ON READ’S TYPE OPERATORS ON HILBERT SPACES

by

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Abstract. — Using Read’s construction of operators without non-trivial invariant subspaces/subsets on $\ell_1$ or $c_0$, we construct examples of operators on a Hilbert space whose set of hypercyclic vectors is “large” in various senses. We give an example of an operator such that the closure of every orbit is a closed subspace, and then, answering a question of D. Preiss, an example of an operator such that the set of its non-hypercyclic vectors is Gauss null. This operator has the property that it is orbit-unicellular, i.e. the family of the closures of its orbits is totally ordered. We also exhibit an example of an operator on a Hilbert space which is not orbit-reflexive.

1. Introduction

Let $X$ be a real or complex infinite-dimensional separable Banach space, and $T$ a bounded operator on $X$. In this paper we will be concerned with the study of the structure of orbits of vectors $x \in X$ under the action of $T$ from various points of view. If $x$ is any vector of $X$, the orbit of $x$ under $T$ is the set $\text{Orb}(x, T) = \{ T^n x : n \geq 0 \}$. The closure of this orbit is denoted by $\overline{\text{Orb}}(x, T)$. The linear orbit of $x$ is the linear span of the orbit of $x$, i.e. the set $\{ p(T)x : p \in \mathbb{K}[\zeta] \}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. When the linear orbit of $x$ is dense, $x$ is said to be cyclic, and $x$ is said to be hypercyclic when the orbit itself is dense. An operator admitting a cyclic (resp. hypercyclic) vector is called cyclic (resp hypercyclic).

The structure of the set $HC(T)$ of hypercyclic vectors for a hypercyclic operator $T \in \mathcal{B}(X)$ has been the subject of many investigations: linear structure ($HC(T)$ always contains a dense linear manifold, see [5], sometimes an infinite-dimensional closed subspace, see [9]), topological structure ($HC(T)$ is a dense $G_\delta$ subset of $X$), measure-theoretic structure (see for instance [7], [2])... In particular, it is interesting to look for operators whose set of hypercyclic (or even cyclic) vectors is as large as possible, especially in the Hilbert space setting. Throughout the paper $H$ will denote a real or complex separable infinite-dimensional Hilbert space. A major open question in operator theory is to know whether,

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given any bounded operator $T$ on $H$, there exists a closed subspace $M$ (resp. a closed subset $F$) which is non-trivial, i.e. $M \neq \{0\}$ and $M \neq H$, and invariant by $T$, i.e. $T(M) \subseteq M$ (resp. with $F$). These problems are known as the Invariant Subspace and the Invariant Subset Problems. If one does not work with operators acting on a Hilbert space, but with operators acting on general separable Banach spaces instead, the question has been answered in the negative by Enflo [6] and Read [16]. Read in particular constructed an operator without non-trivial invariant subspaces in the space $\ell_1$ of summable sequences, and even an operator without non-trivial invariant closed subsets on $\ell_1$ [17]. In other words $HC(T) = \ell_1 \setminus \{0\}$ for this operator. The Invariant Subspace Problem is still open in the reflexive setting, and the closest one could get [18] to this are examples of operators without non-trivial invariant subspaces on some spaces with separable dual, such as $c_0$ for instance.

Our aim in this paper is to present a simplified version of Read’s construction in [17] which is adapted to the Hilbert space setting, and to obtain in this way operators whose orbits have interesting properties: we first construct an example of a Hilbert space operator such that the orbit of every vector $x$ coincides with its linear orbit. This corresponds to the construction of what we call the “(c)-part” in Read’s type operators (see Section 2 for definitions).

**Theorem 1.1.** — There exists a hypercyclic operator on $H$ such that for every vector $x \in H$, the closure of the orbit $\text{Orb}(x, T)$ is a subspace, i.e. the closures of the two sets $\{T^n x ; n \geq 0\}$ and $\{p(T)x ; p \in \mathbb{K}[\zeta]\}$ coincide.

We define in Section 2 the operators which will be needed for the proof of Theorem 1.1, explain the role of the (c)-fan, and then prove Theorem 1.1 in Section 3. This can be seen as the basic construction, and in Section 4 we elaborate on it to prove the next results.

Section 4 is devoted to the study of the set $HC(T)$ from the point of view of geometric measure theory: it is well known and easy to prove that whenever $T$ is hypercyclic on $X$, $HC(T)$ is a dense $G_\delta$ subset of $X$, or equivalently, its complement $HC(T)^c$ is a set of the first category, i.e. a countable union of closed sets with empty interior, so $HC(T)^c$ is a “small” set from this point of view. Increasing the size of $HC(T)$ means having $HC(T)^c$ smaller, and various notions of smallness have been considered in this setting. In particular, Bayart studied in [11] examples of operators such that $HC(T)^c$ was $\sigma$-porous, i.e. a countable union of porous sets. The notion of porosity quantifies the fact that a set has empty interior: a subset $E$ of a Banach space $X$ is called porous if there exists a $\lambda \in [0, 1]$ such that the following is true: for every $x \in E$ and every $\varepsilon > 0$, there exists a point $y \in X$ such that $0 < ||y - x|| < \varepsilon$ and $E \cap B(y, \lambda ||y - x||)$ is empty. A countable union of porous sets is called $\sigma$-porous. We refer the reader to the references [19] or [4] for more information on porous and $\sigma$-porous sets, and their role in questions related to the differentiation of Banach-valued functions.

Bayart constructed in [1] examples of operators $T$ on $F$-spaces such that $HC(T)^c$ was $\sigma$-porous, but on Banach spaces the only operators which were known to have this property were the ones without nontrivial closed invariant subsets. Hence a question of [1] was to know whether it was possible to have a Hilbert space operator $T$ such that $HC(T)^c$
The family \( \mu \). — Theorem 1.3 is interesting in itself and which easily implies Theorem 1.2:

For clarity’s sake we show first in Section 4 that an operator \( T \) and we introduce what we call the “(b)-part” in Read’s examples in order to achieve this.

The proof of Theorem 1.2 requires that we complicate a bit the construction of Section 2, in particular the operator induced by \( T \) on any invariant subspace \( M \) is hypercyclic, i.e. \( M = \overline{\text{Orb}(x,T)} \) for some \( x \in H \).

The term “orbit-unicellularity” comes from the fact that an operator is said to be unicellular if the lattice of its invariant subspaces is totally ordered. When an operator \( T \) is unicellular, every invariant subspace \( M \) of \( T \) is cyclic, i.e. is the closure \( M_x \) of the linear orbit of some vector \( x \in H \), and the unicellularity of \( T \) is equivalent to the fact that for every pair \( (x,y) \) of vectors of \( H \), either \( \overline{\text{Orb}(x,T)} \subseteq \overline{\text{Orb}(y,T)} \) or \( \overline{\text{Orb}(y,T)} \subseteq \overline{\text{Orb}(x,T)} \). In particular the operator induced by \( T \) on any invariant subspace \( M \) of \( H \) is hypercyclic, i.e. \( M = \overline{\text{Orb}(x,T)} \) for some \( x \in H \).

Theorem 1.3. — There exists a bounded operator \( T \) on the Hilbert space \( H \) such that the set \( HC(T)^c \) is a countable union of subsets of closed hyperplanes of \( H \). In particular \( HC(T)^c \) is Gauss null (hence Haar null) and \( \sigma \)-porous.

Recall that a subset \( A \) of \( H \) is said to be Gauss null if for every non-degenerate Gaussian measure \( \mu \) on \( H \), \( \mu(A) = 0 \). Since the \( \mu \)-measure of a closed hyperplane vanishes for every such \( \mu \), \( HC(T)^c \) will clearly be Gauss null.

The proof of Theorem 1.2 requires that we complicate a bit the construction of Section 2, and we introduce what we call the “(b)-part” in Read’s examples in order to achieve this. For clarity’s sake we show first in Section 4 that an operator \( T \) can be constructed with \( HC(T)^c \) Haar null (and \( \sigma \)-porous). Then we show in Section 5 the following result, which is interesting in itself and which easily implies Theorem 1.2.

Theorem 1.3. — There exists a bounded operator \( T \) on \( H \) which is orbit-unicellular: the family \( \overline{\text{Orb}(x,T)} \) of all the closures of its orbits is totally ordered, i.e. for any pair \( (x,y) \) of vectors of \( H \), either \( \overline{\text{Orb}(x,T)} \subseteq \overline{\text{Orb}(y,T)} \) or \( \overline{\text{Orb}(y,T)} \subseteq \overline{\text{Orb}(x,T)} \). In particular the operator induced by \( T \) on any invariant subspace \( M \) of \( H \) is hypercyclic, i.e. \( M = \overline{\text{Orb}(x,T)} \) for some \( x \in H \).

The term “orbit-unicellularity” comes from the fact that an operator is said to be unicellular if the lattice of its invariant subspaces is totally ordered. When an operator \( T \) is unicellular, every invariant subspace \( M \) of \( T \) is cyclic, i.e. is the closure \( M_x \) of the linear orbit of some vector \( x \in H \), and the unicellularity of \( T \) is equivalent to the fact that for every pair \( (x,y) \) of vectors of \( H \), either \( M_x \subseteq M_y \) or \( M_y \subseteq M_x \). See for instance 15 for some examples of unicellular operators. In our case \( \overline{\text{Orb}(x,T)} = M_x \), so \( T \) is in particular unicellular. Let us underline here that the point of Theorem 1.3 is that we are dealing with hypercyclic vectors, and not with cyclic ones: of course there are many operators whose lattice of invariant subspaces it totally ordered. This is the case for the Volterra operator \( V \) on \( L^2([0,1]) \) for instance: each invariant subspace for \( V \) is of the form \( M_t = \{ f \in L^2([0,1]) : f = 0 \ \text{a.e. on } (0,t) \}, \ t \in [0,1] \). In this case the lattice of invariant subspaces is isomorphic to \( \mathbb{R} \) with its natural order. It is even possible that the lattice
of invariant subspaces be countable: this is the case for instance for some weighted unilateral backward shifts on \( \ell_2(\mathbb{N}) \), the Donoghue operators. Here the non trivial invariant subspaces are exactly the finite dimensional spaces \( M_n = \text{sp}[e_0, \ldots, e_n] \), \( n \geq 0 \), where \( (e_n)_{n \geq 0} \) is the canonical basis of \( \ell_2(\mathbb{N}) \). It is worth noting that such a situation cannot occur for an operator whose closure of orbits are subspaces and which is orbit-unicellular. Indeed suppose that \( M \) and \( N \) are two invariant subspaces for \( T \) with \( N \subsetneq M \). As was mentioned in Theorem 1.3 there exist two vectors \( x \) and \( y \) such that \( M = \overline{\text{Orb}}(x, T) \) and \( N = \overline{\text{Orb}}(y, T) \). It is easy to see that the operator induced by \( T \) on the quotient \( M/N \) is hypercyclic, which implies that \( M/N \) is infinite-dimensional. Hence the “gap” between two invariant subspaces of \( T \), if non trivial, is of infinite dimension. This leads to the following observation:

**Proposition 1.4.** — The following dichotomy holds true:

(a) either there exists a bounded operator \( T \) on an infinite-dimensional separable Hilbert space which has no non trivial invariant closed subset;

(b) or every operator acting on an infinite-dimensional separable Hilbert space, whose closure of orbits are subspaces and which is orbit-unicellular, has the following property: there exists a family of (closures of) orbits which is order isomorphic to \( (\mathbb{R}, \leq) \).

In particular such an operator has uncountably many distinct (closures of) orbits.

In Section 6 we give a positive answer to a question of [8] which concerns orbit-reflexive operators on Hilbert spaces. If \( T \in \mathcal{B}(X) \) is a bounded operator on \( X \), \( T \) is said to be orbit-reflexive if whenever \( A \in \mathcal{B}(X) \) is such that \( Ax \) belongs to the closure of \( \text{Orb}(x, T) \) for every \( x \in X \), then \( A \) must belong to the closure of the set \( \{ T^n : n \geq 0 \} \) for the Strong Operator Topology (SOT). In particular, \( A \) and \( T \) must commute. Various conditions are given in [8] under which an operator on a Hilbert space is orbit-reflexive: for instance any contraction on a Hilbert space is orbit-reflexive. The following question is asked in [8]: does there exist an operator on a Hilbert space which is not orbit-reflexive? This question was pointed out to us by Vladimir Müller [11]. We answer it here in the affirmative:

**Theorem 1.5.** — There exists a bounded operator on a Hilbert space which is not orbit-reflexive.

Theorem 1.5 follows from a slight modification of the construction of Section 4. After this paper was submitted for publication, we were informed by Vladimir Müller that a much more simple example of a non orbit-reflexive Hilbert space operator was constructed independently in [12], as well as an example of an operator on the space \( \ell_1(\mathbb{N}) \) which is reflexive but not orbit-reflexive.

We finish this introduction with a comment: we have mentioned previously that the proofs of Theorems 1.1, 1.2 and 1.5 involve operators of Read’s type, and we use the (c)-part and the (b)-part of it. The reader may justly ask about a possible (a)-part: such an (a)-part indeed appears in Read’s constructions in [16], [17] or [18], and it is actually the part which provides the vectors which belong to the closures of all the sets \( \overline{\text{Orb}}(x, T) \).
2. Making all orbits into subspaces: the role of the \((c)\)-fan

We start from the Hilbert space \(l_2(\mathbb{N})\) of square-summable sequences indexed by the set \(\mathbb{N}\) of nonnegative integers, with its canonical basis \((e_j)_{j \geq 0}\). A vector \(x\) of \(l_2(\mathbb{N})\) is as usual said to be \textit{finitely supported} if all but finitely many of its coordinates on the basis \((e_j)_{j \geq 0}\) vanish, and the set of finitely supported vectors will be denoted by \(c_0\). The forward shift \(T\) on \(l_2(\mathbb{N})\) is the operator defined by \(Te_j = e_{j+1}\) for every \(j \geq 0\).

If \((f_j)_{j \geq 0}\) is a sequence of finitely supported vectors such that \(f_0 = e_0\) and \(\text{sp}[f_0, \ldots, f_j] = \text{sp}[e_0, \ldots, e_j]\) for every \(j \geq 1\) (\(f_j\) belongs to \(\text{sp}[e_0, \ldots, e_j]\) and the \(j\)th coordinate of \(f_j\) on the basis \((e_j)_{j \geq 0}\) is non-zero), then one can define on \(c_0\) a new norm associated to the sequence \((f_j)_{j \geq 0}\). For any finite subset \(J\) of \(\mathbb{N}\) and any collection \((x_j)_{j \in J}\) of scalars,

\[
||\sum_{j \in J} x_j f_j|| = \left(\sum_{j \in J} |x_j|^2\right)^{\frac{1}{2}}.
\]

The completion of \(c_0\) under this new norm is a Hilbert space, with the sequence \((f_j)_{j \geq 0}\) as an orthonormal basis. We are going to show that for a suitable choice of the sequence \((f_j)_{j \geq 0}\), the operator \(T\) acting on \(c_0\) extends to a bounded operator on the Hilbert space \(H := H(\{f_j\})\) which satisfies the properties of Theorem 1.1.

We denote by \(K[\zeta]\) the space of polynomials with coefficients in \(K = \mathbb{R}\) or \(\mathbb{C}\), and by \(K_d[\zeta]\) the space of polynomials of degree at most \(d\). For \(p \in K[\zeta]\), \(p(\zeta) = \sum_{k=0}^{d} a_k \zeta^k\), we write as usual \(|p| = \sum_{k=0}^{d} |a_k|\). Let \((d_n)_{n \geq 1}\) be an increasing sequence of positive integers, and for every \(n \geq 1\) let \((p_{k,n})_{1 \leq k \leq k_n}\) be a finite family of polynomials of degree at most \(d_n\) with \(|p_{k,n}| \leq 2\) for every \(1 \leq k \leq k_n\). In the proofs of the theorems, the polynomials \(p_{k,n}\) will have to satisfy some additional properties, the most usual one being that the family \((p_{k,n})_{1 \leq k \leq k_n}\) forms an \(\varepsilon_n\)-net of the ball of radius 2 of \(K_{d_n}[\zeta]\), but since these families will be chosen differently in the proofs of the four theorems, we present for the time being the general construction.

The construction of the vectors \(f_j\), \(j \geq 0\), is to be done by induction, starting from \(f_0 = e_0\). At step \(n\), vectors \(f_j\) will be constructed for \(j \in [\xi_n + 1, \xi_{n+1}]\), where \((\xi_n)_{n \geq 0}\) is a sequence with \(\xi_0 = 0\) which will be chosen to grow very fast. We emphasize that all the constants we are going to construct at step \(n\) are determined by the various constants which are constructed through steps 0 to \(n - 1\). When we say that a certain constant \(C_{\xi_n}\) depends only on \(\xi_n\), it means that it depends only on the construction from steps 0 to \(n - 1\). The construction is done by induction on \(n\), and in all our statements we assume that the construction has been carried out until step \(n - 1\).

There will be two different types of definitions of \(f_j\) for \(j \in [\xi_n + 1, \xi_{n+1}]\), depending on whether \(j\) belongs or not to a collection of intervals called the \textit{fan} (we will later on call it the \((c)\)-fan, to distinguish it from another fan which is going to be introduced afterwards): this fan is a lattice of intervals which we call \textit{working intervals}, and their role is to ensure that every orbit is a linear manifold. The intervals between working intervals we call \textit{lay-off intervals}: on a lay-off interval, \(f_j\) is defined as \(f_j = \lambda_j e_j\), where \(\lambda_j\) is a scalar coefficient which is very large if \(j\) belongs to the beginning of the lay-off interval, and very small if \(j\) belongs to its end, while the quotient \((\lambda_j / \lambda_{j+1})\) is very close to 1. Thus when both \(j\) and
$j + 1$ belong to a lay-off interval, $Tf_j = \lambda_j e_{j+1} = (\lambda_j/\lambda_{j+1})f_{j+1}$ and $T$ acts as a weighted shift. So in a sense, “nothing much happens on the lay-off intervals”, which explains their name. Their role is to prevent “side effects” from the working intervals, which do the real work.

Here are now the precise definition of the vectors $f_j$, $j = \xi_n + 1, \ldots, \xi_{n+1}$. For any finite sub-interval $A$ of $\mathbb{N}$, we denote by $\pi_A$ the projection of $c_{00}$ onto the span of the vectors $f_j$, $j \in A$. Since we will always require that $\text{sp}[f_0, \ldots, f_j] = \text{sp}[e_0, \ldots, e_j]$, $x$ belongs to $c_{00}$ if and only if it is finitely supported in $H$ with respect to $(f_j)_{j \geq 0}$. When we talk of support in the sequel, we will always mean with respect to $(f_j)_{j \geq 0}$: $x$ is supported in $A$ if $x = \sum_{j \in A} x_j f_j$. The norm $\| \cdot \|$ is the norm of $H$.

2.1. Construction of the fan. — Let $c_{1,n} < c_{2,n} < \cdots < c_{k_n,n}$ be an extremely fast increasing sequence of integers with $c_{1,n}$ very large with respect to $\xi_n$. The fan consists of the lattice of all the intervals

$$I_{r_1,r_2,\ldots,r_k} = [r_1 c_{1,n} + r_2 c_{2,n} + \cdots + r_k c_{k_n,n}]$$

where $r_1, \ldots, r_k$ are nonnegative integers belonging to $[0,h_n]$. Here $h_n$ is a very large integer depending only on $\xi_n$, but not on the $c_{k,n}$’s, which will be chosen later on in the proof. If the gaps between the different $c_{k,n}$’s are large enough, all these $k_nh_n$ intervals are disjoint. For $k \in [1,k_n]$, we call $r_k$ the $k$th coordinate of the interval $I_{r_1,r_2,\ldots,r_k}$, and write $|r| = r_1 + \cdots + r_k$.

Let $t \in [1,k_n]$ be the largest integer such that $r_t \geq 1$. We will write $I_{r_1,r_2,\ldots,r_k} = I_{r_1,r_2,\ldots,r_t}$ when there is no risk of confusion. For $j \in I_{r_1,r_2,\ldots,r_t}$, we define $f_j$ to be

$$f_j = \frac{1}{\gamma_n} 4^{1-|r|}(e_j - p_{t,n}(T)e_{j-c_{1,n}}),$$

where $\gamma_n$ is a very small positive number depending only on $\xi_n$, which will be chosen in the sequel. The interest of this definition is twofold: first of all, we can already justify the name of working interval, simply by using the definition of $f_j$ for $j \in I_{0,0,\ldots,0}$ with $r_k = 1$:

**Fact 2.1.** — Let $\delta_n$ be a small positive number. If $\gamma_n$ is small enough, then for every $x$ supported in $[0,\xi_n]$ and every $1 \leq k \leq k_n$,

$$\|T^{c_{k,n}}x - p_{k,n}(T)x\| \leq \delta_n \|x\|.$$  

**Proof.** — Since $\text{sp}[e_0, \ldots, e_{\xi_n}] = \text{sp}[f_0, \ldots, f_{\xi_n}]$, we can write any vector $x$ with support in $[0,\xi_n]$ as $x = \sum_{j=0}^{\xi_n} \alpha_j^{(n)} e_j$. Then $T^{c_{k,n}}x = \sum_{j=0}^{\xi_n} \alpha_j^{(n)} e_{j+c_{k,n}}$. Now $j + c_{k,n}$ belongs to the working interval $[c_{k,n}, c_{k,n} + \xi_n]$, so $f_{j+c_{k,n}} = \gamma_n^{-1}(e_{j+c_{k,n}} - p_{k,n}(T)e_j)$. Hence

$$T^{c_{k,n}}x = \gamma_n \sum_{j=0}^{\xi_n} \alpha_j^{(n)} f_{j+c_{k,n}} + p_{k,n}(T) \sum_{j=0}^{\xi_n} \alpha_j^{(n)} e_j,$$

that is

$$\|T^{c_{k,n}}x - p_{k,n}(T)x\| = \|\gamma_n \sum_{j=0}^{\xi_n} \alpha_j^{(n)} f_{j+c_{k,n}}\| \leq \gamma_n (\sum_{j=0}^{\xi_n} |\alpha_j^{(n)}|^2)^{1/2}.$$
On the space \( F_{\xi_n} = \text{sp}\{f_0, \ldots, f_{\xi_n}\} \), the two norms \( ||x||_0 = (\sum_{j=0}^{\xi_n} |a_j^{(n)}|^2)^{\frac{1}{2}} \) and \( ||x|| \) are equivalent, so there exists a constant \( C_{\xi_n} \), depending only on \( \xi_n \), such that \( ||x||_0 \leq C_{\xi_n} ||x|| \) for every \( x \) supported in \([0, \xi_n]\). Thus \( ||T^{c_k,n}x - p_{k,n}(T)x|| \leq \gamma_n C_{\xi_n} ||x|| \leq \delta_n ||x|| \) if \( \gamma_n \) is small enough.

Hence if the collection \((p_{k,n})\) is “sufficiently dense” among polynomials with \(|p| \leq 2\), Fact 2.1 gives that the orbit of the vector \( x = e_0 \) (and hence of any finitely supported vector \( x \)) contains in its closure any vector \( p(T)x \) with \(|p| \leq 2\). In order to obtain this result for every vector, not only finitely supported ones, one clearly needs to control the behaviour of the quantities \( ||T^{c_k,n}(x - \pi[0,\xi_n]x)|| \) (and then to dispense with the condition \(|p| \leq 2\), but this is not difficult). More precisely, we will need the following proposition, which we shall prove in Section 3:

**Proposition 2.2.** — For every \( n \geq 1 \), every \( 1 \leq k \leq k_n \) and every \( x \in H \) such that \( \pi[0,\xi_n]x = 0 \), \( ||T^{c_k,n}x|| \leq 100 ||x|| \). In other words,

\[
||T^{c_k,n}(x - \pi[0,\xi_n]x)|| \leq 100 ||x - \pi[0,\xi_n]x||
\]

for every \( x \in H \).

Only the intervals \( I_0, \ldots, 0.1 \) are needed for the proof of Fact 2.1 but for the estimates of Proposition 2.2 one needs the whole lattice, and this is why all the other intervals, which could be called “shades” of the basic intervals \( I_0, \ldots, 0.1 \), appear in the definition of the fan.

We finish this section by showing how \( e_j \) can be computed for \( j \) in a working interval by going down the lattice along each successive coordinate:

**Lemma 2.3.** — For every \( \alpha \in [0,\xi_n] \) and every \( k_n \)-tuple \((r_1, \ldots, r_{k_n})\) of integers in \([0, h_n]\),

\[
e_{r_1c_{1,n} + \cdots + r_{l-1}c_{1,n} + \alpha} = \left( \sum_{l=1}^{t} \sum_{s_l=0}^{r_l} \gamma_n 4^{r_1+\cdots+r_{l-1}+(r_t-s_t)l-p_{l,n}(T)s_l+p_{l+1,n}(T)r_{l+1}+\cdots+p_{t,n}(T)r_t} \right) f_{r_1c_{1,n} + \cdots + r_{l-1}c_{1,n}+(r_t-s_t)c_{1,n}+\alpha} + p_{r_1,n}(T)r_1 \cdots p_{r_t,n}(T)r_t e_{\alpha},
\]

where \( t \) is the largest index such that \( r_t \geq 1 \).

**Proof.** — We have

\[
e_{r_1c_{1,n} + \cdots + r_{l-1}c_{1,n} + \alpha} = \gamma_n 4^{r_1} f_{r_1c_{1,n} + \cdots + r_{l-1}c_{1,n} + \alpha} + p_{r_1,n}(T)e_{r_1c_{1,n} + \cdots + (r_t-1)c_{1,n} + \alpha} = \cdots
\]

\[
= \gamma_n 4^{r_1} \sum_{s_l=0}^{r_1} 4^{-s_l} p_{l,n}(T)^{s_l} f_{r_1c_{1,n} + \cdots + (r_t-s_t)c_{1,n} + \alpha} + p_{r_t,n}(T)r_t e_{r_1c_{1,n} + \cdots + r_{l-1}c_{1,n} + \alpha}.
\]

Then we go down in the same way along the \((t-1)\)-coordinate, etc... until there are no more coordinates left.

We will always choose the maximal degree \( d_n \) of the polynomials \( p_{k,n} \) to be small with respect to \( c_{1,n} \): for the proof of Theorem 1.1 we will choose simply \( d_n = n \).
2.2. Construction of $f_j$ for $j$ in a lay-off interval. — The lay-off intervals are the intervals which lie between the working intervals. If we write such an interval as $[r+1, r+s]$ $f_j$ is defined for $j$ in it as $f_j = \lambda_j e_j$, where

$$\lambda_j = 2^{(\frac{1}{4}s+r+1-j)/\sqrt{s}}.$$  

When the length $s$ of such a lay-off interval becomes very large, the coefficients $\lambda_j$ behave in the following way: if $j$ lies in the beginning of the interval, $\lambda_j$ is roughly equal to $2^{\frac{1}{2}\sqrt{s}}$ (very large), and when $j$ is near the end of the lay-off interval, $\lambda_j$ is roughly $2^{-\frac{1}{2}\sqrt{s}}$ (very small). This implies in particular that when $j$ is in the beginning of a lay-off interval, $|e_j|$ is very small, approximately less than $2^{-\frac{1}{2}\sqrt{s}}$. Moreover if $s$ is large, the ratio $\lambda_j/\lambda_{j+1}$ for $j$ and $j+1$ is the lay-off interval becomes very close to 1. Remark that this ratio does not depend on $j$.

Hence the picture at step $n$ is the following: there is first one very large lay-off interval, between $\xi_n + 1$ and $c_{1,n} - 1$, then an alternance of working and lay-off intervals, and at the end a very large lay-off interval between $\lambda_n(c_{1,n} + \ldots, c_{k,n}) + \xi_n + 1$ and $\xi_{n+1}$. Then the length of all the lay-off intervals between working intervals is always comparable to some $c_{k,n}$, the length of the first lay-off interval $[\xi_n + 1, c_{1,n} - 1]$ is comparable to $c_{1,n}$, and the length of the last one is comparable to $\xi_{n+1}$. Since it would make the computations too involved if we were to write each time the precise estimates for $\lambda_j$ or $|e_j|$, we will often write only an approximate estimate which will give the order of magnitude of the quantities involved. When doing this, we will use the symbol $\lesssim$ instead of $\leq$, or $\gtrsim$ instead of $\geq$. For instance for $j$ in the beginning of the lay-off interval $[\xi_n + 1, c_{1,n} - 1]$, let us say $j \in [\xi_n + 1, 2\xi_n + 1]$, we will not write

$$||e_j|| \leq 2^{-\frac{1}{2}\sqrt{\xi_n+1}}(\xi_n+1-j) \leq 2^{-\frac{1}{2}\sqrt{\xi_n+1}}(\xi_n+1-j),$$

but simply $||e_j|| \lesssim 2^{-\frac{1}{2}\sqrt{\xi_n+1}}$, and since $\xi_n$ and $\lambda_n$ are both small with respect to $c_{1,n}$, the estimate $2^{-\frac{1}{2}\sqrt{\xi_n+1}}$ gives the right order of magnitude for $|e_j|$.

2.3. Boundedness of the operator $T$. — In order to show that $T$ is bounded on $H$, we need the following estimates:

**Proposition 2.4.** — Let $(\delta_n)_{n \geq 0}$ be a decreasing sequence of positive numbers going to zero very fast. The vectors $f_j$ can be constructed so that for every $n \geq 0$, assertion (1) below holds true:

(1) if $x$ is supported in the interval $[\xi_n + 1, \xi_{n+1}]$, then

(1a) $||\pi_{[\xi_n+1, \xi_{n+1}]}(Tx)|| \leq (1 + \delta_n)||x||$

(1b) $||\pi_{[0, \xi_n]}(Tx)|| \leq \delta_n||x||$.

Remark that since $[\frac{1}{2}\xi_n + 1, 2\xi_{n+1}]$ can be supposed to be contained in a lay-off interval, it makes sense to write $\pi_{[\xi_n+1, \xi_{n+1}]}(Tx)$, even when $x$ has a non-zero coordinate on $\xi_{n+1}$. If $x = f_{\xi_{n+1}}$ for instance, we know, even if $\lambda_{\xi_{n+1}}$ has not been defined yet, that $Tf_{\xi_{n+1}}$ is a multiple of $f_{\xi_{n+1}+1}$, and thus the projection of $Tx$ on $[\xi_n + 1, \xi_{n+1}]$ is zero.

**Proof.** — Write the vector $x$ as $x = \sum_{j=\xi_{n+1}}^{\xi_{n+1}} x_j f_j$, and its image as $Tx = \sum_{j=\xi_{n+1}}^{\xi_{n+1}} x_j T f_j$. There are four kind of indices $j$ in this sum, with a different expression for $T f_j$ each time.
• Let \( J_1 \) be the set of integers \( j \in [\xi_n + 1, \xi_{n+1}] \) such that \( j \) and \( j + 1 \) belong to a lay-off interval: \( f_j = \lambda_j \psi_j \) and \( f_{j+1} = \lambda_{j+1} \psi_{j+1} \), so that \( T f_j = (\lambda_j/\lambda_{j+1}) f_{j+1} \). If the length of the lay-off interval is very large, \( \lambda_j/\lambda_{j+1} \leq 1 + \delta_n/2 \) for every \( j \in J_1 \), and \( T f_j = \mu_j f_{j+1} \) with \( |\mu_j| \leq 1 + \delta_n/2 \).

• Let \( J_2 \) be the set of integers \( j \in [\xi_n + 1, \xi_{n+1}] \) such that \( j \) and \( j + 1 \) belong to a working interval: then simply \( T f_j = f_{j+1} \).

• Let \( J_3 \) be the set of integers \( j \in [\xi_n + 1, \xi_{n+1}] \) of the form \( j = r_1 c_{1,n} + \cdots + r_t c_{t,n} + \xi_n \): \( j \) is the endpoint of a working interval and \( j + 1 \) is the first point of the next lay-off interval. Then

\[
T f_j = \gamma_n^{-1} 4^{1-|r|} \left( e_{j+1} - p_{t,n}(T) e_{j-c_{1,n}+1} \right).
\]

We have \( \|e_{j+1}\| \leq 2^{-\frac{1}{2} \sqrt{\xi_n}} \). Moreover if we write the polynomial \( p_{t,n} \) as \( p_{t,n}(\xi) = \sum_{u=0}^{d_n} a_u \xi^u \), then \( p_{t,n}(T) e_{j-c_{1,n}+1} = \sum_{u=0}^{d_n} a_u e_{j-c_{1,n}+1+u} \). Now since \( d_n \) is very small with respect to each \( c_{k,n} \), \( j - c_{t,n} + 1 + u \) lies in the beginning of a lay-off interval, and thus \( \|e_{j-c_{1,n}+1+u}\| \leq 2^{-\frac{1}{2} \sqrt{\xi_n}} \), so

\[
\left\| p_{t,n}(T) e_{j-c_{1,n}+1} \right\| \leq 2 \sup_{0 \leq u \leq d_n} \|e_{j-c_{1,n}+1+u}\| \leq 2^{-\frac{1}{2} \sqrt{\xi_n}}.
\]

Hence \( \|T f_j\| \leq \gamma_n^{-1} 2^{-\frac{1}{2} \sqrt{\xi_n}} \) and since \( \gamma_n \) depends only on \( \xi_n \), \( \|T f_j\| \) can be made arbitrarily small for an appropriate choice of \( c_{1,n} \).

• Let \( J_4 \) be the set of integers \( j \in [\xi_n + 1, \xi_{n+1}] \) of the form \( j = r_1 c_{1,n} + \cdots + r_t c_{t,n} - 1 \): \( j \) is the endpoint of a lay-off interval, and \( j + 1 \) is the first point of the next working interval. Then \( T f_j = \lambda_j e_{j+1} \). Using Lemma 2.3 we get that

\[
e_{r_1 c_{1,n} + \cdots + r_t c_{t,n}} = \left( \sum_{l=1}^{t} \sum_{s_l=0}^{r_l} \gamma_n^{4^{r_1+\cdots+r_{l-1}+(r_l-s_l)-1}} p_{l,n}(T)^{s_l} p_{l+1,n}(T)^{(r_l+1)} \cdots p_{t,n}(T)^{r_t} f_{r_1 c_{1,n} + \cdots + r_{l-1} c_{l-1,n} + (r_l-s_l) c_{l,n}} \right) + p_{1,n}(T)^{r_1} \cdots p_{t,n}(T)^{r_t} e_0.
\]

The polynomial \( p_{s_1,\ldots,s_t} = p_{1,n}^{s_1} \cdots p_{t,n}^{s_t} \) has degree at most \( h_n k_n d_n \), and \( |p| \leq 2^{s_1+\cdots+s_t} \). Write \( p_{s_1,\ldots,s_t}(\xi) = \sum_{a_u=0}^{h_n k_n d_n} a_u (s_1,\ldots,s_t) \xi^a \). Then

\[
p_{0,\ldots,0,s_t,r_t+1,\ldots,r_l}(T) f_{r_1 c_{1,n} + \cdots + r_{l-1} c_{l-1,n} + (r_l-s_l) c_{l,n}} = \sum_{u=0}^{h_n k_n d_n} a_u (0,\ldots,0,s_t,r_t+1,\ldots,r_l) T^u f_{r_1 c_{1,n} + \cdots + r_{l-1} c_{l-1,n} + (r_l-s_l) c_{l,n}},
\]

If \( u \leq \xi_n \),

\[
T^u f_{r_1 c_{1,n} + \cdots + r_{l-1} c_{l-1,n} + (r_l-s_l) c_{l,n}} = f_{r_1 c_{1,n} + \cdots + r_{l-1} c_{l-1,n} + (r_l-s_l) c_{l,n} + u}
\]

and if \( \xi_n + 1 \leq u \leq h_n k_n d_n \), then

\[
T^u f_{r_1 c_{1,n} + \cdots + r_{l-1} c_{l-1,n} + (r_l-s_l) c_{l,n}} = T^u f_{r_1 c_{1,n} + \cdots + r_{l-1} c_{l-1,n} + (r_l-s_l) c_{l,n} + \xi_n}
\]
where $1 \leq \alpha \leq h_n k_n d_n$. So if $r' = r_1 c_{1,n} + \cdots + r_{t-1} c_{t-1,n} + (r_1 - s_1) c_{t,n}$, then in the case where $r' \neq 0$,

$$
T^n f_{r_1 c_{1,n} + \cdots + r_{t-1} c_{t-1,n} + (r_1 - s_1) c_{t,n}} = \gamma_n^{-1} 4^{1-|r'|} e_{r_1 c_{1,n} + \cdots + r_{t-1} c_{t-1,n} + (r_1 - s_1) c_{t,n} + \alpha} - p_{e,n}(T) e_{r_1 c_{1,n} + \cdots + r_{t-1} c_{t-1,n} + (r_1 - s_1) c_{t,n} - e_{n} + \alpha}
$$

where $v$ is the largest non-zero coordinate in $r'$. Using exactly the same argument as in the case $j \in J_3$ above, we see that in the case where $\xi_n + 1 \leq u \leq h_n k_n d_n$, then $||T^n f_{r_1 c_{1,n} + \cdots + r_{t-1} c_{t-1,n} + (r_1 - s_1) c_{t,n}}||$ can be made arbitrarily small. When $r' = 0$, $||T^n e_0|| = ||e_u||$, and with $\xi_n + 1 \leq u \leq h_n k_n d_n$, $||e_u||$ can be made arbitrarily small again. Hence

$$
\left| \sum_{l=1}^{t} \sum_{s_l=0}^{t} \gamma_n 4^{r_1 + \cdots + r_{t-1} + (r_1 - s_1) - 1} p_{0,...,0,s_1,...,r_{t-1},r_t}(T) f_{r_1 c_{1,n} + \cdots + r_{t-1} c_{t-1,n} + (r_1 - s_1) c_{t,n}} \right| \\
\lesssim \sum_{l=1}^{t} \sum_{s_l=0}^{t} \gamma_n 4^{r_1 + \cdots + r_{t-1} + (r_1 - s_1) - 1} 2^{s_1 + r_{t-1} + \cdots + r_t} \leq \gamma_n 4^{r_1 - 1} 2k_n.
$$

For the remaining term $||p_{r_1,\ldots,r_t}(T)e_0||$ we proceed as above:

$$||p_{r_1,\ldots,r_t}(T)e_0|| \leq \left( \sum_{u=0}^{h_n k_n d_n} a_{u}^{(r_1,\ldots,r_t)} ||e_u|| \right) \sup_{u \leq h_n k_n d_n} ||e_u|| \leq 2h_n k_n \sup_{u \leq h_n k_n d_n} ||e_u||.
$$

Since $\lambda_j \lesssim 2^{-\frac{1}{2}}$ and neither $h_n$ nor $k_n$ nor $d_n$ depend on $c_{1,n}$, we obtain that $||p_{r_1,\ldots,r_t}(T)e_0||$ can be made arbitrarily small, and hence the same is true for $||Tf_j||$.

- Putting the previous estimates together, we obtain that

$$
||T\left( \sum_{j \in J_1 \cup J_2} x_j f_j \right)||^2 \leq \left( 1 + \frac{\delta_n}{2} \right)^2 \sum_{j \in J_1} |x_j|^2 + \sum_{j \in J_2} |x_j|^2 \leq \left( 1 + \frac{\delta_n}{2} \right)^2 ||x||^2
$$

and

$$
||T\left( \sum_{j \in J_1 \cup J_2} x_j f_j \right)|| \leq \left( \sum_{j \in J_1 \cup J_2} ||T f_j||^2 \right)^{\frac{1}{2}} \left( \sum_{j \in J_1 \cup J_2} |x_j|^2 \right)^{\frac{1}{2}} \leq \delta_n ||x||,
$$

so that $||T x|| \leq (1 + \delta_n)||x||$, and this proves that $||\pi_{[\xi_n+1,\xi_n+1]}(T x)|| \leq (1 + \delta_n)||x||$ which proves (1a). Since $\pi_{[0,\xi_n]} \left( \sum_{j \in J_1 \cup J_2} x_j f_j \right) = 0$, this proves (1b) too.

The boundedness of $T$ follows now easily from Proposition 2.4.

**Proposition 2.5.** — Let $\varepsilon$ be any positive number. If the sequence $(\delta_n)$ corresponding to the construction of Proposition 2.4 goes fast enough to zero, $T$ extends to a bounded operator on $H$ satisfying $||T|| \leq 1 + \varepsilon$. 

Proof. — The proof is by induction on $n$, supposing that $||Tx|| \leq C_n ||x||$ for every $x$ supported in $[0, \xi_n]$. Suppose that $x$ is supported in $[0, \xi_{n+1}]$, and write $Tx$ (which is supported in $[0, \xi_{n+2}]$) as

$$Tx = T(\pi_{[0,\xi_n]}(x)) + T(\pi_{[\xi_n,\xi_{n+1}]}(x)) = \pi_{[0,\xi_n]}(T(\pi_{[0,\xi_n]}(x))) + \pi_{[\xi_n,\xi_{n+1}]}(T(\pi_{[\xi_n,\xi_{n+1}]}(x))).$$

Hence

$$||Tx||^2 = ||\pi_{[0,\xi_n]}(T(\pi_{[0,\xi_n]}(x))) + \pi_{[\xi_n,\xi_{n+1}]}(T(\pi_{[\xi_n,\xi_{n+1}]}(x)))||^2$$

The terms in this expression which remain to be estimated are $||\pi_{[\xi_n,\xi_{n+1}]}(T(\pi_{[0,\xi_n]}(x)))||$ and $||\pi_{[\xi_n,\xi_{n+1}]}(T(\pi_{[\xi_n,\xi_{n+1}]}(x)))||$. The first one is equal to $|x_{\xi_n+1}|^2 ||T\xi_{n+1}||^2 = |x_{\xi_n+1}|^2 (\lambda_{\xi_{n+1}}/\lambda_{\xi_{n+1}+1})^2$, and we can choose $\lambda_{\xi_{n+1}}$ so large that $\lambda_{\xi_{n+1}}/\lambda_{\xi_{n+1}+1} \leq \delta_n$ for instance. We do the same for the last term, and then

$$||Tx||^2 \leq (C_n||\pi_{[0,\xi_n]}(x)|| + \delta_n||\pi_{[\xi_n,\xi_{n+1}]}(x)||)^2$$

which yields that

$$||Tx||^2 \leq ((\max(C_n^2 + \delta_n^2 - 1, 1 + \delta_n^2) + \delta_n^2) ||\pi_{[\xi_n,\xi_{n+1}]}(x)||)^2$$

and the proof of Proposition 2.5 follows by induction.

We finish this section with the following stronger form of Proposition 2.4.

Proposition 2.6. — Given a sequence of positive numbers $(\varepsilon_n)_{n \geq 1}$ which decreases very quickly to zero, the construction of the fans at each step can be conducted in such a way that

(1') if $x$ is supported in the interval $[\xi_n + 1, \xi_{n+1}]$, then for every $m < \xi_n/2$

(1a') $||\pi_{[\xi_n + 1, \xi_{n+1}]}(T^m x)|| \leq (1 + \varepsilon_n)||x||$

(1b') $||\pi_{[0,\xi_n]}(T^m x)|| \leq \varepsilon_n||x||$

(1c') $||\pi_{[\xi_n + 1, \xi_{n+1}]}(T^m x)|| \leq (1 + \varepsilon_n)||x||$.

Proof. — As in the proof of Proposition 2.4, we are going to show that if the construction has been carried out until step $n - 1$, the $c_{j,n}$'s at step $n$ can be chosen so large that (1a') and (1b') hold true at step $n$, as well as (1c') at step $n - 1$. We denote again $F_{\xi_n} = \text{sp}(e_0, \ldots, e_{\xi_n})$. As soon as $c_{1,n}$ is much larger than $\xi_n$, the projection on $[\xi_n + 1, \xi_{n+1}]$ of $T^m(F_{\xi_n})$, $m < \xi_n/2$, consists of vectors supported in the beginning of the lay-off interval $[\xi_n + 1, c_{1,n} - 1]$. This implies that $||\pi_{[\xi_n + 1, \xi_{n+2}]}(T^m x)|| \leq C_{\xi_n} 2^{-\frac{1}{2} \sqrt{\xi_n}} ||x||$ where $C_{\xi_n}$ depends only on the steps $0$ to $n - 1$ while $c_{1,n}$ is very large with respect to $C_{\xi_n}$: this shows that condition (1c') at step $n - 1$ is satisfied.
Denote by $T_{\xi_n}$ the truncated shift on $F_{\xi_n}$ with respect to the vectors $e_j$: $T_{\xi_n}e_j = e_{j+1}$ for $j < \xi_n$ and $T_{\xi_n}e_{\xi_n} = 0$. The proof of Proposition 2.5 shows that one can ensure that $\|T_{\xi_n}\| \leq 2 - \frac{1}{n}$ for instance. The fact that conditions (1a') and (1b') can be fulfilled follows from the statement (P_m) below which we prove by induction:

(P_m): there exists a constant $C_{m,n}$ depending only on the construction until step $n - 1$ such that if properties (1a) and (1b) of Proposition 2.4 at step $n$ are satisfied for some $\delta_n > 0$, then for every $x$ supported in $[\xi_n + 1, \xi_{n+1}]$, $\|\pi_{[\xi_n+1,\xi_{n+1}]}(T^m x)\| \leq (1 + C_{n,m}\delta_n)\|x\|$ and $\|\pi_{[0,\xi_n]}(T^m x)\| \leq C_{m,n}\delta_n\|x\|$.

Once (P_m) is proven, it suffices to choose $\delta_n = \varepsilon_n / \max_{m < \xi_n / 2}(C_{n,m})$. The base of the inductive proof of (P_n) is Proposition 2.4 itself. Assume now that (P_{m-1}) holds true. Write $\pi_{[\xi_n+1,\xi_{n+1}]}(T^m x) = \pi_{[\xi_n+1,\xi_{n+1}]}(T(T^{m-1} x))$. If $y = T^{m-1} x$, $y$ is supported in $[0, \xi_{n+1} - m + 1]$ and we have

$$\pi_{[\xi_n+1,\xi_{n+1}]}(T y) = \pi_{[\xi_n+1,\xi_{n+1}]}(T(\pi_{[\xi_n,\xi_{n+1}]}(y))) + \pi_{[\xi_n+1,\xi_{n+1}]}(T(\pi_{[\xi_{n+1},\xi_{n+1}]}(y)))$$

and

$$\pi_{[0,\xi_n]}(T y) = \pi_{[0,\xi_n]}(T(\pi_{[\xi_n,\xi_{n+1}]}(y))) + \pi_{[0,\xi_n]}(T(\pi_{[\xi_{n+1},\xi_{n+1}]}(y)))$$

Since the vector $\pi_{[\xi_n+1,\xi_{n+1}]}(T y)$ is supported on the first lay-off interval of $[\xi_{n+1} + 1, \xi_{n+2}]$, the operator $T$ acts on it as a weighted shift operator and the projection $\pi_{[0,\xi_n]}(T(\pi_{[\xi_n+1,\xi_{n+1}]}(y)))$, as well as the last term in each of the two displays above is zero. For the other two terms we have (assuming that $\delta_n < 1$)

$$\|\pi_{[\xi_n+1,\xi_{n+1}]}(T y)\| \leq \|T(\pi_{[\xi_n,\xi_{n+1}]}(y))\| + (1 + \delta_n)\|\pi_{[\xi_n+1,\xi_{n+1}]}(y)\| \leq 2C_{m-1,n}\delta_n\|x\| + (1 + \delta_n)\|\pi_{[\xi_n+1,\xi_{n+1}]}(y)\| \leq (1 + (3C_{m-1,n} + 2)\delta_n)\|x\|$$

and

$$\|\pi_{[0,\xi_n]}(T y)\| \leq \|T(\pi_{[\xi_n,\xi_{n+1}]}(y))\| + \delta_n\|\pi_{[\xi_n+1,\xi_{n+1}]}(y)\| \leq 2\|\pi_{[0,\xi_n]}(y)\| + \delta_n(1 + 3C_{m-1,n}\delta_n)\|x\| \leq (3C_{m-1,n} + 1)\delta_n\|x\|,$$

which completes the induction and thus the proof of Proposition 2.6.

3. Estimating $T^{x_{n,k}}$: proof of Theorem 1.1

As was already mentioned before, the crucial step for the proof of Theorem 1.1 is Proposition 2.2. The estimates needed for this are given in Proposition 3.1. 

**Proposition 3.1.** — Let $(\delta_n)_{n \geq 0}$ be a decreasing sequence of positive numbers going to zero very fast. The vectors $f_j$ can be constructed so that for every $n \geq 0$, assertion (2) below holds true:

(2) for any vector $x$ supported in the interval $[\xi_n + 1, \xi_{n+1}]$ and for any $1 \leq k \leq k_n$,

(a) $\|\pi_{[\xi_n+1,\xi_{n+1}]}(T^{x_{k,n}} x)\| \leq 4\|x\|$

(b) $\|\pi_{[0,\xi_n]}(T^{x_{k,n}} x)\| \leq \delta_n\|x\|$
(3) for any $x$ supported in the interval $[0, \xi_n]$ and any $m < \xi_n/2$,
$$ ||\pi_{[\xi_n+1, \xi_{n+1}]}(T^m x)|| \leq \delta_n ||x||. $$

Proof. — • The easy part of the proof is assertion (3): if $x$ is supported in $[0, \xi_n]$ and $m < \xi_n/2$, $T^m x$ is supported in the interval $[0, (3/2) \xi_n]$. Thus if $\xi_{n+1} > (3/2) \xi_n$ then $\xi_{n+1} + 1 > 3 \xi_n$ and the condition on $\xi_{n+1}$ is vacuous. The other case is treated in the following way.

• Fix $1 \leq k \leq \xi_n$. Let us first look at $T^{c,k,n} f_j$ for $j$ in a lay-off interval. Since it would be rather intricate to write down all the possible cases, we give an example of each one of the situations which can occur:

- If $j \in k_c + \xi_n + 1, 2k_c + 1]$, then $T^{c,k,n} f_j = \lambda_j e_{j+c,k}$, and $j+c,k \in [2k_c + \xi_n + 1, 3k_c + 1]$. Thus $e_{j+c,k}$ = $(\lambda_{j+c,k}) f_{j+c,k}$ and since $\lambda_j = \lambda_{j+c,k}$, $T^{c,k,n} f_j = f_{j+c,k}$.

- If $j \in [h_n c_k + \xi_n + 1, c_1, n + h_n c_k - 1]$, then $T^{c,k,n} f_j = (\lambda_j / \lambda_{j+c,k}) f_{j+c,k}$, and $j + c,k \in [c_1, n + h_n c_k + (c_1, n + h_n c_k - 1)]$ which is contained in the beginning of the lay-off interval $[c_1, n + \cdots + h_n c_k + \xi_n + 1, c_1, n + h_n c_k - 1]$ whose length is approximately less than $2\sqrt{c_1, n}$. Hence $|\lambda_j| \approx 2^{-3/2}\sqrt{c_1, n}$ and $|\lambda_{j+c,k}| \approx 2^{\sqrt{c_1, n}}$ so the quotient $\lambda_j / \lambda_{j+c,k}$ is extremely small.

All the situations which can occur reproduce one of these two situations, and we leave the reader to work out the details by himself.

• Then let us consider the case where $j$ belongs to a working interval $I_{r_1, \ldots, r_t}$ with $k \leq t$:

- If $r_k < h_n$, then $T^{c,k,n} f_j = 4f_{j+c,k}$ since $j + c,k$ belongs to $I_{r_1, \ldots, r_{k+1}, \ldots, r_t}$.

- If $r_k = h_n$ and $k < t$, then $j + c,k \in I_{r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + (r_{k+1} + 1) c_{k+1, n} + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + (r_{k+1} + 1) c_{k+1, n} + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + (r_{k+1} + 1) c_{k+1, n} + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + (r_{k+1} + 1) c_{k+1, n} + \xi_n + 1]$. The expression of the lay-off interval $[r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + (r_{k+1} + 1) c_{k+1, n} + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + (r_{k+1} + 1) c_{k+1, n} + \xi_n + 1]$ which is contained in the beginning of the lay-off interval $[r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + (r_{k+1} + 1) c_{k+1, n} + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + h_n c_k + \cdots + r_t c_t, r_1, c_1, n + \cdots + \xi_n + 1]$ if $r_{k+1} < h_n$ (else we have to move over to the first $s$ with $r_s < h_n$ if there is one, or else in the last lay-off interval. We leave this to the reader). So $||e_{j+c,k}|| \leq 2^{-3/2}\sqrt{r_{k+1}, n}$, and since $ ||p_{h_n}(T)|| \leq 2 ||T||^d_n \leq 2^d_n$, for instance, and we get that $||T^{c,k,n} f_j||$ can be made arbitrarily small.

- It is in the case where $r_k = h_n$ and $k = t$ that the condition on $h_n$ appears, and this case has to be worked out carefully: $T^{c,k,n} f_j = \gamma_n^{-1/4} \gamma_t e_{j+c,k} - p_{h_n}(T)e_j$. As before $j + c,k \in [r_1, c_1, n + \cdots + (h_n + 1) c_{k+1, n} + r_1 c_1, n + \cdots + (h_n + 1) c_{k+1, n} + \xi_n + 1]$, the condition on $h_n$ is vacuous. The other case is treated in the following way.

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have to estimate the quantity \( \| \gamma_n^{-1}4^{1-|r|} \sum_{j \in I_{r_1, \ldots, r_n}} x_j p_k, n(T)e_j \|. \) By Lemma 2.3

\[
p_{k,n}(T)e_j = p_{k,n}(T)(e_j)_1 + p_{k,n}(T)(e_j)_2 = \left( \sum_{l=1}^{k} \sum_{s_l=0}^{r_l} \gamma_n 4^{r_1+\cdots+r_{l-1}+(r_l-s_l)-1}\right)
\]

\[
+ p_{1,n}(T)^{r_1} \cdots p_{k,n}(T)^{r_k+1}c_{\alpha}
\]

where \( j = r_1e_{k,1} + \cdots + h_n e_{k,n} + \alpha \) with \( \alpha \in [0, \xi_n] \). The polynomial

\[
p_{l,n}(T)^{s_l}p_{l+1,n}(T)^{r_{l+1}} \cdots p_{k,n}(T)^{r_k+1}
\]

is of degree at most \((h_n + 1)k_d n\) and its modulus is less than \(2^{s_l + r_{l+1} + \cdots + h_n + 1}\). When expanding the expression

\[
p_{l,n}(T)^{s_l}p_{l+1,n}(T)^{r_{l+1}} \cdots p_{k,n}(T)^{r_k+1}f_{r_1e_{k,1} + \cdots + r_{l-1}e_{k,1,n} + (r_l-s_l)c_{k,n} + \alpha}
\]

two kinds of terms appear:

- multiples of \( f_{r_1e_{k,1} + \cdots + r_{l-1}e_{k,1,n} + (r_l-s_l)c_{k,n} + \alpha + u} \) for \( u \leq \xi_n \): this corresponds to “small values” of \( u \), for which

\[
T^u f_{r_1e_{k,1} + \cdots + r_{l-1}e_{k,1,n} + (r_l-s_l)c_{k,n} + \alpha} = f_{r_1e_{k,1} + \cdots + r_{l-1}e_{k,1,n} + (r_l-s_l)c_{k,n} + \alpha + u}.
\]

- multiples of \( T^u f_{r_1e_{k,1} + \cdots + r_{l-1}e_{k,1,n} + (r_l-s_l)c_{k,n} + \alpha} \) for \( \xi_n + 1 \leq u \leq \xi_n + (h_n + 1)k_d n \) corresponding to “large values” of \( u \).

For the first terms the norm can be directly computed, and for the second terms it suffices to notice that the expression of \( T^u f_{r_1e_{k,1} + \cdots + r_{l-1}e_{k,1,n} + (r_l-s_l)c_{k,n} + \alpha} \) involves only vectors \( e_i \) for \( i \) in the beginning of two lay-off intervals between \( (c)\)-fans. Since the length of these intervals is roughly larger than \( c_{k,n} \) which is much larger than \( \xi_n, h_n, k_n, d_n \), \( \| T^u f_{r_1e_{k,1} + \cdots + r_{l-1}e_{k,1,n} + (r_l-s_l)c_{k,n} + \alpha} \| \) is very small. This shows that

\[
\left\| \sum_{j \in I_{r_1, \ldots, r_n}} x_j p_k, n(T)(e_j)_1 \right\| \lesssim \left( \sum_{l=1}^{k} \sum_{s_l=0}^{r_l} \gamma_n 4^{r_1+\cdots+r_{l-1}+(r_l-s_l)-1} \right) + 2^{-\frac{1}{2} \sqrt{c_{1,n}}} \left( \sum_{j \in I_{r_1, \ldots, r_n}} |x_j|^2 \right)^{\frac{1}{2}}
\]

so that

\[
\left\| \gamma_n^{-1}4^{1-|r|} \sum_{j \in I_{r_1, \ldots, r_n}} x_j p_k, n(T)(e_j)_1 \right\|
\]

is approximately less than

\[
2 \left( \sum_{s_k=0}^{h_n} 2^{-s_k} + \sum_{s_{k-1}=0}^{h_n} 2^{-s_{k-1}} + \cdots + 2^{-2(r_2+\cdots+h_n)} \right) \left( \sum_{s_1=0}^{h_n} 2^{-s_1} + 2^{-\frac{1}{2} \sqrt{c_{1,n}}} \right) \left( \sum_{j \in I_{r_1, \ldots, r_n}} |x_j|^2 \right)^{\frac{1}{2}}
\]

which is in turn less than

\[
4 \left( 1 + k_n 2^{-h_n} + 2^{-\frac{1}{2} \sqrt{c_{1,n}}} \right) \left( \sum_{j \in I_{r_1, \ldots, r_n}} |x_j|^2 \right)^{\frac{1}{2}} \leq 5 \left( \sum_{j \in I_{r_1, \ldots, r_n}} |x_j|^2 \right)^{\frac{1}{2}}
\]
if \( h_n \) is large enough with respect to \( k_n \). Then we estimate in the same way
\[
||\gamma_n^{-1}4^{1-|r|} \sum_{j \in I_{t_1, \ldots, t_n}} x_j p_{k_n}(T)(e_j)_2||
\]
which is roughly less than
\[
\gamma_n^{-1}4^{1-|r|}2^{|r|} \left( \sup_{u \leq \xi_n + (k_n + 1)k_n} ||e_u|| \right) \left( \sum_{j \in I_{t_1, \ldots, t_n}} |x_j|^2 \right)^{\frac{1}{2}}.
\]
Since \( c_{1,n} \) is very large with respect to \((h_n + 1)k_n d_n, ||e_u|| \lesssim 2^{-\frac{1}{2}}h_n \) for \( \xi_n + 1 \leq u \leq \xi_n + (h_n + 1)k_n d_n \). Recalling that \(|r| = r_1 + \cdots + h_n\), we get that
\[
||\gamma_n^{-1}4^{1-|r|} \sum_{j \in I_{t_1, \ldots, t_n}} x_j (e_j)_2|| \lesssim c_{\xi_n} 2^{-h_n} \left( \sum_{j \in I_{t_1, \ldots, t_n}} |x_j|^2 \right)^{\frac{1}{2}}
\]
where \( C_{\xi_n} \) depends only on \( \xi_n \), and this is very small if \( h_n \) is large enough. Putting together all the estimates above, we get that
\[
||T^{c_{k,n}} \left( \sum_{j \in I_{t_1, \ldots, t_n}} x_j f_j \right)|| \leq 6 \left( \sum_{j \in I_{t_1, \ldots, t_n}} |x_j|^2 \right)^{\frac{1}{2}}.
\]

- It remains to study the case where \( k > t \): for \( j \in I_{t_1, \ldots, t_t} \),
\[
T^{c_{k,n}} f_j = \gamma_n^{-1}4^{1-|r|}(e_j + c_{k,n} - p_{k,n}(T)e_j + c_{k,n} - c_{l,n}).
\]
Since \( j + c_{k,n} \) belongs to \([r_1 c_{l,n} + \cdots + r_t c_{l,n} + c_{k,n}, r_1 c_{l,n} + \cdots + r_t c_{l,n} + c_{k,n} + \xi_n]\), \( f_j + c_{k,n} = \gamma_n^{-1}4^{1-|r|}(e_j + c_{k,n} - p_{k,n}(T)e_j) \), so we have
\[
T^{c_{k,n}} f_j = 4f_j + c_{k,n} + \gamma_n^{-1}4^{1-|r|}p_{k,n}(T)e_j - \gamma_n^{-1}4^{1-|r|}p_{l,n}(T)e_j + c_{k,n} - c_{l,n}.
\]
Now \( f_j = \gamma_n^{-1}4^{1-|r|}(e_j - p_{l,n}(T)e_j - c_{l,n}) \) so that
\[
p_{k,n}(T)f_j = \gamma_n^{-1}4^{1-|r|}(p_{k,n}(T)e_j - p_{k,n}(T)p_{l,n}(T)e_j - c_{l,n}),
\]
and \( f_j + c_{k,n} - c_{l,n} = \gamma_n^{-1}4^{1-|r|}(e_j + c_{k,n} - c_{l,n} - p_{l,n}(T)e_j - c_{l,n}) \) so that
\[
p_{l,n}(T)f_j + c_{k,n} - c_{l,n} = \gamma_n^{-1}4^{1-|r|}(p_{l,n}(T)e_j + c_{k,n} - c_{l,n} - p_{k,n}(T)p_{l,n}(T)e_j - c_{l,n}).
\]
Hence \( T^{c_{k,n}} f_j = 4f_j + c_{k,n} + p_{k,n}(T)f_j - p_{l,n}(T)f_j + c_{k,n} - c_{l,n} \). We then estimate
\[
||T^{c_{k,n}} \left( \sum_{j \in I_{t_1, \ldots, t_t}} x_j f_j \right)||
\]
as previously, writing \( p_{k,n}(\zeta) = \sum_{u=0}^{d_n} a_u^{(k)} \zeta^u \):
\[
||\sum_{u=0}^{d_n} a_u^{(k)} \sum_{j \in I_{t_1, \ldots, t_t}} x_j T^u f_j|| \leq ||\sum_{u=0}^{d_n} a_u^{(k)} \sum_{\alpha=0}^{\xi_n-u} x_{r_1 c_{l,n} + \cdots + r_t c_{l,n} + \alpha} f_{r_1 c_{l,n} + \cdots + r_t c_{l,n} + \alpha + u}||
\]
\[
+ ||\sum_{u=0}^{d_n} a_u^{(k)} \sum_{\alpha=\xi_n-u+1}^{\xi_n} x_{r_1 c_{l,n} + \cdots + r_t c_{l,n} + \alpha} T^u f_{r_1 c_{l,n} + \cdots + r_t c_{l,n} + \alpha}||
\]
and just as before the first term is less than $2\left(\sum_{j \in I_{r_1, \ldots, r_l}} |x_j|^2\right)^{\frac{1}{2}}$ while the second term is less than $\varepsilon_n \left(\sum_{j \in I_{r_1, \ldots, r_l}} |x_j|^2\right)^{\frac{1}{2}}$ with $\varepsilon_n$ arbitrarily small.

It remains to put all the estimates together, and this finishes the proof. \hfill \Box

We are now ready for the proof of Proposition 2.2.

**Proof of Proposition 2.2** — For $x$ such that $\pi_{[0, \xi_n]}(x) = 0$, let us decompose $T^{c_k, n} x$ as

$$T^{c_k, n} x = \pi_{[0, \xi_n]}(T^{c_k, n}(x)) + \sum_{l=n}^{+\infty} \pi_{[\xi_{l+1}, \xi_{l+1}]}(T^{c_k, n} x)$$

$$= \sum_{k=n}^{+\infty} \pi_{[0, \xi_n]}(T^{c_k, n}(\pi_{[\xi_k+1, \xi_{k+1}]}(x))) + \sum_{l=n}^{+\infty} \sum_{k=l-1}^{+\infty} \pi_{[\xi_{l+1}, \xi_{l+1}]}(T^{c_k, n}(\pi_{[\xi_{k+1}, \xi_{k+1}]}(x))) .$$

Indeed if the sequence $(\xi_n)$ grows fast enough, $T^{c_k, n}(\pi_{[\xi_k+1, \xi_{k+1}]}(x))$ for $k \geq n$ is supported by $[0, \xi_{k+2}]$. So

$$||T^{c_k, n} x||^2 \leq \sum_{k=n}^{+\infty} \sum_{l=n}^{+\infty} \pi_{[\xi_{l+1}, \xi_{l+1}]}(T^{c_k, n}(\pi_{[\xi_{k+1}, \xi_{k+1}]}(x))) \leq \sum_{k=n}^{+\infty} \sum_{l=n}^{+\infty} \pi_{[\xi_{l+1}, \xi_{l+1}]}(T^{c_k, n}(\pi_{[\xi_{k+1}, \xi_{k+1}]}(x))) .$$

Using the inequality $||a_1 + \ldots + a_j||^2 \leq \sum_{i=1}^{j} 2^{j+1-i} ||a_i||^2$ valid for every $j$-tuple $(a_1, \ldots, a_j)$ of vectors of $H$, we get

$$||T^{c_k, n} x||^2 \leq \sum_{k=n}^{+\infty} 2^{k+1-n} ||\pi_{[0, \xi_n]}(T^{c_k, n}(\pi_{[\xi_{k+1}, \xi_{k+1}]}(x)))||^2$$

$$+ \sum_{l=n}^{+\infty} \sum_{k=l-1}^{+\infty} 2^{k+2-l} ||\pi_{[\xi_{l+1}, \xi_{l+1}]}(T^{c_k, n}(\pi_{[\xi_{k+1}, \xi_{k+1}]}(x)))||^2 .$$

This yields

$$||T^{c_k, n} x||^2 \leq 2 ||\pi_{[0, \xi_n]}(T^{c_k, n}(\pi_{[\xi_{n+1}, \xi_{n+1}]}(x)))||^2$$

$$+ \sum_{k=n+1}^{+\infty} 2^{k+1-n} ||\pi_{[0, \xi_n]}(T^{c_k, n}(\pi_{[\xi_{k+1}, \xi_{k+1}]}(x)))||^2$$

$$+ \sum_{k=n-1}^{+\infty} ||\sum_{l=n}^{k+1} 2^{k+2-l} ||\pi_{[\xi_{l+1}, \xi_{l+1}]}(T^{c_k, n}(\pi_{[\xi_{k+1}, \xi_{k+1}]}(x)))||^2 .$$

Now if $k \geq n + 1$, $c_{k, n} < \xi_{k}/2$, so that by (1b')

$$||\pi_{[0, \xi_n]}(T^{c_k, n}(\pi_{[\xi_{k+1}, \xi_{k+1}]}(x)))|| \leq ||\pi_{[0, \xi_k]}(T^{c_k, n}(\pi_{[\xi_{k+1}, \xi_{k+1}]}(x)))|| \leq \delta_k ||\pi_{[\xi_{k+1}, \xi_{k+1}]}(x)|| .$$
If \( n \leq l < k \), by (2b)
\[
||\pi[\xi_{l+1},\xi_{k+1}](T^{c_k,n}(\pi[\xi_{l+1},\xi_{k+1}](x)))|| \leq ||\pi[0,\xi_{k}](T^{c_k,n}(\pi[\xi_{l+1},\xi_{k+1}](x)))||
\leq \delta_k||\pi[\xi_{l+1},\xi_{k+1}](x)||.
\]
If \( k = l \geq n \), then by (2a)
\[
||\pi[\xi_{l+1},\xi_{l+1}](T^{c_k,n}(\pi[\xi_{l+1},\xi_{l+1}](x)))|| \leq (1 + \delta_k)||\pi[\xi_{l+1},\xi_{l+1}](x)||,
\]
and lastly for \( l = k + 1, k \geq n + 1 \), by (3)
\[
||\pi[\xi_{l+1},\xi_{l+1}](T^{c_k,n}(\pi[\xi_{l+1},\xi_{l+1}](x)))|| \leq \delta_{k+1}||\pi[\xi_{l+1},\xi_{l+1}](x)||.
\]
Putting everything together yields that
\[
||T^{c_k,n}x||^2 \leq 2\delta_n^2||\pi[\xi_{n+1},\xi_{n+1}](x)||^2 + \sum_{k=n}^{+\infty} \left( \sum_{l=n}^{k+1} 2^{k+2-l} \right) \delta_k^2||\pi[\xi_{l+1},\xi_{l+1}](x)||^2
+ \sum_{k=n}^{+\infty} 4(1 + \delta_k)^2||\pi[\xi_{l+1},\xi_{l+1}](x)||^2
+ \sum_{k=n}^{+\infty} 2\delta_{k+1}^2||\pi[\xi_{l+1},\xi_{l+1}](x)||^2.
\]
Since \( \sum_{k=n}^{+\infty} ||\pi[\xi_{l+1},\xi_{l+1}](x)||^2 \leq ||x||^2 \), we get that if \( \delta_n \) goes fast enough to zero then \( ||T^{c_k,n}x|| \leq 100||x||\) and we are done.

The road to Theorem 1.1 is now clear.

**Proof of Theorem 1.1.** — Let us choose for every \( n \geq 1 \) the family \( \{p_{k,n}\}_{1 \leq k \leq k_n} \) to form a \( 4^{-\xi_n} \) net of the closed ball of \( \mathbb{K}_n[\xi] \) of radius 2 (we take \( d_n = n \) here). Then for every polynomial \( q \) with \( |q| \leq 2 \) and for every \( n \) greater than the degree of \( q \), there exists a \( k \in [1, k_n] \) such that \( ||p_{k,n}(T) - q(T)|| \leq ||p_{k,n} - q|| \cdot ||T||^n \leq 4^{-\xi_n}2^n \leq 2^{-n} \) if \( ||T|| \leq 2 \) for instance. Let us then estimate for \( x \in H \) the quantity \( ||T^{c_k,n}x - q(T)x|| \):
\[
||T^{c_k,n}x - q(T)x|| \leq ||T^{c_k,n}(x - \pi[0,\xi_{n}])|| + ||T^{c_k,n}\pi[0,\xi_{n}]x - p_{k,n}(T)\pi[0,\xi_{n}]x||
+ ||p_{k,n}(T)\pi[0,\xi_{n}]x - q(T)\pi[0,\xi_{n}]x|| + ||q(T)(x - \pi[0,\xi_{n}]x)||
\leq 100||x - \pi[0,\xi_{n}]x|| + \delta_n||x|| + 2^{-n}||x|| + ||q(T)||||x - \pi[0,\xi_{n}]x||.
\]
Since \( \delta_n \) and \( ||x - \pi[0,\xi_{n}]x|| \) go to zero as \( n \) goes to infinity, we see that \( ||T^{c_k,n}x - q(T)x|| \) can be made arbitrarily small. This implies that for every polynomial \( q \) with \( |q| \leq 2 \), every \( \varepsilon > 0 \) and every \( x \in H \), there exists an integer \( r \) such that \( ||T^r x - p(T)x|| < \varepsilon \). Now if \( p \) is any polynomial with \( |p| \leq 2^j \) for some nonnegative integer \( j \), then for every \( \varepsilon > 0 \) and every \( x \in H \) there exists an integer \( r_j \) such that \( ||T^{r_j}x - 2^{-j}p(T)x|| < \varepsilon 2^{-2j} \). Then \( ||2T^{r_j}x - 2^{-j-1}p(T)x|| < \varepsilon 2^{-2j-1} \), and there exists an integer \( r_{j-1} \) such that \( ||T^{r_{j-1}}x - 2T^{r_j}x|| < \varepsilon 2^{-2j-1} \). Hence \( ||T^{r_{j-1}}x - 2^{-j-1}p(T)x|| < \varepsilon 2^{-2j-1} \). Continuing in this fashion, we obtain an integer \( r_0 \) such that \( ||T^{r_0}x - p(T)x|| < \varepsilon \). Finally, notice that \( e_0 \) is a cyclic vector for \( T \) by construction, so it is in fact a hypercyclic vector. \( \square \)
Remark 3.2. — For the proof of Theorem 1.1 one actually does not need the full complexity of the (c)-fan as presented here. It would be sufficient to consider at each step $n$ only one polynomial $p_n$ and its associated fan consisting of the intervals $[r_c n, r_c n + \xi_n]$, $r \in [0, h_n]$. But we will need to be able to handle several polynomials $p_1, n, \ldots, p_{k_n, n}$ at each step in the proof of Theorem 1.3 and this is why we present the complete (c)-fan already here.

4. Exhibiting hypercyclic vectors: the role of the (b)-fan

Let $x$ be any non-zero vector of $H$ with $||x|| \leq 1$. The starting point of the proofs in [17] or [18] that $x$ must be cyclic for $T$ is the following argument: consider the space $F_{\xi_n} = \text{sp}[e_j : 0 \leq j \leq \xi_n] = \text{sp}[f_j : 0 \leq j \leq \xi_n]$ and the operator $T_{\xi_n}$ on it which is the truncated forward shift on $F_{\xi_n}$: $T_{\xi_n} e_j = e_{j+1}$ for $j \leq \xi_n$ and $T_{\xi_n} e_{\xi_n} = 0$. Write

$$
\pi_{[0, \xi_n]} x = \sum_{j=0}^{\xi_n} \alpha_j^{(n)} e_j = \sum_{j=r_n}^{\xi_n} e_j^{\ast(n)}(x) e_j \quad \text{with} \quad \alpha_j^{(n)} = e_j^{\ast(n)}(x) \neq 0
$$

(since $x$ is non-zero, this is always possible if $n$ is large enough). The functionals $e_j^{\ast(n)}$, $j = 0, \ldots, \xi_n$ are the coordinate functionals with respect to the bases $(e_j)_{j=0, \ldots, \xi_n}$ of $F_{\xi_n}$. Then it is easy to see that the linear orbit of $\pi_{[0, \xi_n]} x$ under $T_{\xi_n}$ is $\text{sp}[e_j : r_n \leq j \leq \xi_n]$. Hence if one of the vectors $e_j$, $r_n \leq j \leq \xi_n$ is sufficiently close to $e_0$ for instance, then there exists a polynomial $p$ of degree less than $\xi_n$ such that $||p(T_{\xi_n}) \pi_{[0, \xi_n]} x - e_0||$ is very small, and the difficulty is to estimate the tail terms in order to show that one must have $||p(T)x - e_0||$ very small too. The obvious way to start this is to estimate $||p(T) \pi_{[0, \xi_n]} x - e_0||$: if $p(\zeta) = \sum_{u=0}^{\xi_n} a_u \zeta^u$ and $0 \leq j \leq \xi_n$, $p(T)e_j = \sum_{u=0}^{\xi_n} a_u e_{j+u}$ and $p(T_{\xi_n}) e_j = \sum_{u=0}^{\xi_n} a_u e_{j+u}$ so that $(p(T) - p(T_{\xi_n})) e_j = \sum_{u=0}^{\xi_n} a_u e_{j+u}$. Hence

$$
||p(T) - p(T_{\xi_n})|| \pi_{[0, \xi_n]} x || = \left| \sum_{j=0}^{\xi_n} \alpha_j^{(n)} \sum_{u=0}^{\xi_n} a_u e_{j+u} \right|
$$

$$
= \left| \sum_{u=0}^{\xi_n} a_u \sum_{j=0}^{\xi_n - u + 1} \alpha_j^{(n)} e_{j+u} \right| \leq |p| C_{\xi_n} \sup_{\xi_n + 1 \leq u \leq 2\xi_n} ||e_{j+u}|| ||x||.
$$

The quantity $\sup_{\xi_n + 1 \leq u \leq 2\xi_n} ||e_{j+u}||$ is very small compared to $\xi_n (\approx 2^{1/\sqrt{\xi_n}})$ with our actual construction, so if $|p|$ is controlled by a constant depending only on $\xi_n$, the quantity $||p(T) - p(T_{\xi_n})|| \pi_{[0, \xi_n]} x ||$ will be very small. The following fact is easy to prove, see the forthcoming Lemma 5.1 for a more precise estimate:

Fact 4.1. — Let $\varepsilon_{\xi_n}$ be a positive constant depending only on $\xi_n$. There exists a constant $C_{\xi_n}$ depending only on $\xi_n$ such that for every $x \in H$, $||x|| \leq 1$, such that $||\alpha_j^{(n)} \varepsilon_{\xi_n}||$, for every $j \in [r_n, \xi_n]$, there exists a polynomial $p$ of degree less than $\xi_n$ with $|p| \leq C_{\xi_n}$ such that

$$
||p(T_{\xi_n}) \pi_{[0, \xi_n]} x - e_j|| \leq \frac{1}{\xi_n}.
$$
Hence with our informal assumptions \( \| p(T_{ξ_n})π[0,ξ_n]x - e_0 \| \) is very small. The next step is to control the tail \( \| p(T)(x - π[0,ξ_n]x) \| \), and for this a natural idea is to use the (c)-fan:

\[
\| T^{c,n}x - e_0 \| \leq 100 \| x - π[0,ξ_n]x \| + \| (T^{c,n} - p_{k,n}(T))π[0,ξ_n]x \| + \| p_{k,n}(T)π[0,ξ_n]x - e_0 \| ,
\]

and then to approximate the polynomial \( p \) by some \( p_{k,n} \) in such a way that \( |p - p_{k,n}| \leq 4^{-ξ_n} \) for instance. But here we run into a difficulty: \( |p_{k,n}| \leq 2 \) for every \( n \) and \( 1 \leq k \leq k_n \), while \( |p| \) may be very large. Since the proof of the uniform estimates for the (c)-fan really requires a uniform bound on the quantities \( |p_{k,n}| \), we have to modify the construction so as to ensure the existence of a polynomial \( q \) with \( |q| \) small such that \( \| (p(T) - q(T))π[0,ξ_n]x \| \) is very small, and then we will be able to approximate \( q \) by \( p_{k,n} \). The (b)-fan is introduced exactly for this purpose: we will see that it ensures that

\[
\left\| \left( \frac{T^{b_n}}{b_n} - I \right)T(π[0,ξ_n]x) \right\| \leq \frac{1}{b_n} \| x \|
\]

where \( b_n \) is very large with respect to \( ξ_n \). Then if the polynomial \( p \) can be written as \( p(ξ) = ζ π_0(ξ) \),

\[
\| (p(T)\frac{T^{b_n}}{b_n} - p(T))π[0,ξ_n]x \| = \| (π_0(T)\frac{T^{b_n}}{b_n} - π_0(T))T(π[0,ξ_n]x) \| \leq |p|2^{ξ_n} \frac{1}{b_n} \| x \| \leq C_{ξ_n} \| x \|
\]

will be extremely small, while \( |q(ξ)| = |p(ξ)| \frac{C_{ξ_n}}{b_n} \| x \| \) will be less than 1 if \( b_n \) is large enough. Then one has to approximate \( q \) by some polynomial \( p_{k,n} \), but here another difficulty appears: the degree of \( q \) is not bounded by \( ξ_n \) anymore, but by \( ξ_n + b_n \), which is much larger, and so one has to modify the fan constructed in Section 2, which we from now on call the (c)-fan, accordingly. We now describe in more details the (b)-fan and the modifications of the (c)-fan.

4.1. Construction of the (b)-fan, modification of the (c)-fan. — The (b)-fan consists of \( ξ_n \) intervals, which are introduced between \( ξ_n + 1 \) and the (c)-fan, and depend on a number \( b_n \) chosen extremely large with respect to \( ξ_n \). The intervals of the (b)-fan are the intervals \([r(b_n + 1), rb_n + ξ_n] \), \( r = 1, \ldots, ξ_n \), and for \( j \) in one of these intervals \( f_j \) is defined as

\[
f_j = e_j - b_ne_{j-b_n}.
\]

The intervals between the (b)-working intervals are lay-off intervals and they are of length approximately \( b_n \), but we modify slightly the definition of \( λ_j \) for \( j \) in a (b)-lay-off interval, just for convenience’s sake: for \( j \in [rb_n + ξ_n + 1, (r+1)b_n - 1] \),

\[
λ_j = 2^{(\frac{1}{b_n}+rb_n+ξ_n+1-j)/\sqrt{κ_n}}
\]

and for \( j \in [ξ_n + 1, b_n] \),

\[
λ_j = 2^{(\frac{1}{b_n}+ξ_n+1-j)/\sqrt{κ_n}}
\]

(instead of using the length of the lay-off interval in the definition we use \( b_n \) which is of the same order of magnitude). The (b)-fan terminates at the index \( ν_n = ξ_n(b_n + 1) \).

We have not yet proved that \( T \) remains bounded with this addition of the (b)-fan, but admitting this for the time being, we can see immediately that \( T^{b_n}/b_n \) is very close to the identity operator on vectors of \( F_ξ \) of the form \( x = \sum_{j=1}^{ξ_n} a_j^{(n)} e_j \), which was one of the reasons for introducing this (b)-fan:
Fact 4.2. — For every $x$ supported in $[0, \xi_n]$, 
\[
\left\| \left( \frac{T^{b_n}}{b_n} - I \right) T(x) \right\| \leq \frac{C_{\xi_n}}{b_n} \|x\|.
\]

Proof. — Write $x = \sum_{j=0}^{\xi_n} \alpha_j^{(n)} e_j$. Then
\[
\left( \frac{T^{b_n}}{b_n} - I \right) T(x) = \sum_{j=0}^{\xi_n} \alpha_j^{(n)} \left( \frac{1}{b_n} e_{j+b_n+1} - e_{j+1} \right)
\]
\[
= \sum_{j=0}^{\xi_n-1} \alpha_j^{(n)} \frac{1}{b_n} f_{j+b_n+1} + \alpha_{\xi_n}^{(n)} \left( \frac{1}{b_n} e_{b_n+\xi_n+1} - e_{\xi_n+1} \right)
\]
because $f_{j+b_n} = e_{j+b_n} - b_n e_j$ for $j \in [1, \xi_n]$. Since $\|e_{b_n+\xi_n+1}\| \leq 2^{-\frac{2}{3}\sqrt{b_n}}$ and $\|e_{\xi_n+1}\| \leq 2^{-\frac{1}{2}\sqrt{b_n}}$, the estimate of Fact 4.2 follows.

Fact 4.2 motivates the introduction of the interval $[b_n + 1, b_n + \xi_n]$, but the role of the “shades” $[r(1+b_n), rb_n + \xi_n]$, $r = 2, \ldots, \xi_n$ which appear afterwards is still obscure at this stage of the construction. The motivation for this will be explained later on.

Let us now explain why we have to modify the (c)-fan: using the previous construction, we have seen that if $q(\zeta) = \frac{c_n}{b_n} p(\zeta) = \frac{c_n+1}{b_n} p_0(\zeta)$, then $|q| < 1$, $q$ is of degree less than $b_n + \xi_n$, so in particular less than $\nu_n$, and $\|q(T)\pi_{[\nu_n]} x - e_0\|$ is very small. Our goal is now to approximate $q$ for $|.|$ by some polynomial $p_{k,n}$. With our actual construction this is impossible, because the degree of $p_{k,n}$ is too large: if for instance we try to estimate $\|T f_{c_{k,n} + \xi_n}\| = \|\gamma_n^{-1} (e_{c_{k,n} + \xi_n + 1} - p_{k,n}(T)e_{\xi_n + 1})\|$, the upper bound we get involves
\[
\gamma_n^{-1} \sup_{0 \leq j \leq \xi_n + b_n} ||e_{\xi_n + 1 + j}||
\]
which is by no means small. So we have to increase the length of the (c)-working intervals from $\xi_n$ to $\nu_n$ (recall that $\nu_n = \xi_n(b_n + 1)$ is the index of the last (b)-working interval), and to chose the family $(p_{k,n})$ as a $4^{-\nu_n}$ net of the unit ball of the set $\mathbb{K}_{\nu_n}[\zeta]$ of polynomials of degree less than $\nu_n$. The (c)-fan starts at $c_{1,n}$ very large with respect to $\nu_n$, and the (c)-working intervals are $[r_1 c_{1,n} + \cdots + c_{k,n}^i k_{k,n} + \nu_n, c_{k,n}^i k_{k,n} + \nu_n]$, $r_1, \ldots, k_{k,n}$ for $i = 1, \ldots, k_{k,n}$. With this definition, the analogue of Fact 2.1 will be:

Fact 4.3. — Let $\delta_n$ be any small positive number. If $\gamma_n$ is small enough, then for every vector $x$ supported in $[0, \nu_n]$ and every $1 \leq k \leq k_{k,n}$,
\[
\|T^{k,n} x - p_{k,n} x\| \leq \delta_n \|x\|.
\]
Notice that $e_0$ remains hypercyclic with this introduction of the (b)-fan.

4.2. Boundedness of $T$, estimates on $T^{k,n}$. — We first have to check that $T$ is still bounded with these modifications. This will follow from Proposition 4.4 below, which is the analogue of our previous Proposition 2.4.

Proposition 4.4. — Let $(\delta_n)_{n \geq 0}$ be a decreasing sequence of positive numbers going to zero very fast. The vectors $f_j$ can be constructed so that for every $n \geq 0$, assertion (1) below holds true:
(1) if \( x \) is supported in the interval \([\nu_n + 1, \nu_{n+1}]\), then

\[
(1a) \quad ||\pi_{[\nu_n + 1, \nu_{n+1}]}(T x)|| \leq (1 + \delta_n)||x||
\]

\[
(1b) \quad ||\pi_{[0,\nu_n]}(T x)|| \leq \delta_n||x||.
\]

**Proof.** — We just outline the points which are different from the proof of Proposition 2.4.

- If \( j = r b_n + \xi_n \), \( T f_j = e_{rb_n + \xi_n + 1} - b_n e_{(r-1)b_n + \xi_n + 1} \). Since \( rb_n + \xi_n + 1 \) and \((r-1)b_n + \xi_n + 1\) are the endpoints of lay-off intervals of length at least roughly \( b_n \), \( ||e_{rb_n + \xi_n + 1}|| \lesssim 2^{-1/2} \sqrt{b_n} \) and \( ||e_{(r-1)b_n + \xi_n + 1}|| \lesssim 2^{-1/2} \sqrt{b_n} \), so \( ||T f_j|| \) can be made arbitrarily small.

- If \( j = r(b_n + 1) - 1 \) is the endpoint of a lay-off interval of type (b), \( T f_j = \lambda_r(b_n+1)-1e_{r(b_n+1)} \).

Now we have a formula for \( e_{r(b_n+1)} \) similar to the one of Lemma 2.3, but much simpler since we go down a one-dimensional lattice, not a multi-dimensional one:

\[
e_{r(b_n+1)} = \sum_{l=0}^{r-1} b_n^{l+1} f_{(r-l)b_n + r} + b_n^{r+1}.
\]

Hence \( ||e_{r(b_n+1)}|| \lesssim b_n^{\xi_n} C_{\xi_n} \) where \( C_{\xi_n} \) depends only on \( \xi_n \). Since \( \lambda_r(b_n+1)-1 \lesssim 2^{-1/2} \sqrt{b_n} \), \( ||T f_j|| \) can be made very small too.

- The proof of the estimates for \( ||T f_{r1c_{k,1} + \ldots + r_{k_n} c_{k_n,n-1}}|| \) and \( ||T f_{r1c_{k,1} + \ldots + r_{k_n} c_{k_n,n-1}}|| \) are exactly the same as in Proposition 2.4, except that \( ||e_{\nu_n+1}|| \) is now involved instead of \( ||e_{\xi_n+1}|| \); \( ||e_{\nu_n+1}|| \lesssim 2^{-1/2} \sqrt{b_n} \) and everything works as previously.

**Corollary 4.5.** — For any \( \varepsilon > 0 \) one can make the construction so that \( T \) is bounded on \( H \) with \( ||T|| \leq 1 + \varepsilon \).

Then Proposition 3.1 becomes

**Proposition 4.6.** — Let \( (\delta_n)_{n \geq 0} \) be a decreasing sequence of positive numbers going to zero very fast. The vectors \( f_j \) can be constructed so that for every \( n \geq 0 \), assertion (2) below holds true:

(2) for any vector \( x \) supported in the interval \([\nu_n + 1, \nu_{n+1}]\) and for any \( 1 \leq k \leq k_n \),

\[
(2a) \quad ||\pi_{[\nu_n + 1, \nu_{n+1}]}(T^{c_{k,n}} x)|| \leq 4||x||
\]

\[
(2b) \quad ||\pi_{[0,\nu_n]}(T^{c_{k,n}} x)|| \leq \delta_n||x||
\]

(3) for any \( x \) supported in the interval \([0,\nu_n]\) and any \( m < \nu_n/2 \),

\[
||\pi_{[\nu_n+1, \nu_{n+1}]}(T^m x)|| \leq \delta_n||x||.
\]

The same argument which is used for the proof of Proposition 2.2 shows that

**Proposition 4.7.** — For every \( n \geq 1 \), every \( 1 \leq k \leq k_n \) and every \( x \in H \) such that \( \pi_{[0,\nu_n]} x = 0 \), \( ||T^{c_{k,n}} x|| \leq 100||x|| \). In other words,

\[
||T^{c_{k,n}}(x - \pi_{[0,\nu_n]} x)|| \leq 100||x - \pi_{[0,\nu_n]} x||
\]

for every \( x \in H \).

**Proof of Proposition 4.6.** — The proof is virtually the same, except that one has additionally to investigate the quantities \( T^{c_{k,n}} f_j \) for \( j \in [\xi_n + 1, \nu_{n+1}] \). This involves no difficulty:
If \( j \) and \( j + c_{k,n} \) belong to the same lay-off interval ([\( \xi_{n+1} + 1, b_{n+1} \]) for instance), \( T^{c_{k,n}} f_j = \lambda_j/\lambda_j + c_{k,n} f_j + c_{k,n} \), and \( \lambda_j/\lambda_j + c_{k,n} \lesssim 2^{\frac{1}{b}}(c_{k,n}/b_{n+1}) \) which can be made arbitrarily close to 1.

- If \( j \) belongs to a lay-off interval ending at the point \( r(b_{n+1} + 1) - 1 \) and \( j + c_{k,n} \) belongs to the working interval \([r(b_{n+1} + 1), r(b_{n+1} + \xi_{n+1})]\), then \( T^{c_{k,n}} f_j = T^c e_{r(b_{n+1} + 1)} \) with 0 \( \leq \alpha \leq c_{k,n} \), so

\[
||T^{c_{k,n}} f_j|| \leq \lambda_j ||T||^{c_{k,n}} ||e_{r(b_{n+1} + 1)}|| \lesssim \lambda_j 2^{c_{k,n}} b_{b_{n+1} + 1} C_{\xi_{n+1}}.
\]

Now \( r(b_{n+1} + 1) - c_{k,n} \leq j \leq r(b_{n+1} + 1) - 1 \) and since \( b_{n+1} \) is very large with respect to \( c_{k,n} \), \( \lambda_j \lesssim 2^{-\frac{1}{b}}(b_{n+1}) \), and thus \( ||T^{c_{k,n}} f_j|| \) can be made very small.

- The argument is exactly the same when \( j \) belongs to a working interval of the (b)-fan, and we omit it.

**4.3. Estimates on \( T^{b_{n+1}} \), construction of some hypercyclic vectors.** — We begin this part by showing that if the \( e_0 \)-coordinate of \( \pi[0,\xi_n]x \) is not too small for infinitely many \( n \)'s, then \( x \) must be hypercyclic. Though not strictly necessary for the proof of Theorem 1.2, this result shows the main idea of the proof, and will allow us to prove easily that \( HC(T)^c \) is Haar null, so we include it.

**Proposition 4.8.** — Let \( x \in H \), \( ||x|| \leq 1 \), be a vector satisfying the following assumption:

(*) for infinitely many \( n \)'s, \( |e_0^{s(n)}(x)| \geq 2^{-n} \), where \( \pi[0,\xi_n]x = \sum_{j=0}^{\xi_n} e_j^{s(n)}(x)e_j \).

Then \( x \) is hypercyclic for \( T \).

**Proof.** — By Fact 2.1, there exists for every \( n \geq 1 \) a constant \( C_{\xi_n} \) such that for every \( y \in F_{\xi_n} \) of the form \( y = \sum_{j=0}^{\xi_n} e_j^{s(n)}(y)e_j \) with \( |e_0^{s(n)}(y)| \geq 2^{-n} \), there exists a polynomial \( p \) of degree less than \( \xi_n \) with \( |p| \leq C_{\xi_n} \) and such that \( \zeta \) divides \( p(\zeta) \) which has the property that

\[
||p(T_{\xi_n})y - e_1|| \leq \frac{1}{\xi_n}.
\]

Write \( p(\zeta) = \zeta p_0(\zeta) \). When \( x \) satisfies (*), we choose an \( n \) such that \( |e_0^{s(n)}(x)| \geq 2^{-n} \) and apply this to the vector \( y = \pi[0,\xi_n]x \). If \( p \) is the polynomial satisfying the above-mentioned properties, then we have seen that

\[
||p_0(T)T(\pi[0,\xi_n]x) - e_1|| \leq \frac{2}{\xi_n}
\]

since \( ||(p(T_{\xi_n}) - p(T))\pi[0,\xi_n]x|| \lesssim C_{\xi_n} \sup_{1 \leq j \leq 2\xi_n} ||e_j|| \lesssim C_{\xi_n} 2^{-\frac{b_{\xi_n}}{b_{n+1}}} \). We now have to make the modulus of the polynomial small, so we take \( q(\zeta) = \frac{\xi_n}{\xi_n} p(\zeta) = \frac{\xi_n}{\xi_n} p_0(\zeta) \): the degree of \( q \) is less than \( \xi_n + b_n \), \( |q| < 1 \), and by Fact 4.2

\[
||q(T)\pi[0,\xi_n]x - e_1|| \leq \frac{3}{\xi_n}.
\]

Let now \( k \leq k_n \) be such that \( |q - p_{k,n}| \leq 4^{-\nu_n} \): then \( ||q(T) - p_{k,n}(T)|| \leq 4^{-\nu_n} ||T||^d \) where \( d \) is the degree of \( q - p_{k,n} \), so \( ||q(T) - p_{k,n}(T)|| \leq 2^{-\nu_n} \) for instance. Hence

\[
||p_{k,n}(T)\pi[0,\xi_n]x - e_1|| \leq \frac{4}{\xi_n}.
\]
This yields that
\[
\|T^{c, n} x - e_1\| \leq \|T^{c, n} - p_{k,n}(T)\pi_{[0, \nu_n]} x\| + \|p_{k,n}(T)\pi_{[\nu_n + 1, \nu_n]} x\| + \|p_{k,n}(T)\pi_{[0, \nu_n]} x - e_1\|
\]
\[
\leq \frac{5}{\xi_n} + \|p_{k,n}(T)\pi_{[\nu_n + 1, \nu_n]} x\| \leq \frac{6}{\xi_n} + \|q(T)\pi_{[\nu_n + 1, \nu_n]} x\|
\]
and the difficulty which remains is to estimate the last term. This is here that we use the fact (which may look a bit strange) that we have approximated \(e_1\) and not \(e_0\), as well as the shades of the (b)-fan: since \(\zeta\) divides \(p(\zeta)\) (because we approximate \(e_1\)), \(q\) can be written as \(q(\zeta) = \frac{1}{k_n} \zeta^{b_n+1} p_0(\zeta)\) with \(|p_0| \leq C_{\xi_n}\) and the degree of \(p_0\) less than \(\xi_n - 1\).

Hence
\[
\|q(T)\pi_{[\nu_n + 1, \nu_n]} x\| = \frac{1}{k_n} T^{b_n+1} p_0(T)\pi_{[\nu_n + 1, \nu_n]} x\| \leq \frac{1}{k_n} C_{\xi_n} 2^{k_n} \|T^{b_n+1} \pi_{[\nu_n + 1, \nu_n]} x\|.
\]

And now the shades of the (b)-fan have been introduced exactly so as to ensure that

**Lemma 4.9.** — For every \(n \geq 1\) and every \(x \in H\),
\[
\|T^{b_n+1} \pi_{[\nu_n + 1, \nu_n]} x\| \leq 2 \|x\|.
\]

Lemma 4.9 allows us to conclude immediately the proof of Proposition 4.8:
\[
\|T^{c, n} x - e_1\| \leq \frac{7}{\xi_n},
\]
and hence \(e_1\) belongs to the closure of the orbit of \(x\). Since \(e_0\) is hypercyclic for \(T\), \(e_1\) is too, and hence \(x\) is hypercyclic.

**Remark 4.10.** — The condition \(|e^{s(n)}_0(x)| \geq 2^{-n}\) can obviously be replaced by any condition of the form \(|e^{s(n)}_0(x)| \geq \varepsilon_{\xi_n}\), where \(\varepsilon_{\xi_n}\) is a small number depending only on \(\xi_n\).

It remains to prove Lemma 4.9.

**Proof of Lemma 4.9** — As in the preceding proofs, we must distinguish several cases.

- If \(j \in [r(b_n + 1), r b_n + \xi_n]\), \(r = 1, \ldots, \xi_n - 1\), \(T^{b_n+1} f_j = \varepsilon_{j+b_n+1} - b_n \varepsilon_{j+1}\), and \(j + b_n + 1 \in [(r + 1)(b_n + 1), (r + 1)b_n + \xi_n]\), so if \(j < rb_n + \xi_n\), \(T^{b_n+1} f_j = f_{j+b_n+1}\). If \(j = rb_n + \xi_n\), \(T^{b_n+1} f_j = \varepsilon_{(r+1)b_n+\xi_n+1} - b_n \varepsilon_{r b_n + \xi_n + 1}\). Since \(\|\varepsilon_{r b_n + \xi_n + 1}\| \leq \frac{2^{-r} \sqrt{b_n}}{\sqrt{\xi_n}}\) for \(r = 1, \ldots, \xi_n\), \(\|T^{b_n+1} f_j\|\) is very small.

- If \(j = \xi_n(b_n + 1), T^{b_n+1} f_j = \varepsilon_{(\xi_n+1)(b_n+1)} - b_n \varepsilon_{\xi_n(b_n+1)+1}\) so \(\|T^{b_n+1} f_j\|\) is very small.

- If \(j \in [r b_n + \xi_n + 1, (r + 1)(b_n + 1) - 1]\), \(r = 1, \ldots, \xi_n - 2\), \(T^{b_n+1} f_j = \lambda_j \varepsilon_{j+b_n+1}\) and \(j + b_n + 1 \in [(r + 1)b_n + \xi_n + 2, (r + 2)(b_n + 1) - 1]\) which is contained in the lay-off interval \([(r + 1)b_n + \xi_n + 2, (r + 2)(b_n + 1) - 1]\). So \(T^{b_n+1} f_j = \lambda_j \varepsilon_{j+b_n+1}\) and \(\|T^{b_n+1} f_j\|\) is very small. Now a straightforward computation shows that \(\lambda_j/\lambda_j + b_n+1 = 2^{1/\sqrt{b_n}}\), which is less than 2 if \(b_n\) is sufficiently large. It is at this point that we use the fact that the definition of the coefficients \(\lambda_j\) for \(j\) in a (b)-lay-off interval involves directly \(b_n\), and not the length of the interval. If \(r = \xi_n - 1\), then \(j + b_n + 1\) belongs to the beginning of the lay-off interval \([\nu_n + 1, c_1, n - 1]\), so \(\lambda_j/\lambda_j + b_n+1 \geq 2^{\frac{1}{\sqrt{b_n}}}\) so \(\lambda_j/\lambda_j + b_n+1 \leq 2^{\frac{1}{\sqrt{b_n}} - \sqrt{c_1,n}}\) is very small.
• If \( j \in [\xi_n + 1, b_n], j + b_n + 1 \in [b_n + \xi_n + 1, 2b_n] \), so again \( T^{b_n + 1}f_j = 2^{1/\sqrt{n}}f_{j + b_n + 1} \). □

### 4.4. The set \( HC(T)^c \) is Haar null.

If \( M \) is any positive integer, let \( E_M \) be the set of vectors \( x \in H \) such that \( ||x|| \leq M \) and there exists an \( n_0 \) such that for every \( n \geq n_0 \), \( |e_0^{(n)}(x)| \leq 2^{-n}M \). Then

\[
HC(T)^c \subseteq \bigcup_{M=1}^{+\infty} E_M.
\]

Indeed if \( x \) is a nonzero vector not in \( HC(T) \), then \( x/||x|| \) does not satisfy assumption (*) of Proposition 4.3, so there exists an \( n_0 \) such that for every \( n \geq n_0 \), \( |e_0^{(n)}(x/||x||)| \leq 2^{-n} \), i.e. \( |e_0^{(n)}(x)| \leq 2^{-n}||x|| \). Hence if \( M \geq ||x|| \), \( x \) belongs to \( E_M \). Since the union of countably many Haar null sets is Haar null, it suffices to show that each \( E_M \) is Haar null.

There are different ways of proving this. A first option is to use a result of Matouskova [10] that every closed convex subset of a separable superreflexive space is Haar null. Or an elementary approach is to exhibit a measure \( m \) such that \( m(x_0 + E_M) = 0 \) for every \( x_0 \in H \). We detail here the second argument. The measures which we consider are non-degenerate Gaussian measures on \( H \): let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a standard probability space, and \( (g_n)_{n \geq 0} \) a sequence of standard independent random Gaussian variables, real or complex depending on whether the Hilbert space \( H \) is supposed to be real or complex. For any sequence \( c = (c_j)_{j \geq 0} \) of non-zero real numbers such that \( \sum_{j \geq 0} |c_j|^2 < +\infty \), consider the random measurable function \( \Phi_c : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow H \) defined by

\[
\Phi_c(\omega) = \sum_{j=0}^{+\infty} c_j g_j(\omega) f_j.
\]

This function is well-defined almost everywhere, and it belongs to all the spaces \( L^p(\Omega) \), \( p \geq 1 \). To each such function \( \Phi_c \) is associated a measure \( m_c \) defined on \( H \) by

\[
m_c(A) = \mathbb{P}(\{\omega \in \Omega : \Phi_c(\omega) \in A\})
\]

for every Borel subset \( A \) of \( H \). This is a Gaussian measure, and since all the \( c_j \)'s are non-zero, its support is the whole space.

**Proposition 4.11.** — For any vector \( x_0 \in H \) and any \( M \geq 1 \), set

\[
B_{x_0,M} = \{\omega \in \Omega ; \text{for infinitely many } n \text{'s, } |e_0^{(n)}(x_0 + \Phi_c(\omega))| \geq 2^{-n}M \}.
\]

Then \( \mathbb{P}(B_{x_0,M}) = 1 \).

**Proof.** — Write \( x_0 = \sum_{j=0}^{+\infty} u_j f_j \). Then \( \pi_{[0,\xi_n]}(x_0 + \Phi_c(\omega)) = \sum_{j=0}^{\xi_n} (u_j + c_j g_j(\omega)) f_j \) for every \( n \geq 0 \). Consider the random variable

\[
X_n(\omega) = \langle e_0^{(n)}, \pi_{[0,\xi_n]}(x_0 + \Phi_c(\omega)) \rangle = \sum_{j=0}^{\xi_n} \langle e_0^{(n)}, (u_j + c_j g_j(\omega)) f_j \rangle,
\]
where for \( x = \sum_{j=0}^{\xi_n} a_j e_j, (e_0^{(n)}, x) = a_0^{(n)} \). Then \( X_n \) is a Gaussian random variable with mean \( m_n = \sum_{j=0}^{\xi_n} a_j^{(n)} f_j \) and variance

\[
\sigma_n = \sqrt{\sum_{j=0}^{\xi_n} |a_j|^2 |(e_0^{(n)}, f_j)|^2} \geq |c_0| |(e_0^{(0)}, f_0)| = |c_0|.
\]

Let us estimate \( P(|X_n| \leq 2^{-n}M) \). If the space \( H \) is real,

\[
P(|X_n| \leq 2^{-n}M) = \int_{-2^{-n}M}^{2^{-n}M} \exp(-\frac{1}{2\sigma_n^2} (t - m_n)^2) \frac{1}{\sigma_n \sqrt{2\pi}} dt \leq \frac{2^{-n+1}M}{\sigma_n \sqrt{2\pi}} \leq \frac{2^{-n}M}{|c_0|}.
\]

If the space \( H \) is complex,

\[
P(|X_n| \leq 2^{-n}M) = \int_{\sqrt{u^2 + v^2} \leq 2^{-n}M} \exp(-\frac{1}{2\sigma_n^2} |u + iv - m_n|^2) \frac{1}{\sigma_n \sqrt{2\pi}} dudv \leq \frac{2^{-2n}M^2}{\sigma_n^2} \leq \frac{2^{-2n}M^2}{2|c_0|^2}.
\]

In both cases the series \( \sum_{n \geq 0} P(|X_n| \leq 2^{-n}M) \) is convergent. By the Borel-Cantelli Lemma, the probability that \( |X_n| \leq 2^{-n}M \) for infinitely many \( n \)’s is zero, and this is exactly the statement of Proposition 4.11.

Theorem 1.2 follows immediately from Propositions 4.8 and 4.11 for any \( x_0 \in H \),

\[
m(-x_0 + E_M) = \mathbb{P}(\{\omega \in \Omega : x_0 + \Phi_c(\omega) \in E_M\}) \leq \mathbb{P}(\{\omega \in \Omega : \exists n_0 \forall n \geq n_0 |e_0^{(n)}(x_0 + \Phi_c(\omega))| \leq 2^{-n}M\}) = 0.
\]

Hence each set \( E_M \) is Haar null.

4.5. The set \( HC(T)^c \) is \( \sigma \)-porous. — It is not difficult to see that \( HC(T)^c \) is also \( \sigma \)-porous in this example. Indeed let \( \tilde{E}_M \) be the set of \( x \in H \) such that \( ||x|| < M \) and there exists an \( n_0 \) such that for all \( n \geq n_0, |e_0^{(n)}(x)| < 2^{-n}M \). Write \( \tilde{E}_M = \bigcup_{n \geq 1} \tilde{E}_M_{n0} \) where \( \tilde{E}_M_{n0} \) is the set of \( x \in H \) such that \( ||x|| < M \) and for every \( n \geq n_0, |e_0^{(n)}(x)| < 2^{-n}M \).

We are going to show that each one of the sets \( \tilde{E}_M_{n0} \) is \( \frac{1}{2} \)-porous. For each \( n \geq 1 \), let \( x_n \in F_{0n}, ||x_n|| = 1 \) be such that \( e_0^{(n)}(x_n) = ||e_0^{(n)}|| \). If we suppose for instance that \( p_{1,n} = 1 \) for every \( n \geq 1 \), then \( f_{c_{1,n}} = \gamma_n^{-1}(e_{c_{1,n}} - e_0) \) so \( e_0^{(n)}(f_{c_{1,n}}) = -\gamma_n^{-1} \) and hence \( ||e_0^{(n)}|| \geq \gamma_n^{-1} \). Thus by choosing \( \gamma_n \) sufficiently small at each step, it is possible to ensure that \( ||e_0^{(n)}|| \geq 2^n \) for every \( n \). So given \( x \in \tilde{E}_M_{n0} \) and \( \varepsilon > 0 \), let \( 0 < \delta < \varepsilon \) be so small that \( ||z|| < M \) for every \( z \) such that \( ||z - x|| < \delta \). Fix \( k \geq n_0 \) such that \( \frac{1}{2}\delta||e_0^{(k)}|| > 2 \cdot 2^{-k}M \) and choose \( y = x + \delta x_k \). Then \( ||y|| < M \) and \( 0 < \delta ||y - x|| < \varepsilon \). Consider \( z \in B(y, \frac{1}{2}\delta ||y - x||) = B(y, \frac{1}{2}) \). Then

\[
||e_0^{(k)}(z)|| \geq ||e_0^{(k)}(y)|| - ||e_0^{(k)}||||z - y|| \geq \left| e_0^{(k)}(x) - \delta ||e_0^{(k)}|| \right| - \frac{\delta}{2} ||e_0^{(k)}|| \geq \delta \frac{||e_0^{(k)}||}{2} - 2^{-k}M > 2^{-k}M.
\]
by our assumption on \( k \). Hence \( z \notin \tilde{E}_{M,n_0} \), and \( B(y, \frac{1}{2}||y-x||) \cap \tilde{E}_{M,n_0} \) is empty. This proves that \( \tilde{E}_{M,n_0} \) is \( \frac{1}{\pi} \)-porous.

5. Orbit-unicellularity of \( T \): proofs of Theorems 1.2 and 1.3

The main step in the proof of Theorem 1.2 is Theorem 1.3, which shows that whenever \( x \) and \( y \) are two vectors of \( H \) of norm 1, either the closure of the orbit of \( x \) is contained in the closure of the orbit of \( y \), or the other way round. In view of Proposition 2.4 this is quite a natural statement: the idea of the proof of Proposition 2.4 is that whenever \( \pi_{[0,\xi_n]}x = \sum_{j=r_n}^{\xi_n} e_j(x)e_j \) with \( e_{r_n}(x) \neq 0 \), then for every vector \( z \) supported in \( [r_n, \xi_n] \) there exists a polynomial \( p \) of degree less than \( \xi_n \) such that \( p(T_{\xi_n})\pi_{[0,\xi_n]}x = z \), and \( |p| \) is controlled by a constant which depends on \( |e_{r_n}(x)| \) (and \( \xi_n \) of course). If our two vectors \( x \) and \( y \) are given:

- either there are infinitely many \( n \)'s such that the first “large” \( e_j \)-coordinate (in a sense to be made precise later) of \( \pi_{[0,\xi_n]}x \) is smaller than the first “large” \( e_j \)-coordinate of \( \pi_{[0,\xi_n]}y \), and in this case there exists infinitely many polynomials \( p_n \) suitably controlled such that \( ||p_n(T_{\xi_n})\pi_{[0,\xi_n]}x - \pi_{[0,\xi_n]}y|| \leq \frac{1}{\xi_n} \) for instance for these \( n \)'s,
- or the first large coordinate appears first in \( \pi_{[0,\xi_n]}y \) infinitely many times, and then \( ||p_n(T_{\xi_n})\pi_{[0,\xi_n]}y - \pi_{[0,\xi_n]}x|| \leq \frac{1}{\xi_n} \).

In the first case \( y \) will belong to the closure of the orbit of \( x \), and in the second case \( x \) will belong to the closure of the orbit of \( y \).

In order to be able to formalise this argument, we have to quantify what it means for an \( e_j \)-coordinate to be “large”, and for this it will be useful to have a precise estimate on \( |p| \) for polynomials \( p \) such that \( p(T_{\xi_n})\pi_{[0,\xi_n]}x = z \) as above in terms of the size of \( |e_{r_n}(x)| \).

**Lemma 5.1.** For every \( n \geq 1 \) there exists a constant \( C_{\xi_n}' \) depending only on \( \xi_n \) such that the following property holds true:

for every vector \( x \) of \( F_{\xi_n} \) of norm 1, \( x = \sum_{j=r_n}^{\xi_n} e_j(x)e_j \) with \( e_{r_n}(x) \neq 0 \), and for every vector \( y \) of norm 1 belonging to the linear span of the vectors \( e_{r_n}, \ldots, e_{\xi_n} \), there exists a polynomial \( p \) of degree less than \( \xi_n \) with

\[
|p| \leq \frac{C_{\xi_n}'}{|e_{r_n}(x)|^{\xi_n-r_n+1}}
\]

such that \( p(T_{\xi_n})x = y \).

**Proof.** If \( p(\zeta) = \sum_{n=0}^{\xi_n} a_n \zeta^n \), then since \( T_{\xi_n}^n e_j = e_{j+u} \) for \( j + u \leq \xi_n \) and \( T_{\xi_n}^n e_j = 0 \) for \( j + u > \xi_n \), we have

\[
p(T_{\xi_n})x = \sum_{u=0}^{\xi_n} a_u \sum_{j=r_n}^{\xi_n} e_j(x)e_{j+u} = \sum_{j=r_n}^{\xi_n} e_j(x) \sum_{u=0}^{\xi_n} a_u e_{u-j} = \sum_{u=r_n}^{\xi_n} \left( \sum_{j=r_n}^{\xi_n} e_j(x) a_{u-j} \right) e_u.
\]
Hence solving the equation \( p(T_{\xi_n})x = y \) boils down to solving the system of \( \xi_n - r_n + 1 \) equations

\[
\sum_{j=r_n}^{u} e_j^{s(n)}(x)a_{u-j} = e_j^{s(n)}(y) \quad \text{for} \quad u = r_n, \ldots, \xi_n.
\]

This can be written in matrix form as

\[
\begin{pmatrix}
  e_{r_n}^{s(n)}(x) & \cdots & e_{r_n+1}^{s(n)}(x) \\
  e_{\xi_n}^{s(n)}(x) & \cdots & e_{\xi_n}^{s(n)}(x)
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{\xi_n-r_n}
\end{pmatrix}
= \begin{pmatrix}
e_{r_n}^{s(n)}(y) \\
e_{r_n+1}^{s(n)}(y) \\
\vdots \\
e_{\xi_n}^{s(n)}(y)
\end{pmatrix}
\]

and if \( M_{\xi_n}(x) \) denotes the square matrix of size \( \xi_n - r_n + 1 \) on the left-hand side, then it is invertible. If we choose

\[
\begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{\xi_n-r_n}
\end{pmatrix}
= M_{\xi_n}(x)^{-1}
\begin{pmatrix}
e_{r_n}^{s(n)}(y) \\
e_{r_n+1}^{s(n)}(y) \\
\vdots \\
e_{\xi_n}^{s(n)}(y)
\end{pmatrix}
\]

then \( p(T_{\xi_n})x = y \). Hence

\[
|p| \leq \|M_{\xi_n}(x)^{-1}\|_{B(\ell_1)} \sum_{j=r_n}^{\xi_n} |e_j^{s(n)}(y)| \leq \|M_{\xi_n}(x)^{-1}\|_{B(\ell_1)}A_{\xi_n}
\]

since \( \|y\| = 1 \), and

\[
\|M_{\xi_n}(x)^{-1}\|_{B(\ell_1)} \leq \frac{B_{\xi_n}}{|e_{r_n}^{s(n)}(x)|_{\xi_n-r_n+1}}
\]

since \( \|x\| = 1 \), which proves Lemma 5.1 \( \square \)

Let now \( x \) and \( y \) be our two vectors of \( H \) with \( \|x\| = \|y\| = 1 \). We will say that the \( e_j \)-coordinate of \( \pi_{[0,\xi_n]}x \) is large if

\[
|e_j^{s(n)}(x)| \geq \frac{1}{C_{\xi_n}^{\xi_n-j+1}}
\]

where \( C_{\xi_n} \) is a constant depending only on \( \xi_n \) which will be chosen later on in the proof. A first point is:

**Fact 5.2.** — Provided the sequence \( (C_{\xi_n}) \) grows fast enough, for every \( x \in H, \|x\| = 1 \), there exists an \( n_0 \) such that for every \( n \geq n_0 \), there exists a \( j \in [0,\xi_n] \) with

\[
|e_j^{s(n)}(x)| \geq \frac{1}{C_{\xi_n}^{\xi_n-j+1}}.
\]

**Proof.** — Suppose on the contrary that for every \( j \in [0,\xi_n], |e_j^{s(n)}(x)| \leq 1/C_{\xi_n}^{\xi_n-j+1} \). Then

\[
\|\pi_{[0,\xi_n]}x\| \leq \sum_{j=0}^{\xi_n} |e_j^{s(n)}(x)| \sup_{0 \leq j \leq \xi_n} |e_j|| \leq \frac{1}{C_{\xi_n}^{\xi_n-1}} \sup_{0 \leq j \leq \xi_n} |e_j||.
\]
If \( \sqrt{C_{\xi}} \geq \sup_{0 \leq j \leq \xi} ||e_j|| \) for instance, \( ||\pi_{[0,\xi]}x|| \leq \sqrt{C_{\xi}}/(C_{\xi} - 1) \), and since \( ||x|| = 1 \) this is impossible if \( n \) is large enough and \( C_{\xi} \) goes fast enough to infinity. 

5.1. Proof of Theorem 1.3. — Denote by \( j_n(x) \) the smallest integer \( j \) in \([0, \xi]\) such that \( |e_j^{(n)}(x)| \geq 1/C_{\xi}^{j_n+1} \). Then either for infinitely many \( n \)'s \( j_n(x) \leq j_n(y) \), or for infinitely many \( n \)'s \( j_n(y) \leq j_n(x) \). In the rest of the proof we suppose that \( j_n(x) \leq j_n(y) \) for infinitely many \( n \)'s and write \( j_n = j_n(x) \):

\[
|e_{j_n}^{(n)}(x)| \geq \frac{1}{C_{\xi}^{j_n+1}}
\]

and for every \( j < j_n \),

\[
|e_{j}^{(n)}(x)| \leq \frac{1}{C_{\xi}^{j_n-j+1}} \quad \text{and} \quad |e_{j}^{(n)}(y)| \leq \frac{1}{C_{\xi}^{j_n-j+1}}.
\]

By Lemma 5.1 applied to the two vectors \( x' = \sum_{j=j_n}^{\xi} e_j^{(n)}(x)e_j \) and \( y' = \sum_{j=j_n}^{\xi} e_j^{(n)}(y)e_j \), there exists a polynomial \( p_n \) of degree less than \( \xi \) with \( |p_n| \leq C_{\xi}^{j_n-j+1} \) such that

\[
\left| p_n(T_{\xi_n}) \left( \sum_{j=j_n}^{\xi} e_j^{(n)}(x)e_j \right) - \sum_{j=j_n}^{\xi} e_j^{(n)}(y)e_j \right| \leq \frac{1}{\xi_n}.
\]

Since

\[
\left| \pi_{[0,\xi]}y - \sum_{j=j_n}^{\xi} e_j^{(n)}(y)e_j \right| = \left| \sum_{j=0}^{j_n-1} e_j^{(n)}(y)e_j \right| \leq \sum_{j=0}^{j_n-1} |e_j^{(n)}(y)| \sup_{0 \leq j \leq j_n} ||e_j||
\]

\[
\leq \frac{\sqrt{C_{\xi}}}{C_{\xi} - 1} \leq \frac{1}{\xi_n}
\]

if \( \sup_{0 \leq j \leq \xi} ||e_j|| \leq \sqrt{C_{\xi}} \) as above and \( C_{\xi} \) grows fast enough, we get

\[
\left| p_n(T_{\xi_n}) \left( \sum_{j=j_n}^{\xi} e_j^{(n)}(x)e_j \right) - \pi_{[0,\xi]}y \right| \leq \frac{2}{\xi_n}
\]

Then

\[
\left| \pi_{[0,\xi]}y - p_n(T_{\xi_n}) \left( \sum_{j=0}^{j_n-1} e_j^{(n)}(x)e_j \right) \right| \leq |p_n|2^{\xi_n} \sum_{j=0}^{j_n-1} |e_j^{(n)}(x)| \sqrt{C_{\xi}}
\]

\[
\leq C_{\xi}^{j_n} C_{\xi}^{\xi - j_n + 1} 2^{\xi_n} \sum_{j=0}^{j_n-1} \frac{1}{C_{\xi}^{j_n-j+1}} \sqrt{C_{\xi}}
\]

\[
\leq C_{\xi}^{j_n} C_{\xi}^{\xi - j_n + 1} 2^{\xi_n} \frac{2}{C_{\xi}^{j_n+2}} \sqrt{C_{\xi}}
\]

\[
\leq \frac{C_{\xi}^{j_n} 2^{\xi_n}}{\sqrt{C_{\xi}}}
\]
Since $C_{\xi_n}$ can be chosen very large with respect to $C'_{\xi_n}$, we can ensure that the quantity on the righthand side is less than $1/\xi_n$, and hence

$$\|p_n(T_{\xi_n})\pi_{[0,\xi_n]}x - \pi_{[0,\xi_n]}y\| \leq \frac{3}{\xi_n}.$$

Now $|p_n|$ is controlled by a constant $D_{\xi_n}$ which depends only on $\xi_n$, and the same argument as in Section 4 (choosing $b_n$ very large with respect to $\xi_n$) shows that

$$\|p_n(T)\pi_{[0,\xi_n]}x - \pi_{[0,\xi_n]}y\| \leq \frac{4}{\xi_n}.$$ 

The polynomial $p_n$ has all the properties we want, except for the fact that $\zeta$ does not necessarily divide $p_n(\zeta)$, so consider $\tilde{p}_n(\zeta) = \zeta p_n(\zeta)$:

$$\|\tilde{p}_n(T)\pi_{[0,\xi_n]}x - T(\pi_{[0,\xi_n]}y)\| \leq \frac{8}{\xi_n},$$

$|\tilde{p}_n| \leq D_{\xi_n}$ and the degree of $\tilde{p}_n$ is less than $\xi_n + 1$ (and not $\xi_n$ as before, but this is not a problem, as will be seen shortly). We take as previously $q_n(\zeta) = \frac{\xi_n}{b_n} \tilde{p}_n(\zeta) = \frac{\xi_n}{b_n} p_n(\zeta)$: $|q_n| < 1$ and the degree of $q_n$ is less than $n_\zeta + 1$. We have

$$\|\left(\frac{\xi_n}{b_n} T_{b_n}(T - p_n(T))\right) \pi_{[0,\xi_n]}x\| \leq \frac{C_{\xi_n}}{b_n} |p_n| 2^{\xi_n}$$

by Fact 4.2 so

$$\|q_n(T)\pi_{[0,\xi_n]}x - \tilde{p}_n(T)\pi_{[0,\xi_n]}x\| \leq \frac{C_{\xi_n}}{b_n} |p_n| 2^{\xi_n + 1} \leq \frac{1}{\xi_n}$$

if $b_n$ is large enough. Thus

$$\|q_n(T)\pi_{[0,\xi_n]}x - T(\pi_{[0,\xi_n]}y)\| \leq \frac{9}{\xi_n}$$

and the proof then goes as in Proposition 2.4 for some $k \in [1, k_n]$,

$$\|T^{ck_n}x - T(\pi_{[0,\xi_n]}y)\| \leq \frac{10}{\xi_n}.$$ 

Since $T(\pi_{[0,\xi_n]}y)$ tends to $Ty$ as $n$ tends to infinity, this shows that $Ty$ belongs to the closure of the orbit of $x$. But since $\overline{Orb}(y, T)$ and $\overline{sp}(p(T)y : p \in K[\zeta])$ coincide, $y$ is a hypercyclic vector for the operator induced by $T$ on $\overline{sp}(p(T)y : p \in K[\zeta])$, and thus $y$ is the limit of some sequence $(T^{q_j}y)$. Hence $y \in \overline{Orb}(x, T)$, which proves that $\overline{Orb}(y, T) \subseteq \overline{Orb}(x, T)$. This finishes the main part of the proof of Theorem 1.3.

We still have to prove that if $M$ is any non trivial invariant subspace of $T$, the operator induced by $T$ on $M$ is hypercyclic. Let $U$ and $V$ be two non empty open subsets of $M$, with $u \in U$, $v \in V$. Since $T$ is orbit-unicellular, either $\overline{Orb}(u, T) \subseteq \overline{Orb}(v, T)$ or $\overline{Orb}(v, T) \subseteq \overline{Orb}(u, T)$. Suppose for instance that we are in the first case: $U$ and $V$ both intersect $\overline{Orb}(v, T) \subseteq M$, so there exist two integers $p$ and $q$, $q > p$, such that $T^p v \in U$ and $T^q v \in V$. Hence $T^{q-p} (U \cap V)$ is non empty. The same argument works if the inclusion of the orbits of $u$ and $v$ is in the reverse direction, and this proves that $T$ acting on $M$ is topologically transitive. The usual Baire Category argument shows then that $T$ acting on $M$ is hypercyclic, which finishes the proof of Theorem 1.3.
5.2. Proof of Theorem 1.2 — The proof of Theorem 1.2 is now easy, and follows the classical argument which shows that any unicellular operator must be cyclic (see for instance 15): let \((x_\alpha)_{\alpha \in A}\) be the family of all non-hypercyclic vectors for \(T\), and for each \(\alpha \in A\) write \(M_\alpha = \overline{\text{orb}(x_\alpha, T)}\) (which is a closed nontrivial subspace of \(T\)). So

\[
HC(T)^c = \bigcup_{\alpha \in A} M_\alpha
\]

is a linear subspace of \(H\). If \(HC(T)^c\) is not dense in \(H\), then it is contained in a closed hyperplane, and \(HC(T)^c\) is clearly Gauss null. So we can suppose that \(HC(T)^c\) is dense in \(H\). Then let \((x_\alpha)_{i \geq 0}\) be a countable subset of \(HC(T)^c\) which is dense in \(HC(T)^c\) (and hence in \(H\)). We are going to show that

\[
HC(T)^c = \bigcup_{i=0}^{+\infty} M_{\alpha_i}.
\]

Let \(\alpha \in A\); we want to show that for some \(i\), \(M_\alpha \subsetneq M_{\alpha_i}\). If this is not true, then by Theorem 1.3 this means that \(M_{\alpha_i} \subseteq M_\alpha\) for every \(i\), hence \(x_{\alpha_i} \in M_\alpha\) for every \(i\). Since \((x_{\alpha_i})_{i \geq 0}\) is dense in \(H\), \(M_\alpha = H\), so \(x_\alpha\) is hypercyclic, a contradiction. Thus \(HC(T)^c = \bigcup_{i=0}^{+\infty} M_{\alpha_i}\) is a countable union of subsets of closed hyperplanes, and hence is Gauss null.

5.3. Proof of Proposition 1.4 — Suppose that every operator on an infinite dimensional separable Hilbert space has a non trivial invariant subspace, and let \(T \in B(H)\) satisfy the assumptions of condition (b) in Proposition 1.4. It is not difficult to see that \(HC(T)^c\) can be written as a strictly increasing union of closures of orbits

\[
HC(T)^c = \bigcup_{n \in \mathbb{Z}} M_n \quad \text{with} \quad M_n \subsetneq M_{n+1} \quad \text{for every} \quad n \in \mathbb{Z}.
\]

Indeed consider the decomposition \(HC(T)^c = \bigcup_{i=0}^{+\infty} M_{\alpha_i}\) obtained in the proof of Theorem 1.2. Take \(M_0 = M_{\alpha_0}\). Since \(M_{\alpha_1} \neq H\) and the sequence \((x_{\alpha_i})\) is dense in \(H\), there exists an \(\alpha_i\) such that \(M_0 \subsetneq M_{\alpha_i}\). Take \(i_1\) to be the smallest integer such that this property holds true, and set \(M_1 = M_{\alpha_{i_1}}\). In the same way let \(j_1\) be the smallest integer such that \(M_{\alpha_{j_1}} \subsetneq M_0\), and set \(M_{-1} = M_{\alpha_{j_1}}\). In this fashion we construct two strictly increasing sequences \((i_n)_{n \geq 1}\) and \((j_n)_{n \geq 1}\) of integers having the property that for every \(i < j_n\), \(M_{\alpha_i} \subsetneq M_{\alpha_{j_n}}\), and for every \(j < j_n\), \(M_{\alpha_{j_{n-1}}} \subsetneq M_{j_n}\) for \(n \geq 1\), we get that this sequence of subspaces is strictly increasing, and that for every \(i \geq 0\) there exists an \(n\) such that \(M_{-n} \subsetneq M_{i_n} \subsetneq M_n\). Hence \(HC(T)^c = \bigcup_{n \in \mathbb{Z}} M_n\). For \(n \in \mathbb{Z}\) set \(\Phi(n) = M_n\). Since \(M_n \subsetneq M_{n+1}\), \(M_{n+1}/M_n\) is non trivial, and by the argument given in the introduction this quotient must be a Hilbert space of infinite dimension. By our assumption, \(T\) acting on \(M_{n+1}/M_n\) has a non trivial invariant subspace. This means that there exists \(M\) invariant for \(T\) such that \(M_n \subsetneq M \subsetneq M_{n+1}\). Set \(\Phi(n + 1/2) = M\). Continuing in this fashion, we can define in the obvious way the subspaces \(\Phi(n + \sum_{k \in I} 2^{-k})\), where \(I\) is any finite subset of the set of positive integers. Clearly if \(n_1 + \sum_{k \in I_1} 2^{-k} < n_2 + \sum_{k \in I_2} 2^{-k}\), \(\Phi(n_1 + \sum_{k \in I_1} 2^{-k}) \subsetneq \Phi(n_2 + \sum_{k \in I_2} 2^{-k})\).

We now wish to extend \(\Phi\) to an increasing and injective application from \(\mathbb{R}\) into the set...
of invariant subspaces (or equivalently orbits) of $T$. For $t \in \mathbb{R}$ set

$$\Phi(t) = \bigcup_{k \in I} \Phi(n + \sum_{k \in I} 2^{-k}),$$

where the union is taken over all the numbers of the form $n + \sum_{k \in I} 2^{-k}$ which are less or equal to $t$. This is clearly an invariant subspace of $T$, and if $t < s$ obviously $\Phi(t) \subseteq \Phi(s)$. If $t < s$, there exist two numbers of the form $n + \sum_{k \in I} 2^{-k}$ such that

$$t < n_1 + \sum_{k \in I_1} 2^{-k} < n_2 + \sum_{k \in I_2} 2^{-k} < s.$$          
Hence $\Phi(t) \subseteq \Phi(n_1 + \sum_{k \in I_1} 2^{-k}) \subseteq \Phi(n_2 + \sum_{k \in I_2} 2^{-k}) \subseteq \Phi(s)$, and thus $\Phi$ is increasing and injective.

6. Orbit-reflexive operators: proof of Theorem 1.5

Let $T$ be the operator constructed in Section 4. In order to show that $T$ is not orbit-reflexive, it suffices to exhibit an operator $A$ which has the property that $Ax \in \overline{\text{Orb}(x, T)}$ for every $x \in H$, but $A$ does not commute with $T$. The natural idea would be to consider $A$ defined by $Ae_0 = 0$ and $Ae_i = e_{i+1}$ for $i \geq 1$. Unfortunately this operator can be unbounded: suppose for instance that $p_{1,n} = 1$: $f_{c_{1,n}} = \gamma_n^{-1}(e_{c_{1,n}} - e_0)$,

$$Af_{c_{1,n}} = \gamma_n^{-1}e_{c_{1,n}+1} = f_{c_{1,n}+1} + \gamma_n^{-1}e_1$$

and thus $A$ is unbounded. A way to circumvent this difficulty is to modify the construction of $T$ and to take for the $p_{k,n}$’s polynomials whose 0-coefficient vanishes: let $(p_{k,n})_{1 \leq k \leq k_n}$ be a $4^{-\nu_n}$-net of the set of polynomials $p$ of degree less than $\nu_n$ such that $|p| \leq 1$ and $p(0) = 0$. Then the definition of $f_j$ for $j \geq 1$ in the (b)- and (c)-working intervals depends only on $e_j$ for $j \geq 1$. Since $Ae_j = Te_j$ for $j \geq 1$, this yields that $Af_j = Tf_j$ for every $j \geq 1$. Hence

**Fact 6.1.** — The operator $A$ is bounded on $H$.

Remark that with this choice of the polynomials $p_{k,n}$, $T$ is no longer hypercyclic. Clearly $A$ and $T$ do not commute, since $TAE_0 = 0$ while $ATE_0 = Ae_1 = e_2$. Theorem 1.5 is a direct consequence of this and the next proposition.

**Proposition 6.2.** — For every $x \in H$, $Ax$ belongs to the closure of the orbit of $x$ under the action of $T$.

**Proof.** — • If $\langle x, e_0 \rangle = 0$, i.e $x = \sum_{j=1}^{+\infty} x_j f_j$, then $Ax = Tx$.

• If $\langle x, e_0 \rangle = \alpha \neq 0$, then for every $n \geq 1$, $\pi_{0,\xi_n} x = \alpha e_0 + \sum_{j=1}^{\xi_n} e_j^{(n)}(x) e_j$, so if $n$ is large enough $|e_j^{(n)}(x)| \geq 2^{-n}$. Hence assumption (*) of Proposition 1.8 is satisfied. Using the notation of the proof of Proposition 1.8

$$||q(T)\pi_{0,\xi_n} x - e_1|| \leq \frac{3}{\xi_n},$$
where \( q \) is of degree less than \( \xi_n + b_n \), \( |q| < 1 \) and \( \zeta^{b_n+1} \) divides \( q(\zeta) \). In particular \( \zeta \) divides \( q(\zeta) \), so with our definition of the polynomials \( p_{k,n} \), there exists a \( k \leq k_n \) such that \( |q - p_{k,n}| \leq 4^{-\nu_n} \). Then the proof of Proposition 4.8 shows that
\[
||T^{c_k,n}x - e_1|| \leq \frac{7}{\xi_n},
\]
and hence \( e_1 \) belongs to the closure of the orbit of \( x \). But the orbit of \( e_1 \) under \( T \) is the linear span \( H_0 \) of the vectors \( f_j \), \( j \geq 1 \). This implies that the closure of \( \text{Orb}(x, T) \) contains \( H_0 \). Since \( Ax \) belongs to \( H_0 \), \( Ax \) belongs to \( \overline{\text{Orb}}(x, T) \), and this finishes the proof of Proposition 6.2.

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