THE DUREN-CARLESON THEOREM IN TUBE DOMAINS OVER
SYMMETRIC CONES

DAVID BÉKOLLÉ, BENOIT F. SEHBA, AND EDGAR L. TCHOUNDJA

Abstract. In the setting of tube domains over symmetric cones, we determine a
necessary and sufficient condition on a Borel measure $\mu$ so that the Hardy space
$H^p$, $1 \leq p < \infty$, continuously embeds in the weighted Lebesgue space $L^q(d\mu)$
with a larger exponent. Finally we use this result to characterize multipliers from
$H^{2m}$ to Bergman spaces for every positive integer $m$.

1. Introduction and statements of the results

Let $T_\Omega = V + i\Omega$ be the tube domain over an irreducible symmetric cone $\Omega$ in the
complexification $V^C$ of a Euclidean space $V$ of dimension $n$. Following the notation
of [13] we denote the rank of the cone $\Omega$ by $r$ and by $\Delta$ the determinant function of
$V$. Letting $V = \mathbb{R}^n$, we have as example of symmetric cone on $\mathbb{R}^n$ the Lorentz cone
$\Lambda_n$ which is a rank 2 cone defined for $n \geq 3$ by

$$
\Lambda_n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_1^2 - \cdots - y_n^2 > 0, \ y_1 > 0\};
$$

the determinant function in this case is given by the Lorentz form

$$
\Delta(y) = y_1^2 - \cdots - y_n^2.
$$

For $0 < q < \infty$ and $\nu \in \mathbb{R}$, let $L^q_\nu(T_\Omega) = L^q(T_\Omega, \Delta^{-\frac{\nu}{r}}(y)dxdy)$ denote the space
of measurable functions $f$ satisfying the condition

$$
\|f\|_{q,\nu} = \|f\|_{L^q_\nu(T_\Omega)} := \left(\int_{T_\Omega} |f(x + iy)|^q \Delta^{-\frac{\nu}{r}}(y)dxdy\right)^{1/q} < \infty.
$$

Its closed subspace consisting of holomorphic functions in $T_\Omega$ is the weighted Bergman
space $A^q_\nu(T_\Omega)$. This space is not trivial i.e $A^q_\nu(T_\Omega) \neq \{0\}$ only for $\nu > \frac{n}{r} - 1$ (see
[10], cf. also [1]). The Bergman projector $P_\nu$ is the orthogonal projector from
the Hilbert-Lebesgue space $L^2_\nu(T_\Omega)$ to its closed subspace $A^2_\nu(T_\Omega)$. The usual (un-
weighted) Bergman space $A^q(T_\Omega)$ corresponds to the case $\nu = \frac{n}{r}$.

Without loss of generality, we may assume that $V = \mathbb{R}^n$ endowed with the standard inner product, and we shall apply this notation in the rest of the paper. By
$H^p(T_\Omega)$, $0 < p < \infty$, we denote the holomorphic Hardy space on the tube domain

\textit{Key words and phrases.} Symmetric cones, Hardy spaces, Bergman spaces.
that is the space of holomorphic functions $f$ such that
\[ \|f\|_{H^p} = \left( \sup_{t \in \Omega} \int_{\mathbb{R}^n} |f(x + it)|^p \, dx \right)^{1/p} < \infty. \]

Let $0 < p, q < \infty$. Our purpose is to characterize those positive Borel measures $\mu$ on $T$ for which the Hardy space $H^p(T)$ is continuously embedded into the Lebesgue space $L^q(T, d\mu)$. We recall that given two Banach spaces of functions $X$ and $Y$ with respective norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, it is said that $X$ continuously embeds into $Y$ ($X \hookrightarrow Y$), if there exists a constant $C > 0$ such that for any $f \in X$,
\[ \|f\|_Y \leq C \|f\|_X. \]

Taking $X = H^p(T)$ and $Y = L^q(T, d\mu)$ in the last inequality, then on the one hand, for all $0 < p, q < \infty$, an obvious example of such a measure $\mu$ is the delta measure $\delta = \delta_{ie}$ at the point $ie$, where $e$ is a fixed point of $\Omega$. On the other hand, testing on the functions
\[ G(z) = G_w(z) := [\Delta^{-\nu}(\frac{z - \bar{w}}{2i})]^\frac{1}{q}, \]
with $w = u + iv \in T$, we obtain that a necessary condition for the embedding of the Hardy space $H^p(T)$ into the Lebesgue space $L^q(T, d\mu)$ is the existence of a positive constant $C_{p,q,\mu}$ such that
\[ \int_{T} |\Delta^{-\nu}(\frac{z - \bar{w}}{2i})| \, d\mu(z) \leq C_{p,q,\mu} \Delta^{-\nu}(\frac{v}{r}) \]
whenever
\[ (\nu + \frac{n}{r})p > 2n - 1. \]

The first result is an extension of a result due to O. Blasco [6] (cf. also [7]) valid on the unit disc, to the case of $T$.

**Theorem 1.1.** Let $\mu$ be a Borel measure on $T$. If $p, q, \nu$ are real numbers satisfying the conditions
\[ (i) \quad 0 < p < q, \quad \frac{q}{p} > 2 - \frac{r}{n}, \]
\[ (ii) \quad (\nu + \frac{n}{r})p > \left(\frac{2n}{r} - 1\right), \]
then
\[ (1) \quad H^p(T) \text{ continuously embeds in } A^q_{\frac{n}{r}(\frac{q}{p} - 1)}(T) \]
if and only if
\[ (2) \quad \text{the condition (1) implies that } H^p(T) \text{ continuously embeds in } L^q(T, d\mu). \]

**Remark 1.2.** (1) For $n = r = 1$ (the case of the upper half-plane, $\Omega = (0, \infty)$), assertion 2. of the theorem was proved by P. Duren [12] (cf. also [11]), using a modification of the argument given by L. Carleson [9] in the case $p = q = 2$; assertion 2. was proved earlier by Hardy and Littlewood [15].
(2) We restrict to the condition $q > p$ and even $\frac{4}{p} > 2 - \frac{\nu}{n}$. Otherwise, the standard Bergman space $A^q_{\frac{p}{p-2}}(T_\Omega)$ is trivial, that is $A^q_{\frac{p}{p-2}}(T_\Omega) = \{0\}$.

The assertion 1. of the theorem is false. Nevertheless, we observed above that there are Borel measures $\mu$ on $T_\Omega$ such that $H^p \hookrightarrow L^q(T_\Omega, d\mu)$ (see the open question in Section 7).

In section 3, we shall prove Theorem 1.1 in a more general form where $\nu$ is a vector of $\mathbb{R}^r$.

Our next result is the following Hardy-Littlewood Theorem.

**Theorem 1.3.** Let $4 \leq p < \infty$. Then $H^2(T_\Omega) \hookrightarrow A^p_{\frac{p}{2r-3}}(T_\Omega)$.

In the case where $r = 2$, it is possible to go below the power $p = 4$. We have exactly the following.

**Theorem 1.4.** Let $r = 2$ and $n = 3, 4, 5, 6$. Then

1. $H^2(T_{\Lambda_3}) \hookrightarrow A^p_{\frac{p}{2r-3}}(T_{\Lambda_3})$ for all $\frac{8}{3} < p < 4$.
2. $H^2(T_{\Lambda_4}) \hookrightarrow A^p_{\frac{p}{2r-3}}(T_{\Lambda_4})$ for all $3 < p < 4$.
3. $H^2(T_{\Lambda_5}) \hookrightarrow A^p_{\frac{p}{2r-3}}(T_{\Lambda_5})$ for all $\frac{16}{5} < p < 4$.
4. $H^2(T_{\Lambda_6}) \hookrightarrow A^p_{\frac{p}{2r-3}}(T_{\Lambda_6})$ for all $\frac{10}{3} < p < 4$.

**Remark 1.5.** For every positive integer $m \geq 2$, it is easy to see that the continuous embedding $H^2(T_{\Omega_m}) \hookrightarrow A^p_{\frac{p}{2r-3}}(T_{\Omega_m})$ implies the continuous embedding $H^{2m}(T_{\Omega_m}) \hookrightarrow A^{mp}_{\frac{mp}{2r-3}}(T_{\Omega_m})$.

Following this remark, we deduce the following corollary from Theorem 1.1, Theorem 1.3 and Theorem 1.4.

**Corollary 1.6.** Let $\mu$ be a positive Borel measure on $T_\Omega$ and $p = 2m$ ($m = 1, 2, \cdots$) be a positive even number. If $q$ is a positive number satisfying one of the two following conditions

(i) $r \geq 2$, $n \geq 3$ and $2p \leq q < \infty$; 
(ii) $r = 2$, $n = 3, 4, 5, 6$ and $2p(1 - \frac{1}{n}) \leq q < 2p$.

and if $\nu$ is a real number satisfying the condition $(\nu + \frac{n}{r})^2 < (\frac{2n}{r} - 1)$, then the condition (7) implies that $H^p(T_\Omega)$ continuously embeds in $L^q(T_\Omega, d\mu)$.

Recall that given two Banach spaces of analytic functions $X$ and $Y$ with respective norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, we say an analytic function $G$ is a multiplier from $X$ to $Y$, if there exists a constant $C > 0$ such that for any $F \in X$,

$$\|FG\|_Y \leq C\|F\|_X.$$ 

We denote by $\mathcal{M}(X,Y)$ the set of multipliers from $X$ to $Y$.

Let $\alpha \in \mathbb{R}$. We denote by $H^\infty_\alpha(T_\Omega)$, the Banach space of analytic functions $F$ on $T_\Omega$ such that

$$\|F\|_{\alpha, \infty} := \sup_{z \in T_\Omega} \Delta(3z)^{\alpha}|F(z)| < \infty.$$
In particular, for $\alpha = 0$, the space $H^\infty_0(T_\Omega)$ is the space $H^\infty$ of bounded holomorphic functions on $T_\Omega$. The above results allow us to obtain the following characterization of pointwise multipliers from $H^2(T_\Omega)$ to $A^p_\nu(T_\Omega)$.

**Theorem 1.7.** Let $4 \leq p < \infty$, $\nu > \frac{n}{p} - 1$. Define $\gamma = \frac{1}{p}(\nu + \frac{n}{p}) - \frac{n}{2p}$. Then for any integer $m \geq 1$, the following assertions hold.

(a) If $\gamma > 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A^m_\nu(T_\Omega)) = H^\infty_\nu(T_\Omega)$.

(b) If $\gamma = 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A^m_\nu(T_\Omega)) = H^\infty(T_\Omega)$.

(c) If $\gamma < 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A^m_\nu(T_\Omega)) = \{0\}$

For $p < 4$, we have under further restrictions the following.

**Theorem 1.8.** Let $2(\frac{2}{p} - \frac{1}{n}) < p < 4$, $\nu > \frac{n}{p} - 1$. Assume that $P_{(\frac{2}{p} - 1)p}$ is bounded on $L^p_{(\frac{2}{p} - 1)p}(T_\Omega)$. Define $\gamma = \frac{1}{p}(\nu + \frac{n}{p}) - \frac{n}{2p}$. Then for any integer $m \geq 1$, the following assertions hold.

(a) If $\gamma > 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A^m_\nu(T_\Omega)) = H^\infty_\nu(T_\Omega)$.

(b) If $\gamma = 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A^m_\nu(T_\Omega)) = H^\infty(T_\Omega)$.

(c) If $\gamma < 0$, then $\mathcal{M}(H^{2m}(T_\Omega), A^m_\nu(T_\Omega)) = \{0\}$

Finally we particularize the previous problems to the tube domain over the light cone $\Lambda_n$. We take advantage of the geometry of this cone to prove the following restricted Hardy-Littlewood Theorem. The point here is that the exponent $p$ is no more restricted to the set of even positive integers and the exponents $p$ and $q$ are just related by the inequality $1 \leq p < q < \infty$. We say that a subset $B$ of the Lorentz cone $\Lambda_n$ is a restricted region with vertex at the origin $O$ if the Euclidean distance of any point of $B$ from $O$ is less than a multiple of the Euclidean distance of that point from the boundary of $\Lambda_n$. We denote $T_B$ the tube domain over $B$.

**Theorem 1.9.** Let $1 \leq p < q < \infty$. Then given each restricted region $B$ of the Lorentz cone $\Lambda_n$ with vertex $O$, there exists a positive constant $C_{p,q}(B)$ such that

$$
\int_{T_B} |F(z)|^q \Delta^{\frac{n}{2} - 1}(y) dx dy \leq C_{p,q}(B) \|F\|_{L^p}^q
$$

for all $F \in H^p(T_\Lambda_n)$.

The plan of this paper is as follows. In section 2, we present some preliminary results. The Blasco Theorem 1.1 is proved in section 3. The Hardy-Littlewood Theorems 1.3 and 1.4 are established in section 4. Section 5 is devoted to the proof of Theorem 1.6. The proof of the restricted Hardy-Littlewood Theorem 1.9 is given in section 6 while in section 7, we pose some open questions related this work.

**Remark 1.10.** In [15], R. Shamoyan and M. Arsenovic investigated the continuous embedding of some generalized Hardy spaces $H^2_\nu(T_\Omega)$ defined in [14] into weighted mixed norm Bergman spaces. The space $H^p_\nu(T_\Omega)$, $0 < p < \infty$, consists of the
functions \( f \) holomorphic on \( T_\Omega \), satisfying

\[
\|f\|_{H^p_\mu} := \left( \sup_{t \in \Omega} \int_{\partial \Omega} \int_{\mathbb{R}^n} |f(x + i(y + t)|^p dx d\mu(y) \right)^\frac{1}{p} < \infty.
\]

In their study, the measure \( \mu \) has the form

\[
d\mu_s(y) = \chi_\Omega(y) \frac{\Delta_s(y)}{\Gamma_\Omega(s)} \Delta_s^\frac{p}{2}(y),
\]

where \( s = (s_1, \ldots, s_r) \in \mathbb{R}^r \) belongs to the so-called Wallach set and is such that \( s_j > 0, \ j = 1, \ldots, r \). The generalized power function \( \Delta_s \) is defined at the beginning of section 2. Contrary to Bergman spaces, the measures \( \mu = \mu_s \) studied by these authors have their support on the boundary \( \partial \Omega \) of \( \Omega \). The usual Hardy spaces \( H^p(T_\Omega) \) correspond to the case where \( \mu = \delta_0 \) (the delta measure) and are outside their scope of application. Their proofs rely heavily on a Paley-Wiener characterization of functions in \( H^2_\mu(T_\Omega) \) proved in [14]. We use the same tool in the proof of Theorem 1.3 (Theorem 4.1): we include this proof for completeness. The reader will also point out that particularly in Theorem 1.9, the exponents \( 1 \leq p < q < \infty \) we consider are more general (\( p \) may be different from 2).

2. Preliminaries and useful results

Materials of this section are essentially from [13]. We give some definitions and useful results.

Let \( \Omega \) be an irreducible open cone of rank \( r \) inside a vector space \( V \) of dimension \( n \), endowed with an inner product \( (\cdot, \cdot) \) for which \( \Omega \) is self-dual. Let \( G(\Omega) \) be the group of transformations of \( \Omega \), and \( G \) its identity component. It is well-known that there exists a subgroup \( H \) of \( G \) acting simply transitively on \( \Omega \), that is every \( y \in \Omega \) can be written uniquely as \( y = ge \) for some \( g \in H \) and a fixed \( e \in \Omega \).

We recall that \( \Omega \) induces in \( V \) a structure of Euclidean Jordan algebra with identity \( e \) such that

\[
\overline{\Omega} = \{ x^2 : x \in V \}.
\]

We can identify (since \( \Omega \) is irreducible) the inner product \( (\cdot, \cdot) \) with the one given by the trace on \( V \):

\[
(x|y) = tr(xy), \ x, y \in V.
\]

Let \( \{c_1, \ldots, c_r\} \) be a fixed Jordan frame in \( V \) and

\[
V = \oplus_{1 \leq i \leq j \leq r} V_{i,j}
\]

be its associated Pierce decomposition of \( V \). We denote by \( \Delta_1(x), \ldots, \Delta_r(x) \) the principal minors of \( x \in V \) with respect to the fixed Jordan frame \( \{c_1, \ldots, c_r\} \).

More precisely, \( \Delta_k(x) \) is the determinant of the projection \( P_kx \) of \( x \) in the Jordan subalgebra \( V^{(k)} = \oplus_{1 \leq i \leq j \leq k} V_{i,j} \). We have \( \Delta = \Delta_r \) and \( \Delta_k(x) > 0, k = 1, \ldots, r \), when \( x \in \Omega \). The generalized power function on \( \Omega \) is defined as

\[
\Delta_s(x) = \Delta_1^{s_1-s_2}(x) \Delta_2^{s_2-s_3}(x) \cdots \Delta_r^{s_r}(x), \ x \in \Omega; \ s \in \mathbb{C}^r.
\]
Since the principal minors $\Delta_k$, $k = 1, \cdots, r$ are polynomials in $V$, they can be extended in a natural way to the complexification $V^C$ of $V$ as holomorphic polynomials we shall denote $\Delta_k \left( \frac{x+iy}{r} \right)$. It is known that these extensions are zero free on $T_\Omega$. So the generalized power functions $\Delta_s$ can also be extended as holomorphic functions $\Delta_s \left( \frac{x+iy}{r} \right)$ on $T_\Omega$.

Next, we recall the definition of the generalized gamma function on $\Omega$:

$$\Gamma_\Omega(s) = \int_\Omega e^{-y\xi} \Delta_s(\xi) \Delta^{-n/r}(\xi) d\xi \quad (s = (s_1, \cdots, s_r) \in \mathbb{C}^r).$$

We set $d := \frac{2(n)r - 1}{r} - 1$. This integral converges if and only if $\Re s_j > (j - 1)\frac{d}{2}$, for all $j = 1, \cdots, r$. Being in this case it is equal to:

$$\Gamma_\Omega(s) = (2\pi)^{\frac{nr}{2}} \prod_{j=1}^r \Gamma \left( s_j - (j - 1)\frac{d}{2} \right)$$

(see Chapter VII of [13]). For $s = (s, \cdots, s)$, $s \in \mathbb{C}$, we simply write $\Gamma_\Omega(s)$ instead of $\Gamma_\Omega(s)$.

We also record the following lemma.

**Lemma 2.1.** Let $s \in \mathbb{C}$ with $\Re s > \frac{n}{r} - 1$. Then for all $y \in \Omega$ we have

$$\int_\Omega e^{-y\xi} \Delta_s^{\frac{n}{r}}(\xi) d\xi = \Gamma_\Omega(s) \Delta^{-s}(y).$$

The beta function of the symmetric cone $\Omega$ is defined by the following integral:

$$B_\Omega(p, q) = \int_{\Omega \cap (\mathbb{E} - \Omega)} \Delta^{p - \frac{n}{r}}(x) \Delta^{q - \frac{n}{r}}(e - x) dx,$$

where $p$ and $q$ are in $\mathbb{C}$. When $\Re p > \frac{n}{r} - 1$ and $\Re q > \frac{n}{r} - 1$, the above integral converges absolutely and

$$B_\Omega(p, q) = \frac{\Gamma_\Omega(p)\Gamma_\Omega(q)}{\Gamma_\Omega(p + q)}$$

(see Theorem VII.1.7 in [13]).

**Lemma 2.2.** Let $p, q \in \mathbb{C}$ with $\Re p > \frac{n}{r} - 1$ and $\Re q > \frac{n}{r} - 1$. Then, for all $y \in \Omega$ we have

$$\int_{\Omega \cap (u - \Omega)} \Delta^{p - \frac{n}{r}}(x) \Delta^{q - \frac{n}{r}}(u - x) dx = B_\Omega(p, q) \Delta^{p + q - \frac{n}{r}}(u).$$

The following is [1, Proposition 3.5].

**Lemma 2.3.** Let $1 \leq p < \infty$ and $\nu > \frac{n}{r} - 1$. Then there is a constant $C > 0$ such that for any $f \in A_p^\nu(T_\Omega)$ the following pointwise estimate holds:

$$|f(z)| \leq C \Delta^{-\frac{n}{r}(\nu + \frac{n}{r})}(3z) ||f||_{p, \nu}, \text{ for all } z \in T_\Omega.$$

We refer to [10] for the following, whose proof relies on the previous lemma.
Lemma 2.4. Let $1 \leq p, q < \infty$, $\alpha, \beta > \frac{n}{r} - 1$. Then $A^p_\alpha(T_\Omega) \hookrightarrow A^q_\beta(T_\Omega)$ if and only if $\frac{1}{p}(\alpha + \frac{n}{r}) = \frac{1}{q}(\beta + \frac{n}{r})$.

From the above lemma, we deduce that to prove Theorem 1.3 it is enough to do this for $p = 4$.

We will make use of Paley-Wiener theory in the next section to prove Theorem 1.3 and Theorem 1.4. The following can be found in [13].

Theorem 2.5. For every $F \in H^2(T_\Omega)$ there exists $f \in L^2(\Omega)$ such that
$$F(z) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_\Omega e^{i(z|\xi)} f(\xi) d\xi, \quad z \in T_\Omega.$$ Conversely, if $f \in L^2(\Omega)$ then the integral above converges absolutely to a function $F \in H^2(T_\Omega)$. In this case, $\|F\|_{H^2} = ||f||_{L^2(\Omega)}$.

In the sequel, we write $V = \mathbb{R}^n$. For the proofs of the following two lemmas, cf. e.g. [16].

Lemma 2.6. Let $s = (s_1, ..., s_r) \in \mathbb{R}^r$ and define
$$I_s(y) := \int_{\mathbb{R}^n} \left| \Delta^{-s} \left( \frac{x + iy}{i} \right) \right| dx \quad \text{for} \quad y \in \Omega.$$ Then $I_s(y)$ is finite if and only if $\Re s_j > (r - j)\frac{d}{2} + \frac{n}{r}$. In this case,
$$I_s(y) = C(s)(\Delta^{-s} \Delta^{\frac{n}{p}})(y).$$ Furthermore, the function $F(z) = F_w(z) = \Delta^{-s}(\frac{z - w}{2i})$ ($w = u + iv$ fixed in $T_\Omega$) is in $H^p(T_\Omega)$ whenever $\Re s_j > \frac{(r-j)\frac{d}{2} + \frac{n}{r}}{p}$. In this case, we have
$$\|F\|_{H^p} = C(s,p)(\Delta^{-s} \Delta^{\frac{n}{p}})(v).$$ The expressions of the constants $C(s)$ and $C(s,p)$ are in terms of generalized gamma functions on the cone $\Omega$.

Lemma 2.7. Let $v \in T_\Omega$ and $s = (s_1, ..., s_r), t = (t_1, ..., t_r) \in \mathbb{C}^r$. The integral
$$\int_\Omega \Delta^{-s}(y + v) \Delta^{-t}(y) dy$$ converges if $\Re t_j > (j-1)\frac{d}{2} - \frac{n}{r}$ et $\Re(s_j - t_j) > \frac{n}{p} + (r-j)\frac{d}{2}$ and is equal to $C_{s,t}(\Delta^{-s+t} \Delta^{\frac{n}{p}})(v)$. In this case this integral is equal to $C_{s,t}(\Delta^{-s+t} \Delta^{\frac{n}{p}})(v)$.

We denote as in [11]
$$L^2_{\nu}(\Omega) = L^2(\Omega; \Delta^{-\nu}(\xi) d\xi).$$ The following Paley-Wiener characterization of the space $A^p_\alpha(T_\Omega)$ can be found in [13].
Theorem 2.8. For every $F \in A^2_\nu(T_\Omega)$ there exists $f \in L^2_{-\nu}(\Omega)$ such that

$$F(z) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Omega} e^{i(z|\xi)} f(\xi) d\xi, \quad z \in T_\Omega.$$  

Conversely, if $f \in L^2_{-\nu}(\Omega)$ then the integral above converges absolutely to a function $F \in A^2_\nu(T_\Omega)$. In this case, $\|F\|_{p, \nu} = \|f\|_{L^2_{-\nu}}$.

The (weighted) Bergman projection $P_\nu$ is given by

$$P_\nu f(z) = \int_{T_\Omega} K_\nu(z, w) f(w) dV_\nu(w),$$

where $K_\nu(z, w) = c_\nu \Delta^{-\nu+\frac{n}{r}}(\frac{z-w}{2i})$ is the Bergman kernel, i.e the reproducing kernel of $A^2_\nu(T_\Omega)$ (see [13]). Here, we use the notation $dV_\nu(w) := \Delta^{-\nu+\frac{n}{r}}(v) du dv$, where $w = u + iv$ is an element of $T_\Omega$. For $\nu = \frac{n}{r}$, we simply write $dV(w)$ instead of $dV_\nu(w)$. The positive Bergman operator $P_\nu^+$ is defined by replacing the kernel function by its modulus in the definition of $P_\nu$.

In the particular case of the tube domain over the Lorentz cone $\Lambda_n$ on $\mathbb{R}^n$, the following theorem is a consequence of results of [2] and the recent $l^2$-decoupling theorem of [8].

Theorem 2.9. Let $\nu > \frac{n}{2} - 1$. Then the Bergman projector $P_\nu$ of $T_{\Lambda_n}$ admits a bounded extension on $L^p_\nu(T_{\Lambda_n})$ if and only if

$$p_\nu' < p < p_\nu := \frac{n + 1}{2} - (1 - \nu)_+. $$

For the other cases we recall the following partial result.

Theorem 2.10. [2], [3]. Let $\Omega$ be a symmetric cone of rank $> 2$. Let $\nu > \frac{n}{r} - 1$. Then the Bergman projector $P_\nu$ of $T_\Omega$ admits a bounded extension on $L^p_\nu(\Omega)$ if

$$q_\nu' < p < q_\nu := 2 + \frac{\nu}{\frac{n}{r} - 1}. $$

We will sometimes face situations where the weight of the projection differs from the weight associated to the space. We then need the following result (see [17]).

Proposition 2.11. Let $1 \leq p < \infty$, $\nu \in \mathbb{R}$, and $\mu > \frac{n}{r} - 1$. Then $P_\mu^+$ is bounded on $L^p_\nu(T_\Omega)$ if and only if $1 < p < q_\nu - 1$ and $\mu p - \nu > (\frac{n}{r} - 1) \max\{1, p - 1\}$.

Definition 2.12. The generalized wave operator $\Box$ on the cone $\Omega$ is the differential operator of degree $r$ defined by the equality

$$\Box_x[e^{i(x|\xi)}] = \Delta(\xi)e^{i(x|\xi)}$$

where $\xi \in \mathbb{R}^n$.

When applied to a holomorphic function on $T_\Omega$, we have $\Box = \Box_z = \Box_x$ where $z = x + iy$.
Theorem 2.13. Let $1 < p < \infty$ and $\nu > \frac{n}{r} - 1$.

1. There exists a positive constant $C$ such that for every $F \in A_p^\nu$,
   \[ \| \varpi F \|_{p,\nu+p} \leq C \| F \|_{p,\nu}. \]

2. If moreover $p \geq 2$, the following two assertions are equivalent.
   (i) $P_\nu$ is bounded on $L^p_{L_\nu^r}(T_\Omega)$.
   (ii) For some positive integer $m$, the differential operator
   \[ \varpi^{(m)} := \varpi \circ \ldots \circ \varpi \quad (m \text{ times}) : A^p_\nu \to A^p_{\nu+m_p} \]
   is a bounded isomorphism.

Let us finish this section by the following result on complex interpolation of Bergman spaces of this setting.

Proposition 2.14. Let $1 \leq p_0 < p_1 < \infty$, $\nu_0, \nu_1 > \frac{n}{r} - 1$. Assume that for some $\mu > \frac{n}{r} - 1$, the projection $P^\mu_\nu$ is bounded on both $L^p_{L_\nu^r}(T_\Omega)$ and $L^{p_1}_{L_\nu^r}(T_\Omega)$. Then for any $\theta \in (0,1)$, the complex interpolation space $[A^p_{\nu_0}, A^p_{\nu_1}]_\theta$ coincides with $A^p_\nu$ with equivalent norms, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{\nu}{p} = \frac{1-\theta}{p_0} \nu_0 + \frac{\theta}{p_1} \nu_1$.

Proof. Consult e.g. [5]. \qed

3. Proof of the Blasco Theorem.

3.1. Proof of Theorem 1.1. By Lemma 2.6, the function
   \[ G(z) = G_w(z) := \left( \Delta_r^{-\frac{\nu_1-\nu_2}{q}} \ldots \Delta_r^{\nu_r-\frac{n}{q}} \Delta_r^{\nu_r-\frac{n}{q}+\frac{p}{r}} \right) \left( \frac{z - \bar{w}}{2i} \right) \]
   with $w = u + iv \in T_\Omega$, belongs to $H^p(T_\Omega)$ if and only if
   \[ (\nu_j + \frac{n}{r}) \frac{p}{q} > (r-j) \frac{d}{2} + \frac{n}{r} \quad (j = 1, \ldots, r). \]

Moreover
   \[ \| G \|_{H^p(T_\Omega)} = C_{p,q} \left( \Delta_r^{-\frac{\nu_1-\nu_2}{q}} \ldots \Delta_r^{\nu_r-\frac{n}{q}} \Delta_r^{\nu_r-\frac{n}{q}+\frac{p}{r}} \right) (v). \]

So for these $\nu$, a necessary condition for the continuous embedding $H^p(T_\Omega) \hookrightarrow L^q(T_\Omega, d\mu)$ to hold is the existence of a positive constant $C_{p,q,\mu}$ such that for every $w = u + iv \in T_\Omega$,

(3) \[ L(w) \leq C_{p,q,\mu} \left( \Delta_r^{-\nu_1-\nu_2} \ldots \Delta_r^{-\nu_{r-1}-\nu_r} \Delta_r^{-\nu_r-\frac{n}{q}+\frac{p}{r}} \right) (v) \]

where
   \[ L(w) := \int_{T_\Omega} \left| \left( \Delta_r^{\nu_1-\nu_2} \ldots \Delta_r^{-\nu_{r-1}-\nu_r} \Delta_r^{-\nu_r-\frac{n}{q}} \right) \left( \frac{z - \bar{w}}{2i} \right) \right| \, d\mu(z). \]

We state Theorem 1.1 in the following more general form.
Lemma 3.2. Let \( q > 0 \) and let \( \nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r \). There exists a positive constant \( C_{q,\nu} \) such that for every \( F \in \mathcal{H}(T_\Omega) \) we have

\[
|F(z)|^q \leq C_{q,\nu} \int_{T_\Omega} \frac{|F(u + iv)|^q \left( \Delta_{j-1}^{\nu_1-\nu_2} \ldots \Delta_{k-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r-\nu_j} \right) (v)}{|(\Delta_{j-1}^{\nu_1-\nu_2} \ldots \Delta_{k-1}^{\nu_{r-1}-\nu_r} \Delta_r^{\nu_r+\nu_j}) (x + iy - \bar{w})|^2} dudv.
\]

Proof of the lemma. We denote \( B(\zeta, \rho) \) the Bergman ball with centre \( \zeta \) and radius \( \rho \). Since \( |F|^q \) is plurisubharmonic, we have

\[
|F(ie)|^q \leq C \int_{B(ie,1)} |F(u + iv)|^q \frac{dudv}{\Delta^{\nu/2} (v)}.
\]
Recall that \( \frac{dudv}{\Delta^{2\mu}(v)} \) is the invariant measure on \( T_{\Omega} \). Let \( z \in T_{\Omega} \) and let \( g \) be an affine automorphism of \( T_{\Omega} \) such that \( g(ie) = z \). We have

\[
|F(z)|^q = |(F \circ g)(ie)|^q 
\leq C \int_{B(ie,1)} |(F \circ g)(u + iv)|^q \frac{dudv}{\Delta^{2\mu}(v)} 
= C \int_{B(z,1)} |F(u + iv)|^q \frac{dudv}{\Delta^{2\mu}(v)}.
\]

We recall that \( |\Delta_j(\frac{z-w}{2i})| \simeq \Delta_j(v) \) for all \( w = u + iv \in B(z,1) \). This implies that

\[
|F(z)|^q \leq C_{q,\nu} \int_{B(z,1)} |\Delta_{j_1}^{\nu_1-\nu_2} \cdots \Delta_{j_r-1}^{\nu_r-\nu_2} \Delta_{r+1}^{\nu_2} (\frac{z+iy-w}{2i})| \frac{dudv}{\Delta^{2\mu}(v)} 
\leq C_{q,\nu} \int_{T_{\Omega}} |\Delta_{j_1}^{\nu_1-\nu_2} \cdots \Delta_{j_r-1}^{\nu_r-\nu_2} \Delta_{r+1}^{\nu_2} (\frac{z+iy-w}{2i})| \frac{dudv}{\Delta^{2\mu}(v)}.
\]

Let us set

\[
I(w) := \int_{T_{\Omega}} \frac{d\mu(z)}{|(\Delta_{j_1}^{\nu_1-\nu_2} \cdots \Delta_{j_r-1}^{\nu_r-\nu_2} \Delta_{r+1}^{\nu_2} (\frac{z+iy-w}{2i})|}
\]

and recall that for \( \nu = (\nu_1, \cdots, \nu_r) \in \mathbb{R}^r \),

\[
\Delta_{\nu-\frac{r}{p}}(v) = (\Delta_{j_1}^{\nu_1-\nu_2} \cdots \Delta_{j_r-1}^{\nu_r-\nu_2} \Delta_{r+1}^{\nu_2} (\frac{z+iy-w}{2i})) (v).
\]

Using the Fubini-Tonelli Theorem, it follows from the previous lemma and the condition \( 3 \) that

\[
\int_{T_{\Omega}} |F(z)|^q d\mu(z) \leq C_{q,\nu} \int_{T_{\Omega}} I(u + iv)|F(u + iv)|^q \Delta_{\nu-\frac{r}{p}}(v) dudv 
\leq C_{p,q,\nu} \int_{T_{\Omega}} \Delta_{\nu-\frac{r}{p}}^2(v) |F(u + iv)|^q dudv.
\]

An application of the assertion 1. of the theorem implies that

\[
\int_{T_{\Omega}} |F(z)|^q d\mu(z) \leq C_{p,q,\nu} \|F\|^q_{H^p}.
\]

This finishes the proof of the implication \((ii) \Rightarrow (i)\).
4. Proofs of the Hardy-Littlewood Theorems.

4.1. Proof of Theorem 1.3. In view of Lemma 2.4, it is sufficient to show the following result.

**Theorem 4.1.** We have that $H^2(T_\Omega) \hookrightarrow A^4(T_\Omega)$.

**Proof.** Given $F$ in $H^2(T_\Omega)$, we would like to show that $F^2$ belongs to $A^2(T_\Omega)$. By Theorem 2.5 there exists $f \in L^2(\Omega)$ such that

$$F(z) = \int_{\Omega} e^{i(z|\xi)} f(\xi) d\xi \quad (z \in T_\Omega).$$

It follows that

$$F^2(z) = \int_{\Omega \times \Omega} e^{i(z|\xi+t)} f(\xi)f(t) d\xi dt = \int_{\Omega} \int_{\Omega \cap (u-\Omega)} e^{i(z|u)} f(u-\xi)f(\xi) d\xi du = \int_{\Omega} e^{i(z|u)} g(u) du,$$

where

$$g(u) = \int_{\Omega \cap (u-\Omega)} f(u-\xi)f(\xi) d\xi.$$

It follows from Theorem 2.8 that to conclude, we only have to show that $g \in L^2 \frac{1}{2}(\Omega)$.

We first estimate $|g(u)|^2$. Using Hölder’s inequality and Lemma 2.2 we obtain

$$|g(u)|^2 \leq \left( \int_{\Omega \cap (u-\Omega)} |f(u-\xi)||f(\xi)| d\xi \right)^2 \leq \left( \int_{\Omega \cap (u-\Omega)} |f(u-\xi)|^2 |f(\xi)|^2 d\xi \right) \times \left( \int_{\Omega \cap (u-\Omega)} d\xi \right) = C\Delta^\frac{\beta}{4}(u) \left( \int_{\Omega \cap (u-\Omega)} |f(u-\xi)|^2 |f(\xi)|^2 d\xi \right).$$

More precisely we have $C = B \left( \frac{n}{2}, \frac{n}{4} \right)$ ($B$ is the beta function). It follows easily that

$$\int_{\Omega} \Delta^\frac{\beta}{4}(u)|g(u)|^2 du \leq C \int_{\Omega} \int_{\Omega \cap (u-\Omega)} |f(u-\xi)|^2 |f(\xi)|^2 d\xi du = C\|f\|^4_{L^2(\Omega)} = C\|F\|^4_{H^2}.$$

The proof is complete. \qed
4.2. Proof of Theorem 1.4 With an application of assertion 1 of Theorem 2.13, we deduce the following that will be useful in the proof of Theorem 1.4.

Corollary 4.2. There exists a constant $C > 0$ such that for any $F \in H^2(T_{\Omega})$,

\[
\left( \int_{T_{\Omega}} |\Delta(\Im z)(\Box F)(z)|^4 dV(z) \right)^{1/4} \leq C \| F \|_{H^2(T_{\Omega})}.
\]

The following is also needed in our proof of Theorem 1.4.

Proposition 4.3. For every positive integer $m$ such that $2m > \frac{n}{r} - 1$, there exists a constant $C_m > 0$ such that for any $F \in H^2(T_{\Omega})$,

\[
\left( \int_{T_{\Omega}} |(\Box(m)^F)(z)|^2 \Delta(\Im z)^{2m - \frac{n}{r}} dV(z) \right)^{1/2} = C \| F \|_{H^2(T_{\Omega})}.
\]

Proof. Let $F \in H^2(T_{\Omega})$. Recall with Theorem 2.5 that there exists $f \in L^2(\Omega)$ such that

\[
F(z) = \int_{\Omega} e^{i(z|\xi)} f(\xi) d\xi \quad (z \in T_{\Omega})
\]

with $\| F \|_{H^2(T_{\Omega})} = \| f \|_{L^2(\Omega)}$. It follows that

\[
\Box(m)^F(z) = \int_{\Omega} e^{i(z|\xi)} \Delta^m(\xi) f(\xi) d\xi.
\]

Using the Plancherel’s formula, we obtain

\[
\int_{\mathbb{R}^n} |\Box(m)^F(x + iy)|^2 dx = \int e^{-2(y|\xi)} |f(\xi)|^2 \Delta(\xi)^{2m} d\xi.
\]

Integrating the latter with respect to $\Delta(y)^{2m - \frac{n}{r}} dy$ and using the definition of the gamma function, we obtain

\[
I := \int_{T_{\Omega}} |\Delta^m(\Im z)(\Box(m)^F)(z)|^2 \Delta(\Im z)^{-n/r} dV(z)
= \int_{\Omega} \int_{\mathbb{R}^n} |\Box(m)^F(x + iy)|^2 \Delta(y)^{2m - \frac{n}{r}} dy dx
= \int_{\Omega} |f(\xi)|^2 \Delta(\xi)^{2m} \left( \int_{\Omega} e^{-2(y|\xi)} \Delta(y)^{2m - \frac{n}{r}} dy \right) d\xi
= C_m \int_{\Omega} |f(\xi)|^2 d\xi.
\]

The latter equality relies on the condition $s = 2m > \frac{n}{r} - 1$ required in Lemma 2.1. □

We use Corollary 4.2 and Proposition 4.3 to deduce the following.
Corollary 4.4. Let \( p \in (2, 4) \). For every positive integer \( m \) such that \( 2m > \frac{n}{r} - 1 \), there exists a constant \( C_m \) such that for any \( F \in H^2(T_\Omega) \),

\[
\left( \int_{T_\Omega} |(\Box^{(m)}F)(z)|^p \Delta(3z)^{mp+(\frac{d}{2}-2)\frac{n}{r}} dV(z) \right)^{1/p} \leq C\|F\|_{H^2(T_\Omega)}.
\]

Proof. Note that by Corollary 4.2 and Proposition 4.3, \( \Box^{(m)} \) defines a bounded operator from \( H^2(T_\Omega) \) to \( A^2_{2m}(T_\Omega) \) and from \( H^2(T_\Omega) \) to \( A^4_{4m+\frac{d}{2}}(T_\Omega) \) respectively. It follows by interpolation that \( \Box^{(m)} \) is bounded from \( H^2(T_\Omega) \) to \( [A^2_{2m}, A^4_{4m+\frac{d}{2}}]_\theta \), \( \theta \in (0, 1) \).

It is easy to check (using Proposition 2.11) that for \( \mu > \frac{n}{r} - 1 \) large, the projector \( P_\mu \) is bounded on both \( L^2_{2m}(T_\Omega) \) and \( L^4_{4m+\frac{d}{2}}(T_\Omega) \). Thus by Proposition 2.14
\[
[A^2_{2m}, A^4_{4m+\frac{d}{2}}]_\theta = A^{mp+(\frac{d}{2}-1)\frac{n}{r}}(T_\Omega).
\]

The proof is complete. \( \square \)

Remark 4.5. Referring to \( \mathbb{B} \), we have shown that \( H^2(T_\Omega) \) continuously embeds into the holomorphic Besov space \( \mathbb{B}_{\frac{d}{2}(\frac{n}{r}-1)}^p(T_\Omega) \) for all \( 2 < p < 4 \).

The following follows from Corollary 4.4 and assertion 2. of Theorem 2.13

Theorem 4.6. Let \( 4 - \frac{2r}{n} < p < 4 \). Assume that \( P_{\frac{d}{2}(\frac{n}{r}-1)}^p \) is bounded on \( L^p_{\frac{d}{2}(\frac{n}{r}-1)}(T_\Omega) \). Then for any integer \( m \geq 1 \), \( H^{2m}(T_\Omega) \hookrightarrow A^{mp}_{\frac{d}{2}(\frac{n}{r}-1)}(T_\Omega) \)

We can now prove Theorem 1.4

Proof of Theorem 1.4. The condition \( p > 4 - \frac{2r}{n} \) is necessary for the non triviality of \( A^p_{\frac{d}{2}(\frac{n}{r}-1)}(T_\Omega) \). By Theorem 4.6, it is enough to check that the Bergman projector \( P_{\frac{d}{2}(\frac{n}{r}-1)}^p \) is bounded on \( L^p_{\frac{d}{2}(\frac{n}{r}-1)}(T_\Omega) \). In view of Theorem 2.9, we first suppose that \( r = 2 \) and \( \frac{n}{2}(\frac{n}{r}-1) < 1 \). The inequality \( \frac{n}{2} - 1 < 1 \) implies that \( n = 3 \). So we have the condition \( \frac{1}{2} < \frac{3}{2}(\frac{n}{r}-1) < 1 \), or equivalently \( \frac{8}{3} < p < \frac{10}{3} \). By Theorem 2.9, the Bergman projector \( P_{\frac{d}{2}(\frac{n}{r}-1)}^p \) is bounded on \( L^p_{\frac{d}{2}(\frac{n}{r}-1)}(T_{\Lambda_3}) \) if \( 4 - \frac{2r}{n} < p < 3p - 4 \), or equivalently if \( p > 4 - \frac{2r}{n} \). Thus the conclusion of Theorem 1.2 is valid in the case \( r = 2, \ n = 3 \).

Still for \( r = 2, \ n = 3, \) we next suppose that \( \frac{3}{2}(\frac{n}{r}-1) \geq 1 \), or equivalently \( p \geq \frac{10}{3} \). By Theorem 2.9, in this case, the Bergman projector \( P_{\frac{d}{2}(\frac{n}{r}-1)}^p \) is bounded on \( L^p_{\frac{d}{2}(\frac{n}{r}-1)}(T_{\Lambda_3}) \) if \( 2 < \frac{8}{3} < p < \frac{10}{3} \). Thus we have the condition always holds. This finishes the proof of Theorem 1.4 in the case \( n = 3 \).

We next suppose that \( r = 2 \) and \( n \geq 4 \). Then \( \frac{n}{2} - 1 \geq 1 \). The condition \( \frac{n}{2}(\frac{n}{r}-1) > \frac{d}{2} - 1 \) is equivalent to \( p > \frac{4(n-1)}{n} \). Moreover by Theorem 2.9, the Bergman projector \( P_{\frac{d}{2}(\frac{n}{r}-1)}^p \) is bounded on \( L^p_{\frac{d}{2}(\frac{n}{r}-1)}(T_{\Lambda_3}) \) if \( \frac{4(n-1)}{n} < p < \frac{8(n-1)+n-1}{2(n-1)} \), or equivalently if \( \frac{4(n-1)}{n} < p < \frac{2n-4}{n-4} \). We must have the inequality \( \frac{4(n-1)}{n} < \frac{2n-4}{n-4} \), which holds if and only if \( n \leq 6 \). This finishes the proof of the continuous embedding \( H^2(T_\Omega) \hookrightarrow \mathbb{B} \).
\( A^p_{\frac{p}{2} - 1}(T_\Omega) \) for \( n = 4, 5, 6 \) respectively for all \( 3 < p < 4, \frac{16}{5} < p < 4 \) and \( \frac{10}{3} < p < 4 \). \( \square \)

5. MULTIPLIERS FROM HARDY SPACES TO BERGMAN SPACES.

Let us now prove Theorem 1.7.

**Proof of Theorem 1.7.** We recall that \( \gamma = \frac{1}{p}(\nu + \frac{n}{mp}) - \frac{n}{2r} \). Let us start by proving the first assertion.

(a): First assume that \( G \in H^\infty_m(T_\Omega) \). Then using Theorem 1.3 we obtain that for any \( F \in H^2(T_\Omega) \),

\[
\int_{T_\Omega} |F(z)G(z)|^{mp} dV(z) \leq \|G\|_{H^\infty_m}^p \int_{T_\Omega} |F(z)|^{mp} \Delta(\Im z)^{(\frac{p}{2} - 1)\frac{n}{mp} - \frac{2}{r}} dV(z)
\]

Conversely, if \( G \in M(H^{2m}(T_\Omega), A^{mp}_\nu(T_\Omega)) \), then by Lemma 2.3 we have a constant \( C > 0 \) such that for any \( F \in H^{2m}(T_\Omega) \),

\[
|F(z)|^{mp} \Delta(\Im z)^{-\frac{1}{mp}(\nu + \frac{n}{mp})} \leq C \|G\|_{H^{\infty}}^p \|F\|_{H^{2m}}^m, \text{ for all } z \in T_\Omega.
\]

We test (7) with the function \( F(z) = F_w(z) = \Delta(\Im w)^{-\frac{n}{mp}} \Delta\left(\frac{z - \bar{w}}{i}\right)^{-\frac{m}{mr}} \) (\( w \) fixed) which is uniformly in \( H^{2m}(T_\Omega) \) by Lemma 2.7 and obtain that there exists \( C > 0 \) such that for all \( z \in T_\Omega \),

\[
|G(z)| \Delta(\Im w)^{\frac{m}{mr}} \Delta\left(\frac{z - \bar{w}}{i}\right)^{-\frac{m}{mr}} \leq C \Delta(\Im w)^{-\frac{1}{mp}(\nu + \frac{n}{mp})}.
\]

Taking in particular \( z = w \) in (8), we obtain that

\[
\Delta(\Im w)^{-\frac{1}{mp}(\nu + \frac{n}{mp})} |G(w)| \leq C
\]

and the constant \( C \) does not depend on \( w \). Thus \( G \in H^\infty_m(T_\Omega) \).

(b): The proof of the necessity part follows as above. For the sufficiency, one observes that in this case, \( \nu = (\frac{p}{2} - 1)\frac{n}{2r} \). It follows using Theorem 1.3 that

\[
\int_{T_\Omega} |F(z)G(z)|^{mp} dV(z) \leq \|G\|_{H^\infty_m}^p \int_{T_\Omega} |F(z)|^{mp} \Delta(\Im z)^{(\frac{p}{2} - 1)\frac{n}{mp} - \frac{2}{r}} dV(z)
\]

\[
\leq C \|G\|_{H^{\infty}}^p \|F\|_{H^{2m}}^m.
\]

(c): It is clear that 0 is multiplier from \( H^{2m}(T_\Omega) \) to \( A^{\nu, \text{mp}}_\nu(T_\Omega) \). Now assume that \( \gamma < 0 \) and that \( G \in M(H^{2m}(T_\Omega), A^{\nu, \text{mp}}_\nu(T_\Omega)) \). Then following exactly the same steps as in the proof of the necessity part in assertion (a), we obtain that there is a constant \( C > 0 \) such that for any \( z \in T_\Omega \),

\[
|G(z)| \leq C \Delta(\Im z)^{\frac{m}{mr} - \frac{1}{mp}(\nu + \frac{n}{mp})}.
\]

As \( \frac{m}{mr} - \frac{1}{p}(\nu + \frac{n}{mp}) > 0 \), we obtain that the right hand side of the last inequality goes to 0 as \( \Delta(y) \to 0 \). Hence \( G(z) = 0 \) for all \( z \in T_\Omega \). The proof is complete. \( \square \)
Proof of Theorem 1.8 This follows as above using Theorem 4.6.

6. The restricted Hardy-Littlewood Theorem

In this section we prove Theorem 1.9. We recall that the Lorentz cone \( \Lambda_n \) is defined by \( \Lambda_n := \{ y = (y_1, y') \in \mathbb{R}^+ \times \mathbb{R}^{n-1} : y_1 > |y'| \} \). We shall rely on the following geometrical lemma.

Lemma 6.1. We write \( d\mu(y) = \frac{\Delta^\beta(y)}{y_1} dy \). Then given \( \beta > \frac{n}{2} - 1 \), there exists a positive constant \( C = C_\beta \) such that

\[
\mu(\{ y \in \Lambda_n : \Delta_n^{\frac{\beta}{2}}(y) < \gamma \}) \leq C \gamma \quad \text{for all} \quad \gamma > 0.
\]

Proof of the Lemma. Using hyperbolic coordinates, an arbitrary point \( y \in \Lambda_n \) can be written as

\[
y = (r \cosh, r \sinh \omega), \quad r > 0, \ t \geq 0, \ \omega \in \mathbb{R}^{n-1}, \ |\omega| = 1.
\]

We use spherical coordinates to write \( \omega \) as

\[
\omega = (\cos \varphi, \sin \varphi) \quad \text{with} \quad 0 \leq \varphi \leq 2\pi \quad \text{if} \quad n = 3,
\]

and

\[
\omega = (\cos \varphi_1, \sin \varphi_1 \cos \varphi_2, ..., a(\varphi), b(\varphi))
\]

where

\[
a(\varphi) := \sin \varphi_1 \sin \varphi_2...\sin \varphi_{n-3} \cos \varphi_{n-2},
\]

\[
b(\varphi) := \sin \varphi_1 \sin \varphi_2...\sin \varphi_{n-3} \sin \varphi_{n-2}
\]

with \( 0 \leq \varphi_j \leq \pi \ (j = 1, ..., n - 3), \ 0 \leq \varphi_{n-2} \leq 2\pi \quad \text{if} \quad n \geq 4. \)

We have \( r^2 = \Delta(y) \) and the Jacobian \( J_n \) of this change of coordinates has absolute value

\[
\frac{|J_n|}{|J_n|} = \begin{cases} 
\frac{r^2 \sinh t}{ \sinh \frac{n-2}{2} t} & \text{if} \ n = 3 \\
\frac{r^{n-1} \sinh \frac{n-2}{2} t \sin \frac{n-3}{2} \varphi_1...\sin \varphi_{n-3}}{\sinh \frac{n-2}{2} t} & \text{if} \ n \geq 4.
\end{cases}
\]

Now we obtain

\[
\mu(\{ y \in \Lambda_n : \Delta_n^{\frac{\beta}{2}}(y) < \gamma \})
\]

\[
= \int_{r^n < \gamma} \frac{r^{n-1} \sinh \frac{n-2}{2} t \sin \frac{n-3}{2} \varphi_1...\sin \varphi_{n-3} \varphi_1...\sin \varphi_{n-3} \varphi_{n-2}}{\sinh \frac{n-2}{2} t} dr dt d\varphi_1...d\varphi_{n-3}d\varphi_{n-2}
\]

\[
= c_n \gamma \int_0^\infty \frac{\sinh \frac{n-2}{2} t}{\sinh \frac{n-2}{2} t} dt.
\]

The latter integral converges when \( \beta > \frac{n}{2} - 1 \). This finishes the proof of the lemma.
Let $1 < p < q < \infty$ and let $\beta > \frac{n}{2} - 1$. We denote by $A_{p,\beta}^q(T_{\Lambda_n})$ the weighted Bergman space on $T_{\Lambda_n}$ defined by

$$A_{p,\beta}^q(T_{\Lambda_n}) := \text{Hol}((T_{\Lambda_n}) \cap L^q(T_{\Lambda_n}, \frac{\Delta^\frac{q}{p}(\frac{q}{p}-2)+\beta}{y_1^{2\beta}} y dx dy)).$$

Obviously this weighted Bergman space contains the standard weighted Bergman space $A_{\nu}^q(T_{\Lambda_n})$, $\nu = \frac{n}{2}(\frac{q}{p} - 1)$.

We deduce the following corollary.

**Corollary 6.2.** The weighted Bergman space $A_{p,\beta}^q(T_{\Lambda_n})$ is not trivial i.e

$$A_{p,\beta}^q(T_{\Lambda_n}) \neq \{0\}.$$

**Proof of the Corollary.** We shall show that given $w = u + iv \in T_{\Lambda_n}$, the function $F(z) := \Delta^{-\frac{\nu}{2}}(\frac{\nu}{2} - i w)$ belongs to $A_{p,\beta}^q(T_{\Lambda_n})$ when $\nu$ is large. By Lemma 2.7 we obtain

$$\int_{\mathbb{R}^n} |F(x + iy)|^q dx = C(q, \nu)\Delta^{-\nu + \frac{n}{2}}(y + v)$$

if $\nu > n - 1$. In the notations of the previous lemma, we write again $d\mu(y) = \frac{\Delta^\beta(y)}{y_1^{2\beta}} dy$. Furthermore

$$L := \int_{T_{\Lambda_n}} |F(x + iy)|^q \frac{\Delta^\frac{q}{p}(\frac{q}{p}-2)+\beta}{y_1^{2\beta}} y dx dy)$$

$$= C(q, \nu) \int_{\Lambda_n} \Delta^{-\nu + \frac{n}{2}}(y + v) \Delta^\frac{q}{p}(\frac{q}{p}-2)(y) d\mu(y)$$

$$= C(q, \nu) \left\{ \sum_{k=1}^{\infty} \int_{2^{-k} < \Delta^\frac{n}{2}(y) \leq 2^{-k+1}} + \sum_{k=0}^{\infty} \int_{2^k < \Delta^\frac{q}{p}(y) \leq 2^{k+1}} \right\}$$

On the one hand we have

$$I := \sum_{k=1}^{\infty} \int_{2^{-k} < \Delta^\frac{q}{p}(y) \leq 2^{-k+1}}$$

$$\leq \Delta^{-\nu + \frac{n}{2}}(v) \sum_{k=1}^{\infty} \int_{2^{-k} < \Delta^\frac{n}{2}(y) \leq 2^{-k+1}} \Delta^\frac{q}{p}(\frac{q}{p}-2)(y) d\mu(y)$$

$$\leq C\Delta^{-\nu + \frac{n}{2}}(v) \sum_{k=1}^{\infty} 2^{-k(\frac{q}{p}-2)} \int_{\Delta^\frac{n}{2}(y) \leq 2^{-k+1}} d\mu(y)$$

$$\leq C\beta \Delta^{-\nu + \frac{n}{2}}(v) \sum_{k=1}^{\infty} 2^{-k(\frac{q}{p}-1)}.$$
The latter inequality follows by the previous lemma and the latter sum converges because \( \frac{q}{p} > 1 \). On the other hand we have

\[
\sum_{k=1}^{\infty} \int_{2^{k+1}<\Delta_n^2(y)<2^{k+1}} \Delta^{-\frac{\nu}{2}}(y) \Delta^{\frac{\beta}{2}}(y) d\mu(y) \\
= \sum_{k=1}^{\infty} 2^k(-\frac{2\nu}{n} + \frac{2\nu}{p} - 1) \Delta^{\frac{\beta}{2}}(y) d\mu(y) \\
\leq C_\beta \sum_{k=1}^{\infty} 2^{k(-\frac{2\nu}{n} + \frac{2\nu}{p})}.
\]

The latter inequality follows by the previous lemma and the latter sum converges if \( \nu \) is chosen sufficiently large. \( \square \)

We observe that for every \( y \in \Lambda_n \), we have \( d(y, \partial \Lambda_n) = \Delta^\frac{1}{2}(y) \). For the proof of Theorem 1.9 it suffices to show the following theorem.

**Theorem 6.3.** Let \( 1 < p < q < \infty \). Then for each \( \beta > \frac{n}{2} - 1 \), there exists a positive constant \( C_{p,q,\beta} \) such that

\[
\int_{T_{\Lambda_n}} \left| F(x + iy) \right|^q \frac{\Delta^\frac{\beta}{2} - 2 + \beta}{y_1^2 \Delta^\frac{\beta}{2} (y)} dxdy \leq C_{p,q,\beta} || F \|^q_{H^p}
\]

for all \( F \in H^p(T_{\Lambda_n}) \).

**Proof.** In the sequel, the notation \( || \cdot ||_p \) stands for the \( L^p \)-norm in \( \mathbb{R}^n \). We record the following well-known facts. For every \( F \in H^p(T_{\Lambda_n}) \), \( p \geq 1 \), the limit \( f(x) = \lim_{y \to 0, \ y \in \Lambda_n} F(x + iy) \) exists in the \( L^p \)-norm; moreover if we call \( P(f) \) the Poisson integral of \( f \) defined by

\[
P(f)(x + iy) = \int_{\mathbb{R}^n} \frac{\Delta^\frac{\beta}{2} (y)}{\Delta^\frac{\beta}{2} \left( x + iy - \xi \right)} f(\xi) d\xi,
\]

we have \( F = P(f) \) and \( || F ||_{H^p} = || f ||_p \). So it is enough to prove that there exists a positive constant \( C_{p,q,\beta} \) such that

\[
\int_{T_{\Lambda_n}} |P(f)(x + iy)|^q \frac{\Delta^\frac{\beta}{2} - 2 + \beta}{y_1^2 \Delta^\frac{\beta}{2} (y)} dxdy \leq C_{p,q,\beta} || f ||^q_{H^p}.
\]

We shall rely on the following lemma.

**Lemma 6.4.** Given \( 1 \leq s \leq q < \infty \), there exists a positive constant \( C_{q,s} \) such that

\[
\int_{\mathbb{R}^n} |P(f)(x + iy)|^s d\xi \leq C_{p,s} || f ||_s \Delta^\frac{s}{2} (y)
\]

for all \( y \in \Lambda_n \) and \( f \in L^s(\mathbb{R}^n) \).
Proof of the lemma. We apply the Young convolution inequality with the parameter \( t \geq 1 \) defined by \( \frac{1}{q} = \frac{1}{s} + \frac{1}{t} - 1 \). We obtain

\[
\left( \int_{\mathbb{R}^n} |P(f)(x + iy)|^q dx \right)^{\frac{1}{q}} \leq ||f||_s \Delta_{\mathbb{R}}^{\frac{2q}{p}}(y) \left( \int_{\mathbb{R}^n} \frac{1}{\Delta_n (\frac{x + iy}{t})} dx \right)^{\frac{1}{q}} \\
\leq C_{q,s} ||f||_s (\Delta^{\frac{n}{p}} \Delta^{\frac{n}{q}})(y) \\
= C_{q,s} ||f||_s \Delta^{\frac{2q(\frac{1}{q} - \frac{1}{s})}{p}}(y).
\]

The latter inequality follows by Lemma 2.7. \( \square \)

We define the operator \( S \) on \( L^p(\mathbb{R^n}, dx), 1 \leq s \leq q, \) by

\[
Sf(y) := \Delta_{\mathbb{R}}^{\frac{n}{p}}(y) ||P(f)(, + iy)||_q (y \in \Lambda_n).
\]

We shall show that \( S \) is a bounded operator from \( L^p(\mathbb{R^n}, dx) \) to \( L^p(\Lambda_n, \frac{\Delta^{\frac{n}{p}}(y)}{y_1^2} dy) \). The conclusion will follow by the Marcinkiewicz interpolation Theorem if we can prove that \( S \) is a weak-type \((1, 1)\) operator and a weak-type \((q, q)\) operator. The estimate \((9)\) of Lemma \(6.4\) gives

\[
\{y \in \Lambda_n : Sf(y) > \lambda\} \subset \{y \in \Lambda_n : C_{p,s} ||f||_s \Delta^{\frac{n}{p}} > \lambda\} \\
= \{y \in \Lambda_n : \Delta^{\frac{n}{p}}(y) < \frac{C_{p,s} ||f||_s}{\lambda} \}.
\]

An application of Lemma \(6.1\) concludes the proof of the theorem. \( \square \)

7. Open Questions

We pose here some questions that arise from this work and for which our methods do not give any answer.

(a) Can Theorem \(1.3\) be extended to the interval \( 4 - \frac{2r}{n} < p \leq 4 \) in the following two cases?

1. \( r = 2 \) and \( n \geq 7; \)
2. \( r \geq 3. \)

(b) Can the restricted Hardy-Littlewood Theorem \(1.9\) be extended to the entire Lorentz cone (unrestricted)?
(c) Can these theorems be extended to general symmetric cones?
(d) What happens when \(1 < \frac{q}{p} \leq 2 - \frac{r}{n}\) in the Duren-Carleson Theorem? Is assertion 2. of Theorem [ ] valid in this case?

References

[1] D. Békollé, A. Bonami, G. Garrigós, C. Nana, M. Peloso and F. Ricci, Lecture notes on Bergman projectors in tube domains over cones: an analytic and geometric viewpoint, IMHOTEP 5 (2004), Exposé I, Proceedings of the International Workshop in Classical Analysis, Yaoundé 2001.
[2] D. Békollé, A. Bonami, G. Garrigós and F. Ricci, Littlewood-Paley decompositions related to symmetric cones and Bergman projections in tube domains, Proc. London Math. Soc. 89 (2004), 317-360.
[3] D. Békollé, A. Bonami, G. Garrigós, F. Ricci and B. Sehba, Hardy-type inequalities and analytic Besov spaces in tube domains over symmetric cones, J. Reine Angew. Math. 647 (2010), 25-56.
[4] D. Békollé, A. Bonami, M. Peloso and F. Ricci, Boundedness of weighted Bergman projections on tube domains over light cones, Math. Z. 237 (2001), 31-59.
[5] D. Békollé, J. Gonessa and C. Nana, Complex interpolation between two weighted Bergman spaces on tubes over symmetric cones, C. R. Acad. Sci. Paris, Ser. I 337 (2003), 13-18.
[6] O. Blasco, A remark on Carleson measures from \(H^p\) to \(L^q(\mu)\) for \(0 < p < q < \infty\), Seminar of Mathematical Analysis 11-19, Colecc. Abierta, 71, Univ. Sevilla Secr. Publ., Seville (2004).
[7] O. Blasco and H. Jarchow, A note on Carleson measures for Hardy spaces, Acta Sci. Math. (Szeged) 71 (2005), No. 1-2, 371-389.
[8] J. Bourgain and C. Demeter, The proof of the \(l^2\)-decoupling conjecture, arXiv:1403.5535v3 [math.CA] 26 Jul 2015.
[9] L. Carleson, Interpolations by bounded analytic functions and the corona problem, Ann. Math. (2) 76 (1962), 547-559.
[10] D. Debertol, Besov spaces and boundedness of weighted Bergman projections over symmetric tube domains, Dottorato di Ricerca in Matematica, Università di Genova, Politecnico di Torino, (April 2003).
[11] P. Duren, Theory of \(H^p\) spaces, Academic Press, New York (1970).
[12] P. Duren, Extension of a theorem of Carleson, Bull. Amer. Math. Soc. 75 (1969), 143-146.
[13] J. Faraut and A. Koranyi, Analysis on symmetric cones. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.
[14] G. Garrigos, Generalized Hardy spaces on tube domains over cones, Colloq. Math. 90 (2001) no. 2, 213-251.
[15] G.H. Hardy, J.E. Littlewood, Some properties of fractional integrals II, Math. Z. 34 (1932), 405-423.
[16] C. Nana, \(L^p,q\)-Boundedness of Bergman Projections in Homogeneous Siegel Domains of Type II, J. Fourier Anal. Appl. 19 (2013), 997-1019.
[17] B. F. Sehba, Bergman type operators in tubular domains over symmetric cones, Proc. Edin. Math. Soc. 52 (2) (2009), 529-544.
[18] R. Shamoyan, M. Arsenovic, Embedding operators and boundedness of the multifunctional operators in tubular domains over symmetric cones, Filomat 25 (4) (2011), 109-126.
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF NGAOUNDÉRÉ, P.O. BOX 454, NGAOUNDÉRÉ, CAMEROON

E-mail address: dbekolle@univ-ndere.cm

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GHANA, P.O. BOX LG62, LEGON, ACCRA, GHANA

E-mail address: bfsehba@ug.edu.gh

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF YAOUNDÉ I, P.O. BOX 812, YAOUNDÉ, CAMEROON

E-mail address: tchoundjaedgar@yahoo.fr