KdV and NLS Equations as Tri-Hamiltonian Systems

J. C. Brunelli

and

Ashok Das

Department of Physics and Astronomy
University of Rochester
Rochester, NY 14627, USA

Abstract

We show that the KdV and the NLS equations are tri-Hamiltonian systems. We obtain the third Hamiltonian structure for these systems and prove Jacobi identity through the method of prolongation. The compatibility of the Hamiltonian structures is verified directly through prolongation as well as through the shifting of the variables. We comment on the properties of the recursion operator as well as the connection with the two boson hierarchy.
1. Introduction

Most integrable models in 0 + 1, 1 + 1 and 2 + 1 dimensions are known to be bi-
Hamiltonian systems [1-5]. These are systems whose dynamical equations can be described
through Hamilton’s equations with respect to two distinct Hamiltonian structures which
are also compatible [5], namely, any linear superposition of the two also defines a Hamil-
tonian structure. Since the Jacobi identity involves a nonlinear relation, compatibility
of Hamiltonian structures is a nontrivial statement. There is only one known 1 + 1 di-
mensional integrable system, namely, the two boson hierarchy or the equation describing
long water waves [6,7-12], which is even a tri-Hamiltonian system. Namely, the dynamical
equations for this system can be written in the Hamiltonian form with respect to three
distinct Hamiltonian structures which are compatible in the sense that any arbitrary, linear
superposition of three of them is also a Hamiltonian structure [13].

It is quite surprising that the two boson hierarchy is the only known integrable system
which is tri-Hamiltonian. This result is even more surprising considering the fact that
several other integrable systems can be embedded into this system [6,7,10,12] and yet they
are only bi-Hamiltonian. This motivated us to examine two of the most familiar integrable
systems – the KdV equation and the nonlinear Schrödinger (NLS) equation – in detail and
show that these systems are tri-Hamiltonian as well. The third Hamiltonian structures for
these systems are highly nontrivial and we use the method of prolongation [14] to verify the
Jacobi identity as well as the compatibility conditions. The paper is organized as follows.
In sec. 2, we derive the third Hamiltonian structure for the KdV equation, prove the Jacobi
identity and compatibility. In sec. 3, we construct the third Hamiltonian structure for the
NLS equations and show that it is a tri-Hamiltonian system as well. In sec. 4, we construct
the three Hamiltonian structures associated with the two boson hierarchy starting from
the NLS equation and present a brief conclusion in sec. 5. In the appendix, we compile a
list of formulae for prolongation which are useful in checking various identities.

2. KdV as a Tri-Hamiltonian System

It is well known that the KdV (Korteweg-de Vries) equation
\[
\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}
\]  
(2.1)
is a bi-Hamiltonian system \[5\]. Namely, let (at equal times)
\[
\{u(x), u(y)\}_1 = \mathcal{D}_1 \delta(x - y) = \frac{\partial}{\partial x} \delta(x - y)
\]
(2.2)
and
\[
H_3 = \int dx \left( \frac{1}{3!} u^3 - \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 \right)
\]
(2.3)
Then, it is easily verified that
\[
\frac{\partial u}{\partial t} = \{u(x), H_3\}_1 = u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}
\]
(2.4)
showing that the KdV equation is Hamiltonian. The anti-symmetry of the Hamiltonian structure, \(\mathcal{D}_1\), is obvious and Jacobi identity is trivially satisfied since this is a constant structure (independent of dynamical variables.). (In modern terminology, one would say that the relation (2.2) describes the U(1) current algebra with \(u(x)\) considered as a current.)

We also note that if we define
\[
\{u(x), u(y)\}_2 = \mathcal{D}_2 \delta(x - y) = \left( \frac{\partial^3}{\partial x^3} + \frac{1}{3} \left( \frac{\partial}{\partial x} u(x) + u(x) \frac{\partial}{\partial x} \right) \right) \delta(x - y)
\]
(2.5)
and
\[
H_2 = \int dx \frac{1}{2} u^2
\]
(2.6)
then, we can again write
\[
\frac{\partial u}{\partial t} = \{u(x), H_2\}_2 = u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}
\]
(2.7)
Namely, the KdV equation is also Hamiltonian with respect to a distinct, second Hamiltonian structure. That \(\mathcal{D}_2\) defines a Hamiltonian structure can be seen as follows. First, the antisymmetry of \(\mathcal{D}_2\) is obvious from the definition in Eq. (2.5). However, because the structure now depends on the dynamical variables, Jacobi identity is no longer automatic. On the other hand, we recognize Eq. (2.5) as defining the Virasoro algebra \[15\] (think of \(u(x)\) as the energy-momentum tensor) and, therefore, Jacobi identity must hold. Compatibility now follows from the simple observation that \(\mathcal{D}_2(u)\) is Hamiltonian for any field.
variable $u$ satisfying Eq. (2.5) and, therefore, $D_2(u + \frac{3}{2}\lambda)$ where $\lambda$ is an arbitrary constant, must also define a Hamiltonian structure. But, by definition (see Eq. (2.5))

$$D_2(u + \frac{3}{2}\lambda) = D_2(u) + \lambda D_1$$

(2.8)

and since an arbitrary linear combination of $D_1$ and $D_2$ defines a Hamiltonian structure, they are compatible. (This result on compatibility can be directly verified as well.)

Let us next note that we can define

$$\{u(x), u(y)\}_3 = D_3\delta(x - y) = \left(\partial^5 + \frac{1}{3}(\partial^3 u + \partial^2 u\partial + \partial u\partial^2 + u\partial^3) + \frac{1}{9}(\partial u^2 + u\partial u + u^2\partial + \partial u\partial^{-1}u\partial)\right)\delta(x - y)$$

(2.9)

with

$$\partial \equiv \frac{\partial}{\partial x}$$

(2.10)

and

$$H_1 = 3 \int dx \ u$$

(2.11)

to obtain

$$\frac{\partial u}{\partial t} = \{u(x), H_1\}_3 = u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

(2.12)

Thus, if we can show that $D_3$ in Eq. (2.9) has the necessary antisymmetry property and satisfies the Jacobi identity, then this would define a third Hamiltonian structure of the KdV equation.

The antisymmetry of $D_3$ is obvious from the definition in Eq. (2.9). Jacobi identity is normally easier to check by examining the closure of the corresponding symplectic form. However, we note that the structure of $D_3$ is highly nontrivial, making it extremely difficult to invert. Thus, we will check Jacobi identity for the Hamiltonian structure, $D_3$, directly, using the method of prolongation. We refer the interested reader to ref.14 (see chapter 7) for details on this method and simply note that in the infinite dimensional space labelled by $(u, u_x, u_{xx}, u_{xxx}, \ldots)$ if we define a bivector

$$\Theta_{D_3} = \frac{1}{2} \int dx \ \theta \wedge D_3 \theta$$

(2.13)
then $\mathcal{D}_3$ would satisfy the Jacobi identity provided

$$ \text{pr} v_{\mathcal{D}_3 \theta}(\Theta_{\mathcal{D}_3}) = 0 \quad (2.14) $$

Here the assumption is that

$$ \theta \neq \theta[u] \quad (2.15) $$

and by definition prolongation acts only on coefficients functionally dependent on $u$.

For the structure $\mathcal{D}_3$ in Eq. (2.9), we note that (The subscript $x$ denotes a derivative with respect to $x$.)

$$ \Theta_{\mathcal{D}_3} = \frac{1}{2} \int dx \left\{ \theta \wedge \theta_{xxxx} + \frac{2}{3} u \theta \wedge \theta_{xxx} - \frac{2}{3} u \theta_x \wedge \theta_{xx} \\
+ \frac{1}{3} u^2 \theta \wedge \theta_x - \frac{1}{9} u \theta_x \wedge (\partial^{-1} u \theta_x) \right\} \quad (2.16) $$

which leads to

$$ \text{pr} v_{\mathcal{D}_3 \theta}(\Theta_{\mathcal{D}_3}) = \frac{1}{2} \int dx \left\{ \frac{2}{3} \text{pr} v_{\mathcal{D}_3 \theta}(u) \wedge (\theta \wedge \theta_{xxx} - \theta_x \wedge \theta_{xx} + u \theta \wedge \theta_x) \\
- \frac{1}{9} \text{pr} v_{\mathcal{D}_3 \theta}(u) \wedge \theta_x \wedge (\partial^{-1} u \theta_x) + \frac{1}{9} u \theta_x \wedge (\partial^{-1} \text{pr} v_{\mathcal{D}_3 \theta}(u) \wedge \theta_x) \right\} \quad (2.17) $$

With

$$\text{pr} v_{\mathcal{D}_3 \theta}(u) = \theta_{xxxx} + \frac{1}{3} (u \theta)_{xxx} + \frac{1}{3} (u \theta_x)_{xx} + \frac{1}{3} u \theta_{xxx} + \frac{1}{3} (u \theta_x)_x + \frac{1}{3} u \theta_{xx} \\
+ \frac{1}{9} (u^2 \theta)_x + \frac{1}{9} u (u \theta)_x + \frac{1}{9} u^2 \theta_x + \frac{1}{9} (u (\partial^{-1} u \theta_x))_x \quad (2.18)$$

it is tedious but straightforward to show that

$$ \text{pr} v_{\mathcal{D}_3 \theta}(\Theta_{\mathcal{D}_3}) = 0 \quad (2.19) $$

This proves that $\mathcal{D}_3$ in Eq. (2.9) satisfies the Jacobi identity and, therefore, defines a third Hamiltonian structure for the KdV equation.

To prove that the three Hamiltonian structures are compatible, we define

$$ \mathcal{D} = \mathcal{D}_3 + \alpha \mathcal{D}_2 + \beta \mathcal{D}_1 \quad (2.20) $$
where $\alpha$ and $\beta$ are arbitrary, independent, constant parameters. By construction $\mathcal{D}$ is antisymmetric since the three Hamiltonian structures are. If we now construct the bivector

$$\Theta_{\mathcal{D}} = \frac{1}{2} \int dx \; \theta \wedge \mathcal{D} \theta = \frac{1}{2} \int dx \; \theta \wedge (\mathcal{D}_3 \theta + \alpha \mathcal{D}_2 \theta + \beta \mathcal{D}_1 \theta) \quad (2.21)$$

then, once again, it is straightforward to show that (We list the formulae for prolongation in the appendix.)

$$pr v_{\mathcal{D} \theta}(\Theta_{\mathcal{D}}) = 0 \quad (2.22)$$

This shows that $\mathcal{D}$ satisfies the Jacobi identity and consequently is a genuine Hamiltonian structure for arbitrary and independent $\alpha$ and $\beta$. Therefore, the three Hamiltonian structures of the KdV equation are compatible making it a tri-Hamiltonian system much like the two boson hierarchy [6].

We note here that the compatibility of the Hamiltonian structures can be seen alternately by shifting the dynamical variable as follows. Note that $\mathcal{D}_3(u)$ defines a Hamiltonian structure for any variable $u$ satisfying the Poisson bracket relation in Eq. (2.9). In particular, if we let

$$u \rightarrow u + \frac{3}{2} \lambda \quad (2.23)$$

where $\lambda$ is an arbitrary constant, $\mathcal{D}_3(u + \frac{3}{2} \lambda)$ defines a hamiltonian structure. On the other hand,

$$\mathcal{D}_3(u + \frac{3}{2} \lambda) = D_3(u) + 2\lambda D_2(u) + \lambda^2 D_1$$

We can identity $\alpha = 2\lambda$ and $\beta = \lambda^2$ and then Eq. (2.24) shows that a linear superposition of the three structures with arbitrary, independent parameters is a Hamiltonian structure leading to compatibility.

We end this section by noting that if we define a recursion operator as

$$R = \partial^2 + \frac{1}{3} u + \frac{1}{3} \partial u \partial^{-1} \quad (2.25)$$

then, it is easy to see that

$$\mathcal{D}_3 = R \mathcal{D}_2 = R^2 \mathcal{D}_1 \quad (2.26)$$

This leads to the vanishing of the Nijenhuis torsion tensor associated with $R$ which is a sufficient condition for integrability [16-19]. We note that since $R$ is a recursion operator
defined from two compatible Hamiltonian structures $D_1$ and $D_2$, the definition in Eq. (2.26) would imply that $D_3$ is Hamiltonian as well [5,18,19]. However, it is not a priori clear that $D_1$, $D_2$ and $D_3$ would define a tri-Hamiltonian system. But we also note that under

\[ u \rightarrow u + \frac{3}{2} \lambda \\
R \rightarrow R + \lambda \]

\[ D_3 \rightarrow (R + \lambda)^2 D_1 = D_3 + 2\lambda D_2 + \lambda^2 D_1 \]

leading once again to the compatibility of the three Hamiltonian structures.

3. NLS Equation as a Tri-Hamiltonian System

In this section let us consider the familiar $1 + 1$ dimensional system described by

\[ \begin{align*}
    i \frac{\partial q}{\partial t} &= -q_{xx} + 2k(q^* q)q \\
    i \frac{\partial q^*}{\partial t} &= q^*_{xx} - 2k(q^* q)q^*
\end{align*} \]

(3.1)

Here $k$ is an arbitrary parameter measuring the strength of the nonlinear interactions and can be set to unity through a rescaling of the dynamical variables $q$ and $q^*$.

The nonlinear Schrödinger equation is also well known to be a bi-Hamiltonian system [5]. Thus, for example, if we define

\[ Q = \begin{pmatrix} q \\ q^* \end{pmatrix} \]

(3.2)

with

\[ \{Q_\alpha(x), Q_\beta(y)\}_1 = (D_1)_{\alpha\beta} \delta(x - y) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \delta(x - y) \quad \alpha, \beta = 1, 2 \]

(3.3)

and

\[ H_3 = -\int dx \left( q_x^* q_x + k(q^* q)^2 \right) \]

(3.4)

then, we obtain

\[ \begin{align*}
    i \frac{\partial q}{\partial t} &= i\{q(x), H_3\}_1 = -q_{xx} + 2k(q^* q)q \\
    i \frac{\partial q^*}{\partial t} &= i\{q^*(x), H_3\}_1 = q^*_{xx} - 2k(q^* q)q^*
\end{align*} \]

(3.5)
This shows that the NLS equation is a Hamiltonian system since the structure $D_1$ in Eq. (3.3) is antisymmetric and satisfies the Jacobi identity (trivially).

We also note that we can define
\[
\{Q_\alpha(x), Q_\beta(y)\}_2 = (D_2)_{\alpha\beta} \delta(x - y)
\]
\[
= \begin{pmatrix}
  kq \partial^{-1} q & \frac{1}{2} \partial - kq \partial^{-1} q^*
  \frac{1}{2} \partial - kq^* \partial^{-1} q & kq^* \partial^{-1} q^*
\end{pmatrix} \delta(x - y)
\]
(3.6)

and
\[
H_2 = i \int dx \ (q^* q_x - q_x^* q)
\]
(3.7)

to obtain
\[
i \frac{\partial q}{\partial t} = i\{q(x), H_2\}_2 = -q_{xx} + 2k(q^* q)q
\]
\[
i \frac{\partial q^*}{\partial t} = i\{q^*(x), H_2\}_2 = q_{xx}^* - 2k(q^* q)q^*
\]
(3.8)
The second bracket structure in Eq. (3.6) is manifestly antisymmetric and is known to satisfy the Jacobi identity. This shows that the nonlinear Schrödinger equation is Hamiltonian with respect to two distinct Hamiltonian structures. Furthermore, these two Hamiltonian structures are known to be compatible making the nonlinear Schrödinger equation a bi-Hamiltonian system.

Let us next define
\[
\{Q_\alpha(x), Q_\beta(y)\}_3 = (D_3)_{\alpha\beta} \delta(x - y)
\]
(3.9)
\[
= -\frac{i}{2} \begin{pmatrix}
  k(\partial q \partial^{-1} q - q \partial^{-1} q \partial) & \frac{1}{2} \partial^2 - k(\partial q \partial^{-1} q^* + q \partial^{-1} q^* \partial)
  \frac{1}{2} \partial^2 - k(\partial q^* \partial^{-1} q + q^* \partial^{-1} q \partial) & -k(\partial q^* \partial^{-1} q^* - q^* \partial^{-1} q^* \partial)
\end{pmatrix} \delta(x - y)
\]
and
\[
H_1 = 4 \int dx \ q^* q
\]
(3.10)

which would give
\[
i \frac{\partial q}{\partial t} = i\{q(x), H_1\}_3 = -q_{xx} + 2k(q^* q)q
\]
\[
i \frac{\partial q^*}{\partial t} = i\{q^*(x), H_1\}_3 = q_{xx}^* - 2k(q^* q)q^*
\]
(3.11)
Therefore, if we can show that $D_3$ defines a Hamiltonian structure, we would have shown that the nonlinear Schrödinger equation is Hamiltonian with respect to three distinct Hamiltonian structures.

To show that $D_3$ is a Hamiltonian structure, we note from the definition in Eq. (3.9) that it is manifestly antisymmetric. The Jacobi identity can also be checked through the method of prolongation in the following way. We note that the dynamical variables in the present case define a two component vector and $D_3$ is a $2 \times 2$ matrix. Correspondingly, let us introduce

$$\theta = \begin{pmatrix} \theta \\ \theta^* \end{pmatrix}$$

and define a bivector as

$$\Theta_{D_3} = \frac{1}{2} \int dx \, \theta^t \wedge D_3 \theta$$

$$= \frac{i}{2} \int dx \left\{ -\frac{1}{2} \theta_{xx} \wedge \theta^* + k(q\theta - q^*\theta^*) \wedge (\partial^{-1}(q\theta_x + q^*\theta^*_x)) \right\}$$

(3.13)

Here $\theta^t$ denotes the transpose of $\theta$. Once again, variables $\theta$ and $\theta^*$ are assumed to be functionally independent of $q$ and $q^*$ and prolongation acts only on functionals of $q$ and $q^*$. Thus, we obtain

$$\text{pr}_v D_3 \theta (\Theta_{D_3}) = 0$$

(3.16)

This shows that the structure, $D_3$, defines a Hamiltonian structure.

To show compatibility of the three Hamiltonian structures, we define as before

$$D = D_3 + \alpha D_2 + \beta D_1$$

(3.17)
where $\alpha$ and $\beta$ are arbitrary, independent constants. By construction, $D$ is antisymmetric since each of the Hamiltonian structures $D_1$, $D_2$ and $D_3$ is. To check Jacobi identity, we again construct a bivector

$$\Theta_D = \frac{1}{2} \int dx \, \theta^t \wedge D \theta = \frac{1}{2} \int dx \left\{ \theta^t \wedge (D \theta + \alpha D_2 \theta + \beta D_1 \theta) \right\}$$  \hspace{1cm} (3.18)

It is, then, tedious but straightforward to check that

$$\mathfrak{pr_v}_D(\Theta_D) = 0$$  \hspace{1cm} (3.19)

This shows that the three Hamiltonian structures $D_1$, $D_2$ and $D_3$ are compatible making the nonlinear Schrödinger equation a tri-Hamiltonian system.

We end this section by noting that if we define a matrix recursion operator as

$$R = \begin{pmatrix}
-\frac{i}{2} \partial + ikq \partial^{-1} q^* & ikq \partial^{-1} q \\
-ikq^* \partial^{-1} q^* & \frac{i}{2} \partial - ikq^* \partial^{-1} q
\end{pmatrix}$$  \hspace{1cm} (3.20)

then, we can write

$$D_3 = R D_2 = R^2 D_1$$  \hspace{1cm} (3.21)

Once again, this would imply that the Nijenhuis torsion tensor associated with $R$ vanishes which is a sufficient condition for integrability [16-19]. Once again since $R$ is constructed from two compatible structures $D_1$ and $D_2$, it would also imply that $D_3$ is Hamiltonian. We also note that $R \rightarrow R + \lambda I$ would provide an alternate way of looking at the compatibility of these structures. However, we have not succeeded in finding a transformation of the dynamical variables which will generate this shift in the recursion operator.

4. Two Boson Hierarchy

It is known that the two boson hierarchy equation [6]

$$\frac{\partial u}{\partial t} = (2h + u^2 - u_x)_x$$
$$\frac{\partial h}{\partial t} = (2uh + h_x)_x$$  \hspace{1cm} (4.1)

yields the nonlinear Schrödinger equation (we will assume $k = 1$) with the field redefinitions [7-10]

$$u = -\frac{q_x}{q}$$
$$h = -q^* q$$  \hspace{1cm} (4.2)
and the coordinate scaling

\[ t \rightarrow it \quad (4.3) \]

The converse is also true, namely, we can obtain the two boson hierarchy from the nonlinear Schrödinger equation through an inverse field redefinition and coordinate transformation. In this section, we will show how we can obtain the three Hamiltonian structures of the two boson hierarchy starting from the structures of the nonlinear Schrödinger equation described in the previous section.

To that end, we define

\[ U = \begin{pmatrix} u \\ h \end{pmatrix} \quad (4.4) \]

and note that with the relations in Eq. (4.2) and the definition in Eq. (3.2), we can think of \( U \) as a functional of \( Q \), namely, \( U[Q] \). Let us next define (We use an abstract operator notation for simplicity. Coordinates can be brought in by taking appropriate matrix elements.)

\[ P_{\alpha\beta} = \frac{\delta U_\alpha}{\delta Q_\beta} = \begin{pmatrix} -\partial e^{(\partial^{-1}u)} & 0 \\ h e^{(\partial^{-1}u)} & -e^{-(\partial^{-1}u)} \end{pmatrix} \quad (4.5) \]

We also define the formal adjoint \[20\] of \( P \) as

\[ P_{\alpha\beta}^* = \begin{pmatrix} e^{(\partial^{-1}u)} \partial & h e^{(\partial^{-1}u)} \\ 0 & -e^{-(\partial^{-1}u)} \end{pmatrix} \quad (4.6) \]

If we now denote the \( 2 \times 2 \) matrix Hamiltonian structure of \( U_\alpha \) as

\[ \{U_\alpha, U_\beta\} = \tilde{D}_{\alpha\beta} \quad (4.7) \]

then, it is easy to see that

\[ \tilde{D} = PDP^* \quad (4.8) \]

where \( D \) is the corresponding Hamiltonian structure for the \( Q_\alpha \)'s.

With the three Hamiltonian structures for the nonlinear Schrödinger equation defined in Eqs. (3.3), (3.6) and (3.9) and the matrices \( P \) and \( P^* \) in Eqs. (4.5) and (4.6), it can now be easily checked that
\[ \tilde{D}_1 = P D_1 P^* = i \left( \begin{array}{cc} 0 & \partial \\ \partial & 0 \end{array} \right) = iD_1 \] (4.9)

\[ \tilde{D}_2 = P D_2 P^* = -\frac{1}{2} \left( \begin{array}{cc} 2\partial & \partial(u - \partial) \\ (\partial + u)\partial & (\partial h + h\partial) \end{array} \right) = -\frac{1}{2}D_2 \] (4.10)

\[ \tilde{D}_3 = P D_3 P^* \\
= -\frac{i}{2} \left( (\partial u + u\partial) (\partial h + h\partial) + \frac{1}{2}(\partial - u)^2 \right) \left( \frac{1}{2}(\partial + u)(\partial h + h\partial) + \frac{1}{2}(\partial h + h\partial)(u - \partial) \right) \]

\[ = -\frac{i}{4}D_3 \] (4.11)

where \( D_1, D_2 \) and \( D_3 \) are the three Hamiltonian structures for the two boson hierarchy as given in ref. 6 (The constant multiples can always be defined away by rescaling the corresponding Hamiltonians.) We note here that the compatibility of the Hamiltonian structures \( D_1, D_2 \) and \( D_3 \) can be easily seen by shifting \( u \rightarrow u + \lambda \) in the context of the two boson theory.

5. Conclusion

We have derived a third Hamiltonian structure for the KdV equation and have shown that all three structures are compatible making KdV a tri-Hamiltonian system. We have also derived a third Hamiltonian structure for the NLS equation and have shown that it is tri-Hamiltonian as well. The proof of Jacobi identity and the compatibility are carried out through the method of prolongation and we have commented on the properties of the recursion operators for these system. We have also shown how the three Hamiltonian structures for the two boson hierarchy can be obtained from those for the NLS equation. We speculate, based on our results, that integrable systems where the Hamiltonian description of the dynamical equations has not exhausted the lowest possible local Hamiltonian, are likely to be tri-Hamiltonian systems.

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Appendix

For completeness we list here all the prolongation formulae used in the text.

*Prolongation Formulae for KdV:*

For all structures $D$ we have

$$\text{pr}_v D\theta(u) = D\theta$$

The prolongations formulae for $D_1$ are

$$D_1\theta = \theta_x$$

$$\Theta_{D_1} = \frac{1}{2} \int dx \theta \wedge \theta_x$$

$$\text{pr}_v D_1\theta(\Theta_{D_1}) = 0 \text{ (trivially)}$$

For $D_2$ they are

$$D_2\theta = \theta_{xxx} + \frac{1}{3}(u\theta)_x + \frac{1}{3}u\theta_x$$

$$\Theta_{D_2} = \frac{1}{2} \int dx \left\{ \theta \wedge \theta_{xxx} + \frac{2}{3}u\theta \wedge \theta_x \right\}$$

$$\text{pr}_v D_2\theta(\Theta_{D_2}) = \frac{1}{3} \int dx \left\{ \text{pr}_v D_2\theta(u) \wedge \theta \wedge \theta_x \right\} = 0$$

And finally for $D_3$ we have

$$D_3\theta = \theta_{xxxx} + \frac{1}{3}(u\theta)_{xxx} + \frac{1}{3}(u\theta_x)_{xx} + \frac{1}{3}(u\theta_{xx})_x + \frac{1}{3}u\theta_{xxx}$$

$$+ \frac{1}{9}(u^2\theta)_x + \frac{1}{9}u(u\theta)_x + \frac{1}{9}u^2\theta_x + \frac{1}{9}(u(\partial^{-1}u\theta_x))_x$$

$$\Theta_{D_3} = \frac{1}{2} \int dx \left\{ \theta \wedge \theta_{xxxx} + \frac{2}{3}u\theta \wedge \theta_{xxx} - \frac{2}{3}u\theta_x \wedge \theta_{xx} \right.$$\n
$$\left. + \frac{1}{3}u^2\theta \wedge \theta_x - \frac{1}{9}u\theta_x \wedge (\partial^{-1}u\theta_x) \right\}$$

$$\text{pr}_v D_3\theta(\Theta_{D_3}) = \frac{1}{2} \int dx \left\{ \frac{2}{3} \text{pr}_v D_3\theta(u) \wedge (\theta \wedge \theta_{xxx} - \theta_x \wedge \theta_{xx} + u\theta \wedge \theta_x) \right.$$\n
$$- \frac{1}{9} \text{pr}_v D_3\theta(u) \wedge \theta_x \wedge (\partial^{-1}u\theta_x) + \frac{1}{9}u\theta_x \wedge (\partial^{-1} \text{pr}_v D_3\theta(u) \wedge \theta_x) \right\} = 0$$
For the compatibility of the structures, we have

\[ D = D_3 + \alpha D_2 + \beta D_1 \]
\[ D\theta = D_3\theta + \alpha D_2\theta + \beta D_1\theta \]

\[ \text{pr} \, v_{D\theta}(\Theta_D) = \text{pr} \, v_{D\theta}(\Theta_{D_3}) + \alpha \text{pr} \, v_{D\theta}(\Theta_{D_2}) + \beta \text{pr} \, v_{D\theta}(\Theta_{D_1}) = 0 \]

**Prolongation Formulae for NLS:**

In this case, \( \theta, Q \) and \( D\theta \) are two-component vectors and

\[ \text{pr} \, v_{D\theta}(Q) = D\theta \]

for any \( D \). \( \text{pr} \, v_{D\theta}(q) \) and \( \text{pr} \, v_{D\theta}(q^*) \), then, would correspond to the first and the second components of \( D\theta \).

The prolongations formulae for \( D_1 \) are

\[ D_1\theta = i \left( \begin{array}{c} \theta^* \\ -\theta \end{array} \right) \]
\[ \Theta_{D_1} = i \int dx \, \theta \wedge \theta^* \]
\[ \text{pr} \, v_{D_1\theta}(\Theta_{D_1}) = 0 \]

For \( D_2 \) they are

\[ D_2\theta = \left( \begin{array}{c} \frac{1}{2} \theta_x^* + \frac{k}{2} q \left( \partial^{-1} (q\theta - q^*\theta^*) \right) \\ \frac{1}{2} \theta_x - \frac{k}{2} q^* \left( \partial^{-1} (q\theta - q^*\theta^*) \right) \end{array} \right) \]
\[ \Theta_{D_2} = \frac{1}{2} \int dx \left\{ \theta \wedge \theta^* + \frac{k}{2} (q\theta - q^*\theta^*) \wedge (\partial^{-1} (q\theta - q^*\theta^*)) \right\} \]
\[ \text{pr} \, v_{D_2\theta}(\Theta_{D_2}) = \frac{k}{2} \int dx \left\{ (\text{pr} \, v_{D_2\theta}(q) \wedge \theta - \text{pr} \, v_{D_2\theta}(q^*) \wedge \theta^*) \wedge (\partial^{-1} (q\theta - q^*\theta^*)) \right\} = 0 \]
Finally, for $\mathcal{D}_3$ we have

$$\mathcal{D}_3 \theta = \begin{pmatrix}
-\frac{i}{4} \theta_{xx} + \frac{ik}{2} q \left( \partial^{-1}(q\theta_x + q^*\theta^*_x) \right) - \frac{ik}{2} \left( q \left( \partial^{-1}(q\theta - q^*\theta^*) \right) \right)_x \\
\frac{i}{4} \theta_{xx} - \frac{ik}{2} q^* \left( \partial^{-1}(q\theta_x + q^*\theta^*_x) \right) - \frac{ik}{2} \left( q^* \left( \partial^{-1}(q\theta - q^*\theta^*) \right) \right)_x
\end{pmatrix}$$

$$\Theta_{\mathcal{D}_3} = -\frac{i}{4} \int dx \left\{ \theta \wedge \theta_{xx} - 2k(q\theta - q^*\theta^*) \wedge \left( \partial^{-1}(q\theta_x + q^*\theta^*_x) \right) \right\}$$

$$\text{pr v}_{\mathcal{D}_3} \theta(\Theta_{\mathcal{D}_3}) = \frac{ik}{2} \int dx \left\{ (\text{pr v}_{\mathcal{D}_3} \theta(q) \wedge \theta - \text{pr v}_{\mathcal{D}_3} \theta(q^*) \wedge \theta^*) \wedge \left( \partial^{-1}(q\theta_x + q^*\theta^*_x) \right) \\
- (q\theta - q^*\theta^*) \wedge \left( \partial^{-1} \left( \text{pr v}_{\mathcal{D}_3} \theta(q) \wedge \theta_x + \text{pr v}_{\mathcal{D}_3} \theta(q^*) \wedge \theta^*_x \right) \right) \right\} = 0$$

For compatibility, we have

$$\mathcal{D} = \mathcal{D}_3 + \alpha \mathcal{D}_2 + \beta \mathcal{D}_1$$

$$\mathcal{D} \theta = \mathcal{D}_3 \theta + \alpha \mathcal{D}_2 \theta + \beta \mathcal{D}_1 \theta$$

$$\text{pr v}_\mathcal{D} \theta(\Theta_{\mathcal{D}}) = \text{pr v}_\mathcal{D} \theta(\Theta_{\mathcal{D}_3}) + \alpha \text{pr v}_\mathcal{D} \theta(\Theta_{\mathcal{D}_2}) + \beta \text{pr v}_\mathcal{D} \theta(\Theta_{\mathcal{D}_1}) = 0$$
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