STATED SL(n)-SKEIN MODULES AND ALGEBRAS

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ABSTRACT. We develop a theory of stated SL(n)-skein modules, $S_n(M, N)$, of 3-manifolds $M$ marked with intervals $N$ in their boundaries. These skein modules, generalizing stated SL(2)-modules of the first author, stated SL(3)-modules of Higgins', and SU(n)-skein modules of the second author, consist of linear combinations of framed, oriented graphs, called $n$-webs, with ends in $N$, considered up to skein relations of the $U_q(sl_n)$-Reshetikhin-Turaev functor on tangles, involving coupons representing the anti-symmetrizer and its dual.

We prove the Splitting Theorem asserting that cutting of a marked 3-manifold $M$ along a disk resulting in a 3-manifold $M'$ yields a homomorphism $S_n(M) \to S_n(M')$ for all $n$. That result allows to analyze the skein modules of 3-manifolds through the skein modules of their pieces.

The theory of stated skein modules is particularly rich for thickened surfaces $M = \Sigma \times (-1, 1)$, in whose case, $S_n(M)$ is an algebra, denoted by $S_n(\Sigma)$. One of the main results of this paper asserts that the skein algebra of the ideal bigon is isomorphic with $O_q(SL(n))$ and it provides simple geometric interpretations of the product, coproduct, counit, the antipode, and the cobraided structure on $O_q(SL(n))$. (In particular, the coproduct is given by a splitting homomorphism.) Additionally, we show that a splitting of a thickened bigon near a marking defines a right $O_q(SL(n))$-comodule structure on $S_n(M)$, or dually, a left $U_q(sl_n)$-module structure. Furthermore, we show that the skein algebra of surfaces $\Sigma_1, \Sigma_2$ glued along two sides of a triangle is isomorphic with the braided tensor product $S_n(\Sigma_1) \otimes S_n(\Sigma_2)$ of Majid. These results allow for geometric interpretation of further concepts in the theory of quantum groups, for example, of the braided products and of Majid’s transmutation operation.

Building upon the above results, we prove that the factorization homology of surfaces $\Sigma$ with $\partial \Sigma = S^1$ with coefficients in the category of representations $Rep U_q(sl_n)$ is equivalent to the category of left modules over $S_n(\Sigma)$. We also discuss the relation of our skein algebras with the quantum moduli spaces of Alekseev-Schomerus.

Another major result of this paper shows that for surfaces with boundary $\Sigma$ every splitting homomorphism is injective and that $S_n(\Sigma)$ is a free module with a basis induced from the Kashiwara-Lusztig canonical bases.

CONTENTS

1. Introduction 4
   1.1. Skein Modules of marked 3-manifolds 4
   1.2. Splitting Homomorphisms 4
   1.3. Basic properties of stated skein modules 5

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Key words and phrases: skein module, skein algebra, surface, Reshetikhin-Turaev invariants, quantum group.
1.4. Stated SL(n)-skein algebras of surfaces
1.5. Geometric interpretation of the cobraided structure on $O_q(SL(n))$
1.6. Relation to Reshetikhin-Turaev theory
1.7. Module and Co-module structures
1.8. Glueing over an ideal triangle
1.9. On injectivity of splitting homomorphism
1.10. Skein algebras of surfaces with boundary
1.11. Kernel of the splitting homomorphism.
1.12. The Image of the Splitting homomorphism
1.13. Relation to Factorization Homology, skein categories and lattice field theory
1.14. Compatibility with stated Kauffman bracket skein modules
1.15. Compatibility with the SU(n)-skein modules
1.16. Compatibility with Higgins’ SL3 skein algebras
1.17. Relation to the Frohman-Sikora SU(3)-skein algebras
1.18. Thanks

2. Quantum groups associated to $sl_n$
2.1. Notations and conventions
2.2. Quantized enveloping algebra $U_q(sl_n)$
2.3. Braiding and the Iwahori-Hecke algebra
2.4. The dual module
2.5. The quantized coordinate algebra $O_q(SL(n))$

3. Review of the Reshetikhin-Turaev theory
3.1. Based $n$-tangles and their operator invariants
3.2. Kernel of functor $RT_0$
3.3. (Unbased) $n$-tangles
3.4. Linear functionals on $U_q(sl_n)$ from $n$-tangles
3.5. Dual operator, orientation reversal invariance
3.6. Annihilators
3.7. Turning right annihilators to left ones
3.8. Kernel of $\Gamma$
3.9. Half-ribbon Hopf algebra

4. Stated SL(n)-skein modules
4.1. Marked 3-manifolds and $n$-webs
4.2. Skein relations for $n$-webs
4.3. Eliminating sinks and sources
4.4. Change of ground ring
4.5. Functoriality
4.6. Grading
4.7. Useful Identities
4.8. Splitting homomorphism
4.9. Reversing orientations of 3-manifolds and of webs
4.10. Marking automorphisms
4.11. Half-twist automorphisms
4.12. Essential uniqueness of the skein relations of $S_n(M, N)$
5. Stated SL(n)-skein algebras of surfaces
5.1. Punctured bordered surface
5.2. Splitting homomorphism for surfaces
5.3. Reflection anti-involution
5.4. Embedding of punctured bordered surfaces
6. Skein algebras of bigon and quantum groups
6.1. Monogon
6.2. Bigon
6.3. Cobraided structure
6.4. Ground ring
6.5. Algebra homomorphism $\mathcal{S}_n(\mathcal{B}) \to \mathcal{O}_q(SL(n))$
6.6. Proof of Theorem 6.1
6.7. Proof that $\mathcal{S}_n(\mathcal{B})$ is a Hopf algebra
6.8. Proof of Theorem 6.3
6.9. Proof of Theorem 6.4
6.10. Proof of Theorem 3.11
6.11. Additional facts
7. Coaction of $\mathcal{O}_q(SL(n))$ on stated skein modules
7.1. Module and Co-module structures
7.2. Boundary relations revisited
7.3. The last among the defining skein relations
7.4. Glueing over an ideal triangle
7.5. Punctured Monogon and Majid’s Transmutation
7.6. On injectivity of splitting homomorphism
7.7. Skein algebras of surfaces with boundary
7.8. Products on skein algebras of surfaces with boundary
7.9. Torus with an arc boundary
8. Kernel and image of the splitting homomorphism.
8.1. Kernel of the Splitting homomorphism
8.2. The Image of the Splitting homomorphism
9. Relation to factorization homology, skein categories, and lattice gauge theory
9.1. Factorization homology
9.2. Skein categories
9.3. Lattice Gauge Theory, Quantum moduli spaces
10. Relation to other known cases
10.1. Compatibility with stated Kauffman bracket skein modules of 3-manifolds
10.2. Compatibility with the SU(n)-skein modules
10.3. Compatibility with Higgins’ SL3 skein algebras
10.4. Relation to the Frohman-Sikora SU(3)-skein algebras
Appendix A. Proof of Proposition 3.13 (A calculation of matrices of $X$.)
References
1. Introduction

In the paper we develop a theory of stated SL($n$)-skein modules of 3-manifolds, building on the theory of stated Kauffman bracket skein algebras developed by the first author with his collaborators, \cite{Le1, CL, BL} (see also \cite{KQ}), and on the notion of the SU($n$)-skein modules of the second author \cite{Si2}. Our theory also generalizes the recent work of Higgins for $n = 3$, \cite{Hi}.

1.1. Skein Modules of marked 3-manifolds. In this paper we will work with a commutative ring of coefficients $R$ with a distinguished invertible $v = q^{\frac{1}{n}}$. A marked 3-manifold is a pair $(M, \mathcal{N})$, where $M$ is a smooth oriented 3-manifold with (possibly empty) boundary $\partial M$ and $\mathcal{N} \subset \partial M$ consists of open intervals, called markings.

An $n$-web $\alpha$ in $(M, \mathcal{N})$ is a disjoint union of an oriented link and a directed ribbon graph whose every vertex is either 1-valent end in $\mathcal{N}$ or an internal $n$-valent sink or source, cf. Fig. 1. Each web is equipped with a transversal vector field called its framing, which at each end $e$ points in the direction of the marking containing $e$, cf. Subsec. 4.1.

![Figure 1. An example of a 3-web with two endpoints in a marking (in red) stated by $i_1, i_2$.](image)

A state of a web $\alpha$ is an assignment of a label from $\{1, \ldots, n\}$ to each of its ends.

The stated SL($n$)-skein module, $\mathcal{S}_n(M, \mathcal{N})$, of $(M, \mathcal{N})$ is the space of all $R$-linear combination of stated $n$ webs in $(M, \mathcal{N})$, up to internal skein relations (43)-(46) and boundary skein relations (47)-(50). These relations mimic those satisfied by the Reshetikhin-Turaev functor on tangles, with $n$-vertices representing the anti-symmetrizer tensor and its dual. More specifically, the internal relations are based on the skein relations of \cite{Si2}. (It may be useful to recall here the premise of \cite{Si2}: that although $n$-webs seem unnecessary from the point of view of study of quantum invariants of links in manifolds, they allow for a very efficient formulation of the necessary skein relations.)

However, our specific relations involve a novel sign modification which leads to a major technical benefit: the half-edges around each $n$-valent vertex have a cyclic ordering only, rather than a linear ordering required in \cite{Si2}, cf. Subsections 3.3 and 3.9. An additional benefit of this modification is that it makes skein relations invariant under the orientation reversal of the webs.

The boundary skein relations of $\mathcal{S}_n(M, \mathcal{N})$ are new and generalize those of \cite{Le1} and \cite{Hi}.

1.2. Splitting Homomorphisms. An important property of stated skein modules is that they behave in a simple manner under the splitting of 3-manifolds along disks. Specifically, for a marked 3-manifold $(M, \mathcal{N})$ with a properly embedded closed disk $D$ in $M - \mathcal{N}$, let $M'$ be $M$ with an open collar neighborhood of $D$ removed. Then $M'$ is a 3-manifolds with copies $D_1, D_2$ of $D$ in its boundary, whose gluing together leads to an epimorphism $pr : M' \to M$. 
Given an oriented open interval $\beta \subset D$, the splitting of $(M, N)$ along $(D, \beta)$, denoted by $\text{cut}_{(D, \beta)}(M, N)$, is the marked 3-manifold $(M', N')$, where $N' = N \cup \beta_1 \cup \beta_2$, where $\beta_i \subset D_i$ are the connected components of $\text{pr}^{-1}(\beta)$, cf. Fig. 2. Note that for any stated $n$-web $\alpha$ in $(M, N)$ transversal to $D$ with $\alpha \cap D \subset \beta$, the inverse image $\text{pr}^{-1}(\alpha)$ is an $n$-web in $(M', N')$ stated at all its ends except those at $\beta_1 \cup \beta_2$. Given any map $s : \alpha \cap \beta \to \{1, \ldots, n\}$ let $\alpha(s)$ be $\text{pr}^{-1}(\alpha)$ with each of its ends $x \in \text{pr}^{-1}(\alpha) \cap (\beta_1 \cup \beta_2)$ stated by $s(\text{pr}(x))$.

![Figure 2. An example of a splitting of an $n$-web (in green) intersecting the splitting disk $D$ twice.](image)

The following result generalizes that of [Le1, BL] for the Kauffman bracket skein modules ($n = 2$) and of [Hi] for $n = 3$: 

**Theorem (Splitting Theorem 4.5).** There is a unique $R$-module homomorphism

$$\Theta_{(D, \beta)} : S_n(M, N) \to S_n(\text{cut}_{(D, \beta)}(M, N))$$

sending every stated $(D, \beta)$-transverse $n$-web $\alpha$ in $(M, N)$ to the sum of all of its lifts,

$$\Theta_{(D, \beta)}(\alpha) = \sum_{s : \alpha \cap \beta \to \{\pm\}} \alpha(s).$$

1.3. **Basic properties of stated skein modules.** We discuss symmetries and other properties of stated skein modules of marked 3-manifolds in Subsections 4.9-4.10. In particular, we observe that for every marked 3-manifold $(M, N)$, the orientation reversal of webs $\alpha \to \overline{\alpha}$ defines an $R$-module automorphism:

$$\overline{\cdot} : S_n(M, N) \to S_n(M, N).$$

Let $(\overline{M}, \overline{N})$ denote $M$ and $N$ with reversed orientations. Let $\overline{R}$ be the ring $R$ with the distinguished element $v^{-1}$ instead of $v$. For an $n$-web $\alpha$ of $(M, N)$ let $\overline{\alpha}$ be the $n$-web in $(\overline{M}, \overline{N})$ obtained from $\alpha$ by negating its framing, $f \to -f$.

**Theorem (Thm. 4.9).** (1) Any ring isomorphism $\kappa : R \to \overline{R}$ sending $v$ to $v^{-1}$ extends to an isomorphism of $R$-modules $\kappa_{(M, N)} : S_n(M, N, R) \isom S_n(M, N, \overline{R})$ sending every stated $n$-web $\alpha$ to $\overline{\alpha}$, where $S_n(M, N, \overline{R})$ is an $R$-module via $\kappa : R \to \overline{R}$.

(2) The composition $\kappa_{(M, N)} \circ \kappa_{(M, N)}$ is the identity on $S_n(M, N, R)$.

For every marking $\beta$ of $(M, N)$ there is an $R$-module automorphism $g_{\beta}$ of $S_n(M, N)$, called a **marking automorphism**, sending stated $n$-webs $\alpha$ to

$$g_{\beta}(\alpha) = \prod_{x \in \alpha \cap \beta} (-1)^{n-1}q^{2s(x)-n-1} \cdot \alpha$$

where $s(x)$ is the state of the endpoint $x$ of $\alpha$.

There is an additional automorphism of $S_n(M, N)$ associated with each marking of $(M, N)$:
Proposition (Prop. 4.11). For any marking \( \beta \) in \( \mathcal{N} \) there exists a unique \( R \)-linear isomorphism, called the half-twist automorphism,

\[
\text{htw}_\beta : \mathcal{S}_n(M, \mathcal{N}) \rightarrow \mathcal{S}_n(M, \mathcal{N})
\]

sending any stated \( n \)-web \( \alpha \) in \( (M, \mathcal{N}) \) with \( k \) endpoints on \( \beta \) to

\[
\text{htw}_\beta \begin{pmatrix}
\vdots \\
i_k \\
i_2 \\
i_1 \\
\end{pmatrix} = \left( \prod_{j=1}^{k}\bar{c}_{ij} \right) \cdot 
\begin{pmatrix}
\vdots \\
i_k \\
\end{pmatrix} = \left( \prod_{j=1}^{k} c_{ij} \right) \cdot 
\begin{pmatrix}
\vdots \\
i_k \\
\end{pmatrix},
\]

where \( H \) is the positive half-twist. (The orientations of the horizontal edges are arbitrary.)

1.4. Stated \( \text{SL}(n) \)-skein algebras of surfaces. The theory of stated \( \text{SL}(n) \)-skein modules is particularly rich for thickened surfaces \( M = \Sigma \times (-1, 1) \). It is most convenient to consider it in the context of punctured bordered surfaces (a pb surfaces for short) which are of the form \( \Sigma = \bar{\Sigma} - \mathcal{P} \), where \( \bar{\Sigma} \) is a compact oriented surface and \( \mathcal{P} \subset \bar{\Sigma} \) is a finite set, called the ideal points of \( \Sigma \), which meets each connected component of \( \partial \bar{\Sigma} \). Then \( \partial \Sigma \) is a union of open intervals. These intervals are called boundary edges.

In each boundary edge \( e \) choose a point \( b_e \). Let \( \mathcal{S}_n(\Sigma) = \mathcal{S}_n(\Sigma \times (-1, 1), \mathcal{N}) \), where \( \mathcal{N} \) is the union of all \( b_e \times (-1, 1) \), each having the natural orientation of the interval \((-1, 1)\).

For the monogon \( \mathcal{M} \), which is the closed disk with a boundary point removed, we have

Theorem (Thm. 6.1). The map \( \mu : R \rightarrow \mathcal{S}_n(\mathcal{M}) \) given by \( \mu(r) = r \cdot \emptyset \) is an \( R \)-algebra isomorphism.

Despite its simple statement, the proof of the above result is non-trivial – see the comment at the end of this subsection.

The bigon, \( \mathcal{B} \), is a closed disk with two boundary points removed. In Lemma 5.5 we show that the \( R \)-algebra \( \mathcal{S}_n(\mathcal{B}) \) is generated by the arcs \( a_{ij} = i \uparrow j \) for \( 1 \leq i, j \leq n \). Splitting \( \mathcal{B} \) along the an interior ideal arc connecting its two ideal vertices defines an algebra \( R \)-homomorphism

\[
\Delta : \mathcal{S}_n(\mathcal{B}) \rightarrow \mathcal{S}_n(\mathcal{B}) \otimes \mathcal{S}_n(\mathcal{B}).
\]

Let \( \epsilon : \mathcal{S}_n(\mathcal{B}) \rightarrow R \) be the composition

\[
\epsilon : \mathcal{S}_n(\mathcal{B}) \xrightarrow{\text{htw}_{e_r}} \mathcal{S}_n(\mathcal{B}) \xrightarrow{\epsilon_1} \mathcal{S}_n(\mathcal{M}) \simeq R,
\]

where \( \text{htw}_{e_r} \) is the half-twist automorphism defined above. On generators,

\[
\epsilon(a^i_j) = \epsilon(a^i_j) = c_j i \uparrow j = \delta_{i,j}.
\]

Let \( \mathcal{O}_q(\text{SL}(n)) \) be the quantized coordinate ring algebra of \( sl_n \). This Hopf algebra is the restricted dual of the quantized enveloping algebra, \( U_q(sl_n) \), cf. [KS 9.2.2]. For technical convenience, we consider \( \mathcal{O}_q(\text{SL}(n)) \) as defined over \( \mathbb{Q}[v^{\pm 1}] \).
Theorem (Thm. 6.3). (a) The algebra \( S_n(\mathfrak{B}) \) has the structure of a Hopf algebra over \( R \) with the coproduct \( \Delta \), the counit \( \epsilon \), and the antipode \( S \) such that
\[
S(a_{ij}^k) = (-q)^{i-j} \tilde{a}_i^j \quad \text{for } i,j = 1, \ldots, n.
\]

(b) The map \( \Psi(u_{ij}^k) = a_{ij}^k \) extends to a unique Hopf algebra isomorphism
\[
O_q(sl(n)) := O_q(SL(n)) \otimes R \xrightarrow{\Psi} S_n(\mathfrak{B}).
\]

The above theorems generalize statements for \( n = 2 \) in \cite{CL} and \( n = 3 \) in \cite{Hi}. However, it is not a straightforward generalization. The proofs of \cite{CL, Hi} relied on specific bases of \( S_n(\Sigma) \) for \( n = 2, 3 \) which can be obtained through the confluence method, cf. \cite{SW}. That method does not work for higher \( n \) and a construction of bases for \( n > 3 \) is an important and still open problem. (We discuss our progress on that problem below.) We were able to establish the above theorem without constructing a basis of \( S_n(\mathfrak{B}) \).

1.5. Geometric interpretation of the cobraided structure on \( O_q(SL(n)) \). The Hopf algebra \( O_q(sl_n; R) \) is dual quasitriangular (see \cite{Maj, Section 2.2}, \cite{KS, Section 10}), also known as cobraided (see e.g. \cite{Kass, Section VIII.5}). This means it has a co-\( R \)-matrix, which is a bilinear form
\[
\rho : O_q(sl_n; R) \otimes O_q(sl_n; R) \rightarrow R
\]
satisfying certain properties, with the help of which one can make the category of \( O_q(sl_n; R) \)-modules a braided category. The following generalizes \cite{CL} Theorem 3.5 from \( n = 2 \) to all \( n \):

Theorem (Thm. 6.4). Under the above identification \( S_n(\mathfrak{B}) \simeq O_q(sl_n; R) \) the co-\( R \)-matrix \( \rho \) has the following geometric description
\[
\rho \left( \begin{array}{cc}
\begin{array}{c}
\circ
\end{array}
\otimes
\begin{array}{c}
\circ
\end{array}
\end{array}\right) = \epsilon \left( \begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}\right),
\]
for any \( x, y \in O_q(sl_n; R) \).

Above we identified \( \mathfrak{B} \) with \([-1, 1] \times (-1, 1) \) by stretching its top and bottom ideal points into horizontal intervals.

1.6. Relation to Reshetikhin-Turaev theory. Sections 2-3 mostly summarize the background in quantum groups and in Reshetikhin-Turaev theory necessary for this paper. Section 2 however also introduces a novel modification of the Reshetikhin-Turaev functor, utilized throughout the paper. Let us briefly describe its connection to stated skein modules.

When the bigon \( \mathfrak{B} \) is identified with \([-1, 1] \times (-1, 1) \), the webs on \( \mathfrak{B} \) can be thought as oriented framed tangles with coupons given by \( n \)-valent sinks and sources. The sign \( \text{sgn}(e) \) of an endpoint \( e \in \partial \alpha \) is positive if the direction of \( \alpha \) goes from left to right at \( e \), and negative otherwise. Let \( \text{sgn}_l(\alpha) \) (respectively: \( \text{sgn}_r(\alpha) \)) be the sequence of signs of left (respectively: right) endpoints of \( \alpha \) appearing from the bottom to the top.

Let \( V = \mathbb{Q}(v)^n \) be the defining representation of \( U_q(sl_n) \) with its standard basis \( \{e_1, \ldots, e_n\} \). Let \( \{f_1, \ldots, f_n\} \) be a basis of \( V^* \) given by \( f_i = (-1)^{i-1}q^{i - \frac{n+1}{2}}e^i \) for \( i = 1, \ldots, n \). For any sign sequence \( \eta = (\eta_1, \ldots, \eta_k) \), let
\[
V^n := V^{\eta_1} \otimes \ldots \otimes V^{\eta_k},
\]
where $V^+ = V$ and $V^- = V^*$. The above bases induce the tensor basis of $V^n$ indexed by the elements of $\{1, \ldots, n\}^k$, cf. Subsec. 3.4.

Note that $V^n$ is a $U_q(sl_n)$-module for every sign sequence $\eta$. The key to our work is a modified version of the Reshetikhin-Turaev functor, introduced in Section 3, which associates to each web $\alpha$ in $\mathcal{B}$ a $U_q(sl_n)$-module homomorphism $\text{RT}(\alpha) : V_{\text{sgn}(\alpha)}^{\text{sgn}(\alpha)} \to V_{\text{sgn}(\alpha)}^{\text{sgn}(\alpha)}$. (It is a sign modification of the standard $U_q(sl_n)$-Reshetikhin-Turaev functor, $\text{[RT]}$, which requires that the half-edges incident to each $n$-valent vertex in $\alpha$ are linearly ordered.)

We show that the benefit of utilizing the basis $\{f_1, \ldots, f_n\}$ of $V^*$ rather than the dual basis to $\{e_1, \ldots, e_n\}$ is that it makes the modified R-T functor values $\text{RT}(\alpha)$ independent of the orientation of tangles.

The following result relates our skein algebra of $\mathcal{B}$ to the Reshetikhin-Turaev theory:

**Proposition (Prop. 6.6).** Let $\alpha$ be an $n$-web $\alpha$ on $\mathcal{B}$ stated by $i = (i_1, \ldots, i_k)$ on the left and $j = (j_1, \ldots, j_l)$ on the right. Then $\varepsilon(\alpha)$ is equal to the $(i, j)$-entry of the matrix of the modified Reshetikhin-Turaev operator $\text{RT}(\alpha)$ of $\alpha$ in the above tensor bases.

1.7. **Module and Co-module structures.** Given a marking $\beta$ of a marked 3-manifold $(M, \mathcal{N})$, consider its closed disk neighborhood $D$ in $\partial M$, disjoint from the other markings of $(M, \mathcal{N})$. By pushing the interior of $D$ inside $M$ we get a new disk $D'$ which is properly embedded in $M$. Splitting $(M, \mathcal{N})$ along $D'$, we get a new marked 3-manifold $(M', \mathcal{N}')$ isomorphic to $(M, \mathcal{N})$, and another marked 3-manifold bounded by $D$ and $D'$. The latter, after removing the common boundary of $D$ and $D'$, is isomorphic to the thickening of the bigon. Hence, this construction yields an $R$-linear splitting map

$$\Delta_\beta : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(M, \mathcal{N}) \otimes \mathcal{O}_q(SL(n)).$$

which defines a right coaction of $\mathcal{O}_q(SL(n))$ on $\mathcal{S}_n(M, \mathcal{N})$. Such coactions at different markings commute.

The Hopf algebra $\mathcal{O}_q(SL(n); \mathbb{Z}[v^\pm 1])$ has a Hopf dual given by a completion $\widehat{U_q(sl_n)}$ of the Lusztig’s integral version, $\widehat{U^L}$, of $U^L$, which is a Hopf algebra over $\mathbb{Z}[v^\pm 1]$, cf. Subsection 3.9 and [Lu1], Sec. 1.3. The duality between these two Hopf algebras turns any right $\mathcal{O}_q(SL(n))$-comodule W to a left $\widehat{U^L}$-module as follows: For $u \in \widehat{U^L}$ and $x \in W$,

$$u * x = \sum x' \langle f'', u \rangle,$$

where $\Delta(x) = \sum x' \otimes f''$ is the $\mathcal{O}_q(SL(n))$-coaction map. We make this left $\widehat{U^L}$-action on $\mathcal{S}_n(M, \mathcal{N})$ explicit in Subsec. 7.1.

The Hopf algebra $\widehat{U^L}$ contains distinguished charmed element $g$ and the half-ribbon element $X \in \widehat{U^L}$. We prove that the action of these elements on $\mathcal{S}_n(M, \mathcal{N})$ at a marking $\beta$ is exactly the marking automorphism $g_\beta$ and the half-twist homomorphism $htw_\beta$ of Subsection 1.3.

1.8. **Glueing over an ideal triangle.** The standard ideal triangle $\mathcal{T} \subset \mathbb{R}^2$ is the closed triangle with vertices $(-1, 0), (1, 0)$ and $(0, 1)$ with these vertices removed. We will denote its sides by $e_1, e_2$, and $\partial_b \mathcal{T}$ as in Figure 3. Suppose $a_1, a_2$ are two distinct boundary edges of...
a (possibly disconnected) pb surface. Define
\[ \Sigma_{a_1 \triangle a_2} = (\Sigma \sqcup \mathcal{T})/\langle e_1 = a_1, e_2 = a_2 \rangle, \]
as in Figure 3. Define the \( R \)-linear homomorphism \( \text{glue}_{a_1, a_2} : S_n(\Sigma) \to S_n(\Sigma_{a_1 \triangle a_2}) \) by continuing the strands of any web \( \alpha \) with endpoints on \( a_1 \) and \( a_2 \) until they reach \( \partial_b \mathcal{T} \), as in Figure 3 (right).

**Figure 3.** Left: The standard ideal triangle \( \mathcal{T} \). Middle: Glueing \( \Sigma \) and \( \mathcal{T} \) by \( a_1 = e_1 \) and \( a_2 = e_2 \) to get \( \Sigma_{a_1 \triangle a_2} \). Right: tangle diagram \( x \in \mathcal{S}_n(\Sigma) \) and its image \( \text{glue}_{a_1, a_2}(x) \in \mathcal{S}_n(\Sigma_{a_1 \triangle a_2}) \)

**Proposition** (Prop. 7.6, see [CL] for \( n = 2 \) and [Hi] for \( n = 3 \)). The map \( \text{glue}_{a_1, a_2} : S_n(\Sigma) \to S_n(\Sigma_{a_1 \triangle a_2}) \) is an \( R \)-linear isomorphism.

We construct an explicit inverse map to \( \text{glue}_{a_1, a_2} \) in Subsec. 7.4.

Although the bijective map \( \text{glue}_{a_1, a_2} \) is not an algebra isomorphism with respect to the standard skein algebra product on \( S_n(\Sigma_{a_1 \triangle a_2}) \), we show in Subsec. 7.4 that it is one with respect to the *self braided tensor product* which we will define right now.

There are two right \( \mathcal{O}_q(\text{SL}(n)) \)-comodule algebra structures on \( S_n(\Sigma) \) given by
\[ \Delta_i := \Delta_{a_i} : \mathcal{S}_n(\Sigma) \to S_n(\Sigma) \otimes \mathcal{O}_q(\text{SL}(n)), \quad i = 1, 2, \]
which commute. Define the \( R \)-linear map \( \Delta : \mathcal{S}_n(\Sigma) \to S_n(\Sigma) \otimes \mathcal{O}_q(\text{SL}(n)) \) by
\[ \Delta(x) = \sum x' \otimes u_1 u_2, \]
in Sweedler’s notation, where
\[ (\Delta_1 \otimes \text{Id}_{\mathcal{O}_q(\text{SL}(n))}) \circ \Delta_2(x) = \sum x' \otimes u_1 \otimes u_2. \]
For \( x, y \in \mathcal{S}_n(\Sigma) \) define a new product by
\[ y \ast x = \sum y' x' \rho(u \otimes w) \]
where
\[ \Delta_2(y) = \sum y' \otimes u, \Delta_1(x) = \sum x' \otimes w. \]
It is proved in [CL] that \( \Delta \) and \( \ast \) together give \( \mathcal{S}_n(\Sigma) \) a right \( \mathcal{O}_q(\text{SL}(n)) \)-comodule algebra structure for \( n = 2 \). That proof extends to all \( n \).

Denote by \( \otimes \mathcal{S}_n(\Sigma) \) the \( R \)-module \( \mathcal{S}_n(\Sigma) \) with this \( \mathcal{O}_q(\text{SL}(n)) \)-comodule algebra structure. On the other hand \( \mathcal{S}_n(\Sigma_{a_1 \triangle a_2}) \) has a right \( \mathcal{O}_q(\text{SL}(n)) \)-comodule algebra structure coming from the boundary edge \( \partial_b \mathcal{T} \). Here is a stronger version of the proposition above.
Theorem (Thm. 7.7). The map $\text{glue}_{a_1,a_2} : \bigotimes \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma_{a_1 \Delta a_2})$ is an isomorphism of right $\mathcal{O}_q(SL(n))$-comodule algebras.

When $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and $a_i \subset \Sigma_i$ for $i = 1, 2$ then each $\mathcal{S}_n(\Sigma_i)$ is a right $\mathcal{O}_q(SL(n))$-comodule algebra via the coaction coming from the edge $a_i$ and $\bigotimes (\mathcal{S}_n(\Sigma))$ is the well-known braided tensor product $\mathcal{S}_n(\Sigma_1)$ and $\mathcal{S}_n(\Sigma_2)$ of the two $\mathcal{O}_q(SL(n))$-module algebras $\mathcal{S}_n(\Sigma_1)$ and $\mathcal{S}_n(\Sigma_2)$ of Majid, cf. [Maj] Lemma 9.2.12.

1.9. On injectivity of splitting homomorphism. A pb surface $\Sigma$ is essentially bordered if every connected component of it has non-empty boundary.

Proposition (Prop. 7.11). Suppose $\Sigma$ is an essentially bordered pb surface. Then for any interior ideal arc $c$ of $\Sigma$, the splitting homomorphism $\Theta_c : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\text{cut}_c \Sigma)$ is injective.

Conjecture (Conj. 7.12). For any punctured bordered surface $\Sigma$ and any interior ideal arc $c$ the splitting homomorphism $\Theta_c$ is injective as well.

The conjecture is true when $n = 2$ by [CL] and for $n = 3$ by Higgins [Hi]. In both cases the proofs rely on explicit bases of $\mathcal{S}_n(\Sigma)$. Proposition 7.11 shows the conjecture is true if $\Sigma$ has non-trivial boundary. We will establish an alternative, weaker version of this conjecture for all pb surfaces in Subsec. 1.11.

1.10. Skein algebras of surfaces with boundary. Let $\Sigma_{g,p}$ denote the surface of genus $g$ with $p - 1$ punctures and $\partial \Sigma_{g,p} = S^1$. Let $\Sigma_{g,p}^*$ be $\Sigma_{g,p}$ with a boundary point removed. Hence, $\Sigma_{0,2}^*$ is a punctured monogon.

Utilizing the results of Subsec. 1.8 we show in Proposition 7.10 that $\mathcal{S}_n(\Sigma_{0,2}^*) \simeq \mathcal{O}_q(SL(n))$ as an $R$-module with the $\mathcal{O}_q(SL(n))$-comodule structure, $\Delta_{\partial \Sigma_n}(\Sigma_{0,2}^*)$, coinciding with the adjoint $\mathcal{O}_q(SL(n))$-coaction on $\mathcal{O}_q(SL(n))$, [Maj] Example 1.6.14. Furthermore, we prove that the product on $\mathcal{S}_n(\Sigma_{0,2}^*)$ coincides with the braided (or, covariantised) product of Majid, [Maj] Example 9.4.10. (That result was shown for $n = 2$ in [CL].) Consequently, our theory provides simple geometric proofs of the associativity of the braided product on $\mathcal{O}_q(SL(n))$ and of $\mathcal{O}_q(SL(n))$ being an $\mathcal{O}_q(SL(n))$-comodule algebra. (The proofs of these facts are quite technical and involved in [Maj].) Furthermore, our theory generalizes these statements to the boundary $\mathcal{O}_q(SL(n))$-coaction on the skein algebra of any essentially bordered punctured surface. We discuss a finite presentation of $\mathcal{S}_n(\Sigma_{0,2}^*)$ in Subsec. 7.5.

Let $\Sigma$ be now any essentially bordered pb surface. A collection $A = \{a_1, \ldots, a_r\}$ of disjoint closed oriented arcs properly embedded in $\Sigma$ is saturated if

(i) each connected component of $\Sigma \setminus \bigcup_{i=1}^r a_i$ contains exactly one ideal point (interior or boundary) of $\Sigma$, and

(ii) $A$ is maximal with respect to the above condition.

Let $U(a_1), \ldots, U(a_n)$ be a collection of disjoint open tubular neighborhood of a saturated collection of arcs $a_1, \ldots, a_n$, respectively. Each $U(a_i)$ is homeomorphic with $a_i \times (-1, 1)$ (by an orientation preserving homeomorphism) and we require that $(\partial a_i) \times (-1, 1) \subset \partial \Sigma$. Let $U(A) = \bigcup_{i=1}^k U(a_i)$.

Theorem (Thm. 7.13). (1) We have $r = r(\Sigma) := \# \partial \Sigma - \chi(\Sigma)$, where $\# \partial \Sigma$ is the number of boundary components of $\Sigma$ and $\chi$ denotes the Euler characteristics.

(2) The embedding $U(A) \hookrightarrow \Sigma$ induces an $R$-module isomorphism $f_A : \mathcal{S}_n(U(A)) \to \mathcal{S}_n(\Sigma)$. 


Note that each $U(a_i) = a_i \times (-1, 1)$ is naturally a directed bigon, with its sides $(\partial a_i) \times (-1, 1)$ oriented in the direction of $(-1, 1)$ and that we have an $R$-linear isomorphism

$$O_q(SL(n)) \otimes^R \Psi \rightarrow \mathcal{S}_n(U(A)) \xrightarrow{f_A} \mathcal{S}_n(\Sigma).$$

Since $O_q(SL(n))$ has a Kashiwara-Lusztig’s canonical basis over $\mathbb{Z}[v^{\pm 1}]$, cf. [Kash, Lu2], we have

**Corollary.** $\mathcal{S}_n(\Sigma)$ is a free $R$-module with a basis given by the image of tensor product of Kashiwara-Lusztig’s canonical bases on $O_q(SL(n)) \otimes^R \Psi$ under $f_A \circ \Psi$.

We apply the above method to show that $\mathcal{S}_n(\Sigma^*) \simeq O_q(SL(n))^\otimes(\text{as an } R\text{-module})$ and to describe the product on it in Subsec. [7.9]. Furthermore, we explain a construction of finite presentations for $\mathcal{S}_n(\Sigma)$, for every essentially bordered $\Sigma$.

1.11. **Kernel of the splitting homomorphism.** Suppose $\Sigma$ is a connected pb surface with an ideal point $p$. Then a trivial ideal arc $c_p$ at $p$ cuts $\Sigma$ into a monogon and of a new pb surface $\Sigma_p$ which has $c_p$ as its boundary edge, see Figure 4.

**Figure 4.** From $\Sigma$ to $\Sigma_p$. Here $p$ is an interior ideal point. The picture when $p$ is a boundary ideal point is similar.

Theorem 8.1 shows that the kernel of

$$\Theta : \mathcal{S}_n(\Sigma) \xrightarrow{\Theta} \mathcal{S}_n(\Sigma_p) \otimes_R \mathcal{S}_n(M) \xrightarrow{\Delta} \mathcal{S}_n(\Sigma_p)$$

does not depend on the choice of $p$. Let us denote it by $K(\Sigma)$. Then the quotient $\bar{\mathcal{S}}_n(\Sigma) := \mathcal{S}_n(\Sigma)/K(\Sigma)$ is called the **projected stated skein algebra of** $\Sigma$. By Proposition 7.11, $K$ is trivial and $\bar{\mathcal{S}}_n(\Sigma) = \mathcal{S}_n(\Sigma)$ if $\partial \Sigma \neq \emptyset$.

**Corollary (Cor. 8.2).** The splitting homomorphism descends to an injective algebra homomorphism

$$\bar{\Theta} : \bar{\mathcal{S}}_n(\Sigma) \rightarrow \bar{\mathcal{S}}_n(\text{cut}_c \Sigma) = \mathcal{S}_n(\text{cut}_c \Sigma).$$

The following is an alternative characterization of projected skein algebras:

**Theorem.** For any $\Sigma$, $p$ and $c_p$ as above, $\bar{\mathcal{S}}_n(\Sigma)$ coincides with the subalgebra of $\mathcal{S}_n(\Sigma_p)$ coinvariant under the coaction $\Delta_{c_p} : \mathcal{S}_n(\Sigma_p) \rightarrow \mathcal{S}_n(\Sigma_p) \otimes \mathcal{S}_n(M)$ at $c_p$:

$$\bar{\mathcal{S}}_n(\Sigma) = \{ x \in \mathcal{S}_n(\Sigma_p) : \Delta_{c_p}(x) = x \otimes 1 \}.$$ 

Let $\Sigma = \overline{\Sigma} - \mathcal{P}$, where $\mathcal{P}$ is a finite subset of compact surface $\overline{\Sigma}$, as in Subsec. 1.5. Generalizing the setup of Subsec. 1.10 consider a collection $A$ of disjoint, oriented, arcs in $\Sigma$, each with endpoints in $\partial \Sigma \cup \mathcal{P}$, satisfying the conditions (i) and (ii) above. We show in Subsec. 8.1 that such $A$ defines an identification of $\bar{\mathcal{S}}_n(\Sigma)$ with $O_q(SL(n))^\otimes$ and, hence, it determines a basis of $\bar{\mathcal{S}}_n(\Sigma)$.
1.12. **The Image of the Splitting homomorphism.** Let $c$ be an interior oriented ideal arc of a pb surface $\Sigma$. Denote the two copies of $c$ in $\text{cut}_c \Sigma$ by $a_1$ and $a_2$. We have the splitting $R$-algebra homomorphism

$$\Theta_c : S_n(\Sigma) \to S_n(\text{cut}_c \Sigma).$$

and $S_n(\text{cut}_c \Sigma)$ is a $O_q(SL(n))$-bi-comodule with the right and left coactions

$$\Delta_{a_1} : S_n(\text{cut}_c \Sigma) \to S_n(\text{cut}_c \Sigma) \otimes O_q(SL(n))$$

$$a_2 \Delta : S_n(\text{cut}_c \Sigma) \to O_q(SL(n)) \otimes S_n(\text{cut}_c \Sigma)$$

respectively, where $O_q(SL(n))$ is identified with the skein algebra of the bigon directed by the orientation of $c$. Recall that the Hochshild cohomology module is defined by

$$HH^0(S_n(\text{cut}_c \Sigma)) = \{ x \in S_n(\text{cut}_c \Sigma) \mid \Delta_{a_1}(x) = \text{fl} \circ a_2 \Delta(x) \},$$

where $\text{fl}$ is the transposition

$$\text{fl} : O_q(SL(n)) \otimes S_n(\Sigma) \to S_n(\Sigma) \otimes O_q(SL(n)), \quad \text{fl}(x \otimes y) = y \otimes x.$$

**Theorem 1.1** ([CL, KQ] for $n = 2$ and [Hi] for $n = 3$). The image of $\Theta_c$ is equal to $HH^0(S_n(\text{cut}_c \Sigma))$.

We prove it by considering the projected version $\Theta_c : \bar{S}_n(\Sigma) \to HH^0(S_n(\text{cut}_c \Sigma))$, which has the same image as non-projected one. Furthermore, we construct an explicit inverse map $\nabla : HH^0(S_n(\text{cut}_c \Sigma)) \to \bar{S}_n(\Sigma)$.

1.13. **Relation to Factorization Homology, skein categories and lattice field theory.** Factorization homology is an invariant of oriented $n$-dimensional manifolds introduced by Beilinson and Drinfeld [BD] in the setting of conformal field theory and then in [Lu, AF, AFT] in the topological context. For $n = 2$, it associates with every surface $\Sigma$ and every pivotal braided category $\mathcal{A}$, a certain category denoted by $\int_{\Sigma} \mathcal{A}$. When the base ring is a field, $\int_{\Sigma} \mathcal{A}$ is equivalent to the category of left modules over a certain algebra $A_{\Sigma}$, [BBJ].

Building upon the above theory of stated skein algebras, we prove that for surfaces $\Sigma$ with $\partial \Sigma = S^1$, the algebra $A_{\Sigma}$ is isomorphic with $S_n(\Sigma^*)$, where $\Sigma^*$ is $\Sigma$ with a boundary point removed.

Although our construction of the stated skein modules was motivated by the rich theory of stated skein modules and algebras developed in this paper, the above result provides a further justification for that construction.

Nonetheless, our theory of stated skein modules generalizes the definition of algebras $A_{\Sigma}$ as they are defined over non-field rings of coefficients and for all 3-dimensional manifolds, with multiple boundary components and multiple markings.

Additionally, our theory is very concrete; see Subsec. 9.1 for further details. Furthermore, our theory allows to construct quantum trace embeddings of the stated skein algebras into quantum tori, over any ground ring, [LY3], which generalize the quantum trace map of Bonahon-Wang ([BW] for surfaces without boundary) and its analogs for surfaces with boundary of the first author and T. Yu [LY2]. In [LY2] there are two versions of quantum trace maps, quantizing respectively the length coordinates and the shear coordinates trace formulas, and both are generalized to the case of $SL_n$ in [LY3]. Embeddings into quantum tori allow to study algebraic properties and representation theory of $S_n(\Sigma)$.
In Subsec. 9.3 we discuss the relation between our skein algebras and the quantum moduli spaces and algebras of quantum observables of Alekseev-Grosse-Schomerus, [AGS] AS and of Buffenoir-Roche [BR1, BR2] for $\mathfrak{g} = \mathfrak{sl}(n)$.

We also believe that our theory determines the skein categories of Walker and Johnson-Freyd, [Wa, JF] for $U_q(\mathfrak{sl}(n))$, cf. Subsect. 9.2.

1.14. **Compatibility with stated Kauffman bracket skein modules.** The stated Kauffman bracket skein algebras (of surfaces) of the first author [Lé1] were generalized to stated skein modules of marked 3-manifolds in [BL] (cf. also [LY]). We are going to prove that these modules are isomorphic with our $SL(2)$-skein modules, $S_2(M, N)$.

To relate these modules to ours, let us replace the variable $q$ of [Lé1] with $q^{1/2}$ and denote the resulted stated Kauffman bracket skein module by $S(M, N)_{q^{1/2}}$. Recall that it is built of non oriented 2-webs without sinks nor sources, stated by sings $\pm$.

**Theorem** (Thm. 10.1). Suppose $(M, N)$ is a marked 3-manifold.

1. There is a unique $R$-linear isomorphism $\Lambda : S(M, N)_{q^{1/2}} \to S_2(M, N)$ which maps framed links $\alpha$ to a stated 2-webs by assigning arbitrary orientation to them, and changing the minus state to 1 and the plus state to 2.

2. The splitting homomorphism of [Lé1, BL] coincides with ours through $\Lambda$.

1.15. **Compatibility with the SU(n)-skein modules.** As mentioned already, a partial motivation for our definition of $S_n(M, N)$ were the $SU(n)$-skein modules of 3-manifolds introduced by the second author in [Si2]. These skein modules are built of based $n$-webs in $M$ which are defined as our $n$-webs in $(M, \emptyset)$, except that the half-edges incident to any of their $n$-valent vertices are linearly ordered. In particular, the based $n$-webs have no endpoints and $SU(n)$-skein modules have no boundary skein relations. In Subsection 10.2 we show that for any 3-manifold $M$ and any $n$ our $S_n(M, \emptyset)$ is isomorphic with the $SU(n)$-skein module of $M$. That isomorphism is straightforward for $n$ odd, but it requires a choice of a spin structure on $M$ for $n$ even.

1.16. **Compatibility with Higgins’ SL3 skein algebras.** In his recent work [Hi] Higgins introduced his version of stated $SL_3$-skein algebras, denoted by $S_q^{SL_3}(\Sigma)$, of punctured bordered surfaces $\Sigma$. His skein algebra is the $R$-module freely generated by 3-webs subject to his system of skein relations. We show that in that our theory recovers Higgins’ for $n = 3$ in Subsection 10.3.

1.17. **Relation to the Frohman-Sikora SU(3)-skein algebras.** C. Frohman and the second author considered in [FS] the “reduced $SU(3)$-skein algebra” of marked surfaces built of unstated 3-webs, subject to the $SU(3)$-skein relations of [Ku], extended by certain boundary skein relations, which depend on an invertible parameter $a \in R$. We denote that algebra by $S_{FS}(\Sigma)$ for the value 1 of that parameter.

For an unstated 3-web $\alpha$ in $\Sigma$, let $\eta_+(\alpha)$ (respectively: $\eta_-(\alpha)$) denote $\alpha$ stated with threes (respectively: ones) at all its ends. In Subsection 10.4 we show that these operations extend to injective homomorphisms $\eta_+, \eta_- : S_{FS}(\Sigma) \to S_3(\Sigma)$. Furthermore, their images are direct summands of $S_3(\Sigma)$. 
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2. Quantum groups associated to $sl_n$

2.1. Notations and conventions. We use the notations $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ for respectively the sets of complex numbers, reals, rationals numbers, and non-negative integers. We emphasize that our $\mathbb{N}$ contains 0. The number $n$ (in $sl_n$) is a fixed integer $\geq 2$.

The ground ring $R$ is a commutative ring with unit containing a distinguished invertible element $v$. The element $q = v^{2n}$ is the usual quantum parameter. The basic example is $R = \mathbb{Z}[v^{\pm 1}]$, the ring of Laurent polynomials in $v$ with integer coefficients. All fractional powers of $q$ in our papers are defined via the obvious integral powers of $v^{1/2n}$.

For a non-negative integer $m$ we define the quantum integer $[m]$ and its factorials by

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]! = \prod_{i=1}^{m} [i], \quad [0]! = 1.$$ 

We will often use the following scalars:

(1) $t_0^{1/2} = q^{\frac{n^2-1}{2n}} := v^{n^2-1}, \quad t = (-1)^{n-1}t_0 = (-1)^{n-1}q^{\frac{n^2-1}{n}}, \quad t^{n/2} = (-1)^{\frac{(n-1)n}{2}}q^{\frac{n^2-1}{2}}$

(2) $a = (-v)^{n(n-1)/2}t^{-n/2} = q^{(1-n)(2n+1)/4} = q^{\frac{1-n^2}{4} - \frac{n^2-n}{4}}, \quad d_i = i - \frac{n+1}{2}$

(3) $c_i = (-1)^{n-i}q^{\frac{n+1}{2} - \frac{i}{2}}q^{\frac{n^2-1}{2n}} = (-1)^{n-i}q^{-d_i}t_0^{1/2}$.

Note that

(4) $\prod_{i=1}^{n} c_i = t^{n/2} = (-1)^{\binom{n}{2}}q^{\frac{n^2-1}{2}}$ and $c_i \cdot c_i = t$, for $i = 1, \ldots, n$.

In fact, our entire theory works (up to a normalization) for any invertible $a, t, c_1, \ldots, c_n$ satisfying Eq. (4), cf. Subsection 4.12. However, our particular choice of these constants makes our theory invariant under the orientation reversal of 3-manifolds and the orientation reversal of webs in it, cf. Corollary 4.8 and Theorem 4.9.

Let $S_n$ the group of permutations of $\{1, \ldots, n\}$. The length of $\sigma \in S_n$ is the minimal number of factors in the decomposition of $\sigma$ into elementary transpositions $(i, i+1), i = 1, \ldots, n-1$. Alternatively, it is

(5) $\ell(\sigma) = |\{(i, j)| 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}|$. 

The conjugate of $i$ is $\bar{i} := n + 1 - i$. The longest element $w_0 \in S_n$ is the permutation $w_0(i) = \bar{i}$.

We use the convention that
\begin{equation}
(-q)^{\ell(\sigma)} = 0 \text{ if } \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \text{ is not a permutation.}
\end{equation}

We also use Kronecker’s delta notation and its sibling:
\begin{equation}
\delta_{i,j} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}, \quad \delta_{j,i} = \begin{cases} 1 & \text{if } j > i \\ 0 & \text{if } j \leq i \end{cases}
\end{equation}

2.2. Quantized envelopping algebra $U_q(sl_n)$. The quantized envelopping algebra $U_q(sl_n)$ is a Hopf algebra over the field $\mathbb{Q}(v)$ with explicit presentation given in [KS] Section 6.1.2.

Let $V = \mathbb{Q}(v)^n$ be the defining representation of $U_q(sl_n)$. Its dual $V^*$ is the simple $U_q(sl_n)$-module with highest weight the $(n-1)$-th fundamental weight. Let $e_1, \ldots, e_n$ be the standard basis of $V$ with $e_n$ being the highest weight vector, see [KS] Section 8.4.1. Let $e^1, \ldots, e^n$ be the dual basis of $V^*$, defined by $e^i(e_j) = \delta_{i,j}.$ As $U_q(sl_n)$ is a Hopf algebra, the category of $U_q(sl_n)$-modules is monoidal. Let $\mathcal{C}_n$ be the full subcategory of $U_q(sl_n)$-modules consisting of objects isomorphic to tensor products of copies of $V$ and $V^*$.

It is known that $\text{Hom}_{U_q(sl_n)}(V^{\otimes n}, \mathbb{Q}(v))$ has dimension 1 and is generated by the $q$-antisymmetrizer $\mathcal{A}_- : V^\otimes n \to \mathbb{Q}(v)$ defined by
\begin{equation}
\mathcal{A}_-(e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}) = t^{n/2}a(-q)^{\ell(\sigma)}, \text{ for any } \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\},
\end{equation}
where $(-q)^{\ell(\sigma)} = 0$ if $\sigma$ is not a permutation (according to Eq. (6)), and $a, t$ are given in Subsection [2.1].

Similarly, $\text{Hom}_{U_q(sl_n)}(\mathbb{Q}(v), V^{\otimes n})$ has dimension 1 and is generated by $\mathcal{A}_+ : \mathbb{Q}(v) \to V^{\otimes n}$, given by
\begin{equation}
\mathcal{A}_+(1) = a \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(n)}.
\end{equation}

2.3. Braiding and the Iwahori-Hecke algebra. The algebra $U_q(sl_n)$ has a topological completion which is a topological ribbon Hopf algebra, making the category of $U_q(sl_n)$-modules a ribbon category, see [CP] [11]. The ribbon structure implies for any two $U_q(sl_n)$-modules $V_1, V_2$ there is a braiding $\mathcal{R}_{V_1,V_2} : V_1 \otimes V_2 \to V_1 \otimes V_2$, which is an invertible $U_q(sl_n)$-morphism satisfying certain conditions discussed below. Let us record here the formula for $\mathcal{R}_{V_1}$, which will be simply denoted by $\mathcal{R}$. An operator $A : V \otimes V \to V \otimes V$ is given by its matrix entries $A_{ij}^{mk} \in \mathbb{Q}(v)$, with $i, j, m, k \in \{1, \ldots, n\}$, which are defined such that
\begin{equation}
A(e_m \otimes e_k) = \sum_{i,j} A_{ij}^{mk} e_i \otimes e_j.
\end{equation}

The braiding $\mathcal{R} : V \otimes V \to V \otimes V$ is given by
\begin{equation}
\mathcal{R}_{ij}^{kl} = R_{ij}^{kl} = q^{-1} \left( q^\delta_{ij,k,l} \delta_{i,j} + (q - q^{-1}) \delta_{j<i} \delta_{i,k} \right),
\end{equation}
cf. [KS] Section 8.4.2(60) and Section 9.2, where $\delta_{j<i} = 1$ if $j < i$ and $\delta_{i<i} = 0$ otherwise, as in Sec. 2.1. We also defined above the operator $\mathcal{R}$, which comes from the universal $R$-matrix.
For an integer $k \geq 2$ and $i = 1, \ldots, k - 1$ we define $\hat{R}_i : V^\otimes k \to V^\otimes k$ by

$$\hat{R}_i = \text{id}^\otimes i-1 \otimes \hat{R} \otimes \text{id}^\otimes k-i-1.$$ 

Then the operators $\hat{R}_i$ satisfy the following relations

$$q^{\frac{1}{2}} \hat{R}_i - q^{-\frac{1}{2}} \hat{R}_i = (q - q^{-1})\text{id}, \quad \text{for } i = 1, \ldots, k - 1$$

$$\hat{R}_i \hat{R}_j = \hat{R}_j \hat{R}_i \quad \text{for } 1 \leq i < j - 1 \leq k - 1$$

$$\hat{R}_i \hat{R}_{i+1} \hat{R}_i = \hat{R}_{i+1} \hat{R}_i \hat{R}_{i+1} \quad \text{for } i = 1, \ldots, k - 2.$$ 

The last equation, known as the braid relation, is a consequence of $\hat{R}$ being induced by the $R$-matrix of $U_q(sl_n)$.

By [Gy] [Si2],

$$\hat{R}_i \circ A_+ = -q^{-\frac{1}{n}-1} A_+, \quad A_- \circ \hat{R}_i = -q^{\frac{1}{n}-1} A_-.$$  

(10)

2.4. The dual module. For $x \in V^*$ and $y \in V$ let $\langle x, y \rangle$ denote $x(y) \in Q(v)$. There is an invertible element $g_0 \in U_q(sl_n)$ called the charmed element, whose action on $V$ is given by

$$g_0(e_i) = q^{2i-n-1} e_i = q^{2d_i} e_i.$$ 

The ribbon structure implies that the following $Q(v)$-linear maps are $U_q(sl_n)$-morphisms:

$$\text{ev} : V^* \otimes V \to Q(v), \quad \text{ev}(e_i \otimes e_j) = \langle e_i, e_j \rangle = \delta_{i,j}$$

$$\text{coev} : Q(v) \to V \otimes V^*, \quad \text{coev}(1) = \sum_{i=1}^{n} e_i \otimes e^i$$

$$\tilde{\text{ev}}_0 : V \otimes V^* \to Q(v), \quad \tilde{\text{ev}}_0(e_i \otimes e^j) = q^{2i-n-1} \delta_{i,j} = \langle e^j, g_0(e_i) \rangle$$

$$\tilde{\text{coev}}_0 : Q(v) \to V^* \otimes V, \quad \tilde{\text{coev}}_0(1) = \sum_{i=1}^{n} q^{n+1-2i} e^i \otimes e_i = \sum_{i=1}^{n} e^i \otimes (g_0)^{-1} e_i.$$ 

2.5. The quantized coordinate algebra $O_q(SL(n))$. The algebra of quantum matrices $O_q(M(n))$ is the associative $\mathbb{Z}[v^\pm 1]$-algebra generated by elements $u^i_j$, for $i, j = 1, 2, \ldots, n$, subject to relations

$$[u_i \otimes u_j] R = R (u_i \otimes u_j),$$

where $R$ is the $R$-matrix given by Eq. (9), and $u \otimes u$ is the $n^2 \times n^2$ matrix with entries $(u \otimes u)_{ij}^{kl} = u^i_j u^k_l$ for $i, j, k, l \in \{1, \ldots, n\}$. We call $u := (u^i_j)$ as well as its images under algebra homomorphisms quantum matrices. Any square submatrix of $u$ is a quantum matrix (of smaller size). The element

$$\det_q(u) := \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} u_{\sigma(1)}^{\sigma(1)} \cdots u_{\sigma(n)}^{\sigma(n)} = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} u_{\sigma(1)}^{\sigma(1)} \cdots u_{\sigma(n)}^{\sigma(n)}$$

is central and called the quantum determinant of the quantum matrix $u$, cf. [KS] 9.2.2.
The quantized coordinate algebra of $SL(n)$, $O_q(SL(n))$ is the quotient $O_q(M(n))/(\det_q u - 1)$. It is a Hopf algebra with comultiplication, counit, and antipode given by

$$
\Delta(u^i_j) = \sum_k u^i_k \otimes u^j_k, \quad \epsilon(u^i_j) = \delta_{i,j}, \quad S(u^i_j) = (-q)^{j-i} M^j_i(u),
$$

where $M^j_i(u)$ is the quantum determinant of the minor of $u$ obtained by removing the $j$-th row and $i$-th column of the matrix $u$, see eg [KS 9.2.2] or [Ta2].

For technical convenience we have defined $O_q(SL(n))$ over the ring $\mathbb{Z}[v^\pm 1]$. The dual $U_q(sl_n)^*$, consisting of all $\mathbb{Q}(v)$-linear maps $U_q(sl_n) \to \mathbb{Q}(v)$, has a $\mathbb{Q}(v)$-algebra structure, dual to the coalgebra structure of $U_q(sl_n)$, and is considered as a $\mathbb{Z}[v^\pm 1]$-algebra via $\mathbb{Z}[v^\pm 1] \hookrightarrow \mathbb{Q}(v)$.

**Proposition 2.1** (Hopf duality between $O_q(SL(n))$ and $U_q(sl_n)$, [Ta1 Sec. 4]). There is a unique pairing $\langle \cdot, \cdot \rangle : O_q(SL(n)) \times U_q(sl_n) \to \mathbb{Q}(v)$, such that $\langle u^i_j, x \rangle = e^i(x(e_j))$ for $x \in U_q(sl_n)$ and $i, j = 1, \ldots, n$.

It induces an embedding of $\mathbb{Z}[v^\pm 1]$-algebras $O_q(SL(n)) \hookrightarrow U_q(sl_n)^*$. Moreover, $O_q(SL(n))$ and $U_q(sl_n)$ are in Hopf duality in the sense that for all $u, u' \in O_q(SL(n))$ and $x, x' \in U_q(sl_n)$

$$
\langle uu', x \rangle = \sum \langle u, x_{(1)} \rangle \langle u', x_{(2)} \rangle, \quad \text{where } \Delta(x) = \sum x_{(1)} \otimes x_{(2)}
$$

$$
\langle u, xx' \rangle = \sum \langle u_{(1)}, x \rangle \langle u_{(2)}, x' \rangle, \quad \text{where } \Delta(u) = \sum u_{(1)} \otimes u_{(2)}
$$

$$
\langle 1, x \rangle = \varepsilon(x), \quad \langle u, 1 \rangle = \varepsilon(u), \quad \langle S(u), x \rangle = \langle u, S(x) \rangle.
$$

### 3. Review of the Reshetikhin-Turaev theory

#### 3.1. Based $n$-tangles and their operator invariants.

Reshetikhin-Turaev theory associates with every ribbon category an operator invariant of ribbon graphs, see [Ta]. Let us make this construction explicit for the category of left $U_q(sl_n)$-modules and a special class ribbon graphs, called based $n$-tangles, defined below.

The cube $Q := [-1, 1] \times (-1, 1)^2$ in the 3-space, see Figure 3(a), has boundary consisting of the right face $\{1\} \times (-1, 1)^2$ and the left face $\{-1\} \times (-1, 1)^2$. The intersection of $Q$ and the XY-plane is $S = [-1, 1] \times (-1, 1) \times \{0\}$. It is depicted as the shaded square in Figure 3(a), with its sides $A_t = \{-1\} \times \{0\} \times (-1, 1)$ and $A_r = \{1\} \times \{0\} \times (-1, 1)$. We will say that vectors of the form $(0, 0, z)$ with $z > 0$ are in the $Z$-direction.

For the sake of the definition below and for later use, we say that directed graph $\alpha$ is **properly embedded** into a 3-manifold $M$, if its set of 1-valent vertices, $\partial \alpha$, coincides with $\alpha \cap \partial M$ and $\alpha$ is transversal to $\partial M$.

**Definition 3.1.** A based $n$-tangle $\alpha$ is a disjoint union of oriented circles and finite directed graphs properly embedded into the cube $Q$, such that

1. Every vertex of $\alpha$ is either 1-valent or $n$-valent. Each $n$-valent vertex is a sink or a source. We denote set of 1-valent vertices, called **endpoints** of $\alpha$, by $\partial \alpha$.
2. Each edge of the graph is a smooth embedding of the closed interval $[0, 1]$ into $Q$.
3. $\alpha$ is equipped with a framing which is a continuous non-vanishing vector field transversal to $\alpha$. In particular, the framing at a vertex is transversal to all incident edges.
Figure 5. (a) the cube $Q$, the square $S$ (shadowed) and its sides $\Lambda_r, \Lambda_l$. (b) an example of a diagram of a based 3-tangle. The order at the source is counterclockwise, beginning with lowest branch. (c) the tensor product $\alpha \otimes \beta$. (d) the composition $\beta \circ \alpha$. (Orientations of $\alpha, \beta$ are not shown.

(4) The set of half-edges at every $n$-valent vertex is linearly ordered.
(5) The endpoints of $\alpha$ lie in $\Lambda_r \cup \Lambda_l$, and the framing at these endpoints has a $\mathbb{Z}$-direction.

We consider based $n$-tangles up to isotopies which are continuous deformations of $n$-webs in their class.

Remark 3.2. Our notion of an $n$-tangle is broader than that of a traditional tangle, because it allows for $n$-valent vertices. It is a version of the notion of an $n$-web of [Si2]. However, we use a different name for it here to distinguish it from $n$-webs which we will introduce in Subsec. 4.1 and which are unbased and have a different framing setup near their boundaries.

Suppose $\alpha$ is a based $n$-tangle. The sign $\text{sgn}(e)$ of an endpoint $e \in \partial \alpha$ is positive if the direction of $\alpha$ goes from left to right at $e$, and negative otherwise. Let $\text{sgn}_l(\alpha)$ (respectively $\text{sgn}_r(\alpha)$) be the sequence of signs of endpoints of $\alpha$ on $\Lambda_l$ (respectively $\Lambda_r$) appearing from the bottom to the top.

Let $\mathcal{S}_n^b$ be the $\mathbb{Z}[v^\pm 1]$-linear monoidal category whose objects $\eta$ are finite sequences of signs $\pm$, and the set of morphisms $\text{Hom}_{\mathcal{S}_n^b}(\eta, \mu)$ is the $\mathbb{Z}[v^\pm 1]$-module freely spanned by isotopy classes of based $n$-tangles $\alpha$ such that $\text{sgn}_r(\alpha) = \eta$ and $\text{sgn}_l(\alpha) = \mu$. Here the tensor product of two sequences $\eta$ and $\mu$ is the concatenation of $\eta$ followed by $\mu$. The empty sequence is the unit. The tensor product $\alpha \otimes \beta$ of two based $n$-tangles is the result of stacking $\beta$ above $\alpha$, as in Figure 5(c). If $\text{sgn}_l(\alpha) = \text{sgn}_r(\beta)$ then the composition $\beta \circ \alpha$ is obtained by placing $\beta$ to the left of $\alpha$ (after an isotopy to match endpoints), as in Figure 5(d).

Let $\mathcal{C}_n$ be the category of left $U_q(sl_n)$-modules isomorphic to tensor products of finite numbers of modules $V$ and $V^*$. The morphisms of $\mathcal{C}_n$ are $U_q(sl_n)$-module homomorphisms. It is a ribbon category. Since based tangles can be viewed as tangles with coupons, in the sense of Reshetikhin-Turaev, their theory ([RT]) defines a monoidal functor $\text{RT}_0 : \mathcal{S}_n^b \to \mathcal{C}_n$ constructed as follows. For a sequence $\eta = (\eta_1, \ldots, \eta_k)$ of signs $\pm$ let

$$\text{RT}_0(\eta) = V^\eta := V^{\eta_1} \otimes \ldots \otimes V^{\eta_k},$$

where $V^+ = V$ and $V^- = V^*$. We also write

$$V_r(\alpha) = V^{\text{sgn}_r(\alpha)}$$
$$V_l(\alpha) = V^{\text{sgn}_l(\alpha)}.$$
We will define values of $RT_0$ for based $n$-tangles through their diagrams. For that purpose, we will identify the XY-plane in $\mathbb{R}^3$ with the pages of this paper and we will point the z-axis towards the reader, as in Figure 5(a). Any based $n$-tangle $\alpha$ can be represented by its diagram obtained by isotoping $\alpha$ first so that its framing is in the Z-direction everywhere and then by putting it in a general position with respect to the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ onto the XY-plane, cf. Figure 5(b). We further assume that the projection of $\alpha$ near an $n$-valent vertex consists of $n$ lines directed from left to right, and that their linear order is counterclockwise beginning with from the lowest line to the highest for the source, and from the highest to the lowest line, for the sink.

Such a projection of $\alpha$ onto the square $S = [-1, 1] \times (-1, 1) \times \{0\}$ (shaded in Figure 5(a)), together with the over/under information at every crossing is called a diagram of $\alpha$.

In Eqs. (13)–(15) we list elementary based $n$-tangles $\alpha$ and the corresponding operators $RT_0(\alpha)$. The associated operators $\text{ev}, \tilde{\text{ev}}_0, \text{coev}, \tilde{\text{coev}}_0$ were defined in Subsection 2.4, while $A_-, A_+$ and $\hat{R}$ were given by Eqs. (7)–(9) respectively. Since every based $n$-tangle can be built of them through tensor products and compositions, these operators totally determine $RT_0$.

(13) \[
\begin{array}{ccc}
 id : V \to V & id : V^* \to V^* & \text{ev} : V^* \otimes V \to R \\
 & & \text{coev} : R \to V \otimes V^*
\end{array}
\]

(14) \[
\begin{array}{ccc}
 \tilde{\text{ev}}_0 : V \otimes V^* \to R & \tilde{\text{coev}}_0 : R \to V^* \otimes V
\end{array}
\]

(15) \[
\begin{array}{ccc}
 \hat{R} : V \otimes V \to V \otimes V & \hat{R}^{-1} : V \otimes V \to V \otimes V & A_+ : R \to V^{\otimes n} \\
 & & A_- : V^{\otimes n} \to R
\end{array}
\]

Based $n$-tangles and the Reshetikhin-Turaev functor on them were considered in [Si2], albeit with a different normalization of $A_{\pm}$.

3.2. Kernel of functor $RT_0$. Recall that a monoidal ideal in a monoidal category $\mathcal{C}$ is a subset $I \subset \text{Hom}(\mathcal{C})$ such that for $x \in I$ and $y \in \text{Hom}(\mathcal{C})$ we have $x \otimes y, y \otimes x \in I$ and $x \circ y, y \circ x \in I$, whenever such compositions can be defined.
The following are well-known elements of ker $RT_0$ (see [Si2]):

\begin{align}
q^n - q^{-n} \quad &\xrightarrow{RT_0} (q - q^{-1}) \\
 RT_0 \quad &\xrightarrow{t_0} \\
 RT_0 \quad &\xrightarrow{\sigma_+} \\
 RT_0 \quad &\xrightarrow{(-q)^{\binom{n}{2}} \sum_{\sigma \in S_n} (-q^{\frac{1}{n}-1})^{\ell(\sigma)}}
\end{align}

where $t_0$ is given by Eq. (1), $\sigma_+$ is the minimum crossing positive braid representing a permutation $\sigma$ and $\ell(\sigma)$ is the length of $\sigma$ defined in Subsection 2.1. Here $x \xrightarrow{RT_0} y$ means $x - y \in \ker(RT_0)$.

**Conjecture 3.3.** The kernel $\ker RT_0$ is the monoidal ideal $I_0$ generated by elements given in Eq. (16)–(19).

For example, the arguments in [Si2] indeed imply that the following identities are consequences of (16)–(19):

\begin{align}
(-1)^{n-1} \quad &\xrightarrow{RT_0} \\
- q^{-(1 + \frac{1}{n})} \quad &\xrightarrow{RT_0} \\
[n - 2]! \quad &\xrightarrow{RT_0} [n - 2]! \frac{n-1}{n} - (-1)^{\binom{n}{2}} q^{-\frac{1}{n}} n^{-2}
\end{align}

where the tangle on the right has $n - 2$ parallel edges in the middle.

Conjecture 3.3 is analogous to Morrison’s conjecture [Mor, Sec. 5.5] which was proved in [CKM]. Note that our formalism of $n$-webs leads to a simpler set of kernel generators than that in [Mor].

3.3. **(Unbased) $n$-tangles.** Now we will consider a modified ribbon structure on the category of $U_q(sl_n)$-modules, giving rise to a new Reshetikhin-Turaev functor, denoted by $RT$, with simpler skein properties. We will explain in Subsection 3.9 how this new ribbon structure comes from the theory of quantized enveloping algebras. For now we simply declare $RT$ to coincide on all elementary based $n$-tangles with $RT_0$, except that we multiply the values of $\tilde{\nu}_0$ and $\tilde{\text{coev}}_0$ of (14) by $(-1)^{n-1}$. Thus, if $D$ is a diagram of a based $n$-tangle $\alpha$ and $\# \downarrow (D)$ is the number of its downward critical points, i.e. points where the tangent is
parallel to the vertical $y$-axis and pointing downward, then

$$
RT(D) = (-1)^{(n-1)\#\downarrow(D)}RT_0(\alpha).
$$

It is an easy exercise to show that $RT(D)$ is an isotopy invariant of based $n$-tangles.

From Eq. (20) it follows that $RT$ is invariant under cyclic changes of an order at a vertex:

$$
RT(\alpha) = \cdots,
$$

where $x \overset{RT}{\longrightarrow} y$ means $RT(x) = RT(y)$.

An $n$-tangle is a based $n$-tangle with the order of half-edges incident to its every $n$-valent vertex is downgraded from a linear order to a cyclic one. (Such cyclic orderings of edges around each vertex are called a ribbon structure.) Eq. (24) shows that $RT$ is an invariant of $n$-tangles. In a diagram of an $n$-tangle the cyclic order at every $n$-valent vertex is the counterclockwise order.

Let $\mathcal{S}_n$ be a monoidal category obtained from $\mathcal{S}_b^n$ by replacing based $n$-tangles with $n$-tangles. Then $RT$ is a $\mathbb{Z}[v^\pm 1]$-linear monoidal function from $\mathcal{S}_n$ to $\mathcal{C}_n$.

By Eqs. (16)–(19), we have

$$
q^{\frac{1}{n}} - q^{-\frac{1}{n}} \overset{RT}{\longrightarrow} (q - q^{-1}) \quad \text{and} \quad \overset{t}{\longrightarrow}
$$

$$
(27) \quad (-1)^{n-1}[n] \overset{RT}{\longrightarrow}
$$

$$
(28) \quad (-q)^{\binom{n}{2}} \sum_{\sigma \in S_n} (-q^{-\frac{1}{n} - 1})^{\ell(\sigma)} \sigma
$$

The following two are consequences of Eqs. (25)–(28):

$$
\overset{RT}{\longrightarrow} -q^{-(1+\frac{1}{n})}
$$

$$
\overset{[n-2]!}{\longrightarrow} [n-2]! q^{-\frac{n-1}{n}} - (1)^{\binom{n}{2}} q^{-\frac{1}{2}} n^{-2}
$$

### 3.4. Linear functionals on $U_q(sl_n)$ from $n$-tangles.

For $x \in \text{Hom}_{\mathcal{S}_n}(\eta, \mu)$ and an element $u \in U_q(sl_n)$ define $RT(x) \circ u : V^{\eta} \to V^{\mu}$ as the composition of the action of $u$ on $V^{\eta}$ with $RT(x)$. Similarly one defines $u \circ RT_0(\alpha) : V^{\eta} \to V^{\mu}$ where $u$ acts on $V^{\mu}$ now. We will use ovals for depicting morphisms of $\mathcal{S}_n$ and rectangles for elements of $U_q(sl_n)$ throughout the
paper. Since $RT(x)$ is a $U_q(sl_n)$-morphism, we have

$$RT(x) \circ u = u \circ RT(x), \quad \text{or, pictorially, } \begin{array}{c|c|c} x & u & \overrightarrow{RT} \end{array}. \quad (31)$$

A vector space equipped with a basis is called **based**. The tensor product of based vector spaces $V_i$ with bases $B_i$ for $i = 1, \ldots, k$, has the natural tensor product basis $\prod_{i=1}^{k} B_i$. A linear operator $A : V_1 \to V_2$ between based vector spaces with bases $\{e_i^{(1)}\}$ and $\{e_j^{(2)}\}$ defines a matrix with elements $A_{ij}^k$, also denoted by $\langle j | A | i \rangle$, such that

$$A(e_i^{(1)}) = \sum_{j} A_{ij}^k e_j^{(2)}, \quad \langle j | A | i \rangle := A_{ij}^k.$$

We consider the $U_q(sl_n)$-module $V$ as a based $\mathbb{Q}(v)$-vector space with basis $\{e_1, \ldots, e_n\}$, defined in Subsec. 2.2. The dual $V^*$ will be considered as a based vector space with the basis $\{f^1, \ldots, f^n\}$ where

$$f^i = c_i e^i, \quad \text{for } i = 1, \ldots, n, \text{ where } c_i \text{ are given by } (3).$$

This basis allows for a simplification of our theory and, in particular, makes the $RT$ functor independent of web orientations, cf. Subsection [3.5]. Now for any sign sequence $\eta$, the vector space $V^\eta$ is based with the **tensor basis**, indexed by $\{1, \ldots, n\}^{|\eta|}$, induced by the above bases of $V$ and $V^*$.

A **right state** (respectively a **left state**) of a morphism $x \in \text{Hom}_{sl_n}(\eta, \mu)$, for some $\eta$ and $\mu$, is an assignment of $i_1, \ldots, i_{|\eta|} \in \{1, \ldots, n\}$ (respectively $j_1, \ldots, j_{|\mu|} \in \{1, \ldots, n\}$) to the right (respectively left) endpoints of $x$, listing them from the bottom to the top. A right (respectively: a left) stated morphism $x$ as above will be denoted by $x = (x, i)$ (respectively: $x = (j, x)$), where $i = \{i_1, \ldots, i_{|\eta|}\} \in \{1, \ldots, n\}^{|\eta|}$ and $j = \{j_1, \ldots, j_{|\mu|}\} \in \{1, \ldots, n\}^{|\mu|}$. Similarly, a **stated morphism** is a triple $x = (j, x, i)$, as above. For such $x$ define a $\mathbb{Q}(v)$-linear function $\Gamma(x) : U_q(sl_n) \to \mathbb{Q}(v)$ whose value at $u \in U_q(sl_n)$ is

$$\langle \Gamma(x), u \rangle := \langle j | RT(x) \circ u | i \rangle = \langle j | u \circ RT(x) | i \rangle.$$

For example, for $x = \emptyset$

$$\Gamma(\emptyset) = \epsilon : U_q(sl_n) \to \mathbb{Q}(v), \quad \text{is the counit.} \quad (33)$$

Let $\mathcal{T}$ be the set of isotopy classes of stated $n$-tangles. The module $\mathbb{Z}[v^{\pm 1}] \mathcal{T}$ over $\mathbb{Z}[v^{\pm 1}]$ freely spanned by $\mathcal{T}$ is an algebra with the product $\alpha \beta = \alpha \otimes \beta$ obtained by placing $\alpha$ below $\beta$ and concatenating the states. We extend $\Gamma$ linearly onto a $\mathbb{Z}[v^{\pm 1}]$-linear map

$$\Gamma : \mathbb{Z}[v^{\pm 1}] \mathcal{T} \to U_q(sl_n)^*,$$

**Proposition 3.4.** The map $\Gamma$ is a $\mathbb{Z}[v^{\pm 1}]$-algebra homomorphism.

**Proof.** Since the $U_q(sl_n)$-action on tensor product of $U_q(sl_n)$-modules is given by the coproduct on $U_q(sl_n)$, for any $n$-tangles $\alpha$ and $\beta$ and any $u \in U_q(sl_n)$ we have

$$(\alpha \otimes \beta) \circ u = \sum (\alpha \circ u') \otimes (\beta \circ u''), \quad \text{where } \Delta(u) = \sum u' \otimes u''.$$

By assigning states, we get

$$\langle \alpha \otimes \beta, u \rangle = \sum \langle \alpha, u' \rangle \langle \beta, u'' \rangle,$$
which means that \( \Gamma \) maps the product \( \alpha \otimes \beta \) to the dual of the coproduct in \( U_q(sl_n) \), which is the product on \( U_q(sl_n)^* \). Hence, \( \Gamma \) is an algebra homomorphism. \( \square \)

3.5. Dual operator, orientation reversal invariance. For a stated morphism \( x = (i, x, j) \) let \( \langle x \rangle \) be the \((i, j)\)-element of the matrix of \( RT(x) \),

\[
\langle x \rangle = \langle i \mid RT(x) \mid j \rangle \in \mathbb{Q}(v).
\]

For a stated \( n \)-tangle, \( \alpha = (i, \alpha, j) \), let \( \bar{\alpha} = (i, \bar{\alpha}, j) \), where \( \bar{\alpha} \) is obtained from \( \alpha \) by reversing the orientation of all its edges and and of all circle components.

Furthermore, let

\[
\alpha^* = \frac{1}{c_i} (c_j) (\bar{\alpha}^*, ro(\alpha), i^*),
\]

where \( ro(\alpha) \) is obtained by \( 180^\circ \) rotation of \( \alpha \),

\[
c_i = \prod_{m=1}^{k} c_{i_m} \quad \text{and} \quad (i_1, \ldots, i_k)^* = (\bar{i}_k, \ldots, \bar{i}_1), \quad \text{where}\; \bar{i} = n + 1 - i.
\]

Pictorially,

\[
\alpha^* = \left( \begin{array}{c|c|c}
\alpha & \bar{\alpha} & \mathbb{R}
\end{array} \right)^* = \left( \begin{array}{c|c|c}
\alpha & \bar{\alpha} & \mathbb{R}
\end{array} \right) = \frac{1}{c_i} (c_j) \left( \begin{array}{c|c|c}
\alpha & \bar{\alpha} & \mathbb{R}
\end{array} \right).
\]

The above operations extend linearly onto \( \mathbb{Z}[v^\pm 1] \mathcal{T} \).

It should be noted that for an \( n \)-tangle \( \alpha \) the operators \( RT(\alpha), RT(\bar{\alpha}) \), and \( RT(ro(\alpha)) \) have different domains and different target spaces. The following statement shows an important benefit the basis \( \{f^1, \ldots, f^n\} \) of \( V^* \):

**Proposition 3.5.** For any stated morphism \( x = (i, x, j) \) one has \( \langle x \rangle = \langle \bar{x} \rangle = \langle x^* \rangle \).

**Proof.** Checking \( \langle \bar{\alpha} \rangle = \langle \alpha \rangle \) for cups and caps is quite easy:

\[
\langle \bar{\alpha} \rangle = \langle \alpha \rangle,
\]

\( (34) \)

(1) The identity \( \langle \alpha^* \rangle = \langle \alpha \rangle \) follows from

\[
\langle j^*|RT(ro(\alpha))|i^* \rangle = \bigg( \prod_{m=1}^{k} c_{i_m} \bigg) \left( \prod_{m=1}^{l} c_{j_m} \right)^{-1} \langle i^*|RT(\alpha)|j^* \rangle,
\]

where the first identity is by an isotopy. For the second identity, we decompose the tangle above along the dashed lines and use the values of caps and cups in Eq. \( (34) \).

(2) To prove \( \langle \bar{\alpha} \rangle = \langle \alpha \rangle \) one needs to check it for the elementary \( n \)-tangles. For cups and caps it have been done in Eq. \( (34) \). The statement for the positive crossing (Eq. \( (15) \)) follows from part (1) and the fact that the \( R \)-matrix formula \( (9) \) is preserved under the involution \( i \leftrightarrow \bar{l}, j \leftrightarrow \bar{k} \). The statement for the sink and the source follows by a straightforward computation. \( \square \)
3.6. **Annihilators.** In this and the next two subsections we will analyze the kernel of $\Gamma : \mathbb{Z}[v^\pm 1] \mathcal{T} \to U_q(sl_n)^\ast$.

An **internal annihilator** is a $S_n$-morphism $x$ such that $RT(x) = 0$. From the definition we have:

**Proposition 3.6.** (a) Internal annihilators form a monoidal ideal in $S_n$.

(b) For any stating $(i, x, j)$ of an internal annihilator $x$, we have $\Gamma((i, x, j)) = 0$.

Eqs. (25)–(28) are internal annihilators, called the **basic internal annihilators**.

For a sequence $\eta$ of signs $\{\pm\}$ let $H_r(\eta)$ be the $\mathbb{Z}[v^\pm 1]$-module freely spanned by right stated $n$-tangles $(\alpha, i)$ such that $V_l(\alpha) = V^\eta$. Note that $H_r(\eta)$ contains $\text{Hom}_{S_n}(\emptyset, \eta)$, and we extend the map $RT$ to $RT : H_r(\eta) \to \text{Hom}_{\mathbb{Q}(v)}(\mathbb{Q}(v), V^\eta)$ by $RT((\alpha, i))(1) = RT(\alpha)(v_i)$, where $v_i$ is the basis vector of $V_r(\alpha)$ with index $i$. For example, for $\eta = (+, -, +, -)$,

$$\alpha = \begin{array}{c}
+ \\
-
\end{array} \begin{array}{c}
+ \\
-
\end{array} \begin{array}{c}
+ \\
-
\end{array} \begin{array}{c}
+
\end{array} \quad \text{and} \quad i = (i_1, i_2),$$

we have

$$RT((\alpha, i)) = e_{i_1} \otimes \widetilde{\text{coev}}_0(1) \otimes f^{i_2} \in V \otimes V^* \otimes V \otimes V^*.$$ 

Note that for $x \in H_r(\eta) \setminus \text{Hom}(\eta, \emptyset)$, the operator $RT(x)$ might not be an $U_q(sl_n)$-morphism. For $y \in H_r(\eta')$ define $x \otimes y \in H_r(\eta \otimes \eta')$ by placing $y$ atop $x$ and by concatenating the right states. For $z \in \text{Hom}_{S_n}(\eta, \mu)$ we can define the composition $z \circ x \in H_r(\mu)$. Clearly

$$RT(x \otimes y) = RT(x) \otimes RT(y) \quad \text{and} \quad RT(x \circ y) = RT(x) \circ RT(y).$$

A **right annihilator** is an element $x \in H_r(\eta)$, for certain $\eta$, such that $RT(x) = 0$. By the above equation and by Eq. (31), we have

**Proposition 3.7.** Let $x \in H_r(\eta)$, $y \in H_r(\eta')$ be right annihilators and let $z \in \text{Hom}(\eta, \mu)$.

(a) For any left state $i$ of $x$ we have $\Gamma((i, x)) = 0$.

(b) All $x \otimes y, y \otimes x$, and $z \circ x$ are right annihilators.

We do not draw the left boundary vertical edge in pictures of right annihilators, to indicate that $RT$ holds for their composition with any web on the left (for which such composition is possible).
Proposition 3.8. We have the following identities for the values of the function \( RT \):

\[
\begin{align*}
(35) & \quad \text{Figure:} \\
\sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} a & = \delta_{j,i} c_i \\
(36) & \quad \text{Figure:} \\
\sum_{i=1}^{n} (c_i)^{-1} & = \delta_{j,i} c_i \\
(37) & \quad \text{Figure:} \\
q^{-\frac{1}{n}} \left( \delta_{j<i} (q - q^{-1}) + q^{\delta_{i,j}} \right) & = \delta_{j,i} c_i \\
(38) & \quad \text{Figure:}
\end{align*}
\]

for any \( i, j = 1, \ldots, n \), where the white circle represents the orientation, left-to-right or right-to-left, and is the same for all white circles in one identity. The black circle stands for the opposite orientation of the white one. The values \( \delta_{j<i}, \delta_{i,j}, a, c_1, \ldots, c_n \) were defined in Subsec. 2.1.

By subtracting the left side from the right side in the equations above we obtain right annihilators which we call basic.

Proof of Proposition 3.8: By applying the total orientation reversion and Proposition 3.5 if necessary, we can assume all the white circles indicate the left-to-right orientation. Identity (35) is the defining equation \( \text{(8)} \) of the operator \( A_+ \), while Identity (38) is the defining equation \( \text{(9)} \) of the braiding. Identities (36) and (37) are consequences of Eq. (34). □

We now define left annihilators. For a sign sequence \( \eta \), let \( H_l(\eta) \) be the \( \mathbb{Z}[v^{\pm 1}] \)-module freely spanned by left-stated \( n \)-tangles \( (i, \alpha) \) such that \( V_r(\alpha) = V^\eta \). Then \( H_l(\eta) \) contains \( \text{Hom}_{\mathbb{Z}_n}(\eta, \emptyset) \), and we extend \( RT \) to

\[
\text{RT} : H_l(\eta) \to \text{Hom}_{\mathbb{Z}[v]}(V^\eta, \mathbb{Q}(v)) \text{ by } \text{RT}(\mu) = \text{coeff}_i(\text{RT}(\alpha)(z)),
\]

for \( z \in V^\eta \), where \( \text{coeff}_i : V^\mu \to \mathbb{Q}(v) \) is coefficient of the basis vector of \( V^\mu \) indexed by \( i \).

A left annihilator is an element \( x \in H_l(\eta) \), for a certain \( \eta \), such that \( \text{RT}(x) = 0 \). In analogy to Proposition 3.7 we have:

Proposition 3.9. Let \( x \in H_l(\eta), y \in H_l(\eta') \) be left annihilators and let \( z \in \text{Hom}(\mu, \eta) \).

(a) For any right state \( \mu \) of \( x \) we have \( \Gamma((x, \mu)) = 0 \).
(b) All \( x \otimes y, y \otimes x, \) and \( x \circ z \) are left annihilators.

3.7. Turning right annihilators to left ones. For an integer \( k \geq 2 \) let \( H_k \) be the positive half-twist of \( k \) strands and let \( \bar{H}_k \) be its inverse:

\[
H_k = \text{Figure:} \quad \bar{H}_k = \text{Figure:}
\]

(Note that \( H_k \) does not twist the framing, which always points towards the reader. This applies to all half-twists considered in this paper.) By abuse of notation, for any \( n \)-tangle \( \alpha \)
with \( k \) left endpoints, denote by \( \bar{H}_k \circ \alpha \) the composition of \( \alpha \) with a version of \( \bar{H}_k \) in which the orientation of some of its components was reversed so that it is composable with \( \alpha \).

Let \( \text{hd} : H_r(\eta) \to H_l(\eta) \) be a \( \mathbb{Z}[v^{\pm 1}] \)-linear map given by
\[
\text{hd}(\alpha, i) = (i, \bar{H} \circ \text{ro}(\alpha)),
\]
where \( \text{ro}(\alpha) \) denotes the \( 180^\circ \) rotation of \( \alpha \) about the center, as before.

In Subsection 3.5, we showed that our basis \( \{ f_i \} \) of \( V^* \) makes the matrices \( R_T(x) \) invariant under the total reversal of orientation of \( x \). It also makes the following statement hold:

**Proposition 3.10.** If \( x \) is a basic right annihilator then \( \text{hd}(x) \) is a left annihilator.

**Proof:** The statement for each basic right annihilator can be checked by a direct computation. The calculation for Relations (36)-(37) follows from the framing change, Eq. (26). The calculation for Relation (38) utilizes Eq. (25). Finally, let us show that the image of (35) under \( \text{hd} \), pictured below, is a left annihilator. (We assume its orientation to the right, since the statement for the opposite orientation follows from Proposition 3.5.)

\[
\text{RT} = a \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \sigma(n) \sigma(2) \sigma(1) \ldots .
\]

Note that since \( H_k \) is invertible in \( \text{End}(V^{\otimes n}) \) we can consider the above equality composed with \( H \) on the right instead. Then, by Eq. (10), it reduces to
\[
(-q^{-\frac{1}{2}} - 1) \frac{n(n-1)}{2} A_{-} = a \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} e^\sigma(1) \otimes \ldots \otimes e^\sigma(n),
\]
which is indeed equivalent to the definition of \( A_{-} \), Eq. (7), since \( (-q^{-\frac{1}{2}} - 1) \frac{n(n-1)}{2} = \frac{t^n - 1}{2} \).

A stronger statement, valid for all right annihilators, will be shown in Subsection 3.9 using a more conceptual approach.

We call the left annihilators of Proposition 3.10 basic.

3.8. Kernel of \( \Gamma \).

**Theorem 3.11** (Proof in Subsection 6.10). The kernel of \( \Gamma : \mathbb{Z}[v^{\pm 1}] T \to U_q(sl_n)^* \) is generated by internal basic annihilators, right basic annihilators, and left basic annihilators.

This means if we begin with the internal, left, and right basic annihilators, and use procedures described in Propositions 3.6, 3.7, and 3.9, we obtain the entire kernel \( \ker \Gamma \). This theorem is analogous to Conjecture 3.3, except that it describes the kernel of a stated version of \( RT \).

3.9. Half-ribbon Hopf algebra. In this subsection we explain conceptually some technical aspects of this paper. In particular, we interpret our sign modification of the Reshetikhin-Turaev functor, Eq. (23), through a modification of the ribbon element in a completion of \( U_q(sl_n) \). Coincidently, that modification leads to a “half-ribbon” element which makes it possible to prove a stronger version of Proposition 3.10 below. Although the content of this subsection is more technical, it is not necessary for the paper.
Proposition 3.12. If $x$ is a right annihilator then $\text{hd}(x)$ is a left annihilator.

We precede the proof with a few preliminaries: The quantized enveloping algebra $U_q(sl_n)$ is a topological ribbon Hopf algebra, meaning it has an $R$-matrix $R$ in a completion of $U_q(sl_n) \otimes U_q(sl_n)$ and a ribbon element $\vartheta_0$ in a completion of $U_q(sl_n)$, satisfying certain conditions. It is proved in [ST] that there is a completion $\widetilde{U}_q(sl_n)$ of $U_q(sl_n)$ having the same $R$-matrix but a new ribbon element $\vartheta$, which acts on $V$ and $V^*$ the same way as $(-1)^{n-1}\vartheta_0$. (That completion was studied in different context also in [Lu4].) Consequently the new charmed element $g$ is $(-1)^{n-1}g_0$ on $V$ and $V^*$, explaining the sign correction in (23), which we used as the definition of $RT$.

Additionally, there is an invertible element $X \in \widetilde{U}_q(sl_n)$, called the half-twist, such that $X^2 = \vartheta$ and the universal $R$-matrix $R$ satisfies

\begin{equation}
R = (X^{-1} \otimes X^{-1})\Delta(X) = ((\text{id} \circ \Delta)(X))(X^{-1} \otimes X^{-1}),
\end{equation}

where $\text{id}(x \otimes y) := y \otimes x$.

For a sign sequence $\eta = (\eta_1, \ldots, \eta_l)$ let $\vec{\eta} = (\eta_l, \ldots, \eta_1)$ and let $\text{rev}_k : V^\eta \to V^{\vec{\eta}}$ be the $R$-linear operator given by $\text{rev}_k(x_1 \otimes \cdots \otimes x_k) = (x_k \otimes \cdots \otimes x_1)$. From Eq. (39) by induction on $k$ we get the following, see [ST] Proposition 4.18: If $H$ stands for $\widetilde{H}$ with an orientation on the strands on the right given by $\eta$ then we have an equality of transformations $V^\eta \to V^{\vec{\eta}}$:

\begin{equation}
RT(H) = \text{rev}_k \circ X^{\otimes k} \circ \Delta^{[k]}(X^{-1}).
\end{equation}

Here $\Delta^{[k]}$ is defined inductively by $\Delta^{[2]} = \Delta$ and $\Delta^{[k+1]} = (\Delta \otimes \text{id}^{\otimes k}) \circ \Delta^{[k]}$.

An additional special feature of the basis $\{f^i\}$ of $V^*$ (besides those discussed already) is:

Proposition 3.13. The actions of $X$ on $V$ and $V^*$ are given by the same matrix

\begin{equation}
X^i_j = \delta_{i,j} c_i.
\end{equation}

As this is not proved in [ST], we give a proof of this result in the appendix. Similarly, the actions of the charmed element $g$ on both $V$ and $V^*$ are given by the same diagonal matrix with entries

\begin{equation}
g^i_j = \delta_{i,j} g_i, \quad \text{where } g_i = (-1)^{n-1}q^{2i-n-1} = (-1)^{n-1}q^{2\ell}.
\end{equation}

Proof of Proposition 3.12. Let $x \in \mathcal{H}_R(\eta)$ for some $\eta$ be a right annihilator. We need to show that $RT(\text{hd}(x)) = 0$. Since $X$ is invertible, this is equivalent to $RT(\text{hd}(x)) \circ X = 0$, which, in turn, is equivalent to:

$$
\langle RT(\text{hd}(x)) \circ X, j \rangle = 0 \quad \text{for all } j = (j_1, \ldots, j_l) \in \{1, \ldots, n\}^l, l := |\eta|.
$$

Suppose $(\alpha,i)$ is a right stated $n$-tangle. By Eq. (40) and then by Eq. (31), we have

\[
\begin{array}{c}
\text{hd}((\alpha,i)) = \frac{i_k}{i_2} \cdot \frac{i_2}{i_1} = \frac{i_2}{i_1} \cdot \frac{i_2}{i_1} \\
\text{rev} \cdot X^{-1} \cdot \text{ro}(\alpha) = \text{rev} \cdot X^{-1} \cdot \text{ro}(\alpha)
\end{array}
\]
By composing with $X$ on the right, then decomposing along the dashed line and using the values of $X^i_j$ from Eq. (41), we get

$$\langle \text{hd}((\alpha, i)) \circ X, j \rangle_{RT} = c_i \langle j^* | (\alpha, i) \rangle$$

where $c_i = \prod_k c_{ik}$. The second identity follows from Proposition 3.5. By linearity,

$$\langle \text{hd}(x) \circ X, j \rangle_{RT} = c_j \langle j^* | x \rangle = 0,$$

for every $j \in \{1, \ldots, n\}^{\partial \eta}$. This proves $\text{hd}(x) \xrightarrow{RT} 0$. □

Remark 3.14. The use of half-ribbon element came up in a discussion of the first author with F. Costantino and J. Korinman. A full fledged theory of stated skein algebra based on half-ribbon category will be developed in an upcoming work [CKL].

4. Stated SL($n$)-skein modules

4.1. Marked 3-manifolds and $n$-webs. A marked 3-manifold is a pair $(M, N)$, where $M$ is a smooth oriented 3-manifold with (possibly empty) boundary $\partial M$ and $N \subset \partial M$ consists of open intervals, called markings. The topological closure of each marking is required to be the closed interval $[0, 1]$, disjoint from the closure of other markings.

Roughly speaking an $n$-web in $(M, N)$ is like $n$-tangle, except that the framing at boundary points is different. Here is the precise definition, where the first four requirements are the same as those in the definition of an $n$-tangle.

Definition 4.1. An $n$-web $\alpha$ in $(M, N)$ is a disjoint union of oriented circles and a finite directed graph properly embedded into $M$ such that

1) Every vertex of $\alpha$ is either 1-valent or $n$-valent. Each $n$-valent vertex is a sink or a source. We denote set of 1-valent vertices, called endpoints of $\alpha$, by $\partial \alpha$.

2) Each edge of the graph is a smooth embedding of the closed interval $[0, 1]$ into $M$.

3) $\alpha$ is equipped with a framing which is a continuous non-vanishing vector field transversal to $\alpha$. In particular, the framing at a vertex is transversal to all incident edges.

4') The set of half-edges at every $n$-valent vertex is cyclically ordered.

5') The endpoints of $\alpha$ lie in $N$ and the framing at these endpoints is the positive tangent vector of $N$.

Webs are considered up to continuous isotopy within their space.

Note that the only difference between the unbased $n$-tangles of Subsection 3.3 and $n$-webs in the cube $Q$ marked with $\Lambda_l, \Lambda_r$ is the framing at their endpoints. The difference explains why half-twists appear in our theory.

The height order on $\partial \alpha$ is the partial order in which two points $x, y \in \partial \alpha$ are comparable if and only if they belong to the same marking, and $x > y$, or $x$ is higher than $y$, if going along the positive direction of the marking we encounter $y$ first. We say $x$ and $y$ are consecutive if there is no $z \in \partial \alpha$ such that $x > z > y$ or $y > z > x$. 
To depict a local part of an $n$-web $\alpha$ we consider the intersection of $\alpha$ with the cube $Q = [-1,1] \times (-1,1)^2$ embedded into $M$, presented in Figure 6(a). The cube $Q$ can be either in the interior of $M$ or its right side, $\{1\} \times (-1,1)^2$, lies in $\partial M$.

- If $Q$ is in the interior of $M$ then we assume that $\alpha \cap Q$ is an $n$-tangle, and depict $\alpha \cap Q$ by its $n$-tangle diagrams on the shaded square, as in Subsection 3.3. In particular, for all drawn diagrams the framing is perpendicular to the page and pointing to the reader, and the cyclic order of half-edges at each $n$-valent vertex is counterclockwise.

**Figure 6.** (a) Cube $Q$ with a marking $\beta$ and a perpendicular line $\beta^\perp$ on the right face. The shaded square $S$ is in the $XY$-plane. (b) An example of a web $\alpha$ with three strands (depicted by different colors) in $\alpha \cap Q$. (c) The projection of $\alpha \cap Q$ onto $S$. (d) A diagram of $\alpha \cap Q$, with the height order indicated by numeric labels: $i > j$ means $i$ is higher than $j$. (e) Another diagram of $\alpha \cap Q$ obtained by using a different height preserving deformation. (f) This is the same diagram of (e), with the height order indicated by the direction of the boundary line.

- In the second case, we assume that $Q \cap \partial M$ is equal to the right face of $Q$, and $Q \cap N$ is a subinterval of a marking $\beta$ depicted pointing in the direction of the $z$-axis, as in Figure 6(a). In the $Q$ coordinates, $\beta \cap \partial M = \{1\} \times \{0\} \times (-1,1)$. Let $\beta^\perp = \Lambda_r$, the right side of the shaded square. As in the previous case, we assume that the framing points to the reader. (Note the difference between webs in $M$ and tangles in $Q$: the boundary points of $\alpha$ in $Q$ are in $\beta$, while the right endpoints tangles in $Q$ are in $\Lambda_r = \beta^\perp$.) By an isotopy, we can bring $\alpha \cap Q$ to a general position with respect to the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$, except that all the points in $\partial \alpha \cap \beta$ project to the same point, see Figure 6(c). To resolve this issue, first we define a **height-preserving deformation of $Q$** as a continuous family of diffeomorphisms $\phi_t : Q \to Q, t \in [0,1]$, supported in a small neighborhood of the right face of $Q$ and preserving the $z$-coordinate, i.e. the height above the page. We use such a height-preserving deformation to bring $\alpha$ to $\alpha'$ whose endpoints on the right face of $Q$ have distinct projections (through $p$). The image $p(\alpha' \cap Q)$ together with the usual over- and undercrossing data and with the linear order of its boundary points on $\beta^\perp$ (induced from the height order) is a **diagram of $\alpha \cap Q$**. For example, Figure 6(d) shows a diagram of Figure 6(b). Note however that a different height-preserving deformation can give rise to different diagram, see for example Figure 6(e). Note that $\alpha'$ is not an $n$-web because its endpoints are on in $N$ in general.

Although the height order of web ends in $\alpha \cap Q$ can be always indicated by integers as in Figure 6(d)-(e), in this paper we will always use the following convention: when presenting $\alpha \cap Q \subset M$ diagrammatically, we will chose a direction of $\beta^\perp$ (indicating it by an arrow...
down or up) and arrange for the height order of web ends in $\alpha \cap Q$ to increase monotonically (without gaps) in the indicated direction. For example, Figure 6(f) indicates the web in part (e). Note that the height order of endpoints of the $\alpha$ outside the drawn part can be arbitrary.

4.2. Skein relations for $n$-webs. A state of an $n$-web $\alpha$ is a map $s : \partial \alpha \to \{1, 2, \ldots, n\}$. The value $s(x)$, for $x \in \partial \alpha$, is called the state of $x$. A web with a state $s$ is called stated.

We will consider stated $n$-webs up to isotopy (in the space of all stated $n$-webs) and denote the set of their isotopy classes by $W_n(M, N)$.

Recall that the ground ring $R$ is commutative and it comes with a distinguished invertible $v = q^{1/2n} \in R$. The stated $SL(n)$-skein module of $(M, N)$, denoted by $S_n(M, N)$, is the quotient of the free $R$-module $RW_n(M, N)$ by the submodule $SkRel_n(M, N)$ generated by the following internal relations (43)-(50), which are the basic internal annihilators, and boundary relations (47)-(50), which comes from the basic right annihilators:

\[
\begin{align*}
(43) & \quad q^{\frac{1}{n}} \begin{array}{c} \vdots \\\vdots \end{array} - q^{-\frac{1}{n}} \begin{array}{c} \vdots \\\vdots \end{array} = (q - q^{-1}) \begin{array}{c} \vdots \\\vdots \end{array} \\
(44) & \quad \begin{array}{c} \vdots \\\vdots \end{array} = t \begin{array}{c} \vdots \\\vdots \end{array} , \quad \text{where } t = (-1)^{n-1} q^{\frac{1}{n}} \\
(45) & \quad \begin{array}{c} \vdots \\\vdots \end{array} = (-1)^{n-1}[n]_q \begin{array}{c} \vdots \\\vdots \end{array} , \quad \text{where } [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \\
(46) & \quad \begin{array}{c} \vdots \\\vdots \end{array} = (-q)^{\binom{n}{2}} \cdot \sum_{\sigma \in S_n} (-q^{-\frac{1-n}{n}})^{\ell(\sigma)} \begin{array}{c} \vdots \\\vdots \end{array} \\
\end{align*}
\]

where the ellipse enclosing $\sigma_+$ is the minimum crossing positive braid representing a permutation $\sigma \in S_n$ and $\ell(\sigma)$ is the length of $\sigma \in S_n$, as before.

The remaining relations in $SkRel_n(M, N)$ take place near markings where we use the convention in Subsection 4.1 about height order. Thus, the bold boundary line of a shaded rectangle is a part of $\beta^\perp$, orthogonal to a marking $\beta$, and if it has a direction, then the endpoints on that part are consecutive in the height order, given by the direction. The height
order outside the drawn part of \( \beta \perp \) can be arbitrary. Here are the boundary relations:

\[
\begin{align*}
(47) & \quad \sigma(n) = a \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \sigma(2) - \sigma(1) \\
(48) & \quad \delta_{j,i} c_i, \\
(49) & \quad \sum_{i=1}^{n} (c_i)^{-1} = \delta_{j,i} c_i, \\
(50) & \quad q^{-\frac{1}{n}} \left( \delta_{j<i} (q - q^{-1}) \delta_{i,j} + q^{\delta_{i,j}} \sigma(i) \right),
\end{align*}
\]

where the values \( \delta_{j<i}, \delta_{i,j}, a, c_1, \ldots, c_n \) were defined in Sec. 2.1 and the small white circles represent an arbitrary direction of the edges (left-to-right or right-to-left), consistent for the entire equation, as before. The black circle represents the opposite direction.

We show in Proposition 7.5 that if \( (n-2)! \) is invertible in \( R \) then Relation (50) is a consequence of Relations (43)-(49).

4.3. Eliminating sinks and sources.

**Proposition 4.2.** For any marked 3-manifold, \( S_n(M,N) \) is spanned by stated \( n \)-webs with no sinks nor sources.

**Proof.** If \( \mathcal{N} = \emptyset \) then the numbers of sinks and sources in any \( n \)-web coincide and they can be eliminated by Relation (46). If \( \mathcal{N} \neq \emptyset \) then sinks and sources can be eliminated by Relation (47). \( \square \)

Nonetheless, the use of sinks and sources in our theory makes it much more manageable.

4.4. Change of ground ring. We will use the notation \( S_n(M,\mathcal{N},R) \) when we need to make the coefficient ring \( R \) explicit. By our assumptions, \( R \) is an algebra over \( \mathbb{Z}[v^\pm 1] \). The right exactness of tensor product gives a natural isomorphism

\[
S_n(M,\mathcal{N},\mathbb{Z}[v^\pm 1]) \otimes_{\mathbb{Z}[v^\pm 1]} R \cong S_n(M,\mathcal{N},R).
\]

Therefore, many properties of \( S_n(M,\mathcal{N},R) \) follow from those of \( S_n(M,\mathcal{N},\mathbb{Z}[v^\pm 1]) \).

4.5. Functoriality. An **embedding** of a marked 3-manifold \( (M,\mathcal{N}) \) into a marked 3-manifold \( (M',\mathcal{N}') \) is an orientation preserving proper embedding \( f : M \hookrightarrow M' \) which maps \( \mathcal{N} \) into \( \mathcal{N}' \) preserving their orientations. Clearly \( f \) induces an \( R \)-module homomorphism \( S_n(f) : S_n(M,\mathcal{N}) \to S_n(M',\mathcal{N}') \) mapping each \( n \)-web \( \alpha \) to \( f(\alpha) \) with its framing transformed by the differential \( f_* : TM \to TM' \). That homomorphism depends only on the isotopy class of \( f \) (in the embeddings). A **morphism** from \( (M,\mathcal{N}) \) to \( (M',\mathcal{N}') \) is an isotopy class of embeddings from \( (M,\mathcal{N}) \) to \( (M',\mathcal{N}') \). Hence, \( S_n(\cdot) \) defines a functor from the category of marked 3-manifolds to the category of \( R \)-modules.
Example 4.3. Let \((M, N)\) be a marked 3-manifold. For any closed subset \(X\) of \(\partial M - N\), its complement \((M - X, N)\) is a marked 3-manifold as well and the natural embedding \(\iota : (M - X, N) \hookrightarrow (M, N)\) is a morphism called a pseudo-isomorphism. It induces an \(R\)-module isomorphism \(\iota_* : \mathcal{S}(M', N) \xrightarrow{\cong} \mathcal{S}(M, N)\).

In this paper we will consider certain geometric operations on 3-manifolds, like cutting and glueing them along disks, which produce new manifolds defined up a diffeomorphisms only. We will address this issue with the aid of the following notion:

A strict isomorphism class of marked 3-manifolds is a family of marked 3-manifolds \((M_i, N_i), i \in I\) equipped with isomorphisms \(f_{ij} : (M_i, N_i) \rightarrow (M_i, N_i)\) for any two indices \(i, j\) such that \(f_{ii} = \text{id}\) and \(f_{jk} \circ f_{ij} = f_{ik}\). For a strict isomorphism class of marked 3-manifolds we can identify all \(R\)-modules \(\mathcal{S}(M_i, N_i)\) via the isomorphisms \(\mathcal{S}(f_{ij})\).

Note that gluing and cutting operations produce strict isomorphism classes of marked 3-manifolds.

For a disjoint union of \(M_1\) and \(M_2\), the map \(\mathcal{S}(M_1, N_1) \otimes \mathcal{S}(M_2, N_2) \rightarrow \mathcal{S}(M_1 \sqcup M_2, N_1 \sqcup N_2)\) sending \(\alpha_1 \otimes \alpha_2\) to \(\alpha_1 \sqcup \alpha_2\) is an isomorphism. We will identify \(\mathcal{S}(M_1 \sqcup M_2, N_1 \sqcup N_2)\) with \(\mathcal{S}(M_1, N_1) \otimes \mathcal{S}(M_2, N_2)\) though this map.

4.6. Grading. For a stated \(n\)-web \(\alpha\) in \((M, N)\) and a marking \(\beta \subset N\) we define the \(\beta\)-degree

\[
\deg_{\beta}(\alpha) = \sum_{x \in \alpha \cap \beta} d_{s(x)} = \sum_{x \in \alpha \cap \beta} \left( s(x) - \frac{n + 1}{2} \right) \in \frac{1}{2} \mathbb{Z},
\]

where \(s(x)\) denotes the state of \(\alpha\) at \(x\). Note that that the \(\beta\)-degree is preserved by the skein relations (43)-(50) and, therefore, it descends to \(\frac{1}{2} \mathbb{Z}\)-valued grading on \(\mathcal{S}(M, N)\).

4.7. Useful Identities. Recall that \(a, t, c_i\) were defined in Subsection 2.1

Proposition 4.4. The following identities hold in any stated skein module \(\mathcal{S}(M, N)\):

\[
[\begin{array}{c}
[1] & [n - 2]! & [n - 2]! & q^{n-1} \\
2 & 3 & 4 & 5 & \ldots
\end{array}] = [n - 2]! q^{\frac{n-1}{2}} - (-1)^{\frac{n}{2}} q^{-\frac{n-1}{2}} [n - 2]!
\]

\[
[\begin{array}{c}
[1] & [n - 2]! & [n - 2]! & q^{n-1} \\
2 & 3 & 4 & 5 & \ldots
\end{array}] = -q^{-(1+\frac{n}{2})}
\]

\[
[\begin{array}{c}
[1] & [n] & [n] & [n] & [n] & \ldots
\end{array}] = a t^{n/2} (-q)^{\ell(\sigma)}
\]

\[
[\begin{array}{c}
[1] & [n] & [n] & [n] & [n] & \ldots
\end{array}] = a (-q)^{\ell(\sigma)}
\]
\begin{align}
\sigma(n) = d & = \frac{a t^{n/2} \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \sigma(n)}{\sigma(2) \sigma(1)}.
\end{align}

(56) \quad \begin{array}{c}
\begin{array}{c}
\text{Proof.}
\end{array}
\end{array}

Proof of (53): The web on the right is non-zero only for \( \tau(i) = \overline{\sigma(i)} \) for \( i = 1, \ldots, n \). Since \( \ell(\tau) = \ell(\sigma) \) then, the above equals
\[
a(-q)^{\ell(\sigma)} \cdot c_1 \cdot \cdots \cdot c_n = a t^{n/2} (-q)^{\ell(\sigma)},
\]
by (54).

Proof of (54): Since
\[
(-q^{-n+1})^{\binom{n}{2}} = (-1)^{\binom{n}{2}} q^{-\frac{n^2-1}{4}},
\]
the statement follows by (53).

Proof of (55): By composing (47) with \( \begin{array}{c}
\begin{array}{c}
\text{on the left, we obtain}
\end{array}
\end{array} \)

\[
(-q^{-n+1})^{\binom{n}{2}} = \sum_{\sigma \in S_n} a(-q)^{\ell(\sigma)} \sigma(n) = \sum_{\sigma \in S_n} a(-q)^{\ell(\sigma)} \sigma(n).
\]
by (52). Now the statement follows by (59) and by rotating these skeins 180°.

**Proof of (56):** The left side equals
\[
\begin{align*}
\begin{array}{c}
\alpha_i \downarrow j = t^{-1}, \\
\beta_i \downarrow j = t^{-1} \delta_{i,j} c_j = t^{-1} \delta_{i,j} c_i = \delta_{i,j} c_i^{-1},
\end{array}
\end{align*}
\]
by (44).

**Proof of (57):** By (44) and (49), \( t^{-1} \begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0.5,1) circle (0.25);
\draw (0.5,0) circle (0.25);
\end{tikzpicture} = \begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0.5,1) circle (0.25);
\draw (0.5,0) circle (0.25);
\end{tikzpicture} = \sum_{i=1}^{n} (c_i)^{-1} \begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0.5,1) circle (0.25);
\draw (0.5,0) circle (0.25);
\end{tikzpicture}.
\]

Now the statement follows by 180° rotation and the fact that \( tc_i^{-1} = c_i \), cf. (4).

**Proof of (58):** \( \begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0.5,1) circle (0.25);
\draw (0.5,0) circle (0.25);
\end{tikzpicture} = \begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0.5,1) circle (0.25);
\draw (0.5,0) circle (0.25);
\end{tikzpicture} = q^{-\frac{1}{n}} \left( \delta_{j<i}(q-q^{-1}) \begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0.5,1) circle (0.25);
\draw (0.5,0) circle (0.25);
\end{tikzpicture} + q^{\delta_{i,j}} \begin{tikzpicture}
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0.5,1) circle (0.25);
\draw (0.5,0) circle (0.25);
\end{tikzpicture} \right) \), by (50). \( \square \)

### 4.8. Splitting homomorphism

As mentioned in Introduction, an important property of stated skein modules is that they behave in a simple manner under the splitting of 3-manifolds along disks. This property, known as the Splitting Theorem, was first proved by [Le1, BL] for the Kauffman bracket skein modules \( (n = 2) \) and then by [Hi] for \( n = 3 \). We formulate it now and prove for all \( n \) below.

Suppose \((M, N)\) is a marked 3-manifold and \( D \) is a properly embedded closed disk in \( M \), disjoint from the closure of \( N \). By removing a collar neighborhood of \( D \) we get a topological 3-manifold \( M' \) whose boundary contains two copies \( D_1 \) and \( D_2 \) of \( D \) such that gluing \( D_1 \) to \( D_2 \) yields \( M \) together with a surjective homomorphism \( pr : M' \to M \). The manifold \( M' \) has a smooth structure with corners. However, these corners can be smoothed out uniquely up to isotopy.

Let \( \beta \subset D \) be an oriented open interval, and \( \beta_1 \subset D_1 \) and \( \beta_2 \subset D_2 \) be preimages of \( \beta \). The **splitting** of \((M, N)\) **along** \((D, \beta)\), denoted by \( \text{cut}_{(D, \beta)}(M, N) \), is the marked 3-manifold \((M', N')\), where \( N' = N \cup \beta_1 \cup \beta_2 \). It is easy to see that \( \text{cut}_{(D, \beta)}(M, N) \) is defined uniquely as a strict isomorphism class, cf. Subsection 4.5.

Let \( \alpha \) be a stated \( n \)-web in \((M, N)\). This web in this subsection is given by a specific embedding and, hence, not considered up to isotopy. It is said to be **\((D, \beta)\)-transverse** if the vertices of \( \alpha \) are not in \( D \), \( \alpha \) is transverse to \( D \), \( \alpha \cap D \subset \beta \), and the framing at every point of \( \alpha \cap \beta \) is a positive tangent vector of \( \beta \). Note that every web \( \alpha \) in \((M, N)\) can be isotoped so that it is **\((D, \beta)\)-transverse**. Suppose in addition that \( \alpha \) is stated. Then the \( n \)-web \( pr^{-1}(\alpha) \) of \((M', N')\) is stated except for its endpoints in \( \beta_1 \cup \beta_2 \), cf. Fig. 7.

Given any map \( s : \alpha \cap \beta \to \{\pm\} \), let \( \alpha(s) \) denote the (partially stated) \( n \)-web \( pr^{-1}(\alpha) \) in \((M', N')\) with additional states \( s(pr(x)) \) for \( x \in pr^{-1}(\alpha) \cap (\beta_1 \cup \beta_2) \). Hence, \( \alpha(s) \) is fully stated. We call \( \alpha(s) \) a **lift** of \( \alpha \). If \( |\alpha \cap \beta| = k \) then \( \alpha \) has \( n^k \) distinct lifts.

**Theorem 4.5.** Let \( D \) be a closed disk properly embedded in a marked 3-manifold \((M, N)\) and let \( \beta \) be an oriented open arc in \( D \). Let \( \text{cut}_{(D, \beta)}(M, N) \) be the splitting of \((M, N)\) along...
Figure 7. An example of a splitting of an $n$-web (in green) intersecting the splitting disk $D$ twice.

$(D, \beta)$, as described above. Then there is a unique $R$-module homomorphism

$$\Theta_{(D, \beta)} : S_n(M, N) \to S_n(\text{cut}_{(D, \beta)}(M, N))$$

sending every stated $(D, \beta)$-transverse $n$-web $\alpha$ in $(M, N)$ to the sum of all of its lifts,

$$\Theta_{(D, \beta)}(\alpha) = \sum_{s : \alpha \cap \beta \to \{\pm\}} \alpha(s). \quad (60)$$

Note that for any arcs $\beta, \beta'$ in $D$ there is an isomorphism between marked 3-manifolds

$$\text{cut}_{(D, \beta)}(M, N) \cong \text{cut}_{(D, \beta')}(M, N),$$

inducing an isomorphism of stated skein modules which commutes the splitting homomorphisms. Consequently, we will often denote $\Theta_{(D, \beta)}$ and $\text{cut}_{(D, \beta)}(M, N)$ by $\Theta_D$ and $\text{cut}_D(M, N)$ when it does not lead to confusion.

**Remark 4.6.** It is easy to see that splitting homomorphisms along any two disjoint splitting disks $D_1$ and $D_2$ commute,

$$\Theta_{D_1} \circ \Theta_{D_2} = \Theta_{D_2} \circ \Theta_{D_1}.$$ 

**Remark 4.7.** By removing a closed subset of $\partial M$ disjoint from $N$ and using the pseudo-isomorphism of Example 4.3, we called it “pseudo”, which may be better than “quasi”, so we don’t have repeating $i$’s. We can apply the theorem to a many cases when $D$ is a closed disk with some closed intervals on its boundary removed. This fact will be useful in Section 5 where we will apply the Splitting Theorem to thickened surfaces $\Sigma \times (-1, 1)$ cut along open disks, $(-1, 1) \times (-1, 1)$.

In general the splitting homomorphism is not injective. For an example in the $n = 2$ case see [CL2]. We will discuss the injectivity and the image of the splitting homomorphism for thickened surfaces in Section 5.

**Proof of the Splitting Theorem:** We identify a closed collar neighborhood of $D$ with the closed cube $\overline{Q} = [-1, 1]^3$ so that $D = \{0\} \times [-1, 1]^2$ and $\beta$ is an open interval subset of $\{0\} \times \{0\} \times [-1, 1]$, as in Figure 8(a). For a stated $(D, \beta)$-transverse $n$-web $\alpha$ let $\Theta(\alpha) \in S_n(\text{cut}_{(D, \beta)}(M, N))$ be the right side of Eq. (60). To prove the theorem we need to show $\Theta(\alpha)$ is invariant under isotopies of $\alpha$.

An ambient isotopy of $\alpha$ in $M$ can be decomposed into a sequence of isotopies, each of which supported in a small neighborhood of $D$ or supported outside of $D$. The latter clearly preserves $\Theta$, so we only need to check invariance of $\Theta$ under isotopies with support in the interior of $\overline{Q}$. By an isotopy outside $D$ we can assume that $\alpha \cap Q$ is an $n$-tangle. To get a diagram of $\alpha \cap Q$ we first use height preserving deformation near $D$ to move $\alpha$ to a general position with respect to the projection onto $[-1, 1]^2 \times \{0\}$ (as always considered in page). The points in $\alpha \cap \beta$, after that deformation, project to points on $\beta \perp$. We will always choose
a height preserving deformation such that the height order on $\beta^\perp$ is given by the direction from the top to the bottom, as in Figure 8(b).

Now we can decompose a diagram of $\alpha$ into elementary tangle diagrams listed in (13)-(15). If $\alpha'$ is isotopic to $\alpha$ by an isotopy in $Q$, then its diagram can be obtained by a sequence of operations moving elementary tangles through $\beta$, and the height exchange move, discussed in point (d) below. Therefore, it is enough to verify that $\Theta(\alpha)$ is preserved by the following four moves:

(a) passing a cap through $\beta$. The invariance of $\Theta(\alpha)$ under this move is a consequence of skein relations (48) and (57):

$$\Theta \left( \begin{array}{c} \includegraphics{cap} \end{array} \right) = \sum_{i,j} \Theta \left( \begin{array}{c} \includegraphics{cap} \end{array} \right) = \sum_{i=1}^{n} c_i \frac{i}{i} = \Theta \left( \begin{array}{c} \includegraphics{cap} \end{array} \right).$$

By the same argument, $\Theta \left( \begin{array}{c} \includegraphics{cap} \end{array} \right) = \Theta \left( \begin{array}{c} \includegraphics{cap} \end{array} \right)$.

(b) passing a sink or a source through $\beta$. The invariance of $\Theta$ under this move is a direct consequence of skein relations (53) and (55):

$$\Theta \left( \begin{array}{c} \includegraphics{sink} \end{array} \right) = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \frac{\sigma(n)}{\sigma(1)} = \Theta \left( \begin{array}{c} \includegraphics{sink} \end{array} \right).$$

(c) passing a positive crossing through $\beta$:

$$\Theta \left( \begin{array}{c} \includegraphics{positive_crossing} \end{array} \right) = \sum_{i,j} q^{\frac{1}{n}} \left( \sum_{j<i} (q - q^{-1}) \frac{i}{j} + q^{\delta_{i,j}} \sum_{i,j} \frac{i}{j} \right) = q^{\frac{1}{n}} \left( \sum_{j<i} (q - q^{-1}) \frac{i}{j} + q^{\delta_{i,j}} \sum_{i,j} \frac{i}{j} \right)$$

$$= \sum_{i,j} q^{\frac{1}{n}} \left( \sum_{j<i} (q - q^{-1}) + q^{\delta_{i,j}} \sum_{i,j} \right) = \Theta \left( \begin{array}{c} \includegraphics{positive_crossing} \end{array} \right),$$

by (50) and (58).
(d) passing a negative crossing through $\beta$ follows from (c) by composing the fragments of diagrams on the left and the right side of the above identity with a negative crossing on their both sides.

(e) height exchange of two consecutive points of $\alpha \cap \beta$ as in Figure 9. The invariance of $\Theta$ under this moves follows from the move in (c) or (d) if the arcs involved have coinciding orientations. If Figure 9 involves arcs in opposite directions, then the left side of the diagram on the right can be decomposed into elementary diagrams and all of them can be moved to the right side by (a)-(d).

Figure 9. Height exchange move

To finish the proof off, observe that $\Theta$ maps the defining relations (43)-(50) to 0 in $S_n(M', N')$, because they are all local and can be moved away from $D$. □

4.9. Reversing orientations of 3-manifolds and of webs. Let $\bar{\alpha}$ denote an $n$-web $\alpha$ with its orientation reversed (and unchanged framing). Since the defining relations (43)-(50) of $S_n(M, N)$ are invariant under the total orientation inversion, we have

**Corollary 4.8.**

$$\bar{\cdot} : S_n(M, N) \to S_n(M, N)$$

is a well defined $R$-module automorphism.

Let $(\bar{M}, \bar{N})$ denote $M$ and $N$ with reversed orientations. Let $\bar{R}$ be the ring $R$ with the distinguished element $v^{-1}$ instead of $v$. For an $n$-web $\alpha$ of $(M, N)$ let $\bar{\alpha}$ be the $n$-web in $(\bar{M}, \bar{N})$ obtained from $\alpha$ by negating its framing, $f \to -f$.

**Theorem 4.9.** (1) Any ring isomorphism $\kappa : R \to \bar{R}$ sending $v$ to $v^{-1}$ extends to an isomorphism of $R$-modules $\kappa_{(M, N)} : S_n(M, N, R) \cong S_n(\bar{M}, \bar{N}, \bar{R})$ sending every stated $n$-web $\alpha$ to $\bar{\alpha}$, where $S_n(\bar{M}, \bar{N}, \bar{R})$ is an $R$-module via $\kappa : R \to \bar{R}$.

(2) The composition $\kappa_{(\bar{M}, \bar{N})} \circ \kappa_{(M, N)}$ is the identity on $S_n(M, N, R)$.

The above isomorphism is called the **orientation reversion isomorphism**. (Note that for some rings $R$, an isomorphism $\kappa : R \to \bar{R}$ as above may not exist or be non-unique.)

**Proof of Theorem 4.9:** By abuse of notation, we define an $R$-linear map $\kappa_{(M, N)}$ first as

$$\kappa_{(M, N)} : RW_n(M, N) \to S_n(\bar{M}, \bar{N}), \quad \kappa_{(M, N)}(\alpha) = \bar{\alpha}.$$

One checks immediately that that map factors through relations (43)-(45).

By our graphical convention, a diagram of $\bar{\alpha}$ near a marking is given by switching all crossings in a diagram $\alpha$ and by reversing of the direction of the vertical line $\beta \perp$. For example, if $\alpha = \begin{array}{c} \alpha \end{array}$ then $\bar{\alpha} = \begin{array}{c} \alpha \end{array}$. It is clear that $\kappa_{(M, N)}$ maps (46) to the equality of Lemma 4.10 (below) and, therefore, it preserves that relation.
To see that $\kappa(M, N)$ factors through (47) substitute $\sigma'$ for $\sigma$ in (55), where $\sigma'(i) = \bar{\sigma}(i)$, for $i = 1, \ldots, n$, and rotate that equation $180^\circ$. Since $\ell(\sigma') = \binom{n}{2} - \ell(\sigma)$, we get

$$t^{n/2}a \sum_{\sigma \in S_n} (-q)\binom{n}{2}^{-\ell(\sigma)} \sigma(n) \cdots \sigma(2) \cdot \sigma(1).$$

Since

$$(-q)\binom{n}{2}t^{n/2} = q\binom{n}{2} + \frac{q^2 - 1}{q} a = a^{-1},$$

by Eq. (2), we get the desired relation

$$t^{n/2}a \sum_{\sigma \in S_n} (-q)^{-\ell(\sigma)} \sigma(n) \cdots \sigma(2) \cdot \sigma(1).$$

Furthermore, $\kappa(M, N)$ maps (48) and (49) to (56) and (57).

Let us show now that $\kappa(M, N)$ factors through (50). We need to verify that $i_j = q^{\frac{1}{n}} \left( \delta_{j<i} (q^{-1} - q) i_j + q^{-\delta_{j<i}} i_j \right)$.

By (43), the left side is

$$q^{\frac{2}{n}} i_j - q^{\frac{1}{n}} (q - q^{-1}) i_j$$

and since $1 - \delta_{j<i} = \delta_{j<i} + \delta_{i,j}$, the above equation reduces to

$$q^{\frac{2}{n}} i_j = q^{\frac{1}{n}} \left( \delta_{j<i} (q - q^{-1}) i_j + q^{\delta_{i,j}} i_j \right),$$

which is (58) rotated $180^\circ$ (and with $i$ and $j$ interchanged).

Hence, we have shown that the above map factors to $\kappa(M, N) : S_n(M, N) \rightarrow S_n(M, N)$.

It is an $R$-module homomorphism by definition.

Part (2) follows immediately from the fact that $S_n(M, N)$ is spanned by webs with no sinks nor sources and that $\kappa(M, N) \circ \kappa(M, N)$ is the identity on those.

Lemma 4.10. One has

$$(-q)\binom{n}{2} \sum_{\sigma \in S_n} (-q)^{-\frac{1}{n}\ell(\sigma)} \sigma(n) \cdots \sigma(2) \cdot \sigma(1),$$

where $\sigma_-$ is the minimal crossing negative braid representing $\sigma \in S_n$. 

Proof. Let $\tau(i) = n + 1 - i$, for $i = 1, \ldots, n$. Then $\tau_-$ is the negative half-twist $n$-braid. By applying it to the left side of (46) we obtain

$$(-q^{\frac{n+1}{n}})^{-\binom{n}{2}} \cdot \sum_{\sigma \in S_n} (-q^{\frac{1-n}{n}})^{\ell(\sigma)} \cdot \sigma_{\rightarrow} \cdot \tau_{\rightarrow} = (-q^{\binom{n}{2}}) \cdot \sum_{\sigma \in S_n} (-q^{\frac{1-n}{n}})^{\ell(\sigma)} \cdot \sigma_{\rightarrow} \cdot \tau_{\rightarrow},$$

by (52). Note that $\tau_{\rightarrow} \sigma_+ = (\tau \sigma)_+$ for every $\sigma \in S_n$ and that by (5),

$$\ell(\sigma) + \ell(\tau \sigma) = \ell(\tau) = \binom{n}{2}.$$

Therefore, by denoting $\tau \sigma$ by $\sigma'$, the right side of (61) reduces to

$$(-q^{\binom{n}{2}}) \cdot \sum_{\sigma' \in S_n} (-q^{\frac{1-n}{n}})^{\binom{n}{2}} \cdot (-q^{\frac{1-n}{n}})^{-\ell(\sigma')} \cdot \sigma'_{\rightarrow} \cdot \tau_{\rightarrow} = (-1)^{\binom{n}{2}} q^{D} \cdot \sum_{\sigma' \in S_n} (-q^{\frac{1-n}{n}})^{\ell(\sigma')} \cdot \sigma'_{\rightarrow} \cdot \tau_{\rightarrow},$$

and, hence, (61) becomes

$$= (-1)^{\binom{n}{2}} q^{D} \cdot \sum_{\sigma' \in S_n} (-q^{\frac{1-n}{n}})^{\ell(\sigma')} \cdot \sigma'_{\rightarrow} \cdot \tau_{\rightarrow},$$

where

$$D = -\frac{n+1}{n} \cdot \binom{n}{2} + \binom{n}{2} + \frac{1-n}{n} \cdot \binom{n}{2} = -\binom{n}{2}.$$

4.10. **Marking automorphisms.** Consider a function $\eta : \{1, \ldots, n\} \to R^*$ such that

$$\prod_{i=1}^{n} \eta(i) = 1 \text{ and } \eta(i) \eta(\bar{i}) = 1 \text{ for every } i,$$

where the bar denotes the conjugation, $\bar{i} = n + 1 - i$, as before. It is easy to see that for every such $\eta$ and every marking $\beta$ in $\mathcal{N}$ there is an $R$-module automorphism $\phi_{\eta, \beta}$ of $\mathcal{S}_n(M, \mathcal{N})$ sending stated $n$-webs $\alpha$ to

$$\phi_{\eta, \beta}(\alpha) = \prod_{x \in \alpha \cap \beta} \eta(s(x)) \cdot \alpha,$$

where $s(x)$ is the state of the endpoint $x$ of $\alpha$. We call $\phi_{\eta, \beta}$ an **marking automorphism** of $\mathcal{S}_n(M, \mathcal{N})$.

When $\eta_i = g_i = (-1)^{n-1} q^{2i-n-1}$, as in Eq. (42), we denote $\phi_{\eta, \beta}$ by $g_{\beta}$.

4.11. **Half-twist automorphisms.**

**Proposition 4.11.** For any marking $\beta$ in $\mathcal{N}$ there exists a unique $R$-linear isomorphism

$$\text{htw}_{\beta} : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(M, \mathcal{N}).$$
sending any stated $n$-web $\alpha$ in $(M, N)$ with $k$ endpoints on $\beta$ to

\begin{equation}
\text{htw}_\beta \left( \begin{array}{c}
\vdots \\
i_k \\
i_2 \\
i_1 \\
\end{array} \right) = \left( \prod_{j=1}^{k} c_{i_j} \right) \cdot \left( \begin{array}{c}
\vdots \\
i_k \\
i_2 \\
i_1 \\
\end{array} \right) = \left( \prod_{j=1}^{k} c_{i_j} \right) \cdot \left( \begin{array}{c}
\vdots \\
i_k \\
i_2 \\
i_1 \\
\end{array} \right),
\end{equation}

where $H$ is the positive half-twist. (The orientations of the horizontal edges are arbitrary.)

We call $\text{htw}_\beta$ the **half-twist automorphism**.

**Remark 4.12.** We will show in Subsection 7.1 that every marking in $N$ defines a left action of a completion $\widetilde{U}^L$ of the Lusztig integral version $U^L$ of $U_q(sl_n)$. That completion contains the half-twist element $X$ and the charmed element $g$ of Subsection 3.9. The automorphisms $\text{htw}_\beta$ and $g_\beta$ coincide with the actions of $X$ and $g$, respectively.

The map $\text{htw}_\beta$ generalizes the inversion along an edge, for the stated Kauffman bracket skein algebras of thickened surfaces in [CL]. However it is the inverse of that map.

The inverse of $\text{htw}_\beta$ is given by,

\begin{equation}
\text{htw}^{-1}_\beta \left( \begin{array}{c}
\vdots \\
i_k \\
i_2 \\
i_1 \\
\end{array} \right) = \left( \prod_{j=1}^{k} c_{i_j} \right) \cdot \left( \begin{array}{c}
\vdots \\
i_k \\
i_2 \\
i_1 \\
\end{array} \right),
\end{equation}

where $\bar{H}$ denotes the negative half-twist, as before.

**Proof of Proposition 4.11:** By abuse of notation, let us first consider a map $\text{htw}_\beta : W_n(M, N) \to S_n(M, N)$ sending any stated $n$-web $\alpha$ in $(M, N)$ with $k$ endpoints on $\beta$ to

\begin{equation}
\text{htw}_\beta \left( \begin{array}{c}
\vdots \\
i_k \\
i_2 \\
i_1 \\
\end{array} \right) = \left( \prod_{j=1}^{k} c_{i_j} \right) \cdot \left( \begin{array}{c}
\vdots \\
i_k \\
i_2 \\
i_1 \\
\end{array} \right).
\end{equation}

Obviously, $\text{htw}_\beta$ preserves the internal skein relations, (43)-(46). It maps (47) to

\begin{equation}
\text{htw}_\beta \left( \begin{array}{c}
\vdots \\
i_k \\
i_2 \\
i_1 \\
\end{array} \right) = a \cdot \left( \prod_{i=1}^{n} c_{i} \right) \cdot \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \cdot \left( \begin{array}{c}
\vdots \\
\sigma(n) \\
\sigma(2) \\
\sigma(1) \\
\end{array} \right).
\end{equation}

The right side equals

\begin{equation}
a \left( \prod_{i=1}^{n} c_{i} \right) \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} = a \left( \prod_{i=1}^{n} c_{i} \right) \cdot (-q)^{\frac{n+1}{n}} \cdot \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \cdot \left( \begin{array}{c}
\vdots \\
\sigma(1) \\
\sigma(2) \\
\sigma(n) \\
\end{array} \right),
\end{equation}
by (52). By (4), \((\prod_{i=1}^{n} c_i \cdot (-q^{-\frac{n+1}{2}})^{\frac{n(n-1)}{2}} = 1\) and, hence, the expression above coincides with the right side of (47) by the substitution \(\sigma \rightarrow \sigma', \) where \(\sigma'(i) \rightarrow \sigma(i), \) for \(i = 1, \ldots, n,\) which does not affect the permutation length. Consequently, \(htw_{\beta}\) preserves (47).

The preservation of the remaining relations, (48)-(50) by \(htw_{\beta}\) is an immediate consequence of the left boundary relations (56) – (58).

This shows that our map descends indeed to an \(R\)-module homomorphism \(htw_{\beta}: \mathcal{S}_n(M, N) \rightarrow \mathcal{S}_n(M, N).\)

Similarly, it is straightforward to show that there is a well defined map \(htw'_{\beta}: \mathcal{S}_n(M, N) \rightarrow \mathcal{S}_n(M, N)\) sending any \(n\)-web \(\alpha\) in \((M, N)\) with \(k\) endpoints on \(\beta\) to

\[
htw'_{\beta} \begin{pmatrix}
\vdots \\
 i_k \\
 i_2 \\
 i_1
\end{pmatrix} = \left( \prod_{j=1}^{k} c_{ij} \right)^{-1} \cdot \begin{pmatrix}
\vdots \\
 i_k \\
 i_2 \\
 i_1
\end{pmatrix}
\]

Since \(htw'_{\beta}\) is an inverse of \(htw_{\beta},\) both are isomorphisms. \(\square\)

### 4.12. Essential uniqueness of the skein relations of \(\mathcal{S}_n(M, N)\)

In the context of our theory it is natural to ask how arbitrary are the constants \(a, c_i\) in Subsection 2.1. For a tuple \(u = (u, u_1, \ldots, u_n)\) of \(n+1\) invertible elements of \(R\) let \(\mathcal{S}_n(M, N; u)\) be defined the same as \(\mathcal{S}_n(M, N),\) with \(c_i\) and \(a\) replaced respectively by \(c'_i = c_i(u_iu_i^{-1})\) and \(a' = a(\prod_{i=1}^{n} u_i)/u,\) and with the right side of (46) multiplied by \(u^2.\) We denote the set of \(n\)-valent vertices of \(\alpha\) by \(V_n(\alpha).\) Then it is easy to see that the map

\[
\alpha \rightarrow \alpha u^{|V_n(\alpha)|} \prod_{x \in \partial\alpha} u_{\varepsilon(x)},
\]

defined on stated \(n\)-webs, extends to an \(R\)-linear isomorphism from \(\mathcal{S}_n(M, N)\) to \(\mathcal{S}_n(M, N; u).\)

One can show that the new stated skein module \(\mathcal{S}_n(M, N; u)\) satisfies the splitting homomorphism if and only if the following holds:

\[
u_i = \pm 1, \quad \prod_{i=1}^{n} u_i = 1 \quad u_iu_i = 1, \quad \text{for every } i.
\]

Furthermore, all properties of \(\mathcal{S}_n(M, N)\) formulated so far have their version for \(\mathcal{S}_n(M, N; u).\)

### 5. Stated SL\((n)\)-skein algebras of surfaces

The theory of stated SL\((n)\)-skein modules is particularly rich for thickened surfaces \(M = \Sigma \times (-1, 1).\) Note that any finite set \(B \subset \partial\Sigma\) defines markings \(N = B \times (-1, 1)\) for which \(\mathcal{S}_n(M, N)\) is an \(R\)-algebra with the product of webs \(\alpha_1 \cdot \alpha_2\) given by stacking \(\alpha_1\) on top of \(\alpha_2\). It is convenient however to represent unmarked boundary components of \(\Sigma\) by punctures and to separate points of \(B\) by ideal boundary points. That leads to the notion of a punctured bordered surface, considered for example in [Le1, CL] already. In particular, a punctured bordered surface encapsulates information about the points \(B\) in it.
5.1. Punctured bordered surface. A punctured bordered surface (a pb surface for short) \( \Sigma \) is an oriented surface with possibly empty boundary \( \partial \Sigma \) such that each connected component of \( \partial \Sigma \) is an open interval. These components are called boundary edges.

For simplicity we will assume that \( \Sigma \) is of finite type in the sense that \( \Sigma = \overline{\Sigma} \setminus \mathcal{P} \), where \( \overline{\Sigma} \) is a compact oriented surface and \( \mathcal{P} \subset \overline{\Sigma} \) is a finite set, called the ideal points of \( \Sigma \). Note that each connected component of \( \partial \overline{\Sigma} \) meets \( \mathcal{P} \). (However, some of the points of \( \mathcal{P} \) may be in the interior of \( \Sigma \).)

An ideal arc in \( \Sigma \) is the image of a proper embedding \( c : (0, 1) \hookrightarrow \Sigma \). This means \( c \) can be extended to an immersion \( \bar{c} : [0, 1] \to \Sigma \) such that \( \bar{c}(0), \bar{c}(1) \in \mathcal{P} \). An ideal arc is trivial if it bounds a disk.

In each boundary edge \( e \) choose a point \( b_e \). Let \( S_n(\Sigma) = S_n(M, N) \), where \( M = \Sigma \times (-1, 1) \) and \( N \) is the union of all \( b_e \times (-1, 1) \), each having the natural orientation of the interval \((-1, 1)\). Since up to a canonical isomorphism, \( S_n(\Sigma) \) does not depend on the specific choice of the points \( b_e \), we do not specify them in our notation. An \( n \)-web in \((M, N)\) is simply called an \( n \)-web over \( \Sigma \).

For stated \( n \)-webs \( \alpha \) and \( \beta \) over \( \Sigma \) let their product \( \alpha \beta \in S_n(\Sigma) \) be the result of stacking \( \alpha \) above \( \beta \). This product turns \( S_n(\Sigma) \) into an \( R \)-algebra.

According to the graphical convention of Subsec. 4.1, an \( n \)-web over \( \Sigma \) is presented by its diagram on \( \Sigma \), which is the projection of \( \alpha \) onto \( \Sigma \) with the over/undercrossing information at every double point. Before projecting, we use height-preserving deformation near the markings \( b_e \times (-1, 1) \) to make the projections of endpoints of \( \alpha \) distinct. As before, the height order at endpoints of the diagram on each boundary edge is part of the diagram.

The orientation of \( \Sigma \) induces orientation on its boundary. When part of \( \Sigma \) is drawn on a page of paper, which is identified with the standard \( XY \)-plane, the orientation of \( \partial \Sigma \) is the counterclockwise direction. A diagram where the height order on a boundary edge \( e \) is given by the orientation of \( e \) induced from that of \( \Sigma \) (respectively: the opposite orientation) is called positively (respectively: negatively) ordered on \( e \).

Given two edges \( e_1, e_2 \) of a pb surface \( \Sigma \), not necessarily connected, the gluing \( \Sigma/(e_1 = e_2) \) is the result of identifying \( e_1 \) with \( e_2 \) via a diffeomorphism \( e_1 \to e_2 \) such that the resulting surface has an orientation induced from that of \( \Sigma \). Such a surface is defined uniquely up to strict isomorphisms.

A pb surface \( \Sigma \) is essentially bordered if every connected component of it has non-empty boundary.

Lemma 5.1. If a pb surface \( \Sigma \) is essentially bordered then \( S_n(\Sigma) \) is spanned by stated \( n \)-web diagrams without any of: sinks, sources, crossings, trivial loops, and trivial arcs.

Proof. Crossings can be eliminated by bringing them to near a boundary edge as in Fig. 10(left), then expressing them as linear combinations of webs of the form Fig. 10(right) by Relation (49), and finally eliminating them by Relation (50). Sinks and sources, trivial loops, and trivial arcs can be eliminated by Relations (47), (45), (48), respectively. \( \Box \)

5.2. Splitting homomorphism for surfaces. Let \( c \) be an ideal arc in the interior of a pb surface \( \Sigma \). The splitting \( \text{cut}_c(\Sigma) \) is a pb surface having two boundary edges \( c_1, c_2 \) such that \( \Sigma = \text{cut}_c(\Sigma)/(c_1 = c_2) \). Let \( \text{pr} : \text{cut}_c(\Sigma) \to \Sigma \) be the natural projection map. An \( n \)-web diagram \( D \) is \( c \)-transverse if \( n \)-valent vertices of \( D \) are not in \( c \) and \( D \) is transerve to \( c \).
Assume that $D$ is a stated $c$-transverse $n$-web diagram. Let $h$ be a linear order on the set $D \cap c$. For a map $s : D \cap c \rightarrow \{1, \ldots, n\}$ let $D(h, s)$ be the stated $n$-web diagram over $\text{cut}_c(\Sigma)$ which is $\text{pr}^{-1}(D)$ with the height order on $c_1 \cup c_2$ induced (via $\text{pr}$) from $h$, and the states on $c_1 \cup c_2$ induced (via $\text{pr}$) from $s$. The Splitting Theorem (Thm. 4.5) for $\Sigma$ becomes

**Theorem 5.2.** Let $c$ be an interior ideal arc of a pb surface $\Sigma$. There is a unique $R$-linear map $\Theta_c : S_n(\Sigma) \rightarrow S_n(\text{cut}_c(\Sigma))$ such that if $D$ is a diagram of a stated $n$-web $\alpha$ over $\Sigma$ which is $c$-transverse and $h$ is any linear order on $D \cap c$, then

$$\Theta_c(\alpha) = \sum_{s : D \cap c \rightarrow \{1, \ldots, n\}} D(h, s).$$

The map $\Theta_c$ is an $R$-algebra homomorphism.

**Proof.** The set $c \times (-1, 1)$ is not a closed disk but we can still use Theorem 4.5, see Remark 4.7. More precisely, let us enlarge the ideal points of $\Sigma$ to open disks in $\overline{\Sigma}$, and embed $\Sigma$ into $\mathbb{R}^3$. Let $\bar{M}$ be the topological closure of $\Sigma \times (-1, 1)$ in $\mathbb{R}^3$.

Then $(M, N)$ is pseudo-isomorphic to $(\bar{M}, \bar{N})$. Applying Theorem 4.5 to split $(\bar{M}, \bar{N})$ along the topological closure of $c \times (-1, 1)$ in $\bar{M}$, we get the $R$-linear map $\Theta_c : S_n(\Sigma) \rightarrow S_n(\text{cut}_c(\Sigma))$ defined in the statement.

From the definition it is clear that $\Theta_c$ is an algebra homomorphism.

5.3. **Reflection anti-involution.** For any pb surface $\Sigma$ we have an involution $\tau : \Sigma \times (-1, 1) \rightarrow \Sigma \times (-1, 1)$, $(x, u) \rightarrow (x, -u)$ which maps webs $\alpha$ to $\tau(\alpha)$ (with their framing transformed by the tangent map $\tau_* : T(\Sigma \times (-1, 1)) \rightarrow T(\Sigma \times (-1, 1))$). Given a stated web $\alpha$ in $\Sigma \times (-1, 1)$ let $\bar{\alpha}$ be $\tau(\alpha)$ with its framing reversed, $f \rightarrow -f$.

**Proposition 5.3.** For any commutative ring $P$ and $R = P[v^{\pm 1}]$ and for any pb surface $\Sigma$, there is a unique $P$-algebra anti-involution $\bar{\cdot} : S_n(\Sigma) \rightarrow S_n(\Sigma)$ such that $\bar{v} = v^{-1}$ and $\bar{\alpha}$ for stated webs $\alpha$ is defined as above.

We call $\bar{\cdot}$ the **mirror reflection** map.

**Proof.** Let $(M, N)$ be defined as in Subsection 5.1 and $(\bar{M}, \bar{N})$ be defined as in Subsection 4.9. Then the mirror reflection map is the composition of the orientation reversion $\kappa_{(M,N)}$ with $\tau$ and hence, a $P$-linear isomorphism sending $v$ to $\bar{v} = v^{-1}$ by Theorem 4.9. It is easy to see that

$$\bar{\alpha} = \alpha \quad \text{and} \quad \overline{\alpha \cdot \alpha'} = \bar{\alpha} \cdot \bar{\alpha'}$$

for stated webs $\alpha, \alpha'$.

If $\alpha$ is a stated $n$-web diagram over $\Sigma$ then $\bar{\alpha}$ is obtained from $\alpha$ by switching all the crossings and reversing the height order on each boundary edge.
5.4. Embedding of punctured bordered surfaces. Consider a proper embedding of a pb surface \( \Sigma_1 \) into \( \Sigma_2 \). Note that it can map several boundary edges of \( \Sigma_1 \) into one boundary edge of \( \Sigma_2 \). For a boundary edge \( b \) of \( \Sigma_2 \) a linear order on the set of boundary edges of \( \Sigma_1 \) mapped into \( b \) is called a \( b \)-order. Fixing it for each \( b \) defines a height ordered embedding \( f : \Sigma_1 \hookrightarrow \Sigma_2 \), inducing an \( R \)-module homomorphism \( f_* : S_n(\Sigma_1) \to S_n(\Sigma_2) \), where \( f_*(\alpha) \) is \( \alpha \) with its height order on each \( b \) determined by the \( b \)-order in addition to the height order of \( \partial \alpha \). If the \( b \)-order is given by the positive (respectively negative) orientation of \( b \), we say \( f_* \) is positively (respectively negatively) induced from \( f \), cf. Fig. 11.

**Figure 11.** Left: A surface \( \Sigma_1 \) with a skein \( x \). Middle: The image of \( x \) under the negative height order embedding of \( \Sigma_1 \) into \( \Sigma_2 \). Right: The image of \( x \) under the positive height order embedding of \( \Sigma_2 \).

Note that \( f_* \) is an \( R \)-algebra homomorphism if and only if each boundary edge of \( \Sigma_2 \) contains the image of at most one boundary edge of \( \Sigma_1 \).

6. Skein algebras of bigon and quantum groups

In this section we prove that the stated skein algebra \( S_n(\mathcal{B}) \) of the bigon, \( \mathcal{B} \), has a natural structure of a co-braided Hopf algebra which is naturally isomorphic to quantized coordinate algebra \( O_q(SL(n)) \). We also show that the stated skein algebra of the monogon, \( \mathcal{M} \), is the ground ring \( R \). Finally, prove Theorem 3.11 which identifies the kernel of the map \( \Gamma \).

6.1. Monogon. Let \( D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \) be the standard disk with the counterclockwise orientation. The monogon \( \mathcal{M} \) is the pb surface obtained by removing the bottom point \((0,-1)\) from \( D \).

**Theorem 6.1** (Proof in Subsec. 6.6.). The stated skein algebra \( S_n(\mathcal{M}) \) of the monogon \( \mathcal{M} \) is isomorphic to the ground ring \( R \) via the map \( \mu : R \to S_n(\mathcal{M}) \) given by \( \mu(r) = r \cdot \emptyset \).

6.2. Bigon. The bigon \( \mathcal{B} \) is the punctured bordered surface obtained from the standard disk \( D \) by removing the top and the bottom points, \((0,1), (0,-1)\). The two edges of \( \mathcal{B} \) are denoted by \( e_l \) and \( e_r \) as in Figure 12. Up to isotopy there are two orientation preserving auto-diffeomorphisms of \( \mathcal{B} \), the identity and the rotation \( \text{rot} \) by 180 degrees about the center of \( \mathcal{B} \). The rotation \( \text{rot} \) induces an algebra involution \( \text{rot}_* : S_n(\mathcal{B}) \to S_n(\mathcal{B}) \).

A directed bigon is an oriented surface diffeomorphic to \( \mathcal{B} \), with one ideal vertex designated as the bottom vertex. Equivalently the direction of a bigon can be specified by choosing the left (or right) edge. We often depict \( \mathcal{B} \) as the square \([-1,1] \times (-1,1)\), as in Figure 12(b). Let \( a_{ij} \) be the stated \( n \)-web over \( \mathcal{B} \) given in Figure 12(c), and let \( \bar{a}_{ij} \) be \( a_{ij} \) with the reverse orientation.

We will now define a Hopf algebra structure on \( S_n(\mathcal{B}) \) geometrically. By splitting \( \mathcal{B} \) along the an interior ideal arc connecting its two ideal vertices we get two directed bigons \( \mathcal{B}_l \) and \( \mathcal{B}_r \).
Figure 12. (a) & (b) Bigon $\mathcal{B}$. (c) Stated arc $a^i_j$, (d) splitting of $\mathcal{B}$ for each the bottom vertex comes from the one of $\mathcal{B}$. The splitting homomorphism becomes an algebra $R$-homomorphism

$$\Delta : S_n(\mathcal{B}) \to S_n(\mathcal{B}) \otimes S_n(\mathcal{B}).$$

The commutativity of the splitting homomorphisms at disjoint ideal arcs shows that $\Delta$ is a co-product. For example, from the definition one has

$$\Delta(a^i_j) = \sum_k a^i_k \otimes a^j_k, \quad \Delta(\bar{a}^i_j) = \sum_k \bar{a}^i_k \otimes \bar{a}^j_k.$$

The natural embedding $\iota : \mathcal{B} \to \mathcal{M}$ (filling in the top ideal point) induces an $R$-linear map $\iota^* : S_n(\mathcal{B}) \to S_n(\mathcal{M})$, where the left edge $e_l$ is higher than the right edge $e_r$. Let $\epsilon : S_n(\mathcal{B}) \to R$ be the composition

$$\epsilon : S_n(\mathcal{B}) \xrightarrow{htw_{er}} S_n(\mathcal{B}) \xrightarrow{\iota^*} S_n(\mathcal{M}) \cong R,$$

where $htw_{er}$ is the half-twist automorphism of Subsection 4.11. Explicitly, for a stated diagram $\alpha$,

$$\epsilon \left( \begin{array}{c} i_k \\ i_2 \\ i_1 \end{array} \right) = \left( \prod_j (c_{ij}) \right) \mu^{-1} \left( \begin{array}{c} \bar{i}_k \\ \bar{i}_2 \\ \bar{i}_1 \end{array} \right),$$

where $\mu : R \to S_n(\mathcal{M})$ is the isomorphism of Theorem 6.1. For example,

$$\epsilon(a^i_j) = \epsilon(\bar{a}^i_j) = c_{ij} \bar{a}^i_j = \delta_{i,j}.$$

Let the $R$-module automorphism $S : S_n(\mathcal{B}) \to S_n(\mathcal{B})$ be the composition

$$S = \text{rot}_{er} \circ htw_{el}^{-1} \circ g_{er} \circ htw_{er},$$

where $g_{er}$ is defined in Section 4.10. Explicitly, for a stated diagram $\alpha$,

$$S \left( \begin{array}{c} i_k \\ i_1 \end{array} \right) = \left( \prod_m c_{im} \right)^{-1} \left( \prod_m c_{jm} \right) \text{rot}(\alpha),$$

where $\text{rot}(\alpha)$ is the result of rotating the planar diagram $\alpha$ about the center of the square by 180 degrees. (Here, we use the fact that $c_{ij} = (-1)^{n-1} q^{2j-n-1} c_{j,i}$.)

For example, we have

$$S(a^i_j) = (-q)^{i-j} \bar{a}^i_j.$$
Remark 6.2. Note that stated \( n \)-tangle diagrams can be identified with the diagrams of stated \( n \)-webs in \( \mathcal{B} \) with the downwards ascending height order on \( \partial_l \mathcal{B} \) and \( \partial_r \mathcal{B} \), i.e. the height order is positive on the left edge but negative on the right edge.

That leads to a natural identification of stated \( n \)-tangles with stated \( n \)-webs in \( \mathcal{B} \), i.e. the height order is positive on the left edge but negative on the right edge.

That leads to a natural identification of stated \( n \)-tangles with stated \( n \)-webs in the thickened bigon, for which the basic internal, right, and left annihilators of Subsec. 3.6 correspond to defining skein relations (43)–(50) of \( S_n(\mathcal{B}) \).

The positive order on the left edge explains why there is a twist in the definition of the operation \( \text{hd} \) which turns right annihilators to basic annihilators of Subsection 3.7.

By this identification \( S \) corresponds to the dual operation \( \alpha \to \alpha^* \) of Subsection 3.5.

Recall that \( \mathcal{B} \) is the standard bigon.

Theorem 6.3 (Proof in Subsec. 6.8). (a) The algebra \( S_n(\mathcal{B}) \) has the structure of a Hopf algebra over \( R \) with the coproduct \( \Delta \), the counit \( \epsilon \), and the antipode \( S \).

(b) The map \( \Psi(u^i_j) = a^i_j \) extends to a unique Hopf algebra isomorphism

\[
\Psi : \mathcal{O}_q(sl_n ; R) \rightarrow S_n(\mathcal{B}).
\]

Here \( \mathcal{O}_q(sl_n ; R) := \mathcal{O}_q(SL(n)) \otimes \mathbb{Z}[v^\pm 1] \) is the algebra \( \mathcal{O}_q(SL(n)) \) of Subsec. 2.5 with the ground ring \( R \).

6.3. Cobraided structure. The Hopf algebra \( \mathcal{O}_q(sl_n ; R) \) is dual quasitriangular (see [Maj, Section 2.2], [KS, Section 10]), also known as cobraided (see e.g. [Kass, Section VIII.5]). This means it has a co-\( R \)-matrix, which is a bilinear form

\[
\rho : \mathcal{O}_q(sl_n ; R) \otimes \mathcal{O}_q(sl_n ; R) \rightarrow R
\]

satisfying certain properties, with the help of which one can make the category of \( \mathcal{O}_q(sl_n ; R) \)-modules a braided category. The following generalizes [CL, Theorem 3.5] from \( n = 2 \) to all \( n \):

Theorem 6.4 (Proof in Subsec. 6.9). Under the identification of \( S_n(\mathcal{B}) \) and \( \mathcal{O}_q(sl_n ; R) \) via the isomorphism \( \Psi \), the co-\( R \)-matrix \( \rho \) has the following geometric description

\[
(68) \quad \rho \left( \begin{array}{c}
\begin{array}{c}
\bigotimes \end{array}
\end{array} \right) \otimes \begin{array}{c}
\begin{array}{c}
\bigotimes \end{array}
\end{array} = \epsilon \left( \begin{array}{c}
\begin{array}{c}
\bigotimes \end{array}
\end{array} \right),
\]

for any \( x, y \in \mathcal{O}_q(sl_n ; R) \).

6.4. Ground ring. The remainder of this section is devoted to proving Theorems 6.1, 6.3, 6.4 and 3.11. Since it is enough to do it for \( R = \mathbb{Z}[v^\pm 1] \), we will assume this ground ring for the rest of this section.

6.5. Algebra homomorphism \( S_n(\mathcal{B}) \to \mathcal{O}_q(SL(n)) \).

Lemma 6.5. The webs \( a^i_j \) for \( i, j \in \{1, \ldots, n\} \), generate \( S_n(\mathcal{B}) \) as an \( R \)-algebra.

Proof. By Lemma 5.1, \( S_n(\mathcal{B}) \) is generated by \( a^i_j \) and \( \bar{a}^i_j \) for \( i, j = 1, \ldots, n \).
Fix \( i, j \) and choose a permutation \( \tau \in S_n \) with \( \tau(1) = i \). By (57) and (54),
\[
\begin{array}{c}
\tau(n) \\
\tau(2) \\
\vdots \\
\tau(j) \\
\end{array}
= c_i
\begin{array}{c}
\tau(n) \\
\tau(n-1) \\
\vdots \\
\tau(1) \\
\end{array}
= a c_i (-q)^{\ell(\tau)} a^i_j.
\]

On the other hand, Eq. (47) expresses the left side in terms of \( a^k_i \)'s:
\[
a \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \begin{array}{c}
\tau(n) \\
\vdots \\
\tau(2) \\
\sigma(1) \\
\end{array} = a \cdot c_j \sum_{\sigma \in S_n: \sigma(1) = j} (-q)^{\ell(\sigma)} a^\tau(2) \cdots a^\tau(n).
\]

Since \( c_j c_i^{-1} = (-q)^{j-i} \), by comparing the two equalities, we have
\[
\tilde{a}^i_j = (-q)^{j-i} \sum_{\sigma \in S_n: \sigma(1) = j} (-q)^{\ell(\sigma) - \ell(\tau)} a^\tau(2) \cdots a^\tau(n)
\]
which shows \( \tilde{a}^i_j \) is in the subalgebra generated by \( a^i_j \).

As the first step towards proving the theorems of this section we will construct an \( R \)-algebra homomorphism \( \Phi: S_n(\mathcal{B}) \to \mathcal{O}_q(SL(n)) \subset U_q(sl_n)^* \).

Let \( \alpha \to T(\alpha) \) be the bijection of Remark 6.2 between the set of isotopy classes of stated n-webs over \( \mathcal{B} \) and stated n-tangles. It extends to an \( R \)-algebra isomorphism \( T: AW_n(\mathcal{B}) \to RT \).

By Remark 6.2, the composition \( \Gamma \circ T : R\mathcal{W}_n(\mathcal{B}) \to U_q(sl_n)^* \) descends to an \( R \)-linear homomorphism \( \Phi: S_n(\mathcal{B}) \to U_q(sl_n)^* \), which by Proposition 3.4 is an algebra homomorphism. From Eq. (33) and Proposition 2.1 we have
\[
\Phi(\emptyset) = \epsilon, \quad \text{counit of } U_q(sl_n),
\]
\[
\Phi(a^i_j) = u^i_j, \quad \text{generators of } \mathcal{O}_q(SL(n)),
\]
for \( i, j = 1, \ldots, n \). As \( u^i_j \) generate \( \mathcal{O}_q(SL(n)) \), Lemma 6.5 and Eq. (71) show that
\[
\Phi(S_n(\mathcal{B})) = \mathcal{O}_q(SL(n)).
\]

6.6. **Proof of Theorem 6.1**. By Lemma 3.1, \( S_n(\mathcal{M}) \) is spanned by the empty n-web. Therefore, the map \( \mu : R \to S_n(\mathcal{M}) \) given by \( \mu(r) = r \cdot \emptyset \) is surjective. By removing the left edge of \( \mathcal{B} \) we get a monogon. This gives an embedding \( \iota : \mathcal{M} \mapsto \mathcal{B} \), which induces an \( R \)-algebra homomorphism \( \iota_* : S_n(\mathcal{M}) \to S_n(\mathcal{B}) \). By Eq. (70) the composition
\[
R \xrightarrow{\mu} S_n(\mathcal{M}) \xrightarrow{\iota_*} S_n(\mathcal{B}) \xrightarrow{\Phi} U_q(sl_n)^*
\]
is an \( R \)-linear map sending 1 to \( \epsilon \). As the free \( R \)-module generated by \( \epsilon \) is a submodule of \( U_q(sl_n)^* \), the composition is injective. Thus \( \mu \) is injective, and hence, bijective. \( \square \).
Theorem 6.1 implies in particular:

\[
\begin{align*}
\sum_{\alpha, j} \epsilon(\alpha) = 0
\end{align*}
\]

\[\tag{72}\]

6.7. **Proof that \( S_n(\mathcal{B}) \) is a Hopf algebra.** We already noted that \( \Delta \) is a coproduct. Since \( \mu \) is an isomorphism, \( \epsilon \) is well defined by Eq. (64). A version of the argument of [CL] shows that \( \epsilon \) is an \( R \)-algebra homomorphism as well: For any webs \( \alpha, \alpha_2 \),

\[
\epsilon(\alpha_1 \alpha_2) = \epsilon \left( \partial \alpha_2 \bigotimes \partial \alpha_1, \partial \alpha_1 \bigotimes \partial \alpha_2 \right) = \prod_{x \in (\alpha_1 \cup \alpha_2) \cap \partial \mathcal{B}} \epsilon_{s(x)} \cdot \mu^{-1} \left( \partial \alpha_1 \bigotimes \partial \alpha_1 \right) \cdot \mu^{-1} \left( \partial \alpha_2 \bigotimes \partial \alpha_2 \right) = \epsilon(\alpha_1) \epsilon(\alpha_2),
\]

where \( \partial \alpha, \partial \alpha \) denote the sequences of left and right side states of \( \alpha \) and the third identity follows from (72).

We also have

\[ (\epsilon \otimes \text{id}) \circ \Delta(x) = x = (\text{id} \otimes \epsilon) \circ \Delta(x) \]

for all \( x \in S_n(\mathcal{B}) \). Indeed, since \( \Delta \) and \( \epsilon \) are algebra homomorphisms, it is enough to verify it for the generators \( x = a^i_j \) and that follows from the explicit values of \( \Delta(a^i_j) \) and of \( \epsilon(a^i_j) \) given by Eqs. (63) and (65).

Consequently, \( (S_n(\mathcal{B}), \Delta, \epsilon) \) is an \( R \)-bialgebra. By (66), \( S \) is \( R \)-algebra anti-isomorphism. Therefore, to prove that \( S \) is an antipode for \( (S_n(\mathcal{B}), \Delta, \epsilon) \) it remains to to be shown that

\[
\sum \tilde{S}(x') x'' = \epsilon(x) = \sum x' \Delta x, \quad \text{where } \Delta x = \sum x' \otimes x''.
\]

As before, it suffices to be verified for the generators \( a^i_j \), \( i, j = 1, \ldots, n \), only. Since \( \Delta(a^i_j) = \sum_k a^i_k \otimes a^j_k \), the left side of (73) reduces to:

\[
\sum_k S(a^i_k) a^j_k = \sum_k \frac{c_k}{c_i} \left( \frac{c_i}{k} \right)^{j} = \sum_k \left( \frac{c_i}{k} \right)^{j} = \delta_{i,j}
\]

by Eqs. (67), (57), and (48). The proof of the right identity of (73) is analogous.

This completes the proof that \( (S_n(\mathcal{B}), \Delta, \epsilon, S) \) is a Hopf algebra.

6.8. **Proof of Theorem 6.3.**

**Proposition 6.6.** Suppose \( \alpha = (i, \alpha, j) \) is a stated \( n \)-web on \( \mathcal{B} \). Then \( \epsilon(\alpha) \) is equal to the matrix element of the corresponding modified Reshetikhin-Turaev operator:

\[
\epsilon(\alpha) = \langle i | RT(T(\alpha)) | j \rangle.
\]

**Proof.** The map \( S_n(\mathcal{B}) \to \mathbb{Q}(v) \) given by \( \alpha \to \langle i | RT(T(\alpha)) | j \rangle \) is clearly an \( R \)-algebra homomorphism whose values on \( a^i_j \) coincide with those of \( \epsilon \), by Eq. (70). \(\square\)
In particular, we have

\[(75)\quad \epsilon \left( \begin{array}{c} l \\ k \\ j \\ i \end{array} \right) = \hat{R}^{kl}_{ij} = R^{lk}_{ij}.\]

Let us show that \(a = (a_{ij}^{kl})\) is a quantum matrix. By isotopy,

\[
\begin{array}{c}
l \\ k \\ j \\ i \\
\hline \\
r \\ s \\ j \\ i
\end{array}
= \begin{array}{c}
l \\ k \\ j \\ i \\
\hline \\
r \\ s \\ j \\ i
\end{array}
\]

Split along the dashed lines (coproduct), then apply the counit

\[
\epsilon \left( \begin{array}{c} l \\ k \\ j \\ i \\
\hline \\
r \\ s \\ j \\ i
\end{array} \right) \cdot \epsilon \left( \begin{array}{c} l \\ k \\ j \\ i \\
\hline \\
r \\ s \\ j \\ i
\end{array} \right) = \epsilon \left( \begin{array}{c} l \\ k \\ j \\ i \\
\hline \\
r \\ s \\ j \\ i
\end{array} \right) \cdot \epsilon \left( \begin{array}{c} l \\ k \\ j \\ i \\
\hline \\
r \\ s \\ j \\ i
\end{array} \right).
\]

Using the value of \(R\) in Eq. \(75\), the above identity becomes

\[
\mathcal{R}(a \otimes a) = (a \otimes a)\mathcal{R},
\]

which is the defining relation Eq. \((11)\) of a quantum matrix. Besides

\[
\det_q(a) = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \left[ \begin{array}{c} n \\ 2 \\ 1 \\ \vdots \\ \sigma(2) \\ \sigma(1) \end{array} \right] 
= a^{-1} \left[ \begin{array}{c} n \\ 2 \\ 1 \\ \vdots \\ \sigma(2) \\ \sigma(1) \end{array} \right] = 1
\]

where the second equality is from Eq. \(47\) and the third is from Eq. \(54\). Hence, the algebra map \(\Psi : \mathcal{O}_q(SL(n)) \rightarrow S_n(\mathfrak{B})\) given by \(\Psi(u_{ij}^k) = a_{ij}^k\) is well-defined.

Since \(\Phi \circ \Psi(a_{ij}^k) = a_{ij}^k\), we have \(\Phi \circ \Psi = \text{id}\). This shows \(\Phi\) is injective, and hence \(\Phi : S_n(\mathfrak{B}) \rightarrow \mathcal{O}_q(SL(n))\) is an algebra isomorphism. By checking the values of \(\Delta, \epsilon,\) and \(S\) on the generators \(a_{ij}^k\) we see that \(\Phi\) is a Hopf algebra homomorphism. This completes the proof of Theorem 6.3. \(\square\)

6.9. Proof of Theorem 6.4

Proof. The co-\(R\)-matrix satisfies the following equalities (stated with Sweedler’s notation for the coproduct):

\[(76)\quad \rho(xy \otimes z) = \sum \rho(x \otimes z')\rho(y \otimes z'')\]

\[(77)\quad \rho(x \otimes yz) = \sum \rho(x' \otimes z)\rho(x'' \otimes y).\]

For \(\mathcal{O}_q(SL(n))\) the values of \(\rho\) are given by (see [Maj, Section 2.2] or [KS, Section 10.1.2]):

\[(78)\quad \rho(u_{ij}^k \otimes 1) = \rho(1 \otimes u_{ij}^k) = \delta_{ij}, \quad \rho(u_{ij}^k \otimes u_{ij}^l) = R_{ij}^{lk},\]

which, together with Relations \(76\) and \(77\), totally determine \(\rho\).

The first part of the proof follows that of [CL]. Let \(\rho'\) be the map defined by the right side of \(68\); we will show that \(\rho' = \rho\). It is enough to show that \(\rho'\) satisfies \(76\), \(77\), and
the initial values (78), all with \( \rho \) replaced by \( \rho' \). We have, where a line labeled by, say \( x \), stands for the stated \( n \)-tangle diagram \( x \),

\[
\rho'(xy \otimes z) = \epsilon \left( \begin{array}{c}
y \\
x \\
z \\
z \\
\end{array} \right)
\]

Splitting the bigon by the vertical middle ideal arc, then using \( \epsilon(u) = \sum \epsilon(u')\epsilon(u'') \),

\[
\rho'(xy \otimes z) = \sum \epsilon \left( \begin{array}{c}
y' \\
x' \\
z' \\
z'' \\
y'' \\
\end{array} \right) \epsilon \left( \begin{array}{c}
y'' \\
x' \\
z' \\
z'' \\
y'' \\
\end{array} \right)
\]

This proves (76) for \( \rho' \). The proof of (77) is similar.

Under the isomorphism, \( u_i^j \) becomes \( a_i^j \). Using Eq. (75), we have

\[
\rho'(a_i^j \otimes a_k^l) = \epsilon \left( \begin{array}{c}
i \\
k \\
j \\
l \\
\end{array} \right) = \mathbf{R}_{ji}^{kl} = \hat{R}_{ji}^{kl},
\]

\[
\rho'(a_i^j \otimes 1) = \rho'(1 \otimes a_i^j) = \epsilon(a_i^j) = \delta_{i,j}
\]

which proves (78), completing the proof of the theorem.

6.10. Proof of Theorem 3.11

Proof. The proof of Theorem 6.3 shows that the kernel of the map \( \Gamma \circ T : RW_n(\mathfrak{B}) \to U_q(sl_n)^* \) is generated by internal relations, boundary relations on the right side, and boundary relations on the left side. Transferring back to \( \Gamma : RT \to U_q(sl_n)^* \) via the isomorphism \( T : RW_n(\mathfrak{B}) \to RT \) we conclude that the kernel of \( \Gamma \) is generated by the basic internal annihilators, basic right annihilators, and basic left annihilators.

6.11. Additional facts.

1. Formula (69), after a simple manipulation, has the form

\[
\tilde{a}_j^i = \det_q M_j^i(a).
\]

where \( M_j^i(a) \) is the submatrix of \( a \) obtained by removing the \( i \)-row and \( j \)-column. Alternatively this formula is a consequence of the (geometric) antipode formula, \( S(a_j^i) = (-q)^{i-j-\gamma_j^i}a_j^i \), combined with the (algebraic) antipode formula, (12), in \( O_q(SL(n)) \).

2. The antipode, given by (66), is equivalent to the dual map of Section 3.5 via \( T \):

\[
T(S(x)) = (T(x))^*.
\]

3. The algebra involution \( \text{rot}_* : O_q(SL(n)) \to O_q(SL(n)) \), induced from the rotation by 180 degrees, is a coalgebra anti-homomorphism. It is easy to show that its dual restrict to an algebra anti-involution \( \text{rot}^* : U_q(sl_n) \to U_q(sl_n) \). One can check that \( \text{rot}^* \) is equal the anti-involution \( \rho \) introduced by Lusztig [Lu3, Chapter 19] in his study of canonical bases of quantized enveloping algebras.
7. Coaction of $\mathcal{O}_q(SL(n))$ on Stated Skein Modules

Similarly to the case $n = 2$ considered in [CL, BL], we are going to show that every marking $\beta$ of a marked 3-manifold $(M,N)$ defines a right coaction of $\mathcal{O}_q(SL(n))$ on $\mathcal{S}_n(M,N)$. Dually, it defines a left-$$\tilde{\mathcal{U}}^L$$ module structure on $\mathcal{S}_n(M,N)$, where $\tilde{\mathcal{U}}^L$ is a completion of the Lusztig integral version $U^L$ of the quantum group $U_q(sl_n)$. We will observe that the actions of the charmed and the half-ribbon elements on $\mathcal{S}_n(M,N)$ coincides with the marking automorphism $g_\beta$ and the half-twist automorphism htw$\beta$ of Subsections 4.10, 4.11, respectively.

The above $\mathcal{O}_q(SL(n))$-coaction will be very important for the further development of the theory of stated skein algebras in the remainder of this paper. For simplicity we assume $R = \mathbb{Z}[v^\pm 1]$ in this section.

7.1. Module and Co-module structures. Suppose $\Sigma$ is a punctured bordered surface and $b$ is a boundary edge. Let $c$ be an interior ideal arc isotopic to $b$. This means that $b$ and $c$ cobound a bigon. By splitting $\Sigma$ along $c$ we get a surface $\Sigma'$ and a directed bigon with $b$ considered its right edge. As $\Sigma'$ is diffeomorphic to $\Sigma$ via a unique up to isotopy diffeomorphism, we identify $\mathcal{S}_n(\Sigma') = \mathcal{S}_n(\Sigma)$. The splitting homomorphism gives an algebra homomorphism

$$\Delta_b : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma) \otimes \mathcal{O}_q(SL(n)).$$

The commutativity of splitting maps and the values of $\epsilon$ on horizontal stated arcs given by (70) imply that $\Delta_b$ is a right coaction of $\mathcal{O}_q(SL(n))$ on $\mathcal{S}_n(\Sigma)$. Moreover, the right coactions at different boundary edges commute. Since $\Delta_b$ in Eq. (79) is an algebra homomorphism, $\mathcal{S}_n(\Sigma)$ is a right comodule-algebra over $\mathcal{O}_q(SL(n))$, as defined in [Kass, Section III.7].

When splitting off a bigon we identify $b$ with its left edge, we get a left $\mathcal{O}_q(SL(n))$-comodule structure on $\mathcal{S}_n(\Sigma)$.

The above construction of the $\mathcal{O}_q(SL(n))$-coactions on $\mathcal{S}_n(\Sigma)$ generalizes to $\mathcal{O}_q(SL(n))$-coactions on stated modules of marked 3-manifolds. Given a marking $\beta$ of a marked 3-manifold $(M,N)$, consider its closed disk neighborhood $D$ in $\partial M$, disjoint from the other markings of $(M,N)$. By pushing the interior of $D$ inside $M$ we get a new disk $D'$ which is properly embedded in $M$. Splitting $(M,N)$ along $D'$, we get a new marked 3-manifold $(M',N')$ isomorphic to $(M,N)$, and another marked 3-manifold bounded by $D$ and $D'$. The latter, after removing the common boundary of $D$ and $D'$, is isomorphic to the thickening of the bigon, with $\beta$ considered its right face marking, as depicted in Figure 6(a). Hence, this construction yields an $R$-linear splitting map

$$\Delta_\beta : \mathcal{S}_n(M,N) \to \mathcal{S}_n(M,N) \otimes \mathcal{O}_q(SL(n)).$$

As in the surface case, this is a right coaction of $\mathcal{O}_q(SL(n))$ on $\mathcal{S}_n(M,N)$, and the right coactions at different markings commute.

The completion $\tilde{U}_q(sl_n)$ of $U_q(sl_n)$ of [ST], cf. Subsection 3.9, has its integral version, $\tilde{\mathcal{U}}^L$, which contains the half-twist element $X$ of Subsection 3.9 see [KT, Comment 3.7]. Equivalently, this is a completion of the Lustig integral version $\mathcal{U}^L$ of $U_q(sl_n)$ [Lu1, Sec. 1.3]. The Hopf algebras $\tilde{\mathcal{U}}^L$ and $\mathcal{O}_q(SL(n))$ are in Hopf duality over $\mathbb{Z}[v^\pm 1]$, which turns any right
\(O_q(SL(n))\)-comodule \(W\) to a left \(\tilde{U}^L\)-module as follows: For \(u \in \tilde{U}^L\) and \(x \in W\),

\[u \ast x = \sum x' \langle f'', u \rangle,\]

where \(\Delta(x) = \sum x' \otimes f''\)
is the \(O_q(SL(n))\)-coaction map.

To make explicit the left action of \(\tilde{U}^L\) on \(S_n(M, N)\) coming the the right coaction \(\Delta^\beta\) we extend the states of an \(n\)-web at marking \(\beta\) as follows. Suppose \(\alpha\) is an \(n\)-web in \((M, N)\) with the sign sequence on \(\beta\) equal to \(\eta = (\eta_1, \ldots, \eta_k) \in \{\pm\}^k\). The set \(\{v_i \mid i \in \{1, \ldots, n\}\}^k\) is the \(R\)-basis of the based module \(V^\eta\). Assume \(\alpha\) is stated at all markings except \(\beta\). For \(x \in V^\eta\) let \((\alpha, x) \in S_n(M, N)\) be defined so that \((\alpha, v_i)\) is \(\alpha\) with states \(i\) on \(\beta\), and the map \(x \rightarrow (\alpha, x)\) is \(R\)-linear. From the definition we have

\[u \ast (\alpha, x) = (\alpha, ux)\]

**Example 7.1.** The action of the charmed element \(g\) on \(S_n(M, N)\) is exactly the map \(g^\beta\) of Subsection 4.10. In fact, \(g\) is a group-like element, i.e. \(\Delta^k(g) = g^\otimes k\) for \(k = 1, 2, \ldots, \) and the actions of \(g\) on the based \(U_q(sl_n)\)-modules \(V^\eta\) and \(V^\eta\) are given by the same diagonal matrix with \(g_1, \ldots, g_n\) on the diagonal, see [42]. That ensures that the action of \(g\) on \(S_n(M, N)\) coincides with the map \(g^\beta\).

**Example 7.2.** The action of the half-ribbon element \(X\) is the half-twist homomorphism \(htw^\beta\) of Subsection 4.11. Indeed, for the positive half-twist \(H\) on \(k\) strands (with arbitrary orientations) by (40) we have

\[RT(H) = \Delta^k(X) \circ (X^{-1})^\otimes k \circ \text{rev}_k.\]

By applying this value of \(RT(H)\) to (62) we obtain

\[htw^\beta\left(\begin{array}{c}
i_k \\
i_2 \\
i_1 \\ \end{array}\right) = \left(\prod_{j=1}^k c_{i_j}ight) \cdot \left(\begin{array}{c}
i_k \\
i_2 \\
i_1 \\ \end{array}\right) = \left(\begin{array}{c}
i_k \\
i_2 \\
i_1 \\ \end{array}\right)
\]

where the second identity follows from (41). This proves the statement.

Formula (80) makes it easy to study \(S_n(M, N)\) as an \(\tilde{U}^L\)-module. For example, one can show that over the field \(Q(v)\) the \(U_q(sl_n)\)-module \(S_n(M, N) \otimes_{\mathbb{Z}[v^\pm 1]} Q(v)\) is integrable and is a direct sum of finite-dimensional simple \(U_q(sl_n)\)-modules. For the case \(n = 2\) see [CL].

### 7.2. Boundary relations revisited

Let \(D = (D, i)\) be an \(n\)-web diagram \(D\) over the bigon \(B\) right stated by \(i = (i_1, \ldots, i_k)\). Assume that \(D\) has \(l\) left endpoints, which are not stated. Suppose further that \(\alpha\) is a stated \(n\)-web in a marked 3-manifold \((M, N)\) and that in a cube \(Q\) which intersects \(N\) at a subinterval of a marking \(\beta\) the intersection \(\alpha \cap Q\) has diagram equal to \(D\), as in the left side of (81). By the property of the counit of the coaction, we have

\[D \leftrightarrow D = \sum_{j_1, \ldots, j_l} \sum_{i_1, \ldots, i_k} i_k \cdot \epsilon(j_1, j_2, \ldots, j_l),\]

where the second identity follows from (41). This proves the statement.
This identity provides a local relation in any stated skein module, called the **D-relation**. By [74], the values of \( \epsilon \) of stated \( n \)-webs are the entries of the matrix describing the Reshetikhin-Turaev operator \( RT(D) \) and are not difficult to calculate. All the boundary relations (47)-(50) are of this type. Since Relations (43)-(50) are sufficient for defining the \( O_q(SL(n)) \)-coaction on \( S_n((M,N)) \), any D-relation is a consequence of these relations.

**Example 7.3.** For \( D = \begin{array}{c} i \\ j \end{array} \), the D-relation is

\[
q^{-\frac{1}{n}} \left( q^{d_{ji}} \begin{array}{c} j \\ i \end{array} + \delta_{ji} \sum_{k>i} (-q)^{k-i}(q - q^{-1}) \begin{array}{c} k \\ i \end{array} \right).
\]

**Example 7.4.** The following relations for \( n = 3 \) will be useful later:

\[
\begin{align*}
\bar{\gamma} & = q^\delta_{i,j} - q^{-1} \\
\tilde{\gamma} & = \delta_{i \neq j} (-q)^{\delta_{i,j}} q^{\frac{17}{16}} i + j - 2 \\
\tilde{\gamma} & = \delta_i, \text{ for } i = 1, 3.
\end{align*}
\]

7.3. The last among the defining skein relations.

**Proposition 7.5.** If \( [n-2]! \) is invertible in \( R \) then the last defining relation, (50), is a consequence of the other defining relations (43)-(49).

**Proof.** Note that Relation (50) is the D-relation for \( D = \begin{array}{c} i \\ j \end{array} \).

Since \( [n-2]! \) is invertible, Identity (51) makes it possible to eliminate all crossings in every diagram,

\[
\begin{array}{c} i \\ j \end{array} = q^{\frac{n-1}{n}} \begin{array}{c} k \\ j \end{array} - (-1)^{\binom{n}{2}} q^{-\frac{1}{n}} \begin{array}{c} n+2 \\ i \end{array}.
\]

For the purpose of this proof, let \( S'_n(M,N) \) be defined as \( S_n(M,N) \), only without Relation (50). The Splitting Theorem holds for \( S'_n(M,N) \), as using (85) we do not have to consider the invariance of the splitting homomorphism under moving a crossing through the splitting disk. Lemma (5.1) holds for \( S'_n(\mathfrak{B}) \) because all crossings of diagrams on \( \mathfrak{B} \) can be eliminated. Consequently, Lemma 6.5 holds as well and the proof of the isomorphism \( S_n(\mathfrak{B}) \simeq O_q(sl_n;R) \) extends to an isomorphism

\[
S'_n(\mathfrak{B}) \simeq S_n(\mathfrak{B}) \simeq O_q(sl_n;R).
\]

Furthermore, every marking \( \beta \) of \( (M,N) \) defines a right coaction \( \Delta'_\beta : S'_n(M,N) \rightarrow S'_n(M,N) \otimes_R O_q(sl_n; R) \) as in previous subsection. Using the coaction, one sees that for every right stated \( n \)-web diagram \( D = (D,i) \) on \( \mathfrak{B} \), the relation (81) is a consequence of the
defining relations for $\mathcal{S}_n'(M, \mathcal{N})$. In particular, for $D = \begin{array}{c} \circ \\ \downarrow \\ \circ \end{array}$, we get the statement of the proposition. □

7.4. Glueing over an ideal triangle. The standard ideal triangle $\mathfrak{I} \subset \mathbb{R}^2$ is the closed triangle with vertices $(-1,0)$, $(0,1)$ and $(0,1)$ with these vertices removed. We will denote its sides by $e_1$, $e_2$, and $\partial_{b}\mathfrak{I}$ as in Figure 13. Suppose $a_1$, $a_2$ are two distinct boundary edges of a (possibly disconnected) pb surface. Define

$$
\Sigma_{a_1 \triangle a_2} = (\Sigma \cup \mathfrak{I}) / (e_1 = a_1, e_2 = a_2),
$$

as in Figure 13. Define the $R$-linear homomorphism $\text{glue}_{a_1, a_2} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma_{a_1 \triangle a_2})$ so that if $\alpha$ is a stated $n$-web diagram over $\Sigma$ with the negative height order on both $a_1$ and $a_2$ then $\text{glue}_{a_1, a_2}(\alpha)$ is the result of continuing the strands of $\alpha$ with endpoints on $a_1$ and $a_2$ until they reach $\partial_{b}\mathfrak{I}$, as in Figure 13 (right). (As usual, the arrows indicate the height order.)

![Figure 13](image)

**Figure 13.** Left: The standard ideal triangle $\mathfrak{I}$. Middle: Glueing $\Sigma$ and $\mathfrak{I}$ by $a_1 = e_1$ and $a_2 = e_2$ to get $\Sigma_{a_1 \triangle a_2}$. Right: tangle diagram $x \in \mathcal{S}_n(\Sigma)$ and its image $\text{glue}_{a_1, a_2}(x) \in \mathcal{S}_n(\Sigma_{a_1 \triangle a_2})$

We are going to show that $\text{glue}_{a_1, a_2}$ is a linear isomorphism and to construct its inverse, $\text{cut}_{a_1, a_2}$.

Let

$$
\varepsilon_{\mathfrak{I}} : \mathcal{S}_n(\mathfrak{I}) \to \mathcal{S}_n(\mathfrak{M}) = R, \quad \varepsilon_{\mathfrak{I}} = \varepsilon \circ \text{fill}_s,
$$

where fill embeds $\mathfrak{I}$ into a bigon by filling in the top vertex of $\mathfrak{T}$ and making the web ends at $e_1$ higher than those at $e_2$. Let us consider also the homomorphism

$$
\text{cut}_{a_1, a_2} : \mathcal{S}_n(\Sigma_{a_1 \triangle a_2}) \to \mathcal{S}_n(\Sigma), \quad \text{cut}_{a_1, a_2} = (\varepsilon_{\mathfrak{I}} \otimes \text{id}_{\mathcal{S}_n(\Sigma)}) \circ \Theta_{a_1} \circ \Theta_{a_2}.
$$

**Proposition 7.6.** The homomorphisms $\text{cut}_{a_1, a_2}$ and $\text{glue}_{a_1, a_2}$ are inverses of each other:

$$
\text{cut}_{a_1, a_2} \circ \text{glue}_{a_1, a_2} = \text{id}_{\mathcal{S}_n(\Sigma)} \quad \text{and} \quad \text{glue}_{a_1, a_2} \circ \text{cut}_{a_1, a_2} = \text{id}_{\mathcal{S}_n(\Sigma_{a_1 \triangle a_2})}.
$$

**Proof.** We follow Higgins’ proof for $n = 3$. It is enough to check the above identities for the diagrams over $\Sigma$ and over $\Sigma_{a_1 \triangle a_2}$ respectively. We have

$$
(\varepsilon_{\mathfrak{I}}) \left(\begin{array}{c} e_1 \\ e_2 \end{array}\right) = \prod_{a=1}^{[j]} (\delta_{ja, ka} c_{ja}^{-1}) \cdot \left(\prod_{b=1}^{[l]} \delta_{ib, kb}\right) \cdot \left(\prod_{d=1}^{[t]} \delta_{ld, td}\right),
$$

where the orientations of arcs in the triangle are arbitrary and $i, s$ and $j, k$ and $l, t$ are any three pairs are state sequences of equal length, $|i| = |s|$, $|j| = |k|$, and $|l| = |t|$.
Let us call diagrams over $\mathcal{T}$ which are like in Eq. (86) without the horizontal arcs, \textbf{vertical}. Since for every $x \in S_n(\Sigma_{a_1 \Delta a_2})$, the skein $\Theta_{a_1} \circ \Theta_{a_2} \circ \text{glue}_{a_1,a_2}(x)$ is a linear combination of diagrams which are vertical on $\mathcal{T}$, the first identity follows.

To prove the second identity observe that each diagram in $S_n(\Sigma_{a_1 \Delta a_2})$ can be positioned so that it intersects $\mathcal{T}$ in disjoint arcs only. By applying Relation (49), it can be presented as a linear combination of diagrams which are vertical on $\mathcal{T}$. By Eq. (86), for each of them we have

$$\text{glue}_{a_1,a_2} \circ (\varepsilon_T(W) \otimes \text{id}_{S_n(\Sigma)}) \circ \Theta_{a_1} \circ \Theta_{a_2}(W) = W.$$ 

Hence, the right identity of Proposition 7.6 follows. $\square$

Although the bijective map $\text{glue}_{a_1,a_2}$ is not an algebra isomorphism with respect to the standard skein algebra product on $S_n(\Sigma_{a_1 \Delta a_2})$, we are going to show that it is one with respect to the \textbf{self braided tensor product} which we will define right now. The $n = 2$ version of this product was considered in [CL].

There are two right $O_q(SL(n))$-comodule algebra structures on $S_n(\Sigma)$ given by

$$\Delta_i := \Delta_{a_i} : S_n(\Sigma) \rightarrow S_n(\Sigma) \otimes O_q(SL(n)), \quad i = 1, 2,$$

which commute.

Define the $R$-linear map $\Delta : S_n(\Sigma) \rightarrow S_n(\Sigma) \otimes O_q(SL(n))$ by

$$\Delta(x) = \sum x' \otimes u_1 u_2,$$

in Sweedler’s notation, where

$$(\Delta_1 \otimes \text{Id}_{O_q(SL(n))}) \circ \Delta_2(x) = \sum x' \otimes u_1 \otimes u_2.$$ 

For $x, y \in S_n(\Sigma)$ define a new product by

$$(87) \quad y \ast x = \sum y' x' \rho(u \otimes w)$$

where

$$\Delta_2(y) = \sum y' \otimes u, \Delta_1(x) = \sum x' \otimes w.$$ 

It is proved in [CL] that $\Delta$ and $\ast$ together give $S_n(\Sigma)$ a right $O_q(SL(n))$-comodule algebra structure for $n = 2$. That proof extends verbatim to all $n$. Denote by $\otimes S_n(\Sigma)$ the $R$-module $S_n(\Sigma)$ with this $O_q(SL(n))$-comodule algebra structure. On the other hand $S_n(\Sigma_{a_1 \Delta a_2})$ has a right $O_q(SL(n))$-comodule algebra structure coming from the boundary edge $\partial_b \mathcal{T}$. Here is a stronger version of Proposition 7.6.

\textbf{Theorem 7.7.} The maps $\text{glue}_{a_1,a_2} : \otimes S_n(\Sigma) \rightarrow S_n(\Sigma_{a_1 \Delta a_2})$, $\text{cut}_{a_1,a_2} : S_n(\Sigma_{a_1 \Delta a_2}) \rightarrow S_n(\Sigma)$ are isomorphisms of right $O_q(SL(n))$-comodule algebras.

\textit{Proof.} The geometric proof of [CL] for $n = 2$ carries over to all $n$ without modification. Here is a sketch. It is enough to show that $f = \text{glue}_{a_1,a_2}$ is an algebra homomorphism. Let $x, y$ be stated $n$-web diagrams.
We present \(yx, f(yx), \) and \(f(y)f(x)\) schematically as in Figure 14. By splitting along the dashed line in the picture of \(f(y)f(x)\) and by using the counit property,

\[
\Delta_2(y) = \sum u, \Delta_1(x) = \sum \rho(x' \otimes w),\]

using co-\(R\)-matrix \(\rho\) of (68).

Thus \(f\) is an algebra homomorphism.

A special case is when \(\Sigma = \Sigma_1 \cup \Sigma_2\) and \(a_i \subset \Sigma_i\) for \(i = 1, 2\). In this case we say that \(\Sigma_{a_1 \triangle a_2}\) is the result of gluing \(\Sigma_1\) and \(\Sigma_2\) over the triangle. Each \(\mathcal{S}_n(\Sigma)\) is a right \(O_q(SL(n))\)-comodule algebra via the coaction coming from the edge \(a_i\). Then \(\otimes(\mathcal{S}_n(\Sigma))\) is the well-known braided tensor product \(\mathcal{S}_n(\Sigma_1)\) and \(\mathcal{S}_n(\Sigma_2)\) of the two \(O_q(SL(n))\)-module algebras \(\mathcal{S}_n(\Sigma_1)\) and \(\mathcal{S}_n(\Sigma_2)\), defined in [Ma] Lemma 9.2.12.

**Example 7.8** (Ideal triangle). Let \(\Sigma_1 = \Sigma_2 = \mathcal{B}\), where \(a_1\) is the right edge of \(\Sigma_1\) and \(a_2\) is the left edge of \(\Sigma_2\). Then \(\Sigma_{a_1 \triangle a_2}\) is the triangle \(\mathcal{T}\). Hence, we have

\[
\mathcal{S}_n(\mathcal{T}) \cong O_q(SL(n)) \otimes O_q(SL(n)),
\]

where each copy of \(O_q(SL(n))\) is a right \(O_q(SL(n))\)-comodule algebra via the coproduct. From here one can easily write down an explicit presentation of the algebra \(\mathcal{S}_n(\mathcal{T})\). Such presentation is used in the work [LY2] on the quantum trace for stated \(SL_n\)-skein algebras.

Let \(\Sigma_{g,p+1}\) be a \(p\)-punctured genus \(g\) surface with a single loop boundary and let \(\Sigma_{g,p}^*\) be \(\Sigma_{g,p}\) with a boundary point removed.

**Example 7.9**. Let \(\mathcal{G} = \{\Sigma_{p,q}; g \geq 0, p \geq 1\}\) be the set of pb surfaces with a single arc boundary, considered up to a homeomorphism. For \(\Sigma_1, \Sigma_2 \in \mathcal{G}\), let \(\Sigma_1 \triangle \Sigma_2\) be the result of gluing over a triangle along \(a_1 = \partial \Sigma_1\) and \(a_2 = \partial \Sigma_2\), as above. Note that the \(\triangle\) operation makes \(\mathcal{G}\) into a monoid with the identity \(\mathcal{M}\). Theorem 7.7 implies that for any \(\Sigma_1, \Sigma_2 \in \mathcal{G}\),

\[
\mathcal{S}_n(\Sigma_1 \triangle \Sigma_2) = \mathcal{S}_n(\Sigma_1) \otimes \mathcal{S}_n(\Sigma_2).
\]

**Figure 15.** The operation \(\Sigma_1 \triangle \Sigma_2\) on two surfaces with a single arc boundaries.
Therefore, $\mathcal{S}_n(\Sigma_{g,p})$ is the braided tensor product of $p-1$ copies of $\mathcal{S}_n(\Sigma_{0,2}^*)$ and $g$ copies of $\mathcal{S}_n(\Sigma_{1,1}^*)$. We will analyze $\mathcal{S}_n(\Sigma_{0,2}^*)$ and $\mathcal{S}_n(\Sigma_{1,1}^*)$ in detail in Subsections 7.5 and 7.9 below and, we will see in particular that

$$\mathcal{S}_n(\Sigma_{0,2}^*) \simeq \mathcal{O}_q(SL(n))$$

and

$$\mathcal{S}_n(\Sigma_{1,1}^*) \simeq \mathcal{O}_q(SL(n))^{\otimes 2}$$

as $R$-modules. Consequently,

$$\mathcal{S}_n(\Sigma_{g,p}) \simeq \mathcal{O}_q(SL(n))^{\otimes (p-1+2g)}$$

as an $R$-module. This statement will be generalized by Theorem 7.13.

7.5. Punctured Monogon and Majid’s Transmutation. By attaching an ideal triangle to the bigon $\mathfrak{B}$ along its left and right edges $e_l$ and $e_r$, as in Figure 16, we obtain the once-punctured monogon, $\Sigma_{0,2} = \mathfrak{B}_{a_1 \triangle a_2}$.

![Figure 16. From the bigon to a punctured monogon by gluing over a triangle.](image)

An algebraic description of the product on $\mathcal{S}_n(\Sigma_{0,2}^*)$ can be derived from that for $\mathcal{O}_q(SL(n))$ by the rule described in Eq. (87). This allows to identify $\mathcal{S}_n(\Sigma_{0,2}^*)$ with Majid’s transmutation of $\mathcal{O}_q(SL(n))$, as we explain now.

Let $tm$ be the composition of the inverse of the half-twist around the left edge with the above triangle gluing map,

$$tm = \text{glue}_{e_l,e_r}^{-1} : \mathcal{O}_q(SL(n)) = \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\Sigma_{0,2}^*).$$

This map can be visualized as follows: Let a stated web $x$ in $\mathfrak{B}$ be represented by a diagram $i \xrightarrow{x} j$, where the left horizontal line represents multiple horizontal edges (of possibly different directions) whose ends on the left are labeled by $i = \begin{pmatrix} i_1 \\ \vdots \\ i_s \end{pmatrix}$ and, similarly, the right horizontal line represents multiple horizontal edges whose ends on the right are labeled by $j = \begin{pmatrix} j_1 \\ \vdots \\ j_t \end{pmatrix}$. Then

$$htw_{e_l}^{-1}(x) = \frac{1}{\prod c_i} \begin{pmatrix} i_1 \\ \vdots \\ i_s \end{pmatrix} \times \begin{pmatrix} j_1 \\ \vdots \\ j_t \end{pmatrix}$$

and

$$tm(x) = \frac{1}{\prod c_i} \begin{pmatrix} i_1 \\ \vdots \\ i_s \end{pmatrix} \times \begin{pmatrix} j_1 \\ \vdots \\ j_t \end{pmatrix}.$$
defines a coaction on $\mathcal{H}$ on itself \textbf{[Maj]\ Example 1.6.14], called the adjoint $\mathcal{H}$-coaction and that there is an associative braided product (or, covariantised)

$$(90) \ x_\cdot y = \sum x_2 y_2 \rho((Sx_1)x_3 \otimes Sy_1),$$

cf. \textbf{[Maj]\ Example 9.4.10]. (This product should not be confused with the braided tensor product of Majid, which we discussed in Subsec. \textbf{7.4]) Furthermore, he showed that $\mathcal{H}$ with the braided product and the adjoint $\mathcal{H}$-coaction is an $\mathcal{H}$-comodule coalgebra. With these structures, $\mathcal{H}$ is denoted by $\mathcal{T} \mathcal{H}$ called the transmutation of $\mathcal{H}$.

**Proposition 7.10.** (1) $tm$ is an isomorphism between the $O_q(SL(n))$-comodule algebra $T O_q(SL(n))$ and $S_n(\Sigma^*_2)$. Hence, $(tm \otimes I) \circ Ad = \Delta \partial \Sigma^*_2$, $\Delta \partial \Sigma^*_2 \circ tm$. (2) $tm$ is a ring isomorphism. Hence, $tm(x)tm(y) = tm(x \cdot y)$.

This statement was proved for $n = 2$ in \textbf{[CL]}.

**Proof of Prop. 7.10**  (1) Let a stated web in $\mathcal{B}$ be given by the diagram $i \overset{x}{\curvearrowleft} j$, as above.

Then $\Delta^2(x) = \sum_{k,l} i \overset{k}{\curvearrowleft} k \otimes k \overset{l}{\curvearrowleft} l \otimes l \overset{j}{\curvearrowleft} j$, where as above, horizontal arcs indicate multiple edges of possibly different directions. Consequently, by Eqs. (89) and (66),

$$(91) \ Ad \left( i \overset{x}{\curvearrowleft} j \right) = \sum_{k,l} \prod_{c_k} k \otimes \prod_{c_l} l \overset{\hat{k} \otimes \hat{l}}{\curvearrowleft} \overset{j}{\curvearrowleft} \overset{i}{\curvearrowleft},$$

where Ro denotes $180^\circ$ rotation in plane, $\hat{i}$ denotes conjugation of components and the inversion of their order: $\hat{i} = \begin{pmatrix} i_1 \\ \vdots \\ i_s \end{pmatrix}$ for $i = \begin{pmatrix} i_1 \\ \vdots \\ i_s \end{pmatrix}$, and $\prod c_i$ denotes $\prod_{t=1}^s c_{i_t}$.

On the other hand, we have

$$\Delta_{\theta \Sigma^*_2,0} \left( \overset{\hat{i}}{\overset{\cdot}{\cdot}} \overset{x}{\cdot} j \right) = \sum_{k,l} k \otimes \hat{k} \overset{l}{\curvearrowleft} \overset{\hat{l}}{\curvearrowleft} \overset{j}{\curvearrowleft} \overset{i}{\curvearrowleft},$$

and, hence,

$$\Delta_{\theta \Sigma^*_2,0} \left( \frac{1}{\prod c_i} \overset{\cdot}{\cdot} \overset{x}{\cdot} j \right) = \sum_{k,l} \prod_{c_k} k \otimes \frac{1}{\prod c_i} \prod_{c_l} l \overset{\hat{k} \otimes \hat{l}}{\curvearrowleft} \overset{j}{\curvearrowleft} \overset{i}{\curvearrowleft}. $$
Since this equality coincides with (91) after replacing $x$ with $tm(x)$ and $Ad$ by $\Delta_{\partial \Sigma \times \partial}$, the statement follows.

(2) Consider stated webs $x = \begin{array}{c} \bullet \\ i \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \bullet \\ j \end{array}$ and $y = \begin{array}{c} \bullet \\ k \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \bullet \\ l \end{array}$ in $\mathfrak{B}$. Then

$$tm(xy) = tm\begin{array}{c} \bullet \\ k \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \bullet \\ l \end{array} \begin{array}{c} \bullet \\ i \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \bullet \\ j \end{array} = \frac{1}{c_i c_k}$$

On the other hand,

$$tm(x)tm(y) = \begin{array}{c} \bullet \\ k \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \bullet \\ l \end{array} \begin{array}{c} \bullet \\ i \end{array} \begin{array}{c} \circ \end{array} \begin{array}{c} \bullet \\ j \end{array}$$

Denoting the stated web diagram on the right by $z \in \mathcal{S}_n(\Sigma_{0,2}, 0, 2)$, we have

$$z = \sum z(1) \varepsilon(z(2)),$$

where $\Delta(z) = \sum z(1) \otimes z(2) \in \mathcal{S}_n(\Sigma_{0,2}, 0, 2) \otimes \mathcal{O}_q(SL(n))$ is the coaction along the dashed line and $\varepsilon$ is the counit in $\mathcal{O}_q(SL(n))$. By applying this identify to (93) we obtain

$$tm(x)tm(y) = \frac{1}{c_i c_k} \sum tm(x(2)y(2))T(x(1), x(3), y(1)),$$

where $T(x(1), x(3), y(1)) = \rho((Sx(1)x(3) \otimes Sy(1)))$, by Theorem 6.4. Consequently,

$$tm(x)tm(y) = tm(x \cdot y),$$

by (90).

Consequently, our theory provides simple geometric proofs of the associativity of the (braided) product on $\mathcal{T}\mathcal{O}_q(SL(n))$ and of $\mathcal{T}\mathcal{O}_q(SL(n))$ being an $\mathcal{O}_q(SL(n))$-comodule algebra. (The proofs of these facts are quite technical and involved in [Maj].) Furthermore, our theory generalizes these statements to the boundary $\mathcal{O}_q(SL(n))$-coaction on the skein algebra of any essentially bordered punctured surface.

Let us discuss generators and relations of $\mathcal{T}\mathcal{O}_q(SL(n))$ now. A reflection equation algebra $A_q(M_n)$ is an $R$-algebra generated by formal variables $x_{ij}$ for $i, j = 1, \ldots, n$ subject to the quadratic relations of the reflection equation:

$$X_2 \hat{R} X_2 \hat{R} = \hat{R} X_2 \hat{R} X_2,$$

where $X = (x_{ij})_{i,j=1,\ldots,n}$, $X_2 = X \otimes Id$, and $\hat{R}$ is the braiding matrix of Subsec. 2.3. (These equations are written explicitly out in [DL, Sec. 3].)

It is proved in [KoS] that $\mathcal{T}\mathcal{O}_q(SL(n))$ is the quotient of the reflection equation algebra by the braided determinant which is the image of the quantum determinant under the linear
isomorphism $\mathcal{TO}_q(SL(n)) \simeq \mathcal{O}_q(SL(n))$ above. An explicit polynomial expression in $x_{ij}$’s for it appears in [JW]. Consequently, that expression together with the relations (94) are a complete set of relations for $\mathcal{TO}_q(SL(n))$.

Let us relate this discussion to $S_n(\Sigma^*_{0,2})$ now. It is straightforward to verify that $tm$ maps the generators $x_{ij} \in \mathcal{TO}_q(SL(n))$ to the arcs which we will denote by $b_{i,j}$. (Then $tm(a_{i,j}) = \frac{1}{c_i}b_{i,j}$ for the generators $a_{i,j}$ for $S_n(\mathfrak{B})$ of Subsec. 6.2. Independently of the above considerations, it is easy to see that $b_{i,j}$’s for $i,j = 1,\ldots,n$ generate $S_n(\Sigma^*_{0,2})$, since any web in $S_n(\Sigma^*_{0,2})$ can be pushed towards the boundary of $\partial \Sigma_{0,2}$ and simplified by the boundary relations to a polynomial expression in $b_{i,j}$’s.) Consequently, the above discussion provides a concrete finite presentation for $S_n(\Sigma^*_{0,2})$.

7.6. On injectivity of splitting homomorphism.

**Proposition 7.11.** Suppose $\Sigma$ is an essentially bordered pb surface. Then for any interior ideal arc $c$ of $\Sigma$, the splitting homomorphism $\Theta_c : S_n(\Sigma) \rightarrow S_n(\text{cut}_c \Sigma)$ is injective.

**Proof.** First assume that an endpoint of $c$ is a boundary ideal point, which is an endpoint of a boundary edge $e \neq c$. In a small neighborhood of $e \cup c$ we can find an interior ideal arc $c'$ such that $e, c, c'$ cobound an ideal triangle. By Theorem [7.7] we see that $\text{cut}_{c,c'}$ is bijective. Since $\Theta_c$ is a component in a composition forming $\text{cut}_{c,c'}$, we conclude that $\Theta_c$ is injective.

Now assume that both endpoints of $c$ are interior ideal points. Since $\Sigma$ is essentially bordered it contains an interior ideal arc $d$, disjoint from $c$, with one endpoint coinciding with an endpoint of $c$ and the other endpoint being a boundary ideal point. By the above case, the splitting map $\Theta_d : S_n(\Sigma) \rightarrow S_n(\text{cut}_d(\Sigma))$ is injective. Since the interior ideal arc $c \subset \text{cut}_d(\Sigma)$ has one endpoint on the boundary, $\Theta_c : \text{cut}_d(\Sigma) \rightarrow \text{cut}_{c,d}(\Sigma)$ is injective. From the commutativity $\Theta_c \circ \Theta_d = \Theta_d \circ \Theta_c$ we conclude that $\Theta_c : S_n(\Sigma) \rightarrow \text{cut}_c(\Sigma)$ is injective. □

**Conjecture 7.12.** For any punctured bordered surface $\Sigma$ and any interior ideal arc $c$ the splitting homomorphism $\Theta_c$ is injective as well.

The conjecture is true when $n = 2$ by [CL] and for $n = 3$ by Higgins [Hi]. In both cases explicit bases of $S_n(\Sigma)$ were used. Proposition [7.11] shows the conjecture is true if $\Sigma$ has non-trivial boundary. Furthermore, the argument of the proof reduces the conjecture to the the empty boundary surfaces with a trivial ideal arc $c$, i.e. an ideal arc bounding a disk in $\Sigma$. Corollary [8.2] will establish a weaker version of this conjecture for all pb surfaces.

7.7. Skein algebras of surfaces with boundary. Let $\Sigma$ be an essentially bordered pb surface. A collection $A = \{a_1,\ldots,a_r\}$ of disjoint compact oriented arcs properly embedded into $\Sigma$ is **saturated** if

(i) each connected component of $\Sigma \setminus \bigcup_{i=1}^r a_i$ contains exactly one ideal point (interior or boundary) of $\Sigma$, and

(ii) $A$ is maximal with respect to the above condition.
Let \( U(a_1), \ldots, U(a_r) \) be a collection of disjoint open tubular neighborhoods of \( a_1, \ldots, a_r \), respectively. Each \( U(a_i) \) is homeomorphic with \( a_i \times (-1, 1) \) (by an orientation preserving homeomorphism) and we require that \((\partial a_i) \times (-1, 1) \subset \partial \Sigma\).

Recall from Subsec. 5.4 that any embedding of pb surfaces \( \Sigma' \subset \Sigma \) together with an ordering on the boundary edges of \( \Sigma' \) in the boundary edges \( b \) of \( \Sigma \), called \( b \)-orders, defines a linear homomorphism \( \mathcal{S}_n(\Sigma') \to \mathcal{S}_n(\Sigma) \). We will show that it is an isomorphism for a saturated system for arcs \( a_1, \ldots, a_r \) and \( \Sigma' = U(A) = \bigcup_{i=1}^r U(a_i) \):

**Theorem 7.13.** Assume \( \Sigma \) is an essentially bordered pb surface and \( A = \{a_1, \ldots, a_r\} \) is a saturated system of arcs.

1. We have \( r = r(\Sigma) := \#\partial \Sigma - \chi(\Sigma) \), where \( \#\partial \Sigma \) is the number of boundary components of \( \Sigma \) and \( \chi \) denotes the Euler characteristics.
2. The embedding \( U(A) \hookrightarrow \Sigma \) with negative \( b \)-orderings for all boundary edges \( b \) of \( \Sigma \), induces an \( R \)-module isomorphism \( f_A : \mathcal{S}_n(U(A)) \to \mathcal{S}_n(\Sigma) \).

Note that each \( U(a_i) = a_i \times (-1, 1) \) is naturally a directed bigon, with its sides \((\partial a_i) \times (-1, 1) \) oriented in the direction of \((-1, 1)\).

**Figure 17.** Examples of saturated systems. Left: \( A = \{a_1\} \) in a punctured monogon, \( \Sigma^{*}_{0,2} \).
Right: \( A = \{a_1, a_2\} \) in an ideal triangle \( \Sigma \) with \( a_1 \) in blue and \( a_2 \) in red.

**Example 7.14.** The saturated systems of Fig. 17 induce the linear isomorphisms

\[
\text{tm : } \mathcal{S}_n(\mathcal{B}) \to \mathcal{S}_n(\Sigma^{*}_{0,2}) \quad \text{and} \quad \mathcal{S}_n(\mathcal{B}) \otimes \mathcal{S}_n(\mathcal{B}) \to \mathcal{S}_n(\Sigma)
\]

of Eq. (88) and Example 7.8.

By the above theorem, for any essentially bordered pb surface we have an \( R \)-linear isomorphism

\[
\mathcal{O}_q(SL(n))^{\otimes r} \xrightarrow{f_A} \mathcal{S}_n(U(A)) \xrightarrow{\mathcal{O}_q} \mathcal{S}_n(\Sigma).
\]

Since \( \mathcal{O}_q(SL(n)) \) has a Kashiwara-Lusztig's canonical basis over \( \mathbb{Z}[v^{\pm 1}] \), cf. [Du] Prop 5.1.1, we have

**Corollary 7.15.** For any essentially bordered pb surface, \( \mathcal{S}_n(\Sigma) \) is a free \( R \)-module with a basis given by the image of tensor product of Kashiwara-Lusztig’s canonical bases on \( \mathcal{O}_q(SL(n))^{\otimes r} \) under \( f_A \mathcal{O}_q \).

**Remark 7.16.** Part (1) implies that condition (ii) in the definition of a saturated system can be replaced by condition \(|A| = r(\Sigma)|\).
Proof of Theorem 7.13. (1) By the maximality of $A$, its arcs cut $\Sigma$ into pieces of the following two types:

![Diagram of two types of pieces](image.png)

whose Euler characteristics are respectively 0 and 1. Hence the Euler characteristic of the result is the number of boundary ideal points, with is $\#\partial \Sigma$. On the other hand each arc cut increases the Euler characteristic by 1. Hence $\chi(\Sigma) + r = \#\partial \Sigma$, proving part (1).

(2) We prove it by induction on $\text{tri}(\Sigma)$ which is the number of ideal triangles in an ideal triangulation of $\Sigma$, defined as follows. Let $\mathcal{E}$ be a maximal collection of nontrivial ideal arcs in $\Sigma$ which are pairwise disjoint and pairwise non-isotopic. The ideal arcs in $\mathcal{E}$ not isotopic to boundary edges split $\Sigma$ into pieces, each is either a monogon, a bigon, or a triangle. Then $\text{tri}(\Sigma)$ is the number of triangles, which is known to be independent on the choice of $\mathcal{E}$. Note that $\text{tri}(\Sigma) = 0$ if and only if $\Sigma$ is a disjoint union of monogons and bigons, and the theorem is true for this case.

Suppose $\text{tri}(\Sigma) > 0$. We can assume that $\Sigma$ is connected.

Lemma 7.17. There is a boundary edge of $\Sigma$ containing at least two endpoints of arcs in $A$.

Proof. Since arcs of $A$ are disjoint and simple, they have $2|A|$ endpoints and it is enough to prove that $2|A| > \#\partial \Sigma$. Assuming otherwise, $2|A| \leq \#\partial \Sigma$, and by part (1) we have $1 \leq \#\partial \Sigma \leq 2\chi(\Sigma)$. The positivity of the Euler characteristic implies that $\chi(\Sigma) = 1$ and $\Sigma$ is a polygon. Then $\#\partial \Sigma \leq 2\chi(\Sigma) = 2$ implies $\Sigma$ is a monogon or a bigon, contradicting the assumption $\text{tri}(\Sigma) > 0$. \qed

Let $b$ be a boundary edge containing at least two endpoints of $A$. Let $p$ the ideal end point of $b$, following the positive direction of $b$. Among all arcs in $A$ having endpoints in $b$ assume $a_1$ has an endpoint closest to $p$. When $\Sigma$ is cut by arcs in $A$, there are two pieces adjacent to $a_1$, one of them, denoted by $P_1$, contains the ideal point $p$. The other piece, denoted by $P_2$, contains an ideal point $p'$. Let $e_1$ be an ideal arc of $\Sigma$ lying in the interior of $P_1 \cup P_2$ connecting $p$ and $p'$ and intersecting $a_1$ once. No other arcs in $A$ intersects $e_1$. Because the geometric intersection of $b$ with all arcs in $A$ is at least 2, $e_1$ cannot be isotopic to $b$. Pushing the union $e_1 \cup b$ slightly into the interior of $\Sigma$ yields an ideal arc $e_2$ such that $e_1, e_2, b$ bounds an ideal triangle $T$, as in Figure 18. After an isotopy we can assume that $A$ is taught with respect to $e_1, e_2, b$ in the sense that for each $a_j \in A$ and each $e \in \{b, e_1, e_2\}$ the number $|a_j \cap e|$ is minimal when we replace $a_j$ by any isotopic arc.

![Figure 18](image.png)

**Figure 18.** Left: The arcs in $A$ are in red, the ideal arcs $b, e_1, e_2$ are in blue. Right: After pulling $e_1, e_2$ and arcs in $A$ taut.
Let $\Sigma'$ be the result of removing $b$ and the interior of $T$ from $\Sigma$, and let $A'$ be the collection $a'_i := a_i \cap \Sigma', i = 1, \ldots, r$. As $|A'| = |A| = r(\Sigma) = r(\Sigma')$, the system $A'$ is saturated for $\Sigma'$. Since each $a'_i$ is a shrinking of $a_i$, there is a natural isomorphism $f_{A \to A'} : S_n(U(A)) \to S_n(U(A'))$. Since $\text{tri}(\Sigma') = \text{tri}(\Sigma) - 1$ the induction hypothesis applies to $\Sigma'$. From the definition we see that $f_A$ is the composition of

$$S_n(U(A)) \xrightarrow{f_{A \to A'}} S_n(U(A')) \xrightarrow{f_{A'}} S_n(\Sigma') \xrightarrow{\text{glue}_{1,2}} S_n(\Sigma).$$

Since each map in this composition is an $R$-linear isomorphism, so is $f_A$. \hfill \Box

We will show now that the assumption about the negativity of all $b$-orderings in Theorem 7.13(2) is unnecessary.

Let us enumerate the boundary edges of $\Sigma$ by $b_1, \ldots, b_s$ for bookkeeping purposes. Let $\sigma_1, \ldots, \sigma_s$ be some $b_1$-,..., $b_s$-orderings of the boundary intervals of $U(A)$ in the boundary intervals of $\Sigma$ and let $f_{A,\sigma_1,\ldots,\sigma_s} : S_n(U(A)) \to S_n(\Sigma)$ be the homomorphism induced by that height ordered embedding.

To relate $f_{A,\sigma_1,\ldots,\sigma_s}$ to $f_A$, note that each $b$-ordering $o$ is obtained by a certain permutation $\sigma$ of the negatively height ordered points $A \cap b$. Let us denote by $\sigma_1, \ldots, \sigma_s$ the permutations corresponding to height orderings $o_1, \ldots, o_s$. Then $f_{A,\sigma_1,\ldots,\sigma_s}(x)$ is induced by the embedding of $U(A)$ into $\Sigma \times (-1,1)$ with the boundary intervals of $U(A)$ braided by $(\sigma_1)_+, \ldots, (\sigma_s)_+$, cf. Figure 11. Let us elaborate on it more detail now.

Let us call skeins of the form $f_A(x_1 \otimes \ldots \otimes x_r) \in S_n(\Sigma)$ pure.

Lemma 7.18. (i) For any braids $\tau_1 \in B_{|A \cap b_1|}, \ldots, \tau_s \in B_{|A \cap b_s|}$ there exists a unique linear transformation

$$\text{braid}_{A,\tau_1,\ldots,\tau_s} : S_n(\Sigma) \to S_n(\Sigma)$$

which braids the endpoints of each pure skein in $S_n(\Sigma)$ in $b_i$ by $\tau_i$, for $i = 1, \ldots, s$. (All skeins are considered with negative $b_i$-orderings for $i = 1, \ldots, s$.)

(ii) Let $\sigma_1, \ldots, \sigma_s$ be permutations corresponding to height orderings $o_1, \ldots, o_s$ on $b_1, \ldots, b_s$. Then for pure $x,

$$f_{A,\sigma_1,\ldots,\sigma_s}(x) = \text{braid}_{A,\sigma_1,\ldots,\sigma_s} \circ f_A.$$ (iii) $(\tau_1, \ldots, \tau_s) \to \text{braid}_{A,\tau_1,\ldots,\tau_s}$ defines a group homomorphism from $B_{|A \cap b_1|} \times \cdots \times B_{|A \cap b_s|}$ to the group of $R$-linear automorphisms of $S_n(\Sigma)$. In particular, each $\text{braid}_{A,\tau_1,\ldots,\tau_s}$ is a linear isomorphism of $S_n(\Sigma)$.

Proof. (i) For braids $\tau_1 \in B_{|A \cap b_1|}, \ldots, \tau_s \in B_{|A \cap b_s|}$ consider the embedding

$$U(a_1) \cup \ldots \cup U(a_n) \subset \Sigma \times (-1,1)$$

modified by the braiding by $\tau_i$ of its components going towards $b_i \subset \Sigma$, for $i = 1, \ldots, s$. This map is considered with the negative height order. It induces a linear map of skein algebras which we denote by $g_{A,\tau_1,\ldots,\tau_s} : S_n(U(A)) \to S_n(\Sigma)$. Then for pure $x$, let

$$\text{braid}_{A,\tau_1,\ldots,\tau_s}(x) = g_{A,\tau_1,\ldots,\tau_s} \circ f_A^{-1}.$$ By Theorem 7.13(2) pure skeins span $S_n(\Sigma)$. Consequently, the condition of (i) determines $\text{braid}_{A,\tau_1,\ldots,\tau_s}$ completely.

(ii) follows from the discussion above Lemma 7.18.
(iii) By definition,
\[ \text{braid}_{A,\tau_1, \ldots, \tau_s} \circ \text{braid}_{A,\tau'_1, \ldots, \tau'_t} = \text{braid}_{A,\tau_1, \ldots, \tau_s \tau'_t}, \]
for any \( \tau_1, \tau'_1 \in B_{[A \cap b_1], \ldots, \tau_s, \tau'_s \in B_{[A \cap b_s]} \). Consequently, each \( \text{braid}_{\tau_1, \ldots, \tau_s} \) is a linear isomorphism of \( S_n(\Sigma) \). Since \( \text{braid}_{A,\text{id}, \ldots, \text{id}} = \text{id} \), the statement follows.

By Theorem 7.13(2) and Lemma 7.18(2) and (3), we have:

**Corollary 7.19.** \( f_{A,o_1, \ldots, o_s} : S_n(U(A)) \to S_n(\Sigma) \) is a linear isomorphism for every \( o_1, \ldots, o_s \).

### 7.8. Products on skein algebras of surfaces with boundary.

In the previous subsection, we discussed \( R \)-module structures of skein algebras only. We will address the algebra products now.

Let \( a_1, \ldots, a_r \) be a saturated system of arcs in \( \Sigma \) as before. Note that the induced linear homomorphism \( S_n(U(a_i)) \to S_n(\Sigma) \) is an algebra homomorphism if and only if \( a_i \) has its ends at different boundary intervals of \( \Sigma \). (We have seen this already in Example 7.14, where the right map of Eq. (93) is an algebra homomorphism on each of the components, \( S_n(\mathcal{B}) \), but the left map \( tm : S_n(\mathcal{B}) \to S_n(\Sigma_{n,2}) \) is not an algebra homomorphism.)

Therefore, for the sake of studying algebra products on \( S_n(\Sigma) \) let us consider modified neighborhoods \( U'(a_i) = U(a_i) \cup V \) for arcs \( a_i \) with both their ends in the same boundary interval, where \( V \) is a tubular neighborhood of the arc of \( \partial \Sigma \) connecting the endpoints of \( a_i \). We assume that \( V \) is small enough so that \( U'(a_i) \) is homeomorphic to a punctured monogon. Note that the transmutation map of is a linear isomorphism \( \text{tm} : S_n(U(a_i)) \to S_n(U'(a_i)) \) by Proposition 7.10. We leave the chosen neighborhoods of the arcs with ends in different components of \( \partial \Sigma \) unchanged, \( U'(a_i) = U(a_i) \).

Let us consider the map
\[
\text{mult}_A : S_n(U'(a_1)) \otimes \ldots \otimes S_n(U'(a_r)) \to S_n(\Sigma), \quad \text{mult}_A(x_1 \otimes \ldots \otimes x_r) = x_1 \cdot \ldots \cdot x_r.
\]

Note that by the transmutation map for arcs in the same component of \( \partial \Sigma \), \( \text{mult}_A \) coincides with \( f_{A,o_1, \ldots, o_s} \) for \( b_1, \ldots, b_s \)-orderings \( o_1, \ldots, o_s \), for which the boundary arcs of \( U'(a_i) \) are higher than the boundary arcs of \( U'(a_j) \) for \( i > j \) in any boundary interval of \( \partial \Sigma \).

Consequently, by the above discussion and by Corollary 7.19:

**Corollary 7.20.** \( \text{mult}_A : S_n(U'(a_1)) \otimes \ldots \otimes S_n(U'(a_r)) \to S_n(\Sigma) \) is an \( R \)-linear isomorphism and an algebra homomorphism on
\[
S_n(U'(a_i)) = R \otimes \ldots \otimes R \otimes S_n(U'(a_i)) \otimes R \otimes \ldots \otimes R \subset S_n(U'(a_1)) \otimes \ldots \otimes S_n(U'(a_r))
\]
for every \( i \) (where each \( R \) is spanned by the appropriate identity element).

\( S_n(\Sigma) \) is not the tensor product of the algebras \( S_n(U'(a_1)), \ldots, S_n(U'(a_n)) \) because elements of different component algebras do not necessarily commute in \( S_n(\Sigma) \).

We have seen in Examples 7.8 and 7.14 already that the skein algebra of the ideal triangle, \( S_n(\Sigma) \) is the braided tensor product \( S_n(\mathcal{B}) \otimes S_n(\mathcal{B}) \). We will see in the next subsection however that our stated skein algebras are not braided products of their component algebras, \( S_n(U'(a_1)), \ldots, S_n(U'(a_r)) \) in general.
7.9. Torus with an arc boundary. Let us apply the approach of the above section to analyze the skein algebra of the torus with an arc boundary, $\mathcal{S}_n(\Sigma_{1,1}^*)$. Fig. 19 shows a torus (in black) with a saturated arc collection: $a_1$ in red and $a_2$ in blue.

By Corollary 7.20,

$$\text{mult}_A : \mathcal{S}_n(\Sigma_{0,2}^*) \otimes \mathcal{S}_n(\Sigma_{0,2}^*) \to \mathcal{S}_n(\Sigma_{1,1}^*), \quad \text{mult}(x \otimes y) = \begin{array}{c}
\end{array}$$

is an $R$-linear isomorphism and an algebra homomorphism on each of the components algebras.

We described a method of finding an algebraic presentation of $\mathcal{S}_n(\Sigma_{0,2}^*)$ in Subsec. 7.5.

The above discussion allows for an algebraic description of the product on $\mathcal{S}_n(\Sigma_{0,2}^*) \otimes \mathcal{S}_n(\Sigma_{0,2}^*)$ (induced from $\mathcal{S}_n(\Sigma_{1,1}^*)$ by $\text{mult}_A$) as follows: By the construction of $\text{mult}_A$,

$$(x \otimes 1) \cdot (x' \otimes 1) = (x \cdot x') \otimes 1, \quad (1 \otimes y) \cdot (1 \otimes y') = 1 \otimes (y \cdot y'), \quad (x \otimes 1) \cdot (1 \otimes y) = x \otimes y$$

in $\mathcal{S}_n(\Sigma_{1,1}^*)$ for $x \in \mathcal{S}_n(U'(a_1))$ and $y \in \mathcal{S}_n(U'(a_2))$. Therefore, to complete the algebraic description of the product in $\mathcal{S}_n(\Sigma_{1,1}^*)$ it remains to consider

$$(1 \otimes y) \cdot (x \otimes 1) = \begin{array}{c}
\end{array}$$

Denoting this diagram by $z$, and applying the identity \text{(94)}, where $\Delta$ is our $\mathcal{O}_q(SL(n))$-coproduct taken with respect to the dashed line, we see that

$$(1 \otimes y) \cdot (x \otimes 1) = \sum x(2) \cdot y(2) \cdot T(x(1), x(3), y(1), y(3)),$$

where $T(x(1), x(3), y(1), y(3))$ is the counit value of the tangle in $\mathcal{B}$ cut off from the diagram above by the dashed line.

The above formulas completely determine the multiplication in $\mathcal{S}_n(\Sigma_{1,1}^*)$ and allow for writing a finite presentation of $\mathcal{S}_n(\Sigma_{1,1}^*)$ in terms of generators and relators. Note that $\text{mult}_A$ in this case is not a braided tensor product of the component algebras $\mathcal{S}_n(U'(a_i))$, for $i = 1, 2$, in the sense of \cite{Maj}. For $R = \mathbb{k}(q)$, the skein algebra $\mathcal{S}_n(\Sigma_{1,1})$ is given by a semi-direct product $U_q(sl(n)) \ltimes \otimes \mathcal{O}_q(SL(n))$ and is called the “elliptic double” of $U_q(sl(n))$, and also the “algebra of quantum differential operators on $SL(n, \mathbb{k})$”, cf. \cite{BBJ} Sec. 6.4.
Remark 7.21. The finite presentations of $S_n(\Sigma_{0,2})$ and $S_n(\Sigma_{1,1}^*)$ (discussed in Subsec. 7.5 and above) induce finite presentations of algebras $S_n(\Sigma_{g,p}^*)$ for all $g \geq 0$, $p > 0$ by the method of Example 7.9.

Furthermore, Corollary 7.20 allows for a generalization of the above method to provide a finite presentation of $S_n(\Sigma)$ in terms of generators and relators for every essentially bordered surface $\Sigma$.

8. Kernel and image of the splitting homomorphism.

8.1. Kernel of the Splitting homomorphism. Suppose $\Sigma$ is a connected pb surface with an ideal point $p$ and a trivial ideal arc $c_p$ at $p$. Then $\text{cut}_{c_p} \Sigma$ is the disjoint union of a monogon $M$ and of a new pb surface $\Sigma_p$ which has $c_p$ as its boundary edge, see Figure 20. Let $K_p(\Sigma)$ be the kernel of the composition 

$$\Theta_p : S_n(\Sigma) \xrightarrow{\Theta_{c_p}} S_n(\Sigma_p) \otimes_R S_n(\mathcal{M}) \xrightarrow{\cong} S_n(\Sigma_p).$$

Explicitly $\Theta_p$ is given as follows. Any stated $n$-web $\alpha$ over $\Sigma$ can be isotoped so that it is disjoint from $c_p$ and, hence, lying in $\Sigma_p$. Then $\Theta_p(\alpha) = \alpha$ as elements of $S_n(\Sigma_p)$.

![Figure 20. From $\Sigma$ to $\Sigma_p$. Here $p$ is an interior ideal point. The picture when $p$ is a boundary ideal point is similar.](image)

Theorem 8.1. For any two ideal points $p$ and $p'$ of a connected punctured bordered surface $\Sigma$ we have $K_p(\Sigma) = K_{p'}(\Sigma)$.

Proof. Assume the two trivial ideal arcs $c_p$ and $c_{p'}$ are disjoint. By splitting both $c_p$ and $c_{p'}$ we get two monogons and a pb surface $\Sigma_{p,p'}$. Let $\Theta : S_n(\Sigma) \to S_n(\Sigma_{p,p'})$ be the composition of the two splittings, first along $c_p$ and then along $c_{p'}$. Since the second one is injective by the above observation, we have $\ker \Theta_p = \ker \Theta$. By switching the order of the splitting we have $\ker \Theta_{p'} = \ker \Theta$. Thus $K_p = K_{p'}$. □

We denote this common ideal by $K(\Sigma)$. The quotient $\tilde{S}_n(\Sigma) := S_n(\Sigma)/K(\Sigma)$ is called the projected stated skein algebra of $\Sigma$. By Proposition 7.11, $K_p$ is trivial and $\tilde{S}_n(\Sigma) = S_n(\Sigma)$ if $\partial \Sigma \neq \emptyset$.

Corollary 8.2. The splitting homomorphism descends to an injective algebra homomorphism 

$$\Theta_c : S_n(\Sigma) \to S_n(\text{cut}_c \Sigma) = S_n(\text{cut}_c \Sigma).$$

Proof. The proof is similar to that of Theorem 8.1. Assume $c$ is disjoint from a trivial arc $c_p$. Since the compositions $\Theta_p \Theta_{c_p}, \Theta_{c_p} \Theta_c : S_n(\Sigma) \to S_n(\text{cut}_c(\Sigma_p))$ coincide and for both of them the second map is injective, $\ker \Theta_{c_p} = \ker \Theta_c$. □
Corollary 8.3. Conjecture 7.12 is equivalent to the projection \( S_n(\Sigma) \rightarrow \overline{S}_n(\Sigma) \) being an isomorphism. (And, hence, this projection is an isomorphism for \( n = 2 \) and \( 3 \).)

In the next subsection we will prove the following:

**Theorem 8.4.** For any \( \Sigma, p \) and \( c_p \) as above, \( \overline{S}_n(\Sigma) \) coincides with the subalgebra of \( S_n(\Sigma_p) \) coinvariant under the coaction \( \Delta_{c_p} : S_n(\Sigma_p) \rightarrow S_n(\Sigma_p) \otimes S_n(\mathcal{B}) \) at \( c_p \):

\[
\overline{S}_n(\Sigma) = \{ x \in S_n(\Sigma_p) : \Delta_{c_p}(x) = x \otimes 1 \}.
\]

**Remark 8.5.** Let \( \Sigma = \Sigma - \mathcal{P} \), where \( \mathcal{P} \) is a finite subset of compact surface \( \Sigma \), as in Subsec. 5.1. Generalizing the setup of Subsec. 7.7, consider a collection \( A \) of disjoint, oriented, arcs in \( \Sigma \), each with endpoints in \( \partial \Sigma \cup \mathcal{P} \), satisfying conditions (i) and (ii) above. Theorem 7.13 and the discussion of the projected stated skein algebra implies that such \( A \) defines an identification of \( \overline{S}_n(\Sigma) \) with \( O_q(SL(n))^{gr} \) and, hence, it determines a basis of \( \overline{S}_n(\Sigma) \).

### 8.2. The Image of the Splitting homomorphism.

Let \( c \) be an interior oriented ideal arc of a pb surface \( \Sigma \). Denote the two copies of \( c \) in \( \text{cut}_c \Sigma \) by \( a_1 \) and \( a_2 \). We have the splitting \( R \)-algebra homomorphism

\[
\Theta_c : S_n(\Sigma) \rightarrow S_n(\text{cut}_c \Sigma),
\]

and \( S_n(\text{cut}_c \Sigma) \) is a \( O_q(SL(n)) \)-bi-comodule with the right and left coactions

\[
\Delta_{a_1} : S_n(\text{cut}_c \Sigma) \rightarrow S_n(\text{cut}_c \Sigma) \otimes O_q(SL(n))
\]

\[
a_2 \Delta : S_n(\text{cut}_c \Sigma) \rightarrow O_q(SL(n)) \otimes S_n(\text{cut}_c \Sigma)
\]

respectively, where \( O_q(SL(n)) \) is identified with the skein algebra of the bigon directed by the orientation of \( c \). Recall that the Hochshild cohomology module is defined by

\[
HH^0(S_n(\text{cut}_c \Sigma)) = \{ x \in S_n(\text{cut}_c \Sigma) | \Delta_{a_1}(x) = \text{fl} \circ a_2 \Delta(x) \},
\]

where \( \text{fl} \) is the transposition

\[
\text{fl} : O_q(SL(n)) \otimes S_n(\Sigma) \rightarrow S_n(\Sigma) \otimes O_q(SL(n)), \quad \text{fl}(x \otimes y) = y \otimes x.
\]

**Theorem 8.6** (See [CL, KQ] for \( n = 2 \) and [Hi] for \( n = 3 \).) The image of \( \Theta_c \) is equal to \( HH^0(S_n(\text{cut}_c \Sigma)) \).

**Proof.** Since the image of \( \Theta_c \) is equal to the image of \( \Theta_c : S_n(\Sigma) \rightarrow S_n(\text{cut}_c \Sigma) \), we can work with projected skein algebras. More specifically, we will assume that one end \( v \) of \( c \) is a boundary ideal point of \( \Sigma \), since we can remove a disk from \( \Sigma \), adjacent to \( v \) and disjoint from \( c \), if necessary. We will present \( c, a_1, a_2 \) in \( \text{cut}_c \Sigma_{a_1 \Delta a_2} \) as in Fig. 21 with \( v \) in the bottom.

![Figure 21](image-url)
Let \((\text{cut}_c \Sigma)_{a_1 \land a_2}\) denote \((\text{cut}_c \Sigma)_{a_1 \triangle a_2} = \partial \Sigma\), for simplicity, where \(\partial \Sigma\) is the bottom edge of \(\Sigma\), as in Subsec. 7.4. Note that we can identify the image of \(S_n((\text{cut}_c \Sigma)_{a_1 \land a_2}) \rightarrow S_n((\text{cut}_c \Sigma)_{a_1 \triangle a_2})\) with \(S_n(\Sigma)\). We will use this identification below.

Let \(\nabla_{a_1, a_2} : S_n(\text{cut}_c \Sigma) \rightarrow S_n((\text{cut}_c \Sigma)_{a_1 \triangle a_2}), \quad \nabla_{a_1, a_2} = \text{glue}_{a_1, a_2} \circ \text{htw}_{a_2}^{-1}\).

It is an isomorphism by Propositions 4.11 and 7.6.

**Lemma 8.7.** (1) \(\nabla_{a_1, a_2} \Theta_c(S_n(\Sigma)) = S_n((\text{cut}_c \Sigma)_{a_1 \land a_2}) \cong S_n(\Sigma)\).

(2) \(\nabla_{a_1, a_2}\) restricted to \(\text{Im } \Theta_c\) is the inverse to \(\Theta_c\).

**Proof.** For every stated web diagram \(D\) on \(\Sigma\) we have

\[
\begin{array}{c}
\theta_d \\
\sum_{i_1, \ldots, i_k} a_1, a_2 \\
\nabla_{a_1, a_2} \\
\sum_{i_1, \ldots, i_k} \prod_{j=1} \gamma_{i_j}^{-1} \\
D
\end{array}
\]

with the last equality by Relation (49). Hence,

\[
\nabla_{a_1, a_2} \Theta_c = \text{id}_{S_n(\Sigma)}.
\]

Since \(\nabla_{a_1, a_2}\) restricted to \(\text{Im } \Theta_c\) is a bijection onto \(S_n(\Sigma)\), this identity implies (2). \(\square\)

For any collection of boundary edges \(X \subset \partial \Sigma\), let \(\text{Im } \iota(\Sigma - X)\) denote the image \(\iota_*(S_n(\Sigma - X))\) in \(S_n(\Sigma)\) of the homomorphism induced by \(\iota : \Sigma - X \hookrightarrow \Sigma\).

**Lemma 8.8.** Let \(b_1, b_2\) be two boundary components of \(\Sigma\) separated by a puncture. Then the embedding

\[
\text{Im } \iota(\Sigma - (b_1 \cup b_2)) \hookrightarrow \text{Im } \iota(\Sigma - b_1) \cap \text{Im } \iota(\Sigma - b_2)
\]

is onto.

**Proof.** Consider an arc \(\gamma\) parallel to \(b_1 \cup b_2\), as in Fig. 22. Then the following diagram

commutes:

\[
\begin{array}{c}
\text{Im } \iota(\Sigma - (b_1 \cup b_2)) \\
\downarrow \Theta_\gamma \\
\text{Im } \iota(\Sigma - b_1) \cap \text{Im } \iota(\Sigma - b_2)
\end{array}
\]

By Proposition 7.11, both homomorphisms \(\Theta_\gamma\) in the diagram are 1-1 and, hence, it is enough to show that the embedding in the bottom line is onto. Since the skein algebra a surface is the tensor product of its connected components, it is enough to show the statement of the lemma for the triangle \(\Sigma\) bounded by \(b_1, b_2\) and \(\gamma\). By Proposition 7.7, \(S_n(\Sigma)\) is isomorphic...
with $S_n(\mathcal{B}_1) \otimes S_n(\mathcal{B}_2)$, as an $R$-module, where $\mathcal{B}_i$ for $i = 1$ are disjoint bigons in $\mathcal{T}$ such that $b_i \subset \partial\mathcal{B}_i$, as in Fig. 22. Through that isomorphism the statement of the lemma reduces to

$$\mathcal{O}_q(SL(n)) \otimes R \cdot 1 \cap R \cdot 1 \otimes \mathcal{O}_q(SL(n)) = S_n(\mathfrak{M}) = R,$$

where 1 is the identify in $\mathcal{O}_q(SL(n))$ and $\mathfrak{M}$ is the monogon $\mathcal{T} - b_1 - b_2$. That follows from the fact that $R \cdot 1$ is a direct summand of $\mathcal{O}_q(SL(n))$, by \cite[Prop. 5.1.1]{Du}. □

Let us continue with the proof of the theorem. To prove that the inclusion

$$\Theta_c(S_n(\Sigma)) \subset \ker (\Delta_{a_1} - \text{fl} \circ_{a_2} \Delta)$$

is an equality, we will show that for any $y \in \ker (\Delta_{a_1} - \text{fl} \circ_{a_2} \Delta)$,

$$x = \nabla_{a_1,a_2}(y) \in S_n((\text{cut}_c \Sigma)_{a_1 \wedge a_2})$$

lies in $S_n((\text{cut}_c \Sigma)_{a_1 \wedge a_2})$. Then

$$\Theta_c(x) = y$$

by Proposition 8.7(2).

As mentioned above, it remains to be shown that $x \in S_n((\text{cut}_c \Sigma)_{a_1 \wedge a_2})$. Recall that $\Delta_{a_1}$ and $\text{fl}_{a_2} \circ \Delta$ map $y$ into $S_n(\text{cut}_c \Sigma) \otimes S_n(\mathcal{B})$, where the left and right edges of $\mathcal{B}$ are denoted by $e_l, e_r$, respectively.

Let $z = \Delta_{a_1}(y) = \text{fl} \circ_{a_2} \Delta(y)$. By Proposition 8.7(2),

$$\nabla_{a_1,b_1}(z) = \nabla_{a_1,b_1}\Delta_{a_1}(y) = y,$$

where $y$ at the end of the above equation is a skein in $\text{cut}_c \Sigma \sqcup \mathcal{B}/(a_1 = b_l)$ identified with $\text{cut}_c \Sigma$. Then

$$\nabla_{b_r,a_2} \nabla_{a_1,b_1}(z) = \nabla_{b_r,a_2}(y) = \nabla_{b_r,a_2}\Theta_c(x) = x,$$

by (96).

By Proposition 8.7(1), $\nabla_{a_1,b_1}\Delta_{a_1}(y)$ belongs to $S_n((\Sigma \sqcup \mathcal{B})_{a_1 \wedge b_l})$. By applying $\nabla_{b_r,a_2}$ to it, we see that

$$x \in S_n((\Sigma \sqcup \mathcal{B})_{a_1 \wedge b_l,b_r \wedge a_2})$$

Since nablas for disjoint pairs of edges commute,

$$x = \nabla_{b_r,a_2} \nabla_{a_1,b_1}(z)$$

and by an analogous argument

$$x \in S_n((\Sigma \sqcup \mathcal{B})_{a_1 \wedge b_l,b_r \wedge a_2})$$

Now the statement follows from Lemma 8.8. □

The construction of the inverse of the splitting map (Lemma 8.7) implies the following:

**Corollary 8.9.** For any union $C$ of ideal boundary arcs of $\Sigma$,

$$\Theta_c(S_n(\Sigma - C)) = \Theta_c(S_n(\Sigma)) \cap S_n(\text{cut}_c \Sigma - C) \text{ in } S_n(\text{cut}_c \Sigma).$$

**Proof.** The inclusion $\subset$ is obvious and the opposite inclusion $\supset$ is obtained by applying the inverse map to $\Theta_c$ of Lemma 8.7. □
Proof of Theorem 8.4:

\[ \overline{\mathcal{S}}_n(\Sigma) \subset \{ x \in \mathcal{S}_n(\Sigma) : \Theta_{c_p}(x) = x \otimes 1 \} \]

is obvious. The opposite inclusion, \( \supset \), is immediate for \( \Sigma = M \): in that case \( p \) is the only vertex of \( \mathcal{M} \), \( \mathcal{M}_{c_p} = \mathcal{B} \), where \( c_p = \partial_i \mathcal{B} \) and if \( \Delta_{c_p} x = x \otimes 1 \) for \( x \in \mathcal{M}_{c_p} \) then by applying \( \varepsilon \otimes 1 \) we obtain \( x = \varepsilon(x)1 \in R \).

More generally, let \( \Sigma \) be a disjoint union of an essentially bordered \( \Sigma' \) and of \( \mathcal{M} \) (with a vertex \( p \) and an arc \( c_p \) as above). Let \( \Delta_{c_p} x = x \otimes 1 \) for \( x \in \Sigma' \sqcup \mathcal{M}_{c_p} \). Then by Theorem 7.13, \( x \) can be written as \( x = \sum_{i=1}^{N} y_i \otimes z_i \), where \( y_1, \ldots, y_N \in \mathcal{S}_n(\Sigma') \) are linearly independent and \( z_1, \ldots, z_N \in \mathcal{S}_n(\mathcal{B}) \). Then \( \Delta_{c_p} z_i = z_i \otimes 1 \) for every \( i \) and, hence, \( z_1, \ldots, z_N \in R \). That concludes the proof of the inclusion \( \supset \) in that case.

Let \( \Theta_{c_p}(x) = x \otimes 1 \) now for some arbitrary \( \Sigma, p, c_p \) (as above) and \( x \in \mathcal{S}_n(\Sigma_p) \). We need to show that \( x \) lies in the image of \( \mathcal{S}_n(\Sigma_p - c_p) \) in \( \mathcal{S}_n(\Sigma_p) \).

Let \( c'_p \) be an arc in \( \Sigma_p \) parallel to \( c_p \), splitting \( \Sigma_p \) into \( \Sigma'_p \) and a bigon bounded by \( c_p \) and \( c'_p \). Then

\[ \Theta_{c_p} \Theta_{c'_p}(x) = \Theta_{c'_p} \Theta_{c_p}(x) = \Theta_{c_p}(x) \otimes 1. \]

Since \( \Theta_{c'_p}(x) \in \mathcal{S}_n(\Sigma'_p \sqcup \mathcal{M}_{c_p}) \) the previous case implies that \( \Theta_{c'_p}(x) \) is of the form \( y \otimes 1 \in \mathcal{S}_n(\Sigma'_p) \otimes \mathcal{S}_n(\mathcal{M}_{c_p}), \) for some \( y \in \mathcal{S}_n(\Sigma'_p) \). By Corollary 8.9 above for \( \mathcal{C} = c_p \), we have \( \Theta_{c'_p}(x) \in \Theta_{c'_p}(\mathcal{S}_n(\Sigma_p - c_p)) \). Since \( \Theta_{c'_p} \) is 1-1, \( x \) lies in \( \mathcal{S}_n(\Sigma_p - c_p) \). \( \square \)

9. Relation to factorization homology, skein categories, and lattice gauge theory

9.1. Factorization homology. Factorization homology was introduced by Beilinson and Drinfeld [BD] in the setting of conformal field theory and then in [Lu, AF, AFT] in the topological context. Given an algebraic object \( \mathcal{A} \) called an \( E_n \)-algebra, it associates to oriented \( n \)-dimensional manifolds (with boundary) \( M \) categories \( \int_M \mathcal{A} \), which are linear over a certain ring of coefficients \( R \).

For \( n = 2 \), the notion of \( \mathcal{E}_2 \) is equivalent to that of a balanced braided category. Important examples of such categories are the categories of finite dimensional representations of quantum groups \( U_q(\mathfrak{g}) \). The factorization homology of surfaces for these categories was studied in [BBJ]. The authors proved in it that if \( \partial \Sigma = S^1 \) then \( \int_M \mathcal{A} \) is equivalent to the category of left modules over a certain algebra \( A_{\Sigma} \) (depending on the Lie algebra \( \mathfrak{g} \) as well).

Theorem 9.1. Let \( R = \mathbb{k}(q) \) for a field \( \mathbb{k} \) and let \( \mathcal{E}_2 \) be the category of type 1 finite dimensional representations of \( U_q(\mathfrak{sl}(n)) \) (over \( R \)). Then \( A_{\Sigma, p, g} \) is isomorphic to \( \mathcal{S}_n(\Sigma^*_{p, g}) \) (as \( R \)-algebras) for every \( p \geq 1, g \geq 0 \).

Our theory of stated skein modules generalize the algebras \( A_{\Sigma} \) as they are defined over non-field rings of coefficients, for all 3-dimensional manifolds, with multiple boundary components and multiple markings. The existence of our stated skein algebras over \( \mathbb{Z}[q^{\frac{1}{n}}] \) will allow for further extensions of this theory – specifically, a construction of embeddings of our stated skein algebras into quantum tori which generalize the quantum trace map of Bonahon-Wang and of the first author, [BW, LE2].
Additionally, our theory is very concrete, unlike the factorization homology and the definition of algebras $A_{\Sigma}$. Specifically, the definition of factorization homology in dimension 2 involves the $(\infty,1)$-category $\text{Mfld}^2$ whose objects are oriented surfaces (with boundary), morphisms are given by their embeddings, 2-morphisms are isotopies between embeddings and higher order morphisms are isotopies between them. It has a monoidal structure given by disjoint embeddings. This category has a full subcategory $\text{Disc}^2$ consisting of disks (with partial boundaries) and their embeddings. One can prove that any pivotal ribbon category defines a symmetric monoidal functor into a symmetric monoidal $(\infty,1)$-category $C$ whose objects are certain presentable categories and its monoidal structure is given by the categorical product. Then $\int_M A$ is the left Kan extension

$$
\begin{array}{c}
\text{Disc}^2 \\
\downarrow \\
\text{Mfld}^2
\end{array} 
\xrightarrow{F} 
\begin{array}{c}
C \\
\downarrow \\
\int_M A
\end{array}
$$

which, in more concrete terms, is a certain colimit in $C$ over all possible embeddings of collections of disks into a given surface.

Finally, the proof of the existence of the algebra $A_{\Sigma}$ for which category of left $A_{\Sigma}$-modules coincides with $\int_M A$ is abstract and non-constructive.

**Proof of Theorem 9.1.** Let $g = sl(n)$. Then the algebra $\mathfrak{g}_A$ of $[BBJ]$ is isomorphic with $S_n(\Sigma^*_0,2)$, cf. $[BBJ$, Sec. 6.1] Furthermore, one can see that $A_{\Sigma,1,1}$ coincides with $S_n(\Sigma^*_1,1)$, by comparing the "gluing pattern" of $\Sigma_{1,1}$ in $[BBJ]$ Thm. 5.11 with ours in Fig. 19(right) or by $[BBJ$, Cor. 6.8].

By $[BBJ]$ Thm. 5.11, $A_{\Sigma,p,g}$ is the braided tensor product of $p−1$ copies of $A_{\Sigma,0,2} = S_n(\Sigma^*_0,2)$ and $g$ copies of $A_{\Sigma,1,1} = S_n(\Sigma^*_1,1)$. Now the statement follows from Example 7.9.$\square$

### 9.2. Skein categories.

Skein categories are categorical analogous of skein algebras introduced by Walker and Johnson-Freyd, $[Wa$, p. 70], $[JF$, Sec. 9]. A framing of a point $p$ on surface $\Sigma$ is a choice of a non-zero vector $v \in T_p \Sigma$. Let $\mathcal{V}$ be a ribbon category, linear over $R$. The ribbon category $\text{Rib}_v(\Sigma)$ has objects given by finite sets of framed, signed disjoint points of $\Sigma$. Its morphisms are $R$-linear combinations of ribbon graphs in $\Sigma \times [0,1]$ (in the sense of Reshetikhin-Turaev) whose edges are decorated with objects of $\mathcal{V}$ and coupons are decorated with intertwiners. The ends of a ribbon graph $\Gamma$ in $\Sigma \times \{0\}$ (respectively: in $\Sigma \times \{1\}$) determine the source and (respectively: the target) of the morphism $\Gamma$.

For an oriented arc $C$, the Reshetikhin-Turaev construction defines a functor $\text{RT} : \text{Rib}_v(C \times [0,1]) \to \mathcal{V}$. (This functor was denoted by $\text{RT}_0$ for the ribbon category $C_0$, of Subsec. 3.1)

The skein category, $\text{Sk}_v(\Sigma)$ is $\text{Rib}_v(\Sigma)$ modulo the relation on morphisms $\sum c_i \Gamma_i \sim 0$, whenever a restriction of $\sum c_i \Gamma_i$ to a certain cube $C \times [0,1] \times [0,1]$ is in the kernel of Reshetikhin-Turaev evaluation, $\text{RT}$.
Cooke proved that for the category $\mathcal{V}$ of finite dimensional representations of a quantum group $U_q(\mathfrak{g})$, the skein category of any surface coincides with its factorization homology $\int_{\Sigma} \mathcal{V}$. Furthermore, Ha proved that for $\mathfrak{g} = sl(2)$ and $R$ a field, the skein category of $\Sigma$ with $\partial \Sigma = S^1$ is equivalent to the category of left modules over $\mathcal{S}_2(\Sigma)$.

Note that this equivalence is quite non-intuitive, as skein categories are built of unstated ribbon graphs with ends in $\Sigma \times \{0, 1\}$ rather than of stated webs with ends in $\partial \Sigma \times (-1, 1)$ considered in stated skein algebras. We believe that the arguments of Ha generalize to stated $SL(n)$-skein algebras as well.

9.3. Lattice Gauge Theory, Quantum moduli spaces. A ciliated graph $\Gamma$ is a finite graph with an additional data specifying for each vertex of $\Gamma$ a linear order of half-edges adjacent to it. Each ciliated graph $\Gamma$ is ribbon and, hence, defines a surface which contracts onto $\Gamma$.

Inspired by an earlier Fock-Rosly’s work [FR], Alekseev-Grosse-Schomerus and Buffenoir-Roche quantized moduli spaces of flat connections on such surfaces in [AGS, AS, BR1, BR2]. Specifically, for each ciliated graph $\Gamma$ and a quantized coordinate Hopf algebra $O_q(G)$ they have defined an $O_q(G)$-comodule $\mathcal{L}(\Gamma)$, called quantum moduli space, quantizing a (properly defined) algebra of functions on the space of flat $G$-connections on $\Gamma$.

Let $\Sigma(\Gamma)$ be a surface without boundary realizing the ribbon structure on $\Gamma$ and let $\Sigma^0(\Gamma)$ be $\Sigma(\Gamma)$ with one of its punctures blown up into a disc, as in Fig. 20. (Hence, $\Sigma^0(\Gamma) = \Sigma(\Gamma)_p$ for some puncture $p$, in the notation of Subsec. 8.1. Note that $\Sigma^0(\Gamma)$ and $\Sigma(\Gamma)$ are uniquely determined up to a homeomorphism.) Then, as observed in [BBJ], the defining equations for $\mathcal{L}(\Gamma)$ coincide with those induced by the glueing patterns of [BBJ]. In other words, quantum moduli spaces coincide with the factorization homology of [BBJ] and, consequently, for $G = SL(n)$,

$$\mathcal{L}(\Gamma) \simeq \mathcal{S}_n(\Sigma^0(\Gamma))$$

as $O_q(G)$-comodule algebras. This result was observed independently by the first author and proved by Ko for $n = 2$.

By Theorem 8.4, the coinvariant subalgebra $\mathcal{L}(\Gamma)^{O_q(SL(n))}$ is isomorphic with our projected skein algebra $\mathcal{S}_n(\Sigma(\Gamma))$. This result generalizes the results of [Ko, BFK] for $n = 2$.

10. Relation to other known cases

10.1. Compatibility with stated Kauffman bracket skein modules of 3-manifolds.

The stated Kauffman bracket skein algebras (of surfaces) of the first author [Lê1] were generalized to stated skein modules of marked 3-manifolds in [BL] (cf. also [LY]). We are going to prove that these modules are isomorphic with our $SL(2)$-skein modules, $\mathcal{S}_2(M, \mathcal{N})$.

To relate these modules to ours, let us replace the variable $q$ of [Lê1] with $q^{1/2}$ and denote the resulted stated Kauffman bracket skein module by $\mathcal{S}(M, \mathcal{N})_{q^{1/2}}$. Let a framed link in $(M, \mathcal{N})$ be any non oriented 2-webs without sinks nor sources, stated by sings $\pm$. By definition $\mathcal{S}(M, \mathcal{N})_{q^{1/2}}$ is the $R$-module freely spanned isotopy classes of framed links subject to the relations [97], [100] below.

**Theorem 10.1.** Suppose $(M, \mathcal{N})$ is a marked 3-manifold.

1. There is a unique $R$-linear isomorphism $\Lambda : \mathcal{S}(M, \mathcal{N})_{q^{1/2}} \rightarrow \mathcal{S}_2(M, \mathcal{N})$ which maps framed links $\alpha$ to stated 2-webs by assigning arbitrary orientations to them, and changing
the minus state to 1 and the plus state to 2.

(2) The splitting homomorphism of \([\mathfrak{Le1}, \mathfrak{BL}]\) coincides with ours through \(\Lambda\).

Proof of Theorem 10.1: Let \(\mathcal{L}(M, N)\) be the set of all stated framed links in \((M, N)\). First let us record the defining relations for \(S(M, N)_{q^{1/2}}:\)

\[
(97) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram1.png}}
\end{array}
\end{array}
= q^{1/2} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram2.png}}
\end{array}
\end{array} + q^{-1/2} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram3.png}}
\end{array}
\end{array}
\]
\]

\[
(98) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram4.png}}
\end{array}
\end{array}
= -(q + q^{-1}) \cdot \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram5.png}}
\end{array}
\end{array}
\]
\]

\[
(99) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram6.png}}
\end{array}
\end{array} = 0, \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram7.png}}
\end{array}
\end{array} = q^{1/4}, \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram8.png}}
\end{array}
\end{array} = -q^{5/4} \text{ (by [Le1] (18)).}
\]
\]

\[
(100) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram9.png}}
\end{array}
\end{array} = q^{-1/4} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram10.png}}
\end{array}
\end{array} - q^{-5/4} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram11.png}}
\end{array}
\end{array}.
\]
\]

(This last equality is a consequence of applying a half-twist to \([\mathfrak{Le1}] (13)].

For convenience, we draw diagrams with the arrow down, rather than up as in \([\mathfrak{BL}, \mathfrak{Le1}]\), to make them compatible with the skein relations of our \(S(M, N)\). Since the half-twist is an invertible operation, they form an alternative set of defining skein relations of the stated skein module of \([\mathfrak{Le1}, \mathfrak{BL}]\).

On the other hand, for \(n = 2\) our skein relations are:

\[
(101) \quad q^{1/2} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram12.png}}
\end{array}
\end{array} - q^{-1/2} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram13.png}}
\end{array}
\end{array} = (q - q^{-1}) \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram14.png}}
\end{array}
\end{array}
\]
\]

\[
(102) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram15.png}}
\end{array}
\end{array} = -q^{3/2} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram16.png}}
\end{array}
\end{array}
\]
\]

\[
(103) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram17.png}}
\end{array}
\end{array} = -(q + q^{-1}) \cdot \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram18.png}}
\end{array}
\end{array}
\]
\]

\[
(104) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram19.png}}
\end{array}
\end{array} = -q \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram20.png}}
\end{array}
\end{array} + q^{1/2} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram21.png}}
\end{array}
\end{array}
\]
\]

\[
(105) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram22.png}}
\end{array}
\end{array} = q^{-5/4} \left( \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram23.png}}
\end{array}
\end{array} - q \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram24.png}}
\end{array}
\end{array} \right)
\]
\]

\[
(106) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram25.png}}
\end{array}
\end{array} = 0, \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram26.png}}
\end{array}
\end{array} = -q^{5/4}, \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram27.png}}
\end{array}
\end{array} = q^{1/4}
\]
\]

\[
(107) \quad \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram28.png}}
\end{array}
\end{array} = -q^{-5/4} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram29.png}}
\end{array}
\end{array} + q^{-1/4} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{diagram30.png}}
\end{array}
\end{array}
\]
\]

(By Proposition 7.5, Relation (50) is redundant.)
Lemma 10.2. The value of any framed link \( T \) in \((M,N)\) considered as a 2-web in \( S_2(M,N) \) does not depend on the orientation of \( T \).

Proof. By (104), (102) and (103) we have

\[
(108) \quad q(q + q^{-1}) + q^{1/2} = .
\]

For any arc with a 2-vertex near its end we have

\[
(109) \quad .
\]

Hence, for any arc we have,

\[
 j - i = j - i = i - j
\]

(Both ends may lie on the same marking in \( N \).) Similarly, by (108), for loops we have

\[
\]

By Lemma 10.2, assigning arbitrary orientations of links defines a map \( \Lambda : RL(M,N) \rightarrow S_2(M,N) \) preserving relations (98)-(100). To see that (97) is preserved as well, we start with the following combinatorial observation:

We say that a crossing \( \) or \( \) in an unoriented framed link \( L \) in \((M,N)\) is of \(-\)type if \( L \) with that smoothing can be oriented so that it is a 2-web with only two 2-vertices, looking like \( \). We define a crossing of \(-\)-type analogously. It is straightforward to verify that every crossing is of one of these two types. (However, it can be of both types simultaneously, if the crossing involves an arc.)

Lemma 10.3. If a crossing \( \) or \( \) in \( L \) is of \(-\)type then for that smoothing of \( L \) we have \( \Lambda(\) = \(\).

Proof. There are two possibilities:

1. the NE end of is connected to the NW or the SW end. Then the statement follows by introducing two 2-valent vertices as in (108).
2. the NE end is connected to the marking. Then one of the SE or SW ends must be connected to a marking and the statement follows by applying (109) twice near the markings.
Suppose that the crossing $\includegraphics{crossing}$ on the left side of (97) is of $\includegraphics{crossing}$-type. Then by the above lemma, $\Lambda$ maps that skein relation to

$$\includegraphics{crossing} = q^{1/2} \includegraphics{crossing} + q^{-1/2} \Lambda \left( \includegraphics{crossing} \right) = q^{1/2} \includegraphics{crossing} + q^{-1/2} \includegraphics{crossing}$$

which coincides with (104) in $S_2(M, N)$. The proof for a crossing of $\includegraphics{crossing}$-type is analogous.

Thus the $R$-linear map $\Lambda : \mathcal{S}(M, N)_{q^{1/2}} \to \mathcal{S}_2(M, N)$ is well-defined.

We prove that $\Lambda$ is an isomorphism by constructing its inverse: Consider first the map $R\mathcal{W}_2(M, N) \to \mathcal{S}(M, N)_{q^{1/2}}$ sending webs $\alpha$ to $(-1)^{|W_2(\alpha)|} \bar{\alpha}$, where $\bar{\alpha}$ is the result of forgetting the orientation and of smoothing all the 2-valent vertices. It is immediate to see that it factors through relations (101)-(107) into a homomorphism $S_2(M, N) \to \mathcal{S}(M, N)_{q^{1/2}}$. Since $\mathcal{S}(M, N)_{q^{1/2}}$ and $S_2(M, N)$ are spanned by links (i.e. webs with no sinks nor sources) and since $\Lambda$ and the above map $S_2(M, N) \to \mathcal{S}(M, N)_{q^{1/2}}$ are inverses of each other on links, the statement follows.

The proof of part (2) is straightforward. □

10.2. Compatibility with the SU(n)-skein modules. In this subsection we are going to show that for any 3-manifold $M$ and any $n$ our skein module $S_n(M, \emptyset)$ is isomorphic with the $SU(n)$-skein module introduced by the second author in [Si2]. That module is built of based $n$-webs in $M$ which are defined as our $n$-webs in $(M, \emptyset)$, except that the half-edges incident to any of their $n$-valent vertices are linearly ordered. We denote the set of all such webs up to isotopy by $\mathcal{W}_n^b(M)$. Let $S_n^b(M)$ be the quotient of the the $R$-module freely generated by $\mathcal{W}_n^b(M)$ subject to relations (16)-(19), which are the internal annihilators of the functor $R T_0$. For an invertible $u \in R$ let $S_n^b(M; u)$ be an $R$-module defined as $S_n^b(M)$, except that the right side of (19) is multiplied by $u$. From the definition we see that $S_n^b(M; u)$ is isomorphic to $S_n^b(M)$ via the map $\alpha \to u^{# \text{sinks}(\alpha)} \alpha$. The $SU_n$-skein module defined in [Si2] is actually $S_n^b(M; (-q)^{n(n-1)/2})$.

Given a based $n$-web $\alpha$, let $\alpha^o$ denote the underlying $n$-web in $S_n(M, N)$. Recall that every oriented 3-manifold has a spin structure.

**Theorem 10.4.** Let $M$ be an oriented 3-manifold.
(a) For odd, the operation $\alpha \to \alpha^o$ on based $n$-webs extends to an isomorphism $S_n^b(M) \cong S_n(M, \emptyset)$.
(b) Every spin structure on $M$ defines a function $s : \mathcal{W}_n^b(M) \to \{1, -1\}$ such that the map $\alpha \to f(\alpha) = (-1)^{s(\alpha)} \alpha^o$ induces a unique $R$-linear isomorphism $S_n^b(M) \cong S_n(M, \emptyset)$.

**Proof.** (a) For odd, Relations (20) which are consequences of the defining relations, (16)-(19), show that a based $n$-web $\alpha$, as an element of $S_n^b(M)$, is determined by $\alpha^o$. Furthermore, the defining relations (16)-(19) coincide with the defining relations (25)-(28).

(b) Let $n$ be even now. Fix a Riemannian metric on $M$ and let $UM$ be a principal $SO(3)$-bundle associated to the tangent bundle of $M$. A section at a point is the group $SO(3)$, which can be identified with the set of all ordered, positively oriented, orthonormal bases $(v_1, v_2, v_3)$ of the tangent space at the point. Any such ordered orthonormal basis is totally determined by the first two vectors. A smooth embedding $a : [0, 1] \to M$ equipped with a
normal vector field defines a lift \( \tilde{a} : [0, 1] \rightarrow UM \) where the first and the second vectors are respectively the velocity vector and the framing vector, normalized to have length 1. For a based \( n \)-web \( \alpha \) define \( s(\alpha) \in \{0, 1\} \) as follows: First isotope \( \alpha \) so that the framing is normal everywhere, and at every \( n \)-valent vertex the \( n \) half edges have the same velocity vector. The latter condition implies the lift of the endpoints at all the half-edges at an \( n \)-valent vertex are the same. As \( n \) is even the lifts of all edges of \( \alpha \) and of all its circle components form a \( \mathbb{Z}/2 \) one-cycle \( \tilde{\alpha} \) of \( UM \). Recall that a spin structure \( s \) of \( M \) can be identified with a cohomology class in \( H^1(UM, \mathbb{Z}/2) \) which is non-trivial at the section at every point of \( M \). Let \( \tilde{s}(\alpha) \) be the evaluation of the spin structure, considered as an element of \( H^1(UM, \mathbb{Z}/2) \) on \( \tilde{\alpha} \). Clearly \( \tilde{s}(\alpha) \) depends only on the isotopy class of \( \alpha \). From the definition, \( \tilde{s}(\alpha) = 1 \) if \( \alpha \) is the trivial loop. If \( \alpha' \) is the result of adding a positive twist to an edge or loop of \( \alpha \) then \( \tilde{s}(\alpha') = -\tilde{s}(\alpha) \).

Hence it extends to a well-defined \( R \)-linear map \( f : S_b^n(M) \rightarrow S_n(M, \emptyset) \).

For the inverse, note that Relations of (20) show that the map \( RW_n(M, \emptyset) \rightarrow S_n^b(M) \) sending \( \alpha \) to \( (-1)^{s(\alpha)}\alpha \) for every based \( n \)-web \( \alpha \) is well-defined. Since \( \tilde{f} \) maps the defining relations (25)-(28) to the defining relations (16)-(19), it descends to a well-defined \( R \)-linear map from \( S_n^b(M, \emptyset) \) to \( S_n(M) \), which is the inverse of \( f \). \( \square \)

10.3. Compatibility with Higgins’ SL3 skein algebras. In his recent work [H] Higgins introduced his version of stated \( SL_3 \)-skein algebras, denoted by \( S_{SL_3}^q(\Sigma) \), of punctured bordered surfaces \( \Sigma \). His skein algebra is the \( R \)-module freely generated by 3-webs stated by \( -1, 0, 1 \), subject to his system of skein relations.

Let us identify Higgins’ states 1, 0, −1 of Higgins with our states 1, 2, 3, respectively.

Theorem 10.5. For any punctured bordered surface \( \Sigma \) there is an isomorphism \( \phi : S_{SL_3}^q(\Sigma) \rightarrow S_3(\Sigma) \) sending every stated 3-web \( \alpha \) to

\[
q^{\frac{3}{2} |V_3(\alpha)|} \cdot \prod_{i=1}^k (q^4(-q)^{-s_i}) \cdot \alpha,
\]

where \( s_1, \ldots, s_k \in \{-1, 0, 1\} \) are Higgins’ states of all edges coming out of the boundary and \( V_3(\alpha) \) is the set of the 3-valent sources and sinks of \( \alpha \).

Hence, \( \phi \) maps Higgins’ \( \begin{array}{c} \text{ } \\ \rightarrow \end{array} s \) to our \( \begin{array}{c} \text{ } \\ \rightarrow \end{array} 2-s \), for \( i = -1, 0, 1 \), and his \( \begin{array}{c} \rightarrow \\ \text{ } \end{array} s \) to our \( \begin{array}{c} \rightarrow \\ \text{ } \end{array} (-q)^{-s}q^s \).

Furthermore, our theory recovers most of Higgins’ for \( n = 3 \). Specifically, Higgins constructed splitting homomorphisms for his skein algebras and an isomorphism \( S_{SL_3}^q(\Sigma) \cong \mathcal{O}_q(sl_3) \). It is straightforward to check that these maps coincide with ours through \( \phi \). Higgins also proved a version of Theorem 7.7 for \( n = 3 \). (However, additionally, he defined bases of his skein algebras \( S_{SL_3}^q(\Sigma) \) for all pb surfaces \( \Sigma \). There appears no easy generalization of
these bases to \( n > 3 \), as the relay on the confluence method of \([SW]\), which works for \( n = 2 \) and 3 only.)

**Proof of Theorem 10.5.** Let \( \mathcal{W}_3^H(\Sigma) \) be the set of all isotopy classes of 3-webs over \( \Sigma \) with states \(-1, 0, 1\). Consider an \( R \)-linear isomorphism \( \tilde{\phi} : R\mathcal{W}_3^H(\Sigma) \to R\mathcal{W}_3(\Sigma) \) given on stated 3-webs \( \alpha \) by (110).

By definition, the Higgins algebra \( S_{SL^3}(\Sigma) \) is \( R\mathcal{W}_3^H(\Sigma) \) modulo by his skein relations \([Hi_{(I1a)-(I4b),(B1)-(B4)}] \) which are the pull back under \( \tilde{\phi} \) of \([51]\) and its reflection, \([45], [82], [83], [54], \) and Relations (111) – (114) below, which we prove to hold in \( S_n(\Sigma) \) now.

By Identity (51),

\[
(111) \quad \begin{array}{c}
\includegraphics{111}
\end{array} = q^{\frac{1}{3}} \begin{array}{c}
\includegraphics{112}
\end{array} - q \begin{array}{c}
\includegraphics{113}
\end{array}.
\]

By capping the skeins of Eq. (111) from the top, we get

\[
(112) \quad \begin{array}{c}
\includegraphics{111}
\end{array} = q^{\frac{1}{3}} \begin{array}{c}
\includegraphics{112}
\end{array} - q \begin{array}{c}
\includegraphics{113}
\end{array} = (q^3 - q(q^2 + 1 + q^{-2})) \begin{array}{c}
\includegraphics{114}
\end{array} = -(q + q^{-1}) \begin{array}{c}
\includegraphics{115}
\end{array}.
\]

Taking the reflection of Eq. (111), we get

\[
(113) \quad \begin{array}{c}
\includegraphics{116}
\end{array} = q^{-\frac{1}{3}} \begin{array}{c}
\includegraphics{112}
\end{array} - q^{-1} \begin{array}{c}
\includegraphics{113}
\end{array}
\]

Hence, we have

\[
(114) \quad \begin{array}{c}
\includegraphics{117}
\end{array} = q^{\frac{1}{3}} \begin{array}{c}
\includegraphics{112}
\end{array} - q \begin{array}{c}
\includegraphics{113}
\end{array} = q^{\frac{1}{3}} \left( q^{-\frac{1}{3}} \begin{array}{c}
\includegraphics{112}
\end{array} - q^{-1} \begin{array}{c}
\includegraphics{113}
\end{array} \right) + q(q + q^{-1}) \begin{array}{c}
\includegraphics{114}
\end{array}.
\]

by Eq. (112).

Hence, we showed that \( \tilde{\phi} \) descends to an \( R \)-linear homomorphism \( \phi : S_{SL^3}(\Sigma) \to S_3(\Sigma) \). The definition of \( \phi \) suggests an obvious inverse homomorphism \( S_3(\Sigma) \to S_{SL^3}(\Sigma) \) and, indeed, one can verify that it is well defined. However, since checking that it respects our relation \([46]\) requires a lengthy calculation, we enclose an alternative proof of \( \phi \) being an isomorphism:

Since it is clearly a surjective algebra homomorphism, it remains to show that \( \phi \) is injective. From the definition it clear that \( \phi \) commutes with the splitting homomorphism. Since \( S_{SL^3}(\Sigma) \) satisfies the splitting homomorphism, \( S_{SL^3}(\Sigma) = \mathcal{O}_q(sl_3) \), and the gluing over a triangle is given by the same isomorphism as described in Theorem 7.7. Theorem 7.13 is also valid with \( S_3 \) replaced by \( S_{SL^3} \). Part (2) of Theorem 7.7 shows that \( \phi \) is an isomorphism when \( \Sigma \) is essentially bordered.
Suppose $\Sigma$ is a connected, having empty boundary, and at least one puncture. Let $c$ be an ideal arc of $\Sigma$. In the commutative diagram

$$
\begin{array}{ccc}
S_q^{SL^3}(\Sigma) & \xrightarrow{\Theta_c} & S_q^{SL^3}(\text{cut}_c \Sigma) \\
\downarrow\phi & & \downarrow\cong \\
S_3(\Sigma) & \xrightarrow{\Theta_c} & S_3(\text{cut}_c \Sigma)
\end{array}
$$

the upper $\Theta_c$ is injective by Higgins result, which forces $\phi$ to be injective.

Consider the remaining case when $\Sigma$ is a closed surface without ideal point. Remove a point $p$ from $\Sigma$ to obtained a pb surface $\Sigma'$ having one puncture. Since for both $S = S_3$ and $S = S_q^{SL^3}$ we have $S(\Sigma) = S(\Sigma')/\text{Rel}$, where $\text{Rel}$ is the relation $\infty = \infty$, we conclude that $\phi : S_q^{SL^3}(\Sigma) \to S_3(\Sigma)$ is an isomorphism. This completes the proof. \hfill $\Box$

10.4. Relation to the Frohman-Sikora SU(3)-skein algebras. Frohman and the second author considered in [FS] the “reduced $SU(3)$-skein algebra” of marked surfaces built of unstated 3-webs, subject to the $SU(3)$-skein relations of [Ku], extended by certain boundary skein relations, which depend on an invertible parameter $a \in R$. We denote that algebra by $S_{FS}(\Sigma, B)$ for the value 1 of that parameter.

For an unstated 3-web $\alpha$ in $\Sigma$, let $\eta_+ (\alpha)$ (respectively: $\eta_- (\alpha)$) denote $\alpha$ stated with 3’s (respectively: 1’s) at all its ends.

**Theorem 10.6.**

(1) For any punctured bordered surface $\Sigma$, the above operations extent to $R$-linear homomorphisms

$$
\eta_{\pm} : S_{FS}(\Sigma) \to S_3(\Sigma).
$$

(2) $\eta_{\pm}$ are embeddings and $\eta_{\pm}(S_{FS}(\Sigma))$ are direct summands of $S_3(\Sigma)$.

**Proof.** (1) $\eta_{\pm}$ maps the internal relations of [FS] to (45), (111)-(114), and the boundary relations (for $a_{FS} = 1$) to

$$
\begin{array}{c}
\begin{array}{c}
\downarrow i = 0 = \begin{array}{c}
\downarrow i \\
\uparrow i
\end{array}, \\
\begin{array}{c}
\downarrow i = 0 = \begin{array}{c}
\downarrow i \\
\uparrow i
\end{array}
\end{array}
\end{array}
\end{array}
$$

for $i = 1, 3$, which are satisfied by (48), (83), and (84).

(2) To prove that we identify $S_3(\Sigma)$ with Higgins’ skein algebra, through Theorem 10.5. Now it is easy to see that $\eta_{\pm}$ map the basis of reduced non-elliptic webs without British highways of [FS] 1-1 into the basis composed of irreducible webs of [Hi]. \hfill $\Box$

**Appendix A. Proof of Proposition 3.13 (A calculation of matrices of $X$.)**

We need to prove Identity (41). The generators The quantized enveloping algebra $U_q(sl_n)$ is generated by $E_i, F_i, K_i^{\pm 1}$ with relations given in [KS]. Its action on $V = \mathbb{Q}(q)^n$ with the standard basis $e_1, \ldots, e_n$ is given by

$$
E_i e_j = \delta_{i,j} e_{i+1}, \quad F_i e_j = \delta_{i,j+1} e_i, \quad K_i e_j = q^{\delta_{i,j+1}-\delta_{i,j}} e_j.
$$

Note that $e_n$ is the highest weight vector.
By definition \([ST]\), a half-ribbon element is an invertible element \(X \in U_q(\mathfrak{sl}_n)\) satisfying
\[
\mathcal{R} = (X^{-1} \otimes X^{-1}) \Delta(X), \quad \text{and} \quad X^2 = \vartheta, \text{the ribbon element.}
\]

In \([ST]\) Section 4, a half-ribbon element, denoted here by \(X_0\), was constructed based on work of Kirillov-Reshetikhin \([KR]\) and Levendorskii-Soibelman \([LS]\).

To calculate the action of \(X_0\), we use the following identities from \([ST]\) Lemma 3.10,
\[
X_0 F_i X_0^{-1} = -E_{n-i}, \quad X_0 K_i X_0^{-1} = K_{n-i}^{-1}, \quad X(T_{w_0}^{-1}(e_n)) = t_0^{1/2} e_n,
\]
where \(T_{w_0} \in U_q(\mathfrak{sl}_n)\) the quantum braid group element corresponding to the longest element \(w_0\) of the symmetric group \(S_n\) whose exact definition is not needed here.

Let \(\delta\) be the sum of all the fundamental weights. From (118) we get
\[
X_0 K_\delta X_0^{-1} = K_{\delta}^{-1}.
\]

It follows that \(X := K_{\delta}^{-1} X_0\) also satisfies (116) and hence is a half-ribbon element. Actually \(X\) is a half-ribbon element considered in \([KR, LS]\).

If \(x \in V\) has weight \(\lambda\), then \(T_{w_0}^{-1}(x)\) has weight \(w_0(\lambda)\). As each weight subspace of \(V\) is 1-dimensional and \(T_{w_0}\) is invertible, we have
\[
T_{w_0}^{-1}(e_n) = ce_1, \quad 0 \neq c \in \mathbb{Q}(v).
\]

By \([KT]\) Proposition 5.9, there are positive integers \(m_1, \ldots, m_k\) and a sequence \(i_1, \ldots, i_k \in \{1, \ldots, n-1\}\) such that
\[
F_i^{(m_1)} \ldots F_i^{(m_k)}(e_n) = T_{w_0}^{-1}(e_n) = ce_1, \quad \text{where} \quad F_i^{(m)} = F_i^m / [m]!
\]

As \(F_i^2 = 0\) on \(V\), all the \(m_j\) must be 1. Since \(F_i e_j\) is either 0 or another \(e_j\) by Eq. (115), we must have \(c = 1\). Hence \(T_{w_0}^{-1}(e_n) = e_1\), and Eq. (119) becomes
\[
X_0(e_1) = t_0^{1/2} e_n.
\]

Applying \(F_{n-1}\) to the above equation and using (117), we get \(X_0 e_2 = -t_0^{1/2} e_{n-1}\). Continue applying \(F_{n-2}, F_{n-3}, \ldots\) and using (117), we get
\[
X_0 e_j = (-1)^{j-1} t_0^{1/2} e_j.
\]

Since \(K_{\delta}^{-1}\) acts on \(V\) by \(K_{\delta}^{-1}(e_i) = q^{\frac{n+1}{2}-i} e_i\), we the matrix of the action of \(X\) on \(V\)
\[
X^i_j = \delta_{i,j} \left( (-1)^{n-i} t_0^{1/2} q^{\frac{n+1}{2}-i} \right) c_i.
\]

The action of \(X\) on a dual space is given by the antipode \(S\). By \([ST]\) Proposition 4.3 we have \(S(X) = g X\). From here one can easily calculate
\[
X e^i = c_i e^j = f^i.
\]

It follows that the action of \(X\) on \(V^*\) in the basis \(\{f^1, \ldots, f^n\}\) is given by \(X^i_j = \delta_{i,j} c_i\). This completes the proof of Identity (41).
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