Abstract. We study how the multiscale-geometric structure of the boundary of a domain \( \Omega \subseteq \mathbb{R}^{d+1} \) relates quantitatively to the behavior of its harmonic measure \( \omega_\Omega \). This has been well-studied in the case that the domain has boundary is Ahlfors regular and is uniformly rectifiable, a property that assumes scale-invariant estimates on the multi-scale flatness of the boundary, measured by the so-called Jones \( \beta \)-numbers. In this note we approach the same problem but without uniform estimates on either the measure of the boundary or on the \( \beta \)-numbers. Firstly, we generalize a result of Garnett, Mourgoglou and Tolsa by showing that domains in \( \mathbb{R}^{d+1} \) whose boundaries are just lower \( d \)-content regular admit Corona decompositions for harmonic measure if and only if the square sum of the generalized Jones \( \beta \)-numbers is finite. Secondly, for semi-uniform domains with Ahlfors regular boundaries, it is known that uniform rectifiability implies harmonic measure is \( A_\infty \) for semi-uniform domains, but now we give more explicit dependencies on the \( A_\infty \)-constant in terms of the uniform rectifiability constant. This follows from a more general estimate that does not assume the boundary to be uniformly rectifiable that relates a log integral of the Poisson kernel to the square sum of \( \beta \)-numbers. For general semi-uniform domains, we also show how to bound the harmonic measure of a subset in terms of that sets Hausdorff measure and the square sum of \( \beta \)-numbers on that set.

Using these results, we give estimates on the fluctuation of Green’s function in a uniform domain in terms of the \( \beta \)-numbers. As a corollary, for bounded NTA domains, if \( B_\Omega = B(x_\Omega, c \text{ diam } \Omega) \) is so that \( 2B_\Omega \subseteq \Omega \), we obtain that

\[
(diam \partial \Omega)^d + \int_{\Omega \setminus B_\Omega} \left| \frac{\nabla^2 G_\Omega(x_\Omega, x)}{G_\Omega(x_\Omega, x)} \right|^2 \text{dist}(x, \Omega^c)^3 \, dx \sim \mathcal{H}^d(\partial \Omega).
\]

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1. INTRODUCTION

1.1. Background. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain and $\omega_{\Omega}$ denote its harmonic measure. This paper continues a long trend of trying to understand the quantitative relationship between the behavior of $\omega_{\Omega}$ and the geometry of the boundary.

One geometric feature of the boundary which has a well-established connection with $\omega_{\Omega}$ is rectifiability. We will say that a measure $\mu$ is $d$-rectifiable if it may be covered up to $\mu$ measure zero by $d$-dimensional Lipschitz graphs, and a set $E \subseteq \mathbb{R}^n$ is $d$-rectifiable if $\mathcal{H}^d|_E$ is a $d$-rectifiable measure, where $\mathcal{H}^d$ denotes $d$-dimensional Hausdorff measure. In one direction, the most general qualitative result says that for $\Omega \subseteq \mathbb{R}^{d+1}$ and $E \subseteq \partial \Omega$, then $\omega_{\Omega}|_E \ll \mathcal{H}^d|_E$ implies $\omega_{\Omega}|_E$ is $d$-rectifiable, and in fact there is $E' \subseteq E$ that is $d$-rectifiable and $\omega_{\Omega}(E \setminus E') = 0$ [AHM+16], which was previously only known for simply connected planar domains [Pom86]. In the reverse direction, there isn’t a result quite as general: for rectifiability to imply absolute continuity, some fatness condition on the boundary is required.

**Definition 1.1.** A domain $\Omega \subset \mathbb{R}^{d+1}$ is said to have large complement if there is $c > 0$ so that

\[(1.1) \quad \mathcal{H}^d_\infty(B \setminus \Omega) \geq cr_B^d \text{ for all } B \text{ centered on } \partial \Omega \text{ with } 0 < r_B < \text{diam } \partial \Omega.\]
We will say that $E \subseteq \mathbb{R}^n$ is lower $d$-content regular if there is $c > 0$ so that
\begin{equation}
\mathcal{H}^d(B \setminus E) \geq c r_B^d \quad \text{for all } B \text{ centered on } E \text{ with } 0 < r_B < \text{diam } E.
\end{equation}

A converse to the aforementioned theorem holds in this setting: if $\Omega \subseteq \mathbb{R}^{d+1}$ has large complement and $\omega_\Omega|_E \subseteq \partial \Omega$ is $d$-rectifiable for some $E \subseteq \partial \Omega$, then $\omega_\Omega|_E \ll \mathcal{H}^d$ [AAM19] (see also [Wu86]). We must caution here that the definition of rectifiability of measures we are using here in describing these results is not standard: Federer rectifiability says a measure $\mu$ is covered up to $\mu$-measure zero by $d$-dimensional Lipschitz images of $\mathbb{R}^d$, as opposed to Lipschitz graphs. This is a really subtle point: When $\mu = \mathcal{H}^d|_E$, then these two notions are equivalent, but this is not so for general measures, even for quite well-behaved measures like doubling measures [GKS10]. It is not true that Federer rectifiability implies $\omega_\Omega$ is absolutely continuous, although it does hold for simply connected planar domains [BJ90]. To guarantee that $\mathcal{H}^d|_\Omega \ll \omega_\Omega$, it is sufficient that $\partial \Omega$ is rectifiable and to assume that the interior is not collapsing near $\partial \Omega$, i.e. $\lim sup_{r \to 0} |\Omega \cap B(x,r)| / |B(x,r)| > 0$ for $\mathcal{H}^d$-a.e. $x \in \partial \Omega$. [ABHM16].

Our interest will be on the quantitative relationship between $\omega_\Omega$ and the geometry of $\Omega$, and in fact many of the results above are deduced using quantitative methods. We will give a short synopsis of these results below, but for a good survey on the state-of-the-art concerning quantitative absolute continuity of harmonic measure, see [Hof19].

For quantitative results, it has been natural to work in the Ahlfors regular setting, since then many of the classical harmonic analytic techniques in Euclidean space can be repeated in this setting.

**Definition 1.2.** We say $E \subseteq \mathbb{R}^{d+1}$ is $C$-Ahlfors $d$-regular (or $C$-AR) if
\begin{equation}
C^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq C r^d \quad \text{for all } x \in E, \ 0 < r < \text{diam } E.
\end{equation}

The quantitative analogue of absolute continuity in this setting are the $A_\infty$ and weak-$A_\infty$ conditions.

**Definition 1.3.** If $\Omega \subseteq \mathbb{R}^{d+1}$ and $\partial \Omega$ is AR, we will say $\omega_\Omega \in A_\infty$ (resp. weak-$A_\infty$) if for every $\varepsilon > 0$ there is $\delta > 0$ so that whenever $B$ is a ball centered on $\partial \Omega$ with $0 < r_B < \text{diam } \partial \Omega$ and $F \subseteq \partial \Omega \cap B$, then $\mathcal{H}^d(F) < \delta r_B^d$ implies $\omega_\Omega^\varepsilon(F) < \varepsilon \omega_\Omega^\varepsilon(B)$ (resp. $\omega_\Omega^\varepsilon(F) < \varepsilon \omega_\Omega^\varepsilon(2B)$) whenever $x \in \Omega \setminus 4B$.

The quantitative analogue of rectifiability is uniform rectifiability.

**Definition 1.4.** A set $E \subseteq \mathbb{R}^n$ is uniformly $d$-rectifiable (UR) if it is $d$-AR and there are constants $\theta, M > 0$ such that for all $x \in E$ and all $0 < r \leq \text{diam } E$ there is an $M$-Lipschitz mapping $g : B_d(0,r) \subseteq \mathbb{R}^d \to \mathbb{R}^n$ such that
\[
\mathcal{H}^d(E \cap B(x,r) \cap g(B_d(0,r))) \geq \theta r^n.
\]

UR sets were introduced by David and Semmes in connection to singular integrals on Ahlfors regular sets (see [DS91] and [DS93]), however they appear very naturally in the study of harmonic measure as we shall see below.
It also turns out that, for quantitative results about harmonic measure, the connectivity of the domain plays an important role. We review some various kinds of connectivity.

**Definition 1.5.** Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an open set.

1. For $x, y \in \Omega$, we say a curve $\gamma \subseteq \Omega$ is a $C$-cigar curve from $x$ to $y$ if
   \[ \min \{ \ell(x, z), \ell(y, z) \} \leq C \text{dist}(z, \Omega^c) \]
   for all $z \in \gamma$, where $\ell(a, b)$ denotes the length of the sub-arc in $\gamma$ between $a$ and $b$. We will also say it has bounded turning if $\ell(\gamma) \leq C |x - y|$. \[
(1.1) \]

2. If there is $x \in \Omega$ such that every $y \in \Omega$ is connected to $x$ by a curve $\gamma$ so that $\ell(y, z) \leq C \text{dist}(z, \Omega^c)$ for all $z \in \Gamma$, we say $\Omega$ is $C$-John.

3. If every pair $x \in \Omega$ and $\xi \in \partial \Omega$ are connected by a $C$-cigar with bounded turning, then we say $\Omega$ is $C$-semi-uniform (SU).

4. One says that that $\Omega$ satisfies the weak local John condition (WLJC) with parameters $\lambda, \theta, \Lambda$ if there are constants $\lambda, \theta \in (0, 1)$ and $\Lambda \geq 2$ such that for every $x \in \Omega$ there is a Borel subset $F \subset B(x, \Lambda \delta \Omega(x)) \cap \partial \Omega$ with $H^d(F) \geq \theta H^d(B(x, \Lambda \delta \Omega(x)) \cap \partial \Omega)$ such that every $y \in F$ can be joined to $x$ by a $\lambda$-cigar curve.

5. If every $x, y \in \Omega$ are connected by a $C$-cigar of bounded turning, we say $\Omega$ is uniform.

6. For a ball $B$ of radius $r_B$ centered on $\partial \Omega$, we say $x \in B$ is an interior/exterior $c$-corkscrew point or that $B(x, cr_B)$ is an interior/exterior $c$-corkscrew ball for $\Omega \cap \text{Bif } B(x, 2cr_B) \subseteq B \cap \Omega$ (or $B(x, 2cr_B) \subseteq B \setminus \Omega$). We say $\Omega$ satisfies the interior $c$-Corkscrew condition if every ball $B$ on $\partial \Omega$ has an interior (or exterior) $c$-corkscrew point.

7. A uniform domain with exterior corkscrews is nontangentially accessible (NTA).

8. An NTA with AR boundary is a chord-arc domain (CAD).

The notion of an NTA domain was introduced by Jerison and Kenig in [JK82]; there they codified many scale invariant properties for harmonic measure that had been known for Lipschitz domains, but their crucial observation was that it was not the Lipschitz structure but the nontangential connectedness that was guaranteeing these properties. Later, Aikawa and Hirata also observed that many of these properties implied good connectivity of the domain as well [Aik06, Aik08, AH08].

Various sufficient [Dah77, DJ90, Sem90] and necessary [HM14, HMIT14, HM15, MT15, AHM+17, HLMN17] conditions have been given for the $A_\infty$ and weak-$A_\infty$ to hold, and we will discuss more of these below. Particular mention should go to [Dah77], who showed that $\omega_\Omega \in A_\infty$ when $\Omega$ is a Lipschitz domain, by which we mean $\omega_\Omega = k d\sigma$ where $\sigma = H^d|_{\partial \Omega}$ and

\[
[\omega]_{A_\infty} := \sup_B \exp \left( \int_B \log \frac{1}{k} d\sigma \right) \int_B k d\sigma < \infty
\]
where the supremum is over all balls $B$ centered on $\partial \Omega$ with $0 < r_B < \text{diam } \partial \Omega$. This is equivalent to the more familiar definition of $A_\infty$ by [Hru84] (see also [Ste93, V.6.6.3]). In fact, Dahlberg showed the stronger reverse Hölder inequality

$$\left( \int_B k^2 d\sigma \right)^{\frac{1}{2}} \lesssim \int_B k d\sigma.$$  

The works of David and Jerison [DJ90] and Semmes [Sem90] who proved $\omega_\Omega \in A_\infty$ when $\Omega$ is a CAD. In their proofs, they actually reduce things to Dahlberg’s theorem by approximating a CAD domain quantitatively from within by Lipschitz domains.

Very recently, however, building on the techniques in these papers, a complete characterization has been obtained. In [Azz17], we showed $\omega \in A_\infty$ if and only if $\Omega$ is SU and $\partial \Omega$ is UR, although shortly after the definitive characterization of weak-$A_\infty$ was obtained (and implies the SU case).

**Theorem 1.6.** [AHMMT19] Let $\Omega \subset \mathbb{R}^{d+1}$, $d \geq 2$, be an open set with AR boundary and interior corkscrew condition. Then $\partial \Omega$ is UR and $\Omega$ satisfies the WLJC if and only if $\omega \in \text{weak } A_\infty$.

Here, weak-$A_\infty$ means that there is $q > 0$ so that if $\omega^x_\Omega = kd \mathcal{H}^d|_{\partial \Omega}$ and $x \in \Omega \setminus 4B$ where $B$ is centered on $\partial \Omega$, and $\sigma = \mathcal{H}^d|_{\partial \Omega}$, then

$$\left( \int_B k^{1+q} d\sigma \right)^{\frac{1}{q+1}} \lesssim \int_{2B} k d\sigma.$$  

With better information about harmonic measure comes better information about the geometry of the boundary. In particular, there is a class of “small constant” results that effectively say that if harmonic measure is “very” much like surface measure, then the boundary must be “very” flat. In [KT97] and [KT03, Main Theorem], for example (and after reviewing the discussion after [HMT10, Definition 4.1.9], which explains how some definitions in these results are equivalent) Kenig and Toro show that if a domain $\Omega \subseteq \mathbb{R}^{d+1}$ whose boundary is Ahlfors regular and sufficiently flat$^1$ and $\omega = kd \mathcal{H}^d|_{\partial \Omega}$, then $\log k \in \text{VMO}$ if and only if the unit normal vector on $\partial \Omega$ is in VMO. This in turn implies that the boundary has very big pieces of Lipschitz graphs with small Lipschitz constant [HMT10, Theorem 4.2.4]. In fact, in [KT97], Kenig and Toro also show that for all $\delta > 0$, if $\partial \Omega$ is Reifenberg flat and $\sigma(B) \leq (1 + \varepsilon)(2r_B)^d$, then for $\varepsilon > 0$ small enough, there is $q > 0$ so that

$$\left( \int_B k^{1+q} \right)^{\frac{1}{1+q}} \leq (1 + \delta) \int_B k.$$  

Moreover, with additional smoothness on $\log k$ comes additional smoothness of the boundary, see for example [Eng16].

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$^1$The flatness condition originally stipulated that the boundary was Reifenberg flat and that the unit normal had sufficiently small BMO norm, although Bortz and Engelstein showed that this latter property implies the former [BE17]
In total, the (weak-)\(A_\infty\) and VMO conditions on harmonic measure are well-studied, and UR plays a crucial role. We mention one last way of describing the quantitative relationship between the geometry and harmonic measure when it is not \(A_\infty\). It may seem a bit ad hoc but is a very convenient form of quantitative absolute continuity. For this result below, we will refer to Christ-David cubes, so see Theorem 2.1 below if you are not familiar with these. For a measure \(\mu\) and a set \(A\), define

\[
\Theta^d_\mu(A) = \frac{\mu(A)}{(\text{diam } A)^d}.
\]

**Definition 1.7** (Corona Decomposition for Harmonic Measure (CDHM)). Let \(\Omega \subseteq \mathbb{R}^{d+1}\) be a domain and \(E = \partial \Omega \subseteq \mathbb{R}^{d+1}\) be lower \(d\)-content regular, \(A > 1 > \tau > 0\), \(\lambda \geq 1\), and let \(Q_0 \in \mathcal{D}\) Suppose there are cubes \(\text{Top}\) in \(Q_0\) and a partition \(\{\text{Tree}(R) : R \in \text{Top}\}\) of the cubes in \(Q_0\) into stopping-time regions so that for each \(R \in \text{Top}\), there is a (interior) corkscrew ball \(B(x_R, c\ell(R)) \subseteq B_R \cap \Omega\) so that for all \(Q \in \text{Tree}(R)\),

\[
\tau \Theta^d_\omega(\lambda B_R) \leq \Theta^d_\omega(\lambda B_Q) \leq A \Theta^d_\omega(\lambda B_R).
\]

For \(Q_0 \in \mathcal{D}\), we let

\[
\text{CDHM}(Q_0, \lambda, A, \tau) = \inf \sum_{R \in \text{Top}} \ell(R)^d.
\]

where the infimum is over all possible decompositions \(\{\text{Tree}(R) : R \in \text{Top}\}\) satisfying the conditions above.

In \([\text{GMT}18, \text{Theorem 1.3}]\), Garnett, Mourgoglou and Tolsa showed that if \(E = \partial \Omega\) is Ahlfors regular and \(\Omega\) has the interior corkscrew property, then for all \(\lambda > 1\) there are \(A, \tau\) so that \(E\) is UR if and only if

\[
\text{CDHM}(Q_0, \lambda, A, \tau) \lesssim \ell(Q_0)^d \text{ for all } Q_0 \in \mathcal{D}.
\]

In other words, while harmonic measure may not be \(A_\infty\) in this theorem, the CDHM is the strongest statement one can make in the AR setting about how the density of \(\omega_{\Omega}\) behaves if the boundary is UR in the absence of any assumptions about connectivity.

The objective of this paper is to try and further quantify the behavior of harmonic measure, both in AR and (more importantly) non-AR settings. In particular, the results mentioned above usually assume some uniform control on harmonic measure or the boundary: the boundary is uniformly rectifiable, or the surface measure satisfies (1.3) uniformly over all balls. The (weak)\(-A_\infty\) condition on harmonic measure and the BMO/VMO conditions on \(\log k\) are also statements that hold uniformly over all balls. We would instead like to study harmonic measure when either there the surface measure, the rectifiable structure, or our estimates on harmonic measure is allowed to vary between balls.
As a motivating example, note that if $\Omega$ is a CAD so that $B(0, c) \subseteq \Omega$ and $\partial \Omega \subseteq B(0, C)$ for some constants $c, C > 0$, say, then the results of David, Jerison and Semmes (and using the scale invariant estimate Lemma 2.15 below) imply that $\omega_\Omega^0$ is $A_\infty$. Now if $\Omega$ is simply uniform with Ahlfors regular boundary but not a CAD, then this can fail since $\omega_\Omega$ may not be $A_\infty$. For instance, recall the Garnett example or 4-corner Cantor set: For $k = 0, 1, 2, 3$, let $f_k(x) = (x/4 + e^{ik\pi/4}1/2\sqrt{2})e^{ik\pi/2}$. Let $K_0 = [-1/2, 1/2]^2$ and for $j > 0$ set

$$K_j := \bigcup_{i=0}^{3} f_i(K_{j-1}).$$

Then $K := \bigcap_{j=0}^\infty K_j$ is an Ahlfors 1-regular set, and $\Omega = K^c$ is a uniform domain so that $\omega_\Omega \perp \mathcal{H}^1$. See Figure 1.

Note however that each of the $K_j^c$ are CADs that are uniformly AR and uniform, but the exterior corkscrew constant is worsening with $j$. In particular, if $\omega_j = \omega_{K_j}^0$ are the respective harmonic measures for these domains with Radon-Nikodym derivatives $k_j$, then

$$(1.5) \quad \exp \left( \int_{\partial K_j^c} \log \frac{1}{k_j} \, dx \right) \int_{\partial K_j^c} k_j \, dx$$

must be going to infinity since the limit of these measures is $\omega_{K^c} \notin A_\infty$. How fast should it go to infinity? If we took our domain to instead be the complement of a finite union of squares covering $K$ of different sizes instead of all being of size $4^{-j}$, how does this affect the integral?

Another natural set of questions is what happens in the non-AR or non-uniform settings? The integrals above may not be useful anymore, but is there another way of quantitatively describing the behavior of harmonic measure? There are plenty of qualitative results in this setting about when $\omega_\Omega$ is absolutely continuous or not, but less about the quantitative behavior.

A first question may be what is the analogue of UR for sets that aren’t AR? Or what geometric quantity or property of the boundary should we use in studying harmonic measure? What we found to be appropriate were $\beta$-numbers, which we now describe.
We first recall Jones’ β-numbers: If \( I \) is a cube in \( \mathbb{R}^d \) and \( E \subseteq \mathbb{R}^d \) is compact, define
\[
\beta_E(I) = \inf_L \sup_{x \in E \cap I} \text{dist}(x, L)/\ell(I)
\]
where the infimum is over all lines. This measures in a scale invariant way how close \( E \) is to being contained in a line in \( I \). Jones showed in [Jon90] for \( d = 2 \) (and Okikiolu for general \( d \) [Oki92]) that the quantity
\[
\text{diam } E + \sum_{I \cap E \neq \emptyset} \beta_E(3I)^2 \ell(I)
\]
is comparable to the length of the shortest curve containing \( E \) (where the sum is over all dyadic cubes \( I \)). This is called the analyst’s traveling salesman theorem. There isn’t a perfect analogue of this result for higher dimensional sets, first of all because it is not clear what the analogue of a curve should be, though there are some results that do generalize this in some sense. For technical reasons, a more suitable β-number is required (see the introduction to [AS18] for a further discussion about why one is needed).

For arbitrary sets \( E \) and \( B \), \( p > 0 \), and a \( d \)-dimensional plane \( L \), define
\[
\beta_{E,B}^d(B, L) = \left( \frac{1}{2r_B} \int_0^1 \mathcal{H}^d_{\infty}(\{x \in B \cap E : \text{dist}(x, L) > tr_B\})t^{p-1}dt \right)^{\frac{1}{p}}
\]
where \( 2r_B = \text{diam } B \), and set
\[
\beta_{E,B}^d(B) = \inf\{\beta_{E,B}^d(B, L) : L \text{ is a } d\text{-dimensional plane in } \mathbb{R}^n\}.
\]

Note that if \( E \) is Ahlfors regular and \( B \) is a ball, then \( \mathcal{H}^d_{\infty} \sim \mathcal{H}^d \), and the above integral is comparable to
\[
\beta_{E,B}^d(B, L) \sim \inf_L \left( \int_B \left( \frac{\text{dist}(x, L)}{r_B} \right)^2 d\mathcal{H}^d|_{E}(x) \right)^{\frac{1}{2}},
\]
that is, the \( L^2 \)-average distance to a \( d \)-dimensional plane.

These numbers are variants of β-numbers defined by David and Semmes in [DS91], where they showed that, if \( E \) is Ahlfors \( d \)-regular, then \( E \) is UR if and only if, for every \( R \in \mathcal{D} \),
\[
\beta_E(R) := \ell(R)^d + \sum_{Q \subseteq R} \beta_{E,Q}^d(3BQ)^2 \ell(Q)^d \lesssim \ell(R)^d.
\]

We will call the sum the linear deviation of \( E \) in \( R \), since it measures how non-flat the set \( E \) is at all scales and locations in \( R \). Thus, UR is equivalent to having uniform control on the linear deviation inside every cube \( R \). However, even if a set \( E \) is not UR, this quantity can still be finite. For example,
\[
\mathcal{H}^d(Q_0) \sim \beta_{\partial Q_0}^d(Q_0) \quad \text{if } \Omega \text{ is NTA and } Q_0 \subseteq \partial \Omega
\]
This follows from [AS18, Corollary III]. Moreover, in [Vil], Villa extends this to more general surfaces than boundaries of NTA domains. Thus, this more resembles the original traveling salesman theorem of Jones mentioned earlier.

The quantity $\beta_E(Q_0)$ still has some significance, even if $E$ is a set where $\beta_E(Q_0)$ and $H^d(Q_0)$ are not comparable. In fact, in [AV19], the author and Villa show that the linear deviation is comparable to many other quantities measuring how much $E$ deviates from satisfying certain geometric properties over all scales and locations. The main results below can be seen as a companion to this paper, where instead we now show how the linear deviation compares to different quantities involving harmonic measure and Green’s function, and it seems to be a natural way of extending some results about harmonic measure to the non-AR setting. So instead of dealing with UR sets below, we will work with the class of sets $E$ where $\beta_E(R)$ is just finite for each cube $R$.

1.2. Main Results. Our first result is a version of Garnett, Mourgoglou and Tolsa’s theorem for lower regular sets.

**Theorem I.** Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an interior $c$-corkscrew domain with lower $d$-content regular boundary and let $\mathcal{D}$ be the Christ-David cubes for $\partial \Omega$. Then for all $\lambda \geq 1$ and for all $A, \tau^{-1}$ sufficiently large depending on $\lambda$, and for all $Q_0 \in \mathcal{D}$,

$$\beta_{\partial \Omega}(Q_0) \sim_{\lambda, A, \tau, c} CDHM(Q_0, \lambda, A, \tau)$$

This has some nicer looking consequences when we know more about our domain. For example, we have the following:

**Theorem II.** Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a SU domain with lower $d$-content regular boundary and let $\mathcal{D}$ be the Christ-David cubes for $\partial \Omega$. Then for $\lambda \geq 1$, for $M, A, \tau^{-1}$ large enough, and $Q_0 \in \mathcal{D}$, if $\omega = \omega^{x_0}_{\Omega}$ where $x_0 \in \Omega \setminus \lambda B_{Q_0}$, there is a partition of the cubes in $Q_0$ into stopping-time regions $\{\text{Tree}(R); R \in \text{Top}\}$ so that

1. $\tau \Theta^d_{\omega}(\lambda B_R) \leq \Theta^d_{\omega}(\lambda B_Q) \leq A\Theta^d_{\omega}(\lambda B_R)$ for all $Q \in \text{Tree}(R)$.
2. We have

$$\sum_{R \in \text{Top}} \ell(R)^d \sim \beta_{\partial \Omega}(Q_0).$$

That is, if the domain is SU, then we can improve over Theorem I by keeping the pole for $\omega$ in our trees the same.

If $\beta_{\partial \Omega}(Q_0) < \infty$, then we can find trees so that the following holds:

3. If $\text{Stop}(R)$ denotes the minimal cubes of $\text{Tree}(R)$ and $Q \in \text{Stop}(R)$, then either $\Theta^d_{\omega}(Q) \sim \tau \Theta^d_{\omega}(\lambda B_R)$ or $\Theta^d_{\omega}(\lambda B_Q) \sim A \Theta^d_{\omega}(\lambda B_R)$ (with implied constants independent of $\tau, \lambda$ and $A$).

We can use this and Theorem II to show how the $\beta$-numbers give estimates on the Poisson kernel for SU domains with AR boundary in the following theorem, which can be seen as an integral form of Theorem II.
Theorem III. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a semi-uniform domain with Ahlfors regular boundary and let $\mathcal{D}$ be the Christ-David cubes. There is $M > 0$ depending on the semi-uniformity and Ahlfors regularity constants so that the following holds. For $Q_0 \in \mathcal{D}$, let $x_0 \in \Omega \setminus MB_{Q_0}$ and $\omega = \omega_{Q_0}^{x_0}$, and suppose $\omega_{Q_0}^{x_0} \ll H^d$. Let $k$ be the Radon-Nikodym derivative of $\omega$ in $Q_0$. Then
\begin{equation}
1 + \int_{Q_0} \log \frac{1}{k} dH^d + \log \left( \int_{Q_0} kdH^d \right) \sim \frac{\beta_{\partial\Omega}(Q_0)}{\ell(Q_0)^d}
\end{equation}
Above, the constants only depend on the semi-uniformity and Ahlfors regularity.

We prove a slightly more general statement than Theorem III in Theorem 5.1 below that allows for the scenario when $\omega_{Q_0}^{x_0} \ll H^d$.

Indeed, a word of caution: the above theorems do not say that, if $\beta_{\partial\Omega}(Q_0) < \infty$, then $\omega_{\Omega} \ll H^d$. In [AM17], we showed with Mourougoulo and Tolsa that there was an NTA domain $\Omega \subseteq \mathbb{R}^{d+1}$ with $H^d(\partial\Omega) < \infty$ so that $\omega_{\Omega} \ll H^d$, meaning there is $E \subseteq \partial\Omega$ with $\omega_{\Omega}(E) > 0 = H^d(E)$. Since $\beta_{\partial\Omega}(R) \sim H^d(\partial\Omega) < \infty$ by [AS18, Corollary III], this means we can still find trees satisfying the conclusions of the above theorems, but the densities $\Theta_d(\lambda B_Q)$ can still diverge on a set of positive harmonic measure. Thus, Theorems I and II cannot be improved to imply absolute continuity of harmonic measure. This counterexample, however, isn’t relevant to Theorem III since its boundary is not Ahlfors regular. Nonetheless, we show that we still cannot conclude absolute continuity in this setting due to the following example:

Theorem IV. There is a domain $\Omega \subseteq C$ so that $\partial\Omega$ is 1-Ahlfors regular, $\beta_{\partial\Omega}(\partial\Omega) < \infty$, and there is $E \subseteq \partial\Omega$ so that $\omega(E) > 0 = H^1(E)$. In particular, $\Omega$ is a uniform domain with 1-rectifiable and 1-Ahlfors regular boundary whose harmonic measure has a singular set.

However, we do have that, under the assumptions of the Theorem I (or any of the theorems above), if $\beta_{\partial\Omega}(R) < \infty$ for all $R \in \mathcal{D}$, then $H^d|_{\partial\Omega} \ll \omega_{\Omega}$. This follows by the main result of [ABHM16] and the fact that $\partial\Omega$ is rectifiable by [AS18, Theorem II].

Theorem III implies that, if $A = e^{-1}$, there are constants $C_1, C_2 > 0$ so that
\begin{equation}
A \exp \left( \frac{C_1 \beta_{\partial\Omega}(Q_0)}{\ell(Q_0)^d} \right) \leq \exp \left( \int_{Q_0} \log \frac{1}{k} dH^d \right) \int_{Q_0} kdH^d \leq A \exp \left( \frac{C_2 \beta_{\partial\Omega}(Q_0)}{\ell(Q_0)^d} \right)
\end{equation}
and this bounds the familiar term that formerly characterized the $A_\infty$ condition. In particular, if $\omega \ll H^d$ and $\beta_{\partial\Omega}(Q_0) \leq C_0 H^d(Q_0)$ for all $Q_0 \in \mathcal{D}$, then this shows $\omega \in A_\infty$ in a much longer way, but now it is more transparent to see how the $A_\infty$-constant in (1.4) depends on the Carleson constant $C_0$: we see $[\omega]_{A_\infty} \leq \exp(C_2C_0)$ where $C_2$ depends only on the Ahlfors regularity of $\partial\Omega$ and the SU constants.

Even if the boundary is not UR, the $\beta$-numbers now allow us to estimate how badly harmonic measure fails to be $A_\infty$: recalling Garnett’s example, one can show
\( \beta_{\partial(K_j)}(K_j) \sim j \), and so the above theorem implies that (1.5) must be growing exponentially in \( j \).

We also obtain a local version of Theorem III, which allows us to estimate the harmonic measure of a subset of the boundary of a SU domain directly in terms of that set’s Hausdorff content and the \( \beta \)-numbers around that set:

**Theorem V.** Let \( \Omega \) be a SU domain with LCR boundary. Let \( Q_0 \in \mathcal{D} \) and \( E \subseteq Q_0 \). Then for \( M \) large enough, \( x_0 \in \Omega \setminus MB_{Q_0} \), and \( \omega = \omega^x_\Omega \)

\[
\frac{\omega(E)}{\omega(Q_0)} \geq \exp \left( -C \frac{\ell(Q_0)^d}{\mathcal{H}_\infty^d(E)} \exp \left( C \sum_{Q \subseteq E \neq \emptyset} \beta_{\partial \Omega}(MB_Q)\ell(Q)^d \right) \right).
\]

Finally, we give a continuous version of the above results for uniform domains which relates the linear deviation of the boundary of a uniform domain to the affine deviation of Green’s function in the domain

**Theorem VI.** Let \( \Omega \subseteq \mathbb{R}^{d+1} \) be a bounded uniform domain with lower \( d \)-content regular boundary. Let \( B_\Omega = B(x_\Omega, c \operatorname{diam} \partial \Omega) \) be so that \( 2B_\Omega \subseteq \Omega \) and \( g = G_\Omega(x_\Omega, \cdot) \). Then

\[
(1.10) \quad (\operatorname{diam} \partial \Omega)^d + \int_{\Omega \setminus B_\Omega} \left| \frac{\nabla^2 g(x)}{g(x)} \right|^2 \operatorname{dist}(x, \Omega^c)^3 \, dx \\
\sim (\operatorname{diam} \partial \Omega)^d + \sum_{Q \subseteq \partial \Omega} \beta_{\partial \Omega}^2(3B_Q)^2\ell(Q)^d.
\]

In particular, if \( \Omega \) is an NTA domain, then

\[
(1.11) \quad (\operatorname{diam} \partial \Omega)^d + \int_{\Omega \setminus B_\Omega} \left| \frac{\nabla^2 g(x)}{g(x)} \right|^2 \operatorname{dist}(x, \Omega^c)^3 \, dx \sim \mathcal{H}^d(\partial \Omega)
\]

The left side measures affine deviation as \( \left| \frac{\nabla^2 g}{g \operatorname{dist}(x, \Omega^c)^2} \right| \) measures in a scale and dilation invariant way how close \( g \) is to being linear around \( x \). This has an analogy with a result of Bishop and Jones in the complex plane: For a conformal mapping \( \phi : \mathbb{D} \to \Omega \), its Schwarzian derivative is defined as

\[
S\phi = \left( \frac{\phi''}{\phi'} \right)' - \frac{1}{2} \left( \frac{\phi''}{\phi'} \right)^2 = g'' - \frac{1}{2} (g')^2 \quad \text{where} \quad g = \log \phi'.
\]

Much like how \( \nabla^2 g/g \) measures how affine \( g \) is (in the sense that \( g \) is affine if this is zero), the Schwarzian derivative of a conformal map \( \phi \) measures how close it is to being a Möbius transformation (that is, \( \phi \) is a Möbius transformation if \( S\phi \) is identically zero).

The following result of Bishop and Jones (see also [GM08, Chapter X, Lemma 6.1 and Theorem 6.2] relates this quantity measuring deviation from being a Möbius map to the size of the boundary.
Theorem 1.8. \cite{BJ94} Let $\Omega \subseteq \mathbb{C}$ be a simply connected planar domain and $\phi : \mathbb{D} \to \Omega$ be the Riemann mapping. If $\partial \Omega$ is rectifiable, then
\[
\text{diam } \Omega + \int_{\Omega} |\phi'(z)||S\phi(z)|^2 \text{dist}(z, \Omega^c)^3 \, dz \lesssim H^1(\partial \Omega).
\]

The opposite inequality holds if $\Omega$ is uniform.

In fact, the papers \cite{BJ90, BJ94} are the first to explore the relationship between $\beta$-numbers and harmonic measure (and conformal mappings).

Also see \cite{DEM18}, where they also study the fluctuation of a certain smoothed form of the distance to the boundary and relate that to the (uniform) rectifiable structure of the boundary for domains whose boundaries have higher codimension.

The motivation for comparing all these quantities to the $\beta$-numbers as opposed to some other quantity is twofold. Firstly, the $\beta$-numbers are more precise to estimate, and they give the most useful information (for example, one can use them directly to construct bi-Lipschitz David-Toro parametrizations, see Lemma 2.8 below). Secondly, in \cite{AV19}, we show that the linear deviation is comparable to a trove of other quantities that measure the multiscale geometry of a set. For example, below we will use one such result that the linear deviation is comparable to the sum of cubes where the BAUP condition fails that were originally developed and studied by David and Semmes for uniformly rectifiable sets \cite{DS91, DS93}. Thus, one can more easily estimate harmonic measure or Green’s function using the above theorems if one knows something about the multiscale geometry. We already saw how this allowed us to compute the integral in (1.9) in the case of Garnett’s example. More generally, due to the BAUP estimate in Theorem 4.8 below and Theorem III above, for every cube $Q \in \mathcal{D}$ around which $\partial \Omega$ does not resemble a finite union of planes, that contributes approximately $\ell(Q)^d$ to the size of the integral in (1.9).

1.3. Outline. Theorem I and II will be proven in Sections 3 and 4 below. In fact, Theorem II can be seen as a corollary of the proof of I and will be proven simultaneously. Showing the $\gtrsim$ estimate in Theorem I is the focus of Section 3. Observe that in the proof of the analogous result in \cite{GMT18} that UR implies the CDHM, they instead prove that $\varepsilon$-approximability implies the CDHM, and then use the result of Hofmann, Martell and Mayboroda \cite{HMM14} that UR implies $\varepsilon$-approximability (see these papers for the definition of $\varepsilon$-approximability). Instead, our main idea is similar to the proofs \cite{Sem90} and \cite{DJ90}: we carefully construct some chord-arc subdomains where we know the $A_\infty$ property holds for harmonic measure and apply the maximum principle to get estimates on how the densities behave. However, some care is needed since unlike \cite{DJ90}, we will require infinitely many such subdomains, and we don’t have nice corona decompositions by interior Lipschitz domains as in \cite{Sem90} or \cite{HMM14} to work with, for example, since we aren’t assuming the boundary is UR. To overcome this, we use the David-Toro parametrizations
from [DT12] to build them from scratch. Ultimately, the set-up for the stopping-time procedure is similar to that in [GMT18], but we need to add in a few more stopping-time conditions.

The \( \lesssim \) estimate in Theorem I is based on the proof of the main result in [HLMN17], however some care is needed since in our setting we do not assume Ahlfors regularity. We will need to use a few new results from [AV19]: the corona decomposition of lower regular sets by Ahlfors regular sets (Lemma 2.7 below) and a generalization of David and Semmes bilateral approximation by planes estimate (Theorem 4.8 below). The former result allows us to effectively pretend that our setting is Ahlfors regular (or at least partition the surface cubes into trees where we can pretend).

The proof of Theorem III is given in Section 5. To prove this, we use Theorem II and perform a martingale-type decompositions similar to those that appear [FKP91] when Fefferman, Kenig, and Pipher study dyadic \( A_\infty \)-weights. We prove Theorem V in the following section and it has a similar proof, but care is needed to account for the lack of Ahlfors regularity. We then move on to Theorem IV in Section 7 by adapting the techniques of Batakis in [Bat96].

Finally, we prove Theorem VI in Section 8. The \( \lesssim \) is mostly the same as the proof of the second part of Theorem I, although things are simpler since we assume uniformity). The \( \gtrsim \) estimate is shown in Section 8.2, and this requires more work. We actually prove a result that holds for more general functions than Green functions, see Theorem 8.2 below.

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2. Preliminaries

2.1. Notation. We will write \( a \lesssim b \) if there is a constant \( C > 0 \) so that \( a \leq Cb \) and \( a \lesssim_t b \) if the constant depends on the parameter \( t \). As usual we write \( a \sim b \) and \( a \sim_t b \) to mean \( a \lesssim b \lesssim a \) and \( a \lesssim_t b \lesssim_t a \) respectively. We will assume all implied constants depend on \( d \) and hence write \( \sim \) instead of \( \sim_d \).

Whenever \( A, B \subset \mathbb{R}^{d+1} \) we define

\[
\text{dist}(A, B) = \inf\{|x - y|; x \in A, y \in B\}, \quad \text{and dist}(x, A) = \text{dist}(\{x\}, A).
\]

Let \( \text{diam} A \) denote the diameter of \( A \) defined as

\[
\text{diam} A = \sup\{|x - y|; x, y \in A\}.
\]

For a domain \( \Omega \) and \( x \in \Omega \), we will write

\[
\delta_\Omega(x) = \text{dist}(x, \partial \Omega).
\]

We let \( B(x, r) \) denote the open ball centered at \( x \) of radius \( r \). For a ball \( B \), we will denote its radius by \( r_B \).
Given two closed sets $E$ and $F$, and $B$ a set we denote 
\[
d_B(E, F) = \frac{2}{\text{diam } B} \max \left\{ \sup_{y \in E \cap B} \text{dist}(y, F), \sup_{y \in F \cap B} \text{dist}(y, E) \right\}
\]

### 2.2. Christ-David Cubes.

**Theorem 2.1.** Let $X$ be a doubling metric space. Let $X_k$ be a nested sequence of maximal $\rho^k$-nets for $X$ where $\rho < 1/1000$ and let $c_0 = 1/500$. For each $n \in \mathbb{Z}$ there is a collection $\mathcal{D}_k$ of “cubes,” which are Borel subsets of $X$ such that the following hold.

1. For every integer $k$, $X = \bigcup_{Q \in \mathcal{D}_k} Q$.
2. If $Q, Q' \in \mathcal{D}_k$ and $Q \cap Q' \neq \emptyset$, then $Q \subseteq Q'$ or $Q' \subseteq Q$.
3. For $Q \in \mathcal{D}_k$, let $k(Q)$ be the unique integer so that $Q \in \mathcal{D}_k$ and set $\ell(Q) = 5\rho^{k(Q)}$. Then there is $\zeta_Q \in X_k$ so that
\[
(2.1) \quad B_X(\zeta_Q, c_0\ell(Q)) \subseteq Q \subseteq B_X(\zeta_Q, \ell(Q))
\]

and 
\[
X_k = \{ \zeta_Q : Q \in \mathcal{D}_k \}.
\]

If $Q \in \mathcal{D}_k$, we let 
\[
\text{Child}(Q) = \{ R \in \mathcal{D}_{k+1} : R \subseteq Q \}.
\]

We recall some facts about stopping-time regions, which can be found in [DS93].

**Definition 2.2.** A tree or stopping-time region is a subcollection $S \subseteq \mathcal{T}$ with a maximal cube $Q(S) \in S$ so that if $Q \in S$ and $Q \subseteq T \subseteq Q(S)$, then $T \in S$, and if whenever a child of $Q \in S$ is not in $S$, then no children of $Q$ are in $S$.

We will usually construct stopping-time regions as follows.

**Lemma 2.3.** Let $R$ be a cube, $\mathcal{C}$ a (possibly empty) collection of subcubes properly contained in $R$, let $\text{Stop}(R)$ be the maximal cubes in $R$ that contain a child in $\mathcal{C}$, and let $\mathcal{T}$ be those cubes in $\mathcal{T}$ that are not properly contained in a cube from $\text{Stop}(R)$. Then $\mathcal{T}$ is a stopping-time region.

**Proof.** The first two properties of being a stopping-time are immediate, so we just verify the last one. Let $Q \in \mathcal{T}$, then there is $S \in \text{Stop}(R)$ with $S \subset Q$. Suppose $Q'$ was a child of $Q$ that was also in $\mathcal{T}$. Then $Q'$ is not properly contained in a cube from $\text{Stop}(R)$, but then neither can any of its siblings, so all of its siblings are in $\mathcal{T}$. □

The last property in the definition may seem odd, but it is to guarantee the following property about minimal cubes. Recall that $Q \in \mathcal{T}$ is a minimal cube for $\mathcal{T}$ if it does not properly contain any cubes from $\mathcal{T}$. Let $z(\mathcal{T})$ be those points in $T$ not contained in any minimal cube. In particular, for $\mathcal{T}$ defined as in the previous lemma, the minimal cubes are exactly $\text{Stop}(R)$. 

**Lemma 2.4.** Let \( \mathcal{T} \) be a stopping-time region with top cube \( T \). Then for all \( x \in T \), there is a smallest cube in \( \mathcal{T} \) containing \( x \) or there are infinitely many cubes from \( T \) containing \( x \), and \( z(\mathcal{T}) \) along with the set of all such minimal cubes partition \( T \).

See [DS93, Page 56].

**2.3. Quantitative Rectifiability.** In this section, we recall a few preliminaries about quantitative rectifiability. We first recall the Analyst’s traveling salesman theorem proven in [AS18] (however see [AV19] for this statement):

**Theorem 2.5.** Let \( 1 \leq d < n \) and \( E \subseteq \mathbb{R}^n \) be lower \((c,d)\)-lower content regular and let \( \mathcal{D} \) denote the Christ-David cubes for \( E \). For a ball \( B \) centered on \( E \) and a \( d \)-dimensional plane \( P \), let

\[
b_{\beta_E}^d(B, P) = d_B(E, P), \quad b_{\beta_E}^d(B) = \inf_P b_{\beta}(B, P)
\]

where the infimum is over all \( d \)-dimensional planes \( P \). Let \( \text{BLWG}(\varepsilon, C_0) = \{ Q : b_{\beta_E}^d(C_0B_Q) \geq \varepsilon \} \). For \( R \in \mathcal{D} \),

\[
\text{BLWG}(R) = \text{BLWG}(R, \varepsilon, C_0) = \sum_{Q \subseteq R} \ell(Q)^d.
\]

and for \( M \geq 3 \),

\[
\beta_{E,M}(R) := \ell(R)^d + \sum_{Q \subseteq R} \beta_{E,M}^d(MB_Q)^2 \ell(Q)^d.
\]

Then for \( R \in \mathcal{D} \),

\[
(2.2) \quad \mathcal{H}^d(R) + \text{BLWG}(R, \varepsilon, C_0) \sim_{n,c,M,C_0,\varepsilon} \beta_E(R).
\]

Note that as these values are comparable for all \( M \), we will denote

\[
\beta_E(R) = \beta_{E,3}(R) \sim_M \beta_{E,M}(R) \quad \text{for all } M \geq 3.
\]

This is a version of the original traveling salesman theorem of Jones [Jon90], which instead had an \( L^\infty \)-\( \beta \)-number, and their square sum was comparable to the shortest curve containing \( E \). This was originally shown in the plane, but was subsequently generalized to Euclidean space [Oki92] and Hilbert space [Sch07].

**Remark 2.6.** To avoid some confusion with notation, the reader will find it helpful to remember that, given a set \( S \), \( \beta(S) \) will denote the usual \( \beta \)-number if \( S \) is a ball and will denote a sum of cubes in \( S \) as \( \beta_{E,M}(R) \) above if \( S \) is a cube.

The following is the main lemma from [AV19].

**Lemma 2.7.** Let \( k_0 > 0 \), \( \vartheta > 0 \), \( d > 0 \) and \( E \) be a closed set that is lower \( d \)-content \( c \)-regular with \( \text{diam } E \sim 1 \). Let \( \mathcal{D}_k \) denote the Christ-David cubes on \( E \) of scale \( k \) and \( \mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k \). Let \( Q_0 \in \mathcal{D}_0 \) and \( \mathcal{D}(k_0) = \bigcup_{k=0}^{k_0} \{ Q \in \mathcal{D}_k : Q \subseteq Q_0 \} \).
Then we may partition $\mathcal{D}(k_0)$ into stopping-time regions $\text{Tree}(R)$ for $R$ from some collection $\text{Top}(k_0) \subseteq \mathcal{D}(k_0)$ with the following properties:

1. We have

$$\sum_{R \in \text{Top}(k_0)} \ell(R)^d \lesssim_{c,d} \mathcal{H}^d(Q_0).$$

2. Given $R \in \text{Top}(k_0)$ and a stopping-time region $\mathcal{T} \subseteq \text{Tree}(R)$ with maximal cube $T$, let $d_\mathcal{T}(x) = \inf_{Q \in \mathcal{T}} (\ell(Q) + \text{dist}(x,Q))$.

For $C_0 > 4$ and $0 < \vartheta \ll C_0^{-1}$, there is a collection $\mathcal{C}$ of disjoint dyadic cubes covering $C_0B_T \cap E$ so that if

$$E(\mathcal{T}) = \bigcup_{I \in \mathcal{C}} \partial_d I,$$

where $\partial_d I$ denotes the $d$-dimensional skeleton of $I$, then the following hold:

(a) $E(\mathcal{T})$ is Ahlfors regular, that is,

$$\mathcal{H}^d(B(x,r) \cap E(\mathcal{T})) \sim_{C_0,\vartheta,d} r^d \text{ for all } x \in E(\mathcal{T}), \ 0 < r < \text{diam } E(\mathcal{T}).$$

(b) We have the containment

$$C_0B_T \cap E \subseteq \bigcup_{I \in \mathcal{C}} I \subseteq 2C_0B_T.$$

(c) $E$ is close to $E(\mathcal{T})$ in $C_0B_T$ in the sense that

$$\text{dist}(x,E(\mathcal{T})) \lesssim \vartheta d_\mathcal{T}(x) \text{ for all } x \in E \cap C_0B_T.$$

(d) The cubes in $\mathcal{C}$ satisfy

$$\ell(I) \sim \vartheta \inf_{x \in I} d_\mathcal{T}(x) \text{ for all } I \in \mathcal{C}.$$

Finally, we recall the David-Toro parametrization theorem [DT12]. We state only a consequence of their result, since their full result is more general. There they used planes associated to balls in their statements, but we would like to use planes associated to cubes. Converting between the two has been done in several papers [AS18, ATT18, AT15], but here we state a converted version for cubes, hopefully so that it doesn’t have to be re-converted in the future. We prove this reformulation in the appendix:

**Lemma 2.8.** Let $E \subseteq \mathbb{R}^n$ be some set with Christ cubes $\mathcal{D}$ in some set $E$. Declare $R \sim Q$ if $C_2^{-1} \ell(Q) \leq \ell(R) \leq C_2 \ell(Q)$ and $\text{dist}(Q,R) \leq C_2 \min\{\ell(Q),\ell(R)\}$. For $\epsilon^{-1} \gg C_1, C_2$, the following holds. Let $S$ be a stopping-time region with top cube $Q(S)$ so that $\zeta_{Q(S)} = 0$ and for all $Q \in S$, there is a $d$-plane $P_Q$ such that

$$\text{dist}(\zeta_Q, P_Q) < \epsilon \ell(Q) \text{ for all } Q \in S.$$
Moreover, if
\[ \varepsilon(Q) = \max_{R \sim Q} \ell(Q)^{-1} \left( \sup_{x \in P_Q \cap C_1 B_Q} \text{dist}(x, P_R) + \sup_{x \in P_R \cap C_1 B_Q} \text{dist}(x, P_Q) \right) \]
and
\[ (2.9) \sum_{Q \subseteq R \subseteq Q(S)} \varepsilon(R)^2 < \varepsilon^2. \]
Then for \( \varepsilon > 0 \) small enough, there is \( g : \mathbb{R}^n \to \mathbb{R}^n \) that is \( C\)-bi-Lipschitz on \( \mathbb{R}^n \) and \( (1 + C \varepsilon^2)\)-bi-Lipschitz when restricted to \( P_Q(S) \) and
\[ (2.10) |g(z) - z| \lesssim \varepsilon \ell(R) \quad \text{for all } z \in \mathbb{R}^n. \]
The surface \( g(P_R) =: \Sigma_S \) is \( C\varepsilon\)-Reifenberg flat so that
\[ (2.11) \text{dist}(Q, \Sigma_Q) \lesssim \varepsilon \ell(Q) \quad \text{for all } Q \in S. \]
If
\[ (2.12) \sup_{x \in 2C_1 B_Q \cap E} \text{dist}(x, P_Q) < \varepsilon \ell(Q) \text{ for all } Q \in S, \]
then
\[ (2.13) \sup_{z \in C_1 B_Q \cap E} \text{dist}(z, \Sigma_R) \lesssim \varepsilon \ell(Q) \text{ for all } Q \in S \]
If
\[ (2.14) \sup_{x \in 2C_1 B_Q \cap P_Q} \text{dist}(x, E) < \varepsilon \ell(Q) \text{ for all } Q \in S, \]
then
\[ (2.15) \sup_{z \in C_1 B_Q \cap \Sigma_R} \text{dist}(z, E) \lesssim \varepsilon \ell(Q) \text{ for all } Q \in S. \]

2.4. Harmonic Measure. For background on harmonic measure and Green’s function, we refer the reader to [AG01].

**Definition 2.9.** For \( K \subset \partial \Omega \), we say that \( \Omega \) has the capacity density condition (CDC) in \( K \) if \( \text{cap}(B(x, r) \cap \Omega^c, B(x, 2r)) \gtrsim r^{d-1} \), for every \( x \in K \) and \( r < \text{diam} K \), and that \( \Omega \) has the capacity density condition if it has the CDC in \( K = \partial \Omega \). Here, \( \text{cap}(\cdot, \cdot) \) stands for the variational \( 2 \)-capacity of the condenser \( (\cdot, \cdot) \) (see [HKM06, p. 27] for the definition).

**Lemma 2.10 ([HKM06, Lemma 11.21]).** Let \( \Omega \subset \mathbb{R}^{d+1} \) be any domain satisfying the CDC condition, \( B \) a ball centered on \( \partial \Omega \) so that \( \Omega \setminus 2B \neq \emptyset \). Then
\[ (2.16) \omega_{\Omega}^x(2B) \geq c > 0 \quad \text{for all } x \in \Omega \cap B, \]
where \( c \) depends on \( d \) and the constant in the CDC.

Using the previous lemma and iterating, it is possible to obtain the following lemma.
Lemma 2.11. [AM18, Lemma 2.3] Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with the CDC, $\xi \in \partial \Omega$ and $0 < r < \text{diam} \partial \Omega / 2$. Suppose $u$ is a non-negative function that is harmonic in $B(\xi, r) \cap \Omega$ and vanishes continuously on $\partial \Omega \cap B(\xi, r)$. Then

\begin{equation}
(2.17) \quad u(x) \lesssim \left( \sup_{y \in B(\xi, r) \cap \Omega} u \right) \left( \frac{|x - \xi|}{r} \right)^\alpha
\end{equation}

where $\alpha > 0$ depends on the CDC constant and $d$.

There are two key facts we will use about Green's function.

Lemma 2.12. [Aik08, Lemma 1] For $x \in \Omega \subseteq \mathbb{R}^{d+1}$ and $\phi \in C^\infty_c(\mathbb{R}^{d+1})$,

\begin{equation}
(2.18) \quad \int \phi d\omega^\Omega_x = \int \Omega \Delta \phi(y) G_\Omega(x, y) dy + \phi(x).
\end{equation}

Lemma 2.13. Let $\Omega \subset \mathbb{R}^{d+1}$ be a CDC domain. Let $B$ be a ball centered on $\partial \Omega$ and $0 < r_B < \text{diam} \partial \Omega$. Then,

\begin{equation}
(2.19) \quad \omega^\Omega_x(4B) \gtrsim r_B^{d-1} G_\Omega(x, y) \quad \text{for all } x \in \Omega \setminus 2B \text{ and } y \in B \cap \Omega,
\end{equation}

The opposite inequality holds if $\Omega$ is also uniform.

This follows quickly from the maximum principle, Lemma 2.10, and the fact that, for $x \in \partial 2B \cap \Omega$ and $y \in B$, $r_B^{d-1} G_\Omega(x, y) \lesssim 1$. For proofs, see [AH08, Lemma 3.5] or [AHM+16, Lemma 3.3].

The following lemma was first shown by Aikawa and Hirata for John domains with the CDC [AH08]; with a minor adjustment, the John condition can be removed [Azz17, Theorem I]

Lemma 2.14. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a CDC domain. Then the following are equivalent:

1. $\omega^\Omega_x$ is doubling, meaning there is a constant $A \geq 2$ and a function $C : (0, \infty) \rightarrow (1, \infty)$ so that, for any ball $B$ centered on $\partial \Omega$ and $\alpha > 0$,

\begin{equation}
(2.20) \quad \omega^\Omega_x(2B) \leq C(\alpha) \omega^\Omega_x(B) \quad \text{for all } x \text{ such that } \text{dist}(x, AB \cap \partial \Omega) \geq \alpha |x - x_B|.
\end{equation}

2. $\Omega$ is semi-uniform.

Lemma 2.15. [Azz17, Theorem II] Let $\Omega$ be a semi-uniform CDC domain, $B$ a ball centered on $\partial \Omega$, and $E \subseteq B \cap \partial \Omega$. Then there is $M > 0$ depending on the CDC and semi-uniformity constants and corkscrew points $x_1$ and $x_2$ in $B \cap \Omega$ so that

\begin{equation}
\omega^\Omega_{x_1}(E) \lesssim \frac{\omega^\Omega_{x_1}(E)}{\omega^\Omega_{x_2}(B)} \lesssim \omega^\Omega_{x_2}(E) \quad \text{for all } x \in \Omega \setminus MB.
\end{equation}

If $\Omega$ is uniform, then we can take $x_1 = x_2$ to be any corkscrew point in $B$.

The last line of the lemma is due to Jerison and Kenig [JK82] for NTA domains, and for general uniform domains this follows from the work of Aikaha and Hirata [AH08].
3. Proof of Theorems I and II, Part I

The aim of this section is to prove the following lemma:

**Lemma 3.1.** Let \( \Omega \subseteq \mathbb{R}^{d+1} \) be an interior corkscrew domain with LCR boundary. Let \( \mathcal{T} \) be a stopping-time region with top cube \( Q_0 \), and BTM (for “bottom”) be the (possibly empty) set of children of the minimal cubes for \( \mathcal{T} \). For \( \lambda \geq 1 \), and for \( A, \tau^{-1} \) sufficiently large, we may find cubes \( \text{Top} \) contained in \( Q_0 \) and a partition of \( \mathcal{T} \) into trees \( \{ \text{Tree}(R) : R \in \text{Top} \} \) so that for each \( R \in \text{Top} \), there is a corkscrew ball \( B(x_R, c\ell(R)) \subseteq B_R \cap \Omega \) so that for all \( Q \in \text{Tree}(R) \),

\[
\tau \Theta^{d}_{\omega \times R}(\lambda B_R) \leq \Theta^{d}_{\omega \times R}(\lambda B_Q) \leq A \Theta^{d}_{\omega \times R}(\lambda B_R).
\]

and for \( M \) large enough,

\[
(3.1) \quad \sum_{R \in \text{BTM}} \ell(R)^d + \sum_{R \in \text{Top}} \ell(R)^d \leq \beta_{\partial \Omega}(\mathcal{T}) := \ell(Q_0)^d + \sum_{Q \in \mathcal{T}} \beta_{\partial \Omega}^2(M B_Q)^2 \ell(Q)^d.
\]

We will write \( \omega = \omega_{\Omega} \) and \( \beta = \beta_{\partial \Omega}^d \) for short.

Let \( k_0 \in \mathbb{N} \) and \( \mathcal{T}(k_0) \) be those cubes in \( \mathcal{T} \) with sidelength at least \( 5 \rho^{k_0} \). Let \( M > 1, \varepsilon > 0 \) and

\[
\text{Bad} = \{ Q \subseteq Q_0 : b_{\partial \Omega}(M B_Q) \geq \varepsilon \} \cap \mathcal{T}(k_0).
\]

Observe that by Theorem 2.5 and Theorem 2.7,

\[
(3.2) \quad \sum_{Q \in \text{Bad}} \ell(Q)^d \lesssim \beta_{\partial \Omega}(Q_0).
\]

For \( R \in \text{Bad} \), we define \( \text{Stop}(R) = \{ \emptyset \} \) and \( \text{Next}(R) \) to be the children of \( R \) that are in \( \mathcal{T}(k_0) \) (so these could be empty).

For \( R \in \mathcal{T}(k_0) \setminus \text{Bad} \), there is \( P_R \) so that \( b_{\partial \Omega}(M B_R, P_R) < \varepsilon \). We can assume without loss of generality that \( P_R \) passes through \( \zeta_R \), since we still have \( b_{\partial \Omega}(M B_R, P_R) \leq 2 \varepsilon \) for \( \varepsilon > 0 \).

Let \( \nu_R \) be the normal vector to \( P_R \), and let \( x_R^\pm = \zeta_R \pm \frac{\ell(R)}{2} \nu_R \). Since \( \Omega \) is a \( c \)-corkscrew domain, for \( \varepsilon > 0 \) small enough, \( \Omega \) must contain either \( B(x_R^+, \ell(R)/8) \).

This is because, since \( b_{\partial \Omega}(M B_R, P_R) < \varepsilon \), every \( z \in \partial \Omega \cap M B_R \) satisfies \( \text{dist}(z, P_R) \leq M \varepsilon \ell(R) \), and so for \( \varepsilon > 0 \) small enough, \( \text{dist}(B(x_R^+, \ell(R)/8), \partial \Omega) > 0 \), and if we let \( B_R^\pm \) be the two components of \( \{ x \in B_R : \text{dist}(x, P_R) \geq \varepsilon \ell(R) \} \) that contain \( x_R^\pm \) respectively, then one of these must be contained in \( \Omega \) since otherwise the largest corkscrew ball in \( B_R \) must have radius at most \( \varepsilon \ell(R) \), which is a contradiction if \( \varepsilon < c \) (recall \( \Omega \) is a \( c \)-corkscrew domain). By changing the \( \pm \) if necessary, we will assume \( x_R = x_R^+ \) is always in \( \Omega \).

Let \( R \in \mathcal{T}(k_0) \). We let \( \text{Stop}(R) \) be the maximal cubes \( Q \in \mathcal{T}(k_0) \setminus \text{Bad} \) which contain a child \( Q' \) for which one the following occurs:

**BTM:** \( Q' \in \text{BTM} \). We call these cubes BTM(R).

**Bad:** \( Q' \in \text{Bad} \). We call these cubes Bad(R).
HD: \( \Theta^d_{\omega^R} (\lambda B_{Q'}) > A \Theta^d_{\omega^R} (\lambda B_R) \), call these cubes \( \text{HD}^+(R) \) and let \( \text{HD}(R) = \text{HD}^+(R) \cup \text{HD}^{-}(R) \). If \( x_R^- \notin \Omega \), we simply let \( \text{HD}^{-}(R) = \emptyset \).

LD: \( \Theta^d_{\omega^R} (Q') < \tau \Theta^d_{\omega^R} (R) \), call these cubes \( \text{LD}^+(R) \) and let \( \text{LD}(R) = \text{LD}^+(R) \cup \text{LD}^{-}(R) \). If \( x_R^- \notin \Omega \), we simply let \( \text{LD}^{-}(R) = \emptyset \).

\( B \beta \): \( Q' \notin \text{Bad} \), but for some fixed \( M > 0 \),

\begin{equation}
\sum_{Q' \subseteq T \subseteq R} \beta(MB_{Q'})^2 \geq 2\varepsilon^2,
\end{equation}

call these cubes \( B \beta(R) \). Note that for such a \( Q' \), since \( Q' \notin \text{Bad} \), we have \( \beta(MB_{Q'}) \leq b \beta(MB_{Q'}) < \varepsilon \), and so

\begin{equation}
2\varepsilon^2 > \sum_{Q' \subseteq T \subseteq R} \beta(MB_{Q'})^2 = \sum_{Q' \subseteq T \subseteq R} \beta(MB_{Q'})^2 - \beta(MB_{Q'})^2 > 2\varepsilon^2 - \varepsilon^2 = \varepsilon^2,
\end{equation}

We let \( \text{Tree}(R) \) denote the cubes in \( \mathcal{T}(k_0) \) contained in \( R \) that are not properly contained in any cube from \( \text{Stop}(R) \) (so \( \text{Stop}(R) \subseteq \text{Tree}(R) \)) and let \( \text{Next}(R) \) denote the children of the cubes in \( \text{Stop}(R) \) that are in \( \mathcal{T}(k_0) \). so \( \text{Next}(R) \) could be empty, for example, if \( R \in \mathcal{G}_{k_0} \), or if \( R \) is a minimal cube for \( \mathcal{T} \).

For \( R \in \text{Bad} \), we let \( \text{Stop}(R) = \{ R \} \) and \( \text{Next}(R) \) denote the children of \( R \) in \( \mathcal{T}(k_0) \) and \( \text{Tree}(R) = \{ R \} \).

Let \( Q_0 \in \text{Top}_0 \), and inductively, if \( R \in \text{Top}_k \), set

\[ \text{Top}_{k+1} = \bigcup_{R \in \text{Top}_k} \text{Next}(R). \]

Note that \( \text{Top}_k = \emptyset \) for all large \( k \). Let \( \text{Top} = \bigcup \text{Top}_k \).

**Remark 3.2.** The tops \( \text{Top} \) and trees \( \text{Tree}(R) \) should really be written \( \text{Top}^{k_0} \) and \( \text{Tree}^{k_0}(R) \) respectively since they depend on \( k_0 \), but we suppress the \( k_0 \) for ease of notation. Notice, however, that the trees are increasing in \( k_0 \), so the final tops we desire will be

\[ \text{Top} = \bigcap_{n > 0} \bigcup_{k_0 \geq n} \text{Top}^{k_0} \]

and for \( R \in \text{Top} \),

\[ \text{Tree}(R) = \bigcup_{k_0 \geq 1} \text{Tree}^{k_0}(R). \]

The purpose of cutting off our cubes at the scale \( k_0 \) is for simplicity, so that our trees are always finite.

Without loss of generality, assume \( P_R = \mathbb{R}^d \) and \( \zeta_R = 0 \). Let \( C_1 = M/2 \) and \( C_2 \) be such that \( M \gg C_2 > 1 \). We can then apply Lemma 2.8 with these constants to \( S = \text{Tree}(R) \). Let \( g_{\text{Tree}(R)} \) be the bi-Lipschitz map and \( g_R(\mathbb{R}^d) = \Sigma_R \) be the surface from the lemma. Since \( \zeta_R = 0 \), by (2.10), \( |g_R(0)| \lesssim \varepsilon \ell(R) \). Let

\[ d_R(x) = \inf \{ \ell(Q) + \text{dist}(x, Q) : Q \in \text{Tree}(R) \}. \]
Let $\Omega_{R,\pm}$ be the component of $\Sigma_R$ containing the corkscrew $x^\pm_R$, we can assume $\Omega_{R,\pm} = g_R(\mathbb{R}^{d+1}_\pm)$. Let $\alpha > 0$ be small, $e_{d+1}$ be the $(d+1)$st standard basis vector, and
\[ U^\pm_R = \{x = x' \pm x_{d+1} e_{d+1} : x' \in \mathbb{R}^d, \ x_{d+1} > \alpha d_R(g(x'))\} \cap B(0, 10\ell(R)). \]
If $x^-_R \not\in \Omega$, then we just set $U^-_R = \emptyset$. Since $d_R$ is Lipschitz, $U^\pm_R$ are disjoint Lipschitz domains, and since $g_R$ is bi-Lipschitz, the domain
\[ \Omega^\pm_R = g_R(U^\pm_R) \]
is a CAD. Also note that, since $\text{Tree}(R) \subseteq \mathscr{D}(k_0)$, we always have $d_R > 0$.

**Lemma 3.3.** If $x^\pm_R \in \Omega$, then $\Omega^\pm_R \subseteq \Omega$.

**Proof.** We will just show this for $\Omega_R = \Omega^+_R$ and $x_R = x^+_R$. We will show that if $y \in \partial \Omega_R$, then $\text{dist}(y, \partial \Omega) > 0$.

Let $G_R = \partial U^+_R \setminus \partial B(0, 10\ell(R))$.

**Case 1:** Suppose first that $y \in g_R(G_R)$. Let $x = g_R^{-1}(y) \in \partial U^+_R$. Then $x = x' + x_{d+1} e_{d+1} \in \mathbb{R}^d \oplus \mathbb{R}$ where $x' \in B(0, 10\ell(R)) \cap \mathbb{R}^d$.

Let $y' = g_R(x') \in \Sigma_R$. Let $Q \in \text{Tree}(R)$ be so that
\[ \ell(Q) + \text{dist}(y', Q) = d_R(y'). \]

Let $\hat{Q} \in \text{Tree}(R)$ be the maximal ancestor of $Q$ so that $\ell(\hat{Q}) \leq d_R(y')$. We claim that
\[ \ell(\hat{Q}) \sim d_R(y'). \]  

Indeed, if $\ell(\hat{Q}) < \ell(R)$, then this is clear since the parent of $\hat{Q}$ (which has comparable size) will have size at least $d_R(y')$. If $\hat{Q} = R$, then because $g_R$ is bi-Lipschitz, dist$(y', R) \leq |y' - \zeta_R| = |y'| \leq |y' - g_R(0)| + |g_R(0)| \lesssim |x' - 0| + \varepsilon \ell(R) \lesssim \ell(R)$, and so we have
\[ \ell(\hat{Q}) \leq d_R(y') \leq \ell(R) + \text{dist}(y', R) \lesssim \ell(R) = \ell(\hat{Q}). \]

This proves the claim. In particular, for $C_1$ large enough, $y' \in \frac{\alpha}{2} B_Q$. Since $g_R$ is bi-Lipschitz, we have
\[ |y - y'| \sim |x - x'| = x_{d+1} = \alpha d_R(g_R(x')) = \alpha d_R(y') \sim \alpha \ell(\hat{Q}). \]  

Hence, for $\varepsilon < \alpha$, and since $y' \in C_1 B\hat{Q} \cap \Sigma_R$, if $z \in \partial \Omega$ is closest to $y$,
\[ |z - y| \leq \text{dist}(y, \partial \Omega) \leq |y - y'| + \text{dist}(y', \partial \Omega) \overset{\text{Claim 1.15}}{\lesssim} \alpha \ell(\hat{Q}). \]  

In particular, for $C_1$ large enough and $\alpha$ small enough, $z \in C_1 B\hat{Q}$, and by (2.13), there is $y_0 \in \Sigma_R$ with $|z - y_0| \lesssim \varepsilon \ell(\hat{Q})$. Let $x_0 = g_R^{-1}(y_0) \in \mathbb{R}^d$. Since $g_R$ is $C$-bi-Lipschitz on $\mathbb{R}^n$, $x_0 \in \mathbb{R}^d$, and $x \in U^+_R$,
\[ |y - y_0| \sim |x - x_0| \geq \text{dist}(x, \mathbb{R}^d) \geq \alpha d_R(g_R(x')) = \alpha d_R(y') \sim \alpha \ell(\hat{Q}) \]
and so for $\varepsilon \ll \alpha$,
\[
\dist(y, \partial \Omega) = |y - z| \geq |y - y_0| - |y_0 - z| \gtrsim \alpha \ell(\hat{Q}) - \varepsilon \ell(\hat{Q}) \gtrsim \alpha d_R(y').
\]

This and (3.7) imply
\[
(3.8) \quad \dist(y, \partial \Omega) \sim \alpha d_R(y') \text{ for } y \in g_R(G_R).
\]
In particular, $\dist(y, \partial \Omega) > 0$ in this case.

**Case 2:** Now suppose $y \in g_R(\partial B(0, 10\ell(R)) \cap U_R^+)$). Let $z \in \partial \Omega$ be closest to $y$.

Since $\dist(g_R(0), R) \ll \varepsilon \ell(R)$, we have that
\[
\dist(y, R) \ll |y - g_R(0)| + \varepsilon \ell(R) \sim |x - 0| + \varepsilon \ell(R) \lesssim \ell(R),
\]
and so $z \in CB_R$ for some $C > 0$. By (2.13), for $C_1 \gg C$, $\dist(z, \Sigma_R) \lesssim \varepsilon \ell(R)$. Let $z' \in \Sigma_R$ be closest to $z$, so $|z - z'| \lesssim \varepsilon \ell(R)$. Hence, $g_R^{-1}(z') \in \mathbb{R}^d$, and so (using the fact that $g_R$ is bi-Lipschitz)
\[
\dist(y, \partial \Omega) = |y - z| \geq |y - z' - C\varepsilon \ell(R)| \gtrsim |x - g_R^{-1}(z')| - C\varepsilon \ell(R)
\]
\[
\geq \dist(x, \mathbb{R}^d) - C\varepsilon \ell(R)
\]
Thus, if $\dist(x, \mathbb{R}^d) \geq \alpha \ell(R)$, then $\varepsilon \ll \alpha$ implies $\dist(y, \partial \Omega) \gtrsim \alpha \ell(R)$. Otherwise, if $\dist(x, \mathbb{R}^d) < \alpha \ell(R)$, then for $\alpha$ small enough, since $x \in \partial B(0, 10\ell(R))$, this implies $|x'| \geq 3\ell(R)$, and so the above inequality and (2.10) imply
\[
\dist(y, \partial \Omega) \gtrsim \dist(x, \mathbb{R}^d) - C\varepsilon \ell(R) \gtrsim |x_{d+1}| - C\varepsilon \ell(R) = \alpha d_R(g_R(x')) - C\varepsilon \ell(R)
\]
\[
\geq \alpha \dist\left(g_R(x'), \frac{2}{1 + C\varepsilon} B_R\right) - C\varepsilon \ell(R)
\]
\[
(2.10) \quad \gtrsim \alpha \dist(x', B(0, 2\ell(R))) - C\varepsilon \ell(R) \gtrsim \alpha \ell(R) - C\varepsilon \ell(R) \gtrsim \alpha \ell(R).
\]
In either case, $\dist(y, \partial \Omega) > 0$. This finishes the proof.

**Lemma 3.4.** We have
\[
(3.9) \quad \sum_{R \in \Top} \ell(R)^d \lesssim \beta_{\partial \Omega}(\mathcal{T}).
\]

**Proof.** We will first get some estimates on the stopped cubes.

**Lemma 3.5.**
\[
(3.10) \quad \sum_{Q \in \HD(R)} \ell(Q)^d \lesssim \lambda A^{-1} \ell(R)^d.
\]

**Proof.** Without loss of generality (and to simplify notation), we assume $\HD(R) = \HD^+(R)$, the general case is similar. If $Q \in \HD(R)$, then $Q$ has a child $Q'$ so that $\Theta_{\omega^x}(\lambda B_{Q'}) > A\Theta_{\omega^x}(\lambda B_R)$. Thus,
\[
(3.11) \quad \Theta_{\omega^x}(\lambda B_Q) \gtrsim \Theta_{\omega^x}(\lambda B_{Q'}) \geq A\Theta_{\omega^x}(\lambda B_R).
\]
By the Vitali covering lemma, we can find disjoint balls \( \lambda B_{Q_j} \) so that
\[
\bigcup_{Q \in \text{HD}(R)} \lambda B_{Q} \subseteq \bigcup_j 5\lambda B_{Q_j}.
\]
Also note that for all \( Q \in \text{HD}(R) \subseteq \text{Tree}(R) \), by (2.13), for \( \varepsilon \) small enough (recall we set \( \delta = 2\varepsilon \))
\[
\text{dist}(\zeta_Q, \Sigma_R) < \frac{c_0}{2}\ell(Q)
\]
where \( \zeta_Q \) is as in Definition 2.1, and so
\[
(3.12) \quad \ell(Q)^d \lesssim \mathcal{H}^d(c_0 B_Q \cap \Sigma_R).
\]
Thus, since the balls \( c_0 B_Q \) are disjoint and \( \lambda B_{Q_j} \subseteq \lambda B_R \) for all \( j \),
\[
\sum_{Q \in \text{HD}(R)} \ell(Q)^d \lesssim \mathcal{H}^d \left( \bigcup_{Q \in \text{HD}(R)} c_0 B_Q \right) \lesssim \mathcal{H}^d \left( \bigcup_j 5\lambda B_{Q_j} \right)
\]
\[
\leq \sum_j \mathcal{H}^d |\Sigma_R (5\lambda B_{Q_j}) \sim_{\lambda} \sum_j \ell(Q_j)^d
\]
\[
\tag{3.11} \lesssim A^{-1} \Theta_{\omega^{x, R}} (\lambda B_R)^{-1} \sum_j \omega^{x, R}(\lambda B_{Q_j})
\]
\[
\leq A^{-1} \Theta_{\omega^{x, R}} (\lambda B_R)^{-1} \omega^{x, R} \left( \bigcup \lambda B_{Q_j} \right)
\]
\[
\leq A^{-1} \Theta_{\omega^{x, R}} (\lambda B_R)^{-1} \omega^{x, R}(\lambda B_R)
\]
\[
= A^{-1} (\text{diam } \lambda R)^d \sim_{\lambda} A^{-1} \ell(R)^d.
\]

Lemma 3.6. There is a universal constant \( \theta > 0 \) so that
\[
(3.13) \quad \sum_{Q \in \text{LD}(R)} \ell(Q)^d \lesssim \tau^\theta \ell(R)^d.
\]

Proof. Again, to simplify notation, we just assume \( \text{LD}(R) = \text{LD}^+(R) \) and \( x_R = x_R^+ \). Recall that for \( Q \in \text{LD}(R) \), there is a child \( Q' \) of \( Q \) so that \( \Theta^d_{\omega^{x, R}}(Q) < \tau \Theta^d_{\omega^{x, R}}(R) \). Let \( \text{LD}'(R) \) be the set of these children. Then we clearly have
\[
\sum_{Q \in \text{LD}(R)} \ell(Q)^d \lesssim \sum_{Q \in \text{LD}'(R)} \ell(Q)^d,
\]
so we will estimate this latter sum. Note that for \( Q \in \text{LD}'(R) \), since \( Q \) has a parent \( \hat{Q} \) with \( b_{\partial \Omega}(MB_{\hat{Q}}, P_{\hat{Q}}) < \varepsilon \), we know by (2.13) that
\[
\text{dist}(\zeta_Q, \partial \Sigma_R) \lesssim \varepsilon \ell(Q)
\]
so there is \( \zeta_Q \in \partial \Sigma_R \) with
\[
|\zeta_Q - \zeta'| \lesssim \varepsilon \ell(Q).
\]

\[
\tag{3.14}
\]
Let

\[ \xi_Q = g_R(g_R^{-1}(\zeta'_Q) + \alpha d_R(\zeta'_Q) e_{d+1}) \in \partial \Omega_R. \]

Note that since \( \zeta_Q \in Q \subseteq R \subseteq B_R, \zeta'_Q \in 2B_R, \) and by (2.10), \( g_R^{-1}(\zeta'_Q) \in 3B_R \) for \( \varepsilon > 0 \) small, thus \( g_R^{-1}(\zeta'_Q) \in G_R. \) Hence, (3.8) implies

\[ \delta_{\Omega}(\zeta_Q) \sim \alpha d_R(\zeta'_Q). \]

Note that \( d_R(\zeta_Q) \leq \ell(Q) \) trivially, but also, if \( T \in \text{Tree}(R) \) is any other cube, then either \( T \cap Q = \emptyset \) (in which case \( \text{dist}(\zeta_Q, T) \geq c_0 \ell(Q) \) by Theorem 2.1) or \( T \supseteq Q \) (in which case \( \ell(T) \geq \ell(Q) \)), and thus in fact \( d_R(\zeta_Q) \geq c_0 \ell(Q), \) so \( d_R(\zeta_Q) \sim \ell(Q). \) By (3.14), this also means that \( d_R(\zeta_Q) \sim \ell(Q). \) Hence,

\[ \delta_{\Omega}(\zeta_Q) \sim \alpha d_R(\zeta'_Q) \sim \alpha \ell(Q). \]

Let \( B^Q = B(\xi_Q, \alpha^2 \ell(Q)). \) Then for \( \alpha \ll c_0, \) the balls \( \{B^Q : Q \in \text{LD}(R)\} \) are disjoint. Moreover, using that \( g_R \) is bi-Lipschitz,

\[
|\xi_Q - \zeta_Q| \leq |\zeta_Q - \zeta'_Q| + |\zeta'_Q - \xi_Q|
\]

\[
\overset{\text{(3.14)}}{\lesssim} \varepsilon \ell(Q) + |g_R^{-1}(\zeta'_Q) - (g_R^{-1}(\zeta'_Q) + \alpha d_R(\zeta'_Q) e_{d+1})|
\]

\[
= \varepsilon \ell(Q) + \alpha d_R(\zeta'_Q) \lesssim \alpha \ell(Q).
\]

Thus, for \( \alpha \) small,

\[
(3.15) \quad B^Q \subseteq \frac{c_0}{4} B_Q \cap \Omega.
\]

Lemma 2.10 implies that

\[
(3.16) \quad \omega_{\cdot \cdot \cdot \cdot}(c_0 B_Q) \gtrsim 1 \quad \text{for} \quad x \in B^Q.
\]

Hence, by the maximum principle, since \( \Omega_R \subseteq \Omega \) is a CAD (and hence \( \partial \Omega_R \) is AR),

\[
\omega_{\Omega_R}^{\cdot \cdot \cdot \cdot} \left( \bigcup_{Q \in \text{LD'}(R)} B^Q \right) \overset{\text{(2.1)}}{\lesssim} \omega_{\Omega}^{\cdot \cdot \cdot \cdot} \left( \bigcup_{Q \in \text{LD'}(R)} c_0 B_Q \right) \leq \sum_{Q \in \text{LD'}(R)} \omega_{\Omega}^{\cdot \cdot \cdot \cdot} (Q)
\]

\[
< \tau \Theta_{\omega^{\cdot \cdot \cdot \cdot}}(R) \sum_{Q \in \text{LD'}(R)} \ell(Q)^d
\]

\[
\overset{\text{(3.12)}}{\lesssim} \tau \Theta_{\omega^{\cdot \cdot \cdot \cdot}}(R) \sum_{Q \in \text{LD'}(R)} \mathcal{H}^d|_{\partial \Omega_R}(B^Q)
\]

\[
= \tau \Theta_{\omega^{\cdot \cdot \cdot \cdot}}(R) \mathcal{H}^d|_{\partial \Omega_R} \left( \bigcup_{Q \in \text{LD'}(R)} B^Q \right)
\]

\[
\lesssim \tau \Theta_{\omega^{\cdot \cdot \cdot \cdot}}(R) \mathcal{H}^d(\partial \Omega_R)
\]

\[
\lesssim \tau \Theta_{\omega^{\cdot \cdot \cdot \cdot}}(R) \ell(R)^d = \tau \omega_{\Omega}^{\cdot \cdot \cdot \cdot}(R) \leq \tau.
\]
By Theorem 1.6 or the main result of [DJ90], $\omega^{x_R}_{\Omega}$ is $A_\infty$-equivalent to $\mathcal{H}^d|_{\partial \Omega_R}$ (with constants only depending on the CAD constants of $\Omega_R$, which don’t depend on any of our parameters apart from $d$). In particular, if $\mathcal{H}^d|_{\partial \Omega_R} = p_R \omega^{x_R}_{\Omega}$, then the function $p_R$ satisfies a reverse Hölder inequality ([Ste93, Section V.5]), that is, there is $1 < p < \infty$ (depending on the CAD constants for $\Omega_R$), so that

$$\left( \int_{\partial \Omega_R} p_R^p d\omega^{x_R}_{\Omega} \right)^{\frac{1}{p}} \lesssim \int_{\partial \Omega_R} p_R d\omega^{x_R}_{\Omega} = \frac{\mathcal{H}^d(\partial \Omega_R)}{\omega^{x_R}_{\Omega}} \sim \ell(R)^d.$$ 

Hence, for any set $F \subseteq \partial \Omega_R$, by Hölder’s inequality, if \( \frac{1}{p} + \frac{1}{p'} = 1 \)

$$\mathcal{H}^d(F) \lesssim \left( \int_{\partial \Omega_R} p_R^p d\omega^{x_R}_{\Omega} \right)^{\frac{1}{p}} \omega^{x_R}_{\Omega}(F)^{\frac{1}{p'}} \lesssim \ell(R)^d \omega^{x_R}_{\Omega}(F)^{\frac{1}{p'}}.$$

Letting $\theta = 1/p'$ and $F = \bigcup_{Q \in \text{LD}'(R)} B^Q$, our previous estimates now give

$$\tau^\theta \ell(R)^d \gtrsim \mathcal{H}^d|_{\partial \Omega_R} \left( \bigcup_{Q \in \text{LD}'(R)} B^Q \right) \sim \sum_{Q \in \text{LD}'(R)} \ell(Q)^d \gtrsim \sum_{Q \in \text{LD}(R)} \ell(Q)^d.$$

\[\square\]

**Lemma 3.7.** For $R \in \text{Top}$,

\[(3.17) \quad \sum_{Q \in B\beta(R)} \ell(Q)^d \lesssim \sum_{T \in \text{Tree}(R)} \beta(MB_T)^2 \ell(T)^d.\]

**Proof.** Using (3.4) and the fact that the cubes in $\text{Stop}(R)$ partition $R$ by Lemma 2.4,

$$\sum_{Q \in B\beta(R)} \ell(Q)^d \leq \varepsilon^{-2} \sum_{Q \in B\beta(R)} \sum_{T \subseteq T \subseteq R} \beta(MB_T)^2 \ell(Q)^d$$

$$\leq \varepsilon^{-2} \sum_{T \in \text{Tree}(R)} \beta(MB_T)^2 \sum_{Q \in \text{Stop}(R)} \ell(Q)^d$$

$$\lesssim \varepsilon^{-2} \sum_{T \in \text{Tree}(R)} \beta(MB_T)^2 \sum_{Q \in \text{Stop}(R)} \mathcal{H}^d(\Sigma_R \cap c_0 B_Q)$$

$$\leq \varepsilon^{-2} \sum_{T \in \text{Tree}(R)} \beta(MB_T)^2 \mathcal{H}^d(\Sigma_R \cap B_T)$$

$$\lesssim \varepsilon^{-2} \sum_{T \in \text{Tree}(R)} \beta(MB_T)^2 \ell(T)^d.$$

\[\square\]
Lemma 3.8. For \( R \in \text{Top} \),

\[
\sum_{Q \in \text{BTM}(R)} \ell(Q)^d \lesssim \ell(R)^d.
\]

**Proof.** Note by (3.12), and since the balls \( c_0 B_Q \) are disjoint for \( Q \in \text{BTM}(R) \),

\[
\sum_{Q \in \text{BTM}(R)} \ell(Q)^d \lesssim \sum_{Q \in \text{BTM}(R)} \mathcal{H}^d(\Sigma_R \cap c_0 B_Q) \leq \mathcal{H}^d(\Sigma_R \cap 2B_R) \lesssim \ell(R)^d.
\]

\[\square\]

Let \( \text{Stop}'(R) = \text{Stop}(R) \backslash \text{BTM}(R) \). Combining all these estimates, we see that

\[
\sum_{Q \in \text{Stop}'(R)} \ell(Q)^d \lesssim (A^{-1} + \tau^\theta) \ell(R)^d + \varepsilon^{-2} \sum_{T \in \text{Tree}(R)} \beta(M_B T)^2 \ell(T)^d + \sum_{Q \in \text{Bad}(R)} \ell(R)^d.
\]

\[
=: I(R)
\]

Note that if \( R \in \text{Top}_k \) for \( k \geq 1 \), then \( R \) is the child of a cube in \( \text{Stop}'(R) \) for some \( R' \in \text{Top}_{k-1} \). Thus,

\[
\sum_{k=0}^\infty \sum_{R \in \text{Top}_k} \ell(R)^d = \ell(Q_0)^d + \sum_{k=1}^\infty \sum_{R \in \text{Top}_k} \ell(R)^d
\]

\[
\lesssim \ell(Q_0)^d + \sum_{k=1}^\infty \sum_{R \in \text{Top}_{k-1}} \sum_{Q \in \text{Stop}'(R)} \ell(Q)^d
\]

\[
\lesssim \ell(Q_0)^d + (A^{-1} + \tau^\theta) \sum_{k=0}^\infty \sum_{R \in \text{Top}_k} \ell(R)^d + \sum_{k=0}^\infty \sum_{R \in \text{Top}_k} I(R)
\]

Thus, for \( A \) large and \( \tau \) small enough, and because the sets \( \text{Tree}(R) \) partition \( \mathcal{T}(k_0) \),

\[
\sum_{k=0}^\infty \sum_{R \in \text{Top}_k} \ell(R)^d \lesssim \sum_{k=0}^\infty \sum_{R \in \text{Top}_k} I(R) \lesssim \varepsilon^{-2} \beta_{\partial \Omega}(\mathcal{T}).
\]

Let \( \text{BTM}(k_0) \) be those cubes \( Q \) from \( \text{BTM} \) that are a child of some cube \( Q' \in \mathcal{T}(k_0) \) (so each cube from \( \text{BTM} \) is in \( \text{BTM}(k_0) \) for some \( k_0 \geq 0 \)). Since the trees partition \( \mathcal{T}(k_0) \), we also see that

\[
\sum_{Q \in \text{BTM}(k_0)} \ell(Q)^d \lesssim \sum_{R \in \text{Top}} \ell(R)^d.
\]

Taking \( k_0 \to \infty \) (and recalling Remark 3.2), this completes the proof of (3.9).

\[\square\]

We record the following Corollary of the proof for the case of semi-uniform domains.
Corollary 3.9. With the same assumptions as Lemma 3.1, if $\Omega$ is also SU and $x \in \Omega \setminus MB_{Q_0}$, for $\lambda \geq 1$, and for $A$, $\tau^{-1}$ sufficiently large, we may find cubes $Top$ contained in $Q_0$ and a partition of $\mathcal{T}$ into trees $\{\text{Tree}(R) : R \in \text{Top}\}$ so that for each $R \in \text{Top}$, if $\omega = \omega^x$,

1. $\tau \Theta^d_\omega(\lambda B_R) \leq \Theta^d_\omega(\lambda B_Q) \leq A \Theta^d_\omega(\lambda B_R)$ for all $Q \in \text{Tree}(R)$.
2. If $\text{Stop}(R)$ denote the minimal cubes of $\text{Tree}(R)$ and $Q \in \text{Stop}(R)$, then either $Q$ as a child in $\lambda B_{\Theta}(Q)$ or $\Theta^d_\omega(\lambda B_Q) \sim A \Theta^d_\omega(\lambda B_R)$ (with implied constants independent of $\tau$, $\lambda$ and $A$).
3. (3.1) holds.

Proof. Assume the same set-up as in the proof of Lemma 3.1. Let $\omega = \omega^x$. Let $R \in \text{Top}$ and $Q \in \text{Tree}(R)$. Let $x_1$ and $x_2$ be the corkscrew points for $\lambda B_R$ in Lemma 2.15 (with $M\lambda$ in place of $M$ and $\lambda B_R$ in place of $B$). Notice that since $b\beta(MB_R) < \varepsilon$, we know that each of these corkscrew points is connected by a Harnack chain in $\Omega$ to either $x_R^\pm$, so without loss of generality, we can assume that $x_1, x_2 \in \{x_R^+\}$. Note that since $\omega$ is doubling, we have

\begin{equation}
\Theta^d_\omega(\lambda B_Q) \sim \lambda \Theta^d_\omega(Q) \quad \text{for all } Q \subseteq Q_0.
\end{equation}

Then Lemma 2.15 and the doubling property for harmonic measure implies that

\[
\frac{\Theta^d_\omega(\lambda B_Q)}{\Theta^d_\omega(\lambda B_R)} = \frac{\ell(R)^d \omega(Q)}{\ell(R)^d \omega(R)} \sim \frac{\ell(R)^d \omega(\lambda B_Q)}{\ell(Q)^d \omega(\lambda B_R)} \leq \frac{\ell(R)^d}{\ell(Q)^d} \omega^x_2(\lambda B_Q) \sim \ell(R)^d \Theta^d_\omega(\lambda B_R) \leq A
\]

and similarly,

\[
\frac{\Theta^d_\omega(\lambda B_Q)}{\Theta^d_\omega(\lambda B_R)} \sim \frac{\ell(R)^d \omega(\lambda B_Q)}{\ell(Q)^d \omega(\lambda B_R)} \geq \frac{\ell(R)^d}{\ell(Q)^d} \omega^x_1(\lambda B_Q) \sim \ell(R)^d \Theta^d_\omega(\lambda B_Q) \geq \tau \ell(R)^d \Theta^d_\omega(\lambda B_R) \quad \text{(2.16)}
\]

Hence,

\begin{equation}
\tau \Theta^d_\omega(\lambda B_R) \lesssim \Theta^d_\omega(\lambda B_Q) \lesssim A \Theta^d_\omega(\lambda B_R) \quad \text{for all } Q \in \text{Tree}(R).
\end{equation}

Now we run a new stopping-time algorithm. Let $C > 0$ be a large constant to be decided later. For $R \subseteq Q_0$, let $\widetilde{HD}(R)$ denote the maximal cubes $Q \subseteq R$ which contain a child $Q'$ so that $\Theta^d_\omega(Q') > CA \Theta^d_\omega(R)$ and $\widetilde{LD}(R)$ be those maximal cubes $Q \subseteq R$ which contain a child $Q'$ so that $\Theta^d_\omega(Q') < C^{-1} \tau \Theta^d_\omega(R)$. Since $\omega$ is doubling, this and (3.20) imply that

$\Theta^d_\omega(Q') \sim CA \Theta^d_\omega(R) \quad \text{for } Q \in \widetilde{HD}(R)$

and

$\Theta^d_\omega(Q') \sim C^{-1} \tau \Theta^d_\omega(R) \quad \text{for } Q \in \widetilde{LD}(R)$. 


Let $\tilde{\text{Next}}(R)$ denote the children of the cubes in $\tilde{\text{Stop}}(R) := \tilde{H}D(R) \cup \tilde{L}D(R)$ that are also in $\mathcal{T}$. Let $\tilde{\text{Top}}_0 = \{Q_0\}$ and for $k \geq 0$ let

$$\tilde{\text{Top}}_{k+1} = \bigcup_{R \in \tilde{\text{Top}}_k} \tilde{\text{Next}}(R).$$

Let $\tilde{\text{Top}} = \bigcup_{k \geq 0} \tilde{\text{Top}}_k$ and for $R \in \tilde{\text{Top}}$, let $\tilde{\text{Tree}}(R)$ be those cubes in $R$ in $\mathcal{T}$ not properly contained in a cube from $\tilde{\text{Stop}}(R)$. Recalling (3.19), it is clear that these trees satisfy (1) and (2), since the minimal cubes in $\tilde{\text{Tree}}(R)$ are either in $\tilde{\text{Stop}}(R)$ or in $\text{BTM}(R')$ for some $R' \in \tilde{\text{Top}}$ and hence has a child in $\text{BTM}$, so we just need to verify (3).

Note that for each $R \in \tilde{\text{Top}}$, there is $R' \in \tilde{\text{Top}}$ so that $R \in \tilde{\text{Tree}}(R')$. Moreover, for $C$ large enough, $\tilde{\text{Stop}}(R) \subseteq \tilde{\text{Tree}}(R')$. Thus, for fixed $R' \in \tilde{\text{Top}}$, the cubes in $\{R \in \tilde{\text{Top}} : R \in \text{Tree}(R')\}$ are disjoint. Thus,

$$\sum_{R \in \tilde{\text{Top}}} \ell(R)^d = \sum_{R' \in \tilde{\text{Top}}} \ell(R)^d \sim_{A,\tau} \sum_{R' \in \tilde{\text{Top}}} \Theta_\omega^d(\lambda B_{R'})^{-1} \sum_{R' \in \tilde{\text{Tree}}(R')} \omega(R)
\leq \sum_{R' \in \tilde{\text{Top}}} \Theta_\omega^d(\lambda B_{R'})^{-1} \omega(R') \leq \sum_{R' \in \tilde{\text{Top}}} (\lambda \ell(R'))^d \lesssim_{\lambda} \beta_{\partial \Omega}(\mathcal{T})$$

The estimate on $\sum_{Q \in \text{BTM}} \ell(Q)^d$ is shown in the same way as before. This concludes the proof.

\[\square\]

4. PROOF OF THEOREMS I AND II: PART II

The goal of this section is to complete the proofs of Theorems I and II.

Combined with Lemma 3.1 and Corollary 3.9, Theorems I and II will follow from the following lemma, whose proof is the objective of this section.

Lemma 4.1. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a interior c-corkscrew domain with lower $d$-content regular boundary and let $\mathcal{T}$ be the Christ-David cubes for $\partial \Omega$. For $\lambda \geq 2$ and for $A, \tau^{-1} > 0$ large enough (depending on $\lambda$), we have that for all $Q_0 \in \mathcal{T}$,

$$\beta_{\partial \Omega}(Q_0) \lesssim_{A,\tau,\lambda} \text{CDHM}(Q_0, \lambda, A, \tau).$$

Remark 4.2. It suffices to prove the lemma when $\lambda = 2$. To see this, let $\tilde{\text{Top}}$ and $\{\text{Tree}(R) : R \in \tilde{\text{Top}}\}$ be as in Definition 1.7 with constants $\lambda \geq 2$ and $A, \tau > 0$. Let $\text{Tree}'(R)$ be those cubes in $\text{Tree}(R)$ where the $N$th generation descendants of their children are in $\text{Tree}(R)$ (we define this in terms of the children to ensure that $\text{Tree}'(R)$ is a stopping-time region). Then for $Q \in \text{Tree}'(R)$,
\[ \Theta_{\omega^x} (2B_Q) \leq (\lambda/2)^d \Theta_{\omega^x} (\lambda B_Q) \leq (\lambda/2)^d A \Theta_{\omega^x} (\lambda B_R) \leq (\lambda/2)^d A (2\lambda \ell(R))^{-d} \]
\[
\leq \frac{CA}{2^d} \Theta_{\omega^x} (2B_R)
\]
and for \( N \) large enough (depending on \( \lambda \)), if \( Q' \in \text{Tree}(R) \) is an \( N \)th generation descendant of \( Q \), then \( \lambda B_{Q'} \subseteq 2B_Q \), and so
\[ \Theta_{\omega^x} (2B_Q) \geq \left( \frac{\lambda \rho^N}{2} \right)^d \Theta_{\omega^x} (\lambda B_{Q'}) \geq \left( \frac{\lambda \rho^N}{2} \right)^d \tau \Theta_{\omega^x} (\lambda B_R) \leq \rho^{dN} \tau \Theta_{\omega^x} (2B_R). \]

Hence, letting \( A' = A \max \{ C/2^d, 1 \} \) and \( \tau' = \rho^{dN} \tau \), we see that
\[ \Theta_{\omega^x} (2B_R)/\Theta_{\omega^x} (2B_Q) \in [\tau', A'] \text{ for all } Q \in \text{Tree}'(R), \]
and so
\[ \text{CDHM}(Q_0, 2, A', \tau') \leq \sum_{R \in \text{Top}} \ell(R)^d. \]

Now let
\[ \text{Top}' = \text{Top} \cup \bigcup_{R \in \text{Top}} \text{Tree}(R) \setminus \text{Tree}'(R) \]
and for \( R \in \text{Top} \) and \( R' \in \text{Tree}(R) \setminus \text{Tree}'(R) \), we simply set \( \text{Tree}'(R') = \{ R' \} \). Then by Definition 1.7, we have
\[ \text{CDHM}(Q_0, 2, A', \tau') \leq \sum_{R \in \text{Top}'} \ell(R)^d \lesssim_N \sum_{R \in \text{Top}} \ell(R)^d \]
and infimizing over all such collections \( \{ \text{Tree}(R) : R \in \text{Top} \} \), we get
\[ \text{CDHM}(Q_0, 2, A', \tau') \lesssim_N \text{CDHM}(Q_0, \lambda, A, \tau). \]

In particular, for \( A, \tau^{-1} \) large enough (so that \( A', (\tau')^{-1} \) are large enough), (4.1) with \( \lambda = 2 \) implies (4.1) for all \( \lambda \geq 2 \).

Before we prove Lemma 4.1, we make a few preliminary estimates comparing different multiscale geometric properties.

The approach is essentially that of [HM15, HLMN17]. There, they use the weak-\( A_\infty \) property to control the behavior of the density of harmonic measure and thus show that Green’s function is often affine, implying that the boundary (outside of a summable set of cubes) is flat from one side. We make “flat from one side” more precise in the following definition.

**Definition 4.3.** Let \( E \subseteq \mathbb{R}^{d+1} \) be a lower \( d \)-regular set and \( \mathcal{D} \) denote the Christ-David cubes for \( E \). For \( \varepsilon > 0 \), we let \( \mathcal{D}_{K_0, \varepsilon} \) denote those cubes \( Q \) for which there is a half-plane \( H_Q \) such that
\[ H_Q \cap \varepsilon^{-2} B_Q \subseteq E^c, \]
\[ \text{dist}(Q, \partial H_Q) \leq K_0 \ell(Q), \]
and

$$\sup_{x \in \partial H_Q \cap x^{-2}B_Q} \text{dist}(x, E) < \varepsilon \ell(Q).$$

Let $\mathcal{D}_{K_0, \varepsilon}(R)$ be those cubes in $\mathcal{D}_{K_0, \varepsilon}$ contained in $R$. Let $\mathcal{D}_{K_0, \varepsilon}(R, K_0, \varepsilon)$ be those cubes in $R$ that are not in $\mathcal{D}_{K_0, \varepsilon}(R)$. If $E$ is also Ahlfors regular, we say it satisfies the Weak Half Space Approximation property (or WHSA) if there are $\varepsilon_0, K_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$, and for any surface cube $R \in \mathcal{D}$,

$$\text{WHSA}(R, K_0, \varepsilon) := \sum_{Q \in \mathcal{D}_{K_0, \varepsilon}(R)} \ell(Q)^d \lesssim_{\varepsilon, K_0} \ell(R)^d$$

In [HM15, Section 5], it was shown that this is equivalent uniform rectifiability for Ahlfors regular sets.

Lemma 4.1 will follow from the earlier remark and the following three lemmas.

**Lemma 4.4.** Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a corkscrew domain with lower $d$-content regular boundary and let $\mathcal{D}$ be the Christ-David cubes for $\partial \Omega$. Then for $\lambda \geq 1$, $R \in \mathcal{D}$, and $A, \tau^{-1} > 0$ large enough

$$\mathcal{H}^d(R) \lesssim_{A, \tau, \lambda} \text{CDHM}(R, \lambda, A, \tau).$$

**Lemma 4.5.** Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a corkscrew domain with lower $d$-content regular boundary and let $\mathcal{D}$ be the Christ-David cubes for $\partial \Omega$. Then for all $K_0 \geq 1$ and $\varepsilon > 0$ small enough,

$$\beta_{\partial \Omega}(R) \lesssim_{K_0, \varepsilon} \mathcal{H}^d(R) + \text{WHSA}(R, K_0, \varepsilon)$$

**Lemma 4.6.** Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a corkscrew domain with lower $d$-content regular boundary and let $\mathcal{D}$ be the Christ-David cubes for $\partial \Omega$. Then for all $\varepsilon^{-1}, A, \tau^{-1} > 0$ large enough, we have that for all $R \in \mathcal{D}$,

$$\text{WHSA}(R, 5, \varepsilon) \lesssim_{K_0, \varepsilon, A, \tau} \text{CDHM}(R, 2, A, \tau).$$

We divide the proofs of these lemmas into subsections:

4.1. **Proof of Lemma 4.4.** Let $Q_0 \in \mathcal{D}$. Without loss of generality, $Q_0 \in \mathcal{D}_0$. Let $\text{Top}$ and $\text{Tree}(R)$ for $R \in \text{Top}$ be as in Definition 1.7. Let $k_0 \in \mathbb{N}$, $\mathcal{D}(k_0) = \{Q \subseteq Q_0 : \ell(Q) \geq \rho^{k_0} \ell(Q_0)\}$, and

$$\text{Top}(k_0) = \text{Top} \cap \mathcal{D}(k_0), \quad \text{Tree}(k_0) = \text{Tree}(R) \cap \mathcal{D}(k_0).$$

For $R \in \text{Top}(k_0)$, let $\text{Stop}(k_0)(R)$ be the minimal cubes in $\text{Tree}(k_0)(R)$ that are in $\mathcal{D}_{k_0}$. For $Q \in \text{Stop}(k_0)(R)$, let $\Delta$ denote the dyadic cubes in $\mathbb{R}^n$ and

$$C(Q) = \{I \in \Delta : I \cap 5\lambda B_Q \neq \emptyset, \ell(I) < \ell(Q) \leq 2\ell(I)\}$$

and

$$E(R) = \bigcup_{Q \in \text{Stop}(k_0)(R)} \bigcup_{I \in C(Q)} I.$$
Note that as $\mathcal{D}(k_0)$ is finite and $\{\text{Tree}_{k_0}(R) : R \in \text{Top}(k_0)\}$ partitions $\mathcal{D}(k_0)$, 

$$
\mathcal{D}_{k_0} \subseteq \bigcup_{R \in \text{Top}(k_0)} \text{Stop}_{k_0}(R).
$$

Hence, since $Q \subseteq \bigcup_{I \in C(Q)} I$, and every $Q \in \mathcal{D}_{k_0}$ is in $\text{Stop}_{k_0}(R)$ for some $R \in \text{Top}(k_0)$,

$$
Q_0 \subseteq \bigcup_{Q \in \mathcal{D}_{k_0}} Q \subseteq \bigcup_{R \in \text{Top}(k_0)} \bigcup_{Q \in \text{Stop}_{k_0}(R)} \bigcup_{I \in C(Q)} I = \bigcup_{R \in \text{Top}(k_0)} E(R).
$$

Let $\lambda B_Q$ be a Vitali subcover of $\{\lambda B_Q : Q \in \text{Stop}_{k_0}(R)\}$, so they are disjoint and $\bigcup_{Q \in \text{Stop}_{k_0}(R)} \lambda B_Q \subseteq \bigcup 5\lambda B_Q$.

Also note that for $Q \in \mathcal{D}_{k_0}$, $\ell(Q) = 5\rho^{k_0}$, and so for $I \in C(Q)$, $\text{diam} I \leq 5\rho^{k_0} \sqrt{n}$. Thus,

$$
\mathcal{H}^d_{5\rho^{k_0} \sqrt{n}}(E(R)) \leq \sum_j \sum_{I \in C(Q_j)} (\text{diam} I)^d \lesssim \lambda \sum_j \ell(Q_j)^d
$$

$$
\lesssim \lambda \tau^{-1} \Theta_{x<R^d} (\lambda B_R)^{-1} \sum_j \omega_x^{x<R}(\lambda B_Q_j)
$$

$$
\leq \tau^{-1} \Theta_{x<R^d} (\lambda B_R)^{-1} \omega_x^{x<R}(5\lambda B_R) \lesssim \lambda \tau^{-1}\ell(R)^d.
$$

Thus,

$$
\mathcal{H}^d_{5\rho^{k_0} \sqrt{n}}(Q_0) \leq \sum_{R \in \text{Top}(k_0)} \mathcal{H}^d_{5\rho^{k_0}}(E(R)) \lesssim_{\tau, \lambda} \sum_{R \in \text{Top}(k_0)} \ell(R)^d \leq \text{CDHM}(R, \lambda, A, \tau).
$$

Letting $k_0 \to \infty$ gives the result and completes the proof of Lemma 4.4.

4.2. Proof of Lemma 4.5. The proof of this lemma will require a bit more theory from quantitative rectifiability. We recall this intermediate geometric property.

**Definition 4.7 (Bilateral Approximation by a Union of Planes (BAUP)).** Let $E \subseteq \mathbb{R}^n$ be lower $d$-content regular. For $\varepsilon, M > 0$, and $R \in \mathcal{D}$, let $\mathcal{Q}^{\text{BAUP}}_{\varepsilon,M}$ denote those cubes $Q$ for which there is a union $U$ of $d$-dimensional planes for which

$$
d_{MBQ}(E, U) < \varepsilon.$$

Let $\mathcal{Q}^{\text{BAUP}}_{\varepsilon,M} = \mathcal{D} \setminus \mathcal{Q}^{\text{BAUP}}_{\varepsilon,M}$, $\mathcal{Q}^{\text{BAUP}}_{\varepsilon,M}(R)$ be those cubes in $\mathcal{Q}^{\text{BAUP}}_{\varepsilon,M}$ contained in $R$, and

$$
\text{BAUP}(R, M, \varepsilon) = \sum_{Q \in \mathcal{Q}^{\text{BAUP}}_{\varepsilon,M}(R)} \ell(Q)^d.
$$

These cubes were introduced by David and Semmes [DS93] who showed that an Ahlfors regular set $E$ is UR if and only if for each $M > 1$ there is $\varepsilon_0$ so that for $0 < \varepsilon < \varepsilon_0$, if

$$
\text{BAUP}(R, M, \varepsilon) \lesssim \ell(R)^d \text{ for all } R \in \mathcal{D}.
$$

In [AV19], a version of this result was shown for just lower regular sets:
Theorem 4.8. [AV19, Theorem 1.4] Let \( E \subseteq \mathbb{R}^n \) be \((c,d)\)-lower content regular. Then for \( M > 1 \) and \( \varepsilon > 0 \) small enough (depending on \( M \) and the lower content regularity constant) and \( R \in \mathcal{D} \),

\[
\beta_E(R) \sim_{\varepsilon,M} \mathcal{H}^d(R) + BAUP(R, M, \varepsilon).
\]

In [HM15, Section 5], the authors show the following (they assume Ahlfors regularity, but an inspection of the proof shows that it is not needed):

Lemma 4.9. Let

\[
D_\varepsilon(Q) = \{ Q' \in \mathcal{D} : \varepsilon^{3/2} \ell(Q') \leq \ell(Q), \; Q' \cap K_0^2 B_Q \neq \emptyset \}.
\]

Let \( \mathcal{B}_{\text{WHSA}} \) be those cubes \( Q \) for which \( D_\varepsilon \cap \mathcal{B}_{\text{WHSA}} \neq \emptyset \). Then

\[
\mathcal{B}_{\text{BAUP}}(\varepsilon, 10) \subseteq \mathcal{B}_{\text{WHSA}}.
\]

Let

\[
\text{WHSA}(R, \varepsilon, K_0) = \sum_{Q \subseteq \mathcal{B}_{\text{WHSA}}(R)} \ell(Q)^d.
\]

Then the previous lemma and Theorem 4.8 imply

\[
\beta_E(R)^d \sim_{\varepsilon} \mathcal{H}^d(R) + BAUP(\varepsilon, 10)(R) \leq \mathcal{H}^d(R) + \text{WHSA}(R, \varepsilon, K_0)
\]

\[
\lesssim_{\varepsilon} \mathcal{H}^d(R) + \text{WHSA}(R, \varepsilon, K_0)
\]

This completes the proof of Lemma 4.5.

4.3. Proof of Lemma 4.6. This proof is modelled after that in [HM15], and Hofmann and Martell attribute this line of attack to [LV07] (there are also some common aspects to the proof of Theorem 1.8). However, we have to take some care since our sets are not Ahlfors regular, but we can fix this by using Lemma 2.7.

We first need a lemma that says if a harmonic function is approximately affine, in some ball, then the boundary is approximately flat near that ball. This has been proved elsewhere before, [HM15, HLMN17] for example, but since those proofs require Ahlfors regularity, we give a different proof here.

Lemma 4.10. Let \( \Omega \subseteq \mathbb{R}^{d+1} \) be a corkscrew domain with lower \( d \)-content regular complement, \( B \) a ball centered on \( \partial \Omega \) with \( 2r_B < \text{diam} \partial \Omega \). Let \( \varepsilon > 0 \) and let \( u \) be a harmonic function on \( \Omega \) that is positive on \( \varepsilon^{-2} B \cap \Omega \) and vanishing continuously on \( \frac{2}{3} B \cap \partial \Omega \). Suppose also that there is a corkscrew ball \( B(y_B, 2cr_B) \subseteq \Omega \cap B \) so that

\[
\sup_{\frac{2}{3} B \cap \Omega} u \lesssim u(y_B).
\]

Also suppose that

\[
\int_{B(y_B, cr_B)} |\nabla^2 u|^2 < \delta \frac{u(y_B)^2}{r_B^4}
\]
Then there is a half space $H_B$ so that
\[ \partial H_B \cap B \neq \emptyset, \quad H_B \cap \varepsilon^{-2}B \subseteq \Omega \]
and
\[ \sup_{x \in \partial H_B \cap \varepsilon^{-2}B} \text{dist}(x, \partial \Omega) < \varepsilon r_B. \]

**Proof.** Without loss of generality, we can assume $B = \mathbb{B}$ and $u(y_B) = 1$. Let $\mathbb{B}' = \varepsilon^{-2}B$. Suppose instead that for all $j$ there are domains $\Omega_j$ and harmonic functions $u_j$ positive on $\mathbb{B}' \cap \Omega_j$, vanishing continuously on $\partial \Omega \cap \mathbb{B}'$, and there is $B_j = B(y_j, c) \subseteq \Omega_j \cap \mathbb{B}'$ such that
\[ \sup_{2\mathbb{B}' \cap \Omega_j} u_j \lesssim 1 \]
and
\[ \int_{B_j} |\nabla^2 u_j|^2 < u_j^2 / j = 1 / j \]
but for every half space $H$, either
\[ H \cap \mathbb{B}' \not\subseteq \Omega_j \]
or
\[ \sup_{x \in \partial H \cap \mathbb{B}'} \text{dist}(x, \partial \Omega_j) \geq \varepsilon. \]

By Lemma 2.11, the $u_j$ are uniformly Hölder on $\frac{3}{2}\mathbb{B}'$, and since they are bounded on $2\mathbb{B}'$, we may pass to a subsequence so that $u_j \to u_\infty$ uniformly on $\frac{3}{2}\mathbb{B}'$ and so that $y_j \to y$. Note that as $u_j \sim 1$ on $B_j$ by Harnack’s inequality for all $j$, we have that $u_j \sim 1$ on $B(y, c/2)$, thus the set $P = \{ x \in \frac{3}{2}\mathbb{B}' : u_\infty > 0 \}$ is nonempty. However, notice that by Cauchy estimates, the second derivatives of $u_j$ also converge uniformly after passing to a subsequence and they do so to the double derivative of $u_\infty$ on $P$. Thus, we must have $\nabla^2 u_\infty \equiv 0$ on $B(y, c/2)$. This implies $u_\infty$ is affine on the connected component $P'$ of $P$ containing $B(y, c/2)$, hence there is a half space $H$ so that $P' \cap \frac{3}{2}\mathbb{B}' = H \cap \frac{3}{2}\mathbb{B}'$. Let $H_{\varepsilon/2} = \{ x \in H : \text{dist}(x, H^c) = \varepsilon/2 \}$.

Since $u_j \to u_\infty$ uniformly on $\mathbb{B}'$ and $u_\infty \gtrsim \varepsilon \frac{1}{j}$ by Harnack’s inequality in $P' \cap \mathbb{B}'$, we know that $u_j > 0$ on $H_{\varepsilon/2} \cap \mathbb{B}'$ for $j$ large enough, that is, $\Omega_j \supseteq H_{\varepsilon/2} \cap \mathbb{B}'$.

Now suppose that for infinitely many $j$ there were $x_j \in \partial H \cap \mathbb{B}'$ so that $B(x_j, \varepsilon) \subseteq \Omega_j$. Passing to a subsequence, $x_j \to x \in \partial H \cap \mathbb{B}'$ and by Harnack’s inequality, $B(x, \varepsilon) \subseteq P'$, but this is impossible since $u_\infty(x) = 0$ as $x \in \partial H \cap \mathbb{B}'$. Thus, for sufficiently large $j$,
\[ \sup_{x \in \partial H \cap \mathbb{B}'} \text{dist}(x, \partial \Omega_j) < \frac{\varepsilon}{2}. \]

Thus,
\[ \sup_{x \in \partial H \cap \mathbb{B}'} \text{dist}(x, \partial \Omega_j) < \varepsilon. \]
The existence of $H_{e}$ now contradicts our assumptions. This proves the lemma. □

Let $Q_{0} \in \mathcal{Q}$. Without loss of generality, $Q_{0} \in \mathcal{Q}_{0}$. Let $\text{Top}_{1}$ and $\text{Tree}_{1}(R)$ for $R \in \text{Top}_{1}$ be as in Definition 1.7 with $\lambda = 2$ so that

\begin{equation}
\sum_{R \in \text{Top}_{1}} \ell(R)^d \leq 2 \cdot \text{CDHM}(Q_{0}, 2, A, \tau).
\end{equation}

Just as in the proof of Lemma 4.4, let $k_{0} \in \mathbb{N}, \mathcal{Q}(k_{0}) = \{Q \subseteq Q_{0} : \ell(Q) \geq \rho^{k_{0}}\ell(Q_{0})\}$, and let

\[ \text{Top}_{1}(k_{0}) = \text{Top}_{1} \cap \mathcal{Q}(k_{0}), \ \text{Tree}_{1}^{k_{0}}(R) = \text{Tree}_{1}(R) \cap \mathcal{Q}(k_{0}). \]

By Definition 1.7, for each $Q \in \text{Tree}_{1}^{k_{0}}(R)$ and $R \in \text{Top}_{1}(k_{0})$, there is a corkscrew point $x_{R} \in B_{R}$ so that

\begin{equation}
\tau \Theta_{u,R}^{d} (2B_{Q}) \leq \Theta_{u,R}^{d} (2B_{Q}) < A \Theta_{u,R}^{d} (2B_{R}).
\end{equation}

Let $\{\text{Tree}_{2}(R) : R \in \text{Top}_{2}(k_{0})\}$ be the stopping-time regions from Lemma 2.7 for $C_{1}, \vartheta^{-1}$ large enough constant we will pick later. Now the sets

\[ \{\text{Tree}_{1}^{k_{0}}(R_{1}) \cap \text{Tree}_{2}(R_{2}) : R_{i} \in \text{Top}_{i}(k_{0})\} \]

partition $\mathcal{Q}(k_{0})$, and if $\text{Top}(k_{0}) = \text{Top}_{1}(k_{0}) \cup \text{Top}_{2}(k_{0})$, then each tree in the above collection has a top cube $R \in \text{Top}(k_{0})$, and we denote that tree $\text{Tree}(R)$ (we drop the $k_{0}$ for convenience, but remember int also depends on $k_{0}$). In particular, (4.3) is still satisfied for $Q \in \text{Tree}(R)$ and $R \in \text{Top}(k_{0})$, and by Lemma 2.7, there are Ahlfors regular sets $E_{R}$ satisfying the conclusion of Lemma 2.7 with respect to $\text{Tree}(R)$. By Lemma 2.7, Lemma 4.4, and Definition 1.7, we have for all $k_{0} \in \mathbb{N}$ that

\begin{equation}
\sum_{R \in \text{Top}(k_{0})} \ell(R)^d \leq \sum_{R \in \text{Top}_{1}} \ell(R)^d + \sum_{R \in \text{Top}_{2}(k_{0})} \ell(R)^d \lesssim \text{CDHM}(Q_{0}, 2, A, \tau) + \mathcal{H}^{d}(Q_{0}) \lesssim \text{CDHM}(Q_{0}, 2, A, \tau).
\end{equation}

**Lemma 4.11.** For $Q \in \mathcal{Q}$, let $\mathcal{W}$ be the Whitney cubes for $\Omega$ and

\[ U_{Q}^{K} = \{I \in \mathcal{W} : I \cap KB_{Q} \neq \emptyset, \ \ell(I) \geq \ell(Q)/K\} \]

and

\[ \Omega_{R} = \bigcup_{Q \in \text{Tree}(R)} U_{Q}^{K}. \]

Then for $K$ large enough, $C_{2} \gg K$, and $\vartheta \ll K^{-1}$, $\partial \Omega_{R}$ is Ahlfors regular.

The proof is exactly the same as [HMM14, Proposition A.2], as for $\vartheta$ small enough, $\Omega_{R}$ will be contained in

\[ \left( \bigcup_{I \in \mathcal{E}_{R}} I \right)^{c}, \]
which is a domain with Ahlfors regular boundary by Lemma 2.7. We leave the details to the reader.

**Lemma 4.12.** For $Q \in \text{Tree}(R)$, there is $z_Q \in 4B_Q$ so that

\[
\delta_Q(z_Q) \sim A, \tau \ell(Q) \quad \text{and} \quad \frac{G_{\Omega}(x_R, z_Q)}{\ell(Q)} \sim A, \tau \Theta_{\omega_{\Omega}^{x_R}}^d(2B_R).
\]

**Proof.** Let $\phi_Q$ be a smooth bump function so that

\[
1_{2B_Q} \leq \phi_Q \leq 1_{4B_Q} \quad \text{and} \quad |\nabla^2 \phi_Q| \lesssim \ell(Q)^{-2}.
\]

Then

\[
\omega_{\Omega}^{x_R}(2B_Q) \leq \int \phi_Q d\omega_{\Omega}^{x_R} \quad \text{(2.18)} = \int G_{\Omega}(x_R, x) \Delta \phi_R dx \lesssim \sup_{4B_R} G_{\Omega}(x_R, \cdot) \ell(Q)^{d-1}.
\]

and so there is $z_Q \in 4B_R$ so that

\[
(4.6) \quad \tau \Theta_{\omega_{\Omega}^{x_R}}^d(2B_R) \leq \Theta_{\omega_{\Omega}^{x_R}}^d(2B_Q) \lesssim \frac{G_{\Omega}(x_R, z_Q)}{\ell(Q)}.
\]

If $\xi_Q \in \partial \Omega$ is the closest point to $z_Q$, then $|\xi_Q - z_Q| \leq |\xi_Q - z_Q| \leq 4\ell(Q)$, and so $z_Q \in B(\xi_Q, 4\ell(Q)) \subseteq 8B_Q$. Thus,

\[
\frac{G_{\Omega}(x_R, z_Q)}{\ell(Q)} \lesssim \sup_{z \in 8B_Q} \left( \frac{G_{\Omega}(x_R, z)}{\ell(Q)} \right)^{2.17} \left( \frac{|z_Q - \xi_Q|}{\ell(Q)} \right)^{2.19} \Theta_{\omega_{\Omega}^{x_R}}^d(32B_Q).
\]

We claim that for any $\lambda \geq 4$,

\[
(4.8) \quad \Theta_{\omega_{\Omega}^{x_R}}^d(\lambda B_Q) \lesssim_{\lambda} A \Theta_{\omega_{\Omega}^{x_R}}^d(2B_R).
\]

Indeed, if $k \in \mathbb{N}$, $Q^k$ is the $k$th ancestor of $Q$ and $Q^k \in \text{Tree}(R)$, then for $k$ large enough depending on $\lambda$,

\[
\Theta_{\omega_{\Omega}^{x_R}}^d(\lambda B_Q) \lesssim_{\lambda} \Theta_{\omega_{\Omega}^{x_R}}^d(2B_{Q^k}) \leq A \Theta_{\omega_{\Omega}^{x_R}}^d(2B_R).
\]

Otherwise, if $R$ is the $j$th ancestor of $Q$ for some $j \leq k$, then

\[
\Theta_{\omega_{\Omega}^{x_R}}^d(\lambda B_R) \lesssim_{\lambda} \ell(R)^{-d} \Theta_{\omega_{\Omega}^{x_R}}^d(2B_R).
\]

and this proves the claim. Thus,

\[
\tau \Theta_{\omega_{\Omega}^{x_R}}^d(2B_R) \lesssim \frac{G_{\Omega}(x_R, z_Q)}{\ell(Q)} \lesssim \left( \frac{\delta_{\Omega}(z_Q)}{\ell(Q)} \right)^{\alpha} \Theta_{\omega_{\Omega}^{x_R}}^d(32B_Q) \lesssim (\frac{\tau}{A}) \frac{1}{\ell(Q)} \delta_{\Omega}(z_Q) \leq 4\ell(Q).
\]

Hence, we have

\[
\left( \frac{\tau}{A} \right)^{1/\alpha} \ell(Q) \lesssim \delta_{\Omega}(z_Q) \leq 4\ell(Q).
\]
Plugging this back into the above inequality also gives the rest of (4.5).

Let \( g_R(x) = G_{\Omega}(x, x_R) \). By (4.5), there is \( c > 0 \) so that if \( B'_Q = B(z_Q, \ell(R)) \), then \( 2B'_Q \subseteq 5B_Q \cap \Omega \) and for \( x \in B'_Q \), by Harnack’s inequality

\[
(4.9) \quad \frac{g_R(x)}{\delta_{\Omega}(x)} \sim \frac{g_R(x)}{\ell(Q)} \sim \Theta_{\omega^x_R}(2B_R) \sim \ell(\Lambda^{2d})
\]

By adjusting the value of \( c \) and the positions of the \( z_Q \) and using Harnack’s inequality, we can assume that \( B'_Q \subseteq \Omega'_R := \Omega_R \setminus B(x_R, \ell(R)) \) for all \( Q \in \text{Tree}(R) \) so that \( R \neq Q \).

Moreover, for \( \lambda \) large enough \( \frac{1}{2} B_Q \supseteq U_Q \), so for all \( x \in U_K \setminus B(x_R, \ell(R)) \),

\[
(4.10) \quad \frac{g_R(x)}{\delta_{\Omega}(x)} \overset{(2.19)}{\lesssim} \Theta_{\omega^x_R}(\lambda B_Q) \overset{(4.8)}{\lesssim} \Theta_{\omega^x_R}(2B_R) \overset{(2.16)}{\lesssim} \ell(R)^{-d}.
\]

Since \( 2|\nabla^2 g_R|^2 = \triangle |\nabla g_R|^2 \), using Green’s formula, the fact that \( g_R \) is harmonic in \( \Omega_R \setminus \{x_R\} \), and the Cauchy estimates

\[
(4.11) \quad |\nabla^2 g_R(x)| \delta_{\Omega}(x), |\nabla g_R(x)| \lesssim \frac{g_R(x)}{\delta_{\Omega}(x)} \text{ for all } x \in \Omega \setminus B(x_R, \ell(R)).
\]

we have

\[
\sum_{Q \in \text{Tree}(R) \setminus \{R\}} \int_{B'_Q} \left| \frac{\nabla^2 g_R}{g_R} \right|^2 \delta_{\Omega}(x)^3 dx \overset{(4.9)}{\sim} \ell(R)^{3d} \sum_{Q \in \text{Tree}(R) \setminus \{R\}} \int_{B'_Q} \left| \nabla^2 g_R \right|^2 g_R dx
\]

\[
\lesssim \ell(R)^{3d} \int_{\Omega_R} \triangle |\nabla g_R|^2 g_R(x) dx
\]

\[
\sim \ell(R)^{3d} \int_{\partial \Omega_R} \left( g_R \frac{d|\nabla g_R|^2}{d\nu} - |\nabla g_R|^2 \frac{dg_R}{d\nu} \right) d\mathcal{H}^d
\]

\[
\overset{(4.11)}{\lesssim} \ell(R)^{3d} \int_{\partial \Omega_R} \frac{g_R(x)^3}{\delta_{\Omega}(x)^3} d\mathcal{H}^d(x)
\]

\[
\overset{(4.10)}{\lesssim} \mathcal{H}^d(\partial \Omega'_R) \sim \ell(R)^d.
\]

In particular, If we let \( NA(R) \) (for ”not affine”) denote those cubes \( Q \in \text{Tree}(R) \) for which

\[
\int_{B'_Q} \left| \frac{\nabla^2 g_R}{g_R} \right|^2 \delta_{\Omega}(x)^4 dx \geq \delta.
\]

Then

\[
\sum_{Q \in NA(R)} \ell(Q)^d \leq \ell(R)^d + \delta^{-1} \sum_{Q \in \text{Tree}(R) \setminus \{R\}} \left( \int_{B'_Q} \left| \frac{\nabla^2 g_R}{g_R} \right|^2 \delta_{\Omega}(x)^4 dx \right) \ell(Q)^d
\]

\[
\lesssim \delta^{-1} \ell(R)^d.
\]
By Lemma 4.10, for $\delta$ small enough (and recalling that $2B'_Q \subseteq 5B_Q$)

$$\mathcal{B}^{\text{WHSA}}_{\varepsilon,5}(Q_0) \cap \text{Tree}(R) \subseteq \text{NA}(R).$$

Thus,

$$\sum_{Q \in \mathcal{B}^{\text{WHSA}}_{\varepsilon,5}(Q_0) \cap \mathcal{D}(k_0)} \ell(Q)^d \leq \sum_{R \in \text{Top}(k_0)} \sum_{Q \in \mathcal{B}^{\text{WHSA}}_{\varepsilon,5}(Q_0) \cap \text{Tree}(R)} \ell(Q)^d \leq \sum_{R \in \text{Top}(k_0)} \sum_{Q \in \text{NA}(R)} \ell(Q)^d \lesssim \delta^{-1} \sum_{R \in \text{Top}(k_0)} \ell(R)^d \lesssim \text{CDHM}(Q_0, 2, A, \tau) \quad (4.4)$$

Letting $k_0 \to \infty$ finally gives

$$\text{WHSA}(Q_0, \varepsilon, 5) = \sum_{Q \in \mathcal{B}^{\text{WHSA}}_{\varepsilon,5}(Q_0)} \ell(Q)^d \lesssim \text{CDHM}(Q_0, 2, A, \tau).$$

This finishes the proof of Lemma 4.6.

4.4. The semi-uniform case. Toward showing the other direction of Theorem II, we will show the following.

Lemma 4.13. Suppose $\Omega$ is SU and \{Tree$_1(R) : R \in \text{Top}_1\} are trees as in the statement of Theorem II with respect to $\omega = \omega_{\mathcal{D}}^{x_0}$ with $x_0 \in \Omega \setminus MB_{Q_0}$ and $Q_0 \in \mathcal{D}$. For $\lambda \geq 2$ and for $A, \tau^{-1} > 0$ large enough (depending on $\lambda$), we have that for all $Q_0 \in \mathcal{D}$,

$$\beta_{\partial \Omega}(Q_0) \lesssim_{A, \tau, \lambda} \sum_{R \in \text{Top}_1} \ell(R)^d.$$

We will require the following lemmas.

Lemma 4.14. With the assumptions of the previous lemma, and for all $\varepsilon^{-1}, A, \tau^{-1} > 0$ large enough, we have that for all $R \in \mathcal{D}$,

$$\text{WHSA}(R, 5, \varepsilon) \lesssim_{\varepsilon, A, \tau} \text{CDHM}(R, 2, A, \tau).$$

This has the same proof as Lemma 4.6, so we omit it.

Lemma 4.15. Suppose $\Omega$ is also SU and \{Tree$_1(R) : R \in \text{Top}_1\} are trees as in the statement of Lemma II with respect to $\omega = \omega_{\mathcal{D}}^{x_0}$ with $x_0 \in \Omega \setminus MB_{Q_0}$ and $Q_0 \in \mathcal{D}$. Then for $Q_0, \tau, A, \varepsilon$ as above,

$$\text{WHSA}(Q_0, \varepsilon, 5) \lesssim \sum_{R \in \text{Top}_1} \ell(R)^d.$$

Proof. We sketch the changes needed in the above proof of Lemma 4.6. Note that if \{Tree$_1(R) : R \in \text{Top}_1\} are as in Lemma II, since harmonic measure is doubling,
we can actually assume $\lambda = 2$ (this changes the constants $A$ and $\tau$ by a constant multiple depending on the doubling constant).

Now define $\text{Top}_1(k_0)$, $\text{Tree}_1^k$, $\text{Top}_2$, $\text{Tree}_2$, $\text{Tree}$ and $\text{Top}(k_0)$ as in the proof of Lemma 4.6. We now have

$$
\sum_{R \in \text{Top}(k_0)} \ell(R)^d \leq \sum_{R \in \text{Top}_1} \ell(R)^d + \sum_{R \in \text{Top}_2} \ell(R)^d
\lesssim \sum_{R \in \text{Top}_1} \ell(R)^d + \mathcal{H}^d(Q_0) \lesssim \sum_{R \in \text{Top}_1} \ell(R)^d
$$

where the last estimate follows from the proof of Lemma 4.4 (where we just replace $\omega^{z_R}$ by $\omega$ everywhere).

We now need a version of Lemma 4.12:

**Lemma 4.16.** For $R \in \text{Top}$, let $\{x^i_R\}_{i=1}^N$ be reference points for $2B_R$ and $g = G_\Omega(x_0, \cdot)$. For $Q \in \text{Tree}(R)$, there is $z_Q \in 4B_Q$ so that

$$
\delta_\Omega(z_Q) \sim_{A, \tau} \ell(Q) \quad \text{and} \quad \sum g(z_Q) \ell(Q) \sim_{A, \tau} \Theta^d_{\omega}(2B_R).
$$

**Proof.** The proof is exactly the same as Lemma 4.12, except that now to prove (4.8), we just use (2.20) since $\Omega$ is SU.

Again, there is $c > 0$ so that if $B'_Q = B(z_Q, c\ell(Q))$, then $2B'_Q \subseteq 5B_Q \cap \Omega$ and for $x \in B'_Q$, by Harnack’s inequality

$$
g(x) = \frac{g(x)}{\delta_\Omega(x)} \sim \frac{g(x)}{\ell(Q)} \sim \Theta^d_{\omega}(2B_R)
$$

By adjusting the value of $c$ and the positions of the $z_Q$, we can assume that $B'_Q \subseteq \Omega'_R := \Omega_R \setminus B(x_R, c\ell(R))$ for all $Q \in \text{Tree}(R)$ so that $R \neq Q$.

Again, for $\lambda$ large enough $\frac{3}{4}B_Q \supseteq U^R_Q$, so for all $x \in U^R_Q \setminus B(x_R, c\ell(R))$,

$$
g(x) = \frac{g(x)}{\delta_\Omega(x)} \overset{(2.19)}{\lesssim} \Theta^d_{\omega}(\lambda B_Q) \overset{(2.20)}{\lesssim} \Theta^d_{\omega}(2B_R).
$$

Also note that (4.11) still holds with $g$ in place of $g_R$. Repeating the same estimates as before, we get

$$
\sum_{Q \in \text{Tree}(R) \setminus \{R\}} \int_{B'_Q} \left| \nabla^2 g \right|^2 \delta_\Omega(x)^3 dx \overset{(4.13)}{\sim} \Theta^d_{\omega}(2B_R)^{-3} \sum_{Q \in \text{Tree}(R) \setminus \{R\}} \int_{B'_Q} |\nabla^2 g|^2 g dx
\lesssim \Theta^d_{\omega}(2B_R)^{-3} \int_{\partial\Omega_R} \frac{g(x)^3}{\delta_\Omega(x)^3} d\mathcal{H}^d(x)
\overset{(4.14)}{\lesssim} \Theta^d_{\omega}(2B_R)^{-3} \Theta^d_{\omega}(2B_R)^3 \mathcal{H}^d(\partial\Omega_R) \sim \ell(R)^d.
$$

The remaining steps are just as in Lemma 4.12.\qed
Now Lemma 4.13 follows from the previous two lemmas and Lemma 4.5.

4.5. Conclusion of proofs of Theorems I and II. We finally remark that Theorem I follows from Lemma 3.1 (with $\mathcal{T}$ equal to all cubes contained in $Q_0$) and 4.1. Theorem II similarly follows from Corollary 3.9 (with $\mathcal{T}$ equal to all cubes contained in $Q_0$) and 4.13.

5. Proof of Theorem III

Theorem III will follow from the following slightly more general result:

**Theorem 5.1.** Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a semi-uniform domain with Ahlfors regular boundary and let $\mathcal{D}$ be the Christ-David cubes. There is $M > 0$ depending on the semi-uniformity and Ahlfors regularity constants so that the following holds. For $Q_0 \in \mathcal{D}$, let $x_0 \in \Omega \setminus M B_{Q_0}$ and $\omega = \omega_{x_0}^0$. Let $k$ be the Radon-Nikodym derivative of $\omega$ in $Q_0$ with respect to $\sigma = \mathcal{H}^d|_{\partial \Omega}$. Then there is $0 < C \lesssim 1$ so that

$$C + \int_{Q_0} \log \frac{1}{k} d\sigma + \log \frac{\omega(Q_0)}{|Q_0|} \sim \frac{\beta_{\partial \Omega}(Q_0)}{\ell(Q_0)^d}.$$  

Above, the constants only depend on the semi-uniformity and Ahlfors regularity.

To see how this implies Theorem III, observe that if $\omega \ll \mathcal{H}^d$ in $Q_0$, then $\omega|_{Q_0} = k d\sigma|_{Q_0}$. Hence, $\omega(Q_0)/|Q_0| = \int_{Q_0} k d\sigma$, and so Theorem 5.1 implies

$$\frac{\beta_{\partial \Omega}(Q_0)}{\ell(Q_0)^d} \sim C + \int_{Q_0} \log \frac{1}{k} d\sigma + \log \int_{Q_0} k d\sigma.$$  

Note that by Jensen’s inequality,

$$\int_{Q_0} \log \frac{1}{k} d\sigma + \log \int_{Q_0} k d\sigma \geq \int_{Q_0} \log \frac{1}{k} d\sigma + \int_{Q_0} \log k d\sigma = \int_{Q_0} \log 1 d\sigma = 0.$$  

Thus,

$$C \leq C + \int_{Q_0} \log \frac{1}{k} d\sigma + \log \int_{Q_0} k d\sigma$$

$$\leq C \left( 1 + \int_{Q_0} \log \frac{1}{k} d\sigma + \log \int_{Q_0} k d\sigma \right)$$

$$\lesssim 1 + \int_{Q_0} \log \frac{1}{k} d\sigma + \log \int_{Q_0} k d\sigma.$$  

This proves Theorem III.

The rest of this section is dedicated to the proof of Theorem 5.1, some of the ideas for which come from the martingale arguments used to study $A_\infty$-weights in \cite[Section 3.18]{FKP91}. The proof will follow from the main lemmas in the following two sections.
5.1. The case assuming \( \beta_{\partial \Omega}(Q_0) < \infty \) a priori.

**Lemma 5.2.** The conclusions of Theorem 5.1 hold assuming \( \beta_{\partial \Omega}(Q_0) < \infty \).

**Proof.** Assume \( \beta_{\partial \Omega}(Q_0) < \infty \). Let \( \text{Top} \) and \( \text{Tree}(R) \) be the cubes and trees from Theorem II and \( \omega = \omega_{Q_0} \) for some \( x_0 \in \Omega \setminus MB_{Q_0} \).

For \( R \in \text{Top} \), let \( \text{Next}(R) \) be the children of the cubes in \( \text{Stop}(R) \). By Theorem II (3) and (2.20), we know that for \( Q \in \text{Next}(R) \), either

(a) \( \Theta_{\omega}(Q) \sim A \Theta_{\omega}(R) \), or
(b) \( \Theta_{\omega}(Q) \sim \tau \Theta_{\omega}(R) \).

Let \( \text{HD}(R) \) and \( \text{LD}(R) \) denote the cubes from the first and second alternatives respectively.

For \( Q \subseteq \partial \Omega \), let \( |Q| = \mathcal{H}^{d}(Q) \) and

\[
\theta_{\omega}^{d}(Q) = \frac{\omega(Q)}{|Q|} \sim \Theta_{\omega}(Q)
\]

where the last comparison follows from Ahlfors regularity of \( \partial \Omega \). Thus, by the doubling property for \( \omega \), there is a constant \( C_0 \) so that

\[
C_0^{-1} A \theta_{\omega}^{d}(R) < \theta_{\omega}^{d}(Q) \leq C_0 A \theta_{\omega}^{d}(R) \quad \text{for all } Q \in \text{HD}(R)
\]

and

\[
C_0^{-1} \tau \theta_{\omega}^{d}(R) < \theta_{\omega}^{d}(Q) \leq C_0 \tau \theta_{\omega}^{d}(R) \quad \text{for all } Q \in \text{LD}(R)
\]

It can be shown using a similar proof to those of [Mat95, Lemmas 2.12 and 2.17] that there is a Borel function \( k \) finite \( \mathcal{H}^d \)-a.e. so that

\[
\omega = k \mathcal{H}^d + \omega_s
\]

where \( \omega_s \perp \omega \) and

\[
\lim_{Q \downarrow \{x\}} \theta_{\omega}^{d}(Q) = k \quad \text{for } \mathcal{H}^d \text{-a.e. } x \in Q_0,
\]

Let

\[
F(R) = R \setminus \bigcup_{Q \in \text{Next}(R)} Q
\]

and define

\[
f_R = \sum_{Q \in \text{Next}(R)} \log \frac{\theta_{\omega}(Q)}{\theta_{\omega}(R)} 1_Q + \frac{k}{\Theta_{\omega}(R)} 1_{F(R)}.
\]

**Lemma 5.3.** For all \( x \in R \),

\[
-|\log \tau/C_0| = \log(\tau/C_0) \leq f_R(x) \leq \log(C_2A).
\]
Proof. Recall from Lemma 2.3 that every \( x \in R \) is either contained in infinitely many cubes from \( \text{Tree}(R) \) or is contained in a cube from \( \text{Stop}(R) \), and since \( \text{Next}(R) \) are the children of the cubes in \( \text{Stop}(R) \), this means \( x \) is contained in a cube from \( \text{HD}(R) \) or \( \text{LD}(R) \). Thus, if \( x \in F(R) \), it is contained in infinitely many cubes \( Q \in \text{Tree}(R) \), and so (5.3) and (1) from Theorem II imply (5.4).

Now if \( x \in Q \in \text{Next}(R) \) and \( Q' \in \text{Stop}(R) \) is the parent of \( Q \), then (5.4) follows from (5.1) and (5.2).

\[ \square \]

Lemma 5.4. The sets \( \{ F(R) : R \in \text{Top} \} \) are disjoint. In particular,

\( (5.5) \sum_{R \in \text{Top}} |F(R)| \leq |Q_0| \).

Proof. To see this, suppose there is \( x \in F(R) \cap F(R') \) with \( R \neq R' \). Then \( x \in R \cap R' \), without loss of generality we can assume \( R \supseteq R' \), so \( R' \subseteq Q \in \text{Next}(R) \), but since \( x \in F(R) \), \( x \) is not contained in any cube from \( \text{Next}(R) \), which is a contradiction. \( \square \)

Note that \( \mathcal{H}^d \)-a.e. \( x \in Q_0 \) is contained in \( F_R \) for some \( R \in \text{Top} \) by (3) of Theorem II and the Ahlfors regularity of \( \partial \Omega \). Thus,

\( (5.6) \log k - \log \theta^d_\omega(Q_0) = \sum_{R \in \text{Top}} f_R \quad \mathcal{H}^d \)-a.e in \( Q_0 \).

Lemma 5.5.

\[ \int_{Q_0} \left( \log k - \log \theta^d_\omega(Q_0) \right) = \sum_{R \in \text{Top}} \int_{Q_0} f_R. \]

Proof. We claim that \( \log k - \log \theta_\omega(Q_0) \) is absolutely integrable. Indeed, by (5.6),

\[ | \log k - \log \theta^d_\omega(Q_0) | \leq \sum_{R \in \text{Top}} | f_R | \]

and so

\[ \int_{Q_0} | \log k - \log \theta^d_\omega(Q_0) | \leq \sum_{R \in \text{Top}} \int_{Q_0} | f_R | \overset{(5.4)}{\lesssim} \tau, A \sum_{R \in \text{Top}} | R | \lesssim \beta \theta_\omega(Q_0), \]

and this proves the claim.

Now we prove the lemma. Let \( \text{Top}_j \) be those cubes in \( \text{Top} \) that are properly contained in \( j \) many cubes from \( \text{Top} \), \( \text{Top}^N = \bigcup_{j=0}^N \text{Top}_j \) and let

\[ f_N = \sum_{R \in \text{Top}^N} f_R. \]

Then

\[ | f_N | \leq \sum_{R \in \text{Top}} | f_R | \lesssim_{\tau, A} \sum_{R \in \text{Top}} \mathbb{1}_R. \]
and the last sum is a nonegative integrable function, thus the dominated convergence theorem implies

\[
\int_{Q_0} \left( \log k - \log \theta^d \omega(Q_0) \right) = \int_{Q_0} \sum_{R \in \text{Top}} f_R = \int_{Q_0} \lim_{N} f_N = \lim_{N} \int_{Q_0} f_N
\]

\[
= \lim_{N} \int_{Q_0} \sum_{R \in \text{Top}^N} f_R = \lim_{N} \sum_{R \in \text{Top}^N} \int_{Q_0} f_R = \sum_{R \in \text{Top}} \int_{Q_0} f_R.
\]

This proves the lemma. □

Thus,

\[
(5.7) \quad \int_{Q_0} \left( \log k - \log \theta^d \omega(Q_0) \right) = \sum_{R \in \text{Top}} \int_{Q_0} f_R \geq -|\log(C_0^{-1}\tau)| \sum_{R \in \text{Top}} |R|
\]

Now we will prove an opposite inequality. Note that

\[
(5.8) \quad \sum_{Q \in \text{HD}(R)} |Q| \sim A^{-1} \theta^d \omega(R)^{-1} \sum_{Q \in \text{HD}(R)} \omega(Q) \leq A^{-1} \theta^d \omega(R)^{-1} \omega(R) = A^{-1} |R|.
\]

And so for some constant \(c\) depending on the Ahlfors regularity,

\[
(5.9) \quad \int_{\bigcup_{Q \in \text{HD}(R)} Q} f_R \leq \sum_{Q \in \text{HD}(R)} \log(C_0 A)|Q| \leq c \frac{\log(C_0 A)}{A} |R|
\]

We will abuse notation here and also write \(\text{LD}(R) = \bigcup_{Q \in \text{LD}(R)} Q\). Then

\[
(5.10) \quad \int_{\bigcup_{Q \in \text{LD}(R)} Q} f_R \leq \sum_{Q \in \text{LD}(R)} \log(C_0 \tau)|Q| = \log(C_0 \tau)|\text{LD}(R)|.
\]

We will pick \(\tau > 0\) small so that

\[
(5.11) \quad \frac{\log(C_0 \tau)}{8} < -1.
\]

Let

\[
\text{Top}_1 = \{ R \in \text{Top} : |\text{LD}(R)| \geq |R|/4 \}, \quad \text{Top}_2 = \text{Top} \setminus \text{Top}_1.
\]

Also note that if \(R \in \text{Top}_1\), then (recalling \(\log(C_0 \tau) < 0\) and \(|\text{LD}(R)| \geq |R|/4\))

\[
(5.12) \quad \int_{\bigcup_{Q \in \text{LD}(R)} Q} f_R \leq \log(C_0 \tau)|\text{LD}(R)| \leq \frac{\log(C_0 \tau)}{4} |R|.
\]
and so (recalling (5.11)) if we pick $A$ large enough so that $c \frac{\log(C_0 A)}{A} < - \frac{\log(C_0 \tau)}{8}$,

$$
\sum_{R \in \text{Top}_1} \left( \int_{\bigcup_{Q \in \text{HD}(R)} Q} f_R + \int_{\bigcup_{Q \in \text{LD}(R)} Q} f_R + \int_{F(R)} f_R \right) \leq \sum_{R \in \text{Top}_1} \left( c \frac{\log(C_0 A)}{A} + \frac{\log(C_0 \tau)}{4} \right) |R| + \log(C_0 A) |Q_0|
$$

(5.13)

$$
\leq \sum_{R \in \text{Top}_1} \frac{\log(C_0 \tau)}{8} |R| + \log(C_0 A) |Q_0| - \sum_{R \in \text{Top}_1} |R| + \log(C_0 A) |Q_0|
$$

Now if $R \in \text{Top}_2$,

$$
|R| = |F(R) \cup LD(R) \cup HD(R)| \leq |F(R)| + \left( C A^{-1} + \frac{1}{4} \right) |R|
$$

and so for $A > 4C$,

(5.14) \hspace{1cm} |R| < 2 |F(R)|.

Thus,

(5.15) \hspace{1cm} \sum_{R \in \text{Top}_2} |R| \leq 2 \sum_{R \in \text{Top}_2} |F_R| \leq 2 |Q_0|.

Hence,

$$
\sum_{R \in \text{Top}_2} \left( \int_{\bigcup_{Q \in \text{HD}(R)} Q} f_R + \int_{\bigcup_{Q \in \text{LD}(R)} Q} f_R + \int_{F(R)} f_R \right) \leq \sum_{R \in \text{Top}_2} \left( c \frac{\log(C_0 A)}{A} |R| + \log(C_0 \tau) |\text{LD}(R)| \right) + \log(C_0 A) |Q_0|
$$

(5.16) \hspace{1cm} \leq \sum_{R \in \text{Top}_2} \left( \frac{2c \log(C_0 A)}{A} + \log(C_0 A) \right) |Q_0| =: C_A |Q_0|.
Thus, noting that $2 + \log(C_0 A) < 2C_A$ for $A$ large,

$$
\int_{Q_0} (\log k - \log \theta^d_{Q_0}(Q)) = \sum_{R \in \text{Top}} \int_{Q_0} f_R = \left( \sum_{R \in \text{Top}_1} + \sum_{R \in \text{Top}_2} \right) f_R
$$

\begin{align*}
&\leq \sum_{R \in \text{Top}_1} |R| + \log(C_0 A)|Q_0| + C_A|Q_0| \\
&\leq \sum_{R \in \text{Top}} |R| + \sum_{R \in \text{Top}_2} |R| + (\log(C_0 A) + C_A)|Q_0| + \\
&\leq \sum_{R \in \text{Top}} |R| + (2 + \log(C_0 A) + C_A)|Q_0| < \sum_{R \in \text{Top}} |R| + 3C_A|Q_0|
\end{align*}

Combining this with (5.7), we get

$$
- \sum_{R \in \text{Top}} |R| - |Q_0| \lesssim \int_{Q_0} (\log k - \log \theta^d_{Q_0}(Q)) \leq - \sum_{R \in \text{Top}} |R| + 3C_A|Q_0|.
$$

Thus, if we subtract $C|Q_0|$ from both sides for large enough $C$, we get

$$
\int_{Q_0} (\log k - \log \theta^d_{Q_0}(Q) - C) \sim - \sum_{R \in \text{Top}} |R|^d - |Q_0|.
$$

Now taking negatives of both sides (and recalling $|R| \sim \ell(R)^d$) gives

$$
\int_{Q_0} \left( \log \frac{1}{k} + \log \theta^d_{Q_0}(Q) + C \right) \sim \sum_{R \in \text{Top}} \frac{|R|}{|Q_0|} + 1 \frac{\beta_{\partial \Omega}(Q_0)}{|Q_0|}.
$$

This proves Theorem 5.1 under the assumption that $\beta_{\partial \Omega}(Q_0) < \infty$. \hfill \Box

### 5.2. The case assuming the log integral is finite a priori.

**Lemma 5.6.** The conclusions of Theorem 5.1 hold assuming

$$
\int_{Q_0} \log \frac{1}{k} d\mathcal{H}^d + \log \frac{\omega(Q_0)}{|Q_0|} < \infty.
$$

**Proof.** Under these assumptions, we’ll show that we still have $\beta_{\partial \Omega}(Q_0) < \infty$, and we can then employ the previous lemma. To do this, it suffices to find a new collection $\{\text{Tree}(R) : R \in \text{Top}\}$ satisfying the conditions of Theorem II.

Recall that $k < \infty \mathcal{H}^d$-a.e. so (5.3) still holds. Run a stopping-time algorithm as follows. Let $C > 1$. For $R \subseteq Q_0$, let $\text{HD}'(R)$ be those maximal cubes in $R$ which have a child $Q'$ for which $\theta^d(Q') > CA\theta^d(R)$, and $\text{LD}'(R)$ be those maximal $Q \subseteq R$ with a child $Q'$ so that $\theta^d_{\omega}(Q') < C^{-1} \theta^d_{\omega}(R)$, let $\text{Stop}(R) = \text{HD}'(R) \cup \text{LD}'(R)$. Let $\text{HD}(R)$ be the children of the cubes in $\text{HD}'(R)$ and $\text{LD}(R)$ be the children of
the cubes in $LD'(R)$ and set $\text{Next}(R) = \text{HD}(R) \cup LD(R)$. For $C$ large enough, by the doubling property, we have that $\theta_\omega(Q) > A\theta_\omega(R)$ for $Q \in \text{HD}(R)$ and $\theta_\omega(Q) < \tau\theta_\omega(R)$ for $Q \in \text{LD}(R)$. By maximality, we also have there is $C_0$ so that $\theta_\omega(Q) < C_0 A\theta_\omega(R)$ for $Q \in \text{HD}(R)$ and $\theta_\omega(Q) \geq C_0^{-1} \tau\theta_\omega(R)$ for $Q \in \text{LD}(R)$.

Let $\text{Top}_0 = \{Q_0\}$ and for $k \geq 0$, let

$$\text{Top}_{k+1} = \bigcup_{R \in \text{Top}_k} \text{Next}(R), \quad \text{Top} = \bigcup_{k \geq 0} \text{Top}_k.$$ 

By Lemma II,

$$\beta_{\partial \Omega}(Q_0) \lesssim \sum_{R \in \text{Top}} |R|$$

and so the lemma will follow once we show

$$\sum_{R \in \text{Top}} |R| \lesssim \int_{Q_0} \log \frac{1}{k} d\mathcal{H}^d + \log \frac{\omega(Q_0)}{|Q_0| < \infty}. \quad (5.17)$$

First observe that for $R \in \text{Top}$,

$$\sum_{Q \in \text{HD}(R)} |Q| < \sum_{Q \in \text{HD}(R)} \theta_\omega^d(R)^{-1} A^{-1} \omega(Q) \leq A^{-1} |R|. \quad (5.18)$$

Let

$$G_R = R \setminus \bigcup_{Q \in \text{Next}(R)} Q, \quad G_N = \bigcup_{k=0}^N \bigcup_{R \in \text{Top}_k} G_R$$

and

$$g_R = \sum_{Q \in \text{Next}(R)} \log \frac{\theta_\omega(R)}{\theta_\omega(Q)} + \log \frac{\theta_\omega(R)}{k} 1_{G_R}.$$ 

Note that if $x \in G_R$, then

$$k(x) \leq CA\theta_\omega(R). \quad (5.19)$$
Since $\theta_\omega(Q) < \tau \theta_\omega(R)$ for $Q \in \text{LD}(R)$ and $\theta_\omega(Q) < C_0 A \theta_\omega(R)$ for $Q \in \text{HD}(R)$,

\[
\log \tau^{-1} \sum_{Q \in \text{LD}(R)} |Q| \leq \sum_{Q \in \text{LD}(R)} |Q| \log \frac{\theta_\omega(R)}{\theta_\omega(Q)}
\]

\[
\leq \sum_{Q \in \text{Next}(R)} |Q| \log \frac{\theta_\omega(R)}{\theta_\omega(Q)} + \sum_{Q \in \text{HD}(R)} |Q| \log \frac{\theta_\omega(Q)}{\theta_\omega(R)}
\]

\[
\leq \sum_{Q \in \text{Next}(R)} |Q| \log \frac{\theta_\omega(R)}{\theta_\omega(Q)} + C_0 A \sum_{Q \in \text{HD}(R)} |Q|
\]

\[
= \int_R g_R + \int_{G_R} \log \frac{k}{\theta_\omega(R)} + \log \frac{A}{A} |R|
\]

\[
\leq \int_R g_R + \log C A |G_R| + \frac{\log A}{A} |R|. \quad (5.19)
\]

In particular, if $\kappa = (\log \tau^{-1})^{-1}$, for $k > 0$, if we pick $A$ large so that $A > 4$ and $\kappa \log \frac{C_0 A}{A} < \frac{1}{4}$,

\[
\sum_{R \in \text{Top}_k} |R| = \sum_{R \in \text{Top}_{k-1}} \sum_{Q \in \text{Next}(R) = \text{HD}(R) \cup \text{LD}(R)} |Q|
\]

\[
\leq \sum_{R \in \text{Top}_{k-1}} \left( A^{-1} |R| + \kappa \left( \int_R g_R + \log C A |G_R| + \frac{\log C_0 A}{A} |R| \right) \right)
\]

\[
< \sum_{R \in \text{Top}_{k-1}} \left( \frac{|R|}{2} + \kappa \left( \int_R g_R + \log C A |G_R| \right) \right)
\]

Thus,

\[
\sum_{k=0}^N \sum_{R \in \text{Top}_k} |R| = |Q_0| + \sum_{k=1}^N \sum_{R \in \text{Top}_k} |R|
\]

\[
\leq |Q_0| + \sum_{k=1}^N \sum_{R \in \text{Top}_{k-1}} \left( \frac{|R|}{2} + \kappa \left( \int_R g_R + \log C A |G_R| \right) \right)
\]

And so
\[
\frac{1}{2} \sum_{k=0}^{N} \sum_{R \in \text{Top}_k} |R| \leq |Q_0| + \sum_{k=0}^{N-1} \sum_{R \in \text{Top}_k} \kappa \left( \int_R g_R + \log C A |G_R| \right)
\]

As before with the \( F_R \), the \( G_R \) are disjoint and so
\[
\sum_{k=0}^{N-1} \sum_{R \in \text{Top}_k} \kappa \log C A |G_R| \leq \kappa \log C A |Q_0|.
\]

So to prove (5.17), we just need to show
\[
\sum_{k=0}^{N-1} \sum_{R \in \text{Top}_k} \int_R g_R \lesssim \int_{Q_0} \log \frac{1}{k} d \mathcal{H}^d + \log \frac{\omega(Q_0)}{|Q_0|}.
\]

Note that
\[
Q_0 = G_{k-1} \sqcup \bigsqcup_{R \in \text{Top}_k} R.
\]

Indeed, if \( x \in Q_0 \) and \( x \not\in \bigcup_{R \in \text{Top}_k} R \), then \( x \in G_R \) for some \( R \in \text{Top}_j \) for some \( j < k \), that is, \( x \in G_{k-1} \). Let
\[
u_k = \sum_{R \in \text{Top}_k} \left( \sum_{Q \in \text{Next}(R)} \frac{\theta_\omega(Q)}{\theta_\omega(R)} 1_Q + \frac{k}{\theta_\omega(R)} 1_{G_R} \right) + 1_{G_{k-1}}.
\]

and
\[
u_k = \sum_{R \in \text{Top}_k} 1_R \int_R \log \frac{1}{u_k} = \sum_{R \in \text{Top}_k} 1_R \int_R g_R.
\]

Note that by [Mat95, Theorem 2.12 (2)],
\[
\int_E k \leq \omega(E) \text{ for } E \text{ Borel.}
\]

Hence, for \( R \in \text{Top}_k \),
\[
\int_R u_k = \theta_\omega^d(R)^{-1} \left( \sum_{Q \in \text{Next}(R)} |Q| \theta_\omega(Q) + \int_{G_R} k \right)
\]
\[
\leq \theta_\omega^d(R)^{-1} \left( \sum_{Q \in \text{Next}(R)} \omega(Q) + \omega(G_R) \right)
\]
\[
= \theta_\omega^d(R)^{-1} \omega(R) = |R|.
\]

Thus, for \( x \in R \in \text{Top}_k \), by Jensen’s inequality,
\[
h_k(x) = \int_R g_R = -\int_R \log u_k \geq -\log \int_R u_k \geq -\log 1 = 0.
\]
and for \( x \in G_{k-1} \), \( h_k(x) = 0 \). Thus, \( h_k \geq 0 \) everywhere. Moreover, \( \sum h_k = \log \frac{1}{k} + \log \theta_\omega(Q_0) \) by (5.3), so the monotone convergence theorem implies that

\[
\sum_{k=0}^{N-1} \int_{R \in \mathcal{T}_p^k} g_R = \sum_{k=0}^{N-1} \sum_{R \in \mathcal{T}_p^k} h_k = \sum_{k=0}^{N-1} \int h_k \leq \sum_{k=0}^\infty \int h_k = \log \frac{1}{k} + \log \theta_\omega(Q_0).
\]

Combined with our earlier estimates, this proves (5.17). \( \square \)

6. Proof of Theorem V

The proof is somewhat similar to that in the previous section. Assuming the conditions of Theorem V, let \( k \in \mathbb{N} \). We can assume \( E \) is a compact subset of \( Q_0 \) and \( Q_0 \in \mathcal{D}_0 \).

We first claim that there is a finite collection of cubes \( \mathcal{C} \) covering \( E \) so that

\[
\mathcal{H}^d_\infty(E) \sim \sum_{Q \in \mathcal{C}} \ell(Q)^d.
\]

To see this, first by definition of Hausdorff content (or rather, dyadic Hausdorff content, since the two are comparable), we may find a collection of disjoint cubes \( \mathcal{C}' \) covering \( E \) so that

\[
\mathcal{H}^d_\infty(E) \sim \sum_{Q \in \mathcal{C}'} \ell(Q)^d.
\]

If \( \mathcal{C}' \) is finite, then we can set \( \mathcal{C} = \mathcal{C}' \). Otherwise, notice that \( \{B^0_Q\}_{Q \in \mathcal{C}'} \) is an open cover of \( E \), and so we can take a finite subcover \( \{B^0_Q\}_{Q \in \mathcal{C}''} \) for some \( \mathcal{C}'' \subseteq \mathcal{C} \) finite. Now let \( \mathcal{C} \) be the maximal cubes \( Q \subseteq Q_0 \) for which there is a sibling \( Q' \) and there is \( Q'' \in \mathcal{C}'' \) so that \( \ell(Q) = \ell(Q') = \ell(Q'') \) and \( Q' \cap B_{Q''} \neq \emptyset \). Then \( \mathcal{C} \) is finite, covers \( E \), and

\[
\mathcal{H}^d_\infty(E) \lesssim \sum_{Q \in \mathcal{C}} \ell(Q)^d \sim \sum_{Q \in \mathcal{C}''} \ell(Q)^d \leq \sum_{Q \in \mathcal{C}''} \ell(Q)^d \sim \mathcal{H}^d_\infty(E).
\]

This proves the claim.

Let \( \mathcal{I} \) be the maximal cubes that contain an element of \( \mathcal{C} \) as a child. Then \( E \subseteq \bigcup_{Q \in \mathcal{I}} \) and so

\[
\mathcal{H}^d_\infty(E) \lesssim \sum_{Q \in \mathcal{I}} \ell(Q)^d \lesssim \sum_{Q \in \mathcal{C}} \ell(Q)^d \sim \mathcal{H}^d_\infty(E).
\]

Let \( \mathcal{B} \) be the maximal cubes that do not intersect a cube in \( \mathcal{I} \) and let \( \mathcal{B}' \) be their children. Then \( \mathcal{B} \cup \mathcal{I} \) is finite. Let \( \mathcal{I} \) be those cubes that contain an element of \( \mathcal{B} \cup \mathcal{I} \). By Lemma 2.3 (with \( \mathcal{C} \) replaced by \( \mathcal{B}' \cup \mathcal{C} \) in our case), \( \mathcal{I} \) is a stopping-time region whose minimal cubes are \( \mathcal{B} \cup \mathcal{I} \).
Let BTM be the children of the minimal cubes. Let $\delta_x$ denote the Dirac mass at $x$ and let
\[
\mu = \sum_{Q \in \text{BTM}} \ell(Q)^d \delta_{Q}.
\]
By Lemma 3.1,
\[
\mu(Q_0) = \sum_{Q \in \text{BTM}} \ell(Q)^d \lesssim \beta_{\partial \Omega}(\mathcal{T}).
\]
Let $5\rho^{k_0}$ be the size of the smallest dyadic cube in BTM and set $\mu^{k_0} = \mu$. We define a collection of cubes FC (for “Frostmann cube”) and measures $\mu^k$ as follows. Suppose $k \leq k_0$ and $\mu^k$ has been defined. Let $Q \in D_{k-1} \cap \mathcal{T}$. If $\mu^k(Q) > 2\ell(Q)^d$, then add $Q$ to FC and define
\[
\begin{align*}
\mu^{k-1}|_Q &= \frac{\ell(Q)^d}{\mu^k(Q)} \mu^k|_Q.
\end{align*}
\]
Note that by induction,
\[
\begin{align*}
2\ell(Q)^d &< \mu^k(Q) \leq \sum_{R \in \text{Child}(Q)} \mu^k(R) \leq \sum_{R \in \text{Child}(Q)} 2\ell(R)^d \lesssim \ell(Q)^d,
\end{align*}
\]
so we have
\[
(6.1) \quad \mu^k|_Q \lesssim \mu^{k-1}|_Q \leq \mu^k|_Q.
\]
Otherwise, we just set $\mu^{k-1}|_Q = \mu^k|_Q$. Continue in this way and set $\nu = \mu^0$.

For $Q \in \text{FC}$, let $n(Q)$ denote the number of cubes in FC properly containing $Q$. If $k(Q)$ is so that $Q \in \mathcal{D}_{k(Q)}$, and if $R \in \text{BTM}$ and $R \subseteq Q$, and $R = Q^0 \subseteq Q^1 \subseteq \ldots \subseteq Q^{n(R)-n(Q)} = Q$ are in FC, then
\[
\mu^{k(Q)}(R) = \mu^{k(R)}(R) \prod_{i=1}^{n(R)-n(Q)} \frac{\ell(Q^i)^d}{\mu^{k(Q^i)+1}(Q^i)} < \frac{\mu^{k(R)}(R)}{2^{n(R)-n(Q)}} = \frac{\ell(R)^d}{2^{n(R)-n(Q)}}.
\]
Thus,
\[
\begin{align*}
\sum_{Q \in \text{FC}} \ell(Q)^d &= \sum_{Q \in \text{FC}} \mu^{k(Q)}(Q) \leq \sum_{Q \in \text{FC}} \sum_{R \in \text{BTM} \cup \{Q\}} \frac{\ell(Q)^d}{2^{n(R)-n(Q)}}
\end{align*}
\]
\[
\leq \sum_{R \in \text{BTM}} \sum_{Q \in \text{FC}} \mu^{k(Q)}(R) < \sum_{R \in \text{BTM}} \sum_{Q \in \text{FC} \setminus R} \frac{\ell(R)^d}{2^{n(R)-n(Q)}} \lesssim \sum_{R \in \text{BTM}} \ell(R)^d \lesssim \beta_{\partial \Omega}(\mathcal{T}).
\]

For $R \in \text{FC}$, let $\text{Stop}_F(R)$ be the maximal cubes in FC properly contained in $R$ and $\text{Tree}_F(R)$ be those cubes contained in $R$ containing a cube from $\text{Stop}_F(R)$. 
Lemma 6.1. For $R \in FC \cup \{Q_0\}$ and $Q \in \text{Tree}_F(R)$,
\begin{equation}
\Theta^d_\nu(Q) \sim \Theta^d(R).
\end{equation}

Proof. Recall that $\nu|_R$ is a multiple of $\mu^{k(R)}|_R$, so it suffices to prove the same statement with $\mu^R := \mu^{k(R)}$ in place of $\nu$. We can assume $Q \neq R$. If $Q \notin \text{Stop}_F(R)$, then $\mu^R(Q) \leq 2\ell(Q)^d$ and if $Q \in \text{Stop}_F(R)$, then $\mu^R(Q) = \ell(Q)^d$. Thus, we just need to focus on showing a lower bound for $\mu^R(Q)$:
\[
\mu^R(Q) = \sum_{T \in \text{Stop}_F(R) \cap Q} \mu^R(T) = \sum_{T \in \text{Stop}_F(R) \cap Q} \ell(T)^d \gtrsim \mathcal{H}^d(Q) \sim \ell(Q)^d.
\]
This proves the lemma. \hfill \Box

Lemma 6.2.
\[
\nu(E) \gtrsim \mathcal{H}^d(E) \exp(-C\beta_\Omega(\mathcal{F})/\mathcal{H}^d(E)).
\]

Proof. For $k \geq 0$, let $\mathcal{S}(k)$ be those cubes in $FC$ that are properly contained in $k$ many cubes in $FC$. Let $\delta \in (0, 1/2)$ and let $N \in \mathbb{N}$ be the maximal integer so that
\[
\sum_{Q \in \mathcal{S}(k)} (2\ell(Q))^d \geq \frac{\mathcal{H}^d(E)}{2} \text{ for all } 1 \leq k \leq N.
\]
Then
\[
N \leq \frac{2}{\mathcal{H}^d(E)} \sum_{k=1}^{N} \sum_{Q \in \mathcal{S}(k)} (2\ell(Q))^d \leq \frac{2}{\mathcal{H}^d(E)} \sum_{Q \in FC} (2\ell(Q))^d \lesssim \frac{\beta_\Omega(\mathcal{F})}{\mathcal{H}^d(E)}.
\]
Thus,
\[
\mathcal{H}^d\left(\bigcap_{Q \in \mathcal{S}(N+1)} Q\right) \leq \sum_{Q \in \mathcal{S}(N+1)} (\text{diam } B_Q)^d = \sum_{Q \in \mathcal{S}(N+1)} (2\ell(Q))^d < \frac{\mathcal{H}^d(E)}{2}.
\]
Hence, if we let $E' = E \setminus \bigcup_{Q \in \mathcal{S}(N+1)} Q$, then $\mathcal{H}^d(E') \geq \mathcal{H}^d(E)/2$. By (6.1), if $Q \in \text{BTM}\setminus \mathcal{S}(N+1)$, then for some $c \in (0, 1)$,
\begin{equation}
\nu(Q) \geq c^{N+1} \mu^k(Q) = c^{N+1} \ell(Q)^d.
\end{equation}
Thus,
\[
\nu(E) \geq \nu(E') = \sum_{Q \in \text{BTM}} \nu(Q) \geq c^{N+1} \sum_{Q \in \text{BTM}} (Q)^d \gtrsim c^{N+1} \mathcal{H}^d(E') \gtrsim c^{N+1} \mathcal{H}^d(E).
\]
\hfill \Box

Let $\mathcal{F}'$ be those cubes in $\mathcal{F}$ that properly contain a cube from $\mathcal{S}(N+1) \cup \text{BTM}$. Let $\text{BTM}'$ be the children of the minimal cubes of $\mathcal{F}'$, so they are each adjacent to a cube in $\mathcal{S}(N+1) \cup \text{BTM}$, and so to a cube in $FC \cup \text{BTM}$. Now we partition $\mathcal{F}'$ into subtrees as follows. For $R \in \mathcal{F}$, let $\text{Stop}'(R)$ be the cubes in $\mathcal{F}'$ that contain cube from $FC \cup \text{Top}$ as a child, and let $\text{Next}'(R)$ be the children of the cubes in
Thus, of Theorem II imply that $C > 0$, let $\text{Top}_k = \{Q_0\}$ and for $k \geq 0$, let $\text{Top}_{k+1} = \bigcup_{R \in \text{Top}_k} \text{Next}'(R)$ and $\text{Top}' = \bigcup_{k \geq 0} \text{Top}_k$. Note that each $R \in \text{Top}'$ either adjacent to a cube in $\text{BTM}' \cup \text{Top} \cup FC$, and recalling that cubes in $\text{BTM}'$ are each adjacent to a cube in $FC \cup \text{BTM}$, we have
\[
\sum_{R \in \text{Top}'} \ell(R)^d \lesssim \sum_{R \in FC \cup \text{Top} \cup \text{BTM}} \ell(R)^d \lesssim \beta(\mathcal{T}).
\]

Let $\theta'(Q) = \omega(Q)/\nu(Q)$,
\[
k = \sum_{Q \in \text{BTM}'} \Theta'(Q) 1_Q,
\]
and for $R \in \text{Top}'$,
\[
f_R = \sum_{Q \in \text{Next}'(R)} \log \frac{\Theta'(Q)}{\Theta'(R)} 1_R.
\]

Note that since each Tree'(R) is contained in some Tree_F(R') for some $R' \in FC \cup \{Q_0\}$ and is also contained in Tree'(R') for some $R \in \text{Top}$, (6.2) and the conclusion of Theorem II imply that
\[
|f_R| \lesssim_{\tau,A} 1.
\]

Thus,
\[
\int_{Q_0} (\log k - \log \Theta'(Q_0)) = \sum_{R \in \text{Top}'} \int_{Q_0} \frac{f_R}{\nu(Q_0)} d\nu \gtrsim_{A,\tau} - \sum_{R \in \text{Top}'} \frac{\nu(R)}{\nu(Q_0)} \gtrsim - \frac{\ell(R)^d}{\ell(Q_0)^d} \gtrsim - \frac{\beta(\mathcal{T})}{\ell(Q_0)^d}.
\]

Following the proof of [Hru84, Theorem 1] up to [Hru84, Equation (7)] (with Lebesgue measure replaced with $\nu$ and $w$ replaced by $k$), and recalling Lemma 6.2, we obtain that there is $C > 1$ so that
\[
\log \frac{C \beta(\mathcal{T})}{\ell(Q_0)^d} \geq \log \frac{\omega(Q_0)}{\omega(E)} \geq \exp \left(-C \beta(\mathcal{T}) \frac{\mathcal{H}_d(E)}{\mathcal{H}_\infty^d(E)} \right) \frac{\mathcal{H}_d(E)}{\ell(Q_0)^d} \log \frac{\omega(Q_0)}{\omega(E)}.
\]

Noting that $\mathcal{H}_\infty^d(E) \lesssim \ell(Q_0)^d$, the above implies (for a slightly larger $C_0 > C$)
\[
\log \frac{\omega(Q_0)}{\omega(E)} \leq \exp \left(C \beta(\mathcal{T}) \frac{\mathcal{H}_\infty^d(E)}{\mathcal{H}_\infty^d(E)} \right) \frac{\ell(Q_0)^d}{\mathcal{H}_\infty^d(E)} \log \frac{C \beta(\mathcal{T})}{\mathcal{H}_\infty^d(E)} \lesssim \frac{\ell(Q_0)^d}{\mathcal{H}_\infty^d(E)} \exp \left(C_0 \beta(\mathcal{T}) \frac{\mathcal{H}_\infty^d(E)}{\mathcal{H}_\infty^d(E)} \right).
\]

Thus, for some $C_1 > 0$,
\[
\frac{\omega(E)}{\omega(Q_0)} \geq \exp \left(-C_1 \frac{\ell(Q_0)^d}{\mathcal{H}_\infty^d(E)} \exp \left(C_0 \beta(\mathcal{T}) \frac{\mathcal{H}_\infty^d(E)}{\mathcal{H}_\infty^d(E)} \right) \right).
\]
In this section we give an example of a domain in $\mathbb{C}$ whose boundary is Ahlfors regular with finite linear deviation but whose harmonic measure has a singular part. The proof is an adaptation of some of the details in [Bat96].

Let $f_0, \ldots, f_3$ be as in the introduction. Let $I_\emptyset = [-1/2, 1/2]^2$ and for a word $\alpha = \alpha_1 \cdots \alpha_n$ with $\alpha_i \in \{0, 1, 2, 3\}$, let

$$f_\alpha = f_{\alpha_n} \circ \cdots \circ f_1 \quad \text{and} \quad I_\alpha = f_\alpha(I_0).$$

See Figure 2

Let $K$ denote the 4-corner cantor set and $\omega = \omega_{K_c}^\infty$ denote the harmonic measure with pole at infinity for $\Omega = \mathbb{C} \setminus K$. We let $|A| = \mathcal{H}^1(A \cap K)$.

By [Bat96, Lemma 2.7], there is $\rho > 1$ so that for all $\alpha$, there is $i$ so that

$$\frac{\omega(I_{\alpha i})}{|I_{\alpha i}|} > \rho \frac{\omega(I_\alpha)}{|I_\alpha|}.$$ 

If we iterate this, this means there is $N$ and a word $\beta$ with $|\beta| = N$ so that

$$\frac{\omega(I_{\alpha \beta})}{|I_{\alpha \beta}|} > 4^N \frac{\omega(I_\alpha)}{|I_\alpha|}.$$ 

For $k \geq 0$, let $D_k = \{I_\alpha : |\alpha| = kN\}$ and for $k \geq 0$, $I_\alpha \in D_k$, and $n \in \mathbb{N}$, let

$$\text{Child}_n(I_\alpha) = \{I_{\alpha \beta} : |\beta| = nN\} \subseteq D_{k+n}, \quad \text{Child}(I_\alpha) := \text{Child}_1(I_\alpha).$$

Let $D = \bigcup D_k$. Thus, the above says that for all $I \in D$ there is $I' \in \text{Child}(I)$ so that

$$\frac{\omega(I')}{|I'|} > 16 \frac{\omega(I)}{|I|}.$$
Thus, by Cauchy-Schwartz, the fact that \((1 - t)^{1/2} \leq 1 - t/2\) for \(t \geq 0\), and since \(\omega(J) \geq c \omega(I)\) for all \(J \in \text{Child}(I)\) (since \(\omega\) is doubling as \(\Omega\) is uniform),

\[
\sum_{J \in \text{Child}(I)} \omega(J)^{1/2} |J|^{1/2} = \sum_{J \in \text{Child}(I)} \omega(J)^{1/2} |J|^{1/2} + \omega(I')^{1/2} |I'|^{1/2}
\]

\[
\leq \left( \sum_{J \in \text{Child}(I)} \omega(J) \right)^{1/2} \left( \sum_{J \in \text{Child}(I)} |J| \right)^{1/2} + \omega(I')^{1/2} \left( \frac{\omega(I)|I|}{16 \omega(I)} \right)^{1/2}
\]

\[
\leq |I|^{1/2} \omega(I)^{1/2} \left( 1 - \omega(I')^{1/2} + \frac{1}{4} \omega(I) \right)
\]

\[
\leq |I|^{1/2} \omega(I)^{1/2} \left( 1 - \frac{1}{4} \omega(I) \right) \leq |I|^{1/2} \omega(I)^{1/2} \left( 1 - c/4 \right) =: \lambda |I|^{1/2} \omega(I)^{1/2}
\]

Let \(\tau > 0\) be small, \(n \in \mathbb{N}\). Since \(|J| = 4^{-N} |I|\) for \(J \in \text{Child}(I)\),

\[
\sum_{J \in \text{Child}(I)} \omega(J)^{1/2} |J|^{1/2} = 4^{N/\tau} |I|^{-\tau} \sum_{J \in \text{Child}(I)} \omega(J)^{1/2} |J|^{1/2}
\]

\[
< 4^{N/\tau} \lambda \omega(I)^{1/2} |I|^{-\tau/2} =: \gamma \omega(I)^{1/2} |I|^{-\tau/2}
\]

where \(\gamma < 1\) for \(\tau\) small enough depending on \(N\) and \(\lambda\). Let

\[
\text{Stop}(I, n) = \left\{ J \in \text{Child}_n(I) : \frac{\omega(J)}{\omega(I)} < \frac{|J|^{1-\tau}}{|I|^{1-\tau}} \right\}.
\]

Then iterating the above estimate, we get

\[
\sum_{J \in \text{Stop}(I, n)} \omega(J) = \sum_{J \in \text{Stop}(I, n)} \omega(J)^{1/2} \omega(J)^{1/2} \leq \frac{\omega(I)^{1/2}}{|I|^{1/2}} \sum_{J \in \text{Stop}(I, n)} \omega(J)^{1/2} |J|^{1-\tau} < \gamma^n \omega(I).
\]

Let

\[
\text{Top}(I, n) = \text{Child}_n(I) \setminus \text{Stop}(I, n).
\]

Then

(7.1)

\[
\sum_{J \in \text{Top}(I, n)} |J| = \sum_{J \in \text{Top}(I, n)} |J|^{1-\tau} |J|^{1-\tau} \leq \sum_{J \in \text{Top}(I, n)} \left( \frac{|I|^{\tau}}{4n N^\tau} \right) \left( |I|^{1-\tau} \omega(J) \right) \leq \frac{|I|}{4n N^\tau}.
\]

Let \(\text{Top}(0) = I_0\). For \(n > 0\), let

\[
\text{Stop}(n) = \bigcup_{I \in \text{Top}(n-1)} \text{Stop}(I, n), \quad \text{Top}(n) = \bigcup_{I \in \text{Top}(n-1)} \text{Top}(I, n).
\]
Then for $n > 0$, 
\[
\sum_{J \in \text{Top}(n)} \omega(J) = \sum_{I \in \text{Top}(n-1)} \sum_{J \in \text{Top}(I,n)} \omega(J) \\
= \sum_{I \in \text{Top}(n-1)} \left( \omega(I) \sum_{J \in \text{Top}(I,n)} \omega(J) \right) \\
> \sum_{I \in \text{Top}(n-1)} \omega(I)(1 - \gamma^n) > \cdots > \omega(I_0) \prod_{k=1}^{n}(1 - \gamma^k)
\]

Thus, taking $n \to \infty$, if we set $G = \bigcap_{n \geq 0} \bigcup_{I \in \text{Top}(n)} I$, we get that $\omega(G) > 0$. Moreover, since $G \subseteq \text{Top}(n)$ for all $n$, 
\[
|G| \leq \sum_{J \in \text{Top}(n)} |J| \leq \sum_{I \in \text{Top}(n-1)} \sum_{J \in \text{Top}(I,n)} |J| < \frac{1}{4^n M^r} \sum_{I \in \text{Top}(n-1)} |I|.
\]

Hence $|G| = 0$, since otherwise, taking the intersection over $\bigcup_{J \in \text{Top}(n)} J$, the above would give $|G| \leq \frac{1}{4^n M^r} |G| < |G|$, a contradiction.

Now let $\eta > 1$ be very close to 1 and 
\[
F = \bigcup_{n \geq 0} \bigcup_{I \in \text{Stop}(n)} \eta I = G \cup \bigcup_{n \geq 0} \bigcup_{I \in \text{Stop}(n)} \eta I
\]

Let $\hat{\Omega} = F^c$ and $\hat{\omega} = \omega_{\hat{\Omega}}^\infty$. See Figure 3

It is a bit annoying but rather straightforward to check that $\hat{\Omega}$ is also uniform, so we omit the details.

**Claim:** $\omega(G) > 0$. We recall a lemma about harmonic measure for uniform domains:

**Lemma 7.1.** [MT15, Theorem 1.3] Let $n \geq 1$, $\Omega$ be a uniform domain in $\mathbb{R}^{d+1}$ and let $B$ be a ball centered on $\partial \Omega$. Let $p_1, p_2 \in \Omega$ be such that $\text{dist}(p_i, B \cap \partial \Omega) > c_0^{-1} r_B$ for $i = 1, 2$. Then, for any Borel set $E \subseteq B \cap \partial \Omega$, 
\[
\frac{\omega^{p_1}(E)}{\omega_{p_1}(B)} \sim \frac{\omega^{p_2}(E)}{\omega_{p_2}(B)}
\]

with the implicit constant depending only on $c_0$ and the uniform behavior of $\Omega$.

This was originally shown for NTA domains in [JK82]. In our setting, since $\partial \Omega$ is 1-Ahlfors regular, if $B = B(0, \text{diam} \partial \Omega)$, then $\omega_0^\infty(\partial \Omega) = 1$. Hence, if $x_j$ is a sequence of points going to infinity so that $\omega_{x_j} / \omega_{x_j}^\infty(\partial \Omega) \to \omega$, 
\[
0 < \omega(G) = \lim_{j \to \infty} \frac{\omega_{x_j}^\infty(G)}{\omega_{x_j}^\infty(\partial \Omega)} \geq \frac{\omega_0^\infty(G)}{\omega_0^\infty(\partial \Omega)} = \omega_0^\infty(G).
\]
The first equality is justified since $G$ is a compact totally disconnected and \cite[Exercise 1.9, p.22]{Mat}. Thus, by Harnack’s principle, since $\Omega$ is connected, $\omega^x(I_G) > 0$ for all $x \in G$.

**Claim:** $\hat{\omega}(G) > 0$. The proof here is modelled after that in the proof of \cite[Lemma 6.3]{JK}.

First we need to show there is $c \in (0, 1)$ so that

\begin{equation}
\omega^x(G^c) \gtrsim 1 \quad \text{for all } x \in \partial \hat{\Omega} \setminus G.
\end{equation}

Note that if $x \in \partial \hat{\Omega} \setminus G$, then $x \in \partial \eta I$ for some square $I \in \text{Stop}(n)$ for some $n$, and so $I \subseteq G^c$. By Lemma 2.10, $\omega^x(I_G) \geq \omega^x(I) \gtrsim 1$. This proves (7.2).

Next, suppose that $\omega^x(G) = 0$. Let $\phi$ be a lower function for $I_G$ with respect to $\Omega$. Then by definition of harmonic measure, $\phi(x) \leq \omega^x(I_G)$ for all $x \in \Omega$, and so by (7.2), for $x \in \partial \hat{\Omega} \setminus G$, there is $c \in (0, 1)$ so that

$$\phi(x) \leq \omega^x(I_G) < c < 1.$$
Thus, \( \phi(x) - c \) is a lower function for \( 1_G \) with respect to \( \hat{\Omega} \), since for all \( \xi \in G \)
\[
\limsup_{x \to \xi} (\phi(x) - c) \leq 1 - c < 1
\]
and for \( \xi \in \partial \hat{\Omega} \setminus G \),
\[
\limsup_{x \to \xi} \phi(x) \leq \omega^\xi_{\hat{\Omega}}(G) - c < c - c = 0.
\]

Hence, \( \phi(x) - c \leq \hat{\omega}^x(G) = 0 \) and so \( \phi(x) \leq c < 1 \) for all \( x \in \hat{\Omega} \). Thus, taking the supremum over all such lower functions, we get
\[
\omega^x_{\hat{\Omega}}(E) = 0
\]
for all \( x \in \Omega \) if and only if
\[
\sup_{x \in \Omega} \omega^x_{\hat{\Omega}}(E) < 1.
\]

Here, \( \partial \infty \Omega = \partial \Omega \cup \{\infty\} \) if \( \Omega \) is unbounded and equals \( \partial \Omega \) otherwise. By [AG01, Example 6.5.6], \( \omega^x_{\Omega}(\{\infty\}) = 0 \). Thus the previous lemma implies \( \omega^x_{\hat{\Omega}}(G) = 0 \), which is a contradiction. This proves the claim.

Finally, we need to show that, if \( \mathcal{D} \) are the surface cubes for \( \partial \hat{\Omega} \), that
\[
\sum_{Q \in \mathcal{D}} \beta_{\partial \Omega}(3B_Q)^2 \ell(Q) < \infty.
\]

Since \( \partial \hat{\Omega} \) is 1-Ahlfors regular, and using the fact that \( \beta_{\partial \Omega}(B') \lesssim \beta_{\partial \Omega}(B) \) whenever \( B' \subseteq B \) and \( r_B \lesssim r_{B'} \), it suffices to show that
\[
\int_{\partial \Omega} \int_0^1 \beta_{\partial \Omega}(B(x, r)) \frac{dr}{r} d\mathcal{H}^1(x) < \infty.
\]

Let \( I \in \text{Top}(n-1) \) and \( J \in \text{Stop}(n, I) \). Note that since \( \partial \lambda J \) is just a square, we know that
\[
\int_{\partial \lambda I} \int_0^{\ell(J)} \beta_{\partial \Omega}(B(x, r)) \frac{dr}{r} d\mathcal{H}^1(x) \lesssim \ell(I).
\]

Then we trivially bound (using that \( J \in \text{Child}_n(I) \) and \( \beta_{\partial \Omega} \leq 1 \))
\[
\int_{\partial \lambda I} \int_{\ell(J)}^{\ell(I)} \beta_{\partial \Omega}(B(x, r)) \frac{dr}{r} d\mathcal{H}^1(x) \lesssim \ell(J) \log \frac{\ell(I)}{\ell(J)} \lesssim \ell(J) n \sim |J| n.
\]
Since $|G| = 0$, 
\[
\int_{\partial\Omega} \int_0^1 \beta_{\partial\Omega}(B(x, r))^2 \frac{dr}{r} d\mathcal{H}^1(x) = \sum_{n > 0} \sum_{I \in \text{Top}(n-1)} \sum_{J \in \text{Stop}(n,J)} \int_{\partial\lambda J} \int_{\ell(I)} \beta_{\partial\Omega}(B(x, r))^2 \frac{dr}{r} d\mathcal{H}^1(x) \leq \sum_{n > 0} \sum_{I \in \text{Top}(n-1)} n|J| \\
\leq 1 + \sum_{n > 0} \sum_{J \in \text{Top}(n-2)} n|J| \leq 1 + \sum_{n > 0} n4^{-nN_\tau} < \infty.
\]

8. Proof of Theorem VI

8.1. Part I: Bounding the oscillation of Green’s function using CDHM. In this section, we prove the following:

Lemma 8.1. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a bounded uniform domain with lower $d$-content regular boundary. Let $Q_0 \in \mathcal{D}$ and $x_0$ be a c-corkscrew point in $B_{Q_0} \cap \Omega$ so that if $B_0 = B(x_0, c\ell(Q_0))$, then $2B_0 \subseteq \Omega \cap B_{Q_0}$. Let $g = G_{\Omega}(x_0, \cdot)$ and
\[
\Omega_{Q_0} = \bigcup_{Q \subseteq Q_0} U^K_Q
\]
Then for $K$ large enough (chosen as in the proof of 4.6)
\[
(8.1) \quad \ell(Q_0)^d + \int_{\Omega_{Q_0} \setminus B_0} \left| \frac{\nabla^2 g}{g} \right|^2 \delta_{\Omega}(x)^3 dx \lesssim \beta_{\partial\Omega}(Q_0).
\]

We assume the same set up as in the proof of Lemma 4.6, but now assuming $\Omega$ is also uniform. Let $\omega = \omega^0_{Q_0}$. Then for $R \in \text{Top}(k_0)$ and $Q \in \text{Tree}(R)$, by the uniform case of Lemma 2.15,
\[
(8.2) \quad 1 \sim_{A, \tau} \frac{\Theta^d_{\omega^0_{Q_0} R}(Q)}{\Theta^d_{\omega^0_{Q_0} R}(R)} = \frac{\omega^\ell_{\Omega}(Q) \ell(R)^d}{\omega^\ell_{\Omega}(R) \ell(Q)^d} \sim \frac{\omega^\ell_{\Omega}(Q) \ell(R)^d}{\omega^\ell_{\Omega}(R) \ell(Q)^d} \sim \Theta^d_{\omega}(Q) \sim \Theta^d_{\omega}(R)
\]
In particular, if we set $g = G_{\Omega}(x_{Q_0}, \cdot)$, then for $x \in U^K_Q$ and $Q \in \text{Tree}(R)$, since $\Omega$ is uniform,
\[
(8.3) \quad \frac{g(x)}{\delta_{\Omega}(x)} \sim_{K} \frac{g(z_Q)}{\delta_{\Omega}(z_Q)} \sim \Theta^d_{\omega}(Q) \sim \Theta^d_{\omega}(R).
\]
Thus, with the same arguments as before, if we set

$$\Omega_{k_0} = \bigcup_{Q \in \mathcal{Q}(k_0)} U^K_Q,$$

then

$$\int_{\Omega_{k_0} \setminus B_0} \left| \frac{\nabla^2 g(x)}{g(x)} \right|^2 \delta_{\Omega}(x)^3 \, dx \leq \sum_{R \in \text{Top}(k_0)} \sum_{Q \in \text{Tree}(R) \cap U^K_Q \setminus B_0} \int_{U^K_Q \setminus B_0} \left| \frac{\nabla^2 g}{g} \right|^2 \delta_{\Omega}(x)^3 \, dx \tag{8.3}$$

$$\cong \sum_{R \in \text{Top}(k_0)} \Theta_{\omega_Q^x} (R)^{-3} \sum_{Q \in \text{Tree}(R) \cap U^K_Q \setminus B_0} \int_{U^K_Q \setminus B_0} \left| \nabla^2 g \right|^2 \, dx$$

$$\sim \sum_{R \in \text{Top}(k_0)} \Theta_{\omega_Q^x} (R)^{-3} \int_{\Omega_R \setminus B_0} \left| \nabla^2 g \right|^2 \, dx$$

$$\leq \sum_{R \in \text{Top}(k_0)} \Theta_{\omega_Q^x} (R)^{-3} \int_{\Omega_R \setminus B_0} \left| \nabla g \right|^2 \left( \frac{g}{d} \frac{d|\nabla g|^2}{dv} - \left| \nabla g \right|^2 \frac{dg}{dv} \right) \, d\mathcal{H}^d \tag{4.11}$$

$$\leq \sum_{R \in \text{Top}(k_0)} \Theta_{\omega_Q^x} (R)^{-3} \int_{\Omega_R \setminus B_0} \delta_{\Omega}(x)^3 \, d\mathcal{H}^d(x) \tag{4.5}$$

$$\lesssim CDHM(Q_0, 2, A, \tau) \lesssim \beta_{\partial \Omega}(Q_0).$$

8.2. Part II: A Theorem on affine deviation of functions in uniform domains.

In this section we prove one half of Theorem VI. In fact, the result holds for a more general class of functions than Green’s function.

**Theorem 8.2.** Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a uniform domain with lower content regular boundary, and $f \in W^{2,2}_{\text{loc}}(\Omega)$ such that $f > 0$ on $\Omega$,

$$f(x) \sim f(y) \text{ whenever } \frac{|x - y|}{\min\{\delta_{\Omega}(x), \delta_{\Omega}(y)\}} \leq \Lambda. \tag{8.4}$$

and

$$|f(x)|1_{B \cap \Omega}(x) \lesssim \|f\|_{L^\infty(B \cap \Omega)} \left| \frac{x - x_B}{r_B} \right|^{\alpha}. \tag{8.5}$$

For $Q_0 \in \mathcal{Q}$, let

$$\Omega(Q_0) = \bigcup_{Q \subseteq Q_0} U^{2\rho^{-1}K}_Q.$$
Then
\[ \int_{\Omega(Q_0)} \left| \frac{\nabla^2 f(x)}{f(x)} \right|^2 \delta_\Omega(x)^3 dx \gtrsim \beta_{\partial\Omega}(Q_0). \]

The rest of this section is dedicated to proving the above theorem.

**Lemma 8.3.** Let \( \Omega \subseteq \mathbb{R}^{d+1} \) be a domain, \( B \) a ball centered on \( \partial\Omega \), and \( f \) a positive function on \( \Omega \) vanishing continuously on \( \partial\Omega \cap 2B \) satisfying (8.4) and (8.5). Then for any corkscrew point \( y \in B \cap \Omega \),
\[
\sup_B f \lesssim f(y).
\]

**Proof.** The proof of this is exactly as in [JK82]. \( \square \)

For a ball \( B \subseteq \Omega \), we let \( f_B = f_B f \). Let
\[ A_B = (\nabla f)_B \cdot (x_B - x) + f_B. \]

Observe that by Poincaré’s inequality,
\[
(8.6) \quad \int_B \left| \frac{f - A_B}{r_B} \right|^2 \lesssim \int_B |\nabla f - (\nabla f)_B| \leq \int_B |\nabla^2 \mathbb{P} f_B|^2
\]

We also let
\[ \gamma(B) = \left( \int_B \left| \frac{\nabla^2 f}{f} \right|^2 r_B^4 \right)^{\frac{1}{2}}. \]

For a ball \( B \), we will write
\[ ||g||_B = \left( \int_B g^2 \right)^{\frac{1}{2}}. \]

Notice that by (8.4) and (8.6), if \( \text{dist}(B, \partial\Omega) \geq cr_B \), then
\[
(8.7) \quad ||f - A_B||_B \lesssim_c \gamma(B) f_B
\]

**Lemma 8.4.** Let \( B_x = B(x, (1 - \varrho)\delta_\Omega(x)) \). Then for \( \varrho > 0 \) small enough and \( \varepsilon \in (0, 1) \) small enough depending on \( \varrho \), \( \gamma(B_x) < \varepsilon \) implies
\[
|\nabla A_{B_x}| = |(\nabla f)_{B_x}| \sim \frac{f(x)}{r_{B_x}} \sim \frac{f(x)}{\delta_\Omega(x)} \sim A_{B_x}(x).
\]

**Proof.** Let \( B = B_x \). Since \( (A_B)_B = f_B \), we see that
\[
|\nabla A_B| \sim \left| \frac{A_B - (A_B)_B}{r_B} \right|_B = \left| \frac{A_B - f_B}{r_B} \right|_B \leq \left| \frac{A_B - f}{r_B} \right|_B + \left| \frac{f - f_B}{r_B} \right|_B \leq \left| \nabla^2 \mathbb{P} f_B \right| + \left| \frac{f}{r_B} \right|_B + \left| \frac{f}{r_B} \right|_B \lesssim \gamma(B) \frac{f(x)}{\delta_\Omega(x)} + \frac{f(x)}{r_B} \lesssim \frac{f(x)}{r_B}.
\]
For the opposite inequality, let $\xi \in \partial \Omega$ be so that $|x - \xi| = \delta_{\Omega}(x) = r_B/(1 - \varrho)$. Then by Lemma 8.3, $f \lesssim \varrho \cdot f(x)$ on $B(\xi, 2\varrho r_B) \cap B_x$. Thus, for $\varrho$ small enough,

$$\frac{f(x)}{r_B} \lesssim \|f - f_B\|_B \lesssim \|\nabla f\|_B \leq \|\nabla f - (\nabla f)_B\|_B + (\nabla f)_B$$

$$\lesssim \|\nabla^2 f B\|_B + \|\nabla A_B\| \lesssim \gamma(B) \frac{f(x_B)}{r_B} + |\nabla A_B| \ll \epsilon \frac{f(x_B)}{r_B} + |\nabla A_B|$$

and so for $\varepsilon$ small enough depending on $\varrho$, this implies

$$\frac{f(x)}{r_B} \lesssim |\nabla A_B|.$$

The last comparison in (8.8) follows from the definition of $A_B$ and Harnack’s inequality.

\[\square\]

Let $\mathcal{W}$ be the set of maximal dyadic cubes $I \subseteq \Omega$ so that $NI \subseteq \Omega$. We will call these the Whitney cubes for $\Omega$. For a Whitney cube $I \in \mathcal{W}$, let $B_I$ be the smallest ball containing $I$ and set $A_I = A_{B_I}$. For $N$ chosen sufficiently large depending on $d, 2B_I \subseteq \Omega$.

For a ball $B \subseteq \Omega$, let $B^* = B_{x_B}$.

**Lemma 8.5.** If $I$ is a Whitney cube and $x_I$ is its center, and $\gamma(B^*_I)$ is small enough, then

$$|\nabla A_I| \sim \frac{f(x_I)}{\ell(I)} \sim A_I(x_I)$$

**Proof.** By the previous lemma,

$$|\nabla A_{B^*_I} - \nabla A_I| = \left| \int_{B_I} (\nabla A_{B^*_I} - \nabla f) \right| \leq \|\nabla A_{B^*_I} - \nabla f\|_{B_I}$$

$$\lesssim \|\nabla^2 f\|_{B_I} \ell(I) \sim \gamma(B^*_I) \frac{f(x_I)}{r_{B^*_I}} \lesssim \gamma(B^*_I) |\nabla A_{B^*_I}|.$$

By picking $\gamma(B^*_I)$ small enough, this implies

$$|\nabla A_I| \sim |\nabla A_{B^*_I}| \lesssim \frac{f(x_I)}{\delta_{\Omega}(x_I)} \sim \frac{f(x_I)}{\ell(I)}.$$

For the last inequality, we just observe that

$$A_I(x_I) = \int_{B_I} f \overset{(8.4)}{=} f(x_I).$$

\[\square\]

**Lemma 8.6.** If $I$ and $J$ are adjacent Whitney cubes and $B_{I,J} = B^*_I \cup B^*_J$. Then for $M \geq 1$,

$$\|A_I - A_J\|_{L^\infty(MB_I)} \lesssim \gamma(B_{I,J}) |\nabla A_I| M \ell(I)$$

**Proof.**
and
\begin{equation}
\frac{|\nabla A_I - \nabla A_J|}{|\nabla A_I|} \lesssim \gamma(B_{I,J})
\end{equation}

**Proof.** Let \( B = B_{I,J} \). We estimate
\[
|\nabla A_I - \nabla A_J| \leq |\nabla A_I - (\nabla f)_B| + |(\nabla f)_B - \nabla A_J| \\
\lesssim \int_B |\nabla f - (\nabla f)_B| \leq ||\nabla f - (\nabla f)_B||_B \\
\lesssim (\text{diam } B)||\nabla^2 f||_B \lesssim \gamma(B) \frac{f_B}{\text{diam } B} \overset{(8.4)}{=} \gamma(B) \frac{f(x_I)}{\ell(I)} \overset{(8.9)}{=} \gamma(B)|\nabla A_I|.
\]

Since \( I \) and \( J \) are adjacent, \( |B_I \cap B_J| \sim |I| \). Thus,
\[
\int_{B_I \cap B_J} |A_I - A_J| \leq \int_{B_I} |A_I - f| + \int_{B_J} |f - A_J| \leq ||A_I - f||_{B_I} + ||A_J - f||_{B_J} \\
\overset{(8.7)}{\lesssim} \gamma(B_I)f_{B_I} \overset{(8.9)}{=} \gamma(B_I)|\nabla A_I|\ell(I).
\]

Thus, there is \( x_0 \in B_I \cap B_J \) so that \( |A_I(x_0) - A_J(x_0)| \lesssim |\nabla A_I|\ell(I) \). Hence, for \( x \in MB_I \),
\[
|A_I(x) - A_J(x)| = |\nabla A_I(x - x_0) + A_I(x_0) - (\nabla A_J(x - x_0) + A_J(x_0))| \\
\lesssim |(\nabla A_I - \nabla A_J)(x - x_0)| + |A_I(x_0) - A_J(x_0)| \\
\overset{(8.12)}{\lesssim} \gamma(B)|\nabla A_I| \cdot |x - x_0| + \gamma(B)|\nabla A_I|\ell(I) \lesssim \gamma(B)|\nabla A_I|M\ell(I).
\]

**Lemma 8.7.** For \( I \in \mathcal{W} \), let \( P_I = \{x : A_I(x) = 0\} \). If \( I, J \in \mathcal{W} \) are adjacent, then for \( M \gg N \) and \( \gamma(B_{I,J}) \) small enough
\[
d_{MB_I}(P_I, P_J) \lesssim \gamma(B_{I,J}).
\]

**Proof.** First we claim that for \( M' \gg N \),
\[
P_I \cap M'B_I \neq \emptyset \text{ and } P_J \cap M'B_I \neq \emptyset.
\]

To see this, recall from the proof of Lemma 8.4 that \( f \lesssim \varrho^\alpha f(x_I) \) on \( B_I^* \cap B(\xi, 2\varrho r_{B_I}) \) and the measure of this set is comparable to \( \ell(I)^d \) (depending on \( \varrho \)), thus there is \( y \in B_I^* \cap B(\xi, 2\varrho r_{B_I}) \) so that
\[
A_I(y) \leq \int_{B_I^* \cap B(\xi, 2\varrho r_{B_I})} |A_I| \lesssim \int_{B_I^* \cap B(\xi, 2\varrho r_{B_I})} |A_I - f| + \varrho^\alpha f(x_I) \\
\overset{(8.7)}{\leq} C_\varrho \gamma(B_I^*)f_{B_I} + \varrho^\alpha f(x_I) \overset{(8.9)}{=} (C_\varrho \gamma(\varrho B_I^*) + \varrho^\alpha) A_I(x_I).
\]

So for \( \varrho \) small enough and \( \gamma(B_I^*) \) small enough depending on \( C_\varrho \) (and hence on \( \varrho \)), \( A_I(y) < \frac{1}{2} A_I(x_I) \). Also, as \( y \in B_I^* \), \( |y - x_I| \lesssim N\ell(I) \). This implies that there is
$z \in \{ A_I = 0 \}$ so that $|x_I - z| \lesssim N \ell(I)$ as well, and this implies the claim for $M'$ large enough.

Now let $M \geq M'$. Let $x \in P_I \cap MB_Q$ and $x' = \pi_{P_I}(x)$. Then
\[
|\nabla A_J| \cdot |x - x'| = |\nabla A_J \cdot (x - x')| = |A_J(x) - A_J(x')| = |A_J(x)|
\]
\[
= |A_J(x) - A_J(x)| \lesssim (B_{I,J}) |\nabla A_I| |\nabla A_J| |M\ell(I)| \lesssim (B_{I,J}) |\nabla A_J| |M\ell(I)|
\]
Dividing both sides by $|\nabla A_J|$ gives
\[
\sup_{x \in P_I \cap MB_I} \text{dist}(x, P_I) \lesssim (B_{I,J}) |\nabla A_I| |M\ell(I)|.
\]
A similar argument with $I$ and $J$ switched finishes the proof.

Now we switch to the dyadic setting by assigning to each surface cube a plane. Let
\[
\gamma(Q) = \left( \int_{U^\kappa_Q} \left| \nabla^2 f \right|^2 \ell(Q)^4 \right)^{\frac{1}{2}}, \quad \gamma'(Q) = \left( \int_{U^{2\kappa^2-\kappa}_Q} \left| \nabla^2 f \right|^2 \ell(Q)^4 \right)^{\frac{1}{2}}.
\]
Let $\delta > 0$. To each cube $Q \in D$, let $I_Q$ be a Whitney cube of maximal size so that (assuming $\delta \ll c_0$)
\[
(8.13) \quad MB_{I_Q} \subseteq \frac{\delta}{2} B_Q \subseteq \frac{c_0}{4} B_Q.
\]
In particular, $\ell(I_Q) \sim_{M,\delta} \ell(Q)$.

Let $C_2 > 1$, and suppose $Q, R \in D$ are such that
\[
C_2 B_Q \cap C_2 B_R \neq \emptyset \quad \text{and} \quad \text{dist}(Q, R) \leq C_2 \min\{\ell(Q), \ell(R)\}.
\]
Since $\Omega$ is uniform, there is a Harnack chain of adjacent Whitney cubes $I_Q = I_1, \ldots, I_n = I_R$ so that $n \lesssim_{C_2} 1$. If we choose $K$ large enough, then
\[
\bigcup_{i=1}^{n-1} B_{I_i, I_{i+1}} \subseteq U_Q^{K/2} \cap U_R^{K/2}.
\]
In particular, applying Lemma 8.7 to each pair $(I_j, I_{j+1})$ gives that, for $C_1$ large enough
\[
(8.14) \quad \ell(Q)^{-1} \left( \sup_{x \in P_Q \cap C_1 B_Q} \text{dist}(x, P_R) + \sup_{x \in P_R \cap C_1 B_Q} \text{dist}(x, P_Q) \right) \lesssim_{C_1, C_2, K} \gamma(Q).
\]

Lemma 8.8. Let $C_1 > 1$, $\delta > 0$. For $K, \epsilon^{-1}$ large enough depending on $\delta$ if $\gamma(Q) < \epsilon$, then (2.12) holds with $E = \partial \Omega$ and $\delta$ in place of $\epsilon$. 
Proof. Let \( x \in 2C_1 B_Q \cap \partial \Omega \). Then there is \( y \in \partial U_Q^K \cap 3C_1 B_Q \) so that \( |x - y| < \frac{1}{K} \ell(Q) \). Let \( I \subseteq U_Q^{K/2} \) be the Whitney cube containing \( y \). Then for \( K \) large, \( B_I^* \subseteq U_Q^K \), and so

\[
|A_Q(y)| = \left| \int_{B_I} A_Q \right| \leq \int_{B_I} (|A_Q - A_{B_I}| + |A_{B_I} - f|) + f
\]

\[
\overset{(8.7)}{\overset{(8.5)}{\leq}} \gamma(U_Q^K) K |\nabla A_Q| \ell(Q) + \gamma(B_I) f_{B_I} + f(x_I) K^{-\alpha}
\]

\[
\overset{\leq}{\leq} (K \varepsilon + K^{-\alpha}) |\nabla A_Q| \ell(Q).
\]

Thus,

\[
\text{dist}(x, P_Q) \leq |x - y| + \text{dist}(y, P_Q) < \frac{1}{K} \ell(Q) + \frac{|\nabla A_Q(y - \pi_{P_Q}(y))|}{|\nabla A_Q|}
\]

\[
= \frac{1}{K} \ell(Q) + \frac{|A_Q(y) - A_Q(\pi_{P_Q}(y))|}{|\nabla A_Q|}
\]

\[
\overset{\leq}{\leq} \frac{1}{K} \ell(Q) + \frac{(K \varepsilon + K^{-\alpha}) |\nabla A_Q| \ell(Q) + 0}{|\nabla A_Q|}
\]

\[
= \frac{1}{K} \ell(Q) + (K \varepsilon + K^{-\alpha}) \ell(Q).
\]

For \( \varepsilon, K^{-1} \) small enough depending on \( \delta \) (and \( \varepsilon \) depending on \( K \)), this proves the lemma. \( \square \)

Let \( k_0 \in \mathbb{N} \). Let \( B_\gamma \) ("big \( \gamma \") denote those cubes \( Q \in \mathcal{D}(k_0) \) for which \( \gamma(Q) \geq \varepsilon \). For \( R \in B_\gamma \), let \( \text{Stop}(R) = \{ R \} \) and \( \text{Next}(R) \) be the children of \( R \). For \( R \in \mathcal{D}(k_0) \) \( \setminus B_\gamma \), let \( \text{Stop}(R) \) be the maximal cubes \( Q \subseteq R \) which have a child \( Q' \) such that

\[
\sum_{Q' \subseteq T \subseteq R} \gamma(T)^2 \geq \varepsilon^2.
\]

Since \( U_Q^K \subseteq U_Q^{2\rho^{-1}K} \), we have that

\[
\gamma(Q') \lesssim \gamma'(Q).
\]

We claim that this implies

\[
\sum_{Q \subseteq T \subseteq R} \gamma'(T)^2 \gtrsim \varepsilon^2.
\]

Indeed, if \( \gamma'(Q) \geq \varepsilon/C \), then this follows since \( Q \in \text{Tree}(R) \) implies

\[
\varepsilon^2/C^2 \leq \gamma'(Q)^2 \leq \sum_{Q \subseteq T \subseteq R} \gamma'(T)^2
\]
Otherwise, if \( \gamma'(Q) < \varepsilon/C \), then by (8.15), \( \gamma(Q') \lesssim \varepsilon/C \), and so

\[
\varepsilon^2 \leq \sum_{Q' \subseteq T \subseteq R} \gamma(T)^2 \lesssim \gamma(Q') + \sum_{Q' \subseteq T \subseteq R} \gamma(T)^2 \lesssim \varepsilon^2/C^2 + \sum_{Q' \subseteq T \subseteq R} \gamma'(T)^2
\]

and this implies (8.16) for \( C > \) large enough.

Let \( \text{Tree}(R) \) denote those cubes in \( \mathcal{D}(k_0) \) be those cubes in \( R \) not properly contained in a cube from \( \text{Stop}(R) \). Let \( \text{Top}_0 = \{Q_0\} \), \( \text{Top}_{k+1} = \bigcup_{R \in \text{Top}_k} \text{Next}(R) \), and \( \text{Top}(k_0) = \bigcup_{k \geq 0} \text{Top}_k \). Observe that if we define \( \varepsilon(Q) \) as in Lemma 2.8, then for \( R \in \text{Top} \) and \( Q \in \text{Tree}(R) \),

\[
(8.14) \quad (8.15) \quad \varepsilon(Q) \lesssim \gamma'(Q).
\]

In particular, these observations and (8.13) imply we can apply Lemma 2.8 (with \( \delta \) in place of \( \varepsilon \) if we pick \( \varepsilon \ll \delta \)) to each stopping-time region \( \text{Tree}(R) \) to obtain a bi-Lipschitz surface \( \Sigma_R \) which, by Lemma 8.7 and (8.13) satisfies (2.11). In particular, by (2.13), for \( \delta > 0 \) small,

\[
\frac{c_0}{2} B_Q \cap \Sigma_R \neq \emptyset.
\]

Since \( \Sigma_R \) is Ahlfors regular, and because the balls \( \{c_0 B_Q : Q \in \text{Stop}(R)\} \) are disjoint by Theorem 2.1, we have that for all \( R \in \text{Top}(k_0) \),

\[
\sum_{Q \in \text{Stop}(R)} \ell(Q)^d \sim \sum_{Q \in \text{Stop}(R)} \mathcal{H}^d(\Sigma_R \cap c_0 B_Q) \\
\sim \varepsilon^{-2} \sum_{Q \in \text{Stop}(R)} \sum_{T \subseteq Q \subseteq R} \gamma'(T)^2 \mathcal{H}^d(\Sigma_R \cap c_0 B_Q) \\
= \varepsilon^{-2} \sum_{T \in \text{Tree}(R)} \gamma'(T)^2 \sum_{Q \in \text{Stop}(R)} \mathcal{H}^d(\Sigma_R \cap c_0 B_Q) \\
\leq \varepsilon^{-2} \sum_{T \in \text{Tree}(R)} \gamma'(T)^2 \mathcal{H}^d(\Sigma_R \cap B_T) \\
\lesssim \varepsilon^{-2} \sum_{T \in \text{Tree}(R)} \gamma'(T)^2 \ell(T)^d.
\]

Thus, for \( k > 0 \),

\[
\sum_{R \in \text{Top}_k \setminus B_T} \ell(R)^d = \sum_{R \in \text{Top}_{k-1}} \sum_{Q \in \text{Next}(R)} \ell(Q)^d \lesssim \sum_{R \in \text{Top}_{k-1}} \sum_{Q \in \text{Stop}(R)} \ell(Q)^d \\
= \sum_{R \in \text{Top}_{k-1}} \sum_{T \in \text{Tree}(R)} \gamma'(T)^2 \ell(T)^d
\]
Therefore, since the $U_Q^{2p^{-1}K}$ have bounded overlap,

$$\sum_{R \in \text{Top}(k_0)} \ell(R)^d = \ell(Q_0)^d + \sum_{k=1}^{\infty} \sum_{R \in \text{Top}_k} \ell(R)^d$$

$$\lesssim \ell(Q_0)^d + \sum_{k=0}^{\infty} \sum_{R \in \text{Top}_{k-1}} \sum_{T \in \text{Tree}(R)} \gamma(T)^2 \ell(T)^d + \sum_{k=1}^{\infty} \sum_{R \in \text{Top}_k \cap B\gamma} \ell(R)^d$$

$$\lesssim \ell(Q_0)^d + \varepsilon^{-2} \sum_{T \subseteq Q_0} \gamma(T)^2 \ell(T)^d + \varepsilon^{-2} \sum_{R \in B\gamma} \gamma'(Q)^2 \ell(Q)^d$$

$$\lesssim \varepsilon^{-2} \sum_{T \subseteq Q_0} \gamma'(T)^2 \ell(T)^d \lesssim \varepsilon \int_{\Omega(Q_0)} \left| \nabla f(x) \right|^2 \delta \Omega(x)^3 dx.$$  

Theorem 8.2 now follows from the following lemma.

**Lemma 8.9.** Suppose that $E$ is a lower $d$-regular set, $\mathcal{D}$ the Christ-David cubes for $E$, and $Q_0 \in \mathcal{D}$, and for each $k_0 \in \mathbb{N}$, let $\text{Top}(k_0)$ be a collection of cubes such that, for each $R \in \text{Top}(k_0)$, there is a stopping-time region $\text{Tree}(R)$ whose top cube is $R$ and a uniformly rectifiable set $\Sigma_R$ so that for $x \in E \cap C_1 B_R$,

$$\text{dist}(x, \Sigma_R) \leq \delta d_R(x) := \delta \inf_{Q \in \text{Tree}(R)} (\ell(Q) + \text{dist}(x, Q)).$$

For $C_1$ large enough and $\delta$ small enough,

$$\ell(Q_0)^d + \beta_E(Q_0) \lesssim \limsup_{k_0 \to \infty} \sum_{R \in \text{Top}(k_0)} \ell(R)^d.$$  

The proof is exactly as that of [AV19, Lemma 4.6].

Thus, to finish the proof of Theorem 8.2, we just need to verify (8.17). Let $R \in \text{Top}(k_0)$ and $x \in C_1 B_R \cap \partial \Omega$. Let $Q \in \text{Tree}(R)$ be so that

$$d_R(x) = \ell(Q) + \text{dist}(x, Q).$$

Let $\hat{Q}$ be the largest ancestor of $x$ so that $\ell(\hat{Q}) < d_R(x)$. Then $\ell(\hat{Q}) \sim d_R(x) > \text{dist}(x, \hat{Q})$, and so for $C_1$ large enough, $x \in C_1 B_{\hat{Q}}$. By (2.13) (which holds by Lemma 2.8 and Lemma 8.8),

$$\text{dist}(x, \Sigma_R) \lesssim \delta \ell(\hat{Q}) \lesssim \varepsilon d_R(x).$$

This finishes the proof.

**APPENDIX A. DAVID-REIFENBERG-TORO PARAMETRIZATIONS**

The rest of this section is dedicated to the proof of Theorem 2.8. We will frequently cite equations and terminology from [DT12] rather than stating their result here, since it is quite long to state.
Without loss of generality, we can assume \( Q(S) \in \mathcal{D}_0, \ell(Q(S)) = 1 \), and \( P_{Q(S)} = \mathbb{R}^d \).

Let \( r_k = 10^{-k} \), and let \( X_k \) be a maximal \( \frac{3}{2} r_k \)-separated set in \( \Xi_k = \{ \zeta_S : Q \in S_k := \mathcal{D}_{s(k)} \cap S \} \)

where \( s(k) \) is such that
\[
(A.1) \quad 5 \rho^{s(k)} \leq r_k/4 < 5 \rho^{s(k) - 1}.
\]

Note that this integer is unique and \( s(0) = 0 \). Recall that \( \ell(Q) = 5 \rho^k \) if \( Q \in \mathcal{D}_k \).

For \( x \in X_k \), let \( Q_k(x) \in S_k \) be so that \( \zeta_{Q_k(x)} = x \). Also let \( x' = \pi_{Q_k(x)}(x) \), \( P_k(x') := P_{Q_k(x)} \), and
\[
X'_k = \{ x' : x \in X_k \}.
\]

Note that \( X_0 \) is a singleton (just the center of \( Q(S) \)), and thus so is \( X'_0 \). Without loss of generality, we can assume \( X'_0 = \{ 0 \} \).

Enumerate \( X'_k = \{ x_{j,k} \}_{j \in J_k} \) and if \( x_{j,k} = x' \), set \( P_{j,k} = P_k(x') \) and \( B_{j,k} = B(x_{j,k}, r_k) \). Since \( X'_0 \) is a singleton, \( X'_0 = \{ x_{0,0} \} \) and \( P_{0,0} = P_{Q(S)} \).

We will show that this collection of points \( X'_k \) and planes \( P_{j,k} \) satisfy the conditions for David and Toro’s theorem [DT12, Theorem 2.5]. First, we show that they form a coherent collection of balls and planes, which requires verifying equations (2.1) and (2.3-2.10) in [DT12] (see [DT12, Definition 2.1]).

By assumption (2.8),
\[
|x - x'| < \varepsilon \ell(Q_k(x)) = 5\varepsilon \rho^{s(k)}.
\]

Then for all \( x', y' \in X'_k \) and \( \varepsilon \) small,
\[
(A.2) \quad |x' - y'| \geq |x - y| - 10\varepsilon \rho^{s(k)} > \frac{3}{2} r_k - \frac{\varepsilon r_k}{2} > r_k.
\]

Thus, \( X'_k \) is an \( r_k \)-separated set for each \( k \geq 0 \) (which is [DT12, Equation (2.1)]).

Next, we want to show [DT12, Equation (2.3)], i.e. that for all \( k > 0 \),
\[
(A.3) \quad X'_k \subseteq V_{k-1}^2 \text{ where } V_k^2 := \{ x : \text{dist}(x, X_k') \leq 2r_k \}.
\]

Let \( x' \in X'_k \), recall that there is a corresponding \( x \in X_k \) and \( |x - x'| \leq \varepsilon \ell(Q) < \rho r_k \) for \( \varepsilon \) small enough.

**Case 1:** If \( r_{k-1}/4 < 5\rho^{s(k) - 1} \), then \( s(k - 1) = s(k) \), so \( \Xi_{k-1} = \Xi_k \). Since \( X_{k-1} \) is a maximally \( \frac{3}{2} r_{k-1} \)-separated set in \( \Xi_{k-1} \) (and hence in \( \Xi_k \)), there is \( y \in X_{k-1} \) so that \( |x - y| < \frac{3}{2} r_{k-1} \). Thus,
\[
|x' - y'| \leq |x - y| + |x' - x| + |y' - y| \leq \frac{3}{2} r_{k-1} + 2\rho r_{k-1} < 2r_{k-1}
\]

and so \( x' \in V_k^2 \).
Case 2: If \( r_{k-1}/2 \geq 5\rho^{s(k)-1} \), then \( s(k-1) = s(k) - 1 \). Let \( R \in S_{k-1} \) be the parent of \( Q_k(x) \in S_k \) (so \( \ell(R) = 5\rho^{s(k)-1} \leq r_{k-1}/4 \)). Then there is \( y \in X_{k-1} \) so that \( |\zeta - y| < \frac{3}{2}r_{k-1} \). Thus, for \( \varepsilon \) small enough,

\[
|x' - y'| \leq |x' - x| + |x - \zeta_R| + |\zeta_R - y| + |y - y'|
\leq \varepsilon \ell(Q) + \ell(R) + \frac{3}{2}r_{k-1} + \varepsilon \ell(R)
\leq (2\varepsilon \rho + 1)\ell(R) + \frac{3}{2}r_{k-1}
\leq (2\varepsilon \rho + 1) r_{k-1}/4 + \frac{3}{2}r_{k-1} < 2r_{k-1}.
\]

These two cases prove (A.3).

Set \( \Sigma_0 = P_{Q(S)} \). Then equations (2.4-2.7) in [DT12] are trivially satisfied (note in particular that as \( s(0) = 0 \), \( X_0 = \{ \zeta_{Q(S)} \} \), and so \( X_0' = \{ \zeta_{Q(S)}' \} \subseteq P_{Q(S)} = \Sigma_0 \).

For \( y \in V_{k}^{11} \), let

\[
\varepsilon_k''(y) = \sup(d_{x_i,100\rho}(P_{j,k},P_{i,\ell}) : j \in J_k, \ell \in \{k-1,k\}, i \in J_\ell, \text{ and } y \in 11B_{j,k}12B_{i,\ell})
\]

and let \( \varepsilon_k''(y) = 0 \) otherwise.

Now \( \varepsilon_k''(x_{j,k}) \lesssim \varepsilon \) by (2.11) for \( C_1 \) and \( C_2 \) large enough, which implies [DT12, Equations (2.8-2.10)], and so \( (\Sigma_0, \Sigma_1^{\ell}, \{ B_{j,k} \}, \{ P_{j,k} \}) \) are a coherent collection of balls and planes. Hence, [DT12, Theorem 2.4] implies there is \( g_S : \mathbb{R}^n \to \mathbb{R}^n \) so that \( \Sigma_S = g_{S}(\mathbb{R}^d) \) is a \((1 + C\varepsilon)\)-Reifenberg flat surface and so that (2.10) holds (recall \( B_{Q(S)} = \mathbb{B} \)) by [DT12, Equation 2.13].

Now we wish to show [DT12, Equation (2.19)], that is,

(A.4) \[
\sum_{k \geq 0} \varepsilon_k(g_S(z))^2 \lesssim \varepsilon^2 \text{ for all } z \in \mathbb{R}^d
\]

since then by [DT12, Theorem 2.5], \( g_S \) is bi-Lipschitz and (after an examination of the details in [DT12, Chapter 8]), \( g_{R|\mathbb{R}^d} = (1 + C\varepsilon^2)\)-bi-Lipschitz.

Let \( y = g_S(z) \) where \( z \in \mathbb{R}^d \). Note that if \( y \in (V_0^{11})^c \), then there are only finitely many \( k \) for which \( \varepsilon_k''(y) \neq 0 \), so (A.4) holds trivially in this case since \( \varepsilon_k'' \lesssim C\varepsilon \) whenever it is nonzero. Hence, we can assume \( y \in V_0^{11} \) by (2.10).

Let \( k(y) \) be the largest integer for which \( y \in V_k^{11} \) (so \( k(y) \geq 0 \) by the previous paragraph), and if such an integer does not exist, let \( k(y) = \infty \). Then dist \((y, X_k') \leq 11r_k \) for all \( 0 \leq k \leq k(y) \).

Let \( x' \in X_k' \) be closest to \( y, Q = Q_k(x') \). Then it is not hard to show

\[
\varepsilon_j(y) \lesssim \varepsilon(R) \text{ for all } j \leq k \text{ and } Q \subseteq R \in S_j.
\]

Thus,

\[
\sum_{j=0}^{\infty} \varepsilon_j(y)^2 = \sum_{j=0}^{k(y)} \varepsilon_j(z) \lesssim \sum_{Q \subseteq R \subseteq Q(S)} \varepsilon(R)^2 < \varepsilon^2
\]

and this proves (A.4).
Suppose (2.12) holds, we will show this implies (2.13). Let \( z \in E \cap C_1B_Q \) for some \( Q \in S \). Assume \( Q \in S_k \) with \( k \geq k_0 \) where \( k_0 \geq 0 \) we will decide shortly.

If \( \pi_Q \) is the projection into \( P_Q \) and \( z' = \pi_Q(z) \), then \( |z - z'| \leq \varepsilon C_1 \ell(Q) \) by (2.12), so \( z' \in 2C_1B_Q \).

Let \( k = k - k_0 \geq 0 \), \( Q' \in S_{k'} \) contain \( Q \), let \( x \in \Xi_{k'} \) be so that \( |x - \zeta_{Q'}| < \frac{3}{2} r_{k'} \), and let \( x_{j,k'} = x' \in X_{k'}' \). Then for \( C_2 \) large enough depending on \( k_0 \), \( Q_{k'}(x) \sim Q \) since \( \ell(Q) \sim k_0 \ell(Q(x)) \) and

\[
\text{dist}(Q, Q_{k'}(x)) \leq \ell(Q') + \varepsilon \ell(Q_{k'}(x)) \leq k_{k_0}/4 + \frac{3}{2} r_{k'} \lesssim r_{k_0} r_k \lesssim \ell(Q).
\]

Since \( z' \in 2C_1B_Q \cap P_Q \subseteq C_1B_{Q'} \) and \( \varepsilon(Q) < \varepsilon \), this means there is \( z'' \in P_{Q_{k'}(x)} = P_{j,k'} \) so that \( |z' - z''| < C_1 \ell(Q_{k'}(x)) \). Moreover,

\[
|z - z''| \leq |z - z'| + |z' - z''| \leq 2C_1 \ell(Q_{k'}(x)).
\]

Also note that since \( z \in C_1B_Q \subseteq \frac{4}{3} B_{Q_{k'}(x)} \) for \( k_0 \) large enough, and since \( Q_{k'}(x), Q' \in S_{k'} \),

\[
|z'' - x_{j,k'}| = |z'' - x'| \leq |z'' - z| + |z - \zeta_{Q'}| + |\zeta_{Q'} - x| + |x - x'|
\leq 2C_1 \ell(Q_{k'}(x)) + \frac{4}{3} \ell(Q') + \frac{3}{2} r_{k'} + \varepsilon \ell(Q')
\leq \left( 2C_1 + \frac{1}{3} + \frac{3}{2} + \frac{\varepsilon}{4} \right) r_{k'} \lesssim 2r_{k'}
\]

if \( \varepsilon \ll C_1^{-1} \). Thus, \( z'' \in 2B_{j,k'} \cap P_{j,k'} \).

By [DT12, Equation (5.3)], \( \text{dist}(z'', \Sigma_k) \lesssim \varepsilon r_{k'} \sim \varepsilon \ell(Q) \). Let \( w \in \Sigma_k \) be closest to \( z'' \). By [DT12, Equation (5.11)], for any \( \ell \geq 0 \),

\[
|\sigma_{\ell}(y) - y| \lesssim \varepsilon r_{\ell} \quad \text{for all } y \in \Sigma_{\ell}
\]

where \( \sigma_{\ell} \) is defined in [DT12, Equation (4.2)] and \( \Sigma_{\ell} \) in [DT12, Equation (5.1)]. Note that \( \Sigma_{\ell} \) is defined in [DT12, Equations (6.1-6.2)]. Iterating this from \( \ell = k' \) with \( y = w \) and taking the limit as \( \ell \to \infty \), we get that

\[
\text{dist}(w, \Sigma_{\ell}) \lesssim \varepsilon r_{k'} \sim k_0 \varepsilon \ell(Q).
\]

Thus,

\[
\text{dist}(z, \Sigma_{\ell}) \lesssim |z - z''| + |z'' - w| + \text{dist}(w, \Sigma_{\ell}) \lesssim (C_1 \varepsilon + \varepsilon) \ell(Q) + \varepsilon \ell(Q) + \varepsilon \ell(Q) \lesssim \varepsilon \ell(Q).
\]

This proves (2.13) when \( k \geq k_0 \). If \( k = k_0 \), then \( Q \sim Q(S) \), so if \( z \in E \cap C_1B_Q \), then \( z' \in 2C_1B_Q \cap P_Q \) just as before, and (2.12) implies \( |z - z'| < 2C_1 \varepsilon \ell(Q) \). Since \( \varepsilon(Q) < \varepsilon \), \( |z' - \pi_{P(Q)}(z')| \lesssim \varepsilon \ell(Q(S)) \sim \varepsilon \ell(Q) \), and (2.10) implies

\[
\text{dist}(\pi_{P(Q)}(z'), \Sigma_R) \leq |\pi_{P(Q)}(z') - g_S(\pi_{P(Q)}(z'))| \lesssim \varepsilon \sim \varepsilon \ell(Q).
\]

Thus,

\[
\text{dist}(z, \Sigma_{\ell}) \lesssim |z - z'| + |z' - \pi_{P(Q)}(z')| + \text{dist}(\pi_{P(Q)}(z'), \Sigma_R)
\lesssim \varepsilon \ell(Q).
\]
This proves (2.13) in every case.

Now suppose (2.14) holds, we will prove (2.15). Let \( z \in C_1 B_Q \cap \Sigma_R \) where \( Q \in S_k \). Then there is \( z_0 \in \mathbb{R}^d \) so that if \( z_k = \sigma_k(z_{k-1}) \), then \( z_k \in \Sigma_k \) and \( z_k \to z \).

Let \( k_0, k', Q', x \), and \( Q_{k'}(x) \in S_k' \) be chosen just as before (assume \( k' \geq k_0 \)), so again \( Q \sim Q_{k'}(x) \) for \( C_2 \) large enough. Again, \( z \in C_1 B_Q \subseteq \frac{4}{3} B_{Q_{k'}(x)} \) for \( k_0 \) large enough, so

\[
|z_{k'} - x'| \leq |z_{k'} - z| + |z - z_Q'\xi| + |z'Q' - x| + |x - x'|
\]

\[\varepsilon r_{k'} + \frac{4}{3}\ell(Q') + \frac{3}{2}r_{k'} + \varepsilon\ell(Q') < 2r_{k'}\]

Hence, \( z_{k'} \in B_{j,k'} \) where \( j \) is so that \( x_{j,k'} = x' \). By [DT12, Equation (5.6)],

\[
|z_{k'} - \pi_{j,k'}(z_{k'})| = \text{dist}(z_{k'}, P_{j,k'}) \lesssim \varepsilon r_{k'} \sim k_0 \varepsilon \ell(Q),
\]

and by [DT12, Equation (5.11)],

\[
|z_{k'} - z| \lesssim \varepsilon r_{k'} \sim k_0 \varepsilon \ell(Q).
\]

Since \( P_{j,k'} \) passes through the center of \( 2B_{j,k'} \) and \( z_{k'} \in B_{j,k'} \), we have

\[
\pi_{j,k'}(z_{k'}) \in 2B_{j,k'} = B(x', 2r_{k'}) \subseteq B(x, 2r_{k'} + \varepsilon \ell(Q_{k'}(x))) \quad \text{(A.1)}
\]

\[
\subseteq B(x, (8\rho + \varepsilon)\ell(Q_{k'}(x))) \subseteq C_1 B_{Q_{k'}(x)}
\]

for \( C_1 \) large enough. Thus, by (2.14),

\[
\text{dist}(\pi_{j,k'}(z_{k'}), E) \lesssim \varepsilon \ell(Q_{k'}(x)) \lesssim k_0 \varepsilon \ell(Q).
\]

Thus,

\[
\text{dist}(z, E) \leq |z - z_{k'}| + |z_{k'} - \pi_{j,k'}(z_{k'})| + \text{dist}(\pi_{j,k'}(z_{k'}), E) \lesssim k_0 \varepsilon \ell(Q).
\]

Now assume \( 0 \leq k \leq k_0 \). Note that by (2.10), if \( y = g_R^{-1}(z) \in P_{Q(S)} \), since \( z \in C_1 B_Q \subseteq C_1 B_{Q(S)} \), \( y \in 2C_1 B_{Q(S)} \) for \( \varepsilon \) small, so (2.14) implies \( \text{dist}(y, E) \lesssim \varepsilon \ell(Q(S)) \sim \varepsilon \ell(Q) \). Thus, by (2.10) again,

\[
\text{dist}(z, E) \leq |z - y| + \text{dist}(y, E) \lesssim \varepsilon + \varepsilon \ell(Q(S)) \sim \varepsilon \ell(Q).
\]

This finishes the proof of Lemma 2.8.

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Jonas Azzam, School of Mathematics, University of Edinburgh, JCMB, Kings Buildings, Mayfield Road, Edinburgh, EH9 3JZ, Scotland.

E-mail address: j.azzam "at" ed.ac.uk