Abstract

In this review paper we give a geometric formulation of the field equations in the Lagrangian and Hamiltonian formalisms of classical field theories (of first order) in terms of multivector fields. This formulation enables us to discuss the existence and non-uniqueness of solutions of these equations, as well as their integrability.

**Key words:** Multivector Fields, Jet Bundles, Connections, Classical Field Theories, Lagrangian and Hamiltonian formalisms.

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1 Introduction

In recent years new developments have been done in the study of multisymplectic Hamiltonian systems and in particular, its application to describe field theories. In this study, multivector fields and their contraction with differential forms are used; and this is an intrinsic formulation of the systems of partial differential equations locally describing the field. Thus, the integrability of such equations; that is, of multivector fields, is a matter of interest. Given a fiber bundle \( \pi : E \to M \), certain integrable multivector fields in \( E \) are equivalent to integrable connections in \( E \to M \) \cite{5}. This result is applied in two particular situations:

- First considering multivector fields in \( J^1E \) (the first-order jet bundle), in order to characterize integrable multivector fields whose integral manifolds are holonomic.
- Second, considering the manifold \( J^{1*}E = \Lambda^m_0 T^* E/\Lambda^m_0 T^* E \) (where \( \Lambda^m_0 T^* E \) is the bundle of \( m \)-forms on \( E \) vanishing by the action of two \( \pi \)-vertical vector fields, and \( \Lambda^m_0 T^* E \equiv \pi^* \Lambda^m T^* M \)), which is also a fiber bundle \( J^{1*}E \to M \). Then, we will take multivector fields in \( J^{1*}E \), in order to characterize those of them being integrable.

From these results we can set the Lagrangian and Hamiltonian equations for multisymplectic models of first-order classical field theories in a geometrical way \cite{3}, \cite{14}, \cite{12}, \cite{17}, in terms of multivector fields; which is equivalent to other formulations using Ehresmann connections in a jet bundle \cite{13}, \cite{19}, or their associated jet fields \cite{4}. This formulation allows us to discuss several aspects of these equations, in particular, the existence and non-uniqueness of solutions. (See also \cite{13}, \cite{14}, where multivector fields are used in a more specific context).

The structure of the work is the following: In section 2 we introduce the terminology and nomenclature concerning with multivector fields in differentiable manifolds and fiber bundles. This is used in Section 3 for setting the field equations for Lagrangian field theories (of first-order) in terms of multivector fields, and for analyzing their characteristic features. Finally, the same study is made in Section 4 for Hamiltonian field theories.

Thoughout this paper \( \pi : E \to M \) will be a fiber bundle (\( \dim M = m \), \( \dim E = N + m \)), where \( M \) is an oriented manifold with volume form \( \omega \in \Omega^m(M) \). \( \pi^1 : J^1E \to E \) is the jet bundle of local sections of \( \pi \), and \( \pi^1 = \pi \circ \pi^1 : J^1E \to M \) gives another fiber bundle structure. \((x^\mu, \rho^A, \nu^A)\) will denote natural local systems of coordinates in \( J^1E \), adapted to the bundle \( E \to M \) (\( \mu = 1, \ldots, m; A = 1, \ldots, N \)), and such that \( \omega = dx^1 \wedge \ldots \wedge dx^m \equiv d^m x \).

Manifolds are real, paracompact, connected and \( C^\infty \). Maps are \( C^\infty \). Sum over crossed repeated indices is understood.

2 Multivector fields in differentiable manifolds

Let \( E \) be a \( n \)-dimensional differentiable manifold. Sections of \( \Lambda^m(TE) \) are called \( m \)-multivector fields in \( E \) (they are contravariant skewsymmetric tensors of order \( m \) in \( E \)). Then, contraction with multivector fields is the usual one for tensor fields in \( J^{1*}E \). We will denote by \( \mathfrak{X}^m(E) \) the set of \( m \)-multivector fields in \( E \).

If \( Y \in \mathfrak{X}^m(E) \), for every \( p \in E \), there exists an open neighbourhood \( U_p \subset E \) and \( Y_1, \ldots, Y_r \in \mathfrak{X}(U_p) \) such that \( Y = \sum_{U_p} f^{i_1 \ldots i_m} Y_{i_1} \wedge \ldots \wedge Y_{i_m} \); with \( f^{i_1 \ldots i_m} \in C^\infty(U_p) \) and \( m \leq r \leq \dim E \).
Then, \( Y \in \mathfrak{X}^m(E) \) is said to be \textit{locally decomposable} if, for every \( p \in E \), there exists an open neighbourhood \( U_p \subset E \) and \( Y_1, \ldots, Y_m \in \mathfrak{X}(U_p) \) such that \( Y \big|_{U_p} = Y_1 \wedge \ldots \wedge Y_m \).

A non-vanishing \( m \)-multivector field \( Y \in \mathfrak{X}^m(E) \) and a \( m \)-dimensional distribution \( D \subset TE \) are \textit{locally associated} if there exists a connected open set \( U \subset E \) such that \( Y \big|_U \) is a section of \( \Lambda^m D \big|_U \). If \( Y, Y' \in \mathfrak{X}^m(E) \) are non-vanishing multivector fields locally associated with the same distribution \( D \), on the same connected open set \( U \), then there exists a non-vanishing function \( f \in C^\infty(U) \) such that \( Y' = fY \). This fact defines an equivalence relation in the set of non-vanishing \( m \)-multivector fields in \( E \), whose equivalence classes will be denoted by \( \{Y\}_U \). Then:

**Theorem 1** There is a one-to-one correspondence between the set of \( m \)-dimensional orientable distributions \( D \) in \( TE \) and the set of the equivalence classes \( \{Y\}_E \) of non-vanishing, locally decomposable \( m \)-multivector fields in \( E \).

(Proof) Let \( \omega \in \Omega^m(E) \) be an orientation form for \( D \). If \( p \in E \) there exists an open neighbourhood \( U_p \subset E \) and \( Y_1, \ldots, Y_m \in \mathfrak{X}(U_p) \), with \( i(Y_1 \wedge \ldots \wedge Y_m) \omega > 0 \), such that \( D\big|_{U_p} = \text{span}\{Y_1, \ldots, Y_m\} \). Then \( Y_1 \wedge \ldots \wedge Y_m \) is a representative of a class of \( m \)-multivector fields associated with \( D \) in \( U_p \). But the family \( \{U_p ; p \in E\} \) is a covering of \( E \); let \( \{U_\alpha ; \alpha \in A\} \) be a locally finite refinement and \( \{\rho_\alpha ; \alpha \in A\} \) a subordinate partition of unity. If \( Y_1^\alpha, \ldots, Y_m^\alpha \) is a local basis of \( D \) in \( U_\alpha \), with \( i(Y_1^{\alpha} \wedge \ldots \wedge Y_m^{\alpha}) \omega > 0 \), then \( Y = \sum_{\alpha} \rho_\alpha Y_1^{\alpha} \wedge \ldots \wedge Y_m^{\alpha} \) is a global representative of the class of non-vanishing \( m \)-multivector fields associated with \( D \) in \( E \).

The converse is trivial because, if \( Y \big|_U = Y_1^1 \wedge \ldots \wedge Y_m^1 = Y_1^2 \wedge \ldots \wedge Y_m^2 \), for different sets \( \{Y_1^1, \ldots, Y_m^1\}, \{Y_1^2, \ldots, Y_m^2\} \), then \( \text{span}\{Y_1^1, \ldots, Y_m^1\} = \text{span}\{Y_1^2, \ldots, Y_m^2\} \).

If \( Y \in \mathfrak{X}^m(E) \) is non-vanishing and locally decomposable; and \( U \subseteq E \) is a connected open set, the distribution associated with the class \( \{Y\}_U \) is denoted \( D_U(Y) \). If \( U = E \) we write \( D(Y) \).

A non-vanishing, locally decomposable multivector field \( Y \in \mathfrak{X}^m(E) \) is said to be \textit{integrable} (resp. \textit{involutive}) if it associated distribution \( D_U(Y) \) is integrable (resp. involutive). Of course, if \( Y \in \mathfrak{X}^m(E) \) is integrable (resp. involutive), then so is every other in it equivalence class \( \{Y\} \), and all of them have the same integral manifolds. Moreover, the Frobenius’ theorem allows us to say that a non-vanishing and locally decomposable multivector field is integrable if, and only if, it is involutive. Nevertheless, in many applications, we have locally decomposable multivector fields \( Y \in \mathfrak{X}^m(E) \) which are not integrable in \( E \); but integrable in a submanifold of \( E \). A (local) algorithm for finding this submanifold has been developed [3].

The particular situation which we will pay attention is the study of multivector fields in fiber bundles. Then, if \( \pi : E \rightarrow M \) is a fiber bundle, we will be interested in the case that the integral manifolds of integrable multivector fields in \( E \) are sections of \( \pi \). Thus, \( Y \in \mathfrak{X}^m(E) \) is said to be \( \pi \)-\textit{transverse} if, at every point \( y \in E \), \( (i(Y)(\pi^* \omega))_y \neq 0 \), for every \( \omega \in \Omega^m(M) \) with \( \omega(\pi(y)) \neq 0 \). Then, if \( Y \in \mathfrak{X}^m(E) \) is integrable, it is \( \pi \)-transverse if, and only if, it integral manifolds are local sections of \( \pi : E \rightarrow M \). In this case, if \( \phi : U \subset M \rightarrow E \) is a local section with \( \phi(x) = y \) and \( \phi(U) \) is the integral manifold of \( Y \) through \( y \), then \( T_y(\text{Im} \phi) = D_y(Y) \).
3 Lagrangian equations in classical field theories

A classical field theory is described by its configuration bundle $\pi: E \to M$; and a Lagrangian density which is a $\pi^1$-semibasic $m$-form on $J^1 E$. A Lagrangian density is usually written as $L = \mathcal{L}(\pi^1 \omega)$, where $\mathcal{L} \in C^\infty(J^1 E)$ is the Lagrangian function associated with $\mathcal{L}$ and $\omega$.

The Poincaré-Cartan $m$ and $(m+1)$-forms associated with the Lagrangian density $\mathcal{L}$ are defined using the vertical endomorphism $\nabla$ of the bundle $J^1 E$:

$$\Theta_\mathcal{L} := i(\nabla)\mathcal{L} + \mathcal{L} \in \Omega^m(J^1 E) \quad ; \quad \Omega_\mathcal{L} := -d\Theta_\mathcal{L} \in \Omega^{m+1}(J^1 E)$$

Then a Lagrangian system is a couple $(J^1 E, \Omega_\mathcal{L})$. The Lagrangian system is regular if $\Omega_\mathcal{L}$ is 1-nondegenerate. In a natural chart in $J^1 E$ we have

$$\Omega_\mathcal{L} = -\frac{\partial^2 \mathcal{L}}{\partial v^B \partial v^A} dv^B \wedge dy^A \wedge d^{m-1}x_\mu - \frac{\partial^2 \mathcal{L}}{\partial y^B \partial v^A} dy^B \wedge dy^A \wedge d^{m-1}x_\mu + \frac{\partial^2 \mathcal{L}}{\partial v^B \partial v^A} v^A dv^B \wedge d^m x + \left( \frac{\partial^2 \mathcal{L}}{\partial y^B \partial v^A} v^A - \frac{\partial \mathcal{L}}{\partial y^B} + \frac{\partial^2 \mathcal{L}}{\partial x^\mu \partial v^B} \right) dy^B \wedge d^m x$$

(1)

(where $d^{m-1}x_\mu \equiv i\left( \frac{\partial}{\partial x^\mu} \right) d^m x$); and the regularity condition is equivalent to $\det \left( \frac{\partial^2 \mathcal{L}}{\partial v^A \partial v^B} (\bar{y}) \right) \neq 0$, for every $\bar{y} \in J^1 E$.

A variational problem can be stated for $(J^1 E, \Omega_\mathcal{L})$ (Hamilton principle): the states of the field are the sections of $\pi$ (denoted by $\Gamma(M, E)$) which are critical for the functional $\mathcal{L}: \Gamma(M, E) \to \mathbb{R}$ defined by $\mathcal{L}(\phi) := \int_M (J^1 \phi)^* \mathcal{L}$, for every $\phi \in \Gamma(M, E)$. These critical sections can be characterized by the condition

$$(J^1 \phi)^* i(X)\Omega_\mathcal{L} = 0 \quad , \text{for every } X \in \mathfrak{X}(J^1 E)$$

In natural coordinates, if $\phi = (x^\mu, y^A(x))$, this condition is equivalent to demanding that the components of $\phi$ satisfy the Euler-Lagrange equations

$$\left. \frac{\partial \mathcal{L}}{\partial y^A} \right|_{J^1 \phi} - \frac{\partial}{\partial x^\mu} \left. \frac{\partial \mathcal{L}}{\partial v^A} \right|_{J^1 \phi} = 0 \quad , \text{for } A = 1, \ldots, N$$

(2)

(For a more detailed description on all these concepts see, for instance, [1], [3], [5], [9], [10], [11], [18], [19]).

The problem of finding these critical sections can be formulated equivalently as follows: to finding a distribution $D$ of $T(J^1 E)$ satisfying that:

- $D$ is integrable (that is, involutive).
- $D$ is $m$-dimensional.
- $D$ is $\pi^1$-transverse.
- The integral manifolds of $D$ are the critical sections of the Hamilton principle.

Then, from the first and second conditions, there exist $X_1, \ldots, X_m \in \mathfrak{X}(J^1 E)$ (in involution), which locally span $D$. Therefore $X = X_1 \wedge \ldots \wedge X_m$ defines a section of $\Lambda^m T(J^1 E)$, that is, a
non-vanishing, locally decomposable multivector field in $J^1E$, whose local expression in natural coordinates is
\[
X = \bigwedge_{\mu=1}^{m} f_{\mu} \left( \frac{\partial}{\partial x^\mu} + F_{\mu}^A \frac{\partial}{\partial y^A} + C_{\mu\rho}^A \frac{\partial}{\partial v_{\rho}^A} \right)
\]
where $f_{\mu}$ are non-vanishing functions. A representative of the class $\{X\}$ can be selected by the condition $i(X)(\tilde{\pi}^1,\omega) = 1$ which, as a particular solution, leads to $f_{\mu} = 1$, for every $\mu$. Furthermore, the third and fourth conditions impose that $X$ is $\tilde{\pi}^1$-transverse, integrable and its integral manifolds are holonomic sections of $\tilde{\pi}^1$.

Bearing this in mind, we want to characterize the integrable multivector fields in $J^1E$ whose integral manifolds are canonical prolongations of sections of $\pi$. So, consider the vector bundle projection $\kappa: T^1J^1E \to TE$ defined by
\[
\kappa(\tilde{y}, \tilde{u}) := T_{\hat{y}}\phi(T_{\hat{y}}\tilde{u}) \quad \text{where} \quad (\tilde{y}, \tilde{u}) \in TJ^1E, \, \phi \in \hat{y}
\]
This projection is extended in a natural way to $\Lambda^m\kappa: \Lambda^mT^1J^1E \to \Lambda^mTE$. Then, a $\tilde{\pi}^1$-transverse multivector field $X \in \mathfrak{X}^m(J^1E)$ is said to be semi-holonomic, or a Second Order Partial Differential Equation, if $\Lambda^m\kappa \circ X = \Lambda^mT\pi^1 \circ X$. In a natural chart in $J^1E$, the local expression of $X$ is
\[
X \equiv \bigwedge_{\mu=1}^{m} f_{\mu} \left( \frac{\partial}{\partial x^\mu} + v_{\mu}^A \frac{\partial}{\partial y^A} + C_{\mu\rho}^A \frac{\partial}{\partial v_{\rho}^A} \right)
\]
where $f_{\mu} \in C^\infty(J^1E)$ are arbitrary non-vanishing functions. On the other hand, $X \in \mathfrak{X}^m(J^1E)$ is said to be holonomic if it is integrable, $\tilde{\pi}^1$-transverse and its integral sections $\psi: M \to J^1E$ are holonomic. Then, it can be proved [6] that a multivector field $X \in \mathfrak{X}^m(J^1E)$ is holonomic if, and only if, it is integrable and semi-holonomic.

Of course, if $X \in \mathfrak{X}^m(J^1E)$ is a semi-holonomic (resp. holonomic) multivector field, everyone in the class $\{X\} \subset \mathfrak{X}^m(J^1E)$ are semi-holonomic (resp. holonomic) too. As local expression of a representative we can take
\[
X \equiv \bigwedge_{\mu=1}^{m} \left( \frac{\partial}{\partial x^\mu} + v_{\mu}^A \frac{\partial}{\partial y^A} + C_{\mu\rho}^A \frac{\partial}{\partial v_{\rho}^A} \right)
\]
Then, given a section $\phi = (x^\mu, f^A)$, if $j^1\phi = \left(x^\mu, f^A, \frac{\partial f^A}{\partial x^\mu}\right)$ is an integral section of this semi-holonomic multivector field, then $v_{\mu}^A = \frac{\partial f^A}{\partial x^\mu}$, and the components of $\phi$ are solution of the system of partial differential equations
\[
C_{\mu\rho}^A \left( x^\mu, f^A, \frac{\partial f^A}{\partial x^\mu} \right) = \frac{\partial^2 f^A}{\partial x^\rho \partial x^\nu}
\]
On the other hand, it can be proved [6] that classes of locally decomposable and $\tilde{\pi}^1$-transverse multivector fields are in one-to-one correspondence with orientable connections in the bundle $\pi: J^1E \to M$ (this correspondence is characterized by the fact that $\mathcal{D}(X)$ is the horizontal subbundle of the connection). For the multivector field [4], the associated Ehresmann connection has the local expression
\[
\nabla = dx^\mu \otimes \left( \frac{\partial}{\partial x^\mu} + v_{\mu}^A \frac{\partial}{\partial y^A} + C_{\mu\rho}^A \frac{\partial}{\partial v_{\rho}^A} \right)
\]

Then \( X \in \mathcal{X}^m(J^1 E) \) is integrable if, and only if, the connection \( \nabla \) associated with the class \( \{ X \} \) is flat; that is, the curvature of \( \nabla \) vanishes everywhere. Thus, the system (5) has solution if, and only if, the following additional system of equations holds (for every \( B, \mu, \rho, \eta \))

\[
0 = G^B_{\eta \mu} - G^B_{\mu \eta} \\
0 = \frac{\partial G^B_{\eta \mu}}{\partial x^\nu} + v^A_\mu \frac{\partial G^B_{\eta \mu}}{\partial y^A} + G^A_\mu \frac{\partial G^B_{\eta \mu}}{\partial v^A_\gamma} - v^A_\eta \frac{\partial G^B_{\mu \rho}}{\partial y^A} - G^A_\eta \frac{\partial G^B_{\mu \rho}}{\partial v^A_\gamma} - C^A_\eta \frac{\partial C^B_{\mu \rho}}{\partial v^A_\gamma}
\]  

(6)

Now, the problem posed by the Hamilton principle can be stated in the following way:

**Theorem 2** Let \((J^1 E, \Omega_L)\) be a Lagrangian system. The critical sections of the Lagrangian variational problem are the integral sections of a class of holonomic multivector fields \( \{ X_L \} \subset \mathcal{X}^m(J^1 E) \), such that

\[
i(X_L)\Omega_L = 0, \quad \text{for every } X_L \in \{ X_L \}
\]

(Proof) The critical sections must be the integral sections of a class of holonomic multivector fields \( \{ X_L \} \subset \mathcal{X}^m(J^1 E) \), as a consequence of the above discussion.

Now, using the local expression (1) of \( \Omega_L \), and taking \( [\ ] \) as the representative of the class of semi-holonomic multivector fields \( \{ X_L \} \), from the relation \( i(X_L)\Omega_L = 0 \) we have that the coefficients on \( dv^A_\mu \), \( dy^A \) and \( dx^\mu \) must vanish. But, for the coefficients on \( dv^A_\mu \) we obtain the identities

\[
0 = \left( v^B_\mu - v^B_\mu \right) \frac{\partial^2 \mathcal{L}}{\partial v^A_\mu \partial v^B_\mu} \quad \text{ (for every } A, \nu \)
\]

meanwhile the condition for the coefficients on \( dy^A \) leads to the system of equations

\[
\frac{\partial^2 \mathcal{L}}{\partial v^B_\nu \partial v^A_\mu} G^B_{\mu \nu} = \frac{\partial \mathcal{L}}{\partial y^A} - \frac{\partial^2 \mathcal{L}}{\partial x^\nu \partial v^A_\mu} - \frac{\partial^2 \mathcal{L}}{\partial y^B \partial v^A_\mu} v^B_\mu \quad (A = 1, \ldots, N)
\]  

(7)

Therefore, if \( j^1 \phi = \left( x^\mu, f^A, \frac{\partial f^A}{\partial x^\nu} \right) \) must be an integral section of \( X_L \), then \( v^A_\mu = \frac{\partial f^A}{\partial x^\mu} \), and hence the coefficients \( G^B_{\mu \nu} \) must satisfy equations (6). As a consequence, the system (6) is equivalent to the Euler-Lagrange equations for the section \( \phi \). Note that, from the above conditions, the coefficients on \( dx^\mu \) vanish identically.

So, in Lagrangian field theories, we search for (classes of) non-vanishing and locally decomposable multivector fields \( X_L \in \mathcal{X}^m(J^1 E) \) such that:

1. The equation \( i(X_L)\Omega_L = 0 \) holds.
2. \( X_L \) are semi-holonomic.
3. \( X_L \) are integrable.

Then we introduce the following nomenclature:

**Definition 1** \( X_L \in \mathcal{X}^m(J^1 E) \) is said to be an Euler-Lagrange multivector field for \( \mathcal{L} \) if it is semi-holonomic and is a solution of the equation \( i(X_L)\Omega_L = 0 \).
Observe that neither the compatibility of the system (7), nor the integrability of (5) are assured. Thus, the existence of Euler-Lagrange multivector fields is not guaranteed in general, and, if they exist, they are not integrable necessarily. Then:

**Theorem 3** (Existence and local multiplicity of Euler-Lagrange multivector fields). Let \((J^1E, \Omega_L)\) be a regular Lagrangian system. Then:

1. There exist classes of Euler-Lagrange multivector fields for \(L\).
2. In a local system these multivector fields depend on \(N(m^2 - 1)\) arbitrary functions.

(Proof)

1. First we analyze the local existence of solutions and then their global extension.

   In a chart of natural coordinates in \(J^1E\), using the local expression (1) of \(\Omega_L\), and taking the multivector field given in (3) (with \(f_\mu = 1\), for every \(\mu\)) as the representative of the class \(\{X_L\}\), from the relation \(i(X_L)\Omega_L = 0\) we have that the coefficients on \(dv^A_\mu\), \(dy^A\) and \(dx^\mu\) must vanish.

   Thus, for the coefficients on \(dv^A_\mu\), we obtain that

   \[
   0 = (F^B_\mu - v^B_\mu) \frac{\partial^2 L}{\partial v^B_\nu \partial v^B_\mu} \quad \text{(for every } A, \nu)\]

   But, if \(L\) is regular, the matrix \(\frac{\partial^2 L}{\partial v^B_\nu \partial v^B_\mu}\) is regular. Therefore \(F^B_\mu = v^B_\mu\) (for every \(B, \mu\)); which proves that if \(X_L\) exists it is semi-holonomic.

   Afterwards, from the condition for the coefficients on \(dy^A\), and taking into account that we have obtained \(F^B_\mu = v^B_\mu\), we obtain the equations (7), which is a system of \(N\) linear equations on the functions \(G^B_\nu^\mu\). This is a compatible system as a consequence of the regularity of \(L\), since the matrix of the coefficients has (constant) rank equal to \(N\) (observe that the matrix of this system is obtained as a rearrangement of rows of the Hessian matrix).

   From the above conditions, we obtain that the coefficients on \(dx^\mu\) vanish identically.

   These results allow us to assure the local existence of (classes of) multivector fields satisfying the desired conditions. The corresponding global solutions are then obtained using a partition of unity subordinated to a covering of \(J^1E\) made of local natural charts.

2. The expression of a semi-holonomic multivector field \(X_L \in \{X_L\}\) is given by (4). So, it is determined by the \(Nm^2\) coefficients \(G^B_\nu^\mu\), which are related by the \(N\) independent equations (7). Therefore, there are \(N(m^2 - 1)\) arbitrary functions.

Now the problem is to finding a class of integrable Euler-Lagrange multivector field, if it exists. So, we can choose from the solutions of this system, those such that \(X_L\) verify the integrability condition; that is, the associated connection \(\nabla_L\) is flat (equations (6)). If the equations (7) and the first group of equations (6) allow us to isolate \(N + \frac{1}{2}Nm(m - 1)\) coefficients \(G^A_\nu^\mu\) as functions on the remaining ones; and the set of \(\frac{1}{2}Nm^2(m - 1)\) partial differential equations (the second group of equations (6)) on these remaining coefficients satisfies the conditions on Cauchy-Kowalewska's theorem (8), then the existence of integrable Euler-Lagrange multivector fields is assured.
• Remark: (Singular Lagrangian systems)

For singular Lagrangian systems, the existence of Euler-Lagrange multivector fields is not assured except perhaps on some submanifold \( I \hookrightarrow J^1E \). Even more, locally decomposable and \( \tilde{\tau}^1 \)-transverse multivector fields, solutions of the field equations can exist (in general, on some submanifold of \( J^1E \)), but none of them being semi-holonomic (at any point of this submanifold). As in the regular case, although Euler-Lagrange multivector fields exist on some submanifold \( S \), their integrability is not assured except perhaps on another smaller submanifold \( I \hookrightarrow S \); such that the integral sections are contained in \( I \). This condition implies that \( \tilde{\tau}^1|_I : I \rightarrow M \) must be onto on \( M \).

The local treatment of the singular case is as follows: starting from (8), and taking the representative obtained by making \( f_\mu = 1 \), for every \( \mu \), we can impose the semi-holonomic condition by making \( P_\mu^A = v_\mu^A \), for every \( A, \mu \). Therefore, we have the system of equations (9) for the coefficients \( G_\mu^A \); but this system is not compatible in general except perhaps in a set of points \( S_1 \subset J^1E \), which is assumed to be a non-empty closed submanifold. Then, there are Euler-Lagrange multivector fields on \( S_1 \), but the number of arbitrary functions on which they depend is not the same as in the regular case, since it depends on the dimension of \( S_1 \) and the rank of the Hessian matrix of \( L \). Next, the tangency condition must be analyze; and finally the question of integrability must be considered as above, but for a submanifold of \( S_1 \).

4 Hamiltonian equations in classical field theories

For the Hamiltonian formalism of field theories, the choice of a multimomentum phase space or multimomentum bundle is not unique (see 8). In this work we take:

\[ J^{1*}E \equiv \Lambda_1^m T^* E / \Lambda_0^m T^* E \]

(where \( \Lambda_0^m T^* E \) is the bundle of \( m \)-forms on \( E \) vanishing by the action of two \( \pi \)-vertical vector fields, and \( \Lambda_0^m T^* E \equiv \pi^* \Lambda^m T^* M \)). We have the natural projections

\[ \tau^1: J^{1*}E \rightarrow E \quad , \quad \tilde{\tau}^1 = \pi \circ \tau^1: J^{1*}E \rightarrow M \]

and we denote by \((x^\mu, y^A, p_\mu^A)\) the natural local systems of coordinates in \( J^{1*}E \) adapted to these bundle structures \((\mu = 1, \ldots, m; A = 1, \ldots, N)\).

For constructing Hamiltonian systems, \( J^{1*}E \) must be endowed with a geometric structure. There are different ways for making this, namely: using Hamiltonian sections, or Hamiltonian densities [3], [8], [14]. So we construct the Hamilton-Cartan \( m \) and \((m+1)\) forms \( \Theta_h \in \Omega^m(J^{1*}E) \), and \( \Omega_h = -d\Theta_h \in \Omega^{m+1}(J^{1*}E) \), which have the local expressions (in an open set \( U \subset J^{1*}E \))

\[
\Theta_h = p_\mu^A dy^A \wedge dx_{\mu-1}^m - H \wedge dx^m \\
\Omega_h = -dp_\mu^A \wedge dy^A \wedge dx_{\mu-1}^m + dH \wedge dx^m
\]

(8)

where \( H \in C^\infty(U) \) is a local Hamiltonian function. A couple \((J^{1*}E, \Omega_h)\) is said to be a Hamiltonian system.

We can state a variational problem for \((J^{1*}E, \Omega_h)\) (Hamilton-Jacobi principle): the states of the field are the sections of \( \tilde{\tau}^1 \) which are critical for the functional \( H(\psi) := \int_M \psi^* \Theta_h \), for every \( \psi \in \Gamma(M, J^{1*}E) \). They are characterized by the condition 8, 9

\[ \psi^* i(X) \Omega_h = 0 \quad , \text{for every } X \in \mathcal{X}(J^{1*}E) \]
In natural coordinates, if \( \psi(x) = (x^\mu, y^A(x), p^\rho_A(x)) \), this condition leads to the system

\[
\frac{\partial y^A}{\partial x^\mu}\bigg|_\psi = \frac{\partial H}{\partial p^\rho_A}\bigg|_\psi ; \quad \frac{\partial p^\mu_A}{\partial x^\mu}\bigg|_\psi = -\frac{\partial H}{\partial y^A}\bigg|_\psi
\]

which is known as the Hamilton-De Donder-Weyl equations.

Let \( (J^1, E, \Omega_h) \) be a Hamiltonian system. The problem of finding critical sections solutions of the Hamilton-Jacobi principle can be formulated equivalently as follows: to finding a distribution \( D \) of \( T(J^1, E) \) satisfying that:

- \( D \) is integrable (that is, involutive).
- \( D \) is \( m \)-dimensional.
- \( D \) is \( \bar{\tau}^1 \)-transverse.
- The integral manifolds of \( D \) are the critical sections of the Hamilton-Jacobi principle.

Then, from the first and the second conditions, there exist \( X_1, \ldots, X_m \in \mathcal{X}(J^1, E) \) (in involution), which locally span \( D \). Therefore \( X = X_1 \wedge \ldots \wedge X_m \) defines a section of \( \Lambda^m T(J^1, E) \), that is, a non-vanishing, locally decomposable multivector field in \( J^1, E \), whose local expression in natural coordinates is

\[
X = \bigwedge_{\mu=1}^m f_\mu \left( \frac{\partial}{\partial x^\mu} + F^A_\mu \frac{\partial}{\partial y^A} + G^\rho_{A\mu} \frac{\partial}{\partial p^\rho_A} \right)
\]

where \( f_\mu \in C^\infty(J^1, E) \) are non-vanishing functions. A representative of the class \( \{X\} \) can be selected by the condition \( i(X)(\bar{\tau}^1, \omega) = 1 \) which, as a particular solution, leads to \( f_\mu = 1 \), for every \( \mu \).

Therefore, the problem posed by the Hamilton-Jacobi principle can be stated in the following way:

**Theorem 4** The critical sections of the Hamilton-Jacobi principle are the sections \( \psi \in \Gamma_c(M, J^1, E) \) such that they are the integral sections of a class of integrable and \( \bar{\tau}^1 \)-transverse multivector fields \( \{X_H\} \subset \mathcal{X}^m(J^1, E) \) satisfying that

\[
i(X_H)\Omega_h = 0 , \quad \text{for every } X_H \in \{X_H\}
\]

(Proof) The critical sections must be the integral sections of a class of integrable and \( \bar{\tau}^1 \)-transverse multivector fields \( \{X_H\} \subset \mathcal{X}^m(J^1, E) \), as a consequence of the above discussion.

Now, using the local expression (8) of \( \Omega_h \); and taking the multivector field (10) (with \( f_\mu = 1 \), for every \( \mu \)) as a representative of the class \( \{X_H\} \), from \( i(X_H)\Omega_h = 0 \) we obtain that the coefficients on \( dp^\mu_A \) must vanish:

\[
0 = F^A_\nu - \frac{\partial H}{\partial p^\nu_A} \quad \text{(for every } A, \nu \text{)}
\]

and the same happens for the coefficients on \( dy^A \):

\[
0 = G^\mu_{A\nu} + \frac{\partial H}{\partial y^A} \quad (A = 1, \ldots, N)
\]
Using these results, the coefficients on $dx^\mu$ vanish identically).

Now, if $\psi(x) = (x^\mu, y^A(x^\nu), p^A_\mu(x^\nu))$ has to be an integral section of $X_H$ then

$$F^A_\mu \circ \psi = \frac{\partial y^A}{\partial x_\mu}; \ G^\mu_{A_\mu} \circ \psi = \frac{\partial p^A_\mu}{\partial x^\mu}$$

and equations (11) and (12) are the Hamilton-De Donder-Weyl equations (3) for $\psi$.

Thus, we search for (classes of) $\tilde{\tau}^1$-transverse and locally decomposable multivector fields $X_H \in \mathfrak{X}^m(J^{1*}E)$ such that:

1. $i(X_H)\Omega_h = 0$ holds.
2. $X_H$ are integrable.

Classes of locally decomposable and $\tilde{\tau}^1$-transverse multivector fields are in one-to one correspondence with connections in the bundle $\tilde{\tau}^1: J^{1*}E \to M$. Then $X_H$ is integrable if, and only if, the curvature of the connection associated with this class vanishes everywhere.

**Definition 2** $X_H \in \mathfrak{X}^m(J^{1*}E)$ will be called a Hamilton-De Donder-Weyl (HDW) multivector field for the system $(J^{1*}E, \Omega_h)$ if it is $\tilde{\tau}^1$-transverse, locally decomposable and verifies the equation $i(X_H)\Omega_h = 0$.

For a Hamiltonian system, the existence of Hamilton-De Donder Weyl multivector fields is guaranteed, although they are not integrable necessarily. In fact:

**Theorem 5** (Existence and local multiplicity of HDW-multivector fields): Let $(J^{1*}E, \Omega_h)$ be a Hamiltonian system. Then

1. There exist classes of HDW-multivector fields $\{X_H\}$.
2. In a local system the above solutions depend on $N(m^2 - 1)$ arbitrary functions.

(Proof)

1. Bearing in mind the proof of Theorem 4, we have that the equations (11) make a system of $Nm$ linear equations which determines univocally the functions $F^A_\nu$, meanwhile the equations (12) are a compatible system of $N$ linear equations on the $Nm^2$ functions $G^\mu_{A_\nu}$. These results assure the local existence. The global solutions are obtained using a partition of unity subordinated to a covering of $J^{1*}E$ made of natural charts.

2. In natural coordinates in $J^{1*}E$, a representative of a class of HDW-multivector fields $X_H \in \{X_H\}$ is given by (11) (with $f_\mu = 1$, for every $\mu$). So, it is determined by the $Nm$ coefficients $F^A_\nu$, which are obtained as the solution of (11), and by the $Nm^2$ coefficients $G^\mu_{A_\nu}$, which are related by the $N$ independent equations (12). Therefore, there are $N(m^2 - 1)$ arbitrary functions.
coefficients assures the existence of an integrable solution. Considering the Hamiltonian equations (12) for the field solution, and applying an integrability algorithm in order to find a submanifold

\[ X \]

coefficients satisfies certain conditions, then the existence of integrable HDW-multivector fields (in A. Echeverría-Enríquez

\[ H \]

be in general less than (where use is made of the Hamiltonian equations). Hence the number of arbitrary functions will remaining ones; and the set of 1

\[ N \]

restrictions (12) and (13) allow us to isolate

\[ N \]

functions is not the same as above (it depends on the dimension of

\[ S \]

must be made for the submanifold

\[ S \]

Finally, the question of integrability must be considered, and similar considerations as above must be made for the submanifold

\[ S \]

For finding a class of integrable HDW-multivector fields (if it exists) we must impose that

\[ X \]

verify the integrability condition: the curvature of the associated connection \( \nabla_X \) vanishes everywhere, that is, the following system of equations holds (for \( 1 \leq \mu < \eta \leq m \))

\[ 0 = \frac{\partial F^B}{\partial x^\mu} + F^A_\mu \frac{\partial F^B}{\partial y^A} + G^\gamma_A \frac{\partial F^B_\eta}{\partial x^\eta} - \frac{\partial F^B_\eta}{\partial x^\mu} - F^A_\eta \frac{\partial F^B}{\partial y^A} - G^\rho_A \frac{\partial F^B}{\partial p^\rho_A} \]

\[ = \frac{\partial^2 H}{\partial x^\mu p^\rho_B} + \frac{\partial H}{\partial p^\rho_A} \frac{\partial^2 H}{\partial y^A p^\rho_B} \]

\[ G^\gamma_A \frac{\partial G^\mu_B}{\partial p^\gamma_A} - \frac{\partial^2 H}{\partial x^\eta \partial p^\rho_B} - \frac{\partial H}{\partial y^A \partial p^\rho_B} - G^\rho_A \frac{\partial^2 H}{\partial p^\rho_A \partial y^A} = \]

\[ 0 = \frac{\partial G^\mu_B}{\partial x^\mu} + F^A_\mu \frac{\partial G^\mu_B}{\partial y^A} + G^\gamma_A \frac{\partial G^\mu_B}{\partial p^\gamma_A} - \frac{\partial G^\mu_B}{\partial x^\eta} - F^A_\eta \frac{\partial G^\mu_B}{\partial y^A} - G^\rho_A \frac{\partial G^\mu_B}{\partial p^\rho_A} \]

\[ = \frac{\partial G^\mu_B}{\partial p^\gamma_A} + \frac{\partial H}{\partial y^A} \frac{\partial G^\mu_B}{\partial p^\gamma_A} + \]

\[ G^\gamma_A \frac{\partial G^\mu_B}{\partial p^\gamma_A} - \frac{\partial G^\mu_B}{\partial x^\eta} - \frac{\partial H}{\partial p^\gamma_A} \frac{\partial G^\mu_B}{\partial y^A} - G^\rho_A \frac{\partial G^\mu_B}{\partial p^\rho_A} \]

\[ (13) \]

\[ (14) \]

(where use is made of the Hamiltonian equations). Hence the number of arbitrary functions will be in general less than \( N(m^2 - 1) \).

As this is a system of partial differential equations with linear restrictions, there is no way of assuring the existence of an integrable solution. Considering the Hamiltonian equations (12) for the coefficients

\[ G^\mu_A \]

\[ H \]

together with the integrability conditions (13) and (14), we have \( N + \frac{1}{2} N m(m - 1) \) linear equations and \( \frac{1}{2} N m^2 (m - 1) \) partial differential equations. Then, if the set of linear restrictions (12) and (13) allow us to isolate \( N + \frac{1}{2} N m(m - 1) \) coefficients

\[ G^\mu_A \]

\[ H \]

\[ G^\mu_A \]

on the remaining ones; and the set of \( \frac{1}{2} N m^2 (m - 1) \) partial differential equations (14) on these remaining coefficients satisfies certain conditions, then the existence of integrable HDW-multivector fields (in \( J^1 * E \)) is assured. If this is not the case, we can eventually select some particular HDW-multivector field solution, and apply an integrability algorithm in order to find a submanifold \( \mathcal{I} \hookrightarrow J^1 * E \) (if it exists), where this multivector field is integrable (and tangent to \( \mathcal{I} \)).

Remarks:

- (Restricted Hamiltonian systems)

There are many interesting cases in field theories where the Hamiltonian field equations are established not in \( J^1 * E \), but rather in a submanifold \( j_0: P \hookrightarrow J^1 * E \), such that \( P \) is a fiber bundle over \( E \) (and \( M \)), and the corresponding projections \( \tau_0^1: P \to E \) and \( \tilde{\tau}_0^1: P \to M \) satisfy that \( \tau^1 \circ j_0 = \tau_0^1 \) and \( \tilde{\tau}^1 \circ j_0 = \tilde{\tau}_0^1 \).

Now, not even the existence of HDW-multivector fields is assured, and an algorithmic procedure in order to obtain a submanifold \( S_f \) of \( P \) where HDW-multivector fields exist, can be outlined. Of course the solution is not unique, in general, but the number of arbitrary functions is not the same as above (it depends on the dimension of \( S_f \)).

Finally, the question of integrability must be considered, and similar considerations as above must be made for the submanifold \( S_f \) instead of \( J^1 * E \).
• (Hamiltonian system associated with a hyper-regular Lagrangian system)

If the Hamiltonian system \((J^1E, \Omega_h)\) is associated with a hyper-regular Lagrangian system, then there exists the so-called Legendre map, which is a diffeomorphism between \(J^1E\) and \(J^1*E\). In this case, it can be proved that, if \(X_L \in \mathfrak{X}^m(J^1E)\) and \(X_H \in \mathfrak{X}^m(J^1*E)\) are multivector fields solution of the Lagrangian and Hamiltonian field equations respectively, then

\[
\Lambda^mTFL \circ X_L = f X_H \circ FL
\]

for some \(f \in C^\infty(J^1*E)\). That is, we have the following (commutative) diagram:

\[
\begin{array}{ccc}
\Lambda^mTJ^1E & \longrightarrow & \Lambda^mTJ^1*E \\
X_L \uparrow & & \uparrow X_H \\
J^1E & \longrightarrow & J^1*E
\end{array}
\]

(we say that the classes \(\{X_L\}\) and \(\{X_H\}\) are \(FL\)-related).

5 Conclusions and outlook

We have used multivector fields in fiber bundles for setting and studying the Lagrangian and Hamiltonian field equations of first-order classical field theories. In particular, we have showed that:

• The field equations for first order classical field theories in the Lagrangian formalism (Euler-Lagrange equations) can be written using multivector fields in \(J^1E\). This description allow us to write the field equations for field theories in an analogous way to the dynamical equations for (time-dependent) Lagrangian mechanical systems.

• The Lagrangian equations can have no integrable solutions in \(J^1E\), for neither regular nor singular Lagrangian systems.

In the regular case, Euler-Lagrange multivector fields (that is, semiholonomic and solution of the equation \(i(X_L)\Omega_L = 0\)) always exist; but they are not necessarily integrable. In the singular case, not even the existence of such an Euler-Lagrange multivector field is assured. In both cases, the multivector field solution (if it exists) is not unique.

• The Hamiltonian field equations can be written using multivector fields in \(J^1*E\) (the multimomentum bundle of the Hamiltonian formalism) in an analogous way to the dynamical equations for (time-dependent) Hamiltonian mechanical systems.

• The field equations \(i(X_H)\Omega_h = 0\), with \(X_H \in \mathfrak{X}^m(J^1*E)\) locally decomposable and \(\tilde{\tau}\)-transverse, have solution everywhere in \(J^1*E\), which is not unique; that is, there are classes of Hamilton-De Donder-Weyl multivector fields which are solution of these equations. Nevertheless, these multivector fields are not necessarily integrable everywhere in \(J^1*E\).

• This multivector field formulation is specially useful for characterizing symmetries, both in the Lagrangian and Hamiltonian formalisms of field theories. First attempts in this subject have been already done, but new developments in this area are expected to be reached in the future.
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