Double Compton scattering in a constant crossed field and approximations used in simulation

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Abstract. The double photon emission of an electron in a constant crossed field has been calculated in [1]. By calculating the process of polarised single-photon emission, it is shown that the double-photon process can be unambiguously separated into two- and one- step subprocesses, also known as sequential and simultaneous double-emission. It is found that the assumption used in simulations of neglecting additional electron spin effects is valid to within a few percent, however the assumption that the two-step process is dominant becomes questionable for shorter background fields.

1. Introduction
The emission of radiation by an accelerated electron or positron is described classically by the process of Thomson scattering [2]. When the energy of the emitted photon in the rest frame of the electron is of the order of the electron rest energy or higher, a quantised theory is necessary to describe the change in emission. When such emission occurs in an intense electromagnetic background field, it is often referred to as “non-linear” Compton scattering”. This was first calculated in monochromatic (hence infinite) plane waves [3, 4] and more recently in finite plane waves [5, 6, 7, 8, 9, 10] with electron spin [11], photon polarisation [12] and external-field polarisation [13, 14] all having been investigated. Experimentally, Compton scattering has been measured in the weakly non-linear regime and recent advances in laser technology have fuelled interest in calculating experimental signals of the two-photon emission process in a pulsed plane-wave background [15, 16] (a review of strong-field QED effects can be found in [17, 18, 19]).

In the current proceedings, we report on results in [1] for the calculation of the two-photon emission process in a constant crossed field. The constant crossed background was chosen at is the one used overwhelmingly in numerical simulations in laser-particle physics, which propagate charged particles using the particle-in-cell method and generate “quantum” events using Monte Carlo techniques [20, 21, 22, 23, 24, 25, 26, 27]. By calculating the two-photon emission rate analytically, the accuracy of various numerical approximations can be assessed. Although the total rate for this process has been found in a calculation for the two-loop electron mass operator [28], using our direct method of calculation, also differential rates can be found that could be used, for example, to calculate the transverse momenta of the emitted particles. Although our analysis is for electrons, the results also apply to positrons. Unless otherwise stated, \( c = \hbar = 1 \) in our system of units.
2. Polarised single Compton scattering

We begin by presenting the highlights of the calculation of single photon emission, when the polarisation of all particles are taken into account. This is the process:

\[ e^- \rightarrow e^- + \gamma \]  

(1)

where \( e^- \) refers to an electron and \( \gamma \) to a photon, in a constant crossed field. Let \( p, s_p, q, s_q, k, \varepsilon_k \) denote the incoming electron momentum and spin, the emitted electron momentum and spin and the photon momentum and polarisation respectively, where \( s_p^2 = s_q^2 = \varepsilon_k^2 = -1 \).

The corresponding Feynman diagram is given in Fig. 1. The electron in a classical plane-wave background is described using the Volkov solution [29] for the wavefunction of the Dirac equation \( \psi_p \) and an on-shell photon by the wavefunction \( A_k \):

\[
\psi_p(x) = \left[ 1 + \frac{e \mathcal{F}[\varphi(x)]}{2(\varkappa \cdot p)} \right] \frac{u_p}{\sqrt{2p^0 V}} e^{iS(p,\varphi(x))} \quad A_k = \frac{\sqrt{4\pi\varepsilon_k} e^{ik \cdot x}}{\sqrt{2k^0 V}},
\]  

(2)

where \( \varkappa, A \) and \( \varphi \) are the wavevector, gauge potential and phase of the external field, \( V \) is the normalisation volume, \( \mathcal{F} = \gamma^\mu \varkappa_\mu \) and \( \gamma^\mu \) are the gamma matrices. The amplitude for the process is then given as usual by:

\[
S\text{fi}(p \rightarrow q + k) = ie \int d^4x \overline{\psi}_q(x) A_k^* \psi_p(x),
\]  

(3)

where the spin of an electron with momentum \( p \) is included using the relation [30]:

\[
u_p\overline{\nu}_p = \frac{1}{2}(\not{p} + m)(1 - \gamma^5 \not{\gamma}).
\]  

(4)

We employed the method developed by Nikishov and Ritus (see [17], and a more detailed application to single Compton scattering in [12]), to calculate Eq. (11). One difference worthy of note was the addition of electron spin. If an arbitrary covariant basis is chosen, divergences seem to appear, associated with violating a symmetry of the total probability in the direction of external electric field. One choice of spin-basis that preserves this symmetry and allows the Nikishov-Ritus method to be straightforwardly applied is:

\[
A(\varphi) = a^{(1)} g^{(1)}(\varphi) + a^{(2)} g^{(2)}(\varphi); \quad \varepsilon_k^{(1,2)} = a^{(1,2)} \frac{k \cdot a^{(1,2)}}{k \cdot \varepsilon_k}; \quad \zeta_p = a^{(2)} - \frac{p \cdot a^{(2)}}{p \cdot \varepsilon_k},
\]  

(5)

where \( a^{(1)} \cdot a^{(2)} = 0, a^{(1)} \cdot a^{(1)} = a^{(2)} \cdot a^{(2)} = -1 \) and \( g^{(2)}(x) = 0 \). The gauge potential of the constant crossed field background can be written \( g^{(1)}(\varphi) = (ma_0/e)\varphi \) where \( a_0 = e|p \cdot F|/m|\varphi \cdot p| \) is a gauge- and relativistic- invariant intensity parameter [31], for \( F \) the Faraday tensor of the external field, which can be written as \( a_0 = mE/\varkappa^0 E_{cr} \) for electric field amplitude \( E \) and critical field \( E_{cr} = m^2/e \). As \( a_0 \) is inversely proportional to background field frequency, in a constant
field it is formally infinite, but total rates can be written in terms of another gauge- and relativistic invariant for a particle with momentum \( p \), namely the “quantum efficiency” parameter \( \chi_p = e|p \cdot F|/m^3 \). One can define the rate for single Compton scattering using \( R_\gamma = P_\gamma/a^0 \int d\varphi \), which the reader notices is independent of the limit \( \kappa_0 \rightarrow 0 \).

In order to interpret the results, let us expand the polarisation and spins as:

\[
\begin{align*}
\varepsilon_k &= c_1 \varepsilon_k^{(1)} + c_2 \varepsilon_k^{(2)} \\
s_p &= \sigma_p \zeta_p & s_q &= \sigma_q \zeta_q
\end{align*}
\]

where \( c_{1,2} \in \{0, 1\} \), \( \sigma_{p,q} \in \{-1, 0, 1\} \). We then find:

\[ R_\gamma = -\frac{\alpha}{\chi_p} \int_0^{\chi_p} d\chi_k \left[ C' \text{Ai}(z) + C'' \text{Ai}'(z) + C_1 \text{Ai}_1(z) \right], \quad z = \left[ \frac{\chi_k}{\chi_p(\chi_p - \chi_k)} \right]^{2/3} \]

where \( \text{Ai}(x) = \frac{1}{\pi} \int_0^\infty dk \cos(kx + k^3/3) \) is the Airy function, \( \text{Ai}'(x) \) its derivative, \( \text{Ai}_1(x) = \int_x^\infty dk \text{Ai}(k) \) and

\[
\begin{align*}
C' &= z \left[ (\sigma_p + \sigma_q)(\chi_k - 2c_k^2 \chi_p) - 2(1 - c_k^2) \sigma_q \chi_k \right] \\
C'' &= \chi_k z^{1/2} \left( 1 - c_\sigma c_\delta \sigma_p \sigma_q \right) + \frac{2}{z} \left( 1 + \sigma_p \sigma_q \right) \left( 1 - c_\sigma c_\delta \right) \\
C_1 &= 1 + \sigma_p \sigma_q - \frac{c_\sigma c_\delta \sigma_p \sigma_q \chi_k^2}{\chi_p(\chi_p - \chi_k)},
\end{align*}
\]

where \( c_k = c_1 = \sqrt{1 - c_2^2} \), \( 2c_\sigma = c_1 + c_2 \) and \( c_\delta = c_2 - c_1 \) have been used to simplify notation.

The analytical form of the rate is of particular interest, as a term containing \( \text{Ai}(z) \), which cannot be written simply in terms of the other Airy functions, appears when the polarisation of all particles is taken into account. The affect on the rate of taking into account particle polarisation is illustrated in Fig. 2. From Fig. 2 we note that the “spin-flip” channels of

![Figure 2](https://example.com/figure2.png)

**Figure 2.** Left: The solid (dashed) lines refer to the rate of polarised single Compton scattering \( R_\gamma \) of an emitted photon in the \( \varepsilon_k^{(1)} \) (\( \varepsilon_k^{(2)} \)) polarisation state and the dotted line refers to the unpolarised rate \( \langle R_\gamma \rangle \). Right: The ratio of the rate of each polarisation channel to the unpolarised rate, using the same line scheme as in the left plot.

an electron changing its polarisation state due to the emission of a photon, are significantly
suppressed compared to the “no-flip” channels. This also holds for the photon polarisation as those channels in which the photon polarisation is parallel to the external field one are favoured over those where there is an effective “flip”. We also note the rate for “no-flip” channels can be more than 300 % the average.

3. Unpolarised double Compton scattering

The calculation for polarised single Compton scattering can be used to better understand the unpolarised rate for double Compton scattering. This is the process:

\[ e^- \rightarrow e^- + \gamma + \gamma' \]  

which can be illustrated using the Feynman diagrams in Fig. 3. The amplitude for this process

\[
S_{fi} = \overrightarrow{S}_{fi} + \overleftarrow{S}_{fi}
\]

with:

\[
\overrightarrow{S}_{fi} = -e^2 \int d^4 x' d^4 x \bar{\psi}_p(x') A_k^{+} G_q(x', x) A_k \psi_p(x),
\]

where the electron propagator in a plane-wave background is given by [17]:

\[
G_q(x', x) = \int \frac{d^4 q}{(2\pi)^4} \left[ 1 + \frac{e \vec{r} \cdot \vec{A}(q)}{2 q^2 - m^2 + i\epsilon} \right] e^{iS(q', q)} \frac{q + m}{q^2 - m^2 + i\epsilon} e^{-iS(q, q)} \left[ 1 + \frac{e \vec{A}(q) \cdot \vec{r}}{2 q \cdot q} \right],
\]

and \(\overleftarrow{S}_{fi}\) is equal to \(\overrightarrow{S}_{fi}\) with the photon positions swapped (due to exchange symmetry). The calculation then proceeds as for single Compton scattering, but is slightly complicated by the exchange term in the amplitude and the presence of a propagator (a detailed example of the Nikishov-Ritus approach being applied to a two-vertex process is given in the calculation of electron-seeded pair creation in [32]). It can be shown (e.g. in [1, 32]) that in a constant crossed field, the probability of emission of two photons can be unambiguously written as a probability containing the sequential process plus the probability of the simultaneous process, which we denote the two- and one-step process respectively:

\[
R_{\gamma\gamma} = R_{\gamma\gamma}^{(2)} + R_{\gamma\gamma}^{(1)};
\]

\[
R_{\gamma\gamma}^{(2)} = \Gamma_{\gamma\gamma}^{(2)} a_0 \int d\varphi_\gamma; \quad R_{\gamma\gamma}^{(1)} = \Gamma_{\gamma\gamma}^{(1)},
\]

where analogous to the previous section, we have defined the rate \(R_{\gamma\gamma} = P_{\gamma\gamma}/a_0 \int d\varphi_\gamma\), \(\varphi_\gamma\) is the phase of the first emission, and \(\Gamma_{\gamma\gamma}^{(1,2)} = \Gamma_{\gamma\gamma}^{(1,2)}(\chi_p)\) depend only on the quantum efficiency parameter of the seed electron \(\chi_p\). Since the two-step process is sequential, there are two phase points on the electron’s trajectory that play a role and in Eq. (13) \(\varphi_\gamma\) is the difference in phase between emissions in the two-step process. In contrast, the one-step process is simultaneous emission and so only a single point of emission, \(\varphi_\gamma\), is relevant.
The interference between exchange diagrams is an order lower in $a_0 \int d\varphi_- = L/L_\chi$, where $L$ is the longitudinal spatial extent of the field and $L_\chi$ is the formation length, and as we are interested in fields of much greater extent than formation length $L_\chi = \lambda E_\text{c}/E$ ($\lambda = 1/m$ is the reduced Compton wavelength), this parameter is formally infinite (later simply assumed large) and the interference terms are neglected. Since we plan to investigate approximations used in simulation, which typically require total rather than differential rates, and since the particle being exchanged is outgoing and hence integrated over, both exchange terms contribute an equal amount and are included in the two- and one-step rates in Eq. (13).

The two-step rate was found to be exactly factorisable as an average over intermediate electron polarisation states, of the twice-iterated rate of single Compton scattering:

$$R_{\gamma\gamma}^{(2)} = \frac{1}{2} \sum_{\sigma_0} \int d\chi_q \frac{\partial R_{\gamma}(\chi_p \rightarrow \chi_k + \chi_q)}{\partial \chi_q} R_{\gamma}(\chi_q \rightarrow \chi_{p'} + \chi_{k'}) \frac{a_0}{2} \int d\varphi_-, \quad (14)$$

where the first factor of a half is an average over the two spin states of the propagating electron, and the second due to time ordering. We note that in choosing the spin states in Eq. (5), which do not precess in the external field, the spin of the electron after the first emission is assumed not to have flipped before the second. If a different spin basis were chosen, which precessed in the external field, it is unclear whether such a straightforward factorisation could be made. (We also mention here, that it is also unclear whether this factorisation is at all possible in a general plane wave.) Photon precession is not included as this is a higher-order affect, first entering at $O(\alpha^2)$ for single-photon emission [33, 34, 35, 36] where $\alpha \approx 1/137$ is the fine-structure constant.

The one-step rate is found to be exclusively negative. This does not contradict the interpretation of the rate as a probability per unit time, as the factor $a_0 \int d\varphi_-$ is formally infinitely large in a constant crossed field and so the positive two-step process being one order higher in this factor, will dominate the one-step process. Later, we will investigate how this changes when, as in simulations, this factor is taken to be finite. An important extra feature in double Compton scattering that is not present in the single version is the appearance of a soft $\chi_k$ singularity in the total probability of the one-step process, which diverges as $\sim 1/\chi_k$ as $\chi_k \rightarrow 0$. This divergence should be cancelled by self-energy corrections, briefly discussed in [28]. The divergence is particularly interesting as it does not seem to be synonymous with the well-known infra-red divergence of emitted soft photons $k^0 \rightarrow 0$ from standard Compton scattering [15, 37, 38], but also appears when the emitted photon is collinear with the external field wavevector. This point is discussed in detail in [39, 40, 41]. A comparison of the one- and two-step processes is given in Fig. 4, where a $\chi_k > 10^{-5}$ cutoff has been chosen to acquire finite results. The comparison of one- and two-step processes was found to be qualitatively the same for cutoffs within $[10^{-7}, 10^{-5}]$. In Fig. 4, the probability for these processes has been compared for an external field of dimension $a_0 \int d\varphi_0 = 1$ ($L = L_\chi$), for a cutoff of $10^{-5}$. It is apparent that for $\chi_p > 0.1$, the magnitude of the one-step process is larger over this length.

4. Approximations used in simulation

Two approximations that are routinely used in the types of numerical simulations of strong-field QED processes mentioned in the introduction are i) neglecting electron spin, ii) neglecting simultaneous processes. Using our results for double Compton scattering in a constant crossed field, both of these could be investigated. Typically, in simulations, the constant crossed field approximation is used, which assumes that a spatio-temporally changing background can be well-approximated as constant during “quantum” processes such as photon emission or pair creation, as long as the scale of variation of the external field $\Delta$ fulfills $\Delta \gg L_\chi$ (we recall...
Figure 4. Left: a plot of the two-step part of double Compton scattering (solid blue line), compared to twice-iterated unpolarised single Compton scattering (red dashed line) and the calculation of Morozov and Ritus; Right: the one-step and two-step processes with cutoff $\chi_k > 10^{-5}$.

$L_* = \lambda E_{cr}/E$ is the formation length). We define the constant crossed field approximation to be when the probability for two-photon emission in such a background is written:

$$P_{\gamma\gamma} \approx P_{ccf}^{(2)} = \frac{1}{2} \sum_{\sigma_1} \int_{\phi_0}^{\phi_\infty} d\varphi_{a\sigma_x'} \int_{\phi_0}^{\phi_{a\sigma_x}} d\varphi_{a\sigma_x} \int_{0}^{\chi_p(\varphi_{a\sigma_x})} d\chi_{\sigma_q} \frac{\partial P_{\gamma} [\chi_{\sigma_q} (\varphi_{a\sigma_x})]}{\partial \chi_{\sigma_q}}$$

(15)

where $\varphi_{a\sigma_x} = \varphi_x a_0(\varphi_x)$ and $\varphi_{a\sigma_x'} = \varphi_{a\sigma_x} a_0(\varphi_{a\sigma_x'})$ and $\chi(\varphi) = \chi[F(\varphi)]$ now depends on the, in general non-constant, background $F(\varphi)$ and $\varphi, \varphi' \in [\phi_0, \phi_\infty]$. If the constant crossed field limit is taken of Eq. (15), then the factorised product in Eq. (14) is recovered.

4.1. Neglecting electron spin

As can be seen from the plot of the two-step process in Fig. 4, when the spin of intermediate electrons is taken into account, the total rate is indistinguishable from when the spin is neglected. To make a fair comparison with numerical simulations, we introduce a cutoff for the minimum $\chi_k$ that a scattered photon can have. Then comparing $P_{ccf}^{(2)}$ from Eq. (15) with a probability that neglects electron spin:

$$P_{ccf}^{(2)} = \frac{1}{2} \int_{\phi_0}^{\phi_\infty} d\varphi_{a\sigma_x'} \int_{\phi_0}^{\phi_{a\sigma_x}} d\varphi_{a\sigma_x} \int_{0}^{\chi_p(\varphi_{a\sigma_x})} d\chi_{\sigma_q} \left\langle P_{\gamma} \right\rangle [\chi_{\sigma_q} (\varphi_{a\sigma_x})] \frac{\partial P_{\gamma} [\chi_{\sigma_q} (\varphi_{a\sigma_x})]}{\partial \chi_{\sigma_q}}$$

(16)

where $\langle \rangle$ indicates an average over polarisations has been performed, we find that even for a cutoff as large as $\chi_k = 0.1$, the relative difference in the expected number of double-photon emissions from including the intermediate electron spin is only of the order of a few percent. When looking at polarised single Compton scattering, we saw that the spin-flip channels were greatly suppressed in Fig. 2 and so it should not be a great surprise that they play little role in determining the double Compton scattering rate. We mention here that the precession of electron spin due to interaction with the external field is already included to all orders in the Volkov solution.
4.2. Neglecting simultaneous processes
It was already seen in Fig. 4 that the magnitude of the probability for the one-step process to occur can be larger than for the two-step process when the total length of the external field $L$ is of the order of the formation length $L_*$ so $L \approx L_*$. If we demand the total probability must be positive, this gives a connection between the seed electron’s $\chi$-parameter and the length of field that can no longer be described by a constant crossed field approximation. If we demand $L \gg \tilde{L}$, where $\tilde{L}$ is the smallest length for which the probability is positive, and we interpret “$\gg$” to be an order of magnitude, we see from Fig. 4, that already for $\chi_p \in [0.1, 1000]$ the minimum acceptable value of $L/L_* \in [10, 300]$. This implies that when $L/L_*$ is smaller than this, current forms of numerical simulation can only reliably reproduce single photon emission for a given background.

We emphasise that single photon emission is dominant in the first stages of an electron’s propagation in an external field, but if $L/L_* \gtrsim 2/\alpha$, two-step double photon emission can begin to become as probable as single photon emission. However, since the asymptotic nature of the simultaneous double-photon emission is identical to single-photon emission, there does not seem to arise a situation where simultaneous double-photon emission can be dominant. Therefore the present calculation implies that if the extent of the external field is large enough, the two-step process is still dominant.

If a measurement were sensitive to the generation of double-emitted photons (perhaps through a different transverse distribution as in [16]), an example of when the simultaneous process could influence the number of photons generated is a comparison of the constant crossed field approximation applied to a pulse with electric field of the form $E = E_0 \cos^4(\pi \varphi/2\Phi) \cos \varphi$ for pulse width $\Phi = \kappa_0 \tau$ and amplitude $E_0$. For a laser of power 500 TW, focussed to a focal width of $10 \mu m$ of 910 nm wavelength with peak $a_0 = 10$, and pulse duration 5 fs counterpropagating with 10 GeV seed electrons. The total and sequential probabilities of double photon emission for these parameters are shown in Fig. 5. In Fig. 5 is an example where a qualitative difference occurs due to acknowledging the existence of the simultaneous process, namely that double-photon emission is predicted to begin half a cycle later than if only the sequential process is included.

![Figure 5](image.png)

Figure 5. An example of suppression of the sequential process of double photon emission by the simultaneous process.
5. Conclusion
We have reported on results from [1] for the calculation of double photon emission in a constant crossed field with an emphasis on the consequences for numerical simulation. First, the results for the rate of electron spin-flipping due to emission of a single photon were presented, where it was found that “flip” channels are suppressed with respect to “no-flip” photon emission in a constant crossed field when a particularly useful non-precessing spin projection is used. Then, results for the calculation of the probability of two-photon emission in a constant crossed field were presented, for which it was shown that the probability can be written in terms of a two- and one-step process. Although the two-step process requires consideration of the intermediate electron’s spin, it was found that the assumption in numerical simulations of using unpolarised rates, was accurate to within the few percent level. If numerical simulations include multiple Compton scattering, then the double Compton scattering results imply the constant crossed field can be safely applied when the smallest length scale of the external field is a couple of orders of magnitude larger than the formation length. For many parameter regimes of interest, the sequential process assumed in simulations is dominant.

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