THE NONCRITICAL $W_\infty$ STRING SECTOR
OF THE MEMBRANE

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December, 1996

ABSTRACT

The exact quantum integrability aspects of a sector of the membrane is investigated. It is found that spherical membranes (in the lightcone gauge) moving in flat target spacetime backgrounds admit a class of integrable solutions linked to $SU(\infty)$ SDYM equations (dimensionally reduced to one temporal dimension) which, in turn, are related to Plebanski 4D SD Gravitational equations. A further rotational Killing-symmetry reduction yields the 3D continuous Toda theory. It is precisely the latter which bears a direct relationship to noncritical $W_\infty$ string theory. The expected critical dimensions for the (super) membrane, $(D = 11)$ and $D = 27$, are easily obtained. This suggests that this particular sector of the membrane’s spectrum (connected to the $SU(\infty)$ SDYM equations) bears a direct connection to a critical $W_\infty$ string spectrum adjoined to a $q = N + 1$ unitary minimal model of the $W_N$ algebra in the $N \to \infty$ limit. Final comments are made about the connection to Jevicki’s observation that the 4D quantum membrane is linked to dilatonic-self dual gravity plus matter. 2D dilatonic (super) gravity was studied by Ikeda and its relation to nonlinear $W_\infty$ algebras from nonlinear integrable deformations of 4D self dual gravity was studied by the author. The full $SU(\infty)$ YM theory remains to be explored as well as the incipient role that noncritical nonlinear $W_\infty$ strings might have in the full quantization program.

PACS : 0465.+e; 02.40.+m

1. Introduction

Recently, exact solutions to $D = 11$ spherical (super) membranes moving in flat target spacetime backgrounds were constructed based on a particular class of reductions of Yang-Mills equations from higher dimensions to four dimensions [1,2]. The starting point was dimensionally-reduced Super Yang-Mills theories based on the infinite dimensional $SU(\infty)$ algebra. The latter algebra is isomorphic to the area-preserving diffeomorphisms of the sphere [3]. In this fashion the super Toda molecule equation was recovered preserving one supersymmetry out of the $N = 16$ expected. The expected critical target spacetime dimensions for the (super) membrane, $D = 27(11)$, was closely related to that of an anomaly-free non-critical (super) $W_\infty$ string theory. A BRST analysis revealed that the spectrum of the membrane should have a relationship to the first unitary minimal model of a $W_N$ algebra adjoined to a critical $W_N$ string in the $N \to \infty$ limit [1]. The class of particular solutions of the dimensionally-reduced $SU(\infty)$ YM equations studied are those of the type proposed by Ivanova and Popov [2] which bears a direct relationship to $SU(\infty)$ instanton solutions in 4D that permits a connection to the $SU(\infty)$ Toda molecule equation after an specific ansatz is made [1].
The Toda theory emerges in other contexts beyond the SDYM sector (the self dual membrane). It makes its appearance in noncritical $W_\infty$ strings; i.e. $W_\infty$ gravity, and in the quantum 4D membrane model studied by Jevicki [23]. A review of $SU(\infty)$ SDYM [1] is described in the next section in connection to the Toda theory. In the final section we analyze in detail the role that noncritical $W_\infty$ strings have in the theory of membranes. The critical dimensions, $D = 27, 11$ are recovered for the bosonic and supersymmetric case. Comments about the role that nonlinear noncritical $W_\infty$ strings in the full theory are made in the conclusion.

II

2.1 $SU(\infty)$ SDYM and the Toda Molecule

Based on the observation that the spherical membrane (excluding the zero modes) moving in $D$ spacetime dimensions, in the light-cone gauge, is essentially equivalent to a $D - 1$ Yang-Mills theory, dimensionally reduced to one time dimension, of the $SU(\infty)$ group (see [8] for references); we look for solutions of the $D = 10$ Yang-Mills equations (dimensionally-reduced to one temporal dimension). For an early review on membranes see Duff [8] and the recent book by Ne’eman and Eizenberg [9].

Marquard et al [10] have shown that the light-cone gauge Lorentz algebra for the bosonic membrane is anomaly free iff $D = 27$. The supermembrane critical dimension was found to be $D = 11$. To this date there is still some controversy about whether or not the (super) membrane is really anomaly free in these dimensions. They may suffer from other anomalies like 3D reparametrization invariance anomalies or global ones. What follows next does not depend on whether or not $D = 27, D = 11$ are truly the critical dimensions. What follows is just a straightforward quantization of a very special class of solutions of the dimensionally-reduced (to one temporal dimension) of $SU(\infty)$ YM equations, and which can be quantized exactly due to their equivalence to the exactly integrable quantum continuous Toda molecule, obtained as a dimensional-reduction of the original continuous 3D Toda theory [4,6] to 2D which is where $W_\infty$ strings live; this clarifies how a 3D membrane can have a connection to a 2D $W_\infty$ string.

We begin with the $D = 10$ YM equations dimensionally reduced to one dimension. Let us focus on the bosonic sector of the theory. The supersymmetric case can be also analyzed via solutions to the supersymmetric Toda theory which has been discussed in detail in the literature. The particular class of solutions one is interested in are those of the type analyzed by Ivanova and Popov. Given:

$$\partial_a F_{ab} + [A_a, F_{ab}] = 0, A_a^\alpha T_\alpha \rightarrow A_a(x^b; q, p), [A_a, A_b] \rightarrow \{A_a, A_b\}_{q,p}. \quad (2.1)$$

where the $SU(\infty)$ YM potentials [5] are obtained by replacing Lie-algebra valued potentials (matrices) by $c$ number functions; Lie-algebra brackets by Poisson brackets w.r.t the two internal coordinates associated with the sphere; and the trace by an integral w.r.t these internal coordinates. In [1] we performed an ansatz following the results of Ivanova and Popov. The $a, b, ... = 8$ are the transverse indices to the membrane after we performed the $10 = 2 + 8$ split of the original $D = 10$ YM equations.

After the dimensional reduction to one dimension is done we found that the following $D = 10$ YM potentials, $A$, -which will be later expressed in terms of the $D = 4$ YM
potentials, \( A_1, A_2, A_3 \) (\( A_0 \) can be gauged to zero)- are one class of solutions to the original \( D = 10 \) equations iff they admit the following relationship:

\[
A_1 = p_1 A_1, A_5 = p_2 A_1, A_3 = p_1 A_3, A_7 = p_2 A_3. \tag{2.2a}
\]
\[
A_2 = p_1 A_2, A_6 = p_2 A_2, A_0 = A_4 = A_8 = A_9 = 0. \tag{2.2b}
\]

where \( p_1, p_2 \) are constants and \( A_1, A_2, A_3 \) are functions of \( x_0, q, p \) only and obey the \( SU(\infty) \) Nahm’s equations:

\[
\epsilon_{ijk} \frac{\partial A_k}{\partial x_0} + \{ A_i, A_j \}_{q,p} = 0. \quad i, j, k = 1, 2, 3. \tag{2.3}
\]

Nahm’s equations are also obtained directly from reductions of \( D = 4 \) Self Dual Yang-Mills equations to one dimension. The temporal variable \( x_o = p_1 X_0 + p_2 X_4 \) has a correspondence, not an identification, with the membrane’s light-cone coordinate : \( X^+ = X^0 + X^{10} \). We refer to Ivanova and Popov and to our results in [1,2] for details.

Expanding \( A_y = \sum A_{yl}(x_o)Y_{l+1}, A_{\bar{y}} = \sum A_{\bar{y}l}(x_o)Y_{l-1} \); and \( A_3 \) in terms of \( Y_{l0} \), the ansatz which allows to recast the \( SU(\infty) \) Nahm’s equations as a Toda molecule equation is [1]:

\[
\{ A_y, A_{\bar{y}} \} = -i \sum_{l=1}^{\infty} \exp[K_{l\ell'}\theta_{l'}]Y_{l0}(\sigma_1, \sigma_2), \quad A_3 = -\sum_{l=1}^{\infty} \frac{\partial \theta_l}{\partial \tau}Y_{l0}. \tag{2.4}
\]

with \( A_y = \frac{A_1+iA_2}{\sqrt{2}}, \quad A_{\bar{y}} = \frac{A_1-iA_2}{\sqrt{2}}. \)

Hence, Nahm’s equations become:

\[
-\frac{\partial^2 \theta_l}{\partial \tau^2} = e^{K_{l\ell'}\theta_{l'}}, \quad l, l' = 1, 2, 3, \ldots. \tag{2.5}
\]

This is the \( SU(N) \) Toda molecule equation in Minkowski form. The \( \theta_l \) are the Toda fields where \( SU(2) \) has been embedded minimally into \( SU(N) \). \( K_{l\ell'} \) is the Cartan matrix which in the continuum limit becomes : \( \delta''(t - t') \) [4]. The solution of the Toda theory is well known to the experts by now. Solving for the \( \theta_l(\tau) \) fields and plugging their values into the first term of eq-(2.4) yields an infinite number of equations -in the \( N \to \infty \) limit- for the infinite number of “coefficients” \( A_{yl}(x_o), A_{\bar{y}l}(x_o) \). This allows to solve for the YM potentials \textit{exactly}. The ansatz [1] automatically yields the coefficients in the expansion of the \( A_3 \) component of the \( SU(\infty) \) YM field given in the second term of (2.4). Upon quantization of the \( SU(\infty) \) YM theory, the first term in eq-(2.4) is replaced by a commutator of two operators and as such the coefficients \( A_{yl}(x_o), A_{\bar{y}l}(x_o) \) become operators as well. The Toda fields become also operators in the Heisenberg representation. We will discuss the quantization of the Toda theory via the \( W_\infty \) codajoint orbit method [25,26] below.

The continuum limit of (2.5) is

\[
-\frac{\partial^2 \theta_l(\tau, t)}{\partial \tau^2} = \exp \left[ \int dt' \delta''(t - t') \theta_l(\tau, t') \right]. \tag{2.6}
\]
Or in alternative form:

\[-\frac{\partial^2 \Psi(\tau,t)}{\partial \tau^2} = \int \delta''(t-t') \exp[\Psi(\tau,t')] \, dt' = \frac{\partial^2 e^\Psi}{\partial t^2}. \quad (2.7)\]

if one sets \( K_{\mu \nu} \theta^\nu = \Psi_\tau. \) The last two equations are the dimensional reduction of the \( 3D \rightarrow 2D \) continuous Toda equation given by Saveliev:

\[\frac{\partial^2 u(\tau,t)}{\partial \tau^2} = -\frac{\partial^2 e^u}{\partial t^2}. \quad i\tau \equiv r = z + \bar{z}. \quad (2.8a)\]

Eq-(2.8a) is referred as the \( SU(\infty) \) Toda molecule whereas

\[\frac{\partial^2 u(z,\bar{z},t)}{\partial z \partial \bar{z}} = -\frac{\partial^2 e^u}{\partial t^2}. \quad (2.8b)\]

is the \( 3D \) continuous Toda equation which can obtained as rotational Killing symmetry reductions of Plebanski equations for Self-Dual Gravity in \( D = 4 \). Eq-(2.8a) is an effective \( 2D \) equation and in this fashion the original \( 3D \) membrane can be related to a \( 2D \) theory ( where the \( W_\infty \) string lives in ) after the light-cone gauge is chosen.

The Lagrangian (and equations) for \( 4D \) SD gravity can be obtained from a dimensional reduction of the \( SU(\infty) \) SDYM (an effective six-dimensional one) [19,24]:

\[\mathcal{L} = \int dzd\bar{z}dyd\bar{y} \frac{1}{2}(\Theta_{,y}\Theta_{,z} - \Theta_{,z}\Theta_{,y}) + \frac{1}{3} \Theta\{\Theta_{,y},\Theta_{,z}\}. \quad (2.9)\]

where \( \Theta(z,\bar{z},y,\bar{y}) \) is Plebanski’s second heavenly form and the Poisson brackets are taken w.r.t \( y,\bar{y} \) variables. A real slice can be taken by setting : \( \bar{z} = z, \bar{y} = y \).

A rotatinal Killing symmetry reduction, \( t \equiv y\bar{y}, \) yields the Lagrangian for the \( 3D \) Toda theory and a further dimensional reduction \( z + \bar{z} = r \) gives the Toda molecule Lagrangian.

From [19] we can find the explicit map between \( \Theta \) and \( \rho(r,t) \)

\[A_y = \Theta_{,y}; \quad A_{\bar{y}} = \Theta_{,\bar{y}}; \quad A_z = -\Theta_{,y} + f(y,\bar{y},z); \quad A_{\bar{z}} = \Theta_{,\bar{y}} + g(y,\bar{y},\bar{z}). \quad (2.10)\]

where \( f,g \) are integration "constants". In the gauge \( A_0 = 0 \Rightarrow A_z = A_{\bar{z}} \) and the two functions, \( f,g \) are constrained to obey the latter condition plus the additional relations stemming from the original SDYM equations. Therefore, \( f,g \) are fully determined.

From our ansatz (2.4) one can read off the correspondence between the Plebanski \( \Theta \) (after the corresponding reductions) and the Toda field \( \rho(r,t) \):

\[
\{\Theta_{,y},\Theta_{,\bar{y}}\} \rightarrow \frac{\partial^2}{\partial t^2} e^\rho; \quad \frac{\partial}{\partial r}(-\Theta_{,y} + f(y,\bar{y})) \rightarrow \frac{\partial^2 \rho}{\partial r^2}; \quad \frac{\partial}{\partial r} A_z = \frac{\partial}{\partial r} A_{\bar{z}}. \quad (2.11a)
\]

And, finally, one makes contact with Savaliev’s Lagrangian of the Toda molecule:

\[\mathcal{L} = \int dt \frac{1}{2} \left(\frac{\partial^2 x}{\partial r \partial t}\right)^2 + e^{(\partial^2 x/\partial t^2)} \rho(r,t) \equiv \frac{\partial^2 x}{\partial t^2}. \quad (2.11b)\]
Eqs-(2.9-2.11) are the essential equations that allows to extract the exact quantization of the Toda theory via the $W_\infty$ coadjoint orbit method described by [25]. Nissimov and Pacheva have shown that induced $W_\infty$ gravity could be seen as a WZNW model. They derived the explicit form of the Wess-Zumino quantum effective action of chiral $W_\infty$ matter coupled to a chiral $W_\infty$ gravity background. The quantum effective action could be expressed as a geometric action on a coadjoint orbit of the deformed group of area-preserving diffs of the cylinder. A “hidden” $SL(\infty, R)$ Kac-Moody algebra was obtained as a consequence of the $SL(\infty, R)$ stationary subgroup of the $W_\infty$ coadjoint orbit. Yamagishi and Chapline earlier [26] proved that an induced 4 $D$ self-dual quantum gravity could be obtained via the $W_\infty$ coadjoint orbit method. An effective quantum action (constructed in the twistor space) was explicitly obtained as an infinite sum of two-dimensional effective Lagrangians with Polyakov two-dim lightcone gauge gravity as its first term (having a hidden $SL(2, R)$ Kac-Moody symmetry). The higher order terms are the result of the central extensions of the $W_\infty$ algebra.

The crucial advantage that the $W_\infty$ coadjoint orbit method has in the quantization of the continuous Toda theory is that an Operator Quantization method is extremely difficult. The ordinary Liouville theory, an $SL(2, R)$ Toda theory, is a notoriously difficult example. Its quantization using operator methods has taken years. Very recently, Fujiwara, Igarashi and Takimoto [21] have shown using exact operator solutions for quantum Liouville theory, based on canonical free field methods, that the exact solutions proposed by Otto and Weight [22] are correct to all orders in the cosmological constant. They found that the hidden (quantum group) exchange algebra found by Gervais and Schnittger [20], $U_q(sl(2))$, was essential in order to maintain locality and the operator form of the field equation. In the continuous Toda case one expects a hidden quantum group $U_q(sl(\infty))$ structure. Not surprisingly, the appearance of the hidden $U_q(sl(\infty))$ must stem from the hidden $SL(\infty, R)$ Kac-Moody algebra associated with the stability subgroup of the $W_\infty$ coadjoint orbit method.

The quantum effective action of the Toda theory is directly obtained from the exact results of [25,26] by the straightforward reduction given in eqs-(2.9-2.11) : one reads off the quantum effective action for the Toda theory directly from the “dictionary” between the $\Theta$ and the $\rho$ established in (2.9-2.11) after performing the Darboux change of coordinates given by Plebanski. It is the latter that expresses the second heavenly form, $\Theta$ in terms of the first heavenly form, $\Omega$. This is needed because the Chapline-Yamagishi quantum effective action is given in terms of $\Omega$.

Having discussed the importance of the Toda theory we proceed to study the role that noncritical $W_\infty$ strings have in the membrane quantization.

2.2 A Membrane Sector as a Non Critical $W_\infty$ String

In [1] we established the correspondence between the target space-times of non-critical $W_\infty$ strings and that of membranes in $D = 27$ dimensions. The supersymmetric case was also discussed and $D = 11$ was retrieved. We shall review in further detail the construction [1].

The relevance of developing a $W_\infty$ conformal field theory (with its quantum group extensions) has been emerging over the past years [28]. It was shown in [11] that the effective induced action of $W_N$ gravity in the conformal gauge takes the form of a Toda
action for the scalar fields and the $W_N$ currents take the familiar free field form. The same action can be obtained from a constrained $WZNW$ model (modulo the global aspects of the theory due to the topology. Tsutsui and Feher have shown that richer structures emerge in the reduction process [29] ) by a quantum Drinfeld-Sokolov reduction process of the $SL(\infty, R)$ Kac-Moody algebra at the level $k$. Each of these quantum Toda actions posseses a $W_N$ symmetry.

The authors [11] coupled $W_N$ matter to $W_N$ gravity in the conformal gauge, and integrating out the matter fields, they arrived at the induced effective action which was precisely the same as the Toda action. It is not surprising that the Toda theory is related to 4 $D$ Self-Dual gravity. In what follows, by $W_N$ string we mean the string associated to $WA_{N-1}$ algebra.

In general, non-critical $W_N$ strings are constructed the same way: by coupling $W_N$ matter to $W_N$ gravity. The matter and Liouville sector (stemming from $W_N$ gravity) of the $W_N$ algebra can be realized in terms of $N-1$ scalars, $\phi_k, \sigma_k$ respectively. These realizations in general have background charges which are fixed by the Miura transformations [12,13]. The non-critical string is characterized by the central charges of the matter and Liouville sectors, $c_m, c_L$. To achieve a nilpotent BRST operator these central charges must satisfy:

$$c_m + c_L = -(n_{gh}) = 2 \sum_{s=2}^{N} (6s^2 - 6s + 1) = 2(N-1)(2N^2 + 2N + 1). \quad (2.12)$$

In the $N \to \infty$ limit a zeta function regularization yields $c_m + c_L = -2$.

The authors [13] have shown that the BRST operator can be written as a sum of nilpotent BRST operators, $Q^N$, and that a nested basis can be chosen either for the Liouville sector or the matter sector but not for both. If the nested basis is chosen for the Liouville sector then [13] found that the central charge for the Liouville sector is:

$$c_L = (N-1)[1 - 2x^2 N(N+1)]. \quad (2.13)$$

were $x$ is an arbitrary parameter which makes it possible to avoid the relation with the $W_N$ minimal models if one wishes to. Later we will show explicitly that the value of $c_L$ coincides precisely with the value of the central charge of a quantum Toda theory obtained from a quantum-Drinfeld-Sokolov reduction of the $SL(\infty, R)$ Kac-Moody algebra at the level $k$ such that $k + N = constant$ (a constant that can be computed exactly) in the $N \to \infty$ limit.

By choosing, if one wishes, $x$ appropriately one can, of course, get the $q^{th}$ unitary minimal models by fixing $x^2$ to be:

$$x_o^2 = -2 - \frac{1}{2q(q+1)}. \quad (2.14)$$

where $q$ is an integer. In this case, since $c_m + c_L = -(n_{gh})$, the central charge for the matter sector must be:

$$c_m = (N-1)(1 - \frac{N(N+1)}{q(q+1)}). \quad (2.15)$$
which corresponds precisely to the $q^{th}$ minimal model of the $W_N$ string as one intended to have by choosing the value of $x_0^2$. In the present case one has the freedom of selecting the minimal model since the value of $q$ is arbitrary. If $q = N$ then $c_m = 0$ and the theory effectively reduces to that of the “critical” $W_N$ string. Conversely, if one chooses for the nested basis that corresponding to the matter sector instead of the Liouville sector, the roles of “matter” and “Liouville” are reversed. One would then have $c_L = 0$ instead.

Noncritical strings involve two copies of the $W_N$ algebra. One for the matter sector and other for the Liouville sector. Since $W_N$ is nonlinear, one cannot add naively two realizations of it and obtain a third realization. Nevertheless there is a way in which this is possible [13]. This was achieved by using the nested sum of nilpotent BRST operators, $Q^N_N$. One requires to have all the matter fields, $\phi_k$; the scalars of the Liouville sector in the nested basis, $\sigma_{n-1}, \ldots, \sigma_{N-1}$ plus the ghost and antighost fields of the spin $n, n + 1, \ldots, N$ symmetries where $n$ ranges between 2 and $N$. Central charges were computed for each set of the nested set of stress energy tensors, $T^N_n$ depending on all of the above fields which appear in the construction of the BRST charges: $Q^N_N$.

In order to find a spacetime interpretation, the coordinates $X^\mu$, must be related to a very specific scalar field of the Liouville sector (since one decided to choose the nested basis in the Liouville sector) and that field is $\sigma^1$. It is this central charge, associated with the scalar field $\sigma^1$, that always appears through its energy momentum tensor in the Miura basis. Because $\sigma^1$ always appears through its energy momentum tensor, it can be replaced by an effective $T_{eff}$ of any conformal field theory as long as it has the same value of the central charge, $c = 1 + 12\alpha^2 \equiv 1 - 12x^2$, where $\alpha$ is a background charge.

$$T(\sigma^1) = -\frac{1}{2} \left( \frac{\partial \sigma^1}{\partial z} \right)^2 - \alpha \frac{\partial^2 \sigma^1}{\partial z^2}. \quad (2.16)$$

In particular, having $D$ worldsheet scalars, $X^\mu$, with a background charge vector, $\alpha_\mu$:

$$T_{eff} = -\frac{1}{2} \partial_z X^\mu \partial_\mu X - \alpha_\mu \partial^2_z X^\mu. \quad c_{eff} = D + 12\alpha_\mu \alpha^\mu = 1 + 12\alpha^2. \quad (2.17)$$

For example, in the critical $W_\infty$ string case, one is bound to the unitary minimal models [12,13] and one must pick for central charge associated with the scalar, $\sigma_1$, the value $\alpha^2 = -x^2 = -x_0^2$ given by (2.14). The explicit value of $c$ of the critical $W_\infty$ string is obtained:

$$c_{crit} = 1 + 12(\alpha_o)^2 = 1 - 12x_0^2 = 1 - 12(-2 - \frac{1}{2q(q+1)}) = 25; q = N \rightarrow \infty. \quad (2.18)$$

In the case of the ordinary critical string, $W_N = W_2$, $q = N = 2$, one has:

$$x_0^2 = -2 - \frac{1}{2q(q+1)} \rightarrow -2 - \frac{1}{12} \Rightarrow c_{eff} = 1 - 12x_0^2 = 1 + 25 = 26 = c_{crit}. \quad (2.19)$$
Since the parameter $x$ in the non-critical string case is an arbitrary parameter that is no longer bound to be equal to $x_o$, the effective central charge in the non-critical $W_N$ string is now $1 - 12x^2$ in contradistinction to the critical $W_N$ string case: $1 - 12x_o^2$. Therefore, if one wishes to make contact with $D = 27$ $X^\mu$ scalars instead of $D = 25$ one can choose $x$ in such a way that it obeys $1 - 12x^2 = (1 - 12x_o^2) + c_{m_o}$ where $c_{m_o}$ will turn out to be the central charge of the $q = N + 1$ unitary minimal model of the $W_N$ algebra. Clearly, if one had chosen $q = N$ instead, from eq-(2.15), one gets that $c_{m_o} \to 0$ and, as expected, the critical $W_N$ string is recovered: $x^2 \to x_o^2 (q = N)$ given by (2.14). If, in addition, one does not wish to break the target space-time Lorentz invariance one cannot have background charges for the $D X^\mu$ coordinates. Therefore, for the case that $q = N + 1 \Rightarrow c_{m_o} = \frac{2(N-1)}{N+1}$ (instead of zero) is obtained from eq-(2.15), and the effective central charge is now:

$$c_{eff} \equiv 1 - 12x^2 = (1 - 12x_o^2) + c_{m_o} = \left[26 - \left(1 - \frac{6}{(N+1)(N+2)}\right)\right] + \left[\frac{2(N-1)}{N+2}\right]$$

then one concludes that $D = 25 + 2 = 27 = c_{eff}$ is recovered in the $N \to \infty$ limit. The reason why one wrote the last term of eq-(2.20) in such a peculiar way will be clarified shortly. In this way we have shown that the expected critical dimension for the bosonic membrane background, $D = 27$, has the same number of $X^\mu$ coordinates as that of a non-critical $W_\infty$ string background if one adjoins the $q = N + 1$ unitary minimal model of the $W_N$ algebra to that of a critical $W_N$ string spectrum in the $N \to \infty$ limit. From eq-(2.20) one also learns that

$$D = 2 + 25 = 27 = 1 - 12x^2 \Rightarrow 2x^2 = -\frac{13}{3}.$$  

which will be important to find the value of the central charge of the Toda theory, below.

There are, of course, many other ways in which one could recover $D = 27$ $X^\mu$ besides the way shown in eq-(2.20). The latter is a particular combination involving the critical $W_\infty$ string with the $q = N + 1$ unitary minimal models. It is important to study the other possibilities. Whatever these may be, these do not preclude the role that non-critical $W_\infty$ strings have in the theory. The physical membrane spectrum has to contain a sector that should be related to a critical $W_\infty$ string adjoined to a $q = N + 1$ unitary minimal model of the $W_N$ algebra in the $N \to \infty$ limit. The full spectrum, moduli space of membrane vacua, etc...is far more vast than the slice furnished in (2.20). Our main point is that $W_\infty$ conformal field theory, with its Kac-Moody extension, $W_\infty$ gravity,... should contain important clues to classify the spectrum and the moduli space of vacua, in the same way that ordinary conformal field theory did for the string. More precisely, we will argue that it is the non-linear extensions of the $W_\infty$ algebra that must be involved if one wishes to relate to Jevicki’s recent results [23].

The critical $W_\infty$ string [12] is a generalization of the ordinary string in the sense that instead of gauging the two-dimensional Virasoro algebra one gauges the higher conformal spin algebra generalization; the $W_\infty$ algebra. The spectrum can be computed exactly and is equivalent to an infinite set of spectra of Virasoro strings with unusual central charges and intercepts [12]. As stated earlier, the critical $W_N$ string (linked to the $A_{N-1}$ algebra) has for central charge the value ($q = N$):

\[c_{eff} = \ldots\]
\[
c = 1 - 12x_o^2 = 26 - \left(1 - \frac{6}{q(q+1)}\right) = 26 - \left(1 - \frac{6}{N(N+1)}\right) = 25 \quad (2.22)
\]
where one has rewritten 25 as 26 - 1 to be able to make contact with the Virasoro unitary minimal models, as well, given by the last term of eq-(2.20). This explains why the last term of (2.20) was written in such a peculiar way. Unitarity is achieved if the conformal-spin two-sector intercept is:

\[
\omega_2 = 1 - \frac{k^2 - 1}{4N(N+1)} \quad 1 \leq k \leq N - 1. \quad (2.23)
\]

A particular example of the above results is that in the ordinary non-critical \((W_2)\) string there are many ways to have \(c = 26\). Choose for arbitrary value \(x^2 = -2\) as opposed to non-arbitrary value of \(x_o^2 = -2 - \frac{1}{12}\) required by the \(q = 2\) Virasoro unitary minimal model. The central charge of the Liouville sector (the nested basis) given by eq-(2.13) reads for \(N = 2\) : \(c_L = 1(1 + 4.2.3) = 25 = 1 - 12x^2 = c_{eff}\), in this particular case the \(c_L = c_{eff}\) and the \(c_m = 26 - c_L = 1\). And viceversa, choosing the matter sector to be in the nested basis, reverses the roles of matter and Liouville, and one has \(c_m = 25; c_L = 1\) which is the standard result that the ordinary \(D = 26\) critical string can be seen as a non-critical string in \(D = 25\) if one adjoins the Liouville mode that plays the role of the extra dimension.

It is not surprising in this picture of non-critical \(W_\infty\) strings and quantization of \(W_\infty\) gravity, to understand why there is the ubiquitous presence of \(W\) symmetry constraints in non-critical strings in \(D \leq 1\) and matrix models in relation to the theory of integrable hierarchies. Paraphrasing [11] : “the partition function of the (multi) matrix model, \(Z\), which is related to the partition function of some low-dimensional string theory, or equivalently, two-dimensional gravity coupled to some \(d \leq 1\) matter system, \(Z\), via the relation \(\mathcal{Z} = log Z\), is a special solution of such integrable hierarchy. Special in the sense that it satisfies an extra constraint known as the string equation. In fact, \(Z\) is itself subject to an infinite number of constraints which form a Virasoro or \(W\) algebra “. Furthermore, \(W\) gravity coupled to \(W\) matter is related to topological coset models [11]. Whithin our picture described here it is no surprise to see the appearance of \(W\) symmetries in non-critical strings. One of the major advantages of \(W\) conformal field theories is that allows the passage! of the string \(c = 1\) barrier.

An important remark is in order : we have to emphasize that one should not confuse \(c_{eff}\) with \(c_m, c_L\) in the same way that one must not confuse \(x^2\) with \(x_o^2\). The ordinary \((W_2)\) string is a very special case insofar that \(c_{eff} = c_m\) or \(c_L\) depending on our choice for the nested basis. The \(D = 27 X^\mu\) spacetime interpretation of the theory is hidden in the stress energy tensor of the \(\sigma^1\) field \(T(\sigma_1) \to T(X^\mu)\) with \(c_{eff} = c(D) = D = 27\). And, in addition to the \(27 X^\mu\), one still has the infinite number of scalars \(\phi_1, \phi_2, \ldots\) and the infinite number of remaining fields \(\sigma_2, \sigma_3, \ldots\) in the Liouville sector. Clearly the situation is vastly more complex that the string.

From eqs-(2.12,2.13) one can infer that the value of the central charge of the matter sector, after a zeta function regularization, is \(c_m = 2 + \frac{1}{24}\). The value of \(c_L = -4 - \frac{1}{24}\). And \(c_{eff} = 27\). The value of \(c_m\) after regularization corresponds to the central charge.
of the first unitary minimal model of the $WA_{n-1}$ after $n$ is analytically continued to a negative value of $n = -146 \Rightarrow c(n) = 2(n-1)/(n+2) = 2 + \frac{1}{24}$ [14]. The value of $c_L$ does not correspond to a minimal model but nevertheless corresponds to a very special value of $c$ where the $WA_{n-1}$ algebra truncates to that of the $W$ algebra associated with non-compact coset models [14] for specific values of the central charge:

$$WA_{n-1} \Rightarrow W(2, 3, 4, 5) \sim \frac{\hat{sl}(2,R)_n}{U(1)}. \quad (2.24)$$

this occurs at the value of $c(n) = 2(1-2n)/(n-2) = -4 - \frac{1}{24}$ for $n = 146$. This is another important clue that $W_\infty$ conformal field theory, with its Kac-Moody algebra extensions, rational and irrational, should reveal to us important information of the membrane spectrum and its moduli space of vacua.

The study of non-critical $W_\infty$ strings is very complicated in general. For example, $W(2, 3, 4, 5)$ strings are prohibitively complicated. One just needs to look into the cohomology of ordinary critical $W_{2,s}$ strings to realize this [12]. Nevertheless there is a way in which one can circumvent this problem when one restricts to the self dual solutions of the membrane. The answer lies in the integrability property of the continuous Toda equation [4]. In the previous subsection we have shown how the exact quantization of the the Toda theory is automatically obtained by a straightforward dimensional reduction of the co-adjoint orbit quantization method described by [25,26]. Furthermore, the quasi-finite highest weight irreducible representations of the $W_{1+\infty}, W_\infty$ algebras [30] allows to classify the co-adjoint orbits associated with these representations.

We are going to proceed and calculate the value of the central charge of the Toda theory without the need to quantize it explicitly! The quantum Toda theory has for central charge the value given in (2.13) for the specific value of $x^2$ found in eq-(2.21). i.e. if the BRST quantization of the continuous Toda action is devoid of $W_\infty$ anomalies the net central charge of the matter plus Toda sector must equal $-2$ as we saw in eq-(2.12); i.e. $c_L = c_{Toda}$ ( after regularization).

The central charge of the quantum $A_{N-1}$ Toda theory obtained from the quantum Drinfeld-Sokolov reduction of the ( noncompact version of $SU(\infty)$) $SL(N, R)$ Kac-Moody algebra at level $k$ [11] is :

$$c_{Toda} = (N - 1)(1 - N(N + 1))(\frac{k + N - 1}{k + N})^2. \quad c_m + c_{toda} = -c_{gh} = c^{crit}. \quad (2.25)$$

Another way of rewriting eq-(2.25) is from the Drinfeld-Sokolov reduction process :

$$c_{DS} = (N - 1) - 12|\beta\rho - \frac{1}{\beta}\rho''|^2; \quad \beta = \frac{1}{\sqrt{k + N}}. \quad (2.26)$$

$\rho, \rho''$ are the Weyl weight vectors of the $A_{N-1}$ Lie algebra and its dual, respectively. One can read now the value of $x^2$ directly from eq-(2.13) and eq-(2.25) by equating $c_{Toda} = c_L$.
The last equation allows us to compute explicitly the value of the coupling constant appearing in the exponential function that gives the interaction potential of the quantum Toda theory [6,15]. The Toda theory is conformally invariant and the conformally-improved stress energy tensor obeys a Virasoro algebra with an adjustable central charge whose value depends on the coupling constant of the exponential potential term appearing in the Toda action: \( c(\beta) \). As pointed out in [11], it turns out that simply replacing the BRST operators by a normal-ordered version does not yield a nilpotent operator. In addition one has to allow for possible (multiplicative) renormalizations of the stress-energy tensor appearing in the BRST charge. This is the origin of the \( \beta = 1/\sqrt{k + N} \) factors.

One may immediately notice that the expression for (2.27) is invariant under the exchange of \( \beta \to 1/\beta \), the exchange of strong/weak coupling, does not alter the value of the central charge. This a good sign consistent with \( S \) duality symmetry of the alleged fundamental description of the membrane/string: \( M, F, \ldots \) theory.

One can now relate the value of the level, \( k \), of the \( SL(N,R) \) Kac-Moody algebra and \( N \) in such a way that \( k + N = \beta^{-2} \) is a finite number when \( N \to \infty \), :

\[
2x^2 = -\frac{13}{3} = (\beta - \frac{1}{\beta})^2 = \left( \frac{1}{\sqrt{k + N}} - \sqrt{k + N} \right)^2 = \frac{(k + N - 1)^2}{(k + N)}.
\] (2.27)

The fact that \( \beta = (k + N)^{-1/2} \) is purely imaginary should not concern us. There exist integrable field theories known as Affine Toda theories whose coupling is imaginary but possesses soliton solutions with real energy and momentum [15]. A natural choice is : \( k = -\infty \) so that \( k + N = \beta^{-2} \) is finite when \( N \to \infty \).

The connection to the unitary Virasoro minimal models was established in eq-(2.22)(set \( q = N + 1 \)) :

\[
D - 2 = 25 = c_{string} - [1 - \frac{6}{q(q + 1)}] = 26 - [1 - \frac{6}{(N + 1)(N + 2)}].
\] (2.29)

This shall guide us in repeating the arguments for the supersymmetric case. Similar arguments leads to \( D = 11 \) in the supermembrane case [1]. The argument proceeds as follows :

Since 10 is the critical dimension of the ordinary superstring the value of the central charge when one has 10 worldsheet scalars and fermions is \( 10(1 + 1/2) = \frac{30}{2} \). In order to have the central charge of a critical super \( W_\infty \) string one requires to have also the central charge of the super Virasoro unitary minimal superconformal models : \( c_{Virasoro} = 3/2 \). The supersymmetric analog of the r.h.s of (2.29) is then :

\[
10(1 + 1/2) - c_{superconformal} = \frac{30}{2} - \frac{3}{2} = \frac{27}{2}.
\] (2.30)
The supersymmetric analog of the term $c_{m_o} = \frac{2(N-1)}{N+1} \rightarrow 2$, is: $2(1+1/2) = 3$. One chooses the parameter $x^2$ in order to make contact with the bosonic sector of the $q = N + 1$ unitary minimal model of the super $W_N$ algebra in the $N \rightarrow \infty$ limit. Writing down the corresponding supersymmetric analog of each single one of the terms appearing in the r.h.s of eq-(2.20), and the same for the l.h.s, one has that $D X^\mu$ and $D \psi^\mu$ (anticommuting spacetime vectors and world sheet spinors) *without* background charges yield a central charge $D(1 + 1/2) = \frac{3D}{2}$; Therefore, the supersymmetric extension of the corresponding terms of eq-(2.20) yields:

$$\frac{3D}{2} = [10(1+1/2) - 3/2] + [2(1+1/2)] = 33/2 \Rightarrow D = 11. \quad (2.31)$$

Concluding, one obtains the expected critical dimensions for the (super) membrane if one *adjoins* a $q = N + 1$ unitary (super) conformal minimal model of the (super) $W_N$ algebra to a critical (super) $W_N$ string spectrum in the $N \rightarrow \infty$ limit. This all suggests that a sector of the physical (super) membrane spectrum could be obtained exactly the same way. Hence, in a heuristic manner, we conjecture that: there is a sector of the physical (super) membrane spectrum that should be related to the non-critical (super) $W_\infty$ string constructed above. Furthermore, the quantum (super) membrane must be related to the quantization of (super) $W_\infty$ gravity. Further arguments that support our conjecture are given below.

What is required now is to quantize, upfront, the membrane and to formulate the no-ghost theorem in order to confirm, if true, our conjecture. This is a very difficult problem. The full-fledge membrane quantization is a more arduous task. As explained in the previous subsection, the self dual sector is just the $SU(\infty)$ Self Dual Yang-Mills theory that can be related to the Toda theory after the dimensional reduction. In view of our findings about interpreting the membrane as a non-critical $W_\infty$ string with $c_{\text{matter}} + c_L = -2$ (eq-(2.12)) and $c_L = c_{\text{Toda}}$, (eqs-(2.13,2.25)), suggests that a large sector of the physical membrane spectrum, in addition to the one obtained by adjoining $q = N + 1 \rightarrow \infty$ unitary minimal $W_N$ conformal matter to a critical $W_\infty$ string, might be obtained by adjoining to the full quantized Toda (or SDYM theory) the remaining infinity of $W_\infty$ conformal scalar matter fields: $\phi_1, \phi_2, \ldots, \phi_k, \ldots$ with the provision that the parameter $x$ is fixed by $c_{\text{eff}}(x) = 1 - 12x^2 = 27$ and $c_m + c_L = -2$. (Similar considerations apply to the supermembrane).

Choosing a different value for $x$ and integer values for $c_{\text{eff}}$ suggests non-critical membrane backgrounds. An important difference between the ordinary critical string and the critical (super) membrane is that the latter requires an infinite number of fields in the Toda sector (a Liouville sector, $c_L$), an infinite number of fields in the matter sector (with central charge $c_m$) and the extra ($D = 11$) $D = 27$ spacetime coordinates ($c_{\text{eff}}$); in contradistinction to the critical string that only requires matter, $c_m = c_{\text{eff}}$ and no Liouville sector. This explains why 4D Self Dual Gravity (which upon dimensional reduction yields the continuous Toda) must be a crucial player in the membrane quantization as pointed out by Jevicki in a different context [23].

Some time ago we were able to show that the $D = 4$ $SU(\infty)$ (super) SDYM equations (an effective 6 dimensional theory) can be reduced to 4D (super) Plebanski’s Self-Dual
Gravitational equations with spacetime signatures (4, 0); (2, 2). The symmetry algebra of $D = 4$ $SU(\infty)$ SDYM is a Kac-Moody extension of $W_{\infty}$ as shown recently by [31]. In particular, new hidden symmetries were found which are affine extensions of the Lorentz rotations. These new symmetries form a Kac-Moody-Virasoro type of algebra. By rotational Killing-symmetry reductions one obtains the $w_{\infty}$ algebra of the continuous Toda theory. For metrics with translational Killing symmetries one obtains the symmetry of the Gibbons-Hawking equations.

The relevance of Kac-Moody extensions of $W_{\infty}$ algebras has also been pointed out by Jevicki who has shown that the 4$D$ membrane in the lightcone gauge yields a four dimensional world volume structure related to dilatonic-self dual gravity plus matter. The quantum theory is defined in terms of a $SU(\infty)$ Kac-Moody algebra. The quantization of 4$D$ self dual gravity via the coadjoint orbit method has a hidden $SL(\infty, R)$ Kac-Moody algebra in the lightcone gauge. This is just the noncompact version of the $SU(\infty)$ Kac-Moody algebra. The presence of matter is also consistent with the presence of the matter fields $\phi_1, ..., \phi_k$ with a central charge $c_m$. The matter terms [23] appear as a current-current interaction, $J^2$, where the four-dimensional field $J(x, t, \sigma^1, \sigma^2)$ found by Jevicki, originated from a $SU(\infty)$ current algebra that accounts for the extra two-indices $\sigma^1, \sigma^2$.

This is similar to what we just found : 4$D$ SD Gravity is a reduction of $D = 4$ $SU(\infty)$ SDYM and the former is reduced to $\to$ the continuous 3$D$ Toda theory. The Toda sector appears in noncritical $W_{\infty}$ strings with the infinite number of scalars ($W_{\infty}$ conformal matter), $\phi_1, \phi_2, ....$. This construction must be related to the membrane’s spectrum. What is left is the presence of the scalar dilaton $\alpha(x, t)$ [23]. 2$D$ dilatonic supergravity was studied by Ikeda within the context of a nonlinear gauge theory principle : one does not have a Lie bracket structure. The nonlinear gauge principle allowed the author to construct a non-linear bracket that led to nonlinear $W_{\infty}$ algebras directly from nonlinear integrable perturbations of 4$D$ self dual gravity. Hence, it seems that nonlinear (but still integrable) perturbations of self dual gravity span a richer sector than $W_{\infty}$ strings so it is very plausible that it is nonlinear noncritical $W_{\infty}$ strings that bear a closer connection to the full membrane. What is warranted is to establish, if possible, the relationship between the matter sector of Jevicki’s Hamiltonian and ours. Integrable but linear deformations of self dual gravity were studied by Strachan [32]. The original $W_{\infty}$ algebra was constructed by [16]. The interpretation of such algebras as Moyal bracket deformations of the Poisson structure associated with the area-preserving diffs was found by [17,18].

This completes the review of [1]. We hope that we’ve clarified the interplay between 3$D$ and 2$D$ that appears after the light-cone gauge for the membrane is chosen and the importance that noncritical linear (nonlinear) $W_{\infty}$ strings; i.e. $W_{\infty}$ conformal field theory must have in the understanding of the membrane. Quantum Group extensions will come as no surprise. $W_{\infty}$ symmetries in string theory were also discussed by Zaikov [7].

Acknowledgements

We thanks G. Sudarshan for advice. To Y. Ne’eman for having introduced me to membranes long before their current fashion; and to the World Laboratory, Lausanne, Switzerland for financial support.

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