Effective action for the homogeneous radion in brane cosmology

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Abstract

We consider cosmological two-brane models with AdS bulk, for which the radion, i.e. the separation between the two branes, is time dependent. In the case of two de Sitter branes (including Minkowski branes as a limiting case), we compute explicitly, without any approximation, the effective four-dimensional action for the radion. With the scale factor on-shell, this provides the non-perturbative dynamics for the radion. We discuss the differences between the dynamics derived from the four-dimensional action with the scale factor off-shell and the true five-dimensional dynamics.

I. INTRODUCTION

Intense activity has followed the recent suggestion that our universe could be embedded in a higher dimensional spacetime, while exhibiting the usual law of gravity at least in the range of scales which have been probed by gravity experiments. In the particular case of a single extra dimension, a lot of research has been inspired by the Randall-Sundrum models [1,2] based on branes in an Anti de Sitter (AdS) five-dimensional bulk spacetime. It has been shown explicitly in [3] that usual gravity (up to corrections of order $\mu^2 r^2$, where $\mu$ is the AdS
mass scale) is indeed recovered in the single brane model whereas, in the two-brane models, one gets a Brans-Dicke type gravity. This cannot be compatible with observations if we live on the negative tension brane (the case of interest to solve the hierarchy problem) unless one invokes a stabilization mechanism for the radion, i.e. the interbrane separation, such as the one suggested in [4]. Whatever the specific mechanism, the outcome is usually presented as producing an effective potential for the radion which can be seen as a four-dimensional scalar field.

In the cosmological context, it has been possible to solve exactly the five-dimensional Einstein equations when the bulk includes energy only in the form of a cosmological constant [5]. Whatever the relative motion of the branes, the radion does not appear explicitly since the expansion law of each brane is given, independently of the other, by the unconventional Friedmann equation (which follows from the Israel junction conditions)

\[
H^2 = \frac{\kappa^4}{36} \rho^2 - \mu^2 + \frac{C}{a^4},
\]

(1)

where \( H, \ a \) and \( \rho \) are, respectively, the Hubble parameter, the scale factor and the total energy density of each brane, while \( C \) is an integration constant (analogous to the Schwarzschild mass).

At first sight, the two results mentioned above do not seem to be compatible: on the one hand, a four–dimensional perspective yielding a Brans-Dicke type gravity with the radion in the rôle of the Brans-Dicke scalar field, on the other hand a five–dimensional analysis showing that the Friedmann equation in our brane is independent of the radion.

The purpose of the present work is to present a detailed analysis of the dimensional reduction of a two-brane model at the level of the variational problem, i.e. starting from the five-dimensional action and integrating over the extra-dimension to obtain an effective four-dimensional action. We restrict our analysis to homogeneous systems, but we do not require any restriction on the radion velocity (spacetime fluctuations of the radion, although at the linearized level, have been considered in [6] for Minkowski and in [7] for de Sitter branes).

Equipped with the full effective four-dimensional action for homogeneous fields, our
analysis is then twofold. We first focus on the dynamics of the radion in the cosmological background given by the unconventional Friedmann equation. Since we have not assumed the radion velocity to be small, we get an action which includes the full nonlinear dynamics of the radion, and from which we can recover the equations of motion for the radion obtained in [8] by writing the junction conditions for a moving brane.

We then consider the four-dimensional action as a variational problem for the full system (radion plus gravity and not only the radion), and explore in which regimes the resulting dynamics is a good approximation of the true five-dimensional dynamics. Since the latter is exactly known, we can quantify the deviation between the true and the “effective” dynamics. Our results illustrate the dangers of extending the four-dimensional intuition to systems which are intrinsically five-dimensional, as already pointed out in [9]. In some sense, our analysis enables us to go beyond the moduli approximation (which consists in promoting free parameters of degenerate solutions into four-dimensional fields) used recently in [10] in the context of brane cosmology, and provides a quantitative delimitation of its range of validity.

Our plan is the following. We start, in section 2, with a description of the model and the definition of our coordinate system. In section 3, we compute the effective action by integrating explicitly over the extra dimension. The following section analyses the resulting dynamics of the radion. Section 5 is devoted to a comparison between the four-dimensional effective dynamics and the true dynamics. And we give our conclusions in the final section.

II. BULK METRIC

We consider a portion of the five-dimensional Anti-de Sitter spacetime with cosmological constant $\Lambda \equiv -6 \mu^2$, bounded by two “parallel”, spatially homogeneous and isotropic three-branes. The fifth dimension is made effectively periodic by assuming a mirror (orbifold) symmetry across each of the branes.

The purpose of this paper is to derive the four-dimensional effective theory for this
system from the point of view of an observer in the brane corresponding to our universe, which we call $\mathcal{B}_0$. For this reason, rather than using a coordinate system in which the metric is manifestly static \cite{11}, we will prefer to use a Gaussian normal (GN) coordinate system based on $\mathcal{B}_0$, in which the metric has the form

$$ds^2 = g_{AB}dx^A dx^B = -n(t, \tilde{y})^2 dt^2 + a(t, \tilde{y})^2 \delta_{ij} dx^i dx^j + d\tilde{y}^2,$$

and where our brane-universe $\mathcal{B}_0$ is always at $\tilde{y}_0 = 0$. We will moreover assume that the energy densities in $\mathcal{B}_0$ and in the second brane $\mathcal{B}_1$, respectively $\sigma_0$ and $\sigma_1$, are constants. We thus avoid the delicate point of defining an action for a brane with a generic perfect fluid as matter.

The general (cosmological) solution to the Einstein equations for the above system, in the GN coordinates, is well known \cite{5}. In what follows, we will assume that the Schwarzschild-type constant ($\mathcal{C}$ in eq. (1)) is zero. This means that we choose the bulk to be strictly AdS rather than Schwarzschild-AdS \cite{11}. This choice simplifies the expression for the bulk metric, that acquires the form

$$n(t, \tilde{y}) = N(t) A(\mu \tilde{y}), \quad a(t, \tilde{y}) = a_0(t) A(\mu \tilde{y}), \quad A(\xi) \equiv \cosh \xi - \eta_0 \sinh |\xi|,$$

where we have introduced the dimensionless quantity $\eta_0 = \kappa^2 \sigma_0 / (6 \mu)$ related to the energy density in our brane-universe. In an analogous way, we define $\eta_1 = \kappa^2 \sigma_1 / (6 \mu)$.

We will allow here the second brane to move with respect to the frame defined by eq. (2). Its position at any time $t$ will be given by

$$\tilde{y}_1 = R(t),$$

where the function $R(t)$ represents the (homogeneous) radion, which is thus defined as the proper distance between the two branes in the GN coordinate system defined by eq. (2). We will always assume that the GN coordinate system does not break down before reaching the second brane, which means that $\mathcal{B}_1$ is within the horizon.

Usually one prefers to express the size of the extra dimension, and thus its time dependence, in the metric components rather than in a time-dependent coordinate for the
boundary of the extra-dimension. We can proceed similarly by introducing a new coordinate $y$ defined as

$$\tilde{y} = y \mathcal{R}(t),$$  \hspace{1cm} (5)$$

such that the two branes are now at fixed positions, respectively $y_0 = 0$ and $y_1 = 1$. The metric (2) now reads

$$ds^2 = - \left( n^2 - \dot{\mathcal{R}}^2 y^2 \right) dt^2 + a^2 dx^2 + 2 \mathcal{R} \dot{\mathcal{R}} y dy dt + \mathcal{R}^2 dy^2.$$  \hspace{1cm} (6)$$

One can notice that, not only the metric component along the fifth dimension is explicitly time dependent, but off-diagonal components also appear. This explicit dependence of the metric on the radion velocity is usually claimed to be ignorable under the assumption that this velocity is small. As we will see later, this is justified only in very specific regimes.

III. EFFECTIVE ACTION

In this section we derive the effective four-dimensional action of the above system as seen by an observer on $B_0$. The total five-dimensional action includes the action for each brane, the bulk Einstein-Hilbert action (with a cosmological constant term), but also an extra term, usually called the Gibbons-Hawking [12] term (involving the trace $K$ of the extrinsic curvature tensor on the space boundaries), in order to take proper care of the boundary terms at $y = 0$ and $y = 1$. The total five-dimensional action thus reads

$$S = \frac{1}{2\kappa^2} \int d^5 x \sqrt{-g} \left( (5)R - 2\Lambda \right) + \frac{1}{\kappa^2} \sum_{a=0,1} \int d^4 x \sqrt{-h_a} K - \sum_{a=0,1} \sigma_a \int d^4 x \sqrt{-h_a},$$  \hspace{1cm} (7)$$

where $h_a$ denotes the determinant of the induced metric on each of the branes. Substituting the metric ansatz (3) in this five-dimensional action we get a functional of $n(t, y)$, $a(t, y)$ and $\mathcal{R}(t)$.

The four–dimensional effective action is defined as the result of the integration of the above expression (7) over the fifth dimension. This requires knowledge of the explicit dependence on $y$ of each term in the five-dimensional action. It is thus necessary at this stage
to replace the metric components \(n(t, y)\) and \(a(t, y)\) by their explicit form given in (3), leaving the variables \(a_0(t)\), \(N(t)\) and \(\mathcal{R}(t)\) as unspecified functions of time. The resulting four-dimensional effective action can be expressed as

\[
S = \frac{1}{\kappa^2 \mu} \int d^4x \, N a_0^3 \left[ (^{(4)} R \psi_1(\mathcal{R}) + 12 \mu^2 (\psi_2(\mathcal{R}) + \psi_3(\mathcal{R})) + 
+ 3 \mu A_1^3 \left( \frac{\dot{a}_0}{N a_0} + \mu \frac{A_1'}{A_1 N} \right) \ln \left( \frac{N A_1 - \dot{\mathcal{R}}}{N A_1 + \mathcal{R}} \right) + 6 \mu A_1^2 \frac{\dot{a}_0}{N a_0} \frac{\dot{\mathcal{R}}}{N} + \n- \kappa^2 \mu \sigma_0 - \kappa^2 \mu \sigma_1 A_1^4 \sqrt{1 - \frac{\dot{\mathcal{R}}^2}{N^2 A_1^2}} \right] \right),
\]

with \(A_1 \equiv A(\mu \mathcal{R})\) and \(A_1' \equiv A'(\mu \mathcal{R})\), while the \(\psi\)'s are dimensionless functions of \(\mu \mathcal{R}\) defined by

\[
\psi_1(\mathcal{R}) = \int_0^{\mu \mathcal{R}} d\xi A(\xi)^2, \quad \psi_2(\mathcal{R}) = \int_0^{\mu \mathcal{R}} d\xi A(\xi)^2 A'(\xi)^2, \quad \psi_3(\mathcal{R}) = \int_0^{\mu \mathcal{R}} d\xi A(\xi)^4,
\]

and \((^{(4)} R)\) is the (homogeneous) four-dimensional Ricci scalar

\[
(^{(4)} R) = \frac{6}{N^2} \left( \frac{\dot{a}_0^2}{a_0^2} + \frac{\ddot{a}_0}{a_0} - \frac{\dot{a}_0}{a_0} \frac{\dot{N}}{N} \right).
\]

It can be useful to integrate (8) by parts in time in order to get rid of \(\ddot{a}_0\) and \(\dot{N}\).

The action (8) is the main result of this work. It depends on the three time-dependent functions \(N(t)\), \(a_0(t)\) and \(\mathcal{R}(t)\) and it contains, without any approximation, the full dynamics of the radion, as will be checked explicitly in the next section. This result shows that if the radion can indeed be considered as a scalar field from the four-dimensional point of view, its full nonlinear dynamics is governed by a rather unfamiliar type of action.

IV. EQUATIONS OF MOTION

As the analysis carried out in the next section will show in detail, the four-dimensional effective action (8), if considered as embodying the dynamics of the variables \(N(t)\), \(a_0(t)\) and \(\mathcal{R}(t)\), will not yield, in general, the exact dynamics both of the radion and of the scale factor on our brane. Indeed, the dynamics of the latter is governed by the five-dimensional...
Einstein equations, and their content is partially lost in the four-dimensional reduction of the action. In particular, the unconventional Friedmann equation (equation (1) with $C = 0$),

$$H_0^2 \equiv \left( \frac{\dot{a}_0}{N a_0} \right)^2 = \frac{\kappa^4}{36} \sigma_0^2 - \mu^2 = \mu^2 \left( \eta_0^2 - 1 \right),$$

(11)

which characterizes the evolution of the scale factor in our brane-universe $B_0$, does not follow from the four-dimensional action (8).

However, once we put gravity on-shell, i.e. once we assume that the scale factor is a solution of (11), the effective four-dimensional dynamical equations will be able to yield the exact dynamics of the radion field. We will now show that this is indeed the case by comparing the equations of motion obtained from the variation of the action with respect to $N$ and $R$ with the equations of motion for the radion obtained in [8] directly from the junction conditions.

As usual, the lapse function $N$ is not a physical degree of freedom as it corresponds to the arbitrariness in the definition of time. The variation of the action with respect to it yields a first integral, which corresponds in ordinary cosmology to the (first) Friedmann equation. In our case the Friedmann equation reads

$$H_0^2 \psi_1 + 2 \mu^2 (\psi_2 + \psi_3) + \mu \left( H_0 + \mu \frac{\mathcal{A}'}{\mathcal{A}^2} \frac{\mathcal{R}}{1 - (\mathcal{R} / \mathcal{A}^2)^2} \right) = \frac{\kappa^2 \mu}{6} \sigma_0 + \frac{\kappa^2 \mu}{6} \sigma_1 \frac{\mathcal{A}^4}{\sqrt{1 - (\mathcal{R} / \mathcal{A}^2)^2}},$$

(12)

where we have set $N = 1$ after variation.

If we now impose that the expansion rate of $B_0$ is given by eq. (11), then the constraint (12) simplifies to give

$$\mu \frac{\mathcal{A}'}{\mathcal{A}^2} + H_0 \frac{\mathcal{R}}{\mathcal{A}^2} = \mu \eta_1 \sqrt{1 - \frac{\mathcal{R}^2}{\mathcal{A}^2}},$$

(13)

where we have used the identities $(\eta_0^2 - 1) \psi_1 = \psi_2 - \psi_3$, $3 \psi_2 + \psi_3 = \mathcal{A}_1 \mathcal{A}' + \eta_0$, and $\mathcal{A}^2 - \mathcal{A}^2 = \eta_0^2 - 1$, which follow from the definitions (3) and (8). This equation corresponds exactly to the result of [8], obtained by writing directly the junction conditions for a moving brane.
The equation of motion for the radion itself, obtained by variation of the action (8) with respect to \( \mathcal{R} \), looks at first rather cumbersome, but can be remarkably simplified, using (13), to yield finally

\[
\ddot{\mathcal{R}} + 3 H_0 \left( 1 - \frac{\dot{\mathcal{R}}^2}{\mathcal{A}_1^2} \right) \dot{\mathcal{R}} + \mu \mathcal{A}_1 \dot{\mathcal{A}}_1' \left( 4 - 5 \frac{\dot{\mathcal{R}}^2}{\mathcal{A}_1^2} \right) = 4 \mu \mathcal{A}_1^2 \eta_1 \left( 1 - \frac{\dot{\mathcal{R}}^2}{\mathcal{A}_1^2} \right)^{3/2},
\]

which is also in agreement with the results of [8]. We can therefore conclude that, if we impose the condition (11), we recover the exact dynamics of the radion. As we will discuss in the next section, however, the condition (11) does not emerge from the dynamical equations of the four dimensional system.

As a particular situation, one can consider the case where the second brane is not moving, i.e. \( \dot{\mathcal{R}} = 0 \). This implies, in the case of two de Sitter branes, that the second brane must be located at the equilibrium position \( \mathcal{R}_{eq} \), defined by the condition

\[
\left( \frac{\mathcal{A}_1'}{\mathcal{A}_1} \right)_{\mathcal{R} = \mathcal{R}_{eq}} = \eta_1 .
\]

The linearized dynamics around this equilibrium position can be obtained from the action eq. (8) by substituting \( \mathcal{R} = \mathcal{R}_{eq} + \delta \mathcal{R} \left( t \right) \), and keeping terms up to the second order in the perturbation \( \delta \mathcal{R} \). It is then easy to obtain the canonically normalized radion \( \varphi_c \) as

\[
\varphi_c \simeq \sqrt{-\sigma_1} \mathcal{A}_1 \delta \mathcal{R} .
\]

The effective mass can also be read from the second-order action and one recovers the familiar result \( m_{eff}^2 = -4 H_0^2 \).

As we mentioned at the end of section 2, ignoring the dependence of the metric (8) on \( \dot{\mathcal{R}} \), as is usually done in the moduli approximation, leads after expansion in powers of \( \dot{\mathcal{R}} \) to a different action for the radion, difference which shows up in the \( \dot{\mathcal{R}}^2 \) term. One can show that the discrepancy is negligible, in the regime near equilibrium, if the condition

\[
\eta_0^2 - 1 \ll \frac{\mathcal{A}_1^2}{\mu} \frac{\kappa^2 |\sigma_1|}{\mu}
\]

is satisfied. This turns out to be the case for the regime \( \eta_0 = 1 \) considered in [11].
We finally notice that it is possible to rewrite eq. (13) as an equation for the scale factor on the brane \( B_1 \), which reads

\[
H_1^2 = \mu^2 \left( \eta_1^2 - 1 \right),
\]

where the Hubble parameter on the second brane is given in terms of \( a_0 \) and \( R \) by the expression

\[
H_1 = \frac{1}{A_1 \sqrt{1 - \frac{\dot{R}^2}{A_1^2}}} \left( H_0 + \mu \frac{A'_1}{A_1} \right).
\]

Two equivalent descriptions of the two-brane system are thus possible. One consists in parametrizing the two branes by their scale factor \( a_0 \) and \( a_1 \) respectively, in which case the dynamics is described by the two unconventional Friedmann equations (11) and (18), which are completely independent. The second description, directly related to the four-dimensional effective point of view, consists in choosing the scale factor \( a_0 \) and the radion as degrees of freedom of the theory. The five-dimensional setup can then be ignored, the radion appearing as a four-dimensional scalar field in our usual four-dimensional spacetime, but the memory of the five-dimensional setup, which is embodied in the unconventional Friedmann equation, has to be added as an additional constraint.

V. VALIDITY OF THE FOUR-DIMENSIONAL APPROACH?

In this section we analyze the coupled dynamics of the scale factor and radion field that one would naively deduce from the four–dimensional action (8) considering the three functions \( N(t) \), \( a_0(t) \) and \( R(t) \) as dynamical variables for the variational problem. This means that we do not impose eq. (11) as an external constraint.

The corresponding system is analogous to that of the scale factor and a scalar field in four-dimensional FLRW cosmology, their dynamics being determined by a coupled system of second order differential equations, in addition to the constraint which comes from the variation of the action with respect to \( N \). Thus, at a fiducial initial time \( t_* \), one must specify,
for example, the values of $a_0(t_*)$ (which is in fact arbitrary because of the rescaling property of the system), $R(t_*)$ and $\dot{R}(t_*)$ while $\dot{a}_0(t_*)$ is deduced from the constraint equation. This must be contrasted with the full five-dimensional dynamics, which is described by eqs. (11) and (13), where the only quantity to be specified at an initial time $t_*$ is $R(t_*)$, since $\dot{R}(t_*)$ is determined by eq. (13). We can thus see that the action (8) generates more solutions than the true (five-dimensional) solutions which belong to a subspace characterized by the unconventional Friedmann equation (11).

Keeping this in mind, let us however examine in more detail the “theory” suggested by the four-dimensional action (8). We will not write here the second order differential equations, which are rather cumbersome, but the inspection of the Friedmann equation (12) is already instructive in itself. A first remarkable feature of eq. (12) is that the energy densities of the two branes enter linearly in the Friedmann equation, the energy density of the second brane being corrected by both a warping effect and a Lorentz factor due to the motion of the brane. This linear behaviour is of course familiar in ordinary cosmology but might appear more surprising in brane cosmology where the brane energy density enters quadratically. As remarked in the previous section, the correct behaviour is recovered only if the unconventional Friedmann law is imposed by hand on one of the two branes.

One can give the Friedmann equation (12) a more familiar aspect by introducing the effective static potential for the radion. It can be read directly from the action (8) and its expression is given by

$$\kappa^2 V_{\text{stat}}(R) = -12 \mu (\psi_2 + \psi_3) + \kappa^2 \sigma_0 + \kappa^2 \sigma_1 A^4_1. \quad (20)$$

The total effective potential for the radion can then be deduced by including gravity. It is given by the expression

$$\kappa^2 V_{\text{tot}}(R) = \kappa^2 V_{\text{stat}} - 12 \frac{H_0^2}{\mu} \psi_1. \quad (21)$$

One can check that its extremum yields the equilibrium position $R_{eq}$ defined above in (15).

Using the static potential (20), one can now rewrite the Friedmann equation (12) in the form
\[ 3 H_0^2 = \kappa_4^2 \left[ V_{\text{stat}} - \frac{6}{\kappa^2} \left( H_0 + \mu \frac{A_1'}{A_1} \hat{R} \right) \frac{\hat{R}^2}{1 - (\hat{R}/A_1)^2} - \sigma_1 A_1^4 \left( 1 - \frac{1}{\sqrt{1 - (\hat{R}/A_1)^2}} \right) \right], \quad (22) \]

where the effective four-dimensional gravitational coupling is defined by

\[ \kappa_4^2 \equiv \frac{\kappa^2 \mu}{2 \psi_1 (\hat{R})}. \quad (23) \]

Let us now consider an expansion in the parameter \( \hat{R}/A_1 \), which means that the velocity of the radion measured by an observer on the second brane is small. Keeping terms only up to \( \hat{R}^2 \) on the right hand side, one then finds in addition to the potential, a kinetic contribution for the radion, which reads

\[ \kappa_4^2 \left[ -\frac{12}{\kappa^2} \mu A_1' A_1 + \sigma_1 A_1^2 \right] \frac{\hat{R}^2}{2} \quad (24) \]

as well as a coupling between the radion velocity and the Hubble parameter. The latter term can be understood by observing that the action (3) describes a Brans-Dicke type theory. In such theories, the scalar curvature in the Lagrangian is multiplied by a scalar field, say \( \Psi \). Then an extra term of the form \(-3 H_0 \dot{\Psi}/\Psi\) appears on the right-hand side of the Friedmann equation. In our case, \( \Psi = \psi_1 \) and \( \dot{\psi}_1 = A_1^2 \hat{R} \), which accounts for the term that appears in eq. (22). With the above expansion, the Friedmann equation acquires a familiar look, but in the more general case where the radion velocity is not assumed to be small, all the terms are modified by corrections due to the Lorentz factor \((1 - (\hat{R}/A_1)^2)^{-1/2}\). Nevertheless, even when the radion velocity is small, the four dimensional system yields a space of solutions that is larger than the actual space of solutions of the full five-dimensional dynamics.

In fact, things are slightly more subtle because the most general Friedmann equation resulting from the five-dimensional analysis is not (11), but includes a radiation-like Weyl term as well (see eq. (1)) with \( C \) as an integration constant. One can thus wonder if the extra freedom among the initial conditions of the four-dimensional system can somehow mimic this constant of integration of the five-dimensional dynamics. It turns out that this is the case in the particular Randall-Sundrum limit (i.e. with critical branes, \( \eta_0 = 1 \) and \( \eta_1 = -1 \)) and in the slow-velocity approximation, as was shown explicitly in [10] by deriving an effective action for the moduli consisting of the scale factors of the two branes.
Here, we can reconsider this question from a more general point of view. To compare the four-dimensional dynamics with the true five-dimensional dynamics, we consider the quantity

$$\chi \equiv \dot{H}_0 + 2 \left[ H_0^2 - \mu^2 \left( \eta_0^2 - 1 \right) \right].$$  \hspace{1cm} (25)$$

When the brane energy density is constant, as is the case here, this quantity is exactly zero for the true five-dimensional dynamics $\mathcal{I}$, even when one has a radiation-like term (i.e. $\mathcal{C} \neq 0$ in $\mathcal{I}$). The value of $\chi$ obtained from the four-dimensional system will therefore represent directly the deviation from the true dynamics.

Let us first consider the Randall-Sundrum regime, for which $\eta_0 = 1$ and $\eta_1 = -1$. Expanding with respect to the brane velocity $\dot{R}$, one obtains from the equations of motion that

$$\dot{H}_0 + 2H_0^2 = \frac{3 + 4A_1 + A_1^2}{2(1-A_1)^2} \mu^2 \dot{R}^4 + \mathcal{O} \left( \dot{R}^6 \right).$$  \hspace{1cm} (26)$$

The above formula shows that, at order $\mathcal{O}(\dot{R}^2)$, the four-dimensional effective action yields the expansion law of a radiation-dominated universe: the constant $\mathcal{C}$ is in fact mimicked by a constant term proportional to $\dot{R}(t_\ast)^2$.

There is another regime where the four-dimensional dynamics can mimic the true five-dimensional dynamics. This is when the radion is near its equilibrium point, defined by (15). Keeping only terms up to second order in time derivatives in the equations of motion, one finds

$$\chi = \mu^2 \frac{(\eta_0^2 - 1) \dot{R}^2}{A_1^2 - 2 \eta_1 \psi_1}.$$  \hspace{1cm} (27)$$

The four-dimensional dynamics thus approximates the true dynamics near the equilibrium point if

$$\epsilon_0 \dot{R}^2 \ll H_0^2/\mu^2,$$  \hspace{1cm} (28)$$

with $\epsilon_0 = \eta_0 - 1 \ll 1$. 

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The analysis of the equilibrium regime also gives a solution to the apparent contradiction, mentioned in the introduction, between the Brans-Dicke type behaviour of the effective four-dimensional theory (corroborated by the analysis of the fluctuations [1]) and the unconventional Friedmann equations, which are independent of the radion, i.e. of the Brans-Dicke field (see also a recent discussion on this problem in [13] from a different perspective). This can be seen in the low energy limit $\epsilon_0 \ll 1$, where one gets approximately the usual Friedmann equation with

$$\tilde{\kappa}_4^2 = \kappa^2 \mu,$$  \hspace{1cm} (29)

in contrast with the gravitational coupling found in (23) which depends explicitly on the radion.

The explanation of this paradox comes from the observation that, in the effective theory, Brans-Dicke gravity couples to the energy densities on both branes, as is explicit in the Friedmann equation (22), which for $\dot{\mathcal{R}} = 0$ reduces to

$$3 H_0^2 = \kappa_4^2 V_{stat},$$

with the potential $V_{stat}$ containing an average sum of the energy densities. When the system is close to the Randall-Sundrum regime, one finds that

$$\kappa^2 V_{stat} = 6 \mu \left( \epsilon_0 + \epsilon_1 e^{-4\mu \mathcal{R}} \right),$$

where we have defined $\epsilon_1 \equiv \eta_1 + 1 \ll 1$. As seen above, the four-dimensional effective theory approximates well the true five–dimensional dynamics only close to the equilibrium point, which requires a fine–tuning between the tension on the two branes, the relation being

$$\epsilon_1 = -\epsilon_0 e^{2\mu \mathcal{R}}.$$  \hspace{1cm} (32)

This coupling between $\epsilon_1$ and $\epsilon_0$ exactly cancels the dependence of $\kappa_4^2$ on the radion, and the gravitational coupling to only the matter in our brane does not depend on the radion as expected.
VI. CONCLUSIONS

We have computed the four-dimensional effective action, which governs the full non-perturbative dynamics of the homogeneous radion. The correct dynamics for the radion, including corrections due to the Lorentz factor, is obtained in all cases only if one imposes “by hand” the non-conventional Friedmann equation for the brane scale factor.

Our work emphasizes the fact that one cannot in general use a four-dimensional approach to describe a setup which is intrinsically five-dimensional, even if our universe is, in this type of model, a four-dimensional manifold where ordinary matter is confined. This is reminiscent of the realization that the Friedmann equation in the brane must include a $\rho^2$ term because of the five-dimensional nature of gravity [14]. Noticing that, in the case of the Friedmann equation, one recovers the usual four-dimensional form at low energy, i.e. when our brane-universe is close to Minkowski, one could argue that the four-dimensional and five-dimensional approaches are equivalent at low energies. However, this can sometimes be misleading as shown in this work, since at low energy, in a two-brane system, one requires to fine-tune the brane tensions in such a way that gravity looks purely tensorial rather than scalar–tensor if one demands the four–dimensional effective cosmological dynamics to reproduce the actual five-dimensional one.

In the present work, by considering a system simple enough but yielding a non trivial effective potential for the radion, we have been able to compare explicitly the effective and true dynamics. The space of effective solutions is larger than the space of true solutions corresponding to the five-dimensional ansatz, because one constraint is lost in the integration of the action over the fifth dimension. We have shown that in a restrictive range of parameters, namely for critical branes or for a radion near equilibrium point, and with a slow radion in both cases, the two dynamics are compatible (in these two regimes, the effective solutions coincide with the enlarged space of five-dimensional solutions which allow for Weyl radiation even if there solutions were not included in the ansatz before integration of the action). We were also able to evaluate quantitatively the deviation of the effective dynamics with respect
to the true dynamics.

By extension, our analysis suggests that studying brane cosmology in a five-dimensional setup appears unavoidable as soon as one wishes to explore regimes far from the quasi-static limit. And, indeed, these regimes might produce new and potentially interesting effects, which cannot be seen in the already thoroughly explored four-dimensional models.

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