Old and New No Go Theorems on Interacting Massless Particles in Flat Space

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Abstract

We review model independent arguments showing that massless particles interacting with gravity in a Minkowski background space can have at most spin two. These arguments include a classic theorem due to Weinberg, as well as a more recent extension of the Weinberg-Witten theorem. A puzzle arising from an apparent counterexample to these theorems is examined and resolved.

1 Introduction

Interacting theories of high spin particles are notoriously fraught with problems. In particular, when particles are massless and spacetime is Minkowski, there seem to exist obstacles to the very existence of consistent interactions [1]. My contribution to the proceeding of this conference shall review old and new no go theorems on interacting high-spin theories. These theorems are surprisingly strong when combined together and rule out any interaction between particles of spin higher than 2 and any matter that interacts with gravity. The same theorems also show that there exists only one massless spin two particle interacting with gravity, i.e. the graviton itself.

2 Old and New No Go Theorems

Weinberg 1964

An important obstruction to consistent interactions of high-spin massless particles was derived in 1964 by Weinberg [2] (see also [3]) using general properties of the S-matrix. His result was extended to Fermions and specifically to supersymmetric theories in [4, 5].

Weinberg considers an S-matrix element with $N$ external particles of four-momentum $p_i$, $i = 1, \ldots, N$ and one massless spin-$s$ particle of momentum $q$ and polarization vector $\epsilon^{m_1 \ldots m_s}(q)$. In the soft limit $q \to 0$, the interaction of a single spin-$s$ particle is described by a matrix element of a spin-$s$ current $J_{m_1 \ldots m_s}$. Moreover, since in the soft limit the scattering of the particle is elastic, the matrix element obeys

$$\lim_{q \to 0} \langle pA|J_{mnr}|A'p'\rangle = g\delta_{AA'}p_mp_np_r...$$

(1)

Here $A, A'$ denote all quantum numbers other than momentum for the initial/final hard particle –i.e. the particle that either absorbs or emits the soft, spin-$s$ particle. When the spin-$s$ particle

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These theorems assume a mild form of locality that in our view is indispensable to ensure that a theory is unitary and causal. By relaxing these requirements, interacting theories seem to exist [6]. We shall briefly discuss later their potential problems.
ends on an external line, the S-matrix exhibits a pole $1/pq$. So, up to regular terms it factorizes as (see fig. 1)

$$S(p_1, \ldots, p_N, q, \epsilon) \approx \sum_{i=1}^{N} g^i p^i_{m_1} \cdots p^i_{m_s} \epsilon^{m_1 \cdots m_s}(q) \frac{1}{2pq} S(p_1 \cdots p_N).$$

(2)

Figure 1: Factorization of S-matrix amplitude in the soft limit

The polarization vector is transverse and traceless, so it gives a redundant description of the massless particle, which has only two physical polarizations. Redundancy is eliminated by demanding that the S-matrix be independent of spurious polarizations

$$\epsilon_{\text{spurious}}^{m_1 \cdots m_s}(q) \equiv g^{(m_1} \eta^{m_2 \cdots m_s)}(q), \quad q_n \eta^{n m_1 \cdots m_s-2}(q) = \eta^{n m_1 \cdots m_s-3}(q) = 0.$$  

(3)

The factorization eq. (2) implies that spurious polarizations decouple only when

$$\sum_{i} g_i p_i^{m_1} \cdots p_i^{m_{s-1}} = 0, \quad \forall p_i.$$  

(4)

For generic momenta this equation has a nonzero solution only in two cases:

$s = 1$ In this case eq. (4) reduces to $\sum_{i} g_i = 0$, i.e. to conservation of charge.

$s = 2$ Here eq. (4) becomes $\sum_{i} p_i^{m} = 0$ and $g_i = \kappa$. The first equation enforces energy-momentum conservation, while the second gives the principle of equivalence: all particles must interact with the massless spin two with equal strength $\kappa$.

For $s > 2$ eq. (4) has no solution for generic momenta.

This argument shows that only scalars, vectors and one spin two particle can give rise to long-distance interactions. There can exist only one graviton with long range interactions because if there were two, then Weinberg’s theorem would say that the first would interact with all matter with a universal constant, say $g$, while the second would interact with a universal constant $g'$. Now, if one calls $J_{mn}$ the current associated with the first graviton and $J'_{mn}$ that associated with the second, then the current $g' J_{mn} - g J_{mn}$ describes a “non-graviton,” which does not produce long range interactions with matter. The real graviton is associated with the current $g J_{mn} + g' J_{mn}$. This argument extends trivially to $N > 1$ massless spin-two particles.

Weinberg’s theorem was extended to Fermions in [4, 5]. There, it was shown that massless Fermions only up to spin 3/2 interact at low energies. Both [2] and [4, 5] rely on the existence of processes in which the number of spin-$s$ particles changes by one unit. This is necessary to generate long-range interactions, but it leaves out the possibility of interacting high-spin particles.

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2Spin 3/2 Fermions were also shown to interact as the supersymmetric partners of the graviton, i.e. the gravitini of supergravity theory.
particles with a nonzero conserved charge. In particular, particles interacting only with the graviton according to the principle of equivalence are still allowed.

One may think that another way out of Weinberg’s theorem is to soften the interaction of \( s > 2 \) particles by appropriate powers of the soft momentum \( q_m \) in such a manner as to cancel the offending pole in eq. (2). If this were true, high-spin particles would not generate long-range forces, but they could still interact..... but not even this is true. When combined with another, more recent no go theorem [7], Weinberg’s theorem completely excludes interactions between \( s > 2 \) particles and any object that interacts with gravity –and also proves that only one interacting massless graviton exists.

The no go theorem of ref. [7] is an extension of the famous Weinberg Witten theorem [8]. It can be obtained by studying the matrix element

\[
\lim_{p' \to p} \langle \epsilon p | T_{mn} | \epsilon' p' \rangle,
\]

which describes the absorption (or emission) of a soft graviton by an on-shell massless spin-\( s \) particle, described by the polarization tensors \( \epsilon, \epsilon' \), with initial momentum \( p \) and final momentum \( p' \). Since the particle is on shell before and after interacting with the graviton, the graviton momentum \( q = p' - p \) is off-shell and space-like.

When the polarization tensor \( \epsilon' = \epsilon \) describes a physical state, one must have

\[
\lim_{p' \to p} \langle \epsilon p | T_{mn} | \epsilon' p' \rangle = p_m p_n,
\]

This is a special case of Eq. (1). The coupling constant \( g = 1 \) is the same for all states because of Weinberg’s theorem.

The Weinberg Witten Theorem and Beyond

When the polarization tensor \( \epsilon' \) describes a spurious state, the matrix element [5] must not contribute to a physical scattering amplitude. One obvious way to achieve this is for the matrix element itself to vanish

\[
\lim_{p' \to p} \langle \epsilon p | T_{mn} | \epsilon' p' \rangle = 0 \quad \text{when } \epsilon' \text{ is spurious.}
\]

As shown in [8], this condition implies that the spin-\( s \) massless particles can have at most spin one. We refer to that reference for an elegant proof of this statement.

Eq. (7) guarantees decoupling of spurious states, but it can be weakened. The reason is that the graviton in (5) is off-shell and so it contributes a factor \( \Pi_{mn}^{pq}(q) \) (the graviton propagator) to the scattering amplitude. If the matrix element is proportional to the graviton linearized equations of motion [3] as in

\[
\langle \epsilon p | T_{mn} | \epsilon' p' \rangle \propto L_{mn}^{pq}(p - p') \Delta_{pq}(q),
\]

then it contributes to the amplitude a factor

\[
\langle \epsilon p | T_{mn} | \epsilon' p' \rangle \Pi_{pq}^{mn}(q) \propto \Pi_{pq}^{mn} L_{mn}^{rs}(q) \Delta_{rs}(q) = \Delta_{pq}(q).
\]

If \( \Delta_{pq}(q) \) is analytic near \( q = 0 \), the spurious states contribution may be canceled by a contact term [4]. So, eq (8) is the most general necessary condition for consistency of a high-spin massless particle interacting with gravity.

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3Schematically denoted here by \( L_{mn}^{rs} h_{pq} = 0 \), with \( h_{mn} \), the metric fluctuation around Minkowski space.

4In Field theory language this is equivalent to say that the change in the action due to the linear gauge transformation of the spin-\( s \) field can be canceled by a change in the graviton field.
Ref. \[7\] gives a proof of this statement, here we shall only sketch it out (for Fermions, when it is technically a bit simpler).

The matrix element \(\langle \epsilon, p + q | T_{mn} | \epsilon', p \rangle\) is bilinear in \(\epsilon, \epsilon'\) and it otherwise depends only on the momenta. For spin \(s\), the minimum set of spurious states needed to write a nonzero conserved, symmetric tensor is given by Dirac spinor-tensors \(\epsilon_{\alpha, m_1...m_n}(p), s = n + 1/2\). They are symmetric in the vector indices \(m_1, ..., m_n\) and satisfy the constraints

\[
\rho \epsilon_{m_1, ..., m_n}(p) = 0, \quad p^m \epsilon_{m_1, ..., m_n}(p) = 0, \quad \gamma^m \nu_{m_1, ..., m_n}(p). \tag{10}
\]

We are interested in initial and final states with the same physical helicity +\(s\), so we impose on the representatives of the initial state (\(\epsilon\)) and final state (\(\epsilon'\)) the equations

\[
\gamma^5 \epsilon_{m_1, ..., m_n}(p) = \epsilon_{m_1, ..., m_n}(p), \quad \gamma^5 \epsilon'_{m_1, ..., m_n}(p + q) = \epsilon'_{m_1, ..., m_n}(p + q). \tag{11}
\]

In the kinematic configuration of interest, there exist two independent light-like vectors: \(p\) and \(p + q\). The space-like vector \(q\) can be used to define \(n + 1\) algebraically independent spinor-tensors

\[
\epsilon_{m_1...m_k}(p) \equiv q^{m_k+1}...q^{m_n} \epsilon_{m_1...m_n}(p), \quad k = 0, ..., n. \tag{12}
\]

Constraints \[10\][11] and the on-shell condition on momenta, \(p^2 = (p + q)^2 = 0\), vastly reduce the possible terms in the matrix element of interest. A short reflection suffices to convince oneself that its most general form is

\[
\langle \epsilon, p + q | T_{mn} | \epsilon', p \rangle = \sum_{k=0}^{n} A^k \epsilon^k(p + \alpha^k q)(\gamma_n)\epsilon^k + \sum_{k=1}^{n} B^k \epsilon(k_m \gamma_n)\epsilon^{k-1} + \sum_{k=1}^{n} C^k(\epsilon_n \gamma_n)\epsilon^{k}. \tag{13}
\]

The coefficients \(A^k, B^k, C^k\) and \(\alpha^k\) are functions of \(q^2\) which in principle can be singular at \(q^2 = 0\). A first constraint on the singularity is due to the principle of equivalence, that demands

\[
\lim_{q \to 0} \langle \epsilon, p + q | T_{mn} | \epsilon', p \rangle = p_m p_n. \tag{14}
\]

This equation implies

\[
\lim_{q \to 0} A^n(q) = 1, \tag{15}
\]

\[
\lim_{q \to 0} A^k(q) q^{2(n-k)} = 0, \quad k < n, \tag{16}
\]

\[
\lim_{q \to 0} \alpha^k(q) A^k(q) q^{2(n-k)-1} = 0, \tag{17}
\]

\[
\lim_{q \to 0} B^k(q) q^{2(n-k)+1} = 0, \tag{18}
\]

\[
\lim_{q \to 0} C^k(q) q^{2(n-k)+1} = 0. \tag{19}
\]

Conservation of \(T_{mn}\) implies that the matrix element \[13\] is divergenceless

\[
q^m \langle \epsilon, p + q | T_{mn} | \epsilon', p \rangle = 0. \tag{20}
\]

This yields the further constraints

\[
A^k (\alpha^k - 1/2) q^2 + B^{k+1} + C^{k+1} = 0, \quad k = 0, ..., n - 1, \quad \lim_{q \to 0} \alpha^n(q) = 1/2. \tag{21}
\]

Though not strictly necessary to prove our result, eq. \[21\] is useful since it simplifies the structure of the matrix element. In particular, together with the mass-shell conditions \[10\] it makes the matrix element transverse and traceless.
In reality, constraints (15-19) are too weak, because if any of the coefficients $A^k, B^k, C^k$ and $\alpha^k A^k$ had a singularity $1/q^2$, then vertex (13) would imply the existence of another massless spin 2 field (it couples to a transverse-traceless vertex!) which mixes linearly with the graviton. This linear mixing contradicts Weinberg’s uniqueness theorem. It also violates the principle of equivalence—which we assumed (and need) to prove or theorem—either because it implies the existence of a second massless graviton that couples only to some type of matter, or because it re-sums to give the graviton a mass. A singularity stronger than $1/q^2$ is even worse since it implies the existence of a spin two ghost mixing linearly with the ordinary graviton (see fig. 2).

![Figure 2: A singular vertex implies the existence of an additional massless particle mixing with the graviton.](image)

We have introduced extra polarizations to ensure that the $T_{mn}$ matrix element transforms covariantly. Now we must check under which conditions spurious polarizations do indeed decouple. Spurious states have the form

$$\epsilon^s_{m_1...m_n}(p) = p_{(m_1}\eta_{m_2...m_n)}(p),$$

where $\eta_{m_1...m_{n-1}}$ is on shell, transverse and gamma-transverse. For the spurious state (22), the spinor-tensors given in eq. (12) have the form

$$\epsilon^k_{s m_1...m_k}(p) = p_{(m_1}\eta_{m_2...m_k)}(p) - (n-k)\frac{q^2}{2}\eta^k_{m_1...m_k}, \quad \eta^k_{m_1...m_k} \equiv q^{m_{k+1}}q^{m_n}\eta^k_{m_1...m_n}.$$  (23)

Matrix element (13) is transverse and traceless, therefore the decoupling condition (8) simplifies to

$$\langle \epsilon, p + q | T_{mn} | \epsilon^s, p \rangle = q^2 \Delta_{mn}(q).$$  (24)

Substitution of eqs. (22,23) into eq. (13) then yields a set of recursion relations among the coefficients $A^k, ..., C^k$:

$$-kA^k - \frac{q^2}{2}(n+1-k)A^{k-1} + C^k = O(q^2), \quad k = 1, ..., n;$$

$$-k\alpha^k A^k - \frac{q^2}{2}(n+1-k)\alpha^k A^{k-1} = O(q^2), \quad k = 1, ..., n;$$

$$-(k-1)B^k - \frac{q^2}{2}(k+2-k)B^{k-1} = O(q^2), \quad k = 2, ..., n;$$

$$-(k-1)C^k - \frac{q^2}{2}(k+1-k)C^{k-1} = O(q^2), \quad k = 2, ..., n.$$

For instance $A^k(q) = A^k_0(q)/q^2$, $A^k(q) = 0$ at $q^2 = 0$. 
As we have seen earlier, no coefficient in eq. (13) can be more singular than \(1/q^2\). So in particular
\[
\lim_{q \to 0} q^2 C^1(q) = \lim_{q \to 0} q^2 A^0(q) = 0.
\] (29)
Recursion relations (25, 28) then imply
\[
\lim_{q \to 0} A^n(q) = 0, \quad n > 1,
\] (30)
in contradiction with the equivalence principle, which implies \(A^n(0) = 1\) [see eq. (15)].

This completes our proof: only when spurious polarizations decouple from the cubic vertex (13), a chance exists for massless high-spin fields to interact with gravity, but decoupling contradicts the universality of gravitational interactions.

Our argument rules out interactions for Fermions of spin \(s > 3/2\). It still allows for gravitational interactions of spin 3/2 particles. This is not surprising since supergravity theories provide many examples of massless spin 3/2 particles consistently interacting with gravity and other fields.

When the argument is repeated for Bosons, it shows that particles of spin higher than 2 cannot interact with gravity.

Combining Old and New Theorems

The “new” theorem just described here does not yet forbid the existence of some exotic matter, which does not interact with gravity according to the principle of equivalence. But such possibility is ruled out by combining our “new” theorem with Weinberg’s “old” one. Consider indeed an amplitude with one soft graviton and three particles with arbitrary momentum: two “exotic” particles of spin \(s \geq 2\), and one graviton. It factorizes as in Eq. (2) [see figure (3)]. The soft graviton couples with exotic matter with coupling constant \(g = 0\) because of our new theorem, but it couples with the graviton with \(g = 1\). This contradicts Weinberg’s old theorem, so the amplitude must vanish. The argument can be repeated with \(N\) “exotic” and \(M\) “standard” particles, and the result is similar: the amplitude factorizes into one containing only exotic particles times one containing only normal ones. So, exotic particles cannot interact at all with standard ones. The result can be summarized in field theory language as follows: call \(\psi\) the exotic fields and \(\phi\) the standard ones (including the graviton); then, the above argument states that the action factorizes completely as
\[
S[\phi, \psi] = S_1[\phi] + S_2[\psi]
\] (31)
Notice that the action of the exotic particles, \(S_2[\psi]\), does not depend on any standard field, not even on the metric. So, by any meaningful definition of the word “exist,” exotic particles do not.

3 A Counterexample that Isn’t

An excellent way to test the robustness of a no go theorem is to search for possible counterexamples. Here we shall examine an apparent one and explain why in reality it does not contradict our theorem [9].

In Anti de Sitter spaces it is possible to construct interacting theories of massless high spin states. These theories were pioneered by Fradkin and Vasiliev in [10]. Modern reviews on this vast subject are e.g. [11, 12]; their relations to and ways out of no go theorems have been reviewed in [13]. A puzzle arises now if we cut off AdS\(_5\) space in a Vasiliev theory; in other words, if we extend the Randall Sundrum 2 (RS2) [14] construction to high-spin theories.

\(^6\) Vasiliev’s theories “evade” no go theorems because such theories are defined in Anti de Sitter space instead of Minkowski space. The cosmological constant of AdS introduces a new mass scale that allows for Lagrangian interactions, singular in the flat space limit, that decouple spurious polarizations.
Figure 3: “Exotic” particles cannot interact at all with gravity.

Even for Vasiliev theories, a crucial property of the Randall-Sundrum construction holds: all states with \( k^2 = 0 \) and a normalizable 5D wave function appear as massless particles in 4D Minkowski space. These particles interact with each others and with the graviton. Moreover, they are invariant under all the spin-\( s \) gauge symmetries needed to decouple spurious polarizations, because this is the defining property of the Fradkin-Vasiliev construction \([10]\). Finally, their interactions with the graviton obeys the principle of equivalence \([7]\). But this contradicts all our no go theorems! So, something in the previous line of reasoning must be incorrect. One possibility is that high-spin fields do not give normalizable modes at \( k^2 = 0 \). This possibility is excluded by computing the induced 4D kinetic term. Explicitly, for a spin-\( s \) field, \( \psi_{m_1...m_s}(z, x) \), \( m_i = 0, 1, 2, 3, 4 \), the \( k^2 = 0 \) wave function must behave as \( \psi_{m_1...m_s}(z, x) = z^E \psi_{m_1...m_s}(x) \) and the \( m_i = 4 \) components can be set to zero with a gauge choice. Since \( \psi_{\mu_1...\mu_s}(x) \) is a rank-\( s \) covariant tensor \([8]\), the relation between \( E \) and the scaling dimension \( \Delta \) is \( E = 4 - \Delta - s \) \([9]\). Thus, the 5D kinetic term of the massless mode for rank-\( s \) field becomes

\[
\int d^4xdz \sqrt{-g} g^{m_1n_1}...g^{m_s+n_s+1} \partial_{m_1} \psi_{m_2...m_{s+1}}(z, x) \partial_{n_1} \psi_{m_2...m_{s+1}}(z, x) + ..... \\
\propto \int d^4z \frac{2z^{8-2s-2\Delta}}{z^{-3s}} \int d^4x \partial_{\mu_1} \psi_{\mu_2...\mu_{s+1}}(x) \partial_{\mu_1} \psi_{\mu_2...\mu_{s+1}}(x) + ..... 
\]

\( \text{This is seen best by using the duality between Vasiliev’s theory and free three-dimensional } O(N) \text{ models conjectured in } [15] \text{ and further elaborated and tested in } [16]. \text{ In the 3D picture, the principle of equivalence for high-spin particles follows from the fact that correlators of high-spin currents with } T_{mn} \text{ obey standard Ward identities (see also } [17]). \)

\( \text{In this subsection, 4D indices are denoted by Greek lowercase letters while Latin lowercase letters denote 5D indices.} \)
The dimension $\Delta$ is $s + 2$ thus, the integral in $dz$ in eq. (32) converges for all spins $s > 1$.

The problem arises with interactions, as it may have been expected; specifically with the spin-$s$ gauge invariance of interactions, starting at the first nontrivial order (i.e. cubic order in the fields). When we stated that the Fradkin-Vasiliev construction ensured gauge invariance of the full interacting action under spin-$s$ gauge transformations, we implicitly assumed that the gauge transformations were normalizable near $z = 0$. This is the correct boundary condition for $AdS_5$, but not for $AdS_5$ cut off at $z = \epsilon$. In the latter case, it is precisely the (cutoff) non-normalizable gauge transformations that become the gauge transformations of 4D massless particles. It is they that ensure the decoupling of 4D spurious polarizations. It is also they that may not leave the action invariant, because for them it is no longer legitimate to integrate by part and discard boundary terms.

To be specific, let us start by writing down the inhomogeneous part of the spin-$s$ gauge transformation

$$\delta \psi_{m_1 \ldots m_s}(z, x) = D_{(m_1 \epsilon_{m_2 \ldots m_s})} \zeta(z, x).$$

The 4D gauge transformations are those that leave the field $\psi_{m_1 \ldots m_s}(z, x) = z^F \hat{\psi}_{m_1 \ldots m_s}(x)$ in the gauge $\psi_{4m_2 \ldots m_s} = 0$. This condition constrains the gauge parameter to have the form

$$\epsilon_{(m_1 \ldots m_{s-1})} = z^{2s-2} \hat{\epsilon}_{(m_1 \ldots m_{s-1})}.$$

Next, decompose the bulk action into a free quadratic part plus an interacting part. The quadratic part is, schematically,

$$S_2 = \int d^4 x dz \psi(z, x)(L \psi)(z, x) + \int d^4 x \hat{\psi}(x)(B \hat{\psi})(x).$$

The kinetic term of the bulk quadratic action has been denoted here by $L$, while all boundary terms needed to enforce Neumann boundary conditions have been called $B$. These terms ensure that when $\hat{\psi}$ obeys the 4D equations of motion of a free massless spin-$s$ particle, then the action is stationary $\text{even under variation that do not vanish at the boundary } z = \epsilon$, such as those given by eq. (33) with gauge parameter (34).

Interactions arise first at cubic order. One universal interaction term that is always present involves two spin-$s$ fields and a metric fluctuation $h_{mn}(z, x) \equiv L^{-2} z^2 g_{mn} - \eta_{mn}$. Schematically the action is

$$S = S_2 + S_3, \quad S_3 = \int d^4 x dz V[\psi(x), \psi(x), h(x)].$$

The local cubic interaction $V[\psi(x), \psi(x), h(x)]$ is a sum of two terms: $V_m + V_{FV}$. The first one comes from covariantizing the action $S_2$ using the minimal coupling procedure and expanding to linear order in $h_{mn}$. The second is the Fradkin-Vasiliev (FV) vertex $10$, which is needed to ensure consistency of the equations of motion to cubic order (see $18$ for a recent derivation of this vertex).

Now a crucial observation is that if the fields $\psi$ are on shell, i.e. if they obey both the 5D bulk equations as well as the 4D equations of motion, then the quadratic action is invariant under arbitrary variations $\delta \psi$. So, under a full non-linear gauge variation and up to quadratic order in the fields, the change in $S_2 + S_3$ reduces to the change in $S_3$:

$$\delta S = \int d^4 x dz \{ V[\delta \psi(x), \psi(x), h(x)] + V[\psi(x), \delta \psi(x), h(x)] \} ,$$

with $\delta \psi$ given in eq. (33).

To find out where the problem lays with the would be massless high-spin states, we consider now the explicit case of spin 3 particles.

The cubic FV vertex for two $s = 3$ particles and a graviton can be found e.g. in $19$ eq. (14). It simplifies dramatically in the gauge $\psi_{4mn}(z, x) = 0, h_{4m}(z, x) = 0$, especially when the
$s = 3$ field $\psi_{\mu\nu\rho}$ is on shell and the metric fluctuation $h_{\mu\nu}$ is independent of $z$. These are the field configurations we need to show the problem with interactions. On such configuration the FV vertex contains only one term involving derivatives w.r.t. $z$ [compare with 19 eq. (14)]:

$$
\int d^4x dz V_{FV}[\psi, \psi, h] = -\frac{3}{\Lambda} \int d^5x dz \frac{L^5}{5!} w_{\alpha\beta\gamma\delta} D^2 \psi^{\mu\alpha\beta} D_z \psi^{\gamma\delta}.
$$

(38)

Here $\Lambda$ and $w^{\alpha\beta\gamma\delta}$ are, respectively, the 5D cosmological constant and the linearized Weyl tensor. Inserting the gauge variation (33) into eq. (38), the only way to cancel the resulting term is to integrate by part in $dz$. Integration by part produces a term that cancels against lower-dimension terms upon using the free $\psi$ equations of motion. But it also produces the boundary term

$$
-\frac{3}{\Lambda} \int d^4x \frac{L^5}{5!} w_{\alpha\beta\gamma\delta} D^{(\mu} \epsilon^{\alpha\beta)} T \leftrightarrow D_z \psi^{\gamma\delta}.
$$

(39)

This term would have been zero on a metrically complete AdS$_5$ space and for a normalizable gauge variation. On cutoff AdS$_5$ instead, it does not vanish. Moreover, since the only field that is not on shell with respect to the 4D equations of motion is the metric fluctuation, the only chance to cancel (39) is by a local variation of $h_{\mu\nu}$. For this to be possible, eq. (39) would have to vanish when $h_{\mu\nu}$ is on shell. But eq. (39) is proportional to the Weyl tensor, which does not vanish on shell!

Therefore, the gauge symmetry (33) is anomalous to first order in the gravitational interactions.

This is in exact agreement with our no go theorems. As for any anomalous gauge symmetry, the only way to escape algebraic inconsistency is for the high spin field to acquire a mass. Luckily, a mass counterterm is natural in the RS2 construction, since it can be introduced simply by modifying the term $B$ in eq. (35). So, a RS2-type construction is possible even for Vasiliev high-spin theories but the resulting dynamical gravity plus matter in 4D Minkowski space does not contain massless particles of spin higher than 2.

### 4 End Note on Locality

Recently, a theory of massless high spin theory has been proposed in [6], that evades our theorems. It does so by having quartic Lagrangian vertices that contain non-local terms $\propto 1/pq$, which cancel the spurious polarization contribution to physical $S$-matrix amplitudes. Because of this non-locality it seems difficult to reconcile this theory with both unitarity and causality (see e.g. [20]).

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