Higher level BGG reciprocity for current algebras*

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Abstract

We prove the Ext-orthogonality between thick and thin Demazure modules of the twisted affinization of a simple Lie algebra. This yields a higher level analogue of the Bernstein-Gelfand-Gelfand (BGG) reciprocity for twisted current algebras for each positive integer, that recovers the original one (established by Bennett, Berenstein, Chari, Ion, Khoroshkin, Loktev, and Manning) as its level one case. We also establish the branching properties about the both versions of Demazure modules and provide a new interpretation of level restricted generalized Kostka polynomials in terms of symmetric polynomials.

Introduction

For an affine Kac-Moody algebra $\tilde{g}$ over $\mathbb{C}$, we have its non-negative part $\tilde{g}_{\geq 0}$ that contains a simple Lie algebra $g$ whose Dynkin diagram is obtained by removing the zero-th node from the Dynkin digram of $\tilde{g}$ ([28]). Let $P$ denote the set of weights of $g$, and let $P^+$ be the set of dominant weights of $g$. Chari-Pressley [10] defined the local and global Weyl modules of $\tilde{g}_{\geq 0}$, denoted by $W_\lambda$ and $W_\lambda (\lambda \in P^+)$, respectively. Let $V_\lambda (\lambda \in P^+)$ be the irreducible finite-dimensional $g$-module with its highest weight $\lambda$. Bennett, Berenstein, Chari, Ion, Khoroshkin, Loktev, and Manning [2, 1, 9] showed the Ext-orthogonality

$$\text{Ext}^i(W_\lambda, W_\mu^*) \cong \begin{cases} \mathbb{C} & (i = 0, V_\lambda \cong V_\mu^*) \\ 0 & (else) \end{cases},$$

(0.1)

that can be understood as an analogue of the BGG reciprocity. The graded characters of $W_\lambda$ and $W_\lambda$ are proportional up to a constant in $\mathbb{Z}[\lbrack q \rbrack]$, and represent a Macdonald polynomial ([37]) specialized at $t = 0$.

If we replace the subalgebra $\tilde{g}_{\geq 0}$ with the Iwahori subalgebra (= the upper triangular part in the sense of Kac-Moody algebras) of $\tilde{g}$, then we have the modules $D_\lambda$ and $D_\lambda$ indexed by $\lambda \in P$ with the properties similar to (0.1) offered in [16, 14]. In addition, their graded characters are given as

$$\text{gch} D_\lambda = E_\lambda(q, 0), \quad \text{gch} D_\lambda \in c_\lambda \cdot E_{-\lambda}(q^{-1}, \infty) \quad c_\lambda \in \mathbb{Z}[\lbrack q \rbrack]^\times,$$

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where $E_\lambda(q,t)$ is a non-symmetric Macdonald polynomial ([12]) and $E_\lambda^*(q,t)$ is its character conjugate.

As pointed out by Sanderson [46] and Ion [24], the module $D_\lambda$ can be obtained as the Demazure module of a level one integrable representation of $\tilde{\mathfrak{g}}$ whenever $\tilde{\mathfrak{g}}$ is the twisted affinization of a simple Lie algebra. In fact, the module $D_\lambda$ can also be understood as a version of the Demazure module in this case, and hence (0.1) can be viewed as an affine analogue of the duality between the Demazure modules extending the classical result of van der Kallen [50] (cf. Mathieu [39] and Polo [45]). From this aspect, we should be able to generalize the above results to an arbitrary Demazure module of an integrable highest weight module of a Kac-Moody algebra. However, the proof in the finite case does not generalize naively to the infinite-dimensional setting. The main reason is that there are two versions of Demazure modules for an arbitrary Kac-Moody algebra, and these two versions (essentially) coincide only for finite types. Outside of finite types, a proper formulation of (0.1) always involves both of the two versions of Demazure modules, one is finite-dimensional and the other is infinite-dimensional.

The finite-dimensional Demazure modules, that we refer as the thin Demazure modules (these are “the” Demazure modules in the literatures, see e.g. [36]) are obtained as successive applications of Demazure functors from one-dimensional modules. This fact is quite helpful in their analysis (see Remark 4.2). Infinite-dimensional Demazure modules, that we refer as the thick Demazure modules, are much more untractable as they are only linked to irreducible highest weight integrable representations, and are not linked from some finite-dimensional objects through successive applications of Demazure functors. In fact, as Demazure functors lose information, we had practically no effective means to analyze the precise module-theoretic structure of thick Demazure modules.

For the twisted affinization of a simple Lie algebra, the level one case discussed above have perspective coming from the theory of Macdonald polynomials. Moreover, they are enough to derive the Rogers-Ramanujan identity and its generalizations ([13, 14]). In addition, their higher level analogues are pursued numerically as the level restricted generalized Kostka polynomials by combinatorial methods, and some of their descriptions are known ([21, 22, 47, 48, 35]). Thus, it has been desirable to extend (0.1) to the higher level case (at least) since the appearance of [13, 9, 16, 14].

In this paper, we make use of [14, 33] to improve the situation: Let $\mathcal{C}$ be the category of $[\tilde{\mathfrak{g}}_{\geq 0}, \tilde{\mathfrak{g}}_{\geq 0}]$-modules that is semi-simple when viewed as $\mathfrak{g}$-modules. In particular, we dismiss the central charge and grading.

**Theorem A** (≃ Theorems 3.2, 3.3, and 9.6). For each $k \in \mathbb{Z}_{>0}$ and $\lambda \in P^+$, we have two modules $W^{(k)}_\lambda$ and $W^{(k)}_\lambda$ with the following properties:

1. The modules $W^{(k)}_\lambda$ and $W^{(k)}_\lambda$ are $\tilde{\mathfrak{g}}_{\geq 0}$-modules whose heads are $V_\lambda$;

2. The module $W^{(k)}_\lambda$ is obtained as the thin Demazure module of an irreducible integrable highest weight module of $\tilde{\mathfrak{g}}$ with its level $k$;

3. We have:

$$\text{Ext}_i^C([W^{(k)}_\lambda], (W^{(k)}_\mu)^*) \cong \begin{cases} \mathbb{C} & (i = 0, V_\lambda \cong V_\mu^*) \\ 0 & (\text{else}) \end{cases}$$

(0.2)
4. The module $W^{(k)}_\lambda$ is filtered by $\{W^{(k+1)}_\mu\}_{\mu \in P^+}$, and the module $W^{(k+1)}_\lambda$ is filtered by $\{W^{(k)}_\mu\}_{\mu \in P^+}$.

5. For each $\mu \in P^+$, we have

$$\langle P_\lambda : W^{(k)}_{\mu} \rangle = [W^{(k)}_{\mu} : V_\lambda] \quad \text{and} \quad \langle W^{(k+1)}_\lambda : W^{(k)}_{\mu} \rangle = (W^{(k)}_{\mu} : W^{(k+1)}_\lambda),$$

where $(M : N)$ denote the multiplicity count of $N$ in a filtration of $M$, and $P_\lambda$ is the projective cover of $V_\lambda$ in $C$.

We also present the non-symmetric counterparts of Theorem A in §3. We remark that the $k = 1$ case of (0.2) is (0.1), and their non-symmetric counterparts are contained in [14]. Theorem A asserts that we have a structure on $C$ similar to highest weight categories [15]. However, the natural ordering that makes $C$ into a highest weight category must distinguish grading shifts of $W^{(k)}_\lambda$ as essentially different objects. This is inconsistent with our formulation (as our index set is the set of weights of $\tilde{g}$ instead of $g$, that has no grading part), and hence we do not pursue this issue here.

Except for the $k = 1$ case and the first two items, the only previously known case of Theorem A is the existence of a filtration of $W^{(k)}_\lambda$ by $\{W^{(k+1)}_\lambda\}_{\lambda \in P^+}$ when $g$ is of types ADE (Joseph-Naoi [27, 40], see also [11]). In particular, Theorem A is new even for $g = \mathfrak{sl}(2)$ in its full generality. Let us add that our proof here (and consequently our proof of Corollary B below) is independent of Joseph-Naoi.

Note that we can compute $(W^{(k)}_\mu : W^{(k+1)}_\lambda)$ by counting highest weight elements in a suitable tensor product (Demazure) crystals (Joseph [26]). There is also another combinatorial formula for $g = \mathfrak{sl}(2)$ by Biswal-Kus [3]. In this sense, Theorem A yields the first effective mean to analyze the structure of infinite-dimensional Demazure modules studied in [31, 13, 33, 14] for an arbitrary level since the description in [26, §3.3] cannot capture $(W^{(k+1)}_\lambda : W^{(k)}_\mu)$ (as these modules do not contain highest weight vectors).

Let us point out that for a fixed $\lambda \in P^+$, we have $W^{(k)}_\lambda = V_\lambda$ and $W^{(k)}_\lambda$ is isomorphic to a level $(1 - k)$ integrable lowest weight module of $\tilde{g}$ for $k \gg 0$ (Lemma 9.1). Thus, some of the modules in Theorem A are well-known.

In any case, our consideration resolves a rather long-standing speculation in this particular case (generalizing [27] that holds in simply-laced cases):

**Theorem B** (≈ Corollary 9.4). Every finite-dimensional Demazure module of $\tilde{g}$ admits a filtration whose successive quotient is isomorphic to a Demazure module of $g$ by restriction.

As a bonus of our consideration, we identify the level $k$ restricted Kostka polynomials (in the sense of [35]) as a subfamily of the branching polynomials arising from Theorem A (Corollary 9.8 and Remark 9.10). It follows that the graded characters of our modules offer a clear meaning of level restricted generalized Kostka polynomials in the language of symmetric polynomials:

**Corollary C** (≈ Corollary 9.9 and Remark 9.10). Let $k \in \mathbb{Z}_{>0}$. Let

$$\sum_{\lambda \in P^+} X_\lambda(q) \cdot \text{ch} V_\lambda \quad X_\lambda(q) \in \mathbb{Z}[q]$$
be the character of a tensor product $B$ of Kirillov-Reshetikhin crystals of level $\leq k$ ([29, §3.3]). We expand as

$$ \sum_{\lambda \in P^+} X_\lambda(q) \cdot \text{ch} V_\lambda = \sum_{\lambda \in P^+} X^{(k)}_\lambda(q) \cdot \text{gch} W^{(k+1)}_\lambda \quad X^{(k)}_\lambda(q) \in \mathbb{Z}[q], $$

where $\text{gch}$ denote the graded character (that reduces to the ordinary character $\text{ch}$ by setting $q = 1$). Then, the level $k$ restricted generalized Kostka polynomials of $B$ are precisely $X^{(k)}_\lambda(q)$ for $\lambda \in P^+$ such that $\lambda$ can be understood as a level $k$ integrable highest weight.

The level shift here may be explained by the following branching formula:

**Corollary D** (\(=\) Corollary 9.11). Let $k \in \mathbb{Z}_{>0}$. For each $\lambda, \mu \in P^+$ such that $\lambda$ can be understood as a level $k$ integrable highest weight, we have

$$ (W^{(k)}_\mu : W^{(k+1)}_\lambda) = \begin{cases} 1 & (\lambda \sim_k \mu) \\ 0 & \text{(else)} \end{cases}, $$

where $\lambda \sim_k \mu$ means that $\mu$ is an extremal weight of the level $k$ integrable highest weight module of $\hat{\mathfrak{g}}$ with its highest weight $\lambda$.

**Example E.** Assume that $\mathfrak{g} = \mathfrak{sl}(2)$, whose fundamental weight is denoted by $\varpi$. We have

$$ \text{gch} W^{(1)}_{2\varpi} = \text{gch} W^{(2)}_{2\varpi} + q \text{gch} W^{(2)}_0 \quad \text{and} \quad \text{gch} W^{(1)}_{3\varpi} = \text{gch} W^{(2)}_{3\varpi} + q^2 \text{gch} W^{(2)}_2, $$

$$ \text{gch} W^{(2)}_{3\varpi} = \text{gch} W^{(3)}_{3\varpi} + q \text{gch} W^{(3)}_\varpi \quad \text{and} \quad \text{gch} W^{(2)}_{4\varpi} = \text{gch} W^{(3)}_{4\varpi} + q^2 \text{gch} W^{(3)}_0, $$

$$ \text{gch} W^{(3)}_{3\varpi} = \text{gch} W^{(4)}_{3\varpi} + q \text{gch} W^{(4)}_2, \ldots. $$

Note that $q^4$ and $q^8$ in

$$ \text{gch} W^{(1)}_{4\varpi} = \text{gch} W^{(2)}_{4\varpi} + (q^2 + q^3) \text{gch} W^{(2)}_{2\varpi} + q^4 \text{gch} W^{(2)}_0 $$

$$ \text{gch} W^{(1)}_{6\varpi} = \text{gch} W^{(2)}_{6\varpi} + (q^3 + q^4 + q^5) \text{gch} W^{(2)}_{4\varpi} + (q^6 + q^7 + q^8) \text{gch} W^{(2)}_2 + q^6 \text{gch} W^{(2)}_0, $$

coming from Corollary D, implies that $W^{(2)}_0$ appears in $W^{(1)}_{4\varpi}$ and $W^{(1)}_{6\varpi}$ only at the socles. Theorem A 5) asserts

$$ \text{gch} W^{(2)}_0 = \text{gch} W^{(1)}_0 + q \text{gch} W^{(1)}_{2\varpi} + q^4 \text{gch} W^{(1)}_{4\varpi} + q^9 \text{gch} W^{(1)}_{6\varpi} + \cdots $$

$$ \text{gch} W^{(2)}_{2\varpi} = \text{gch} W^{(1)}_{2\varpi} + (q^2 + q^3) \text{gch} W^{(1)}_{4\varpi} + (q^6 + q^7 + q^8) \text{gch} W^{(1)}_{6\varpi} + \cdots. $$

The proof of Theorem A is by induction on level $k$ based on the level one case provided in [14]. The crucial inputs in our proof of (0.2) are the lifting theorem (Theorem 5.1) of a morphism to a Demazure module, a characterization of (global) Weyl modules ([14, 32]), and the adjoint property of the Demazure functors ([16, 39]). In addition, we make use of the structure of special Demazure modules (Proposition 6.2) proved by case-by-case analysis. We remark that Theorem A yields its quantum group analogue (cf. [34, §2.2]).

The organization of this paper is as follows: We present some preliminary results in §1. We additionally prove some preparatory module-theoretic results in §2. Then, we exhibit a series of statements that includes Theorem A 3)
and their non-symmetric analogues in §3, whose level one cases are contained in [14]. We provide a characterization of Demazure modules that is useful to prove Theorem A in §4. We present our lifting theorem and its consequences in §5. We exhibit some ext \(-1\)-vanishing results used in the sequel in §6. We deduce some consequences of §5 and §6 in §7. We prove the assertions of §3 in §8. We prove Theorem A 4)–5) and provide some applications in §9.

1 Preliminaries

We work over the field of complex numbers. A graded vector space always refer to a \(\mathbb{Z}\)-graded vector space whose degree \(n\)-part is denoted by \(V_n\). For a graded vector space, we set

\[
g\text{dim} V := \sum_{m \in \mathbb{Z}} q^m \dim V_m.
\]

General references of this section are Kac [28] and Kumar [36].

1.1 Lie algebras and its root systems

Let \(\mathfrak{g}\) be a simple Lie algebra over \(\mathbb{C}\), with a fixed Cartan subalgebra \(\mathfrak{h}\) and a Borel subalgebra \(\mathfrak{b} \supseteq \mathfrak{h}\). We set \(n := [\mathfrak{b}, \mathfrak{b}]\). We have root space decompositions

\[
n = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad n^- := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g} \oplus n^-,
\]

where \(\Delta^+ \subset \mathfrak{h}^*\) is the set of positive roots and \(\mathfrak{g}_\alpha\) is the root space of \(\alpha \in \mathfrak{h}^*\). Let \(r\) be the lacing number of \(\mathfrak{g}\), defined as the ratio of the norm of the short and long roots of \(\mathfrak{g}\). Namely, we have \(r = 1\) (\(\mathfrak{g}\) is of types ADE), 2 (\(\mathfrak{g}\) is of types BCF), or 3 (\(\mathfrak{g}\) is type \(G_2\)).

Let \(\Pi \subset \Delta^+\) be the set of simple roots that we enumerate as \(\Pi = \{\alpha_i\}_{i \in \mathbb{I}}\), and let \(\vartheta \in \Delta^+\) be the highest root. For each \(\alpha \in \Delta\), we have its coroot \(\alpha^\vee \in \mathfrak{h}^*\). We set \(\Pi^\vee := \{\alpha^\vee\}_{\alpha \in \Pi}\). The reflection \(s_\alpha \in \text{GL}(\mathfrak{h}^*)\) with respect to \(\alpha^\vee\) generates the Weyl group \(W\) of \(\mathfrak{g}\), whose length function with respect to \(\{s_i\}_{i \in \mathbb{I}}\) (where we set \(s_i \equiv s_{\alpha_i}\)) is denoted by \(\ell\). Let \(w_0 \in W\) be the longest element. We have \(-w_0 \vartheta = \vartheta\) since \(\vartheta\) is the non-zero smallest dominant root. Let \(Q := \sum_{i \in \mathbb{I}} \mathbb{Z} s_i \subset \mathfrak{h}^*\) be the root lattice, and let \(Q^+ := \sum_{i \in \mathbb{I}} \mathbb{Z}_{\geq 0} s_i \subset Q\) be its positive submonoid. We set \(Q^\vee := \sum_{i \in \mathbb{I}} \mathbb{Z} s_i^\vee \subset \mathfrak{h}^*\) to be the coroot lattice. The set of (dominant) integral weights of \(\mathfrak{g}\) is defined as:

\[
P := \{\lambda \in \mathfrak{h}^* \mid \alpha^\vee(\lambda) \in \mathbb{Z} \forall \alpha \in \Delta\} \quad \text{and} \quad P^+ := \{\lambda \in P \mid \alpha^\vee(\lambda) \in \mathbb{Z}_{\geq 0} \forall \alpha \in \Delta^+\}.
\]

Let \(\{\pi_i\}_{i \in \mathbb{I}} \subset P\) denote the set of fundamental weights of \(\mathfrak{g}\), i.e. we have \(\alpha_i^\vee(\pi_j) = \delta_{ij}\). Each \(\lambda \in P^+\) defines an irreducible finite-dimensional \(\mathfrak{g}\)-module \(V_\lambda\) with a unique \(n\)-fixed vector \(v_\lambda\) of \(\mathfrak{h}\)-eigenvalue \(\lambda\) up to scalar.

Let

\[
\Delta_{af} := (\Delta \times r\mathbb{Z}\delta) \cup \bigcup_{1 \leq i < r} (W, \theta + i\delta + r\mathbb{Z}\delta) \cup \mathbb{Z}_{\neq 0} \delta
\]

be the twisted affine root system of \(\Delta\) with its positive part \(\Delta^+ \subset \Delta^+_{af}\). We set \(\alpha_0 := -\theta + \delta\), \(\Pi_{af} := \Pi \cup \{\alpha_0\}\), and \(I_{af} := I \cup \{0\}\). We set \(W_{af} := W \ltimes Q\) and
call it the affine Weyl group. It is a reflection group generated by \( \{ s_i \mid i \in I_{af} \} \), where \( s_0 \) is the reflection with respect to \( \alpha_0 \). This equips \( W_{af} \) a length function \( \ell : W_{af} \to \mathbb{Z}_{\geq 0} \) extending that of \( W \). For \( w \in W_{af} \), a reduced expression of \( w \) is a sequence \( s_{i_1} \ldots s_{i_t} \) such that

\[
\ell = \ell(w), \quad w = s_{i_1} \ldots s_{i_t}.
\]

For each \( w, v \in W_{af} \), we define \( w \leq v \) if and only if a(n ordered) subsequence of a reduced expression of \( v \) defines a reduced expression of \( w \). The embedding \( Q \to W_{af} \) defines a translation element \( t_\gamma \) for each \( \gamma \in Q \) whose normalization is \( t_{-\delta} = s_0 s_0 \).

### 1.2 Twisted affine Lie algebras

Let \( \tilde{g} \) be the twisted affinization of \( g \), that is the affine Kac-Moody algebra with its set of roots \( \Delta_{af} \). These are types

\[
A_{1}^{(1)}, D_{1+3}^{(1)}, E_{6}^{(1)}, E_{8}^{(1)}, \text{ and } A_{2\ell+3}^{(2)}, D_{\ell+3}^{(2)}, E_{6}^{(2)}, D_{4}^{(3)}
\]

for \( \ell \in \mathbb{Z}_{>0} \) in [28]. In particular, we have \( g \subset \tilde{g} \) as the Lie subalgebra obtained by deleting the node 0 from the Dynkin diagram of \( \tilde{g} \) [28, Chap. 4 Table Aff].

We have the triangular decomposition using Lie subalgebras

\[
\tilde{g} = \tilde{n} \oplus \tilde{h} \oplus \tilde{n}^- \supset n \oplus h \oplus n^- = g,
\]

where we require

\[
\tilde{n} = \bigoplus_{\alpha \in \Delta_{af}^+} \tilde{g}_\alpha, \quad \tilde{h} = \tilde{h} \oplus \mathbb{C}d = h \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \tilde{n}^- = \bigoplus_{\alpha \in \Delta_{af}^-} \tilde{g}_\alpha
\]

such that \( [K, \tilde{g}] = 0, [h, \tilde{h}] = 0 \), and \( [\tilde{h}, \tilde{n}] \subset \tilde{n} \). Note that \( \tilde{g}_\alpha \) for each \( \alpha \in \Delta_{af} \) is defined as the \( (\text{ad} \tilde{h}) \)-eigenspace through an inclusion

\[
\Delta_{af} \subset (h \oplus \mathbb{C}d)^+ \subset \tilde{h}^+
\]

such that \( h \) is the dual element of \( d \) and \( \Delta \subset h^+ \subset (h \oplus \mathbb{C}d)^+ \). We extend the definition of the coroot by requiring \( \alpha^{\vee} \in h \subset \tilde{h} \) for \( \alpha \in \Delta \) and \( \alpha_0^{\vee} = K - \delta^{\vee} \).

We also set

\[
\tilde{g} := [\tilde{g}, \tilde{g}], \quad \tilde{h} := \tilde{h} \oplus \tilde{n}, \quad \text{and} \quad \tilde{n}^- := \tilde{h} \oplus \tilde{n}^-.
\]

Let \( \tilde{g}_{\geq 0} \) and \( \tilde{g}_{\leq 0} \) denote the direct sum of non-negative and non-positive \( d \)-eigenspaces of \( \tilde{g} \), respectively. They are naturally Lie subalgebras of \( \tilde{g} \).

For \( i \in I_{af} \), we find standard Kac-Moody generators \( E_i \in \tilde{g}_{\alpha_i} \) and \( F_i \in \tilde{g}_{-\alpha_i} \) ([28, §1.2]) such that \( [E_i, F_i] = \alpha_i^{\vee} \in \tilde{h} \). We have the Chevalley involution \( \theta \) on \( \tilde{g} \) such that \( \theta(E_i) = F_i, \theta(F_i) = E_i \) \( (i \in I_{af}) \) and \( \theta(d) = -d \). Let \( \{ \Lambda_i \}_{i \in I_{af}} \subset \tilde{h}^+ \) be the set of fundamental weights of \( \tilde{g} \), and we fix \( \rho \in \tilde{h}^+ \) such that

\[
\Lambda_i(\alpha_j^{\vee}) = \delta_{ij}, \quad \rho(\alpha_j^{\vee}) = 1, \quad \Lambda_i(d) = 0, \quad \rho(d) = 0 \quad i, j \in I_{af}.
\]

We set \( P_{af} := \bigoplus_{i \in I_{af}} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta \) and \( P_{af}^+ := \bigoplus_{i \in I_{af}} \mathbb{Z}_{\geq 0} \Lambda_i \oplus \mathbb{Z} \delta \). The inclusion \( h \subset \tilde{h} \) induces a surjection \( \varphi : P_{af} \to P \) such that \( \overline{\lambda}_i = \varphi_i \) \( (i \in I) \) and \( \overline{\lambda}_0 = 0 = \overline{\delta} \).
We have an inclusion $P \hookrightarrow P_{af}$ obtained by extending $\omega_i \mapsto \Lambda_i - \langle \vartheta_i, \omega_i \rangle \Lambda_0$ ($i \in I$) as a map of abelian groups (note that we have $\overline{\Lambda} = \lambda$ for $\lambda \in P$). The affine Weyl group $W_{af}$ acts on $P_{af}$ by letting $s_i$ ($i \in I_{af}$) act by the reflection with respect to $\alpha_i^\vee$.

For each $k \in \mathbb{Z}$, we set $P_{af,k} := \{ \Lambda \in P_{af} \mid K(\Lambda) = k \}$ and

$$P^+_k := \{ \Lambda \in P_{af} \mid \Lambda(K) = k, \Lambda(\alpha_i^+) \geq 0 \ \forall i \in I_{af} \} \subset P_{af}^+.$$  

An element of $P_{af,k}$ is called a level $k$ weight, and an element of $P^+_k$ is called a level $k$ dominant weight. Let $k \in \mathbb{Z}_{>0}$. The set $P^+_k$ is finite modulo the action of $\mathbb{Z}\delta$. For each $\Lambda \in P^+_k$, we have a (level $k$) highest weight integrable representation $L(\Lambda)$ with its highest weight vector $v_\Lambda$. For each $w \in W_{af}$, the highest weight $w\Lambda$-part of $L(\Lambda)$ is one-dimensional, and hence the vector $v_{w\Lambda} \in L(\Lambda)$ is determined uniquely up to scalar. In view of the fact that $P^+_k$ is the fundamental domain of the $W_{af}$-action on $P_{af,k}$ ([28, Corollary 10.1]), each $\lambda \in P$ uniquely defines $\Lambda \in P^+_k$ such that

$$\lambda + k\Lambda_0 = w\Lambda \text{ for some } w \in W_{af}. \quad (1.1)$$

This defines the level $k$ thin Demazure module as

$$D^{(k)}_\Lambda := U(\tilde{\mathfrak{b}})v_{w\Lambda} \subset L(\Lambda).$$

Here we remark that $\dim D^{(k)}_\Lambda < \infty$ since its character is obtained by a finitely many applications of Demazure operators ([36, 8.1.17 Proposition]). We similarly define the level $k$ thick Demazure module as:

$$L(\Lambda)^w := U(\tilde{\mathfrak{b}})v_{w\Lambda} \subset L(\Lambda).$$

We twist the $\tilde{\mathfrak{b}}$-module structure on $L(\Lambda)^w$ by the Chevalley involution $\theta$, and denote the resulting $\mathfrak{b}$-module by $\tilde{D}^{(k)}_\Lambda$. We may also denote these $\mathfrak{b}$-module structure of $L(\Lambda)^w$ and $L(\Lambda)$ by $\theta L(\Lambda)^w$ and $\theta L(\Lambda)$ whenever it is appropriate (and hence $\tilde{D}^{(k)}_\Lambda \equiv \theta L(\Lambda)^w$). The both of two modules $D^{(k)}_\Lambda$ and $\tilde{D}^{(k)}_\Lambda$, regarded as $\mathfrak{b}$-modules, are $\mathfrak{h}$-semisimple and have cyclic generating vectors of $\mathfrak{h}$-weight $\lambda$.

As $\theta L(\Lambda) \ (\Lambda \in P^+_k)$ has level $-k$ as a $\mathfrak{g}$-module, we might consider $\tilde{D}^{(k)}_\Lambda$ as a submodule of the level $-k$ irreducible lowest weight integrable representation of $\mathfrak{g}$.

**Definition 1.1** (Cherednik order). For each $\lambda \in P$, let $\lambda_+, \lambda_- \in P$ denote the unique elements in $(W\lambda \cap P^+)$ and $(W\lambda \cap -P^+)$, respectively. Let $\preceq$ be the partial order on $P$ defined as:

$$\lambda \preceq \mu \iff \lambda_- \in \mu_- + Q_+ \text{ or } \lambda = \mu_- \text{ and } \lambda \in \mu + Q_+.$$  

We write $\lambda \prec \mu$ if $\lambda \preceq \mu$ and $\lambda \neq \mu$. We set

$$\Sigma(\lambda) := \{ \mu \in P \mid \mu \preceq \lambda \} \text{ and } \Sigma_*(\lambda) := \{ \mu \in P \mid \mu \prec \lambda \}.$$  

The partial order $\preceq$ equips $P_{af}$ a preorder induced by the projection to $P$. We define the shifted action of $W_{af}$ on $\mathbb{R} \otimes \mathbb{Z} P$ as

$$w(\lambda) := w(\lambda + \Lambda_0) - \Lambda_0 \quad \lambda \in \mathbb{R} \otimes \mathbb{Z} P, \ w \in W_{af}.$$
Note that we have $\lambda_+ = w_0 \lambda_-$.  

**Lemma 1.2** ([14] §2.2.1). For each $\lambda \in P$, we have  
\[ \Sigma(\lambda) = (\text{Conv } \Sigma(\lambda)) \cap (\lambda + Q), \]  
where Conv denotes the convex hull in $\mathbb{R} \otimes_{\mathbb{Z}} P$. \hfill \square

**Proposition 1.3** ([14] §4.1.1). Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P^+_k$ and $w, v \in W_{af}$ such that $w \leq v$, we have $w\Lambda \preceq v\Lambda$. \hfill \square

**Theorem 1.4** ([29] see also [33] Theorem C). Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P^+_k$ and $w, v \in W_{af}$ such that $w \leq v$, we have $L(\Lambda)^w \supset L(\Lambda)^v$ as $\mathfrak{b}^-$-modules. This inclusion is an equality if and only if $w\Lambda = v\Lambda$. \hfill \square

Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P^+_k$ and $w \in W_{af}$, we set  
\[ \text{gr}^w \cdot L(\Lambda) := \frac{L(\Lambda)^w}{\sum_{v > w, w\Lambda \neq v\Lambda} L(\Lambda)^v}. \]

We assume $\lambda + k\Lambda_0 = w\Lambda$ in addition. We denote the $\mathfrak{h}$-module obtained by twisting the action on $\text{gr}^w \cdot L(\Lambda)$ by the Chevalley involution by $\mathbb{D}^{(k)}_{\lambda}$. Since $\text{gr}^w \cdot L(\Lambda)$ is a quotient of $L(\Lambda)^w$ as $\mathfrak{b}^-$-modules, we deduce that $\mathbb{D}^{(k)}_{\lambda}$ is a quotient of $\mathbb{D}^{(k)}_0$ as $\mathfrak{b}$-modules for each $\lambda \in P$. In particular, $\mathbb{D}^{(k)}_0$ is a cyclic $\mathfrak{b}$-module generated by a $\mathfrak{h}$-weight vector of weight $\lambda$.

### 1.3 Categories of representations

We set $Q^+_q := \sum_{\alpha \in \Delta^+_q} \mathbb{Z}_{\geq 0}\alpha \subset P_{af}$. For an inclusion $q \subset p$ of Lie algebras with $q$ abelian, let us denote by $(p,q)$ the category of $p$-modules equipped with semi-simple $q$-action.

A $\tilde{\mathfrak{b}}_{\geq 0}$-module $M$ is said to be $\mathfrak{g}$-integrable if $M$ decomposes into a direct sum of finite-dimensional $\mathfrak{g}$-modules (by restriction).

**Definition 1.5** (Category $\mathcal{B}$). A $\tilde{\mathfrak{h}}$-module $M$ is said to be graded if $M$ is $\tilde{\mathfrak{h}}$-semisimple and all of its eigenvalues belong to $P_{af}$, and each $\tilde{\mathfrak{h}}$-weight space is at most countable dimension. Let $\Psi(M) \subset P_{af}$ denote the set of $\tilde{\mathfrak{h}}$-weights of $M$ with non-zero weight vectors. The module $M$ is said to be bounded if the $d$-degrees of $M$ is bounded from the bottom, and each $d$-weight space of $M$ is finite-dimensional. Let $\mathcal{B}$, $\mathcal{B}_{\text{bdd}}$, and $\mathcal{B}_0$ be the fullsubcategories of the category of $\tilde{\mathfrak{b}}$-modules consisting of graded $\tilde{\mathfrak{b}}$-modules, bounded graded $\tilde{\mathfrak{b}}$-modules, and finite-dimensional graded $\tilde{\mathfrak{b}}$-modules, respectively.

**Definition 1.6** (Category $\mathcal{C}$). A graded $\tilde{\mathfrak{g}}_{\geq 0}$-module $M$ is a $\tilde{\mathfrak{g}}_{\geq 0}$-module that restricts to a graded $\mathfrak{b}$-module by restriction. Let $\mathcal{C}$ be the fullcategory of the category of graded $\tilde{\mathfrak{g}}_{\geq 0}$-modules consisting of $\mathfrak{g}$-integrable modules. Let $\mathcal{C}_{\text{bdd}}$ and $\mathcal{C}_0$ denote the fullsubcategories of $\mathcal{C}$ consisting of modules whose restrictions to $\mathfrak{b}$ belong to $\mathcal{B}_{\text{bdd}}$ and $\mathcal{B}_0$, respectively.

A simple object in $\mathcal{B}$ is a character of $\tilde{\mathfrak{h}}$, and a simple object of $\mathcal{C}$ is isomorphic to $V_\lambda (\lambda \in P^+_1)$ up to a character twist. Note that a module $M$ in $\mathcal{B}_{\text{bdd}}$ or $\mathcal{C}_{\text{bdd}}$ is generated by its head $\mathfrak{h} M$.
For each $\Lambda \in P_{af}$, we set

$$Q_{\Lambda} := U(\tilde{b}) \otimes_{U(\tilde{b})} C_{\Lambda}. \tag{1.2}$$

The module $Q_{\Lambda}$ is the projective cover of $C_{\Lambda}$ in $\mathcal{B}$.

For each $\lambda \in P^+$, the $\mathfrak{g}$-module $V_{\lambda}$ can be regarded as a graded $\tilde{\mathfrak{g}}_{\geq 0}$-module via the natural surjection $\tilde{\mathfrak{g}}_{\geq 0} \to \mathfrak{g}$ annihilating the positive $d$-degree parts. We set

$$P_{\lambda} := U(\tilde{\mathfrak{g}}_{\geq 0}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0})} V_{\lambda} \in \mathcal{C} \tag{1.2},$$

for each $\lambda \in P^+$. The module $P_{\lambda}$ is the projective cover of $V_{\lambda}$ in $\mathcal{C}$ ([8]). For $\Lambda \in P_{af}$ such that $\Lambda = \lambda$, we set $P_{\Lambda} := P_{\lambda} \otimes C_{\Lambda-\lambda}$. We also set $V_{\Lambda} := V_{\lambda} \otimes C_{\Lambda-\lambda}$, that we regard as a ($\mathfrak{g} + \tilde{\mathfrak{h}}$)-module.

All the six categories above are abelian, and equipped with a functor $q_m$ ($m \in \mathbb{Z}$) corresponding to tensoring with $C_m$. Using this, we set

$$\hom_{\mathcal{C}}(M, N) := \bigoplus_{m, l \in \mathbb{Z}} \hom_{\mathcal{C}}(q^m M, N \otimes C_{l\Lambda_0}), \tag{1.2}$$

where $\mathcal{C}$ is one of our categories and $M, N$ are their objects. We similarly define

$$\text{ext}^i_{\mathcal{C}}(M, N) := \bigoplus_{m, l \in \mathbb{Z}} \text{ext}^i_{\mathcal{C}}(M, q^{-m} N \otimes C_{l\Lambda_0}) \tag{1.2},$$

using the (graded) projective resolution of $M$. We regard these spaces as graded vector spaces via the $d$-degree twists ($= \mathfrak{q}$-twists). Despite of the conventional isomorphism of graded vector spaces

$$\text{ext}^i_{\mathcal{C}}(M, N) \cong \text{ext}^i_{\mathcal{C}}(M, N \otimes C_{\Lambda_0}), \tag{1.2}$$

we usually keep track of the natural action of $K$ as it is helpful to understand the idea in many cases.

For $M \in \mathcal{B}_{\text{bdd}}$ or $\mathcal{C}_{\text{bdd}}$, we define

$$M^\vee := \bigoplus_{\Lambda \in P_{af}} \hom_{\mathcal{C}}(C_{\Lambda}, M)^*. \tag{1.2}$$

We have $M^\vee \in \mathcal{B}$ or $\mathcal{C}$, respectively. In addition, we have an isomorphism

$$\hom_{\mathcal{B}}(M_1, M_2) \cong \hom_{\mathcal{B}}(M_1^\vee, M_2^\vee) \quad M_1 \in \mathcal{B}_{\text{bdd}}, M_2 \in \mathcal{B}_0. \tag{1.2}$$

The isomorphism (1.2) is functorial, and we can cover a module in $\mathcal{B}_{\text{bdd}}$ to eliminate an individual element in higher extensions. Hence (1.2) extends to an isomorphism between higher exts ([20, Proposition 2.2.1]).

Consider a decreasing separable filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \bigcap_{r \geq 0} M_r = 0 \tag{1.2}$$

of objects in one of the above categories. Assume that we can find $r \in \mathbb{Z}$ for each $m \in \mathbb{Z}$ such that the $d$-degree of $\Psi(M_r)$ is concentrated in $\geq m$. Then, we have

$$\lim_{\leftarrow \substack{r \rightarrow \infty\n}} M/M_r = M \tag{1.2}$$

in that category. This particularly mean that we do have the completion of $M$ with respect to the $d$-grading, but it can be different from that in the category of all modules.
1.4 BGG resolution and Demazure functors

For each \( i \in I_{af} \), we set \( \tilde{p}_i := \tilde{b} \oplus \tilde{\gamma}_{-\alpha_i} \). We also set

\[
\mathfrak{sl}(2, i) := \mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee \oplus \mathbb{C}F_i \subseteq \tilde{p}_i.
\]

It is standard that \( \mathfrak{sl}(2, i) \subseteq \tilde{p}_i \subseteq \tilde{\mathfrak{g}} \) are Lie subalgebras. A \( \tilde{p}_i \)-module is \( \mathfrak{sl}(2, i) \)-integrable if it is a direct sum of finite-dimensional \( \mathfrak{sl}(2, i) \)-modules by restriction. The Demazure functor \( D_i \) sends a module \( M \in \mathfrak{B} \) to the maximal \( \mathfrak{sl}(2, i) \)-integrable quotient of \( U(\tilde{p}_i) \otimes_{U(\tilde{\mathfrak{g}})} M \) (as \( \tilde{p}_i \)-modules). We regard \( D_i \) as an auto-functor on \( \mathfrak{B} \), that preserves \( \mathfrak{B}_0 \).

For each \( \Lambda \in P_{af} \), we define the Verma modules as

\[
M(\Lambda) := U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}})} C_{\Lambda}, \quad \text{and} \quad \delta M(\Lambda) := U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}})} C_{-\Lambda}.
\]

We have the the BGG resolution of \( L(\Lambda) \ (\Lambda \in P_{af}^+ \) ), that reads as

\[
\cdots \to \bigoplus_{\ell(w) = 2} M(w(\Lambda + \rho) - \rho) \to \bigoplus_{\ell(w) = 1} M(w(\Lambda + \rho) - \rho) \to M(\Lambda) \to L(\Lambda) \to 0
\]

by [36, 9.1.3 Theorem]. Since \( M(\Lambda') \ (\Lambda' \in P_{af}^+ \) is a free \( U(\tilde{\mathfrak{g}}^-) \)-module of rank one, each \( \delta M(\Lambda') \) is a projective object in \( \mathfrak{B} \). Thus, the \( \delta \)-twist of \( (1.3) \) offers a projective resolution of \( \delta L(\Lambda) \).

**Theorem 1.7** (Joseph [25]). Let \( i, j \in I_{af} \).

1. We have a natural functor \( \text{Id} \to D_i \);
2. We have an isomorphism of functors \( D_i \to D_i \circ D_i \);
3. The functor \( D_i \) is right exact, and its left derived functor satisfies \( \mathbb{L}^m D_i \equiv 0 \) for \( m 
eq 0, -1 \) as (universal) \( \delta \)-functors on \( \mathfrak{B} \);
4. If \( M \in \mathfrak{B}_0 \) is obtained as the restriction of a \( \mathfrak{sl}(2, i) \)-integrable \( \tilde{p}_i \)-module, then we have a natural isomorphism of functors:

\[
M \otimes D_i(\bullet) \cong D_i(M \otimes \bullet);
\]
5. If we have \( m \in \mathbb{Z}_{>0} \) such that

\[
\overbrace{s_is_js_i\cdots}^m = \overbrace{s_js_is_i\cdots}^m, \quad \text{then we have} \quad \overbrace{D_iD_iD_i\cdots}^m \cong \overbrace{D_jD_iD_jD_i\cdots}^m.
\]

For each \( w \in W_{af} \), we have a reduced expression

\[
w = s_{i_1} \cdots s_{i_{\ell}}, \quad i_1, \ldots, i_{\ell} \in I_{af}.
\]

Thanks to Theorem 1.7 5), the functor

\[
D_w := D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_{\ell}} \quad (1.4)
\]

does not depend on a reduced expression of \( w \). A module \( M \in \mathfrak{B} \) admits an integrable \( g \)-action if and only if \( M \cong D_{w_0}(M) \), that is equivalent to \( D_i(M) = M \) for each \( i \in I \).

The categories \( \mathfrak{B}_0 \) and \( \mathfrak{C}_0 \) admit the duality \( * \). It induces the functor \( D_i^\dagger := \ast \circ D_i \circ \ast \ (i \in I_{af}) \), for which we can consider its right derived functor. As \( * \) is an involution, we have the functor \( D_i^\dagger(w \in W_{af}) \) defined by replacing each \( D_i \) with \( D_i^\dagger \) in \( (1.4) \).
Theorem 1.8 ([16] Proposition 5.7, cf. Mathieu [39] Lemma 8). For each $M \in \mathfrak{B}$ and $N \in \mathfrak{B}$, we have a functorial isomorphism
\[
\text{ext}^i_\mathfrak{B}(L^\bullet \mathcal{D}_i(M), N) \cong \text{ext}^i_\mathfrak{B}(M, R^\bullet \mathcal{D}^i_\mathfrak{B}(N)) = \text{ext}^i_\mathfrak{B}(M, (\mathbb{L}^\bullet \mathcal{D}_i(N^*))^*),
\]
where all terms are understood to be the hypercohomologies. \hfill \qed

2 Some module-theoretic results

Keep the setting of the previous section.

Proposition 2.1 (Cartan-Eilenberg [7]). For each $M \in \mathfrak{C}$ and $N \in \mathfrak{C}_0$, we have
\[\text{Ext}^\bullet_\mathfrak{C}(M, N) = \text{Ext}^\bullet_\mathfrak{C}(M, N).\]

Proof. We set $\widetilde{\mathfrak{g}}_{>0} := \ker (\widetilde{\mathfrak{g}}_{>0} \to (\mathfrak{g} + \mathfrak{h})).$ The Hochschild-Serre spectral sequence [7, Chapter XVI §6] claims that
\[E_2^{q,p} := \text{Ext}^q_{(\mathfrak{g} + \mathfrak{h})(\text{C}, \text{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N))} \Rightarrow \text{Ext}^{q+p}_{(\mathfrak{g}, \mathfrak{h})}(M, N)
\]
holds for $M, N \in (\mathfrak{b}, \mathfrak{h})$. In case $M, N \in \mathfrak{C}$, we find that $\text{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)$ also admits an action of $\mathfrak{g}$, that is $\mathfrak{g}$-integrable since $U(\widetilde{\mathfrak{g}}_{>0})$ is $\mathfrak{g}$-integrable by the adjoint $\mathfrak{g}$-action. In particular, we have
\[\text{Ext}^{>0}_{(\mathfrak{b} + \mathfrak{h}, \mathfrak{b})}(\text{C}, \text{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)) = 0 = \text{Ext}^{>0}_{(\mathfrak{g} + \mathfrak{h}, \mathfrak{b})}(\text{C}, \text{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)) \quad \text{and}
\]
\[\text{Hom}_{(\mathfrak{b} + \mathfrak{h}, \mathfrak{b})}(\text{C}, \text{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)) = \text{Hom}_{(\mathfrak{g} + \mathfrak{h}, \mathfrak{b})}(\text{C}, \text{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N))
\]
in this case by the complete reducibility of integrable $\mathfrak{g}$-modules in $(\mathfrak{b}, \mathfrak{h})$. Therefore, we have
\[\text{Ext}^\bullet_\mathfrak{C}(M, N) = \text{Ext}^\bullet_\mathfrak{C}(M, N)
\]
as required. \hfill \qed

Theorem 2.2 (see Kumar [36] Chapter VIII, [14] Proposition 4.16). Let $k \in \mathbb{Z}_{>0}$. For each $\lambda \in P$ and $i \in I_{af}$, we have
\[\mathbb{L}^\bullet \mathcal{D}_i(D_{\lambda}(k)) = \left\{\begin{array}{ll}
\mathcal{Q}^{\delta_{\lambda,n}(q^{\nu,\lambda} - k)} & (\delta_{\lambda,n}(s_{\lambda + k\lambda}) > \lambda) \\
D_{\lambda}(k) & (\delta_{\lambda,n}(s_{\lambda + k\lambda}) \leq \lambda)
\end{array}\right.. \]

Theorem 2.3 (Kashiwara [29], see also [33] Theorem C). Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P_i^+$, $w \in W_{af}$, and $i \in I_{af}$, we have
\[\mathbb{L}^\bullet \mathcal{D}_i(\theta L(\Lambda)^w) = \left\{\begin{array}{ll}
\theta L(\Lambda)^{s_{w}w} & (s_{w}w < w) \\
\theta L(\Lambda)^w & (s_{w}w > w)
\end{array}\right.. \]

Corollary 2.4. Keep the setting of Theorem 2.3. For a subset $S \subset W_{af}$, we set
\[M := \sum_{w \in S} \theta L(\Lambda)^w \quad \text{and} \quad N := \bigcap_{w \in S} \theta L(\Lambda)^w.
\]

For each $i \in I_{af}$, we have
\[L^{<0} \mathcal{D}_i(M) = 0 = L^{<0} \mathcal{D}_i(N), \quad M \subset \mathcal{D}_i(M), \quad \text{and} \quad N \subset \mathcal{D}_i(N).
\]
Proof. In [29], the key argument is the string property of Demazure crystal, that is closed under taking sum or intersection. As the containment relations and the intersections of Demazure crystals correspond to the containment relations and the intersections between the corresponding Demazure modules (cf. [33]), we conclude the result.

Corollary 2.5. Keep the setting of Theorem 2.3. The module $\theta L(\Lambda)^w$ is $g$-stable if and only if $w\Lambda$ is dominant.

Proof. The only if part of the condition follows since we need $U(b^-)v_{w\Lambda} = U(g)v_{w\Lambda}$, that can occur only if $w\Lambda \in P^+$. The if part of the condition follows as $D_i(\theta L(\Lambda)^w) = \theta L(\Lambda)^w$ holds for each $i \in I$ if $w\Lambda \in P^+$ by Proposition 1.3 and Theorem 1.4, in addition to Theorem 2.3.

For each $\lambda \in P^+$ such that $\lambda_- + k\Lambda_0 = w\Lambda$ for some $w \in W_{af}$ and $\Lambda \in P^+_k$, we set $W_\lambda^{(k)} := D_{-\lambda_-}^{(k)}$ and

$$W_{-\lambda_-}^{(k)} := \sum_{v \in W_{af}, \lambda_- \prec (\Lambda v)_{-\lambda_-}} \theta L(\Lambda)^w.$$  \hspace{1cm} (2.1)

Lemma 2.6. Let $k \in \mathbb{Z}_{>0}$. For each $\lambda \in P^+$, the modules $W_\lambda$ and $W_\lambda$ are $\tilde{g}_{\geq 0}$-modules generated by $V_\lambda$.

Proof. The assertions follow as the both modules are cyclic as $\tilde{b}$-module with a cyclic vector of $h$-weight $\lambda_-$, and it generates $V_\lambda$ as (a unique) $g$-module (that has lowest weight $\lambda_-$).

The following result is a straightforward generalization of [14]:

Proposition 2.7. Let $k \in \mathbb{Z}_{>0}$. For each $\lambda \in P^+$, the module $W_\lambda^{(k)}$ admits a filtration by $\{D^{(k)}_{\mu} \}_{\mu \in W_\lambda}$ as a graded $b$-module such that each module appears exactly once in the associated graded.

Proof. In view of Proposition 1.3, we find that $W_\lambda$ is the only possible $h$-weights of shape $-w\Lambda$, where the extremal weight vector $v_{w\Lambda} \in L(\Lambda)$ contributes to $W_\lambda^{(k)}$ through subquotients. Since the sums of Demazure submodules of an integrable highest weight module form a distributive lattice with respect to intersections ([32, Theorem C]), we conclude the result.

Lemma 2.8. Let $k \in \mathbb{Z}_{>0}$. For each $\lambda \in P^+$, we have

$$W_\lambda^{(k)} \cong L^\bullet \mathcal{D}_{w_0}(D^{(k)}_\lambda).$$

Proof. By Theorem 2.3 and Corollary 2.4, we find a surjection

$$L^\bullet \mathcal{D}_{w_0}(D^{(k)}_\lambda) \equiv \mathcal{D}_{w_0}(D^{(k)}_\lambda) \longrightarrow W_\lambda^{(k)}$$

as a quotient of thick Demazure module by a proper submodule spanned by the sum of smaller thick Demazure modules. To see their coincidence, it suffices to see that $w' \leq v'$ holds when $w \leq v$ ($w, v \in W_{af}$), where $w', v' \in W_{af}$ is the maximal length representative of the coset $W \backslash W_{af}$ of $w$ and $v$, respectively. This is a standard property of the Bruhat order that follows from [36, 1.3.14 Lemma], for example.
Theorem 2.9 (Fourier-Littelmann-Manning-Senesi [17, 18]). For each $\lambda \in P^+$, the module $W_{\lambda}^{(1)} \otimes \mathbb{C}_{\lambda_0}$ is a self-extension of $W_{\lambda}^{(1)} \otimes \mathbb{C}_{-\lambda_0}$ such that

$$\text{end}_q(W_{\lambda}^{(1)} \otimes \mathbb{C}_{\lambda_0})$$

is a graded polynomial ring with positive $d$-degree generators. In addition, $W_{\lambda}^{(1)} \otimes \mathbb{C}_{\lambda_0}$ is a projective object in the fullsubcategory of $\mathcal{C}$ whose object $M$ satisfies $\overline{\mathcal{P}}(M) \leq \lambda$.

The following result is [14, Theorem 5.2], rewritten along the lines of [32, Theorem 4.13]:

Theorem 2.10 ([14, 32]). For each $\lambda \in P^+$, we have

$$L^\bullet D_{-\beta}(W_{\lambda}^{(1)} \otimes \mathbb{C}_{\lambda_0}) \cong q^{-\langle \beta, \lambda \rangle} W_{\lambda}^{(1)} \otimes \mathbb{C}_{\lambda_0}$$

for each $\beta \in (Q \cap P^+) \subset Q^\vee$.

Corollary 2.11. Keep the setting of Theorem 2.10. We apply an affine Dynkin diagram automorphism to $(W_{\lambda}^{(1)} \otimes \mathbb{C}_{\lambda_0})$ and denote $W$ the resulting module. Then, we have

$$L^\bullet D_{w_0}(W) \cong q^m W_{\lambda'}^{(1)} \otimes \mathbb{C}_{\lambda_0} \quad m \in \mathbb{Z}, \lambda' \in P^+.$$

Proof. Combining Theorem 2.9 and Theorem 2.10, we find

$$\bigcup_{w \in W} D_w(W_{\lambda}^{(1)} \otimes \mathbb{C}_{\lambda_0})$$

satisfies the same universal property as the level zero extremal weight module of $\tilde{g}$ ([29, §8]). Thus, $W_{\lambda}^{(1)}$ must be a Demazure module of an extremal weight module of $\tilde{g}$ ([30]). This yields a bijection between $\{W_{\lambda}^{(1)}\}_{\lambda \in P^+}$ and $g$-stable Demazure modules of level zero extremal weight modules (up to grading shifts). As the notion of Demazure module of a level zero extremal weight module is invariant under the diagram automorphism, we conclude the assertion by the $g$-symmetrization.

Let us consider the $\mathbb{Q}(q)$-vector space

$$B := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q]} \lim_{\leftarrow m} \bigoplus_{\lambda \in P} (\mathbb{Z}[q] / (q^{m+1})) e^\lambda.$$

We also set $C$ as the space of $W$-invariants of $C$, where $w \in W$ sends $q^m e^\lambda \mapsto q^m e^{w\lambda}$ for $m \in \mathbb{Z}, \lambda \in P$. These two vector spaces are equipped with the topology induced by the inverse system.

For each $M \in \mathcal{B}_{\text{bdd}}$ or $M \in \mathcal{C}_{\text{bdd}}$, we set

$$\text{gch} M := \sum_{\lambda \in P, m \in \mathbb{Z}} q^m e^{\lambda} \dim \text{Hom}_{\mathfrak{h} + \mathfrak{c}(d)}(C_{\lambda + m\delta}, M).$$

In particular, we have $\text{gch} M = \text{gch} (M \otimes \mathbb{C}_{k\lambda_0})$ for $k \in \mathbb{Z}$.  

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Lemma 2.12. For each $M \in \mathcal{B}_{\text{bdd}}$ or $M \in \mathcal{C}_{\text{bdd}}$, we have $\text{gch}M \in \mathcal{B}$ or $\text{gch}M \in \mathcal{C}$, respectively.

Proof. This follows from the definitions of our categories. \qed

Proposition 2.13 ([16] Appendix). For each $M \in \mathcal{B}_{\text{bdd}}$ and $N \in \mathcal{B}_0$, we set

$$(M, N)_{EP} := \sum_{i \in \mathbb{Z}} \text{gdim} \text{ext}^i_{\mathcal{B}}(M, N^*).$$

Then, it extends to a $\mathbb{Z}[q^{\pm 1}]$-antibilinear map

$$[\mathcal{B}_{\text{bdd}}] \times [\mathcal{B}_0] \rightarrow \mathbb{Q}(\langle q^{-1} \rangle),$$

where $[\mathcal{C}]$ denotes the Grothendieck group of an abelian category $\mathcal{C}$. \qed

Lemma 2.14. Let $k \in \mathbb{Z}_{>0}$. We have:

1. $\{\text{gch} D^{(k)}_{\mu}(\lambda) \}_{\lambda \in P}$ and $\{\text{gch} D^{(k)}_{\mu}(\lambda) \}_{\lambda \in P}$ are topological $\mathbb{Q}(\langle q \rangle)$-bases of $\mathcal{B}$;

2. $\{\text{gch} W^{(k)}_{\mu}(\lambda) \}_{\lambda \in P^+}$ and $\{\text{gch} W^{(k)}_{\mu}(\lambda) \}_{\lambda \in P^+}$ are topological $\mathbb{Q}(\langle q \rangle)$-bases of $\mathcal{C}$.

Proof. An integrable lowest module of $\tilde{\mathfrak{g}}$ is a quotient of $P_\lambda$ for some $\lambda \in P^+$ as it is a $\mathfrak{g}$-integrable quotient of a Verma module (see [28, §9.2]). Being a subquotient of an integrable lowest weight module of $\tilde{\mathfrak{g}}$ whose $\tilde{\mathfrak{h}}$-weights are contained in $P_{af}$, we find that both of the modules $D^{(k)}_{\mu}(\lambda)$ and $W^{(k)}_{\mu}(\lambda)$ are subquotients of $P_\lambda$. Thus, their graded characters belong to $\mathcal{B}$ or $\mathcal{C}$, respectively.

By Lemma 2.6, we have

$$\text{gch} W^{(k)}_{\mu}(\lambda) \equiv \text{gch} W^{(k)}_{\mu}(\lambda) \equiv \text{gch} V_{\lambda} \mod q \quad \lambda \in P^+.$$  

Since $D^{(k)}_{\mu}(\lambda) \subset W_{\mu+}^{(k)}$ and $D_{\mu+}^{(k)}$ stratifies $W_{\mu+}$ for each $\mu \in P$, we find

$$(\text{gch} D^{(k)}_{\mu})(q = 0) \in \left( \sum_{\lambda \in \Sigma(\mu)} \mathbb{Z} e^\lambda \right) \supset (\text{gch} D^{(k)}_{\mu})(q = 0).$$

Hence, the $d$-degree zero part of all the four graded modules define a basis of the constant part of $\mathcal{B}$ or $\mathcal{C}$ with respect to the variable $q$ modulo the positive degree part. Thus, we repeatedly expand elements of $\mathcal{B}$ or $\mathcal{C}$ from the lower $q$-degree part to see that they are topological bases of $\mathcal{B}$ or $\mathcal{C}$, respectively. \qed

Theorem 2.15 ([14] Theorem 4.19). For each $\lambda, \mu \in P$, we have

$${\text{ext}}^i_{\mathcal{B}}((D^{(1)}_{\lambda} \otimes \mathbb{C}_{\Lambda_0}, (D^{(1)}_{\mu} \otimes \mathbb{C}_{-\Lambda_0})^*) = \begin{cases} \mathbb{C} & (i = 0, \lambda + \mu = 0) \\ 0 & (\text{else}) \end{cases}$$

(2.2)

In addition, we have $\Psi(D^{(1)}_{\lambda}) \subset \Sigma(\lambda)$. \qed

Corollary 2.16. For each $\lambda, \mu \in P$ such that $-\mu \not\preceq -\lambda$, we have

$${\text{ext}}^{i > 0}_{\mathcal{B}}((D^{(1)}_{\lambda}, D^{(1)}_{\mu}) = 0.$$

In addition, we have $-\Psi(D^{(1)}_{\lambda}) \subset \Sigma(-\lambda)$. 14
Proof. We have

\[ gch D^{(1)}_\lambda = E_\lambda(q, 0), \quad \text{and} \quad (gch D^{(1)}_\lambda) \uparrow = c_\lambda E_{-\lambda}(q^{-1}, \infty), \quad c_\lambda \in \mathbb{Z}[q]^\times \]

by [14, Proposition 4.17] and [14, Corollary 4.21], where \( \uparrow \) sends \( e^\gamma \) (\( \gamma \in P \)) to \( e^{-\gamma} \). In particular, we find

\[ \Psi(D^{(1)}_\lambda) = -\Psi(D^{(1)}_{-\lambda}) \subset -\Sigma(-\lambda). \]

In view of the fact that \( D^{(1)}_\mu \) is decomposed into \( q^* \mathbb{C}_\gamma \) for \( \gamma \preceq \mu \) by a finite application of the short exact sequence, we deduce

\[ \text{ext}^0(\mathbb{D}_\lambda^{(1)} \otimes C_{\lambda_0}, (C_{\gamma - \lambda_0})^*) = 0 \quad \gamma \not\geq -\lambda. \]

Therefore, we conclude the assertion.

Proposition 2.17 ([32] Lemma 4.4, see also [25, 38]). Assume that \( g = sl(2, \mathbb{C}) \). Let \( D \) be the Demazure functor with respect to \( b \). Let \( M \) be a \( b \)-module in which the \( h \)-action is semisimple. If we have \( L^{-1}D_i(M) = 0 \) and we have a \( b \)-module embedding

\[ M \subset D(M), \]

then \( M \) admits a filtration whose associated graded is the direct sum of modules of the following two types:

1. irreducible finite-dimensional module of \( sl(2, \mathbb{C}) \);
2. one-dimensional \( b \)-module \( \mathbb{C}_\mu \) with \( \langle \alpha^\vee, \mu \rangle \geq 0 \).

Proof. The assumption \( L^{-1}D(M) = 0 \) guarantees that \( gch D(M) \) is given by the Demazure operator applied to \( gch M \). Thus, the assumption of [32, Lemma 4.4] is satisfied.

3 Main Theorems

Keep the setting of the previous section. We state the main assertions of this paper here. The proofs of these assertions occupy the subsequent sections.

Definition 3.1 (Filtrations). Let \( k \in \mathbb{Z}_{>0} \).

A module \( M \in \mathbb{B}_{\text{bdd}} \) admits a \( D^{(k)} \)-filtration (resp. a \( D^{(k)} \)-filtration) if there is a decreasing separable filtration

\[ M = M_0 \supset M_1 \supset M_2 \supset \cdots, \quad \bigcap_{i \geq 1} M_i = 0 \]

as \( \mathfrak{b} \)-modules such that for each \( i \in \mathbb{Z}_{\geq 0} \), the quotient module \( M_i/M_{i+1} \) is isomorphic to \( q^{m_i} D^{(k)}_{\mu_i} \) (resp. \( q^{m_i} D^{(k)}_{\mu_i} \)) for some \( m_i \in \mathbb{Z} \) and \( \mu_i \in P \) up to the twists by \( C_{\lambda_0} \) (\( l \in \mathbb{Z} \)).

A module \( M \in \mathcal{C}_{\text{bdd}} \) admits a \( W^{(k)} \)-filtration (resp. a \( \mathcal{W}^{(k)} \)-filtration) if there is a decreasing separable filtration

\[ M = M_0 \supset M_1 \supset M_2 \supset \cdots, \quad \bigcap_{i \geq 1} M_i = 0 \]
Similarly, let $M$ be the collection of elements $(M : X_\lambda)_q \in \mathbb{Z}[[q]]$ as

$$\text{gch } M = \sum_{\lambda} (M : X_\lambda)_q \cdot \text{gch } X_\lambda,$$

where $\lambda$ runs over $P$ or $P^+$ in accordance with $X = \mathbb{D}^{(k)}, D^{(k)}$ or $\mathbb{W}^{(k)}, W^{(k)}$, respectively. In view of Lemma 2.14, these elements are determined uniquely.

They count the graded occurrence of $X_\lambda$ in $M$ with respect to the $X$-filtration. Similarly, let $[M : V_\lambda]_q \in \mathbb{Z}_{\geq 0}[[q]]$ denote the graded occurrence of $V_\lambda$ in $M$ for each $M \in \mathcal{C}_{\text{bdd}}$. In particular, we have $\text{gch } M = \sum_{\lambda} [M : V_\lambda]_q \cdot \text{ch } V_\lambda$. We might replace the subscript $q$ with other variable (like $q^{-1}$) to represent a substitution.

**Theorem 3.2 (Level $k$-duality).** Let $k \in \mathbb{Z}_{\geq 0}$. We have:

1. For each $\lambda, \mu \in P$, we have

$$\text{ext}^i_{2\mathbb{B}}(\mathbb{D}^{(k)}_\lambda, (D^{(k)}_\mu)^*) \cong \begin{cases} \mathbb{C} & (i = 0, \lambda + \mu = 0) \\ 0 & (\text{else}) \end{cases}. \quad (3.1)$$

2. For each $\lambda, \mu \in P^+$, we have

$$\text{ext}^i_{2\mathbb{B}}(\mathbb{W}^{(k)}_\lambda, (W^{(k)}_\mu)^*) \cong \begin{cases} \mathbb{C} & (i = 0, \lambda + \mu^- = 0) \\ 0 & (\text{else}) \end{cases}. \quad (3.2)$$

**Theorem 3.3 (Level $k$-criterion of filtrations).** Let $k \in \mathbb{Z}_{\geq 0}$. We have:

1. A module $M \in \mathcal{B}_0$ admits a $D^{(k)}$-filtration if and only if

$$\text{ext}^1_{2\mathbb{B}}(\mathbb{D}^{(k)}_\lambda, M^*) \cong 0 \quad \lambda \in P. \quad (3.3)$$

Similarly, $M \in \mathcal{B}_{\text{bdd}}$ admits a $D^{(k)}$-filtration if and only if

$$\text{ext}^1_{2\mathbb{B}}(M, (D^{(k)}_\lambda)^*) \cong 0 \quad \lambda \in P. \quad (3.4)$$

2. A module $M \in \mathcal{C}_0$ admits a $W^{(k)}$-filtration if and only if

$$\text{ext}^1_{2\mathbb{B}}(\mathbb{W}^{(k)}_\lambda, M^*) \cong 0 \quad \lambda \in P^+. \quad (3.5)$$

Similarly, $M \in \mathcal{C}_{\text{bdd}}$ admits a $W^{(k)}$-filtration if and only if

$$\text{ext}^1_{2\mathbb{B}}(M, (W^{(k)}_\lambda)^*) \cong 0 \quad \lambda \in P^+. \quad (3.6)$$

In each of the above cases, all the higher exts also vanish.

The $k = 1$ cases of Theorem 3.2 1) and Theorem 3.3 are [14, Theorem 4.19 and Theorem 5.9]. The $k = 1$ case of Theorem 3.2 2) follows from the $k = 1$ case of Theorem 3.2 by the last paragraph of the proof of Theorem 3.2. Thus, we assume the validity of Theorem 3.2 and Theorem 3.3 for strictly smaller $k$ in the course of their proofs.
Corollary 3.4. Let $k \in \mathbb{Z}_{>0}$ and assume Theorem 3.3 for level $k$. For $M, N \in C_{\text{bdd}}$ such that $M \oplus N$ admits a $W^{(k)}$-filtration, so is $M$. Similar assertion holds for $M, N \in B_{\text{bdd}}$ with respect to the $D^{(k)}$-filtration.

Proof. This is an immediate consequence of Theorem 3.3 as $\text{ext}^i_C$ and $\text{ext}^i_B$ commutes with finite direct sums.

4 A characterization of Demazure modules

Keep the setting of the previous section.

Lemma 4.1. For each $\lambda \in P$ and $k \in \mathbb{Z}_{\geq 2}$, the module $D^{(k)}(\lambda)$ is a quotient of $D^{(k-1)}(\lambda) \otimes C_{\lambda_0}$ in $B$. In addition, we have

$$\Psi(D^{(k)}(\lambda)) \subset \Psi(D^{(1)}(\lambda)) \subset \Sigma(\lambda).$$

Proof. Note that a defining set of equations of $D^{(k)}(\lambda)$ is of the form

$$e_{\alpha}^{\max\{-\langle \alpha^\vee, \lambda \rangle, 0\} + 1} \mathbf{v} = 0 \quad \alpha \in \Delta_+,$$

$$e_{\alpha + m \delta}^{\max\{-\langle \alpha^\vee, \lambda \rangle - mk, 0\} + 1} \mathbf{v} = 0 \quad \alpha \in \Delta, m \in \mathbb{Z}_{>0},$$

where $\mathbf{v}$ is the cyclic vector (of $\mathfrak{h}$-weight $\lambda$) by [25, 3.4 Theorem]. From this, it is evident that $D^{(k)}(\lambda)$ is a quotient of $D^{(k-1)}(\lambda)$. In particular, we have $\Psi(D^{(k)}(\lambda)) \subset \Psi(D^{(1)}(\lambda))$. The containment $\Psi(D^{(1)}(\lambda)) \subset \Sigma(\lambda)$ follows as the character of $D^{(1)}(\lambda)$ is identified with non-symmetric Macdonald polynomials ([24]) and the corresponding estimate is a part of the characterization of non-symmetric Macdonald polynomials in [12, (4.4)].

Remark 4.2. Joseph [25] deals only semisimple Lie algebra $\mathfrak{g}$. It generalizes to the case of thin Demazure modules of a Kac-Moody algebra without modification. As the proof in [25, §4] rely on the induction from the one-dimensional case, its proof does not extend to the case of thick Demazure modules.

Lemma 4.3. Let $k \in \mathbb{Z}_{>0}$ and assume Theorem 3.3 for level $k$. Let $M, N \in B_{\text{bdd}}$ be two $D^{(k)}$-filtered modules equipped with a surjection $f : M \to N$. Then, $\ker f$ also admits a $D^{(k)}$-filtration. Similar assertions hold also for the $D^{(k)}$, $W^{(k)}$, or $W^{(k)}$-filtrations.

Proof. Since the proofs of assertions for all the cases are similar, we concentrate into the first case. The long exact sequence applied to the short exact sequence

$$0 \to \ker f \to M \to N \to 0$$

yields a (part of the) long exact sequence

$$\text{ext}^i_B(M, (D^{(k)}(\lambda))^*) \to \text{ext}^i_B(\ker f, (D^{(k)}(\lambda))^*) \to \text{ext}^{i+1}_B(N, (D^{(k)}(\lambda))^*)$$

for each $i \geq 0$. By Theorem 3.3, we deduce

$$0 = \text{ext}^i_B(M, (D^{(k)}(\lambda))^*) \to \text{ext}^i_B(\ker f, (D^{(k)}(\lambda))^*) \to \text{ext}^{i+1}_B(N, (D^{(k)}(\lambda))^*) = 0$$

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for $i > 0$, and therefore
\[
\text{ext}_{B}^{\geq 0}(\ker f, (D^{(k)}_{\lambda})^{*}) = 0.
\]
This yields the assertion by Theorem 3.3.

**Corollary 4.4.** Let $k \in \mathbb{Z}_{>0}$, and assume Theorem 3.2 and Theorem 3.3 for level $k$. We have:

1. If $M \in \mathfrak{B}_{\text{bdd}}$ admits a $\mathbb{D}^{(k)}$-filtration, then we have
   \[
   (M : \mathbb{D}_{\lambda}^{(k)})_{q-1} = \text{gdim} \hom_{B}(M, (D_{\lambda}^{(k)})^{*}) \quad \lambda \in P.
   \]
   Similarly, if $M \in \mathfrak{B}_{0}$ admits a $D^{(k)}$-filtration, then we have
   \[
   (M : D_{\lambda}^{(k)})_{q-1} = \text{gdim} \hom_{B}(\mathbb{D}_{\lambda}^{(k)}, (M)^{*}) \quad \lambda \in P;
   \]
2. If $M \in \mathfrak{C}_{\text{bdd}}$ admits a $\mathbb{W}^{(k)}$-filtration, then we have
   \[
   (M : \mathbb{W}_{\lambda}^{(k)})_{q-1} = \text{gdim} \hom_{B}(\mathbb{W}_{\lambda}^{(k)}, (M)^{*}) \quad \lambda \in P_{+},
   \]
   Similarly, if $M \in \mathfrak{C}_{0}$ admits a $W^{(k)}$-filtration, then we have
   \[
   (M : W_{\lambda}^{(k)})_{q-1} = \text{gdim} \hom_{B}(\mathbb{W}_{\lambda}^{(k)}, (M)^{*}) \quad \lambda \in P_{+}.
   \]

**Proof.** Since the proofs of assertions for all the cases are similar (and easier for the cases of $W^{(k)}$ and $D^{(k)}$), we concentrate into the case of $D^{(k)}$-filtration. Since $M \in \mathfrak{B}_{\text{bdd}}$, we have a quotient $f : M \to q^{m}D_{\mu}^{(k)}$ for some $m \in \mathbb{Z}$ and $\mu \in P$. By Lemma 4.3, we find
\[
\text{hom}_{B}(\ker f, (D_{\lambda}^{(k)})^{*}) \oplus q^{-m}C^{\delta_{\lambda}+\mu, 0} \cong \text{hom}_{B}(M, (D_{\lambda}^{(k)})^{*}) \quad (4.1)
\]
and
\[
\text{ext}_{B}^{\geq 0}(\ker f, (D_{\lambda}^{(k)})^{*}) \equiv 0 \quad \lambda \in P.
\]
Thus, we inductively find a sequence of quotients of $M$
\[
\cdots \to M_{3} \to M_{2} \to M_{1} \quad (4.2)
\]
such that each term is $\mathbb{D}^{(k)}$-filtered. In view of $M \in \mathfrak{B}_{\text{bdd}}$, we can rearrange the sequence (4.2) if necessary to find $i_{m}$ for each $m \in \mathbb{Z}$ such that $\ker(M \to M_{i})$ is concentrated in the $d$-degree $> m$ for $i > i_{m}$. This forces $M = \varinjlim_{i_{m}} M_{i}$, and hence the assertion follows by a repeated application of (4.1) as required.

**Proposition 4.5.** Let $k \in \mathbb{Z}_{>0}$. We assume Theorem 3.2 for level $k$ when $k \geq 2$. For each $\lambda \in P$, a proper (nonzero) quotient $M$ of $\mathbb{D}_{\lambda}^{(k)}$ or $D_{\lambda}^{(k)}$ in $\mathfrak{B}$ satisfies
\[
\bigoplus_{\mu \in P} \text{ext}_{B}^{1}(M, (D_{\mu}^{(k)})^{*}) \neq 0 \quad \text{or} \quad \bigoplus_{\mu \in P} \text{ext}_{B}^{1}(\mathbb{D}_{\mu}^{(k)}, M^{*}) \neq 0, \quad (4.3)
\]
respectively. In addition, we can find an extension of $M$ by the socle of $(D_{\mu})^*$ or $(\mathbb{D}_{\mu})^*$ that realizes some non-zero class in (4.3). For each $\lambda \in P^+$, a proper quotient $M$ of $W_{\lambda}^{(k)}$ or $W_{\lambda}^{(k)}$ in $\mathcal{C}$ satisfies

$$
\bigoplus_{\mu \in P^+} \text{ext}^1_{\mathbb{B}}(M, (W_{\mu}^{(k)})^*) \neq 0 \quad \text{or} \quad \bigoplus_{\mu \in P^+} \text{ext}^1_{\mathbb{B}}(W_{\mu}^{(k)}, M^*) \neq 0,
$$

respectively.

**Proof.** Assume that $M$ is a proper quotient of $\mathbb{D}_{\lambda}^{(k)}$ with its kernel ker. The module ker is generated by its head, that contains $q^{m}C_{\gamma-k\Lambda_0}$ for some $m \in \mathbb{Z}$ and $\gamma \in P$. We have an extension $M^+$ of $M$ by $q^{m}C_{\gamma-k\Lambda_0}$, that is also a quotient of $\mathbb{D}_{\lambda}^{(k)}$ as ker $\neq 0$. Here $q^{m}C_{\gamma-k\Lambda_0}$ appears in the socle of $q^{m}(D_{-\gamma})^*$. Thus, obtain the module $M^\sharp$ by the following short exact sequence:

$$
0 \rightarrow q^{m}C_{\gamma-k\Lambda_0} \rightarrow M^+ \oplus q^{m}(D_{-\gamma})^* \rightarrow M^\sharp \rightarrow 0,
$$

where the map from $q^{m}C_{\gamma-k\Lambda_0}$ is the anti-diagonal embedding. We have an induced short exact sequence

$$
0 \rightarrow q^{m}(D_{-\gamma})^* \rightarrow M^\sharp \rightarrow M \rightarrow 0. \tag{4.4}
$$

It suffices to show that (4.4) does not split in order to see $\text{ext}^1_{\mathbb{B}}(M, (D_{-\gamma})^*) \neq 0$. Thus, we assume that (4.4) splits to deduce contradiction. We have a non-trivial map $M^\sharp \rightarrow q^{m}(D_{-\gamma})^*$, that lifts to a map from $M^+ \oplus q^{m}(D_{-\gamma})^*$ by the precomposition. By examining the socle, we have a non-zero map $M^+ \rightarrow q^{m}(D_{-\gamma})^*$. This induces a non-zero map $\mathbb{D}_{\lambda}^{(k)} \rightarrow q^{m}(D_{-\gamma})^*$ by the precomposition. Thanks to Theorem 3.2, that holds by assumption $(k \geq 2)$ or [14] $(k = 1)$, such a map exists only if $\lambda = \gamma$, and its image is isomorphic to the head of $\mathbb{D}_{\lambda}^{(k)}$. In particular, the image must be one-dimensional. This space can appear in the generator of ker only if $M = \{0\}$. Thus, we have a contradiction on the assumption of $M$. Thus, we conclude the case $M$ is a proper quotient of $\mathbb{D}_{\lambda}^{(k)}$ with our desired extension class given by $M^\sharp$.

The case $M$ is a proper quotient of $\mathbb{D}_{\lambda}^{(k)}$ is dual to the previous case in view of the duality isomorphism (1.2) and the Yoneda interpretation of $\text{ext}^1$.

The cases $M$ is a proper quotient of $W_{\lambda}^{(k)} = \mathbb{L}^{\bullet}D_{w_0}(\mathbb{D}_{\lambda}^{(k)})$ or $W_{\lambda}^{(k)} = D_{w_0}^{(k)}$ reduces to one of the above two cases since we have $\mathbb{L}^{\bullet}D_{w_0}(M) = M$ and hence Theorem 1.8 introduces $D_{w_0}$ in each of the factor.

5 A lifting theorem

Keep the setting of the previous section.

**Theorem 5.1.** Let $M \in \mathfrak{B}_{\text{add}}$ and $N \in \mathfrak{B}_0$ be modules on which $K$ acts by $-k$ and $k$, respectively. We suppose

$$
M \subset \mathcal{D}_1(M), N \subset \mathcal{D}_1(N) \quad \text{and} \quad \ll_{\mathcal{D}_1}(M) = 0 = \ll_{\mathcal{D}_1}(N)
$$

for some $i \in \mathfrak{I}_{\text{af}}$, and $N$ has a $\mathfrak{h}$-cyclic $\mathfrak{h}$-eigenvector $v$. Then, we have a surjection:

$$
\text{hom}_{\mathfrak{B}}(\mathcal{D}_1(M), N^*) \twoheadrightarrow \text{hom}_{\mathfrak{B}}(M, N^*).
$$
Remark 5.2. Theorem 5.1 holds for an arbitrary Kac-Moody algebra.

Proof of Theorem 5.1. If we have \( D_i(M) \cong M \) or \( D_i(N) \cong N \), then the map must be an isomorphism by Theorem 1.8 and \( D_i^2 \cong D_i \).

By (1.2), we consider the dual statement

\[
\text{hom}_B(N, D_i(M)^\vee) \rightarrow \text{hom}_B(N, M^\vee).
\]  

(5.1)

By Proposition 2.17 and the isomorphism \( U(g, \lambda) \cong \mathbb{C}[E_i] \), we conclude that \( N \) decomposes into a direct sum of indecomposable \((\mathbb{C}E_i + \mathfrak{h})\)-modules such that each indecomposable direct summand is an irreducible \( \mathfrak{sl}(2, i) \)-module twisted by a one-dimensional \( \mathfrak{h} \)-module with its weight \( \lambda \) such that \( \langle \alpha_i^\vee, \lambda \rangle \geq 0 \). We find that \( M \) also admits the direct sum decomposition with the same types of direct summands.

Consider a non-zero \( \mathfrak{h} \)-module map

\[
f : N \rightarrow M^\vee.
\]

Let \( B \) be an indecomposable \((\mathbb{C}E_i + \mathfrak{h})\)-module direct summand of \( N \), regarded as a \((\mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee)\)-module. The \((\mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee)\)-module \( B \) has head \( \mathbb{C}_r \) and socle \( \mathbb{C}_s \), where \( r \leq s \) are integers.

Let \( L \) be an indecomposable \((\mathbb{C}E_i + \mathfrak{h})\)-module direct summand of \( M \), regarded as a \((\mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee)\)-module. The \((\mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee)\)-module \( L \) is an irreducible \( \mathfrak{sl}(2, i) \)-module twisted by a \( \mathfrak{h} \)-weight \( \lambda \) with \( \langle \alpha_i^\vee, \lambda \rangle \geq 0 \). Thus, \( L^\vee \subset M^\vee \) is twisted by \(-\lambda \). Let \( f_L \) be the composition map

\[
N \rightarrow M^\vee \rightarrow L^*.
\]

Being the quotient of \( B \) as a \((\mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee)\)-module, its image \( f_L(B) \) is either zero or has head \( \mathbb{C}_r \) and socle \( \mathbb{C}_s \), where \( s \leq r \). By using the natural embedding \( N \subset D_i(N) \), we find the \( \mathfrak{sl}(2, i) \)-submodule \( \tilde{B} \subset D_i(N) \) generated by \( B \). We have \( D_i(B) = \tilde{B} \) by their characterizations, and hence the module \( \tilde{B} \) is the direct sum of irreducible \( \mathfrak{sl}(2, i) \)-modules with their highest weights

\[
s \varpi, (s - 2) \varpi, \ldots, (|r| + 2) \varpi, |r| \varpi.
\]  

(5.2)

We examine whether the map \( f_L \) extends from \( B \) to \( \tilde{B} \):

In case \( s' > |r| \), then we have a unique irreducible \( \mathfrak{sl}(2, i) \)-direct summand of \( \tilde{B} \) (with its highest weight \( s' \varpi \)) that injects into \( L^* \) whose image contains \( f_L(B) \) (since \( s' \varpi \), and hence \(-s' \varpi \), is a weight of \( L^* \)).

In case \( s' \leq |r| \), then we have a unique irreducible \( \mathfrak{sl}(2, i) \)-direct summand of \( \tilde{B} \) (with its highest weight \( |r| \varpi \)) that maps onto \( f_L(B) \subset N^* \).

In both of the above two cases, we have a non-zero map from \( \tilde{B} \) to \( L^* \).

Claim A. If \( v \in B \) has \( \alpha_i^\vee \)-eigenvalue \( t \), then we have

\[
v = \sum_{|t| \leq t'} v_{t'},
\]

where \( v_{t'} \) is a non-zero \( \alpha_i^\vee \)-eigenvector with eigenvalue \( t \) inside the irreducible \( \mathfrak{sl}(2, i) \)-module with its highest weight \( t' \varpi \) in (5.2).
Proof. Since $B$ is an indecomposable $(\mathcal{C}E_i \oplus \mathbb{C}\alpha_i^\vee)$-module, it suffices to prove the assertion when $v$ is a cyclic vector that has $\alpha_i^\vee$-eigenvalue $r$. In case some of the irreducible $\mathfrak{sl}(2,i)$-module with its highest weights $(5.2)$ does not contribute to $v$, then the $\mathfrak{sl}(2,i)$-span of $B$ is strictly smaller than $\tilde{B}$. This violates the characterization of the Demazure functor, and hence it is impossible. Thus, all possible contributions of $v$ must be non-zero as required.

We return to the proof of Theorem 5.1. By Claim A, the above non-zero map $\tilde{B} \to L^\ast$ can be rearranged uniquely to recover $f_L$ by restriction (to $B$).

Now we set $B_0 := U(\mathcal{C}E_i) v$. This is a direct summand of $N$ as $(\mathcal{C}E_i + \tilde{h})$-modules by the $\tilde{h}$-weight consideration. In particular, its $\mathfrak{sl}(2,i)$-span $\tilde{B}_0 \subset \mathcal{D}_i(N)$ admits a lifting $\tilde{f}: \tilde{B}_0 \to M^\vee$ of $(\mathcal{C}E_i + \tilde{h})$-modules, that recovers the map $f$ (restricted to $B_0$) by restriction. Let

$$u := \bigoplus_{\beta \in \Delta_i \setminus \{\alpha_i\}} \tilde{g}_\beta \subset \tilde{h}.$$ 

Since we have $U(\tilde{h}) = U(u) \otimes U(\mathcal{C}E_i)$, we can regard $U(u) \otimes_\mathcal{C} B_0$ and $U(u) \otimes_\mathcal{C} \tilde{B}_0$.

as $\tilde{h}$-modules that are projective as $U(u)$-modules. In addition, the module $u$ is a $\mathfrak{sl}(2,i)$-integrable $\tilde{p}_i$-module by inspection. Therefore, we find

$$U(u) \otimes_\mathcal{C} B_0 \equiv U(u) \otimes_\mathcal{C} \mathcal{D}_i(B_0) \simeq \mathcal{D}_i(U(u) \otimes_\mathcal{C} B_0).$$

In particular, we have surjections

$$U(u) \otimes_\mathcal{C} B_0 \longrightarrow N, \quad U(u) \otimes_\mathcal{C} \tilde{B}_0 \longrightarrow \mathcal{D}_i(N)$$

as $\tilde{h}$-modules. We set

$$I := \ker(U(u) \otimes_\mathcal{C} B_0 \longrightarrow N).$$

In view of the above description of the indecomposable direct summands as $(\mathcal{C}E_i + \tilde{h})$-modules, we find a short exact sequence

$$0 = L^{-1}\mathcal{D}_i(N) \to \mathcal{D}_i(I) \to U(u) \otimes_\mathcal{C} \tilde{B}_0 \to \mathcal{D}_i(N) \to 0.$$ 

Thanks to Theorem 1.8, we see that giving a $\tilde{h}$-module map

$$\tilde{f}: \mathcal{D}_i(N) \longrightarrow M^\vee$$

is the same as giving a $\tilde{h}$-module map

$$\tilde{f}' : \mathcal{D}_i(N) \longrightarrow \mathcal{D}_i(M)^\vee.$$ 

Since $\mathcal{D}_i(M)^\vee \to M^\vee$ is a surjection by assumption, we need to construct a map $\tilde{f}'$ from $f$, that is to construct a map

$$\tilde{\psi} : U(u) \otimes_\mathcal{C} \tilde{B}_0 \longrightarrow \mathcal{D}_i(M)^\vee.$$ 

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such that $\tilde{\psi}(D_i(I)) = 0$. Since the map $f$ lifts to the map

$$\psi : U(u) \otimes C B_0 \rightarrow M^\vee$$

such that $\psi(I) = 0$, the lift

$$\tilde{\psi}' : U(u) \otimes \tilde{B}_0 \rightarrow M^\vee$$

of $\psi$ obtained by the universal lift of a specific lift from $B_0$ to $\tilde{B}_0$ (as $(C E_i + \tilde{h})$-module) satisfies $\tilde{\psi}'(I) = 0$. The map

$$\tilde{\psi}'' : U(u) \otimes \tilde{B}_0 \rightarrow D_i(\mathcal{M})^\vee$$

obtained from $\tilde{\psi}'$ by the above correspondence must satisfy $\tilde{\psi}''(D_i(I)) = 0$ as the functor $D_i$ must send a zero map to a zero map. Hence, the map $\tilde{\psi}''$ can be employed as our desired lift $\tilde{f}'$, and hence $\tilde{f}$ as required.

**Corollary 5.3.** Let $k, l \in \mathbb{Z}_{\geq 0}$ be such that $k \geq l$. Let $M \in \mathfrak{B}_{\text{odd}}$ be a module on which $K$ acts by $-k$, and we have

$$M \subset D_i(M), \quad \text{and} \quad L^{<0}D_i(M) = 0$$

for some $i \in \mathcal{I}_{\text{af}}$. For each $\mu \in P$, we have a surjection:

$$\text{hom}_{\mathfrak{B}}(D_i(M), (D_\mu^{(i)} \otimes C_{(k-l)\Lambda_0})^*) \rightarrow \text{hom}_{\mathfrak{B}}(M, (D_\mu^{(i)} \otimes C_{(k-l)\Lambda_0})^*)$$

**Proof.** For each $i \in \mathcal{I}_{\text{af}}$, we have $D_\mu^{(i)} \subset D_i(D_\mu^{(i)}) \equiv L^{\bullet}D_i(D_\mu^{(i)})$ by Theorem 2.2. Thus, we conclude the assertion for the case $i \neq 0$. For the case $i = 0$, we apply Proposition 2.17 to deduce the assumption of Theorem 5.1 for $D_\mu^{(i)} \otimes C_{(k-l)\Lambda_0}$ to conclude the assertion.

6 ext\(^1\)-vanishing results

Keep the setting of the previous section.

**Proposition 6.1.** Let $k \in \mathbb{Z}_{\geq 1}$. For each $\Lambda \in P^+_k$ and $\mu \in P^+$, we have

$$\text{ext}^{1}_{\mathfrak{B}}(\theta L(\Lambda), (W^{(k)}_\mu)^*) = 0.$$  \hspace{1cm} (6.1)

**Proof.** Let us find $\Lambda' \in P^+_k$ and $w \in W_{\text{af}}$ such that

$$W^{(k)}_\mu = \mathcal{D}_w(\mathcal{C}_{\Lambda'}).$$

We rewrite (6.1) as

$$\text{ext}^{1}_{\mathfrak{B}}(\theta L(\Lambda), (W^{(k)}_\mu)^*) = \text{ext}^{1}_{\mathfrak{B}}(\theta L(\Lambda), (\mathcal{D}_w(\mathcal{C}_{\Lambda'}))^*)$$

$$\cong \text{ext}^{1}_{\mathfrak{B}}(\mathcal{D}_w^{-1}(\theta L(\Lambda)), (\mathcal{C}_{\Lambda'})^*)$$

$$= \text{ext}^{1}_{\mathfrak{B}}(\theta L(\Lambda), (\mathcal{C}_{\Lambda'})^*)$$

by Theorem 1.8 and Theorem 2.3. In view of the BGG-resolution (1.3), we conclude that

$$\text{ext}^{1}_{\mathfrak{B}}(\theta L(\Lambda), \mathcal{C}_{-, \Lambda'}) = 0$$

by $\Lambda' \notin \{s_i(\Lambda + \rho) - \rho + m\delta \mid i \in \mathcal{I}_{\text{af}}, m \in \mathbb{Z}\}$. This completes the proof.
Proposition 6.2. Let $k \in \mathbb{Z}_{\geq 2}$. For each $\Lambda \in P^+_k$ such that $\langle \vartheta^\vee, \Lambda \rangle = k$, there exists $w \in W_{af}$ and $\Lambda' \in P^+_{(k-1)}$ such that

$$D_{sw}(C_{\Lambda'})/D_w(C_{\Lambda'}) \cong C_{(\Lambda-\Lambda_0)},$$

(6.2)

Proof. We have

$$s_0(\Lambda - \Lambda_0) = \Lambda - \Lambda_0 + \alpha_0$$

by $\langle \alpha_0^\vee, \Lambda - \Lambda_0 \rangle = 0 - 1 = -1$. In case $s_0(\Lambda - \Lambda_0) \in P^+_{(k-1)}$, we have

$$D_A^{(k-1)} \cong C_{(\Lambda-\Lambda_0)} \oplus C_{s_0(\Lambda-\Lambda_0)}.$$

Hence, the second assertion holds if every $\gamma$ is not of type $A_1$ in the below.

We assume $s_0(\Lambda - \Lambda_0) \notin P^+_{(k-1)}$ in the below. Thus, possible $j \in I_{af}$ that satisfies $\langle \alpha_j^\vee, s_0(\Lambda - \Lambda_0) \rangle < 0$ is $j = 1, n$ (type $A_n$, according to [6, P206 (VI)] with $n = l$) or $j = i$ for a uniquely determined $i$ (other cases) by inspection ([28, Table Aff]). In case $\langle \alpha_j^\vee, \Lambda \rangle > 0$ for all of these possible $j$, we have necessarily $s_0(\Lambda - \Lambda_0) \in P^+_{(k-1)}$. Hence, we have $\langle \alpha_j^\vee, \Lambda \rangle = 0$ for $i = 1$ or $n$ in type $A_n$, and $\langle \alpha_i^\vee, \Lambda \rangle = 0$ in the other cases by assumption. We set

$$S^+ := \{ j \in I_{af} \mid \langle \alpha_j^\vee, \Lambda \rangle = 0 \}$$

and let $S \subset S^+$ be the maximal subset that forms a connected subdiagram of the Dynkin diagram of $\mathfrak{g}$ that contains $0$. We have $S \subseteq I_{af}$ as $k > 0$. Let $g_S$ be the simple Lie subalgebra of $\mathfrak{g}$ generated by $\{E_j, \alpha_j^\vee, F_j \}_{j \in S}$.

The subdiagram that appears in this construction is types ABCDE by inspection. In all cases, the weight $\langle \Lambda - \Lambda_0 \rangle$ is an anti-dominant miniscule weight (corresponding to the vertex $0$) of $g_S$. Hence we have a minimal length $w \in (s_j \mid j \in S) \subset W_{af}$ such that $w^{-1}(\Lambda - \Lambda_0)$ is a dominant miniscule (fundamental) weight of $g_S$.

Claim B. Assume that $w^{-1}(\Lambda - \Lambda_0) \in P^+_{(k-1)}$. We have $\ell(s_0 w) < \ell(w)$ and

$$C_{\Lambda-\Lambda_0} = D_w(C_{w^{-1}(\Lambda-\Lambda_0)})/D_{s_0 w}(C_{w^{-1}(\Lambda-\Lambda_0)}).$$

Proof. Since $w^{-1}(\Lambda - \Lambda_0) \in P^+_{(k-1)}$

$$s_0 w w^{-1}(\Lambda - \Lambda_0) = (\Lambda - \Lambda_0) + \alpha_0 = w w^{-1}(\Lambda - \Lambda_0) + \alpha_0,$$

we deduce $s_0 w < w$. This implies $\ell(s_0 w) < \ell(w)$.

We set $W_S := (s_j \mid j \in S)$. Since the weight $\langle \Lambda - \Lambda_0 \rangle$ is miniscule for $g_S$, every weight space of $D_w(C_{w^{-1}(\Lambda-\Lambda_0)})$ is one-dimensional with its weight $\langle v(\Lambda - \Lambda_0) \rangle$ ($v \in W_S$). Hence, the second assertion holds if every $v(\Lambda - \Lambda_0) \neq (\Lambda - \Lambda_0)$ ($v \in W_S$) gives rise to a non-zero weight space of $D_{s_0 w}(C_{w^{-1}(\Lambda-\Lambda_0)})$. As $\langle \Lambda - \Lambda_0 \rangle$ is antidominant for $g_S$, we have a minimal sequence $i_1, i_2, \ldots, i_t \in S$ such that

$$(s_{i_1} s_{i_2} \cdots s_{i_t}) v(\Lambda - \Lambda_0) = (\Lambda - \Lambda_0)$$

if $v(\Lambda - \Lambda_0) \neq (\Lambda - \Lambda_0)$. By the minimality of $t$, we have $i_1 = 0$ and hence $s_{i_2} \cdots s_{i_t} v(\Lambda - \Lambda_0) = s_0(\Lambda - \Lambda_0)$. Again by the minimality of $t$, we conclude

$$C_{v(\Lambda - \Lambda_0)} \subset (D_{s_{i_2} \cdots s_{i_t}} \circ \cdots \circ D_{s_{i_1}})(D_{w^{-1}}(C_{w^{-1}(\Lambda-\Lambda_0)})) = D_{s_0 w}(C_{w^{-1}(\Lambda-\Lambda_0)}).$$

Hence, the above $v(\Lambda - \Lambda_0)$ is a weight of $D_{s_0 w}(C_{w^{-1}(\Lambda-\Lambda_0)})$ as required. \(\square\)
We return to the proof of Proposition 6.2. The remaining problem is whether $w^{-1}(\Lambda - \Lambda_0) \in P^+(k-1)$, that we examine case-by-case. We remark that if $j \in (I_{af} \setminus S)$ is adjacent to a vertex of $S$ in the extended Dynkin diagram, then we have $\langle \alpha_j^+, \Lambda \rangle \geq 1$, and $\langle \alpha_j^+, \Lambda \rangle \geq 2$ if $|I_{af}| - |S| = 1$ and

$$\langle \vartheta^+, \varpi_j \rangle = 1 \iff \vartheta^+ \in \alpha_j^+ + \sum_{j \neq r \in I} Z_{\geq 0} \alpha_r^+ \quad (6.3)$$

by $k \geq 2$. The condition (6.3) is automatic when $g$ is type $A$, or when $g$ is of types BCD and $j \in I$ is an index of a short root connected to a unique another vertex in the (non-extended) Dynkin diagram by inspection.

In case $S$ is type $A_l$ and there are at most one adjacent vertex to $0 \in S$ (in the subdiagram), we enumerate as

$$S = \{1, 2, \ldots, l\} \quad \text{with} \quad 1 = 0, 2 = i$$

such that $j, (j - 1) \in S$ are adjacent (here we understand $i = 1$ or $n$ when $g$ is type $A_n$). Then, $(\Lambda - \Lambda_0)$ is the lowest weight of its vector representation of type $A_l$. We have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \sum_{j=1}^l \alpha_j.$$ We have $\langle \alpha_j^+, \alpha_j \rangle = 0, -1$ when $j \in S$ and $t \in (I_{af} \setminus S)$. If $j \in (I_{af} \setminus S)$ is adjacent to two simple roots of $S$ (that is maximal possible), then $g$ is type $A_n$ and $|I_{af}| - |S| = 1$. Therefore, we find $w^{-1}(\Lambda - \Lambda_0) \in P^+(k-1)$ by $k \geq 2$.

In case $S$ is type $A_l$ and there are two adjacent vertices to $0 \in S$ (in the subdiagram), we enumerate as

$$S = \{-r_1, -1, 0, 1, 2, \ldots, r_2\}$$

such that $j, (j - 1) \in S$ are adjacent. Then, $(\Lambda - \Lambda_0)$ is the miniscule representation of type $A_{r_1+r_2+1}$, and $g$ is type $A_n$. We have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \alpha_{-r_1} + 2\alpha_{-r_1+1} + \cdots + \alpha_{r_2}.$$ As $j \in (I_{af} \setminus S)$ that is adjacent to $S$ in the extended Dynkin diagram is adjacent to $-r_1$ or $r_2$, we find $w^{-1}(\Lambda - \Lambda_0) \in P^+(k-1)$ by $k \geq 2$.

In case $S$ is type $D_l$ and $i \in S$ is adjacent to three vertices (in the subdiagram), then we enumerate as

$$S = \{1, 2, \ldots, l\} \quad \text{with} \quad (l - 2) = i, l = 0$$

and $j, (j - 1) \in S$ with $j < l$ and $(l, l - 2)$ are adjacent. This happens only when $g$ is of types CD. Here, $(\Lambda - \Lambda_0)$ is the lowest weight vector of a half-spin representation. In view of [6, P209 (VI)], we have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \varpi_l + \varpi_{l-1} = \alpha_1 + 2\alpha_2 + \cdots. \quad (6.4)$$

A vertex in $(I_{af} \setminus S)$ can be connected only to $1, 2 \in S$ in the extended Dynkin diagram, and the latter case occurs only when $|I_{af}| - |S| = 1$ and $g$ is type $D$. In addition, $\alpha_1$ must be a short root. Therefore, we find $w^{-1}(\Lambda - \Lambda_0) \in P_k^-$.
In case $S$ is type $D_l$ and $i \in S$ is adjacent to two vertices, then we enumerate as
\[ S = \{1, 2, \ldots, l\} \quad \text{with} \quad 2 = i, 1 = 0 \]
and $j, (j - 1) \in S$ with $j < l$ and $(l - 2, l)$ are adjacent. This happens only when $g$ is of type $DE$. Here, $(\Lambda - \Lambda_0)$ is the lowest weight of a minuscule representation. In view of [6, P209 (VI)], we have
\[ w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = 2\varpi_1 = 2\alpha_1 + 2\alpha_2 + \cdots + \alpha_{l-1} + \alpha_l. \quad (6.5) \]
In case $g$ is type $D$, we have $|I_{a(l)}|-|S|=1$ and the unique vertex $(I_{a(l)} \setminus S)$ is connected to 2 in the extended Dynkin diagram (and the corresponding root lengths are the same). In case $g$ is type $E$, other vertices are connected only to $(l - 1)$ or $l$. From these, we find $w^{-1}(\Lambda - \Lambda_0) \in P_{k-1}^+$. In case $S$ is type $B_l$, we enumerate as
\[ S = \{1, 2, \ldots, l\} \quad \text{with} \quad (l - 1) = i, l = 0 \]
in accordance with [6, P202 (IV)]. In this case, $l$ corresponds to the short root, and this occurs only when $g$ is of type $B$. In this case, $(\Lambda - \Lambda_0)$ is the lowest weight of a minuscule representation of $g_S$. Thus, we have
\[ w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = 2\varpi_1 = \alpha_1 + 2\alpha_2 + \cdots . \]
A vertex of $(I_{a(l)} \setminus S)$ can be connected only to 1 in the extended Dynkin diagram, and corresponding roots have different lengths only when $|I_{a(l)}|-|S|=1$ (in this case $\alpha_1$ is the long root). Therefore, we find $w^{-1}(\Lambda - \Lambda_0) \in P_{k}^+$ by $k \geq 2$. In case $S$ is type $C_l$, we enumerate as
\[ S = \{1, 2, \ldots, l\} \quad \text{with} \quad 1 = 0, 2 = i \]
such that $j, (j - 1)$ with $j \geq 1$ are adjacent and $l$ corresponds to the long root (this occurs only when $g$ is of types $CF$). Here, $(\Lambda - \Lambda_0)$ is the lowest weight of the vector representation of $g_S$. Thus, we have
\[ w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \alpha_l + \sum_{j=1}^{l-1} 2\alpha_j. \]
Here, $\alpha_l$ and $\alpha_j$ ($l \in I_{a(l)} \setminus S$) have the same length when $l$ is connected to $t$ in the extended Dynkin diagram. In case $l \neq j \in S$ is connected to $t \in (I_{a(l)} \setminus S)$ in the extended Dynkin diagram, then our $g$ is type $C$, $|I_{a(l)}|-|S|=1$, and $\alpha_j$ and $\alpha_l$ have the same length. Thus, we find $w^{-1}(\Lambda - \Lambda_0) \in P_{k-1}^+$ by $k \geq 2$. In case $S$ is type $E_6$, we find that $g$ is type $E_6$. We enumerate as
\[ S = \{1, 2, \ldots, 6\} \quad \text{with} \quad 1 = 0, 3 = i \]
in accordance with [6, P218 (IV)]. In this case, $(\Lambda - \Lambda_0)$ is the lowest weight of its minuscule representation. Thus, we have
\[ w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \varpi_1 + \varpi_6 = 2\alpha_1 + 2\alpha_2 + \cdots . \]
In addition, 2 is the only vertex in $S$ that is adjacent to the unique vertex $j$ of $(I_{a(l)} \setminus S)$ in the extended Dynkin diagram. We have $\langle \varpi^V, \varpi_j \rangle = 1$ by [6, P218 (IV)]. From this, we find $w^{-1}(\Lambda - \Lambda_0) \in P_{k-1}^+$ by $k \geq 2$. 25
In case $S$ is type $E_7$, we find that $g$ is type $E_7$. We enumerate as
\[ S = \{1, 2, \ldots, 7\} \quad \text{with} \quad 7 = 0, 6 = 1 \]
in accordance with [6, P216 (IV)]. In this case, $(\Lambda - \Lambda_0)$ is the lowest weight of its miniscule representation. We have
\[ w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = 2\alpha_7 = 2\alpha_1 + 3\alpha_2 + \cdots. \]
In addition, 1 is the only vertex in $S$ that is adjacent to the unique vertex of $(I_{af} \setminus S)$ in the extended Dynkin diagram. We have $\langle \vartheta^\vee, \varpi_j \rangle = 1$ by [6, P216 (IV)]. Thus, we find $w^{-1}(\Lambda - \Lambda_0) \in P^+_k$ by $k \geq 2$.
These yield the assertion.

**Theorem 6.3.** Let $k \in \mathbb{Z}_{\geq 2}$. For each $\Lambda \in P^+_k$ and $\mu \in P^+$, we have
\[ \text{ext}^1_2(\vartheta L(\Lambda), (W^{(k-1)}_\mu \otimes C_{\Lambda_0})^*) = 0. \quad (6.6) \]

**Proof.** We prove (6.6) by induction on $k$. By [5], we find $\Lambda' \in P^+_k$ and $\Lambda_i \in P^+_1$ such that
\[ L(\Lambda) \subset L(\Lambda') \otimes L(\Lambda_i). \]
Thus, (6.6) is equivalent to
\[ \text{ext}^1_2(\vartheta L(\Lambda') \otimes \vartheta L(\Lambda_i), (W^{(k-1)}_\mu \otimes C_{\Lambda_0})^*) = 0 \quad (6.7) \]
for $\Lambda' \in P^+_k$, $\Lambda_i \in P^+_1$, and $\mu \in P^+$.

Here $\vartheta L(\Lambda_i)$ admits a decreasing separable filtration whose associated graded is the direct sum of grading shifts of $\{W^{(1)}_\lambda\}_{\lambda \in P_k}$ by Proposition 6.1 and Theorem 3.3 for $k = 1$ (see [14, Theorem 4.7] for their constructions). Thus, it suffices to prove
\[ \text{ext}^1_2(\vartheta L(\Lambda') \otimes W^{(1)}_\lambda, (W^{(k-1)}_\mu \otimes C_{\Lambda_0})^*) = 0 \quad \Lambda' \in P^+_k, \lambda, \mu \in P^+. \quad (6.8) \]

In view of the BGG resolution (1.3), the ext$^1$-group appearing in (6.8) has bounded $d$-grading from the above. In view of Theorem 2.9, we have
\[ W^{(1)}_\lambda = \lim_{m \to \infty} \left( W^{(1)}_\lambda \otimes \text{end}_C(W^{(1)}_\lambda) \right), \]
where $\text{end}_C(W^{(1)}_\lambda)_{\leq m}$ is the degree $m$ truncation of the endomorphism ring. As each of the terms are finite self-extension of $W^{(1)}_\lambda \otimes \mathbb{C}_{-2\Lambda_0}$, it is enough to prove
\[ \text{ext}^1_2(\vartheta L(\Lambda') \otimes W^{(1)}_\lambda \otimes \mathbb{C}_{-2\Lambda_0}, (W^{(k-1)}_\mu \otimes C_{\Lambda_0})^*) = 0 \quad \Lambda' \in P^+_k, \lambda, \mu \in P^+. \quad (6.9) \]

We find $w \in W_{af}$ and $\Lambda'' \in P^+_k$ such that
\[ W^{(k-1)}_\mu = D_w(C_{\Lambda''}). \]

We rearrange $w \in W_{af}$ if necessary to assume that $w$ is the longest element in the double coset $WwW \subset W_{af}$. We set $W_\lambda := W^{(1)}_\lambda \otimes \mathbb{C}_{-\Lambda_0}$. 

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By Theorem 1.8, we rewrite (6.9) as:

\[
\text{ext}_{B}^{1}\left(θL(Λ') ⊗ W_{Λ}^{(1)} ⊗ C_{-Λ_0}, (W_{μ}^{(k-1)})^{*}\right) = \text{ext}_{B}^{1}\left(θL(Λ') ⊗ W_{Λ}, (D_{w}(C_{Λ''}))^{*}\right) \\
\cong \text{ext}_{B}^{1}\left(θL(Λ') ⊗ D_{w-1}(W_{Λ}), (C_{Λ''})^{*}\right) \\
\cong \text{ext}_{B}^{1}\left(θL(Λ') ⊗ W_{Λ}, (C_{Λ''})^{*}\right),
\]

(6.10)

where we have neglected the effect of the grading shift coming from Theorem 2.10 in the last term.

In case \(k = 2\) (the initial case of the induction), we have

\[
\text{ext}_{B}^{1}\left(θL(Λ') ⊗ W_{Λ}, (C_{Λ''})^{*}\right) = \text{ext}_{B}^{1}\left(θL(Λ'), (W_{Λ} ⊗ C_{Λ''})^{*}\right)
\]

(6.11)

for \(Λ', Λ'' ∈ P_{1}^{+}\). We apply an affine Dynkin diagram automorphism to rearrange \(Λ''\) into \(Λ_0\). Then, we can apply \(D_{w_0}\) to the second factor by Theorem 1.8 as (an affine Dynkin diagram automorphism twist of) \(θL(Λ')\) is a \(\hat{g}\)-module (as being an integrable \(\hat{g}\)-module). By Corollary 2.11, the vanishing of (6.11) is equivalent to

\[
\text{ext}_{B}^{1}\left(θL(Λ'), (W_{Λ'} ⊗ C_{Λ''})^{*}\right) = \text{ext}_{B}^{1}\left(θL(Λ'), (W_{Λ'}^{(1)})^{*}\right) = 0
\]

(6.12)

for every \(Λ' ∈ P_{1}^{+}\) and \(Λ' ∈ P_{1}^{+}\). This is the contents of Proposition 6.1.

Let us find \(w' ∈ W_{af}\) and \(Λ_j ∈ P_{1}^{+}\) such that

\[W_{Λ}^{(1)} = D_{w'}(C_{Λ_j})\]

up to grading shifts.

We further rewrite (6.10) as:

\[
\text{ext}_{B}^{1}\left(θL(Λ') ⊗ W_{Λ}, (C_{Λ''})^{*}\right) \cong \text{ext}_{B}^{1}\left(θL(Λ') ⊗ W_{Λ}^{(1)}, (C_{Λ''-Λ_0})^{*}\right) \\
= \text{ext}_{B}^{1}\left(θL(Λ') ⊗ D_{w'}(C_{Λ_j}), (C_{Λ''-Λ_0})^{*}\right) \\
\cong \text{ext}_{B}^{1}\left(θL(Λ'), (L^1D_{(w')^{-1}}(C_{Λ''-Λ_0}) \otimes C_{Λ_j})^{*}\right)
\]

(6.13)

Therefore, if the complex

\[L^1D_{(w')^{-1}}(C_{Λ''-Λ_0})\]

defines a genuine Demazure module, then we deduce

\[
\text{ext}_{B}^{1}\left(θL(Λ'), (L^1D_{(w')^{-1}}(C_{Λ''-Λ_0}) \otimes C_{Λ_j})^{*}\right) = 0
\]

from the level \((k - 1)\) case of (6.6), by applying an affine Dynkin diagram automorphism that sends \(Λ_j\) to \(Λ_0\) and introducing \(D_{w_0}\) to the RHS by Theorem 1.8 (thanks to Corollary 2.11). This is the case when \(Λ'' - Λ_0 \not∈ P_{k-1}^{+}\). Thus, we concentrate into the case when \(Λ'' - Λ_0 \not∈ P_{k-1}^{+}\) (but \(Λ'' \in P_{k}^{+}\)).

By Proposition 6.2, we have a short exact sequence

\[0 → D_{Λ''-Λ_0}^{(k-1)} → D_{Λ''-Λ_0}^{(k)} → C_{Λ''-Λ_0} → 0\]

(6.14)
We have deduced (6.12). This proceeds the induction and we conclude the assertion.

Applying an affine Dynkin diagram automorphism that sends $\Lambda_0$ to $\Lambda_0$ and introducing $D_{w_0}$ to the second factor, we can use (6.6) for level $(k-1)$ to deduce

$$\text{ext}^1_{\mathfrak{g}}(\theta L(\Lambda_1) \otimes W^{(1)}_\lambda, (D^{(k-1)}_{\mathfrak{g}})^*) = 0.$$ 

Now we apply $\text{hom}_{\mathfrak{g}}(\theta L(\Lambda') \otimes D_{w}((\mathfrak{C}_{\Lambda_0})^*), (\bullet)^*)$ to (6.14). Then, the vanishing of (6.13), namely

$$\text{ext}^1_{\mathfrak{g}}(\theta L(\Lambda') \otimes W^{(1)}_\lambda, (\mathfrak{C}_{\lambda''-\Lambda_0})^*) = 0,$$

is equivalent to the surjectivity of the map $\psi$

$$\text{hom}_{\mathfrak{g}}(\theta L(\Lambda') \otimes W^{(1)}_\lambda, (D^{(k-1)}_{\mathfrak{g}})^*) \xrightarrow{\psi} \text{hom}_{\mathfrak{g}}(\theta L(\Lambda') \otimes W^{(1)}_\lambda, (D^{(k-1)}_{\mathfrak{g}+\alpha_0})^*)$$

$$\xrightarrow{\text{ext}^1_{\mathfrak{g}}(\theta L(\Lambda') \otimes W^{(1)}_\lambda, (\mathfrak{C}_{\lambda''})^*)} \text{ext}^1_{\mathfrak{g}}(\theta L(\Lambda_1) \otimes W^{(1)}_\lambda, (D^{(k-1)}_{\mathfrak{g}})^*) = 0.$$ 

We have $W^{(1)}_\lambda \subset D_0(W^{(1)}_\lambda)$ and $\mathbb{L}^{\infty}D_0(W^{(1)}_\lambda) = 0$ as $W^{(1)}_\lambda$ is a Demazure module of level one. In view of Theorem 1.7, we have

$$\theta L(\Lambda') \otimes W^{(1)}_\lambda \subset \mathbb{L}^\bullet D_0(\theta L(\Lambda') \otimes W^{(1)}_\lambda) \cong \theta L(\Lambda') \otimes \mathbb{L}^\bullet D_0(W^{(1)}_\lambda).$$

We have $\mathbb{L}^\bullet D_0(D^{(k-1)}_{\mathfrak{g}+\alpha_0}) = D^{(k-1)}_{\mathfrak{g}}$. By Theorem 1.8, the map $\psi$ is the same as

$$\text{hom}_{\mathfrak{g}}(D_0(\theta L(\Lambda') \otimes W^{(1)}_\lambda), (D^{(k-1)}_{\mathfrak{g}+\alpha_0})^*) \rightarrow \text{hom}_{\mathfrak{g}}(\theta L(\Lambda_1) \otimes W^{(1)}_\lambda, (D^{(k-1)}_{\mathfrak{g}})^*).$$

Hence, Corollary 5.3 asserts that $\psi$ must be surjective. Therefore, (6.6) for level $(k-1)$ implies (6.6) for level $k$ with its initial case (6.12). This proceeds the induction and we conclude the assertion. 

**Corollary 6.4.** Let $k \in \mathbb{Z}_{\geq 1}$. For each $\Lambda \in P^+_k$ and $\mu \in P$, we have

$$\text{ext}^1_{\mathfrak{g}}(\theta L(\Lambda), (D^{(k)}_{\mu})^*) = 0.$$ 

In addition, we have

$$\text{ext}^1_{\mathfrak{g}}(\theta L(\Lambda), (D^{(k-1)}_{\mu} \otimes \mathfrak{C}_{\lambda_0})^*) = 0 \quad k \geq 2.$$ 

(6.15)

**Proof.** We have

$$D_{w_0}(\theta L(\Lambda)) = \theta L(\Lambda), \quad D_{w_0}(D^{(k)}_{\mu} \otimes \mathfrak{C}_{(k-1)\lambda_0}) = W^{(k)}_{\mu} \otimes \mathfrak{C}_{(k-1)\lambda_0}$$

by Theorem 2.3 and the definition. Hence, the assertion follows from the combination of Proposition 6.1 and Theorem 6.3, Proposition 2.1, and Theorem 1.8. 

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7 Some filtration results

Keep the setting of the previous section.

**Proposition 7.1.** Let $k, l \in \mathbb{Z}_{>0}$ be such that $k \geq l$. We assume Theorem 3.3 for level $< k$. For each $\lambda \in P^+$, the module $\mathbb{D}_\lambda^{(k)}$ admits a $\mathbb{D}_\lambda^{(l)}$-filtration.

**Proof.** The case $k = l$ is trivial by the definition. Thus, we assume $k > l$ in the below. In view of Theorem 3.3, that holds for level $< k$ by assumption, it suffices to prove

$$\text{ext}^1_{\mathcal{B}}(\mathbb{D}_\lambda^{(l)}(\mathcal{D}_\mu^{(l-1)} \otimes \mathcal{C}_{\lambda_0}^*)) \equiv 0 \quad \lambda, \mu \in P$$

(7.1)

for each $2 \leq l < k$. Thus, we assume $k = l$ in the below.

For $\Lambda \in P_+^*$, let $W^\Lambda_{af} \subset W^\Lambda_{af}$ denote the set of maximal length coset representatives of $W^\Lambda_{af}/\{w \in W^\Lambda_{af} \mid w\Lambda = \Lambda\}$. For each $\Lambda \in P_+^*$, $m \in \mathbb{Z}_{\geq 0}$, and $i \in I_{af}$, we set

$$L^m(\Lambda) := \sum_{\ell(v) \geq m} \theta L(\Lambda)^v$$

and

$$L_i^m(\Lambda) := L^{m+1}(\Lambda) + \sum_{\ell(v) = m, s_i \cdot v < v} \theta L(\Lambda)^v,$$

where $v$ runs over $W^\Lambda_{af}$ subject to the conditions. We have

$$L_i^* \mathcal{D}_i(L^m(\Lambda)) = \sum_{w \in W^\Lambda_{af}, \ell(w) = m-1, s_i \cdot w > w} \theta L(\Lambda)^w + L^m(\Lambda).$$

by Theorem 2.3 and Corollary 2.4. In view of Theorem 1.4 (cf. [33, Theorem C]), we deduce

$$\frac{\mathbb{D}_i(L^m(\Lambda))}{L^m(\Lambda)} \cong \bigoplus_{w \in W^\Lambda_{af}, \ell(w) = m-1, s_i \cdot w > w} \mathbb{D}^{(k)}_{w\Lambda}. \quad (7.2)$$

It follows that $\mathbb{D}_w^* \mathcal{D}_i(L^m(\Lambda)) = L_i^* \mathcal{D}_i(L^m(\Lambda))$ since $s_i \cdot w < v$ implies $s_i \cdot v > v$ for each $v \in W^\Lambda_{af}$.

We prove the following assertion by induction on $m$:

($\star$)$_1^m$ \quad $L^m(\Lambda)$ admits a $\mathbb{D}^{(k-1)}$-filtration;

($\star$)$_2^m$ \quad $\mathbb{D}^{(k)}_{w\Lambda}$ admits a $\mathbb{D}^{(k-1)}$-filtration for every $w \in W^\Lambda_{af}$ such that $\ell(w) < m$.

The assertion ($\star$)$_1^0$ holds by Corollary 6.4 and Theorem 3.3 for level $(k - 1)$ as ($\star$)$_2^0$ is a void condition. We assume ($\star$)$_1^m$. In particular, (7.2) implies that $\mathbb{D}_i(L_i^m(\Lambda))$ admits a $\mathbb{D}^{(k-1)}$-filtration by ($\star$)$_2^m$.

By Corollary 5.3, we have

$$\text{hom}_B(\mathbb{D}_i(L_i^m(\Lambda)), (D_{\mu}^{(k-1)} \otimes \mathcal{C}_{\lambda_0}^*)) \rightarrow \text{hom}_B(L_i^m(\Lambda), (D_{\mu}^{(k-1)} \otimes \mathcal{C}_{\lambda_0}^*))$$

for each $i \in I_{af}$.

We have a (part of the) long exact sequence

$$\text{hom}_B(\mathbb{D}_i(L_i^m(\Lambda)), (D_{\mu}^{(k-1)} \otimes \mathcal{C}_{\lambda_0}^*)) \rightarrow \text{hom}_B(L_i^m(\Lambda), (D_{\mu}^{(k-1)} \otimes \mathcal{C}_{\lambda_0}^*)) \rightarrow \text{ext}^1_{\mathcal{B}}(\mathbb{D}_i(L_i^m(\Lambda)), (D_{\mu}^{(k-1)} \otimes \mathcal{C}_{\lambda_0}^*)) = 0$$

(7.2)
admits a $\mathcal{D}$-associated graded is the direct sum of character twists of $\mathcal{D}$. We conclude (\star) by applying Proposition 7.1 repeatedly, it is refined into a $\mathcal{D}$-sequence to build up $\mathcal{D}$. For each $\Lambda$, we have $\mathcal{D}$. By (\star), we can decompose each of $\mathcal{L}$ into $\mathcal{D}$-sequences. These short exact sequences yield
$$\text{ext}_2^\bullet (\theta L(\Lambda)^w, \mathcal{C}_{\mu+k\Lambda}^\ast) \equiv 0 \quad \mu \not\preceq w\Lambda.$$ 
by applying (7.4). This is the first assertion.

The second assertion follows from the first assertion by using the short exact sequences to build up $\mathcal{D}$. For level $\mathcal{D}$, we deduce that $\mathcal{D}$ for level $(k)$. We have a short exact sequence
$$0 \rightarrow L^m(\Lambda) \rightarrow \mathcal{D} (L^m(\Lambda)) \rightarrow \mathcal{D} (L^m(\Lambda)) \rightarrow 0.$$ 
From this, we deduce that $\mathcal{D}(L^m(\Lambda))$ admits a $\mathcal{D}^{(k)}$-filtration by Theorem 3.3. Hence, the induction proceeds and we have (\star).

**Theorem 7.2.** Let $k \in \mathbb{Z}_{>0}$. Assume that Theorem 3.3 holds for level $< k$. For each $\Lambda \in P^+_k$ and $w \in W_{\text{af}}$, we have
$$\text{ext}_2^\bullet (\theta L(\Lambda)^w, \mathcal{C}_{\mu+k\Lambda}^\ast) \equiv 0 \quad \mu \not\preceq w\Lambda.$$ 
In particular, we have
$$\text{ext}_2^\bullet (\theta L(\Lambda)^w, \mathcal{C}_{\mu+k\Lambda}^\ast) \equiv 0 \quad \mu \not\preceq w\Lambda$$ 
for each $k' \in \mathbb{Z}_{>0}$.

**Proof.** The module $\theta L(\Lambda)^w$ admits a $\mathcal{D}^{(k)}$-filtration by its definition. By applying Proposition 7.1 repeatedly, it is refined into a $\mathcal{D}^{(k)}$-filtration. In view of the fact that $\theta L(\Lambda)^w$ has cyclic $\mathfrak{b}$-generator with $\mathfrak{b}$-weight $-w\Lambda$, we find that its associated graded is the direct sum of character twists of $\mathcal{D}_{\mu}^{(1)}$, where $w\Lambda \preceq -\nu$ by Corollary 2.16. In view of Theorem 3.2 for $k = 1$, we find
$$\text{ext}_2^\bullet (\theta L(\Lambda)^w, (\mathcal{D}_{\mu}^{(1)} \otimes \mathcal{C}_{(k-1)\Lambda})^\ast) \equiv 0 \quad \mu \not\preceq w\Lambda.$$ 
Thanks to Lemma 4.1, we can decompose each of $\{q^m \mathcal{C}_{\mu}^\ast\}_{\mu \not\preceq w\Lambda, m \in \mathbb{Z}}$ into $\{q^m \mathcal{C}_{\mu}^\ast\}_{\mu \not\preceq w\Lambda, m \in \mathbb{Z}}$ by a finitely many repeated application of short exact sequences. These short exact sequences yield
$$\text{ext}_2^\bullet (\theta L(\Lambda)^w, \mathcal{C}_{\mu+k\Lambda}^\ast) \equiv 0 \quad \mu \not\preceq w\Lambda.$$ 
by applying (7.4). This is the first assertion.

The second assertion follows from the first assertion by using the short exact sequences to build up $\mathcal{D}$. For level $(k)$, we deduce that $\mathcal{D}$ for level $(k-1)$. We have a short exact sequence
$$0 \rightarrow L^m(\Lambda) \rightarrow \mathcal{D} (L^m(\Lambda)) \rightarrow \mathcal{D} (L^m(\Lambda)) \rightarrow 0.$$ 
From this, we deduce that $\mathcal{D}(L^m(\Lambda))$ admits a $\mathcal{D}^{(k)}$-filtration by Theorem 3.3. Hence, the induction proceeds and we have (\star).
8 Proofs of main theorems

Keep the setting of the previous section. This section is entirely devoted to the proofs of Theorem 3.2 and Theorem 3.3. As our proof is by induction on level $k$ (whose initial case reduces to [14]), we assume the validity of Theorem 3.2 and Theorem 3.3 for level $< k$.

**Proof of Theorem 3.2 for level $k$.** We prove the first assertion. Let $\Lambda, \Lambda' \in P_k^+$ and $w, v \in W_{st}$ such that $\mu + k\Lambda_0 = v\Lambda'$. We can rearrange $w, v$ if necessary to assume that $w$ is the smallest element among all element that yields the same $w\Lambda$, and $v$ is the largest element among all $v$ with $\mu + k\Lambda_0 = v\Lambda'$. We have

$$\text{ext}^*_{\mathfrak{B}}(\theta(L(\Lambda)^w, (D_{\mu}^{(k)})^*) \cong \text{ext}^*_{\mathfrak{B}}(\theta(L(\Lambda)^w, \mathcal{D}_v(\mathcal{C}_\Lambda)^*)$$

by Theorem 1.8. Thanks to Theorem 2.3, we have

$$\mathcal{D}_{v^{-1}}(\theta(L(\Lambda)^w) = \theta(L(\Lambda))$$

if and only if a reduced expression of $v^{-1}$ contains a subexpression whose product yields $w^{-1}$. This occurs precisely when $w^{-1} \leq v^{-1}$ by the definition of the Bruhat order. This is equivalent to $w \leq v$. Hence, we conclude

$$\text{ext}^*_{\mathfrak{B}}(\theta(L(\Lambda)^w, (D_{\mu}^{(k)})^*) \cong \begin{cases} \text{ext}^*_{\mathfrak{B}}(\theta(L(\Lambda), (\mathcal{C}_\Lambda)^*)) & (w \leq v) \\ \text{ext}^*_{\mathfrak{B}}(\theta(L(\Lambda)^w, (\mathcal{C}_\Lambda)^*)) & (\text{else}) \end{cases}$$

(8.1)

for some $e \neq u \in W_{st}$ such that $u\Lambda \neq \Lambda$ by Proposition 1.3. By Theorem 7.2 and the fact that $W_P^\pm$ is closed under taking smaller elements with respect to $\prec$, we find

$$\text{ext}^*_{\mathfrak{B}}(\theta(L(\Lambda)^u, (\mathcal{C}_\Lambda)^*)) \equiv 0 \quad \text{if} \quad u\Lambda \neq \Lambda \text{ for every } v \in W.$$

Applying Proposition 7.1 to $\theta(L(\Lambda)^u$ (that requires Theorem 3.3 for level $< k$), we obtain

$$\text{ext}^*_{\mathfrak{B}}(\theta(L(\Lambda)^u, (\mathcal{C}_\Lambda)^*)) \equiv 0 \quad u \in W$$

whenever $\Lambda' - \Lambda_0 \in P_{k-1}^+$ by Theorem 3.2 for level $(k - 1)$ and $\mathcal{C}_{\Lambda'} \cong D_{\mathcal{X'}}^{(k)} \cong D_{\mathcal{X'}}^{(k-1)} \otimes \mathcal{C}_{\Lambda_0}$. In case $\Lambda' - \Lambda_0 \notin P_{k-1}^+$, we have necessarily $\langle \mathcal{X'}, \mathcal{X} \rangle = k$. In this case, we use Proposition 6.2 to deduce

$$\text{ext}^i_{\mathfrak{B}}(\theta(L(\Lambda)^u, (D_{\mathcal{X'}}^{(k-1)} \otimes \mathcal{C}_{\Lambda_0})^*) \rightarrow \text{ext}^{i+1}_{\mathfrak{B}}(\theta(L(\Lambda)^u, (\mathcal{C}_\Lambda)^*)) \rightarrow \text{ext}^{i+1}_{\mathfrak{B}}(\theta(L(\Lambda)^u, (D_{\mathcal{X'}}^{(k-1)} \otimes \mathcal{C}_{\Lambda_0})^*)$$

(8.2)

for each $i \geq 0$. Thanks to Proposition 7.1 and Theorem 3.2 for level $(k - 1)$, we deduce that the middle term of (8.2) is zero for $i > 1$. Here we have

$$\text{hom}_{\mathfrak{B}}(\theta(L(\Lambda)^u \otimes \mathcal{C}_{\Lambda_0}, (D_{\mathcal{X'}}^{(k-1)} \otimes \mathcal{C}_{\Lambda_0})^*) \rightarrow \text{hom}_{\mathfrak{B}}(\theta(L(\Lambda)^u \otimes \mathcal{C}_{\Lambda_0}, (D_{\mathcal{X'}}^{(k-1)} \otimes \mathcal{C}_{\Lambda_0})^*)$$

by Theorem 5.1 and Proposition 2.17. Hence, the middle term of (8.2) vanishes also for $i = 1$. In particular, we find

$$\text{ext}^*_{\mathfrak{B}}(\theta(L(\Lambda)^u, (\mathcal{C}_\Lambda)^*)) \equiv 0 \quad u \in W$$

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whenever $\Lambda' - \Lambda_0 \notin P^w_{\leq -1}$. Since $^\theta L(\Lambda)^w$ is a cyclic $\mathfrak{b}$-module whose head is $C_{-\lambda \Lambda}$, the corresponding hom also vanishes, we conclude

$$\operatorname{ext}^i_{\mathfrak{g}}(\theta L(\Lambda)^w, (C_{\lambda \Lambda})^*) \equiv 0 \quad w \in W \text{ such that } u \Lambda \neq \Lambda.'$$

Summarizing the above, we have

$$\operatorname{ext}^i_{\mathfrak{g}}(\theta L(\Lambda)^w, (D_{\mu}^{(k)})^*) \cong \begin{cases} \mathbb{C} & (i = 0, w \leq v, \Lambda = \Lambda') \\ 0 & \text{(else)} \end{cases}, \quad (8.3)$$

We have

$$\Lambda' = w(\Lambda + \rho) - \rho \quad w \in W_{af}$$

if and only if $w = e$ and $\Lambda = \Lambda'$ since $\rho(K) = h^\vee > 0$ (the dual Coxeter number).

We have $P^k + h^\vee \Lambda_0 \subset P^k_{(k + h^\vee)}$, and $P^k_{(k + h^\vee)}$ is the fundamental domain of the action of $W_{af}$ on $P^k_{(k + h^\vee)}$. Therefore, we have

$$\operatorname{ext}^i_{\mathfrak{g}}(\theta L(\Lambda)^w, (D_{\mu}^{(k)})^*) \cong \begin{cases} \mathbb{C} & (i = 0, w \leq v, \Lambda = \Lambda') \\ 0 & \text{(else)} \end{cases}, \quad (8.4)$$

from the BGG resolution (1.3). Since the maps in (8.4) are induced by the RHS of (8.3) through applications of the Demazure functors, the inclusion $^\theta L(\Lambda)^w \subset ^\theta L(\Lambda)^{w'} (w' \in W_{af})$ induces an isomorphism

$$\operatorname{hom}_{\mathfrak{g}}(^\theta L(\Lambda)^w, (D_{\mu}^{(k)})^*) \cong \operatorname{hom}_{\mathfrak{g}}(^\theta L(\Lambda)^{w'}, (D_{\mu}^{(k)})^*)$$

whenever $w \leq v$. Thus, taking the associated graded of the filtration induced from the thick Demazure modules implies

$$\operatorname{ext}^i_{\mathfrak{g}}(D_{\lambda}^{(k)}, (D_{\mu}^{(k)})^*) \cong \begin{cases} \mathbb{C} & (i = 0, \lambda + \mu = 0) \\ 0 & \text{(else)} \end{cases}, \quad (8.5)$$

that is the first assertion.

We prove the second assertion. By Proposition 2.1, we have

$$\operatorname{ext}^i_{\mathfrak{g}}(W_{\lambda}^{(k)}, (W_{\mu}^{(k)})^*) = \operatorname{ext}^i_{\mathfrak{g}}(W_{\lambda}^{(k)}, (D_{\mu}^{(k)})^*) \quad \lambda, \mu \in P^+.$$  

Here we have $W_{\mu}^{(k)} = D_{w_0 \mu}^{(k)}$, and $W_{\lambda}^{(k)}$ is filtered by $D_{w_0 \lambda}^{(k)}$ with multiplicity one by Proposition 2.7. Hence, we have

$$\operatorname{ext}^i_{\mathfrak{g}}(W_{\lambda}^{(k)}, (W_{\mu}^{(k)})^*) \cong \bigoplus_{\nu \in W_{\lambda}} \operatorname{hom}_{\mathfrak{g}}(D_{\nu}^{(k)}, (D_{\mu}^{(k)})^*)$$

by the $\operatorname{ext}^0$-vanishing part of the first assertion. This further implies

$$\operatorname{ext}^i_{\mathfrak{g}}(W_{\lambda}^{(k)}, (W_{\mu}^{(k)})^*) \cong \begin{cases} \mathbb{C} & (\lambda = \mu, i = 0) \\ 0 & \text{(else)} \end{cases}$$

by the hom-part of the first assertion and Schur’s lemma. This is the second assertion.
Proof of Theorem 3.3 for level \( k \). We can assume the validity of Theorem 3.2 for level \( k \).

In case the module admits one of the four types of filtrations, Theorem 3.2 asserts the desired Ext-vanishing by taking the limits. If we have a vanishing of \( \Ext^{>0}(M,N) \) (for some modules \( M,N \)), then we have a vanishing of \( \Ext^1(M,N) \). Thus, it is enough to prove that the \( \Ext^1 \)-vanishing with respect to one of the four series of modules implies the existence of the desired filtration.

Since the cases of \( \mathbb{D}^{(k)} \) and \( \mathbb{W}^{(k)} \), and the cases \( D^{(k)} \) and \( W^{(k)} \) are similar, we concentrate into the cases of \( \mathbb{D}^{(k)} \) and \( D^{(k)} \) and leave the detail of the cases of \( \mathbb{W}^{(k)} \) and \( W^{(k)} \) to the readers (cf. Proof of Proposition 4.5).

By \( M \in \mathcal{B}_\text{bdd} \), the module \( M \) is generated by its head. Hence, we find a projective cover \( f_1 : Q \to M \) that has bounded \( d \)-gradings from the below. For a direct summand of \( Q \) of shape \( q^n Q_{\mu+kA_0} \) (\( n \in \mathbb{Z}, \mu \in P \)), we also find a map \( f_2 : q^n Q_{\mu+kA_0} \to q^n D_{\mu}^{(k)} \). Then, we have the maximal quotient

\[
M' := q^m Q_{\mu+kA_0} / ( \ker f_1 \cap q^m Q_{\mu+kA_0} + \ker f_2 ),
\]

that is also the maximal quotient of \( q^m D_{\mu}^{(k)} \) that admits a (compatible) surjection from \( M \). Let \( M'' := \ker (M \to M') \).

We assume that \( M' \) is a proper quotient of \( q^m D_{\mu}^{(k)} \) to deduce contradiction. We have \( \Ext^2_{\mathcal{B}}(\mathbb{D}^{(k)}_\gamma, (M')^*) \neq 0 \) for some \( \gamma \in P \) by Proposition 4.5. We have a (part of the) long exact sequence

\[
\hom_{\mathcal{B}}(\mathbb{D}^{(k)}_\gamma, (M'')^*) \to \Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\gamma, (M')^*) \to \Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\gamma, M^*).
\]

Our non-zero element of \( \Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\gamma, (M')^*) \) afforded by Proposition 4.5 also represents an extension of \( M' \) by a one-dimensional \( \mathfrak{b} \)-module of shape \( \mathbb{C}_{-\gamma+m'\delta} \) (\( m' \in \mathbb{Z} \)). If this extension is also a quotient of \( M \), then it violates the maximality of \( M' \). Therefore, we have an element of \( \Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\gamma, (M')^*) \) that does not come from \( \hom_{\mathcal{B}}(\mathbb{D}^{(k)}_\lambda, (M'')^*) \). This implies \( \Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\gamma, M^*) \neq 0 \), that is a contradiction. Therefore, this case cannot occur.

As a consequence, we have \( M' = q^m D_{\mu}^{(k)} \). We have a (part of the) long exact sequence

\[
\Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\lambda, M^*) \to \Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\lambda, (M'')^*) \to \Ext^2_{\mathcal{B}}(\mathbb{D}^{(k)}_\lambda, (D_{\mu}^{(k)})^*)
\]

for every \( \lambda \in P \). Since we have \( \Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\lambda, M^*) = 0 \) and \( \Ext^2_{\mathcal{B}}(\mathbb{D}^{(k)}_\lambda, (D_{\mu}^{(k)})^*) = 0 \), we conclude

\[
\Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\lambda, (M'')^*) = 0 \quad \lambda \in P.
\]

Thus, the question reduces to \( M'' \) with \( \dim M'' < \dim M \). Hence the induction on \( \dim M \) yields the result for the case of \( M \in \mathcal{B}_0 \) such that

\[
\Ext^1_{\mathcal{B}}(\mathbb{D}^{(k)}_\lambda, M^*) = 0 \quad \lambda \in P.
\]

We consider the case \( M \in \mathcal{B} \) such that

\[
\Ext^1_{\mathcal{B}}(M, (D_{\lambda}^{(k)})^*) = 0 \quad \lambda \in P.
\]
Since $M$ is generated by its head, we find a projective cover $f_1: Q \to M$. For a direct summand of $Q$ of shape $q^m Q_{\mu + k\lambda_0}$ ($m \in \mathbb{Z}, \mu \in P$), we also find a map $f_2: q^m Q_{\mu + k\lambda_0} \to q^m \mathbb{D}_{\mu}^{(k)}$. Then, we have the maximal quotient

$$M' := q^m Q_{\mu + k\lambda_0}/((\ker f_1 \cap q^m Q_{\mu + k\lambda_0}) + \ker f_2),$$

that is also the maximal quotient of $q^m \mathbb{D}_{\mu}^{(k)}$ that admits a (compatible) surjection from $M$. Let $M' := \ker (M \to M')$.

We assume that $M'$ is a proper quotient of $q^m \mathbb{D}_{\mu}^{(k)}$ to deduce contradiction. We have $\text{ext}^1_{\mathbb{D}}(M', (D_{\gamma}^{(k)})^*) \neq 0$ for some $\gamma \in P$ by Proposition 4.5. We have a (part of the) long exact sequence

$$\text{hom}_{\mathbb{D}}(M'', (D_{\gamma}^{(k)})^*) \to \text{ext}^1_{\mathbb{D}}(M', (D_{\gamma}^{(k)})^*) \to \text{ext}^1_{\mathbb{D}}(M, (D_{\gamma}^{(k)})^*).$$

Our non-zero element of $\text{ext}^1_{\mathbb{D}}(M', (D_{\gamma}^{(k)})^*)$ afforded by Proposition 4.5 also represents an extension of $M'$ by a one-dimensional $\mathfrak{b}$-module of shape $C_{-\gamma + m' \delta}$ ($m' \in \mathbb{Z}$). If this extension is also a quotient of $M$, then it violates the maximality of $M'$. Therefore, we have an element of $\text{ext}^1_{\mathbb{D}}(M', (D_{\gamma}^{(k)})^*)$ that does not come from $\text{hom}_{\mathbb{D}}(M'', (D_{\gamma}^{(k)})^*)$. This implies $\text{ext}^1_{\mathbb{D}}(M, (D_{\gamma}^{(k)})^*) \neq 0$, that is a contradiction. Therefore, this case cannot occur.

As a consequence, we have $M' = q^m \mathbb{D}_{\mu}^{(k)}$. We have a (part of the) long exact sequence

$$\text{ext}^1_{\mathbb{D}}(M, (D_{\lambda}^{(k)})^*) \to \text{ext}^1_{\mathbb{D}}(M'', (D_{\lambda}^{(k)})^*) \to \text{ext}^2_{\mathbb{D}}(\mathbb{D}_{\mu}^{(k)}, (D_{\lambda}^{(k)})^*)$$

for every $\lambda \in P$. Since we have $\text{ext}^1_{\mathbb{D}}(M, (D_{\lambda}^{(k)})^*) = 0$ and $\text{ext}^2_{\mathbb{D}}(\mathbb{D}_{\mu}^{(k)}, (D_{\lambda}^{(k)})^*) = 0$, we conclude

$$\text{ext}^1_{\mathbb{D}}(M'', (D_{\lambda}^{(k)})^*) = 0 \quad \lambda \in P.$$

Thus, the question reduces to $M''$. As the construction of $M''$ works for each head of $M$ and $M \in \mathfrak{B}_{\text{bdd}}$, we find a decreasing filtration

$$M = M_0 \supset M'' = M_1 \supset M_2 \supset \cdots$$

such that each associated graded piece is isomorphic to some $q^m \mathbb{D}_{\mu}^{(k)}$ ($m \in \mathbb{Z}, \mu \in P$) and there exists a number $t_m$ for each $m \in \mathbb{Z}$ such that

$$\text{hom}_{\mathfrak{C}_\mathfrak{d}}(C_{m\delta}, M_t) = 0 \quad t > t_m.$$

This implies that

$$M = \lim_{\leftarrow t} M/M_t \in \mathfrak{B}_{\text{bdd}}$$

that satisfies

$$\text{ext}^1_{\mathbb{D}}(M, \mathbb{D}_{\lambda}^{(k)}) = \lim_{\leftarrow t} \text{ext}^1_{\mathbb{D}}(M/M_t, \mathbb{D}_{\lambda}^{(k)}) = 0 \quad \lambda \in P$$

by the Yoneda interpretation. Thus, we conclude the result for the case $M \in \mathfrak{B}_{\text{bdd}}$ such that

$$\text{ext}^1_{\mathbb{D}}(M, (D_{\lambda}^{(k)})^*) = 0 \quad \lambda \in P.$$

These complete the proof. \qed
9 Branching rules

Keep the setting of the previous section.

Lemma 9.1. Let \( k \in \mathbb{Z}_{>0} \) and \( \lambda \in P^+ \). In case \( \langle \vartheta \vee, \lambda \rangle < k \), we have an isomorphism

\[
\mathcal{W}_{\lambda}^{(k)} \otimes \mathbb{C}_{\Lambda_0} \cong \theta L(-\lambda_+ + (k-1)\Lambda_0)
\]
as \( \tilde{b} \)-modules.

Proof. The kernel of the surjection

\[
\theta L(-\lambda_+ + k\Lambda_0) \rightarrow \mathcal{W}_{\lambda}^{(k)} \quad (9.1)
\]
is spanned by \( \theta L(-\lambda_+ + k\Lambda_0)u \), where \( u \) runs over elements in \( \bigcup_{\beta \neq 0} Wt_{\beta} W \). In particular, the kernel of (9.1) contains the submodule generated by (the \( \theta \)-twist of) \( v_s(0) \) as \( \tilde{b} \)-modules.

Hence, the defining equations of \( \theta L(-\lambda_+ + (k-1)\Lambda_0) \) offered by (1.3) is satisfied by \( \mathcal{W}_{\lambda}^{(k)} \) realized as a quotient of \( \theta L(-\lambda_+ + k\Lambda_0) \). In other words, we have a surjection

\[
\theta L(-\lambda_+ + (k-1)\Lambda_0) \rightarrow \mathcal{W}_{\lambda}^{(k)} \otimes \mathbb{C}_{\Lambda_0} .
\]

(9.2)

We have \( s_0(-\lambda_- + k\Lambda_0) \neq -\lambda_- + k\Lambda_0 \) since

\[ -\langle \vartheta \vee, \lambda_- \rangle = \langle \vartheta \vee, \lambda \rangle \leq (k-1) < k. \]

Note that \( s_0 \not\leq w \in W_{af} \) if and only if \( w \notin W \). By Corollary 2.5, every \( g \)-stable proper thick Demazure submodule \( L(\Lambda)^w \) of \( L(\Lambda) \) satisfies \( w \not\in W \), and hence satisfies \( w \geq s_0 \). Thus, the kernel of (9.1) is contained in \( L(\Lambda)^{s_0} \) by taking \( \Lambda = -\lambda_- + k\Lambda_0 \). Therefore, we conclude that (9.2) is an isomorphism as required.

Theorem 9.2. Let \( k \in \mathbb{Z}_{>0} \) and \( \lambda \in P \). We have:

1. The graded \( \tilde{b} \)-module \( D(\lambda)^{k+1} \) admits a \( \mathbb{D}(k) \)-filtration;

2. The graded \( \tilde{b} \)-module \( D(\lambda)^k \) admits a \( \mathbb{D}(k+1) \)-filtration.

In addition, we have

\[
[\mathbb{D}_{\lambda}^{(k+1)} : \mathbb{D}_{\mu}^{(k)}]_q = (D_{\mu}^{(k)} : D_{\lambda}^{(k+1)})_q , \quad \mu \in P.
\]

Remark 9.3. We remark that Theorem 9.2 2) is contained in Joseph [27] (see also Naoi [40, Remark 4.15]) when \( g \) is of types ADE.

Proof of Theorem 9.2. In view of Theorem 3.3, the first two assertions are equivalent to

\[
\text{ext}^1_{\mathbb{D}}(\mathbb{D}_{\lambda}^{(k+1)}, (D_{\mu}^{(k)})^*) \equiv 0, \quad \lambda, \mu \in P, \quad (9.3)
\]

that is (7.1).

The second assertion follows as we have

\[
\langle \mathbb{D}_{\lambda}^{(k+1)}, D_{\mu}^{(k+1)} \rangle_{EP} = \delta_{\lambda, \mu} = \langle \mathbb{D}_{\lambda}^{(k)}, D_{\mu}^{(k)} \rangle_{EP} , \quad \lambda, \mu \in P
\]

by Theorem 3.2, and hence the transition matrices between their graded characters are transpose to each other by Lemma 2.14.

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Corollary 9.4. Let \( k \in \mathbb{Z}_{>0} \) and \( \lambda \in P \). The module \( D^{(k)}_{\lambda} \), regarded as a \( b \)-module, admits a filtration by the Demazure modules of \( g \).

Proof. Applying Theorem 9.2 repeatedly, we obtain a \( D^{(l)} \)-filtration of \( D^{(k)}_{\lambda} \) such that \( \langle \theta^\vee, \mu^+ \rangle < l \) for every \( \mu \in \Psi(D^{(k)}_{\lambda}) \). Then, its filtration piece \( D^{(l)}_{\nu} \) (\( \nu \in P \), up to grading shifts) is contained in \( V_{\nu^+} \) as \( (\nu^+ + l\Lambda_0) \in P^+ \), and it is a Demazure module of \( g \). Therefore, we conclude the result.

Lemma 9.5. Let \( k \in \mathbb{Z}_{>0} \). For each \( \lambda \in P^+ \), the module \( P^{(k)}_{\lambda} \) admits a \( \Psi^{(k)} \)-filtration.

Proof. We have \( \text{ext}^1_{\mathfrak{g}}(P_{\lambda}, (W^{(k)}_{\mu})^*) = 0 \) for each \( \nu \in P^+ \). Hence, \( P_{\lambda} \) admits a \( \Psi^{(k)} \)-filtration by Theorem 3.3.

Theorem 9.6. Let \( k \in \mathbb{Z}_{>0} \) and \( \lambda \in P^+ \). We have:

1. The graded \( \mathfrak{g}_{\geq 0} \)-module \( W^{(k+1)}_{\lambda} \) admits a \( \Psi^{(k)} \)-filtration;
2. The graded \( \mathfrak{g}_{\geq 0} \)-module \( W^{(k)}_{\lambda} \) admits a \( W^{(k+1)} \)-filtration.

In addition, we have

\[
(P_{\lambda} : \Psi^{(k)}_{\mu})_q = [W^{(k)}_{\mu} : V_{\lambda}]_q \quad \text{and} \quad (\Psi^{(k+1)}_{\lambda} : \Psi^{(k)}_{\mu})_q = (W^{(k)}_{\mu} : W^{(k+1)}_{\lambda})_q \quad (9.4)
\]

for each \( \mu \in P^+ \).

Proof. In view of Theorem 3.3, the first two assertions are equivalent to

\[
\text{ext}^0_{\mathfrak{g}}(\Psi^{(k+1)}_{\lambda}, (W^{(k)}_{\mu})^*) = 0 \quad \lambda, \mu \in P^+. \quad (9.5)
\]

Since we have \( W^{(k)}_{\mu} = D^{(k)}_{\mu_0} = \bigotimes \cdot D^{(k)}_{\mu_0} \) and \( \Psi^{(k)}_{\lambda} = \bigotimes \cdot D^{(k)}_{\lambda_0} \), Theorem 1.8 reduces (9.5) to a special case of (9.3).

The second assertion follows as we have

\[
\langle \Psi^{(k+1)}_{\lambda}, W^{(k+1)}_{\mu} \rangle_{EP} = \delta_{\lambda, \mu} = \langle \Psi^{(k)}_{\lambda}, W^{(k)}_{\mu} \rangle_{EP}
\]

by Theorem 3.2 and Proposition 2.1, and hence the transition matrices between their graded characters are transpose to each other by Lemma 2.14.

Remark 9.7. The proofs of Lemma 9.1 and Corollary 9.4 imply

\[ V_{\lambda} = W^{(k)}_{\lambda} \quad \text{for} \quad k \gg 0 \quad \text{and} \quad P_{\lambda} = \lim_{k \to \infty} W^{(k)}_{\lambda} \quad \text{for each} \quad \lambda \in P^+ . \]

Hence we can regard as

\[
\lim_{k \to \infty} (\bullet : W^{(k)}_{\mu})_q = [\bullet : V_{\mu}]_q \quad \text{and} \quad \lim_{k \to \infty} (\Psi^{(k)}_{\lambda} : \bullet)_q = (P_{\lambda} : \bullet)_q ,
\]

whenever they make senses.
Corollary 9.8. Assume that $g$ is of types ADE. Let $k \in \mathbb{Z}_{>0}$ and $\Lambda \in P^+_k$ such that $\langle d, \Lambda \rangle = 0$. For each $\mu \in P^+$, the level $k$ restricted Kostka polynomial $X^{(k)}_{\mu, \Lambda}(q) \in \mathbb{Z}[q]$ defined in [35, §5] (originally due to [44, 48]) satisfies
\[ X^{(k)}_{\mu, \Lambda}(q) = (W^{(1)}_\mu : W^{(k+1)}_\Lambda)_q \in \mathbb{Z}_{\geq 0}[q]. \tag{9.6} \]

Corollary 9.9. Let $k \in \mathbb{Z}_{>0}$. Let $W \in \mathfrak{c}_0$ be a $W^{(k)}$-filtered module such that $K$ acts by $k$ and
\[ \bigcup_{w \in \mathcal{W}_{af}} \mathcal{D}_w(W) \cong \bigoplus_{\Lambda \in P^+_k} L(\Lambda)^{\otimes m_\Lambda(W)} m_\Lambda(W) \in \mathbb{Z}. \]

Then, for each $\lambda \in P^+$, we have
\[ \sum_{\Lambda = \lambda} m_\Lambda(W)q^{\langle d, \Lambda \rangle} = (W^{(k+1)}_\lambda : W^{(1)}_\mu)_q. \tag{9.7} \]

Remark 9.10. 1) As the RHS of (9.6) makes sense for every $\Lambda \in P^+_k$ such that $\overline{\Lambda} \in P^+$, Corollary 9.8 embeds the set of level $k$ restricted Kostka polynomials into the following family of polynomials indexed by $\lambda, \mu \in P^+_+$:
\[ (W^{(k+1)}_\lambda : W^{(1)}_\mu)_q \in \mathbb{Z}_{\geq 0}[q]. \tag{9.4} \]

2) The ADE assumption in Corollary 9.8 comes from [35] and it is an artificial restriction. In fact, the RHS belongs to $\mathbb{Z}_{\geq 0}[q]$ in general by Theorem 9.6.

3) Assume that $g$ is not of types $E_7E_8F_4$. Then, $gch W^{(l)}_{\overline{\Lambda}} (l \leq k, \lambda \in P^+)$ is the character of the tensor product $B$ of Kirillov-Reshetikhin crystals of level $l$ ([49, Theorem 5.1] and [19, 42, 4, 43]). It follows that the LHS of (9.7) is the character of the set of level $k$ restricted highest weight elements of $B$ ([23, (3.9)] or [35, Definition 5.5]). We have more general tensor product of Kirillov-Reshetikhin crystals (and a recipé to construct the corresponding module $W$) such that the LHS of (9.7) is given by the character of the set of level $k$ restricted highest weight elements ([41]). At least in these cases, (9.7) provides a module-theoretic interpretation of level $k$-restricted generalized Kostka polynomials beyond Corollary 9.8 ([44, 21, 22, 23]).

Proof of Corollary 9.8. By [35, Corollary 5.12 and Corollary 3.6], we find
\[ X^{(k)}_{\mu, \Lambda}(q) = (^gL(\Lambda) : W^{(1)}_{\mu, \Lambda})_q. \]

In view of Lemma 9.1, we have
\[ (^gL(\Lambda) : W^{(1)}_{\mu, \Lambda})_q = (W^{(k+1)}_{\Lambda, \mu} : W^{(1)}_{\mu, \Lambda})_q. \]

Now we deduce
\[ (W^{(k+1)}_{\Lambda, \mu} : W^{(1)}_{\mu, \Lambda})_q = (W^{(k+1)}_{\mu} : W^{(1)}_{\Lambda})_q = (W^{(1)}_{\mu} : W^{(k+1)}_{\Lambda})_q \in \mathbb{Z}_{\geq 0}[q] \]
by Theorem 9.6. \qed
Proof of Corollary 9.9. By Theorem 2.2, we deduce $L^{\ast}D_w(W) \cong D_w(W)$ for each $w \in W_{af}$. By Theorem 1.8 and Theorem 3.2, we have
\[
\text{ext}_C^i(\theta L(\lambda + k\Lambda_0), W^*) \cong \text{ext}_C^i(\theta L(\lambda + k\Lambda_0), D_w(W^*))
\]
\[
\cong \lim_{\rightarrow} \text{ext}_C^i(\theta L(\lambda + k\Lambda_0), D_w(W^*))
\]
\[
= \bigoplus_{\Lambda \in P_+} \text{ext}_C^i(\theta L(\lambda + k\Lambda_0), \theta L(\Lambda)) \oplus \mathfrak{m}(\Lambda)
\]
\[
= \bigoplus_{\Lambda \in P_+} \text{ext}_C^i(\theta L(\lambda + k\Lambda_0), D_w(C_\Lambda)^*) \oplus \mathfrak{m}(\Lambda)
\]
\[
\cong \bigoplus_{\Lambda \in P_+} \text{ext}_C^i(\theta L(\lambda + k\Lambda_0), C_\Lambda^*) \oplus \mathfrak{m}(\Lambda)
\]
\[
\cong \begin{cases} 
\bigoplus_{w=\lambda} C_{\Lambda}^*(-m, \lambda) & (i = 0) \\
0 & (i > 0). 
\end{cases}
\]

Here we used the BGG resolution (1.3) to derive the last isomorphism.

In view of Lemma 9.1, we apply Proposition 4.4 to deduce
\[
gdim \text{hom}_C(\theta L(\lambda + k\Lambda_0), W^*) = gdim \text{hom}_C(\mathcal{W}_{\lambda - \alpha}^{(k+1)}; W^*) = (W : W_\lambda^{(k+1)})_{q^{-1}}.
\]
Equating these two interpretations of $gdim \text{hom}_C(\theta L(\lambda + k\Lambda_0), W^*)$ yields the assertion by replacing $q^{-1}$ with $q$.

Corollary 9.11. Let $k \in \mathbb{Z}_{\geq 0}$ and $\lambda, \mu \in P^+$. In case $\langle \theta \nu, \lambda \rangle < k$, we have
\[
(W_\rho^{(k)} : W_\lambda^{(k+1)})_q = \begin{cases} 
q^m & (\exists w \in W_{af} \text{ s.t. } \mu + k\Lambda_0 - m\delta = w(\lambda + k\Lambda_0)) \\
0 & (\text{else}) 
\end{cases}
\]

In particular, $W_\lambda^{(k+1)}$ appears only in the socle of $W_\mu^{(k)}$ (with respect to the $W^{(k+1)}$-filtration).

Proof. By Lemma 9.1 and the definition of $\mathcal{W}^{(k)}$, we have
\[
(\mathcal{W}_\lambda^{(k+1)} : \mathcal{W}_\mu^{(k)})_q = (\theta L(-\lambda - k\Lambda_0) : \mathcal{W}_\mu^{(k)})_q
\]
\[
= \begin{cases} 
q^m & (\exists w \in W_{af} \text{ s.t. } \mu + k\Lambda_0 - m\delta = w(\lambda + k\Lambda_0)) \\
0 & (\text{else}) 
\end{cases}
\]

Applying Theorem 9.2 yields the multiplicity count. We have $W_\lambda^{(k)} = V_\lambda$ by $\lambda + k\Lambda_0 \in P_+^k$. Since $V_\lambda \subset W_\lambda^{(k+1)} \subset W_\lambda^{(k)}$, we have
\[
V_\lambda = W_\lambda^{(k+1)} = W_\lambda^{(k)} \subset q^{-m}W_\mu^{(k)} \subset L(\lambda + k\Lambda_0)
\]
as $\tilde{b}$-module. This asserts that $W_\lambda^{(k+1)}$ appears in the socle of $W_\lambda^{(k)}$, and it is the only contribution of $W_\lambda^{(k+1)}$ in the $W^{(k+1)}$-filtration of $W_\lambda^{(k)}$ as required.

Remark 9.12. For each $k \in \mathbb{Z}_{\geq 0}$ and $\lambda \in P_+$ such that $\langle \theta \nu, \lambda \rangle < k$, we have a short exact sequence of $\mathfrak{b}$-modules
\[
0 \to C_\lambda \to D_0(C_{\lambda + k\Lambda_0}) \otimes C_{-k\Lambda_0} \to q^{-1}D_0(C_{\lambda + \theta^{(k+1)}(k+1)\Lambda_0}) \otimes C_{-(k+1)\Lambda_0} \to 0.
\]

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Applying $D_{w_0}$ to this short exact sequence, we find a short exact sequence

$$0 \to q^{\bullet}W^{(k)}_\lambda \to W^{(k)}_\mu \to W^{(k+1)}_\mu \otimes C_{-\lambda_0} \to 0$$

for some $\bullet \in \mathbb{Z}$, where $\mu = (\lambda - \langle \vartheta^\vee, \lambda \rangle \vartheta + k\vartheta)$. The multiplicity count of Corollary 9.11 generalizes this occurrence to an arbitrary $\mu \in P^+$.

The following result is the higher level analogue of the main result of [35]:

**Corollary 9.13.** Let $k, l \in \mathbb{Z}_{>0}$ be such that $k \geq (l-1)$. For each $\Lambda \in P_k^+$, the module $\theta L(\Lambda)$ admits a $W^{(1)}$-filtration.

**Proof.** Combine Lemma 9.1 and Theorem 9.6 1).

**Corollary 9.14.** Let $k, l \in \mathbb{Z}_{>0}$. For each $\lambda \in P$, the module $W^{(k)}_\lambda$ admits a $W^{(1)}$-filtration.

**Proof.** By Proposition 7.1, we have

$$\text{ext}^1_k(W^{(k)}_\lambda, (W^{(1)}_\mu \otimes C_{(k-1)\Lambda_0})^*) = 0 \quad \lambda, \mu \in P^+.$$ 

In particular, we find that $W^{(1)}_\mu$ admits a $W^{(k)}$-filtration, and $W^{(k)}_\lambda$ admits a $W^{(1)}$-filtration by Theorem 3.3. As $W^{(1)}_\lambda$ admits a $W^{(1)}$-filtration, we conclude the result. 

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**References**

[1] Matthew Bennett, Arkady Berenstein, Vyjayanthi Chari, Anton Khoroshkin, and Sergey Loktev. Macdonald polynomials and BGG reciprocity for current algebras. *Selecta Math. (N.S.)*, 20(2):585–607, 2014.

[2] Matthew Bennett, Vyjayanthi Chari, and Nathan Manning. BGG reciprocity for current algebras. *Adv. Math.*, 231(1):276–305, 2012.

[3] Rekha Biswal and Deniz Kus. A combinatorial formula for graded multiplicities in excellent filtrations. *Transform. Groups*, 26(1):81–114, 2021.

[4] Rekha Biswal and Travis Scrimshaw. Existence of Kirillov-Reshetikhin crystals for multiplicity-free nodes. *Publ. Res. Inst. Math. Sci.*, 56(4):761–778, 2020.

[5] Maegan K. Bos and Kailash C. Misra. An application of crystal bases to representations of affine Lie algebras. *Journal of Algebra*, 173:436–458, 1995.

[6] Nicolas Bourbaki. *Elements of Mathematics. Lie Groups and Lie algebras. Chapters 4–6*. Springer-Verlag, Berlin, 2002.

[7] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

[8] Vyjayanthi Chari and Jacob Greenstein. Current algebras, highest weight categories and quivers. *Adv. Math.*, 216(2):811–840, 2007.

[9] Vyjayanthi Chari and Bogdan Ion. BGG reciprocity for current algebras. *Compos. Math.*, 151(7):1265–1287, 2015.
[10] Vyjayanthi Chari and Andrew Pressley. Weyl modules for classical and quantum affine algebras. *Represent. Theory*, 5:191–223 (electronic), 2001.

[11] Vyjayanthi Chari, Lisa Schneider, Peri Shereen, and Jeffrey Wand. Modules with demazure flags and character formulae. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 10(032):16pp, 2014.

[12] Ivan Cherednik. Nonsymmetric Macdonald Polynomials. *International Mathematics Research Notices*, 2(10), 1995.

[13] Ivan Cherednik and Boris Feigin. Rogers-Ramanujan type identities and Nil-DAHA. *Adv. Math.*, 248:1050–1088, 2013.

[14] Ivan Cherednik and Syu Kato. Nonsymmetric Rogers-Ramanujan sums and thick Demazure modules. *Adv. in Math.*, 374:Article number 107335, 2020.

[15] Ivan Cherednik. Nonsymmetric Macdonald Polynomials. *International Mathematics Research Notices*, 2(10), 1995.

[16] Ivan Cherednik and Syu Kato. Nonsymmetric Rogers-Ramanujan sums and thick Demazure modules. *Adv. in Math.*, 374:Article number 107335, 2020. arXiv: 1703.04108.

[17] G. Fourier and P. Littelmann. Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions. *Adv. Math.*, 211(2):566–593, 2007.

[18] G. Fourier, N. Manning, and P. Senesi. Global weyl modules for the twisted loop algebra. *Abh. Math. Semin. Univ. Hamb.*, 83:53–82, 2013.

[19] Ghislain Fourier, Anne Schilling, and Mark Shimozono. Demazure structure inside Kirillov-Reshetikhin crystals. *J. Algebra*, 309(1):386–404, 2007.

[20] Alexander Grothendieck. Sur quelques points d’algèbre homologique. *Tohoku Math. J. (2)*, 9:119–221, 1957.

[21] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 243–291. Amer. Math. Soc., Providence, RI, 1999.

[22] Goro Hatayama, Anatol N. Kirillov, Atsuo Kuniba, Masato Okado, Taichiro Takagi, and Yusuhiko Yamada. Character formulae of \( \hat{\mathfrak{sl}}_n \)-modules and inhomogeneous paths. *Nuclear Phys. B*, 536(3):575–616, 1999.

[23] Goro Hatayama, Atsuo Kuniba, Masato Okado, Taichiro Takagi, and Zengo Tsuboi. Paths, crystals and fermionic formulae. In *MathPhys odyssey, 2001*, volume 23 of *Prog. Math. Phys.*, pages 205–272. Birkhäuser Boston, Boston, MA, 2002.

[24] Bogdan Ion. Nonsymmetric Macdonald polynomials and Demazure characters. *Duke Math. J.*, 116(2):299–318, 2003.

[25] Anthony Joseph. On the Demazure character formula. *Ann. Sci. École Norm. Sup. (4)*, 18(3):389–419, 1985.

[26] Anthony Joseph. A decomposition theorem for Demazure crystals. *J. Algebra*, 265(2):562–578, 2003.

[27] Anthony Joseph. Modules with a Demazure flag. In *Studies in Lie theory*, volume 243 of *Progress in Mathematics*. Birkhäuser Boston, Boston, MA, 2006.

[28] Victor G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[29] Masaki Kashiwara. The crystal base and Littelmann’s refined Demazure character formula. *Duke Math. J.*, 71(3):839–858, 1993.

[30] Masaki Kashiwara. Level zero fundamental representations over quantized affine algebras and Demazure modules. *Publ. Res. Inst. Math. Sci.*, 41(1):223–250, 2005.

[31] Masaki Kashiwara and Mark Shimozono. Equivariant K-theory of affine flag manifolds and affine Grothendieck polynomials. *Duke Math. J.*, 148(3):501–538, 2009.

[32] Syu Kato. Demazure character formula for semi-infinite flag varieties. *Math. Ann.*, 371(3):1769–1801, 2018, arXiv:1605.0279.

[33] Syu Kato. Frobenius splitting of thick flag manifolds of Kac-Moody algebras. *Int. Math. Res. Not. IMRN*, 2020(17):5401–5427, 2020. arXiv:1707.03773.
[34] Syu Kato. Frobenius splitting of Schubert varieties of semi-infinite flag manifolds. *Forum of Mathematics, Pi*, 9:e5, 2021.

[35] Syu Kato and Sergey Loktev. A Weyl module stratification of integrable representations. *Comm. Math. Phys.*, 368:113–141, 2019. arXiv:1712.03508.

[36] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002.

[37] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[38] Olivier Mathieu. Formules de caractères pour les algèbres de Kac-Moody générales. *Astérisque*, 159–160:1–267, 1988.

[39] Olivier Mathieu. Frobenius action on the $B$-cohomology. In *Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)*, volume 7 of *Adv. Ser. Math. Phys.* World Sci. Publ., Teaneck, NJ, 1989.

[40] Katsuyuki Naoi. Weyl modules, Demazure modules and finite crystals for non-simply laced type. *Adv. in Math.*, 229(2):875–934, 2012.

[41] Katsuyuki Naoi. Demazure crystals and tensor products of perfect Kirillov-Reshetikhin crystals with various levels. *J. Algebra*, 374:1–26, 2013.

[42] Katsuyuki Naoi. Existence of Kirillov-Reshetikhin crystals of type $G_2^{(1)}$ and $D_4^{(3)}$. *J. Algebra*, 512:47–65, 2018.

[43] Katsuyuki Naoi and Travis Scrimshaw. Existence of Kirillov-Reshetikhin crystals for near adjoint nodes in exceptional types. *J. Pure Appl. Algebra*, 225(5):Paper No. 106593, 38, 2021.

[44] Masato Okado. mimeo.

[45] Patrick Polo. Variétés de Schubert et excellentes filtrations. In *Orbites unipotentes et représentations, III*, number 173-174 in Astérisque. 1989.

[46] Yasmine B. Sanderson. On the connection between Macdonald polynomials and Demazure characters. *J. Algebraic Combin.*, 11(3):289–275, 2000.

[47] A. Schilling and S. O. Warnaar. Inhomogeneous lattice paths, generalized Kostka polynomials and $A_{n-1}$-supernomials. *Commun. Math. Phys.*, 202:359–401, 1999.

[48] Anne Schilling and Mark Shimozono. Fermionic formulas for level-restricted generalized Kostka polynomials and coset branching functions. *Comm. Math. Phys.*, 220(1):105–164, 2001.

[49] Anne Schilling and Peter Tingley. Demazure crystals, Kirillov-Reshetikhin crystals, and the energy function. *Electron. J. Combin.*, 19(2):Paper 4, 42, 2012.

[50] Wilberd van der Kallen. Longest weight vectors and excellent filtrations. *Math. Zast.*, 201(1):19–31, 1989.