Quantum collision theory in flat bands

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Abstract
We consider quantum scattering of particles in media exhibiting strong dispersion degeneracy. In particular, we study flat-banded lattices and linearly dispersed energy bands. The former constitute a prime example of single-particle frustration while the latter show degeneracy at the few- and many-particle level. We investigate both impurity and two-body scattering and show that, quite generally, scattering does not occur, which we relate to the fact that transition matrices vanish on the energy shell. We prove that scattering is instead replaced by projections onto band-projected eigenstates of the interaction potential. We then use the general results to obtain localized flat band states that are eigenstates of impurity potentials with vanishing eigenvalues in one-dimensional flat bands and study the particular case of a sawtooth lattice. We also uncover the relation between certain solutions of one-dimensional systems that have been categorized as ‘strange’, and the scattering states in linearly dispersed continuum systems.

Keywords: flat bands, scattering theory, few-body physics

1. Introduction

The behavior of quantum systems is to a large extent controlled by the competition between the kinetic energy and the interaction energy. The former tends to drive a system towards delocalization in order to minimize the spatial variation of quantum states, while the latter tends to drive the system towards localization in the minima of the interaction potential. This is the case irrespective of whether one is interested in few- or many-body systems, and whether one studies particles on a lattice or in the continuum. For instance, in few-body physics bound states are more likely in lower spatial dimension due to reduced kinetic energy [1–4]. For many-body systems, there is also a strong dependence on the spatial dimension when considering collective quantum states such as superconductors and condensates as the coherence of many-body systems becomes relatively weaker in lower dimension [5–7].

Another path to the study of kinetic versus interaction energy in quantum systems is via changes to the dispersion relation, i.e. to the particular form that the kinetic energy takes as a function of the momentum. By changing the dispersion relation from the usual quadratic form one may influence the competition of effects and thus explore a richer variety of quantum behavior. Here we are interested in few-body physics with non-standard dispersion relations. The first step towards an understanding of such systems is the determination of the scattering properties of particles, i.e. single particle scattering off an impurity that is represented by a given potential and two particles scattering off each other.

An interesting and important case is the limit where the dispersion is essentially flat in the sense that the single particle energy is independent of the momentum or quasi-momentum of the particle. In the context of lattices, this is called a flat band. The lack of kinetic energy means that interactions are decisive, and one may expect to see strong localization in such systems. However, superfluidity and kinetic behavior can be induced by interactions [8–12]. This sort of band can be achieved in many different lattices including the dice lattice [13–16], stub lattice (1D Lieb lattice) or diamond lattices [17], the sawtooth lattice [18, 19], Kagome lattices [20], and Lieb lattices [21]. For a theoretical discussion on how to construct general flat band lattices see [22]. Flat bands have been considered in many-body physics...
of the Hubbard model as a driver for ferromagnetism on so-called line graphs [23] of which the Kagome lattice is one example [24], and more generally if the flat band has more states than there are electrons in the system [25]. Lattices that contain flat bands have been realized experimentally using a diverse variety of physical systems including stub lattices with optical cavities [26], sawtooth [27], Kagome [29], and Lieb lattices [30] with photonic lattices, Kagome [28] and Lieb [31] lattices with cold atoms, and most recently the Lieb lattice realization using electronic surface states on an appropriately doped surface has been reported [32].

Flat bands are a very interesting case of the general phenomenon of non-trivial dispersion relations, and systems that display such behavior are attracting much attention at the moment. A particularly prominent example is graphene, a two-dimensional material with a linear dispersion around the Fermi energy, i.e. a behavior expected of particles in the relativistic regime (or massless particles). This feature can be generalized to three-dimensional materials with linear dispersions which are typically called either Dirac or Weyl semimetals depending on whether time-reversal or parity symmetry is broken or not. In fact, a new subfield has emerged studying these so-called Dirac materials [33]. These systems are very interesting also from a few-body point of view due to their non-trivial scattering properties [34–37]. We note that truly flat bands are also of great interest in the context of graphene [38, 39].

Within the field of cold atoms, a recent direction of great interest is the production of artificial gauge potentials [40]. In particular, different kinds of spin–orbit coupling known mostly from condensed-matter contexts, Rashba/Dresselhaus spin–orbit coupling, have been explored extensively both theoretically and experimentally [41, 42]. In the few-body community there has also been a flurry of activity in spin-orbit coupled systems with its non-trivial dispersion. In particular, the scattering properties have been discussed recently [43–48] as well as the effects of spin–orbit coupling on systems that are in a harmonic trap [49–54]. Furthermore, the three-body physics of this systems has also been found to be very rich [55–57].

In this paper we study potential scattering and two-body scattering for particles that either live on a lattice with a flat single-particle dispersion, or that are in a continuum system with a purely linear dispersion. We find in general that there is no scattering, strictly speaking, in these systems, and scattering states can be written as a projection of the incident Bloch waves onto the null subspace of the interaction potential, thereby generalizing the continuum results for intrabranch collisions in a Luttinger liquid [58], where the single-branch Hamiltonian, having a purely linear dispersion, commutes with total momentum and the two-body eigenstates are therefore eigenstates of the interaction potential. We also construct localized eigenstates for these systems in one-dimension and apply our general results to a particular flat-banded system—the so-called sawtooth lattice.

2. Potential scattering

We begin by considering the simplest non-trivial scenario consisting of a single particle colliding with a short-range potential in an otherwise periodic lattice structure. We denote the non-interacting Hamiltonian by $\hat{H}_0$ and the ‘impurity’ potential by $\hat{V}$. We further assume, without loss of generality, that the system possesses a finite number of bands labeled by $\sigma = 1, \ldots, n$, one of them being flat, namely $\sigma_0$, while the other $n - 1$ bands, i.e. $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, are in principle dispersive. We denote the different energy dispersions by $E_{\sigma_j}(\mathbf{k})$, with $\mathbf{k}$ in the first Brillouin zone (1BZ), while their corresponding eigenstates (Bloch waves), are denoted by $|\sigma_j, \mathbf{k}\rangle$, such that

$$\hat{H}_0|\sigma \mathbf{k}\rangle = E_{\sigma}(\mathbf{k})|\sigma \mathbf{k}\rangle. \quad (1)$$

We further assume that the flat band is gapped, that is, $E_{\sigma_0}(\mathbf{k}) \equiv E_{\sigma_0} = E_{\sigma_0}(\mathbf{k})$ for $j = 1, \ldots, n - 1$ and $\forall \mathbf{k} \in 1BZ$.

In order to evaluate the collisional properties of the system, we need to solve the Lippmann–Schwinger equation for the $T$-matrix at (complex) energy $\varepsilon$, which reads

$$\hat{T}(\varepsilon) = \hat{V} + \hat{V}G^{(0)}(\varepsilon)\hat{T}(\varepsilon). \quad (2)$$

Above, $G^{(0)}(\varepsilon) \equiv (\varepsilon - \hat{H}_0)^{-1}$ is the non-interacting Green’s function at energy $\varepsilon$. We work directly in the infinite volume limit, and in the quasi-momentum representation. We denote the volume of the 1BZ by $\Omega$, and normalize the single-particle states such that $\langle \sigma \mathbf{k}'|\sigma \mathbf{k}\rangle = \Omega \delta_{\sigma_j,\sigma} \delta(\mathbf{k}' - \mathbf{k})$. With these definitions, a resolution of the identity can be written as

$$\hat{1} = \sum_{\tau = 0}^{n-1} \frac{1}{\Omega} \int_{1BZ} d\varepsilon \langle \tau \mathbf{q}|\tau \mathbf{q}\rangle. \quad (3)$$

The equations satisfied by the elements of the $T$-matrix are then obtained by inserting equation (3) into the Lippmann–Schwinger equation, which yields

$$\langle \sigma \mathbf{k}'|\hat{T}(\varepsilon)|\sigma \mathbf{k}\rangle = \langle \sigma \mathbf{k}'|\hat{V}|\sigma \mathbf{k}\rangle + \sum_{\tau = 0}^{n-1} \frac{1}{\Omega} \int_{1BZ} d\varepsilon \langle \sigma \mathbf{k}'|\hat{V}G^{(0)}(\varepsilon)|\tau \mathbf{q}\rangle \langle \tau \mathbf{q}|\hat{T}(\varepsilon)|\sigma \mathbf{k}\rangle. \quad (4)$$

Until here, we have simply written down the Lippmann–Schwinger equation of potential scattering for a general multiband system. We now proceed to studying scattering of a particle with incident quasi-momentum $\mathbf{k}$ and incident band index $\sigma_0$, that is, on the flat band. To this end, we go on the energy shell by setting $\varepsilon \rightarrow \varepsilon_0 = E_{\sigma_0} + i\eta$, with $\eta$ an infinitesimal imaginary part of the energy that is taken to 0 ($\eta \rightarrow 0^+$) at the end of the calculation. Separating the contribution $\tau = \sigma_0$ from the rest of the sum on the second line of equation (4), we can re-write it as

$$\langle \sigma \mathbf{k}'|\hat{T}(\varepsilon_0)|\sigma \mathbf{k}\rangle = \langle \sigma \mathbf{k}'|\hat{V}|\sigma \mathbf{k}\rangle + \sum_{\tau = \sigma_0}^{n-1} \frac{1}{\Omega} \int_{1BZ} d\varepsilon \langle \sigma \mathbf{k}'|\hat{V}G^{(0)}(\varepsilon)|\tau \mathbf{q}\rangle \langle \tau \mathbf{q}|\hat{T}(\varepsilon_0)|\sigma \mathbf{k}\rangle + \frac{1}{\Omega} \int_{1BZ} d\varepsilon \langle \sigma \mathbf{k}'|\hat{V}|\sigma_0\mathbf{q}\rangle \langle \sigma_0\mathbf{q}|\hat{T}(\varepsilon_0)|\sigma \mathbf{k}\rangle. \quad (5)$$

Notice that on the second line of the above equation we have
used our assumption that the flat-band is gapped. It is now immediate to realize that, for equation (5) to have a non-trivial solution, the on-shell T-matrix elements of the form 
\[ \langle \sigma_n, q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle \]
must vanish as \( \sim \eta \) for \( \eta \to 0^+ \). In order to get rid of the seemingly divergent term in equation (5), we define
\[ \langle \sigma_n, q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle = \langle \sigma_n, q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle, \quad (6) \]
and therefore equation (5) reads
\[ \langle \sigma'\kappa' \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle = \langle \sigma'\kappa' \mid \hat{V} \rangle \mid \sigma_k \rangle \]
\[ + \sum_{\tau = \sigma_n} \frac{1}{\Omega} \int_{1BZ} dq \langle \sigma'\kappa' \mid \hat{V} \rangle \mid \tau q \rangle \langle \tau q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle \]
\[ + \frac{1}{\Omega} \int_{1BZ} dq \langle \sigma'\kappa' \mid \hat{V} \rangle \mid \sigma_n, q \rangle \langle \sigma_n, q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle. \quad (7) \]
Consider now, in equation (7), the matrix elements with \( \sigma' = \sigma_n \). After using the definition (6), and taking the limit \( \eta \to 0^+ \), we obtain
\[ - \langle \sigma_n, k' \mid \hat{V} \rangle \mid \sigma_k \rangle = \sum_{\tau = \sigma_n} \frac{1}{\Omega} \int_{1BZ} dq \langle \sigma_n, k' \mid \hat{V} \rangle \langle \tau q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle \]
\[ + \frac{1}{\Omega} \int_{1BZ} dq \langle \sigma_n, k' \mid \hat{V} \rangle \langle \sigma_n, q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle. \quad (8) \]
The final system of equations that must be solved for the on-shell T-matrix is therefore given by equations (7) for \( \sigma' = \sigma_n \), and equation (8) when \( \sigma' \neq \sigma_n \).

We now have all that is necessary to study collisions in flat bands. However, just with the Lippmann–Schwinger set of equations (7) and (8) the physics is not very transparent. We can easily see, from equations (8) and (6), that \( \langle \sigma_n, q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle \) satisfies
\[ - \langle \sigma_n, k' \mid \hat{V} \rangle \mid \sigma_k \rangle = \frac{1}{\Omega} \int_{1BZ} dq \langle \sigma_n, k' \mid \hat{V} \rangle \langle \sigma_n, q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle. \quad (9) \]
To prove the above, we only needed to use that \( \langle \tau q \mid \hat{T}(z_0) \rangle \mid \sigma_k \rangle = 0 \), which is granted by equation (6). In the projected flat band subspace, equation (9) can be rewritten as
\[ - \hat{V} = \hat{V} \hat{I}(z_0). \quad (10) \]
We diagonalise \( \hat{V} \) in the flat band subspace in order to solve equation (10). We choose a unitary operator \( \hat{U} \) that diagonalizes \( \hat{V} \). The corresponding orthogonal set of eigenfunctions are \( \mid \sigma_n, \alpha_k \rangle \) and their eigenvalues are denoted by \( V(\alpha_k) \). This means
\[ \mid \sigma_n, \alpha_k \rangle = \hat{U} \mid \sigma_n, k \rangle. \quad (11) \]
Equation (10) becomes
\[ - \Omega V(\alpha_q) \delta(q - q) = V(\alpha_q) \langle \sigma_n, \alpha_q \mid \hat{T}(z_0) \rangle \mid \sigma_n, \alpha_q \rangle. \quad (12) \]
For a short-range potential \( \hat{V} \), most of its eigenvalues vanish. We define the set of vanishing eigenvalues as
\[ I_0 = \{ q \in 1BZ: V(\alpha_q) = 0 \}, \quad (13) \]
and \( I' = 1BZ - I_0 \). For \( q \) and \( q' \) in the set \( I_0 \), the matrix elements \( \langle \sigma_n, \alpha_q \mid \hat{T}(z_0) \rangle \mid \sigma_n, \alpha_q \rangle \) are arbitrary, as is seen from equation (12). If \( q' \) is in \( I' \), then \( \langle \sigma_n, \alpha_q \mid \hat{T}(z_0) \rangle \mid \sigma_n, \alpha_q \rangle = 0 \). For \( q \) in \( I' \), if \( q \neq q' \) we have that \( \langle \sigma_n, \alpha_q \mid \hat{T}(z_0) \rangle \mid \sigma_n, \alpha_q \rangle \) is arbitrary, while if \( q = q' \) we have \( \langle \sigma_n, \alpha_q \mid \hat{T}(z_0) \rangle \mid \sigma_n, \alpha_q \rangle = -2\pi \).

We now shall calculate the scattering states. We split the scattering state \( \mid \psi \rangle \) into incident \( \mid \psi_0 \rangle \) and scattered \( \mid \psi_s \rangle \) waves, as
\[ \mid \psi \rangle = \mid \psi_0 \rangle + \mid \psi_s \rangle. \quad (14) \]
The relation between scattered and incident waves in terms of the T-matrix is given by
\[ \mid \psi_s \rangle = \hat{G}(z_0) \hat{T}(z_0) \mid \psi_0 \rangle. \quad (15) \]
The incident state is nothing but a flat band state, \( \mid \psi_0 \rangle = \mid \sigma_n, k \rangle \). The scattered wave has the form
\[ \langle \sigma_n, k' \mid \hat{T}(z_0) \rangle \mid \sigma_n, k \rangle = \frac{1}{i\hbar} \langle \sigma_n, k' \mid \hat{T}(z_0) \rangle \mid \sigma_n, k \rangle, \quad (16) \]
where in the last step we have used equation (6). We now set all the arbitrary matrix elements discussed above to a constant value \( \gamma \). Using the T-matrix calculated above, and after some algebraic manipulations, we find that the total scattering wave function is given by
\[ \mid \psi \rangle = \mid \sigma_n, k \rangle + \int_{1BZ} \frac{d\kappa}{\Omega} \mid \sigma_n, \kappa \rangle \left[ \sum_{\alpha_q} \int_{L_0} \frac{d\kappa'}{\Omega} \langle \sigma_n, \kappa' \mid \sigma_n, \alpha_q \rangle \right] \]
\[ \times \int_{1BZ} \frac{d\kappa'}{\Omega} \langle \sigma_n, \kappa' \mid \sigma_n, \alpha_q \rangle \mid \sigma_n, \kappa \rangle \]
2.1. Localized states

We turn our attention now to localized states in flat-banded one-dimensional lattices. These occur generally in highly frustrated systems, and correspond to a change of basis from usual Bloch waves. The physics of localized states is easy to grasp: since all Bloch waves share the same energy, any arbitrary superposition thereof is an eigenstate, too. If the flat band has a non-vanishing density of states\(^5\), we can integrate over all flat band modes with particular weights in order to obtain a localized eigenstate. It is simple to visualize this scenario by considering a toy model. Assume that a particle in a one-dimensional lattice has a single dispersionless ‘band’, and the eigenstates are plane waves with momentum \(k\). The corresponding flat band can be taken to have zero energy \((E(k) = 0)\). The following states are also eigenstates

\[
\hat{\delta}_{k_0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \exp(ikx)\exp(-i\xi x),
\]

and are obviously localized. This model, as unphysical as it seems, already contains all we need to know about localized states.

We begin by considering a general one-dimensional system with a flat band, labeled by \(\sigma_n\). We introduce a probe impurity modeled by a potential \(V\). As is clear from equation (20), scattering states on the flat band satisfy \(\hat{V}\psi = 0\). As before, we denote the Bloch waves by \(|\sigma_n k\rangle\), with \(k \in 1BZ\), and the Brillouin zone volume by \(\Omega\). We perform a change of basis as follows

\[
|\sigma_n\psi\rangle = \frac{2}{\Omega} \int_{-\pi/2}^{\pi/2} dk y(k)|\sigma_n k\rangle.
\]

The condition \(|\sigma_n\psi\rangle \in 1BZ\) is satisfied if, for all \(q\),

\[
\int_{-\pi/2}^{\pi/2} dk y(k)\langle \sigma_n q|\hat{V}|\sigma_n k\rangle = 0.
\]

The matrix elements of \(\hat{V}\) are, by construction, given by

\[
\langle \sigma_n q|\hat{V}|\sigma_n k\rangle = A(q)A(k) \sum_{x=-\infty}^{\infty} f_x(q)^* f_x(k)^* \exp(ikx),
\]

where \(A(k)\) is a normalization constant (which we have chosen to be real and positive), while \(f_x(k)\) are \(x\)-dependent functions. We are free to choose probe impurity potentials consisting of zero-range potentials acting on a given site \(x_0\), which we label by \(V_{x_0}^{(0)}\), and have the form (see equation (24))

\[
\langle \sigma_n q|V_{x_0}^{(0)}|\sigma_n k\rangle \propto A(k)f_{x_0}(k)^* \exp(ikx_0),
\]

where we have dropped the irrelevant factors that depend on \(q\). With these considerations, if \(f_{x_0}(k)\) has no nodes, equation (23) for the zero range potential \(V_{x_0}^{(0)}\) has solutions of the form

\[
\gamma_l(k) = \exp(ikx_0) [A(k)f_{x_0}(k)]^{-1}\exp(\xi x), \quad \ell \in \mathbb{Z} - \{0\}.
\]

If, on the other hand, \(f_{x_0}(k_*) = 0\) for some \(k_*\), then obviously \(V_{x_0}^{(0)}|\sigma_n k_*\rangle = 0\) and one solution is given by

\[
y(k) \propto \delta(k - k_0).\]

Other solutions can be constructed by eliminating this state from the expansion.

2.2. Example: sawtooth lattice

We consider an impurity located at \(x = 0\) in the so-called sawtooth lattice in one-dimension. The single-particle Schrödinger equation is given by [19]

\[
t \sum_{\mu=\pm 1} \left[\sqrt{2} \psi(x + \mu) + \frac{1}{2} \psi(x + 2\mu)\right] = E\psi(x),
\]

where \(x\) is integer lattice sites, and we have set the lattice spacing \(d = 1\). This system has two energy bands, one of which is flat while the other one is dispersive. These are given by

\[
E_0(k) = -2t, \quad E_1(k) = 2t(1 + \cos 2k).
\]

The eigenfunctions in the flat band are given by Bloch’s theorem, i.e.

\[
\psi_\ell(x) = \phi_\ell(x)\exp(ikx),
\]

where \(\phi_\ell(x)\) and \(\phi_\ell(1)\) are given by

\[
\phi_\ell(0) = \frac{1}{\sqrt{1 + 2\cos^2 k}}, \quad \phi_\ell(1) = -\sqrt{2}\cos(k)\phi_\ell(0).
\]

We see that the only points at which the ‘issue’ with the nodes happens is at \(k = \pm \pi/2\). In that case, \(\phi_\ell(0)\) is a constant while \(\phi_\ell(1)\) vanishes. That is, the eigenfunctions in the flat band with \(k = \pm \pi/2\) vanish at all odd sites, and those do not see an impurity located at an odd site. For an impurity located at an even site, such as the case we are considering, the eigenfunctions are all non-zero at even sites and there is no such issue with the nodes.

It is now straightforward to calculate eigenstates of the impurity potential with zero eigenvalue. We obtain

\[
y_\ell(k) = \sqrt{1 + 2\cos^2 k}\exp(ik\ell), \quad \ell \in \mathbb{Z} - \{0\}.
\]

In the position representation, these eigenfunctions \(\alpha_\ell(x)\) get the form

\[
\alpha_\ell(x) = \int_{-\pi/2}^{\pi/2} dk \exp(2\pi \ell i x), \quad x \text{ even}
\]

\[
\alpha_\ell(x) = -\sqrt{2} \int_{-\pi/2}^{\pi/2} dk \cos k \exp(2\pi \ell i x), \quad x \text{ odd}.
\]

Interestingly, the wave functions as constructed above immediately become, as one would naïvely expect, completely localized, and form the so-called ‘V-states’ of the sawtooth lattice [19].

\[
\alpha_\ell(x) = \pi \delta_{x, -2\ell} - \frac{\pi}{\sqrt{2}} \left( \delta_{x, -2\ell+1} + \delta_{x, -2\ell-1} \right),
\]

which exclude the \(\ell = 0\) term—the only state that overlaps with the impurity. If we add the \(\ell = 0\) state to the above set of eigenstate (which can be obtained by moving the impurity to \(x = 2\)), we actually recover the localized flat-band basis [59].
3. Two-body problems

We consider now two or more particles interacting with each other via a two-body potential \( \hat{V} \). We study two distinct scenarios below. Firstly, we generalize, very briefly, the result of the previous section on potential scattering for two particles in single-particle flat bands. We then move on to study collision theory for one-dimensional systems with linear dispersions, which is in many ways analogous to particles in flat bands.

3.1. Flat bands

We first consider the two-body problem in a system admitting a flat band. We use the same notations and definitions of section 2. The difference here is the fact that the potential \( \hat{V} \) is here a two-body interaction. The two-body scattering states are calculated in a similar way as in section 2. The two-body non-interacting states can be written as \( |\sigma_A k_A, \sigma_B k_B\rangle \), denoting particle \( A \) in state \( |\sigma_A k_A\rangle \) and particle \( B \) in state \( |\sigma_B k_B\rangle \). We assume that the two-body interaction only depends on the relative distance between the two particles, and therefore total quasi-momentum \( \mathbf{K} = k_A + k_B \) is conserved. For ease of notation, we write

\[
|\sigma_A k_A, \sigma_B k_B\rangle \equiv |\sigma_A \sigma_B k\rangle_K, \tag{36}
\]

where \( \mathbf{k} = (k_B - k_A)/2 \) is the relative momentum. Conservation of total quasi-momentum implies

\[
\mathbf{k}' \langle \sigma' A \sigma' B | \hat{T} (z_0) | \sigma_A \sigma_B \rangle_K \equiv \delta (\mathbf{K} - \mathbf{K}')
\times \mathbf{k} \langle \sigma' A \sigma' B | \hat{T} (z_0) | \sigma_A \sigma_B \rangle_K, \tag{37}
\]

and similarly for the interaction matrix elements. In equation (37), \( \langle \psi' | \hat{T} | \psi \rangle \) denotes reduced matrix elements for the \( T \)-matrix, after separation of center of mass and relative coordinates. Notice that, on a lattice, these matrix elements do depend on the conserved total quasi-momentum \( \mathbf{K} \). Now we can proceed in almost exactly the same manner as in section 2, where the on-shell scattering is now defined by the matrix elements

\[
\mathbf{k} \langle \sigma_\alpha \sigma_\alpha | \hat{T} (z_0) | \sigma_\alpha \sigma_\alpha \rangle_K = i \eta
\times \mathbf{k} \langle \sigma_\alpha \sigma_\alpha | \hat{T} (z_0) | \sigma_\alpha \sigma_\alpha \rangle_K. \tag{38}
\]

There are two formal differences between two-body scattering and potential scattering, however. Firstly, the assumption that band \( \sigma_\alpha \) is either the only flat band or that there are no other flat bands \( \sigma_\beta \) and \( \sigma_\gamma \) such that \( E_{\sigma_\beta} + E_{\sigma_\gamma} = 2E_{\sigma_\alpha} \) is necessary. Otherwise, an extra infinite set of states bring the system on the energy shell. The other difference, after assuming that the above does not hold, lies in the fact that a countably finite number of states \( |\sigma_\alpha \sigma_\beta \rangle_K \) may have non-interacting energies of \( 2E_{\sigma_\alpha} \). This does not bring any further problems, but in such case, the infinitesimal \( + i \eta \) must be added to the Green’s function (see the second line of equation (5), where we could safely omit this) and the extra states must be included in the calculation. Notice that this ‘point’ degeneracy is always present in standard scattering theory. For simplicity, and without loss of generality, we assume that \( E_{\sigma_\alpha} (K/2 + k) + E_{\sigma_\alpha} (K/2 - k) = 2E_{\sigma_\alpha} \) holds for every \( |\sigma_\alpha \sigma_\beta \rangle_K \) for a given total quasi-momentum \( \mathbf{K} \). The final result follows easily now from the analysis in section 2 and reads (see equation (20))

\[
|\psi\rangle = \hat{P}_0 |\sigma_A k_A, \sigma_B k_B\rangle. \tag{39}
\]

Above, the set \( E_0 \) is just the degenerate eigenspace of \( \hat{V} \) associated with eigenvalue zero on the projected flat band subspace. We can generalize the two-body result to any number of particles, provided we are at zero density or, more generally, at densities below the flat band critical density \( \nu_c \) for which \( \hat{V} \) would have no vanishing eigenvalues in the projected subspace. We point out that, even if the two restrictions mentioned above do not hold, the states (39) are still scattering states of the system.

3.2. Linear dispersions

In one-dimensional many-body systems, dispersion relations for collective excitations are typically linearized around the Fermi points \( \pm k_F \). This lead to Tomonaga’s [60] and Luttinger’s models [61]. While Tomonaga solved the many-body problem at finite densities correctly, Luttinger found a solution to his model which is only correct if the Fermi sea is not filled. Note that in Luttinger’s model the filled Fermi sea corresponds to infinite density, and Luttinger’s solutions are actually correct at finite densities, although they do not give the ground state\(^6\), at finite densities. The infinite density limit of Luttinger’s model was later solved by Mattis and Lieb in [62]. Mattis and Sutherland gave later on further solutions for one-dimensional linearly dispersed fermions, which they called ‘strange’ [63]. These correspond to localized wave functions that minimize the potential energy. We show here, from a scattering-theoretical point of view, that these ‘strange’ solutions are in one-to-one correspondence with flat-band scattering states. The so-called strange solutions, for the two-body system that we are here interested in, are easiest to understand in the position representation. This formalism is due to Mattis and Sutherland [63]. The Hamiltonian of the system is given by

\[
H = -i \hbar v_0 (\partial_x + \partial_x^2) + V(x_1 - x_2). \tag{40}
\]

In terms of center of mass \( X = (x_1 + x_2)/2 \) and relative \( x = x_1 - x_2 \) coordinates, the Hamiltonian can be written as

\[
H = -i \hbar v_0 \partial_X + V(x), \tag{41}
\]

which obviously commutes with both \( \hat{X} \) and \( \hat{x} \). The solutions to the stationary Schrödinger equation \( H \Psi = E \Psi \) read

\[
\Psi (X, x) = e^{iKX} \psi (x), \tag{42}
\]

where \( \psi \) satisfies

\[
V(x) \psi (x) = (E - \hbar v_0 K) \psi (x). \tag{43}
\]

Since scattering states have energies equal to the non-interacting energies, i.e. \( E = \hbar v_0 K \), in order for scattering states to satisfy (43), we require

\[
\psi (x) = e^{-iKx} \psi (x). \tag{44}
\]

In particular, the ‘strange’ scattering states are those \( \mu \) for which

\[
V(x) \psi (x) = (E - \hbar v_0 K) \psi (x). \tag{45}
\]

The above equation is the eigenvalue equation for the scattering problem. The solutions to this equation are called ‘strange’ solutions, and they correspond to localized wave functions that minimize the potential energy. We show here, from a scattering-theoretical point of view, that these ‘strange’ solutions are in one-to-one correspondence with flat-band scattering states. The so-called strange solutions, for the two-body system that we are here interested in, are easiest to understand in the position representation. This formalism is due to Mattis and Sutherland [63]. The Hamiltonian of the system is given by

\[
H = -i \hbar v_0 (\partial_x + \partial_x^2) + V(x_1 - x_2). \tag{40}
\]

In terms of center of mass \( X = (x_1 + x_2)/2 \) and relative \( x = x_1 - x_2 \) coordinates, the Hamiltonian can be written as

\[
H = -i \hbar v_0 \partial_X + V(x), \tag{41}
\]

which obviously commutes with both \( \hat{X} \) and \( \hat{x} \). The solutions to the stationary Schrödinger equation \( H \Psi = E \Psi \) read

\[
\Psi (X, x) = e^{iKX} \psi (x), \tag{42}
\]

where \( \psi \) satisfies

\[
V(x) \psi (x) = (E - \hbar v_0 K) \psi (x). \tag{43}
\]

Since scattering states have energies equal to the non-interacting energies, i.e. \( E = \hbar v_0 K \), in order for scattering states to satisfy (43), we require

\[
\psi (x) = e^{-iKx} \psi (x). \tag{44}
\]
exist, we need $V$ to have a strict finite range. Obviously, scattering states are highly degenerate.

We now show how this problem can be recast in terms of flat-band scattering. The single-particle dispersion reads
\[ E(k) = \hbar v_0 k, \]
and we work first without momentum cutoffs $\pm \Lambda$, that is, we take the limit $\Lambda \to \infty$ before we try to find the stationary scattering states. The two-body interaction $V$ is assumed to conserve total momentum. Then, any two particles with total momentum $K = k_1 + k_2$ have the same energy $E(K) = \hbar v_0 K$ regardless of their relative momentum $k = (k_1 - k_2)/2$. Therefore, for each $K$ we effectively have a potential scattering problem in a flat band. If the interaction has vanishing eigenvalues (that is, if it has nodes or has finite range), then scattering states exist and they have the form of equation (39). If we now assume that $V$ has finite range, that is, $V(x_1 - x_2) = 0$ for $|x_1 - x_2| > R$, then there are infinitely many scattering states. Since these all share the same energy, for fixed $K$, an arbitrary superposition of them also has the same energy. Hence, if $|\psi_{k, K-k}\rangle = \hat{P}_K|k, K-k\rangle$, the following vector $|\phi_K\rangle$ is also an eigenstate
\[ |\phi_K\rangle = \int_{-\infty}^{\infty} dk_i \phi_K(k_i)|\psi_{k, K-k}\rangle. \tag{45} \]
In particular, we can choose a completely co-localized relative wave function, and have the simple form
\[ |\phi_K\rangle = |K\rangle |x\rangle, \tag{46} \]
for all $x$ for which $V(x) = 0$. Obviously, all eigenstates, including those that are not scattering states, are eigenstates of the potential, and an orthonormal basis of them is given by the position eigenstates. These are what Mattis and Sutherland called ‘strange’ solutions in [63]. One of the reasons for this name was simply that they do not reduce to Slater determinants (for fermions) constructed with plane waves in the limit of zero interaction. As we see now from our perspective, this is not a problem, since, because of degeneracy, plane waves can be trivially constructed as superpositions of position eigenstates. The other, more physical reason, is that from these solutions one cannot construct the ground state of the infinitely filled Fermi sea or, equivalently for finite numbers of particles, the ground state with a momentum cutoff at $-\Lambda$. However, this ground state, which is easily obtained by means of bosonization, is also an eigenstate of the interaction. To see this, let us write down the second-quantized Hamiltonian for fermions in the momentum representation with cutoffs at $k = \pm \Lambda$,
\[ H = \hbar v_0 \sum_{k = -\Lambda}^{\Lambda} k c_k^\dagger c_k + \frac{1}{2\pi} \sum_{k, k', q} V(q) c_{k+q}^\dagger c_{k'-q}^\dagger c_{k'} c_k, \tag{47} \]
where $c_k$ ($c_k^\dagger$) annihilates (creates) a fermion with momentum $k$. The interaction part of the Hamiltonian conserves total momentum. Clearly, for $N$ fermions, the state of lowest total momentum $K = k_1 + \cdots + k_N$ corresponds to the Fermi sea $|F\rangle = \prod_{i=1}^{N} c_i^\dagger |0\rangle$, and is unique. Since $[\hat{K}, \hat{H}] = 0$, we know that $\hat{H}|F\rangle$ is also an eigenstate of $\hat{K}$ with momentum $K$. But by the uniqueness of $|F\rangle$, they must be equal up to a multiplicative factor. Thus $|F\rangle$ is an eigenstate of $\hat{H}$. Using the fact that $|F\rangle$ is an eigenstate of the non-interacting Hamiltonian implies that $|F\rangle$ is an eigenstate of the interaction.

4. Discussion

We have considered what standard scattering theory has to say about collisions in highly degenerate, or flat bands. We have found that, while the on-shell $T$-matrix vanishes in general linearly with energy, i.e. $\tilde{T}_{\text{on-shell}} \sim i\eta$ for $\eta \to 0^+$, it has a non-trivial structure given by $\tilde{T}$ which can be used to extract scattering states. We have also found that, for incident particles in the flat band, interband transitions are forbidden, and scattering states only exist if the interaction potential has at least one vanishing eigenvalue. Due to the extreme energy degeneracy, scattering states are not uniquely defined, and we have shown how a particular choice for certain arbitrary $T$-matrix elements yields physically intuitive stationary scattering states. These are simply given by the projection of the incident Bloch waves onto the null subspace of the interaction potential. Using this fact, we have found how to construct localized states in flat bands that are scattering states off impurity potentials in one-dimension, and particularized to a sawtooth lattice. We have also considered linearly dispersed continuum systems and showed how our scattering states are related to the strange solutions of one-dimensional quantum field theories found by Sutherland and Mattis [63].

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