Canard Doublet in a Lotka-Volterra type model

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Abstract.

New types of canard solutions that arise in the modified Lotka-Volterra equations are discussed.

Let us consider the two-dimensional system

\[ \dot{x} = F(x, y) = x(a - f(y)), \quad \varepsilon \dot{y} = G(x, y, \delta) = y^p(-b + x(g(y) - \delta h(y))). \] (1)

Here \( p \geq 1, a, b > 0 \) and \( f(0) = 0, g'(y) > 0, g(0) \geq 0, h'(y) > 0, h(0) \geq 0, \delta \) and \( \varepsilon \) are small parameters, and

\[ \lim_{y \to \infty} g(y)/h(y) = 0. \] (2)

This system has been suggested in [5] as a model of some predator-prey interactions. Here the term \( g(y) \) describes the facilitation between the predators, and \( h(y) \) describes competition between the predators. The relationship (2) means that the competition dominates when the predator population is high. Since \( \delta \) is supposed to be small, the cooperation dominates as long as the predator population is low or moderate. The small parameter \( \varepsilon \) manifests that the predators’ metabolism is significantly faster than that of the prey; this is the case, for instance, in the case of Bacteria-Phage interaction. The values \( p > 1 \) are important in the context of Bacteria-Phage interaction, see [3, 1].

System (1) has the unique positive equilibrium

\[ y_* = f^{-1}(a), \quad x_*(\delta) = b/(g(y_*) - \delta h(y_*)) \] (3)

if \( g(y_*) - \delta h(y_*) > 0 \). Here \( f^{-1} \) denotes the inverse function: \( x_* \) is the unique solution of the equation \( f(y) = a \). Let us define

\[ \delta_* = \min \left\{ \frac{g(y_*)}{h(y_*)}, \frac{g'(y_*)}{h'(y_*)} \right\}. \]

For \( g(y_*) - \delta h(y_*) > 0 \) the linearization of (1) at equilibrium is

\[ \dot{u} = -f'(y_*)v, \]
\[ \dot{v} = y_*(g(y_*) - \delta h(y_*))u + y_*(g'(y_*) - \delta h'(y_*))v. \]
The equilibrium is a sink if
\[ g'_y(y_*) - \delta h'_y(y_*) > 0, \]
and it is a source, if
\[ g'_y(y_*) - \delta h'_y(y_*) < 0. \]
In particular the standard Andronov-Hopf bifurcation occurs at
\[ \delta = \delta_H = g'_y(y_*)/h'_y(y_*). \]

Let us recall details of the behaviour of the solution for small \( \delta \) [5]. Denote by \( Q_\gamma \), \( 0 < \gamma < 1 \) the square
\[ \{(x, y) : \gamma < x, y < 1/\gamma \}. \]
The smaller is \( \gamma \), the larger is the square \( Q_\gamma \), and the union of all squares coincides with the open positive quadrant \( x, y > 0 \).

**Theorem 1**

(i) For \( 0 < \delta < \delta_* \), the equilibrium (3) is a source, and each trajectory of system (1), except the equilibrium, converges to a periodic trajectory (which may depend on an initial condition).

(ii) For any \( \gamma > 0 \) there exists \( \delta(\gamma) \leq \delta_* \), such that for \( 0 < \delta < \delta(\gamma) \), the square \( Q_\gamma \) belongs to the area bounded by any periodic trajectory of system (1).

(iii) For any \( 0 < \delta < \delta_* \), there exists \( \gamma(\delta) \) such that all periodic trajectories are located within \( Q_{\gamma(\delta)} \).

We emphasize, that we do not guarantee the uniqueness of a periodic trajectory in item (i). If there are many periodic solutions, than there is a solution whose trajectory is \( T_I \) the innermost, and another solution whose trajectory \( T_O \) is the outermost. Then the solutions with an initial condition within the area bounded by the curve \( T_I \) approach this curve as time increases, and solutions with an initial condition outside of the area bounded by the curve \( T_O \) approach this curve. As usual there exists a periodic solution which is orbitally stable in the Ljapunov sense.

The situation described in the theorem above is similar to the phenomenon of Hopf bifurcation at infinity. Existence of a continuous branch of cycles for the values of parameter \( 0 < \delta < \delta_* \) may also be proven.

Below, we discuss the limit behaviour of the periodic solutions at \( \varepsilon \to 0 \) for a fixed \( \delta \), which satisfies \( 0 < \delta < \delta_* \). For \( \delta_H < \delta < \delta_* \), under the usual technical conditions the unique equilibrium of the system is globally stable.

For a fixed \( 0 < \delta < \delta_H \), and for sufficiently small \( \varepsilon > 0 \), we observe relaxation cycles, which also involve the phenomenon of delayed loss of stability. Let us denote \( P(y) = -g(y) + \delta h(y) \), \( y \geq 0 \). We suppose that \( P'(0) < 0 \), and that the function \( P(y) \) is unimodal: it strictly decreases on some interval \([0, m]\) and it strictly increases on \([m, \infty)\), for instance \( P''(y) > 0 \), \( y > 0 \). As an instructive example we mention \( g(y) = c + \alpha y, \ b(y) = y^2 \) for any positive \( c, \alpha \).

The function \( Q(y) = \frac{b}{P(y)} \) has the unique minimum \( \mu = \min Q(y) \) taken at \( y = m \), and for \( y \geq m \) the inverse function \( Q^{-1}(x) \) is well defined. We need one more auxiliary number. Consider the equation \(-\tau + e^{\alpha T} = -T + e^{\alpha T} \) where \( \tau = \frac{1}{a} \ln \frac{P(\mu)}{b} \). This equation has the unique solution \( T > \tau \), and we define \( z = \frac{b}{P(\mu)} e^{\alpha T} \).

Now we consider the closed curve \( \Gamma_* \), which consists of:

(i) the horizontal segment with the endpoints \((\mu, 0), (z, 0)\);
(ii) the vertical segment with the endpoints \((\mu, 0), (\mu, m)\);
(iii) the vertical segment with the endpoints \((z, 0)), (z, Q^{-1}(z))\);
(iv) the graph \(\Gamma_s\) of the function \(Q^{-1}(x), \mu \leq x \leq z\).

Essentially the limit behaviour of periodic solutions at \(\varepsilon \to 0\) is described by the following statement.

**Proposition 1**  The graphs of periodic solutions are converging to \(\Gamma_s\) as \(\varepsilon \to 0\).

**Figure 1.** Numerical results (1–4) against the theoretical limits (5)

Figure 1 illustrates this assertion for the case

\[
\begin{align*}
f(y) &= y + y^2, \quad g(y) = 1 + y, \quad h(y) = y^2, \quad \delta = 1/2. \tag{4}
\end{align*}
\]

The left figure 1(a) plots the phase diagrams of the solutions \((x(t), y(t))\) for \(p = 1\), and for the values \(\varepsilon = 0.2\) (graph 1), \(\varepsilon = 0.02\) (graph 2) and \(\varepsilon = 0.002\) (graph 3) against the theoretical limit (graph 5). The right figure 1(b) plots the phase diagrams of solutions for \(p = 2\), and for the values \(\varepsilon = 0.2\) (graph 1), \(\varepsilon = 0.02\) (graph 2), \(\varepsilon = 0.002\) (graph 3) and \(\varepsilon = 0.0002\) (graph 4) against the theoretical limit (graph 5).

Figure 2(a) plots the theoretical limits for the case

\[
\begin{align*}
f(y) &= y + y^2, \quad g(y) = 1 + y, \quad h(y) = y^2. \tag{5}
\end{align*}
\]

for \(\delta = 1/2\) (graph 1), \(\delta = 1/5\) (graph 2) and \(\delta = 1/10\) (graph 3). Figure 2(b) plots the theoretical limits, at \(\varepsilon \to 0\), of the \(y\) population as the time series for the case \(\varepsilon = 0.1\).

An interesting question is the evolution of the relaxation cycle when simultaneously \(\varepsilon\) is small and \(\delta\) is close to \(\delta_H\). To describe this evolution in detail we need some auxiliary definitions.

Some special solutions of singularly perturbed ordinary differential equations are called canards. We first recall in a convenient form the corresponding definitions.

Let us consider a two-dimensional autonomous system:

\[
\begin{align*}
\dot{x} &= f(x, y, \alpha), \quad \varepsilon \dot{y} = g(x, y, \alpha), \tag{6}
\end{align*}
\]

where \(x, y\) are scalar functions of time, \(\varepsilon\) is a small positive parameter, and \(f, g\) are sufficiently smooth scalar functions. The set of points

\[
S_\alpha = \{(x, y) : g(x, y, \alpha) = 0\}
\]

of the phase plane is called a slow curve of the system.

We will make the following assumptions:
1) The curve $S_\alpha$ consists of regular points, i.e. at every point $(x, y) \in S_\alpha$

$$[g_x(x, y, \alpha)]^2 + [g_y(x, y, \alpha)]^2 > 0$$

holds.

2) Singular points, i.e. points at which $g_y(x, y, \alpha) = 0$, are isolated on $S_\alpha$.

3) At any singular point the following inequality holds: $g_{yy} \neq 0$.

**Definition 1** A singular point $A$ of the slow curve $S_\alpha$ is called a jump point \[4\] if

$$\text{sgn} [g_y(A)g_x(A)f(A)] = 1.$$

Parts of $S_\alpha$ which contain only regular points are called regular. A regular part of $S_\alpha$, all points of which satisfy the inequality

$$g_y(x, y, \alpha) < 0 \quad (g_y(x, y, \alpha) > 0),$$

is called attractive (repulsive).

**Definition 2** Trajectories which at first pass along the attractive part of the slow curve, then jump in the direction of another attractive part of the slow curve, and pass along this attractive part of the slow curve are called relaxation trajectories.

**Definition 3** Trajectories which at first pass along the attractive part of the slow curve and then continue for a while along the repulsive part of the slow curve are called canards or duck-trajectories.

The canards and the corresponding values of the parameter $\alpha$ allow asymptotic expansions in powers of the small parameter $\varepsilon$. Near the slow curve the canards are exponentially close, and have the same asymptotic expansion in powers of $\varepsilon$. An analogous assertion is true for corresponding parameter values $\alpha$. Namely, any two values of the parameter $\alpha$ for which canards exist have the same asymptotic expansions, and the difference between them is given by $\exp(-1/c\varepsilon)$ where $c$ is some positive number.

Now we introduce new types of canards, which are important in the context of this article.

**Definition 4** Trajectories which at first pass along an attractive part of the slow curve, and after that jump in the direction of another attractive part of the slow curve, pass along this attractive part of the slow curve, then continue for a while along a repulsive part of the slow curve are called relaxation canards.
To illustrate this definition, consider the following system

\[
\dot{x} = 1 - y + \alpha, \quad \varepsilon \dot{y} = y(-y^2 + 2y + x - 2).
\]  

(7)

The slow curve of this system consists of the horizontal line \( y \equiv 0 \) with attractive part \( x < 2 \) and repulsive part \( x > 2 \), and the parabola \(-y^2 + 2y + x - 2 = 0\) with attractive parts \( y > 1\), \( y < 0\), and repulsive part \( 0 < y < 1 \) (see Fig. 3). Thus, the slow curve of (7) has two different attractive/repulsive sets.

If \( \alpha \in (0, 1) \) then the unique equilibrium with \( x = 1 + \alpha^2 \) and \( y = 1 + \alpha \) lies on the repulsive part \( 0 < y < 1 \) of the slow curve and the system under consideration possesses a periodic trajectory which is a relaxation canard. It is interesting that this trajectory is a periodic canard for \( \forall \alpha \in (0, 1) \) and it is not necessary to choose the value of \( \alpha \) in a special way. The relaxation canard is pictured in Fig. 4.

**Definition 5** Trajectories which at first pass along an attractive part of the slow curve, then continue for a while along a repulsive part of the slow curve and after that jump towards the another attractive/repulsive set of the slow curve and follow it, are called canard doublets.

To illustrate this definition, consider the system (7), in which the value of the parameter \( \alpha \) is chosen in a special way \( \alpha = \varepsilon \mu(\varepsilon) \) with \( \mu \simeq -0.25 \) (see Fig. 5).

Turning back to the modified Lotka-Volterra model, consider the special case, when one variable \( y \) is faster than the other variable \( x \). We can use singularly perturbed differential systems for modelling such phenomena. The biologically relevant case is “Fast Predators – Slow Preys” and the corresponding system is

\[
\dot{x} = x(a - y - y^2 + \varepsilon \mu), \quad \varepsilon \dot{y} = y^p(-b + x(1 + y - \delta y^2)),
\]

with \( p > 1 \).
Figure 4. Relaxation canard

Figure 5. Canard doublet
Figure 6. Slow curve in Lotka-Volterra model

Figure 7. Relaxation canards in Lotka-Volterra model for $p = 1$
Figure 8. Relaxation canards in Lotka-Volterra model for $p = 2$

Figure 9. Canard doublet in Lotka-Volterra model for $p = 1$
The slow curve is presented in Fig. 6. For \( p = 1 \) it consists of horizontal line \( y \equiv 0 \) with attractive part \( x < b \) and repulsive part \( x > b \), and the curve \(-b + x(1 + y - \delta y^2) = 0\) with attractive parts \( y > 1/2\delta \), \( y < 0 \), and repulsive part \( 0 < y < 1/2\delta \). In the case \( p = 2 \) we have the same slow curve, but the horizontal line \( y \equiv 0 \) has attractive part \( x < 2 \) and repulsive part \( x > 2 \) for \( y > 0 \), and the curve \(-b + x(1 + y - \delta y^2) = 0\) has the attractive part \( y > 1/2\delta \), \( y < 0 \), and repulsive part \( 0 < y < 1/2\delta \). We can observe here the phenomena of a relaxation canard (see Fig. 7, 8) and a canard doublet (see Fig. 9, 10).

The following statements are true.

**Theorem 2** Let \( 0 < a < 1/2\delta + 1/4\delta^2 \), then there is \( \varepsilon_0 \), such that for every \( \varepsilon \in (0, \varepsilon_0) \) the system (8) has a relaxation canard.

**Theorem 3** There is \( \varepsilon_0 \) such that for every \( \varepsilon \in (0, \varepsilon_0) \) and any fixed positive \( b \) and \( \delta \), there exist \( a = a^*(\varepsilon) \), \( a^*(0) = 1/2\delta \) and a canard doublet corresponding to this parameter value.

**Acknowledgments**

This work was supported by Grants 07-01-00169a and 06-01-72552 of the Russian Foundation of Basic Researches.

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