A DUALITY BETWEEN VERTEX SUPERALGEBRAS $L_{-3/2}(\mathfrak{osp}(1\vert 2))$ AND $\mathcal{V}^{(2)}$ AND GENERALIZATION TO LOGARITHMIC VERTEX ALGEBRAS

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Abstract. We introduce a subalgebra $\mathcal{F}$ of the Clifford vertex superalgebra $(bc$ system) which is completely reducible as a $L^{Vir}(-2,0)$–module, $C_2$–cofinite, but it is not conformal and it is not isomorphic to the symplectic fermion algebra $\mathcal{S}F(1)$. We show that $\mathcal{S}F(1)$ and $\mathcal{F}$ are in an interesting duality, since $\mathcal{F}$ can be equipped with the structure of a $\mathcal{S}F(1)$–module and vice versa.

Using the decomposition of $\mathcal{F}$ and a free-field realization from [3], we decompose $L_k(\mathfrak{osp}(1\vert 2))$ at the critical level $k = -3/2$ as a module for $L_k(\mathfrak{sl}(2))$. The decomposition of $L_k(\mathfrak{osp}(1\vert 2))$ is exactly the same as of the $N = 4$ superconformal vertex algebra with central charge $c = -9$, denoted by $\mathcal{V}^{(2)}$. Using the duality between $\mathcal{F}$ and $\mathcal{S}F(1)$, we prove that $L_k(\mathfrak{osp}(1\vert 2))$ and $\mathcal{V}^{(2)}$ are in the duality of the same type. As an application, we construct and classify all irreducible $L_k(\mathfrak{osp}(1\vert 2))$–modules in the category $\mathcal{O}$ and the category $\mathcal{R}$ which includes relaxed highest weight modules. We also describe the structure of the parafermion algebra $N_{-3/2}(\mathfrak{osp}(1\vert 2))$ as a $N_{-3/2}(\mathfrak{sl}(2))$–module.

We extend this example, and for each $p \geq 2$, we introduce a non-conformal vertex algebra $\mathcal{A}_n^{(p)}$ and show that $\mathcal{A}_n^{(p)}$ is isomorphic to the doublet vertex algebra as a module for the Virasoro algebra. We also construct the vertex algebra $\mathcal{V}_n^{(p)}$ which is isomorphic to the logarithmic vertex algebra $\mathcal{V}^{(p)}$ as a module for $\hat{\mathfrak{sl}}(2)$.

1. Introduction

In the representation theory of vertex algebras a special emphasis was put on $C_2$–cofinite, non-rational vertex algebras. Recently these vertex algebras are also called logarithmic vertex algebras since they appeared in logarithmic conformal field theory (cf. [17], [16]). Basic examples are the triplet vertex algebras $\mathcal{W}_n^{(p)}$ (cf. [6], [19], [20]) and the symplectic fermion vertex superalgebra $\mathcal{S}F(d)$ (cf. [1], [25]). The triplet vertex algebra $\mathcal{W}_n^{(p)}$ has a simple current extension $\mathcal{A}_n^{(p)}$ (cf. [17]), which is called the doublet vertex algebra. Note that for $p = 2$, $\mathcal{A}_n^{(2)} \cong \mathcal{S}F(1)$.

Dualities like the Kazama-Suzuki duality and the S-duality, have recently appeared in the papers on logarithmic vertex algebras, affine and $\mathcal{W}$–algebras (cf. [5], [14], [13], [18]). In the current paper, we begin a study of a different duality which relates the logarithmic vertex algebras with some non-conformal vertex algebras, like affine vertex superalgebras at the critical level.

Let $\mathcal{F}$ be a Clifford vertex superalgebra generated by the charged fermionic fields $\Psi^\pm$ with non-trivial $\lambda$-bracket $[\Psi^+\Psi^-] = 1$. Then $\mathcal{S}F(1)$ is realized as:

$$\mathcal{S}F(1) = \text{Ker}_F \int \Psi^-(z)dz.$$ 

The even part of $\mathcal{S}F(1)$ is isomorphic to the triplet vertex algebra $\mathcal{S}F(1)^+ \cong \mathcal{W}^{(p)}$ with $p = 2$ (cf. [1], [6]).
The triplet vertex algebras and the symplectic fermion of rank one \( \mathcal{SF}(1) \) are completely reducible modules for the Virasoro algebra. Let \( L^{\text{vir}}(c, h) \) denotes the irreducible, highest weight module for the Virasoro algebra of central charge \( c \) and highest weight \( h \). Then

\[
\mathcal{SF}(1) = \bigoplus_{n=0}^{\infty} (n + 1)L^{\text{vir}}(-2, \frac{n^2 + n}{2}).
\]

A \( \widehat{\mathfrak{sl}}(2) \)-version of logarithmic vertex algebras were introduced in \([2]\) and studied also in \([4]\). These algebras are called the \( V(p) \)-algebras. The \( V(p) \)-algebras are completely reducible modules for the affine Lie algebra \( \widehat{\mathfrak{sl}}(2) \) at level \(-2 + \frac{1}{p}\). It was proved that the quantum Hamiltonian reduction functor \( H_{DS} \) sends \( V(p) \) to \( A(p) \).

A very interesting case is \( p = 2 \) when the vertex superalgebra \( V(2) \) carries the structure of the small \( N=4 \) superconformal algebra with central charge \( c_1/2 = -9 \) (cf. \([2]\)). This vertex superalgebra has also appeared in the four-dimensional super Yang-Mills theory in physics (cf. \([9]\)).

Recall that the small \( N=4 \) superconformal algebra at central charge \( c_k = -6(k + 1) \) is realised as the minimal affine \( W \)-algebra \( W_k(\mathfrak{psl}(2|2), f_\theta) \) (cf. \([23]\)). So in our case \( V(2) = W_k(\mathfrak{psl}(2|2), f_\theta) \) with \( k = 1/2 \). Then we have:

\[
V(2) = \bigoplus_{n=0}^{\infty} (n + 1)L^{\widehat{\mathfrak{sl}}(2)}_{-3/2}(n\omega_1).
\]

In the current paper we shall investigate two non-conformal vertex algebras \( \overline{F} \) and \( L_{-3/2}(\mathfrak{osp}(1|2)) \) such that

\[
\overline{F} = \bigoplus_{n=0}^{\infty} (n + 1)L^{\text{vir}}(-2, \frac{n^2 + n}{2}),
\]

\[
L_{-3/2}(\mathfrak{osp}(1|2)) = \bigoplus_{n=0}^{\infty} (n + 1)L^{\widehat{\mathfrak{sl}}(2)}_{-3/2}(n\omega_1).
\]

\( L_{-3/2}(\mathfrak{osp}(1|2)) \) is the unique graded simple quotient of the universal affine vertex superalgebra \( V^{-3/2}(\mathfrak{osp}(1|2)) \) at the critical level. This vertex superalgebra is not conformal, but it has a vertex subalgebra isomorphic to \( L_{-3/2}(\mathfrak{sl}(2)) \). Thus, we solve the branching rule problem for the embedding \( \mathfrak{sl}(2) \hookrightarrow \mathfrak{osp}(1|2) \) at level \( k = -3/2 \). Quite surprisingly, we get the decomposition \((1.4)\) which coincides with \((1.2)\). Note that the Sugawara Virasoro vector of \( L_{-3/2}(\mathfrak{sl}(2)) \) does not define the structure of a vertex operator superalgebra on \( L_{-3/2}(\mathfrak{osp}(1|2)) \).

The realization of \( L_k(\mathfrak{osp}(1|2)) \) was presented in \([3]\), so that \( L_k(\mathfrak{osp}(1|2)) \hookrightarrow L_{c_k}^{\text{ns}} \otimes \Pi(0)^{1/2} \), where \( L_{c_k}^{\text{ns}} \) is the simple \( N=1 \) Neveu-Schwarz vertex superalgebra, and \( \Pi(0)^{1/2} \) is a half-lattice vertex algebra. A critical version of the realization was also presented in \([3]\).

In the current paper, we investigate the critical level case in more details. We get a homomorphism \( f : V^{-3/2}(\mathfrak{osp}(1|2)) \rightarrow \overline{F} \otimes \Pi(0)^{1/2} \). The vertex algebra \( \overline{F} \) is realized as

\[
\overline{F} = \text{Ker}_F \int \Phi(z)dz,
\]
where $\Phi_2(z) = \frac{1}{\sqrt{2}}(\Psi^+(z) - \Psi^-(z))$ is the neutral fermionic field. We show that $Im(f)$ is the simple vertex superalgebra:

$$L_{-3/2}(osp(1|2)) = \text{Ker}_F \otimes \Pi(0)^{1/2} S^{osp}$$

where

$$S^{osp} = \int e^{2\hat{\Phi}^+(z)dz} : F \otimes \Pi(0)^{1/2} \rightarrow F_{tw} \otimes \Pi_\nu$$

is a screening operator defined in Subsection 4.3.

We prove the following results which can explain the coincidences which we have noticed above:

- The vertex superalgebra $F$ has the structure of an irreducible $\mathcal{SF}(1)$–module and vice versa (cf. Proposition 3.4).
- The vertex superalgebra $L_{-3/2}(\mathfrak{g})$ has the structure of an irreducible $\mathcal{V}(2)$–module and vice versa (cf. Theorem 5.1).

Table 1. Correspondences between irreducible $\mathcal{V}(2)$–modules and $L_{-3/2}(osp(1|2))$–modules (see Sect. 6)

| Vertex algebra | $\mathcal{V}(2)$ | $L_{-3/2}(osp(1|2))$ |
|----------------|-----------------|---------------------|
| ordinary modules | $\mathcal{V}(2)$ | $L_{-3/2}(osp(1|2))$ |
| category $\mathcal{O}$ | $\mathcal{V}(2)$ | $L_{-3/2}(osp(1|2))$ |
| Indecomposable modules in $\mathcal{O}$ | $\mathcal{M}^N_{-1}$ | $L_{-3/2}(osp(1|2))$ |
| Relaxed h.w. modules | $L_{-3/1}(U_{-1})$ | $L_{-3/2}(osp(1|2))$ |

It is important to note that modules in the same lines are isomorphic as $L_{-3/2}(\mathfrak{sl}(2))$–modules.

In this article, we show that this interesting connection can be extended on a broader class of logarithmic vertex algebras. For each $p \geq 2$, in Section 7 we construct non-conformal vertex algebras $A^{(p)}_{\text{new}}$ and $\mathcal{V}^{(p)}_{\text{new}}$ and show that

- $A^{(p)}_{\text{new}} \cong A^{(p)}_\text{new}$; $\mathcal{W}^{(p)}_{\text{new}} \cong \mathcal{W}^{(p)}$ as modules for the Virasoro algebra;
- $\mathcal{V}^{(p)}_{\text{new}} \cong \mathcal{V}^{(p)}$ as a modules for the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$.

The proof is based on the construction of an explicit Virasoro (resp. $\widehat{\mathfrak{sl}}(2)$)–isomorphism.

2. Clifford vertex superalgebra $F$ and its twisted modules

We assume that the reader is familiar with basic concepts in the theory of vertex algebras and the representation theory of affine Lie algebras and the Virasoro algebras. Let $(V,Y,1)$ be a vertex superalgebra (cf. [26], [27]). The derivation in the vertex superalgebra $V$ is denoted by $D$. Let $L^{Vir}(c,0)$ be the simple Virasoro vertex algebra of central charge $c$. Let $L^{Vir}(c,h)$ denotes the irreducible, highest weight module for the Virasoro algebra of central charge $c$ and highest weight $h$. 
2.1. **Vertex superalgebra \( F \) and its Virasoro vectors.** The Clifford vertex superalgebra \( F \) is the universal vertex superalgebra generated by the odd fields \( \Phi_i, i = 1, 2 \), and the following \( \lambda \)-brackets:

\[
[(\Phi_i)_\lambda \Phi_j] = \delta_{i,j}.
\]

The vertex superalgebra \( F \) has the structure of an irreducible module for the Clifford algebra \( Cl(A) \) associated to the vector superspace \( A = \mathbb{C}\Phi_1 \oplus \mathbb{C}\Phi_2 \) with generators \( \Phi_i(r), r \in \frac{1}{2} + \mathbb{Z} \) and anti-commutation relations

\[
[\Phi_i(r), \Phi_j(s)]_\pm = \delta_{r+s,0}\delta_{i,j}.
\]

As a vector space \( F = \bigwedge (\Phi_i(1/2 - n) \mid n \in \mathbb{Z}_{>0}, i = 1, 2) \).

Let

\[
\Psi^+ = \frac{1}{\sqrt{2}}(\Phi_1 + \sqrt{-1}\Phi_2), \quad \Psi^- = \frac{1}{\sqrt{2}}(\Phi_1 - \sqrt{-1}\Phi_2).
\]

Then

\[
\Phi_1 = \frac{1}{\sqrt{2}}(\Psi^+ + \Psi^-), \quad \Phi_2 = \frac{1}{\sqrt{-2}}(\Psi^+ - \Psi^-).
\]

By the boson-fermion correspondence we have that \( F \cong V_{2\alpha} \), where \( \langle \alpha, \alpha \rangle = 1 \) and \( \Psi^\pm = e^{\pm \alpha} \). Therefore

\[
\Phi_1 = \frac{1}{\sqrt{2}}(e^\alpha + e^{-\alpha}), \quad \Phi_2 = \frac{1}{\sqrt{-2}}(e^\alpha - e^{-\alpha}).
\]

\[
\omega_{sf} = D\Psi^+ \Psi^- = \frac{1}{2} \left( \Phi_1(-\frac{3}{2})\Phi_1(-\frac{1}{2}) + \Phi_2(-\frac{3}{2})\Phi_2(-\frac{1}{2}) \right) + \frac{\sqrt{-1}}{2} \left( \Phi_2(-\frac{3}{2})\Phi_1(-\frac{1}{2}) - \Phi_1(-\frac{3}{2})\Phi_2(-\frac{1}{2}) \right)
\]

\[
D\Psi^+ \Psi^+ = \frac{1}{2} \left( \Phi_1(-\frac{3}{2})\Phi_1(-\frac{1}{2}) - \Phi_2(-\frac{3}{2})\Phi_2(-\frac{1}{2}) \right) + \frac{\sqrt{-1}}{2} \left( \Phi_2(-\frac{3}{2})\Phi_1(-\frac{1}{2}) + \Phi_1(-\frac{3}{2})\Phi_2(-\frac{1}{2}) \right)
\]

Note that \( \omega_{sf} \) is a Virasoro vector of central charge \( c = -2 \) in \( F \), \( D\Psi^+ \Psi^+ \) is a commutative vector.

Let \( L_{sf}(n) = (\omega_{sf})_{n+1} \). Then

\[
\omega = \omega_{sf} + D\Psi^+ \Psi^+ = \sqrt{-1}\Phi_2(-\frac{3}{2})\Phi_1(-\frac{1}{2}) + \Phi_1(-\frac{3}{2})\Phi_1(-\frac{1}{2})1
\]

\[
= \frac{1}{2}\alpha(-1)^2 + \frac{1}{2}(-2) + e^{2\alpha}
\]

is a Virasoro vector in \( F \cong V_{2\alpha} \) of central charge \( c_{1,2} = -2 \). Let \( L(n) = \omega_{n+1} \).

2.2. **Symplectic fermion vertex superalgebra \( SF(1) \).** The symplectic fermion vertex algebra \( SF(1) \) is defined as

\[
SF(1) = \ker_F \int \Psi^-(z)dz.
\]

As a vector space,

\[
SF(1) = \bigwedge (\Psi^-(r + \frac{1}{2}), \Psi^+(r - \frac{1}{2}) \mid r \in \mathbb{Z}_{>0}).
\]
The vertex superalgebra $\mathcal{SF}(1)$ is a simple, $C_2$–cofinite $\mathbb{Z}_{\geq 0}$–graded vertex operator superalgebra with conformal vector $\omega_{sf}$ (cf. [1]). It is freely generated by the fields $a^+ = \Psi^-, a^- = D\Psi^+$ of conformal weights one. Set $a^+(n) = \Psi^-(n+1/2), \ a^-(n) = -n\Psi^+(n-1/2)$. Then we have

\begin{equation}
[a^+(n), a^+(m)]_+ = 0, \ [a^+(n), a^-(m)]_+ = n\delta_{n+m,0}.
\end{equation}

The PBW basis of $\mathcal{SF}(1)$ is given by

$$a^-(-m_1) \cdots a^-(-m_r)a^+(-n_1) \cdots a^-(-n_s)1$$

where $r, s \in \mathbb{Z}_{\geq 0}, m_i, n_i \in \mathbb{Z}_{>0}, m_1 > \cdots > m_r \geq 1, n_1 > \cdots > n_s \geq 1$.

In other words $\mathcal{SF}(1)$ is the universal affine vertex superalgebra of level 1 associated to the Lie superalgebra $\mathfrak{psl}(1|1)$, i.e.

$$\mathcal{SF}(1) = L_1(\mathfrak{psl}(1|1)) = V^1(\mathfrak{psl}(1|1)).$$

So in order to construct a $\mathcal{SF}(1)$–module, it is enough to construct a restricted $\mathfrak{psl}(1|1)$–module of level 1.

2.3. Twisted module $F^{tw}$. Let $\Theta$ be the automorphism of the vertex superalgebra $F$ such that

$$\Phi_i \mapsto -\Phi_i, i = 1, 2.$$ 

Then a $\Theta$–twisted $F$–module is realised as

$$(F^{tw}, Y_{tw}) := (F, Y(\Delta(\alpha/2, z), z)).$$

$F^{tw}$ is realized as $V_{2\alpha+\alpha/2}$. Note that $v_{tw} = e^{\alpha/2}$ is a singular vector for the Virasoro algebra such that

$$U(Vir).v_{tw} \cong L^{Vir}(-2, -\frac{1}{8}).$$

3. Vertex superalgebra $\overline{F}$ and its duality with $\mathcal{SF}(1)$

In this section we introduce the subalgebra $\overline{F}$ of $F$ with a Virasoro vector $\omega$ of central charge $c_{1,2} = -2$ defined by [21]. $\overline{F}$ is not a vertex operator superalgebra since $L(-1) \neq D$, but it shares similar properties as symplectic fermion vertex superalgebra $\mathcal{SF}(1)$. In particular, we will show that $\overline{F}$ is isomorphic to $\mathcal{SF}(1)$ as a module for the Virasoro algebra. This case will be a motivated example for introducing non-conformal duals of logarithmic vertex algebras in Section 7.

Let $G = \Phi_2(1/2)$. Define the following vertex subalgebra of $F$:

\begin{equation}
\overline{F} = \text{Ker}_F G = \cap \left( \Phi_1(-r + \frac{1}{2}), \Phi_2(-r - \frac{1}{2}) \mid r \in \mathbb{Z}_{>0} \right).
\end{equation}

Proposition 3.1. We have:

1. $\overline{F}$ is a simple vertex superalgebra.
2. $\overline{F}$ is strongly generated by $\Phi_1(z)$ and $\partial_z \Phi_2(z)$
3. $\overline{F}$ is $C_2$–cofinite, non-rational vertex superalgebra.

Proof. The assertions (1) and (2) are clear. Using the basis (3.1) of $\overline{F}$ we get

$$\overline{F}/C_2(\overline{F}) = \text{span}_C \{ 1, \Phi_1, D\Phi_2, : \Phi_1 D\Phi_2 : \} ,$$

where for $v \in \overline{F}$, we write $\overline{v} = v + C_2(\overline{F}) \in \overline{F}/C_2(\overline{F})$. 

Finally, as in the case of symplectic fermion, we see that $F$ is indecomposable, but reducible $\mathcal{F}$-module. Therefore $\mathcal{F}$ is non-rational vertex superalgebra.  

Recall that $\omega_{sf}$ is a Virasoro vector of central charge $c = -2$ in $F$, $D\Psi^+\Psi^+ = \epsilon^{2\alpha}$ is a commutative vector. Then

$$\omega = \omega_{sf} + D\Psi^+\Psi^+ = \sqrt{-1}\Phi_2(-\frac{3}{2})\Phi_1(-\frac{1}{2})1 + \Phi_1(-\frac{3}{2})\Phi_1(-\frac{1}{2})1$$

is a Virasoro vector in $\mathcal{F}$ of central charge $c = -2$.

The conformal vector $\omega$ does not define on $\mathcal{F}$ the structure of a vertex operator superalgebra since the derivation $D = L_{sf}(-1)$ on $\mathcal{F}$ is not $L(-1)$. Moreover, $Q = L(-1) - D = (D\Psi^+\Psi^+)_0$ is a well defined operator which commutes with the action of the Virasoro algebra $L(n) = L_{sf}(n) + (D\Psi^+\Psi^+)_{n+1}$, since it commutes with $L_{sf}(n)$ (cf. [6]) and obviously with $(D\Psi^+\Psi^+)_{n+1}$.

The proof of the following lemma is clear.

**Proposition 3.2.** We have:

- \([Q, Q] = 0\).
- \(Q\) is a derivation on the vertex superalgebra $\mathcal{F}$ such that $[Q, L(n)] = 0$ for $n \in \mathbb{Z}$.

Define the following operators

$$\varphi^-(n) = a^-(n) = -\frac{n}{\sqrt{2}} (\Phi_1(n - \frac{1}{2}) + \sqrt{-1}\Phi_2(n - \frac{1}{2}))$$

$$\varphi^+(n) = -\sqrt{2} (n\Phi_1(n + \frac{1}{2}) + \sqrt{-1}(n + 1)\Phi_2(n + \frac{1}{2}))$$

$$= -(2n + 1)\Psi^+(n + \frac{1}{2}) + \Psi^-(n + \frac{1}{2})$$

Note that

$$\omega = \varphi^+(-1)\varphi^-(-1)1.$$ 

Let $\varphi^\pm(z) = \sum_{n \in \mathbb{Z}} \varphi^\pm(n)z^{-n-1}$. Then

$$\varphi^+(z) = z\sqrt{2} (\partial_z \Phi_1(z) + \sqrt{-1}\partial_z \Phi_2(z)) + \sqrt{2}\Phi_1(z)$$

$$\varphi^-(z) = \frac{\sqrt{2}}{2} (\partial_z \Phi_1(z) + \sqrt{-1}\partial_z \Phi_2(z))$$

This implies:

$$\Phi_1(z) = \frac{1}{\sqrt{2}} (\varphi^+(z) - 2z\varphi^-(z))$$

$$\partial_z \Phi_2(z) = -\sqrt{2}\varphi^-(z) - \frac{\sqrt{-1}}{\sqrt{2}}\partial_z (\varphi^+(z) - 2z\varphi^-(z)).$$

By a direct calculation we have:

**Lemma 3.3.** We have:

$$[L(n), \varphi^\pm(m)] = -m\varphi^\pm(n + m), \quad [\varphi^\pm(m), \varphi^\pm(n)]_+ = 0, \quad [\varphi^+(n), \varphi^-(m)]_+ = n\delta_{n+m,0}.$$  

**Proof.** By a direct calculation we get:

$$[L(n), \Phi_1(m + \frac{1}{2})] = -(n + 1 + 2m)\Phi_1(n + m + \frac{1}{2}) - \sqrt{-1}(n + m + 1)\Phi_2(n + m + \frac{1}{2})$$

$$[L(n), \Phi_2(m + \frac{1}{2})] = -m\sqrt{-1}\Phi_1(n + m + \frac{1}{2})$$
This implies
\[
\begin{align*}
[L(n), \varphi^-(m)] &= -m\varphi^-(n + m), \\
[L(n), \varphi^+(m)] &= \sqrt{2}m(-n + 1 + 2m)\Phi_1(n + m + \frac{1}{2}) - \sqrt{-1}(n + m + 1)\Phi_2(n + m + \frac{1}{2}) + \sqrt{2}(m + 1)\Phi_1(n + m + \frac{1}{2}) \\
&= -m\varphi^+(n + m).
\end{align*}
\]

Let \(\tilde{SF}(1)\) be the vertex superalgebra generated by local fields \(\varphi^+(z), \varphi^-(z)\) acting on \(\tilde{F}\).

**Proposition 3.4.** We have:

1. \(\tilde{SF}(1)\) is isomorphic to \(SF(1)\).
2. The vertex superalgebra \(\tilde{F}\) has the structure of an irreducible \(SF(1)\)–module, denoted by \((\tilde{F}, \tilde{Y}_{SF}(1)(\cdot, z))\), which is uniquely determined by \(\tilde{Y}_{SF}(1)(a^\pm, z) = \varphi^\pm(z)\). The action of the Virasoro field is \(\tilde{Y}_{SF}(1)(\omega, z) = L(z)\).
3. The vertex superalgebra \(SF(1)\) has the structure of an irreducible \(\tilde{F}\)–module, denoted by \((SF(1), Y_{SF}(1)(\cdot, z))\), determined by formulas \((3.4)-(3.5)\).

**Proof.** Lemma 3.3 together with the fact that \(SF(1) = V^1(\mathfrak{psl}(1|1))\) (cf. Subsection 2.2) implies that \(\tilde{SF}(1) \cong SF(1)\). So (1) holds. Then using the theory of local fields from [27] we get map \(a^\pm(z) \mapsto \varphi^\pm(z)\) which uniquely define on \(\tilde{F}\) the structure of a \(SF(1)\)–module. Assume that \(\tilde{F}\) is reducible \(SF(1)\)–module, with a submodule \(U \neq \tilde{F}\). Then the relations \((3.4)-(3.5)\) show that \(U\) is also a \(\tilde{F}\)–submodule. But this is not possible since \(\tilde{F}\) is simple vertex superalgebra. This proves the assertion (2). The proof of (3) is completely analogous. \(\square\)

Since \(\tilde{F} \cong SF(1)\) as a \(SF(1)\)–module, and since the Virasoro field \(L_{sf}(z)\) acts on \(\tilde{F}\) as \(L(z)\), the results from [1] and [6] directly imply:

**Theorem 3.5.** For each \(n \in \mathbb{Z}_{>1}\) and \(0 \leq j \leq n\) we have:

1. \(w_{nj} := Q^n\varphi^+(n)\cdots\varphi^+(1)\) is a singular vector in \(\tilde{F}\).
2. \(Q^n\varphi^+(n)\cdots\varphi^+(1) = \nu\varphi^-(n)\cdots\varphi^-(1)\) for certain \(\nu \neq 0\).

As a \(L^{Vir}(-2, 0)\)–module:

\[
\tilde{F} = \bigoplus_{n=0}^{\infty}(n + 1)L^{Vir}(-2, \frac{n^2 + n}{2}).
\]

The vertex superalgebra \(\tilde{F}\) has the canonical parity automorphism \(\sigma\) such that

\[
\Phi_i \mapsto -\Phi_i, i = 1, 2.
\]

Let

\[
\tilde{F}^+ = \{v \in \tilde{F} \mid \sigma(v) = v\}.
\]

**Remark 3.6.** Note that \(\tilde{F}\) is not isomorphic to the symplectic fermion vertex superalgebra \(SF(1)\), although these two vertex superalgebras are isomorphic as \(L^{Vir}(-2, 0)\)–modules.

In Section 7 we shall construct an explicit \(L^{Vir}(-2, 0)\)–isomorphism between symplectic fermion vertex superalgebra \(SF(1)\) and \(\tilde{F}\). Note that \(SF(1)^+\) is isomorphic to the triplet vertex algebra \(W^{(p)}\) for \(p = 2\).

The following result are special case of Theorem 7.1 in the case \(p = 2\).
Proposition 3.7. The mappings
\[ \Omega = \exp[e^{2\pi i}]_{SF(1)} : SF(1) \to \mathcal{F}, \]
\[ \Omega = \exp[e^{2\pi i}]_{W(2)} : W(2) \to \mathcal{F}^+, \]
are $L^{vir}(-2, 0)$-isomorphisms. Moreover, as operators of $End(F)$ we have
\[ \Omega L_{sf}(n) = L(n)\Omega, \quad \Omega a^\pm(n) = \varphi^\pm(n)\Omega. \]

4. Realization of $L_{-\frac{3}{2}}(\mathfrak{osp}(1|2))$ and its consequences

In this section we first recall a realisation from [3] which gives a homomorphism $\Phi : V^k(\mathfrak{osp}(1|2)) \to \mathcal{F} \otimes \Pi(0)^{1/2}$ at $k = -3/2$. We prove that the image of this homomorphism is the simple vertex superalgebra $L_k(\mathfrak{osp}(1|2))$. We also construct the screening operator $S_{osp}$ for this realisation. By using the decomposition of $\mathcal{F}$ as a $L^{vir}(-2, 0)$–module from Section 4.1 we obtain the decomposition of $L_k(\mathfrak{osp}(1|2))$ as a $L_k(\mathfrak{s}(2))$–module.

4.1. Affine vertex superalgebra $V^k(\mathfrak{osp}(1|2))$. Recall that $\mathfrak{g} = \mathfrak{osp}(1|2)$ is the simple complex Lie superalgebra with basis $\{e, f, x, y\}$ such that the even part $\mathfrak{g}_0 = \text{span}_\mathbb{C}\{e, f, h\} \cong \mathfrak{s}(2)$ and the odd part $\mathfrak{g}_1 = \text{span}_\mathbb{C}\{x, y\}$. The anti-commutation relations are given by
\[ [e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f \]
\[ [h, x] = x, \quad [e, x] = 0, \quad [f, x] = -y \]
\[ [h, y] = -y, \quad [e, y] = -x, \quad [f, y] = 0 \]
\[ \{x, y\} = 2e, \quad \{x, y\} = h, \quad \{y, x\} = -2f. \]

Choose the non-degenerate super-symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ such that non-trivial products are given by
\[ (e, f) = (f, e) = 1, \quad (h, h) = 2, \quad (x, y) = -(y, x) = 2. \]

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K + \mathbb{C}d$ be the associated affine Kac-Moody Lie superalgebra, where $d$ is the degree operator and $K$ is the central element. Let $\tilde{\mathfrak{g}} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K$. Let $V^k(\mathfrak{g})$ be the associated universal affine vertex superalgebra, and $L_k(\mathfrak{g})$ be its unique simple $d$–graded quotient. As usual, we identify $x \in \mathfrak{g}$ with $x(-1)1$.

Define also the parafermion vertex algebra $N_k(\mathfrak{g}) = \{v \in L_k(\mathfrak{g}) \mid h(n)v = 0 \quad \forall n \in \mathbb{Z}_{\geq 0}\}$. The representation theory of the affine vertex superalgebra $L_k(\mathfrak{g})$ at admissible levels was recently studied in [12, 13, 29, 22, 30]. In the positive integer level case, $N_k(\mathfrak{g})$ is generated by $N_k(\mathfrak{g}_0)$ and a primary vector of weight three (cf. [24]). In the present paper shall study the structure and representation theory of $L_k(\mathfrak{g})$ and $N_k(\mathfrak{g})$ at the critical level $k = -3/2$.

4.2. Realization. We shall here recall some consequences of the realization of the vertex superalgebra $V^\frac{3}{2}(\mathfrak{g})$ from [3].

Consider the lattice vertex algebra $\Pi(0)^{1/2} = M(1) \otimes \mathbb{C}[\mathbb{Z}^2_\mathbb{Z}]$ as in [3] (see also [10, 28]). Let $g = \exp[\pi i d(0)]$ be the automorphism of order two of $\Pi(0)^{1/2}$. Set $\mu = \frac{d}{2} + \frac{c}{4}, \nu = \frac{d}{2} - \frac{c}{4}$. For $\lambda \in \mathbb{C}$ and $r \in \mathbb{Z}$, we define
\[ \Pi^{1/2}_{(r)}(\lambda) := \Pi^{1/2}(0).e^{r\mu + \lambda c}. \]

Then $\Pi^{1/2}_{(r)}(\lambda)$ is an untwisted $(g$–twisted) $\Pi(0)^{1/2}$–module if $r$ is even (if $r$ is odd) (see [3 Section 4.1]).
Theorem 4.1. Let \( k = -3/2 \). There exists a non-trivial homomorphism

\[
\Phi : V^k(\mathfrak{g}) \to F \otimes \Pi(0)^{1/2}
\]

such that

\[
e \mapsto e^c,
\]

\[
h \mapsto 2\mu(-1),
\]

\[
f \mapsto [(k + 2)\omega - \nu(-1)^2 - (k + 1)\nu(-2)] e^{-c}
\]

\[
x \mapsto \sqrt{2}\Phi_1(-\frac{1}{2}) e^{c/2}
\]

\[
y \mapsto \sqrt{2} \left[ -\frac{i}{2} \Phi_2(-\frac{3}{2}) + \Phi_1(-\frac{1}{2})\nu(-1) + \frac{2k + 1}{2} \Phi_1(-\frac{3}{2}) \right] e^{-c/2}.
\]

where \( \omega \) is the Virasoro vector of \( F \) of central charge \( c_{1,2} = -2 \).

By using the boson-fermion correspondence (cf. [26]), one can derive expressions for odd generators \( x, y \) in \( V_{Z} \alpha \otimes \Pi(0)^{1/2} \):

Lemma 4.2. We have:

\[
(4.1) \quad x = e^{\alpha+c/2} + e^{-\alpha+c/2},
\]

\[
(4.2) \quad y = [\nu(-1) - \frac{1}{2}\alpha(-1)] e^{-\alpha-c/2} + [\nu(-1) - \frac{3}{2}\alpha(-1)] e^{\alpha-c/2}
\]

Let \( \omega^0_{\text{sug}} \) be the Sugawara Virasoro vector in \( V^k(\mathfrak{g}_0) \). Then

\[
\omega_{\text{sug}} = \Phi(\omega^0_{\text{sug}}) = \omega + \frac{1}{2}c(-1)d(-1) - \frac{1}{2}d(-2) + \frac{k}{4}c(-2).
\]

Set \( L_{\text{sug}}(n) = (\omega_{\text{sug}})_{n+1} \). We have:

\[
(4.3) \quad L_{\text{sug}}(-1) - D = Q.
\]

4.3. Screening operators. Note that \( \Pi_{\nu} := V(0)^{1/2\nu} \) is a twisted \( V(0)^{1/2} \)-module. Moreover \( F^{tw} \otimes \Pi_{\nu} \) is a twisted \( F \otimes V(0)^{1/2} \)-module. One shows that \( F^{tw} \otimes \Pi_{\nu} \) is an untwisted \( V^k(\mathfrak{g}) \)-module. Let \( s = e^{\frac{1}{2}+\nu} = v^{tw} \otimes e^{\nu} \), and \( S^{\text{osp}} = \int Y(s, z)dz = s_0 \).

Lemma 4.3. \( S^{\text{osp}} \) is the screening operator and commutes with the action of \( \hat{\mathfrak{g}} \).

Proof. Note that \( v^{tw} \) is a highest weight vector of conformal weight \(-1/8\) which for \( c_{1,2} = -2 \) correspond to the singular vector denoted by \( v_{2,1} \) in [3]. Therefore \( s = v_{2,1} \otimes e^{\nu} \) and \( S^{\text{osp}} \) coincides with the \( \mathfrak{g}_0 \) screening operator from [3] and therefore it commutes with the action of \( \hat{\mathfrak{g}}_0 \). It remains to prove that \( S^{\text{osp}} \) commutes with operators \( x(n) \) and \( y(n) \) for \( n \in \mathbb{Z} \).

By a direct calculation and using expressions for \( x, y \) from Lemma 4.2, we get

\[
(4.4) \quad S^{\text{osp}} x = S^{\text{osp}} y = 0.
\]
Indeed, using standard calculation in lattice vertex algebras we get
\[ S^{osp} x = e_0^{\alpha/2+\nu} (e^{\alpha+c/2} + e^{-\alpha+c/2}) \]
\[ = e_0^{\alpha/2+\nu} e^{-\alpha+c/2} = 0. \]
\[ S^{osp} y = e_0^{\alpha/2+\nu} \left[ \nu(-1) + \frac{1}{2} \alpha(-1) \right] e^{-\alpha-c/2} \]
\[ + e_0^{\alpha/2+\nu} \left[ \nu(-1) + k\alpha(-1) \right] e^{\alpha-c/2} \]
\[ = \left[ \nu(-1) + \frac{1}{2} \alpha(-1) \right] e^{-\alpha/2-c/2+\nu} \]
\[ - e_0^{\alpha/2+\nu} e^{-\alpha-c/2} = 0. \]

Using the commutator formula, we get
\[ [S^{osp}, x(n)] = (S^{osp} x)_n = 0, \] \[ [S^{osp}, y(n)] = (S^{osp} y)_n = 0. \]

The proof follows. \(\square\)

Since the operator \(S^{osp}\) coincides with the screening operator for \(\hat{g}_0\) from [3] and [4], we have the following consequence of [4] Lemma 1:

**Lemma 4.4.** Let \(w_n\) be the highest weight vector of \(L^{Vir}_{-2, \frac{n(n+1)}{2}}\). Then
\[ L_{\frac{3}{2}}^{\hat{g}_0}(n\omega_1) = \operatorname{Ker}_{L^{Vir}_{-2, \frac{n(n+1)}{2}} \otimes \Pi(0)^{1/2}} S^{osp} = L_{-3/2}(\hat{g}_0). (w_n \otimes e^{\frac{\alpha}{2}}). \]

**Theorem 4.5.** We have:
(1) \(L_{\frac{3}{2}}(\hat{g}) = \operatorname{Ker}_{L^{Vir}_{-2, \frac{n(n+1)}{2}} \otimes \Pi(0)^{1/2}} S^{osp}\).
(2) As a \(L_{\frac{3}{2}}(\hat{g}_0)\)–module:
\[ L_{\frac{3}{2}}(\hat{g}) = \bigoplus_{n=0}^{\infty} (n+1) L_{\frac{3}{2}}^{\hat{g}_0}(n\omega_1), \]
where \(\omega_1\) is the fundamental dominant weight of \(\hat{g}_0\).

**Proof.** Assume that \(w\) is a \(\hat{g}_0\)–singular vector in \(\mathcal{F} \otimes \Pi(0)^{1/2}\) with dominant integral weight with respect to \(\hat{g}_0\). Using Lemma 4.4 we see that \(w\) must have the form \(w_n \otimes e^{\frac{\alpha}{2}}\) for \(n \in \mathbb{Z}_{>0}\) such that \(w_n\) is a singular vector for the Virasoro algebra with highest weight \(\frac{n(n+1)}{2}\). So \(w_n = w_{n,j}\) for certain \(0 \leq j \leq n\). Then using the same arguments as in [4] Proposition 3 we get that \(\operatorname{Ker}_{\mathcal{F} \otimes \Pi(0)^{1/2}} S^{osp}\) is simple. The assertion (1) holds.

The proof of the decomposition in (2) follows from the screening realization in (1), Lemma 4.3 and the decomposition of the vertex superalgebra \(\mathcal{F}\) from Theorem 5.3. Alternatively, the assertion follows directly using the decomposition of \(\mathcal{V}^{(2)}\) as \(L_{-3/2}(\hat{g}_0)\)–module and Theorem 5.5 below. \(\square\)

5. Coincidences and duality between \(L_{-3/2}(osp(1|\overline{2}))\) and \(\mathcal{V}^{(2)}\)

The small \(N = 4\) superconformal algebra is realized as the minimal affine \(W\)–algebra \(\mathcal{W}_k(\mathfrak{psl}(2|2), f_0)\) (cf. [23]). We shall denote \(\mathcal{W}_{1/2}(\mathfrak{psl}(2|2), f_0)\) by \(\mathcal{V}^{(2)}\), since it belongs to the series of vertex algebras \(\mathcal{V}^{(p)}\) defined in [2] and investigated in detail in [4].

Recall (cf. [2], [4]) that
\[ \mathcal{V}^{(2)} = \operatorname{Ker}_{S\mathcal{F}(1) \otimes \Pi(0)^{1/2}} S^{N=4} \]
where

\[ S^{N=4} = \int e^{\frac{2}{3} + \nu} (z) dz : SF(1) \otimes \Pi(0)^{1/2} \to F_{tw} \otimes \Pi_\nu. \]

Recall that there is a conformal embeddings \( L_{-3/2}(g_0) \hookrightarrow \mathcal{V}(2) \) and

\[ \mathcal{V}(2) = \bigoplus_{n=0}^{\infty} (n+1) L_{-\frac{3}{2}}^{g_0}(n \omega_1). \]

Then Theorem 4.5 implies that vertex superalgebras \( \mathcal{V}(2) \) and \( L_{-3/2}(g) \) are isomorphic as \( L_{-3/2}(g_0) \)-modules.

There are some other coincidences between \( \mathcal{V}(2) \) and \( L_{-3/2}(g) \). By [3] we have screening operator

\[ S = \int e^{\frac{2}{3} + \nu} (z) dz : F \otimes \Pi(0)^{1/2} \to F_{tw} \otimes \Pi_\nu, \]

such that

\[ S|_{SF(1) \otimes \Pi(0)^{1/2}} = S^{N=4}, \quad S|_{\overline{SF}(1) \otimes \Pi(0)^{1/2}} = S^\text{osp}. \]

This shows that the \( N = 4 \) superconformal algebra \( \mathcal{V}(2) \) and \( L_{-3/2}(g) \) are described as the kernels of the restrictions of the same screening operator. We have:

\[ \mathcal{V}(2) = \text{Ker}_{F \otimes \Pi(0)^{1/2}} S \cap \text{Ker}_{F \otimes \Pi(0)^{1/2}} \Psi^{-}(1/2), \]

\[ L_{-3/2}(g) = \text{Ker}_{F \otimes \Pi(0)^{1/2}} S \cap \text{Ker}_{F \otimes \Pi(0)^{1/2}} \Phi^{(1/2)}_2. \]

Thus, both algebras are intersection of kernels of two screening operators acting on \( F \otimes \Pi(0)^{1/2} \), the screening \( S \) is identical for both algebras, and only difference are the fermionic screenings acting on \( F \).

Using Proposition 3.4 we can prove a stronger result which says that \( L_{-3/2}(g) \) can be equipped with the structure of a \( \mathcal{V}(2) \)-module. By Proposition 3.4(1), the vertex superalgebra \( \overline{SF}(1) \cong SF(1) \) is realised as the vertex superalgebra generated by local fields \( \varphi^\pm(z) \) acting on \( \overline{F} \).

Then we have the vertex superalgebra homomorphism \( \mathcal{V}(2) \to \overline{SF}(1) \otimes \Pi(0)^{1/2} \). Denote by \( \overline{\mathcal{V}}^{(2)} \) the image of this homomorphism. More precisely, since \( \mathcal{V}(2) \) is a simple vertex superalgebra, we have \( \overline{\mathcal{V}}^{(2)} \cong \mathcal{V}(2) \) as vertex superalgebras.

Applying the realisation from [2] (see also [1]) we get that \( \overline{\mathcal{V}}^{(2)} \) is generated by

- \( \mathfrak{sl}(2) \) generators \( e, f, h \) which generate \( L_{-3/2}(\mathfrak{sl}(2)) \), Sugawara Virasoro vector \( \omega_{\text{sug}} \) (identified with the fields in \( \overline{SF}(1) \otimes \Pi(0)^{1/2} \));
- four odd primary fields \( G^\pm(z), \overline{G}^\pm(z) \), which are expressed as
  \[ G^+(z) = \varphi^+(z)e^\frac{\varphi}{(3)}(z), \quad \overline{G}^+(z) = 2\varphi^-(z)e^\frac{\varphi}{3}(z), \]
  \[ G^-(z) = f(0)G^+(z), \quad \overline{G}^-(z) = -f(0)\overline{G}^+(z). \]

**Theorem 5.1.**
(1) $L_{-3/2}(\mathfrak{g})$ has the structure of an irreducible $\mathcal{V}^{(2)}$–module, denoted by $(L_{-3/2}(\mathfrak{g}), \mathcal{Y}^{N=4}_{L_{-3/2}(\mathfrak{g})}(\cdot, z))$, uniquely determined by
\[
\mathcal{Y}^{N=4}_{L_{-3/2}(\mathfrak{g})}(v, z) = v(z) \quad v \in L_{-3/2}(\mathfrak{g}_0),
\]
\[
G^+(z) = \mathcal{Y}^{N=4}_{L_{-3/2}(\mathfrak{g})}(G^+, z) = x(z) - z\left(\frac{d}{dz}x(z) - (L_{\text{sug}}(-1)x)(z)\right),
\]
\[
\overline{G}^+(z) = \mathcal{Y}^{N=4}_{L_{-3/2}(\mathfrak{g})}(\overline{G}^+, z) = -\frac{d}{dz}x(z) + (L_{\text{sug}}(-1)x)(z),
\]
\[
G^-(z) = \mathcal{Y}^{N=4}_{L_{-3/2}(\mathfrak{g})}(G^-, z) = -y(z) + z\left(\frac{d}{dz}y(z) - (L_{\text{sug}}(-1)y)(z)\right),
\]
\[
\overline{G}^-(z) = \mathcal{Y}^{N=4}_{L_{-3/2}(\mathfrak{g})}(\overline{G}^-, z) = -\frac{d}{dz}y(z) + (L_{\text{sug}}(-1)y)(z),
\]
where for $v \in L_{-3/2}(\mathfrak{g}_0)$, we set $v(z) = \mathcal{Y}_{L_{-3/2}(\mathfrak{g})}(v, z)$.

(2) $\mathcal{V}^{(2)}$ has the structure of an (irreducible) $L_{-3/2}(\mathfrak{g})$–module, denoted by $(\mathcal{V}^{(2)}, \mathcal{Y}^g_{\mathcal{V}^{(2)}}(\cdot, z))$, which is uniquely determined by
\[
\mathcal{Y}^g_{\mathcal{V}^{(2)}}(v, z) = v(z), \quad v \in L_{-3/2}(\mathfrak{g}_0),
\]
\[
x(z) = \mathcal{Y}^g_{\mathcal{V}^{(2)}}(x, z) = G^+(z) - z\overline{G}^+(z),
\]
\[
y(z) = \mathcal{Y}^g_{\mathcal{V}^{(2)}}(y, z) = -G^-(z) - z\overline{G}^-(z),
\]
where for $v \in \mathcal{V}^{(2)}$, we set $v(z) = \mathcal{Y}_{\mathcal{V}^{(2)}}(v, z)$.

Proof. Recall the formula (1.3) which gives $L_{\text{sug}}(-1) = D + Q$, where $D$ is the derivation on the vertex superalgebra $L_{-3/2}(\mathfrak{g})$ and $Q$ is an operator on $F$. Therefore
\[
L_{\text{sug}}(-1)x = Dx + Qx = Dx + \sqrt{2}(D\Phi_1 + \sqrt{-1}D\Phi_2)e^{{\Phi_1}'},
\]
which implies that
\[
\sqrt{2}(D\Phi_1 + \sqrt{-1}D\Phi_2)e^{{\Phi_1}'} = L_{\text{sug}}(-1)x - Dx.
\]

Now we apply formulas (5.1)–(5.3) and get
\[
G^+(z) = \left(z\sqrt{2} \left(\partial_z \Phi_1(z) + \sqrt{-1}\partial_z \Phi_2(z)\right) + \sqrt{2}\Phi_1(z)\right)e^{{\Phi_1}'}(z)
\]
\[
(5.1) \quad = x(z) - z\left(\frac{d}{dz}x(z) - (L_{\text{sug}}(-1)x)(z)\right)
\]
\[
(5.2) \quad \overline{G}^+(z) = -\left(\frac{d}{dz}x(z) - (L_{\text{sug}}(-1)x)(z)\right).
\]

Since $G^-(z), \overline{G}^-(z)$ are obtained from (5.1)–(5.2) by applying the operator $f(0)$, we conclude that all fields $G^\pm(z)$ and $\overline{G}^\pm(z)$ act on $L_{-3/2}(\mathfrak{g})$. Therefore $L_{-3/2}(\mathfrak{g})$ is a $\mathcal{V}^{(2)}$–module.

Assume that $L_{-3/2}(\mathfrak{g})$ is not irreducible $\mathcal{V}^{(2)}$–module. Then it has a proper submodule $0 \neq W \subsetneq L_{-3/2}(\mathfrak{g})$. By using formulas (5.1)–(5.2) we get
\[
(5.3) \quad x(z) = G^+(z) - z\overline{G}^+(z),
\]
which implies that $W$ is invariant for the field $x(z)$. Since $y(z) = -f(0)x(z)$, we get
\[
(5.4) \quad y(z) = -G^-(z) - z\overline{G}^-(z),
\]
implying that $W$ is also invariant for $y(z)$. Therefore $W$ is $\mathfrak{g}$–invariant, which is a contradiction. This proves the assertion (1).
The action of the Casimir $\Omega$ study of the category $R$ irreducible $V$ formulas (5.3)-(5.4), which proves that $\mathcal{Y}(\mathfrak{g})$ is a $L_{-3/2}(\mathfrak{g})$-module. Since $G^+(z)$ and $G^+(z)$ can be also expressed from $x(z)$ using formulas (5.1)-(5.2) we conclude that $\mathcal{Y}(\mathfrak{g})$ is an irreducible $L_{-3/2}(\mathfrak{g})$-module.

Remark 5.2. The $N = 4$ superconformal vertex algebra $\mathcal{Y}(\mathfrak{g})$ has appeared in the $N = 4$ super Yang-Mills theory in physics. In the recent paper [11], the authors found a very interesting exact vector spaces isomorphism between $\mathcal{Y}(\mathfrak{g})$ and the doublet vertex algebra $A(\mathfrak{g})$. The result from the present paper shows that there is another vector space isomorphism to the vertex algebra associated to $\mathfrak{osp}(1|2)$ at the critical level.

6. A correspondence between $\mathcal{Y}(\mathfrak{g})$ and $L_{-3/2}(\mathfrak{g})$-modules

Recent development in the representation theory of affine vertex algebras motivated the study of the category $\mathcal{R}$ of modules (see [22, Section 2] for a formal definition) which includes:

- Ordinary modules (also called the category $KL_k$),
- Highest weight and lowest weight modules,
- Relaxed highest weight modules ([2], [3], [22]).

Assume that $V = \mathcal{Y}(\mathfrak{g})$ or $V = L_{-3/2}(\mathfrak{g})$. In our case one can show that $V$-module $W$ is in the category $\mathcal{R}$ if and only if $W$ is $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded:

$$W = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} W(n)$$

and if $h(0)$ acts on each graded component $W(n)$ semi-simply with finite-dimensional weight components.

The irreducible modules in the category $\mathcal{R}$ can be obtained using Zhu’s algebra theory. For any $A(\mathcal{Y}(\mathfrak{g}))$-module (resp. $A(L_{-3/2}(\mathfrak{g}))$-module) $U$, let $L^N=4(U)$ (resp. $L^6_{-3/2}(U)$) denote the corresponding $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex superalgebra module obtained using Zhu’s theory. The irreducible $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded $\mathcal{Y}(\mathfrak{g})$-modules were classified in [2]. We will classify and construct the irreducible $L_{-3/2}(\mathfrak{g})$-modules in the category $\mathcal{R}$ and find explicit correspondence between irreducible $\mathcal{Y}(\mathfrak{g})$-modules and $L_{-3/2}(\mathfrak{g})$-modules.

6.1. The representation theory of $\mathcal{Y}(\mathfrak{g})$: revisited. Let $U_{\mu,r}$, $\mu, r \in \mathbb{C}$, be the weight $\mathfrak{g}_0 = \mathfrak{sl}(2)$-module with basis: $E_i, i \in \mathbb{Z}$ and the $\mathfrak{g}_0$-action is given by

$$eE_i = E_{i-1}, hE_i = -(2r + 2i - \mu)E_i, fE_i = -(r + i + 1)(r + i - \mu)E_{i+1}.$$ 

The action of the Casimir $\Omega$ is $\mathfrak{g}_0$-module $U_{\mu,r} = \frac{\mu(\mu+2)}{2}$Id.

The representation theory of $\mathcal{Y}(\mathfrak{g}) = L_{\mathfrak{c}=-9}^{N=4}$ was studied in [2]. The vertex operator superalgebra $\mathcal{Y}(\mathfrak{g})$ has, up to a isomorphism or parity reversing:

1. One irreducible module in the category of ordinary modules: the vertex operator superalgebra $\mathcal{Y}(\mathfrak{g})$ itself.
2. Two irreducible modules in the category $\mathcal{O}$: $\mathcal{Y}(\mathfrak{g})$ and the highest weight module $L_{\mathfrak{c}=-9}^{N=4}(U_{-1})$, where the top component is irreducible highest weight $\mathfrak{sl}(2)$-module with highest weight $-\omega_1$, where $\omega_1$ is the fundamental dominant weight of $\mathfrak{sl}(2)$. 


(3) The irreducible relaxed highest weight modules \( L^{N=4}(U_{-1,r}) \), \( r \notin \mathbb{Z} \), where the top component is isomorphic to \( U_{-1,r} \).

These modules are explicitly realised in [2]. Using embedding \( \mathcal{V}^{(2)} \hookrightarrow F \otimes \Pi(0)^{1/2} \) we get a slightly reformulated result:

**Proposition 6.1.** [2]

1. There exist a non-split extension of \( \mathcal{V}^{(2)} \)-modules
   \[
   0 \rightarrow \mathcal{V}^{(2)} \rightarrow \mathcal{M}^{N=4}_{N=1} \rightarrow L^{N=4}(U_{-1}) \rightarrow 0,
   \]
   where \( \mathcal{M}^{N=4}_{N=1} \) is a highest weight \( \mathcal{V}^{(2)} \)-module with highest weight vector \( e^{\alpha - c/2} \).

2. Assume that \( r \notin \mathbb{Z} \). We have:
   \[
   L^{N=4}(U_{-1,r}) = \mathcal{V}^{(2)} e^{\alpha/2 - \mu - (r+1/4)c}.
   \]
   The character of \( L^{N=4}(U_{-1,r}) \) is given by
   \[
   \text{ch}[L^{N=4}(U_{-1,r})](q,z) = \text{Tr}_q^{L_{sug}(0)} z^h(0) = z^{-2r} \delta(z^2) \prod_{n=1}^{\infty} (1 + q^{n-3/2} z^{-1})^{\prod_{n=1}^{\infty} (1 + q^2 z^{n/2})} / \prod_{n=1}^{\infty} (1 - q^n)^2.
   \]

3. As a \( \mathcal{V}^{(2)} \)-modules we have
   \[
   F^{tw} \otimes \Pi_{-1}^{1/2}(r + 1/4) \cong L^{N=4}(U_{-1,r}) \oplus L^{N=4}(U_{-1,r+1/2}).
   \]

**Proof.** In [2] we proved that
\[
\mathcal{M}^{N=4}_{N=1} = \mathcal{V}^{(2)} e^{-\delta}, \quad L^{N=4}(U_{-1,r}) = \mathcal{V}^{(2)} e^{\beta - \delta - r(\alpha + \beta)},
\]
where \( \delta, \alpha, \beta \) are as in [4] Section 2] in the case \( p = 2 \). Direct calculation shows that
\[
-\delta = \alpha - c/2, \beta - \delta - r(\alpha + \beta) = \alpha/2 - \mu - (r + 1/4)c.
\]

Next we consider the twisted \( F \otimes \Pi(0)^{1/2} \)-module \( W = F^{tw} \otimes \Pi_{-1}^{1/2}(r+1/4) \) as un twisted \( \mathcal{V}^{(2)} \)-module. The top component \( W_{top} \) is isomorphic to the direct sum of \( g_0 \)-modules:
\[
W_{top} = U_{-1,r} \oplus U_{-1,r+1/2}.
\]

Then the representation theory of the vertex algebra \( \mathcal{V}^{(2)} \) from [2] implies that
\[
W \cong L^{N=4}(U_{-1,r}) \oplus L^{N=4}(U_{-1,r+1/2}).
\]

The proof follows. \( \square \)

6.2. **The representation theory of \( L_{-3/2}(\mathfrak{osp}(1|2)) \).** The universal affine vertex algebra \( V^{-3/2}(\mathfrak{g}) \) contains a singular vector
\[
T = \left( e(-1) f(-1) + f(-1) e(-1) + \frac{1}{2} h(-1)^2 + \frac{1}{2} (x(-1) y(-1) - y(-1) x(-1)) \right) 1,
\]
therefore the Zhu’s algebra \( A(L_{-3/2}(\mathfrak{g})) \) is a quotient of the associative algebra \( U(\mathfrak{g})/\langle \Omega^8 \rangle \), where \( \langle \Omega^8 \rangle \) is the two-sided ideal generated by the Casimir central element
\[
\Omega^8 = ef + fe + \frac{1}{2} h^2 + \frac{1}{2} (yx - xy).
\]
Let $\Sigma = xy - yx + 1/2$ be the super Casimir. We have (see [30]: Section 2): 
\[ \Omega^g = \frac{1}{2} \Sigma^2 - \frac{1}{8}. \]

Then on every $\mathbb{Z}_{\geq 0}$-graded $L_{-3/2}(\mathfrak{g})$-module $W$ we must have:
\[ \Omega^g \equiv 0 \quad \text{on } W_{top}. \]

This implies that $\Sigma^2 = \frac{1}{4} \text{Id}$ and therefore:
\[ W_{top} = W_{top}^+ \oplus W_{top}^- \]

such that
\[ \Sigma \equiv \pm \frac{1}{2} \text{Id} \quad \text{on } W_{top}^\pm. \]

We get:
\[ \Omega^g|W_{top}^+ = 0, \quad \Omega^g|W_{top}^- = -\frac{1}{2} \text{Id}. \]

Since on $W_{top}$ we have $L_{sug}(0) \equiv \Omega^g$, we get the following important lemma:

**Lemma 6.2.** Let $W$ be a $\mathbb{Z}_{\geq 0}$-graded $L_{-3/2}(\mathfrak{g})$-module. Then $W_{top} = W_{top}^+ \oplus W_{top}^-$ such that:
\[ L_{sug}(0) \equiv 0 \quad \text{on } W_{top}^+, \quad L_{sug}(0) \equiv -\frac{1}{2} \text{Id} \quad \text{on } W_{top}^-.

Denote by $U_{\mathfrak{g}}(t \omega_1)$ the irreducible $\mathfrak{g}$-module with highest weight $t \omega_1$, and the corresponding $V_{-3/2}(\mathfrak{g})$-module by $L_{-3/2}(t \omega_1)$.

**Proposition 6.3.** The set:
\[ \{ L_{-3/2}(\mathfrak{g}), L_{-3/2}^0(-\omega_1) \} \]

is a complete set of irreducible $L_{-3/2}(\mathfrak{g})$-modules in the category $\mathcal{O}$. There exist a highest weight module $\mathcal{M}_{-1}$ such that the following extension of $L_{-3/2}(\mathfrak{g})$-modules is non-split:
\[ 0 \to L_{-3/2}(\mathfrak{g}) \to \mathcal{M}_{-1}^g \to L_{-3/2}^g(-\omega_1) \to 0. \]

**Proof.** Define $\mathcal{M}_{-1}^g = L_{-3/2}(\mathfrak{g})e^{\alpha-c/2}$. Then $\mathcal{M}_{-1}^g$ is a highest weight $L_{-3/2}(\mathfrak{g})$-module, and therefore its simple quotient $L_{-3/2}^g(-\omega_1)$ is also $L_{-3/2}(\mathfrak{g})$-module.

Assume that $L_{-3/2}^g(t \omega_1)$ is a $L_{-3/2}(\mathfrak{g})$-module. Then
\[ \Omega^g|U_{\mathfrak{g}}(t \omega_1) \equiv 0 \quad \implies \quad \Omega^g|U_{\mathfrak{g}}(t \omega_1) \equiv 0 \quad \implies \quad \Omega^g|U_{\mathfrak{g}}(t \omega_1) \equiv 0 \quad \implies \quad \Omega^g|U_{\mathfrak{g}}(t \omega_1) \equiv 0 \quad \implies \quad \Omega^g|U_{\mathfrak{g}}(t \omega_1) \equiv 0 \quad \implies \quad t(t + 1) = 0.

Therefore $t = 0$ or $t = -1$. The case $t = 0$ corresponds to the vertex operator superalgebra $L_{-3/2}(\mathfrak{g})$, and $t = -1$ to $L_{-3/2}(-\omega_1)$.

Note that in $\mathcal{M}_{-1}^g$ we have $x(0)e^{\alpha-c/2} = 1$, we have that $L_{-3/2}(\mathfrak{g})$ is a submodule of $\mathcal{M}_{-1}^g$. It remains to prove that the quotient module $W = \mathcal{M}_{-1}^g/L_{-3/2}(\mathfrak{g})$ is irreducible.

If $W$ is not simple, the $W$ must contain a proper submodule $W'$ which by Lemma 6.2 should contain vectors of $L_{sug}(0)$-conformal weights 0 or $-1/2$.

Since as a $L_{-3/2}(\mathfrak{g}_0)$-modules
\[ \mathcal{M}_{-1}^g \cong \mathcal{M}_{-1}^{\mathfrak{g}_0} \cong L_{-3/2}(\mathfrak{g}), \]
we conclude that $W'$ can not have vectors of $L_{sug}(0)$-conformal weights 0 or $-1/2$. A contradiction. The proof follows.

□
Now we want to construct and classify irreducible relaxed highest weight $L_{-3/2}(\mathfrak{g})$-modules. The top components of these modules are irreducible $\mathfrak{g}$-modules on which $\Omega^g$ acts trivially. It is not difficult to classify these modules.

**Lemma 6.4.** Assume that $U$ is an irreducible infinite-dimensional $\mathfrak{g}$-module with 1-dimensional weight spaces such that $\Omega^g | U \equiv 0$. Then $U$ is isomorphic (up to parity reversing) to exactly one of the following modules:

- highest weight module $U^g(\omega_1)$;
- lowest weight module $(U^g(\omega_1))^*$;
- module $U^g_0(r)$ with basis $E_i, E_{i+1/2}, i \in \mathbb{Z}$ and $\mathfrak{g}$-action defined by
  \[
e E_i = E_{i-1}, \quad h E_i = -(2i+2r+1)E_i, \quad f E_i = -(r+i+1)^2E_{i+1}
  \]
  \[
e E_{i-1/2} = E_{i-3/2}, \quad h E_{i-1/2} = -(2i+2r)E_{i-1/2}, \quad f E_{i-1/2} = -(r+i+1)(r+i)E_{i+1/2}
  \]
  \[x E_i = E_{i-1/2}, \quad y E_i = -(r+i+1)E_{i+1/2}
  \]
  \[x E_{i-1/2} = E_{i-1}, \quad y E_{i-1/2} = -(r+i)E_i.
  \]

As an $\mathfrak{sl}(2)$-module: $U^g_0(r) \cong U_{-1,r} \oplus U_{0,r}$.

**Proof.** We already proved in Proposition 6.3 that $U^g(\omega_1)$ is the unique irreducible infinite-dimensional highest weight module annihilated by $\Omega^g$. Arguments for lowest weight modules are completely dual. Next we assume that $U$ is neither lowest nor highest weight. Using Lemma 6.2 we get that $U$ is as $\mathfrak{g}_0$-module a direct sum of two weight modules with 1-dimensional weight spaces on which $\Omega^{g_0}$ acts by zero or $-\frac{1}{2} \text{Id}$. So:

for certain $r$ and $s$. But one gets that $U_{-1,r} \oplus U_{0,s}$ is a $\mathfrak{g}$-module if and only if $r \equiv s \mod(\mathbb{Z})$.

The proof follows.

Let $L^g_{-3/2}(U_0(r))$ be the irreducible $V_{-3/2}(\mathfrak{g})$-module whose top component is $U^g_0(r)$.

**Theorem 6.5.** Assume that $r \notin \mathbb{Z}$. Then $L^g_{-3/2}(U_0(r))$ is a $L_{-3/2}(\mathfrak{g})$-module and it is realised as

$\quad L^g_{-3/2}(U_0(r)) = L_{-3/2}(\mathfrak{g}).e^{\alpha/2 - \mu - (r+1/4)c}$.

The basis of the top component $U^g_0(r)$ is given by

$\quad E_i = e^{\alpha/2 - \mu - (r+i+5/4)c}, \quad E_{i-1/2} = e^{-\alpha/2 - \mu - (r+i+3/4)c}$.

As a $L_{-3/2}(\mathfrak{g})$-module we have:

$\quad F_{tw} \otimes \Pi^{1/2}_1(r + \frac{1}{2}) \cong L^g_{-3/2}(U_0(r)) \oplus L^g_{-3/2}(U_0(r + \frac{1}{2}))$.

**Proof.** Let $W = L_{-3/2}(\mathfrak{g}).e^{\alpha/2 - \mu - (r+i+5/4)c}$. By direct calculation we get:

$\quad e(0)E_i = E_{i-1}, \quad e(0)E_{i-1/2} = E_{i-3/2}$

$\quad h(0)E_i = -(2i+2r+1)E_i, \quad h(0)E_{i-1/2} = -(2i+2r)E_{i-1/2}$

$\quad f(0)E_i = -(r+i+1)^2E_{i+1}, \quad f(0)E_{i-1/2} = -(r+i+1)(r+i)E_{i+1/2}$

$\quad x(0)E_i = E_{i-1/2}, \quad y(0)E_i = -(r+i+1)E_{i+1/2}$

$\quad x(0)E_{i-1/2} = E_{i-1}, \quad y(0)E_{i-1/2} = -(r+i)E_i$

So $W$ is a cyclic $\mathbb{Z}_{\geq 0}$-graded $L_{-3/2}(\mathfrak{g})$-module, whose top component $W_{top}$ is isomorphic to the irreducible $\mathfrak{g}$-module $U^g_0(r)$. If $W$ is not irreducible, then there is a proper submodule
$Z \subseteq W$ which intersects $W_{\text{top}}$ trivially. By using Lemma 6.2 we conclude that $Z$ has vectors of $L_{\text{sug}}(0)$-conformal weights 0 or $-1/2$. But since $W \cong L^{N=4}(U_{-1,r})$ as $\mathfrak{g}_0$-module, using character formula for $L^{N=4}(U_{-1,r})$ (cf. Proposition 6.1) we see that all vectors of conformal weights 0 or $-1/2$ should correspond to $U_{-1,r} \oplus U_{0,r} \cong W_{\text{top}}$ which intersect $Z$ trivially. A contradiction. Therefore $W$ is irreducible. \hfill \Box

7. Generalization to logarithmic vertex algebras

In this section we will see that correspondences

\[ \mathcal{S}F(1) \leftrightarrow \mathcal{F}, \quad \mathcal{V}(2) \leftrightarrow L_{-3/2}(osp(1|2)) \]

can be extended to a larger family of logarithmic vertex algebras.

Consider the generalized lattice vertex algebra $\tilde{V}_L$ associated to the lattice

\[ \tilde{L} = \mathbb{Z}\gamma/2, \quad \langle \gamma, \gamma \rangle = 2p. \]

Recall [7] that the doublet vertex algebra is defined as:

\[ \mathcal{A}^{(p)} = \ker \tilde{V}_L \tilde{Q}, \]

where

\[ \tilde{Q} = e_0^{-\gamma/p} = \int e^{-\gamma/p}(z)dz. \]

Doublet algebra has the Virasoro vector

\[ \omega^{(p)} = \frac{1}{4p} \gamma(-1)^2 + \frac{p-1}{2p} \gamma(-2) \]

of central charge $c_{p,1} = 1 - \frac{6(p-1)^2}{p}$ and derivation $Q = e_0^\gamma = \int e^\gamma(z)dz$. Let

\[ L_{st}(z) = Y(\omega^{(p)}, z) = \sum_{n \in \mathbb{Z}} L_{st}(n) z^{-n-2}. \]

Define a new Virasoro vector of central charge $c_{p,1}$:

\[ \omega = \omega^{(p)}_{\text{new}} = \omega^{(p)} + e_0^\gamma = \frac{1}{4p} \gamma(-1)^2 + \frac{p-1}{2p} \gamma(-2) + e_0^\gamma. \]

Let $L(n) = \omega_{n+1}$, and

\[ (7.1) \quad v = e^{-\gamma/p} - \frac{1}{p-1} e_0^{\gamma-\gamma/p}. \]

Then we have:

\[ L(0)v = e^{-\gamma/p} + e^{\gamma-\gamma/p} \]
\[ L(-1)v = Dv + \gamma(-1)e^{\gamma-\gamma/p} = Dv + \frac{p}{p-1} De^{\gamma-\gamma/p} \]
\[ = D(e^{-\gamma/p} + e^{\gamma-\gamma/p}) \]

This implies that

\[ \tilde{Q}_{\text{new}} = e_0^{-\gamma/p} - \frac{1}{p-1} e_0^{\gamma-\gamma/p} \]

is a screening operator. Define new generalized vertex algebra

\[ \mathcal{A}^{(p)}_{\text{new}} = \ker \tilde{V}_L \tilde{Q}_{\text{new}}. \]
In the case $p = 2$, we get $A^{(p)} = SF(1)$ and $A_{\text{new}}^{(p)} = \mathcal{F}$ and we proved that $SF(1)$ and $\mathcal{F}$ are isomorphic as $L^{\text{Vir}}(-2,0)$–modules.

**Theorem 7.1.** $A^{(p)}$ and $A_{\text{new}}^{(p)}$ are isomorphic as $L(c_{p,1},0)$–modules and

\[
A_{\text{new}}^{(p)} = \bigoplus_{n=0}^{\infty} (n+1)L(c_{p,1}, \frac{n(p+2p-2)}{4}).
\]

The isomorphism is given by

\[
\Omega|_{A^{(p)}} : A^{(p)} \rightarrow A_{\text{new}}^{(p)}
\]

where $\Omega$ as an operator $V_L$ given by

\[
\Omega = \exp[e_1^\gamma] = \sum_{n=0}^{\infty} \frac{(e_1^\gamma)^n}{n!}.
\]

**Proof.** Note that $\Omega$ is invertible and $\Omega^{-1} = \exp[-e_1^\gamma]$. Since

\[
[Q, e_1^\gamma] = -(\frac{\gamma-1}{p}e_1^{\gamma-\frac{\gamma}{p}})_1 = -\frac{1}{p-1}(De_1^{\gamma-\frac{\gamma}{p}})_1 = \frac{1}{p-1}e_1^{\gamma-\frac{\gamma}{p}},
\]

we get for $n \geq 1$:

\[
\frac{(e_1^\gamma)^n}{n!} Q = Q\frac{(e_1^\gamma)^n}{n!} - \frac{1}{p-1}e_1^{\gamma-\frac{\gamma}{p}}(e_1^\gamma)^{n-1}.
\]

This implies

\[
\Omega Q = Q_{\text{new}}\Omega,
\]

\[
\Omega^{-1}Q_{\text{new}} = Q_{\text{new}}^{-1}
\]

Since

\[
L(n)\Omega = \Omega L_{sl}(n),
\]

we get that

\[
\Omega|_{A^{(p)}} : A^{(p)} \rightarrow A_{\text{new}}^{(p)}
\]

is an isomorphism of $L^{\text{Vir}}(c_{p,1},0)$–modules. The proof of the decomposition (7.2) follows from the decomposition of $A^{(p)}$ as a direct sum of irreducible $L^{\text{Vir}}(c_{p,1},0)$–modules (cf. [6]. \[7. \[8]. \]

Recall that the $\mathbb{Z}_2$–orbifold of $A^{(p)}$ is the triplet vertex algebra $W^{(p)} = \text{Ker}_{\mathbb{Z}_2}Q$ (cf. [6]) generated by

\[
\omega^{(p)}, \quad F = e^{-\gamma}, \quad H = QF, \quad E = Q^2F.
\]

Define:

\[
W_{\text{new}}^{(p)} = \text{Ker}_{\mathbb{Z}_2}Q_{\text{new}}.
\]

**Theorem 7.2.** $W_{\text{new}}^{(p)}$ is generated by

\[
\omega_{\text{new}}^{(p)}, F_{\text{new}} = \Omega F, H_{\text{new}} = \Omega H, E_{\text{new}} = \Omega E.
\]

**Proof.** The proof is similar to that of [6] Proposition 1.3]. We know that $W_{\text{new}}^{(p)}$ is as a module for the Virasoro algebra isomorphic to $W^{(p)}$ and it is generated by the following singular vectors:

\[
\Omega Q^j e^{-n\gamma} = Q^j e^{-n\gamma} + Q^j e_1^\gamma e^{-n\gamma} + \cdots,
\]
for $n \in \mathbb{Z}_{\geq 0}$, $0 \leq j \leq 2n$. This implies that
\[ \Omega Q^j e^{-n \gamma} = Q^j e^{-n \gamma} + z'_j, \quad z'_j \in V_{Z\gamma}, \quad Q^{2n-j} z'_j = 0. \]

Let $Z_n$ be the Virasoro module generated by singular vectors
\[ \Omega Q^j e^{-m \gamma}, \quad m \leq n, j \geq 0. \]

Then
\[ W^{(p)}_{\text{new}} = \bigcup_{n \in \mathbb{Z}} Z_n. \]

Let $U$ be the vertex subalgebra of $W^{(p)}_{\text{new}}$ generated by $F_{\text{new}}, H_{\text{new}}, E_{\text{new}}, \omega_{\text{new}}^{(p)}$.

We will show by induction that $Z_n \subset U$ for every $n \in \mathbb{Z}_{>0}$. For $n = 1$, the claim holds. Assume that $Z_n \subset U$. Set $\ell = -2np - 1$. We use the following relations in $W^{(p)}$ proved in \cite[Proposition 1.3]{6}:
\begin{align*}
F_{\ell} e^{-n \gamma} &= \nu_1 e^{-(n+1)\gamma}, \\
E_{\ell} Q^{2n} e^{-n \gamma} &= \nu_2 Q^{2n+2} e^{-(n+1)\gamma}, \\
H_{\ell} Q^j e^{-n \gamma} &= C_j Q^{j+1} e^{-(n+1)\gamma} + v'_j, \\
& \quad v'_j \in \text{Ker}_{V_{Z\gamma}} Q^{2n+1-j},
\end{align*}

$j = 1, \ldots, 2n, \nu_1, \nu_2, C_j \neq 0$. These relations imply that in $W^{(p)}_{\text{new}}$ we have:
\begin{align*}
(F_{\text{new}})_{\ell} \Omega e^{-n \gamma} &= \nu_1 \Omega e^{-(n+1)\gamma} + v_0, v_0 \in \text{Ker}_{V_{Z\gamma}} Q^{2n+2} \\
(E_{\text{new}})_{\ell} \Omega Q^{2n} e^{-n \gamma} &= \nu_2 \Omega Q^{2n+2} e^{-(n+1)\gamma}, \\
(H_{\text{new}})_{\ell} \Omega Q^j e^{-n \gamma} &= C_j \Omega Q^{j+1} e^{-(n+1)\gamma} + v_j, \\
& \quad v_j \in \text{Ker}_{V_{Z\gamma}} Q^{2n+1-j}.
\end{align*}

We conclude that $\Omega Q^j e^{-(n+1)\gamma} \in Z_n$ for $j = 0, \ldots, 2n + 2$. These relations imply that $Z_{n+1} \subset U$. The claim now follows by induction. Therefore $U = W^{(p)}_{\text{new}}$. \hfill \Box

Next we consider the vertex algebra $A^{(p)}_{\text{new}} \otimes \Pi(0)^{1/2}$. Then we have the vertex algebra homomorphism $L_{-2+\frac{1}{p}}(\mathfrak{sl}(2)) \rightarrow A^{(p)}_{\text{new}} \otimes \Pi(0)^{1/2}$ with screening operator $S^{(p)} = s^{(p)}_0$, where $s^{(p)} = e^{\frac{2}{p} + \nu}$. We define:
\[ \gamma^{(p)}_{\text{new}} = \text{Ker}_{A^{(p)}_{\text{new}} \otimes \Pi(0)^{1/2}} S^{(p)}. \]

In the case $p = 2$ we already proved that $\gamma^{(2)}_{\text{new}} = L_{-3/2}(\mathfrak{osp}(1|2))$ and that $L_{-3/2}(\mathfrak{osp}(1|2))$ is isomorphic to $\gamma^{(2)}$ as $L_{-3/2}(\mathfrak{sl}(2))$–module. Next results extends this for $p > 2$.

**Theorem 7.3.** Assume that $p \geq 2$. We have:
\[ \overline{\Omega} = \Omega \otimes \text{Id} |_{\gamma^{(p)}} : \gamma^{(p)} \rightarrow \gamma^{(p)}_{\text{new}} \]
is an isomorphism of $L_{-2+\frac{1}{p}}(\mathfrak{sl}(2))$–modules.

**Proof.** First we notice that $\overline{\Omega}$ commutes with the action of $S^{(p)}$ and therefore it defines a linear bijection $\gamma^{(p)} \rightarrow \gamma^{(p)}_{\text{new}}$. The claim now follows from the relation:
\[ x_{\text{new}}(n) \overline{\Omega} = \overline{\Omega} x(n) \quad x \in \mathfrak{sl}(2). \]

\hfill \Box
8. The Structure of the Parafermion Vertex Algebra $N_{-3/2}(\mathfrak{osp}(1|2))$

First consider the parafermion vertex algebras of $\mathcal{V}^{(p)}$ and $\mathcal{V}_{\text{new}}^{(p)}$. Let $M_h(1)$ be the Heisenberg vertex algebra generated by $h$. Let

\[ \mathcal{N}^{(p)} = \text{Com}(M_h(1), \mathcal{V}^{(p)}) = \{ v \in \mathcal{V}^{(p)} | h(n)v = 0, n \geq 0 \}, \]

\[ \mathcal{N}_{\text{new}}^{(p)} = \text{Com}(M_h(1), \mathcal{V}_{\text{new}}^{(p)}) = \{ v \in \mathcal{V}_{\text{new}}^{(p)} | h(n)v = 0, n \geq 0 \}. \]

Since the operator $\Omega = \Omega \otimes \text{Id}$ commutes with operators $h(n), n \in \mathbb{Z}$, Theorem 7.3 directly implies:

**Corollary 8.1.** We have:

\[ \overline{\Omega}|_{\mathcal{N}^{(p)}} : \mathcal{N}^{(p)} \to \mathcal{N}_{\text{new}}^{(p)} \]

is an isomorphism of $N_{-2+\frac{1}{p}}(\mathfrak{g}_0)$–modules. In particular, as $N_{-2+\frac{1}{p}}(\mathfrak{g}_0)$–modules we have

\[ \mathcal{N}^{(p)} \cong \mathcal{N}_{\text{new}}^{(p)} \cong \bigoplus_{n=0}^\infty (2n+1)N_{-2+\frac{1}{p}}(2n\omega_1). \]

The most interesting case is $p = 2$, since then $\mathcal{N}_{\text{new}}^{(2)} = N_{-3/2}(\mathfrak{g})$. Let us now determine the generators of $N_{-3/2}(\mathfrak{g})$.

If $U_1$ and $U_2$ are vector subspaces of the vertex algebra $V$, denote by

\[ U_1 \cdot U_2 = \text{span}_\mathbb{C}\{u_nv \mid u \in U_1, v \in U_2\} \]

the fusion product of $U_1$ and $U_2$. If $U_1, U_2$ are modules for a vertex subalgebra $V_0$ of $V$, then $U_1 \cdot U_2$ is also a $V_0$–module.

The next lemma follows from the proof of Theorem 7.1 in the case $p = 2$.

**Lemma 8.2.**

1. $\mathcal{W}_{\text{new}}^{(2)} = \mathcal{F}^+$ is a simple vertex algebra strongly generated by $\omega, w_{2,1}, w_{2,0}$.

2. Let $U_{n,j}$ be the Virasoro module generated by $w_{n,j}$. Then we have:

\[ U_{2n+2,2n+2} \subset U_{2,2} \cdot U_{2n,2n}, \quad U_{2n+2,0} \subset U_{2,0} \cdot U_{2n,0}, \quad U_{2n+2,j+1} \subset U_{2,1} \cdot U_{2n,j} \]

where $j = 0, \ldots, 2n$.

**Theorem 8.3.** $N_{-3/2}(\mathfrak{g})$ is generated by $N_{-3/2}(\mathfrak{g}_0)$ and three primary vectors $Z_{2,0}, Z_{2,1}, Z_{2,2}$ of conformal weight 4.

**Proof.** Let $W_{n,j} := L_{-3/2}(\mathfrak{g}_0).(w_{n,j} \otimes e^{\frac{\pi}{\sqrt{2}}}) \cong L_{-3/2}(n\omega_1)$. Using Lemma 8.2 we get the following fusion rules between $L_{-3/2}(\mathfrak{g}_0)$–submodules of $L_{-3/2}(\mathfrak{g})$:

\[ W_{2n+2,2n+2} \subset W_{2,2} \cdot W_{2n,2n}, \quad W_{2n+2,0} \subset W_{2,0} \cdot W_{2n,0}, \quad W_{2n+2,j+1} \subset W_{2,1} \cdot W_{2n,j}, \]

where $j = 0, \ldots, 2n$. Let $N_{n,j} = \{ v \in W_{n,j} \mid h(n)v = 0 \ \forall n \geq 0 \}$. Note that $N_{n,j} = 0$ if $n$ is odd. Using restriction of the fusion rules (8.1) to the parafermion algebra, we get the following fusion rules between $N_{-3/2}(\mathfrak{g}_0)$–modules inside of $N_{-3/2}(\mathfrak{g})$:

\[ N_{2n+2,2n+2} \subset N_{2,2} \cdot N_{2n,2n}, \quad N_{2n+2,0} \subset N_{2,0} \cdot N_{2n,0}, \quad N_{2n+2,j+1} \subset N_{2,1} \cdot N_{2n,j}, \]

where $j = 0, \ldots, 2n$. 

Using Corollary 8.1 we get the following decomposition:

$$N_{-\frac{3}{2}}(g) = \bigoplus_{n=0}^{\infty} 2n \bigoplus_{j=0}^{2n} N_{2n,j}.$$ 

Now relation (8.2) easily implies that $N_{-\frac{3}{2}}(g)$ is generated by $N_{-\frac{3}{2}}(g_0)$ and $N_{2,j}, j = 0, 1, 2$. Moreover, $N_{2n,j}$ is an irreducible $N_{-\frac{3}{2}}(g_0)$–module generated by a highest weight vector which we denote by $Z_{2n,j}$. Thus, $N_{-\frac{3}{2}}(g_0)$ is generated by $N_{-\frac{3}{2}}(g_0)$ and three highest weight vectors $Z_{2,j}, j = 0, 1, 2$. □

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