Noncommutative Space Corrections on the Klein-Gordon and Dirac Oscillators Spectra

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Abstract

We consider the influence of a noncommutative space on the Klein-Gordon and the Dirac oscillators. The nonrelativistic limit is taken and the \( \theta \)-modified Hamiltonians are determined. The corrections of these Hamiltonians on the energy levels are evaluated in first-order perturbation theory. It is observed a total lifting of the degeneracy to the considered levels. Such effects are similar to the Zeeman splitting in a commutative space.
I. INTRODUCTION

The idea of a noncommutative space-time has been under intense investigation in recent years, since its resurgence in connection with string theory [1]. The subject has received a great deal of attention and many studies have been developed involving different respects (see e.g. Refs. [2] and [3] for reviews on noncommutativity in quantum field theory and quantum mechanics, respectively).

In the context of quantum mechanics, the noncommutative space can be implemented by the coordinate operators $\hat{x}^i$ and the conjugate momenta $\hat{p}^i$, satisfying commutation relations:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \quad [\hat{x}^i, \hat{p}^j] = i\hbar \delta^{ij}, \quad [\hat{p}^i, \hat{p}^j] = 0,$$

(1)

where $\theta^{ij}$ is a real and antisymmetric parameter matrix with dimension of length squared. It is shown in Ref. [4] that the relation between noncommutative and commutative variables can be obtained by a linear transformation:

$$\hat{x}^i = x^i - \frac{1}{2\hbar} \theta^{ij} p^j, \quad \hat{p}^i = p^i,$$

(2)

with $x^i$ and $p^i$ satisfying the commutation relations as the usual commutative space.

A well-known result involving the spatial noncommutativity is the splitting on the energy spectrum of the hydrogen atom [5, 6]. Other interesting result is the connection of the chiral oscillator with the noncommutative space, as was shown in Ref. [7]. In that work, the chiral oscillator is determined from the usual harmonic oscillator such that the Poisson brackets between the coordinates exhibit a similar structure to that in Eq. (1). Some applications of the chiral oscillator, such as its connection with the electric-magnetic duality and the Zeeman effect were then reported.

In another paper [8], the noncommutative versions of the Klein-Gordon and Dirac oscillators were initially discussed. In both cases, the relativistic equations of motion in a noncommutative space were determined and their similarities with those of a particle in a commutative space with a constant magnetic field have been reported. As claimed by the authors, in the case of a particle with spin the problem is not exactly soluble in three spatial dimensions. This fact, in the context of a perturbative treatment, opens the possibility of investigating the issue concerning the nonrelativistic corrections, induced by spatial noncommutativity on the energy spectrum of the system under consideration. Furthermore, it
is interesting to answer the question whether such corrections are able to completely lift the degeneracy of the energy levels.

The present work has as its main goal to examine the noncommutativity effects on the Klein-Gordon and Dirac oscillators, with special emphasis to the nonrelativistic limit and possible modifications on the energy levels. Our calculations are performed in the framework of the degenerate perturbation theory, by considering the $\theta$-modification as a first-order perturbation. We show that in both cases, the spatial noncommutativity is able to completely remove the degeneracy of the levels analyzed. This correction is similar to the Zeeman effect and proportional to the $\theta$-parameter magnitude.

Our paper is organized as follows. In Sec. 2, we study the noncommutativity effects on the nonrelativistic limit of the Klein-Gordon oscillator. The corrections to the energy levels are evaluated by first-order perturbation theory. In Sec. 3, we extend our investigation to the Dirac oscillator. In Sec. 4, we present our conclusions.

II. ENERGY CORRECTIONS OF THE KLEIN-GORDON OSCILLATOR IN A NONCOMMUTATIVE SPACE

The equation that describes the behaviour of a relativistic spin-zero particles is the well-known Klein-Gordon equation. For a free particle in 4-dimensional space-time it reads $(\Box + m^2c^2/\hbar^2)\varphi = 0$, where $m$ is the rest mass of the particle and $c$ the velocity of light. The Klein-Gordon oscillator can be obtained through the following non-minimal substitution in the free equation,

$$c^2 (p + im\omega x) \cdot (p - im\omega x) \varphi(x) = (W^2 - m^2c^4) \varphi(x),$$

where $p = -i\nabla$, $\omega$ is the oscillator frequency and we take the time dependence of the wave function to be $\varphi(x,t) = \varphi(x) \exp(-iWt/\hbar)$.

Now, let us consider the space noncommutativity version of the equation above. As shown in Ref. [8], it can be represented as

$$c^2 \left[ p + im\omega \left( x + \frac{\theta \times p}{2\hbar} \right) \right] \cdot \left[ p - im\omega \left( x + \frac{\theta \times p}{2\hbar} \right) \right] \varphi = (W^2 - m^2c^4) \varphi,$$

where $\theta_i = \frac{1}{2}\varepsilon_{ijk}\theta_{jk}$ is the constant noncommutativity parameter. Let us write Eq. (4) more
explicitly,
\[
c^2 \left[ p^2 + m^2 \omega^2 r^2 - 3m\omega - \frac{m^2 \omega^2}{\hbar} \theta \cdot L + \frac{m^2 \omega^2}{4\hbar^2} (\theta \times \mathbf{p})^2 \right] \varphi = (W^2 - m^2 c^4) \varphi, \quad (5)
\]
where \( r = \sqrt{\mathbf{x} \cdot \mathbf{x}} \) and \( \mathbf{L} = \mathbf{x} \times \mathbf{p} \) is the orbital angular momentum of the particle.

To set out the form and physical meaning of the possible effects related to the noncommutative space, we shall confine ourselves to the nonrelativistic regime. In this case, the energy is concentrated mainly in the mass of the particle and we can write \( W = E + mc^2 \).

Therefore, in the nonrelativistic limit: \( W^2 - m^2 c^4 \approx 2mc^2E \) for \( E \ll mc^2 \) being the nonrelativistic energy. Taking this limit in Eq. (5) and dividing through by \( 2mc^2 \) one achieves to the following nonrelativistic equation for \( \varphi(x) \),
\[
\left[ \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2 - \frac{3}{2} \hbar\omega - \frac{m\omega^2}{2\hbar} \theta \cdot \mathbf{L} + \frac{m\omega^2}{8\hbar^2} (\theta \times \mathbf{p})^2 \right] \varphi = E \varphi. \quad (6)
\]

The first three terms into brackets contains the well-known Hamiltonian of the nonrelativistic harmonic oscillator added by a constant term, whereas the other two terms constitute the \( \theta \)-dependent Hamiltonian (\( \hat{H}^\theta \)). It should be noted that the linear term in \( \theta \) is very similar to the interaction between the magnetic field and the magnetic dipole moment associated with the orbital angular momentum. The quadratic \( \theta \)-term can also be interpreted as an electric dipole-dipole interaction, where \( \mu_e \propto \theta \times \mathbf{p} \) \[8, 9\].

Our main objective is to evaluate the first-order corrections on the energy spectrum yielded by \( \hat{H}^\theta \), into the framework of the perturbation theory. The first thing to do is to write the exact eigenfunctions and the eigenvalues of the unperturbed Hamiltonian (\( \hat{H}_{\text{ho}} \)) \[10\]:
\[
\varphi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi),
\]
\[
R_{nl}(r) = A_{nl} \exp \left( -\frac{m\omega r^2}{2\hbar} \right) \left[ \frac{m\omega}{\hbar} \right]^{1/2} \frac{L_{n+1/2}(m\omega r^2)}{r},
\]
\[
A_{nl} = \left[ \frac{m\omega}{\hbar\pi} \right]^{1/2} \frac{2^n+2l+1}{(2n+2l+1)!!},
\]
where \( A_{nl} \) is the normalization constant, \( Y_{lm}(\theta,\phi) \) are standard spherical harmonics and \( L_{n+1/2}(x) \) (with \( n = 0, 1, 2 \ldots \)) are the associated Laguerre polynomials (see Ref. \[11\] for definition). Hence, the stationary states \( \varphi_{nlm} \) are also eigenstates of \( \hat{L}^2 \) and \( \hat{L}_z \).
It follows from Eq. (7), that the energy only depends on the quantum number \( N = 2n + l \) and the levels with \( N \geq 1 \) are degenerate. Thus, according to degenerate perturbation theory, it is necessary to diagonalize the matrix \( \langle n' l' m'_l | \hat{H}^\theta | nlm \rangle \) inside each of the degenerate subspaces of \( \hat{H}_{ho} \). The first-order energy corrections are the eigenvalues of this matrix. To be more specific, we shall calculate the corrections to \( N = 0, 1, 2 \).

First of all, let us note that the matrix element associated with the term \( \hat{H}_1^\theta = -(m\omega^2/2\hbar)\theta \cdot \mathbf{L} \) is clearly diagonal and generates a Zeeman-like shift. For the case of the noncommutative \( \theta \)-vector aligned along the \( z \)-axis: \( \theta = \theta x \hat{\mathbf{z}} \) (which it is accomplished by a rotation or a redefinition of coordinates), one obtains

\[
\langle n'l'm'|\hat{H}^\theta_1|nlm \rangle = -\frac{m\omega^2}{2\hbar}\langle n'l'm'|\theta \cdot \mathbf{L}|nlm \rangle
\]

\[
= -\frac{m\omega^2}{2\hbar} \int_0^\infty r^2 R_{n'l'}(r)R_{nl}(r)dr \int_0^{4\pi} Y_{l'm'}^* \theta_z \hat{L}_z \left| \theta_z \right| Y_{lm},d\Omega
\]

\[
= -\frac{m\omega^2\theta_z}{2} \delta_{n'l'} \delta_{l'm'} \, \, (11)
\]

where we have taken into account the eigenvalue equation \( \hat{L}_z Y_{lm} = \hbar m Y_{lm} \) and the orthogonality relation between the eigenfunctions. The magnitude order of this correction is \( m\omega^2 \theta_z^2/2 \).

Moreover, the diagonal elements of \( \hat{H}_2^\theta = m\omega^2(\theta \times \mathbf{p})^2/8\hbar^2 \) can be evaluated as following:

\[
\langle nlm|\hat{H}^\theta_2|nlm \rangle = \frac{m\omega^2}{8\hbar^2}\langle nlm|(\theta \times \mathbf{p})^2|nlm \rangle
\]

\[
= \frac{m\omega^2}{8\hbar^2} \left[ \langle nlm|\theta_z^2 p^2|nlm \rangle - \langle nlm|\theta \cdot \mathbf{p}^2|nlm \rangle \right] \, \, (12)
\]

According to Eq. (7), the first term in (12) can be written as

\[
\langle nlm|p^2|nlm \rangle = \langle nlm|\left[2m\varepsilon_{nl} - m^2\omega^2 r^2\right]|nlm \rangle
\]

\[
= m\hbar \omega \left(2n + l + \frac{3}{2}\right), \, \, (13)
\]

with \( \varepsilon_{nl} = \hbar \omega (2n + l + 3/2) \). We have used the well-known relation \( \int_0^\infty dx e^{-x} x^{\alpha+1} \left[ L_n^\alpha(x) \right]^2 = \frac{\Gamma(n+\alpha+1)}{n!} (2n + \alpha + 1) \).

The second term in Eq. (12) is a bit more complicated. If we take into account that \( \mathbf{p} = \frac{m}{\hbar}[\mathbf{x}, \frac{p^2}{2m}] \), then

\[
\langle nlm|\theta \cdot \mathbf{p}^2|nlm \rangle = -\frac{m^2}{\hbar^2} \langle nlm|\left[\theta \cdot \mathbf{p^2}\right]|\theta \cdot \mathbf{x}, \frac{p^2}{2m}|nlm \rangle
\]

\[
= \frac{m^2\theta^2}{\hbar^2} \sum_{n',l',m'} (\varepsilon_{nl} - \varepsilon_{n'l'})^2 \left| \langle n'l'm'|r \cos \theta |nlm \rangle \right|^2, \, \, (14)
\]
where we have applied the closure relation to the basis \{nlm_\ell\} between the commutators. So the angular integration is performed by means of result

\[
\langle l' m' | \cos \theta | l m \rangle = \left( \frac{(l - m_\ell + 1)(l + m_\ell + 1)}{(2l + 1)(2l + 3)} \right)^{1/2} \delta_{m',m} \delta_{l',l+1}
\]

\[
+ \left( \frac{(l - m_\ell)(l + m_\ell)}{(2l - 1)(2l + 1)} \right)^{1/2} \delta_{m',m} \delta_{l',l-1}.
\]

The remaining radial integration can be explicitly calculated by using the recurrence relations for the associated Laguerre polynomials [11]:

\[
xL_{k+1}^{n} = (n + k + 1)L_{n}^{k} - (n + 1)L_{n+1}^{k} \]

and

\[
L_{k-1}^{n} = L_{n}^{k} - L_{n+1}^{k}.
\]

These results enable us to write the diagonal elements of \(\hat{H}_2^{\theta}\) in the form:

\[
\langle nlm | \hat{H}_2^{\theta} | nlm \rangle = \frac{m^2 \omega^3 \theta^2}{8 \hbar} \left( \frac{2n + l + 3}{2} \right) \times \left[ 1 - \left( \frac{(l - m_\ell + 1)(l + m_\ell + 1)}{(2l + 1)(2l + 3)} + \frac{(l - m_\ell)(l + m_\ell)}{(2l - 1)(2l + 1)} \right) \right],
\]

with multiplicative factor of strength \(m^2 \omega^3 \theta^2 / 8 \hbar\). As a remarkable result, we have seen a factor of \(\hbar\) in the denominator to the earlier expression. On the other hand, it has been supposed in Ref. [12] that the noncommutative length scale is of order \(\theta \leq 10^{-30} m^2\). In this manner, the two terms \(\hat{H}_1^{\theta}\) (\(\theta\)-linear) and \(\hat{H}_2^{\theta}\) (\(\theta\)-quadratic) must be treated on an equal footing. In summary, the total energy shift is due the whole matrix \(\hat{H}^{\theta} = \langle n'l'm'|\hat{H}_1^{\theta} + \hat{H}_2^{\theta}|nlm\rangle\).

Furthermore, it is easy to see that \([\hat{H}^{\theta}, \hat{L}_z] = 0\), but it does not occur with \(\hat{L}^2\). Consequently, the perturbation can mix states with different values of orbital angular momentum, but the matrix elements are non-zero only between states with the same value of \(m_\ell\).

Now, we must calculate the various matrix elements. For this goal, it is convenient to define:

\[
\alpha \equiv m \omega^2 \theta_z, \quad \beta \equiv \frac{m^2 \omega^3 \theta^2}{\hbar}.
\]

• \(N = 0; n = l = 0\).

The ground state \(\langle E_{N=0} = 0 \rangle\) is non-degenerate; the first-order correction only shifts the energy as a whole by a quantity:

\[
\langle 000 | \hat{H}^{\theta} | 000 \rangle = \frac{\beta}{8}.
\]
\[ N = 1; \ n = 0, \ l = 1 \text{ and } m_l = 0, \pm 1. \]

The first excited state \( (E_{N=1} = \hbar \omega) \) is three-fold degenerate. The \( 3 \times 3 \) matrix representing \( \hat{H}^\theta \) is diagonal:

\[
\hat{H}^\theta = \begin{pmatrix}
\frac{\alpha}{2} + \frac{\beta}{4} & 0 & 0 \\
0 & \frac{\beta}{8} & 0 \\
0 & 0 & -\frac{\alpha}{2} + \frac{\beta}{4}
\end{pmatrix}.
\] (19)

\[ N = 2; \ n = 0, \ l = 2, \ m_l = 0, \pm 2, \pm 1, \text{ or } n = 1, \ l = m_l = 0. \]

The second excited state \( (E_{N=2} = 2\hbar \omega) \) is six-fold degenerate. The \( 6 \times 6 \) matrix representing \( \hat{H}^\theta \) can be written (the basis vectors are arranged in the order \(|0, 2, -1\rangle, |0, 2, 0\rangle, |0, 2, 1\rangle, |0, 2, 2\rangle, |1, 0, 0\rangle\)):

\[
\hat{H}^\theta = \begin{pmatrix}
\alpha + \frac{3\beta}{8} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\alpha}{2} + \frac{\beta}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5\beta}{24} & 0 & 0 & \frac{\beta}{6\sqrt{2}} \\
0 & 0 & 0 & -\frac{\alpha}{2} + \frac{\beta}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha + \frac{3\beta}{8} & 0 \\
0 & 0 & \frac{\beta}{6\sqrt{2}} & 0 & 0 & \frac{7\beta}{24}
\end{pmatrix},
\] (20)

whose eigenvalues are \( \{-\frac{\alpha}{2} + \frac{\beta}{4}, \frac{\alpha}{2} + \frac{\beta}{4}, -\alpha + \frac{3\beta}{8}, \alpha + \frac{3\beta}{8}, \frac{\beta}{8}, \frac{3\beta}{8}\}\).

The splitting of the energy levels of the Klein-Gordon oscillator are shown in Fig. 1 as a function of the noncommutative \( \theta \)-parameter.

As a result, we have observed that the noncommutative space effects, closed by \( \hat{H}^\theta \), yielded effective shifts on the Klein-Gordon oscillator spectrum. This result indicates the complete breakdown of the degeneracy, with the energy corrections depending on \( n, l, \) and \( m_l \) quantum numbers. Further, if we take a vanishing \( \theta \), we retrieve the typical result of a commutative space.

It is worth mentioning that the Klein-Gordon oscillator in a noncommutative space admits an exact solution when we choose another basis of eigenfunctions to take advantage of the symmetry of the problem. In fact, one can rewrite Eq. (5) in Cartesian coordinates as \( (\theta = \theta_z) \):

\[
H \left| \varphi \right\rangle = (H_{xy} + H_z) \left| \varphi \right\rangle = (W^2 - m^2 c^4) \left| \varphi \right\rangle,
\] (21)
where:

\[ H_{xy} = c^2 \left[ \left( 1 + \frac{m^2 \omega^2 \theta^2}{4 \hbar^2} \right) (p_x^2 + p_y^2) + m^2 \omega^2 (x^2 + y^2) \right], \]  
(22)

\[ H_z = c^2 \left[ p_z^2 + m^2 \omega^2 z^2 - 3m \hbar \omega - \frac{m^2 \omega^2 \theta}{\hbar} \hat{L}_z \right]. \]  
(23)

To obtain the energy eigenvalues in Eq. (21), we defined the operators \( a_\pm \) and \( a_z \) in the following form:

\[ a_\pm = \frac{1}{\sqrt{2}} \left[ \lambda (x \pm iy) + \frac{i}{\lambda \hbar} (p_x \pm ip_y) \right], \]  
(24)

\[ a_z = \frac{1}{\sqrt{2}} \left( \lambda z + \frac{i}{\lambda \hbar} p_z \right), \]  
(25)

with \( \lambda = \sqrt{m \omega / \hbar} \). Now, it is not difficult to see that \( H \) can be expressed in terms of the number operators \( N_\pm = a_\dagger \pm a_\pm \) and \( N_z = a_z \dagger a_z \) as follows [8]:

\[ H_{xy} = 2mc^2 \hbar \omega \sqrt{1 + \frac{m^2 \omega^2 \theta^2}{4 \hbar^2}} (N_+ + N_- + 1), \]  
(26)

\[ H_z = 2mc^2 \hbar \omega \left( N_z + \frac{1}{2} \right) - 3mc^2 \hbar \omega - \frac{m^2 c^2 \omega^2 \theta}{\hbar} h (N_- - N_+). \]  
(27)

The common eigenvectors \( |n_+, n_-, n_z \rangle \) of \( H \) and \( \hat{L}_z \) can be obtained by methods similar to the conventional harmonic oscillator. The relevant point here is that this exact result is compatible with our perturbative analysis. Indeed, the exact energy levels corresponding to Eq. (21) can be explicitly written as

\[ W^2 - m^2 c^4 = 2mc^2 \left[ \hbar \omega \sqrt{1 + \frac{m^2 \omega^2 \theta^2}{4 \hbar^2}} (n_+ + n_- + 1) \right. \]
\[ \left. + \hbar \omega \left( n_z + \frac{1}{2} \right) - \frac{3}{2} \hbar \omega - \frac{m^2 \omega^2 \theta}{2} \right] (n_- - n_+), \]  
(28)

where \( n_+, n_- \) and \( n_z \) are positive integers or zero associated with \( N_+, N_- \) and \( N_z \) respectively.

A comparison with the perturbation method can easily be made by taking the following approximations:

\[ W^2 - m^2 c^4 \approx 2mc^2 E \quad \text{and} \quad \sqrt{1 + \frac{m^2 \omega^2 \theta^2}{4 \hbar^2}} \approx 1 + \frac{m^2 \omega^2 \theta^2}{8 \hbar^2}, \]  
(29)

which along with (28) leads us to

\[ E = \hbar \omega (n_+ + n_- + n_z) - \frac{\alpha}{2} (n_- - n_+) + \frac{\beta}{8} (n_+ + n_- + 1), \]  
(30)
with $E$ being the nonrelativistic energy and $\alpha, \beta$ defined as in (17). Now, we can identify $N = 2n + l = n_+ + n_- + n_z$ and as result, the energy levels are not degenerate and the corrections induced by the noncommutativity are the same as those obtained earlier. For example, if $N = 0$ then automatically we have $n_+ = n_- = n_z = 0$ and $E = \beta/8$, in complete agreement with (18).

III. ENERGY CORRECTIONS OF THE DIRAC OSCILLATOR IN A NONCOMMUTATIVE SPACE

The relativistic wave equation for free fermions in 4-dimensional space-time is the usual Dirac equation \( (i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0 \). In order to get the Dirac oscillator, we introduce an external potential by a non-minimal coupling through the replacement \( p \rightarrow p - im\omega \beta x \):

\[
(c\alpha \cdot (p - im\omega \beta x) + \beta mc^2) \psi(x, t) = W \psi(x, t),
\]

where \( \psi(x, t) = \psi(x) \exp(-iWt/\hbar) \).

As before, the Dirac oscillator equation in a noncommutative space is given by

\[
\left[ c\alpha \cdot \left( p - im\omega \beta \left( x + \frac{\theta \times p}{2\hbar} \right) \right) + \beta mc^2 \right] \psi = W \psi.
\]

Following standard procedure, the equation for the upper component of \( \psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \) can be written as:

\[
c^2 \left[ p^2 + m^2 \omega^2 r^2 - 3m\hbar \omega - \frac{4m\omega}{\hbar} S \cdot L - \frac{m^2 \omega^2}{\hbar} \theta \cdot (L + 2S) \right. \\
\left. + \frac{2m\omega}{\hbar^2} (S \times p) \cdot (\theta \times p) + \frac{m^2 \omega^2}{4\hbar^2} (\theta \times p)^2 \right] \varphi = (W^2 - m^2 c^4) \varphi,
\]

where \( L \) is the orbital angular momentum, \( S = (\hbar/2)\sigma \) is the spin and \( r = \sqrt{x \cdot x} \). The above equation has no exact solution [8], thus, a perturbative approach is needed.

As in the previous discussion, we are interested in the nonrelativistic limit of (33). Here, this restriction implies at the following nonrelativistic \( \theta \)-modified Hamiltonian for the Dirac

\[\text{1 The Dirac matrices are written as: } \gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^i = \beta \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \text{with } \sigma^i = (\sigma_x, \sigma_y, \sigma_z) \text{ being the usual Pauli matrices.} \]
oscillator:
\[
\hat{H} = \left\{ \left[ \frac{p^2}{2m} + \frac{m\omega^2 r^2}{2} - \frac{3\hbar \omega}{2} S \cdot L \right]
\right. \\
\left. - \frac{m\omega^2}{2\hbar} \theta \cdot (L + 2S) + \frac{\omega}{\hbar} (S \times p) \cdot (\theta \times p) + \frac{m\omega^2}{8\hbar^2} (\theta \times p)^2 \right\}.
\]

(34)

We see that the ordinary Hamiltonian of the Dirac oscillator \((\hat{H}_{DO})\) appears between brackets. The other terms compose the \(\theta\)-dependent Hamiltonian \((\hat{H}^\theta)\). Clearly, if the spin \(S\) is ignored, we recover the Eq. (6) for the Klein-Gordon oscillator. Moreover, the term associated with \(\theta \cdot (L + 2S)\) is very similar to the correction that leads to the anomalous Zeeman effect [7].

Our purpose is to determine the contribution of \(\hat{H}^\theta\) on the energy spectrum of \(\hat{H}_{DO}\). Since we have now the presence of terms involving the spin operator, it is more suitable to work with the eigenstates common of \(L^2\), \(S^2\), \(J^2\) and \(J_z\), where \(J = L + S\) is the total angular momentum. In particular, it is easy to verify that \([\hat{H}_{DO}, J] = 0\). In this way, the corresponding eigenfunctions of \(\hat{H}_{DO}\) can be split into \(\psi_{nljm} = R_{nl}(r)\Omega_{jm}^l(\theta, \phi)\), with \(n, l, j, m_j\) being the associated quantum numbers. The radial components are the same that in (9), with the angular part of the wave function being given by

\[
\Omega_{jm}^l = \left( \pm \sqrt{\frac{l \pm m_j + \frac{1}{2}}{2l + 1}} Y_{l \pm m_j}^{m_j - \frac{3}{2}}(\theta, \phi) \right),
\]

(35)

where \(Y_{l \pm m_j}^{m_j \pm \frac{1}{2}}\) is the spherical harmonic function, with \(l \geq 0\), \(j = l \pm \frac{1}{2}\) and \(-j \leq m_j \leq j\).

The energy spectrum of \(\hat{H}_{DO}\) can be written as [10]:

\[
E = \begin{cases} 
(N - j + 1/2)\hbar\omega = 2n\hbar\omega & \text{if } j = l + \frac{1}{2} \\
(N + j + 3/2)\hbar\omega = (2n + 2l + 1)\hbar\omega & \text{if } j = l - \frac{1}{2}
\end{cases}.
\]

(36)

It should be noted the remarkable amount of degeneracy found in the previous expression. For \(j = l + \frac{1}{2}\), the energy depends only the values of \(n\). Since \(l\) is any positive integer or zero, the degeneracy of this energy level is infinite. For \(j = l - \frac{1}{2}\), the energy depends on the sum \(k = n + l\) with \(k \geq 1\), but now the degeneracy remains finite, increasing with the \(k\) value. Furthermore, if \(j = \frac{1}{2}\) \((l = 0)\) the energy value is the same that in (7) and all states are two-fold degenerate (with fixed \(n\)). When compared with the nonrelativistic Klein-Gordon oscillator, the previous analysis shows the non-trivial effect induced by spin-orbit coupling on the energy levels of the system.
Finally, as in the Sec. 2, we shall calculate the \( \theta \)-modifications on the energy levels by determining the eigenvalues of the matrix \( \hat{H}^\theta = \langle n' l' j' m_j' | \hat{H}^\theta | n l j m_j \rangle \) where

\[
\hat{H}^\theta = -\frac{m_\omega^2}{2\hbar} \theta \cdot (L + 2S) + \frac{\omega}{\hbar^2} (S \times p) \cdot (\theta \times p) + \frac{m_\omega^2}{8\hbar^2} (\theta \times p)^2 ,
\]

with non-vanishing elements only when \( m_j' = m_j \) (it is not difficult to see that now \( [\hat{J}_z, \hat{H}^\theta] = 0 \)).

To do so, we confine ourselves to the case where \( j = l - \frac{1}{2} \)\( (l \neq 0) \) and \( k = n + l = 1, 2, 3 \). The case \( j = l + \frac{1}{2} \) is more complicated because in principle, we have to diagonalize an infinite matrix. Thus, we obtain:

- **\( k = 1 ; n = 0, l = 1, j = \frac{1}{2} \) and \( m_j = \pm \frac{1}{2} \).**

This energy level \( (E_{k=1} = 3\hbar \omega) \) is two-fold degenerate (essential degeneracy). The \( 2 \times 2 \) matrix representing \( \hat{H}^\theta \) is diagonal:

\[
\hat{H}^\theta = \begin{pmatrix}
\alpha + \frac{5\beta}{24} & 0 \\
0 & -\alpha + \frac{5\beta}{24}
\end{pmatrix} ,
\]

where \( \alpha \) and \( \beta \) are defined as in (17).

- **\( k = 2 ; n = 0, l = 2, j = \frac{3}{2} \) and \( m_j = \pm \frac{3}{2}, \pm \frac{1}{2} \) or \( n = 1, l = 1, j = \frac{1}{2} \) and \( m_j = \pm \frac{1}{2} \).**

This energy level \( (E_{k=2} = 5\hbar \omega) \) is six-fold degenerate (essential and accidental degeneracies). The \( 6 \times 6 \) matrix representing \( \hat{H}^\theta \) is diagonal too:

\[
\hat{H}^\theta = \begin{pmatrix}
2\alpha + \frac{7\beta}{20} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2\alpha}{3} + \frac{2\beta}{30} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{2\alpha}{3} + \frac{7\beta}{30} & 0 & 0 & 0 \\
0 & 0 & 0 & -2\alpha + \frac{7\beta}{20} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{5\alpha}{3} + \frac{3\beta}{8} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{5\alpha}{3} + \frac{3\beta}{8}
\end{pmatrix} .
\]

- **\( k = 3 ; n = 0, l = 3, j = \frac{5}{2} \) and \( m_j = \pm \frac{5}{2}, \pm \frac{3}{2}, \pm \frac{1}{2} \) or \( n = 1, l = 2, j = \frac{3}{2} \) and \( m_j = \pm \frac{3}{2}, \pm \frac{1}{2} \) or \( n = 2, l = 1, j = \frac{1}{2} \) and \( m_j = \pm \frac{1}{2} \).**

This energy level \( (E_{k=3} = 7\hbar \omega) \) is twelve-fold degenerate and the \( 12 \times 12 \) matrix representing \( \hat{H}^\theta \) is non-diagonal. Taking into account the eigenfunctions of \( \hat{H}_{DO} \), it is
possible to show that the eigenvalues of $\hat{H}^\theta$ are all non-degenerate and have the form:

$$\left\{ -\frac{9\alpha}{5} + \frac{99\beta}{280}, -\frac{9\alpha}{5} + \frac{99\beta}{280}, -\frac{14\alpha}{15} + \frac{11\beta}{30}, -\frac{14\alpha}{15} + \frac{11\beta}{30}, -3\alpha + \frac{27\beta}{56}, \right.$$

$$\left. \frac{1}{840} \left( 1232\alpha + 349\beta - 2\sqrt{132496\alpha^2 + 38584\alpha\beta + 3103\beta^2} \right), \right.$$

$$\left. \frac{1}{840} \left( 1232\alpha + 349\beta + 2\sqrt{132496\alpha^2 + 38584\alpha\beta + 3103\beta^2} \right), \right.$$

$$\left. \frac{1}{840} \left( -1232\alpha + 349\beta - 2\sqrt{132496\alpha^2 - 38584\alpha\beta + 3103\beta^2} \right), \right.$$

$$\left. \frac{1}{840} \left( -1232\alpha + 349\beta + 2\sqrt{132496\alpha^2 - 38584\alpha\beta + 3103\beta^2} \right) \right\}. \quad (40)$$

Such as in the nonrelativistic Klein-Gordon oscillator, the spatial noncommutativity is able to modify the fine structure of the spectrum, with total lifting of the degeneracy on the energy levels considered. It is important to point out here the difference between these results and those reported in Ref. [6], for nonrelativistic hydrogen atom. In the latter, the degeneracy is only partially removed by the noncommutativity. The corresponding splits of the energy levels to the Dirac oscillator are shown in Fig. 2 as a function of the noncommutative $\theta$-parameter.

IV. CONCLUSIONS

In this paper, we have studied the effects of spatial noncommutativity on the energy spectrum of the Klein-Gordon and Dirac oscillators. Indeed, the nonrelativistic limit has been worked out and the $\theta$-modified Hamiltonians (derived from the Bopp shift) were determined. In both systems, the first-order corrections induced by spatial noncommutativity were able to completely remove the degeneracy of the energy levels analyzed. In the case of the Dirac oscillator, we observed the presence of terms depending on the spin operator and the noncommutative $\theta$-parameter, implying similar modifications to the anomalous Zeeman effect. It was also found that if the limit $\theta \to 0$ is taken, then we recover the results of the commutative case. Once that the Dirac oscillator has been extensively explored in the literature (see Ref. [14] for a review on the subject), we expect that the above results can be used to set up new bounds to the $\theta$-parameter magnitude.
Finally, we would like to point out that the same calculations are possible for the noncommutative Kemmer Oscillator \cite{15}. Moreover, in a recent paper the spin noncommutativity (different from the canonical) has been proposed \cite{16}. Thus, an extension of our work in this new context would be interesting.

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Figure 1. Energy shift to the Klein-Gordon oscillator. When \( \theta \geq 0.7 \), some additional degeneracies appear. It is assumed that \( m\omega/\hbar = 1 \).

Figure 2. Energy shift to the Dirac oscillator. When \( \theta \geq 0.4 \), some additional degeneracies appear. It is assumed that \( m\omega/\hbar = 1 \).