Multi-Branch Matching Pursuit
with applications to MIMO radar

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Abstract—We present an algorithm, dubbed Multi-Branch Matching Pursuit (MBMP), to solve the sparse recovery problem over redundant dictionaries. MBMP combines three different paradigms: being a greedy method, it performs iterative signal support estimation; as a rank-aware method, it is able to exploit signal subspace information when multiple snapshots are available; and, as its name foretells, it leverages a multi-branch (i.e., tree-search) strategy that allows us to trade-off hardware complexity (e.g. measurements) for computational complexity. We derive a sufficient condition under which MBMP can recover a sparse signal from noiseless measurements. This condition, named MB-coherence, is met when the dictionary is sufficiently incoherent. It incorporates the number of branches of MBMP and it requires fewer measurements than other conditions (e.g. the Neuman ERC or the cumulative coherence). As such, successful recovery with MBMP is guaranteed for dictionaries that do not satisfy previously known conditions.

Index Terms—Sparse recovery algorithm, compressive sensing, support estimation, matching pursuit, exact recovery condition.

I. INTRODUCTION

LINEAR inverse problems can be found throughout engineering and the mathematical sciences. Usually these problems are ill-conditioned or underdetermined, so that regularization must be introduced in order to obtain meaningful solutions. Sparsity constraints have emerged as a fundamental type of regularizer, and in the last decade, an enormous body of work has been generated around the theory of compressed sensing [1]. Radar has been among the many areas where compressive sensing has found application, and in particular, sparse recovery has been effectively applied to multiple input multiple output (MIMO) radar [2], [3], [4]. To fit sparse recovery in localization applications, one generates a grid of possible targets’ locations and an associated unknown vector of responses, such that only locations associated with targets are non-zero. Therefore, the localization problem aims to recover the support of such unknown vector (non-zero elements of the vector).

Compressive sensing seeks to recover an \( n \times l \) matrix \( \mathbf{X} \) from a small number of linear observations \( \mathbf{Y} = \mathbf{A} \mathbf{X} \) (possibly corrupted by noise), where the \( m \times n \) matrix \( \mathbf{A} \), with \( m < n \), is commonly referred to as measurement matrix or dictionary, and its columns are called atoms. While the linear system is highly underdetermined (\( m < n \)), the inverse problem still has a unique solution if \( \mathbf{X} \) is sparse, i.e., it has only \( K \) non-zero norm rows out of \( n \) (with \( K \leq m < n \)). In this case, the problem of recovering the signal \( \mathbf{X} \) from \( \mathbf{Y} \) can be cast as a non-convex combinatorial \( \ell_0 \)-norm problem, i.e., \( \min_{\mathbf{X}} \| \mathbf{Y} - \mathbf{A} \mathbf{X} \|_F \), s.t. \( \| \mathbf{X} \|_0 \leq K \), where \( \| \mathbf{X} \|_0 \) counts the number of non-zero norm rows of \( \mathbf{X} \). In the following, we will refer to rows of \( \mathbf{Y} \) as measurements, and to the columns of \( \mathbf{Y} \) as snapshots. The \( \ell_0 \)-norm problem is known also under other names, such as sparse approximation or highly nonlinear approximation [5], and it can be related to the Deterministic Maximum Likelihood (DML) estimator [6], [7]. Both \( \ell_0 \)-norm minimization and DML problems require a multi-dimensional search with exponential complexity [8], which is infeasible in practical scenarios. A core algorithmic question arises for a given class of dictionaries, how does one design a fast algorithm that provably recovers a \( K \)-sparse input signal?

Finding conditions that guarantee correct recovery with practical algorithms has been an active topic of research and one of the underpinnings of compressive sensing theory. Compressive sensing theory [1] shows that it is possible to recover any \( K \)-sparse signal \( \mathbf{X} \) using a practical algorithms (e.g., the relaxation of the \( \ell_0 \)-norm to an \( \ell_1 \)-norm, called Basis Pursuit (BP) or LASSO [9]), if the measurement matrix \( \mathbf{A} \) satisfies specific properties. For instance, a correct solution is guaranteed, if the matrix is sufficiently incoherent (as measured by the cumulative coherence [10]) or if it satisfies the restricted isometry property (RIP). Such properties are satisfied with high probability for a wide class of random measurement matrices (e.g. Gaussian, Bernoulli, or partial Fourier), as long as a sufficient number of measurements is available (e.g. \( m > \beta K \log n \) for some constant \( \beta \) [1], but they may not hold when the measurement matrix is structured (e.g., in MIMO radar [3]).

While BP (or LASSO) has strong recovery guarantees, its complexity is still considerably high for real-world implementations. As a result, many other methods have been proposed, and the area is still very active. These methods target a complexity reduction (from BP) using sophisticated convex optimization theory concepts [11], [12], [13], [14], graphical methods [15], reweighting family [16], [17], the M-FOCUSS algorithm [13], local solutions of non-convex relaxations, such as the \( \ell_p \)-norm (with \( p < 1 \)) [19], or simple, but effective, matching pursuit strategies (also known as greedy algorithms) that estimate the support one index at a time. The latter family includes Orthogonal Matching Pursuit (OMP) [20], Order Recursive Matching Pursuit (ORMP) (which is also known as Orthogonal Least Squares) [21] and Rank Aware-Orthogonal Regularized Matching Pursuit (RA-ORMP) [22].

In some extensions of the matching pursuit, at each iteration, more than one index is added to the provisional support. Notable examples are CoSaMP [23] and IHT [24]. See [1], [5] for an overview of sparse recovery algorithms.
This paper illustrates the MBMP algorithm, first proposed in [25], which builds upon the low complexity matching pursuit by leveraging a multi-branch (i.e., tree-search) strategy. Similar to MBMP, matching pursuit has been used in conjunction with tree-search strategies to improve reconstruction performance. Tree-search strategies based on matching pursuit are proposed in [26], [27], and multi-branch generalizations of OMP appear in [28], [29]. In these works no multi-branch based recovery guarantee is provided. Recently, another multi-branch generalization of OMP, called Multipath Matching Pursuit (MMP), was proposed in [30] together with a recovery guarantee based on RIP. However, such guarantee does not improve upon the RIP guarantee of BP. Moreover, whereas tree-search algorithms in the literature focus on the SMV setup, MBMP addresses the general MMV setup where, being rank aware, it takes advantage of the signal subspace information. To avoid possible confusion, we remark that MBMP can address the recovery of any sparse signal, as it does not impose an additional structure on the sparse signals (e.g., tree-structured dictionary [31]).

This work expands the literature by formulating recovery guarantees for MBMP in a noisy setup: (i) A sufficient condition under which MBMP recovers any sparse signal belonging to a given support; (ii) A sufficient condition under which MBMP can recover any K-sparse signal. Condition (i), named Multi-Branch Exact Recovery Condition (MB-ERC), generalizes the well-known Tropp’s ERC [10] to a multi-branch algorithm. Condition (ii), named MB-coherence, generalizes Neuman ERC [32] to a multi-branch algorithm. MB-coherence is met when the dictionary is sufficiently incoherent and it provides a guideline to design the multi-branch structure of MBMP. In contrast to other recovery guarantees for tree-structure algorithms (e.g., MMP), both MB-ERC and MB-coherence conditions improve the state-of-the-art in the sense that they enable to guarantee MBMP success for dictionaries that do not satisfy previously known conditions (e.g., ERC or Neuman ERC). Due to its ability to trade-off measurements with computational complexity, MBMP is particularly well suited to applications in which measurements are very expensive, such as in radar applications where the number of measurements is commensurate with the number of antenna elements.

The rest of the paper is organized as follows: Section II introduces the sparse recovery problem; Section III details the proposed algorithm; in Section IV we develop recovery guarantees for MBMP; Section V contains numerical results to demonstrate the potential of the MBMP algorithm in the MIMO radar sparse localization framework. Section VI provides the conclusions.

The following notation is used: boldface denotes matrices (uppercase) and vectors (lowercase); for a matrix A, A (i, j) denotes the element at i-th row and j-th column. The complex conjugate operator is (·)*, the transpose operator is (·)T, the complex conjugate-transpose operator is (·)∗H, and the pseudo-inverse operator is (·)+. For a full rank matrix X ∈ ℂm×n with m ≥ n, we have X† = (XH X)−1 XH. The Frobenius norm of X is ∥X∥F, and the ℓ1-induced norm is ∥X∥1 = max{i} |X(i,j)| and the ℓ∞-induced norm is ∥X∥∞ = max{j} |X(i,j)|. Given a set S of indices, |S| denotes its cardinality, AS is the sub-matrix obtained by considering only the columns indexed in S, and ΠSA = I−A*S A†S is the orthogonal projection matrix onto the null space of A*S. Given two sets of indices, S and S', S\S' contains the indices of S which are not present in S'. We define the support S of a matrix X as the set of non-zero norm rows, and we define ∥X∥0 = |S|. We say that X is K-sparse if ∥X∥0 ≤ K.

II. SPARSE RECOVERY PROBLEM

In a noiseless setting, sparse recovery seeks the sparsest solution to a linear system of equations [5]:

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_0 \quad \text{s.t.} \quad \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}. \quad (1)$$

This setup is known as Single Measurement Vector (SMV), highlighting the fact that a single vector of measurements y is available. More generally, when multiple measurement vectors have the same support, the setting is known as Multiple Measurement Vectors (MMV) or joint sparse. In this case, the model is \( \boldsymbol{Y} = \boldsymbol{AX} \), where \( \boldsymbol{Y} \in \mathbb{C}^{m \times l} \) is the observed signal matrix, \( \boldsymbol{A} \in \mathbb{C}^{m \times l} \) is the measurement matrix and the matrix \( \boldsymbol{X} \in \mathbb{C}^{n \times l} \) is the unknown signal. The unknown signal \( \boldsymbol{X} \) is sparse since it has only \( K \leq n \) non-zero norm rows. The MMV sparse recovery problem is to estimate the sparse matrix \( \boldsymbol{X} \). It has been shown [11] that, under certain conditions on the matrix \( \boldsymbol{A} \) and the sparsity \( K \), the sparse matrix \( \boldsymbol{X} \) can be recovered from linear measurements \( \boldsymbol{Y} \) by solving the nonconvex \( l_0 \)-norm problem:

$$\min_{\boldsymbol{X}} \|\boldsymbol{X}\|_0 \quad \text{s.t.} \quad \boldsymbol{Y} = \boldsymbol{AX}, \quad (2)$$

where \( \|\boldsymbol{X}\|_0 \) counts the number of non-zero norm rows of \( \boldsymbol{X} \). In this work, we assume that \( \text{rank} (\boldsymbol{A}) > 2K - \text{rank} (\boldsymbol{A}) + 1 \), where \( \text{rank} (\boldsymbol{A}) \) is the smallest number of linearly dependent columns of the matrix \( \boldsymbol{A} \). This is a necessary and sufficient condition for \( \boldsymbol{Y} = \boldsymbol{AX} \) to uniquely determine any K-sparse matrix \( \boldsymbol{X} \) [22], [33].

In the presence of noise, the measurements comply with

$$\boldsymbol{Y} = \boldsymbol{AX} + \boldsymbol{E} \quad (3)$$

where \( \boldsymbol{E} \in \mathbb{C}^{m \times l} \) is the noise term. In this scenario, the sparse matrix \( \boldsymbol{X} \) can be recovered by solving a relaxation of (2),

$$\min_{\boldsymbol{X}} \|\boldsymbol{X}\|_0 \quad \text{s.t.} \quad \|\boldsymbol{Y} - \boldsymbol{AX}\|_F \leq \epsilon. \quad (4)$$

where the parameters \( \epsilon \) and \( K \) depend on prior information, e.g., noise level or signal sparsity.

In the following, we detail the MBMP algorithm to address (4) when the sparsity level \( K \) is known. In scenarios when \( K \) is unknown and only \( \epsilon \), or \( \nu \), are available, MBMP can be used to solve the Lagrangian formulation (min\( X \) \( \|Y - AX\|_F \))
\[ \nu \|X\|_0, \) or the residual constrained formulation \( \min_X \|X\|_0 \]
\[ \text{s.t. } \|Y - AX\|_F \leq \epsilon, \) with minor modifications to the algorithm’s termination criteria and support selection \[34\].

While MBMP addresses problem \[4\] for any measurement matrix \( A \), in this work we focus on radar (e.g., target localization) applications. In general, target localization consists of two stages: detection and estimation \[7\]. While detection is a process that inherently relies on a single target point of view, and deals with lower SNR levels, estimation builds on detection by seeking to improve the accuracy of localization for detected targets. In this work, we adopt an estimation point-of-view, which assumes that the sparsity level \( K \) (e.g., number of targets) is known, and requires a medium to high SNR level. We formalize the latter condition by assuming that the SNR is sufficient to guarantee that the support of the combinatorial problem \(4\) solution coincides with the true support. As problem \(4\) can be related to the DML estimator, this assumption implies that such estimator achieves the Cramér-Rao bound \[6\]. Our goal is to guarantee a similar value of \( \nu \).

In order to detail MBMP, it is instructive to first reformulate problem \(4\) in terms of the support \( S \) of the solution \( X \). In particular, \(4\) is equivalent to
\[ \min_S \|\Pi_S^\perp Y\|_F \text{ s.t. } |S| \leq K. \tag{5} \]

The reformulation follows by noticing that the minimization with respect to \( X \) in \(4\) can be separated into the minimization with respect to the support \( S \) and the minimization with respect to the actual non-zero value of \( X \). In particular, assuming that the spark condition is satisfied (i.e., \( \text{spark}(A) > 2K - \text{rank}(X) + 1 \)), for a given support \( S \), the optimal non-zero value of \( X \) is given by the least square solution: \( X_S = \Pi_S^\perp Y \). This reduces problem \(4\) to problem \(5\).

III. Multi-Branch Matching Pursuit

Here we introduce MBMP, a multi-branch algorithm, which belongs to the matching pursuit family and aims to solve problem \(5\). We first discuss previous algorithms, and then detail MBMP.

A. Matching pursuit

We start by providing an overview of matching pursuit \(1\). This strategy starts with an empty provisional support \( C = \emptyset \), and then adds a new index to \( C \) at each iteration, based on a selection strategy. For example, in OMP, the index \( g \) that maximizes \( \|a_g^H \Pi_{\mathcal{A}_C} Y\|_2 \) is selected. This selection strategy may be refined in two ways: a dictionary refinement and a subspace refinement.

The dictionary refinement applies when a non-empty provisional support \( C \) is already available. In this case, instead of using the original dictionary’s atoms, the current dictionary is projected on the orthogonal subspace of \( \mathcal{A}_C \), i.e., \( \bar{a}_g^C \equiv \Pi_{\mathcal{A}_C} a_g \), and each atom is renormalized according to,
\[ \bar{a}_g^C \equiv \begin{cases} \frac{a_g^H}{\|a_g^H\|_2} & \text{if } \|a_g^H\|_2 > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{6} \]

The subspace refinement is possible in an MMV scenario (when \( \text{rank}(X) > 1 \)). In such case, rather than evaluating the norm of the inner product \( \|a_g^H \Pi_{\mathcal{A}_C} Y\|_2 \), we may use an orthonormal basis \( U \) of \( \Pi_{\mathcal{A}_C} Y \), and compute \( \|a_g^H U\|_2 \). The matrix \( U \) is also known as the signal subspace.

Depending on how refinement strategies are combined (dictionary refinement and/or subspace refinement), four different algorithms are obtained. Three of them have been already introduced in the literature \[22\]: if dictionary and residual refinements are not used, we have the Simultaneous Orthogonal Matching Pursuit (SOMP), which extends OMP to the general MMV scenario; if dictionary refinement is not used, but residual refinement is used, we have RA-OMP (which is not fully rank-aware); finally, if both dictionary and residual refinements are used, we get the best algorithm, namely RA-ORMP, which is fully rank-aware. In particular, the so-called “rank awareness” means that, assuming \( \text{spark}(A) > 2K - \text{rank}(X) + 1 \) and considering a noiseless scenario, whenever the received signal \( Y \) is full rank, RA-ORMP recovers the correct support with probability one.

B. MBMP

The proposed MBMP algorithm generalizes RA-ORMP by including a multi-branch structure. In particular, it is possible to visualize RA-ORMP as a chain of nodes, depicted in Fig. \(1\)(a). Node A is tagged with an empty support. A new index is selected following a chosen selection strategy, and it becomes the provisional support of node B. To solve \(5\), this procedure is repeated until level \( K + 1 \) is reached.

Instead of a chain of nodes, the MBMP algorithm may be visualized as a tree of nodes as shown in Fig. \(1\)(b), where each node is allowed to have multiple children (node A is the parent of nodes B, C and D; B is the parent of E and F). For instance, in Fig. \(1\)(b), node A has 3 branches, resulting in 3 nodes at level 2. Node A is tagged with an empty support. Then, the index \( g \) that maximizes \( \|a_g^H U\|_2 \) (where \( U \) is the signal subspace) becomes the provisional support of node B. While RA-ORMP doesn’t have any other node at level 2, with MBMP, the index \( g \) that gives the second largest value of \( \|a_g^H U\|_2 \) is assigned to the provisional support of node C.
Similarly, the index $g$ that gives the third largest value of $\|a_g^H U\|_2$ is assigned to the provisional support of node D. One of these atom indices will necessarily be part of the solution returned by the algorithm. Then, MBMP continues to populate nodes at level 3. For example, consider node B. Since node B has two branches, it has two children. Following the selection strategy, two new indices are selected. Each of these is added to the provisional support of node B and used to tag node E and F, respectively. This procedure is performed for all nodes at level 2 (i.e., nodes C and D), thus populating nodes G, H, I, and J. The process stops when all nodes at level $K + 1$ have been populated. The support $C$ achieving the minimum value of $\|\Pi_{A_C} Y\|_F$ is elected as the solution to (5).

The MBMP tree depends on the number of levels and on the number of branches at each level (assumed constant for nodes within the same level of the tree). The MBMP structure can be specified using a vector $d \triangleq [d_1, \ldots, d_K]$ referred to as branch vector: $d_i$ represents the number of branches of each node at level $i$. For instance, the tree in Fig. 1(a) has $d = [1, 1]$ while the tree in Fig. 1(b) has $d = [3, 2]$ (node A at level 1 has $d_1 = 3$ branches, and each node at level 2 (i.e., B, C, and D) possesses $d_2 = 2$ branches). We call root node the node at level 1 (i.e., node A in Fig. 1), and $U = \text{orth} (Y)$ denotes an estimate of the signal subspace (see [7], [22] for an overview of signal subspace estimation).

The pseudo-code of the MBMP algorithm is detailed in the following table.

| Algorithm 1 Multi-branch matching pursuit algorithm |
|---------------------------------------------------|
| **Input**: $Y \in \mathbb{C}^{m \times 1}$, $A \in \mathbb{C}^{m \times n}$, and $d \in \mathbb{N}^K$ |
| **Output**: Support of approximate solution to problem (5) |
| 1: Initialize root node (tagged with $C = \emptyset$ and $\hat{C} = \emptyset$) |
| 2: Set $U = Y$, $f_{opt} = +\infty$ |
| 3: for $\forall$ node without children at level $i \leq K$ |
| 4: if $i > 1$: Set $U = \text{orth} (\Pi_{A_C} Y)$ |
| 5: for $j = 1, \ldots, d_i$ |
| 6: $g_j \in \arg \max_g \|\hat{a}_g (C, \ldots, g_{j-1})\|_2 ||\Pi_{A_C} Y\|_F$ |
| 7: Tag a new child node with: $C = C \cup \{g_j\}$, $\hat{C} = \hat{C} \cup \{g_1, \ldots, g_j\}$ |
| 8: if $|C| = K$ and $||\Pi_{A_C} Y||_F < f_{opt}$: |
| Set $S = C$, and $f_{opt} = ||\Pi_{A_C} Y||_F$ |
| 9: end |
| 10: end |
| 11: Return support $S$ |

We finally note that nodes at level $K$ need only $d_K = 1$ branch. This is because any additional branch would be tagged with provisional support $C$ that cannot minimize the objective function of problem (5).

### C. Computational Complexity

Given an $m \times n$ matrix $A$ and an $m \times l$ matrix $Y$, the computational requirements of MBMP depend on the specific implementation details, the structure of the measurement matrix $A$ and the branch vector $d = [d_1, \ldots, d_K]$ (MBMP has 1 node at level 1 and $\prod_{j<i} d_j$ nodes at level $i$). Due to the variability of the computation costs of applying the transform $A^H$ (ranging from $O(n \log (m))$ for an FFT-type operations to $O(nml)$ for unstructured matrices), we denote with $F$ the computational cost associated with performing $A^H U$ without specifying an associated number of flops. Furthermore, to perform residual refinement, a practical implementation of MBMP would also need to incorporate an estimate of the signal subspace, and we denote $R$ the relative cost. For a node at level $i$, other operations performed by MBMP are: selecting the $d_i$ largest inner products, which is known as the “selection problem” [55] and can be solved using $O(n)$ flops; the dictionary refinement, which costs $2m(n-i+1)$ flops; the update of the projection matrix $\Pi_{A_K}$, which requires $2m$ flops, and the computation of the residual, that needs $ml$ flops. An efficient implementation of both the dictionary refinement and the projection matrix update is obtained by applying a QR factorization [1].

Summarizing, the first node requires $F + R + O(n)$ flops, since the dictionary refinement and the projection matrix update are not performed. Any node at level $i$ (with $2 \leq i \leq K$) requires $F + R + O(n) + 2m(n-i+1)+ml$ flops. Finally, a node at level $K+1$ requires $m(l+2)$ flops to update the projection matrix and to compute the residual norm.

As a rule of thumb, the complexity of MBMP scales approximately with the number of nodes in the first $K$ levels of MBMP tree (i.e., all nodes except those at level $K+1$). Therefore, while RA-ORMP complexity is proportional to $K$, the complexity of MBMP with branch vector $d = [d_1, \ldots, d_K]$ scales approximately with $1 + \sum_{i=2}^{K} \prod_{j<i} d_j$. For example, the complexity of MBMP with branch vector $d = [2, 2, 2, 1]$ is approximately $31/5 = 6.2$ times that of RA-ORMP. This aspect will be further investigated in the numerical results. It is worth mentioning that, due to the tree-structure, the MBMP algorithm lends itself to a parallel implementation. Indeed, if multiple processors are available, although the total number of MBMP operations remains the same, most of them can be performed in parallel, reducing the total algorithm’s execution time.

### IV. Recovery Guarantees for MBMP

In this section, we develop recovery guarantees for MBMP. Throughout this section, the measurement matrix $A$ is a given deterministic matrix. MBMP is executed with a branch vector $d \triangleq [d_1, \ldots, d_K]$ of length $K$. The information available to the recovery algorithm includes $Y$, $A$, and $K$. Moreover, as MBMP is executed, provisional supports, denoted $C_i$, are available at all nodes of level $i$. By convention, $C_1 = \emptyset$, since at level 1, no provisional support is available. Finally, we say that MBMP succeeds in recovering a $K$-sparse $X$ if one node at level $K+1$ is tagged with the correct support of $X$, denoted with $S^*$, which is assumed to be the (global optimal) solution of problem (5).

The road map of this section is as follows: We start by reviewing Tropp’s ERC [10]. This condition considers signals with a specific support $S^*$. This restriction enables to obtain recovery guarantees for pursuit algorithms (e.g. BP, OMP,
ORMP, and RA-ORMP), By generalizing ERC to a multi-branch algorithm, we formulate in Definition 1 the MB-ERC. Theorem 1 relies on the MB-ERC to provide a sufficient condition that guarantees successful recovery with MBMP. Similar to ERC, MB-ERC is non-constructive, since it focuses only on signals with a specific support \( S^* \). To overcome this limitation, in Definition 2 we introduce the MB-coherence condition for multi-branch algorithms. Using the MB-coherence, Theorem 2 specifies a sufficient condition that guarantees the recovery of any \( K \)-sparse signal \( X \) using MBMP. Interestingly, in the noiseless setup, the MB-coherence condition can be seen as the multi-branch generalization of the Neuman ERC (or weak ERC) [52], which improves upon the cumulative coherence condition proposed in [10].

### A. MB-ERC

We first overview the ERC, which characterizes the ability of practical algorithms to recover sparse signals supported on a specific support \( S^* \). For a given support \( S^* \) and for a matrix \( A \), the ERC is formulated [10]

\[
\max_{g \notin S^*} \left\| A^\top_{S} a_g \right\|_1 < 1. \tag{7}
\]

This condition addresses linear systems of equations of the form \( A_{S^*} x = a_g \), where \( a_g \) is a column from \( A \) that is outside the support \( S^* \). The ERC states that the minimum (\( _2 \)-energy solution to all these systems should have an \( l_2 \)-length smaller than 1. The importance of the ERC stems from its strong connection to the success of pursuit techniques. In particular, ERC is a sufficient condition for successful recovery via RA-ORMP (as shown in [22]) and thus for MBMP as well. ERC is also sufficient for correct recovery via OMP, ORMP and BP in the SMV setup (see [10] and [57]).

Next, we proceed to introduce MB-ERC, which generalizes ERC to a multi-branch algorithm and leads to a stronger sufficient condition to guarantee the success of MBMP. In contrast to RA-ORMP, in which each node has only one child, the number of children of each node of MBMP is specified by the branch vector \( d = [d_1, \ldots, d_K] \), where \( d_i \) is the number of branches at each node at level \( i \). As a result, MB-ERC is a function of \( d_i \). To proceed, it is convenient to define the \( d_i \)-max operator. Explicitly, given a positive integer \( d_i \) and a real vector \( z \) (where its elements are indexed by \( g \)), \( d_i \)-max operation is defined as

\[
d_i \max_{g \notin \overset{\wedge}{S}^*} (z) = \text{the } d_i \text{-largest entry among the indices of } z \text{ outside the support } S^*. \tag{8}
\]

In the SMV setup, we assume by convention that \( U = y \), while, in the MMV setup, \( U = \text{orth} (\overset{\wedge}{A}_{C_i}, Y) \) is an estimate of the signal subspace given a provisional support \( C_i \). OIR is the square-root of the ratio between the largest energy of \( U^H \overset{\wedge}{A}_{C_i} \) among indices outside \( S^* \setminus C_i \) and the largest energy of \( U^H \overset{\wedge}{a}_g \) over the indices inside \( S^* \). Since the definition of the OIR depends on unknown quantities (e.g., the support \( S^* \)), it must be estimated.

**Definition 1** (MB-ERC). Consider a support \( S^* \), a matrix \( A \), a positive integer \( d_i \), and a correct provisional support \( C_i \subset S^* \). Let \( S \triangleq S^* \setminus C_i \) be the set of indices yet to be identified. The MB-ERC\((S^*, C_i, d_i)\) is defined as

\[
d_i \max_{g \notin \overset{\wedge}{S}^*} \left( \left\| \left( \overset{\wedge}{A}_{S^*} \right)^\top \overset{\wedge}{a}_g \right\|_1 \right) < 1 - \text{OIR}, \tag{9}
\]

where \( \text{OIR} \) is defined in (8).

MB-ERC generalizes ERC to a multi-branch algorithm and to a noisy setup. In particular, in a noiseless setup (OIR = 0), MB-ERC\((S^*, 0, 1)\) (i.e., (9) at level 1 with \( d_1 = 1 \) branches) reduces to ERC in (7). By using MB-ERC, we can guarantee success of MBMP for any signal \( X \) supported on \( S^* \).

**Theorem 1** (Recovery of any signal supported on \( S^* \)). Let \( X \) be an unknown, \( K \)-sparse matrix of rank \( r \), with known support \( S^* \), and \( A \) be full rank with normalized columns and \( \text{spark} (A) > 2K - r + 1 \). Let \( Y = AX + E \) be the noisy data with OIR given by (8). If the MB-ERC in (9) is met for all nodes at levels \( i = 1, \ldots, K - 1 \), then MBMP with branch vector \( d = [d_1, \ldots, d_K - 1, 1] \) is guaranteed to recover \( X \) successfully.

**Proof:** See Appendix A.

Theorem 1 formulates a sufficient condition for MBMP to recover sparse signals supported on a specific support \( S^* \). In the next subsection, by removing the knowledge of \( S^* \), we obtain a condition that guarantees MBMP successful recovery for any \( K \)-sparse signal.

### B. MB-coherence condition

A disadvantage of both MB-ERC and ERC is that they require the knowledge of the true support \( S^* \), hardly available in practice. This implies that to check if a measurement matrix \( A \) satisfies MB-ERC (or ERC), one has to compute the conditions for all \( \binom{n}{K} \) possible supports \( S^* \) of cardinality \( K \), which is usually prohibitive even for small values of \( K \). To overcome this limitation, we develop a practical condition that guarantees recovery via MBMP for any \( K \)-sparse signal \( X \).

The main problem with MB-ERC and ERC is the presence of the pseudo-inverse. As shown in [10], by using standard norm inequalities to upper bound ERC, it is possible to obtain practical conditions that include only inner products rather than the pseudo-inverse operator. These conditions rely on the notion of coherence of a measurement matrix \( A \), defined as
\( \mu (A) \triangleq \max_{x,j} |a_H^T a_j| \) [10], and on the notion of cumulative coherence (also known as Babel's function [38]), defined as \( \bar{\mu}(K, A) \triangleq \max_{S, |S| = K} \max_{g \in S} \| A_H^T a_g \|_1 \) [10]. Using these definitions, it was shown in [10] that the ERC holds for any \( K \)-sparse signal \( X \), if either the coherence satisfies
\[
\mu (A) < \frac{1}{2K - 1}
\] (10)
or if the cumulative coherence satisfies
\[
\bar{\mu}(K, A) + \bar{\mu}(K, A) < 1.
\] (11)

A condition that requires fewer measurements is called Neuman ERC (or weak ERC). It was proposed in [32], and can be stated as:
\[
\max_{S, |S| = K} \left( \max_{g \in S} \| A_H^T a_g \|_1 + \max_{g \notin S} \| A_H^T a_g \|_1 \right) < 2(1 - \text{NSR}),
\] (12)
where the Noise-to-Signal Ratio (NSR) is defined in [32]. Similarly to the OIR, NSR depends on unknown quantities (e.g., signal and noise realizations) and must be estimated. As shown in [32], condition (12) may be used to guarantee correct recovery of any \( K \)-sparse signal using BP.

The number of measurements required to guarantee correct recovery can be further reduced by capturing the multi-branch structure of MBMP. Indeed, we now develop a condition, dubbed MB-coherence, which guarantees recovery of any \( K \)-sparse signal using MBMP, while requiring less measurements than (12) for a multi-branched algorithm. Considering a provisional support \( C_i \), as before, we denote \( \bar{A}_{C_i} \triangleq \{ a_{C_i}, g \notin C_i \} \) the associated refined measurement matrix. For the sake of notation, in the definition, we drop the superscript \( C_i \) and we use \( \bar{A}_S \) instead.

**Definition 2 (MB-coherence).** Consider a matrix \( A \), integers \( K \) and \( d_i \), a provisional support \( C_i \), and OIR defined in (8), with \( \text{OIR} < 1 \). Let \( k \triangleq K - |C_i| \). The MB-coherence \( (C_i, d_i) \) is defined as
\[
\max_{S, |S| = k} \left( \max_{g \in S} \| \bar{A}_S^H \bar{a}_g \|_1 + \frac{d_i \max_{g \in S} \left( \| \bar{A}_g^H \bar{a}_C \|_1 \right)}{1 - \text{OIR}} \right) < 2.
\] (13)

A key aspect of the MB-coherence condition is that it includes only inner products among columns of the matrix \( \bar{A}_{C_i} \) (as opposed to MB-ERC in (9) which incorporates the pseudo-inverse operator). This enables to practically compute the smallest integer \( d_i \) such that the MB-coherence condition (13) is met, as discussed in Appendix C.

By using the MB-coherence condition, it is possible to obtain a sufficient condition to guarantee that MBMP recovers any \( K \)-sparse signal \( X \):

**Theorem 2 (Recovery of any \( K \)-sparse signal).** Let \( X \) be an unknown, \( K \)-sparse matrix of rank \( r \), and \( A \) be full rank with normalized columns and \( \text{spark}(A) > 2K - r + 1 \). Let \( Y = AX + E \) be the noisy data with OIR given by (8). If the MB-coherence condition in (13) is met for all nodes at levels \( i = 1, \ldots, K - 1 \), then MBMP with branch vector \( d = [d_1, \ldots, d_{K - 1}, 1] \) is guaranteed to recover \( X \) successfully.

**Proof:** See Appendix B.

Theorem 2 guarantees correct recovery of any \( K \)-sparse signal using MBMP. Furthermore, in a noiseless case (when \( \text{OIR} = \text{NSR} = 0 \), MB-coherence(\( \emptyset, 1 \)) (i.e., (13) at level 1 with \( d_1 = 1 \) branches) reduces to (12). Since the \( d_{\max} \) operator is decreasing in \( d_i \), MB-coherence(\( \emptyset, d_1 \)) with \( d_1 > 1 \) guarantee MBMP success for dictionaries that do not satisfy Neuman ERC. In the numerical results section, this point will be further explored.

**C. Discussion**

A key aspect highlighted by the theoretical results above is that increasing the number of branches of MBMP does not only allow us to reduce the number of measurements, but it enables to tolerate higher noise levels. In particular, consider MB-ERC in (9) (MB-coherence in (13)). Since the \( d_{\max} \) operator is decreasing in \( d_i \), one can preserve the validity of MB-ERC (MB-coherence) even if the noise level increase (i.e., larger OIR) by increasing \( d_i \). This point will be further analyzed in the numerical results section.

Additionally, Theorem 1 (Theorem 2) reads as the intersection of the conditions MB-ERC(\( S^*, C_i, d_i \)) (MB-coherence(\( C_i, d_i \))) for all nodes of the MBMP tree at levels \( i = 1, \ldots, K - 1 \). These requirements can be considerably simplified in two situations. According to [37, Lemma 2], MB-ERC(\( S^*, \hat{C}, 1 \)) implies MB-ERC(\( S^*, C, 1 \)) whenever \( C \subset \hat{C} \subset S^* \). For example, MB-ERC(\( S^*, \emptyset, 1 \)) implies MB-ERC(\( S^*, C, 1 \)) for any \( C \subset S^* \). More generally, it can be shown that MB-ERC(\( S^*, \hat{C}, d \)) (MB-coherence(\( C, d \))) implies MB-ERC(\( S^*, C, d \)) (MB-coherence(\( C, d \))) whenever \( \hat{C} \subset C \) and \( d \leq d_\hat{C} \). Let a node be tagged with support \( \hat{C} \), the condition \( \hat{C} \subset C \) is satisfied for any support \( C \) of a descendant of such node (i.e., children, children of children, etc.). This implies that Theorem 1 (Theorem 2) requires MB-ERC (MB-coherence) only at level 1 (root node) and at nodes with a smaller number of branches than their parents. As a concrete example, if \( d_i = d_1 \) for \( i = 1, \ldots, K - 1 \), Theorem 1 (Theorem 2) requires only MB-ERC(\( S^*, \emptyset, 1 \)) (MB-coherence(\( \emptyset, d_1 \))) (thus requiring a similar complexity as Neuman ERC). Equivalently, for MBMP with branch vector \( d = [d_1, \ldots, 1, 1] \), Theorem 1 (Theorem 2) requires MB-ERC (MB-coherence) conditions only for nodes at level 1 and 2, for a total of \( d_1 + 1 \) conditions to be checked. Another situation where we can simplify these conditions is in a noiseless setup when \( \text{rank}(X) > 1 \). In this scenario, Theorem 1 (Theorem 2) requires MB-ERC (MB-coherence) only at level \( i \) with 1 \( \leq i \leq K - \text{rank}(X) \), since at level \( i > K - \text{rank}(X) \), MBMP is guaranteed to take correct decisions thanks to the rank aware property.

Given a matrix \( A \), we would like to design the number of branches of MBMP to guarantee recovery of any \( K \)-sparse signals for some targeted sparsity level \( K \). An application of Theorem 2 is to provide an upper bound on the number of branches needed by each node of MBMP. Consider level 1 of MBMP. By choosing \( d_1 \) as the smallest integer such that (13) holds at level 1, we guarantee that at least one node at level 2 has a support \( C_2 \) such that \( C_2 \subset S^* \). In general, for
each node at level \( i \), we compute the refined measurement matrix \( \tilde{A}^C_i \equiv \{ \tilde{a}^C_{j}, j \in C_i \} \), and we select \( d_j \) to satisfy (13) at level \( i \). The process continues until \( d_{K-1} \) is set at level \( K-1 \), since nodes at level \( K \) need only \( d_K = 1 \) branch. Moreover, from the discussion above, if at some node, (13) holds with a given \( d_j \), then, at any children of such node, the number of branches \( d_j \) needed to meet (13) obeys \( d_j \leq d_j \). This implies that, if at some node (13) holds with \( d_j = 1 \), we can set \( d_j = 1 \) branch for all children of such node without requiring additional conditions.

V. Numerical Results

In this section, we present numerical results to illustrate the guarantees obtained in Section IV and to investigate the performance of the proposed MBMP algorithm. Although MBMP may solve the problem (5) for any type of measurement matrix \( A \), in this section we apply MBMP to perform direction-of-arrival (DOA) estimation in a MIMO radar system where spatial compressive sensing (3) is employed. We start by introducing the MIMO radar spatial compressive sensing setup.

A. MIMO radar setup

We model a MIMO radar system (see Fig. 2), where \( N \) sensors collect a finite train of \( l \) pulses. Each pulse consists of \( M \) orthogonal spread spectrum waveforms of length \( M \) chips. Each one of the waveforms is sent by one of the \( M \) transmitters and returned from \( K \) stationary targets. We assume that transmitters and receivers form (possibly overlapping) linear arrays of equal aperture \( Z/2 \), respectively (\( Z \) is normalized in wavelength units): the \( i \)-th transmitter is at position \( Z \xi_i /2 \), where \( \xi_i \in [-0.5, 0.5] \) for \( i = 1, \ldots, M \) on the \( x \)-axis; the \( j \)-th receiver is at position \( Z \eta_j /2 \), where \( \eta_j \in [-0.5, 0.5] \) for \( j = 1, \ldots, N \). The targets’ positions are assumed constant over the observation interval of \( l \) pulses.

The purpose of the system is to determine the DOA angles to targets of interest, which translate to recover the unknown signal support. We consider targets associated with a particular range and Doppler bin. Targets in adjacent range-Doppler bins contribute interference to the bin of interest. The assumption of a common range bin implies that all waveforms are received with the same time delay after transmission. Targets are assumed in the far-field, meaning that a target’s DOA parameter \( \theta = \sin \vartheta \) (where \( \vartheta \) is the DOA angle) is constant across the array. Following (3), the DOA estimation problem can be cast within a sparse localization framework. Neglecting the discretization error, it is assumed that the target possible locations comply with a grid of \( n \) points \( \phi_{1:n} \) (with \( n \gg K \)). By defining the \( MN \times n \) matrix

\[
A = [a(\phi_1), \ldots, a(\phi_n)]
\]

where \( a(\phi) \equiv c(\theta) \otimes b(\theta) \) with \( b(\theta) = [\exp(j2\pi Z_\theta \xi_1), \ldots, \exp(j2\pi Z_\theta \xi_M)]^T \) the receiver steering vector and \( c(\theta) = [\exp(j2\pi Z_\theta \xi_1), \ldots, \exp(j2\pi Z_\theta \xi_M)]^T \) the transmitter steering vector, the signal model is expressed as (3). In particular, the unknown matrix \( X \in \mathbb{C}^{n \times l} \) contains the targets locations and gains. The support of \( X \) corresponds to grid points with a target (see (3) for further details).

Spatial compressive sensing assumes that the elements’ positions are random variables (described by the probability density functions (pdf) \( p(\xi) \) and \( p(\zeta) \)). Following the setup discussed in (3), we chose \( p(\xi) \) and \( p(\zeta) \) as uniform distributions, and \( \phi_{1:n} \) as a uniform grid of \( 2Z \)-spaced points in the range \([-1, 1]\). This implies that the number of grid points is \( n = Z + 1 \) (columns of the measurement matrix \( A \)).

In this section, the target gains are given by \( x_{k,p} = \exp(-j\varphi_{k,p}) \), with \( \varphi_{k,p} \) drawn i.i.d., uniform over \([0, 2\pi)\), for all \( k = 1, \ldots, K \) (where \( K \) is the number of targets) and \( p = 1, \ldots, l \) (where \( l \) is the number of snapshots). The noise (see (3)) is assumed to be distributed as \( \text{vec}(E) \sim \mathcal{CN}(0, \sigma^2 I) \) (where \( \text{vec}(\cdot) \) is the vectorization operator) and the SNR is defined as \( 10 \log_{10} \left( \min_{k,p} |x_{k,p}|^2 \right) - 10 \log_{10} (\sigma^2) \), which in our setup reduces to \(-10 \log_{10} (\sigma^2) \), since \( |x_{k,p}| = 1 \) \( \forall k, p \). From the definition of the measurement matrix \( A \), its columns all have norms equal to \( \sqrt{MN} \). Throughout the numerical results, the columns of \( A \) are normalized to unit norm.

B. Numerical experiments

We start by exploring the guarantee obtained in Section IV using the MB-coherence. We investigate numerically the trade-off between the number of measurements and number of branches \( d_k \) at level 1 of MBMP (which relates to the algorithm’s complexity) in order to meet the MB-coherence \((\theta, d_1)\) condition (13) at level 1 in a noiseless setup (OIR = 0), i.e.,

\[
\max_{S, |S| = K} \left( \max_{g \in S} \| A_S^H a_g \|_1 + d_1 \max_{g \in S} \left( \| A_S^H a_g \|_1 \right) \right) < 2,
\]

where \( A^C_1 = A \) since \( C_1 = 0 \). As discussed in Section IV condition (13) is sufficient to guarantee the correct recovery of \( K \)-sparse signal \( X \) with rank \( r \) using MBMP with branch vector \( d = [d_1, \ldots, d_{K-1}, 1, \ldots, 1] \), where \( d_i = d_1 \) for \( i = 2, \ldots, K - r \).

We generate several realizations of the MIMO radar measurement matrix \( A \in \mathbb{C}^{MN \times n} \) (as defined in (13)), and for each realization we test whether (13) holds, the probabilities of meeting the coherence condition in (10) and the cumulative coherence condition in (11) are also plot as references (notice that the case \( d_1 = 1 \) reduces to the Neuman ERC).
plots the probability of meeting condition (15) as a function of the number of measurements $MN$ for different value of $d_1$. The MIMO radar measurement matrix $A \in \mathbb{C}^{MN \times n}$ defined in (14) is employed. Signal sparsity is $K = 3$ and $n = 501$.

To investigate the typical recovery behavior of MBMP, we present numerical results for the non-uniform recovery setting (i.e., at each realization, the matrix $A$ and the signal $X$ are drawn independently at random), and we explore the localization performance in the presence of noise comparing MBMP with other SMV and MMV algorithms. For the SMV setting, we implement target localization using LASSO applying the algorithm proposed in (14). In addition, we implement the discrete version of beamforming (which, in the SMV setup, identifies the support’s elements as the $K$ indices $g$ that maximize $|a_g^H y|$), ORMP, CoSaMP and FOCUSS (18). For the MMV scenario, we compare MBMP with RA-ORMP, M-FOCUSS, and the discrete version of MUSIC (which identifies the support’s elements as the $K$ indices $g$ that maximize $\|a_g^H U\|_2$, where $U = \text{orth}(Y)$ is an estimate of the signal subspace (22)). As stated above, MBMP with $d = [1, \ldots, 1]$ reduces to ORMP (RA-ORMP) in the SMV (MMV) scenario.

We define a support recovery error event when the estimated support does not coincide with the true one. For algorithms that return an estimate $\hat{X}$ of the sparse signal $X$ (e.g., LASSO and M-FOCUSS), the support is identified as the $K$ largest norm rows of the signal $\hat{X}$. We further assume that the noise variance.
error of SNR than MUSIC: for instance, to achieve a probability of error of $10^{-3}$, MUSIC requires SNR = 47 dB, while MBMP with $d = [2, 2, 2, 2, 1]$ achieves the same probability of error with just 20 dB. This gain is ascribed to the iterative signal support estimation performed by MBMP, which differs from the non-iterative support estimation performed by MUSIC.

In Fig. 6, we address an MMV setting ($l = 5$) and we investigate the probability of support recovery error as a function of the SNR. We set the number of antenna elements $M = N = 4$, $l = 5$ and $K = 5$ targets with $|x_{k,l}| = 1$ for all $k$ and $l$. SNR is 20 dB.

In Fig. 7, we analyze the probability of support recovery error as a function of the number of rows $MN$ of $A$. SMV setup ($l = 1$). The system settings are $Z = 250$, $n = 251$ and $K = 5$ targets with $|x_k| = 1$ for all $k$. SNR is 20 dB.

$\sigma^2$ is known, since this information is needed by LASSO and M-FOCUSS. The virtual aperture is set to $Z = 250$ (thus $n = 251$ grid/points), and numerical results were obtained for $K = 5$ targets.

In Fig. 5 we address an MMV setting ($l = 5$) and we investigate the probability of support recovery error as a function of the SNR. We set the number of antenna elements $M = N = 4$. The figure supports the theoretical findings of Section 1 that increasing the number of MBMP branches for MBMP translates into an SNR gain. In addition, MBMP has performance superior to both M-FOCUSS and MUSIC. The floor incurred by M-FOCUSS is due to the inability of this method to exploit the signal subspace information (i.e., it is not rank aware [22]). In addition, MBMP requires a much smaller SNR than MUSIC: for instance, to achieve a probability of error of $10^{-3}$, MUSIC requires SNR = 47 dB, while MBMP with $d = [2, 2, 2, 2, 1]$ achieves the same probability of error with just 20 dB. This gain is ascribed to the iterative signal support estimation performed by MBMP, which differs from the non-iterative support estimation performed by MUSIC.

In Fig. 6 we fix the number of snapshots ($l = 5$), the SNR (20 dB), and we illustrate the probability of support recovery error as a function of the number of measurements $MN$ (number of rows of the matrix $A$). We evaluate five different element configurations: $(M, N) = (3, 3), (4, 4), (5, 5), (6, 6)$ and $(7, 7)$. It can be seen that, by increasing the complexity of MBMP, the probability of error can be decreased even when we use a limited number of antenna elements (e.g., MBMP with $d = [2, 2, 2, 2, 1]$ achieves a probability of error close to $10^{-5}$ with $MN = 25$). Moreover, in all cases, MBMP performs much better than MUSIC.

In Fig. 7 we analyze the probability of support recovery error as a function of the number of measurements $MN$ in an SMV setting ($l = 1$). We evaluate six different configurations: $(M, N) = (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)$ and $(10, 10)$, and keep the SNR = 20 dB. In an SMV setting, MUSIC cannot be applied since the signal is not full-rank (rank $(X) = 1 < K$). In addition to MBMP and FOCUSS (the SMV version of M-FOCUSS), we performed target DOA recovery using beamforming, LASSO and CoSaMP. From Fig. 7 it can be seen that beamforming is not well suited to the sparse recovery framework, incurring in a very high probability of error as compared to sparse recovery methods. Moreover, although in a SMV scenario the signal subspace is not available, MBMP still provides competitive performance with respect to other algorithm. Comparing Fig. 6 and Fig. 7 it can be appreciated that by having multiple snapshots ($l > 1$) and using MBMP, the number of antenna elements can be dramatically reduced.

Finally, we also analyze the complexity of MBMP with respect to other CS algorithms. Fig. 8 plots the average runtime in seconds as a function of the number of measurements (rows of the matrix $A$) in an SMV setting ($l = 1$). First,
it can be seen how, by properly setting the branch vector of
MBMP, we can adjust the MBMP complexity. Moreover, as
discussed above, the figures shows that MBMP complexity
scales proportionally with the number of nodes in the first
K levels of MBMP tree. In particular, the average run-time of
MBMP with \( d = [2, 1, 1, 1, 1] \) is slightly less than double
(9/5) that of MBMP with \( d = [1, 1, 1, 1, 1] \), while the average
run-time of MBMP with \( d = [2, 2, 2, 1, 1] \) is approximately
31/5 = 6.2 times that of MBMP with \( d = [1, 1, 1, 1, 1] \). Fur-
thermore, although the computational complexity of MBMP is
exponential in \( K \), in the scenario at hand with \( K = 5 \), MBMP
has a smaller, or comparable, complexity to that of LASSO,
FOCUSS and CoSaMP, while providing better performance
(e.g., see Fig. [7]). We also remark that, whereas OMP com-
plexity is smaller that ORMP (i.e., MBMP with \( d = [1, 1, 1, 1, 1] \)
in an SMV setup), we build MBMP around RA-ORMP in
order to take full advantage of the rank-aware property in
a MMV setup. This is because, in radar applications, it is
common to have several snapshots and the ability to use the
signal subspace information improves performance.

VI. CONCLUSIONS

We develop the MBMP algorithm for sparse recovery,
and derive a sufficient condition under which MBMP can
recover any sparse signal belonging to a given support. We
then introduce the MB-coherence, and apply it to derive a
sufficient condition under which MBMP can recover any \( K \)
sparse signal. This condition enables to guarantee the success
of the proposed MBMP for dictionaries that do not satisfy
previously known conditions based on coherence or on cumu-
lagative coherence. Furthermore, we demonstrate by numerical
examples that MBMP supports trading off measurements (e.g.
antenna elements) for computational complexity. Both theo-
retical guarantees and numerical results illustrate that MBMP
enables recovery with fewer measurements than other practical
algorithms.

VII. APPENDIX

For the sake of notation, in the Appendix we drop the
superscript \( C_i \) from \( \bar{A}_S \) and we use \( \bar{A}_S \).

A. Proof of Theorem 7

We start by proving that, given a node at level \( i \) tagged with
a correct provisional support \( C_i \subset S^* \), if MB-ERC(\( S^*, C_i, d_i \))
in \( (9) \) holds then at least one of the \( d_i \) branches of the node
successfully selects an index \( g \) from the correct support set \( g \in S \triangleq S^* \setminus C_i \). We follow similar steps as in the proof that ERC
is sufficient for RA-ORMP given in \( [22] \). The only differences
are: (i) the use of the \( d_{\text{max}} \) operator; (ii) the use of the refined
dictionary \( \bar{A}_S C_i \triangleq \{ \bar{a}_g^C, g \notin C_i \} \) when a provisional support
\( C_i \) is available; (iii) the use of the OIR to address a noisy
scenario.

Similar to other MP techniques, but with the key difference
of the \( d_{\text{max}} \) operator, in order to guarantee that at least one
of the \( d_i \) branches of the considered node successfully selects
an atom \( \bar{a}_g^C \) from the remaining correct indices \( g \in S \), we
require the following

\[
\frac{d_i \cdot \max_{g \in S} \left\| U^H \bar{a}_g^C \right\|_2}{\max_{g \in S} \left\| U^H \bar{a}_g^C \right\|_2} < 1, \tag{16}
\]

where \( U = \text{orth} (\Pi_{\bar{A}_S}^C, Y) \). Since \( U = \Pi_{\bar{A}_S}^C U + \Pi_{\bar{A}_g}^C U \),
by using standard norm inequalities, we can upper bound the
numerator of \( (16) \) as

\[
d_i \cdot \max_{g \in S} \left\| U^H \bar{a}_g^C \right\|_2 \leq d_i \cdot \max_{g \notin S} \left\| U^H \Pi_{\bar{A}_S}^C \bar{a}_g^C \right\|_2 + \max_{g \notin S} \left\| U^H \Pi_{\bar{A}_g}^C \bar{a}_g^C \right\|_2. \tag{17}
\]

By using \( (17) \) and the definition of OIR in \( (8) \), the left-hand
side of \( (16) \) can be upper bounded as

\[
\frac{d_i \cdot \max_{g \in S} \left\| U^H \bar{a}_g^C \right\|_2}{\max_{g \in S} \left\| U^H \bar{a}_g^C \right\|_2} \leq \frac{d_i \cdot \max_{g \notin S} \left\| U^H \Pi_{\bar{A}_S}^C \bar{a}_g^C \right\|_2}{\max_{g \notin S} \left\| U^H \bar{a}_g^C \right\|_2} + \text{OIR.} \tag{18}
\]

By using standard norm inequalities as in \( [22] \), the first term
of the right-hand side of \( (18) \) can be upper bounded as

\[
\frac{d_i \cdot \max_{g \notin S} \left\| U^H \Pi_{\bar{A}_S}^C \bar{a}_g^C \right\|_2}{\max_{g \notin S} \left\| U^H \bar{a}_g^C \right\|_2} \leq d_i \cdot \max_{g \notin S} \left\| \bar{A}_S^C \bar{a}_g^C \right\|_1. \tag{19}
\]

Using \( (19) \) into inequality \( (18) \), we can conclude that, if \( (9) \)
holds, then \( (16) \) is guaranteed to hold too. Therefore at least
one of the \( d_i \) branches of the considered node successfully
selects an index \( g \) from the correct support set \( g \in S \).

It remains to prove that, if MB-ERC(\( S^*, C_i, d_i \)) holds for
any node at level \( i = 1, \ldots, K - 1 \), then MBMP with
branch vector \( d = [d_1, \ldots, d_{K-1}, 1] \) is guaranteed to recover
\( X \) from the measurements \( Y = AX + E \). To prove this,
note that if MB-ERC(\( S^*, C_i, d_i \)) holds for any node at level
\( i = 1, \ldots, K - 1 \), it follows that a chain of correct decisions
exists along the MBMP tree: MB-ERC holds for the first node,
thus at least one node at level 2 has a correct provisional support. Considering such node, since MB-ERC holds there, it will select a correct index in at least one branch, and we have a node at level 3 with correct provisional support, and so on up to level $K$. Finally, a node at level $K$ tagged with a correct provisional support $C_K \subset S^*$ selects the index yielding the smallest residual, which achieves the global optimal solution to (3), concluding the proof.

B. Proof of Theorem 2

We start by showing that, given a node at level $i$ tagged with a correct provisional support $C_i$, the MB-coherence($C_i$, $d_i$) in (13) implies MB-ERC($S^*$, $C_i$, $d_i$) in (9), for any support $S^* \triangleq S \cup C_i$ of cardinality $K$. To achieve this, we use standard arguments (e.g., as in [10]) and the properties of the $d_{\max}$ operator. In details, by using the definition of pseudo-inverse and introducing the $d_{\max}$ operator, the left-hand-side of (9) can be upper bounded as

$$d_{\max}_{g \in S} \left( \left\| \bar{A}^H S \bar{a}^C_g \right\|_1 \right) \leq \frac{d_{\max}_{g \in S} \left( \left\| \bar{A}^H S \bar{a}^C_g \right\|_1 \right)}{2 - \max_{g \in S} \left( \left\| \bar{A}^H S \bar{a}^C_g \right\|_1 \right)}.$$

(20)

It follows that MB-ERC($S^*$, $C_i$, $d_i$) holds for any support $S^* \triangleq S \cup C_i$ of cardinality $K$, if

$$\max_{|S| = k} \frac{d_{\max}_{g \in S} \left( \left\| \bar{A}^H S \bar{a}^C_g \right\|_1 \right)}{2 - \max_{g \in S} \left( \left\| \bar{A}^H S \bar{a}^C_g \right\|_1 \right)} < 1 - \text{OIR},$$

(21)

where $k \triangleq K - |C_i|$. This can be manipulated to obtain (13), thus establishing that the MB-coherence($C_i$, $d_i$) condition (13) implies MB-ERC($S^*$, $C_i$, $d_i$). The claim of the theorem follows by invoking Theorem 1.

C. Testing for MB-coherence

We develop a practical way to find the smallest integer $d_i$ such that the MB-coherence($C_i$, $d_i$) in (13) is met. The following proposition relates the MB-coherence condition to an integer program, which can be solved using discrete optimization techniques [36]. We denote $q_g$ as the $g$-th column of $Q \triangleq \left( \bar{A}^C \right)^H \bar{A}^C$ ($\langle \rangle$ is the element-wise absolute value):

**Proposition 1.** Let $\gamma \triangleq \frac{1}{1 - \text{OIR}} > 1$ and $k \triangleq K - |C_i|$. The smallest integer $d_i$ such that the MB-coherence($C_i$, $d_i$) in (13) holds is given by the optimal value of the objective function

$$\max_{S^*, y, \gamma} \left( \begin{array}{c} 1 + \sum_{l=1}^{n} z_l \\ (q_j + \gamma q_0)^T (s + y) \geq y_j + z_g \quad \forall g \neq j \\ \sum_{l=1}^{n} y_l = k - 1 \\ s_l + z_l \leq 1 \quad \forall l \end{array} \right).$$

(22)

s.t.

$$\begin{array}{l}
\sum_{l=1}^{n} s_l = k - 1 \\
\sum_{l=1}^{n} y_l = 1 \\
s_l, y_l, z_l \in \{0, 1\} \\
\forall l
\end{array}.$$

Proof: Because of space limitation, we provide a sketch of the proof. In particular, the proof follows by exploiting the one-to-one correspondence between a set with $k$ elements out of $n$, and its characteristic vector (i.e., a binary vector with $k$ ones and $n - k$ zeros). Let $d_i$ be the smallest integer such that (13) holds. Then we have a support $S$ of cardinality $k$, an index $j \in S$, and a set $G$, such that $|G| = d_i - 1$, $S \cap G = \emptyset$, and $\left\| \bar{A}^H S \bar{a}^C_g \right\|_1 + \gamma \left\| \bar{A}^H S \bar{a}^C_j \right\|_1 \geq 2 \forall g \in G$. Given such index $j$, and the sets $S$ and $G$, we can consider the associated characteristic (binary) vectors $s$, $y$, and $z$ (i.e., $y_1 = 1$ iff $l = j$; $y_i = 1$ iff $l \in S \setminus j$; and $z_1 = 1$ iff $l \in G$). Since $d_i = 1 + |G| = 1 + \sum_{l=1}^{n} z_l$, it follows that the vectors $s$, $y$, and $z$ maximize problem (22). The converse is obtained by reversing the above argument, concluding the proof.

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