**ABSTRACT:** Excitons in parabolically confined planar quantum dots with a transverse magnetic field have been studied in various model systems. The correlations between e−h, e−e, and h−h have been incorporated in terms of exact, simply elegant, and absolutely terminating finite summed Lauricella functions which eliminate the secular divergence problem and pave way for a comprehensive understanding of certain exotic phenomena of various two-dimensional regular and irregular quantum dots. A simple yet highly accurate and exact variational wave function in terms of Whittaker-M function extensible to multie excitonic systems has been propounded. We have also presented a formulation extending the size of the systems to triexcitonic (e−e−h/e−h−h), biexcitonic (e−h−e−h), and multie excitonic (“N” e−h pair) planar dots by mono-, di-, quadru-, and octopole expansions. As a benchmark, we have examined the energy spectra, level-spacing statistics, heat capacities (C_v at 1 K), and magnetization (T ≈ 0–1 K) of He/SiO_2/BN/GaAs model systems for different lateral confinements, magnetic fields, mass ratios of e−h, and dielectric constants (ε).

**INTRODUCTION**

Quantum dots (QDs) formulate a very fundamental area of interest in mesoscopic physics because of their exotic applications in electronic and optoelectronic technologies such as new-generation QD lasers, quantum information processing etc. In two-dimensionally confined systems such as graphene or boron nitride (BN), the unusual transport phenomena yield various applications such as composite materials, electric batteries, field emitters, solid-state gas sensors etc. Often referred to as “artificial atoms”, the unique features of these artificially simulated structures arise from the controlled modification of their density of states achieved by reducing the dimensions through confinement from 3-D to zero-dimension. Moreover, exciton formation upon the binding of an electron and a hole, electron–phonon, and e−e interactions in QDs has opened another frontier of as yet elusive phenomena like fractional quantum Hall effect, Meissner effect, Josephson effect, Kondo effect, and Coulomb blockade etc.1−13 Unlike composite fermions, the response of excitons to gate voltage (lateral confinement) and magnetic field (transverse confinement) is susceptible to change depending upon the individual masses of electrons and holes. Eventually, the inclusion of attractive or scattering correlations existing between e−h, e−e, and h−h pairs further shuffles bound states to a greater extent, making the Hamiltonian nontrivial. A gamut of theoretical and experimental investigations into such systems has revealed and put the interplay between the quantum confinement, magnetic field, and the Coulomb correlations at the focal point of scientific curiosity.14−27 In the present work, we have examined a strongly correlated excitonic Hamiltonian of a 2-D QD with a due cyclotron field where the Coulomb interaction has been treated analytically.18−21 The response of excitonic systems with varying e−h mass ratios in the presence of a magnetic field is also undertaken, along with the zero and nonzero confining potential in different dielectric media. We have extended e−h correlations by invoking multipole expansion to multie excitonic systems such as biexciton and triexciton which have gravitated current research areas.22 In the multie excitonic systems, the regime where Coulomb correlations and magnetic field-induced contributions to energy are comparable, a peculiar pattern is observed for different effective masses, indicating that the Born–Oppenheimer approximation holds true for excitons with heavy holes. We have applied the formalism to the model systems from low dielectric to high dielectric media like He/SiO_2/BN/GaAs and presented the cumulative effects of the competing factors on the energy-level spectra, heat capacities (C_v), and magnetization (M).

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With the canonical momenta of e−h proposed by Chakraborty et al., the spin-free Hamiltonian of a 2-D multie exciton QD confined by a harmonic potential \( \omega_{0,k} \) of both regular and irregular anisotropic superlattices/QDs (lattice point “\( \vec{R}_k \)”) with transverse magnetic field \( \omega \) and dielectric constant \( \epsilon \) in au can be described as

\[
H = \sum_{i=1}^{N} \left\{ \frac{1}{2m_i} \left( \frac{\vec{p}_i}{c} - \frac{\epsilon}{c} \vec{A}(r_i) \right)^2 + \frac{1}{2}m_i\omega_{0,k}^2 \right\} + \frac{1}{2} \sum_{k=1}^{K} (\vec{r}_k - \vec{R}_k)^2 + \frac{1}{2} \sum_{i,j=1}^{N} \sum_{\substack{\alpha=1,2 \atop \beta=1,2}} \epsilon_{ij}^{\alpha\beta} \frac{\vec{r}_i \cdot \vec{r}_j}{r_{ij}} \tag{1}\]

where \( m_i \) and \( \epsilon_i \) are the mass and the charge in au on the \( i \)-th charge carrier (either an electron or a hole depending on the composition of the multie exciton system). Anisotropy often emerges in the excitonic QD systems as a consequence of irregular geometry, electronegativity difference between constituent atoms, and/or applied gate voltage. This anisotropy is well accounted for and incorporated in different confinements \( (\omega_{0,k}(s)) \) around lattice points “\( \vec{R}_k \)”, which contains information about the detailed geometry of superlattices/QDs. As the charge carriers (electron/hole) orbit around a unique origin, both the diagonal and off-diagonal Coulomb/exchange interactions are enhanced for such simple systems; that is, it is preferred to introduce anisotropy of different orders to make the Hamiltonian more holistic. Although the hexagonal honeycomb structure of graphene offers the best possible template of isotropicity in the two-dimensional configuration, transition of each individual charge carrier (e/h) bearing an effective mass to massless Dirac fermions occurs at Dirac points which ultimately yields a zero-band gap in graphene, and the system demands for a Dirac-relativistic treatment, which is beyond the scope of this paper, as the current discussion is wholly dedicated to the treatment of many body quantum systems in a generic nonrelativistic regime. Apart from graphene, BN (analogue of graphene) and TMDs (transition-metal dichalcogenides) also offer an important class of 2-D-confined excitonic systems. Both BN and TMDs represent a class of anisotropic layer compounds which are characterized by strong in-plane bonding and weak out-of-plane interactions. However, the degree of anisotropy in TMDs is more acute than in BN. The effective masses of charge carriers and the dielectric constant of TMDs are dependent on the crystal direction and the number of layers, respectively, making them extremely anisotropic. The values of effective masses of charge carriers (electron/hole) differ greatly for different crystal directions even in the in-plane configuration. TMDs include a whole array of different compounds like MoS\(_2\), MoSe\(_2\), WS\(_2\), WSe\(_2\), etc., all of which are extremely anisotropic and have crystal lattice directiondependent properties like anisotropic effective mass and a layer-dependent dielectric constant. Although BN also possesses anisotropy, the effective mass of charge carriers is isotropic, that is, not dependent on layer numbers or lattice directions, and in the present study, BN has been treated as an isotropic 2-D-confined system \( (\omega_c = \omega_h) \). This is the first step in treating anisotropy, a complete description of which calls for a separate paper in itself. Therefore, in the current paper, we limit our discussion to isotropic nonrelativistic 2-D-confined excitonic systems.

**Exciton Systems.** For a 2-D exciton under a unique origin, that is, \( (\vec{R}_k = 0) \), the Hamiltonian appears as

\[
H = \frac{1}{2m_e} \left( \frac{\vec{p}_e}{c} + \frac{\epsilon}{c} \vec{A}(r_e) \right)^2 + \frac{1}{2}m_e\omega_c^2 r_e^2 + \frac{1}{2}m_h \left( \frac{\vec{p}_h}{c} - \frac{\epsilon}{c} \vec{A}(r_h) \right)^2 + \frac{1}{2}m_h\omega_h^2 r_h^2 - \frac{\epsilon^2}{4\pi\epsilon_0 \epsilon c} \left( \vec{r}_e - \vec{r}_h \right)^2 \tag{2}
\]

Introducing of relative and center-of-mass co-ordinates as \( \vec{R} = (\vec{r} + \vec{r}^0) \), \( \vec{r} = \vec{r}_e - \vec{r}_h \), where \( m = m_h/m_e \) and the symmetric gauge of cyclotron field for both electron and hole: \( \vec{A}_e = -\vec{A}_h = \vec{A} = \vec{B} \times (\vec{r}_e - \vec{r}_h) \) transforms the full Hamiltonian to

\[
H = \frac{(1 + \alpha)}{\alpha} \left[ H_c - \frac{\alpha}{1 + \alpha} \frac{e^2}{c} \right] + \frac{1}{1 + \alpha} H_R^0 + H_0 \tag{3}
\]

where \( H_c = \frac{a\omega_c^2}{(1 + \alpha)} (\omega_c^2 - \omega_h^2) \vec{R} \cdot \vec{r} - \frac{2e}{m_c(1 + \alpha)} \vec{B} \cdot \vec{A}_c \), \( \omega_R^2 = 1 + \alpha (\omega_c^2 + \omega_h^2) \). In atomic units, \( h = c = 1 \), \( 1/4\pi\epsilon_0 = 1 \), \( H_0^0 \) can be written as

\[
\frac{1}{2} \left[ \frac{r^{1/2}}{r} \frac{d}{dr} r^{1/2} + \frac{1}{r^2} \frac{d}{d\theta} r^2 + \frac{1}{4} \right] + \frac{(\alpha - 1)}{\alpha + 1} \omega_{c,s} \left[ \frac{\partial}{\partial \theta} + \frac{r^2}{2} \left( \frac{m_c}{2} \omega_{c,s}^2 + \frac{\alpha_{c,s}^2}{4} \right) \right] x_{n,m}(r, \theta) \tag{4}
\]

where \( \omega_{c,s} = eB/c \) and \( \Omega = \sqrt{\frac{m_c}{2} \omega_{c,s}^2 + \frac{\alpha_{c,s}^2}{4}} \). The Hamiltonian for the system can be segregated into two parts, namely, unperturbed \( (H_0^0 + H_R^0) \) and perturbed \( (H' = -\frac{1}{c^2} + H_R) \). The ansatz wave function is chosen to be \( x_{n,m}(r, \theta) = \xi^0_{n,m}(r) \left( \begin{array}{c} \cos \frac{m \pi}{2} \theta \\ \sin \frac{m \pi}{2} \theta \end{array} \right) \), \( m = 0, \pm 1, \pm 2, \ldots \), where \( u(r) \) is a radial part and \( e^{i\phi} \) is the angular part. Eigenvalues of \( H^0 \) and \( H_R^0 \) are \( E_{n,m}^0 \) and \( E_{n,m}^R \), respectively. Solving for \( H^0 \) and \( E_{n,m}^R \) yields the following radial equation:

\[
E_{n,m} = m E_{n,m}^0 \tag{5}
\]
\[
\begin{align*}
\left\{ \frac{d^2}{dr^2} + \left( \frac{1}{4} - m^2 \right) \frac{1}{r^2} - m \left( \frac{\alpha - 1}{\alpha + 1} \right) \omega^2_r - \Omega^2_r r^2 \\
+ 2m_r E_{n,m} \right\} u(r) = 0
\end{align*}
\]

(6)

On substituting \( x = \Omega_r r \) with \( u(r) \Rightarrow U(x) \), the above equation transforms as

\[
\begin{align*}
4x \frac{d^2}{dx^2} + 2 \frac{d}{dx} + \left( -x + \frac{1}{4} - m^2 \right) \frac{1}{x} - \left( \frac{\alpha - 1}{\alpha + 1} \right) \\
\frac{m \omega^2 x}{\Omega_r} + 2m_r E_{n,m} \right\} U(x)
\end{align*}
\]

(7)

Transformation \( U(x) = x^{-1/4} M(x) \) leads to

\[
\begin{align*}
M^*(x) + \left\{ \left( \frac{1}{4} - \frac{m^2}{x} \right) - \frac{1}{4} + \frac{\lambda}{x} \right\} M(x) = 0
\end{align*}
\]

(8)

where \( M(x) \) is Whittaker-M function and \( \lambda = \left( \frac{2m_r E_{n,m} - m \omega^2_x}{4\Omega_r} \right) \). Comparison of the above equation with the self-adjoint differential equation of the Whittaker-M function yields the corresponding eigenvalue as \( E_{n,m}^0 = (2n + \nu + \frac{1}{2}) \Omega_{nm} + \frac{(\alpha - 1)m \omega_x}{\alpha + 1} \), and the normalized eigenfunction is given by

\[
\xi_{n,m}(x, \theta) = N_{n,m} x^{-\nu/2} M_{\nu(n)}(x) e^{-x/2}
\]

where \( N_{n,m} = \sqrt{\frac{(2\nu + 1)!!}{\nu(\nu + 1)!!}} \), \( \nu = \nu + n + \frac{1}{2} \), and \( 2\nu = \nu_m \). The eigensolutions of \( H_e^R \) can be evaluated in a similar manner which furnishes the corresponding eigenvalue as \( E_{N,M}^0 = (2N + 1 + \nu)(1 + \nu) \omega^R \). Integrals of the Coulomb interaction between the e–h pair can be expressed as (\( N \) and \( M \) conserved)

\[
I_1 = \frac{N_{n,m} N_{n,m}^*}{2 \sqrt{\Omega_e}} \int_0^\infty g^* g_{n,m}^* \frac{x^{(\nu - 1/2)}\xi_{n,m}^*(x, \theta)}{x^{(\nu - 1/2)}\xi_{n,m}(x, \theta)} dx
\]

(9)

The above integral is estimated in terms of diagonal and off-diagonal matrix elements using standard integrals (likely to be unfeasible for H-like systems).

\[
I_2 = \int_0^\infty x^{(\nu - 1/2)} e^{-x} g_{\nu,n-1/2}(a e) M_{\nu,n-1/2}(a e) dx
\]

\[
= a_1^\nu a_2^\nu b^{\nu + M} \Gamma(\nu + M)
\]

(10)

which satisfies the conditions \( \text{Re}(\nu + M) > 0 \) and \( \text{Re}(b + \frac{1}{2}a_1 + \frac{1}{2}a_2) > 0 \); here, for above integral, \( b = 0 \), \( a_1 = a_2 = 1 \), \( A = \frac{1}{2} (a_1 + a_2) = 1 \), \( \gamma_1 - \frac{1}{2} = \gamma_2 - \frac{1}{2} = 1 \), \( M = \gamma_1 + \gamma_2 = \nu_m + 1, \rho = -1/2 \), and \( F_2 \) is the Lauricella function.

Diagonal Term \((n_1 = n_2 = n; m_1 = m_2 = m)\).18–20

\[
I_1 = \frac{N_{n,m}^2}{2 \sqrt{\Omega_e}} \int_0^\infty \left[ M_{\nu,n}(z) \right]^2 z^{(-3/2)} dz
\]

(11)

where \( 2\nu = 0, 1, 2, \ldots \), \( \kappa - \nu = 1/2 = 0, 1, 2 \ldots \), which further reduces to

\[
I_1 = \frac{\Omega_e}{e \nu! (\nu + 1)!} I \left[ \nu + 1, \frac{1}{2} \right] \times \sum_{\nu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{(\nu + 1/2)!}{(\nu + 1)! (\nu + 1)!} !\\
\]

(12)

where for any number “\( n \)”, the Pochhammer symbol \((n)_0 = 1 \).

Off-Diagonal Term \((m_1 = m_2 = m)\).18–20

\[
I_2 = \int \left[ \xi_{n,m}(x, \theta) \right] \frac{a_m (a_k^2 - a_h^2)^{1/2}}{(1 + \alpha)} \xi_{n,m}(x, \theta) dx
\]

(13)

The cross term \( H_e \) is evaluated as shown below

\[
I_2 = \int \left[ \xi_{n,m}(x, \theta) \right] \frac{a_m (a_k^2 - a_h^2)^{1/2}}{(1 + \alpha)} \xi_{n,m}(x, \theta) dx
\]

(14)

\[
= \frac{a_m (a_k^2 - a_h^2)}{(1 + \alpha)}
\]

(15)

In the above integral, the radial dependence can be calculated by using the standard integral [eq 10].30 Therefore, the integral transforms into

\[
I_2 = \int \left[ N_{n,m} M_{k,e} \right] \frac{x^{1/2}}{2 \Omega_e} \left[ N_{n,m} M_{k,e} \right] dx
\]

(16)
where $\text{Re}(\rho + M) > 0$ and $\text{Re}(b \pm \frac{1}{2}a_1 \pm \frac{i}{2}a_2) > 0$; here, for above integral, $b = 0, a_1 = a_2 = 1, A = \frac{1}{2}(a_1 + a_2) = 1, \gamma_1 = \gamma_2 = \frac{1}{2}, M = \gamma_1 + \gamma_2 = \text{lm} + 1, \rho = 1/2,$ and $F_2$ is the Lauricella function.\(^7\)

The angular dependence of the integral is evaluated in the following manner

\[
\left< \frac{e^{iM\theta_2}}{\sqrt{2\pi}} \frac{e^{i\theta_2}}{\sqrt{2\pi}} \cos(\Theta_R - \theta_2) \frac{e^{iM\theta_2}}{\sqrt{2\pi}} \frac{e^{i\theta_2}}{\sqrt{2\pi}} \right> = \frac{1}{2} \left\{ \delta_{M_0, M_2+1} \times \delta_{m_1+m_2} + \delta_{M_1+1, M_2} \times \delta_{m_0,m_2+1} \right\}
\]

Thus, $I_2$ becomes

\[
I_2 = \frac{am}{(1 + \alpha)} (\omega_2^2 - \omega_h^2) 
\times \left\{ \frac{\Omega_1}{n_1!(m_1!)^2 n_2!(m_2!)^2} \frac{|m_1 + l| + 3}{2} \right\}
\times \left\{ \frac{\Omega_2}{N_1!(M_1!)^2 N_2!(M_2!)^2} \frac{|M_1 + l| + 3}{2} \right\}
\times \left\{ \frac{\Omega_3}{N_1!(M_1!)^2 N_2!(M_2!)^2} \frac{|M_1 + l| + 3}{2} \right\}
\times \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \delta_{ij} \sum_{k=0}^{N_0} \frac{\Omega_k (N_k + |M_k|)! (N_k + |M_k|)!}{(M_k + l)! (M_k + l + 1)!} \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \delta_{ij} (N_l)! (M_l)!^2
\times \delta_{M_0, M_2+1} \times \delta_{m_1+m_2} + \delta_{M_1+1, M_2} \times \delta_{m_0,m_2+1} \right\}
\]

**Multiexciton Systems.** A lot of investigative/experimental studies into the multiexciton systems such as bi- and triexcitons have been done. The characteristic which makes the multiexcitonic systems like bi- and triexcitons very interesting and unique is the presence of unprecedented Coulombic/exchange correlations with respect to applied gate voltage and magnetic field. However, the exact solution to the Coulomb/exchange correlation beyond one electron has yet not been available for a long time. The discussion about multiexcitonic systems has been provided as a stepping stone to extend the current methodology beyond an $e^{-h}$ pair. The current formalism, in any way, is not limited to be applicable to only the model systems discussed here. However, it is a universal approach applicable with equal ease to any system of interest within the nonrelativistic 2-D-confined regime. In the four model systems (He, SiO₂, BN, and GaAs) under study, the different values of dielectric constant ($\epsilon$) lead to different expressions of the relative strength of Coulombic/exchange correlations. As the Coulomb correlation scales to a factor of $1/\epsilon$, different signatures for energy emerge, from the study of which a more concise evaluation of the field parameters required to fabricate various nanoscale devices and technological applications can be estimated. Also, the stabilization introduced in the systems on account of Coulomb/exchange correlations is a major factor that determines the overall stability and lifetime of the multieuctionic systems, where an exciton is the fundamental building block.

**Isotropic Multiexcitonic Systems.** For a 2-D multiexciton under a unique origin ($\vec{R}_{k-l} = 0$), that is, the isotropic system, the Hamiltonian appears as

\[
H = \sum_{i=1}^{N} \left\{ -\frac{1}{2 m_i} \frac{\vec{p}_i^2}{c^2} + \frac{1}{2} m_i \omega_0^2 \gamma_i^2 \right\} + \frac{1}{\epsilon} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{\epsilon_{ij}}{\gamma_i^2}
\]

\[
H = \sum_{i=1}^{N} H_i + \sum_{i=1}^{N} \sum_{j \neq i}^{N} H_{ij}
\]

where $H_{ij} = \frac{1}{\epsilon} \frac{1}{\gamma_i^2}$, $m_i$ and $\epsilon_i$ denote the mass and the charge in au on the $i$-th charge carrier (either an electron or a hole as specified by the composition of the multiexciton system under consideration). The correlation term ($H_{ij}$) can be expanded using the well-known multipole expansion.\(^3\) The above Hamiltonian of an isotropic multiexciton system can be solved by invoking the well-known multipole expansion\(^1\) for handling each term of Coulomb interactions wherein $\gamma_i$ is expanded in the neighborhood of $\vec{r}_i$

\[
V = \sum_{p=0}^{\infty} V^{(p)}
\]

Here, $V$ is the total Coulomb correlations between all $i$-th and $j$-th charge carriers, with

\[
V^{(0)} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{\epsilon_{ij}}{\gamma_i^2}
\]

\[
V^{(1)} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{\epsilon_{ij} \vec{r}_i \vec{r}_j}{\gamma_i^2 \vec{r}_i^2}
\]

\[
V^{(2)} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{\epsilon_{ij}}{\gamma_i^2} \left[ \frac{3(\vec{r}_i \cdot \vec{r}_j)^2}{\vec{r}_i^2 \vec{r}_j^2} - \frac{\vec{r}_i^2 \vec{r}_j^2}{\vec{r}_i^2 \vec{r}_j^2} \right]
\]

Because the methodology proposed by us deals with the generic co-ordinates of all the individual charge carriers, formation of Slater determinants will automatically reveal that the Pauli exclusion principle is obeyed.
Triexciton ($e^-e^-e^+h^+e^-h^+h^+$). For the case of a 2-D triexciton, the Hamiltonian is given by

$$H = \sum \sum \sum \left( \frac{1}{2m_\ell} \left( \frac{e}{c} A(r_\ell) \right)^2 + \frac{1}{2} m_\ell \omega_\ell^2 \right) + \frac{1}{2} \left( \sum \sum \sum \frac{\epsilon_{\ell j}}{r_{\ell j}} \right)$$

(24)

The Coulomb correlation is solved by invoking a multipole expansion method

$$V = \sum_{p=0}^{\infty} V^{(p)}$$

where

$$V^{(0)} = \frac{1}{2} \sum_{i=1}^{3} \sum_{j \neq i}^{3} \frac{\epsilon_{ij}}{|r_{ij}|}$$

$$V^{(1)} = \frac{1}{2} \sum_{i=1}^{3} \sum_{j \neq i}^{3} \frac{\epsilon_{ij} r_{ij} n_{ij}^2}{|r_{ij}|^3}$$

$$V^{(2)} = \frac{1}{2} \sum_{i=1}^{3} \sum_{j \neq i}^{3} \frac{\epsilon_{ij} (3 r_{ij} n_{ij}^2 - r_{ij}^2)}{|r_{ij}|^5}$$

... = ... 

(25)

The variational principle is employed for solving the Coulomb correlation in the monopole term [eq 25] using the standard integral [eq 10]. The term accounting for the dipole [eq 26] with $p = 1$ can be expressed as

$$V^{(1)} = \frac{1}{2} \left( \frac{\epsilon_{i j} n_{i j}^2}{|r_{i j}|^3} \right)$$

(28)

into which the insertion of the discretized closure relation

$$\sum_{k=0}^{\infty} \langle \xi_k | \langle \xi_{i j} |$$

facilitates the evaluation of dipole-type and Coulomb-type integrals using eq 10 recursively. The quadrupole factor [eq 27] can be solved by adopting a similar approach.

Biexciton ($e^-h^-e^+h^+$). The Hamiltonian of a biexciton in 2-D QDs is represented as

$$H = \sum \sum \sum \left( \frac{1}{2m_\ell} \left( \frac{e}{c} A(r_\ell) \right)^2 + \frac{1}{2} m_\ell \omega_\ell^2 \right) + \frac{1}{2} \left( \sum \sum \sum \frac{\epsilon_{\ell j}}{r_{\ell j}} \right)$$

(29)

The case of a biexciton is also handled in a similar way to a triexciton. For the case of a biexciton, the Coulomb correlation expanded in multipole series arises as
The biexcitonic dipole term is given by

\[
V = \sum_{p=0}^{\infty} V^{(p)}
\]

where

\[
V^{(0)} = \frac{1}{2} \sum_{i=1}^{4} \sum_{j \neq i} \varepsilon \mathbf{p}_{ij} \cdot \mathbf{r}_{ij}
\]

\[
V^{(1)} = \frac{1}{2} \sum_{i=1}^{4} \sum_{j \neq i} \frac{\varepsilon \mathbf{p}_{ij} \cdot \mathbf{r}_{ij}}{|r_{ij}|^3}
\]

\[
V^{(2)} = \frac{1}{2} \frac{1}{2} \sum_{i=1}^{4} \sum_{j \neq i} \frac{3(\mathbf{r}_{ij} \cdot \mathbf{r}_{ij})^2 - \mathbf{r}_{ij}^2}{|r_{ij}|^5}
\]

... = ...

The biexcitonic dipole term is given by

\[
V^{(1)} = \frac{1}{2} \left( \frac{\varepsilon \mathbf{e}_1 \cdot \mathbf{r}_1 - \mathbf{r}_1}{|r_1|^3} + \frac{\varepsilon \mathbf{e}_2 \cdot \mathbf{r}_2 - \mathbf{r}_2}{|r_2|^3} + \frac{\varepsilon \mathbf{e}_3 \cdot \mathbf{r}_3 - \mathbf{r}_3}{|r_3|^3} + \frac{\varepsilon \mathbf{e}_4 \cdot \mathbf{r}_4 - \mathbf{r}_4}{|r_4|^3} \right)
\]

\[
+ \frac{\varepsilon \mathbf{e}_1 \cdot \mathbf{r}_2 + \varepsilon \mathbf{e}_2 \cdot \mathbf{r}_1}{|r_1|^3} + \frac{\varepsilon \mathbf{e}_3 \cdot \mathbf{r}_4 + \varepsilon \mathbf{e}_4 \cdot \mathbf{r}_3}{|r_3|^3} + \frac{\varepsilon \mathbf{e}_1 \cdot \mathbf{r}_4 + \varepsilon \mathbf{e}_4 \cdot \mathbf{r}_1}{|r_1|^3}
\]

\[
+ \frac{\varepsilon \mathbf{e}_2 \cdot \mathbf{r}_3 + \varepsilon \mathbf{e}_3 \cdot \mathbf{r}_2}{|r_2|^3} + \frac{\varepsilon \mathbf{e}_4 \cdot \mathbf{r}_1 + \varepsilon \mathbf{e}_1 \cdot \mathbf{r}_3}{|r_3|^3}
\]

which can be solved by inserting a discretized closure relation as done in the case of the triexciton.

**RESULT AND DISCUSSION**

A product basis span of 55 × 36 = 1980 is considered for carrying out diagonalization. The four model systems, that is, He, SiO\textsubscript{2}, BN, and GaAs have been chosen to study a whole spectrum of dielectric constants. These systems encompass the window of the dielectric from 1.0 to 12.35, thereby giving a comprehensive reflection of the effect of changing the dielectric on the Coulomb correlations as the later scale to a factor of 1/\(\epsilon\). A systemic study of the interplay of \(\omega_o = \omega_h = \omega_o\) (zero and nonzero), that is, isotropic ambience, \(\omega_c\), \(\epsilon\), e−h/e−e/h−h interactions, \(m_o\) and \(m_h\) on the energy-level diagram has been done. A change in \(\omega_o\) that is, an increase or decrease in \(\omega_o\) indicates a change in the size of the QD which in turn points toward a decrease or increase, respectively, in interactions...
among the charge carriers. Level-spacing statistics, magnetization ($T \approx 1$ K), and $C_{\alpha}$ are employed for investigating the rearrangement of states where a transverse magnetic field modifies overall confinement frequency (hybrid magneto-electric confinement frequency, $\Omega_{\alpha}$). Inclusion of $(a-1)\alpha m_{\omega}^{2}/2m_{e}$ does not cause appreciable crossings and anticrossings among lower-bound states for interacting systems. The inferences from the results indicate that the lower states do not show much of mixing with the onset of magnetic field [Figures 1–4] because both the Coulomb and the exchange integrals stabilize the system. The relative extent of stabilization depends on the strength of both applied $\omega_{c}$ and $\omega_{\alpha}$. In the regime of low magnetic field, the Coulomb interaction dominates [Figures 1b,c, 2b,c, 3b,c, and 4b,c] whereas in the high-magnetic regime, the stabilization due to the exchange interaction additionally drives the system toward Landau levels. This tendency is further strengthened at stronger confinement [Figures 1e,f, 2e,f, 3e,f, and 4e,f]. In the regime of moderate magnetic field, a glance at the elements of correlation matrices indicates that the Coulomb interaction (diagonal matrix elements) is a stronger stabilizing force than the exchange integral (off-diagonal matrix elements) [Table 1] which can be approximated for states with a higher angular momentum ($m$) as

$$\left\langle \frac{e^{2}}{r_{0,m}}(r) \right\rangle \approx \frac{1}{e} \frac{\sqrt{\Omega_{\alpha}}}{m_{l}l + \frac{1}{2}}$$

The $e^{-h/e-e/h-h}$ interactions decrease with increasing angular momentum $(m_{l})$ which limits to $1/\sqrt{|m_{l}|}$ for larger $m_{l}$ virtually mapping onto Laughlin’s results and existing literature at high $\omega_{c}$. An interesting revelation that comes to the fore from the study of all the systems (He, SiO$_{2}$, BN, and GaAs) of varying dielectric constants at zero confinement and approximately zero $\omega_{c}$ is that even in presence of the Coulomb interaction, the ground state exhibits extremely high degeneracy which is lifted with the onset of magnetic field [Figure 1(g–i)]. Such splitting of energy levels by $\omega_{\alpha}$ is reminiscent of the Zeeman effect in hydrogenic systems. For a particular system, the energy spectrum does not change appreciably on moving from $\alpha = 10.0$ to 100.0, which validates the Born–Oppenheimer approximation.

**Role of Effective Mass ($m_{*}$) and Dielectric Constant ($\varepsilon$)**: The spectra of states for a variety of materials such as He, SiO$_{2}$, BN, and GaAs with $m_{e} = 1.0, 0.42, 0.7$, and 0.07 au and $\varepsilon = 1.0, 3.9, 7.1$, and 12.35, respectively, are examined as a function of $\omega_{\alpha}$ with zero ($\omega_{c} = 0.0$ au) and nonzero confinement ($\omega_{c} = 0.002$ and 0.0002 au). To a first approximation, the energy of the interacting systems as a function of high cyclotron frequency, $m_{*}$, and hybrid magneto-electric confinement frequency ($\Omega_{\alpha}$) can be represented as

$$E_{0,m} \approx (m_{l}l + 1) \left( \frac{\alpha + 1}{\alpha} \right) \sqrt{\omega_{c}^{2} + \frac{\alpha \omega_{c}^{2}}{4m_{e}^{2}}} + \left( \frac{\alpha - 1}{\alpha} \right) m_{l}l + \frac{1}{e} \frac{\sqrt{\Omega_{\alpha}}}{m_{l}l + \frac{1}{2}}$$

For noninteracting systems with nonzero confinement ($\omega_{c} = 0.002$ and 0.0002 au), there is no appreciable change in the

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**Figure 3. Energy-level diagram of an exciton in the 2-D QD as a function of magnetic field ($\alpha$) for BN ($\varepsilon = 7.1, m_{e} = 0.7$ au) (a,d—noninteractive system for $\omega_{c} = 0.0002$ and 0.002 au, respectively) (b,e—interactive system with $\alpha = 1.0$ for $\omega_{c} = 0.0002$ and 0.002 au, respectively) (c,f—interactive system with $\alpha = 10.0$ for $\omega_{c} = 0.0002$ and 0.002 au, respectively) (g—interactive system with $\alpha = 0.0$ au with $\omega = 0.0$ au with $\alpha = 1.0, 10.0, 100.0$, respectively).**

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energy spectra with change in $m_e$ [Figures 1(a,d)–4(a,d)]. For interacting systems with nonzero confinement ($\omega_o = 0.002$ and 0.0002 au), two kinds of states arise which can be classified into hydrogenic states (HS) and harmonic oscillator/Gaussian states (GS). These states are stabilized to different extents depending on the ratio of $m_e\omega_o/\omega_c$ ($\approx 0.034$ au), leading to the emergence of bunch states which favor the formation of excitons from Fermi-sea [Figures 1(b,c,e,f)–4(b,c,e,f)]. With the onset of magnetic field, the previously potent stabilizing Coulomb interaction is overpowered by stabilization due to magnetic field on moving from $\alpha = 1.0$ to 10.0 [Figures 1(b,c,e,f)–4(b,c,e,f)]. The details of the aforementioned trends will be discussed subsequently.

Exciton in a He QD ($m_e = 1.0$ au, $e = 1.0$). An exciton in a He QD represents a strongly correlated system, as a dielectric constant of 1.0 implies that the separation between charge carriers is reduced resulting in increased Coulombic (exchange) interactions, a direct evidence of which is furnished by the energy-level diagrams of excitons in the He QD at different $\omega_o$. Even at different confinement frequencies and magnetic fields, the resulting bound states for the excitonic system in the He QD predominantly exhibit a hydrogenic character of the state. A change in the confinement frequency changes the spatial extent/size of the QD. Upon increasing $\omega_o$ from 0.0002 to 0.002 au, the size of the QD confining the same charge (carrying the same amount of charge) is reduced. A reduction in the size also points toward an increased amount of interactions among charge carriers which is also proved to be the case by the pattern of energy in energy-level spectra. The study of energy-level spectra for an exciton in the He QD has been carried out at different values of $\alpha$, $\alpha$ being the ratio of effective mass of hole to electron signifies whether the exciton is a heavy hole or a light hole. An increase in $\alpha$ indicates a stronger attractive interaction contributing to the stability of the system. This can also be understood by consulting the energy-

| $\omega_o$ (au)$^a$ | $(n,m_1; n,m_2)$ | matrix elements |
|---------------------|-----------------|----------------|
| 0.0                 | 0–4; 0–4        | 0.01532615     |
|                     | 0–3; 0–3        | 0.01751560     |
|                     | 0–3; 1–3        | 0.00437890     |
|                     | 0–3; 2–3        | 0.00207709     |
|                     | 0–3; 0–2        | 0.02101872     |
|                     | 0; 0            | 0.05654992     |
|                     | 0–4; 0–4        | 0.02725417     |
|                     | 0–3; 0–3        | 0.03141763     |
|                     | 0–3; 1–3        | 0.00778691     |
| 0.005               | 0–3; 2–3        | 0.00069365     |
|                     | 0–2; 0–2        | 0.03737715     |
|                     | 0; 0            | 0.09967240     |
|                     | 0–4; 0–4        | 0.08396807     |
|                     | 0–3; 0–3        | 0.09596352     |
|                     | 0–3; 1–3        | 0.02399088     |
| 0.01                | 0–3; 2–3        | 0.01137987     |
|                     | 0–2; 0–2        | 0.11515621     |
|                     | 0; 0            | 0.30708325     |

$^a$1 au = $2.5 \times 10^3$ T.
level spectra for a fixed $\omega_o$ at different values of $\alpha$. For a better understanding of the energy-level spectra, the terms contributing to the total energy of the system must be cautiously borne in mind:

$$E = (2n + l\ell + 1)\frac{\Omega_o}{m_e} + \left(\frac{\alpha - 1}{\alpha + 1}\right)\frac{\omega_m}{2m_e} + (2N + l\ell + 1).$$

For the terms containing $\Omega_o$ and $\omega_m$, an increase in their respective values contributes positively toward the stabilization of the system. For a noninteracting exciton of the 2-D He system, the ground state energy increases with the increasing magnetic field for $\omega_o = 0.0002$ and 0.002 au, respectively, typical of a Fock–Darwin spectrum [Figure 1a,d]. For the interacting system at a weaker field strength, the Coulomb (exchange) correlation is the major stabilizing force. The magnetic field facilitates stabilization both through Coulomb and exchange correlations, but the magnitude of stabilization induced by the exchange correlation exceeds that of Coulomb, leading to faster fall in energy for HS [Figure 1b,c,e,f]. The energy of all states is lowered on increasing $\alpha$ from 1.0 to 10.0, but a further increase in $\alpha = 100.0$ seems to cause no significant change in energy spectra. In case of $\alpha = 1.0$, the contribution due to $\left(\frac{\alpha - 1}{\alpha}\right)\frac{\omega_m}{2m_e}$ term is negated; hence, the fall of energy is not as steep as compared with $\alpha = 10.0$ and 100.0. In case of zero confinement ($\omega_o = 0.0$ au), a striking feature that emerges from the energy-level spectra [Figure 1g–i] is the existence of negative-bound states with a steep fall in the stabilization energy even at weaker magnetic field. This feature reveals a very important point regarding excitons in 2-D QDs, which is that even when the confinement frequency is made almost zero, the system does not fly off, rather, the attractive interactions among the charge carriers lead to the generation of a potential providing stability for the existence of HS. A value of $\omega_o$ implies that the confining potential, that is, the gate voltage applied across the dot, has been removed, but because of the presence of a relatively weak (when compared to cases with nonzero $\omega_o$) magnetic field, the Coulombic correlations among charge carriers lead to the formation of hydrogenic-bound states. There is, however, not much significant change in the signature of energy-level spectra on increasing $\alpha$ (10.0 to 100.0).

Exciton in the SiO\textsubscript{2} QD ($m_e = 0.42$ au, $\epsilon = 3.9$). For a noninteracting exciton in the SiO\textsubscript{2} system, the ground state energy increases with the increase in the magnetic field similar to He with delayed crossing and anticrossing [Figure 2a,d]. In interacting SiO\textsubscript{2} excitonic systems, the stabilization due to the Coulomb (exchange) interaction induced on account of $\omega_o$ and/or $\omega_e$ exhibits analogous behavior to He except for the particular amounts of stabilization brought into picture by effective mass through the $\left(\frac{\alpha - 1}{\alpha}\right)\frac{\omega_m}{2m_e}$ factor [Figure 2b,c,e,f]. The effect of increased $\epsilon$ in case of the exciton in SiO\textsubscript{2} can be immediately gauged from the decrease in the energy gap between few lowest states and the higher excited states. The relative energy gap between the HS and the GS is less [Figure 2b,c] as compared to He at lower confinement ($\omega_o = 0.0002$ au) which further reduces, leading to an overlap between the states if the confinement of the system is raised to $\omega_o = 0.002$ au [Figure 2e,f]. For zero confinement, only few HS appear to be forming negative-bound states, whereas the excited states do not exhibit the same which can be attributed to a higher dielectric constant [Figure 2g–i].
Exciting in the BN QD ($m_e = 0.7$ au, $\epsilon = 7.1$). In case of a noninteracting BN excitonic system, the ground state energy increases with rapid crossing and anticrossing upon raising the magnetic field, and the stabilization due to $\left(\frac{a-1}{a}\right)_{\text{min}}$ is 0.6 times that of SiO$_2$ on account of higher effective mass [Figure 3a,d]. The energy gap between ground and higher excited states is further quenched [Figure 3b,e] in interacting systems at lower confinement ($\omega_0 = 0.0002$ au). At high confinement ($\omega_0 = 0.002$ au), shuffling of the HS and the GS leading to crossing and anticrossing is further intensified with both increasing magnetic field and $\alpha$ (1.0 to 10.0) [Figure 3c,f]. However, no significant change in the spectra is observed upon increasing $\alpha$ from 10.0 to 100.0. At zero confinement, apart from the fewest lowest states, all excited states increase in energy with increasing magnetic field and $\alpha$ [Figure 3g–i].

Exciting in the GaAs QD ($m_e = 0.7$ au, $\epsilon = 12.35$). For a noninteracting excitonic system in the GaAs QD, the ground state energy increases upon raising the magnetic field while exhibiting quick crossing and anticrossing among levels. The stabilization due to $\left(\frac{a-1}{a}\right)_{\text{min}}$ is enhanced 10 and 6 times that of BN and SiO$_2$, respectively, on account of poor effective mass [Figure 4a,d]. In case of interacting systems, the effective mass leads to quick crossings and anticrossings of states with increasing magnetic field. The cumulative effect of both an exceptionally high dielectric and poor effective mass quenches the energy gap, leading to an overlap of ground and other excited states even at lower confinement ($\omega_0 = 0.0002$ au) [Figure 4b,e,b,f]. The significant crossings and anticrossings exhibited in case of $\omega_0 = 0.0002$ and 0.002 au are also enhanced with increasing $\alpha$ from 1.0 to 10.0 [Figure 4c,f]. However, no appreciable change is observed on moving $\alpha$ from 10.0 to 100.0. At zero confinement ($\omega_0 = 0.0$ au), the number of HS becomes less while the number of GS increases in comparison to He, SiO$_2$ and BN excitonic systems [Figure 4g–i]. The profound effect of extremely high $\epsilon$ is reflected by these GS which indicate that for GaAs, the Coulombic correlations are substantially muted because of the high dielectric constant in comparison to the He QD system. Relative stabilization of all HS including the ground state also becomes negligible.

Level-Spacing Statistics. For an exciton in the 2-D harmonic dot with a due cyclotron frequency to investigate the degree of disorder in the energy-level spectrum, level-spacing statistics were studied for all systems with varying dielectric constants, that is, He, SiO$_2$, BN, and GaAs at $\omega_0 = 0.0$, 0.0002, and 0.002 au. Uncorrelated levels give rise to exponential or Poisson distribution (PS ($\approx be^{-\omega_0}$) as stated in accepted literature $^{41-43}$ The level statistics remain unchanged for varying confinement frequencies for different systems. In spite of crossings and anticrossings between levels, the level repulsions are not significant enough to give rise to phase transitions [Figure 5a–d]. We have presented here the level-spacing statistics for all systems, that is, He, SiO$_2$, BN, and GaAs at a particular $\omega_0 = 0.002$ au and $\alpha = 1.0$; the signature is not altered significantly on changing the confinement frequency and hence to keep matters simple, we have refrained from giving all graphs. The level-spacing statistics also do not change much with changing $\alpha$ from 1.0 to 10.0, 100.0, and 1000.0.

Thermodynamic Properties. From the energy-level spectra and the level-spacing statistics, thermodynamic quantities like specific heat ($C_v$) and magnetization ($M$) have been evaluated. These thermodynamic properties implicitly carry the qualitative information regarding the material properties. The present work on $C_v$ and $M$ is based on two-states problem, that is, the investigation of interaction among the ground and the first excited states in the limit of low temperature. Thus, specific heat and magnetization directly reflect the response of the system as a function of transverse magnetic field. The equations relating the energy and thermodynamic properties are mentioned in the subsequent sections.

Heat Capacity ($C_v$). From the energy-level spectra values and the level-spacing statistics, thermodynamic quantities like specific heat ($C_v$) and magnetization ($M$) have been evaluated. These thermodynamic properties implicitly carry the qualitative information regarding the material properties. The equations relating the energy and the thermodynamic properties are mentioned below. For an exciton in a 2-D QD at $\omega_0 = 0.0$, 0.002, and 0.00002 au for systems with varying dielectric constants viz. He ($\epsilon = 1.0$), SiO$_2$ ($\epsilon = 3.9$), BN ($\epsilon = 7.1$), and GaAs ($\epsilon = 12.35$), heat capacities ($C_v$) are examined as a function of $\omega_0$ at $T \approx 1$ K approximated for HS and GS by the following relation $^{44,45}$

$$C_v \approx \frac{e^2\epsilon^2e^{-\epsilon / x^2}}{(1 + e^{-\epsilon / x}^2)} + \frac{e^2\epsilon e^{-\epsilon}}{(1 + e^{-\epsilon} / x^2)^2}$$

(36)
where $\epsilon = \Delta E_{1}/2k_B T$ and $\Delta E_{1} = (E_1 - E_0)$. As indicated by the lack of level repulsions in the level-spacing statistics, $C_v$ is independent of $\omega_c$ for all systems [Figure 6], which is substantiated by the fact that there is no mixing of ground HS with the other GS. Only one heat capacity versus $\omega_c$ graph is shown for representational purposes.

**Magnetization ($M$).** For noninteractive and interactive systems at $\omega_0 = 0.0, 0.002$, and $0.00002$ au at low temperature limit ($T \approx (0-1)$ K), magnetization of ground hydrogenic and first excited GS is evaluated using the following form:

$$ E = yE_0 + (1 - y)E_1 $$

(37)
\[ M = -\frac{\partial E}{\partial \omega_c} \]  

(38)

where \( y = 1/(1 + e^{-\epsilon}) \), \( \epsilon_1 = \Delta E / k_B T \), \( \Delta E_1 = (E_1 - E_0) \), and \( E \) is the energy at specific \( \omega_c \). Magnetization shows exponential decay for all systems considered in the present study at different confinement frequencies.

Figure 9. Energy-level diagram of a triexciton in the 2-D QD as a function of magnetic field (\( \omega_c \)) for GaAs (\( \epsilon = 12.35, m_e = 0.07 \) au) (a,b—interactive system for \( \omega_o = 0.0002 \) au with \( \alpha = 1.0 \) and 10.0, respectively) (c,d—interactive system for \( \omega_o = 0.002 \) au with \( \alpha = 1.0 \) and 10.0, respectively).

Figure 10. Energy-level diagram of a biexciton in the 2-D QD as a function of magnetic field (\( \omega_c \)) for GaAs (\( \epsilon = 12.35, m_e = 0.07 \) au) (a,b—interactive system for \( \omega_o = 0.0002 \) au with \( \alpha = 1.0 \) and 10.0, respectively) (c,d—interactive system for \( \omega_o = 0.002 \) au with \( \alpha = 1.0 \) and 10.0, respectively).
Exciton, Triexciton, and Biexciton in the He QD ($m_e = 1.0 \text{ au}$, $\epsilon = 1.0$) and the GaAs QD ($m_e = 0.07 \text{ au}$, $\epsilon = 12.35$). As the number of interactions among the charge carriers on moving from exciton to biexciton increases, it leads to a corresponding increase in the energy of the system. Additional stabilization is introduced in the system because of enhanced exchange correlations. The He QD represents the lower end of the dielectric window for which we have conducted the study in excitonic systems. A multieexcitonic system in the He QD represents a strongly correlated system as $\epsilon = 1.0$. The various terms that contribute to energy along with the Coulomb (exchange) correlations are

$$E = (2n + ml + 1) \frac{\Omega^2}{m_e} + \left( \frac{\alpha - 1}{\alpha + 1} \right) \frac{\omega_m}{2m_e}$$

$$+ (2N + |M| + 1) \omega_k.$$

At lower confinement, that is, $\omega_k = 0.0002 \text{ au}$ and $\alpha = 1.0$, the energy gap between the few lowest and the higher excited states begins to quench as one moves from exciton and triexciton to biexciton [Figures 7 and 8a,b]. A highly appreciable energy gap between HS and GS in case of an exciton is quenched significantly for a triexciton which is further quenched, leading to an overlap of HS and GS in the biexciton. The effect of raising $\alpha$ from 1.0 to 10.0 results in an addition to the energy due to nonzero $\left( \frac{\alpha - 1}{\alpha + 1} \right) \frac{\omega_m}{2m_e}$. The energy increases on increasing the confinement frequency $\omega_k$ from 0.0002 to 0.002 $\text{ au}$ as expected, as the contribution due to both $(2n + ml + 1) \frac{\Omega^2}{m_e}$ and $\left( \frac{\alpha - 1}{\alpha + 1} \right) \frac{\omega_m}{2m_e}$ is increased [Figures 7 and 8c,d]. The major stabilizing force that leads to the stabilization of HS is the attractive Coulombic interaction. The energy of the states increases with the increase in magnetic field. A feature that emerges from scanning across the exciton, triexciton, and biexciton He QD is that the energy states do not exhibit crossings and anticrossings among different levels [Figures 7 and 8a–d]. The extent of stabilization due to exchange correlations differs from GS to HS. For biexcitons, in case of lower dielectric QDs, for example, He, GS are stabilized more than HS which is evident from acute and exaggerated crossings/anticrossings among both bound states, whereas for triexcitons, a shallow band gap is observed. The trend is found to be similar in the case of higher dielectric QDs (GaAs) but the band gap for the triexciton and mixing of states for the biexciton are less pronounced [Figures 9 and 10a–d].

**CONCLUSIONS**

In case of excitonic systems of a wide variety of materials with varying dielectric constants, a striking feature that has emerged from this study is that the unique ground state is rarely involved in crossings and anticrossings. Also, the number of lower bunching states along with the ground state is found to be 11 which is invariant for all systems (He, SiO$_2$, BN, and GaAs) in the given chosen basis. For all the systems considered in the present study at $\omega_k = 0.0 \text{ au}$, the density of states is analogous to HS. Also, such hydrogenic-bound states increase in number with increasing $\alpha$. As the dielectric constant of the system is increased, the states show a transformation from HS to GS; this observation is strengthened by the fact that on muting the oscillatory transverse field, the system collapses into a hydrogenic system. In case of He with $\epsilon = 1.0$, the states at $\omega_k = 0.0 \text{ au}$ resemble purely HS, whereas at the other end, in case of GaAs with $\epsilon = 12.35$, the states have more of GS character than HS. The impact of the Coulombic interaction on the electronic structure of excitonic QDs confined in a harmonic potential with transverse magnetic field is astonishing. The competitive behavior of the hybrid magnetoelectric confinement frequency ($\Omega_0$), $\omega_k$, $\epsilon$, $m_e$, and $m_h$ with each other leads to a criss-cross pattern of the energy spectrum. Moreover, $e^{-h/\epsilon-e^{-h/\epsilon}}$ interactions are higher for states having lower angular momentum ($m$). Level-spacing statistics, specific heat ($T \approx 1 \text{ K}$), and magnetization ($T \approx (0–1) \text{ K}$) also give clear affirmation of it. It is also observed that under limiting case of high angular momentum, the Coulomb correlation employed in our methodology virtually maps onto Laughlin’s speculations.

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**Notes**

The authors declare no competing financial interest.

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