Bandits with adversarial scaling

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Abstract

We study adversarial scaling, a multi-armed bandit model where rewards have a stochastic and an adversarial component. Our model captures display advertising where the click-through-rate can be decomposed to a (fixed across time) arm-quality component and a non-stochastic user-relevance component (fixed across arms). Despite the relative stochasticity of our model, we demonstrate two settings where most bandit algorithms suffer. On the positive side, we show that two algorithms, one from the action elimination and one from the mirror descent family are adaptive enough to be robust to adversarial scaling. Our results shed light on the robustness of adaptive parameter selection in stochastic bandits, which may be of independent interest.

1 Introduction

The multi-armed bandit setting is the cleanest paradigm to capture the tension between exploring information about the underlying system and exploiting the most profitable actions based on the current information. A decision-maker (or learner) repeatedly selects among a set of k actions also referred to as arms, earns the reward of the selected arm, and obtains feedback only about it. This creates a direct trade-off between learning about the performance of underexplored actions and earning the reward from those that seem the most profitable. This trade-off is prominent in applications such as display advertising where a platform needs to repeatedly select which ad to show in response to a particular pageview. The arms therefore correspond to the competing ads and, upon being shown, each ad can result in a click or not. We use this as a running example and assume that the goal of the learner is to maximize the total number of clicks (in practice, there are other goals such as revenue and user-experience, but we ignore those for simplicity).

Modeling this problem, one soon realizes that the two classical multi-armed bandit approaches fail to capture the essence of the setting. The two main approaches in the multi-armed bandit learning literature assume that the rewards obtained from each pageview are either completely adversarial or coming from identical and independent distributions (i.i.d.). However, clearly some ads have better quality and are consistently more clickable than others, so assuming a fully adversarial model seems to ignore a lot of useful structure in the data. On the other extreme, stochastic models assume that each ad has a fixed probability of being clicked whenever displayed (this is typically referred to as click-through-rate). In practice, click-through-rates are known to vary due to various factors: time of the day, day of the week, seasonalities (e.g. users tend to click more before Christmas and on Black

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Friday) but those factors affect ads uniformly. Tracà and Rudin [TR15] propose a model where the mean \( \mu_t(a) \) reward of an arm \( a \) at time \( t \) is a product

\[
\mu_t(a) = q^t \cdot \theta(a)
\]

of the intrinsic quality of the arm and a seasonality term \( q^t \). In the display ads example, \( q^t \) is the clickiness of the user behind his page-view and \( \theta(a) \) is the intrinsic quality of the ad. The motivation of Tracà and Rudin [TR15] comes from retail: \( \theta(a) \) is the effect certain action has in the store (the product price, which items are on sale...) and \( q^t \) is the number of customers in the store in that day. In their model, the seasonality effect \( q^t \) is known to the algorithm.

We depart from their model by assuming we don’t have access to \( q^t \). In display ads, the clickiness of users can be a function of a large number of covariates and it affects users in complex ways: certain users click more during the evening while other users click more during the day. Clickiness of users in China is affected by Chinese New Year while clickiness of users in the US is affected by Thanksgiving. Estimating all those patterns is costly so it is useful to have models that exploit the structure of \( \mu_t(a) \) without having to estimate the seasonality/clickness model directly.

To obtain more robust algorithms we assume \( q^t \) is adversarially chosen and hence the name adversarial scaling. If we think of \( q^t \) as a seasonality effect, there is not an adversary per se in the motivation, but rather a very complex pattern which we choose not to assume anything about to get robust algorithms. However, \( q^t \) can also model some actually adversarial effects. For example, an advertiser may create a botnet that does not click anything in the initial time, i.e. \( q^t = 0 \), aiming to prolong the algorithm’s exploration and hence her number of displays. This attack is very simple to run as it does not actively do anything and causes a cold start to the algorithm as the latter needs to appropriately disregard these insignificant samples.

As we demonstrate in this paper, despite the inherent stochasticity in these settings (best arm is better at each single round), we show that most stochastic algorithms perform poorly. This naturally generates the following question:

**What makes bandit algorithms robust to adversarial scaling attacks?**

**Our contribution.** Tackling this question, we demonstrate that adaptive parameter selection is essential for robustness to adversarial scaling. In particular, we show a modification of the classical Action Elimination algorithm [EMM06] which we term AAEAS that uses the reward of the algorithm as a proxy for the number of rounds that matter, i.e. the total scaling \( \sum_t q^t \). This allows it to adapt to the intrinsic qualities \( \theta(a) \) without being misled by the adversarial scaling in a way similar to the self-confident learning technique developed for small-loss bounds in adversarial bandits. The resulting guarantees also hold with high probability which is an advantage of this method.

Aiming to understand whether this robustness is satisfied more broadly, we then focus on the Online Mirror Descent with Log-Barrier algorithm of Foster et al. [FLL+16]. Interestingly, when combined with the neat doubling trick of Wei and Luo [WL18], this algorithm (then termed BROAD) seamlessly adapts to the total scaling. Its resulting guarantees hold only for the weaker pseudo-regret notion but are stronger by a factor of \( k \) and the empirical performance is also enhanced.

To complement our study, we show two very simple attacks manage to make, to the best of our knowledge, all other stochastic algorithms perform ineffectively (detailed comparison in Section 4). Surprisingly these attacks work even against the recent breakthrough, Online Mirror Descent with Tsallis entropy [ZS19]. The latter has optimal regret guarantee for both stochastic, adversarial
setting as well as regimes in between and is therefore considered as the best algorithm for such intermediate settings.

The first attack is a purely stochastic setting where all the arms have fixed but really small means $\mu(a)$ where $\mu(a^*)$ is the mean of the best action. This can be viewed as adversarial scaling with $\theta(a) = \frac{\mu(a)}{\mu(a^*)}$ and $q^t = \mu(a^*)$. The bounds for these algorithms scales inversely to the absolute difference $\mu(a^*) - \mu(a)$. In contrast, AAEAS and BROAD have bounds based on the normalized difference $1 - \frac{\mu(a)}{\mu(a^*)}$ which provides great improvement when the means are small as typical in click-through-rates for display advertising.

The second attack is a cold start attack where in some initial period, the $q^t = 0$; this can arise because either maliciously or organically as we discuss in the end of Section 4.2. The number of rounds that have passed becomes a rather irrelevant quantity in this case as what really matters is the $\sum q^t$. However, all algorithms other than AAEAS and BROAD use this number of rounds (many of them in ways that seem not fixable). The cold start therefore leads them to a really bad prior state and they take a very long time to recover. Surprisingly, the most effective stochastic algorithm Thompson Sampling [AG17] is prone to this attack when the cold start is really short (see Figure 4).

Related work. Our paper lies in a broader line of work that tries to achieve enhanced guarantees when data exhibit a particular nice structure. The main structure that has been utilized is the data being i.i.d. across time where many algorithms achieve logarithmic guarantees at the existence of a large gap in $\mu(a^*) - \mu(a)$. Since we compare to these algorithms, we elaborately review them in Section 4. Beyond stochasticity, other approaches aim to enhance adversarial guarantees when some nice properties exist in the data. Example properties include small variance of the losses [HK09, WL18], small effective loss range [CBS18], small variation in the losses across rounds [BLLW19], small loss of the best arm [AAGO06, Neu15, FLL+16, LST18, AZBL18], second-order excess loss [WL18], locally perturbed adversarial inputs [SS17] among others.

Finally, our model can be cast as a rank-1 assumption on the structure of the reward function. Rank-1 assumptions have been explored in various ways in the bandits literature, for example the rank-1 bandit model [KKS+16] and factored bandits [ZS18] assume that the action space have a cartesian product and hence the reward in each period can be represented as a matrix over which the rank-1 assumption is made. In other words, this rank-1 structure is on the space of actions unlike our model which is in the space of actions cross time. Another common setting where rank-1 assumptions are made is in semi-parametric bandits [KWS18] where the loss function is a product of unknown parameters and features. A difference with respect to our model is that the features are observed by the learner while in our model the quality parameter $q^t$ is never observed by the learner.

2 Model

We study a multi-armed bandit setting with $k$ arms where each arm $a \in [k]$ is associated with an intrinsic mean parameter $\theta(a) \in [0,1]$ which is unknown to the learner. We define $F^t(a)$ as the distribution of the reward of arm $a$ in round $t$ and we assume that it has positive measure only on the interval $[0,1]$. The distributions are adaptively selected by the adversary subject to the constraint
that the means \( \mu_t(a) = \mathbb{E}_{r \sim F_t(a)}[r] \) must satisfy a rank-1 constraint, described below. Formally, the protocol between the learner and the adversary at each round \( t = 1..T \) is as follows:

- The learner chooses a distribution \( p^t \) over the \( k \) arms.
- The adversary chooses an adversarial quality parameter \( q^t \in [0, 1] \) and distributions \( F^t(a) \) supported in \([0, 1]\) with mean \( \mu^t(a) = q^t \cdot \theta(a) \).
- Rewards \( r^t(a) \sim F^t(a) \) are drawn.
- Learner draws \( a^t \sim p^t \) and observes \( r^t(a^t) \).

For ease of presentation, we assume that the highest intrinsic mean is equal to 1, i.e. \( \max_a \theta(a^*) = 1 \). This can be done by appropriately scaling down all the adversarial qualities replacing \( \theta(a) \) by \( \frac{\theta(a)}{\max_a \theta(a^*)} \) and \( q^t \) by \( q^t \cdot \max_a \theta(a^*) \).

\[ \text{Regret} = \max_{a \in [k]} \mathbb{E} \left[ T \sum_{t=1}^{T} r^t(a^*) - r^t(a^t) \right] = T \sum_{t=1}^{T} \mu^t(a^*) - \mu^t(a^t) = T \sum_{t=1}^{T} q^t \cdot \Delta(a^t), \]

where \( \Delta(a) = \theta(a^*) - \theta(a) = 1 - \theta(a) \). In other words, the pseudo-regret is worst-case over the sequence \( q(t) \) but it is in expectation over the draws \( r^t(a) \sim F^t(a) \).

3 Algorithms robust to adversarial scaling

In this section, we show that two algorithms, one that we introduce from the active arm elimination family (Section 3.1) and one existing from the mirror descent family (Section 3.2) achieve the desired robustness to adversarial scaling. A common property of both algorithms is their adaptivity on the number of rounds; they do not scale with the number of rounds which is easily targeted by adversarial scaling attacks.\(^1\) In the next section, we show that other algorithms with improved guarantees in the stochastic regime that do not enjoy such adaptivity have their performance severely compromised at the presence of adversarial scaling.

3.1 Active Arm Elimination with Adversarial Scaling

The algorithm we introduce is based on Active Arm Elimination of Even-Dar et al. [EMM06], but is appropriately adapted to handle the adversarial scaling of the rewards.

Classical Active Arm Elimination keeps a set of active arms (initially all arms) and selects arms in a round-robin fashion among the active arms, updating the empirical mean of the selected arm

\(^1\)The need for such adaptivity was only known in adversarial settings prior to our work.
as well as a confidence interval around it. The latter ensures that when samples are i.i.d. \( q^t \) same across rounds), with high probability, the actual mean lies within the confidence interval (this comes from an application of a Chernoff bound). Once the confidence intervals cease overlapping, we are confident that the dominated arm is not the arm with the highest intrinsic quality, thus it is safe to eliminate it from the active arms and never select it again.

Trying to extend this approach to adversarial scaling, one encounters some difficulties. One issue is that classic Active Arm Elimination selects arms deterministically. The adversary can easily cause linear regret to any deterministic policy by setting \( q^t = 0 \) anytime the algorithm is about to pull the optimal arm. The natural way to get around this difficulty is to pull arms in the active set randomly. A more serious issue is that the number of samples (arm pulls) is no longer a meaningful quantity. The adversary may provide many samples initially with quality \( q^t = 0 \). These samples do not help in informing our estimates about where the intrinsic qualities \( \{ \theta(a) \} \) lie; therefore treating these as real samples can provide a misleading picture of the confidence intervals. Ideally, we would like to use as effective samples the total adversarial quality at rounds where we selected each arm \( a \), i.e. \( \sum_{t:a^t = a} q^t \). Since we do not have access to this quantity, we need to design our confidence intervals in a way that will be robust to this adversarial scaling.

The main idea behind robustifying our confidence intervals is to use the reward of our algorithm as a proxy of the total effective samples of each arm. At the rounds that we selected arm \( a^* \), the reward of the algorithm is, in expectation, equals to the effective samples of \( a^* \) (since \( \theta(a^*) = 1 \)). Since we select each non-eliminated arm with equal probability, the total reward of the algorithm \( S \) serves as proxy for the effective rounds of each arm (\( S/k \) is a lower bound and \( S \) is an upper bound). Our algorithm, formalized in Algorithm 1, uses the proxy \( S \) instead of the unknown effective samples to construct confidence intervals for the arms; in fact, there is a single confidence interval \( CB(S) \).

**Algorithm 1** Active Arm Elimination with Adversarial Scaling (AAEAS)

Initialize the set of active arms \( A = [k] \), the aggregate reward for each arm \( R(a) = 0 \), and the total reward collected by the algorithm \( S = 0 \).

for \( t = 1 \ldots T \) do

Select \( a^t \) randomly across the set of active arms \( A \) and earn reward \( r^t(a^t) \)

Update the total reward earned:

\[
S \leftarrow S + r^t(a^t)
\]

Update empirical reward of selected action:

\[
R(a^t) \leftarrow R(a^t) + r^t(a^t).
\]

Eliminate arms based on algorithm-induced confidence intervals, i.e. remove all \( a' \) from \( A \) if

\[
R(a') + CB(S) < \max_{a \in A} R(a)
\]

for confidence bound (setting \( \delta' = \frac{(k+1)T}{\delta} \)):

\[
CB(S) := 2\sqrt{\max(4S \log(2/\delta'), 16k \log^2(2/\delta'))}
\]

end for
Theorem 3.1. The AAEAS algorithm (Algorithm 1) run with $\delta = \frac{1}{T}$ has pseudo-regret at most:

$$O \left( \sum_{a \neq a^*} k \log \frac{kT}{\Delta(a)} \right)$$

The proof follows the standard active arm elimination analysis but replaces the samples of each arm by the proxy-created confidence interval. For completeness, we provide the Chernoff bound which we use in our analysis.

Lemma 3.2 (standard Chernoff bound). Let $\{X_i\}$ be independent random variables in $[0, 1]$, $\mu = E[\sum_i X_i]$, $\epsilon > 0$:

$$\Pr \left[ \sum_i X_i - \mu \geq \epsilon \mu \right] \leq 2e^{-\frac{\mu \epsilon^2}{3}}$$

Proof of Theorem 3.1. To prove the guarantee, we need to ensure two properties: i) the arm $a^*$ with highest intrinsic quality, with high probability, never gets eliminated and ii) we can bound the regret incurred by each suboptimal arm.

Our analysis is based on a few events occurring and we bound the failure probability of these events. We separate the time horizon in phases where a phase $\phi$ begins when exactly $\phi$ arms have been eliminated; denote by $\tau_\phi$ the round that the $\phi$-th arm gets eliminated. First, the empirical reward $R(a)$ that we experience for each arm is close to its expected reward $\bar{R}(a)$, that is, for each arm $a$ and each round $t$, with probability $1 - \delta'$:

$$R(a) \in \left[ \bar{R}(a) \pm \sqrt{2\bar{R}(a) \log(2/\delta')} \right]$$

where $\bar{R}(a) = \sum_{\phi=0}^{k-1} \sum_{\tau=\tau_\phi+1} \frac{\theta(a) \cdot q_\tau}{k - \phi}$

This follows by a Chernoff bound (Lemma 3.2), since the rewards at round $t$ are supported in $[0, 1]$. In a similar fashion, we can provide a similar confidence bound for the reward of our algorithm. For any round $t$, with probability $1 - \delta'$:

$$S \in \left[ \bar{S} \pm \sqrt{2\bar{S} \log(2/\delta')} \right] \quad \text{for} \quad \bar{S} = \sum_a \bar{R}(a)$$

Setting all failure probabilities to $\delta' = \frac{\delta}{(k+1)T}$, the probability that any of them fails is at most $\delta$. In the remainder of the proof, we assume that none of these bounds fails.

First property. First, we establish that, when these bounds do not fail, the arm $a^*$ with the highest intrinsic mean does not become eliminated. Since the mean for arm any $a \neq a^*$ is at most the mean for $a^*$, equation (1) implies:

$$R(a) - R(a^*) \leq 2\sqrt{2\bar{R}(a^*) \log(2/\delta')}$$

We now show that, at every round, this difference is covered by the confidence bounds of our algorithm, i.e:

$$\text{CB}(S) \geq 2\sqrt{2\bar{R}(a^*) \cdot \log(2/\delta')}$$

(3)
If \( \bar{R}(a^*) \leq 8k\log(1/\delta') \) then \([3]\) holds as the second term of \( \text{CB}(S) \) is \( 2\sqrt{16k\log^2(2/\delta')} \geq 2\sqrt{2\bar{R}(a^*) \cdot \log(2/\delta')} \). Otherwise, \([2]\) and the optimality of \( a^* \) imply:

\[
S \geq \bar{S} - \sqrt{2\bar{S}\log(2/\delta')} \geq \bar{R}(a^*) - \sqrt{2k\bar{R}(a^*) \log(2/\delta')} \geq \frac{\bar{R}(a^*)}{2}
\]

Hence, \([3]\) holds as the first term of \( \text{CB}(S) \) is \( 2\sqrt{4S\log(2/\delta')} \geq 2\sqrt{2\bar{R}(a^*) \cdot \log(2/\delta')} \). Since, in both cases, \([3]\) holds, arm \( a^* \) does not become eliminated.

**Second property.** Now we bound the regret coming from each suboptimal arm. Let’s consider the contribution to regret of arm \( a \neq a^* \) with gap \( \Delta(a) \). First we bound the difference of empirical rewards of two arms. By \([1]\) and the fact that \( \theta(a^*) = 1 \) we have that:

\[
R(a^*) - R(a) \in \left[ \Delta(a) \cdot \bar{R}(a^*) \pm 2\sqrt{2\bar{R}(a^*) \log(1/\delta')} \right]
\]

We want to argue that once \( \bar{R}(a^*) \) is large enough, the difference between the two arms \( R(a^*) - R(a) > \text{CB}(S) \), which leads to arm \( a \) getting eliminated and not contributing further regret. Let \( T(a) \) denote the time of the elimination of arm \( a \). Since arm \( a \) is selected with equal probability with other active arms until then, the total expected regret from \( a \) is at most:

\[
\Delta(a) \cdot \sum_{\phi = 0}^{k-1} \min(\tau_{a+1}, T(a)) \sum_{\tau_{\phi}+1} \frac{q_t}{k - \phi} \leq \Delta(a) \cdot \bar{S}^{T(a)}
\]

where \( \bar{S}^{T(a)} \) denotes the value of \( \bar{S} \) at round \( T(a) \); recall that the latter is defined in \([2]\). We will argue that arm \( a \) is eliminated by the time we have:

\[
\bar{S}^{T(a)} > \frac{128k \log(2/\delta')}{\Delta(a)^2},
\]

In order for an arm to be eliminated the difference \( R(a^*) - R(a) \) needs to be relatively large. This difference can be expressed as:

\[
R(a^*) - R(a) \geq \bar{R}(a^*) - \bar{R}(a) - 2\sqrt{2(\bar{R}(a^*)) \log(2/\delta')}
\]

\[
= \Delta(a) \cdot \bar{R}(a^*) - 2\sqrt{2\bar{R}(a^*) \log(2/\delta')}
\]

\[
\geq \Delta(a) \cdot \frac{\bar{S}^{T(a)}}{k} - 2\sqrt{2\frac{\bar{S}^{T(a)}}{k} \log(2/\delta')}.
\]

The first inequality follows by applying \([1]\) for both arm \( a \) and \( a^* \), and noting that \( \bar{R}(a^*) \geq \bar{R}(a) \). The second inequality is since \( \theta(a^*) - \theta(a) = \Delta(a) \). The third inequality holds since the function \( x \mapsto \Delta(a)x - 2\sqrt{2x \log(2/\delta')} \) is increasing for \( x \geq \frac{2\log(2/\delta')}{\Delta(a)^2} \) which holds in \([5]\).

We can re-arrange the terms in equation \([5]\) and obtain:

\[
\Delta(a) \frac{\bar{S}^{T(a)}}{2k} \geq 2\sqrt{2\frac{\bar{S}^{T(a)}}{k} \log(2/\delta')}
\]
Combining (6) and (7) with the fact that $\Delta(a) \leq 1$, it holds:

$$R(a^*) - R(a) \geq \frac{S^T(a)}{2k} \Delta(a) \geq \sqrt{32S^T(a)\log(2/\delta')}.$$  

The latter term is greater than the second term of the confidence bound $CB(S)$. Comparing to the first, denoting $S^T(a)$ the value of $S$ at $T(a)$, it holds that $S^T(a) \leq 2\tilde{S}^T(a)$ by (2) and (5). Hence, the latter RHS also dominates the first term of the confidence bound $CB(S)$. As a result, if arm $a$ was not eliminated by then, it gets eliminated when the expected reward of the algorithm becomes $\tilde{S}^T(a) = \frac{128k\log(2/\delta')}{\Delta(a)^2}$; by (4) this implies that the expected contribution of arm $a$ to the regret is at most $\frac{128k\log(2/\delta')}{\Delta(a)^2}$. Summing across all suboptimal arms $a \neq a^*$ and setting $\delta = 1/T$ completes the proof. 

\[\square\]

**Remark 3.3.** A simple bound on the expected regret from each arm is $\Delta(a) \cdot \sum_t q(t) \leq \Delta(a) \cdot T$ so as usual so if there are arms very close to the optimal arm (or more than one optimal arm) then the proof directly implies the bound of: $O\left(\sum_a \min\left(\Delta(a) \cdot \sum_t q(t), \frac{k\log(k \sum_t q(t))}{\Delta(a)}\right)\right)$

**Remark 3.4.** While the proof is written in expectation over draws $r^t(a_t) \sim F_t^a$ all arguments are high-probability arguments. The bound therefore can be converted to a high probability regret bound (as usual in stochastic bandits) with the difference that instead of capping the performance of each arm by $\Delta(a) T$ as in the previous remark, we cap it by $\sqrt{T}$. Formally, we obtain that with probability $1 - \delta$ we obtain the following bound on actual regret:

$$O\left(\sum_a \min\left(\sqrt{T}, \frac{k \cdot \log(kT/\delta)}{\Delta(a)}\right)\right)$$

### 3.2 Online Mirror Descent with Log-Barrier

The second algorithm that we show to be robust to adversarial scaling lies in the mirror descent family and has a stronger regularizer, log-barrier. It was initially suggested by Foster et al. [FLL+16] for a fixed learning rate $\eta$, who proved that it attains first-order bounds for pseudo-regret. The update on the probabilities is the following (Algorithm 3 in [FLL+16] adapted to rewards):

$$p_t(a_t^{t-1}) = \frac{p_t^{t-1}(a_t^{t-1})}{1 - \eta r^t(a_t^{t-1}) + \gamma p_t^{t-1}(a_t^{t-1})}$$

$$p_t(a) = \frac{p_t^{t-1}(a)}{1 + \gamma p_t^{t-1}(a)} \quad \forall a \neq a_t^{t-1} \quad (8)$$

where $\gamma \geq 0$ is such that $p$ is a valid probability distribution.

Via using a neat doubling trick to update the learning rate $\eta$, Wei and Luo [WL18] showed that, in fact, this algorithm can also attain stochastic guarantees (Algorithm 3 in [WL18] for the particular doubling trick). In particular, the algorithm halves the learning rate and restarts once:

$$\sum_{\tau=t_{\ell R}}^{t} \sum_{a=1}^{k} p^\tau(a)^2 (\hat{r}^\tau(a) - r^\tau(a_t^\tau))^2 \geq \frac{k \ln T}{3\eta^2},$$

where $t_{\ell R}$ is the time of the last restart and $\hat{r}^\tau(a) = r^\tau(a)/p^\tau(a) \cdot 1[a = a_t^\tau]$ is the importance sampling estimator. Their algorithm is a particular instantiation of a more general framework they termed $BROAD$. 

8
We show that the BROAD algorithm (with no modification) is robust to adversarial scaling. The proof follows from replacing the potential the potential function in the proof of Theorem 10 in [WL18] to a potential that accommodates adversarial scaling. The remaining arguments are essentially the same as the ones used by [WL18] in their analysis of the stochastic setting.

**Theorem 3.5.** Online Mirror Descent with Log-Barrier with the above doubling (also known as BROAD) has pseudo-regret at most $O\left(\frac{k \log T}{\Delta}\right)$ where $\Delta$ is the minimum non-zero gap on intrinsic means.

We will use the following results proved in [WL18]. The first lemma corresponds to equation (27) in their paper and the second to equation (29). They are restated here for rewards instead of losses.

**Lemma 3.6 ([WL18]).** There is a constant $C$ such that for $p_t^i$, $r_t^i$ and $\hat{r}_t^i$ in the BROAD algorithm it holds that:

$$
\mathbb{E}\left[\sum_{t=1}^{T} r_t^i(a^*) - r_t^i(a)\right] \leq C k \ln T + C \cdot \sqrt{(k \ln T) \mathbb{E}\left[\sum_{t=1}^{T} \sum_{a=1}^{k} (p_t^i(a))^2 (\hat{r}_t^i(a) - r_t^i(a))^2\right]}
$$

**Lemma 3.7 ([WL18]).** Again in the context of BROAD, it holds that:

$$
\mathbb{E}_{a_t \in p_t^i}\left[\sum_{a=1}^{k} p_t^i(a)^2 (\hat{r}_t^i(a) - r_t^i(a))^2\right] \leq \max_a \mathbb{E}[r_t^i(a)] \cdot 2 \mathbb{E}[1 - p_t^i(a^*)]
$$

**Proof of Theorem 3.5.** The pseudo-regret of the algorithm can be expressed as:

$$
\sum_{t=1}^{T} \mathbb{E}[r_t^i(a^*) - r_t^i(a)] = \sum_{t=1}^{T} \mathbb{E}\left[\sum_{a=1}^{k} p_t^i(a)(r_t^i(a^*) - r_t^i(a))\right] 
\geq \mathbb{E}\left[\sum_{t=1}^{T} \sum_{a \neq a^*} p_t^i(a) q_t^i \Delta\right] = \Delta \mathbb{E}\left[\sum_{t=1}^{T} q_t^i(1 - p_t^i(a^*))\right] \tag{9}
$$

where the inequality holds by noting that the difference in the means is at least $q_t^i \cdot \Delta$.

Note that in the adversarial scaling setting, we can bound the term $\max_a \mathbb{E}[r_t^i(a)]$ in Lemma 3.7 by $q_t^i$ obtaining:

$$
\mathbb{E}_{a_t \in p_t^i}\left[\sum_{a=1}^{k} p_t^i(a)^2 (\hat{r}_t^i(a) - r_t^i(a))^2\right] \leq 2 q_t^i \cdot \mathbb{E}[1 - p_t^i(a^*)] \tag{10}
$$

By Lemma 3.6 as well as (10) and (9), setting as potential function $H = \mathbb{E}\left[\sum_{t=1}^{T} q_t^i(1 - p_t^i(a^*))\right]$ it holds that:

$$
H \Delta \leq \sqrt{(k \ln T) H + k \ln T}
$$

which leads to $H \leq \frac{K \ln T}{\Delta^2}$ and concludes the proof. \qed
4 Attacks against other stochastic algorithms

Besides AAE and BROAD which we previously discussed, there are few other algorithms available offering \( \log T/\Delta \) type of guarantees for stochastic bandits. In this sections we discuss how those perform in adversarial scaling settings. We describe two adversarial scaling attacks: (a) small means; and (b) cold-start. Besides AAEAS and BROAD we show that the remaining alternatives perform poorly in either of those cases. The algorithms we consider are:

- **Upper Confidence Bound (UCB)** [ACBF02] keeps track of the total reward \( r(a) \) of each arm and the number of times \( n(a) \) each arm was pulled. For each arm we compute the upper confidence bound of each arm as
  \[
  \text{Ucb}(a) = r(a) / n(a) + \sqrt{\log t / n(a)}
  \]
  and deterministically pull the arm with largest \( \text{Ucb} \). As we previously mentioned, every algorithm that deterministically selects an arm can be easily fooled by adversarial scaling by setting \( q^t = 0 \) when the algorithm is about to pull the optimal arm. We will see it is also tricked by much simpler (i.e. less adaptive) attacks.

- **Thompson Sampling (TS)** [AG17] is more easily described for the Bernoulli case where rewards are in \{0, 1\}. The algorithm keeps a Beta\((n_0(a), n_1(a))\) prior for each arm \( a \) initially set with \( n_0(a) = n_1(a) = 1 \). In each round, the algorithm takes a sample from each prior, chooses the arm \( a \) with largest sampled value, observes the reward \( r \in \{0, 1\} \) and updates the prior by increasing \( n_r(a) \) by 1.

- **EXP3++** [SS14]: While traditional EXP3 algorithms of [ACBFS95] don’t offer \( \log T/\Delta \) guarantees in stochastic settings, this modification does by introducing an exploration parameter tuned for each arm as a function of its past comparative performance. Each arm is explored with probability given by such parameter and with remaining probability a standard EXP3 algorithm is run. This algorithm retains the EXP3 guarantees in the adversarial regime.

- **Tsallis Entropy** [ZS19]: A recent breakthrough result provides an optimal algorithm (up to constants) for both stochastic and adversarial bandits via a standard mirror descent regularized by the Tsallis entropy. Remarkably, this algorithm requires no special tuning and no deviation from the standard mirror descent paradigm. The algorithm computes an unbiased estimator \( \tilde{r}(a) \) of the reward of each arm and then samples an arm from the probability distribution in the solution of the following maximization problem:
  \[
  \max_{p \in \Delta} \sum_a \tilde{r}(a) \cdot p(a) + \sqrt{\frac{4}{\sqrt{t}} \left[ \sum_a \sqrt{p(a)} - \frac{1}{2} p(a) \right]}
  \]

In Figure 1 we compare those algorithms in a purely stochastic instance with large means. As usually noted in the literature, the performance of Thompson Sampling is vastly superior than all other algorithms. On this instance, UCB, Tsallis and BROAD have similar performance, EXP3++ is somewhat worse followed by AAE and AAEAS which are notably worse. This is expected as they are the least adaptive. It is good to keep those in mind as we compare their performance on certain adversarial scaling scenarios.

4.1 Small means

Even in the absence of adversarial scaling the AAEAS and BROAD algorithm start outperforming other algorithms as the means become smaller and smaller. Consider a purely stochastic instance (i.e. \( q^t = 1 \) for all \( t \)) with only two arms with means \( 1 \geq \theta_1 > \theta_2 \geq 0 \) and let \( \Delta = \theta_1 - \theta_2 \). The
traditional pseudo-regret bound obtained by stochastic bandits algorithms is $O(\log(T)/\Delta)$. The bound obtained by AAEAS and BROAD on the other hand is $O(\theta_1 \cdot \log(T)/\Delta)$, which follows by viewing this purely stochastic problem as an instance with adversarial scaling with $q^t = \theta_1$ and two arms with $\theta_1' = 1$ and $\theta_2' = \theta_1 q^t$. In Figure 2 we compare the same algorithms in an instance with two Bernoulli arms having means 0.005 and 0.001. We see that while the performance of AAEAS and BROAD is unaffected by scaling the means down, the performance of UCB, AAE, Tsallis and EXP3++ degrades despite the fact that the relative strength of both arms remains almost the same.

Figure 1: Comparison of different bandit algorithms for a purely stochastic instance ($q^t = 1, \forall t$) with two arms with means $\mu = [0.5, 0.8]$. The cumulative in each round is the average of 100 runs of the each algorithm.

Figure 2: A purely stochastic instance ($q^t = 1, \forall t$) with two arms with small means $\mu = [0.005, 0.001]$. The cumulative in each round is the average of 100 runs of the each algorithm.

A direct consequence of Theorem 3.1 and Theorem 3.5 is the following (we treat the number of arms $k$ as a constant in order to unify the statements for AAEAD and BROAD):

**Corollary 4.1.** Given a stochastic bandit instance where rewards are supported in $[0,1]$ with means $\theta(a)$, then the performance of AAEAS and BROAD: $O(\max_a \theta_a \cdot \sum_a \log(T)/\Delta(a))$.

Interestingly, the performance of Thompson Sampling seems unaffected by adversarial scaling. It is not clear to us how to generalize the proof of [AG17] to explain the good performance of Thompson Sampling in the small mean regimes. We leave understanding this phenomenon as an open problem.
Figure 3: Comparison of different bandit algorithms on an extreme cold start instance: $q^t = 0$ for $t < t_0 = 10^7$ and $q^t = 1$ afterwards. Arm means are $\mu = [0.5, 0.8]$. We only plot the rounds after $t_0$ since the loss up to that point is zero.

4.2 Cold start attack

A very effective type of attack against randomized algorithms (even Thompson Sampling) is the cold start attack, where for the first $t_0$ periods, the adversary chooses $q^t = 0$ giving the algorithm the impression it has pulled a lot of arms although there are effectively no pulls. In Figure 3 we exhibit an extreme form of this attack, where we have a very long cold-start period ($t_0 = 10^7$) and from then on we have a standard stochastic instance with means 0.5 and 0.8. The performance of all algorithms except AAEAS and BROAD is severely hurt. In fact this performance degradation can become arbitrarily bad as $t_0 \to \infty$. This happens for different reasons depending on the type of algorithm:

- For confidence-bound based algorithms like UCB and AAE, the cold start attack produces the impression that the arms have much smaller mean than they actually have. Since the confidence bounds scale with the inverse square root of the empirical gap, the exploration phase can be arbitrarily extended as $t_0$ grows to infinity.
- Tsallis and EXP3++ are mirror-descent based algorithms that have a learning rate schedule that depends directly on the number of rounds ($\eta_t = 1/\sqrt{t}$ in either case). For those the cold-start will cause the learning rate to start at $1/\sqrt{t_0}$ which is much smaller than the learning rate that would be required for a stochastic instance with larger means. Note that while BROAD is also based on mirror descent, the learning rate is adaptively tuned based on the rewards and it not directly depending on the numbers of rounds. This ability of adaptive tuning also enables first-order bounds for BROAD [FLL+16] and stems from the strong log-barrier regularizer that effectively deals with the variance in the second-order term. In contrast, Tsallis and EXP3++ have weaker regularizers and do not admit first-order bounds; for the same reason, we believe that it is unlikely that their learning rate can be appropriately tuned to circumvent this issue.
- Thompson Sampling is based on keeping a prior on the means of the arms. A large number of cold-start periods leads the algorithm to a state with very skewed priors. Once we reach the end of the cold-start period at $t_0$ each arm will have a beta distribution around $\text{Beta}(t_0/k, 1)$ instead of $\text{Beta}(1, 1)$.

In Figure 4 we compare Thompson Sampling and AAEAS in an instance with a very small number of cold start rounds ($t_0 = 25$) and then run the algorithm for another $T = 30000$ rounds. Interestingly, we see that, even when the cold start is really small, the effect of the attack is long-lasting, which
Figure 4: Comparing AAEAS with Thompson Sampling on a cold start instance ($t_0 = 25$) with means $\mu = [0.5, 0.8]$. The y-axis is the cumulative loss of the algorithm averaged over 100 runs.

provides powerful evidence that the phenomenon described is, in fact, an actual concern when deploying Thompson Sampling. We also note that this phenomenon does not even need to be caused due to an adversarial source. It could well occur that the initial samples are less effective because, e.g., initially the advertised product is not yet well established which leads customers to prefer alternative options. In fact, recent work has suggested that, under competition, learning algorithms generally suffer from such an effect due to the exploration they need to perform in the beginning; see [MSW18] and [ALSW19] for a relevant discussion.

4.3 Discussion on algorithms not run

Finally, we would like to mention that there are a few other algorithms with logarithmic stochastic guarantees that we decided to not include. The original best of both worlds algorithm SAO [BS12] and its follow-up SAPO [AC16] are rooted in AAE and switch to EXP3 if some test fails. Similarly, multi-layer AAE [LMPL18] provides a way to robustify AAE to adversarial corruptions. Since the instances are mostly stochastic and therefore the tests are not expected to fail, their performance is strictly inferior to the one of AAE. Another line of work focuses on stochastic algorithms from the EXP3 family that can effectively select non-stationarities in the environment, e.g. the R.EXP3 algorithm [BGZ14]. These algorithms again come with horizon-dependent learning rates and are expected to suffer similarly with Tsallis and EXP3++ (that belong to the same family). Finally, for the same reason, we did not run the recent HYBRID algorithm [ZLW19] which extends upon Tsallis but again uses horizon-dependent learning rates.

5 Conclusion

In this work, we suggest a new intermediary model between stochastic and adversarial bandits where an adversary can rescale all rewards in a given round by the same factor. This is inspired by learning click-through-rates in advertising which are typically a product between the intrinsic quality of the ad and the clickability of the user. We show that two adaptive algorithms are robust to this adversarial scaling and provide two natural attacks that demonstrate that other stochastic algorithms are not. In particular, our work suggests the importance of adaptivity that has been not been discussed in the context of stochastic bandits.
There are two nice open questions coming from our work. AAEAS is weaker by a factor of $k$ compared to BROAD; we believe that our analysis is tight in that matter but it would be interesting to see if some alternative modification of AAE can remove this dependence. Moreover, Thompson Sampling, although ineffective for the cold start attack, has very good performance in the small means attack; this suggests that its analysis could become more tight to scale with the ratio of the means (rather than their difference). We leave both of these questions to future work.

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