MAXIMIZING TORSIONAL RIGIDITY ON RIEMANNIAN MANIFOLDS

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Abstract. Let \((M, g)\) be a \(n\)-dimensional Riemannian manifold and \(\Omega\) be any compact connected domain in \(M\). We study the problem of finding the maxima of the functional \(E(\Omega)\) (known as torsional rigidity associated to \(\Omega\)) among all domains of prescribed volume \(v\). Our results show that for a given Riemannian manifold which is strictly isoperimetric at one of its points the maximum of such functional is realized by the geodesic ball centered at this point. More generally, we prove estimates for the functional \(E(\Omega)\) by comparison with symmetrized domains. We also investigate on finding sharp upper bounds for the functional \(E(\Omega)\), under certain conditions on the geometry of \((M, g)\) and of \(\Omega\). Finally we find an universal upper bound for \(E(\Omega)\) in terms of the isoperimetric Cheeger constant.

1. Introduction

Let \((M, g)\) be a \(n\)-dimensional Riemannian manifold (compact or not), \(d\) be the associated Riemannian distance and \(dv_g\) the associated Riemannian measure. Let \(\Omega\) be any compact connected domain in \(M\), with smooth boundary \(\partial \Omega\) (by this, in the case where \(M\) is compact, we also intend that the interior of \(M \setminus \Omega\) is a non empty open set). Let us denote by \(\Delta\) the Laplacian\(^1\) on \(M\) associated to the Riemannian metric \(g\), and let \(f_\Omega\) be a solution of the following Dirichlet problem

\[
\begin{align*}
\Delta f &= 1 \quad \text{on} \quad \Omega \\
f &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

\(^1\) If \(f\) is a smooth function on \(M\) then \(\Delta f = -\text{Trace}(\nabla df)\) and so the Euclidean Laplacian writes \(\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\).
Let $C^\infty_c(\Omega)$ be the space of $C^\infty$ functions with compact support in the interior of $\Omega$ and let $H^2_{1,c}(\Omega)$ be its completion with respect to the Hilbert (Sobolev) norm $\|f\|_{H^2_{1,c}(\Omega)} = (\|f\|_{L^2(\Omega)}^2 + \|
abla f\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$. As $f_\Omega$ is regular and vanishes on $\partial\Omega$, it is not hard to see that $f_\Omega \in H^2_{1,c}(\Omega)$. Moreover, $f_\Omega(x) > 0$ for any $x \in \Omega$.

On the space $H^2_{1,c}(\Omega)$ let us consider the functional $E_\Omega$ defined by

$$E_\Omega(f) = \frac{1}{\text{Vol}(\Omega)} \left( 2 \int_\Omega f \, dv - \int_\Omega |\nabla f|^2 \, dv \right).$$

(2)

Computing the first variation of $E_\Omega$ at the point $f$, we get that

$(f$ is a critical point of $E_\Omega) \iff (f$ is a solution of (1)).

The existence of (at least) one solution $f_\Omega$ of (1) proves that the functional $f \mapsto E_\Omega(f)$ (defined on $H^2_{1,c}(\Omega)$) admits at least one critical point. Moreover, the functional $E_\Omega$ is strictly concave hence it admits a unique critical point, which is its (unique) absolute maximum and, consequently, problem (1) admits $f_\Omega$ as a unique solution.

Therefore one can give the following definition (see [18] and references therein for more details).

**Definition 1.1.** Let $\Omega \subset M$ as above. The torsional rigidity $E_\Omega$ of $\Omega$ is the value

$$E(\Omega) = E_\Omega(f_\Omega) = \max_{f \in H^2_{1,c}(\Omega)} (E_\Omega(f)).$$

Consider the functional $E : \Omega \to \mathbb{R}$ restricted to the set of domains $\Omega \subset M$ with smooth boundary and prescribed volume $v$. It is known that its critical points are the harmonic domains namely those domains $\Omega \subset M$ such that the function $\|
abla f_\Omega(x)\|$ is constant on the boundary $\partial\Omega$. In the literature, the proof of this assertion is based on a Brownian motion probabilistic argument (see [18], Proposition 2.1). Recall also the classical fact that, in a Riemannian manifold $(M, g)$ which is harmonic at one of its point $x_0$ (see next section for the definition), every geodesic ball centered at $x_0$ is a harmonic domain. Hence one has the following fundamental question.

**Question 1.2.** On a Riemannian manifold $(M, g)$ which is harmonic at some of its points $x_0$ is every harmonic domain a geodesic ball centered at $x_0$?

With respect to this question, when $(M, g)$ is a real space form (and hence harmonic at each of its points) the following results hold true. J.

\textsuperscript{2}See Remark 3.4 in Section 3 below for the explanation of the appearance of $\text{Vol}(\Omega)$ in the definition of $E_\Omega$.

\textsuperscript{3}When $\Omega$ is a domain of the Euclidean plane, $E(\Omega)$ is the torsional rigidity of a beam whose cross section is $\Omega$. 
Serrin ([21]) proved that every harmonic domain of \((\mathbb{R}^n, \text{can.})\) is a ball. S. Kumaresan and J. Prajapat ([16]) extended Serrin’s result, proving that every harmonic domain of \((\mathbb{H}^n, \text{can.})\) is a geodesic ball and that every harmonic domain of \((S^n, \text{can.})\) whose closure is contained in an hemisphere is a geodesic ball. On the other hand it is not true that every harmonic domain in \((S^n, \text{can.})\) is a geodesic ball (semi-classical counterexamples are given by tubular neighbourhoods, in \(S^3\), of some geodesic circle \(S^1\) and, more generally, by domains with isoparametric boundary in \(S^n\)) and so the previous question has, in general, a negative answer.

One of the aims of the present paper is to address the problem of studying the maxima (instead of the critical points) of the functional \(\Omega \mapsto E(\Omega)\) among all domains of prescribed volume \(v\) (obviously such maxima are harmonic domains).

The following represents our first result. It shows that, for a given Riemannian manifold \((M, g)\) which is strictly isoperimetric at one of its points (see Definition 2.3 in Section 2.2 below), the geodesic ball centered at this point realizes the maximum of the torsional rigidity \(E(\Omega)\) and moreover a maximum is a geodesic ball.

**Theorem 1.3.** Let \((M, g)\) be a Riemannian manifold which is isoperimetric at some point \(x_0 \in M\) and let \(\Omega\) be any compact domain with smooth boundary in \(M\). Let \(\Omega^*\) be the geodesic ball of \((M, g)\) centered at \(x_0\) such that \(\text{Vol}(\Omega^*) = \text{Vol}(\Omega)\), then

\[E(\Omega) \leq E(\Omega^*).\]

Moreover, if \((M, g)\) is strictly isoperimetric at \(x_0\) then the equality \(E(\Omega) = E(\Omega^*)\) is realized if and only if \(\Omega\) is isometric to \(\Omega^*\).

An immediate consequence of Theorem 1.3 is that, on the Euclidean space, on the Hyperbolic space and on the canonical sphere, the geodesic balls are the domains which realize the maximum of the functional \(\Omega \mapsto E(\Omega)\) among all domains of prescribed volume \(v\) (this was already known for domains in the Euclidean space, in the Hyperbolic space and in the canonical open hemisphere, see [13] and also [7]). The present result extends this property to the whole sphere and to domains in some manifolds of revolution described in Section 2.1 below.

The proof of Theorem 1.3 is a particular case of the following Theorem 1.4. The reader is referred to Section 3 below for the definition (Definition 3.1) of symmetric domain \(\Omega^*\) of a given domain \(\Omega\) and of pointed isoperimetric model space \((M^*, g^*, x^*)\) of a Riemannian manifold \((M, g)\) (PIMS in the sequel).

**Theorem 1.4.** Let \((M, g)\) be a Riemannian manifold and let \((M^*, g^*, x^*)\) be a PIMS for \((M, g)\). Let \(\Omega\) be any compact domain with smooth boundary in \(M\) and let \(\Omega^*\) be its symmetrized domain. Then

\[E(\Omega) \leq E(\Omega^*).\]
Moreover, if \((M^*, g^*, x^*)\) is a strict PIMS for \((M, g)\) then the equality 
\(E(\Omega) = E(\Omega^*)\) is realized if and only if \(\Omega\) is isometric to \(\Omega^*\).

From Theorem 1.4 the following question naturally arises.

**Question 1.5.** Can we find sharp universal upper bounds \(C(v)\) for the torsional rigidity \(E(\Omega)\) which are independent on the geometry of \((M, g)\) (except for some a priori bounds on its curvature and diameter) and on the geometry of the domain \(\Omega \subset M\) (provided that this domain has prescribed volume \(v\))? 

In order to attack this question one needs to find a unique “universal’ strict PIMS for all the Riemannian manifolds that belong to a given class.

In the noncompact case one has the following well-known conjecture which is called Cartan–Hadamard’s conjecture (or Aubin’s conjecture) in the literature. We recall that a Cartan–Hadamard manifold is a complete simply connected Riemannian manifold with non positive sectional curvature.

**Conjecture 1.** The Euclidean \(n\)-dimensional space \(E^n\), pointed in any point \(x^* \in E^n\), is a strict PIMS for every Cartan-Hadamard manifold.

This conjecture is known to be true when the dimension \(n\) is equal to 2 (it is a classical fact, using the Gauss-Bonnet formula, proved for the first time by A. Weil in [23]), in dimension 4 (it was proved by C. B. Croke [8], using Santalo’s formula) and in dimension 3 (it is a more recent proof by B. Kleiner [15]). In higher dimensions, Conjecture 1 is still open.

Using these results we immediately get the following corollary of Theorem 1.4 which provides an answer to Question 1.5 in the noncompact case.

**Corollary 1.6.** Let \((M, g)\) be a Cartan–Hadamard manifold of dimension \(n \leq 4\). For every compact domain \(\Omega \subset M\) with smooth boundary, one has 
\[E(\Omega) \leq E(\Omega^*),\]
where \(\Omega^*\) is the Euclidean ball with the same volume as \(\Omega\). Moreover the equality 
\(E(\Omega) = E(\Omega^*)\) is realized if and only if \(\Omega\) is isometric to an Euclidean ball.

Notice that, if the conjecture 1 was true in every dimension \(n\), the Corollary 1.6 would be automatically true in any dimension.

In the compact case one has the celebrated Gromov’s isoperimetric inequality and its generalization due to P. Béard, G. Besson and S. Gallot (see respectively Theorems 4.1 and 4.6). Using these isoperimetric results and a Theorem of G. Perelman (Theorem 4.9) we obtain the following result which gives an answer to Question 1.5 in the compact case. Moreover it shows that a manifold has the same geometry and topology as a sphere by the knowledge of the value of the torsional rigidity of one of its domains.

**Theorem 1.7.** For every complete, connected Riemannian manifold \((M, g)\) whose Ricci curvature satisfies \(\text{Ric}_g \geq (n - 1).g\), for every compact domain
with smooth boundary $\Omega$ in $M$, let $\Omega^*$ be a geodesic ball of the canonical sphere $(S^n, g_0)$ such that \[
\frac{\text{Vol}(\Omega^*, g_0)}{\text{Vol}(\mathbb{S}^n, g_0)} = \frac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)},\]
then
\[E(\Omega) \leq E(\Omega^*).\]

Moreover,

(i) if there exists some domain $\Omega \subset M$ such that $E(\Omega) = E(\Omega^*)$ then $(M, g)$ is isometric to $(\mathbb{S}^n, g_0)$ and $\Omega$ is isometric to $\Omega^*$.

(ii) If there exists some domain $\Omega \subset M$ such that $E(\Omega) > \left(1 - \int_0^{\pi/2} \frac{\varepsilon(n, \kappa^2)}{\sin t} (t)^{n-1} dt \int_0^{\pi/2} (\sin t)^{n-1} dt\right)^{\frac{2}{n}} E(\Omega^*)$,

(where $-\kappa^2$ is a lower bound for the sectional curvature of $(M, g)$ and where $\varepsilon(n, \kappa)$ is the Perelman constant described in Theorem 4.9) then $M$ is diffeomorphic to $\mathbb{S}^n$.

Our last result is the following Theorem 1.8, where we provide a sharp universal bound for the torsional rigidity of any domain of a compact Riemannian manifold $(M, g)$ in terms of its Cheeger isoperimetric constant $H(M, g)$. Let us recall that $H(M, g)$ is defined by

\[H(M, g) = \inf_{\Omega} \left(\frac{\text{Vol}_{n-1}(\partial\Omega)}{\min(\text{Vol}(\Omega), \text{Vol}(M \setminus \Omega))}\right),\]

where $\Omega$ runs in the set of all domains with smooth boundary in $M$.

**Theorem 1.8.** Let $(M, g)$ be any compact Riemannian manifold and let $\Omega$ be any compact domain with smooth boundary in $M$ such that $\text{Vol}(\Omega) \leq \frac{1}{2} \text{Vol}(M)$. Then $E(\Omega) \leq \frac{1}{H(M, g)^2}$.

This inequality is sharp: at the end of the paper we shall exhibit examples of sequences of Riemannian manifolds $(M, g_\epsilon)$, $0 < \epsilon < 1$ and of domains $\Omega_\epsilon \subset M$ such that $E(\Omega_\epsilon) H^2(M, g_\epsilon) \to 1$ as $\epsilon \to 0$.

The paper is organized as follows. In Section 2 we recall the definition of manifolds which are harmonic and isoperimetric at a point and provide examples of non standard isoperimetric Riemannian manifolds. In Section 3 we give the definition of PIMS for a given manifold $(M, g)$ and we prove Theorems 1.3 and 1.4. The main tool in the proof of this theorem is the Theorem of symmetrization (Theorem 3.3) which gives precise relationships between the integrals that appear in the definition of $E(\Omega)$ when calculated on $\Omega$ and on its symmetrized $\Omega^*$. In Section 4 we investigate how to compare torsional rigidities of domains of two different compact manifolds. This will allow us to prove Theorem 1.7. This section ends with the proof of Theorem 1.8 and of its sharpness (Proposition 4.12).
2. HARMONIC AND ISOPERIMETRIC MANIFOLDS AT ONE POINT

We briefly recall the definition of harmonic manifolds (the reader is referred to [5], [9] and references therein for details). Let us recall that there exists, in any Riemannian manifold \((M, g)\), and for any point \(x_0 \in M\), a closed subset of measure zero, the cut-locus of \(x_0\) (denoted by \(\text{Cut}(x_0)\)) such that the exponential map \(\exp_{x_0}\) is a diffeomorphism from an open subset \(U_{x_0}\) of the tangent space \(T_{x_0}M\) onto \(M \setminus \text{Cut}(x_0)\). Let \(S_{x_0}\) be the (Euclidean) unit sphere of the Euclidean space \((T_{x_0}M, g_{x_0})\) and let us define the open subset \(\tilde{U}_{x_0} \subset \mathbb{R}_{+}\times S_{x_0}\) as the pull-back of \(U_{x_0}\) by the map \((t, v) \mapsto t.v\) from \(\mathbb{R}_{+}\times S_{x_0}\) to \(T_{x_0}M\); this provides a generalization of the usual “polar coordinates” by the notion of normal coordinates \(\phi\):

\[
\phi : \left\{ \begin{array}{l}
\tilde{U}_{x_0} \to U_{x_0} \to M \setminus \text{Cut}(x_0) \\
t (v) \mapsto t.v \mapsto \exp_{x_0}(t.v)
\end{array} \right.
\]

In this coordinates system, let us write the Riemannian measure at the point \(\phi(t, v)\) as

\[
\phi^* dv_g = \theta(t, v) \, dt \, dv,
\]

where \(dv\) is the canonical measure of the canonical sphere \(S_{x_0}\). This defines \(\theta(t, v)\) as the density of the measure \(\phi^* dv_g\) with respect to the measure \(dt \, dv\).

**Definition 2.1.** Let \((M, g)\) be a Riemannian manifold and \(x_0\) be a point in \(M\), \((M, g)\) is said to be harmonic at \(x_0\) iff the following two conditions are satisfied:

- \(U_{x_0}\) is equal to \(T_{x_0}M\) or to an open ball of the Euclidean space \((T_{x_0}M, g_{x_0})\) (and thus there exists some \(\beta \in [0, +\infty)\) such that \(\tilde{U}_{x_0} = [0, \beta [\times S_{x_0}\).
- for every \(t \in ]0, \beta [\), \(\theta(t,v)\) does not depend on \(v\) and will be, in this case, denoted by \(\theta(t)\).

**Definition 2.2.** A Riemannian manifold \((M, g)\) is said to be harmonic iff it is harmonic at each of its points.

For example, spaces of revolution are harmonic at their pole(s), but they are generally not harmonic in the sense of the Definition 2.2. See Section 2.1 below.

**Definition 2.3.** Let \((M, g)\) be a Riemannian manifold and \(x_0\) a point of \(M\). The manifold \((M, g)\) is said to be isoperimetric at \(x_0\) if it is harmonic at \(x_0\) and if, for any compact domain \(\Omega \subset M\) with smooth boundary, the geodesic ball \(\Omega^*\) centered at \(x_0\) with the same volume as \(\Omega\) satisfies

\[\text{Vol}(B(x_0, r)) = \text{Vol}(\Omega),\]

where \(r = \int_0 r(t) \, dt\) is a continuous strictly increasing function. We do not really need harmonicity to prove that but, in this case, the proof is simpler.

\[^4\text{Such a domain } \Omega^* \text{ writes } B(x_0, R_0), \text{ where } R_0 \text{ is the solution of the equation } \text{Vol}(B(x_0, r)) = \text{Vol}(\Omega), \text{ this solution exists and is unique because, being } (M, g) \text{ harmonic at } x_0, \text{ the function } r \mapsto \text{Vol}(B(x_0, r)) = \text{Vol}_{n-1}(S^{n-1}) \int_0^r \theta(t) \, dt \text{ is a continuous strictly increasing function. We do not really need harmonicity to prove that but, in this case, the proof is simpler.}\]
Vol_{n-1}(\partial \Omega^*) \leq Vol_{n-1}(\partial \Omega) ; the same manifold is said to be strictly isoperimetric at \( x_0 \) if, moreover, the equality occurs iff \( \Omega \) is isometric to \( \Omega^* \).

The Euclidean space, the Hyperbolic Space and the Sphere are strictly isoperimetric at every point (for proofs see for instance [6]). These examples are the only known examples (up to homotheties) of Riemannian manifolds which are isoperimetric at every point. If we only require the Riemannian manifolds to be isoperimetric at (at least) one point, we get much more examples. In fact, some non standard spaces of revolution are isoperimetric at one pole, as in the following example.

2.1. Examples of nonstandard Riemannian manifolds which are isoperimetric at some point. A (noncompact) space of revolution \((M, g)\) with only one pole \(x_0\) is such that \((M \setminus \{x_0\}, g)\) is isometric to \([0, +\infty[ \times S^{n-1}, \) endowed with a Riemannian metric of the type \((dt)^2 + b(t)^2 g_{S^{n-1}}, \) where \(b\) is a smooth strictly positive function whose extension to \([0, +\infty[\) satisfies \(b(0) = 0\) (and \(b'(0) = 1\) if we want the metric to be regular at \(x_0\)), where \(g_{S^{n-1}}\) is the canonical metric of the sphere \(S^{n-1},\) and where \(\{0\} \times S^{n-1}\) is identified with the point \(x_0\).

A (compact) space of revolution \((M, g)\) with two poles \(x_0\) and \(x_1\) is such that \((M \setminus \{x_0, x_1\}, g)\) is isometric to \([0, L] \times S^{n-1}, \) endowed with a Riemannian metric of the type \((dt)^2 + b(t)^2 g_{S^{n-1}, \) where \(b\) is a smooth strictly positive function whose extension to \([0, L]\) satisfies \(b(0) = b(L) = 0\) (and \(b'(0) = 1, b'(L) = -1\) if we want the metric to be regular at \(x_0\) and \(x_1\) ) and where \(\{0\} \times S^{n-1}\) (resp. \(\{L\} \times S^{n-1}\) ) is identified with the point \(x_0\) (resp. with the point \(x_1\)). It is not hard to see that a space of revolution is harmonic at any of its poles.

We now show that some (nonstandard) spaces of revolution are isoperimetric at their poles. The first example is given by a 2-dimensional cylinder \([0, +\infty[ \times S^1, \) equipped with \(g_{S^1}\) with 1 hemisphere glued to the boundary \(\{0\} \times S^1 \) (resp. with 2 hemispheres respectively glued to the boundaries \(\{0\} \times S^1 \) and \(\{L\} \times S^1 \) ). Other examples are given by the paraboloid of revolution \(z = x^2 + y^2\) or the hyperboloid of equation \(x^2 + y^2 - z^2 = -1, z > 0\) in \(\mathbb{R}^3\) (isoperimetric at their pole).

More generally, a large class of nonstandard examples is given by the

**Theorem 2.4.** ([19, Theorem 1.2]) Consider the plane \(\mathbb{R}^2\) equipped with a complete and rotationally invariant Riemannian metric \(g\) such that the Gauss curvature is positive and a strictly decreasing function of the distance from the origin. Then \((\mathbb{R}^2, g)\) is isoperimetric at the origin.

**Remark 2.5.** Notice that it is not true that every space of revolution is isoperimetric at its pole: for example let us consider the hypersurface of revolution \(S\) in \(\mathbb{R}^3\) of equation \(x^2 + y^2 + (|z| + \cos R)^2 = 1\), whose poles are \(x_0 = (0, 0, 1 - \cos R)\) and \(x_1 = -x_0\); then the geodesic ball \(B(x_0, R)\) is

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5 A short history of these proofs is given in [6], Section 10.4, see also Section 8.6.
the subset \( \{(x, y, z) \in S : z > 0\} \) and \( \partial B(x_0, R) \) is the circle \( x^2 + y^2 = \sin^2 R \), \( z = 0 \) whose length is \( 2\pi \sin(R) \). The plane \( y = 0 \) separates the surface \( S \) in two symmetric domains \( \Omega_1 \) and \( \Omega_2 \), which have the same area as \( B(x_0, R) \) and whose boundary is the union of two arcs of circle of length \( 2R \), we thus have

\[
\text{length}(\partial \Omega_1) = 4R < 2\pi \sin(R) = \text{length}(\partial B(x_0, R)).
\]

3. The symmetrization of a function, the theorem of symmetrization and the proofs of Theorems 1.3 and 1.4

Let \( (M, g) \) and \( (M^*, g^*) \) be two Riemannian manifolds such that \( \text{Vol}(M, g) \) and \( \text{Vol}(M^*, g^*) \) are both infinite or both finite. Let us define the constant \( \alpha(M, M^*) \) by

\[
\alpha(M, M^*) = \begin{cases} 
1 & \text{if } \text{Vol}(M, g) \text{ and } \text{Vol}(M^*, g^*) \text{ are both infinite,} \\
\frac{\text{Vol}(M, g)}{\text{Vol}(M^*, g^*)} & \text{if } \text{Vol}(M, g) \text{ and } \text{Vol}(M^*, g^*) \text{ are both finite.}
\end{cases}
\]

**Definition 3.1.** Let \( x^* \) be a fixed point of \( M^* \).

a) For any compact domain \( \Omega \subset M \) with smooth boundary, one defines its symmetrized domain \( \overline{\Omega} \) (around the point \( x^* \)) as the geodesic ball of \( (M^*, g^*) \), centered at \( x^* \), such that \( \text{Vol}(\overline{\Omega}) = \alpha(M, M^*)^{-1} \text{Vol}(\Omega) \).

b) \( (M^*, g^*, x^*) \) is said to be a pointed isoperimetric model space (PIMS) for \( (M, g) \) if, for any compact domain \( \Omega \subset M \), with smooth boundary, the symmetrized domain \( \overline{\Omega} \) satisfies the isoperimetric inequality \( \text{Vol}_{n-1}(\partial \overline{\Omega}) \geq \alpha(M, M^*) \text{Vol}_{n-1}(\partial \Omega^*) \); the same manifold is said to be a strict PIMS if, moreover, the equality occurs iff \( \Omega \) is isometric to \( (\Omega^*, g^*) \).

**Remark 3.2.** When the two manifolds have different finite volumes (i.e. when \( \alpha(M, M^*) \neq 1 \)), we are compelled to make the assumption \( \text{Vol}(\Omega^*) = \alpha(M, M^*)^{-1} \text{Vol}(\Omega) \) (which, in this case means that the relative volumes \( \text{Vol}(\Omega)/\text{Vol}(M, g) \) and \( \text{Vol}(\Omega^*)/\text{Vol}(M^*, g^*) \) are equal) instead of the usual assumption \( \text{Vol}(\Omega^*) = \text{Vol}(\Omega) \). In fact, if the symmetrized domain \( \overline{\Omega} \) is defined by the equality \( \text{Vol}(\overline{\Omega}) = \text{Vol}(\Omega) \), it is hopeless to expect some bound from below for \( \text{Vol}_{n-1}(\partial \overline{\Omega}) \) in terms of \( \text{Vol}_{n-1}(\partial \Omega^*) \). Indeed, if \( \text{Vol}(M, g) > \text{Vol}(M^*, g^*) \), then \( \overline{\Omega} \) would not exist when \( \Omega \) is such that \( \text{Vol}(M^*, g^*) < \text{Vol}(\Omega) < \text{Vol}(M, g) \); on the other hand, if \( \text{Vol}(M, g) < \text{Vol}(M^*, g^*) \) and if \( \Omega = M \setminus B(x, \varepsilon) \), where \( \varepsilon \) is arbitrarily small and \( B(x, \varepsilon) \) is any geodesic ball of radius \( \varepsilon \) in \( (M, g) \), we get that \( \text{Vol}(M^* \setminus \Omega^*) = \text{Vol}(M^*, g^*) - \text{Vol}(M, g) + O(\varepsilon^n) \) and \( \text{Vol}(\partial \Omega^*) \) does not go to zero while

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\( \text{The symmetrized domain } \overline{\Omega} = B(x^*, R), \text{ where } R \text{ is the solution of the equation } \text{Vol}(B(x^*, r)) = \alpha(M, M^*)^{-1} \text{Vol}(\Omega), \text{ always exists and is unique because } \alpha(M, M^*)^{-1} \text{Vol}(\Omega) \in ]0, \text{Vol}(M^*, g^*)[ \text{ and } r \mapsto \text{Vol}(B(x^*, r)) \text{ is a continuous strictly increasing function whose image is the closure of } ]0, \text{Vol}(M^*, g^*)[ \text{ in } ]0, +\infty[. \)
Vol(∂Ω) = O(ε^{n−1}) goes to zero when ε → 0. It is thus impossible to bound Vol_{n−1}(∂Ω) from below in terms of Vol_{n−1}(∂Ω^*)

Let (M, g) be a Riemannian manifold and (M^*, g^*, x^*) be a PIMS for (M, g). Let Ω ⊂ M be any compact domain with smooth boundary. Let Ω^* be its symmetrized domain in the sense of Definition 3.1.

Let f be any smooth nonnegative function on Ω which vanishes on the boundary, we denote by Ω_t (or by \{f > t\}) the set of points x ∈ Ω such that f(x) > t. Let us denote by \{f = t\} the set of points x ∈ Ω such that f(x) = t; notice that the set of critical points of f is compact and thus its image S(f) by f is compact and, by Sard’s theorem, it has Lebesgue measure zero in [0, sup f]. For any regular value t of f, namely for any t ∈ [0, sup f] \ S(f) the set \{f = t\} is a smooth submanifold of codimension 1 in M, which is equal to ∂Ω_t. For any t ∈ [0, sup f], let us define Ω^*_t as the symmetrized domain of Ω_t, i.e. the geodesic ball B(x^*, R(t)) whose radius R(t) is chosen in such a way that Vol(B(x^*, R(t))) = α(M, M^*)^{-1} Vol(Ω_t).

When t = sup f, then Ω_{sup f} is empty, and thus R(sup f) = 0.

The function t ↦ A(t) := Vol(Ω_t) is strictly decreasing because, when 0 ≤ t < t' ≤ sup f, the set \{x ∈ X : t < f(x) < t'\} is a nonempty open set of nonzero volume; a consequence is that the function t ↦ R(t) is also strictly decreasing.

We then define f^* : Ω^* → ℝ^+, the symmetrized of f in such a way that \{f^* > t\} = Ω^*_t, namely, we decide that f^* = f ◦ ρ, where ρ(x) = d^*(x^*, x), where d^* is the Riemannian distance on M^* associated to g^*, and where \( \tilde{f} : [0, R_0] → [0, sup f] \) is defined by

\[
\tilde{f}(r) := \inf (\{R^{-1}(0, r)\}) = \inf\{t ∈ [0, sup f] : R(t) ≤ r\} = \inf\{t : A(t) ≤ α(M, M^*) VolB(x^*, r)\}
\]

We now state the Theorem of symmetrization which represents the main tool for the proof of our main results. Symmetrization methods have their origin in J. Steiner’s works. The following classical application to functional analysis (also called rearrangements) generalizes to Riemannian manifolds ideas of G. Talenti.

**Theorem 3.3.** Let (M, g) be a Riemannian manifold and (M^*, g^*, x^*) be a PIMS for (M, g). Let Ω ⊂ M be a compact domain with smooth boundary and f be any smooth nonnegative function on Ω which vanishes on its boundary. Let f^* be the symmetrized function, constructed as above on the symmetrized geodesic ball Ω^* of (M^*, g^*), centered at the point x^*. Then

(i) f^* is Lipschitz (with Lipschitz constant ||∇f||_{L^∞}) and thus f^* lies in \( H^2_{loc}(Ω^*, g^*) \).

\(^7\)Thus t ↦ R(t) is well defined (for every t) and injective. However, it is generally not surjective nor continuous, moreover the measure of the set [0, R_0] \ Image(R) is generally not zero. This is one of the main problems when studying the regularity of \( \tilde{f} \), and thus of f^*. 
(ii) \( \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} f(x)^p \, dv_g(x) = \frac{1}{\text{Vol}(\Omega^*)} \int_{\Omega^*} (f^*(x))^p \, dv_{g^*}(x) \) for every \( p \in [1, +\infty[ \),

(iii) \( \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} \| \nabla f(x) \|^2 \, dv_g(x) \geq \frac{1}{\text{Vol}(\Omega^*)} \int_{\Omega^*} \| \nabla f^*(x) \|^2 \, dv_{g^*}(x) \). If, moreover, \((M^*, g^*)\) is a strict PIMS for \((M, g)\) then equality holds iff the set \( \{ f > 0 \} \subset (\Omega^*, g^*) \) is isometric to the set \( \{ f^* > 0 \} \subset (\Omega^*, g^*) \).

The proof of this theorem can be obtained by following the same lines as in \cite{2} and \cite{11} (see also \cite{1}). We point out that one of the main tools of the proof of the theorem of symmetrization given in \cite{2} and \cite{11}, namely the \textit{coarea formula}, is not entirely correct in these references, though this has no consequences on the results proved in these papers. The correct version of the coarea formula (see for instance \cite{6}), pp. 104-107) is the following:

**Coarea formula:** Let \((M, g)\) be a Riemannian manifold and let \( f : M \to \mathbb{R} \) be a smooth function. Then, for any measurable function \( \varphi \) on \( M \), one has\(^8\)

\[
\int_{M} \varphi(x) \| \nabla f(x) \| \, dv_g(x) = \int_{\inf f}^{\sup f} \left( \int_{f^{-1}(\{t\})} \varphi(x) \, da_t(x) \right) \, dt. \tag{4}
\]

Formula (4) together with the corrected proof of the theorem of symmetrization will appear in a forthcoming survey paper.

**Proof of Theorem 3.4:** Let \( f_\Omega \) be the unique solution of the problem (1) on the domain \( \Omega \), let \((f_\Omega)^*\) be the corresponding symmetrized function. By Theorem of Symmetrization 3.3 (ii) and (iii) we get

\[
\mathcal{E}(\Omega) = E_\Omega(f_\Omega) = \frac{1}{\text{Vol}(\Omega)} \left( 2 \int_{\Omega} f_\Omega \, dv_g - \int_{\Omega} |\nabla f_\Omega|^2 \, dv_g \right) \leq \frac{1}{\text{Vol}(\Omega^*)} \left( 2 \int_{\Omega^*} (f_\Omega)^* \, dv_{g^*} - \int_{\Omega^*} |\nabla (f_\Omega)^*|^2 \, dv_{g^*} \right) = E_{\Omega^*}((f_\Omega)^*). 
\]

Let us recall that the \textit{torsional rigidity} of the domain \( \Omega^* \) is the value \( \mathcal{E}(\Omega^*) = \max_{u \in H^2_1(\Omega^*)} (E_{\Omega^*}(u)) \). Since by (i) of Theorem 3.3 \((f_\Omega)^* \in H^2_1(\Omega^*, g^*)\) it follows

\[
\mathcal{E}(\Omega^*) \geq E_{\Omega^*}((f_\Omega)^*) \geq \mathcal{E}(\Omega).
\]

Let us suppose that \( \mathcal{E}(\Omega^*) = \mathcal{E}(\Omega) \), then all the inequalities are equalities, in particular

\[
\int_{\Omega} |\nabla f_\Omega|^2 \, dv_g = \alpha(M, M^*) \int_{\Omega^*} |\nabla (f_\Omega)^*|^2 \, dv_{g^*}.
\]

\(^8\)By \( \int_{\inf f}^{\sup f} \) we intend the integral on \( [\inf f, \sup f] \setminus \mathcal{S}(f) \) (because \( \mathcal{S}(f) \) has measure zero). Moreover, as we only integrate with respect to regular values \( t \) of \( f \), \( \{ f = t \} \) is a submanifold of codimension 1 in \( M \) and \( da_t \) is the \((n-1)\)-dimensional Riemannian measure on \( \{ f = t \} \) (viewed as a Riemannian submanifold of \( (M, g) \)).
and \( E_{\Omega^*}((f_{\Omega^*})^*) = \mathcal{E}(\Omega^*) \). Thus, since the set \( \{ f_{\Omega} > 0 \} \) coincides with the interior of \( \Omega \), from the equality case of Theorem 3.3 it follows that \( \Omega^* \) is isometric to \( \Omega \).

**Proof of Theorem 1.3.** As we have already pointed out in the introduction the proof follows immediately by Theorem 1.4. Indeed, Definition 2.3 implies that \((M, g, x_0)\) is a PIMS for \((M, g)\) itself in the sense of the Definition 3.1, and the constant \( \alpha(M, M^*) \) is, in this case, always equal to 1, because either \((M, g)\) has infinite volume, either the quotient of the volumes of the manifold and of the model space is equal to 1, because these two spaces coincide. □

**Remark 3.4.** Let us return to the definition of the torsional rigidity \( \mathcal{E}(\Omega) = E_{\Omega}(f_{\Omega}) \). Two possible definitions of this functional can be found in the classical literature: the one we considered here, i.e.

\[
\mathcal{E}(\Omega) = \frac{1}{\text{Vol}(\Omega)} \left( 2 \int_{\Omega} f_{\Omega} dv_g - \int_{\Omega} |\nabla f_{\Omega}|^2 dv_g \right) = \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} f_{\Omega} dv_g
\]

and, more frequently, the functional

\[
\tilde{\mathcal{E}}(\Omega) = 2 \int_{\Omega} f_{\Omega} dv_g - \int_{\Omega} |\nabla f_{\Omega}|^2 dv_g = \int_{\Omega} f_{\Omega} dv_g.
\]

The critical or maximal domains (among all domains of prescribed volume) for the two functionals \( \Omega \to \mathcal{E}(\Omega) \) and \( \Omega \to \tilde{\mathcal{E}}(\Omega) \) being the same, what is the interest of considering the first functional instead of the second one? In fact, as \( f_{\Omega}(x) \) is the “mean exit time” for the paths of the Brownian motion issued from \( x \), \( \mathcal{E}(\Omega) \) is the mean value of this “mean exit time” with respect to all possible initial points \( x \in \Omega \), thus it still has some physical and stochastic meaning. Moreover if, on the same domain \( \Omega \), we change the Riemannian metric \( g \) in the homothetic metric \( \lambda^2 g \), a direct computation gives:

\[
\mathcal{E}(\Omega, \lambda^2 g) = \lambda^2 \mathcal{E}(\Omega, g) \quad \text{and} \quad \tilde{\mathcal{E}}(\Omega, \lambda^2 g) = \lambda^{n+2} \tilde{\mathcal{E}}(\Omega, g),
\]

thus \( \mathcal{E}(\Omega, g) \) has the same homogeneity as the Riemannian metric \( g \).

But the main reason to prefer \( \mathcal{E}(\Omega) \) to \( \tilde{\mathcal{E}}(\Omega) \) is Theorem 1.4 which provides a direct and simple comparison of the type \( \mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*) \), while, with respect to \( \tilde{\mathcal{E}} \), the comparison writes \( \tilde{\mathcal{E}}(\Omega) \leq \alpha(M, M^*) \tilde{\mathcal{E}}(\Omega^*) \), where \( \Omega \) is any domain on a Riemannian manifold (resp. on a compact Riemannian manifold \((M, g)\)) and \( \Omega^* \) is a geodesic ball of the same volume (resp. of the same relative volume) on a model-space \((M^*, g^*, x^*)\).

**4. The proofs of Theorem 1.7 and Theorem 1.8.**

In order to prove Theorem 1.7 (see the end of this section) which represents our main comparison between torsional rigidities in two different compact manifolds (we remind the reader that the noncompact case has been treated in Corollary 1.6 of the Introduction), we need some known isoperimetric inequalities (see Theorem 4.1, Theorem 4.6 and Theorem 4.9).
and, along the way, we also deduce some comparison results for the torsional rigidity (Corollary 4.3 and Corollary 4.8 respectively).

Revisiting Paul Lévy’s work [17] (applied to convex bodies in the Euclidean space), M. Gromov ([13]) proved the following celebrated isoperimetric inequality:

**Theorem 4.1.** For every Riemannian manifold $(M, g)$ whose Ricci curvature satisfies $\text{Ric}_g \geq (n-1).g$, for every compact domain with smooth boundary $\Omega$ in $M$, let $\Omega^*$ be a geodesic ball of the canonical sphere $(\mathbb{S}^n, g_0)$ such that

$$\frac{\text{Vol}(\Omega^*, g_0)}{\text{Vol}(\mathbb{S}^n, g_0)} = \frac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)},$$

then

$$\frac{\text{Vol}_{n-1}(\partial \Omega)}{\text{Vol}(M, g)} \geq \frac{\text{Vol}_{n-1}(\partial \Omega^*)}{\text{Vol}(\mathbb{S}^n, g_0)}.$$

Moreover, this last inequality is an equality if and only if $\Omega$ is isometric to $\Omega^*$. In other words, for any $x_0 \in \mathbb{S}^n$, $(\mathbb{S}^n, g_0, x_0)$ is a strict PIMS for all the Riemannian manifolds $(M, g)$ which satisfy $\text{Ric}_g \geq (n-1).g$.

**Remark 4.2.** Theorem 4.1 is evidently sharp, because the canonical sphere $(\mathbb{S}^n, g_0)$ satisfies its assumption “$\text{Ric}_g \geq (n-1).g$” (actually $\text{Ric}_{g_0} = (n-1).g_0$), thus the Theorem applies when $(M, g) = (\mathbb{S}^n, g_0)$, and because the isoperimetric inequality given by the Theorem 4.1 is an equality when $(M, g) = (\mathbb{S}^n, g_0)$ and when $\Omega$ is a geodesic ball of $(\mathbb{S}^n, g_0)$.

Applying Theorems 1.4 and 4.1, we obtain:

**Corollary 4.3.** For every Riemannian manifold $(M, g)$ whose Ricci curvature satisfies $\text{Ric}_g \geq (n-1).g$, for every compact domain with smooth boundary $\Omega$ in $M$, let $\Omega^*$ be a geodesic ball of the canonical sphere $(\mathbb{S}^n, g_0)$ such that

$$\frac{\text{Vol}(\Omega^*, g_0)}{\text{Vol}(\mathbb{S}^n, g_0)} = \frac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)},$$

then $E(\Omega) \leq E(\Omega^*)$. Moreover, the equality $E(\Omega) = E(\Omega^*)$ is realized if and only if $\Omega$ is isometric to $\Omega^*$.

**Remark 4.4.** It is easy to extend Corollary 4.3 to every Riemannian manifold $(M, g)$ whose Ricci curvature satisfies $\text{Ric}_g \geq K(n-1).g$ (with $K > 0$): in fact the Riemannian manifold $(M, K.g)$ then satisfies $\text{Ric}_{K.g} \geq (n-1).(K.g)$ and we can thus apply Theorem 4.1 to the Riemannian manifold $(M, K.g)$; for every compact domain with smooth boundary $\Omega$ in $M$, if $\Omega^*$ is a geodesic ball on the Euclidean sphere $\mathbb{S}^n(\frac{1}{\sqrt{K}})$ of radius $\frac{1}{\sqrt{K}}$ and if $\Omega^{**}$ is a geodesic ball of the canonical sphere $\mathbb{S}^n(1) = (\mathbb{S}^n, g_0)$ such that

$$\frac{\text{Vol}(\Omega^{**})}{\text{Vol}(\mathbb{S}^n, g_0)} = \frac{\text{Vol}(\Omega^*)}{\text{Vol}(\mathbb{S}^n(\frac{1}{\sqrt{K}}))} = \frac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)},$$

then, by (5) and Theorem 4.3

$$E(\Omega, g) = \frac{1}{K} E(\Omega, Kg) \leq \frac{1}{K} E(\Omega^{**}, g_0) = E(\Omega^{**}, \frac{1}{K}g_0) = E(\Omega^*),$$
where the last equality deduces from the fact that \((S^n, \frac{1}{\sqrt{K}} g_0)\) is isometric to \(S^n(\frac{1}{\sqrt{K}})\) and that this isometry maps \(\Omega^*\) onto \(\Omega^*\).

**Remark 4.5.** Corollary \([3]\) is sharp in the following sense: for every \(\beta \in [0, 1[,\) if we consider the set \(\mathcal{W}_\beta\) of all domains \(\Omega\) in all the Riemannian manifolds \((M, g)\) whose Ricci curvature satisfies \(\text{Ric}_g \geq (n - 1)g\) such that \(\text{Vol}(\Omega, g) / \text{Vol}(M, g) = \beta\) then the geodesic ball \(\Omega^*\) of the canonical sphere \((S^n, g_0)\) such that \(\text{Vol}(\Omega^*, g_0) / \text{Vol}(S^n, g_0) = \beta\) is an element of \(\mathcal{W}_\beta\).

If we consider the functional \(\Omega \mapsto \mathcal{E}(\Omega)\) restricted to \(\mathcal{W}_\beta\), then \(\Omega^*\) is the point where this functional attains its maximum.

Theorem 4.1 was improved and generalized to the case where the Ricci curvature satisfies \(\text{Ric}_g \geq (n - 1)g\) and diameter \((M, g) \leq D\) is given by the Euclidean sphere of radius \(R(K, D)\) (PIMS at any point) where \(R(K, D)\) is defined by

\[
R(K, D) = \begin{cases} 
\frac{1}{\sqrt{K}} \left( \int_0^{\frac{\pi}{2}} (\cos t)^{n-1} dt \right)^{\frac{1}{n}} & \text{if } K > 0 \\
\frac{1}{\pi} \left( \int_0^{\frac{\pi}{2}} (\cos t)^{n-1} dt \right)^{-\frac{1}{n}} D & \text{if } K = 0 \\
\frac{1}{\sqrt{|K|}} \max \left( \int_0^{D \sqrt{|K|}} (\cosh 2t)^{\frac{n-1}{2}} dt, \int_0^{D \sqrt{|K|}} (\cosh 2t)^{\frac{n-1}{2}} dt \right)^{\frac{1}{n}} & \text{if } K < 0
\end{cases}
\]

In other terms, for every compact domain with smooth boundary \(\Omega\) in \(M\), if \(\Omega^*\) is a geodesic ball on the Euclidean sphere \(S^n(R(K, D))\) of radius \(R(K, D)\) and if \(\Omega^{**}\) is a geodesic ball of the canonical sphere \((S^n(1) = (S^n, g_0))\) such that

\[
\frac{\text{Vol}(\Omega^{**})}{\text{Vol}(S^n, g_0)} = \frac{\text{Vol}(\Omega^*)}{\text{Vol}(S^n(R(K, D)))} = \frac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)}
\]

then

\[
\frac{\text{Vol}_{n-1}(\partial \Omega)}{\text{Vol}(M, g)} \geq \frac{\text{Vol}_{n-1}(\partial \Omega^*)}{\text{Vol}(S^n(R(K, D)))} = \frac{1}{R(K, D)} \frac{\text{Vol}_{n-1}(\partial \Omega^{**})}{\text{Vol}(S^n, g_0)}.
\]

**Remark 4.7.** This theorem is sharp, because the canonical sphere satisfies its assumptions “\(\text{Ric}_g \geq (n - 1)g\)” and “diameter \(\leq \pi\)”, thus the Theorem \(4.6\) applies to any domain \(\Omega \subset S^n\) when \((M, g) = (S^n, g_0)\), with the values \(K = 1\) and \(D = \pi\) of the constants, and then the isoperimetric inequality \([3]\) given by the theorem \(4.6\) is an equality when \((M, g) = (S^n, g_0)\) and
when $\Omega$ is a geodesic ball because, in this case, $R(K, D) = R(1, \pi) = 1$. Moreover, under the assumptions “$\text{Ric}_g \geq (n-1)g$” and “$(M, g)$ not isometric to $(S^n, g_0)$”, Myers’ theorem (and its equality case) implies that diameter$(M, g) < \pi$, and thus we can apply the Theorem 4.6 with the values $K = 1$ and $D < \pi$ of the constants, which implies that, under these assumptions, $R(K, D) < 1$. The isoperimetric inequality (6) is then strictly better than the one of the canonical sphere. Let us also remark that the smaller $R(K, D)$ is, the better is the isoperimetric inequality (6) given by the Proposition 4.6.

**Corollary 4.8.** For any $K \in \mathbb{R}$, for any $n$-dimensional Riemannian manifold $(M, g)$ which satisfies $\text{Ric}_g \geq (n-1)Kg$ and diameter$(M, g) \leq D$, for every compact domain with smooth boundary $\Omega$ in $M$, if $\Omega^*$ is a geodesic ball on the Euclidean sphere $S^n(R(K, D))$ and if $\Omega^{**}$ is a geodesic ball of the canonical sphere $S^n(1) = (S^n, g_0)$ such that

$$\frac{\text{Vol}(\Omega^{**})}{\text{Vol}(S^n, g_0)} = \frac{\text{Vol}(\Omega^*)}{\text{Vol}(S^n(R(K, D)))} = \frac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)},$$

then

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*) = R(K, D)^2 \mathcal{E}(\Omega^{**}). \quad (7)$$

**Proof:** Theorem 4.6 shows that the Euclidean sphere $S^n(R(K, D))$ of radius $R(K, D)$ is a PIMS for the Riemannian manifold $(M, g)$. For every compact domain with smooth boundary $\Omega$ in $M$, if $\Omega^*$ is a geodesic ball on the Euclidean sphere $S^n(R(K, D))$ of radius $R(K, D)$ and if $\Omega^{**}$ is a geodesic ball of the canonical sphere $S^n(1) = (S^n, g_0)$ such that

$$\frac{\text{Vol}(\Omega^{**})}{\text{Vol}(S^n, g_0)} = \frac{\text{Vol}(\Omega^*)}{\text{Vol}(S^n(R(K, D)))} = \frac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)},$$

then Theorem 4.4 implies that

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^*) = R(K, D)^2 \mathcal{E}(\Omega^{**}),$$

where the last equality deduces from the fact that the sphere of radius $R(K, D)$ is isometric to $(S^n, R(K, D)^2g_0)$ and from formula (5). $\square$

We now recall an inequality due to G. Perelman [20] (which is an improvement of a previous result of S. Ilias [14]).

**Theorem 4.9.** Let $(M, g)$ be a $n$-dimensional compact Riemannian manifold. Assume that $M$ is not diffeomorphic to $S^n$, that $\text{Ric}_g \geq (n-1)g$ and that the sectional curvature of $(M, g)$ is $\leq -\kappa^2$. Then there exists a constant $\varepsilon(n, \kappa) > 0$ such that diameter$(M, g) \leq \pi - \varepsilon(n, \kappa)$. 

Remark 4.10. By applying the Theorem 4.6 with the values $K = 1$ and $D = \pi - \varepsilon(n, \kappa)$ of the constants, which implies that, under these assumptions,

$$R(K, D) = R(1, \pi - \varepsilon(n, \kappa)) = \left(1 - \frac{\int_0^{\frac{\varepsilon(n, \kappa)}{2}} (\sin t)^{n-1} dt}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-1} dt}\right)^{\frac{1}{n}}, \quad (8)$$

we observe that, with respect to the isoperimetric inequality of the canonical sphere, the isoperimetric inequality on $(M, g)$ induced by (6) is improved by some factor which is bounded far from 1.

We are now in the position to prove Theorem 1.7 (notice that it improves Corollary 4.3).

**Proof of Theorem 1.7.** Applying Theorem 4.6 (with the values $K = 1$ and $D = \text{diameter}(M, g)$ of the constants) we prove that the Euclidean sphere $S^n(R(1, D))$ of radius $R(1, D) = R(1, \text{diameter}(M, g))$ is a PIMS (at any point) for the Riemannian manifold $(M, g)$. For every compact domain with smooth boundary $\Omega$ in $M$, if $\Omega^0$ is a geodesic ball on the Euclidean sphere $S^n(R(1, D))$ of radius $R(1, D)$ and if $\Omega^*$ is a geodesic ball of the canonical sphere $S^n(1) = (S^n, g_0)$ such that

$$\frac{\text{Vol}(\Omega^*)}{\text{Vol}(S^n, g_0)} = \frac{\text{Vol}(\Omega^0)}{\text{Vol}(S^n(R(1, D)))} = \frac{\text{Vol}(\Omega, g)}{\text{Vol}(M, g)}$$

then Theorem 1.4 implies that

$$\mathcal{E}(\Omega) \leq \mathcal{E}(\Omega^0) = R(1, D)^2 \mathcal{E}(\Omega^*), \quad (9)$$

where the last equality deduces from the fact that the sphere of radius $R(1, D)$ is isometric to $(S^n, R(1, D)^2 g_0)$ and from formula (5).

Let us first suppose that $(M, g)$ is not isometric to $(S^n, g_0)$, then Myers’ theorem (and its equality case) implies that $\text{diameter}(M, g) < \pi$, and thus that $R(1, D) < 1 \quad (if \quad D = \text{diameter}(M, g) \quad \text{) by the definition of} \quad R(K, D)$. Using the fact that $R(1, D) < 1$ in the inequality (9), we conclude that, if $(M, g)$ is not isometric to $(S^n, g_0)$, then $\mathcal{E}(\Omega) < \mathcal{E}(\Omega^*)$ for every compact domain with smooth boundary $\Omega$ in $M$, which proves the part (i) of the Theorem 1.7.

If we now suppose that $M$ is not diffeomorphic to $S^n$, we know, by Theorem 4.9, that $\text{diameter}(M, g) \leq \pi - \varepsilon(n, \kappa)$ in this case and thus that we can choose the value $D = \pi - \varepsilon(n, \kappa)$ for the upper bound of the diameter of $(M, g)$. Using the inequality (9) (and the formula (8) for the computation of $R(1, \pi - \varepsilon(n, \kappa))$, we get that

$$\mathcal{E}(\Omega) \leq \left(1 - \frac{\int_0^{\frac{\varepsilon(n, \kappa)}{2}} (\sin t)^{n-1} dt}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-1} dt}\right)^{\frac{2}{n}} \mathcal{E}(\Omega^*)$$
for every compact domain with smooth boundary $\Omega$ in $M$, which proves part (ii) of Theorem 1.7. □

In order to prove Theorem 1.8 we need the following:

**Lemma 4.11.** Let $(M, g)$ be a Riemannian manifold. Then, for any compact domain $\Omega \subset M$ and for any smooth nonnegative function $f$ on $\Omega$ which vanishes on $\partial \Omega$, one has:

$$\int_{\Omega} f dv_g = \int_{0}^{\sup f} A(t) dt,$$

where $A(t) = \text{Vol}(\Omega_t)$ and where $\Omega_t$ denotes the set of points $x \in \Omega$ such that $f(x) > t$.

**Proof:** Let $t_i = \frac{1}{N} \sup f$ (for every $i \in \{0, \ldots N\}$). The function $t \mapsto A(t) = \text{Vol}(\Omega_t)$ being strictly decreasing, we have

$$\sum_{i=0}^{N-1} t_i (A(t_i) - A(t_{i+1})) \leq \int_{\Omega} f dv_g \leq \sum_{i=0}^{N-1} t_{i+1} (A(t_i) - A(t_{i+1})). \quad (10)$$

Let $S_N^+$ (resp. $S_N^-$) denote the right (resp. the left) hand side of (10). This is an approximation from above (resp. from below) of the integral $\int_{\Omega} f dv_g$. As $0 \leq S_N^+ - S_N^- \leq \int_{0}^{\sup f} A(t) dt$ (and to $\int_{\Omega} f dv_g$ by (10)). □

**Proof of Theorem 1.8** By the definition of $E(\Omega)$ and by Lemma 4.11, we have

$$\text{Vol}(\Omega) E(\Omega) = \int_{\Omega} f \Omega dv_g = \int_{[0, \sup f] \setminus S(f)} A(t) dt, \quad (11)$$

where $A(t) = \text{Vol}(\Omega_t)$ and where $\Omega_t$ denotes the set of points $x \in \Omega$ such that $f_\Omega(x) > t$. For every regular value $t$ of $f_\Omega$ one has:

$$A(t) \leq \text{Vol}(\Omega) \leq \text{Vol}(M, g)/2$$

and thus, by the definition of Cheeger’s isoperimetric constant,

$$\text{Vol}_{n-1}(\partial \Omega_t) \geq H(M, g) A(t).$$

From this and from (11) we deduce

$$\text{Vol}(\Omega) E(\Omega) \leq \frac{1}{H(M, g)} \int_{[0, \sup f_\Omega] \setminus S(f)} \text{Vol}_{n-1}(\partial \Omega_t) dt = \frac{1}{H(M, g)} \int_{\Omega} |\nabla f_\Omega| dv_g,$$

where, in the last equality, we have used the coarea formula. Thus, by Cauchy–Schwarz inequality

$$\text{Vol}(\Omega) E(\Omega) \leq \frac{1}{H(M, g)} (\text{Vol}(\Omega))^{\frac{1}{2}} \left( \int_{\Omega} |\nabla f_\Omega|^2 dv_g \right)^{\frac{1}{2}}$$

and hence, since $E(\Omega) = \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} |\nabla f_\Omega|^2 dv_g$, one gets $(E(\Omega))^{\frac{1}{2}} \leq \frac{1}{H(M, g)}$ which ends the proof of the theorem. □
On the sharpness of Theorem 1.8. The following proposition shows that Theorem 1.8 is sharp.

**Proposition 4.12.** There exists a family of n-dimensional compact Riemannian manifolds \((M, g_ε)\), \(0 < ε < 1\), such that for every \(β \in [\frac{1}{n}, \frac{1}{2}]\) there exists a compact domain \(Ω \subset M\) and a universal constant \(B = B(n)\) such that

\[
\frac{\text{Vol}(Ω, g_ε)}{\text{Vol}(M, g_ε)} = β
\]

and

\[
E(Ω, g_ε) ≥ \frac{(1 - ε)(1 - B√ε)^2}{H(M, g_ε)^2}.
\]

**Proof.** Let \(R, δ, ε\) be positive real numbers such that \(δ > \frac{1}{n}\) and \(ε ≤ \frac{1}{Rδ} < 1\). Define two other positive real numbers \(λ, η\) by:

\[
λ = \frac{e^δR}{ε} \sqrt{1 - ε^2δ^2 e^{-2δR}}, \quad η = \frac{1}{λ} \arctan(\frac{λ}{δ}).
\]

Consider the compact manifold \(M\) obtained as the quotient of \([- (R + η), R + η] × S^{n-1}\) by identifying all the points \(\{ R + η \} × S^{n-1} \) (resp. \(\{ - (R + η) \} × S^{n-1}\)) to a single point \(x_0\) (resp. \(x_1\)) (see Subsection 2.4 above). Denote by

\[
π : [- (R + η), R + η] × S^{n-1} \to M
\]

the corresponding quotient map. We endow \(M\) with the metric \(g_ε\) defined at the point \((t, v) \in [- (R + η), R + η] × S^{n-1}\) by:

\[
g_ε = (dt)^2 + b_ε(t)^2 g_0,
\]

where \(b_ε\) is the even function on \([- (R + η), R + η]\] given by

\[
b_ε(t) = \begin{cases} 
ε e^{-δt} & \text{if } t ∈ [0, R], \\
ε e^{-δR (\frac{\sin(λ(η + R + t))}{\sin(λη)})} & \text{if } t ∈ [R, R + η].
\end{cases}
\]

It is easily seen that \(b_ε\) is \(C^1\) on \(]0, R + η[\) and that

\[
b_ε(-(R + η)) = b_ε(R + η) = 0, \quad b'(R + η) = -1, \quad b'(-(R + η)) = 1.
\]

Moreover we have

\[
b_ε(t) ≤ ε e^{-δt}, \quad t ∈ [0, R + η].
\]

For \(r ∈ [0, R + η]\) consider the compact domain

\[
Ω_r = π([r, R + η] × S^{n-1}) \subset M,
\]

we claim that

\[
\frac{\text{Vol}_{n-1}(∂Ω_r, g_ε)}{\text{Vol}(Ω_r, g_ε)} ≥ (n - 1)δ, \quad r ∈ [0, R + η].
\]

Indeed, on the one hand, when \(r ∈ [0, R]\), (14) yields

\[
\frac{\text{Vol}_{n-1}(∂Ω_r, g_ε)}{\text{Vol}(Ω_r, g_ε)} = \frac{b_ε(r)^{n-1}}{f_r^{R+η} b_ε(t)^{n-1} dt} ≥ \frac{e^{-(n-1)δr}}{f_r^{R+η} e^{-(n-1)δt} dt} ≥ (n - 1)δ.
\]
On the other hand, when \( r \in [R, R + \eta] \), by setting \( \tilde{r} = R + \eta - r \) we get
\[
\frac{\text{Vol}_{n-1}(\partial \Omega_r, g_\varepsilon)}{\text{Vol}(\Omega_r, g_\varepsilon)} = \frac{\sin[\lambda(R + \eta - r)]^{n-1}}{\int_r^{R+\eta} \sin[\lambda(R + \eta - t)]^{n-1} dt} \geq \frac{\lambda \sin[\lambda \tilde{r}]^{n-1}}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-1} dt} \geq \lambda \geq (n - 1)\delta,
\]
where the second inequality comes from the fact that the function \( \xi \mapsto \frac{(\sin \xi)^{n-1}}{\int_0^\xi (\sin t)^{n-1} dt} \) is decreasing on the interval \([0, \frac{\pi}{2}]\) and \( \lambda \tilde{r} \leq \lambda \eta = \arctan(\frac{\lambda}{\delta}) < \frac{\pi}{2} \)
and where the last inequality deduces from the definition of \( \lambda \) and from the assumption \( \varepsilon \leq \frac{1}{\delta_0} \). Hence the claim (15) is proved. Therefore, if we define
\[
H_{\text{rad}}(M, g_\varepsilon) := \inf_{r \in [0, R + \eta]} \frac{\text{Vol}_{n-1}(\partial \Omega_r, g_\varepsilon)}{\text{Vol}(\Omega_r, g_\varepsilon)}
\]
we get
\[
H_{\text{rad}}(M, g_\varepsilon) \geq (n - 1)\delta.
\]
By the method developed in Appendix A.4 of [11], we can find a universal constant \( B = B(n) \) such that
\[
(1 - B\sqrt{\varepsilon})(n - 1)\delta \leq (1 - B\varepsilon)H_{\text{rad}}(M, g_\varepsilon) \leq H(M, g_\varepsilon).
\]
Consider now the test function \( f: \Omega_r \to \mathbb{R}^+ \)
\[
f(t, v) = u(t) = \begin{cases} \frac{t-r}{(n-1)\delta} & \text{if } t \in [r, R], \\ \frac{R-r}{(n-1)\delta} & \text{if } t \in [R, R + \eta]. \end{cases}
\]
By a straightforward computation we get:
\[
\int_{\Omega_r} |\nabla f|^2 dv_{g_\varepsilon} < \frac{1}{(n - 1)^2 \delta^2} \text{Vol}(\Omega_r, g_\varepsilon), \tag{17}
\]
and, using (14), and setting \( \omega_{n-1} = \text{Vol}(S^{n-1}, g_0) \)
\[
\int_{\Omega_r} f^2 dv_{g_\varepsilon} \leq \frac{\omega_{n-1} e^{n-1}}{(n - 1)^2 \delta^2} \int_r^{R+\eta} (t-r)^2 e^{-(n-1)\delta t} dt < +\infty.
\]
As \( f \) is piecewise \( C^1 \) and vanishes on \( \partial \Omega_r \) (because \( u(r) = 0 \)), we get that
\[
f \in H^2_{1,c}(\Omega_r). \tag{18}
\]
Using (14) we also have:
\[
\frac{\omega_{n-1} e^{n-1}}{(n - 1)\delta} e^{-(n-1)\delta r} \left( 1 - e^{-(n-1)\delta (R-r)} \right) \leq \text{Vol}(\Omega_r, g_\varepsilon) \leq \frac{\omega_{n-1} e^{n-1}}{(n - 1)\delta} e^{-(n-1)\delta r}
\]
Integrating by parts, we get:
\[
\int_{\Omega_r} f dv_{g_\varepsilon} \geq \frac{\omega_{n-1} e^{n-1}}{(n - 1)\delta} \int_r^R (t-r) e^{-(n-1)\delta t} dt \geq \frac{\omega_{n-1} e^{n-1}}{(n - 1)\delta} e^{-(n-1)\delta r} \left[ 1 - (1 + (n - 1)\delta (R-r)) e^{-(n-1)\delta (R-r)} \right] \tag{19}
\]
Combining this last equality with \((19)\) it follows that:
\[
\frac{1}{\text{Vol}(\Omega_r, g_\varepsilon)} \int_{\Omega_r} f \, dv_{g_\varepsilon} \geq \frac{1 - (1 + (n - 1)\delta(R - r)) \, e^{-(n-1)\delta(R-r)}}{(n - 1)^2\delta^2}.
\]
As \(f \in H^2_{1,\varepsilon}(\Omega_r)\) one may apply Definition \((14)\) which gives (with the help of \((20)\) and \((17)\)):
\[
\mathcal{E}(\Omega_r, g_\varepsilon) \geq \frac{2 \int_{\Omega_r} f \, dv_{g_\varepsilon} - \int_{\Omega_r} |\nabla f|^2 \, dv_{g_\varepsilon}}{\text{Vol}(\Omega_r, g_\varepsilon)} \geq \frac{1 - 2 [1 + (n - 1)\delta(R - r)] \, e^{-(n-1)\delta(R-r)}}{(n - 1)^2\delta^2}.
\]
Let \(A_\varepsilon\) be the unique solution of the equation
\[
2(1 + x)e^{-x} = \varepsilon.
\]
For every \(R \geq \frac{2A_\varepsilon}{(n-1)\delta}\) and \(0 \leq r \leq \frac{R}{2}\) we have \((n - 1)\delta(R - r) \geq A_\varepsilon\) and thus
\[
2 [1 + (n - 1)\delta(R - r)] \, e^{-(n-1)\delta(R-r)} \leq \varepsilon.
\]
Using \((16)\) we thus get (when \(R \geq \frac{2A_\varepsilon}{(n-1)\delta}\) and \(0 \leq r \leq \frac{R}{2}\)):
\[
\mathcal{E}(\Omega_r, g_\varepsilon) \geq \frac{1 - \varepsilon}{(n - 1)^2\delta^2} \geq \frac{(1 - \varepsilon)(1 - B\sqrt{\varepsilon})^2}{H(M, g_\varepsilon)^2},
\]
which proves \((13)\). As \(x \mapsto \frac{x}{1+x}\) is increasing on \([0, +\infty[\), we have (again by \((14)\)):
\[
v(r) := \frac{\text{Vol}(\Omega_r, g_\varepsilon)}{\text{Vol}(M, g_\varepsilon)} = \frac{\int_0^{R+\eta} b_\varepsilon(t)^{n-1} \, dt}{2 \int_0^{R+\eta} b_\varepsilon(t)^{n-1} \, dt} \leq \frac{\int_0^{+\infty} e^{-(n-1)\delta t} \, dt}{2 \int_0^{+\infty} e^{-(n-1)\delta t} \, dt} \leq \frac{1}{2} e^{-(n-1)\delta r}.
\]
As \(v(0) = \frac{1}{2}\) in order to prove \((12)\), it is enough to show that \(v(\frac{R}{2}) \leq \frac{1}{2} e^{-(n-1)\delta \frac{R}{2}} \leq \frac{1}{4}\) which follows immediately from \((21)\) with \(r = \frac{R}{2}\). This concludes the proof of the proposition. \(\Box\)

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