The recently introduced concept of synthetic dimensions allows for the realization of higher-dimensional topological phenomena in lower-dimensional systems. In this paper, we propose a setup where synthetic dimensions arise in mesoscopic hybrid devices and discuss how they provide a natural route to topological states. We demonstrate this for the current induced into a closed one-dimensional Aharonov-Bohm ring by the interaction with a dynamic mesoscopic magnet. The quantization of the magnetic moment provides a synthetic dimension that complements the charge motion around the ring. We present a direct mapping that places the combined ring-magnet system into the class of quantum Hall models and demonstrate that topological features, combined with the magnet’s anisotropy, can lead to clear signatures in the persistent current of the single-particle ground state. Our synthetic-dimension model also extends to the many-electron case, where the collective electronic motion couples with the magnet.

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reinterpreting each spin state \( m \) as a different lattice site along a synthetic dimension, as sketched in Fig. 1(b). As we will show, this analogy goes far. The combined magnetoelectronic states form flattened-out bands with nonchiro numbers and chiral edge states according to the bulk-boundary correspondence. These chiral edge states have a simple interpretation in terms of a robust mesoscopic spin-orbit locking and leave direct signatures in the persistent current in the ground state of the combined system.

**Hamiltonian of the hybrid system.** Before illustrating the mapping to a synthetic dimension, we introduce the Hamiltonian of the hybrid ring-magnet system [Fig. 1(a)]. We shall assume that the magnetic moment of the nanomagnet is aligned with its spin so the flux induced into the ring is proportional to the spin operator \( S_z \), which at the same time is taken to align with one of the principal axes of the magnet’s matrix of inertia. The spin quantum numbers \( m \) introduced above then correspond to the eigenvalues of \( S_z \). In a symmetric gauge, the vector potential felt by a charged particle on the ring has \( A_\varphi \propto \rho S_z \), where \( \rho \) is the ring radius. In appropriate units, the Hamiltonian for a single electron then takes the simple form

\[
H = \frac{1}{2M\rho^2} (-i\nabla_\varphi - \gamma S_z)^2 + a S_x^2 + \beta S_y,
\]

where the kinetic-energy term includes the effect of the magnetic flux via minimal coupling with strength \( \gamma \). The remaining terms describe the dynamics of the magnet as a rigid body, where \( a S_x^2 \) determines its anisotropy while additional quadratic terms can be partially eliminated using the fact that \( S_x^2 + S_y^2 + S_z^2 \) is a constant (thus absorbing a constant into the energy). The term \( \beta S_y \) induces a preferred orientation of the nanomagnet and breaks the full integrability of the system [31].

**Mapping to a synthetic dimension.** We map the Hamiltonian Eq. (1) to a 2D quantum Hall system by re-interpreting the eigenstates \(| m \rangle \) of \( S_z \) as different lattice sites along a synthetic dimension [Fig. 1(b)]. In this language, the \( S_z \) and \( S_y \) operators lead to “hoppings” along the synthetic dimension, as they can be expressed by ladder operators as \( S_y = (S_+ - S_-)/2i \) and

\[
S_y = (S_+ - S_-)/2i \quad \text{with}
\]

\[
S_+ | m \rangle = \sqrt{s(s + 1) - m(m + 1)} | m + 1 \rangle \equiv 2m | m + 1 \rangle,
\]

\[
S_- | m \rangle = \sqrt{s(s + 1) - m(m - 1)} | m - 1 \rangle \equiv 2m | m - 1 \rangle.
\]

Using these relations, the Hamiltonian Eq. (1) can be cast into a second-quantized form,

\[
\hat{H} = \sum_m \int_0^{2\pi} d\varphi \left[ \frac{(i\nabla_\varphi - \gamma m)c_{\varphi,m}^\dagger( - i\nabla_\varphi - \gamma m)c_{\varphi,m}}{2M\rho^2}
+ \alpha \left( t_m + t_{m-1} \right) c_{\varphi,m}^\dagger c_{\varphi,m} + \alpha (t_{m+1} - t_m) c_{\varphi,m+1}^\dagger c_{\varphi,m} + \text{H.c.} \right]
+ \beta (- i t_{m+1} c_{\varphi,m}^\dagger c_{\varphi,m} + \text{H.c.})
\]

where, \( c_{\varphi,m}^\dagger (c_{\varphi,m}) \) creates (annihilates) an excitation in the state indexed by \( \varphi \) and \( m \). Note that this excitation describes a single effective “particle” that physically corresponds to the occupation of a composite state \(| \varphi, m \rangle \), with the electron at angular position \( \varphi \) on the ring and the nanomagnet in the spin state \( m \).

Reinterpreting both \( m \) and \( \varphi \) as spatial coordinates, the Hamiltonian Eq. (2) is analogous to that of a single particle on a 2D cylinder [Fig. 1(b)]. The dimension spanned by \( \varphi \) is continuous and periodic \((\varphi + 2\pi = \varphi)\), while that spanned by \( m \) is discrete with a finite number of lattice sites, \( N = 2\pi + 1 \). Under this mapping, the first line in the Hamiltonian Eq. (2) describes the kinetic energy along the \( \varphi \) direction, whereas the second and third lines provide the on-site potential energy as well as hoppings (kinetic energy) along the spin \( (m) \) direction. Furthermore, the factor of \(- \gamma m\) appearing in the kinetic energy in the \( \varphi \) direction can be recognized as a vector potential in the Landau gauge, corresponding to a uniform synthetic magnetic field of strength \( \gamma \) through the \( \varphi - m \) plane. Thus, this hybrid system mimics a particle moving on a cylinder in the presence of a uniform magnetic field.

**Relationship to standard quantum hall systems.** As we verify below, our model Eq. (2) therefore belongs to the class of 2D quantum Hall systems with energy bands characterized by nontrivial topological Chern numbers. However, unlike most standard quantum Hall Hamiltonians, our system is continuous in one direction and discrete in the other. This is known as a “coupled wire” configuration, as was first introduced theoretically as a tool for constructing fractional quantum Hall states [32]. At a single-particle level, such coupled wire models have also recently become of interest due to experimental proposals for ultracold atoms [33] and, in the context of synthetic dimensions, for coupled optical cavities [19].

Compared to previously studied coupled wire models, however, our system Eq. (2) has unusual nearest- and next-nearest-neighbour hoppings along the spin direction. For example, these hopping amplitudes are nonuniform, meaning that translational invariance is broken even away from the synthetic “edges” of the system at \( m = \pm s \). Such hopping anisotropy is often a feature of synthetic dimensions [1,10], and topological properties, such as the existence of chiral edge states, are expected to be robust provided that the anisotropy is sufficiently weak [1,10], as we now also demonstrate here.

![FIG. 1](Image)
Topological properties. To calculate the bulk topological properties and anticipate the key signatures of the synthetic magnetic field, we first assume that the hopping along the spin direction is uniform, in order to make analytical arguments. To proceed, we introduce the angular momentum quantum number  \( l_\varphi \) around the ring, which is constrained to take only integer values due to the periodicity of  \( \varphi \). Assuming uniform hopping,  \( t_m \approx t \), the Hamiltonian is

\[
\hat{H} \approx \sum_{l_\varphi, m} \left[ \frac{(l_\varphi - y m)^2}{2 M \rho^2} \hat{c}^{\dagger}_{l_\varphi, m} \hat{c}_{l_\varphi, m} + \alpha t^2 \hat{c}^{\dagger}_{l_\varphi, m} \hat{c}_{l_\varphi, m+1} + \text{h.c.} \right],
\]

where \( \hat{c}^{\dagger}_{l_\varphi, m} \) is the Fourier transform of \( \hat{c}^{\dagger}_{\varphi, m} \) along the  \( \varphi \) direction, and we have omitted a constant energy offset term. This Hamiltonian does not couple operators with different \( l_\varphi \). We thus have, for a given value of \( l_\varphi \), a one-dimensional tight-binding Hamiltonian along the spin direction, where the first term in Eq. (3) is a harmonic trapping potential centered at  \( m = l_\varphi / y \) and the other terms are hopping terms. This is reminiscent of the ordinary Landau-level problem, except that the spin direction,  \( m \), and the momentum,  \( l_\varphi \), are both discrete. Note that, in general, \( l_\varphi / y \) is not an integer, and thus the center of the trapping potential falls between lattice sites in spin direction. Due to the discreteness of the spin direction, the energy levels are not completely flat, although, practically, as we see below, we observe very flat energy levels for low-lying states in the bulk of the synthetic 2D system. The emergence of these flat Landau levels is a key signature of the synthetic magnetic field.

If we regard the energy levels as a function of \( l_\varphi \) as energy bands, we can calculate their topological Chern numbers analytically, finding that the Chern number of every band is always equal to one regardless of the values of  \( \alpha \) and  \( \beta \) [31]. This behavior agrees with previous studies of coupled wire models [19,32,33], where \( \alpha = 0 \). The nonzero value of the Chern number implies that, from the bulk-boundary correspondence, between the  \( n \)th and  \( (n + 1) \)th bulk energy levels, there are  \( n \) chiral edge modes at each boundary  \( m = \pm s \). Physically, these edge modes correspond to spin-dependent chiral currents flowing around the ring; as the largest and smallest spin states serve as the edges of the synthetic spin dimension, the edge-mode chirality means that there are currents mainly propagating states  \( m = s \) flowing in one direction around the ring, and currents with  \( m = -s \) flowing in the opposite direction. This is the second key signature of the synthetic magnetic field.

While we arrived at these expectations assuming uniform hopping  \( t_m \approx t \), the result is more general. In the original Hamiltonian Eq. (2), the momentum  \( l_\varphi \) is also a good quantum number. As we reintroduce nonuniformity of the hoppings, the bands as a function of  \( l_\varphi \) can become deformed, but as long as band gaps do not close, topological properties, such as the number of chiral edge modes which appear at  \( m = \pm s \), do not change. Nonetheless, the anisotropy of the full model leaves characteristic fingerprints, as we demonstrate below.

Energy spectrum. To illustrate the topological robustness of the edge states in the full anisotropic model Eq. (2), we plot the energy spectrum in Fig. 2. Figure 2(a) shows the energy spectrum as a function of  \( l_\varphi \), normalized by the total spin  \( s = 10 \) with  \( \alpha = 1.5 / (2 M \rho^2) \),  \( \beta = 0 \), and  \( \gamma = 1.2 \). The low-lying levels with small  \( |l_\varphi| \) show weak dispersion; these are almost-flat Landau levels. Since \( \beta = 0 \), states with even  \( m \) and odd  \( m \) do not couple, so these levels are nearly twofold degenerate. Plotted in dashed lines are ideal bulk Landau-level energies calculated from Eq. (3), assuming  \( t_m \) to be equal to  \( t_0 \) and  \( l_\varphi \) = 0. They agree well with the energy spectrum for low-lying states, verifying that the introduction of nonuniform hopping  \( t_m \) does not alter the basic phenomenon of bulk Landau levels. Analytical details of the Landau-level spacing and the associated cyclotron frequency are derived in the Supplemental Material [31].

Away from the central region, we see that Landau levels split into two and eventually move up in energy, corresponding to chiral edge states, with states concentrated around  \( m = s \) propagating in one direction and those concentrated around  \( m = -s \) propagating in the opposite direction. To clarify the nature of these chiral edge states, we plot the spin-expectation value of these eigenstates  \( \langle S_z \rangle \) in color in Fig. 2(b). States in red are concentrated on the edge at  \( m = s \), and states in blue are on the other edge at  \( m = -s \), showing that the propagating states are edge states in the synthetic dimension. Since each bulk Landau level eventually moves up in energy and becomes an edge state as  \( |l_\varphi| \) increases, the net number of chiral-edge modes propagating in the same direction is equal to the number of bands below the energy. This observation is in agreement with the fact introduced above that each Landau level has a Chern number of one. We note that at higher energies the propagation direction in the edge states is reversed. This is a finite-size effect, as although the Chern number of each band is one when calculated in the bulk, this assumed an infinite number of bands. For a finite number of  \( 2s + 1 \) bands, the Chern numbers must sum to zero, leading to the observed reversal at large energies.

Including a nonzero  \( \beta \) splits the nearly twofold degeneracy of the Landau levels, but otherwise does not significantly alter
If the magnetic flux is increased by a flux quantum, as is typical also reflects their topological nature. The robustness of these edge modes in the energy spectrum flat, and each bulk level still leads to one chiral edge mode. The many-body Hamiltonian can then be written as

\[ H_{\text{MB}} = \frac{1}{2M_{\text{tot}}\rho} \left( -i\nabla_{\text{CM}} - N\left( \gamma S_z + \frac{\Phi_{\text{ext}}}{\Phi_0} \right) \right)^2 + \alpha S_z^2 + \beta S_z^4 + H_{\text{ext}}, \]

where \( M_{\text{tot}} = N\mathcal{M} \) is the total mass of electrons, \( -i\nabla_{\text{CM}} \equiv -i\nabla_{\psi_0} - i\nabla_{\psi_1} \cdots - i\nabla_{\psi_N} \) represents their total angular momentum, and \( H_{\text{ext}} \) is the Hamiltonian of their relative motion, which is not affected by the presence of the nanomagnet. The important observation is that the nanomagnet only couples to the collective motion of the electrons, whilst the relative motion is completely decoupled. Physically, this means that the characteristic edge states and spin-switching persistent-current jumps will now emerge from the center-of-mass motion of electrons around the ring. This should facilitate the experimental observation and interpretation of the described effects, in analogy to what has previously been found for persistent currents [26,27], which can be accurately described in a noninteracting single-particle picture.

In conclusion, we have demonstrated that hybrid nanomagnetic-electronic mesoscopic systems can display topological effects based on the interpretation of the nanomagnet spin as a synthetic dimension. The key signatures are flat Landau-level bands and spin-polarized chiral currents. In the presence of an additional external magnetic flux, the magnet’s anisotropy leads to spin-switching behavior in the persistent current of the single-particle ground state. While we have focused on the mesoscopic interpretation of the system, where such a magnet could be, e.g., a molecular magnet [28–30], it is worthwhile mentioning that the components are the mean spin of the magnet, indicating that the ground-state persistent current is associated with large spin-polarizations. This can be understood by noting that the lowest energy states in Fig. 2 are modes localized at edges of the spin synthetic dimension. Although the existence of edge modes is a general characteristic of quantum Hall systems, their dominance in the ground state is a new feature due to the magnetic anisotropy; for \( \alpha > 0 \), as in Fig. 3(a), the anisotropy favors extremal spin values, as it generates an inverted harmonic trap \( \alpha(g^{2m} + t_{m-1}) = \alpha(s(s + 1) - m^2) \) in the synthetic dimension Eq. (2) as well as nonuniform hopping terms. Instead, if \( \alpha < 0 \), the ground state and hence the persistent current are localized around the middle of the synthetic dimension, as shown in Fig. 3(b).

Another clear and unusual signature of the edge modes is in the spin-switching jumps of the persistent current in Fig. 3(a), occurring whenever \( \Phi_{\text{ext}}/\Phi_0 = n/2 \), with \( n \) being any integer; in Fig. 3(b), by contrast, the persistent current only jumps when \( n \) is an odd integer. Physically, these jumps correspond to parameters where the ground state becomes doubly degenerate and then switches between different values of \( \mathcal{L}_\psi \), as shown in Figs. 3(c) and 3(d), where we plot the lowest energy states for panels (a) and (b), respectively. We provide analytical considerations for the ground-state transitions in the limit \( \alpha = \beta = 0 \) in Ref. [31].

Many-electron case. Let us consider the case where there are \( N \) electrons, each coupled to the same nanomagnet according to Hamiltonian Eq. (4), whilst the electron-electron interactions only depend on the relative position of electrons. The many-body Hamiltonian can then be written as

\[ H_{\text{MB}} = \frac{1}{2M_{\text{tot}}\rho^2} \left( -i\nabla_{\text{CM}} - N\left( \gamma S_z + \frac{\Phi_{\text{ext}}}{\Phi_0} \right) \right)^2 + \alpha S_z^2 + \beta S_z^4 + H_{\text{ext}}. \]
generic and can be realized in other ways, in particular using synthetic quantum engineering. For instance, we can envisage quantum-optical realizations in which the large spin arises from the collective interactions of two-level systems as in a Dicke model [34], the flux is induced via an artificial gauge field [35], and similar atom-optical realizations in which the described effects may serve as robust tools for quantum control [36,37].

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