On whether zero is in the global attractor of the 2D Navier–Stokes equations

Ciprian Foias¹, Michael S Jolly², Yong Yang¹,³ and Bingsheng Zhang¹

¹ Department of Mathematics, Texas A&M University, College Station, TX 77843, USA
² Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
E-mail: foias@math.tamu.edu, msjolly@indiana.edu, yytamu@math.tamu.edu and bszhang@math.tamu.edu

Received 6 January 2014, revised 19 June 2014
Accepted for publication 17 September 2014
Published 15 October 2014

Abstract
The set of nonzero external forces for which the zero function is in the global attractor of the two-dimensional Navier–Stokes equations is shown to be meagre in a Fréchet topology. A criterion in terms of a Taylor expansion in complex time is used to characterize the forces in this set. This leads to several relations between certain Gevrey subclasses of $C^\infty$ and a new upper bound for a Gevrey norm of solutions in the attractor, valid in the strip of analyticity in time.

Keywords: Navier–Stokes equations, global attractor, analyticity in time
Mathematics Subject Classification: 35Q30, 76D05, 34G20, 37L05, 37L25

1. Introduction
A challenge posed by P Constantin [2] is to find a simple proof that zero is in the global attractor $\mathcal{A}$ of the 2D Navier–Stokes equations (NSEs) if and only if the external force $g$ is zero. A related and perhaps equally challenging problem is to find sharp lower bounds on the energy in cases where we know $0 \notin \mathcal{A}$. A bound which is probably far from sharp can be found in [4]. Such a lower bound can have implications for turbulent flow because a direct cascade of energy is indicated by a large quotient of average enstrophy to average energy [8].

In this paper, we show that the set of nonzero forces for which $0 \in \mathcal{A}$ is meagre (of the first Baire category in a Fréchet topology (see theorem 6.3)). In doing so we establish several relations between certain Gevrey subclasses of $C^\infty$ (see theorems 5.2, 5.4). We also prove a new upper bound for a Gevrey norm of solutions in the attractor, valid for all $\xi$ in the strip

³ Author to whom any correspondence should be addressed.
S(δ) of time-analyticity (see theorem 7.3). Moreover, using complex time analytic techniques from [10], we present a concrete criterion that is both sufficient and necessary for 0 ∈ ∂A. We demonstrate the use of this criterion to prove that zero is not in the global attractor in the particular case of Kolmogorov forcing (where g is in an eigenfunction of the Stokes operator).

2. General preliminaries

We consider the NSEs with periodic boundary conditions in Ω = [0, L]²

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= F, \\
\nabla \cdot u &= 0, \\
u_0(x), \\
\int_{\Omega} u \, dx &= 0, \\
\int_{\Omega} F \, dx &= 0,
\end{aligned}
\]

where \( u : \mathbb{R}^2 \to \mathbb{R}^2 \), \( p : \mathbb{R}^2 \to \mathbb{R} \) are unknown Ω-periodic functions, \( \nu > 0 \) is the viscosity of the fluid, \( L > 0 \) is the period, \( p \) is the pressure and \( F \) is the ‘body’ force (see [3, 12, 13] for more details). The phase space \( H \) is defined as the subspace of \([L^2(\Omega)]^2\) consisting of all elements in the closure of the set of \( \mathbb{R}^2 \)-valued trigometric polynomials \( v \) satisfying \( \nabla \cdot v = 0 \) and \( \int_{\Omega} v(x) \, dx = 0 \).

The scalar product in \( H \) is taken to be

\[
(u, v) = \int_{\Omega} u(x) \cdot v(x) \, dx
\]

with associated norm \( |u| = (u, u)^{1/2} \).

Let \( \mathcal{P} : [L^2(\Omega)]^2 \to [L^2(\Omega)]^2 \) be the orthogonal projection (called the Helmholtz–Leray projection) with range \( H \), and define the Stokes operator \( A = -\mathcal{P} \Delta (=-\Delta, \text{under periodic boundary conditions}) \), which is positive, self-adjoint with a compact inverse. As a consequence, the real Hilbert space \( H \) has an orthonormal basis \( \{\omega_j\}_{j=1}^\infty \) eigenfunctions of \( A \), namely, \( A\omega_j = \lambda_j \omega_j \) with \( 0 < \lambda_1 = \left(\frac{2\pi}{L}\right)^2 \leq \lambda_2 \leq \lambda_3 < \cdots \). The powers \( A^\sigma \) are defined by

\[
A^\sigma v = \sum_{j=1}^\infty \lambda_j^\sigma (v, \omega_j) \omega_j, \quad \sigma \in \mathbb{R},
\]

where \((\cdot, \cdot)\) is the \( L^2 \)-scalar product. The domain of \( A^\sigma \) is denoted by \( \mathcal{D}(A^\sigma) \), where

\[
\mathcal{D}(A^\sigma) = \left\{ v \in H : \sum_{j=1}^\infty \lambda_j^{2\sigma} (v, \omega_j)^2 < \infty \right\}.
\]

Finer points regarding \( \mathcal{D}(A) \) and Sobolev spaces are provided in appendix C. The system (2.1) can be written as a differential equation

\[
\frac{du}{dt} + \nu Au + B(u, u) = g, \quad u \in H,
\]

where the bilinear operator \( B \) and the driving force \( g \) are defined as \( B(u, v) = \mathcal{P}((u \cdot \nabla)v) \) and \( g = \mathcal{P} F \), respectively.

Under periodic boundary conditions, we may express an element \( u \in H \) as a Fourier series expansion

\[
u_0(x) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \hat{u}(k)e^{i\kappa k \cdot x},
\]
where \( \kappa_0 = \lambda_1^{1/2} = \frac{2\pi}{T} \), \( \hat{u}(0) = 0 \), \((\hat{u}(k))^* = \hat{u}(-k) \), \( k \cdot \hat{u}(k) = 0 \). Parseval’s identity reads as
\[
|u|^2 = L^2 \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \hat{u}(k) \cdot \hat{v}(-k) = L^2 \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |\hat{u}(k)|^2,
\]
or more generally
\[
(u, v) = L^2 \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \hat{u}(k) \cdot \hat{v}(-k).
\]

The following inequalities will be repeatedly used in this paper
\[
\kappa_0 |u| \leq |A^{1/2}u|, \quad \text{for } u \in D(A^{1/2}),
\]
\[
|u|_{L^2} \leq C_{2, \Omega}|u|^{1/2} A^{1/2}u |^{1/2}, \quad \text{for } u \in D(A^{1/2}),
\]
\[
|u|_\infty \leq c_A |u|^{1/2} A u |^{1/2}, \quad \text{for } u \in D(A),
\]
known, respectively, as the Poincaré, Ladyzhenskaya and Agmon inequalities. Both \( c_L \) and \( c_A \) are absolute constants.

We recall that the global attractor \( A \) of the NSE is the collection of all elements \( u_0 \) in \( H \) for which there exists a solution \( u(t) \) of the NSE, for all \( t \in \mathbb{R} \), such that \( u(0) = u_0 \) and \( \sup_{t \geq 0} |u(t)| < \infty \).

To give an equivalent definition of \( A \), we need to recall several concepts. First, as is well known, for any \( u_0 \in H \), \( g \in H \), there exists a unique continuous function \( u \) from \( \{0, \infty\} \) to \( H \) such that \( u(0) = u_0 \), \( u(t) \in D(A) \), \( t \in (0, \infty) \) and \( u \) satisfies the NSE for all \( t \in (0, \infty) \). Therefore, one can define the map \( S(t) : H \to H \) by
\[
S(t) u_0 = u(t),
\]
where \( u(\cdot) \) is as above. Since \( S(t_1) S(t_2) = S(t_1 + t_2) \), the family \( \{S(t)\}_{t \geq 0} \) is called the ‘solution’ semigroup. Furthermore, a compact set \( \mathcal{B} \) is called absorbing if for any bounded set \( \mathcal{B} \subset H \) there is a time \( \tilde{t} = \tilde{t}(\mathcal{B}) \geq 0 \) such that \( S(t) \mathcal{B} \subset \mathcal{B} \) for all \( t \geq \tilde{t} \). The attractor can be now defined by the formula
\[
A = \bigcap_{t \geq 0} S(t) \mathcal{B},
\]
where \( \mathcal{B} \) is any absorbing compact subset of \( H \). For other equivalent definitions of \( A \), see sections 3.1, 4.1 in [9].

We now consider the NSE with complexified time and the corresponding solutions in \( H_C \) as in [6], [10] and [11]. We recall that
\[
H_C = \{ u + iv : u, v \in H \},
\]
and that \( H_C \) is a Hilbert space with respect to the following inner product:
\[
(u + iv, w^* + iw^*)_{H_C} = (u, w^*_H) + (v, w^*_H) + i[(v, w^*_H) - (u, w^*_H)],
\]
where \( u, u^*, v, v^* \in H \). The extension \( A_C \) of \( A \) is given by
\[
A_C (u + iv) = Au + iAv,
\]
for \( u, v \in D(A) \); thus \( D(A_C) = D(A_C) \subset D(A_C) \). Similarly, \( B(\cdot, \cdot) \) can be extended to a bounded bilinear operator from \( D(A^{1/2}) \times D(A^{1/2}) \) to \( H_C \) by the formula
\[
B_C (u + iv, w^* + iw^*) = B(u, w^*_H) - B(v, w^*_H) + i[B(u, w^*_H) + B(v, w^*_H)],
\]
for \( u, v \in D(A^{1/2}) \), \( w, w^* \in D(A) \).

The NSE in complex time is defined as
\[
\frac{du(\zeta)}{d\zeta} + vAu(\zeta) + B_C (u(\zeta), u(\zeta)) = g,
\]
where \( \zeta \in \mathbb{C} \), \( u(\zeta) \in H_C \) and \( \frac{du(\zeta)}{d\zeta} \) denotes the derivatives of \( H_C \)-valued analytic function \( u(\zeta) \).
3. Specific preliminaries

In this section, we first recall the definition of the class $C(\sigma)$ introduced in our previous paper [7]. We also collect the properties regarding the class $\bigcup_{\sigma > 0} C(\sigma)$. The class $C(\sigma)$ is defined to be a subset of $C^\infty([0, L]^2) \cap H$ for which every element $u \in C(\sigma)$ has a specified growth rate for the powers of the operator $A$ applied to $u$

$$C(\sigma) := \{ u \in C^\infty([0, L]^2) \cap H : \exists \ c_0 = c_0(u) \in \mathbb{R}, \ \frac{|A^2 u|^2}{|\partial_x^2 u|^2} \leq c_0 e^{\pi \alpha^2}, \ \forall \alpha \in \mathbb{N} \}. \quad (3.1)$$

In this definition we allow only $\alpha \in \mathbb{N}$; however, as shown in section 11 of [7], we could actually extend this definition to allow $\alpha$ to take any real numbers without changing the class $C(\sigma)$. We stress that the constant $c_0 \in \mathbb{R}$ in the definition of the class $C(\sigma)$ depends on $u$.

To make our presentation more self-contained, we include some of the relevant results from [7]. The first result gives some consequences of zero belonging to the attractor.

**Theorem 3.1.** If $0 \in \mathcal{A}$, then both the attractor $\mathcal{A}$ and the force $g$ will be in the class $C(\sigma)$; namely

$$0 \in \mathcal{A} \Rightarrow \mathcal{A} \subset C(\sigma_0) \quad \text{and} \quad g \in C(\sigma_1), \quad (3.2)$$

for some $\sigma_1 > \sigma_0 > 0$, where $\sigma_0$ and $\sigma_1$ both depend on the force $g$.

In particular, we have the following estimates.

**Theorem 3.2.** If $0 \in \mathcal{A}$, then there exist fixed positive constants $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, C(g), \beta_1, \beta_2, \delta_1, \delta_2, \delta_3$ such that the analytic extension of any solution $u(t), \ t \in \mathbb{R}$ in $\mathcal{A}$ satisfies for any $\alpha \in \mathbb{N}$

$$|A^\alpha u(\xi)| \leq \tilde{R}_{\alpha+1}^2 |k_0^{\alpha+1}|, \quad \forall \xi \in \mathcal{S}(\delta_u) := \{ \xi \in \mathbb{C} : |\Im(\xi)| < \delta_u \},$$

where for $\alpha > 3$,

$$\tilde{R}_{\alpha+1} \leq C(g) \beta_1 \beta_2^\alpha \delta_3, \quad \delta_u = \delta_3.$$

For the sake of completeness, explicit expressions for $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, C(g), \beta_1, \beta_2, \delta_1, \delta_2, \delta_3$ are recalled in appendix A. The next result from [7] merely states a simple hierarchy of the spaces $C(\sigma), \ \sigma \in \mathbb{R}^+$. 

**Proposition 3.3.** For the family of classes $\{C(\sigma)\}_{\sigma > 0}$, we have,

$$C(\sigma_1) \subset C(\sigma_2), \quad \forall \sigma_1 < \sigma_2.$$

The union of the classes $C(\sigma)$ is a proper subset of $C^\infty$.

**Theorem 3.4.** $\bigcup_{\sigma > 0} C(\sigma) \subset C^\infty([0, L]^2 \cap H)$

The following ‘all for one, one for all’ law states that the attractor cannot be only partially contained in the union of these classes.

**Theorem 3.5.** If $\mathcal{A} \cap \bigcup_{\sigma > 0} C(\sigma) \neq \emptyset$, then $\mathcal{A} \subset \bigcup_{\sigma > 0} C(\sigma)$.

4. Constantin–Chen Gevrey classes

In this section, we give the definition for the general Constantin–Chen Gevrey ($C^2\mathcal{G}$) classes [1].

Given a function $\phi(\chi)$ with the following properties:

1. $\phi'(\chi) > 0,$
2. $\phi''(\chi) < 0,$

for all $\chi \in [1, \infty)$, we define the general Constantin–Chen Gevrey ($C^2\mathcal{G}$) class $E(\phi)$ as the collection of all $u \in C^\infty([0, L]^2) \cap H$ for which $|e^{\phi(\chi)/\chi^2}|u|$ is finite, that is,
In this section we will investigate the relation between the class $C^5$. The main results are stated in theorem 5.2 and theorem 5.4.

Definition 4.1. $E(\phi) = \{u \in H : |e^{\phi(k_{\chi}^{-1}A^2)}u| < \infty\}$.

where

$$(e^{\phi(k_{\chi}^{-1}A^2)}u)(k) := e^{\phi(k)}\hat{u}(k), \quad \forall k \in \mathbb{Z}^2 \setminus \{0\}.$$ 

A typical example of a $C^2\mathcal{G}$ class is $\tilde{\phi}(\chi) = \beta \ln \chi$, for $\beta > 0$. Actually, $E(\tilde{\phi}) = H \cap \mathcal{D}(A^{3/2})$. This typical $C^2\mathcal{G}$ class is used in [7] to prove the following two estimates for the bilinear term $[(B(u, v), A^\gamma w)]$, where $u \in \mathcal{D}(A^2), v \in \mathcal{D}(A^3), w \in \mathcal{D}(A^\gamma), \gamma > 3$.

Lemma 4.2. Let $u \in \mathcal{D}(A^2), v \in \mathcal{D}(A^3), w \in \mathcal{D}(A^\gamma), \alpha > 3$, then

$$|(B(u, v), A^\alpha w)| \leq 2^\alpha c_A \left(|u|^{1/2}A^1 u + |A^2 w||v|^{1/2}|A^{3/2} v|^{1/2}\right)|A^\alpha w|.$$ 

If, moreover, $u \in \mathcal{D}(A^2)_C, v \in \mathcal{D}(A^3)_C, w \in \mathcal{D}(A^\gamma)_C$, then

$$|(B(u, v), A^\alpha w)| \leq 2^{3\alpha/2} c_A \left(|u|^{1/2}A^1 u + |A^2 w||v|^{1/2}|A^{3/2} v|^{1/2}\right)|A^\alpha w|.$$ 

5. $C(\sigma)$ and $E(\phi_b)$

In this section we will investigate the relation between the class $C(\sigma)$ and $E(\phi_b)$, where

$$\phi_b(\chi) = b[\ln(\chi + e)]^2, \quad b > 0.$$ 

(5.1)

The main results are stated in theorem 5.2 and theorem 5.4.

For convenience, we take the following notation.

Definition 5.1. For $b > 0$, we define $E_b := E(\phi_b), E^b u := e^{\phi_b(k_{\chi}^{-1}A^2)}u, |u|_b := |E^b u|.$

In our previous paper [7], we have obtained the following result.

Theorem 5.2 (see remark 7.6 in [7]). If $v \in E_b$ for some $b > 0$, then

$$v \in C\left(\frac{1}{2b}\right).$$

The ‘reverse’ inclusion relation between the classes $E_b$ and $C(\sigma)$ is given in theorem 5.4.

Proposition 5.3. If $u \in C(\sigma)$ for some $\sigma > 0$, i.e.

$$\exists c_0 > 0, \quad \text{s.t.} \quad |A^\sigma u|^2 \leq c_0 e^{\alpha^*} (\nu e_{\sigma})^2, \quad \forall \alpha \in \mathbb{N},$$ 

(5.2)

then for fixed $\epsilon \in [0, 1]$, there exists $b := \frac{1}{\pi^2 \alpha^*}$ such that

$$|e^{b[\ln^\epsilon(k_{\chi}^{-1}A^{1/2})]} u| < \infty.$$ 

(5.3)

In particular, we have

$$|e^{b[\ln^\epsilon(k_{\chi}^{-1}A^{1/2})]} u|^2 \leq \frac{4}{3} c(\epsilon)|u|^2 + c_1^1 (c_1|A^1 u|)^2 v^2 k_0^{1/2},$$ 

(5.4)

where

$$c(\epsilon) := e^{2b[\ln^\epsilon(\sigma^\epsilon)]}, \quad c_1 = \sum_{m \geq 2} \frac{1}{e^m} = \frac{1}{e^2 - e}.$$
Proof. First, by the definition of $e^{(|\ln(\epsilon)|^{1/2})u}$, we have
\[
|e^{2\ln(|\epsilon|^{1/2})u}|^2 = \sum_{m=0}^{\infty} \sum_{e^m \leq |k| < e^{m+1}} e^{2\ln(|k|+e)} |\hat{u}(k)|^2
\]
\[
= \sum_{m=0}^{\infty} \sum_{m \geq 2} =: I_1 + I_2.
\]

For $I_1$, it is easy to see that
\[
I_1 = \sum_{m=0}^{\infty} \sum_{e^m \leq |k| < e^{m+1}} e^{2\ln(|k|+e)} |\hat{u}(k)|^2
\]
\[
\leq e^{2\ln(|\epsilon|^{1/2})u} \sum_{k \in \mathbb{Z} \setminus \{0\}} |\hat{u}(k)|^2
\]
\[
= e^{2\ln(|\epsilon|^{1/2})u} |u|^2
\]
\[
= c(\epsilon) |u|^2 < \infty,
\]
while for $I_2$, using the definition of the class $C(\sigma)$ and Young’s inequality we can infer
\[
I_2 = \sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} e^{2\ln(|k|+e)} |\hat{u}(k)|^2
\]
\[
= \sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} (|k| + e)^{2\ln(|k|+e)} |\hat{u}(k)|^2
\]
\[
\leq \sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} (|k| + e)^{2b(m+2)} |\hat{u}(k)|^2
\]
\[
\leq \sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} |k|^{4b(m+2)} |\hat{u}(k)|^2
\]
\[
= \sum_{m \geq 2} |A^{b(m+2)}(P_{m+1} - P_m)u|^2 \frac{4^{2b(m+2)}}{3^{1/2}}
\]
\[
\leq \sum_{m \geq 2} |A^{2b(m+2)}u||P_{m+1} - P_m|u|^2 \frac{4^{2b(m+2)}}{3^{1/2}}
\]
\[
\leq v \sum_{m \geq 2} c_0^{1/2} e^{|P_{m+1} - P_m|u|^2} \frac{4^{2b(m+2)}}{3^{1/2}}
\]
\[
\leq v \sum_{m \geq 2} c_0^{1/2} e^{2\sigma b \epsilon m^2} \frac{4^{2b(m+2)}}{3^{1/2}} |(P_{m+1} - P_m)u|^2 \frac{4^{2b(m+2)}}{3^{1/2}}
\]
\[
\leq v \left( \sum_{m \geq 2} c_0^{1/2} e^{2\sigma b \epsilon m^2} \frac{4^{2b(m+2)}}{3^{1/2}} \right)^{1/4} \left( \sum_{m \geq 2} |(P_{m+1} - P_m)u|^2 \right)^{3/4}
\]
\[
= v^{1/4} J_{21}^{1/2} J_{22}^{3/4}.
\]

We now derive estimates for $I_{21}$ and $I_{22}$. For $I_{21}$, we obtain
\[
I_{21} = \sum_{m \geq 2} \sum_{e^m \leq |k| < e^{m+1}} |(P_{m+1} - P_m)u|^2
\]
\[
\leq \sum_{m \geq 0} \sum_{e^m \leq |k| < e^{m+1}} |(P_{m+1} - P_m)u|^2
\]
\[
= \sum_{m \geq 0} \sum_{e^m \leq |k| < e^{m+1}} |\hat{u}(k)|^2
\]
\[c_0^2 \sum_{m \geq 0} \sum_{m \geq 0} e^{2\sigma b^2 \|k\| + \epsilon} |\hat{u}(k)|^2 \leq c_0^2 \sum_{m \geq 0} \sum_{m \geq 0} e^{2\sigma b^2 \|k\| + \epsilon} |\hat{u}(k)|^2,\]

since

\[2\epsilon \leq 1 + \epsilon, i.e. \epsilon \leq 1.\]

Defining \(b\) as

\[2\epsilon = 1 + \epsilon, \quad i.e. \quad \epsilon = \frac{1}{2+\epsilon},\]

we immediately obtain

\[I_{21} \leq c_0^2 \sum_{m \geq 0} \sum_{m \geq 0} e^{2\epsilon \|k\| + \epsilon} |\hat{v}(k)|^2 \]

\[= c_0^2 e^{\epsilon \|k\| \epsilon^1/2} |u|^2.\]

For \(I_{22}\), we set \(v = A^{1/2} u\) and apply Hölder’s inequality as follows:

\[I_{22} = \sum_{m \geq 2} \|A^{-1/2} (P_{m+1} - P_m) A^{1/2} u\|^{2/3} \]

\[= \sum_{m \geq 2} \left( \sum_{m \geq 2} \frac{1}{k_0^1} \sum_{m \leq k \leq m+1} |\hat{v}(k)|^2 \right)^{1/3} \]

\[\leq \frac{1}{k_0^{2/3}} \sum_{m \geq 2} \frac{1}{e^{2m/3}} \left( \sum_{m \leq k \leq m+1} |\hat{v}(k)|^2 \right)^{1/3} \]

\[\leq \frac{1}{k_0^{2/3}} \left( \sum_{m \geq 2} \frac{1}{e^m} \right)^{2/3} \left( \sum_{m \geq 2} \sum_{m \leq k \leq m+1} |\hat{v}(k)|^2 \right)^{1/3} \]

\[\leq c_1 \frac{\|A^{1/2} u\|}{k_0} < \infty.\]

Therefore, we have

\[\|e^{\epsilon \|k\| \epsilon^1/2} u\|^2 \leq I_1 + I_2 \]

\[\leq I_1 + I_2^{1/4} I_2^{3/4} \]

\[= I_1 + \left(\|e^{\epsilon \|k\| \epsilon^1/2} u\|^2 \right)^{1/4} \left( c_0^{1/2} I_2^{1/4} \right) \]

\[\leq I_1 + c_1^{2/3} I_2^{2/3} \]

\[= \frac{4}{3} I_1 + \frac{4}{3} \left( c_0 \epsilon \right)^2 + \left( c_0 \epsilon \right)^2 \frac{4}{3} \frac{1}{k_0^{2/3}} \|A^{1/2} u\|^{2/3}.\]
Theorem 5.4. If \( u \in C(\sigma) \), then \( u \in E_b \), where \( b := \frac{1}{64\sigma} \).

In corollary 7.5 of [7], we have proved that if \( 0 \in A \), then \( g \in C(\frac{1}{2}\ln \beta_3) \), where \( \beta_3 \) is defined in theorem 3.2. Applying theorem 5.4, we obtain the following result.

Corollary 5.5. If \( 0 \in A \), then \( g \in E_b \), where \( b = \frac{1}{160\ln \beta_3} \).

6. The topological properties of the ‘all for one, one for all law’ classes

In this section we use the space \( \mathcal{F} := C^\infty \cap H \) with the Fréchet topology defined by the following metric

\[
d(u, v) := \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha} \frac{|A^\alpha(u - v)|}{1 + |A^\alpha(u - v)|}.
\]

Let

\[
E_{b,n} := \{ u \in E_b, |u|_b \leq n \}.
\]

Lemma 6.1. \( E_{b,n} \) is nowhere dense in \( (\mathcal{F}, d) \).

Proof. First, we prove \( i_b : E_{b,n} \to (\mathcal{F}, d) \) is compact. Clearly, for all \( \alpha \in \mathbb{N} \), there exist a constant \( c_\alpha \) such that

\[
|A^\alpha u| \leq c_\alpha |u|_b.
\]

For any sequence \( \{u_n\} \subset E_{b,n} \), we have that \( \{|A^\alpha u_n|\} \) is bounded by (6.3). Therefore, there exists a subsequence \( \{u_{n_m}\} \) which is convergent in \( D(A^\alpha) \). Since this is true for any fixed \( \alpha \in \mathbb{N} \), by the diagonal process, we obtain a subsequence, denoted by the same notation \( \{u_{n_m}\} \) for convenience, which is convergent for any \( \alpha \in \mathbb{N} \). Hence it is convergent in \( C^\infty \) with the metric \( d(\cdot, \cdot) \). Therefore, \( i_b \) is compact. Then, it follows that \( i_b(E_{b,n}) \) is compact in \( (\mathcal{F}, d) \).

Secondly, suppose \( i_b(E_{b,n}) \) is not nowhere dense. Then there exists a ball \( B(x_0, \epsilon) \subset i_b(E_{b,n}) \). Clearly, \( B(x_0, \epsilon) \subset i_b(E_{b,n}) \). Clearly, \( B(x_0, \epsilon) \) is compact. If \( x_0 = 0 \), this contradicts the extension of the classical Riesz’s Lemma for normed spaces to locally convex topological vector space, since \( C^\infty \) is infinite-dimensional space.

If \( x_0 \neq 0 \), consider a convex open neighbourhood \( N_{x_0} \subset B(x_0, \epsilon) \). We have that \( N_{x_0} \) and \( -N_{x_0} \) both are compact and convex. Let \( f : N_{x_0} \times (-N_{x_0}) \ni (x_1, x_2) \mapsto \frac{x_1}{2} + \frac{x_2}{2} \in C^\infty \cap H \).

Clearly, \( f \) is continuous. Therefore the range \( R(f) \) is compact. Since \( \frac{1}{2}(N_{x_0} + (-N_{x_0})) \) is an open neighbourhood of 0 in \( R(f) \). For the same reason as above, we get a contradiction. \( \square \)

Due to the above lemma, it follows that

Theorem 6.2. \( \cup_{n=1}^{\infty} \cup_{n=1}^{\infty} E_{b,n} \) is of first (Baire) category in \( (\mathcal{F}, d) \).

From the above theorem and corollary 5.5, we have that the conjecture that \( g \neq 0 \) implies \( 0 \notin A \) is ‘almost’ true in the following sense.

Theorem 6.3. If \( 0 \in A \), then \( g \in E_b \) (is defined in corollary 5.5) where \( E_b \) has the property that \( E_b = \cup_{n=1}^{\infty} E_{b,n} \) is of first (Baire) category in \( (\mathcal{F}, d) \).
7. Dynamical properties of the NSE in $E_b$

This section is devoted to getting a new estimate for solutions in the global attractor with the norm $|·|_{E_b}$ in the strip $\mathcal{S}(\delta)$, where

$$\mathcal{S}(\delta) := \{ \zeta \in \mathbb{C} : |\Im(\zeta)| < \delta \}. \quad (7.1)$$

First, for the nonlinear term $B(·, ·)$ we need the following estimate.

**Lemma 7.1.** If $u \in E_b$, then $B(u, u) \in E_b$ and

$$|E_b B(u, u)| \leq \frac{C A L}{k_0^{2b} \ln 2} \left( |A^{\ln 2} u|^{1/2} |A^{1+b \ln 2} u|^{1/2} |A^{1/2} E^b u| \right. \right.$$  
$$+ \left. |A^{1/2+2b \ln 2} u|^{1/2} |A^{3/2+2b \ln 2} u|^{1/2} |E^b u| \right), \quad (7.2)$$

where

$$c_3 = e^{b \ln 2} (1 + e)^{2b \ln 2}. \quad (7.3)$$

**Proof.** By definition, for any $w \in E_b$, we have

$$I := (E_b B(u, u), w) = (B(u, u), E_b w)$$

$$= L^2 \sum_{h, j, k \in \mathbb{Z}^2 \setminus \{0\}} (\hat{u}(h) \cdot j)(\hat{u}(j) \cdot \hat{w}(k)) e^{b \ln(|h| e)})^2$$

$$= L^2 \sum_{h, j, k \in \mathbb{Z}^2 \setminus \{0\}} \cdots + L^2 \sum_{h, j, k \in \mathbb{Z}^2 \setminus \{0\}} \cdots$$

$$|h| \leq |j| \quad |h| > |j|$$

$$=: I_1 + I_2.$$

For $I_1$, it is easy to check that $\psi(x) = [\ln(c+x)]^2 - [\ln(d+x)]^2$ is decreasing for $e \leq d < c$ so that

$$e^{b \ln(|h| e)})^2 - b \ln(|h| e)})^2 \leq e^{b \ln(|h| + e)})^2 - b \ln(|h| + e)})^2 \leq e^{b \ln(2|h| + e)})^2 - b \ln(|h| + e)})^2.$$

Moreover, since

$$\left( \ln(2|h| + e) \right)^2 - \left( \ln(|h| + e) \right)^2 = \ln \frac{2|h| + e}{|h| + e} \ln(2|h| + e)/(|h| + e) \quad \leq \ln 2 \ln(2 + 2 \ln(|h| + e)), $$

it follows that

$$I_1 = L^2 \sum_{h, j, k \in \mathbb{Z}^2 \setminus \{0\}} (\hat{u}(h) \cdot j)(\hat{u}(j) \cdot \hat{w}(k)) e^{b \ln(|h| e)})^2 - b \ln(|j| e)})^2$$

$$\leq L^2 \sum_{h, j, k \in \mathbb{Z}^2 \setminus \{0\}} (\hat{u}(h) \cdot j)(\hat{u}(j) \cdot \hat{w}(k)) e^{b \ln 2}(|h| + e)^{2b \ln 2}$$

$$\leq L^2 \sum_{h, j, k \in \mathbb{Z}^2 \setminus \{0\}} (\hat{u}(h) \cdot j)(\hat{u}(j) \cdot \hat{w}(k)) e^{b \ln 2} (1 + e)^{2b \ln 2} |h|^{2b \ln 2}$$

$$= \frac{C_3 A L}{k_0^{2b} \ln 2} \left( |A^{\ln 2} u|^{1/2} |A^{1+b \ln 2} u|^{1/2} |A^{1/2} E^b u| \right). \quad (7.4)$$

2763
By estimating $I_2$ in the same way, just replacing the right-hand side of the first equality of (7.4) by

$$L^2 \sum_{h, j, k \in \mathbb{Z} \setminus \{0\}, \ h+j+k=0, \ |j| \leq |h|} (\hat{E}(h) \cdot j)(\hat{G}(j) \cdot \hat{w}(k))e^{b \ln(|k|+\varepsilon)^2 - b \ln(|h|+\varepsilon)^2}, \tag{7.5}$$

we obtain

$$I_2 \leq \frac{C_3 A}{k_0^{2/3}} |A^{1/2+b \ln 2} u|^{1/2} |A^{3/2+b \ln 2} u|^{1/2} |E^h u||w|. \tag{7.6}$$

Since (7.4) and (7.6) are true for arbitrary $w$, then we infer (7.2). □

Using interpolation, it is easy to obtain the following estimates on $A^\alpha u$ for any $\alpha > 0$ in the strip $S(\delta)$.

**Lemma 7.2.** Suppose $\alpha \geq 3$, and $\alpha \in [\frac{7 + \varepsilon}{2}, \frac{7 + 2\varepsilon}{2})$. Then

$$|A^\alpha u| \leq \tilde{R}_a v_0^\alpha, \tag{7.7}$$

where

$$\tilde{R}_a^2 = C(g) \beta_1^4 (6\alpha + 3\gamma) \beta_2^{-\gamma^2 + (4\alpha - 1)\gamma + 11\alpha}, \tag{7.8}$$

and $C(g), \beta_1, \beta_2$ are defined in theorem 3.2.

**Proof.** By interpolation, we have

$$|A^\alpha u| \leq |A^\frac{\gamma}{2} u|^{\gamma + 1 - 2\alpha} |A^{\frac{\gamma}{2}} u|^{2\alpha - \gamma}. \tag{7.9}$$

Using theorem 3.2, it follows that

$$|A^\alpha u|^2 \leq [C(g) \beta_1^4 \beta_2^{\gamma^2 + 2\gamma + 2(y + 1)\gamma + 2(\gamma + 1)]^{2\alpha - \gamma} v^2 k_0^{2\alpha}$$

$$= C(g) \beta_1^4 (6\alpha + 3\gamma) \beta_2^{-\gamma^2 + (4\alpha - 1)\gamma + 11\alpha} v^2 k_0^{2\alpha}$$

$$= \tilde{R}_a^2 v^2 k_0^{2\alpha}. \tag{7.10}$$

□

Using lemmas 7.1 and 7.2, we are ready to obtain an estimate of $|u|_b$ in the strip $S(\delta)$.

**Theorem 7.3.** If $0 \in A$ and if $u(t)$, $t \in \mathbb{R}$ is any solution of the NSE in $A$, then $u(t)$ satisfies

$$|u(\zeta)|_b \leq \tilde{R}_{new} v, \quad \forall \zeta \in S(\delta), \tag{7.11}$$

where $\delta = \delta_3 (\delta_3$ is defined in appendix A),

$$\tilde{R}_{new}^2 := e^{M_1 \delta v_k^2} (\frac{1}{2} c_2 \tilde{R}_1^2 + c_1 (c_1 \tilde{R}_1)^2) + M_2 (e^{M_1 \delta v_k^2} - 1), \tag{7.12}$$

and

$$M_1 := 4(c_3 c_A)^2 \tilde{R}_{new}^2 + 2\sqrt{2} c_3 c_A \tilde{R}_{new}^{2/3} + 2, \quad M_2 := \frac{2\sqrt{2} |A^{-\frac{\gamma}{2}} E^h g|^2}{(2\sqrt{2} (c_3 c_A)^2 \tilde{R}_{new}^{2/3} + 2 c_3 c_A \tilde{R}_{new}^{2/3}) v^2 k_0^{2\alpha}}. \tag{7.13}$$
**Proof.** It is shown in remark 7.3 in [7] that if \( 0 \in A \), then there exists a constant \( \beta_1 \) such that
\[
u(t_0) \in C_0(\ln \beta_1), \quad \forall t_0 \in \mathbb{R}.
\]
(7.14)
Applying corollary 5.4 and proposition 5.3, we obtain that
\[
u(t_0) \in \mathcal{E}_0(\ln \beta_1)^{-1},
\]
(7.15)
and
\[
|E^b u(t_0)|^2 \leq \frac{4}{3} c_2 |u|^2 + c_3^2 (c_1 |A^\frac{1}{2} u|)^2 v^2 |\kappa_0^2|^{-2}
\]
(7.16)
where
\[
c_2 := c(\epsilon)_{\epsilon=1} = e^{2h(\ln(c^2+\epsilon))}.
\]
(7.17)
Taking the inner product of (2.8) with \( E^{2h} u \) we obtain
\[
\frac{1}{2} \frac{d}{d\rho} |E^b u(t_0 + \rho e^{i\theta})|^2 + \nu \cos \theta |A|^{1/2} E^b u|^2 \leq |A^{\frac{1}{2}} E^b u||A^{-\frac{1}{2}} E^b g| + |(E^b B(u, u), E^b u)|,
\]
where \( \theta \in [-\pi/4, \pi/4] \). Using lemma 7.1, we have that
\[
\frac{1}{2} \frac{d}{d\rho} |E^b u(t_0 + \rho e^{i\theta})|^2 + \nu \cos \theta |A|^{1/2} E^b u|^2 \leq \frac{c_3 A}{k_0^{2b in \frac{2}{2}}} |A^{\frac{1}{2}}E^b u|^2 \|A^{\frac{1}{2}} E^b u||E^b u|^2
\]
\[
+ \frac{c_3 A}{k_0^{2b in \frac{2}{2}}} |A^{\frac{1}{2}}E^b u|^2 |A^{\frac{1}{2}} E^b u||E^b u|^2 \|A^{\frac{1}{2}} E^b u||E^b u|^2
\]
\[
\leq \eta_1 + \eta_2 |E^b u|^2,
\]
where
\[
\eta_1 = \frac{2 \sqrt{2}}{\nu} |A^{-\frac{1}{2}} E^b g|^2,
\]
(7.18)
and
\[
\tilde{\eta}_2 := \frac{2 \sqrt{2}}{\nu} \left( \frac{c_3 A}{k_0^{2b in \frac{2}{2}}} \right)^2 |A^{\frac{1}{2}}E^b u|^2 \|A^{\frac{1}{2}} E^b u||E^b u|^2
\]
\[
+ \frac{2c_3 A}{k_0^{2b in \frac{2}{2}}} |A^{\frac{1}{2}}E^b u|^2 |A^{\frac{1}{2}} E^b u||E^b u|^2 \|A^{\frac{1}{2}} E^b u||E^b u|^2.
\]
For \( \tilde{\eta}_2 \), applying the Poincaré inequality and lemma 7.2, we have the following estimate:
\[
\tilde{\eta}_2 \leq \frac{2 \sqrt{2}}{\nu} \left( \frac{c_3 A}{k_0^{2b in \frac{2}{2}}} \right)^2 \left( \frac{1}{k_0} \|A^{\frac{1}{2}}E^b u|^2 \right) + \frac{2c_3 A}{k_0^{2b in \frac{2}{2}}} \left( \frac{1}{k_0} \right) |A^{\frac{1}{2}} E^b u|^2
\]
(7.19)
\[
\leq \frac{2 \sqrt{2}}{\nu} \left( \frac{c_3 A}{k_0^{2b in \frac{2}{2}}} \right)^2 \left( \frac{1}{k_0} \|A^{\frac{1}{2}}E^b u|^2 \right) + \frac{2c_3 A}{k_0^{2b in \frac{2}{2}}} \left( \frac{1}{k_0} \right) \tilde{R}_{\tau+2b ln^2} \tilde{R}_{\tau+2b ln^2} \kappa_0^2 \kappa_0^2
\]
\[
= \left[ 2 \sqrt{2} (c_3 A)^2 \tilde{R}_{\tau+2b ln^2} + 2c_3 A \tilde{R}_{\tau+2b ln^2} \right] \kappa_0^2 \kappa_0^2 \eta_2,
\]
where \( \tilde{R}_{\tau+2b ln^2} \) is defined as in (7.8).
Then we have
\[ \frac{d}{d\rho} |E^b u(t_0 + \rho e^{i\theta})|^2 \leq \eta_1 + \eta_2 |E^b u|^2. \]  
(7.20)

It follows from Gronwall’s inequality that
\[ |E^b u(t_0 + \rho e^{i\theta})|^2 \leq e^{\sqrt{2} \delta_0} |E^b u(t_0)|^2 + \frac{\eta_1}{\eta_2} (e^{\sqrt{2} \delta_0^2} - 1). \]  
(7.21)

Plugging (7.16), (7.18), (7.19) into (7.21), we obtain that
\[ |E^b u(t_0 + \rho e^{i\theta})|^2 \leq e^{M_1 \delta_0} \left[ \frac{4}{3} c_2 \tilde{R}_i^2 + c_0 (c_1 \tilde{R}_i)^2 v^2 + M_2 (e^{M_1 \delta_0^2} - 1) \right] v^2 = \tilde{R}_{\text{new}} v^2, \]

where \( \tilde{R}_{\text{new}} \) is defined in (7.11). Since \( t_0 \in \mathbb{R} \) and \( \theta \in [-\pi/4, \pi/4] \) are arbitrary, it follows that (7.10) holds for all \( \zeta \in S(\delta) \).

These are only \textit{a priori} estimates. Using a Galerkin approximation on a countable system of time intervals which exhausts the real time axis, it can be shown that they are also valid for any solution in \( A \).

\[ \square \]

8. An explicit criterion

In section 6, we found that generically 0 is not in the attractor \( A \) since if \( 0 \in A \), then \( g \) must be in the set \( E_b \) which is of first category (see theorem 6.3). One immediately asks the following question: if \( g \in E_b \), will \( 0 \in A \)? We partially answer this question by presenting a concrete criterion that is both sufficient and necessary for \( 0 \in A \).

To present our result, we need some preparation. First, theorem 7.3 tells us we can choose \( \delta > 0 \) and \( M > 0 \), such that for every \( u_0 \in A \), \( S(t)u_0 \) is extendable to a holomorphic function on \( S(\delta) = \{ z \in \mathbb{C} : |z| < \delta \} \) with values in \( E_b \), and \( |S(t)u_0|_b \leq M \) for all \( t \in S(\delta) \).

Let \( u_0 = 0 \in A \); let \( u(t) = S(t)u_0 \) be the solution of the NSE; we use the conformal mapping (see [10])
\[ \phi : S(\delta) \to \Delta = \{ T \in \mathbb{C} : |T| < 1 \} \]
defined by the following formula:
\[ T = \phi(t) = \frac{\exp(\pi t/2\delta) - 1}{\exp(\pi t/2\delta) + 1}, \quad t \in S(\delta) \]

with inverse given by
\[ t = \phi^{-1}(T) = \frac{2\delta}{\pi} [\log(1 + T) - \log(1 - T)]. \]

The function \( U(T) = u(t) \) satisfies the ODE
\[ \frac{dU}{dT} = \delta_0 \psi(T) [g - vAU - B(U, U)], \quad T \in \Delta \]  
(8.1)

with initial value
\[ U(0) = u_0, \]

where
\[ \psi(T) = \frac{1}{2} \left( \frac{1}{1 + T} + \frac{1}{1 - T} \right) = \frac{1}{1 - T^2} \]

and \( \delta_0 = 4\delta/\pi \).
By the analyticity of the function \( U(T) \), we may express it in a Taylor series
\[
U(T) = U_0 + U_1 T + U_2 T^2 + \cdots.
\] (8.2)
Note that \( U_0 = u_0 \). The convergence radius of the series (8.2) is at least 1 if \( u_0 \in A \), and it may be less than 1 if \( u_0 \notin A \).

Combining the series expansion form (8.2) for \( U(T) \) and the ODE (8.1), we obtain
\[
\frac{d}{dT} \left( \infty \sum_{n=0}^{\infty} U_n T^n \right) = \delta_0 g - \nu A \infty \sum_{n=0}^{\infty} U_n T^n - \infty \sum_{n=0}^{\infty} \sum_{h+k=n} B(U_h, U_k)
\]
from which we obtain the following criterion for \( 0 \in A \):

**Theorem 8.1.** \( 0 \in A \) if and only if the Taylor series
\[
\infty \sum_{n=0}^{\infty} U_n T^n, \quad T \in \Delta
\] (8.3)
converges in \( \| \cdot \| \) for all \( T \in \Delta \) and the sum \( U(T) = \infty \sum_{n=0}^{\infty} U_n T^n \), for \( |T| < 1 \), satisfies an estimate \( |U(T)|_b \leq M \), for some \( M > 0 \), where \( U_n \) are computed recursively according to
\[
U_0 = 0, \quad U_1 = \delta_0 g, \quad U_2 = -\frac{\nu \delta_0^2}{2} A g
\] and for \( n \geq 2 
\[
U_{n+1} = \frac{n}{n+1} U_{n-1} - \frac{\nu \delta_0}{n+1} A U_n - \frac{\delta_0}{n+1} \sum_{h+k=n} B(U_k, U_h),
\] (8.4)

**Remark 8.2.** Several remarks are in order.
(1) Note that all the \( U_n \)'s defined in the theorem 8.1 depend only on \( g \).
(2) The application of the criterion given in theorem 8.1 does not seem to be an easy task in general. We illustrate its use in the next section in the special case of forcing a single eigenvector of \( A \).

**9. The case of Kolmogorov forcing**

We now use the criterion given in theorem 8.1 to show that if the force \( g \neq 0 \) is an eigenvector of the Stokes operator \( A \), with corresponding eigenvalue \( \lambda > 0 \), then \( 0 \) cannot be in \( A \).

If \( 0 \in A \), where \( Ag = \lambda g \), then noting that (by theorem B.1 in appendix B)
\[
B(g, g) = 0,
\] (9.1)
the following lemma immediately follows from the recursive relation (8.4) given in theorem 8.1.

**Lemma 9.1.** For the coefficients \( U_n \), we have
\[
U_n = p_n(\lambda) g, \quad n = 1, 2, 3, \ldots,
\]
where \( p_n(\cdot) \) are polynomials satisfying the following relations:
\[
p_1(\lambda) = \delta_0, \quad (9.2)
\]
\[
p_2(\lambda) = -\frac{\nu \lambda}{2}\delta_0^2, \quad (9.3)
\]
\[
p_{N+1}(\lambda) = \frac{N-1}{N+1} p_{N-1}(\lambda) - \frac{\nu \delta_0 \lambda}{N+1} p_N(\lambda), \quad N = 2, 3, \ldots.
\] (9.4)
**Proof.** By theorem 8.1, we can obtain (9.2) and (9.3) easily. Assume by induction that $U_n = p_n(\lambda)g$ is valid for all $n \leq N$, where $N \geq 2$. Then by (8.4),

$$(N + 1)U_{N+1} = (N - 1)U_{N-1} - v\delta_0 AU_N - \delta_0 \sum_{h+k=N} B(U_h, U_k)$$

$$= (N - 1)p_{N-1}(\lambda)g - v\delta_0 \lambda p_N(\lambda)g - \delta_0 \sum_{h+k=N} p_h(\lambda)p_k(\lambda)B(g, g)$$

$$= (N - 1)p_{N-1}(\lambda)g - v\delta_0 \lambda p_N(\lambda)g.$$

Therefore,

$$U_{N+1} = p_{N+1}(\lambda)g,$$

where,

$$p_{N+1}(\lambda) = \frac{N - 1}{N + 1} p_{N-1}(\lambda) - \frac{v\delta_0 \lambda}{N + 1} p_N(\lambda).$$

The proof is completed by the induction hypothesis. □

From the above lemma and theorem 8.1, we conclude that if $0 \in \mathcal{A}$, then the solution $u(t)$ is of a special form, namely, $u(t) = \phi(t)g$, where $\phi(t)$ is a bounded real-valued function on $\mathbb{R}$. Clearly the function $\phi(t)$ must satisfy the following ODE:

$$\frac{d\phi}{dt} + v\lambda \phi = 1,$$

from which it follows that

$$\phi(t) = \frac{1}{v\lambda} + \left(\phi(0) - \frac{1}{v\lambda}\right) e^{-v\lambda t}.$$

Boundedness of the solution $u(t)$ for all negative time implies that $\phi(0) = \frac{1}{v\lambda}$, and hence $u(t) \equiv \frac{\delta_0}{v\lambda}$. This contradicts $u(0) = \phi(0)g = 0$. Therefore, in this case, using the criterion and dynamics analysis, we obtain that 0 is not in $\mathcal{A}$.

**Acknowledgments**

This work was supported in part by NSF grant numbers DMS-1109638 and DMS-1109784.

**Appendix A**

The bounds in theorem 3.2 are found recursively in [7], starting with

$$\tilde{R}_1 = \sqrt{2}G, \quad \tilde{R}_2 = \left(\frac{3(\sqrt{2} \cdot 16 \cdot 24^6 c_1 L)^{2/3}}{4(2c_1^2 + c_A)^{4/3}} G^6 + 4R_2^2\right)^{1/2}, \quad \tilde{R}_3 = \frac{4N_2^{1/3}}{vK_0\delta_2^2},$$

where

$$R_2 = 2137c_1^6 G^3, \quad N_2 = R_2^2 + \frac{2\delta_2\tilde{R}_2^2}{vK_0\delta_1} + 16(2c_1^2 + c_A)^2 vK_0\delta_2 \tilde{R}_1 \tilde{R}_2^2.$$
\[ \delta_2 = \min \left\{ \delta_1, 16^{-1} \left[ (2c_L^2 + c_A)^4 \tilde{R}_1^4 \left( \frac{\nu k_0^2}{8 \delta_1} \right)^{\frac{1}{4}} + (2c_L^2 + c_A)^4 (v k_0^2)^2 \tilde{R}_1^2 \tilde{R}_2^2 \right]^{\frac{1}{2}} \right\}, \quad \delta_1 = \frac{\delta_2}{2}. \]

The other constants are given by
\[ C(g) = C_1 C_2 \tilde{R}_3^3 \beta_2^{-19/2}, \quad \beta_1 = e^{2 \sqrt{2} \nu k_0^2 C_1 b}, \]
\[ \beta_2 = \max \left\{ \frac{72 \sqrt{2}}{\pi^2}, c_A^2 \tilde{R}_1 \tilde{R}_2 \right\}, \quad \beta_3 = \max \left\{ \frac{1024 \sqrt{2}}{\pi^2}, c_A^3 \tilde{R}_1 \tilde{R}_2 \right\}, \quad b = \frac{1}{96 \ln \beta_3}, \]
where
\[ C_1 := \prod_{\gamma = 3}^{\infty} (1 + \epsilon_\gamma), \quad C_2 := 3^3 2^{-7} e_A^2 \tilde{R}_1^2 \prod_{\gamma = 3}^{\infty} (1 + \eta_\gamma), \]
\[ C_3 := 4 \left( 2 e_A^2 \tilde{R}_1 \tilde{R}_2 + 2 e_A \sqrt{\tilde{R}_1 \tilde{R}_3} \right), \]
\[ \epsilon_\gamma = \frac{1}{2 \sqrt{2} \gamma_0 \nu k_0^2} + \frac{\sqrt{2}}{\Gamma_0^2 k_0^2 \delta_0^2} + \frac{\pi^2}{72 \nu^2 k_0^2 \delta_0^2}, \quad \eta_\gamma = \frac{\sqrt{\tilde{R}_1 \tilde{R}_3}}{2^{y+2} e_A \tilde{R}_1 \tilde{R}_2}, \]
and
\[ \Gamma_\gamma := 2 e_A^2 \tilde{R}_1 \tilde{R}_2 \left( 2^{y+2} e_A \tilde{R}_1 \tilde{R}_2 + \sqrt{\tilde{R}_1 \tilde{R}_3} \right). \]

**Appendix B**

**Theorem B.1.** If \( Ag = \lambda g \), then
\[ B(g, g) = 0. \]

**Proof.** In appendix A of [5], it was shown that
\[ (B(Au, u), v) = (B(v, u), Au), \quad \text{for } u, v \in D(A). \] (B.1)
Now, if \( Ag = \lambda g \), then clearly \( g \in D(A) \), and thus by (B.1)
\[ \lambda(B(g, g), v) = (B(Ag, g), v) = (B(v, g), Ag) = \lambda(B(v, g), g). \]
Using the well-known fact that \( (B(v, g), g) = 0 \), we get \( B(g, g), v) = 0 \), for all \( v \in D(A) \). Since \( D(A) \) is dense in \( H \), we have \( B(g, g) = 0 \). \( \square \)

**Appendix C**

We recall that if a bounded open set \( \Omega \) has the strong local Lipschitz property, then any element \( \phi \in H^2(\Omega) = W^{2,2}(\Omega) \) coincides almost everywhere in \( \Omega \) with a function \( \tilde{\phi} \in C(\Omega) \) (see theorem 5.4, Part II on page 98 in the book [14]).

In case \( \Omega = [0, L]^2 \), it is obvious that \( \Omega \) has a locally Lipschitz boundary, hence it has the strong local Lipschitz property (see page 66 in [14] for the definition of strong local Lipschitz property and the remark on page 67 immediately after the definition). Therefore, the following
definition is consistent: $\phi$ is periodic in $x_1, x_2$ (with period $L > 0$), if $\tilde{\phi}$ is periodic in $x_1, x_2$ (with period $L > 0$), i.e.

$$
\tilde{\phi}(0, x_2) = \tilde{\phi}(L, x_2), \quad \forall \ x_2 \in [0, L],
$$

$$
\tilde{\phi}(x_1, 0) = \tilde{\phi}(x_1, L), \quad \forall \ x_1 \in [0, L].
$$

We can now define the domain $D(A)$ of the Stokes operator $A$ as the set of the vector-valued functions $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H^2(\Omega)^2$ satisfying the following conditions:

(i) $\tilde{u}_1, \tilde{u}_2$ are periodic in $x_1, x_2$ with period $L$;

(ii) $\int_{\Omega} u_j \, dx_1 \, dx_2 = \int_{\Omega} \tilde{u}_j \, dx_1 \, dx_2 = 0, \forall \ j = 1, 2$;

(iii) $u$ is divergence-free in $\Omega$, i.e.

$$
\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \quad \text{in } \Omega,
$$

where the partial derivatives are in the weak sense. (Actually these derivatives are classical almost everywhere in $\Omega$. See theorem 2.1.4, page 44 in [15], and use the well-known Fubini theorem).

References

[1] Chen W 1994 New a priori estimates in Gevrey class of regularity for weak solutions of 3D Navier–Stokes equations Differ. Integral Eqns 7 101–7

[2] Constantin P 2004 private communication

[3] Constantin P and Foias C 1989 Navier–Stokes Equations (Chicago Lectures in Mathematics) (Chicago, IL: University of Chicago)

[4] Dascaliuc R, Foias C and Jolly M S 2005 Relations between energy and enstrophy on the global attractor of the 2D Navier–Stokes equations J. Dyn. Diff. Eqns 17 643–736

[5] Dascaliuc R, Foias C and Jolly M S 2010 Estimates on enstrophy, palinstrophy, and invariant measures for 2D turbulence J. Diff. Eqns 248 792–819

[6] Foias C, Hoang L and Nicolaenko B 2007 On the helicity in 3D periodic Navier–Stokes equations: part 1. The nonstatistical case Proc. Lond. Math. Soc. 94 53–90

[7] Foias C, Jolly M S, Lan R, Rupam R, Yang Y and Zhang B 2014 Time analyticity with higher norm estimates for the 2D Navier–Stokes equations J. Appl. Math. arXiv:1312.0929

[8] Foias C, Jolly M S, Manley O P and Rosa R 2002 Statistical estimates for the Navier–Stokes equations and the Kraichnan theory of 2-D fully developed turbulence ISSN=0022-4715 J. Stat. Phys. 108 591–645

[9] Foias C, Manley O P, Rosa R and Temam R 2001 Navier–Stokes Equations and Turbulence (Cambridge: Cambridge University Press)

[10] Foias C, Jolly M S and Kukavica I 1996 Localization of attractors by their analytic properties Nonlinearity 9 15–65

[11] Li D and Sinai Y G 2008 Blow ups of complex solutions of the 3D Navier–Stokes system and renormalization group method J. Euro. Math. Soc. 10 267–313

[12] Temam R 1983 Navier–Stokes Equations and Nonlinear Functional Analysis (CBMS-NSF Regional Conference Series in Applied Mathematical) (Philadelphia, PA : Society for Industrial and Applied Mathematics)

[13] Temam R 1997 Infinite dimensional dynamical systems in mechanics and physics Applied Mathematical Sciences (Berlin: Springer)

[14] Adams R A 1975 Sobolev Spaces (New York: Academic Press)

[15] Ziemer W P 1989 Weakly Differentiable Functions: Sobolev Space and Functions of Bounded Variation (New York: Springer)