Controllability of Impulsive Non–Linear Delay Dynamic Systems on Time Scale

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ABSTRACT In this paper, we obtain the results of controllability for first order impulsive non–linear time varying delay dynamic systems and Hammerstein impulsive system on time scale, using the theory of fixed points such as Banach fixed point theorem combined with Lipchitz conditions and non linear functional analysis. We also provide examples to support our theoretical results.

INDEX TERMS Controllability, impulses, delay, time scale.

I. INTRODUCTION

The theory of differential equations with impulses has been well utilized in mathematical modeling. In real life problems, there are numerous procedures and phenomena that are characterized by the fact that at certain occasions they experienced sudden changes in their states. These procedures are exposed to short-term perturbations and is known as impulsive effects in the system. Impulsive differential equations can be used to represent various real world problems containing variation in its state. Recently, the theory of differential equations with impulses has influenced many author’s attention. For more details we refer the work of Samoilenko and Perestyuk [1], Lakshmikantham et al. [2] and Rogovchenko [3].

On the other hand, in control theory, controllability is a mathematical problem which consists of determining a specific control parameter to steers the solutions of the problem throughout the process. Due to many applications of control in practical problems, controllability received an increasing interest [4]–[8]. In the last few decades, impulsive control has attracted the interest of many researchers [9]. Such control arises naturally in a wide variety of phenomena, such as orbital transfer of satellite [10], ecosystems management [12], synchronization in chaotic secure communication systems [13] and control of money supply in a financial market [14]. For more results on the controllability of impulsive systems one can see the work of Shubov et al. [15] and Park et al. [16]. Impulsive systems with time-delay describe the models of practical processes where both dependence on the past and instantaneous disturbances are observed. The interaction between the impulsive effects and the time-delay makes it rather difficult to analyze controllability of such systems [17], [18].

The idea of time scale was firstly introduced by Hilger [19]. He introduced this theory in 1988, as a unification of discrete and continuous calculus. Due to its importance, this theory has been well utilized in differential and difference equations. For details of mentioned topic, see [20]–[24] and the books [25], [26]. More recently, many researchers discussed the existence, uniqueness, stability and controllability of abstract equations on time scale. In [27], Lupulescu and Zada discussed the basic concepts of impulsive linear systems and studied the solutions of linear impulsive dynamic systems on time scale. For the controllability results of dynamical systems on time scale we recommend [28]–[30], [35].

Recently, Shah et al. [31] studied the existence of solutions and Hyers-Ulam stability of first order delay dynamic systems on time scale. The concept of Hyers-Ulam
stability [32]–[34], is related to the difference between the exact and approximate solutions of the considered problem, and thus this concept can be used in approximation theory and numerical analysis.

In this paper, we investigate the models presented in [31] for controllability. In fact, we study the controllability results for delay dynamic systems of the form:

$$\begin{align*}
\chi^A(\theta) &= A(\theta)\chi(\theta) + B(\theta)u(\theta) + H(\theta, \chi(\theta), \chi(v(\nu))), \\
\chi(\theta) &= \alpha(\theta), \\
\chi(0) &= \chi_0,
\end{align*}$$

(1)

and of impulsive non–linear delay dynamic system of the form:

$$\begin{align*}
\chi^A(\theta) &= M(\theta)\chi(\theta) + N(\theta)u(\theta) + H(\theta, \chi(\theta), \chi(v(\nu))), \\
\chi(\theta) &= \alpha(\theta), \\
\chi(0) &= \chi_0,
\end{align*}$$

(2)

where $T$ is a time scale with $0, b, \theta_k, \in T$. $A(\theta)$ and $M(\theta)$ is a family of linear bounded operators which is continuous and piecewise continuous on $I$, respectively. $H : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions. Also $\chi(\theta) = \lim_{\tau \to 0^+} \chi(\theta_k + \tau)$ and $\chi(\theta) = \lim_{\tau \to 0^+} \chi(\theta_k - \tau)$ be the right and left side limits of $\chi(\theta)$ at $\theta_k$, where $\theta_k$ are not isolated points and satisfies

$$\theta_0 < \theta_1 < \theta_3 < \cdots < \theta_m < \theta_m+1 = \theta_l < +\infty.$$

Moreover, $v : T \to T \cup [-\lambda, 0)T$ will be a continuous and delay function such that $v(\theta) \leq \theta$. $B$ and $N$ are bounded linear operators from a Banach space $U$ to $\mathbb{R}^n$ and $u(\cdot)$ is control function given in $L^2(I, U)$.

This paper is organized as follows. In first and second sections, we provide introduction, basic notations and definitions which are required for the main results. In third and fourth sections, we give the main results of controllability for systems (1) and (2), respectively. In last section, an example is provided to utilize the applicability of obtained results.

II. PRELIMINARIES

Here, we recall fundamental definitions, lemmas and some notations, which will be utilized throughout the paper.

Let $(Y, \| \cdot \|)$ be a Banach space and $B(Y)$ be the space of all linear and bounded operators on $Y$. Furthermore, $PC(I, Y)$ is a Banach space of all piecewise continuous functions from $I$ to $Y$, induced with the norm, $\|y\|_{PC} = sup_{t\in I} \|y(t)\|$.

The time scale $T$ is a non-empty subset of real numbers. The backward and forward jump operators $\rho : T \to T$ and $\sigma : T \to T$ are respectively defined as: $\rho(s) = sup\{t \in T : t < s\}$ and $\sigma(s) = inf\{t \in T : t > s\}$. Also, $t$ is called right or left dense if $t < sup\{T\}$ and $\sigma(t) = t$ or $t > inf\{T\}$ and $\rho(t) = t$, respectively. The point $t$ is called the dense point if at the same time it is right as well as left dense. A function is said to be regulated if its right hand limit exists at all right dense points and left hand limit exists at all left dense points in $T$. A function $f : T \to Y$ is said to be rd-continuous, if it is regulated and continuous at all right-dense points. The derived form of $T$ is denoted by $T^c$, and is defined as: $T^c = T \setminus \{\max T\}$ if $\max T$ exists, otherwise $T^c = T$.

**Remark 1:** Throughout this paper, we consider that $T \not\subseteq \mathbb{Z}$, where $T$ is a time scale and $\mathbb{Z}$ is the set of integers. Also, the impulses $\Delta\chi(\theta_k)$ on the isolated points are assumed to be zero.

The $\Delta$–derivative and $\Delta$–integral of $f : T \to \mathbb{R}$ are respectively defined as

$$f^\Delta(t) = \lim_{s \to t, s \neq \theta(t)} \frac{f(\theta(t)) - f(s)}{\theta(t) - s}, \quad t \in T^c,$$

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \forall a, b \in T,$$

where $F^\Delta = f$ on $T^c$.

**Definition 1:** A function $\chi \in PC(I, Y)$ is called a mild solution of (1) if it satisfies $\chi(0) = \chi_0$ and the following equation

$$\chi(\theta) = \alpha(0) + \Psi_A(\theta, 0)\chi_0 + \int_0^\theta \Psi_A(\theta, \Theta(s))[B(s)u(s) + H(s, \chi(s), \chi(v(s)))]\Delta s. \quad (3)$$

**Definition 2:** A function $\chi \in PC(I, Y)$ is called a mild solution of (2) if it satisfies $\chi(0) = \chi_0$ and the following equation

$$\chi(\theta) = \alpha(0) + \Psi_M(\theta, 0)\chi_0 + \int_0^\theta \Psi_M(\theta, \Theta(s))[N(s)u(s) + H(s, \chi(s), \chi(v(s)))]\Delta s + \sum_{j=1}^{i} I_j(\chi(\theta^-_j)). \quad (4)$$

III. MAIN RESULTS

In the following sections, we discuss the controllability of (1) and (2) on time scale.

**Definition 3:** The systems (1) and (2) are controllable on the interval $I$, if for every $\chi_0, \chi_1 \in Y$, there exists a control $u \in L^2(I, U)$, such that the mild solutions $\chi(t)$ of (1) and (2) corresponding to $u$ satisfies $\chi(0) = \chi_0$ and $\chi(b) = \chi_1$.

A. CONTROLLABILITY OF EQUATION (1)

To analyze the controllability results for equation (1). The following presumptions will be needed:

(A1) The function $H$ is continuous and satisfying the following

- $\|\mathbf{H}(\theta, x_1, x_2) - \mathbf{H}(\theta, y_1, y_2)\| \leq \sum_{i=1}^{2} L\|x_i - y_i\|, \quad L > 0,$
  for all $t \in T$ and $x_i, y_i \in \mathbb{R}^n, i \in [1, 2]$;

- $\|\mathbf{H}(\theta, x, y)\| \leq C_M$, where $C_M$ is a positive constant.

(A2) For some $C_\varepsilon > 0$, we have $\|\Psi_A(\theta, \Theta(s))\| \leq C_\varepsilon; \quad \theta, s \in T$;
\((A_3)\) \(2C_cL\theta < 1;\)
\((A_4)\) The linear operator \(W : L^2(I, U) \rightarrow Y\) given by
\[
Wu = \int_0^b \Psi_A(b, \Theta(s))B(s)u(s)\Delta s
\]
has a bounded invertible operator \(W^{-1}\), which takes values in \(L^2(I, U) / \ker W\). Also, \(B\) is continuous operator from \(U\) to \(Y\) and there exists \(M_1, M_2 > 0\), such that \(||W^{-1}|| \leq M_1\) and \(||B|| \leq M_2\).

**Theorem 1:** If the conditions \((A_1) - (A_3)\) hold, then equation (1) has a unique solution in \(C(T \cup [-\lambda, 0]_T, \mathbb{R}^n)\).

**Proof:** Define the operator \(\Upsilon : C(T \cup [-\gamma, 0]_T, \mathbb{R}^n) \rightarrow C(T \cup [-\lambda, 0]_T, \mathbb{R}^n)\) by
\[
(\Upsilon \chi)(\theta) = \begin{cases} \alpha(\theta), & \theta \in [-\lambda, 0]_T, \\ \alpha(0) + \Psi_A(\theta, 0)\chi_0 + \int_0^\theta \Psi_A(\theta, \Theta(s)) [B(s)u(s) + H(s, \chi(s), \chi'(v(s)))] \Delta s, & \theta \in T \cup [-\lambda, 0]_T. \end{cases}
\]

Now, for any \(\chi_1, \chi_2 \in C(T \cup [-\lambda, 0]_T, \mathbb{R}^n)\) and \(\forall \, \theta \in [-\lambda, 0]_T\), we have
\[
\left\| (\Upsilon \chi_1)(\theta) - (\Upsilon \chi_2)(\theta) \right\| = 0.
\]
For \(\theta \in T \cup [-\lambda, 0]_T\), we conclude that
\[
\left\| (\Upsilon \chi_1)(\theta) - (\Upsilon \chi_2)(\theta) \right\| = \left\| \int_0^\theta [\Psi_A(\theta, \Theta(s))(H(s, \chi_1(s), \chi_1'(v(s))) - H(s, \chi_2(s), \chi_2'(v(s))))] \Delta s \right\|
\leq \int_0^\theta \|\Psi_A(\theta, \Theta(s))\|\left\| (H(s, \chi_1(s), \chi_1'(v(s))) - H(s, \chi_2(s), \chi_2'(v(s)))) \right\| \Delta s
\leq 2\|\chi_1 - \chi_2\| \int_0^\theta C_cL \Delta s
\leq \|\chi_1 - \chi_2\| \left(2C_cL\right).
\]

Using the Picard operator on \(C(T \cup [-\lambda, 0]_T, \mathbb{R}^n)\) and the presumption \((A_3)\), \(\Upsilon\) is strictly contractive. So, the operator \(\Upsilon\) has a unique fixed point (FP). Hence (1) has a unique solution in \(C(T \cup [-\lambda, 0]_T, \mathbb{R}^n)\).

**Lemma 1:** If the assumptions \((A_1) - (A_4)\) hold, then for \(\theta \in [0, b]\), the control function
\[
u(\theta) = W^{-1}\left[\chi_b - \chi(0) - \Psi_A(b, 0)\chi_0\right]
- \int_0^b \Psi_A(b, \Theta(s))H(s, \chi(s), \chi'(v(s))) \Delta s\]
steers the state function \(\chi(\theta)\) from initial state to final state at \(\theta = b\). Also, the estimate for the control function \(\|u(\theta)\|\) is
\[
M_u = M_1 \left[\|\chi_b\| + (1 + C_c)\|\chi_0\| + C_M C_c b\right].
\]

**Proof:** By substituting \(\theta = b\) in the mild solution (3) of the system (1), we get
\[
\chi(b) = \chi(0) + \Psi_A(b, 0)\chi_0 + \int_0^b \Psi_A(b, \Theta(s))[B(s)u(s) + H(s, \chi(s), \chi'(v(s)))] \Delta s
= \chi(0) + \Psi_A(b, 0)\chi_0 + Wu(s)
+ \int_0^b \Psi_A(b, \Theta(s))H(s, \chi(s), \chi'(v(s))) \Delta s
= \chi(0) + \Psi_A(b, 0)\chi_0 + \left[\chi_b - \chi(0) - \Psi_A(b, 0)\chi_0\right]
- \int_0^b \Psi_A(b, \Theta(s))H(s, \chi(s), \chi'(v(s))) \Delta s
+ \int_0^b \Psi_A(b, \Theta(s))H(s, \chi(s), \chi'(v(s))) \Delta s
= \chi_b.
\]
Hence, the control function (5) steers the state function \(\chi(\theta)\) from initial state \(\chi_0\) to final state \(\chi_b\) at time \(\theta = b\).

Also,
\[
\|u(\theta)\| \leq W^{-1}\left|\chi_b - \chi(0) - \Psi_A(b, 0)\chi_0\right|
- \int_0^b \Psi_A(b, \Theta(s))H(s, \chi(s), \chi'(v(s))) \Delta s\]
\leq M_1 \left[\|\chi_b\| + \|\chi(0)\| + C_c\|\chi_0\| + C_M \int_0^b C_c \Delta s\right]
\leq M_1 \left[\|\chi_b\| + (1 + C_c)\|\chi_0\| + C_M C_c b\right]
= M_u.
\]

**Theorem 2:** Assume that the assumptions \([A1] - [A4]\) are fulfilled. Then the system (1) is controllable on \(I\), provided that
\[
2LC_c\theta(1 + C_cM_1M_2b) < 1.
\]

**Proof:** Define an operator \(T : \Omega \rightarrow \Omega\) by
\[
(T \chi)(\theta) = \chi(0) + \Psi_A(\theta, 0)\chi_0 + \int_0^\theta \Psi_A(\theta, \Theta(s))[B(s)u(s) + H(s, \chi(s), \chi'(v(s)))] \Delta s,
\]
where \(\Omega = \{C(T \cup [-\lambda, 0]_T, \mathbb{R}^n)\}\).

For \(\theta \in I\) and \(\chi \in \Omega\), we have
\[
\|(T \chi)(\theta)\| \leq \|\chi(0)\| + \Psi_A(\theta, 0)\chi_0 + \int_0^\theta \Psi_A(\theta, \Theta(s))[B(s)u(s) + H(s, \chi(s), \chi'(v(s)))] \Delta s\]
\leq (1 + C_c)\|\chi_0\| + (M_2M_u + C_M) \int_0^\theta C_c \Delta s
\leq (1 + C_c)\|\chi_0\| + (M_2M_u + C_M)\theta C_c.
\]
Also, for $\mu_1, \mu_2 \in \Omega$ and $\theta \in I$,
\[
\|(T\mu_1)\theta - (T\mu_2)\theta\|
\]
\[
= \left\| \int_0^\theta \Psi_A(\theta, \Theta(s))B(s)u(s)
+ H(s, \mu_1(s), \mu_1(v(s)))\Delta s - \int_0^\theta \Psi_A(\theta, \Theta(s))B(s)u(s)
+ H(s, \mu_2(s), \mu_2(v(s)))\Delta s \right\|
\]
\[
= \left\| \int_0^\theta \Psi_A(\theta, \Theta(\tau))B\Phi W^{-1}
\times \left[ \int_0^b \Psi_A(b, \Theta(s))(H(s, \mu_1(s), \mu_1(v(s)))
+ H(s, \mu_2(s), \mu_2(v(s))))\Delta s \right] \Delta \tau
\]
\[
+ \int_0^\theta \Psi_A(\theta, \Theta(s))H(s, \mu_1(s), \mu_1(v(s)))\Delta s
\]
\[
- \int_0^\theta \Psi_A(\theta, \Theta(s))H(s, \mu_2(s), \mu_2(v(s)))\Delta s \right\| \Delta s
\]
\[
\leq M_1M_2C_e \int_0^\theta \left[ C_2L_\mu \mu_1 - \mu_2 \|b\] \Delta \tau
\]
\[
+ C_2L_\mu \mu_1 - \mu_2 \|b \theta
\]
\[
\leq 2LC_e \theta (1 + C_eM_1M_2b) \|\mu_1 - \mu_2\|.
\]
Hence
\[
\|(T\mu_1)\theta - (T\mu_2)\theta\| \leq L_\alpha \|\mu_1 - \mu_2\|,
\]
where
\[
L_\alpha = 2LC_e \theta (1 + C_eM_1M_2b).
\]

Thus, $T$ is a contraction operator. So, by a Banach FP theorem, $T$ has a unique FP on $I$. Therefore, system (1) has a solution on $I$ and hence we conclude that the system (1) is controllable on $I$.

**B. CONTROLLABILITY OF EQUATION (2)**

Before proving the controllability result for equation (2), let us assume the following presumptions:

(B1) $I_{k} : \mathbb{R}^n \to \mathbb{R}^n$ is such that

- $\|I_{k}(\mu_1) - I_{k}(\mu_2)\| \leq M_k \|\mu_1 - \mu_2\|$, $M_k > 0$, for all $\mu_1, \mu_2 \in \mathbb{R}^n$ and $k \in \{1, 2, \ldots, m\}$,

- $\|I_{k}(x)\| \leq P_1$, $P_1 > 0$, for all $x \in \mathbb{R}^n$;

(B2) $\|\Psi_M(\theta, \Theta(s))\| \leq C_M$, for some $C \geq 0$, where $\theta, s \in \mathbb{T}$;

(B3) \[
\sum_{j=1}^m M_j + 2C_Mr \theta < 1;
\]

hold.

**Theorem 3:** If the presumptions (A1) and (B1) – (B3) hold, then equation (2) has a unique solution in $PC(T \cup [-\lambda, 0]_{\mathbb{N}}, \mathbb{R}^n) \cap PC^1(T, \mathbb{R}^n)$.

**Proof:** Define an operator $\Upsilon : PC(T \cup [-\lambda, 0]_{\mathbb{N}}, \mathbb{R}^n) \to PC(T \cup [-\lambda, 0]_{\mathbb{N}}, \mathbb{R}^n)$ by

\[
\|(\Upsilon \chi)(\theta)\|
\]
\[
\leq \sum_{j=1}^m \left\|I_{j}(\chi(\theta_j^+)) - I_{j}(\chi(\theta_j^-))\right\| + \int_0^\theta \left\|\Psi_M(\theta, \Theta(s))\right\| \Delta s
\]
\[
\leq \sum_{j=1}^m M_j \|\chi_1 - \chi_2\| + 2\|\chi_1 - \chi_2\| \int_0^\theta C_M r \theta \Delta s
\]
\[
\leq \|\chi_1 - \chi_2\| \left( \sum_{j=1}^m M_j + 2C_Mr \theta \right).
\]

Using Picard operator and assumption (B3), $\Upsilon$ is strictly contractive on $PC(T \cup [-\lambda, 0]_{\mathbb{N}}, \mathbb{R}^n)$. The operator $\Upsilon$ has a unique FP. Hence (2) has a unique solution in $PC(T \cup [-\lambda, 0]_{\mathbb{N}}, \mathbb{R}^n) \cap PC^1(T, \mathbb{R}^n)$. □

**Lemma 2:** If the assumtions (B1) – (B3) as well as (A1) and (A4) hold, then for $t \in [0, b]$, the control function

\[
u(\theta) = W^{-1}\left[ f(0) - \sum_{j=1}^i I_j(\chi(\theta_j^-)) - \Psi_M(b, 0)\chi_0
\]
\[
- \int_0^\theta \Psi_M(b, \Theta(s))H(s, \chi(s), \chi(v(s)))\Delta s \right](\theta)
\]

steers the state function $\chi(t)$ from initial state to final state at $\theta = b$. Also, the estimate for the control function $u(\theta)$ is $M_u$ where

\[
M_u = M_1 \left[ \|x_0\| + \|\chi(0)\| + P_1i + C_M\|\chi_0\| + bC_McM \right].
\]

**Proof:** By substituting $\theta = b$ in the mild solution (4) of the system (2), we get

\[
\chi(b) = \chi(0) + \sum_{j=1}^i I_j(\chi(\theta_j^-)) + \Psi_M(b, 0)\chi_0
\]
\[
+ \int_0^b \Psi_M(b, \Theta(s))[N(s)u(s) + H(s, \chi(s), \chi(v(s)))\Delta s
\]
Hence, the control function (6) steers the state function \( \chi(\theta) \) from initial state \( \chi_0 \) to final state \( \chi_b \) at time \( \theta = b \). Also,

\[
\|u(\theta)\| = \|W^{-1}[\chi_b - \chi(0) - \sum_{j=1}^{i} \int_{\theta_j}^{\theta} \Psi_M(b, \Theta(s)) H(s, \chi(s), \chi(v(s))) \Delta s + \int_{0}^{b} \Psi_M(b, \Theta(s)) \times H(s, \chi(s), \chi(v(s))) \Delta s]\| \\
\leq \|W^{-1}[\|\chi_b\| + \|\chi(0)\| + \sum_{j=1}^{i} \int_{\theta_j}^{\theta} \Psi_M(b, \Theta(s)) \times H(s, \chi(s), \chi(v(s))) \Delta s]\| \\
+ \|\Psi_M(0)\| \chi_0 + \int_{0}^{b} \|\Psi_M(b, \Theta(s)) \times H(s, \chi(s), \chi(v(s))) \Delta s\| \\
\leq M_1[\|\chi_b\| + \|\chi(0)\| + P_1 t + C_M |\chi_0| + b C_M C_M] \\
= M_u.
\]

**Theorem 4:** Assume that the presumptions (A1) and \([B1] - [B3]\) are fulfilled. Then (2) is controllable on \( I \), provided that

\[
\left((M_1 M_2 M_1 + 1) 2 C_M L \theta + (M_1 M_2 M_1 + 1) \sum_{j=1}^{i} C_j \right] < 1.
\]

**Proof:** Define an operator \( \Gamma : \Omega \rightarrow \Omega \) by

\[
(\Gamma \chi)\theta = \chi(0) + \Psi_M(\theta, 0) \chi_0 + \int_{0}^{\theta} \Psi_M(\theta, \Theta(s))[N(s)u(s) + H(s, \chi(s), \chi(v(s))) \Delta s + \int_{\theta}^{\theta} \Psi_M(\Theta(s))[N(s)u(s) + H(s, \chi(s), \chi(v(s))) \Delta s + \sum_{j=1}^{i} \int_{\theta_j}^{\theta} \Psi_M(b, \Theta(s)) \times H(s, \chi(s), \chi(v(s))) \Delta s]\] \]

where \( \Omega = \{C(T \cup [-\lambda, 0]) \subset \mathbb{R}^n\} \).

For \( \theta \in I \) and \( \chi \in \Omega \), we have

**\( \| (\Gamma \chi)\theta \| = \| \chi(0) + \sum_{j=1}^{i} \int_{\theta_j}^{\theta} \Psi_M(b, \Theta(s)) \times H(s, \chi(s), \chi(v(s))) \Delta s + \int_{0}^{\theta} \Psi_M(\theta, \Theta(s))[N(s)u(s) + H(s, \chi(s), \chi(v(s))) \Delta s + \sum_{j=1}^{i} \int_{\theta_j}^{\theta} \Psi_M(b, \Theta(s)) \times H(s, \chi(s), \chi(v(s))) \Delta s]\| \] \]

Also, for \( \mu_1, \mu_2 \in \Omega \) and \( \theta \in I \),

\[
\| (\Gamma \mu_1)\theta - (\Gamma \mu_2)\theta \| \\
= \left[\left(1 + \Psi_M(\theta, 0)\right) \chi_0 + \sum_{j=1}^{i} \int_{\theta_j}^{\theta} \Psi_M(\theta, \Theta(s)) \times H(s, \mu_1(s), \mu_1(v(s))) \Delta s + \int_{0}^{\theta} \Psi_M(\theta, \Theta(s))[N(s)u(s) + H(s, \mu_1(s), \mu_1(v(s))) \Delta s + \sum_{j=1}^{i} \int_{\theta_j}^{\theta} \Psi_M(b, \Theta(s)) \times H(s, \mu_1(s), \mu_1(v(s))) \Delta s]\] \]

\[
\leq M_1 M_2 M_1 \int_{0}^{\theta} [C_M 2 L \|\mu_1 - \mu_2\| \theta \] \]

\[
+ \sum_{j=1}^{i} C_j \|\mu_1 - \mu_2\| \] \]

\[
+ \sum_{j=1}^{i} C_j \|\mu_1 - \mu_2\| + C_M 2 L \|\mu_1 - \mu_2\| \theta \] \]

\[
\leq M_1 M_2 M_1 (b2 C_M L + \sum_{j=1}^{i} C_j) \theta \|\mu_1 - \mu_2\| \] \]

\[
+ \sum_{j=1}^{i} C_j \|\mu_1 - \mu_2\| + 2 L C_M \|\mu_1 - \mu_2\| \theta \] \]
\[
\leq \left[ (M_1 M_2 C \theta b + 1) 2 C M \theta + (M_1 M_2 C \theta + 1) \sum_{j=1}^{i} C_j \right] \times \| \mu_1 - \mu_2 \|
\]

Hence
\[
\|(\Gamma \mu_1) \theta - (\Gamma \mu_2) \theta \| \leq L_\beta \| \mu_1 - \mu_2 \|
\]

where
\[
L_\beta = \left[ (M_1 M_2 C \theta b + 1) 2 C M \theta + (M_1 M_2 C \theta + 1) \sum_{j=1}^{i} C_j \right].
\]

Thus, \( \Gamma \) is a contraction operator. So, by a Banach FP theorem, \( \Gamma \) has a unique FP on \( I \). Therefore, system (2) has a solution on \( I \) and hence we conclude that the system (2) is controllable on \( I \). \( \square \)

**C. CONTROLLABILITY OF HAMMERSTEIN IMPULSIVE SYSTEM**

In this section, we establish the controllability results for integro-differential equation with impulses which is known as Hammerstein impulse system.

\[
\begin{cases}
\chi^\Delta (\theta) = C(\theta) \chi (\theta) + D(\theta) u(\theta) \\
+ G(\theta, \chi (\theta), \chi (v(\theta))) \\
\times \int_0^\theta g(s, \theta) H(s, \chi (\theta), \chi (v(\theta))) \Delta s, \\
\theta \in I, \theta \neq \theta_k, \\
\chi (\theta) = \alpha (\theta), \quad \theta \in [-\lambda, 0]_T, \\
\Delta \chi (\theta_k) = \chi (\theta_k^+) - \chi (\theta_k^-), \quad k = 1, \ldots, m, \\
\chi (\theta_0) = \chi_0.
\end{cases}
\]

where \( C \) and \( D \) are linear bounded operators and \( u \) is a control parameter. Also, \( G, H, \chi_k, \alpha \) and the kernel \( g \) are continuous functions and are defined as:

\[
G : T \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad H : T \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n.
\]

Moreover, \( v : T \rightarrow T \cup [-\lambda, 0] \) with \( v(\theta) \leq \theta \), is a continuous delay function.

To analyze the results of controllability for (7), the following assumptions will be required:

(B1) For some positive constants \( C_\delta, \delta, G_1 \) and \( \xi \), we have

- \( \| \Psi \chi (\theta, \Theta (s)) \| \leq C_\delta, \| g(\theta, s) \| \leq \xi, \)

- \( \| G(s, \chi_1(s), \chi_1(v(s))) - G(s, \chi_2(s), \chi_2(v(s))) \| \leq \delta, \)

- \( \| G(s, \chi_1(s), \chi_1(v(s))) \| \geq G_1, \quad \text{for every} \quad \theta, s \in T; \)

(B2) \( \sum_{j=1}^m M_j + C_\delta \xi L \theta^2 < 1. \)

**Theorem 5:** If conditions (A1), (B1) and (C1) – (C2) hold, then equation (2) has a unique solution in \( PC(I) \cup [-\lambda, 0]_T, \mathbb{R}^n. \)

**Proof:** Consider an operator \( \Upsilon : PC(T \cup [-\lambda, 0]_T, \mathbb{R}^n) \) by

\[
(\Upsilon \chi)(t) = \begin{cases}
\chi(t), & \theta \in [-\lambda, 0]_T, \\
\alpha(\theta) + \int_0^\theta \Psi \chi (\theta, \Theta (s)) \\
\times \left[ D(s)u(s) + G(s, \chi (\theta), \chi (v(\theta))) \right] \Delta s, & \theta \in (0, \theta_1), \\
\alpha(\theta_0) + \int_0^\theta \Psi \chi (\theta, \Theta (s)) \\
\times \left[ D(s)u(s) + G(s, \chi (\theta), \chi (v(\theta))) \right] \Delta s, & \theta \in (t_i, t_{i+1}].
\end{cases}
\]

We see that for any \( \chi_1, \chi_2 \in PC(T \cup [-\lambda, 0]_T, \mathbb{R}^n) \) and for all \( \theta \in [-\lambda, 0]_T \), we have

\[
\| (\Upsilon \chi_1)(\theta) - (\Upsilon \chi_2)(\theta) \| = 0. \quad \text{For} \quad \theta \in (\theta_m, \theta_{m+1}] , \text{simple calculation shows that}
\]

\[
\left\| (\Upsilon \chi_1)(\theta) - (\Upsilon \chi_2)(\theta) \right\| \leq \sum_{j=1}^m \left\| f_j^{*} (\chi_1(\theta_j^+) - \chi_2(\theta_j^+)) \right\| + \int_0^\theta \| \Psi \chi (\theta, \Theta (s)) \| \\
\times \left( \left| G(s, \chi_1(s), \chi_1(v(s)) - G(s, \chi_2(s), \chi_2(v(s))) \right| \\
\times \int_0^\theta \| g(s, r) \| \| H(r, \chi_1(r), \chi_1(v(r))) \| \\
\times \int_0^\theta \| H(r, \chi_2(r), \chi_2(v(r))) \| \right) \| \right\|\Delta r \Delta s
\]

\[
\leq \sum_{j=1}^m M_j \left\| \chi_1(\theta_j^-) - \chi_2(\theta_j^-) \right\| \\
+ \int_0^\theta C_\delta \xi L \| \chi_1(r) - \chi_2(r) \| \Delta r \Delta s \\
+ \int_0^\theta C_\delta \xi L \| \chi_1(v(r)) - \chi_2(v(r)) \| \Delta r \Delta s
\]

\[
\leq \sum_{j=1}^m M_j \left\| \chi_1 - \chi_2 \right\| + 2 \| \chi_1 - \chi_2 \| \int_0^\theta C_\delta \xi L \Delta r \Delta s
\]

\[
\leq \| \chi_1 - \chi_2 \| \left( \sum_{j=1}^m M_j + C_\delta \xi L \theta^2 \right).
\]

From (B3), \( \Upsilon \) is contractive and so it is a Picard operator on \( PC(T \cup [-\lambda, 0]_T, \mathbb{R}^n). \) The operator \( \Upsilon \) has a unique FP which is the unique solution of (7) in \( PC(I) \cup [-\lambda, 0]_T, \mathbb{R}^n. \) \( \square \)
Lemma 3: If the assumptions (B1), (C1) – (C2) as well as \((A_1)\) and \((A_4)\) hold, then for \(\theta \in [0, b]\), the control function

\[
u(\theta) = W^{-1} \left[ x_b - x(0) - \sum_{j=1}^{i} I_j(x(\theta_j^{-})) - \Psi_C(b, 0)x_0 \right. \\
- \left. \int_0^{b} \Psi_C(\theta, \Theta(s))G(s, \chi(s), \chi(v(s))) \int_0^{s} g(s, r) \times H(r, \chi(r), \chi(v(r))) d\tau d\delta \right] \]

steers the state function \(\chi(\theta)\) from initial state to final state at \(\theta = b\). Also, the estimate for the control function \(u(\theta)\) is \(M_u\) where

\[
M_u = M_1 \left[ \|x_b\| + \|x(0)\| + \sum_{j=1}^{i} C_j + \|s\| \xi G_1 C_M C_C \right].
\]

Proof: By substituting \(\theta = b\) in the mild solution of the system (7), we get

\[
\chi(b) = \chi(0) + \sum_{j=1}^{i} I_j(x(\theta_j^{-})) + \Psi_C(b, 0)x_0 + W_b(s)
\]

\[
+ \int_0^{b} \Psi_C(b, \Theta(s))[D(s)u(s) + G(s, \chi(s), \chi(v(s))) \times \int_0^{s} g(s, r)H(r, \chi(r), \chi(v(r))) d\tau d\delta] ds
\]

\[
= \chi(0) + \sum_{j=1}^{i} I_j(x(\theta_j^{-})) + \Psi_C(b, 0)x_0 + \left[ x_b - x(0) \right. \\
- \left. \Psi_C(b, 0)x_0 - \int_0^{b} \Psi_C(b, \Theta(s))G(s, \chi(s), \chi(v(s))) \times \int_0^{s} g(s, r)H(r, \chi(r), \chi(v(r))) d\tau d\delta \right]
\]

\[
\times \left[ - \sum_{j=1}^{i} I_j(x(\theta_j^{-})) \right] + \left[ \int_0^{b} \Psi_C(b, \Theta(s))G(s, \chi(s), \chi(v(s))) \times \int_0^{s} g(s, r)H(r, \chi(r), \chi(v(r))) d\tau d\delta \right]
\]

\[
= x_b.
\]

Hence, the control function (9) steers the state function \(\chi(\theta)\) from initial state \(x_0\) to final state \(x_b\) at time \(\theta = b\). Also,

\[
\|u(\theta)\| = \left\| W^{-1} \left[ x_b - x(0) - \sum_{j=1}^{i} I_j(x(\theta_j^{-})) - \Psi_C(b, 0)x_0 \right. \\
- \left. \int_0^{b} \Psi_C(b, \Theta(s))G(s, \chi(s), \chi(v(s))) \times \int_0^{s} g(s, r)H(r, \chi(r), \chi(v(r))) d\tau d\delta \right] \right\|
\]

\[
\leq W^{-1} \left[ \|x_b\| + \|x(0)\| + \sum_{j=1}^{i} I_j(x(\theta_j^{-})) \right. \\
+ \left. \|\Psi_C(b, 0)x_0\| + \int_0^{b} \|\Psi_C(b, \Theta(s))G(s, \chi(s), \chi(v(s))) \times \int_0^{s} g(s, r)H(r, \chi(r), \chi(v(r))) d\tau d\delta \right] \| \|s\| \xi G_1 C_M C_C \right].
\]

Theorem 6: Assume that the presumptions (B1), (C1) – (C2) as well as \((A_1)\) are fulfilled. Then (7) is controllable on \(I\), provided that

\[
1 > \left( \sum_{j=1}^{i} C_j \right)^{\theta} + \sum_{j=1}^{i} C_j + \|s\| \xi 2L \frac{\theta^2}{2} \times M_1 M_2 C.
\]

Proof: Define an operator \(\hat{\Gamma}: \Omega \rightarrow \Omega\) by

\[
(\hat{\Gamma})\chi = \chi(0) + \sum_{j=1}^{i} I_j(x(\theta_j^{-})) + \Psi_C(b, 0)x_0 + W_b(s)
\]

\[
+ \int_0^{b} \Psi_C(b, \Theta(s))[D(s)u(s) + G(s, \chi(s), \chi(v(s))) \times \int_0^{s} g(s, r)H(r, \chi(r), \chi(v(r))) d\tau d\delta] ds,
\]

where \(\Omega = \{C(T \cup [-\lambda, 0] \cap \mathbb{R}^n)\}\).

For \(\theta \in I\) and \(\chi \in \Omega\), we have

\[
\|\hat{\Gamma}\chi\| = \|\chi(0) + x_0 + \int_0^{\theta} G(s, \chi(s), \chi(v(s))) \times \int_0^{\theta} g(s, r)H(r, \chi(r), \chi(v(r))) d\tau d\delta \|
\]

\[
\leq (1 + C_1)\|x_0\| + \|s\| \xi \int_0^{\theta} C_1 d\delta + \sum_{j=1}^{i} I_j(x(\theta_j^{-})) d\delta.
\]

Also, for \(\mu_1, \mu_2 \in \Omega\) and \(\theta \in I\),

\[
\|\hat{\Gamma}\mu_1\| - \|\hat{\Gamma}\mu_2\| = \left\| \chi(0) + \sum_{j=1}^{i} I_j(\mu_1(\theta_j^{-})) + \Psi_C(b, 0)x_0 + \int_0^{\theta} G(s, \chi(s), \chi(v(s))) \times \int_0^{\theta} g(s, r)H(r, \chi(r), \chi(v(r))) d\tau d\delta \right\|
\]

Thus, $\tilde{\Gamma}$ is a contraction operator. So, by a Banach fixed point theorem, $\tilde{\Gamma}$ has a unique FP on $\tilde{\Gamma}$. Therefore, system (2) has a solution on $I$ and hence we conclude that the system (2) is controllable on $I$.

Example 1: We consider the partial differential equation on time scale $\mathbb{T}$ in the following form

$$\frac{\partial}{\partial \theta}(Y(\theta, \eta)) = c(\theta, \eta) \frac{\partial^2}{\Delta \eta^2} Y(\theta, \eta) + b(\eta)W(\theta, \eta)$$
$$+ G(\theta, Y(\theta, \eta), Y(v(\eta), \eta)), \quad \theta \in [0, b]_\mathbb{T},$$
$$Y(\theta, \pi) = Y(\theta, 0), \quad \theta \in [0, b]_\mathbb{T},$$
$$\Delta Y(\theta, \eta) = \Upsilon_k(\theta, \eta), \quad k = 1, 2, \ldots, p.$$}

where $c(\theta, \eta)$ is a continuous function. Define an operator $A(\theta)$ by $A(\theta)y = c(\theta, \eta) \frac{\partial^2}{\Delta \eta^2} y$ and $B$ by $Bu(\theta) = b(\eta)W(\theta, \eta)$, $\eta \in [0, \pi]_\mathbb{T}$ and $b(\eta) \in L^2[0, \pi]_\mathbb{T}$. With the above formulations and $Y = L^2[0, \pi]_\mathbb{T}$ the equation (10) can be written in abstract form

$$\chi^\Delta(\theta) = A(\theta)\chi(\theta) + H(\theta, \chi(\theta), \chi(v(\theta))) + Bu(\theta), \quad \theta \in [0, b]_\mathbb{T},$$
$$\Delta \chi(\theta) = \Upsilon_k(\chi(\theta^{-})), \quad k = 1, 2, \ldots, p.$$}

$$\chi(0) = \chi_0,$$ (11)

where $\chi(\theta) = Y(\theta, \cdot)$ and $H(\theta, \chi(\theta), \chi(v(\theta))) = G(\theta, Y(\theta, \eta), Y(v(\eta), \eta)), \eta \in [0, \pi]_\mathbb{T}$. Suppose that the functions $H(\theta, \chi(\theta), \chi(v(\theta)))$ satisfies the conditions of Theorem 2, it can be concluded that the equation (10) is controllable.

Example 2: Consider the following non-linear system with impulsive condition

$$\chi^\Delta(\theta) = A(\theta)\chi(\theta) + H(\theta, \chi(\theta), \chi(v(\theta))) + Bu(\theta), \quad \theta \in [0, 3]_\mathbb{T},$$
$$\Delta \chi(\theta) = \Upsilon_k(\chi(\theta^{-})), \quad k = 1, 2, \ldots, p.$$}

$$\chi(0) = 1,$$ (12)

where $A(\theta) = -2$, $\theta_0 = 0$, $\theta_1 = \frac{3}{2}$ and $\theta_2 = b = 3$.

$H(\theta, \chi(\theta), \chi(v(\theta))) = \frac{1+\sin(x(\theta))}{(\theta+2)^2}, \quad \Upsilon_k(\chi(\theta^{-})) = \frac{2+\sin(x(\theta^{-}))}{k_1+3}$. Here $\Psi_{A}(\theta, \Theta(\sigma)) = e_p(\theta, \Theta(\sigma)), \quad e_p(\theta, \Theta(\sigma)) = e^{p(0-\Theta(\sigma))}$. Clearly, if we choose $B = 1$, then the system (12) becomes

$$\chi^\Delta(\theta) = A(\theta)\chi(\theta) + H(\theta, \chi(\theta), \chi(v(\theta))) + u(\theta), \quad \theta \in [0, 3]_\mathbb{T},$$
$$\Delta \chi(\theta) = \Upsilon_k(\chi(\theta^{-})), \quad k = 1, 2, \ldots, p.$$}

$$\chi(0) = 1.$$ (13)

Also, we have

$$W = \int_0^3 e_3(\theta, \Theta(\sigma)) \Delta s = \frac{1}{2}(1 - e^{-2}(3, 0)) = \frac{1}{2}(1 - e^{-6})$$

Hence, $W$ is invertible. Thus we conclude that the system (12) is controllable.
IV. CONCLUSION
In this paper, we established the controllability of systems (1) and (2). Also, we studied the controllability of Hammerstein impulsive system (7) on time scale. We used the theory of FP combined with non linear functional analysis to establish the main results. The considered problems are new for controllability on time scale. We believe that the obtained results will be valuable and give remarkable contribution to the existing literature on the topic.

CONFLICTS OF INTEREST
The author declares that there is no competing interest.

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