Tracking Control for FES-Cycling based on Force Direction Efficiency with Antagonistic Bi-Articular Muscles

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Abstract—A functional electrical stimulation (FES)-based tracking controller is developed to enable cycling based on a strategy to yield force direction efficiency by exploiting antagonistic bi-articular muscles. Given the input redundancy naturally occurring among multiple muscle groups, the force direction at the pedal is explicitly determined as a means to improve the efficiency of cycling. A model of a stationary cycle and rider is developed as a closed-chain mechanism. A strategy is then developed to switch between muscle groups for improved efficiency based on the force direction of each muscle group. Stability of the developed controller is analyzed through Lyapunov-based methods.

I. INTRODUCTION

In the human body, coordinated firing of motor neurons activates skeletal muscles which generate torques about the body’s joints, and thereby, produce complex motions. Neurological disorders that damage the motor neurons can lead to paresis or paralysis and impaired motion. Specifically, people suffering from upper motor neuron disorders like stroke paresis or paralysis and impaired motion. Specifically, people suffering from upper motor neuron disorders like stroke, grasping and releasing etc. (1), recovering some functional motion (e.g., walking (1), standing (2), grasping and releasing (3), etc.).

Cycling induced by FES has been reported as physiologically and psychologically beneficial for people suffering from disorders affecting the muscles of the lower limbs (4); however, FES-cycling is metabolically inefficient and produces less power output than able-bodied cycling (5). Previous studies have used various design and control techniques to address these shortcomings. Chen et al. (6) used a model-free fuzzy logic controller for FES-cycling. Gfööhler and Lugner (7) considered an optimized stimulation pattern of leg muscles by FES. In (8), the influence of a number of individual parameters on the optimal stimulation pattern and power output during FES-cycling was investigated. Eser et al. (9) examined the relation between stimulation frequency and power output for cycling by trained SCI patients. Kim et al. (10) proposed a feedback control system for FES-cycling, focusing on automatically determining stimulation patterns for multiple muscle groups.

The aforementioned results provide useful methods for FES-cycling from a practical perspective, though explicit analysis of FES-cycling control from a theoretical point of view has been limited to linear approximations of the nonlinear cycle-rider system. Some recent studies (11)–(13) have focused on the development of RISE-based FES controllers and the associated analytical stability analysis for tracking of a human knee joint in the presence of a nonlinear uncertain muscle model with non-vanishing additive disturbances. However, these previous works have only considered knee joint dynamics.

In this paper, we consider tracking control for FES-cycling based on force direction efficiency derived from using antagonistic bi-articular muscles. Antagonistic bi-articular muscles, which pass over two adjacent joints and therefore act on the both joints simultaneously, are considered as one of the most important mechanisms of the human body associated with motion (14). Based on the antagonistic bi-articular muscle model, a stimulation pattern is derived for the gluteal, quadriceps femoris, hamstrings, and gastrocnemius muscle groups which aims to improve efficiency by maintaining a pedal force that is tangent to the pedal path. Considering the bi-articular muscle effects and controlling the pedal force direction may prove to increase FES-cycling power output and efficiency. A RISE-based controller and associated stability analysis are developed for an uncertain nonlinear cycle-rider system in the presence of an unknown time-varying disturbance, and semi-global asymptotic tracking of the desired trajectories is guaranteed provided sufficient control gain conditions are satisfied.

II. BICYCLE MODEL

A stationary cycle and rider can be modeled as a closed-chain mechanism (15). Consider a three degree-of-freedom (DOF) holonomic mechanical multi-body system $\Sigma$ as shown in Fig. 1 which consists of a collection of rigid bodies described as

$$\Sigma': M'(q')\ddot{q'} + C'(q', \dot{q'})\dot{q'} + g'(q') = 0, \quad (1)$$

where $q' = [q_1, q_2, q_3]^T \in \mathbb{R}^3$ represents the hip, knee, and crank angles, respectively, $M'(q') \in \mathbb{R}^{3 \times 3}$ is the inertia matrix, $C'(q', \dot{q'})\dot{q'} \in \mathbb{R}^3$ represents the centrifugal and Coriolis terms, and $g'(q') \in \mathbb{R}^3$ is the gravity term.

From Fig. 1 the scleronomic holonomic constraints are

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given by

\[ C : \phi(q') = \begin{bmatrix} l_1C_1 + l_2C_{12} - l_3C_3 - c_x \\ l_1S_1 + l_2S_{12} - l_3S_3 - c_y \end{bmatrix} = 0 \quad (2) \]

where \( l_i \ (i = 1, 2, 3) \) are the lengths of the thigh, shank and crank, \( c_x \) and \( c_y \) are the coordinates of the center of the crank, \( S_{ij} \) is defined as \( S_{ij} := \sin(q_i + q_j) \), and \( C_{ij} \) is defined as \( C_{ij} := \cos(q_i + q_j) \).

**Assumption 1.** From (2) and the physical relationships associated with the seated cyclist, the hip and knee angles are constrained to the regions \( \pi < q_2 < 2\pi \) and \( \pi < q_1 + q_2 < 2\pi \).

In the subsequent development, only the crank angle \( q_3 \) is assumed to be measurable. Hence, a parameterization for the generalized coordinates \( q \) is developed as

\[ q' \mapsto q = \alpha(q') = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} q'. \quad (3) \]

From Theorem 1 in [15], the equation of motion of the constrained system expressed in terms of the independent generalized coordinate \( q \) is obtained by combining

\[
\begin{cases}
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = 0 \\
\dot{q} = \mu(q')q' \\
q' = \sigma(q)
\end{cases}
\quad (4)
\]

to yield

\[ \Sigma : \quad M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \quad (5) \]

where \( \tau \in \mathbb{R} \) is the torque about the crank, \( \mu(q') \) is expressed by using the constraints in (2) and the parameterization in (3), and \( \sigma(q) \) can be derived by solving the constraints \( C \) in (2). Detailed expressions for \( \mu(q') \) and \( \sigma(q) \) are in Appendix A.

### III. INPUT FORCE

The human thigh model can be divided into three pairs of antagonistic muscles as depicted in Fig. 2 where two groups consist of antagonistic mono-articular muscles and one group consists of antagonistic bi-articular muscles. The antagonistic mono-articular muscles that span the hip joint consist of three extensor muscles denoted by \( e_1 \) and two flexor muscles denoted by \( f_1 \). The antagonistic mono-articular muscles that span the knee joint consist of a flexor muscle denoted by \( f_2 \) and three extensor muscles denoted by \( e_2 \). Antagonistic bi-articular muscles span both the hip and the knee joint and consist of \( f_{3e} \) and \( f_{3f} \) where \( f_{3e} \) flexes the hip and extends the knee, and \( f_{3f} \) extends the hip and flexes the knee.

The resulting force at the pedal depends on the combination of the active muscle forces. Moreover, as shown in Fig. 2 the directions of \( \vec{F}_{f1} \) and \( \vec{F}_{e1} \) coincide with the direction of the shank, the direction of \( \vec{F}_{f2} \) and \( \vec{F}_{e2} \) pass through the hip joint \( J_1 \) and the directions of \( \vec{F}_{f3e} \) and \( \vec{F}_{e3f} \) are parallel to the thigh. The torque produced at the joint(s) the muscle spans is defined as

\[ \tau_i := \Omega_iu_i, \quad \Omega_i := \zeta_i\eta_i\cos(a_i), \quad (6) \]

where \( \zeta_i \in \mathbb{R} \) denotes a positive moment arm with changes with the crank angle [16], [17]. \( a_i \in \mathbb{R} \) is defined as the pennation angle between the tendon and the muscle which changes with the crank angle [11]. \( \eta_i \in \mathbb{R} \) is an unknown function that relates the applied voltage to muscle fiber force which changes with the crank angle and velocity, and \( u_i \in \mathbb{R} \) is the control voltage input applied across each muscle group.

**Assumption 2:** The moment arm \( \zeta_i \) is assumed to be a positive, bounded, second order differentiable function such that its first and second time derivatives are bounded if \( q^k \in \mathbb{L}_\infty \), where \( q^k \) denotes the \( k \)th time derivative of \( q \) for \( k = 0, 1, 2 \) [16]. Similarly, the function \( \eta_i \) is assumed to be a positive, bounded, second order differentiable function such that its second time derivative is bounded if \( q^k \in \mathbb{L}_\infty \) for \( k = 0, 1, 2, 3 \) [18].

**Assumption 3:** For each bi-articular muscle, the torque acting on each of the two joints is assumed to be equal. The forces at the pedal \( F = [F_x \ F_y]^T \) are related to the joint...
Moreover, the joint torques can be represented as follows [19]:

\[
R_f = \begin{bmatrix}
\frac{1}{l_1 S_0} & \frac{1}{l_2 S_0} & \frac{1}{l_2 S_1} & \frac{1}{l_2 S_2}
\end{bmatrix}
\theta_{f1} = q_1 + q_2 - \pi,
\theta_{f2} = \tan^{-1}\left(\frac{l_1 S_1 + l_2 S_1}{l_1 C_1 + l_2 C_1}\right),
\theta_{f3} = q_1 - \pi,
\theta_{e1} = q_1 + q_2,
\theta_{e2} = \tan^{-1}\left(\frac{l_1 S_1 + l_2 S_1}{l_1 C_1 + l_2 C_1}\right),
\theta_{e3} = q_1.
\]

and \( \theta_i \) is the direction of the force at the pedal. Note that \( \theta_i \) and \( \theta_{e3} \) make use of the geometric relation

\[
S_0 = \sqrt{l_1^2 + l_2^2 + 2l_1 l_2 C_2}
\]

where \( g_0 = 2\pi - (q_1 + q_2 + \tan^{-1}(l_1 S_1 + l_2 S_1)/(l_1 C_1 + l_2 C_1)).
\]

While healthy individuals may be able to activate individual muscles during voluntary contractions, it is difficult to selectively activate individual muscles during external FES with transcutaneous electrodes if the muscles are in close proximity to each other (e.g., the pair of vastus intermedius, vastus lateralis and vastus medialis \( e_2 \) and rectus femoris \( f_{e3} \), the pair of biceps femoris short head \( f_2 \) and biceps femoris long head, semimembranosus and semitendinosus \( f_{e3} \)). Moreover, deep muscles (e.g., psoas major and iliacus \( f_1 \)) cannot be activated by transcutaneous stimulation without also activating the superficial muscles. Therefore, we consider the quadriceps femoris muscle group which contains \( e_2 \) and \( f_{e3} \), and the hamstrings muscle group which contains \( f_2 \) and \( e_3 \) as shown in Fig. 3. Additionally, the gastrocnemius, \( f_3 \), is a flexor muscle for the knee joint and is used to modify the direction of force. Hereafter, we consider the following four muscle groups: Gluteus Maximus, Hamstrings, Gastrocnemius, and Quadriceps.

The forces acting at the pedal for each muscle group are expressed as

\[
\vec{F}_{\text{Glut}} = \vec{F}_{e1},
\vec{F}_{\text{Ham}} = \vec{F}_{f2} + \vec{F}_{e3},
\vec{F}_{\text{Gast}} = \vec{F}_{f4},
\vec{F}_{\text{Quad}} = \vec{F}_{e2} + \vec{F}_{e3},
\]

where \( \vec{F}_{f4} \) is similar to \( \vec{F}_{f2} \) (i.e., \( R_{f2} = R_{f2} \) and \( \theta_{f4} = \theta_{f2} \)). The crank torque can be expressed in terms of the muscle forces as

\[
\tau = (\vec{F}_{\text{Glut}} + \vec{F}_{\text{Ham}} + \vec{F}_{\text{Gast}} + \vec{F}_{\text{Quad}}) \times \vec{r}_3
\]

\[
d - M_c(q) + M_v(q)
\]

where \( M_c(q) \in \mathbb{R} \) and \( M_v(q) \in \mathbb{R} \) are elastic [20] and viscous moments [21] defined as

\[
M_c(q) = \mu(q)^T \begin{bmatrix}
-k_{11}e^{-k_{12}q_1}(q_1 - k_{13})
-k_{21}e^{-k_{22}q_2}(q_2 - k_{23})
0
\end{bmatrix}
\]

\[
M_v(q) = \mu(q)^T \begin{bmatrix}
b_{11} \tanh(-b_{12}q_1) - b_{13}q_1
b_{21} \tanh(-b_{22}q_2) - b_{23}q_2
0
\end{bmatrix}
\]
where $k_{11}, \ldots, k_{23} \in \mathcal{R}$ and $b_{11}, \ldots, b_{23} \in \mathcal{R}$ are unknown constants, $l_3$ is defined as

$$l_3 = l_3 \begin{bmatrix} C_3 \\ S_3 \end{bmatrix},$$

(26)

and $d$ is an unknown bounded disturbance from unmodeled dynamics. Combining (5) and (23) yields

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \left(\sum_{i \in S} \bar{\Omega}_i u_i \times l_3\right) - d + M_e(q) + M_e(\dot{q})$$

(27)

where $S = \{\text{Glut, Ham, Gast, Quad}\}$ and

$$\bar{\Omega}_{\text{Glut}} := R_{e1}\Omega_{e1} \begin{bmatrix} C_{12} \\ S_{12} \end{bmatrix},$$

$$\bar{\Omega}_{\text{Ham}} := R_{f2}\Omega_{f2} \begin{bmatrix} C_{\theta f2} \\ S_{\theta f2} \end{bmatrix} - R_{ef3}\Omega_{ef3} \begin{bmatrix} C_1 \\ S_1 \end{bmatrix},$$

$$\bar{\Omega}_{\text{Gast}} := R_{f4}\Omega_{f4} \begin{bmatrix} C_{\theta f4} \\ S_{\theta f4} \end{bmatrix},$$

$$\bar{\Omega}_{\text{Quad}} := R_{e2}\Omega_{e2} \begin{bmatrix} C_{\theta e2} \\ S_{\theta e2} \end{bmatrix} + R_{f e3}\Omega_{f e3} \begin{bmatrix} C_1 \\ S_1 \end{bmatrix}.$$  

(29)

Given the natural muscle redundancy, a transformation is developed as

$$u_i = \chi_i u, \text{ } i \in S$$

(32)

where $u \in \mathcal{R}$ is the control input, and $\chi_i \in [0, 1]$ is the designed activation ratio used to control force direction. The position of the pedal exists inside of the quadrilateral which is constructed by the force directions of the four muscle groups as shown in Fig. 4, and thus, the resulting force can be selected to be in any direction by altering the relative activation of the muscle groups. Because there exists infinitely many combinations by which three or more muscle groups can result in the same desired force direction, only two muscle groups are activated at any given time in this approach. The designed activation ratios are selected to satisfy the following relationships:

$$\chi_i + \chi_j = 1, \chi_k = 0, \chi_l = 0, \sin \theta = 1$$

(33)

where $(i, j) \in \{(\text{Glut, Ham}), (\text{Ham, Gast}), (\text{Gast, Quad}), (\text{Quad, Glut})\}$ and $(k, l) \in S \neq i, j,$ and $\theta$ is the angle between the direction of the combination of the muscle forces $\sum_{i \in S} \chi_i \Omega_i$ and the crank $l_3$. The constraint on $\theta$ in (33) is designed such that the resulting combination of muscle forces is tangent to the crank. In other words, $\chi_i$ is designed to improve the efficiency of the cycling by ensuring that the resulting combination of muscle forces only contributes to the forward movement of the crank. By using (32), (27) can be expressed as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \left(\sum_{i \in S} \chi_i \bar{\Omega}_i u_i \right) - d + M_e(q) + M_e(\dot{q})$$

(34)

where $\Omega_\chi = \left\| \sum_{i \in S} \chi_i \bar{\Omega}_i \right\| l_3.$

To design $\chi_i$ and satisfy the constraint on $\theta$ in (33), the magnitude and direction must be known for $\Omega_{\text{Glut}}, \Omega_{\text{Ham}}, \Omega_{\text{Gast}},$ and $\Omega_{\text{Quad}}.$ The directions of $\bar{\Omega}_{\text{Glut}}$ and $\bar{\Omega}_{\text{Gast}}$ can be obtained analytically as a function of the crank angle $\theta$. However, $\bar{\Omega}_{\text{Ham}}$ and $\bar{\Omega}_{\text{Quad}}$ consist of multiple muscles where the force directions are known but the relative magnitudes of the forces are unknown, and thus, the directions of $\bar{\Omega}_{\text{Ham}}$ and $\bar{\Omega}_{\text{Quad}}$ have to be estimated numerically from experimental data. Further, the relative magnitudes of $\bar{\Omega}_{\text{Glut}}, \bar{\Omega}_{\text{Ham}}, \bar{\Omega}_{\text{Gast}},$ and $\bar{\Omega}_{\text{Quad}}$ are unknown functions of the crank angle and crank velocity, and thus, the activation ratio $\chi_i$ must be designed based on experimental data.

**Assumption 4**: The first and second partial derivatives of $\chi_i$ with respect to the crank angle and crank velocity are assumed to exist and are bounded. Thus from (28), (31) and Assumption 2, the first and second partial derivatives of $\Omega_\chi$ are bounded such that $q^k \in L_\infty$ for $k = 0, 1, 2, 3,$ and $\Omega_\chi$ is assumed to be a bounded function.

From Assumption 2, $\Omega_{\chi}, i \in T', T' := \{e_1, e_2, f_2, f e_3, f e_3, f_3, f_4\}$ is bounded such that $\xi_i > \Omega_\chi > \varepsilon_i > 0, i \in T'$ where $\xi_i, \varepsilon_i \in \mathcal{R}$ are positive constants. Further, from Assumptions 1 and 2, $\Omega_\chi$ is bounded such that $\Omega_{\chi} > \Omega_\chi > \varepsilon_\chi > 0$ where $\Omega_{\chi}, \varepsilon_\chi \in \mathcal{R}$ are positive constants.

**IV. STABILITY ANALYSIS**

The position error is defined as

$$e_1 = q_d - q$$

(35)

where $q_d$ is the desired crank angle which is designed such that $q_d, q^k_d \in L_\infty,$ where $q^k_d$ denotes the $k$th time derivative of $q_d$ for $k = 1, 2, 3, 4.$ To facilitate the subsequent analysis, the filtered tracking errors $e_2, r \in \mathcal{R}$ are defined as

$$e_2 = e_1 + \alpha_1 e_1$$

(36)

$$r = \dot{e}_2 + \alpha_2 e_2$$

(37)

where $\alpha_1, \alpha_2 \in \mathcal{R}$ are selectable positive constants. By using (35)–(37), the crank dynamics in (34) can be transformed as follows

$$M(q)\ddot{r} = M(q)(\ddot{q}_d + \alpha_1 \dot{e}_1 + \alpha_2 e_2) + C(q, \dot{q})\dot{q}$$

$$- M_e(q) + M_e(\dot{q}) + d - \Omega_{\chi} u$$

$$W + d - \Omega_{\chi} u$$

(38)

where $W$ is defined as

$$W := M(q)(\ddot{q}_d + \alpha_1 \dot{e}_1 + \alpha_2 e_2) + C(q, \dot{q})\dot{q}$$

$$- M_e(q) + M_e(\dot{q}) + g(q).$$

(39)

After multiplying (38) by $\Omega_{\chi}^{-1}$, the following dynamics can be obtained.

$$M_{\Omega}\ddot{q} + q_d = W_\Omega - u + d_{\Omega}$$

(40)

2 Analytic solutions of $q_3$ at $\chi_{\text{Glut}} = 1$ and $\chi_{\text{Gast}} = 1$ are shown in Appendix B.
where $M_{\Omega}(q, \dot{q})$, $W_{\Omega}$ and $d_{\Omega}$ are defined as

\[ M_{\Omega}(q, \dot{q}) := \Omega^{-1}_{\alpha} M(q) \]

\[ W_{\Omega} := \Omega^{-1}_{\alpha} W \]

\[ = M_{\Omega}(q, \dot{q})(\dot{q} = \alpha_1 e_1 + \alpha_2 e_2) + C_{\Omega}(q, \dot{q}) \]

\[-M_{\xi}(q, \dot{q}) - M_{\xi}(q, \dot{q}) + g_{\Omega}(q, \dot{q}) \]

\[ d_{\Omega} := \Omega^{-1}_{\alpha} d. \]

From Assumptions 1, 2, 4 and the facts that $M \leq M_{\Omega} \leq \overline{M}$ where $M$ and $\overline{M}$ are positive constants, we have that

\[ M_{\Omega} \leq M_{\Omega} \leq \overline{M}_{\Omega}, \quad (41) \]

where $M_{\Omega}$, $\overline{M}_{\Omega} \in \mathcal{R}$ are positive constants. Also, the following auxiliary terms are defined:

\[ S_d := M_{d\Omega}\dot{q}_d + C_{d\Omega}\dot{q}_d - M_{d\Omega} + g_{d\Omega} + d_{d\Omega} \]

\[ M_{d\Omega} := M_{d\Omega}(q, \dot{q}_d), \quad C_{d\Omega} := C_{d\Omega}(q, \dot{q}_d) \]

\[ M_{\xi d} := M_{\xi d}(q, \dot{q}_d), \quad M_{\xi d} := M_{\xi d}(q, \dot{q}_d) \]

\[ g_{d\Omega} := g_{d\Omega}(q, \dot{q}_d), \quad d_{d\Omega} := d_{d\Omega}(q, \dot{q}_d). \]

To facilitate the stability analysis, the time derivative of $d_{\Omega}$ can be determined as

\[ M_{\Omega}(q, \dot{q}) \dot{r} = -\dot{M}_{\Omega}(q, \dot{q}) r + \dot{W}_{\Omega} - \dot{u} + \dot{d}_{\Omega} \]

\[ = -\frac{1}{2} \dot{M}_{\Omega}(q, \dot{q}) r + N - \dot{u} - e_2 \]

\[ = -\frac{1}{2} \dot{M}_{\Omega}(q, \dot{q}) r + \tilde{N} + N_d - \dot{u} - e_2 \quad (42) \]

where $N$, $N_d$ and $\tilde{N} \in \mathcal{R}$ denote the following unmeasurable auxiliary terms

\[ N := \dot{W}_{\Omega} + e_2 - \frac{1}{2} \dot{M}_{\Omega}(q, \dot{q}) r + \dot{d}_{\Omega} \]

\[ N_d := \dot{S}_d \]

\[ \tilde{N} := N - N_d. \]

By applying the Mean Value Theorem, $\tilde{N}$ can be upper bounded by state-dependent terms as

\[ \|\tilde{N}\| \leq \rho(\|z\|) \|z\| \quad (43) \]

where $z \in \mathcal{R}^3$ is defined as

\[ z := [e_1 \ e_2 \ r]^T \quad (44) \]

and $\rho(\|z\|)$ is some positive, nondecreasing function [22]. By the design of the desired trajectory, $N_d$ can be upper bounded as

\[ \|N_d\| \leq \zeta_{N_d}, \quad \|\dot{N}_d\| \leq \zeta_{\dot{N}_d}, \quad (45) \]

where $\zeta_{N_d}, \zeta_{\dot{N}_d} \in \mathcal{R}$ are known positive constants.

The voltage control input is designed as [11]

\[ u = (k_s + 1)(e_2 - e_2(0)) + \nu \]

\[ \dot{v} = (k_s + 1)\alpha_2 e_2 + \beta \text{sgn}(e_2), \quad v(0) = v_0 \quad (47) \]

where $\nu$ is the generalized Filippov solution to $\dot{v}$, $v_0$ is some initial condition, $k_s$, $\beta \in \mathcal{R}$ are positive, constant control gains, and $\text{sgn}(\cdot)$ denotes the signum function.

To facilitate the subsequent stability analysis, $y$ and $Q$ are defined as

\[ y := \begin{bmatrix} z \ \sqrt{P} \end{bmatrix}, \quad Q := \begin{bmatrix} \alpha_1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \alpha_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (48) \]

where $P \in \mathcal{R}$ is the Filippov solution to

\[ \dot{P} = -r^T (N_d - \beta \text{sgn}(e_2)), \quad (49) \]

\[ P(0) = \beta |e_2(0)| - e_2(0) N_d(0). \quad (50) \]

\[ \text{Theorem 1:} \quad \text{The control law of (46) yields semi-global tracking in the sense that} \]

\[ |e_1| \to 0 \text{ as } t \to \infty \quad (51) \]

for the region of attraction $\mathcal{D}_z$

\[ \mathcal{D}_z = \left \{ y | \rho \left( \sqrt{\frac{\lambda_2}{\lambda_1}} \|y\| \right) < 2\sqrt{\lambda_{\min}(Q)} k_s \right \} \quad (52) \]

where $\lambda_1 := \frac{1}{2} \min \{1, M_{\Omega} \}$, $\lambda_2 := \max \{ \frac{1}{2} \overline{M}_{\Omega}, 1 \}$, and $\lambda_{\min}(Q)$ denotes the minimum eigenvalue of $Q$, provided $\alpha_1$, $\alpha_2$, $\beta$ and $k_s$ are selected according to the following sufficient conditions:

\[ \alpha_1 \alpha_2 > \frac{1}{4}, \quad (53) \]

\[ \beta > \left( \zeta_{N_d} + \frac{1}{\alpha_2} \zeta_{\dot{N}_d} \right), \quad (54) \]

\[ k_s > \frac{1}{4 \lambda_{\min}(Q)} \rho(\|z(0)\|)^2, \quad (55) \]

where $\zeta_{N_d}$ and $\zeta_{\dot{N}_d}$ were introduced in (45).

\[ \text{Proof:} \quad \text{The proof for Theorem 1 closely follows the proof given in [13]. The proof details are provided in Appendix C.} \]
Differentiating (56) with respect to time yields
\[ \psi_q(q) \dot{q} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \dot{q} \] (57)
where
\[ \psi_q(q') := \partial \psi(q') \partial q' = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} - l_2 S_{12} & l_4 S_3 & l_1 C_1 + l_2 C_{12} & l_2 C_{12} & -l_3 C_3 & 0 & 0 & 1 \end{bmatrix}. \]

Therefore \( \mu(q') \) is obtained as
\[ \mu(q') = \psi_q^{-1}(q') \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \] (58)
where \( \det(\psi_q) = l_1 l_2 S_2 \neq 0 \) except for \( q_2 = n\pi, \ n \in \mathbb{Z}. \)

From (2) and (61),
\[ -q_1 \] and \( q_1 \) by Assumption 1, i.e., the knee joint angle \( q_2 \) never equals \( n\pi, \ n \in \mathbb{Z}. \)

By solving the constraints \( C \) in (2), \( q_1 \) and \( q_2 \) can be represented as functions of \( q_3 \) as
\[ q_1 = \cos^{-1} \left( \frac{l_1^2 + (l_3 C_3 + c_x)^2 + (l_3 S_3 + c_y)^2 - l_2^2}{2l_1 \sqrt{(l_3 C_3 + c_x)^2 + (l_3 S_3 + c_y)^2}} \right) + \tan^{-1} \left( \frac{l_3 S_3 + c_y}{l_3 C_3 + c_x} \right) \] (59)
\[ q_2 = \cos^{-1} \left( \frac{l_1^2 + l_2^2 - (l_3 C_3 + c_x)^2 - (l_3 S_3 + c_y)^2}{2l_1 l_2} \right) + \pi. \] (60)
The expressions in (59) and (60) yield the parameterization \( \sigma(q) \).

B. Analytic Solution of \( q_3 \) for \( \chi_{\text{Glut}} = 1 \) and \( \chi_{\text{Gast}} = 1 \)

This appendix develops on analytic solutions of \( q_3 \) at \( \chi_{\text{Glut}} = 1 \) and \( \chi_{\text{Gast}} = 1 \). The crank angle which satisfies that \( \bar{F}_{\text{Glut}} \) and \( l_3 \) cross at right angles is denoted by \( q_{\text{Glut}} \). In other words, \( q_{\text{Glut}} \) equals \( q_3 \) which satisfies
\[ q_3 - \frac{\pi}{2} = q_1 + q_2. \] (61)

From (2) and (61),
\[ q_{\text{Glut}} = \sin^{-1} \left( \frac{l_3^2 + l_2^2 - l_1^2 + c_x^2 + c_y^2}{-2 \sqrt{(c_x l_3 - c_y l_2)^2 + (c_y l_3 + c_x l_2)^2}} \right) - \varphi_1 + 2n\pi, \ n \in \mathbb{Z}, \] (62)
where
\[ \varphi_1 := \tan^{-1} \left( \frac{c_x l_3 - c_y l_2}{c_y l_3 + c_x l_2} \right) + \pi. \] (63)

In a similar way, \( q_{\text{Gast}} \) is defined as a crank angle when \( \bar{F}_{\text{Gast}} \) and \( l_3 \) cross at right angles. In other words, \( q_{\text{Gast}} \) equals \( q_3 \) which satisfies
\[ q_3 - \frac{\pi}{2} = \tan^{-1} \left( \frac{l_1 S_1 + l_2 S_{12}}{l_1 C_1 + l_2 C_{12}} \right) = \tan^{-1} \left( \frac{l_3 S_3 + c_y}{l_3 C_3 + c_x} \right), \] (64)
where (2) was utilized. From (64),
\[ q_{\text{Gast}} = \sin^{-1} \left( \frac{l_3}{\sqrt{c_x^2 + c_y^2}} \right) - \varphi_2 + 2n\pi, \ n \in \mathbb{Z}, \] (65)
where
\[ \varphi_2 := \tan^{-1} \left( \frac{c_x}{c_y} \right) + \pi. \] (66)

C. Proof of Theorem 7

Proof: Consider the following positive definite continuously differentiable function
\[ V(y) = \frac{1}{2} y^T M_\Omega y + \frac{1}{2} \alpha_1^T e_1 + \frac{1}{2} \alpha_2^T e_2 + P. \] (67)

Integrating (49) indicates that
\[ P(t) - P(0) = - \int_0^t \alpha_2 e_2(\tau) \left( N_d(\tau) - \beta \text{sgn}(e_2(\tau)) \right) d\tau - \int_0^t \frac{d e_2(\tau)}{d\tau} \left( N_d(\tau) - \beta \text{sgn}(e_2(\tau)) \right) d\tau = - \int_0^t \alpha_2 e_2(\tau) \left( N_d(\tau) - \beta \text{sgn}(e_2(\tau)) \right) d\tau \]
\[ -e_2(t) N_d(t) - e_2(0) N_d(0) + \beta |e_2(t)| - |e_2(0)| \]
\[ = \int_0^t \alpha_2 e_2(\tau) \left( \beta \text{sgn}(e_2(\tau)) - N_d(\tau) - \frac{1}{\alpha_2} \frac{d N_d(\tau)}{d\tau} \right) d\tau \]
\[ -e_2(t) N_d(t) + e_2(0) N_d(0) + \beta |e_2(t)| - |e_2(0)| \geq \int_0^t \alpha_2 |e_2(\tau)| \left( \beta - |N_d(\tau)| - \frac{1}{\alpha_2} |\frac{d N_d(\tau)}{d\tau}| \right) d\tau \]
\[ + |e_2(t)| \left( \beta - |N_d(t)| \right) - |e_2(0)| N_d(0) \] (68)

Based on the sufficient condition in (54), (55) and (68) indicate that \( P(t) \geq 0 \), and (67) satisfies the following inequalities:
\[ \lambda_1 ||y||^2 \leq V \leq \lambda_2 ||y||^2. \] (69)

The time derivative of (67) exists almost everywhere (a.e.), i.e., for almost all \( t \in [0, \infty) \), and \( \dot{V} \in \dot{V} \) where
\[ \dot{V} := \sum_{\xi \in \partial V} \xi^T K \left[ e_1^T e_2^T r^T \frac{1}{2} P^{-\frac{1}{2}} \dot{P} 1 \right]^T \] (70)
and \( \partial V \) is the generalized gradient of \( V \). Since \( V \) is continuously differentiable, (70) can be rewritten as
\[ \dot{V} \subset \nabla V^T K \left[ e_1^T e_2^T r^T \frac{1}{2} P^{-\frac{1}{2}} \dot{P} 1 \right]^T \] (71)
where \( \nabla V := \left[ e_1^T e_2^T r^T M_\Omega 2 P^{-\frac{1}{2}} \frac{1}{2} r^T M_\Omega r \right]^T \). Using \( K[\cdot] \) from (23), (71) yields
\[ \dot{V} \subset e_1^T (e_2 - \alpha_1 e_1) + e_2^T (r - r \alpha_2 e_2) \]
\[ + r^T \left( -\frac{1}{2} M_\Omega(q) r + \dot{N} + N_d - (k_s + 1) \dot{e}_2 \right) \]
\[ - (k_s + 1) \alpha_2 e_2(t) - \beta K \left[ \text{sgn}(e_2) \right] - e_2 \]
\[ + K \left[ \dot{P} \right] + \frac{1}{2} r^T M_\Omega r \] (72)
By substituting $\hat{P}$ from (49), (72) can be transformed into

$$
\dot{V} \subset \epsilon_1^T (e_2 - \alpha_1 e_1) - \alpha_2 e_2^2 + r^T (\tilde{N} + N_d - (k_s + 1)r - \beta K [\text{sgn}(e_2)]) \\
+ K [-r^T (N_d - \beta \text{sgn}(e_2))]
= \epsilon_1^T (e_2 - \alpha_1 e_1) - \alpha_2 e_2^2 + r^T \tilde{N} - (k_s + 1)r - \beta K [\text{sgn}(e_2)]
+ r^T \beta K [\text{sgn}(e_2)].
$$

(73)

Eq. (73) can be further upper bounded as

$$
\dot{V} \leq -\alpha_1 e_1^2 + \epsilon_1^T e_2 - \alpha_2 e_2^2 + r^T \tilde{N} - (k_s + 1)r^2
= r^T \tilde{N} - k_s r^2 - \tilde{z}^T Q z
$$

(74)

where the set in (73) reduces to the scalar inequality in (74) because the right-hand side is continuous a.e., i.e., the right-hand side is continuous except for the Lebesgue negligible set of times when

$$
r^T (\beta K [\text{sgn}(e_2)] - \beta K [\text{sgn}(e_2)]) \neq \{0\}.
$$

By using Eq. (43), the term $r^T \tilde{N}$ can be upper bounded as

$$
||r^T \tilde{N}|| \leq \rho(||z||) ||z|| ||r||
$$

(75)

to obtain

$$
\dot{V} \leq -\lambda_{\text{min}}(Q) ||z||^2 + \rho(||z||) ||z|| ||r|| - k_s ||r||^2.
$$

(76)

By completing the squares,

$$
\dot{V} \leq -\lambda_{\text{min}}(Q) ||z||^2 - k_s \left(||r||^2 - \frac{\rho(||z||)||z||}{2k_s}\right)^2
+ \frac{\rho(||z||)^2||z||^2}{4k_s}
\leq -\left(\lambda_{\text{min}}(Q) - \frac{\rho(||z||)^2||z||^2}{4k_s}\right) ||z||^2.
$$

(77)

From (77), it follows that

$$
\dot{V} \leq -\gamma ||z||^2, \quad \forall y \in D
$$

(78)

where $\gamma \in \mathcal{R}$ is some positive constant, and $D := \{y \in \mathbb{R}^{3+1} \mid \rho(||y||) < 2\sqrt{\lambda_{\text{min}}(Q)k_s}\}$. From the inequalities in (69) and (78), $V \in \mathcal{L}_\infty$, and hence, $e_1, e_2$, and $r \in \mathcal{L}_\infty$. The remaining signals in the closed-loop dynamics can be proven to be bounded. By the definition of $z$ in (44) and $U$ in (78), $U$ can be shown to be uniformly continuous. Then, the region of attraction $D_z$ can be expanded arbitrarily by increasing $k_s$. By invoking Corollary 1 in (24), $\gamma ||z||^2 \to 0$ as $t \to \infty$, $\forall y(0) \in D_z$. Based on the definition of $z$, $e_1 \to 0$ as $t \to \infty$, $\forall y(0) \in D_z$. 

References

[1] V. Nekoukar and A. Erfanian, “A Decentralized Modular Control Framework for Robust Control of FES-Activated Walker-Assisted Paraplegic Walking Using Terminal Sliding Mode and Fuzzy Logic Control,” IEEE Trans. on Biomedical Engineering, Vol. 59, No. 10, pp. 2818–2827, 2012.

[2] R. Kamnik, J. Q. Shi, R. Murray-Smith and T. Bajd “Nonlinear Modeling of FES-Supported Standing-Up in Paraplegia for Selection of Feedback Sensors,” IEEE Trans. on Neural Systems and Rehabilitation Engineering, Vol. 13, No. 1, pp. 40–52, 2005.

[3] A. J. Westerveld, A. C. Schouten, P. H. Veltink and H. van der Kooij, “Selectivity and Resolution of Surface Electrical Stimulation for Grasp and Release,” IEEE Trans. on Neural Systems and Rehabilitation Engineering, Vol. 20, No. 1, pp. 94–101, 2012.

[4] C.-W. Peng, S.-C. Chen, C.-H. Lai, C.-J. Chen, C.-C. Chen, J. Mizrahi and Y. Handa, “Review: Clinical Benefits of Functional Electrical Stimulation Cycling Exercise for Subjects with Central Neurological,” Journal of Medical and Biological Engineering, Vol. 31, No. 1, pp. 1–11, 2011.

[5] K.J. Hunt, J. Fang, J. Saengsuan, M. Grob and M. Laubacher, “On the Efficiency of FES Cycling: A Framework and Systematic Review,” Technology and Health Care, Vol. 20, No. 5, pp. 395–422, 2012.

[6] J.-J. Chen, N.-Y. Yu, D.-G. Huang, B.-T Ann, and G.-C. Chang, “Applying Fuzzy Logic to Control Cycling Movement Induced by Functional Electrical Stimulation,” IEEE Trans. on Rehabilitation Engineering, Vol. 5, No. 2, pp. 158–169, 1997.

[7] M. Gföller and P. Lugner, “Cycling by Means of Functional Electrical Stimulation,” IEEE Trans. on Rehabilitation Engineering, Vol. 8, No. 2, pp. 233–243, 2000.

[8] M. Gföller and P. Lugner, “Dynamic Simulation of FES-Cycling: Influence of Individual Parameters,” IEEE Trans. on Neural Systems and Rehabilitation Engineering, Vol. 12, No. 4, pp. 398–405, 2004.

[9] P. C. Eser, N. de N. Donaldson, H. Knecht and E. Stiess, “Influence of Different Stimulation Frequencies on Power Output and Fatigue During FES-Cycling in Recently Injured SCI People,” IEEE Trans. on Neural Systems and Rehabilitation Engineering, Vol. 11, No. 3, pp. 236–240, 2003.

[10] C.-S. Kim, G.-M. Eom, K. Hase, G. Khang, G.-R. Tack, J.-H. Yi and J.-H. Jun, “Simulation Pattern-Free Control of FES Cycling: Simulation Study,” IEEE Trans. on Systems, Man, and Cybernetics—Part C, Vol. 38, No. 1, pp. 125–134, 2008.

[11] N. Sharma, K. Stegath, C. M. Gregory and W. E. Dixon, “Nonlinear Neuromuscular Electrical Stimulation Tracking Control of a Human Limb,” IEEE Trans. on Neural Systems and Rehabilitation Engineering, Vol. 17, No. 6, pp. 576–584, 2009.

[12] N. Sharma, C. M. Gregory, M. Johnson and W. E. Dixon “Closed-Loop Neural Network-Based NMES Control for Human Limb Tracking,” IEEE Trans. on Control Systems Technology, Vol. 20, No. 3, pp. 712–724, 2012.

[13] R. J. Downey, T.-H. Cheng and W. E. Dixon “Tracking Control of Human Limb during Asynchronous Neuromuscular Electrical Stimulation,” Proc. of the 52nd IEEE Conf. on Decision and Control, 2013 (to appear).

[14] N. Hogan, “Impedance Control: An Approach to Manipulation: Part I—Theory,” “Part II—Implementation” and “Part III—Applications,” Journal of Dynamic Systems, Measurement, and Control, Vol. 107, No. 1, pp. 1–24, 1985.

[15] F. H. Ghorbal, O. Chetelat, R. Gunawardana and R. Longchamp, “Modeling and Set Point Control of Closed-Chain Mechanisms: Theory and Experiment,” IEEE Trans. on Control Systems Technology, Vol. 8, No. 5, pp. 801–815, 2000.

[16] J. L. Krevolin, M. G. Pandy and J. C. Pearce, “Moment Arm of the Patellar Tendon in the Human Knee,” Journal of Biomechanics, Vol. 37, No. 5, pp. 785–788, 2004.

[17] W. L. Buford, Jr., F. M. Ivey, Jr., J. D. Malone, R. M. Patterson, G. L. Peare, D. K. Nguyen and A. A. Stewart, “Muscle Balance at the Knee–Moment Arms for the Normal Knee and the ACL-Minus Knee,” IEEE Trans. on Rehabilitation Engineering, Vol. 5, No. 4, pp. 367–379, 1997.

[18] T. Watanabe, R. Futami and N. Hoshimiya, “An Approach to a Muscle Model with a Stimulus Frequency-Force Relationship for FES Applications,” IEEE Trans. on Rehabilitation Engineering, Vol. 7, No. 1, pp. 12–18, 1999.
[19] I. Nara, M. Kumamoto, et al., Bi-articular Muscles: Motion control and rehabilitation. Igaku-Shoin, 2008.

[20] M. Ferrarin and A. Pedotti, “The Relationship Between Electrical Stimulus and Joint Torque: A Dynamic Model,” IEEE Trans. on Rehabilitation Engineering, Vol. 8, No. 3, pp. 342–352, 2000.

[21] T. Schauer, N.-O. Negard, F. Previdi, K.J. Hunt, M.H. Fraser, E. Ferchland and J. Raisch, “Online Identification and Nonlinear Control of the Electrically Stimulated Quadriceps Muscle,” Control Engineering Practice, Vol. 13, No. 9, pp. 1207–1219, 2005.

[22] B. Xian, D. M. Dawson, M. S. de Queiroz and J. Chen, A Continuous Asymptotic Tracking Control Strategy for Uncertain Nonlinear Systems, IEEE Trans. on Automatic Control, Vol. 49, No. 7, pp. 1206–1211, 2004.

[23] B. E. Paden and S. S. Sastry, “A Calculus for Computing Filippov’s Differential Inclusion with Application to the Variable Structure Control of Robot Manipulators,” IEEE Trans. on Circuits and Systems, Vol. 34, No. 1, pp. 73–82, 1987.

[24] N. Fischer, R. Kamalapurkar and W. E. Dixon “LaSalle-Yoshizawa Corollaries for Nonsmooth Systems,” IEEE Trans. on Automatic Control, Vol. 58, No. 9, pp. 2333–2338, 2013.