WHO CHANGES THE STRING COUPLING?

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ABSTRACT

In general bosonic closed string backgrounds the ghost-dilaton is not the only state in the semi-relative BRST cohomology that can change the dimensionless string coupling. This fact is used to establish complete dilaton theorems in closed string field theory. The ghost-dilaton, however, is the crucial state: for backgrounds where it becomes BRST trivial we prove that the string coupling becomes an unobservable parameter of the string action. For backgrounds where the matter CFT includes free uncompactified bosons we introduce a refined BRST problem by including the zero-modes “x” of the bosons as legal operators on the complex. We argue that string field theory can be defined on this enlarged complex and that its BRST cohomology captures accurately the notion of a string background. In this complex the ghost-dilaton appears to be the only BRST-physical state changing the string coupling.

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1. Introduction and Summary

The soft-dilaton theorem is an old result in critical string theory. It is stated as a property of string amplitudes for on-shell vertex operators. A physical string amplitude involving a zero-momentum dilaton is written in terms of derivatives, with respect to the dimensionless coupling and the slope parameter, of the string amplitude with the dilaton suppressed [1]. It is natural to ask what does this result tell us about the role of the dilaton field in the string action. There has been much work on the role of the dilaton in effective field theory limits of strings. Our interest here is on the role of the dilaton in the complete string action. This line of work began with Yoneya [2] who investigated the dilaton theorem using light cone string field theory. Subsequent studies [3,4,5] considered the dilaton in the context of covariantized light-cone string field theories.

A dilaton state has a component built by acting on the vacuum by operators from the ghost sector. This component is called the ghost-dilaton $|D_g\rangle$, and its relevance for the on-shell dilaton theorem was studied in Ref.[6]. This work was extended recently to prove the off-shell “ghost-dilaton theorem” in covariant quantum closed string field theory [7,8]. This result states that an infinitesimal shift along the zero-momentum ghost-dilaton changes the quantum string action, or more precisely, the path integral string measure, in a way equivalent to a shift in the dimensionless string coupling. This work showed concretely that conformal field theories and string backgrounds are not in one to one correspondence: while the ghost-dilaton deforms the string background, it does not deform the conformal field theory underlying the string background. The string background has a parameter which is absent in the conformal field theory.

A study of the ghost-dilaton alone is not enough to understand how the value of the string coupling can be changed in string theory. In critical string theory the zero-momentum “matter-dilaton” also shifts the coupling constant. Moreover, the properties of the ghost-dilaton depend on the matter sector of the conformal theory. In critical string theory it is a nontrivial BRST-physical state, but in two-dimensional string theory, for example, it becomes BRST-trivial. If the ghost-dilaton is trivial one may think that the string coupling cannot be changed. This is not correct, a shift of the ghost dilaton will always shift the string coupling. What happens is that the string coupling becomes unobservable. Thus the ghost-dilaton plays a fundamental role: if it is trivial the background has no string coupling constant parameter. This is one of the main results of this paper.

Another point we develop in detail is the analysis of conformal field theory deformations and string background deformations induced by the dimension $(1,1)$ primary field $\partial X \cdot \bar{\partial} X$. In critical string theory this is the matter-dilaton. We first consider the analog of this state in a conformal theory containing a single scalar field $X$ living on the real line. We ask if $\partial X \bar{\partial} X$ deforms the conformal theory. The deformation involves integrating the two-form $\partial X \bar{\partial} X dz \wedge d\bar{z}$ over the surface, and this two form can be written as the exterior derivative
\[ d(X\partial Xd\bar{z}) \]. Stokes theorem cannot be used directly because \((X\partial X)\) is not primary, and therefore \((X\partial Xd\bar{z})\) is not a true one-form. Apart from a piece that can be absorbed by a redefinition of the basis of the conformal theory, we show that the deformation amounts to a scaling of correlators with a factor proportional to the Euler number of the underlying surface. Since the conformal theory has non-vanishing central charge, correlators depend on the scale factor of the metric and the deformation simply corresponds to a variation of this scale factor. Strictly speaking, the conformal field theory has been deformed: the correlators of the two theories on any fixed surface cannot be made to agree with a redefinition of the basis of states.

If the matter conformal theory includes twenty six scalars and is coupled to the ghost system, the operator \(\partial X \cdot \bar{\partial} X\) does not deform the conformal theory. This result has been verified earlier to various degrees of completeness in interesting works by Mende \[9\] and Mahapatra \textit{et al.}\[10\]. Indeed, following \[7\] the deformation can be absorbed by a change of basis generated by the ghost-number operator. For integrated correlators this cannot be done, implying that the matter dilaton alters the string coupling. We emphasize, however, that the matter dilaton does not change the value of the dimensionful slope parameter \(\alpha'\).

In addition to the dilaton we also consider the graviton trace \(G\). This state, physical only at zero momentum, is the linear combination of the ghost-dilaton and the matter-dilaton that does not change the string coupling. The graviton trace can be written as \(Q\) acting on the state \(|\xi\rangle\) created by \((cX \cdot \bar{\partial} X - \bar{c}X \cdot \partial X)\). This state is usually considered illegal since it uses the operator \(X\) which is not a scaling field of the conformal theory. We argue in this paper that \(G\) is legally BRST trivial. There is an immediate issue with this interpretation. The failure of \(G\) to decouple \[11\], easily verified in \(\langle G, \text{phys, phys} \rangle \neq 0\), seems to be in contradiction with the claim that \(G\) is BRST trivial: by contour deformation the BRST operator that occurs in \(G\) can be made to act on the physical states giving zero. We show that this is not a correct argument, the point being that correlators involving \(X\) are distributions. Careful use of delta functions confirms that \(G\) does not decouple.

Refining the discussion of Mahapatra \textit{et al.}\[10\] and, in agreement with them, we claim that the graviton trace \(G\) does not change any physical property of the string background. On the face of it this seems puzzling given the failure of decoupling for \(G\). There is no contradiction, however. Strictly speaking, a state leaves the physics of the background invariant if it appears as the inhomogeneous term of a nonlinear string field transformation that can be verified to leave the string action invariant. A state capable of having such behavior need not decouple. For \(G\) this is possible if correlators of operators including \(X\)'s can be defined, and, once defined, obey the standard properties of sewing and action of the BRST operator. We discuss these matters and argue that the requisite nonlinear field transformation is simply a gauge transformation with gauge parameter \(|\xi\rangle\).

Once we accept \(X\) in the gauge parameters we must accept it in the physical states as well. Having a larger set of gauge parameters, we lose some physical states; having a larger state space, we may also gain some new physical states. We formalize this setup via a new
refined cohomology problem: BRST cohomology in the extended (semirelative) complex where $X$ and powers of it are accepted as legal operators. The cohomology of this complex appears to capture accurately the idea of a string background: we lose the states that do not change the string background, as the graviton trace, and we gain no states describing new physics. In this cohomology the matter-dilaton is the same as the ghost-dilaton. Similarly, in two-dimensional string theory we show that the states in the semirelative cohomology that are trivial in the extended complex do not appear to change the string background. A detailed computation of BRST cohomology at various ghost numbers in the extended complex of critical string theory will be presented elsewhere [12].

This paper is organized as follows. In section two we set up notation and conventions. We discuss zero momentum physical states, and the uses of the $X^\mu(z, \bar{z})$ field operator. In section three we define the extended BRST complex, explain the failure of $\mathcal{G}$ to decouple, and argue that the usual BRST action on correlators holds in the extended complex. We summarize the properties of the new BRST cohomology following Ref.[12]. In section four we consider CFT deformations induced by the matter dilaton, and derive formulae for the integration of insertions of matter dilatons over spaces of surfaces. We use these results in section five to give a complete proof of the dilaton theorem in closed string field theory. In section six we prove that whenever the ghost-dilaton is BRST trivial the string coupling constant is not observable. Finally, in section seven we illustrate some of our work in the context of two-dimensional string theory.

2. Zero momentum states and uses of the $X^\mu(z, \bar{z})$ operator.

In this section we begin by enumerating the zero-momentum physical states in critical string theory. This enables us to set our conventions and definitions for the dilaton, the ghost-dilaton, the matter dilaton, and the graviton trace. We then elaborate on the definition of the $X^\mu$ field operator and how it can be used to write all zero momentum states, with the exception of the ghost-dilaton, as BRST trivial states. The ghost-dilaton can be also be written in BRST trivial form, but, as is well known, in this case the gauge parameter is not annihilated by $b_0^-$. Finally, we discuss charges that can be constructed using currents that involve the field operator $X^\mu$.

2.1. Zero momentum physical states in critical string theory

In this section we will list the ghost number two physical states of critical bosonic closed strings at zero momentum. These states are defined as cohomology classes of the semirelative BRST complex. We will find that only some of those states can be obtained as a zero momentum limit of physical states that exist for non-zero momentum (states corresponding to massless particles). This is a familiar phenomenon noted, for example, in Ref.[13] for the case
of the fully relative closed string BRST cohomology. The present section will also serve the purpose of setting up definitions and conventions.

Let \( |\Psi\rangle \) be the string field state and \( Q \) the BRST operator. Physical states are defined by

\[
Q |\Psi\rangle = 0,
\]

up to gauge transformations

\[
\delta |\Psi\rangle = Q |\Lambda\rangle.
\]

Here both \( |\Psi\rangle \) and \( |\Lambda\rangle \) must be annihilated by \( b_0^- = b_0 - \bar{b}_0 \). To look for states that can be physical at zero momentum the relevant part of the string field is

\[
|\Psi\rangle = E_{\mu\nu} c_1^{\alpha_{-1}} \bar{c}_{1} \bar{\alpha}_{-1} |p\rangle
- \overline{A}_\mu c_0^+ c_1^{\alpha_{-1}} |p\rangle + A_\mu c_0^+ \bar{c}_1 \bar{\alpha}_{-1} |p\rangle
+ F c_1 c_{-1} |p\rangle - \overline{F} \bar{c}_1 \bar{c}_{-1} |p\rangle + \cdots
\]

(2.3)

where \( c_0^+ = (1/2)(c_0 + \bar{c}_0) \), and that of the gauge parameter

\[
|\Lambda\rangle = \varepsilon_\mu c_1^{\alpha_{-1}} |p\rangle - \overline{\varepsilon}_\mu \bar{c}_1 \bar{\alpha}_{-1} |p\rangle + \varepsilon c_0^+ |p\rangle + \cdots.
\]

(2.4)

Let us work at zero momentum \( p_\mu = 0 \). Equation (2.1) gives \( A_\mu = \overline{A}_\mu = 0 \) and Eq.(2.2) gives us the gauge transformations \( \delta F = -\delta \overline{F} = -\phi \). It follows that at zero momentum the \( d^2 \) degrees of freedom of \( E_{\mu\nu} \) are unconstrained, the combination \( F + \overline{F} \) is gauge invariant and unconstrained, and \( F - \overline{F} \) can be gauged away. This gives a total of \( d^2 + 1 \) nontrivial BRST physical states in the semirelative complex. For \( p_\mu \neq 0 \), it is well known that there are \( (d-2)^2 \) nontrivial BRST states for each value of momentum satisfying \( p^2 = 0 \). These considerations indicate that we have \( (4d-3) \) states that are only physical at zero momentum. These states are called discrete states.

The \( (d^2 + 1) \) zero-momentum physical states correspond to the following CFT fields

\[
D_g \equiv \frac{1}{2} (c \partial^2 c - \bar{c} \bar{\partial}^2 \bar{c}),
\]

\[
D^{\mu\nu} \equiv c \partial X^\nu \bar{\partial} X^\mu.
\]

(2.5)

The state associated to \( D_g \) is called the ghost-dilaton and we will refer to the state associated to the trace \( \eta_{\mu\nu} D^{\mu\nu} \) as the matter dilaton. In addition we identify two relevant linear combinations of the ghost and the matter dilaton. The first combination is the zero-momentum dilaton

\[
D \equiv \eta_{\mu\nu} D^{\mu\nu} - D_g,
\]

(2.6)

which is the zero-momentum limit of the scalar massless state called the dilaton. It is recognized as such because the corresponding spacetime field transforms as a scalar under gauge
transformations representing diffeomorphisms. The second state is the “graviton trace”

\[ G \equiv \eta_{\mu\nu} D^\mu D^\nu - \frac{d}{2} D_g. \]  (2.7)

This state corresponds to the trace of the graviton field in the convention where the gravity action is of the form \( \int \sqrt{g} R dx \) without a factor involving the dilaton (see, for example, Ref.[14]).

It is of interest to consider the gauge transformation generated by \( |\Lambda\rangle \) for the case when we set \( \varepsilon = 0 \). We find

\[ \begin{align*}
\delta E_{\mu\nu} &= p_{\mu} \varepsilon_{\nu} + p_{\nu} \varepsilon_{\mu}, \\
\delta F &= -p^\mu \varepsilon_{\mu}, \\
\delta \bar{F} &= -p^\mu \bar{\varepsilon}_{\mu}.
\end{align*} \]  (2.8)

Transforming to coordinate space we obtain the linearized gauge transformations

\[ \begin{align*}
\delta E_{\mu\nu}(x) &= \partial_{\mu} \varepsilon_{\nu}(x) + \partial_{\nu} \varepsilon_{\mu}(x), \\
\delta F(x) &= -\partial^\mu \varepsilon_{\mu}, \\
\delta \bar{F}(x) &= -\partial^\mu \bar{\varepsilon}_{\mu}.
\end{align*} \]  (2.9)

Now consider the gauge parameters

\[ \begin{align*}
\varepsilon_{\nu}(x) &= C_{\mu\nu} x^\mu, \\
\bar{\varepsilon}_{\mu}(x) &= C_{\mu\nu} x^\nu,
\end{align*} \]  (2.10)

where \( C_{\mu\nu} \) is a matrix of constants. Equation (2.9) implies that the following constant field configurations are pure gauge:

\[ E_{\mu\nu}(x) = 2C_{\mu\nu}, \quad F(x) = \bar{F}(x) = C^\mu_{\mu}. \]  (2.11)

In string field theory the coefficient \( E_{\mu\nu}(p) \) in Eq.(2.3) should be interpreted as a Fourier component of the space-time field \( E_{\mu\nu}(x) \). The above spacetime constant field configurations must correspond to zero momentum states. The corresponding string field in (2.3) should be expected to be BRST trivial. This requires that

\[ D^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} D_g \equiv G^{\mu\nu}, \]  (2.12)

is BRST trivial

\[ G^{\mu\nu} = -\{ Q, \xi^{\mu\nu} \}. \]  (2.13)

In the ordinary closed string BRST complex there is no state \( \xi^{\mu\nu} \) satisfying (2.13). It is necessary to extend the BRST complex to include states corresponding to field configurations growing linearly in space-time. An example is the state \( \lim_{z \to 0} X^\mu(z, \bar{z})|0\rangle \), which requires the consideration of \( X^\mu(z, \bar{z}) \) as a field operator. We analyze this next.
2.2. Definition of the $X^\mu$ operator

In the ordinary CFT of 26 free bosons only derivatives of $X^\mu(z, \bar{z})$ appear as conformal fields. These derivatives have the mode expansions

\[ i \partial X^\mu(z, \bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\alpha^\mu_n}{z^{n+1}}, \quad i \bar{\partial} X^\mu(z, \bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{\alpha}^\mu_n}{\bar{z}^{n+1}}, \tag{2.14} \]

where the $\alpha^\mu_n$’s are operators with commutation relations

\[ [\alpha^\mu_m, \alpha^\nu_n] = m \eta^{\mu\nu} \delta_{n,m}, \quad [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m \eta^{\mu\nu} \delta_{n,m}. \tag{2.15} \]

Formally integrating Eq. (2.14) we find

\[ X^\mu(z, \bar{z}) = X^\mu_0 - 2i \alpha^\mu_0 \log |z| + \sum_{n \neq 0} \frac{i \alpha^\mu_n}{n z^n} + \sum_{n \neq 0} \frac{i \bar{\alpha}^\mu_n}{n \bar{z}^n}. \tag{2.16} \]

The zero mode operator $X^\mu_0$, which appears in (2.16) as a constant of integration is not specified by (2.14) and has to be defined independently. It is standard to interpret it as the position operator for the center of mass, and taken to commute with all $\alpha$’s except the momentum operator $\alpha^\mu_0$

\[ [X^\mu_0, \alpha_n^\nu] = i \eta^{\mu\nu}. \tag{2.17} \]

This interpretation is usually justified by canonical analysis of the two-dimensional quantum field theory. Eqs. (2.16) and (2.17) provide an abstract definition of $X^\mu(z, \bar{z})$ as an element of a Lie algebra.

2.3. A gauge parameter involving $X^\mu(z, \bar{z})$

Now we will try to use the $X^\mu$ field to find a gauge parameter which will generate $G^{\mu\nu}$. Equations (2.4) and (2.10) suggest that the proper candidate is

\[ \xi^{\mu\nu} = \frac{1}{2} \left( c : X^\nu \partial X^\mu : - \bar{\sigma} : X^\mu \bar{\partial} X^\nu : \right). \tag{2.18} \]

We have to explain what the products like $: X^\mu \partial X^\nu :$ or $: X^\nu \bar{\partial} X^\mu :$ mean. Normal ordering amounts to placing annihilation operators to the right of creation operators. In our case it is not clear how to order the product of $X^\mu_0$ and $\alpha_0$. We adopt the following definition

\[ : X^\nu \partial X^\mu : (z, \bar{z}) \equiv \oint \frac{R (\partial X^\mu(w)X^\nu(z, \bar{z})) dw}{w - z} \frac{1}{2\pi i}, \tag{2.19} \]

where $R$ denotes the necessary radial ordering. Note that the integral is contour independent because $\partial X^\mu$ is a holomorphic field. As usual, to evaluate the integral we replace the contour
around $z$ by two constant radius contours, one with $|w| > |z|$, and the other with $|w| < |z|$. Because of radial ordering one must use different expansions for $1/(w - z)$ in the two contours. A small calculation gives

$$:X^\mu \partial X^\nu: (z, \bar{z}) = X^\mu (z, \bar{z}) \sum_{n \geq 0} \frac{-i \alpha^\nu_n}{z^{n+1}} + \sum_{n < 0} \frac{-i \alpha^\nu_n}{z^{n+1}} X^\mu (z, \bar{z}), \quad (2.20)$$

which shows that the momentum zero mode $\alpha^\mu_0$ appears to the right of the coordinate zero mode $X^\mu_0$.

We now calculate the action of the BRST operator on $\xi^{\mu\nu}$:

$$\{ Q, \xi^{\mu\nu}(z, \bar{z}) \} = \oint \left( T_m(z)c(z) + :c(z)\partial c(z): \right) \eta^{\mu\nu} (z, \bar{z}) \frac{dz}{2\pi i} \xi^{\mu\nu} (z, \bar{z})$$

Using Wick's theorem * to expand the operator product under the integral, we obtain

$$\{ Q, \xi^{\mu\nu} \} = \frac{1}{4} \eta^{\mu\nu} (c\partial^2 c - \bar{c}\partial^2 \bar{c}) - c\partial X^\mu \bar{\partial} X^\nu, \quad (2.21)$$

and, as expected,

$$G^{\mu\nu} = -\{ Q, \xi^{\mu\nu} \}. \quad (2.22)$$

The ghost dilaton, the first state in (2.5), is a nontrivial state in the semirelative cohomology, but in the absolute complex it can be represented as [6]

$$D_g = -\{ Q, \chi_g \}, \quad (2.23)$$

where $\chi_g$ is given by

$$\chi_g = -\frac{1}{2} (\partial c - \bar{\partial} \bar{c}). \quad (2.24)$$

The state $\chi_g$ does not belong to $\widehat{\mathcal{H}}$ since it is not annihilated by $b_0^-$. Using $\chi_g$ we can represent the matter states of (2.5) as $\{ Q, \cdot \}$. Indeed, let

$$\chi^{\mu\nu} \equiv \xi^{\mu\nu} + \frac{1}{2} \eta^{\mu\nu} \chi_g. \quad (2.25)$$

Unlike $\chi_g$ and $\xi^{\mu\nu}$, the state $\chi^{\mu\nu}_m$ can be verified to be a $(1, 1)$ primary. Now combining

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* As explained in Ref.[15] Wick's theorem for composite operators is valid when normal ordering is defined as in (2.19).
Eq. (2.23) with Eq. (2.21) we obtain

\[ D^{\mu\nu} = \{Q, \chi^{\mu\nu}\}, \tag{2.26} \]

and conclude that \( \chi^{\mu\nu} \) is the gauge parameter generating the matter states of (2.5). We therefore conclude that all semirelative cohomology states at zero momentum can be represented as \( \{Q, \cdot\} \) when we allow gauge parameters using the \( X \) operator and/or violating the \( b_0^- = 0 \) condition. This information is summarized in the following table.

| \( D_g = \frac{1}{2}(c\partial^2 c - \overline{c}\partial^2 c) \) | \( D_g = \{Q, \chi_g\} \) | \( \chi_g = -\frac{1}{2}(\partial c - \overline{c}\partial) \) |
| --- | --- | --- |
| \( \xi^{\mu\nu} = \{Q, \chi^{\mu\nu}\} \) | \( \chi^{\mu\nu} = \frac{1}{2}(\partial c\partial X^\mu - \overline{c}\partial X^\nu) \) |
| \( G^{\mu\nu} = D^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} D_g \) | \( G^{\mu\nu} = \{Q, \xi^{\mu\nu}\} \) | \( \xi^{\mu\nu} = \frac{1}{2}(cX^\nu \partial X^\mu - \overline{c}X^\mu \partial X^\nu) \) |
| \( D = D^\mu_{\mu} - D_g \) | \( D_g = \{Q, \chi_D\} \) | \( \chi_D = \chi^\mu_{\mu} - \chi_g \) |
| \( \chi_{D} = \chi^\mu_{\mu} - \chi_g \) | \( \chi_{D} = \chi^\mu_{\mu} + (\frac{d-2}{2}) \chi_g \) |

2.4. Properties of \( :X^\nu \partial X^\mu : dz \)

It is of interest to consider some additional properties of \( :X^\nu \partial X^\mu : dz \) and \( :X^\mu \overline{\partial} X^\nu : dz \). Both of them have an exterior derivative proportional to the two-form \( \partial X^\mu \overline{\partial} X^\nu dz \wedge d\overline{z} \)

\[ -d \left( :X^\nu \partial X^\mu : dz \right) = d \left( :X^\nu \overline{\partial} X^\mu : dz \right) = \partial X^\mu \overline{\partial} X^\nu dz \wedge d\overline{z}. \]  \tag{2.27}

Nevertheless \( :X^\nu \partial X^\mu : dz \) is not a one form because the field \( :X^\nu \partial X^\mu : \) is not primary

\[ T(z) :X^\nu \partial X^\mu : (w, \overline{w}) = \frac{-\eta^{\mu\nu}}{(z-w)^2} + \frac{X^\nu \partial X^\mu : (w, \overline{w})}{(z-w)^2} + \frac{\partial(\cdot X^\nu \partial X^\mu:) (w, \overline{w})}{z-w} + \cdots. \] \tag{2.28}

As a consequence we have an anomalous transformation law under analytic maps

\[ :X^\nu \partial X^\mu : (z', \overline{z}') dz' = \left( :X^\nu \partial X^\mu : (z, \overline{z}) - \frac{1}{2} \eta^{\mu\nu} \frac{d^2 z/dz'^2}{(dz/dz')^2} \right) dz. \] \tag{2.29}

See [16] for details. Similar results hold for \( :X^\mu \overline{\partial} X^\nu : d\overline{z} \).
This implies that \( :X^\nu \partial X^\mu : dz \) and/or \( :X^\mu \bar{\partial} X^\nu : d\bar{z} \) cannot be integrated unambiguously over a contour unless we fix a coordinate in the vicinity of it. Nevertheless, we can show that an integral over a contractible path does not change if we make a coordinate transformation which is holomorphic inside the path. Indeed, according to Eq. (2.29), when we make a holomorphic change of coordinate the anomalous piece which appears in the transformation law is a holomorphic (or antiholomorphic) one-form whose integral over a contractible path is zero. Thus the integrals \( \oint_\gamma :X^\nu \partial X^\mu : dz \) and \( \oint_\gamma :X^\mu \bar{\partial} X^\nu : d\bar{z} \) are well defined if \( \gamma \) is contractible but they still depend on the choice of the contour \( \gamma \) itself. For later use we define

\[
\mathcal{D}_X^{\mu\nu} = - \oint_{|z|=1} :X^\nu \partial X^\mu: (z,\bar{z}) \frac{dz}{2\pi i} = \oint_{|z|=1} :X^\mu \bar{\partial} X^\nu: (z,\bar{z}) \frac{d\bar{z}}{2\pi i},
\]

where the equality follows by use of (2.27). Using Eq. (2.20) we rewrite the above as

\[
\mathcal{D}_X^{\mu\nu} = - \oint_{|z|=1} \frac{dz}{2\pi i} X^\nu(z,\bar{z}) \sum_{n \geq 0} \frac{-i}{z^{n+1}} \alpha^\mu_n z^{n+1} \left( z,\bar{z} \right) - \oint_{|z|=1} \frac{dz}{2\pi i} \sum_{n < 0} \frac{-i}{z^{n+1}} \alpha^\mu_n X^\nu(z,\bar{z}) .
\]

The integration is done using Eq. (2.16). Since we integrate over the unit circle the logarithm in (2.16) vanishes, and we obtain

\[
\mathcal{D}_X^{\mu\nu} = iX^\nu_0 \alpha^\mu_0 - \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n \bar{\alpha}^\nu_n .
\]

This operator appeared earlier in the dilaton theorem analysis of refs. [2,3].

3. Extended BRST complex

In this section we define an extended BRST complex where the coordinate zero mode \( X_0 \) acts as a linear operator. We will show how the BRST operator \( Q^{\text{ext}} \) acts on this complex and explain why BRST exact states in this complex may not decouple from physical correlations. We will explain, through an example why the usual BRST action on correlators holds in the extended complex. This, together with the fact that sewing also holds in the extended complex, implies that string field theory is well defined in the extended complex. The explicit computation of the BRST cohomology in this complex will be given in Ref. [12]. The present section concludes with a review of the results of this computation.
3.1. Definition of the Extended Complex

We define the extended space of states at any given momentum $p$ as a tensor product of the original state space $\hat{H}_p$ with the space of polynomials of $D$ variables $x^\mu$:

$$\hat{H}_p^{\text{ext}} = \mathbb{C}[x^\mu] \otimes \hat{H}_p.$$  \hfill (3.1)

A state in this complex is written in the form $P \otimes v$ where $P \in \mathbb{C}[x^\mu]$ is a polynomial in $x^\mu$, and $v \in \hat{H}_p$ is a vector from the original state space. The operator $X_0^\mu$ acts by multiplying the polynomial by $x^\mu$. All the mode operators, except for $\alpha_0^\mu$, act as they acted on $\hat{H}_p$. For $\alpha_0^\mu$ it is natural to define

$$\alpha_0^\mu P \otimes v = p^\mu P \otimes v - i\eta^{\mu\nu} \frac{\partial P}{\partial x^\nu} \otimes v,$$ \hfill (3.2)

preserving in this way the commutation relation $[X_0^\mu, \alpha_0^\nu] = i\eta^{\mu\nu}$. With the matter Virasoro generators written as

$$L_n = \frac{1}{2} \sum_m \eta_{\mu\nu} :\alpha_m^\mu \alpha_{n-m}^\nu :,$$ \hfill (3.3)

the BRST charge reads

$$Q^{\text{ext}} = \sum_n c_n L_{-n} - \frac{1}{2} \sum_{m,n} (m - n) :c_{-m}c_{-n}b_{m+n} : + \text{a.h.}$$ \hfill (3.4)

Although equations (3.3) and (3.4) have the standard form, when acting on the extended complex we must use Eq. (3.2). For the BRST operator this implies that

$$Q^{\text{ext}} P \otimes v = P \otimes Qv - i \frac{\partial P}{\partial p^\mu} \otimes \sum_{n=-\infty}^{\infty} (c_n \alpha_{-n}^\mu + c_n \bar{\alpha}_{-n}^\mu) v - \eta^{\mu\nu} \frac{\partial^2 P}{\partial x^\mu \partial x^\nu} \otimes c_0^+ v.$$ \hfill (3.5)

3.2. The failure of BRST decoupling

The analysis of section two shows that the states $G^{\mu\nu}$ are BRST trivial in the extended complex. Indeed, as written in the table at the end of section 2.3, $G^{\mu\nu} = \{ Q, \xi^{\mu\nu} \}$, where $\xi^{\mu\nu}$ contains an explicit $X$ operator. Correlators involving an explicit $X$ operator should be evaluated using $X^\mu(z) = -i \frac{\partial}{\partial p^\mu} \exp(i p X(z)) \bigg|_{p=0}$, which operationally means evaluating the correlator with $X^\mu(z)$ replaced by $\exp(i p X(z))$ and evaluating the derivative $-i \frac{\partial}{\partial p^\mu}$ of the resulting correlator at $p = 0$. 

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In general, the correlation functions of a BRST trivial state with physical states are known to vanish. Indeed, let $|\phi\rangle = -Q|\chi\rangle$ be BRST trivial, and $|\psi_k\rangle$ for $k = 1, \ldots, n$, be BRST physical $(Q|\psi_k\rangle = 0)$. It follows by contour deformation that the BRST operator can be taken to act on the physical states giving

$$\langle \phi \psi_1 \cdots \psi_n \rangle = \sum_{k=1}^{n} \langle \chi \psi_1 \cdots Q\psi_k \cdots \psi_n \rangle = 0.$$  

(3.6)

On the other hand, consider the three point function of the BRST trivial zero-momentum state $G^{\mu\nu}$ with two tachyons $\tau_p(z, \bar{z}) = c\bar{c}\exp(ipX)(z, \bar{z})$. One readily verifies that for $\mu \neq \nu$

$$\langle G^{\mu\nu}(z_1, \bar{z}_1) \tau_p(z_2, \bar{z}_2) \tau_q(z_3, \bar{z}_3) \rangle = 2p^\mu p^\nu |z_2 - z_3|^{4-2p^2}(2\pi)^d\delta^d(p + q),$$  

(3.7)

and observes that this is not zero for on-shell tachyons ($p^2 = q^2 = 2$), in apparent contradiction (3.6). Since the computation leading to (3.7) is beyond doubt we must find why the usual argument for decoupling fails.

The problem with (3.6) is that when $\phi = G^{\mu\nu}$, the field $X^\mu$ is present in the correlators under the sum, and we cannot make sense of their on-shell values. This is because correlators are not functions but rather distributions. In the absence of $X$ fields, however, we associate an ordinary function to a correlator because every correlator can be represented as an ordinary function of momenta times the standard momentum conserving $\delta$-distribution

$$\langle \psi_1\psi_2 \cdots \psi_n \rangle = F(p_1, p_2, \ldots, p_n) (2\pi)^d\delta^d(p_1 + p_2 + \cdots + p_n).$$  

(3.8)

When we say that the correlator $\langle \psi_1\psi_1 \cdots \psi_n \rangle$ vanishes for particular values of momenta what we really mean is that the function $F$ vanishes. On the other hand, if the fields $\psi_i$ contain $X$ without derivatives, correlators have a more general structure

$$\langle \psi_1\psi_2 \cdots \psi_n \rangle = F(p_1, \ldots, p_n) (2\pi)^d\delta^d(p_1 + \cdots + p_n)$$

$$+ F^\mu(p_1, \ldots, p_n) (2\pi)^d\delta^d_{\mu}(p_1 + \cdots + p_n) + \cdots.$$  

(3.9)

where $\delta^d_{\mu}(p) \equiv \frac{d}{dp^\mu}\delta^d(p)$, and the dots indicate possible terms with higher derivatives of delta functions. A function $F^\mu$ that vanishes on-shell can contribute to the correlator if its derivative does not vanish on shell. This implies that the right hand side of Eq. (3.6) need not vanish when the correlator contains an $X$. We will illustrate this with an example. Rather than using the field $G^{\mu\nu}$, the point can be made by considering open string theory where the analogous field is the zero-momentum photon $c\partial X^\mu = \{Q, X^\mu\}$. We therefore examine the correlator of
this state with two tachyons
\[
\langle c \partial X^\mu(x) c e^{ip_1 X(x_1)} c e^{ip_2 X(x_2)} \rangle = -i p_1^\mu |x_2 - x_1|^{2-p_2^2} (2\pi)^d \delta^d(p_1 + p_2),
\] (3.10)
where \(x, x_1, x_2\) denote the insertion points on the real axis. On the other hand, using the BRST property we are led to write
\[
\langle c \partial X^\mu(x) c e^{ip_1 X(x_1)} c e^{ip_2 X(x_2)} \rangle = -\langle X^\mu(x) \{Q, ce^{ip_1 X(x_1)} \} ce^{ip_2 X(x_2)} \rangle \\
+ \langle X^\mu(x) ce^{ip_1 X(x_1)} \{Q, ce^{ip_2 X(x_2)} \} \rangle.
\] (3.11)
Using \(\{Q, ce^{ipX(z)}\} = (p_2^2 - 1) c \partial ce^{ipX(z)}\), we obtain
\[
(I) = \langle X^\mu(x) \{Q, ce^{ip_1 X(x_1)} \} ce^{ip_2 X(x_2)} \rangle = \left(\frac{p_1^2}{2} - 1\right) |x_1 - x_2|^{2+p_1 p_2} \langle X^\mu(x) e^{ip_1 X(x_1)} e^{ip_2 X(x_2)} \rangle,
\] (3.12)
and the remaining matter correlator is evaluated using the prescription given at the beginning of the section
\[
(I) = -i \left(\frac{p_2^2}{2} - 1\right) |x_1 - x_2|^{2+p_1 p_2} \left[p_1^\mu \log \frac{|x - x_1|}{|x - x_2|} (2\pi)^d \delta^d(p_1 + p_2) + (2\pi)^d \delta^d(p_1 + p_2)\right].
\] (3.13)
In this expression, the term including the ordinary delta function is unchanged under the simultaneous exchanges \(x_1 \leftrightarrow x_2\) and \(p_1 \leftrightarrow p_2\), and thus back in (3.11) we obtain
\[
\langle c \partial X^\mu(x) ce^{ip_1 X(x_1)} ce^{ip_2 X(x_2)} \rangle = \frac{i}{2} \left(p_1^2 - p_2^2\right) |x_1 - x_2|^{2+p_1 p_2} (2\pi)^d \delta^d(p_1 + p_2),
\] (3.14)
where \(p^\mu = p_1^\mu + p_2^\mu\) and \(q^\mu = p_1^\mu - p_2^\mu\). Since \(xf(x)\delta'(x) = -f(0)\delta(x)\) we find
\[
\langle c \partial X^\mu(x) ce^{ip_1 X(x_1)} ce^{ip_2 X(x_2)} \rangle = -i p_1^\mu |x_1 - x_2|^{2-p_2^2} (2\pi)^d \delta^d(p_1 + p_2),
\] (3.15)
in agreement with (3.10), and confirms the failure of decoupling. This example also illustrates that with proper treatment of distributions the BRST property of correlators holds in the extended complex. There is no problem with the contour deformation arguments that are used to prove the BRST property. This will also be the case when we deal with integrated correlators. The sewing property of correlators is also preserved in the extended complex. This follows from the definition of correlators when a field \(X\) is present and the fact that the sewing ket is not changed. Since the consistency of string field theory depends only on the proper BRST action on correlators and sewing, the above arguments indicate that there is no difficulty in defining string field theory on the extended complex.
3.3. Cohomology of $Q^{\text{ext}}$

While the extended BRST complex is larger than the original one, a priori, this has no immediate implication for the cohomologies. When we extend a complex we increase both the number of BRST closed states and the number of BRST trivial states. As we saw earlier, some zero momentum states that were physical in the original complex are trivial in the extended one. On the other hand there might be new solutions to $Q |\Psi\rangle = 0$ in the new complex. The cohomology of the extended complex $\hat{H}^{\text{ext}}$ is presented in Ref. [12]. Indeed, we lose some states, in particular states that do not change the physics of the background, and states of peculiar ghost numbers. We gain some states, but the new states can be understood in terms of the old complex.

When we have a physical state $|v, p\rangle$ which remains physical under continuous variations of the momentum $p$, we can easily construct physical states in the extended complex by taking linear combinations of $|v, p_i\rangle$ where all $p_i \approx p$ are on-shell. Adjusting the coefficients we can get as many derivatives with respect to $p$ as we want which we can interpret as factors of $X_0^\mu$. Since mass-shells are not flat, in general we get nontrivial combinations of states with different numbers of $X_0^\mu$. At non-zero momentum and ghost number two ($G = 2$), all new physical states can be obtained from standard states by the above limiting procedure [12].

Let us recall the structure of the semirelative cohomology. At non-zero momentum $p$ the cohomology at $G = 2$ and at $G = 3$ can be represented by the states $c_1 \bar{c}_1 |v, p\rangle$, and $(c_0 + \bar{c}_0)c_1 \bar{c}_1 |v, p\rangle$ respectively, where $|v, p\rangle$ is a dimension $(1, 1)$ primary matter state. All $G = 3$ states are trivial in the extended complex. Indeed, using Eq. (3.5) we can write

$$1 \otimes (c_0 + \bar{c}_0)c_1 \bar{c}_1 |v, p\rangle = Q^{\text{ext}} \frac{p \cdot x}{p^2} \otimes c_1 \bar{c}_1 |v, p\rangle - \sum \frac{p_\mu}{p^2} \otimes c_1 \bar{c}_1 (c_{-n} \bar{a}_n^\mu + \bar{c}_{-n} \bar{c}_n^\mu) |v, p\rangle.$$ 

The last sum must be $Q$ trivial because, being annihilated by $Q$, $b_0$ and $\bar{b}_0$, it would represent a non-trivial relative cohomology class at $G = 3$. Such a class does not exist.

Calculation of the cohomology of the extended complex at zero momentum is more delicate [12]. In the standard semirelative case the physical states go as follows. At $G = 2$ we have the $(d^2 + 1)$ states of section three, and at $G = 3$ the $(d^2 + 1)$ states obtained by multiplying the $G = 2$ states by $(c_0 + \bar{c}_0)$. There are $2d$ states at $G = 1$ and at $G = 4$, and one state at $G = 0$ ($SL(2, \mathbb{C})$ vacuum) and at $G = 5$. In the extended complex there are no zero-momentum physical states at $G > 2$. There is an infinite tower of $G = 2$ states which can be described as different limits of linear combination of massless states as all momenta are taken to zero. There is only one $G = 0$ state, the $SL(2, \mathbb{C})$ vacuum, and there are $d(d+1)/2$ physical states at $G = 1$ which contain no more than one $X_0^\mu$. These are precisely the states that generate Poincare symmetry. While in the standard complex one gets the states that generate translations, the states generating Lorentz transformations are missing. They appear properly in the extended complex.
4. CFT Deformations and the Matter Dilaton

In this section we give a detailed analysis of the operator $\partial X \bar{\partial} X$ and its effect on conformal theories that include a free field $X$ living on the open line. There are two cases of interest. In the first one, the conformal theory includes the ghost system and has total central charge of zero. We derive identities that show that $\partial X \bar{\partial} X$ induces a trivial deformation of the CFT, the ghosts playing a crucial role here.

We then consider the second case, when this operator appears in the context of the $c = 1$ matter conformal field theory of the free field (no extra ghosts). We use the definition of a $c \neq 0$ conformal theory in the operator formalism to show that the deformation is not strictly trivial. The detailed analysis shows that the deformation in question can be mostly eliminated by a change of basis in the conformal theory, but the scale of the world sheet metric is changed by the deformation.

The above results are certainly not controversial for the case of zero central charge. The triviality of the operator in question has been argued earlier at various levels of detail. In ref.[10], for example, the usual argument that such perturbation can be redefined away from the conformal field theory lagrangian is reviewed, along with a discussion from the viewpoint of gauge transformations in string field theory. In ref.[9] the deformation of the two-point function of stress-tensors is investigated explicitly. For the case of non-zero central charge our result appears to be new.

We then turn to integrals of string forms over moduli spaces of Riemann surfaces and consider their deformation by the insertion of the matter operator in question. The resulting integral expressions will be needed in section six in order to establish the complete dilaton theorems.

4.1. The operator $\partial X \bar{\partial} X$ in $c = 0$ CFT

Here we answer the following question: does the $(1,1)$ primary field :$\partial X^{\mu} \bar{\partial} X^{\nu}$ : define a non-trivial deformation of a conformal field theory ? Being a $(1,1)$ primary we can write the corresponding deformation by integrating the field over the surface minus unit disks [17, 18]

$$\delta^{\mu\nu} \langle \Sigma_{g,n} | \equiv \frac{1}{2\pi i} \int_{\Sigma_{g,n} - \cup D_k} \langle \Sigma_{g,n+1}(z,\bar{z})|\partial X^{\mu} \bar{\partial} X^{\nu} \rangle \, dz \wedge d\bar{z} , \quad (4.1)$$

This deformation is trivial if there is an operator $O^{\mu\nu}$ such that

$$\delta^{\mu\nu} \langle \Sigma_{g,n} | = -\langle \Sigma_{g,n} | \sum_{k=1}^{n} O^{\mu\nu}(k) , \quad (4.2)$$

since this means that the deformation can be absorbed by a change of basis in the CFT, a
change induced by the operator \( \mathcal{O}^{\mu\nu} \). We will show that \( \mathcal{O}^{\mu\nu} \) is given by
\[
\mathcal{O}^{\mu\nu} = D_X^{\mu\nu} + \frac{1}{6} \eta^{\mu\nu} G, \tag{4.3}
\]
where \( D_X^{\mu\nu} \) was defined in (2.30) and \( G \) is the total ghost number operator. This shows that the operator \( \partial X^\mu \partial X^\nu \) induces a trivial conformal field theory deformation.

We now give a simple proof of the above assertion. Note that the operator-valued two form that is being integrated can be written as
\[
\partial X^\mu \partial X^\nu : dz \wedge d\bar{z} = d \left[ \frac{1}{2} : X^\mu \partial X^\nu : dz - \frac{1}{2} : X^\nu \partial X^\mu : dz + \frac{6}{2} (G(z)dz - \overline{G}(\bar{z})d\bar{z}) \right], \tag{4.4}
\]
where \( A^{\mu\nu}(z) \) and \( \overline{A}^{\mu\nu}(\bar{z}) \) are arbitrary holomorphic or antiholomorphic operators, and are therefore annihilated by the exterior derivative. They are important, however. The left hand side of the above equation is a well defined two-form, but the expression within brackets on the right hand side is not a well-defined one-form unless the \( A \) operators are suitably chosen. This is the case because the operators \( X^\mu \partial X^\nu : \) and \( \partial X^\mu X^\nu : \) are not primary. We can obtain primary operators by choosing non-primary \( A \) operators. We take
\[
\frac{1}{2} \sum_{k=1}^{n} \oint_{D_k} \left\{ \cdots : \partial X^\mu \partial X^\nu : dz \wedge d\bar{z} \right\}, \tag{4.6}
\]
becomes
\[
\frac{1}{2\pi i} \sum_{k=1}^{n} \oint_{D_k} \left\{ \cdots : \partial X^\mu \partial X^\nu : dz - \partial X^\mu \partial X^\nu : d\bar{z} \right\}, \tag{4.7}
\]
where the contour integrals are over the boundaries of the disks oriented as such. We now recognize that the contour integrals simply represent a single operator acting on each puncture, one at a time. The operator is just
\[
\frac{1}{2\pi i} \oint_{|z|=1} \left\{ \cdots : \partial X^\mu \partial X^\nu : dz - \partial X^\mu \partial X^\nu : d\bar{z} \right\} = \left( \mathcal{D}_X^{\mu\nu} + \frac{6}{2} G \right). \tag{4.8}
\]
This concludes our proof of the triviality of the deformation.
4.2. The operator $\partial X \bar{\partial} X$ in $c \neq 0$ CFT

If we have any matter conformal theory coupled to the ghost conformal theory, the ghost number operator $G$ acting on surface states will give

$$\langle \Sigma_{g,n} | \sum_{k=1}^{n} G^{(k)} = 6(1 - g) \langle \Sigma_{g,n} |. \quad (4.9)$$

Using this result we recognize that the result of the previous subsection showing the triviality of the deformation can be written as

$$\delta \langle \Sigma_{g,n} | = \frac{1}{2\pi i} \int_{\Sigma_{g,n} - \bigcup D_k} \langle \Sigma_{g,n+1}(z, \bar{z}) | \partial X^\mu \bar{\partial} X^\nu \rangle \, dz \wedge d\bar{z}$$

$$= - \langle \Sigma_{g,n} | \sum_{k=1}^{n} D_{X}^{\mu\nu(k)} - (1 - g) \eta^{\mu\nu} \langle \Sigma_{g,n} |, \quad (4.10)$$

In this form, of course, the triviality of the deformation is not manifest since the second term in the right hand side is not written as a sum of linear operators acting on the surface state.

We now claim that equation (4.10) applies for the case when the matter conformal theory does not include the ghost conformal theory. By construction, (4.10) applies when the total conformal theory is the $c = 1$ matter theory coupled to the ghosts. The surface states in this total conformal theory are the tensor product of the surface states in the two separate conformal theories $\langle \Sigma_{g,n} | = \langle \Sigma_{g,n}^{c=1} | \otimes \langle \Sigma_{g,n}^{gh} |$. Since the operators in the right hand side of (4.10) are ghost-independent it is clear that we can factor out the ghost part $\langle \Sigma_{g,n}^{gh} |$ of the surface state. Since the insertion in the left hand side carries no ghost dependence the additional puncture is deleted in the ghost sector $\langle \Sigma_{g,n+1}(z, \bar{z}) | \partial X^\mu \bar{\partial} X^\nu \rangle = \langle \Sigma_{g,n+1}^{c=1}(z, \bar{z}) | \partial X^\mu \bar{\partial} X^\nu \rangle \otimes \langle \Sigma_{g,n}^{gh} |$. It follows that we can factor the ghost sector out of equation (4.10) totally, and we find

$$\delta \langle \Sigma_{g,n}^{c=1} | = \frac{1}{2\pi i} \int_{\Sigma_{g,n} - \bigcup D_k} \langle \Sigma_{g,n+1}^{c=1}(z, \bar{z}) | \partial X^\mu \bar{\partial} X^\nu \rangle \, dz \wedge d\bar{z}$$

$$= - \langle \Sigma_{g,n}^{c=1} | \sum_{k=1}^{n} D_{X}^{\mu\nu(k)} - (1 - g) \eta^{\mu\nu} \langle \Sigma_{g,n}^{c=1} |. \quad (4.11)$$

We can now address the issue of triviality. As mentioned earlier, the first term in the right hand side is just a similarity transformation. The second term is not. To give an interpretation to that term we recall the scaling properties of $c \neq 0$ CFT. Under a scale change of the metric
on the surface the correlators change as
\[ \langle \cdots \rangle_{g e^\sigma} = \exp \left[ \frac{c}{48 \pi} S_L(\sigma; g) \right] \langle \cdots \rangle_g \] (4.12)
where
\[ S_L(\sigma; g) = \int d^2 \xi \sqrt{g} \left( \frac{1}{2} g^{\alpha \beta} \partial_\alpha \sigma \partial_\beta \sigma + R(g) \sigma \right) \] (4.13)
For constant \( \sigma \) we get
\[ S_L(\sigma; g) = \sigma \int d^2 \xi \sqrt{g} R(g) = 4 \pi \sigma (1 - g), \]
where \( g \) is the genus of the surface. This shows that for an infinitesimal scaling parameter \( \sigma \) the correlators scale as
\[ \langle \cdots \rangle_{g e^\sigma} = \left( 1 + \frac{c}{12} \sigma (1 - g) \right) \langle \cdots \rangle_g . \] (4.14)
This shows that the last term in (4.11) corresponds to a constant scale deformation of the metric on the two-dimensional surface. The deformation induced by \( \partial X \bar{\partial} X \) is not completely trivial.

4.3. Generalization to spaces of surfaces

We must now extend the discussion of the \( c = 0 \) case to include spaces of surfaces. Since we will use the ghost sector in a nontrivial fashion we use the ghost number two primary states \( D^{\mu \nu} \) defined in (2.5). Rather than integrating the matter dilaton over a single surface \( \Sigma \), an operation that we can denote as \( f_{D^{\mu \nu}}(\mathcal{K} \Sigma) \), we want to consider the object \( f_{D^{\mu \nu}}(\mathcal{K} \mathcal{A}) \), where \( \mathcal{A} \) is a space of surfaces. We claim that the following result holds
\[ f_{D^{\mu \nu}}(\mathcal{K} \mathcal{A}) + f_{\chi^{\mu \nu}}(\mathcal{L} \mathcal{A}) = \frac{1}{2} \eta^{\mu \nu} (2 g - 2 + n) f(\mathcal{A}) , \] (4.15)
where \( \chi^{\mu \nu} \) is the state defined in (2.25) and whose main property is that upon action by the BRST operator it gives us the matter dilaton state. The purpose of the present subsection is to prove equation (4.15).

We begin the proof of the above relation by evaluating \( f_{D^{\mu \nu}}(\mathcal{K} \mathcal{A}) \). Since the matter dilaton state can be written as \( |D^{\mu \nu}\rangle = -Q|\xi^{\mu \nu}\rangle + \frac{1}{2} \eta^{\mu \nu} |D_g \rangle \) we find
\[ f_{D^{\mu \nu}}(\mathcal{K} \mathcal{A}) = -\frac{1}{n!} \int_{\mathcal{K} \mathcal{A}} \langle \Omega^{[\mathcal{A}+2]g,n+1} |Q|\xi^{\mu \nu}\rangle + \frac{1}{2} \eta^{\mu \nu} f_{D_g}(\mathcal{K} \mathcal{A}) . \] (4.16)
Using the relation \( f_{D_g}(\mathcal{K} \mathcal{A}) + f_{\chi_g}(\mathcal{L} \mathcal{A}) = (2 g - 2 - n) f(\mathcal{A}) [7] \), we rewrite the above as
\[ f_{D^{\mu \nu}}(\mathcal{K} \mathcal{A}) + \frac{1}{2} \eta^{\mu \nu} f_{\chi_g}(\mathcal{L} \mathcal{A}) = -\frac{1}{n!} \int_{\mathcal{K} \mathcal{A}} \langle \Omega^{[\mathcal{A}+2]g,n+1} |Q|\xi^{\mu \nu}\rangle + \frac{1}{2} \eta^{\mu \nu} (2 g - 2 + n) f(\mathcal{A}) . \] (4.17)
By virtue of (2.25), we see that (4.17) implies the desired result (4.15) if

\[ f_{\xi^{\mu\nu}}(\mathcal{L}A) = \frac{1}{n!} \int_{\mathcal{K}_A} \langle \Omega^{[A+2]g,n+1}|Q|\xi^{\mu\nu} \rangle. \tag{4.18} \]

We must therefore establish this equation. Our first step is to replace the BRST operator, which is only acting on the additional puncture, by a sum of BRST operators acting on all punctures. The right hand side then becomes

\[ \frac{1}{n!} \int_{\mathcal{K}_A} \langle \Omega^{[A+2]g,n+1}|\sum_{k=1}^{n+1} Q^{(k)}|\xi^{\mu\nu} \rangle - \frac{1}{n!} \int_{\mathcal{K}_A} \langle \Omega^{[A+2]g,n+1}|\xi^{\mu\nu} \rangle \sum_{k=1}^{n} Q^{(k)}. \]

The second integral can be seen to vanish identically. It involves the motion of the insertion over fixed surfaces, and thus includes two antighost insertions that have the property of annihilating the vacuum state. Since the \( \xi^{\mu\nu} \) state only has a single ghost operator acting on the vacuum, the two antighost insertions will annihilate it. The first integral is rewritten by using the BRST property of forms and Stokes’s theorem

\[ \frac{1}{n!} \int_{\mathcal{K}_A} d \langle \Omega^{[A+1]g,n+1}|\xi^{\mu\nu} \rangle = \frac{1}{n!} \int_{\partial(\mathcal{K}_A)} \langle \Omega^{[A+1]g,n+1}|\xi^{\mu\nu} \rangle. \tag{4.19} \]

We now recall that \( \partial(\mathcal{K}_A) = \mathcal{K}(\partial A) + \mathcal{L}A \). The integral over \( \mathcal{K}(\partial A) \) vanishes for exactly the same reason as quoted in the above paragraph; two antighosts insertions for position that annihilate the state. Thus the above term is simply

\[ \frac{1}{n!} \int_{\mathcal{L}A} \langle \Omega^{[A+1]g,n+1}|\xi^{\mu\nu} \rangle = f_{\xi^{\mu\nu}}(\mathcal{L}A). \]

This establishes the correctness of (4.18), and as a consequence concludes our proof of (4.15).
5. Complete Dilaton Theorem

In this section we write a general hamiltonian that induces string field diffeomorphisms relevant for the dilaton theorem. Such hamiltonian will take a form similar to that of Ref. [7] and will allow us to treat in a uniform way the dilaton, the ghost-dilaton, the matter-dilaton and the graviton-trace states. We explain what kind of deformations these various states produce, and emphasize that none of them leads to a change of the slope parameter $\alpha'$. We then establish the complete dilaton theorem for critical closed string field theory. The main point is that the complete dilaton state can be written as

$$ |D\rangle = -Q|\xi^\mu \rangle + \frac{(d-2)}{2}|D_g\rangle, \quad (5.1) $$

and therefore in the cohomology of the extended complex the complete dilaton is just proportional to the ghost-dilaton

$$ |D\rangle \approx \frac{(d-2)}{2}|D_g\rangle. \quad (5.2) $$

In the extended complex only the ghost-dilaton changes the string coupling, and the above equation indicates that the complete dilaton changes the string coupling with a proportionality factor $(d-2)$.

5.1. A General Hamiltonian

What we have in mind here is writing a hamiltonian $U$ that generates a diffeomorphism $F$ of the string field via a canonical transformation

$$ F : |\Psi\rangle \rightarrow |\Psi\rangle + dt \{ |\Psi\rangle, U \}. \quad (5.3) $$

Associated to such canonical transformation we can imagine that a parameter $\lambda$ of the string measure is shifted. This is expressed as

$$ F^\ast \left\{ d\mu(\lambda) \exp \left( \frac{2}{\hbar} S(\lambda) \right) \right\} = d\mu(\lambda + d\lambda) \exp \left( \frac{2}{\hbar} S(\lambda + d\lambda) \right), \quad (5.4) $$

namely, the diffeomorphism pulls back the relevant measure of the theory with parameter $\lambda$ to the measure of the theory with parameter $\lambda + d\lambda$. In order to express the requirement (5.4)
explicitly we use the following two relations [19]

\[
F^*(d\mu(\lambda)) = \frac{\rho(\lambda)}{\rho(\lambda + d\lambda)} d\mu(\lambda + d\lambda) \left(1 + 2 dt \Delta U\right), \\
F^*\{S(\lambda)\} = S(\lambda) + dt \{S(\lambda), U\},
\]

where \(d\mu(\lambda) = \rho(\lambda) \prod d\psi\). Equation (5.4) then reduces to

\[
\left(\frac{d\lambda}{dt}\right) \cdot \frac{d}{d\lambda} \left(S + \frac{1}{2} \hbar \ln \rho\right) = \{S, U\} + \hbar \Delta U \equiv \hbar \Delta_S U.
\]

If the right hand side of the above equation is zero for some specific hamiltonian \(U\), we must conclude that the diffeomorphism does not change anything in the string field measure. The diffeomorphism is then a symmetry transformation.

The hamiltonian we will introduce depends on a pair of states \(O\) and \(\chi\) related by a BRST operator:

\[
\mathcal{O} + \{Q, \chi\} = 0 \quad \Rightarrow \quad |\mathcal{O}\rangle = -Q|\chi\rangle.
\]

We demand that \(|\mathcal{O}\rangle \in \hat{\mathcal{H}}\), namely \(b_0^-|\mathcal{O}\rangle = L_0^-|\mathcal{O}\rangle = 0\), but \(|\chi\rangle\) need not be annihilated by \(b_0^-\), and may involve the coordinate operator. Neither state needs to be primary. We define

\[
U_{\mathcal{O}, \chi} = U_{\mathcal{O}}^{[0,2]} - f_\mathcal{O}(\mathcal{B}_>^\prime) + f_\chi(\mathcal{V}_{0,3} + \{\mathcal{B}_{0,3}, \mathcal{V}\}).
\]

Note that this hamiltonian may involve a state \(\chi\) which is not annihilated by \(b_0^-\) since this state only appears inserted on three punctured spheres and there is no problem defining the phase of a local coordinate on such collection of surfaces (see [7]). Using this hamiltonian we will be able to treat ghost and matter dilatons in a similar fashion. We now follow [7] to compute the right hand side of (5.6). Since the computation is rather similar, we will be brief. The first term of the right hand side gives

\[
\{S, U_{\mathcal{O}, \chi}\} = \frac{1}{\kappa} \{S, U_{\mathcal{O}}^{[0,2]}\} + \{S, f_\chi(\mathcal{V}_{0,3})\} \\
- \frac{1}{\kappa} \{S, f_\mathcal{O}(\mathcal{B}_>)\} + \{S, f_\chi(\{\mathcal{B}_{0,3}, \mathcal{V}\})\}.
\]

Making use of the identities given in Eqs. (2.47) and (2.48) of Ref.[7] we can rewrite (5.9) as

\[
\{S, U_{\mathcal{O}, \chi}\} = \frac{1}{\kappa} f_\mathcal{O}(\mathcal{V}) - f_\mathcal{O}(\mathcal{V}_{0,3}) - f_\chi(\{\mathcal{V}, \mathcal{V}_{0,3}\}) + \frac{1}{\kappa} f_\mathcal{O}\left(\partial \mathcal{B}_> + \{\mathcal{V}, \mathcal{B}_>\}\right) \\
- f_\mathcal{O}(\{\mathcal{B}_{0,3}, \mathcal{V}\}) - f_\chi\left(\partial \mathcal{B}_{0,3}, \mathcal{V} + \{\mathcal{V}, \mathcal{B}_{0,3}, \mathcal{V}\}\right),
\]

where each row in (5.10) is equal to the corresponding row in (5.9). Using the Jacobi identity in the fourth row, the geometrical recursion relations, and the definition \(\partial \mathcal{B}_{0,3} = \mathcal{V}_{0,3} - \mathcal{V}_{0,3}\),
we obtain
\[
\{ S, U_{\mathcal{O}, \chi} \} = \frac{1}{\kappa} f_O \left( \partial \mathcal{B} > + \{ \mathcal{V}, \mathcal{B} \} + \mathcal{V} > \right) + f_X \left( \{ \mathcal{B}_0, 3, \hbar \Delta \mathcal{V} \} - \{ \mathcal{V}'_0, \mathcal{V} \} \right).
\] (5.11)

A similar calculation gives
\[
\hbar \Delta U_{\mathcal{O}, \chi} = - hf_X (\Delta \mathcal{V}_{0,3}) + \frac{1}{\kappa} f_O (\hbar \Delta \mathcal{B} >) - f_X (\{ \mathcal{B}_0, 3, \hbar \Delta \mathcal{V} \}).
\] (5.12)

Equations (5.12) and (5.11) must be added to give the right hand side of (5.6). Doing this, and using the recursion relations
\[
\partial \mathcal{B} > = \kappa \overline{\mathcal{V}} - \hbar \Delta \mathcal{B} > - \{ \mathcal{V}, \mathcal{B} \} - \mathcal{V} > + \hbar \kappa \mathcal{V}_{1,1},
\] (5.13)

for the $\mathcal{B}$ spaces we finally find
\[
\Delta_S U_{\mathcal{O}, \chi} = f_O (\overline{\mathcal{K}} \mathcal{V}) + f_X (\overline{\mathcal{C}} \mathcal{V}) + \hbar \left[ f_O (\mathcal{V}_{1,1}) - f_X (\Delta \mathcal{V}_{0,3}) \right].
\] (5.14)

Note that by definition, the term $\overline{\mathcal{C}} \mathcal{V}$ does not include vertices with zero punctures. Writing out the above equation more explicitly we have
\[
\Delta_S U_{\mathcal{O}, \chi} = \sum_{g, n \geq 1} f_O (\overline{\mathcal{K}} \mathcal{V}_{g,n}) + f_X (\overline{\mathcal{C}} \mathcal{V}_{g,n})
+ \sum_{g \geq 2} f_O (\overline{\mathcal{K}} \mathcal{V}_{g,0})
+ \hbar \left[ f_O (\mathcal{V}_{1,1}) - f_X (\Delta \mathcal{V}_{0,3}) \right].
\] (5.15)

### 5.2. Application to the various dilaton-like states

In this section we will use the general hamiltonian (5.8) and its basic property (5.14) to calculate the effect of shifts of the ghost dilaton $D_g$, the matter dilaton $D^\mu_\mu$, the true dilaton $D$, and the graviton trace $\mathcal{G}_t$. We begin with the case of the ghost-dilaton, fully analyzed in Refs.[7,8], as a way to use the present general formalism.

**Ghost-Dilaton.** Since $D_g + \{ Q, \chi_g \} = 0$, we consider the hamiltonian $U_{D_g, \chi_g}$ (a simpler form
of the ghost-dilaton hamiltonian will be given in the next section). The identities

\[ f_{D_g}(\mathcal{K}V_{g,n}) + f_{\chi_g}(\mathcal{L}V_{g,n}) = (2g - 2 + n)f(V_{g,n}), \]  

established in Ref. [7], and the identities

\[ f_{D_g}(\mathcal{K}V_{g,0}) = (2g - 2)f(V_{g,0}), \quad g \geq 2, \]
\[ f_{D_g}(\mathcal{V}_{1,1}) = f_{\chi_g}(\Delta V_{0,3}) = 0, \]  

established in [8] imply that (5.15) yields

\[ \Delta S U_{D_g,\chi_g} = \sum_{g,n} \bar{h}g^{2g-2+n}(2g - 2 + n)f(V_{g,n}). \]  

Since the string action is given by

\[ S = S_{0,2} + f(V) + \bar{h}S_{1,0} \]  

and the measure \( \ln \rho \) are all coupling constant independent we can write (5.18) as

\[ \Delta S U_{D_g,\chi_g} = \kappa \frac{d}{d\kappa} \left( S + \frac{1}{2} \bar{h} \ln \rho \right). \]  

This equation shows that the ghost dilaton changes the coupling constant \( \kappa \). Comparing with (5.6) we see that \( \frac{d\kappa}{dt} = \kappa \), or \( \kappa = \kappa_0 e^t \). Here \( t \) plays the role of the vacuum expectation value of the ghost-dilaton.

**Matter dilaton.** Since \( D^{\mu\nu} + \{Q, \chi^{\mu\nu}\} = 0 \), we are led to consider the hamiltonian \( U_{D^{\mu\nu},\chi^{\mu\nu}} \). This time (4.15) is relevant in the form

\[ f_{D^{\mu\nu}}(\mathcal{K}V_{g,n}) + f_{\chi^{\mu\nu}}(\mathcal{L}V_{g,n}) = \frac{1}{2} \eta^{\mu\nu}(2g - 2 + n)f(V_{g,n}). \]  

Moreover we claim that

\[ f_{D^{\mu\nu}}(\mathcal{K}V_{g,0}) = \frac{1}{2} \eta^{\mu\nu}(2g - 2)f(V_{g,0}), \quad g \geq 2. \]  

This follows from \( |D^{\mu\nu} = -Q[\xi^{\mu\nu}] + \frac{1}{2} \eta^{\mu\nu}|D_y\rangle \), the first equation in (5.17), and the vanishing of \( f_Q[\mathcal{K}V_{g,0}] \). On the other hand

\[ f_{\xi^{\mu\nu}}(\mathcal{V}_{1,1}) - f_{\xi^{\mu\nu}}(\Delta \mathcal{V}_{0,3}) = 0, \]  

by virtue of (2.12) and Stokes theorem which is valid here because \( b_0^- \) annihilates the state.
Since $D_{\mu\nu} = G_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} D_g$, it now follows that

$$f_{D_{\mu\nu}}(\mathcal{V}_{1,1}) = f_{G_{\mu\nu}}(\mathcal{V}_{1,1}) + \frac{1}{2} \eta_{\mu\nu} f_{D_g}(\mathcal{V}_{1,1}),$$

(Using (5.17) and (5.22))

$$f_{\chi_{\mu\nu}}(\Delta \mathcal{V}_{1,3}) - \frac{1}{2} \eta_{\mu\nu} f_{\chi_g}(\Delta \mathcal{V}_{0,3}),$$

(Using (2.25))

$$f_{\chi_{\mu\nu}}(\Delta \mathcal{V}_{0,3}),$$

(Using (5.17))

and therefore

$$f_{D_{\mu\nu}}(\mathcal{V}_{1,1}) - f_{\chi_{\mu\nu}}(\Delta \mathcal{V}_{0,3}) = 0.$$  

(5.24)

The computation of $\Delta S U_{D_{\mu\nu}, \chi_{\mu\nu}}$ is now straightforward. The terms in the right hand side of (5.15) have been evaluated in Eqs.(5.20),(5.21), and (5.24). We then find

$$\Delta S U_{D_{\mu\nu}, \chi_{\mu\nu}} = \frac{1}{2} \eta_{\mu\nu} \cdot \kappa \frac{d}{d\kappa} \left(S + \frac{1}{2} h \ln \rho\right).$$

(5.25)

Note that the off-diagonal states ($\mu \neq \nu$) have no effect whatsoever, they ought to be interpreted as generating gauge transformations. Each one of the $d$ diagonal states change the coupling constant. In particular, for the trace state we have

$$\Delta S U_{D_{\mu\mu}, \chi_{\mu\mu}} = \frac{d}{2} \cdot \kappa \frac{d}{d\kappa} \left(S + \frac{1}{2} h \ln \rho\right).$$

(5.26)

The only effect of a shift of the matter dilaton $D_{\mu}^\mu$ is a shift of the string coupling, with a strength proportional to the number of noncompact dimensions.

The complete dilaton. This state is written as $D = D_{\mu}^\mu - D_g = -\{Q, \chi_{\mu}^\mu - \chi_g\}$. Therefore

$$\Delta S U_{D_{\mu\mu}, \chi_{\mu\mu}} = \Delta S U_{D_{\mu\mu}, \chi_{\mu\mu}} - \Delta S U_{D_g, \chi_g} = \left(\frac{d-2}{2}\right) \cdot \kappa \frac{d}{d\kappa} \left(S + \frac{1}{2} h \ln \rho\right),$$

(5.27)

where use was made of (5.26) and (5.19). This equation shows that the only effect of a dilaton shift is changing the string coupling with a strength proportional to the number of noncompact dimensions minus two. This is the complete dilaton theorem in critical bosonic closed string field theory.

It is sometimes said that the effect of a dilaton is to change the dimensionless string coupling and the slope parameter $\alpha'$. We believe such statements are at best misleading. We see in the above discussion that $\Delta S U$ amounts to just changing the dimensionless string coupling. The slope parameter is the only dimensionful parameter in the string theory, and, as such, it is not
really a parameter but a choice of units. There is no invariant meaning to a change in $\alpha'$. If the theory had another dimensionful parameter, say a compactification radius $R$, then there is a new dimensionless ratio $R/\sqrt{\alpha'}$ that can be changed in the theory. One can view such change, if so desired, as a change of the dimensionful radius of compactification, or equivalently, as a change in $\alpha'$. Still, it should be remembered that only changes in dimensionless couplings have invariant meaning.

**Graviton Trace.** The final case of interest is that of the states $G^{\mu\nu}$ written as

$$G^{\mu\nu} = D^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}D_g = -\{Q, \chi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\chi_g\}.$$  

(5.28)

Therefore

$$\Delta_S U_{G^{\mu\nu},\xi^{\mu\nu}} = \Delta_S U_{D^{\mu\nu},\chi^{\mu\nu}} - \frac{1}{2}\eta^{\mu\nu}\Delta_S U_{D_g,\chi_g} = 0,$$  

(5.29)

where use was made of Eqs.(5.25) and (5.19). This shows that none of the $G^{\mu\nu}$ states deforms the string background. In particular, the graviton trace $G$ does not deform the string background. Note that the hamiltonian $U_{G^{\mu\nu},\xi^{\mu\nu}}$ defines a string field transformation whose inhomogeneous term is indeed $G^{\mu\nu}$, and leaves the string measure invariant. Our discussion of the extended complex indicates that a completely equivalent field transformation would be a gauge transformation generated by $\xi^{\mu\nu}$.

### 6. The Relevance of the Ghost-dilaton

In this section we wish to consider the case when the ghost-dilaton becomes a trivial state in the standard semirelative BRST cohomology. This is not a hypothetical situation, it happens in $D = 2$ string theory, as will be reviewed in sect.8. If the ghost dilaton is trivial it may seem that it cannot change the coupling constant, leaving the possibility that other states change it. This is not the way things work out. Inspection of Ref.[7] reveals that the ghost-dilaton shifts the string coupling whether or not it is BRST trivial. It then seems clear that the string coupling should not be an observable. We give a proof that this is the case. Explicitly this means the following: while the string coupling is a parameter appearing in the string action its value can be changed by a string field redefinition having no inhomogeneous term.

If the ghost dilaton is trivial, it can be written as $|D_g\rangle = -Q|\eta\rangle$ with $|\eta\rangle$ a legal state in the standard semirelative complex. If the two-form associated to the motions of the state $|\eta\rangle$ vanishes, as is the case for $D = 2$ strings, the field redefinition changing the string coupling is simply a homogeneous field redefinition. This will be the case whenever $|\eta\rangle$ involves a single ghost field acting on the vacuum. If this is not the case, the field redefinition is nonlinear; while it lacks an inhomogeneous term at the classical level, there may be $\hbar$-dependent inhomogeneous terms. We do not know of an example where $|\eta\rangle$ is this complicated. If this happens the string coupling might not be completely unphysical at the quantum level.
6.1. Simplifying the ghost-dilaton Hamiltonian

In this subsection we examine again the ghost-dilaton Hamiltonian and show that it can be simplified considerably. The simplified Hamiltonian will be of utility to show that a trivial ghost-dilaton implies an unphysical coupling constant.

The ghost-dilaton Hamiltonian reads

\[ H = \{0, 2\} - f \chi_g(\mathcal{V}_0 + \{\mathcal{B}_0, \mathcal{V}\}). \]  

We now show that the last term in this Hamiltonian, involving \( \chi_g \), can be replaced by a simpler term involving the ghost-dilaton. Consider the evaluation of \( \Delta S f \chi_g(\mathcal{B}_0) \)

\[ \Delta S f \chi_g(\mathcal{B}_0) = \Delta f \chi_g(\mathcal{B}_0) + \{ S, f \chi_g(\mathcal{B}_0) \}, \]

where we made use of the relation \( f \chi_g(\mathcal{V}'_0) = 0 \), which follows from sect.6.2 of Ref.[7]. Rearranging the terms in the equation we write

\[ f \chi_g(\mathcal{V}_0 + \{\mathcal{B}_0, \mathcal{V}\}) = -f \chi_g(\mathcal{B}_0) + f \chi_g(\Delta \mathcal{B}_0) + \Delta S f \chi_g(\mathcal{B}_0). \]

Constant terms are irrelevant for Hamiltonians since they do not generate transformations. We can therefore drop the second term in the above right hand side. Moreover, the third term in the right hand side can also be dropped since it is annihilated by \( \Delta S \). It follows that we can replace \( f \chi_g(\mathcal{V}_0 + \{\mathcal{B}_0, \mathcal{V}\}) \) by \( -f \chi_g(\mathcal{B}_0) \) in the dilaton Hamiltonian. We thus find that

\[ H = \{0, 2\} - f \chi_g(\mathcal{B}), \]

is a Hamiltonian equivalent to the original one, and by a slight abuse of notation we denote it with the same symbol. This Hamiltonian is completely analogous to the background independence Hamiltonians found in Ref.[19]. It is straightforward to verify that this Hamiltonian has the desired properties. One computes

\[ \Delta S H = \{ S, \{0, 2\} \} - \Delta f \chi_g(\mathcal{B}) - \{ S, f \chi_g(\mathcal{B}) \}, \]

where use was made of the recursion relations (5.13) together with \( \partial \mathcal{B}_0 = \mathcal{V}'_0 - \mathcal{V}_0 \). In the last step we used \( f \chi_g(\mathcal{V}'_0) = 0 \), which follows from \( f \chi_g(\mathcal{V}'_0) = 0 \) and the BRST property.
and, of the result of [8] that the two-form associated to the ghost-dilaton vanishes identically on the moduli space of once punctured tori. The fact that the ghost-dilaton hamiltonian can be written in the standard background independence form was not anticipated in [7] because both the identity \( f_{D_9}(V'_{0,3}) = 0 \), and the understanding of the behavior of dilatons at genus one were missing.

### 6.2. TRIVIALITY OF THE GHOST-DILATON AND THE STRING COUPLING

The ghost-dilaton theorem in string field theory, as established in Refs.[7,8] holds true whether or not the ghost-dilaton is BRST trivial or not. The ghost-dilaton hamiltonian will always have the effect of changing the string coupling. This ghost-hamiltonian produces a string field redefinition that includes an inhomogeneous term, a shift along the ghost-dilaton state. If the ghost-dilaton is trivial in the standard semirelative BRST cohomology, then it can be written as \( |D_g\rangle = -Q|\eta\rangle \), where \( |\eta\rangle \) is a standard vector in the closed string field theory state space. It then follows that there is another hamiltonian, the hamiltonian corresponding to a gauge transformation, that also has the property of inducing a shift along the direction of the ghost-dilaton. This gauge hamiltonian \( U_G \) reads

\[
U_G = \Delta_S U^{[0,2]}_\eta = U^{[0,2]}_{D_9} + f_\eta(V),
\]

where the gauge invariance property follows from \( \Delta_S U_G = 0 \). Since the gauge hamiltonian induces no change in the string action, it follows that the hamiltonian

\[
U_F \equiv U_D - U_G = -f_{D_9}(\mathcal{B}) - f_\eta(V),
\]

still shifts the string coupling constant. The term \( U^{[0,2]}_{D_9} \) inducing the shift along the ghost dilaton is absent in \( U_F \) and therefore the hamiltonian \( U_F \) is a hamiltonian that induces a field redefinition without physical import; it does not change the vacuum expectation value of the string field. The fact that the string coupling parameter in the action can be changed by a field redefinition without an inhomogeneous term implies that the string coupling is unphysical. As we will see now, strictly, and in all generality, this is only the case for genus zero, or the \( h \)-independent part of the string field redefinition generated by \( U_F \). In order to appreciate this point we now obtain a simple expression for the hamiltonian \( U_F \).

Our calculation begins by simplifying the expression for \( f_{D_9}(\mathcal{B}) \) in the ghost-dilaton hamiltonian by taking into account that \( |D_g\rangle = -Q|\eta\rangle \). We consider

\[
\{S, f_\eta(\mathcal{B})\} = -f_Q(\mathcal{B}) - f_\eta \left( \partial \mathcal{B} + \{V, \mathcal{B}\} \right). \tag{6.8}
\]

Using \( \Delta f_\eta(\mathcal{B}) = -f_\eta(\Delta \mathcal{B}) \), we rewrite the above equation as

\[
-f_{D_9}(\mathcal{B}) = -f_\eta \left( \partial \mathcal{B} + \{V, \mathcal{B}\} + \Delta \mathcal{B} \right) - \Delta_S f_\eta(\mathcal{B}). \tag{6.9}
\]
Using the recursion relations (5.13) we find

\[-f_D(B) = -f_\eta \left( \mathcal{V}'_{0,3} + \mathcal{K} \nu - \nu \right) - f_\eta \left( \Delta \mathcal{B}_{0,3} + \mathcal{V}_{1,1} \right) - \Delta S f_\eta(B) . \tag{6.10}\]

Since the left hand side is to be used in a hamiltonian we can drop the second term, being a constant, and the last term, being annihilated by \( \Delta S \). It follows that back in (6.7) the hamiltonian \( U_F \) can be taken to be

\[ U_F = -f_\eta \left( \mathcal{V}'_{0,3} + \mathcal{K} \nu \right) . \tag{6.11} \]

This is the simplest form of the hamiltonian. We see that at genus zero the hamiltonian is quadratic, and thus generates a field redefinition without an inhomogeneous term. Thus, at genus zero it is completely clear that the string coupling is unphysical. At higher genus there are, in principle, non-vanishing inhomogeneous terms arising from the surfaces \( \mathcal{K} \nu_{g,1} \). This might mean that the string coupling is not fully unphysical at the quantum level. More likely, it may be that whenever the ghost-dilaton is trivial the two-form corresponding to the motion of the state \( |\eta \rangle \) vanishes. This happens, for example, when each term in \( |\eta \rangle \) only has one ghost field acting on the vacuum, as is the case in \( D = 2 \) string theory. If the two-form vanishes the contribution from \( \mathcal{K} \nu \) vanishes as well, and the hamiltonian \( U_F = -f_\eta \left( \mathcal{V}'_{0,3} \right) \) simply generates a homogeneous field redefinition

\[ \delta |\Psi \rangle_1 = -\{ f_\eta \left( \mathcal{V}'_{0,3} \right) , |\Psi \rangle_1 \} = \left[ \langle V'_{123} |\eta \rangle_3 |S_{11'} \rangle \right] |\Psi \rangle_2 . \tag{6.12} \]

The linear operator acting on the string field has the interpretation of a contour integral of the conformal field operator \( \eta(z, \bar{z}) \) using local coordinates induced by the special puncture in the string vertex \( \mathcal{V}'_{0,3} \).

It is a straightforward calculation using \( \bar{\mathcal{L}} = -\{ \mathcal{V}'_{0,3} , \nu \} \) to show that acting on the first part of the hamiltonian (6.11) the operator \( \Delta S \) gives

\[ \Delta S \left( -f_\eta \left( \mathcal{V}'_{0,3} \right) \right) = f_D(\mathcal{V}'_{0,3}) + f_\eta(\Delta \mathcal{V}'_{0,3}) - f_\eta(\bar{\mathcal{L}} \nu) , \tag{6.13} \]

In the last step we have used the vanishing of \( f_\eta(\Delta \mathcal{V}'_{0,3}) \) which is readily established

\[ f_\eta(\Delta \mathcal{V}'_{0,3}) = f_\eta \left( \Delta \mathcal{V}_{0,3} - \partial \mathcal{B}_{0,3} \right) = f_\eta \left( -\partial \left[ \mathcal{V}_{1,1} - \mathcal{B}_{0,3} \right] \right) = f_D(\mathcal{V}_{1,1} - \mathcal{B}_{0,3}) = 0 . \tag{6.14} \]

Using once more the vanishing of the ghost-dilaton two-form on the moduli space of punctured
tori. It follows from (6.13) and (6.11) that in general
\[ \Delta_S U_F = -f_\eta (\overline{\mathcal{V}}) - \Delta_S f_\eta (\mathcal{K} \mathcal{V}). \] (6.15)
Whenever the two-form associated with moving the state \(|\eta\rangle\) vanishes we see that the effect of \(U_F\) reduces to inserting the state \(\eta\) (via \(\mathcal{L}\)) on all the coordinate disks of the string vertices. One can readily verify that for an arbitrary \(|\eta\rangle\) the relation \(\Delta_S f_\eta (\mathcal{K} \mathcal{V}) = f_Q \eta (\mathcal{K} \mathcal{V}) - f_\eta (\mathcal{L} \mathcal{V})\) holds. This confirms that (6.15) is equivalent to \(\Delta_S U_F = f_{Dg} (\mathcal{K} \mathcal{V})\) as befits a hamiltonian that must change the string coupling.

### 7. Application to D=2 String Theory.

In this section we consider the case of \(D = 2\) string theory as an illustration of the ideas developed in the previous sections of this paper. We analyze zero-momentum physical states that are candidates for deformations of the string background. The semirelative BRST-physical states are seen to be trivial in the extended complex and should not deform the string background. We discuss why they do not appear to deform the conformal theory. Our analysis here is a refinement of that of Mahapatra, Mukherji and Sengupta [10]. We argue that states in the absolute cohomology that are not annihilated by \(b_0\) do not lead to CFT deformations, thus evading a possible conflict with background independence. Finally, noting that in this background the ghost-dilaton is BRST trivial, we explain how to apply our earlier considerations showing that the string coupling is unobservable.

#### 7.1. Zero-momentum states and CFT deformations

Two dimensional string theory is based on a matter CFT including a Liouville field \(\varphi(z, \bar{z})\) and a field \(X(z, \bar{z})\). The holomorphic matter energy-momentum tensor reads \(T_m = -\frac{1}{2} \partial X \partial X - \frac{1}{2} \partial \varphi \partial \varphi + \sqrt{2} \partial^2 \varphi\). Due to the background charge, the field \(\varphi\) does not transform as a scalar. Under general analytic coordinate changes \(z'(z)\)
\[ \varphi(z', \bar{z'}) = \varphi(z, \bar{z}) - 2\sqrt{2} \ln \left| \frac{dz'}{dz} \right|. \] (7.1)
In this string theory an important operator \(a(z)\) [20,21] is obtained by taking the commutator of the BRST operator with the field \(\varphi\)
\[ a(z) \equiv \frac{1}{\sqrt{2}} \{ Q, \varphi(z, \bar{z}) \} = \partial c + \frac{1}{\sqrt{2}} c \partial \varphi. \] (7.2)
The operator \(a(z)\) is trivial in semirelative cohomology of the extended complex, but it is nontrivial in the standard semirelative cohomology.
Let us consider the absolute cohomology BRST physical states at ghost number two that can be formed without using exponentials of the free fields. The space of such states is spanned by a total of six states \([21]\), the first three of which are states in the semirelative cohomology

\[
S_1 = \frac{1}{\sqrt{2}} c\bar{c} \partial X \bar{\partial} \varphi + c\partial X(\partial c + \bar{\partial} \bar{c}) , \\
S_2 = \frac{1}{\sqrt{2}} c\bar{c} \bar{\partial} X \partial \varphi + \bar{c}\bar{\partial} X(\partial c + \partial \bar{c}) , \\
S_3 = c\bar{c} \partial X \bar{\partial} X ,
\]

and the other three are states that do not obey the semirelative condition

\[
A_1 = \frac{1}{\sqrt{2}} c\bar{c} \partial X \bar{\partial} \varphi + c\partial X(\bar{\partial} \bar{c} - \partial c) , \\
A_2 = \frac{1}{\sqrt{2}} c\bar{c} \bar{\partial} X \partial \varphi + \bar{c}\partial X(\partial c - \bar{\partial} \bar{c}) , \\
A_3 = a \cdot \bar{a}.
\]

The states in (7.3) are the only states in the semirelative cohomology at ghost number two under the condition of zero momenta \([21]\).

Let us begin our analysis with the first two semirelative states. We first observe that they are trivial in in the extended complex

\[
S_1(z, \bar{z}) = \{Q, s_1\}, \quad s_1 = -\frac{1}{\sqrt{2}} c \varphi \partial X ,
\]

with an exactly analogous statement holding for \(S_2\). This indicates that such states do not deform the string background. Indeed, since the BRST property holds for the class of states containing \(\varphi\), we may simply use the state \(s_1\) as the gauge parameter in a string field gauge transformation.

As further confirmation that the string background is not changed, let us now see that if we try to use the state \(S_1\) to deform the underlying CFT the only possibility seems to be zero deformation. In order to use \(S_1\) to deform the CFT we must find the corresponding coordinate invariant two-form. We introduce a metric \(\rho\) on the Riemann surface and a brief computation using \([7]\) gives

\[
S_1^{[2]} = -\frac{1}{\sqrt{2}} dz \wedge d\bar{z} \left[ \partial X \bar{\partial} (\varphi - 2\sqrt{2} \ln \rho) \right].
\]

We see that only the first term in \(S_1\) has contributed to the result. Moreover, \(\varphi\) has been replaced by the coordinate invariant combination \((\varphi - 2\sqrt{2} \ln \rho)\). While the two-form is well-defined (it is coordinate invariant), it is not Weyl invariant. The difficulty of obtaining a
coordinate invariant and Weyl invariant two-form was pointed out in [21]. At that time it
was not clear how to obtain the two-form associated to arbitrary states. We now see
that there is no Weyl-invariant two-form and thus no well-defined way to integrate the two-form on
Riemann surfaces.

The analysis cannot stop here. A similar situation happens for the ghost-dilaton: its two-
form is not Weyl independent. This dependence drops out of surface integrals when we add the
integral of a suitable one-form over the boundary of the region of integration. That one-form is
the one associated to the state \( |\chi_g\rangle \) which acted by the BRST operator gives the ghost-dilaton.
We therefore consider the state \( s_1 \) defined in (7.5) and construct the corresponding one-form.
We find
\[
s_1^{[1]} = \frac{1}{\sqrt{2}} dz (\varphi - 2\sqrt{2}\ln \rho) \partial X.
\] (7.7)
This one-form is coordinate invariant but it is not Weyl invariant. One readily verifies that
\( S_1^{[2]} = ds_1^{[1]} \), as expected from (7.5). In the case of the ghost-dilaton the gauge parameter
is not annihilated by \( b_0^- \), and the one-form, sensitive to the phase of the local coordinates,
is difficult to define globally. This time the state is annihilated by \( b_0^- \) and the one-form is phase independent. If we attempt to use the one-form to cancel out the Weyl dependence of
the integral of the two-form, the relation \( S_1^{[2]} = ds_1^{[1]} \) holding globally, and Stokes theorem
will imply that we get a total result of zero. This concludes our plausibility argument for the
absence of a nontrivial CFT deformation induced by the semirelative states \( S_1 \) and \( S_2 \). The
analysis of \( S_3 \) will be done shortly.

Consider now the states that are not annihilated by \( b_0^- \). These states, being outside the
closed string state space, do not correspond to linearized solutions of the string equations of
motion, and therefore do not represent deformations of string backgrounds. One can ask if such
states can deform the CFT. If this were the case we would have a problem with background
independence; we would have two nearby conformal theories giving rise to two string theories
that are not related by a shift of the string field. As we will explain now we believe it is unlikely
that states which are not annihilated by \( b_0^- \) define CFT deformations.

Consider the first state listed in (7.4). The associated two-form is found to be given by
\[
\mathcal{A}_1^{[2]} = -\frac{1}{\sqrt{2}} dz \wedge d\bar{z} \left[ \partial X \bar{\partial} (\varphi - 2\sqrt{2}i\theta) \right],
\] (7.8)
where \( \theta \) is the phase of the quantity \( a(z_0, \bar{z}_0) \) appearing in the definition of the local coordinate:
\( z - z_0 = a(z_0, \bar{z}_0)w + O(w^2) \). The deformation induced by this state of the CFT partition
function on a fixed surface would be given by integrating the above two-form over the complete
surface. Due to the non-zero Euler number, the phase of the local coordinate cannot in general
be defined globally throughout the surface and the integral is not well defined. It seems very
unlikely that one can define a nontrivial CFT deformation using the states in the absolute
cohomology that are not annihilated by \( b_0^- \).
7.2. The coupling constant in $D = 2$ strings

The ghost-dilaton, always trivial in absolute cohomology, becomes trivial in semirelative cohomology for the background defining $D = 2$ string theory. Indeed, one readily verifies that

$$c\partial^2 c - \bar{c}\partial^2 \bar{c} = -\frac{1}{\sqrt{2}} \{ Q, c\partial \varphi - \bar{c}\partial \bar{\varphi} \}. \tag{7.9}$$

Note that $c\partial \varphi - \bar{c}\partial \bar{\varphi}$ is fully legal; it is a state in the standard semirelative complex. Not only is the ghost-dilaton absent in $D = 2$ string theory, but now the last semirelative state $S_3 = c\bar{c}\partial X\bar{\partial}X$, is recognized to be trivial in the extended complex. This is because $S_3$ is equivalent to the ghost-dilaton in the extended complex.

Let us see explicitly why the string coupling is unobservable in this background. A change of coupling constant in string field theory amounts to scaling the string forms as

$$\langle \Omega^{[d]g,n} \rangle \rightarrow \langle \Omega^{[d]g,n} \rangle \left(1 - \epsilon \{2 - 2g - n\}\right), \tag{7.10}$$

where $d$ is the degree of the form. On the other hand in $D = 2$ strings

$$\langle \Sigma_{g,n} \rangle \sum_{i=1}^n \oint \frac{dz}{2\pi i} J^{(i)}(z) = -2\sqrt{2} (2 - 2g) \langle \Sigma_{g,n} \rangle \cdot J^{(i)}(z) = \partial \varphi^{(i)}(z). \tag{7.11}$$

Since this current has no ghost dependence, an identical relation holds for string forms. Since one can always add to the above right hand side a contribution proportional to $n$ by adding a constant to the charge associated to $J$, we see that the deformation (7.10) can be implemented by a similarity transformation induced by $J$. This means that a homogeneous string field redefinition changes the coupling constant making it unobservable. Indeed, the background we are considering, called the linear dilaton vacuum, has a coordinate dependent string coupling. The coupling is not observable since a shift of coupling is equivalent to a translation along the $\varphi$ coordinate.
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