Can gravity appear due to polarization of instantons in the SO(4) gauge theory?

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Abstract. Conventional non-Abelian SO(4) gauge theory is able to describe gravity provided the gauge field possesses a specific polarized vacuum state. In this vacuum the instantons and antiinstantons have a preferred direction of orientation. Their orientation plays the role of the order parameter for the polarized phase of the gauge field. The interaction of a weak and smooth gauge field with the polarized vacuum is described by an effective long-range action which is identical to the Hilbert action of general relativity. In the classical limit this action results in the Einstein equations of general relativity. Gravitational waves appear as the mode describing propagation of the gauge field which strongly interacts with the oriented instantons. The Newton gravitational constant describes the density of the considered phase of the gauge field. The radius of the instantons under consideration is comparable with the Planck radius.

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1. Introduction

I wish to present arguments favoring a scenario in which gravity can arise as a particular effect in the framework of the conventional Yang-Mills gauge theory \[1\] formulated in flat space-time. The idea was first reported in \[2\], this paper presents it in detail. We will suppose that the Lagrangian of the theory describes the usual degrees of freedom of gauge theory, i.e. gauge bosons interacting with fermions and scalars, while there are no geometrical degrees of freedom on the Lagrangian level. In particular, the Lagrangian does not contain gravitons. Our goal is to show that all geometrical objects necessary for the gravitational phenomenon can originate from the gauge degrees of freedom if a particular nontrivial vacuum state, which we will call the vacuum with polarized instantons develops in the \(SO(4)\) gauge theory.

Let us first describe the main idea in general terms. To do this let us imagine that in some conventional gauge theory there develops a particular nontrivial vacuum state. Its nature is discussed below in detail. At the moment let us only assume that this nontrivial vacuum has a very strong impact on the properties of excitations of the gauge field. Instead of the usual spin-1 gauge boson there appears quite new massless spin-2 excitation. Certainly this is a very peculiar property, but if it exists, it provides a possibility to identify this massless mode with the graviton. Furthermore one can hope to evaluate all manifestations of gravity from an effective theory which should describe propagation of this low-energy mode. In the classical limit one can hope also that this effective theory should reproduce the Einstein equations of general relativity. Developing this line of reasoning one should keep in mind a sufficiently long list of important relevant questions. The most urgent ones are listed below.

1. In which gauge group one should look for the effect?
2. What is the origin of the necessary nontrivial vacuum, roughly speaking from which fields should it be constructed?
3. What is the nature of the order parameter which governs the nontrivial phase? In particular, which symmetry should it possess?
4. What is the nature of the low-energy excitations in this vacuum? As said above, this excitation should be the massless spin-2 mode.
5. What is the effective low-energy theory governing classical propagation of this massless spin-2 mode?
6. How important are the quantum corrections? One should hope that they neither confine the spin-2 mode, nor supply it with the mass in the infrared region.
7. What is the origin of the phase transition into the necessary vacuum state?
8. Which phase transitions separate the necessary vacuum from conventional vacua of gauge theory?

All these questions are concentrating around the two major problems. One of them
is the existence of the necessary vacuum. Possible models in which it can arise have been considered in recent Refs.\cite{3,4}. The other problem is physical manifestations of the nontrivial vacuum, granted it exists. This later problem is in the focus of this paper.

Several reasons make this study necessary. Firstly, the physical nature of the excitations proves to be very interesting. Secondly, one should know precisely the properties of the excitations in order to construct and tune a model in which the necessary ground state appears. Thirdly, even in such a long known phase as the QCD vacuum the properties of the vacuum state are not fully understood up to now. Instantons are known to play an important role there, but verification of this fact remains an interesting as well as nontrivial problem \cite{5}. This paper proposes to consider a new phase in gauge theory in which instantons play an important role. It could also prove to be sufficiently complicated. To begin such a study one should fully realize its purpose. Having these arguments in mind I address in this paper some questions from the above list in detail, while only qualitatively discussing others.

The idea discussed can suggest a novel approach to the old problem of quantization of gravity having several important advantages. One of them is the renormalizability which is guaranteed from the very beginning because on the basic Lagrangian level the theory is the usual renormalizable gauge theory. It is important also that the basic idea is sufficiently simple being formulated by means of a pure field theory. It needs the single tool, the nontrivial condensate, to explain the single phenomenon, gravity. The string theory \cite{6}, which for a long period of time has been considered as one of the most advanced candidates for description of quantum gravity offers much more complicated approach.

One of the popular modern developments in quantum gravity was originated in \cite{7}, where it was argued that the variables of gauge theory can be used for description of geometrical objects of the Riemann geometry. In this approach it is assumed that the geometrical degrees of freedom do exist on the basic Lagrangian level. In contrast, in this paper we suppose that geometry does not manifest itself on the Lagrangian level, providing only an effective description for some particular low energy degrees of freedom of gauge field.

Let us now describe shortly the answers which this paper proposes to some of the listed above questions. First, the gauge group has to be SO(4), or bigger one. Furthermore, the phase considered may be described in terms of nontrivial topological excitations of the gauge field. In this sense the condensate is composed of the SO(4) gauge field. The necessary phase can be described in terms of the BPST instanton \cite{8}. An instanton is known to possess eight degrees of freedom. Four of them give its position, one is its radius. The rest three describe its orientation which plays a crucial role in our consideration. In the usual phases of gauge theory, for example in the QCD vacuum orientations of instantons are arbitrary. In this paper we discuss a phase in which
instantons have a preferred direction of orientation. This means that a mean value of orientation is nonzero. A possible way to visualize this vacuum, to find a simple physical analogy, is to compare it with the usual ferromagnetic or antiferromagnetic phases in condensed matter physics in which spins of atoms or electrons have preferred orientation. The instantons constituting this vacuum will be called “polarized instantons” or “the condensate of polarized instantons”. The vacuum itself will be referred to as “the polarized vacuum”. The density of the condensate of polarized instantons is described by a length parameter which depends on radiiuses and separations of the polarized instantons. This length parameter proves to be equal to the Planck radius, which is only possible if radiiuses of the considered instantons are comparable with the Planck radius. The existence of this length parameter permits the Newton gravitational constant to appear in the theory as the inverse density of the condensate. Thus the main gravitational parameter absent in the initial Lagrangian is introduced into the theory by the nontrivial vacuum state. In Section 4 we will return to the questions listed above and discuss them in more detail.

It is necessary to mention that the existence of a state with polarized instantons does not come into contradiction with a general principle of gauge invariance which in the context considered is known as the Elitzur theorem [9], see also Refs.[10, 11]. The theorem forbids existence of any vacuum state in which local gauge invariance is broken spontaneously by a non-gauge-invariant mean value of a field. The orientation of an instanton is a gauge invariant parameter. Therefore polarization of instantons does not break spontaneously gauge invariance, and the vacuum state with polarized instantons is not forbidden.

We will mainly restrict our discussion to the dilute gas approximation for instantons, mainly to secure simplicity of consideration. In this approximations the gases of instantons and antiinstantons can coexist. We will suppose that both gases of instantons and antiinstantons are polarized. The mean values of matrixes which describe orientations of polarized instantons and antiinstantons play the role of the order parameter for the considered phase. It will be argued that this order parameter should have a particular symmetry. Namely orientations of all polarized topological excitations should be described by one $6 \times 6$ matrix which belongs to $SO(3, 3)$.

The main result of this paper can be formulated as the following statement. Suppose that instantons and antiinstantons in the vacuum state of the $SO(4)$ gauge theory have preferred orientations which are described by a matrix from the $SO(3, 3)$ group. Then long-wave excitations above this vacuum are the spin-2 massless particles which on the classical level are described by the Hilbert action and the Einstein equations of general relativity.
2. Instantons in the external field

Consider Euclidean formulation of the $SO(4)$ gauge theory. The gauge algebra for $SO(4)$ gauge group consists of two $su(2)$ gauge subalgebras, $so(4) = su(2) + su(2)$. The instantons and antiinstantons can belong to any one of these two available $su(2)$ gauge subalgebras. It is convenient to choose the generators for one $su(2)$ gauge subalgebra to be $(-1/2)\eta^{aij}$ and the generators for the other one to be $(-1/2)\bar{\eta}^{aij}$. To distinguish between these two subalgebras we will refer to them as $su(2)\eta$ and $su(2)\bar{\eta}$.

Symbols $\eta^{aij}, \bar{\eta}^{aij}$ are the usual 't Hooft symbols, $a = 1, 2, 3, i, j = 1, \cdots, 4$ which give a full set of $4 \times 4$ matrixes antisymmetric in $ij$. In this notation the gauge potential and the strength of the gauge field are

\begin{align}
A^{ij}_\mu &= -\frac{1}{2}(A^a_\mu \eta^{aij} + \bar{A}^a_\mu \bar{\eta}^{aij}), \\
F^{ij}_{\mu\nu} &= -\frac{1}{2}(F^a_{\mu\nu} \eta^{aij} + \bar{F}^a_{\mu\nu} \bar{\eta}^{aij}),
\end{align}

where $A^a_\mu$ and $F^a_{\mu\nu}$ belong to $su(2)\eta$ and $\bar{A}^a_\mu, \bar{F}^a_{\mu\nu}$ belong to $su(2)\bar{\eta}$. The Yang-Mills action reads

\begin{equation}
S = \frac{1}{4g^2} \int F^{ij}_{\mu\nu}(x) F^{ij}_{\mu\nu}(x) \, d^4x.
\end{equation}

The Latin indexes $i, j = 1, \cdots, 4$ label the isotopic indexes, while the Greek indexes $\mu, \nu = 1, \cdots, 4$ label the indexes in Euclidean coordinate space. Remember that we consider the usual gauge field theory in flat space-time. For the chosen normalization of generators the relation between the gauge potential and the field strength reads

\begin{equation}
F^{ij}_{\mu\nu} = \partial_\mu A^{ij}_\nu - \partial_\nu A^{ij}_\mu + A^{ik}_\mu A^{kj}_\nu - A^{ik}_\nu A^{kj}_\mu.
\end{equation}

For our purposes it is important to consider an interaction of nontrivial topological excitations, instantons and antiinstantons with an external gauge field which has trivial topological structure. Consider first a single instanton in an external gauge field. It is sufficient to assume that this external field is weak and smooth, i.e. it is weaker than the field of the instanton inside the instanton and varies much more smoothly than the instanton field which sharply decreases outside the instanton radius. Thus formulated problem was first considered by Callan, Dashen and Gross in \cite{13} where it was shown that the interaction of an instanton with the external field is described by an effective action

\begin{equation}
\Delta S = \frac{2\pi^2 \rho^2}{g^2} \eta^{a\mu\nu} D^{ab} F^{b}_{\mu\nu}(x_0).
\end{equation}

Here $\rho$ and $x_0$ are the radius and position of the instanton. The matrix $D^{ab} \in SO(3)$ describes the orientation of the instanton in the $su(2)$ gauge subalgebra where the instanton belongs (suppose for example that it is $su(2)\eta$). Its definition is given in
Appendix A, see (A8). $F^{a}_{\mu\nu}(x)$ is the gauge field in the subalgebra where the instanton belongs. This field has to be taken in the singular gauge \cite{12}.

The interaction of an antinstanton with an external field is described similarly. The only distinction is that it produces the corresponding term with the 't Hooft symbol $\eta^{a\mu\nu}$ instead of $\bar{\eta}^{a\mu\nu}$ which stands in \cite{9}. Notice that we use $\eta^{a\mu\nu}, \bar{\eta}^{a\mu\nu}$ as generators of rotations in the coordinate space, while $\eta^{aij}, \bar{\eta}^{aij}$ generate rotations in the isotopic space. These two sets of symbols are defined in different spaces, but numerically they are identical of course.

It is worth to stress several interesting and important properties of the action (5). First of all it has a very unusual structure being linear in the external field. At first sight this fact looks as a paradox because an instanton provides a minimum for the action and one could anticipate only quadratic terms in the weak field to appear. The paradox is resolved by the fact that the linear term arises from the region of large separations from the instanton where the external field exceeds the instanton field, resulting in breaking down of the naive perturbation theory. This question is discussed in detail in Appendix A which develops the approach of \cite{13} to cover the case of several overlapping instantons. The action (5) is derived starting from the usual Yang-Mills action in which the external field is considered as a perturbation. The contribution to the action linear in the external field is found as an integral over a sphere which is well separated from instantons. An alternative way to derive the action (5) was suggested in \cite{14} where an instanton was considered as an effective source for gauge bosons. In this approach the linear term in the action arises due to the large probability of creating small-momenta bosons by an instanton.

Furthermore, it is important to stress that the orientation of an instanton is a gauge invariant parameter. A convenient way to define the orientation is given by the general n-instanton ADHM solution of Ref.\cite{15}, for a review see \cite{16}. This solution is briefly discussed in Appendix A where Eq. (A8) defines the matrix of the instanton orientation $D^{ab}$ in terms of variables of the ADHM construction. The fact that the orientation of an instanton is a gauge invariant parameter is closely connected with gauge invariance of the action (5). This action is derived from the usual Yang-Mills action, from which it inherits gauge invariance. The invariance is guaranteed by the mentioned explicit requirement that the field $F^{a}_{\mu\nu}(x_{0})$ has to be considered in the singular gauge. This fixing of the gauge indicates that there is no spontaneous breaking of the local symmetry.

The last comment addresses the behaviour of several instantons. It is found in Appendix A that they interact with the external field as individual objects. This statement is trivial when the dilute gas approximation is valid, but surprisingly it stands for any separation between instantons, even if they strongly overlap. In this paper we will not push this argument further on restricting our consideration by the dilute gas approximation. Still the independent interaction of instantons with the external field
gives a hope that final conclusions of the paper can be more reliable than the dilute gas approximation which is used to derive them.

The discussion of the action (3) given above permits us now to address the following important for us problem. Suppose that there is a number of instantons and antiinstantons which belong to either $su(2)\eta$ or $su(2)\bar{\eta}$ gauge algebras and satisfy the dilute gas approximation. Suppose also that there is an additional gauge field $F_{\mu\nu}^{ij}(x)$ which has a trivial topological structure, is weaker than the fields of the instantons and varies smoothly for the distances which characterize separations and radiiuses of instantons. In this situation we immediately deduce that the influence of the external field on instantons and antiinstantons can be described by the effective action which is equal to the sum of terms of the type of (3) which describe independent interaction of instantons and antiinstantons with the field. Using definition (2) one can present the corresponding action in the following form

$$\Delta S = -\frac{\pi^2}{g^2} \sum_k \eta^{A\mu\nu} \eta^{Bij} T_k^{AB} \rho_k^2 F_{\mu\nu}^{ij}(x_k).$$

(6)

Here an index $k$ runs over all available instantons and antiinstantons which have radiiuses and coordinates $\rho_k$ and $x_k$. To simplify notation the 't Hooft symbols are enumerated as 6-vectors $\eta^A = (\eta^a, \bar{\eta}^b)$, $A = 1, \ldots, 6$; $a, b = 1, 2, 3$. More precisely this definition means that

$$\eta^{Aij} = \eta^{aij}, \quad \eta^{A\mu\nu} = \eta^{a\mu\nu}, \quad \text{if} \quad A = a = 1, 2, 3,$$

$$\eta^{Aij} = \bar{\eta}^{bij}, \quad \eta^{A\mu\nu} = \bar{\eta}^{b\mu\nu}, \quad \text{if} \quad A - 3 = b = 1, 2, 3.$$

(7)

To describe an orientation of every (anti)instanton it is convenient to introduce the $6 \times 6$ matrix $T_k^{AB}, \quad A, B = 1, \ldots, 6$

$$T_k^{AB} \equiv T_k = \begin{pmatrix} C_k & D_k \\ D_k & \bar{C}_k \end{pmatrix}$$

(8)

as a set of four $3 \times 3$ matrixes $C_k, \bar{C}_k, D_k, \bar{D}_k$. For any given $k$-th (anti)instanton only one of these four matrixes is essential while the other three are equal to zero. This nonzero matrix belongs to $SO(3)$ and describes the orientation of the $k$-th topological object in the gauge algebra where it belongs. For example, if the $k$-th object is an antiinstanton in the $su(2)\eta$ gauge subalgebra, then we assume that $C_k \in SO(3)$ describes its orientation in the $su(2)\eta$ while $\bar{C}_k, D_k, \bar{D}_k = 0$. Similarly, $D_k$ describes the orientation if the $k$-th object is an instanton $\in su(2)\eta$ ($C_k, D_k, \bar{D}_k = 0$), $\bar{D}_k$ describes the orientation if the antiinstanton $\in su(2)\bar{\eta}$ is considered ($C_k, \bar{C}_k, D_k = 0$), and $C_k$ describes orientation if the instanton $\in su(2)\bar{\eta}$ is considered ($C_k, D_k, \bar{D}_k = 0$). These definitions can be presented in a short symbolic form

$$T_k = \begin{pmatrix} \text{antiinstanton} \in su(2)\eta & \text{instanton} \in su(2)\eta \\ \text{antiinstanton} \in su(2)\bar{\eta} & \text{instanton} \in su(2)\bar{\eta} \end{pmatrix}.$$

(9)
Using definitions (7),(8) as well as Eq.(2) it is easy to verify that every term in Eq.(6) describes an interaction of some (anti)instanton with the external field in accordance with (5).

Let us address now the question of the behavior of the ensemble of instantons in the vacuum state assuming that there exists a weak and smooth topologically trivial gauge field \( F_{\mu\nu}(x) \). One can derive the effective action which describes interaction of the vacuum with this field averaging Eq.(6) over short-range quantum fluctuations in the vacuum. The result can be presented as the effective action

\[
\Delta S = - \int \eta^{\mu\nu} \eta^{\alpha\beta} M^{AB}(x) F_{\mu\nu} F_{\alpha\beta} \, d^4x.
\]

where the matrix \( M^{AB}(x) \) is

\[
M^{AB}(x) = \pi^2 \left( \frac{1}{g^2} \rho^2 T^{AB} \right) n(\rho, T, x).
\]

The brackets \( \langle \rangle \) here describe averaging over quantum fluctuations whose wavelength is shorter than a typical distance describing variation of the external field. These fluctuations in the dilute gas approximation for instantons should include averaging over positions, radiuses and orientations of instantons. In Eq.(11) \( n(\rho, T, x) \) is the concentration of (anti)instantons which have the radius \( \rho \) and the orientation described by the matrix \( T \equiv T^{AB} \). In the usual vacuum states the concentration of instantons does not depend on the orientation, \( n(\rho, T, x) \equiv n(\rho, x) \). In that case an averaging over orientations gives the trivial result \( M^{AB}(x) \equiv 0 \), as mentioned in [14]. The main goal of this paper is to investigate what happens if the concentration \( n(\rho, T, x) \) does depend on the orientation \( T = T^{AB} \) providing the nonzero value for the matrix \( M^{AB}(x) \).

From (11) one can anticipate that the consequences should be interesting because there appears the unusual term linear in the gauge field in the action. To clarify this point we are to be more specific about properties of the matrix \( M^{AB}(x) \) assuming that it satisfies the following three conditions. First we will assume that it is a nondegenerate matrix with positive determinant. It is convenient to present this assumption as a statement that the matrix \( M(x) \) is proportional to the unimodular \( 6 \times 6 \) matrix \( M(x) \)

\[
M^{AB}(x) = \frac{1}{4} f M^{AB}(x), \quad \det M(x) = 1.
\]

Second, we suppose that \( f \) introduced above is a positive constant

\[
f = \text{const} > 0.
\]

This means that the determinant \( \det M(x) = (f/4)^6 \) is a constant. Third, we postulate that the matrix \( M^{AB}(x) \) possesses a particular symmetry property, namely that it belongs to the \( SO(3,3) \) group,

\[
M^{AB}(x) \in SO_{+}(3,3).
\]
This means that $M = M^{AB}(x)$ should satisfy identity
\[ M \Sigma M^T = \Sigma. \] (15)

Here the matrix $\Sigma^{AB}$ is defined as
\[ \Sigma = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & -\hat{1} \end{pmatrix}, \] (16)

where numbers with hats represent $3 \times 3$ diagonal matrices. Notation $SO_{+}(3,3)$ is used in (14) to distinguish the subset of matrixes $M \in SO(3,3)$ which can be transformed into the unity matrix $1$ by a continuous function in $SO(3,3)$, see Appendix Appendix B.

Conditions (12),(13),(14) are to be considered as the main assumptions made in this paper about properties of the vacuum state of the $SO(4)$ gauge theory. Now we can clarify the meaning of the term “polarization of instantons” which was introduced above intuitively. We say that instantons in the vacuum state are polarized if the matrix $M^{AB}$ which defines the mean values of orientations of (anti)instantons in (11) is nonzero. We will say that the polarization of instantons has the $SO(3,3)$ symmetry, or equivalently there is the $SO(3,3)$ polarization of instantons, if Eqs.(12),(13),(14) are valid. Let us mention once more that the orientation of an instanton is a gauge invariant parameter. Therefore its nonzero mean value does not come into contradiction with the Elitzur theorem \[ \text{[9]}. \]

In the following consideration we postulate that there is the $SO(3,3)$ polarization of instantons in the vacuum. This assumption at the moment looks rather bizarre, but later development will show its advantages. Firstly, in the next Section \[ \text{[3]} \] we will verify that it makes excitations above the vacuum identical to gravitational waves. This is exactly what we are looking for. Then in Section \[ \text{[4]} \] we discuss what is known about a way to justify assumptions (12),(13),(14) in the framework of gauge theory.

In the usual phases of gauge theory the mean values of matrixes describing orientations of (anti)instantons vanish. Therefore one can interpret Eqs.(12),(13),(14) as the statement that there exists the new nontrivial phase of the $SO(4)$ gauge theory. The matrix $M(x)$ plays the role of the order parameter for this phase. The (anti)instantons which contribute to the nontrivial value of the matrix $T^{AB}$ in (11) can be looked at as a specific condensate. The constant $f$ characterizes the density of this condensate. The dimension $\text{cm}^{-2}$ of this constant supplies the theory with the dimensional parameter which later on will be interpreted as the Newton gravitational constant, see Eq.(44).

In order to examine the consequences of assumptions (12),(13),(14) it is very instructive to re-write (11) in another form. With this purpose let us take into consideration that there exists a relation between matrixes belonging to $SO(3,3)$ and matrixes belonging to $SL(4)$ groups. It states that for any $M^{AB} \in SO_{+}(3,3)$, $A, B =$
1, · · · , 6 there exists some matrix $H^{i\mu} \in SL(4)$, $i, \mu = 1, \cdots, 4$ satisfying

$$H^{i\mu}H^{j\nu} - H^{i\nu}H^{j\mu} = \frac{1}{2} \eta^{A\mu} \eta^{B\nu} M^{AB}. \tag{17}$$

This matrix if defined uniquely up to a sign factor $\pm H^{i\mu}$. The reversed statement is also true: for any given $H^{i\mu} \in SL(4)$ Eq. (17) defines the matrix $M^{AB}$ which belongs to $SO_+(3,3)$. Moreover, it can be shown that (17) gives an isomorphism $SL(4)/(\pm 1) \equiv SO_+(3,3)$. All these statements are verified in Appendix A.

Identifying $M^{AB}(x) = M^{AB}$ one finds from (17) $H^{i\mu}(x) \in SL(4)$. This new matrix is well defined by the matrix $M^{AB}(x)$ and hence can be considered as another representation of the order parameter for polarized instantons. Substituting $H^{i\mu}(x)$ defined by (17) into (12) one finds that the action (10) can be rewritten in the following useful form

$$\Delta S = -f \int H^{i\mu}(x)H^{j\nu}(x)F^{ij}_{\mu\nu}(x) d^4x. \tag{18}$$

Up to now our consideration was fulfilled in the orthogonal coordinates which describe the considered flat space-time. In these coordinates the Lagrangian of the gauge theory is formulated. It is instructive however to present the action (18) in arbitrary coordinates. Under the coordinate transformation $x^\mu \rightarrow x'^\mu$ the gauge field obviously transforms as

$$F^{ij}_{\mu\nu}(x) \rightarrow F^{ij}_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} F^{ij}_{\lambda\rho}(x). \tag{19}$$

Moreover, a coordinate transformation does not affect the action. From this one deduces that the matrix $H^{i\mu}(x)$ is transformed by the coordinate transformation as

$$H^{i\mu}(x) \rightarrow h^{i\mu}(x') = \frac{\partial x'^\mu}{\partial x^\lambda} H^{i\lambda}(x). \tag{20}$$

We chose different notation for the transformed matrix calling it $h^{i\mu}(x)$ in order to distinguish it from the unimodular matrix $H^{i\mu}(x)$. According to (20) the determinant of the transformed matrix depends on the coordinate transformation

$$\det[h^{i\mu}(x')] = \det \left[ \frac{\partial x'^\mu}{\partial x^\lambda} \right]. \tag{21}$$

From Eqs. (19), (20) one deduces that in arbitrary coordinates $x$ the action (18) can be presented in the following form

$$\Delta S = -f \int h^{i\mu}(x)h^{j\nu}(x)F^{ij}_{\mu\nu}(x) \det h(x) d^4x. \tag{22}$$

It is convenient for further discussion to introduce notation in which $h^{i\mu}(x)$ is understood as the matrix inverse to $h^{i\mu}(x)$, i.e.

$$h^{i\mu}(x)h^{j\mu}(x) = \delta_{ij}. \tag{23}$$
The determinant in Eq.(22) is defined as a determinant of this inverse matrix. Thus the factor $\det h(x)$ in (22) simply accounts for the variation of the phase volume under the coordinate transformation

$$ \det h(x') d^4x' \equiv \det[ h^i_\mu(x')] d^4x' = \det \left[ \frac{\partial x^\mu}{\partial x'^\nu} \right] d^4x' = d^4x, \tag{24} $$

Here $x$ and $x'$ are the orthogonal and arbitrary coordinates respectively.

Summarizing, it is shown in this Section that (22) gives the effective action which describes the interaction of a weak and smooth gauge field $F^{ij}_{\mu\nu}(x)$ with polarized instantons.

3. The Riemann geometry and the Einstein equations

Let us consider in this Section excitations above the polarized vacuum in the classical approximation. It is clear that these excitations should have very interesting properties because a variation of the gauge field results in the contribution to the action (22) which is linear in the field. This is in contrast to the standard quadratic behavior of the Yang-Mills action (3). We will assume that the fields considered vary on the macroscopic distances, say $\sim 1$cm, and their magnitude can be roughly estimated as $|F^{ij}_{\mu\nu}| \sim 1/cm^2$. We will see below that the constant $f$ which was defined in Eqs.(11),(12) is large, $f \sim 1/r_P^2$, where $r_P$ is the Planck radius. This shows that for weak fields the integrand in the Yang-Mills action (3) is suppressed compared to (22) by a drastic factor

$$ |F^{ij}_{\mu\nu}|/f \sim (r_P/1cm)^2 = 10^{-64}. \tag{25} $$

This estimate demonstrates that our first priority is to take into account the action (22) which describes interaction of the weak field with polarized instantons, neglecting the Yang-Mills action (3). We will return to this point in Section 4.

Thus we suppose that properties of low-energy excitations of the gauge field can be described by the action (22). The fact that the field $F^{ij}_{\mu\nu}(x)$ varies on the macroscopic scale makes it weak and smooth on the microscopic level. These properties of the field mean that it reveals trivial topological structure on the microscopic level. In contrast, the matrix $h^{i\mu}(x)$ describes those degrees of freedom of the gauge field which are associated with instantons and therefore have highly nontrivial microscopic topological structure. Thus $F^{ij}_{\mu\nu}(x)$ and $h^{i\mu}(x)$ describe quite different topological structures. Their different topology enables us to consider them as two sets of independent variables, or modes. This makes the action (22) a functional of these variables

$$ \Delta S = \Delta S(\{A^{ij}_{\mu}(x)\}, \{h^{i\mu}(x)\}), $$
where \( A_{ij}^{\mu}(x) \) is the vector potential of the external field \( F_{\mu\nu}^{ij}(x) \). We see that classical equations for the functional (22) read

\[
\frac{\delta(\Delta S)}{\delta A_{ij}^{\mu}(x)} = 0, \\
\frac{\delta(\Delta S)}{\delta h^{i\mu}(x)} = 0.
\]

From Eq.(22) one finds that the first classical equation (26) results in

\[
\nabla_{ik}^{\mu} \left[ \left( h^{k\mu}(x) h^{ij}(x) - h^{k\nu}(x) h^{ij}(x) \right) \det h(x) \right] = 0.
\]

Here \( \nabla_{ij}^{\mu} = \partial_{\mu} \delta_{ij} + A_{ij}^{\mu}(x) \) is the covariant derivative in the external gauge field. The second classical equation which follows from Eqs.(22),(27) reads

\[
h^{ij}(x) F_{ij}^{\mu\nu}(x) - \frac{1}{2} h^{i\mu}(x) h^{k\lambda}(x) h^{j\nu}(x) F_{k\lambda\nu}(x) = 0.
\]

Here \( h^{i\mu}(x) \) is defined in Eq.(23). In order to present Eqs.(28),(29) in a more convenient form let us define three quantities,

\[
g^{\mu\nu}(x) = h^{i\mu}(x) h^{i\nu}(x), \\
\Gamma_{i\mu\nu}^{\lambda}(x) = h^{i\lambda}(x) h^{j\mu}(x) A_{ij}^{\nu}(x) + h^{i\lambda}(x) \partial_{\nu} h_{ij}^{i}(x), \\
R_{\rho\mu\nu}^{\lambda}(x) = h^{i\lambda}(x) h_{ij}^{\rho}(x) F_{ij}^{\mu\nu}(x).
\]

Remember that space-time under consideration is basically flat. Therefore Eqs.(30),(31),(32) just define the left-hand sides in terms of \( A_{ij}^{\mu}(x) \) and \( h^{i\mu}(x) \). Remarkably, the classical Eqs.(28),(29) for the gauge field supply these definitions with an interesting geometrical content. In order to see this let us notice that after simple calculations the first classical Eq.(28) may be presented in the following form

\[
\Gamma_{\rho\mu\nu}^{\sigma}(x) = \frac{1}{2} g^{\sigma\tau}(x) \left[ \partial_{\lambda} g_{\tau\mu}(x) + \partial_{\mu} g_{\lambda\tau}(x) - \partial_{\tau} g_{\lambda\mu}(x) \right].
\]

Here the matrix \( g^{\mu\nu}(x) \) is defined as

\[
g^{\mu\nu}(x) = h^{i\mu}(x) h^{i\nu}(x),
\]

which according to (23) makes it inverse to \( g_{\mu\nu}(x) \)

\[
g^{\mu\lambda}(x) g_{\lambda\nu}(x) = \delta_{\mu\nu}.
\]

The form of Eq.(33) is identical to the usual relation which expresses the Christoffel symbol in terms of the Riemann metric for some Riemann geometry [17]. Moreover, using (33),(30),(31), and (32) it is easy to verify that the quantity \( R_{\rho\mu\nu}^{\lambda}(x) \) can be presented in terms of \( g_{\mu\nu}(x) \) as well

\[
R_{\rho\mu\nu}^{\lambda}(x) = \partial_{\rho} \Gamma_{\lambda\nu}^{\lambda} - \partial_{\nu} \Gamma_{\lambda\rho}^{\lambda} + \Gamma_{\rho\nu}^{\sigma} \Gamma_{\sigma\lambda}^{\lambda} - \Gamma_{\rho\sigma}^{\sigma} \Gamma_{\sigma\lambda}^{\lambda}.
\]
One recognizes in this relation the usual connection between the Riemann tensor and the Riemann metric. Furthermore, it follows from (33) that \( g_{\mu\nu}(x), \Gamma^\lambda_{\mu\nu}(x) \) and \( R^\sigma_{\lambda\mu\nu}(x) \) possess the symmetry properties which are usual in the Riemann geometry

\[
\begin{align*}
g_{\mu\nu}(x) &= g_{\nu\mu}(x), \quad (37) \\
\Gamma^\lambda_{\mu\nu}(x) &= \Gamma^\lambda_{\nu\mu}(x), \quad (38) \\
R^\sigma_{\lambda\rho\mu\nu}(x) &= R^\sigma_{\lambda\rho\nu\mu}(x) = R^\rho_{\lambda\sigma\mu\nu}(x) = R^\mu_{\lambda\sigma\rho\nu}(x), \quad (39)
\end{align*}
\]

where

\[
R^\sigma_{\lambda\rho\mu\nu}(x) = g_{\sigma\lambda}(x)R^\lambda_{\rho\mu\nu}(x).
\]

Eqs.(33),(36), (37),(38) and (39) show that \( g_{\mu\nu}(x) \) can be considered as a metric for some Riemann geometry with the Christoffel symbol \( \Gamma^\lambda_{\mu\nu}(x) \) and the Riemann tensor \( R^\lambda_{\rho\mu\nu}(x) \).

Consider now the second classical equation (29). With the help of Eqs.(30),(32) it is easy to verify that it can be presented in the following form

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \quad (40)
\]

Here

\[
\begin{align*}
R_{\mu\nu}(x) &= R^\lambda_{\mu\lambda\nu}(x), \quad (41) \\
R(x) &= g^{\mu\nu}(x)R_{\mu\nu}(x). \quad (42)
\end{align*}
\]

These definitions show that \( R_{\mu\nu}(x) \) and \( R(x) \) are the Ricci tensor and the curvature of the Riemann geometry based on the metric \( g_{\mu\nu}(x) \). Hence, Eq.(40) proves to be identical to the Einstein equations of general relativity in the absence of matter.

We come to the interesting result. Remember that the matrix \( h^i_{\mu}(x) \) describes the orientation of (anti)instantons in the considered nontrivial vacuum, while \( A^{ij}_{\mu}(x) \) is the potential of the weak gauge field. Both these quantities describe properties of the gauge field. We have demonstrated above that the first classical condition (26) for the gauge field can be expressed as a statement that particular combinations (30),(31),(32) of \( h^i_{\mu}(x) \) and \( A^{ij}_{\mu}(x) \) are identical to the Riemann metric, the Christoffel symbol, and the Riemann tensor for some Riemann space. The second classical equation (27) proves to be identical to the Einstein equations for this Riemann metric.

The Einstein equations imply in particular that there appear excitations called gravitational waves. They possess zero mass and spin-2, exactly what we have been looking for. In the considered above picture these excitations arise from variables which describe the particular degrees of freedom of the gauge field. Remember that these variables can be considered as the two modes. One of them describes a weak topologically trivial gauge field. The other one describes polarization of instantons.
The strong interaction of these two modes (22) mixes them and the resulting spin-2 excitation should be considered as a coherent propagation of the two interacting modes.

It is instructive to consider the action (22) when the first classical Eq.(26) is valid. It is clear from (30),(32) that the integrand of the action (22) proves be proportional to the integrand of the usual Hilbert action of general relativity [17] which has the following form in Euclidean formulation

\[ S_H = -\frac{1}{16\pi k} \int R(x)[\text{det} g(x)]^{1/2} d^4x. \]  

(43)

One can consider the two actions (22) and (43) as same quantity if the Newton gravitational constant \( k \) is connected with the constant \( f \) which characterize the density of the polarized condensate of instantons

\[ k = \frac{1}{16\pi f}. \]  

(44)

This relation demonstrates that \( f = 2/r_P^2 \), thus supporting estimation (25). Remember that the constant \( f \) introduced in (11),(12) depends on the typical radiuses and separations of the polarized instantons. Relation (44) shows that we should assume that these radiuses and separations are comparable with the Planck radius.

The consideration given verifies that the long-range excitations of the gauge field proves to be the massless spin-2 particles. Their classical propagation is described by the Einstein equation of general relativity. The corresponding effective action turns out to be identical to the Hilbert action of general relativity. These facts altogether permit one to identify the found excitations with gravitational waves. This indicates that gravity arises in the framework of the gauge theory. It is very important that the dynamics of general relativity, its action and equations of motion, originate directly from the dynamics of the gauge field. All these results follow directly from assumptions (11)-(16) which were interpreted above as the SO(3, 3) polarization of instantons.

The picture developed above remains valid until the gauge field varies smoothly on the radius of polarized instantons. Under this condition the action (22) remains valid. As was mentioned, radiuses of polarized instantons are comparable with the Planck radius. It makes it necessary for gravitational waves to have the wavelength larger then the Planck radius. For shorter wavelengths the term (22) does not manifest itself in the action. This means that the interaction of short wavelength gauge field with the polarized instantons is suppressed. Therefore in this high-energy region the gauge field is described by the conventional Yang-Mills action and reveals its usual properties. In particular its excitations are spin-1 gauge bosons, while gravitons do not exist. Thus gravity manifests itself only for energies below the Planck energy, while for higher energies it does not exist.

Remember that \( h^{\mu \nu}(x) \) was introduced in Section 3 as an order parameter which describes the polarization of instantons. It is interesting that the Riemann structure
defined in (30),(31),(32) makes it possible another physical interpretation of this quantity. Namely, $h^{i\mu}(x)$ can be identified with the vierbein, the quantity which generally speaking is well known in general relativity [17]. The way of deriving the Einstein equation (33) from the two classical equations is similar to the Palatini method, see Ref. [17], formulated with the help of the vierbein formalism. Our consideration reveals however an important subtlety. In the usual vierbein formulation the physical nature of the space where the index $i$ of the matrix $h^{i\mu}(x)$ belongs does not play a substantial role. This index can be considered purely as a label [17]. In contrast, in our approach this index belongs to the isotopic space which plays the most important central role. In this space gauge transformations of the considered $SO(4)$ gauge theory are defined. There is however a price for this new physical interpretation of the index $i$. It is important for us to rely on the Euclidean formulation. An attempt to use the Minkowsky formulation as a starting point meets a difficulty. Really, in Euclidean formulation a connection between the metric and the vierbein can be presented as

$$g^{\mu\nu}(x)h^i_\mu(x)h^j_\nu(x) = \delta_{ij},$$

where Eqs.(30),(33) were used. The delta-symbol in the right hand side here simply shows that the isotopic space is the 4D Euclidean space. In contrast, if one would attempt to develop an approach used above starting from the Minkowsky coordinate space then connection between the metric and the vierbein would look [17]

$$g^{\mu\nu}(x)h^i_\mu(x)h^j_\nu(x) = g^{(0)ij},$$

where $g^{(0)} = \text{diag}(1,1,1,-1)$ is the Minkowsky metric. Thus the space which in our approach should be considered as an isotopic space for some gauge theory acquires the structure of the $3+1$ Minkowsky space. Gauge transformations in this space look as $SO(3,1)$ transformations belonging to the noncompact Lorentz group. As is well known, the gauge theory for noncompact groups is not unitary and therefore is poorly defined [18]. Thus the gauge formulation discussed above becomes questionable if the Minkowsky space would be used a starting point.

We conclude that an attempt to continue the vierbein into real space-time presents a problem. Fortunately, one can avoid it. One can use first Euclidean formulation, deriving the Einstein equations (33) as was discussed above. These equations are formulated entirely in terms of metric, neither the vierbein nor any isotopic index manifest themselves in these equations explicitly. This fact permits one to fulfill continuation of the final Einstein equations into real space-time by continuation of the metric without reference to the vierbein formalism.

In summary, it is shown in this Section that the Einstein equations of general relativity arise from of the $SO(4)$ gauge theory.
4. Discussion of results

The main results of this paper is a set of conditions which are to be fulfilled in order to derive the effect of gravity by means of conventional gauge theory. Let us formulate these results presenting them as comments to the list of questions posed in the Introduction.

1. It is shown above that one can hope to deduce the effect of gravity from the gauge theory if the gauge group is $SO(4)$.

2. The necessary vacuum is shown to include the polarized instantons. There is a simple physical reason explaining a necessity for this vacuum state. Compare the Yang-Mills action (3) with the Hilbert action (43). The striking difference between them is the power of the field in their integrand. The Yang-Mills action is quadratic in the gauge field strength. In contrast the Hilbert action (43) can be considered as a quantity which is linear in the Riemann tensor due to an obvious relation $R = g^{\mu\nu} R_{\mu\lambda\nu}$. At the same time there is the long-known resemblance between the basic properties of the gauge field strength and the Riemann tensor, see for example Ref. [21]. This paper proposes to make this resemblance identity in a sense explained in (32). This proposal can only be accomplish if some phenomenon in gauge theory produces a term in the action which is linear in the field strength. Eq.(5) shows that an instanton provides just the necessary effect. Moreover, an ensemble of polarized instantons results in the action (22) which proves to be identical to the Hilbert action. This makes polarized instantons be so special for gravity.

3. It is argued above that the role of the order parameter for the phase with polarized instantons should is played by mean orientations of instantons and antiinstantons. The orientation for an (anti)instanton is described by $3 \times 3$ matrixes. Therefore in the $SO(4)$ gauge theory there can arise four $3 \times 3$ matrixes describing mean orientations of all available instantons and antiinstantons. The main assumption of this paper is that these four matrixes can be described by one $6 \times 6$ matrix which belongs to $SO(3,3)$. It is important to mention that one cannot afford something less than that, the order parameter in the picture suggested has to be an $SO(3,3)$ matrix. To see this one can reverse arguments of Section 3. The gravitational field is known to be described by the vierbein which can be looked at as a $SL(4)$ matrix. Therefore to describe gravity by means of gauge theory one needs to express the vierbein, the local $SL(4)$ matrix in terms of some variables of gauge theory. Eq.(17) provides such a possibility, provided there exists the $SO(3,3)$ order parameter. This consideration shows in particular that the gauge theory should be based on the $SO(4)$ group (or bigger one) which possesses sufficient number of different (anti)instantons in order to develop the necessary $SO(3,3)$ order parameter.

4. Excitations above the considered vacuum state are found to be identical to gravitational waves. These excitations are described in terms of two modes of the
gauge field. One of them is a weak topologically trivial gauge field $A^{ij}_\mu(x)$. The other mode describes orientations of instantons $h^{i\mu}(x)$. Eq.(22) shows that there is the strong interaction between these modes which results in their mixing. Thus gravitational waves should be considered as a coherent propagation of the two interacting modes describing two different degrees of freedom of the gauge field.

5. We verified that low energy degrees of freedom of the gauge field are described by the variables of the Riemann geometry if the polarized vacuum exists. The low-energy action (22) is found to be identical to the Hilbert action of general relativity. This makes the classical equations be identical to the Einstein equations of general relativity. Eq.(14) shows that the relevant instantons should have radiuses comparable with the Planck radius. This fact restricts the energies from above. For energies well below the Planck radius the long-wave-length action (22) is valid, resulting in existence of gravitational waves. For energies higher than the Planck limit Eq.(22) becomes not applicable, the weak gauge field interacts with instantons no more and one should describe the gauge field by the usual Yang-Mills action. This shows that gravitational waves exist only for low energies, while high energy behavior of the gauge field should be described by the gauge bosons.

6. We have not considered quantum corrections above. From experience in QCD one could anticipate that they might be dangerous for the scenario considered. Even if the gauge constant is small for some short distances, quantum corrections in the QCD vacuum are known to make it rise for larger distances [22, 23, 24] eventually resulting in confinement. One should keep in mind, however, that in the considered polarized vacuum the role of quantum corrections differ qualitatively from the QCD vacuum. Remember that the rise of the gauge coupling constant in the QCD vacuum is related to those quantum fluctuations whose wavelength is the longer the larger the distances considered. The point is that the long range quantum fluctuations of the gauge field in the considered vacuum are strongly affected by polarized instantons. This makes the quantum corrections different from the ones in the QCD vacuum. This is an important issue and it is worth to repeat it in terms of the basic excitations. The perturbation series in the QCD vacuum is known to become non-reliable for large distances due to large contribution from virtual long-wavelength gauge bosons. The divergence of the perturbation theory is usually considered as an indication that for large distances the quantum corrections become very important. In the polarized vacuum this picture does not work because, as we verified above, low-energy spin-1 gauge bosons simply do not exist in this vacuum. There are spin-2 gravitational waves instead of them. Definitely this argument deserves to be considered in much more detail, which puts it outside the main stream of this paper.

7. Eq.(14) postulates the $SO(3, 3)$ symmetry for the order parameter. This proves to be a very demanding assumption. To illustrate this statement consider the simplest
possible case of zero gravitational field. Notice that choosing the Galilean coordinates one can always eliminate the gravitational field locally. This makes the case of the zero field be of interest for nonzero gravitational fields as well. In the absence of gravitational field the vierbein can be chosen to be a constant $SO(4)$ matrix $h^{\mu \nu}(x) = H^{\mu \nu} \in SO(4)$. Eq. (17) shows that the matrix $M^{AB}(x)$ in this case is a constant $SO(3) \times SO(3)$ matrix

$$M = \begin{pmatrix} C & \hat{0} \\ \hat{0} & \hat{C} \end{pmatrix},$$

where $C, \hat{C} \in SO(3)$. This means according to (1) that the antiinstantons in the $su(2)\eta$ subalgebra must be polarized, their polarization is described by the orthogonal matrix $C$. Similarly, the instantons in the $su(2)\bar{\eta}$ subalgebra are also polarized, their polarization is described by the orthogonal matrix $\hat{C}$. In contrast, the instantons in the $su(2)\eta$ and antiinstantons in the $su(2)\bar{\eta}$ subalgebras are not polarized. Thus the polarization of (anti)instantons is to be not trivial even in the absence of the gravitational field.

To meet the requirement (45) one should develop a $SU(2)$ gauge theory model in which instantons are polarized, while antiinstantons remain not polarized. The candidate for such a model was proposed in Refs. [3, 4]. In this model there arises interaction between instantons which makes their identical orientation more probable. This interaction originates from single-fermion loop correction when interaction of fermions with the given instantons as well as with particular scalar condensates is taken into account. There is no interaction of this type between antiinstantons. The interaction between instantons provides a possibility for a phase transition into the state with polarized instantons. An existence of the polarized phase has been verified in [4] in the mean field approximation. The order parameter for polarized instantons has the $SO(3)$ symmetry as is necessary to satisfy Eq. (45).

Thus the mentioned model gives a hope that the phase described by the simplest $SO(3) \times SO(3)$ order parameter (45) can exist. To move further on one should overcome two limitations of the model. Firstly, the way is to be found to make the ensemble of instantons and antiinstantons in the $SO(4)$ gauge theory to develop the condensate governed by the more sophisticated $SO(3,3)$ order parameter. Secondly, the model of [3, 4] relies upon existence of the scalar condensates transformed by the gauge group. These condensates make the gauge field acquire a mass through the Higgs mechanism. Therefore there is a danger that the mass of the gauge field would result in creation of the mass for gravitational waves. One should find a way to avoid this undesirable property.

Thus, either development of the model [3, 4] or possibly a new more sophisticated model is necessary. The present paper clearly sets the requirements for such model.

The only information available on possible phase transitions into the state with polarized instantons comes from the mentioned above model [3, 4] in which it was found
that in the mean field approximation there exists the first order phase transition into the polarized state.

Our last comment is on the dilute gas approximation for instantons. It is shown in Appendix A that even closely located instantons interact with the external field as individual objects. This fact may be considered as an indication that the picture developed above for the gas of instantons may be valid for the instanton liquid as well.

5. Conclusion

The conventional approach to general relativity postulates that the theory is to be written in geometrical terms. Postulating then the Hilbert action one derives the Einstein equations. In this paper we discuss a possibility for a novel approach. Space-time is supposed to be basically flat. In this space-time the $SO(4)$ gauge theory is formulated. Assuming that in this theory there develops the particular phase, called polarized instantons we find that for low energies the gauge field can be described in terms of the Riemann geometry. There are two low-energy modes of the gauge field which play the important role. One of them accounts for topologically trivial sector of the theory, the other one describes the polarization of instantons. These two modes strongly interact. Their interaction originates from the conventional Yang-Mills action, but for weak fields it can be presented in the form of the Hilbert action from which one derives the Einstein equations describing gravitational waves. Thus the gravitational wave arises as a mixing of the two modes of the gauge field.

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Appendix A. Interaction of instantons with an external field

Let us show that instantons interact with an external field as individual objects. It means that the action describing the interaction of several instantons with a weak slowly varying external gauge field is given by a sum of terms of Eq.(5) type describing interactions of different instantons with the field. This statement is trivial when the dilute gas approximation is valid. Surprisingly it remains valid for any configuration of instantons, even if they are overlapping.

To describe the closely situated instantons one needs to consider the $k$-instanton
general solution [13], for the review see [13]. It is sufficient for our purposes to suppose that all instantons are in some $su(2)$ subalgebra. Let us introduce a quaternion as $q = q_\mu \tau_\mu^+,$ where $q_\mu, \mu = 1, \cdots, 4$ is an arbitrary vector. In this notation $q^+ = q_\mu \tau_\mu^-$, where $\tau_\mu^\pm = (\pm i \vec{\tau}, 1)$. Consider the $(k + 1) \times k$ matrix $M_{s,t}(x), s = 1, \cdots, k + 1, t = 1, \cdots, k$ which has quaternionic matrix elements. $M(x)$ is a linear function of the quaternion of coordinates $x = x_\mu \tau_\mu^+$

$$M(x) = B - Cx,$$

where $B$ and $C$ are $x$-independent $(k + 1) \times k$ quaternionic matrixes. They must be chosen so that the condition

$$M^+(x)M(x) = R(x)$$  \hspace{1cm} (A1)

be fulfilled for any $x$. Here $R(x)$ is a non-degenerate $k \times k$ real matrix. The matrix $C$ can be chosen to be

$$C_{1t} = 0, \quad C_{1+s,t} = \delta_{st}, \quad s, t = 1, \cdots, k.$$  \hspace{1cm} (A2)

(The matrix $M(x) = M_{s,t}(x)$ which plays the central role in the ADHM construction should not be confused with the matrix $M(x) = M^{AB}(x)$ defined in (12) to describe polarization of instantons.) Having $M(x)$ one can find the $k + 1$ quaternionic vector $N(x)$ which satisfies equations

$$M^+(x)N(x) = 0,$$  \hspace{1cm} (A3)

$$N^+(x)N(x) = 1.$$  \hspace{1cm} (A4)

Then the vector-potential defined in the quaternion representation as

$$A_\mu(x) = \frac{1}{2i} A^a_\mu(x) \tau^a = N^+(x) \partial_\mu N(x)$$  \hspace{1cm} (A5)

results in the general self-dual gauge field for $k$ instantons

$$F_{\mu\nu}(x) = \frac{1}{2i} F^a_{\mu\nu}(x) \tau^a$$  \hspace{1cm} (A6)

$$= 2iN^+(x)C \eta^{a\mu\nu} \tau^a R^{-1}(x)C^+ N(x).$$

To simplify notation in the following consideration consider the case of two instantons when the matrixes $M(x), C$ in the ADHM solution have an explicit simple form

$$M(x) = \begin{pmatrix} q_1 & q_2 \\ y_1 - x & b \\ b & y_2 - x \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$  \hspace{1cm} (A7)

where $q_i, \ i = 1, 2$ are the quaternions describing the radiuses $\rho_i = |q_i|$ and orientations of the two instantons. The orientations of instantons may be considered as $SU(2)$
matrices $n_i = q_i/|q_i|$. An equivalent definition of the orientation may be given in terms of a matrix $D^{ab} \in SO(3)$, see Eq.(4). The relation
\begin{equation}
\tau^a D^{ab}_i = n_i \tau^b n_i^+ \tag{A8}
\end{equation}
defines these $SO(3)$ orientation matrices for the two instantons in terms of $q_i$. The positions of the instantons are given by $y_i$, and $b$ is
\begin{equation}
b = \frac{y_{12}}{2|y_{12}|^2} \left( q_2^+ q_1 - q_1^+ q_2 \right), \quad y_{12} = y_1 - y_2. \tag{A9}
\end{equation}
In the following consideration we will need the instanton field in the region $x \gg |y_{12}|, \rho_1, \rho_2$.

Eq.(A3) for two instantons reads
\begin{equation}
(x - y_1)^+ N_2(x) = q_1^+ N_1(x) + b^+ N_3(x), \tag{A11}
\end{equation}
\begin{equation}
(x - y_2)^+ N_3(x) = q_2^+ N_1(x) + b^+ N_2(x). \tag{A12}
\end{equation}
The approximate solution of this equation in the region (A10) is
\begin{equation}
N_2(x) \simeq \frac{x - y_1}{|x - y_1|^2} q_1^+ N_1(x), \tag{A13}
\end{equation}
\begin{equation}
N_3(x) \simeq \frac{x - y_2}{|x - y_2|^2} q_2^+ N_1(x), \tag{A14}
\end{equation}
where the terms $\sim 1/|x - y_i|^2$ are omitted in the lhs’s. From these relations and the normalization condition (A4) one finds $|N_1(x)| \simeq 1$. A choice
\begin{equation}
N_1(x) = 1, \tag{A15}
\end{equation}
fixes the singular gauge. The matrix $R(x)$ for two instantons in the region (A10) is simplified to be
\begin{equation}
R(x) \simeq \begin{pmatrix}
\frac{|x - y_1|^2}{2} & 0 \\
0 & \frac{|x - y_2|^2}{2} 
\end{pmatrix}. \tag{A16}
\end{equation}
Substituting (A13),(A14),(A15),(A16) in Eqs.(A3),(A6) one finds that the potential and the field created by the two instantons in the region (A10) is equal to the sum of potentials and fields created by individual instantons
\begin{equation}
A^a_{\mu\nu}(x) = \sum_{i=1,2} A^a_{\mu\nu,i}(x), \tag{A17}
\end{equation}
\begin{equation}
F^a_{\mu\nu}(x) = \sum_{i=1,2} F^a_{\mu\nu,i}(x). \tag{A18}
\end{equation}
Here $A^a_{\mu\nu,i}(x), F^a_{\mu\nu,i}(x)$ are given by the known expressions which in the region (A10) read
\begin{equation}
A^a_{\mu\nu,i}(x) = 2\tilde{\eta}^{\mu\nu} D^a_i \frac{(x - y_i)_{\mu\nu}}{|x - y_i|^2}, \tag{A19}
\end{equation}
\begin{equation}
F^a_{\mu\nu,i}(x) = -4M_{\mu\nu,i} M_{\mu\nu,i} \tilde{\eta}^{\mu'\nu'} D^a_i \frac{1}{|x - y_i|^4} \rho_i^2. \tag{A20}
\end{equation}
Here $M_{\mu\nu,i} = \delta_{\mu\nu} - 2(x - y_i)_\mu(x - y_i)_\nu/|x - y_i|^2$. Thus in the region $|y_i|\ll 1$, the instantons contribute independently to the gauge potential and the gauge field even for small separations of instantons.

Consider now the interaction of the external gauge field $[F_{\mu\nu}^a]_{\text{ext}}$ with two instantons generalizing the approach of [13] originally proposed for a single instanton. The field is supposed to be weak enough $|F_{\mu\nu}^a|_{\text{ext}} \ll 1/(g^2 \rho_i^2)$, $1/(g^2 \rho_2^2)$. It is also supposed to vary slowly, $r|\partial_\lambda [F_{\mu\nu}^a]_{\text{ext}}| \ll |F_{\mu\nu}^a|_{\text{ext}}$, where $r = \max(\rho_1, \rho_2, |y_{12}|)$. Consider the sphere of radius $R \gg r$ centered at $x_0 = (y_1 + y_2)/2$ surrounding the instantons. Let us choose the radius of the sphere $R$ to be so small that the variation of the external field inside the sphere is negligible. At the same time the radius $R$ is to be large enough to make the external field outside the sphere larger than the field created by the instantons. Let us use the Landau gauge for the potential of the external field

$$[A^a_\mu(x)]_{\text{ext}} = -\frac{1}{2}(x - x_0)_\nu [F^a_{\mu\nu}(x_0)]_{\text{ext}}. \tag{A21}$$

Consider the action of the gauge field

$$S = \frac{1}{4g^2} \int F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) d^4 x. \tag{A22}$$

The instantons give large contribution to the action $S_{\text{ins}} = 2 \times (8\pi^2/g^2)$ which comes mainly from the inner-sphere region of integration in (A22). Let us find now the contribution which the inner-sphere region gives to the action (A22) where the external field can be considered as a perturbation. From the definition

$$S_I(\{F\}) = \frac{1}{4g^2} \int_{|x-x_0|<R} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) d^4 x$$

one finds

$$\delta S_I = \frac{1}{g^2} \int_{|x-x_0|<R} \nabla_\mu \left([A^a_\nu(x)]_{\text{ext}}\right) [F^a_{\mu\nu}(x)]_{\text{ins}} d^4 x. \tag{A23}$$

Here $\delta S_I = S_I(\{F_{\text{ins}} + F_{\text{ext}}\}) - S_I(\{F_{\text{ins}}\})$. Integrating by parts and using the classical equation $\nabla_\mu [F^a_{\mu\nu}(x)]_{\text{ins}} = 0$ for the field of instantons one finds from (A23)

$$\delta S_I = \frac{1}{g^2} \int [A^a_\nu(x)]_{\text{ext}} [F^a_{\mu\nu}(x)]_{\text{ins}} dS_\mu, \tag{A24}$$

where $\int dS_\mu$ denotes integration over the sphere. Substituting (A18),(A20),(A21) in Eq.(A24) one finds the contribution of the inner-sphere region to the action

$$\delta S_I = \frac{\pi^2}{g^2} \sum_{i=1,2} \rho_i^2 \tilde{\eta}^{\mu\nu} D^a_i F^b_{\mu\nu}(x_i). \tag{A25}$$

One can consider similarly the contribution $\delta S_E$ of the outer-sphere region where the field of instantons plays the role of a perturbation. The integration again is reduced
to integration over the sphere and the final result is identical to (A23), $\delta S_E = \delta S_I$. Deriving it one should neglect the integral over 4-dimensional outer-sphere region

$$\int_{|x-x_0| > R} [A_\nu(x)]^{\text{ins}} \nabla_\mu [F^\alpha_{\mu\nu}(x)]^{\text{ext}} d^4 x \to 0. \tag{A26}$$

There are two possible reasons for this. If the external field is strong in some region, then it is supposed to satisfy there the classical equation $\nabla_\mu [F^\alpha_{\mu\nu}(x)]^{\text{ext}} = 0$. The weak field is supposed to decrease smoothly at infinity. If $l$ is a large length parameter governing this field, $|F^\alpha_{\mu\nu}]^{\text{ext}} | \sim 1/l^2$, $|\nabla_\rho [F^\alpha_{\mu\nu}(x)]^{\text{ext}} | \sim 1/l^3$, then the integral in (A26) is estimated as $\int_{x \sim a} d^4 x \sim (\rho_1^2 + \rho_2^2)/l^2 \ll 1$. Note that the decrease of the external field should be slower than the decrease of the instantons field to ensure that everywhere in the outer-sphere region the external field exceeds the instanton field.

Summing contributions of both inner-region and outer-region one finds

$$\Delta S = \frac{2\pi^2}{g^2} \sum_i \rho_i^2 \eta^{\alpha\mu\nu} D_i^{ab} F_{\mu\nu}(x_i). \tag{A27}$$

The external field considered above was supposed not to vary dramatically on the distance of instanton separation $|y_{12}|$. This condition was assumed in order to take into consideration the case of small separations where the obtained result is surprising. If the separation is larger then radiuses, than two instantons may certainly be considered as individual objects and (5) may be applied to every instanton separately. In this case there is no need to suppose that the external field remains constant, it may vary from one instanton to the other. This fact permits one to chose the argument of the external field in (A27) as $x_i$. Eq.(A27) is verified above for the case of two instantons mainly to simplify notation. A similar consideration shows that it remains valid for any number of instantons provided the external field is weaker than the instanton field in the vicinity of each instanton and its variation on their radiuses is negligible. The instantons separations may be arbitrary.

Remember that an instanton provides a minimum for the classical action for small variations of the field. The action (5) reveals a nontrivial behavior, it is linear in the field. This property is explained by the fact that the external field is stronger than the field of instantons in the outer-sphere region. Therefore in this region the perturbation theory breaks down. As a result there appears the surface-term (A24) in the action which describes the integration over a 3-dimensional surface separating the region where the external field is strong. It is this term that gives rise to the action (A27). In the approach developed in [14] the surface-term does not manifest itself explicitly. There exists however the field of free gauge bosons created by the instanton. Their radiation field exceeds the field of the instanton at infinity.

In conclusion, we verified that instantons interact with the external field independently even if their separations are small. Notice that this result demonstrates very clearly that the number of parameters in the $k$-instanton ADHM-construction is...
equal to $8k - 3$ and therefore this construction describes the most general configuration of instantons. In Ref.[19] this statement was proved using more sophisticated arguments.

**Appendix B. SO(3, 3) $\cong$ SL(4)**

Let us describe properties of the $SO(3, 3)$ group, in particular a relation which connects the $SO(3, 3)$ and $SL(4)$ groups. An existence of this relation is known in mathematical literature, see for example Ref.[20], where it is mentioned. The goal of this Section is to show that this relation has the form defined by (17).

Notice first that any matrix $M^{AB}$, $A, B = 1, 2, \cdots, 6$, $M \in SO(3, 3)$ may be presented in the following form

$$M = \left( \begin{array}{cc} C & \hat{0} \\ \hat{0} & \bar{C} \end{array} \right) \left( \begin{array}{cc} \sigma U & \bar{\sigma} V \\ \bar{\sigma} V & \sigma U \end{array} \right) \left( \begin{array}{cc} D^T & \hat{0} \\ \hat{0} & \bar{D}^T \end{array} \right).$$

(B28)

Here $C, \bar{C}, D, \bar{D} \in SO(3)$ and $U, V$ are diagonal $3 \times 3$ non-negative matrices

$$U_{ij} = U_i \delta_{ij}, \quad V_{ij} = V_i \delta_{ij}, \quad U_i, V_i \geq 0,$$

satisfying condition

$$U^2 - V^2 = \hat{1}.$$  

$\sigma, \bar{\sigma}$ are the numbers: $\sigma = \pm 1, \bar{\sigma} = \mp 1$. Numbers with a hat are used to represent $3 \times 3$ matrixes. The 15 parameters governing a matrix belonging to $SO(3, 3)$ are: 3 parameters for every one of the four $SO(3)$ matrixes in (B28) plus 3 parameters governing the matrixes $U, V$.

It is clear that if $\sigma = 1$ in (B28) then $M$ may be transformed into the unity $6 \times 6$ matrix $\mathbf{1}$ by the continuum transformation, i.e. there exists the continuum function $M(s)$, $M(s) \in SO(3, 3)$, $s \in [0, 1]$, $M(0) = M$, $M(1) = \mathbf{1}$. Therefore $\sigma = 1$ implies $M \in SO_+(3, 3)$. If $\sigma = -1$, then this transformation is impossible, this case is denoted as $M \in SO_-(3, 3)$. The proof of Eq.(B28) requires simple conventional algebraic transformations based on (E).

Consider now a matrix $H^{ij} \in SL(4)$ and construct from it the quantity $H^{im}H^{jn} - H^{in}H^{jm}$. The later one is obviously antisymmetric in $i, j$ as well as in $m, n$. Therefore it can be expanded in a series of the 't Hooft symbols which provide the full basis for antisymmetric matrixes. It is convenient to present this expansion with the help of $\eta^{Aij}, A = 1, 2, \cdots, 6$, symbols defined in (7). Then the expansion reads

$$H^{im}H^{jn} - H^{in}H^{jm} = \frac{1}{2} M^{AB} \eta^{Anm} \eta^{Bij},$$

(B29)

The coefficients $M^{AB}$ in the rhs may be considered as a $6 \times 6$ matrix $M$, see Eq.(17). Let us show that $M \in SO_+(3, 3)$ and that Eq.(7) describes an isomorphism $SL(4)/(\pm 1) \equiv SO_+(3, 3)$.
The symbols $\eta^{Aij}$ and $\tilde{\eta}^{Aij} = (1/2)\epsilon^{ijkl}\eta^{Akl}$ satisfy relations which follow directly from the usual relations for the ’t Hooft symbols

$$\eta^{Aij} \eta^{Bij} = \tilde{\eta}^{Aij} \tilde{\eta}^{Bij} = 4\delta_{AB},$$  \hspace{1cm} \text{(B30)}

$$\eta^{Aij} \tilde{\eta}^{Bij} = 4\Sigma_{AB},$$  \hspace{1cm} \text{(B31)}

where $\Sigma$ is defined in (13). Using (B30) one can present (B29) in the equivalent useful form

$$H^{im}H^{jn}\eta^{Amn} = M^{AB}\eta^{Bij}.$$  \hspace{1cm} \text{(B32)}

Let us define by $\eta'^{A}_{ij}$ the lhs of this equation $\eta'^{A}_{ij} = H^{im}H^{jn}\eta^{Amn}$. Then using the condition $\det H = 1$ it is easy to verify that (B31) remains valid for the primed symbols as well, $\eta'^{A}_{ij} \tilde{\eta}'^{B}_{ij} = 4\Sigma_{AB}$. Then from (B32) one finds restrictions on the matrix $M$ which show that $M \in SO(3,3)$. Thus (B29) gives a function $H \rightarrow M$ from $SL(4)$ into $SO(3,3)$. From (B32) it is easy to deduce that this function is a homomorphism: if $H_1 \rightarrow M_1$, $H_2 \rightarrow M_2$ then $H_1H_2 \rightarrow M_2M_1$.

Let us show now that for any $M \in SO_+(3,3)$ there exists $H \in SL(4)$ satisfying (B29), while for $M \in SO_-(3,3)$ this equation cannot be satisfied. With this purpose consider representation (B28) for some $M \in SO(3,3)$. Notice first that if $M$ has the form

$$M = M_C = \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}, \quad C, \bar{C} \in SO(3),$$

then (B32) may certainly be solved with respect to the matrix $H = H_C$ which in this case belongs to $SO(4)$. Really, if $C = \exp(-\phi^a t^a)$, $\bar{C} = \exp(-\bar{\phi}^a t^a)$, where $t^a$ are the generators of $SO(3)$, then

$$(H_C)_{ij} = \pm(\exp(\phi^a \eta^a + \bar{\phi}^a \bar{\eta}^a))_{ij} \in SO(4)$$  \hspace{1cm} \text{(B33)}

satisfies (B32). For this case Eq. (B32) describes the well-known homomorphism $SO(4) \rightarrow SO(3) \times SO(3)$.

In order to find $H$ for any $M \in SO(3,3)$ it is necessary now to consider the second factor in the rhs of (B28)

$$M_0 = \begin{pmatrix} \sigma U & \bar{\sigma} V \\ \bar{\sigma} V & \sigma U \end{pmatrix}.$$  \hspace{1cm} \text{(B34)}

For $M = M_0$ Eq. (B29) can be solved directly in respect to $H = H_0$. This is an easy task because $H_0$ proves to be diagonal

$$H_0 = \pm \frac{1}{H_{0,44}} \begin{pmatrix} \sigma U & 0 \\ 0 & \bar{\sigma} V \end{pmatrix}.$$  \hspace{1cm} \text{(B35)}

The symbols $\eta^{Aij}$ and $\tilde{\eta}^{Aij} = (1/2)\epsilon^{ijkl}\eta^{Akl}$ satisfy relations which follow directly from the usual relations for the ’t Hooft symbols

$$\eta^{Aij} \eta^{Bij} = \tilde{\eta}^{Aij} \tilde{\eta}^{Bij} = 4\delta_{AB},$$  \hspace{1cm} \text{(B30)}

$$\eta^{Aij} \tilde{\eta}^{Bij} = 4\Sigma_{AB},$$  \hspace{1cm} \text{(B31)}

where $\Sigma$ is defined in (13). Using (B30) one can present (B29) in the equivalent useful form

$$H^{im}H^{jn}\eta^{Amn} = M^{AB}\eta^{Bij}.$$  \hspace{1cm} \text{(B32)}

Let us define by $\eta'^{A}_{ij}$ the lhs of this equation $\eta'^{A}_{ij} = H^{im}H^{jn}\eta^{Amn}$. Then using the condition $\det H = 1$ it is easy to verify that (B31) remains valid for the primed symbols as well, $\eta'^{A}_{ij} \tilde{\eta}'^{B}_{ij} = 4\Sigma_{AB}$. Then from (B32) one finds restrictions on the matrix $M$ which show that $M \in SO(3,3)$. Thus (B29) gives a function $H \rightarrow M$ from $SL(4)$ into $SO(3,3)$. From (B32) it is easy to deduce that this function is a homomorphism: if $H_1 \rightarrow M_1$, $H_2 \rightarrow M_2$ then $H_1H_2 \rightarrow M_2M_1$.

Let us show now that for any $M \in SO_+(3,3)$ there exists $H \in SL(4)$ satisfying (B29), while for $M \in SO_-(3,3)$ this equation cannot be satisfied. With this purpose consider representation (B28) for some $M \in SO(3,3)$. Notice first that if $M$ has the form

$$M = M_C = \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}, \quad C, \bar{C} \in SO(3),$$

then (B32) may certainly be solved with respect to the matrix $H = H_C$ which in this case belongs to $SO(4)$. Really, if $C = \exp(-\phi^a t^a)$, $\bar{C} = \exp(-\bar{\phi}^a t^a)$, where $t^a$ are the generators of $SO(3)$, then

$$(H_C)_{ij} = \pm(\exp(\phi^a \eta^a + \bar{\phi}^a \bar{\eta}^a))_{ij} \in SO(4)$$  \hspace{1cm} \text{(B33)}

satisfies (B32). For this case Eq. (B32) describes the well-known homomorphism $SO(4) \rightarrow SO(3) \times SO(3)$.

In order to find $H$ for any $M \in SO(3,3)$ it is necessary now to consider the second factor in the rhs of (B28)

$$M_0 = \begin{pmatrix} \sigma U & \bar{\sigma} V \\ \bar{\sigma} V & \sigma U \end{pmatrix}.$$  \hspace{1cm} \text{(B34)}

For $M = M_0$ Eq. (B29) can be solved directly in respect to $H = H_0$. This is an easy task because $H_0$ proves to be diagonal

$$H_0 = \pm \frac{1}{H_{0,44}} \begin{pmatrix} \sigma U & 0 \\ 0 & \bar{\sigma} V \end{pmatrix}.$$  \hspace{1cm} \text{(B35)}
Condition $\det H = 1$ results in $H_{0,44}^2 = \det(\sigma U - \sigma V)$. One concludes from here that there is a solution $H_0$ when $\sigma = +1$, no solution exists for $\sigma = -1$. One finds the matrix $H$ presenting it in a form similar to (B28) for $M$

$$H = H_D H_0 H_T^T. \tag{B35}$$

A way to derive $H_C, H_D \in SO(4)$ is shown in (B33), and $H_0 \in SL(4)$ is found in (B34) with $\sigma = +1$. Eq. (B35) proves explicitly that (B29) gives a homomorphism $SL(4) \rightarrow SO_+(3,3)$. It follows from Eqs. (B34), (B33) that a kernel of this homomorphism is $\pm 1$. Thus (B29) gives an isomorphism $SL(4)/(\pm 1) \equiv SO_+(3,3)$. The 15 parameters governing $SL(4)$ matrix are: 6 parameters for each of the two $SO(4)$ matrixes in (B35) plus 3 parameters $U_i$ governing $H_0$ according to (B34).

In conclusion, Eq. (17) is true.

References

[1] C. N. Yang, R. Mills, Phys. Rev. 96, 191 (1954).
[2] M. Yu. Kuchiev, Europhys. Lett. 28, 539 (1994).
[3] M. Yu. Kuchiev, Europhys. Lett. 30, 255 (1995).
[4] M. Yu. Kuchiev, Phys. Rev. D53, 6946 (1996).
[5] D. Shuryak, Physics Reports, 264, 357 (1996).
[6] M. B. Green, J. H. Schwarz, E. Witten. Superstring theory (Cambridge, New York, 1987).
[7] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986).
[8] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. 59B, 85 (1975).
[9] S. Elitzur, Phys. Rev. D12, 3978 (1975).
[10] A. M. Polyakov, Gauge Fields and Strings (Harwood Academic Publishers, Chur, London, Paris, New-York, Melbourne, 1987).
[11] C. Hamer, in Statistical Mechanics and Field Theory, Proceedings of the Seventh Physics Summer School Eds. V. V. Bazhanov and C. J. Burden, (The Australian National University, 1994), pp. 253-278.
[12] G. ’t Hooft, Phys. Rev. D14, 3432 (1976).
[13] C. Callan, K. Dashen, and D. Gross, Phys. Rev. D17, 2717 (1978); Phys. Rev. D19, 1826 (1979).
[14] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Usp. Fiz. Nauk 136, 553 (1982).
[15] M. F. Atiah, N. J. Hintchin, V. G. Drinfeld, and Y. I. Manin, Phys. Lett. A65, 185 (1978).
[16] M. K. Prasad, Physica D1, 167 (1980).
[17] L. D. Landau, E. M. Lifshitz, The Classical Theory of Fields (Pergamon Press, Oxford and New York, 1971).
[18] S. L. Glashow and M. Gell-Mann, Ann. of Phys. 15, 437 (1961).
[19] N. H. Christ, E. J. Weinberg and N. K. Stanton, Phys. Rev. D18, 2013 (1978).
[20] R. Gilmore, Lie Groups, Lie Algebras, and some of their applications, (John Willey & Sons, New York, London, Sydney, Toronto, 1974).
[21] L. D. Faddeev and A. A. Slavnov, *Gauge fields: introduction to quantum theory* (Benjamin/Cummings, London, 1980).

[22] I. B. Khriplovich, *Soviet Journal of Nuclear Physics*, 10, 235 (1970).

[23] D. J. Gross and F. Wilczek, *Phys. Rev. Lett.* 30, 1343 (1973); *Phys. Rev. D* 8, 3633 (1973).

[24] H. Politzer, *Phys. Rev. Lett.* 30, 1346 (1973), *Phys. Rep.* 14C, 129, (1974).