Twisted IBL$_\infty$-algebra and string topology:
First look and examples

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Abstract
Cieliebak & Fukaya & Latschev proposed to twist the canonical IBL-
structure on cyclic cochains of $H_{dR}(M)$ for a closed oriented manifold $M$
with a Maurer-Cartan element $n$ built up from Chern-Simons like integrals
associated to trivalent ribbon graphs. They conjectured that this construc-
tion gives a chain model for Chas-Sullivan string topology. In this text, we
assume that the integrals converge and explicitly compute the case of $S^n$,
supporting the conjecture. We generalize this computation and show that
the twist with $n$ is often trivial.

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1 Introduction and summary

An \(\text{IBL}_\infty\)-algebra is essentially a collection of multilinear operations \(q_{klg}\) with \(k\) inputs, \(l\) outputs and “genus” \(g\) satisfying certain relations; in particular, \(q_{110}\) is a boundary operator, and the pair \(q_{210}, q_{120}\) induces the structure of an involutive Lie bialgebra on the homology of \(q_{110}\). It was introduced in [10] and applications to string topology, symplectic field theory and higher genus Lagrangian Floer theory were proposed.

This text is an attempt to understand the application to string topology. The idea was to carry out some explicit computations according to the plan sketched in [10, Section 13] and test the string topology conjecture (see below).

The following results from [10, Corollary 11.9] are our starting point (precise definitions of all the notions will be given in Section 2; our \(\text{IBL}_\infty\)-algebras will be strict and filtered in the terminology of [10]):

(A) For a finite-dimensional cyclic cochain complex \((V, P, m_1)\) of degree \(2 - n\), there is a canonical \(d\text{IBL}\)-structure \(p_{110}, p_{210}, p_{120}\) of bidegree \((n - 3, 2)\) on the degree shifted dual cyclic bar complex

\[
C(V) := B^*_\text{cyc} V[2 - n] \simeq \bigoplus_{k \geq 1} (V[1]^\otimes k/\text{cyc})', [2 - n],
\]

where cyc stands for cyclic permutations with the Koszul sign, ’ denotes the graded dual and [·] the degree shift. This structure is denoted by \(d\text{IBL}(C(V))\).

(B) Let \((\mathcal{H}, P, m_1) \subset (V, P, m_1)\) be a subcomplex such that the restriction of \(P\) to \(\mathcal{H}[1]\) is non-degenerate. We apply (A) to \((\mathcal{H}, P, m_1)\) to get the canonical \(d\text{IBL}\)-algebra \(d\text{IBL}(C(\mathcal{H})) = (C(\mathcal{H}), q_{110}, q_{210}, q_{120})\). Suppose that \(\pi : V[1] \rightarrow V[1]\) is a projection to \(\mathcal{H}[1]\) which satisfies

\[
\pi \circ m_1 = m_1 \circ \pi \quad \text{and} \quad P(\pi(v_1), v_2) = P(v_1, \pi(v_2))
\]

for all \(v_1, v_2 \in V[1]\), and let \(\iota : \mathcal{H}[1] \rightarrow V[1]\) be the inclusion. A linear map \(G : V[1] \rightarrow V[1]\) of degree \(-1\) such that

\[
m_1 \circ G + G \circ m_1 = \iota \circ \pi - \mathbb{1}_{V[1]} \quad \text{and} \quad P(G(v_1), v_2) = (-1)^{|v_1|} P(v_1, G(v_2))
\]

for all \(v_1, v_2 \in V[1]\) induces the \(\text{IBL}_\infty\)-homotopy equivalence

\[
f = (f_{klg}) : d\text{IBL}(C(V)) \rightarrow d\text{IBL}(C(\mathcal{H}))
\]

such that \(f_{110} : C(V)[1] \rightarrow C(\mathcal{H})[1]\) is the map given by the precomposition
with $i$ in every component. We recall from [10] that $f_{klg}: E_k C(V) \to E_l C(H)$ is a linear map between exterior powers.

The map $f_{klg}$ is constructed as a sum of contributions coming from isomorphism classes of ribbon graphs ($\equiv$ multigraphs with a cyclic ordering of half-edges at every internal vertex) with $k$ internal vertices, $l$ boundary components and genus $g$. To compute the contribution of a labeled ribbon graph $\Gamma$ to the value

$$f_{klg}(\Psi_1 \otimes \cdots \otimes \Psi_k)(W_1 \otimes \cdots \otimes W_l)$$

for $\Psi_1, \ldots, \Psi_k \in B^c \Gamma V[3-n]$ and $W_1, \ldots, W_l \in B^c \Gamma H[3-n]$, we decorate the $i$-th internal vertex of $\Gamma$ with $\Psi_i$, external vertices lying on the $i$-th boundary component with components $v_{i1}, \ldots, v_{im} \in V[1]$ of $W_i = s(v_{i1} \otimes \cdots \otimes v_{im}/cyc)$, where $s$ is a formal symbol of degree $n-3$, and internal edges with the Schwartz kernel $G$ of $\mathcal{G}$ with respect to $\mathcal{P}$. Decorated ribbon graphs are then evaluated in a consistent way to obtain real numbers (see Appendix A for an invariant kernel $G$ component with components $\Psi_i$ for the value $n$, $\Gamma$, genus $g$). For details see [10, Proposition 12.5 and Theorem 12.9] for a coordinate version of this construction.

We will also use the following results from [10, Section 10] about deformations of IBL$_\infty$-algebras:

(C) If in addition to (A) there is the product $m_2: V[1] \otimes V[1] \to V[1]$ making $(V, m_1, m_2)$ into a cyclic dga, then $(-1)^{n-2} m_2^+$ defines a canonical Maurer-Cartan element $m := (m_{10})$ for dIBL$(C(V))$. The twisted IBL$_\infty$-algebra is again a dIBL-algebra of bidegree $(n-3, 2)$; it is denoted by dIBL$^m(C(V))$ and satisfies

$$dIBL^m(C(V)) = (C(V), \mathbf{p}_1^{m} = \mathbf{p}_{110} + \mathbf{p}_{210} \circ_1 \mathbf{m}_{10}, \mathbf{p}_2^{m} = \mathbf{p}_{210}, \mathbf{p}_3^{m} = \mathbf{p}_{120} = \mathbf{p}_{120}).$$

(D) The IBL$_\infty$-morphism $f$ from (B) can be used to pushforward $m$ and obtain the Maurer-Cartan element $n = (n_{lg})$ for dIBL$(C(H))$. The twist by $n$ is an IBL$_\infty$-algebra of bidegree $(n-3, 2)$; it is denoted by dIBL$^n(C(H))$ and satisfies

$$dIBL^n(C(H)) = (C(H), q_{11}^{n} = q_{110} + q_{210} \circ_1 n_{10}, q_{210}^{n} = q_{210}, q_{120}^{n} = q_{120} + q_{210} \circ_1 n_{20}, \text{ plus the higher operations } q_{110}^n = q_{210} \circ_1 n_{10}).$$

This IBL$_\infty$-algebra is IBL$_\infty$-homotopy equivalent to dIBL$^m(C(V))$ via the twisted IBL$_\infty$-morphism

$$f^m = (f_{klg}^m): dIBL^m(C(V)) \to dIBL^n(C(H)).$$

The pushforward Maurer-Cartan element $n = f_\ast m$ can be expressed as a
sum of contributions of isomorphism classes of trivalent ribbon graphs \( m_2^+ \) has namely three inputs), where a labeled ribbon graph \( \Gamma \) is decorated with \( m_2^+ \) at internal vertices, with the components of the \( i \)-th argument of \( n_g \), i.e., elements of \( \mathcal{H}(V)[1] \), at the \( i \)-th boundary component and with \( G \) at internal edges. Note that whereas (A) – (C) can be formulated without completions, infinite sums appear in \( n_{lg} \), and hence filtration and completions necessarily come into play.

The application to string topology of an oriented closed manifold \( M \) of dimension \( n \) comes from studying generalizations of (A) – (D) to the infinite-dimensional cyclic dga \((\Omega^*(\mathcal{M}), \mathcal{P}, m_1, m_2)\). Here \( \Omega^*(\mathcal{M}) \) is the de Rham complex of \( \mathcal{M} \) and the maps \( \mathcal{P} : \Omega(\mathcal{M})[1]^{\otimes 2} \to \mathbb{R} \), \( m_1 : \Omega(\mathcal{M})[1] \to \Omega(\mathcal{M})[1] \) and \( m_2 : \Omega(\mathcal{M})[1]^{\otimes 2} \to \Omega(\mathcal{M})[1] \) are defined for all \( \eta, \eta_1, \eta_2 \in \Omega(\mathcal{M}) \) as follows:

\[
\begin{align*}
\mathcal{P}(\theta \eta_1, \theta \eta_2) &:= (-1)^{\eta_1} \int_{\mathcal{M}} \eta_1 \wedge \eta_2, \\
m_1(\theta \eta) &:= \theta d \eta, \\
m_2(\theta \eta_1, \theta \eta_2) &:= (-1)^{\eta} \theta(\eta_1 \wedge \eta_2),
\end{align*}
\]

where \( d \) is the de Rham differential, \( \wedge \) the wedge product, \( \theta \) a formal symbol of degree \(-1\) and \( \eta_1 \) in the exponent denotes the form-degree of \( \eta_1 \). By picking a Riemannian metric on \( \mathcal{M} \), we obtain the subcomplex of harmonic forms

\((\mathcal{H}^*(\mathcal{M}), \mathcal{P}, m_1 \equiv 0)\)

with the projection \( \pi_H : \Omega(M) \to \mathcal{H}(\mathcal{M}) \) coming from the Hodge decomposition. This cyclic cochain complex shall be taken as the subcomplex in (B).

From technical reasons stemming from the fact that the non-degenerate pairing \( \mathcal{P} \) on \( \Omega(\mathcal{M})[1] \) is not perfect, one has to restrict the construction in (A) to the subspace \( \mathcal{B}^{\text{cyc}}_{\text{cyc}} \Omega(\mathcal{M})_\infty \) of elements with a smooth Schwartz kernel. Then (A) and (B) work in the setting of the so called Fréchet IBL\( \infty \)-algebras introduced in [10, Section 13]. However, the element \( m_{10} \in \mathcal{B}^{\text{cyc}}_{\text{cyc}} \Omega(\mathcal{M})[3-n] \), which translates into the Chern-Simons term

\[
m^+_2(\theta \eta_1, \theta \eta_2, \theta \eta_3) := (-1)^{\eta} \int_{\mathcal{M}} \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{for all } \eta_1, \eta_2, \eta_3 \in \Omega(\mathcal{M}),
\]

does not define the canonical Maurer-Cartan element \( m \) in (C) directly because \( m^+_2 \notin \hat{\mathcal{B}}^{\text{cyc}}_{\text{cyc}} \Omega(\mathcal{M})_\infty \). This also means that one cannot use (D) to conclude the existence of the pushforward Maurer-Cartan element \( n \).

Nevertheless, it was proposed to define \( n \) formally using the summation over trivalent ribbon graphs as in the finite-dimensional case. We call such \( n \) a formal pushforward Maurer-Cartan element. In order to compute the contribution of a labeled trivalent ribbon graph \( \Gamma \) with \( k \) internal vertices, \( l \) boundary components
and genus $g$ to the value

$$n_g(\Omega_1 \otimes \cdots \otimes \Omega_l),$$

where $\Omega_i = s_\omega_i$ for $\omega_1, \ldots, \omega_l \in B_{\text{cyc}}^{\omega_i}H(M)$, one starts by decorating internal vertices with integration variables $x_1, \ldots, x_k$ on the $k$-fold product $M \times \cdots \times M$, external vertices on the $i$-th boundary component with the components $\alpha_{i1}, \ldots, \alpha_{is_i} \in H(M)[1]$ of $\omega_i$ and internal edges with the Green kernel $G$. In this setting, $G$ becomes the Schwartz kernel of $G$ in the sense of pseudo-differential operators; this $G$ is necessarily singular at the diagonal $\Delta$, so that we have only $G \in \Omega^{n-1}(M \times M \setminus \Delta)$. One then takes the wedge product of all forms in the decorated graph in the order and with the sign deduced from the labeling of $\Gamma$ and computes the integral over $x_1, \ldots, x_k$. Similar integrals appear in perturbative Chern-Simons quantum field theory.

Because of the singularity of $G$ at $\Delta$, the integrand described above is smooth only on the $k$-th configuration space of $M$. It is not clear that all the integrals converge and that the resulting $n_g$ are well-defined and satisfy the Maurer-Cartan equation. The idea of work in progress [12] of K. Cieliebak and E. Volkov is to use iterated spherical blow-ups of the diagonals to resolve the singularities and obtain integrals of smooth forms on compact manifolds with corners; this guarantees integrability. The Maurer-Cartan equation for $n = (n_g)$ is then proven with the help of Stokes’ formula and by showing that the contributions of hidden codimension-1 faces cancel. This method is similar to the method from [2] and [3], where Feynman integrals of perturbative Chern-Simons theory were considered.

Having $n$, the twisted IBL$_\infty$-algebra $d\text{IBL}^n(C(H(M)))$, which can be equivalently written as $d\text{IBL}^n(C(H_{\text{dR}}(M)))$ using the Hodge isomorphism $H(M) \simeq H_{\text{dR}}(M)$, should satisfy the following conjecture:

**String topology conjecture** (Conjecture 1.12 in [10]). Let $M$ be a closed oriented manifold of dimension $n$ and $H_{\text{dR}}(M)$ its de Rham cohomology. Then there exists an IBL$_\infty$-structure on (a suitable version of) $B_{\text{cyc}}^{\omega_i}H_{\text{dR}}[2 - n]$ whose homology equals the cyclic cohomology of the de Rham complex of $M$.

The idea is that the $S^1$-equivariant homology of the free loop space $H^*_\omega(LM)$ is isomorphic to a version of Connes’ cyclic cohomology of the de Rham algebra $H^*_\omega(\Omega^*(M))$, at least for simply-connected $M$. The precise relation will be established in yet another work in progress [12] of K. Cieliebak and E. Volkov using a chain-map coming from a cyclic version of Chen’s iterated integrals. Now, a suitable degree shift of $H^*_\omega(\Omega^*(M))$ is isomorphic to the homology of the boundary operator $\partial^{\Omega}_{10}$ of the only formally defined IBL-algebra $d\text{IBL}^n(C(\Omega(M)))$, which is according to (D) (formally) quasi-isomorphic to $d\text{IBL}^n(C(H(M)))$ via the twisted IBL$_\infty$-morphism $f^m$.

The space $H^*_\omega(LM)$ is equipped with an IBL-structure coming from the Chas-
Sullivan string bracket $m_2$ and string cobracket $c_2$; these operations were defined geometrically on suitably transverse smooth chains in [9] and [8], respectively. The natural question is: How is the IBL-structure $m_2$, $c_2$ related to the IBL-structure $q_{210}^n$, $q_{120}^n$ induced on $H_*^C(LM)$ via the isomorphism from the string topology conjecture? The extended string topology conjecture asserts that these structures agree, and hence the operations $q_{210}^n$, $q_{120}^n$ defined on cyclic cochains provide a chain model for $m_2$, $c_2$. Based on our observations and explicit computations, we formulate an up-to-date version of the string topology conjecture for simply-connected manifolds (see Conjecture 3.33).

A large part of this text consists of setting up the algebraic base for the work with $dIBL^n(C(H(M)))$. In addition to repeating the theory from [10] in a slightly different formalism, we also include the following topics:

- A formula for the partial composition $\circ_s$ in terms of operations of the canonical associative bialgebra on the symmetric algebra (Definition 2.15); formulas for $q_{5l19}^n$ (Proposition 2.45).
- Definition of the cyclic cohomology of $A_\infty$-algebras (Definition 2.34) and its relation to the homology of $q_{5110}^n$ (Proposition 2.47); definitions of the reduced versions (Definitions 2.37, 2.48 and 2.49) and their relation to the unreduced versions (Propositions 2.51 and 2.38).
- An invariant formulation of the evaluation of labeled ribbon graphs (Definition A.1 and Proposition A.2); formal analogy of the finite-dimensional and the de Rham case which we use to obtain signs for the definition of $n$ (Proposition A.6).
- Definition of the Green kernel (Definition 3.5) and of the formal pushforward Maurer-Cartan element $n$ (Definition 3.19).

Our first result is an explicit computation of $dIBL^n(C(H_{dR}(S^n)))$ by finding a particular Green kernel and showing that all integrals which contribute to $n$ vanish for $n \geq 3$; for $n = 1$, there is a non-vanishing integral whose value we compute (see Section 4.1); for $n = 2$, the existence of a non-vanishing integral remains open.

**Theorem A** (Explicit computation for $S^n$). Consider the round sphere $S^n \subset \mathbb{R}^{n+1}$. Define $1 := \theta 1$, $v := \theta \text{Vol} \in H_{dR}(S^n)[1]$, where Vol is the volume form, $1$ the constant one and $\theta$ a formal symbol of degree $-1$. The following holds for

\footnote{In fact, $\sigma_3$ is geometrically defined only on the homology relative to constant loops and $m_2$ does not always restrict to it.}
the homology of the twisted boundary operator $q^n_{110}$:

$$
\mathbb{H}^n(C(H_{dR}(S^n)))[1] := H(\hat{\Phi}_{\text{cyc}} H_{dR}(S^n)[3-n], q^n_{110})
=$$

$$
= \begin{cases} 
\langle sv^{2\ast}, s_1^{2j-1\ast} | i, j \geq 1 \rangle & \text{for } n \geq 3 \text{ odd}, \\
\langle sv^{2i-1\ast}, s_1^{2j-1\ast} | i, j \geq 1 \rangle & \text{for } n \text{ even}, \\
\langle s \sum_{k=1}^{\infty} c_k v^{k\ast}, s_1^{2j-1\ast} | c_k \in \mathbb{R}, j \geq 1 \rangle & \text{for } n = 1.
\end{cases}
$$

Here $\langle \rangle$ denotes the linear span over $\mathbb{R}$, $\ast$ the dual and $s$ is a formal symbol of degree $n - 3$. The product $q^n_{210}$ vanishes on $\mathbb{H}^n$ except for the following relations for $n \geq 3$ odd

$$q^n_{210}(s_1^\ast \otimes sv^{k\ast}) = q^n_{210}(sv^{k\ast} \otimes s_1^\ast) = -(k-1)v^{k-1\ast}
$$

and the following relations for $n = 1$:

$$q^n_{210}(s_1^\ast \otimes s \sum_{k=1}^{\infty} c_k v^{k\ast}) = -s \sum_{k=1}^{\infty} kc_{k+1} v^{k\ast}.
$$

The coproduct $q^n_{120}$ as well as all higher operations $q^n_{110}$ vanish on $\mathbb{H}^n$ in every dimension $n$. For $S^1$, we have $q^n_{120} \neq q^n_{120}$ on the chain level; i.e., the twisting is non-trivial. For $n \neq 2$, all higher operations vanish on the chain level.

If we mod out $s_1^{2j-1\ast}$, i.e., if we consider the point-reduced version, then, after dropping $s$, the results agree with the string topology of $M$ relative to one constant loop and with Chas-Sullivan operations. The only exception is $M = S^1$.

This supports the string topology conjecture for simply-connected manifolds and provides a counterexample for non-simply connected manifolds.

Our second result generalizes the previous explicit computation and shows that in many cases, the twists with $n$ and $m$ coincide. Its proof is a combination of facts from Section 3.4.

**Theorem B** (Triviality of the twist with $n$ on the chain level). Let $M$ be a closed oriented $n$-manifold. There exists a Green kernel $G$ such that the following holds for the twisted IBL$_{\infty}$-structure $dIBL^n(C(H_{dR}(M)))$:

1. For the basic operations $q^n_{110} = q_{210} \circ_1 n_{10}$, $q^n_{210} = q_{210}$, $q^n_{120} = q_{120} + q_{210} \circ_1 n_{20}$, we have:
   a. If $H^1_{dR}(M) = 0$, then $n_{20} = 0$, and hence $q^n_{120} = q_{120}$.
   b. If $M$ is geometrically formal, then $n_{10} = m_{10}$, and hence $q^n_{110} = q^n_{110}$.
      (In fact, if in addition $H^1_{dR}(M) = 0$, then $n = m$, at least for $n \neq 2$.)
2. For the higher operations $q^n_{110} = q_{210} \circ_1 n_{20}$ with $(l, g) \neq (1, 0)$, $(2, 0)$, we have $n_{1g} = 0$, and hence $q^n_{110} = 0$ with the possible exception of surfaces and 3-manifolds with $H^1_{dR}(M) \neq 0$. 

8
In our future work, we plan to concentrate on the following:

1. We would like to improve Theorem 3 by showing that the higher operations for $S^2$ vanish. If this is the case, then the statement that all higher operations vanish for every manifold $M$ with $H^1_{dR}(M) = 0$ is true.

2. For a formal simply-connected manifold $M$, we would like to investigate whether $d\text{IBL}^n(C(H^1_{dR}(M)))$ and $d\text{IBL}^m(C(H^1_{dR}(M)))$ are homotopy equivalent as $\text{IBL}_{\infty}$-algebras. If not, we would like to understand the obstruction.

3. We would like to compute $d\text{IBL}^n(C(H^1_{dR}(M)))$ for surfaces $\Sigma_g$ with $g \geq 1$ and formulate a string topology conjecture for non-simply connected manifolds.

4. We would like to know whether the Schwartz kernel $G_{\text{std}}$ of $G_{\text{std}} = -d^*\Delta^{-1}$ (the so called standard Green kernel), where $d^*$ is the codifferential and $\Delta$ the Hodge-de Rham Laplacian, extends smoothly to a blow-up. If yes, then it is a canonical Green kernel for which the statement of Theorem 3 holds.

5. We would like to define a generalization of an $\text{IBL}_{\infty}$-algebra—a weak, non-reduced $\text{IBL}_{\infty}$-algebra with a gauge group—and understand its precise relation to perturbative Chern-Simons theory within the $\text{BV}$-formalism.

In the end, let us summarize some existing work on $\text{IBL}_{\infty}$-algebras which helped us to understand $\text{IBL}_{\infty}$-algebras in broader context: In [30], they find an $\text{IBL}_{\infty}$-structure in open-closed string field theory. In [14], they view $\text{IBL}_{\infty}$-algebras as algebras over a certain Frobenius properad. In [26], they consider $\text{IBL}_{\infty}$-algebras as a particular case of $\text{BV}_{\infty}$- or, more generally, $\text{MV}$-algebras.

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2 Algebraic structures

In Section 2.1, we recall weight-grading (Definition 2.1), Koszul sign (Definition 2.2), degree shift (Definition 2.3), filtrations (Definition 2.8) and completions (Definition 2.9). We prove the Künneth formula for completed symmetric cohomology (Proposition 2.13).

In Section 2.2, we review basics of IBL∞-algebras from [10]. We define the exterior algebra $E_C$ over a graded vector space $C$ as the symmetric algebra $S_C[1]$ (Definition 2.14) and use the operations $\mu$ and $\Delta$ of the structure of an associative bialgebra on $S(C[1])$ to give explicit formulas for the partial compositions $\circ_{h_1,\ldots,h_k}$ (Definition 2.15). We use the compositions to define the notion of an IBL∞-algebra $(q_{klg})$ on $C$ (Definition 2.17), a Maurer-Cartan element $(n_{lg})$ (Definition 2.19) and twisted operations $(q^n_{klg})$ (Definition 2.20). We mention that an IBL-algebra according to our definition is an odd degree shift of a classical IBL-algebra (Proposition 2.18). We define the induced IBL-structure on homology (Definition 2.21), briefly discuss the BV-formalism (Remark 2.22) and mention weak IBL∞-algebras (Remark 2.23). Finally, we summarize the situation for twisted dIBL-algebras (Proposition 2.24) and briefly discuss higher operations (Remark 2.25).

In Section 2.3, we define the (weight-reduced) dual cyclic bar-complex $B^*_{cyc}V$ of a graded vector space $V$ (Definition 2.26) and introduce some notation (Notation 2.28). We then summarize some facts about the completions $\hat{B}^*_{cyc}V$ and $\hat{E}_kB^*_{cyc}V$ (Proposition 2.31). We define the notion of a cyclic $A_\infty$-structure on $V$ (Definition 2.32) and its Hochschild and cyclic cohomology (Definition 2.34). We recall strict units and strict augmentations (Definition 2.35), define the reduced dual cyclic bar complex $B^*_{cyc,red}V$ (Definition 2.37) and sketch a proof of the fact that the cyclic cohomology is a direct sum of the reduced cyclic cohomology and the cyclic cohomology of the ground field (Proposition 2.38). We relate our version of the cyclic cohomology for dga's to the classical version from [24] (Proposition 2.39). We also show that the reduced spaces for a simply connected and connected $V$ are complete (Proposition 2.40).

In Section 2.4, we review the construction of the canonical dIBL-structure $dIBL(C(V))$ (Definition 2.42) and the canonical Maurer-Cartan element $m$ (Definition 2.43) starting from a cyclic dga $(V, P, m_1, m_2)$. We give formulas for the operations $(q^n_{klg})$ of the IBL∞-algebra $dIBL^n(C(V))$ twisted by a Maurer-Cartan element $n$ (Proposition 2.45). We consider the $A_\infty$-structure induced on $V$ by $n_{10}$ (Definition 2.46) and relate its cyclic cohomology to the homology of $q^n_{110}$ (Proposition 2.47). We define the reduced canonical dIBL-algebra $dIBL(C_{red}(V))$ (Definition 2.48) and the notion of a strictly reduced Maurer-Cartan element (Definition 2.49). The twisted IBL∞-structure then splits into the reduced part and the part generated by $1^*$, which we can explicitly compute (Proposition 2.51).
2.1 Gradings, degree shifts and completions

We will work with vector spaces over $\mathbb{R}$, possibly infinite-dimensional, graded by the degree $d \in \mathbb{Z}$ and the weight $k \in \mathbb{N}_0$.

**Definition 2.1 (Weight-graded vector spaces).** A graded vector space is a vector space $W$ together with a collection of subspaces $W^d \subset W$ for all $d \in \mathbb{Z}$ such that

$$W = \bigoplus_{d \in \mathbb{Z}} W^d.$$  

Elements of $W^d$ are called homogenous of degree $d$; given $w \in W^d$, we denote the degree of $w$ by $|w| := d$.

A linear map of graded vector spaces $f : W_1 \to W_2$ is called homogenous of degree $|f| \in \mathbb{Z}$ if it holds

$$f(W^d_1) \subset W^{d+|f|}_2 \quad \text{for all} \quad d \in \mathbb{Z}. \quad (4)$$

A weight-graded vector space is a graded vector space $W$ together with a collection of subspaces $W^d_k \subset W$ for all $k \in \mathbb{N}_0$ and $d \in \mathbb{Z}$ such that

$$W^d = \bigoplus_{k \in \mathbb{N}_0} W^d_k \quad \text{for all} \quad d \in \mathbb{Z}.$$  

We define the weight-$k$ component by

$$W_k := \bigoplus_{d \in \mathbb{Z}} W^d_k \quad \text{for all} \quad k \in \mathbb{N}_0.$$  

If $W^d_0 = 0$ for all $d \in \mathbb{Z}$, we say that $W$ is weight-reduced. We define the weight-reduced subspace of a weight-graded vector space $W$ by

$$\bar{W} := \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k \in \mathbb{N}_0} W^d_k.$$  

We consider the following versions of the dual space of $W$:

$$W^* := \{ \psi : W \to \mathbb{R} \text{ linear} \} \ldots \text{linear dual},$$

$$W' := \bigoplus_{d \in \mathbb{Z}} \prod_{k \in \mathbb{N}_0} W^{d_k} \ldots \text{graded dual},$$

$$W'' := \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k \in \mathbb{N}_0} W^{d_k} \ldots \text{weight-graded dual}. \quad (5)$$

We identify $W'$ with the subspace of $W^*$ generated by homogenous maps and $W''$ with the subspace of $W^*$ generated by maps which are non-zero only on finitely
many $W_k^d$; hence, we have

$$W'' \subset W' \subset W^*.$$  

The grading convention for $W'$ is the cohomological grading convention, which differs from the convention (4) for maps $f : W \to \mathbb{R}$ by the degree reversal (see Definition 2.3).

**Definition 2.2 (Koszul sign).** Let $k \geq 1$, and let $\sigma \in \mathbb{S}_k$ be a permutation on $k$ elements. For $i = 1, \ldots, k$, let $a_i$ and $b_i$ be graded symbols of degrees $|a_i|$ and $|b_i|$, respectively. We denote by

$$\varepsilon(\sigma, a) \quad \text{and} \quad \varepsilon(a, b)$$

the Koszul signs of the transformations

$$a_1 \cdots a_k \mapsto a_{\sigma^{-1}} \cdots a_{\sigma^{-1}} \quad \text{and} \quad a_1 \cdots a_kb_1 \cdots b_k \mapsto a_1b_1 \cdots a_kb_k,$$

respectively. Here $\sigma^{-1} := \sigma^{-1}(i)$. The Koszul sign is computed by permuting the left-hand side to the right-hand side using transpositions of two adjacent elements such that whenever we transpose two graded symbols, e.g., $a_i \leftrightarrow a_j$, we multiply with $(-1)^{|a_i||a_j|}$.

We emphasize that the Koszul sign depends only on the initial and the final order of the graded symbols; not on the sequence of transpositions.

**Definition 2.3 (Degree shift and grading reversal).** Let $A \in \mathbb{Z}$. The degree shift by $A$ is a functor which associates to a graded vector space $W$ the graded vector space $W[A]^d := W[d + A]$ for all $d \in \mathbb{Z}$.

There is the canonical degree shift morphism

$$W \longrightarrow W[A]$$

of degree $-A$ mapping $W^d$ identically to $W[A]^{d - A}$. We view this morphism as multiplication from the left with a formal symbol $s_A$ of degree $|s_A| = -A$, so that (6) is given by $w \in W \mapsto s_Aw \in W[A]$.

Given graded vector spaces $W_1$, $W_2$ and constants $A_1$, $A_2 \in \mathbb{Z}$, we associate to a morphism $f : W_1 \to W_2$ its degree shift $f : W_1[A_1] \to W_2[A_2]$ by defining

$$f(s_{A_1}w) := s_{A_2}f(w) \quad \text{for all} \ w \in W_1.$$  

Notice that if $f : W_1 \to W_2$ has degree $|f|$, then $f : W_1[A_1] \to W_2[A_2]$ has degree $|f| + A_1 - A_2$.

The grading reversal $r$ is a functor which associates to a graded vector space $W$...
the graded vector space \( r(W) \) with

\[
r(W)^d := W^{-d} \quad \text{for all } d \in \mathbb{Z}.
\]

There is the canonical morphism \( W \to r(W) \) mapping \( W^d \) identically to \( W^{-d} \) for every \( d \in \mathbb{Z} \). The degree reversal of a morphism \( f : W_1 \to W_2 \) is the morphism \( f : r(W_1) \to r(W_2) \) defined by conjugating \( f \) with the canonical morphism. If \( |f| \) is the degree of \( f : W_1 \to W_2 \), then \(-|f|\) is the degree of \( f : r(W_1) \to r(W_2) \).

In our main reference [10], they view \( W \) and \( W[A] \) as one vector space with two different gradings \( \deg(\cdot) \) and \( |\cdot| \), respectively; these are related by

\[
|w| = \deg(w) - A \quad \text{for all homogenous } w \in W.
\]

On the other hand, we think of \( W \) and \( W[A] \) as two different graded vector spaces and never use the same symbol for an element \( w \in W \) and its degree shift \( s_A w \in W[A] \). It allows us to use just one notation \( |\cdot| \) for the gradings on both \( W \) and \( W[A] \). However, in order to preserve compatibility with [10], we will also sometimes use the notation \( \deg(w) \) (in the exponent just \((-1)^w\) for the degrees on \( W \)).

For graded vector spaces \( W_1, \ldots, W_k \) and constants \( A_1, \ldots, A_k \in \mathbb{Z} \), we identify

\[
W_1[A_1] \otimes \cdots \otimes W_k[A_k] \simeq (W_1 \otimes \cdots \otimes W_k)[A_1 + \cdots + A_k]
\]

using the Koszul convention for the tensor product; for homogenous elements \( w_1 \in W_1, \ldots, w_k \in W_k \), it reads

\[
s_{A_1} w_1 \otimes \cdots \otimes s_{A_k} w_k = \varepsilon(s_A, w) s_{A_1} \cdots s_{A_k} w_1 \otimes \cdots \otimes w_k. \tag{8}
\]

If \( A_1 = \cdots = A_k = A \) is fixed in the context, which is our usual case, we omit the subscript \( A \) and write just \( s \).

In the case of the multilinear map \( f : W_1 \otimes \cdots \otimes W_k \to V_1 \otimes \cdots \otimes V_l \), the combination of (7) and (8) gives for \( f : W_1[A_1] \otimes \cdots \otimes W_k[A_k] \to V_1[B_1] \otimes \cdots \otimes V_l[B_l] \) the following:

\[
f(s_{A_1 + \cdots + A_k} w_1 \otimes \cdots \otimes w_k) = s_{B_1 + \cdots + B_l} f(w_1 \otimes \cdots \otimes w_k). \tag{9}
\]

Remark 2.4 (Why is this sign convention bad?). Let us illustrate that (9) is not compatible with the following standard Koszul rule:

\[
(K) \quad (f_1 \otimes f_2)(w_1 \otimes w_2) = (-1)^{|f_2||w_1|} f_1(w_1) \otimes f_2(w_2).
\]
On one hand, we get

\[(f_1 \otimes f_2)(s^2 w_1 \otimes w_2) \overset{\text{(K)}}{=} s^2(f_1 \otimes f_2)(w_1 \otimes w_2)
= (-1)^{|f_2||w_1|} s^2 f_1(w_1) \otimes f_2(w_2)
= (-1)^{|f_2||w_1| + A(|f_1| + |w_1|)} sf_1(w_1) \otimes sf_2(w_2).\]

On the other hand, we get

\[(f_1 \otimes f_2)(s^2 w_1 \otimes w_2) \overset{\text{(S)}}{=} (-1)^{A|w_1|}(f_1 \otimes f_2)(sw_1 \otimes sw_2)
= (-1)^{A|w_1| + |f_2|(|A| + |w_1|)} f_1(sw_1) \otimes f_2(sw_2)
= (-1)^{A|w_1| + |f_2|(|A| + |w_1|)} sf_1(w_1) \otimes sf_2(w_2).\]

The results differ by \((-1)^{A(|f_1| + |f_2|)}\). Therefore, we cannot use (K) to identify \(\text{Hom}(W_1, V_1) \otimes \text{Hom}(W_2, V_2)\) with a subspace of \(\text{Hom}(W_1 \otimes W_2, V_1 \otimes V_2)\) in general. We will rather define an ad-hoc pairing in the case where we need it (see Definition \ref{definition:29}).

Another caveat is that in the case of the tensor product, the degree shift by \(A_1\) followed by the degree shift by \(A_2\) is not the same as the degree shift by \(A_1 + A_2\). Indeed, we compute

\[(s_{A_1 + A_2} w_1) \otimes (s_{A_1 + A_2} w_2) = (s_{A_2} s_{A_1} w_1) \otimes (s_{A_2} s_{A_1} w_2)
= (-1)^{A_2(A_1 + |w_1|)} s_{A_2}^2 (s_{A_1} w_1) \otimes (s_{A_1} w_2)
= (-1)^{A_2 A_1 + (A_1 + A_2)|w_1|} s_{A_2}^2 s_{A_1}^2 (w_1 \otimes w_2)
= (-1)^{A_2 A_1 + (A_1 + A_2)|w_1|} s_{A_2}^2 s_{A_1}^2 (w_1 \otimes w_2),\]

which differs by \((-1)^{A_1 A_2}\) from the direct degree shift by \(A_1 + A_2\). Therefore, we have to always remember the vector spaces which we started with and the sequence of degree shifts.

Note that we also have the unnatural \(s_{A_1} s_{A_2} = s_{A_2} s_{A_1}\) due to (S).

\section*{Remark 2.5 (Is there a better sign convention?)}

The author originally respected the Koszul rule for the algebra with formal symbols and considered the following map \(\sigma^k : W[A]^{\otimes k} \to V[A]^{\otimes l}\) as the degree shift of \(f : W^{\otimes k} \to V^{\otimes l}\):

\[(s_1^k \sigma^k f)(s^k w_1 \otimes \cdots \otimes w_k) = (-1)^{k|A| + \frac{1}{2} k(k-1)A} s^j f(w_1 \otimes \cdots \otimes w_k).\] (10)

Here \(\sigma\) denotes the “inverse” of \(s\) with \(|\sigma| = -|s|\), \(s_1 f = s^j \circ f\) is the post-composition, \(\sigma^k f = (-1)^{k|A|/l} f \circ \sigma^k\) the pre-composition, and the sign \(\varepsilon(s, \sigma) = (-1)^{\frac{1}{2} k(k-1)A}\) comes from the “collision” \(s_1 \cdots s_k s_1 \cdots s_k \mapsto \sigma s_1 s_1 \cdots \sigma s_k s_k\).

However, the author did not manage to reprove the theory in \textbf{10} using (10) (because of too many “external” signs appearing and a problem with disconnected
graphs). A motivation to try a different sign convention was to explain some artificial signs in [10] and formulate their coordinate constructions invariantly in order to generalize them to the “continuous” de Rham case.

It might be possible to deduce a “universal” sign convention “respecting” the Koszul rules by considering the category of chain complexes and graded morphisms $\mathcal{C}$ as the category enriched in the closed monoidal category of chain complexes and chain maps of degree 0. One can then define the enriched degree shift functor $s_A : \mathcal{C} \to \mathcal{C}$, embed $\mathcal{C}^\otimes_k \subset \mathcal{C}$ using $(K)$ and study enriched natural transformations in the algebra of functors consisting of tensor products and compositions of $s_A$, $\text{Hom}(\cdot, \cdot)$ and the dual $\ast$. 

Definition 2.6 (Permutations). For $k \geq 1$ and $\sigma \in S_k$ (:= the group of permutations on $k$ elements), we define the action of $\sigma$ on $W^\otimes_k$ by

$$\sigma(w_1 \otimes \cdots \otimes w_k) := \varepsilon(\sigma, w)w_{\sigma^{-1}_1} \otimes \cdots \otimes w_{\sigma^{-1}_k} \quad (11)$$

for all homogenous $w_1, \ldots, w_k \in W$.

Notice that the $i$-th vector is permuted to the $\sigma_i$-th place — this is the “active” convention for permutations.

Definition 2.7 (Symmetric algebra). Let $T(V) := \bigoplus_{k \geq 0} V^\otimes_k$ be the tensor algebra over a graded vector space $V$. The symmetric algebra over $V$ is defined by $S(V) := \bigoplus_{k \geq 0} S_k(V)$, where

$$S_k(V) := V^\otimes_k / \sum_{\sigma \in S_k} \text{Im}(\mathbb{1} - \sigma) \quad (=: S_k{-}\text{coinvariants}).$$

It is a weight-graded vector space with components denoted by $(S_k V)^d$ for all $d \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. Note that $S_0 V = \mathbb{R}$ has degree 0 by definition. Consider the canonical projection $\pi : T(V) \longrightarrow S(V)$

$$v_1 \otimes \cdots \otimes v_k \longmapsto v_1 \cdots v_k.$$ 

The dot $\cdot$ indicates the symmetric product. If $v_i \in V$ are homogenous, we call $v_1 \cdots v_k$ a generating word; we have

$$v_1 \cdots v_k = \varepsilon(\sigma, v)v_{\sigma^{-1}_1} \cdots v_{\sigma^{-1}_k} \quad \text{for every } \sigma \in S_k.$$

Let $\iota : S(V) \to T(V)$ be the section of $\pi$ defined by

$$\iota(v_1 \cdots v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \varepsilon(\sigma, v)v_{\sigma^{-1}_1} \otimes \cdots \otimes v_{\sigma^{-1}_k}.$$
We use it to identify $S(V)$ with the subspace of symmetric tensors

$$
i(S_k(V)) = \bigcap_{\sigma \in S_k} \ker(1 - \sigma) \subset T_k(V) \quad (=: S_k\text{-invariants}).$$

**Definition 2.8 (Filtrations).** Let $W$ be a graded vector space. A filtration of $W$ is a collection of linear subspaces $F_\lambda W \subset W$ for $\lambda \in \mathbb{R}$ such that we have either

- $F_{\lambda_1} W \subset F_{\lambda_2} W$ for all $\lambda_1 \leq \lambda_2$ $\iff$ increasing filtration,
- $F_{\lambda_1} W \supset F_{\lambda_2} W$ for all $\lambda_1 \leq \lambda_2$ $\iff$ decreasing filtration.

We will assume that our filtrations are graded in the following sense:

- **exhaustive** $\iff \bigcup_{\lambda \in \mathbb{R}} F_\lambda W = W$;
- **Hausdorff** $\iff \bigcap_{\lambda \in \mathbb{R}} F_\lambda W = 0$;
- **Z-gapped** $\iff F_\lambda W = F_{\lfloor \lambda \rfloor} W$ for all $\lambda \in \mathbb{R}$;
- **bounded from below** $\iff \exists \lambda \in \mathbb{R} : F_\lambda W = 0$;
- **bounded from above** $\iff \exists \lambda \in \mathbb{R} : F_\lambda W = W$.

Given a graded vector space $W$ filtered by a Z-gapped filtration $F_\lambda W$, we associate to it the bi-graded vector space $\text{gr}(W) = \bigoplus_{d,\lambda \in \mathbb{Z}} \text{gr}(W)_\lambda^d$ called the graded module whose components are given as follows:

$$\forall d, \lambda \in \mathbb{Z} : \quad \text{gr}(W)_\lambda^d := \begin{cases} F_\lambda W^d / F_{\lambda-1} W^d & \text{for increasing } F_\lambda W, \\ F_{\lambda-1} W^d / F_{\lambda} W^d & \text{for decreasing } F_\lambda W. \end{cases}$$

We naturally extend a filtration over degree shifts, graded duals, direct sums, tensor products and symmetric products as follows:

$$F_\lambda W[A]_\lambda^d := F_\lambda W^{d+A},$$

$$F_\lambda (W^*)^d := \{ \psi \in W^{d*} \mid \psi|_{F_\lambda W} = 0 \},$$

$$F_\lambda (\bigoplus_{i \in I} W_i)^d := \bigoplus_{i \in I} F_\lambda W_i^d.$$
\[ F_{\lambda}(W_1 \otimes \cdots \otimes W_k)^d := \bigoplus_{d_1, \ldots, d_k \in \mathbb{Z}, \lambda_1 + \cdots + \lambda_k = \lambda} F_{\lambda_1}W_1^{d_1} \otimes \cdots \otimes F_{\lambda_k}W_k^{d_k}, \]

\[ F_{\lambda}(S_k V)^d := \pi(F_{\lambda}(V \otimes^k)^d), \]

where \( \pi : T(V) \to S(V) \) is the canonical projection. If \((W, \partial)\) is a filtered chain complex, we filter the homology as follows:

\[ \forall \lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad F_{\lambda}H_d(W, \partial) := \{ \alpha \in H_d(C, \partial) | \exists w \in \alpha : w \in F_{\lambda}W^d \}. \]

**Definition 2.9 (Completions).** Let \( W \) be a graded vector space filtered by a decreasing filtration \( F_{\lambda}W \). The filtration degree of \( w \in W \) is defined by

\[ \|w\| := \sup \{ \lambda \in \mathbb{R} | w \in F_{\lambda}W \}. \]

The filtration degree of a linear map \( f : W_1 \to W_2 \) is defined by

\[ \|f\| := \sup \{ \lambda \in \mathbb{R} | \|f(w)\| \geq \|w\| + \lambda \ \forall w \in W_1 \}. \]

We say that the filtration degree is finite if \( \|f\| > -\infty \). Note that \( \|0\| = \infty \).

The completion of \( W \) is the graded vector space

\[ \hat{W} := \bigoplus_{d \in \mathbb{Z}} \hat{W}^d, \]

where for all \( d \in \mathbb{Z} \) we define

\[ \hat{W}^d := \left\{ \sum_{i=0}^{\infty} w_i \mid \forall i \in \mathbb{N}_0 : w_i \in W^d, \|w_i\| \to \infty \ \text{as} \ i \to \infty \right\}/\sim. \]

Here \( \sum_{i=0}^{\infty} w_i \sim \sum_{i=0}^{\infty} w'_i \) if and only if \( \| \sum_{i=0}^{n} (w_i - w'_i) \| \to \infty \) as \( n \to \infty \).

The completion \( \hat{W} \) is canonically filtered by the filtration \( F_{\lambda} \hat{W} \) defined as follows:

\[ \forall \lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad F_{\lambda} \hat{W}^d := \left\{ \sum_{i=0}^{\infty} w_i \mid \forall i \in \mathbb{N}_0 : w_i \in F_{\lambda} W^d \right\}. \]

We denote the completion of \( W_1 \otimes \cdots \otimes W_k \) by \( W_1 \hat{\otimes} \cdots \hat{\otimes} W_k \) and the completion of \( S_k V \) by \( \hat{S}_k V \).

A map \( f : W_1 \to W_2 \) of finite filtration degree extends continuously to a

\[ \text{In fact,} \ \hat{W} \text{is the inverse limit } \lim_{\leftarrow} \mathbb{R}(W/F_{\lambda}W) \text{ in the category of graded vector spaces and } \hat{W}^d \text{ the inverse limit } \lim_{\leftarrow} \mathbb{R}(W^d/F_{\lambda}W^d) \text{ in the category of vector spaces. As a side-remark, if we forget the grading on } \hat{W}, \text{ we might also consider } \lim_{\leftarrow} \mathbb{R}(W/F_{\lambda}W), \text{ which would be a vector space containing } \hat{W} \text{ as a subspace.} \]
linear map $f : \hat{W}_1 \to \hat{W}_2$; this extension is defined by

$$f\left(\sum_{i=0}^{\infty} w_i\right) := \sum_{i=0}^{\infty} f(w_i) \quad \text{for all} \quad \sum_{i=0}^{\infty} w_i \in \hat{W}.$$  

Remark 2.10 (Completed tensor product). Using Proposition 2.11 below, one can show that the completed tensor product $\hat{\otimes}$ is associative and that $W_1 \hat{\otimes} \hat{W}_2 \simeq \hat{W}_1 \hat{\otimes} \hat{W}_2$. By refining this argument, one can show that $\hat{S}_k V \simeq \hat{S}_k V$ for any $k \in \mathbb{N}$.

A weight-graded vector space $W$ is canonically filtered by weights:

$$\forall \lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad \mathcal{F}_\lambda W^d := \bigoplus_{k \leq \lambda} W^d_k. \quad (12)$$

This filtration is $\mathbb{Z}$-gapped, exhaustive, Hausdorff, increasing and bounded from below. The induced filtration on the graded dual $W'$ is $\mathbb{Z}$-gapped, Hausdorff, decreasing and bounded from above (and thus automatically exhaustive). It holds $\text{gr}(W) \simeq W$, and it is easy to see from (5) that the canonical map $W'' \to W'$ induces the isomorphism $\hat{W}'' \simeq W'$.

We also see that the condition $(WG0) : \quad \forall d \in \mathbb{Z} \exists J \subset \mathbb{N}_0, |J| < \infty \forall k \in \mathbb{N}_0 \setminus J : \quad W^d_k = 0$

is equivalent to $W'' = W'$.

A useful tool to compare completions is the following proposition:

**Proposition 2.11** ([15, Proposition 7.3.7], Isomorphism criterion). Let $W_1$ and $W_2$ be graded vector spaces filtered by $\mathbb{Z}$-gapped filtrations which are decreasing and bounded from above. Suppose that $f : W_1 \to W_2$ is a filtration preserving homogenous linear map. Then the continuous extension $\hat{f} : \hat{W}_1 \to \hat{W}_2$ is an isomorphism if and only if the induced map $\text{gr}(W_1) \to \text{gr}(W_2)$ is an isomorphism.

**Proof.** The implication from the right to the left is obtained from the diagram

$$
\begin{array}{c}
0 \longrightarrow \text{gr}(W_1)_{\lambda} \longrightarrow W_1/\mathcal{F}_\lambda W_1 \longrightarrow W_1/\mathcal{F}_{\lambda-1} W_1 \longrightarrow 0 \\
\downarrow f \quad \downarrow f \quad \downarrow f \\
0 \longrightarrow \text{gr}(W_2)_{\lambda} \longrightarrow W_2/\mathcal{F}_\lambda W_2 \longrightarrow W_2/\mathcal{F}_{\lambda-1} W_2 \longrightarrow 0
\end{array}
$$

by induction using the definition of $\hat{W}$ as the inverse limit of $W/\mathcal{F}_\lambda W$ (see Footnote [2] on page 17).

For a graded vector space $W$ filtered by a $\mathbb{Z}$-gapped filtration, consider the
following conditions:

(WG1): \( \forall \lambda \in \mathbb{Z} \ \exists I \subset \mathbb{Z}, |I| < \infty \ \forall d \in \mathbb{Z}\setminus I : \quad \text{gr}(W)_\lambda^d = 0, \)

(WG2): \( \forall d, \lambda \in \mathbb{Z} : \quad \dim(\text{gr}(W)_\lambda^d) < \infty. \) (13)

**Lemma 2.12** (Completion of symmetric powers of the graded dual). Let \( W \) be a graded vector space filtered by an exhaustive \( \mathbb{Z} \)-gapped filtration \( F_\lambda W \) which is increasing and bounded from below. If (WG1) & (WG2) are satisfied, then the natural map \( S_k(W') \to (S_k W)' \) induces the isomorphism

\[
\hat{S}_k(W') \cong (S_k W)' \quad \text{for every } k \in \mathbb{N}.
\]

Note that we filter graded duals by the induced filtration from Definition 2.8.

**Proof.** The natural map \( S_k(W') \to (S_k W)' \) is clearly filtration preserving, and hence it extends continuously to a map of completions. The target space \( (S_k W)' \) is already complete (the dual space \( W' \) is complete, provided that the filtration of \( W \) is exhaustive), and thus we obtain the map \( \hat{S}_k(W') \to (S_k W)' \). According to Proposition 2.11, this map is an isomorphism if and only if the induced map \( \text{gr}(S_k(W')) \to \text{gr}((S_k W)') \) is. This is shown by the following computation (the maps involved are natural in at least one direction):

\[
\frac{F_\lambda(W^\otimes k')^d}{F_{\lambda+1}(W^\otimes k')^d} \cong \left( \bigoplus_{|\lambda|=d} \sum_{|\lambda|=\lambda+1} F_{\lambda_1} W_{d_1} \otimes \cdots \otimes F_{\lambda_k} W_{d_k} \right)^* \cong \left( \bigoplus_{|\lambda|=d} \sum_{|\lambda|=\lambda} F_{\lambda_1} W_{d_1} \otimes \cdots \otimes F_{\lambda_k} W_{d_k} \right)^*
\]

\[
\cong \left( \bigoplus_{|\lambda|=\lambda} \left( \frac{F_{\lambda+1} W_{d_1}}{F_{\lambda_1} W_{d_1}} \otimes \cdots \otimes \frac{F_{\lambda_k+1} W_{d_k}}{F_{\lambda_k} W_{d_k}} \right) \right)^*
\]

\[
\text{Z\text{-}gapped} & \quad \text{bounded below} \quad \Rightarrow \quad \cong \left( \bigoplus_{|\lambda|=|\lambda|=d} \sum_{|\lambda|=\lambda} \left( \frac{F_{\lambda+1} W_{d_1}}{F_{\lambda_1} W_{d_1}} \otimes \cdots \otimes \frac{F_{\lambda_k+1} W_{d_k}}{F_{\lambda_k} W_{d_k}} \right) \right)^*
\]

(WG2) \( \Rightarrow \quad \sum_{|\lambda|=\lambda} \left( \frac{F_{\lambda+1} W_{d_1}}{F_{\lambda_1} W_{d_1}} \otimes \cdots \otimes \frac{F_{\lambda_k+1} W_{d_k}}{F_{\lambda_k} W_{d_k}} \right)^* \]

\[
\cong \left( \bigoplus_{|\lambda|=\lambda} \left( \frac{F_{\lambda+1} W_{d_1}}{F_{\lambda_1} W_{d_1}} \otimes \cdots \otimes \frac{F_{\lambda_k+1} W_{d_k}}{F_{\lambda_k} W_{d_k}} \right) \right)^*
\]
\[ \simeq \frac{F_\lambda(W'^{\otimes k})}{F_{\lambda + 1}(W'^{\otimes k})}d. \]

In fact, this computation shows that \( \hat{T}_k(W') \simeq (T_kW)' \). The conclusion for \( S_k \) follows by checking that the maps above are \( S_k \)-equivariant. \qed

Given a chain complex \((W, \partial)\), the boundary operator \( \partial \) induces the boundary operator \( \partial_k : W^{\otimes k} \to W^{\otimes k} \) for all \( k \in \mathbb{N} \); for all \( w_1, \ldots, w_k \in W \), it is defined by

\[
\partial_k(w_1 \otimes \cdots \otimes w_k) := \sum_{i=1}^{k} (-1)^{|w_1| + \cdots + |w_{i-1}|} w_1 \otimes \cdots \otimes \partial w_i \otimes \cdots \otimes w_k. \tag{14}
\]

The map \( \partial_k \) is clearly \( S_k \)-equivariant, and thus induces the boundary operator \( \partial_k : S_kW \to S_kW \).

**Proposition 2.13** (Künneth formula for completed symmetric cohomology). Let \((W, \partial)\) be a \( \mathbb{Z} \)-graded chain complex over \( \mathbb{R} \) filtered by an exhaustive \( \mathbb{Z} \)-gapped filtration \( F_\lambda W \) which is increasing and bounded from below. Consider the dual cochain complex \((W', \partial := \partial^*)\). Suppose that \( \partial \) has finite filtration degree, so that \( \partial_k : S_kW' \to S_kW' \) extends continuously to \( \partial_k : \hat{S}_kW' \to \hat{S}_kW' \) for every \( k \in \mathbb{N} \). If (WG1) \& (WG2) are satisfied, then the natural map \( S_kH(W', \partial) \to \hat{H}(S_kW', \partial_k) \) induces the isomorphism

\[
\hat{S}_kH(W', \partial) \simeq \hat{H}(S_kW', \partial_k) \quad \text{for all } k \in \mathbb{N}.
\]

**Proof.** The natural map \( S_kH(W', \partial) \to \hat{H}(S_kW', \partial_k) \) is clearly filtration preserving, and hence it extends continuously to a map of completions. The target space \( \hat{H}(S_kW', \partial_k) \) is already complete (the homology of a complete space is complete), and hence we obtain the map \( \hat{S}_kH(W', \partial) \to \hat{H}(S_kW', \partial_k) \). The following facts are easy to verify:

1. The isomorphism from Lemma 2.12 is an isomorphism of cochain complexes

   \[
   (\hat{S}_kW', \partial_k) \simeq ((S_kW')', \partial_k').
   \]

2. If the filtration on \( W \) satisfies (WG1) and (WG2), then the filtration on \( H(W) \) also satisfies (WG1) and (WG2), respectively. Consequently, Lemma 2.12 holds for symmetric powers of \( H(W, \partial)' \) as well.

3. The Künneth formula \( H(W'^{\otimes k}) \simeq H(W')^{\otimes k} \) implies \( H(S_kW) \simeq S_kH(W) \) for any \( \mathbb{Z} \)-graded chain complex \( W \) over \( \mathbb{R} \).

4. We have \( (H(W))' \simeq H(W') \) over \( \mathbb{R} \) by the universal coefficient theorem.
Now, we compute
\[
\begin{align*}
H(\hat{S}_k W', \partial_k) & \simeq H((S_k W', \partial_k)' \simeq H(S_k W, \partial)\simeq (S_k H(W, \partial))' \\
\simeq \hat{S}_k (H(W, \partial)' \simeq \hat{S}_k H(W', d).
\end{align*}
\]
This proves the proposition. 

2.2 Basics of IBL∞-algebras

Definition 2.14 (Exterior algebra). Given a graded vector space \( C \) over \( \mathbb{R} \), we define the exterior algebra over \( C \) by
\[
E_C := S(C[1]).
\]
The weight-\( k \) component is denoted by \( E_k C \) and the weight-reduced part by \( \bar{E} C \).

If \( C \) is in addition filtered, then \( E_k C \) is filtered by the induced filtration and its completion is denoted by \( \hat{E}_k C \).

We have the product \( \mu : E_C \otimes E_C \to E_C \) and coproduct \( \Delta : E_C \to E_C \otimes E_C \) defined by
\[
\begin{align*}
\mu(c_{11} \ldots c_{1k} \otimes c_{21} \ldots c_{2k'}) & := c_{11} \ldots c_{1k}c_{21} \ldots c_{2k'} \quad \text{and} \\
\Delta(c_1 \ldots c_k) & := \sum_{k_1, k_2 \geq 0} \sigma \in S_{k_1, k_2} \epsilon(\sigma, c_1) c_{\sigma^{-1}_1} \ldots c_{\sigma^{-1}_{k_1}} \otimes c_{\sigma_1+1} \ldots c_{\sigma_{k_1+k_2}}
\end{align*}
\]
for all homogenous \( c_{ij} \), \( c_i \in C[1] \) and \( k, k' \geq 0 \), respectively, where \( S_{k_1, k_2} \subset \hat{S}_{k_1+k_2} \) denotes the set of shuffle permutations with blocks of lengths \( k_1 \) and \( k_2 \).

These operations satisfy relations of an associative bialgebra (see [25]):
\[
\begin{align*}
\text{Ass. bialg.} & \quad \left\{ \begin{array}{l}
\mu \circ (\mathbb{1} \otimes \mu) = \mu \circ (\mu \otimes \mathbb{1}), \\
(\mathbb{1} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathbb{1}) \circ \Delta, \\
\Delta \circ \mu = (\mu \otimes \mu) \circ (\mathbb{1} \otimes \tau \otimes \mathbb{1}) \circ (\Delta \otimes \Delta).
\end{array} \right.
\end{align*}
\]

Here \( \tau : C_1 \otimes C_2 \to C_2 \otimes C_1, c_1 \otimes c_2 \mapsto (-1)^{|c_1||c_2|} c_2 \otimes c_1 \) denotes the twist map.

We will use the bialgebra calculus (:= relations [15]) to write down explicit formulas for the operations \( o_{h_1, \ldots, h_r} \) which were briefly introduced in [10]: these operations take symmetric maps \( f_1, \ldots, f_r \) and connect \( h_1, \ldots, h_r \) of their outputs to the inputs of a symmetric map \( f \) in all possible ways, so that the result, which we denote by \( f \circ_{h_1, \ldots, h_r} (f_1, \ldots, f_r) \), becomes a symmetric map again.
Definition 2.15 (Partial compositions). Let $C$ be a graded vector space. For $i, j \geq 0$, we denote by

$$
\begin{align*}
\pi_i : EC \rightarrow E_i C, \quad &i : E_i C \rightarrow EC, \\
1_i : E_i C \rightarrow E_i C, \quad &\tau_{i,j} : E_i C \otimes E_j C \rightarrow E_j C \otimes E_i C
\end{align*}
$$

the components of the canonical projection $\pi$, the canonical inclusion $i$, the identity $1$ and the twist map $\tau$, respectively. We also set

$$
\Delta_{i,j} := (\pi_i \otimes \pi_j) \circ \Delta \circ \iota_{i+j} \quad \text{and} \quad \mu_{i,j} := \pi_{i+j} \circ \mu \circ (\iota_i \otimes \iota_j).
$$

For $k', k_1, l', l_1 \geq 0$, let $f : E_{k'} C \rightarrow E_{l'} C$ and $f_1 : E_{k_1} C \rightarrow E_{l_1} C$ be linear maps, and let $0 \leq h \leq \min(k', l_1)$. We set

$$
k := k' + k_1 - h \quad \text{and} \quad l := l' + l_1 - h
$$

and define the composition of $f$ and $f_1$ at $h$ common outputs to be the linear map $f \circ_h f_1 : E_{k'} C \rightarrow E_{l'} C$ given by

$$
f \circ_h f_1 := \mu_{(l_1-h) \circ (f \otimes 1_{l_1-h}) \circ (\mu_{h,k'1} \otimes 1_{l_1-h}) \circ (1_{h} \otimes \tau_{l_1-h,k'1})} \circ (\Delta_{h,l_1-h} \otimes 1_{k'1}) \circ (f_1 \otimes 1_{k'1}) \circ \Delta_{1,1}
$$

(16)

More generally, we define the composition of $f : E_{k'} \rightarrow E_{l'}$ with $r \geq 1$ linear maps $f_i : E_{k_i} \rightarrow E_{l_i}$ with $k_i, l_i \geq 0$ for $i = 1, \ldots, r$ at $0 \leq h_i \leq l_i$ common outputs such that $h := h_1 + \cdots + h_r \leq k'$ as follows. We set

$$
k := k' + k_1 + \cdots + k_r - h \quad \text{and} \quad l := l' + l_1 + \cdots + l_r - h
$$

and define $f \circ_{h_1, \ldots, h_r} (f_1, \ldots, f_r) : E_{k'} C \rightarrow E_{l'} C$ by

$$
f \circ_{h_1, \ldots, h_r} (f_1, \ldots, f_r) := \mu \circ (f \otimes 1) \circ (\mu \otimes 1) \circ (1 \otimes \tau)
$$

$$
\circ (\bigl[(\mu^{(r)} \otimes \mu^{(r)}) \circ (F_{h_1, \ldots, h_r} \otimes 1) \circ \sigma_r \circ \Delta^{\otimes r}\bigr] \otimes 1)
$$

$$
\circ (f_1 \otimes \cdots \otimes f_r \otimes 1) \circ \Delta^{(r+1)},
$$

(17)

where we have:

- The operation $\mu^{(r)}$ is the “product with $r$ inputs” and the operation $\Delta^{(r)}$ the “coproduct with $r$ outputs”; they are defined by

$$
\mu^{(r)} := \mu(1 \otimes \mu) \cdots (1 \otimes r-2 \otimes \mu), \quad \mu^{(1)} := 1,
$$

$$
\Delta^{(r)} := (1^{\otimes r-2} \otimes \Delta) \cdots (1 \otimes \Delta) \Delta, \quad \Delta^{(1)} := 1,
$$

22
• $F_{h_1,\ldots,h_r} := (\epsilon_h, \pi_{h_1}) \otimes \cdots \otimes (\epsilon_h, \pi_{h_r})$.

• The permutation $\sigma_r \in S_{2r}$ is given by

$$\sigma_r : (1, 2, \ldots, 2r) \mapsto (1, r+1, \ldots, r, 2r).$$

• The symbols $f$ and $f_i$ inside the formula denote the trivial extensions of $f$ and $f_i$, respectively; we extend a linear map $f : E_k C \to E_l C$ trivially to $f : EC \to EC$ by defining $f(E_k C) = 0$ for $i \neq k'$.

Remark 2.16 (On partial compositions). (i) Defining $f \circ_{h_1,\ldots,h_r} (f_1, \ldots, f_r)$ : $E_k C \to E_l C$ using [17] makes sense because the right hand side is a trivial extension of its component $E_k C \to E_l C$. In fact, all $\mu, \Delta, \pi, i$ in [17] can be replaced with $\mu_{i,j}, \Delta_{i,j}, \pi_{i,j}$ for unique $i, j$, so that trivial extensions do not have to be used at all. In this way, it can be seen that [16] is indeed a special case of [17].

(ii) If $h = k' = l_1$, then $f \circ_h f_1 = f \circ f_1$.

(iii) It holds $f \circ_0 f_1 = (-1)^{|f| |f_1|} f_1 \circ_0 f$ and

$$f \circ_{h_1,\ldots,h_r} (f_1, \ldots, f_r) = e(\sigma, f) f \circ_{h_{r-1},\ldots,h_{r-r}} (f_{\sigma_1^{-1}}, \ldots, f_{\sigma_r^{-1}}).$$

(iv) Consider the (“non-trivial”) extension $\hat{f} := \mu f 1 \Delta : EC \to EC$ and the symmetric product $f_1 \odot \cdots \odot f_r := \mu^{(r)} f_1 \odot \cdots \odot f_r \Delta^{(r)} : EC \to EC$. The proof of the following formulas appearing in [10] is now an exercise on the bialgebra calculus:

$$f \circ_{h_1,\ldots,h_{r-1},0} (f_1, \ldots, f_r) = f \circ_{h_1,\ldots,h_{r-1}} (f_1, \ldots, f_{r-1}) \odot f_r,$$

$$f \circ \hat{f}_1 = \sum_{h=0} \hat{f} \circ_h f_1,$$

$$\hat{f} \circ (f_1 \odot \cdots \odot f_r) = \sum_{h_1,\ldots,h_r \geq 0 \atop h_1 + \cdots + h_r = k'} f \circ_{h_1,\ldots,h_r} (f_1, \ldots, f_r).$$

(18)

We also have the “weak associativity”

$$\sum_{0 \leq h_2 \leq \min(f_1^-, f_2^-) \atop h_1 + h_2 = h} f_1 \circ_{h_1} (f_2 \circ_{h_2} f_3) = \sum_{0 \leq h_1 \leq \min(f_1^+, f_2^-) \atop h_1 + h_2 = h} (f_1 \circ_{h_1} f_2) \circ_{h_2} f_3$$

(19)

for every $0 \leq h \leq \min(k_1 + k_2 + k_3, l_1 + l_2 + l_3)$, where $f^+$ denotes the number of inputs and $f^-$ the number of outputs of $f$. The weak associativity of $\circ_h$ can be proven using the associativity of $\circ$ and the second relation of [18].
If $C$ is filtered by a decreasing filtration, then the bialgebra operations extend continuously to
\[
\mu : \hat{E}_{k_1}C \otimes \hat{E}_{k_2}C \longrightarrow \hat{E}_{k_1+k_2}C \quad \text{and}
\Delta : \hat{E}_{k}C \longrightarrow \bigoplus_{l_1,l_2 \geq 0 \atop l_1+l_2 = k} \hat{E}_{l_1}C \otimes \hat{E}_{l_2}C
\]
for all $k_1, k_2, k \in \mathbb{N}_0$ because they preserve the filtration degree (see [15] for a similar construction). Next, if $f_1 : \hat{E}_{k_1}C \rightarrow \hat{E}_{l_1}C$ and $f_2 : \hat{E}_{k_2}C \rightarrow \hat{E}_{l_2}C$ have finite filtration degrees, then $f_1 \otimes f_2 : \hat{E}_{k_1}C \otimes \hat{E}_{k_2}C \rightarrow \hat{E}_{l_1}C \otimes \hat{E}_{l_2}C$ has finite filtration degree too, and hence it extends continuously to $f_1 \otimes f_2 : \hat{E}_{k_1}C \otimes \hat{E}_{k_2}C \rightarrow \hat{E}_{l_1}C \otimes \hat{E}_{l_2}C$. Using these facts, we can canonically extend Definition 2.15 to maps $f : \hat{E}_dC \rightarrow \hat{E}_dC$ and $f_l : \hat{E}_kC \rightarrow \hat{E}_lC$ of finite filtration degrees. The resulting map $f \circ_{h_1,\ldots,h_r} (f_1,\ldots,f_r) : \hat{E}_C \rightarrow \hat{E}_C$ will have finite filtration degree too. Moreover, the formulas in Remark 2.16 will still hold.

We will now rephrase the definitions of an IBL$\infty$-algebra, a Maurer-Cartan element and twisted operations from [10] in terms of $\circ_{h_1,\ldots,h_r}$.

**Definition 2.17 (IBL$\infty$-algebra).** Let $C$ be a graded vector space equipped with a decreasing filtration, and let $d \in \mathbb{Z}$ and $\gamma \geq 0$ be fixed constants. An **IBL$\infty$-algebra** of bidegree $(d,\gamma)$ on $C$ is a collection of linear maps $q_{klg} : \hat{E}_kC \rightarrow \hat{E}_lC$ for all $k,l \geq 1$, $g \geq 0$ which are homogenous, of finite filtration degree and satisfy the following conditions:

1) $|q_{klg}| = -2d(k + g - 1) - 1.$

2) $\|q_{klg}\| \geq \gamma\chi_{klg}$, where $\chi_{klg} := 2 - 2g - k - l$.

3) The **IBL$\infty$-relations** hold: for all $k,l \geq 1$, $g \geq 0$, we have
\[
\sum_{h=1}^{g+1} \sum_{k_1,k_2,l_1,l_2 \geq 1 \atop k_1+k_2 = k+h \atop l_1+l_2 = l+h \atop g_1,g_2 \geq 0} q_{k_2,l_2,g_2} \circ_h q_{k_1,l_1,g_1} = 0. \tag{20}
\]

We denote a given IBL$\infty$-algebra structure on $C$ by IBL$\infty(C)$; i.e., we write IBL$\infty(C) = (C, (q_{klg})$).

If $q_{klg} \equiv 0$ for all $(k,l,g) \neq (1,1,0)$, $(2,1,0)$, $(1,2,0)$, then we call IBL$\infty(C)$ a **dIBL$\infty$-algebra** and denote it by dIBL$\infty(C)$. If in addition $q_{110} \equiv 0$, then we have an **IBL$\infty$-algebra** IBL$\infty(C)$. If the operations on the completed exterior powers $\hat{E}_C$ arise as continuous extensions of operations $q_{klg} : E_kC \rightarrow E_lC$, then we call the IBL$\infty$-algebra **completion-free** and denote $C$ together with the operations $q_{klg} : E_kC \rightarrow E_lC$ by IBL$\infty^0(C)$.
The acronym IBL stands for an involutive Lie bialgebra. It follows namely from the IBL∞-relations (20) that for IBL(C) = (C, q_{210}, q_{120}) the following holds:

\[
\begin{align*}
0 &= q_{210} \circ_{1} q_{210} \quad \leftarrow \text{Jacobi id.} \\
0 &= q_{120} \circ_{1} q_{120} \quad \leftarrow \text{co-Jacobi id.} \\
0 &= q_{120} \circ_{1} q_{210} + q_{210} \circ_{1} q_{120} \quad \leftarrow \text{Drinfeld id.} \\
0 &= q_{210} \circ_{2} q_{120} \quad \leftarrow \text{Involutivity}
\end{align*}
\]

Proposition 2.18 (Odd degree shift of an IBL-algebra). Let (C, q_{210}, q_{120}) be an IBL-algebra of degree \(d\) from Definition 2.17, and let \(\tilde{q}_{210} : C \rightarrow C^\otimes 2\) and \(\tilde{q}_{120} : C \rightarrow C^\otimes 3\) be the linear maps defined by

\[
\begin{align*}
\tilde{q}_{210}(x_1 \otimes x_2) &:= q_{210}(\pi(\theta^2 x_1 \otimes x_2)) \quad \text{and} \\
\tilde{q}_{120}(x) &:= \iota(q_{120}(\theta x))
\end{align*}
\]  

for all \(x_1, x_2, x \in C\), where \(\iota : S_2(C[1]) \rightarrow C[1]^\otimes 2\) is the section of \(\pi : C[1]^\otimes 2 \rightarrow S_2(C[1])\) from Definition 2.7 and \(\theta\) is a formal symbol of degree \(|\theta| = -1\). Then the degrees satisfy

\[
|\tilde{q}_{210}| = |q_{210}| - 1 = -2d - 2 \quad \text{and} \quad |\tilde{q}_{120}| = |q_{120}| + 1 = 0,
\]

the operations \(\tilde{q}_{210}\) and \(\tilde{q}_{120}\) are graded antisymmetric, i.e., we have

\[
\tilde{q}_{210} \circ \tau = -\tilde{q}_{210} \quad \text{and} \quad \tau \circ \tilde{q}_{120} = -\tilde{q}_{120}
\]

for the twist map \(\tau\), and the relations

\[
\begin{align*}
0 &= q_{210} \circ (\tilde{q}_{210} \otimes 1) \circ (1^\otimes 3 + t_3 + t_2^3), \\
0 &= (1^\otimes 3 + t_3 + t_2^3) \circ (\tilde{q}_{120} \otimes 1) \circ \tilde{q}_{120}, \\
0 &= x_1 \cdot \tilde{q}_{120}(x_2) - (-1)^{x_1 x_2} x_2 \cdot \tilde{q}_{120}(x_1) - \tilde{q}_{120}(q_{210}(x_1, x_2)), \\
0 &= \tilde{q}_{210} \circ \tilde{q}_{120},
\end{align*}
\]

hold for all \(x_1, x_2 \in C\). Where \(t_3 \in S_3\) denotes the cyclic permutation with \(t_3(1) = 2\) acting on \(C^\otimes 3\) and we define

\[
x \cdot (y_1 \otimes y_2) := \tilde{q}_{210}(x, y_1) \otimes y_2 + (-1)^{x y_1} y_1 \otimes \tilde{q}_{210}(x, y_2)
\]

for all \(x, y_1, y_2 \in C\).

Proof. The proof is a lengthy but straightforward computation. \(\Box\)

Definition 2.19 (Maurer-Cartan element). A Maurer-Cartan element for an IBL∞-algebra IBL∞(C) from Definition 2.17 is a collection \(n := (n_\ell)_{\ell \geq 1, g \geq 0}\) of
(a) The term \( q_{k_1 l_1 g_1} \circ_{h} q_{k_2 l_2 g_2} \) in the IBL\( \infty \)-equation \((20)\).

(b) The term \( q_{k' l' g'} \circ_{h_1,...,h_r} (n_{l_1 g_1},...,n_{l_r g_r}) \) in the Maurer-Cartan equation \((22)\). We remark that the contour of the surface corresponding to \( q_{k' l' g'} \) starts on the left and continues to the right along the dotted line behind the two trivial cylinders.

(c) The term \( q_{k' l' g'} \circ_{h_1,...,h_r} (n_{l_1 g_1},...,n_{l_r g_r}) \) in the twisted operation \((23)\). The remark to Figure (b) applies too.

Figure 1: Graphical representation of compositions appearing in Definitions 2.17, 2.19 and 2.20 as gluing of connected Riemannian surfaces. The figure is to be read from the top to the bottom, the empty cylinder represents the identity, and the resulting surface must be connected. We emphasize that the gluing is not associative (c.f., weak associativity \((19)\)).

**Elements** \( n_{l g} \in \hat{E}_l C \) which are homogenous, of finite filtration degree and satisfy the following conditions:

1) \( |n_{l g}| = -2d(g - 1) \).

2) \( ||n_{l g}|| \geq \gamma \chi_{l g} \) with \( > \) for \((l, g) = (1, 0), (2, 0) \) (see Definition \((2.17)\) for \( \chi_{l g} \)).

3) The **Maurer-Cartan equation** holds: for all \( l \geq 1, g \geq 0 \), we have

\[
\sum_{r \geq 1} \frac{1}{r!} \sum_{l' k' l_1,...,l_r \geq 1, k_1,...,k_r \geq 1, g_1,...,g_r \geq 0, h_1,...,h_r \geq 1, l_1+...+l_r+l'-k'=l, g_1+...+g_r+g'+k'=g+r, h_1+...+h_r-k'=0} q_{k' l' g'} \circ_{h_1,...,h_r} (n_{l_1 g_1},...,n_{l_r g_r}) = 0,
\]

where we view \( n_{l g} \) as a linear map \( n_{l g} : \hat{E}_0 C = \mathbb{R} \to \hat{E}_l C \) with \( n_{l g}(1) = n_{l g} \).

26
Definition 2.20 (Twisted operations). In the setting of Definition 2.19, the twisted operations \( q_{klg}^n : \hat{E}_k C \to \hat{E}_l C \) for \( k, l \geq 1, g \geq 0 \) are defined by

\[
q_{klg}^n = \sum_{r \geq 0} \frac{1}{r!} \sum_{k', l', j_1, \ldots, j_r \geq 1 \atop g', g_1, \ldots, g_r \geq 0 \atop h_1, \ldots, h_r \geq 1 \atop l_1 + \cdots + l_r + l' - k' = l - k \atop g_1 + \cdots + g_r + g' + k' = g + r + k \atop h_1 + \cdots + h_r - k' = -k} \prod_{i=1}^{r} q_{k' i' l'} \circ_{h_1, \ldots, h_r} (n_{1, g_1}, \ldots, n_{r, g_r}).
\]

(23)

In [10, Proposition 9.3], they prove that \( (q_{klg}^n)_{k,l \geq 1, g \geq 0} \) is again an IBL\(_\infty\)-algebra of bidegree \((d, \gamma)\) on \(C\) — the twisted IBL\(_\infty\)-algebra. We denote it by IBL\(_\infty\)\(_C\).

Let \( (q_{klg}) \) be an IBL\(_\infty\)-algebra on \(C\). The boundary operator \( q_{110} : C[1] \to C[1] \) induces the boundary operator \( \partial_k : E_k C \to E_k C \) for every \( k \in \mathbb{N} \) (see [14]). Because of the finite filtration degree, \( \partial_k \) continuously extends to \( \partial_k : \hat{E}_k C \to \hat{E}_k C \). The following is easy to see using [10]:

\[
q_{klg} \circ 1 \cdot q_{110} = q_{klg} \circ \partial_k, \\
q_{110} \circ 1 \cdot q_{klg} = \partial \circ q_{klg}.
\]

Because \( q_{klg} \) are odd (:= have odd degree), we have

\[
[\partial, q_{klg}] := \partial \circ q_{klg} - (-1)^{\partial q_{klg}} q_{klg} \circ \partial_k
= \partial_k \circ q_{klg} + q_{klg} \circ \partial_k
= q_{110} \circ 1 \cdot q_{klg} \circ 1 \cdot q_{110}.
\]

With this notation, the IBL\(_\infty\)-relations (20) for \((k, l, g) = (2, 1, 0)\) and \((1, 2, 0)\) become \([\partial, q_{210}] = 0\) and \([\partial, q_{120}] = 0\), respectively. Therefore, \( q_{210} \) and \( q_{120} \) descend to the homology.

Definition 2.21 (Homology and the induced IBL-algebra). We define the homology of an IBL\(_\infty\)-algebra IBL\(_\infty\)(\(C\)) by

\[
\mathbb{H}(C)[1] := H(\hat{C}[1], q_{110}).
\]

It is a graded vector space with the induced filtration. If the canonical map \( E_k \mathbb{H}(C) \to H(\hat{E}_k C, \partial_k) \) induces the isomorphism \( \hat{E}_k \mathbb{H}(C) \simeq H(\hat{E}_k C, \partial_k) \), then the induced maps

\[
q_{210} : \hat{E}_2 \mathbb{H}(C) \to \hat{E}_1 \mathbb{H}(C) \quad \text{and} \quad q_{120} : \hat{E}_1 \mathbb{H}(C) \to \hat{E}_2 \mathbb{H}(C)
\]

define an IBL-structure on \( \mathbb{H}(C) \) — the induced IBL-algebra on homology.

If \( n \) is a Maurer-Cartan element for IBL\(_\infty\)(\(C\)), we denote by \( \mathbb{H}_n(C) \) the homology of IBL\(_\infty\)(\(C\)).
Remark 2.22 (BV-formalism). Consider the weight-reduced exterior algebra $\mathcal{E}C$. Let $\mathcal{E}C[[h]]$ and $\mathcal{E}C((h))$ be the spaces of power and Laurent series in a formal variable $h$ of degree $|h| = 2d$ with coefficients in $\mathcal{E}C$, respectively, where $\mathcal{E}C$ is a suitable completion of $\mathcal{E}C$. Operations of an IBL$_\infty$-algebra on $C$ can be encoded in a degree $-1$ operator $\Delta : \mathcal{E}C[[h]] \to \mathcal{E}C[[h]]$ called the BV$_\infty$-operator, while the data of a Maurer-Cartan element $(n_l g)$ give rise to an operator $e^n : \mathcal{E}C[[h]] \to \mathcal{E}C((h))$ called the exponential of $n$. These operators are given by

$$\Delta := \sum_{i \geq 0} \Delta_i + 1 \cdot h^i \quad \text{and} \quad e^n := \sum_{j \in \mathbb{Z}} (e^n)_j h^j,$$

where the maps $\Delta_i, (e^n)_j : \mathcal{E}C \to \mathcal{E}C$ for $i \geq 1, j \in \mathbb{Z}$ are defined by

$$\Delta_i := \sum_{k \geq 1, g \geq 0} \sum_{l \geq 1} \hat{q}_{klg} \quad \text{and} \quad (e^n)_j := \sum_{r=0}^{\infty} \frac{1}{r! \cdot g_1 \cdots g_r} \sum_{g_1 + \cdots + g_r = j} \sum_{l_1, \ldots, l_r \geq 1} n_{l_1 g_1} \odot \cdots \odot n_{l_r g_r}.$$

It can be shown that the IBL$_\infty$-relations (20) and the Maurer-Cartan equation (22) are equivalent to

$$\Delta \circ \Delta = 0 \quad \text{and} \quad \Delta \circ e^n = 0,$$

respectively, and that the BV$_\infty$-operator $\Delta^a$ for the twisted IBL$_\infty$-structure $(q^n_{klg})$ satisfies

$$\Delta^a = e^{-n} \circ \Delta \circ (e^n \cdot).$$

The notation $(e^n \cdot)$ means that we take the input $\cdot$ and multiply it, using the extension of $\mu$ to $\mathcal{E}C[[h]]$, with $e^n$ evaluated at $1 \in \mathcal{E}_0 C = \mathbb{R}$. These facts were shown in [10] using (18).

Remark 2.23 (Weak IBL$_\infty$-algebras). A possible generalization of the IBL$_\infty$-theory above is to allow $k = 0$ and $l = 0$, so that $EC$ must be used instead of $\mathcal{E}C$ in Remark 2.22. Such structures would be called weak IBL$_\infty$-algebras.

In our application in string topology, a canonical dIBL$_\infty$-algebra dIBL$(C)$ with a natural Maurer-Cartan element $n$ are given, and we want to study dIBL$_n(C)$; in particular, we are interested in $\mathbb{H}^n(C)$, IBL$(\mathbb{H}^n(C))$ and possible higher operations on $\mathbb{H}^n(C)$ induced by $q^n_{klg}$ (these are not chain maps in general). The following proposition summarizes some observations in this situation:

Proposition 2.24 (Twist of a dIBL$_\infty$-algebra). Let $dIBL(C) = (C, q_{110}, q_{210}, q_{120})$

\footnote{One has to check that the compositions (23) and (25) are well-defined and pick a suitable completion $\mathcal{E}C$ so that all the constructions work. The details will be discussed in [10].}
be a dIBL-algebra, and let $n = (n_{lg})$ be a Maurer-Cartan element. The Maurer-Cartan equation \(^{(22)}\) reduces to the following:

$$0 = q_{110} \circ_1 n_{lg} + q_{120} \circ_1 n_{l-1,g} + q_{210} \circ_2 n_{l+1,g-1} + \frac{1}{2} \sum_{\substack{l_1,l_2 \geq 1 \\ g_1,g_2 \geq 0 \\ l_1 + l_2 = l + 1 \\ g_1 + g_2 = g}} q_{210} \circ_{1,1} (n_{l_1 g_1}, n_{l_2 g_2}) \quad \forall l \geq 1, g \geq 0.$$ 

In particular, the “lowest” equation is given by

$$(l, g) = (1, 0) : \quad q_{110}(n_{10}) + \frac{1}{2} q_{210}(n_{10}, n_{10}) = 0. \quad \text{(26)}$$

This can be visualized as

\[ 0 = n_{10} + \frac{1}{2} n_{10} \]

The twisted IBL\(\infty\)-algebra dIBL\(n\)(C) consists of the operations $q^n_{110}$, $q^n_{210}$ and $q^n_{120}$, which we call the basic operations, and of the operations $q^n_{1 l g}$ for $(l, g) \in \mathbb{N} \times \mathbb{N} \times \{ (1, 0), (2, 0) \}$, which we call the higher operations. These operations are given by

$$q^n_{110} = q_{110} + q_{210} \circ 1 n_{10}, \quad q^n_{210} = q_{210}, \quad q^n_{120} = q_{120} + q_{210} \circ 1 n_{20}, \quad q^n_{1 l g} = q_{210} \circ 1 n_{lg}.$$ 

This can be visualized as

$q^n_{110} = \quad + \quad n_{10} \quad q^n_{210}$

$q^n_{210} = \quad q^n_{210}$

*In [10, Definition 2.4.], they define a partial ordering on the signatures $(k, l, g)$. \(\text{\[29\]}\)
The IBL\(_\infty\)-relations satisfied by \((q_{k lg}^n)\) read for all \(l \geq 1, g \geq 0\) as follows:

\[
\begin{align*}
(3, 1, 0) : & \quad 0 = q_{210}^n \circ_1 q_{210}^n, \\
(2, l, g) : & \quad 0 = q_{l g}^n \circ_1 q_{210}^n + q_{210}^n \circ_1 q_{l g}^n, \\
(1, l, g) : & \quad 0 = \sum_{l_1, l_2 \geq 1, g_1, g_2 \geq 0, l_1 + l_2 = l + 1, g_1 + g_2 = g} q_{l_1 g_1}^n \circ_1 q_{l_2 g_2}^n + q_{210}^n \circ 2 q_{1, l + 1, g - 1}^n. 
\end{align*}
\]

We call the relations for \((k, l, g) = (1, 1, 0), (2, 1, 0), (1, 2, 0), (3, 1, 0), (1, 3, 0), (2, 2, 0), (1, 1, 1)\) basic relations because they contain all compositions of basic operations. In the order above, they read:

\[
\begin{align*}
0 = q_{110}^n \circ_1 q_{110}^n, \\
0 = q_{110}^n \circ_1 q_{210}^n + q_{210}^n \circ_1 q_{110}^n, \\
0 = q_{110}^n \circ_1 q_{120}^n + q_{120}^n \circ_1 q_{110}^n, \\
0 = q_{210}^n \circ_1 q_{210}^n, & \quad \leftarrow \text{Jacobi identity} \\
0 = q_{120}^n \circ_1 q_{120}^n + q_{110}^n \circ_1 q_{130}^n + q_{130}^n \circ_1 q_{110}^n, & \quad \leftarrow \text{co-Jacobi id. up to htpy.} \\
0 = q_{120}^n \circ_1 q_{210}^n + q_{210}^n \circ_1 q_{120}^n, & \quad \leftarrow \text{Drinfeld identity} \\
0 = q_{210}^n \circ 2 q_{120}^n + q_{111}^n \circ_1 q_{110}^n + q_{110}^n \circ_1 q_{111}^n. & \quad \leftarrow \text{Involutivity up to htpy.}
\end{align*}
\]

The last four equations can be visualized as

\[
0 = \begin{array}{c}
\text{Diagram of last four equations.}
\end{array}
\]
Proof. The proof is clear by specializing (20), (22) and (23).

Remark 2.25 (Higher operations). We see from Proposition 2.24 that if \( q^n_{120} \circ_1 q^n_{120} = 0 \) and \( q^n_{120} \circ_2 q^n_{120} = 0 \), then \([\partial^n, q^n_{130}] = 0\) and \([\partial^n, q^n_{111}] = 0\), respectively, and hence the operations \( q^n_{130} : \hat{E}_1^H \to \hat{E}_3^H \) and \( q^n_{111} : \hat{E}_1^H \to \hat{E}_3^H \) are well-defined (provided that the assumption of Definition 2.21 holds). Likewise, the higher operation \( q^n_{1lg} \) defines a map \( \hat{E}_1^H \to \hat{E}_l^H \), provided that the following equation holds:

\[
q^n_{210} \circ_2 q^n_{1f+1,g-1} + \sum_{\substack{l_1,l_2 \geq 1 \\
(l_1l_2=0) \\
(l_1+l_2=a+1) \\
g_1+g_2=g \\
(l_1,g_1) \neq (1,0)} q^n_{l_1g_1} \circ_1 q^n_{l_2g_2} = 0.
\]

This expression is just the left-over after subtracting the commutator \([q^n_{1lg}, q^n_{110}] = q^n_{110} \circ_1 q^n_{1lg} + q^n_{1lg} \circ_1 q^n_{110}\) from \([27]\).

2.3 Dual cyclic bar complex and cyclic cohomology

Definition 2.26 (Bar complexes). Let \( V \) be a graded vector space. The bar- and dual bar-complex of \( V \) are the weight-graded vector spaces defined by

\[
B_*V := \bar{T}(V[1]) \quad \text{and} \quad B^*V := (B_*V)'',
\]

respectively, where \( \bar{T}V := \bigoplus_{k=1}^{\infty} V^\otimes k \) is the weight-reduced tensor algebra. For every \( k \in \mathbb{N} \), let \( t_k \in S_k \) be the cyclic permutation \( t_k : (1, \ldots, k) \mapsto (2, \ldots, k, 1) \).
so that for all $v_1, \ldots, v_k \in V[1]$ we have
\[ t_k(v_1 \otimes \cdots \otimes v_k) = (-1)^{|v_k|(|v_1| + \cdots + |v_{k-1}|)} v_k \otimes v_1 \otimes \cdots \otimes v_{k-1}. \]

We set
\[ t := \sum_{k=1}^\infty t_k : B_* V \to B_* V. \]

The cyclic bar-complex is defined by
\[ B^cyc_* V := B_* V / \text{Im}(1 - t). \]

We denote the image of $v_1 \otimes \cdots \otimes v_k \in B_* V$ under the canonical projection $\pi : B_* V \to B^cyc_* V$ by $v_1 \ldots v_k$. If $v_i \in V[1]$ are homogenous, then $v_1 \ldots v_k$ is called a generating word; we have
\[ v_1 \ldots v_k = (-1)^{|v_k|(|v_1| + \cdots + |v_{k-1}|)} v_k v_1 \ldots v_{k-1}. \]

We define the section $\iota : B^cyc_* V \to B_* V$ of $\pi$ by
\[ \iota(v_1 \ldots v_k) := \frac{1}{k} \sum_{i=0}^{k-1} t_k^i (v_1 \otimes \cdots \otimes v_k) =: t_k \circ \cdots \circ t_k \text{ i-times} \]
and use it to identify $B^cyc_* V$ with the subspace $\text{Im} \iota = \ker(1 - t) \subset B_* V$ consisting of cyclic symmetric tensors.

We define the dual cyclic bar-complex by
\[ B^\ast_{\text{cyc}} V := \{ \psi \in B^\ast V \mid \psi \circ t = \psi \}. \]

Remark 2.27 (Non-weight-reduced bar complex). In fact, our $B^\ast_{\text{cyc}} V$ is weight-reduced. The non-weight-reduced version would be $B^\ast_{\text{cyc}} V \oplus \mathbb{R}$ with $\mathbb{R}$ of degree 0. This might play a role in the theory of weak $A_{\infty}$-algebras (:= operation $\mu_0$ added; c.f., Definition 2.32), and it might also be possible to consider IBL$_{\infty}$-algebras on non-weight-reduced cyclic cochains (c.f., Section 2.4). This may be discussed more in [19].

Notice that $\psi \in B^\ast V$ is homogenous of degree $|\psi| \in \mathbb{Z}$ if and only if for all homogenous $v_1, \ldots, v_k \in V[1]$ the following implication holds:
\[ |v_1| + \cdots + |v_k| \neq |\psi| \implies \psi(v_1 \otimes \cdots \otimes v_k) = 0. \]

This is the cohomological grading convention.

Notation 2.28 (Degree shifts of bar complexes). Let $A \in \mathbb{Z}$. In the following, we write $B^\ast_{\text{cyc}} V$, but the convention applies to all complexes from Definition 2.26.
We denote by $s_A$ and $\theta$ the formal symbols of degrees

$$|s_A| = -A \quad \text{and} \quad |\theta| = -1,$$

respectively. The degree shift $V \mapsto V[1]$ will be realized as multiplication with $\theta$ and the degree shift $B^*_\text{cyc} V \mapsto B^*_\text{cyc} V[A]$ as multiplication with $s_A$. In addition, the following notation will be used consistently:

- $\tilde{v} \in V \mapsto v = \theta \tilde{v} \in V[1]$
  
  To clarify this, given $\tilde{v} \in V$, then $v$ automatically means $v = \theta \tilde{v} \in V[1]$, and the other way round. Recall that the degree of $\tilde{v} \in V$ is denoted by $\deg(\tilde{v})$ or simply by $\tilde{v}$ in the exponent, e.g., $(-1)^{\tilde{v}}$.

- $\psi \in B^*_\text{cyc} V \mapsto \Psi = s_A \psi \in B^*_\text{cyc} V[A]$. 

- A generating word of $B^*_\text{cyc} V$ of weight $k$ will be denoted by the symbol $w$ and written as $w = v_1 \cdots v_k$, where $v_i = \theta \tilde{v}_i \in V[1]$. A generating word of $E_k B^*_\text{cyc} V$ is an element $w_1 \cdots w_k \in E_k B^*_\text{cyc} V$ such that each $w_i$ is a generating word of $B^*_\text{cyc} V$.

- $w \in B^*_\text{cyc} V \mapsto W = s_A w \in B^*_\text{cyc} V[A]$.

We abbreviate

$$B^*_\text{cyc} V[A] := (B^*_\text{cyc} V)[A].$$

In contrast to this, we would write $B^*_\text{cyc}(V[A])$ for the dual cyclic bar-complex of $V[A]$. We also identify $(B^*_\text{cyc} V[A])[1] = B^*_\text{cyc} V[A + 1]$ in $EB^*_\text{cyc} V[A]$.

**Definition 2.29** (Pairing of tensor powers of bar complexes). For every $A \in \mathbb{Z}$ and $k \in \mathbb{N}$, we define the pairing as follows:

$$(B^* V[A])^{\otimes k} \otimes (B_* V[A])^{\otimes k} \longrightarrow \mathbb{R}$$

$$(\Psi_1 \otimes \cdots \otimes \Psi_k, W_1 \otimes \cdots \otimes W_k) \longrightarrow \psi_1(w_1) \cdots \psi_k(w_k). \quad (28)$$

This means that we evaluate elements from the left-hand side on the elements from the right-hand side in this way without any signs (see the discussion in Remark 2.35). We extend the pairing by $0$ if the number of $\Psi_i$‘s and the number of $W_i$’s differ.

**Remark 2.30** (Dual bar complex and dual of the bar complex). Because the pairing (28) is non-degenerate, we can embed the space on the left into the the linear dual of the space on the right. From Definition 2.26 we have $B^*_\text{cyc} V \subset B^* V$, and $B^*_\text{cyc} V$ is identified with $\text{Im} \; \iota \subset B_* V$. Therefore, we can restrict (28) to obtain the pairing of $B^*_\text{cyc} V$ and $B^*_\text{cyc} V$. It is easy to see that for any $\psi \in B^*_\text{cyc} V$
and any generating word \(v_1 \ldots v_k \in B^{\text{ cyc}} V\), we have

\[
\psi(v_1 \ldots v_k) = \psi(v_1 \otimes \cdots \otimes v_k).
\]

The subspace of \((B^{\text{ cyc}} V)^*\) corresponding to \(B^* \text{ cyc} V\) is then precisely \((B^{\text{ cyc}} V)^*\).

More generally, for every \(k \in \mathbb{N}\), the spaces \(E_k B^* \text{ cyc} V\) and \(E_k B^{\text{ cyc}} V\) are embedded into \((B^{\text{ cyc}} V[1])^\otimes k\) and \((B^{\text{ cyc}} V[1])^\otimes k\), respectively, using \(\iota\) and \(\pi\) from Definition 2.7. Therefore, the restriction of (28) gives the pairing of \(E_k B^* \text{ cyc} V\) and \(E_k B^{\text{ cyc}} V\). It is easy to see that for any generating word \(w_1 \cdots w_k \in E_k B^{\text{ cyc}} V\) and any \(\psi_1 \cdots \psi_k \in E_k B^* \text{ cyc} V\), we have

\[
(\psi_1 \cdots \psi_k)(w_1 \cdots w_k) = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \epsilon(\sigma, w) \psi_1(w_{\sigma^{-1} 1}) \cdots \psi_k(w_{\sigma^{-1} k}).
\]

The subspace of \((E_k B^* \text{ cyc} V)^*\) corresponding to \(E_k B^* \text{ cyc} V\) lies in \((E_k B^{\text{ cyc}} V)^*\); it is equal to \((E_k B^{\text{ cyc}} V)^*\), provided that \(V\) is finite-dimensional.\footnote{The problem is that if \(\dim(V) = \infty\), then \((V \otimes V)^* \neq V^* \otimes V^*\).}

The weight-graded vector spaces \(B_* V\) and \(B^* \text{ cyc} V\) are canonically filtered by the filtration by weights (12). Their weight-graded duals \(B^* V\) and \(B^{\text{ cyc}} V\) are filtered by the dual filtrations and the exterior powers \(E_k B^* V\) and \(E_k B^{\text{ cyc}} V\) by the induced filtration from Definition 2.8.

**Proposition 2.31** (Completed dual cyclic bar complex). Let \(V\) be a graded vector space and \(A \in \mathbb{Z}\). The filtration of \(B^* \text{ cyc} V\) dual to the weight-filtration of \(B^* \text{ cyc} V\) is \(\mathbb{Z}\)-gapped, Hausdorff, decreasing and bounded from above. Moreover, the following holds:

\[
\dim(V) < \infty \implies (WG1) \& (WG2) \text{ are satisfied.}
\]

The same holds for the induced filtration of \(E_k B^* \text{ cyc} V[A]\).

In the sense of Remark 2.30, we have

\[
\hat{B}^* \text{ cyc} V \simeq (B^* \text{ cyc} V)' \quad \text{and} \quad \hat{E}_k B^* \text{ cyc} V[A] \subset (E_k B^* \text{ cyc} V[A + 1])',
\]

where “\(\simeq\) holds if \(V\) is finite-dimensional.

The filtration degree of \(\Psi \in \hat{E}_m B^* \text{ cyc} V[A]\) satisfies

\[
\|\Psi\| = \min \{k \in \mathbb{N}_0 \mid \exists W \in (E_m B^* \text{ cyc} V[A])_k : \Psi(W) \neq 0\}.
\]

**Proof.** The proof is clear. \(\square\)

**Definition 2.32** (Cyclic \(\Lambda_\infty\)-algebra). A graded vector space \(V\) together with a pairing

\[
\mathcal{P} : V[1] \otimes V[1] \to \mathbb{R}
\]
of degree $d \in \mathbb{Z}$ and a collection of homogenous linear maps

$$\mu_k : V[1]^{\otimes k} \to V[1] \quad \text{for } k \geq 1$$

is called a cyclic $\Lambda_\infty$-algebra of degree $d$ if the following conditions are satisfied:

1. The pairing $\mathcal{P}$ is non-degenerate and graded antisymmetric; i.e., we have
   $$\mathcal{P}(v_1, v_2) = (-1)^{1+|v_1||v_2|} \mathcal{P}(v_2, v_1) \quad \text{for all } v_1, v_2 \in V[1].$$

2. The degrees satisfy $|\mu_k| = 1$ for all $k \geq 1$.

3. The $\Lambda_\infty$-relations are satisfied: for all $k \geq 1$, we have
   $$\sum_{k_1, k_2 \geq 1} \sum_{p=1}^{k_1} \mu_{k_1} \circ_{p} \mu_{k_2} = 0,$$
   where for all $p = 1, \ldots, k$ and $v_1, \ldots, v_k \in V[1]$ we define
   $$(\mu_{k_1} \circ_{p} \mu_{k_2})(v_1, \ldots, v_k) := (-1)^{|v_1|+\cdots+|v_{p-1}|} \mu_{k_1}(v_1, \ldots, v_{p-1}, v_p, \ldots, v_{k_2-1}, v_{k_2} \ldots, v_k).$$

4. The operations $\mu_k^+ : V[1]^{\otimes k+1} \to \mathbb{R}$ defined by
   $$\mu_k^+ := \mathcal{P} \circ (\mu_k \otimes 1)$$
   for all $k \geq 1$ are cyclic symmetric; i.e., we have
   $$\mu_k^+ \circ l_{k+1} = \mu_k^+.$$

We denote by $\hat{\mathcal{P}} : V \otimes V \to \mathbb{R}$ and $\hat{\mu}_k : V^{\otimes k} \to \mathbb{R}$ the operations before the degree shift; i.e., for all $k \geq 1$ and $\tilde{v}_1, \ldots, \tilde{v}_k \in V$ with $v_i = \theta \tilde{v}_i$, we have

$$\hat{\mathcal{P}}(\tilde{v}_1, \tilde{v}_2) := (-1)^{|\tilde{v}_1||\tilde{v}_2|} \mathcal{P}(v_1, v_2) \quad \text{and}$$

$$\hat{\mu}_k(\tilde{v}_1, \ldots, \tilde{v}_k) := \varepsilon(\theta, \tilde{v}) \mu_k(v_1, \ldots, v_k).$$

We define $\hat{\mu}_k^+ : V^{\otimes k+1} \to \mathbb{R}$ similarly.

If $\mu_k \equiv 0$ for all $k \geq 2$, then $(V, \mathcal{P}, \mu_1)$ is called a cyclic cochain complex. If $\mu_k \equiv 0$ for all $k \geq 3$, then $(V, \mathcal{P}, \mu_1, \mu_2)$ is called a cyclic dga. We use the same terminology but omit “cyclic” if there is no pairing $\mathcal{P}$ and 1) and 4) are thus irrelevant.

Remark 2.33 (A difference in sign conventions). Our definition of $\mu_k^+$ differs from the definition of $m_k^+$ in \cite{10} Definition 12.1] by a sign. To compensate this, we
have to add this artificial sign in the definitions of Maurer-Cartan elements later; e.g., in Definition 2.43 or in the formula (100).

\[\text{Definition 2.34 (Cyclic (co)homology of } A_{\infty}\text{-algebras). Let } A = (V, (\mu_k)) \text{ be an } A_{\infty}\text{-algebra. For every } k \geq 1, \text{ we consider the maps } b^k, R^k : V[1]^\otimes k \to B_* V \text{ given by} \]
\[
b^k := \sum_{j=1}^{k} \sum_{i=0}^{k-j} \mu_j \otimes t_{k-j+1} \circ (\mu_j \otimes 1_{k-j}) \circ t_{i}, \quad \text{and} \]
\[
R^k := \sum_{j=1}^{k} \sum_{i=1}^{j-1} \mu_j \otimes t_{k-j} \circ t_{i}, \quad \text{(30)}
\]

respectively, and define the following maps \(B_* V \to B_* V\):
\[
b' := \sum_{k=1}^{\infty} b^k, \quad R := \sum_{k=2}^{\infty} R^k \quad \text{and} \quad b := b' + R.
\]

We denote by \(b^* : \hat{B}^* V = (B_* V)' \to \hat{B}^* V\) the dual map to \(b : B_* V \to B_* V\). The following holds:
\[
|b| = 1 \quad (|b^*| = -1), \quad b \circ b = 0 \quad \text{and} \quad b(1-t) = (1-t)b'. \quad \text{(31)}
\]

From the last equation we see that \(b\) restricts to \(B_{\text{cyc}}^* V = B_* V / \text{Im}(1-t)\). We define the following graded vector spaces:
\[
D_\lambda^*(V) := r(B_{\text{cyc}}^* V)[1], \quad D^*(V) := r(\hat{B}^* V)[1],
\]
\[
D_\lambda^*(V) := r(B_{\text{cyc}}^* V)[1], \quad D_\lambda^*(V) := r(\hat{B}_{\text{cyc}}^* V)[1].
\]

For instance, we have
\[
D_\lambda^*(V) = r(\hat{B}_{\text{cyc}}^* V)^{q+1} = (\hat{B}_{\text{cyc}}^* V)^{-q-1} \quad \text{for all } q \in \mathbb{Z}.
\]

Then \((D_\lambda^*(V), b)\) and \((D^*(V), b^*)\) are chain complexes and \((D^*(V), b^*)\) and \((D_\lambda^*(V), b^*)\) the dual cochain complexes, respectively. We define the following \((\co)\)homologies:
\[
\text{HH}_\lambda(A; \mathbb{R}) := H(D_\lambda^*(V), b), \quad \text{HH}^*(A; \mathbb{R}) := H(D^*(V), b^*),
\]
\[
\text{H}_\lambda^*(A; \mathbb{R}) := H(D_\lambda^*(V), b), \quad \text{H}^*_\lambda(A; \mathbb{R}), := H(D^*_\lambda(V), b^*).
\]

We call \(\text{HH}_\lambda\) the Hochschild homology and \(\text{H}^*_\lambda\) the cyclic homology of the \(A_{\infty}\)-algebra \(A\). We call \(\text{HH}^*\) the Hochschild cohomology and \(\text{H}^*_\lambda\) the cyclic cohomology of \(A\).

\[\text{6The facts (31) are generally known in some form (see [27] or [23]). We also show them in [19] using a graphical formalism which simplifies computations.}\]
For a dga $\mathcal{A} = (V, \mu_1, \mu_2)$, we have for all $v_1, \ldots, v_k \in V[1]$ the formula

$$b(v_1 \ldots v_k) = \sum_{i=1}^{k} (-1)^{|v_i| + \cdots + |v_{i-1}|} v_1 \ldots \mu_1(v_i) \ldots v_k$$

$$+ \sum_{i=1}^{k-1} (-1)^{|v_i| + \cdots + |v_{i-1}|} v_1 \ldots \mu_2(v_i, v_{i+1}) \ldots v_k$$

$$+ (-1)^{|v_k|(|v_1| + \cdots + |v_{k-1}|)} \mu_2(v_k, v_1) v_2 \ldots v_{k-1}.$$

**Definition 2.35** (Strict units and strict augmentations). Let $\mathcal{A} = (V, (\mu_k))$ be an $A_\infty$-algebra. A non-zero homogenous element $1 \in V[1]$ with $|1| = -1$ is called a strict unit for $\mathcal{A}$ if the following holds:

$$\mu_2(1, v) = (-1)^{|v|+1} \mu_2(v, 1) = v \quad \forall v \in V[1],$$

$$\mu_k(v_1, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_k) = 0 \quad \forall k \neq 2, 1 \leq i \leq k, v_j \in V[1].$$

The pair $(\mathcal{A}, 1)$ is called a strictly unital $A_\infty$-algebra.

A strictly unital $A_\infty$-algebra $(\mathcal{A}, 1)$ is called strictly augmented if it is equipped with a linear map $\epsilon : V[1] \to \mathbb{R}[1]$ which satisfies

$$\epsilon(1_V) \equiv 1_{\mathbb{R}}, \quad \epsilon \circ \mu_1 = 0 \quad \text{and} \quad \epsilon \circ \mu_2 = \mu_2 \circ (\epsilon \otimes \epsilon),$$

where $1_{\mathbb{R}}$ is the strict unit for $\mathbb{R}$ endowed with the standard multiplication. The map $\epsilon$ is called a strict augmentation.

If the homological dga $H(\mathcal{A}) := (H(V, \tilde{\mu}_1, \mu_1 \equiv 0, \mu_2)$ of $\mathcal{A}$ is strictly unital and strictly augmented, then $\mathcal{A}$ is called homologically unital and homologically augmented, respectively. A strictly unital and strictly augmented cochain complex $(V, \mu_1, 1, \epsilon)$ is called just augmented.

We denote by $u : \mathbb{R}[1] \to V[1]$ the injective linear map defined by $u(1_{\mathbb{R}}) := 1_V$, and by $u^* : B^*_cyc V \to B^*_cyc \mathbb{R}$ and $\epsilon^* : B^*_cyc \mathbb{R} \to B^*_cyc V$ the precompositions with $u_{\otimes k}$ and $\epsilon_{\otimes k}$ in every weight-$k$ component, respectively.

**Remark 2.36** (On units and augmentations). (i) A strict unit $1_V$ for $\mathcal{A}$ induces an $A_\infty$-morphism $(u_k) : \mathbb{R} \to V$ given by $u_1(1_{\mathbb{R}}) := 1_V$ and $u_k \equiv 0$ for all $k \geq 2$. A (general) augmentation of $(\mathcal{A}, 1_V)$ is by definition any $A_\infty$-morphism $(\epsilon_k) : V \to \mathbb{R}$ such that $(\epsilon_k) \circ (u_k) \equiv 1$ as $A_\infty$-morphisms (see [20]). Strict augmentations are precisely the maps $\epsilon_1$ coming from augmentations $(\epsilon_k)$ with $\epsilon_k \equiv 0$ for all $k \geq 2$.

(ii) As for $(V, \mu_1, 1, \epsilon)$, we need the chain map $\epsilon$ to provide the splitting of the
short exact sequence of chain complexes
\[
0 \to \mathbb{R}[1] \xrightarrow{u} V[1] \xrightarrow{\text{coker}(u)} 0,
\]
so that we get \( H(V) \simeq H_{\text{red}}(V) \oplus \mathbb{R} \), where \( H_{\text{red}}(V) := H(\text{coker}(u)) \). If \((V, \mu_1)\) is non-negatively graded and we are given an injective chain map \( u : \mathbb{R}[1] \to V[1] \) (=: the classical augmentation), then one can show that such \( \varepsilon \) always exists. \( \triangleleft \)

**Definition 2.37** (Reduced dual cyclic bar complex). Let \((A, \mu)\) be a strictly unital \(A_{\infty}\)-algebra. Consider the injection \( \iota : B_* V \to B_* V, v_1 \otimes \cdots \otimes v_k \mapsto \varepsilon \otimes v_1 \otimes \cdots \otimes v_k \). We define the reduced dual cyclic bar-complex by
\[
B_{\text{cyc,red}}^* V := \{ \psi \in B_{\text{cyc}}^* V \mid \psi \circ \iota = 0 \}.
\]
Under the assumption of strict unitality, \( b^* \) preserves \( B_{\text{cyc,red}}^* V \), and hence we can consider the reduced cyclic cochain complex
\[
D_{\lambda,\text{red}}^*(V) := r(B_{\text{cyc}}^* V)[1]
\]
and define the reduced cyclic cohomology of \( A \) by
\[
H_{\lambda,\text{red}}^*(A; \mathbb{R}) := H(D_{\lambda,\text{red}}^*(V), b^*).
\]

**Proposition 2.38** (Reduction to the reduced cyclic cohomology). Let \( A = (V, \mu) \) be an \( A_{\infty}\)-algebra with a strict unit 1 and a strict augmentation \( \varepsilon \). Then the inclusions \( B_{\text{cyc,red}}^* V, \varepsilon^*(B_{\text{cyc}}^* \mathbb{R}) \subseteq B_{\text{cyc}}^* V \) induce the decomposition
\[
H_{\lambda}^*(A; \mathbb{R}) \simeq H_{\lambda,\text{red}}^*(A; \mathbb{R}) \oplus H_{\lambda}^*(\mathbb{R}; \mathbb{R}).
\]
Here we have
\[
H_{\lambda}^q(\mathbb{R}; \mathbb{R}) = \begin{cases} 
1^{q+1} & \text{for } q \geq 0 \text{ even}, \\
0 & \text{for } q > 0 \text{ odd and } q < 0,
\end{cases}
\]
where \( 1^i : [\mathbb{R}[1]]^{\oplus i} \to \mathbb{R} \) is defined by \( 1^i(1^i) = 1 \).

**Sketch of the proof.** The maps \( \varepsilon^* : D_{\lambda}(\mathbb{R}) \to D_{\lambda}(V) \) and \( u^* : D_{\lambda}(V) \to D_{\lambda}(\mathbb{R}) \) are chain maps with \( u^* \circ \varepsilon^* = 1 \). Therefore, we have the sequence of cochain complexes
\[
0 \to D_{\lambda,\text{red}}(V) \xrightarrow{u^*} D_{\lambda}(V) \xrightarrow{\varepsilon} D_{\lambda}(\mathbb{R}) \to 0,
\]
which is exact everywhere except for the middle, and where $\epsilon^*$ is a splitting map.

The idea of [24] is to replace these cochain complexes with quasi-isomorphic bicomplexes consisting of normalized Hochschild cochains $\bar{D}(V)$ such that the sequence becomes exact. The work then reduces to proving that $\bar{D}(V)$ computes $\text{HH}(A;\mathbb{R})$; a variant of this result for $A_\infty$-algebras was proven in [23]. A detailed proof in our formalism will be provided in [19].

We will now compare our version of the cyclic cohomology of a dga $(V,\mu_1,\mu_2)$ to a version based on [24, Section 5.3.2]. Let $b, \delta : TV \to TV$ be the linear maps defined for all $\tilde{v}_1, \ldots, \tilde{v}_k \in V$ by

$$
\tilde{b}(\tilde{v}_1 \otimes \cdots \otimes \tilde{v}_k) := \sum_{i=1}^{k-1} (-1)^{i-1} \tilde{v}_1 \otimes \cdots \otimes \tilde{\mu}_2(\tilde{v}_i, \tilde{v}_{i+1}) \otimes \cdots \otimes \tilde{v}_k + \sum_{i=1}^{k-1} (-1)^{k-1+i} \tilde{\mu}_2(\tilde{v}_i, \tilde{v}_1) \otimes \cdots \otimes \tilde{\mu}_2(\tilde{v}_k, \tilde{v}_1) \otimes \cdots \otimes \tilde{\mu}_2(\tilde{v}_k, \tilde{v}_1) \otimes \cdots \otimes \tilde{v}_{k-1},
$$

$$
\tilde{\delta}(\tilde{v}_1 \otimes \cdots \otimes \tilde{v}_k) := \sum_{i=1}^{k} \sum_{j=1}^{k} (-1)^{k+1+i} \tilde{v}_1 \otimes \cdots \otimes \tilde{\mu}_1(\tilde{v}_i) \otimes \cdots \otimes \tilde{v}_k.
$$

For all $q \geq 0$, we define

$$
\tilde{D}_q(V) := \bigoplus_{d \in \mathbb{Z}} (V^\otimes_k)^d
$$

and $\tilde{\partial} : \tilde{D}_{q+1}(V) \to \tilde{D}_q(V)$ by

$$
\tilde{\partial}(\tilde{v}_1 \cdots \tilde{v}_k) = \tilde{b}(\tilde{v}_1 \cdots \tilde{v}_k) + (-1)^{k+1} \tilde{\delta}(\tilde{v}_1 \cdots \tilde{v}_k).
$$

It can be checked that $\tilde{\partial} \circ \tilde{\partial} = 0$ and $\tilde{\partial}(\text{Im}(1-\tilde{t})) \subset \text{Im}(1-\tilde{t})$, so that $\tilde{\partial}$ induces a boundary operator on the chain complexes

$$
\tilde{D}_*(V) := \bigoplus_{q \in \mathbb{Z}} \tilde{D}_q(V) \quad \text{and} \quad \tilde{D}_*^\lambda(V) := \tilde{D}_*(V) / \text{Im}(1-\tilde{t}).
$$

Here, we have $\tilde{t}(\tilde{v}_1 \cdots \tilde{v}_k) := (-1)^{k+|\tilde{v}_1|+\cdots+|\tilde{v}_k-1|} \tilde{v}_1 \cdots \tilde{v}_k$. We call $(\tilde{D}_*(V), \tilde{\partial})$ the classical Hochschild complex and $(\tilde{D}_*^\lambda(V), \tilde{\partial})$ the classical cyclic complex of the dga $(V,\mu_1,\mu_2)$. The chain complex $(\tilde{D}_*(V), \tilde{\partial})$ is the total
complex of the bicomplex

\[
\begin{array}{ccccc}
(V^\otimes 3)^1 & \xleftarrow{\delta} & (V^\otimes 3)^0 & \xleftarrow{\delta} & (V^\otimes 3)^{-1} \\
(V^\otimes 2)^1 & \xleftarrow{\delta} & (V^\otimes 2)^0 & \xleftarrow{\delta} & (V^\otimes 2)^{-1} \\
V^1 & \xleftarrow{\delta} & V^0 & \xleftarrow{\delta} & V^{-1}
\end{array}
\]

which differs from the bicomplex \[24\] Equation (5.3.2.1) by the reversed grading and by the fact that it lies in the whole upper half-plane and not just in the first quadrant. Their convention for a dga is namely \(|\tilde{\mu}_1| = -1\), whereas ours is \(|\tilde{\mu}_1| = 1\), and they consider \(\mathbb{N}_0\)-grading, whereas we have \(\mathbb{Z}\)-grading.

**Proposition 2.39** (The classical case). Let \(A = (V, \mu_1, \mu_2)\) be a dga. Then the degree shift map

\[
U : \tilde{D}_q(V) \longrightarrow D_0(V),
\]

\[
v_1 \otimes \cdots \otimes v_k \longmapsto \varepsilon(\theta, \tilde{v}) v_1 \otimes \cdots \otimes v_k,
\]

where we denote \(v_i = \theta \tilde{v}_i\), is an isomorphism of the chain complexes \((\tilde{D}_q(V), \bar{\partial}) \simeq (D_0(V), b)\) and \((\tilde{D}^\Lambda_q(V), \bar{\partial}) \simeq (D^\Lambda_0(V), b)\), respectively.

**Proof.** First of all, for the degrees holds \(|\tilde{\mu}_j| = 2 - j\) for every \(j \geq 1\). For every \(j, k, l \geq 1\) such that \(j + l \leq k + 1\) and for every \(\tilde{v}_1, \ldots, \tilde{v}_k \in V\), we compute

\[
[U^{-1}(\mathbb{I}^{l-1} \otimes \mu_j \otimes \mathbb{I}^{k-j-l+1})U] (\tilde{v}_1 \cdots \tilde{v}_k)
= (-1)^{1-l+2(j-l)} \tilde{v}_1 \cdots \tilde{v}_j \tilde{\mu}_j (\tilde{v}_{j+1} \cdots \tilde{v}_k)
\]

where we use the Koszul convention \((f_1 \otimes f_2)(v_1 \otimes v_2) = (-1)^{|f_2||v_1|} f_1(v_1) \otimes f_2(v_2)\). Using this, we obtain

\[
U^{-1}b^k U = \sum_{j=1}^k \sum_{i=0}^{k-1} (-1)^{j+i(k+1)} t_{k-j-1} \tilde{\mu}_j \otimes \mathbb{I}^{k-j} t_{j}^{-1}
\]

and

\[
U^{-1}R^k U = \sum_{j=1}^k \sum_{i=1}^{j-1} (-1)^{(i+j)(k+1)} (\tilde{\mu}_j \otimes \mathbb{I}^{k-j}) t_{i}^{k-j}.
\]

It is now easy to check that \(U^{-1} \circ b \circ U = \bar{\partial}\).

If \(k \in \mathbb{N}\) is a weight and \(d \in \mathbb{Z}\) a degree such that \(k - d - 1 = q\) for some \(q \in \mathbb{Z}\), we have schematically \(U : (k, d) \mapsto (k, d - k) = (k, -q - 1)\). Therefore, \(U\) preserves the grading of chain complexes. This finishes the proof. \(\square\)
Proposition 2.40 (Reduced cochains are complete in 0,1-connected case).
Suppose that $V = \bigoplus_{d \geq 0} V^d$ is a non-negatively graded vector space with $V^0 = \langle 1 \rangle$ for some $1 \in V$ $(=: V$ is connected) and $V^1 = 0$ $(=: V$ is simply-connected). Then for all $m \geq 1$, we have

$$\hat{E}_m B^{\text{cyc,red}}_V = E_m B^{\text{cyc,red}}_V.$$ 

Proof. Let $\tilde{V} := \bigoplus_{d \geq 2} V^d$. We clearly have $B^{\text{cyc,red}}_V \simeq B^{\text{cyc}}_\tilde{V}$. Since $\tilde{V}[1]$ is positively graded, we have $(B^{\text{cyc}}_\tilde{V})^k_d = 0$ whenever $k > d$. Therefore, a map $\Psi \in \hat{E}_m \tilde{V}$, which is non-zero only on finitely many homogenous components of $B^{\text{cyc}}_\tilde{V}[1] \otimes^m$, will be non-zero only on finitely many weights. This implies that $\Psi \in E_m \tilde{V}$. \hfill \Box

Remark 2.41 (Universal coefficient theorem). Because $(D^*_\lambda(V), b^*)$ is dual to $(D^*_\lambda(V), b)$ as a chain complex and because we work over $\mathbb{R}$, the universal coefficient theorem gives

$$H^q_\lambda(A, b^*) \simeq [H^q_\lambda(A, b)]^* \quad \text{for all } q \in \mathbb{Z}.$$ 

Suppose that we have found closed homogenous elements $(w_i)_{i \in I} \subset D^*_\lambda(V)$ for some index set $I$ which induce a basis of $H^*_\lambda(A; \mathbb{R})$. For every $i \in I$, we define the linear map $w^*_i : D^*_\lambda(V) \to \mathbb{R}$ by prescribing

$$w^*_i(w_j) = \delta_{ij} \quad \text{for all } j \in I$$

and $w^*_i \equiv 0$ on $\text{Im } b$ and on a complement of $\ker(b)$ in $D^*_\lambda(V)$. Then $(w^*_i)_{i \in I} \subset D^*_\lambda(V)$ are closed homogenous elements which generate linearly independent cohomology classes in $H^*_\lambda(A; \mathbb{R})$; if we denote $I_q := \{ i \in I \mid w_i \in C^*_q(V) \}$, then we can write

$$H^q_\lambda(A; \mathbb{R}) = \left\{ \sum_{i \in I_q} \alpha_i w^*_i \mid \alpha_i \in \mathbb{R} \right\} \quad \text{for all } q \in \mathbb{Z}. \quad \checkmark$$

2.4 Canonical dIBL-structure on cyclic cochains

In this section, we will consider a finite-dimensional cyclic dga $(V, \mathcal{P}, m_1, m_2)$ of degree $2 - n$ for some $n \in \mathbb{N}$. This means that for all $v_1, v_2, v_3 \in V[1]$, the
We define the tensor
\[ \mathcal{P}(v_1, v_2) = (-1)^1|v_1||v_2|\mathcal{P}(v_2, v_1), \]
cyclic dga (cyclic cochain complex)

\[
\begin{aligned}
\mathcal{P}(v_1, v_2) &= (-1)^1|v_1||v_2|\mathcal{P}(v_2, v_1), \\
m_1(m_1(v_1)) &= 0, \\
m_1^+(v_1, v_2) &= (-1)^1|v_1||v_2|m_1^+(v_2, v_1), \\
m_1(m_2(v_1, v_2)) &= -m_2(m_1(v_1), v_2) \\
&\quad - (-1)^1|v_1|m_2(v_1, m_1(v_2)), \\
m_2(m_2(v_1, v_2), v_3) &= (-1)^1|v_1|+1m_2(v_1, m_2(v_2, v_3)), \\
m_2^+(v_1, v_2, v_3) &= (-1)^1|v_1|+|v_2|m_2^+(v_3, v_1, v_2).
\end{aligned}
\] (32)

The facts (A) and (C) from the Introduction apply, and we get the canonical dIBL-algebra dIBL(B\text{cyc}$^*$V[2 − n]) of bidegree (n − 3, 2) and the canonical Maurer-Cartan element \( \mathbf{m} = (\mathbf{m}_{10}) \). We will denote
\[ C(V) := B_{\text{cyc}}^*V[2 − n] \]
and call it the space of cyclic cochains on \( V \). If \( V \) is fixed, we will write just \( C \).

**Definition 2.42** (The canonical dIBL-algebra). Let \((V, \mathcal{P}, m_1)\) be a cyclic cochain complex of degree \( 2 - n \) which is finite-dimensional. Let \( (e_0, \ldots, e_m) \subset V[1] \) be a basis of \( V[1] \), and let \((e^0, \ldots, e^m)\) be the dual basis with respect to \( \mathcal{P} \); this means that
\[ \mathcal{P}(e_i, e^j) = \delta_{ij} \quad \text{for all } i, j = 0, \ldots, m. \]
We define the tensor \( T = \sum_{i,j=0}^m T^{ij} e_i \otimes e_j \in V[1]^{\otimes 2} \) by
\[ T^{ij} = (-1)^{|e_i|}\mathcal{P}(e^i, e^j) \quad \text{for all } i, j = 0, \ldots, m. \] (33)

The canonical dIBL-algebra on \( C(V) \) is the quadruple
\[ \text{dIBL}(C(V)) := (C(V), q_{110}, q_{210}, q_{120}), \]
where the operations \( q_{110}, q_{210}, q_{120} \) are defined for all \( \psi, \psi_1, \psi_2 \in \hat{B}_{\text{cyc}}^*V \) and generating words \( w = v_1 \ldots v_k, w_1 = v_{11} \ldots v_{1k_1}, w_2 = v_{21} \ldots v_{2k_2} \in B_{\text{cyc}}^*V \) with \( k, k_1, k_2 \geq 1 \) as follows:

- **The dIBL-boundary operator** \( q_{110} : \hat{E}_1 C \rightarrow \hat{E}_1 C \) of degree \( |q_{110}| = -1 \) is defined by
\[
q_{110}(\psi)(sw) := s \sum_{i=1}^k (-1)^{|v_i|+\ldots+|v_{i-1}|} \psi(v_1 \ldots v_{i-1}m_1(v_i)v_{i+1} \ldots v_k).
\] (34)

\footnote{See Appendix A for the invariant meaning of \( T \) as the Schwartz kernel of \( \pm 1 \).}

42
• The product $q_{210} : \hat{E}_2 C \rightarrow \hat{E}_1 C$ of degree $|q_{210}| = -2(n-3) - 1$ is written schematically as

$$q_{210}(s^2 \psi_1 \otimes \psi_2)(sw) := \sum \varepsilon(w \mapsto w^1 w^2) (-1)^{|e_i||w^1|} T^{ij} \psi_1(e_i w^1) \psi_2(e_j w^2)$$

and defined “algorithmically” as follows:

For every cyclic permutation $\sigma \in S_k$, consider the tensor

$$\sigma(w) := \varepsilon(\sigma, w) v_{\sigma_1^{-1}} \otimes \cdots \otimes v_{\sigma_k^{-1}},$$

and split it into two parts $w^1$ and $w^2$ of possibly zero length such that $v_{\sigma_1^{-1}} \otimes \cdots \otimes v_{\sigma_k^{-1}} = w^1 \otimes w^2$. Feed $w^1$ and $w^2$ into $\psi_1$ and $\psi_2$ preceded by $e_i$ and $e_j$, respectively, and multiply the result with the sign $(-1)^{|e_i||w^1|}$, which is the Koszul sign to order

$$e_i e_j w^1 w^2 \mapsto e_i w^1 e_j w^2.$$

Finally, sum over all $\sigma \in S_k$, all splittings of $\sigma(w)$ and all indices $i, j = 0, \ldots, m$. The sign $\varepsilon(\sigma, w)$ is denoted by $\varepsilon(w \mapsto w^1 w^2)$ to indicate the splitting.

• The coproduct $q_{120} : \hat{E}_1 C \rightarrow \hat{E}_2 C$ of degree $|q_{120}| = -1$ is written schematically as

$$q_{120}(s\psi)(s^2 w_1 \otimes w_2) = \frac{1}{2} \sum \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) (-1)^{|e_i||w_1^1|} T^{ij} \psi(e_i w_1^1 e_j w_2^1)$$

and defined “algorithmically” as follows:

For all cyclic permutations $\sigma \in S_{k_1}$ and $\mu \in S_{k_2}$, denote $w_1^1 := \sigma(w_1)$ and $w_2^1 := \mu(w_2)$ and let $\varepsilon(w_1 \mapsto w_1^1)$ and $\varepsilon(w_2 \mapsto w_2^1)$ be the corresponding Koszul signs, respectively. Feed $w_1^1$ and $w_2^1$ into $\psi$ in the indicated order interleaved by $e_i$ and $e_j$ and multiply the result with the sign $(-1)^{|e_i||w_1^1|}$, which is the Koszul sign to order

$$e_i e_j w_1^1 w_2^1 \mapsto e_i w_1^1 e_j w_2^1.$$

Finally, sum over all $\sigma \in S_{k_1}, \mu \in S_{k_2}$ and all indices $i, j = 0, \ldots, m$.

The operations are extended continuously to the completion.

**Definition 2.43** (The canonical Maurer-Cartan element). Let $(V, P, m_1, m_2)$ be a finite-dimensional cyclic dga of degree $2 - n$. The canonical Maurer-Cartan element $m$ for $dIBL(C(V))$ consists of only one element $m_{10} \in \hat{E}_1 C$ of degree
$|m_{10}| = 2(n - 3)$ which is defined by

$$m_{10}(s v_1 v_2 v_3) := (-1)^{n-2} \mu^+_2(v_1, v_2, v_3) \quad \text{for all } v_1, v_2, v_3 \in V[1]$$

on the weight-three component of $B^{cy}_c V[3-n]$ and extended by 0 to other weight-$k$ components.

Remark 2.44 (On canonical dIBL-structure).  (i) Elements of the completion $\hat{C}(V)$ which are not in $C(V)$ will be called long cyclic cochains. Because there are no infinite sums in Definition 2.42, dIBL$(C)$ is completion-free. Clearly, the twist dIBL$(C)$ remains completion-free as long as $n_l g \in E_l C$ for all $l, g$.

(ii) The constructions of $q_{210}$ and $q_{120}$ do not depend on the choice of a basis and can be rephrased in terms of summation over ribbon graphs (see Example A.5).

(iii) According to Proposition 2.31, the filtration on $C(V)$ satisfies (WG1) & (WG2), and hence the IBL-structures IBL$(\mathbb{H}(C))$ and IBL$(\mathbb{H}^m(C))$ are well-defined (see Definition 2.21).<

Proposition 2.45 (Formulas for twisted operations). Let dIBL$(C(V))$ be the canonical dIBL-algebra for a finite-dimensional cyclic cochain complex $(V, P, m_1)$ of degree $2 - n$, and let $n = (n_l g)$ be a Maurer-Cartan element. Then for all $l \geq 1, g \geq 0, \Psi \in \hat{B}_c V[3-n]$ and generating words $W_1, \ldots, W_l \in B^{cy}_c V[3-n]$, we have

$$([q_{210} \circ n_l g](\Psi))(W_1 \otimes \cdots \otimes W_l)$$

$$= \sum_{j=1}^l \sum \epsilon'(w_j \mapsto w_j^1 w_j^2) T^{a b}(se_a w_j^1) n_l g(W_1 \otimes \cdots \otimes W_{j-1} \otimes (se_b w_j^2)$$

$$\otimes W_{j+1} \otimes \cdots \otimes W_l),$$

(35)

where the sum without limits is the sum in Definition 2.42 for $q_{210}$ and $\epsilon'$ is the Koszul sign of the following operation:

$$(se_a w_j^1) W_1 \ldots W_{j-1} (s w_j^1 w_j^2) W_{j+1} \ldots W_l$$

$$\mapsto (se_a w_j^1) W_1 \ldots W_{j-1} (se_b w_j^2) W_{j+1} \ldots W_l.$$

In particular, for $l = 1, g \geq 0$ and $W \in B^{cy}_c V[3-n]$, we have

$$(q_{210} \circ n_l g)(W) = (-1)^{n-3} \sum T^{a b} \epsilon(w \mapsto w^1 w^2) n_l g(se_a w^1) \psi(e_b w^2),$$

(36)
and for \( l = 2, g \geq 0 \) and \( W_1, W_2 \in B^{\text{cy}}V[3-n] \), we have

\[
[(\theta_{210} \circ n_{2g})(\Psi)](W_1 \otimes W_2) = (-1)^{(n-3)(|\Psi|+1)} \left[ \sum_{a,b} T^{ab} \xi(w_1 \mapsto w_1^1 w_1^2) (-1)^{|e_b|} \psi(se_a w_1^1) n_{20}(se_b w_1^2 \otimes W_2) \right. \\
\left. + (-1)^{|W_1||W_2|} \sum_{a,b} T^{ab} \xi(w_2 \mapsto w_2^1 w_2^2) (-1)^{|e_b|} \psi(se_a w_2^1) n_{20}(se_b w_2^2 \otimes W_1) \right].
\]

(37)

**Proof.** Let us first discuss the completions. Given \( n_{0g} \in \hat{E}_l C \), we can write it as \( n_{0g} = \sum_{i=1}^{\infty} \Phi_1 \cdots \Phi_i \) with generating words \( \Phi_1 \cdots \Phi_i \in E_l C \) of weights approaching \( \infty \). The canonical extension of \( \sigma_h \) to maps with finite filtration degree commutes with convergent infinite sums, and hence we have \( q_{klg} \circ_{h} n_{0g} = \sum_{i=1}^{\infty} q_{klg} \circ_{h} (\Phi_1 \cdots \Phi_i) \). Therefore, it suffices to prove the formulas for generating words \( \Phi_1 \cdots \Phi_i \in E_l C \).

From (16), we get for every \( \Psi, \Phi_1, \ldots, \Phi_l \in C \) the equation

\[
[q_{210} \circ (\Phi_1 \cdots \Phi_l)](\Psi) = \sum_{i=1}^{l} (-1)^{|\Phi_i|(|\Phi_1| + \cdots + |\Phi_{i-1}|)} q_{210}(\Psi, \Phi_i) \Phi_1 \cdots \hat{\Phi}_i \cdots \Phi_l,
\]

where \( \Phi_1 \cdots \Phi_l \) on the left-hand-side is considered as a map \( E_0 C = R \rightarrow E_l C \) mapping 1 to \( \Phi_1 \cdots \Phi_l \). For \( W_1, \ldots, W_l \in B^{\text{cy}}V[3-n] \) and \( \sigma \in S_l \), we use

\[
[s(\Phi_1 \otimes \cdots \otimes \Phi_l)](W_1 \otimes \cdots \otimes W_l) = (\Phi_1 \otimes \cdots \otimes \Phi_l)[s^{-1}(W_1 \otimes \cdots \otimes W_l)]
\]

and Definition 2.42 to get

\[
(q_{210} \circ (\Phi_1 \cdots \Phi_l)](\Psi)) (W_1 \otimes \cdots \otimes W_l) =
\]

\[
= \sum_{i=1}^{l} (-1)^{|\Phi_i|(|\Phi_1| + \cdots + |\Phi_{i-1}|)} \frac{1}{l!} \sum_{\sigma \in S_l} \xi(\sigma^{-1}, W) [q_{210}(\Psi, \Phi_i)](W_{\sigma_i})
\]

\[
\Phi_1(W_{\sigma_2}) \cdots \hat{\Phi}_i(W_{\sigma_i}) \cdots \Phi_l(W_{\sigma_l})
\]

\[
= \sum_{i=1}^{l} (-1)^{|\Phi_i|(|\Phi_1| + \cdots + |\Phi_{i-1}|)} \frac{1}{l!} \sum_{\sigma \in S_l} \xi(\sigma^{-1}, W)(-1)^{|s|} \psi
\]

\[
\xi(w_\sigma \mapsto w_\sigma^1 w_\sigma^2) (-1)^{|e_b|} \psi(se_a w_\sigma^1) \Phi_i(e_b w_\sigma^2)
\]

\[
\Phi_1(W_{\sigma_2}) \cdots \hat{\Phi}_i(\emptyset) \cdots \Phi_l(W_{\sigma_l})
\]

\[
=: (+),
\]
where $\hat{\Phi}_i(\emptyset)$ means omission of the corresponding term. Consider the bijection

$$I : \{1, \ldots, l\} \times S_l \rightarrow \{1, \ldots, l\} \times S_l$$

$$(i, \sigma) \mapsto \left( j := \sigma_1, \mu := \left( \begin{array}{cccc} 1 & \ldots & i - 1 & i \\ \sigma_2 & \ldots & \sigma_i & \sigma_{i+1} & \ldots & \sigma_l \end{array} \right) \right).$$

Given $(i, \sigma) \in \{1, \ldots, l\} \times S_l$ and $b \in \{1, \ldots, m\}$, let $(j, \mu) := I(i, \sigma)$ and

$$W' := W_1 \otimes \cdots \otimes W_{j-1} \otimes (se_kw_j^2) \otimes W_{j+1} \otimes \cdots \otimes W_l.$$

Suppose that $(\Phi_1 \otimes \cdots \otimes \Phi_l)(W') \neq 0$. We compute the Koszul sign $\varepsilon(\mu^{-1}, W')$ in the following way:

$$W' \mapsto (-1)^{(|w_j^1| + |e_a| + |W_j|)(|W_1| + \cdots + |W_{j-1}|)(se_kw_j^2)}W_1 \cdots \hat{W}_j \cdots W_l$$

$$\mapsto (-1)^{(|w_j^1| + |e_a|)}(-1)^{\varepsilon^{-1}(\sigma, W)}W_{\sigma_2} \cdots W_{\sigma_l}.$$

Using this, we can rewrite $(\ast)$ as

$$(\ast) = (-1)^{|s| \Psi} \sum_{j=1}^l \varepsilon(w_j \mapsto w_j^1 w_j^2) (-1)^{|e_a| |w_j^1|} T^{ab} \Psi(e_a w_j^1)$$

$$\varepsilon_1 \frac{1}{m} \sum_{\mu \in S_l} \varepsilon(\mu^{-1}, W') \Phi_1(W_{\mu_1}) \cdots \Phi_l(W_{\mu_l})$$

$$= \sum_{j=1}^l \sum \varepsilon(w_j \mapsto w_j^1 w_j^2) (-1)^{|s| \Psi + |e_a| |w_j^1| + (|w_j^1| + |e_a|)(|W_1| + \cdots + |W_{j-1}|)} T^{ab} \Psi(se_a w_j^1) (\Phi_1 \cdots \Phi_l)(W_1 \otimes \cdots \otimes W_{j-1} \otimes (se_kw_j^2) \otimes W_{j+1} \otimes \cdots W_l).$$

Finally, we use

$$T^{ab} \neq 0 \implies |e_a| + |e_b| = n - 2$$

to write

$$|s| \Psi = |s|(|w_j^1| + |e_a|) = (n - 3)(|w_j^1| + n - 2 - |e_b|)$$

$$= |s|(|w_j^1| + |e_b|) \mod 2,$$

and the formula (35) follows.
As for \([36]\), we first compute \(\varepsilon'\) for \(l = 1\) as follows:

\[
\ln^{-1}(\varepsilon') = |w_1||e_6| + ((|e_6| + |w_1|)|s| = |w_1|^2|w_2| + |s||e_6|
\]

\[
= |w_1|^2|w_2| + |e_6||e_6| \mod 2.
\]

Using this, we obtain

\[
[q_{210} \circ n_{10}](\Psi)(V) = \sum \varepsilon'(w \mapsto w_1^2)T^{ab}b \Psi(se_a w^1) n_{10}(se_b w^2)
\]

\[
= (-1)^{|s|} \sum \varepsilon(w \mapsto w^2 w^1)T^{ba}a n_{10}(se_b w^2) \Psi(se_a w^1),
\]

\[
e(w \mapsto w_1^2 w^2) = (-1)^{|w_1||w_2|} e(w \mapsto w_1^2 w^2)
\]

which implies \([36]\).

The proof of \([37]\) is a combination of the same arguments. \(\square\)

We will now relate homology of the twisted boundary operator \(q_{110}^{\natural}\) to cohomology of an \(\mathcal{A}_{\infty}\)-algebra on \(V\) induced by \(n_{10}\).

**Definition 2.46 (\(\mathcal{A}_{\infty}\)-operations and compatible Maurer-Cartan element).** Let \((V, \mathcal{P}, m_1)\) be a finite-dimensional cyclic cochain complex of degree \(2 - n\), and let \(n = (n_{10})\) be a Maurer-Cartan element for \(d\text{IBL}(C(V))\). We define the operations \(\mu_k : V[1]^k \to V[1]\) for all \(k \geq 1\) by

\[
\mu_k(v_1, \ldots, v_k) := (-1)^{n-3} \sum_{i,j} T^{ij} n_{10}(se_i v_1 \ldots v_k) e_j
\]

for all \(v_1, \ldots, v_k \in V[1]\), where \(T^{ij}\) is the matrix from Definition 2.42.

If \((V, \mathcal{P}, m_1, m_2)\) is in addition a cyclic dga and \(m\) the canonical Maurer-Cartan element for \(d\text{IBL}(C(V))\), then we say that \(n\) is compatible with \(m\) if

\[
n_{10}(sv_1 v_2 v_3) = m_{10}(sv_1 v_2 v_3) \quad \text{for all } v_1, v_2, v_3 \in V[1].
\]

**Proposition 2.47 (Twisted boundary operator \(q_{110}^{\natural}\) and \(\mathcal{A}_{\infty}\)-cyclic cohomology).** In the setting of Definition 2.46, the triple \(\mathcal{A}_n(V) := (V, \mathcal{P}, (\mu_k))\) is a cyclic \(\mathcal{A}_{\infty}\)-algebra. We always have \(\mu_1 = m_1\), and if \(n\) is compatible with \(m\) for a cyclic dga \((V, \mathcal{P}, m_1, m_2)\), then also \(\mu_2 = m_2\).

The following holds for the homologies:

\[
\mathbb{H}^n_\bullet(C(V)) = r(H_\bullet^\ast(A_n(V); \mathbb{R}))[3 - n].
\]

**Proof.** First of all, according to Definition 2.19 we must have \(\|n_{10}\| > 2\), and

47
with respect to \( A \). This implies \( \mu_1 = m_2 \).

Now, let \( e_0, \ldots, e_m \) be a basis of \( V[1] \) and let \( e_0^\dagger, \ldots, e_m^\dagger \) be the dual basis with respect to \( \mathcal{P} \). For all \( k \geq 2 \) and \( v_1, \ldots, v_k \in V[1] \), we compute the following:

\[
\mathcal{P}(\mu_k(v_1, \ldots, v_k), v_{k+1}) = (-1)^{n-3} \sum_{i,j} (-1)^{|e_i|} \mathcal{P}(e_i, e_j) n_{10}(se_i v_1 \ldots v_k) \mathcal{P}(e_j, v_{k+1}) \\
= (-1)^{n-2} \sum_{i,j} (-1)^{|e_i|} n_{10}(se_i v_1 \ldots v_k) \mathcal{P}(e_i, e_j) \mathcal{P}(v_{k+1}, s v_i e_k) \\
= (-1)^{n-2} n_{10}(sv_1 \ldots v_{k+1}).
\]

Therefore, we have

\[ n_{10} = (-1)^{n-2} \sum_{k \geq 2} \mu_k^+. \]

In this case, [10] Proposition 12.3] asserts that the \( A_\infty \)-relations [29] for \( (\mu_k)_{k \geq 1} \) are equivalent to the “lowest” Maurer-Cartan equation [26] for \( n_{10} \). The degree condition \( |\mu_k| = 1 \) and the cyclic symmetry of \( \mu_k^+ \) are easy to check. Therefore, \( A_n(V) \) is a cyclic \( A_\infty \)-algebra.

As for the compatibility with \( m \), we have for all \( v_1, v_2 \in V[1] \) the following:

\[
m_2(v_1, v_2) = \sum_i \mathcal{P}(e_i, m_2(v_1, v_2)) e_i \\
= \sum_i (-1)^{|e_i|} T^{ij} \mathcal{P}(e_i, m_2(v_1, v_2)) e_j \\
= \sum_i (-1)^{1+(n-2)|e_i|} T^{ij} \mathcal{P}(m_2(v_1, v_2), e_i) e_j \\
= \sum_i (-1)^{n-3} T^{ij} m_{10}(sv_1 v_2 e_i) e_j \\
= \mu_2(v_1, v_2).
\]
We will now clarify the relation to the cyclic cohomology of \( A_n(V) \). Recall from Proposition 2.24 that \( q_{110}^\Psi = q_{110}(\Psi) + q_{210}(n_{10}, \Psi) \) for \( \Psi \in \tilde{B}_{\text{cyc}}^* V[3-n] \), where the first term is given by (34) and the second by (36). Consider now \( b^j_k \) and \( R^k \) from (30), whose sum gives the Hochschild boundary operator \( b \). Using the cyclic symmetry, we can rewrite a summand of \( b^j_k \) for \( j = 1, \ldots, k \) and \( i = 0, \ldots, k - j \) applied to a generating word \( v_1 \ldots v_k \in B^*_{\text{cyc}} V \) as follows:

\[
[t^i_{k-j+1} \circ (\mu_j \otimes \mathbb{1}^k-j) \circ t^{-i}_k](v_1 \ldots v_k) = \\
= (-1)^{|v_1|+\cdots+|v_i|} v_1 \ldots v_i \mu_j(v_{i+1} \ldots v_{i+j}) v_{i+j+1} \ldots v_k \\
= \varepsilon(w \mapsto w^1 w^2) \mu_j(v_{i+1} \ldots v_{i+j}) v_{i+j+1} \ldots v_k v_1 \ldots v_i \tag{38}
\]

Clearly, summing (38) over \( j = 1 \) and \( i = 0, \ldots, k - 1 \) gives the dual to \( q_{110} \).

For \( j = 2, \ldots, k \), we can write (38) as

\[
(\varepsilon(w \mapsto w^1 w^2) T^j \iota_{10}(w^1 w^1)) \varepsilon_j w^2.
\]

Therefore, the sum over \( j = 2, \ldots, k \) and \( i = 0, \ldots, k - j \) gives the part of the dual to \( q_{210}(n_{10}, \Psi) \) corresponding to the cyclic permutations \( \sigma \in S_k \) with \( \sigma_1 = 1, j+1, \ldots, k \). The rest, i.e., the cyclic permutations with \( \sigma_1 = 2, \ldots, j \), is obtained analogously from the summands \( (\mu_j \otimes \mathbb{1}^k-j) \circ t^i_k \) of \( R^k \) for \( j = 2, \ldots, k \) and \( i = 1, \ldots, j - 1 \). We conclude that \( q_{110}^\Psi : \tilde{B}_{\text{cyc}}^* V[3-n] \to \tilde{B}_{\text{cyc}}^* V[3-n] \) is a degree shift of \( b^* : \tilde{B}_{\text{cyc}}^* V \to \tilde{B}_{\text{cyc}}^* V \). As for the gradings, we have:

\[
 r(D_{\lambda}(V))[3-n] = (D_{\lambda}(V))^{i+3-n} = (\tilde{B}_{\text{cyc}}^* V)^{i+3-n} = (\tilde{B}_{\text{cyc}}^* V)^{i+3-n} = \tilde{B}_{\text{cyc}}^* V[2-n].
\]

This finishes the proof.

We will now turn to units and augmentations.

**Definition 2.48 (Reduced canonical dIBL-algebra).** Let \( (V, P, m_1, 1, \varepsilon) \) be an augmented cyclic cochain complex of degree \( 2-n \) from Definition 2.33. We define the space of reduced cyclic cochains on \( V \) by

\[
C_{\text{red}}(V) := B_{\text{cyc,red}}^* V[2-n].
\]

We define the reduced canonical dIBL-algebra by

\[
dIBL(C_{\text{red}}(V)) := (C_{\text{red}}(V), q_{110}, q_{210}, q_{120}),
\]

where \( q_{110}, q_{210}, q_{120} \) are restrictions of the operations of \( dIBL(C(V)) \).
Definition 2.49 (Strictly reduced Maurer-Cartan element). In the setting of Definition 2.48, we call a Maurer-Cartan element \( n = (n_{lg}) \) for \( \text{dIBL}(C(V)) \) strictly reduced if \( n_{lg} \in \tilde{E}_l C_{\text{red}}(V) \) for all \( (l, g) \neq (1, 0) \) and if the \( A_\infty \)-algebra \( (A_n(V), 1, \varepsilon) \) induced by \( n_{10} \) is strictly unital and strictly augmented. Given a strictly reduced Maurer-Cartan element \( n \), we can define the twisted \( \text{dIBL} \) \( \infty \)-algebra \( \text{dIBL}^n(C_{\text{red}}(V)) = (C_{\text{red}}(V), (q^n_{klg})) \), where \( q^n_{klg} \) are the restrictions of the operations of \( \text{dIBL}^n(C(V)) \). We denote the homology of \( \text{dIBL}^n(C_{\text{red}}) \) by \( H^n(C_{\text{red}}) \) or \( H^n_{\text{red}}(C) \).

Remark 2.50 (On strictly reduced Maurer-Cartan element). (i) We see that the \( \text{dIBL} \) \( \infty \)-algebra \( \text{dIBL}^n(C_{\text{red}}) \) is a subalgebra of \( \text{dIBL}^n(C) \), which means that the inclusion \( C_{\text{red}} \rightarrow C \) induces the following commutative diagram for all \( k, l \geq 1, g \geq 0 \):

\[
\begin{array}{ccc}
\hat{E}_k C & \xrightarrow{q^n_{klg}} & \hat{E}_l C \\
\downarrow & & \downarrow \\
\hat{E}_k C_{\text{red}} & \xrightarrow{q^n_{klg}} & \hat{E}_l C_{\text{red}}.
\end{array}
\]

We denote this fact by \( \text{dIBL}^n(C_{\text{red}}) \subset \text{dIBL}^n(C) \).

(ii) The canonical Maurer-Cartan element \( m \) of a strictly augmented strictly unital dga \( (V, m_1, m_2, 1, \varepsilon) \) is strictly reduced (this follows from Proposition 2.47).

(iii) In the situation of Definition 2.49 we denote

\[ \tilde{V}[1] := \ker(\varepsilon), \]

so that \( V = \tilde{V} \oplus (1) \). We use the canonical projection \( \pi : V \rightarrow \tilde{V} \) to identify \( \tilde{B}_n^\ast \tilde{V} \xrightarrow{\pi^*} B_n^\ast \tilde{V}_{\text{cyc,red}} \) via the componentwise pullback \( \pi^* \). In this way, we obtain the \( \text{IBL}_\infty \)-algebras \( \text{dIBL}(C(\tilde{V})) \) and \( \text{dIBL}^n(C(\tilde{V})) \), which are isomorphic to \( \text{dIBL}(C_{\text{red}}(V)) \) and \( \text{dIBL}^n(C_{\text{red}}(V)) \), respectively.

In the following list, we sum up our main reasons for considering units, augmentations and reduced Maurer-Cartan elements. Suppose that we are in the situation of Definition 2.49 then:

- Proposition 2.38 implies the splitting

\[ H^n(C)[1] = H^n(C_{\text{red}})[1] \oplus \langle s \varepsilon^q | s \in \mathbb{N} \rangle. \]  

(39)

Here \( i^* \in B_n^\ast \tilde{V} \) is the componentwise pullback of \( i^* \in B_n^\ast (\mathbb{R}) \). To

8The latter option suggests that it might be possible to define the reduced homology with the induced \( \text{IBL} \)-algebra even if \( n \) is not strictly reducible, e.g., if \( (A_n(V), 1, \varepsilon) \) is only homologically unital and augmented.
get this, we used
\[ H^n_x(C_{\text{red}}) = r(H^*_x,\text{red}(\mathcal{A}_n))[3 - n], \]
which can be seen by redoing the proof of Proposition 2.47 with reduced cochains.

- The subalgebra \( \text{dIBL}^n(C_{\text{red}}) \subset \text{dIBL}^n(C) \) induces the subalgebra
  \[ \text{IBL}(H^n_x(C_{\text{red}})) \subset \text{IBL}(H^n_x(C)), \]
  and any higher operation \( q_{11}^n \) which induces a map \( \hat{E}_1H(C_{\text{red}}) \to \hat{E}_lH(C_{\text{red}}) \) as well.

- If \( V \) is non-negatively graded, connected and simply-connected, then we have \( \hat{E}_kC_{\text{red}} \simeq E_kC_{\text{red}} \) for all \( k \in \mathbb{N}_0 \) by Proposition 2.40, and hence \( \text{dIBL}^n(C_{\text{red}}) \) is completion-free.

**Proposition 2.51 (Operations on units).** Suppose that \( (V, \mathcal{P}, m_1, \varepsilon) \) is a finite-dimensional augmented cyclic cochain complex of degree \( 2 - n \) such that \( n \geq 1 \), and let \( n \) be a strictly reduced Maurer-Cartan element for \( \text{dIBL}(C(V)) \). The following relations are the only relations containing \( i^* \) which may be non-zero on the homology \( H^n_x(C) \):

For all \( \Psi \in C_{\text{red}}(V) \) and \( l \geq 1, g \geq 0 \), we have
\[
q_{210}(s^* \otimes \Psi) = (-1)^{(n-2)|\Psi|} q_{210}(\Psi \otimes s^*) = (-1)^{n-2} \Psi \circ \iota_v \quad \text{and} \quad q_{11}^n(s^*) = -n_{lg} \circ \iota_v,
\]
where \( \iota_v \) is defined as follows:

- The element \( v \in V[1] \) is the unique vector such that \( \mathcal{P}(1, v) = 1 \) and \( v \perp V[1] \) with respect to \( \mathcal{P} \). Note that \( |v| = n - 1 \) and that such \( v \) always exists due to non-degeneracy.

- We start by defining \( \iota_v : B^x_{\text{cyc}}V \to B^x_{\text{cyc}}V \) by
\[
\iota_v(v_1 \ldots v_k) := \sum_{i=1}^k (-1)^{|v|(|v_1| + \cdots + |v_{i-1}|)} v_1 \ldots v_{i-1} v_i v_i \ldots v_k
\]
for all generating words \( v_1 \ldots v_k \in B^x_{\text{cyc}}V \). Next, for all \( k \geq 1 \), we define \( \iota_v : (B^x_{\text{cyc}}V)^\otimes k \to (B^x_{\text{cyc}}V)^\otimes k \) by
\[
\iota_v(w_1 \otimes \cdots \otimes w_k) := (-1)^{|v|k} \sum_{j=1}^k (-1)^{|v|(|w_1| + \cdots + |w_{j-1}|)} w_1 \otimes \cdots \otimes w_{j-1} \otimes \iota_v(w_j) \otimes w_{j+1} \otimes \cdots \otimes w_k
\]
for all generating words \( w_1, \ldots, w_k \in \mathcal{B}_q^* V \). Finally, we take the degree shift \( \iota_v : (\mathcal{B}_q^* V[3-n])^\otimes k \to (\mathcal{B}_q^* V[3-n])^\otimes k \) according to the degree shift convention [7].

Proof. Pick a basis \((e_0, \ldots, e_m)\) of \( V[1] \) such that \( e_0 = 1 \) and \( \tilde{V}[1] = (e_1, \ldots, e_m) \). If \( (e^0, \ldots, e^m) \) is the dual basis, then we have \( v = e^0 \). We will often use the following relation:

\[
\sum_{j=0}^m T^{1j} e_j = \sum_{j=0}^m (-1)^{1j} \mathcal{P}(v, e^j) e_j = -v. \quad (40)
\]

We consider only those generating words \( w = v_1 \ldots v_k \) of \( \mathcal{B}_q^* V \) with either \( v_i \in \tilde{V} \) for each \( i \) (shortly \( w \in \mathcal{B}_q^* V \)) or \( v_i = 1 \) for each \( i \) with \( k \) odd (i.e., \( w = 1^{2j-1} \) for some \( j \)). Let \( w_1, \ldots, w_1 \) with \( w_j = v_{j1} \ldots v_{jk} \) denote such generating words. Clearly, if \( \Phi \in \tilde{E}_0 C(V) \) is a \( n_{10}^* \)-closed element which vanishes on all \( w_1 \otimes \cdots \otimes w_1 \), then (39) implies that \( [\Phi] = 0 \) in \( \tilde{E}_0 \mathcal{H}(C) \).

For \( \Psi \in C_{\text{red}}(V) \) and \( q \geq 1 \) odd, we compute using (40) the following:

\[
\begin{align*}
\mathfrak{q}_{210}(s^2 1^n q^\ast \otimes \psi)(sw) &= \sum \varepsilon(w \mapsto w^1 w^2) (-1)^{(n-1)|w^1|} T^{1j} 1^n q^\ast (1w^1) \psi(e_j w^2) \\
&= -\sum \varepsilon(w \mapsto w^1 w^2) (-1)^{(n-1)|w^1|} 1^n q^\ast (1w^1) \psi(vw^2) \\
&=: (\ast).
\end{align*}
\]

Now, in order to get \((\ast) \neq 0\), we need either \( q = 1 \) and \( w \in \mathcal{B}_q^* \tilde{V} \), in which case

\[
(\ast) = -\sum \varepsilon(w \mapsto w^1 w^2) \psi(vw^2)
\]

\[
= -\sum_{j=1}^k (-1)^{|v_j|} \psi(v_{j1} + \cdots + v_{j-1}) \psi(v_{j1} \ldots v_{j-1} v v_{j+1} \ldots v_k)
\]

\[
= -(\psi \circ \iota_v)(w) = (-1)^{n-2}(\psi \circ \iota_v)(w),
\]

or \( q > 1 \) odd and \( w = 1^n q^{-1} \), in which case

\[
(\ast) = \sum \varepsilon(w \mapsto w^1 w^2) 1^n q^\ast (1q^\ast) \psi(v)
\]

\[
= \psi(v) \sum_{j=1}^{q-1} (-1)^j
\]

\[
= 0.
\]

Next, because \( n \geq 1 \), we get \( T^{11} = 0 \), and hence

\[
\mathfrak{q}_{120}(1^n q^\ast) = 0 \quad \text{for all} \quad q \in \mathbb{N}
\]

52
on the chain level. Therefore, we have $q_{1l} n_{lg} = q_{210} \circ_1 n_{lg}$ for all $l \geq 1$, $g \geq 0$, and using Proposition 2.45 and (40), we obtain

$$[(q_{210} \circ_1 n_{lg})(1^{q*})](W_1 \otimes \cdots \otimes W_l)$$

$$= - \sum_{j=1}^{l} \sum_{\varepsilon} \varepsilon(w_j \mapsto w_j^1 w_j^2) 1^{q*} (1 w_j^1) n_{lg}(W_1 \otimes \cdots \otimes W_{j-1} \otimes (sv w_j^2) \otimes W_{j+1} \otimes \cdots \otimes W_l)$$

$$=: (**).$$

In order to get $(**) \neq 0$, we need either $q = 1$ and $w_j \in B^{\mathcal{V} \otimes \mathcal{V}}$ for all $j$, in which case

$$(**) = - \sum_{j=1}^{l} \sum_{\varepsilon} (-1)^{\varepsilon}(w_1 \mapsto w_1^1 w_1^2) 1^{q*} (1 w_1^1) n_{lg}(W_1 \otimes \cdots \otimes W_{j-1} \otimes (sv w_1^2) \otimes W_{j+1} \otimes \cdots \otimes W_l)$$

$$= -(n_{lg} \circ \varepsilon)(W_1 \otimes \cdots \otimes W_l),$$

or $q > 1$ odd and $w_j = 1^{q-1}$ for some $j$, in which case

$$(**) = - \sum_{1 \leq j \leq l \atop w_j = 1^{q-1}} \varepsilon'(\sum_{i=1}^{q-1} (-1)^i) n_{lg}(W_1 \otimes \cdots \otimes W_{j-1} \otimes (sv) \otimes W_{j+1} \otimes \cdots \otimes W_l)$$

$$= 0.$$

The only relation left to check is

$$q_{210}(s_1 q^{1*}, s_1 q^{1*}) = 0 \text{ for all } q_1, q_2 \in \mathbb{N}.$$ 

However, this is easy to see, and the proof is done. \hfill\square
3 Twisted $\text{IBL}_\infty$-structure and string topology

In Section 3.1 we consider the cyclic dga’s $\Omega(M)$, $\text{H}_{dR}(M)$ and $\text{H}(M)$ for a closed oriented $n$-manifold $M$ (Proposition 3.2) and apply the theory from Section 2.4 to the last two, which are finite-dimensional.

In Section 3.2 we define the Green kernel $G$ (Definition 3.5). It is a primitive to the Schwartz kernel $H$ of the harmonic projection $\pi_H$ (see Proposition 3.8) outside the diagonal and extends smoothly to the spherical blow-up of the diagonal. These ideas come from an early version of [12]. We consider conditions (G1)–(G5) on a linear operator $G$ and its Schwartz kernel $G$ (see p. 63) and show that $G$ satisfying all these conditions always exists (Proposition 3.11). We also mention the standard Green kernel $G_{\text{std}}$ (see (55)), which might be a canonical Green kernel satisfying (G1)–(G5).

In Section 3.3 we review ribbon graphs, labelings, compatibility of the order and orientation of internal edges, and the edge and vertex order from [10] (Definitions 3.14, 3.16, 3.17 and 3.18). We then define $n$ as a signed sum of integrals of products of Green kernels and harmonic forms which are associated to labeled trivalent ribbon graphs (Definition 3.19). We do not show that these integrals converge and that $n$ satisfies the Maurer-Cartan equation, but we do show all other properties of a Maurer-Cartan element (Lemma 3.20 and Proposition 3.23). We define the $Y$-graph, trees, circular graphs, vertices of types $A$, $B$, $C$ and their contributions $A_{\alpha_1,\alpha_2}$, $B_{\alpha}$, $C$, respectively (Definitions 3.21 and 3.24).

In Section 3.4 we observe that vanishing of some special vertices in the graphs implies $m_9 = m_g$. For example, if all graphs, except for the $Y$-graph, with 1 at an external vertex vanish, which holds if $G$ satisfies (G4) and (G5) (Proposition 3.26), then all higher operations $q_{11g}^n$ vanish on the chain level in dimensions $n > 3$ (Proposition 3.25). Next, if all graphs with an $A$-vertex vanish, then $n_{10} = m_{10}$, and hence $q_{110}^n = q_{110}^m$ (Proposition 3.27). We show that $n = m$ for simply-connected geometrically formal manifolds with $n \neq 2$ (Proposition 3.29). Using the results of [12], we argue that the chain complexes of $q_{110}^n$ and $q_{110}^m$ are quasi-isomorphic provided $M$ is simply-connected and formal (Proposition 3.31).

In Section 3.5 we recall basic facts about the Chas-Sullivan operations $m_2$ and $\sigma_2$ on the $S^1$-equivariant homology of the free loop space and formulate a version of the string topology conjecture for simply-connected manifolds (Conjecture 3.33).
3.1 Canonical dIBL-structures on $C(H_{\text{dR}}(M))$

Let $M$ be an oriented closed Riemannian manifold of dimension $n$. We consider the following graded vector spaces:

$\Omega^*(M)$ . . . smooth de Rham forms,

$\mathcal{H}^*(M)$ . . . harmonic forms,

$H^*_{\text{dR}}(M)$ . . . de Rham cohomology.

Since $M$ is fixed, we often write just $\Omega$, $\mathcal{H}$ and $H_{\text{dR}}$. We consider the Hodge decomposition $\Omega = \mathcal{H} \oplus \text{Im } d \oplus \text{Im } d^*$, where $d$ is the de Rham differential and $d^*$ the codifferential. We call the corresponding projection

$$\pi_H : \Omega^*(M) \rightarrow \mathcal{H}^*(M)$$

the harmonic projection and the induced isomorphism $\pi_H : H_{\text{dR}} \rightarrow \mathcal{H}$ mapping a cohomology class into its unique harmonic representative the Hodge isomorphism.

**Notation 3.1** (Updated notation for bar complexes). We use Notation 2.28 for $V = \Omega$, $\mathcal{H}$, $H_{\text{dR}}$ and $A = n - 3$ with the following changes:

$\tilde{v} \sim \eta \in V$, $v \sim \alpha \in V[1]$, $w \sim \omega \in B^*_c V$, $w \sim \mathcal{P} \in B^*_c V[n - 3]$.

We use the formal symbols $s$ and $\theta$ with $|s| = n - 3$ and $|\theta| = -1$, so that $\alpha = \theta \eta$ and $\Omega = s \omega$.

**Proposition 3.2** (De Rham cyclic dga’s). Let $M$ be an oriented closed Riemannian manifold of dimension $n$. The quadruple $(\Omega(M), \mathcal{P}, m_1, m_2)$ with the operations from (3) is a cyclic dga of degree $2 - n$. For the operations before the degree shift, we have

$$\tilde{m}_1(\eta_1) = d\eta_1,$$

$$\tilde{m}_2(\eta_1, \eta_2) = \eta_1 \wedge \eta_2,$$

$$\tilde{\mathcal{P}}(\eta_1, \eta_2) = \int_M \eta_1 \wedge \eta_2 : (\eta_1, \eta_2),$$

where $d$ is the de Rham differential, $\wedge$ the wedge product and $\tilde{\mathcal{P}}$ the intersection pairing. The operations restrict to $H_{\text{dR}}(M)$ and make $(H_{\text{dR}}(M), \mathcal{P}, m_1 \equiv 0, m_2)$ into a cyclic dga. If we define $\mu_1 \equiv 0$ and

$$\mu_2(\alpha_1, \alpha_2) := \pi_H(m_2(\alpha_1, \alpha_2)) \text{ for all } \alpha_1, \alpha_2 \in \mathcal{H}(M)[1],$$

then $(H(M), \mathcal{P}, \mu_1, \mu_2)$ is a cyclic dga as well, and $\pi_H : H_{\text{dR}} \rightarrow \mathcal{H}$ is an isomorphism of cyclic dga’s. All three dga’s $\Omega$, $H_{\text{dR}}$ and $\mathcal{H}$ are strictly unital and strictly augmented with the unit $1 := \theta 1 \in \Omega[1]$, where $1$ is the constant one.
Proof. The relations follow from the classical properties of $d$ and $\wedge$ and from the Stokes’ theorem for oriented closed manifolds. The Poincaré duality asserts that $(\cdot, \cdot)$ is non-degenerate on $H_{dR}$ and $\mathcal{H}$, and thus they are cyclic dga’s as well. The fact that $\pi_{\mathcal{H}} : H_{dR} \to \mathcal{H}$ is an isomorphism of vector spaces follows from the Hodge theory. As for compatibility with the product, given $\eta_1, \eta_2 \in \mathcal{H}$, then $\eta_1 \wedge \eta_2$ is closed, and since $\ker d = \mathcal{H} \oplus \text{Im} d$, we see that $\pi_{\mathcal{H}}(\eta_1 \wedge \eta_2) = \eta_1 \wedge \eta_2 + d\eta$ for some $\eta \in \Omega$ is a harmonic representative of the cohomology class $[\eta_1 \wedge \eta_2] = [\eta_1] \wedge [\eta_2]$. Unitality is obvious, and the construction of an augmentation map clear. Note that a strict augmentation for $\Omega(M)$ is the evaluation at a point, for instance.

The facts (A) and (C) from the Introduction apply to the cyclic dga’s $\mathcal{H}$ and $H_{dR}$ (not to $\Omega$ because it is infinite-dimensional!), and we get the canonical dIBL-algebras $\text{dIBL}(C(\mathcal{H}))$ and $\text{dIBL}(C(H_{dR}))$ of bidegrees $(n - 3, 2)$ with the canonical Maurer-Cartan element $m = (m_{10})$. The Hodge isomorphism induces an isomorphism of these dIBL-algebras, and hence we can use $\mathcal{H}$ and $H_{dR}$ interchangeably. We have $q_{110} \equiv 0$, and hence $\text{dIBL}(C(\mathcal{H}))$ is, in fact, an IBL-algebra. However, we will denote it by $\text{dIBL}$ and call it a dIBL algebra as a reminder of the canonical dIBL-structure. The canonical Maurer-Cartan element $m$ satisfies

$$m_{10}(\alpha_1 \alpha_2 \alpha_3) = (-1)^{n-2+\eta_2} \int_M \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{for all } \alpha_1, \alpha_2, \alpha_3 \in H[1]. \quad (43)$$

We get the canonical twisted dIBL-algebra $\text{dIBL}^m(C(\mathcal{H}))$ from (2) with, in general, non-trivial boundary operator $q^m_{110}$ whose homology is the cyclic homology of $H_{dR}$ up to degree shifts.

### 3.2 Green kernel $G$

We will use fiberwise integration and spherical blow-ups, which we now recall.

**Definition 3.3** (Fiberwise integration). Let $pr : E \to B$ be a smooth oriented fiber bundle with an oriented fiber $F$ over an oriented manifold $B$ with $\partial B = \emptyset$. We orient $E$ as $F \times B$. Let $\Omega_c(E)$ denote the space of forms with compact support and $\Omega_{cv}(E)$ the space of forms with compact vertical support. For any $\kappa \in \Omega_{cv}(E)$, let $\int^F \kappa \in \Omega(B)$ be the unique smooth form such that

$$\int_E \kappa \wedge pr^* \eta = \int_B \left( \int^F \kappa \right) \wedge \eta \quad \text{for all } \eta \in \Omega_c(B).$$

**Definition 3.4** (Spherical blow-up). Let $X$ be a smooth $n$-dimensional manifold and $Y \subset X$ a smooth $k$-dimensional submanifold. The blow-up of $X$ at $Y$ is as a set defined by

$$\text{Bly} X := X \setminus Y \cup P^+ NY,$$
where $P^+NY$ is the real oriented projectivization of the normal bundle $NY$ of $Y$ in $X$. This means that $P^+NY$ is the quotient of $\{v \in NY \mid v \neq 0\}$ by the relation $v \sim av$ for all $a \in (0, \infty)$. The blow-down map is defined by

$$\pi : \text{Bl}yX \rightarrow X$$

$$p \in X \setminus Y \mapsto p,$$

$$[v]_p \in P^+NY \mapsto p.$$  

In the following, we will equip the blow-up with the structure of a smooth manifold with boundary such that its interior becomes diffeomorphic to $X \setminus Y$ via the blow-down map and the boundary becomes $P^+NY$. Consider an adapted chart $(U, \psi)$ for $Y$ in $X$ with $\psi(U) = \mathbb{R}^n$ and $\psi(U \cap Y) = \{(0, y) \mid y \in \mathbb{R}^k\}$. It induces the bijection

$$\tilde{\psi} : \text{Bl}_{U \cap Y}U \rightarrow [0, \infty) \times \mathbb{S}^{n-k-1} \times \mathbb{R}^k$$

$$p \in U \setminus Y \mapsto \left(\left[\pi_1 \psi(p)\right], \frac{\pi_1 \psi(p)}{\left[\pi_1 \psi(p)\right]}, \frac{\pi_2 \psi(p)}{\left[\pi_1 \psi(p)\right]}\right),$$

$$[v] \in P^+NY \mapsto \left(0, \frac{\pi_1 \psi(v)}{\left[\pi_1 \psi(v)\right]}, \frac{\pi_2 \psi(v)}{\left[\pi_1 \psi(v)\right]}\right),$$

where $\pi_1$ and $\pi_2$ are the canonical projections to the factors of $\mathbb{R}^{n-k} \times \mathbb{R}^k$. Notice that we have the canonical inclusion $\text{Bl}_{U \cap Y}U \subset \text{Bl}yX$. It can be checked that for any two overlapping adapted charts $(U_1, \psi_1)$ and $(U_2, \psi_2)$, the transition function $\tilde{\psi}_1 \circ \tilde{\psi}_2^{-1}$ is a diffeomorphism of manifolds with boundary. Therefore, we can use the charts $(\text{Bl}_{U \cap Y}U, \tilde{\psi})$ to define a smooth atlas on $\text{Bl}yX$. If $X$ is oriented, we orient $\text{Bl}yX$ so that $\pi$ restricts to an orientation preserving diffeomorphism of the interior.

An important fact is that if $X$ is compact, then $\text{Bl}yX$ is compact.

We are interested in the case when $X = M \times M$ for an oriented closed manifold $M$ and $Y = \Delta := \{(m, m) \mid m \in M\}$ is the diagonal. Given a chart $\varphi : U \rightarrow \mathbb{R}^n$ on $M$, the following is a smooth chart on $\text{Bl}_{\Delta}(M \times M)$:

$$\tilde{\varphi} : \text{Bl}_{\Delta}(U \times U) \rightarrow [0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n$$

$$(x, y) \in (U \times U) \setminus \Delta \mapsto (r, w, u) := \left(\frac{1}{2}|\varphi(x) - \varphi(y)|, \frac{\varphi(x) - \varphi(y)}{|\varphi(x) - \varphi(y)|}, \frac{1}{2}(\varphi(x) + \varphi(y))\right),$$

$$[(v, -v)]_{(x, x)} \mapsto \left(0, \frac{d\varphi_x(v)}{d\varphi_x(-v)}, \varphi(x)\right).$$

The inverse relations for $r > 0$ read

$$\varphi(x) = u + wr \quad \text{and} \quad \varphi(y) = u - wr.$$
We will denote by $M_i$ the $i$-th factor of $M \times M$; i.e., we will write $M \times M = M_1 \times M_2$. We denote the corresponding projection by $\text{pr}_i$. We define $\tilde{\text{pr}}_i := \text{pr}_i \circ \pi$, where $\pi : \text{Bl}_\Delta(M \times M) \to M \times M$ is the blow-down map. We also identify $(M \times M) \setminus \Delta$ with the interior of $\text{Bl}_\Delta(M \times M)$ via $\pi$.

The map $\tilde{\text{pr}}_2 : \text{Bl}_\Delta(M \times M) \to M_2$ is an oriented fiber bundle with fiber $\text{Bl}_*^*(M_1)$, which is the blow-up of $M_1$ at a point (we shall assume that $M$ is connected). The fiberwise integration along $\tilde{\text{pr}}_2$ will be denoted by $\int^{\text{Bl}_*^*(M_1)}$.

**Definition 3.5** (Green kernel). Let $M$ be an oriented closed $n$-dimensional Riemannian manifold. Consider the harmonic projection $\pi_H$ from (41), and let $i_H : H(M) \hookrightarrow \Omega(M)$ be the inclusion. A smooth $(n-1)$-form $G$ on $(M \times M) \setminus \Delta$ is called a Green kernel if the following conditions are satisfied:

1. The form $G$ admits a smooth extension to $\text{Bl}_\Delta(M \times M)$. More precisely, the pullback $(\pi|_{\text{int}})^* G$ along the blow-down map restricted to the interior is a restriction of a smooth form on $\text{Bl}_\Delta(M \times M)$. We denote this form by $G$ again by uniqueness.

2. The operator $\mathcal{G} : \Omega^*(M) \to \Omega^{*-1}(M)$ defined by

$$\mathcal{G}(\eta) := \int^{\text{Bl}_*^*(M_1)} G \wedge \tilde{\text{pr}}_1^* \eta \quad \text{for all } \eta \in \Omega(M)$$

(45)

satisfies

$$d \circ \mathcal{G} + \mathcal{G} \circ d = i_H \circ \pi_H - \mathbb{1}. \tag{46}$$

Any homogenous linear operator $\mathcal{G} : \Omega^*(M) \to \Omega^{*-1}(M)$ satisfying (46) will be called a Green operator.

3. For the twist map $\tau : M \times M \to M \times M$ defined by $(x,y) \mapsto (y,x)$, the following symmetry property holds:

$$\tau^* G = (-1)^n G. \tag{47}$$

**Remark 3.6** (On Green kernel). (i) Given a homogenous linear operator $\mathcal{G} : \Omega^*(M) \to \Omega^{*-1}(M)$, if there is a $G \in \Omega^{*-1}(\text{Bl}_\Delta(M \times M))$ such that (45) holds, then it is unique.

(ii) Because $\tau : M \times M \to M \times M$ preserves $\Delta$, it extends to a diffeomorphism $\tilde{\tau}$ of $\text{Bl}_\Delta(M \times M)$. The condition (47) is then equivalent to $\tilde{\tau}^* \tilde{G} = (-1)^n \tilde{G}$ for the extension $\tilde{G}$ of $G$ to $\text{Bl}_\Delta(M \times M)$. We denote both extensions by $\tau$ and $G$, respectively.
(iii) Using the intersection pairing \((\cdot, \cdot)\), we have

\[
(G(\eta_1), \eta_2) = \int_{\text{Bl}_\Delta(M \times M)} G \wedge \tilde{\pr}_2^* \eta_1 \wedge \tilde{\pr}_1^* \eta_2
\]

and

\[
(\eta_1, G(\eta_2)) = (-1)^n \int_{\text{Bl}_\Delta(M \times M)} \tau^* G \wedge \tilde{\pr}_2^* \eta_1 \wedge \tilde{\pr}_1^* \eta_2
\]

for all \(\eta_1, \eta_2 \in \Omega(M)\). This implies the following:

\[
\tau^* G = (-1)^n G \iff (G(\eta_1), \eta_2) = (-1)^n (\eta_1, G(\eta_2)) \quad \forall \eta_1, \eta_2 \in \Omega(M).
\]

(iv) Because \(\text{Bl}_\Delta(M \times M)\) is compact, \(G \in \Omega(\text{Bl}_\Delta(M \times M))\) induces an \(L^1\)-integrable form on \(M \times M\).

(v) In the literature, the term “Green operator” often denotes a generalized inverse of an elliptic pseudo-differential operator, e.g., of the Laplacian \(\Delta\). This is not what we mean here.

We will now prove three propositions which will allow us to rewrite (46) equivalently as a differential equation for \(G\) on \(M \times M \setminus \Delta\).

**Proposition 3.7** (Identities for fiberwise integration). In the situation of Definition 3.3, assume that \(F\) has a boundary \(\partial F\). We orient \(\partial F\) using \(T_pF = N(p) \oplus T_p \partial F\) for \(p \in \partial F\), where \(N\) is an outward pointing normal vector field. The following formulas hold for all \(\kappa \in \Omega_{cv}(E)\) and \(\eta \in \Omega_c(B)\):

- **The projection formula**

  \[
  \int_F (\kappa \wedge \pi^* \eta) = \left( \int_F \kappa \right) \wedge \eta,
  \]

- **Stokes’ formula**

  \[
  (-1)^F d \int_F \kappa = \int_F d\kappa - \int_{\partial F} \kappa,
  \]

  where \(F\) in the exponent denotes the dimension of \(F\).

**Proof.** The projection formula is proven by a straightforward calculation from the definition.

As for Stokes’ formula, we get the oriented fiber bundle \(\partial E \to B\) with fiber \(\partial F\) by restricting an oriented trivialization of \(E\). There are two orientations...
∂E — as the total space of ∂E → B and as the boundary of E. They agree due to our orientation convention. Using standard Stokes’ theorem, we get
\[-1] \int_B d\left( \int_F \kappa \right) \wedge \eta = (-1)^{\kappa+1} \int_E \kappa \wedge d\pi^* \eta
\[-1] = \int_E (d\kappa \wedge \pi^* \eta - d(\kappa \wedge \pi^* \eta))
\[-1] = \int_B \left( \int_F d\kappa \right) \wedge \eta - \int_{\partial E} \kappa \wedge \pi^* \eta
\[-1] = \int_B \left( \int_F d\kappa - \int_{\partial F} \kappa \right) \wedge \eta.

This proves the proposition. \(\square\)

In what comes next, we will view the canonical projection \(pr_2 : M_1 \times M_2 \to M_2\) as an oriented fiber bundle such that the orientation of the total space agrees with the product orientation. The fiberwise integration for this bundle will be denoted by \(\int_{M_1}\).

**Proposition 3.8** (Schwartz kernel of the harmonic projection). Let \(M\) be an oriented closed \(n\)-dimensional Riemannian manifold. Let \(\nu_1, \ldots, \nu_m\) be a homogenous basis of \(H(M)\) which is orthonormal with respect to the \(L^2\)-inner product
\[(\eta_1, \eta_2)_{L^2} := \int_M \eta_1 \wedge \pi^* \eta_2 \quad \text{for} \quad \eta_1, \eta_2 \in \Omega(M),\]
where \(\pi^*\) denotes the Hodge star. The smooth form \(H \in \Omega^n(M \times M)\) defined by
\[H := \sum_{i=1}^m (-1)^{m-i} pr_1^* (\ast \nu_i) \wedge pr_2^* (\nu_i) \quad (49)\]
satisfies the following properties:

(a) For all \(\eta \in \Omega(M)\), we have
\[\pi_H(\eta) = \int_{M_1} H \wedge pr_1^* \eta.\]

(b) The form \(H\) is closed and Poincaré dual to \(\Delta \subset M \times M\).

(c) The following symmetry condition is satisfied:
\[\tau^* H = (-1)^n H. \quad (50)\]

Proof. (a) For the purpose of the proof, we denote \(H(\eta) := \int_{M_1} H \wedge pr_1^* \eta.\) For
every $k = 1, \ldots, m$, we use the projection formula to compute

$$\mathcal{H}(\nu_k) = \sum_{i=1}^{m} (-1)^{\nu_i n + \nu_k n} \int_M \text{pr}_1^*(\ast \nu_i \wedge \nu_k) \wedge \text{pr}_2^*(\nu_i)$$

$$= \sum_{i=1}^{m} (-1)^{\nu_i (n + \nu_k) + \nu_k (n + \nu_k)} (\int_M \nu_k \wedge \ast \nu_i) \nu_i$$

$$= \nu_k.$$

It is easy to see that $\mathcal{H}(\eta) \in \mathcal{H}(M)$ for all $\eta \in \Omega(M)$. Therefore, $\mathcal{H}$ is a projection to $\mathcal{H}(M)$. Relations $\mathcal{H}(d\eta) = \mathcal{H}(d^\ast \eta) = 0$ for all $\eta \in \Omega(M)$ follow from the second line of the computation above with $\nu_k$ replaced by $d\eta$ and $d^\ast \eta$ using that $\text{Im } d^\ast \oplus \text{Im } d$ is $L^2$-orthogonal to $\mathcal{H}(M)$. We see that $\mathcal{H} = \pi_H$.

(b) Using $d \circ \mathcal{H} = \mathcal{H} \circ d = 0$ and Stokes’ theorem, we get

$$\int_M dH \wedge \text{pr}_1^* \eta = (-1)^n d\mathcal{H}(\eta) - \mathcal{H}(d\eta) = 0 \quad \text{for all } \eta \in \Omega(M).$$

It follows that $dH = 0$. Using the Künneth formula, we can write a given $\kappa \in \Omega(M \times M)$ with $d\kappa = 0$ as $\kappa = \text{pr}_1^* \eta_1 \wedge \text{pr}_2^* \eta_2 + d\eta$ for some $\eta_1, \eta_2 \in \mathcal{H}(M)$ and $\eta \in \Omega(M)$. Then

$$\int_{M \times M} H \wedge \kappa = \int_{M \times M} H \wedge \text{pr}_1^* \eta_1 \wedge \text{pr}_2^* \eta_2$$

$$= \int_M \mathcal{H}(\eta_1) \wedge \eta_2$$

$$= \int_M \eta_1 \wedge \eta_2 = \int_\Delta \kappa.$$ 

This shows that $H$ is Poincaré dual to $\Delta$.

(c) It follows from the Hodge decomposition that

$$(\pi_H(\eta_1), \eta_2) = (\pi_H(\eta_1), \pi_H(\eta_2)) = (\eta_1, \pi_H(\eta_2)) \quad \text{for all } \eta_1, \eta_2 \in \Omega(M). \quad (51)$$

As in (iii) of Remark 3.6, one shows that this is equivalent to (50).

Proposition 3.9 (Differential condition). Let $M$ be an oriented closed $n$-dimensional Riemannian manifold. For $G \in \Omega^{n-1}(\text{Bl}_\Delta(M \times M))$, the following claims are equivalent:

1. The operator $\mathcal{G} : \Omega^\ast(M) \to \Omega^{n-1}(M)$ defined by $\mathcal{G}(\eta) := \int_{\text{Bl}_1 M} G \wedge \text{pr}_1^* \eta$ for $\eta \in \Omega(M)$ is a Green operator.

2. It holds $dG = (-1)^n H$ on $(M \times M) \backslash \Delta$. \quad (52)
Proof. Before we begin, note that (52) is equivalent to the equation \( d\tilde{G} = (-1)^n \pi^* H \) on Bl\( \Delta \)(\( M \times M \)) for the extension \( \tilde{G} \) of \( G \); we denote \( \tilde{G} \) by \( G \) and \( \pi^* H \) by \( H \) by uniqueness.

We will first prove 2) \( \Rightarrow \) 1). Using Stokes’ formula, we get for every \( \eta \in \Omega(M) \) the following:

\[
\begin{align*}
    dG(\eta) &= d \int^{\text{Bl}_{M_1}} G \wedge \tilde{pr}_1^* \eta \\
    &= (-1)^n \left( \int^{\text{Bl}_{M_1}} G \wedge \tilde{pr}_1^* \eta - \int^{\partial\text{Bl}_{M_1}} G \wedge \tilde{pr}_1^* \eta \right) \\
    &= \pi_H(\eta) - G(d\eta) + \int^{\partial\text{Bl}_{M_1}} (-1)^{n+1} G \wedge \tilde{pr}_1^* \eta.
\end{align*}
\]

Since \( \tilde{pr}_1 = \tilde{pr}_2 \) on \( \partial\text{Bl}_{\Delta}(M \times M) \), we get with the help of the projection formula the following:

\[
\int^{\partial\text{Bl}_{M_1}} G \wedge \tilde{pr}_1^* \eta = \int^{\partial\text{Bl}_{M_1}} G \wedge \tilde{pr}_2^* \eta = \left( \int^{\partial\text{Bl}_{M_1}} G \right) \wedge \eta.
\]

We will show that the 0-form \( \int^{\partial\text{Bl}_{M_1}} G \) is constant \( (-1)^n \). Stokes’ formula implies

\[
\int^{\partial\text{Bl}_{M_1}} G = \int^{\text{Bl}_{M_1}} dG = (-1)^n \int^{\text{Bl}_{M_1}} H.
\]

Using that \( H \) is Poincaré dual to \( \Delta \), we get for every \( \eta \in \Omega^n(M) \) the following:

\[
\begin{align*}
    \int_M \left( \int^{\text{Bl}_{M_1}} H \right) \wedge \eta &= \int_{\text{Bl}_{\Delta}(M \times M)} H \wedge \tilde{pr}_2^* \eta \\
    &= \int_{M \times M} H \wedge \tilde{pr}_2^* \eta \\
    &= \int_{\Delta} \tilde{pr}_2^* \eta = \int_M 1 \wedge \eta.
\end{align*}
\]

The implication follows.

We will now prove 1) \( \Rightarrow \) 2). Assume that (46) holds and that \( G \) extends smoothly to the blow-up. Denote

\[
K := (-1)^n dG - H \quad \text{and} \quad L := -1 + \int^{\partial\text{Bl}_{(M_1)}} (-1)^n G.
\]

Notice that \( L \) is a function on \( M \). From the previous computations, we deduce that

\[
\int^{\text{Bl}_{(M_1)}} K \wedge \tilde{pr}_1^* \eta = L \eta \quad \text{for all} \ \eta \in \Omega(M),
\]

62
and hence
\[
\int_{\text{Bl}_\Delta(M \times M)} K \wedge \tilde{\mu}_1(\eta_1) \wedge \tilde{\mu}_2(\eta_2) = \int_M L \eta_1 \wedge \eta_2 \quad \text{for all } \eta_1, \eta_2 \in \Omega(M).
\]

If \( K(x, y) \neq 0 \) for some \((x, y) \in (M \times M) \setminus \Delta, \) we can choose \( \eta_1, \eta_2 \) with disjoint supports such that the left-hand side is non-zero. This is a contradiction. Consequently, we have \( K \equiv 0. \)

In general, the Schwartz kernel of a linear operator \( G : \Omega(M) \to \Omega(M) \) is a distributional form \( G \) on \( M \times M \) which satisfies
\[
G(\eta)(x) = \int_{y \in M_1} G(y, x) \eta(y) \quad \text{for all } \eta \in \Omega(M) \text{ and } x \in M_2.
\]

We consider the following conditions on \( G \) and \( G: (G1) \)\( \) The Schwartz kernel \( G \) of \( G \) is a restriction of a smooth form on \( \text{Bl}_\Delta(M \times M). \)
\( (G2) \) \( d \circ G + G \circ d = \iota_{\mathcal{H}} \circ \pi_{\mathcal{H}} - \mathbb{1}. \)
\( (G3) \) \( (G(\eta_1), \eta_2) = (-1)^\eta_1(\eta_1, G(\eta_2)) \) for all \( \eta_1, \eta_2 \in \Omega(M). \)
\( (G4) \) \( G \circ \pi_{\mathcal{H}} = \pi_{\mathcal{H}} \circ G = 0. \)
\( (G5) \) \( G \circ G = 0. \)

Clearly, \( (G1)-(G3) \) are equivalent to \( G \) being a Green kernel from Definition 3.5. Conditions \( (G4) \) and \( (G5) \) play a crucial role in the vanishing results for the formal pushforward Maurer-Cartan element \( n \) in Section 3.4 — the more conditions are satisfied, the more vanishing we get.

The following lemma will be used in the proof of the upcoming proposition.

**Lemma 3.10.** Let \( G_1, G_2 \) be two linear operators \( \Omega(M) \to \Omega(M) \) with Schwartz kernels \( G_1, G_2 \in \Omega(\text{Bl}_\Delta(M \times M)). \) Then \( G := G_1 \circ G_2 \) is a smoothing operator, i.e., its Schwartz kernel \( G \) is a smooth form on \( M \times M. \)

**Proof.** It holds \( G(x_1, x_2) = \pm \int G_2(x_1, x)G_1(x, x_2). \) The lemma follows from properties of convolution. See \[19\] for details.

A version of the following proposition can be found in [7].

**Proposition 3.11 (Existence of special Green operator).** Every oriented closed Riemannian manifold \( M \) admits an operator \( G : \Omega(M) \to \Omega(M) \) which satisfies (\( G1)-(G5) \).

\[^9\text{We may consider such class of } G \text{'s, e.g., pseudo-differential operators, such that } G \text{ exists and is unique (c.f., the well-known Schwartz kernel theorem).} \]
Proof of Proposition 3.11. Because $H$ is Poincaré dual to $\Delta$, we have for any closed $\kappa \in \Omega_c((M \times M) \setminus \Delta)$ the following:

$$\int_{(M \times M) \setminus \Delta} H \wedge \kappa = \int_{M \times M} H \wedge \kappa = \int_\Delta \kappa = 0.$$  

Poincaré duality for non-compact oriented manifolds (see [5]) implies that $H$ is exact on $(M \times M) \setminus \Delta$. Because a manifold with boundary is homotopy equivalent to its interior, the restriction of the blow-down map induces an isomorphism $\pi_*: H^*((M \times M) \setminus \Delta) \to H^*(\text{Bl}_\Delta(M \times M))$. It follows that $(-1)^n \pi^* H$ admits a primitive $G \in \Omega(\text{Bl}_\Delta(M \times M))$. According to Proposition 3.9, the corresponding $G$ satisfies (G1) and (G2).

If we define
definition

$$\tilde{G} := \frac{1}{2} (G + (-1)^n \tau^* G) \in \Omega^{n-1}(\text{Bl}_\Delta(M \times M)),$$

then $\tilde{G}$ satisfies $\tau^* \tilde{G} = (-1)^n \tilde{G}$ and is still a primitive to $(-1)^n \pi^* H$. Proposition 3.9 and (48) imply that the corresponding $\mathcal{G}$ satisfies (G1)–(G3).

Given $\mathcal{G}$ satisfying (G1)–(G3), we will now show that we can arrange (G4). Let us define

$$\hat{\mathcal{G}} := (\mathbb{I} - \pi_H) \circ \mathcal{G} \circ (\mathbb{I} - \pi_H).$$

Then $\hat{\mathcal{G}}$ is a Green operator because

$$d \circ \hat{\mathcal{G}} + \hat{\mathcal{G}} \circ d = (\mathbb{I} - \pi_H) \circ (d \circ \mathcal{G} + \mathcal{G} \circ d) \circ (\mathbb{I} - \pi_H) = \mathbb{I} - \pi_H.$$

Using (51) and (48), we see that $\hat{\mathcal{G}}$ satisfies (G3). Using the intersection pairing and Proposition 3.8, we can write

$$\pi_H(\eta) = \sum_{i=1}^m (-1)^{(n+1)\nu_i} \eta_i \nu_i \quad \text{for all } \eta \in \Omega(M),$$

and hence we have for all $\eta_1, \eta_2 \in \Omega(M)$ the following:

$$\mathcal{G}(\pi_H(\eta_1), \eta_2) = \sum_{i=1}^m (-1)^{(n+1)\nu_i} \eta_i (\mathcal{G}(\nu_i), \eta_2)$$

$$= \sum_{i=1}^m (-1)^{(n+1)\nu_i} \int_{M \times M} \text{pr}_1^*(\nu_i) \wedge \text{pr}_2^*(\mathcal{G}(\nu_i)) \wedge \text{pr}_1^*(\eta_1) \wedge \text{pr}_2^*(\eta_2).$$

It follows that the Schwartz kernel of $\mathcal{G} \circ \pi_H$ is the smooth form

$$\mathcal{K}_{\mathcal{G} \circ \pi_H} := \sum_{i=1}^m (-1)^{(n+1)\nu_i} \text{pr}_1^*(\nu_i) \wedge \text{pr}_2^*(\mathcal{G}(\nu_i)).$$

64
Moreover, if we replace \( G \) with \( \pi H \circ G \), we get the smooth Schwartz kernel \( K_{\pi H \circ G} \circ (\pi H \circ G) \circ \pi H \). In the same way, but now using in addition (48), we can write

\[
(\pi H(G(\eta_1)), \eta_2) = (-1)^{|i|} (\eta_1, G(\pi H(\eta_2))) = (-1)^{|i| n_2} (G(\pi H(\eta_2)), \eta_1)
\]

\[
= \sum_{i=1}^{m} (-1)^{|i| n_2 + (i+1) \nu_i} \int_{M \times M} \text{pr}_1^*(*\nu_i) \wedge \text{pr}_2^*(G(\nu_i)) \wedge \text{pr}_1^*(*\eta_i) \wedge \text{pr}_2^*(\eta_i)
\]

\[
= \sum_{i=1}^{m} (-1)^{|i| n_2 + (i+1) \nu_i} \int_{M \times M} \text{pr}_2^*(*) \wedge \text{pr}_1^*(G(\nu_i)) \wedge \text{pr}_1^*(\nu_i) \wedge \text{pr}_2^*(\eta_i),
\]

where in the last equality we pulled back the integral along the twist map. It follows that the Schwartz kernel of \( \pi H \circ G \) is the smooth form

\[
K_{\pi H \circ G} := \sum_{i=1}^{m} (-1)^{n_2} \nu_i \text{pr}_1^*(G(\nu_i)) \wedge \text{pr}_2^*(*) \wedge \text{pr}_2^*(\nu_i).
\]

The Schwartz kernel of \( \tilde{G} = G - \pi H \circ G - G \circ \pi H + \pi H \circ G \circ \pi H \) is then

\[
\tilde{G} = G - \pi^* K_{G \circ \pi H} - \pi^* K_{\pi H \circ G} + \pi^* K_{\pi H \circ G \circ \pi H},
\]

which is a smooth form on \( Bl_\Delta(M \times M) \). Therefore, \( \tilde{G} \) satisfies (G1)–(G4).

Given \( G \) satisfying (G1)–(G4), we will show that we can arrange (G5). The trick from [7] is to define

\[
\tilde{G} = G \circ dG.
\]

Applying (G1) and (G2) repeatedly, we compute

\[
d G \circ dG = d G \circ -d dG = d G \circ -dG \circ + d dG
\]

\[
= d G \circ -d dG
= d G \circ G - G \circ dG,
\]

and hence

\[
\tilde{G} = G - dG \circ G \circ dG.
\]

Clearly, \( \tilde{G} \) satisfies (G1) and (G2). As for (G3), we compute

\[
(\eta_1, \tilde{G} \eta_2) = (-1)^{|i|} (G \eta_1, G \eta_2) = (d G \eta_1, \eta_2) = (-1)^{|i|} (G \eta_1, \eta_2).
\]

As for (G5), we have

\[
\tilde{G} G = G \circ d(G \circ dG) = G \circ dG \circ -G \circ d(G \circ dG)
\]

\[
= G \circ dG \circ -dG \circ G + dG \circ G \circ dG = 0.
\]

In order to show (G4), we have to compute the Schwartz kernel of \( dG \circ G \circ dG \). By Lemma 3.10, the Schwartz kernel \( T \) of \( T := G \circ G \circ dG \) is a smooth form on \( M \times M \).
Therefore, Stokes’ formula without the boundary term applies, and we get
\[
(dT\eta) = d \int_{M_1} T \wedge d\pi_1^*(\eta) = \int_{M_1} dT \wedge d\pi_1^*(\eta) = (-1)^T \int_{M_1} d_1 dT \wedge \pi_1^*(\eta).
\]
Here \( d_1 : \Omega(M \times M) \to \Omega(M \times M) \) is the operator defined in local coordinates by
\[
d_1(f(x,y) \, dx^I \, dy^J) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x,y) \, dx^i \, dx^I \, dy^J.
\]
It follows that the Schwartz kernel \( \tilde{G} \) of \( \tilde{G} \) satisfies
\[
\tilde{G} = G + (-1)^n d_1 dT
\]
and is a smooth \((n-1)\)-form on \( Bl_\Delta(M \times M) \). Conditions (G1)–(G5) are satisfied. \( \square \)

**Remark 3.12 (Property (G5) in dimensions 1 and 2).** In dimension 1, every operator of degree \(-1\) satisfies (G5) from degree reasons. In dimension 2, every operator satisfying (G1) and (G2) satisfies (G5) as well, which follows from (53) and (54). \( \triangleright \)

**Remark 3.13 (The standard Green kernel).** Consider the Hodge-de Rham Laplacian \( \Delta = d \circ d^* + d^* \circ d : \Omega(M) \to \Omega(M) \) and its “Green operator” \( G_\Delta \) of degree 0 (see (v) of Remark 3.6 for the collision of terminology) which was defined in [33, Definition 6.9] by
\[
G_\Delta := (\Delta |_{\mathcal{H}(M)^\perp})^{-1} \circ \pi_{\mathcal{H}(M)^\perp},
\]
where \( \perp \) denotes the \( L^2 \)-orthogonal complement. We introduce the standard Green operator by
\[
G_{std} := -d^* G_\Delta. \tag{55}
\]
Using the properties of \( G_\Delta, d \) and \( d^* \), one can show that \( G_{std} \) satisfies (G2)–(G5) (this will be shown in [19]).

As for (G1), the author was able to show it for flat manifolds (:= locally isometric to \( \mathbb{R}^n \)) by transforming the following formula inspired by [16] to blow-up coordinates and explicitly computing the integral and limit:
\[
G_{std} = -\lim_{t \to 0} \int_t^\infty \frac{1}{2} d^* K_t \, d\tau,
\]
where \( K_t(x,y) = \sum_i (-1)^n e_i e^{-\lambda_i t}(e_i(x)) \wedge e_i(y) \) is the heat kernel of \( \Delta \) and \( e_i \) the \( L^2 \)-orthonormal eigenbasis of \( \Delta \) with eigenvalues \( \lambda_i \) (the signs come from our convention for fiberwise integration, c.f., [49]). \( \triangleright \)
3.3 Formal pushforward Maurer-Cartan element n

We first recall ribbon graphs and their labelings based on \[10\].

**Definition 3.14 (Ribbon graph).** A graph \( \Gamma \) is a quadruple \((V, H, \mathcal{V}, \mathcal{E})\), where \( V \) is a finite set of vertices, \( H \) a finite set of half-edges, \( \mathcal{V} : H \to V \) the “vertex map” and \( \mathcal{E} : H \to H \) with \( \mathcal{E} \circ \mathcal{E} = \mathbb{1} \) and without fixed points the “edge map”. The preimage \( \mathcal{E}^{-1}(h_1) = \{h_1, h_2\} \) for some \( h_1, h_2 \in H \) is called an edge; the set of edges is denoted by \( E \). We assume that the graphs are connected, i.e., that for any \( v_1, v_2 \in V \) there exists a path in \( E \) connecting \( v_1 \) to \( v_2 \).

A ribbon graph is a graph \( \Gamma \) which is equipped with a free transitive action \( \mathbb{Z}_d(v) \recht V^{-1}(v) \) for every \( v \in V \), where \( d(v) := |V^{-1}(v)| \) is the valency of \( v \). We denote by \( N : H \to H \) the bijection induced by \( 1 \in \mathbb{Z}_d(v) \) for every \( v \in V \).

For a ribbon graph \( \Gamma \), consider the set of sequences \((h_n)_{n \in \mathbb{Z}} \subset H\) such that the following conditions hold:

\[
\forall n \in \mathbb{Z} : \quad h_{n+1} = \begin{cases} \mathcal{E}(h_n) & n \text{ even}, \\ N(h_n) & n \text{ odd}. \end{cases}
\]

Two such sequences \((h_n)_{n \in \mathbb{Z}} \) and \((h'_n)_{n \in \mathbb{Z}} \) are equivalent if and only if there exist \( n_0, n'_0 \in \mathbb{Z} \) both even or both odd such that \( h_{n_0} = h'_{n_0} \). An equivalence class \([ (h_n)_{n \in \mathbb{Z}} ] \) is called a boundary (or a boundary component) of \( \Gamma \). The set of boundaries of \( \Gamma \) is denoted by \( \partial \Gamma \).

An IE ribbon graph is a ribbon graph \( \Gamma \) together with the decomposition \( V = V_{\text{int}} \sqcup V_{\text{ext}} \) into internal and external vertices \( V_{\text{int}} \) and \( V_{\text{ext}} \) such that \( d(v) = 1 \) for all \( v \in V_{\text{ext}} \), respectively. This decomposition induces the decomposition \( E = E_{\text{int}} \sqcup E_{\text{ext}} \), where an edge \( e \) is internal if it connects two internal vertices and is external otherwise. We allow only graphs with at least one internal vertex. We often identify an external vertex with its unique adjacent half-edge or the unique adjacent external edge; we call either of these an external leg. For any \( b \in \partial \Gamma \), we define the valency of \( b \) by

\[
s(b) := |\mathcal{V}(b) \cap V_{\text{ext}}|,
\]

where \( \mathcal{V}(b) = \{\mathcal{V}(h_n) \mid n \in \mathbb{Z}\} \). We also have the free transitive \( \mathbb{Z}_{s(b)} \)-action on \( \mathcal{V}(b) \cap V_{\text{ext}} \) mapping \( v \in \mathcal{V}(b) \cap V_{\text{ext}} \) to the next external vertex in the sequence \( \mathcal{V}(h_n)_{n \in \mathbb{Z}} \). We will denote this action by \( N \) again.

We say that an IE ribbon graph \( \Gamma \) is reduced if \( s(b) \geq 1 \) for all \( b \in \partial \Gamma \).

The following letters will be used to denote the numerical invariants of a
graph:

\[ k \ldots \text{the number of internal vertices}, \]
\[ s \ldots \text{external vertices}, \]
\[ l \ldots \text{boundary components}, \]
\[ e \ldots \text{internal edges}. \]

Moreover, we define the genus \( g \in \mathbb{N}_0 \) so that the following Euler formula holds:

\[ k - e + l = 2 - 2g. \tag{56} \]

We denote by \( \text{RG}_{klg} \) the set of isomorphism classes of connected IE ribbon graphs with fixed \( k, l, g \). We let \( \text{RG}^{\tiny \text{red}}_{klg} \subset \text{RG}_{klg} \) be the subset of reduced graphs. For \( m \in \mathbb{N}_0 \), we denote by \( \text{RG}^{(m)}_{klg} \subset \text{RG}_{klg} \) the set of isomorphism classes of connected IE ribbon graphs with all internal vertices \( m \) valent, i.e., with \( d(v) = m \) for all \( v \in V_{\text{int}} \).

The notation \( \Gamma \in \text{RG}_{klg} \) means that \( \Gamma \) is a representative of an equivalence class \( [\Gamma] \in \text{RG}_{klg} \).

**Remark 3.15 (On ribbon graphs).**

(i) An \( m \)-valent ribbon graph with \( m \geq 2 \) has a unique decomposition \( V = V_{\text{int}} \sqcup V_{\text{ext}} \), and hence we can omit writing IE.

(ii) In this text, we will use only reduced ribbon graphs. Non-reduced ribbon graphs may play a role in the extension of the theory of \( \text{dIBL}^n(C(H)) \) to non-reduced cyclic cochains or in the weak \( \text{IBL}_\infty \)-theory (see Remarks 2.23 and 2.27).

**Definition 3.16 (Labeling).** A labeling of an IE ribbon graph \( \Gamma \) is the triple \( L = (L_1, L_2, L_3) \), where \( L_i \) have the following meanings:

- The symbol \( L_1 \) represents an ordering of internal vertices (\( =: L_1^v \)) and of boundary components (\( =: L_1^b \)). Given \( L_1 \), we write \( V_{\text{ext}} = \{v_1, \ldots, v_k\} \), \( \partial \Gamma = \{b_1, \ldots, b_l\} \) and denote \( d_i := d(v_i) \) and \( s_j := s(b_j) \).

- The symbol \( L_2 \) represents an ordering and orientation of internal edges. Given \( L_2 \), we write \( E_{\text{int}} = \{e_1, \ldots, e_e\} \) and \( e_i = \{h_{i,1}, h_{i,2}\} \) for \( h_{i,1}, h_{i,2} \in H \).

- The symbol \( L_3 \) represents an ordering of half-edges at every internal vertex (\( =: L_3^v \)) and of external vertices at every boundary component (\( =: L_3^b \)), both compatible with the ribbon structure (\( =: \mathbb{Z}_m \)-actions). Given \( L_3 \), we write \( \mathcal{V}^{-1}(v) = \{h_{v,1}, \ldots, h_{v,d(v)}\} \) and \( \mathcal{V}(b) \cap V_{\text{ext}} = \{v_{b,1}, \ldots, v_{b,s(b)}\} \) with \( \mathcal{N}(h_{v,i}) = h_{v,i+1} \) and \( \mathcal{N}(v_{b,j}) = v_{b,j+1} \) for all \( i, j \), respectively.
We sometimes call \( L_i \) partial labelings and \( L \) a full labeling. A ribbon graph \( \Gamma \) together with a labeling \( L \) is called a labeled ribbon graph.

Given a ribbon graph \( \Gamma \), one can construct an oriented surface with boundary \( \Sigma_\Gamma \) — the thickening of \( \Gamma \) — in the obvious way and a closed oriented surface \( \hat{\Sigma}_\Gamma \) by gluing oriented disks to the oriented boundaries of \( \Sigma_\Gamma \). If partial labelings \( L_1 \) and \( L_2 \) are given, we obtain the following chain complex with oriented chain groups (vector spaces over \( \mathbb{R} \)):

\[
C_2 := \langle b_1, \ldots, b_l \rangle \xrightarrow{\partial_2} C_1 := \langle e_1, \ldots, e_e \rangle \xrightarrow{\partial_1} C_0 := \langle v_k, \ldots, v_1 \rangle. \tag{57}
\]

Here \( b_i \) stands for the oriented disc glued to the \( i \)-th boundary component of \( \Sigma_\Gamma \) and now being mapped into \( \hat{\Sigma}_\Gamma \), \( e_i \) stands for the \( 1 \)-simplex in \( \hat{\Sigma}_\Gamma \) corresponding to the \( i \)-th internal edge, \( v_i \) stands for the \( 0 \)-simplex in \( \hat{\Sigma}_\Gamma \) corresponding to the \( i \)-th internal vertex, and the boundary map \( \partial \) is the “geometric” boundary operator. The homology of this chain complex is isomorphic to the singular homology \( \text{H}(\hat{\Sigma}) := \text{H}(\hat{\Sigma}_\Gamma; \mathbb{R}) \).

The orientation of \( C_i \) (:= the order of generators in (57)) induces naturally an orientation of \( \text{H}(\hat{\Sigma}_\Gamma) \). The construction from \cite{10} Appendix A) is as follows. We pick complements \( H_i \) of \( \text{Im}(\partial_{i+1}) \) in \( \text{ker}(\partial_i) \) and complements \( V_i \) of \( \text{ker}(\partial_i) \) in \( C_i \) and write

\[
C_2 = V_2 \oplus H_2 \xrightarrow{\partial_2} C_1 = V_1 \oplus H_1 \oplus \text{Im}(\partial_2) \xrightarrow{\partial_1} C_0 = \text{Im}(\partial_1) \oplus H_0.
\]

We orient \( V_i \) arbitrarily and transfer the orientation to \( \text{Im}(\partial_i) \) via \( \partial_i : V_i \xrightarrow{\text{can}} \text{Im}(\partial_i) \). Then, assuming the direct sum orientation, orienting \( H_i \) is equivalent to orienting \( C_i \), and we obtain the orientation of \( \text{H}(\hat{\Sigma}_\Gamma) \) via the canonical projection \( \pi : H_i \xrightarrow{\text{can}} \text{H}_i(\hat{\Sigma}_\Gamma) = \text{ker}(\partial_i)/\text{Im}(\partial_{i+1}) \). This construction does not depend on the choices of complements and orientations of \( V_i \).

**Definition 3.17** (Compatibility of \( L_1 \) and \( L_2 \)). Given a ribbon graph \( \Gamma \) with partial labelings \( L_1 \) and \( L_2 \), we say that \( L_2 \) is compatible with \( L_1 \) if the orientation on \( \text{H}(\hat{\Sigma}_\Gamma) \) induced by (57) agrees with the canonical orientation

\[
\text{H}(\hat{\Sigma}_\Gamma) = \langle v_1 + \cdots + v_k \rangle \oplus \text{H}_1(\hat{\Sigma}_\Gamma) \oplus \langle b_1 + \cdots + b_l \rangle,
\]

where \( \text{H}_1(\hat{\Sigma}_\Gamma) \) is oriented using the canonical symplectic intersection form.

Given a labeled IE ribbon graph \( \Gamma \), the set of half-edges adjacent to internal vertices \( V^{-1}(V_{\text{int}}) \) can be ordered in two ways corresponding to writing

\[
2e + (s_1 + \cdots + s_l) = d_1 + \cdots + d_k.
\]

This leads to the following definition.
Definition 3.18 (Edge order and vertex order). For a labeled IE ribbon graph $\Gamma$, we define the following two orders on the set of half-edges $H$:

- **Edge order:** The first $2e$ half-edges $h_{e,i,j}^c$ are the ones from internal edges; they are ordered according to $L_2$. They are followed by blocks of $s_1, \ldots, s_l$ half-edges $h_{b,i,j}^b$ which come from the boundary components $i = 1, \ldots, l$, respectively, and which are ordered according to $L_3^b$ inside the blocks. Schematically, we have

  $$(h_{e,1,1}^c \ldots h_{e,1,2}^c)(h_{e,1,1}^b \ldots h_{e,s_1}^b) \ldots (h_{e,1,1}^b \ldots h_{e,s_l}^b).$$

- **Vertex order:** It consists of blocks of $d_1, \ldots, d_k$ half-edges $h_{v,i,j}^v$ which come from internal vertices $1, \ldots, k$, and which are ordered according to $L_3^v$ inside the blocks. Schematically, we have

  $$(h_{v,1,1}^v \ldots h_{v,1,d_1}^v) \ldots (h_{v,k,1}^v \ldots h_{v,k,d_k}^v).$$

We denote by $\sigma_L \in S_{|H|}$ the permutation from the edge to the vertex order which is constructed such that the $i$-th half-edge in the edge order is the same as the $\sigma_L(i)$-th half-edge in the vertex order.

From now on, we will consider only reduced trivalent ribbon graphs $\text{RG}^{(3)}_{k|g}$ with $k, l \geq 1, g \geq 0$. We will often use the equation

$$2e + s = 3k. \quad (58)$$

Definition 3.19 (Formal pushforward Maurer-Cartan element). Let $M$ be an oriented closed Riemannian manifold, and let $G \in \Omega^n-1(\text{Bl}_\Delta(M \times M))$ be a Green kernel from Definition 3.5. The formal pushforward Maurer-Cartan element $\mathbf{n}$ is the collection of

$$n_lg \in \hat{E}_{lC}(H(M)) \quad \text{for all } l \geq 1, g \geq 0$$

defined on generating words $\omega_l = \alpha_{i_1} \ldots \alpha_{i_s} \in \text{B}^\text{cyc}_S(M)$, where $\alpha_{i,j} = \theta \eta_{i,j}$ with $\eta_{i,j} \in H(M)$ for $s_i \geq 1$ and $i = 1, \ldots, l$, by the formula

$$n_lg(s^l \omega_1 \otimes \cdots \otimes \omega_l) := \frac{1}{|\Gamma|} \sum_{[\Gamma] \in \text{\textit{RG}}^{(3)}_{k|g}} (-1)^{s(k,l)+P(\omega)} \sum_{L_1, L_3^b} (-1)^{s_L} I(\sigma_L). \quad (59)$$

which we explain as follows:

- The second sum is over all partial labelings $L_1$ and $L_3^b$ of a representative $\Gamma$ of $[\Gamma]$. In every summand, we complete $L_1$ and $L_3^b$ to a full labeling
\[ L = (L_1, L_2, L_3) \] by picking an arbitrary \( L_3 \) and an arbitrary \( L_2 \) compatible with \( L_1 \).

- Suppose that \( \Gamma \) and \( L_1 \) are admissible with respect to the input \( \omega_1, \ldots, \omega_l \); this means that \( \Gamma \) has \( l \) boundary components and that the \( i \)-th boundary component has valency \( s_i \) for every \( i = 1, \ldots, l \). In this case, denoting \( \sigma = \sigma_L \), we define

\[
I(\sigma_L) := \int_{x_1, \ldots, x_k} G(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \cdots G(x_{\xi(\sigma_{2e-1})}, x_{\xi(\sigma_{2e})}) 
\eta_{11}(x_{\xi(\sigma_{2e+1})}) \cdots \eta_{s_1}(x_{\xi(\sigma_{2e+s})}),
\]

where \( \xi : \{1, \ldots, 3k\} \to \{1, \ldots, k\} \) is the function defined by

\[ \xi(3j - 2) = \xi(3j - 1) = \xi(3j) := j \]

for all \( j = 1, \ldots, k \), \( s = s_1 + \cdots + s_l \), \( \eta(x_i) \) denotes the pullback of \( \eta \) along the canonical projection \( \pi : M^{\times k} \to M \) to the \( i \)-th component \( M_i \), \( G(x_i, x_j) \) denotes the pullback of \( G \) along \( \pi_i \times \pi_j : M^{\times k} \to M_i \times M_j \), and \( \int_{x_1, \ldots, x_k} \) denotes the integral of an \( nk \)-form over \( k \) copies of \( M \).

If \( \Gamma \) and \( L_1 \) are not admissible, then we set \( I(\sigma_L) := 0 \).

- \( s(k, l) := k + kl(n - 1) + \frac{1}{2} k(k - 1)n \mod 2 \)
- \( P(\omega) := \sum_{i=1}^l \sum_{j=1}^{s_i} (s - s_1 - \cdots - s_{i-1} - j) \eta_{ij} \mod 2 \)

In order to show that \( \eta_L \) is well-defined and that the collection \( (\eta_L) \) satisfies Definition 2.19 for \( \text{dIBL}(\mathcal{C}(\mathcal{H}(M))) \), there are several things to check:

1. The integral \( I(\sigma_L) \) converges.
2. The sums are finite.
3. The product \( (-1)^{s_L} I(\sigma_L) \) is independent of the choice of \( L_3 \) and \( L_2 \) compatible with \( L_1 \).
4. The sum over labelings is independent of the chosen representative \( \Gamma \) in an isomorphism class from \( R\mathcal{G}^{(3)}_{klg} \).
5. The map \( \eta_L : B_{kcl}^{\infty}(\mathcal{H}(M))[3 - n]^{\otimes l} \to \mathbb{R} \) is graded symmetric on permutations of its inputs \( s\omega_i \).
6. The map \( \eta_L \) is graded symmetric on cyclic permutations of the components \( \alpha_{ij} \) of each \( \omega_i \).
7. The degree condition 1) from Definition 2.19 holds with \( d = n - 3 \).
8. The filtration-degree condition 2) from Definition 2.19 holds with \( \gamma = 2 \).
(9) The Maurer-Cartan equation \[22\] holds.

Conditions 1) and 9) will be proven in \[12\] using the theory of iterated blow-ups. In this text, we will take 1) and 9) for granted.

Lemma 3.20. Assuming 1), the conditions 2) – 8) hold.

Proof. As for 2), the fixed input \(\omega_1, \ldots, \omega_l\) fixes the number \(s\) of external vertices of \(\Gamma\) by admissibility. Expressing \(e\) from (56) and plugging it in (58) gives

\[k = s + 2l + 4g - 4.\]  

We see that all parameters are fixed. Now, there is only finitely many elements with fixed \(s\) in \(\mathcal{R}_{klg}^{(3)}\), and each of them has only finitely many labelings. Therefore, the sums are finite.

As for 3), we have to consider the orientation of the complex \[57\]. Clearly, if two \(L_2\)'s are compatible with \(L_1\), then they differ by an even number of the following operations: a transposition of two internal edges or a change of the orientation of an internal edge. The former operation introduces no sign in \((-1)^{\sigma_L}\) but generates the sign \((-1)^{n-1}\) in \(I(\sigma_L)\) from swapping the corresponding \(G\)'s. The latter operation induces the sign \(-1\) in \((-1)^{\sigma_L}\) and the sign \((-1)^{n}\) in \(I(\sigma_L)\) from the symmetry \(G(x,y) = (-1)^n G(y,x)\). Because the overall signs in \((-1)^{\sigma_L} I(\sigma_L)\) are the same, an even number of these operations preserves \((-1)^{\sigma_L} I(\sigma_L)\). This implies the independence of an \(L_2\) compatible with \(L_1\). A change in \(L_3\) produces no sign in \((-1)^{\sigma_L}\) because every internal vertex is trivalent and a cyclic permutation of an odd number of elements is even. The integral \(I(\sigma_L)\) remains unchanged because the change in \(\sigma_L\) is compensated by the composition with \(\xi\). Independence of the choice of \(L_3\) follows.

As for 4), every isomorphism of ribbon graphs \(\Gamma \rightarrow \Gamma'\) induces the bijection \(L \mapsto L'\) of compatible labelings such that \(\sigma_L = \sigma_{L'}\) (\(L'\) is the “pushforward” labeling). The independence of the choice of a representative of \([\Gamma]\) follows.

As for 5), let \(\mu \in S_{2e}\) be a permutation of the inputs \(s\omega_1, \ldots, s\omega_l\). The set of graphs which admit an admissible labeling is the same for both \(n_{ij}(s^i\omega_1 \otimes \cdots \otimes \omega_l)\) and \(n_{ij}(s^i\omega_{\sigma_i^{-1}} \otimes \cdots \otimes \omega_{\sigma_i^{-1}})\); we will pick one such \(\Gamma\) and study the admissible labelings \(L\) and \(L'\), respectively. We write \(\eta_i = \eta_{i_1} \ldots \eta_{i_{2e}}\) and \(\Omega_i = s\omega_i\) for all \(i, j,\) and denote by \(I'(\sigma_{L'})\) the integral in the definition of \(n_{ij}(s^i\omega_{\mu_i^{-1}} \otimes \cdots \otimes \omega_{\mu_i^{-1}})\). Let \(\tilde{\mu} \in S_{3k}\) be the permutation which acts as the identity on \(1, \ldots, 2e\) and as the block permutation determined by \(\mu\) on \(2e + 1, \ldots, 2e + s\) divided into \(l\)
blocks of lengths $s_1, \ldots, s_l$. For any $\sigma \in \mathbb{S}_{3k}$, we have

$$I'(\sigma) = \int_{x_1, \ldots, x_k} G(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \cdots G(x_{\xi(\sigma_{2k-1})}, x_{\xi(\sigma_{2k})})$$

$$\eta_{\mu_1}^{-1}(x_{\xi(\sigma_{2k+1})}) \cdots \eta_{\mu_l}^{-1}(x_{\xi(\sigma_{2k+l})})$$

$$= \varepsilon(\mu, \eta) \int_{x_1, \ldots, x_k} G(x_{\xi(\sigma_{2k+1})}, x_{\xi(\sigma_{2k+2})}) \cdots G(x_{\xi(\sigma_{2k+l})}, x_{\xi(\sigma_{2k+l+1})})$$

$$\eta_{\sigma_1}(x_{\xi(\sigma_{2k+1})}) \cdots \eta_{\sigma_l}(x_{\xi(\sigma_{2k+l+1})})$$

$$= \varepsilon(\mu, \eta) I(\sigma \circ \tilde{\mu}).$$

The precomposition with $\tilde{\mu}$ corresponds to a bijection $(L_1, L_2) \mapsto (L_1', L_2')$ of partial labelings for $n_g(s\omega_1 \ldots \omega_l)$ and $n_g(s\omega_{\mu_1} \otimes \cdots \otimes \omega_{\mu_l})$, respectively. However, if $L_2$ is compatible with $L_1$, then in order to get an $L_2'$ compatible with $L_1'$, the labeling $L_2$ has to be altered by as many operations of switching two internal edges or changing the orientation of an internal edge as there are transpositions in $\mu$. We explained in the proof of 3) that this produces the sign $(-1)^{(n-1)\mu}$ in $(-1)^{\sigma_L} I(\sigma_{L'})$. Therefore, after the choice of compatible $L_2$ and $L_2'$, we have

$$(-1)^{\sigma_L} I'(\sigma_{L'}) = (-1)^{(n-1)\mu} (-1)^{\beta} \varepsilon(\mu, \eta) (-1)^{\sigma_L} I(\sigma_L).$$

If we view $\eta$ as $\eta_{11} \cdots \eta_{s_k}$, we can understand $(-1)^{\beta}$ as the Koszul sign $\varepsilon(\theta, \eta)$. Similarly, we write $(-1)^{\beta} \varepsilon(\theta, \mu(\eta))$, where we first view $\eta$ as $\eta_1 \otimes \cdots \otimes \eta_l$ to apply $\mu$ and then as the list of components $\eta_{ij}$ to compute the Koszul sign (this is a little ambiguity in our notation). If we denote by $\pi$ the permutation of $1, \ldots, s$ permuting the $l$ blocks of lengths $s_1, \ldots, s_l$ according to $\mu$, then $\pi$ has the same sign as $\tilde{\mu}$, and the decomposition of $\varepsilon(\theta, \mu(\eta))$ into the moves

$$\theta_1 \cdots \theta_s \eta_{\mu_1}^{-1} \cdots \eta_{\mu_l}^{-1} \eta_{s_1} \cdots \eta_{s_l} \xrightarrow{(1)} \theta_{\mu_1} \cdots \theta_{\mu_l} \eta_{11} \cdots \eta_{s_k} \xrightarrow{(2)} \theta_{\mu_1} \eta_{11} \cdots \theta_{\mu_l}, \eta_{s_k} \xrightarrow{(3)} \theta_1 \eta_{\mu_1}^{-1} \cdots \theta_s \eta_{\mu_l}^{-1} \eta_{s_k}$$

shows that

$$(-1)^{\beta} \varepsilon(\mu, \eta) (-1)^{\beta} \varepsilon(\mu, \omega).$$

Using this, we write

$$(-1)^{\beta} \varepsilon(\mu, \omega)(-1)^{\sigma_L} I'(\sigma_{L'}) = \varepsilon(\mu, \omega)(-1)^{(n-1)\mu} (-1)^{\beta} \varepsilon(\mu, \omega).$$

73
and compute
\[
\nu_G(\Omega_{\mu_1^{-1}} \otimes \cdots \otimes \Omega_{\mu_l^{-1}})
\]
\[
= \varepsilon(\mu(s), \mu(\omega))\nu_G(s^i \omega_{\mu_1^{-1}} \otimes \cdots \otimes \omega_{\mu_l^{-1}})
\]
\[
= \varepsilon(\mu(s), \mu(\omega))(-1)^{|s|\mu} \varepsilon(\mu, \omega)\nu_G(s^i \omega_1 \otimes \cdots \otimes \omega_l)
\]
\[
= \varepsilon(\mu(s), \mu(\omega)) - (-1)^{|s|\mu} \varepsilon(\mu, \omega)\varepsilon(s, \omega)\nu_G(s \omega_1 \otimes \cdots \otimes s \omega_l)
\]
\[
= \varepsilon(\mu, \omega)\nu_G(\Omega_1 \otimes \cdots \otimes \Omega_l).
\]
We used \(|s| = n - 1 \mod 2\), and the last equality follows from the decomposition of \(\varepsilon(\mu, \Omega)\) into the moves
\[
s_1 \omega_1 \ldots s_l \omega_l \stackrel{(3)}{\longrightarrow} s_1 \ldots s_l \omega_1 \ldots \omega_l \stackrel{(2)}{\longrightarrow} s_{\mu_1^{-1}} \ldots s_{\mu_l^{-1}} \omega_{\mu_1^{-1}} \ldots \omega_{\mu_l^{-1}} \stackrel{(1)}{\longrightarrow} s_{\mu_1^{-1}} \omega_{\mu_1^{-1}} \ldots s_{\mu_l^{-1}} \omega_{\mu_l^{-1}}.
\]
This proves the symmetry of \(\nu_G\).

As for 6), fix an \(i = 1, \ldots, l\) and let \(\mu \in S_{s_i}\) be a cyclic permutation permuting the components of \(\omega_i = \alpha_{s_i} \ldots \alpha_{s_i} \omega_i\). Similarly to the previous case, we denote by \(\hat{\mu}\) the corresponding permutation of \(1, \ldots, 3k\) and get a bijection \((L_1, L_2) \mapsto (L_1', L_2')\) of admissible labelings of a given graph \(\Gamma\) for \(n_{s_i}(s^i \omega_1 \otimes \cdots \otimes \alpha_{s_1} \ldots \alpha_{s_i} \ldots \otimes \omega_l)\) and \(n_{s_i}(s^i \omega_1 \otimes \cdots \otimes \alpha_{s_1} \ldots \alpha_{s_i} \ldots \otimes \omega_l)\), respectively. This time, there is no change in \(L_1\), and thus we can take \(L_2' = L_2\), producing no sign. Therefore, we have
\[
(-1)^{s_{\sigma_L'}} I'(\sigma_{L'}) = (-1)^{\hat{\beta}} \varepsilon(\mu, \eta_i)(-1)^{s_{\sigma_L}} I(\sigma_L),
\]
where \(\varepsilon(\mu, \eta_i)\) comes from permuting the forms in \(I'(\sigma_{L'})\). Further, we deduce
\[
(-1)^{P(\mu(\omega))} = (-1)^{\hat{\beta}} \varepsilon(\mu, \eta_i)(-1)^{P(\omega)} \varepsilon(\mu, \omega_i),
\]
and hence
\[
\nu_G(s^i \omega_1 \otimes \cdots \otimes \alpha_{i \mu_1^{-1}} \ldots \alpha_{i \mu_s^{-1}} \otimes \cdots \otimes \omega_l)
\]
\[
= \varepsilon(\mu, \omega_i)\nu_G(s^i \omega_1 \otimes \cdots \otimes \alpha_{s_1} \ldots \alpha_{s_s} \otimes \cdots \otimes \omega_l).
\]
This shows the symmetry of \(\nu_G\) on cyclic permutations of the components of \(\omega_i\).

As for 7), suppose that \(n_{\eta}(s^i \omega_1 \otimes \cdots \otimes \omega_l) \neq 0\), and let \(D\) denote the total form-degree of the input \(\eta_{t_1}, \ldots, \eta_{s_i} \in \mathcal{H}(M); \) i.e., we define
\[
D := \deg(\eta_{t_1}) + \cdots + \deg(\eta_{s_i}) + \cdots + \deg(\eta_{t_1}) + \cdots + \deg(\eta_{s_i}).
\]
Clearly, we must have
\[ nk = (n - 1)e + D, \tag{62} \]
where the left-hand side is the dimension of \( M \times k \) and the right-hand side the form-degree of the integrand of \( I(\sigma_L) \). If we plug in \( e \) from (56) and \( k \) from (61), we get
\[
D = nk - (n - 1)e
= nk - (n - 1)(k + l + 2g - 2)
= k - (n - 1)(l + 2g - 2)
= s + 2l + 4g - 4 - (n - 1)(l + 2g - 2)
= s - (n - 3)(l + 2g - 2).
\]
It follows that
\[
|n_{lg}| = |s'| + |\omega_1| + \cdots + |\omega_l| = l(n - 3) + D - s = 2(n - 3)(g - 1).
\]
This is exactly the degree from Definition 2.19.

As for 8), if \( n_{lg}(s'\omega_1 \otimes \cdots \otimes \omega_l) \neq 0 \), then
\[
s = k - 2l - 4g - 4 \geq 1 + 2(2 - 2g - l) = 1 + 2\chi_{otg},
\]
and hence \( n_{lg} \in F_{1+2\chi_{otg}}\hat{E}_lC \) for the filtration induced from the dual of the filtration of \( B_{*}^{\text{cyb}}H \) by weights. Therefore, we get
\[
\|n_{lg}\| \geq 1 + 2\chi_{otg} > 2\chi_{otg} \quad \text{for all } l \geq 1, g \geq 0.
\]
This finishes the proof.

**Definition 3.21** (Vertices of types A, B, C and some special graphs). Let \( \Gamma \in \text{RG}_{\mu}^{3} \) be a trivalent ribbon graph and \( v \) its internal vertex. We say that \( v \) is of type A, B or C if it is connected to precisely 1, 2 or 3 internal vertices, respectively (see Figure 4). The graph \( \Gamma \) is called (see Figures 2 and 3):

- a **tree** if \( [\Gamma] \in \text{RG}_{k_{10}}^{3} \) for some \( k \geq 1 \);
- **circular** if \( [\Gamma] \in \text{RG}_{k_{20}}^{3} \) for some \( k \geq 1 \);
- the **Y-graph** is the unique tree with \( k = 1 \);
- an **Ok-graph** if \( \Gamma \) is circular with \( k \) internal vertices and no A-vertex.

We denote the Y-graph simply by \( Y \).

**Remark 3.22** (On A, B, C vertices and special graphs). We observe the following:

(i) A trivalent graph \( \Gamma \neq Y \) has each internal vertex of type A, B or C.
Figure 2: A tree and a circular graph. Internal vertices are denoted with a full dot and external vertices with an empty dot.

Figure 3: The Y-graph and an $O_6$-graph.

(ii) The term $n_{10}$ is a sum over trees, and the term $m_{10}$ is the contribution of the Y-graph to $n_{10}$ (see Proposition 3.23 below). The term $n_{20}$ is a sum over circular graphs.

We will also denote by $A$, $B$, $C$ the numbers of internal vertices of the corresponding type. Under the change of variables

\begin{align*}
    s &= 2A + B, \\
    e &= B + \frac{1}{2}A + \frac{3}{2}C, \\
    k &= A + B + C,
\end{align*}

the trivalent formula \eqref{eq:trivalent} becomes trivial and the Euler formula \eqref{eq:euler} becomes

\begin{equation}
    C - A = 2l - 4 + 4g.
\end{equation}

**Proposition 3.23** (Formal pushforward Maurer-Cartan element). The pushforward Maurer-Cartan element $n = (n_g)$ is a Maurer-Cartan element for $dIBL(\mathcal{H}(M))$ which is compatible with $m$. In particular, the $A_\infty$-algebra $\mathcal{H}(M)_n$ is homologically unital and augmented.

**Proof.** The fact that $n$ is a Maurer-Cartan element for $dIBL(\mathcal{H}(M))$ follows from Lemma 3.20 assuming 1) and 9) from 12.

As for the compatibility with $m$, the only graph contributing to $n_{10}(s\alpha_1\alpha_2\alpha_3)$ is the $Y$-graph with $k = 1$. The group $\text{Aut}(Y)$ consists of three rotations, and there is only one possible $L_1$, no $L_2$ and three $L_3$. In Definition 3.19 we get $s(1, 1) = n - 2$, $(-1)^{nL} = 1$ because a cyclic permutation of an odd number of
elements is even, and also $P(\alpha_1\alpha_2\alpha_3) = \eta_2$. Finally, we compute

$$n_{10}(s\alpha_1\alpha_2\alpha_3) = \frac{1}{3} (-1)^{n-2+\eta_2} \sum_{L} \int_{x} \alpha_1(x_{\xi(\sigma_1)})\alpha_2(x_{\xi(\sigma_2)})\alpha_3(x_{\xi(\sigma_3)})$$

$$= (-1)^{n-2+\eta_2} \int_{M} \eta_1 \wedge \eta_2 \wedge \eta_3$$

$$= m_{10}(s\alpha_1\alpha_2\alpha_3).$$

**Definition 3.24** (Contributions of A, B, C vertices). Consider an internal vertex of type A, B or C as in Figure 4. We define the following smooth forms on $M$, $M \times 2$ and $M \times 3$, respectively:

$$A_{\alpha_1,\alpha_2}(y) := \int_{x} G(y,x)\eta_1(x)\eta_2(x),$$

$$B_{\alpha}(y_1,y_2) := \int_{x} G(y_1,x)G(x,y_2)\eta(x),$$

$$C(y_1,y_2,y_3) := \int_{x} G(x,y_1)G(x,y_2)G(x,y_3).$$

### 3.4 Results about vanishing of $n$

In the situation of Definition 3.19, let $\Gamma \in R_{klg}^{(3)}$ be a reduced trivalent ribbon graph, $L = (L_1, L_2, L_3)$ its labeling, $x_i$ the integration variable associated to the $i$-th internal vertex, $G(x_i, x_j)$ the Green kernel on the oriented internal edge between $x_i$ and $x_j$, and $\alpha_{ij} \in H(M)[1]$ the harmonic form on the $j$-th external vertex on the $i$-th boundary component. Recall that we denote by $\omega_i = s\alpha_{i1} \ldots \alpha_{iK_i}$ the $i$-th input of $n_{lg}$ and by $D$ the total form-degree of all inputs.

By saying “a graph vanishes” we mean that $I(\sigma_L) = 0$ in the given context.

**Proposition 3.25** (Vanishing of graphs with 1). In the setting of Definition 3.19 suppose that the following condition is satisfied:

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The definitions can be made precise in local coordinates. Smoothness of $A_{\alpha_1,\alpha_2}$ is clear, smoothness of $B_{\alpha}$ follows from Lemma 3.10, and smoothness of $C$ can be shown by a similar argument.
Every graph \( \Gamma \in \text{RG}_{klg}^{(3)} \), \( \Gamma \neq Y \) which has \( \sigma = \theta \in \mathcal{H}(M)[1] \) at an external vertex vanishes.

Then \( n \) is strictly reduced, and the following holds depending on the dimension \( n \):

(a) For \( n > 3 \): All graphs which are not trees or circular vanish. Therefore, \( n_{lg} = 0 \) for all \( (l, g) \neq (1, 0), (2, 0) \), and it follows that all higher operations \( q^n_{l1g} \) vanish on the chain level.

(b) For \( n = 3 \): A tree vanishes unless all \( \eta_1, \ldots, \eta_s \) are one-forms. Therefore, \( n_{10}(s \alpha_1 \ldots \alpha_s) \neq 0 \) implies \( \text{deg}(\eta_i) = 1 \) for all \( i \).

(c) For \( n < 3 \): All trees except for \( Y \) vanish. Therefore, we have \( n_{10} = m_{10} \), and consequently \( q^n_{110} = q^n_{110} \).

Moreover, we have

(d) A circular graph vanishes unless all \( \eta_{11}, \ldots, \eta_{2s_2} \) are one-forms. Therefore, \( n_{20}(s^2 \alpha_{11} \ldots \alpha_{1s_1} \otimes \alpha_{21} \ldots \alpha_{2s_2}) \neq 0 \) implies \( \text{deg}(\eta_{ij}) = 1 \) for all \( i, j \).

In addition to \( (V_1) \), suppose that \( H^1_{dR}(M) = 0 \). Then:

(e) All circular graphs vanish. Therefore, we have \( n_{20} = 0 \), and consequently \( q^n_{120} = q^n_{120} \).

(f) For \( n \leq 6 \): All trees except for \( Y \) vanish. Therefore, we have \( n_{10} = m_{10} \), and consequently \( q^n_{110} = q^n_{110} \).

Proof. The proof is just combinatorics with \( D \). Suppose that a trivalent ribbon graph \( \Gamma \neq Y \) does not vanish on the input \( \omega_1, \ldots, \omega_l \). Because all external vertices of \( \Gamma \) are adjacent to an \( A \)-vertex or a \( B \)-vertex, the assumption \( (V_1) \) implies \( D \geq s \), where \( s \) is the total number of external vertices. A combination of (62) and (58) yields

\[ nk - (n - 1)e = D \geq s = 3k - 2e \quad \iff \quad (n - 3)k \geq (n - 3)e. \]

(a) For \( n > 3 \), we get \( k \geq e \), which implies that \( \Gamma \) is either a tree or a circular graph.

(b) If \( \Gamma \) is a tree, then \( s = k + 2 \) and \( e = k - 1 \). From (62) we get

\[ D = nk - (n - 1)(k - 1) = k + n - 1. \] (65)

Now \( D \) is the sum of \( s = k + 2 \) form-degrees \( \text{deg}(\eta_{ij}) > 0 \), and hence (58) for \( n = 3 \) implies that \( \text{deg}(\eta_{ij}) = 1 \) for all \( i, j \).

(c) For \( n < 3 \), we get \( e \geq k \), which implies that \( \Gamma \) is not a tree.
(d) If $\Gamma$ is a circular graph, then $e = k = s$, and we get using (62) that
\[ D = nk - (n-1)k = k. \]
Here $D$ is the sum of $s = k$ form-degrees $\deg(\eta_{ij}) > 0$, and hence $\deg(\eta_{ij}) = 1$ for all $i, j$.
We will now assume, in addition, that $H^1(M) \simeq H^1_{dR}(M) = 0$.
(e) We must have $D \geq 2s$, which is in contradiction with $D = s$ for a circular graph. Therefore, $n_{20} = 0$.
(f) Finally, for a tree $\Gamma \neq Y$, we have
\[ k + n - 1 = D \geq 2s = 2(k + 2) \iff n - 5 \geq k. \]
This finishes the proof of the proposition.

**Proposition 3.26** (Green kernel with (G4) and (G5)). *In the setting of Definition 3.19, suppose that the Green kernel $G$ satisfies (G4) and (G5). Then the condition $(V_1)$, and hence Proposition 3.25 holds.*

**Proof.** It is easy to see that $A_{\alpha_1,\alpha_2} = G(\eta_1 \wedge \eta_2)$ for all $\alpha_1, \alpha_2 \in H(M)[1]$, and that $-B_1$ is the Schwartz kernel of $G \circ G$. Therefore, (G4) and (G5) imply $A_{\alpha_1,1} = 0$ and $B_1 = 0$, respectively.

As for the integral $I(\sigma_L)$, one has to apply the Fubini theorem in order to integrate out single vertices $A_{\alpha_1,1}$ and $B_1$. This step relies on $L^1$-integrability of the integrand which follows from [12] (the integrand comes from a smooth form on a compact manifold with corners).

**Proposition 3.27** (Vanishing of $A$-vertices). *In the setting of Definition 3.19, suppose that the following condition is satisfied:

$(V_3)$ Every graph with an $A$-vertex vanishes.

Then we have $n_{10} = m_{10}$, and the only contribution to $n_{20}(s_2^1\alpha_{11} \cdots \alpha_{1s_1} \otimes \alpha_{21} \cdots \alpha_{2s_2})$ comes from $O_k$-graphs with $k = s_1 + s_2 = D$.*

**Proof.** The only trees and circular graphs which are not excluded by the assumption are the $Y$-graph and $O_k$-graphs, respectively (the external branches contract).

The condition on form-degrees is obtained as in the proof of Proposition 3.25.

To argue that $I(\sigma_L) = 0$, we again need $L^1$-integrability as in the proof of Proposition 3.26.

**Remark 3.28** (Integrability for trees). Given a tree, we can start at a leaf and write $I(\sigma_L)$ as an iterative integral of contributions $A_{\alpha_1,\alpha_2}$ for $\alpha_1, \alpha_2 \in \Omega(M)$. These are smooth forms, and hence integrability is guaranteed. Therefore, the result $n_{10} = m_{10}$ is independent of the convergence results from [12]. \(\triangleq\)
Proposition 3.29 (1-connected geometrically formal manifolds). Let $M$ be a geometrically formal $n$-manifold and $G$ a Green kernel satisfying (G4) and (G5) (it exists by Proposition 3.11). If $H^1_{dR}(M) = 0$, then the following holds:

$(n \neq 2)$ All $Y \neq \Gamma \in RG_{kg}$ with $k, l \geq 1, g \geq 0$ vanish, and hence $n = m$.

$(n = 2)$ All $Y \neq \Gamma \in RG_{kl0}$ with $k, l \geq 1$ vanish, and hence $n_{l0} = m_{l0}$ for all $l \geq 1$.

Proof. Given $\eta_1, \eta_2 \in H$, geometric formality implies $\eta_1 \wedge \eta_2 \in H$, and hence $A_{\alpha_1, \alpha_2} = G(\eta_1 \wedge \eta_2) = 0$. We see that $(V_1)$ and $(V_A)$ are satisfied, and hence the implications of Propositions 3.25 and 3.27 hold. The claim for $n > 3$ follows.

As for $n = 3$, Poincaré duality implies $H^2_{dR}(M; \mathbb{R}) = 0$. Therefore, the total form-degree $D$ satisfies $D = n B$, where $B$ is the number of $B$-vertices. We see using (64) that (62) is equivalent to

$$B + \frac{1}{2}(3 - n)C = D = n B \iff (n - 1)B = \frac{1}{2}(3 - n)C. \quad (66)$$

It follows that $B = 0$, and hence all reduced graphs vanish.

As for $n = 2$, we get from (66) and (64) that $B \geq l$ is equivalent to $g \geq 1$. \hfill \square

Remark 3.30 ($A_\infty$-homotopy transfer). In [12], it will be shown that the $A_\infty$-algebra $H(M)_n = (H(M), (\mu_k))$ induced by $n_{10}$ agrees with the $A_\infty$-algebra obtained by the $A_\infty$-homotopy transfer

$$\left( \begin{array}{c} \Omega(M) \\ m_1, m_2 \end{array} \right) \sim \sim \Rightarrow \left( \begin{array}{c} H(M) \\ \mu_1 \equiv 0, \mu_2 = \pi_H m_2(\iota_H, \iota_H), \mu_3, \ldots \end{array} \right)$$

using the homotopy retract (see [32])

$$\varpi \quad \left( \Omega(M), m_1 \right) \xleftarrow{\pi_H} \left( H(M), m_1 \equiv 0 \right).$$

The operation $\mu_k$ of the transferred $A_\infty$-structure is computed as a sum over planar trees with a root and $k$ leaves decorated by $\iota_H$ at the leaves, $\pi_H$ at the root and $G$ at the internal edges (see [1]). The result of [12] is plausible because the part of $n_{10}$ contributing to $\mu_k$ is a sum over trivalent ribbon trees with $k + 1$ leaves.

In [12], they will also show that $\iota_1 := \iota_H : H \to \Omega$ extends to an $A_\infty$-quasi-isomorphism $(\iota_k)_{k \geq 1}$ from $(H, (\mu_k))$ to $(\Omega, m_1, m_2)$. The induced chain map on the dual cyclic bar complexes is then the map $\varpi^{110}$ coming from the $IBL_\infty$-theory in the Introduction. \hfill $\triangleleft$
Proposition 3.31 (Twisted boundary operator for formal manifolds). In the setting of Definition 3.19, suppose that $M$ is formal in the sense of rational homotopy theory. Then there is a quasi-isomorphism

$$h_{110} : (\hat{B}_c^* \mathcal{H}_dR(M)[3-n], q^n_{110}) \longrightarrow (\hat{B}_c^* \mathcal{H}(M)[3-n], q^n_{110}).$$

Proof. Formality of $M$ is equivalent to the existence of a zig-zag of dga-quasi-isomorphisms (see [32])

$$(H_dR(M), m_1 \equiv 0, m_2) \cdots \cdots \cdots \cdots \cdots \cdots (\Omega(M), m_1, m_2).$$

Because a dga-quasi-isomorphism has a homotopy inverse in the category of $A_\infty$-algebras, we get a direct $A_\infty$-quasi-isomorphism

$$(g_k) : (\Omega(M), m_1, m_2) \longrightarrow (H_dR(M), m_1 \equiv 0, m_2).$$

Precomposing with $(\iota_k)$ from Remark 3.30 we get the $A_\infty$-isomorphism

$$(h_k) : (\mathcal{H}(M), (\mu_k)) \longrightarrow (H_dR(M), m_1 \equiv 0, m_2).$$

This induces the quasi-isomorphism $h_{110}$ of the corresponding cyclic cochain complexes (see [19] for details).

Remark 3.32 (On formality). Geometrically formal manifolds include $S^n$, $\mathbb{C}P^n$ and Lie groups (see [22]). Any geometrically formal manifold is formal. Every simply-connected manifold of dimension at most 6 is formal (see [28]).

3.5 Conjectured relation to string topology

Given a smooth connected oriented $n$-dimensional manifold $M$, we consider the equivariant homology of the free loop space $LM := \{ \gamma : S^1 \to M \text{ continuous} \}$ with respect to the reparametrization action of $S^1$. It is defined as the singular homology of the Borel construction $L_{S^1}M := E S^1 \times_{S^1} LM := (E S^1 \times LM)/S^1$, where $E S^1 = S^\infty \to B S^1 = \mathbb{C}P^\infty$ is a model for the universal bundle for $S^1$, and we quotient out the diagonal action. We denote this homology by

$$\mathcal{H}_{S^1}(LM) := H_*(L_{S^1}M).$$

The “geometric versions” of the homologies were defined in [9] as the degree shifts

$$\mathcal{H}(LM) := H(LM)[n] \text{ and } \mathcal{H}(LM) := H^{S^1}(LM)[n].$$
There is the loop product $\bullet : \mathbb{H}(LM)^{\otimes 2} \to \mathbb{H}(LM)$ of degree 0 which makes $\mathbb{H}(LM)$ into a graded commutative dga. There is also the loop coproduct $\tau : \mathbb{H}(LM) \to \mathbb{H}(LM)^{\otimes 2}$ of degree $1 - 2n$ which is graded cocommutative and coassociative and is a derivation of $\bullet$. The geometric construction of $\bullet$ and $\tau$ on transverse smooth chains in $LM$ was described in [9] and [5], respectively. Here, the symbol $\mathbb{H}(LM)$ stands for the degree shifted relative homology

$$\mathbb{H}(LM) := H(LM,M)[n]$$

with respect to constant loops $M \hookrightarrow LM$. The geometric construction of $\tau$ does not work on the whole $\mathbb{H}(LM)$ because of the phenomenon of “vanishing of small loops” depicted in [11, Figure 4, p.13].

The projection $ES^1 \times LM \to L_{S^1} M$ is an $S^1$-principal bundle and thus induces a Gysin sequence. This sequence written using the geometric versions reads

$$\ldots \longrightarrow \mathbb{H}_i \xrightarrow{E} \mathcal{H}_i \xrightarrow{\cap c} \mathcal{H}_{i-2} \xrightarrow{\mathcal{M}} \mathbb{H}_{i-1} \longrightarrow \ldots,$$

where the map $\mathcal{M}$ adds a marked point in each string in a family in all possible positions, the map $E$ erases the marked point of each string in a family, $c \in H^2_{S^1}(LM)$ is the Euler class of the circle bundle and $\cap$ the cap product.

The string bracket $\tilde{m}_2 : \mathcal{H}(LM)^{\otimes 2} \to \mathcal{H}(LM)$ and the string cobracket $\tilde{c}_2 : \mathcal{H}(LM) \to \mathcal{H}(LM)^{\otimes 2}$ are defined by

$$\tilde{m}_2 := E \circ \bullet \circ \mathcal{M}^{\otimes 2} \quad \text{and} \quad \tilde{c}_2 := E^{\otimes 2} \circ \nu \circ \mathcal{M}.$$

Here, the symbol $\mathcal{H}(LM)$ stands for the degree shifted relative $S^1$-equivariant homology

$$\mathcal{H}(LM) := H^{S^1}(ES^1 \times_{S^1} LM, ES^1 \times_{S^1} M)[n],$$

$$= : \mathbb{H}^{S^1}(LM).$$

Because $|\mathcal{M}| = 1$ and $|E| = 0$, we have for all $\xi \in \mathcal{H}(LM)$ and $\xi_1$, $\xi_2 \in \mathcal{H}$ the relations

$$\tilde{m}_2(\xi_1, \xi_2) = (-1)^{|\xi_1|} \mathcal{E}(\mathcal{M}(\xi_1) \bullet \mathcal{M}(\xi_2)),$$

$$\tilde{c}_2(\xi) = \sum \mathcal{E}(\nu^1) \otimes \mathcal{E}(\nu^2),$$

where we write $\nu(\mathcal{M}(\xi)) = \sum \nu^1 \otimes \nu^2$. The operations $\tilde{m}_2$ and $\tilde{c}_2$ have degrees 2 and $2 - 2n$ with respect to the grading on $\mathcal{H}(LM)$, respectively. In fact, we will consider $\tilde{m}_2$ and $\tilde{c}_2$ given by [68] as operations on the even degree shift $H^{S^1}(LM)[2 - n] = \mathcal{H}(LM)[2 - 2n]$, which have degrees $2(2 - n)$ and 0, respectively. The symbols $\tilde{m}_2$ and $\tilde{c}_2$ will denote their degree shifts to $H^{S^1}(LM)[3 - n]$, which have degrees of an IBL-algebra from Definition 2.17.
In work in progress [13], they consider the map
\[ I_{\lambda,*}: H^\lambda_{\ast - 1}(\Omega^*(M)) \to H^\lambda_{\ast}(LM; \mathbb{R}) \]
defined on the chain level as a cyclic version of Chen’s iterated integrals. Recall that \( H^\lambda_{\ast - 1}(\Omega^*) = H_*(B_\lambda^*, \Omega^*, b^*) \), where \( b : B_* \Omega = \bigoplus_{k \geq 1} \Omega[1] \otimes^k \to B_* \Omega \) is the Hochschild differential of the de Rham dga \((\Omega^*, m_1, m_2)\), and the grading on \( H^\lambda_* \) was chosen such that \( H^\lambda_0(\Omega^*) \simeq H^{\lambda, c}_*(\Omega^*) \) for the classical cyclic homology of a dga. They prove in [13] that if \( M \) is simply-connected, then the map \( I_{\lambda,*} \) induces an isomorphism \( H^\lambda_{\ast - 1}(\Omega^*(M)) \simeq H^\lambda_{\ast, \text{red}}(LM) \), where
\[ H^\lambda_{\star, \text{red}}(LM) := H^\lambda_{\ast}(E S^1 \times_S^\ast LM, E S^1 \times_S^\ast \{x_0\}) \]
is the reduced \( S^1 \)-equivariant cohomology with respect to a base point \( x_0 \in M \) (the constant loop at \( x_0 \)). Dualizing their map, we obtain the isomorphism
\[ H^{\ast - 1}_{\lambda, \text{red}}(\Omega^*(M)) \simeq H^\ast_{\text{red}}(LM; \mathbb{R}). \tag{69} \]

Suppose from now on that \( M \) is closed. Pick a Riemannian metric and a Green kernel \( G \in \Omega^{n-1}(BL_\lambda(M \times M)) \). We will assume that \( G \) satisfies (G1)–(G5) from Section 3.2 so that the formal pushforward Maurer-Cartan element \( n \) is strictly reduced, and hence the twisted reduced IBL\(_{\text{red}}\)-algebra \( dIBL^n(\eta(C(H))) \) and the induced IBL-algebra IBL\(_{\text{red}}(\eta(C(H))) \) are well-defined. Recall that \( H^n_\lambda(C(H)) = H^n_{\lambda - 3, *}(H_n) \), where \( H_n \) is the \( A_\infty \)-algebra on \( H \) twisted by \( n_{10} \). From [12], we have
\[ H^n_\lambda(H^*(M)_{\text{red}}) \simeq H^n_\lambda(\Omega^*(M)). \tag{70} \]

A combination of (69) and (70) gives the following version of the string topology conjecture from the Introduction.

**Conjecture 3.33** (String topology conjecture for simply-connected manifold). Let \( M \) be an oriented closed manifold of dimension \( n \). There is a chain map
\[ (C^n_{\text{sing}}(E S^1; M; \mathbb{R}), \partial) \to (\tilde{B}^*_\text{cyc}(H(M), q^n_{110})), \]
where \( C^n_{\text{sing}} \) denotes the (smooth) singular chain complex and \( \partial \) the standard boundary operator, which, if \( M \) is simply-connected, satisfies the following:

- It induces an isomorphism \( H^\ast_{\text{red}}(LM; \mathbb{R})[2 - n] \simeq H^n_{\text{red}}(C(H(M))) \).
- It intertwines \( m_2 \) on \( H^2(\mathbb{R}; \mathbb{R}) \) and \( \varphi_{210} \).
- The pullback of \( q^n_{120} \) to \( H^3_{\text{red}}(LM; \mathbb{R}) \) is compatible with \( \varphi_2 \) on \( H^3(LM; \mathbb{R}) \) under the morphism induced by the inclusion \( (LM, x_0) \to (LM, M) \).

**Remark 3.34** (On string topology conjecture). (i) The conjecture can be interpreted as follows. There is an IBL-structure on \( H^3_{\text{red}}(LM; \mathbb{R}) \) compatible
with Chas-Sullivan operations, and the \( \text{dIBL}_\infty^n(C_{\text{red}}(\mathcal{H}(M))) \) is its chain model.

(ii) The loop coproduct \( \tau \) is geometrically defined only on \( \bar{H}^{S^1}(LM) \); the conjecture thus provides an extension of \( \mathfrak{c}_2 \) to \( H^{S^1,\text{red}}(LM) \). In \([4]\), it is shown that the geometric definition of \( \tau \) can be extended to \( H(LM) \) for manifolds with zero Euler characteristic, i.e., \( \chi(M) = 0 \). This extension depends on the choice of a non-vanishing vector field on \( M \). By homotopy invariance (see (v) below), our extension of \( \mathfrak{c}_2 \) should not depend on the Green kernel \( G \).

(iii) The loop product \( \bullet \) is geometrically defined on \( H(LM) \); however, it does not always induce an associative product on \( H_{\text{red}}(LM) = H(LM,x_0) \). Indeed, the examples of \( T^2 \) (see \([4]\)) and \( S^3 \) (see \([9]\)) show that \( H(x_0;\mathbb{R}) \subset H(LM;\mathbb{R}) \) is not an ideal with respect to \( \bullet \). By \([31]\), this does not happen when \( \chi(M) \neq 0 \), and hence, in this case, \( \bullet \) restricts to \( H(LM,x_0;\mathbb{R}) \).

(iv) The computation for \( S^n \) with \( n \geq 2 \) and the computation for \( \mathbb{C}P^n \) in Section 4 support the conjecture. The computation for \( S^1 \) in Section 4.3 provides a counterexample for non-simply-connected \( M \). In \([19]\), surfaces of genus \( g \geq 1 \) will be considered.

(v) We expect that if \( M_1 \) and \( M_2 \) are homotopy equivalent, then the \( \text{IBL}_\infty \)-algebras \( \text{dIBL}_n^\infty(C(\mathcal{H}_{\text{dR}}(M_1))) \) and \( \text{dIBL}_n^\infty(C(\mathcal{H}_{\text{dR}}(M_2))) \) are \( \text{IBL}_\infty \)-homotopy equivalent.

\( \triangleright \)
4 Explicit computations

In Section 4.1, we solve the differential equation for the Green kernel $G$ for $S^n$ (Proposition 4.2) using the Relative Poincaré Lemma (Lemma 4.1). In the rest of the section, we will be showing that $G$ satisfies all properties of the Green kernel (Proposition 4.10); the most work is to show that $G$ extends smoothly to the blow-up (Proposition 4.9). Another Green kernel for $S^1$ can be obtained in an alternative simple way by writing $S^1 = \mathbb{R}/\mathbb{Z}$, and there are nice geometric formulas for $G$ for $S^2$ (Example 4.3).

In Section 4.2, we use $G$ from Section 4.1 to compute the formal pushforward Maurer-Cartan element $n$ for $S^n$ (Proposition 4.21). We first prove that the condition $(V_i)$ from Proposition 3.25 is satisfied (Lemma 4.12) and then perform combinatorics with degrees to show vanishing of some more integrals (Proposition 4.13). In fact, all the integrals vanish for $S^n$ with $n \geq 3$, and the only non-vanishing integrals for $S^1$ are the $O_k$-graphs with even $k$. We compute these integrals explicitly together with all signs and combinatorial coefficients required to obtain $n_{20}$ (Lemmas 4.16, 4.17, 4.18 and 4.19). There might be some non-vanishing integrals associated to reduced graphs for $S^2$ as well as some non-vanishing integrals associated to graphs without external vertices for $S^3$; however, the simplest examples vanish (Remarks 4.14 and 4.15).

In the remaining Sections 4.3 and 4.4, we compute IBL$(\mathbb{H}^n(C(H(M))))$ and the higher operations $q^m_n$ on $\mathbb{H}^n$ for $M = S^n$, $\mathbb{C}P^n$. As soon as we argue that $n_{10} = m_{10}$ due to geometric formality, the computation of $\mathbb{H}^n(C(H(S^n)))$ and $\mathbb{H}^n(C(H(\mathbb{C}P^n)))$ is an easy exercise in cyclic homology. The operations for $S^{2m}$ and $\mathbb{C}P^n$ vanish for degree reasons (Remark 4.22). Therefore, the integrals from Section 4.2 help only in the case of $S^{2m-1}$. We compare our results to Chas-Sullivan string topology from [4] and confirm Conjecture 3.33 for $S^n$ with $n \geq 2$ and for $\mathbb{C}P^n$.

4.1 Computation of $G$ for $S^n$

The standard Riemannian volume form on the round sphere $S^n \subset \mathbb{R}^{n+1}$ is the restriction of the following closed form on $\mathbb{R}^{n+1}\{0\}$:

$$\text{Vol}(x) := \frac{1}{|x|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i+1} x^i \overline{dx_i} \cdots \overline{dx_i} \cdots \overline{dx_{n+1}}.$$ 

Here $\overline{dx_i}$ means that $dx_i$ is omitted. We denote the Riemannian volume of $S^n$ by

$$V := \int_{S^n} \text{Vol}.$$
The $n$-form $H$ from Proposition 3.8 reads
\[ H = \frac{1}{V} (pr_1^* \text{Vol} + (-1)^n pr_2^* \text{Vol}) . \]

According to Proposition 3.9, the equation which we want to solve reads
\[ dG = \frac{1}{V} ((-1)^n pr_1^* \text{Vol} + pr_2^* \text{Vol}) . \] (71)

We denote
\[ \tilde{G} := VG \quad \text{and} \quad \tilde{H} := VH. \]

The following lemma will be used to construct a solution to (71).

**Lemma 4.1 (Relative Poincaré Lemma).** Let $M$ be a smooth oriented manifold and $\psi : [0,1] \times M \to M$ a smooth map. Consider the operator $T : \Omega^*(M) \to \Omega^{*-1}(M)$ defined by
\[
T(\eta) := \int_{[0,1]} \psi^* \eta \quad \text{for all } \eta \in \Omega(M),
\]
where we integrate along the fiber of the oriented fiber bundle $pr_2 : [0,1] \times M \to M$. Then we have
\[ d \circ T + T \circ d = \psi_1^* - \psi_0^* . \]

*Proof.* Stokes’ formula from Proposition 3.7 gives
\[ d \int_{[0,1]} \psi^* \eta = - \left( \int_{[0,1]} d\psi^* \eta - \int_{[0,1]} \psi^* \eta \right) = - \int_{[0,1]} \psi^* d\eta + \psi_1^* \eta - \psi_0^* \eta \]
for all $\eta \in \Omega(M)$.

**Proposition 4.2 (Solution to (71)).** For all $(x,y) \in (\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$, let
\[ G(x,y) := (-1)^n \sum_{k=0}^{n-1} g_k(x,y) \omega_k(x,y), \] (72)
where
\[
g_k(x,y) := \int_0^1 \frac{t^k(t-1)^{n-1-k}}{(2t(t-1)(1+x \cdot y)+1)^{\frac{n+k}{2}}} dt \] (73)
and
\[
\omega_k(x,y) := \frac{1}{k!} \frac{1}{(n-1-k)!} \sum_{\sigma \in S_{n+1}} (-1)^{\sigma} x^{\sigma_1} y^{\sigma_2} dx^{\sigma_3} \cdots dx^{\sigma_{2+k}} \cdots dy^{\sigma_{n+k}} . \]
(74)

The form (72) is a smooth solution to (71) on $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$. 86
Figure 5: Retraction $\psi_t = (\psi^1_t, \psi^2_t)$. A point of $\mathbb{S}^n \times \mathbb{S}^n$ is visualized as a pair of points on $\mathbb{S}^n$.

**Proof.** Define the set

$$N := (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \setminus \{(x, ax) \mid x \in \mathbb{R}^{n+1}, a > 0\}.$$  

It is an open thickening of $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$ in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \Delta$. Consider the smooth deformation retraction

$$\psi : [0, 1] \times N \rightarrow N,$$

$$(t, x, y) \mapsto \psi_t(x, y) := (x, (1 - t)y - tx)$$

with

$$\psi_0(x, y) = (x, y) \quad \text{and} \quad \psi_1(x, y) = (x, -x) \quad \text{for all} \ (x, y) \in N.$$  

The retraction is depicted in Figure 5. Denote by $A : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$, $x \mapsto -x$ the antipodal map. It is easy to see that

$$A^*\text{Vol} = (-1)^{n+1}\text{Vol},$$

and hence

$$\psi^*_1 \tilde{H} = \psi^*_1 \text{pr}^*_1 \text{Vol} + (-1)^n \psi^*_1 \text{pr}^*_2 \text{Vol} = \text{pr}^*_1 \text{Vol} + (-1)^n \text{pr}^*_1 A^*\text{Vol} = 0.$$  

Define

$$G := (-1)^{n+1} \int_{[0, 1]} \psi^* H.$$  

(75)
Let $T : \Omega^*(N) \to \Omega^{*-1}(N)$ be the cochain homotopy from Lemma 1.1 associated to $\psi$. Because $dH = 0$, we get
\[ dG = (-1)^{n+1}dT(H) = (-1)^{n+1}(dT + Td)H = (-1)^{n+1} (\psi_1^* - \psi_0^*)H = (-1)^n H. \]
For every $i = 1, \ldots, n+1$, we have
\[ \psi^*(dx^i) = dx^i \quad \text{and} \quad \psi^*(dy^i) = (1 - t) dy^i - t dx^i - (y^i + x^i) \, dt. \]
We compute
\[
(-1)^{n+1} \int_{[0,1]}^{[0,1]} \psi^* H = - \int_{[0,1]}^{[0,1]} \psi^* \text{pr}_2^* \text{Vol}
\]
\[ = \int_{[0,1]}^{[0,1]} \sum_{i=1}^{n+1} (-1)^i \frac{(1 - t) y^i - tx^i}{1 - t} y - tx^{n+1} (\psi^*(dy^i \cdots dy^i \cdots dy^{n+1}))
\]
\[ = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} (x^i y^j - y^i x^j) \int_{[0,1]}^{[0,1]} \frac{dt}{1 - t} \psi^*(dy^i \cdots dy^i \cdots dy^i \cdots dy^{n+1})
\]
\[ = (-1)^n \sum_{k=0}^{n-1} \left( \int_0^1 t^k(t-1)^{n-1-k} \, dt \right) \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} (x^i y^j - y^i x^j)
\]
\[ \sum_{\sigma : \{1, \ldots, n+1\} \to \{1, \ldots, i, \ldots, j, \ldots, n+1\}} (-1)^{\sigma} dx^\sigma_1 \cdots dx^\sigma_{k+1} \cdots dx^\sigma_{n+1}. \]

The formulas (73) and (74) are obtained from this by writing
\[ |(1 - t) y - tx|^2 = 2t(t - 1)(1 + x \cdot y) + 1 \]
in the denominator of the integrand and by simple combinatorics in the form part, respectively. Smoothness of $G$ on $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$ follows from the expression (75).  

Note that $g_k$ are smooth functions on $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$.

**Example 4.3** (Green kernel for $S^1$ and $S^2$). (a) Let
\[ \alpha : (S^1 \times S^1) \setminus \Delta \to (0, 2\pi) \]
be the smooth function assigning to a pair $(x, y) \in (S^1 \times S^1) \setminus \Delta$ the counterclockwise angle from $x$ to $y$. Let $\alpha_1, \alpha_2 \in [0, 2\pi)$ be such that $x = \cos(\alpha_1)e_1 + \sin(\alpha_1)e_2$ and $y = \cos(\alpha_2)e_1 + \sin(\alpha_2)e_2$ for the standard Euclidean basis $e_1, e_2$ of $\mathbb{R}^2$. It
is easy to see that

\[ \alpha(x, y) = \begin{cases} 
\alpha_2 - \alpha_1 & \text{if } \alpha_1 < \alpha_2, \\
\alpha_2 - \alpha_1 + 2\pi & \text{if } \alpha_1 > \alpha_2.
\end{cases} \]

Therefore, we get

\[ d\alpha = d\alpha_2 - d\alpha_1 = -2\pi H \quad \text{on } (S^1 \times S^1) \setminus \Delta. \]

On the other hand, we can compute \( G \) from \((72)\) as follows. Using the substitution \( u = 2t - 1 \), we get for all \( x, y \in S^1 \) with \( x \neq \pm y \) the following:

\[ g_0(x, y) = \int_0^1 \frac{dt}{2t(t - 1)(1 + x \cdot y) + 1} = \int_{-1}^1 \frac{du}{1 + x \cdot y} \left( \frac{1}{u^2 + 1} \right) \]

\[ = \frac{2}{\sqrt{1 - (x \cdot y)^2}} \arctan \left( \frac{1 + x \cdot y}{1 - x \cdot y} \right) \]

\[ = \frac{\pi - \arccos(x \cdot y)}{\sqrt{1 - (x \cdot y)^2}} = \frac{\pi - \arccos(x \cdot y)}{|x^2y^2 - x^2y^1|} = \frac{\pi - \alpha(x, y)}{x^1y^2 - x^2y^1}. \]

The third from last equality can be obtained by trigonometric considerations and the second from last equality by an algebraic manipulation with the denominator. We will explain the last equality. Consider the matrix

\[ R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

representing the counterclockwise rotation by \( \frac{\pi}{2} \). The function \( \arccos : (-1, 1) \to (0, \pi) \) satisfies

\[ \arccos(x \cdot y) = \begin{cases} 
\alpha(x, y) & \text{if } y \cdot Rx > 0, \\
2\pi - \alpha(x, y) & \text{if } y \cdot Rx < 0.
\end{cases} \]

The last equality becomes clear when we notice that \( x^1y^2 - x^2y^1 = y \cdot Rx \).

Finally, we have \( \omega_0(x, y) = x^1y^2 - x^2y^1 \), and hence

\[ 2\pi G(x, y) = -g_0(x, y)\omega_0(x, y) = \alpha(x, y) - \pi = \pi - \alpha(y, x). \]

(b) For \( n = 2 \), we get the formulas

\[ g_0(x, y) = -g_1(x, y) = \frac{1}{x \cdot y - 1} \quad \text{and} \]

\[ \omega_0(x, y) = (x^2y^1 - x^1y^2) \, dy^1 + (x^3y^1 - x^1y^3) \, dy^2 + (x^1y^2 - x^2y^1) \, dy^3 \]
\[ \sum_{i=1}^{3} (x \times y)^i \, dy^i. \]

The formula for \( \omega_1(x, y) \) is obtained from the formula for \( \omega_0(x, y) \) by replacing \( dy \) with \( dx \).

Consider the diagonal action of the orthogonal group \( O(n+1) \) on \( \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \) by matrix multiplication.

**Proposition 4.4 (Symmetries of \( G \)).** Consider \( G \) from Proposition 4.2. For all \( R \in O(n+1) \), we have

\[ R^*G = (-1)^R G, \]

where \( (-1)^R = \det(R) \). Moreover, if \( \tau \) denotes the twist map, then

\[ \tau^*G = (-1)^n G. \]

**Proof.** We will use the thickening \( N \), the antipodal map \( A \) and the expression (75) for \( G \) from the proof of Proposition 4.2.

It is easy to check that both \( \tau \) and \( R \) preserve \( N \). Let \( \tilde{\tau} \) and \( \tilde{R} \) be the isomorphisms of the fiber bundle \( \text{pr}_2 : [0,1] \times N \to N \) given by

\[ \tilde{\tau}(t, x, y) := (1-t, y, x) \quad \text{and} \quad \tilde{R}(t, x, y) := (t, Rx, Ry) \]

for all \((t, x, y) \in [0,1] \times N\). Then \( \tilde{\tau} \) covers \( \tau \) and \( \tilde{R} \) covers \( R \). A simple computation directly from Definition 3.3 shows that the fiberwise integration commutes with the pullback along a bundle morphism if the bundle map and the base map are both either orientation preserving or reversing. In our case, we have

\[ (-1)^{\tau + \tilde{\tau}} = -1 \quad \text{and} \quad (-1)^{R + \tilde{R}} = 1. \]

Using this and the equation

\[ \text{pr}_2 \circ \psi \circ \tilde{\tau} = A \circ \text{pr}_2 \circ \psi, \]

we get firstly

\[ \tau^* \int_{[0,1]} \psi^* \hat{H} = - \int_{[0,1]} \tilde{\tau}^* \psi^* \text{pr}_2^* \text{Vol} \]

\[ = - \int_{[0,1]} \psi^* \text{pr}_2^* A^* \text{Vol} \]

\[ = (-1)^n \int_{[0,1]} \psi^* \text{pr}_2^* \text{Vol} \]

\[ = (-1)^n \int_{[0,1]} \psi^* \hat{H}. \]
and secondly
\[
R^* \int_{[0,1]} \psi^* H = \int_{[0,1]} R^* \psi^* H = \int_{[0,1]} \psi^* R^* H = (-1)^{n+1} \int_{[0,1]} \psi^* H.
\]
This proves the proposition.

Both diffeomorphisms $R$ and $\tau$ preserve $\Delta$, and hence they extend to diffeomorphisms of $\text{Bl}_\Delta(S^n \times S^n)$. If also $G$ extends, then the statement of Proposition 4.4 holds for $G$ on $\text{Bl}_\Delta(S^n \times S^n)$.

In the rest of the section, we will be proving that $G$ extends smoothly to $\text{Bl}_\Delta(S^n \times S^n)$. This is a local problem at the boundary, where we introduce the following radial coordinates. Define the set
\[
X := \{(r, \eta, x) \in [0, \infty) \times S^n \times S^n \mid \eta \cdot x = 0\},
\]
and let $\kappa : X \to \text{Bl}_\Delta(S^n \times S^n)$ be the map defined by
\[
\kappa(r, \eta, x) := \begin{cases} 
(x, \frac{x + r\eta}{|x + r\eta|}) \in (S^n \times S^n)\setminus \Delta & \text{for } r > 0, \\
[(-\eta, \eta)] \in P^+ N_{(x,x)} \Delta & \text{for } r = 0.
\end{cases}
\]
For the upcoming computations, it is convention to define the map $\gamma : \mathbb{R} \to (-1, 1)$ by
\[
\gamma(r) := \frac{r}{\sqrt{1 + r^2} + 1} \quad \text{for all } r \in \mathbb{R}.
\]
It is a diffeomorphism with inverse $r = \frac{2\gamma}{1 - \gamma}$.

**Lemma 4.5 (Parametrization of the collar neighborhood).** The subset $X \subset \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a submanifold with boundary, and the map $\kappa : X \to \text{Bl}_\Delta(S^n \times S^n)$ is an embedding onto a neighborhood of $\partial \text{Bl}_\Delta(S^n \times S^n)$.

**Proof.** The set $X$ is a Cartesian product of $[0, \infty)$ and a regular level set; therefore, it is a submanifold with boundary. The inclusion $S^n \times S^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ induces an embedding of manifolds with boundary $\text{Bl}_\Delta(S^n \times S^n) \subset \text{Bl}_\Delta(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$. Consider the global chart $\tilde{\pi} : \text{Bl}_\Delta(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \to [0, \infty) \times S^n \times \mathbb{R}^{n+1}$ from (14) induced by the identity. We have
\[
Y := \tilde{\pi}(\text{Bl}_\Delta(S^n \times S^n)) = \{(\tilde{r}, w, u) \in [0, \infty) \times S^n \times \mathbb{R}^{n+1} \mid |u|^2 + \tilde{r}^2 = 1, \ w \cdot u = 0\},
\]
where we denote $r$ on $Y$ by $\tilde{r}$ in order to distinguish it from $r$ on $X$. It suffices to prove the claim for the map $\mu := \tilde{\pi} \circ \kappa : X \to Y$. For $(r, \eta, x) \in X$, we compute
\[
\mu(r, \eta, x) = \left(\frac{\gamma}{\sqrt{1 + \gamma^2}}, \frac{1}{\sqrt{1 + \gamma^2}}(\gamma x - \eta), \frac{1}{1 + \gamma^2}(x + \gamma \eta)\right).
\]

91
This formula defines a smooth map of $\mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. It is a local diffeomorphism because its Jacobian is non-vanishing:

$$|D\mu| = \frac{\partial r}{\partial r} \left( \frac{\partial w}{\partial \eta} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} \frac{\partial u}{\partial \eta} \right)^{n+1} = (-1)^{n+1}(1 + \gamma^2)^{-\frac{n+4}{2}} \frac{\partial \gamma}{\partial r}.$$

Moreover, the map $\mu$ is injective, maps $X$ into $Y$ and $\partial X$ onto $\partial Y$. The claim follows.

Consider the action of $O(n+1)$ on $X$ defined by

$$R \cdot (r, \eta, x) := (r, R\eta, Rx) \quad \text{for all } (r, \eta, x) \in X \text{ and } R \in O(n+1).$$

Via $\kappa$, this agrees with the diagonal action of $O(n+1)$ on $\text{Bl}_\Delta(S^n \times S^n)$. Denote

$$G' := \kappa^* G \in \Omega^{n-1}(\text{Int}(X)).$$

From Proposition 4.4 we get

$$R^* G' = (-1)^n G' \quad \text{for all } R \in O(n+1). \quad (76)$$

Consider the smooth curve (see Figure 6)

$$\zeta' : [0, \infty) \rightarrow X$$

$$r \mapsto (r, e_n, e_{n+1}).$$

We have the following lemma.

**Lemma 4.6** (Smooth extension along the curve). The form $G'$ extends smoothly to $X$ if and only if the map $G' \circ \zeta' : (0, \infty) \rightarrow \Lambda^{n-1}T^* X$ extends smoothly to the interval $[0, \infty)$.

**Proof.** As for the non-trivial implication, let $(0, \eta_0, x_0) \in X$ be a boundary point. Pick vectors $v_1, \ldots, v_{n-1} \in \mathbb{R}^{n+1}$ so that the vectors $v_1, \ldots, v_{n-1}, \eta_0, x_0$ are
linearly independent, and define the set

\[ U := \{ (r, \eta, x) \in X \mid v_1, \ldots, v_{n-1}, \eta, x \text{ are linearly independent} \}. \]

It is an open neighborhood of \((0, \eta_0, x_0)\) in \(X\). Applying the Gram-Schmidt orthogonalization to \(v_1, \ldots, v_{n-1}, \eta, x\), we find a smooth map \(R : U \to O(n+1)\) such that

\[ R(r, \eta, x) \cdot (r, \eta, x) = (r, e_n, e_{n+1}) \quad \text{for all } (r, \eta, x) \in U. \]

The equation (76) implies

\[ G'(r, \eta, x) = (-1)^n R(r, \eta, x)^* (G'(r, e_n, e_{n+1})) \quad \text{for all } (r, \eta, x) \in \text{Int}(U), \]

where \(R(r, \eta, x)^* : \Lambda^* T^* X \to \Lambda^* T^* X\) is the smooth cotangential map which is induced by the diffeomorphism \(R(r, \eta, x) : X \to X\), and which maps the fiber over \(z \in X\) to the fiber over \(R(r, \eta, x)^{-1}z\). By the assumption, all maps in the composition are smooth in their arguments. The lemma follows.

Lemma 4.7 (Local expression at the boundary). On the interval \((0, \infty)\), we have

\[ \tilde{G}' \circ \zeta' = (-1)^{n+1} (1 + \gamma^2)^{\frac{-n}{4}} \sum_{k=0}^{n-1} \gamma^{n-k} (h_k \circ \gamma) (\nu_k \circ \zeta'), \]

where the functions \(h_k : (0, 1) \to \mathbb{R}\) are defined by

\[ h_k(\gamma) := \int_{-1}^{1} \frac{(u + \gamma^2)^k (u - 1)^{n-1-k}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} \, du \quad \text{for all } \gamma \in (0, 1) \]

and the forms \(\nu_k \in \Omega(X)\) are defined by

\[ \nu_k(r, x, \eta) := \frac{1}{k!(n-1-k)!} \sum_{\sigma \in S_{n-1}} (-1)^{\sigma} dx^{\sigma_1} \cdots dx^{\sigma_k} dy^{\sigma_{k+1}} \cdots dy^{\sigma_{n-1}}. \]

Proof. We start with the following formula from the proof of Proposition 4.2:

\[ \tilde{G} = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} (x^i y^j - y^i x^j) \int_{[0,1]} \frac{dt \psi^*(dy^1 \cdots dy^i \cdots dy^j \cdots dy^{n+1})}{|(1-t)y - tx|^{n+1}}. \]

We restrict to the points \((x, y) = \kappa(r, e_n, e_{n+1})\) with \(r > 0\). There, we have

\[ x^1 = \cdots = x^n = 0, \quad x^{n+1} = 1, \]
\[ y^1 = \cdots = y^{n-1} = 0, \quad y^n = \frac{2\gamma}{1 + \gamma^2}, \quad y^{n+1} = \frac{1 - \gamma^2}{1 + \gamma^2}. \]
Under the substitution $u = 2t - 1$, we get

$$|(1-t)y-tx|^2 = \frac{4t(t-1)}{1+\gamma^2} + 1 = \frac{u^2 + \gamma^2}{1+\gamma^2}.$$ 

We make the following preliminary computations:

$$x^iy^j - y^ix^j = 0 \quad \text{for } 1 \leq i \leq n-1 \text{ and } i < j \leq n+1,$$

$$x^ny^{n+1} - y^nx^{n+1} = -\frac{2\gamma}{1+\gamma^2},$$

$$\kappa^*(dy^i) = \frac{1}{1+\gamma^2}((1-\gamma^2)dx^i + 2\gamma d\eta^i) \quad \text{for } 1 \leq i \leq n-1.$$

We plug these in the formula for $\tilde{G}$ and get

$$\tilde{G}'(\zeta(r)) = 2\gamma(1+\gamma^2)^{\frac{n-1}{2}} \int_0^{[0,1]} dt \frac{\prod_{i=1}^{n-1}((1-t)\kappa^*(dy^i) - t dx^i)}{(u^2 + \gamma^2)^{\frac{n+1}{2}}}$$

$$= (-1)^{n+1}(1+\gamma^2)^{-\frac{n-1}{2}} \int_{[-1,1]} du \frac{\prod_{i=1}^{n-1}((u+\gamma^2)dx^i + \gamma(u-1)d\eta^i)}{(u^2 + \gamma^2)^{\frac{n+1}{2}}}$$

$$= (-1)^{n+1}(1+\gamma^2)^{-\frac{n-1}{2}} \sum_{k=0}^{n-1} \gamma^{n-k} \left( \int_{-1}^1 \frac{(u+\gamma^2)^k(u-1)^{n-1-k}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} du \right) \nu_k.$$

The lemma follows.

**Lemma 4.8** (Integrals depending on parameter). Let $n \in \mathbb{N}$, and let $l = 0, 1, \ldots, n-1$. The function $F_{n,l} : (0, \infty) \to \mathbb{R}$ defined by

$$F_{n,l}(t) := \int_{-1}^1 \frac{u^{n-l}t^l}{(u^2 + t^2)^{\frac{n+1}{2}}} du \quad \text{for all } t \in (0, \infty) \quad (77)$$

extends smoothly to $\mathbb{R}$.

**Proof.** We have

$$F_{1,0}(t) = 2\arctan\left(\frac{1}{t}\right) = \pi - 2\arctan(t) \quad \text{for all } t \in (0, \infty).$$

The right-hand side is a smooth function on $\mathbb{R}$.

For $n \geq 2$, we deduce the recursive formula

$$F_{n,0}(t) = \frac{1}{n-1} ((n-2)F_{n-2,0}(t) + \frac{2t^{n-2}}{(1+t^2)^{\frac{n+1}{2}}}).$$

If $l$ is odd, then $F_{n,l} \equiv 0$ for all $n$ because the integrand of (77) is odd as a function of $u$. 

94
For $n \geq 3$ and even $2 \leq l \leq n - 1$, we deduce yet another recursive formula

$$F_{n,l}(t) = \frac{1}{n-l} \left((l-1)F_{n,l-2}(t) - \frac{2n-l}{(1+t^2)^{n-l+1}}\right).$$

The claim for all $F_{n,l}$ follows by induction.

**Proposition 4.9** (Smooth extension to the boundary). The form $G$ from (72) extends smoothly to $Bl_\Delta(S^n \times S^n)$.

**Proof.** According to Lemmas 4.5 and 4.6, it suffices to show that the curve $G' \circ \zeta' : (0, \infty) \to \Lambda^{n-1}T^*X$ extends smoothly to $[0, \infty)$. Lemma 4.7 gives an expression for $G' \circ \zeta'$ as a linear combination of smooth forms $\nu_k \in \Omega^{n-1}(X)$ with coefficients $\gamma^{n-k}(h_k \circ \gamma)$ for $k = 0, \ldots, n - 1$ multiplied by the overall coefficient $(-1)^n(1 + \gamma^2)^{-\frac{n-1}{2}}$. We expand

$$\gamma^{n-k}(h_k \circ \gamma) = \sum_{a=0}^{k} \sum_{b=0}^{n-k} (-1)^{n-k-a} \binom{k}{a} \binom{n-1-k}{b} \int_{-1}^{1} \frac{\gamma^{n+k-2a} u^{a+b}}{(u^2 + \gamma^2)^{\frac{n-1}{2}}} \, du$$

and notice that we can write

$$\int_{-1}^{1} \frac{\gamma^{n+k-2a} u^{a+b}}{(u^2 + \gamma^2)^{\frac{n-1}{2}}} \, du = \gamma^{k-a+b}(F_{n,a+b} \circ \gamma)$$

for the function $F_{n,l}$ from (77) with $l := a+b$. Because $0 \leq l \leq n - 1$, Lemma 4.8 asserts that $F_{n,l}$ extends smoothly to $[0, \infty)$. Because $k-a+b \geq 0$, the entire coefficient at $\nu_k$ extends smoothly to $[0, \infty)$ for every $k = 0, \ldots, n - 1$. The lemma follows.

We summarize our results in the following proposition:

**Proposition 4.10** (Green kernel for $S^n$). The form $G$ from (72) defines a Green kernel for $S^n$ satisfying Definition 3.3. Moreover, we have the symmetries

$$R^*G = (-1)^R G \quad \text{for all } R \in O(n+1) \text{ and}$$

$$\tau^*G = (-1)^n G.$$

**Proof.** The proposition is a summary of Propositions 4.2, 4.4 and 4.9.

**Remark 4.11** (Better notation due to R. Bryant, see [6]). Pick an oriented basis $e_1, \ldots, e_{n+1}$ of $\mathbb{R}^{n+1}$ as generators of the exterior algebra $\Lambda^*(\mathbb{R}^{n+1})$, and view $x, y, dx, dy$ as $\Lambda^*(\mathbb{R}^{n+1})$-valued forms on $\mathbb{R}^{n+1}$. For example, we view $x$ as the map $x : \mathbb{R}^{n+1} \to \sum_{i=1}^{n+1} x^i e_i \in \Lambda^1(\mathbb{R}^{n+1})$ and $dx$ as the map $x : \mathbb{R}^{n+1} \to \sum_{i=1}^{n+1} (dx_i)_x e_i \in \Lambda^1(\mathbb{R}^{n+1})$. There is a natural wedge product on the space of $\Lambda^*(\mathbb{R}^{n+1})$-valued forms. If $\omega$ is a top-form, we denote by $[\omega]$ the
Figure 7: The $Y$-graph for $S^n$.

The coefficient of $\omega$ at $e_1 \wedge \cdots \wedge e_{n+1}$. Then it holds

$$\omega_k(x, y) = \frac{1}{k!} \frac{1}{(n - 1 - k)!} [x \wedge y \wedge (dx)^k \wedge (dy)^{n-1-k}].$$

Note that if we view $e_1$ as odd variables, then $[\cdot]$ corresponds to the odd integration $\int De(\cdot)$. It would be interesting to know whether this notation simplifies some proofs, especially if Lemma 4.12 can be deduced from abstract algebraic facts or rules valid for odd integration.

4.2 Computation of $n$ for $S^n$

We recall from Definition 3.19 that the formal pushforward Maurer-Cartan element $n$ is computed as a sum over trivalent ribbon graphs decorated with the Green kernel $G$ at internal edges, integration variables $x_i$ at internal vertices and, in the case of $S^n$, with $1$ or $v$ at external vertices.

The canonical Maurer-Cartan element $m$ is the contribution of the $Y$-graph (see Figure 7), and it is easy to see that

$$m_{10}(sv) = (-1)^n m_{10}(s1v) = m_{10}(s1v) = (-1)^{n-2}.$$  

Throughout this section, we will be in the setting of Definition 3.19. In particular, $\Gamma \in RG_{klg}^{(3)}$ is a ribbon graph, $L$ its compatible labeling admissible with respect to an input $\omega_1, \ldots, \omega_l$ and $I(\sigma_L)$ the corresponding integral.

**Lemma 4.12** (Condition (V_1) holds). Consider $S^n$ with the Green kernel $G$ from (72). Then every graph $\Gamma \neq Y$ with 1 at an external vertex vanishes.

**Proof.** The only contribution of an $A$-vertex which does not vanish for degree reasons is

$$A_{v,1}(y) = \int G(x, y) \text{Vol}(x).$$

From the symmetry of $G$ and $\text{Vol}$ under the action of $O(n + 1)$, we get

$$R^* A_{v,1} = (-1)^R A_{v,1} \quad \text{for all } R \in O(n + 1).$$

Therefore, it suffices to check that $A_{v,1}(e_1) = 0$, where $e_1, \ldots, e_{n+1}$ denotes the
standard basis of $\mathbb{R}^{n+1}$. Evaluation of (74) at $(x, e_1)$ gives

$$\omega_0(x, e_1) = \frac{1}{(n-1)!} \sum_{\sigma \in S_{n+1}} (-1)^{\sigma} x^{\sigma_1} dy^{\sigma_2} \cdots dy^{\sigma_{n+1}}.$$ 

Therefore, we get

$$A_{\mathbf{v}, \mathbf{e}_1} = (-1)^n \int_x g_0(x \cdot e_1) \omega_0(x, e_1) \text{Vol}(x)$$

$$= \sum_{j=2}^{n+1} (-1)^{n+j+1} \left( \int_x g_0(x^1 x^j \text{Vol}(x)) \right) dy^2 \cdots dy^j \cdots dy^{n+1},$$

where we view $g_0$ as a function of $x \cdot y$. For every $j = 2, \ldots, n+1$, consider the orientation reversing diffeomorphism

$$I_j : S^n \to S^n$$

$$(x^1, \ldots, x^{n+1}) \mapsto (x^1, \ldots, -x^j, \ldots, x^{n+1}).$$

Then we have

$$\int_x g_0(x^1 x^j \text{Vol}(x)) = - \int_x I_j^* \left( g_0(x^1 x^j \text{Vol}(x)) \right) = - \int_x g_0(x^1)(-x^j)(-\text{Vol}(x)),$$

and it follows that $A_{\mathbf{v}, \mathbf{e}_1} = 0$.

Let us now consider the contribution of a $B$-vertex with $1$:

$$B_1(y, z) = \int_x G(y, x)G(x, z) = (-1)^n \int_x G(y, x)G(x, z).$$

For $n = 1$, the degree of $G(y, x)G(x, z)$ is 0, and hence $B_1 = 0$ trivially. Suppose that $n \geq 2$. As in the case of $A_{\mathbf{v}, \mathbf{e}_1}$, we get that

$$R^* B_1 = (-1)^R B_1 \quad \text{for all } R \in O(n+1).$$

Therefore, it suffices to check that $B_1(e_1, c_1 e_1 + c_2 e_2) = 0$ for all $(c_1, c_2) \in S^1$.

We have

$$B_1(e_1, c_1 e_1 + c_2 e_2) = (-1)^n \sum_{a=1}^{n-1} \int_x g_a(x^1)g_{n-a}(c_1 x^1 + c_2 x^2)\omega_a(x, e_1)$$

$$\omega_{n-a}(x, c_1 e_1 + c_2 e_2).$$

We will show that for every $a = 1, \ldots, n-1$ we can write

$$\mu_a(x) := \omega_a(x, e_1)\omega_{n-a}(x, c_1 e_1 + c_1 e_2) = \left( \sum_{i=3}^{n+1} \pm x^i \text{Vol}(x) \right) \eta_a(y, z) \quad (78).$$
for some form $\eta_\nu(y, z)$. Then, using the same argument as for $A_{\nu, 1}$, we will have

$$
\int_x g_a(x^1)g_{n-a}(c_1x^1 + c_2x^2)x^1\text{Vol}(x)
= - \int_x I^*_x (g_a(x^1)g_{n-a}(c_1x^1 + c_2x^2)x^1\text{Vol}(x))
= - \int_x g_a(x^1)g_{n-a}(c_1x^1 + c_2x^2)(-x^1)(-\text{Vol}(x))
$$

for all $3 \leq i \leq n + 1$, and hence $B_1(e_1, c_1e_1 + c_2e_2) = 0$.

In order to show (78), we have to study the product of $\omega_1$'s. From (74) we we get

$$
\omega_n(x, y)\omega_{n-a}(x, z) = \frac{1}{a!(n-1-a)!(n-a)!(a-1)!} \sum_{\sigma, \mu \in S_{n+1}} (-1)^{\sigma+\mu}x^{\sigma_1}y^{\sigma_2}z^{\mu_2}
$$

$$
dx^{\sigma_3} \cdots dx^{\sigma_{n+a}} dy^{\sigma_{n+2}} \cdots dy^{\sigma_{n+2n-a}} dz^{\mu_{n+2}} \cdots dz^{\mu_{n+a+1}}.
$$

(79)

In order to simplify this expression, we decompose $\sigma \in S_{n+1}$ as

$$
\sigma = \sigma^5 \circ \sigma^4 \circ \sigma^3 \circ \sigma^2 \circ \sigma^1,
$$

(80)

where $\sigma^1, \ldots, \sigma^5 \in S_{n+1}$ are permutations defined as follows:

- The permutation $\sigma^1$ is a shuffle permutation $\sigma^1 \in S_{2+a,n-a-1}$ such that its first block denoted by $\sigma^1(1) = (\sigma_1^1, \ldots, \sigma_{2+a}^1)$ is equal to the ordered set $\{\sigma_1, \ldots, \sigma_{2+a}\}$. The second block $\sigma^1(2)$ is then the ordered set $\{\sigma_{3+a}, \ldots, \sigma_{n+1}\}$, which will be denoted by $J_\sigma$.

- The permutation $\sigma^2$ acts on the block $\sigma^1(1)$ by moving $\sigma_2$ in front. We denote the new block $\sigma^1(1) \backslash \{\sigma_2\}$ by $I_\sigma$, so that we can write $\sigma^2 : \sigma^1(1) \mapsto (\sigma_2, I_\sigma)$.

- The permutation $\sigma^3$ acts on the block $I_\sigma$ by moving $\sigma_1$ in front. Together with the previous step we get $\sigma^3(1) \mapsto (\sigma_2, \sigma_1, I_\sigma \backslash \{\sigma_1\})$.

- The permutation $\sigma^4$ is a transposition of $\sigma_1$ and $\sigma_2$.

- The permutation $\sigma^5$ is determined by the pair $(\sigma^5_1, \sigma^5_2) \in S_a \times S_{n-1-a}$ of permutations $\sigma^5_1$ and $\sigma^5_2$ acting on blocks $I_\sigma \backslash \{\sigma_1\}$ and $J_\sigma$ to get $(\sigma_3, \ldots, \sigma_{2+a})$ and $(\sigma_{3+a}, \ldots, \sigma_{n+1})$, respectively.

We define the decomposition $\mu^1, \ldots, \mu^5$ for $\mu \in S_{n+1}$ from (74) analogously.
We distinguish the two cases left:

The following implication holds:

where

\[
\frac{1}{a!(n-1-a)!(a-1)!} \sum_{\sigma^1,\ldots,\sigma^5,\mu^1,\ldots,\mu^5} (-1)^{\sigma^1+\cdots+\sigma^5+\mu^1+\cdots+\mu^5} x^{\sigma^1} x^{\sigma^2} y^{\sigma^2} z^{\mu^2} dx^{\sigma^2(I_\sigma \setminus \{\sigma_1\})} dx^{\mu^2(I_\mu \setminus \{\mu_1\})} dy^{\sigma^2(J_\sigma)} dz^{\mu^2(J_\mu)}
\]

\[= - \sum_{\sigma^1,\mu^1} (-1)^{\sigma^1+\mu^1} \left( \sum_{\sigma^2,\mu^2} (-1)^{\sigma^2+\mu^2} \sum_{\sigma^3,\mu^3} (-1)^{\sigma^3+\mu^3} x^{\sigma^1} x^{\mu_1} y^{\sigma^2} z^{\mu^2} \right) dx^{I_\sigma \setminus \{\sigma_1\}} dx^{I_\mu \setminus \{\mu_1\}} dy^{J_\sigma} dz^{J_\mu},
\]

where \(-1\) comes from \((-1)^{\sigma^1}\) and \(\sigma^5\) is compensated by permutations of forms. For fixed \(\sigma^1\) and \(\mu^1\), consider the coefficient at \(dy^{J_\sigma} dz^{J_\mu}\) in the brackets. If we evaluate it at \(y = e_1, z = c_1 e_1 + c_2 e_2\), we get

\[= \text{I} c_1 \sum_{\sigma^3,\mu^3} (-1)^{\sigma^3+\mu^3} x^{\sigma^1} x^{\mu_1} dx^{I_\sigma \setminus \{\sigma_1\}} dx^{I_\mu \setminus \{\mu_1\}}
\]

\[+ (-1)^{\mu_2} c_2 \sum_{\sigma^3,\mu^3} (-1)^{\sigma^3+\mu^3} x^{\sigma^1} x^{\mu_1} dx^{I_\sigma \setminus \{\sigma_1\}} dx^{I_\mu \setminus \{\mu_1\}},
\]

where \((-1)^{\mu_2} = -1\) if and only if \(1 \in I_\mu\).

More generally, for multiindices \(I_1, I_2 \subset \{1, \ldots, n+1\}\) of lengths \(a + 1\) and \(n - a + 1\), respectively, consider the sum

\[S(I_1, I_2) := \sum_{i_1, i_2} (-1)^{(i_1, I_1) + (i_2, I_2)} x^{i_1} x^{i_2} dx^{I_1 \setminus \{i_1\}} dx^{I_2 \setminus \{i_2\}}, \tag{81}
\]

where \((i_j, I_j)\) is the number of transpositions required to move \(i_j\) in front of \(I_j\). The following implication holds:

\[S(I_1, I_2) \neq 0 \implies 1 \leq |I_1 \cap I_2| \leq 2.
\]

We distinguish the two cases left:

Case \(I_1 \cap I_2 = \{i, j\}\) with \(i < j\): We get

\[S(I_1, I_2) = (-1)^{(i, I_1) + (j, I_2)} x^i x^j dx^{I_1 \setminus \{i\}} dx^{I_2 \setminus \{j\}} + (-1)^{(j, I_1) + (i, I_2)} x^j x^i dx^{I_1 \setminus \{j\}} dx^{I_2 \setminus \{i\}}.
\]
Case I

We will prove that the signs alternate, and hence

The signs clearly alternate. A symmetric argument holds when

Now assume that

Suppose that

in (81) with

Therefore, for some signs \pm, we can write

We will prove that the signs alternate, and hence \( S(I_1, I_2) = \pm x^t \text{Vol}(x) \).

Suppose that \( j, j + 1 \in I_1 \) for some \( j \in \{1, \ldots, n\} \). The two summands in (81) with \( (i_1, i_2) = (j, i) \) and \( (i_1, i_2) = (j + 1, i) \), respectively, give

The signs clearly alternate. A symmetric argument holds when \( j, j + 1 \in I_2 \).

Now assume that \( j \in I_1 \) and \( j + 1 \in I_2 \). The two summands in (81) which have \( (i_1, i_2) = (j, i) \) and \( (i_1, i_2) = (i, j + 1) \), respectively, give

100
\[ = (-1)^{(i;1)+(i;1\setminus(j))+(j+1;2)} x^i \, dx^j \, dx^{j+1} \, dx_{I_1 \setminus \{i,j\}} \, dx_{I_2 \setminus \{j+1\}} \\
+ (-1)^{(i;1)+(i;1\setminus(j))+(j+1;2)} x^i \, dx^j \, dx^{j+1} \, dx_{I_1 \setminus \{i,j\}} \, dx_{I_2 \setminus \{j+1\}} \]

The signs alternate again. A symmetric argument holds for \( j \in I_2 \) and \( j+1 \in I_1 \).

Back to the original problem, we have \( I = S(I_{\sigma}, I_{\mu}) \) with \( I_{\sigma}, I_{\mu} \subset \{2, \ldots, n+1\} \). It follows that the first case applies, and hence \( I = 0 \). We have \( II = S(I_{\sigma}, I_{\mu}) \) with \( I_{\sigma} \subset \{2, \ldots, n+1\} \) and \( I_{\mu} \subset \{1,2, \ldots, n+1\} \). It follows that either the first case or the second case with \( i \geq 3 \) applies. This proves (78). Consequently, we get \( B_1 = 0 \) also for \( n \geq 2 \).

We summarize the consequences in the following proposition. The main argument is the same as in the proof of Proposition 3.29.

**Proposition 4.13** (Vanishing of graphs for \( S^n \)). Consider \( S^n \) with the Green kernel (72). Only the following trivalent ribbon graphs \( \Gamma \neq Y \) do not necessarily vanish:

\( (n = 1) \): The \( O_k \)-graph with \( k \in 2\mathbb{N} \) internal vertices of type B with \( v \) at the external vertex (see Figure 11).

\( (n = 2) \): It must hold \( A = 0 \), \( C = 2B \) and all B vertices must have \( v \) at the external vertex. Moreover, if \( \Gamma \) is reduced, it must have \( g \geq 1 \).

\( (n = 3) \): There is no external vertex and \( 4 \mid C \) holds.

\( (n > 3) \): All graphs vanish.

**Proof.** Lemma 4.12 implies that \( A = 0 \) and that the total form-degree \( D \) satisfies \( D = nB \). Therefore, we get from (66) the following: for \( n > 3 \) there is neither a B-vertex nor a C-vertex; for \( n = 3 \), there is no B-vertex; for \( n = 2 \), we have \( C = 2B \); and for \( n = 1 \), there is no C-vertex.

Consider the pullback of \( I(\sigma_L) \) along the (multi)diagonal action of an \( R \in O(n+1) \) with \( \det(R) = -1 \) on \( (S^n)^{\times k} \). We get schematically

\[ \int_{(S^n)^{\times k}} G^e \text{Vol}^n = (-1)^{k+e+s} \int_{(S^n)^{\times k}} G^e \text{Vol}^n. \]

Therefore, \( k + e + s \) has to be even. If we plug-in from (65), we get

\[ k + e + s = \begin{cases} 3B & \text{for } n = 1, \\ 8B & \text{for } n = 2, \\ \frac{5}{2}C & \text{for } n = 3. \end{cases} \]
A non-vanishing reduced graph must have $B \geq l$. For $n = 2$, so that $C = 2B$, the formula (64) gives $g \geq 1$. \hfill \Box

Remark 4.14 (Graphs for $S^2$). The simplest possibly non-vanishing graph for $S^2$ has $A = 0$, $B = 1$, $C = 2$. If it is reduced, we must have $l = g = 1$, and hence it will contribute to $n_{11}$. Up to an isomorphism, there is only one such graph, which we denote by $P_1$ (see Figure 8). However, we see that the pair of internal vertices $x_1$ and $x_2$ is connected by two edges, which implies that $P_1 = 0$. Indeed, $G(x,y)$ has odd degree, and hence we have

$$G(x,y)G(y,x) = G(x,y)^2 = 0$$

by the symmetry on the pullback along the twist map. It follows that $n_{11} = 0$.

The second simplest possibly non-vanishing reduced graph is the graph $P_2$ from Figure 8. Let

$$\eta(x_1, x_2, x_3, x_4, x_5) := G(x_1, x_2)G(x_1, x_3)G(x_4, x_2)G(x_4, x_3)G(x_3, x_5)$$

$$G(x_2, x_3)\text{Vol}(x_5)$$

denote the form in the integrand coming from the part of the graph on the right-hand side of the vertical axis going through $x_1$, $x_4$. If $\tau_{1,4}$ denotes the exchange of $x_1$ and $x_4$, then clearly $\tau_{1,4}^*\eta = \eta$ because the graph is symmetric.

---

\[11\] We recall from Section 3.3 that the notation $G(x_i, x_j)$ means $(\pi_i \times \pi_j)^*G$ and not just the evaluation at $(x_i, x_j)$.
with respect to the horizontal axis going through \( x_5, x_6 \). Using this, we compute

\[
\int_{x_1, x_2, x_3, x_4, x_5, x_6} v(x_6)G(x_1, x_6)G(x_4, x_6)\eta(x_1, x_2, x_3, x_4, x_5)
\]

\[
= \int_{\tau_1, \tau_2(x_1, x_2, x_3, x_4, x_5, x_6)}^{\tau_1, \tau_2(x_1, x_2, x_3, x_4, x_5, x_6)} v(x_6)G(x_1, x_6)G(x_4, x_6)\eta(x_1, x_2, x_3, x_4, x_5)
\]

\[
= \int_{x_4, x_2, x_3, x_1, x_5, x_6} v(x_6)G(x_4, x_6)G(x_1, x_6)\eta(x_4, x_2, x_3, x_1, x_5)
\]

\[
= - \int_{x_1, x_2, x_3, x_4, x_5, x_6} v(x_6)G(x_1, x_6)G(x_4, x_6)\eta(x_1, x_2, x_3, x_4, x_5),
\]

where the minus sign comes from switching the first two \( G \)'s. We see that \( P_2 \) vanishes. The other variants with \( x_5 \) moved on the edge \( x_3, x_4 \) and \( x_2, x_4 \) vanish by a similar argument using the compositions \( \tau_{1,1} \circ \tau_{5,6} \) and \( \tau_{1,2} \circ \tau_{5,6} \), respectively. We conclude that \( n_{21} = 0 \), and hence \( q_{121} = 0 \).

We sum up some general observations about the integrals for \( S^2 \):

- We have \( B_i \neq 0 \) and \( C \neq 0 \) for the corresponding forms.
- We have the multiplication formula (c.f., Example 4.3)

\[
\omega_1(x, y)\omega_1(x, z) = x \cdot (y \times z) \text{Vol}(x).
\]

- The number \((-1)^{n_x}I(\sigma_L)\) does not depend on the choice of \( L_1 \) provided a compatible \( L_2 \) is chosen.
- It holds \( \sum_{L_1} (-1)^{n_x}I(\sigma_L) = 0 \) whenever there is a boundary component with even number of \( v \)'s.
- If there is a \( B \)-vertex \( x \) such that the underlying graph (after forgetting the ribbon structure) is symmetric on the reflection along an axis going through \( x \), then \( I(\sigma_L) = 0 \).

\[
\text{Remark 4.15 (Graphs for } S^3) \text{. For } S^3, \text{ we consider the non-reduced graphs } K_1 \text{ and } K_2 \text{ and the tadpole graph from Figure 9}. \text{ The graphs } K_1 \text{ and } K_2 \text{ appear in the definition of the Chern-Simons topological invariant in } [21] \text{ (with a gauge group). The corresponding integrals from our theory vanish "algebraically", i.e., at the level of wedge products of } \omega_i. \text{ Indeed, every summand in } K_1 \text{ contains}
\]

\[
\omega_a(x_1, x_1) = 0 \quad \text{for some } a = 0, 1, 2,
\]

and, for degree reasons, the form part of \( K_2 \) can contain only

\[
\omega_1(x_1, x_2)^3 = 0 \quad \text{or} \quad \omega_0(x_1, x_2)\omega_1(x_1, x_2)\omega_2(x_1, x_2) = 0.
\]
Figure 9: Graphs $K_1$ and $K_2$ from the Chern-Simons theory and the tadpole graph with $(l,g) = (2,0)$ for $n = 3$.

The tadpole graph contains only

$$\omega_2(x_1, x_3)\omega_1(x_1, x_2)\omega_2(x_2, x_3) = 0.$$  

Equations in Remarks 4.14 and 4.15 were checked by the computer. The program for Wolfram Mathematica 10.4 will be made available at [18].

We will now compute $n_{20}$ for $S^1$, which according to Proposition 4.13 consists only of contributions from the $O_k$-graphs with $k$ even. By analogy with the finite dimensional case (see Appendix A), we expect that the number $(-1)^{|\sigma|}I(\sigma_L)$ does not depend on $L$. All inputs are namely the same and the degrees even, i.e., $|m_i^2| = -2$, $|\theta^2G| = -2$ and $|v| = 0$.

We fix $s_1, s_2 \geq 1$ such that $k = s_1 + s_2$ is even and make the ansatz

$$n_{20}(sv^{s_1} \otimes sv^{s_2}) := \varepsilon(s_1, s_2)C(s_1, s_2)I(k),$$

where $I(k)$ is the integral

$$\frac{1}{V_k} \int_{x_1, \ldots, x_k} G(x_1, x_2) \cdots G(x_{k-1}, x_k)G(x_k, x_1)Vol(x_1) \cdots Vol(x_k), \quad (82)$$

$\varepsilon(s_1, s_2)$ a sign and $C(s_1, s_2)$ a combinatorial coefficient to be determined.

We fix a circle in the plane with $k$ points (= internal vertices) and denote by $O(s_1, s_2)$ the set of ribbon graphs constructed by attaching external legs from which $s_1$ points in the interior and $s_2$ in the exterior, or the other way round, so that $O(s_1, s_2) = O(s_2, s_1)$ (see Figure 11). Recall that the ribbon structure is induced from the counterclockwise orientation of the plane. It is easy to see
that all graphs in $O(s_1, s_2)$ admit a labeling which is admissible with respect to $sv^{s_1} \otimes sv^{s_2}$, and that $O(s_1, s_2)$ contains a representative of every such $O_k$-graph.

**Lemma 4.16 (Integral for the $O_k$-graph for $S^1$).** For every even $k \geq 2$, the integral $I(k)$ is equal to

$$(-1)^{k+1} \frac{1}{2\pi} \sum_{i=2,4,\ldots,k} \frac{1}{i} \sum_{i_1+\ldots+i_r=k-1 \atop i_1,\ldots,i_r \in 2\mathbb{N}, r \in \mathbb{N}} (-1)^r \frac{1}{(i_1+1)! \cdots (i_r+1)!}.$$  

(83)

**Proof.** Denote $\bar{G}(x, y) := -2\pi G$. For all $k$, $l \geq 1$, we consider the more general integral

$$I(k, l) := \int_{x_1, \ldots, x_k} \bar{G}(x_1, x_2) \cdots \bar{G}(x_{k-1}, x_k) \bar{G}(x_k, x_1)^l \Vol(x_1) \cdots \Vol(x_k).$$

Taking the pullback along $(x_1, x_2, \ldots, x_{k-1}, x_k) \mapsto (x_k, x_{k-1}, \ldots, x_2, x_1)$ and using the antisymmetry of $\bar{G}(x, y)$, we get $I(k, l) = 0$ whenever $k + l$ is even.

We will compute $I(k, 1)$ for $k \in 2\mathbb{N}$ from a recursive relation which arises from successive integration.

For the recursion step, we need to evaluate the integral

$$\int_y \bar{G}(x, y) \bar{G}(y, z)^l \Vol(y)$$

for fixed $(x, z) \in (S^1 \times S^1)\setminus \Delta$. Pick the chart $g : S^1 \setminus \{z\} \to (-\pi, \pi)$ defined by

$$g(y) = \bar{G}(y, z) = \pi - \alpha(y, z) \text{ for } y \in S^1 \setminus \{z\},$$

where the angle $\alpha$ was defined in Example 4.3. It holds $dg(y) = \Vol(y)$ and

$$\bar{G}(x, y) = \begin{cases} \bar{G}(x, z) - g(y) - \pi & \text{for } -\pi < g(y) < \bar{G}(x, z), \\ \bar{G}(x, z) - g(y) + \pi & \text{for } \bar{G}(x, z) < g(y) < \pi. \end{cases}$$

We compute

$$\int_y \bar{G}(x, y) \bar{G}(y, z)^l \Vol(y) = \int_{-\pi}^{\pi} (\bar{G}(x, z) - g)^l dg - \pi \int_{-\pi}^{\pi} g^l dg + \pi \int_{-\pi}^{\pi} \bar{G}(x, z)^l dg$$

$$= 2\pi \int_{-\pi}^{\pi} \bar{G}(x, z)^l dg - \bar{G}(x, z)^l + \bar{G}(x, z)^{l+1}$$

$$= \frac{\pi^{l+1}}{l+2} - \bar{G}(x, z)^{l+1}$$

for $l$ odd.

From now on, $\int$ will stand for the Riemannian integral, i.e., $\int f := \int f \Vol$.
for a function $f$. We compute

$$I(2, l) = \int_{x_1, x_2} \tilde{G}(x_1, x_2) \tilde{G}(x_2, x_1)^l = -\int_{y_2} \tilde{G}(y, z)^l + 1 = -2\pi \int_{-\pi}^{\pi} g^{l+1} \, dg$$

$$= \begin{cases} 0 & \text{for } l \text{ even}, \\ -\frac{4\pi^{l+3}}{l+2} & \text{for } l \text{ odd}. \end{cases}$$

For $k \geq 4$ even and $l$ odd, we compute

$$I(k, l) = \frac{2\pi}{l+1} \int_{x_1, \ldots, x_{k-1}} \tilde{G}(x_1, x_2) \cdots \tilde{G}(x_{k-2}, x_{k-1})$$

$$= \frac{-4\pi^2}{(l+1)(l+2)} \int_{x_1, \ldots, x_{k-2}} \tilde{G}(x_1, x_2) \cdots \tilde{G}(x_{k-3}, x_{k-2})$$

$$= \frac{4\pi^2}{(l+1)(l+2)} (-\pi^{l+1} I(k-2, 1) + I(k-2, l+2)).$$

For the second equality, we used $\int_{x_1} \tilde{G}(x_1, x_2) = 0$ to show that the term multiplied by $\frac{\pi^{l+1}}{l+2}$ vanishes. It follows that

$$I(k, 1) = \frac{(2\pi)^{k-2}}{(k-1)!} I(2, k-1) - \sum_{l=2, 4, \ldots, k-2} \frac{(2\pi^2)^{k-l}}{(k-l+1)!} I(l, 1)$$

$$= -\frac{k(2\pi^2)^k}{(k+1)!} - \sum_{l=2, 4, \ldots, k-2} \frac{(2\pi^2)^{k-l}}{(k-l+1)!} I(l, 1) \quad \text{for all } k = 2, 4, \ldots$$

This is a recursive equation of the form $a_k = c_k + \sum_{l=1}^{k-1} d_{k-l} a_l$. Its solution is $a_k = \sum_{i=1}^{k} c_i D_{k-i}$ with $D_0 := 1$ and $D_i = \sum d_{i_1} \cdots d_{i_r}$, where we sum over all $r = 1, \ldots, i$ and $i_1, \ldots, i_r \in \mathbb{N}$ such that $i_1 + \cdots + i_r = i$. Therefore, we get

$$I(k, 1) = -\frac{(2\pi^2)^k}{k+1} \sum_{i=2, 4, \ldots, k} \frac{i}{(i+1)!} \sum_{\substack{i_1 + \cdots + i_r = k-i \\
i_1, \ldots, i_r \in 2\mathbb{N}, r \in \mathbb{N}}} \frac{1}{(i_1+1)!} \cdots (i_r+1)!.$$ 

The result has to be multiplied by $(-1)^k (2\pi)^{-2k}$ in order to get $I(k)$. 

**Lemma 4.17** (Independence of labeling). The summand $(-1)^{\sigma_L} I(\sigma_L)$ in the definition of $\eta_{20}(sv^{s_1} \otimes sv^{s_2})$ for $S^1$ is independent of the choice of $\Gamma \in O(s_1, s_2)$ and its labeling $L$ which is compatible and admissible with respect to the input.

**Proof.** Pick $\Gamma \in O(s_1, s_2)$ and its admissible labeling $L$. Let $L'$ be an other admissible labeling of $\Gamma$. We distinguish the following situations:
Suppose that $L$ and $L'$ differ by a permutation $\mu$ in $L^b_3$. A similar argument as in the proof of Lemma 3.20 shows that $(-1)^{\sigma_{L'}} = (-1)^{\sigma_L}$ and $I(\sigma'_{L}) = (-1)^{\mu} I(\sigma_L)$, where the sign in the integral comes from the permutation of Vol's, which have form-degree 1. Hence $(-1)^{\sigma_{L'}} I(\sigma_{L'}) = (-1)^{\sigma_L} I(\sigma_L)$.

Suppose that the boundaries are permuted, i.e., that $L$ and $L'$ differ in $L^b_1$. Notice that $s_1 = s_2$ because otherwise one of $L$ or $L'$ would not be admissible. The sign from changing $L^b_1$ cancels as in the previous case.

Suppose that $L$ and $L'$ differ in $L^b_2$. It was explained in the proof of Lemma 3.20 that a single change of $L_2$ induces the sign $(-1)^{n-1} = 1$ in $(-1)^{\sigma_L} I(\sigma_L)$.

A cyclic permutation in $L^v_3$ induces a sign neither in $(-1)^{\sigma_L}$ nor in $I(\sigma_L)$.

A permutation $\mu$ in $L^v_1$ induces $(-1)^{\mu}$ in $(-1)^{\sigma_L}$ and a change in $I(\sigma_L)$, which can be realized by taking the pullback along $\mu : (x_1, \ldots, x_k) \mapsto (x_{\mu_1}, \ldots, x_{\mu_k})$. However, the sign of the Jacobian is $(-1)^{\mu}$, which cancels the sign from $(-1)^{\sigma_L}$.

Next, we prove the independence of $\Gamma \in O(s_1, s_2)$. Let $L$ be an admissible and compatible labeling of $\Gamma$. Pick two adjacent internal vertices with external legs pointing to different regions, i.e., one to the interior of the circle and the other to the exterior. Suppose that the vertices are labeled by $v_1$ and $v_2$ and the legs by $l_1$ and $l_2$, respectively. Let $\Gamma' \in O(s_1, s_2)$ be the graph with the two legs turned inside out (see Figure 10). We can construct an admissible and compatible labeling $L'$ of $\Gamma'$ by making the following changes to $L$: The new leg at $v_1$ will be labeled by $l_2$ and the new leg at $v_2$ by $l_1$. The cyclic orderings at $v_1$ and $v_2$, respectively, have to be modified by a transposition in order to get compatibility with the new ribbon structure. All other labelings can be copied from $L$. In total, we get

$(-1)^{\sigma_L - \sigma_{L'}} = -1$.

This sign is compensated by swapping the one-forms in $I(\sigma_L)$:

$$\text{Vol}(x_{v_1}) \ldots \text{Vol}(x_{v_2}) \leftrightarrow \text{Vol}(x_{v_2}) \ldots \text{Vol}(x_{v_1}).$$
The independence of $\Gamma \in O(s_1, s_2)$ follows from the fact that we can span the entire $O(s_1, s_2)$ by repeating the swap-of-legs operation.

**Lemma 4.18** (Sign). We have

$$\varepsilon(s_1, s_2) = (-1)^{s_1+1}.$$ 

**Proof.** By Lemma 4.17 in order to compute $(-1)^{s_2} I(\sigma_L)$, we can pick $\Gamma^* \in O(s_1, s_2)$ and its admissible and compatible labeling $L^*$ from Figure 11. We abbreviate $\sigma_0 = \sigma_{L^*}$. The corresponding integral (60) reads

$$I(\sigma_0) = \frac{1}{V_k} \int \prod_{x_1, \ldots, x_k} G(x_{k-1}, x_k) \cdots G(x_1, x_2) G(x_k, x_1) \text{Vol}(x_{s_1}) \cdots \text{Vol}(x_{s_1+1}) \cdots \text{Vol}(x_k) \text{Vol}(x_{s_1+1}) \cdots \text{Vol}(x_k).$$

It differs from $I(k)$ from 62 in the order of $G$’s and Vol’s. A reordering produces the sign

$$(-1)^{\frac{1}{2}s_1(s_1-1)}.$$ 

We will compute $(-1)^{\sigma_0}$ by ordering half-edges from the edge order back to the vertex order while looking at Figure 11. The steps are as follows:

- Transpose half-edges at internal vertices so that the first half-edge goes inside the vertex and the third outside with respect to the counterclockwise orientation. This gives $(-1)^{\sigma}$.
Figure 12: The mirror isomorphism $M : 1 \ldots k \mapsto \bar{k} \ldots \bar{1}$ is a composition of the inversion and the counterclockwise rotation by one place.

- Permute external legs so that $v_i$ is at $x_i$ for all $i = 1, \ldots, k$. This gives $(-1)^{\frac{s_1(s_1-1)}{2}}$.

- Permute internal edges so that $G_i$ starts at the third half-edge of $x_i$ and ends at the first half-edge of $x_{i+1}$ for all $i = 1, \ldots, k-1$. This does not produce any sign as swapping of two $G$'s requires two transpositions.

- At this point, we have the permutation

$$
\begin{pmatrix}
1 & 2 & \ldots & 2(e-1) & 2(e-1) & 2e-1 & 2e & 2e+1 & \ldots & 3k \\
3 & 4 & \ldots & 3k-3 & 3k-2 & 3k & 1 & 2 & \ldots & 3k-1
\end{pmatrix}.
$$

We interpret the last line as $G_1 \ldots G_k v_1 \ldots v_k$ and permute it to the sequence $v_1 G_1 v_2 G_2 \ldots v_k G_k$, which does not produce any sign. We end up with

$$
\sigma'_0 = \begin{pmatrix}
1 & 2 & 3 & \ldots & 3k-1 & 3k \\
2 & 3 & 4 & \ldots & 3k & 1
\end{pmatrix}.
$$

It is now easy to see that

$$
(-1)^{\sigma'_0} = (-1)^{3k-1}.
$$

In total, we get

$$
(-1)^{\sigma_0} = (-1)^{s_1 + \frac{1}{2}s_1(s_1-1)+k+1}.
$$

As for the other signs in Definition 3.19, we have $s(k,l) = k + \frac{1}{2}k(k-1)$ and $P(s^k) = \frac{1}{2}k(k-1)$. There is no sign from $s^k v^{s_1} \otimes v^{s_2} = sv^{s_1} \otimes sv^{s_2}$ since $|s| = -2$. Multiplying everything together, we get $\epsilon(s_1,s_2)$.

Lemma 4.19 (Combinatorial coefficient). It holds

$$
C(s_1,s_2) = \frac{1}{2}akl \binom{k-1}{s_1}.
$$

\[ \square \]
Proof. Every isomorphism of ribbon graphs $\Gamma$ and $\Gamma'$ from $O(s_1, s_2)$ is a composition of the clockwise rotation ($r$) for $r \in \mathbb{Z}$ and the mirror operation $M$ defined as follows: If $1, \ldots, k$ label internal vertices in the clockwise direction starting from the north-pole, then the result of $M$ is $\tilde{k}, \ldots, 1$, where $\tilde{i}$ means that the external leg is reversed (see Figure [12]). These operations satisfy

$$(r + k) = (r), \ (r)(-r) = 1, \ M^2 = 1, \ (r)M = M(-r),$$

and hence generate a group $G$ which is isomorphic to the dihedral group $\mathbb{Z}_k \rtimes \mathbb{Z}_2$.

The orbit space $O(s_1, s_2)/G$ is in $1 : 1$ correspondence with isomorphism classes of admissible $O_k$-graphs and $\text{Aut}(\Gamma)$ is in $1 : 1$ correspondence with $\text{Stab}(\Gamma)$. From the orbit-stabilizer formula, we get

$$\sum_{[\Gamma] \text{ admiss.}} \frac{1}{|\text{Aut}(\Gamma)|} = \sum_{[\Gamma] \in O(s_1, s_2)/G} \frac{1}{|\text{Stab}(\Gamma)|} = \sum_{\Gamma \in O(s_1, s_2)} \frac{1}{|\text{Orb}(\Gamma)||\text{Stab}(\Gamma)|}$$

$$= \frac{|O(s_1, s_2)|}{|G|} = \frac{1}{2k} \binom{k}{s_1} \begin{cases} 1 & \text{for } s_1 = s_2, \\ 2 & \text{for } s_1 \neq s_2. \end{cases}$$

The two cases are compensated in the sum over labelings: For $s_1 = s_2$, both labelings $L_1$ are admissible, and hence we get the factor 2.

Next, we multiply by $k!s_1(k-s_1)$, which is the number of $L_1$ and $L_2$. There is also the factor $\frac{1}{n} = \frac{1}{2}$. Multiplying everything together, we get $C(s_1, s_2)$. $\square$

Before we summarize the results of our computations (see Proposition 4.21 below), we show directly that $n$ is a Maurer-Cartan element.

Lemma 4.20 (Maurer-Cartan equation for $S^n$). Consider $S^n$ with the Green kernel from (72). The collection $(n_{ig})$ satisfies the Maurer-Cartan equation (22) for dIBL($C(H(S^n))$).

Proof. We will show that for every $l \geq 1, g \geq 0$ all summands in the relation corresponding to $(l, g)$ vanish. The summands for $(l, g) = (1, 0)$ are $q_{110}(n_{10})$ and $\frac{1}{2}q_{210}(n_{10}, n_{10})$, and the summand for $(l, g) = (2, 0)$ is $q_{120}(n_{10})$. The first term vanishes trivially as $q_{110} = 0$, while the other two terms vanish by [10] Proposition 12.5] because $n_{10} = m_{10}$ is the canonical Maurer-Cartan element. For $(l, g) \neq (1, 0)$, we have the following four situations:

$q_{210} \circ_2 n_{1g}, l \geq 2$: Let $\Psi = \Psi_1 \cdots \Psi_l \in E_l C$ be a summand of $n_{1g}$. From Proposition 4.13 it follows that the summands can be chosen such that $\Psi_1, \ldots, \Psi_l \in B_{\text{cyc}, \text{red}}^{H(S^n)}[3-n]$, i.e., such that $\Psi_i$ evaluates to 0 whenever $i$ is a part of its argument. From Definition 2.13 we compute

$$q_{210} \circ_2 (\Psi_1 \cdots \Psi_l) = \sum_{\sigma \in S_2, l-2} \varepsilon(\sigma, \Psi)q_{210}(\Psi_{\sigma_1^{-1}} \cdots \Psi_{\sigma_l^{-1}}).$$

110
We clearly have $q_{210}(\Psi_{\sigma^{-1}} \circ \Psi_{\tau^{-1}}) = 0$ because $q_{210}$ feeds 1 into one of its inputs. It follows that $q_{210} \circ_2 n_{tg} = 0$.

$q_{210} \circ_{1,1} (n_{1g_1} \circ n_{2g_2})$, $(l, g_1) \neq (1, 0)$: A similar argument as above.

$q_{120} \circ_1 n_{tg}$, $(l, g) \neq (1, 0)$: A similar argument as above using that $q_{120}$ also feeds 1 into its input.

$q_{210} \circ_{1,1} (n_{10} \circ n_{2g})$, $(l, g) \neq (1, 0)$: As in the case of $q_{210} \circ_2 n_{tg}$, let $\Psi_1, \ldots, \Psi_l \in B_{\text{Cyc-red}} H(S^n)[3-n]$. Recall that we write $\Omega_i = s\omega_i \in B^{2c}_H(S^n)[3-n]$ and $\Omega = \Omega_1 \circ \cdots \circ \Omega_l$. From Definition 2.15, we compute

$$[q_{210} \circ_{1,1} (n_{10} \circ \Psi_1) \cdots \cdots \Omega_l)] = \sum_{\mu \in S_l} \sum_{i=1}^l (-1)^{(|\Psi_i| + \cdots + |\Psi_{l-1}|)} q_{210}(n_{10} \Psi_i) \cdots \cdots \Omega_i \cdots \cdots \Omega_l.)$$

For every $i = 1, \ldots, l$, we have

$$q_{210}(n_{10} \cdot \Psi_i)(\Omega) = q_{210}(n_{10} \circ \Psi_i)(\Omega)$$

$$= -\sum \varepsilon(\omega \mapsto \omega_1 \omega_2) \Omega_i \cdots \cdots \Omega_l.$$

This can be non-zero only if $\omega = 1v^{s-1}$ for some $s \geq 2$ (up to a cyclic permutation). For this input, we get

$$q_{210}(n_{10} \circ \Psi_i)(s1v^{s-1})$$

$$= -[\varepsilon(1v^{s-1} \mapsto 1v^{s-1})(s1v^{s-1})](s1v^{s-1})$$

$$+ [\varepsilon(1v^{s-1} \mapsto 1v^{s-2})(s1v^{s-1})](s1v^{s-1})$$

$$= (-1)^{n-3}(1 + (-1)^{n+1})\Psi_i(s1v^{s-1}).$$

The prefactor in brackets is 0 for $n$ odd or $s$ even, whereas $v^{s-1} = 0$ for $n$ even and $s$ odd. Therefore, we have $q_{210} \circ_{1,1} (n_{10} \circ n_{g}) = 0$.  

**Proposition 4.21** (Formal pushforward Maurer-Cartan element for $S^n$). Consider the round sphere $S^n$ with the Green kernel (72). The formal pushforward Maurer-Cartan element $n$ is a strictly reduced Maurer-Cartan element for
dIBL\((C(H(S^n)))\) which satisfies
\[
n_{10} = m_{10} \quad \text{for all } n \in \mathbb{N}
\]

plus the following properties depending on the dimension:

\((n = 1)\): It holds \(n_{10} = 0\) for all \(l \geq 1, g \geq 0\) such that \((l, g) \neq (1, 0), (2, 0)\); the only non-trivial relation for \(n_{20}\) is
\[
n_{20}(sv^{s_1} \otimes sv^{s_2}) = (-1)^{s_1} + 1 \frac{1}{2} s_1 k! \begin{pmatrix} k - 1 \\ s_1 \end{pmatrix} I(k), \quad (84)
\]
where \(s_1, s_2 \geq 1\) are such that \(k = s_1 + s_2\) is even, and \(I(k)\) is given by \((83)\).

\((n = 2)\): It holds \(n_{10} = 0\) for all \(l \geq 2\). We also have \(n_{11} = 0\).

\((n \geq 3)\): It holds \(n_{10} = 0\) for all \(l \geq 1, g \geq 0\) such that \((l, g) \neq (1, 0)\).

Notice that \(n_{20} \notin E_2 C(H(S^1))\), i.e. \(n_{20}\) is a long cochain, because it is non-zero in infinitely many weights.

### 4.3 Twisted IBL\(_{\infty}\)-structure for \(S^n\)

Let \(e_0, e_1\) be the basis of \(H(S^n)[1]\) defined by
\[
e_0 := 1 := \theta 1, \quad e_1 := v := \frac{1}{\text{Vol}} \theta \text{Vol}.
\]
The degrees satisfy
\[
|\theta| = -1, \quad |v| = n - 1.
\]
The matrix of the pairing \(P\) with respect to the basis \(e_0, e_1\) reads
\[
P = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.
\]
The dual basis \(e^0, e^1\) to \(e_0, e_1\) with respect to \(P\) is thus
\[
e^0 = v, \quad e^1 = (-1)^n 1.
\]
It follows that the matrix \((T^{ij})\) from \((83)\) satisfies
\[
(T^{ij}) = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
We clearly have
\[ \hat{B}_{\text{cyc,red}}^* \mathcal{H}(S^1) = \left\{ \sum_{k=1}^{\infty} c_k v^{k*} \mid c_k \in \mathbb{R} \right\}, \]
where \( v^{k*} \) is the dual to the cyclic word \( v^k = v \ldots v \) of length \( k \). Observe that the cyclic symmetry gives
\[ v^i = (-1)^{(n-1)(i-1)} v^i \quad \text{for all } i \geq 1. \]

Therefore, \( v^{i*} = 0 \) holds if both \( n \) and \( i \) are even.

For \( n \geq 2 \), the vector space \( \mathcal{H}(S^n) \) is connected and simply-connected, and Proposition 2.40 implies that there are no long reduced cyclic cochains (i.e., we have only finite sums of \( v^{k*} \)'s).

The product \( \mu_2 : \mathcal{H}[1] \otimes \mathcal{H}[1] \rightarrow \mathcal{H}[1] \) from (42) has the following matrix with respect to the basis 1, \( v \):
\[ \mu_2 = \begin{pmatrix} 1 & v \\ (-1)^n v & 0 \end{pmatrix}. \]

Because \( \mu_2(v, v) = 0 \), we get
\[ \mathbb{H}^m(C_{\text{red}}(\mathcal{H}(S^n))[1]) = \begin{cases} \langle sv^{i*} \mid i \geq 1 \rangle & \text{for } n \geq 3 \text{ odd}, \\
\langle sv^{2i-1*} \mid i \geq 1 \rangle & \text{for } n \text{ even}, \\
\{ s \sum_{k=1}^{\infty} c_k v^{k*} \mid c_k \in \mathbb{R} \} & \text{for } n = 1. \end{cases} \]

Because we are in the strictly unital and strictly augmented case, we obtain
\[ \mathbb{H}^m(C)[1] = \begin{cases} \langle sv^{i*}, s_1^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \geq 3 \text{ odd}, \\
\langle sv^{2i-1*}, s_1^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \text{ even}, \\
\{ s \sum_{k=1}^{\infty} c_k v^{k*}, s_1^{2j-1*} \mid c_k \in \mathbb{R}, j \geq 1 \} & \text{for } n = 1. \end{cases} \]

The canonical IBL-operations can be written as
\[ q_{210}(s^2 \psi_1 \otimes \psi_2) = -\sum \varepsilon(\omega \mapsto \omega^1 \omega^2)[(-1)^{(n-1)|\omega^1|}\psi_1(e\omega^1) \psi_2(e_1 \omega^2) + (-1)^{|\omega|}\psi_1(e_1 \omega^1) \psi_2(e\omega^2)], \]
\[ q_{120}(s\psi)(s^2 \omega_1 \otimes \omega_2) = -\frac{1}{2} \sum \varepsilon(\omega_1 \mapsto \omega^1_1)\varepsilon(\omega_2 \mapsto \omega^2_2)[(-1)^{(n-1)|\omega^1_1|}\psi(e\omega^1_1 e\omega^2_1) + (-1)^{|\omega^1|}\psi(e_1 \omega^1_1 e_1 \omega^2_1)]] \]
for all \( \psi, \psi_1, \psi_2 \in \hat{B}_{\text{cyc}}^* \mathcal{H} \) and generating words \( \omega, \omega_1, \omega_2 \in B_{\text{cyc}}^* \mathcal{H} \). For all \( k \),
\( k_1, k_2 \geq 1 \), we have

\[
q_{210}((sv^{k_1 *}) \cdot (sv^{k_2 *})) = 0 \quad \text{and} \quad q_{120}(sv^{k_*}) = 0
\]

because both \( q_{210} \) and \( q_{120} \) feed 1 into their inputs. For the canonically twisted reduced IBL-algebra, this implies the following:

\[
\text{IBL}(\mathbb{H}^m(C_{\text{red}})) = (\mathbb{H}^m(C_{\text{red}}), q_{210} \equiv 0, q_{120} \equiv 0) \quad \text{for all } n \in \mathbb{N}.
\]

By Proposition 2.51, the only possibly non-zero relation of IBL(\( \mathbb{H}^m(C) \)) is

\[
q_{210}(s1^* \otimes sv^{k_*})
\]

\[
= (-1)^{n-2} s(v^{k_*} \circ \iota_v)
\]

\[
= (-1)^{n-2} (\sum_{i=1}^{k-1} (-1)^i v^{k-1*}) = \begin{cases} 
-(k-1)v^{k-1*} & \text{for odd } n, \\
0 & \text{for even } n.
\end{cases}
\]

The reason for 0 for even \( n \) is that either \( k \) is odd, in which case \( \sum_{i=1}^{k-1} (-1)^i = 0 \), or \( k \) is even, in which case \( v^{k_*} = 0 \). Therefore, for the canonically twisted IBL-algebra, we have

\[
\text{IBL}(\mathbb{H}^m(C)) = (\mathbb{H}^m(C), q_{210}, q_{120} \equiv 0) \quad \text{for all } n \in \mathbb{N},
\]

where \( \mathbb{H}^m(C) \) is given by (85) and \( q_{210} \) satisfies the following:

\( n \) even: \( q_{210} \equiv 0 \).

\( n \geq 3 \) odd: The non-trivial relations are

\[
q_{210}(s1^* \otimes sv^{k_*}) = q_{210}(sv^{k_*} \otimes s1^*) = -(k-1)v^{k-1*} \quad \text{for } k \geq 2.
\]

\( n = 1 \): The non-trivial relations are

\[
q_{210} \left( s1^* \otimes s \sum_{k=1}^{\infty} c_k v^{k_*} \right) = -s \sum_{k=1}^{\infty} k c_{k+1} v^{k_*} \quad \text{for } c_k \in \mathbb{R}.
\]

Recall that the twist by \( m \) does not produce any higher operation \( q_{119}^m \).

We will now consider \( \text{dIBL}^n_C(\mathcal{H}(S^n)) \). Recall that \( q_{110}^n = q_{210} \circ_1 n_{10} \), \( q_{210}^n = q_{210}^n \) and \( q_{120}^n = q_{120} + q_{210} \circ_1 n_{20} \). By Proposition 4.21, we have \( n_{10} = m_{10} \) for all \( n \in \mathbb{N} \) and \( n_{20} = 0 \) for all \( n \geq 2 \). It follows that \( q_{110}^n = q_{110}^m \) for all \( n \in \mathbb{N} \) and that the only non-trivial twist may occur in \( q_{120}^m \) for \( S^3 \). Using (37), we get
for all $\psi \in \tilde{B}_C^{\text{cy}}(\mathbb{S}^n)$ and generating words $\omega_1, \omega_2 \in B_C^{\text{cy}}(\mathbb{S}^n)$ the following:

$$(q_{210} \circ 1_n)(s\omega) (86)$$

\[\left.\begin{array}{l}
(q_{210} \circ 1_n)(s\omega) (86) \\
= (-1)^{n-2} \sum e(\omega_1 \mapsto \omega_1^2 \omega_2) \psi(1\omega_1^2) n_20(s\omega \omega_2^2 \otimes s\omega_2) \\
+ (-1)^{(n-3+|\omega|)(n-3+|\omega|/2)} \sum e(\omega_2 \mapsto \omega_2^2 \omega_2^2) \psi(1\omega_2^2) n_20(s\omega \omega_2^2 \otimes s\omega_2) \\
\end{array}\right]_{(86)}.
\]

In this paragraph, we suppose that $n = 1$ and compute $q_{120}^n$. Clearly, $(q_{210} \circ 1_n)(s\omega k) = 0$ for all $k \geq 1$ since $1$ is fed into $v^k$. A non-zero evaluation of $(q_{210} \circ 1_n)(s\omega k)$ for some $k \geq 1$ odd is possible only on $s_1^{(k-1)v^{k1}} \otimes sv^{k2}$ for $k_1, k_2 \geq 0$ (up to a transposition of arguments and their cyclic permutation). If $k > 1$, only the first summand of $(86)$ contributes, and we get

$$(q_{210} \circ 1_n)(s\omega k) (86)$$

\[\left.\begin{array}{l}
(q_{210} \circ 1_n)(s\omega k) (86) \\
= (-1)^{n-2} e(1^{k-1} v^{k1} \mapsto \omega_1 \omega_2) \psi(1\omega_1^2) n_20(s\omega \omega_2 \otimes sv^{k2}) \\
= (-1)^{n-2} k^* (11^{k-1}) n_20(sv^{k1} \otimes sv^{k2}) \\
= -n_20(sv^{k1} \otimes sv^{k2}).
\end{array}\right]
\]

According to Proposition 4.21, this is non-zero if and only if $k_1 + k_2$ is odd. It follows that

$$q_{120}^n \neq q_{120}^n = q_{120}^n$$

on the chain level for $\mathbb{S}^1$.

However, the chains $s_1^{(k-1)v^{k1}} \otimes sv^{k2}$ for $k > 1$ do not survive to the homology (c.f., 85). The only possibility is thus $k = 1$. In this case, both summands of $(86)$ contribute, and using $(84)$, we get for all $k_1, k_2 \geq 1$ the following:

$$(q_{210} \circ 1_n)(s\omega k) (86)$$

\[\left.\begin{array}{l}
(q_{210} \circ 1_n)(s\omega k) (86) \\
= (-1)^{n-2} e(v^{k1} \mapsto v^{0v^{k1}}) \psi(1) n_20(sv^{k1} \otimes sv^{k2}) \\
+ (-1)^{(n-3+k_1(n-1))(n-3+k_2(n-1))} \sum e(v^{k2} \mapsto v^{0v^{k2}}) \psi(1) \\
\end{array}\right]_{(86)}
\]

\[\left.\begin{array}{l}
= (-1)^{n-2} k_1 n_{20}(sv^{k1} \otimes sv^{k2}) - k_2 n_{20}(sv^{k2} \otimes sv^{k1}) \\
= -\frac{1}{2} (k_1 + k_2 + 1) ! I(k_1 + k_2 + 1) \left[ (-1)^{k_1} k_1 (k_1 + 1) \binom{k_1 + k_2}{k_1 + 1} + (-1)^{k_2} k_2 (k_2 + 1) \binom{k_1 + k_2}{k_2 + 1} \right] \\
= -\frac{1}{2} (k_1 + k_2 + 1) ! k_1 k_2 \binom{k_1 + k_2}{k_1} I(k_1 + k_2 + 1) \left[ (-1)^{k_1} + (-1)^{k_2} \right].
\end{array}\right]
\]

Denoting $k := k_1 + k_2 + 1$, we have that $(-1)^{k_1} + (-1)^{k_2} = 0$ for $k$ even and
$I(k) = 0$ for $k$ odd. Therefore, $(\ast) = 0$ for any $k_1, k_2 \geq 1$. This implies that

$$q^n_{120} = q^n_{1120} = q_{120}$$
onumber

on the homology for $S^1$.

We conclude that the twisted IBL-algebra satisfies

$$\text{IBL}(H^n(C(H(S^n)))) = \text{IBL}(H^n(C(H(S^n)))) \quad \text{for all } n \in \mathbb{N}.$$ 

As for the higher twisted operations, combining Proposition 2.24 and Proposition 4.21, we see that for $S^n$ with $n \in \mathbb{N}\setminus\{2\}$ all higher operations $q^n_{1l}$ vanish already on the chain level. For $n = 2$, we have that $q^n_{1l} = 0$ for all $l \geq 3$ and $q^n_{111} = 0$ on the chain level. However, we did not prove that all higher operations vanish on the chain level. As for the operations induced on the homology, the graded vector space $H^n(C(H(S^n)))$ is concentrated in even degrees and $q^n_{1l}$ are odd (see Definition 2.17). Therefore, all higher operations vanish also on $H^n(C(H(S^n)))$.

The string topology $H^*_s(S^n)$ and the string operations $m_2$ and $c_2$ were computed in [4] for all $n \in \mathbb{N}$. We review their results and basic ideas below:

We will consider even spheres first. The minimal model for the Borel construction $L^s_\mathbb{R}S^{2m}$ for $m \in \mathbb{N}$ is denoted by $\Lambda^s(2, m)$ — it is the free graded commutative dga (=cdga) over $\mathbb{R}$ generated by homogenous vectors $x_1, y_1, x_2, y_2, u$ of degrees

$$|x_1| = 2m, \quad |y_1| = 2m - 1, \quad |x_2| = 4m - 1, \quad |y_2| = 2(2m - 1), \quad |u| = 2,$$

whose differential $d$ satisfies

$$dy_1 = 0, \quad dx_1 = y_1u, \quad dy_2 = -2x_1y_1, \quad dx_2 = x_1^2 + y_2u.$$ 

The minimal model for the loop space $L^s\mathbb{R}S^{2m}$ is the dga $\Lambda(2, m)$ which is obtained from $\Lambda^s(2, m)$ by setting $u = 0$. A computation (see [4] Theorem 3.6) gives the following for all $m \in \mathbb{N}$:

$$H^*(L^s\mathbb{R}S^{2m}; \mathbb{R}) \cong H_*(\Lambda(2, m), d) = \langle y^2_1x_1 - 2iy_1x_2y^i_2, 1 \mid i, j \in \mathbb{N}_0 \rangle,$$

$$H^*_s(L^s\mathbb{R}S^{2m}; \mathbb{R}) \cong H_*(\Lambda^s(2, m), d) = \langle y^2_1, u^j \mid i, j \in \mathbb{N}_0 \rangle,$$

where $y^0_2 := u^0 := 1$ is the unit in $\Lambda^s(2, m)$ and $\langle \cdot \rangle$ denotes the linear span over $\mathbb{R}$.

Clearly, the cohomology groups are degree-wise finite-dimensional, and hence, using the universal coefficient theorem, they are isomorphic to the corresponding homology groups. We can thus identify $H_*(L^s\mathbb{R}S^{2m}; \mathbb{R})$ and $H^*_s(L^s\mathbb{R}S^{2m}; \mathbb{R})$ with the vector spaces on the right hand side of (87). We have $H^*_s_k = \langle u^k \rangle$ for all $k \in \mathbb{N}_0$, and hence the multiplication with $u$ induces an isomorphism $H^*_s_{2k} \cong H^*_s_{2k+2}$. This corresponds to the cap product with the Euler class in (87), and exactness of the
sequence implies $M(H^k_{2k}) = E(H^k_{2k}) = 0$. Using this and degree considerations, we get $m_2 = c_2 = 0$.

We will now consider odd spheres with $n ≥ 3$. The minimal model for $L_{S^1}S^{2m+1}$ for $m ∈ N$ is denoted simply by $Λ(x, y, u)$ — it is the free cdga on homogenous vectors $x, y, u$ of degrees

$$|x| = 2m + 1, \quad |y| = 2m, \quad |u| = 2,$$

such that

$$dx = yu, \quad dy = du = 0.$$ We get immediately

$$H^*(L_{S^1}^{2m+1}; R) ≃ ⟨x^i, y^j | i, j ∈ N⟩,$$

$$H_{S^1}^*(L_{S^1}^{2m+1}; R) ≃ ⟨y^i, u^j, α^k | i, j ∈ N, k ∈ Z\{0\}⟩,$$

and we can again identify $H^*_*$ and $H_{S^1}^*$ with the vector spaces on the right hand side. Clearly, $H^*_{2k-1} = 0$ for all $k ∈ N$, and hence $m_2 = c_2 = 0$ for degree reasons (the operations are odd).

We will now consider the circle $S^1$. For every $i ∈ Z$, let $α_i : S^1 → S^1$ and $θ_i : S^1 → L_{S^1}$ be the maps defined by

$$α_i(z) := z^i \quad \text{and} \quad θ_i(w) := wα_i \quad \text{for all } w, z ∈ S^1 ⊂ C.$$ By examining the equivariant homology of connected components of $L_{S^1}$ containing $α_i$ separately as in [4, Section 2.1.4], we get

$$H_*(L_{S^1}; R) = ⟨α_i, θ_j | i, j ∈ Z⟩,$$

$$H_{S^1}^*(L_{S^1}; R) = ⟨u^i, θ_0u^j, α_k | i, j ∈ N, k ∈ Z\{0\}⟩,$$

where $u$ corresponds to the Euler class and

$$|u| = 2, \quad |θ_i| = 1, \quad |α_i| = 0$$

are the degrees in the singular chain complex. On [4, p. 21] they show that the string cobracket $c_2$ is 0 and that all non-trivial relations for the string bracket $m_2 : H^S^1(L_{S^1})[2] ⊗ 2 → H^S^1(L_{S^1})[2]$ are the following:

$$m_2(σ_{α_k}, σ_{α_{-k}}) = k^2sθ_0 \quad ∀k ∈ N.$$ We will now compare the reduced IBL-structures motivated by Conjecture 3.33. The point-reduced versions $H^*_n ∨ (L_{S^n})$ for $n ≥ 2$ are obtained from $H^*_n(L_{S^n})$ by deleting $w^i$. We have the following isomorphisms of graded vector
spaces:

\[
H^p_\ast(C_{\text{red}}(H(S^n)))[1] \to H^{3,\text{red}}_\ast(LS^n)[3-n]
\]

\[
sv^i \mapsto sy^i \quad \text{for } n > 1 \text{ odd,}
\]

\[
sv^{2i+1} \mapsto sy_{1y_2}^i \quad \text{for } n \text{ even.}
\]

Because all operations are trivial, it induces the isomorphism

\[
\text{IBL}(H^p_\ast(C_{\text{red}}(H(S^n)))) \simeq \text{IBL}(H^{3,\text{red}}_\ast(LS^n)[2-n]) \quad \text{for } n \geq 2.
\]

For \( n = 1 \), the reduced homology is seemingly different.

**Remark 4.22 (Triviality for degree reasons).** Both \( H^p_\ast(LS^{2m-1})[3-n] \) and \( H^p_\ast(C(H(S^{2m})))[1] \) are concentrated in even degrees, and hence any IBL\(_\infty\)-structure must be trivial for degree reasons. On the other hand, \( H^p_\ast(LS^{2m}) \) and \( H^p_\ast(C(H(S^{2m-1})))[1] \) have both even and odd degrees, and hence an additional argument is needed to prove vanishing of the IBL-structure. This is not the case of the reduced homology, which is again concentrated in even degree. ◄

### 4.4 Twisted IBL\(_\infty\)-structure for \( \mathbb{C}P^n \)

Let \( K \in \Omega^2(\mathbb{C}P^n) \) be the Fubini–Study Kähler form on \( \mathbb{C}P^n \) (see [17, Examples 3.1.9]). The powers of \( K \) are harmonic\(^{12}\) and we get easily

\[
H(\mathbb{C}P^n) = \langle 1, K, \ldots, K^n \rangle.
\]

We denote the Riemannian volume of \( \mathbb{C}P^n \) by

\[
V := \frac{1}{\int_{\mathbb{C}P^n} K^n}.
\]

Consider the basis \( e_0, \ldots, e_n \) of \( H(\mathbb{C}P^n)[1] \) defined for all \( i = 0, \ldots, n \) by

\[
e_i := \frac{k^i}{(n!)^\frac{1}{2}}, \quad \text{where} \quad k^i := \theta K^i.
\]

The matrix of the pairing \( \mathcal{P} \) from (3) with respect to the basis \( e_0, \ldots, e_n \) reads:

\[
(\mathcal{P}^{ij}) = \begin{pmatrix}
0 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 0
\end{pmatrix}
\]

---

\(^{12}\)This follows by induction on the power of \( K \) using the fact that, on a general Kähler manifold \( M \), the Lefschetz operator \( L : \Omega(M) \to \Omega(M) \) defined by \( L(\eta) := \eta \wedge K \) for all \( \eta \in \Omega(M) \) commutes with the Hodge–de Rham Laplacian \( \Delta \) (see [17, Chapter 3]).
The basis \( e^0, \ldots, e^n \) dual to \( e_0, \ldots, e_n \) with respect to \( \mathcal{P} \) thus satisfies

\[ e^i = e_{n-i} \quad \text{for all } i = 0, \ldots, n. \]

Therefore, the following holds for the matrix \((T^{ij})\) from (33):

\[ (T^{ij}) = -(\mathcal{P}^{ij}). \]

For all \( 1 \leq i, j, k \leq n \), we have

\[ \mu_2(e_i, e_j) = e_{i+j} \quad \text{and} \quad m_{10}(se_i e_j e_k) = \delta_{i+j+k,0}. \]

For \( \psi, \psi_1, \psi_2 \in \hat{\mathcal{B}}_{cyc,0} \mathcal{H} \) and generating words \( \omega, \omega_1, \omega_2 \in \mathcal{B}_{cyc,0} \mathcal{H} \), we have

\[ q_{210}(s^2 \psi_1 \otimes \psi_2)(s \omega) = -\sum_{i=0}^{n} \sum_{j=0}^{n} \varepsilon(\omega \mapsto \omega^i \omega^j)(-1)^{|\omega^i|} \psi_1(\varepsilon(\omega^1) \psi_2(\varepsilon(\omega^2) (s \omega)^{n-i} \omega^2)), \]

\[ q_{120}(s \psi)(s^2 \omega_1 \otimes \omega_2) = -\sum_{i=0}^{n} \sum_{j=0}^{n} \varepsilon(\omega_1 \mapsto \omega_1^i) \varepsilon(\omega_2 \mapsto \omega_2^j)(-1)^{|\omega^i|} \psi(\varepsilon(\omega_1^1) \psi_1(\varepsilon(\omega_2^1) \psi_2(\varepsilon(\omega_2^2) (s \omega)^{n-i} \omega^2)). \]

The cyclic homology of \( \mathcal{H}(\mathbb{C}P^n) \) is that of the truncated polynomial algebra

\[ A := \mathbb{R}[x]/(x^{n+1}) \quad \text{with} \quad \deg(x) = 2. \]

The computation of \( H^\lambda_{cyc}(A) \) for \(|x| = 0\) over a field is the goal of [24, Exercise 4.1.8.] or [24, Exercise 9.1.1]. The case of \(|x| = d\) can be solved by taking suitable degree shifts in the proposed projective resolution which is used to compute \( \text{HH}(A) \). Unfortunately, using a non-canonical projective resolution, we lose the concrete form of the cyclic cycles and obtain just the following result (the full computation will be provided in [19]):

For all \( i = 1, \ldots, n \) and \( k \in \mathbb{N}_0 \), there are cycles \( \tilde{t}_{2k+1,i} \in \tilde{D}_i(A) \) of weights \( 2k + 1 \) and degrees \( d(i + (n + 1)k) \) which form a basis of \( H^\lambda_{cyc}(A) \). We apply the degree shift \( U : \tilde{D}_n(A) \to D_n(A) \) from Proposition 2.39 to get the generators

\[ t_{w,i} := U(\tilde{t}_{w,i}) \in D^\lambda_n(\mathcal{H}(\mathbb{C}P^n)) \]

of weights \( w \) and degrees \( 2i + (w - 1)n - 1 \), so that

\[ H^\lambda(\mathcal{H}(\mathbb{C}P^n)) = \langle t_{w,i}, 1^w \mid w \in \text{odd}, i = 1, \ldots, n \rangle. \]

By the universal coefficient theorem we have \( H^\lambda = (H^\lambda)^\vee \) with respect to the grading by the degree. Given \( d \in \mathbb{Z} \), the equation \( d = 2i + (w - 1)n - 1 \) has only finitely many solution \( (w, i) \in \mathbb{N} \times \{1, \ldots, n\} \), and hence we get

\[ \mathbb{H}_{\mathcal{H}}^n(C(\mathcal{H}(\mathbb{C}P^n))) = \langle s^w t_{w,i}, s^w t_{w,i}^\vee \mid w \in \text{odd}, i = 1, \ldots, n \rangle, \quad (88) \]
where $t^*_{w,i}$ and $w^* \in B_{cy}^r \mathcal{H}$ are the duals to $t_{w,i}$ and $w$, respectively (see Remark 2.41). Notice that both $|s^*_{w,i}|$ and $|s^w|$ are even since $|s| = 2n - 3$.

Because $\mathbb{C}P^n$ is geometrically formal, Proposition 3.29 implies that $m_{10} = m_{10}$. Because $\mathbb{H}^m(C)$ is concentrated in even degrees and because a general IBL$_\infty$-operation $q_{k+l}$ is odd, all operations vanish on the homology. Therefore, for the twisted IBL-algebras we have

$$\text{IBL}(\mathbb{H}^m(C)) = \text{IBL}(\mathbb{H}^m(C)) = (\mathbb{H}^m(C), q_{210} \equiv 0, q_{120} \equiv 0),$$

where $\mathbb{H}^m(C)$ is given by (88).

According to [4, Section 3.1.2], the minimal model for the Borel construction $L_0: \mathbb{C}P^n$ is the cdga $\Lambda S_1^{n+1}$, which is freely generated (over $\mathbb{R}$) by the homogenous vectors $x_1, x_2, y_1, y_2, u$ of degrees

$$|x_1| = 2, \quad |x_2| = 2n + 1, \quad |y_1| = 1, \quad |y_2| = 2n, \quad |u| = 2,$$

and whose differential $d$ satisfies

$$dy_1 = 0, \quad dx_1 = y_1 u, \quad dy_2 = -(n + 1)x_1^ny_1, \quad dx_2 = x_1^{n+1} + y_2 u.$$

By [4] Theorem 3.6, the string cohomology $H^{0}_{*}(L_0 C\mathbb{P}^n; \mathbb{R}) \simeq H_*(\Lambda S_1^{n+1}, d)$ satisfies for all $m \in \mathbb{N}_0$ the following:

$$H^{0}_{*}(L_0 C\mathbb{P}^n; \mathbb{R}) = \begin{cases} \langle u^p \rangle & \text{if } m = 2j, \\ \langle y_1 y_2^p x_1^q \rangle & \text{if } m = 2j + 1. \end{cases}$$

The right-hand side can be identified with $H^{0}_{*}(L_0 C\mathbb{P}^n; \mathbb{R})$ by the universal coefficient theorem. According to [4] Proposition 3.7, we have $m_2 = 0$ and $c_2 = 0$. We conclude that the map

$$H^{0}_{*}(C_{red}(H(\mathbb{C}P^n)))[1] \longrightarrow H^{0}_{*}(L_0 C\mathbb{P}^n; \mathbb{R})[3 - n]$$

$$s_{2k+1} \longrightarrow \text{sgn} y_2^k x_1^{l-1}$$

for $k \geq 0$ and $l = 1, \ldots, n$

induces an isomorphism of IBL-algebras

$$\text{IBL}(H^{0}_{*}(C_{red}(H(\mathbb{C}P^n)))) \simeq \text{IBL}(H^{0}_{*}(L_0 C\mathbb{P}^n; \mathbb{R})[3 - n]).$$
A Evaluation of labeled ribbon graphs

In this appendix, we define the propagator $P$ and the graph pairing $\langle \cdot, \cdot \rangle^P_\Gamma$ (Definition A.1), which encapsulates the contribution of a ribbon graph $\Gamma$ to the map $f_{klg} : (B^*\text{cyc} V)^{\otimes k} \to (B^*\text{cyc} V)^{\otimes l}$ defined as a sum of contributions of ribbon graphs (Proposition A.2). Such maps were already defined in [10, Section 11] using coordinates; here we use an invariant framework inspired by [29]. As an example, we work out in details expressions for the canonical dIBL-operations $q_{210}$ and $q_{120}$ (Example A.5). We also explain the technicality of identifying symmetric maps with maps on symmetric powers (Remark A.3).

Next, we define the notion of an algebraic Schwartz kernel (Definition A.4) and show that the matrix $(T_{ij})$ from Definition 2.42 corresponds to the Schwartz kernel of the identity $1$ up to a sign. Assuming that the Green kernel $G$ from Definition 3.5 is algebraic, we deduce the signs in Definition 3.19 using the formula from [10, Remark 12.10] for the genuine pushforward Maurer-Cartan element $n$ in the finite-dimensional case. Establishing the formal analogy between the de Rham case and the finite-dimensional case is our main application of the invariant framework. Finally, we sketch how to obtain signs for the Fréchet dIBL-structure on $\Omega(M)$ (Remark A.7).

Throughout this appendix, we will use Notation 2.28 without further remarks.

**Definition A.1 (Propagator & graph pairing).** Let $V$ be a graded vector space. The tensor $P \in V[1]^{\otimes 2}$ is called a **propagator** if it satisfies the following symmetry condition:

$$\tau(P) = (-1)^{|P|} P.$$  

(89)

The map $\tau$ is the twist map defined by $\tau(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} v_2 \otimes v_1$ for all $v_1, v_2 \in V[1]$.

For a ribbon graph $\Gamma \in \mathbb{RG}_{klg}$ and its labeling $L$, consider the permutation $\sigma_L$ from Definition 3.18. It acts on tensor powers according to Definition 2.6 and thus defines the map

$$\sigma_L : (V[1]^{\otimes 2})^{\otimes e} \otimes V[1]^{\otimes s_1} \otimes \cdots \otimes V[1]^{\otimes s_k} \rightarrow V[1]^{\otimes d_1} \otimes \cdots \otimes V[1]^{\otimes d_k},$$

where $d_i$ and $s_i$ are the valencies of internal vertices $1, \ldots, k$ and boundary components $1, \ldots, l$, respectively, and $e$ is the number of internal edges. We extend $\sigma_L$ by $0$ to other combinations of tensor powers. The graph pairing

$$\langle \cdot, \cdot \rangle^P_\Gamma : (B^*\text{cyc} V)^{\otimes k} \otimes (B^*\text{cyc} V)^{\otimes l} \rightarrow \mathbb{R}$$

is defined for all $\psi_1, \ldots, \psi_k \in B^*\text{cyc} V$ and generating words $w_i = v_{i_1} \cdots v_{i_m}$,
with \( v_{ij} \in V[1] \) for \( m_i \in \mathbb{N} \) and \( i = 1, \ldots, l \) by the following formula:

\[
\langle \psi_1 \otimes \cdots \otimes \psi_k, w_1 \otimes \cdots \otimes w_l \rangle_{\Gamma}^P := \sum_{L_1, L_3} \langle \psi_1 \otimes \cdots \otimes \psi_k, \sigma_L(P \otimes \epsilon \otimes \Pi) \otimes (v_{11} \otimes \cdots \otimes v_{1m_1}) \otimes \cdots \otimes (v_{l1} \otimes \cdots \otimes v_{lm_l}) \rangle_{\Gamma},
\]

where we use the pairing from Definition 2.29 and in every summand an \( L_2 \) compatible with \( L_1 \) and an \( L_3 \) are chosen arbitrarily to get a full labeling \( L \) of \( \Gamma \).

The graph pairing extends to \( \langle \cdot, \cdot \rangle_{\Gamma}^P : \hat{T}B^*_{\text{cyc}} V \otimes \hat{T}B^*_{\text{cyc}} V \to \mathbb{R} \).

**Proposition A.2.** In the setting of Definition A.1, we denote \( w = w_1 \otimes \cdots \otimes w_l \) and \( \psi = \psi_1 \otimes \cdots \otimes \psi_k \) and have the following:

(a) The number \( \psi(\sigma_L(P \otimes \epsilon \otimes w)) \) does not depend on the choice of \( L_3 \) and an \( L_2 \) compatible with \( L_1 \). Moreover, \( \langle \cdot, \cdot \rangle_{\Gamma}^P \) does not depend on the representative of \( [\Gamma] \in \mathcal{R}G_{klg} \).

(b) If \( V \) is finite-dimensional, then for every \( k, l \geq 1, g \geq 0 \) there is a unique linear map

\[
f_{klg} : (B^*_{\text{cyc}} V)^{\otimes k} \to (B^*_{\text{cyc}} V)^{\otimes l}
\]

such that

\[
f_{klg}(\psi_1 \otimes \cdots \otimes \psi_k)(w_1 \otimes \cdots \otimes w_l) = \frac{1}{l!} \sum_{[\Gamma] \in \mathcal{R}G_{klg}} \frac{1}{|\text{Aut}(\Gamma)|} \langle \psi_1 \otimes \cdots \otimes \psi_k, w_1 \otimes \cdots \otimes w_l \rangle_{\Gamma}^P.
\]

(c) The following holds for the map \( f_{klg} \) from b):

- It is homogenous of degree

\[
|f_{klg}| = -|P|(k + l - 2 + 2g). \quad (90)
\]

- The filtration degree satisfies

\[
\|f_{klg}\| \geq -2(k + l - 2 + 2g). \quad (91)
\]

- For all \( \eta \in S_k \) and \( \mu \in S_k \), we have

\[
\eta \circ f_{klg} \circ \mu = (-1)^{|P|(|\eta + \mu|)} f_{klg}. \quad (92)
\]

**Proof.** (a) Let us denote by \( \tilde{i} \) and \( ij \) the operations on \( L_2 \) given by \( e_i \mapsto -e_i \) and \( e_i \leftrightarrow e_j \), respectively. An even number of these operations does not change
the orientation of $[57]$. Their effect in $\sigma_L$ acting on $P^{\otimes e} \otimes w$ is

$$\tilde{i} : P_i \mapsto \tau(P_i) = (-1)^{|P|} P_i \quad \text{and} \quad ij : P_i \ldots P_j \mapsto (-1)^{|P|} P_j \ldots P_i.$$ 

Therefore, an even number of them does not change $\sigma_L(P^{\otimes e} \otimes w)$. This proves the independence of the choice of a compatible $L_2$. The independence of the choice of $L_2^\kappa$ is clear since $\psi_i$ are cyclic symmetric.

An isomorphism of ribbon graphs $\eta : \Gamma \to \Gamma'$ induces the map of compatible labelings $L \mapsto L' = \eta^* L$ such that $\sigma_L = \sigma_{L'}$. The independence of the choice of a representative of $[\Gamma]$ follows.

(b) Suppose that $\psi = \psi_1 \otimes \cdots \otimes \psi_k$ with $\psi_i \in (B^*_\cyc V)^{e_i}$, where $e_i \in \mathbb{N}$ and $c_i \in \mathbb{Z}$ for $i = 1, \ldots, k$. A general element of $(B^*_\cyc V)^{\otimes k}$ is then a finite linear combination of such $\psi$'s.

First of all, let us argue that the sum $\sum_{[\Gamma] \in RG\ell g}$ is finite. The number of internal edges $e$ is fixed from (56). Therefore, the number of contributing graphs $(V_{int}, E_{int})$ is finite. In order to bound the number of external vertices, we notice that $d_1 = r_1, \ldots, d_k = r_k$ must hold for $\psi(\sigma_L(P^{\otimes e} \otimes w))$ to be non-zero. Therefore, the sum is finite.

We now have the linear functional

$$f_{klg}(\psi) := \frac{1}{\Gamma} \sum_{[\Gamma] \in RG\ell g} \frac{1}{|\Aut(\Gamma)|} \langle \psi | \cdot \rangle^P : (B^*_\cyc V)^{\otimes l} \to \mathbb{R}$$

and need to show that $f_{klg}(\psi) \in (B_{\cyc}^* V)^{\otimes l} \subset (B_{\cyc}^* V)^{\otimes l*}$. Because $V$ is finite-dimensional, the weight-filtration of $B^*_\cyc V$ satisfies (WG1) & (WG2) (see (13) and Proposition 2.31), and hence we have

$$(B^*_\cyc V)^{\otimes l} = (B^*_\cyc V)^{\mu \otimes l} = ((B^*_\cyc V)^{\otimes l})''$$

for the weight-graded duals. Therefore, it suffices to show that $f_{klg}(\psi)$ vanishes on all but finitely many degrees and weights of $(B^*_\cyc V)^{\otimes k}$. However, the relation $f_{klg}(\psi)(w) \neq 0$ for a generating word $w \in (B^*_\cyc V)^{\otimes k}$ implies

$$|w| = |\psi| - e|P| \quad \text{and} \quad k(w) = k(\psi) - 2e,$$

where $k$ denotes the weight, and hence $f_{klg}(\psi) \in (B^*_\cyc V)^{\otimes l}$ indeed holds.

(c) The formulas (90) and (91) follow from (93) and (56).

As for the symmetry (92), suppose that $L$ and $L'$ are compatible labelings of the same graph $\Gamma$ such that $L_1'$ differs from $L_1$ by a permutation $\mu \in S_k$ of internal vertices and a permutation $\eta \in S_l$ of boundary components. Viewing $\mu$
\[ \sigma_L'(P^{\otimes e} \otimes w) = (-1)^{|P|} \mu \sigma_L(P^{\otimes e} \otimes \eta(w)). \]

The sign comes from the difference of \( L_2 \) and \( L'_2 \) which compensates the change of the orientation of \((57)\) caused by \( \mu \) and \( \eta \).

Given \( \mu \in S_k \) and \( \psi = \psi_1 \otimes \cdots \otimes \psi_k \in (B^*_{\text{cyc}} V)^{\otimes k} \), it is easy to see that

\[ \varepsilon(\mu, \Psi) = \varepsilon(\mu(s), \mu(\psi))\varepsilon(s, \psi)\varepsilon(\mu, \psi), \]

where \( \Psi = (s\psi_1) \otimes \cdots \otimes (s\psi_k) \in (B^*_{\text{cyc}} V[A])^{\otimes k} \) and \( \varepsilon(\mu, s) = (-1)^{|s|} \). If \( A = -|P| \), then we get from \((92)\) that the degree shift \( f_{klg} : (B^*_{\text{cyc}} V[A])^{\otimes k} \to (B^*_{\text{cyc}} V[A])^{\otimes l} \) has the following symmetries:

\[ \forall \mu \in S_k, \eta \in S_l : \; \eta \circ f_{klg} \circ \mu = f_{klg}. \quad (94) \]

Note that the degrees satisfy

\[ |f_{klg}| = |f_{klg}| + (k - l)A. \quad (95) \]

**Remark A.3 (Symmetric maps versus maps on symmetric powers).** In the situation above, we define \( \tilde{f}_{klg} \) as the unique map such that the solid lines of the following diagram commute:

\[ \begin{array}{ccc}
(B^*_{\text{cyc}} V[A])^{\otimes k} & \xrightarrow{f_{klg}} & (B^*_{\text{cyc}} V[A])^{\otimes l} \\
\downarrow \pi & & \downarrow \pi \\
S_k B^*_{\text{cyc}} V[A] & \xrightarrow{\tilde{f}_{klg}} & S_l B^*_{\text{cyc}} V[A].
\end{array} \]

The symmetry condition \((94)\) provides the existence of \( \tilde{f}_{klg} \) and implies commutativity of the dotted diagram as well. Moreover, for all \( \psi_1, \ldots, \psi_k \in B^*_{\text{cyc}} V \) and \( w_1, \ldots, w_l \in B^*_{\text{cyc}} V \), we have

\[ \tilde{f}_{klg}(s^k \psi_1 \cdots \psi_k)(s^l w_1 \cdots w_l) = f_{klg}(s^k \psi_1 \otimes \cdots \otimes \psi_k)(s^l w_1 \otimes \cdots \otimes w_l), \]

where we use the pairing from Definition 2.29. We denote \( \tilde{f}_{klg} \) again by \( f_{klg} \). \( \triangleq \)

**Definition A.4 (Algebraic Schwartz kernel).** Let \( V \) be a graded vector space and \( \mathcal{P} : V \otimes V \to \mathbb{R} \) a non-degenerate pairing on \( V \). We extend \( \mathcal{P} \) to a non-degenerate pairing \( \mathcal{P} : V^{\otimes k} \otimes V^{\otimes k} \to \mathbb{R} \) for \( k \geq 1 \) by setting

\[ \mathcal{P}(v_{11} \otimes \cdots \otimes v_{1k}, v_{21} \otimes \cdots \otimes v_{2k}) := \varepsilon(v_1, v_2)\mathcal{P}(v_{11}, v_{21}) \cdots \mathcal{P}(v_{1k}, v_{2k}) \]

for all \( v_{11}, \ldots, v_{1k}, v_{21}, \ldots, v_{2k} \in V \), where \( \varepsilon \) is the Koszul sign (see Definition 2.29).\]
tion 2.2). For \( k = 0 \), we let \( P : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R} \) be the multiplication on \( \mathbb{R} \).

For \( k, l \geq 0 \), we say that \( K_L \in V^{\otimes k+1} \) is the algebraic Schwartz kernel of a linear operator \( L : V^{\otimes k} \to V^{\otimes l} \) if the following is satisfied:

\[
\forall w_1 \in V^{\otimes k}, w_2 \in V^{\otimes l} : \quad P(L(w_1), w_2) = P(K_L, w_1 \otimes w_2). \tag{96}
\]

We usually omit writing “algebraic” if it is clear from the context (i.e., if we do not consider any “extensions” of \( V^{\otimes k} \)).

In the situation of Definition A.4, let \((e_i) \subset V\) be a basis and \((e_i^*)\) its dual basis such that \( P(e_i, e_j^*) = \delta_{ij} \). We define the coordinates \( K_{ij}^L \in \mathbb{R} \) and \( L^{ij} \in \mathbb{R} \) by

\[
K_{ij}^L = \sum_{i,j} K_{ij}^L e_i \otimes e_j \quad \text{and} \quad L^{ij} := P(L(e_i), e_j). \tag{97}
\]

From (96) we have

\[
K_{ij}^L = (-1)^{(|L|+1)(|P|+|e_i|)} L^{ij} \quad \text{for all } i, j. \tag{97}
\]

From now on, we will be in the situation of (A) and (B) in the Introduction; in particular, we put \( V[1] \) in place of \( V \) in Definition A.4. Let \( K_1 \in V[1]^{\otimes 2} \) be the Schwartz kernel of the identity \( \mathbb{I} : V[1] \to V[1] \) and \( K_G \in V[1]^{\otimes 2} \) the Schwartz kernel of the cochain homotopy \( G : V[1] \to V[1] \). From (97), we get

\[
K_{ij}^G = G^{ij} \quad \text{and} \quad K_1^{ij} = (-1)^{|e_i|+|P|} P(e_i, e_j) \quad \text{for all } i, j.
\]

We see that the tensor \( T = \sum_{i,j} T^{ij} e_i \otimes e_j \) from (63) can be expressed as

\[
T = (-1)^{n-2} K_1.
\]

This is the invariant meaning of \( T \). Note that the degrees satisfy

\[
|T| = n - 2 \quad \text{and} \quad |K_G| = n - 3.
\]

The assumption (I) on \( G \) is equivalent to graded antisymmetry of the bilinear form \( G^+ := P \circ (G \otimes \mathbb{I}) : V[1]^{\otimes 2} \to \mathbb{R} \). This is further equivalent to

\[
\tau(K_G) = (-1)^{|K_G|} K_G.
\]

Therefore, \( K_G \) satisfies (89), and hence it can be used as a propagator for the construction of \( f_{k|l|} \) for every \( k, l \geq 1, g \geq 0 \). We have from (92) that the degree shift \( f_{k|l|} : (B_{\text{cyc}} V[3-n])^{\otimes k} \to (B_{\text{cyc}}^* V[3-n])^{\otimes l} \) is symmetric. Moreover,
using (90), (91) and (95), we obtain

\[ |f_{k\ell g}| = -2d(k + g - 1), \]
\[ ||f_{k\ell g}|| \geq \gamma(2 - 2g - k - t), \]

where \((d, \gamma) = (n - 3, 2)\). These are the degree and filtration conditions on an IBL\(_{\infty}\)-morphism from [10] Definition 2.8 and (8.3)]. As a matter of fact, our \(f = (f_{k\ell g})_{k, \ell \geq 0, g \geq 1}\) is precisely the IBL\(_{\infty}\)-homotopy from [10] Theorem 11.3.

Graded antisymmetry of \(P\) is equivalent to

\[ \tau(T) = (-1)^{|T| + 1}T. \]

Visibly, \(T\) does not satisfy (89). Nevertheless, we can still use it to define \(f_{210}\) and \(f_{120}\) since the corresponding graphs \(\Gamma\) (see Figure 13) have only one internal edge \(e\), and, for a given \(L_1\), there is a unique compatible \(L_2\) determined by the orientation of \(e\) (see Example A.5 for the compatibility condition). As for the symmetry of the resulting maps, a transposition of internal vertices or boundary components in (57) can be compensated only by \(e \mapsto -e\), which produces \((-1)^{|T| + 1}\) (c.f., the proof of Proposition A.2 (a)). Therefore, if we shift the degrees by \(A = -|T| + 1 = n - 3\), we obtain symmetric maps \(q_{210} : (\mathrm{B}_c^\ast V[A]\otimes^2 \rightarrow \mathrm{B}_c^\ast V[A]\) and \(q_{120} : \mathrm{B}_c^\ast V[A] \rightarrow (\cdots \mathrm{B}_c^\ast V[A])\otimes^2\). We show in Example A.5 below that these operations agree with those defined in Definition 2.42.

**Example A.5 (The canonical dBL-operations).** We have

\[
\begin{align*}
\psi_1 \otimes \psi_2)(w) &= \frac{1}{2!} \sum_{T \in \text{RG}_{210}} \frac{1}{|\text{Aut}(T)|} \langle \psi_1 \otimes \psi_2 | w \rangle_T^P, \\
\psi_1 \otimes \psi_2)(w) &= \frac{1}{2!} \sum_{T \in \text{RG}_{120}} \frac{1}{|\text{Aut}(T)|} \langle \psi_1 \otimes \psi_2 | w \rangle_T^P.
\end{align*}
\]

(98)

We parametrize \(\text{RG}_{210}\) by the ribbon graphs \(\Gamma_{k_1, k_2}\) with \(1 \leq k_1 \leq k_2\) and \(\text{RG}_{120}\) by the ribbon graphs \(\Gamma_{s1, s2}\) with \(0 \leq s_1 \leq s_2\); these graphs are depicted in Figure 13. We have \(\text{RG}_{210} = \text{RG}_{210}\setminus\{[1_1, 1]\}\) and \(\text{RG}_{120} = \text{RG}_{120}\setminus\{[0, 0], [0, 1]\}\).

We also have

\[ |\text{Aut}(\Gamma_{k_1, k_2})| = \begin{cases} 1 & \text{if } k_1 \neq k_2, \\ 2 & \text{if } k_1 = k_2, \end{cases} \]

and likewise for \(\Gamma_{s1, s2}\). We fix labelings \(L^b_3\) and parametrize \(L^b_3\) by \(c = 1, \ldots, k_1 + k_2 - 2\) for \(\Gamma_{k_1, k_2}\) and by \(c_1 = 1, \ldots, s_1\) and \(c_2 = 1, \ldots, s_2\) for \(\Gamma_{s1, s2}\) as it is indicated in Figure 13.

There are two possible labelings \(L^a_2\) for \(\Gamma_{k_1, k_2}\) and two possible labelings \(L^a_1\) for \(\Gamma_{s1, s2}\); this is the only freedom in choosing a full labeling \(L\) because \(L_3\) is fixed and \(L_2\) is just the orientation of the single internal edge, which is uniquely
determined by \( L_1 \). For both \( \Gamma_{k_1,k_2} \) and \( \Gamma^{s_1,s_2} \), we will denote the two possible full labelings by \( L^1 \) and \( L^2 \). They can be depicted as follows:

\[
\begin{array}{c|cc}
  & \Gamma_{k_1,k_2} & \Gamma^{s_1,s_2} \\
L^1 & 1 \rightarrow 2 & 2 \rightarrow 1 \\
L^2 & 1 \rightarrow 2 & 2 \rightarrow 1 \\
\end{array}
\]

(99)

Let us check that the indicated \( L^1 \) and \( L^2 \) are compatible. For the complexes \( C_2 \rightarrow C_1 \rightarrow C_0 \) from (57), we have the following:

\[
\begin{align*}
\Gamma_{k_1,k_2} : & \quad \langle b \rangle \xrightarrow{\partial_2=0} \langle e \rangle \xrightarrow{\partial_1} \langle v_2-v_1 \rangle \oplus \langle v_1+v_2 \rangle, \\
\Gamma^{s_1,s_2} : & \quad \langle b_1-b_2 \rangle \oplus \langle b_1+b_2 \rangle \xrightarrow{\partial_2=0} \langle e \rangle \xrightarrow{\partial_1} \langle v \rangle.
\end{align*}
\]

As for \( \Gamma_{k_1,k_2} \), the basis \( v_2-v_1, v_1+v_2 \) of \( C_0 \) is positively oriented with respect to the basis \( v_2, v_1 \). Therefore, \( e \) has to be oriented such that \( \partial_1 e = v_2 - v_1 \); i.e., it is a path from \( v_1 \) to \( v_2 \). As for \( \Gamma^{s_1,s_2} \), the basis \( b_1-b_2, b_1+b_2 \) of \( C_2 \) is positively oriented with respect to \( b_1, b_2 \). Therefore, \( e \) has to be oriented such that \( e = \partial_2(b_1-b_2) \). Recall that we orient the boundary of a 2-simplex by the “outer normal first” convention. We conclude that the labelings from (99) are indeed compatible.

As for \( f_{210} \), the permutations \( \sigma_1 := \sigma_{L^1} \) and \( \sigma_2 := \sigma_{L^2} \) corresponding to the labelings \( L^1 \) and \( L^2 \), respectively, read

\[
\sigma_1 = \begin{pmatrix}
1 & 2 \\
1 & k_1+1
\end{pmatrix}
\begin{pmatrix}
\cdots & c+2 & \cdots \\
\cdots & 2 & \cdots
\end{pmatrix}
_{\overbrace{k_1+k_1-2}}
\quad \text{and} \quad
\sigma_2 = \begin{pmatrix}
1 & 2 \\
1 & k_2+1
\end{pmatrix}
\begin{pmatrix}
\cdots & c+2 & \cdots \\
\cdots & k_2+2 & \cdots
\end{pmatrix}
_{\overbrace{k_1+k_2-2}}
\]

127
We use these facts to rewrite (98) as follows:

\[ \sigma_1 : V^\otimes 2 \otimes V^\otimes s \rightarrow V^\otimes k_1 \otimes V^\otimes k_2, \quad e_i e_j w \mapsto e_i w^1 e_j w^2, \]

\[ \sigma_2 : V^\otimes 2 \otimes V^\otimes s \rightarrow V^\otimes k_2 \otimes V^\otimes k_1, \quad e_i e_j w \mapsto e_j w^2 e_i w^1, \]

where \( w^1 = w_{c+1} \cdots w_{c+k_1-2}, \ w^2 = w_{c+k_1-1} \cdots w_{c+k_1+k_2-3} \) and \( s := k_1 + k_2 - 2 \).

Defining \( \tilde{w}^1 := w^2 \) and \( \tilde{w}^2 := w^1 \), The Koszul sign of \( \sigma_2 \) can be written as

\[ \varepsilon(w \mapsto \tilde{w}^1 w^2)(-1)^{|w_1|(|e_j|+|w_i|)} = \varepsilon(w \mapsto \tilde{w}^1 w^2)(-1)^{|\tilde{w}^1| |e_j|}. \]

We use these facts to rewrite (98) as follows:

\[
f_{210}(\psi_1 \otimes \psi_2)(w) = \sum_{1 \leq k_1 < k_2, i,j} \sum_{k(w^1) = k_1 - 1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1| |e_j|} \psi_1(e_i w^1) \psi_2(e_j w^2)
+ \sum_{k(w^1) = k_2 - 1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1| |e_j|} \psi_1(e_i w^2) \psi_2(e_j w^1)
+ \sum_{1 < k_1 = k_2} \frac{1}{2} \left( \sum_{k(w^1) = k_1 - 1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1| |e_j|} \psi_1(e_i w^1) \psi_2(e_j w^2)
+ \sum_{k(w^1) = k_2 - 1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1| |e_j|} \psi_1(e_i w^2) \psi_2(e_j w^1) \right)
= \sum_{k_1, k_2 \geq 1, k(w^1) = k_1 - 1, k_1 + k_2 \geq 2} \sum_{k(w^2) = k_2 - 1} T^{ij} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1| |e_j|} \psi_1(e_i w^1) \psi_2(e_j w^2).
\]

This coincides with the formula from Definition 2.42.

As for \( f_{120} \), the permutations \( \sigma_1 := \sigma_{L^1} \) and \( \sigma_2 := \sigma_{L^2} \) corresponding to the labelings \( L^1 \) and \( L^2 \), respectively, read

\[
\sigma_1 = \begin{pmatrix}
1 & 2 & \ldots & c_1 + 2 & \ldots & c_2 + s_1 + 2 & \ldots \\
1 & s_1 + 2 & \ldots & 2 & \ldots & s_1 + 3 & \ldots \\
& s_1 & & s_2 & & & \\
\end{pmatrix}
\quad \text{and}
\sigma_2 = \begin{pmatrix}
1 & 2 & \ldots & c_2 + 2 & \ldots & c_1 + s_2 + 2 & \ldots \\
1 & s_1 + 3 & \ldots & 2 & \ldots & & \\
& s_2 & & s_1 & & & \\
\end{pmatrix},
\]

where the underbracketed blocks denote cyclic permutations of consecutive indices on the corresponding boundary component. We see that

\[
\sigma_1 : V^\otimes 2 \otimes V^\otimes s_1 \otimes V^\otimes s_2 \rightarrow V^\otimes k, \quad e_i e_j w_1 w_2 \mapsto e_i w_1^1 e_j w_2^1,
\]

\[
\sigma_2 : V^\otimes 2 \otimes V^\otimes s_2 \otimes V^\otimes s_1 \rightarrow V^\otimes k, \quad e_i e_j w_1 w_2 \mapsto e_j w_2^1 e_i w_1^1.
\]

128
where \( w_1^i \) denotes a cyclic permutation and \( k := s_1 + s_2 + 2 \). The Koszul sign of \( \sigma_2 \) can be written as

\[
(-1)^{|\varepsilon_1||\varepsilon_2|+|w_1||w_2|+|\varepsilon_1||w_2|} \varepsilon(w_1 \mapsto w_1^i)\varepsilon(w_2 \mapsto w_2^i)
\]

\[
= (-1)^{|\varepsilon_1|+|w_1||\varepsilon_j|+|w_2||\varepsilon_i|+|\varepsilon_1||w_1||\varepsilon_2||w_2|} \varepsilon(w_1 \mapsto w_1^i)\varepsilon(w_2 \mapsto w_2^i).
\]

We use this fact and the cyclic symmetry of \( \psi \) to rewrite (98) as follows:

\[
f_{120}(\psi)(w_1 \otimes w_2) = \sum_{0 \leq s_1 < s_2} \left( \delta^{k(w_1)=s_1}_{k(w_2)=s_2} \sum T^{ij}\varepsilon(w_1 \mapsto w_1^j)\varepsilon(w_2 \mapsto w_2^i)(-1)^{|w_1||\varepsilon_1|}
\]

\[
\psi(e_i w_1^i e_j w_2^j) + \delta^{(k(w_1)=s_1)}(k(w_2)=s_2) \frac{1}{2} \sum T^{ij}\varepsilon(w_1 \mapsto w_1^j)\varepsilon(w_2 \mapsto w_2^i)(-1)^{|w_1||\varepsilon_1|}
\]

\[
\psi(e_i w_1^i e_j w_2^j) + \sum T^{ij}\varepsilon(w_1 \mapsto w_1^j)\varepsilon(w_2 \mapsto w_2^i)(-1)^{|w_1||\varepsilon_1|}
\]

\[
\psi(e_i w_1^i e_j w_2^j).
\]

This coincides with the formula from Definition 2.42.

We will now establish a formal analogy between the finite-dimensional and the de Rham case, which will explain the signs in Definition 3.19.

The finite-dimensional case. Consider the situation of (A) – (D) in the Introduction. To recall briefly, we have a finite-dimensional cyclic dga \((V, \mathcal{P}, m_1, m_2)\) and a subcomplex \(\mathcal{H} \subset V\) such that there is a projection \(\pi : V[1] \to \mathcal{H}[1]\) chain homotopic to \(1\) via a chain homotopy \(\mathcal{G} : V[1] \to V[1]\). Using \(m_2\), one constructs the canonical Maurer-Cartan element \(m\) for \(dIBL(C(V))\). Using the algebraic Schwartz kernel \(K_G\) of \(\mathcal{G}\), one constructs the IBL\(_\infty\)-quasi-isomorphism \(I = (\mathcal{G}_\infty) : dIBL(C(V)) \to dIBL(C(\mathcal{H}))\). The Maurer-Cartan element \(m\) is then pushed forward along \(I\) to obtain the Maurer-Cartan element \(n := f_*m\) for \(dIBL(C(\mathcal{H}))\) (see 10 Lemma 9.5). The formula for \(n\) given in 10 Remark 12.10 reads

\[
n_{\text{d}}(s^i w_1 \otimes \cdots \otimes w_l)
\]

\[
= \frac{1}{l!} \sum_{[\Gamma] \in \text{Comm}^{(l)}_k} \frac{1}{|\text{Aut}(\Gamma)|}(1)^{k(n-2)}((m_2^+)_{[\Gamma]} \otimes w_1 \otimes \cdots \otimes w_l)^{K_{g\Gamma}}.
\]

Here the artificial sign \((-1)^{k(n-2)}\) is added because our sign conventions for \(m_2^+\) differ (see Remark 2.33).
The de Rham case. We are in the setting of Definition 3.19. To recall briefly, we have the cyclic dga \((\Omega(M), \mathcal{P}, m_1, m_2)\), the subspace of harmonic forms \(\mathcal{H} \subset \Omega\), the harmonic projection \(\pi_{\mathcal{H}} : \Omega \to \mathcal{H}\), and a Green kernel \(G \in \Omega(Bl_{\Delta}(M \times M))\), which is the Schwartz kernel of a chain homotopy \(\mathcal{G} : \Omega \to \Omega\) between \(\pi_{\mathcal{H}}\) and \(\mathbb{1}\). In analogy with the finite-dimensional case, the canonical Maurer-Cartan element for \(dIBL\) satisfies \(m_{(2)} = (-1)^{n-2}m_{(2)}^+\) with \(m_{(2)}^+ = \mathcal{P}(m_2 \otimes \mathbb{1})\). Because \(\dim(\Omega) = \infty\), Definition 2.42 does not give the canonical \(dIBL\)-structure on \(C(\Omega)\), and hence we have neither \(\mathfrak{f}\) nor \(\mathfrak{n}\) in the standard sense.

In order to deduce the formal analogy, we embed \(\Omega(M)^{\otimes 2}\) into \(\Omega(Bl_{\Delta}(M \times M))\) using the external wedge product \((\eta_1, \eta_2) \mapsto \tilde{\pi}_1^* \eta_1 \wedge \tilde{\pi}_2^* \eta_2\) and suppose that the Green kernel \(G\) satisfies \(G \in \Omega^{\otimes 2}\). This never happens, so what follows is just a formal computation.

**Proposition A.6.** In the de Rham case, suppose that \(G \in \Omega(M)^{\otimes 2}\). Then (100) reduces to (59).

**Proof.** Consider the intersection pairing \(\hat{\mathcal{P}}\) and its degree shift \(\mathcal{P}\) (see Proposition 3.2). According to Definition A.4, they extend to pairings on \(\Omega(M)^{\otimes k}\) and \(\Omega(M)^{[1]}^{\otimes k}\) for all \(k \geq 1\), respectively. For all \(\eta_1, \eta_2, \eta_{21}, \eta_{22} \in \Omega(M)\), we have:

\[
\begin{align*}
\mathcal{P}(\theta^2 \eta_{11} \otimes \eta_{12}, \theta^2 \eta_{21} \otimes \eta_{22}) &= (-1)^{\eta_1 + \eta_{21}} \mathcal{P}(\theta \eta_{11} \otimes \theta \eta_{12}, \theta \eta_{21} \otimes \theta \eta_{22}) \\
&= (-1)^{\eta_1 + \eta_{21} + (1+\eta_{12})(1+\eta_{21})} \mathcal{P}(\theta \eta_{11}, \theta \eta_{21}) \mathcal{P}(\theta \eta_{12}, \theta \eta_{22}) \\
&= (-1)^{1+\eta_{12} \eta_{21}} \hat{\mathcal{P}}(\eta_{11}, \eta_{21}) \hat{\mathcal{P}}(\eta_{12}, \eta_{22}) \\
&= -\hat{\mathcal{P}}(\eta_{11} \otimes \eta_{12}, \eta_{21} \otimes \eta_{22}).
\end{align*}
\]

One can also check that

\[
\hat{\mathcal{P}}(\eta_{11} \otimes \eta_{12}, \eta_{21} \otimes \eta_{22}) = \int_{x,y} \eta_{11}(x) \eta_{12}(y) \eta_{21}(x) \eta_{22}(y).
\]

For the Green operator \(\mathcal{G} : \Omega(M) \to \Omega(M)\) and its Green kernel \(G \in \Omega(M)^{\otimes 2}\), we have the following:

\[
\forall \eta_1, \eta_2 \in \Omega(M) : \hat{\mathcal{P}}(\mathcal{G}(\eta_1), \eta_2) = \int_{x,y} G(x,y) \eta_1(x) \eta_2(y) = \hat{\mathcal{P}}(G, \eta_1 \otimes \eta_2).
\]

From this and (101), we obtain

\[
\mathcal{P}(\mathcal{G}(\eta_1), \eta_2) = \mathcal{P}(\theta \mathcal{G}(\eta_1), \theta \eta_2) = (-1)^{\eta_1} \hat{\mathcal{P}}(\mathcal{G}(\eta_1), \eta_2) = (-1)^{\eta_1} \mathcal{P}(\theta^2 \mathcal{G}, \theta^2 \eta_1 \otimes \eta_2) = \mathcal{P}(\theta^2 \mathcal{G}, \theta \eta_1 \otimes \eta_2).
\]
Therefore, the element $\theta^2 G \in V[1]^{\otimes 2}$ corresponds to the Schwartz kernel $K_G$ of $G : V[1] \to V[1]$. We write this correspondence as

$$K_G \in V[1]^{\otimes 2} \sim \theta^2 G \in \text{Bl}_\Delta(M \times M)[2].$$

Let us check that $\theta^2 G$ satisfies (89). First of all, if we embed $\Omega(M)^{\otimes k}$ into $\Omega(M^{\times k})$ using the external wedge product $\eta_1 \otimes \cdots \otimes \eta_k \mapsto \pi^* \eta_1 \wedge \cdots \wedge \pi^*_k \eta_k =: \eta(x_1) \wedge \cdots \wedge \eta(x_k)$, then for all $\eta_1, \ldots, \eta_k \in \Omega(M)$ we have

$$\sigma(\eta_1 \otimes \cdots \otimes \eta_k)(x_1, \ldots, x_k) = \eta_1(x_{\sigma_1}) \wedge \cdots \wedge \eta_k(x_{\sigma_k}),$$

where the action on the left-hand side is given by (11). Now, the symmetry property (17) implies

$$\tau(\theta^2 G) = -\theta^2 \tau^*(G) = (-1)^{n+1} \theta^2 G = (-1)^{\theta^2 G} \theta^2 G.$$

Therefore, the symmetry condition (89) is indeed satisfied.

Let $\Gamma \in KG^{(3)}_{klp}$, and let $L$ be a labeling of $\Gamma$. We abbreviate $\sigma := \sigma_L \in S_{3k}$. Given $\eta_j \in \Omega(M)$ for $j = 1, \ldots, s_i$ and $i = 1, \ldots, l$, where $s_i$ is the valency of the $i$-th boundary component, we set $\eta = \eta_1 \otimes \cdots \otimes \eta_{s_i}$, $\eta = \eta_1 \otimes \cdots \otimes \eta_i$, $\alpha_{ij} = \theta \eta_{ij}$, $\omega_i = \alpha_{ij} \circ \otimes \alpha_{ks_i}$ and $\omega = \omega_1 \otimes \cdots \otimes \omega_l$. We denote $s := s_1 + \cdots + s_l$, so that $3k = 2e + s$, where $e$ is the number of internal edges. We have

$$(m_2^+)^{\otimes 2} (\sigma((\theta^2 G)^{\otimes e} \otimes \omega)) = \varepsilon(\theta, \eta) (m_2^+)^{\otimes 2k} (\sigma((\theta^2 G)^{\otimes e} \otimes \theta^* \eta))$$

$$= (-1)^{se(n-1)} \varepsilon(\theta, \eta) (m_2^+)^{\otimes 2k} (\sigma((\theta^2 G)^{\otimes e} \otimes \eta))$$

$$= (-1)^{\sigma + se(n-1)} \varepsilon(\theta, \eta) (m_2^+)^{\otimes 2k} (\theta^2 G^{\otimes e} \otimes \eta),$$

where $\varepsilon(\theta, \eta)$ is the Koszul sign to order $\theta^* \eta_1 \ldots \eta_{s_i} \mapsto \theta \eta_1 \ldots \theta \eta_{s_i}$ and $m_2^+ : \Omega(M)[1]^{\otimes 3} \to \mathbb{R}$ is given by $m_2^+ = \mathcal{P}(m_2 \otimes 1)$. We denote $\kappa := G^{\otimes e} \otimes \eta = \kappa_1 \otimes \cdots \otimes \kappa_{3k}$, $\kappa_i \in \Omega(M)[1]$ and compute

$$(m_2^+)^{\otimes 2k} (\theta^{\theta 2} \sigma(\kappa))$$

$$= \varepsilon(\sigma, \kappa) (m_2^+)^{\otimes 2k} (\theta^{\theta 2} \kappa_{\sigma_1} \otimes \cdots \otimes \kappa_{\sigma_{3k}})$$

$$= (-1)^{\frac{1}{2}k(k-1)n} \varepsilon(\sigma, \kappa) (m_2^+)^{\otimes 2k} (\theta^3 (\kappa_{\sigma_1} \otimes \kappa_{\sigma_{2}} \otimes \kappa_{\sigma_{3}}) \otimes \cdots \otimes \theta^3 (\kappa_{3k-2} \otimes \kappa_{3k-1} \otimes \kappa_{3k}))$$

$$= (-1)^{\frac{1}{2}k(k-1)n+\kappa_1 \cdots \kappa_{3k-1}} \varepsilon(\sigma, \kappa) (m_2^+)^{\otimes 2k} ((\theta \kappa_{\sigma_1} \otimes \theta \kappa_{\sigma_2} \otimes \cdots \otimes \theta \kappa_{\sigma_{3k-2}} \otimes \theta \kappa_{\sigma_{3k-1}} \otimes \theta \kappa_{\sigma_{3k}})).$$
Next, using the formula \( \Box \) for \( m_2^+ \), we get

\[
(m_2^+)^\otimes k \left( (\theta \kappa_{\sigma_1} \otimes \theta \kappa_{\sigma_2} \otimes \theta \kappa_{\sigma_3}) \otimes \cdots \otimes (\theta \kappa_{\sigma_1} \otimes \theta \kappa_{\sigma_2} \otimes \theta \kappa_{\sigma_3}) \right)
\]

\[
= (-1)^{\kappa_{\sigma_1} + \cdots + \kappa_{\sigma_3} - 1} \left( \int_{x_1} \kappa_{\sigma_1}^{-1}(x_1) \kappa_{\sigma_2}^{-1}(x_1) \kappa_{\sigma_3}^{-1}(x_1) \cdots \right)
\]

\[
= (-1)^{\kappa_{\sigma_1} + \cdots + \kappa_{\sigma_3} - 1} \int_{x_1, \ldots, x_k} \kappa_{\sigma_1}^{-1}(x_1) \kappa_{\sigma_2}^{-1}(x_1) \kappa_{\sigma_3}^{-1}(x_1) \cdots \kappa_{\sigma_{2k-2}}(x_{2k-2}) \kappa_{\sigma_{2k-1}}(x_{2k-1}) \kappa_{\sigma_{2k}}(x_{2k}),
\]

where \( \xi(3j - 2) = \xi(3j - 1) = \xi(3j) = j \) for \( j = 1, \ldots, k \) (see Definition 3.19).

In total, we have

\[
(m_2^+)^\otimes k \left( (\theta^2 \mathcal{G})^{\otimes e} \otimes \omega \right)
\]

\[
= \varepsilon_1 \varepsilon_2 \varepsilon_3 \int_{x_1, \ldots, x_k} G(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \cdots G(x_{\xi(\sigma_{2k-1})}, x_{\xi(\sigma_{2k})})
\]

\[
\alpha_{11}(x_{\xi(\sigma_{2k+1})}) \cdots \alpha_{1k}(x_{\xi(\sigma_{2k+1})}),
\]

where

\[
\varepsilon_1 \varepsilon_2 \varepsilon_3 = (-1)^{s + e(n-1) + \frac{1}{2} k(k-1)n} \varepsilon(\theta, \eta).
\]

Using (56), (58) and \( \varepsilon(\theta, \eta) = (-1)^{P(\omega)} \), we get the total sign

\[
(-1)^{k(n-2)} \varepsilon_1 \varepsilon_2 \varepsilon_3 = (-1)^{s(k,l) + s + P(\omega)},
\]

where \( (-1)^{k(n-2)} \) is the artificial sign from \( \Box \). This proves the proposition. \( \square \)

Remark A.7 (Signs for the Fréchet dIBL-structure on \( \Omega(M) \)). In [10, Section 13], they consider the weight-graded nuclear Fréchet space \( B_\infty \Omega(M) \subset B_\infty \Omega(M) \) generated by \( \varphi \in B_\infty \Omega(M) \) which have a smooth Schwartz kernel \( K_\varphi \in \Omega(M \times k) \); they showed that there is a canonical Fréchet dIBL-structure on \( B_\infty \Omega(M) \). In order to deduce the signs, we can consider the subspace \( B_{\text{alg}} \Omega(M) \subset B_\infty \Omega(M) \) generated by \( \varphi \in B_{\text{alg}} \Omega(M) \) with an algebraic Schwartz kernel \( K_\varphi \in \Omega(M \times k) \), rewrite (58) in terms of \( K_\varphi \) and extend the obtained formulas to \( B_{\text{alg}} \Omega(M) \). This may be done in [19]. \( \square \)
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