Asynchronous Games 3
An Innocent Model of Linear Logic

Paul-André Melliès

Equipe Preuves Programmes Systèmes
CNRS et Université Denis Diderot
Paris, France

Abstract
Since its early days, deterministic sequential game semantics has been limited to linear or polarized fragments of linear logic. Every attempt to extend the semantics to full propositional linear logic has bumped against the so-called Blass problem, which indicates (misleadingly) that a category of sequential games cannot be self-dual and cartesian at the same time. We circumvent this problem by considering (1) that sequential games are inherently positional; (2) that they admit internal positions as well as external positions. We construct in this way a sequential game model of propositional linear logic, which incorporates two variants of the innocent arena game model: the well-bracketed and the non well-bracketed ones.

Keywords: Game semantics, linear logic, categorical models.

Foreword
This paper does not simply introduce our innocent model of propositional linear logic. It also explains in detail the conceptual stages which brought it to existence. We hope that this presentation will satisfy a categorically-minded audience. The paper is organized in six sections. We start by recalling André Joyal’s category $\mathcal{Y}$ of Conway games and winning strategies (Section 1). We prove that the category $\mathcal{Y}$ does not have binary products (Section 2). This fact is well-known, but the proof does not appear anywhere in full details. We then reduce the Blass problem to the fact that the linear continuation monad $A \mapsto ((A \multimap \bot) \multimap \bot)$ is strong but not commutative on Conway games. Finally, after a crash course on asynchronous games (Section 4), we construct a linear continuation monad equivalent to the identity functor, by
allowing internal positions in our games. This circumvents the Blass problem, and defines a model of linear logic (Section 5). We conclude (Section 6).

1 Introduction: Conway games

Twenty-five years ago, André Joyal realized after a lecture by John H. Conway on surreal numbers, that he could construct a category $Y$ with Conway games as objects, and winning strategies as morphisms, composed by sequential interaction. The construction appears in an article of 7 pages, written in French, and published in 1977 in the *Gazette des Sciences Mathématiques du Québec* [7]. Since it is extremely difficult to get a copy of the *Gazette* today, we find useful to recall below André Joyal’s construction of the category $Y$ of Conway games.

Before explaining the category, it may be worth discussing briefly what makes the category $Y$ so interesting today. Two reasons at least. Historically, it is a precursor of game semantics for proof-theory and programming languages. Conceptually, it is a self-dual category of sequential games. We are particularly interested in this last point here. The categories of games considered today are generally symmetric monoidal closed, with a tensor product (noted $\otimes$) and a monoidal closure (noted $\rightarrowtail$). Except for a few exceptions, they are not self-dual. In contrast, the category $Y$ is $\ast$-autonomous, that is, symmetric monoidal closed, with a dualizing object $\bot$ making the canonical morphism:

$$A \rightarrow ((A \rightarrow \bot) \rightarrow \bot)$$

an isomorphism in the category $Y$, for every Conway game $A$. Since we are looking for game models of full propositional linear logic, and since linear logic is based on a duality between proofs and counter-proofs, we find extremely instructive to study the category $Y$ more closely. For the reader’s comfort, we will recast the original set-theoretic formulation of Conway games [7] in a graph-theoretic style. This choice is also made in the recent account of (money) games by André Joyal [8]. This may not be the best presentation, but it clarifies the connections with our own game-theoretic model of linear logic, given in Section 5.

**Conway games.** A Conway game is an oriented graph $(V, E, \lambda)$ consisting of a set $V$ of vertices, a set $E \subseteq V \times V$ of edges, and a function $\lambda : E \rightarrow \{-1, +1\}$ associating a polarity $-1$ or $+1$ to every edge of the graph. The vertices are called the positions of the game, and the edges its moves. Intuitively, a move $m \in E$ is played by Player when $\lambda(m) = +1$ and by Opponent when $\lambda(m) = -1$. As is usual in graph-theory, we write $x \to y$ when $(x, y) \in E$. 
and call path any sequence of positions $s = (x_0, x_1, ..., x_k)$ in which $x_i \rightarrow x_{i+1}$ for every $i \in \{0, ..., k - 1\}$. In that case, we write $s : x_0 \rightarrow x_k$ to indicate that $s$ is a path from the position $x_0$ to the position $x_k$.

In order to be a Conway game, the graph $(V, E, \lambda)$ is required to verify two additional properties:

- the graph is rooted: there exists a position $*$ called the root of the game, such that for every other position $x \in V$, there exists a path from the root $*$ to the position $x$:

  $$* \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \cdots \rightarrow x_k \rightarrow x,$$

- the graph is well-founded: every sequence of positions

  $$* \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots$$

starting from the root is finite.

A path $s = (x_0, x_1, ..., x_k, x_{k+1})$ is called alternating when:

$$\forall i \in \{1, ..., k - 1\}, \quad \lambda(x_i \rightarrow x_{i+1}) = -\lambda(x_{i-1} \rightarrow x_i).$$

A play is defined as a path $s : * \rightarrow x$ starting from the root. The set of plays of a Conway game $A$ is denoted $P_A$.

**Winning strategies.** A strategy $\sigma$ of the Conway game $(E, V, \lambda)$ is defined as a set of alternating plays such that, for every positions $x, y, z, z_1, z_2$:

(i) the empty play ($*$) is element of $\sigma$,
(ii) every play $s \in \sigma$ starts by an Opponent move, and ends by a Player move,
(iii) for every play $s : * \rightarrow x$, for every Opponent move $x \rightarrow y$ and Player move $y \rightarrow z$,

$$\quad * \xrightarrow{s} x \rightarrow y \rightarrow z \in \sigma \quad \Rightarrow \quad * \xrightarrow{s} x \in \sigma,$$

(iv) for every play $s : * \rightarrow x$, for every Opponent move $x \rightarrow y$ and Player moves $y \rightarrow z_1$ and $y \rightarrow z_2$,

$$\quad * \xrightarrow{s} x \rightarrow y \rightarrow z_1 \in \sigma \text{ and } * \xrightarrow{s} x \rightarrow y \rightarrow z_2 \in \sigma \quad \Rightarrow \quad z_1 = z_2.$$

Thus, a strategy is a set of plays closed under even-length prefix (Clause 3) and deterministic (Clause 4). A strategy $\sigma$ is called winning when for every play $s : * \rightarrow x$ element of $\sigma$ and every Opponent move $x \rightarrow y$, there exists
a position \( z \) and a Player move \( y \to z \) such that the play
\[
* \to s \to x \to y \to z
\]
is element of the strategy \( \sigma \). Note that the position \( z \) is unique in that case, by determinism. We write \( \sigma : A \) to mean that \( \sigma \) is a winning strategy of \( A \).

**Duality and tensor product.** The dual \( A^\perp \) of a Conway game \( A = (V, E, \lambda) \) is the Conway game \( A^\perp = (V, E, -\lambda) \) obtained by reversing the polarities of moves. The tensor product \( A \otimes B \) of two Conway games \( A \) and \( B \) is the Conway game defined below:

- its positions are the pairs \((x, y)\) noted \( x \otimes y \) of a position \( x \) of the game \( A \) and a position \( y \) of the game \( B \),
- its moves from a position \( x \otimes y \) are of two kinds:
  \[
  x \otimes y \to \begin{cases} 
  u \otimes y & \text{if } x \to u, \\
  x \otimes v & \text{if } y \to v,
  \end{cases}
  \]
  - the move \( x \otimes y \to u \otimes y \) is noted \((x \to u) \otimes y \) and has the polarity of the move \( x \to u \) in the game \( A \); the move \( x \otimes y \to x \otimes v \) is noted \( x \otimes (y \to v) \) and has the polarity of the move \( y \to v \) in the game \( B \).

Every play \( s \) of the tensor product \( A \otimes B \) of two Conway games \( A \) and \( B \) may be projected to a play \( s_{|A} \in P_A \) and to a play \( s_{|B} \in P_B \). The Conway game \( 1 = (\emptyset, \emptyset, \lambda) \) has an empty set of positions and moves.

**The category \( Y \) of Conway games.** The category \( Y \) has Conway games as objects, and winning strategies of \( A^\perp \otimes B \) as morphisms \( A \to B \). The identity strategy \( \text{id}_A : A^\perp \otimes A \) copycats every move received in one component \( A \) to the other component. The composite of two strategies \( \sigma : A^\perp \otimes B \) and \( \tau : B^\perp \otimes C \) is the strategy \( \tau \circ \sigma : A^\perp \otimes C \) obtained by letting the strategies \( \sigma \) and \( \tau \) react to a Player move in \( A \) or to an Opponent move in \( C \), possibly after a series of internal exchanges in \( B \).

A formal definition of identities and composition is also possible, but it requires to introduce a few notations. A play is called \( \text{legal} \) when it is alternating and when it starts by an Opponent move. The set of legal plays is denoted \( L_A \). The set of legal plays of even-length, or equivalently ending by a Player move, is denoted \( L_A^{\text{even}} \). The identity of the Conway game \( A \) is the strategy below:
\[
\text{id}_A = \{ s \in L_A^{\text{even}}_{A^\perp \otimes A} \mid \forall t \in L_A^{\text{even}}_{A^\perp \otimes A}, t \text{ is prefix of } s \Rightarrow t_{|A^\perp} = t_{|A} \}.
\]
The composite of two strategies $\sigma : A^\perp \otimes B$ and $\tau : B^\perp \otimes C$ is the strategy of $\tau \circ \sigma : A^\perp \otimes C$ below:

$$\tau \circ \sigma = \{ s \in L_{A^\perp \otimes C}^{\text{even}} \mid \exists t \in P_{A \otimes B \otimes C}, t|_{A,B} \in \sigma, t|_{B,C} \in \tau, t|_{A,C} = s \}.$$  

The tensor product between Conway games gives rise to a bifunctor on the category $Y$, which makes the category $Y$ $\ast$-autonomous, that is, symmetric monoidal closed, with a dualizing object noted $\perp$. The category $Y$ is more than just $\ast$-autonomous: it is compact closed, in the sense that there exists an isomorphism $(A \otimes B)^\perp \cong A^\perp \otimes B^\perp$ natural in $A$ and $B$. As in any compact closed category, the dualizing object $\perp$ is isomorphic to the identity object of the monoidal structure, in that case the Conway game $1$. Thus, the monoidal closure $A^\perp \otimes \perp$ is isomorphic to $A^\perp$, for every Conway game $A$.

2 Key observation: the category $Y$ does not have binary products

The category $Y$ has been rediscovered at the beginning of the 90's in the context of linear logic and programming language semantics. As a $\ast$-autonomous category, the category $Y$ defines a model of Multiplicative Linear Logic (MLL). In this model, every closed formula $F$ of MLL is interpreted as a Conway game $[F]$; and every proof $\pi$ of the formula $F$ is interpreted as a winning strategy $[\pi]$ of the Conway game $[F]$. This interpretation provides a precise and lively picture of proofs, understood as symbolic device interacting during cut-elimination.

Because MLL is only a small fragment of linear logic, many authors have tried to adapt the category $Y$ in order to capture larger or more interesting fragments of the logic. One particularly resistant fragment is Multiplicative Additive Linear Logic (MALL) which is MLL extended with the additive connectives $\oplus$ and $\&$ and constants $0$ and $\top$. Every $\ast$-autonomous category with finite products defines a model of MALL. Alas, the category $Y$ does not have binary products. To our knowledge, the proof of this well-known fact appears nowhere in the litterature. We thus give it below, after introducing the subcategory $Y^-$ of negative Conway games.

Negative Conway games. A Conway game $A$ is called negative when every nonempty play of $A$ starts by an Opponent move. The category $Y^-$ is defined as the full subcategory of $Y$, whose objects are the negative Conway games. The category $Y^-$ is symmetric monoidal closed. The symmetric monoidal structure is inherited from the category $Y$, while the monoidal closure of $Y^-$ is slightly different. The category $Y^-$ is introduced here because it has finite
products. The terminal object of the category is the Conway game 1. The cartesian product of two negative Conway games $A$ and $B$ is the negative Conway game noted $A \& B$, and defined below:

- its set of positions is the disjoint sum of the set of positions of $A$ and the set of positions of $B$, in which the two roots $\ast_A$ and $\ast_B$ of $A$ and $B$ are identified as the root $\ast_{A \& B}$ of $A \& B$. This construction is similar to lifted sum in domain theory,

- its Opponent moves from the root position $\ast_{A \& B}$ are of two kinds:

  $$\ast_{A \& B} \rightarrow \begin{cases} x & \text{if } \ast_A \rightarrow x \text{ in the Conway game } A, \\ y & \text{if } \ast_B \rightarrow y \text{ in the Conway game } B, \end{cases}$$

- its moves from a position $x$ in the component $A$ are exactly the moves from $x$ in the Conway game $A$, with the same polarity:

  $$x \rightarrow y \text{ in the game } A \& B \iff x \rightarrow y \text{ in the game } A.$$ 

- its moves from a position $x$ in the component $B$ are exactly the moves from $x$ in the Conway game $B$, with the same polarity.

It is not difficult to see that the game $A \& B$, equipped with the accurate projection strategies $A \& B \rightarrow A$ and $A \& B \rightarrow B$, defines a cartesian product of $A$ and $B$ in the category $Y^-$. The end of the section is devoted to the proof that:

**Proposition 2.1** The category $Y$ does not have binary products.

**Proof.** The forgetful functor $U : Y^- \rightarrow Y$ has a right adjoint $\text{Neg} : Y \rightarrow Y^-$ which associates to every Conway game $A = (V, E, \lambda)$ the negative Conway game $\text{Neg}(A) = (V', E', \lambda)$ obtained by removing every Player move starting from the root $\ast$:

$$E' = E \setminus \{ (\ast, x) \in E \mid \lambda(\ast \rightarrow x) = +1 \},$$

then removing every position in $V$ not accessible from the root in the graph $(V, E')$:

$$V' = \{ x \in V \mid \text{there exists a path in } (V, E') \text{ from the root to } x \}.$$ 

As a right adjoint, the functor $\text{Neg}$ preserves limits. We proceed by contradiction, and suppose that every pair of Conway games $A$ and $B$ has a cartesian product noted $A \times B$ in the category $Y$. Then, the image $\text{Neg}(A \times B)$ of
this product is isomorphic to the cartesian product \( \text{Neg}(A) \& \text{Neg}(B) \) in the category \( Y^- \).

Now, a Conway game \( A \) is called positive when its dual \( A^\perp \) is negative. We claim that the cartesian product of two positive Conway games \( A \) and \( B \) in \( Y \) is positive \( A \times B \). Note that a Conway game \( A \) is positive iff \( \text{Neg}(A) = 1 \). The negative game \( \text{Neg}(A \times B) \) associated to the product of two positive games \( A \) and \( B \) is equal to \( \text{Neg}(A) \& \text{Neg}(B) = 1 \& 1 = 1 \). The game \( A \times B \) is thus positive, as claimed.

Let \( Y^+ \) denote the full subcategory of \( Y \) consisting of positive Conway games. Since \( Y^+ \) is a full subcategory of \( Y \), we have just established that if \( Y \) has binary products, then \( Y^+ \) has binary products as well. We conclude our proof of Proposition 2.1 by showing that \( Y^+ \) does not have binary products.

Consider the negative game \( \mathbb{B} \) interpreting the booleans, with four positions \( \ast, q, \text{true}, \text{false} \), an Opponent move \( \ast \rightarrow q \) and two Player moves \( q \rightarrow \text{true} \) and \( q \rightarrow \text{false} \). Let \( X = \mathbb{B}^\perp \) denote the positive game obtained by dualizing \( \mathbb{B} \). Consider two positive games \( A \) and \( B \), and suppose that \( A \times B \) is their cartesian product in \( Y^+ \). Let the morphism \( \sigma_{\text{true}} : X \rightarrow A \) in the category \( Y^+ \) denote the smallest strategy of \( B \otimes A \) containing the play:

\[
\ast_{\mathbb{B}} \otimes \ast_A \rightarrow q \otimes \ast_A \rightarrow \text{true} \otimes \ast_A.
\]

Similarly, let \( \tau_{\text{bool}} : X \rightarrow B \) denote the smallest strategy of \( \mathbb{B} \otimes B \) containing the play:

\[
\ast_{\mathbb{B}} \otimes \ast_B \rightarrow q \otimes \ast_B \rightarrow \text{bool} \otimes \ast_B,
\]

where bool is either the position true or false. Let \( \langle \sigma_{\text{true}}, \tau_{\text{false}} \rangle : X \rightarrow A \times B \) denote the unique morphism in \( Y^+ \) such that

\[
\sigma_{\text{true}} = \pi_1 \circ \langle \sigma_{\text{true}}, \tau_{\text{false}} \rangle \quad \tau_{\text{false}} = \pi_2 \circ \langle \sigma_{\text{true}}, \tau_{\text{false}} \rangle
\]

for \( \pi_1 : A \times B \rightarrow A \) and \( \pi_2 : A \times B \rightarrow B \) the projections associated to the Conway games \( A \) and \( B \) in \( Y^+ \). A careful inspection of (1) establishes that the strategy \( \langle \sigma_{\text{true}}, \tau_{\text{false}} \rangle \) contains a play of the form:

\[
\ast_{\mathbb{B}} \otimes \ast_{A \times B} \rightarrow q \otimes \ast_{A \times B} \rightarrow q \otimes x.
\]

for a position \( x \) and a Player move \( \ast \rightarrow x \) of the game \( A \times B \); and that the strategy \( \pi_1 : (A \times B)^\perp \otimes A \) contains a play of the form

\[
\ast_{A \times B} \otimes \ast_A \rightarrow x \otimes \ast_A \rightarrow y_1 \otimes \ast_A
\]

for a position \( y_1 \) and an Opponent move \( \ast \rightarrow y_1 \) of the game \( A \times B \). Similarly, the strategy \( \pi_2 : (A \times B)^\perp \otimes B \) contains a play of the form

\[
\ast_{A \times B} \otimes \ast_B \rightarrow x \otimes \ast_B \rightarrow y_2 \otimes \ast_B
\]
for a position $y_2$ and an Opponent move $* \rightarrow y_2$ of the game $A \times B$. Note that the two positions $y_1$ and $y_2$ may be equal in $A \times B$. Now, let $\nu : X \rightarrow A \times B$ denote the smallest strategy containing the play:

\[ *_B \otimes *_{A \times B} \rightarrow q \otimes *_{A \times B} \rightarrow \text{true} \otimes *_{A \times B}. \]

And let $\nu' : X \rightarrow A \times B$ denote the smallest strategy containing all the plays of the form:

\[ *_B \otimes *_{A \times B} \rightarrow q \otimes *_{A \times B} \rightarrow q \otimes x \rightarrow q \otimes y \rightarrow \text{true} \otimes y \]

for $y$ a position such that $x \rightarrow y$ is an Opponent move of $A \times B$. The two equalities

\[ \pi_1 \circ \nu = \sigma_{\text{true}} = \pi_1 \circ \nu' \]

\[ \pi_2 \circ \nu = \tau_{\text{true}} = \pi_2 \circ \nu'. \]

follow immediately from these definitions. So, there exists more than one morphism $X \rightarrow A \times B$ making the cartesian diagrams commute for $\sigma_{\text{true}} : X \rightarrow A$ and $\tau_{\text{true}} : X \rightarrow B$. We conclude that the category $Y^+$ does not have binary products. This concludes the proof of Proposition 2.1.

\[ \square \]

**Remark:** there is another more direct way to establish that the category $Y$ does not have finite products, which is to show that the category $Y$ does not have a terminal object. This alternative argument is less conclusive however, since it is possible to add formally an initial and a terminal object to the category $Y$, without breaking self-duality.

### 3 A categorical formulation of the Blass problem

We have just seen in section 2 that

- the category $Y$ is $*$-autonomous but does not have finite products,
- its subcategory $Y^-$ of negative Conway games is symmetric monoidal closed and has all products.

This explains why game semantics is generally more concerned with variants of the category $Y^-$ than with variants of the category $Y$. Prima facie, self-duality is less important than cartesianity in order to interpret a programming language built on top of the $\lambda$-calculus. Besides, it is much simpler to interpret the exponential modality $!$ of linear logic in the category $Y^-$ (or a variant) than in the category $Y$. By starting from the category $Y^-$, one obtains a model of Intuitionistic Linear Logic (ILL) whose categorical axiomatization ensures that
the kleisli category associated to the comonad $!$ is cartesian closed, and thus defines a model of the simply-typed $\lambda$-calculus with products, see [14,5,10] among other works. By selecting among variants of the category $Y^-$, and among variants of the comonad, one generates a wide range of models of the $\lambda$-calculus, some of them capturing the essence of particular syntactic programming languages (cf. the full abstraction results).

The methodology is nice and fruitful. We claim however that the lack of self-duality of the category $Y^-$ is a serious conceptual limitation of game semantics. Our ambition here is to clarify the foundations of the subject, by reunderstanding $Y^-$ as part of a larger $\ast$-autonomous category $Z$ with products and coproducts. In this section, we try to deduce the general shape of the category $Z$ from a categorical reformulation of the so-called Blass problem. We proceed by keeping the symmetry between the category $Y^-$ and its opposite category $Y^+$ as far as possible, in order to let unexpected structures emerge from the symmetry. This prepares Section 5, in which we construct a candidate for the category $Z$, a category of asynchronous games and innocent strategies.

First adjunction between lifting functors. We start our analysis by the so-called lifting functor $\downarrow : Y^- \to Y^+$ which associates to every negative Conway game $A$, the positive Conway game $\downarrow A$ defined below:

- a position of $\downarrow A$ is a position of $A$ or a new position $\ast$,
- the only move from $\ast$ is the Player move $\ast \to \ast A$ to the root $\ast A$ of $A$,
- the moves from a position in $A$ are the same as in $A$, with same polarities.

By duality, there is a lifting functor $\uparrow : Y^+ \to Y^-$ defined by the equation $\uparrow A = (\downarrow (A^\perp))^\perp$. Interestingly, the functor $\downarrow$ is left adjoint to the functor $\uparrow$. What this adjunction means on Conway games is very simple. Consider a negative Conway game $A$, and a positive Conway game $B$. The elements of $Y^+(\downarrow A, B)$ and of $Y^-(A, \uparrow B)$ are the winning strategies of $\uparrow (A^\perp) \otimes B$ and the winning strategies of $A^\perp \otimes \uparrow B$, respectively. Note that both $A^\perp$ and $B$ are positive Conway games. So, the plays starting by an Opponent move are the same in the Conway games $\uparrow (A^\perp) \otimes B$ and $A^\perp \otimes \uparrow B$: in each case, the dummy move followed by a play in $A^\perp \otimes B$. This induces a bijection between the set of strategies $Y^+(\downarrow A, B)$ and $Y^-(A, \uparrow B)$ which is natural in $A$ and $B$. From this follows that $\downarrow$ is left adjoint to $\uparrow$. This adjunction induces a monad on $Y^-$ and a comonad on $Y^+$, obtained by lifting every game twice. Note that variants of the monad on $Y^-$ have been already observed, typically in the litterature on arena games.
**Second adjunction between lifting functors.** From now on, we focus on another adjunction \( \uparrow \dashv \downarrow \) which follows from the adjunction \( \downarrow \dashv \uparrow \), and which plays a fundamental role in the formulation of games as continuation passing style models. Note that the category \( Y^+ \) has coproducts, since its opposite category \( Y^- \) has products. From this follows that the functor \( \downarrow \) factors as:

\[
Y^- \xrightarrow{(1)} \Sigma Y^- \xrightarrow{(2)} Y^+
\]

where \( \Sigma Y^- \) is the free completion of \( Y^- \) with respect to coproducts. This completion is also called the *family construction* in [2]. We recall that:

- an object of \( \Sigma Y^- \) is a family \( \{ A_i \mid i \in I \} \) of negative Conway games \( A_i \), indexed by the elements of a set \( I \),
- a morphism \( \{ A_i \mid i \in I \} \to \{ B_j \mid j \in J \} \) consists of a *reindexing function* \( f : I \to J \) and of a winning strategy \( \sigma_i : A_i \to B_{f(i)} \), for each index \( i \in I \).

Dually, the lifting functor \( \uparrow \) factors as:

\[
Y^+ \xrightarrow{(3)} \Pi Y^+ \xrightarrow{(4)} Y^-
\]

where \( \Pi Y^+ \) is the free completion of \( Y^- \) with respect to products. Note that the category \( \Pi Y^+ \) is the opposite of the category \( \Sigma Y^- \).

By composing the resulting functors together, one obtains two new “lifting” functors \( \uparrow \) and \( \downarrow \) defined below:

\[
\uparrow : \Sigma Y^- \xrightarrow{(2)} Y^+ \xrightarrow{(3)} \Pi Y^+, \quad \downarrow : \Pi Y^+ \xrightarrow{(4)} Y^- \xrightarrow{(1)} \Sigma Y^-.
\]

Our notation \( \uparrow \) and \( \downarrow \) for the lifting functors indicates already that we consider \( \Sigma Y^- \) as a category of positive games, and \( \Pi Y^+ \) as a category of negative games. Typically, we like to think of an object of \( \Sigma Y^- \), presented as a family \( \{ A_i \mid i \in I \} \) of negative games, as a positive game whose initial moves by Player are the indices \( i \in I \). We come back to this point later in the Section.

Interestingly, the functor \( \uparrow \) is left adjoint to the functor \( \downarrow \). Indeed, consider a family \( A = \{ A_i \mid i \in I \} \) of negative Conway games, and a family \( B = \{ B_j \mid j \in J \} \) of positive Conway games. The family \( A \) is transported (or lifted) by \( \uparrow \) to the singleton family consisting of the positive Conway game \( \bigoplus_i A_i \), where \( \bigoplus \) denotes the coproduct in \( Y^+ \). Dually, the family \( B \) is transported (or lifted) by \( \downarrow \) to the singleton family consisting of the negative
Conway game $\&_j \uparrow B_j$. Now, we have a series of bijections between sets:

$$\Sigma Y^-(A, \downarrow B) \cong \Pi_{i \in I} Y^-(A_i, \&_{i \in I} \uparrow B_j) \text{ by definition of } \Sigma Y^-,$$

$$\cong \Pi_{(i,j) \in I \times J} Y^-(A_i, B_j) \text{ because } \& \text{ is product in } Y^-,$$

$$\cong \Pi_{(i,j) \in I \times J} Y^+(\downarrow A_i, B_j) \text{ thanks to the adjunction } \downarrow \dashv \uparrow,$$

$$\cong \Pi_{j \in J} Y^+= \left( \oplus_{i \in I} \downarrow A_i, B_j \right) \text{ because } \oplus \text{ is coproduct in } Y^+,$$

$$\cong \Pi Y^+\left( \uparrow A, B \right) \text{ by definition of } \Sigma Y^+,$$

whose naturality in $A$ and $B$ is easily established.

$\Sigma Y^-$ as a linear continuation category. As free completion of a symmetric monoidal closed category with products, the category $\Sigma Y^-$ is symmetric monoidal closed. The functor:

$$(- \otimes -) : \Sigma Y^- \times \Sigma Y^- \longrightarrow \Sigma Y^-$$

is defined on the families of positive Conway games, as follows:

$$(2) \{A_i \mid i \in I\} \otimes \{B_j \mid j \in J\} = \{A_i \otimes B_j \mid (i, j) \in I \times J\}.$$  

The monoidal closure $A \rightarrow{} B$ is defined as follows:

$$(3) \{A_i \mid i \in I\} \rightarrow{} \{B_j \mid j \in J\} = \{\&_{i \in I}(A_i \otimes B^f(i)) \mid f \in I \rightarrow J\}.$$  

So, the initial Player moves of the Conway game $A \rightarrow{} B$ (equivalently, the indices of the family $A \rightarrow{} B$) are the set-theoretic functions $f$ from the set $I$ of initial Player moves in $A$, to the set $J$ of initial Player moves in $B$. This way of defining the initial moves of $A \rightarrow{} B$ does not fit in with the general philosophy of game semantics, which is to avoid “extensional” constructions like set-theoretic function spaces. Quite fortunately, one may specialize the construction to the case where $B = \bot$ is the singleton family with the empty Conway game 1 as unique element. This defines what one calls a linear continuation category, that is, a symmetric monoidal category with finite coproducts distributive over the tensor product, and an exponentiable object $\bot$. Besides, the resulting endofunctor $A \mapsto (A \rightarrow{} \bot)$ of the category $\Sigma Y^-$ coincides with the endofunctor $A \mapsto \downarrow (A^\bot)$.

$\Sigma Y^-$ and $\Pi Y^+$ as categories of central maps. We have indicated that we like to think of the category $\Sigma Y^-$ as a category of positive Conway games. This is justified by the existence of the functor $\Sigma Y^- \longrightarrow Y^+$ mentioned earlier, which transports every family $\{A_i \mid i \in I\}$ of negative games to the positive game with initial moves the indices $i \in I$, followed by the plays of $A_i$. The functor is faithful, and injective of objects. The category $\Sigma Y^-$ is thus
isomorphic to its image in the category $Y^+$, which we note $Y^{+-}$.

The category $Y^{+-}$ may be defined directly as follows. The objects of $Y^{+-}$ are the Conway games in which:

- every initial move in a play is by Player,
- every second move in a play is by Opponent.

The morphisms $A \to B$ of $Y^{+-}$ are the winning strategies $\sigma : A^\perp \otimes B$ such that, for every Player move $*_{A} \to x$ in $A$, there exists a Player move $*_{B} \to y$ in $B$, such that the play $*_{A} \otimes *_{B} \to x \otimes *_{B} \to x \otimes y$ is element of the strategy $\sigma$.

Dually, the functor $\Pi Y^+ \to Y^-$ defines an isomorphism of categories $\Pi Y^+ \cong Y^{--}$ where $Y^{--}$ is defined as the opposite category of $Y^{+-}$. It is not difficult to see that the resulting functors:

$$\uparrow : Y^{+-} \to Y^{--}, \quad \downarrow : Y^{--} \to Y^{+-}.$$ 

coincide with the lifting functors $\uparrow$ and $\downarrow$ restricted to the subcategories $Y^{+-}$ and $Y^{--}$ of $Y^+$ and $Y^-$, respectively. This justifies our notations for $\uparrow$ and $\downarrow$.

Now, let $Y^{--}$ denote the full subcategory of $Y^-$ with the objects of $Y^{--}$, and let $Y^{++}$ denote the full subcategory of $Y^+$ with the objects of $Y^{--}$. By construction, the category $Y^{++}$ is opposite to the category $Y^{--}$.

There is a crucial observation to make here: the category $Y^{--}$ is the co-Kleisli category over the category $Y^{--}$, induced by the comonad $\uparrow \downarrow$. It is not difficult indeed to check that the set $Y^{--}(A, B)$ of morphisms between two negative Conway games $A$ and $B$ of $Y^{--}$, is equal to the set $Y^{--}(\downarrow A, \downarrow B)$ of morphisms in the category $Y^{--}$. This implies that the category $Y^{--}$ is the category of continuations associated to the category $Y^{--}$.

The category $Y^{--}$ thus defines what Peter Selinger calls a (linear) control category in [15]. The category $Y^{--}$ is the category of central maps associated to this control category $Y^{--}$. This is the key to understand together the family construction by Samson Abramsky and Guy McCusker in [2], the polarized presentation of games by Olivier Laurent in [9], and the representation theorem of control categories by Peter Selinger in [15].

The adjunction $\uparrow \downarrow \downarrow$ simulates synchronization. After this long discussion, we are ready to clarify the computational meaning of the adjunction $\uparrow \downarrow \downarrow$. Suppose that $A$ denotes a positive Conway game in $Y^{+-}$, and $B$ a negative Conway game in $Y^{--}$. Every element of $Y^{+-}(A, \downarrow B)$ is a strategy $\sigma$ of $A^\perp \otimes \downarrow B$ which waits for an Opponent move $m : *_{A} \to x$ in $A^\perp$, plays the dummy move in $\downarrow B$ after receiving $m$, waits for an Opponent move $n : *_{B} \to y$ in $B$, and carries on after receiving $n$. Symmetrically, every element of $Y^{--}(\uparrow A, B)$ is a strategy $\tau$ of $\uparrow A^\perp \otimes B$ which waits for an Opponent
move \( n : \ast_B \to y \) in \( B \), plays the dummy move in \( \downarrow A^\perp \) after receiving \( n \), waits for an Opponent move \( m : \ast_A \to x \) in \( A^\perp \), and carries on after receiving \( m \). In both cases, the strategy \( \sigma \) or \( \tau \) waits for the two inputs \( m : \ast_A \to x \) and \( n : \ast_B \to y \), then carries on. In that way, the two strategies \( \sigma \) and \( \tau \) implement the synchronized input of \( m \) in \( A \) and \( n \) in \( B \): the strategy \( \sigma \) simulates synchronization of \( A \) and \( B \) by asking in \( A \) then in \( B \) (in the call-by-value order) whereas the strategy \( \tau \) asks in \( B \) then in \( A \) (in the call-by-name order).

**The Conway game** \( A \to B \). This discussion on synchronization has a categorical counterpart. The functor associated to the adjunction \( \uparrow \dashv \downarrow \):

\[
(Y^{+-})^{\text{op}} \times Y^{-+} \to \text{Set},
\]

factorizes as a functor on Conway games:

\[
(- \to -) : (Y^{+-})^{\text{op}} \times Y^{-+} \to Y^{-+}
\]

postcomposed to the global element functor \( Y^{-+} \to \text{Set} \) which associates to every negative Conway game its set of winning strategies. The Conway game \( A \to B \) is defined just as \( A^\perp \otimes B \) except that the initial Opponent moves are pairs \((m, n)\) of a Player move in \( A \) and an Opponent move in \( B \).

**The Blass problem.** The definition of the Conway game \( A \to B \) coincides with the definition given by Andrea Blass in his game-theoretic account of linear logic [3]. Interestingly, the synchronization of the initial moves is precisely what leads (apparently) to the so-called *Blass problem*. The problem is the following one: there seems to be a natural way to build a category of negative and positive games — unfortunately, this natural construction does work, because it defines to a non-associative structure, see the comprehensive account by Samson Abramsky in [1].

The Blass problem may be reformulated categorically in the following way. As any profunctor, the functor (5) induces a category \( Y_* \) with Conway games of \( Y^{+-} \) and \( Y^{-+} \) as objects, and:

- the morphisms of \( Y^{+-} \) between two positive Conway games \( A \) and \( B \),
- the morphisms of \( Y^{-+} \) between two negative Conway games \( A \) and \( B \),
- the strategies of \( A \to B \) from a positive game \( A \) to a negative game \( B \),
- no morphism from a negative game \( A \) to a positive game \( B \).

The composition law of the category \( Y_* \) is deduced from the composition laws of the categories \( Y^{+-} \) and \( Y^{-+} \), as well as from the functor (5). Associativity is ensured by the bifunctoriality of (5).

The Blass problem arises when one tries to replace the two categories \( Y^{+-} \) and \( Y^{-+} \) in the construction of \( Y_* \), by their kleisli categories \( Y^{+-} \) and \( Y^{-+} \).
Suppose indeed that one tries to compose a morphism \( h_A : A' \rightarrow A \) in the kleisli category \( Y^{+\cdot} \), a strategy \( \sigma : A \rightarrow B \), and a morphism \( \sigma : B \rightarrow B' \) in the co-kleisli category \( Y^{-\cdot} \). This amounts to extending the functor (6) to a functor
\[
(\_ - - \_ - \leftarrow - - \_ - \leftarrow) : (Y^{+\cdot})^{\text{op}} \times Y^{-\cdot} \rightarrow Y^{-\cdot}.
\]
The Blass problem amounts to the fact that there is no such functor (7) but only a functor:
\[
(\_ - - \_ - \leftarrow - - \_ - \leftarrow) : (Y^{+\cdot})^{\text{op}} \otimes Y^{-\cdot} \rightarrow Y^{-\cdot}.
\]
where \((Y^{+\cdot})^{\text{op}} \otimes Y^{-\cdot}\) is a variant of \((Y^{+\cdot})^{\text{op}} \times Y^{-\cdot}\) without the interchange law between composition and tensor product, see [13] for a definition. In other words, the equality:
\[
(id_A \leftarrow h_B) \circ (h_A \leftarrow id_B) = (h_A \leftarrow id_B) \circ (id_A \leftarrow h_B)
\]
is not necessarily verified.

Now, observe that the functors (6) and (2) are related by the natural isomorphism \( A \rightarrow B = (A \otimes B^\bot)^\bot \). Thus, extending the functor \( \rightarrow \) from the categories \( Y^{+\cdot} \) and \( Y^{-\cdot} \) to their kleisli categories \( Y^{+\cdot} \) and \( Y^{-\cdot} \), is just like extending the functor \( \otimes \) from \( Y^{+\cdot} \) to its kleisli category \( Y^{+\cdot} \). This enables to apply this well-known fact of the theory of monads, see [6,13], that the functor \( \otimes \) defines a premonoidal structure on \( Y^{+\cdot} \) because the linear continuation monad \( \downarrow\uparrow \) on the category \( Y^{+\cdot} \), is strong but not commutative.

Towards the category \( Z \). We have just reduced Blass problem to the property that the linear continuation monad \( A \mapsto ((A \rightarrow \bullet \perp) \rightarrow \bullet \perp) \) is strong but not commutative. This provides us with a recipe to get a model of linear logic: find an analogue of the category \( Y^{+\cdot} \) in which the linear continuation monad \( A \mapsto ((A \rightarrow \bullet \perp) \rightarrow \bullet \perp) \) would be \textit{commutative}. More than that: in order to obtain a \( * \)-autonomous category, we want this linear continuation monad to be equivalent (as a monad) to the identity. The category of asynchronous games introduced in Section 5 is designed precisely for that purpose.

4 A crash course on asynchronous game semantics

In this section, we recall the definitions of asynchronous games and innocent strategy given in [12]. The original definition of asynchronous game is adapted in three ways. First, we consider asynchronous games with a well-founded event structure, in order to relate them to Conway games. This is only a detail of presentation, since all our definitions work perfectly well in non well-founded asynchronous games. We also add an incompatibility relation \( # \)
between the moves of the game, in order to interpret the additive connectives and constants of linear logic. Finally, we polarize every position of the game with a payoff in \([-\infty, -1, +1, \infty]\) in order to distinguish between Player positions \((+1, +\infty)\) and Opponent positions \((-1, -\infty)\) as well as between internal positions \((+\infty, -\infty)\) and external positions \((+1, -1)\).

**Event structures.** An event structure \((M, \leq, \#)\) is a partially ordered set \((M, \leq)\) of events equipped with a binary symmetric irreflexive relation \(#\) verifying:

- the set \(m \downarrow = \{ n \in M \mid n \leq m \}\) is finite for every event \(m \in M\),
- \(m \# n \leq p\) implies \(m \# p\) for every events \(m, n, p \in M\).

Two events \(m, n \in M\) are called incompatible when \(m \# n\), and compatible otherwise. Two moves \(m\) and \(n\) are called independent when they are compatible, and different. We write \(m \perp n\) in that case.

**Positions.** A position of an event structure \(A\) is a finite downward closed subset of \((M_A, \leq_A)\), consisting of pairwise compatible events. The set of positions of \(A\) is denoted \(D_A\).

**The positional graph.** Every event structure \(A\) induces a graph \(G_A\) whose nodes are the positions \(x, y \in D_A\), whose edges \(m : x \rightarrow y\) are the events verifying \(y = x + \{m\}\), where + indicates a disjoint union, that is, \(y = x \cup \{m\}\) and the move \(m\) is not element of \(x\). An event structure is called well-founded when the graph \(G_A\) is well-founded.

**Asynchronous games.** An asynchronous game \(A = (M_A, \leq_A, \#_A, \lambda_A, \kappa_A)\) is a well-founded event structure \((M_A, \leq_A, \#_A)\) whose events are called the moves of the game, equipped with a polarity function \(\lambda_A : M_A \rightarrow \{-1, +1\}\) on moves, and a payoff function \(\kappa_A : D_A \rightarrow \{-\infty, -1, +1, +\infty\}\) on positions. A move with polarity \(+1\) (resp. \(-1\)) is called a Player (resp. Opponent) move. A Player (resp. Opponent) position is a position with payoff in \(+1, +\infty\) (resp. in \([-1, -\infty]\)). An external (resp. internal) position is a position with payoff in \(+1, -1\) (resp. in \(+\infty, -\infty\)).

**The underlying Conway game.** The positional graph attached to the asynchronous game \(A\) defines a Conway game \(S_A\), in which the polarity of a move \(x \rightarrow y\) is given by the polarity of the underlying move \(m\) such that \(y = x + \{m\}\) in the asynchronous game \(A\). For simplicity, we write \(P_A\) instead of \(P_{S_A}\) for the set of plays of \(S_A\). There is more structure in \(S_A\) than in a usual Conway game, since every position has a payoff, and moves may be
permuted in plays, as explained below. The set of external positions of \( G_A \) is denoted \( \mathcal{D}_A^o \).

**Homotopy.** Given two paths \( s, s' : x \longrightarrow y \) in \( G_A \), we write \( s \sim^1 s' \) when the paths \( s \) and \( s' \) are of length 2, with \( s = m \cdot n \) and \( s' = n \cdot m \) for two moves \( m, n \in \mathcal{M}_A \). The *homotopy equivalence* \( \sim \) between paths is defined as the least equivalence relation containing \( \sim^1 \), and closed under composition. We also use the notation \( \sim \) in our diagrams to indicate that two (necessarily independent) moves \( m \) and \( n \) are permuted. The word homotopy is justified mathematically by the work on directed homotopy by Philippe Gaucher and Eric Goubault [4]. Indeed, every asynchronous game defines a directed simplicial set, in which directed homotopy between paths coincides with our permutation equivalence \( \sim \).

**Strategy.** A strategy \( \sigma \) of an asynchronous game is a strategy of the underlying Conway game \( G_A \), such that, moreover, every play \( s : * \longrightarrow x \) in the strategy \( \sigma \) has its target position \( x \) of positive payoff: +1 or +\( \infty \). A strategy \( \sigma \) of \( A \) is winning when it is winning in the underlying Conway game \( G_A \). We write \( \sigma : A \) when \( \sigma \) is a winning strategy of the asynchronous game \( A \).

**Innocence.** We reformulate in [12] the usual notion of innocence found in arena games, as follows. A strategy \( \sigma \) is called *innocent*, when it is *side consistent* and *forward consistent* in the following sense.

**Backward consistency.** A strategy \( \sigma \) is *backward consistent* (see Figure 1) when for every play \( s_1 \in P_A \), for every path \( s_2 \), for every moves \( m_1, n_1, m_2, n_2 \in \mathcal{M}_A \), it follows from

\[
s_1 \cdot m_1 \cdot n_1 \cdot m_2 \cdot n_2 \cdot s_2 \in \sigma \quad \text{and} \quad \neg (n_1 \vdash_A m_2) \quad \text{and} \quad \neg (m_1 \vdash_A m_2)
\]

that

\[
\neg (n_1 \vdash_A n_2) \quad \text{and} \quad \neg (m_1 \vdash_A n_2) \quad \text{and} \quad s_1 \cdot m_2 \cdot n_2 \cdot m_1 \cdot n_1 \cdot s_2 \in \sigma.
\]

**Forward consistency.** A strategy \( \sigma \) is *forward consistent* (see Figure 2) when for every play \( s_1 \in P_A \) and for every moves \( m_1, n_1, m_2, n_2 \in \mathcal{M}_A \), it follows from

\[
s_1 \cdot m_1 \cdot n_1 \in \sigma \quad \text{and} \quad s_1 \cdot m_2 \cdot n_2 \in \sigma \quad \text{and} \quad m_1 \vdash m_2 \quad \text{and} \quad m_2 \vdash n_1
\]

that

\[
m_1 \vdash n_2 \quad \text{and} \quad n_1 \vdash n_2 \quad \text{and} \quad s_1 \cdot m_1 \cdot n_1 \cdot m_2 \cdot n_2 \in \sigma.
\]
**Positional strategy.** A strategy $\sigma : A$ is called *positional* when for every two plays $s_1, s_2 : *_A \rightarrow x$ in the strategy $\sigma$, and every path $t : x \rightarrow y$ of $G_A$, one has:

$$s_1 \sim s_2 \text{ and } s_1 \cdot t \in \sigma \Rightarrow s_2 \cdot t \in \sigma.$$ 

**Proposition 4.1** ([12]) *Every innocent strategy $\sigma$ is positional.*

Note that every positional strategy is characterized by the set of positions of $\mathcal{D}_A$ it reaches, defined as $\sigma^* = \{x \in \mathcal{D}_A, \exists s \in \sigma, s : *_A \rightarrow x\}$.

5 **An innocent model of full propositional linear logic**

**Lifting of asynchronous games.** The lifting $\downarrow A$ of any asynchronous game $A$ is the asynchronous game defined by lifting the set of moves $\mathcal{M}_A$ with
an Opponent move \( m \), and giving the internal and Player payoff \( +\infty \) to the root \( \star_{A} \) of the asynchronous game \( \Downarrow A \). The operation \( \Uparrow A \) is defined dually.

**Tensor product of asynchronous games.** The tensor product \( A \otimes B \) of two asynchronous games

\[
A = (M_A, \leq_A, \#_A, \lambda_A, \kappa_A) \quad \text{and} \quad B = (M_B, \leq_B, \#_B, \lambda_B, \kappa_B)
\]

is defined by a disjoint sum of polarized event structures

\[
(M_A + M_B, \leq_A + \leq_B, \#_A + \#_B, \lambda_A + \lambda_B).
\]

The underlying Conway game of \( A \otimes B \) is thus equal to the tensor product of the underlying Conway games of \( A \) and \( B \). The payoff \( \kappa_{A \otimes B}(x \otimes y) \) of a position \( x \otimes y \) is given by the table below, in which the payoffs \( \kappa_A(x) \) and \( \kappa_B(y) \) appear in the first row and first column.

|   | -\infty | -1 | +1 | +\infty |
|---|---------|----|----|---------|
| -\infty | -\infty | -\infty | -\infty | -\infty |
| -1 | -\infty | -\infty | -1 | +\infty |
| +1 | -\infty | -1 | +1 | +\infty |
| +\infty | -\infty | +\infty | +\infty | +\infty |

Note that the table is symmetric in \( A \) and \( B \), and that the tensor product of an internal position with another position is always internal.

**Linear games.** A linear game is defined as a pair \( \{ \pi \mid A_i \mid i \in I \} \) consisting of a polarity \( \pi \in \{+1, -1\} \) and of a family \( (A_i)_{i \in I} \) of asynchronous games indexed by \( I \). A position of \( A \) is defined as a position of any of the asynchronous games \( A_i \). The component of a position of \( A \) is the asynchronous game \( A_i \) in which it appears. A position of \( A \) is called initial when it is the root of its component \( A_i \). Every initial position in \( A \) is required to have a positive payoff when \( \pi = +1 \), and a negative payoff when \( \pi = -1 \). A linear game is called negative when \( \pi = -1 \) and positive when \( \pi = +1 \).

**Lifting.** The lifting of a negative game \( A = \{-1 \mid A_i \mid i \in I\} \) is the positive game \( \Downarrow A = \{+1 \mid \&_{i \in I} \Downarrow A_i\} \) consisting of a unique asynchronous game \( \&_{i \in I} \Downarrow A_i \) with polarized event structure the disjoint sum of the polarized event structures of the \( \Downarrow A_i \)'s, with all moves in \( \Downarrow A_i \) and \( \Downarrow A_j \) incompatible when \( i \neq j \). Note that the underlying Conway game of \( \&_{i \in I} \Downarrow A_i \) is the cartesian product of the underlying Conway game of each \( \Downarrow A_i \) in the category \( Y^- \).
The category $\mathbb{Z}$-product of two linear games $A \otimes B$ is deduced by the de Morgan equality: $A \otimes B = (A^\perp \otimes B^\perp) ^\perp$ where duality is defined as expected. Note that the linear game $A \otimes B$ is always positive, and that the linear game $A \otimes B$ is always negative.

**Strategies.** A strategy $\sigma$ of a negative linear game $\{−1 \mid A_i \mid i \in I\}$ is defined as a family $\{\sigma_i \mid i \in I\}$ of strategies $\sigma_i$ of the asynchronous game $A_i$. The strategy $\sigma$ is innocent (resp. winning) when each strategy $\sigma_i$ is innocent (resp. winning).

**External equivalence.** The main idea underlying our model is that two innocent strategies should be identified when they meet the same external positions. The set of external positions of the strategy $\sigma = \{\sigma_i \mid i \in I\}$ on a negative linear game $\{−1 \mid A_i \mid i \in I\}$ is the family $\sigma^o = \{\sigma_i^* \cap D_{A_i}^o \mid i \in I\}$. Two innocent strategies $\sigma$ and $\tau$ of a linear game $A$ are called externally equivalent when $\sigma^o = \tau^o$. We write this $\sigma \simeq_A \tau$.

**The category $\mathbb{Z}$.** The category $\mathbb{Z}$ has linear games as objects, and $\simeq$-equivalence classes of winning innocent strategies of $A \triangleright \triangleright \otimes B$ as morphisms from $A$ to $B$.

**Proposition 5.1** The category $\mathbb{Z}$ is $*$-autonomous and has all products.

We indicate briefly how morphisms $\sigma : A \longrightarrow B$ behave in the category $\mathbb{Z}$.

When $A$ and $B$ are positive, the morphism $\sigma$ is a strategy (modulo $\simeq$) of $A^\perp \otimes \downarrow B$ which thus waits for an initial position $(x, \ast)$ in $A^\perp \otimes \downarrow B$, then plays either $(x, \ast) \rightarrow (x, y)$ where $y$ is an initial position of $B$, or $(x, \ast) \rightarrow (x', \ast)$ where $x'$ is a position of payoff $−\infty$ in $A$. The payoff condition on $x'$ follows from the definition of a strategy $\sigma$, which says that, if played, the position $(x', \ast)$ is necessarily of Player payoff. Since the payoff of $\ast$ is $−\infty$ in $\downarrow B$,
the payoff of \( x' \) is necessarily \( +\infty \). Call a linear game external when all its positions are external. In that case, a morphism between two external positive games \( A \) and \( B \) behaves in the same way as a central map on Conway games, discussed in Section 3. That is, after receiving the initial position \( x \) of \( A \), the strategy \( \sigma \) plays necessarily an initial position \( y \) in \( B \). Interestingly, the monoidal closure of two such external positive games \( A \) and \( B \) in the category \( Z \) is the non-external and negative game \( A \oplus B \) in which Opponent plays an initial position of \( A \), then Player answers with a Player move in \( B \). Thus, a strategy \( A \otimes B \rightarrow C \) in which \( A, B, C \) are external positive, waits for a pair \((x, y)\) of initial positions in \( A \) and \( B \), then plays an initial position in \( C \). Exactly the same can be said of a strategy \( A \rightarrow B \oplus C \). This improves the definition of monoidal closure (4) in a very nice way, since the set-theoretic definition of (4) is simply replaced by altering the order in which Player and Opponent appear in \( A \bullet B \).

When \( A \) and \( B \) are negative, the situation is dual to the previous one.

When \( A \) is positive and \( B \) is negative, the strategy \( \sigma \) (modulo \( \cong \)) waits for a pair of an initial position in \( A \) and an initial position in \( B \), then plays a move in \( A \) or in \( B \).

When \( A \) is negative and \( B \) is positive, the strategy \( \sigma \) (modulo \( \cong \)) plays an initial position in \( A \) or an initial position in \( B \), as long as this position is internal. Indeed, the strategy \( \sigma \) is forbidden to play an external position \( x \) in \( A \) or \( y \) in \( B \) since the resulting position \((x, *)\) or \((*, y)\) would be of payoff \(-\infty \) in the negative game \( \uparrow (A) \downarrow \oplus \uparrow B \).

This ensures that every positive game \( A \) is isomorphic to the negative game \( \uparrow A \). From this follows that \( Z \) has all products, since the full subcategory of negative games in \( Z \) is easily shown to have products, in the same way as the categories \( Y^- \) or \( \Pi Y^+ \).

**A model of linear logic.** The category \( Z \) may be equipped with a linear exponential \( ! \) constructed according to the group theoretic ideas of [11] in order to establish that:

**Proposition 5.2** The category \( Z \) together with the linear exponential \( ! \) defines a model of full propositional linear logic, in the sense of [14,5,10].

Besides, the category \( Z \) incorporates two well-known variants of the innocent arena game model: the well-bracketed and the non well-bracketed ones. More precisely, there are structure preserving functors \( F \) (resp. \( G \)) from the category of arena games and well-bracketed (resp. non well-bracketed) innocent strategies, to the category \( Z \). The two functors \( F \) and \( G \) differ mainly
in the way they translate the boolean arena (noted `bool`). The linear game $F(\text{bool})$ is the positive game $1 \oplus 1$ with two external positions, equivalently the negative game $\uparrow (1 \oplus 1)$ obtained by lifting $1 \oplus 1$ with an internal position $\ast$. The linear game $G(\text{bool})$ is the negative game obtained by lifting $1 \oplus 1$ with an external position $\ast$. The functor $F$ is full, but not faithful, whereas the functor $G$ is full and faithful.

This may be illustrated as follows. The left and right implementations of the `and` operator of type $X = \text{bool} \to \text{bool} \to \text{bool}$ are interpreted as different strategies $\sigma_1$ and $\sigma_2$ in the category of well-bracketed strategies. The functor $F$ transports them to strategies $F(\sigma_1)$ and $F(\sigma_2)$ identified in the linear game $F(X)$, because they hit the same set of external positions in $F(X)$. On the other hand, the functor $G$ transports $\sigma_1$ and $\sigma_2$ to strategies $G(\sigma_1)$ and $G(\sigma_2)$ not identified in the linear game $G(X)$ because all the positions in $G(X)$ are external. Intuitively, the external positions track the “terminal states” in the game $F(X)$, and all the “intermediate states” in the game $G(X)$.

6 Conclusion and future work

The equality $A = \downarrow \uparrow A$ is reminiscent of the Reidemeister moves in tangle diagrams. By imposing the equality, we identify enough strategies in order to obtain a model of propositional linear logic. We conjecture that the category $Z$ (or a close variant) provides a fully complete model of propositional linear logic. In a related line of research, we would like to understand the relationship between the category $Z$ and the free bicompletion of the singleton category, with respect to limits and colimits, as characterized and popularized by André Joyal.

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