Locating a complex inhomogeneous medium with an approximate factorization method

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Abstract

Consider the inverse problem of scattering of time-harmonic acoustic waves by an inhomogeneous medium with complex refractive index. We show that an approximate factorization method can be applied to reconstruct the support of the complex inhomogeneous medium from the far-field data. Numerical examples are also provided to illustrate the practicability of the inversion algorithm.

Keywords: Approximate factorization method, inverse scattering, far-field pattern, inhomogeneous medium.

1 Introduction

In this paper, we study the inverse problem of recovering an inhomogeneous medium with complex refractive index from the far-field data. This problem occurs in lots of areas of application such as radar and sonar, medical imaging and non-destructing testing. Precisely, let an open bounded obstacle $D$ denote the inhomogeneous medium with a $C^2$-smooth boundary $\partial D$. Assume that $\overline{D} = \bigcup_{j=1}^{K} \overline{D}_j$ with $D_{j_1} \cap D_{j_2} = \emptyset$ if $j_1 \neq j_2$. Assume further that $D$ is filled with an inhomogeneous material characterized by the refractive index $n(x) \in L^\infty(D)$ with $\text{Re}[n(x)] > 1$ or $\text{Re}[n(x)] < 1$ in $D_l$ $(1 \leq l \leq K)$, $|\text{Re}[n(x)] - 1| \geq c$ in $D$ for some positive constant $c$, $\text{Im}[n(x)] \geq 0$ in $D$, and the exterior $\mathbb{R}^3 \setminus \overline{D}$ is filled with a homogeneous material with the refractive index $n(x) = 1$. It should be remarked that we shall in the current paper consider the case of complex refractive index, that is, there at least exists two subdomains $D_{l_1}, D_{l_2}$ such that $\text{Re}[n(x)] > 1$ in $D_{l_1}$ and $\text{Re}[n(x)] < 1$ in $D_{l_2}$ with $1 \leq l_1 \neq l_2 \leq K$. Then the scattering of time-harmonic acoustic waves by the complex inhomogeneous medium $D$ can be modeled by the

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inhomogeneous Helmholtz equation

\[ \triangle u(x) + k^2 n(x) u(x) = 0 \quad \text{in } \mathbb{R}^3. \]  

(1.1)

Here, \( k > 0 \) is the wave number and \( u = u^i + u^s \) denotes the total field with the incident wave \( u^i \) and the scattered field \( u^s \), where \( u^s \) satisfies the Sommerfeld radiation condition

\[ \frac{\partial u^s}{\partial |x|} - ik u^s = O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty. \]  

(1.2)

Moreover, it is known that the scattered field \( u^s \) has the asymptotic behavior [7]

\[ u^s(x) = \frac{e^{ik|x|}}{4\pi|x|} u_\infty(\tilde{x}) + O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty, \]  

(1.3)

uniformly for all \( \tilde{x} = x/|x| \), where \( u_\infty \) is known as the far-field pattern of \( u^s \), which is an analytic function defined on \( \mathbb{S}^2 := \{ x \in \mathbb{R}^3 : |x| = 1 \} \). In the present paper, we consider \( u^i \) to be the incident plane wave which is given by \( u^i = u^i(x;d) := e^{ikx \cdot d} \), where \( d \in \mathbb{S}^2 \) is the incident direction. Accordingly, the total field, the scattered field and the far-field pattern are denoted as \( u(x;d), u^s(x;d) \) and \( u_\infty(\tilde{x};d) \), respectively.

By using a variational approach, it can be easily shown that the problem \( \text{(1.1)-(1.2)} \) has a unique solution (see, e.g., [7] or [22] for the case when \( D \) contains buried objects inside). In the current paper, we are interested in the inverse problem of reconstructing the shape and location of the inhomogeneous medium \( D \) from a knowledge of the far-field pattern \( u_\infty \) for incident plane waves. The uniqueness of this inverse problem has been established in [25] for the case when \( n \) is an unknown constant, in [23] for the case when \( n \) is an unknown piecewise constant, and in [11, 16, 24] for other related inverse medium scattering problems.

In this paper, we study the factorization method as an analytic as well as a numerical tool to reconstruct the shape and location of the inhomogeneous medium \( D \) with complex refractive index \( n(x) \). For the case when \( \text{Re}[n(x)] > 1 \) or \( \text{Re}[n(x)] < 1 \) in \( D \), based on a Lippmann-Schwinger integral equation method, [13] proved the validity of the factorization method for recovering the inhomogeneous obstacle \( D \). Recently, a factorization method has been developed in [26] in determining a penetrable obstacle \( D \) with unknown buried objects inside in the case when the solution is discontinuous across the interface \( \partial D \), that is, \( u_+ = u_- \), \( \partial_\nu u_+ = \lambda \partial_\nu u_- \) on \( \partial D \) for \( \lambda \neq 1 \). However, the method used in [26] can not be applied to the case when the solution is continuous across the interface \( \partial D \), that is, \( \lambda = 1 \) (see [26] Remark 2.5)). To overcome this difficulty, in [22] an approximate factorization method was proposed to solve the same inverse problem as that in [26] for the case when the solution is continuous across the interface \( \partial D \). However, the factorization method in [13, 22, 26] depends closely on the assumption that \( \text{Re}[n(x)] > 1 \) or \( \text{Re}[n(x)] < 1 \) in \( D \). Therefore, the techniques developed in [13, 22, 26] can not be directly extended to deal with the case when \( D = \bigcup_{j=1}^{K} D_j \) with \( \text{Re}[n(x)] > 1 \) in \( D_i \) and...
Re\[n(x)\] < 1 in \(D_1\) for some \(1 \leq l_1 \neq l_2 \leq K\) which is the case of the inverse problem under consideration. The reader is referred to [2, 8, 19] for applications of factorization method for the scattering by diffraction gratings, to [18] for the photonics and rough surfaces problems, to [9] and [27] for the cases of the conductive boundary condition and the generalized impedance boundary condition, and to [17, 28, 29] for the fluid-solid interaction problems. See also [9] for the rigorous mathematical justification of the factorization method with near-field data. For more detailed overview of the factorization method, we refer to the monograph [15] and the references therein, where many related inverse problems for different kinds of partial differential equations are studied by using this method.

For the inverse medium scattering problems, there are also lots of different reconstruction algorithms; see, e.g., [14] for the music-algorithm method, [21] for the singular sources method, [11, 10, 30] for the iteration method and [4, 5, 12, 20] for the linear sampling method.

In the present paper, we are motivated by [13, 22] to solve the inverse problem of locating the inhomogeneous medium by developing an approximate factorization method in the case when the medium is filled with an inhomogeneous material characterizing by the complex refractive index. Due to the close dependence of the classical factorization method on the complex refractive index in \(D\), we attempt to construct a sequence of perturbed operators \(F_m\) of the far-field operator \(F\) in a suitable way such that \(F_m\) satisfies the Range Identity in [15, Theorem 2.15] for each \(m \in \mathbb{N}_+\). Consequently, we can reconstruct the shape and location of medium \(D\) from the spectral data of \(F_m\) for each \(m \in \mathbb{N}_+\). Relying on the construction of \(F_m\), we can easily show that \(\|F_m - F\|_{L^2(S^2)} \to 0\) as \(m \to \infty\). Thus the exact far-field data \(F\) can be regraded as a sufficiently small perturbation of \(F_{m_0}\) for some large enough \(m_0 \in \mathbb{N}_+\). This implies that, for the noise level \(\delta\), the noisy operator \(F_\delta\) for \(F\) is also a small perturbation of the noisy operator \(F_{m_0}^{\delta}\) for \(F_{m_0}\). Therefore, the shape and location of the medium \(D\) can be numerically reconstructed by using the spectral data of \(F\) and \(F_\delta\). Numerical examples that carried out later indeed demonstrate the practicability of the inversion algorithm.

The remaining part of this paper is organized as follows. In section 2, we propose an approximate factorization method for our inverse problem of locating the inhomogeneous medium with complex refractive index. Numerical examples are provided to illustrate the efficiency of the inversion algorithm in section 3. Some remarks are also given at the end of section 3.

## 2 Approximate Factorization Method

In this section, we shall develop an approximate factorization method to study the inverse problem in determining the shape and location of an inhomogeneous medium with complex refractive index. For simplicity we only consider the case when \(\overline{D} = \overline{D}_1 \cup \overline{D}_2\) with \(n = n_1\) in \(D_1\) satisfying that \(\text{Re}[n_1(x)] - 1 \geq c\), \(\text{Im}[n_1(x)] \geq 0\) and \(n = n_2\) in \(D_2\) satisfying that \(\text{Re}[n_2(x)] - 1 \leq -c\),
Im\[n_2(x)\] ≥ 0 for some positive constant c. We first consider the following general problem

\[
\begin{cases}
\Delta w + k^2 w = 0, & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\
\Delta w + k^2 n_1 w = k^2 (1 - n_1) f_1, & \text{in } D_1, \\
\Delta w + k^2 n_2 w = k^2 (1 - n_2) f_2, & \text{in } D_2, \\
w(x)|_{\partial D} = 0, & \text{on } \partial D_1 \cup \partial D_2, \\
\frac{\partial w}{\partial |x|} - ik w = O\left(\frac{1}{|x|^2}\right), & \text{as } |x| \to \infty,
\end{cases}
\] (2.1)

where \(f_1 \in L^2(D_1), f_2 \in L^2(D_2)\). Let \(u\) be the total field of the problem (1.1)-(1.2) corresponding to the incident field \(u^i = e^{ikx \cdot d}\), it then follows that \(w := u - u^i\) satisfies the problem (2.1) with \(f_1 = u^i|_{D_1}\) and \(f_2 = u^i|_{D_2}\). We now introduce the solution operator we need to prove the denseness of the range of \(G\).

Secondly, we have the following lemma.

**Lemma 2.1.** The solution operator \(G\) is compact with dense range in \(L^2(\mathbb{S}^2)\).

**Proof.** Firstly, the compactness of the operator \(G\) follows easily from the interior regularity results of elliptic equations. Secondly, we need to prove the denseness of the range of \(G\) in \(L^2(\mathbb{S}^2)\), it suffices to show that the \(L^2\)-adjoint operator \(G^*\) of \(G\) is injective.

Assume that \(w\) is the solution of the problem (2.1) with the data \((f_1, f_2)^T \in Y\), \(w_\infty\) is the far-field pattern of the solution \(w\) and \(\tilde{w}\) is the total field of the problem (1.1)-(1.2) with the incident field

\[
\tilde{w}^i(y) = \int_{\mathbb{S}^2} e^{-ikd \cdot \varphi(d)} ds(d), \quad \text{for } y \in \mathbb{R}^3.
\]

It then follows from the Green’s theorem that

\[
w_\infty(d) = \int_{\partial D} \left( \frac{\partial e^{-ikd \cdot y}}{\partial \nu(y)} w(y) - e^{-ikd \cdot y} \frac{\partial w(y)}{\partial \nu(y)} \right) ds(y),
\]

which combines with the definition of the incident field \(\tilde{w}^i\) further implies that, for \(\varphi \in L^2(\mathbb{S}^2)\),

\[
(G(f_1, f_2)^T, \varphi)_{L^2(\mathbb{S}^2)} = \int_{\partial D} \left( \frac{\partial \tilde{w}^i}{\partial \nu} - \tilde{w} \frac{\partial w}{\partial \nu} \right) ds.
\] (2.2)

Notice that

\[
\int_{\partial D} \left( \frac{\partial \tilde{w}^s}{\partial \nu} w - \tilde{w} \frac{\partial w}{\partial \nu} \right) ds = 0,
\]

where \(\tilde{w}^s := \tilde{w} - \tilde{w}^i\) is the scattered field of the problem (1.1). We then derive that

\[
\int_{\partial D} \left( \frac{\partial \tilde{w}^i}{\partial \nu} w - \tilde{w} \frac{\partial w}{\partial \nu} \right) ds = \int_{\partial D} \left( \frac{\partial \tilde{w}}{\partial \nu} w - \tilde{w} \frac{\partial w}{\partial \nu} \right) ds
\]

\[
= \int_{D_1} k^2(n_1 - 1)f_1 \tilde{w} dx + \int_{D_2} k^2(n_2 - 1)f_2 \tilde{w} dx.
\]
This together with \(2.2\) yields that

\[
G^* \varphi = \begin{pmatrix} k^2(\overline{w}_1 - 1) \overline{w}_1, k^2(\overline{w}_2 - 1) \overline{w}_2 \end{pmatrix}^T
\]

with \(\tilde{w}_j := \tilde{w}|_{D_j}, j = 1, 2\). Let now \(G^* \varphi = 0\), then \(\tilde{w}_1 = 0 \text{ in } D_1, \tilde{w}_2 = 0 \text{ in } D_2\), which further implies \(\tilde{w}|_{D} = \frac{\partial \tilde{w}}{\partial \nu}|_{D} = 0\), this together with Holmgren’s uniqueness theorem gives \(\tilde{w} = \tilde{w}^i + \tilde{w}_s = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}\). Since \(\tilde{w}^i\) does not satisfy the radiation condition if \(\varphi \neq 0\), we obtain that \(\tilde{w}^i = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}\). It then follows from Theorem 3.19 in [7] that \(\varphi = 0\), and thus \(G^*\) is injective, which proves the lemma. \(\square\)

Introduce the far-field operator \(F : L^2(S^2) \rightarrow L^2(S^2)\) by

\[
(Fg)(\tilde{x}) = \int_{S^2} u_{\infty}(\tilde{x}, d) g(d) ds(d) \quad \text{for } g \in L^2(S^2),
\]

where \(u_{\infty}\) is the far-field pattern of the scattered field \(u^s\) of the problem (1.1)-(1.2) with the incident wave \(u^i = e^{i k x \cdot d}\). Define the incident operator \(H : L^2(S^2) \rightarrow Y\) by \(H = (H_1, H_2)^T\) with

\[
(H_1 g)(x) = \int_{S^2} e^{i k x \cdot d} g(d) ds(d) \quad \text{for } x \in D_1, \quad (2.5)
\]

\[
(H_2 g)(x) = \int_{S^2} e^{i k x \cdot d} g(d) ds(d) \quad \text{for } x \in D_2. \quad (2.6)
\]

It then follows from the superposition principle and the definition of the operator \(G\) that \(F = GH\).

In order to derive the factorization of the far-field operator \(F\), we next introduce the operator \(V_j\) that defined as follows: for \(\varphi_j \in L^2(D_j)\),

\[
(V_j \varphi_j)(x) = \int_{D_j} \Phi(x, y) \varphi_j(y) dy \quad \text{for } x \in D_j, \ j = 1, 2 \quad (2.7)
\]

and the restriction operators \(V_j^{(m)} := V_j|_{D_m}, (j, m = 1, 2)\). Here, \(\Phi(x, y) = \frac{e^{i k |x-y|}}{4 \pi |x-y|}\) is the fundamental solution of Helmholtz equation \(\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^3 \setminus \{y\}\). Then we define the operator \(T : Y \rightarrow Y\) by

\[
T = \begin{pmatrix} q_1 I_{D_1} - V_1^{(1)} & -V_2^{(1)} \\ -V_1^{(2)} & q_2 I_{D_2} - V_2^{(2)} \end{pmatrix}, \quad (2.8)
\]

where for \(j = 1, 2\), \(q_j := \frac{1}{k^2(n_j-1)}\) and \(I_{D_j}\) are the identity operators on \(L^2(D_j)\).

We have the following lemma on the property of the operator \(T\).

**Lemma 2.2.** The operator \(T\) is invertible and

\[
T^{-1} = T_1^{-1} + T_{com} \quad (2.9)
\]

with \(T_1\) to be an invertible operator given by

\[
T_1 = \begin{pmatrix} q_1 I_{D_1} & 0 \\ 0 & q_2 I_{D_2} \end{pmatrix}
\]

and the compact part \(T_{com} = -T_1^{-1} T_2 T^{-1}\) where \(T_2\) is a compact operator.
Proof. It is easily checked that the operator $T$ defined by (2.8) can be divided into two parts

$$T = \begin{pmatrix} q_1I_{D_1} & 0 \\ 0 & q_2I_{D_2} \end{pmatrix} - \begin{pmatrix} V_1^{(1)} & V_2^{(1)} \\ V_1^{(2)} & V_2^{(2)} \end{pmatrix} =: T_1 + T_2.$$  

Clearly, $T_1$ is invertible on $Y$ and $T_2$ is compact on $Y$. This yields that the operator $T$ is of Fredholm-type with index 0. Now let $T\varphi = 0$ for $\varphi = (\varphi_1, \varphi_2)^T \in Y$. We define the function

$$w(x) = \int_{D_1} \Phi(x, y)\varphi_1(y)dy + \int_{D_2} \Phi(x, y)\varphi_2(y)dy \quad \text{for } x \in \mathbb{R}^3.$$  

Then it follows from the properties of the operator $V_j, j = 1, 2$ (see e.g. [7, Section 8.2]) that $w$ is a solution of the problem (2.1) with the data $f_1 = f_2 = 0$. Thus, the uniqueness of (2.1) ensures that $w = 0$ in $\mathbb{R}^3$. So, using the properties of the operator $V_j$ ($j = 1, 2$) again, we derive that $\Delta w + k^2w = -\varphi_1$ in $D_1$ and $\Delta w + k^2w = -\varphi_2$ in $D_2$ which yields that $\varphi_1 = \varphi_2 = 0$. Therefore, the invertibility of $T$ follows from the Fredholm alternative. Finally, by a direct calculation and the compactness of $T_2$, one can obtain the assertion (2.9) and the compactness of $T_{com}$. This ends the proof of the lemma.

Now, we present the factorization of the far-field operator $F$.

**Theorem 2.3.** Let the far-field operator $F$ be defined by (2.4). Then we have the following factorization

$$F = H^*T^{-1}H,$$  

where $H^*$ is the adjoint operator of the incident operator $H$.

**Proof.** It can be easily proved that the adjoint $H^*$ of $H$ satisfies: for $\varphi = (\varphi_1, \varphi_2)^T \in Y$,

$$(H^*\varphi)(d) = \int_{D_1} e^{-ikd\cdot y}\varphi_1(y)dy + \int_{D_2} e^{-ikd\cdot y}\varphi_2(y)dy \quad \text{for } d \in S^2,$$

which is the far-field pattern of the function $w$ defined by (2.10). Then we derive that $w$ solves the problem (2.1) with the data

$$f_1 := q_1\varphi_1 - [V_1^{(1)}\varphi_1 + V_2^{(1)}\varphi_2], \quad f_2 := q_2\varphi_2 - [V_1^{(2)}\varphi_1 + V_2^{(2)}\varphi_2].$$  

So that $H^* = GT$, which together with Lemma 2.2 yields $G = H^*T^{-1}$. This combines with the fact $F = GH$ leads to the factorization that $F = H^*T^{-1}H$, which completes the proof of the Theorem.

We now introduce an auxiliary operator $\tilde{H}_1 : L^2(S^2) \mapsto H^*(\partial D_1)$ defined by

$$\tilde{H}_1\varphi(x) = \int_{S^2} e^{ikx\cdot d}\varphi(d)ds(d), \quad \text{for } x \in \partial D_1$$  

(2.13)
and the compact operator $L : H^\frac{1}{2}((\partial D_1)) \to L^2(D_1)$ with $Lh = w|_{D_1}$, where $w$ is a solution of the problem

$$
\begin{align*}
\begin{cases}
\Delta w + k^2 w &= 0, &\text{in } D_1, \\
w &= h, &\text{on } \partial D_1,
\end{cases}
\end{align*}
$$

(2.14)

for $h \in H^\frac{1}{2}((\partial D_1))$. In the following, we always assume that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $D_1$. So, the above problem (2.14) is well posed and consequently, the operator $L$ is well defined.

It is noted that $L\tilde{H}_1 = H_1$ and thus

$$
H = \begin{pmatrix} L & 0 \\ 0 & I_{D_2} \end{pmatrix} \begin{pmatrix} \tilde{H}_1 \\ H_2 \end{pmatrix} := A\tilde{H}.
$$

(2.15)

Based on (2.15), we define a series of perturbation operators $F_m$ by

$$
F_m := F + \rho_m \tilde{H}_1^* N_{i,\partial D_1} \tilde{H}_1
$$

(2.16)

with $\rho_m > 0$ for every $m \in \mathbb{N}$, which satisfies that $\rho_m \to 0$ as $m \to \infty$. Here, $N_{i,\partial D_1}$ is defined by

$$
(N_{i,\partial D_1}\varphi_1)(x) = \frac{\partial}{\partial \nu(x)} \int_{\partial D_1} \frac{\partial \Phi(i;x,y)}{\partial \nu(y)} \varphi_1(y) ds(y), \quad \text{for } x \in \partial D_1,
$$

where $\Phi(i;x,y)$ is a fundamental solution of the special Helmholtz equation $\Delta u - u = 0$. Notice that $\tilde{H}_1$ is well-defined, it then follows

$$
\|F_m - F\|_{L^2(S^2)} = \|\rho_m \tilde{H}_1^* N_{i,\partial D_1} \tilde{H}_1\|_{L^2(S^2)} = \rho_m \|\tilde{H}_1^* N_{i,\partial D_1} \tilde{H}_1\|_{L^2(S^2)} \to 0, \quad \text{as } m \to \infty.
$$

(2.17)

We define an auxiliary matrix $J_m$ by

$$
J_m := \begin{pmatrix} \rho_m N_{i,\partial D_1} & 0 \\ 0 & 0 \end{pmatrix}.
$$

This, together with (2.3) leads to the factorization for the perturbation far-field operator $F_m$ as

$$
F_m = \tilde{H}^* (A^* T^{-1} A + J_m) \tilde{H}
$$

$$
= \tilde{H}^* \left[ \begin{pmatrix} \rho_m N_{i,\partial D_1} & 0 \\ 0 & \frac{1}{q_2} I_{D_2} \end{pmatrix} + \begin{pmatrix} L^*(\frac{1}{q_1} I_{D_1}) & 0 \\ 0 & 0 \end{pmatrix} + A^* T_{\text{com}} A \right] \tilde{H}
$$

$$
=: \tilde{H}^* (M_m + M_{\text{com}}) \tilde{H}.
$$

(2.18)

It is obvious that $M_{\text{com}}$ is compact on $\tilde{Y} := H^\frac{1}{2}((\partial D_1)) \times L^2(D_2)$ and $-\text{Re} M_m$ is coercive on $\tilde{Y}$, i.e., there exists $c_0 > 0$ with $-\langle \text{Re} M_m \varphi, \varphi \rangle \geq c_0 \|\varphi\|^2$ for all $\varphi \in \tilde{Y}$ because $\text{Re}(n_2) - 1 < -c$ in $D_2$ for some positive constant $c$ and the operator $-N_{i,\partial D_1}$ is coercive (see e.g. [15] Theorem 1.26]).

For $z \in \mathbb{R}^3$, define the function $\phi_z(\tilde{x}) = e^{-ik\tilde{x} \cdot z}$ with $\tilde{x} \in \mathbb{S}^2$. Then we next prove the fact that $z \in D \iff \phi_z \in R(\tilde{H}^*)$. To this end, we first need to show the following lemma.
Lemma 2.4. For \( z \in \mathbb{R}^3 \), we have that
\[
z \in D \iff \phi_z \in R(G).
\]

Proof. Assume first \( z \in D \) and thus there exists a closed ball \( B_\delta(z) \) centered at \( z \) with radius \( \delta > 0 \) such that \( B_\delta(z) \subset D \). Then we choose a cut-off function \( \chi \in C^\infty(\mathbb{R}^3) \) with \( \chi(t) = 1 \) for \( |t| \geq \delta \) and \( \chi(t) = 0 \) for \( |t| \leq \delta/2 \) and define a function \( w(x) \) by
\[
w(x) := \chi(|x - z|)\Phi(x, z) = \chi(|x - z|)\frac{e^{ik|x-z|}}{4\pi|x-z|} \quad \text{in} \ \mathbb{R}^3.
\]
Obviously, \( w \in C^\infty(\mathbb{R}^3) \) and \( w(x) = \Phi(x, z) \) for \( |x - z| \geq \delta \). Indeed, for \( x \in D_j \ (j = 1, 2) \), we have
\[
\Delta w + k^2 n_j w = \Phi \Delta \chi + \chi \Delta \Phi + 2 \nabla \chi \cdot \nabla \Phi + k^2 n_j \chi \Phi =: k^2(1 - n_j)g_j \quad \text{in} \ D_j.
\]
Then \( g_j \in L^2(D_j) \), this combines with the unique solvability of the problem (2.1) implies that \( w \) is the solution of (2.1) with the data \( (f_1, f_2)^T = (g_1, g_2)^T \). So we immediately get \( G(g_1, g_2)^T = w_\infty = \phi_z \) and consequently \( \phi_z \in R(G) \).

On the other hand, let \( z \notin D \) and assume that there exists \( (\tilde{f}_1, \tilde{f}_2)^T \in Y \) such that \( G(\tilde{f}_1, \tilde{f}_2)^T = \phi_z \). Let \( \tilde{w} \) be the solution of the problem (2.1) with the data \( (f_1, f_2)^T = (\tilde{f}_1, \tilde{f}_2)^T \) and \( \tilde{w}_\infty \) be the far-field pattern of \( \tilde{w} \). Then \( \tilde{w}_\infty = \phi_z \). It follows from Rellich’s Lemma and unique continuation theorem that \( \tilde{w}(x) = \Phi(x, z) \) in \( \mathbb{R}^3 \setminus (\bar{D} \cup \{z\}) \). However, this is a contradiction because \( \|\tilde{w}\|_{H^1(B_\delta(z))} < \infty \) and \( \|\Phi(\cdot, z)\|_{H^1(B_\delta(z))} = \infty \), where \( B_\delta(z) \) is chosen to be a sufficiently small ball centered at \( z \). The proof of this lemma is thus completed.

It is noted that, from the proof of Theorem 2.3 the solution operator \( G \) and the incident operator \( H \) satisfy \( H^* = GT \) and \( G = H^*T^{-1} \). This implies that \( R(H^*) = R(G) \), thus we have the following lemma. The proof is easily obtained, hence we omit it.

Lemma 2.5. It holds that
\[
z \in D \iff \phi_z \in R(H^*).
\]

Combining the above lemmas yields the following theorem.

Theorem 2.6. \( \tilde{H}^* \) is compact with dense range in \( L^2(\mathbb{S}^2) \) and
\[
z \in D \iff \phi_z \in R(\tilde{H}^*).
\]

Proof. From [7, Theorem 3.19], we can easily obtain that \( \tilde{H} \) is injective. Then the compactness and denseness of \( \tilde{H}^* \) easily follow from the compactness and injectivity of \( \tilde{H} \). Assume \( z \in D \), it is seen from Lemma 2.5 that \( \phi_z \in R(H^*) \). Let \( Y' \) denote the adjoint of \( Y \). So there exists \( \varphi \in Y' \)
such that \( \phi_z = H^* \varphi \). Notice that \( H^* = \tilde{H}^* A^* \) from (2.15) and thus \( \phi_z = \tilde{H}^*(A^* \varphi) \), which gives \( \phi_z \in R(\tilde{H}^*) \).

On the other hand, let \( z \notin D \) and assume on the contrary that there exists \( \varphi = (\varphi_1, \varphi_2)^T \in \tilde{Y}' \) such that \( \tilde{H}^* \varphi = \phi_z \). Here, \( \tilde{Y}' \) denotes the adjoint of \( \tilde{Y} \). Then from Rellich’s Lemma and unique continuation theorem, it can be obtained that

\[
\int_{\partial D_1} \Phi(\cdot,y)\varphi_1(y)dy + \int_{D_2} \Phi(\cdot,y)\varphi_2(y)dy = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus (\overline{D} \cup \{z\}). \tag{2.19}
\]

However, this is a contradiction because the left-hand of (2.19) belongs to \( H^1(B_\delta(z)) \) but the right-hand of (2.19) does not belong to \( H^1(B_\delta(z)) \), where \( B_\delta(z) \) is chosen to be a sufficiently small ball centered at \( z \). This proves the theorem.

To proceed further we need to introduce the following interior transmission eigenvalue.

**Definition 2.7.** [Definition 4.7] \( k^2 \) is called an interior transmission eigenvalue if there exists \( (u,w) \in H_0^1(D) \times L^2(D) \) with \( (u,w) \neq (0,0) \) and a sequence \( w_j \in H^2(D) \) with \( w_j \to w \) in \( L^2(D) \) and \( \triangle w_j + k^2 w_j = 0 \) in \( D \) and \( (u,w) \) satisfies

\[
\int_D (\nabla u \cdot \nabla \varphi - k^2 n w \varphi) \, dx = k^2 \int_D (n-1) w \varphi \, dx \quad \text{for all} \quad \varphi \in H^1(D). \tag{2.20}
\]

It should be remarked that the eigenvalue problem (2.20) has been studied by Kirsch in [15] when \( n \) is real-valued and \( n(x) > 1 \) or \( n(x) < 1 \) in \( D \), where it was proved that (2.20) has at most a countable number of eigenvalues \( k^2 > 0 \). In the current paper, we always assume that \( k^2 > 0 \) is not an interior transmission eigenvalue under the case when \( \text{Re}[n_1(x)] - 1 \geq c \) in \( D_1 \) and \( \text{Re}[n_2(x)] - 1 < -c \) in \( D_2 \) for some positive constant \( c \).

In order to show the main theorem of the factorization method for our perturbation far-field pattern \( F_m \) that derived in (2.18), we need to prove the following theorem.

**Theorem 2.8.** Let \( \tilde{M}_m = M_m + M_{com} \) and assume that \( k^2 > 0 \) is neither an interior transmission eigenvalue in the sense of Definition 2.7 nor a Dirichlet eigenvalue of \(-\triangle\) in \( D_1 \). Then

(i) \( \text{Re} \tilde{M}_m = \tilde{M}_m^{(1)} + \tilde{M}_m^{(2)} \), where \( -\tilde{M}_m^{(1)} \) is coercive and \( \tilde{M}_m^{(2)} \) is a compact operator;

(ii) \( \text{Im} \tilde{M}_m \) is strictly positive on \( R(\tilde{H}) \), i.e., \( \text{Im} \langle \tilde{M}_m \varphi, \varphi \rangle > 0 \) for all \( \varphi \in R(\tilde{H}) \) with \( \varphi \neq 0 \).

**Proof.** (i) The assertion (i) follows easily from the properties of \( M_m \) and \( M_{com} \) (see the sentences under (2.18)).

(ii) Since \( N_{i,\partial D_1} \) is self-adjoint, we obtain \( \langle J_m \varphi, \varphi \rangle = 0 \) for all \( \varphi \in R(\tilde{H}) \) and

\[
\text{Im} \langle \tilde{M}_m \varphi, \varphi \rangle = \text{Im} \langle (A^* T^{-1} A + J_m) \varphi, \varphi \rangle = \text{Im} \langle T^{-1} A \varphi, A \varphi \rangle = \text{Im} \langle (T^{-1} A \varphi), (T^{-1} A \varphi) \rangle = \text{Im} \langle (T^{-1} A \varphi), (T^{-1} A \varphi) \rangle - \text{Im} \langle T (T^{-1} A \varphi), (T^{-1} A \varphi) \rangle
\]

9
for all \( \varphi \in \overline{R(H)} \). The fact \( \varphi \in \overline{R(H)} \) implies that there exists \( g_p \in L^2(\mathbb{S}^2) \) for \( p \in \mathbb{N} \) such that \( \tilde{H}g_p \to \varphi \) as \( p \to \infty \). With the aid of \( A\tilde{H} = H \), we derive \( T^{-1}\ast Hg_p \to T^{-1}\ast A\varphi \) as \( p \to \infty \). Then using \( H^* = GT \) that obtained in the proof of Theorem 2.3, we have \( T^{-1}\ast Hg_p = G^*g_p \). Therefore, the assertion (ii) is equivalent to

\[
\text{Im}(T\psi, \psi) < 0 \quad \text{for all} \quad \psi \in \overline{R(G^*)} \quad \text{with} \quad \psi \neq 0.
\]

Firstly, we prove that

\[
\text{Im}(T\psi, \psi) \leq 0 \quad \text{for all} \quad \psi \in \overline{R(G^*)} \quad \text{with} \quad \psi \neq 0. \tag{2.21}
\]

Define two functions \( w_1 \) and \( w_2 \) by

\[
w_1(x) := \int_{D_1} \Phi(x, y)\psi_1(y)dy, \quad \text{for} \quad x \in \mathbb{R}^3,
\]

\[
w_2(x) := \int_{D_2} \Phi(x, y)\psi_2(y)dy, \quad \text{for} \quad x \in \mathbb{R}^3
\]

and \( w := w_1 + w_2 \) with \( \psi = (\psi_1, \psi_2)^T \), we then obtain

\[
\langle T(\psi_1, \psi_2)^T, (\psi_1, \psi_2)^T \rangle_{Y \times Y'} = \langle (f_1, f_2)^T, (\psi_1, \psi_2)^T \rangle_{Y \times Y'}
\]

\[
= (q_1\psi_1, \psi_1)_D - (w_1, \psi_1)_D - (w_2, \psi_1)_D - (w_1, \psi_2)_D + (q_2\psi_2, \psi_2)_D - (w_2, \psi_2)_D
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{2.22}
\]

where \( (f_1, f_2)^T \in Y' \) is defined by (2.12) with \( (\varphi_1, \varphi_2)^T \) replaced by \( (\psi_1, \psi_2)^T \). In view of the fact that \( \text{Im}[n_1(x)] \geq 0 \) and \( \text{Im}[n_2(x)] \geq 0 \), we can easily observe that \( \text{Im}I_1 \leq 0 \) and \( \text{Im}I_5 \leq 0 \). By the definition of the function \( w_1 \) and the Green’s theorem we derive

\[
I_2 = \int_{D_1} w_1(\Delta w_1 + k^2 w_1)dx
\]

\[
= \int_{\partial D} \frac{\partial w_1}{\partial \nu} w_1 ds - \int_{D_1} (|\nabla w_1|^2 - k^2|w_1|^2) dx
\]

\[
= \int_{\partial B_R} \frac{\partial w_1}{\partial r} w_1 ds - \int_{B_R} (|\nabla w_1|^2 - k^2|w_1|^2) dx, \tag{2.23}
\]

where \( B_R \supset D \) is a ball centered at 0 with radius \( R \). It then follows from the Sommerfeld radiation condition that

\[
\text{Im}I_2 = \text{Im} \left( \lim_{R \to \infty} \int_{\partial B_R} \frac{\partial w_1}{\partial r} w_1 ds \right) = -\frac{k}{(4\pi)^2} \int_{\mathbb{S}^2} |w_{1,\infty}|^2 ds, \tag{2.24}
\]

where \( w_{1,\infty} \) is the far-field pattern of \( w_1 \). Similarly, we can show that

\[
\text{Im}I_6 = \text{Im} \left( \lim_{R \to \infty} \int_{\partial B_R} \frac{\partial w_2}{\partial r} w_2 ds \right) = -\frac{k}{(4\pi)^2} \int_{\mathbb{S}^2} |w_{2,\infty}|^2 ds \tag{2.25}
\]
as well as
\[
\text{Im}(I_3 + I_4) = \text{Im} \left[ \lim_{R \to \infty} \left( \int_{\partial B_R} \frac{\partial \bar{w}_2}{\partial r} w_1 - \frac{\partial \bar{w}_1}{\partial r} w_2 ds \right) \right]
\]
\[
= - \frac{2k}{(4\pi)^2} \text{Re} \left[ \int_{\mathbb{S}^2} w_{1,\infty} \bar{w}_{2,\infty} ds \right],
\]
where \(w_{2,\infty}\) is the far-field pattern of \(w_2\). Denote by \(w_\infty\) the far-field pattern of \(w\). Then the Cauchy-Schwarz inequality together with (2.22)-(2.26) yields
\[
\text{Im}(T(\varphi_1, \varphi_2)^T, (\varphi_1, \varphi_2)^T)_{Y \times Y} \leq - \frac{k}{(4\pi)^2} \|w_\infty\|^2_{L^2(\mathbb{S}^2)} \leq 0,
\]
which proves the assertion (2.24).

Secondly, let \(\psi^{(0)} = (\psi_1^{(0)}, \psi_2^{(0)})^T \in \overline{R(G^*)}\) such that \(\text{Im}(T\psi^{(0)}, \psi^{(0)}) = 0\). We define
\[
w^{(0)}(x) = \int_{D_1} \Phi(x, y)\psi_1^{(0)}(y) dy + \int_{D_2} \Phi(x, y)\psi_2^{(0)}(y) dy \]
and \(w^{(0)}\) to be the far-field pattern of \(w^{(0)}\). Then it follows from (2.27) that \(w^{(0)} = 0\) and thus \(w^{(0)} = 0\) in \(\mathbb{R}^3 \setminus \overline{\mathcal{T}}\) due to Rellich’s Lemma. Hence, using this and the fact that \(w^{(0)} \in H^2_{\text{loc}}(\mathbb{R}^3)\), we obtain \(w^{(0)}|_D \in H^1_0(D)\) and \(\partial w^{(0)}/\partial s = 0\) on \(\partial D\). Since \(\psi^{(0)} \in \overline{R(G^*)}\) there exists \(\psi_j \in G^* g_j\) such that \(\psi_j \to \psi^{(0)}\) in \(L^2(D)\) as \(j \to \infty\). Further it follows from (2.3) that
\[
(\tilde{q}_1\tilde{w}_j|_{D_1}, \tilde{q}_2\tilde{w}_j|_{D_2})^T \to \tilde{\psi}^{(0)} \quad \text{in} \quad L^2(D) \quad \text{as} \quad j \to \infty,
\]
where \(\tilde{q}_l = k^2(n_l - 1), l = 1, 2\) and \(\tilde{w}_j\) is the total field of the problem (1.1)-(1.2) corresponding to the incident field \(\tilde{w}_j^{(0)}(x) := \int_{\mathbb{S}^2} e^{-ikd}g_j(d) ds(d)\). We now define
\[
w^{(0)}(x) = \int_{D_1} \Phi(x, y)\tilde{q}_1(y)\tilde{w}_1(y) dy + \int_{D_2} \Phi(x, y)\tilde{q}_2(y)\tilde{w}_2(y) dy
\]
with \(\tilde{w}_i := \tilde{w}_j|_{D_i}, i = 1, 2\). Then we have \(w_j \to w^{(0)}\) in \(H^1(D)\) as \(j \to \infty\).

It is noted that \(\triangle w_j + k^2 w_j = -\tilde{q} \tilde{w}_j\) in \(D\) with \(\tilde{q} := \tilde{q}_l\) in \(D_l, l = 1, 2\). Then by the Green’s representation theorem, we derive
\[
\tilde{w}_j(x) = \int_{\partial D} \left\{ \frac{\partial \tilde{w}_j(y)}{\partial \nu(y)} \Phi(x, y) - \tilde{w}_j(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) + \int_D \Phi(x, y)\tilde{q}(y)\tilde{w}_j(y) dy \quad \text{in} \quad D.
\]
This in combination with the definition of \(w_j\) in (2.29) yields
\[
\tilde{w}_j(x) - w_j(x) = \int_{\partial D} \left\{ \frac{\partial \tilde{w}_j(y)}{\partial \nu(y)} \Phi(x, y) - \tilde{w}_j(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) = \int_{\mathbb{S}^2} e^{-ikd}g_j(d) ds(d) := v_j(x) \quad \text{in} \quad D.
\]
Then we conclude that
\[ v_j \to \frac{1}{q} \tilde{w}_j^{(0)} - w^{(0)} \quad \text{in} \quad L^2(D) \] (2.31)
due to the fact that \( \tilde{w}_j \to \frac{1}{q} \tilde{w}_j^{(0)} \) and \( w_j \to w^{(0)} \) in \( L^2(D) \). Setting \( v := \frac{1}{q} \tilde{w}_j^{(0)} - w^{(0)} \), then
\[ (w^{(0)}, v) \in H^1_0(D) \times L^2(D), \quad v_j \in H^2(D) \]
and \( w_j \to w^{(0)} \) in \( L^2(D) \). Setting \( v := \frac{1}{q} \tilde{w}_j^{(0)} - w^{(0)} \), then
\[ (w^{(0)}, v) \in H^1_0(D) \times L^2(D), \quad v_j \in H^2(D) \]
satisfying
\[ \Delta v_j + k^2 v_j = 0 \quad \text{in} \quad D, \quad v_j \to v \quad \text{in} \quad L^2(D) \]
and
\[ w_j \to w^{(0)} \quad \text{in} \quad L^2(D). \]
From this and the fact that \( \partial w^{(0)}/\partial \nu = 0 \) on \( \partial D \), it follows that \( (w^{(0)}, v) \) satisfy (2.20) with \( u \) and \( w \) replaced by \( w^{(0)} \) and \( v \), respectively. Since \( k^2 > 0 \) is not an interior transmission eigenvalue in the sense of Definition 2.7, it then follows that \( (w^{(0)}, v) \) has to vanish in \( D \). Thus \( \psi^{(0)} = 0 \) due to the fact that \( \Delta w^{(0)} + k^2 w^{(0)} = -\tilde{\psi}^{(0)} \) in \( D \). This, together with the assertion (2.21), proves the statement (ii) of this lemma.

Finally, the Range Identity in [15, Theorem 2.15] in combination with Theorems 2.3, 2.6 and 2.8 gives the following main theorem in this section.

**Theorem 2.9.** Assume that the conditions presented in Theorem 2.8 hold true. Then
\[ z \in D \iff \phi_z \in R(F_{m,\#}^\bullet) \]
\[ \iff W_m(z) := \left[ \sum_j \frac{|\langle \phi_z, \psi_j^{(m)} \rangle_{L^2(\Sigma)}|^2}{\lambda_j^{(m)}} \right]^{-1} > 0 \]
for every fixed \( m \in \mathbb{N} \), where \( \{\lambda_j^{(m)}; \psi_j^{(m)}\}_{j \in \mathbb{N}} \) is an eigen-system of the self-adjoint operator \( F_{m,\#} := |\text{Re} F_m| + |\text{Im} F_m| \).

**Remark 2.10.** Since the classical factorization method can not be directly applied to deal with our inverse problem associated with the complex refractive index. Then we instead construct a sequence of perturbed operators \( F_m \) by (2.16) of the far-field operator \( F \). It is shown in this section that \( F_m \) has a factorization satisfying the Range Identity in [15, Theorem 2.15] for every \( m \in \mathbb{N} \). Consequently, the support of the inhomogeneous medium \( D \) can be recovered from the spectral data of \( F_{m,\#} \) for every \( m \in \mathbb{N} \). We point out that, due to (2.17), if \( m_0 \) is sufficiently large then the exact operator \( F_{\#} \) can be regarded as a sufficiently small perturbation of \( F_{m_0,\#} \) and the noisy operator \( F_{\#}^\delta \) with the noise level \( \delta \) of \( F_{\#} \) can also be regarded as a sufficiently small perturbation of \( F_{m_0,\#}^\delta \) with the noise level \( \delta \). Based on the above discussions, in the numerical examples presented in the next section, we just use the spectral data of \( F \) and \( F_{\#}^\delta \) to numerically reconstruct the shape and location of \( D \).

**Remark 2.11.** Theorem 2.9 remains true for the two-dimensional case. The proof is similar with minor modifications.
Curve type | Parametrization:  
--- | ---  
Kite shaped | $x(t) = (\cos t + 0.65 \cos 2t - 0.65, 1.5 \sin t), \ t \in [0, 2\pi]$  
Rounded square | $x(t) = (1/2)(\cos^3 t + \cos t, \sin^2 t + \sin t), \ t \in [0, 2\pi]$  
Rounded triangle | $x(t) = (2 + 0.3 \cos 3t)(\cos t, \sin t), \ t \in [0, 2\pi]$  

| Table 1: Parametrization of the curve |

## 3 Numerical examples

In this section, numerical experiments in two dimensions are carried out to demonstrate the efficiency of the approximate factorization method. To generate the synthetic far-field data, we make use of the finite element method on a truncated domain enclosed by a PML layer with uniform meshes (see e.g. [6] for the PML technique). Further, the far-field data $u_\infty(\vec{x};d)$ are discretized for a finite number of observation directions $\hat{x}_r \in S^1$ and incident directions $d_s \in S^1$ with $r, s = 1, 2, \ldots, M$, which are equidistantly distributed on the unit circle $S^1$. Thus the measured data are obtained as the matrix $F_M = (u_\infty(\vec{x}_r;d_s))_{1 \leq r, s \leq M} \in \mathbb{C}^{M \times M}$. Then the indicator function $W(z)$ for the far-field operator $F$ is approximated as follows:

$$W_M(z) = \left[ \sum_{p=1}^{M} \frac{1}{\lambda_p} \left| \sum_{q=1}^{M} \phi_{p,q} \psi_{p,q} \right| \right]^{-1} \quad \text{for} \ z \in \mathbb{R}^2,$$

(3.1)

where $\{\phi_{z,q}\}_{q=1}^M$ is the discretization of the test function $\phi_z$ and $\{\lambda_p; \psi_p\}_{p=1}^M$ is the eigen-system of the self-adjoint matrix $F_{M,\#} := |\text{Re}(F_M)| + |\text{Im}(F_M)|$ with $\psi_p = (\psi_{p,q})_{q=1}^M$. From Theorem 2.9, it is expected that $W_M(z)$ is much bigger for $z \in D$ than that for $z \notin D$.

In each examples, we will also show the reconstructed results for the approximate factorization method from noisy data. For the noisy data, a complex-valued noise matrix $X$ is added to the data matrix $F_M$, where $X = (x_{rs})_{1 \leq r, s \leq M}$ with $x_{rs} = \xi_{rs} + i\zeta_{rs}$ and $\xi_{rs}, \zeta_{rs}$ are normally distributed random numbers in $[-1, 1]$. Then the perturbed matrix with noisy level $\delta > 0$ can simulated as follows:

$$F_M^\delta := F_M + \delta \frac{X}{\|X\|_2} \|F_M\|_2,$$

(3.2)

$$(F_M^\delta)_{\#} := |\text{Re}(F_M^\delta)| + |\text{Im}(F_M^\delta)|.$$  

(3.3)

Accordingly, the truncated indicator function $W_M(z)$ can be computed from the eigen-system of the perturbed matrix $(F_M^\delta)_{\#}$ which is similar as (3.1).

In the following examples, we set $M = 64$, $k = 5$ and the test curves for the boundary $\partial D$ are given in Table 1. The indicator function $W_N(z)$ is plotted against the sampling point $z \in \mathbb{R}^2$.

**Example 1.** In this example, we consider the case when $\partial D$ is a rounded triangle-shaped boundary and the refractive index $n$ in $D$ is given by

$$n(x) = \begin{cases} 2 + 2i & \text{for} \ x \in D_1, \\ 0.5 + 2i & \text{for} \ x \in D_2, \end{cases}$$

13
where $D_1 = \{(x_1, x_2) \in D : x_2 > 0\}$ and $D_2 = \{(x_1, x_2) \in D : x_2 < 0\}$. See Figure 1(a) for the physical configuration. The reconstruction results of the boundary $\partial D$ are presented in Figure 1 by using the far-field data without noise, with 5% noise and with 10% noise, respectively.

Example 2. In this example, we consider the case when $\partial D$ is a rounded square-shaped boundary and the refractive index $n$ in $D$ is given by

$$n(x) = \begin{cases} 2 + 2i & \text{for } x \in D_1, \\ 0.5 + 2i & \text{for } x \in D_2, \end{cases}$$

where $D_1 = \{(x_1, x_2) \in D : x_1 < 0\}$ and $D_2 = \{(x_1, x_2) \in D : x_1 > 0\}$. See Figure 2(a) for the physical configuration. The reconstruction results of the boundary $\partial D$ are presented in Figure 2 by using the far-field data without noise, with 5% noise and with 10% noise, respectively.

Example 3. In this example, we consider the case when $\partial D$ is a kite-shaped boundary and
Figure 2: Reconstruction of rounded square-shaped boundary $\partial D$. The refractive index is given by $n(x) = 2 + 2i$ in $D_1$ and $n(x) = 0.5 + 2i$ in $D_2$. 

(a) Physical configuration  
(b) $k=5$, no noise  
(c) $k=5$, 5% noise  
(d) $k=5$, 10% noise
the refractive index \( n \) in \( D \) is given by

\[
n(x) = \begin{cases} 
0.5 + 2i & \text{for } x \in D_1, \\
2 + 2i & \text{for } x \in D_2,
\end{cases}
\]

where \( D_1 = \{(x_1, x_2) \in D : x_2 > 0\} \) and \( D_2 = \{(x_1, x_2) \in D : x_2 < 0\} \). See Figure 3(a) for the physical configuration. The reconstruction results of the boundary \( \partial D \) are presented in Figure 3 by using the far-field data without noise, with 5% noise and with 10% noise, respectively.

![Figure 3: Reconstruction of kite-shaped boundary \( \partial D \). The refractive index is given by \( n(x) = 0.5 + 2i \) in \( D_1 \) and \( n(x) = 2 + 2i \) in \( D_2 \).](image)

From the above three examples and the other cases carried out but not presented here it can be seen that the shape and location of the obstacle \( D \) is numerically reconstructed from the spectral data of the far-field operator for the case of an inhomogeneous medium with complex refractive index. This indeed verifies the theoretical analysis of the approximate factorization method that presented in Section 2. In the future, motivated by this work, we hope to investigate
the exact factorization of the far-field operator. Moreover, we plan to extend our result to the case of electromagnetic scattering problems, which is more challenging.

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