Space-Time Localisation for the Dynamic $\Phi^4_3$ Model

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Abstract  
We prove an a priori bound for solutions of the dynamic $\Phi^4_3$ equation. This bound provides a control on solutions on a compact space-time set only in terms of the realisation of the noise on an enlargement of this set, and it does not depend on any choice of space-time boundary conditions.

We treat the large- and small-scale behaviour of solutions with completely different arguments. For small scales we use bounds akin to those presented in Hairer’s theory of regularity structures. We stress immediately that our proof is fully self-contained, but we give a detailed explanation of how our arguments relate to Hairer’s. For large scales we use a PDE argument based on the maximum principle. Both regimes are connected by a solution-dependent regularisation procedure.

The fact that our bounds do not depend on space-time boundary conditions makes them useful for the analysis of large-scale properties of solutions. They can, for example, be used in a compactness argument to construct solutions on the full space and their invariant measures. © 2020 The Authors. *Communications on Pure and Applied Mathematics* published by Wiley Periodicals LLC

1 Introduction

The aim of this article is to derive a priori bounds for the three-dimensional stochastic quantisation equation, also known as the dynamic $\Phi^4_3$ model. This model is—at least formally—given by the nonlinear stochastic partial differential equation (SPDE)

\[(\partial_t - \Delta)u = -u^3 + \xi,\]

where $\xi$ is the space-time white noise over $\mathbb{R} \times \mathbb{R}^d$. In our main result, Theorem 2.1, we show a bound on the solution in the case $d = 3$ on a compact space-time set that depends only on a finite number of explicit polynomials in the Gaussian noise on a slightly larger space-time set. In particular, our bound does not depend on any space-time boundary conditions.

The main difficulty when working with (1.1) is the roughness of the driving noise $\xi$, which in turn makes the solution irregular and the interpretation of nonlinear terms nontrivial. It is now well-understood that solutions are distribution...
valued in spatial dimension \( d \geq 2 \), and the nonlinearity has to be renormalised, which loosely speaking corresponds to replacing (1.1) by

\[(\partial_t - \Delta)u = -u^3 + "\infty" u + \xi.\]  

The theory of singular SPDEs of this type has been revolutionised in recent years, starting with Hairer’s theory of regularity structures [14] and the theory of paracontrolled distributions developed independently by Gubinelli, Imkeller, and Perkowski [13]. Hairer’s theory permits the development of a stable small-scale theory, i.e., a local-in-time existence theory on compact spatial domains, for a large class of SPDEs satisfying a scaling property called subcriticality. This notion corresponds exactly to super-renormalisability in quantum field theory and equation (1.1) satisfies it for spatial dimension \( d < 4 \). In this approach, solutions are constructed in two steps. In the first step, a finite number of terms in a perturbative expansion of the solution based on the noise are constructed using probabilistic methods. In the second step, the actual solutions are sought in a space of distributions that are locally well approximated by these stochastic terms. The renormalisation procedure is treated in the probabilistic step, and makes strong use of stochastic cancellations, while the second step is purely deterministic. Hairer’s work created a lot of activity. We mention in particular the works by Catellier and Chouk [5] and Kupiainen [16], who produced similar short-time existence and uniqueness results for (1.1) using the method of paracontrolled distribution and renormalisation group.

The theory of regularity structures is by now well-developed and permits the analysis of a range of equations that are much more singular than the dynamic \( \Phi^4_3 \) model. It has been applied to (1.1) in “4−\( \delta \)” dimensions [3] sec. 2.8.2, to the sine-Gordon model in the full subcritical regime [6], the evolution of a random string on a manifold [4], and the constructions of two-dimensional gauge theories [7]. However, the arguments currently available are insufficient to go beyond a short-time existence theory in any of these equations. For example, the construction of solutions to (1.1) in [14] does not make use of the “good” sign of the nonlinear term \(-u^3\) and would work equally if it were replaced with a \(+u^3\). Solutions for this modified equation are expected to blow up in finite time.

For (1.1) in dimension 3 the problem of passing from a local to a global solution theory has been largely overcome in a series of very recent works starting with [19], where (1.1) was studied on the torus \( \mathbb{T}^3 \) and a priori estimates were obtained that ruled out the possibility of finite time blowup. In [12] a priori estimates for solutions on the full space \( \mathbb{R}^3 \) were shown; see also [1, 11] for an analysis of the invariant measures based on similar ideas. All of these articles worked in the framework of paracontrolled distribution rather than regularity structures.

In this article we present a completely different technique to derive a priori estimates within the framework of regularity structures. We show a space-time version of the “coming down from infinity” property; i.e., we provide a bound on the solution on a compact space-time set that depends on the realisation of the noise on a
slightly larger set but does not depend on the behaviour of the solution elsewhere, making full use of the strong nonlinear damping term \(-u^3\). This local dependence makes this bound extremely useful when analysing the behaviour of solutions on large scales.

A main interest of this approach is the technique itself. Its advantages are that we effectively separate the argument for small and large scales by dealing with a family of regularised equations for large scales and use (an appropriate restatement) of the theory of regularity structures to analyse the small scales. This results in a relatively short argument compared to previous works and has the potential to work for a much larger class of equations. We want to stress that our argument is fully self-contained and does not make use explicitly of any of the results in [14]. In fact, in both the statement of our main result and its proof we fully avoid the terminology of this theory, i.e., the notions of model, modeled distribution, structure group, etc., but give a direct statement of all of the required bounds. This is possible, because the algebra involved in the small-scale solution theory of (1.1) is still not too complex, and we hope that our direct approach makes the presentation more clear. We do, however, include a separate section in which we translate our main estimates into the regularity structure terminology.

Finally, we would like to mention that in our companion paper [18] we have implemented our approach in the case of one-dimensional reaction-diffusion equations. Even in this much more regular case where no renormalisation enters, a priori bounds that do not depend on space-time boundary conditions seem to have been unknown.

This paper is structured as follows. Section 2 contains the elements needed to state our main result, Theorem 2.1, starting with the definitions of the proper Hölder spaces in Section 2.1. Section 2.2 presents the setting in which we solve equation (1.1) and our main result. The outline of the proof and the different lemmas required are presented in Section 2.3. We then explain the close connection between our setting and Hairer’s theory of regularity structures in Section 3, where the full regularity structure for the \(\Phi^4_3\) equation is presented. Section 4 contains the proof of Theorem 2.1 and Section 5 contains the proof of the lemmas presented in Section 2.3.

2 Setting and Main Result

2.1 Measuring Regularity

As usual when dealing with parabolic equations, regularity will be measured with respect to the metric

\[
d((t, x), (\bar{t}, \bar{x})) = \max \left\{ \sqrt{|t - \bar{t}|}, |x - \bar{x}| \right\},
\]

where \(| \cdot |\) denotes the supremum norm on \(\mathbb{R}^3\). We introduce the parabolic ball of center \(\bar{z} = (t, x)\) and radius \(R\) in this metric \(d\), looking only into the past:

\[
B(\bar{z}, R) = \left\{ z = (\bar{t}, \bar{x}) \in \mathbb{R} \times \mathbb{R}^3, d(\bar{z}, \bar{x}) < R, \bar{t} < t \right\}.
\]
We define the parabolic boundary of a set $D$ accordingly, as the set of points $z \in \partial D$ such that for any $r$, $B(z, r) \not\subseteq \partial D$. For $R > 0$ we define $D_R \subseteq D$ as the set at distance $R$ from the parabolic boundary. Set $P = (0, 1) \times (-1, 1)^3$, then we have

$$P_R := (R^2, 1) \times \{x : |x| < (1 - R)\}.$$  

Note that for $R' < R \leq 1$ we have for any domain $D$, $D_{R'} + B(0, R' - R) \subseteq D_R$. For $\alpha \in (0, 1)$, we define the Hölder seminorm $[\cdot]_\alpha$:

$$[u]_\alpha := \sup_{z \neq \bar{z} \in \mathbb{R}^3} \frac{|u(z) - u(\bar{z})|}{d(z, \bar{z})^\alpha}.$$  

For $\alpha \in (1, 2)$, we define the Hölder seminorm $[\cdot]_\alpha$:

$$[u]_\alpha := \sup_{z \neq \bar{z} \in \mathbb{R}^3} \frac{|u(z) - u(\bar{z}) - \nabla u(z) \cdot X(z - \bar{z})|}{d(z, \bar{z})^\alpha},$$  

where $\nabla$ refers to the spatial gradient, and we introduce the function $X$ that is the projection onto space coordinates. We will often deal with functions $U(z, \bar{z})$ of two variables generalising the increments of $u(z) - u(\bar{z})$ in (2.5) above. In this case we define for $\alpha \in (1, 2)$

$$[U]_\alpha := \sup_{z \in \mathbb{R}^3} \inf_{v(z) \in \mathbb{R}^3} \sup_{\bar{z} \neq \bar{z} \in \mathbb{R}^3 \setminus \{z\}} \frac{|U(z, \bar{z}) - v(z) \cdot X(z - \bar{z})|}{d(z, \bar{z})^\alpha}.$$  

The infimum over functions $v$ is attained when $v(z)$ is the spatial gradient in the second coordinate of $U$ at point $(z, \bar{z})$. We often work with norms that only depend on the behaviour of functions/distributions on a fixed subset of space-time: if $B \subseteq \mathbb{R} \times \mathbb{R}^3$ is a bounded set, then we define the local $\alpha$-Hölder seminorm $[\cdot]_{\alpha, B}$ as in (2.4) with the supremum restricted to $z, \bar{z} \in B$. The use of a third index $r$ as in $[\cdot]_{\alpha, B, r}$ indicates that the supremum is restricted to $z$ and $\bar{z}$ at distance at most $r$. Similarly, $\|\cdot\|$ denotes the $L^\infty$ norm on the whole space $\mathbb{R} \times \mathbb{R}^3$ and $\|\cdot\|_B$ the norm of the restriction of the function to $B$, and for a function of two variables, $\|\cdot\|_{B, r}$ is the norm restricted to $z, \bar{z} \in B$ with $d(z, \bar{z}) \leq r$.

From now on, $x, y, \text{ and } z$ will always denote a generic space-time variable.

We work with a Besov-Hölder type norm to measure negative regularity. These norms are usually defined by measuring the rate of blowup when testing the distribution with a smooth approximation of the Dirac delta. Different definitions boil down to different choices of approximations (e.g., convolution with rescaled smooth kernels in [14] def. 3.7, projection in Fourier space in Littlewood-Paley theory [2], or wavelet basis [17]), and different choices usually require slightly different proofs of key properties such as multiplicative inequalities and Schauder estimates. We need our testing operation to commute with the heat operator, which makes the convolution against rescaled kernels natural. Our definition is strongly inspired by the choice of smooth kernel satisfying the semigroup property with respect to the scaling parameter first introduced in [21]. This semigroup property allows us to effectively connect regularisations at different scales and thus makes
the proof of the reconstruction theorem very convenient. However, an additional twist is required. For us it is important to be able to define local norms that only depend on properties of distributions on a compact set. This makes it most convenient to work with a compactly supported kernel in the definition of the norm. But the kernel used in [21] does not have this property. Wavelet bases, on the other hand, permit a convenient transition from one scale to another and can consist of compactly supported functions, but unfortunately the projection on these basis functions do not commute with differential operators. The following simple construction yields a kernel which is compactly supported and enjoys a version of the semigroup property for dyadic scales that is enough to prove the reconstruction theorem.

We fix a nonnegative smooth function \( \hat{\Phi}(x) \in [0, 1] \) for all \( x \in \mathbb{R} \times \mathbb{R}^3 \) and with integral 1. Setting \( \hat{\Phi}(t, x) = T^{-5} \Phi \left( \frac{t}{T}, \frac{x}{T} \right) \), we now define \( \Psi_{T,n} = \Phi_{T2^{-n}} \) and \( \Psi_T = \lim_{n \to \infty} \Psi_{T,n} \) so that \( \Psi_T = \Phi_{T/2} \ast \Phi_{T/2} \). The convergence can be checked easily. \( \Psi_{T,n} \) and \( \Psi_{T+n} \) are nonnegative and smooth, symmetric in space and with support \( B(0, 1) \) and \( B(0, 1 - 2^{-n}) \). We define the operator \( \ast_T \) by convolution with \( \Psi_T \), and \( \ast_{T,n} \) by convolution with \( \Psi_{T,n} \) for \( n \geq 1 \). \( \ast_T \) is the identity. Since \( \Psi_{T,n+m} = \Psi_{T,n} \ast \Phi_{T2^{-m}} \), we have

\[
(\ast_{T,n}) \ast (\ast_{T,n}) = (\ast_{T2^{-n}}) \ast (\ast_{T,n}) \tag{2.7}
\]

Taking \( m \) to infinity in this, or equivalently noticing that \( \Psi_T = \Phi_{T,n} \ast \Phi_{T2^{-n}} \), we have the desired relation between dyadic scales

\[
(\ast_T) = (\ast_{T2^{-n}}) \ast (\ast_{T,n}) \tag{2.8}
\]

We then define the local \( C^{\alpha} \) norm of a distribution \( \theta \) for \( \alpha < 0 \) as

\[
[\theta]_{\alpha,C} = \sup_{T \leq 1} \| (\theta)_{T} \| C^{T^{-\alpha}}. \tag{2.9}
\]

It is proven in [2, theorem 2.34] that for a similar quantity, in the case where \( C \) is a torus of size 1, this corresponds to the classical Besov norm \( B^{\alpha}_{\infty, \infty} \). In our case, \( [\theta]_{\alpha,C} \) depends on the distribution \( \theta \) on \( C + B(0, 1) \) since \( \Psi \) has support in \( B(0, 1) \).

Furthermore, we mention the scaling estimates, for \( n \in \mathbb{N} \cup \{ \infty \} \) and \( \alpha > -5 \),

\[
\int |\Psi_{T,n}(x - y)|d(x, y)^{\alpha}dy \leq T^{\alpha}, \tag{2.10}
\]

\[
\int |\nabla \Psi_{T,n}(x - y)|d(x, y)^{\alpha}dy \leq T^{\alpha-1}.
\]

Here and in the rest of the paper, “\( \leq \)” denotes a bound that holds up to a multiplicative constant. This immediately implies that for any \( h \in C^{\alpha}, \alpha \in (0, 2) \), and for any bounded set \( C \), we have

\[
\| h_T - h \|_{C_{T}} \leq T^{\alpha} \sup_{z \in C_{T}} [h]_{\alpha, B(z, T)} = T^{\alpha} [h]_{\alpha, C, 2T}. \tag{2.11}
\]
Indeed, since $\Psi$ is symmetric in space we have \( \int \Psi(y)X(y)dy = 0 \) where $X$ denotes the projection onto space coordinates, and for all $x \in C$:

\[
(h_T - h)(x) = \int \Psi_T(x - y)(h(y) - h(x))dy
= \int \Psi_T(x - y)(h(y) - h(x) - 1_{|y| > 1})V h(x) \cdot X(y - x)dy
\leq [h]_{a, B(x, T)} \int \Psi_T(x - y)d(x, y)^a dy.
\]

For products of function, we will sometimes be using the following notational convention:

\[
(fg)_T(x) - f(x)g_T(x) = ((f - f(x))g)_T(x).
\]

The presence of the variable means that we evaluate the function there first, and the absence means that the convolution variable is used.

### 2.2 Main Result

We will work with a regularised version of (1.1) throughout; i.e., we assume that $u$ is a smooth function that on $P$ satisfies

\[
(\partial_t - \Delta)u = -u^3 + \zeta + (3C_1 - 9C_2)u,
\]

for real-valued parameters $C_1, C_2$. Thus, throughout this article, we never have to address the question of how a given expression has to be interpreted to make sense. The main application we have in mind is the case where $\zeta = \xi$, i.e., a regularisation of the white noise at scale $\delta$ and where $C_1$ and $C_2$ are defined as the expectations of certain polynomials in $\xi$ that diverge like $\frac{1}{\delta}$ and $\log \delta^{-1}$ as the regularisation is removed. However, in our analysis these values only enter in the assumptions on the “trees” (see (2.14), (2.17), and (2.19)), and their precise values do not appear.

Despite dealing with smooth functions we stress that all of our estimates are stable in the limit $\delta \to 0$, where $\xi$ can only be measured as a distribution of regularity $-\frac{5}{2}$ and $u$ as a distribution of regularity $-\frac{1}{2}$. We will freely use the convention to speak of “distributions” when we refer to smooth functions that can only be measured in a distributional norm in this limit.

We first introduce several polynomials in $\xi$ that are used in the local description of the solution to (2.12). These are (essentially) the same objects that appear in Hairer’s small-scale solution theory for (2.12), and we use his convention to denote these objects by trees.

We start by fixing an $\varepsilon > 0$, which will always be assumed to be “sufficiently small.” The first tree is $1$ is assumed to satisfy the pointwise identity on $P$

\[
(\partial_t - \Delta)1 = \zeta,
\]
and we assume a control in the $C^{-\frac{1}{2}-\varepsilon}$ norm. The constant $C_1$ appears in the following definitions:

\begin{equation}
\mathcal{V} := t^2 - C_1, \quad \mathcal{W} := t^3 - 3C_1 t,
\end{equation}

and these distributions will be measured in the norms of $C^{-1-2\varepsilon}$ and $C^{-\frac{3}{2}-3\varepsilon}$, respectively. We also introduce symbols of higher order: we assume that $\mathcal{V}$ and $\mathcal{W}$ satisfy the pointwise identity on $P$

\begin{equation}
(\partial_t - \Delta)\mathcal{V} = \mathcal{V}, \quad (\partial_t - \Delta)\mathcal{W} = \mathcal{W}.
\end{equation}

As expected with the heat operator, we assume that the regularity is increased by 2, i.e., $\mathcal{V} \in C^{1-2\varepsilon}$ and $\mathcal{W} \in C^{\frac{1}{2}-3\varepsilon}$.

Finally, we introduce the trees $\mathcal{T}$ denoting the product of $\mathcal{V}$ with $\mathcal{W}$, $\mathcal{Y}$, and $\mathcal{Z}$, and for these we will need bounds on the quantities

\begin{align}
[\mathcal{V}]_{2\varepsilon} &= \sup_{x \in P} \sup_{T < 1} T^{2\varepsilon} \left| \int X(y-x)\mathcal{V}(y)\mathcal{W}_T(y-x)dy \right|, \\
[\mathcal{W}]_{4\varepsilon} &= \sup_{x \in P} \sup_{T < 1} T^{4\varepsilon} \left| \int ((\mathcal{V}(y) - \mathcal{V}(x))\mathcal{W}(y) - C_2)\mathcal{W}_T(y-x)dy \right|, \\
[\mathcal{Y}]_{4\varepsilon} &= \sup_{x \in P} \sup_{T < 1} T^{4\varepsilon} \left| \int (\mathcal{Y}(y)\mathcal{Y}(y) - \mathcal{Y}(x)\mathcal{Y}(y))\mathcal{Y}_T(y-x)dy \right|, \\
[\mathcal{Z}]_{5\varepsilon} &= \sup_{x \in P} \sup_{T < 1} T^{\frac{1}{2}+5\varepsilon} \left| \int ((\mathcal{Z}(y) - \mathcal{Z}(x))\mathcal{Z}(y) - 3C_2 \mathcal{Z}(y))\mathcal{Z}_T(y-x)dy \right|.
\end{align}

We will work with the function $v := u - 1$, which satisfies

\begin{align}
(\partial_t - \Delta)v &= -u^3 + (3C_1 - 9C_2)u = -(v + 1)^3 + (3C_1 - 9C_2)(v + 1) \\
&= -v^3 - 3v^2 + 1 - 3v(t^2 - C_1) - (t^3 - 3C_1 t) - 9C_2(v + 1) \\
&= -v^3 - 3v^2 + 1 - 3v\mathcal{V} - \mathcal{W} - 9C_2(v + 1).
\end{align}

The fact that the constant $C_1$ disappears in this expansion was already noted in [8], and that was enough to define solutions in dimension 2, where the constant $C_2$ is unnecessary. For stating the theorem, we need to introduce one more notation, the numbers $n_\tau$, which count the number of “leaves” in a given tree $\tau$. It is given explicitly by

\begin{align*}
n_i = 1, \quad n_v = n_v = n_\tau = 2, \quad n_\mathcal{V} = n_\mathcal{T} = 3, \quad n_\mathcal{W} = n_\mathcal{Y} = 4, \quad n_\mathcal{Z} = 5.
\end{align*}

The main result can now be stated.

**Theorem 2.1.** If $v$ solves (2.20) pointwise on $P$, then we have:

\begin{equation}
\|v\|_{P_R} \leq C \max \left\{ \frac{1}{R}, \left[ \tau \right]^\frac{1}{\eta(1-\varepsilon)} \right\}, \tau \in L,
\end{equation}

where $R$ is the radius of the ball $B(x_0, R)$.
where $L = \{1, V, V, Y, \Psi, \dot{V}, \dot{\Psi}, \Phi\}$, $|\tau|$ is the regularity in which we measure the tree $\tau$ in the way explained above.

**Remark 2.2.** As stated above, the bounds we assume on the “trees” are (almost – see the following Remarks 2.4 and 2.5) identical to those appearing as input into the analytic part of [14]. The particular form of the $x$-dependent “counterterms” $-X(x)\Psi(y)$ in (2.16), $\Phi(x)\Psi(y)$ in (2.17), $-\Psi(x)\Psi(y)$ in (2.18) and $\Psi(x)\Psi(y)$ in (2.19) corresponds exactly to the “positive renormalisation” or re-centering procedure of the trees performed there. See Section 3 for a more detailed discussion of positive renormalisation in the theory of regularity structures.

In the case where $D$ is a regularised white noise and where $C_1 = \mathbb{E}t(y)^2$ and $C_2 = \mathbb{E}Y(y)\Psi(y)$, e.g. for $y = (1,0)$, uniform-in-$\delta$-bounds on the various norms were obtained in [14, sec. 10]. We stress that in this low-regularity situation the convergence of these terms as $\delta \to 0$ is highly non-obvious, even after renormalisation. The calculations use probabilistic tools and strongly rely on stochastic cancellations.

The estimates in [14, sec. 10] actually yield bounds on the moments of all of these terms, so that our main result (2.21) implies bounds on moments of the solution. Since $\tau$ is a random variable in the (inhomogeneous) Wiener chaos of order $n_\tau$ over the Gaussian noise, $[\tau]_{\gammaR}$ has Gaussian tails:

$$\mathbb{E}\left[\exp(\lambda [\tau]_{\gammaR}^2)\right] < \infty$$

for some $\lambda > 0$. Hence for $x = \frac{\lambda}{\tau^{1/2}}$ we get

$$\mathbb{E}\left[\exp(x \|v\|_{P_R}^{1-2e})\right] < \infty.$$  

**Remark 2.3.** One of the main motivations to consider (1.1) is to use the Markovian dynamics described by it to study its invariant measure, the Euclidean $\Phi^4_3$ quantum field theory. In order to link this Euclidean (imaginary time) field theory to a real time field theory, this measure should satisfy certain properties, the Osterwalder-Schrader axioms [10, section 6.1]. Our bound (2.22) immediately transfers to this invariant measure. Unfortunately, these stretched exponential moments just fall short of the exponential bounds required for the analyticity axiom.

**Remark 2.4.** Hairer’s convention in the definition of the symbols $\Theta$ in (2.13), $\Psi$, and $\dot{\Psi}$ in (2.15) differs slightly from ours: instead of assuming that these objects satisfy a partial differential equation as we do, he defines them using an integral condition, i.e.,

$$\Theta(x) = \int_{\mathbb{R} \times \mathbb{R}^3} K(x, y) \xi(y) dy,$$

for a singular integral kernel $K$. This kernel $K$ is essentially the Gaussian heat kernel, but it is postprocessed to make it compactly supported and to integrate to 0 against polynomials up to a certain degree. After this postprocessing, $K$ is not associated to a differential operator any more, and in this definition $\Theta$ and the other stochastic terms are not characterised by a (simple) PDE. This is in line with the
general philosophy pursued in [14] to view (1.1) as an integral equation using the mild formulation rather than a differential equation.

**Remark 2.5.** Continuing the discussion of the symbols Φ, Ψ, and Ψ', we point out that in (2.13) and (2.15), we do not impose boundary conditions, but only that a certain PDE holds pointwise on P. There is thus some choice in how these objects are defined and our main result, the estimate (2.21), holds uniformly over all of these choices. This is also the reason why the symbols Ψ and Ψ' appear in the list L. For many choices of boundary conditions, Schauder theory would imply that [Ψ]_{-2} ≤ [Ψ]_{-1} and [Ψ]_{-3} ≤ [Ψ]_{-2} so that these symbols could be removed from L.

A natural choice would be to impose Dirichlet boundary conditions on the parabolic boundary of P in (2.13) and (2.15), and in this case such a Schauder estimate holds indeed. Moreover, with this choice one would have the nice property that all of the objects on the right-hand side only depend on the realisation of ζ on P, which would be in line with a “space-time Markov property.” This nice choice has the slight disadvantage that (in the case where ζ = ξt is a regularised white noise) the negative renormalisation would have to be modified reflecting the boundary conditions, which would lead to x-dependent C1 and C2 in (2.14), (2.17), and (2.19), and then an extra term would have to be added in (2.12) in order to make the renormalisation of the original equation x-independent. Such a construction could certainly be implemented, but we refrain from doing so here (see [9, 22] for discussion of similar boundary issues).

**Remark 2.6.** The spatial dimension d = 3 only enters our analysis through the regularity assumptions on the “trees”. The various |τ| are all derived from the parabolic regularity of the white noise in 1 + 3 time-space dimensions, which is −d/2−. The actual PDE arguments we present do not rely on a specific choice of d.

### 2.3 Outline of Proof

One of the key ideas behind the theory of regularity structures is the following scaling argument:

\[ \hat{u}(t, x) = \lambda^{d/2 - 1} u(\lambda^2 t, \lambda x) \]

is the scaling under which the stochastic heat equation \((\partial_t - \Delta) u = \xi\) is invariant in law. For the \(\Phi^4\) equation, the nonlinearity \(-u^3\) scales like \(-\lambda^{4-d}\hat{u}^3\). In dimension less than 4, this term formally vanishes on small scales, i.e., when \(\lambda\) goes to 0. This property is called subcriticality in Hairer’s theory and corresponds to super-renormalisability in quantum field theory. This observation suggests that in order to control the behaviour of \(u\) on “small scales,” one should use the heat operator and treat the nonlinearity as a perturbation. This is precisely how a small-scale local solution theory is built in [14]. The sign of the nonlinearity \(-u^3\) is not used in this argument. The argument for large scales, on the other hand, clearly has to rely on the “good term” \(-u^3\) and should not use the smoothing of the heat operator too much.
We have already seen that as a perturbation of the linear equation, \( v = u - \tau \) satisfies
\[
(\partial_t - \Delta) v = -v^3 - 3v^2\tau - 3v\nabla v - \nabla - 9C_2(v + \tau).
\]
To control large scales, we apply the regularising operator \((\cdot)_T\) for some \( T \) to be chosen below, and we get the equation
\[
(\partial_t - \Delta) v_T = -(v_T)^3 - 3(v^2\tau)_T - 3(v\nabla)_T - 9C_2(v_T + \tau) - (\nabla)_T \\
+ ((v_T)^3 - (v^3)_T).
\]
This equation is not closed in terms of \( v_T \), and we will require control on the commutator \((v_T)^3 - (v^3)_T\) and on the products \((v^2\tau)_T\) and \((v\nabla)_T\). These are bounded in the small-scale theory. For large-scale bounds, we use the following lemma:

**Lemma 2.7.** Let \( u \) be a continuous function defined on \([0, 1] \times [-1, 1]^3\), for which the following holds pointwise in \((0, 1] \times (-1, 1)^3\):
\[
(\partial_t - \Delta) u = -u^3 + g(u, \tau),
\]
where \( g \) is a bounded function. We have the following pointwise bound on \( u \), for all \((t, x) \in (0, 1] \times (-1, 1)^3\):
\[
|u(x, t)| \leq C \max \left\{ \frac{1}{\min\{\sqrt{i}, (1 - x_i), (1 + x_i), i = 1, 2, 3\}}, \|g\|^{\frac{1}{2}} \right\}
\]
for some independent constant \( C \).

This lemma is a simplified version of [18, theorem 4.4], and the proof (in Section 5.3) is based on the maximum principle. It is the only part of the argument that makes use of the fact that \( u \) is a scalar field and not vector valued. The rest of the proof would go through in the vector-valued case, and we expect that it is possible to find a vector-valued replacement for Lemma 2.7 as well.

In order to close the estimate obtained from Lemma 2.7 we require a bound that allows us to control high-order regularity of \( v \) in terms of the \( L^\infty \) norm. The classical method would consist of using the Schauder estimate of the form [15, theorem 8.9.2]
\[
[u]_{\delta + 2, D_R} \lesssim R^2[(\partial_t - \Delta)]_{\delta, D} + \|u\|_{D_R}
\]
for solutions of the inhomogeneous heat equation. Then if the right-hand side depends on a lower-order norms of \( u \) it can be absorbed into the left-hand side. We perform such an argument in the case where usual Hölder norms are replaced by the norms of "modeled distributions" (which depend on the underlying noise \( \zeta \)).

First, power counting suggests that \( v + \nabla \) has better regularity than \( v \) (namely \( 1 - 2\varepsilon \)) and that this would be enough to define \( v^2\tau = v(v + \nabla)\tau - \nabla \tau \) (assuming that we can construct \( \nabla \tau \)), but not enough to define \( v\nabla \). The next idea to get even better description of solution by explicit stochastic terms is to freeze coefficients
at base point, and to look at local expansions that depend on that base point. The expansion of \( v \) in around base point \( x \) goes as follows:

\[
(2.28) \quad v(y) \sim v(x) - (\Psi(y) - \Psi(x)) - 3v(x)(\Psi(y) - \Psi(x)).
\]

We introduce the following function of two variables based on this local description:

\[
(2.29) \quad U(x, y) = v(y) - v(x) + \Psi(y) - \Psi(x) + 3v(x)(\Psi(y) - \Psi(x)).
\]

The regularity of \( U \), as defined in (2.6), is expected to be higher than \( 1 \). This better description is indeed enough to define \( v \). The core observation is the following abstract reconstruction theorem, which is a variant of \cite[Theorem 3.10]{14} and \cite[Prop. 1]{20}.

**Theorem 2.8 (Reconstruction).** Let \( y > 0 \) and \( A \) be a finite subset of \((-\infty, y]\). Let \( T \in (0, 1) \) and \( x \in \mathbb{R} \times \mathbb{R}^3 \). For a function \( F: B(x, T)^2 \to \mathbb{R} \) assume that for all \( \beta \in A \) there exist constants \( C_{\beta} > 0 \) and \( \gamma_{\beta} \geq y \) such that for all \( t \in (0, T) \) and for all \( x_1, x_2 \in B(x, T - t) \),

\[
(2.30) \quad \left| \int \Psi_t(x_2 - y)(F(x_1, y) - F(x_2, y)) dy \right| \leq \sum_{\beta \in A} C_{\beta} d(x_1, x_2)^{\gamma_{\beta} - \beta t^\beta}.
\]

Then \( f : y \mapsto F(y, y) \) satisfies

\[
(2.31) \quad \left| \int \Psi_T(x - y)(F(x, y) - f(y)) dy \right| \leq \sum_{\beta \in A} C_{\beta} T^\gamma_{\beta},
\]

where \( \ll \) represents a bound up to a multiplicative constant depending only on \( y \) and \( A \).

As a consequence of this theorem, we get the following bounds on the products.

**Lemma 2.9.** The following bound on \( v^2 \) holds:

\[
(2.32) \quad |(v^2)_T(x)| \leq T^{\frac{1}{2} - 3\epsilon} \| v \|_{B(x, T)} [v + \Psi_{1 - 2\epsilon, B(x, T)}]_{\frac{1}{2} - \epsilon} + T^{\frac{1}{2} - 7\epsilon} [v]_{\frac{1}{2} - 3\epsilon, B(x, T)} [\Psi]_{\frac{1}{2} - 3\epsilon} + [v]_{-4\epsilon} + \| v \|_{B(x, T)} [\Psi]_{-4\epsilon} T^{-4\epsilon}.
\]

**Lemma 2.10.** The following bound on \( v \Psi \) holds:

\[
(2.33) \quad |((v - v(x))\Psi)_T(x) + 3C_2(v_T + 1_T)(x)| \leq T^{\frac{1}{2} - 7\epsilon} \left( [v]_{\frac{1}{2} - 3\epsilon, B(x, T)} [\Psi]_{-4\epsilon} + [U]_{\frac{1}{2} - 5\epsilon, B(x, T)} [\Psi]_{-1 - 2\epsilon} + [v]_{\frac{1}{2} - 5\epsilon, B(x, T)} [\Psi]_{-2\epsilon} \right)
\]

\[
+ [\Psi]_{\frac{1}{2} - 5\epsilon} T^{-\frac{1}{2} - 5\epsilon} + T^{-4\epsilon} \| v \|_{B(x, T)} [\Psi]_{-4\epsilon} + \| v \|_{B(x, T)} [\Psi]_{-2\epsilon} T^{-2\epsilon}.
\]
where $U$ is as introduced in (2.29) and $v$ is optimal in the definition (2.6).

The proof of Theorem 2.8 and Lemmas 2.9 and 2.10 can be found in Section 5.2. To bound the quantities appearing in the right-hand side of these lemmas, we introduce our version of the Schauder estimate.

**Lemma 2.11.** Let $1 < \kappa < 2$ and $A \subset (-\infty, \kappa]$ finite. Let $U$ be a bounded function of two variables defined on a domain $D \times D$ such that $U(x, x) = 0$ for all $x$. Let $d_0 > 0$ and assume that for any $0 < d \leq d_0$ and $L \leq \frac{d}{4}$ there exists a constant $M_{D_d, L}^{(1)}$ such that for all base points $x \in D_d$ and length scales $T \leq L$, it holds that

\[
T^2 \| (\partial_t - \Delta) U_T(x, \cdot) \|_{B(x, L)} \leq M_{D_d, L}^{(1)} \sum_{\beta \in A} T^\beta L^{\kappa - \beta}.
\]

Assume furthermore that for $L_1, L_2 \leq \frac{d}{4}$ there exists a constant $M_{D_d, L_1, L_2}^{(2)}$ such that for any $x \in D_d$, for any $y \in B(x, L_1)$, and for any $z \in B(y, L_2)$ the following “three-point continuity” holds:

\[
|U(x, z) - U(x, y) - U(y, z)| \leq M_{D_d, L_1, L_2}^{(2)} \sum_{\beta \in A} d(y, x)^\beta d(z, y)^{\kappa - \beta}.
\]

Additionally, define $M_{D_d, \frac{d}{4}}^{(1)} := \sup_{d \leq d_0} d^\kappa M_{D_d, \frac{d}{4}}^{(1)}$ and $M_{D_d, \frac{d}{4}, \frac{d}{4}}^{(2)} := \sup_{d \leq d_0} d^\kappa M_{D_d, \frac{d}{4}, \frac{d}{4}}^{(2)}$. Then

\[
\sup_{d \leq d_0} d^\kappa [U]_{k, D_d} \lesssim M_{D_d}^{(1)} + M_{D_d}^{(2)} + \sup_{d \leq d_0} \| U \|_{D_d, d}.
\]

Here and in the proof, “$\lesssim$” denotes a bound that holds up to a multiplicative constant that only depends on $\kappa$ and $A$.

**Corollary 2.12.** Fix $0 < d \leq d_0$ such that $D_d \neq \emptyset$. Assume that $D_d$ satisfies a spatial interior cone condition with parameters $r_d > 0$ and $\lambda \in (0, 1)$; i.e., for all $r \in [0, r_d)$, for all $x \in D_d$, and for any vector $v \in \mathbb{R}^3$, there exists $y \in D_d$ such that $d(x, y) = r$

\[
|v \cdot X(y - x)| \geq \lambda |v| d(x, y).
\]

Then for an optimal function $v$ in (2.36), for all $r \in [0, r_d)$,

\[
\lambda \| v \|_{D_d} \leq [U]_{k, D_d} r^{\kappa - 1} + \| U \|_{D_d, r} r^{-1}.
\]

If (2.35) holds for all $x, y, z \in D_d$, we have for $r \leq r_d$,

\[
[v]_{k-1, D_d} \lesssim [U]_{k, D_d} + M_{D_d, \frac{d}{4}, \frac{d}{4}}^{(2)} + r^{-\kappa} \| U \|_{D_d, r}.
\]

Here and in the proof, “$\lesssim$” denotes a bound that holds up to a multiplicative constant that only depends on $\kappa$. 
Note that if $D = P_R$ for $R < \frac{1}{2}$, the interior cone condition holds for $D_d$ with $r_d = \frac{1}{2} - d$ and $\lambda = \frac{\sqrt{2}}{2}$, uniformly in $R$. The proof of the lemma and its corollary are in Section 5.1.

The Schauder estimate and the reconstruction lemmas can be combined with the large-scale bound into a self-consistent bound that can be iterated leading to our main result, Theorem 2.1. This argument can be found in Section 4.

3 Translation to the Language of Regularity Structures

Although our argument is not formulated using the terminology of the theory of regularity structures, the analysis of the small-scale behaviour, Theorem 2.8, Lemmas 2.9 and 2.10 as well as the Schauder estimate Lemma 2.11 builds on the same key ideas as this theory. We now provide a translation of how the lemmas that appear in our paper can be stated in terms of the central objects introduced in the theory of regularity structures such as the models, modeled distributions, and the abstract integration operator. This section is aimed at those interested in this theory but plays no role in the results presented in this paper, beyond explaining why the assumptions we make in Section 2.2 are reasonable in this framework.

We begin by recalling the setup in [14]: In [14, def. 2.1] a regularity structure is defined as a triple $(\mathcal{A}, T, G)$, consisting of an index set $\mathcal{A} \subset \mathbb{R}$, a graded vector space $T = \bigoplus_{\alpha \in \mathcal{A}} T_\alpha$, and a group $G$ of linear transformations acting on $T$ with some additional properties. In this framework, the local description of the solution $u$ is encoded by replacing the scalar-valued function/distribution $u$ by modeled distribution, which is a function $\mathcal{U} : \mathbb{R} \times \mathbb{R}^3 \to T$ for a certain purpose-built regularity structure. To build this structure one first introduces some symbols, namely
\[
\{1, 1, \Psi, \Psi\} \cup \{X_i : i = 1, 2, 3\}.
\]

At this level these blue symbols are completely abstract objects, but of course they ultimately represent the functions/distributions appearing in the local description. To each of these symbols $\tau$ one associates a homogeneity $|\tau| \in \mathbb{R}$, namely,
\[
|1| = 0, \quad |X_i| = 1, \quad |1| = -\frac{1}{2} - \epsilon, \quad |\Psi| = \frac{1}{2} - 3\epsilon, \quad |\Psi| = 1 - 2\epsilon.
\]

The space $T$ is then defined as the finite-dimensional space
\[
T = \bigoplus_{\tau \in \{1, X_i, 1, \Psi, \Psi\}} \mathbb{R}\tau,
\]
and $\mathcal{A}$ is defined to be the set of homogeneities of these symbols. It turns out that the modeled distribution $\mathcal{U}$ takes the form
\[
\mathcal{U}(x) = \mathcal{T} + v(x) \mathcal{1} - \mathcal{\Psi} - 3v(x)\mathcal{\Psi} - v(x) \cdot X,
\]
for some functions $v$ and $\mathcal{T}$ (which of course coincide with our functions $v$ and $\mathcal{T}$). For our analysis we choose to work with a local description for $v = \mathcal{U} - \mathcal{T}$, which
in the notation of regularity structures would take the form

\[ V(x) = v(x) \mathbf{1} - \Psi - 3v(x)\mathcal{V} - \nu(x) X; \]

i.e., the only difference with respect to (3.1) is that the term \( \mathbf{1} \) is removed. Equation (3.2) should be viewed as an abstract counterpart of our equation (2.28). For us it is more convenient to work with \( v \) rather than \( u \) to get good bounds on the error term \( (v^3) - (v^3) \). We argue below that the regularity assumption we impose on \( V \) is equivalent to the condition imposed on \( \mathcal{U} \) in [14].

Just like our main result, Theorem 2.1, the solution theory using regularity structures requires a perturbative expansion as an input. There this expansion is encoded in the notion of a model [14, def. 2.17]. To each of the symbols, one associates a function/distribution \( \Pi \tau \) corresponding exactly to our definitions (2.13) and (2.15), i.e.,

\[
\begin{align*}
\Pi \mathbf{1}(y) &= 1, & \Pi X_i(y) &= y_i, & \Pi \mathbf{1}(y) &= \mathbf{1}(y), \\
\Pi \Psi(y) &= \Psi(y), & \Pi \mathcal{V}(y) &= \mathcal{V}(y).
\end{align*}
\]

A key idea of the theory is to not work with these distributions directly, but with centered or positively renormalised objects, \( \Pi_x \tau \) indexed by a base point in \( x \in \mathbb{R} \times \mathbb{R}^3 \). The right notion of regularity for the modeled distributions \( \mathcal{U} \) and \( \mathcal{V} \) is then defined in term of this recentering procedure.

For the symbols we have introduced so far, the centering is relatively simple and amounts to subtracting the value at the base point for the symbols of strictly positive homogeneity:

\[
\begin{align*}
\Pi_x \mathbf{1}(y) &= 1, & \Pi_x X_i(y) &= y_i - x_i, & \Pi_x \mathbf{1}(y) &= \mathbf{1}(y), \\
\Pi_x \mathcal{V}(y) &= \mathcal{V}(y) - \mathcal{V}(x), & \Pi_x \Psi(y) &= \Psi(y) - \Psi(x).
\end{align*}
\]

The reason why one works with these centered objects is that one has good control over their behaviour as the argument approaches the base point \( x \). This is encoded in the formula [14, eq. (2.15)],

\[ (\Pi_x \tau, \phi^n_x) \lesssim \lambda|\tau|, \]

where \( \phi \) is a smooth test function rescaled to scale \( \lambda \) and centered at the base point \( x \). This corresponds exactly to our regularity assumption on the Hölder norms of the objects; see Section 2.2 (our scale is called \( T \) rather than \( \lambda \) and the test function is called \( \Psi \) rather than \( \phi \)).

In order to connect the centering procedure to the functions \( \mathcal{U} \) and \( \mathcal{V} \) and to formulate the right continuity condition, it is useful to introduce the structure group \( G \). In the current context this group is simply the five-dimensional group of all linear transformations \( F \) on \( T \) of the form

\[
\begin{align*}
F \mathbf{1} &= \mathbf{1}, & F X_i &= x_i + a_i \mathbf{1}, & a_i \in \mathbb{R}, & F \mathbf{1} &= \mathbf{1}, \\
F \Psi &= \Psi + b \mathbf{1}, & b \in \mathbb{R}, & F \mathcal{V} &= \mathcal{V} + c \mathbf{1}, & c \in \mathbb{R},
\end{align*}
\]

(3.4)
but this group will be enlarged as more symbols are introduced below. For each $x \in \mathbb{R} \times \mathbb{R}^3$ we define $F_x \in G$ by

\[
F_x \mathbf{1} = \mathbf{1}, \quad F_x X_i(y) = X_i - x_i \mathbf{1}, \quad F_x \Psi = \Psi - \Psi(x) \mathbf{1},
\]

so that one gets

\[
\Pi_x \tau = \Pi F_x \tau.
\]

Now, for $x, y \in \mathbb{R} \times \mathbb{R}^3$ we set

\[
\Gamma_{xy} = F_{y}^{-1} \circ F_{x},
\]

and we trivially have the identity, cf. [14, def. 2.17].

The continuity assumption on $\mathcal{U}$ and $\mathcal{V}$ is formulated in terms of the translation operators $\Gamma_{xy}$. $\mathcal{U}$ is said to be a modeled distribution of order $\gamma$ if

\[
\| \mathcal{U}(x) - \Gamma_{xy} \mathcal{U}(y) \|_\beta \lesssim d(x, y)^{\gamma - \beta},
\]

where $\| \cdot \|_\beta$ refers to the component in $T_\beta$. It is easy to check that for both, $\mathcal{U}$ defined by (3.1) and $\mathcal{V}$ defined by (3.2), this condition translates precisely into the “modeledness conditions”

\[
\begin{align*}
|v(y) - v(x) + \Psi(y) - \Psi(x) - v(x) \cdot X(y - x) & + 3v(x)(\Psi(y) - \Psi(x))| \lesssim d(x, y)^\gamma, \\
|v(y) - v(x)| & \lesssim d(x, y)^{\gamma - 1}, \\
|v(y) - v(x)| & \lesssim d(x, y)^{\gamma - 1 + 2\varepsilon},
\end{align*}
\]

and this condition for $\gamma = \frac{3}{2} - 5\varepsilon$ corresponds exactly to the regularity assumptions on $\mathcal{U}, v,$ and $\psi$ we work with.

The main feature of the space of modeled distributions is that although expansions like (3.1) are ultimately used as good local descriptions of distributions, one can multiply them as if they were of positive regularity, provided one can expand the action of the model to new symbols that are seen as products of the symbols introduced earlier. For equation (1.1) one has to get a bound on $u^3 = (v + 1)^3 = v^3 + 3v^2 1 + 3v \Psi + \Psi$. We aim to bound this in terms of:

- a high-regularity norm on $v$, namely the $\mathcal{D}^{\gamma}$ norm of the modeled distribution $\mathcal{V}$, which is defined as the smallest possible constant in the inequalities (3.6);
- the low regularity $L^\infty$ norm $\| v \|$;
- the bounds on the various stochastic terms.

The term $v^3$ can immediately be bounded by $\| v \|^3$ and $\Psi$ is a stochastic term that does not involve $v$. The only terms that require work are $3v^2 1$ and $3v \Psi$. The distribution $\Psi$ has regularity $-1 - 2\varepsilon$, so a description of $v$ to order $\gamma > 1 + 2\varepsilon$ is
required. Such a description is precisely provided by \((3.2)\). One now defines new symbols
\[(3.7) \quad \{\Psi, \mathcal{V}, \Psi, \Psi, \Psi, \mathcal{V}\},\]
associates to them a homogeneity using the rule \(|\tau| = |\tau| + |\xi|\), and simply defines a new modeled distribution for the local description of \(v\mathcal{V}\) by
\[(3.8) \quad \mathcal{V}(x) = v(x)\mathcal{V} + \Psi - 3v(x)\Psi - v(x)\mathcal{V}.\]
This definition becomes substantial by extending the model \((\Pi, \Gamma_{xy})\) to these new symbols. One would like to extend the operator \(\Pi\) to these products simply by defining locally
\[\Pi(x)(\tau\xi)(y) = (\Pi(x\tau)(y))(\Pi(x\xi)(y)),\]
but such a definition may not be meaningful when the regularisation is removed. Fortunately, there is some flexibility at this level. The main requirements for multiplication to be well-behaved are only that \((3.3)\) and \((3.5)\) remain valid for the new symbols and additionally that one has the identity
\[\Gamma_{xy}(\tau\xi) = (\Gamma_{xy}\tau)(\Gamma_{xy}\xi).\]
It is here that the positive renormalisation, and hence the action of the structure group \(G\), becomes more involved than subtracting the value at a base point and the condition \(|\tau\xi| = |\tau| + |\xi|\) becomes strictly stronger than Hölder regularity. For example \(\Pi \Psi\) is a distribution of regularity \(-1 - 2\varepsilon\) but its homogeneity is strictly larger, namely \(-\frac{1}{2} - 5\varepsilon\). The condition \((3.3)\) states that near any base point \(x\), \(\Pi \Psi\) is well described by a \(\Psi(x)\Pi\mathcal{V}\) up to an error of order \(-\frac{1}{2} - 5\varepsilon\), which is strictly stronger than a bound on the \(C^{-1-2\varepsilon}\) norm of it. Our definitions \((2.14)\) and the assumed bounds \((2.19), (2.17), \text{and} (2.16)\) correspond exactly to the definitions for \(\Pi\) and the bound \((3.3)\) in \([14]\). The only difference is that Hairer defines the trees \(\Gamma, \mathcal{V}\), and \(\Psi\) using the inverse heat operator with some cutoff at large scales and the appropriate right-hand side. We only assume that they satisfy the heat equation pointwise without imposing any boundary conditions, but we additionally impose some natural regularity bounds, as explained in Remark \(2.5\). Combining Hairer’s multiplication theorem \([14\text{, theorem } 4.7]\) and his reconstruction theorem \([14\text{, theorem } 3.10]\) then yields the estimate
\[\|\mathcal{R}(\mathcal{V}) - \Pi(x)(\mathcal{V})\varphi^\lambda\| \lesssim \lambda^{\frac{1}{2} - 2\varepsilon}\|\mathcal{V}\|_{\mathcal{V}}\|\Pi\|,\]
where \(\|\Pi\|\) is the smallest possible constant in all of the assumed bounds on the model. This is essentially the statement of our Lemma \(2.10\) up to a few points:
- Some of the terms in \(\Pi(x)(\mathcal{V})\) are removed from the left-hand side of \((2.33)\)
  and added to the right-hand side using the triangle inequality.
- We prove these estimates “by hand” without using the algebraic machinery discussed above and in particular without introducing the group \(G\) to organise the various continuity assumptions. More precisely, Theorem \(2.8\).
is a condensed version of [14, theorem 3.10], which contains the key analytic estimate but assumes the output of the algebraic machinery. In the case of (1.1), the algebraic manipulations are not too complex and can be done directly quite easily, and that is precisely what we do in the proof of Lemma 2.10.

Along the way we keep track of the precise norms needed in each term, rather than compiling them in \( \|V\|_{D^V} \) and \( \|\Pi\| \). This added level of detail is important for us, especially when determining the exact exponents of each tree appearing in our final estimate (2.21).

The treatment of the term \( v^2 \) goes along similar lines. As \( 1 \) has better regularity than \( V \), namely \(-\frac{1}{2} - \varepsilon\), a local description of \( v^2 \) is only required to order \( > \frac{1}{2} + \varepsilon \), and this is provided by

\[
V^2(x) = v^2(x) 1 - 2v(x)\Psi,
\]

which in turn prompts us to define

\[
\mathcal{V}^2(x) = v^2(x) 1 - 2v(x)\Psi.
\]

Again, our assumption (2.18) corresponds exactly to the homogeneity condition (3.3) in [14], and our Lemma 2.9 is obtained by combining the multiplication and reconstruction theorem and applying the triangle inequality, this time to remove the term corresponding to \( \Pi \cdot v^2 \) from the left-hand side completely. The Hölder norm \( [v + \Psi]_{1 - 2\varepsilon} \) that appears on the right-hand side of our estimate (2.32) corresponds to the norm of the modeled distribution one obtains by removing the terms \(-3v(x)\Psi - v(x)\cdot X\), which are not necessary here, from the definition of \( V \) in (3.2).

The last ingredient from the theory of regularity structures concerns the heat operator. For us, the gain of regularity for solutions to the heat equation is expressed in Lemma 2.11 and this corresponds to [14, theorem 5.12]. As stated above in (3.6), we seek a local description of the solution \( v \) of order \( \gamma = \frac{3}{2} - 5\varepsilon \). The heat operator \( (\partial_t - \Delta)^{-1} \) is a 2-regularising operator (\( \beta = 2 \) in Hairer’s theory), and thus it seems reasonable that a local description of the right-hand side of (2.20) up to order \(-\frac{1}{2} - 5\varepsilon \) is required as input for the Schauder lemma. Therefore we work with

\[
\mathcal{W}(x) = v(x)\Psi + \Psi.
\]

At this point we slightly deviate from Hairer’s approach: his Schauder lemma [14, theorem 5.12] assumes that the right-hand side is a modeled distribution of strictly positive order, because he applies the reconstruction operator as part of the argument. We circumvent this by viewing the reconstruction as an extra input to the theorem. Imposing that \( \mathcal{W} \) is a modeled distribution of order \(-\frac{1}{2} - 5\varepsilon \) [14, def. 3.1] translates precisely to our three-point continuity condition (2.35), and our smallness assumption (2.34) corresponds to [14, eq. 5.42], which in the notation of this section would be

\[
(3.9) \quad \left| \left| (R \mathcal{W} - \Pi_{\mathcal{X}} \mathcal{W}, \phi^\lambda_{\mathcal{X}}) \right| \right| \lesssim \lambda^{-\frac{1}{2} - 5\varepsilon}.
\]
The exact statement of the assumption (2.35) appears slightly stronger than (3.9) because of the $L^\infty$ norm on the left-hand side and the extra parameter $L$, but in practice the seemingly stronger bound can be obtained easily from the weaker bound using the triangle inequality and some lower-order regularity information.

In the framework of regularity structure, the operator that encodes the integration of a modeled distribution is described as the sum of three operators. The first operator $I$ acts pointwise on the modeled distributions by a shift of coefficients. The action on the trees is

$$I\Psi = \Psi, \quad TV = V.$$

The continuity of the coefficients for a modeled distribution is transferred accordingly under the action of $I$. In our setting, this is also automatic and follows from our assumptions, as explained in Remark 2.5.

The nontrivial part of the integration happens on the levels 1 and X, which is encoded in Hairer’s theory in the operators $J$ and $N$. We have again a direct translation, although we do not need to split the operator.

$$N|V(x) = ((\partial_t - \Delta)^{-1}(-v^3 - 3v^2 y - 3(y - u(x)\Psi)|_{y = x})1 + u(x)X,$$

$$J(x)|V = (3v(x)\Psi - \Psi)1.$$

The differences between our approach and the one adopted by Hairer is that in the spirit of [21] we use a kernel-free approach, and we have a special treatment of the boundary on the levels 1 and X. We are also more precise in our final bounds in the sense that, as in the definition of $\mathcal{V}V$, we keep track of the precise norms needed in each term.

## 4 Proof of Theorem 2.1

### 4.1 Assumption

We assume that the bound of Theorem 2.1 in terms of powers of trees does not hold on a domain $D = P_R$, and use that assumption to prove that then a bound in $\frac{1}{R}$ holds. This assumption is not necessary for the proof but it simplifies the computations greatly by allowing us to replace all occurrences of norms of trees by powers of $\|v\|_D$. In particular, using this convention the key estimate (4.19) takes a particularly simple form. Our assumption is stated as follows:

\[
\forall \tau \in L \quad [\tau]_{V, c} \leq c \|v\|_D^{n_\tau(\frac{1}{2} - \epsilon)}
\]

for some constant $c < 1$ that we will tune later, according to conditions suggested by equations (4.17) and (4.31). With these assumptions, Lemmas 2.9 and 2.10 can be restated as, for any $x$ with $B(x, T) \in D$,

\[
|v^2 1_T(x)| \leq c \left( |v + \Psi|_{1 - 2\epsilon, B(x, T)} T^{\frac{1}{2} - 3\epsilon} \|v\|_D^{\frac{3}{2} - \epsilon} \right.
\]

\[
+ |v|_{\frac{1}{2} - 3\epsilon, B(x, T)} T^{\frac{1}{2} - 7\epsilon} \|v\|_D^{2 - 4\epsilon}
\]

\[
+ c \left( T^{-\frac{1}{2} - \epsilon} \|v\|_D^{\frac{5}{2} - \epsilon} + T^{-4\epsilon} \|v\|_D^{3 - 4\epsilon} + T^{\frac{1}{2} - 7\epsilon} \|v\|_D^{-7\epsilon} \right).
\]
and
\[
\left\| (v - v(x))\nabla x T(x) + 3C_2(v_T + 1_T)(x) \right\|
\leq cT^{\frac{1}{2} - \frac{1}{2} - 7\varepsilon} \left[ \left\| v \right\|_D^{\frac{5}{2} - 4\varepsilon} + \left\| U \right\|_{1 - 3\varepsilon, B(x,T)} \right]^{\frac{1}{2} - 3\varepsilon, B(x,T)}
+ \left[ v \right]_{1 - 3\varepsilon, B(x,T)} \right\|_D^{\frac{5}{2} - 4\varepsilon}
+ \varepsilon \left( \left\| v \right\|_D^{\frac{5}{2} - 4\varepsilon} T^{-\frac{1}{2} - 5\varepsilon} + T^{-4\varepsilon} \left\| v \right\|_D^{\frac{5}{2} - 4\varepsilon} \right) + \left\| v \right\|_{B(x,T)} \right\|_D^{\frac{5}{2} - 4\varepsilon} T^{-2\varepsilon}.
\]

(4.3)

4.2 Applying Lemma 2.11

For any domain \( D \), for \( x, L, T \), and \( d \) such that \( B(x, L + T) \subseteq D_d \), we prove the following bound, which is the first condition to apply Lemma 2.11:
\[
\left\| (\partial_t - \Delta)U(x, \cdot, T) \right\|_{B(x, L)}
\leq \left\| v \right\|_D^3 + L^{\frac{1}{2} - 3\varepsilon} T^{-1 - 2\varepsilon} c[v]_{1 - 3\varepsilon, D_{d,L}} \left\| v \right\|_D^{\frac{5}{2} - 4\varepsilon}
+ \left[ v \right]_{1 - 3\varepsilon, B(x,T)} \right\|_D^{\frac{5}{2} - 4\varepsilon}
+ \left( T^{-\frac{1}{2} - \varepsilon} \left\| v \right\|_D^{\frac{5}{2} - 4\varepsilon} + T^{-4\varepsilon} \left\| v \right\|_D^{\frac{5}{2} - 4\varepsilon} T^{-\frac{1}{2} - 7\varepsilon} \right) \right]^{\frac{5}{2} - 4\varepsilon}
+ \left( \left\| v \right\|_D^{3 - 4\varepsilon} \right) + \left( \left\| v \right\|_D^{\frac{5}{2} - 4\varepsilon} \right) + \left\| v \right\|_{B(x,T)} \right\|_D^{\frac{5}{2} - 4\varepsilon} T^{-2\varepsilon},
\]

(4.4)

where \( v \) is the optimal function in the definition of \( [U]_{1 - 3\varepsilon, D_{d,L}} \).

Let \( x \) be an arbitrary point in \( D_{d,L} \), and \( y \) a point in \( B(x, L) \subseteq D_d \). We have
\[
(\partial_t - \Delta)U(x, \cdot, T)(y) = \int \Psi_T(z - y)(\partial_t - \Delta)U(x, z)dz
= -(v^3)T(y) - 3(v^2)T(y) - 3((v - v(x))\nabla)_T(y) - 9C_2(v_T + 1_T)(y)
= -(v^3)T(y) - 3(v^2)T(y) - 3((v - v(x))\nabla)_T(y)
- 3((v - v(y))\nabla)_T(y) - 9C_2(v_T + 1_T)(y).
\]

We bound some terms of this expression by the previous bounds (4.2) and (4.3) and the remaining ones as follows:
\[
\left\| v^3 \right\|_D^3 \leq \left\| v \right\|_D^3 B(y, T) \leq \left\| v \right\|_D^3 B_d,
\]
\[
\left\| (v - v(x))\nabla \right\|_T(y) \leq \ell(x, y) \left[ v \right]_{1 - 3\varepsilon, D_{d,L}} T^{-1 - 2\varepsilon} \left[ \Psi \right] - 1 - 2\varepsilon,
\]

\[
\leq c(v)_{1 - 3\varepsilon, D_{d,L}} \left\| v \right\|_D^{\frac{5}{2} - 4\varepsilon} T^{-2\varepsilon}.
\]

(4.5)

(4.6)
This proves (4.4).

The three-point continuity on $U$ holds as follows. For any $x \in D_d$, for any $y \in B(x, \frac{d}{4})$, and for any $z \in B(y, \frac{d}{4})$,

$$|U(x, y) - U(x, z) - U(z, y)| = 3|v(x) - v(z)||\gamma(y) - \gamma(z)|$$

(4.7)

$$\leq 3[v]_{1/2-3\epsilon, D_d, d/2} \gamma_1 \gamma_1 2d(x, z)^{1/2-3\epsilon} d(y, z)^{1-2\epsilon}$$

$$\leq 3c[v]_{1/2-3\epsilon, D_d, d/2} \|v\|_{D}^{1-2\epsilon} d(x, z)^{1/2-3\epsilon} d(y, z)^{1-2\epsilon}$$

Lemma 2.11 applies to $U$ with $\kappa = \frac{3}{2} - 5\epsilon$. Note that in the bound we see powers of $T$ higher than $T^{-1/2-5\epsilon}$, but we use the fact that $T \lesssim d$ to make up for that. After a few simplifications we get

\[ \sup_{d \leq d_0} d^{1/2-5\epsilon}[U]_{1/2-5\epsilon, D_d} \]

\[ \lesssim \sup_{d \leq d_0} \left( d^2 \|v\|_D^3 + d^{1/2-5\epsilon} c[v]_{1/2-3\epsilon, D_d, d} \|v\|_{D}^{1-2\epsilon} \right. \]

\[ + c(|v + \gamma|_{1-2\epsilon, D_d, d} d^{3/2-3\epsilon} \|v\|_{D}^{3-\epsilon} \]

\[ + d^{1/2-\epsilon} \|v\|_{D}^{5-\epsilon} + d^{3/2-7\epsilon} \|v\|_{D}^{7-7\epsilon} \right) \]

(4.8)

\[ + c d^{5-7\epsilon} (v)_{1/2-3\epsilon, D_d, d} \|v\|_{D}^{7-4\epsilon} \]

\[ + \left( [U]_{1/2-5\epsilon, D_d, d} + [v]_{1/2-5\epsilon, D_d, d}\|v\|_{D}^{1-2\epsilon} \right) \]

\[ + c \left( \|v\|_{D}^{5-5\epsilon} d^{1/2-5\epsilon} + d^{2-4\epsilon} \|v\|_{D}^{3-4\epsilon} \right) \]

\[ + \|v\|_{D_d} c \|v\|_{D}^{1-2\epsilon} d^{2-2\epsilon} + \|U\|_{D_d, d}. \]

### 4.3 Simplifications

Our goal in this section is to produce bounds on the seminorms $[v]_{1/2-3\epsilon, D_d, d}$, $[v + \gamma]_{1-2\epsilon, D_d, d}$, and $[U]_{(3/2)-5\epsilon, D_d}$ that depend only on $\|v\|_D$, in particular independent of each other. We introduce the following elementary bounds, which
can be deduced from triangle inequalities and application of Assumption 4.1:

$$\|v\|_{1-3\epsilon, D_d, d} \leq d^{1-2\epsilon} [U]_{1-5\epsilon, D_d}^{1/2} + [\mathcal{Y}]_{1-3\epsilon}^{1/2} + 3d^{1+\epsilon} \|v\|_{D_d} \|\mathcal{Y}\|_{1-2\epsilon} + d \|v\|_{D_d}$$

(4.9)

$$\leq d^{1-2\epsilon} [U]_{1-5\epsilon, D_d}^{1/2} + c \|v\|_{D_d}^{1/2} + \epsilon + \frac{1}{2} + \epsilon \|\mathcal{Y}\|_{1-2\epsilon} + d \|v\|_{D_d} + d^{1+\epsilon} \|v\|_{D_d}$$

and

$$\|v + \mathcal{Y}\|_{1-2\epsilon, D_d, d}$$

(4.10)

$$\leq d^{1-3\epsilon} [U]_{1-5\epsilon, D_d}^{1/2} + 3 \|v\|_{D_d} \|\mathcal{Y}\|_{1-2\epsilon} + d^{2\epsilon} \|v\|_{D_d}$$

$$\leq d^{1-3\epsilon} [U]_{1-5\epsilon, D_d}^{1/2} + 3c \|v\|_{D_d}^{2-2\epsilon} + d^{2\epsilon} \|v\|_{D_d}.$$

and we recall that from Corollary 2.12 we have the two bounds, assuming $d \in (0, \ell_d]$.

(4.11) $$\|v\|_{D_d} \leq [U]_{1-5\epsilon, D_d}^{1/2} + \|U\|_{D_d, d} \leq d^{1-3\epsilon} [U]_{1-5\epsilon, D_d}^{1/2} + \|U\|_{D_d, d} d^{-1}$$

and

(4.12) $$\|v\|_{1-3\epsilon, D_d} \leq [U]_{1-3\epsilon, D_d}^{1/2} + \|U\|_{1-2\epsilon, D_d}^{1/2} d^{1-3\epsilon} + \|U\|_{D_d, d} \leq d^{-1} \|v\|_{D_d} + \|U\|_{D_d, d}^{1-3\epsilon} + \|U\|_{D_d, d}^{1-2\epsilon} + \|U\|_{D_d, d}^{1-2\epsilon}.$$

We will also be using the bound

(4.13) $$\|U\|_{D_d, d} \leq 2 \|v\|_{D_d} + \|U\|_{D_d, d}^{1-3\epsilon} + 3 \|v\|_{D_d} \|\mathcal{Y}\|_{1-2\epsilon, d}^{1-2\epsilon} + \|U\|_{D_d, d}^{1-3\epsilon} + 3 \|v\|_{D_d}^{2-2\epsilon} + d \|v\|_{D_d}^{2-2\epsilon}.$$

By combining the bounds above to get bounds in terms of $\|v\|_{D_d}$ and $[U]_{1-5\epsilon, D_d}$ only, we get the following bounds (in order of logical deduction):

(4.14) $$\|v\|_{D_d} \leq d^{1-3\epsilon} [U]_{1-5\epsilon, D_d}^{1/2} + d^{-1} \|v\|_{D_d} + c d^{-1} \|v\|_{D_d}^{1-3\epsilon} + \|v\|_{D_d}^{2-2\epsilon} + d^{-1} \|v\|_{D_d}^{2-2\epsilon}.$$

$$\leq d^{1-3\epsilon} [U]_{1-5\epsilon, D_d}^{1/2} + c d^{-1} \|v\|_{D_d}^{1-3\epsilon} + \|v\|_{D_d}^{2-2\epsilon} + d^{-1} \|v\|_{D_d}^{2-2\epsilon}.$$

(4.15) $$\frac{1}{2} - 3\epsilon, D_d, d \leq d^{1-2\epsilon} [U]_{1-5\epsilon, D_d}^{1/2} + d^{-1} \|v\|_{D_d} + c d^{1-3\epsilon} \|v\|_{D_d}^{2-2\epsilon} + d^{1+3\epsilon} \|v\|_{D_d}^{2-2\epsilon} + c d^{1+3\epsilon} \|v\|_{D_d}^{2-2\epsilon}.$$
\[
\begin{align*}
\frac{1}{2} - 5\varepsilon, D_d & \lesssim [U]^{\frac{1}{2} - 5\varepsilon, D_d} (1 + c d^{1 - 2\varepsilon \|v\|^{1 - 2\varepsilon}}) \\
& \quad + c^2 \left( \|v\|^\frac{5}{2} D_d + d^{\frac{1}{2} + \varepsilon} \|v\|^\frac{3 - 4\varepsilon}{D_d} \right) \\
& \quad + d^{-\frac{3}{2} + 5\varepsilon \|v\| D_d} + c d^{-1 + 2\varepsilon \|v\|^\frac{3}{2} - 3\varepsilon} + c d^{-\frac{1}{2} + 3\varepsilon \|v\|^\frac{3}{2} - 2\varepsilon}.
\end{align*}
\] (4.16)

We inject those in the right-hand side of (4.8), and we bound positive powers of \(d\) by powers of \(d_0\). A few computations allow us to reduce the result to the following expression:

\[
\begin{align*}
sup_{d \leq d_0} d^{\frac{3}{2} - 5\varepsilon} [U]^{\frac{1}{2} - 5\varepsilon, D_d} & \lesssim c \sum_h d_0^h \|v\|^{h + 1} \\
& \quad + c \sup_{d \leq d_0} d^{\frac{1}{2} - 5\varepsilon} [U]^{\frac{1}{2} - 5\varepsilon, D_d} \sum_l d_0^l \|v\|^{l}.
\end{align*}
\] (4.17)

where the index of the sum \(h\) is taken in a finite subset of \([0, \frac{9}{2} - 11\varepsilon]\) and the index \(l\) in \(\{1 - 2\varepsilon, \frac{3}{2} - \varepsilon, 2 - 4\varepsilon\}\). If we set

\[
d_0 = \|v\|^{-1},
\] (4.18)

we see that we can get rid of \(\sup_{d \leq d_0} d^{\frac{3}{2} - 5\varepsilon} [U]^{\frac{1}{2} - 5\varepsilon, D_d}\) in the right-hand side under a first smallness condition on \(c\) depending on the constant implicit in \(\lesssim\). Note that if (4.18) defines \(d_0\) too large, it means that we already have a bound on \(\|v\|_D\). If the smallness condition on \(c\) is satisfied, we have

\[
\begin{align*}
sup_{d \leq d_0} d^{\frac{3}{2} - 5\varepsilon} [U]^{\frac{1}{2} - 5\varepsilon, D_d} & \lesssim c \|v\|_D.
\end{align*}
\] (4.19)

In this equation and in the following, \(\lesssim\) does not depend on this first condition on \(c\). Applying this to equations (4.14) to (4.16) gives

\[
\begin{align*}
sup_{d \leq d_0} d \|v\|_D & \lesssim c \|v\|_D, \\
sup_{d \leq d_0} d^{1 - 2\varepsilon} [v + \mathcal{Y}]^{-1 - 2\varepsilon, D_d} & \lesssim c \|v\|_D, \\
sup_{d \leq d_0} d^{\frac{1}{2} - 3\varepsilon} [v]^{-\frac{1}{2} - 3\varepsilon, D_d} & \lesssim c \|v\|_D, \\
sup_{d \leq d_0} d^{\frac{3}{2} - 5\varepsilon} [v]^{-\frac{1}{2} - 5\varepsilon, D_d} & \lesssim c \|v\|_D.
\end{align*}
\] (4.20)-(4.23)

Applying estimates (4.19) and (4.20)-(4.23) to (4.2) and (4.3), we have, for \(d \leq d_0\) and \(B(x, T) \in D_d\),

\[
\begin{align*}
|(v^2 T(x))| & \lesssim c^2 \left( d^{-1 + 2\varepsilon} T^{\frac{1}{2} - 3\varepsilon \|v\|^\frac{5}{2} - \varepsilon} + d^{-\frac{1}{2} + 3\varepsilon} T^{\frac{3}{2} - 7\varepsilon \|v\|^\frac{3 - 4\varepsilon}{D} + d^{\frac{1}{2} + 3\varepsilon} T^{\frac{3}{2} - 7\varepsilon \|v\|^\frac{3 - 4\varepsilon}{D}}
\right) \\
& \quad + c \left( T^{-\frac{1}{2} - 3\varepsilon \|v\|^\frac{5}{2} - \varepsilon} + T^{-4\varepsilon \|v\|^\frac{3 - 4\varepsilon}{D}} + T^{-\frac{1}{2} + 3\varepsilon \|v\|^\frac{3 - 4\varepsilon}{D} + T^{-\frac{1}{2} - 3\varepsilon \|v\|^\frac{3 - 4\varepsilon}{D}} \right).
\] (4.24)
and

\[ |(v \nabla)_T(x) + 3C_2(v_T + 1_T)(x)| \]
\[ \leq c \|v\|_{L^1}^2 \frac{2-2\varepsilon}{T-1-2\varepsilon} \]
\[ + c2^1 \frac{1}{1-7}(d-\frac{1}{2}+3\varepsilon)\|v\|_{L^3}^{3} + d-\frac{1}{2}+5\varepsilon\|v\|_{L^3}^{2} \]
\[ + c\left(\|v\|_{L^3}^{-2} \frac{T-\frac{1}{2}-5\varepsilon}{T} + T^{-4}\|v\|_{L^3}^{3} \right) \]
\[ + c2^2 d^{-1} \|v\|_{L^1}^{2} \frac{T^{-2-2\varepsilon}}{T^{-2\varepsilon}}. \]

(4.25)

In this last estimate, we have used the triangle inequality to get \( \|v\|_{L^1} \) out of the left-hand side, and then used the assumption (4.1) to bound it.

### 4.4 Application of Lemma 2.7

We now go back to the original equation, and start to study large scale. We convolve equation (2.20) with \( \Psi_T \):

\[ \partial_t - \Delta (v)_T = -(v_T)^3 - 3(v^2)_T - 3(v \nabla)_T - 9C_2(v_T + 1_T) - (\Psi)_T \]
\[ + ((vT)^3 - (v^3)_T). \]

Lemma 2.7 implies that for all \( r > 0 \) and \( 0 < R' < R \) such that \( r + R < \frac{T}{2} \), we have

\[ \|(v)_T\|_{L^1} \leq \max \left\{ \frac{1}{R - R'}, \|(v)^3_T\|_{L^1}^\frac{1}{3}, \|(v^2)_T\|_{L^1}^\frac{1}{3}, \|(v \nabla)_T\|_{L^1}^\frac{1}{3}, \|(\Psi)_T\|_{L^1}^\frac{1}{3} \right\}. \]

(4.27)

The goal now is to balance the commutator and the renormalized powers of the noise term by choosing the parameter \( T \) appropriately. We first mention that applying (2.11) and then assumption (4.1) gives the bound

\[ \|v\|_{L^1} \leq \max \left\{ \frac{1}{R - R'}, T^{\frac{1}{2} - \varepsilon}, \|v\|_{L^1}^{\frac{1}{2} - \varepsilon}, \|(v^2)_T\|_{L^1}^\frac{1}{3}, \|(v \nabla)_T\|_{L^1}^\frac{1}{3} \right\}. \]

(4.28)
We need estimates on the commutator \((v_T)^3 - (v^3)_T\). This is easily obtained as \(v\) is \(C^{1-3\epsilon}\), using the moment bounds (2.10) and (2.11). For any \(z \in P_{r+R'}\),

\[
((v_T)^3 - (v^3)_T)(z) = \int \Psi_T(z - \overline{z}) \left((v_T(z)^3 - v(\overline{z})^3)\right) d\overline{z}
\]

\[
= \int \Psi_T(z - \overline{z}) \int_0^1 (v_T(z) - v(\overline{z}))^2 d\lambda d\overline{z}
\]

\[
\leq 3 \|v\|_{B(z,T)}^2 \int \Psi_T(z - \overline{z})(v_T(z) - v(z) + v(z) - v(\overline{z})) d\overline{z}
\]

\[
\leq 3 \|v\|_{B(z,T)}^2 \int \Psi_T(z - \overline{z}) \left(T^{\frac{1}{2} - 3\epsilon} + d(z, \overline{z})^{\frac{1}{2} - 3\epsilon} \right) \left[1\right]_{\frac{1}{2} - 3\epsilon, B(z,T)} d\overline{z}
\]

\[
\leq 6 \|v\|_{B(z,T)}^2 T^{\frac{1}{2} - 3\epsilon}[v]_{\frac{1}{2} - 3\epsilon, B(z,T)}.
\]

Since this is true for all \(z \in P_{r+R'}\),

\[
(4.29) \quad \|(v_T)^3 - (v^3)_T\|_{P_{r+R'}} \lesssim \|v\|_{P_{r+R'-T}}^2 T^{\frac{1}{2} - 3\epsilon}[v]_{\frac{1}{2} - 3\epsilon, P_{r+R'-T}, 2T}.
\]

In conclusion of this step,

\[
\|v\|_{P_{r+R}} \lesssim \max \left\{ \frac{1}{R - R'}, T^{\frac{1}{2} - 3\epsilon}[v]_{\frac{1}{2} - 3\epsilon, P_{r+R'-T}, 2T},
\right.
\]

\[
T^{\frac{1}{2} - \epsilon} \|v\|_{P_{r+R'-T}}^2 [v]_{\frac{1}{2} - 3\epsilon, P_{r+R'-T}, 2T},
\]

\[
(4.30) \quad \|(v^2)_T\|_{P_{r+R'}}^2, \|(v \Psi)_T\| + 3C_2(v_T + 1_T)\|_{P_{r+R'}}^2,
\]

\[
\|v\|_{P_{r+R}}^2 \frac{1}{P_{r+R'}} \frac{1}{P_{r+R'}} \right\}.
\]

### 4.5 Choice of Scale

We now apply assumption (4.1) with \(\tau = \Psi\) and the results from the previous steps, (4.22), (4.24), and (4.25), to equation (4.30). In (4.30) we choose

\[
R' = d_0 \quad \text{and} \quad T = \frac{d_0}{k}
\]

for some \(k > 2\) to be specified. Recall that \(d_0 = \frac{1}{\|v\|_{P_{r'}}}\), as set in (4.18). In the left-hand side of (4.22), (4.24), and (4.25), we make the particular choice

\[
d = d_0 \frac{k - 1}{k}.
\]
Since $k > 2$ we have $d \sim d_0$, and we also have $T + d = d_0$ so $\| v \|_{P_{r' + T}} = \| v \|_{P_{D'}}$. Equation (4.30) simplifies to

$$\| v \|_{P_{r + T}} \leq C \max \left\{ \frac{1}{R - R'}, \| v \|_{P_r}, \| v \|_{P_{r'}} k^{-\frac{1}{2} + \epsilon}, \| v \|_{P_{r'}} k^{-\frac{1}{2} + \epsilon}, \| v \|_{P_{r'}} k^{-\frac{1}{2} + \epsilon}, \| v \|_{P_{r'}} k^{-\frac{1}{2} + \epsilon}, \right\}$$

(4.31)

for some constant $C > 1$. We see that we can choose $k$ large and then impose another smallness condition on $c$ to get

$$\| v \|_{P_{r + T}} \leq \max \left\{ \frac{1}{R - R'}, \frac{1}{2} \| v \|_{P_{r}} \right\}$$

(4.32)

4.6 Iterating the Result

If we have $R \geq 2R'$, then we can rewrite equation (4.32) for $r = 0$ as

$$\| v \|_{P_R} \leq \frac{4C}{\| v \|_{P}}.$$  

(4.33)

The first argument of the maximum (4.33) is equal to the second one for

$$R = R_1 := \frac{4C}{\| v \|_{P}}.$$  

This is not in contradiction with $R \geq 2R' = \frac{2}{\| v \|_{P}}$ as $C > 1$. We now define a finite set $0 = R_0 < \cdots < R_N = \frac{1}{2}$ by setting

$$R_{n+1} - R_n = 4C \| v \|_{P_{R_n}}^{-1},$$

as long as the times $R_{n+1}$ defined this way stay strictly less than $\frac{1}{2}$. We terminate the sequence once this algorithm produces a $R_{n+1} \geq \frac{1}{2}$, in which case we set $R_{n+1} = R_N = \frac{1}{2}$, or once assumption (4.1) does not hold for $D = R_n$. Note that $4C \| v \|_{P_{R_n}}^{-1}$ is increasing in $n$, so the sequence necessarily terminates after finitely many steps. Equation (4.32) applied with $r = R_{n-1}$ for $n = 1, \ldots, N$ then gives the bounds for smaller and smaller parabolic boxes,

$$\| v \|_{P_{R + R_{n-1}}} \leq \max \left\{ \frac{2C}{R}, \frac{1}{2} \| v \|_{P_{R_{n-1}}} \right\}$$

(4.34)

hence for $R = R_n - R_{n-1}$,

$$\| v \|_{P_{R_n}} \leq \frac{1}{2} \| v \|_{P_{R_{n-1}}}.$$  

(4.35)

We now show that the bound (2.21) in Theorem 2.1 holds for all $R = R_n, n \in \{0, \ldots, N\}$. If assumption (4.1) does not hold for $D = R_N$, it is immediate, in the
other case for \( k \leq n, \|v\|_{P_{R_n}} \leq \|v\|_{P_{R_k}} 2^{k-n} \) and hence

\[
R_n = \sum_{k=0}^{n-1} R_{k+1} - R_k = \sum_{k=0}^{n-1} 4C \|v\|_{P_{R_k}} \leq 4C \|v\|_{P_{R_n}} \sum_{k=0}^{n-1} 2^{k-n} \leq \|v\|_{P_{R_n}}^{-1}.
\]

(4.36)

For the end point \( R_N \) we have either \( R_{N-1} \geq \frac{1}{4} \) or \( R_{N} - R_{N-1} \geq \frac{1}{4} \). In the first case we invoke (4.36) for \( n = N - 1 \), and in the second case we invoke the definition of \( R_{n+1} - R_n \), in both cases yielding a bound on \( \|v\|_{P_{R_{N-1}}} \). Finally, for values \( R \in (R_n, R_{n+1}) \), we use the definition of \( R_{n+1} - R_n \):

\[
R \leq R_{n+1} - R_n + R_n \leq \|v\|_{P_{R_{n+1}}} + R_n \leq \|v\|_{P_{R_{n}}}^{-1}.
\]

This concludes the proof of the theorem.

5 Proof of the Lemmas

5.1 Proof of Lemma 2.11

This proof is inspired from the one in [20, prop. 2], our contribution being the introduction of blowup at the boundaries of the domain instead of an assumption of periodicity.

Step 1. We claim that for all base points \( x \) and scales \( T, R, \) and \( L \) with \( R \leq L \) and such that \( B(x, L) \subset D \), it holds that

\[
\inf_l \|U_T(x, \cdot) - l\|_{B(x, R)} \leq \frac{R^2}{L^2} \inf_l \|U_T(x, \cdot) - l\|_{B(x, L)} + L^2 M^{(1)}_{\{x\}, L} \sum_{\beta \in A} T^{\beta-2} L^{k-\beta},
\]

(5.1)

where the infimum runs over all affine functions \( l(y) = C \cdot X(y-x) + c \). To prove this, we define a decomposition \( U_T(x, \cdot) = u_\ast + u_* \) where \( u_\ast \) is the solution to

\[
(\partial_t - \Delta) u_\ast = 1_{B(x, L)}(\partial_t - \Delta) U_T(x, \cdot).
\]

with Dirichlet boundary conditions. By standard estimates for the heat equation [15, cor. 8.1.5],

\[
\|u_\ast\|_{B(x, L)} \leq L^2 \| (\partial_t - \Delta) U_T(x, \cdot) \|_{B(x, L)} \leq L^2 M^{(1)}_{\{x\}, L} \sum_{\beta \in A} T^{\beta-2} L^{k-\beta}.
\]

(5.2)

As \( (\partial_t - \Delta) u_* = 0 \) on \( B(x, L) \) for \( \partial \in \{\partial_t, \partial_i \partial_j\} \) a differential operator of order 1 in time or 2 in space,

\[
\|\partial u_*\|_{B(x, R)} \leq L^{-2} \|u_* - l_*\|_{B(x, L)}
\]
for any affine function \( l_\succ \), where we used \( R \lesssim \frac{\ell}{2} \), and the fact that the differential operators used cancel the spatial linear functional. Next we define a concrete affine function \( l_\prec \) via \( l_\prec(y) := u_\prec(x) + \nabla u_\prec(x) \cdot X(y - x) \) and observe, using Taylor’s formula,

\[
\|u_\prec - l_\prec\|_{B(x, R)} \leq R^2 \|Du_\prec\|_{B(x, R)} \leq \frac{R^2}{L^2} \|u_\prec - l_\succ\|_{B(x, L)} \\
\leq \frac{R^2}{L^2} \|U_T(x, \cdot) - l_\succ\|_{B(x, L)} + \|u_\succ\|_{B(x, L)}.
\]

Using the triangle inequality once more and (5.2) gives

\[
\|U_T(x, \cdot) - l_\succ\|_{B(x, R)}
\leq \|u_\prec - l_\prec\|_{B(x, R)} + \|u_\succ\|_{B(x, R)}
\leq \frac{R^2}{L^2} \|U_T(x, \cdot) - l_\succ\|_{B(x, L)} + \|u_\succ\|_{B(x, L)}
\leq \frac{R^2}{L^2} \|U_T(x, \cdot) - l_\succ\|_{B(x, L)} + L^2 M^{(1)}_{\{x\}, L} \sum_{\beta \in A} T^{\beta - 2} L^{\beta - \beta},
\]

which implies (5.3).

**Step 2.** We claim that for all base points \( x \) and scales \( T \) and \( L \), it holds that

\[
(5.3) \quad \|U_T(x, \cdot) - U(x, \cdot)\|_{B(x, R)} \leq M^{(2)}_{\{x\}, R, T} \sum_{\beta \in A} R^\beta T^{\alpha - \beta} + T^\epsilon [U]_{x, B(x, R), T}.
\]

Indeed, since \( \Psi \) is symmetric, it integrates to 0 against linear functions; hence for any \( y \in B(x, R) \), we have

\[
[U_T(x, y) - U(x, y)]
= \left| \int \Psi_T(y - z) (U(x, z) - U(x, y)) dz \right|
= \inf_{v(y)} \left| \int \Psi_T(y - z) (U(x, z) - U(x, y) - v(y) \cdot X(z - y)) dz \right|
+ \int \Psi_T(y - z) (U(y, z) - v(y) \cdot X(z - y)) dz
\leq M^{(2)}_{\{x\}, R, T} \sum_{\beta \in A} d(y, x)^\beta \int \Psi_T(y - z) d(z, y)^{\epsilon - \beta} dz
+ \left( \sup_{y \in B(x, R)} \inf_{v(y)} \sup_{z \in B(y, T)} d(y, z)^{-\epsilon} |U(y, z) - v(y) \cdot X(z - y)| \right)
\times \int \Psi_T(y - z) d(z, y)^\epsilon dz.
\]
Step 3. We prove

\[
\sup_{R \leq \frac{d}{2}} \inf_{l} R^{-\kappa} \| U(x, \cdot) - l \|_{B(x, R)} \leq \sum_{\beta \in A} \left( M_{\{x\}, \frac{d}{2}}^{(1)} \epsilon^{-4+2\beta-\kappa} + M_{\{x\}, \frac{d}{2}}^{(2)} \epsilon^{-\kappa} \right) + \epsilon^{2-2\kappa} \frac{d^{-\kappa}}{2-\kappa} \| U(x, \cdot) \|_{B(x, \frac{d}{2})} \leq \frac{2}{2-\kappa} \leq \frac{2}{2-\kappa} \tag{5.1} \]

Multiplying equation (5.1) by \( R^{-\kappa} \) and fixing the length ratios \( R = \epsilon L = \epsilon^{-1} T \) for some \( \epsilon \leq \frac{1}{2} \) to be fixed below, we get for any point \( x \in D_d \) and length \( L \leq \frac{d}{2} \),

\[
R^{-\kappa} \inf_{l} \| U_T(x, \cdot) - l \|_{B(x, R)} \leq \epsilon^{2-\kappa} L^{-\kappa} \inf_{l} \| U_T(x, \cdot) - l \|_{B(x, L)} + \sum_{\beta \in A} M_{D_d, L}^{(1)} \epsilon^{-4+2\beta-\kappa}. \]

Taking the supremum over \( L \leq \frac{d}{2} \) while keeping the ratios \( R = \epsilon L = \epsilon^{-1} T \) fixed, we get

\[
\sup_{R \leq \frac{d}{2}} \inf_{l} R^{-\kappa} \| U_T(x, \cdot) - l \|_{B(x, R)} \leq \epsilon^{2-\kappa} \inf_{l} \| U_T(x, \cdot) - l \|_{B(x, L)} + \sum_{\beta \in A} M_{D_d, L}^{(1)} \epsilon^{-4+2\beta-\kappa}. \]

The last term is bounded by

\[
\epsilon^{2-\kappa} \left( \frac{\epsilon d}{2} \right)^{-\kappa} \| U_T(x, \cdot) \|_{B(x, \frac{d}{2})} \leq \epsilon^{2-2\kappa} \frac{d^{-\kappa}}{2-\kappa} \| U(x, \cdot) \|_{B(x, \frac{d}{2})} \leq \frac{2}{2-\kappa} \| U(x, \cdot) \|_{B(x, \frac{d}{2}) (1+\epsilon^2)} \tag{5.2} \]

Hence we have

\[
\sup_{R \leq \frac{d}{2}} \inf_{l} R^{-\kappa} \| U_T(x, \cdot) - l \|_{B(x, R)} \leq \epsilon^{2-\kappa} \inf_{l} \| U_T(x, \cdot) - l \|_{B(x, L)} + \sum_{\beta \in A} M_{D_d, L}^{(1)} \epsilon^{-4+2\beta-\kappa} + \epsilon^{2-2\kappa} \frac{d^{-\kappa}}{2-\kappa} \| U(x, \cdot) \|_{B(x, \frac{d}{2}) (1+\epsilon^2)}. \]
where the ratios between $L$ and $T$ and between $R$ and $T$ are fixed only within the supremum operators. Applying equation (5.3) gives

$$
sup_{R \leq \frac{c^2d}{2}} R^{-\kappa} \inf_{l} \| U(x, \cdot) - l \|_{B(x, R)} \leq \sup_{R \leq \frac{c^2d}{2}} R^{-\kappa} \inf_{l} \| U_T(x, \cdot) - l \|_{B(x, R)} + M^{(2)}_{\{x, \frac{c^2d}{2}, \frac{c^2d}{2}\}} \sum_{\beta \in A} e^{x-\beta} + e^{\kappa}[U]_{\kappa, B(x, \frac{c^2d}{2}, \frac{c^2d}{2})}$$

$$+ \sup_{R \leq \frac{c^2d}{2}} \inf_{l} \| U_T(x, \cdot) - l \|_{B(x, R)}$$

$$\leq \sup_{R \leq \frac{c^2d}{2}} \inf_{l} \| U_T(x, \cdot) - l \|_{B(x, R)}$$

$$\leq \sum_{\beta \in A} \left( M^{(1)}_{\{x, \frac{c^2d}{2}, \frac{c^2d}{2}\}, \frac{c^2d}{2}} e^{-4+2\beta-\kappa} + M^{(2)}_{\{x, \frac{c^2d}{2}, \frac{c^2d}{2}\}} e^{\kappa-\beta} \right)$$

$$+ \sup_{L \leq \frac{c^2d}{2}} \inf_{l} \| U(x, \cdot) - l \|_{B(x, L)}$$

The last term on the right-hand side can now be absorbed into the left-hand side for $\epsilon$ sufficiently small, giving the bound (5.4).

**Step 4.** We prove that

$$\sup_{d \leq d_0} d^{\kappa}[U]_{\kappa, d} \lesssim \sum_{\beta \in A} \left( M^{(1)}_{\{x, \frac{c^2d}{2}, \frac{c^2d}{2}\}, \frac{c^2d}{2}} e^{-4+2\beta-\kappa} + M^{(2)}_{\{x, \frac{c^2d}{2}, \frac{c^2d}{2}\}} e^{\kappa-\beta} \right)$$

$$+ (\epsilon^{-\kappa} + \epsilon^{2-2\kappa}) \sup_{d \leq d_0} \| U \|_{D_{\kappa, d}}.$$

We first argue that we can change the order of the supremum and the infimum in $\sup_{R \leq \frac{c^2d}{2}} R^{-\kappa} \inf_{l} \| U(x, \cdot) - l \|_{B(x, R)}$. Since $U(x, x) = 0$ it is clear that one can restrict to $l(x) = 0$; hence $l(y) = C(x, R) \cdot X(y - x)$. We argue that $C$ may be chosen independently of $R$. Let $C_R$ be the (near) optimal constant for the radius $R$. Then

$$R^{-(\kappa-1)} |C_R - C_R| \lesssim \sup_{R \leq \frac{c^2d}{2}} R^{-\kappa} \inf_{l} \| U(x, \cdot) - l \|_{B(x, R)}.$$


Since \( \kappa > 1 \), this can be extended by summation to all \( R \leq \frac{\epsilon d}{2} \), thus there exists a near optimal constant \( C \) independent of \( \rho \). We then have

\[
\inf_{\nu(x)} \sup_{x \neq y \in B(x, \frac{\epsilon d}{2})} d(x, y)^{-\kappa} |U(x, y) - \nu(x) \cdot X(y - x)| \\
\leq \inf_{l} \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \| U(x) - l \|_{B(x, R)} \\
\leq \sup_{R \leq \frac{\epsilon d}{2}} R^{-\kappa} \inf_{l} \| U(x) - l \|_{B(x, R)}.
\]

Therefore, if we take the supremum over \( x \in D_{d} \) in equation (5.4) and then multiply it by \( d^{\kappa} \) and take the supremum over \( d \), we get

\[
\sup_{d \leq d_{0}} d^{\kappa} \sup_{x \in D_{d}} \inf_{\nu(x)} \sup_{x \neq y \in B(x, \frac{\epsilon d}{2})} d(x, y)^{-\kappa} |U(x, y) - \nu(x) \cdot X(y - x)| \\
\leq \sup_{d \leq d_{0}} d^{\kappa} \sum_{\beta \in A} \left( \frac{M^{(1)}_{D_{d}, \frac{\epsilon d}{2}}}{d_{0}} \epsilon^{-4 + 2\beta - \kappa} + \frac{M^{(2)}_{D_{d}, \frac{\epsilon d}{2}}}{d_{0}} \epsilon^{\kappa - \beta} \right) \\
+ \epsilon^{2 - 2\kappa} \sup_{d \leq d_{0}} \| U \|_{D_{d}, d} \\
+ \epsilon^{\kappa} \sup_{d \leq d_{0}} d^{\kappa} \sup_{x \in D_{d}} \sup_{y \in B(x, \frac{\epsilon d}{2})} \left[ \inf_{y \neq z} \sup_{z \in B(y, \frac{\epsilon d}{2})} d(y, z)^{-\kappa} |U(y, z) - \nu(y) \cdot X(z - y)| \right].
\]

The last term can be absorbed into the left-hand side for \( \epsilon \) small enough since for \( y \in B(x, \frac{\epsilon d}{2}) \) we have \( d(y, \delta D) \geq d(1 - \frac{\epsilon}{2}) \) and consequently for \( z \in B(y, \frac{\epsilon d}{2}) \) we have \( d(y, z) \leq \frac{\epsilon d(y, \delta D)}{2(1 - \frac{\epsilon}{2})} \leq \frac{\epsilon d(y, \delta D)}{2} \), which gives

\[
\sup_{d \leq d_{0}} d^{\kappa} \sup_{x \in D_{d}} \inf_{\nu(x)} \sup_{x \neq y \in B(x, \frac{\epsilon d}{2})} d(x, y)^{-\kappa} |U(x, y) - \nu(x) \cdot X(y - x)| \\
\leq \sup_{d \leq d_{0}} d^{\kappa} \sum_{\beta \in A} \left( \frac{M^{(1)}_{D_{d}, \frac{\epsilon d}{2}}}{d_{0}} \epsilon^{-4 + 2\beta - \kappa} + \frac{M^{(2)}_{D_{d}, \frac{\epsilon d}{2}}}{d_{0}} \epsilon^{\kappa - \beta} \right) \\
+ \epsilon^{2 - 2\kappa} \sup_{d \leq d_{0}} \| U \|_{D_{d}, d}.
\]
We conclude the proof of (5.4) by extending to all \( y \in D_d \) with the following argument:

\[
\sup_{x \in D_d} \inf_{x \neq y \in B(x, \epsilon_d)} \sup_{x \in D_d} d(x, y)^{-\kappa} |U(x, y) - v(x) \cdot X(y - x)| \\
\leq \sup_{x \in D_d} \inf_{x \neq y \in B(x, \epsilon_d)} d(x, y)^{-\kappa} |U(x, y) - v(x) \cdot X(y - x)| \\
+ \sup_{x \in D_d} \inf_{x \neq y \in D_d \setminus B(x, \epsilon_d)} d(x, y)^{-\kappa} |U(x, y) - v(x) \cdot X(y - x)| \\
\leq \sup_{x \in D_d} \inf_{x \neq y \in B(x, \epsilon_d)} \frac{|U(x, y) - v(x) \cdot X(y - x)|}{d(x, y)^{\kappa}} + \left( \frac{\epsilon d}{2} \right)^{-\kappa} \|U\|_{D_d, d}.
\]

**Proof of Corollary 2.12**

From the definition of \([U]_{k,D}\) in (2.6) used with variables \( x, y \in D_d \) and with triangle inequalities, we get

\[
|v(x) \cdot X(y - x)| \leq [U]_{k,D} d(x, y)^{\kappa} + \|U\|_{D_d, d(x,y)}.
\]

Applying the interior cone condition for \( r \in [0, r_d] \) gives the existence of some \( y \) such that

\[
\lambda |v(x)| d(x, y) \leq [U]_{k,D} d(x, y)^{\kappa} + \|U\|_{D_d, r},
\]

which proves (2.37).

Using again the definition of \([U]_{k,D}\) with variables \( x, y \) and \( y, z \in D_d \) and with triangle inequalities, we get

\[
|U(x, y) - U(x, z) - U(z, y) - (v(x) - v(z)) \cdot X(y - z)| \\
\leq [U]_{k,D} (d(x, y)^{\kappa} + d(y, z)^{\kappa} + d(x, z)^{\kappa}).
\]

We combine this with the three-point continuity condition (2.35) and assume that

\[
r > d(x, z) = d(y, z) \geq \frac{d(x,y)}{2}
\]

to get

\[
|(v(x) - v(z)) \cdot X(y - z)| \leq d(x, z)^{\kappa} \left( [U]_{k,D} + M^{(2)}_{D_d, \frac{d}{\lambda}} \right).
\]

Choosing finally \( y \) such that \( |(v(x) - v(z)) \cdot X(y - z)| \geq \lambda |v(x) - v(z)||d(y, z)| \)
gives (2.38) for \( d(x, y) \leq r \). For \( d(x, y) \geq r \), we have

\[
d(x, y)^{-\kappa + 1} |v(x) - v(y)| \leq 2r^{-\kappa + 1} \|v\|_{D_d}.
\]

Applying (2.37) gives, for \( d(x, y) \geq r \),

\[
d(x, y)^{-\kappa + 1} |v(x) - v(y)| \leq [U]_{k,D} + r^{-\kappa} \|U\|_{D_d, r}.
\]
5.2 Proof of the Reconstruction

In this section we will use the following notations for $f$ a function of one variable and $F$ a function of two variables:

$$[F, (\cdot)T](x) = \int \Psi_T(x-y) F(x, y) dy.$$  \hspace{1cm} (5.5)

Proof of Theorem 2.8

This is the only place where our particular choice of convolution kernel is crucial. It allows the use of the following factorisation:

$$[F, (\cdot)T_{2^{-n}}](x_1) - ([F, (\cdot)T_{2^{-n-1}}]_{T_{2^{-n}}, 1}(x_1)]
= \left| \int \int \Psi_{T_{2^{-n-1}}}(x_2-y) \Phi_{T_{2^{-n-1}}}(x_1-x_2)(F(x_1, y) - F(x_2, y)) dy \, dx_2 \right|
\leq \sum_{\beta \in A} C_\beta \int \Phi_{T_{2^{-n-1}}}(x_1-x_2) d(x_1, x_2)^{\gamma n - \beta} (T_{2^{-n-1}})^{\beta} \, dx_2
\leq \sum_{\beta \in A} C_\beta (T_{2^{-n-1}})^{\gamma \beta}.

This proves the convergence of $[F, (\cdot)T_{2^{-n}}]$ to $f : y \mapsto F(y, y)$ and justifies the bound following the telescopic sum, obtained once more thanks to the semigroup property of our kernel:

$$\left| [F, (\cdot)T] - [F, (\cdot)T_{2^{-n}}]_{T, N-1} \right|
= \left| \sum_{n=0}^{N} (\left[ [F, (\cdot)T_{2^{-n}}] - [F, (\cdot)T_{2^{-n-1}}]_{T_{2^{-n}}, 1} \right]_{T, 0} \right|
\leq \sum_{n=0}^{N} \sum_{\beta \in A} C_\beta (T_{2^{-n-1}})^{\gamma \beta} \lesssim \sum_{\beta \in A} C_\beta T^{\gamma \beta},

where the constant in “$\lesssim$” depends only on $\gamma$ (in particular not on $N$), thus proving the theorem.

Proof of Lemma 2.9

To obtain a bound on $(v^2)T(y)$ we implement the following expansion:

$$(v^2)T(x) = (v^2)T(x) - v(x)^2 1_T(x) - 2v(x)((\nabla(x) - \nabla)1)T(x)
+ v(x)^2 1_T(x) + 2v(x)((\nabla(x) - \nabla)1)T(x).$$

From the bound (2.18) we have

$$|v(x)^2 1_T(x)| \leq \|v \|^2_{B(x, T)} [1]_{-\frac{1}{2}-\epsilon} T^{-\frac{1}{2}-\epsilon}$$  \hspace{1cm} (5.6)

$$|v(x)((\nabla(x) - \nabla)1)T(x)| \leq \|v \|_{B(x, T)} [\mathcal{R}]_{-4\epsilon} T^{-4\epsilon}.  \hspace{1cm} (5.7)$$
To bound the remaining part, we will apply Theorem 2.8. To that end we set

\[(5.8)\]
\[F(x_1, y) = v(x_1)^2 t(y) + 2v(x_1)(\Psi(x_1) - \Psi(y)) t(y).\]

Then

\[F(x_1, y) - F(x_2, y)\]
\[= (v(x_1) + v(x_2))(v(x_1) - v(x_2) + \Psi(x_1) - \Psi(x_2)) t(y)\]
\[+ (v(x_1) - v(x_2))(\Psi(x_1) - \Psi(x_2)) t(y)\]
\[+ 2(v(x_1) - v(x_2))(\Psi(x_1) - \Psi(y)) t(y).
\]

By definition of \(\Psi\) (2.18) this gives, for \(x_1, x_2 \in B(x, T - t)\),

\[
\left| \int \Psi_t(x_1 - y)(F(x_1, y) - F(x_2, y)) dy \right| \leq 2\|v\|_{D_d}[v + \Psi]_1 - 2\epsilon, B(x, T) d(x_1, x_2)^{1 - 2\epsilon} [\frac{1}{2} - \epsilon]^T \epsilon^{-\frac{1}{2}} - \epsilon
\]
\[+ [v]_{\frac{1}{2} - 3\epsilon, B(x, T)} [\Psi]_{\frac{1}{2} - 3\epsilon} d(x_1, x_2)^{1 - 6\epsilon} [\frac{1}{2} - \epsilon]^T \epsilon^{-\frac{1}{2}} - \epsilon
\]
\[+ [v]_{\frac{1}{2} - 3\epsilon, B(x, T)} d(x_1, x_2)^{\frac{1}{2} - 3\epsilon} [\Psi]_{-4\epsilon} [\epsilon^{-4\epsilon}].
\]

Hence by Theorem 2.8 we have the bound

\[
|v(x_1)^2 t(x) - v(x_2)^2 t(x) + 2v(x)((\Psi(x) - \Psi(y)) t(x))|
\leq T^{\frac{1}{2} - 3\epsilon} 2\|v\|_{D_d}[v + \Psi]_1 - 2\epsilon, B(x, T) [\frac{1}{2} - \epsilon]^T \epsilon^{-\frac{1}{2}} - \epsilon
\]
\[+ T^{\frac{1}{2} - 7\epsilon} v_{\frac{1}{2} - 3\epsilon, B(x, T)} [\Psi]_{\frac{1}{2} - 3\epsilon} [\frac{1}{2} - \epsilon]^T \epsilon^{-\frac{1}{2}} - \epsilon + [\Psi]_{-4\epsilon}.
\]

Together with bounds (5.6) to (5.7), we get Lemma 2.9.

**Proof of Lemma 2.10**

The last quantity we need to bound is

\[
((v - v(x))\Psi)_T(x) + 3C_2(v_T(x) + 1_T(x))
\]
\[= [\tilde{U} v(\cdot) T(x) - 3C_2(v - v_T)(x) - ((\Psi - \Psi(x))\Psi - 3C_2) 1_T(x)\]
\[- 3v(x)(\Psi - \Psi(x))\Psi - 3C_2) 1_T(x) + v(x) \cdot (X(y - x)\Psi)_T(x),
\]

where \(\tilde{U}(x, y) = U(x, y) - v(x) \cdot X(y - x)\) and \(v\) is optimal in the definition of \([U]_1 + \epsilon, D\). From the bounds (2.19), (2.17), and (2.16), we have

\[
|((\Psi - \Psi(x))\Psi - 3C_2) 1_T(x)| \leq [\Psi]_{\frac{1}{2} - 5\epsilon} T^{-\frac{1}{2} - 5\epsilon},
\]
\[|v(x)(\Psi - \Psi(x))\Psi - 3C_2) 1_T(y)| \leq \|v\|_{B(x, T)} [\Psi]_{-4\epsilon} T^{-4\epsilon},
\]
\[|v(x) \cdot (X(y - x)\Psi)_T(x)| \leq \|v\|_{B(x, T)} [\Psi]_{-2\epsilon} T^{-2\epsilon}.
\]
To bound the remaining part, we will apply Theorem 2.8; to that end we set
\[
F(x_1, y) = (v(x_1) + \Psi(x_1) - \Psi(y) + 3v(x_1)(\Psi(x_1) - \Psi(y)) \\
- v(x_1) \cdot X(x_1 - y)\Psi(y) - 3C_2(v(x_1) - v(y)).
\]
(5.12)

Then for \(x_1, x_2 \in B(x, T-t)\),
\[
F(x_1, y) - F(x_2, y) = (3(v(x_1) - v(x_2))((\Psi(y) - \Psi(x_2))\Psi(y) - C_2) \\
+ U(x_1, x_2)\Psi(y) - (v(x_1) - v(x_2)) \cdot X(y - x_2)\Psi(y).
\]

By definition of \(\Psi\) (2.17) and \(\Psi\) (2.16), this gives
\[
\int \Psi_{x_1 - y}(F(x_1, y) - F(x_2, y))dy \\
\leq 3[u]^{\frac{1}{2} - 3\varepsilon, B(x, T)}d(x_1, x_2)^{\frac{1}{2} - 3\varepsilon}[\Psi]^{-4\varepsilon} - 4\varepsilon \\
+ [U]^{\frac{1}{2} - 5\varepsilon, B(x, T)}d(x_1, x_2)^{\frac{1}{2} - 5\varepsilon}[\Psi]^{-1 - 2\varepsilon} - 1 - 2\varepsilon \\
+ [v]^{\frac{1}{2} - 5\varepsilon, B(x, T)}d(x_1, x_2)^{\frac{1}{2} - 5\varepsilon}[\Psi]^{-2\varepsilon} - 2\varepsilon.
\]

Hence by Theorem 2.8 we have the bound \(v\Psi\) such that
\[
\|\tilde{U}_\Psi, (\cdot)_T\|_{X} = 3C_2(v - v_T)(x)\| \\
(5.13) \leq T^{\frac{1}{2} - 7\varepsilon} [3[u]^{\frac{1}{2} - 3\varepsilon, B(x, T)}[\Psi]^{-4\varepsilon} + [U]^{\frac{1}{2} - 5\varepsilon, B(x, T)}[\Psi]^{-1 - 2\varepsilon} \\
+ [v]^{\frac{1}{2} - 5\varepsilon, B(x, T)}[\Psi]^{-2\varepsilon}].
\]

Together with bounds (5.10) to (5.11), we get Lemma 2.10.

### 5.3 Proof of Lemma 2.7

This proof is a version of the proof of theorem 4.4 in our companion paper, [18], specialised to cubic nonlinearity. This specialisation makes the proof significantly simpler. We only prove the bound for the positive part of \(u\). The bound for the negative part follows by symmetry. Let \(\eta\) be a continuous function defined on \(\mathbb{R}_+ \times [-1, 1]^3\), \(C^2\) and strictly positive on the interior and such that \(\eta = 0\) on the boundary. Either \(u\eta\) attains its maximum on \(\mathbb{R}_+ \times [-1, 1]^3\) at some point \(z_0 \in (0, 1) \times (-1, 1)^3\), or it is nonpositive, in which case \(u \leq 0\) in \([0, 1] \times [-1, 1]^3\). Assuming this is not the case, we get that at the maximum point, \(0 = \nabla (u\eta)(z_0)\), i.e.,
\[

\nabla u = - \frac{\nabla \eta}{\eta} u.
\]
(5.14)
If \( z_0 \in \{ 1 \} \times (-1, 1)^3 \), then \( \partial_t u \eta(z_0) \geq 0 \). Else, \( \partial_t u \eta(z_0) = 0 \). Additionally, \( \Delta u \eta(z_0) \leq 0 \) and therefore at the maximum we have

\[
0 \leq (\partial_t - \Delta)(u \eta) = \eta(\partial_t - \Delta)u + u(\partial_t - \Delta)\eta - 2\nabla u \cdot \nabla \eta \tag{5.14.2.20}
- \eta(u^3 - g(u, z)) + u \left( (\partial_t - \Delta)\eta + 2 \frac{[\nabla \eta]^2}{\eta} \right). \tag{5.14}
\]

Assume \( \eta \) satisfies the following inequality:

\[
(\partial_t - \Delta)\eta \eta + 2 \frac{[\nabla \eta]^2}{\eta^2} \leq \frac{1}{2\eta^2}. \tag{5.15}
\]

Then we get

\[
u^2 \leq \frac{\eta^{-2}}{2} + \frac{\|g\|}{u} \leq 2 \max \left\{ \frac{\eta^{-2}}{2}, \frac{\|g\|}{u} \right\}. \tag{5.16}
\]

If the maximum on the right-hand side is realised by the first term, then at \( z_0 \), \( u \eta \leq 1 \). If the maximum is realised by the second term, then it has to be bigger than the first one:

\[
\frac{\eta^{-2}}{2} \leq \frac{\|g\|}{u} \Rightarrow u \eta \leq 2\eta^3 \|g\|.
\]

We then have that at \( z_0 \), \( u \eta \leq 2 \) under the condition \( \eta \leq \|g\|^{-1/3} \). In both cases, we obtain that \( u \leq \frac{\eta}{\sqrt{\eta}} \) on all of \( (0, 1] \times (-1, 1)^3 \). With a choice of \( \eta \) also satisfying the inequality (5.15), we obtain good bounds on the function \( u \). We choose the following for \( \eta(x, \tau) = \frac{\lambda}{\lambda \|g\|^\frac{1}{3} + \frac{1}{\sqrt{\eta}} + \sum_{l=1}^{d} \frac{1}{1 + x_l} + \frac{1}{1 - x_l}} \), and we continuously extend with the value 0 on the boundary of the domain. This choice of \( \eta \) guarantees a bound on \( u \) that is related to the distance from the boundary of \( [0, 1] \times [-1, 1]^d \), independently of the boundary conditions. Indeed,

\[
\frac{2d + 1}{\lambda \min \{ \sqrt{\eta}, 1 + x_l, 1 - x_l \}} \geq \frac{1}{\eta} - \|g\|^\frac{1}{3} \geq \frac{1}{\lambda \min \{ \sqrt{\eta}, 1 + x_l, 1 - x_l \}}.
\]

It also satisfies \( 0 \leq \eta \leq \|g\|^{-1/3} \). We compute the derivatives to check (5.15),

\[
\partial_\tau \eta = \frac{1}{2\lambda \tau} \eta^2, \quad \partial_t \eta = \frac{1}{\lambda} \left( \frac{1}{(1 + x_l)^2} - \frac{1}{(1 - x_l)^2} \right) \eta^2,
\]

and

\[
\partial_t^2 \eta = -\frac{2x_l}{\lambda} \left( \frac{1}{(1 + x_l)^3} + \frac{1}{(1 - x_l)^3} \right) \eta^2 + \frac{2}{\lambda^2} \left( \frac{1}{(1 + x_l)^2} - \frac{1}{(1 - x_l)^2} \right)^2 \eta^3.
\]
Some of these terms cancel and we have the bound, using (5.18),
\[
2\eta (\partial_t - \Delta) \eta + 4|\nabla \eta|^2 = \frac{1}{\lambda t^2} \eta^3 + 4 \sum_{i=1}^{\lambda} \frac{x_i}{\lambda} \left( \frac{1}{(1 + x_i)^3} + \frac{1}{(1 - x_i)^3} \right) \eta^3 \leq 25\lambda^2.
\]
Therefore, by taking \( \lambda \leq \frac{1}{t^2} \), we have proved Lemma 2.7.

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