ON THE UNIVALENCE OF POLYHARMONIC MAPPINGS

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ABSTRACT. A $2p$-times continuously differentiable complex valued function $f = u + iv$ in a simply connected domain $\Omega$ is polyharmonic (or $p$-harmonic) if it satisfies the polyharmonic equation $\Delta^p f = 0$. Every polyharmonic mapping $f$ can be written as $f(z) = \sum_{k=1}^{p} |z|^{2(p-1)} G_{p-k+1}(z)$, where each $G_{p-k+1}$ is harmonic. In this paper we investigate the univalence of polyharmonic mappings on linearly connected domains and the relation between univalence of $f(z)$ and that of $G_{p}(z)$. The notion of stable univalence and logpolyharmonic mappings are also considered.

1. Introduction

A $2p$-times continuously differentiable complex valued function $f = u + iv$ in a simply connected domain $\Omega$ is polyharmonic (or $p$-harmonic) if it satisfies the polyharmonic equation

$$\Delta^p f = 0,$$

where $p \geq 1$ is an integer and $\Delta$ represents the Laplacian operator

$$\Delta := 4 \frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

and

$$\Delta^p f = \underbrace{\Delta \ldots \Delta}_{p} f = \Delta^{p-1} \Delta f.$$

Clearly when $p = 1$, $f$ is harmonic and when $p = 2$, $f$ is biharmonic. The properties of univalent harmonic mappings have been studied by many authors (see [17][20][22][24]). One the most fundamental articles on univalent harmonic mappings is due to Clunie-Shiel and Small ([17]).

Biharmonic mappings and their univalence have been investigated recently by several authors (see [1][10]). Biharmonic functions arise in many physical situations, in fluid dynamics and elasticity problems which have many applications in engineering and biology (see [21][25][27]). In addition biharmonic mappings are closely related to the theory of Laguerre minimal surfaces([7]). More recently, properties of polyharmonic functions are investigated. We refer to ([11][13][15][28]) for many interesting results on polyharmonic mappings.

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If $\Omega \subset \mathbb{C}$ is a simply connected domain, then it is easy to see that (see [11])
every polyharmonic mapping $f$ can be written as
$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$
where each $G_{p-k+1}$ is harmonic, that is $\Delta G_{p-k+1} = 0$ for $k \in \{1, ..., p\}$.
This is known as the Almansi expansion (see [5]).

Throughout we consider polyharmonic functions defined on the unit disk $D = \{ z : |z| < 1 \}$.

**Definition 1.** A domain $\Omega \subset \mathbb{C}$ is linearly connected if there exists a constant $M < \infty$ such that any two points $w_1, w_2 \in \Omega$ are joined by a path $\gamma$, $\gamma \subset \Omega$, of length $\ell(\gamma) \leq M|w_1 - w_2|$.

Such a domain is necessarily a Jordan domain, and for piecewise smoothly bounded domains, linear connectivity is equivalent to the boundary having no inward-pointing cusps.

In [16], Chuaqui and Hernandez, considered the relationship between the harmonic mapping $f = h + \overline{g}$ and its analytic factor $h$ on linearly connected domains. They show that if $h$ is an analytic univalent function, then every harmonic mapping $f = h + \overline{g}$ with dilatation $|\omega| < c$ is univalent if and only if $h(D)$ is linearly connected.

In [6], Abdulhadi and El Hajj showed analogous results for biharmonic functions. In this paper we generalise these results for polyharmonic mappings. Moreover some results are obtained for logpolyharmonic mappings. Recently properties of logbiharmonic and logpolyharmonic mappings have been investigated (See [11, 15]). Many physical problems are modelled by logbiharmonic mappings particularly those arising from fluid flow theory.

The property of stable univalence is also considered.

2. Main Results

A complex valued function $f : \Omega \rightarrow \mathbb{C}$ is said to belong to the class $C^1(\Omega)$ if $\Re f$ and $\Im f$ have continuous first order partial derivatives in $\Omega$. We denote the Jacobian of $f$ by
$$J_f = |f_z|^2 - |f_{\overline{z}}|^2.$$
We also denote
$$\lambda_f = |f_z| - |f_{\overline{z}}|,$$
and
$$\Lambda_f = |f_z| + |f_{\overline{z}}|,$$
We then have
$$J_f = \lambda_f \Lambda_f.$$
We first start by noting that the results in [6], hold for functions of the form $f(z) = |z|^{2(p-1)} G(z) + K(z)$, where $G$ is not necessarily harmonic. (If $G$ is harmonic these classify as a special family of polyharmonic mappings).
**Theorem 1.** Let \( f(z) = |z|^{2(p-1)}G(z) + K(z) \), where \( G \in C^1(\mathbb{D}) \) (not necessarily harmonic) and \( K \) harmonic. If \( K \) is univalent and \( K(\mathbb{D}) \) is a linearly connected domain with constant \( M \), and if
\[
\frac{2(p-1)|G| + \Lambda_G}{|\lambda_K|} < \frac{1}{M},
\]
then \( f(z) \) is univalent. Moreover, if
\[
\frac{2(p-1)|G| + \Lambda_G}{|\lambda_K|} \leq C < \frac{1}{M},
\]
then \( f(\mathbb{D}) \) is a linearly connected domain.

**Proof.** Let \( H(z) = |z|^{2(p-1)}G(z) \). We define
\[
\varphi = H \circ K^{-1}.
\]
Given \( w \in K(\mathbb{D}) \), we claim \( w + \varphi(w) \) is univalent.

Assume \( w + \varphi(w) \) is not univalent, then there exists \( w_1 \neq w_2 \) such that
\[
\varphi(w_2) - \varphi(w_1) = w_1 - w_2.
\]
Let \( \gamma \) be a path in \( K(\mathbb{D}) \) joining \( w_1, w_2 \) such that \( l(\gamma) \leq M|w_2 - w_1| \). Then
\[
|\varphi(w_2) - \varphi(w_1)| \leq \left| \int_{\gamma} \varphi_w dw + \varphi_{\overline{w}}d\overline{w} \right| \leq \int_{\gamma} (|\varphi_w| + |\varphi_{\overline{w}}|)|dw|.
\]

But
\[
\varphi_w = H_z(K^{-1})_w + H_{\overline{z}}(K^{-1})_{\overline{w}}
\]
\[
\varphi_{\overline{w}} = H_z(K^{-1})_{\overline{w}} + H_{\overline{z}}(K^{-1})_{\overline{w}}.
\]

Differentiating \( K^{-1}(K(z)) = z \), we show that
\[
(K^{-1})_wK_z + (K^{-1})_{\overline{w}}(\overline{K_z}) = 1
\]
\[
(K^{-1})_wK_{\overline{z}} + (K^{-1})_{\overline{w}}(\overline{K_{\overline{z}}}) = 0.
\]

Solving above system we get,
\[
(K^{-1})_w = \frac{\overline{K_z} - |K_{\overline{z}}|^2}{|K_z|^2 - |K_{\overline{z}}|^2},
\]
and,
\[
(K^{-1})_{\overline{w}} = \frac{-K_z - |K_{\overline{z}}|^2}{|K_z|^2 - |K_{\overline{z}}|^2}.
\]
It follows
\[
|\varphi_w| + |\varphi_{\overline{w}}| \leq \frac{|H_z|}{|K_z|^2 - |K_{\overline{z}}|^2}(|\overline{K_z} + |K_{\overline{z}}|) + \frac{|H_{\overline{z}}|}{|K_z|^2 - |K_{\overline{z}}|^2}(|\overline{K_z} + |K_{\overline{z}}|)
\]
\[
= \frac{|H_z| + |H_{\overline{z}}|}{|K_z| - |K_{\overline{z}}|}.
\]
where $z = K^{-1}(w) \in D$. But

$$H_z = (p - 1)z^{p-2}z^{p-1}G + |z|^{2(p-1)}G_z,$$

and

$$H_\gamma = (p - 1)z^{p-2}z^{p-1}G + |z|^{2(p-1)}G_\gamma,$$

hence,

$$|\varphi(w_2) - \varphi(w_1)| \leq \sup_{D} \frac{2(p-1)|G| + |G_\gamma| + |G_\gamma|}{|K_z| - |K_\gamma|} |dw| < \frac{1}{M} l(\gamma) < |w_2 - w_1|$$

which is a contradiction. Therefore $f(z)$ is univalent.

For the second part of theorem, given $w \in \Omega = K(\mathbb{D})$, we let $\Psi(w) = w + \varphi(w)$, where $\varphi = H \circ K^{-1}$, and $H = |z|^{2(p-1)}G$. Since $K$ is univalent, we may look at $R = f(\mathbb{D})$, as the image of $\Omega = K(\mathbb{D})$ under the mapping $\Psi$, and we show $\Psi(\Omega)$ is linearly connected. Let $\varsigma_1 = \Psi(w_1), \varsigma_2 = \Psi(w_2), w_1, w_2 \in \Omega$. Since $K(\mathbb{D})$ is a linearly connected domain, then there exists a curve $\gamma \subset \Omega$ satisfying $l(\gamma) \leq M|w_2 - w_1|$.

Let $\Gamma = \Psi(\gamma)$.

We have showed that

$$|\varphi_w| + |\varphi_\gamma| \leq \frac{2(p-1)|G| + \Lambda G}{|\lambda_K|} < C,$$

It follows

$$|\psi_w| + |\psi_\gamma| \leq 1 + |\varphi_w| + |\varphi_\gamma| < 1 + C.$$

Hence we have,

$$l(\Gamma) = \int_{\gamma} |\psi_w| + |\psi_\gamma| |dw| < (1 + C)l(\gamma) \leq (1 + C)M|w_2 - w_1|.$$

But,

$$|\varsigma_1 - \varsigma_2| = |w_1 - w_2 + \varphi(w_1) - \varphi(w_2)| \geq |w_1 - w_2| - |\varphi(w_1) - \varphi(w_2)|$$

$$\geq |w_1 - w_2| - \int_{\gamma} (|\varphi_w| + |\varphi_\gamma|) dw$$

$$> |w_1 - w_2| - CL(\gamma) \geq (1 - CM)|w_1 - w_2|$$

It follows,

$$l(\Gamma) \leq \frac{(1 + C)M}{1 - CM}|\varsigma_1 - \varsigma_2|$$

and so $f(\mathbb{D})$ is linearly connected with constant $\frac{(1+C)M}{1-CM}$. $\square$

In particular, for the special case where $p = 2$ we have the following corollary :

**Corollary 1.** Let $f(z) = |z|^2G(z) + K(z)$, where $G \in C^1(\mathbb{D})$ (not necessarily harmonic). If $K$ is univalent harmonic and $K(\mathbb{D})$ is a linearly connected domain with constant $\lambda$, and if

$$\frac{2|G| + \Lambda G}{|\lambda_K|} < \frac{1}{M},$$

then $f(z)$ is univalent.
then \( f(z) \) is univalent. Moreover, if
\[
2|G| + \frac{\Lambda G}{|\lambda_K|} \leq C < \frac{1}{M},
\]
then \( f(\mathbb{D}) \) is a linearly connected domain.

We are now ready to prove the theorem for polyharmonic functions. In fact, we will state the theorem in its most general form and the polyharmonic functions will be a special case.

**Theorem 2.** Let
\[
f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)
\]
where \( G_{p-k+1} \in C^1(D) \) (not necessarily harmonic) \( k = \{2, ..., p\} \) (In particular \( f \) is polyharmonic). If \( G_p \) is univalent harmonic and \( G_p(\mathbb{D}) \) is a linearly connected domain with constant \( M \), and if
\[
\sum_{k=1}^{p-1} 2k|G_{p-k}| + (k-1)\Lambda_{G_{p-k}} < \frac{1}{M},
\]
then \( f(z) \) is univalent. Moreover, if
\[
\sum_{k=1}^{p-1} 2k|G_{p-k}| + (k-1)\Lambda_{G_{p-k}} \leq C < \frac{1}{M},
\]
then \( f(\mathbb{D}) \) is a linearly connected domain.

**Proof.** We rewrite \( f(z) \) as \( f(z) = |z|^2G(z) + G_p(z) \), where
\[
G(z) = \sum_{k=1}^{p-1} |z|^{2(k-1)} G_{p-k}(z).
\]

Since
\[
G_z = \sum_{k=1}^{p-1} (k-1)z^{k-2}z^{k-1}G_{p-k} + |z|^{2(k-1)}(G_{p-k})_z
\]
and
\[
G_{\overline{z}} = \sum_{k=1}^{p-1} (k-1)z^{k-1}z^{-2}G_{p-k} + |z|^{2(k-1)}(G_{p-k})_{\overline{z}}
\]
We get that
\[
\frac{2|G| + |G_z| + |G_{\overline{z}}|}{|\lambda_{G_p}|} \leq \frac{\sum_{k=1}^{p-1} 2k|G_{p-k}| + (k-1)((G_{p-k})_z + |(G_{p-k})_{\overline{z}}|)}{|\lambda_{G_p}|} < \frac{1}{M},
\]
(or for the proof of the second part of the theorem \( \leq C < \frac{1}{M} \).)

Therefore by Corollary 1, we get that \( f \) is univalent and \( f(\mathbb{D}) \) is a linearly connected domain.

The following corollary is deduced as a special case of theorem 2.
Corollary 2. Let
\[ f(z) = \sum_{k=1}^{p} |z|^{2(k-1)}G_{p-k+1}(z) \]
where \( G_{p-k+1} \in C^1(D) \) (not necessarily harmonic) \( k = \{2, ..., p\} \) (In particular \( f \) is polyharmonic). If \( G_p \) is univalent harmonic and \( G_p(D) \) is a convex domain, and if
\[ \sum_{k=1}^{p-1} 2k|G_{p-k}| + (k-1)\Lambda G_{p-k} < 1, \]
then \( f(z) \) is univalent. Moreover, if
\[ \sum_{k=1}^{p-1} 2k|G_{p-k}| + (k-1)\Lambda G_{p-k} \leq C < 1, \]
then \( f(D) \) is a convex domain.

In [6], the following theorem that allows to conclude the univalence of \( K \) from the univalence of \( f \) was proved:

Theorem 3 ([6, theorem 3]). Let \( f(z) = |z|^2G(z) + K(z) \) be a biharmonic function in the unit disk \( D \). Suppose \( f \) is univalent and \( f(D) \) is a linearly connected domain with constant \( M \) and satisfies
\[ \frac{2|G| + \Lambda G}{|\lambda_f|} < \frac{1}{M}, \]
then \( K(z) \) is univalent. Moreover, if
\[ \frac{2|G| + \Lambda G}{|\lambda_f|} \leq C < \frac{1}{M}, \]
then \( K(D) \) is a linearly connected domain.

More generally, we have

Theorem 4. Let \( f(z) = |z|^{2(p-1)}G(z) + K(z) \), where \( G \in C^1(D) \) (not necessarily harmonic) and \( K \) harmonic. If \( K \) is univalent and \( K(D) \) is a linearly connected domain with constant \( M \), and if
\[ \frac{2(p-1)|G| + \Lambda G}{|\lambda_f|} < \frac{1}{M}, \]
then \( k(z) \) is univalent. Moreover, if
\[ \frac{2(p-1)|G| + \Lambda G}{|\lambda_f|} \leq C < \frac{1}{M}, \]
then \( k(D) \) is a linearly connected domain.
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Proof. Let \( H(z) = |z|^{2(p-1)}G(z) \). We define
\[
\varphi = H \circ f^{-1}.
\]

Given \( w \in f(D) \), we claim \( \psi(w) = w - \varphi(w) \) is univalent.

Assume \( w - \varphi(w) \) is not univalent, then there exists \( w_1 \neq w_2 \) such that
\[
\varphi(w_1) - \varphi(w_2) = w_1 - w_2.
\]

Let \( \gamma \) be a path in \( K(D) \) joining \( w_1, w_2 \) such that \( l(\gamma) \leq M|w_2 - w_1| \). We proceed as in the proof of Theorem 1 to show that
\[
|\varphi(w_1) - \varphi(w_2)| \leq \left| \int_\gamma \varphi_w \, dw + \varphi_\bar{w} \, d\bar{w} \right|
\]
\[
\leq \int_\gamma (|\varphi_w| + |\varphi_\bar{w}|) \, |dw|
\]
\[
\leq \int_\gamma \sup_{D} \frac{2(p-1)|G| + |G_z| + |G_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \, |dw|
\]
\[
< \frac{1}{M} l(\gamma)
\]
\[
< |w_2 - w_1|
\]
which is a contradiction. Therefore \( K(z) \) is univalent.

For the second part of theorem, given \( w \in \Omega = f(D) \), we let \( \psi(w) = w - \varphi(w) \) and we show \( \psi(\Omega) \) is linearly connected, where \( \Omega = f(D) \). Let \( \varsigma_1 = \psi(w_1), \varsigma_2 = \psi(w_2), w_1, w_2 \in \Omega \). Since \( f(D) \) is a linearly connected domain, then there exists a curve \( \gamma \subset \Omega \) satisfying \( l(\gamma) \leq M|w_2 - w_1| \).

Let \( \Gamma = \psi(\gamma) \). We proceed as in the proof of Theorem 1 and we show that
\[
l(\Gamma) \leq \frac{(1 + C)M}{1 - CM} |\varsigma_1 - \varsigma_2|
\]
and so \( k(D) \) is linearly connected with constant \( \frac{(1 + C)M}{1 - CM} \). \( \square \)

In a similar fashion as in the proof of Theorem 2, we generalize the theorem 3 in [6] to the following :

**Theorem 5.** Let
\[
f(z) = \sum_{k=1}^{p} |z|^{2(k-1)}G_{p-k+1}(z)
\]
where \( G_{p-k+1} \in C^1(D) \) (not necessarily harmonic), \( k = \{2, \ldots, p\} \). Suppose \( f \) is univalent and \( f(D) \) is a linearly connected domain with constant \( M \) and satisfies
\[
\frac{\sum_{k=1}^{p-1} 2k|G_{p-k}| + (2(k-1))(|\Delta G_{p-k}|)}{|\lambda_f|} < \frac{1}{M},
\]

then $G_p(z)$ is univalent. Moreover, if
\[
\sum_{k=1}^{p-1} 2k |G_{p-k}| + (2(k - 1))(\|\Lambda G_{p-k}\|) \leq C < \frac{1}{M},
\]
then $G_p(\mathbb{D})$ is a linearly connected domain.

Proof. We rewrite $f(z)$ as
\[
f(z) = |z|^2 G(z) + G_p(z),
\]
where
\[
G(z) = \sum_{k=1}^{p-1} |z|^{2(k-1)} G_{p-k}(z).
\]

Since
\[
G_z = \sum_{k=1}^{p-1} (k-1) z^{k-1} z^{k-1} G_{p-k} + |z|^{2(k-1)} (G_{p-k})_z
\]
and
\[
G_{\overline{z}} = \sum_{k=1}^{p-1} (k-1) z^{k-1} \overline{z}^{k-2} G_{p-k} + |z|^{2(k-1)} (G_{p-k})_{\overline{z}}
\]
We get that
\[
2|G| + |G_z| + |G_{\overline{z}}| \leq \sum_{k=1}^{p-1} 2k |G_{p-k}| + (k - 1)(\|G_{p-k}\| + |(G_{p-k})_z| + |(G_{p-k})_{\overline{z}}|) < \frac{1}{2M}.
\]
Therefore by theorem 3 in [6], $f$ is univalent. \qed

The case where $f$ is convex is a corollary:

Corollary 3. Let
\[
f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)
\]
where $G_{p-k+1} \in C^1(\mathbb{D})$ (not necessarily harmonic), $k = \{2, ..., p\}$. Suppose $f$ is univalent, $f(\mathbb{D})$ is a convex domain. and satisfies
\[
\sum_{k=1}^{p-1} 2k |G_{p-k}| + (2(k - 1))(\|\Lambda G_{p-k}\|) < \frac{1}{2},
\]
then $G_p(z)$ is univalent. Moreover, if
\[
\sum_{k=1}^{p-1} 2k |G_{p-k}| + (2(k - 1))(\|\Lambda G_{p-k}\|) \leq C < \frac{1}{2},
\]
then $G_p(\mathbb{D})$ is a convex domain.

We next prove a stable univalence property for polyharmonic mappings onto linearly connected domains.
Theorem 6. Let \( f(z) = |z|^{2(p-1)}G(z) + K(z) \), where \( G \in C^1(\mathbb{D}) \) (not necessarily harmonic) and \( K \) harmonic. If \( f \) is univalent and \( f(\mathbb{D}) \) is a linearly connected domain with constant \( M \), and if

\[
\frac{2(p-1)|G| + \lambda G}{|f'_z|} < \frac{1}{2M},
\]

then \( f_a(z) = a|z|^{2(p-1)}G(z) + K(z) \) is univalent for any \( a \), such that \( |a| < 1 \). Moreover, if

\[
\frac{2(p-1)|G| + \lambda G}{|\lambda_f|} \leq C < \frac{1}{2M},
\]

then \( f_a(\mathbb{D}) \) is a a linearly connected domain.

Proof. We note that

\[
f_a(z) = a|z|^{2(p-1)}G(z) + K(z) = a|z|^{2(p-1)}G(z) + f(z) - |z|^{2(p-1)}G(z) = f(z) + (a - 1)|z|^{2(p-1)}G(z)
\]

Suppose \( f_a(z) \) is not univalent, then there exists \( z_1 \neq z_2 \) such that

\[
f(z_2) - f(z_1) = (1 - a)(H(z_2) - H(z_1)),
\]

where \( H(z) = |z|^{2(p-1)}G(z) \).

Let \( w = f(z) \) and \( \varphi = H \circ f^{-1} \), we get \( w_2 - w_1 = (1 - a)(\varphi(w_2) - \varphi(w_1)) \). So

\[
|w_2 - w_1| \leq 2|\varphi(w_2) - \varphi(w_1)|.
\]

As in the proof of theorem 1, we have

\[
|\varphi(w_2) - \varphi(w_1)| \leq \int_\gamma \sup_{D} \frac{2(p-1)|G| + |G_z| + |G_{zz}|}{||f'_z| - |f'_z||} |dw| < \frac{1}{2M} l(\gamma) < \frac{|w_2 - w_1|}{2}
\]

which is a contradiction.

Therefore \( f_a(z) \) is univalent.

Next we assume

\[
\frac{2(p-1)|G| + \lambda G}{|\lambda_f|} \leq C < \frac{1}{2M},
\]

we let \( \psi(w) = w + (a - 1)\varphi(w) \) and we show \( \psi(\Omega) \) is linearly connected.where \( \Omega = f(\mathbb{D}) \). Let \( \varsigma_1 = \psi(w_1), \varsigma_2 = \psi(w_2), w_1, w_2 \in \Omega \). Since \( f(\mathbb{D}) \) is a linearly connected domain, then there exists a curve \( \gamma \subset \Omega \) satisfying \( l(\gamma) \leq M|w_2 - w_1| \).

Let \( \Gamma = \psi(\gamma) \). We proceed as in the proof of Theorem 1 and we show that

\[
l(\Gamma) \leq \int_\gamma (|\psi'_w| + |\psi'_w|) dw < (1 + |a - 1|C)l(\gamma) \leq (1 + 2C)M|w_2 - w_1|.
\]

But,

\[
|\varsigma_1 - \varsigma_2| \geq |w_1 - w_2| - |1 - a||\varphi(w_1) - \varphi(w_2)| > |w_1 - w_2| - 2C l(\gamma) \geq (1 - 2CM)|w_1 - w_2|
\]
It follows, $$l(\Gamma) \leq \frac{(1 + 2C)M}{1 - 2CM} |\varsigma_1 - \varsigma_2|$$ and so $k(\mathbb{D})$ is linearly connected with constant $\frac{(1 + 2C)M}{1 - 2CM}$.

since $f$ is univalent and $f(\mathbb{D})$ is a linearly connected domain, it follows by theorem 3 $K(\mathbb{D})$ is a linearly connected domain. Hence by theorem 1, $f_\alpha(\mathbb{D})$ is a linearly connected domain. □

**Corollary 4.** Let $f(z) = |z|^{2(p-1)}G(z) + K(z)$, where $G \in C^1(\mathbb{D})$ (not necessarily harmonic) and $K$ harmonic. If $f$ is univalent and $f(\mathbb{D})$ is a convex domain, and if

$$\frac{2(p-1)|G| + \Lambda_G}{|\lambda_f|} < \frac{1}{2},$$

then $f_\alpha(z) = a|z|^{2(p-1)}G(z) + K(z)$ is univalent for any $a$, such that $|a| < 1$. Moreover, if

$$\frac{2(p-1)|G| + \Lambda_G}{|\lambda_f|} \leq C < \frac{1}{2},$$

then $f_\alpha(\mathbb{D})$ is a convex domain.

### 3. log $p$-HARMONIC MAPPINGS

A log harmonic mapping defined on $\mathbb{D}$ is a solution of the non linear elliptic partial differential equation

$$\overline{\mu_f} \frac{\partial f}{\partial \overline{z}} = f_z, \quad f(0) = 0$$

where the second dilation $\mu$ is analytic in $\mathbb{D}$ such that $|\mu(z)| < 1$. In general the solution of this equation is not necessarily univalent. For example, $f(z) = |z|^4z^4$ is logharmonic but not univalent in $\mathbb{D}$. We say that $f$ log $p$-harmonic if $\log f$ is $p$-harmonic. Throughout “log” denotes the principal branch of the logarithm. It can be easily shown that every log $p$-harmonic function in a simply connected domain $\Omega$ has the form

$$f(z) = \Pi_{k=1}^p (g_{p-k+1}(z)) |z|^{2(k-1)},$$

where all $g_{p-k+1}(z)$ are non vanishing log harmonic mappings in $\Omega$ for $k = \{1, ..., p\}$.

**Theorem 7.** Let

$$f(z) = \Pi_{k=1}^p (g_{p-k+1}(z)) |z|^{2(k-1)},$$

where all $g_{p-k+1}(z)$ are non vanishing log harmonic mappings in $\mathbb{D}$, $k = \{1, ..., p\}$. If $\log g_p$ is univalent and $\log g_p(\mathbb{D})$ is a linearly connected domain with constant $M$, and if

$$\frac{|g_p| \sum_{k=1}^{p-1} 2k|g_{p-k}|| \log g_{p-k} + (k - 1)\Lambda g_{p-k}}{|\lambda_{g_p}|} < \frac{1}{M},$$

then $f_\alpha(\mathbb{D})$ is a convex domain.
then \( f(z) \) is univalent. Moreover, if
\[
\frac{|g_p| \sum_{k=1}^{p-1} 2k|g_{p-k}|| \log g_{p-k}| + (k - 1)\Lambda g_{p-k}}{|\lambda_{g_p}|} \leq C < \frac{1}{M},
\]
then \( G_p(D) \) is a linearly connected domain.

**Proof.** Define \( F = \log f \) and \( G_{p-k} = \log g_{p-k} \) for all \( g_{p-k}, k = \{1, \ldots, p\} \). Then \( F \) is \( p \)-harmonic and all \( G_{p-k} \) are harmonic. Moreover, \( G_p \) is univalent, \( G_p(D) \) is a linearly connected domain with constant \( M \), and
\[
\frac{\sum_{k=1}^{p-1} 2k|G_{p-k}| + (k - 1)(|G_{p-k}|_z + |G_{p-k}|_\zeta)}{|(G_p)_z - |(G_p)\zeta|} = \frac{|g_p| \sum_{k=1}^{p-1} 2k|g_{p-k}|| \log g_{p-k}| + (k - 1)(|g_{p-k}|_z + |g_{p-k}|_\zeta)}{|(g_p)_z - |(g_p)\zeta|}
\]
\[
< \frac{1}{M},
\]
and so \( F(z) \) is univalent by Theorem 2. Therefore \( f \) is univalent. \( \square \)

**Theorem 8.** Let
\[
f(z) = \prod_{k=1}^{p} (g_{p-k+1}(z))^{2(k-1)},
\]
where all \( g_{p-k+1}(z) \) are non vanishing log harmonic mappings in \( D \), \( k = \{1, \ldots, p\} \). If \( \log g_p \) is univalent and \( \log g_p(D) \) is convex, and if
\[
\frac{|g_p| \sum_{k=1}^{p-1} 2k|g_{p-k}|| \log g_{p-k}| + (k - 1)\Lambda g_{p-k}}{|\lambda_{g_p}|} < 1,
\]
then \( f(z) \) is univalent. Moreover, if
\[
\frac{|g_p| \sum_{k=1}^{p-1} 2k|g_{p-k}|| \log g_{p-k}| + (k - 1)\Lambda g_{p-k}}{|\lambda_{g_p}|} \leq C < 1,
\]
then \( G_p(D) \) is a convex domain.

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