EQUIPARTITIONS OF MEASURES IN $\mathbb{R}^4$

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Abstract. A well-known problem of B. Grünbaum (1960) asks whether for every continuous mass distribution (measure) $d\mu = f \, dm$ on $\mathbb{R}^n$ there exist $n$ hyperplanes dividing $\mathbb{R}^n$ into $2^n$ parts of equal measure. It is known that the answer is positive in dimension $n = 3$ (see H. Hadwiger (1966)) and negative for $n \geq 5$ (see D. Avis (1984) and E. Ramos (1996)). We give a partial solution to Grünbaum’s problem in the critical dimension $n = 4$ by proving that each measure $\mu$ in $\mathbb{R}^4$ admits an equipartition by 4 hyperplanes, provided that it is symmetric with respect to a 2-dimensional affine subspace $L$ of $\mathbb{R}^4$.

Moreover we show, by computing the complete obstruction in the relevant group of normal bordisms, that without the symmetry condition, a naturally associated topological problem has a negative solution. The computation is based on Koschorke’s exact singularity sequence (1981) and the remarkable properties of the essentially unique, balanced binary Gray code in dimension 4; see G. C. Tootill (1956) and D. E. Knuth (2001).

1. Introduction

A notorious open problem in geometric combinatorics and discrete and computational geometry is the question whether each continuous mass distribution $\mu$ in $\mathbb{R}^4$ admits an equipartition by hyperplanes. This is an “essentially 4-dimensional” problem by the classification of V. Klee [Klee99], indicating that the answer is known and positive in dimensions $\leq 3$ and negative in dimensions $\geq 5$. Recall that a collection $H_1, H_2, \ldots, H_n$ of hyperplanes in $\mathbb{R}^n$ is by definition an equipartition for a mass distribution (measure) $\mu$ if each of the $2^n$ “orthants” associated to $\{H_j\}_{j=1}^n$ contains the fraction $1/2^n$ of the total mass. In other words, $\mu(H^\epsilon) = 1/2^n \mu(\mathbb{R}^n)$ for each $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$, where $H^\epsilon := H_1^{\epsilon_1} \cap \ldots \cap H_n^{\epsilon_n}$ is the “orthant” associated to $\epsilon$ and $H_i^0$ (respectively $H_i^1$) is the positive (respectively negative) closed halfspace associated to the hyperplane $H_i$.

Progress in the general problem was slow after B. Grünbaum posed the question in 1960, [Grün60]. The case $n = 1$ is trivial, and the next case $n = 2$ is a consequence of the “mean value theorem”, so the first important progress was made by H. Hadwiger who showed in [Had66] that equipartitions exist for $n = 3$. D. Avis showed in [Avis84] that there exist non-equipartitionable mass distributions for $n \geq 5$. In the meantime many related questions were formulated and many of them solved ([BM01], [BM02], [Mak01], [Ram96], [YZ92]), a connection with discrete and computational geometry was established ([YY85], [YDEP89]) and the subject grew...
into a separate branch of geometric combinatorics [Ziv04]. Nevertheless, the 4-equipartition problem itself has resisted all attempts and remains one of the central open problems in the field.

In this paper we prove (Theorem 5.1) that a measure $\mu$ in $\mathbb{R}^4$ admits a 4-equipartition if it is symmetric with respect to a 2-plane $L$ in $\mathbb{R}^4$. Moreover we demonstrate in Theorem 5.9 by computing the obstruction in the relevant normal bordism group, that there are no “obvious” topological obstacles for the existence of non-equipartitionable measures. This result may be an indication that such a peculiar measure does exist in $\mathbb{R}^4$ and it is an intriguing question whether the “topological counterexample”, provided by Theorem 5.9, can be turned into a genuine counterexample.

2. THE CS/TM-scheme

The configuration space/test map scheme [Ziv04] has emerged as one of the key principles for the application of topological methods in geometric combinatorics and discrete and computational geometry. The basic idea can be outlined as follows.

One starts with a configuration space or manifold $M_P$ of all candidates for the solution of a geometric/combinatorial problem $P$. The next step is a construction of a test map $f : M_P \to V_P$ which measures how far is a given candidate configuration $C \in M_P$ from being a solution. More precisely, there is a subspace $Z$ of the test space $V_P$ such that a configuration $C \in M_P$ is a solution if and only if $f(C) \in Z$.

The inner symmetries of the problem $P$ typically show up at this stage. This means that there is a group $G$ of symmetries of $M_P$ which acts on $V_P$, such that $Z$ is a $G$-invariant subspace of $V_P$, which turns $f : M_P \to V_P$ into an equivariant map. If a configuration $C$ with the desired property $f(C) \in Z$ does not exist, then there arises an equivariant map $f : M_P \to V_P \setminus Z$. The final step is to show by topological methods that such a map does not exist.

The reader can follow the genesis of the method in the review papers [Alon88], [Bar93], [Bjö91], [Ziv96], [Ziv98], [Ziv04] and see how the solutions of well-known combinatorial problems such as Kneser’s conjecture (L. Lovász [Lov78]), “the splitting necklace problem” (N. Alon [Alon87]), the Colored Tverberg problem (R. Živaljević, S. Vrećica, [ZV92], [VZ94]) etc. eventually led to the formulation and codification of the general principle.

2.1. The equipartition problem. Our first choice for the configuration space suitable for the equipartition problem is the manifold of all ordered collections $H = (H_1, \ldots, H_n)$ of oriented hyperplanes in $\mathbb{R}^n$.

Suppose that $e : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is the embedding defined by $e(x) = (x, 1)$. Each oriented hyperplane $H$ in $e(\mathbb{R}^n) \cong \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\} \subset \mathbb{R}^{n+1}$ is obtained as an intersection $H = e(\mathbb{R}^n) \cap H'$ for a unique oriented, $(n+1)$-dimensional subspace $H' \subset \mathbb{R}^{n+1}$. The oriented subspace $H'$ is determined by the corresponding orthogonal unit vector $u \in S^n \subset \mathbb{R}^{n+1}$, so the natural environment for collections $H = (H_1, \ldots, H_n)$, and our second choice for the configuration space is $M_P := (S^n)^n$. The group which acts on the configuration manifold $M_P$ is the reflection group $W_n := (\mathbb{Z}/2)^n \rtimes S_n$ where $S_n$ permutes the factors while the subgroup $(\mathbb{Z}/2)^n$ is in charge of the antipodal actions on individual spheres. The action of $W_n$ on $M_P := (S^n)^n$ is not free, so our third choice for the configuration space associated
to the equipartition problem is
\begin{equation}
M^\delta_P = (S^n)_P^\delta := \{ x \in (S^n)^n \mid x_i \neq \pm x_j \text{ for } i \neq j \}.
\end{equation}

This space is a relative of the (standard) “configuration space” $F_m(S^n) := \{ x \in (S^n)^m \mid x_i \neq x_j \text{ for } i \neq j \}$ [CaHu01]. It has already appeared in combinatorics, for example in [FZ02], where it is referred to as the “signed configuration space”.

The associated “orbit configuration space” $(S^n)_P^\delta/W_n$ can be identified as a submanifold of the symmetric product $SP^n(RP^n)$ of the projective space $RP^n$. This is the reason why we occasionally denote this quotient by $SP^n(RP^n)$ and view its elements as unordered collections of $n$ distinct lines in $\mathbb{R}^{n+1}$.

The test space $V = V_P$ is defined as follows. If $\mu$ is a measure defined on $\mathbb{R}^n$, let $\mu'$ be the “push-down” measure induced on $\mathbb{R}^{n+1}$ by the embedding $e : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$, $\mu'(A) := \mu(e(\mathbb{R}^n) \cap A)$. An $n$-tuple $(u_1, \ldots, u_n) \in (S^n)^n$ of unit vectors determines an $n$-tuple $\mathcal{H} = (H_1', \ldots, H_n')$ of oriented $(n+1)$-dimensional subspaces of $\mathbb{R}^{n+1}$. The $n$-tuple $\mathcal{H}$ dissects $\mathbb{R}^{n+1}$ into $2^n$-orthants $\text{Ort}_{\beta}(\mathcal{H})$ which are naturally indexed by 0-1 vectors $\beta \in \{0,1\}^n$. Let $b_\beta : M_P \to \mathbb{R}$ be the function defined by $b_\beta(\mathcal{H}) := \mu'(\text{Ort}_{\beta}(\mathcal{H})) = \mu(\text{Ort}_{\beta}(\mathcal{H}) \cap e(\mathbb{R}^n))$. Let $B_\mu : (S^n)^n \to \mathbb{R}^2^n$ be the function defined by $B_\mu(\mathcal{H}) = (b_\beta(\mathcal{H}))_{\beta \in \{0,1\}^n}$. The test space $V = V_P \cong \mathbb{R}^{2^n}$ has a natural action of the group $W_n := (\mathbb{Z}/2)^n \rtimes S_n$ such that the map $B_\mu$ is $W_n$-equivariant. Note that the real $W_n$-representation $V$, restricted to the subgroup $(\mathbb{Z}/2)^n \hookrightarrow W_n$, reduces to the regular representation $\text{Reg}((\mathbb{Z}/2)^n)$ of the group $(\mathbb{Z}/2)^n$.

The “zero” subspace $Z_P$ is defined as the trivial, 1-dimensional $W_n$-representation $V_0$ contained in $V$. Let $U = U_n$ be the complementary $W_n$-representation, $U_n \cong V/V_0$ and $A_\mu : (S^n)^n \to U_n$ the induced, $W_n$-equivariant map. By the construction we have the following proposition which says that $A_\mu$ is a genuine test map for the $\mu$-equipartition problem.

**Proposition 2.1.** An $n$-tuple $\mathcal{H} = (H_1, \ldots, H_n) \in M_P = (S^n)^n$ of oriented hyperplanes in $\mathbb{R}^n$ is an equipartition of a measure $\mu$ defined on $\mathbb{R}^n$ if and only if $A_\mu(\mathcal{H}) = 0$.

**Corollary 2.2.** If there does not exist a $W_n$-equivariant map $A : M_P \to U_n \setminus \{0\}$, then each positive, continuous mass distribution (measure) $d\mu = f \, dm$, where $dm$ is the Lebesgue measure, admits an equipartition by $n$ hyperplanes.

**Remark 2.3.** The assumption that $\mu$ is a measure absolutely continuous to the Lebesgue measure on $\mathbb{R}^n$ is unnecessarily restrictive. All we need is the continuity of the test map $A_\mu : (S^n)^n \to U_n$, a condition satisfied by a very large class of measures having the desired continuity properties. Notable examples of interesting measures that are not in this class are counting measures $\nu_D$ of finite sets $D \subset \mathbb{R}^n$ defined by $\nu_D(X) := |D \cap X|$. Note however, that all equipartition results can be suitably extended to weak limits of continuous measures; cf. [MVZ] for a general set up. An example of such a result applying to counting measures is Corollary 5.2 from Section 5.1.

2.2. **Singular sets** $\Sigma_\mu$. It is a well-known fact that for a free $G$-space $P$ and a $G$-representation $V$, there does not exist a $G$-equivariant map $f : P \to V \setminus \{0\}$ if and only if the associated vector bundle $V \to P \times_G V \to P/G$ does not admit a non-zero, continuous cross-section; cf. Proposition I.7.2 in [Dieck87].

A well-known approach to the last question, applicable in the case when $P$ is a free $G$-manifold, is the singularity approach; cf. [Kosch81]. Given a $G$-map
For a measure $\mu$, the singularity set $\Sigma(\mu)$ is defined by $\Sigma(\mu) := h^{-1}(0)$. In the case when $h$ is transverse to $0 \in V$, the singularity set $\Sigma(h)$ is a $G$-manifold.

If $h = A_\mu$ is the test map of a measure $\mu$, then the associated singularity set $\Sigma_\mu := \Sigma(A_\mu)$ is simply the set of all solutions to the equipartition problem for $\mu$. In this case $\Sigma_\mu$ is often referred to as the solution set (manifold) of $\mu$.

The singularity manifold $\Sigma(h)$ of a map $h$, sometimes accompanied by the additional information recording the behavior of $h$ in the tubular neighborhood of $\Sigma(h)$ (the normal data), can be used for computation of an associated obstruction element in a suitable group of bordisms. The singularity manifold and the associated normal data together yield a very strong obstruction invariant which is in a number of important cases complete in the sense that an equivariant map exists if and only if this obstruction vanishes. The reader is referred to [Kosch81] for the general theory.

2.3. Equipartitions of planar measures. The case $n = 2$ of the equipartition problem is well known and usually deduced from the “mean value theorem”. Nevertheless we briefly review this case by offering a different proof which serves as a fairly good illustration of the general ideas applied in this paper. According to the CS/TM-scheme, as presented in Section 2.1, the problem is to prove that there does not exist a $W_2$-equivariant map $f : M_P \to U_2 \setminus \{0\}$, where $W_2 = \mathbb{D}_8$ is the dihedral group, $M_P = S^2 \times S^2$, and $U_2$ is the 3-dimensional real representation of $W_2$, described in Section 2.1.

One can establish a slightly stronger statement that there does not exist a $W_2$-equivariant map $f : M_P^0 \to U_2 \setminus \{0\}$ where $M_P^0 = (S^2)_2^2 = S^2 \times S^2 \setminus \{(x, y) \mid x = y \, \text{or} \, x = -y\}$. The advantage of $(S^2)_2^2$ over $(S^2)^2$ is that the former is a free $W_2$-space.

Let us see how the singularity approach works in the case of planar equipartitions. For a generic measurable set $A \subset \mathbb{R}^2$, the singularity $\Sigma_A$ of $A$, that is, the collection of all pairs $(L_1, L_2)$ of oriented lines in $\mathbb{R}^2$ which form an equipartition for $A$, is a 1-dimensional $W_2$-manifold. For example if $A$ is a unit disc $D$, the singularity $\Sigma_D$ is a union of 4 circles. Here we do not make precise what is meant by a generic measure. Instead we naively assume, for the sake of this example, that there exists such a notion of genericity for measurable sets/measures so that each measurable set $A$ can be well approximated by generic measures. Moreover, we assume that for any two measurable sets $A$ and $B$ there exists a path of generic measures $\mu_t$, $t \in [0, 1]$, so that $\mu_0$ is an approximation of $A$, $\mu_1$ is an approximation for $B$ and the solution set

$$
\Sigma_{\{\mu_t\}_{t \in [0, 1]}} := \{(L_1, L_2; t) \mid (L_1, L_2) \text{ is an equipartition for } \mu_t \} \subset (S^2)_2^2 \times [0, 1]
$$

is a 2-dimensional manifold (bordism) connecting solution sets for measures $\mu_0$ and $\mu_1$. The group $\Omega_1(\mathbb{D}_8)$ of classes of 1-dimensional, free $\mathbb{D}_8$-manifolds is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and the $\mathbb{D}_8$-solution manifold $\Sigma_D$, associated to the unit disc in $\mathbb{R}^2$, is easily shown to represent a nontrivial element in this group. It immediately follows that for any measurable set $A \subset \mathbb{R}^2$, or more generally for any continuous mass distribution, the singularity $\Sigma_A$ is nonempty. Indeed, suppose $\Sigma_A = \emptyset$. Let $\{\mu_t\}_{t \in [0, 1]}$ be a path of generic measures such that $\mu_0$ approximates $A$ and $\mu_1$ approximates $D$. If these approximations are sufficiently good, we deduce that the solution set $\Sigma_{\mu_0}$ is empty and that $[\Sigma_{\mu_1}]$ and $[\Sigma_D]$ represent the same element in
Ω₁(𝒟₈). This is a contradiction since Σₘ₁ = ∂(Σ) where Σ = Σₘ₁₋₁⊂[0,1], i.e., Σᵣ would represent a trivial element in Ω₁(𝒟₈).

Remark 2.4. It is worth noting that the scheme outlined above, if applicable, shows that a general equipartition problem can be solved by a careful analysis of the singularity set of a well-chosen, particular measure/measurable set, the unit disc 𝒟² in our example above. Unfortunately, in higher dimensions the unit balls do not represent generic measures; i.e., their solution manifolds are very special and cannot be used for the evaluation of the relevant obstruction elements. Instead, for this purpose one can use measures distributed along a convex curve Γ in ℝⁿ; see Section 3.

3. FROM CONVEX CURVES TO GRAY CODES

3.1. Convex curves. A simple, smooth curve Γ in ℝⁿ is called convex if the total multiplicity of its intersection with an affine hyperplane does not exceed n. Convex curves are classical objects appearing in many different fields including the real algebraic geometry (“ecstatic points” of curves), the theory of convex polytopes (neighborly polytopes), interpolations of functions (Tschebycheff systems), linear ordinary differential equations (disconjugate equations), etc.; see [Ami98, Arn96, Copp71, Scho54, SeSh96], and the references in these papers. Standard examples of convex curves are the “moment curve”, or the rational normal curve Mₙ := {(t,t²,...,tⁿ) | t ∈ ℝ} ⊂ ℝⁿ and the standard trigonometric curve Γ₂ₙ := {(cos t, sin t, cos 2t, ..., cos nt, sin nt) | t ∈ [0,2π]} ⊂ ℝ²ⁿ.

The importance of convex curves for the equipartition problem stems from the fact that they minimize the number of intersections with hyperplanes. As a consequence, each collection ℱ = {ℱ₁,...,ℱₙ} of n hyperplanes in ℝⁿ has at most n² 2 intersection points with Γ. It follows that if n = 2d is even, and Γ is a simple, closed convex curve, then ℱ divides Γ in at most n² 2 arcs. Suppose that µ is a measure concentrated on a closed, convex curve Γ ⊂ ℝ²ᵈ. Then if ℱ is an equipartition for µ, then n² ≥ 2ⁿ, i.e. n ≤ 4. This explains why the dimension 4 is so special in this context.
3.2. **Gray codes.** Gray codes arise in an attempt to describe the solution manifold (singularity set) $\Sigma_\mu$ of a measure $\mu$ distributed along a closed convex curve $\Gamma \subset \mathbb{R}^4$. Our preferred example of such a curve is $\Gamma_4 = \{(z, z^2) \mid |z| = 1\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$. From here on we assume that $d\mu' = d\theta'$, $\theta' = \arg(z)$, is the “arc length” measure on the circle $C = \{z \in \mathbb{C} \mid |z| = 1\}$ and $d\mu = d\theta$ is the associated measure on $\Gamma_4$, where $\theta = \theta' \circ \pi_1$ and $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ is the first projection. A similar analysis can be carried out for other measures concentrated on $\Gamma_4$. Each collection $P = \{p_1, p_2, p_3, p_4\}$ of 4 points in $\Gamma_4$ is contained in a unique hyperplane $H \subset \mathbb{R}^4$; hence the combinatorics of $\mu$-equipartitions can be read off from the circle $C$ (see Figures 2, 3 and 4). As a consequence, an equipartition $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ of the measure $d\theta$ is essentially a division of the circle $C$ into 16 arcs of equal length and associating each of the 16 division points $\{x_j\}_{j=0}^{15}$, where $x_j = \epsilon^j x_0$ and $\epsilon = \exp(2\pi i/16)$, to one of the hyperplanes $H_i$. Note that not all maps $\alpha : \{x_j\}_{j=0}^{15} \to \mathcal{H}$ are allowed; cf. [Ram96], p. 157. Each of the arcs $[x_j, x_{j+1}]$ belongs to an orthant coded by an associated 4-bit word $\beta_j \in \{0, 1\}^4$. Each of the 4-bit words $\beta \in \{0, 1\}^4$ appears exactly once in the cyclic order inherited from the order of the intervals $[x_j, x_{j+1}]$. Moreover, moving from one orthant to another along the curve $\Gamma_4$, that is, from the interval $[x_j, x_{j+1}]$ to the consecutive interval $[x_{j+1}, x_{j+2}]$, changes only one bit at a time. It follows that the sequence $\{\beta_j\}_{j=0}^{15}$ of 4-bit words forms a so-called Gray code ([Knu01], [Ram96]). For a graph theorist, a Gray code is a *Hamiltonian path* on a hypercube $\{0, 1\}^n$. For an engineer, a Gray code is a device useful for converting digital signals into analog information and vice versa, [Knu01].

![Figure 2. The unique balanced 4-bit Gray code](image)

Our next observation is that Gray codes arising in the equipartitions of the convex curve $\Gamma_4$ are quite special. Indeed, each hyperplane $H_i$ intersects the convex curve $\Gamma_4$ precisely 4 times implying that the code must have the same number of bit
changes in each of four coordinate tracks. Such codes are called balanced; see Figure 2. A Gray code with these properties was originally discovered by G.C. Tootill [Toot56]. One of its remarkable properties is that it is unique up to a permutation of coordinate tracks; see [Gil58] or [Knu01, Exercise 56]. In particular if one reads the code clockwise the same code is obtained, except that the second and the third track interchange places; see Figures 2–4.

There is one more attractive way to describe this code. As a variation on the theme of the play “Quad” by S. Beckett [Quad] where “... Four actors, whose colored hoods make them identifiable yet anonymous, accomplish a relentless closed-circuit drama ...” (R. Frielings), one can design a scheme for a play based on the balanced Gray code. In this scheme the stage begins and ends empty, 4 actors enter and exit one at a time, running through all 16 possible subsets, and each actor is supposed to enter (leave) the stage precisely 2 times; see [Knu01, Exercise 65] (attributed to B. Stevens) for a similar idea.

3.3. The solution set $\Sigma_\theta$. The analysis from Sections 3.1 and 3.2 allows us to describe the solution manifold (singular set) $\Sigma_\theta$ of the measure $\theta$ on the convex curve $\Gamma_4$, arising from the “arc length” measure on the unit circle $C \subset \mathbb{C}$; see Section 3.2. This solution set is a $W_4$-invariant subset of the configuration space $M_\delta^q = (S^4)^4_\delta$.

Suppose that $H = \{H_1, H_2, H_3, H_4\} \in \Sigma_\theta$. Each hyperplane $H_j$ intersects the curve $\Gamma_4$ in four points, vertices of a 3-simplex $\sigma_j \subset H_j$. The first projection map $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ maps the curve $\Gamma_4$ to the unit circle $C \subset \mathbb{C}$. The images $P_j := \pi_1(\sigma_j)$ of $\sigma_j$ are convex polygons in the plane displayed in Figure 4 (B). Figure 3 depicts essentially the same polygons in the order corresponding to one of the possible $4! = 24$ orders of hyperplanes in $H$.

![Figure 3. An element of $\Sigma_\theta$.](image)

By taking into account the (chosen) orientation of hyperplanes $H_j$ and the induced orientations of simplices $\sigma_j \subset H_j$, we arrive at the conclusion that Figure 3 is a fairly accurate description of an element $H = \{H_1, H_2, H_3, H_4\} \in \Sigma_\theta$. This element belongs to an oriented circle $\Sigma \subset \Sigma_\theta$ of all solutions, obtained essentially by rotating Figure 3 counterclockwise. All other circles in $\Sigma_\theta$ are obtained by the action of the group $W_4$, in other words by changing the orientations of simplices $\sigma_j$, performed by the subgroup $(\mathbb{Z}/2)^4$, and by permuting the circumcircles of polygons.
In other words the permutation of polygons is achieved by permuting the circles, the tracks in the balanced Gray code.

For example if we move Figure 3 clockwise, we obtain the circle \( \xi(\Sigma) \subset \Sigma \theta \) where \( \xi : [4] \rightarrow [4] \) is the permutation which keeps tracks 1 and 4 fixed and interchanges tracks 2 and 3. We see this as a manifestation of the symmetry of diagrams in Figures 3 and 4 with respect to the vertical axes.

Figure 4. ... and the associated element in \( \Sigma_\theta/W_4 \)

Let us turn to the question of non-degeneracy of the solution manifold \( \Sigma_\theta \). This is by definition the condition that the associated test map \( A_\theta : (S^4)_4^4 \rightarrow U_4 \) is transverse to \( 0 \in U_4 \). Suppose that \( H \in \Sigma_\theta \) and assume that \( \{x_j\}_{j=0}^{15} \) are the associated division points; see Section 3.2. Then for a small positive real number \( \epsilon > 0 \), the angles \( y_j \in (x_j - \epsilon, x_j + \epsilon) \) can be used as the coordinates on \( (S^4)_4^4 \) in the neighborhood of \( H \in (S^4)_4^4 \). The associated tangent vectors are \( \partial/\partial y_j \in T_H((S^4)_4^4) \). Similarly, the functions \( b_\beta \) introduced in Section 2.1 are coordinates on \( V \); consequently the functions \( c_j := b_{\beta_{j+1}} - b_{\beta_j} \), where \( \{\beta_j\}_{j=0}^{15} \) is the sequence defined in Section 3.2, are coordinates on \( U_4 \). The proof is completed by the observation that \( dA_\theta(\partial/\partial y_j) = -\partial/\partial c_j \).

For future reference we record an essential part of this analysis in the following proposition.

**Proposition 3.1.** The solution manifold \( \Sigma_\theta \subset M^\delta_P = (S^4)_4^4 \) of all equipartitions for the measure \( d\theta \) on the convex curve \( \Gamma_4 \) is a (non-degenerate) 1-dimensional \( W_4 \)-manifold which has \( |W_4| = 2^4! \) connected components. The quotient manifold \( \Sigma' := \Sigma_\theta/W_4 \) is a circle in the manifold \( (S^4)_4^4/W_4 \cong SP^m(RP^4) \subset SP^m(SP^m(X) := X^m/S_m \) is the symmetric product of X and \( SP^m(X) \) its subspace of “square-free” divisors.

**Remark 3.2.** Figure 4 (B) symbolically represents an element of the circle \( \Sigma' = \Sigma_\theta/W_4 \). The orientations of the polygons are forgotten and all 4 circles, representing different tracks in the balanced Gray code visible in Figure 4 (A), are “merged together” in Figure 4 (B). This emphasizes the fact that the order of the polygons (associated coordinate tracks) is no longer important.
4. From Gray codes to normal bordisms

According to Corollary 2.2, the 4-equipartition problem is closely related to the question if there exists a \( W_4 \)-equivariant map \( f : (S^4)^4_3 \to U_4 \setminus \{0\} \). In turn, this is equivalent to the question if the vector bundle

\[
E : U_4 \to (S^4)^4_3 \times W_4 U_0 \to (S^4)^4_3/W_4
\]

admits a non-zero, continuous cross-section. A complete topological obstruction \( \omega \) for the existence of such a section lives in the normal bordism group \( \Omega_1(M; E - TM) \).\[Kosch81\], where \( TM \) is the tangent bundle of the manifold \( M := (S^4)^4_3/W_4 \) and \( E \) is the bundle \((4.1)\). This group can be computed from the Koschorke’s exact singularity sequence,\[Kosch81\] Theorem 9.3. The computation of the obstruction \( \omega \in \Omega_1(M; E - TM) \) is based on this sequence and the analysis of the singularity \( \Sigma_\theta \) of the measure \( \theta \) distributed along a closed, convex curve \( \Gamma_4 \subset \mathbb{R}^4 \); see Section 3.

4.1. Koschorke’s exact singularity sequence. A consequence of Koschorke’s exact singularity sequence,\[Kosch81\] Section 7, is a short exact singularity sequence,\[Kosch81\] Theorem 9.3, involving low dimensional normal bordism groups \( \Omega_j(X; \phi) \) and \( \Omega_j(X; \phi) \) for \( j \leq 4 \), where \( \phi = \phi^+ - \phi^- \) is a virtual vector bundle over \( X \). The final fragment of this sequence has the following form:

\[
\delta_2 : \Omega_2(X; \phi) \xrightarrow{j_2} \tilde{\Omega}_2(X; \phi) \xrightarrow{\sigma_{\phi}j_2} \mathbb{Z}/2 \xrightarrow{\delta_1} \Omega_1(X; \phi) \xrightarrow{j_1} \tilde{\Omega}_1(X; \phi) \to 0.
\]

We are interested in this sequence in the case when \( X = M = (S^4)^4_3/W_4 \) and \( \phi = \phi^+ - \phi^- = E - TM \). More precisely, our objective is to evaluate the obstruction element \( \omega \in \Omega_1(M; E - TM) \) defined in Section 4.

4.2. The image of \( \delta_1 \). According to\[Kosch81\], the image \( \delta_1(1) \) of the generator \( 1 \in \mathbb{Z}/2 \) “... can be represented by the unit circle with constant map and the standard parallelization, suitably stabilized ...”. In this section we show that the image \( \delta_1(1) \) coincides with the obstruction element \( \omega \in \Omega_1(M; E - TM) \). This is a consequence of the following proposition.

Proposition 4.1. \( f_1(\omega) = 0 \).

Proof. The configuration space \((S^4)^4_3\) is simply connected; hence the projection \((S^4)^4_3 \to (S^4)^4_3/W_4 \) is a universal covering map. Let \( \Sigma \) be a circle, a connected component of the solution manifold \( \Sigma_\theta \), and let \( \Sigma' := \Sigma/\mathbb{Z}_4 \subset (S^4)^4_3/W_4 \). Suppose that \( \xi \) is the orientation line bundle of the virtual bundle \( E - TM \). Then, by definition of \( \tilde{\Omega}_1(X; \phi) \),\[Kosch81\] Section 9, the image \( f_1(\omega) \) of \( \omega \) in \( \tilde{\Omega}_1(X; E - TM) \) is determined by \( \Sigma' \) and the restriction \( \xi|\Sigma' \) of \( \xi \) on \( \Sigma' \).

Since the circle \( \Sigma' \) can be lifted to the circle \( \Sigma \), we conclude that \( \Sigma' \) is contractible; hence \( \xi|\Sigma' \) is a trivial bundle. It follows that there exists a map \( g : D^2 \to M \) such that \( \partial(D^2) \) is mapped bijectively to \( \Sigma' \) and the extension of the bundle \( g^* (\xi|\Sigma') \) to the trivial bundle over \( D^2 \). Hence, \( f_1(\omega) \) is a trivial element in \( \tilde{\Omega}_1(X; E - TM) \). \( \square \)

4.3. The image of \( \sigma \circ j_2 \). In this section we focus on the calculation of the image of the map \( \tilde{\Omega}_1(X, \phi) \xrightarrow{\sigma \circ j_2} \mathbb{Z}/2 \) in the Koschorke’s singularity exact sequence \( (4.2) \). By definition, an element \( \alpha = [N, g, \sigma g, \sigma o] \in \tilde{\Omega}_1(X, \phi) \) is mapped to \( \sigma \circ j_2(\alpha) := g^*(w_3(\phi))(N) \), where \( w_3(\phi) \) is the second Stiefel-Whitney class of the virtual bundle \( \phi = \phi^+ - \phi^- \) and \( [N] \in H_2(N, \mathbb{Z}/2) \) is the fundamental class of the surface \( N \). From here we conclude that the elements \( \alpha = [N, g, \sigma g, \sigma o] \) such that both \( g^*(\phi^+) \) and \( g^*(\phi^-) \)
are trivial vector bundles can be ignored; in particular, we ignore those elements where $g$ is homotopic to a constant map. Recall the exact sequence
\begin{equation}
(4.3) \quad \mathcal{N}_2 \rightarrow \mathcal{N}_2(X) \rightarrow H_2(X, \mathbb{Z}/2) \rightarrow 0,
\end{equation}
where $\mathcal{N}_2(X)$ is the group of unoriented bordisms and $\mathcal{N}_2 := \mathcal{N}_2(\ast)$. It follows that in the evaluation of the image of $\sigma \circ \delta_2$ we are allowed to pick a representative $\alpha = [N, g, \text{or}]$ in each of the homology classes $x \in H_2(X, \mathbb{Z}/2)$.

The following standard lemma reduces the calculation of $w_2(\phi)[N]$ to the calculations of Steifel numbers of individual bundles $\phi^+$ and $\phi^-$.

**Lemma 4.2.**

\[ w(\phi) = w(\phi^+) \cdot w(\phi^-)^{-1} \]
\[ = (1 + w_1(\phi^+) + w_2(\phi^+) + \ldots)(1 + w_1(\phi^-) + w_2(\phi^-) + \ldots) \]
\[ = 1 + A_1 + A_2 + \ldots, \]

where

\[ A_1 = w_1(\phi^+) + w_1(\phi^-) \]
\[ A_2 = w_2(\phi^+) + w_1(\phi^+)w_1(\phi^-) + w_1(\phi^-)^2 + w_2(\phi^-) \]

are terms of graduation 1 and 2 respectively. Consequently if $\phi = \phi^+ - \phi^-$ is a virtual vector bundle over a surface $N$, then

\[ w_2(\phi)[N] = w_2(\phi^+)[N] + (w_1(\phi^+)w_1(\phi^-))[N] + w_1(\phi^-)^2[N] + w_2(\phi^-)[N]. \]

In our case $X = SP^4_\delta(RP^4)$. The following lemma, an easy consequence of Poincaré duality, allows us to search for surfaces $N$ representing nontrivial homology 2-classes in the symmetric product $SP^4(RP^4)$.

**Lemma 4.3.** There is an isomorphism $H_2(SP^4_\delta(RP^4)) \cong H_2(SP^4(RP^4))$ of homology groups, induced by the inclusion map $SP^4_\delta(RP^4) \hookrightarrow SP^4(RP^4)$.

It is easy to see that $H_2(SP^4(RP^4)) \cong H_2(SP^\infty(RP^4))$. By the Dold-Thom theorem, [Ha02,]

\[ SP^\infty(RP^4) \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}/2, 3) \times K(\mathbb{Z}/2, 4). \]

From here we deduce that $H_2(SP^4(RP^4)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

It is not difficult to describe surfaces $N_1$ and $N_2$, embedded in $SP^4_\delta(RP^4)$, $SP^4(RP^4)$, representing the generators of the homology group $H_2(SP^4_\delta(RP^4))$.

Suppose that $e_1$ and $e'_1$ are two disjoint circles embedded in $RP^4$, both representing the nontrivial element in $H_1(RP^4; \mathbb{Z}/2)$. Let $e_2 \cong RP^2$ be a projective plane embedded in $RP^4$, representing the generator of $H_2(RP^4; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Finally suppose that $e_3, e_4, e_5$ are three distinct points in $RP^4$ such that $* \notin e_1 \cup e'_1 \cup e_2$ for each $i = 1, 2, 3$. As usual, elements of the symmetric product $SP^m(X)$ are thought of as positive “divisors”, i.e., commutative and associative formal sums $D = n_1 x_1 + \ldots + n_k x_k$ where $n_i \in \mathbb{N}, x_i \in X$ and $\sum n_i = m$. By definition let

\begin{equation}
N_1 := *_1 + *_2 + e_1 + e'_1 \quad \text{and} \quad N_2 := *_1 + *_2 + *_3 + e_2.
\end{equation}

In other words $N_1 \cong S^1 \times S^1$ is a torus embedded in $SP^4_\delta(RP^4)$ where $D \in N_1 \Leftrightarrow D = *_1 + *_2 + x + y$ for some $x \in e_1$ and $y \in e'_1$. Similarly, $N_2 \cong RP^2$ consists of all divisors of the form $D = *_1 + *_2 + *_3 + x$ for some $x \in e_2$.

In our case $\phi^+ = E$ and $\phi^- = T(SP^4_\delta(RP^4))$. We focus our attention on the bundles $\phi_i^+ := \phi^+|N_i$ and $\phi_i^- := \phi^-|N_i$ for $i = 1, 2$. Recall that $E \cong (S^4)^4 \times_{W_4} U$ where $U = U_4$ is the 15-dimensional representation of $W_4$ described in Section 2.1.

If $\pi : (S^4)^4 \rightarrow SP^4_\delta(RP^4)$ is the projection map, then $Z_i := \pi^{-1}(N_i)$ is a free,
W_4-submanifold of (S^4)_3 and \( \phi_i^+ \cong Z_i \times W_4 \). It is not difficult to describe these manifolds.

The connected component of \( Z_1 \) is a torus \( T^2 = S^1 \times S^1 \subset (S^4)_3 \), and the stabilizer of \( T^2 \) is the group \( H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \subset (\mathbb{Z}/2)^4 \subset W_4 = (\mathbb{Z}/2)^4 \ltimes S_4 \) where \( H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) acts on \( T^2 = S^1 \times S^1 \) by the product action. It follows that \( Z_1 \cong T^2 \times H_1 \mathbb{R} \) and

\[
\phi_1^+ \cong Z_1 \times W_4 \mathbb{R} \cong (T^2 \times H_1 \mathbb{R}) \times W_4 \mathbb{R} \cong T^2 \times H_1 \mathbb{R}.
\]

Similarly, the connected component of \( Z_2 \) is the sphere \( S^2 \subset (S^4)_3 \) and its stabilizer is the group \( H_2 = \mathbb{Z}/2 \subset (\mathbb{Z}/2)^4 \subset W_4 \) where \( H_2 \) acts on \( S^2 \) by the antipodal action. It follows that \( Z_2 \cong S^2 \times H_2 \mathbb{R} \) and

\[
\phi_2^+ \cong Z_2 \times W_4 \mathbb{R} \cong (S^2 \times H_2 \mathbb{R}) \times W_4 \mathbb{R} \cong S^2 \times H_2 \mathbb{R}.
\]

Keeping in mind that the restriction \( \text{Res}_{Z^4_3}(U) \) is the regular (real) representation of \( \mathbb{Z}^{\oplus 4} \), minus the trivial 1-dimensional representation, it is easy to identify the bundles \( \phi_1^+ = T^2 \times H_1 \mathbb{R} \) and \( \phi_2^+ = S^2 \times H_2 \mathbb{R} \). As a preliminary step, let us describe some canonical line bundles over \( N_1 \cong T^2/H_1 \cong S^1/\mathbb{Z}/2 \times S^1/\mathbb{Z}/2 \cong T^2 \) and \( N_2 \cong S^2/H_2 \cong RP^2 \).

There are 4 real, 1-dimensional representations of \( H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). If \( \omega_1 \) and \( \omega_2 \) are the generators of \( H_1 \), then \( L_{\omega_1 \omega_2} \), \( \omega_1, \omega_2 \in \{-1, +1\}, \) is the representation characterized by the condition \( \omega_1(v) = \epsilon_1 v \) for each \( v \in L_{\omega_1 \omega_2} \). Let \( \lambda_{\omega_1 \omega_2} := T^2 \times H_1 \) \( L_{\omega_1 \omega_2} \) be the associated line bundle where \( \epsilon_i \in \{0, 1\} \) and \((-1)^{\epsilon_1} = \epsilon_i \). For example \( \lambda_{00} \) is the trivial bundle, usually denoted by \( \epsilon \), while \( \lambda_{11} \) is the Cartesian product of 2 canonical bundles over \( RP^1 \cong S^1 \). Let \( \gamma = \gamma_n \) be the canonical line bundle over \( RP^n \). The decompositions of bundles \( \phi_1^+ \) and \( \phi_2^+ \) into line bundles is recorded in the following statement.

**Proposition 4.4.**

\[
\phi_1^+ = T^2 \times H_1 \mathbb{R} \cong \epsilon^{\oplus 3} \oplus \lambda_{01}^{\oplus 4} \oplus \lambda_{05}^{\oplus 4} \oplus \lambda_{11}^{\oplus 4},
\]

\[
\phi_2^+ = S^2 \times H_2 \mathbb{R} \cong \epsilon^{\oplus 7} \oplus \gamma^{\oplus 8}.
\]

The next step is the identification of the restrictions \( \phi_1^- := TM|_{N_1} \) and \( \phi_2^- := TM|_{N_2} \) of the tangent bundle \( TM = T(SP_3^4(RP^4)) \) on the surfaces \( N_1 \) and \( N_2 \) respectively.

**Lemma 4.5.** For each point \( D = p_1 + p_2 + p_3 + p_4 \in SP_3^4(RP^4) \) there is an isomorphism

\[
T_D(SP_3^4(RP^4)) \cong \bigoplus_{i=1}^4 T_{p_i}(RP^4).
\]

**Lemma 4.6 [INS74].** Let \( \gamma_n \) be the canonical line bundle over the real projective space \( RP^n \) and \( \epsilon \) the trivial line bundle. Then there is an equality of virtual vector bundles \( T(RP^n) = \gamma_0^{(n+1)} - \epsilon \). As a consequence, the total Stiefel-Whitney class of the restriction bundle \( \xi = T(RP^n)|_{RP^{m-1}} \) is

\[
w(\xi) = (1 + t)^{n+1} = 1 + \binom{n+1}{1} t + \binom{n+1}{2} t^2 + \ldots + \binom{n+1}{m} t^m.
\]

**Proposition 4.7.**

\[
\phi_1^- = TM|_{N_1} \cong \epsilon^{\oplus 6} \oplus \lambda_{01}^{\oplus 5} \oplus \lambda_{05}^{\oplus 5},
\]

\[
\phi_2^- = TM|_{N_2} \cong \epsilon^{\oplus 11} \oplus \gamma^{\oplus 5}.
\]
Lemma 4.2 and the first equality from Corollary 4.8, implies that

\[ \phi^+ - \phi^- = \lambda_{11} - \lambda_{01} - \lambda_{10} - \epsilon^3, \]

\[ \phi^+_2 - \phi^-_2 = \gamma^3 - \epsilon^4. \]

**Corollary 4.8.**

\[ \phi^+_1 - \phi^-_1 = \lambda_{11}^2 - \lambda_{01} - \lambda_{10} - \epsilon^3, \]

\[ \phi^+_2 - \phi^-_2 = \gamma^3 - \epsilon^4. \]

**Proposition 4.9.** \( \text{Im}(\sigma \circ j_2) = \mathbb{Z}/2. \)

**Proof.** It is a basic fact that \( H^*(RP^2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[t]/(t^3) \) and \( H^*(T^2; \mathbb{Z}/2) \cong \Lambda[a, b], \) where \( \Lambda[a, b] \) is a \( (\mathbb{Z}/2) \)-exterior algebra generated by two elements of degree 1. The first two characteristic classes of line bundles \( \lambda_{1}, \lambda_{2} \) and \( \gamma, \) defined over \( N_1 = T^2 \) and \( N_2 = RP^2 \) respectively, are

\[ w_2(\lambda_{1}, \lambda_{2}) = t_1^2, \quad w_1(\gamma) = t, \quad w_2(\lambda_{1}, \lambda_{2}) = w_2(\gamma) = 0. \]

From here we deduce that the total Stiefel-Whitney classes of bundles \( \phi^+_1 \) and \( \phi^-_1 \) are respectively \( w(\lambda_{11}^2) = 1 \) and \( w(\lambda_{01} + \lambda_{10}) = 1 + a + b + ab. \) This, together with Lemma 4.2 and the first equality from Corollary 4.8 implies that

\[ w_2(\phi^+_1 - \phi^-_1) = w_1(\lambda_{01} + \lambda_{10})^2 + w_2(\lambda_{01} + \lambda_{10}) = ab. \]

Similarly,

\[ w_2(\phi^+_2 - \phi^-_2) = w_2(\gamma^3) = t^2. \]

In other words,

\[ w_2(\phi^+_1 - \phi^-_1)[N_1] = w_2(\phi^+_2 - \phi^-_2)[N_2] = 1, \]

and we finally conclude that \( \text{Im}(\sigma \circ j_2) = \mathbb{Z}/2. \)

\[ \square \]

5. Results and proofs

5.1. Measures admitting a 2-plane of symmetry. In this section we show that each measure with a 2-dimensional plane of symmetry admits a 4-equivariant.

**Theorem 5.1.** Suppose that \( \mu \) is a measure on \( \mathbb{R}^4 \) admitting a 2-dimensional plane of symmetry in the sense that for some 2-plane \( L \subset \mathbb{R}^4 \) and the associated reflection \( R_L : \mathbb{R}^4 \to \mathbb{R}^4, \) for each measurable set \( A \subset \mathbb{R}^4, \mu(A) = \mu(R_L(A)). \) Then \( \mu \) admits a 4-equivariant.

**Proof.** Without loss of generality we assume that \( L = C_{(2)} \) in the decomposition \( \mathbb{R}^4 \cong C^2 \cong C_{(1)} \oplus C_{(2)}. \) In that case the reflection \( R = R_L \) is the map described by \( R(z_1, z_2) = (-z_1, z_2); \) in particular, \( R \) is a symmetry of the convex curve (Section 2.1) \( \Gamma_4 = \{(z, z^2) \in C^2 \mid |x| = 1 \}. \)

In pursuit of an equivariant of a general measure in \( \mathbb{R}^4, \) we introduced in Section 2.1 the configuration space \( (S^4)_2^3 = \{v \in \mathbb{R}^4 \mid v_i \neq \pm v_j \text{ for } i \neq j \}. \) Recall that the group \( W_4 = (\mathbb{Z}/2)^4 \times S_4 \) acts freely on this configuration space and that the problem of 4-equivariant was reduced, Corollary 2.2 to the question of the existence of a \( W_4 \)-equivariant map \( f : (S^4)_2^3 \to U_4 \setminus \{0\}, \) where \( U_4 \) is the 15-dimensional, real representation of \( W_4 \) defined in Section 2.1.
If the measure $\mu$ admits an additional symmetry, e.g. if it admits a plane of symmetry $L$ such that the associated reflection $R_L$ keeps $\mu$ invariant, it is natural to enlarge the group $W_4$ by this transformation.

Assume as before that $\mathbb{R}^4$ is identified to the affine subspace $\mathbb{R}^4 + e_5 \subset \mathbb{R}^5$. The isometry $R_L : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is extended to the unique isometry $\hat{R}_L$ of $\mathbb{R}^5$ such that $\hat{R}_L(e_5) = e_5$.

Let $\mathbb{Z}/2$ be the group generated by $\hat{R}_L$. Define the enlarged group of symmetries as the direct product $G := W_4 \times \mathbb{Z}/2$. The action of $W_4$ on $(S^4)_A^\Delta$ can be extended to the action of the group $G$ by the requirement that $\hat{R}_L(v_1, v_2, v_3, v_4) = (\hat{R}_L(v_1), \hat{R}_L(v_2), \hat{R}_L(v_3), \hat{R}_L(v_4))$. In order to make this action free, let us introduce an even smaller configuration space

$$(5.1) \quad (S^4)^\Delta := (S^4)^\Delta \setminus F,$$

where $F \subset (S^4)^\Delta$ is the subset of points $v = (v_1, v_2, v_3, v_4)$ such that the stabilizer $\text{Stab}_G(v)$ is a non-trivial group.

Note that $\hat{R}_L$ is the reflection with respect to the 3-plane $L + Re_5 \subset \mathbb{R}^5$. Consequently, $v \in (S^4)^\Delta$ is in $F$ if (up to a permutation of coordinates $v_i$) either

$$\hat{R}_L(v_1) = v_2, \hat{R}_L(v_3) = v_4 \quad \text{or} \quad \hat{R}_L(v_1) = v_1, \hat{R}_L(v_2) = v_2, \hat{R}_L(v_3) = v_4.$$

\[ \square \]

**Key observation.** The solution manifold $\Sigma_\theta \subset (S^4)^\Delta$ of all 4-equipartitions of the convex curve $\Gamma_4$, defined in Section 3.3, is $\hat{R}_L$-invariant. Moreover $\Sigma_\theta \subset (S^4)^\Delta$, or in other words the $\mathbb{Z}/2$-action on $\Sigma_\theta$ induced by $\hat{R}_L$ is free. Indeed, $\hat{R}_L$ acts on the element displayed in Figure 3 by rotation through the angle of 180°.

Denote by $\lambda$ both the non-trivial, 1-dimensional real representation of $\mathbb{Z}/2$ and the associated, 1-dimensional $G$-representation induced by the projection homomorphism $G \rightarrow \mathbb{Z}/2$. Similarly, $U_4$ is both the 15-dimensional real $W_4$-representation defined in Section 2.1 and the representation induced by the projection homomorphism $G \rightarrow W_4$.

According to results of Section 2.1 for the proof of the theorem it is sufficient to show that a $W_4$-equivariant map $f : (S^4)^\Delta \rightarrow U_4$ must have a zero. This is a consequence of the following stronger result.

**Claim:** There does not exist a $G$-equivariant map $f : (S^4)^\Delta \rightarrow S(U_4 \oplus \lambda)$, where $S(U_4 \oplus \lambda)$ is the $G$-invariant unit sphere in $U_4 \oplus \lambda$. In other words each $G$-invariant map $f : (S^4)^\Delta \rightarrow U_4 \oplus \lambda$ has a zero.

**Proof of the Claim.** The claim is equivalent to the statement that the vector bundle $\xi : (S^4)^\Delta \times_G (U_4 \oplus \lambda) \rightarrow (S^4)^\Delta / G$ does not admit a non-zero continuous cross-section. For this it is sufficient to show that the top Stiefel-Whitney class $w_n(\xi)$ is non-zero. By duality this is equivalent to the fact that for some (each) smooth cross-section $s : (S^4)^\Delta / G \rightarrow (S^4)^\Delta \times_G (U_4 \oplus \lambda)$, transverse to the zero section $Z$, the number of elements in $s^{-1}(Z) \subset (S^4)^\Delta / G$ is odd. In the language of equivariant maps, this is equivalent to the statement that for some smooth, $G$-equivariant map $s = (\phi, \psi) : (S^4)^\Delta \rightarrow U_4 \oplus \lambda$, transverse to $0 \in E \oplus \lambda$, the number of $G$-orbits in $s^{-1}(0)$ is odd.

Define $\phi : (S^4)^\Delta \rightarrow U_4$ as the restriction of the test map $A_\theta : (S^4)^\Delta \rightarrow U_4$ introduced in Section 2.1 where $d\theta$ is the “arc length” measure on $\Gamma_4 = \{(z, z^2) \in \mathbb{C}^2 \mid |z| = 1\}$ introduced in Section 3.2.
Since $A_\theta$ is $W_4$-equivariant and $A_\theta \circ \tilde{R}_L = A_\theta$, we conclude that $\phi$ is $G$-equivariant. The orbit space $SP_4^d(RP^4) := (S^4)^4 / W_4$ is a smooth manifold. Note, Section 3 that the unique balanced Gray code allowed us to identify the “solution manifold” of all equipartitions of the curve $\Gamma_4 \subset \mathbb{R}^4$ as the circle $\Sigma' \subset SP_4^d(RP^4)$. Moreover, this solution manifold was shown to be non-degenerate, Proposition 5.1.1 in the sense that the associated test map $A_\theta$ is transverse to $0 \in U_4$.

The $\mathbb{Z}/2$-action on $SP_4^d(RP^4)$, induced by the involution $\tilde{R}_L$, is free. Let $\psi' : SP_4^d(RP^4) \rightarrow \lambda$ be a $\mathbb{Z}/2$-equivariant, smooth map such that $0 \in \lambda$ is not a critical value of the restriction $\psi'' := \psi'|\Sigma'$ of $\psi'$ on the circle $\Sigma'$. Define $\psi$ as the composition of $\psi'$ with the natural projection map $(S^4)^4 / \Delta \rightarrow SP_4^d(RP^4)$.

Then it is not difficult to check that $s = (\phi, \psi)$ is a smooth, $G$-equivariant map such that $0 \in \lambda$ is not one of its critical values.

Let us show that the number of $G$-orbits in the set $s^{-1}(0)$ is always an odd number. Note that $s^{-1}(0) = \phi^{-1}(0) \cap Z(\psi)$ where $Z(\psi) := \psi^{-1}(0)$ is the zero set of $\psi$. Since $\phi^{-1}(0)/G = \phi^{-1}(0)/W_4 = \Sigma'$, we observe that the number of $G$-orbits in the set $s^{-1}(0)$ is equal to the number of $\mathbb{Z}/2$-orbits in the zero set $Z(\psi'')$ of the $\mathbb{Z}/2$-equivariant map $\psi'' : \Sigma' \rightarrow \lambda$. Since $0$ is not a critical value of $\psi''$, the proof is completed by an elementary observation that a $\mathbb{Z}/2$-equivariant map $p : S^1 \rightarrow \lambda$, transverse to $0 \in \lambda$, must have an odd number of $\mathbb{Z}/2$-orbits.

**Corollary 5.2.** Suppose that $D$ is a finite set of $16d$ distinct points in $\mathbb{R}^4$ which is symmetric with respect to a 2-plane $L \subset \mathbb{R}^4$. Then there exists a collection $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ of distinct hyperplanes such that each of 16 associated open orthants contains not more than $d$ points from the set $D$. Moreover, if $D$ is in general position in the sense that no 5 points belong to the same hyperplane, then each of the open orthants contains at least $d - 16$ elements from $D$. In particular, if $D$ is a not necessarily symmetric set of $16d$ distinct points in $\mathbb{R}^4$, then for some collection $\mathcal{H}$ of hyperplanes, each of the associated open orthants contains not more than $2d$ points from $D$.

**Remark 5.3.** Note that the proof of Theorem 5.1.1 is based on the “accident” that the convex curve $\Gamma_4$ and the associated solution manifold $\Sigma'$ are $\mathbb{Z}/2$-spaces, where $\mathbb{Z}/2$ is the group generated by the reflection $\tilde{R}_L$. As a consequence $\Sigma'$ determines a nontrivial element in the group $\Omega(\mathbb{Z}/2)$ of $\mathbb{Z}/2$-bordisms, [CoF16], which means that one should be able to repeat the pattern of the proof of the equipartition theorem in the planar case; see Section 2.3. Our proof essentially follows this idea. For example the definition of $\psi'$ corresponds to the choice of a non-trivial $\mathbb{Z}/2$-equivariant, line bundle on the circle $\Sigma'$, a step used in the proof that $\Sigma'$ defines a non-trivial element in $\Omega(\mathbb{Z}/2)$.

**5.2. Measures admitting a center or a 3-plane of symmetry.** For completeness we include proofs of 4 equipartition results for measures in $\mathbb{R}^4$ which admit a center or a 3-plane of symmetry; see Proposition 5.2.6 and Proposition 5.2.8. These results formally resemble Theorem 5.1.1 but there are important differences. Both Propositions 5.2.6 and 5.2.8 are easily deduced from Hadwiger’s result about 3-equipartitions of measures in $\mathbb{R}^3$; see Theorem 5.4. Contrary to this, the proof of Theorem 5.1.1 relies on the existence of a unique, balanced, 4-bit binary Gray code, so this result appears to be an essential 4-dimensional phenomenon.

**Theorem 5.4 (Had66).** Each continuous mass distribution $\mu$ in $\mathbb{R}^3$ admits a 3-equipartition, that is, a collection $H_1, H_2, H_3$ of three planes in $\mathbb{R}^3$ dissecting the
ambient space into eight octants of equal measure. Moreover, the first of these planes can be chosen to contain two arbitrary points \( A, B \in \mathbb{R}^3 \) prescribed in advance.

Remark 5.5. Hadwiger [Had66] deduced Theorem 5.4 from the result that any two measures in \( \mathbb{R}^3 \) admit a simultaneous equipartition by 2 hyperplanes. Both results of Hadwiger can be proved, along the lines of the CS/TM-scheme, by an analysis of the set of all equipartitions for measures with compact support, concentrated on the convex curve \( M_3 = \{(t, t^2, t^3) \mid t \in \mathbb{R}\} \); see [Ziv98, Proposition 4.9].

Proposition 5.6 ([Zie04]). Suppose that \( \mu \) is a continuous mass distribution in \( \mathbb{R}^4 \) which has a center of symmetry \( O \). Then \( \mu \) admits a 4-equipartition. Moreover, one of the hyperplanes can be chosen in advance as an arbitrary 3-plane passing through the center of symmetry \( O \).

Proof. Choose a “halving” hyperplane \( H_1 \) for \( \mu \). One can assume that \( O \in H_1 \). Let \( \pi : \mathbb{R}^4 \to H_1 \) be the orthogonal projection and let \( \nu \) be the measure on \( H_1 \) defined by \( \nu(A) := \mu(\pi^{-1}(A) \cap H_1^+) \) where \( H_1^+ \) is a closed halfspace bounded by \( H_1 \). Find 2-planes \( P_2, P_3, P_4 \) in \( H_1 \) which form a 3-equipartition for \( \nu \) such that \( O \in P_i \) for each \( i \). This is always possible by Theorem 5.4. Then \( \mathcal{H} = \{H_1, H_2, H_3, H_4\} \) is a 4-equipartition for \( \mu \) where \( H_j := \pi^{-1}(P_j) \) for \( j \geq 2 \) is the hyperplane orthogonal to \( H_1 \) at \( P_j \).

□

Corollary 5.7. There does not exist a centrally symmetric, convex closed curve in \( \mathbb{R}^4 \).

Proof. The “arc length” measure on such a curve would be centrally symmetric. According to Proposition 5.6 its space of 4-equipartitions is “fibered” over the Grassmannian of all affine 3-planes in \( \mathbb{R}^4 \); hence it is at least 4-dimensional. This is a contradiction. Indeed, by the results of Section 3 the solution manifold of all 4-equipartitions of a measure concentrated on a closed, convex curve in \( \mathbb{R}^4 \) must be 1-dimensional. □

Proposition 5.8. Suppose that \( \mu \) is a continuous mass distribution in \( \mathbb{R}^4 \) which has a 3-plane of symmetry. Then there exists a 4-equipartition for \( \mu \).

Proof. The proof is similar to the proof of Proposition 5.6 so we omit the details. □

5.3. Topological “counterexample”.

Theorem 5.9. There exists a \( W_4 \)-equivariant map \( f : (S^4)^4 \to S(U_4) \).

Proof. By [Kosch81, §3], a \( W_4 \)-equivariant map \( f : (S^4)^4 \to S(U_4) \) exists if and only if the obstruction \( \omega \in \Omega_1(\mathcal{M}; E - TM) \) in the corresponding group of normal bordisms vanishes; cf. Section 4. By Proposition 4.1 \( \omega \) is in the image of the map \( \delta_1 \). By Proposition 4.9 the map \( \sigma \circ j_2 \) is onto and the result follows from the exactness of the sequence (4.2). □

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