The Solitary Wave in Advanced Nuclear Energy System

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Abstract — The solitary wave naturally arises in many areas of mathematical physics, including in nonlinear optics, plasma physics, quantum field theory, and fluid mechanics. In the past few years, for an advanced nuclear energy system, a particular class of traveling wave reactor called the Constant Axial shape of Neutron flux, nuclide number densities and power shape During Life of Energy production (CANDLE) reactor has been proposed, and an analytical solution has been desired since it could reveal the global characters of the solution. In this study, from the perspective of the solitary wave, the analytical solution of this advanced nuclear energy system is demonstrated through coupling the one-group neutron diffusion equation with the burnup equation. The tanh-function method is applied to solve that nonlinear partial differential equation. The relationship between the velocity of the solitary wave, wave amplitude, or neutron flux and the evolution of the nuclide is revealed by the analytical method. The results demonstrate that the neutron flux is proportional to the wave velocity. The results also imply that the amplitude of the neutron flux is proportional to the square root of the diffusion coefficient but is inversely proportional to the initial $^{235}\text{U}$ density.

Keywords — Nonlinear system, solitary wave, CANDLE burnup.

Note — Some figures may be in color only in the electronic version.

I. INTRODUCTION

In 1834, Russell discovered the solitary wave in a nonlinear system, which he described in his “Report on Waves.” A solitary wave propagates without any temporal evolution in shape or size when it moves at a constant speed and conserves amplitude, shape, and velocity. A solitary wave naturally arises in many areas of mathematical physics, including in nonlinear optics, plasma physics, quantum field theory, and fluid mechanics. The classical example of an equation yielding solitary wave solutions is the Korteweg–De Vries equation, which is a model of waves on shallow water surfaces. The solitary wave also can be observed in an advanced nuclear energy system, i.e., in the traveling wave reactor class called the Constant Axial shape of Neutron flux, nuclide number densities and power shape During Life of Energy production (CANDLE) reactor. If the solitary wave is a solution of a nonintegrable equation, then it is not a soliton. The CANDLE traveling wave reactor strongly depends on in situ breeding and burning; therefore, there is no $N$-soliton solution for this system.

The concept of a completely automated nuclear reactor for long-term operation was proposed by Teller. This reactor core comprises an ignition region and a breeding region and is quite different from a conventional reactor. The neutrons leaking from the ignition region are captured by fertile fuel that is subsequently converted into fissile fuel in the breeding region. The breeding region is adopted with thorium material or depleted uranium, and 50% of the fertile material is utilized without any reprocessing. The self-stabilizing criticality waves in such a reactor were presented by Van Dam. The analytical model with reactivity feedback was illustrated through introducing a parabolic burnup function, which is the most simple form, and the ignition condition for a criticality wave was provided. Based on Van Dam’s model, a two-group diffusion model coupled with simplified burnup equations was investigated for a one-dimensional burnup drift wave problem.

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The feasibility of creating self-organizing breeding/burning waves was investigated by Fomin. At the starting phase, the requirements for wave initiation and evolution in space and time were discussed through a coupling diffusion transient model with burnup equations. A breed-and-burn strategy in a fast reactor with optimized starter fuel was devised by Huang. Optimization of the starter fuel was performed to reduce the initial positive excess reactivity swing and to flatten the power distribution, and the results show that broadening the ignition region is effective to reduce fuel enrichment and improve the operation performance during the starting phase.

The particular class of traveling wave reactor called the CANDLE reactor was first illustrated by Sekimoto et al. The CANDLE reactor could be fed by natural or depleted uranium and has a variety of attractive characteristics. The equilibrium state of the CANDLE burnup was discussed through solving numerically neutronic diffusion equations coupled with burnup equations. In the equilibrium state, the distributions of nuclide densities, neutron flux, and power density remain constant shapes, and the same constant speed remains for constant power operation. The equilibrium state of the CANDLE reactor can also be considered as a solitary wave formed by the neutron flux starting to propagate steadily.

For the solitary wave, if and only if the nonlinear effects cancel out dispersive effects can the solitary wave remain. The dispersive effects are represented by the leakage term in a nonlinear partial differential equation (PDE). Consequently, dealing with the nonlinear effects caused by neutron fission and absorption in the medium is key to producing and propagating the solitary wave. In this study, the nonlinear PDE was constructed through a coupling neutron diffusion equation with a burnup equation. Usually, an analytical solution is difficult to attain with a nonlinear PDE. However, if the nonlinear term in the above PDE was expanded by the Taylor series, then the tanh-function method could be applied to solve this PDE. The necessary boundary condition and simplification were adopted to obtain the analytical solution. The results show that the neutron fluxes, the neutron fluxes, and the evolution of the nuclide density can be presented as the form of a solitary wave. Even though numerical solutions were widely developed to solve the PDE, analytical solutions are still desired since the numerical approaches have to be recalculated for every set of parameters, but analytical approaches could reveal the global characters of the solution.

II. NONLINEAR PDE

II.A. Burnup Equations

For the heavy nuclides $I$, the general burnup equations could be written as

$$\frac{\partial N_I}{\partial t} = -\sigma_a^I \Phi N_I + \sigma_c^{I-1} \Phi N_{I-1} - \gamma_I N_I + \gamma_{I-1} N_{I-1},$$

where

$N_I =$ atomic density of nuclide $I$

$\Phi =$ neutron flux

$\sigma_a^I =$ microscopic absorption cross section of nuclide $I$

$\sigma_c^I =$ microscopic capture cross section of nuclide $I$

$\gamma_I =$ radioactive decay constant of nuclide $I$

$I =$ mother nuclide of $I$.

In an actual situation, for $^{238}\text{U}$$\rightarrow^{240}\text{Pu}$ conversion chains, the production of $^{239}\text{Pu}$ (e.g., by the neutron capture of $^{238}\text{U}$) and decay processes should be considered, but they are omitted here for the sake of simplicity. In the $^{238}\text{U}$$\rightarrow^{239}\text{Pu}$ conversion chain, some radioactive decay processes could be neglected, such as $^{239}\text{U}$ and $^{239}\text{Np}$, since inclusion of them makes little difference because of their short half-lives, 23.5 min and 2.35 days, respectively, compared to several years. Therefore, $^{238}\text{U}$$\rightarrow^{239}\text{U}$$\rightarrow^{239}\text{Np}$$\rightarrow^{239}\text{Pu}$$\rightarrow^{240}\text{Pu}$ conversion chains can be simplified as $^{238}\text{U}$$\rightarrow^{239}\text{Pu}$$\rightarrow^{240}\text{Pu}$ (or fission products). There would be some loss of accuracy even with this simplification, but this would not change the physical inheritance and would make the issue easier. For $^{238}\text{U}$, the burnup equation can be expressed as

$$\frac{\partial N_8}{\partial t} = -\sigma_a^8 \Phi N_8. \quad (2)$$

The solution of this differential equation is as follows:

$$N_8 = N_{8,0} e^{-\sigma_a^8 \Psi}, \quad (3)$$

where

$$\Psi = \int_{t_0}^{t} \Phi dt.$$
and where \( N_{8,0} \) is the initial atomic density of \(^{238}\text{U}\).

Similarly, for \(^{239}\text{Pu}\), burnable poisons (BPs), and fission products, the burnup equation can be expressed as

\[
\frac{\partial N_9}{\partial t} = -\sigma_{d9}\Phi N_9 + \sigma_{s8}\Phi N_8 ,
\]

(4)

\[
\frac{\partial N_{BP}}{\partial t} = -\sigma_{dBP}\Phi N_{BP} ,
\]

(5)

and

\[
\frac{\partial N_{FP}}{\partial t} = \sum_i \sigma_{fi} N_i \Phi .
\]

(6)

The solutions of these differential equations are the following:

\[
N_9 = N_{9,0} e^{-\sigma_{d9}\Psi} + N_{8,0} \sigma_{s8} \left( e^{-\sigma_{d9}\Psi} - e^{-\sigma_{d8}\Psi} \right),
\]

(7)

\[
N_{BP} = N_{BP,0} e^{-\sigma_{dBP}\Psi},
\]

(8)

and

\[
N_{FP} = \sum_i \sigma_{fi} N_i \Psi ,
\]

(9)

where \( N_{9,0} \) denotes the initial atomic density of \(^{239}\text{Pu}\) and nuclide \( i \) could be chosen as \(^{238}\text{U}\) and \(^{239}\text{Pu}\) for the fission reaction. The value of \( N_{9,0} \) can be equal to zero for the CANDLE reactor in the breeding region since the production of \(^{239}\text{Pu}\) is only through the neutron capture of \(^{238}\text{U}\). It should be emphasized that these solutions can be expressed as exponential functions even if \(^{240}\text{Pu}\) and \(^{241}\text{Pu}\) in the burnup chain are taken into account. The exponential functions can still be expressed as the Taylor series, which provides one way of analytically solving the nonlinear PDE.

II.B. Neutron Diffusion Theory

The one-group neutron diffusion model plays an important role in reactor theory even though it is sufficiently simple. This simple diffusion model coupled with the burnup equation is also sufficiently realistic to reveal the producing and propagating of the solitary wave. The one-group neutron diffusion can be expressed here\(^{19}\) as

\[
D \frac{\partial^2 \Phi}{\partial x^2} + \left( \nu \Sigma_f - \Sigma_a \right) \Phi = \frac{1}{v} \frac{\partial \Phi}{\partial t} ,
\]

(10)

where

\[
D = \text{neutron diffusion coefficient}
\]

\[
\Sigma_a = \text{macroscopic absorption cross section}
\]

\[
\nu = \text{average neutron number per fission}
\]

\[
\Sigma_f = \text{macroscopic fission cross section}
\]

\[
v = \text{neutron speed}.
\]

Furthermore,

\[
\Sigma_a = \sum_i N_i \sigma_{ai}, \quad \nu \Sigma_f = \sum_i \nu_i N_i \sigma_{fi},
\]

\[
D = \frac{1}{2 \Sigma_{ff}},
\]

where \( N_i \) could be substituted into the solution of the burnup equations, such as \( N_8, N_9, \) etc., and \( \Sigma_{ff} \) is the macroscopic transport cross section. In terms of Eqs. (7), (8), and (9), all the expressions of \( N_i \) contact with the factor \( e^{-\sigma_{d9}\Psi} \). Hence, the second term of Eq. (10) on the left side \( \left( \nu \Sigma_f - \Sigma_a \right) \) multiplied by \( \Phi \) is a nonlinear term in PDE. Only these nonlinear effects cancel out dispersive effects, and the solitary wave can propagate over large distances but without dissipation. The \( \left( \nu \Sigma_f - \Sigma_a \right) \) term is equal to

\[
\left( \nu \Sigma_f - \Sigma_a \right) = \left( \nu_0 N_9 \sigma_{f9} + \nu_8 N_8 \sigma_{f8} + \nu_0 N_0 \sigma_{f0} \right)
\]

\[
- \left( N_8 \sigma_{a8} + N_9 \sigma_{a9} + N_0 \sigma_{a0} \right)
\]

\[
+ N_{FP} \sigma_{aFP} + N_{BP} \sigma_{dBP} .
\]

(11)

All the \( N_i \) terms \((i = 238, 239, 240, \text{BP})\) are the function of the exponential form and can be expanded as the Taylor series. The \(^{239}\text{Pu}\) fissions caused by fast neutrons are dominant compared with \(^{238}\text{U}\) fissions and \(^{240}\text{Pu}\) fissions in the fast neutron spectrum. Therefore, \(^{238}\text{U}\) fissions and \(^{240}\text{Pu}\) fissions are neglected.

For the nuclides \(^{239}\text{Pu}, ^{240}\text{Pu}, \) and \(^{241}\text{Pu}, \) the capture cross section decreases sharply in the fast spectrum.\(^{20}\) The magnitudes of the capture cross sections of \(^{239}\text{Pu}, ^{240}\text{Pu}, \) and \(^{241}\text{Pu} \) are similar in the fast range. Therefore, the quantities of \(^{239}\text{Pu}, ^{240}\text{Pu}, \) and \(^{241}\text{Pu} \) could be used for balance of the importance. The study\(^{7}\) in which the complete conversion chain was kept shows that the magnitude of the nuclide number density of \(^{239}\text{Pu}\) is \( \sim 10\% \) of that of the initial \(^{238}\text{U}, ^{240}\text{Pu} \) is only \( \sim 2\% \) of that of the initial \(^{238}\text{U}, \) and of \(^{241}\text{Pu} \) is only \( \sim 2\% \) of that of the initial \(^{238}\text{U} \). Therefore, \(^{239}\text{Pu} \) is a dominant fissile nuclide
during these nuclides. Consequently, it is estimated that the impacts of the simplifications is about 20% through comparing $^{240}$Pu with $^{239}$Pu.

All the terms in Eq. (11) taken into account for the calculation are possible and can also improve the accuracy, but taking all the terms increases the complication of solving the nonlinear PDE. One of the convenient ways to consider the impact of fission products is that fission products are grouped into $^{238}$U by introducing a coefficient to revise the effective absorption cross section, but the total absorption reaction rates keep constant. Therefore, neutron absorption without fission products is adopted to derive the analytical solution concisely.

Appendix A shows in detail the terms $N_0\sigma_{eff}$ and $N_{FP}$ $\sigma_{eff}$ taken into account for the calculation. For the sake of simplicity, only the $\nu_0 N_0 \sigma_{eff}$, $N_0 \sigma_{rh}$ terms remain to clearly reveal the propagating of the solitary wave. Consequently, the nonlinear PDE can be rewritten as

$$D \frac{\partial^2 \Phi}{\partial x^2} + (\nu_0 N_0 \sigma_{rh} - N_0 \sigma_{rh}) \Phi = \frac{1}{v} \frac{\partial \Phi}{\partial t}$$  \hspace{1cm} (12)

and

$$F(\Psi) = (\nu \Sigma_f - \Sigma_a)$$

$$= N_0 \left[ \left( \frac{\nu_0}{\sigma_{rh}} - \frac{\sigma_{rh}}{\sigma_{eff}} \right) a^{- \sigma_{eff}} \Psi 
+ \left( \frac{\sigma_{rh}}{\sigma_{rh}} - \frac{\sigma_{rh}}{\sigma_{eff}} \right) a^{- \sigma_{eff}} \Psi \right]$$  \hspace{1cm} (13)

where $e^{-\sigma_{eff}} \Psi$ and $e^{-\sigma_{rh}} \Psi$ can be expanded by the second-order Taylor series and omit here the higher-order terms. This approximation could be available since the magnitude of $\sigma_{rh}$ is several barns and of $\Psi$ is 10$^{20}$ cm$^{-2}$ in a fast spectrum type of CANDLE reactor. For a thermal spectrum type of CANDLE reactor, this approximation could also be available since $\sigma_{rh} \Psi$ is still small even though $\sigma_{rh}$ increases but $\Psi$ decreases. Therefore, the value of $\sigma_{eff} \Psi$ is small so that $e^{-\sigma_{eff}} \Psi$ can be expanded by the second-order Taylor series:

$$e^{-\sigma_{eff}} \Psi = 1 + (-\sigma_{eff} \Psi) + \frac{1}{2} (-\sigma_{eff} \Psi)^2$$

and

$$e^{-\sigma_{rh}} \Psi = 1 + (-\sigma_{rh} \Psi) + \frac{1}{2} (-\sigma_{rh} \Psi)^2.$$  

Surely, adopting the second-order Taylor series would provide the higher accuracy for the smaller $\sigma_{eff} \Psi$.

III. ANALYTICAL SOLUTION AND DISCUSSION

III.A. Analytical Solution

In this section, the processes of pursuing the analytical solution of the nonlinear PDE are illustrated in detail. Usually, it is difficult to have an analytical solution for the nonlinear PDE. It has been discovered that this type of PDE would have an analytical solution by using the tanh-function method if the nonlinear terms are expanded by the Taylor series. Although analytical solution methods such as the inverse scattering transform, the homogeneous balance method, and the $\left( \frac{\xi}{\eta} \right)$-expansion method were developed, in most cases those analytical methods are difficult to handle and require a thorough knowledge of its properties and possibilities. The tanh-function method is a common powerful method for solving nonlinear equations, and it plays an important role in problems where reaction, dispersive effects, diffusion and/or convection.

According to the mentioned neutron diffusion theory, neutron diffusion equation could be written as

$$D \frac{\partial^2 \Phi}{\partial x^2} + F(\Psi) \Phi = \frac{1}{v} \frac{\partial \Phi}{\partial t}$$  \hspace{1cm} (14)

where $F(\Psi)$ has been defined as Eq.(13). The analytical solution of Eq.(14) would be performed as the following:

\textbf{Step I:} Use the traveling wave transformation,

$$\Phi(x, t) = \Phi(c(x - ut)) = \Phi(\eta), \quad (u > 0, \ c > 0)$$  \hspace{1cm} (15)

and

$$\frac{1}{v} \frac{\partial \Phi(x, t)}{\partial t} = - \frac{cu}{v} \frac{d\Phi(\eta)}{d\eta}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = c^2 \frac{d^2 \Phi(\eta)}{d\eta^2},$$  \hspace{1cm} (16)

where

$u$ = phase speed of wave

$c$ = wave numbers, and it represents wave with characteristic width $c^{-1}$

$\eta$ = coordinate.
Therefore, Eq. (14) can be transformed into the ordinary differential equation

\[ c^2 D \frac{d^2 \Phi(\eta)}{d\eta^2} + \frac{c u}{v} \frac{d \Phi(\eta)}{d\eta} + F(\Psi)\Phi(\eta) = 0 . \]  

(17)

**Step 2:** Assume that the solution of Eq. (17) has the form

\[ \Phi = \sum_{j=0}^{n} a_j T^j, \quad T = \tanh(\eta) , \]  

(18)

where

\[ a_j = \text{constant coefficient} \]

\[ T = \text{hyperbolic tangent function}. \]

Integer \( n \) depends on the balance between the highest-order derivatives and the nonlinear terms.

**Step 3:** The highest order of derivatives terms \( \frac{d^2 \Phi}{d\eta^2} \) is \( n + 2 \), and the highest order of nonlinear terms \( F(\Psi)\Phi \) is \( 2(n - 1) + n \). The proof is presented in Appendix B. Consequently, we obtain \( n = 2 \), and then \( \Phi \) could be expressed as

\[ \Phi = a_0 + a_1 T + a_2 T^2 . \]

Now, the boundary conditions \( \Phi(\pm \infty) = 0 \) are taken into account in this expression. Even though the neutron flux distribution of the CANDLE reactor is not always limited to the narrow region, in this research, only the practical case in which the CANDLE reactor is constrained in the narrow region was investigated. In the CANDLE reactor, when a solitary wave is limited to the narrow region rather than being distributed in the entire region, the value of \( \Phi(\pm \infty) \) should be zero since the neutrons have a finite mean free path in the medium. The research\(^7\) shows that if the core height is set at 8 m, then the characteristic of the CANDLE reactor could be obtained. If the core height is less than 8 m or the value of the \( \tanh \)-function is not small enough, the boundary condition would not be suitable as well.

By applying the boundary condition and according to Fig. 1, two expressions, \( \Phi(\infty) = a_0 + a_1 + a_2 = 0 \) and \( \Phi(-\infty) = a_0 - a_1 + a_2 = 0 \), could be obtained separately. Subtracting or adding the two expressions, the coefficients would be solved. Therefore, coefficient \( a_1 \) should be zero; not only this but also \( a_0 \) should be equal to \(-a_2\), and then the boundary conditions could be yielded. Consequently,

\[ \Phi = -a_2 + a_2 T^2, \quad (a_2 < 0) . \]  

(19)

Coefficient \( a_2 \) should be negative real numbers since the neutron flux should be positive and finite real numbers.

Neutron fluence \( \Psi \) can be obtained by the integral of \( \Phi \):

\[ \Psi = \int \Phi(\eta) dt = -\frac{a_2}{c u} (1 - T) . \]  

(20)

Appendix B shows the proof. Substitute Eq. (20) into Eq. (13), and the expression can be written as follows:

\[ F(\Psi) = N_{8,0} \left( (C_8 - C_9) + \frac{-a_2(1 - T)}{c u} (C_9 \sigma_{o9} - C_8 \sigma_{o8}) + \frac{1}{2} \left( \frac{-a_2(1 - T)}{c u} \right)^2 (C_8 (\sigma_{o8})^2 - C_9 (\sigma_{o9})^2) \right) , \]  

(21)

where

\[ C_8 = \left( \frac{0.9 \sigma_{o8} \sigma_{o9}}{\sigma_{o9} - \sigma_{o8}} - \sigma_{o8} \right) \]

and

\[ C_9 = 0.9 \frac{\sigma_{o8} \sigma_{o9}}{\sigma_{o9} - \sigma_{o8}} = C_8 + \sigma_{o8} . \]

**Step 4:** Substitute Eqs. (19), (20), and (21) into Eq. (17), and collect all the terms with the same power \( T^i \) (where \( i = 0, 1, 2, 3, 4 \)):

\[ c^2 (1 - T^2) \left[ -2T(a_1 + 2a_2 T) + 2a_2 (1 - T^2) \right] 
+ \frac{c u}{D v} (1 - T^2)(a_1 + 2a_2 T) + F(\Psi) \frac{\Phi}{D} = 0 , \]  

(22)
$T^3$ coefficient:

$$
\begin{align*}
T^3 & = -\frac{2cu_2}{Du} - \frac{a_1^2 C_8 \sigma_{d8} N_{8,0}}{c Du} - \frac{a_1^2 C_8 \sigma_{a8}^2 N_{8,0}}{c^2 Du^2} \\
& \quad + \frac{a_1^2 C_9 \sigma_{a9} N_{8,0}}{c Du} + \frac{a_1^2 C_9 \sigma_{a9}^2 N_{8,0}}{c^2 Du^2} \\
& = 0, \quad (a_1 = 0) ,
\end{align*}
$$

and

$$
T^4 \text{ coefficient:}

$$

$$
\begin{align*}
T^4 & = \frac{6c^2 a_2 + \frac{a_1^2 C_8 \sigma_{a8}^2 N_{8,0}}{2c^2 Du^2} - \frac{a_1^2 C_8 \sigma_{a8}^2 N_{8,0}}{2c^2 Du^2}}{a_2} = 0 ,
\end{align*}
$$

Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $a_n$ ($n = 0, 1, 2$). Because of $a_1$ determined by the boundary condition, the $T^3$ and $T^4$ coefficients of these polynomials were selected. Solving the equation system, we can construct a variety of analytical solutions for Eq. (22). Solving the $T^4$ and $T^3$ coefficients of these polynomials, Eqs. (23), (24), and (25) can be obtained separately:

$$
a_2 = -\frac{2\sqrt{3}Du c^2}{\sqrt{N_{8,0}} \sqrt{-C_8 \sigma_{a8} + C_9 \sigma_{a9}^2}} \quad (a_2 < 0) , \quad (23)
$$

$$
a_2 = \frac{2\sqrt{3}Du c^2}{\sqrt{N_{8,0}} \sqrt{-C_8 \sigma_{a8} + C_9 \sigma_{a9}^2}} \quad (a_2 > 0) , \quad (24)
$$

and

$$
a_2 = \frac{2c^2 u (-u + 6c Du)}{v (C_8 \sigma_{a8} - C_9 \sigma_{a9}) N_{8,0}} . \quad (25)
$$

Equation (24) should be rejected because $a_2$ should be less than zero to yield the boundary condition due to $C_9 > C_8$ and $\sigma_{a9} > \sigma_{a8}$. The solution of Eq. (25) could not always yield the boundary condition ($a_2 < 0$); therefore, it could be rejected.

Consequently, neutron flux $\Phi$ has the analytical solution

$$
\Phi(x, t) = \frac{2\sqrt{3}Du c^2}{\sqrt{N_{8,0}} \sqrt{-C_8 \sigma_{a8} + C_9 \sigma_{a9}^2}} \quad \left(1 - \tanh^2 (c(x - ut))\right) ,
$$

where $C_8$, $C_9$, and $C_\beta$ would be redefined as in Appendix A. Comparing with Eq. (26), one only needs to substitute $C_8$ for $C_8$ in Eq. (26), and then the impact of fission products could be taken into account automatically, where $C_8$ could be defined as follows:

$$
C_\delta = \left( C_8 \sigma_{a8}^2 + 2C_\beta \sigma_{a8} - 2C_\beta \sigma_{a9} \right)/\sigma_{a8}^2 .
$$

The results indicate that the differences are slight if $N_\sigma \sigma_{a9}$ and $N_{FP} \sigma_{aFP}$ are taken into account, but this would not change the trends.

For a given medium, Eq. (23) demonstrates that wave velocity $u$ is proportional to wave amplitude $a_2$ as a result of the nonlinear character of the wave, and Eq. (23) also shows that wave velocity $u$ is proportional to $N_{8,0}$ since $D$ is proportional to $1/N_{8,0}$ if the neutron flux is fixed. These conclusions are consistent with those of Van Dam and Sekimoto et al. The characteristic for a soliton is the wave velocity depending on the wave amplitude, which is kept constant.

What is more, Eq. (23) also reveals the relationship between the wave amplitude and the microscopic cross section of the nuclides $^{238}$U and $^{239}$Pu. Equation (23) explicitly shows that the neutron flux is independent of neutron speed $v$; however, both diffusion coefficient $D$ and the microscopic cross section depend on the neutron speed implicitly. The parameters of the solitary wave can be easily adjusted to meet the requirements in terms of Eq. (23) or Eq. (26). It should be mentioned that a critical reactor can operate at any flux level; hence, the magnitude of the flux depends on the power level of the core.

Free coefficient $c^2$ is still unknown and could be determined by thermal power output $P$ to normalize the neutron flux:

$$
P = \int_{-\infty}^{\infty} Q_f \sigma_{\gamma} \Phi(\eta) d\xi
$$

$$
= \frac{4\sqrt{3}Du \sigma_{\gamma}}{Q_f \sqrt{N_{8,0}} \sqrt{-C_8 \sigma_{a8} + C_9 \sigma_{a9}^2} - 2C_\beta \sigma_{a8} + 2C_\beta \sigma_{a9}} ,
$$

where $Q_f$ is the energy produced per fission (taken as 200 MeV).
The neutron fluence can be represented as

\[
\Psi(\eta) = \frac{-a_2}{cU} (1 - \tanh(\eta)) \\
= \frac{2\sqrt{3}Dc}{\sqrt{N_{8.0}} \sqrt{-C_8\sigma_{f8}^2 + C_9\sigma_{p8}^2 - 2C_{fp}\sigma_{f8} + 2C_{fp}\sigma_{p8}}} (1 - \tanh(\eta)).
\] (27)

III.B. Discussion

III.B.1. Profile of Neutron Flux and Neutron Fluence

It should be mentioned that Eqs. (23) and (26) will provide us with the profile of the neutron flux since free coefficient \( c \) is still unknown. The magnitude of the neutron flux or nuclide density should be determined by the thermal power output of the reactor core. For a given medium, Eq. (26) shows that the profile of the solitary wave would be lanky if the neutron flux is increased, which means the solitary wave would have a higher peak but be distributed in a smaller region due to increased wave number \( c \). Equation (26) also implies that the amplitude of the neutron flux is proportional to the square root of the diffusion coefficient but is inversely proportional to the initial \( ^{238}U \) density since diffusion coefficient \( D \) is inversely proportional to the initial \( ^{238}U \) density and other materials. This illustrates that reducing the fuel density increases the amplitude of the neutron flux, but the total reaction rate should stay the same if the power is constant; therefore, the changing makes the solitary wave lanky. The larger amplitude of the solitary wave is not desirable since it notably increases the power peaking factor. Oppositely, the lower fuel density would broaden the distribution of power. One of the concerns for the CANDLE reactor is that the high power peaking factor presents a great challenge for reactor safety. This study provides insight into resolving this issue.

Table I shows the selected parameters taken from Sekimoto et al.\(^7\) and Chen\(^27\) to apply and verify the equations. In Table I, even though diffusion coefficient \( D \) contains some information about the medium, in this simplified model, the BPs and coolant would be ignored. The term \( \Phi_{\text{max}} \) is the maximum neutron flux under a given condition, and here it is applied to one dimension rather than multiple dimensions. According to the provided parameters, wave number \( c \), which was considered the only free parameter, could be determined in terms of Eq. (26), and then, the profiles of the neutron flux, neutron fluence, and fuel density, from Figs. 2 through Fig. 6, could be drawn according to the analytical solutions. Figure 2 shows that the profile of the neutron flux is a bell-shaped solitary wave that is drifting as time goes on.

Equation (27) indicates that the neutron fluence is inversely proportional to the initial \( ^{238}U \) density, which denotes that the lower fuel density would have higher neutron fluence. Equation (27) also shows that the maximum neutron fluence has nothing to do with the wave speed and depends on the characteristics of the medium.

III.B.2. Profile of Fuel Burnup and Evolution of Nuclides

One of the notable merits for the CANDLE reactor is that the fuel burnup can be as high as 400 GWd/t loading with only depleted uranium or natural uranium in the breeding region. The fuel burnup is linearly proportional to the neutron fluence. Therefore, the burnup should have the same profile as Fig. 3. The trend of the burnup solitary wave profile coincides well with the previous study\(^13\) that was performed with the Monte Carlo method coupled ORIGEN burnup code. On the other hand, the

| Parameter | Value |
|-----------|-------|
| \( D \)   | 1.470 cm |
| \( \Phi_{\text{max}} \) | 3.0 \( \times \) 10\(^{15} \) cm\(^{-2} \) s\(^{-1} \) |
| \( N_{8.0} \) | 0.01221 \( \times \) 10\(^{22} \) cm\(^{-3} \) |
| \( \omega_f, \sigma_f, \sigma_{\text{n}}, \sigma_{\text{br}} \) | barn |
| \( ^{238}U \) | 0.142, 0.051, 0.404, 0.352, 8.181 (b) |
| \( ^{239}Pu \) | 5.878, 2.007, 2.481, 0.474, 8.593 (b) |
| Fission products | 0, 0, 0.4973, 0.4973, 11.92 (b) |
| \( u \) | 1.1 \( \times \) 10\(^{-7} \) cm/s |

TABLE I

Selected One-Group Parameters of Fast Neutron Spectrum
Fig. 2. Profile of neutron flux is bell-shaped solitary wave.

Fig. 3. Profile of neutron fluence is antikink solitary wave.

Fig. 4. Evolution of $^{238}$U as kink solitary wave.
very high fuel burnup also presents a great challenge due to fuel and cladding radiation damage for the nuclear engineering. This analytical solution of the solitary wave points out that the fuel burnup can be reduced through increasing the initial fuel density due to increasing neutron fluence.

The evolutions of nuclides play a vital role in maintaining the solitary wave. The solutions of the burnup equations reveal such evolutions. Applying the Taylor series, the nuclide $^{238}\text{U}$ can be written as

$$N_8 = N_{8,0} \left( 1 - \Psi \sigma_{a8} + \frac{1}{2} \Psi^2 \sigma_{a8}^2 \right). \quad (28)$$

Substituting Eq. (20) into Eq. (28), one obtains

$$N_8 = N_{8,0} \left[ 1 - \left( \frac{\sigma_{a8} a_2}{cu} - \frac{\sigma_{a8} a_2}{cu} \tanh(\eta) \right) + \frac{1}{2} \left( \frac{\sigma_{a8} a_2}{cu} - \frac{\sigma_{a8} a_2}{cu} \tanh(\eta) \right)^2 \right]. \quad (29)$$

Similarly, the evolution of $^{239}\text{Pu}$ can be expressed as

$$N_9 = N_{9,0} \frac{\sigma_{a9}}{\sigma_{a8} - \sigma_{a8}} \left[ \exp \left( -\frac{a_2 \sigma_{a8}}{cu} - \frac{a_2 \sigma_{a8}}{cu} \tanh(\eta) \right) - \exp \left( -\frac{a_2 \sigma_{a9}}{cu} - \frac{a_2 \sigma_{a9}}{cu} \tanh(\eta) \right) \right]. \quad (30)$$

The evolutions of the $^{238}\text{U}$ and $^{239}\text{Pu}$ profiles are displayed in Figs. 4 and 5. The trends of the evolutions of $^{239}\text{Pu}$ also match well with the previous studies.\textsuperscript{6,13,27} Even though
the analytical process introduces some simplifications that will cause relative error, it could be acceptable.

The results also could be checked and verified by dimensional analysis. Because of the lack of input data, the quantities of the density of $^{238}\text{U}$ and $^{239}\text{Pu}$ could not be checked and verified completely compared with previous studies; however, the trends of $^{238}\text{U}$ varying from $10^{22}/\text{cm}^3$ to $10^{21}/\text{cm}^3$ and $^{239}\text{Pu}$ varying from $10^{21}/\text{cm}^3$ to $10^{20}/\text{cm}^3$ could be checked and verified. The $^{238}\text{U}$ density depends on the microscopic absorption cross section and neutron fluence. The evolution of $^{239}\text{Pu}$, breeding from $^{238}\text{U}$, shows that there is a plutonium peak value toward the propagating direction. Comparing Fig. 4 with Fig. 5, the stored density of $^{239}\text{Pu}$ is approximately ten times less than the feeding $^{238}\text{U}$. In terms of Figs. 2 and 5, it can be found that the locations of the maximum neutron flux advance with the location of the maximum density of $^{239}\text{Pu}$. Figure 6 shows the evolution of the fission products, and its behavior is similar to the neutron fluence.

**IV. CONCLUSIONS**

The analytical solutions of an advanced nuclear energy system, i.e., the traveling wave reactor or the CANDLE reactor, are presented through coupling a one-dimensional, one-group neutron diffusion equation with a burnup equation, and the tanh-function method is employed to solve this nonlinear PDE. In order to obtain the analytical solutions, some necessary simplifications are adopted. Based on solid mathematics, the analytical solutions of neutron fluxes, neutron fluences, and the evolution of the nuclide density can be obtained and presented as different solitary waves. Finally, parameters are selected to apply and verify the analytical solutions.

The density of $^{238}\text{U}$ decreases sharply for the central region due to the neutron flux peak. The density of $^{239}\text{Pu}$ increases along burnup at the initial stage, but it later increases sharply and reaches the maximum value due to the neutron flux peak, and then, the density of $^{239}\text{Pu}$ reduces with the $^{238}\text{U}$. The results demonstrate that the neutron flux is proportional to the wave velocity but is inversely proportional to the fuel density and microscopic absorption cross section. The profile of the neutron flux is a bell-shaped solitary wave modified by the wave velocity, fuel density, and microscopic absorption cross section. In addition, the neutron fluence and the fuel density of $^{238}\text{U}$ and $^{239}\text{Pu}$ show as a solitary wave. The maximum neutron fluence is independent of the wave velocity and depends on the property of the medium. The analytical solutions clearly provide important insights into the physical phenomena of an advanced nuclear energy system. In the future, the analytical solutions of multidimensional with thermal-hydraulic feedbacks will be investigated to extend the scenarios. Moreover, the fundamental principle abstracted from in situ breeding and burning also could be extended to population changes, flame propagation, and so on.

**APPENDIX A**

This appendix shows how to obtain the analytical solution briefly of the absorption reaction of $^{239}\text{Pu}$ and absorption reaction of fission products.

For the calculation, $N_0\sigma_{a\theta} + N_{FP}\sigma_{a\text{FP}}$ was taken into account:

$$F(\Psi) = (\nu\Sigma_f - \Sigma_a) = \nu_0 N_0 \sigma_{f\theta} - N_0 \sigma_{a\theta} - N_{FP} \sigma_{a\text{FP}}$$

$$= N_{8,0} \left[ \left( \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} - \sigma_{a9} \right) e^{-\sigma_{a9} \Psi} - \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} e^{-\sigma_{a8} \Psi} \right]$$

$$- N_{8,0} \left[ \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} \left( e^{-\sigma_{a9} \Psi} - e^{-\sigma_{a8} \Psi} \right) - N_{8,0} \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} \left( e^{-\sigma_{a9} \Psi} - e^{-\sigma_{a8} \Psi} \right) \sigma_{f\theta} \sigma_{a\text{FP}} \right]$$

$$= N_{8,0} \left[ \left( \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} - \sigma_{a9} - \sigma_{a8} \right) e^{-\sigma_{a9} \Psi} - \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} e^{-\sigma_{a8} \Psi} + \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} \sigma_{f\theta} \sigma_{a\text{FP}} \right] e^{-\sigma_{a9} \Psi}$$

$$- N_{8,0} \left[ \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} \left( \frac{\sigma_{f\theta} - \sigma_{a9} - \sigma_{a8}}{\sigma_{a9} - \sigma_{a8}} \sigma_{f\theta} - \sigma_{a8} - \sigma_{a9} \right) e^{-\sigma_{a9} \Psi} - \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} \left( \frac{\sigma_{f\theta} - \sigma_{a9} - \sigma_{a8}}{\sigma_{a9} - \sigma_{a8}} \sigma_{f\theta} - \sigma_{a8} - \sigma_{a9} \right) e^{-\sigma_{a8} \Psi} \right]$$

$$= N_{8,0} \left[ \left( \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} - \sigma_{a9} - \sigma_{a8} \right) e^{-\sigma_{a9} \Psi} - \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} e^{-\sigma_{a8} \Psi} + \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} \sigma_{f\theta} \sigma_{a\text{FP}} \right] e^{-\sigma_{a9} \Psi}$$

$$- N_{8,0} \left[ \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} \left( \frac{\sigma_{f\theta} - \sigma_{a9} - \sigma_{a8}}{\sigma_{a9} - \sigma_{a8}} \sigma_{f\theta} - \sigma_{a8} - \sigma_{a9} \right) e^{-\sigma_{a9} \Psi} - \frac{\sigma_{c8}}{\sigma_{a9} - \sigma_{a8}} \left( \frac{\sigma_{f\theta} - \sigma_{a9} - \sigma_{a8}}{\sigma_{a9} - \sigma_{a8}} \sigma_{f\theta} - \sigma_{a8} - \sigma_{a9} \right) e^{-\sigma_{a8} \Psi} \right]$$

$$\approx 0$$

(A.1)
where \( F(\Psi) \), \( A \), \( B \), \( C_{FP} \), \( C_8 \), and \( C_9 \) are defined as follows:

\[
F(\Psi) = N_{8,0}(C_8A - C_9B - C_{FP}\Psi(1 - \sigma_{a8}\Psi) + C_{FP}\Psi(1 - \sigma_{a9}\Psi)) ,
\]

\[
A = 1 - \sigma_{a8}\Psi + \frac{1}{2}(\sigma_{a8}\Psi)^2 ,
\]

\[
B = 1 - \sigma_{a9}\Psi + \frac{1}{2}(\sigma_{a9}\Psi)^2 ;
\]

\[
C_{FP} = \frac{\sigma_{a8}}{\sigma_{a9} - \sigma_{a8}} \sigma_{a9} \sigma_{aFP} ;
\]

\[
C_8 = \frac{\sigma_{a8}}{\sigma_{a9} - \sigma_{a8}} \left( \psi_{a9} - \frac{\sigma_{a9} - \sigma_{a8}}{\sigma_{a8}} \sigma_{a8} - \sigma_{a9} \right) ;
\]

\[
C_9 = \frac{\sigma_{a9}}{\sigma_{a9} - \sigma_{a8}} \left( \psi_{a9} - \sigma_{a9} \right) ;
\]

\[
D \frac{\partial^2 \Phi}{\partial x^2} + F(\Psi)\Phi = \frac{1}{\nu} \frac{\partial \Phi}{\partial t} .
\]

Substitute \( F(\Psi) \) into Eq. (A.6), and apply the tanh-method:

\[
2c^2a_2 - \frac{a_2C_6N_{8,0}}{D} + \frac{a_2C_9N_{8,0}}{D} - \frac{a_2^2C_8\sigma_{a8}N_{8,0}}{cDu} - \frac{a_2^2C_{FP}\sigma_{a8}N_{8,0}}{c^2Du^2} - \frac{a_2^2C_8\sigma_{a8}^2N_{8,0}}{2c^2Du^2} + \frac{a_2^2C_9\sigma_{a9}N_{8,0}}{cDu} + \frac{a_2^2C_{FP}\sigma_{a9}N_{8,0}}{c^2Du^2} + \frac{a_2^2C_9\sigma_{a9}^2N_{8,0}}{c^2Du^2} - \frac{2a_2^3C_9\sigma_{a9}^3N_{8,0}}{c^2Du^3} + T^2 \left( -8c^2a_2 - \frac{a_2C_6N_{8,0}}{D} - \frac{a_2C_9N_{8,0}}{D} - \frac{a_2^2C_8\sigma_{a8}N_{8,0}}{cDu} - \frac{a_2^2C_{FP}\sigma_{a8}N_{8,0}}{c^2Du^2} - \frac{a_2^2C_8\sigma_{a8}^2N_{8,0}}{2c^2Du^2} + \frac{a_2^2C_9\sigma_{a9}N_{8,0}}{cDu} + \frac{a_2^2C_{FP}\sigma_{a9}N_{8,0}}{c^2Du^2} + \frac{a_2^2C_9\sigma_{a9}^2N_{8,0}}{c^2Du^2} - \frac{2a_2^3C_9\sigma_{a9}^3N_{8,0}}{c^2Du^3} \right) + 
\]

\[
T^3 \left( - \frac{2cua_2}{D} - \frac{a_2^2C_8\sigma_{a8}N_{8,0}}{cDu} - \frac{a_2^2C_{FP}\sigma_{a8}N_{8,0}}{c^2Du^2} - \frac{a_2^2C_8\sigma_{a8}^2N_{8,0}}{c^2Du^2} + \frac{a_2^2C_9\sigma_{a9}N_{8,0}}{cDu} + \frac{a_2^2C_{FP}\sigma_{a9}N_{8,0}}{c^2Du^2} + \frac{a_2^2C_9\sigma_{a9}^2N_{8,0}}{c^2Du^2} - \frac{2a_2^3C_9\sigma_{a9}^3N_{8,0}}{c^2Du^3} \right) + 
\]

\[
T^4 \left( 6c^2a_2 + \frac{a_2^2C_{FP}\sigma_{a8}N_{8,0}}{c^2Du^2} + \frac{a_2^2C_8\sigma_{a8}^2N_{8,0}}{2c^2Du^2} - \frac{a_2^2C_9\sigma_{a9}N_{8,0}}{c^2Du^2} - \frac{a_2^2C_{FP}\sigma_{a9}N_{8,0}}{2c^2Du^2} - \frac{a_2^2C_9\sigma_{a9}^2N_{8,0}}{2c^2Du^2} \right) = 0.
\]

Collecting again all terms with the same power \( T^j \) \( (j = 0,1,2,3,4) \) and performing some algebra, the solution can be obtained:

\[
a_2 = - \frac{2\sqrt{3}Du}{\sqrt{N_{8,0}^2 - C_8\sigma_{a8}^2 + C_9\sigma_{a9}^2 - 2C_{FP}\sigma_{a8} + 2C_{FP}\sigma_{a9}}} .
\]
APPENDIX B

The parameter $n$ would be determined by balancing the linear terms of highest order with the nonlinear terms.

For the hyperbolic tangent function $\tanh(\eta)$, it has the following formula:

$$\int \tanh^n(\eta)d\eta = -\frac{\tanh^{n-1}(\eta)}{n-1} + \int \tanh^{n-2}(\eta)d\eta .$$

Because of $\Phi = \sum_{j=0}^{n} a_j \tanh^j(\eta)$, the highest order of $\Phi$ is $n$.

The highest order of $\Psi$ is $O(\Psi)$:

$$O(\Psi) = O\left(\int \Phi dt\right) = O\left(-\frac{1}{u} \int \tanh^n(\eta)d\eta\right) = n - 1 .$$

Consequently, $O(\Psi^2) = 2(n - 1)$. After $F(\Psi)$ expands by the second-order Taylor series, we can obtain

$$O(\Phi) = 2(n - 1) .$$

Therefore, the highest order of nonlinear term $F(\Psi)\Phi$ is $2(n - 1) + n$. The highest order of derivatives term $\frac{d^2\Phi}{d\eta^2}$ is $n + 2$. It can be explained as follows:

$$T = \tanh(\eta) ,$$

$$\frac{d\Phi}{d\eta} = (1 - T^2) \frac{d\Phi}{dT} ,$$

and

$$\frac{d^2\Phi}{d\eta^2} = (1 - T^2) \left(-2T \frac{d\Phi}{dT} + (1 - T^2) \frac{d^2\Phi}{dT^2}\right) ,$$

where $\Phi = \sum_{j=0}^{n} a_j T^j$; that is, $O(\Phi) = n$. Thus, the following could be obtained.

$$O\left(\frac{d^2\Phi}{d\eta^2}\right) = n + 2 .$$

The highest order of the nonlinear terms and the highest order of the derivatives term should stay in balance; thus, $2(n - 1) + n = n + 2$. Consequently, $n$ should be equal to 2.

The neutron fluence is calculated as follows:

$$\Psi = \int \Phi(\eta')d\eta = \int_{-\infty}^{\eta} \left(-a_2 + a_2(\tanh(\eta))^2\right) d\eta$$

$$= -\frac{a_2}{cu} \int_{-\infty}^{\eta} \left(1 - (\tanh(\eta))^2\right) d\eta$$

$$= \frac{a_2}{cu} \int_{-\infty}^{\eta} \left(\text{Sech}^2(\eta')\right) d\eta'$$

$$= -\frac{a_2}{cu} (1 - \tanh(\eta)) .$$ (B.1)

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