Research Article

A Study of Third and Fourth Hankel Determinant Problem for a Particular Class of Bounded Turning Functions

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In this present paper, a new generalized class $R_{p,q}$ from the family of function with bounded turning was introduced by using $(p,q)$-derivative operator. Our aim for this class is to find out the upper bound of third- and fourth-order Hankel determinant. Moreover, the upper bounds for two-fold and three-fold symmetric functions for this class are also obtained.

1. Introduction and Motivation

In order to better explain the terminology included in our key observations, some of the essential relevant literatures on geometric function theory need to be provided and discussed here. We start with symbol $\mathcal{A}$ which represents the class of holomorphic functions in the region of open unit disc $\mathbb{D} = \{z: |z| < 1\}$ and satisfy the relationship $g(0) = g'(0) - 1 = 0$ for $g \in \mathcal{A}$. That is, if $g \in \mathcal{A}$, then it has the following Taylor series form:

$$g(z) = z + \sum_{k=2}^{\infty} \delta_k z^k, \quad (z \in \mathbb{D}). \quad (1)$$

Also, let $\Psi \subset \mathcal{A}$ represent all univalent functions in $\mathbb{D}$. Next, we are going to define the most useful class of geometric function theory known as the class $\mathcal{P}$ of Carathéodory functions and is defined as; a holomorphic function $h$ belongs to $\mathcal{P}$ if it satisfies $\Re(h(z)) > 0$ along with the series expansion:

$$h(z) = 1 + \sum_{k=1}^{\infty} d_k z^k, \quad (z \in \mathbb{D}). \quad (2)$$

Using the Carathéodory functions family, we consider the following basics subclasses of $\Psi$ as

$$\Delta^* = \left\{ g(z) \in \mathcal{A}: zg'(z) \in \mathcal{P}, \quad (z \in \mathbb{D}) \right\},$$

$$\mathcal{K} = \left\{ g(z) \in \mathcal{A}: \frac{g'(z)}{g'(z)} \in \mathcal{P}, \quad (z \in \mathbb{D}) \right\}, \quad (3)$$

$$\mathcal{R} = \left\{ g(z) \in \mathcal{A}: g'(z) \in \mathcal{P}, \quad (z \in \mathbb{D}) \right\}.$$

The investigation of $q$-calculus ($q$ stands for quantum) fascinated and inspired many scholars due its use in various areas of the quantitative sciences. Jackson [1, 2] was among the key contributors of all the scientists who introduced and developed the $q$-calculus theory. Just like $q$-calculus was used in other mathematical sciences, the formulations of this idea are commonly used to examine the existence of various structures of function theory. Though the link between certain geometric nature of the analytic function and the $q$-derivative operator was established by the authors in [3], but, for the usage of $q$-calculus in function theory, a solid and comprehensive
foundation is given in [4] by Srivastava. After this development, many researchers introduced and studied some useful operators in $q$-analog with the applications of convolution concepts. For example; Kanas and Răducanu [5] established the $q$-differential operator and then examined the behavior of this operator in function theory. This operator was generalized further for multivalent analytic functions by Arif et al. [6]. Analogous to $q$-differential operator, Arif et al. and Khan et al. contributed the integral operators for analytical and multivalent functions in [7, 8], respectively. Similarly, in the article [9], the authors developed and analyzed the operator in $q$-analog for meromorphic functions. Also, see the survey type article [10] on quantum calculus and their applications. With the use of these operators, many researchers were contributed some good articles in this direction in the field of geometric function theory, see [11, 12].

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The substitution of $q$ by $q/p$ in $q$-calculus has given us $(p,q)$-calculus which is the extension of $q$-calculus. Chakrabarti and Jagannathan [13] considered $(p,q)$-integer.

For $0 < q < p < 1$ and $z \neq 0$, the $(p,q)$-derivative operator $D_{p,q}g$ of the function $g(z) \in \mathcal{A}$ is defined as

$$D_{p,q}g(z) = \frac{g(pz) - g(qz)}{(p-q)z}.$$  \hspace{1cm} (4)

From the use (1) and (4), the following formulae are easily obtained:

$$D_{p,q}(g_1(z) + g_2(z)) = D_{p,q}g_1(z) + D_{p,q}g_2(z),$$  \hspace{1cm} (5)

and for a constant $c$,

$$D_{p,q}(cg(z)) = cD_{p,q}g(z).$$  \hspace{1cm} (6)

Here, we also note that $D_{p,q}g(z) = g'(z)$ when we take $p = 1$ and $q \rightarrow 1^-$. 

$$\lim_{q \rightarrow 1^-} D_{p,q}(g(z)) = \lim_{q \rightarrow 1^-} \frac{g(z) - g(qz)}{(1-q)z} = g'(z).$$  \hspace{1cm} (7)

Further by using (1) and (4), we have

$$D_{p,q}(g(z)) = 1 + \sum_{\lambda=2}^{\infty} [\lambda]_{p,q}\delta_\lambda z^{\lambda-1},$$  \hspace{1cm} (8)

with

$$[\lambda]_{p,q} = \frac{p^{\lambda} - q^{\lambda}}{p-q} = p^{\lambda-1}\left(1 + \sum_{i=1}^{\lambda-1} \left(\frac{q}{p}\right)^i\right).$$  \hspace{1cm} (9)

Let us define the class $\mathcal{R}_{p,q}$, which consists all the functions $g \in \mathcal{A}$, satisfying

$$\text{Re}(D_{p,q}(g(z))) > 0, \quad z \in \mathbb{U}.$$  \hspace{1cm} (10)

The Hankel determinant $\mathcal{H}D_{n\lambda}(g)$ with $n, \lambda \in \mathbb{N} = \{1,2, \ldots\}$ and $\delta_1 = 1$ for a function $g \in \Psi$ of the series form (1) was given by Pommerenke [14, 15], as

$$\mathcal{H}D_{n\lambda}(g) = \begin{vmatrix} \delta_1 & \delta_{\lambda+1} & \cdots & \delta_{\lambda+n-1} \\ \delta_{\lambda+1} & \delta_{\lambda+2} & \cdots & \delta_{\lambda+n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\lambda+n-1} & \delta_{\lambda+n} & \cdots & \delta_{\lambda+2n-2} \end{vmatrix}.$$  \hspace{1cm} (11)

In particular, the following determinants are known as the first-, second-, and third-order Hankel determinants, respectively.

$$\mathcal{H}D_{2,1}(g) = \begin{vmatrix} 1 & \delta_2 \\ \delta_2 & \delta_3 \end{vmatrix} = \delta_3 - \delta_2^2,$$

$$\mathcal{H}D_{2,2}(g) = \begin{vmatrix} 1 & \delta_2 & \delta_3 \\ \delta_2 & \delta_3 & \delta_4 \end{vmatrix} = \delta_4 - \delta_4^2,$$

$$\mathcal{H}D_{3,1}(g) = \begin{vmatrix} 1 & \delta_2 & \delta_3 & \delta_4 \\ \delta_2 & \delta_3 & \delta_4 & \delta_5 \end{vmatrix} = \delta_5 - \delta_4^2.$$  \hspace{1cm} (12)

There are comparatively few observations in literature in relation to the Hankel determinant for the function $g$ which belongs to the general family $\Psi$. For the function $g \in \Psi$, the best established sharp inequality is $|\mathcal{H}D_{2,1}(g)| \leq \mu \sqrt{\lambda}$, where $\mu$ is the absolute constant, which is due to Hayman [16]. Further for the same class $\Psi$, it was obtained in [17] that

$$|\mathcal{H}D_{2,2}(g)| \leq \mu_1, \quad \text{for} \quad 1 \leq \mu_1 \leq \frac{11}{3},$$

$$|\mathcal{H}D_{3,1}(g)| \leq \mu_2, \quad \text{for} \quad \frac{4}{9} \leq \mu_2 \leq \frac{32 + \sqrt{285}}{15}.$$  \hspace{1cm} (13)

In a given family of functions, the problem of calculating the bounds, probably sharp, of Hankel determinants attracted the minds of several mathematicians. For example, the sharp bound of $|\mathcal{H}D_{2,2}(g)|$, for the subfamilies $\mathcal{H}$, $\mathcal{S}^*$, and $\mathcal{R}$ (family of bounded turning functions) of the set $\Psi$, was calculated by Janet et al. [18, 19]. These estimates are

$$|\mathcal{H}D_{2,2}(g)| \leq 1, \quad \text{for} \quad f \in \mathcal{S}^*,$$

$$|\mathcal{H}D_{2,2}(g)| \leq \frac{4}{9}, \quad \text{for} \quad f \in \mathcal{R}.$$  \hspace{1cm} (14)

For the following two families, $\mathcal{S}^*(\beta) (0 \leq \beta < 1)$ of star-like functions of order $\beta$ and for $\delta^* \mathcal{S}^*(\beta) (0 < \beta \leq 1)$ of strongly star-like functions of order $\beta$, the authors computed in [20, 21] that $|\mathcal{H}D_{2,2}(g)|$ is bounded by $\left(1 - \beta^*\right)^*\delta^2$, respectively. The exact bound for the family of Ma-Minda star-like functions was measured in [22], see also [23]. For more work on $|\mathcal{H}D_{2,2}(g)|$, see references [24–28].

It is quite clear from the formulae given in (12) that the calculation of $|\mathcal{H}D_{3,1}(g)|$ is far more challenging compared...
with finding the bound of $|\mathcal{HD}_{3,1}(g)|$. Babalola was the first mathematician who investigated the bounds of third-order Hankel determinant for the families of $\mathcal{K}$, $\mathcal{A}$, and $\mathcal{R}$ in an article [29] published in 2010. Using the same approach, later, several authors [30–34] published their articles regarding $|\mathcal{HD}_{3,1}(g)|$ for certain subfamilies of analytic and univalent functions. After this study, Zaprawa [35] improved the findings of Babalola in 2017 by applying a new methodology. He obtained the following bounds:

$$|\mathcal{HD}_{3,1}(g)| \leq \begin{cases} \frac{49}{540} & \text{for } g \in \mathcal{K}, \\ 1, & \text{for } g \in \mathcal{A}, \\ \frac{41}{60} & \text{for } g \in \mathcal{R}. \end{cases} \quad (15)$$

He argued that such limits are indeed not the best. Later in 2018, Kwon et al. [36] strengthened the Zaprawa’s result for $g \in \mathcal{A}$ and showed that $|\mathcal{HD}_{3,1}(g)| \leq \left(\frac{8}{9}\right)$ and this bound was further improved by Zaprawa et al. [37] in 2021. They got $|\mathcal{HD}_{3,1}(g)| \leq \left(\frac{5}{9}\right)$ for $g \in \mathcal{A}$. Recently in 2018, Kowalczyk et al. [38] and Lecko et al. [39] succeeded in finding the sharp bounds of $|\mathcal{HD}_{3,1}(g)|$ for the families $\mathcal{K}$ and $\mathcal{A}$ (1/2), respectively, where $\mathcal{A}$ (1/2) indicate the starlike functions family of order 1/2. These results are given as

$$|\mathcal{HD}_{3,1}(g)| \leq \begin{cases} \frac{4}{135} & \text{for } g \in \mathcal{K}, \\ \frac{1}{9} & \text{for } g \in \mathcal{A}. \end{cases} \quad (16)$$

The estimation of fourth Hankel determinant $|\mathcal{HD}_{4,1}(g)|$ for the bounded turning functions has been obtained by Arif et al. [40] and they proved the following bounds for $g \in \mathcal{R}$:

$$|\mathcal{HD}_{4,1}(g)| \leq 0.78050. \quad (17)$$

After that, Kaur and Singh [41] proved fourth Hankel determinant for bounded turning function of order $\alpha$. For more contributions, see [42–46]. Recently, Srivastava et al. [47] consider a family of normalized analytic functions with bounded turnings in the open unit disk which are connected with the cardioid domains and they obtained the estimates of fourth Hankel determinant.

2. A Set of Results

In order to investigate $|\mathcal{HD}_{3,1}(g)|$, we need the following results.

**Lemma 1** (see [48]). If $h(z) \in \mathcal{P}$, having the form (2), then

$$\begin{align*}
[d_{1}] |2; & \text{ for } \lambda \in \mathbb{N}, \\
[d_{1+k}-\mu d_{1}] & \leq 2; \text{ for } 0 \leq \mu \leq 1, \\
[d_{1}^{2}d_{3}] & \leq 4; \text{ for } p + q = r + s.
\end{align*} \quad (18)$$

**Theorem 1** (see [40]). Let $g(z) = z + \sum_{k=2}^{\infty} \delta_{k}z^{k} \in \mathcal{P}$ and for real $\mu$,

$$\delta_{3}^{2}(\delta_{3} - \mu \delta_{2}^{2}) = \begin{cases} 4(3 - 4\mu), & \text{for } \mu \leq \frac{5}{8}; \\
\frac{1}{2(2\mu - 1)}, & \text{for } \mu \in \left[\frac{3}{8}, \frac{5}{8}\right]; \\
\frac{1}{4(1 - \mu)}, & \text{for } \mu \in \left[\frac{3}{8}, \frac{7}{8}\right]; \\
4(4\mu - 3), & \text{for } \mu \geq \frac{7}{8}. \end{cases} \quad (21)$$

3. Bounds of Third Hankel Determinant

The third Hankel determinant $|\mathcal{HD}_{3,1}(g)|$ is a polynomial of four variables, as:

$$|\mathcal{HD}_{3,1}(g)| = (\delta_{3}\delta_{5} - \delta_{3}^{2}) + \delta_{2}(\delta_{1}\delta_{4} - \delta_{2}\delta_{3}) + \delta_{3}(\delta_{2}\delta_{4} - \delta_{3}^{2}). \quad (22)$$

In order to solve $|\mathcal{HD}_{3,1}(g)|$, we need to know the correspondence between $g$ and $h \in \mathcal{P}$.

$$g(z) \in \mathcal{P} \iff D_{p,q}g(z) \in \mathcal{P}. \quad (23)$$

Thus,

$$\left(1 + \sum_{k=2}^{\infty} \left[\frac{1}{\lambda} p_{q} \delta_{k}z^{k-1}\right]\right) = \left(1 + \sum_{k=1}^{\infty} d_{k}z^{k}\right). \quad (24)$$

By simplifying, we yield

$$\delta_{3} = \frac{d_{k-1}}{[\lambda]_{p,q}}. \quad (25)$$

Now, using the above coefficients in (22), we obtain

$$\mathcal{HD}_{3,1}(g) = \frac{1}{[3]_{p,q}[5]_{p,q}} d_{2}d_{4} + \frac{1}{[4]_{p,q}} d_{2}^{2} + \frac{2}{[2]_{p,q}[3]_{p,q}[4]_{p,q}} d_{1}d_{3}d_{4} - \frac{1}{[2]_{p,q}[5]_{p,q}} d_{2}^{2}d_{4} - \frac{1}{[3]_{p,q}} d_{3}^{2}. \quad (26)$$
Rearranging the above terms,

\[
\mathcal{H}_3,1 (g) = \frac{1}{[2]_{pq}[3]_{pq}[4]_{pq}} d_4 (d_2 - d_1^2) - \frac{1}{[4]_{pq}} d_3 (d_3 - d_1 d_2) + \frac{1}{[3]_{pq}[4]_{pq}} d_3 (d_4 - d_2^2)
- \left( \frac{2}{[2]_{pq}[3]_{pq}[4]_{pq}} - \frac{1}{[4]_{pq}} \right) d_2 (d_4 - d_1 d_3)
+ \left( \frac{1}{[3]_{pq}[5]_{pq}} - \frac{2}{[2]_{pq}[5]_{pq}} - \frac{1}{[4]_{pq}} \right) d_3 d_4.
\] (27)

Using triangular inequality and the results (18) and (19) of lemma of Section 2, we get

\[
|\mathcal{H}_3,1 (g)| \leq 4 \left( \frac{4}{[2]_{pq}[3]_{pq}[4]_{pq}} + \frac{1}{[3]_{pq}[5]_{pq}} - \frac{1}{[4]_{pq}} \right).
\] (28)

Remark 1. As we know by (7), if \( p = 1 \) and \( q \to 1^- \), then the above result (28) coincides with Zaprawa [35].

4. Bounds of Fourth Hankel Determinant

Firstly, \( \mathcal{H}_{4,1} (g) \) is the fourth Hankel determinant of the form (11) with six coefficients which can be written in the form,

\[
\mathcal{H}_{4,1} (g) = \delta_1 \mathcal{H}_{3,1} (g) - \delta_2 \Lambda_1 + \delta_3 \Lambda_2 - \delta_4 \Lambda_3,
\] (29)

where \( \Lambda_1, \Lambda_2, \) and \( \Lambda_3 \) are third-order determinants given as

\[
\Lambda_1 = (\delta_1 \delta_6 - \delta_4 \delta_3) - \delta_2 (\delta_2 \delta_6 - \delta_3 \delta_2) + \delta_4 (\delta_2 \delta_6 - \delta_3 \delta_1);
\] (30)

\[
\Lambda_2 = (\delta_1 \delta_6 - \delta_2 \delta_2) - \delta_2 (\delta_2 \delta_6 - \delta_3 \delta_5) + \delta_3 (\delta_3 \delta_5 - \delta_4 \delta_3);
\] (31)

\[
\Lambda_3 = \delta_2 (\delta_6 \delta_1 - \delta_5^2) - \delta_3 (\delta_6 \delta_3 - \delta_5 \delta_3) + \delta_4 (\delta_6 \delta_3 - \delta_5^2).
\] (32)

Theorem 2. If \( g \in \mathcal{R}_{pq} \), then

\[
\Lambda_1 = \frac{1}{[3]_{pq}[6]_{pq}} d_4 d_2 - \frac{1}{[4]_{pq}[5]_{pq}} d_3 d_4 - \frac{1}{[2]_{pq}[6]_{pq}} d_2^3 - \frac{1}{[3]_{pq}[4]_{pq}} d_3^2 + \frac{1}{[2]_{pq}[3]_{pq}[5]_{pq}} d_3^2 d_4 + \frac{1}{[2]_{pq}[4]_{pq}} d_3 d_4^2
- \frac{1}{[2]_{pq}[3]_{pq}[4]_{pq}} d_3^2 d_3;
\]

\[
\Lambda_2 = \frac{1}{[4]_{pq}[6]_{pq}} d_4 d_2 - \frac{1}{[5]_{pq}} d_3^2 - \frac{1}{[2]_{pq}[3]_{pq}[6]_{pq}} d_3 d_4^2 + \frac{1}{[2]_{pq}[4]_{pq}[5]_{pq}} d_3 d_4^2 + \frac{1}{[3]_{pq}[5]_{pq}} d_3^2 d_4
- \frac{1}{[3]_{pq}[4]_{pq}} d_3^2 d_3;
\]

\[
\Lambda_3 = \frac{1}{[2]_{pq}[4]_{pq}[6]_{pq}} d_4 d_2 d_5 - \frac{1}{[2]_{pq}[5]_{pq}} d_3 d_4^2 - \frac{1}{[3]_{pq}[6]_{pq}} d_3^2 d_3 - \frac{2}{[3]_{pq}[4]_{pq}[5]_{pq}} d_3 d_4 d_4 - \frac{1}{[4]_{pq}} d_3^3.
\] (37)
Now, rewrite the above equations as follows:

\[
\Lambda_1 = \frac{1}{[2]_{p,q}^2} d_4 (d_2 - d_1^2) + \frac{1}{[3]_{p,q}^3} d_4 (d_4 - d_2^2) - \frac{1}{[2]_{p,q}^2} d_3 (d_4 - d_3 - d_1)
\]

\[
+ \left( \frac{1}{[4]_{p,q}^4} + \frac{1}{[3]_{p,q}^4} - \frac{1}{[2]_{p,q}^4} \right) d_4 (d_2 - d_1^2)
\]

\[
+ \left( \frac{1}{[4]_{p,q}^4} + \frac{1}{[3]_{p,q}^4} - \frac{1}{[2]_{p,q}^4} \right) d_2 (d_2 - d_1^2)
\]

\[
+ \left( \frac{1}{[6]_{p,q}^6} + \frac{1}{[5]_{p,q}^6} - \frac{1}{[4]_{p,q}^6} - \frac{1}{[3]_{p,q}^6} + \frac{1}{[2]_{p,q}^6} + \frac{1}{[1]_{p,q}^6} \right) d_2 d_3 d_2.
\]

\[
\Lambda_2 = \frac{1}{[2]_{p,q}^2} d_4 (d_3 - d_1^2) - \frac{1}{[3]_{p,q}^3} d_4 (d_4 - d_2^2) + \frac{1}{[3]_{p,q}^4} d_3 (d_3 - d_1^4)
\]

\[
- \left( \frac{1}{[2]_{p,q}^4} [4]_{p,q}^4 [5]_{p,q}^4 - \frac{1}{[3]_{p,q}^4} [4]_{p,q}^4 [5]_{p,q}^4 \right) d_4 (d_2 - d_1^2)
\]

\[
+ \left( \frac{1}{[4]_{p,q}^4} [6]_{p,q}^4 - \frac{1}{[2]_{p,q}^4} [3]_{p,q}^4 [6]_{p,q}^4 - \frac{1}{[3]_{p,q}^4} [4]_{p,q}^4 [5]_{p,q}^4 + \frac{1}{[2]_{p,q}^4} [4]_{p,q}^4 [5]_{p,q}^4 + \frac{1}{[5]_{p,q}^4} [4]_{p,q}^4 \right) d_2 d_3 d_2.
\]

\[
\Lambda_3 = \frac{1}{[3]_{p,q}^3} d_4 (d_4 - d_2^2) - \frac{1}{[2]_{p,q}^2} d_4 (d_2 - d_1^2) + \frac{1}{[4]_{p,q}^4} d_3 (d_3 - d_2^2)
\]

\[
- \frac{1}{[4]_{p,q}^4} d_2 (d_2 - d_1^2) + \frac{1}{[2]_{p,q}^2} d_4 (d_4 - d_1^2) - \left( \frac{2}{[3]_{p,q}^4} [4]_{p,q}^4 [5]_{p,q}^4 - \frac{1}{[4]_{p,q}^4} \right) d_4 (d_2 - d_1^2)
\]

\[
+ \left( \frac{1}{[6]_{p,q}^6} + \frac{2}{[3]_{p,q}^4} [4]_{p,q}^4 [5]_{p,q}^4 - \frac{1}{[3]_{p,q}^6} + \frac{1}{[2]_{p,q}^4} [4]_{p,q}^4 \right) d_2 d_3 d_2.
\]

Using the triangular inequality with the inequalities (18) and (19) of lemma on the above equations, we obtain

\[
|\Lambda_1| \leq 4 \left( \frac{2}{[3]_{p,q}^4} [4]_{p,q}^4 + \frac{1}{[4]_{p,q}^4} [5]_{p,q}^4 + \frac{1}{[3]_{p,q}^4} [6]_{p,q}^4 \right):
\]

\[
|\Lambda_2| \leq 4 \left( \frac{2}{[2]_{p,q}^4} [4]_{p,q}^4 [5]_{p,q}^4 + \frac{1}{[3]_{p,q}^4} [6]_{p,q}^4 + \frac{1}{[4]_{p,q}^4} [5]_{p,q}^4 \right):
\]

\[
|\Lambda_3| \leq 4 \left( \frac{2}{[2]_{p,q}^4} [4]_{p,q}^4 [6]_{p,q}^4 + \frac{1}{[3]_{p,q}^4} [4]_{p,q}^4 [5]_{p,q}^4 \right).
\]

Now, using the values (39)–(41) and (28) along with the inequality $|\delta_3| \leq (2/[3]_{p,q})$ in (29), we get our desired result.

5. **Bounds of $H_{\mathcal{K}^1}(g)$ for Two-Fold and Three-Fold Symmetric Functions**

$n$-Fold symmetric function consists all those functions $g$ which satisfy the following condition:

\[
g(\varepsilon z) = g(z), \quad \forall z \in \mathbb{U},
\]

where $\varepsilon = \exp(2\pi i/n)$. The set of univalent functions with $n$-fold symmetry (that is $\Psi^n$) has the expansion of the form,

\[
\Psi^n = \left\{ g(z) \in \Psi; g(z) = z + \sum_{j=1}^{\infty} \delta_{nj+1} \varepsilon^{nj+1}, \quad z \in \mathbb{U} \right\}
\]

Furthermore, univalent function $g(z) \in \Psi^n$ belongs to $\mathcal{K}^{(n)}_{p,q}$ if and only if

\[
D_{p,q}(g(z)) = h(z) \quad \text{with} \quad h \in \mathcal{K}^{(n)},
\]

whereas
and function \( g(z) \in \Phi \) if \( \text{Re}(zg'(z)/g(z)) > 0, z \in \mathbb{U} \).

Now, if \( g \in \Psi^3 \), then \( g(z) = z + \delta_1 z^4 + \delta_2 z^7 + \cdots \); hence, \( \mathcal{H}(g) = \delta_1^2 (\delta_2^2 - \delta_3^2) \). In the same way, if \( g \in \Psi^2 \), then \( g(z) = z + \delta_1 z^5 + \delta_2 z^8 + \cdots \); clearly, we can see that two-fold symmetric functions are odd. So, \( \mathcal{H}(g) = \delta_2 \delta_3 \delta_4 - \delta_3^2 \delta_2^2 - \delta_3^2 \).

**Theorem 3.** If \( g(z) \) is three-fold symmetric bounded turning function, that is, \( g(z) \in \Phi^{(3)} \), then

\[
|\mathcal{H}(g)| \leq \frac{1}{7|\mathcal{P}|}.
\]

**Proof.** Firstly, consider that \( g(z) \in \Phi^{(3)} \), then \( g(z) = \sum_{j=1}^{\infty} a_j z^{3j} + \sum_{j=1}^{\infty} b_j z^{4j} \) such that \( (a_j, b_j) = D_{p,q} \). By the definition of two-fold symmetric function, the Hankel determinant can be written as

\[
\mathcal{H}(g) = \delta_1 \delta_2 \delta_3 \delta_4 - \delta_2^2 \delta_3^2 - \delta_3^2 \delta_4^2 - \delta_4^2.
\]

**Theorem 4.** If \( g(z) \) is two-fold symmetric bounded turning function, that is, \( g(z) \in \Phi^{(2)} \), then

\[
|\mathcal{H}(g)| \leq \frac{1}{7|\mathcal{P}|}.
\]

**Proof.** By the definition of two-fold symmetric function, the Hankel determinant can be written as

\[
\mathcal{H}(g) = \delta_1 \delta_2 \delta_3 \delta_4 - \delta_2^2 \delta_3^2 - \delta_3^2 \delta_4^2 - \delta_4^2.
\]
Now, with the help of lemma in Section 2, we get our desired result as asserted by the statement. □

**Remark 2.** For \( p = 1, q \to 1^- \) results of (29), (45) and (43) will coincide with results derived in [40].

## Data Availability

The required data are included in this article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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