ON THE ENTIRE FUNCTIONS FROM THE LAGUERRE–PÓLYA I CLASS HAVING THE INCREASING SECOND QUOTIENTS OF TAYLOR COEFFICIENTS

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Abstract. We prove that if \( f(x) = \sum_{k=0}^{\infty} a_k x^k \), \( a_k > 0 \), is an entire function such that the sequence \( Q := \left( \frac{a_{2k}}{a_{2k-1} a_{2k+1}} \right)_{k=1}^{\infty} \) is non-decreasing and \( \frac{a_2}{a_0 a_2} \geq 2 \sqrt{2} \), then all but a finite number of zeros of \( f \) are real and simple. We also present a criterion in terms of the closest to zero roots for such a function to have only real zeros (in other words, for belonging to the Laguerre–Pólya class of type I) under additional assumption on the sequence \( Q \).

1. Introduction

In this paper, we study conditions under which special entire functions with positive Taylor coefficients have only real zeros. First, we need the definition of two well-known classes of entire functions: the Laguerre–Pólya class and the Laguerre–Pólya class of type I.

Definition 1. A real entire function \( f \) is said to be in the Laguerre–Pólya class, written \( f \in \mathcal{L} - \mathcal{P} \), if it can be expressed in the form

\[
f(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left( 1 - \frac{x}{x_k} \right) e^{xx_k^{-1}},
\]

where \( c, \alpha, \beta, x_k \in \mathbb{R} \), \( x_k \neq 0 \), \( \alpha \geq 0 \), \( n \) is a nonnegative integer and \( \sum_{k=1}^{\infty} x_k^{-2} < \infty \).

A real entire function \( f \) is said to be in the Laguerre–Pólya class of type I, written \( f \in \mathcal{L} - \mathcal{P} I \), if it can be expressed in the following form

\[
f(x) = cx^n e^{\beta x} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{x_k} \right),
\]

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where \( c \in \mathbb{R}, \beta \geq 0, x_k > 0, n \) is a nonnegative integer, and \( \sum_{k=1}^{\infty} x_k^{-1} < \infty \). As usual, the product on the right-hand side can be finite or empty (in the latter case the product equals 1).

These classes are important for the theory of entire functions since the polynomials with only real zeros (only real and nonnegative zeros) converge locally uniformly to these and only these functions. The following prominent theorem states an even stronger fact.

**Theorem A** (E. Laguerre and G. Pólya, see, for example, [7, p. 42–46]) and [21, chapter VIII, §3]).

(i) Let \( (P_n)_{n=1}^{\infty}, P_n(0) = 1 \), be a sequence of complex polynomials having only real zeros which converges uniformly on the disc \(|z| \leq A, A > 0\). Then this sequence converges locally uniformly to an entire function from the \( \mathcal{L} - \mathcal{P} \) class.

(ii) For any \( f \in \mathcal{L} - \mathcal{P} \) there exists a sequence of complex polynomials with only real zeros which converges locally uniformly to \( f \).

(iii) Let \( (P_n)_{n=1}^{\infty}, P_n(0) = 1 \), be a sequence of complex polynomials having only real negative zeros which converges uniformly on the disc \(|z| \leq A, A > 0\). Then this sequence converges locally uniformly to an entire function from the class \( \mathcal{L} - \mathcal{P}I \).

(iv) For any \( f \in \mathcal{L} - \mathcal{P}I \) there is a sequence of complex polynomials with only real nonpositive zeros which converges locally uniformly to \( f \).

For interesting properties and characterizations of the Laguerre–Pólya class and the Laguerre–Pólya class of type I, see [28, p. 100], [29] or [26, Kapitel II] (also see the survey [27] on the zero distribution of entire functions, its sections and tails). Note that for a real entire function (not identically zero) of the order less than 2 having only real zeros is equivalent to belonging to the Laguerre–Pólya class. Also, for a real entire function with positive coefficients of the order less than 1 having only real zeros is equivalent to belonging to the Laguerre–Pólya class of type I.

Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be an entire function with real nonzero coefficients. We define the quotients \( p_n \) and \( q_n \):

\[
p_n = p_n(f) := \frac{a_{n-1}}{a_n}, \quad n \geq 1,
\]

\[
q_n = q_n(f) := \frac{p_n}{p_{n-1}} = \frac{a_{n-1}^2}{a_{n-2} a_n}, \quad n \geq 2.
\]
From these definitions it readily follows that

\[ a_n = \frac{a_0}{p_1 p_2 \cdots p_n}, \quad n \geq 1, \]

\[ a_n = a_1 \left( \frac{a_1}{a_0} \right)^{n-1} \frac{1}{q_2^{n-1} q_3^{n-2} \cdots q_m^{n-m}}, \quad n \geq 2. \]

It is not trivial to understand whether a given entire function has only real zeros. In 1926, J. I. Hutchinson found the following sufficient condition for an entire function with positive coefficients to have only real zeros.

**Theorem B** (J. I. Hutchinson, [8]). Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k, \ a_k > 0 \) for all \( k \). Then \( q_n(f) \geq 4 \), for all \( n \geq 2 \), if and only if the following two conditions are fulfilled:

(i) The zeros of \( f(z) \) are all real, simple and negative, and

(ii) the zeros of any polynomial \( \sum_{k=m}^{n} a_k z^k, \ m < n, \) formed by taking any number of consecutive terms of \( f(z) \), are all real and non-positive.

For some extensions of Hutchinson’s results see, for example, [4, §4].

The entire function \( g_a(z) = \sum_{k=0}^{\infty} z^k a^{-k^2}, \ a > 1 \), known as a partial theta-function, was investigated in many works and has a prominent role. We can also observe that \( q_n(g_a) = a^2 \) for all \( n \). The survey [31] by S. O. Warnaar contains the history of investigation of the partial theta-function and some of its main properties.

In [9] it is shown that for every \( n \geq 2 \) there exists a constant \( c_n > 1 \) such that \( S_n(z, g_a) := \sum_{j=0}^{n} z^j a^{-j^2} \in \mathcal{L} - \mathcal{P} \iff a^2 \geq c_n \).

**Theorem C** (O. Katkova, T. Lobova, A. Vishnyakova, [9]). There exists a constant \( q_\infty \) \((q_\infty \approx 3.23363666\ldots)\) such that:

1. \( g_a(z) \in \mathcal{L} - \mathcal{P} \iff a^2 \geq q_\infty; \)
2. \( g_a(z) \in \mathcal{L} - \mathcal{P} \iff \text{there exists } x_0 \in (-a^3, -a) \text{ such that } g_a(x_0) \leq 0; \)
3. for a given \( n \geq 2 \) we have \( g_{a,n}(z) \in \mathcal{L} - \mathcal{P} \iff \text{there exists } x_n \in (-a^3, -a) \text{ such that } g_{a,n}(x_n) \leq 0; \)
4. \( 4 = c_2 > c_4 > c_6 > \cdots \) and \( \lim_{n \to \infty} c_{2n} = q_\infty; \)
5. \( 3 = c_3 < c_5 < c_7 < \cdots \) and \( \lim_{n \to \infty} c_{2n+1} = q_\infty. \)

There is a series of works by V. P. Kostov dedicated to the interesting properties of zeros of the partial theta-function and its derivative (see [11–19]). A wonderful paper [20] among the other results explains the role of the constant \( q_\infty \) in the study of the set of entire functions with positive coefficients having all Taylor truncations with only real zeros.
A. D. Sokal in [30] studies the leading roots of the partial theta-function. A formal power series
\[ f(x, y) = \sum_{n=0}^{\infty} \alpha_n x^n y^{n(n-2)/2}, \]
(where the coefficients \((\alpha_n)_{n=0}^{\infty}\) belong to a commutative ring with identity element, and \(\alpha_0 = \alpha_1 = 1\)) is considered as a formal power series in \(y\) whose coefficients are polynomials in \(x\). A. D. Sokal defines the ”leading root” of \(f\) as a unique formal power series \(x_0(y) \in R[y]\) which satisfies the equation \(f(x_0(y), y) = 0\). The coefficientwise positivity of \(-x_0(y)\) was proved. Moreover, all the coefficients of \(1/x_0(y)\) and \(1/x_0(y)^2\) after the constant term 1 are strictly negative, except for the vanishing coefficient of \(y^3\) for the latter case.

In [10], the following questions are investigated: whether the Taylor sections of the function \(\prod_{k=1}^{\infty} (1 + \frac{z}{a^k})\), where \(a > 1\), and \(\sum_{k=0}^{\infty} \frac{z^k}{k! a^{k^2}}\) belong to the Laguerre–Pólya class of type I. In [3] and [2], some important special functions with increasing sequence of the second quotients of Taylor coefficients are studied.

B. He in [6] considers the entire function as follows
\[ A_q^{(\alpha)}(a; z) = \sum_{k=0}^{\infty} \frac{(a; q)_k q^{\alpha k^2} z^k}{(q; q)_k}, \]
where \(\alpha > 0\), \(0 < q < 1\) and
\[ (a; q)_n = \begin{cases} 
1, & n = 0 \\
\frac{(1 - a q^2)(n \geq 1)}{\prod_{j=1}^{n-1} (1 - a q^2)(n \geq 1)} 
\end{cases} \]
is the q-shifted factorial. The entire function \(A_q^{(\alpha)}(a; z)\) defined as above is the generalisation of Ramanujan entire function and the Stieltjes-Wigert polynomial which have only real positive zeros. The paper gives a partial answer to Zhang’s question: under what conditions the zeros of the entire function \(A_q^{(\alpha)}(a; z)\) are all real.

In [22] it is proved that if \(f(z) = \sum_{k=0}^{\infty} a_k z^k\), \(a_k > 0\) for all \(k\), is an entire function such that \(q_2 \geq q_3 \geq q_4 \geq \cdots\), and \(\lim_{n \to \infty} q_n(f) = b \geq q_\infty\), then \(f \in \mathcal{L} - \mathcal{P}\).

In [23] it is proved that if \(f(z) = \sum_{k=0}^{\infty} a_k z^k\), \(a_k > 0\) for all \(k\), is an entire function such that \(q_2 \leq q_3 \leq q_4 \leq \cdots\), and \(\lim_{n \to \infty} q_n(f) = c < q_\infty\), then the function \(f\) does not belong to the Laguerre–Pólya class.
The first author studied a special function related to the partial theta-function and the Euler function, $f_a(z) = \sum_{k=0}^{\infty} z^k \left( \frac{a^k+1}{(a^k+1)(a^{k-1}+1)\cdots(a+1)} \right)$, $a > 1$, which is also known as the $q$-Kummer function $_1\phi_1(q; -q; q, -z)$, where $q = 1/a$ (see [5], formula (1.2.22)), and found the conditions for it to belong to the Laguerre–Pólya class (see [25]).

A. Baricz and S. Singh in [1] investigated the Bessel functions. The Hurwitz theorem on the exact number of nonreal zeros was extended for the Bessel functions of the first kind. In addition, the results on zeros of derivatives of Bessel functions and the cross-product of Bessel functions were obtained.

It turns out that for many important entire functions with positive coefficients $f(z) = \sum_{k=0}^{\infty} a_k z^k$ (for example, partial theta-function from [9], functions from [3] and [2], the $q$-Kummer function $_1\phi_1(q; -q; q, -z)$, and others) the following two conditions are equivalent:

(i) $f$ belongs to the Laguerre–Pólya class of type I, and
(ii) there exists $z_0 \in [-a, 0]$ such that $f(z_0) \leq 0$.

In our previous work, we have obtained a new necessary condition for an entire function to belong to the Laguerre–Pólya class of type I in terms of the closest to zero roots.

**Theorem D** (T. H. Nguyen, A. Vishnyakova, [24]). Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ for all $k$, be an entire function. Suppose that the quotients $q_n(f)$ satisfy the following condition: $q_2(f) \leq q_3(f) \leq q_4(f) \leq \cdots$. Then all but a finite number of zeros of $f$ are real and simple.

Our first result concerns the possible number of nonreal zeros of a real entire function whose sequence of the second quotients of Taylor coefficients is non-decreasing.

**Theorem 1.1.** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0, k = 0, 1, 2, \ldots$, be an entire function such that $2\sqrt{2} \leq q_2(f) \leq q_3(f) \leq q_4(f) \leq \cdots$. Then all but a finite number of zeros of $f$ are real and simple.

In connection with the theorem above, we formulate the following conjecture.

**Conjecture.** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0, k = 0, 1, 2, \ldots$, be an entire function such that $1 < q_2(f) \leq q_3(f) \leq q_4(f) \leq \cdots$. Then all but a finite number of zeros of $f$ are real and simple.

The second result of our paper is the following criterion for belonging to the Laguerre–Pólya class of type I for real entire functions with the regularly non-decreasing sequence of second quotients of Taylor
coefficients. To clarify the statement of the next theorem we need the following lemma from [24].

Lemma 1.2. (Lemma 1.2 from [24], cf. Lemma 2.1 from [23]). If 
\[ f(z) = \sum_{k=0}^{\infty} a_k z^k, a_k > 0, \] 
belongs to \( \mathcal{L} - \mathcal{P} I \), then 
\[ q_3(q_2 - 4) + 3 \geq 0. \]
In particular, if \( q_3 \geq q_2 \), then \( q_2 \geq 3 \).

So, if we investigate whether a real entire function, with the non-decreasing sequence of second quotients of Taylor coefficients, belongs to the Laguerre–Pólya class of type I, then the necessary condition is \( q_2 \geq 3 \). Our main result is the following theorem.

Theorem 1.3. Let
\[ f(z) = \sum_{k=0}^{\infty} a_k z^k, a_k > 0, \]
for all \( k \), be an entire function. Suppose that \( 3 \leq q_2(f) \leq q_3(f) \leq q_4(f) \leq \ldots \). Suppose also that if there is an integer \( j_0 \geq 2 \), such that \( q_{j_0}(f) < 4 \), and \( q_{j_0+1}(f) \geq 4 \), then \( q_{j_0-1}(f) \geq 0.525 \), or \( q_{j_0}(f) \geq 3.4303 \). Then \( f \in \mathcal{L} - \mathcal{P} I \) if and only if there exists \( z_0 \in [-a_1/a_2, 0] \) such that \( f(z_0) \leq 0 \).

Unfortunately, at the moment we do not know whether the additional assumptions in the above theorem are essential.

2. Proof of Theorem 1.1

Proof. To prove Theorem 1.1 we need the following Lemma.

Lemma 2.1. Let
\[ f(x) = \sum_{k=0}^{\infty} (-1)^k a_k x^k, a_k > 0, \]
be an entire function such that \( 2\sqrt{2} \leq q_2(f) \leq q_3(f) \leq q_4(f) \leq \ldots \). For an arbitrary integer \( j \geq 2 \) we define
\[ \rho_j(f) := q_2(f) q_3(f) \cdots q_j(f) \sqrt{q_{j+1}(f)}. \]
Then, for all sufficiently large \( j \), the function \( f \) has exactly \( j \) zeros on the disk \( \{ z : |z| < \rho_j(f) \} \) counting multiplicities.

Proof. For simplicity, we will write \( q_j \) instead of \( q_j(f) \) and \( \rho_j \) instead of \( \rho_j(f) \). We have
\[ f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{q_2 q_3^{-1} q_3^{-2} \cdots q_k}, \]
where the sequence \( q_2, q_3, \ldots \) is non-decreasing. We now dissect the above sum as
\[ \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{q_2 q_3^{-1} q_3^{-2} \cdots q_k} = \left( \sum_{k=0}^{j-3} \sum_{k=j-2}^{j+2} \sum_{k=j+3}^{\infty} \right) := \Sigma_{j-3}(z) + g_j(z) + \Sigma_{j+3}(z), \]
where
\[ g_j(z) = \left( \sum_{k=j-2}^{j+1} \frac{(-1)^k z^k}{q_2^{k+1}z^{k+2} \cdots q_j} + \frac{(-1)^j z^{j+2}}{q_2^{j+1} \cdots q_j^{j+2} q^q_{j+1} q^q_{j+2}} \right) \]
\[ + \frac{(-1)^j z^{j+2}}{q_2^{j+1} \cdots q_j^{j+2}} \left( \frac{1}{q_2^{j+1} \cdots q_j^{j+2} q^q_{j+1} q^q_{j+2}} \right) \]
\[ =: \tilde{g}_j(z) + \xi_j(z). \]

By the definition of \( \rho_j \) we have \( q_2 q_3 \cdots q_j < \rho_j < q_2 q_3 \cdots q_j q_{j+1} \). We get
\[ (4) \quad (-1)^j g_j(\rho_j e^{i\theta}) = e^{i(j-2)\theta} q_2 q_3 \cdots q_j^{j-3} q_j^{-2} q^{j-2} q_{j+1}. \]
\[ (1 - e^{i\theta} q_j \sqrt{q_j + 1} + e^{2i\theta} q_j q_{j+1} - e^{3i\theta} q_j \sqrt{q_j + 1} + e^{4i\theta} q_j q_{j+1} - 1) \]
\[ = e^{i(j-2)\theta} q_2 q_3 \cdots q_j^{j-3} q_j^{-2} q^{j-2} q_{j+1}. \]
\[ (1 - e^{i\theta} q_j \sqrt{q_j + 1} + e^{2i\theta} q_j q_{j+1} - e^{3i\theta} q_j \sqrt{q_j + 1} + e^{4i\theta}) \]
\[ + e^{i(j+2)\theta} q_2 q_3 \cdots q_j^{j-3} q_j^{-2} q^{j-2} q_{j+1} \left( q_j q_{j+1} - 1 \right) \]
\[ = \tilde{g}_j(\rho_j e^{i\theta}) + \xi_j(\rho_j e^{i\theta}). \]

Our aim is to show that for every sufficiently large \( j \) the following inequality holds:
\[ \min_{0 \leq \theta \leq 2\pi} |\tilde{g}_j(\rho_j e^{i\theta})| > \max_{0 \leq \theta \leq 2\pi} |f(\rho_j e^{i\theta}) - \tilde{g}_j(\rho_j e^{i\theta})|, \]
so that the number of zeros of \( f \) in the circle \( \{ z : |z| < \rho_j \} \) is equal to the number of zeros of \( \tilde{g}_j \) in the same circle. Later in the proof, we also find the number of zeros. First, we find \( \min_{0 \leq \theta \leq 2\pi} |\tilde{g}_j(\rho_j e^{i\theta})| \).
\[ (5) \quad \tilde{g}_j(\rho_j e^{i\theta}) = e^{i(j-2)\theta} q_2 q_3 \cdots q_j^{j-3} q_j^{-2} q^{j-2} q_{j+1}. \]
\[ (e^{-2i\theta} - e^{-i\theta} q_j \sqrt{q_j + 1} + q_j q_{j+1} - e^{i\theta} q_j \sqrt{q_j + 1} + e^{2i\theta}) \]
\[ = e^{i(j-2)\theta} q_2 q_3 \cdots q_j^{j-3} q_j^{-2} q^{j-2} q_{j+1}. \]
\[ (2 \cos 2\theta - 2 \cos \theta q_j \sqrt{q_j + 1} + q_j q_{j+1}) \]
\[ =: e^{i(j-2)\theta} q_2 q_3 \cdots q_j^{j-3} q_j^{-2} q^{j-2} q_{j+1} \cdot \psi_j(\theta). \]

We consider \( \psi_j(\theta) \) as following
\[ \psi_j(\theta) = \tilde{\psi}_j(t) := 4t^2 - 2q_j \sqrt{q_j + 1} t + (q_j q_{j+1} - 2) \].
where \( t := \cos \theta \), and where we have used that \( \cos 2\theta = 2t^2 - 1 \).

The vertex of the parabola is \( t_j = q_j \sqrt{q_j+1}/4 \). Under our assumptions, \( 2\sqrt{2} \leq q_2 \leq q_3 \leq \ldots \), so that \( q_j \sqrt{q_j+1}/4 \geq q_2 \sqrt{q_2}/4 \geq 2\sqrt{2}/2 \sqrt{2}/4 = 1 \), and we have \( t_j \geq 1 \). Hence,

\[
\min_{t \in [-1,1]} \tilde{\psi}_j(t) = \tilde{\psi}_j(1) = 2 - 2q_j \sqrt{q_j+1} + q_j q_{j+1} = q_j \sqrt{q_j+1}(\sqrt{q_j+1} - 2) + 2.
\]

If \( q_{j+1} \geq 4 \), then \( q_j \sqrt{q_j+1}(\sqrt{q_j+1} - 2) + 2 > 0 \). If \( q_{j+1} < 4 \), then

\[
q_j \sqrt{q_j+1}(\sqrt{q_j+1} - 2) + 2 \geq q_j+1 \sqrt{q_j+1}(\sqrt{q_j+1} - 2) + 2 = q_j^2 - 2q_j+1 \sqrt{q_j+1} + 2.
\]

Denote by \( y = \sqrt{q_j+1} \geq 0 \), and \( g(y) = y^4 - 2y^3 + 2 \). It is easy to calculate that \( \min_{y \geq 0} g(y) = g(\frac{3}{2}) = \frac{\sqrt{3}}{16} > 0 \). Therefore, we get

\[
2 - 2q_j \sqrt{q_j+1} + q_j q_{j+1} > 0.
\]

Thus, \( \tilde{\psi}_j(t) > 0 \) for all \( t \in [-1,1] \). Consequently, we have obtained the estimate from below:

\[
\min_{0 \leq t \leq 2\pi} |\tilde{g}_j(\rho_j e^{i\theta})| \geq q_2 q_3^2 \cdots q_{j-3} q_{j-2} q_{j-1} q_j^3 q_j^{j-2} q_{j+1} \cdot \\
\left( 2 - 2q_j \sqrt{q_j+1} + q_j q_{j+1} \right).
\]

Second, we estimate the modulus of \( \Sigma_1 \) from above. We have

\[
|\Sigma_1(\rho_j e^{i\theta})| \leq \sum_{k=0}^{j-3} q_k^2 q_{k+1}^4 \cdots q_{j-2} q_{j-1} q_j^{j-3} q_j^{j-2} q_j^{j-1} q_j^3 q_{j+1} = \\
(\text{we rewrite the sum from right to left})
\]

\[
= \left( q_2 q_3^2 \cdots q_{j-3} q_{j-2} q_{j-1} q_j^4 q_j^{j-3} q_j^{j-2} q_j^{j-1} q_j q_j^3 q_j^{j-2} q_j^{j-1} q_j^{j-3} q_j^{j-2} q_j^{j-1} q_j^3 q_{j+1} + \cdots \right)
\]

\[
= q_2 q_3^2 \cdots q_{j-3} q_{j-2} q_{j-1} q_j^4 q_j^{j-3} q_j^{j-2} q_j^{j-1} q_j q_j^3 q_j^{j-2} q_j^{j-1} q_j^{j-3} q_j^{j-2} q_j^{j-1} q_j^3 q_{j+1} + \cdots \\
\cdot \left( 1 + \frac{1}{q_{j-2} q_{j-1} \sqrt{q_j+1}} + \frac{1}{q_{j-3} q_{j-2} q_{j-1} q_j^2 q_j \sqrt{q_j+1}} + \cdots \right)
\]

\[
\leq q_2 q_3^2 \cdots q_{j-3} q_{j-2} q_{j-1} q_j^4 q_j^{j-3} q_j^{j-2} q_j^{j-1} q_j^3 q_j^{j-2} q_j^{j-1} q_j^3 q_{j+1} \cdot \\
\frac{1}{1 - q_2 q_3 q_{j-2} q_{j-1} q_j \sqrt{q_j+1}}
\]

(we estimate the finite sum from above by the sum of the infinite geometric progression). Finally, we obtain

\[
|\Sigma_1(\rho_j e^{i\theta})| \leq \\
q_2 q_3^2 \cdots q_{j-3} q_{j-2} q_{j-1} q_j^4 q_j^{j-3} q_j^{j-2} q_j^{j-1} q_j^3 q_j^{j-2} q_j^{j-1} q_j^3 q_{j+1} \cdot \\
\frac{1}{1 - q_2 q_3 q_{j-2} q_{j-1} q_j \sqrt{q_j+1}}.
\]
Next, the estimation of $|\Sigma_2(\rho e^{i\theta})|$ from above can be made analogously:

$$|\Sigma_2(\rho e^{i\theta})| \leq \sum_{k=j+3}^{\infty} \frac{q_2^k q_3^k \cdots q_j^k}{q_2^{k-1} q_3^{k-2} \cdots q_k} \left( \frac{q_2 q_3^2 \cdots q_j^{j-1} q_{j+1}^{j-3}}{q_{j+2}^2 q_{j+3}} \right).$$

The latter can be estimated from above by the sum of the geometric progression, so, we obtain

$$|\Sigma_2(\rho e^{i\theta})| \leq \frac{q_2 q_3^2 \cdots q_{j+1}^{j-3}}{q_{j+2}^2 q_{j+3}} \cdot \frac{1}{1 - \frac{1}{\sqrt{q_{j+1} q_{j+2} q_{j+3} q_{j+4}}}}. \tag{9}$$

Note that

$$|\xi_j(\rho e^{i\theta})| = q_2 q_3^2 \cdots q_{j-2}^2 q_{j-1}^2 q_j^{j-2} q_{j+1}^{j-3} \left( 1 - q_j^{-1} q_{j+1}^{-1} \right) \cdot \left( 2 - 2q_j \sqrt{q_{j+1}} + q_j q_{j+1} \right) \cdot \frac{1}{1 - \frac{1}{\sqrt{q_{j-1} q_{j-2} q_{j-3} q_{j+4}}}}.$$

Therefore, the desired inequality $\min_{0 \leq \theta \leq 2\pi} |\tilde{g}_j(\rho e^{i\theta})| > \max_{0 \leq \theta \leq 2\pi} |f(\rho e^{i\theta}) - \tilde{g}_j(\rho e^{i\theta})|$ follows from

$$q_2 q_3^2 \cdots q_{j-2}^2 q_{j-1}^2 q_j^{j-2} q_{j+1}^{j-3} \left( 1 - q_j^{-1} q_{j+1}^{-1} \right) \cdot \left( 2 - 2q_j \sqrt{q_{j+1}} + q_j q_{j+1} \right) \cdot \frac{1}{1 - \frac{1}{\sqrt{q_{j-1} q_{j-2} q_{j-3} q_{j+4}}}}.$$

Or, equivalently,

$$q_{j-1} q_j \sqrt{q_{j+1}} \left( 2 - 2q_j \sqrt{q_{j+1}} + q_j q_{j+1} \right) >$$

$$\frac{1}{1 - \frac{1}{q_{j-1} q_{j-2} q_{j-3} q_{j+4}}} + \frac{q_{j-1} q_j^2}{q_{j+1} q_{j+2}^2} \cdot \frac{1}{1 - \frac{1}{\sqrt{q_{j+1} q_{j+2} q_{j+3} q_{j+4}}}} + q_{j-1} q_j \sqrt{q_{j+1}} (1 - q_j^{-1} q_{j+1}^{-1}). \tag{10}$$

Since, under our assumptions $q_2 \leq q_3 \leq q_4 \leq \ldots$, the sequence $(q_j)_{j=2}^{\infty}$ has the limit, that is finite or infinite. At first, we consider the
case when this limit is finite and put $\lim_{j \to \infty} q_j = a$, $a \geq 2\sqrt{2}$. We firstly investigate the limiting inequality
\[
(11) \quad a^2 \sqrt{a} (2 - 2a \sqrt{a} + a^2) > \frac{1}{1 - \frac{1}{a^3 \sqrt{a}}} + \frac{1}{1 - \frac{1}{a^3 \sqrt{a}}} + a^2 \sqrt{a} \cdot 0.
\]
Equivalently,
\[
2 - 2a \sqrt{a} + a^2 > \frac{2a}{a^3 \sqrt{a} - 1}.
\]
Set $\sqrt{a} = b$, then we obtain \((2 - 2a^3 b^2 + b^4)(b^7 - 1) > 2b^3\), or \(b^{11} - 2b^{10} + 2b^7 - b^4 + 2b^3 - 2b^2 - 2 > 0\).

We have found the roots of the polynomial on the left-hand side of the inequality using the computer, and its greatest real root is less than 1.47. Thus, the inequality is fulfilled for $b > 1.47$, and, therefore, for $a > 2.17$. Under our assumptions, $a \geq 2\sqrt{2} > 2.51$, so the inequality (11) is valid. Whence, for the case when the sequence $(q_j)_{j=2}^{\infty}$ has the finite limit, the inequality (10) is valid for all $j$ being large enough.

Now we consider the case when $\lim_{j \to \infty} q_j = +\infty$. The inequality (10) follows from the
\[
(12) \quad q_{j-1} q_j \sqrt{q_{j+1}} \left( 2 - 2q_j \sqrt{q_{j+1}} + q_j q_{j+1} \right) > \frac{1}{1 - \frac{1}{q_{j-2} q_{j-1} \sqrt{q_{j+1}}} + \frac{1}{1 - \frac{1}{\sqrt{q_{j+1} q_{j+2} \sqrt{q_{j+3}}}}}} + q_{j-1} q_j \sqrt{q_{j+1}},
\]
or
\[
2 - 2q_j \sqrt{q_{j+1}} + q_j q_{j+1} > \frac{1}{q_{j-1} q_j \sqrt{q_{j+1}}} \cdot \frac{1}{1 - \frac{1}{q_{j-2} q_{j-1} \sqrt{q_{j+1}}} + 1} + \frac{1}{1 - \frac{1}{q_{j+1} q_{j+2} \sqrt{q_{j+3}}}} + 1.
\]
The left-hand side of the inequality above tends to infinity, and the right-hand side tends to 1. So, the last inequality is valid for all $j$ being large enough. Whence, for the case when the sequence $(q_j)_{j=2}^{\infty}$ has the infinite limit, the inequality (10) is valid for all $j$ being large enough.

Consequently, we have proved that for all $j$ being large enough $\min_{0 \leq \theta \leq 2\pi} |\tilde{g}_j(\rho_j e^{i\theta})| > \max_{0 \leq \theta \leq 2\pi} |f(\rho e^{i\theta}) - \tilde{g}_j(\rho_j e^{i\theta})|$, so the number of zeros of $f$ in the circle \(\{z : |z| < \rho_j\}\) is equal to the number of zeros of $\tilde{g}_j$ in this circle.

In the next stage of the proof, it remains to find the number of zeros of $\tilde{g}_j$ in the circle \(\{z : |z| < \rho_j\}\). We have
\[ \tilde{g}_j(z) = \sum_{k=j-2}^{j+1} \frac{(-1)^k z^k}{q_2^k q_3^{k-2} \cdots q_k} + \frac{(-1)^{j+2} z^{j+2}}{q_2^{j+1} q_3^j \cdots q_j^{-1} q_{j+1}^1}. \]

Let us use the denotation \( w = z \rho_j^{-1} \), so that \( |w| < 1 \). This yields

\[ \tilde{g}_j(\rho_j w) = (-1)^{j-2} w^{j-2} q_2 q_3^{2} \cdots q_j^{j-2} q_{j-1} \cdot q_j^{j-1} \cdot (1 - q_j \sqrt[q_j+1]{w + q_j q_{j+1} w^2 - q_j \sqrt[q_j+1]{w^3 + w^4}). \]

It follows from (6) that \( \tilde{g}_j \) does not have zeros on the circumference \( \{ z : |z| = \rho_j \} \), whence \( \tilde{g}_j(\rho_j w) \) does not have zeros on the circumference \( \{ w : |w| = 1 \} \). Since \( P_j(w) = 1 - q_j \sqrt[q_j+1]{w + q_j q_{j+1} w^2 - q_j \sqrt[q_j+1]{w^3 + w^4} \) is a self-reciprocal polynomial in \( w \), we can conclude that \( P_j \) has exactly two zeros in the circle \( \{ w : |w| < 1 \} \). Hence, \( \tilde{g}_j(z) \) has exactly \( j \) zeros in the circle \( \{ z : |z| < \rho_j \} \), and we have proved the statement of Lemma 2.1.

Lemma 2.1 is a simple corollary of Lemma 2.1.

\[ \square \]

3. Proof of Theorem 1.3

Without loss of generality, we can assume that \( a_0 = a_1 = 1 \), since we can consider a function \( \psi(z) = a_0^{-1} f(a_0 a_1^{-1} z) \) instead of \( f(z) \), due to the fact that such rescaling of \( f \) preserves its property of having real zeros and preserves the second quotients: \( q_n(\psi) = q_n(f) \) for all \( n \). For brevity, during the proof we write \( p_n \) and \( q_n \) instead of \( p_n(f) \) and \( q_n(f) \). Then, we have

\[ f(z) = 1 + z + \sum_{k=2}^{\infty} \frac{z^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1} q_k}. \]

Further, during the proof, we need the inequalities related to the roots of the function \( f \). So, for the convenience of dealing with inequalities, we are going to consider the positive roots. Thus, we consider the entire function

\[ \varphi(z) = f(-z) = 1 - z + \sum_{k=2}^{\infty} \frac{(-1)^k z^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1} q_k} \]

instead of \( f \).

In [24], it was proved that if \( \varphi \in \mathcal{L} - \mathcal{P} \) and \( q_2(f) \leq q_3(f) \), then there exists \( z_0 \in (0; \frac{a_1}{a_2}] = (0, q_2] \) such that \( \varphi(z_0) \leq 0 \) (see Theorem D.
in the introduction). It remains to prove the inverse statement. To do this we need the following lemma.

**Lemma 3.1.** According to Lemma 3.1, we denote by \( \rho_k = \rho_k(q) := q_2(q_3(q) \cdots q_k(q)\sqrt{q_{k+1}(q)}) \), \( k \in \mathbb{N} \). Under the assumptions of Theorem 1.3, for every \( k \geq 2 \) the following inequality holds:

\[
(-1)^k \varphi(q) \geq 0.
\]

**Proof.** Since \( \rho_k \in (q_2q_3 \cdots q_k, q_2q_3 \cdots q_kq_{k+1}) \), we have

\[
1 < \rho_k < \frac{\rho_k^2}{q_2} < \cdots < \frac{\rho_k^k}{q_2^{k-1}q_3^{k-2} \cdots q_k},
\]

and

\[
\frac{\rho_k^{k-1}}{q_2^{k-1}q_3^{k-2} \cdots q_k} > \frac{\rho_k^{k+1}}{q_2^{k+1}q_3^{k} \cdots q_k} > \frac{\rho_k^{k+2}}{q_2^{k+2}q_3^{k+1} \cdots q_k} > \cdots.
\]

Therefore, we get for \( k \geq 2 \)

\[
(-1)^k \varphi(q) = \sum_{j=k-3}^{k+3} 0
\]

and it is sufficient to prove that for every \( k \geq 2 \) we have \( \mu_k(q) \geq 0 \). After factoring out \( (\rho_k^{k-3})(q_2^{k-4}q_3^{k-5} \cdots q_k) \) the desired inequality takes the form:

\[
-1 + \frac{\rho_k}{q_2q_3 \cdots q_k} = \frac{\rho_k^2}{q_2^2q_3 \cdots q_{k-2}q_{k-1}} + \frac{\rho_k^3}{q_2^3q_3^2 \cdots q_{k-3}q_{k-2}q_{k-1}} q_{k+1} q_{k+2} q_{k+3}
\]

or, using that \( \rho_k = q_2q_3 \cdots q_k \sqrt{q_{k+1}} \)

\[
\nu_k(q) := -1 + q_{k-1}q_k \sqrt{q_{k+1}} - 2q_{k-1}q_kq_{k+1} + q_{k-1}q_kq_{k+1} \sqrt{q_{k+1}} q_{k+2} q_{k+3} \geq 0.
\]

We observe that

\[
\nu_k(q) = (q_{k-1}q_k \sqrt{q_{k+1}} - 1) + q_{k-1}q_kq_{k+1}(\sqrt{q_{k+1}} - 2) + q_{k-1}q_kq_{k+1} \sqrt{q_{k+1}} q_{k+2} q_{k+3}.
\]
At first, we consider the case when $q_{k+1} \geq 4$. Then we have $(q_{k-1} q_k \sqrt{q_{k+1}} - 1) > 0$, $q_{k-1} q_k^2 q_{k+1} (\sqrt{q_{k+1}} - 2) \geq 0$, and $q_{k-1} q_k^2 q_{k+1} (1 - \frac{1}{\sqrt{q_{k+1} q_{k+2} q_{k+3}}}) > 0$. Thus, in the case $q_{k+1} \geq 4$ the desired inequality $\nu_k(\rho_k) \geq 0$ is proved.

Now we consider the case when $q_{k+1} < 4$ and either $q_{k+2} < 4$ (so that $\frac{q_{k+2}}{q_{k+2} - 1} \geq \frac{3}{4} \geq 0.525$, or $q_{k+2} \geq 4$ and $\frac{q_{k+2}}{q_{k+2} - 1} \geq 0.525$.

After rearranging we get

$$\nu_k(\rho_k) = q_{k-1} q_k \sqrt{q_{k+1}} - 2 q_{k-1} q_k^2 q_{k+1} + q_{k-1} q_k \sqrt{q_{k+1}} \left(1 + \frac{q_k}{q_{k+2}}\right) - 2 \geq 0.$$

Since $q_k$ are non-decreasing in $k$, it is easy to see that $(q_{k-1} q_k^2)/(q_{k+2} q_{k+3}) \leq 1$, hence, it is sufficient to prove the following inequality

$$q_{k-1} q_k^2 q_{k+1} \sqrt{q_{k+1}} - 2 q_{k-1} q_k^2 q_{k+1} + q_{k-1} q_k \sqrt{q_{k+1}} \left(1 + \frac{q_k}{q_{k+2}}\right) - 2 \geq 0.$$

Under our assumptions that $q_k$ are non-decreasing in $k$ and $q_2 \geq 3$, we have $2 < \frac{2}{9} q_{k-1} q_k$, and we can observe that

$$q_{k-1} q_k^2 q_{k+1} \sqrt{q_{k+1}} - 2 q_{k-1} q_k^2 q_{k+1} + q_{k-1} q_k \sqrt{q_{k+1}} \left(1 + \frac{q_k}{q_{k+2}}\right) - 2 \geq q_{k-1} q_k^2 q_{k+1} \sqrt{q_{k+1}} - 2 q_{k-1} q_k^2 q_{k+1} + q_{k-1} q_k \sqrt{q_{k+1}} \left(1 + \frac{q_k}{q_{k+2}}\right) - 2 \geq q_{k-1} q_k^2 q_{k+1} \sqrt{q_{k+1}} - 2 q_{k-1} q_k^2 q_{k+1} + q_{k-1} q_k \sqrt{q_{k+1}} \left(1 + \frac{q_k}{q_{k+2}}\right) - 2 \geq 0.$$

So, we need to check that for all $k \geq 2$

$$q_k q_{k+1} \sqrt{q_{k+1}} - 2 q_k q_{k+1} + 1.525 \sqrt{q_{k+1}} - \frac{2}{9} = q_k q_{k+1} \left(\sqrt{q_{k+1}} - 2\right) + 1.525 \sqrt{q_{k+1}} - \frac{2}{9} \geq 0.$$

Since $q_k$ is non-decreasing in $k$, we get

$$q_k q_{k+1} \sqrt{q_{k+1}} - 2 q_k q_{k+1} + 1.525 \sqrt{q_{k+1}} - \frac{2}{9} \geq q_k^2 q_{k+1} \sqrt{q_{k+1}} - 2 q_k^2 q_{k+1} + 1.525 \sqrt{q_{k+1}} - \frac{2}{9}.$$

Set $\sqrt{q_{k+1}} = t$, $t \geq 0$, then we obtain the following inequality

$$t^5 - 2t^4 + 1.525t - \frac{2}{9} \geq 0.$$

This inequality holds for $t \geq 1.73051$ (we used numerical methods to find that the greatest real root of the polynomial on the left-hand side is less than 1.73051), so it follows that it holds for $q_{k+1} \geq 2.99466$....
Thus, in the case $q_{k+1} < 4$ and either $q_{k+2} < 4$, or $q_{k+2} \geq 4$ and $\frac{q_k}{q_{k+2}} \geq 0.525$ the desired inequality $\nu_k(\rho_k) \geq 0$ is proved.

It remains to consider the case when $q_{k+1} < 4$, $q_{k+2} \geq 4$, and $q_{k+1} \geq 3.4303$. We have

\[
\nu_k(\rho_k) := (q_{k-1}q_k\sqrt{q_{k+1}} - 1) + q_{k-1}q_k^2q_{k+1}(\sqrt{q_{k+1}} - 2) + q_{k-1}q_k^2\sqrt{q_{k+1}}(1 - \frac{1}{\sqrt{q_{k+1}q_{k+2}q_{k+3}}}) \geq (q_{k-1}q_k\sqrt{q_{k+1}} - 1) + q_{k-1}q_k^2q_{k+1}(\sqrt{q_{k+1}} - 2) - \frac{q_{k-1}q_k}{9} + q_{k-1}q_k^2q_{k+1}(\sqrt{q_{k+1}} - 2) = q_{k-1}q_k((\sqrt{q_{k+1}} - \frac{1}{3})) + q_kq_{k+1}(\sqrt{q_{k+1}} - 2).
\]

We want to show that

\[
(\sqrt{q_{k+1}} - \frac{1}{9}) + q_kq_{k+1}(\sqrt{q_{k+1}} - 2) \geq 0.
\]

Since $\sqrt{q_{k+1}} - 2 < 0$, and $q_k \leq q_{k+1}$, the last inequality follows from

\[
(\sqrt{q_{k+1}} - \frac{1}{9}) + q_{k+1}^2(\sqrt{q_{k+1}} - 2) \geq 0.
\]

Denote by $t = \sqrt{q_{k+1}}$, we get the inequality

\[
t^5 - 2t^4 + t - \frac{1}{9} \geq 0.
\]

We have found the roots of the polynomial on the left-hand side of the inequality using the computer, and its greatest real root is less than 1.8521. Thus, this inequality valid for $q_{k+1} \geq 3.4303$. So, in the case when $q_{k+1} < 4$, $q_{k+2} \geq 4$, and $q_{k+1} \geq 3.4303$ the desired inequality $\nu_k(\rho_k) \geq 0$ is also proved.

Lemma 3.1 is proved. \(\square\)

Suppose that there exists $z_0 \in (1, q_2)$, such that $\varphi(z_0) \leq 0$. Then, by Lemma 3.1 we have for every $k \geq 2$:

\[
\varphi(0) > 0, \varphi(z_0) \leq 0, \varphi(\rho_2) \geq 0, \varphi(\rho_3) \leq 0, \ldots, (-1)^k \varphi(\rho_k) \geq 0.
\]

So, for every $k \geq 2$ the function $\varphi$ has at least $k - 1$ real zeros on the disk $\{z : |z| < \rho_k\}$. By Lemma 2.1 the function $\varphi$ has exactly $k$ zeros on the disk $\{z : |z| < \rho_k\}$ for $k$ being large enough. So, for all $k$ being large enough all the zeros of $\varphi$ on the disk $\{z : |z| < \rho_k\}$ are real. Thus, if there exists $z_0 \in (1, q_2)$, such that $\varphi(z_0) \leq 0$, then all the zeros of $\varphi$ are real, thus $\varphi \in \mathcal{L} - \mathcal{P}I$.

Theorem 1.3 is proved.

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