Estimating Sparse Discrete Distributions Under Local Privacy and Communication Constraints

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Abstract

We consider the task of estimating sparse discrete distributions under local differential privacy and communication constraints. Under local privacy constraints, we present a sample-optimal private-coin scheme that only sends a one-bit message per user. For communication constraints we present a public-coin scheme based on random hashing functions, which we prove is optimal up to logarithmic factors. Our results show that the sample complexity only depends logarithmically with the ambient dimension, thus providing significant improvement in sample complexity under sparsity assumptions. Our lower bounds are based on a recently proposed chi-squared contraction method.

1 Introduction

Estimating distributions from data samples is a central task in statistical inference. In several modern applications data is generated from distributed sources including cell phones, wireless sensors, and smart healthcare devices. Access to such data is subject to severe “local information constraints”, such as communication and energy constraints, privacy concerns. For several statistical inference tasks, including distribution estimation, privacy and communication constraints lead to significant degradation in utility (see Section 1.3 for a detailed discussion). In several applications, such as web-browsing, genomics, and language modeling, the distribution is often supported over a small unknown subset of the domain. Motivated by the utility gain in high dimensional statistics under sparsity assumptions [33], we study the problem of estimating sparse discrete distributions under privacy and communication constraints.

1.1 Notations and problem set-up

Let \([k] := \{1, 2, ..., k\}\) and \(\Delta_k := \{p \in [0, 1]^k: \sum_{x \in [k]} p(x) = 1\}\) be the set of all distributions over \([k]\). For \(p \in \Delta_k\) and \(S \subset [k]\), let \(p^S\) be the vector restricted on indices in \(S\). Independent samples \(X_1, \ldots, X_n\) from an unknown \(p \in \Delta_k\) are observed by \(n\) users, where user \(i\) observes \(X_i\). User \(i\) sends a message \(Y_i = W_i(X_i)\) to a central server, where \(W_i : [k] \to \mathcal{Y}\) is a randomized mapping (channel) with

\[
W_i(y \mid x) = \Pr(Y_i = y \mid X_i = x).
\]

We consider privacy and communication constrained messages in this paper, which can be enforced by restricting \(W_i\)s to belong to a class \(\mathcal{W}\) of allowed channels.

**Local Differential Privacy (LDP).** A channel \(W : [k] \to \mathcal{Y} = \{0, 1\}^*\) is \(\epsilon\)-LDP if

\[
\sup_{y \in \mathcal{Y}} \sup_{x, x' \in [k]} \frac{W(y \mid x)}{W(y \mid x')} \leq e^{\epsilon}. \tag{1}
\]
\[ W = \{ W : W \text{ is } \varepsilon\text{-LDP} \} \] is the set of all \( \varepsilon\text{-LDP} \) channels.

**Communication constraints.** Let \( \ell < \log k \), and \( W = \{ W : [k] \to \mathcal{Y} = \{0,1\}^{\ell} \} \) be the set of channels that output \( \ell \)-bit messages, and thus characterize communication constraints.

![Diagram of Distributed Inference with Simultaneous Message Passing (SMP) Protocol](image)

**Figure 1:** Distributed inference with simultaneous message passing (SMP) protocol.

**Distribution estimation.** Let \( d : \triangle_k \times \triangle_k \to \mathbb{R}_+ \) be a distance measure. A distribution estimation protocol under constraints \( W \) is a set of channels \( W^n := W_1, W_2, \ldots, W_n \in W^n \) and an estimator \( \hat{p} : \mathcal{Y}^n \to \triangle_k \). Upon observing the \( n \) output messages \( Y^n := (Y_1, Y_2, \ldots, Y_n) \), the central server outputs an estimate \( \hat{p}(Y^n) \) of the underlying unknown distribution \( p \). Let \( \alpha > 0 \) be an accuracy parameter. The minimax sample complexity for estimation is

\[
SC(\alpha, \triangle_k, W, d) := \min_n \left\{ \min_{\hat{p}} \min_{W^n \in W^n} \max_{p \in \triangle_k} \Pr\left(d(p, \hat{p}(Y^n)) \leq \alpha \right) \right\},
\]

the fewest number of samples for which we can estimate \( p \) up to \( \alpha \) accuracy with probability at least 0.9. We will use the total variation distance, \( d_{TV}(\hat{p}, p) := \sup_{S \subseteq [k]} |\hat{p}(S) - p(S)| = \|p - \hat{p}\|_1 / 2 \) as the distance measure.

**Sparse distribution estimation.** Let \( s \leq k/100 \) and

\[ \triangle_{k,s} := \{ p \in \triangle_k : \|p\|_0 \leq s \} \]

be the distributions in \( \triangle_k \) with support size at most \( s \). Let

\[
SC(\alpha, \triangle_{k,s}, W, d_{TV}) := \min_n \left\{ \min_{\hat{p}} \min_{W^n \in W^n} \max_{p \in \triangle_{k,s}} \Pr\left(d_{TV}(p, \hat{p}(Y^n)) \leq \alpha \right) \right\}
\]

denote the sample complexity of estimating \( s \)-sparse distributions to total variation distance \( \alpha \).

In this paper, we consider simultaneous message passing (SMP) protocols (noninteractive schemes) where all the messages are sent simultaneously (see Figure 1). SMP protocols are broadly classified as private-coin and public-coin protocols. In private-coin schemes the channels

\[ \text{Sparsity larger than } k/100 \text{ gives same answers as the non-sparse case.} \]
$W_j$ are independent. In the more general public-coin schemes, the channels are chosen based on a function of a public randomness $U$ observed by all the users and the server. Private-coin protocols are a strict subset of public-coin protocols. We refer the readers to [3] for detailed definitions of these protocols.

1.2 Previous results and our contribution

Discrete distribution estimation under communication constraints is well studied, and it is now known that for $\ell$-bit communication constraints $\mathcal{W}_\ell$, and $\varepsilon$-LDP channels $\mathcal{W}_\varepsilon$ (for $\varepsilon = O(1)$),

$$\text{SC}(\alpha, \mathcal{W}_\ell, d_{\text{TV}}) = \Theta\left(\frac{k^2}{\alpha^2 \min\{k, 2^\ell\}}\right), \quad \text{SC}(\alpha, \mathcal{W}_\varepsilon, d_{\text{TV}}) = \Theta\left(\frac{k^2}{\alpha^2 \varepsilon^2}\right). \tag{2}$$

Plugging $\ell = \log k$ in the first equation gives the centralized sample complexity of $\Theta(k/\alpha^2)$. Note that for $\varepsilon = O(1)$ and $\ell = O(1)$, the sample complexity increases by a factor $k$ from the centralized setting.

We now present our results on estimating distributions in $\mathcal{W}_{\Delta_k, s}$, the set of $s$ sparse distributions in $\mathcal{W}_{\Delta_k}$. Our first result is a complete characterization of the sample complexity under $\varepsilon$-LDP up to constant factors.

**Theorem 1.** For $\varepsilon = O(1)$ and $\alpha \in (0, 1)$,

$$\text{SC}(\alpha, \mathcal{W}_{\Delta_k, s}, \mathcal{W}_\varepsilon, d_{\text{TV}}) = \Theta\left(\frac{s^2 \max\{\log(k/s), 1\}}{\alpha^2 \varepsilon^2}\right).$$

Moreover, there exists a private-coin protocol with one-bit privatized messages that achieves the upper bound. The algorithm runs in nearly linear time in $n$ and $k$.

A few remarks are in order. This result shows that sparsity $s$ is the effective domain size up to logarithmic factors. While an additional log $k$ factor is slightly simpler to obtain, a more involved technique in sparse estimation based on a covering set argument is used to establish the upper bound with the optimal overhead factor of log($k/s$). The lower bound is obtained by applying the chi-squared contraction bounds to a new construction of distributions.

We now present our sample complexity bounds under communication constraints. Unlike LDP, our bounds are off by logarithmic factors in various parameter regimes. Resolving this gap and obtaining the tight bounds is an open question.

**Theorem 2** (Upper bound for $\ell$-bit constraint). For $\alpha \in (0, 1)$, and channel family $\mathcal{W}_\ell$, the sample complexity for learning distributions in $\mathcal{W}_{\Delta_k, s}$ satisfies,

$$\text{SC}(\alpha, \mathcal{W}_{\Delta_k, s}, \mathcal{W}_\ell, d_{\text{TV}}) = O\left(\frac{s^2 \max\{\log(k/s), 1\}}{\alpha^2 \min\{2^\ell, s\}}\right),$$

$$\text{SC}(\alpha, \mathcal{W}_{\Delta_k, s}, \mathcal{W}_\ell, d_{\text{TV}}) = \Omega\left(\max\left\{\frac{s^2 \max\{\log(k/s), 1\}}{\alpha^2 \min\{2^\ell, s\}}, \frac{s^2 \max\{\log(k/s), 1\}}{\alpha^2 \ell}\right\}\right).$$

**Organization.** The remainder of the paper is organized as follows. We discuss related works in Section 1.3. We present algorithms and proofs for sparse estimation under LDP and communication constraints in Section 2 and Section 3 respectively.
1.3 Related work

Distribution estimation has a rich literature (see e.g., [7, 32, 15, 16], and references therein). There has been recent interest in distributed distribution estimation under communication and privacy constraints. For estimating discrete distributions under communication constraints, the optimal sample complexity is established in [24, 2, 9], and for local privacy constraints in [19, 21, 25, 35, 10, 6, 5]. [4, 3] unify both constraints under the framework of distributed inference under local information constraints, where optimal bounds are obtained under both non-interactive and interactive protocols. [14] considers the trade-off between privacy and communication constraints and provides optimal bounds in all parameter regimes. [28, 1] study discrete distribution estimation under different privacy constraints on the symbols.

A closely related problem is heavy hitter detection under LDP constraints [12, 11], where no distributional assumption on the data is made. A modification of their heavy hitters algorithms provides a sub-optimal $O(s^2 \log k/\alpha^2 \epsilon^2)$ sample complexity for $\epsilon$-LDP distribution estimation.

Statistical inference with sparsity assumption has been studied extensively for decades. The closest to our works are the Gaussian sequence model and high dimensional linear regression [17, 29, 20]. In these applications, it is assumed that the observations are linear transforms of the underlying parameter plus independent Gaussian noises on each dimension. In Section 2.1, it can be seen that using Algorithm 1, the histogram of observations can also be seen as a linear transform of the parameter of interest, however, with dependent noises on each dimension. We borrow ideas from these works in proving the upper bound. However, the lower bound part requires new proofs due to the dependency structure.

A few recent works study sparse estimation under information constraints. [19] and [34] consider the 1-sparse case and study mean estimation and linear regression under LDP constraints respectively. [18] provides lower bounds for sparse Gaussian mean estimation under LDP constraints via communication complexity. [8] considers estimating the mean of product Bernoulli distribution when the mean vector is sparse, which is different from the $k$-ary setting considered in this paper. [31] considers the problem of detecting the biased coordinate of product Bernoulli distributions under communication constraints, which can be viewed as a 1-sparse detection problem. [36, 22, 13, 24] consider sparse Gaussian mean estimation under communication constraints ([22, 13] consider interactive protocols with the goal of bounding the total amount of communication from all users). It was shown that under a fixed communication budget, the rate still scales linearly with the ambient dimension of the problem instead of the logarithmic dependence in the discrete case considered in this paper.

2 Sparse estimation under LDP constraints

In this section we will establish the sample complexity of sparse distribution estimation under LDP constraints.

In Section 2.1, we present a private-coin algorithm where each user sends only one-bit messages. The algorithm has two steps, which are detailed in Algorithm 1.

1. Players send one-bit messages using the private-coin Hadamard Response algorithm from [5].
2. The server projects a vector obtained from these messages onto $\triangle_{k,s}$ to obtain the final estimate.

We note that this algorithm is similar to that in [5, 10] where in the projection step they project onto $\triangle_k$ to estimate distributions without sparsity assumptions. While the algorithm is simple, to obtain the tight upper bounds, our analysis relies on a standard but involved covering-based techniques in sparse estimation. We also remark that a sample-optimal scheme can also be obtained using the popular RAPPOR mechanism [21, 25], which has higher communication overhead. We present this algorithm in the appendix for completeness.
In Section 2.2, we present a matching lower bound to prove the optimality of the aforementioned algorithm. The proof relies on applying the recently developed chi-squared contraction method [4] and a variant of Fano's inequality in [20].

**Algorithm 1: 1-bit Hadamard Response with Projection**

**Input:** \(X_1, \ldots, X_n\) i.i.d. from \(p \in \Delta_{k,s}\), the sparsity parameter \(s\).

**Output:** \(\hat{p} \in \Delta_{k}\) : an estimate of \(p\)

1. Let \(K = 2^{\lceil \log_2((k+1)) \rceil}\) be the smallest power of 2 more than \(k\).
2. For \(y \in [K]\), let \(B_y := \{x \in [K] : H_K(x, y) = 1\}\) be the rows where the \(y\)th column has 1.
3. Divide the \(n\) users into \(K\) sets \(S_1, \ldots, S_K\) deterministically by assigning all \(i \equiv j \mod K\) to \(S_j\) for \(i \in [n]\).
4. \(\forall j \in [K]\) and \(\forall i \in S_j\), the distribution of message \(Y_i\) is
   \[
   \Pr(Y_i = 1) = \begin{cases} 
   \frac{e^\varepsilon}{e^\varepsilon + 1}, & X_i \in B_j, \\
   \frac{1}{e^\varepsilon + 1}, & \text{otherwise,}
   \end{cases}
   \] (3)
   namely if \(H_K(X_i, j) = 1\), we send 1 with higher probability than 0.
5. Let \(\hat{t} := (\hat{t}_1, \ldots, \hat{t}_K)\) where \(\forall j \in [K]\), \(\hat{t}_j := \frac{1}{|S_j|} \sum_{i \in S_j} Y_i\) is the fraction of messages from \(S_j\) that are 1.
6. Compute intermediate estimates for
   \[
   \tilde{p}_K := \frac{e^\varepsilon + 1}{K(e^\varepsilon - 1)}H_K(2\hat{t} - 1_K).
   \] (4)
7. Keep the first \(k\) elements of \(\tilde{p}_K\), i.e., \(\bar{p} := \tilde{p}_K^{[k]}\) and project it onto \(\Delta_{k,s}\).
   \[
   \hat{p} := \min_{p \in \Delta_{k,s}} \|\bar{p} - p\|_2^2 .
   \]

### 2.1 Upper bounds under LDP constraints

We now establish the sample and time complexity of Algorithm 1.

Steps 1–6 of Algorithm 1 are identical to [5], who showed a time complexity of \(\tilde{O}(n + k)\). For the final step, where we project the vector \(\tilde{p}_K\) onto \(\Delta_{k,s}\), we can use [26, Algorithm 1], which runs in \(\tilde{O}(k)\) time, proving the overall time complexity.

Algorithm 1 uses Hadamard matrices. For \(m\) that is a power of two, let \(H_m\) be the \(m \times m\) Hadamard matrix with entries in \([-1, 1]\). The privacy guarantee of the algorithm follows from (3) which obeys (1). A key property we use is the following claim from [5], which shows a relationship between underlying distribution and the message distributions.

**Claim 1** ([5]). In (3), let \(t_j := \Pr(Y_i = 1 \mid i \in S_j)\) for \(j \in [K]\). Let \(t := (t_1, \ldots, t_K)\). Let \(p_K\) be the distribution over \([K]\) obtained by appending \(K - k\) zeros to \(p\). Then,
   \[
   p_K = \frac{(e^\varepsilon + 1)}{K(e^\varepsilon - 1)}H_K(2t - 1_K).
   \] (4)
By definition of \( \hat{p} \), we have \( \| \hat{p} - \hat{p} \|_2^2 \leq \| \hat{p} - p \|_2^2 \). Hence
\[
\| \hat{p} - p \|_2^2 \geq \| \hat{p} - \hat{p} \|_2^2 = \| \hat{p} - p \|_2^2 + \| p - \hat{p} \|_2^2 + 2 \langle \hat{p} - p, p - \hat{p} \rangle.
\]
Rearranging the terms, we have
\[
\| \hat{p} - p \|_2^2 \leq 2 \langle \hat{p} - p, \hat{p} - p \rangle.
\]  
We bound the right hand side by analyzing the projection step (Step 7), and using Claim 1. The proof of the lemma is from standard ideas from high dimensional sparse regression [29], and is provided in Appendix A.

**Lemma 3.**
\[
\langle \hat{p} - p, \hat{p} - p \rangle \leq \frac{25(e^\varepsilon + 1) \sqrt{s \log(2k/s)}}{(e^\varepsilon - 1) \sqrt{n}} \| \hat{p} - p \|_2.
\]
We can now prove the sample complexity bound as follows.
\[
d_{TV}(\hat{p}, p) = \frac{1}{2} \| \hat{p} - p \|_1 \leq \frac{1}{2} \sqrt{2s} \| \hat{p} - p \|_2 \leq \frac{40s \sqrt{\log(2k/s)} e^\varepsilon + 1}{\sqrt{n} e^\varepsilon - 1}.
\]  
where (6) applies Cauchy-Schwarz inequality on the 2s-sparse vector \( \hat{p} - p \), and (7) is from plugging Lemma 3 in (5). Plugging in \( d_{TV}(\hat{p}, p) = \alpha \), and using \( e^\varepsilon - 1 = O(\varepsilon) \) for \( \varepsilon = O(1) \) gives us the desired sample complexity bound of \( n = O(s^2 \max\{\log(k/s), 1\}/\alpha^2 \varepsilon^2) \).

### 2.2 Lower bound under LDP constraints

We now prove the sample complexity lower bound in Theorem 1 using the chi-squared contraction method in [4] and an extension of Fano’s method from [20].

For simplicity of analysis we add 0 to the underlying domain and consider distributions over \([k] \cup \{0\}\). Let \( Z_{k,s} \subseteq \{0, 1\}^k \) be all k-ary binary strings with \( s \) one’s. Then, \( |Z_{k,s}| = \binom{k}{s} \). We will restrict to \( P_{k,s} := \{z \in Z_{k,s} \} \), where \( p_z \) is described below for \( z \in Z_{k,s} \):
\[
p_z(x) = \begin{cases} 1 - 8\alpha, & \text{for } x = 0, \\ \frac{8\alpha x}{s}, & \text{for } x = 1, \ldots, k. \end{cases}
\]
(8)
Since \( s \) of the \( z_x \)’s are one, \( \sum_{x=1}^{k} p_z(x) = 8\alpha \) and \( p_z \) is a valid distribution.

Let \( Z \) be a uniform random variable over \( Z_{k,s} \). Let \( Y^n := (Y_1, \ldots, Y_n) \) be the output of an \( \varepsilon \)-LDP scheme whose input are \( X^n = (X_1, \ldots, X_n) \), drawn i.i.d. from \( p_Z \), and \( \hat{p} \) is such that \( \Pr (d_{TV}(p, \hat{p}(Y^n)) \leq \alpha) \geq 0.9 \). In other words, we can estimate distributions in \( P_{k,s} \) to within \( \alpha \) in total variation distance with probability at least 0.9. Let \( \hat{Z} \in Z_{k,s} \) be such that \( p_{\hat{Z}} \) is the distribution in \( P_{k,s} \) closest to \( \hat{p}(Y^n) \) in \( d_{TV} \). Then, we have
\[
\frac{4\alpha}{s} d_{\text{Ham}}(Z, \hat{Z}) = d_{TV}(p_Z, p_{\hat{Z}}) \leq d_{TV}(\hat{p}, p_{\hat{Z}}) + d_{TV}(p_{\hat{Z}}, \hat{p}) \leq 2d_{TV}(p_Z, \hat{p}).
\]
Since \( \Pr (d_{TV}(p_Z, \hat{p}(Y^n)) \leq \alpha) \geq 0.9 \), we have \( \Pr \left( d_{\text{Ham}}(Z, \hat{Z}) \leq s/2 \right) \geq 0.9 \), which implies using the estimator \( \hat{p} \), we can estimate the underlying \( Z \) to within Hamming distance \( s/2 \). We now state a form of Fano’s inequality from [20], adapted to our setting.
Lemma 4 (Corollary 1 [20]). Let $Z_{k,s} \subseteq \{0,1\}^k$ and $Z$ be uniformly distributed over $Z_{k,s}$. For $t \geq 0$, define the maximum neighborhood size at radius $t$

$$N^\text{max}_t := \max_{z \in Z} \{|z' \in Z : d_{\text{Ham}}(z, z') \leq t\},$$

to be the maximum number of elements of $Z_{k,s}$ in a Hamming ball of radius $t$. If $|Z_{k,s}| \geq 2N^\text{max}_t$, then for any Markov chain $Z - Y^n - \hat{Z}$,

$$\Pr \left( d_{\text{Ham}}(\hat{Z}, Z) > t \right) \geq 1 - \frac{I(Z; Y^n) + \log 2}{\log |Z_{k,s}| - \log N^\text{max}_t}.$$  \hspace{1cm} (9)

Substituting $t = s/2$, and using $\Pr \left( d_{\text{Ham}}(Z, \hat{Z}) \leq s/2 \right) \geq 0.9$ with this lemma gives

$$I(Z; Y^n) + \log 2 > 0.9 \left( \log |Z_{k,s}| - \log N^\text{max}_{s/2} \right).$$  \hspace{1cm} (10)

Using chi-squared contraction bounds from [4], we upper bound $I(Z; Y^n)$ in the next lemma.

Lemma 5. Let $Z$ be uniformly drawn from $Z_{k,s}$ and $Y^n$ be the outputs of $n$ users,

$$I(Z; Y^n) = O \left( \frac{n\alpha^2(e^\varepsilon - 1)^2}{s} \right).$$  \hspace{1cm} (11)

We lower bound $\log |Z_{k,s}| - \log N^\text{max}_{s/2}$ using standard Gilbert-Varshamov type arguments.

Lemma 6. Let $1 \leq s \leq k/100$, then

$$\log |Z_{k,s}| - \log N^\text{max}_{s/2} \geq \frac{s}{8} \log \left( \frac{k}{s} \right).$$  \hspace{1cm} (12)

Plugging these two bounds in (10) gives the tight sample complexity lower bound for sparse distribution estimation under $\varepsilon$-local differential privacy.

We now prove these lemmas.

Proof of Lemma 5. For a distribution $q$ over $[k] \cup \{0\}$ and a channel $W$, let $q^W$ denote output distribution $Y$ when $X \sim q$ and $Y = W(X)$. Then,

$$q^W(y) = \sum_{x \in [k] \cup \{0\}} W(y | x)q(x).$$  \hspace{1cm} (13)

Let $p_0$ be the average of all distributions in $\mathcal{P}_{k,s}$. Then $p_0(0) = 1 - 8\alpha$, and $p_0(x) = 8\alpha/k$ for $x = 1, \ldots, k$. We will use [4] to bound the maximum value of $I(Z; Y^n)$ in terms of the $\chi^2$-divergence between the output distributions induced by distributions in $\mathcal{P}_{k,s}$ and by $p_0$ and as follows:

$$I(Z; Y^n) \leq n \cdot \max_{W \in \mathcal{W}_e} \mathbb{E}_Z \left[ \chi^2(p_Z^W, p_0^W) \right]$$

$$= n \cdot \max_{W \in \mathcal{W}_e} \sum_y \mathbb{E}_Z \left[ \left( \sum_{x=1}^k (p_Z(x) - p_0(x))W(y | x) \right)^2 \right],$$  \hspace{1cm} (14)

where $\mathcal{W}_e$ is the set of all channels.

(15)
where (14) is from the chi-squared contraction bound, and (15) is by using (13) in the definition of $\chi^2$-divergence.²

For an $\varepsilon$-LDP channel $W \in \mathcal{W}_\varepsilon$, let $W_{\min}^y := \min_x W(y \mid x)$. By (1), we have $W(y \mid x) = W_{\min}^y + \eta_x^y \cdot W_{\min}^y$ for some $0 \leq \eta_x^y \leq \varepsilon - 1$. Furthermore, for $z \in \mathcal{Z}_{k,s}$, by the definition of $p_z$, $p_z(x) - p_0(x) = 8\alpha \left( \frac{z_x}{s} - \frac{1}{k} \right)$, and $\sum_x z_x = s$, thus giving

$$\sum_x (p_z(x) - p_0(x))W(y \mid x) = 8\alpha \sum_x \left( \frac{z_x}{s} - \frac{1}{k} \right) (W_{\min}^y + W_{\min}^y \eta_x^y) = 8\alpha W_{\min}^y \sum_x \left( \frac{z_x}{s} - \frac{1}{k} \right) \eta_x^y.$$  

Since $Z$ is uniformly distributed over $\mathcal{Z}_{k,s}$, elementary computations show that $E[Z_x] = E[Z_x^2] = s/k$, and for $x_1 \neq x_2 \in [k]$, $E[Z_{x_1}Z_{x_2}] = \binom{k-2}{s-2} / \binom{k}{s} = \frac{s(s-1)}{k(k-1)}.

Therefore,

$$E_Z \left[ \left( \sum_x (p_Z(x) - p_0(x))W(y \mid x) \right)^2 \right]$$

$$= 64\alpha^2 W_{\min}^y \sum_{x_1 \neq x_2} E_Z \left[ \frac{1}{k^2} - \frac{Z_{x_1} + Z_{x_2}}{sk} + \frac{Z_{x_1}Z_{x_2}}{s^2} \right] \eta_x^{y_{x_1}} \eta_x^{y_{x_2}}$$

$$= 64\alpha^2 W_{\min}^y \sum_x \left( \frac{1}{sk} - \frac{1}{k^2} \right) (\eta_x^y)^2 + \sum_{x_1 \neq x_2} E_Z \left[ \frac{1}{k^2} + \frac{s-1}{sk(k-1)} \right] \eta_x^{y_{x_1}} \eta_x^{y_{x_2}}$$

$$\leq 64\alpha^2 W_{\min}^y \left( \frac{(\max_x \eta_x^y)^2}{s} \right),$$

and

$$E_Z \left[ \sum_x (p_Z(x) - p_0(x))W(y \mid x) \right] \leq 64\alpha^2 \left( \frac{(\max_x \eta_x^y)^2}{s} \right) \cdot W_{\min}^y.$$  

Using $\sum_y W_{\min}^y \leq 1$, and $\eta_x^y \leq \varepsilon - 1$, we obtain

$$E[Z \cdot (p_Z^W, p_0^W)] = O \left( \frac{\alpha^2 (\varepsilon - 1)^2}{s} \right).$$

Combining with (15), this completes the proof. □

**Proof of Lemma 6.** Let $z \in \mathcal{Z}_{k,s}$. A vector in $\mathcal{Z}_{k,s}$ that is at most $s/2$ away from $z$ in Hamming distance can be obtained as follows: Fix $s/2$ coordinate in $z$ that are 1, and from the remaining $k - s/2$ coordinates, choose $s/2$ coordinates and make them 1. Therefore,

$$N_{s/2}^{\max} \leq \binom{s}{s/2} \binom{k}{s/2} \binom{s/2}{s/2}.$$  

Recall that $|\mathcal{Z}_{k,s}| = \binom{k}{s}$. Using Stirling’s approximation for binomial coefficients³, we get

$$\log \frac{|\mathcal{Z}|}{N_{s/2}^{\max}} \geq \log \frac{\binom{k}{s}}{\binom{s/2}{s/2} \binom{k-s/2}{s/2}} \geq \log \frac{\binom{\frac{k}{2}}{s/2}}{(2e)^{s/2} \binom{\frac{k}{2} - s/2}{s/2}} \geq \log \frac{\binom{k}{s}}{(2e)^{s/2} \binom{\frac{k}{2} - s/2}{s/2}} = \frac{s}{2} \log \left( \frac{k}{4e^2 s} \right),$$

which is at least $\frac{s}{2} \log \frac{k}{s}$ when $s \leq k/100$. □

² $\chi^2(p, q) := \sum_x (p(x) - q(x))^2 / q(x)$

³ For $1 \leq s \leq k$, we have $\binom{\frac{k}{2}}{s} \leq \binom{k}{s} \leq \binom{\frac{k}{2}}{s}$.  

8
3 Sparse estimation under communication constraints

We now prove guarantees of Theorem 2 and establish upper and lower bounds for estimating for communication constrained sparse discrete distribution estimation. In Section 3.1, we propose an algorithm that requires public randomness with sample complexity given in Theorem 2. Designing a private-coin protocol for estimation is an open question. In Section 3.2 we establish the lower bounds.

3.1 Upper bounds under communication constraints

Note that the sample complexity upper bound in Theorem 2 has min \{2^L, s\}, which equals s when \( \ell \geq \log s \). We therefore only consider \( \ell \leq \log s \) since if \( \ell > \log s \), we can just use \( \log s \) bits and get the same bound.

Our first step is to use public randomness to design hash functions at the users. A randomized mapping \( h: [k] \to [2^\ell] \) is a random hash function if \( \forall x \in [k], y \in [2^\ell], \Pr (h(x) = y) = \frac{1}{2^\ell} \).

The scheme. Let \( h_1, \ldots, h_n \) be \( n \) independent hash functions, available at the users and at the server. User \( i \)'s \( \ell \) bit output is \( Y_i = h_i(X_i) \in [2^\ell] \). The probability of \( x \in [k] \) being in the preimage of user \( i \)'s message is,

\[
\Pr (Y_i = h_i(x)) = p(x) + (1 - p(x)) \frac{1}{2^\ell} = p(x) \left( 1 - \frac{1}{2^\ell} \right) + \frac{1}{2^\ell} =: b(x). \quad (16)
\]

The estimator. Upon receiving messages \( Y_1, \ldots, Y_n \), the estimator is as follows,

1. The first \( n/2 \) messages are used to obtain a set \( T \subseteq [k] \) with \( |T| = O(s) \) such that with high probability \( p(T) > 1 - \alpha/2 \).
2. With the remaining messages we estimate \( p(x) \) for \( x \in T \).

We now describe and analyze the two steps.

Step 1. Let \( Y_1, \ldots, Y_{n/2} \) be the first \( n/2 \) messages, where \( Y_i = h_i(X_i) \). For \( x = 1, \ldots, k \), let

\[
M(x) := |\{i : h_i(x) = Y_i, 1 \leq i \leq n/2\}|
\]

be the number of these messages in the first half whose preimage \( x \) belongs to. Let \( T \subseteq [k] \) be the set of symbols with the largest \( |T| = 2s \) values of \( M(x) \).s. If \( p(x) \) is large we expect \( M(x) \) to be large. In particular, we show that for sufficiently large \( n \), the probability of symbols not in \( T \) is small.

Lemma 7. There is a constant \( C_1 \) such that for \( n = C_1 \cdot s^2 \log(k/s)/(\alpha^2 \min\{2^\ell, s\}) \) with probability at least 0.95, \( p(T) := \sum_{x \in T} p(x) \geq 1 - \alpha/2 \).

Step 2. For \( x = 1, \ldots, k \), let \( N(x) := |\{i : h_i(x) = Y_i, n/2 < i \leq n\}| \) be the number of messages in the second half such that \( x \) belongs to the preimage of \( Y_i \). Our final estimator is given by

\[
\hat{p}(x) = \begin{cases} 
\frac{(2^\ell N(x)/(n/2)) - 1}{2^{\ell-1}}, & \text{if } x \in T \\
0, & \text{otherwise}
\end{cases}
\quad (17)
\]

The following lemma shows that \( \hat{p} \) converges to \( p \) over \( T \).

Lemma 8. There is a constant \( C_2 \) such that for \( n = C_2 \cdot s^2 \log(k/s)/(\alpha^2 \min\{2^\ell, s\}) \) with probability at least 0.95,

\[
\sum_{x \in T} |\hat{p}(x) - p(x)| \leq \frac{\alpha}{2}.
\]

Combining Lemma 7 and Lemma 8, by the union bound we get that with \( n \) samples, with probability at least 0.9, \( \|\hat{p} - p\|_1 \leq \alpha \), establishing upper bound of Theorem 2.
3.2 Lower bounds under communication constraints

We now prove the sample complexity lower bound in Theorem 2. The first term of $\Omega(\frac{s^2}{\alpha \min\{|2^{y^2}, s^2\}})$ follows from (2) and holds even with the knowledge of the support $S$.

We prove the second term by considering the construction $\mathcal{P}_{k,s}$ as in Section 2.2, and bound (10). The lower bound on $\log(|Z|/N_{\text{max}})$ is the same as from Lemma 5. Analogous to Lemma 6, we will now bound the mutual information $I(Z;Y^n)$ by $O(n\alpha 2^s/s)$ as follows. As in (15), we have the following bound

$$I(Z;Y^n) \leq n \cdot \max_{W \in \mathcal{W}_n} \sum_y \frac{E_Z \left[ \left( \sum_{x=1}^{k} (p_Z(x) - p_0(x))W(y|x) \right)^2 \right]}{E_{X \sim p_0}[W(y|X)]},$$

and we now bound it for each $y \in [2^s]$. Similar to the expansion in proving the LDP lower bounds, we have

$$E_Z \left[ \left( \sum_{x=1}^{k} (p_Z(x) - p_0(x))W(y|x) \right)^2 \right] = 64\alpha^2 E_Z \left[ \sum_{x_1, x_2} \left( \frac{1}{k^2} - \frac{Z_{x_1} + Z_{x_2}}{sk} + \frac{Z_{x_1} Z_{x_2}}{s^2} \right)W(y|x_1)W(y|x_2) \right] = 64\alpha^2 \sum_{x=1}^{k} \left( \frac{1}{sk} - \frac{1}{k^2} \right)W(y|x)^2 + 64\alpha^2 \sum_{x_1 \neq x_2} \left( \frac{1}{k^2} + \frac{s-1}{sk(k-1)} \right)W(y|x_1)W(y|x_2) \leq 64\alpha^2 \sum_{x=1}^{k} \left( \frac{1}{sk} - \frac{1}{k^2} \right)W(y|x)^2.$$

Note that $E_{X \sim p_0}[W(y|X)] = (1 - 8\alpha)W(y|0) + \sum_{x=1}^{k} \frac{8\alpha}{s}W(y|x)$. Hence,

$$E_Z \left[ \chi^2(p_Z^{W}, p_0^{W}) \right] \leq 64\alpha^2 \sum_{y} \frac{\left( \sum_{x=1}^{k} \left( \frac{1}{sk} - \frac{1}{k^2} \right)W(y|x)^2 \right)}{(1 - 8\alpha)W(y|0) + \sum_{x=1}^{k} \frac{8\alpha}{s}W(y|x)} \leq 64\alpha^2 \left( \frac{1}{sk} - \frac{1}{k^2} \right) \sum_{y} \frac{\left( \sum_{x=1}^{k} W(y|x)^2 \right)}{\sum_{x=1}^{k} \frac{8\alpha}{s}W(y|x)} \leq \frac{8\alpha}{s} \sum_{y} \frac{\sum_{x=1}^{k} W(y|x)^2}{\sum_{x=1}^{k} W(y|x)} \leq \frac{8\alpha}{s} 2^s,$$

where we used that $W(y|x)^2 \leq W(y|x)$, proving the lower bound.

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A Proof of Lemma 3

We will use the following bound on the covering number of $s$-sparse vectors.

**Claim 2** ([29]). Let $S(r,2s) = \{ \xi \in \mathbb{R}^k : \|\xi\|_2 \leq r, \|\xi\|_0 \leq 2s \}$. There exists a $\rho r$-covering $C_r \subset S(r,2s)$ of $S(r,2s)$ with size

$$|C_r| = N(S(r,2s), \rho r) \leq \left( \frac{k}{2s} \right) \left( \frac{1}{\rho} \right)^{2s}.$$
Let $C_1$ be a $\rho$-covering of $S(1, 2s)$ with size $N(S(1, 2r), \rho)$. Let $r = \|\hat{p} - p\|_2$, then $\hat{p} - p \in S(r, 2s)$, and one can obtain a covering $C_r$ of $S(r, 2s)$ by multiplying each vector in $C_1$ by $r$.

Since $\hat{p} - p$ is 2s-sparse, let $\xi^*$ be the closest point in $C_r$ to $\hat{p} - p$. Then we can bound the right hand side of (5).

\[
\langle \hat{p} - p, \hat{p} - p \rangle \leq |\langle \hat{p} - p, \xi^* \rangle| + |\langle \hat{p} - p, p - \xi^* \rangle| \\
\leq \max_{\xi \in C_r} |\langle \hat{p} - p, \xi \rangle| + \rho r \|\hat{p} - p\|_2 \\
= r \max_{\xi \in C_r} |\langle \hat{p} - p, \xi / r \rangle| + \rho r \|\hat{p} - p\|_2.
\]

$\hat{p} - p$ is the first $k$ entries of $\hat{p}_K - p_K$, and by Claim 1, we have

\[
\hat{p}_K - p_K = \frac{2(e^\varepsilon + 1)}{K(e^\varepsilon - 1)} H_K(\hat{t} - t).
\]

Note that in Algorithm 1, $\forall j \in [K]$, $\hat{t}_j$ is the average of $|S_j| \geq K/(2n)$ i.i.d. Bernoulli random variables with the mean satisfying $E \hat{t}_j = t$. Hence $\forall i \in [K]$, $\hat{t}_j - t_j$ is a zero-mean sub-Gaussian random variable with variance proxy at most $\frac{K^2}{2n}$. Moreover, $\hat{t}_j$’s are independent since $S_j$’s are disjoint. Note for all $\forall \xi \in C_1$,

\[
\langle \hat{p}_K - p_K, \xi \rangle = \frac{2(e^\varepsilon + 1)}{K(e^\varepsilon - 1)} \xi^T H_K(\hat{t} - t),
\]

which are linear combinations of $(\hat{t}_j - t_j)$’s. Since $\hat{p} - p$ is the first $k$ entries of $\hat{p}_K - p_K$, we have $\forall \xi \in C_1$, $\langle \hat{p}_K - p_K, \xi \rangle$ is also sub-Gaussian (see Corollary 1.7 in [30]) with variance proxy at most

\[
\frac{K}{2n} \left\| \frac{2(e^\varepsilon + 1)}{K(e^\varepsilon - 1)} \xi^T H_K \right\|_2^2 = \frac{2}{nK} \left( \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \right)^2 \|\xi^T H_K\|_2^2 \\
= \frac{1}{n} \left( \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \right)^2 \|\xi\|_2^2
\]

(19)

where (19) follows since $H_K^T H_m = m\hat{u}_k$ by the orthogonality of Hadamard matrices.

Therefore using maximal inequalities of sub-Gaussian random variables [30, Theorem 1.14], with probability at least 19/20, we have

\[
\max_{\xi \in C_1} |\langle \hat{p}_K - p_K, \xi \rangle| < \sigma \sqrt{14 \log |C_1|}.
\]

(20)

By the utility guarantee of Hadamard Response [5], with probability at least 19/20,

\[
\|\hat{p} - p\|_2^2 \leq 20E \left[ \|\hat{p} - p\|_2^2 \right] \leq \frac{40k(e^\varepsilon + 1)^2}{n(e^\varepsilon - 1)^2} \quad \Longrightarrow \quad \|\hat{p} - p\|_2 \leq 7 \sqrt{\frac{e^\varepsilon + 1}{e^\varepsilon - 1}} \sqrt{\frac{k}{n}}.
\]

(21)

By union bound, conditioned on (20) and (21), which happens with probability at least 9/10,

\[
\langle \hat{p} - p, \hat{p} - p \rangle \leq \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \sqrt{\frac{56 \log |C_1|}{n}} + 7 \rho \sqrt{e^\varepsilon + 1} \sqrt{\frac{k}{n}} \\
\leq \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \sqrt{\frac{112 s \log \left( \frac{ek}{2s} \right)}{\rho} + 112 s \log \frac{1}{\rho} + 7 \rho \sqrt{e^\varepsilon + 1} \sqrt{\frac{k}{n}}}
\]

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Taking $\rho = \sqrt{\frac{2}{s}}$,
\[
\langle \hat{p} - p, \hat{p} - p \rangle \leq 25 \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \frac{r\sqrt{s \log(2k/s)}}{\sqrt{n}} = 25 \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \frac{s \log(2k/s)}{\sqrt{n}} \|\hat{p} - p\|_2.
\]
Combining with (5),
\[
\|\hat{p} - p\|_2 \leq \frac{50(e^\varepsilon + 1) \sqrt{s \log(2k/s)}}{(e^\varepsilon - 1)\sqrt{n}}.
\]
Since $\hat{p} - p$ is $2s$-sparse, by Cauchy-Schwarz,
\[
d_{TV}(\hat{p}, p) = \frac{1}{2} \|\hat{p} - p\|_1 \leq \frac{1}{2} \sqrt{2s} \|\hat{p} - p\|_2 \leq \frac{40s\sqrt{\log(2k/s)} e^\varepsilon + 1}{\sqrt{n} e^\varepsilon - 1}.
\]
Setting the right hand side as $\alpha$ yields the desired sample complexity bound.

\section*{B Proof of Lemma 7 and Lemma 8}

\textit{Proof of Lemma 7.} $M(x)$ is distributed as a Binomial $Bin(n/2, b(x))$. Hence, $\text{E}[M(x)] = n \cdot b(x)/2$. By (16) and (17) we have $\text{E}[\hat{p}(x)] = p(x)$. Using the variance formula of Binomials, we know $\text{Var}(M(x)) = n \cdot b(x) \cdot (1 - b(x))/2 \leq n \cdot b(x)/2$.

Set $\gamma = 1 - 1/2^d$ and $\beta = 1/2^d$, then $b(x) = \gamma \cdot p(x) + \beta$. We will use the following multiplicative Chernoff bound.

\textbf{Lemma 9} (Multiplicative Chernoff bound [27]). Let $Y_1, \ldots, Y_n$ be independent random variables with $Y_i \in 0, 1$, and $Y = Y_1 + \cdots + Y_n$, and $\mu = \text{E}[Y]$. Then for $\tau > 0$,
\[
\text{Pr}(Y \geq (1 + \tau)\mu) \leq e^{-\frac{\tau^2}{2}} \text{, } \text{Pr}(Y \geq (1 - \tau)\mu) \leq e^{-\frac{\tau^2}{2}}.
\]

Let $S := \{x : p(x) > 0\}$. Therefore, for $x \in [k] \setminus S$, $p(x) = 0$. By Lemma 9,
\[
\text{Pr}(M(x) \geq \frac{n}{2} \beta + \sqrt{3n\beta \log \frac{k}{s}}) \leq \left(\frac{s}{k}\right)^2.
\]

Let $E$ be the event that at most $s$ symbols in $[k] \setminus S$ appear at least $M^* := \frac{n}{2} \beta + \sqrt{3n\beta \log \frac{k}{s}}$ times. By Markov’s inequality,
\[
\text{Pr}(E^c) = \text{Pr}\left(\left| x \in [k] \setminus S : M(x) \geq \frac{n}{2} \beta + \sqrt{3n\beta \log \frac{k}{s}} \right| > s \right) \leq \frac{s}{k} \leq \frac{1}{100}.
\]

We condition on $E$ in the remainder of the proof. Note that it suffices to show
\[
\text{E}[p(T^c) \mid E] := \sum_x p(x) \text{Pr}(x \text{ not selected} \mid E) \leq \frac{\alpha}{50},
\]
since if (23) is true, by Markov inequality,
\[
\text{Pr}\left(p(T^c) > \frac{\alpha}{2} \mid E\right) \leq \frac{1}{25},
\]
which, combined with (22), implies Lemma 7.

Next we prove (23). Conditioned on $E$, a symbol $x$ is not selected after the first stage only if it appears at most $M^*$ times, which implies $\text{Pr}(x \text{ not selected} \mid E) \leq \text{Pr}(M(x) \leq M^* \mid E).$
Moreover, since $\forall x_1 \in S, x_2 \notin S$, $M(x_1)$ and $M(x_2)$ are independent, we have $M(x_1)$ is independent of event $E$. Thus:

$$\mathbb{E}[p(T^c) \mid E] = \sum_x p(x) \Pr(x \text{ not selected} \mid E) \leq \sum_x p(x) \Pr(M(x) \leq M^*) .$$

Next we divide symbols into three sets based on their probability mass: $A = \{ x \in [k] : p(x) \leq \frac{\alpha}{600} \}$, $B = \{ x \in [k] : \frac{\alpha}{600} < p(x) \leq \beta/\gamma \}$ and $C = \{ x \in [k] : p(x) > \beta/\gamma \}$. For set $A$, we have:

$$\sum_{x \in A} p(x) \Pr(M(x) \leq M^*) \leq \sum_{x \in A} p(x) \leq \frac{\alpha}{60} . \quad (24)$$

Next we bound the sum over set $B$ and $C$. $\forall x \in B \cup C, p(x) > \alpha/60s$. In the rest of the proof, we set the constant $C_1 = 700000$. For $n = C_1 \cdot s^2 \log(k/s)/(\alpha^2 \min\{2^\ell, s\})$,

$$\frac{n}{4} \gamma p(x) - \sqrt{3n\beta \log \frac{k}{s}} \geq 0 .$$

Hence,

$$\mathbb{E}[M(x) - M^*] = \frac{n}{2} \gamma p(x) - \sqrt{3n\beta \log \frac{k}{s}} \geq \frac{n\gamma p(x)}{4} ,$$

Using Lemma 9,

$$\Pr(M(x) \leq M^*) = \Pr(M(x) \leq \mathbb{E}[M(x)] - (\mathbb{E}[M(x)] - M^*))$$

$$\leq \exp \left( \left. \left( \frac{\gamma p(x)^2}{\gamma p(x) + \beta} \right)^n \frac{\alpha}{2} (\gamma p(x) + \beta) \right) \right)$$

$$= \exp \left( \left. \left( \frac{\gamma p(x)^2}{8(\gamma p(x) + \beta)} \right) \right) \right)$$

If $x \in C$, i.e., $p(x) > \beta/\gamma$, we have

$$\Pr(M(x) \leq M^*) \leq \exp \left( \left. \left( - \frac{\gamma^2 p(x)^2 n}{16 \gamma p(x)} \right) \right) \right) \leq \exp \left( \left. \left( \frac{\gamma p(x)n}{16} \right) \right) \leq \frac{16}{\gamma p(x)n} \leq \frac{16}{n\beta} \leq \frac{\alpha}{1000} ,$$

where we use $n\beta > C_1 s^2 \log(k/s)/(\alpha^2 2^\ell) \geq C_1/\alpha^2$ when $s \geq 2^\ell$. This implies

$$\sum_{x \in C} p(x) \Pr(M(x) < M^*) \leq \sum_{x \in A} p(x) \frac{\alpha}{1000} \leq \frac{\alpha}{1000} . \quad (25)$$

When $x \in B$, i.e., $p(x) \leq \beta/\gamma$,

$$\Pr(M(x) \leq M^*) \leq \exp \left( \left. \left( - \frac{\gamma^2 p(x)^2 n}{16 \beta} \right) \right) \right) .$$

Now let $p(x) = (1 + \zeta_x)\alpha/60s$ where $\zeta_x > 0$, we have

$$\frac{\gamma^2 p(x)^2 n}{16 \beta} \geq 2(1 + \zeta_x)^2 \log \frac{k}{s} .$$

$$\sum_{x \in C} p(x) \Pr(M(x) \leq M^*) \leq \frac{\alpha}{60s} \sum_{x \in C} (1 + \zeta_x) \exp \left( \left. \left( -2(1 + \zeta_x)^2 \log \frac{k}{s} \right) \right) \right) \leq \frac{\alpha}{500} . \quad (26)$$

Combining (24), (25) and (26), we get (23), and thus proving the lemma. \qed
Obtain \( \hat{p}(x) \). Construct the set \( T \subset [n] \) using RAPPOR: Each user randomizes its sample using RAPPOR, and \( n \) \( (1-2\epsilon) \) elements are privatized data using RAPPOR. We first describe the high level idea of the algorithm for LDP estimation. All users send their privatized data using RAPPOR [21, 25]. As in Section 3.1, we use the first half of privatized samples to estimate a subset \( T \subseteq [k] \) which has a similar form as (16). Hence setting \( \beta = 1/(e^\epsilon + 1) \) and \( \gamma = (e^\epsilon - 1)/(e^\epsilon + 1) \), we can follow the steps in the communication constrained setting and obtain with probability at least \( 9/10 \), \( d_{TV}(\hat{p}, p) \leq \alpha \) using

\[
n = O \left( \beta \frac{s^2 \log \frac{k}{\delta}}{\alpha^2 \gamma^2} \right) = O \left( \frac{s^2 \max \{\log (k/s), 1\}}{\alpha^2 \epsilon^2} \right),
\]

when \( \epsilon = O(1) \).

**Proof of Lemma 8.** Note that \( N(x) \) and \( M(x) \) are identically distributed, and therefore, \( M(x) \) is distributed \( Bin(n/2, b(x)) \), and \( E[N(x)] = n \cdot b(x)/2 \), and \( Var(N(x)) \leq n \cdot b(x)/2 \).

\[
E[(\hat{p}(x) - p(x))^2] = \left( \frac{2 \cdot 2^\epsilon}{n(2^\epsilon - 1)} \right)^2 \cdot Var(N(x)) \leq \frac{2}{n} \left( \frac{2^\epsilon}{2^\epsilon - 1} \right)^2 b(x).
\]

Now, note that \( \sum_{x \in T} b(x) = p(T) (1 - 1/2^\epsilon) + |T|/2^\epsilon \), and therefore,

\[
E \left[ \| \hat{p}^T - p^T \|_2^2 \right] = \sum_{x \in T} E \left[ (\hat{p}(x) - p(x))^2 \right] \leq \frac{2}{n} \left( \frac{2^\epsilon}{2^\epsilon - 1} \right)^2 \sum_{x \in T} b(x) \leq \frac{2(|T| + 2^\epsilon) 2^\epsilon}{n(2^\epsilon - 1)^2}.
\]

Using Jensen’s inequality and Cauchy-Schwarz,

\[
E \left[ \| \hat{p}^T - p^T \|_1 \right] \leq \sqrt{E \left[ \| \hat{p}^T - p^T \|_1^2 \right]} \leq \sqrt{|T| \cdot E \left[ \| \hat{p}^T - p^T \|_2^2 \right]} \leq \frac{4s2^\epsilon (2^\epsilon + 2s)}{n(2^\epsilon - 1)^2}.
\]

Setting \( C_2 = 6400 \), the lemma follows by Markov’s inequality.

\[\square\]

**C An LDP estimation scheme using RAPPOR.**

We first describe the high level idea of the algorithm for LDP estimation. All users send their privatized data using RAPPOR [21, 25]. Details are described in Algorithm 2.

Algorithm 2: Sparse estimation using RAPPOR

1. **Input:** \( n \) i.i.d. samples from unknown \( s \)-sparse \( p \).
2. Each user randomizes its sample using RAPPOR: Each sample \( X_i \) is first mapped to a one-hot vector \( Z_i \in \{0, 1\}^k \) which has a 1 at the \( X_i \)’th coordinate and 0’s elsewhere. Then each bit is flipped independently with probability \( 1/(e^\epsilon/2 + 1) \) to obtain \( Y_i \in \{0, 1\}^k \).
3. Compute \( M := [M(1), \ldots, M(k)] = \sum_{i=1}^{n/2} Y_i \) using the first \( \frac{n}{2} \) samples.
4. Construct the set \( T \subseteq [n] \) by keeping the 2s symbols with highest \( M(x) \)’s.
5. Obtain \( \hat{p} \): estimate the distribution over \( T \) using the remaining \( \frac{n}{2} \) samples.

We note that

\[
E[M(x)] = \frac{n}{2} \left( p(x) \frac{e^\epsilon - 1}{e^\epsilon + 1} + \frac{1}{e^\epsilon + 1} \right),
\]

which has a similar form as (16). Hence setting \( \beta = 1/(e^\epsilon + 1) \) and \( \gamma = (e^\epsilon - 1)/(e^\epsilon + 1) \), we can follow the steps in the communication constrained setting and obtain with probability at least \( 9/10 \), \( d_{TV}(\hat{p}, p) \leq \alpha \) using

\[
n = O \left( \beta \frac{\gamma^2 s^2 \log \frac{k}{\delta}}{\alpha^2 \gamma^2} \right) = O \left( \frac{\gamma s^2 \max \{\log (k/s), 1\}}{\alpha^2 \epsilon^2} \right),
\]

when \( \epsilon = O(1) \).