On graph theory Mertens’ theorems

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Abstract

In this paper, we study graph-theoretic analogies of the Mertens’ theorems by using basic properties of the Ihara zeta-function. One of our results is a refinement of a special case of the dynamical system Mertens’ second theorem due to Sharp and Pollicott.

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1 Introduction

Throughout this paper, we use the notation in the textbook [11] of Terras for graph theory and the (Ihara) zeta-function, and we often refer to basic facts included in this textbook.

In 1874, Mertens proved the so-called Mertens’ first/second/third theorems (the equalities (5)(13)(14) in [7], respectively). In 1991, Sharp studied the dynamical-systemic analogues of Mertens’ second/third theorems (Theorem 1 in [10]), and in 1992, Pollicott improved the error terms in the theorems of Sharp as follows (Theorem and Remark in [9]):

• Dynamical system Mertens’ second theorem: For a hyperbolic (and so geodesic) flow (which is not necessarily topologically weak-mixing) restricted to a basic set with closed orbits $\tau$ of least period $\lambda(\tau)$ and topological entropy $h > 0$, as $x \to \infty$,

$$\sum_{N(\tau) \leq x} \frac{1}{N(\tau)} = \log(\log x) + \gamma + \log\left(\frac{\text{Res}_{s=1} \zeta(s)}{\zeta(s)}\right) - \sum_{\tau} \sum_{n \geq 2} \frac{1}{n} \cdot \frac{1}{N(\tau)^n} + O\left(\frac{1}{\log x}\right),$$

where $N(\tau) = e^{h\lambda(\tau)}$, $\gamma$ is the Euler-Mascheroni constant,

$$\zeta(s) = \prod_{\tau} \left(1 - \frac{1}{N(\tau)^s}\right)^{-1},$$

and $\text{Res}_{s=1} \zeta(s)$ denotes the residue of $\zeta$ at $s = 1$.

• Dynamical system Mertens’ third theorem: For the same flow, as $x \to \infty$,

$$\prod_{N(\tau) \leq x} \left(1 - \frac{1}{N(\tau)}\right) = \frac{e^{-\gamma}}{\text{Res}_{s=1} \zeta(s)} \cdot \frac{1}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$
(For the notation of dynamical systems, see the textbook [8] of Parry-Pollicott.) Note that Sharp and Pollicott did not explicitly write the dynamical system Mertens’ first theorem.

From the second theorem of Pollicott, the constant term (so-called Mertens constant) can be explicitly known, but the coefficients of $1/\log^k x$ cannot be computed. Our purpose in this paper is to present graph-theoretic analogies of the Mertens’ theorems whose coefficients can be explicitly known. So, our second theorem is a refinement of a special case of the theorem due to Pollicott in the sense that the coefficients of $1/\log^k x$ can be computed.

In the rest of this section, we introduce the notation of graph theory, next recall the notation and properties of the (Ihara) zeta-function, and last state the main theorem.

First, we recall the notation of graphs. Let $X$ be an undirected graph with vertex set $V$ of $\nu := |V|$ and edge set $E$ of $\epsilon := |E|$. Simply, such a graph $X$ is denoted by $X := (V, E)$.

A directed edge (or an arc) $a$ from a vertex $u$ to a vertex $v$ is denoted by $a = (u, v)$, and the inverse of $a$ is denoted by $a^{-1} = (v, u)$. The origin (resp. terminus) of $a$ is denoted by $o(a) := u$ (resp. $t(a) := v$).

We can direct the edges of $X$, and label the edges as follows:

$$\tilde{E} := \{e_1, e_2, \ldots, e_\ell, e_{\ell+1} = e_1^{-1}, e_{\ell+2} = e_2^{-1}, \ldots, e_{2\ell} = e_\ell^{-1}\}.$$  

A path $C = a_1 \cdots a_s$, where the $a_i$ are directed edges, is said to have a backtrack (resp. tail) if $a_{j+1} = a_j^{-1}$ for some $j$ (resp. $a_s = a_1^{-1}$), and a path $C$ is called a cycle (or closed path) if $o(a_1) = t(a_s)$. The length $\ell(C)$ of a path $C = a_1 \cdots a_s$ is defined by $\ell(C) := s$.

A cycle $C$ is called prime (or primitive) if it satisfies the following:

- $C$ does not have backtracks and a tail;
- no cycle $D$ exists such that $C = D^f$ for some $f > 1$.

The equivalence class $[C]$ of a cycle $C = a_1 \cdots a_s$ is defined as the set of cycles

$$[C] := \{a_1a_2 \cdots a_{s-1}a_s, a_2 \cdots a_{s-1}a_sa_1, \ldots, a_s a_1a_2 \cdots a_{s-1}\},$$

and an equivalence class $[P]$ of a prime cycle $P$ is called prime. Let $\Delta_X$ and $\pi_X(n)$ denote

$$\Delta = \Delta_X := \gcd\{\ell(P) : [P] \text{ is a prime equivalence class in } X\},$$

$$\pi(n) = \pi_X(n) := |\{[P] : [P] \text{ is a prime equivalence class in } X \text{ with } \ell(P) = n\}|.$$  

Throughout this paper, we always assume that $X$ is a finite, connected, noncycle and undirected graph without degree-one vertices, and we denote by a symbol $[P]$ a prime equivalence class.

Next, we recall the zeta-function of $X = (V, E)$. The (Ihara) zeta-function of $X$ is defined as follows (the equality (9) in [4], and also see Definition 2.2 in [11]):

$$Z_X(u) := \prod_{[P]} (1 - u^{\ell(P)})^{-1}$$

with $|u|$ sufficiently small, where $[P]$ runs through all prime equivalence classes in $X$. The radius of convergence of $Z_X(u)$ is denoted by $R_X$. Note that $0 < R_X < 1$ since $X$ is a noncycle graph (see, for example, page 197 in [11]).

Let $W = W_X := (w_{ij})$ denote the edge adjacency matrix of a graph $X$, that is, a $2\epsilon \times 2\epsilon$ matrix defined by

$$w_{ij} := \begin{cases} 
1 & \text{if } t(e_i) = o(e_j) \text{ and } e_j \neq e_i^{-1} \text{ for } e_i, e_j \in \tilde{E}, \\
0 & \text{otherwise} 
\end{cases}$$
In this paper, our main theorem is:

**Main Theorem.** Suppose that \( X = (V, E) \) is a finite, connected, noncycle and undirected graph without degree-one vertices. Set

\[
a = a(N) := \left\lfloor \frac{N}{\Delta_X} \right\rfloor \Delta_X = N - \left\lfloor \frac{N}{\Delta_X} \right\rfloor \Delta_X,
\]

where \([x]\) (resp. \(\{x\}\)) denotes the integer (resp. fractional) part of the real number \(x\), and thus \(0 \leq a(N) < \Delta_X\). Then, the following items (1) (2) (3) hold:

1. **(Graph theory Mertens’ first Theorem)** As \(N \to \infty\),

\[
\sum_{n \leq N} n \cdot \pi_X(n) R_X^n = N - a(N) + A_X + K_X + O\left(\left(\rho_X R_X\right)^N\right)
\]

\[= \left\lfloor \frac{N}{\Delta_X} \right\rfloor \Delta_X + A_X + K_X + O\left(\left(\rho_X R_X\right)^N\right),\]

where the constants \(A_X\), \(K_X\) and \(\rho_X\) are defined by

\[
A_X := \sum_{\lambda \in \text{Spec}(W), \ |\lambda| < 1/R_X} \frac{\lambda R_X}{1-\lambda R_X}, \quad K_X := \sum_{n=1}^{\infty} \left( n \cdot \pi_X(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right) R_X^n,
\]

and

\[
\rho_X := \max\{|\lambda| : \lambda \in \text{Spec}(W), \ |\lambda| < 1/R_X\},
\]

respectively. (The convergence of \(K_X\) is shown in Section 3.)

2. **(Graph theory Mertens’ second Theorem)** As \(N \to \infty\),

\[
\sum_{n \leq N} \pi_X(n) R_X^n = \log N + \gamma + \log C_X - H_X
\]

\[+ \sum_{s=1}^{k} \left( a^s + s \sum_{m=0}^{s-1} \binom{s-1}{m} \frac{a^m B_{s-m} \Delta_X^{s-m}}{s-m} \right) \frac{1}{N^s} + O\left(\frac{1}{N^{k+1}}\right),
\]

for each \(k \geq 1\), where \(\gamma\) is the Euler-Mascheroni constant, \(B_s\) are the \(s\)-th Bernoulli numbers defined by

\[
\frac{t}{e^t - 1} = \sum_{s=0}^{\infty} B_s \frac{t^s}{s!},
\]

and the constants \(C_X\) and \(H_X\) are defined by

\[
C_X := -\frac{1}{R_X} \text{Res}_{u=R_X} Z_X(u), \quad H_X := -\sum_{n \geq 1} \frac{1}{n} \left( n \cdot \pi_X(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right) R_X^n,
\]

respectively. (The convergence of \(H_X\) is shown in Section 3.)

In particular \((k = 0)\), as \(N \to \infty\),

\[
\sum_{n \leq N} \pi_X(n) R_X^n = \log N + \gamma + \log C_X - H_X + O\left(\frac{1}{N}\right).
\]
(3) (Graph theory Mertens’ third Theorem, [2]) As $N \to \infty$, 
\[
\prod_{n \leq N} (1 - R_X^n)^{\pi_X(n)} = \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N} \left( 1 + O \left( \frac{1}{N} \right) \right).
\]

Note that our first theorem (1) is a refinement of a result in our previous paper [2] in the sense that the constant term $A_X + K_X$ is explicitly written, and note that our second theorem (2) is a refinement of a special case of the result due to Pollicott (Theorem (i) and Remark in [9]) in the sense that all the coefficients of $1/N$ can be explicitly computed. Our proofs in this paper are elementary (without the theory of the Ihara prime zeta-function which is studied in [2]), and moreover they are completely different from previous proofs.

Our theorems (1)(2) can be simplified under the assumption which all the degrees of vertices are greater than 2: If $X$ is bipartite, then $\Delta_X = 2$, and so $a(2N) = 0$ or $a(2N+1) = 1$. Otherwise, $\Delta_X = 1$, and therefore $a(N) = 0$. (See, for details, Proposition 3.2 in [5].)

The contents of this paper are as follows. In the next section, we first prove a key lemma, which plays an important role in the proof of the main theorem, and next introduce the constants in the main theorem. In Section 3 we give the proof of the main theorem.

## 2 Key Lemma

In this section, in order to show the theorem, we introduce a key lemma and two constants.

The following facts are often used in this paper.

**Fact.** Suppose that $X = (V, E)$ satisfies the same conditions as the main theorem.

1. (Theorem 1.4 in [3], and also see Theorem 8.1 (3) in [11]) The poles of $Z_X(u)$ on the circle $|u| = R_X$ have the form $R_X e^{2\pi i a/\Delta_X}$, where $a = 1, 2, \ldots, \Delta_X$.

2. (Orthogonality relation, for example, see Exercise 10.1 in [11])

\[
\sum_{a=1}^{\Delta_X} e^{2\pi i a n/\Delta_X} = \begin{cases} 
\Delta_X & \text{if } \Delta_X | n; \\
0 & \text{otherwise.}
\end{cases}
\]

3. (Two-term determinant formula, [3] and [1], and also see (4.4) in [11]) The zeta-function $Z_X(u)$ can be written as

\[
Z_X(u) = \frac{1}{\det(J_{2e} - Wu)} = \prod_{\lambda \in \text{Spec}(W)} (1 - \lambda u)^{-1}.
\]

The following key lemma plays an important role in the proof of the main theorem.

**Key Lemma.** Suppose that $X = (V, E)$ satisfies the same conditions as the main theorem.

(a) As $N \to \infty$, 
\[
\sum_{n=1}^{N} \sum_{\lambda \in \text{Spec}(W)} (\lambda R_X)^n = \left[ \frac{N}{\Delta_X} \right] \Delta_X + A_X + O \left( (\rho_X R_X)^N \right),
\]

\[
\text{where } [x] \text{ denotes the integer part of the real number } x.
\]
(b) As $N \to \infty$,

\[
\sum_{n=1}^{N} \frac{1}{n} \sum_{\lambda \in \text{Spec}(W)} (\lambda R_X)^n = \sum_{n=1}^{\lfloor N/\Delta \rfloor} \frac{1}{n} \log C_X + \log \Delta_X + O\left((\rho_X R_X)^N\right).
\]

(c) ([2]) Let $0 < \alpha < 1/2$ be a real number, and fix it. Then, there exists a natural number $N_0 = N_0(\alpha)$ such that for any $n \geq N_0$,

\[
\left| n \cdot \pi_X(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right| < 2 \epsilon \left(1 - \frac{1}{R_X}\right)^{(1-\alpha)n}.
\]

(d) (cf. Section 2 in [6]) Set

\[
a = a(n) := N - \left\lfloor \frac{N}{\Delta_X} \right\rfloor \Delta_X,
\]

and thus $0 \leq a(N) < 1$. Then,

\[
\sum_{n=1}^{\lfloor N/\Delta \rfloor} \frac{1}{n} = \log N - \log \Delta_X + \gamma
\]

\[
- \sum_{s=1}^{k} \left( \frac{a^s}{s} + \sum_{m=0}^{s-1} \left( \frac{a^m B_{s-m} \Delta_X^{s-m}}{s-m} \right) \right) \frac{1}{N^s} + O\left(\frac{1}{N^{k+1}}\right)
\]

for each $k \geq 1$.

**Proof.** In this proof, we abbreviate the suffix $X$, that is, $R = R_X$, $\Delta = \Delta_X$, etc.

Let $\{a_n\}$ be a sequence of real numbers with $0 < a_n \leq 1$ for any $n$. Then, it follows from Facts (1)(2)(3) that we obtain the equality

\[
\sum_{n=1}^{N} a_n \sum_{\lambda \in \text{Spec}(W), \ |\lambda| = 1/R} (\lambda R_X)^n = \sum_{n=1}^{N} a_n \sum_{\lambda \in \text{Spec}(W), \ |\lambda| < 1/R} e^{-2\pi i an/\Delta} = \Delta \sum_{n=1}^{\lfloor N/\Delta \rfloor} a_n \Delta.
\]  

(1)

Moreover, it follows by the triangle inequality that we obtain the inequality

\[
\left| \sum_{n>N} a_n \sum_{\lambda \in \text{Spec}(W), \ |\lambda| = 1/R} (\lambda R_X)^n \right| \leq \sum_{n>N} a_n \sum_{\lambda \in \text{Spec}(W), \ |\lambda| < 1/R} (|\lambda| R_X)^n
\]

\[
< 2 \epsilon \sum_{n>N} (\rho R)^n = \frac{2 \epsilon \rho R}{1 - \rho R} (\rho R)^N.
\]  

(2)

In the proofs of the items (a)(b), we use the equality (1) and the inequality (2).

(a) Note that

\[
\sum_{n=1}^{N} \sum_{|\lambda| = 1/R} (\lambda R)^n - \left\lfloor \frac{N}{\Delta} \right\rfloor \Delta = \sum_{n=1}^{\lfloor N/\Delta \rfloor} \Delta - \left\lfloor \frac{N}{\Delta} \Delta \right\rfloor = 0
\]
by the equality (1). On the other hand, note that
\[
\left| \sum_{n=1}^{N} \sum_{|\lambda| < 1/R} (\lambda R)^n - A \right| = \left| \sum_{n>N} \sum_{|\lambda| < 1/R} (\lambda R)^n \right| < \frac{2\epsilon \rho R}{1 - \rho R} (\rho R)^N
\]
from the inequality (2). By combining these, the item (a) follows from the triangle inequality.

(b) Set the sums
\[
S_1(N) := \sum_{n=1}^{N} \frac{1}{n} \sum_{|\lambda| = 1/R} (\lambda R)^n \quad \text{and} \quad S_2(N) := \sum_{n=1}^{N} \frac{1}{n} \sum_{|\lambda| < 1/R} (\lambda R)^n.
\]

First, we consider the sum \(S_1(N)\). It follows from the equality (1) that
\[
S_1(N) = \sum_{n=1}^{N} \frac{1}{n} \sum_{|\lambda| = 1/R} (\lambda R)^n = \sum_{n=1}^{[N/\Delta]} \frac{1}{n}.
\]

Next, we compute the sum \(S_2(N)\). We now consider the constant defined by
\[
F := \log \prod_{|\lambda| < 1/R} \frac{1 - u}{1 - \lambda R} \left( = - \sum_{|\lambda| < 1/R} \log(1 - \lambda R) = \sum_{|\lambda| < 1/R, n \geq 1} \frac{1}{n} (\lambda R)^n \right),
\]
and then we obtain \(F = \log C_X + \log \Delta\). This is proved as follows: Note that
\[
C_X = \lim_{u \uparrow R} \frac{(R - u)Z_X(u)}{R} = \lim_{u \uparrow R} \left( 1 - \frac{u}{R} \right) \prod_{\lambda \in \text{spec}(W), \lambda \neq 1/R} \frac{1}{1 - \lambda u} = \lim_{u \uparrow R} \prod_{\lambda \in \text{spec}(W), \lambda \neq 1/R} \frac{1}{1 - \lambda u} \prod_{\lambda \in \text{spec}(W), \lambda \neq 1/R} \frac{1}{1 - \lambda R}
\]
by the definition of \(C_X\) and Facts (1)(3). It is well known that
\[
\sum_{n=0}^{\Delta - 1} X^n = \frac{X^{\Delta - 1} - 1}{X - 1} = \prod_{a=1}^{\Delta - 1} \left( X - e^{-2\pi ia/\Delta} \right), \quad \text{and so} \quad \Delta = \prod_{a=1}^{\Delta - 1} \left( 1 - e^{-2\pi ia/\Delta} \right).
\]

Combining these equalities, we obtain
\[
\prod_{|\lambda| < 1/R} \frac{1}{1 - \lambda R} = \prod_{\lambda \in \text{spec}(W), \lambda \neq 1/R} \frac{1}{1 - \lambda R} \cdot \prod_{|\lambda| = 1/R, \lambda \neq 1/R} (1 - \lambda R) = \prod_{\lambda \in \text{spec}(W), \lambda \neq 1/R} \frac{1}{1 - \lambda R} \cdot \prod_{a=1}^{\Delta - 1} \left( 1 - e^{-2\pi ia/\Delta} \right) = C_X \cdot \Delta,
\]
and thus \(F = \log C_X + \log \Delta\).

It follows from the inequality (2) that we obtain the inequality
\[
|S_2(N) - F| = \left| \sum_{|\lambda| < 1/R} \sum_{n > N} \frac{1}{n} (\lambda R)^n \right| < \frac{2\epsilon (\rho R)^{N+1}}{1 - \rho R},
\]
that is, 
\[ S_2(N) = F + O ((\rho R)^N) = \log C_X + \log \Delta + O ((\rho R)^N). \]

Hence, by combining the above results, we obtain
\[
\sum_{n=1}^{N} \frac{1}{n} \sum_{\lambda \in \text{Spec}(W)} (\lambda R)^n = S_1(N) + S_2(N)
\]
\[
= \sum_{n=1}^{[N/\Delta]} \frac{1}{n} + \log C_X + \log \Delta + O ((\rho R)^N).
\]

(c) Let \( \mu(n) \) denote the Möbius function. Note that \( \sum_{d|n} |\mu(d)| \leq n \). It is known that
\[
\pi(n) = \frac{1}{n} \sum_{d|n} \mu(d) N_{n/d}, \quad \text{and} \quad N_n = \sum_{\lambda \in \text{Spec}(W)} \lambda^n
\]
(see (10.3) and (10.4) in [11], respectively). Combining these equalities, we obtain
\[
n \cdot \pi(n) = \sum_{\lambda \in \text{Spec}(W)} \sum_{d|n} \mu(d) \lambda^{n/d},
\]
and therefore
\[
\left| n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right| = \left| \sum_{\lambda \in \text{Spec}(W)} \sum_{d|n} \mu(d) \lambda^{n/d} \right| \\
\leq \sum_{\lambda \in \text{Spec}(W)} \sum_{d|n, \ d \geq 2} |\mu(d)| \cdot |\lambda|^{n/d} \leq \sum_{\lambda \in \text{Spec}(W)} \sum_{d|n, \ d \geq 2} |\mu(d)| \cdot |\lambda|^{n/2} \\
< n \sum_{\lambda \in \text{Spec}(W)} \left( \frac{1}{R} \right)^{n/2} \leq 2\epsilon n \left( \frac{1}{R} \right)^{n/2}.
\]

On the other hand, since \( 0 < R < 1 \) and \( 0 < \alpha < 1/2 \) by our assumptions, there exists a natural number \( N_0 = N_0(\alpha) \) such that for any \( n \geq N_0 \),
\[
n \leq \left( \frac{1}{R} \right)^{(1/2-\alpha)n}, \quad \text{and so} \quad n \left( \frac{1}{R} \right)^{n/2} \leq \left( \frac{1}{R} \right)^{(1-\alpha)n}.
\]
Hence, for any \( n \geq N_0 \),
\[
\left| n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right| \leq 2\epsilon \left( \frac{1}{R} \right)^{(1-\alpha)n},
\]
and the assertion of the item (c) follows.

(d) It is known from the equality (9) in [6] that
\[
\left| \int_{[N/\Delta]}^{\infty} \frac{P_{2k+1}(x)}{x^{2k+2}} \, dx \right| = O \left( \frac{1}{N^{2k+1}} \right) \quad \left( = O \left( \frac{1}{N^{2k+1}} \right) \right),
\]
where \( P_{2k+1}(x) \) is a periodic Bernoulli polynomial. Note that \( [N/\Delta] = (N-a)/\Delta \). Recall that the \((2s-1)\)-th Bernoulli numbers \( B_{2s-1} \) \((s \geq 1)\) are given by
\[
B_1 = -1/2 \quad \text{and} \quad B_{2s-1} = 0 \quad (s \geq 2).
\]
Then, it follows from the equality (7) in [6] and the above equality (3) that
\[
\sum_{n=1}^{[N/\Delta]} \frac{1}{n} - \gamma = \log \left[ \frac{N}{\Delta} \right] + \frac{1}{2[N/\Delta]} - \sum_{s=1}^{k} \frac{B_{2s}}{2s[N/\Delta]^{2s}} + O \left( \frac{1}{N^{k+1}} \right),
\]
and therefore
\[
\sum_{n=1}^{[N/\Delta]} \frac{1}{n} - \gamma = \log \left[ \frac{N}{\Delta} \right] - \sum_{s=1}^{k} \frac{B_{s}}{s[N/\Delta]^{s}} + O \left( \frac{1}{N^{k+1}} \right),
\]
which implies
\[
\sum_{n=1}^{[N/\Delta]} \frac{1}{n} - \gamma = \log \left[ \frac{N}{\Delta} \right] - \sum_{s=1}^{k} \frac{B_{s}}{s[N/\Delta]^{s}} + O \left( \frac{1}{N^{k+1}} \right),
\]
On the other hand, since the inequality
\[
\binom{s-1+m}{m} = \frac{s-1+m}{m} \cdot \frac{s-2+m}{m-1} \cdot \ldots \cdot \frac{s+1}{2} \cdot \frac{1}{s^{m}}
\]
holds, we obtain the inequalities
\[
\sum_{s=1}^{k} \frac{B_{s}}{s[N/\Delta]^{s}} \sum_{m>k-s} \binom{s-1+m}{m} \left( \frac{a}{N} \right)^{m} \leq \sum_{s=1}^{k} \frac{B_{s}}{s[N/\Delta]^{s}} \sum_{m>k-s} \left( \frac{sa}{N} \right)^{m} \leq \sum_{s=1}^{k} \frac{B_{s}}{s[N/\Delta]^{s}} \left( \frac{sa}{N} \right)^{k-s+1} \frac{1}{1-sa/N}
\]
that is,
\[
\sum_{s=1}^{k} \frac{B_{s}}{s[N/\Delta]^{s}} \sum_{m>k-s} \binom{s-1+m}{m} \left( \frac{a}{N} \right)^{m} = O \left( \frac{1}{N^{k+1}} \right).
\]
Hence, combining the equalities (4) (5), we obtain
\[
\sum_{n=1}^{[N/\Delta]} \frac{1}{n} = \log N - \log \Delta + \gamma
\]
and the assertion of the item (d) follows after an elementary computation. □
By using KeyLemma (c), we can prove the convergences of constants.

**Lemma 1.** Suppose that $X = (V, E)$ satisfies the same conditions as the main theorem.

1. The series
   \[ K_X = \sum_{n \geq 1} \left( n \cdot \pi_X(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right) R^n_X \]
   is convergent.

2. The series
   \[ H_X = -\sum_{n \geq 1} \frac{1}{n} \left( n \cdot \pi_X(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right) R^n_X \]
   is convergent. Moreover,
   \[ H_X = \sum_{|P|} \sum_{m \geq 2} \frac{1}{m} R^m(P) \]
   holds.

**Proof.** Let $\{a_n\}$ be a sequence of real numbers with $0 < a_n \leq 1$ for any $n$. Note that

\[
\left| \sum_{n \geq N_0} a_n \left( n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right) R^n \right| \leq \sum_{n \geq N_0} \left| n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right| R^n
\]

\[
< 2 \epsilon \sum_{n \geq N_0} \left( \frac{1}{R} \right)^{(1-\alpha)n} R^n
\]

\[
\leq 2 \epsilon \sum_{n \geq 1} R^n = \frac{2 \epsilon R^\alpha}{1 - R^\alpha}
\]

by KeyLemma (c). Hence, the convergences of the items (1)(2) hold from this inequality.

Next, we show the equality of the item (2). Assume that $|u| < R$. It follows from Fact (3) and the definition of $Z_X(u)$ that

\[
\sum_{n \geq 1} \sum_{\lambda \in \text{Spec}(W)} \frac{\lambda^n}{n} u^n = \sum_{\lambda \in \text{Spec}(W)} \log(1 - \lambda u)^{-1} = \log Z_X(u) = \sum_{|P|} \log(1 - u^{\ell(P)})^{-1}
\]

\[
= \sum_{|P|} \sum_{m \geq 1} \frac{1}{m} u^{m\ell(P)} = \sum_{|P|} u^{\ell(P)} + \sum_{|P|} \sum_{m \geq 2} \frac{1}{m} u^{m\ell(P)}
\]

\[
= \sum_{n \geq 1} \pi(n) u^n + \sum_{|P|} \sum_{m \geq 2} \frac{1}{m} u^{m\ell(P)},
\]

and therefore

\[
-\sum_{n \geq 1} \frac{1}{n} \left( n \cdot \pi_X(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right) u^n = \sum_{|P|} \sum_{m \geq 2} \frac{1}{m} u^{m\ell(P)}. \tag{6}
\]
On the other hand, by the graph theory prime-number theorem (see Theorem 10.1 in [11]), the radius of convergence of the function
\[ P(u) = \sum_{\mathcal{P}} u^{e_\mathcal{P}} = \sum_{n \geq 1} \pi(n) u^n \]
is equal to \( R \). Note that \( R^2 < R \) since \( 0 < R < 1 \). Then,
\[
\sum_{\mathcal{P}} \sum_{m \geq 2} \frac{1}{m} R^{m e_\mathcal{P}} \leq \sum_{\mathcal{P}} \sum_{m \geq 2} R^{m e_\mathcal{P}} = \sum_{\mathcal{P}} \frac{R^{2 e_\mathcal{P}}}{1 - R^{e_\mathcal{P}}} \leq \frac{1}{1 - R} \sum_{\mathcal{P}} R^{2 e_\mathcal{P}} \leq \frac{1}{1 - R} P(R^2) < +\infty.
\]
Hence, since both sides of the equality (6) are also convergent for \( u = R \), the assertion follows by the uniqueness of analytic continuation (namely, the principle of uniqueness).

3 Proof of the main theorem

In this section, we show the main theorem.

Proof. (The main theorem) (1) Assume that \( N \) is sufficiently large. Then, it follows from KeyLemma (c) that we obtain
\[
\left| \sum_{n=1}^{N} n \cdot \pi(n) R^n - \sum_{\mathcal{P}} \sum_{\lambda \in \text{Spec}(W)} (\lambda R)^n - K \right| = \left| \sum_{n > N} \left( n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right) R^n \right| \leq \sum_{n > N} \left| n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right| R^n < 2\epsilon \sum_{n > N} \left( \frac{1}{R} \right)^{(1-\alpha)n} R^n = 2\epsilon R^{\alpha n} \frac{R^{N+1}}{1 - R^\alpha},
\]
and therefore by KeyLemma (a), we obtain
\[
\sum_{n=1}^{N} n \cdot \pi(n) R^n = \sum_{\mathcal{P}} \sum_{\lambda \in \text{Spec}(W)} (\lambda R)^n + K + O((\rho R)^N)
\]
\[
= \left[ \frac{N}{\Delta} \right] \Delta + A + K + O((\rho R)^N)
\]
as \( N \to \infty \). Hence, the assertion of the item (1) follows.
(2) Suppose that \(N\) is sufficiently large. Then, it follows from KeyLemma (c) that

\[
\left| \sum_{n=1}^{N} \pi(n)R^n - \sum_{n=1}^{N} \frac{1}{n} \sum_{\lambda \in \text{Spec}(W)} (\lambda R)^n + H \right| = \left| \sum_{n>N} \left( \frac{\pi(n) - 1}{n} \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right) R^n \right|
\]

\[
\leq \sum_{n>N} \left| n \cdot \pi(n) - \sum_{\lambda \in \text{Spec}(W)} \lambda^n \right| R^n
\]

\[
< 2\epsilon \sum_{n>N} \left( \frac{1}{R} \right)^{(1-\alpha)n} R^n
\]

\[
= 2\epsilon \sum_{n>N} R^{\alpha n} = \frac{2\epsilon R^{\alpha(n+1)}}{1-R^\alpha},
\]

and therefore we obtain

\[
\sum_{n=1}^{N} \pi(n)R^n = \sum_{n=1}^{N} \frac{1}{n} \sum_{\lambda \in \text{Spec}(W)} (\lambda R)^n - H + O \left( R^{\alpha N} \right)
\]

as \(N \to \infty\). Hence, it follows from KeyLemmas (b)(d) that the assertion holds.

(3) Assume that \(N\) is sufficiently large, and define the following functions:

\[
H_{\leq N} = \sum_{n \leq N} \pi(n) \sum_{m \geq 2} \frac{1}{m} R_{X}^{mn}, \quad \text{and} \quad H_{> N} = \sum_{n>N} \pi(n) \sum_{m \geq 2} \frac{1}{m} R_{X}^{mn}.
\]

Note that \(H = H_{\leq N} + H_{> N}\) by Lemma \(\Box\) (2). It follows from the graph theory prime-number theorem (see Theorem 10.1 in [11]) that there exists a constant \(c_1 > 0\) such that for any \(n > N\),

\[
\pi(n) < \frac{c_1}{R^n}.
\]

Recall that \(0 < R < 1\) since \(X\) is a noncycle graph. Then, we obtain

\[
H_{> N} = \sum_{n>N} \pi(n) \sum_{m \geq 2} \frac{1}{m} R_{X}^{mn} < c_1 \sum_{n>N} \sum_{m \geq 2} R^{(m-1)n}
\]

\[
= c_1 \sum_{n>N} \frac{R^n}{1-R^n} < c_1 \frac{1}{1-R} \sum_{n>N} R^n = \frac{c_1 R}{(1-R)^2} R^N,
\]

and therefore \(H_{> N} = O(R^N)\).

From the item (2) and the above result, we obtain

\[
\sum_{n \leq N} \pi(n)R^n + H_{\leq N} = \log N + \gamma + \log C_X - H_{> N} + O \left( \frac{1}{N} \right)
\]

\[
= \log N + \gamma + \log C_X + O \left( \frac{1}{N} \right).
\]
Since the left-hand side of the above equality is equal to
\[ \sum_{n \leq N} \pi(n) R^n + H \leq N = \sum_{n \leq N} \pi(n) \sum_{m=1}^{\infty} \frac{1}{m} R^{mn} \]
\[ = - \sum_{n \leq N} \pi(n) \log (1 - R^n) = - \log \left( \prod_{n \leq N} (1 - R^n)^{\pi(n)} \right), \]
we obtain
\[ \prod_{n \leq N} (1 - R^n)^{\pi(n)} = \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N} \exp \left( O \left( \frac{1}{N} \right) \right) = \frac{e^{-\gamma}}{C_X} \cdot \frac{1}{N} \left( 1 + O \left( \frac{1}{N} \right) \right), \]
and the assertion of the item (3) follows.

Remark. (1) When \( k = 0 \), our second theorem just corresponds to a special case of the second theorem due to Pollicott (see Section 1 in this paper). This is proved as follows:

In our case, the topological entropy is equal to the constant \( h = -\log R > 0 \). We now define \( u = R_s \), \( N(P) = e^{h\ell(P)} = R^{-\ell(P)} \) and \( x = e^{hN} \). Then, the left-hand side is equal to
\[ \sum_{n \leq N} \pi(n) R^n = \sum_{\ell(P) \leq N} R^{\ell(P)} = \sum_{N(P) \leq x} \frac{1}{N(P)}. \]

On the other hand, the right-hand side can be transformed as follows. Note that
\[ C_X = -\frac{1}{R} \cdot \lim_{u \to R} (u - R) Z_X(u) \]
\[ = -\frac{1}{R} \cdot \lim_{s \to 1} \frac{R^s - R}{s - 1} \cdot \lim_{s \to 1} (s - 1) Z_X(R^s) = h \cdot \text{Res}_{s=1} Z_X(R^s). \]

It follows from Lemma 7 (2) that
\[ H_X = \sum_{P} \sum_{n \geq 2} \frac{1}{n} R^{n\ell(P)} = \sum_{P} \sum_{n \geq 2} \frac{1}{n} \cdot \frac{1}{N(P)^n}. \]

By combining the above results, we obtain
\[ \log N + \gamma + \log C_X - H_X + O \left( \frac{1}{N} \right) \]
\[ = \log (\log x) + \gamma + \log \left( \text{Res}_{s=1} Z_X(R^s) \right) - \sum_{P} \sum_{n \geq 2} \frac{1}{n} \cdot \frac{1}{N(P)^n} + O \left( \frac{1}{\log x} \right). \]

Hence, we obtain
\[ \sum_{N(P) \leq x} \frac{1}{N(P)} = \log (\log x) + \gamma + \log \left( \text{Res}_{s=1} Z_X(R^s) \right) - \sum_{P} \sum_{n \geq 2} \frac{1}{n} \cdot \frac{1}{N(P)^n} + O \left( \frac{1}{\log x} \right). \]

(2) The error term \( O(1/N) \) in our second theorem can not be replaced by \( o(1/N) \) since in general, the coefficient \( \Delta/2 - a(N) \) of \( 1/N \) is not equal to zero.
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