Statistical null-controllability of stochastic nonlinear parabolic equations

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Abstract

In this paper, we consider forward stochastic nonlinear parabolic equations, with a control localized in the drift term. Under suitable assumptions, we prove the small-time global null-controllability, with a truncated nonlinearity. We also prove the “statistical” local null-controllability of the true system. The proof relies on a precise estimation of the cost of null-controllability of the stochastic heat equation and on an adaptation of the source term method to the stochastic setting. The main difficulty comes from the estimation of the nonlinearity in the fixed point argument due to the lack of regularity (in probability) of the functional spaces where stochastic parabolic equations are well-posed. This main issue is tackled through a truncation procedure. As relevant examples that are covered by our results, let us mention the stochastic Burgers equation in the one dimensional case and the Allen-Cahn equation up to the three-dimensional setting.

Keywords: Local null-controllability, observability, semilinear stochastic parabolic equations, stochastic source term method, Lebeau-Robbiano method.

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1 Introduction

1.1 Main result

Let \( T > 0 \) be a positive time, \( \mathcal{D} \) be a sufficiently smooth bounded, connected, open subset of \( \mathbb{R}^n \), with \( 1 \leq n \leq 3 \), whose boundary is denoted by \( \Gamma := \partial \mathcal{D} \) and \( \mathcal{D}_0 \) be a nonempty open subset of \( \mathcal{D} \). We introduce the notation \( \chi_{\mathcal{D}_0} \), for the characteristic function of the set \( \mathcal{D}_0 \).

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space on which a one-dimensional standard Brownian motion \( \{W(t)\}_{t \geq 0} \) is defined such that \( \{\mathcal{F}_t\}_{t \geq 0} \) is the natural filtration generated by \( W(\cdot) \) augmented by all the \( \mathbb{P} \)-null sets in \( \mathcal{F} \). Let \( X \) be a Banach space, for every \( p \in [1, +\infty] \), we introduce

\[
L^p_T(0, T; X) := \{ \phi : \phi \text{ is an } X\text{-valued } \mathcal{F}_t\text{-adapted process on } [0, T] \text{ and } \phi \in L^p([0, T] \times \Omega; X) \},
\]

endowed with the canonical norm and we denote by \( L^p_T(\Omega; C([0, T]; X)) \) the Banach space consisting on all \( X \)-valued \( \mathcal{F}_t \)-adapted process \( \phi(\cdot) \) such that \( \mathbb{E}\left(\|\phi(\cdot)\|_{C([0, T]; X)}^2\right) < \infty \), also equipped with the canonical norm.

We consider the stochastic semilinear heat equation

\[
\begin{aligned}
\frac{dy}{dt} &= (\Delta y + \chi_{\mathcal{D}_0} h + f(y, \nabla y))dt + (ay + g(y))dW(t) & \text{in } (0, T) \times \mathcal{D}, \\
y &= 0 & \text{on } (0, T) \times \Gamma, \\
y(0, \cdot) &= y_0 & \text{in } \mathcal{D}.
\end{aligned}
\]  

where \( a \in \mathbb{R} \) and \( f, g \) satisfy the following hypothesis.

Assumption 1.1. There exist \( \alpha, \beta, \gamma \in \mathbb{R} \) such that

\[
\forall(s, u) \in \mathbb{R} \times \mathbb{R}^n, \quad f(s, u) = \alpha s^p + \beta s^q u, \quad p > 1, \quad q \geq 1 \quad \text{for } n = 1,
\]

\[
= \alpha s^p, \quad p > 1 \quad \text{for } n = 2,
\]

\[
= \alpha s^p, \quad p \in (1, 3] \quad \text{for } n = 3,
\]

\[
g(s) = \gamma s^r, \quad r > 1 \quad \text{for } n = 1,
\]

\[
= 0, \quad \text{for } n = 2,
\]

\[
= 0, \quad \text{for } n = 3.
\]

In the controlled system (1), \( y \) denotes the state while \( h \) denotes the control, whose support is localized in \( \mathcal{D}_0 \). We are interested in the null-controllability at time \( T > 0 \) of (1), that is to say we wonder if there exists a control \( h \) such that the solution \( y \) of (1) satisfies \( y(T, \cdot) = 0 \) in \( \mathcal{D} \), a.s.

Before stating the main results of the paper, let us introduce some notations. First, we define

\[
\forall t \in [0, T), \quad \hat{\rho}(t) = \exp\left(-C/(T - t)\right),
\]

where the constant \( C > 0 \) will be defined later in the paper (see Section 2.4 below) and will only depend on \( \mathcal{D}, \mathcal{D}_0, a, p, q \) and \( r \).

We introduce the functional space: for every \( t \in [0, T] \),

\[
X_t := \left\{ y \in C([0, t]; H^1_0(\mathcal{D})) \cap L^2(0, t; H^2(\mathcal{D})) : \sup_{0 \leq s \leq t} \left\| \frac{y(s)}{\hat{\rho}(s)} \right\|_{H^1_0(\mathcal{D})} + \left( \int_0^t \left\| \frac{y(s)}{\hat{\rho}(s)} \right\|^2_{H^2(\mathcal{D})} ds \right)^{1/2} < +\infty \right\},
\]
endowed with the corresponding norm.

For each $R > 0$, defining $\varphi_R \in C^\infty_0(\mathbb{R}^+)$ such that

$$\varphi_R(s) = \begin{cases} 1, & s \leq R, \\ 0, & s \geq 2R, \end{cases} \quad \text{and} \quad \|\varphi_R\|_\infty \leq C/R,$$

we introduce the truncated semilinearities $f_R$ and $g_R$ defined as follows

$$\forall(t, x, y) \in [0, T] \times \mathcal{D} \times X_T, \quad f_R(t, x, y) = \varphi_R(\|y\|_{X_T})f(y(t, x), \nabla y(t, x)), \quad \text{and} \quad g_R(t, x, y) = \varphi_R(\|y\|_{X_T})g(y(t, x)).$$

For convenience, from now we will abridge the notation in $f_R(y, \nabla y), g_R(y)$ and we introduce the corresponding semilinear heat equation

$$\begin{cases} dy = (\Delta y + \chi_D h + f_R(y, \nabla y))dt + (ay + g_R(y))dW(t) & \text{in } (0, T) \times \mathcal{D}, \\ y = 0 & \text{on } (0, T) \times \Gamma, \\ y(0, \cdot) = y_0 & \text{in } \mathcal{D}. \end{cases}$$

Now, we state our two main results.

**Theorem 1.2.** Let $T > 0$. There exists $R > 0$ sufficiently small such that for every initial data $y_0 \in L^2(\Omega, \mathcal{F}_0; H^1(\mathcal{D}))$, there exists a control $h \in L^2_T(0, T; L^2(\mathcal{D}))$, such that the solution $y$ of (8) satisfies $y(T, \cdot) = 0$ in $\mathcal{D}$, a.s. Moreover, we have the following estimate

$$E\left(\|y\|_{X_T}^2\right) \leq C^2E\left(\|y_0\|_{H^1_0(\mathcal{D})}^2\right),$$

for a positive constant $C > 0$ depending only on $T$, $\mathcal{D}$, $\mathcal{D}_0$, $a$, $\alpha$, $\beta$, $p$, $q$.

**Theorem 1.3.** Let $\epsilon > 0$ and $T > 0$ be given. Then, there exists $\delta = \delta(\epsilon) > 0$ such that for every initial data $y_0 \in L^2(\Omega, \mathcal{F}_0; H^1(\mathcal{D}))$ verifying $\|y_0\|_{L^2(\Omega, \mathcal{F}_0; H^1(\mathcal{D}))} \leq \delta$, there exists a control $h \in L^2_T(0, T; L^2(\mathcal{D}_0))$ such that the solution $y$ of (8) satisfies $y(T, \cdot) = 0$ in $\mathcal{D}$, a.s.

$$\mathbb{P}\left(\{f_R(y, \nabla y) = f(y, \nabla y)\} \cap \{g_R(y) = g(y)\}\right) \geq 1 - \epsilon.$$ 

Before continuing, let us make some comments on Theorem 1.2 and Theorem 1.3.

- **Theorem 1.2** is a small-time global null-controllability result for the equation (8) which corresponds to (1), with truncated semilinearities $f_R$ and $g_R$. Remark that the parameter of truncation $R$ is taken sufficiently small then if the solution $y$ of (8) is too big in the space $X_T$, $f_R$ and $g_R$ vanish.

- **Theorem 1.3** is a “small-time statistical local null-controllability” for the equation (1). Indeed, we justify this new terminology as follows. Given any small time $T > 0$ and a small constant $\epsilon > 0$, we can find a ball of size $\delta > 0$ such that, with a confidence level $1 - \epsilon$, we can steer any initial data smaller than $\delta$ for system (1) to zero. One could compare Theorem 1.3 to the results obtained in [GHV14, Theorem 4.6] about global existence for stochastic Euler equations in the three dimensional case. Indeed, they prove that for every $\epsilon > 0$ and any given deterministic initial condition, the probability that particular solutions never blow up is bigger than $1 - \epsilon$. 

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• We may wonder if local null-controllability holds for (1), i.e. if there exists $\delta > 0$ such that for every initial data $y_0 \in L^2(\Omega, F_0; H^1(D))$, $\|y_0\|_{L^2(\Omega, F_0; H^1_0(D))} \leq \delta$, one can find a control $h \in L^2_T(0, T; L^2(D_0))$ such that the solution $y \in L^2_T(\Omega; C([0, T]; L^2(D)))$ of (1) satisfies $y(T, \cdot) = 0$ a.s. This is an interesting open question and new ideas have to be introduced in order to solve this problem.

• Related to the comments above, we shall emphasize that by our method, the existence and uniqueness of the solution to the uncontrolled equation (1) is not a prerequisite for studying its controllability. Actually, it is well-known that without imposing any growth, sign condition, or monotonicity condition on the nonlinear terms, the solutions may not exist globally and blow-up in finite time might occur (see, e.g., the seminal work [Par79], the newer references [Zha09, DKZ19], and the references within for some results and remarks in this direction). In turn, our method restricts to a truncated case and yields the existence of a solution in the weighted space $X_T$ which by construction implies the controllability constraint $y(T, \cdot) = 0$ in $D$, a.s.

• Among the physical examples that our results cover, let us quote the stochastic Allen-Cahn equation with $f(y) = y - y^3$, up to the change of variable $y \leftarrow e^{t}y$ and the Burgers equation in the one dimensional case, i.e. $f(y, \partial_x y) = -y \partial_x y$. We refer to [DPD99] where the optimal control of the stochastic Burgers equation is studied. Note also that the multiplicative noise term $g(y)dW(t)$ can represent the existence of external perturbations or a lack of knowledge of certain physical parameters. Its importance is well-known in physics and biology, see for instance, [KS10, MMQ11, WXZZ16, KY20].

• Let us mention an open problem that could be addressed in the future. Consider the stochastic Navier-Stokes equation for $n = 2, 3$,

\[
\begin{cases}
   dy = (\Delta y - y \cdot \nabla y - \nabla p + \chi_{D_0}h)dt + (ay)dW(t) & \text{in } (0, T) \times D, \\
   \text{div } y = 0 & \text{in } (0, T) \times D, \\
   y = 0 & \text{on } (0, T) \times \Gamma, \\
   y(0, \cdot) = y_0 & \text{in } D.
\end{cases}
\]

We may wonder if (11) is statistically locally null-controllable? To prove this type of result, a good strategy seems to first prove the null-controllability of the Stokes equation, with one control localized in the drift term, by combining the proofs in [CSL16] and [LÎ1]. Then, one could adapt the method present in this paper to deal with the nonlinear term $y \cdot \nabla y$. Difficulties will appear by estimating this nonlinear term due to the fact that $H^1(D)$ does not embed in $L^\infty(D)$ for $n \geq 2$. Probably, one should work in $W^{1,p}(D)$, which embeds in $L^\infty(D)$ for $p > 2$.

1.2 Bibliographical comments

In the deterministic setting, the (small-time) null-controllability of the heat equation has been proved independently in the seminal papers [LR95] and [F196]. Both proofs rely on Carleman estimates. The local null-controllability of semilinear parabolic equations is also established in [F196, Chapter 4] by a linearization argument. Then, in [FCZ00] and [Bar00], the global null-controllability for slightly superlinear heat equations is obtained.

The study of null-controllability of stochastic linear heat equations was first performed in [BRT03]. In particular, the authors remark that the null-controllability of forward equations was a challenging topic, this is why results have been established in different settings. In [TZ09],
the authors prove the null-controllability of forward parabolic equations by introducing two controls, one localized in the drift term and another in the diffusion term. This result was obtained thanks to Carleman estimate for backward parabolic equations. Then, in [L11], the control in the diffusion term is removed, assuming that the coefficients of the parabolic operator do not depend on the spatial variable. The strategy of obtaining such a result relies on the Lebeau-Robbiano method [LR95] adapted to the stochastic setting.

In the nonlinear setting, the result of null-controllability for semilinear parabolic equations was deemed as a difficult problem even for globally Lipschitz nonlinearities (see [TZ09, Remark 2.6]), due to the lack of compactness. Nonetheless, in our recent work [HSLBP20] we have overcome this difficulty by presenting a new Carleman estimate and a Banach fixed point procedure, where compactness is not needed. In spite of this new result, the question of how to address the controllability for semilinear equations where the global condition for the nonlinearity is dropped is still open. Due to the fact that maximal regularity arguments for stochastic parabolic equations give only regularity in time and space but not in probability, the nonlinearity is difficult to estimate in suitable spaces in a fixed-point procedure. For this reason, in this paper, we study some local controllability properties.

1.3 Strategy of proof

In this part, we explain the proof of Theorem 1.3 that we split into different main steps. Note that Theorem 1.2 would be actually a byproduct of the proof of Theorem 1.3.

- First, we linearize the equation (1) around 0 to obtain a stochastic (linear) heat equation. By [L11, Theorem 1.1], we know that this equation is small-time (globally) null-controllable. Moreover, by working a little bit more, we are able to prove that the cost of null-controllability in time $T > 0$, denoted by $C_T$, behaves as $C_T \leq \exp(C/T) > 0$ where the constant $C > 0$ depends on $D, D_0$ and $a$, see Section 2.1 below.

- Secondly, we employ an adaptation of the well-known source term method of [LTT13] to the stochastic setting in order to prove the null-controllability of the stochastic linear heat equation with a source term exponentially decreasing as $t \to T$, see Section 2.2 below. We remark that as a byproduct of this stochastic source term method is a new observability estimate for the backward heat equation, see Section 2.3 below.

- To conclude the proof of Theorem 1.3, the application of a Banach fixed point strategy is not straightforward. Indeed, maximal regularity arguments for stochastic parabolic equations give us only regularity in time and space, but not in probability. This leads to some trouble for estimating the semilinearity $f(y)$. This is why we first replace the semilinearity $f$ by the truncated nonlinearity $f_R$, defined in (6), for which we are able to perform a Banach fixed point argument for $R > 0$ sufficiently small. All of this actually leads to the proof of Theorem 1.2.

- The conclusion of Theorem 1.3 will follow from Theorem 1.2 and Markov’s inequality.

2 Null-controllability result for the linearized stochastic heat equation

2.1 An estimate of the control cost for the stochastic heat equation

For a given positive time $\tau > 0$, we introduce the notations $Q_\tau = D \times (0, \tau)$, $\Sigma_\tau = \Gamma \times (0, \tau)$. 
We linearize (1) around 0 and obtain
\[
\begin{cases}
dy = (\Delta y + \chi_{D_0} h) dt + (ay) dW(t) & \text{in } Q_T, \\
y = 0 & \text{on } \Sigma_T, \\
y(0, \cdot) = y_0 & \text{in } D.
\end{cases}
\] (12)

We have the following result.

**Proposition 2.1.** For every \( \tau > 0 \), \( y_0 \in L^2(\Omega, F_0; L^2(D)) \), there exists \( h \in L^2(0, \tau; L^2(D_0)) \) such that \( y(\tau) = 0 \) in \( D \), a.s. Moreover, we have the following estimate
\[
E \left( \int_{D_0 \times (0, \tau)} |h|^2 \, dx \, dt \right) \leq C_{\tau} E \left( \|y_0\|^2_{L^2(D)} \right),
\] (13)
where \( C_{\tau} = C e^{C/\tau} \) with a positive constant \( C > 0 \) only depending on \( D, D_0 \) and \( a \).

This result is actually already known in the literature. It was established in a slightly different framework in [Li11, Theorem 1.1] (see also [LL18, Theorem 1.1] for a more general case of coupled systems). Nonetheless, in such works, the control cost is not made explicit and in their current form the results are not suitable for our purposes.

Below, we give a description of the main parts needed to achieve the proof of Proposition 2.1 and we pay special attention at the end in the dependence of \( T \) for obtaining the constant \( C_T \). For this reason, in what follows, we always assume that \( T \in (0, 1) \).

**Proof of Proposition 2.1.** The proof is based on the classical Lebeau-Robbiano strategy introduced in [LR95] and for the sake of presentation we follow the methodology in [LB19, Section 3]. We split the proof in three main steps.

**Step 1: A controllability result for low frequencies.** We consider the unbounded linear operator in \( L^2(D) \) given by \((-\Delta, H^2(D) \cap H^1_0(D))\). Let \((\lambda_k)_{k \geq 1}\) and \((\phi_k)_{k \geq 1}\) be the corresponding eigenvalues and (normalized) eigenfunctions, i.e., \(-\Delta \phi_k = \lambda_k \phi_k\) and \((\phi_k, \phi_l) = \delta_{k,l}\). It is clear that \((\phi_k)_{k \geq 1}\) is an orthonormal basis of \( L^2(D) \). For \( \lambda > 0 \), we define the finite dimensional space \( E_\lambda = \left\{ \sum_{k \leq \lambda} c_k \phi_k : c_k \in \mathbb{R} \right\} \subset L^2(D) \) and we denote by \( \Pi_{E_\lambda} \) the orthogonal projection from \( L^2(D) \) to \( E_\lambda \).

The first part consists in obtaining an observability inequality for the adjoint system
\[
\begin{cases}
dz = -(\Delta z + \bar{z}) dt + \bar{z} dW(t) & \text{in } Q_T, \\
z = 0 & \text{on } \Sigma_T, \\
z(\tau, \cdot) = z_\tau \in E_\lambda & \text{in } D.
\end{cases}
\] (14)

The result is the following.

**Lemma 2.2.** There exists \( C > 0 \) such that for every \( \tau \in (0, T) \), \( \lambda \geq \lambda_1 \), and \( z_\tau \in L^2(\Omega, F_\tau; E_\lambda) \), the solution \( z \) to (14) satisfies
\[
E \left( \|z(0)\|^2_{L^2(D)} \right) \leq \frac{C}{\tau} e^{C\sqrt{\lambda}} E \left( \int_{D_0 \times (0, \tau)} |z|^2 \, dx \, dt \right).
\]

This result can be proved as in [Li11, Proposition 2.1] with only minor modifications, so we omit it. By means of a classical duality argument, Lemma 2.2 yields a partial controllability result for the forward system
\[
\begin{cases}
dy = (\Delta y + h) dt + y dW(t) & \text{in } Q_T, \\
y = 0 & \text{on } \Sigma_T, \\
y(0, \cdot) = y_0 & \text{in } D.
\end{cases}
\] (15)
Lemma 2.3. There exist constants $C, C_2 > 0$ such that for every $\tau \in (0, T)$, $\lambda \geq \lambda_1$, $y_0 \in L^2(\Omega, F_\tau; L^2(D))$, there exists a control $h_\lambda \in L^2_T(0, T; L^2(D_0))$ verifying

$$
\|h_\lambda\|_{L^2_T(0, T; L^2(D_0))} \leq \frac{C}{\tau} e^{C\sqrt{\lambda}} \mathbb{E} \left( \|y_0\|_{L^2(D)}^2 \right),
$$

such that the corresponding controlled solution $y$ to (15) satisfies

$$
\Pi_\lambda(y(\tau)) = 0, \quad \text{in } D, \quad \text{a.s.,}
$$

and

$$
\mathbb{E} \left( \|y(\tau\|_{L^2(D)}^2 \right) \leq \left( C_2 + \frac{C_2}{\tau} e^{C_2\sqrt{\lambda}} \right) \mathbb{E} \left( \|y_0\|_{L^2(D)}^2 \right).
$$

As in the previous case, Lemma 2.3 can be proved by following [L11, Proposition 2.2] with a few minor adjustments, so we skip the proof.

Step 2: The Lebeau-Robbiano iterative method. The second part of the method relies on a time-splitting iterative procedure (see e.g., [LRL12, Section 6.2]). Here, we will argue slightly different as compared to [L11, Section 3] which will allow us to track in a simple way the dependency of the constants with respect to $T$. In the remainder of this section, the constants $C, C', C_2, \ldots$, are independent of $T$ and may vary from line to line.

We split the time interval $[0, T] = \bigcup_{k \in \mathbb{N}} [a_k, a_{k+1}]$ where $a_k$ is defined recursively, i.e., $a_0 = 0$ and $a_{k+1} = a_k + 2T_k$, where $T_k = T/2^{k+2}$ with $k \in \mathbb{N}$. Also, for some constant $M > 0$ sufficiently large (which will be fixed later on), we define $\mu_k = M^{2k}$.

The control strategy can be roughly described as:

- **Active period.** If $t \in (a_k, a_k + T_k)$, we take the control $h_\lambda$ and the corresponding controlled solution $y$ to (15) provided by Lemma 2.3 where we select $\lambda = \mu_k$.

- **Passive period.** If $t \in (a_k + T_k, a_{k+1})$, we set $h \equiv 0$ and use the dissipation properties of the system.

In more detail, during the active period, we take $\lambda = \mu_k$ and by Lemma 2.3 we know that there exists $h_k := h_{\mu_k}$ such that

$$
\mathbb{E} \left( \|y(a_k + T_k)\|_{L^2(D)}^2 \right) \leq \left( C_2 + \frac{C_2}{T_k} e^{C_2\sqrt{\lambda}} \right) \mathbb{E} \left( \|y(a_k)\|_{L^2(D)}^2 \right)
$$

$$
\leq \frac{C_2}{T} e^{C_2\sqrt{\lambda}} \mathbb{E} \left( \|y(a_k)\|_{L^2(D)}^2 \right)
$$

and

$$
\Pi_{\mu_k} (y(a_k + T_k)) = 0, \quad \text{a.s.}
$$

In the passive period of control, we will prove that the solution decays exponentially and will provide a suitable bound with an explicit dependency of $T$. This point is different from [L11], where Itô’s formula and a direct computation is performed. Instead, we will use the properties of the heat semigroup $S(t) := e^{t\Delta}$, Burkholder-Davis-Gundy and Gronwall’s inequality to deduce the required inequality.

More precisely, for $t \in (a_k + T_k, a_{k+1})$, $h(t) \equiv 0$, so the solution to (15) writes as

$$
y(t) = S(t - a_k - T_k)y(a_k + T_k) + \int_{a_k + T_k}^t S(t - s)y(s)dW(s), \quad \text{a.s.}
$$
Then taking the $L^2$-norm and expectation on both sides, we get
\[
\mathbb{E} \left( \|y(t)\|^2_{L^2(D)} \right) \\
\leq C \mathbb{E} \left( \| S(t - a_k - T_k) y(a_k + T_k) \|^2_{L^2(D)} \right) + C \mathbb{E} \left( \left\| \int_{a_k + T_k}^t S(a_k + 1 - s) y(s) dW(s) \right\|^2_{L^2(D)} \right) \\
=: I_1 + I_2.
\]

(19)

We proceed to estimate $I_1$ and $I_2$. For the first term, using that $\|S(t)\psi\|_{L^2(D)} \leq e^{-\mu t} \|\psi\|_{L^2(D)}$ for all $\psi \in L^2(D)$ such that $\Pi_\mu(\psi) = 0$, we have from (18), that
\[
I_1 \leq C e^{-2\mu_k(t-a_k-T_k)} \mathbb{E} \left( \|y(a_k + T_k)\|^2_{L^2(D)} \right). 
\]

(20)

For the second one, using a Burkholder-Davis-Gundy type inequality (see e.g. [LR15, Thm. 6.1.2]) and the fact that $\|S(t)\|_{L^1(D)} \leq C$, we obtain
\[
I_2 \leq C \mathbb{E} \left( \sup_{\tau \in [a_k + T_k, t]} \left\| \int_{a_k + T_k}^\tau S(\tau - s) y(s) dW(s) \right\|^2_{L^2(D)} \right) \\
\leq C \int_{a_k + T_k}^t \mathbb{E} \left( \|S(a_k + 1 - s) y(s)\|^2_{L^2(D)} \right) ds \\
\leq C \int_{a_k + T_k}^t \mathbb{E} \left( \|y(s)\|^2_{L^2(D)} \right) ds. 
\]

(21)

Hence, using estimates (20)–(21) in (19) and employing Gronwall inequality, we deduce
\[
\mathbb{E} \left( \|y(t)\|^2_{L^2(D)} \right) \leq C e^{-2\mu_k(t-a_k-T_k)} \mathbb{E} \left( \|y(a_k + T_k)\|^2_{L^2(D)} \right) (1 + e^{C(t-a_k-T_k)}).
\]

Thus, particularizing the previous estimate with $t = a_{k+1}$, the identity $a_{k+1} = a_k + 2T_k$ and taking into account that $T \in (0, 1)$, we deduce the existence of a constant $C' > 0$ independent of $T$ such that
\[
\mathbb{E} \left( \|y(a_{k+1})\|^2_{L^2(D)} \right) \leq C' e^{-C'M2^{k+1}T_k} \mathbb{E} \left( \|y(a_k + T_k)\|^2_{L^2(D)} \right). 
\]

(22)

Noting that $2^{2k+1}T_k = 2^kT/2$, we obtain from (22) and (17)
\[
\mathbb{E} \left( \|y(a_{k+1})\|^2_{L^2(D)} \right) \leq C' \frac{C_2}{T} e^{-C'M2^kT + C_2\sqrt{M}2^k} \mathbb{E} \left( \|y(a_k)\|^2_{L^2(D)} \right)
\]
whence
\[
\mathbb{E} \left( \|y(a_{k+1})\|^2_{L^2(D)} \right) \leq \left( \frac{C_2}{T} \right)^{k+1} e^{\sum_{j=0}^k (-C'M2^kT + C_2\sqrt{M}2^k)} \mathbb{E} \left( \|y_0\|^2_{L^2(D)} \right) \\
\leq e^{C_2/T + (C_2\sqrt{M} - C'MT)2^{k+1}} \mathbb{E} \left( \|y_0\|^2_{L^2(D)} \right). 
\]

(23)

Once again, taking $M > 0$ large enough such that $C_2\sqrt{M} - C'MT < 0$ (for instance $M \geq 2(C_2/C'T)^2$), we deduce from (23) that $\lim_{k \to +\infty} \mathbb{E} \left( \|y(a_k)\|_{L^2(D)} \right) = 0$, which together with (18) implies $y(T) = 0$ in $D$, a.s.

**Step 3: Conclusion.** We define the control $h$ by gluing all the controls $(h_k)_{k \in \mathbb{N}}$. Notice that this control is an element of $L^2_T(0,T;L^2(D_0))$. Moreover, we have
\[
\|h\|^2_{L^2_T(0,T;L^2(D_0))} = \sum_{k=0}^{+\infty} \|h_k\|^2_{L^2_T(a_k,a_k+T_k;L^2(D_0))}.
\]
Using the estimate on the control (16) on each subinterval \((a_k, a_k + T_k)\) together with (23), we get
\[
\|h_k\|^2_{L^2_T(a_k, a_k + T_k; L^2(D_0))} \leq \frac{C}{T_k} e^{C\sqrt{M}T^2 + (C_2\sqrt{M} - C'\sqrt{MT})^2k} \mathbb{E} \left( \|y_0\|^2_{L^2(D)} \right)
\]
for \(k \geq 1\) and
\[
\|h_0\|^2_{L^2_T(0, T_0; L^2(D_0))} \leq \frac{C}{T_0} e^{C\sqrt{M}T} \mathbb{E} \left( \|y_0\|^2_{L^2(D)} \right).
\]
Therefore, using the three above estimates and recalling the definition of \(T_k\), we obtain
\[
\|h\|^2_{L^2_T(0, T; L^2(D_0))} \leq \left( C T^{-1} e^{C\sqrt{M}} + \sum_{k \geq 1} C_2^k T^{-1} e^{(C_2\sqrt{M} - C'\sqrt{MT})^2k} \right) \mathbb{E} \left( \|y_0\|^2_{L^2(D)} \right).
\]
Taking \(M\) large enough such that \(C_2\sqrt{M} - C'\sqrt{MT}/2 = -C''/T\), with \(C'' > 0\), we obtain the the above expression that
\[
\|h\|^2_{L^2_T(0, T; L^2(D_0))} \leq C e^{C/T} \int_0^{+\infty} \frac{\sigma}{T} e^{-C'' \sigma^2 T} d\sigma \mathbb{E} \left( \|y_0\|^2_{L^2(D)} \right) \leq C e^{C/T} \mathbb{E} \left( \|y_0\|^2_{L^2(D)} \right)
\]
which yields the desired result (13).

\[\square\]

2.2 Stochastic source term method

From Proposition 2.1, we have an estimate for the control cost \(C_T\) of the equation (12) where \(C_T = C e^{C/T}\) is defined in (13). Then we fix \(M > 0\) such that \(C_T \leq M e^{M/T}\) and we introduce the weight
\[
\forall t > 0, \ C(t) = M e^{M/t}.
\]
We introduce the notation
\[
s = \min(p, q + r + r) > 1,
\]
where \(p\) and \(q\) are defined in (2) and \(r\) is defined in (3).

Let \(Q \in (1, \sqrt{2})\) and \(P > Q^*/(2 - Q^*)\). We define the weights
\[
\forall t \in [0, T), \ \rho_0(t) := M^{-p} \exp \left( -\frac{MP}{(Q^*/2 - 1)(T - t)} \right),
\]
\[
\forall t \in [0, T), \ \rho(t) := M^{-1-p} \exp \left( -\frac{(1 + P)Q^* M}{(Q^*/2 - 1)(T - t)} \right).
\]

For appropriate source terms \(F, G \in L^2_T(0, T; L^2(D))\), we consider
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\text{d}y &= (\Delta y + \chi_D h + F) \text{d}t + (a y + G) \text{d}W(t) & \text{in } Q_T, \\
y &= 0 & \text{on } \Sigma_T, \\
y(0, \cdot) &= y_0 & \text{in } D.
\end{array} \right.
\end{aligned}
\]

We define associated spaces for the source term, the state and the control
\[
\mathcal{S} := \left\{ S \in L^2_T(0, T; L^2(D)) : \frac{S}{\rho} \in L^2_T(0, T; L^2(D)) \right\},
\]
\[
\mathcal{Y} := \left\{ y \in L^2_T(0, T; L^2(D)) : \frac{y}{\rho_0} \in L^2_T(0, T; L^2(D)) \right\},
\]
\[
\mathcal{H} := \left\{ h \in L^2_T(0, T; L^2(D)) : \frac{h}{\rho_0} \in L^2_T(0, T; L^2(D)) \right\},
\]

9
From the behaviors near \( t = T \) of \( \rho \) and \( \rho_0 \), we deduce that each element of \( \mathcal{S}, \mathcal{Y}, \mathcal{H} \) vanishes at \( t = T \).

We have the following result.

**Proposition 2.4.** For every \( y_0 \in L^2(\Omega, \mathcal{F}_0; L^2(\mathcal{D})) \) and \( F, G \in \mathcal{S} \), there exists a control \( h \in \mathcal{H} \) such that the corresponding controlled solution \( y \) to (12) belongs to \( \mathcal{Y} \). Moreover, there exists a positive constant \( C > 0 \) depending only on \( T, \mathcal{D}, \mathcal{D}_0, a, p, q \) and \( r \) such that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left\| \frac{y(t)}{\rho_0(t)} \right\|^2 \right) + \mathbb{E} \left( \int_0^T \int_{\mathcal{D}_0} \left| \frac{h(t)}{\rho_0(t)} \right|^2 \, dx \, dt \right) \leq C \mathbb{E} \left( \|y_0\|_{L^2(\mathcal{D})}^2 + \int_0^T \left[ \left\| \frac{F(t)}{\rho(t)} \right\|_{L^2(\mathcal{D})}^2 + \left\| \frac{G(t)}{\rho(t)} \right\|_{L^2(\mathcal{D})}^2 \right] \, dt \right). \tag{28}
\]

In particular, since \( \rho_0 \) is a continuous function satisfying \( \rho_0(T) = 0 \), the above estimate implies

\[ y(T) = 0 \quad \text{in } \mathcal{D}, \text{ a.s.} \]

**Proof.** In the following, the constants \( C > 0 \) can vary from line to line, they are independent of the parameters \( k \) and \( n \).

For \( k \geq 0 \), we define \( T_k = T - \frac{T}{Q_k^{2+r}} \). We easily have the following relation between the weights defined in (24), (25) and (26)

\[
\rho_0(T_{k+2}) = \rho(T_k) \gamma(T_{k+2} - T_{k+1}). \tag{29}
\]

For \( k \geq 0 \), we consider the equation

\[
\begin{cases}
\begin{aligned}
  dy_1 &= (\Delta y_1 + F)dt + (ay_1 + G) \, dW(t) & \quad \text{in } (T_k, T_{k+1}) \times \mathcal{D}, \\
y_1 &= 0 & \quad \text{on } (T_k, T_{k+1}) \times \Gamma, \\
y_1(T_k) &= 0 & \quad \text{in } \mathcal{D}.
\end{aligned}
\end{cases} \tag{30}
\]

We introduce the sequence of random variables \( \{a_k\}_{k \geq 0} \)

\[ a_0 = y_0 \text{ and } a_{k+1} = y_1(T_{k+1}). \]

For \( k \geq 0 \), we also consider the equation

\[
\begin{cases}
\begin{aligned}
  dy_2 &= (\Delta y_2 + \chi_{\mathcal{D}_0} \, h_k)dt + (ay_2) \, dW(t) & \quad \text{in } (T_k, T_{k+1}) \times \mathcal{D}, \\
y_2 &= 0 & \quad \text{on } (T_k, T_{k+1}) \times \Gamma, \\
y_2(T_k) &= a_k & \quad \text{in } \mathcal{D}.
\end{aligned}
\end{cases} \tag{31}
\]

Observe that due to the regularity of the solution of (30), each \( a_k, k \geq 0 \), is \( \mathcal{F}_{T_k} \)-measurable and belongs to \( L^2(\Omega \times \mathcal{D}) \). Hence, system (31) is well posed for each \( h_k \in L^2_T(\mathcal{T}k; T_{k+1}; L^2(\mathcal{D})) \) thanks to Lemma A.1.

According to Proposition 2.1, we can construct a control \( h_k \in L^2_T(\mathcal{T}k; T_{k+1}; L^2(\mathcal{D})) \) such that

\[ y_2(T_{k+1}) = 0, \quad \text{a.s.} \]

and the following estimates holds

\[
\mathbb{E} \left( \int_{T_k}^{T_{k+1}} \int_{\mathcal{D}_0} |h_k(x,t)|^2 \, dx \, dt \right) \leq \gamma^2(T_{k+1} - T_k) \mathbb{E} \left( \|a_k\|_{L^2(\mathcal{D})}^2 \right). \tag{32}
\]
By Lemma A.1 applied to (30), there exists $C_0 > 0$ such that
\[
E\left(\|a_{k+1}\|_{L^2(D)}^2\right) \leq C_0 E\left(\int_{T_k}^{T_{k+1}} \left[\|F(t)\|_{L^2(D)}^2 + \|G(t)\|_{L^2(D)}^2\right] dt\right)
\]
(33)

By using (32), (33), the fact that $\rho$ is a non-increasing, deterministic function and (29), we have
\[
E\left(\int_{T_{k+1}}^{T_{k+2}} \int_{D_0} \left|h_{k+1}(x,t)\right|^2 dx dt\right)
\]
\[
\leq C_0 \gamma^2 (T_{k+2} - T_{k+1}) E\left(\int_{T_k}^{T_{k+1}} \left[\left\|F(t)\right\|_{L^2(D)}^2 + \left\|G(t)\right\|_{L^2(D)}^2\right] dt\right)
\]
\[
\leq C_0 \gamma^2 (T_{k+2} - T_{k+1}) \rho^2 (T_k) E\left(\int_{T_k}^{T_{k+1}} \left[\left\|F(t)\right\|_{L^2(D)}^2 + \left\|G(t)\right\|_{L^2(D)}^2\right] dt\right)
\]
\[
\leq C_0 \gamma^2 (T_{k+2} - T_{k+1}) \rho^2 (T_k) E\left(\int_{T_k}^{T_{k+1}} \left[\left\|F(t)\right\|_{L^2(D)}^2 + \left\|G(t)\right\|_{L^2(D)}^2\right] dt\right)
\]
\[
\leq C_0 \rho_0^2 (T_{k+2}) E\left(\int_{T_k}^{T_{k+1}} \left[\left\|F(t)\right\|_{L^2(D)}^2 + \left\|G(t)\right\|_{L^2(D)}^2\right] dt\right),
\]
so we deduce
\[
E\left(\int_{T_{k+1}}^{T_{k+2}} \int_{D_0} \left|h_{k+1}(x,t)\right|^2 dx dt\right) \leq C_0 E\left(\int_{T_k}^{T_{k+1}} \left[\left\|F(t)\right\|_{L^2(D)}^2 + \left\|G(t)\right\|_{L^2(D)}^2\right] dt\right).
\]
(34)

Since $\rho_0$ is a non-increasing, deterministic function and using (34), we have
\[
E\left(\int_{T_{k+1}}^{T_{k+2}} \int_{D_0} \left|h_{k+1}(x,t)\right|^2 dx dt\right) \leq C_0 E\left(\int_{T_k}^{T_{k+1}} \left[\left\|F(t)\right\|_{L^2(D)}^2 + \left\|G(t)\right\|_{L^2(D)}^2\right] dt\right).
\]
(35)

Let $n \in \mathbb{N}^*$. From (35), we have
\[
E\left(\int_{T_k}^{T} \int_{D_0} \sum_{k=0}^{n} 1_{[T_{k+1}, T_{k+2})} (t) \left|h_{k+1}\right|^2 dx dt\right)
\]
\[
\leq C_0 E\left(\int_{T_k}^{T} \sum_{k=0}^{n} 1_{[T_{k}, T_{k+1})} (t) \left[\left\|F(t)\right\|_{L^2(D)}^2 + \left\|G(t)\right\|_{L^2(D)}^2\right] dt\right).
\]
(36)

From (32) at $k = 0$ and recalling that $a_0 = y_0$, we have
\[
E\left(\int_{D_0} \left|h_0\right|^2 dx dt\right) \leq \gamma^2 (T_1) E\left(\|y_0\|_{L^2(D)}^2\right),
\]
so
\[
E\left(\int_{D_0} \left|h_0\right|^2 dx dt\right) \leq \gamma^2 (T_1) \rho_0 (T_1) E\left(\|y_0\|_{L^2(D)}^2\right).
\]
(37)

Putting together (36) and (37) yields the existence of a constant $C > 0$ independent of $n$ such that
\[
E\left(\int_{D_0} \left|h_0\right|^2 dx dt\right) + E\left(\int_{T_k}^{T} \int_{D_0} \sum_{k=0}^{n} 1_{[T_{k+1}, T_{k+2})} (t) \left|h_{k+1}\right|^2 dx dt\right)
\]
\[
\leq CE\left(\|y_0\|_{L^2(D)}^2 + \int_{T_k}^{T} \sum_{k=0}^{n} 1_{[T_{k}, T_{k+1})} (t) \left[\left\|F(t)\right\|_{L^2(D)}^2 + \left\|G(t)\right\|_{L^2(D)}^2\right] dt\right).
\]
Finally, using Lebesgue’s convergence theorem, we can pass to the limit \( n \to \infty \) and obtain

\[
E \left( \int_0^T \frac{|h|^2}{\rho_0} \, dx \, dt \right) \leq C E \left( \|y_0\|^2_{L^2(D)} + \int_0^T \left[ \left\| \frac{F(t)}{\rho(t)} \right\|^2_{L^2(D)} + \left\| \frac{G(t)}{\rho(t)} \right\|^2_{L^2(D)} \right] \, dt \right). \tag{38}
\]

where we have set \( h := \sum_{k=0}^{\infty} h_k \).

Applying Itô’s rule to \( y := y_1 + y_2 \) for \( t \in [T_k, T_{k+1}) \), we get

\[
\begin{align*}
\begin{cases}
dy = (\Delta y + \chi_{D_0} h_k + F) \, dt + (y + G) \, dW(t), & \text{in } (T_k, T_{k+1}) \times \mathcal{D}, \\
y = 0, & \text{on } (T_k, T_{k+1}) \times \Gamma, \\
y(T_k) = a_k & \text{in } \mathcal{D}.
\end{cases}
\end{align*}
\]

(39)

Note that by construction \( y \) is continuous at \( T_k \) for all \( k \geq 0 \), therefore by using (39), \( y \) is a solution to (12).

On the other hand, by Lemma A.1 applied to (39), we have for \( k \geq 1 \)

\[
E \left( \sup_{T_k \leq t \leq T_{k+1}} \|y(t)\|^2_{L^2(D)} \right) \leq C_0 E \left( \|a_k\|^2_{L^2(D)} + \int_{T_k}^{T_{k+1}} \left[ \|\chi_{D_0} h_k(t)\|^2_{L^2(D)} + \|F(t)\|^2_{L^2(D)} + \|G(t)\|^2_{L^2(D)} \right] \, dt \right)
\]

where we have used that \( L^2(D) \subset H^{-1}(D) \). Then using (32) and (33) to estimate in the above equation yields

\[
E \left( \sup_{T_k \leq t \leq T_{k+1}} \|y(t)\|^2_{L^2(D)} \right) \leq C_0 \left[ \int_{T_k}^{T_{k+1}} \left[ \|F(t)\|^2_{L^2(D)} + \|G(t)\|^2_{L^2(D)} \right] \, dt \right) + (C_0 + \gamma^2 (T_{k+1} - T_k)) E \left( \|a_k\|^2_{L^2(D)} \right)
\]

\[
\leq C \gamma^2 (T_{k+1} - T_k) E \left( \int_{T_{k-1}}^{T_{k+1}} \left[ \|F(t)\|^2_{L^2(D)} + \|G(t)\|^2_{L^2(D)} \right] \, dt \right) \tag{40}
\]

From identity (29) we get

\[
E \left( \sup_{T_k \leq t \leq T_{k+1}} \|y(t)\|^2_{L^2(D)} \right) \leq C \gamma^2 (T_{k+1} - T_k) \rho_0^2 (T_{k-1}) E \left( \int_{T_{k-1}}^{T_{k+1}} \left[ \left\| \frac{F(t)}{\rho(t)} \right\|^2_{L^2(D)} + \left\| \frac{G(t)}{\rho(t)} \right\|^2_{L^2(D)} \right] \, dt \right)
\]

\[
\leq C \rho_0^2 (T_{k+1}) E \left( \int_{T_k}^{T_{k+1}} \left[ \left\| \frac{F(t)}{\rho(t)} \right\|^2_{L^2(D)} + \left\| \frac{G(t)}{\rho(t)} \right\|^2_{L^2(D)} \right] \, dt \right),
\]

so by using that \( \rho_0 \) is non-increasing, we have

\[
E \left( \sup_{T_k \leq t \leq T_{k+1}} \left\| \frac{y(t)}{\rho_0(t)} \right\|^2_{L^2(D)} \right) \leq C E \left( \int_{T_k}^{T_{k+1}} \left[ \left\| \frac{F(t)}{\rho(t)} \right\|^2_{L^2(D)} + \left\| \frac{G(t)}{\rho(t)} \right\|^2_{L^2(D)} \right] \, dt \right). \tag{40}
\]

Moreover, arguing as before, it is not difficult to establish that

\[
E \left( \sup_{0 \leq t \leq T_1} \left\| \frac{y(t)}{\rho_0(t)} \right\|^2_{L^2(D)} \right) \leq C \left( \|y_0\|^2_{L^2(D)} + \int_{T_0}^{T_1} \left[ \left\| \frac{F(t)}{\rho(t)} \right\|^2_{L^2(D)} + \left\| \frac{G(t)}{\rho(t)} \right\|^2_{L^2(D)} \right] \, dt \right). \tag{41}
\]
Let \( n \in \mathbb{N} \). From (40) and (41), we have

\[
\begin{align*}
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left\| \frac{y(t)}{\rho(t)} \right\|^2 \right) + \sum_{k=1}^{n} \mathbb{E} \left( \sup_{T_k \leq t \leq T_{k+1}} \left\| \frac{y(t)}{\rho(t)} \right\|^2 \right) \\
\leq \tilde{C} \mathbb{E} \left( \| y_0 \|^2_{L^2(\mathcal{D})} + \sum_{k=1}^{n} \int_0^T 1_{[T_{k-1}, T_k)} \left[ \left\| \frac{F(t)}{\rho(t)} \right\|^2_{L^2(\mathcal{D})} + \left\| \frac{G(t)}{\rho(t)} \right\|^2_{L^2(\mathcal{D})} \right] dt \right)
\end{align*}
\]  

(42)

where \( \tilde{C} > 0 \) is uniform with respect to \( n \). Letting \( n \to \infty \) in (42) yields

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left\| \frac{y(t)}{\rho(t)} \right\|^2 \right) \leq \tilde{C} \mathbb{E} \left( \| y_0 \|^2_{L^2(\mathcal{D})} + \int_0^T \left[ \left\| \frac{F(t)}{\rho(t)} \right\|^2_{L^2(\mathcal{D})} + \left\| \frac{G(t)}{\rho(t)} \right\|^2_{L^2(\mathcal{D})} \right] dt \right). 
\]  

(43)

Finally, combining (38) and (43) gives the desired result. This concludes the proof.

\[\square\]

2.3 A byproduct: a new observability estimate for backward parabolic equation

We introduce the backward parabolic equation

\[
\begin{align*}
z &= -(\Delta z + \bar{z} + \bar{F})dt + a\bar{x}dW(t) \quad \text{in } Q_T, \\
z &= 0 \quad \text{on } \Sigma_T, \\
z(T, \cdot) &= z_T \quad \text{in } \mathcal{D}.
\end{align*}
\]  

(44)

where \( \bar{F} \in L^2_T(0, T; L^2(\mathcal{D})) \) and \( z_T \in L^2(\Omega, \mathcal{F}_T; L^2(\mathcal{D})) \).

Under these conditions, by using [Zho92, Theorem 3.1], the equation (44) admits a unique solution \((z, \bar{z}) \in [L^2_T(\Omega; C([0, T]; L^2(\mathcal{D}))) \cap L^2_T(0, T; H^1(\mathcal{D}))] \times L^2_T(0, T; L^2(\mathcal{D}))\).

From a classical duality argument (see e.g. [Cor07, Lemma 2.48 & Theorem 2.44]) and the duality between (27) and (44), we have as a consequence of the null-controllability result stated in Proposition 2.4 the following observability inequality.

**Corollary 2.5.** For every \( \bar{F} \in L^2_T(0, T; L^2(\mathcal{D})) \) and \( z_T \in L^2(\Omega, \mathcal{F}_T; L^2(\mathcal{D})) \), the solution \((z, \bar{z})\) to (44) satisfies

\[
\begin{align*}
\mathbb{E} \left( \int_{\mathcal{D}} |z(0)|^2 \, dx \right) + \mathbb{E} \left( \int_{Q_T} |\rho z|^2 \, dx \, dt \right) + \mathbb{E} \left( \int_{Q_T} |\rho \bar{z}|^2 \, dx \right) \\
\leq C \mathbb{E} \left( \int_{Q_{\mathcal{D}_0}} |\rho_0 z|^2 \, dx \, dt + \int_{Q_T} |\rho_0 \bar{F}|^2 \, dx \right)
\end{align*}
\]  

(45)

where \( C > 0 \) is the constant appearing in (28).

Estimate (45) looks like the classical observability inequality for the forward stochastic heat equation shown in [TZ09, Theorem 2.3] proved by means of Carleman estimates. However, our proof is far from Carleman-based strategies and an important difference can be pointed out. Unlike [TZ09, Eq. (1.6)], in our estimate the process \( \bar{z} \) stays on the left-hand side of the inequality which allows us to consider only one observation term. Although similar estimates with one observation can be obtained, see [Li11], the incorporation of \( \bar{z} \) on the left-hand side enables us to study more general control problems in the linear setting which are not covered by previous results. Moreover, this will enable us to study some controllability properties for systems with a nonlinear diffusion term.
2.4 Regular controlled trajectories

The next proposition gives more information on the regularity of the controlled trajectory obtained in Proposition 2.4. We define the weight \( \hat{\rho} \) such that \( \hat{\rho}(T) = 0 \), satisfying the inequalities

\[
\rho_0 \leq C \hat{\rho}, \quad \rho \leq C \hat{\rho}, \quad |\hat{\rho}'| \rho_0 \leq C \hat{\rho}^2, \quad \hat{\rho}^2 \leq C \rho.
\]

(46) \hspace{1cm} (47)

For instance, one can take

\[
\hat{\rho}(t) = \exp\left(-\frac{M \zeta}{(Q^{s/2} - 1)(T - t)}\right), \quad \text{with} \quad \frac{(1 + P)Q^s}{2} < \zeta < P.
\]

Proposition 2.6. For every \( y_0 \in L^2(\Omega, \mathcal{F}_0; H^1_0(\mathcal{D})) \), \( F \in \mathcal{S}, \; G \in \mathcal{S} \) such that \( \nabla G \in \mathcal{S} \), then there exists a control \( \hat{h} \in \mathcal{H} \), such that the solution \( y \) of (12) satisfies the following estimate

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \left\| \frac{y(t)}{\hat{\rho}(t)} \right\|^2_{H^1_0(\mathcal{D})} \right) + \mathbb{E}\left( \int_0^T \left\| \frac{y(t)}{\hat{\rho}(t)} \right\|^2_{H^2(\mathcal{D})} dt \right) \leq C \mathbb{E}\left( \left\| y_0 \right\|^2_{H^1_0(\mathcal{D})} + \int_0^T \left\| \frac{F(t)}{\rho(t)} \right\|^2_{L^2(\mathcal{D})} dt + \left\| \frac{G(t)}{\rho(t)} \right\|^2_{H^1(\mathcal{D})} dt \right),
\]

(48)

where \( C \) is a positive constant depending only on \( T, \mathcal{D}, D_0, a, p, q \) and \( r \).

The proof of Proposition 2.6 is a straightforward adaptation of [LTT13, Proposition 2.8]. We sketch it briefly. Let us consider a control \( h \in \mathcal{H} \) and \( y \in \mathcal{Y} \) the corresponding controlled solution provided by Proposition 2.4. We define \( w := \frac{y}{\hat{\rho}} \) and by means of Itô’s formula we readily deduce that \( w \) verifies

\[
dw = \left( \Delta w + \chi_{\mathcal{D}} \frac{h}{\hat{\rho}} + \frac{F}{\hat{\rho}} - \frac{\hat{\rho}'|\rho_0 y}{\hat{\rho}^2 |\rho_0} \right) dt + \left( aw + \frac{G}{\hat{\rho}} \right) dW(t) \quad \text{in} \quad Q_T
\]

and the conclusion follows from applying the maximal regularity estimate of Lemma A.1 and using estimates (46).

Remark 2.7. For each \( y_0 \in L^2(\Omega, \mathcal{F}_0; H^1_0(\mathcal{D})) \), \( F \in \mathcal{S}, \; G \in \mathcal{S} \) such that \( \nabla G \in \mathcal{S} \), by classical arguments, see [LTT13, Proposition 2.9], we can fix a control \( h \in \mathcal{H} \) such that \( y \) satisfies (48), by choosing among those the unique minimizer of the functional

\[
h \mapsto \left\| h \right\|^2_{\mathcal{H}} + \mathbb{E}\left( \sup_{0 \leq t \leq T} \left\| \frac{y(t)}{\hat{\rho}(t)} \right\|^2_{H^1_0(\mathcal{D})} \right) + \mathbb{E}\left( \int_0^T \left\| \frac{y(t)}{\hat{\rho}(t)} \right\|^2_{H^2(\mathcal{D})} dt \right).
\]

3 The fixed point argument

The goal of this section is to prove Theorem 1.2 and Theorem 1.3.

3.1 Proof of the global null-controllability result for the truncated equation

Proof of Theorem 1.2. We split the proof into three main steps:

- First, we prove some Lipschitz type estimate on \( f \);
- then, we see how the previous estimate translates for \( f_R \),
• finally, we employ a Banach fixed-point argument to prove Theorem 1.2.

To simplify, we will only treat the case \( n = 3 \), i.e. \( f(y) = \alpha y^p \), with \( 1 < p \leq 3 \) and \( g(y) = 0 \). The other cases can be treated in a similar way, see Remark 3.1 below.

The constants that will appear may vary from line to line but are independent of the parameter \( R > 0 \).

**Step 1: A Lipschitz estimate for \( f \).**

Consider \( y_1, y_2 \in X_T \). The goal of this step is to prove the following estimate

\[
\left\| \frac{f(y_1) - f(y_2)}{\rho} \right\|_{L^2(D)} \leq C \left( \left\| \frac{y_1}{\rho} \right\|_{H^1(D)}^{p-1} + \left\| \frac{y_2}{\rho} \right\|_{H^1(D)}^{p-1} \right) \left\| \frac{y_1 - y_2}{\rho} \right\|_{H^2(D)}.
\]  

(49)

First, we have by using \( |a^p - b^p| \leq C|a - b|(|a|^{p-1} + |b|^{p-1}) \), Hölder’s estimate and Minkowski’s inequality

\[
\|\alpha y_1^p - \alpha y_2^p\|_{L^2(D)} \leq C \|y_1 - y_2\|_{L^\infty(D)} \left\| |y_1|^{p-1} + |y_2|^{p-1} \right\|_{L^2(D)} \\
\leq C \|y_1 - y_2\|_{L^\infty(D)} \left( \|y_1\|_{L^2(p-1)(D)}^{p-1} + \|y_2\|_{L^2(p-1)(D)}^{p-1} \right).
\]

So, by using \( H^2(D) \hookrightarrow L^\infty(D) \) and \( H^1(D) \hookrightarrow L^6(D) \hookrightarrow L^{2(p-1)}(D) \) because \( n = 3 \) and \( p \leq 3 \), we deduce

\[
\|\alpha y_1^p - \alpha y_2^p\|_{L^2(D)} \leq C \|y_1 - y_2\|_{H^2(D)} \left( \|y_1\|_{H^1(D)}^{p-1} + \|y_2\|_{H^1(D)}^{p-1} \right).
\]  

(50)

Since the weights \( \rho \) and \( \tilde{\rho} \) are \( x \)-independent, we can incorporate them in (50) to obtain

\[
\left| \rho \right| \left\| \frac{f(y_1) - f(y_2)}{\rho} \right\|_{L^2(D)} \leq C|\tilde{\rho}|^p \left( \left\| \frac{y_1}{\rho} \right\|_{H^1(D)}^{p-1} + \left\| \frac{y_2}{\rho} \right\|_{H^1(D)}^{p-1} \right) \left\| \frac{y_1 - y_2}{\rho} \right\|_{H^2(D)},
\]

which leads to (49), using (47).

**Step 2: A Lipschitz estimate for \( f_R \).**

We borrow some ideas from [Lia14] and [Gao17]. We recall that the space \( X_t \) is defined in (4). Without loss of generality, we assume that

\[
\|y_2\|_{X_t} \leq \|y_1\|_{X_t}.
\]  

(51)

Using the definition of \( f_R \), see (6), and triangle inequality we have

\[
\left\| \frac{f_R(y_1) - f_R(y_2)}{\rho} \right\|_{L^2(D)} = \left\| \frac{\varphi_R(y_1) f(y_1) - \varphi_R(y_2) f(y_2)}{\rho} \right\|_{L^2(D)} \leq I_1 + I_2,
\]  

(52)

where

\[
I_1 := \left\| \frac{\varphi_R(y_1) f(y_1) - \varphi_R(y_2) f(y_2)}{\rho} \right\|_{L^2(D)},
\]  

(53)

\[
I_2 := \left\| \frac{\varphi_R(y_1) - \varphi_R(y_2)}{\rho} \right\|_{L^2(D)}.
\]  

(54)

From the definition (5), (51) and the mean value theorem, we have

\[
|\varphi_R(y_1) - \varphi_R(y_2)| = (C/R) \|y_1\|_{X_t} - \|y_2\|_{X_t} \mathbf{1}_{\{\|y_2\|_{X_t} \leq 2R\}}.
\]  

(55)
Thus, using (55) in (54) and the triangle inequality, we get
\[ I_2 \leq (C/R) \| y_1 - y_2 \|_{X_t} \left\| \frac{f(y_2)}{\rho} \right\|_{L^2(D)} \chi_{\{\|y_2\|_{X_t} \leq 2R\}}. \]  
(56)

So from (56) and $H^1(D) \hookrightarrow L^p(D) \hookrightarrow L^{2p}(D)$ because $p \leq 3$, we have
\[ I_2 \leq (C/R) \| y_1 - y_2 \|_{X_t} \left\| \frac{y_2}{\rho} \right\|_{H^1(D)} \chi_{\{\|y_2\|_{X_t} \leq 2R\}} \leq CR^{p-1} \| y_1 - y_2 \|_{X_t} \chi_{\{\|y_2\|_{X_t} \leq 2R\}}. \]  
(57)

For $I_1$ defined in (53), we use the definition of $\varphi_R$ in (5), the estimate (49) established in Step 1 and (51) to get
\[ I_1 \leq C \left( \left\| \frac{y_1}{\rho} \right\|_{H^1(D)}^{p-1} + \left\| \frac{y_2}{\rho} \right\|_{H^1(D)}^{p-1} \right) \left\| y_1 - y_2 \right\|_{H^2(D)} \chi_{\{\|y_2\|_{X_t} \leq 2R\}} \leq CR^{p-1} \left\| y_1 - y_2 \right\|_{H^2(D)} \chi_{\{\|y_2\|_{X_t} \leq 2R\}}. \]  
(58)

Finally, we combine (52), (58) and (57)
\[ \left\| \frac{f_R(y_1) - f_R(y_2)}{\rho} \right\|_{L^2(D)} \leq CR^{p-1} \left( \| y_1 - y_2 \|_{X_t} + \left\| \frac{y_1 - y_2}{\rho} \right\|_{H^2(D)} \right). \]  
(59)

Step 3: A Banach fixed-point argument.
Let $y_0 \in L^2(\Omega, \mathcal{F}_0; H^1_0(D))$.
We introduce the following mapping
\[ \mathcal{N} : F \in \mathcal{S} \mapsto f_R(y) \in \mathcal{S}, \]
where $y$ is the solution to (12) defined in Proposition 2.6 and Remark 2.7.

First, we show that $\mathcal{N}$ is well-defined. By taking the square of (59) with $y_2 = 0$, then integrating in time between 0 and $T$ then taking the expectation, we obtain
\[ \mathbb{E} \left( \int_0^T \left\| \frac{f_R(y(t))}{\rho(t)} \right\|_{L^2(D)}^2 \, dt \right) \leq C^2 R^{2(p-1)} \mathbb{E} \left( T \| y \|_{X_T}^2 + \| y \|_{X_T}^2 \right) \leq C^2 R^{2(p-1)} \mathbb{E} \left( \| y \|_{X_T}^2 \right). \]  
(60)

Note that we have used $\| \cdot \|_{X_t} \leq \| \cdot \|_{X_T}$ for every $t \in [0, T]$ in (60). Then using the estimate (48), we have
\[ \mathbb{E} \left( \int_0^T \left\| \frac{f_R(y(t))}{\rho(t)} \right\|_{L^2(D)}^2 \, dt \right) \leq C^2 R^{2(p-1)} \mathbb{E} \left( \| y_0 \|_{H^1(D)}^2 + \int_0^T \left\| \frac{F(t)}{\rho(t)} \right\|_{L^2(D)}^2 \, dt \right) < +\infty, \]
which translates into $f_R(y) \in \mathcal{S}$.

Secondly, we show that $\mathcal{N}$ is a strictly contraction mapping. By taking the square of (59) and arguing as in (60), we obtain
\[ \mathbb{E} \left( \int_0^T \left\| \frac{f_R(y_1(t)) - f_R(y_2(t))}{\rho(t)} \right\|_{L^2(D)}^2 \, dt \right) \leq C^2 R^{2(p-1)} \mathbb{E} \left( \| y_1 - y_2 \|_{X_T}^2 \right), \]
then using the estimate (48) with \( y_0 = 0 \), we have
\[
E \left( \int_0^T \left\| \frac{f_R(y_1) - f_R(y_2)}{\rho} \right\|_{L^2(\mathcal{D})}^2 \right) \leq C^2 R^{2(p-1)} E \left( \int_0^T \left\| \frac{F_1 - F_2}{\rho} \right\|_{L^2(\mathcal{D})}^2 \right),
\]
which translates into
\[
\| \mathcal{N}(F_1) - \mathcal{N}(F_2) \|_\mathcal{S} \leq C^2 R^{2(p-1)} \| F_1 - F_2 \|_\mathcal{S}. 
\]
So taking \( R \) such that
\[
C^2 R^{2(p-1)} < 1, 
\]
we deduce from (61) that \( \mathcal{N} \) is a strictly contraction mapping of the Banach space \( \mathcal{S} \) so \( \mathcal{N} \) admits a unique fixed point \( F \). By calling \( y \) the trajectory associated to this source term \( F \), we remark that \( y \) is the solution to (8).

Moreover, we observe from (48) and (60) that
\[
E \left( \left\| y \right\|_{X_T}^2 \right) \leq C^2 E \left( \left\| y_0 \right\|_{H_0^1(\mathcal{D})}^2 + \int_0^T \left\| \frac{f_R(y(t))}{\rho(t)} \right\|_{L^2(\mathcal{D})}^2 \, dt \right) 
\leq C^2 \left( E \left\| y_0 \right\|_{H_0^1(\mathcal{D})}^2 + R^{2(p-1)} E \left\| y \right\|_{X_T}^2 \right),
\]
so taking \( R \) sufficiently small if necessary, we can assume that \( C^2 R^{2(p-1)} < 1 \), then
\[
E \left( \left\| y \right\|_{X_T}^2 \right) \leq C^2 E \left( \left\| y_0 \right\|_{H_0^1(\mathcal{D})}^2 \right),
\]
which leads to the expected estimate (9). This concludes the proof of Theorem 1.2. □

**Remark 3.1.** The case \( n = 2 \), i.e. \( f(y) = \alpha y^p \) with \( p \in (1, +\infty) \) could be treated as follows.
For Step 1, using that \( H^2(\mathcal{D}) \hookrightarrow L^\infty(\mathcal{D}) \) and \( H^1(\mathcal{D}) \hookrightarrow L^{2(p-1)}(\mathcal{D}) \), we can obtain (50) so (49) holds. For Step 2, using \( H^1(\mathcal{D}) \hookrightarrow L^{2p}(\mathcal{D}) \), we can obtain (57) so (59) holds.

For the case \( n = 1 \), i.e. \( f(y, \partial_x y) = \alpha y^p + \beta y^q \partial_x y, g(y) = \gamma y^r \), using \( H^1(\mathcal{D}) \hookrightarrow L^\infty(\mathcal{D}) \), we can prove the following estimates,
\[
\| \beta y_1^q \partial_x y_1 - \beta y_2^q \partial_x y_2 \|_{L^2(\mathcal{D})} \leq C \| y_1 - y_2 \|_{H^1(\mathcal{D})} \left( \| y_1 \|^q_{H^1(\mathcal{D})} + \| y_2 \|^q_{H^1(\mathcal{D})} \right),
\]
\[
\| \gamma y_1^r - \gamma y_2^r \|_{H^1(\mathcal{D})} \leq C \| y_1 - y_2 \|_{H^1(\mathcal{D})} \left( \| y_1 \|^{r-1}_{H^1(\mathcal{D})} + \| y_2 \|^{p-1}_{H^1(\mathcal{D})} \right),
\]
then the Lipschitz estimate (49) in Step 1 becomes
\[
\left\| \frac{f(y_1, \partial_x y_1) - f(y_2, \partial_x y_2)}{\rho} \right\|_{L^2(\mathcal{D})} + \left\| \frac{g(y_1) - g(y_2)}{\rho} \right\|_{H^1(\mathcal{D})} \leq C \left( \left\| \frac{y_1}{\rho} \right\|_{H^1(\mathcal{D})}^q + \left\| \frac{y_1}{\rho} \right\|_{H^1(\mathcal{D})}^{r-1} + \left\| \frac{y_2}{\rho} \right\|_{H^1(\mathcal{D})}^{p-1} \right) \left\| \frac{y_1 - y_2}{\rho} \right\|_{H^2(\mathcal{D})}.
\]

For obtaining an estimate similar to (57) for \( f \) and \( g \), we use again the embedding of \( H^1(\mathcal{D}) \) in \( L^\infty(\mathcal{D}) \) to obtain
\[
(C/R) \| y_1 - y_2 \|_{X_t} \left\| \frac{\beta y_1^q \partial_x y_2}{\rho} \right\|_{L^2(\mathcal{D})} \chi_{\{\|y_2\|_{X_t} \leq 2R\}} \leq (C/R) \| y_1 - y_2 \|_{X_t} \left\| \frac{y_2}{\rho} \right\|_{H^1(\mathcal{D})}^{p+1} \chi_{\{\|y_2\|_{X_t} \leq 2R\}} \leq CR^q \| y_1 - y_2 \|_{X_t} \chi_{\{\|y_2\|_{X_t} \leq 2R\}},
\]

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and
\[
\frac{(C/R) \| y_1 - y_2 \|_{X_t}}{\rho} \leq (C/R) \frac{\| y_2 \|_{H^1(D)}^{\prime}}{\rho} \chi\{\| y_2 \|_{X_t} \leq 2R\} \leq (C/R) \frac{\| y_2 \|_{H^1(D)}^{\prime}}{\rho} \chi\{\| y_2 \|_{X_t} \leq 2R\}.
\]

so (59) in Step 2 should be replaced by
\[
\frac{|f_R(y_1, \partial_x y_1) - f_R(y_2, \partial_x y_2)|}{\rho} \leq C(R^p + R^{p-1}) \left( \| y_1 - y_2 \|_{X_t} + \frac{\| y_2 \|_{H^1(D)}^{\prime}}{\rho} \right).
\]

The rest of the proof is analogous.

**Remark 3.2.** Observe that in the right hand side of (48), we have to estimate the $H^1$-norm of source term in the diffusion while we only have to estimate the $L^2$-norm of source term in the drift. This is why we can prove Theorem 1.2 and Theorem 1.3 only for the case $g(y, \nabla y) = \gamma y^r$ with $\gamma \in \mathbb{R}$, $r \in (1, +\infty)$ and $n = 1$.

**Remark 3.3.** In the previous proof, see for instance Step 2, by looking at (56) and (57), another possibility might be to estimate as follows
\[
I_2 \leq (C/R) \frac{\| y_1 - y_2 \|_{X_t}}{\rho} \frac{\| y_2 \|_{H^2(D)}^{\prime}}{\rho} \chi\{\| y_2 \|_{X_t} \leq 2R\},
\]

using $H^2(D) \hookrightarrow L^{2p}(D)$ which holds for every $p \in [1, +\infty]$ because $n \leq 3$. With this type of estimate, at first glance, it seems that we can treat nonlinearities $\alpha y^r$, for every $p \in (1, +\infty)$, but the problems comes from that we do not have $\| \frac{\partial}{\partial t} \|_{H^2(D)} \leq \| y \|_{X_t}$, see the definition of the norm $X_t$ in (4) so one cannot obtain (57) with this strategy.

### 3.2 Proof of the statistical local null-controllability result

We are now in position to prove Theorem 1.3.

**Proof of Theorem 1.3.** Let $T > 0$ and $\epsilon > 0$ be given. Let $R$ as in Theorem 1.2 and $C^2$ as in equation (9). Let us fix $y_0 \in L^2(\Omega, \mathcal{F}_0; H^1_0(D))$ such that
\[
\| y_0 \|_{L^2(\Omega, \mathcal{F}_0; H^1_0(D))} \leq \delta,
\]
where $\delta > 0$ verifying
\[
\frac{C^2 \delta^2}{R^2} \leq \epsilon.
\]

Thanks to Theorem 1.2, we know that there exists a control $h \in \mathcal{H}$, such that the solution $y$ of (8) satisfies $y(T, \cdot) = 0$ a.s. and the estimate (9) holds. Notice that this result is independent of the size of the initial datum.

By Markov’s inequality, (9) and (64), we have
\[
P\left( \| y \|_{X_T}^2 > R^2 \right) \leq \frac{\mathbb{E}\left( \| y \|_{X_T}^2 \right)}{R^2} \leq \frac{C^2\mathbb{E}\left( \| y_0 \|_{H^1_0(D)}^2 \right)}{R^2} \leq \frac{C^2 \delta^2}{R^2},
\]
so by (65), we deduce
\[
P\left( \| y \|_{X_T} \leq R \right) \geq 1 - \epsilon.
\]

Using the fact that $\sup_{t \in [0, T]} \| \cdot \|_{X_t} = \| \cdot \|_{X_T}$, we easily deduce (10). This concludes the proof of Theorem 1.3. \qed
4 Remarks on the case of the backward equation

The strategy introduced in the previous sections can be used to deal with the controllability of semilinear backward equations. Most of the arguments can be adapted and only minor adjustments are needed. To fix ideas, let us consider the system given by

\[
\begin{cases}
dz = - (\Delta z + \bar{f}(z) + \sigma \Sigma + \chi_{D_0} h) dt + \Sigma dW(t) & \text{in } Q_T, \\
z = 0 & \text{in } \Sigma_T, \\
z(T) = z_T & \text{in } D,
\end{cases}
\]

where \( z_T \) is a given initial datum, \( \bar{f} \) is a suitable nonlinear function and \( \sigma \in \mathbb{R} \). Notice that the function \( \bar{f} \) only depends on the variable \( z \). This is due to some technical reasons that we shall explain in more detail in Remark 4.4. To simplify, we take \( \bar{f}(z) = z^2 \), but other polynomial nonlinearities could be considered.

As for the forward system, the idea is to find a control \( h \in \mathcal{L}_2^2(0, T; L^2(D)) \) such that \( z(0, \cdot) = 0 \), a.s. First, we linearize (66) around 0 to obtain

\[
\begin{cases}
dz = - (\Delta z + \chi_{D_0} h + \sigma \Sigma) dt + \Sigma dW(t) & \text{in } Q_T, \\
z = 0 & \text{in } \Sigma_T, \\
z(T) = z_T & \text{in } D.
\end{cases}
\]

For each initial datum \( z_T \in \mathcal{L}_2^2(0, T; L^2(D)) \), system (67) admits a unique solution \((z, \Sigma) \in \mathcal{L}_2^2(\Omega; C([0, T]; L^2(D))) \cap \mathcal{L}_2^2(0, T; H^1_0(D)) \times \mathcal{L}_2^2(0, T; L^2(D))\) (see Lemma A.2).

Following Section 2, the first thing to do is to obtain a controllability result for the linear equation (67). By duality, this can be done by obtaining a suitable observability inequality for its adjoint system. In this case, it is not difficult to see that the adjoint is given by

\[
\begin{cases}
dr = \Delta r dt + \sigma r dW(t) & \text{in } Q_T, \\
r = 0 & \text{on } \Sigma_T, \\
r(0) = r_0 & \text{in } D,
\end{cases}
\]

where \( r_0 \in \mathcal{L}_2^2(\Omega, \mathcal{F}_0; L^2(D)) \). The observability inequality for the above system can be deduced by using the Carleman estimate in [Liu14, Thm. 1.1]. In more detail, we have that there exists a constant \( C > 0 \) such that

\[
\mathbb{E} \left( \int_D |r(T)|^2 \right) \leq C_T \mathbb{E} \left( \int_0^T \int_D |r|^2 \, dx \, dt \right)
\]

for all \( r_0 \in \mathcal{L}_2^2(\Omega, \mathcal{F}_0; L^2(D)) \). A close inspection to the proof of [Liu14, Thm. 1.1] allows to conclude that the constant \( C_T \) is of the form \( C e^{C/T} \) where \( C > 0 \) only depends on \( \sigma \). With this, we have the following result.

**Theorem 4.1.** For every \( T > 0 \), \( z_T \in \mathcal{L}_2^2(\Omega, \mathcal{F}_T; L^2(D)) \), there exists \( h \in \mathcal{L}_2^2(0, T; D_0) \) such that \( y(0) = 0 \) in \( D \), a.s. Moreover, we have the following estimate

\[
\mathbb{E} \left( \iint_{D_0 \times (0, T)} |h|^2 \, dx \, dt \right) \leq C_T \mathbb{E} \left( \|z_T\|_{L^2(D)}^2 \right)
\]

where \( C_T = C e^{C/T} \) with \( C > 0 \) only depending on \( D, D_0 \) and \( \sigma \).
Proposition 4.2. For every $t \in [0, T)$, \[ \tilde{y}(t) = M e^{\frac{M}{t}}. \]

Observe that this weight blows-up as $t \to T^-$. For some parameters $Q \in (1, \sqrt{2})$ and $P > Q^2/(2 - Q^2)$, we define the weights

\[ \forall t \in (0, T], \quad \tilde{\rho}(t) := M^{-1-P} \exp \left( -\frac{(1+P)Q^2M}{(Q-1)t} \right) \]

and

\[ \forall t \in (0, T], \quad \tilde{\rho}_0(t) := M^{-P} \exp \left( -\frac{PM}{(Q-1)t} \right) \]

Notice that these weights are very similar to (25)–(26), however this time they are strictly increasing and they vanish as $t \to 0^+$. For an appropriate source term $\tilde{F} \in L^2_T(0, T; L^2(D))$, we consider

\[
\begin{cases}
 dz = (-\Delta z + \chi_{D_0} h + \tilde{F}) dt + \bar{\rho} dW(t) & \text{in } Q_T, \\
 z = 0 & \text{on } \Sigma_T, \\
 z(T, \cdot) = z_T & \text{in } D.
\end{cases}
\]

(69)

We define associated spaces for the source term, the state and the control as follows

\[
\tilde{S} := \left\{ S \in L^2_T(0, T; L^2(D)) : \frac{S}{\tilde{\rho}} \in L^2_T(0, T; L^2(D)) \right\},
\]

\[
\tilde{Y} := \left\{ y \in L^2_T(0, T; L^2(D)) : \frac{y}{\tilde{\rho}_0} \in L^2_T(0, T; L^2(D)) \right\},
\]

\[
\tilde{H} := \left\{ h \in L^2_T(0, T; L^2(D)) : \frac{h}{\tilde{\rho}_0} \in L^2_T(0, T; L^2(D)) \right\}.
\]

From the behaviors near $t = 0$ of $\tilde{\rho}$ and $\tilde{\rho}_0$, we deduce that each element of $\tilde{S}$, $\tilde{Y}$, $\tilde{H}$ vanishes at $t = 0$.

We have the following result.

Proposition 4.2. For every $z_T \in L^2(\Omega, \mathcal{F}_T; L^2(D))$ and $\tilde{F} \in \tilde{S}$, there exists a control $h \in \tilde{H}$ such that the corresponding controlled solution $y$ to (69) belongs to $\tilde{Y}$. Moreover, there exists a positive constant $C > 0$ depending only on $T$, $D$, $\mathcal{D}_0$, such that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left\| \frac{z(t)}{\tilde{\rho}_0(t)} \right\|^2 \right) + \mathbb{E} \left( \int_0^T \int_{\mathcal{D}_0} \left\| \frac{h}{\tilde{\rho}_0} \right\|^2 dx dt \right) \leq C \mathbb{E} \left( \left\| z_T \right\|^2_{L^2(D)} + \int_0^T \left\| \frac{\tilde{F}(t)}{\tilde{\rho}(t)} \right\|^2_{L^2(D)} dt \right).
\]

(70)

In particular, since $\tilde{\rho}_0$ is a continuous function satisfying $\tilde{\rho}_0(0) = 0$, the above estimate implies

\[ z(0) = 0 \quad \text{in } D, \ 	ext{a.s.} \]
The proof is similar to the one of Proposition 2.4 and can be adapted just by taking into account the definitions of \( \tilde{\rho} \), \( \tilde{\rho}_0 \), the identity \( \tilde{\rho}_0(t) = \tilde{\rho}(Q^2t) \tilde{\gamma}(T + (1 - Q)t) \) and the first regularity estimate of Lemma A.2. For brevity, we omit it.

As in Section 2.4, once this result have been established, a more regular controlled trajectory can be obtained. Indeed, defining a weight \( \tilde{\rho} \) such that \( \tilde{\rho}(0) = 0 \) verifying

\[
\tilde{\rho}_0 \leq C \tilde{\rho}, \quad \tilde{\rho} \leq C \tilde{\rho}, \quad |\tilde{\rho}'| \tilde{\rho}_0 \leq C \tilde{\rho}^2,
\]

we can prove the following result by using the maximal regularity estimate in Lemma A.2.

**Proposition 4.3.** For every \( z_T \in L^2(\Omega, \mathcal{F}_T; H^1_0(D)) \) and \( \tilde{F} \in \tilde{S} \), then there exists an unique control \( h \) of minimal norm in \( \tilde{H} \), such that the solution \( y \) of (69) satisfies the following estimate

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} \frac{\| z(t) \|_{H^1_0(D)}^2}{\tilde{\rho}(t)} \right) + \mathbb{E}\left( \int_0^T \frac{\| z(t) \|_{H^2(D)}^2}{\tilde{\rho}(t)} \, dt \right) + \mathbb{E}\left( \int_0^T \frac{\| \tilde{\gamma}(t) \|_{H^1(D)}^2}{\tilde{\rho}(t)} \, dt \right) \leq C \mathbb{E}\left( \| y_0 \|_{H^2_0(D)}^2 + \int_0^T \left\| \tilde{F}(t) \tilde{\rho}(t) \right\|_{L^2(D)}^2 \, dt \right),
\]

(71)

where \( C > 0 \) is only depends on \( T, D, \text{ and } D_0 \).

We conclude this section remarking that the analysis of the semilinear case can be carried out as in Section 3, we just need to adapt the analysis of the truncated nonlinearity. This can be done by considering the space

\[
\tilde{X}_t := \left\{ z \in [C([0, t]; H^1_0(D)) \cap L^2(0, t; H^2(D))]: \sup_{0 \leq s \leq t} \frac{\| z(s) \|_{H^1_0(D)}}{\tilde{\rho}(s)} + \left( \int_0^t \frac{\| z(s) \|_{H^2(D)}}{\tilde{\rho}(s)} \right)^{1/2} < +\infty \right\}.
\]

Thus, following the notation of Section 3, it is not difficult to see that most arguments can be readily adapted. For our example, we can discover that

\[
\left\| f_R(z_1) - f_R(z_2) \right\|_{\tilde{\rho}} \leq CR \left( \| z_1 - z_2 \|_{\tilde{X}_t} + \frac{\| z_1 - z_2 \|}{\tilde{\rho}} \right)_{H^2}
\]

where \( R > 0 \) is small and \( f_R(z) = \varphi_R(\| z \|_{\tilde{X}_t})f(z) \), with \( \varphi_R \) as in (5). Of course, this can be generalized to consider some other polynomial nonlinearities but it is not the goal here.

Then, we can do a fixed point argument analogous to Step 3 in Section 3.1 and obtain results for the global controllability case with truncated nonlinearity (cf. Theorem 1.2) and the statistical local null-controllability case (cf. Theorem 1.3) for the semilinear backward system (66). For brevity, we omit the details.

**Remark 4.4.** One may wonder why we cannot consider a more general nonlinearity of the form \( f(z, \varpi) = z^2 + \varpi^2 \). Of course, we can change the the space \( X_t \) and include the process \( \varpi \in L^2(0, T; H^1(D)) \) in its definition. However, observe that this only adds an \( L^2 \)-in-time estimate for \( \varpi \), which does not allow us to obtain a nice Lipschitz estimate for the nonlinearity. This is closely related to Remark 3.3 and the fact that we cannot estimate, for instance, \( \left\| \varpi(s) \right\|_{H^1(D)}^2 \leq \int_0^s \left\| \varpi(s) \right\|_{H^1(D)}^2 \, ds \).
A Regularity results

The following result concerns some regularity estimates for forward stochastic parabolic equations. In a slightly more general form they are due to Krylov and Rozovskii [KR77]. We follow the presentation of [Zho92, Proposition 2.1].

Lemma A.1. Let $\tau \in (0, 1)$. We have the following energy estimates for (27) (with $h \equiv 0$).

a) Assume that $F \in L^2_F(0, \tau; H^{-1}(D))$, $G \in L^2_F(0, \tau; L^2(D))$, and $y_0 \in L^2(\Omega, \mathcal{F}_\tau; L^2(D))$, then (27) has a unique solution $y \in L^2_F(0, \tau; H^1_0(D)) \cap L^2_F(\Omega; C([0, \tau]; L^2(D)))$. Moreover, there exists a positive constant $C_0$ independent of $\tau$, $F$, $G$ and $y_0$ such that

$$
\mathbb{E} \left( \sup_{0 \leq t \leq \tau} \|y(t)\|_{L^2(D)}^2 \right) + \mathbb{E} \left( \int_0^\tau \|y(t)\|_{H^1_0(D)}^2 \, dt \right) \\
\leq C_0 \mathbb{E} \left( \|y_0\|_{L^2(D)}^2 + \int_0^\tau \|F(t)\|_{H^{-1}(D)}^2 + \|G(t)\|_{L^2(D)}^2 \, dt \right).
$$

b) Assume that $F \in L^2_F(0, \tau; L^2(D))$, $G \in L^2_F(0, \tau; H^1(D))$, and $y_0 \in L^2(\Omega, \mathcal{F}_\tau; H^1_0(D))$, then (27) has a unique solution $y \in L^2_F(0, \tau; H^2(D)) \cap L^2_F(\Omega; C([0, \tau]; H^1_0(D)))$. Moreover, there exists a positive constant $C_0$ independent of $\tau$, $F$, $G$ and $y_0$ such that

$$
\mathbb{E} \left( \sup_{0 \leq t \leq \tau} \|y(t)\|_{H^1_0(D)}^2 \right) + \mathbb{E} \left( \int_0^\tau \|y(t)\|_{H^2(D)}^2 \, dt \right) \\
\leq C_0 \mathbb{E} \left( \|y_0\|_{H^1(D)}^2 + \int_0^\tau \|F(t)\|_{L^2(D)}^2 + \|G(t)\|_{H^1(D)}^2 \, dt \right).
$$

In the following result, we present some regularity estimates for backward stochastic parabolic equations. We refer to [Zho92, Theorem 3.1] for a more general result.

Lemma A.2. Let $\tau \in (0, 1)$. We have the following energy estimates for (67) (with $h \equiv 0$).

a) Assume that $\tilde{F} \in L^2_F(0, \tau; H^{-1}(D))$ and $z_\tau \in L^2(\Omega, \mathcal{F}_\tau; L^2(D))$, then (67) has a unique solution $(z, \tilde{z}) \in [L^2_F(0, \tau; H^1_0(D)) \cap L^2_F(\Omega; C([0, \tau]; L^2(D)))) \times L^2_F(0, \tau; L^2(D))$. Moreover, there exists a positive constant $C_0$ independent of $\tau$, $\tilde{F}$ and $z_\tau$ such that

$$
\mathbb{E} \left( \sup_{0 \leq t \leq \tau} \|z(t)\|_{L^2(D)}^2 \right) + \mathbb{E} \left( \int_0^\tau \|z(t)\|_{H^1_0(D)}^2 + \|\tilde{z}(t)\|_{L^2(D)}^2 \, dt \right) \\
\leq C_0 \mathbb{E} \left( \|z_\tau\|_{L^2(D)}^2 + \int_0^\tau \|\tilde{F}(t)\|_{H^{-1}(D)}^2 \, dt \right).
$$

b) Assume that $\tilde{F} \in L^2_F(0, \tau; L^2(D))$ and $z_\tau \in L^2(\Omega, \mathcal{F}_\tau; H^1_0(D))$, then (67) has a unique solution $(z, \tilde{z}) \in [L^2_F(0, \tau; H^2(D)) \cap L^2_F(\Omega; C([0, \tau]; H^1_0(D)))) \times L^2_F(0, \tau; H^1(D))$. Moreover, there exists a positive constant $C_0$ independent of $\tau$, $\tilde{F}$ and $z_\tau$ such that

$$
\mathbb{E} \left( \sup_{0 \leq t \leq \tau} \|z(t)\|_{H^1_0(D)}^2 \right) + \mathbb{E} \left( \int_0^\tau \|z(t)\|_{L^2(D)}^2 + \|\tilde{z}(t)\|_{H^1(D)}^2 \, dt \right) \\
\leq C_0 \mathbb{E} \left( \|z_\tau\|_{H^1_0(D)}^2 + \int_0^\tau \|\tilde{F}(t)\|_{L^2(D)}^2 \, dt \right).
$$
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