Nest algebras in an arbitrary vector space

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Abstract

We examine the properties of algebras of linear transformations that leave invariant all subspaces in a totally ordered lattice of subspaces of an arbitrary vector space. We compare our results with those that apply for the corresponding algebras of bounded operators that act on a Hilbert space.

1 Introduction

The study of triangular forms for operators has long been an important part of the theory of non-self-adjoint operators and operator algebras. See [1] for a detailed account. In [5] Ringrose introduced the terms ‘nest’ and ‘nest algebra’. For Ringrose a nest \( \mathcal{N} \) is a complete, totally ordered sublattice of the lattice of all closed subspaces of a Hilbert space \( \mathcal{H} \) that contains the trivial subspaces \( \{0\} \) and \( \mathcal{H} \). The corresponding nest algebra \( \text{Alg}\mathcal{N} \) is algebra of all operators on \( \mathcal{H} \) that leave invariant each of the subspaces in \( \mathcal{N} \).

In this paper we examine totally ordered lattices of subspaces of an arbitrary vector space and the associated operator algebras. Here a nest \( \mathcal{N} \) in a vector space \( \mathcal{X} \) is a complete, totally ordered sublattice of the lattice of all subspaces of \( \mathcal{X} \) that contains the trivial subspaces \( \{0\} \) and \( \mathcal{X} \). The corresponding nest algebra \( \text{Alg}\mathcal{N} \) is algebra of all operators on \( \mathcal{X} \) that leave invariant each of the subspaces in \( \mathcal{N} \). We obtain results concerning the finite rank operators in \( \text{Alg}\mathcal{N} \) that mirror those that apply in the Hilbert space case. We also examine the Jacobson radical of \( \text{Alg}\mathcal{N} \) and obtain a simple characterization when the nest satisfies a descending chain condition. We also show that the same characterization of the Jacobson radical holds for other types of nest algebras.

1.1 Complete distributivity

The lattice operations \( \wedge \) and \( \vee \) in \( \mathcal{S}(\mathcal{X}) \), the lattice of all subspaces of the vector space \( \mathcal{X} \), are intersection and linear span. In particular, if \( \mathcal{M} \) and \( \mathcal{N} \) are

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subspaces of $\mathfrak{X}$, $\mathcal{M} \lor \mathcal{N} = \text{span}\{\mathcal{M}, \mathcal{N}\} = \{x + y : x \in \mathcal{M}, y \in \mathcal{N}\}$. However in a totally ordered sublattice the lattice operations are simply the set operations $\cap$ and $\cup$. So any nest $\mathfrak{N}$ is completely distributive (see [1]).

Suppose that $\mathfrak{N}$ is a nest in $\mathfrak{X}$. For each $x \in \mathfrak{X}$ we define

$$\mathfrak{N}(x) = \bigcap\{\mathcal{M} \in \mathfrak{N} : x \in \mathcal{M}\} \text{ and } \mathfrak{N}(x) = \bigcup\{\mathcal{M} \in \mathfrak{N} : x \notin \mathfrak{N}\}. \tag{1}$$

It follows easily from (1) that $x \in \mathcal{N} \iff \mathfrak{N}(x) \subseteq \mathcal{N}$ and $x \notin \mathcal{N} \iff \mathcal{N} \subseteq \mathfrak{N}(x) \tag{2}$

**Lemma 1** The join-irreducible elements of the completely distributive lattice $\mathfrak{N}$ are the subspaces of the form $\mathfrak{N}(x)$ where $x$ is any non-zero vector in $\mathfrak{X}$.

**Proof.** Suppose that $x \neq 0$, and that $\mathfrak{N}(x) \subseteq \bigcup\{\mathcal{N} : \mathcal{N} \in \mathfrak{N}\#\}$ where $\mathfrak{N}\# \subseteq \mathfrak{N}$.

Then $x \in \bigcup\{\mathcal{N} : \mathcal{N} \in \mathfrak{N}\#\}$ by (2). So $x \in \mathcal{N}$ for some $\mathcal{N} \in \mathfrak{N}\#$, and it follows from (2) that $\mathfrak{N}(x) \subseteq \mathcal{N}$. So $\mathfrak{N}(x)$ is join-irreducible.

Suppose now that $\mathcal{N}$ is a join-irreducible subspace in $\mathfrak{N}$. Clearly $\mathcal{N} = \bigcup\{\mathfrak{N}(x) : x \in \mathcal{N}\}$, and so $\mathcal{N} \subseteq \mathfrak{N}(x)$ for some $x \in \mathcal{N}$. So $\mathcal{N} = \mathfrak{N}(x)$. ■

**Remark 2** Complete distributivity distinguishes the vector space case from the Hilbert space case. Some of the most interesting nests of closed subspaces of a Hilbert space are ‘continuous’, have no join-irreducible elements, and are not completely distributive.

## 2 Finite rank operators

The rank of an operator in $\mathcal{L}(\mathfrak{X})$ is the dimension of its range. In this section we examine the properties of operators in a nest algebra $\mathcal{A} = \text{Alg}\mathfrak{N}$ whose ranks are finite. Let $\mathcal{R}$ denote the set of finite-rank operators in $\mathcal{L}(\mathfrak{X})$. Various authors have investigated the properties of $\mathcal{R} \cap \mathcal{A}$ in the Hilbert space context. For example, Erdos proved [2] that if $\mathcal{N}$ is a nest of closed subspaces of a Hilbert space then the strong closure of $\mathcal{R} \cap \mathcal{A}$ is $\mathcal{A}$.

Rank-one operators also have an important role in the Hilbert space context. Suppose that $T \in \mathcal{R}_1$, where $\mathcal{R}_1$ denotes the set of all rank-one operators in $\mathcal{L}(\mathfrak{X})$. Then there exists $y \in \mathfrak{X}$ such that for all $x \in \mathfrak{X}$, $Tx = \phi(x)y$ where $\phi(x) \in \mathbb{F}$. Since $T$ is linear the map $x \rightarrow \phi(x)$ is a linear functional of $\mathfrak{X}$. Let $\mathfrak{X}'$ denote the algebraic dual of $\mathfrak{X}$, i.e., the set of all linear maps from $\mathfrak{X}$ into $\mathbb{F}$.

Each rank-one operator on $\mathfrak{X}$ has the form $x \otimes \varphi$, where $x \in \mathfrak{X}$, $\varphi \in \mathfrak{X}'$, and $(x \otimes \varphi)(y) = \varphi(y)x$ for all $y \in \mathfrak{X}$.

The following lemma characterizes the rank-one operators in $\mathcal{A}$.

**Lemma 3** Suppose that $x \in \mathfrak{X}$ and $\varphi \in \mathfrak{X}'$. Then $x \otimes \varphi \in \mathcal{R}_1 \cap \mathcal{A}$ if and only if $\mathfrak{N}_-(x) \subseteq \ker \varphi$.

**Proof.** First suppose that $x \otimes \varphi \in \mathcal{R}_1 \cap \mathcal{A}$, and that $y \in \mathfrak{N}_-(x)$. Since $\mathfrak{N}_-(x) \subseteq \mathcal{A}$, $(x \otimes \varphi)(y) = \varphi(y)x \in \mathfrak{N}_-(x)$. Since $x \notin \mathfrak{N}_-(x)$ it follows that $\varphi(y) = 0$. So $\mathfrak{N}_-(x) \subseteq \ker \varphi$. 


Now suppose that $\mathcal{N}_-(x) \subseteq \ker \varphi$ and that $N \in \mathcal{N}$. If $N \subseteq \mathcal{N}(x)$ then $N \subseteq 
abla_-(x)$ and $(x \otimes \varphi)N = \{0\} \subseteq N$. If $\mathcal{N}(x) \subseteq N$ then $(x \otimes \varphi)N = \text{span } x \subseteq \mathcal{N}(x) \subseteq N$. So $x \otimes \varphi \in \mathcal{R}_1 \cap \mathcal{A}$. □

2.1 Reflexivity of $\mathcal{N}$

For any subset of $\mathcal{A}$ of $\mathcal{L}(\mathcal{X})$ let $\text{Lat}\mathcal{A}$ denote the sublattice of $\mathcal{S}(\mathcal{X})$ consisting of all subspaces of $\mathcal{X}$ that are invariant under each of the operators in $\mathcal{A}$. We shall show that

$$\mathcal{N} = \text{Lat}(\mathcal{R}_1 \cap \mathcal{A}),$$

from which it follows that $\mathcal{N}$ is reflexive, i.e. $\mathcal{N} = \text{Lat Alg}\mathcal{N}$.

Longstaff shows in (3) that (3) holds in the Hilbert space context.

The following lemma will be useful to establish the reflexivity of $\mathcal{N}$.

Lemma 4 If $x$ and $y$ are non-zero vectors in $\mathcal{X}$ and $y \in \mathcal{N}(x)$, then there exists $R \in \mathcal{R}_1 \cap \mathcal{A}$ such that $Rx = y$.

Proof: Since $y \in \mathcal{N}(x)$, $\mathcal{N}(y) \subseteq \mathcal{N}(y) \subseteq \mathcal{N}(x)$. So $x \not\in \mathcal{N}(y)$, and hence there exists $\varphi \in \mathcal{X}'$ such that $\varphi(x) = 1$ and $\mathcal{N}(y) \subseteq \ker \varphi$. Then $R = y \otimes \varphi \in \mathcal{R}_1 \cap \mathcal{A}$ and $Rx = \varphi(x)y$. □

Theorem 5 $\mathcal{N}$ is reflexive.

Proof: We shall show that $\mathcal{N} = \text{Lat}(\mathcal{R}_1 \cap \mathcal{A})$. Clearly $\mathcal{N} \subseteq \text{Lat}(\mathcal{R}_1 \cap \mathcal{A})$. Suppose that $N \in \text{Lat}(\mathcal{R}_1 \cap \mathcal{A})$. It is enough to show that $N \in \mathcal{N}$.

Suppose that $x$ and $y$ are non-zero vectors in $N$ and $\mathcal{N}(x)$ respectively. So by Lemma 4 there exists $R \in \mathcal{R}_1 \cap \mathcal{A}$ such that $Rx = y$. Since $N \in \text{Lat}(\mathcal{R}_1 \cap \mathcal{A})$, it follows that $y \in N$, and hence $\mathcal{N}(x) \subseteq N$.

Clearly $N \subseteq \bigcup \{\mathcal{N}(x) : x \in N\}$, and so

$$N \subseteq \bigcup \{\mathcal{N}(x) : x \in N\} \subseteq N$$

So $N = \bigcup \{\mathcal{N}(x) : x \in N\} \in \mathcal{N}$, as required. □

2.2 Finite rank idempotents

A simple calculation shows that $(x_1 \otimes \varphi_1)(x_2 \otimes \varphi_2) = \varphi_1(x_2)(x_1 \otimes \varphi_2)$. So $x \otimes \varphi$ is idempotent if and only if $\varphi(x) = 1$.

The following lemma concerning rank-one idempotents in $\mathcal{A}$ will be useful.

Lemma 6 Suppose that $M$ is a finite-dimensional subspace of $\mathcal{X}$. Then $M = \text{ran } P$ for some idempotent $P \in \mathcal{A}$. Furthermore, $P$ is the sum of $n$ rank-one idempotents in $\mathcal{A}$, where $n = \text{dim } M$.

Proof: The proof is by induction on $\text{dim } M$. First suppose that $\text{dim } M = 1$, and choose a non-zero vector $x \in M$. Now choose $\varphi \in \mathcal{X}'$ such that $\mathcal{N}(x) \subseteq \ker \varphi$ and $\varphi(x) = 1$. Such a $\varphi$ exists because $x \not\in \mathcal{N}(x)$. Then $x \otimes \varphi$ is the required idempotent.

Now suppose that $n = \text{dim } M > 1$ and that the result is true for all subspaces of $\mathcal{X}$ with dimension less than $n$. Choose a non-zero vector $y \in M$ and a
subspace $M^\#$ of $M$ such that $M^\#$ and span $y$ are complementary subspaces of $M$, i.e. $M = M^\# + \text{span } y = M$ and $M^\# \cap \text{span } y = \{0\}$. By the induction hypothesis there exists an idempotent $P^\# \in \mathcal{A}$ such that $\text{ran } P^\# = M^\#$, and rank-one idempotents $P_1, P_2, \cdots, P_{n-1}$ in $\mathcal{A}$ such that $P^\# = P_1 + P_2 + \cdots + P_{n-1}$.

Let $x = y - P^\# y$. Then $0 \neq x \in M$ and $P^# x = 0$. Suppose that $x = u + v$, where $u \in \mathcal{R}(x)_-$ and $v \in M^\#$. Then $P^# x = P^# u + P^# v$, i.e. $0 = P^# u + v$, since $M^\# = \text{ran } P^#$ and $P^#$ is idempotent. So $x = u - P^# u$. Since $P^# \in \mathcal{A}$, it follows that $u - P^# u \in \mathcal{N}(x)_-$. Since $x \notin \mathcal{N}(x)_-$ we have a contradiction. So $x \notin \mathcal{R}(x)_- + M^\# = \mathcal{R}(x)_- + \text{ran } P^#$, and hence there exists $\varphi \in \mathfrak{X}'$ such that

\[ \varphi(x) = 1, \quad \text{and } \mathcal{R}(x)_- + \text{ran } P^\# \subseteq \ker \varphi. \]

Let $P_n = x \otimes \varphi$. Then $P_n$ is idempotent since $\varphi(x) = 1$, and $P_n \in \mathcal{A}$ since $\mathcal{R}(x)_- \subseteq \ker \varphi$. Furthermore, $P^# P_n = P^# x \otimes \varphi = 0$, and $P_n P^# = x \otimes \varphi P^# = 0$ since $\text{ran } P^# \subseteq \ker \varphi$. Now let $P = P^# + P_n$. Then

\[ P^2 = (P^#)^2 + P^# P_n + P_n P^# + P_n^2 = P^# + P_n = P, \]

and $\text{ran } P = \text{ran } P^# + \text{ran } P_n = M^# + \text{span } x = M$, as required. \hfill \blacksquare

### 2.3 Rank decomposition

Lemma 6 provides an easy proof of a rank-decomposition property of finite rank operators in the nest algebra $\mathcal{A}$.

**Theorem 7** Suppose that $T$ is a finite rank operator in $\mathcal{A}$. Then $T$ is the sum of $n$ rank-one operators in $\mathcal{A}$, where $n = \text{rank } T$.

**Proof.** By Lemma 6, $\text{ran } T = \text{ran } P$ for some idempotent $P$ in $\mathcal{A}$. Furthermore $P = P_1 + P_2 + \cdots + P_n$ where each $P_k$ is a rank-one idempotent in $\mathcal{A}$. Let $T_k = P_k T$ for $1 \leq k \leq n$. Then $T_k \in \mathcal{A}$ and $\text{rank } T_k \leq 1$ for each $k$. Furthermore,

\[ T = PT = \sum_{k=1}^{n} P_k T = \sum_{k=1}^{n} T_k. \]

This is the required decomposition. The Hilbert version of this result was proved by Ringrose (12). \hfill \blacksquare

**Remark 8** The proof of Theorem 7 is easily modified to show that if $T$ is a finite rank operator in $\mathcal{I}$, where $\mathcal{I}$ is a left ideal in $\mathcal{A}$, then $T$ is the sum of $n$ rank-one operators in $\mathcal{I}$, where $n = \text{rank } T$.

### 2.4 Density

Lemma 6 also provides an easy proof of a density property of the linear span of rank-one operators in $\mathcal{A}$. First we introduce a special topology on $\mathcal{L}(\mathfrak{X})$. 

Definition 9 The set of all subsets of $\mathcal{L}(\mathfrak{X})$ of the form

$$\mathcal{U}(T, x) = \{ S \in \mathcal{L}(\mathfrak{X}) : Sx = Tx \},$$

where $x \in \mathfrak{X}$ and $T \in \mathcal{L}(\mathfrak{X})$, is a set of subbasic neighbourhoods of $T$ for the strict topology on $\mathcal{L}(\mathfrak{X})$.

Theorem 10 The span of the rank-one operators in $\mathcal{A}$ is strictly dense in $\mathcal{A}$.

Proof. Suppose that $T \in \mathcal{A}$ and that $\mathcal{F}$ is a finite subset of $\mathfrak{X}$. Let $\mathcal{R}_1^\# \cap \mathcal{A}$ denote the span of $\mathcal{R}_1 \cap \mathcal{A}$. We need to show that there exists $S \in \mathcal{R}_1^\# \cap \mathcal{A}$ such that $Sx = Tx$ for all $x \in \mathcal{F}$.

By Lemma 7 span $\mathcal{F} = \text{ran} \ P$ for some idempotent $P \in \mathcal{A}$. Furthermore, $P$ is the sum of $n$ rank-one idempotents in $\mathcal{A}$, where $n = \text{dim span} \ \mathcal{F}$. Let $T_k = TP_k$ for $1 \leq k \leq n$. Then $T_k \in \mathcal{A}$ and $\text{rank} \ T_k \leq 1$ for each $k$. So $S = \sum_{k=1}^n T_k \in \mathcal{R}_1^\# \cap \mathcal{A}$. Furthermore, for each $x \in \text{span} \ \mathcal{F}$,

$$Tx = TPx = \sum_{k=1}^n TP_k x = Sx,$$

as required. ■

Remark 11 The proof of Theorem 10 is easily modified to show that if $T$ is a finite rank operator in $\mathcal{I}$, where $\mathcal{I}$ is a right ideal in $\mathcal{A}$, then $T$ is the sum of $n$ rank-one operators in $\mathcal{I}$, where $n = \text{rank} \ T$.

3 Dual nests

For any subset $M$ of $\mathfrak{X}$, let $M^\perp$ denote the annihilator of $M$, i.e.

$$M^\perp = \{ \varphi : \varphi \in \mathfrak{X}' \text{ and } M \subseteq \ker \varphi \}$$

Suppose that $\mathfrak{N}$ is a nest of subspaces of $\mathfrak{X}$, and that $\mathfrak{N}^\perp = \{ M^\perp : M \in \mathfrak{N} \}$. We call $\mathfrak{N}^\perp$ the dual of the nest $\mathfrak{N}$. Since the map $M \mapsto M^\perp$ is order reversing, i.e. $M_1 \subseteq M_2 \iff M_1^\perp \supseteq M_2^\perp$, $\mathfrak{N}^\perp$ is a linearly ordered family of subspaces of $\mathfrak{X}'$ that is anti-order isomorphic to $\mathfrak{N}$.

We are interested in the issue of completeness of $\mathfrak{N}^\perp$.

Lemma 12 For any family $\{ M_\alpha : \alpha \in \Psi \}$ of subspaces in $\mathfrak{N}$,

$$\bigcap_{\alpha \in \Psi} M_\alpha^\perp = \left( \bigcup_{\alpha \in \Psi} M_\alpha \right)^\perp \quad \text{and} \quad \bigcup_{\alpha \in \Psi} M_\alpha^\perp \subseteq \left( \bigcap_{\alpha \in \Psi} M_\alpha \right)^\perp$$

Proof. Suppose that $\varphi \in \mathfrak{X}'$. It is easy to see that

$$\varphi \in \bigcap_{\alpha \in \Psi} M_\alpha^\perp \iff M_\alpha \subseteq \ker \varphi \text{ for all } \alpha \in \Psi \iff \varphi \in \left( \bigcup_{\alpha \in \Psi} M_\alpha \right)^\perp.$$  

Similarly, if $\varphi \in \bigcup_{\alpha \in \Psi} M_\alpha^\perp$ then $M_\alpha^\perp \subseteq \ker \varphi$ for some $\alpha^\# \in \Psi$. It follows that $\bigcap_{\alpha \in \Psi} M_\alpha \subseteq \ker \varphi$, i.e. $\varphi \in \left( \bigcap_{\alpha \in \Psi} M_\alpha \right)^\perp$. ■
Corollary 13. $\mathcal{R}^\perp$ is complete if and only if $\bigcup_{\alpha \in \Psi} M_\alpha = (\bigcap_{\alpha \in \Psi} M_\alpha)^\perp$ for each family $\{M_\alpha : \alpha \in \Psi\}$ of subspaces in $\mathcal{R}$.

**Proof.** In the light of Lemma 13 it is sufficient to show that if $\mathcal{R}^\perp$ is complete and $\{M_\alpha : \alpha \in \Psi\}$ is a family of subspaces in $\mathcal{R}$, then $\bigcap_{\alpha \in \Psi} M_\alpha^\perp \subseteq \bigcup_{\alpha \in \Psi} M_\alpha^\perp$.

If $\mathcal{R}^\perp$ is complete, $\bigcup_{\alpha \in \Psi} M_\alpha^\perp = M_\#^\perp$ for some $M_\# \in \mathcal{R}$. Suppose that $\alpha_0 \in \Psi$. Then $M_{\alpha_0}^\perp \subseteq \bigcup_{\alpha \in \Psi} M_\alpha^\perp = M_\#^\perp$, and so $M_\# \subseteq M_{\alpha_0}$. Therefore $M_\# \subseteq \bigcap_{\alpha \in \Psi} M_\alpha$, and so $\bigcap_{\alpha \in \Psi} M_\alpha^\perp \subseteq \bigcup_{\alpha \in \Psi} M_\alpha^\perp$, as required. □

Example 14. Suppose that $\mathcal{X} = c_00(\mathbb{N})$, the vector space of all finitely non-zero $\mathbb{F}$-valued sequences. Then $\mathcal{X}'$ can be regarded as the vector space of all $\mathbb{F}$-valued sequences. If $f = (f(k))_{k=1}^\infty \in \mathcal{X}$ and $\varphi = (\varphi(k))_{k=1}^\infty \in \mathcal{X}'$, then $\varphi(f) = \sum_{k=1}^\infty \varphi(k)f(k)$.

For each $n \in \mathbb{N}$, let $M_n = \{f \in \mathcal{X} : \text{supp } f \subseteq \{1, 2, 3, \ldots, n\}\}$, where $\text{supp}(f(k))_{k=1}^\infty = \{k : f(k) \neq 0\}$, and let

$$\mathcal{R} = \{\{0\}, M_1, M_2, M_3, \ldots, \mathcal{X}\}.$$ 

Then $\mathcal{R}$ is a complete, totally ordered family of subspaces of $\mathcal{X}$, i.e. $\mathcal{R}$ is a nest.

Note that $M_n^\perp = \{\varphi \in \mathcal{X}' : \text{supp } \varphi \subseteq \{n + 1, n + 2, n + 3, \ldots\}\}$. It is easy to see that $\mathcal{R}^\perp = \{\mathcal{X}', M_1^\perp, M_2^\perp, M_3^\perp, \ldots, \{0\}\}$ is a complete, totally ordered family of subspaces of $\mathcal{X}'$, i.e. $\mathcal{R}^\perp$ is a nest.

Example 15. Suppose that $\mathcal{X} = c_00(\mathbb{N})$ as in Example 14, and let

$$\mathcal{R}_\# = \{\mathcal{X}, M_1^\#, M_2^\#, M_3^\#, \ldots, \{0\}\},$$

where $M_n^\# = \{f \in \mathcal{X} : \text{supp } f \subseteq \{n + 1, n + 2, n + 3, \ldots\}\}$ for each $n \in \mathbb{N}$. Then $\mathcal{R}_\#$ is a complete, totally ordered family of subspaces of $\mathcal{X}$, i.e. $\mathcal{R}_\#$ is a nest.

Note that $(M_n^\#)^\perp = M_n = \{\varphi \in \mathcal{X}' : \text{supp } \varphi \subseteq \{1, 2, 3, \ldots, n\}\}$ as in Example 14. So $(M_1^\#)^\perp, (M_2^\#)^\perp, (M_3^\#)^\perp, \ldots$ is a strictly increasing sequence in $(\mathcal{R}_\#)^\perp$, and $\bigcup_{n=1}^\infty (M_n^\#)^\perp = \mathcal{X} \notin (\mathcal{R}_\#)^\perp$. So $(\mathcal{R}_\#)^\perp$ is not complete.

The nest $\mathcal{R}_\#$ in Example 15 has a strictly decreasing, infinite sequence of subspaces, i.e., it is not well-ordered. The following lemma shows that this is the key to the incompleteness of $(\mathcal{R}_\#)^\perp$.

Lemma 16. Suppose that $\mathcal{R}$ is a complete nest of subspaces of a vector space $\mathcal{X}$. Then $\mathcal{R}^\perp$ is complete if and only if $\mathcal{R}$ is well-ordered.

**Proof.** First suppose that $\mathcal{R}$ is well-ordered, and that $\{M_\alpha : \alpha \in \Psi\}$ is a family of subspaces in $\mathcal{R}$. In the light of Corollary ?? it is sufficient to show that $\bigcap_{\alpha \in \Psi} M_\alpha^\perp \subseteq \bigcup_{\alpha \in \Psi} M_\alpha^\perp$.

Since $\mathcal{R}$ is well-ordered, $\bigcap_{\alpha \in \Psi} M_\alpha = M_\alpha^\#$ for some $\alpha^\# \in \Psi$. So

$$\left(\bigcap_{\alpha \in \Psi} M_\alpha\right)^\perp = M_\alpha^\# \subseteq \bigcup_{\alpha \in \Psi} M_\alpha^\perp,$$ as required.
Now suppose that $\mathcal{N}$ is not well-ordered, and that $M_1, M_2, M_3, \cdots$ is a strictly decreasing infinite sequence of subspaces in $\mathcal{N}$. For each $n \in \mathbb{N}$ choose $x_n$ such that $x_n = M_n \setminus M_{n+1}$. Then \{x_1, x_2, x_3, \cdots\} is a linearly independent set and span\{x_1, x_2, x_3, \cdots\} \cap M_\infty = \{0\}$, where $M_\infty = \cap_{n=1}^\infty M_n$. So there exists $\varphi \in X'$ such that 
\[ \varphi(x_n) = 1 \text{ for each } n \in \mathbb{N} \text{ and } M_\infty \subseteq \ker \varphi \] 
It follows easily from (4) that $\varphi \in M_\infty^\perp \setminus (\bigcup_{n=1}^\infty M_n^\perp)$. So 
\[ \bigcup_{n=1}^\infty M_n^\perp \subset M_\infty^\perp \] 

Suppose that $\bigcup_{n=1}^\infty M_n^\perp \in \mathcal{N}^\perp$, i.e. $\bigcup_{n=1}^\infty M_n^\perp = M^\perp$ for some $M \in \mathcal{N}$. Then $M_n^\perp \subseteq M^\perp$ and $M \subseteq M_n$ for each $n \in \mathbb{N}$. So $M \subseteq M_\infty$, and hence $M_\infty^\perp \subseteq M^\perp$. But this contradicts (5), and so there is no such subspace $M$ in $\mathcal{N}$. So $\mathcal{N}^\perp$ is not complete.

4 The Jacobson radical

Suppose that $\mathcal{R}$ is a ring with identity 1. The Jacobson radical $\text{Rad} \mathcal{R}$ is the intersection of all maximal left ideals of $\mathcal{R}$. It is also the intersection of all maximal right ideals of $\mathcal{R}$. See (3). A more useful characterisation of $\text{Rad} \mathcal{R}$ is the following:

**Proposition 17** Suppose that $T \in \mathcal{R}$. The following are equivalent:

1. $T \in \text{Rad} \mathcal{R}$
2. $1 - AT$ is invertible in $\mathcal{R}$ for each $A \in \mathcal{A}$
3. $1 - AT$ is invertible in $\mathcal{R}$ for each $A \in \mathcal{A}$

**Definition 18** Suppose that $\mathcal{N}$ is a nest on $X$ and that $\mathcal{A} = \text{Alg} \mathcal{N}$. The strictly triangular ideal $\mathcal{A}_-$ is defined by 
\[ \mathcal{A}_- = \{ T : T \in \mathcal{A} \text{ and } Tx \in \mathcal{N}(x)_- \text{ for all } x \in X \} \]

**Lemma 19** Suppose that $\mathcal{N}$ is a nest on $X$ and that $\mathcal{A} = \text{Alg} \mathcal{N}$. Then 
\[ \text{Rad} \mathcal{A} \subseteq \mathcal{A}_- \]

**Proof.** Suppose that $T \in \mathcal{A} \setminus \mathcal{A}_-$. Then $Tx \notin \mathcal{N}(x)_-$ for some $x \in X$. Choose $\varphi \in X'$ such that $\varphi(Tx) = 1$ and $\mathcal{N}(x)_- \subseteq \ker \varphi$. It follows from (3) that $x \otimes \varphi \in \mathcal{A}$.

Now $(1 - (x \otimes \varphi)T)x = x - \varphi(Tx)x = 0$. So $1 - (x \otimes \varphi)T$ is not invertible and so $T \notin \text{Rad} \mathcal{A}$ by Proposition 17.

We now seek conditions which are either necessary or sufficient for the equality of the radical $\text{Rad} \mathcal{A}$ and the strictly triangular ideal $\mathcal{A}_-$. The notion of local nilpotence will be useful.
Definition 20 We say that $T \in \mathcal{L}(\mathfrak{X})$ is nilpotent at $x \in \mathfrak{X}$ if $T^n x = 0$ for sufficiently large $n$. We say that $T$ is locally nilpotent if it is nilpotent at each $x \in \mathfrak{X}$.

Lemma 21 If each $T \in \mathcal{A}_-$ is locally nilpotent, then $\text{Rad} \mathcal{A} = \mathcal{A}_-$.

Proof. Suppose that $T \in \mathcal{A}_-$ and that $A \in \mathcal{A}$. Then $AT \in \mathcal{A}_-$ and hence is locally nilpotent by assumption.

Let $S = 1 + \sum_{n=1}^{\infty} (AT)^n$. The sum $S$ is well-defined as an operator in $\mathcal{L}(\mathfrak{X})$, because the local nilpotence of $AT$ ensures that for each $x \in \mathfrak{X}$ the series $\sum_{n=1}^{\infty} (AT)^n x$ has only finitely many non-zero terms. If $x \in M$ for some $M \in \mathfrak{M}$, it is clear that $Sx \in M$. So $S \in \mathcal{A}$. Furthermore, it is easy to see that $S(1 - AT) = (1 - AT)S = 1$. So $S$ is the inverse of $1 - AT$ in $\mathcal{A}$, and hence $T \in \text{Rad} \mathcal{A}$. ■

Lemma 22 If $\mathfrak{M}$ is well-ordered then each $T \in \mathcal{A}_-$ is locally nilpotent.

Proof. Suppose that $T \in \mathcal{A}_-$ is not locally nilpotent. Then there exists $x \in \mathfrak{X}$ such that $T^n x \neq 0$ for all $n \in \mathbb{N}$. Since $T \in \mathcal{A}_-$, for each $n \in \mathbb{N}$,

$$\mathfrak{M}(T^{n+1}x) \subseteq T(\mathfrak{M}(T^n x)) \subseteq (\mathfrak{M}(T^n x))_+ \subseteq \mathfrak{M}(T^n x)$$

So $\mathfrak{M}(T^n x) : n = 1, 2, 3, \ldots$ is a strictly decreasing, infinite sequence of subspaces in $\mathfrak{M}$, and hence $\mathfrak{M}$ is not well-ordered. ■

Corollary 23 If $\mathfrak{M}$ is well-ordered then $\text{Rad} \mathcal{A} = \mathcal{A}_-$.

The following result shows that for dual nests, well-ordering is not essential for the equality of the radical and the strictly triangular ideal.

Theorem 24 Suppose that $\mathfrak{M}$ is a nest of subspaces of a vector space $\mathfrak{X}$ whose order type is $\omega$, the first infinite ordinal. Then $\mathfrak{M}^\perp$ is a nest of subspaces of $\mathfrak{X}^\perp$, whose order type is anti-isomorphic to $\omega$, and $(\text{Alg} \mathfrak{M}^\perp)_- = \text{Rad} (\text{Alg} \mathfrak{M}^\perp)$.

Proof. In view of Lemma 19 it is sufficient to show that $\mathcal{A}_- = \text{Rad} \mathcal{A}$, where $\mathcal{A} = \text{Alg} \mathfrak{M}^\perp$.

Let $M_0 = \{0\}$, and for each $n > 0$ let $M_n$ denote the immediate successor of $M_{n-1}$ in $\mathfrak{M}$. Since the order type of $\mathfrak{M}$ is $\omega$, $\bigcup_{n=1}^{\infty} M_n = \mathfrak{X}$.

Suppose that $T \in \mathcal{A}_-$ and that $\varphi \in M_n^\perp$. Then $T\varphi \in \mathfrak{M}^\perp(\varphi)_- \subseteq \mathfrak{M}^\perp(\varphi) \subseteq M_n^\perp$. Since $M_{n+1}^\perp$ is the immediate predecessor of $M_n^\perp$ in $\mathfrak{M}^\perp$, it follows that $T\varphi \in M_{n+1}$, and so $T(M_{n+1}) \subseteq M_n^\perp$.

Suppose that $A \in \mathcal{A}$. Then $AT \in \mathcal{A}_-$ and so $AT(M_n^\perp) \subseteq M_{n+1}^\perp$ for each $n \geq 0$ and so $(AT)^n(\mathfrak{X}) = (AT)^n(M_n^\perp) \subseteq M_n^\perp$ for each $n \geq 0$.

Let $S = 1 + \sum_{n=1}^{\infty} (AT)^n$. The sum $S$ is well-defined as an operator in $\mathcal{L}(\mathfrak{X}^\perp)$ because, for each $x \in \mathfrak{X}$ and each $\varphi \in \mathfrak{X}^\perp$, the series $\sum_{n=1}^{\infty} (AT)^n(\varphi)(x)$ has only finitely many non-zero terms. (To see this note that $x \in M_{n^\#}$ for some $n^\# \geq 0$, and $(AT)^n\varphi(x) = 0$ if $n \geq n^\#$.) Furthermore $S(1 - AT)\varphi(x) = (1 - AT)S\varphi(x) = \varphi(x)$, and so $S = (1 - AT)^{-1}$. Finally, it is easy to check that $S(M_n^\perp) \subseteq M_n^\perp$ for each $n \geq 0$ and so $S \in \mathcal{A}$. So $T \in \text{Rad} \mathcal{A}$, and hence $\mathcal{A}_- \subseteq \text{Rad} \mathcal{A}$. It follows from Lemma 19 that $\mathcal{A}_- = \text{Rad} \mathcal{A}$.
4.1 An example

The nest \( \mathfrak{N} \) defined in Example 14 satisfies the conditions of Theorem 23, and so \((\text{Alg } \mathfrak{N})_\perp = \text{Rad}(\text{Alg } \mathfrak{N})\). Note that \( \mathfrak{N} \) is not well-ordered. It does, however, satisfy the ascending chain condition, i.e. each subset of \( \mathfrak{N} \) contains a maximal element.

**Definition 25** Suppose that \( \mathfrak{X}_1 \) and \( \mathfrak{X}_2 \) are vector spaces over the same field \( \mathbb{F} \), and that \( \mathfrak{N}_k \) is a nest of subspaces of \( \mathfrak{X}_k \) for \( k \in \{1, 2\} \). The ordinal sum \( \mathfrak{N}_1 + \mathfrak{N}_2 \) is a nest of subspaces of \( \mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2 \) defined by

\[
\mathfrak{N}_1 + \mathfrak{N}_2 = \{ N \ominus \{0\} : N \in \mathfrak{N}_1 \} \cup \{ \mathfrak{X}_1 + N : N \in \mathfrak{N}_2 \}
\]

Let \( \mathcal{A} = \text{Alg}(\mathfrak{N}_1 + \mathfrak{N}_2) \) and let \( \mathcal{A}_k = \text{Alg } \mathfrak{N}_k \) for \( k \in \{1, 2\} \). Every \( T \) in \( \mathcal{L}(\mathfrak{X}) \) has an operator matrix,

\[
T = \begin{pmatrix}
A_1 & B \\
C & A_2
\end{pmatrix}
\]

relative to the decomposition \( \mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2 \). It is easy to check that

\[
T \in \mathcal{A} \text{ if and only if } A_k \in \mathcal{A}_k \text{ for } k \in \{1, 2\} \text{ and } C = 0, \text{ and (6)}
\]

\[
T \in \mathcal{A}_- \text{ if and only if } A_k \in (\mathcal{A}_k)_- \text{ for } k \in \{1, 2\} \text{ and } C = 0. \text{ (7)}
\]

**Lemma 26** With the above notation and \( C = 0 \),

\[
T \in \text{Rad } \mathcal{A} \text{ if and only if } A_k \in \text{Rad } \mathcal{A}_k \text{ for } k \in \{1, 2\}, \text{ and (8)}
\]

\[
\text{Rad } \mathcal{A} = \mathcal{A}_- \text{ if and only if } \text{Rad } \mathcal{A}_k = (\mathcal{A}_k)_- \text{ for } k \in \{1, 2\}. \text{ (9)}
\]

\textbf{Proof.} A simple matrix computation shows that if

\[
\begin{pmatrix}
D & E \\
0 & F
\end{pmatrix} = \begin{pmatrix}
A_1 & B \\
0 & A_2
\end{pmatrix}^{-1}
\]

if and only if \( D = A_1^{-1}, F = A_2^{-1} \) and \( E = -A_1^{-1}BA_2^{-1} \). So

\[
\begin{pmatrix}
A_1 & B \\
0 & A_2
\end{pmatrix}^{-1} \in \mathcal{A}
\]

if and only if \( A_1^{-1} \in \mathcal{A}_1 \) and \( A_2^{-1} \in \mathcal{A}_2 \). Statement (9) is now obvious. Statement (9) follows from (7) and (8). \( \Box \)

**Example 27** Let \( \mathfrak{X} = \mathfrak{N} \oplus \mathfrak{N} \), where \( \mathfrak{N} \) is the vector space of all \( \mathbb{F} \)-valued sequences. Let

\[
\mathfrak{N}_1 = \{ \mathfrak{N}, M_1^+, M_2^+, M_3^+, \ldots, \{0\} \}
\]

where \( M_n^+ = \{ \varphi \in \mathfrak{N} : \text{ supp } \varphi \subseteq \{n+1, n+2, n+3, \ldots\} \} \), as in Example 14, and let

\[
\mathfrak{N}_2 = \{ \{0\}, (M_1^\#)_+, (M_2^\#)_+, (M_3^\#)_+, \ldots, \mathfrak{N} \}
\]

where \( (M_n^\#)_+ = \{ \varphi \in \mathfrak{N} : \text{ supp } \varphi \subseteq \{1, 2, 3, \ldots, n\} \} \), as in Example 15.

Note that \( \mathfrak{N}_1 = \mathfrak{N}^+ \), where \( \mathfrak{N} \) is as defined in Example 14. Since \( \mathfrak{N}_1 \) is well-ordered with order type \( \omega \), it follows from Theorem 23 that \( \text{Rad } \mathcal{A}_1 = (\mathcal{A}_1)_- \). Note also that \( \mathfrak{N}_2 \) is well-ordered, i.e. it satisfies the descending chain condition. So by Corollary 23 \( \text{Rad } \mathcal{A}_2 = (\mathcal{A}_2)_- \). So by Lemma 26 \( \text{Rad } (\mathfrak{N}_1 + \mathfrak{N}_2) = (\mathfrak{N}_1 + \mathfrak{N}_2)_- \).
But $\mathfrak{N}_1 + \mathfrak{N}_2$ satisfies neither the ascending chain condition nor the ascending chain condition. Its order type is $1 + \omega^* + \omega + 1$, i.e. the order type of $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, where $\mathbb{Z}$ denote the set of integers, and it contains both strictly decreasing and strictly increasing infinite sequences of subspaces.

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