Gain Scheduling Controller Synthesis for Control Moment Gyroscope Using Improved Approximation

Toru Inaba *, Gan Chen **, and Isao Takami **

Abstract: This paper presents gain scheduling (GS) control of a variable speed control moment gyroscope (VSCMG) based on sum of squares (SOS). Nonlinear motion equations of the VSCMG are complicated because they contain many trigonometric functions of angles of gimbals. The dynamics varies depending on the angles of the gimbals. In this study, the difficulty of control design of the VSCMG is solved by two methods. First, the complicated nonlinear model is transformed to the linear parameter varying model, such that the linear control method can be applied, to make control design easy by using a proposed approximation method. The sine function and the cosine function are generally approximated by the first-order Taylor series expansion in ordinary controller synthesis. The model obtained by the first-order Taylor series expansion ignores the nonlinear dynamics. But, in this study, those nonlinear functions are highly accurately approximated by using the proposed approximation method. The proposed approximation method is based on a high-order Taylor series expansion and a high-order Padé approximation. By using redundant representations, the synthesis condition can be reduced to polynomially parameter-dependent linear matrix inequalities (PDLMIs). Second, GS controller whose gains depend on the angles is applied. The polynomially PDLMIs can be relaxed to finite design conditions based on matrix SOS polynomials. The GS controller is designed by solving the finite SOS conditions. By using those methods, GS controller depending on the nonlinearities is designed. The effectiveness of the proposed controller is illustrated by simulations and experiments.

Key Words: linear parameter-varying model, linear matrix inequalities, sum of squares polynomials, robust semidefinite programming, variable speed control moment gyroscope.

1. Introduction

A control moment gyroscope (CMG) is known as an attitude control device for space crafts. CMGs are attractive actuators because the maximum torque generated by CMGs is dozens of times compared with the torque by conventional actuators called reaction wheel. In this study, the control design of a single-variable speed CMG (VSCMG) is treated as a fundamental study for cooperation control of CMGs. The VSCMG consists of a variable speed reaction wheel and a single gimbal [1],[2]. It is not easy to adopt general linear control methods to the VSCMG, because the dynamics of the VSCMG varies depending on an attitude of the spacecraft. Nonlinear control methods are adopted to the controller design of the VSCMG. Recently, gridding-based gain scheduling (GS) controller design based on a linear parameter varying (LPV) system is reported to solve the nonlinear dynamics [3]–[6], while robust stability of the system is guaranteed by solving many linear matrix inequalities (LMIs).

This paper presents a GS controller design of the VSCMG based on sum of squares (SOS). The VSCMG which has the multi-freedom actuator is able to control two-degrees-of-freedom (DOF) attitude, but the controller design is complicated because the motion equations include many nonlinearities such as the sine and the cosine of the gimbal’s angle $\theta(t)$. Nonlinearities such as $\sin(\theta(t))$ and $\cos(\theta(t))$ in motion equations are generally approximated by the first-order Taylor series expansion to obtain a simplified linear model. However, the simplified linear model is invalid when the nonlinearities significantly affect the dynamics of the controlled plant. An LPV model with the varying parameter $\sin(\theta(t))$ is introduced in this study. An approximation depending on $\sin(\theta(t))$ is applied only to $\cos(\theta(t))$. In our previous study [7], the nonlinearity $\cos(\theta(t))$ is approximated by not the first-order but the third-order Taylor series expansion to obtain an LPV model which do not ignore the nonlinear dynamics.

In this study, in order to conduct a more accurate approximation, the third-order Padé approximation is applied. Furthermore, the coefficients of the approximation function is optimized by using the method of least-squares under the region of the varying parameter. A highly accurate LPV model including rational functions of the varying parameter is derived by the proposed approximation method. The LPV model including the rational functions of the varying parameter is transformed to an equivalent one which has only fourth-order polynomials of the varying parameter by using descriptor representation. By using those redundant representation, the synthesis condition can be reduced to not the parameter-dependent linear matrix inequalities (PDLMIs) including the rational functions but polynomially PDLMIs. In this study, the GS controller is designed to adjust gains depending on the nonlinear dynamics of the VSCMG. The GS controller is designed based on matrix SOS polynomials, because the polynomially PDLMIs can be
relaxed to finite matrix SOS polynomials conditions, naturally. The performance of the GS controller designed by the proposed approximation method is higher than the performance of a GS controller designed by an ordinary approximation method. The effectiveness of the proposed GS controller is shown by simulations and experiments comparing with an ordinary GS controller.

2. Attitude Control Model of a VSCMG

2.1 Hardware and System Restriction

In this study, Model 750 CMG produced by Educational Control Products (ECP) is used as a controlled plant. Model 750 CMG has the variable speed wheel. The schematic diagram of Model 750 CMG is shown in Fig. 1. It consists of four rigid bodies which are Rotor1, Gimbal2, Gimbal3, and Gimbal4. These bodies rotate around Axis1, 2, 3, and 4. Gimbal2 which is equipped with Rotor1 is a VSCMG. Gimbal3 and Gimbal4 are controlled bodies. Here, let \( q_1, q_2, q_3 \), and \( q_4 \) be the angles of Rotor1, Gimbal2, Gimbal3, and Gimbal4, respectively. The angular velocity of Rotor1, Gimbal2, Gimbal3, and Gimbal4 are \( \dot{q}_1, \dot{q}_2, \dot{q}_3 \), and \( \dot{q}_4 \), respectively. Let \( \tau_1 \) and \( \tau_2 \) be the input torque for Rotor1 and Gimbal2, respectively.

![Fig. 1 Model 750 control moment gyroscope.](image)

The VSCMG which is used in this study has a hardware restriction on the motion range of Gimbal2. The motion range of Gimbal2 is \( \pm \pi/2 \) rad from the vertical position to Gimbal3. Rotor1 and Gimbal2 are driven by DC motors, while Gimbal3 and Gimbal4 have no drive source. Gimbal3 and Gimbal4 are driven by reaction torque and gyroscopic precession, respectively. Each torque is generated by the VSCMG. The gyroscopic precession is largest when the position of Gimbal2 is rapidly tilted around the vertical position to Gimbal3. Note that the dynamics of the VSCMG depends on the position of Gimbal2. We would like to deal with the nonlinear dynamics in a restricted motion range of Gimbal2, because it is difficult to ensure the global stability of the nonlinear dynamics. The motion range of Gimbal2 is assumed as follows:

\[
-\pi/6 \leq q_2 \leq \pi/6.
\]

(1)

An attitude of Gimbal3 is given by the angles \( q_3 \) and \( q_4 \). The attitude is regarded as an attitude of a spacecraft. Note that the attitude of Gimbal3 also affects the dynamics of the controlled plant. To permit the variety of attitudes, the motion range of Gimbal3 have to be large. However, the motion range of Gimbal3 is restricted, because the system has singular points depending on the attitude of Gimbal3. The singular points are the positions of \( \pm \pi/2 \) rad from the vertical position to Gimbal4. The motion range of Gimbal3 is assumed as follows:

\[
-5\pi/12 \leq q_3 \leq 5\pi/12.
\]

In this study, our aims are the tracking control of the angles \( q_1 \) and \( q_4 \) and the stabilization under the ranges (1) and (2).

2.2 Nonlinear Model

In this subsection, motion equations of Model 750 CMG are derived. The physical constants of Model 750 CMG are shown in Table 1. Here, \( I_i, J_i, \) and \( K_i (i = 1, 2, 3, 4) \) are the moment of inertia of Gimbal/Rotori with respect to Axis2, Axis3(Axis1), and Axis4 at the attitude in Fig. 1, respectively. If the angular velocity \( \dot{q}_i \) is large enough compared with other angular velocity \( \dot{q}_2, \dot{q}_3, \) and \( \dot{q}_4 \), a simplified nonlinear mathematical model can be derived [3]. Thus, motion equations of Model 750 CMG are represented as follows:

\[
\begin{bmatrix}
\dot{b}_3 \\
0 \\
-b_1 \sin q_3 \\
-b_2 \sin q_2 \cos q_3 \\
-b_2 \sin q_2 \cos q_3 \\
-b_2 \sin q_2 \cos q_3 \\
0 \\
-q_1 b_1 \sin q_2 \\
-q_1 b_1 \sin q_2 \\
-q_1 b_1 \sin q_2 \\
\dot{q}_1 b_1 \sin q_2 \\
\dot{q}_1 b_1 \sin q_2 \\
\dot{q}_1 b_1 \sin q_2 \\
0 \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\end{bmatrix}
\]

As can be seen in (3), there exist nonlinearities such as \( \sin q_i \) and \( \cos q_i (i = 2, 3) \).

2.3 Linear Parameter Varying (LPV) Modeling

In this subsection, linear parameter varying (LPV) modeling with the highly approximate accuracy is explained. First, the most ordinary approximation is discussed. Those nonlinear terms \( \sin q_i \) and \( \cos q_i \) in the system (3) are generally approximated by the first-order Taylor series expansion at an equilibrium point to obtain a simplified LPV system. The most ordinary approximation is as follows:

\[
\begin{cases}
\sin q_i \approx q_i \\
\cos q_i \approx 1.
\end{cases}
\]

(4)

The simplified LPV model by the ordinary approximation (4) cannot represent the nonlinear dynamics when the varying
range of the parameter \( q_i \) is large enough. Then, a high-order polynomial approximation is naturally introduced to deal with the nonlinear dynamics. To avoid that the system (3) becomes too complicated, the nonlinear terms \( \sin q_2 \) and \( \cos q_2 \) are approximated by the first order and the second order, respectively. To deal with the largely varying range (2), the nonlinear terms \( \sin q_3 \) and \( \cos q_3 \) are approximated by the third order and the second order, respectively. Those higher order Taylor approximations are written as follows:

\[
\begin{align*}
\sin q_2 & \approx q_2, \\
\cos q_2 & \approx 1 - q_2^2/2, \\
\sin q_3 & \approx q_3 - q_3^3/6, \\
\cos q_3 & \approx 1 - q_3^2/2.
\end{align*}
\]  

(5)  

Second, an approximation with the high accuracy is introduced. To deal with the largely varying range (2), Padé approximation is applied to the nonlinear terms \( \sin q_1 \) and \( \cos q_1 \). The nonlinear terms \( \sin q_1 \) and \( \cos q_1 \) are approximated by the third-order and the second-order Padé approximations, respectively. Those approximations with the high accuracy are written as follows:

\[
\begin{align*}
\sin q_3 &= \frac{60q_3 - q_3^3}{60 + 3q_3^2}, \\
\cos q_3 &= \frac{12 - 5q_3^2}{12 + q_3^2}.
\end{align*}
\]  

(8)  

(9)  

In control design of CMG, there are not research results that trigonometric functions are approximated by rational functions as far as the authors know. The approximate accuracy is significantly improved, but the system produced by the approximations (8) and (9) becomes too complicated.

To avoid this complexity, we propose new approximation with high accuracy. When the nonlinearities of the sine function in (3) are replaced as \( \alpha_i := \sin q_i \), the nonlinearities of the cosine function are represented by the square roots as \( \cos q_i = \sqrt{1 - \alpha_i^2} \). An approximation is carried out only to \( \cos q_i \) by using a rational function of \( \alpha_i \) that is \( \sin q_i \). The square roots can be approximated by rational functions of the reasonable degree depending on the ranges (1) and (2). The square root \( \sqrt{1 - \alpha^2} \) is approximated by the second-order Taylor series expansion to avoid that the system (3) becomes too complicated. The other square root \( \sqrt{1 - \alpha^2} \) is more accurately approximated by the third-order Padé approximation to cover the largely varying range (2). Those proposed approximations are represented as follows:

\[
\begin{align*}
\sin q_2 & = \alpha_2, \\
\cos q_2 & = 1 - \alpha_2^2/2, \\
\sin q_3 & = \alpha_3, \\
\cos q_3 & = (1 - \alpha_3^2)/(1 - \alpha_3^2/2).
\end{align*}
\]  

(10)  

(11)  

Hence, the constant terms of approximations for \( \sin q_i \) and \( \cos q_i \) must be 0 and 1, respectively. For example, consider to approximate (11) by \( (1 - c_1q_i^2)/(1 - c_2q_i^2) \). The coefficients \( c_1 \) and \( c_2 \) are obtained as follows:

\[
\begin{align*}
(c_1, c_2) &= \text{argmin}_{c_1, c_2} \sum_{k=0}^{N} \frac{1 - c_1\sin(q_3k)^2}{1 - c_2\sin(q_3k)^2} - \sqrt{1 - \alpha(q_3k)^2},
\end{align*}
\]  

(12)  

\[
\alpha(q_3k) := \sin(q_3k), q_{3k} := k(q_{3\text{max}} - q_{3\text{min}})/N + q_{3\text{min}}, \\
q_{3\text{min}} = -5\pi/12, q_{3\text{max}} = 5\pi/12, N = 15000.
\]

By using the optimized coefficients, the higher order Taylor approximations (5)–(7) are rewritten as follows:

\[
\begin{align*}
\sin q_2 & \approx 0.9728q_2, \\
\cos q_2 & \approx 1 - 0.4819q_2^2, \\
\sin q_3 & \approx 0.9945q_3 - 0.1514q_3^3, \\
\cos q_3 & \approx 1 - 0.4512q_3^2.
\end{align*}
\]  

(13)  

(14)  

(15)  

The approximations with the high accuracy (8) and (9) are rewritten as follows:

\[
\begin{align*}
\sin q_3 &= (0.9358q_1 - 0.1060q_1^3)/(0.9357 + 0.0504q_1^3), \\
\cos q_3 &= (1 - 0.4090q_1^2)/(1 + 0.0933q_1^2).
\end{align*}
\]  

(16)  

(17)  

The proposed approximations (10) and (11) are rewritten as follows:

\[
\begin{align*}
\sin q_2 & = \alpha_2, \\
\cos q_2 & = 1 - 0.5251a_2^2 =: \beta_\mu(a_2), \\
\sin q_3 & = \alpha_3, \\
\cos q_3 & = (1 - 0.8943a_3^2)/(1 - 0.4401a_3^2) =: \beta_\mu(a_3).
\end{align*}
\]  

(18)  

(19)  

The true values of the square root and the values of the approximations \( \beta_\mu(a_2) \) and \( \beta_\mu(a_3) \) are illustrated in Fig. 2. The dotted curve, the dashed and dotted curve, and the solid curve show the true values of the square root, the values of the approximation \( \beta_\mu(a_2) \), and the values of the approximation \( \beta_\mu(a_3) \), respectively. The maximum relative error between the true value and the value of the approximation (18) is 0.31% in the range (1) by using the optimized coefficients, while the maximum relative error between the true value and the value of the approximation (10) is 1.04% in the range (1). The maximum relative error between the true value and the value of the approximation (19) is 8.55% in the range (2) by using the optimized coefficients, while the maximum relative error between the true value...
and the value of the approximation (11) is 51.49% in the range (2). Those maximum relative errors are reduced by using the method of least-squares.

Here, let $x := [q_2 \; \dot{q}_1 \; \dot{q}_1]^T$, $u := [\tau_1 \; \tau_2]^T$, and $\alpha := [\alpha_2 \; \alpha_3]^T$ be the state variable vector, the control input vector, and the varying parameter vector, respectively. By using the proposed approximations (18) and (19), the linear parameter varying (LPV) system is represented as follows:

$$ E(\alpha) \dot{x} = A(\alpha) x + B(\alpha) u, \quad (20) $$

$$ E(\alpha) = \begin{bmatrix} b_1 & 0 \\ -b_1 \alpha_3 & -b_1 \alpha_2 \beta_p(\alpha_2) \beta_\iota(\alpha_3) \\ 0 & -b_1 \alpha_3 \end{bmatrix}, $$

$$ A(\alpha) = \begin{bmatrix} \dot{q}_1 \beta(\alpha_2) \beta_\iota(\alpha_3) \\ \dot{q}_1 \beta(\alpha_2) \beta_\iota(\alpha_3) \\ 0 \\ \dot{q}_1 \beta(\alpha_2) \beta_\iota(\alpha_3) \\ 0 \\ \dot{q}_1 \beta(\alpha_2) \beta_\iota(\alpha_3) \\ 0 \end{bmatrix}, $$

$$ B(\alpha) = \begin{bmatrix} 1 \\ -\beta_p(\alpha_2) \\ -\beta_p(\alpha_2) \\ 0 \end{bmatrix}. $$

There exist the rational functions (terms of $\beta_\iota(\alpha_3)$) of the varying parameter $\alpha_3$ in the matrices $E(\alpha)$, $A(\alpha)$, and $B(\alpha)$ in (20). It is not easy to reduce controller synthesis conditions for the system (20) to polynomially parameter-dependent linear matrix inequalities (PDLMIs).

### 2.4 Redundant Representation

Descriptor representation is applied to the system (20) to obtain an LPV model which does not include the rational function of the varying parameter $\alpha_3$. The LPV model (20) is transformed to an equivalent system with fourth-order polynomials of the varying parameters $\alpha$ by using the following methods. First, the redundant descriptor representation is applied. The varying parameters $\alpha$ are integrated into a new matrix $\hat{A}$ by introducing a redundant descriptor variable vector as $\hat{x} := [x^T \; \dot{x}^T \; \tau_1^T]^T$. The redundant descriptor representation is obtained as follows:

$$ \dot{\hat{x}} = \hat{A}(\alpha) \hat{x} + \hat{B} u, \quad (21) $$

$$ \hat{E} = \begin{bmatrix} I_4 & 0_{4 \times 4} \\ 0_{4 \times 4} & 0_{4 \times 4} \\ \hat{A}(\alpha) & \hat{A}(\alpha) \end{bmatrix}, \quad \hat{A}(\alpha) = \begin{bmatrix} 0_{4 \times 3} & I_4 \\ A(\alpha) & E_d(\alpha) \end{bmatrix}, $$

$$ \hat{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, $$

$$ E_d(\alpha) := \begin{bmatrix} -E(\alpha) & -\beta_p(\alpha_2) \\ 0 & 0 \end{bmatrix}. $$

There still exist rational terms $\beta(\alpha_3)$ in the fifth, sixth and seventh rows of $\hat{A}(\alpha)$. To eliminate the rational terms in the fifth row, pre-multiply $T$ to (21):

$$ T\dot{\hat{x}} = T\hat{A}(\alpha) \hat{x} + T\hat{B} u, \quad (22) $$

$$ T = \text{diag}[1, 1, 1, 1, \beta_p(\alpha_3), 1, 1, 1]. \quad (23) $$

To eliminate the rational terms in sixth and seventh rows in (22), $T\hat{A}(\alpha)$ is decomposed as follows:

$$ T\hat{A}(\alpha) = A_\iota(\alpha) - B_\iota(\alpha) C_\iota(\alpha)^{-1} C_\iota(\alpha), \quad (24) $$

$$ A_\iota(\alpha) = \begin{bmatrix} 0_{4 \times 3} & I_4 \\ A'(\alpha) & E'_d(\alpha) \end{bmatrix}, $$

$$ B_\iota(\alpha) = \begin{bmatrix} 0 & 0 & 0 & 0 & \beta_p(\alpha_2) & 0 \\ 0 & 0 & 0 & 0 & \beta_p(\alpha_2) \end{bmatrix}, $$

$$ C_\iota(\alpha) = \begin{bmatrix} -q_1 \beta(\alpha_2) & 0 & 0 & b_{12} \alpha_2 \beta_\iota(\alpha_3) \\ 0 \end{bmatrix}. $$

The redundant descriptor representation is applied for the sake of the space, $\beta_p(\alpha_2)$ and $\beta_p(\alpha_3)$ are replaced by $\beta_p$ and $\beta_p$, respectively. Let the descriptor variable $\hat{x}_d$ as follows:

$$ \hat{x}_d := \begin{bmatrix} \hat{x} \\ \hat{\tau}_6 \end{bmatrix}, \quad \hat{\tau}_6 := -D_d(\alpha)^{-1} C_d(\alpha) \hat{x}. \quad (25) $$

By using the matrices $A_k$, $B_k$, $C_k$, and $D_k$ satisfying (24), the redundant system is rewritten as follows:

$$ \hat{E}_d \hat{x}_d = \hat{A}_d(\alpha) \hat{x}_d + \hat{B}_d(\alpha) u, \quad (26) $$

$$ \hat{E}_d = \begin{bmatrix} \hat{E} \\ 0_{8 \times 2} \\ 0_{8 \times 2} \end{bmatrix}, \quad \hat{A}_d(\alpha) = \begin{bmatrix} A_\iota(\alpha) & B_\iota(\alpha) \\ C_\iota(\alpha) & D_\iota(\alpha) \end{bmatrix}, $$

$$ \hat{B}_d(\alpha) = \begin{bmatrix} 0 & 0 \end{bmatrix}, $$

As can be seen in (26), there exist the fourth-order polynomials of the varying parameter $\alpha$.

The size of the matrix $A$ in (20) is expanded from $A \in \mathbb{R}^{3 \times 3}$ to $\hat{A}_d \in \mathbb{R}^{8 \times 9}$ by the redundant descriptor representation. The sizes of the expanded matrix $\hat{A}_d$ produced by the ordinary approximations and the proposed approximations are shown in Table 2. Approximation methods which keep a certain approximate accuracy under the ranges (1) and (2) are shown in the first column of Table 2. The second column of Table 2 represents the order of the varying parameter in the system by each approximation method.
approximation method. Those expanded matrices \( \hat{A}_i \) shown in Table 2 include the fourth-order polynomials of the varying parameter. As can be seen in Table 2, the degree of expanding sizes becomes lower by using the proposed method of (18) and (19) even if the complicated approximation including the rational functions is used, because the order of varying parameter in the system is reduced from the eighth order to the fifth order. In later sections, we synthesize the controller by solving sum of squares (SOS) conditions numerically. However, the size of \( \hat{A}_i \) by the proposed method of (18) and (19) is equal to the size of \( \hat{A}_i \) by the proposed method of (13), (14), and (15). The maximum relative errors between the true value and the approximated value produced by the ordinary method and the proposed method are shown in Tables 3 and 4. As can be seen in Table 3, the maximum relative error of the proposed method is 2.51 points lower than the maximum relative error of the ordinary method. Note that the proposed method has no error for the sine function. As can be seen in Table 4, the maximum relative error of the proposed method is 4.34 points lower than the maximum relative error of the ordinary method. Consequently, the approximate accuracy is improved, and the degree of expanding sizes is reduced by using the proposed approximations.

Table 3 Maximum relative errors \((q_2) \leq \pi/6\).

| approximation method | sine (%) | cosine (%) | total points |
|----------------------|----------|------------|--------------|
| (13) (ordinary)      | 2.72     | 0.10       | 2.82         |
| (18) (proposed)      | 0        | 0.31       | 0.31         |

Table 4 Maximum relative errors \((q_1) \leq 5\pi/12\).

| approximation method | sine (%) | cosine (%) | total points |
|----------------------|----------|------------|--------------|
| (14), (15) (ordinary)| 0.55     | 12.34      | 12.89        |
| (19) (proposed)      | 0        | 8.55       | 8.55         |

3. Controller Design Based on Sum of Squares

The varying parameter vector \( \alpha \in \mathbb{R}^2 \) is chosen as a scheduling parameter vector to design a gain scheduling (GS) controller. The purpose of this study is to achieve attitude control of Gimbal3 and Gimbal4. The angles \( q_3 \) and \( q_4 \) are added to the state variable vector \( x \) in (20). Let \( q_3^\text{ref} \) and \( q_4^\text{ref} \) be given constant references for the angles \( q_3 \) and \( q_4 \), respectively. The output equation is defined as follows:

\[
y = Cx, \tag{27}
\]

where \( y = [q_3 \ q_4]^T \), \( C = [I_2 \ 0_{2 \times 1}] \). \( x := [q_3 \ q_4 \ x^T]^T \). To follow the reference without steady-state error, the servo system is adopted as

\[
u = K(\alpha)\left[\int_0^\infty (y^\text{ref} - y) \, dt \right]_x, \tag{28}
\]

where \( y^\text{ref} := [q_3^\text{ref} \ q_4^\text{ref}]^T \). The GS controller consists of not the redundant descriptor variable vector \( x_e \), but of the original state variable vector \( x_c \).

In this study, the linear quadratic (LQ) optimal control is adopted to evaluate an initial-value response. A cost function is introduced as

\[
J = \int_0^\infty \left( \hat{x}_e^T Q \hat{x}_e + u^T Ru \right) \, dt, \tag{29}
\]

where \( \hat{x}_e := \left[ \int_0^t (q_3^\text{ref} - q_3) \, dt \ 0_{2 \times t} \right]^T \), \( Q = Q^T \geq 0 \), \( R > 0 \). The redundant descriptor system can be obtained as follows:

\[
\dot{\hat{x}}_e = \left[ I_7 \ 0_{7 \times 6} \right] \hat{x}_e + \left[ 0_{2 \times 4} \tilde{B}_e(\alpha)^T \right]^T. \tag{30}
\]

The matrix \( \tilde{A}_e \) can be represented by using (26) and (27), but the detail is omitted for the sake of the space.

In this section, design conditions based on matrix sum of squares (SOS) polynomials are discussed. The SOS conditions are generally produced by polynomially parameter-dependent linear matrix inequalities (PDLMIs) [10],[11]. To obtain the PDLMIs, the parameter sets of the scheduling parameters \( \alpha \) are required. Let \( \Omega_\alpha \) and \( \Omega_\delta \) be the parameter sets of the scheduling parameters \( \alpha \) and its derivatives \( \dot{\alpha} \), respectively. If the varying range is assumed such as \( |\alpha| \leq \tilde{q}_1 \), then the parameter range is defined as \( |\alpha| \leq \tilde{q}_1 \). The region of the derivative \( \dot{\alpha} \) is chosen as \( |\dot{\alpha}| \leq \tilde{q}_2 \). Those sets \( \Omega_\alpha \) and \( \Omega_\delta \) are defined as follows:

\[
\Omega_\alpha := \{ \alpha \in \mathbb{R} : \ g_j(\alpha) \geq 0 \ (j = 1, 2) \}, \tag{31}
\]

\[
\Omega_\delta := \{ \alpha \in \mathbb{R} : \ g_j(\alpha) \geq 0 \ (j = 3, \ldots, 6) \}, \tag{32}
\]

\[
\begin{align*}
g_1(\alpha) &= (\alpha_2 + \sin \tilde{q}_1)(\sin \tilde{q}_2 - \alpha_2), \\
g_2(\alpha) &= (\alpha_3 + \sin \tilde{q}_1)(\sin \tilde{q}_3 - \alpha_3), \\
g_3(\alpha) &= \dot{\alpha}_2 + \dot{\alpha}_2, \\
g_4(\alpha) &= \dot{\alpha}_3 - \dot{\alpha}_2, \\
g_5(\alpha) &= \dot{\alpha}_3 + \dot{\alpha}_3, \\
g_6(\alpha) &= \dot{\alpha}_3 - \dot{\alpha}_3.
\end{align*}
\]

By using the sets \( \Omega_\alpha \) and \( \Omega_\delta \), polynomially PDLMIs for the redundant descriptor system (30) are introduced as follows [12]:

**Lemma 3.1** If there exist polynomial matrices \( X_1(\alpha) \) and \( Y_1(\alpha) \) satisfying the following PDLMIs, then the redundant descriptor system (30) is stabilized by the state feedback \( u = Y_1(\alpha)X_1^{-1}(\alpha)x_e \).

\[
F_1(\xi, \alpha) := \begin{bmatrix}
\gamma I & 0 \\
I & X_1(\alpha)
\end{bmatrix} > 0 \quad (\forall \alpha \in \Omega_\alpha), \tag{33}
\]

\[
F_2(\xi, \alpha, \dot{\alpha}) := \begin{bmatrix}
- \left[ H(\alpha) - \hat{E}_e(\alpha)X(\alpha) \right] & 0 \\
QX_1(\alpha) & I & 0 \end{bmatrix} > 0 \quad \forall (\alpha, \dot{\alpha}) \in \Omega_\alpha \times \Omega_\delta, \tag{34}
\]

\[
X(\alpha) := \begin{bmatrix}
X_1(\alpha) & 0_{n_x} \\
X_{21}(\alpha) & X_{22}(\alpha)
\end{bmatrix}, \\
Y(\alpha) := \begin{bmatrix}
Y_1(\alpha) & 0_{2 \times n_x}
\end{bmatrix}, \\
\Phi(\alpha) := \tilde{A}_e(\alpha)X(\alpha) + \tilde{B}_e(\alpha)Y(\alpha). \tag{35}
\]

Through the minimization of the upper bound \( \gamma \), the cost function \( J \) is minimized.
Note that \( \xi \) is the vector of the decision variables \( X_{00}, Y_{00}, \) and \( \gamma \). Let the polynomial matrices \( X(\alpha) \) and \( Y(\alpha) \) be the second-order of the scheduling parameter \( \alpha_i \) as follows:

\[
X(\alpha) := X_{00}(\alpha) + a_2 X_{10}(\alpha) + a_3 X_{01}(\alpha) + a_4 X_{11}(\alpha) + a_5 X_{20}(\alpha),
\]

\[
Y(\alpha) := Y_{00}(\alpha) + a_2 Y_{10}(\alpha) + a_3 Y_{01}(\alpha) + a_4 Y_{11}(\alpha) + a_5 Y_{20}(\alpha).
\]  

The design conditions as the polynomially PDLMI (33)-(34) can be relaxed to sum of squares (SOS) conditions. In previous studies, the robust stability to a system containing multiple parameters is shown by the proposed approximation method of (18) and (19). As can be seen in Table 7, the upper bound of \( J \) is minimized. Those results are shown in Table 7.

Table 5 Design conditions.

| controller                  | approximation | deg X | deg Y |
|-----------------------------|---------------|-------|-------|
| ordinary+RLQ                | (13), (14), (15) | 0th   | 0th   |
| proposed+RLQ                | (18), (19)    | 3rd   | 3rd   |
| ordinary+GS(CL)            | (13), (14), (15) | 0th   | 0th   |
| proposed+GS(CL)            | (18), (19)    | 3rd   | 3rd   |
| ordinary+GS(PDL)           | (13), (14), (15) | 2nd   | 2nd   |
| proposed+GS(PDL)           | (18), (19)    | 2nd   | 2nd   |

Table 6 Assumed ranges of varying parameters.

| parameter | range |
|-----------|-------|
| angle \( q_2 \) (rad) | \(|q_2| \leq \pi/6\) |
| angle \( q_3 \) (rad) | \(|q_3| \leq 5\pi/12\) |
| velocity \( q_1 \) (rad/s) | 40 (constant) |
| velocity \( q_2 \) (rad/s) | \(|q_2| \leq 1.0\) |
| velocity \( q_3 \) (rad/s) | \(|q_3| \leq 1.0\) |

Table 7 Upper bound of \( J \).

| controller                  | \( \gamma \) |
|-----------------------------|--------------|
| ordinary+RLQ                | 172.75       |
| proposed+RLQ                | 68.12        |
| ordinary+GS(CL)            | 61.86        |
| proposed+GS(CL)            | 49.49        |
| ordinary+GS(PDL)           | 27.87        |
| proposed+GS(PDL)           | 22.65        |

As can be seen in Table 7, the upper bound by “proposed+GS” is about 82% of that of “ordinary+GS.” The effectiveness of an appropriate approximation for the sine and cosine functions can be verified by comparing ordinary+RLQ/GS(CL)/GS(PDL) and proposed+RLQ/GS(CL)/GS(PDL), respectively. The effectiveness of gain scheduling is verified by comparing ordinary/proposed+RLQ and ordinary/proposed+GS(CL), respectively. The effectiveness of using a parameter dependent Lyapunov matrix is verified by comparing ordinary/proposed+GS(CL) and ordinary/proposed+GS(PDL), respectively.
4.1 Simulation

By the results in [13], the influence of friction disturbance cannot be ignored. Let $f_{\text{c,}n}$ and $f_{\text{v,}n}$ ($n = 1, \ldots, 4$) be the coefficients of Coulomb friction and viscous friction for each body such as Rotor1, Gimbal2, Gimbal3, and Gimbal4. The friction disturbance $F_n$ for each body is given as follows:

$$F_n = f_{\text{c,}n}\text{sign}(q_n) + f_{\text{v,}n}q_n.$$  

(40)

Simulations are carried out with the friction disturbance (40).

The simulation results are shown in Figs. 3 and 4 when the step references $q_3^{\text{ref}} = 1.0$ rad and $q_4^{\text{ref}} = 1.0$ rad are given for the angles $q_3$ and $q_4$, respectively. The thick dash-and-dotted line, thin dash-and-dotted line, thick dashed line, thick dotted line, thin dashed line, and thin solid line indicate the responses by “ordinary+RLQ”, “proposed+RLQ”, “ordinary+GS(CL)”, “proposed+GS(CL)”, “ordinary+GS(PDL)”, and “proposed+GS(PDL)”, respectively. As can be seen in Figs. 3 and 4, the responses by “ordinary+GS(PDL)” and “proposed+GS(PDL)” are improved compared with the responses by “ordinary+RLQ”, “proposed+RLQ”, “ordinary+GS(CL)” and “proposed+GS(CL)”, respectively. The response $q_4$ by “proposed+GS(PDL)” follows the reference without the error, while the response $q_3$ by “ordinary+GS(PDL)” does not converge. The settling time of “proposed+RLQ” is faster than that of “ordinary+RLQ” because the minimization of the cost function (29) is successful in the controller design of “proposed+RLQ” comparing with the controller design of “ordinary+RLQ.” The effectiveness of the proposed method of (18) and (19) is verified by this simulation results.

4.2 Experiment

The online calculation of GS controller $u = Y_1(\alpha)X_{11}^{-1}(\alpha)\tilde{x}$, is not available in experiments because of the software restriction of the experimental unit and the calculation load to CPU. The simplified GS controller is approximated as follows:

$$u_{\text{approx}} = (K_{00} + \alpha_2K_{10} + \alpha_3K_{01})$$  

(41)

$$+ \alpha_2^2K_{20} + \alpha_2\alpha_3K_{11} + \alpha_3^2K_{02})$$

$$+ \alpha_2^2K_{30} + \alpha_2^2\alpha_3K_{21} + \alpha_2\alpha_3^2K_{12} + \alpha_3^3K_{03})\tilde{x}.$$  

The gain of the GS controller consists of the third-order matrix polynomials of the varying parameter $\alpha$. The elements of those matrices $K_{ij}(q,r)$ ($q, r = 0, \ldots, 3$) are determined by using the method of least-squares based on the rational GS controller $u = K(\alpha)\tilde{x} = Y_1(\alpha)X_{11}^{-1}(\alpha)\tilde{x}$.

For example, consider to approximate (1, 1) element of the gain $K(\alpha)$. Let $K_{11}(\alpha_2, \alpha_3)$ and $K_{11}(\alpha_2, \alpha_3) = \tilde{\alpha}e^T$ be (1, 1) element of the gain $K(\alpha)$ and its approximation, respectively:

$$c = [k_{00}, k_{10}, k_{20}, k_{11}, k_{02}, k_{30}, k_{21}, k_{12}, k_{03}],$$

$$\tilde{\alpha} = [1, \alpha_2, \alpha_3, \alpha_2^2, \alpha_2\alpha_3, \alpha_3^2, \alpha_2^3, \alpha_2^2\alpha_3, \alpha_2\alpha_3^2, \alpha_3^3].$$

The approximated gain $c$ is derived as follows:

$$c = \text{argmin} \sum_{j=0}^{M} \sum_{k=0}^{N} (K_{11}(\alpha_2, \alpha_3), \alpha_3(q_{3k})) - K_{11}(\alpha_2, \alpha_3(q_{3k}))^2.$$  

(42)

$$\alpha_2(q_{2j}) = \sin(q_{2j}), q_{2j} = \frac{j(q_{2\text{max}} - q_{2\text{min}})}{M} + q_{2\text{min}},$$

$$q_{2\text{min}} = -\pi/6, q_{2\text{max}} = \pi/6, M = 12,$$

$$\alpha_3(q_{3k}) = \sin(q_{3k}), q_{3k} = k(q_{3\text{max}} - q_{3\text{min}})/N + q_{3\text{min}},$$

$$q_{3\text{min}} = -5\pi/12, q_{3\text{max}} = 5\pi/12, N = 36.$$  

The simulation results by the rational GS controller $u = Y_1(\alpha)X_{11}^{-1}(\alpha)\tilde{x}$ and the simplified GS controller (41) and the experimental results are shown in Figs. 5 and 6 when the step references $q_3^{\text{ref}} = 1.0$ rad and $q_4^{\text{ref}} = 1.0$ rad are given for the angles $q_3$ and $q_4$, respectively. The dotted line, the dashed line and the solid line show the simulation with the rational GS controller, the simulation with the simplified/approximated controller and experimental result, respectively. As can be seen in Figs. 5 and 6, the responses by the simplified GS controller are almost similar to the responses by the rational GS controller. The responses by the proposed controller follow the references without steady-state error even if there exist some unmodelled frictions. The reliability of the proposed controller is verified...
by the experimental results because the responses of the experiments are similar to the responses of the simulations.

5. Conclusion

This paper presents the GS controller design based on SOS for the VSCMG which has the nonlinear dynamics. The more accurate linear parameter varying (LPV) model is derived by using Padé approximation and least square approximation. The LPV model including the rational functions of the varying parameter is transformed to the equivalent LPV model including the fourth-order polynomials by using the descriptor representation. The synthesis condition of the GS controller is described by using PDLMIs. The PDLMIs are relaxed to the sufficient conditions based on matrix SOS polynomials. The performance of the GS controller designed by the proposed approximation is improved compared with a GS controller designed by an ordinary approximation. Finally, the effectiveness of the proposed GS controller is shown by simulations and experiments comparing with the ordinary controller.

References

[1] H. Schaub, S.R. Vadari, and J.L. Junkins: Feedback control law for variable speed control moment gyros, Journal of the Astronautical Sciences, Vol. 46, No. 3, pp. 307–328, 1998.
[2] V.S. Prabhakaran, A.K. Sanyal, F. Leve, and N.H. McClamroch: Geometric mechanics based modeling of the attitude dynamics and control of spacecraft with variable speed control moment, Proc. ASME 2013 Dynamic Systems and Control Conference, Paper No. DSCC2013-4033, 2013.
[3] H. Abbas, A. Ali, S.M. Hashemi, and H. Werner: LPV gain-scheduled control of a control moment gyroscope, Proc. 2013 American Control Conference, pp. 6841–6846, 2013.
[4] H. Abbas, A. Ali, S.M. Hashemi, and H. Werner: LPV state-feedback control of a control moment gyroscope, Control Engineering Practice, Vol. 24, pp. 129–137, 2014.
[5] J. Theis, C. Radisch, and H. Werner: Self-scheduled control of a gyroscope, Proc. 19th IFAC World Congress, pp. 6129–6134, 2014.
[6] C. Hoffmann and H. Werner: LFT-LPV modeling and control of a control moment gyroscope, Proc. IEEE Conference on Decision and Control, pp. 5328–5333, 2015.
[7] T. Inaba, M. Chintatsu, G. Chen, and I. Takami: Robust control of control moment gyroscope with friction disturbance, Proc. 7th International Conference on Information Technology and Electrical Engineering, pp. 559–564, 2015.
[8] Y. Fujisaki and Y. Oishi: Guaranteed cost regulator design: A probabilistic solution and a randomized algorithm, Automatica, Vol. 43, No. 2, pp. 317–324, 2007.
[9] H. Ichihara and M. Kawata: Gain scheduling control of an arm-driven inverted pendulum based on sum of squares: Comparison with a SDRE method, Proc. 18th IFAC World Congress, pp. 9613–9617, 2011.
[10] H. Ichihara and M. Kawata: Attitude control of acrobot by gain scheduling control based on sum of squares, Proc. 2010 American Control Conference, pp. 6636–6643, 2010.
[11] C.W. Scherer and C.W.J. Hol: Matrix sum-of-squares relaxations for robust semi-definite programs, Mathematical Programming, Vol. 107, pp. 189–211, 2006.
[12] G. Chen: System analysis using redundancy of descriptor representation, Proc. 2004 IEEE International Symposium on Computer Aided Control System Design, pp. 231–236, 2004.
[13] S. Washizu, C. Murai, I. Takami, and G. Chen: Nonlinear control for first-order nonholonomic system with hardware restriction and disturbance, Proc. 10th Asian Control Conference, TS2.4-6-2, 2015.

Topu Inaba
He received his B.S. and M.S. degrees from Nanzan University, Japan, in 2015 and 2017, respectively. His research interests include control systems design.

Gan Chen (Member)
He received his M.S. and Dr. Eng. degrees from the Graduate School of Engineering of Kyoto University. He is an Associate Professor at the Faculty of Science and Engineering of Nanzan University. His research interests are related to robust control. He is a member of IEEE.

Isao Takami (Member)
He received his B.S., M.S., and Dr. Eng. degrees from Kyoto University, Kyoto, Japan in 1972, 1974, and 1986, respectively. He joined Mitsubishi Heavy Industries, Ltd. in 1974, and with Nanzan University, Nagoya, Japan in 2002, respectively. He is currently a Professor at the Faculty of Science and Engineering of Nanzan University. His research interests include analysis and synthesis of large scaled systems. He is a member of IEEE.