EXISTENCE OF TRAVELING WAVEFRONT FOR DISCRETE BISTABLE COMPETITION MODEL

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Abstract. We study traveling wavefront solutions for a two-component competition system on a one-dimensional lattice. We combine the monotonic iteration method with a truncation to obtain the existence of the traveling wavefront solution.

1. Introduction. In this paper, we study the following two-component lattice dynamical system (LDS):

\begin{align*}
\frac{du_j}{dt} &= (u_{j+1} + u_{j-1} - 2u_j) + au_j(1 - u_j - kv_j), \\
\frac{dv_j}{dt} &= d(v_{j+1} + v_{j-1} - 2v_j) + bv_j(1 - v_j - hu_j),
\end{align*}

(1.1)

where \( u_j = u_j(t) \), \( v_j = v_j(t) \), \( t \in \mathbb{R} \), \( j \in \mathbb{Z} \), \( d > 0 \), \( h > 1 \), \( k > 1 \), \( b > 0 \) and \( a > 0 \). This model arises in the study of strong competition of two species in a habitat which is divided into discrete niches. Here the unknowns \( u_j, v_j \) are the populations of species \( u, v \) at niches \( j \), respectively, constants \( a, b \) are the birth rates and \( h, k \) are the competition coefficients of species \( u, v \). We assume that \( h, k > 1 \) so that these two species are of strong competition. By a renormalization, the carrying capacities are taken to be 1 for both species. Also, the diffusion (or migration) rates are taken to be 1 for \( u \) and \( d \) for \( v \).

Moreover, the system also can be regarded as a spatial discrete version of the following reaction-diffusion system:

\begin{align*}
\frac{du}{dt} = u_{xx} + au(1 - u - kv), \\
\frac{dv}{dt} = dv_{xx} + bv(1 - v - hu),
\end{align*}

(1.2)

where \( u = u(x,t) \), \( v = v(x,t) \), \( x \in \mathbb{R} \) and \( t \in \mathbb{R} \).

We are interested in the traveling fronts of (1.1) connecting \((0,1)\) and \((1,0)\), i.e., a solution of (1.1) in the form \( u_j(t) = U(j + ct) \) and \( v_j(t) = V(j + ct) \) for all \((j,t) \in \mathbb{Z} \times \mathbb{R} \) for some wave speed \( c \in \mathbb{R} \) and wave profile \((U,V)\) with the boundary condition

\begin{align*}
(u_j, v_j)(t) \to (0,1) \text{ as } j \to -\infty, \quad (u_j, v_j)(t) \to (1,0) \text{ as } j \to \infty,
\end{align*}

(1.3)

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for all \( t \in \mathbb{R} \). In the case when \( c \neq 0 \), the problem (1.1) and (1.3) is reduced to find 
\((U, V) \in C^1(\mathbb{R}) \times C^1(\mathbb{R})\) such that \( 0 \leq U, V \leq 1 \) on \( \mathbb{R} \),
\[
\begin{cases}
  cU' = D_2[U] + aU(1 - U - kV) & \text{in } \mathbb{R}, \\
  cV' = dD_2[V] + bV(1 - V - hU) & \text{in } \mathbb{R},
\end{cases}
\] (1.4)
with \((U, V)(-\infty) = (0, 1)\) and \((U, V)(+\infty) = (1, 0)\), where
\[
D_2[\phi](\xi) := \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi).
\]
When \( c = 0 \), it is reduced to find a pair of sequences \( \{(U(j), V(j))\}_{j \in \mathbb{Z}} \) such that
\[
D_2[U] + aU(1 - U - kV) = 0, \quad dD_2[V] + bV(1 - V - hU) = 0 \quad \forall j \in \mathbb{Z}.
\]
Note that the speed \( h \) is an unknown to be determined. To study the existence of traveling front, it is more convenient to work on \((U, W)\), where \( W := 1 - V \). We also let
\[
f(U, W) := aU[1 - U - k(1 - W)], \quad g(U, W) := b(1 - W)(hU - W).
\]
Then \( 0 \leq U, W \leq 1 \) on \( \mathbb{R} \) and (1.4) is equivalent to
\[
\begin{cases}
  cU' = D_2[U] + f(U, W) & \text{in } \mathbb{R}, \\
  cW' = dD_2[W] + g(U, W) & \text{in } \mathbb{R},
\end{cases}
\] (1.5)
with \((U, W)(-\infty) = (0, 0)\) and \((U, W)(+\infty) = (1, 1)\), where \( h, k > 1 \) and \( a, b, d > 0 \) are always assumed.

Note that as we discuss the relative work for the traveling front, we have four situations determined by the competition coefficient \( h \) and \( k \). As \( 0 < k < 1 < h \) (or \( 0 < h < 1 < k \)), the species \( u \) is superior than the species \( v \). We expect the species \( u \) would take over the territory of the species \( v \). Here we refer the work of Okubo, Maini, Williamson and Marry [10], Hosono [6, 7] and Kan-on [8] for the continuum problem (1.2). For the discrete version, we refer the reader the work of Guo and Wu [4]. As \( h > 1, k > 1 \), both equilibria \((0, 1)\) and \((1, 0)\) are stable and so we have the bistable nonlinearity. Then the chance of who can take whose territory depends on their initial data. The reader can see the work of Gardner [3] and Conley and Gardner [2] for the continuum case. As \( 0 < h, k < 1 \), we call this case as co-existence with weak competition which means the two species can be happy to live together. Please see the work of Tang and Fife [11] for more detail.

In this paper, we shall always assume that \( h, k > 1 \) and the aim is to study the existence of a traveling wavefront. Motivated by the work [1] considering a discrete periodic media for bistable dynamics, we extend the method from a single equation to a system. In the beginning, we transfer the system to an integral system and by choosing an appropriate integral factor, we have two monotonic operators. Using the iterated monotone method, we first obtain the existence of the monotone solution for the truncated problem. Extending to the whole domain, we have the theorem as follows.

**Theorem 1.1.** Fix \( 0 < a < 1 \). There exists a solution \((c, U, W)(\xi)\) of (1.5) such that either
\[
(0, 0) < (U, W)(y) \leq (U, W)(\xi) < (1, 1) \text{ if } y \leq \xi,
\] (1.6)
\[
\max\{U(0), W(0)\} = a \text{ if } c \leq 0; \quad \min\{U(0), W(0)\} = a \text{ if } c \geq 0.
\] (1.7)
or it is a semi-trivial solution with the following two alternatives:

(i) \( U \equiv 0, \ W(0) = a,\)
\[0 = W(-\infty) < W(y) \leq W(\xi) < W(\infty) = 1 \text{ for } y \leq \xi; \quad (1.8)\]

(ii) \( W \equiv 1, \ U(0) = a,\)
\[0 = U(-\infty) < U(y) \leq U(\xi) < U(\infty) = 1 \text{ for } y \leq \xi. \quad (1.9)\]

Moreover, \( c < 0 \) for the case (1.8) and \( c > 0 \) for the case (1.9).

Hereafter we denote that \((a_1, a_2) \leq (b_1, b_2)\) means \(a_1 \leq b_1\) and \(a_2 \leq b_2\). Also, \((a_1, a_2) < (b_1, b_2)\) means \(a_1 < b_1\) and \(a_2 < b_2\). The proof of Theorem 1.1 is based on the method of [1]. Applying this method to our system may produce the existence of semi-trivial solution as indicated in Theorem 1.1. The main difficulty here is due to the coupling of two equations so that there are more cases to be analyzed in the proof of Theorem 1.1. Moreover, unlike the situation of a single equation, we were unable to determine the sign of the speed. To the author’s knowledge, the question of determining the sign of the speed for PDE case (1.2) is still open. However the phenomenon of propagation failure occurs when we replace the diffusion coefficient of the first equation in (1.1) by \( \bar{d} > 0 \) and assume that both \( d \) and \( \bar{d} \) are sufficiently small. For this aspect, we refer the reader to [5] (and [9] for the case of a single equation).

When the former case in Theorem 1.1 occurs, we have the exact tail behavior \((U, W)(-\infty) = (0, 0)\) and \((U, W)(+\infty) = (1, 1)\) so that we have the existence of a traveling front for our problem. We state the theorem as follows.

**Theorem 1.2.** If \((U, W)\) is a nontrivial solution of (1.5) constructed as the first alternative in Theorem 1.1, then \((U, W)\) is monotone and satisfies \((U, W)(-\infty) = (0, 0) < (U, W)(\xi) < (1, 1) = (U, W)(\infty)\) for all \( \xi \in \mathbb{R} \).

The plan of this paper is as follows. In the next section, we give the proof of Theorem 1.1. Then we prove the existence theorem (Theorem 1.2) in section 3.

2. **Proof of Theorem 1.1.** As \( c \neq 0 \), given a constant \( \mu > 0 \) large enough, we define

\[
H_1(U, W)(\xi) = \mu U(\xi) + D_2[U](\xi) + f(U, W)(\xi),
\]

\[
H_2(U, W)(\xi) = \mu W(\xi) + dD_2[W](\xi) + g(U, W)(\xi).
\]

Then (1.5) is reduced to solving the following integral equations:

\[
U(\xi) = T_1(c, U, W)(\xi) := \int_{-\infty}^{0} e^{\mu s} H_1(U, W)(\xi + cs)ds
\]
\[= \int_{-c-\infty}^{\xi} \frac{1}{c} e^{\mu(y-\xi)/c} H_1(U, W)(y)dy,\]

\[
W(\xi) = T_2(c, U, W)(\xi) := \int_{-\infty}^{0} e^{\mu s} H_2(U, W)(\xi + cs)ds
\]
\[= \int_{-c-\infty}^{\xi} \frac{1}{c} e^{\mu(y-\xi)/c} H_2(U, W)(y)dy.\]
where \( \mu > 0 \) is chosen sufficiently large so that the above integrals are well-defined in \( \mathbb{R} \) and the following monotonic property holds, i.e.,
\[
0 \leq U_1(t) \leq U_2(t) \leq 1, \quad 0 \leq W_1(t) \leq W_2(t) \leq 1 \quad \text{in} \quad \mathbb{R}
\]
\[
\Rightarrow \quad T_1(c, U_1, W_1)(t) \leq T_1(c, U_2, W_2)(t), \quad T_2(c, U_1, W_1)(t) \leq T_2(c, U_2, W_2)(t) \quad \text{in} \quad \mathbb{R}.
\]
Following [1], for each \( n \in \mathbb{N} \), we consider the following truncated problem:
\[
cU' = D_2[U] + f(U, W) \quad \text{in} \quad (-n, n),
\]
\[
cW' = dD_2[W] + g(U, W) \quad \text{in} \quad (-n, n)
\]
with the exterior conditions:
\[
U(\xi) = W(\xi) = 1, \quad \forall \xi \in (n, +\infty),
\]
\[
U(\xi) = W(\xi) = 0, \quad \forall \xi \in (-\infty, -n).
\]
Since the solution of the above truncated problem is discontinuous at \( x = \pm n \), it is more convenient to consider the following system of integral equations
\[
U(\xi) = T_n^1(c, U, W)(\xi), \quad W(\xi) = T_n^2(c, U, W)(\xi),
\]
\[
0 \leq U(\xi), W(\xi) \leq 1 \quad \forall \xi \in \mathbb{R},
\]
where \( T_n^1(c, U, W)(\xi) := P_n T_1(c, U, W)(\xi) \) and \( T_n^2(c, U, W)(\xi) := P_n T_2(c, U, W)(\xi) \) with the project operator defined by
\[
P_n T(\xi) := \begin{cases} 
0 & \text{if } \xi < -n, \\
T(\xi) & \text{if } \xi \in [-n, n], \\
1 & \text{if } \xi > n.
\end{cases}
\]

**Remark 1.** When \( c = 0 \), the problem (1.5) with \( c = 0 \) is reduced to solving the following equations:
\[
U(\xi) = T_1(U, W) := \int_{-\infty}^{0} e^{\mu s} H_1(U, W)(\xi) \, ds,
\]
\[
W(\xi) = T_2(U, W) := \int_{-\infty}^{0} e^{\mu s} H_2(U, W)(\xi) \, ds.
\]
Moreover, the integral system associated with the truncated problem of (2.1)-(2.2) with \( c = 0 \) and (2.3)-(2.4) is the following integral equations:
\[
U(\xi) = T_n^1(U, W)(\xi), \quad W(\xi) = T_n^2(U, W)(\xi) \quad \forall \xi \in \mathbb{R},
\]
where \( T_n^1(U, W)(\xi) := P_n T_1(U, W)(\xi) \) and \( T_n^2(U, W)(\xi) := P_n T_2(U, W)(\xi) \).

We now prove the following existence result for the truncated problem.

**Lemma 2.1.** Let \( c \neq 0 \) be fixed. For each \( n \in \mathbb{N} \), there exists a unique function \((U^n, W^n)\) from \( \mathbb{R} \) to \([0, 1] \times [0, 1]\) that satisfies (2.5) and has the following properties:
\[
(1) \quad U^n(\cdot), W^n(\cdot) \in C^1((-n, n)).
\]
\[
(2) \quad (U^n)'(\cdot) > 0 \quad \text{and} \quad (W^n)'(\cdot) > 0 \quad \text{on} \quad (-n, n).
\]

**Proof.** We divide the proof into two parts.

1. **Existence.** Set
\[
(U^0_0, W^0_0)(\xi) = (0, 0), \quad (U^0_1, W^0_1)(\xi) = (1, 1),
\]
\[
(U^0_k, W^0_k)(\xi) = \{T^0_1(c, U^0_{k-1}, W^0_{k-1}))(\xi), T^0_2(c, U^0_{k-1}, W^0_{k-1}))(\xi)\},
\]
\[
(U^n_k, W^n_k)(\xi) = \{T^n_1(c, U^n_{k-1}, W^n_{k-1}))(\xi), T^n_2(c, U^n_{k-1}, W^n_{k-1}))(\xi)\}.
\]
From the monotonic property, we have
\[
(0, 0) \leq (U^n_{*k}, W^n_{*k})(\xi) \leq (U^n_{*k}, W^n_{*k})(\xi) \\
\leq (U^n_{k-1}, W^n_{k-1})(\xi) \leq (U^n_{k-1}, W^n_{k-1})(\xi) \leq (1, 1).
\]
It follows from the monotonicity property that there exist \((U^n_{*n}, W^n_{*n})(\xi)\) and \((U^n_{*n}, W^n_{*n})(\xi)\) such that
\[
\lim_{k \to \infty} (U^n_{*k}, W^n_{*k})(\xi) = (U^n_{*n}, W^n_{*n})(\xi), \\
\lim_{k \to \infty} (U^n_{k-1}, W^n_{k-1})(\xi) = (U^n_{k-1}, W^n_{k-1})(\xi).
\]
Using Lebesgue’s Dominated Convergence Theorem, it is easy to see that \((U^n_{*n}, W^n_{*n})(\xi)\) and \((U^n_{*n}, W^n_{*n})(\xi)\) are solutions of (2.5). It is also easy to see that they are in \(C^1(-n, n)\).

Notice that
\[
(0, 0) \leq (U^n_{*n}, W^n_{*n})(\xi) \leq (U^n_{*n}, W^n_{*n})(\xi) \leq (1, 1).
\]
Claim \(U^n_{*n}(\xi) < 1 \forall \xi \leq n\). Assume not. Then there is a \(\xi_1 < n\) such that \(U^n_{*n}(\xi_1) = 1\) and \(U^n_{*n}(\xi) < 1 \forall \xi < \xi_1\). Note that from the monotonicity property of \(H_i(U, W)(\xi), i = 1, 2, H_1(U^n_{*n}, W^n_{*n})(\xi) < H_1(1, 1)(\xi) = \mu \forall \xi < \xi_1\). Moreover, \(\forall \xi \in [\xi_1, \xi_1 + 1/2], H_1(U^n_{*n}, W^n_{*n})(\xi) = \mu + U^n_{*n}(\xi - 1) - 1 - ak(1 - W^n_{*n}(\xi)) < \mu\).
Hence \(H_1(U^n_{*n}, W^n_{*n})(\xi) < \mu \forall \xi < \xi_1 + 1/2\). Then
\[
1 = U^n_{*n}(\xi_1) = \int_{-\infty}^{0} e^{\mu^*}H_1(U^n_{*n}, W^n_{*n})(\xi_1 + cs) \, ds < \int_{-\infty}^{0} \mu e^{\mu^*} \, ds = 1.
\]
This is a contradiction and so \(U^n_{*n}(\xi) < 1 \forall \xi \leq n\). Similarly, \(W^n_{*n}(\xi) < 1 \forall \xi \leq n, \ U^n_{*n}(\xi) > 0 \forall \xi \geq -n\) and \(W^n_{*n}(\xi) > 0 \forall \xi \geq -n\). Hence we have for all \(\xi \in [-n, n]\)
\[
(0, 0) < (U^n_{*n}, W^n_{*n})(\xi) \leq (U^n_{*n}, W^n_{*n})(\xi) < (1, 1).
\]
Notice that the first derivatives of \(U^n_{*n}(\xi), U^n_{*n}(\xi), W^n_{*n}(\xi)\) and \(W^n_{*n}(\xi)\) for \(\xi \in (-n, n)\) are nonnegative. Claim that any one of the above four derivatives is positive for \(\xi \in (-n, n)\). Here we only show that \(U^n_{*n}(\xi) > 0\) for \(\xi \in (-n, n)\). The others can be proven similarly. Note that for any solution \((U, W)\) of (2.5), in distribution sense, \((U', W')(n) = (1 - U(n), 1 - W(n))\delta\) and \((U', W')(n) = (U(n), W(n))\), where \(\delta\) is the Dirac Mass. Then for any \(\xi \in (-n, n)\), differentiating the integral equation (2.5) and using the definition of \(\mu\),
\[
U^n_{*n}(\xi) \geq \int_{-\infty}^{0} e^{\mu^*}U^n_{*n}(\xi + cs) \, ds \begin{cases} \frac{U^n_{*n}(-n)e^{-\mu(n+c)}}{1 - U^n_{*n}(n)e^{-\mu(n-c)}} & \text{if } c > 0, \\ \frac{U^n_{*n}(n)e^{-\mu(n-c)}}{1 - U^n_{*n}(n)e^{-\mu(n-c)}} & \text{if } c < 0. \end{cases}
\]
Hence \(U^n_{*n}(\xi) > 0 \forall \xi \in (-n, n)\). Thus we have derived the property (2).

2. Uniqueness. Let
\[
h_1 := \inf \{h \mid U^n_{*n}(\xi + h) \geq U^n_{*n}(\xi) \forall \xi \in \mathbb{R}\}, \\
h_2 := \inf \{h \mid W^n_{*n}(\xi + h) \geq W^n_{*n}(\xi) \forall \xi \in \mathbb{R}\}.
\]
Then \(h_1, h_2 \in [0, 2n]\). We have four possibilities: (I) \(h_1 = 0 = h_2\), (II) \(h_1 > 0 = h_2\), (III) \(h_2 > 0 = h_1\), (IV) \(h_1 > 0, h_2 > 0\). To show the uniqueness, we need to show that the only possibility is (I). To exclude the other possibilities, we only consider the case (IV). The other cases can be treated similarly.
Suppose that $h_1 \geq h_2 > 0$. Note that, by the continuity of $U^n$, $U^{*n}$, $W^*_n$ and $W^{*n}$ on $\mathbb{R}\setminus\{-n,n\}$, we have
\[
0 \leq \limsup_{\xi \to h_1} \{T_1(c, U^n, W^n)(\xi + \xi_1) - T_1(c, U^{*n}, W^{*n})(\xi)\} = T_1(c, U^n, W^n)(\xi + h_1) - T_1(c, U^{*n}, W^{*n})(\xi),
\]
and so
\[
0 \leq \limsup_{\xi \to h_2} \{T_2(c, U^n, W^n)(\xi + \xi_2) - T_2(c, U^{*n}, W^{*n})(\xi)\} = T_2(c, U^n, W^n)(\xi + h_2) - T_2(c, U^{*n}, W^{*n})(\xi).
\]
Then, after the projection, $U^n(\xi + h_1) - U^{*n}(\xi) \geq 0$ and $W^*_n(\xi + h_2) - W^{*n}(\xi) \geq 0$ \forall $\xi \in \mathbb{R}$. Moreover, since $W^n(\xi + h_1) - W^{*n}(\xi) \geq W^*_n(\xi + h_2) - W^{*n}(\xi) \geq 0$ and from the construction of $\mu$, we have
\[
H_1(U^*_n, W^*_n)(\xi + h_1) - H_1(U^{*n}, W^{*n})(\xi) \geq U^*_n(\xi + h_1) - U^{*n}(\xi)
\]
for all $\xi \in \mathbb{R}$. Hence for all $\xi \in \mathbb{R}$,
\[
U^*_n(\xi + h_1) - U^{*n}(\xi) \geq \int_{-\infty}^{0} e^{\mu s}\{U^*_n(\xi + h_1 + cs) - U^{*n}(\xi + cs)\}ds.
\]
Notice that $U^*_n(\xi + h_1 + cs) - U^{*n}(\xi + cs) > 0$ \forall $\xi \in \mathbb{R}$ for $cs \in (-n - \xi - h_1, n - \xi) \cup [-n - \xi - h_1, -n - \xi]$. Thus $U^*_n(\xi + h_1) - U^{*n}(\xi) > 0$ \forall $\xi \in [-n, n - \frac{h_1}{2}]$. Hence there exists $\epsilon \in (0, \frac{h_1}{2})$ such that $U^*_n(\xi + h_1 - \epsilon) - U^{*n}(\xi) \geq 0$ \forall $\xi \in [-n, n - \frac{h_1}{2}]$. Moreover, $U^*_n(\xi + h_1 - \epsilon) - U^{*n}(\xi) = U^*_n(\xi + h_1 - \epsilon) \geq 0$ \forall $\xi < -n$ and $U^*_n(\xi + h_1 - \epsilon) - U^{*n}(\xi) = 1 - U^{*n}(\xi) \geq 0$ \forall $\xi > n - \frac{h_1}{2}$. Hence $U^*_n(\xi + h_1 - \epsilon) - U^{*n}(\xi) \geq 0$ \forall $\xi \in \mathbb{R}$. This contradicts with the definition of $h_1$ and so $h_1 = 0 = h_2$. Here we can set $U_1(\xi) := U^*_n(\xi) = U^{*n}(\xi)$ and $W_1(\xi) := W^*_n(\xi) = W^{*n}(\xi)$. The case of $h_2 \geq h_1 > 0$ is similar. Hence we have the uniqueness. The proof is completed.

\[\Box\]

**Remark 2.** When $c = 0$, (2.5) has a minimal solution ($U^n, W^n$) and a maximal solution ($U^{*n}, W^{*n}$). Notice that ($U^n, W^n$) and ($U^{*n}, W^{*n}$) are nondecreasing and are constant in $(b, b + 1)$ for each $b \in \mathbb{Z}$.

Next, we prove the following monotonicity in $c$ of solutions of (2.5).

**Lemma 2.2.** (i) Fix $n \in \mathbb{N}$. Let $c_1 < c_2$ and ($U^{n,c_1}, W^{n,c_1}$) be a solution of (2.5) with $c = c_i$, $i = 1, 2$, then $U^{n,c_2}(\xi) < U^{n,c_1}(\xi)$ \forall $\xi \in [-n, n]$.

(ii) Let ($U^n, W^n$) and ($U^{*n}, W^{*n}$) be the minimal solution and maximal solution to (2.5) with $c = 0$ respectively. Then for all $\xi \in \mathbb{R}\setminus\mathbb{Z}$,
\[
\lim_{c \to 0} (U^{n,c}, W^{n,c})(\xi) = (U^{*n}, W^{*n})(\xi),
\]
and
\[
\lim_{c \to 0} (U^{n,c}, W^{n,c})(\xi) = (U^*_n, W^*_n)(\xi).
\]

**Proof.** (i) Consider $0 \neq c_1 < c_2$. Then \forall $\xi \in [-n, n]$
\[
T_1(c_2, U^{n,c_1}, W^{n,c_1})(\xi) = \int_{-\infty}^{0} e^{\mu s}H_1(U^{n,c_1}, W^{n,c_1})(\xi + c_2 s)ds < \int_{-\infty}^{0} e^{\mu s}H_1(U^{n,c_1}, W^{n,c_1})(\xi + c_1 s)ds = U^{n,c_1}(\xi),
\]
and

\[\Box\]
and similarly, \( T_2^n(c_2, U^{n,c_1}, W^{n,c_1})(\xi) < W^{n,c_1}(\xi) \). Let \((U_0, W_0) = (U^{n,c_1}, W^{n,c_1})(\xi)\).
Set
\[
(U_{k+1}, W_{k+1})(\xi) = (T_1^n(c_2, U_k, W_k)(\xi), T_2^n(c_2, U_k, W_k)(\xi))
\]
for all \( \xi \in [-n, n] \). Then we have \((U_{k+1}, W_{k+1})(\xi) < (U_k, W_k)(\xi) \forall \xi \in [-n, n], \forall k\).
Hence there exist \( \bar{U} \) and \( \bar{W} \) such that \((U_k, W_k) \to (\bar{U}, \bar{W})\) and \((\bar{U}, \bar{W}) = (T_1^n(c_2, \bar{U}, \bar{W}), T_2^n(c_2, \bar{U}, \bar{W}))\) for all \( \xi \in [-n, n] \). When \( c_2 \neq 0 \), by the uniqueness as in Lemma 2.1, \((\bar{U}, \bar{W}) = (U^{n,c_2}, W^{n,c_2})\). Hence \((U^{n,c_2}, W^{n,c_2})(\xi) < (U^{n,c_1}, W^{n,c_1})(\xi) \forall \xi \in [-n, n] \). For the case \( c_2 = 0 \), using the same idea of the proof for the uniqueness of Lemma 2.1, we observe that \((U^{n,c}, W^{n,c}) \geq (U^{n,c}, W^{n,c})\) for any \( c < 0 \). Then we have \((U^{n,c_1}, W^{n,c_1})(\xi) > (U^{n,c}, W^{n,c})(\xi) \forall \xi \in [-n, n] \). The case for \( c_1 < c_2 \neq 0 \) is similar.

Now we prove the assertion (ii). Since \( U^{n,c} \) and \( W^{n,c} \) are monotonic in \( c \), passing to a subsequence if necessary, there exists \((\bar{U}, \bar{W}) = \lim_{c \to 0}(U^{n,c}, W^{n,c}).\) Note that \((\bar{U}, \bar{W}) \geq (U^{n,c}, W^{n,c})\). Since \((\bar{U}, \bar{W})(\xi)\) is nondecreasing, there is a countable set \( S \) such that \((\bar{U}, \bar{W})(\xi)\) is continuous in \( \mathbb{R} \setminus S \). Set \( S_0 := \{ \xi + i | \xi \in S, i = -1, 0, 1 \} \).
Then \( \forall \xi \in [-n, n] \setminus S_0 \),
\[
\tilde{U}(\xi) = \lim_{c \to 0} U^{n,c}(\xi) = \lim_{c \to 0} \int_{-\infty}^{0} e^{\mu s} H_1(U^{n,c}, W^{n,c})(\xi + cs)\,ds = \frac{1}{\mu} H_1(\bar{U}, \bar{W})(\xi),
\]
and similarly, \( \tilde{W}(\xi) = \frac{1}{\mu} H_2(\bar{U}, \bar{W})(\xi). \) Thus \( 0 = D_2[\bar{U}] + f(\bar{U}, \bar{W}) = dD_2[\bar{W}] + g(\bar{U}, \bar{W}) \forall \xi \in [-n, n] \setminus S_0 \).
Set \( (\bar{U}, \bar{W}) = \begin{cases} (\bar{U}, \bar{W}), & \forall \xi \in \mathbb{R} \setminus S_0, \\ (U^{n,c}, W^{n,c}), & \forall \xi \in S_0. \end{cases} \)
Then \((\bar{U}, \bar{W})\) is a solution of (2.5) with \( c = 0 \). Hence \((\bar{U}, \bar{W})(\xi) \leq (U^{n,c}, W^{n,c})(\xi) \forall \xi \in \mathbb{R} \setminus S_0 \). Therefore, we have \((\bar{U}, \bar{W})(\xi) = (U^{n,c}, W^{n,c})(\xi) \forall \xi \in \mathbb{R} \setminus S_0 \). Moreover, since \((\bar{U}, \bar{W})(\xi)\) is nondecreasing and \((U^{n,c}, W^{n,c})(\xi)\) is constant in \( (b, b + 1) \forall b \in \mathbb{Z} \),
\[
\lim_{c \to 0} (U^{n,c}, W^{n,c})(\xi) = (\bar{U}, \bar{W})(\xi) = (U^{n,c}, W^{n,c})(\xi)
\]
for all \( \xi \in \mathbb{R} \setminus \mathbb{Z} \). Using a similar proof, we have \( \lim_{c \to 0}(U^{n,c}, W^{n,c})(\xi) = (U^{n,c}, W^{n,c})(\xi) \).
The lemma follows.

Define
\[
\begin{align*}
\mathcal{N}_1[c, U, W](\xi) &= eU'(\xi) - D_2[U](\xi) - f(U, W)(\xi), \\
\mathcal{N}_2[c, U, W](\xi) &= eW'(\xi) - dD_2[W](\xi) - g(U, W)(\xi).
\end{align*}
\]
The following lemma gives some useful bounds of solutions for the later purpose.

**Lemma 2.3.** Let \( e^c = (1 + d)(e + e^{-1} + ak + bh) \), then
(i) For any \( c \geq e^c \), \((U^{n,c}, W^{n,c})(0) \leq (e^{-n}, e^{-n})\).
(ii) For any \( c \leq -e^c \), \((U^{n,c}, W^{n,c})(0) \geq (1 - e^{-n}, 1 - e^{-n})\).

**Proof.** We first prove the assertion (i). Let \( \bar{U}(\xi) = \bar{W}(\xi) = \min\{1, e^{x-n}\} \).
Then
\[
\mathcal{N}_i[c, \bar{U}, \bar{W}](\xi) \geq 0
\]
for any \( \xi > n, i = 1, 2 \), and for any \( \xi < n \)
\[
\begin{align*}
\mathcal{N}_1[c, \bar{U}, \bar{W}](\xi) &\geq e^{x-n} \{ c - [e + e^{-1} + a] \} \geq 0, \\
\mathcal{N}_2[c, \bar{U}, \bar{W}](\xi) &\geq e^{x-n} \{ c - [d(e + e^{-1}) + bh] \} \geq 0.
\end{align*}
\]
if \( c \geq c^* \). Hence for any \( c \geq c^* \), integrating \( e^{\xi N_i}[c, \overline{U}, \overline{W}](y) \geq 0 \) \( i = 1, 2 \) from \( -\infty \) to \( \xi \), we have \( (\overline{U}, \overline{W})(\xi) \geq (T_1(c, \overline{U}, \overline{W})(\xi), T_2(c, \overline{U}, \overline{W})(\xi)) \) for any \( \xi \in \mathbb{R} \).

After a projection, \( P_n(\overline{U}, \overline{W})(\xi) \geq (T_{n1}^*(c, \overline{U}, \overline{W})(\xi), T_{n2}^*(c, \overline{U}, \overline{W})(\xi)) \) \( \forall \xi \in \mathbb{R} \). Since \( \overline{U}(\xi) = \overline{W}(\xi) = 1 \) \( \forall \xi \notin n \) and \( (\overline{U}, \overline{W})(\xi) > (0, 0) \) \( \forall \xi \leq n \), \( (\overline{U}, \overline{W})(\xi) \geq P_n(\overline{U}, \overline{W})(\xi) \).

Thus for any \( c \geq c^* \),

\[
(T_{n1}^*(c, 0, 0)(\xi), T_{n2}^*(c, 0, 0)(\xi)) \leq (T_{n1}^*(c, \overline{U}, \overline{W})(\xi), T_{n2}^*(c, \overline{U}, \overline{W})(\xi)) \leq P_n(\overline{U}, \overline{W})(\xi) \leq (\overline{U}, \overline{W})(\xi).
\]

Following a similar proof of Lemma 2.1, \((U_{n,c}^*, W_{n,c}^*)(\xi) \leq (\overline{U}, \overline{W})(\xi) \) \( \forall \xi \in \mathbb{R} \) if \( c \geq c^* \). Hence \((U_{n,c}^*, W_{n,c}^*)(0) \leq (\overline{U}, \overline{W})(0) = (e^{-n}, e^{-n}) \) if \( c \geq c^* \).

The proof for the assertion (ii) is similar by considering

\[
\hat{U}(\xi) = \hat{W}(\xi) = \max(0, 1 - e^{-(\xi + n)}).
\]

Then the lemma follows.

With these properties of solutions to truncated problem, we first prove the existence and monotonicity property of solutions to (1.5).

**Proof of Theorem 1.** There are three possibilities.

(i) \( \lim \inf_{n \to \infty} (U_{n}^*, W_{n}^*)(0) < (a, a) \),

(ii) \( \lim \sup_{n \to \infty} (U_{n}^*, W_{n}^*)(0) > (a, a) \),

(iii) \( \lim \inf_{n \to \infty} \max\{U_{n}^*(0), W_{n}^*(0)\} > a > \lim \sup_{n \to \infty} \min\{U_{n}^*(0), W_{n}^*(0)\} \).

Consider the case (i). There there exists a sequence \( n_l \) with \( n_l \to \infty \) as \( l \to \infty \) such that \((U_{n_l,c}^*, W_{n_l,c}^*)(0) < (a + \frac{1}{l})(1, 1) \). Since \((U_{n,c}^*, W_{n,c}^*)(0) \geq (1 - e^{-n})(1, 1) \) for each \( l \), we can find a sequence \( c_l \in (-c^*, 0) \) such that \( \max\{U_{n_l,c_l}(0), W_{n_l,c_l}(0)\} = a + \frac{1}{l} \). Letting \( l \to \infty \) and passing to a subsequence if necessary, we have \( c := \lim_{l \to \infty} c_l \in [-c^*, 0] \) and \((U_{c}^*, W_{c}^*)(\xi) := \lim_{l \to \infty} (U_{n_l,c_l}^*, W_{n_l,c_l}^*)(\xi) \) with \( \max\{U_{c}^*(0), W_{c}^*(0)\} = a \). Then we can easily see that \((U_{c}^*, W_{c}^*)(\xi) \) is a solution of (1.5).

When both \( U_{c} \equiv 0 \) and \( W_{c} \equiv 0 \), (1.6) follows from the strong maximum principle. If \( U_{c} \equiv 0 \) (so that \( U_{c}^*(0) = 0 \)), then \( W_{c}^*(0) = a \) and \( W_{c} \) satisfies the equation

\[
{cW'} = dD_2[W] + b(1 - W)(-W).
\]

Hence (1.8) follows by applying the strong maximum principle to (2.6). Note that by the monotonicity of \( W_{c}^* \), we have \( W_{c}^*(\pm \infty) \) exist such that \( W_{c}^*(\infty) = 0 \) and \( W_{c}^*(0) = 1 \). Moreover, by integrating the equation (2.6) from \( -\infty \) and \( \infty \), we obtain that

\[
c = -b \int_{-\infty}^{\infty} W_{c}^*(\xi)[1 - W_{c}^*(\xi)]d\xi < 0.
\]

If \( W_{c} \equiv 0 \) (so that \( W_{c}^*(0) = 0 \)), then \( U_{c}^*(0) = a \) and \( U_{c} \) satisfies the equation

\[
cU' = D_2[U] + aU(1 - U - k).
\]

But, this is impossible. In fact, by monotonicity the limits \( U_{c}^*(\pm \infty) \) exist such that \( U_{c}^*(\pm \infty) \in \{0, 1 - k\} \) and \( U_{c}^*(\infty) \leq a \leq U_{c}^*(\infty) \). This leads to a contradiction, due to the assumption \( k > 1 \). This proves the theorem for the case (i).

By a similar discussion, case (ii) leads to the conclusion of the theorem for \( c \geq 0 \) and \( \min\{U_{c}^*(0), W_{c}^*(0)\} = a \).

Consider the case (iii). Define \( \alpha_n \) and \( \beta_n \) as

\[
\alpha_n := \sup\{\xi | (U_{n}^*, W_{n}^*)(\xi) \leq (a, a), \ \xi \in \mathbb{R}\},
\]

\[
\beta_n := \inf\{\xi | (U_{n}^*, W_{n}^*)(\xi) \geq (a, a), \ \xi \in \mathbb{R}\}.
\]
Note that $\alpha_n \in [-n, 0]$ and $\beta_n \in [0, n]$. Denote
\[
(U^n_1, W^n_1)(\xi) := (U^{*n}, W^{*n})(\xi + \alpha_n), \\
(U^n_2, W^n_2)(\xi) := (U^n_2, W^n_2)(\xi + \beta_n).
\]

Then we have
\[
(U^n_1, W^n_1)(\xi) = (1, 1) \forall \xi > -\alpha_n + n, \quad (U^n_2, W^n_1)(\xi) = (1, 1) \forall \xi > -\beta_n + n, \\
(U^n_1, W^n_2)(\xi) = (0, 0) \forall \xi < -\alpha_n - n, \quad (U^n_2, W^n_2)(\xi) = (0, 0) \forall \xi < -\beta_n - n, \\
(0, 0) < (U^n_1, W^n_1)(\xi) < (1, 1) \forall \xi \in [-\alpha_n - n, -\alpha_n + n], \\
(0, 0) < (U^n_2, W^n_2)(\xi) < (1, 1) \forall \xi \in [-\beta_n - n, -\beta_n + n].
\]

Moreover, notice that $-\alpha_n + n \to \infty$, $-\beta_n - n \to -\infty$ as $n \to \infty$ and there are three possibilities:

(I) $\limsup_{n \to \infty} (\alpha_n + n) = \infty$, 
(II) $\limsup_{n \to \infty} (-\beta_n + n) = \infty$, 
(III) $\limsup_{n \to \infty} (\alpha_n + n) < \infty$ and $\limsup_{n \to \infty} (-\beta_n + n) < \infty$.

For the case (I), there exists $n_l$ with $n_l \to \infty$ as $l \to \infty$ such that $\lim_{l \to \infty} (\alpha_{n_l} + n_l) = \infty$ and $(U, W)(\xi) := \lim_{l \to \infty} (U^{n_l}, W^{n_l})(\xi) \forall \xi \in \mathbb{R}$ where $(U^{n_l}, W^{n_l})$ is a subsequence of $(U^n_1, W^n_1)$. Note that $(U, W)$ is a solution of (1.5) with $c = 0$ satisfying (1.6) and (1.7).

For the case (II), the discussion is similar as case (I).

Consider the case (III). Since $(U^{*n}, W^{*n})$ and $(U^n, W^n)$ are constant in $(b, b + 1)$ \forall $b \in \mathbb{Z}$, $\alpha_n, \beta_n \in \mathbb{Z}$. Then there exists a sequence $n_l \in \mathbb{Z}$ with $n_l \to \infty$ as $l \to \infty$ and nonnegative integers $A$ and $B$ such that $\alpha_{n_l} + n_l = A$, $-\beta_{n_l} + n_l = B$, $(U^1, W^1)(\xi) := \lim_{l \to \infty} (U^{n_l}, W^{n_l})(\xi)$ and $(U^2, W^2)(\xi) := \lim_{l \to \infty} (U^{n_l-1}, W^{n_l-1})(\xi)$, where $(U^1, W^1)(\xi)$ satisfies (1.5) with $c = 0$ for $\xi \in [-A, \infty)$, and $(U^2, W^2)(\xi)$ satisfies (1.5) with $c = 0$ for $\xi \in (-\infty, B]$. Moreover,
\[
(U^1, W^1)(\xi) = (0, 0) \forall \xi < -A, \\
(U^1, W^1)(\xi) \leq (a, a) \forall \xi \leq 0, \\
\max\{U^1(\xi), W^1(\xi)\} \geq a \forall \xi \geq 0, \\
\min\{U^2(\xi), W^2(\xi)\} \leq a \forall \xi \leq 0.
\]

Notice that $\forall \xi < -A$,
\[
-D_2[U^1](\xi) - f(U^1, W^1)(\xi) = -U^1(\xi + 1) \leq 0, \\
-dD_2[W^1](\xi) - g(U^1, W^1)(\xi) = -dW^1(\xi + 1) \leq 0,
\]
and $\forall \xi > B$,
\[
-D_2[U^2](\xi) - f(U^2, W^2)(\xi) = 1 - U^2(\xi - 1) \geq 0, \\
-dD_2[W^2](\xi) - g(U^2, W^2)(\xi) = d(1 - W^2(\xi - 1)) \geq 0,
\]
hence $(U^1, W^1)$ is a subsolution and $(U^2, W^2)$ is a supersolution of (1.5) with $c = 0$. Moreover, $(U^1, W^1)(\xi) \leq (U^2, W^2)(\xi + A + B - 1) \forall \xi \in \mathbb{R}$, since $(U^1, W^1)(\xi) = (0, 0) \forall \xi < -A$ and $(U^2, W^2)(\xi + A + B + 1) = (1, 1) \forall \xi > -A + 1$. Set $(U_0, W_0) := (U^1, W^1)$ and $(U_k, W_k) := (T_1(U_{k-1}, W_{k-1}), T_2(U_{k-1}, W_{k-1}))$. Then
\[
(U^1, W^1)(\xi) \leq (U_1, W_1)(\xi) \leq \cdots \leq (U_{k-1}, W_{k-1})(\xi) \leq (U_k, W_k)(\xi) \leq (U^2, W^2)(\xi + A + B + 1), \forall \xi \in \mathbb{R}.
\]
Hence we have \((U, W) := \lim_{k \to \infty} (U_k, W_k)\) as a solution of (1.5) with \(c = 0\). Moreover, by a strong comparison,
\[
(U^1, W^1)(\xi) < (U, W)(\xi) < (U^2, W^2)(\xi + A + B + 1).
\]
Then we have \(\max\{U(\xi), W(\xi)\} \geq a \forall \xi \geq 0\) and \(\min\{U(\xi), W(\xi)\} \leq a \forall \xi \leq -A - B - 1\) and so after a translation, \((U, W)(\xi)\) satisfies (1.6) and (1.7).

3. **Existence of a traveling wavefront.** We shall prove Theorem 1.2 in this section. It remains to show the tail behavior for a nontrivial solution \((U, W)\) at \pm \infty. We see that \((U, W)(-\infty)\) and \((U, W)(\infty)\) are stationary points of (1.5). Hence \((0, 0), (0, 1), (U^0, W^0)\) and \((1, 1)\) are the only four candidates for \((U, W)(-\infty)\) and \((U, W)(\infty)\). Here \((U^0, W^0) = (\frac{k-1}{hk-1}, \frac{h(k-1)}{hk-1})\) is a solution of \(1 - U - k(1 - W) = 0\) and \(hU - W = 0\). Moreover, from the above theorem, \((U, W)(-\infty) < (1, 1), (U, W)(\infty) > (0, 0)\). Thus \((U, W)(-\infty) = (0, 0)\) or \((U^0, W^0)\) and \((U, W)(\infty) = (1, 1)\) or \((U^0, W^0)\). Here we want to show that \((U, W)(-\infty) = (0, 0)\) and \((U, W)(\infty) = (1, 1)\).

To proceed further, we denote
\[
\alpha = \min\{\frac{(k-1)U^0}{4(k+1)}, \frac{1-W^0}{4}, \frac{1-U^0}{4}, \frac{W^0}{4}\},
\]
\[
\xi_{1,n,c} = \min\{\xi \mid (U_{n,c}^{n,c}, W^{n,c})(\xi) \geq (U^0 - \alpha, W^0 - \alpha)\},
\]
\[
\xi_{2,n,c} = \max\{\xi \mid (U_{n,c}^{n,c}, W^{n,c})(\xi) \leq (U^0 + \alpha, W^0 + \alpha)\},
\]
\[
\gamma = \limsup_{n \to \infty} \sup_{c \in [-c^*, c^*]} (\xi_{2,n,c}^{n,c} - \xi_{1,n,c}^{n,c}).
\]

**Lemma 3.1.** \(\gamma < \infty\).

**Proof.** Assume \(\gamma = \infty\). Then there exist \(n_l\) and \(c_l\) with \(n_l \to \infty, c_l \in [-c^*, 0) \cup (0, c^*]\) such that \(\lim_{l \to \infty} (\xi_{2,n_l,c_l}^{n_l,c_l} - \xi_{1,n_l,c_l}^{n_l,c_l}) = \infty\). Passing to a subsequence if necessary, we have
\[
c := \lim_{l \to \infty} c_l,
\]
\[
(U_1, W_1)(\xi) := \lim_{l \to \infty} (U_{n_l,c_l}^{n_l,c_l}, W^{n_l,c_l})(\xi + \xi_{1,n_l,c_l}^{n_l,c_l}),
\]
\[
(U_2, W_2)(\xi) := \lim_{l \to \infty} (U_{n_l,c_l}^{n_l,c_l}, W^{n_l,c_l})(\xi + \xi_{2,n_l,c_l}^{n_l,c_l}),
\]
\[
A := \lim_{l \to \infty} (n_l + \xi_{1,n_l,c_l}^{n_l,c_l}) \in (0, \infty],
\]
\[
B := \lim_{l \to \infty} (n_l - \xi_{2,n_l,c_l}^{n_l,c_l}) \in (0, \infty].
\]
Note that from the definition of \(\xi_{1,n,c}^{n,c}\) and \(\xi_{2,n,c}^{n,c}\), and the assumption of \(\gamma = \infty\),
\[
(U^0 - \alpha, W^0 - \alpha) \leq (U_1, W_1)(\xi) \leq (U^0 + \alpha, W^0 + \alpha) \forall \xi \geq 0,
\]
\[
(U^0 - \alpha, W^0 - \alpha) \leq (U_2, W_2)(\xi) \leq (U^0 + \alpha, W^0 + \alpha) \forall \xi \leq 0,
\]
\[
(U_1, W_1)(\xi) = (0, 0) \forall \xi < -A, \quad (U_2, W_2)(\xi) = (1, 1) \forall \xi > B,
\]
\[
(U_1, W_1)(\xi) = (T_1(c, U_1, W_1)(\xi), T_2(c, U_1, W_1)(\xi)) \forall \xi \geq -A
\]
\[
(U_2, W_2)(\xi) = (T_1(c, U_2, W_2)(\xi), T_2(c, U_2, W_2)(\xi)) \forall \xi \leq B.
\]
Then there exist a positive constant \(\beta\) such that
\[
U_1(\xi + 1) - U_1(\xi) \geq 2\beta, \quad W_1(\xi + 1) - W_1(\xi) \geq 2\beta \forall \xi \in [0, 1];
\]
\[
U_2(\xi + 1) - U_2(\xi) \geq 2\beta, \quad W_2(\xi + 1) - W_2(\xi) \geq 2\beta \forall \xi \in [-2, -1].
\]
Hence we can find $n_l$ and $c_l$ for $l$ large enough, writing $(n_l, c_l)$ simply as $(n, c)$, such that $\forall \xi \in [\xi_1^{n,c}, \xi_1^{n,c}+1) \cup [\xi_2^{n,c} - 2, \xi_2^{n,c} - 1]$,

$$U^{n,c}(\xi + 1) - U^{n,c}(\xi) \geq \beta \quad \text{and} \quad W^{n,c}(\xi + 1) - W^{n,c}(\xi) \geq \beta.$$ 

Set $Z_1(\xi) := U^{n,c}(\xi + 1) - U^{n,c}(\xi)$ and $Z_2(\xi) := W^{n,c}(\xi + 1) - W^{n,c}(\xi)$. As $c \neq 0$, $U^{n,c}(\xi)$ and $W^{n,c}(\xi)$ are continuous on $[\xi_1^{n,c}, \xi_2^{n,c}]$, thus we can find $\eta := \min\{Z_i(\xi) \mid \xi \in [\xi_1^{n,c}, \xi_2^{n,c} - 1] \ i = 1, 2\}$. We say $\eta = Z_i(\xi_0) > 0$, for some $i = 1, 2$ and $\xi_0 \in [\xi_1^{n,c}, \xi_2^{n,c} - 1]$. Here we assume $i = 1$ and the case for $i = 2$ is similar.

Then there are two cases for $\xi_0$.

(i) $\xi_0 \in (\xi_1^{n,c} + 1, \xi_2^{n,c} - 2)$.

(ii) $\xi_0 \in [\xi_1^{n,c}, \xi_2^{n,c} - 1] \cup [\xi_2^{n,c} - 2, \xi_2^{n,c} - 1]$.

For the case (i), $\xi_0$ is a local minimum point of $Z_1(\xi)$ in $(\xi_1^{n,c}, \xi_2^{n,c} - 1)$. Hence

$$0 = cZ_1'(\xi_0) = c\{(U^{n,c})'(\xi_0 + 1) - (U^{n,c})'(\xi_0)\} \geq f(U^{n,c}(\xi_0 + 1), W^{n,c}(\xi_0 + 1)) - f(U^{n,c}(\xi_0), W^{n,c}(\xi_0)) = \frac{\partial f}{\partial U}(U_\theta, W_\theta)(U^{n,c}(\xi_0 + 1) - U^{n,c}(\xi_0)) + \frac{\partial f}{\partial W}(U_\theta, W_\theta)(W^{n,c}(\xi_0 + 1) - W^{n,c}(\xi_0)) \geq a\eta(1 - 2U_\theta - k(1 - W_\theta) + kU_\theta) \geq a\eta(1 - 2(U_0 + \alpha) - k[1 - (W_0 - \alpha)] + k(U_0 - \alpha)) = a\eta(k - 1)U_0 - 2\alpha k(1 + k) > 0,$$ 

here we use the Mean Value Theorem for two variables and $U_\theta \in (U^{n,c}(\xi_0), U^{n,c}(\xi_0 + 1))$ and $W_\theta \in (W^{n,c}(\xi_0), W^{n,c}(\xi_0 + 1))$. This leads to a contradiction. For the case (ii), then $\eta \geq \beta$. Hence for $K$ the biggest integer which is less than $\xi_2^{n,c} - \xi_1^{n,c} - 1$,

$$\beta \leq \frac{\sum_{k=0}^{K}(U^{n,c}(\xi_1^{n,c} + k + 1) - U^{n,c}(\xi_1^{n,c} + k))}{K}.$$

Then we have $K\beta \leq U^{n,c}(\xi_2^{n,c} - 1) - U^{n,c}(\xi_1^{n,c}) \leq 2\alpha$ and so $K \leq \frac{2\alpha}{\beta}$. This is a contradiction with $\gamma = \infty$. Hence $\gamma < \infty$. \hfill \Box

Finally, we prove the existence theorem for solutions of (1.5) with boundary conditions at $\pm\infty$.

**Proof of Theorem 2.** Let $(U, W)(\xi)$ be a nontrivial solution constructed from Theorem 1.1, that is, there is a sequence of $(U^{n_l,c_l}, W^{n_l,c_l})$ converging to $(U, W)$. Notice that we have known that $(0, 0) < (U, W)(\xi) < (1, 1)$ for all $\xi \in \mathbb{R}$, so here we only need to show that $(U, W)(-\infty) = (0, 0)$ and $(U, W)(\infty) = (1, 1)$. Suppose that $(U, W)(\infty) \neq (1, 1)$. Then $(U, W)(\infty) = (U_0, W_0)$. Hence there exists $\xi_0 \gg 1$ such that

$$(U, W)(\xi) \geq (U_0 - \alpha/2, W_0 - \alpha/2)$$

for all $\xi \geq \xi_0$. Moreover, since $(U^{n_l,c_l}, W^{n_l,c_l})(\xi) \to (U, W)(\xi)$ as $l \to \infty$ for all $\xi \in \mathbb{R}$, there exists $l_0 \gg 1$ such that $(U^{n_l,c_l}, W^{n_l,c_l})(\xi_0) \geq (U(\xi_0) - \alpha/2, W(\xi_0) - \alpha/2) \geq (U_0 - \alpha, W_0 - \alpha)$ for all $l \gg l_0$. Thus, by the monotonicity of $(U^{n_l,c_l}, W^{n_l,c_l})$,

$$(U^{n_l,c_l}, W^{n_l,c_l})(\xi) \geq (U^{n_l,c_l}, W^{n_l,c_l})(\xi_0) \geq (U_0 - \alpha, W_0 - \alpha)$$

for all $\xi \geq \xi_0$ and $l \geq l_0$. Notice that $\xi_{l_0}^{n_l,c_l} \leq \xi_0$ for all $l \geq l_0$. Meanwhile,

$$(U^{n_l,c_l}, W^{n_l,c_l})(\xi) \leq (U^{0} + \alpha/2, W^{0} + \alpha/2)$$

for all $\xi \leq -\xi_0$ and $l \geq l_0$. Notice that $\xi_{l_0}^{-n_l,c_l} \leq -\xi_0$ for all $l \geq l_0$. Meanwhile,

$$(U^{n_l,c_l}, W^{n_l,c_l})(\xi) \geq (U^{0} - \alpha/2, W^{0} - \alpha/2)$$

for all $\xi \leq -\xi_0$ and $l \geq l_0$. Notice that $\xi_{l_0}^{-n_l,c_l} \geq -\xi_0$ for all $l \geq l_0$. Meanwhile,
for all $\xi \in \mathbb{R}$ and for $l$ large enough. Hence $\xi_2^{n_l, c_l} = \infty$ for $l$ large. Then $\xi_2^{n_l, c_l} - \xi_1^{n_l, c_l} = \infty$ for $l \gg 1$. That leads to a contradiction with $\gamma < \infty$. Therefore, $(U, W)(\infty) = (1, 1)$. The proof for $(U, W)(-\infty) = (0, 0)$ is similar. \hfill \Box

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