_\infty)^\simeq \]
of the underlying $\infty$-groupoids.

Proof. For an $\infty$-operad $Q^\otimes$ over a perfect operator category, it suffices to show that the equivalence in Theorem 2.26 restricts to natural equivalences
\[ \text{Mon}^\text{bilax}_{(Q,O),P}(\text{Cat}_\infty)^\simeq \simeq \text{Mon}_Q(\text{Mon}^\text{bilax}_{(O,P)}(\text{Cat}_\infty))^\simeq \simeq \text{Mon}^\text{bilax}_{(Q,O),P}(\text{Cat}_\infty)^\simeq. \]
By Lemma 3.19 we have equivalences
\[ \text{Mon}_Q(\text{Mon}^\text{bilax}_{(O,P)}(\text{Cat}_\infty))^\simeq \simeq \text{Mon}^\text{lax}_Q(\text{Mon}^\text{bilax}_{(O,P)}(\text{Cat}_\infty))^\simeq \simeq \text{Mon}^\text{bilax}_{(Q,O),P}(\text{Cat}_\infty)^\simeq. \]
The other equivalence can be proved similarly. □

Corollary 3.22. The equivalence in Corollary 2.26 restricts to a natural equivalence
\[ \text{Mon}^\text{lax}_O(\text{Cat}_\infty)^\simeq \simeq \text{Mon}^\text{oplax}_{O_{\text{rev}}}(\text{Cat}_\infty)^\simeq \]
of the underlying $\infty$-groupoids.

By Corollary 3.22 a coCartesian $O$-monoidal $\infty$-category canonically determines a Cartesian $O_{\text{rev}}$-monoidal $\infty$-category, and vice versa, by taking dual fibrations.

Next, we give a fiberwise criterion on bilax monoidal functors.

Proposition 3.23. Let $f: C^\otimes \to D^\otimes$ be a bilax $(O^\otimes, (P^\otimes)^{\text{op}})$-morphism between $(O, P)$-monoidal $\infty$-categories. Then $f$ is a bilax $(O, P)$-monoidal functor if and only if the following two conditions hold:

(a) $f_{(s',t)}$ is a lax $O_1$-monoidal functor for each $(s', t) \in \prod O_{s,t}^\otimes \times \prod (P^\otimes)^{\text{op}}$.

(b) $f_{(s',t)}$ is an oplax $P_1$-monoidal functor for each $(s, t') \in \prod O^\otimes \times \prod (P_{s,t'})^{\text{op}}$.

Proof. It is clear that if $f$ is a bilax $(O, P)$-monoidal functor, then (a) and (b) holds. Thus, it suffices to show that if $f$ satisfies (a) and (b), then $f$ is a bilax $(O, P)$-monoidal functor.

We may assume that $l(O) + l(P) > 1$. We suppose that $l(O) > 0$. The case $l(P) > 0$ can be proved similarly.

We let $p_1: C^\otimes \to O_1^\otimes$ and $q_1: D^\otimes \to O_1^\otimes$ be the projection. Then $f$ is bilax $(O, P)$-monoidal if

(1) $f$ takes $p_1$-coCartesian morphisms over inert morphisms of $O_{s,t}^\otimes$ to $q_1$-coCartesian morphisms, and

(2) $f_x: C_x^\otimes \to D_x^\otimes$ is a bilax $(O_{\geq 2}, P)$-monoidal functor for every $x \in O_1^\otimes$. By Lemma 3.19, (1) is equivalent to the condition that $f_y$ is lax $O_1$-monoidal for every $y \in \prod O_{s,t}^\otimes \times (P_{s,t'})^{\text{op}}$.

Thus, by induction on $l(O)$, $f$ is bilax $(O, P)$-monoidal if $f$ satisfies (a) and (b). □

4. ADJoints of (op)lax monoidal functors

In this section we study adjoints of (op)lax monoidal functors between higher monoidal $\infty$-categories. For this purpose, we study adjoints of (op)lax morphisms between $S$-(co)Cartesian fibrations in [1]. In [12] we prove the main theorem (Theorem 4.7) which says that the $\infty$-category of coCartesian $O$-monoidal $\infty$-categories and right adjoint lax $O$-monoidal functors is equivalent to the opposite of the $\infty$-category of Cartesian $O_{\text{rev}}$-monoidal $\infty$-categories and left adjoint oplax $O_{\text{rev}}$-monoidal functors.
4.1. Adjoints of (op)lax morphisms between S-(co)Cartesian fibrations. In this subsection we study adjoints of (op)lax morphisms between S-(co)Cartesian fibrations. We show that the ∞-category of coCartesian S-fibrations and right adjoint lax S-morphisms is equivalent to the opposite of the ∞-category of Cartesian S_{rev}^{op}-fibrations and left adjoint oplax S_{rev}^{op}-morphisms.

Definition 4.1. Let $S = (S_1, \ldots , S_n)$ be a finite sequence of ∞-categories. We write

$$\coCart_{/S}^{\text{lax}, R}(\Cat_{\infty})$$

for the wide subcategory of $\coCart_{/S}^{\text{lax}}(\Cat_{\infty})$ spanned by those lax S-morphisms $f : X \to Y$ over $\prod S$ that satisfy the following condition: For each $s \in \prod S$, the restriction $f_s : X_s \to Y_s$ is a right adjoint functor.

Dually, we write

$$\Cart_{/S}^{\text{oplax}, L}(\Cat_{\infty})$$

for the wide subcategory of $\Cart_{/S}^{\text{oplax}}(\Cat_{\infty})$ spanned by those oplax S-morphisms $f : X \to Y$ over $\prod S$ that satisfy the following condition: For each $s \in \prod S$, the restriction $f_s : X_s \to Y_s$ is a left adjoint functor.

The goal of this subsection is to prove the following theorem.

Theorem 4.2. There is a natural equivalence

$$\coCart_{/S}^{\text{lax}, R}(\Cat_{\infty}) \simeq \Cart_{/S_{\text{rev}}^{\text{op}}}^{\text{oplax}, L}(\Cat_{\infty})^{\text{op}}$$

of ∞-categories, which is given on objects by taking dual fibrations and on morphisms by taking adjoints fiberwise.

We recall that a biCartesian fibration is both a Cartesian fibration and a coCartesian fibration. See [11, §4.7.4] for the relationship between biCartesian fibrations and adjoint functors. In particular, if a coCartesian fibration $p : X \to S$ in which the induced functor $X_s \to X_{s'}$ admits a right adjoint for each morphism $s \to s'$ in S, then $p$ is a biCartesian morphism. Dually, if a Cartesian fibration $q : Y \to T$ in which the induced functor $Y_t \to Y_{t'}$ admits a left adjoint for each morphism $t' \to t$ in $T$, then $q$ is a biCartesian fibration.

Notation 4.3. We write

$$\bCart_{/U}^{\text{bilax}}(\Cat_{\infty})$$

for the full subcategory of $\Cat_{\infty}/U$ spanned by biCartesian fibrations over $U$.

Lemma 4.4. There are natural equivalences

$$\Cart_{/U}^{\text{oplax}}(\coCart_{/S}^{\text{lax}, R}(\Cat_{\infty})) \simeq \coCart_{/U}^{\text{lax}}(\bCart_{/U}^{\text{bilax}}(\Cat_{\infty})),$$

$$\coCart_{/U}^{\text{lax}}(\Cart_{/S}^{\text{oplax}, L}(\Cat_{\infty})) \simeq \Cart_{/S}^{\text{oplax}}(\bCart_{/U}^{\text{bilax}}(\Cat_{\infty}))$$

of ∞-categories.

Proof. The first equivalence follows from the fact that the both sides are full subcategories of $\Mix_{/S \times U}(\Cat_{\infty})$ spanned by those mixed fibrations $p : X \to \prod S \times U$ such that $p_s : X_s \to U$ is a coCartesian fibration for each $s \in \prod S$. The second equivalence can be proved similarly. □

Proof of Theorem 4.2. First, we shall show that there exists a natural equivalence

$$\Map_{\Cat_{\infty}}(U, \coCart_{/S}^{\text{lax}, R}(\Cat_{\infty})) \simeq \Map_{\Cat_{\infty}}(U, \Cart_{/S_{\text{rev}}^{\text{op}}}^{\text{oplax}, L}(\Cat_{\infty})^{\text{op}})$$

of ∞-groupoids for any ∞-category $U$. 

By Lemmas 2.7 and 2.8 we have natural equivalences
\[\text{Map}_{\text{Cat}_{\infty}}(U, \text{coCart}^{\text{lax,R}}_{/S} (\text{Cat}_{\infty})) \simeq \text{Cart}_{/U}^{\text{lax,R}}(\text{coCart}^{\text{lax,R}}_{/S} (\text{Cat}_{\infty}))^{\ast}\]
\[\simeq \text{Cart}_{/U}^{\text{op}}(\text{coCart}^{\text{lax,R}}_{/S} (\text{Cat}_{\infty}))^{\ast}\]

Similarly, by Lemmas 2.2 and 2.3 we have natural equivalences
\[\text{Map}_{\text{Cat}_{\infty}}(U, \text{Cart}^{\text{op,plax,L}}_{/S_{\text{rev}}} (\text{Cat}_{\infty})^{\ast}) \simeq \text{Map}_{\text{Cat}_{\infty}}(U^{\ast}, \text{Cart}^{\text{op,plax,L}}_{/S_{\text{rev}}} (\text{Cat}_{\infty}^{\ast}))\]
\[\simeq \text{coCart}_{/U}^{\text{op}}(\text{Cart}^{\text{op,plax,L}}_{/S_{\text{rev}}} (\text{Cat}_{\infty}))^{\ast}\]
\[\simeq \text{coCart}_{/U}^{\text{lax}}(\text{Cart}^{\text{op,plax,L}}_{/S_{\text{rev}}} (\text{Cat}_{\infty}))^{\ast}\]

By Lemma 4.4 and Corollary 2.28 we obtain the desired equivalence.

By the Yoneda Lemma, we obtain an equivalence between the \(\infty\)-categories \(\text{coCart}^{\text{lax,R}}_{/S} (\text{Cat}_{\infty})\) and \(\text{Cart}^{\text{op,plax,L}}_{/S_{\text{rev}}} (\text{Cat}_{\infty})^{\ast}\). We can verify that this equivalence is given on objects by taking dual fibrations and on morphisms by taking adjoints fiberwise. \(\square\)

4.2. Monoidal adjoints of higher monoidal \(\infty\)-categories. Let \(O^{\otimes} = (O_1^{\otimes}, \ldots, O_n^{\otimes})\) be a finite sequence of \(\infty\)-operads over perfect operator categories. In this subsection we show that the \(\infty\)-category of \(\text{coCartesian} \ O\text{-monoidal} \ \infty\text{-categories}\) and right adjoint lax \(O\text{-monoidal}\) functors is equivalent to the opposite of the \(\infty\)-category of Cartesian \(O_{\text{rev}}\text{-monoidal} \ \infty\text{-categories}\) and left adjoint oplax \(O_{\text{rev}}\text{-monoidal}\) functors (Theorem 4.7).

Definition 4.5. We define an \(\infty\)-category
\[\text{Mon}_{O}^{\text{lax,R}}(\text{Cat}_{\infty})\]
to be the wide subcategory of \(\text{Mon}_{O}^{\text{lax}}(\text{Cat}_{\infty})\) spanned by those lax \(O\text{-monoidal}\) functors \(f : C^{\otimes} \to D^{\otimes}\) that satisfy the following condition: For each \(x \in \prod O\), the restriction \(f_x : C_x \to D_x\) is a right adjoint functor.

Dually, we define an \(\infty\)-category
\[\text{Mon}_{O}^{\text{op,plax,L}}(\text{Cat}_{\infty})\]
to be the wide subcategory of \(\text{Mon}_{O}^{\text{op,plax}}(\text{Cat}_{\infty})\) spanned by those oplax \(O\text{-monoidal}\) functors \(f : C^{\otimes} \to D^{\otimes}\) that satisfy the following condition: For each \(x \in \prod O^{\ast}\), the restriction \(f_x : C_x \to D_x\) is a left adjoint functor.

Remark 4.6. Notice that \(\text{Mon}_{O}^{\text{lax,R}}(\text{Cat}_{\infty})\) is a subcategory of \(\text{coCart}_{/O_{\otimes}}^{\text{lax,R}}(\text{Cat}_{\infty})\). This follows from the fact that the restriction \(f_x : C_x^{\otimes} \to D_x^{\otimes}\) for \(x = (x_1, \ldots, x_n) \in \prod O^{\otimes}\) with \(p(x_i) = I_i\) \((1 \leq i \leq n)\) is equivalent to a product of \(f_x\) for \(x = (x_{1,i}, \ldots, x_{n,i}) \in \prod O\) over \(i_1 \in |I_1|, \ldots, i_n \in |I_n|\).

Similarly, \(\text{Mon}_{O}^{\text{op,plax,L}}(\text{Cat}_{\infty})\) is a subcategory of \(\text{Cart}_{/(O_{\otimes})^{\ast}}^{\text{op,plax,L}}(\text{Cat}_{\infty})\).

The following is the main theorem of this paper.

Theorem 4.7. The equivalence of Theorem 4.2 restricts to an equivalence
\[\text{Mon}_{O}^{\text{lax,R}}(\text{Cat}_{\infty}) \simeq \text{Mon}_{O_{\text{rev}}}^{\text{op,plax,L}}(\text{Cat}_{\infty})^{\ast}\]
of \(\infty\)-categories, which is given on objects by taking dual fibrations and on morphisms by taking adjoints fiberwise.

Corollary 4.8. The left adjoint of a lax \(O\text{-monoidal}\) functor between \(\text{coCartesian} \ O\text{-monoidal} \ \infty\text{-categories}\) is canonically an oplax \(O_{\text{rev}}\text{-monoidal}\) functor between the corresponding Cartesian \(O_{\text{rev}}\text{-monoidal} \ \infty\text{-categories}\), and vice versa.
Proof of Theorem 4.7. We will prove the theorem by induction on \( l(\mathbf{O}) \). When \( l(\mathbf{O}) = 0 \), this follows from Theorem 4.2 for \( \mathbf{S} = \emptyset \).

Suppose \( l(\mathbf{O}) > 0 \). First, we note that an equivalence

\[
\text{coCart}^{\text{lax}, R}_{/\mathbf{S}}(\text{Cat}_{\infty}) \cong \text{Cart}^{\text{oplax}, L}_{/\mathbf{S}_{\text{rev}}} \cong \text{Cart}^{\text{oplax}, L}_{/\mathbf{S}_{\text{rev}}} \cong \text{Cart}^{\text{oplax}, L}_{/\mathbf{S}_{\text{rev}}}
\]

induced by the equivalence in Theorem 4.2 restricts to an equivalence

\[
\text{Mon}^{\text{lax}, R}_{/\mathbf{O}}(\text{Cat}_{\infty}) \cong \text{Mon}^{\text{oplax}, L}_{/\mathbf{O}}(\text{Cat}_{\infty}) \cong \text{Mon}^{\text{oplax}, L}_{/\mathbf{O}}(\text{Cat}_{\infty}) \cong \text{Mon}^{\text{oplax}, L}_{/\mathbf{O}}(\text{Cat}_{\infty})
\]

of \( \infty \)-groupoids.

Thus, by symmetry, it suffices to show the following claim: Suppose that \( f : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes} \) is a lax \( \mathbf{O} \)-monoidal functor between coCartesian \( \mathbf{O} \)-monoidal \( \infty \)-categories. We assume that \( f_x \) is a right adjoint functor for each \( x \in \prod \mathbf{O} \). Let \( f : (\mathcal{D}^{\otimes})^{\vee} \to (\mathcal{C}^{\otimes})^{\vee} \) be a corresponding oplax \( \mathbf{O}_{\text{rev}} \) morphism under the equivalence in Theorem 4.2.

By Proposition 4.22 it suffices to show that \( g_x : (\mathcal{D}^{\otimes})^{\vee}_x \to (\mathcal{C}^{\otimes})^{\vee}_x \) is an oplax \( \mathbf{O}_i \)-monoidal functor for each \( x \in \prod \mathbf{O} \). Let \( x \to x' \) be an inert morphism of \( \mathbf{O}_i \). Since \( f \) is a lax \( \mathbf{O} \)-monoidal functor, \( f_x : \mathcal{C}^{\otimes}_x \to \mathcal{D}^{\otimes}_x \) is a lax \( \mathbf{O}_i \)-monoidal functor. This implies that there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}^{\otimes}_{(x,x)} & \xrightarrow{f_{(x,x)}} & \mathcal{D}^{\otimes}_{(x,x)} \\
\downarrow & & \downarrow \\
\mathcal{C}^{\otimes}_{(x,x')} & \xrightarrow{f_{(x,x')}} & \mathcal{D}^{\otimes}_{(x,x')} 
\end{array}
\]

Since the vertical arrows are projections, the above commutative diagram is left adjointable. Hence we obtain a commutative diagram

\[
\begin{array}{ccc}
(\mathcal{C}^{\otimes})^{\vee}_{(x,x)} & \xleftarrow{g_{(x,x)}} & (\mathcal{D}^{\otimes})^{\vee}_{(x,x)} \\
\downarrow & & \downarrow \\
(\mathcal{C}^{\otimes})^{\vee}_{(x,x')} & \xleftarrow{g_{(x,x')}} & (\mathcal{D}^{\otimes})^{\vee}_{(x,x')} 
\end{array}
\]

This means that \( g_x \) is an oplax \( \mathbf{O}_i \)-monoidal functor. \( \square \)

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