Stable monopole-antimonopole string background in \(SU(2)\) QCD

Y. M. Cho \(^{a,b}\) and D. G. Pak \(^{c}\)

\(^{a}\) C. N. Yang Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794, USA

\(^{b}\) School of Physics, College of Natural Sciences, Seoul National University, Seoul 151-742, Korea

\(^{c}\) Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea

Motivated by the instability of the Savvidy-Nielsen-Olesen (SNO) vacuum we make a systematic search for a stable magnetic background in pure \(SU(2)\) QCD. It is shown that a pair of axially symmetric monopole-antimonopole strings is stable, provided that the distance between the two strings is less than a critical value. As far as we understand, the pair of stable magnetic vortex-antivortex strings, and show that the first one becomes stable provided the distance between two strings is less than a critical value. The existence of a stable monopole-antimonopole string background strongly supports that a magnetic condensation of monopole-antimonopole pairs can generate a dynamical symmetry breaking, and thus the magnetic confinement of color in QCD.

PACS numbers: 12.38.-t, 11.15.-q, 12.38.Aw, 11.10.Lm

Keywords: stable magnetic background, magnetic confinement

1. Introduction

It has long been argued that the monopole condensation could explain the confinement of color through the dual Meissner effect \(^{\text{1}}\). Indeed, if one assumes the monopole condensation, one could easily argue that the ensuing dual Meissner effect guarantees the confinement \(^{\text{2}}\). There have been many attempts to prove this scenario in QCD \(^{\text{3,4}}\). Unfortunately, the earlier attempts had encountered the well known problem of Savvidy-Nielsen-Olesen (SNO) vacuum instability \(^{\text{5}}\). In fact the effective action of QCD obtained with the SNO vacuum develops an imaginary part, which implies that the vacuum is unstable \(^{\text{6}}\). This instability of the magnetic condensation has been widely accepted and never been convincingly revoked.

Recently it has been shown that, if one uses a proper infra-red regularization which respects causality, the imaginary part in the effective action can be removed \(^{\text{3}}\). In addition, a numerical evidence for the stable magnetic condensation has been found in lattice simulation \(^{\text{6}}\).

The purpose of this paper is to search for a stable classical magnetic background in \(SU(2)\) model of QCD. We analyze the stability of two classical magnetic backgrounds, a pair of axially symmetric monopole-antimonopole strings and a pair of magnetic vortex-antivortex strings, and show that the first one becomes stable provided the distance between two strings is less than a critical value. As far as we understand, the pair of axially symmetric monopole-antimonopole strings constitutes a first explicit example of a stable magnetic background in QCD. More importantly, the result can serve as a strong indication that a monopole-antimonopole condensation can provide a stable vacuum in QCD.

2. Instability of Wu-Yang monopole background

Let us start with a gauge invariant Abelian projection in \(SU(2)\) model of QCD which includes explicitly a topological degree of freedom expressed by the unit isotriplet \(\hat{n}\). We decompose the gauge potential into the restricted part (valence gluon) \(\hat{X}_\mu\) \(^{\text{2}}\),

\[
\hat{X}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} + \hat{X}_\mu = \hat{A}_\mu + \hat{X}_\mu,
\]

\[
(\hat{n}^2 = 1, \quad \hat{n} \cdot \hat{X}_\mu = 0),
\]

Notice that the restricted potential is precisely the connection which leaves \(\hat{n}\) invariant under the parallel transport,

\[
\hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g \hat{A}_\mu \times \hat{n} = 0.
\]

The restricted potential \(\hat{A}_\mu\) has a dual structure which can be seen from the field strength decomposition

\[
\hat{F}_{\mu\nu} = (F_{\mu\nu} + H_{\mu\nu}) \hat{n},
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

\[
H_{\mu\nu} = -\frac{1}{g} \hat{n} \times (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu \hat{C}_\nu - \partial_\nu \hat{C}_\mu,
\]

where \(\hat{C}_\mu\) is the “magnetic” potential \(^{\text{2}}\).

With the decomposition \(^{\text{11}}\) one has

\[
\hat{F}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{D}_\mu \hat{X}_\nu - \hat{D}_\nu \hat{X}_\mu + g \hat{X}_\mu \times \hat{X}_\nu,
\]

so that the Lagrangian can be written as follows

\[
\mathcal{L} = -\frac{1}{4} \hat{F}_{\mu\nu}^2 - \frac{1}{4} (\hat{D}_\mu \hat{X}_\nu - \hat{D}_\nu \hat{X}_\mu)^2 - \frac{g}{2} \hat{F}_{\mu\nu} \cdot (\hat{X}_\mu \times \hat{X}_\nu) - \frac{g^2}{4} (\hat{X}_\mu \times \hat{X}_\nu)^2.
\]
With the gauge fixing condition \( \hat{D}_\mu \tilde{X}^\mu = 0 \) the one-loop correction to the effective action reduces to the form

\[
\exp(i\Delta S) = \text{Det}^{-1} K_{\mu\nu} \text{Det}^2 M_{FP},
\]

\[
K_{\mu\nu} = g_{\mu\nu} \hat{D}\hat{D} + 2i(F_{\mu\nu} + H_{\mu\nu}),
\]

\[
M_{FP} = \bar{D}\hat{D}, \quad \bar{D}_\mu = \partial_\mu + ig(A_\mu + \tilde{C}_\mu),
\]

(6)

where the operators \( K_{\mu\nu} \) and \( M_{FP} \) originate from the functional integration over the off-diagonal gluon and Faddeev-Popov ghost respectively (the contribution from integration over the quantum part of the Abelian field \( A_\mu \) is trivial). For arbitrary static magnetic background \( F_{\mu\nu} + H_{\mu\nu} \) one can simplify the one-loop correction \( \tilde{D}^2 \).

\[
\Delta S = i \ln \text{Det}(\bar{D}^2 + 2a) + i \ln \text{Det}(\bar{D}^2 - 2a), (7)
\]

where \( a = g\sqrt{\frac{1}{2} H^2_{\mu\nu}} \), hereafter, for the brevity of notation, we employ a single notation \( H_{\mu\nu} \) for the additive combination \( F_{\mu\nu} + H_{\mu\nu} \).

Before we discuss the stability of monopole-antimonopole pair, we first review the instability of the Wu-Yang monopole because two problems are closely related \( \tilde{D} \). The Wu-Yang monopole solution of charge \( q/g \) is described by \( \tilde{E} \) and \( \tilde{C} \). The Wu-Yang monopole solution of charge \( q/g \) is described by \( \tilde{E} \) and \( \tilde{C} \).

\[
\tilde{A}_\mu = \frac{1}{g} \hat{n} \times \partial_\mu \hat{n},
\]

\[
\hat{n} = \left( \frac{\sin \theta \cos q\phi}{\sin \theta \sin q\phi}, \frac{\cos q\phi}{\cos \theta} \right),
\]

(8)

where \( (r, \theta, \phi) \) is the spherical coordinates and \( q \) is an integer monopole charge. In the Abelian formalism it is more convenient to describe the monopole in terms of the magnetic potential \( \tilde{C}_\mu \). This implies the components of the magnetic potential \( \tilde{C}_\mu \) and the magnetic field strength \( \tilde{H} \) to be as follows

\[
\tilde{C}_\mu = \frac{q}{gr}(\cos \theta - 1)\partial_\mu \phi,
\]

\[
\tilde{H}_{ij} = \frac{q}{g} \epsilon_{ij3} x^3,
\]

(9)

We use the parametrization for the magnetic potential \( \tilde{C} \) slightly different from the parametrization in \( \tilde{E} \) where the magnetic potential is defined on two coordinate patches.

To study the stability of the monopole background we should consider an operator obtained by taking the second variation of the classical Lagrangian with respect to small fluctuations of the field \( \tilde{X}_\mu \). The operator is identical to the operator \( K_{\mu\nu} \). \( \tilde{L} \) and the problem of finding unstable modes is reduced to calculation of the scalar functional determinants in \( \tilde{C} \)

\[
\text{Det} K = \text{Det}(-\bar{D}^2 + 2a), (10)
\]

here, \( a = \frac{q}{g^2} \) is given by the magnetic field strength of the Wu-Yang monopole. The absence or presence of negative modes of the operator \( K \) implies stability or instability of the classical background against small fluctuations of the gauge potential. To calculate the eigenvalues of the operator \( K \) one can write down the eigenvalue equation as the following Schrödinger type equation for a complex scalar field \( \Psi \) which plays a role of wave function

\[
K \Psi(r, \theta, \phi) = E \Psi(r, \theta, \phi),
\]

\[
K = -\Delta - \frac{iq}{r^2 \sin^2 \theta} \cos \theta \partial_\phi + \frac{q^2}{r^2} \cot^2 \theta \pm 2 \frac{q}{r^2},
\]

(11)

where \( \hat{L} \) is the angular momentum operator. Notice that here the \( \pm \) signatures represent two spin orientations of the valence gluon.

Substituting the separated solution

\[
\Psi(r, \theta, \phi) = R(r)Y(\theta, \phi),
\]

(12)

into \( \tilde{C} \) one obtains the equation for the angular eigenfunction \( Y(\theta, \phi) \)

\[
\left( \hat{L}^2 - \frac{2i q \cos \theta}{\sin^2 \theta} \partial_\phi + \frac{q^2 \cot^2 \theta}{\sin^2 \theta} \right) Y(\theta, \phi) = \lambda Y(\theta, \phi).
\]

(13)

Moreover, with

\[
Y(\theta, \phi) = \sum_{m=\pm\infty}^{+\infty} \Theta_m(\theta) \Phi_m(\phi),
\]

\[
\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp(im\phi).
\]

(14)

one can reduce \( \tilde{C} \) to

\[
\left( -\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\phi) + \frac{(m+q \cos \theta)^2}{\sin^2 \theta} \right) \Theta = \lambda \Theta.
\]

(15)

This is exactly the eigenvalue equation for the monopole harmonics which has been well-studied in the literature \( \tilde{E} \). From the equation one obtains the following expression for the monopole harmonics and the corresponding eigenvalue spectrum

\[
Y_{qjm}(\theta, \phi) = \Theta_{qjm}(\theta) \Phi_m(\phi),
\]

\[
\Theta_{qjm}(\theta) = (1 - \cos \theta)^{\gamma_+} (1 + \cos \theta)^{\gamma_-} P_k(\cos \theta),
\]

\[
\gamma_{\pm} = \frac{|m \pm q|}{2},
\]

\[
\lambda = j(j+1) - q^2,
\]

\[
j = k + \gamma_+ + \gamma_-, \quad k = 0, 1, 2, \ldots
\]

(16)

where \( P_k(x) \) is the Legendre polynomial of order \( k \). The quantum number \( j \) is analogous to the orbital angular
momentum quantum number \( l \) of the standard spherical harmonics \( Y_{lm} \), except that here \( j \) starts from a non-zero integer value for a non-vanishing monopole charge \( q \).

Now, consider the equation for the radial eigenfunction

\[
\left( \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) - \frac{1}{r^2} \left[j(j+1) - q^2 + \frac{2q}{r^2} + E \right] \right) R(r) = 0.
\]

(17)

With \( R(r) = \frac{1}{r} \chi(r) \) one obtains

\[
\left( \frac{d^2}{dr^2} - \frac{1}{r^2} \left[j(j+1) - q^2 \pm 2q \right] + E \right) \chi(r) = 0.
\]

(18)

The equation (18) is nothing but the one-dimensional Schrödinger equation with the attractive potential \(-1/r^2\). The solution to this equation has a continuous eigenvalue spectrum for both positive and negative energies, and leads to the radial eigenfunction \( R(r) \) which behaves like

\[
R(r) \sim \sin \log \left( \sqrt{E/|r|} + \text{const} \right) \sqrt{r}
\]

(21)

near the origin. The solution has an infinite number of zeros approaching the point \( r = 0 \). This can be interpreted as a valence gluon moving around the monopole and falling down to the center.

One can observe that for the lowest energy states, \((j = 1)\), the undesired attractive potential proportional to \(-1/r^2\) in \(18\) vanishes when \( q = 0 \). This can serve as a hint that one might expect the absence of negative modes for a magnetic background with zero monopole charge. The simplest magnetic configuration with a total vanishing monopole charge can be realized as a Wu-Yang monopole-antimonopole pair. Unfortunately, since near the location of the (anti-)monopole we still have the attractive potential part \(-1/r^2\) one can verify that the Wu-Yang monopole-antimonopole pair has to be unstable.

3. Axially symmetric monopole string

The axially symmetric monopole string can be regarded as an infinite string carrying homogeneous monopole charge density along the string. The magnetic field strength of the axially symmetric monopole string can be written in a simple form in the cylindrical coordinates \((\rho, \phi, z)\)

\[
\hat{C}_\mu = 0, \quad A_\mu = -\alpha(z + \tau) \partial_\mu \phi, \quad \hat{H} = \frac{\alpha}{\rho} \hat{\rho},
\]

(22)

where \( \alpha \) is the monopole charge density and \( \tau \) is an arbitrary constant which represents the translational invariance of the magnetic field along the \( z \)-axis. Just like the monopole solution the above monopole string forms a singular classical solution of SU(2) QCD. In the following we will assume \( \alpha = 1 \) and \( \tau = 0 \) for simplicity, since this will not affect the stability analysis.

Let us consider the eigenvalue problem for the operator \( K \)

\[
K \Psi(\rho, \phi, z) = E \Psi(\rho, \phi, z).
\]

(23)

With

\[
\Psi = \sum_{m=\infty}^{\infty} F_m(\rho, z) \Phi_m(\phi),
\]

(24)

and repeating the steps of the previous section we obtain the following eigenvalue equation

\[
F_{\rho \rho} + \frac{1}{\rho} F_\rho + F_{zz} - \left[ \frac{(m+\pm 2)}{\rho^2} + \frac{2}{\rho} - E \right] F = 0.
\]

(25)

By shifting \( z \) to \( z + m \) one can put \( m = 0 \). The quantum mechanical potential of this equation behaves like \( \pm 2/\rho \) near \( \rho = 0 \). So we still have an undesired attractive potential \(-2/\rho\). This implies two things. First, the attractive interaction of the axially symmetric monopole string background is less severe than the attractive interaction of the spherically symmetric monopole background. So we can expect the absence of continuous negative energy spectrum for the axially symmetric monopole string background. Secondly, the attractive potential \(-2/\rho\) tells that the monopole string background must still be unstable, because it indicates the existence of discrete bound states with negative energy.

To confirm this we make a qualitative estimate of the negative energy eigenvalues of \(25\). We look for a solution which has the form

\[
F(\rho, z) = \sum_{n=0}^{\infty} f_n(\rho) Z_n(x), \quad Z_n(x) = \exp(\frac{x^2}{2}) H_n(x), \quad x = \frac{z}{\sqrt{\rho}},
\]

(26)
where $H_n(x)$ is the Hermite polynomial. Notice that $Z_n(x)$ forms a complete set of eigenfunctions of the harmonic oscillator,

$$\left(\frac{d^2}{dx^2} - x^2\right)Z_n(x) = -(2n + 1)Z_n(x).$$

Substituting (26) into (25) we obtain

$$\sum_{n=0}^{\infty} \left\{ \left( \frac{d^2 f_n}{d\rho^2} + \frac{1}{\rho} \frac{d f_n}{d\rho} + \frac{z^2}{\rho^2} \frac{df_n}{d\rho} \right) H_n - \frac{z}{\rho} \cdot \frac{df_n}{d\rho} + \frac{d H_n}{dx} \right\}$$

$$+ f_n \left[ \frac{2}{4\rho^4} - \frac{z}{4\rho^2} \left( 1 - \frac{z^2}{\rho^2} \frac{d H_n}{dx} \right) \right]$$

$$+ \left( \frac{z^4}{4\rho^4} - \frac{z^3}{4\rho^3} \frac{d}{d\rho} + \frac{z^2}{\rho^3} - \frac{2n + 1 \pm 2}{\rho} + E \right) f_n H_n \right\} = 0.$$ (28)

Using the recurrence relations and orthogonality properties of Hermite polynomials one can derive the equations for $f_n(\rho)$

$$\left( \frac{d^2}{d\rho^2} + \frac{2n + 3}{\rho} \frac{d}{d\rho} + \frac{4n^2 - 2n - 1}{16\rho^2} \right.$$  

$$- \frac{2n + 1 \pm 2}{\rho} + E \right) f_n$$

$$= \frac{-1}{64\rho^2} f_n - \frac{1}{32\rho^2} f_n - \frac{1}{4\rho} \left( \frac{d}{d\rho} + \frac{n - 1}{4\rho} \right) f_{n-2}$$

$$+ \frac{3}{16\rho^2} f_{n-1} + \frac{3(n + 1)^2}{8\rho^2} f_{n+1}$$

$$+ \frac{(n + 1)(n + 2)}{\rho} \left( \frac{d}{d\rho} - \frac{3n + 2}{4\rho} \right) f_{n+2}$$

$$+ \frac{(n + 1)(n + 2)(n + 3)}{4\rho^2} \left( f_{n+3} - 3(n + 4) f_{n+4} \right).$$ (29)

where $f_n = 0$ for negative integer $n$. So we have infinite number of equations for infinite number of unknown functions $f_n(\rho)$.

Notice that the left hand side of the last equation contains a second order differential operator with the quantum mechanical potential

$$U = \frac{2n + 1 \pm 2}{\rho}.$$ (30)

The potential becomes attractive only if $n = 0$, so that in a first approximation we expect that the negative energy eigenvalues will originate mainly from the lowest bound state with $n = 0$ of the harmonic oscillator part. So, by neglecting all $f_n$ with $n \neq 0$ we can still get an approximate qualitative solution for $f_0$. In such an approximation the equation reduces to a simple one

$$\left( \frac{d^2}{d\rho^2} + \frac{3}{2\rho} \frac{d}{d\rho} + \frac{1}{\rho} - \frac{1}{16\rho^2} \right. + E \left. \right) f_0 = 0.$$ (31)

The solution to this equation has a new integer quantum number $k$,

$$f_{0,k}(\rho) = \rho^s \exp\left( -\sqrt{|E_k|} \rho \right) \sum_{l=0}^{l=k} a_l \rho^l,$$

$$s = \sqrt{2} - 1, $$

$$E_k = \frac{1}{(2k + 2s + 3/2)^2},$$

$$a_{l+1} = \frac{\sqrt{E_k}(2l + 2s + 3/2) - 1}{(l + 1)(l + 2s + 3/2)} a_l.$$ (32)

With this we may express the corresponding eigenfunction $\Psi_k$ as

$$\Psi_k(\rho, \phi, z) = N_k \exp\left( -\frac{z^2}{2\rho} \right) f_{0,k}(\rho),$$ (33)

where $N_k$ is a normalization constant. One can find the lowest energy eigenvalues

$$E_0 = -0.343..., $$

$$E_1 = -0.073..., $$

$$E_2 = -0.031..., $$

$$E_3 = -0.017..., $$

$$E_4 = -0.011....$$ (34)

This confirms that the axially symmetric monopole string background is indeed unstable.

Surprisingly, we find that the approximate solution (33) can also be obtained as an exact solution of variational method with the trial function $\tilde{F}$ of the form

$$\tilde{F}(\rho, z) = N \rho^s \exp\left( -\beta_k \rho - \frac{\gamma z^2}{2\rho} \right) \sum_{l=0}^{l=k} a_l \rho^l,$$ (35)

where $s, \beta_k, \gamma, a_l$ are treated as variational parameters. In other words, the variational minimum of the energy functional with the above trial function is provided exactly by the solution (33).

To quantify the accuracy of our approximate analytic solution we solve numerically the starting equation (24) (with the lower negative sign and $m = 0$). The obtained numerical solution for $\Psi$ have the same essential singularity structure as in (33). The corresponding lowest energy eigenvalues

$$E_0 = -0.545..., $$

$$E_1 = -0.093..., $$

$$E_2 = -0.036..., $$

$$E_3 = -0.019..., $$

$$E_4 = -0.011....$$ (36)

can confirm that the solution (33) provides a good qualitative estimation of the energy spectrum to analyse the vacuum
stability of the axially symmetric monopole string. What is more important is that we can apply the structure of that solution to the analysis of the stability problem for the monopole-antimonopole string configuration in the subsequent section.

4. A stable magnetic background

The main idea how to construct a stable magnetic background is quite clear. Consider a pair of axially symmetric monopole and anti-monopole strings which are orthogonal to the xy-plane and separated by a distance $a$. Due to the opposite directions of the magnetic fields of the monopole and anti-monopole strings the attractive part of the quantum mechanical potential $U(\rho)$ in the eigenvalue equation falls down as $U(\rho) \to O(-a/\rho^2)$ when $\rho \to \infty$. This allows the centrifugal potential to be competitive with the attractive part and prevail for small enough values of $a$. By decreasing the distance $a$ we can decrease the effective size of the quantum mechanical potential well, so that the bound state energy levels will be pushed out from the well. This, with the positive asymptotics of the potential at infinity, implies that the bound states will have disappeared completely at some finite critical value of $a$.

To show this, consider a pair of axially symmetric monopole and anti-monopole strings located at $(\rho = a/2, \phi = 0)$ and $(\rho = a/2, \phi = \pi)$ in cylindrical coordinates. The magnetic field strengths $\vec{H}_\pm$ for the monopole and anti-monopole strings are defined as follows

$$\vec{H}_\pm = \pm \frac{\alpha}{\rho} \hat{\rho}_\pm,$$

$$\rho_\pm = \rho \pm \frac{a}{2},$$

$$\rho_\pm^2 = \rho^2 \pm a \cos \phi + \frac{a^2}{4},$$

where $\hat{\rho}$ is the two-dimensional vector starting from the anti-monopole string to the monopole string in xy-plane. From now on we will assume $\alpha = 1$ without loss of generality. The total magnetic field is given by

$$\vec{H} = \vec{H}_+ + \vec{H}_-,$$

$$H_\rho = \frac{\rho - a \cos \phi}{2\rho_+^2} - \frac{\rho + a \cos \phi}{2\rho_-^2},$$

$$H_\phi = \frac{\alpha \rho (r_+^2 + \frac{a^2}{4}) \sin \phi}{\rho_+^2 \rho_-^2}, \quad H_z = 0,$$

$$H = \sqrt{H_\rho^2 + \frac{1}{\rho^2} H_\phi^2} = \frac{a}{\rho_+ \rho_-}.$$  

One can express the corresponding energy functional in terms of $H_\rho, H_\phi$ components

$$A_\mu = z H_\phi \partial_\mu \rho - \rho z H_\rho \partial_\mu \phi.$$  

The eigenvalue equation for the operator $K$ takes the form

$$\left\{-\frac{1}{\rho} \partial_\rho (\rho \partial_\rho) - \frac{1}{\rho^2} \partial_\phi^2 - \partial_\rho^2 - 2i \frac{z}{\rho} (H_\phi \partial_\rho - H_\rho \partial_\phi) + z^2 H^2 \pm 2H \right\} \Psi(\rho, \phi, z) = E \Psi(\rho, \phi, z).$$  

(40)

The equation can be interpreted as a Schroedinger equation for a massless gluon in the magnetic field of monopole and anti-monopole string pair.

Let us analyse the equation qualitatively first. We will concentrate on the potentially dangerous term $-2H$ in (40). The singularities of the term $z^2 H^2$ determine the essential singularities of the differential equation. One can extract the leading factor of the solution and look for a finite solution for the ground state in the form similar to (33)

$$\Psi(\rho, \phi, z) = (\pi \rho_+ \rho_-)^{-\frac{3}{4}} \exp \left(-\frac{z^2}{2 \rho_+ \rho_-} F(\rho, \phi) \right),$$  

(41)

where $F(\rho, \phi)$ is normalized by

$$\int |F(\rho, \phi)|^2 \rho d\rho d\phi = 1.$$  

(42)

The solution describes a wave function localized mainly near the string pair. The wave function vanishes exactly on the axes of the strings. This implies that the ground state has a non-zero orbital angular momentum which provides a centrifugal potential as we will see later.

The lowest negative eigenvalue of this equation can be obtained by variational method by minimizing the corresponding energy functional

$$E = \int \rho \Psi^* \left[-\frac{1}{\rho} \partial_\rho (\rho \partial_\rho) - \frac{1}{\rho^2} \partial_\phi^2 - \partial_\rho^2 - 2i \frac{z}{\rho} (H_\phi \partial_\rho - H_\rho \partial_\phi) + z^2 H^2 \pm 2H \right] \Psi d\rho dz d\phi.$$  

(43)

Now, with

$$F(\rho, \phi) = \sum_{-\infty}^{\infty} f_m(\rho) \Phi_m(\phi),$$  

(44)

one may suppose that the main contribution to the ground state energy comes from the first term of Fourier expansion with $m = 0$. With this we can perform the integration over $z$-coordinate and simplify the above expression to

$$E = \int \rho f(\rho) \left(-\partial_{\rho \rho} - \frac{1}{\rho} \partial_\rho + U(\rho, \phi) \right) f(\rho) d\rho d\phi,$$

$$U(\rho, \phi) = \rho \rho_+ \rho_- - \frac{2a \rho_+ \rho_-}{2 \rho_+^2 \rho_-^2},$$  

(45)

where $f(\rho) = f_0(\rho)$ and $U(\rho, \phi)$ is an effective potential. Since the energy eigenvalues decrease with decreasing the
parameter \( a \), to study the features of the potential at small \( a \) we make the following rescaling

\[
\rho \to a\rho, \quad f \to f/a, \quad E \to E/a^2.\]  

Under this rescaling the potential near the origin can be approximated to

\[
U(\rho, \phi) \to -4a + (8 - 16a \cos 2\phi)\rho^2.\]  

So that the potential reduces to a two-dimensional harmonic oscillator potential whose depth decreases as \( a \) goes to zero. This implies that the negative energy eigenvalues can disappear for some small values of \( a \) if the asymptotics of the potential at infinity becomes positive. To see this we perform the integration over the angle variable \( \phi \) in the energy functional, and with the change of variable

\[
f(\rho) = \chi(\rho)/\sqrt{\rho},\]  

we obtain the following equation which minimizes the energy,

\[
\left[-\frac{d^2}{d\rho^2} + V(\rho)\right]\chi(\rho) = E\chi(\rho),
\]

\[
V(\rho) = -\frac{1}{4\rho^2} + \frac{8\rho^2}{\sqrt{(a^4 - 16\rho^2)^2}}
- \frac{8a}{\pi \sqrt{(a^2 - 4\rho^2)^2}} K\left(-\frac{16 a^2 \rho^2}{(a^2 - 4\rho^2)^2}\right),
\]

where \( K(x) \) is the complete elliptic integral of the first kind. Notice that the first term in the potential does not produce bound states because the potential \(-\frac{\kappa}{\rho^2}\) leads to negative energy eigenvalues only for \( \kappa > 1/4 \). From the last equation we obtain the asymptotic behavior of the potential \( V(\rho) \) near space infinity

\[
V(\rho) \simeq \left(\frac{1}{4} - a\right) \frac{1}{\rho^2}.
\]

This tells that the potential becomes positive when the distance \( a \) becomes less than the critical value \( a_{cr} \) (in the unit 1/\( a \))

\[
a < a_{cr} = \frac{1}{4}.
\]

To check the analytic estimate of the critical value \( a_{cr} \) we solve numerically the original equation \( (41) \). Since the wave function \( \Psi(\rho, \phi, z) \) becomes spread to long distance for small energy eigenvalues we extend sufficiently the upper limit of \( \rho \) in the domain \((0 < \rho < \rho_{upper}, 0 < \phi < 2\pi)\) and increase the mesh near the location of the monopole and antimonopole strings. The extrapolation of the results from finite small energy eigenvalues to zero gives the following critical value within 6% of accuracy

\[
a_{cr} \simeq 0.246.
\]

The numerical result confirms that qualitatively the approximate solution \( (11) \) describes the correct physical picture. In particular, this tells that a pair of monopole and antimonopole strings becomes a stable magnetic background if the distance between two strings is small enough.

### 5. On stability of vortex-antivortex pair

Recently an alternative mechanism of confinement has been proposed which advocates the condensation of magnetic vortices \([14]\). However, it has been known that the magnetic vortex configuration is unstable \([15]\). So it would be interesting to study the stability of the vortex-antivortex pair. In this section we study the stability of a special vortex-antivortex configuration.

Let us start with a single magnetic vortex given by

\[
\tilde{H} = \frac{1}{\rho} \frac{\partial}{\partial \rho}, \quad \mu = \rho \partial_{\mu} \phi.
\]

Notice that, unlike the monopole string \( (22) \), the vortex configuration is not a classical solution of the system. But since such a type of configuration multiplied by an appropriate profile function has been studied in the framework of various approaches, we will consider that special vortex configuration in the following.

One can write down the corresponding eigenvalue equation of the operator \( \tilde{K} \)

\[
\left[-\frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} - \frac{2}{\rho} \frac{\partial}{\partial \phi} \pm 2\tilde{H}\right] \Psi(\rho, \phi, z) = E \Psi(\rho, \phi, z).
\]

The equation becomes separable in all three variables. With factorization

\[
\Psi = \sum_{-\infty}^{+\infty} f_m(\rho) g(z) \varphi_m(\phi), \quad g(z) = 1,
\]

one obtains the following ordinary differential equation for \( f(\rho) \) from \( (52) \),

\[
\left(-\frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} + (1 + \frac{m}{\rho})^2 \pm \frac{2}{\rho} - E\right) f(\rho) = 0.
\]

The bound states are possible for the potential \(-2/\rho \) with non-positive integer \( m \), in which case the corresponding solution can be obtained

\[
f_{n,m}(\rho) = \rho^{[m]} e^{-\sqrt{1-E_{n,m}} \cdot u_{n,m}(\rho)},
\]

\[
u_{n,m}(\rho) = \sum_{k=0}^{n} \beta_{k}^{n,m} \rho^{k},
\]

\[
a_{k+1}^{n,m} = \sqrt{1-E_{n,m}(2k+2|m|+1) - 2 + 2m} a_{k}^{n,m},
\]

\[
E_{n,m} = 1 - \frac{4(1-m)^2}{(2n+2|m|+1)^2},
\]

\[
n = 0, 1, 2, \ldots; \quad m = 0, -1, -2, \ldots
\]

The magnetic vortex configuration is unstable \([15]\). So it would be interesting to study the stability of the vortex-antivortex pair. In this section we study the stability of a special vortex-antivortex configuration.
Clearly, the ground state has a negative energy $E_{0,0}$, which tells that the vortex configuration is unstable.

There is a principal difference between the axially symmetric monopole string and the vortex configuration. The ground state of the monopole string has a non-trivial centrifugal potential but the ground state eigenfunction $f_{0,0}(\rho)$ of the vortex configuration corresponds to an $S$-state, which implies the absence of the centrifugal potential. This plays a crucial role in the existence of the negative energy eigenstates in the case of vortex-antivortex pair.

The vortex-antivortex background is described in a similar manner as the monopole-antimonopole string pair. The eigenvalue equation corresponding to the operator $\mu$ in the form

$$A_\mu = \frac{a}{2} \sin(\frac{1}{\rho^2} + \frac{1}{\rho^4}) \partial_{\mu} \rho + \left[ \frac{\rho}{\rho^2} \left( \cos(\phi) - \frac{\rho}{\rho^2} \sin(\phi) \right) \right] \partial_{\phi} \rho, \quad \rho \equiv r,$$

$$\bar{H} = \left( \frac{1}{\rho^2} + \frac{1}{\rho^4} \right) \frac{\partial}{\partial \rho^2} + \frac{\partial}{\partial \phi} (A_\mu \partial_{\mu} + \frac{A_\phi}{\rho^2} \partial_{\phi}) + A_\mu^2 \pm 2 \bar{H} \right] F(\rho, \phi, z) = EF(\rho, \phi, z).$$

The numerical analysis of the equation shows that there is no critical value for the parameter $a$, so that the negative energy eigenvalues exist for any small $a$. Qualitatively one can see this from the effective potential $V = A_\mu^2 - 2H$. After averaging over the angle variable one can find the asymptotic expansion of the potential near the origin and infinity

$$V(\rho) \approx \begin{cases} 8\pi - \frac{64}{a^2} \rho - \frac{16\pi}{a^2} \rho^2 & (\rho \equiv 0), \\ - \left( 4 - \frac{\pi}{a} \right) \frac{2a}{\rho^2} & (\rho \equiv \infty). \end{cases}$$

This shows that there is no centrifugal term which could prevent the appearance of bound states for small $a$. Whether the instability problem can be overcome with a more complicating configuration of the vortex-antivortex remains an open question.

In conclusion, we have shown that the axially symmetric monopole-antimonopole string background is stable under the small field fluctuation if the distance between two strings becomes less than the critical value $a_{cr} \approx 1/4$. The existence of the stable classical magnetic background implies that “a spaghetti of monopole-antimonopole string pairs” could generate a stable vacuum condensation. This would allow a magnetic confinement of color in QCD.

Acknowledgements

One of the authors (YMC) thanks G. Sterman for the kind hospitality during his visit to YITP. Another author (DGP) thanks K.-I. Kondo and N. I. Kochelev for useful discussions. The work is supported in part by the ABRL Program of KOSEF (R14-2003-012-01002-0) and by the BK21 Project of the Ministry of Education.

[1] Y. Nambu, Phys. Rev. D10 (1974) 4262; S. Mandelstam, Phys. Rep. 23C (1976) 245; A. Polyakov, Nucl. Phys. B120 (1977) 429; G. ’t Hooft, Nucl. Phys. B190 (1981) 455.
[2] Y. M. Cho, Phys. Rev. D21 (1980) 1080; Phys. Rev. Lett. 46 (1981) 302; Phys. Rev. D23 (1981) 2415; Z. Ezawa and A. Iwashiki, Phys. Rev. D25 (1982) 2681.
[3] G. K. Savvidy, Phys. Lett. B71 (1977) 133; N. K. Nielsen and P. Olesen, Nucl. Phys. B144 (1978) 376;
[4] A. Yildiz and P. Cox, Phys. Rev. D21 (1980) 1095; M. Claudson, A. Yildiz, and P. Cox, Phys. Rev. D22 (1980) 2022; W. Dittrich and M. Reuter, Phys. Lett. B128 (1983) 321; M. Reuter, M. G. Schmidt, and C. Schubert, Ann. Phys. 259 (1997) 313; C. Floey, Phys. Rev. D28 (1983) 1425; S. K. Blau, M. Visser, and A. Wipf, Int. J. Mod. Phys. A6 (1991) 5409; V. Schanbacher, Phys. Rev. D26 (1982) 489.
[5] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B156 (1979) 1; N. K. Nielsen and P. Olesen, Nucl. Phys. B160 (1979) 380.
[6] H. Leutwyler, Nucl. Phys. B179 (1981) 129; Phys. Lett. B96 (1980) 154; C. Rajiakos, Phys. Lett. B100 (1981) 471.
[7] Y. M. Cho, H. W. Lee, and D. G. Pak, Phys. Lett. B 525 (2002) 347; Y. M. Cho and D. G. Pak, Phys. Rev. D65 (2002) 074027; Y. M. Cho and M. L. Walker, Mod. Phys. Lett. A19, 2707 (2004); K.-I. Kondo, Phys.Lett. B600 (2004) 287.
[8] S. Kato, K.-I. Kondo, T. Murakami, A. Shibata, T. Shinohara, hep-ph/0504054.
[9] R. A. Brandt and F. Neri, Nucl. Phys. B 161 (1979) 253.
[10] T. T. Wu and C. N. Yang, Phys. Rev. D12 (1975) 3845.
[11] Y. M. Cho, Phys. Rev. Lett. 44 (1980) 1115; Phys. Lett. B115 (1982) 125.
[12] T. T. Wu and C. N. Yang, Nucl. Phys. B 107 (1976) 365; Phys. Rev. D 16 (1977) 1018.
[13] L. D. Landau and E. M. Lifshitz, Course of Theoretical Physics: Vol. 3, Quantum Mechanics, (Pergamon Press, 1977).
[14] D. Diakonov and M. Maul, Phys. Rev. D66 (2002) 096004; J. D. Lange, M. Engelhardt and H. Reinhardt, Phys. Rev. D68 (2003) 025001.
[15] M. Bordag, Phys. Rev. D67 (2003) 065001.