The Yrast Line of a Rapidly Rotating Bose Gas: Gross-Pitaevskii Regime

Elliott H. Lieb

Departments of Mathematics and Physics, Princeton University, Princeton, NJ 08544, USA

Robert Seiringer

Department of Physics, Princeton University, Princeton, NJ 08544, USA

Jakob Yngvason

Fakultät für Physik, Universität Wien, and Erwin Schrödinger Institute for Mathematical Physics, 1090 Vienna, Austria

We consider an ultracold rotating Bose gas in a harmonic trap close to the critical angular velocity so that the system can be considered to be confined to the lowest Landau level. With this assumption we prove that the Gross-Pitaevskii energy functional accurately describes the ground state energy of the corresponding \( N \)-body Hamiltonian with contact interaction provided the total angular momentum \( L \) is much less than \( N^2 \). While the Gross-Pitaevskii energy is always an obvious variational upper bound to the ground state energy, a more refined analysis is needed to establish it as an exact lower bound. We also discuss the question of Bose-Einstein condensation in the parameter range considered. Coherent states together with inequalities in spaces of analytic functions are the main technical tools.

I. INTRODUCTION

A Bose gas rotating in a harmonic trap has a critical rotation speed above which the trap cannot confine it against centrifugal forces. If the trapping potential equals \( \frac{1}{2} (\omega_1^2 r^2 + \omega_3 x_3^2) \), with \( m \) the particle mass and \( r = \sqrt{x^2 + x_3^2} \) the distance from the axis of rotation, then the critical angular velocity is \( \omega_\perp \). In a reference frame rotating with angular velocity \( \Omega \) the Hamiltonian for one particle is

\[
\frac{p^2}{2m} + \frac{m}{2} (\omega_1^2 r^2 + \omega_3 x_3^2) - \Omega L
\]

with \( L \) the component of the angular momentum along the rotation axis. It is convenient and instructive to complete the square and write (1) as

\[
\frac{1}{2m} (p - A)^2 + \frac{m \omega_1^2}{2} x_3^2 + (\omega_\perp - \Omega) L
\]

where \( A = m \omega_\perp (x_3, -x_1, 0) \). In the rapidly rotating case, where \( 0 < \omega_\perp - \Omega \ll \min(\omega_1, \omega_3) \), it is natural to restrict the allowed wave functions to the ground state space of the first two terms in (2), which we denote by \( \mathcal{H} \), and this restriction will be made in this paper. The space \( \mathcal{H} \) consists of functions in the lowest Landau level (LLL) for motion in the plane perpendicular to the axis of rotation, multiplied by a fixed Gaussian in the \( x_3 \)-direction. Apart from the irrelevant additive constants \( 2 \hbar \omega_\perp \) and \( \hbar \omega_3 \), the kinetic energy in \( \mathcal{H} \) is simply

\[
\omega L
\]

with \( \omega = \omega_\perp - \Omega > 0 \). Note that \( L \) is non-negative for functions in \( \mathcal{H} \). Its eigenvalues are \( 0, \hbar, 2\hbar, \ldots \).

To characterize the functions in the space \( \mathcal{H} \), it is natural to introduce complex notation, \( z = x_1 + ix_2 \). Functions in \( \mathcal{H} \) are of the form

\[
f(z) \exp \left( -\frac{m \omega_1}{2 \hbar} |x|^2 - \frac{m \omega_3}{2 \hbar} |x_3|^2 \right)
\]

with \( f \) an analytic function. All the freedom is in \( f \) since the Gaussian is fixed. If the trapping potential in the \( x_3 \)-direction is not quadratic, the Gaussian in the \( x_3 \)-variable has to be replaced by the appropriate ground state wave function.

A fancy way of saying this is that our Hilbert space \( \mathcal{H} \) consists of analytic functions on the complex plane \( \mathbb{C} \) with inner product given by

\[
\langle \phi | \psi \rangle = \int_\mathbb{C} \phi(z)^* \psi(z) e^{-|z|^2} \, dz,
\]

where \( dz \) is short for \( dx_1 \, dx_2 \). For simplicity we choose units such that \( m = \hbar = \omega_\perp = 1 \). The eigenfunction of the angular momentum \( L \) corresponding to the eigenvalue \( n \) is simply \( z^n \). In other words, \( L = z \partial_z \). We remark that the expectation value of \( r^2 = |z|^2 \) in this state is \( n + 1 \).

For a system of \( N \) bosons, the appropriate wave functions are analytic and symmetric functions of the bosons coordinates \( z_1, \ldots, z_N \). The Hilbert space is thus \( \mathcal{H}_N = \otimes_{\text{symm}}^N \mathcal{H} \). The kinetic energy is simply \( \omega \) times the total angular momentum.

In addition to the kinetic energy, there is pairwise interaction among the bosons. It is assumed to be short range compared to any other characteristic length in the system, and can be modeled by \( g \sum_{1 \leq i < j \leq N} \delta(z_i - z_j) \) for some coupling constant \( g > 0 \). Physically, this \( g \) is proportional to \( a \omega_3^{1/2} \) where \( a \) is the scattering length of the three-dimensional interaction potential. On the full,
original, Hilbert space \( \otimes_{\text{symm}}^N L^2(\mathbb{R}^2) \), a \( \delta \)-function as a repulsive interaction potential is meaningless. On the subspace \( H_N \) the matrix elements of \( \delta(z_i - z_j) \) make perfect sense, however, and define a bounded operator \( \delta_{ij} \). Using the analyticity of the wave functions this operator is easily shown to act as
\[
(\delta_{12}\psi)(z_1, z_2) = \frac{1}{2\pi} \psi(((z_1 + z_2)/2, (z_1 + z_2)/2),
\]
which takes analytic functions into analytic functions. Its matrix elements in a two-particle function \( \psi(z_1, z_2) \) are
\[
\langle \psi | \delta_{12} | \psi \rangle = \int_C |\psi(z, z)|^2 e^{-2|z|^2} dz.
\]
The dimensional reduction from three to two dimensions for the \( N \)-body problem and the restriction to the LLL is, of course, only reasonable if the interaction energy per particle is much less than the energy gap \( 2\hbar \omega_L \) between Landau levels and the gap \( k \omega_1 \) for the motion in the \( x_3 \)-direction. For a dilute gas the interaction energy per particle is of the order \( a \rho \) where \( \rho \) is the average three-dimensional density \([1]\). Provided \( Ng/\omega \) is not small we can estimate \( \rho \) by noting that the effective radius \( R \) of the system can be obtained by equating \( a \rho \sim Na/(R^2\omega_3^{1/2}) \sim Ng/R^2 \) with the kinetic energy \( \omega R \sim \omega R^2 \). This gives \( R \sim (Ng/\omega)^{1/4} \) and the condition for the restriction to the LLL becomes
\[
(Ng\omega)^{1/2} \ll \min\{\omega_\perp, \omega_3\}. \tag{8}
\]
The physics of rapidly rotating ultracold Bose gases close to the LLL regime has been the subject of many theoretical and experimental investigations in recent years, starting with the papers \([2, 3, 4, 5, 6]\). The recent reviews \([2, 4, 5, 10]\) contain extensive lists of references on this subject. On the experimental side we mention in particular the papers \([11, 12, 13]\) that report on experiments with rotational frequencies exceeding 99% of the trap frequency.

### II. MODEL AND MAIN RESULT

The discussion in the Introduction leads us to the following well-known model (see, e.g., \([2, 7, 10]\)) for \( N \) bosons with repulsive short-range pairwise interactions:
\[
H = \omega \sum_{i=1}^N L_i + g \sum_{1 \leq i < j \leq N} \delta_{ij}. \tag{9}
\]
It acts on analytic and symmetric functions of \( N \) variables \( z_i \in \mathbb{C} \). The angular momentum operators are \( L_i = z_i \partial_{z_i} \), and \( \delta_{ij} \) acts as in \([4]\). The parameters \( \omega \) and \( g \) are assumed to be positive. The rigorous derivation of this model from the 3D Schrödinger equation for particles interacting with short, but finite range potentials will be presented elsewhere \([14]\).

Note that the two terms in \( H \) commute with each other, and hence can be diagonalized simultaneously. An exact eigenfunction with total angular momentum \( L_{\text{tot}} = \sum_{i=1}^N L_i = N(N - 1) \) is the Laughlin state \( \prod_{i<j}(z_i - z_j)^2 \) for which the interaction vanishes, of course. For smaller \( L \) the interaction is strictly positive. It is expected that the lowest eigenvalue of the interaction is of the order \( N \) when \( L_{\text{tot}} \) is of the order \( N^2 \). The lower boundary of the joint spectrum of interaction and angular momentum has been called the yrast curve \([15]\), a term originating in nuclear physics. Upper bounds on this curve can be found by variational calculations, and a host of trial functions with many interesting properties related to the Fractional Quantum Hall Effect (FQHE) have been employed for this purpose, see, e.g., \([6, 10, 16, 18, 20, 21]\). A challenging problem is to establish the missing reliable lower bounds.

In this paper we consider the case when \( L_{\text{tot}} \) is much smaller than \( N^2 \), i.e., we investigate one corner of the asymptotics of the yrast region. In this case, we shall show that the ground state energy of \( H \) is well approximated (rigorously so in the \( N \to \infty \) limit) by the Gross-Pitaevskii energy. The latter energy is the minimum of the Gross-Pitaevskii functional
\[
E_{\text{GP}}(\phi) = \omega \langle \phi | L | \phi \rangle + \frac{g}{2} \int_C |\phi(z)|^4 e^{-2|z|^2} dz \tag{10}
\]
over all analytic functions \( \phi \) with \( \langle \phi | \phi \rangle = N \). This can also be viewed as a Hartree approximation to the many-body system, where one takes the expectation value of \( H \) with a simple product function \( \prod_{i=1}^N \phi(z_i) \) (and ignores a factor \( N - 1/N \)). The minimization problem for \([10]\) and the properties of the minimizers, in particular their vortex structure, have been studied in some detail by many authors including \([5, 22, 23, 24, 25]\).

To state our result precisely, let \( E_0(N, \omega, g) \) be the ground state energy of \( H \), and let \( E_{\text{GP}}(N, \omega, g) \) be the Gross-Pitaevskii energy. We will find a positive, finite constant \( C \) such that
\[
E_{\text{GP}}(N, \omega, g) \geq E_0(N, \omega, g) \geq E_{\text{GP}}(N, \omega, g) \left[ 1 - C \left( \frac{g}{Ng} \right)^{1/10} \right] \tag{11}
\]
for all \( N, \omega \) and \( g \) such that \( gN/\omega \) is bounded below by some (arbitrary) fixed constant. More precisely, for any \( c > 0 \) there exists a \( C < \infty \) such that \([11]\) holds if \( gN/\omega \geq C \). In particular, this implies that the ratio of \( E_0 \) and \( E_{\text{GP}} \) is close to one if \( g \ll N\omega \) and \( g \gtrsim N^{-1} \omega \).

Note that, by simple scaling, \( E_{\text{GP}}(N, \omega, g) = NaE_{\text{GP}}(1, 1, gN/\omega) \). Hence the contrary case of small \( gN/\omega \) is not particularly interesting; as \( gN/\omega \to 0 \) one obtains a non-interacting gas. On the other hand, if \( gN/\omega \) is not small the kinetic and interaction energy are of the same order of magnitude, which, according to the previous back-of-the-envelope analysis, implies \( E_{\text{GP}}(N, \omega, g) \sim N \sqrt{Ng/\omega} \). Hence the total angular momentum, which is obtained by taking the derivative of
the energy with respect to \( \omega \), is of the order \( N \sqrt{Ng/\omega} \), which is much less than \( N^2 \) if and only if \( g \ll N\omega \). Hence our bound (11) covers the whole parameter regime \( L_{\text{tot}} \ll N^2 \).

We note also that in terms of the filling factor \( \nu = N^2/(2L_{\text{tot}}) \) \( \nu \) the parameter regime \( g/(N\omega) \ll 1 \) corresponds to \( \nu \gg 1 \). In contrast, the Laughlin wave function has filling factor \( \nu = 1/2 \) for \( N \) large. Between these two extremes rich physics related to the FQHE is expected \[10\], but this regime is apparently still out of experimental reach.

Before giving the proof of (11), we shall discuss some of its implications for the yrast line. Recall that the yrast energy \( I_0(L) \) is defined as the ground state energy of \( \sum_{i<j} \delta_{ij} \) in the sector of total angular momentum \( L \).

What (11) says is that

\[
\omega L + g I_0(L) \geq E^{GP}(N, \omega, g) \left[ 1 - C \left( \frac{g}{N\omega} \right)^{1/10} \right]
\]

for any \( L \geq 0 \). As \( Ng/\omega \) gets large, it is expected that

\[
E^{GP}(N, \omega, g) \approx b N \sqrt{Ng/\omega}
\]

to leading order, with \( b \approx 0.57 \). An upper bound with this value of \( b \) was actually derived in \[29\], but a lower bound with the same \( b \) is still open. We remark that it is easy to derive a lower bound with \( b = \sqrt{8/\pi} \approx 0.53 \). This follows using \( \langle \phi | L | \phi \rangle = \int |\phi(z)|^2 |z|^2 |z|^2 - 1 | e^{-|z|^2} \, dz \) and minimizing \( \int |\omega| |z|^2 |\phi(z)|^2 e^{-|z|^2} + (g/2) |\phi(z)|^4 e^{-2|z|^2} \, dz \) over all \( \phi \), dropping the LLL condition. Thus, \( 0.53 \leq b \leq 0.57 \). Using our bound (12) we deduce from (13) that

\[
I_0(L) \geq \frac{b^2 N^3}{4L} \quad \text{for} \quad N \ll L \ll N^2.
\]

For \( 2 \leq L \leq N \) it is well known that \( I_0(L) = N(4\pi)^{-1}(N - 1 - L/2) \) \[3, 4, 17, 18, 19\]. Our bound (14) reproduces this result for large \( N \) since, as we shall prove below,

\[
E^{GP}(N, \omega, g) = \frac{N^2 g}{4\pi} \quad \text{if} \quad g \leq \frac{8\pi \omega}{N},
\]

which is just the GP energy of the constant function. In particular, the zero angular momentum state is a minimizer of the GP functional for \( gN \leq 8\pi \omega \). Eqs. (12) and (14) imply that

\[
I_0(L) \geq \frac{N}{4\pi} \left[ N \left[ 1 - C \left( \frac{8\pi \omega}{N^2} \right)^{1/10} \right] - \frac{1}{2}L \right]
\]

for all \( L \geq 0 \). In order to prove (14), we note that the GP energy (10) equals a certain two-particle energy, namely

\[
E^{GP}(\phi) = \frac{1}{2} \left\langle \phi \otimes \phi \left| \frac{\omega}{N} (L_1 + L_2) + g \delta_{12} \phi \otimes \phi \right. \right\rangle.
\]

The operator \( \delta_{12} \) commutes with \( L_1 + L_2 \), and satisfies \( \delta_{12}^2 = (2\pi)^{-1} \delta_{12} \). That is, \( 2\pi \delta_{12} \) is a projection operator. We claim that

\[
\langle \phi \otimes \phi | L_1 + L_2 | \phi \otimes \phi \rangle \geq 4 \langle \phi \otimes \phi | 1 - 2\pi \delta_{12} | \phi \otimes \phi \rangle
\]

for any \( \phi \in \mathcal{H} \). This clearly implies (14). To see (16), we can decompose the two-particle function \( \phi(z_1) \phi(z_2) \) into a sum \( \sum_{n \geq 0} \psi_n(z_1, z_2) \) of functions of given total angular momentum \( n \). For the terms with \( n \geq 4 \) the bound (10) certainly holds. Also for \( n = 0 \) and \( n = 1 \) it holds, since \( 1 - 2\pi \delta_{12} = 0 \) in this case. We thus have to consider only the cases \( n = 2 \) and \( n = 3 \). On the subspace with \( n = 2 \), \( (1 - 2\pi \delta_{12}) \phi \otimes \phi \) is proportional to the normalized function \( \eta_2(z_1, z_2) = (z_1 - z_2)^2/\sqrt{8} \). It is straightforward to check that \( \langle \phi \otimes \phi | \eta_2 \rangle \leq ||\psi_2||/\sqrt{2} \) which implies the desired result. Similarly, on the subspace with \( n = 3 \), \((1 - 2\pi \delta_{12}) \phi \otimes \phi \) is proportional to \( \eta_3(z_1, z_2) = (z_1 - z_2)^2(z_1 + z_2)/\sqrt{16} \). One checks that \( ||\phi \otimes \phi | \eta_3 \rangle \leq ||\psi_3|| \sqrt{3/4} \), which is what is needed.

Everything now depends on the proof of (11) and the most of the rest of this paper is devoted to this task. The first inequality in (11) follows easily by taking product wave functions as trial wave functions, as mentioned earlier. Hence it remains to show the second inequality in (11), which is the lower bound on \( E_0 \).

For the proof we shall employ the technique of coherent states and c-number substitutions \[27, 28\] that was used in \[29\] to solve a related problem, namely to derive the Gross-Pitaevskii equation from the three-dimensional many-body Hamiltonian of a rotating Bose gas away from the critical rotational frequency. This was done in the Gross Pitaevskii limit where both \( \omega \) and \( Ng \) are fixed and order unity. The situation discussed in the present paper is partly simpler than that in \[29\], where the interaction was described by an arbitrary repulsive potential of short range instead of a contact interaction. On the other hand we now have to face new problems. In the present setting the coupling parameter \( Ng/\omega \) can vary with \( N \) (as long as it is \( \ll N^2 \)) and this fact requires considerably more delicate estimates on matrix elements of the interaction between two-particle states than in \[29\]. In particular, our bound is asymptotically exact in the whole "Thomas–Fermi" regime. In order to make the proof more transparent it will be divided into seven steps.

### III. DERIVATION OF THE MAIN INEQUALITY

#### A. Step 1

As a first step towards deriving a lower bound on the ground state energy of \( H \), we consider a slightly bigger Hamiltonian \( H' \) which is constructed in the following way. The reason for introducing \( H' \) is that the addi-
tional small positive term is needed to compensate negative terms that will occur at a later stage of the proof.

For integers $J$, let

$$
\chi_J(L) = \begin{cases} 
0 & \text{for } L \leq J \\
\exp \left[ -\frac{1}{8} (\sqrt{L} - \sqrt{J+1})^2 \right] & \text{for } L \geq J + 1.
\end{cases}
$$

(17)

Pick an integer $J_0 \geq 1$. We first show that there exists a $J$ with $J_0 \leq J < 2J_0$ such that

$$
\left\langle \sum_{i=1}^{N} \chi_J(L_i) \right\rangle \leq 2^{3/2} \left( 1 + 4\sqrt{\pi} \right) \frac{E_0(N, \omega, g)}{\omega J^{3/2}},
$$

(18)

where $\langle \cdot \rangle$ denote the expectation value in the ground state of $H$. To see this, note that

$$
\sum_{J \geq 0} \chi_J(L) \leq 1 + 2 \int_0^{\sqrt{L}} \exp \left[ -\frac{1}{8} (\sqrt{L} - s)^2 \right] s \, ds 
\leq 4 \sqrt{\pi L}.
$$

In particular, since $\chi_J(0) = 0$, we have $\sum_J \chi_J(L) \leq (1 + 4\sqrt{\pi}) \sqrt{L}$ for all integers $L \geq 0$. This implies that

$$
\frac{1}{J_0} \sum_{J=J_0}^{2J_0-1} \chi_J(L) \leq (1 + 4\sqrt{\pi}) \frac{\sqrt{L}}{J_0} \leq 2^{3/2} \left( 1 + 4\sqrt{\pi} \right) \frac{L}{J_0^{3/2}},
$$

where we have used that $L \geq J_0$ in order that the left side be non-zero, and $J_0 \geq J/2$. The expectation value of $\sum_{i=1}^{N} L_i$ is bounded by the total energy divided by $\omega$. Hence the bound (18) holds on average for $J_0 \leq J < 2J_0$ and must thus hold for at least one such $J$.

We can now pick an $\eta > 0$ and consider the modified Hamiltonian

$$
H' = H + \eta J^{3/2} \omega \sum_{i=1}^{N} \chi_J(L_i).
$$

By (18) its ground state energy is bounded from above by $E_0(1 + b \eta)$, with $b = 2^{3/2} \left( 1 + 4\sqrt{\pi} \right)$. In other words, $E_0$ is bounded from below by the ground state energy of $H'$ divided by $(1 + b \eta)$. We choose $\eta = (g/N\omega)^{1/10}$.

### B. Step 2

In order to derive a lower bound on $H'$, we shall first extend it to Fock space in the usual way. We do this in order to be able to utilize coherent states and the lower and upper symbols of the extended operator.

Let $\varphi_j(z) = z^j (\pi j!)^{-1/2}$ denote the normalized eigenfunctions of the angular momentum operator $L$ with eigenvalue $j \in \mathbb{N}$. Let $a_j^\dagger$ and $a_j$ denote the corresponding creation and annihilation operators. On Fock space, consider the operator

$$
H = \omega \sum_{i=1}^{N} \left[ i + \eta J^{3/2} \chi_J(i) \right] a_i^\dagger a_i
$$

$$
+ \frac{g}{2} \sum_{i,j,k} \langle \varphi_i \otimes \varphi_j | \delta | \varphi_k \otimes \varphi_{i+j-k} \rangle a_i^\dagger a_j^\dagger a_k a_{i+j-k}
$$

$$
+ \mu \left( \sum_{i \geq 0} a_i^\dagger a_i - N \right)^2
$$

(19)

for some $\mu > 0$. For simplicity we denote $\delta_{12}$ simply by $\delta$. Note that because of conservation of angular momentum all other matrix elements of $\delta$ vanish. The Hamiltonian $H'$ can be viewed as the restriction of $H$ to the subspace containing exactly $N$ particles. The value of $\mu$ is irrelevant, since the term multiplying $\mu$ vanishes in this subspace. In particular, the ground state energy of $H'$ is bounded from below by the ground state energy of $H$, for any value of $\mu$.

We introduce coherent states for all angular momentum states up to $J$. That is, in Eq. (19) we normal-order all the $a_i^\dagger$ and $a_i$ in the usual way (which is only relevant for the term multiplying $\mu$ since all other terms are already normal ordered) and then replace all $a_i$ by complex numbers $\zeta_i$ and all $a_i^\dagger$ by the conjugate numbers $\zeta^*_i$ for $0 \leq i \leq J$. The resulting operator on the Fock space generated by the modes $> J$ is called the lower symbol of $H$ and will be denoted by $h_{\ell}(<\zeta>$), where $\zeta$ stands for $(\zeta_0, \ldots, \zeta_J)$. Note that $h_{\ell}(<\zeta>$) does not conserve the particle number, which explains why it was necessary to embed our $N$-particle Hilbert space into Fock space.

The ground state energy of $H$ is bounded from below by the ground state energy of the upper symbol $h_u(\zeta)$ when minimized over all parameters $\zeta$. The upper symbol is given in terms of the lower symbol as

$$
h_u(\zeta) = e^{-\sum_{i=0}^{J} \partial_{\zeta_i} \partial_{\zeta^*_i} h_{\ell}(\zeta)} = h_{\ell}(\zeta) + U_1(\zeta) + U_2(\zeta),
$$

(20)

where

$$
U_1(\zeta) = -\sum_{i=0}^{J} \partial_{\zeta_i} \partial_{\zeta^*_i} h_{\ell}(\zeta)
$$

equals

$$
- \omega \sum_{i \leq J} i - 2g \sum_{0 \leq i \leq J} \langle \varphi_i \otimes \varphi_i | \delta | \varphi_i \otimes \varphi_i \rangle
$$

$$
- 2g \sum_{0 \leq i < J \leq J} \langle \varphi_i \otimes \varphi_i | \delta | \varphi_i \otimes \varphi_j \rangle a_i^\dagger a_j
$$

$$
- \mu \left( 2 \| \varphi_0 \|^2 - 2N + 2N^> + 1 \right) (J + 1) + 2 \| \varphi_0 \|^2
$$

and

$$
U_2(\zeta) = \frac{1}{2} \left( \sum_{i=0}^{J} \partial_{\zeta_i} \partial_{\zeta^*_i} \right) h_{\ell}(\zeta)
$$

$$
= g \sum_{0 \leq i, j \leq J} \langle \varphi_i \otimes \varphi_j | \delta | \varphi_i \otimes \varphi_j \rangle + \mu (J + 1) (J + 2).
$$

Here, $\phi_0 = \sum_{j=0}^{J} \zeta_j \varphi_j$, $\| \phi_0 \|^2 = \langle \phi_0 | \phi_0 \rangle = \sum_{j=0}^{J} \| \zeta_j \|^2$ and $N^> = \sum_{k>J} a_k^\dagger a_k$. Note that in a Taylor expansion of the exponential in (20) only the first three terms contribute since $h_{\ell}(<\zeta>$) is a polynomial of degree four.
For a lower bound, we can use the fact that $U_2(\vec{\zeta}) \geq 0$. Moreover, to bound the various terms in $U_1(\vec{\zeta})$ we can use the fact that $\sum_{i \leq J} |\varphi_i(z)|^2 \leq \pi^{-1}e^{z^2}$. It is then easy to see that $U_1(\vec{\zeta})$ is bounded from below as

\[
U_1(\vec{\zeta}) \geq -\frac{\omega}{2}(J(J+1) - \left(\frac{2g}{\pi} + 2\mu(J+2)\right) \|\phi_{\vec{\zeta}}\|^2 - 2N> \left[\mu(J+1) + \frac{g}{\pi}\right].
\]

In order for the first term to be much smaller than the GP energy $E^{GP}(N,g,\omega) \sim N\sqrt{Ng}\omega$ it is clear that we can not replace all modes by a c-number but must rather require that $J \ll (g/N\omega)^{1/4}$.

C. Step 3

As we have explained in the previous step, $H'$ is bounded from below by the minimum over $\vec{\zeta}$ of the ground state energy of $h_u(\vec{\zeta})$. Suppose the minimum is attained at some $\vec{\zeta}_0$, and let $\langle \cdot \rangle_0$ denote the corresponding ground state expectation value in the Fock space of the modes $> J$. The next step is to use a simple lower bound on $h_u(\vec{\zeta}_0)$ to bound the $\eta$-dependent term in $H$ involving the modes $> J$.

Let us choose the parameters $\mu$ and $J$ in following way. Recall from step 1 that $J_0 \leq J < 2J_0$, with $J_0$ an arbitrary integer that we are free to choose. We take $J_0 = \lfloor N(g/N\omega)^{3/10} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part of a number $t > 0$, and $\mu = \omega J/(4N)$. The proportionality constants here are chosen more for convenience than for optimality. Note that $J_0$ is a big number if $g/(N\omega) \ll 1$ and $gN/\omega \gtrsim 1$.

Recall that $\eta = (g/N\omega)^{1/10}$. We claim that, for small $g/N\omega$,

\[
\eta \left| \sum_{i > J} \chi_J(i)a_i^\dagger a_i \right|_0 \leq \frac{2E^{GP}(N,\omega,g)}{\omega J^{3/2}}.  \tag{22}
\]

This bound is similar to (18) except for the additional factor $\eta$ on the left side and $E^{GP}$ replacing $E_0$ on the right side. In order to prove (22), we can use the positivity of the interaction to obtain the following lower bound on $h_u(\vec{\zeta})$. The lower symbol $h_l(\zeta)$ is bounded from below as

\[
h_l(\vec{\zeta}) \geq \sum_{i > J} \chi_J(i)a_i^\dagger a_i + \omega(J+1)N>
\]

\[
+ \mu \left((\|\phi_{\vec{\zeta}}\|^2 - N)^2 + 2N>(\|\phi_{\vec{\zeta}}\|^2 - N)
\]

\[
+ (N>)^2 + \|\phi_{\vec{\zeta}}\|^2 \right].  \tag{23}
\]

In particular, using (21),

\[
h_u(\vec{\zeta}) \geq \eta\omega J^{3/2} \sum_{i > J} \chi_J(i)a_i^\dagger a_i + \mu \left((\|\phi_{\vec{\zeta}}\|^2 - N)^2 + N>(J+1)(\omega - 2\mu) - \frac{2g}{\pi} - 2\mu N
\]

\[
- N \left(\frac{2g}{\pi} + 2\mu(J+2)\right) - \omega J(J+1)
\]

\[
- \left(\frac{2g}{\pi} + 2\mu(J+2)\right) (\|\phi_{\vec{\zeta}}\|^2 - N).  \tag{24}
\]

With the aid of Schwarz’s inequality, we obtain

\[
h_u(\vec{\zeta}) \geq \eta\omega J^{3/2} \sum_{i > J} \chi_J(i)a_i^\dagger a_i + \mu \left((\|\phi_{\vec{\zeta}}\|^2 - N)^2 + N>(J+1)(\omega - 2\mu) - \frac{2g}{\pi} - 2\mu N
\]

\[
- N \left(\frac{2g}{\pi} + 2\mu(J+2)\right) - \omega J(J+1)
\]

\[
- \left(\frac{2g}{\pi} + 2\mu(J+2)\right) (\|\phi_{\vec{\zeta}}\|^2 - N).  \tag{24}
\]

For our choice of $\mu$ and $J$, $\mu \ll \omega$ and $g \ll \omega J$ for small $g/N\omega$. Moreover, $\mu N = \omega J/4$. Hence the second term on the right side of (24) is positive for small $g/N\omega$. Moreover, since $E^{GP}(N,\omega,\eta) \sim N\sqrt{Ng}\omega$ for $Ng \gtrsim \omega$, as remarked earlier, all the terms in the last line of (24) are much smaller than $E^{GP}(N,\omega,\eta)$ if $g \ll N\omega$ and, in particular, are bounded from below by $-E^{GP}/2$. Since the ground state energy of $h_u(\vec{\zeta}_0)$ is bounded above by $E_0(N,\omega,\eta)(1 + b\eta) \leq 1.5E^{GP}(N,\omega,\eta)$, Inequality (22) follows.

D. Step 4

We now give a more refined lower bound on $h_l(\vec{\zeta})$. Instead of dropping all the interaction terms, as we did in the previous step in order to obtain the bound (23), we use the fact that

\[
\delta \geq P_J \delta P_J \otimes P_J + P_J \otimes P_J \delta(P_J \otimes Q_J + Q_J \otimes P_J)
\]

\[
+ (P_J \otimes Q_J + Q_J \otimes P_J) \delta P_J \otimes P_J,  \tag{25}
\]

where $Q_J = \mathbb{1} - P_J$ and $P_J$ denotes the projection onto the $J + 1$ dimensional subspace of $H$ spanned by eigenfunctions of the angular momentum with $L \leq J$. Inequality (25) is a simple Schwarz inequality and follows from positivity of $\delta$ and the fact that $P_J \otimes P_J \delta Q_J \otimes Q_J = 0$. Using
We apply this to our case, where \( c_k \) denotes the multiplication operator \( \rho \zeta \chi_j^{-1}(L) \rho \zeta \chi_j \), we see that \( h_k(\zeta) \) is bounded from below by

\[
h_k(\zeta) \geq E^{GP}(\phi \zeta) + \omega \sum_{j > J} \langle \phi \zeta | j \rangle a_j \\
+ \mu \left( \| \phi \zeta \|^2 - N \right)^2 + 2N^\geq(\| \phi \zeta \|^2 - N) \\
+ \langle \phi \zeta \rangle^2 \| \phi \zeta \| \right] \\
+ g \sum_{k > J} \langle \phi \zeta \otimes \phi \zeta | \delta \phi \zeta \otimes \phi \zeta \rangle a_k + \text{h.c.}
\]

For the remaining terms in the upper symbol \( h_k(\zeta) \) we proceed as in the previous step. We retain \( \mu(\| \phi \zeta \|^2 - N)^2/2 \) for later use, however, and arrive at

\[
h_k(\zeta) \geq E^{GP}(\phi \zeta) + \frac{\mu}{2} \left( \| \phi \zeta \|^2 - N \right)^2 \\
+ g \sum_{k > J} \langle \phi \zeta \otimes \phi \zeta | \delta \phi \zeta \otimes \phi \zeta \rangle a_k + \text{h.c.} \\
- N \left( \frac{2g}{\pi} + 2\mu(J + 2) \right) - \frac{\omega J}{2}(J + 1) \\
- \frac{\left( \frac{2g}{\pi} + 2\mu(J + 2) \right)^2}{2\mu}.
\]

The right side of (29) contains the desired quantity \( E^{GP}(\phi \zeta) \). The second term guarantees that \( \| \phi \zeta \|^2 \) is close to \( N \). All the terms in the last two lines are small compared to \( E^{GP}(N, \omega, g) \) for our choice of \( \mu \) and \( J \). Hence we are left with giving a bound on the third term in (29), which is linear in creation and annihilation operators. It is this bound that requires most effort; it will be completed in the remaining three steps.

E. Step 5

A simple Schwarz inequality shows that, for any sequence of complex numbers \( c_k \) and positive numbers \( e_k \),

\[
\sum_k \left( c_k a_k + c_k^* a_k^\dagger \right) \leq \sum_k \frac{|c_k|^2}{e_k} + \sum_k e_k c_k a_k^\dagger a_k.
\]

We apply this to our case, where \( c_k = - \langle \phi \zeta \otimes \phi \zeta | \delta \phi \zeta \otimes \phi \zeta \rangle \) for \( k > J \), and \( c_k = 0 \) otherwise. We pick some \( \kappa > 0 \) and choose \( e_k = \kappa^{-1} \chi_j(k) \) for \( k > J \), with \( \chi_j(k) \) given in (17). Then

\[
\sum_k \frac{|c_k|^2}{e_k} = \kappa \langle \phi \zeta | \rho \zeta \chi_j^{-1}(L) \rho \zeta | \phi \zeta \rangle,
\]

where \( \rho \zeta \) denotes the multiplication operator \( \rho \zeta(z) = |\phi(z)|^2 e^{-|z|^2/2} \) and

\[
\chi_j^{-1}(L) \equiv \sum_{k = J + 1}^\infty \frac{1}{\chi_j(k)} |\varphi_k \rangle \langle \varphi_k |.
\]

(For simplicity, we abuse the notation slightly, since \( \rho \zeta \) does not leave \( \mathcal{H} \) invariant; \( \langle \varphi_k | \rho \zeta | \phi \zeta \rangle \) makes perfectly good sense, however.)

In Step 7 below we shall show that

\[
\langle \phi \zeta | \rho \zeta \chi_j^{-1}(L) \rho \zeta | \phi \zeta \rangle \leq 6 \int_C |\phi \zeta(z)|^4 e^{-2|z|^2} \, dz \overset{3/2}{\leq} \int_C \langle \phi \zeta(z)|^4 e^{-2|z|^2} \, dz \overset{3/2}{\leq} \frac{1}{\kappa N \omega J^2/2}\int_C |\phi \zeta(z)|^4 e^{-2|z|^2} \, dz.
\]

This inequality may seem surprising at first sight. The function \( \rho \zeta | \phi \zeta \rangle \) contains angular momenta up to \( L \leq 2J \), and \( \chi_j^{-1}(2J) \) grows exponentially with \( J \). As will become clear below, however, \( |\langle \varphi_k | \rho \zeta | \phi \zeta \rangle|^2 \) is exponentially small unless \( k \) is smaller than roughly \( J + \sqrt{J} \). Since \( \chi_j^{-1}(J + \sqrt{J}) \) is bounded independently of \( J \), this makes (29) plausible.

Altogether, (27) – (29) imply that

\[
\sum_k \left( c_k a_k + c_k^* a_k^\dagger \right) \leq 6 \kappa \int_C |\phi \zeta(z)|^4 e^{-2|z|^2} \, dz \overset{3/2}{\leq} \sum_k c_k a_k^\dagger a_k.
\]

We have shown in Step 3 above that

\[
\langle \sum_k c_k a_k^\dagger a_k \rangle \geq 2 \frac{E^{GP}(N, \omega, g)}{\kappa N \omega J^3/2}
\]

in the ground state of \( h_k(\zeta) \). After optimizing over \( \kappa \) we get the following lower bound on \( - \sum_k (c_k a_k + c_k^* a_k^\dagger) \) (valid as an expectation value in the ground state):

\[
-4\sqrt{3} \int_C |\phi \zeta(z)|^4 e^{-2|z|^2} \, dz \geq 6 \beta \left( \frac{E^{GP}(N, \omega, g)}{\kappa N \omega J^3/2} \right)^{1/2}.
\]

Finally, using Hölder’s inequality this expression is bounded from below by

\[
-\frac{\beta}{2} \int_C |\phi \zeta(z)|^4 e^{-2|z|^2} \, dz \geq -\frac{65}{\beta^3} \left( \frac{E^{GP}(N, \omega, g)}{\kappa N \omega J^3/2} \right)^{1/2}.
\]

for arbitrary \( \beta > 0 \).

F. Step 6

The analysis in the preceding steps has shown that

\[
h_k(\zeta) \geq (1 - \beta) E^{GP}(\phi \zeta) + \frac{\mu}{2} (\| \phi \zeta \|^2 - N)^2 \\
- N \left( \frac{2g}{\pi} + 2\mu(J + 2) \right) - \frac{\left( \frac{2g}{\pi} + 2\mu(J + 2) \right)^2}{2\mu} \\
- \frac{\omega J}{2}(J + 1) + g \frac{65}{\beta^3} \left( \frac{E^{GP}(N, \omega, g)}{\kappa N \omega J^3/2} \right)^{1/2}.
\]

(30)
As mentioned above, we choose $\mu = \omega J/(4N)$ and $J \sim N(g/N\omega)^{3/10}$. Moreover, we make the choice $\eta = (g/N\omega)^{1/10}$. Then all the error terms are small compared to $E^{GP}(N,\omega,g)$, namely they are of the order $(g/N\omega)^{1/10}E^{GP}$.

Since the ground state energy of $h(\tilde{\zeta})$ is bounded above by $E^{GP}(N,\omega,g)(1+b_n) \leq 1.5E^{GP}(N,\omega,g)$ and all the error terms in (30) are bounded by $E^{GP}/2$ for small $g/(N\omega)$, we see that

$$\|\phi_\zeta\|^2 - N \leq \left(\frac{2E^{GP}(N,\omega,g)}{\mu}\right)^{1/2} \sim N\left(\frac{g}{N\omega}\right)^{1/10}$$

for $\zeta = \tilde{\zeta}_0$ and hence

$$(1 - \beta)E^{GP}(|\phi_\zeta|) + \frac{\mu}{2} (\|\phi_\zeta\|^2 - N)^2 \geq E^{GP}(N,\omega,g) \left(1 - C\left(\frac{g}{N\omega}\right)^{1/10}\right)$$

for some constant $C > 0$. This yields the lower bound in (31).

**G. Step 7**

The only thing left to do is to prove Inequality (29). For convenience, we shall introduce the notation

$$\|\phi\|_p = \left(\int_C |\phi(z)|^p e^{-(p/2)|z|^2} dz\right)^{1/p}$$

for $1 \leq p < \infty$. We can bound

$$\langle \phi_\zeta|\rho_\zeta\chi_j^{-1}(L)\rho_\zeta|\phi_\zeta\rangle \leq \|\phi_\zeta\|^2 \|\sqrt{\chi_j^{-1}(L)}\sqrt{\rho_\zeta}\|.$$  

The latter norm is an operator norm on $L^2(C, e^{-|z|^2}dz)$ (defined as the maximum expectation value), and equals the operator norm of $\sqrt{\chi_j^{-1}(L)}\rho_\zeta\chi_j^{-1}(L)$ on $\mathcal{H}$.

Next, we derive a pointwise bound on $\rho_\zeta(z) = |\phi_\zeta(z)|^2 e^{-|z|^2}$. Let $P_j(z,z') \sum_{j=0}^{\infty} |\varphi_j(z)|^2 \varphi_j(z')$ denote the integral kernel of the projection onto the subspace of $\mathcal{H}$ with angular momentum $L \leq J$. With the aid of H"older’s inequality,

$$|\phi_\zeta(z)| = \left|\int_C P_j(z,z')\phi_\zeta(z') e^{-|z'|^2} dz'\right| \leq \|\phi_\zeta\| \|P_j(z, \cdot)\|_{4/3}.$$  

Another application of H"older’s inequality yields

$$\|P_j(z, \cdot)\|_{4/3} \leq \|P_j(z, \cdot)\|_{(4-3p)/(4-2p)}^{(4-3p)/(4-2p)} \|P_j(z, \cdot)\|_p$$

for any $1 \leq p \leq 4/3$.

The $L^p$ norms with $p \neq 2$ are not given by simple expressions, however. In the following, we will show that they can be bounded above by a quantity independent of $J$. More precisely, for any $1 < p < \infty$, we shall show that

$$\|P_j(z, \cdot)\|_p \leq \frac{(2\pi/p)^{1/p}}{\pi\sin(\pi/p)} e^{\frac{|z|^2}{2}}$$

independently of $J$. In order to see this, note that, for fixed $z$ and $|z'|$, the function $\theta \mapsto P_j(z,|z'| e^{i\theta})$ is obtained from $P_\infty(z,|z'| e^{i\theta})$ by restricting the Fourier components to $0 \leq j \leq J$. This restriction is a bounded operation (uniformly in $J$) on $L^p(T)$ for $1 < p < \infty$ [30, Ch. II, Sec. 1], and hence $\|P_j(z, \cdot)\|_p$ is bounded by a constant times $\|P_\infty(z, \cdot)\|_p$. In fact, an upper bound on the optimal constant in this inequality is given by the norm of the Riesz projection (viewed as an operator from $L^p(T)$ to itself), which is known to equal $1/\sin(\pi/p)$ [31].

Hence we have the bound

$$\|P_j(z, \cdot)\|_p \leq \frac{1}{\sin(\pi/p)} \|P_\infty(z, \cdot)\|_p = \left[\frac{(2\pi/p)^{1/p}}{\pi\sin(\pi/p)}\right] e^{\frac{|z|^2}{2}}.$$  

The last equality follows from the fact that $P_\infty(z, z') = \pi^{-1} e^{-zz'^*}$.

Let $c_p$ denote the constant in square brackets in (31) taken to the power $2p/(4 - 2p)$. We have thus shown that, for $1 < p \leq 4/3$ and $\alpha(p) = (4 - 3p)/(4 - 2p)$

$$\rho_\zeta(z) \leq c_p \|\phi_\zeta\|^2 \left(\sum_{j=0}^{\infty} |\varphi_j(z)|^2 e^{-|z|^2}\right)^{\alpha(p)}.$$  

A possible choice is $p = 6/5$, in which case $c_p = (10/3) \sqrt{2}/(5/3\pi)^{1/4}\approx 4.02$ and $\alpha(p) = 1/4$. Hence, for any function $|\psi\rangle = \sum_{k \geq J+1} a_k |\varphi_k\rangle$, we have that

$$\langle \psi|\sqrt{\chi^{-1}}(L)\rho_\zeta\sqrt{\chi^{-1}}(L)|\psi\rangle \leq 4.02 \|\phi_\zeta\| \sum_{k \geq J+1} |a_k|^2 \exp\left[\frac{1}{8}(\sqrt{k} - \sqrt{J+1})^2\right] \times \int_C |\varphi_k(z)|^2 \left(\sum_{j=0}^{\infty} |\varphi_j(z)|^2 e^{-|z|^2}\right)^{1/4} e^{-|z|^2} dz.$$  

(32)

Here we have used the fact that our upper bound on $\rho_\zeta(z)$ is radial, i.e., depends only on $|z|$, and hence there are no off-diagonal terms on the right side. An application of Jensen’s inequality for the concave function $t \mapsto t^{1/4}$ implies that

$$\int_C |\varphi_k(z)|^2 \left(\sum_{j=0}^{\infty} |\varphi_j(z)|^2 e^{-|z|^2}\right)^{1/4} e^{-|z|^2} dz \leq \left(\int_C |\varphi_k(z)|^2 e^{-|z|^2} \left(\sum_{j=0}^{\infty} |\varphi_j(z)|^2 e^{-|z|^2}\right) dz\right)^{1/4}.$$  

From Stirling’s formula \(32\) it follows that \(j! \geq (j/e)^j/\sqrt{2\pi j}\) for any \(j \geq 0\). Using this bound it is easy to see that

\[
|\varphi_j(z)|^2 e^{-|z|^2} \leq \frac{1}{\pi \sqrt{1 + j}} e^{-\left(|z| - \sqrt{3}j\right)^2}.
\]

Similarly,

\[
|\varphi_k(z)|^2 e^{-|z|^2} \leq \frac{3^{3/2}}{\pi \sqrt{8 e |z|}} e^{-\left(|z| - \sqrt{k}\right)^2/2}
\]

for \(k \geq 1\). Hence

\[
\int_0^\infty |\varphi_j(z)|^2 |\varphi_k(z)|^2 e^{-2|z|^2} \, dz \\
\leq \frac{3^{3/2}}{\pi \sqrt{2 e \sqrt{1 + j}}} \int_0^\infty e^{-(x - \sqrt{3}j)^2} e^{-(x - \sqrt{k})^2} \, dx \\
\leq \frac{3^{3/2}}{\sqrt{4\pi e \sqrt{1 + j}}} e^{-(\sqrt{3} - \sqrt{k})^2/2}.
\]

To obtain the last inequality, we simply extended the integral to the whole real axis. The sum over \(j\) can be bounded from above by the integral

\[
\sum_{j=0}^J \frac{1}{\sqrt{1 + j}} e^{-(\sqrt{3} - \sqrt{k})^2/2}
\]

\[
\leq \int_0^{J+1} e^{-(\sqrt{3} - \sqrt{k})^2/2} \, ds \\
\leq 2 \int_0^\infty e^{-(t + \sqrt{3} - \sqrt{k})^2/2} \, dt \\
\leq 2 \int_{-\infty}^\infty e^{-(t + \sqrt{3} - \sqrt{k})^2/2} e^{-t(\sqrt{3} - \sqrt{k})} \, dt \\
= \sqrt{8\pi} e^{-(\sqrt{3} - \sqrt{k})^2/2}.
\]

The one-fourth root of this expression cancels exactly the exponential factor in \(32\), and we arrive at \(29\).

### IV. BOSE-EINSTEIN CONDENSATION

The technique employed for the energy bounds can be used to show that the model \(9\) exhibits Bose-Einstein condensation in the ground state if \(N \to \infty\) with \(gN/\omega\) fixed, in complete analogy to the proof of the corresponding result in \(29\). The nonuniqueness of the minimizers of the GP functional \(10\) due to breaking of rotational symmetry makes the precise statement a little complicated, but in brief the result is as follows: If \(\Psi_N\) is a sequence of ground states of \(21\) for \(N = 1, 2, \ldots\) and \(\gamma_N\) are the corresponding normalized 1-particle density matrices, then any limit \(\gamma\) of the sequence \(\gamma_N\) as \(N \to \infty\) with the coupling constant \(gN/\omega\) fixed is a convex combination of projectors onto normalized minimizers of the GP functional \(10\) for the given coupling constant. A precise formulation is given in Theorem 2 in \(29\). A key step in the proof is to extend the bounds on the ground state energy of \(H\) to perturbed operators \(H^{(S)} = H + \sum_{i=1}^N S_i\), where \(S\) is a bounded operator on the 1-particle space \(\mathcal{H}\) and \(S_i\) the corresponding operator on the \(i\)-th factor in \(\otimes_{\text{symm}}^N \mathcal{H}\). These bounds lead to the inequalities

\[
\min_\phi \langle \phi | S | \phi \rangle \leq \text{Tr} \, S \gamma \leq \max_\phi \langle \phi | S | \phi \rangle
\]

for any limit \(\gamma\) of the sequence of density matrices \(\gamma_N\), where the max resp. min is taken over all normalized minimizers \(\phi\) of \(10\) with coupling parameter \(gN/\omega\). With the aid of some arguments from convex analysis one can then conclude as in \(29\) that \(\gamma\) has a representation in terms of projectors \(|\phi\rangle \langle \phi|\) onto normalized minimizers \(\phi\) of the GP functional. The appropriate extension of this result to the Thomas-Fermi limit is still an open problem.

### V. CONCLUSION

We analyzed the ground state energy of a rapidly rotating Bose gas in a harmonic trap, under the usual assumption that the particles are restricted to the lowest Landau level, that the two-body interaction is a \(\delta\)-function, and that the number of particles, \(N\), is very large. For the limiting situation we prove, rigorously, that the Gross-Pitaevskii energy, appropriately modified to the lowest Landau level, is exact provided \(g \ll N\omega\). Here, \(g\) is the strength of the \(\delta\)-function interaction, and \(\omega\) is the difference between the trapping and rotation frequencies.

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