JOHNSON’S HOMOMORPHISMS AND
THE ARAKELOV-GREEN FUNCTION

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Abstract. Let $\pi : C_g \to M_g$ be the universal family of compact Riemann
surfaces of genus $g \geq 1$. We introduce a real-valued function on the moduli
space $M_g$ and compute the first and the second variations of the function. As
a consequence we relate the Chern form of the relative tangent bundle $T_{C_g/M_g}$
induced by the Arakelov-Green function with differential forms on $C_g$ induced
by a flat connection whose holonomy gives Johnson’s homomorphisms on the
mapping class group.

Introduction

Let $\pi : C_g \to M_g$ be the universal family of compact Riemann surfaces of genus
$g \geq 1$. The orbifold fundamental group $\pi_1^{orb}(M_g)$ of the moduli space $M_g$ is the
mapping class group for a closed surface of genus $g$. Johnson [6] introduced a series
of nested subgroups of $\pi_1^{orb}(M_g)$ and a homomorphism on each of these subgroups.
Today they are called Johnson’s homomorphisms. The first one of the subgroups
is the Torelli group $I_g$, the kernel of the action of the mapping class group on the
homology of the surface. Moreover he proved the free part of the abelianization
$T_g^{\text{abel}}$ is given by the first Johnson homomorphism [7]. Morita extended the first
Johnson homomorphism to a crossed homomorphism on the whole group $\pi_1^{orb}(M_g)$,
and proved that his extension yields all of the Morita-Mumford classes on the
moduli space $M_g$ [11][12].

It is an important subject to study differential geometry of the moduli space $M_g$
through Johnson’s homomorphisms. Harris [5] defined the harmonic volume of a
compact Riemann surface. This can be interpreted as an analytic counterpart of the
first Johnson homomorphism. Let $L$ be the Hodge line bundle on the moduli space $M_g$.
The first Morita-Mumford class $e_1$ is twelve times the Chern class $c_1(L)$. Hain and Reed took the pullback of the biextension line bundle [3] along the harmonic
volume to construct a Hermitian line bundle $B$ on $M_g$, isomorphic to $L^{\otimes (8g+4)}$.
Comparing the Hermitian metric on $B$ with the standard metric on $L$, they defined
and studied a real-valued function $\beta_g : M_g \to \mathbb{R}$ [4].

In our previous paper [9] we introduced a flat connection on a vector bundle
over the space $M_{g,1} := T_{C_g/M_g} \setminus (0\text{-section})$ whose holonomy gives all of John-
son’s homomorphisms for the mapping class group $\pi_1(M_{g,1})$. The first term of the
connection form is exactly the first variation of (pointed) harmonic volumes. By Morita’s recipe [12] the connection form induces canonical 2-forms $e_j$ on $C_g$ and $e_j'$ on $M_g$ representing the Chern class of the relative tangent bundle $T_{C_g/M_g}$ and

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the first Morita-Mumford class $e_1$, respectively. The 2-form $e^J$ corresponds to the Hermitian bundle $B$.

Let $\{\psi_i\}_{i=1}^g$ be an orthonormal basis of the holomorphic 1-forms; $\sum_{i=1}^g \int_C \psi_i \wedge \overline{\psi_j} = \delta_{ij}, 1 \leq i, j \leq g$. Then the 2-form $B := \sum_{i=1}^g \psi_i \wedge \overline{\psi_i}$ is a volume form on $C$, independent of the choice of the orthonormal basis $\{\psi_i\}_{i=1}^g$. The 2-form $e^J$ is related to the volume form $B$. It restricts to $B$

$$e^J|_C = (2 - 2g)B \in A^2(C)$$
on any Riemann surface $C$ regarded as a fiber of the universal family $\pi : \mathcal{C}_g \rightarrow \mathcal{M}_g$ ([4, 5]). Here $A^q(\ )$ means the space of the $q$-forms.

Arakelov made use of the volume form $B$ in his study of arithmetic surfaces [1]. The Arakelov-Green function $G$ on a compact Riemann surface $C$ is defined by $G(P_0, P_1) := \exp(-4\pi h(P_1)(P_1)), P_0, P_1 \in C$, where the function $h_{P_0}$ is the Green function with respect to the volume form $B$ ([1, 2]). We may regard the Arakelov-Green function $G$ as a function defined on the fiber product $\mathcal{C}_g \times \mathcal{M}_g \mathcal{C}_g$. Then the differential form $e^A := \frac{1}{2\pi \sqrt{-1}} \partial \overline{\partial} \log G$ (diagonal) represents the Chern class of the line bundle $T_{\mathcal{C}_g/\mathcal{M}_g}$, since the normal bundle of the diagonal in the product $\mathcal{C}_g \times \mathcal{M}_g \mathcal{C}_g$ is exactly the relative tangent bundle $T_{\mathcal{C}_g/\mathcal{M}_g}$. As was observed by Arakelov [1, 2], we have

$$e^A|_C = (2 - 2g)B \in A^2(C)$$

for any $C$.

In this paper we will give an explicit function which links $e^J$ and $e^A$ together. We define

$$a_g(C) := -\sum_{i,j=1}^g \int_C \psi_i \wedge \overline{\psi_j} \Phi(\overline{\psi_i} \wedge \psi_j)$$

for any compact Riemann surface $C$, where $\Phi : A^2(C) \rightarrow A^0(C)$ is the Green operator with respect to the volume form $B$ ([1, 3]). It is independent of the choice of the orthonormal basis $\{\psi_i\}_{i=1}^g$. If $g \geq 2$, $a_g(C)$ is a positive real number (Corollary [1, 2]). Hence we obtain a real-valued function $a_g : \mathcal{M}_g \rightarrow \mathbb{R}$. Then we have

**Theorem 0.1** ( = Theorem [3, 1]).

$$e^A - e^J = \frac{-2\sqrt{-1}}{2g(2g + 1)} \partial \overline{\partial} a_g.$$

The 2-form $e^J$ is induced by the exterior derivative of the second term of the flat connection. This fact simplifies the proof, which requires only the first variation of the function $a_g$.

Next we study the 2-form $e^J$. Moreover we consider the integral along the fiber $e^J_{\text{fiber}} := \int_{\text{fiber}} (e^J)^2 \in A^2(\mathcal{M}_g)$, which also represents the first Morita-Mumford class $e_1$. The difference $e^J_{\text{fiber}} - e^J \in A^2(\mathcal{M}_g)$ is null-cohomologous, but does not vanish as a differential form. Then we have

**Theorem 0.2** ( = Theorem [6, 1]).

$$\frac{-2\sqrt{-1}}{2g(2g + 1)} \partial \overline{\partial} a_g = \frac{1}{(2g - 2)^2} (e^J_{\text{fiber}} - e^J).$$
To prove the theorem, we compute explicitly $e^{E_1}$, $e^{J_1}$ and the second variation of $a_g$. The latter half of this paper is devoted to these computations.

As a corollary of these two theorems we obtain

**Corollary 0.3.**

$$e^A - e^J = \frac{1}{(2g-2)^2}(e^{E_1} - e^{J_1}) \in A^2(\mathbb{C}_g).$$

Thus the Chern form $e^A$ induced by the Arakelov-Green function is expressed in terms of differential forms induced by the flat connection whose holonomy gives Johnson’s homomorphisms on the mapping class group.

The author does not know any of further properties of the function $a_g$. It would be interesting if we could find its explicit relations with other real-valued functions on $\mathbb{M}_g$ including Faltings’ delta function [2] and Hain-Reed’s beta function [4].

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1. A real-valued function $a_g$ on the moduli space $\mathbb{M}_g$.

We begin by recalling some basic notions on Riemann surfaces. In the second half of this section we define the real-valued function $a_g$ on the space $\mathbb{M}_g$.

Let $g \geq 1$ be an integer, $C$ a compact Riemann surface of genus $g$.

The Hodge $*$-operator $*$ : $T^*_R C \rightarrow T_R C$ on the real cotangent bundle $T_R C$ depends only on the complex structure of $C$. The $-\sqrt{-1}$-eigenspace in $(T^*_R C) \otimes \mathbb{C}$ is the holomorphic cotangent bundle $T^* C$, and the $\sqrt{-1}$-eigenspace the antiholomorphic cotangent bundle $T^{\ast \ast} C$.

We denote by $A^q(C)$ the complex-valued $q$-currents on $C$ for $0 \leq q \leq 2$. The operator $*$ decomposes the space $A^1(C)$ into the $\pm \sqrt{-1}$-eigenspaces

$$A^1(C) = A^{1,0}(C) \oplus A^{0,1}(C),$$

where $A^{1,0}(C)$ is the $-\sqrt{-1}$-eigenspace and $A^{0,1}(C)$ the $\sqrt{-1}$-eigenspace. Throughout this paper we denote by $\varphi'$ and $\varphi''$ the $(1,0)$- and the $(0,1)$-parts of $\varphi \in A^1(C)$, respectively, i.e.,

$$\varphi = \varphi' + \varphi'', \quad *\varphi = -\sqrt{-1}\varphi' + \sqrt{-1}\varphi''.$$ 

If $\varphi$ is harmonic, then $\varphi'$ is holomorphic and $\varphi''$ anti-holomorphic.

Let $H$ denote the complex first homology group of $C$, $H_1(C; \mathbb{C})$, which admits the intersection pairing

$$m : H \otimes H \rightarrow \mathbb{C}, \quad X \otimes Y \mapsto m(X \otimes Y) = X \cdot Y.$$
The dual $H^*$ is the first complex cohomology group $H^1(C; \mathbb{C})$. Consider the map $H^* \to A^1(C)$ assigning to each cohomology class the harmonic 1-form representing it. The map can be regarded as a $H$-valued real harmonic 1-form $\omega(1) \in A^1(C) \otimes H$.

Let $\{X_i, X_{g+i}\}_{i=1}^g$ be a symplectic basis of $H$

$$X_i \cdot X_{g+j} = \delta_{ij}, \quad X_i \cdot X_j = X_{g+i} \cdot X_{g+j} = 0, \quad 1 \leq i, j \leq g,$$

and $\{\xi_i, \xi_{g+i}\}_{i=1}^g \subset A^1(C)$ the basis of the harmonic 1-forms dual to $\{X_i, X_{g+i}\}_{i=1}^g$. Then we have

$$\omega(1) = \sum_{i=1}^g \xi_i X_i + \xi_{g+i} X_{g+i} \in A^1(C) \otimes H.$$ 

In particular, if $\{\psi_i\}_{i=1}^g$ is an orthonormal basis of the holomorphic 1-forms

$$(1.1) \quad \frac{\sqrt{-1}}{2} \int_C \psi_i \wedge \overline{\psi_j} = \delta_{ij}, \quad 1 \leq i, j \leq g,$$

then we obtain

$$(1.2) \quad \omega(1) = \sum_{i=1}^g \psi_i \overline{Y_i} + \overline{\psi_i Y_i},$$

where $\{Y_i, Y_{g+i}\}_{i=1}^g \subset H_C$ is the dual basis of the symplectic basis $\{[\psi_i], \frac{\sqrt{-1}}{2} [\overline{\psi_i}]\}_{i=1}^g$ of $H^* = H^1(C; \mathbb{C})$.

Since the complete linear system of the canonical divisor on the complex algebraic curve $C$ has no basepoint, the 2-form

$$(1.3) \quad B := \frac{1}{2g} m(\omega(1) \wedge \omega(1)) = \frac{1}{2g} \omega(1) \cdot \omega(1) = \frac{\sqrt{-1}}{2g} \sum_{i=1}^g \psi_i \wedge \overline{\psi_i}$$

is a volume form on $C$ with $\int_C B = 1$.

We denote by $\hat{\Phi} = \hat{\Phi}_C : A^2(C) \to A^0(C)$ the Green operator with respect to the volume form $B$. We have

$$(1.4) \quad d \ast d \hat{\Phi}(\Omega) = \Omega - (\int_C \Omega) B,$$

$$(1.5) \quad \int_C \hat{\Phi}(\Omega) B = 0$$

for any $\Omega \in A^2(C)$. The Hodge decomposition on the 1-currents on $C$ is given by

$$(1.6) \quad \varphi = \mathcal{H}\varphi + \ast d\overline{\varphi} + d\overline{\varphi} \ast \varphi$$

for any $\varphi \in A^1(C)$. Here $\mathcal{H}$ is the harmonic projection and satisfies

$$(1.7) \quad \mathcal{H}\varphi = -\omega(1) \cdot \left(\int_C \omega(1) \wedge \varphi \right) = -\left(\int_C \varphi \wedge \omega(1) \right) \cdot \omega(1).$$

If $\varphi'$ is a $(1,0)$-current, then

$$(1.8) \quad \varphi' = \mathcal{H}\varphi' + 2 \ast \partial \overline{\varphi} \varphi' = \mathcal{H}\varphi' - 2\sqrt{-1} \partial \overline{\varphi} \varphi'.$$

Let $\delta_{P_0} \in A^2(C)$ be the delta current at $P_0 \in C$. We define

$$(1.9) \quad h = h_{P_0} := -\hat{\Phi}(\delta_{P_0}).$$

$G(P_0, P_1) := \exp(-4\pi h_{P_0}(P_1))$ is the Arakelov-Green function. We have

$$\frac{1}{2\pi \sqrt{-1}} \partial \overline{\partial} \log G(P_0, ) = B - \delta_{P_0}. $$
If $\Omega$ is a smooth 2-form on $C$, then $\hat{\Phi}(\Omega)$ is smooth. We have
\[
\int_C \hat{\Phi}(\Omega') \Omega = \int_C \Omega' \hat{\Phi}(\Omega)
\]
for any $\Omega' \in A^2(C)$. In particular, if $\Omega' = \delta_{P_0}$, then
\[
\hat{\Phi}(\Omega)(P_0) = -\int_C h_{P_0} \Omega = \frac{1}{4\pi} \int_C \log G(P_0,.) \Omega.
\]

Now we introduce the function $a_g$ on the moduli space $M_g$. For $[C] \in M_g$ we define
\[
a_g(C) := (m \otimes m) \int_C \omega_1 \hat{\Phi}(\omega_1 \wedge \omega_1) \omega_1 = \int_C \omega_1 \cdot \hat{\Phi}(\omega_1 \wedge \omega_1) \cdot \omega_1.
\]

From (1.4) we obtain $\Omega = (\int_C \Omega) B \in CB$. It follows from the lemma

**Lemma 1.1.** We have
\[
\int_C \Omega \hat{\Phi}(\Omega) \leq 0
\]
for any smooth 2-form $\Omega$. Moreover we have $\int_C \Omega \hat{\Phi}(\Omega) = 0$ if and only if $\Omega \in CB$.

**Proof.** By a straightforward computation we have
\[
\int_C \Omega \hat{\Phi}(\Omega) = -\sqrt{-1} \int_C \partial \hat{\Phi}(\Omega) \wedge \overline{\partial} \hat{\Phi}(\Omega) - \sqrt{-1} \int_C \partial \hat{\Phi}(\Omega) \wedge \overline{\partial} \hat{\Phi}(\Omega).
\]
This implies the first half of the lemma.

Assume $\int_C \Omega \hat{\Phi}(\Omega) = 0$. Then $\partial \hat{\Phi}(\Omega) = \overline{\partial} \hat{\Phi}(\Omega) = 0$, so that $\hat{\Phi}(\Omega)$ is constant.

From (1.4) we obtain $\Omega = (\int_C \Omega) B \in CB$. □

It follows from the lemma

**Corollary 1.2.**
\[
a_g(C) > 0, \quad \text{if} \quad g \geq 2.
\]
In the case $g = 1$ we have $a_1(C) = 0$.

2. **The First Variation of the Function $a_g$.**

In this section we study the first variations of the function $a_g$ and the Green function $h$.

Let $C$ be a compact Riemann surface of genus $g \geq 1$. We define the map $M : H^4 \to \mathbb{C}$ by
\[
M(Z_1 Z_2 Z_3 Z_4) := (m \otimes m)(Z_2 Z_3 Z_4 Z_1) = (Z_2 \cdot Z_3)(Z_4 \cdot Z_1)
\]
for $Z_i \in H$. Here and throughout this paper we omit the symbol $\otimes$ frequently, so that we write simply $Z_1 Z_2 Z_3 Z_4$ for $Z_1 \otimes Z_2 \otimes Z_3 \otimes Z_4$. Then
\[
a_g(C) = -M \int_C \hat{\Phi}(\omega_1 \wedge \omega_1) \omega_1 \wedge \omega_1.
\]
Let $I \in H^{\otimes 2}$ denote the intersection form. If $\{X_i, X_{g+i}\}_{i=1}^g \subset H$ is a symplectic basis, then

$$I = \sum_{i=1}^g X_i X_{g+i} - X_{g+i} X_i.$$  

For $Z, Z_1, Z_2 \in H$ it is easy to see

$$\begin{align*}
(2.2) & \quad (m \otimes 1_H)(ZI) = (1_H \otimes m)(IZ) = -Z, \\
(2.3) & \quad M(I Z_1 Z_2) = M(Z_1 Z_2 I) = m(Z_1 Z_2) = Z_1 \cdot Z_2.
\end{align*}$$

We denote by $\bar{\omega} : H \to H$ the transpose of the Hodge $*$-operator on $H^\ast = H^1(C; \mathbb{C})$. It is clear that

$$\omega(I) = \omega'(1) + \omega''(1), \quad \omega'(1) = \sum_{i=1}^g \psi_i Y_i \in A^1(C) \otimes H',$$

where $\{\psi_i\}_{i=1}^g$ is the orthonormal basis of the holomorphic 1-forms in (1.1). In particular,

$$\ast \omega(1) = \bar{\omega}(1).$$

Consider a $C^\infty$ family of compact Riemann surfaces $C_t, t \in \mathbb{R}, |t| \ll 1$, with $C_0 = C$. The family $\{C_t\}$ is trivial as a $C^\infty$ fiber bundle over an interval near $t = 0$, so that we obtain a $C^\infty$ family of $C^\infty$ diffeomorphisms $f^t : C \to C_t$ with $f^0 = 1_C$. In general, if $\emptyset = \emptyset_t$ is a “function” in $t \in \mathbb{R}, |t| \ll 1$, then we write simply

$$\begin{align*}
\emptyset & := \bigg|_{t=0} \emptyset_t.
\end{align*}$$

For example, we denote

$$\begin{align*}
(\mu(f))^\ast & := \bigg|_{t=0} d \mu(f^t).
\end{align*}$$

Here $\mu(f^t)$ is the complex dilatation of the diffeomorphism $f^t$. Let $z$ be a complex coordinate on $C$, and $\zeta$ on $C_t$. The complex dilatation $\mu(f^t)$ is defined locally by

$$\mu(f^t) = \mu(f^t)(z) \frac{d}{dz} \otimes d\zeta = \frac{(\zeta \circ f^t)\zeta}{(\zeta \circ f^t)z} \frac{d}{dz} \otimes d\zeta,$$

which does not depend on the choice of the coordinates $z$ and $\zeta$. In this paper we write more simply

$$\mu = (\mu(f))^\ast.$$

The Dolbeault cohomology class $[\mu] \in H^1(C; \mathcal{O}_C(TC))$ is exactly the tangent vector $\frac{d}{dt}|_{t=0} [C_t] \in T[C]M_g$.

We define a linear operator $S = S[\mu] : A^1(C) \to A^1(C)$ by

$$S(\varphi) = S(\varphi') + S(\varphi'') := -2\varphi' \mu - 2\varphi'' \overline{\mu},$$

for $\varphi = \varphi' + \varphi'', \varphi' \in A^{1,0}(C), \varphi'' \in A^{0,1}(C)$. The first variation of $\ast_t := f^{t*} \ast_{C_t}$ is given by

$$\begin{align*}
\ast = \ast S = -S : A^1(C) \to A^1(C).
\end{align*}$$
As to the harmonic form \( \omega_{(1)}^t := f^* \omega_{(1)}^C \), we have

**Lemma 2.1.**

\[ \omega_{(1)}^t = -d\hat{\Phi}d \ast S\omega_{(1)}. \]

**Proof.** Since \( \omega_{(1)}^t \) is cohomologous to \( \omega_{(1)}^0 \), there exists a function \( u \) such that \( \omega_{(1)}^t = du \). Differentiating \( d \ast t \omega_{(1)}^t = 0 \), we get

\[ d \ast du = d \ast \omega_{(1)}^t = -d \ast S\omega_{(1)}. \]

Hence \( u \equiv -\hat{\Phi}d \ast S\omega_{(1)} \) modulo the constant functions. \( \square \)

In this paper we write

\( \Omega_0 := \omega_{(1)} \wedge \omega_{(1)} \in A^2(C) \otimes H^{\otimes 2} \)

\( K_0 := \hat{\Phi}(\Omega_0) = \hat{\Phi}(\omega_{(1)} \wedge \omega_{(1)}) \in A^0(C) \otimes H^{\otimes 2} \)

\( \nu_0 := \ast \partial K_0 = \ast \hat{\Phi}(\Omega_0) \in A^{1,0}(C) \otimes H^{\otimes 2} \).

The first variation of \( a_g \) is given by the following.

**Theorem 2.2.** Let \( \Xi \) be the quadratic differential defined by

\[ \Xi := M \left( (\ast \partial K_0)(\ast \partial K_0) + 4(\ast \hat{\Phi}(\ast dK_0 \wedge \omega_{(1)}))\omega_{(1)}' \right) \]

Then we have

\[ a_g' = -\Re \left( 4\sqrt{-1} \int_C \Xi \mu \right). \]

We denote by

\[ \hat{T} := \prod_{m=0}^{\infty} H^{\otimes m} \]

the completed tensor algebra generated by \( H \). Let \( \Omega = \{ \Omega_t \} \), \( t \in \mathbb{R}, |t| \ll 1 \), be a family of 2-forms with values in the algebra \( \hat{T} \). Assume \( A := \int_C \Omega \in \hat{T} \) is constant, and denote \( K := \hat{\Phi}(\Omega) \). Then we have

**Lemma 2.3.**

\[ (\int_C K\Omega)' = \int_C (\hat{\Omega} K + K \hat{\Omega}) - \int_C (\hat{B} AK + K \hat{B} A) - \int_C (d\ast dK)K. \]

**Proof.** Since \( \Omega = d \ast dK + BA \) and \( \int_C KB = 0 \), we have

\[ \int_C \hat{K} \Omega = \int_C \hat{K} d \ast dK + \int_C \hat{\Omega} - \int_C K \hat{B} A \]

\[ = \int_C (\hat{\Omega} - \hat{B} A - d\ast dK)K - \int_C K \hat{B} A. \]

Hence

\[ (\int_C K\Omega)' = \int_C \hat{K} \Omega + \int_C K \hat{\Omega} \]

\[ = \int_C (\hat{\Omega} K + K \hat{\Omega}) - \int_C (\hat{B} AK + K \hat{B} A) - \int_C (d\ast dK)K, \]

as was to be shown. \( \square \)
Let \( \theta \) and \( \varphi \) be real 1-forms on \( C \) with values in \( \hat{T} \). Then we have
\[
\int_C \ast \theta \wedge S \varphi = \Re \left( 4\sqrt{-1} \int_C \theta' \varphi' \mu \right).
\]
In fact, we have
\[
\int_C \ast \theta \wedge S \varphi = -2 \int_C \ast \theta \wedge (\varphi' \mu + \varphi'' \mu) = 2\sqrt{-1} \int_C (\theta' - \theta'') \wedge (\varphi' \mu + \varphi'' \mu).
\]

Proof of Theorem 2.2. Since \( H' \) and \( H'' \) are isotropic in \( H \),
\[
M(\ast \hat{\partial} \hat{\Phi}(\nu_0 \wedge \omega''(1)) \omega'(1)) = M(\ast \hat{\partial} \hat{\Phi}(\Omega_0) \wedge \omega(1)) \omega'_1).
\]
Hence we have
\[
\Xi = M \left( \nu_0 \nu_0 + 4(\ast \hat{\partial} \hat{\Phi}(\nu_0 \wedge \omega''(1))) \omega'(1) \right).
\]
Now applying Lemma 2.3 to \( a_g = -M \int_C K_0 \Omega_0 \), we have
\[
- \hat{a}_g = 2M \int_C K_0 \hat{\Omega}_0 \hat{K}_0 - 2M \int_C \hat{B} \hat{K}_0 \hat{l} - M \int_C (d^s dK_0) K_0.
\]
The first term of the RHS in (2.11) is
\[
2M \int_C \hat{\Phi}(\Omega_0) \wedge \omega(1) \wedge \omega''(1) + 2M \int_C \hat{\Phi}(\Omega_0) \wedge \omega_1 \wedge \omega''(1),
\]
which can be calculated.

From (2.3) the second term vanishes. In fact,
\[
M \int_C \hat{B} \hat{K}_0 \hat{l} = m \int_C \hat{B} \hat{K}_0 = 2g \int_C \hat{B} \hat{\Phi}(B) = 0.
\]
The third term is
\[
- M \int_C (d^s dK_0) K_0 = - M \int_C (\hat{d} K_0) dK_0 = M \int_C dK_0 \hat{d} dK_0
\]
and
\[
- M \int_C \ast dK_0 \wedge S \ast dK_0 = M \int_C \ast dK_0 \wedge \ast S \ast dK_0.
\]
Hence from (2.9)
\[
- \hat{a}_g = M \left( \int_C \ast dK_0 \wedge S \ast dK_0 + 4 \int_C \ast \hat{\Phi}(\ast dK_0 \wedge \omega(1)) \wedge S \omega(1) \right)
\]
and
\[
\Xi = \Re \left( 4\sqrt{-1} \int_C \Xi \mu \right),
\]
as was to be shown. \( \square \)
Next we study the first variation of the Green function \( h = -\hat{\Phi}(\delta_{P_0}) \). Let 
\((C_t, P^t_0), t \in \mathbb{R}, |t| < 1\), a \( C^\infty \) curve on the universal family \( \mathcal{C}_g \). We can choose a
family of diffeomorphisms \( f^t : (C, P_0) \to (C_t, P^t_0) \) such that \( f^t \) is complex analytic near \( P_0 \) for sufficiently small \( t \). Then \( \mu = (\mu(f))^* \) vanishes near the point \( P_0 \).

We compute \( \dot{h} = \frac{df}{dt} \bigg|_{t=0} f^t \cdot \delta_{P_0} \). Since \( f^t \cdot \delta_{P_0} = \delta_{P_0} \) is constant, we have
\[
d \ast \dot{h} = \dot{B} - d \ast dh.
\]
The RHS is smooth near \( P_0 \) since \( \mu \) and \( d \ast dh \) vanish near \( P_0 \). Hence \( \dot{h} \) is smooth near \( P_0 \) so that we may take the value of \( \dot{h} \) at the point \( P_0, \dot{h}(P_0) \).

From what we have discussed we may apply Lemma 2.3 to the 2-current \( \Omega = -\delta_{P_0} \). Then we have \( \delta_{P_0}^\ast = 0, K = h_{P_0} \), \( A = -1 \) and \((\int C, K \Omega) = \int C, K' \Omega = -\dot{h}(P_0) \). Hence we obtain
\[
\dot{h}(P_0) = -2 \int_C \dot{h} \cdot B + \int_C \dot{h} (d \ast dh).
\]

**Theorem 2.4.** Let \( \Upsilon \) be the quadratic differential defined by
\[
\Upsilon := (\ast d\bar{h})(\ast dh) + \frac{2}{g} \ast \partial \Phi(\ast dh \wedge \omega(1)) \cdot \omega'(1)
\]
Then we have
\[
\dot{h}(P_0) = -\Re \left( 4 \sqrt{-1} \int_C \Upsilon \cdot \mu \right).
\]

**Proof.** We have \( \dot{B} = \frac{1}{g} \omega(1) \cdot \omega'(1) \) since \( B = \frac{1}{2g} \omega(1) \cdot \omega(1) \). As to the first term in (2.12) we have
\[
g \int_C \dot{h} \cdot B = \int_C h \omega(1) \cdot \omega'(1) = - \int_C h \omega(1) \cdot d \Phi \ast S \omega(1)
\]
\[
= - \int_C \ast d \Phi(dh \wedge (\omega(1)) \cdot \omega(1)) = \int_C \ast d \Phi(\ast dh \wedge (\omega(1))) \cdot S \omega(1)
\]
\[
= \int_C \ast d \Phi(\ast dh \wedge (\omega(1))) \cdot \omega(1) = \int_C \ast d \Phi(\ast dh \wedge (\omega(1))) \cdot \omega(1) \cdot \mu.
\]
The last line follows from (2.20).

The second term is equal to
\[
\int_C h (d \ast dh) = - \int_C dh \wedge \ast dh = \int_C dh \wedge S \ast dh
\]
\[
= - \int_C \ast dh \wedge S \ast dh = -\Re \left( 4 \sqrt{-1} \int_C (\ast dh)(\ast dh) \cdot \mu \right).
\]
This completes the proof. \( \square \)

### 3. The flat connection for Johnson’s homomorphisms.

Now we regard the Arakelov-Green function \( G(P_0, P_1) := \exp(-4\pi h_{P_0}(P_1)) \) as a function on the fiber product \( \mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \). The normal bundle of the diagonal map \( \mathbb{C}_g \to \mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g \) is equal to the relative tangent bundle \( T_{\mathbb{C}_g/\mathbb{M}_g} \). Hence the \((1,1)\)-form on \( \mathbb{C}_g \)
\[
e^A := \frac{1}{2 \pi \sqrt{-1}} \partial \bar{\partial} \log G
\]

\text{diagonal}
represents the first Chern class of the bundle $T_{\mathcal{C}_g/M_g}$. Since $\frac{1}{2\pi \sqrt{-1}} \log G(P_0) = 2\sqrt{-1} \log h_{P_0}$, we have
\begin{equation}
(3.1) \quad e^A_{[C, P_0]} = -2\sqrt{-1} \partial \bar{\partial} h_{P_0}.
\end{equation}

In this section we review a flat connection introduced in the previous paper [9], whose holonomy gives Johnson’s homomorphisms on the mapping class group. The connection induces a $(1, 1)$-form $e^J$ on $\mathbb{C}_g$ representing the first Chern class of $T_{\mathcal{C}_g/M_g}$ following Morita’s recipe [12]. We relate the second term of the connection form with the first variations $\partial a_g$ and $\delta h(P_0)$ computed in §2. As a consequence we obtain

**Theorem 3.1.**
\begin{equation}
(3.5) \quad * dh = * d\Phi(B) = \frac{1}{2g} m * d\Phi(\Omega_0)
\end{equation}

Now we define $\omega(n)$, $n \geq 2$, by
\begin{equation}
\omega(n) := * d\Phi \left( \sum_{p=1}^{n-1} \omega(p) \wedge \omega(n-p) \right)
\end{equation}
inductively on $n$, and $\omega := \sum_{n=1}^{\infty} \omega(n) \in A^1(C) \hat{\otimes} \hat{T}$. Here $\hat{\otimes}$ means the completed tensor product. Then we have the (modified) integrability condition
\begin{equation}
d\omega = \omega \wedge \omega - I \delta_{P_0},
\end{equation}
where $I \in H^{\otimes 2}$ is the intersection form in [2.1]. This means the 1-form $\omega$ defines a flat connection on the vector bundle $(C \setminus \{P_0\}) \times \hat{T}$. Its holonomy is the harmonic Magnus expansion, an embedding of the fundamental group of $C \setminus \{P_0\}$ with a tangential basepoint towards $P_0$ into the multiplicative group of the algebra $\hat{T}$ [9].
In the case \( n = 2 \) we have
\[
\omega(2) = \ast d\Phi(\Omega_0) = \ast d\hat{\Phi}(\Omega_0) + I \ast dh = \ast dK_0 + I \ast dh.
\]
from (3.4).

**Lemma 3.2.**
\[
(m \otimes m + M)(\omega'_2) = 2\omega'_2 + 2\omega'_3 = \Xi + 2g(2g + 1)\Upsilon.
\]

**Proof.** We have \( mK = 2g \tilde{\Phi}(B) = 0 \) so that \( m\omega(2) = 2g \ast dh \). It follows from (3.7)
\[
(m \otimes m + M)(\omega'_2) = M(\ast dK_0)(\ast dh) + (m \otimes m + M)(I)(\ast dh)(\ast dh) = M(\ast dK_0)(\ast dh) + 2g(2g + 1)(\ast dh)(\ast dh).
\]
Since \( \omega(2) \wedge \omega(1) \) is d-exact,
\[
\omega'_3 = \ast d\hat{\Phi}(\omega(2) \wedge \omega(1)) + \ast d\hat{\Phi}(\omega(1) \wedge \omega(2)).
\]
Hence
\[
(m \otimes m + M)(\omega'_3) = 2(2g + 1) \ast d\hat{\Phi}(\ast dh \wedge \omega(1)) \cdot \omega'_3 + 2M(\ast dK_0 \wedge \omega(1) \wedge \omega(1)).
\]
This completes the proof. \( \square \)

We introduce the operator \( N : \hat{T} \to \hat{T} \) defined by
\[
N|_{H^\otimes n} := \sum_{k=0}^{n-1} \binom{1}{2} \cdots \binom{n-1}{n} \cdot \binom{n}{n}.
\]
As was shown in [3, §7], the covariant tensor \( N(\omega^\prime) \) is a meromorphic quadratic differential with a unique pole at \( P_0 \), and so it is regarded as a \((1,0)\)-cotangent vector at \( [C, P_0, v] \in \mathcal{M}_{g,1} \) in a natural way. We denote by \( \eta_n \) the \((n+2)\)-nd graded component of \( N(\omega^\prime) \). By abuse of notation we denote by \( H^\otimes n \) the vector bundle
\[
H^\otimes n := \bigotimes_{[C] \in \mathcal{M}_{g}} H_1(C; \mathbb{C})^\otimes n
\]
over the moduli space \( \mathcal{M}_g \). \( \eta_n := \eta_n^1 + \eta_n^2 + \eta_n^3 \) is a twisted real 1-form with values in the vector bundle \( \mathcal{H}^\otimes(n+2) \) on \( \mathcal{M}_{g,1} \). The real twisted 1-form \( \eta := \sum_{n=1}^{\infty} \eta_n \) induces a flat connection on \( H^\otimes n \) and the holonomy gives all of Johnson’s homomorphisms on the mapping class group \( \pi_1(M_{g,1}) \). Here we denote \( \tilde{T}_2 := \prod_{n=1}^{\infty} H^\otimes(n+2) \).

Now we look at \( \eta_2 = N(\omega'_1(\omega'_3 + \omega'_2(\omega'_2 + \omega'_3)) \wedge \omega'_1) \). By Lemma 3.2 we have
\[
(m \otimes m)\eta_2 = \sqrt{-1}(2g + 1)\sqrt{-1}(\partial \tilde{a} + 2g(2g + 1)\partial dh(P_0).
\]
Hence \( \partial (m \otimes m)\eta_2 = 0 \) and \( d(m \otimes m)\eta_2 = 2\hat{\partial}(m \otimes m)\eta'_2 \). Consequently

**Corollary 3.3.**
\[
d(m \otimes m)\eta_2 = -2\sqrt{-1}(\hat{\partial} \tilde{a} + 2g(2g + 1)e^A).
\]
From the flatness of $\eta$ we have $d\eta_1 = 0$. Moreover $\eta_1$ may be regarded as a real
1-form on $\mathbb{C}_g$. We identify $\Lambda^3 H$ with a submodule of $H^{\otimes 3}$ by the embedding
\[(3.8) \quad \Lambda^3 H \to H^{\otimes 3}, \quad Z_1 \wedge Z_2 \wedge Z_3 \mapsto \sum_{\sigma \in S_3} (\mathrm{sgn} \, \sigma) Z_{\sigma(1)} Z_{\sigma(2)} Z_{\sigma(3)}.
\]

As was proved in [9, §8], the cohomology class $-[\eta]$ is equal to the first extended
Johnson homomorphism $\tilde{k} \in \mathcal{H}^1(C_g; \Lambda^3 H)$ introduced by Morita [11].

We define $\tilde{M} : H^{\otimes 6} \to \mathbb{C}$ by
\[(3.9) \quad \tilde{M}(Z_1 Z_2 Z_3 W_1 W_2 W_3) := (Z_1 \cdot W_1)(Z_2 \cdot W_2)(Z_3 \cdot W_3), \quad Z_i, W_i \in H,
\]

$M_1$ and $M_2 : \Lambda^3 H \otimes \Lambda^3 H \to \mathbb{C}$ by
\[M_1 := (m \otimes m \otimes m)_{|\Lambda^3 H \otimes \Lambda^3 H} \quad \text{and} \quad M_2 := \tilde{M}_{|\Lambda^3 H \otimes \Lambda^3 H}
\]respectively. Then Morita [12] proved
\[\text{Theorem 3.4 (12, Theorems 5.1 and 5.8).}
\]

\[-\frac{1}{2g(2g+1)}(M_1 + M_2)(\tilde{k}^{\otimes 2}) = e(\cdot = c_1(T\mathcal{C}_g/\mathcal{M}_g)) \in H^2(\mathcal{C}_g; \mathbb{C}).
\]

\[\frac{1}{2g+1}(-3M_1 + 2(g-1)M_2)(\tilde{k}^{\otimes 2}) = e_1 \in H^2(\mathcal{C}_g; \mathbb{C}).
\]

Here $e_1$ is the first Morita-Mumford class [13, 10] on the moduli space $\mathcal{M}_g$.

Following this theorem we define
\[(3.10) \quad e^J := -\frac{1}{2g(2g+1)}(M_1 + M_2)(\eta^{\otimes 2})
\]
\[(3.11) \quad e_1^J := \frac{1}{2g+1}(-3M_1 + 2(g-1)M_2)(\eta^{\otimes 2}).
\]

These are closed 2-forms on $\mathbb{C}_g$ representing the cohomology classes $e$ and $e_1$, respectively. As will be shown in [13] $e_1^J$ can be regarded as a 2-form on $\mathcal{M}_g$.

Moreover we define $M_3$ and $M_4 : H^{\otimes 6} \to H^{\otimes 4}$ by
\[(3.12) \quad M_3(Z_1 Z_2 Z_3 W_1 W_2 W_3) := (Z_3 \cdot W_1)Z_1 Z_2 W_2 W_3,
\]
\[(3.13) \quad M_4(Z_1 Z_2 Z_3 W_1 W_2 W_3) := (Z_3 \cdot W_2)W_1 Z_1 Z_2 W_3,
\]
respectively. Then we have
\[(3.14) \quad (m \otimes m)M_3|_{\Lambda^3 H \otimes \Lambda^3 H} = M_1, \quad \text{and} \quad (m \otimes m)M_4|_{\Lambda^3 H \otimes \Lambda^3 H} = M_2.
\]

It follows from the flatness of $\eta$
\[d\eta_2 = (M_3 + M_4)(\eta^{\otimes 2})
\]
[9, Lemma 2.4]. Hence we obtain
\[(3.15) \quad (m \otimes m)d\eta_2 = (M_1 + M_2)(\eta^{\otimes 2}) = -2g(2g+1)e^J.
\]

Theorem 3.1 follows from Corollary 3.3 and (3.14).

The residue of the quadratic differential $(m \otimes m)(\eta^J)$ at the pole $P_0$ is $-\frac{1}{8\pi^2}2g(2g+1)$. This also implies $-\frac{1}{4g(2g+1)}(m \otimes m)d\eta_2$ represents the first Chern form of the relative tangent bundle $T_{\mathcal{C}_g/\mathcal{M}_g}$.
4. Integration along the fiber.

Now we introduce another 2-form on $\mathcal{M}_g$ 
\[
e_1^F := \int_{\text{fiber}} (e^J)^2
\]
representing the first Morita-Mumford class $e_1 \in H^2(\mathcal{M}_g; \mathbb{C})$. To simplify the situation we compute
\[
E_1^F := \frac{1}{4(2g-2)} \int_{\text{fiber}} M_1(\eta_1 \otimes \eta_2)^2 = \frac{1}{4} \int_{\text{fiber}} (m \otimes m)(\eta_1^H)^2
\]
instead of $e_1^F$. Here we denote
\[
\eta_1^H := \frac{1}{2g-2}(m \otimes 1)\eta_1
\]
It is clear that
\[
\frac{1}{2g-2} \int_{\text{fiber}} M_1(\eta_1 \otimes \eta_2)^2 = m(\eta_1^H \otimes \eta_2).
\]
Since
\[
M_1(\eta_1 \otimes \eta_2) = -e_1^J + 2g(2-2g)e^J,
\]
we have
\[
E_1^F = \frac{g^2}{(2g-2)^2}e_1^F - \frac{g}{(2g-2)^2}e_1^J
\]
\[
\left[ E_1^F \right] = \frac{g}{4(4g-1)}e_1.
\]

Let $\lambda$ and $\mu$ be Beltrami differentials, or equivalently elements in $C^\infty(C; TC \otimes T^*C)$. $\lambda$ and $\mu$ are regarded as tangent vectors of $\mathcal{M}_g$ at $[C]$. We define
\[
\ell^\lambda := 2\Phi d * (\omega(1) \lambda) \in A^0(C) \otimes H'
\]
\[
L^\lambda := \int_C \ell^\lambda \omega(1) \wedge \omega(1) \in H^{\otimes 3}
\]
\[
(4.2) \quad c^\lambda := \frac{1}{1-g}(m \otimes 1)(L^\lambda) = \frac{1}{1-g} \int_C (\ell^\lambda \cdot \omega(1))\omega(1) \in H.
\]

The purpose of this section is to prove

**Theorem 4.1.** The 2-form $E_1^F$ is a $(1,1)$-form on the moduli space $\mathcal{M}_g$, and we have
\[
E_1^F(\lambda, \mu) = 2 \int_C (\ell^\lambda \cdot \omega(1))(\omega(1) \cdot \overline{\mu}) + 2g \int_C (\ell^\lambda \cdot \overline{\mu}) B + (2-2g)e^\lambda \cdot \overline{\mu}
\]
for any $\lambda$ and $\mu \in C^\infty(C; TC \otimes T^*C)$.

We denote by $q$ the $(1,0)$-part of $-\eta_1^H$
\[
q := \frac{1}{2-2g}(m \otimes 1)N(\omega(1)\omega(2) + \omega(2)\omega(1)),
\]
which is a meromorphic quadratic differential on $C$ with a unique pole at $P_0$.

**Lemma 4.2.**
\[
\int_C q \lambda = \ell^\lambda(P_0) + c^\lambda
\]
for any $\lambda \in C^\infty(C; TC \otimes T^*C)$. 

Proof. From (3.5) and (3.7) we have

\[ q = \frac{2}{1 - g}(m \otimes 1)(\omega'(1)\nu_0) - 2\omega'(1) \ast \partial h. \]  

In general, for any 2-current \( \Omega \) and smooth 1-form \( \varphi \), we have

\[ \int_{C}^{*} d\tilde{\Phi}(\Omega) \wedge \varphi = \int_{C}^{*} \Omega \tilde{d} \ast \varphi \]
\[ \int_{C} \varphi \wedge d\tilde{\Phi}(\Omega) = -\int_{C} (\tilde{d} \ast \varphi) \Omega. \]

Hence

\[ 2(m \otimes 1) \int_{C} (\omega'(1)\nu_0) = -2(m \otimes 1) \int_{C} (\omega'(1) \lambda) \ast d\tilde{\Phi}(\omega(1) \wedge \omega(1)) \]
\[ = (m \otimes 1) \int_{C} \ell^\lambda \omega(1) \wedge \omega(1) = (1 - g)c^\lambda, \]
\[ 2 \int_{C} (\omega'(1) \ast \partial h) \lambda = 2 \int_{C} (\omega'(1) \lambda) \ast d\tilde{\Phi}(\delta P_0) = -\ell^\lambda(P_0). \]

Consequently \( \int_{C} q \lambda = c^\lambda + \ell^\lambda(P_0) \), as was to be shown. \( \square \)

**Lemma 4.3.**

\[ \int_{C} q\overline{\varphi}V' = (\omega'(1)V')(P_0) \]

for any \( V' \in C^\infty(C; TC) \). In other words, as a \((1,0)\)-form on \( \mathbb{C}_g \), \( q \) restricts to \( \omega'(1) \) on the fiber \( C \) of the universal family \( \pi : \mathbb{C}_g \to M_g \).

**Proof.** Let \( z \) be a complex coordinate on \( C \) centered at \( P_0 \). We have \(-2 \ast \partial h \sim \frac{1}{2\pi \sqrt{-1}} \frac{dz}{z} \) near \( P_0 \). Since \( q \) is integrable on \( C \) and holomorphic on \( C \setminus \{ P_0 \} \),

\[ \int_{C} q\overline{\varphi}V' = -\lim_{\epsilon \to 0} \int_{|z| = \epsilon} d(q\overline{\varphi}V') = \lim_{\epsilon \to 0} \int_{|z| = \epsilon} q\overline{\varphi}V' \]
\[ = \frac{1}{2\pi \sqrt{-1}} \lim_{\epsilon \to 0} \int_{|z| = \epsilon} (\omega'(1)V') \frac{dz}{z} = (\omega'(1)V')(P_0). \]

This proves the lemma. \( \square \)

By (4.1) we obtain

\[ e^\lambda|_{C} = \frac{2 - 2g}{2g} m(\eta_1^H \otimes 2)|_{C} = \frac{2 - 2g}{2g} m(\omega(1) \wedge \omega(1)) = (2 - 2g)B. \]

**Proof of Theorem 4.1.** The 1-forms \( q \) and \( \omega'(1) \) have values in \( H' \). Since \( H' \) is isotropic, we have \( m(\omega'(1)\omega'(1)) = m(\omega'(1)\eta) = m(q\eta) = 0 \). This means the (2,0)- and the (0,2)-parts of \( E_{1}^{F} \) vanish.

From what we have discussed above it follows

\[ E_{1}^{F}(\lambda, \overline{\pi}) \]
\[ = (m \otimes m) \left( \int_{C} \left( \int_{C} q \lambda \right) \left( \int_{C} \overline{\eta} \right) \omega(1) \omega(1) - 2 \int_{C} \left( \int_{C} q \lambda \right) \omega(1) \left( \int_{C} \overline{\eta} \right) \omega(1) \right) \]
\[ = \int_{C} \left( \int_{C} q \lambda \right) \cdot (2\Omega_0 - 2gBI) \cdot \left( \int_{C} \overline{\eta} \right) \]
\[ = \int_{C} (\ell^\lambda + c^\lambda) \cdot (2\Omega_0 - 2gBI) \cdot (\overline{\eta} + \overline{\eta}'). \]
Since $\int_C \ell^\lambda B = 0$, we have
\[
\begin{aligned}
\ell^\lambda \cdot (\int_C 2\Omega_0 - 2gBJ) \cdot \overline{\nu} &= \ell^\lambda \cdot (2 - 2g)I \cdot \overline{\nu} = (2g - 2)\ell^\lambda \cdot \overline{\nu}, \\
\ell^\lambda \cdot (\int_C (2\Omega_0 - 2gBJ) \cdot \overline{\nu}) &= 2\ell^\lambda \cdot \int_C \omega(1) \wedge \omega(1) \cdot \overline{\nu} = (2 - 2g)\ell^\lambda \cdot \overline{\nu}, \\
(\int_C \ell^\lambda \cdot (2\Omega_0 - 2gBI)) \cdot \overline{\nu} &= (2 - 2g)\ell^\lambda \cdot \overline{\nu}.
\end{aligned}
\]
Hence we obtain
\[
E^F_1(\lambda, \overline{\nu}) = 2 \int_C (\ell^\lambda \cdot \omega(1)) (\omega(1) \cdot \overline{\nu}) + 2g \int_C (\ell^\lambda \cdot \overline{\nu}) B + (2 - 2g)\ell^\lambda \cdot \overline{\nu}.
\]
This completes the proof. \hfill \Box

In order to compare $E^F_1$ with the 2-form $e^I_1$ we prove

**Lemma 4.4.**
\[
E^F_1(\lambda, \overline{\nu}) = -2 \int_C \overline{\nu} \cdot (\ell^\lambda \omega(1) - \omega(1) \ell^\lambda) \cdot \omega''(1) + (2 - 2g)\ell^\lambda \cdot \overline{\nu}
\]

**Proof.**
\[
\begin{aligned}
\int_C (\ell^\lambda \cdot \omega(1)) (\omega(1) \cdot \overline{\nu}) + & g \int_C (\ell^\lambda \cdot \overline{\nu}) B \\
= & \int_C (\ell^\lambda \omega(1)) (\ell^\lambda \omega''(1)) - \int_C (\ell^\lambda \cdot \overline{\nu}) (\omega(1) \ell^\lambda \omega''(1)) \\
= & - \int_C \overline{\nu} \cdot (\ell^\lambda \omega(1) - \omega(1) \ell^\lambda) \cdot \omega''(1).
\end{aligned}
\]
\hfill \Box

5. The 2-form $e^I_1$.

In this section we compute $e^I_1(\lambda, \overline{\nu})$ for $\lambda, \mu \in C^\infty(C; TC \otimes \overline{TC})$. We begin by a review on the module $\Lambda^3 H$ and $Sp(H)$-invariant linear forms on $\Lambda^3 H \otimes \Lambda^3 H$. We regard $H$ as a submodule of $\Lambda^3 H$ through the injection $q^H : Z \in H \mapsto Z \wedge I = N(ZI) \in \Lambda^3 H$. If we define $p^H := \frac{1}{2g - 2}(m \otimes 1)|_{\Lambda^3 H} : \Lambda^3 H \to H$, we have $p^H q^H = 1_H$. Following [12] we write
\[
U := \text{Coker } q^H = \bigwedge^3 H / H.
\]
We denote the natural projection by $p^U : \bigwedge^3 H \to U$. The module $U$ is identified with $\text{Ker } p^U \subset \bigwedge^3 H$. We denote by $q^U : U \to \text{Ker } p^U \subset \bigwedge^3 H$ the natural injection. As was proved in [39, §8],
\[
\eta^U_I := q^U p^U \eta_I
\]
can be regarded as a 1-form on the moduli space $M_g$ with values in the vector bundle $\bigwedge^3 H$.

The map $\widehat{M} : H^{\otimes 6} \to \mathbb{C}$ in (3.3) satisfies
\[
\widehat{M}((Z_1 Z_2 Z_3)(Z_4 I)) = (Z_1 \cdot Z_4)(Z_2 \cdot Z_3)
\]
for any $Z_i \in H$. 
Lemma 5.1. For any \( z = Z_1 \wedge Z_2 \wedge Z_3 \) and \( w = W_1 \wedge W_2 \wedge W_3 \in \bigwedge^3 H \) we have
\[
\widehat{M}(q^U p^U z)(q^U p^U w) = (M_2 - \frac{3}{2g - 2} M_1)(zw).
\]

Proof. Denote \( Z^H := p^H z, W^H := p^H w, z^H := q^H Z^H \) and \( w^H := q^H W^H \). We have \( q^U p^U z = z - z^H \) and \( q^U p^U w = w - w^H \). It is clear that \( \widehat{M}(zw) = M_2(zw) \). By straightforward computation using (5.1) we obtain
\[
\widehat{M}(zw)^H = \widehat{M}(z^H w^H) = \frac{3}{2g - 2} M_1(zw)
\]
and
\[
\widehat{M}(z^H w^H) = 3\widehat{M}(N(Z^H I)W^H) = \frac{3}{2g - 2} M_1(zw).
\]

Hence
\[
\widehat{M}(q^U p^U z)(q^U p^U w) = \widehat{M}(z - z^H)(w - w^H) = M_2(zw) - \frac{3}{2g - 2} M_1(zw),
\]
as was to be shown. \( \square \)

By Lemma 5.1 and (3.11) we obtain
\[
e_1 = \frac{2g - 2}{2g + 1} \widehat{M}((\eta_1^U)^{\otimes 2}).
\]

Denote by \( Q_0 \) the \((1,0)\)-part of \( \eta_1^U \)
\[
Q_0 := N(\omega_1^I \omega_{(2)}^I + \omega_{(2)}^I \omega_1^I) + N(qI),
\]
which has values in \( \bigwedge^2 H \wedge H^\mu \). Since \( H^\nu \) and \( H^\mu \) are isotropic, \( e_1 \) is a \((1,1)\)-form. We have
\[
(5.2) \quad \frac{2g + 1}{2(2g - 2)} e_1^I(\lambda, \mu) = \widehat{M} \left( \left( \int_C Q_0 \lambda \right) \left( \int_C Q_0 \mu \right) \right)
\]
for any \( \lambda \) and \( \mu \in C^\infty(C; TC \otimes T^*C) \).

Lemma 5.2.
\[
\int_C Q_0 \lambda = N(L^\lambda + c^I I) \in \bigwedge^3 H \subset H^{\otimes 3}.
\]

Proof. By (5.3) and (3.7)
\[
Q_0 = 2N(\omega_1^I \omega_{(2)}^I - \omega_{(2)}^I \wedge \nu \Omega_{(0)} + \frac{1}{1 - g} \omega_{(1)}^I \cdot \nu_0 I)
\]
\[
= 2N(\omega_1^I \nu_0 + \frac{1}{1 - g} \omega_{(1)}^I \cdot \nu_0 I).
\]

Moreover
\[
2 \int_C \omega_{(1)}^I \nu_0 \lambda = -2 \int_C \omega_{(1)}^I \lambda \wedge d\Phi(\Omega_{(0)}) = \int_C \ell^\lambda \omega_{(1)}^I \wedge \omega_{(1)} = L^\lambda.
\]

Hence
\[
\int_C Q_0 \lambda = N(L^\lambda + \frac{1}{1 - g} (m \otimes 1)(L^\lambda I)) = N(L^\lambda + c^I I),
\]
as was to be shown. \( \square \)
For simplicity we write

\[ E'_1 := \frac{2g + 1}{6(2g - 2)} e'_1. \]

Then we have

**Lemma 5.3.**

\[ E'_1 (\lambda, \overline{\mu}) = \hat{M}((NL^\lambda)\overline{L^\mu}) + (2 - 2g)c^\lambda \cdot \overline{e^\mu}. \]

**Proof.** By (5.2) we have

\[ E'_1 (\lambda, \overline{\mu}) = \hat{M}((NL^\lambda)\overline{L^\mu}) + \hat{M}((NL^\lambda)\overline{\cal{C}^\mu}) - \hat{M}((NL^\mu)\overline{L^\lambda}) + \hat{M}((L^\nu)\overline{\cal{C}^\mu}). \]

The fourth term in the RHS is \((2g - 2)c^\lambda \cdot \overline{e^\mu}.\) From (5.1) the second term is

\[ \hat{M}(N(\int_C \ell^\lambda \omega_{(1)} \wedge \omega_C)) = (\int_C \ell^\lambda (\omega_{(1)} \cdot \omega_{(1)})) \cdot \overline{e^\mu} + 2 \int_C (\ell^\lambda \cdot \omega_{(1)})(\omega_{(1)} \cdot \overline{e^\mu}) \]

\[ = 2g(\int_C \ell^\lambda B) \cdot \overline{e^\mu} + 2((m \otimes 1)L^\lambda) \cdot \overline{e^\mu} = (2 - 2g)c^\lambda \cdot \overline{e^\mu}. \]

Similarly the third term is equal to \((2 - 2g)c^\lambda \cdot \overline{e^\mu}.\) This proves the lemma. \(\square\)

The amounts \(L^\lambda\) and \(c^\lambda\) depend only on \(\lambda\) and the surface \(C.\) This means \(\eta'_1\) and \(e'_1\) can be regarded as differential forms on the space \(M_g.\)

Moreover we obtain

**Proposition 5.4.**

\[ E'_1 (\lambda, \overline{\mu}) = -2 \int_C \overline{e^\mu} \cdot \cal{H}(\ell^\lambda \omega_{(1)} - \omega_{(1)}') \cdot \omega_{(1)}' + (2 - 2g)c^\lambda \cdot \overline{e^\mu}. \]

**Proof.** We have

\[ \hat{M}(L^\lambda \overline{L^\mu}) \]

\[ = \hat{M}(\int_C \ell^\lambda \omega_{(1)} \wedge \omega_{(1)'})(\int_C \overline{\ell^\mu} \omega_{(1)} \wedge \omega_{(1)}) + \hat{M}(\int_C \ell^\lambda \omega_{(1)}' \wedge \omega_{(1)}) - \hat{M}(M(\ell^\lambda \overline{L^\mu})) \]

\[ = 2\hat{M}(\int_C \ell^\lambda \omega_{(1)} \wedge \omega_{(1)'})(\int_C \overline{\ell^\mu} \omega_{(1)} \wedge \omega_{(1)}) \]

\[ = 2\hat{M}(\int_C \ell^\lambda \omega_{(1)} \wedge \omega_{(1)'})(\int_C \overline{\ell^\mu} \omega_{(1)} \wedge \omega_{(1)}') \]

\[ = -2\hat{M}(\int_C \cal{H}(\ell^\lambda \omega_{(1)}') \omega_{(1)'} \overline{\ell^\mu}). \]

Hence

\[ \hat{M}((NL^\lambda)\overline{L^\mu}) \]

\[ = \hat{M}(L^\lambda \overline{L^\mu}) - 2\hat{M}(\int_C \omega_{(1)}' \ell^\lambda \wedge \omega_{(1)}')(\int_C \overline{\ell^\mu} \omega_{(1)} \wedge \omega_{(1)}) \]

\[ = \hat{M}(L^\lambda \overline{L^\mu}) - 2M(\int_C \omega_{(1)}' \ell^\lambda \wedge \omega_{(1)}')(\int_C \omega_{(1)} \wedge \omega_{(1)}') \]

\[ = \hat{M}(L^\lambda \overline{L^\mu}) + 2M(\int_C \cal{H}(\ell^\lambda \omega_{(1)}) \omega_{(1)}' \overline{\ell^\mu}). \]

\[ = 2M(\int_C \cal{H}(\omega_{(1)}' \ell^\lambda - \ell^\lambda \omega_{(1)}) \omega_{(1)}' \overline{\ell^\mu}). \]
as was to be shown. □

Finally we compute the 2-form on \( M_g \)

\[
(5.3) \quad E^D_1 := E^F_1 - E^I_1 = \frac{g^2}{(2g - 2)^2} \epsilon^F_1 - \frac{2g^2 + 2g - 1}{3(2g - 2)^2} \epsilon^I_1
\]

representing \( \frac{1}{12} e_1 \).

**Lemma 5.5.**

\[
E^D_1(\lambda, \mu) = 4M \int_C \mathcal{H}(\omega'_1(\lambda) \wedge \omega'_1(\lambda) \hat{\Phi} d * (\omega''_1(\lambda) \overline{\nu} - \overline{\nu} \omega''_1(\lambda)).
\]

**Proof.** By \((1.6)\) we have

\[
(5.4) \quad \partial \ell^\lambda = 2\partial \hat{\Phi} d * (\omega'_1(\lambda) = \omega'_1(\lambda) - \mathcal{H}(\omega'_1(\lambda).
\]

Hence by \((1.8)\)

\[
(1 - \mathcal{H})(\ell^\lambda \omega'_1(\lambda) - \omega'_1(\ell^\lambda)
= -2\sqrt{-1} \partial \hat{\Phi}(\ell^\lambda \omega'_1(\lambda) - \ell^\lambda)
= -2\sqrt{-1} \partial \hat{\Phi}(\omega'_1(\lambda) - \mathcal{H}(\omega'_1(\lambda)) \omega'_1(\lambda) + \omega'_1(\omega'_1(\lambda) - \mathcal{H}(\omega'_1(\lambda)))
= 2\sqrt{-1} \partial \hat{\Phi}(\mathcal{H}(\omega'_1(\lambda) \omega'_1(1) + \omega'_1(\mathcal{H}(\omega'_1(\lambda)))
\]

From Lemmas \([4.4]\) and \([5.4]\) we have

\[
E^D_1(\lambda, \mu) = -2 \int_C \overline{\nu} \cdot (1 - \mathcal{H})(\ell^\lambda \omega'_1(\lambda) - \omega'_1(\ell^\lambda) \cdot \omega''_1(1)
= 4 \int_C \overline{\nu} \cdot \hat{\Phi}(\mathcal{H}(\omega'_1(\lambda) \omega'_1(1) + \omega'_1(\mathcal{H}(\omega'_1(\lambda)) \cdot \omega''_1)
= 4M \int_C \overline{\nu} \cdot \hat{\Phi}(\mathcal{H}(\omega'_1(\lambda) \omega'_1(1) + \omega'_1(\mathcal{H}(\omega'_1(\lambda)) \omega''_1(\nu)
= 4M \int_C \overline{\nu} \cdot \hat{\Phi}(\mathcal{H}(\omega'_1(\lambda) \omega'_1(1) \cdot \omega''_1(\nu) - \overline{\nu} \omega''_1(1)
= 4M \int_C \mathcal{H}(\omega'_1(\lambda) \omega'_1(1) \hat{\Phi} d * (\omega''_1(\nu) - \overline{\nu} \omega''_1(1).
\]

The last line follows from \((4.4)\).

By similar computation we have

\[
(5.5) \quad E^D_1(\lambda, \mu) = 4\sqrt{-1} M \int_C \mathcal{H}(\omega'_1(\lambda) \omega'_1(1) \hat{\Phi} (\mathcal{H}(\omega''_1(\lambda) \nu) \omega'_1(1) + \omega''_1(1) \mathcal{H}(\omega''_1(\nu))
\]

6. THE SECOND VARIATION OF THE FUNCTION \( a_g \).

This section is devoted to proving

**Theorem 6.1.**

\[
\frac{-2\sqrt{-1}}{2g(2g + 1)} \partial \hat{\Phi} a_g = \frac{1}{(2g - 2)^2} (\epsilon^F - \epsilon^I_1).
\]
In the setting of (2.7) we denote by ⋆ the antiholomorphic part of the variation ⋆. By Theorem 2.2 we have
\[(\partial\bar{\partial}a_\ell)(\lambda, \mu) = 2\sqrt{-1} \int_C \Xi \lambda\]
for any \(\lambda\) and \(\mu \in C^\infty(C; TC \otimes \bar{TC})\). From (2.10) the quadratic differential \(\Xi\) is given by
\[\Xi = M(\nu_0 \nu_0) + 4M \ast \bar{\Phi}(\nu_0 \wedge \nu''(1))\nu'(1)
= M(\nu_0 \nu_0) + 4M(\nu_1 \nu'(1)).\]
Here we write simply
\[\nu_1 = \ast \partial\bar{\Phi}(\nu_0 \wedge \nu''(1)).\]
From Lemma 6.2 it follows
\[(6.2) \quad \omega^0 = d\bar{\Phi}d \ast (2\omega''(1)\mu) = d\bar{\mu}^0.\]
Hence we have
\[(6.3) \quad \Omega^0 = \omega''(1) \wedge \omega(1) + \omega(1) \wedge \omega^0 = d(\bar{\mu}^0 \omega(1) - \omega(1)\bar{\mu}^0),\]
\[(6.4) \quad \bar{B} = \frac{1}{2g} m \bar{\Omega}_0 - \frac{1}{g} d(\bar{\mu} \cdot \omega(1)).\]
Let \(\Omega = \{\Omega^1\}, t \in \mathbb{R}, |t| \ll 1\), be a family of 2-forms with values in the algebra \(\hat{T}\). Assume \(A := \int_C \Omega \in \hat{T}\) is constant, and denote \(\nu := \ast \partial\bar{\Phi}(\Omega)\). Then we have
\[\text{Lemma 6.2.}\]
\[\bar{\nu} = \frac{1}{2} (\int_C \Omega \bar{\mu}^0 \cdot \omega(1) + \ast \partial\bar{\Phi} \bar{\Omega} - \frac{1}{g} A \ast \partial\bar{\Phi} d(\bar{\mu} \cdot \omega(1)).\]
\[\text{Proof.} \quad \text{Differentiating } (\ast + \sqrt{-1})\nu = 0, \text{ we get } 0 = \hat{\nu} + (\ast + \sqrt{-1})\nu = -2\sqrt{-1}\nu_1 + 2\sqrt{-1}(\nu''). \text{ Hence } (\nu'')'' = \nu_1, \text{ and so } \hat{\nu} + (\nu'')' = \hat{\nu} + 2 \ast \partial\bar{\Phi} \hat{\nu}. \]
Since \(\int_C \nu \wedge \omega(1) = 0\) and \(d\nu = \partial\nu = \frac{1}{2} - \frac{1}{4} AB\), we have
\[\hat{\nu} = - (\int C \nu \wedge \omega(1)) \cdot \omega(1) = (\int C \nu \wedge \omega(1) \cdot \omega(1))
= (\int C (d\nu \bar{\mu}^0) \cdot \omega(1)) = (\int C (d\nu \bar{\mu}^0) \cdot \omega(1)) = \frac{1}{2} (\int_C \Omega \bar{\mu}^0 \cdot \omega(1).\]
The last line follows from \(\int_C B\bar{\mu}^0 = 0\). Hence we obtain
\[\bar{\nu} = \hat{\nu} + 2 \ast \partial\bar{\Phi} \hat{\nu} = \hat{\nu} + \ast \partial\bar{\Phi} \hat{\nu} - A \hat{B}
= \frac{1}{2} \left( \int_C \Omega \bar{\mu}^0 \cdot \omega(1) + \ast \partial\bar{\Phi} \hat{\nu} - \frac{1}{g} A \ast \partial\bar{\Phi} d(\bar{\mu} \cdot \omega(1)) \right),\]
as was to be shown. \(\square\)

Differentiating \((\ast - \sqrt{-1})\omega(1) = -2\sqrt{-1}\omega''(1)\), we obtain from \(\omega(1) = 2\sqrt{-1}\omega''(1)\bar{\mu}\)
and (6.2)
\[-2\sqrt{-1}\omega''(1)) \wedge = \ast_1 \omega(1) + (\ast - \sqrt{-1})\omega_1\]
= \[2\sqrt{-1}\omega''(1)\bar{\mu} - 4\sqrt{-1}\partial\bar{\Phi}d \ast (\omega''(1)\bar{\mu}) = 2\sqrt{-1}\hat{H}(\omega''(1)\bar{\mu}),\]
so that
\[(6.5)\quad (\omega'_1)^\circ = -\mathcal{H}(\omega''_1 \overline{T}).\]

Hence
\[(6.6)\quad M \int_C \nu_1 (\omega'_1)^\circ \lambda = -M \int_C *\partial \Phi (\nu_1 \wedge \omega''_1) \mathcal{H}(\omega''_1 \overline{T}) \lambda = 0,\]
since $H'$ and $H''$ are isotropic. Applying Lemma 6.2 to $\nu_1$ we have
\[
4M \int_C (\nu_1 \omega'_1)^0 \lambda = 4M \int_C \overset{\circ}{\nu}_1 \omega'_1 \lambda
= 2M \left( \int_C \nu_0 \wedge \omega(1) \overline{T} \right) \cdot \left( \int_C \omega'_1 \omega'_1 \lambda \right)
+ 4M \int_C *\partial \Phi (\nu_0 \wedge \omega(1)^0) \omega'_1 \lambda
= 2M \left( \int_C \nu_0 \wedge \omega(1) \overline{T} \right) \cdot \left( \int_C \omega'_1 \omega'_1 \lambda \right) + 2M \int_C (\nu_0 \wedge \omega(1)) \omega'_1 \lambda
= 2M \left( \int_C \nu_0 \wedge \omega(1) \overline{T} \right) \cdot \left( \int_C \omega'_1 \omega'_1 \lambda \right) + 2M \int_C \nu_0 (d\overline{T}) \omega'_1 \lambda
+ 2M \int_C \overset{\circ}{\nu}_0 \omega'_1 \lambda
\]

Now we compute the third term $2M \int_C \overset{\circ}{\nu}_0 \omega(1) \omega'_1 \lambda$. From Lemma 6.2 applied to $\nu_0$ it follows
\[(6.8)\quad \overset{\circ}{\nu}_0 = \frac{1}{2} \left( \int_C \omega(1) \wedge \omega(1) \overline{T} \right) \cdot \omega(1) + *\partial \Phi d(\overline{T} \omega(1) - \omega(1) \overline{T}) - \frac{1}{g} I *\partial \Phi d(\overline{T} \cdot \omega(1)).\]

We have
\[
M \left( \int_C \omega(1) \wedge \omega(1) \overline{T} \right) \cdot \left( \int_C \omega(1) \wedge \omega(1) \overline{T} \right) = -M \left( L^p L^p \right)
= -\frac{1}{2} \widehat{M}(NL^p L^p) + \frac{1}{2} \widehat{M}^p L^p L^p.
\]

Since $H'$ and $H''$ are isotropic,
\[
2M \int_C *\partial \Phi d(\overline{T} \omega(1) - \omega(1) \overline{T}) \omega(1) \omega'_1 \lambda = 2M \int_C *\partial \Phi d(\overline{T} \omega'_1 \omega(1)) \omega''_1 \omega'_1 \lambda
= M \int_C \overline{T} \omega'_1 \omega''_1 \omega'_1 \lambda + M \left( \int_C \overline{T} \omega'_1 \omega(1) \right) \cdot \left( \int_C \omega'_1 \omega''_1 \lambda \right)
= g \int \omega'_1 \cdot \overline{T} B - \frac{1}{2} \widehat{M}(L^p L^p).
\]

On the other hand, from (4.2),
\[
(g - 1) \omega(1) \wedge \omega(1) \wedge \omega(1) \wedge (\omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1))
= -\int \mathcal{H} (\overline{T} \cdot \omega(1)) \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1)
= -\int (\overline{T} \cdot \omega(1)) (\omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1)) + 2 \int *\partial \Phi d(\overline{T} \cdot \omega(1)) \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1) \cdot \omega(1)
\]
Hence
\[-\frac{2}{g}M \int_C I \ast \hat{\Phi} d(\ell^\nu) \cdot \omega(1) \omega(1) = -\frac{2}{g} \int_C \ast \hat{\Phi} d(\ell^\nu) \cdot \omega(1) \omega(1) = \frac{(g-1)^2}{g} \lambda \cdot c^2 \nu + \frac{1}{g} \int_C (\ell^\lambda \cdot \omega(1)) (\omega(1) \cdot \ell^\nu) .\]

Consequently we obtain
\[2M \int_C \nu_0 \omega(1) = \frac{1}{2} M(NL^\nu) \ell^\nu + \frac{1}{g} \int_C \ell^\lambda \cdot \ell^\nu B = \frac{1}{g} \int_C (\ell^\lambda \cdot \omega(1)) (\omega(1) \cdot \ell^\nu) .\]

Next we compute \( M(\nu_0 \nu_0) \hat{\Phi} \). Here we remark \( M(I \nu_0) = M(\nu_0) \hat{\Phi}(\omega(1)) \omega(1) = 2g \ast \hat{\Phi} B = 0 \). From (6.8)
\[M(\nu_0 \nu_0) \hat{\Phi} = 2M(\nu_0 \nu_0) \hat{\Phi} .\]

The second term is equal to
\[4M(\ast \hat{\Phi} d(\ell^\nu \omega(1)) \nu_0) = 2M(\ell^\nu \omega(1) \nu_0) - 2M(\hat{\Phi}(\ell^\nu \omega(1)) \nu_0) .\]

Since \( \omega(1) \lambda = \hat{\Phi}(\omega(1)) \lambda + \ell^\lambda \), we have
\[2 \int_C \omega(1) \nu_0 \lambda = -2 \int_C \omega(1) \lambda \nu_0 = \int_C \ell^\lambda \omega(1) \lambda + \omega(1) = L^\lambda .\]

Hence the sum of the first and the fourth terms in (6.10) is
\[\frac{1}{2} M(L^\nu L^\lambda) + M(L^\nu \cdot L^\lambda) = -\frac{1}{2} M(NL^\nu) \ell^\nu .\]

The second term in (6.7) and the third in (6.10) are
\[2M \int_C \nu_0(d\ell^\nu) = 2M \int_C \nu_0 \ell^\nu B = 2M \int_C (\nu_0) \ell^\nu B .\]
The first in (6.7) and the second in (6.10) are

\[
2M \left( \int_C \nu_0 \omega'(1) \overline{\ell^B} \right) \cdot \left( \int_C \omega'(1) \omega'(1) \lambda \right) + 2M \int_C \nu_0 \overline{\ell^B} H'(\omega'(1) \lambda) = 2M \left( \int_C \nu_0 (\overline{\omega'(1) \overline{\ell^B}}) \right) \cdot \left( \int_C \omega'(1) \omega'(1) \lambda \right) = 2M \left( \int_C \omega'(1) \omega'(1) \lambda \right) \cdot \left( \int_C \omega'(1) \omega'(1) \lambda \right) + 2M \int_C H'(\omega'(1) \lambda) \omega'(1) \overline{\Phi d} * (\omega''(1) \overline{\ell^B} - \overline{\ell^B} \omega''(1)) = 2M \left( \int_C \omega'(1) \omega'(1) \lambda \right) \cdot \left( \int_C \omega'(1) \omega'(1) \lambda \right) + 1/2 \: E^D_1(\lambda, \overline{\mu}).
\]

Consequently we obtain

\[
\int_C \frac{g}{2} \lambda = -\hat{M}(NL^\lambda) \overline{\ell^B} + (g + 1) \int_C \ell^\lambda \cdot \overline{\ell^B} B + \frac{g + 1}{g} \int_C (\ell^\lambda \cdot \omega'(1))(\omega'(1) \cdot \overline{\ell^B}) + \frac{1}{2} E^D_1(\lambda, \overline{\mu}) + \frac{(g - 1)^2}{g} (\ell^\lambda \cdot \overline{\ell^B}) = -E^F_1(\lambda, \overline{\mu}) + \frac{2g + 1}{2g} E^F_1(\lambda, \overline{\mu}) + \frac{1}{2} E^D_1(\lambda, \overline{\mu}) = \frac{(2g + 1)2g}{4(2g - 2)} (e_1^F - e_1^J),
\]

which means

\[
-\frac{2\sqrt{-1}}{2g(2g + 1)} \partial \overline{\partial a_g} = \frac{1}{(2g - 2)^2} (e_1^F - e_1^J).
\]

This completes the proof of Theorem 6.1.

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