ASYMPTOTIC BEHAVIORS OF FUNDAMENTAL SOLUTION
AND ITS DERIVATIVES RELATED TO SPACE-TIME
FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Let \( p(t, x) \) be the fundamental solution to the problem
\[
\partial_t^\alpha u = -(-\Delta)^\beta u, \quad \alpha \in (0, 2), \ \beta \in (0, \infty).
\]
In this paper we provide the asymptotic behaviors and sharp upper bounds of \( p(t, x) \) and its space and time fractional derivatives
\[
D_n^x (-\Delta)^\gamma D^\sigma_t I^\delta_t p(t, x), \quad \forall \ n \in \mathbb{Z}_+, \ \gamma \in [0, \beta], \ \sigma, \delta \in [0, \infty),
\]
where \( D_n^x \) is a partial derivative of order \( n \) with respect to \( x \), \( (-\Delta)^\gamma \) is a fractional Laplace operator and \( D^\sigma_t \) and \( I^\delta_t \) are Riemann-Liouville fractional derivative and integral respectively.

1. Introduction

Let \( \alpha \in (0, 2), \ \beta \in (0, \infty) \) and \( p(t, x) \) be the fundamental solution to the space-time fractional equation
\[
\partial_t^\alpha u = \Delta^\beta u, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d
\]
with \( u(0, x) = u_0(x) \) (and \( \partial_t u(0, x) = 0 \) if \( \alpha > 1 \)). Here \( \partial_t^\alpha \) denotes Caputo fractional derivative and \( \Delta^\beta := -(-\Delta)^\beta \) is the fractional Laplacian. The fractional time derivative of order \( \alpha \in (0, 1) \) can be used to model the anomalous diffusion exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena (see \[25, 28\]), and the fractional spatial derivative describes long range jumps of particles. The fractional wave equation \( \partial_t^\alpha u = \Delta u \) with \( \alpha \in (1, 2) \) governs the propagation of mechanical diffusive waves in viscoelastic media (see \[23, 33\]). Equation (1.1) has been an important topic in the mathematical physics related to non-Markovian diffusion processes with a memory \([26, 27]\), in the probability theory related to jump processes \([5, 6, 24]\) and in the theory of differential equations \([7, 8, 17, 31, 36]\).

The aim of this paper is to present rigorous and self-contained exposition of fundamental solution \( p(t, x) \). More precisely, we provide asymptotic behaviors and upper bound of
\[
D_n^x (-\Delta)^\gamma D^\sigma_t I^\delta_t p(t, x), \quad \forall \ n \in \mathbb{Z}_+, \ \gamma \in [0, \beta], \ \sigma, \delta \in [0, \infty),
\]
where \( I^\delta_t \) and \( D^\sigma_t \) denotes the Riemann-Liouville fractional integral and derivative respectively.

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There are certainly considerable works dealing with explicit formula for the fundamental solutions and their asymptotic behaviors (see e.g. \[9, 10, 11, 12, 13, 20, 22\]). However, only few of them cover the derivative estimates of the fundamental solutions. In \[9, 10, 17, 20\], upper bounds of \([1.2]\) were obtained for \(\beta = 1, \sigma = 1 - \alpha, \) and \(\gamma = 0\). Also, in \[13\] asymptotic behavior for the case \(\alpha = 1, \beta \in (0, 1)\) and \(\sigma = 0\) was obtained. Note that it is assumed that either \(\alpha = 1\) or \(\beta = 1\) in \([9, 10, 13, 17, 20]\), and moreover spatial fractional derivative \(\Delta^p\) and time fractional derivative \(D_t^p\) are not obtained in \([9, 10, 17, 20]\) and \([13]\) respectively. Our result substantially improves these results because we only assume \(\alpha \in (0, 2)\) and \(\beta \in (0, \infty)\) and we provide two sides estimates of both space and time fractional derivatives of arbitrary order.

Our approach relies also on the properties of some special functions including the Fox H functions. However, unlike most of previous works, we do not use the method of Mellin, Laplace and inverse Fourier transforms which, due to the non-exponential decay at infinity of the Fox H functions, require restrictions on the space dimension \(d\) and other parameters \(\beta, \delta, \gamma,\) and \(\sigma\) (see Remark 5.2).

Below we give two main applications of our results. First, consider the non-homogeneous fractional evolution equation

\[
\partial_t^\alpha u = \Delta^\beta u + f, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \tag{1.3}
\]

with \(u(0, x) = 0\) and additionally \(\partial_t u(0, x) = 0\) if \(\alpha > 1\). One can show (see e.g. \[9, 17, 20\]) that the solution to the problem is given by

\[
Gf(t, x) = \int_0^t \int_{\mathbb{R}^d} Q_\alpha(t-s, x-y)f(s, y)dyds
\]

where

\[
Q_\alpha(t, x) := \begin{cases} D_1 t^{-\alpha}p(t, x) & : \alpha \in (0, 1) \\ I^{\alpha-1}_t p(t, x) & : \alpha \in (1, 2). \end{cases}
\]

It turns out (see \[18, 19\])) that to obtain the \(L_q(L_p)\)-estimate

\[
\|\Delta^\beta Gf\|_{L_q((0,T),L_p(\mathbb{R}^d))} \leq N\|f\|_{L_q((0,T),L_p(\mathbb{R}^d))}, \quad p, q > 1 \tag{1.4}
\]

it is sufficient to show that for \(a, b > 0\)

\[
\int_a^\infty \int_{|x| \geq b} |\Delta^\beta Q_\alpha(t, x)| dxdt \leq N\frac{a^\alpha}{b^{\beta}}, \\
\int_a^\infty \int_{\mathbb{R}^d} |\partial_t \Delta^\beta Q_\alpha(t, x)| dxdy \leq \frac{N}{\alpha}, \quad \int_a^\infty \int_{\mathbb{R}^d} |\nabla \Delta^\beta Q_\alpha(t, x)| dxdt \leq Na^{-\frac{\alpha}{\beta}}.
\]

One can use our estimates to prove the above three inequalities, and therefore (1.4) can be obtained as a corollary. Our second application is the \(L_p\)-theory of the stochastic partial differential equations of the type

\[
\partial_t^\alpha u = \Delta u + \partial_t^{\alpha+\sigma} \int_0^t g(s, x)dW_s,
\]

where \(\sigma < \frac{1}{2}\) and \(W_t\) is a Wiener process defined on a probability space \((\Omega, dP)\).

One can show (see \[51\]) that the solution to this problem is given by the formula

\[
u(t, x) = \int_0^t \left( \int_{\mathbb{R}^d} P_t(t-s, x-y)g(s, y)dy \right) dW_s.
\]
Here \( P_\sigma(t, x) \) is defined as
\[
P_\sigma(t, x) := \begin{cases} 
I_{t, \infty}^\sigma p(t, x) & : \sigma \leq 0 \\
D_t^\sigma p(t, x) & : \sigma > 0.
\end{cases}
\]
As has been shown for the case \( \alpha = 1 \) (see [15, 16, 21, 35]), sharp estimates of \( D_t^\sigma \Delta^\gamma P_\sigma \) can be used to obtain \( L_p \)-estimate
\[
\| \Delta^\gamma u \|_{L_p(\Omega \times (0, T) \times \mathbb{R}^d)} \leq N \| g \|_{L_p(\Omega \times (0, T) \times \mathbb{R}^d)}.
\]
The detail of (1.5) will be given for \( \gamma \leq 2 \left( 1 - \frac{2\alpha}{\alpha} \right) \) in a subsequent paper.

The rest of the article is organized as follows. In Section 2 we state our main results, Theorems 2.1, 2.2, and 2.3. In Section 3 we present the definition of the Fox H functions and their several properties. For the convenience of the reader, we repeat the relevant material and demonstration in [14] and [17], thus making our exposition self-contained. Section 4 contains asymptotic behaviors at zero and infinity of the Fox H function. In Section 5 we present explicit representation of fundamental solutions and their fractional and classical derivatives. Finally, in Section 6 we prove our main results.

We first introduce some definitions related to the fractional calculus. Let \( \beta \geq 0 \). For a function \( u \in L_1(\mathbb{R}^d) \), we write \( \Delta^\beta u = f \) if there exists a function \( f \in L_1(\mathbb{R}^d) \) such that
\[
\mathcal{F} \{ f(\cdot) \} (\xi) = |\xi|^{2\beta} \mathcal{F} \{ u(\cdot) \} (\xi).
\]
For \( u \in L_1((0, T)) \), the Riemann-Liouville fractional integral of the order \( \alpha \in (0, \infty) \) is defined as
\[
I_0^\alpha u := \int_0^t (t - s)^{\alpha - 1} u(s) \, ds, \quad t \leq T.
\]
One can easily check
\[
I_t^\alpha I_t^\beta = I_t^{\alpha + \beta}, \quad \forall \alpha, \beta \geq 0.
\]

2. Main results
Let \( n \in \mathbb{N} \) and \( n - 1 \leq \alpha < n \). The Riemann-Liouville fractional derivative \( D_t^\alpha \) and the Caputo fractional derivative \( \partial_t^\alpha \) are defined as

\[
D_t^\alpha u := \left( \frac{d}{dt} \right)^n \left( I_t^{n-\alpha} u \right)
\]

\[
\partial_t^\alpha u := D_t^{\alpha-(n-1)} \left( u^{(n-1)}(t) - u^{(n-1)}(0) \right).
\]

By definition \((2.2)\) for any \( \alpha \geq 0 \) and \( u \in L_1((0, T)) \),

\[
D_t^\alpha I_t^\alpha u = u.
\]

Using \((2.1)-(2.3)\), one can check

\[
\partial_t^\alpha u = D_t^\alpha \left( u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) \right), \quad n - 1 \leq \alpha < n.
\]

Thus \( D_t^\alpha u = \partial_t^\alpha u \) if \( u(0) = u^{(1)}(0) = \ldots = u^{(n-1)}(0) = 0 \). For more information on the fractional derivatives, we refer the reader to \([29, 32]\).

For \( \sigma \in \mathbb{R} \) we define Riemann-Liouville fractional operator \( D_t^\sigma \) as

\[
D_t^\sigma := \begin{cases} 
D_t^{|\sigma|} & : \sigma > 0 \\
I_t^{|\sigma|} & : \sigma < 0.
\end{cases}
\]

Then by \((2.1)\) and \((2.2)\), for any \( \alpha, \beta \geq 0 \)

\[
D_t^\alpha I_t^\beta = D_t^{\alpha-\beta}.
\]

The Mittag-Leffler function \( E_{\alpha,\beta}(z) \) is defined as

\[
E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \Re[\alpha] > 0
\]

for \( z \in \mathbb{C} \) and we write \( E_{\alpha}(z) = E_{\alpha,1}(z) \) for short. Using the equality

\[
D_t^\sigma I_t^\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - \sigma)} t^{\alpha-\sigma}, \quad \forall \sigma \in \mathbb{R}, \alpha \geq 0
\]

one can check that for \( t > 0 \)

\[
D_t^\sigma E_{\alpha}(-\lambda t^\alpha) = t^{-\sigma} E_{\alpha,1-\sigma}(-\lambda t^\alpha), \quad \sigma \in \mathbb{R}, \lambda \geq 0,
\]

and that for any constant \( \lambda \),

\[
\varphi(t) := E_{\alpha}(-\lambda t^\alpha)
\]

satisfies \( \varphi(0) = 1 \) (also \( \varphi'(0) = 0 \) if \( \alpha > 1 \)) and

\[
\partial_t^\alpha \varphi = -\lambda \varphi, \quad t > 0.
\]

Let \( \alpha \in (0, 2) \) and \( \beta \in (0, \infty) \). By taking the Fourier transform to the equation

\[
\partial_t^\alpha u = \Delta^\beta u, \quad t > 0, \quad u(0, x) = u_0(x), \quad (\text{and } u'(0, x) = 0 \text{ if } \alpha > 1)
\]

one can formally get

\[
\mathcal{F} \{ u(t, \cdot) \} = E_{\alpha}(-|\xi|^{2\beta} t^\alpha) \mathcal{F} \{ u_0 \}.
\]

Therefore, to obtain the fundamental solution, it is needed to find an integrable function \( p(t, x) \in L_1(\mathbb{R}^d) \) satisfying

\[
\mathcal{F} \{ p(t, \cdot) \} = E_{\alpha}(-|\xi|^{2\beta} t^\alpha). \quad (2.5)
\]
Denote

$$M(t, x) := |x|^{2\beta} t^{-\alpha}.$$  

In the following theorems we give the asymptotic behaviors of $D^n_x \Delta^\gamma D^n_t p(t, x)$ as $M \to 0$ and $M \to \infty$. We also provide upper bounds when $M \leq 1$ and $M \geq 1$.

Firstly, we consider the case $M \to \infty$.

**Theorem 2.1.** Let $\alpha \in (0, 2)$, $\beta \in (0, \infty)$, $\gamma \in [0, \infty)$, $\sigma \in \mathbb{R}$ and $n \in \mathbb{N}$. There exists a function $p(t, \cdot) \in L_1(\mathbb{R}^d)$ satisfying (2.5). Furthermore, the following asymptotic behaviors hold as $M \to \infty$:

1. If $\beta \in \mathbb{N}$, then for some constant $c > 0$ depending only on $d, n, \alpha, \beta, \sigma$

   $$|D^\gamma D_t^\sigma p(t, x)| \leq |x|^{-\sigma} \exp \left\{ -c|x|^{2\beta} t^{-\frac{\alpha}{\alpha - \beta}} \right\}$$

   and

   $$|D^\gamma D_t^\sigma p(t, x)| \leq |x|^{-\sigma} \exp \left\{ -c|x|^{2\beta} t^{-\frac{\alpha}{\alpha - \beta}} \right\}. \quad (2.6)$$

2. If $\alpha = 1$, $\beta \notin \mathbb{N}$, and $\sigma = 0$,

   $$|\Delta^\gamma D_t^\sigma p(t, x)| \sim \begin{cases} t|\gamma|^{-d-2\gamma-2\beta} : \gamma \in \mathbb{Z}_+ \\ |\gamma|^{-d-2\gamma} : \gamma \in [0, \infty) \setminus \mathbb{Z}_+ \end{cases} \quad (2.7)$$

   and

   $$|D^\gamma \Delta^\gamma D_t^\sigma p(t, x)| \leq \begin{cases} t|\gamma|^{-d-2\gamma-2\beta-n} : \gamma \in \mathbb{Z}_+ \\ |\gamma|^{-d-2\gamma-n} : \gamma \in [0, \infty) \setminus \mathbb{Z}_+ \end{cases}.$$

3. If $\gamma \in (0, \beta) \setminus \mathbb{N}$,

   $$|\Delta^\gamma D_t^\sigma p(t, x)| \sim |x|^{-\sigma} t^{d-2\gamma}$$

   and

   $$|D^\gamma \Delta^\gamma D_t^\sigma p(t, x)| \leq |x|^{-d-2\gamma-n} t^{-\sigma}. \quad (2.8)$$

4. If $\beta \notin \mathbb{N}$ and $\gamma \in [0, \beta) \cap \mathbb{Z}_+$,

   $$|\Delta^\gamma D_t^\sigma p(t, x)| \sim |x|^{-d-2\gamma-2\beta+\alpha} \quad (2.8)$$

   and

   $$|D^\gamma \Delta^\gamma D_t^\sigma p(t, x)| \leq |x|^{-d-2\gamma-2\beta-n} t^{-\sigma+\alpha}.$$

5. If $\gamma = \beta$ and $d \geq 2$,

   $$|\Delta^\beta D_t^\sigma p(t, x)| \sim \begin{cases} |x|^{-d-4\beta-\sigma+\alpha} : \beta \in \mathbb{N} \text{ or } \sigma \in \mathbb{N} \\ |x|^{-d-2\beta-\sigma} : \text{otherwise} \end{cases} \quad (2.9)$$

   and

   $$|D^\beta \Delta^\beta D_t^\sigma p(t, x)| \leq \begin{cases} |x|^{-d-4\beta-n-\sigma+\alpha} : \beta \in \mathbb{N} \text{ or } \sigma \in \mathbb{N} \\ |x|^{-d-2\beta-n-\sigma} : \text{otherwise}. \end{cases} \quad (2.10)$$

6. If $\gamma = \beta$, $\sigma + \alpha \in \mathbb{N}$, and $d = 1$, then (2.9) and (2.10) hold.
Remark 2.2. (i) Note that \( D^n_x \Delta^\gamma \mathcal{D}_t^\alpha p(t, x) \) has exponential decay as \( M \to \infty \) only when \( \beta \) is a positive integer.

(ii) Let \( x \neq 0 \). Then by \( \ref{eq:2.6} \) and \( \ref{eq:2.8} \) with \( \gamma = 0 \), \( D^n_x \Delta^\gamma \mathcal{D}_t^\alpha p(t, x) \to 0 \) as \( t \to 0 \) if either \( \beta \) is a positive integer or \( \sigma < \alpha \).

(iii) If \( \gamma = \beta \) and \( d = 1 \), then we additionally assumed \( \sigma + \alpha \in \mathbb{N} \). Without this extra condition we had a trouble in using Fubini’s theorem in our proof.

(iv) Note that we have only upper bounds of \( D^n_x \Delta^\gamma \mathcal{D}_t^\alpha p(t, x) \) unless \( n = 0 \). This is because, for instance, \( \mathcal{D}_t p(t, x) \) is of type \( x'g(t, x) \) (see \( \ref{eq:5.1} \)) and becomes zero if \( x' = 0 \). Hence we can not have positive lower bound of \( \mathcal{D}_t p(t, x) \) for such \( x \).

Secondly, we consider the case \( M \to 0 \).

Theorem 2.3. Let \( \alpha, \beta, \gamma, \sigma \) be given as in Theorem \( \ref{thm:2.1} \) and \( n \in \mathbb{N} \). Then the following asymptotic behaviors hold as \( M \to 0 \):

(i) If \( \gamma \in [0, \beta) \) and \( \sigma + \alpha \notin \mathbb{N} \),

\[
|\Delta^\gamma \Delta^\alpha \mathcal{D}_t^\alpha p(t, x)| \sim \begin{cases} \frac{t^{-\sigma - \frac{\alpha(4d+2)}{2d}}}{\rho} & : \gamma < \beta - \frac{d}{2} \\ |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} (1 + |\ln |x||^{\frac{2\gamma}{\beta}} t^{-\alpha}) & : \gamma = \beta - \frac{d}{2} \\ |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} & : \gamma > \beta - \frac{d}{2} \end{cases} \tag{2.11}
\]

and

\[
|D^n_x \Delta^\gamma \Delta^\alpha \mathcal{D}_t^\alpha p(t, x)| \leq \begin{cases} |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} (1 + |\ln |x||^{\frac{2\gamma}{\beta}} t^{-\alpha}) & : \gamma < 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} & : \gamma = 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} & : \gamma > 2\beta - \frac{d}{2} - 1 \end{cases} \tag{2.12}
\]

(ii) If \( \gamma \in [0, \beta) \) and \( \sigma + \alpha \in \mathbb{N} \),

\[
|\Delta^\gamma \Delta^\alpha \mathcal{D}_t^\alpha p(t, x)| \sim \begin{cases} t^{-\sigma - \frac{\alpha(4d+2)}{2d}} & : \gamma < 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} (1 + |\ln |x||^{\frac{2\gamma}{\beta}} t^{-\alpha}) & : \gamma = 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} & : \gamma > 2\beta - \frac{d}{2} - 1 \end{cases} \tag{2.13}
\]

and

\[
|D^n_x \Delta^\gamma \Delta^\alpha \mathcal{D}_t^\alpha p(t, x)| \leq \begin{cases} |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} (1 + |\ln |x||^{\frac{2\gamma}{\beta}} t^{-\alpha}) & : \gamma < 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} & : \gamma = 2\beta - \frac{d}{2} - 1 \\ |x|^{-d-2\gamma + 2\beta t^{-\sigma - \alpha}} & : \gamma > 2\beta - \frac{d}{2} - 1 \end{cases} \tag{2.14}
\]

(iii) If \( \alpha = 1 \) and \( \sigma = 0 \),

\[
|\Delta^\gamma p(t, x)| \sim t^{-\frac{d+2\gamma}{2}}, \hspace{1em} |D^n_x \Delta^\gamma p(t, x)| \leq |x|^{-d-2\gamma + 2\beta t^{-\alpha}}\frac{d+2\gamma}{2}.
\]

(iv) If \( \gamma = \beta \) and \( d \geq 2 \),

\[
|\Delta^\beta \mathcal{D}_t^\alpha p(t, x)| \sim \begin{cases} t^{-\sigma - \frac{\alpha}{2\beta}} & : \frac{d}{2} \leq \beta \\ |x|^{-d+2\beta t^{-\sigma - 2\alpha}} (1 + |\ln |x||^{\frac{2\beta}{\beta}} t^{-\alpha}) & : \frac{d}{2} = \beta \\ |x|^{-d+2\beta t^{-\sigma - 2\alpha}} & : \frac{d}{2} > \beta \end{cases} \tag{2.15}
\]

and

\[
|D^n_x \Delta^\beta \mathcal{D}_t^\alpha p(t, x)| \leq \begin{cases} |x|^{-d+2\beta t^{-\sigma - 2\alpha}} (1 + |\ln |x||^{\frac{2\beta}{\beta}} t^{-\alpha}) & : \frac{d}{2} \leq 1 < \beta \\ |x|^{-d+2\beta t^{-\sigma - 2\alpha}} & : \frac{d}{2} + 1 = \beta \\ |x|^{-d+2\beta t^{-\sigma - 2\alpha}} & : \frac{d}{2} + 1 > \beta \end{cases} \tag{2.16}
\]
(v) If $\gamma = \beta$, $\sigma + \alpha \in \mathbb{N}$, and $d = 1$, then (2.15) and (2.16) hold.

Next we give the upper estimates when $M \geq 1$ and $M \leq 1$.

**Theorem 2.4.** (i) Assertions (i)-(iv) of Theorem 2.1 also hold for $M \geq 1$ if “∼” is replaced by “≤”.
(ii) Assertions (i)-(v) of Theorem 2.3 also hold for $M \leq 1$ if “∼” is replaced by “≤”.

The proofs of Theorems 2.1, 2.3 and 2.4 are given in Section 6.

**Remark 2.5.** Let $\alpha = 1$ and $\beta \in (0, \infty)$, and $\sigma = 0$. Then by Theorem 2.4

$$|D^n_x \Delta^\gamma p(t, x)| \lesssim \begin{cases} t|x|^{-d-\gamma-n} : \gamma \in \mathbb{Z}_+ \\ |x|^{-d-\gamma-n} : \gamma \notin \mathbb{Z}_+ \end{cases}$$

holds for $|x|^2 \geq t$. Also, by Theorem 2.4, for $|x|^2 \leq t$,

$$|\Delta^\gamma p(t, x)| \lesssim t^{-\frac{d+\alpha}{2}}, \quad |D^n_x \Delta^\gamma p(t, x)| \lesssim |x|^{2-n} t^{-\frac{d+\alpha+2}{2}}.$$

These estimates cover the results of [15, Lemma 3.1, 3.3] and [13, Corollary 1].

**Remark 2.6.** Theorem 2.4 also implies the results of [9, Proposition 5.1, 5.2]. Let $\beta = 1$ and take $|n| \geq 1$. For $|x|^2 \geq t^\alpha$.

$$|D^n_x \Delta^\gamma p(t, x)| \lesssim |x|^{-d-n} t^{-\gamma} \exp \left\{ -c t^{-\frac{\alpha}{2}} |x|^{2\alpha} \right\}.$$ 

Also, (cf. (2.11), (2.12), and (2.14)),

$$|p(t, x)| \lesssim \begin{cases} t^{-d/2} : d \geq 3 \\ t^{-\alpha}(1 + |\ln |x||^2 t^{-\alpha}) : d = 2 \\ |x|^{-d+2-t^{-\alpha}} : d = 1 \end{cases}$$

and

$$|D^n_x p(t, x)| \lesssim |x|^{-d+2-nt^{-\alpha}}, \quad |D^n_x \Delta p(t, x)| \lesssim |x|^{-d+2-nt^{-2\alpha}},$$

$$|D^n_x \Delta^\gamma p(t, x)| \lesssim \begin{cases} |x|^{-d+4-n} t^{-\gamma-1} : d \geq 3 \\ |x|^{-n} t^{-1}(1 + |\ln |x||^2 t^{-\alpha}) : d = 2 \\ |x|^{-n} t^{-1} : d = 1 \end{cases}$$

hold for $|x|^2 \leq t^\alpha$.

### 3. The Fox H function

In this section, we introduce the definition and some properties of the Fox H function. We refer to [14] for further information.
3.1. **Definition.** Let $\Gamma(z)$ denote the gamma function which can be defined (see [2 Section 1.1]) for $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ as

$$\Gamma(z) = \lim_{n \to \infty} \frac{n!n^z}{z(z + 1) \cdots (z + n)}.$$  

Note that $\Gamma(z)$ is a meromorphic function with simple poles at the nonpositive integers. From the definition, for $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$,

$$z\Gamma(z) = \Gamma(z + 1),$$  

and it holds that

$$\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \quad \prod_{k=0}^{m-1} \Gamma(z + \frac{k}{m}) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2} - mz} \Gamma(mz).$$  

One can easily check that for $k \in \mathbb{Z}_+$,

$$\text{Res}_{z=-k}[\Gamma(z)] = \lim_{z \to -k} (z + k)\Gamma(z) = \frac{(-1)^k}{k!},$$

where $\text{Res}_{z=z_0}[f(z)]$ denotes the residue of $f(z)$ at $z = z_0$. From the Stirling’s approximation

$$\Gamma(z) \sim \sqrt{2\pi e^{\frac{1}{2}} \log z} e^{-z}, \quad |z| \to \infty,$$

it follows that

$$|\Gamma(a + ib)| \sim \sqrt{2\pi |a|^{a-\frac{1}{2}} e^{-a \pi[1 - \text{sign}(a)]b/2}}, \quad |a| \to \infty$$  

and

$$|\Gamma(a + ib)| \sim \sqrt{2\pi |b|^{b-\frac{1}{2}} e^{-b \pi[1 - \text{sign}(b)]a/2}}, \quad |b| \to \infty.$$  

Let $m, n, \nu, \mu$ be fixed integers satisfying $0 \leq m \leq \mu$, $0 \leq n \leq \nu$. Assume that real parameters $\epsilon_1, \ldots, \epsilon_\nu$, $\delta_1, \ldots, \delta_\nu$ and positive real parameters $\gamma_1, \ldots, \gamma_\nu$, $\delta_1, \ldots, \delta_\nu$ are given such that

$$\max_{1 \leq j \leq m} \left( -\frac{\delta_j}{\epsilon_j} \right) < \min_{1 \leq j \leq n} \left( \frac{1 - \epsilon_j}{\gamma_j} \right).$$  

For each $k \in \mathbb{Z}_+$, we set

$$\epsilon_{j,k} = \frac{1 - \epsilon_j + k}{\gamma_j}, \quad \delta_{j,k} = -\frac{\delta_j + k}{\delta_j},$$

which constitute

$$P_1 = \{\delta_{j,k} \in \mathbb{R} : j \in \{1, \ldots, m\}, k \in \mathbb{Z}_+\},$$

$$P_2 = \{\epsilon_{j,k} \in \mathbb{R} : j \in \{1, \ldots, n\}, k \in \mathbb{Z}_+\}.$$  

Note that $P_1 \cap P_2 = \emptyset$ by (3.5). We arrange the elements of $P_1$ and $P_2$ as follows:

$$P_1 = \{\delta_0 > \delta_1 > \delta_2 > \cdots\}, \quad P_2 = \{\epsilon_0 < \epsilon_1 < \epsilon_2 < \cdots\}.$$  

For the above parameters, define

$$H(z) := \frac{\prod_{j=1}^{m} \Gamma(\delta_j + \delta_j z) \prod_{j=1}^{n} \Gamma(1 - \epsilon_j - \gamma_j z)}{\prod_{j=m+1}^{0} \Gamma(\epsilon_j + \gamma_j z) \prod_{j=n+1}^{0} \Gamma(1 - \epsilon_j - \gamma_j z)}.$$
Note that $P_1$ and $P_2$ are sets of poles of $\mathcal{H}(z)$. To describe the behavior of $\mathcal{H}(z)$ as $|z| \to \infty$, we set

$$
\alpha^* := \sum_{i=1}^{n} \gamma_i - \sum_{i=n+1}^{\nu} \gamma_i + \sum_{j=1}^{m} \delta_j - \sum_{j=m+1}^{\mu} \delta_j,
$$

and

$$
\Lambda := \sum_{j=1}^{\mu} \delta_j - \sum_{j=1}^{\nu} \gamma_j - \sum_{j=1}^{\mu} \delta_j.
$$

Due to (3.3),

$$
|\mathcal{H}(a+ib)r^{-\alpha-i\beta}| \sim \left( \frac{c}{a} \right)^{-\omega a} \left( \frac{\eta}{r} \right)^{a \Lambda}, \quad r \in (0, \infty) \quad (3.7)
$$
as $a \to \infty$ and

$$
|\mathcal{H}(a+ib)r^{-\alpha-i\beta}| \sim \left( \frac{c}{|a|} \right)^{\omega a} \left( \frac{r}{\eta} \right)^{-|\alpha|} |a|^\Lambda, \quad r \in (0, \infty) \quad (3.8)
$$
as $a \to -\infty$. By (3.4), it follows that

$$
|\mathcal{H}(a+ib)r^{-\alpha-i\beta}| \sim \left( \frac{e}{|b|} \right)^{-\omega a} \left( \frac{\eta}{r} \right)^{a |b|^\Lambda} e^{-\alpha |b|\pi/2}, \quad r \in (0, \infty) \quad (3.9)
$$
as $|b| \to \infty$.

The Fox H function $H_{\nu\mu}^{mn}(r)$ $(r > 0)$ is defined via Mellin-Barnes type integral in the form

$$
H_{\nu\mu}^{mn}(r) := H_{\nu\mu}^{mn} \left[ \begin{array}{c} r \\ \mathcal{F} \\
\mathcal{G}
\end{array} \right] := H_{\nu\mu}^{mn} \left[ \begin{array}{c} (\xi_1, \gamma_1) \\ (\delta_1, \delta_1) \\
(\xi_\nu, \gamma_\nu) \\ (\delta_\nu, \delta_\nu)
\end{array} \right] := \frac{1}{2\pi i} \int_L \mathcal{H}(z) r^{-z} dz. \quad (3.10)
$$

In (3.10), $L$ is the infinite contour which separates all the poles in $P_1$ to the left and all the poles in $P_2$ to the right of $L$. Precisely, we choose $L$ as follows:

(i) if $\omega > 0$, then $L = L_{\mathcal{H}a}$, which is a left loop situated in a horizantal strip (or left Hankel contour), runs from $-\infty + \ell h_1$ to $\ell + \ell h_1$, and then to $\ell + \ell h_2$ and finally terminates at the point $-\infty + \ell h_2$ with $-\infty < h_1 < 0 < h_2 < \infty$ and

$$
\tilde{\delta}_0 < \ell < \tilde{c}_0, \quad (3.11)
$$

(ii) if $\omega < 0$, then $L = L_{\mathcal{H}a}^+$, which is a right loop situated in a horizantal strip (or right Hankel contour), runs from $+\infty + \ell h_1$ to $\ell + \ell h_1$, and then to $\ell + \ell h_2$ and finally terminates at the point $+\infty + \ell h_2$ with $-\infty < h_1 < 0 < h_2 < \infty$ and

(iii) if $\omega = 0$, then $L = L_{\mathcal{H}a}^-$ and $L = L_{\mathcal{H}a}^+$ and for $r \in (0, \eta)$ and $r \in (\eta, \infty)$ respectively.

The following proposition shows that integral (3.10) is well defined and independent of the choice of $h_1, h_2$, and $\ell \in (\tilde{\delta}_0, \tilde{c}_0)$. 
**Proposition 3.1** ([14, Theorem 1.2]). Assume (3.9) and choose the contour $L$ as above. Then Mellin-Barnes integral (3.10) makes sense and it is an analytic function of $r \in (0, \infty)$ and of $r \in (0, \eta) \cup (\eta, \infty)$ if $\omega \neq 0$ and $\omega = 0$ respectively. Furthermore,

(i) if $\omega > 0$, then

$$H_{\nu\mu}^{mn}(r) = \sum_{k=0}^{\infty} \text{Res}_{z=\delta_k} [\mathcal{H}(z)r^{-z}]$$

(3.12)

(ii) if $\omega < 0$, then

$$H_{\nu\mu}^{mn}(r) = -\sum_{k=0}^{\infty} \text{Res}_{z=\delta_k} [\mathcal{H}(z)r^{-z}]$$

(3.13)

(iii) if $\omega = 0$, then (3.12) and (3.13) hold for $r \in (0, \eta)$ and $r \in (\eta, \infty)$ respectively.

**Proof.** (i) Let $\omega > 0$ and $r \in (0, \infty)$. Choose $h_1 < 0$, $h_2 > 0$, and $\ell \in \mathbb{R}$ so that (3.11) holds. Take a sufficiently large $p \in \mathbb{Z}^+_+$ so that $\delta_p < 0$. Then there exists a real number $M = M(p) > 0$ such that

$$-\delta_p < M < -\delta_{p+1}.$$  

(3.14)

Define a closed rectangular contour $C^M$ which can be decomposed into four lines

$$C_M = L_1 \cup L_2 \cup L_3 \cup L_4$$

where

$$L_1 := \{ z \in \mathbb{C} : \Re[z] = \ell, h_1 \leq \Im[z] \leq h_2 \},$$

$$L_2 := \{ z \in \mathbb{C} : \Re[z] = -M, h_1 \leq \Im[z] \leq h_2 \},$$

$$L_3 := \{ z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = h_1 \},$$

$$L_4 := \{ z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = h_2 \}.$$  

Note that $\mathcal{H}(z)r^{-z}$ is a meromorphic function on $z \in \mathbb{C} \setminus P_1 \cup P_2$ and

$$\int_{C^M} |\mathcal{H}(z)r^{-z}| |dz| < \infty.$$  

(3.15)

Due to (3.8) ($i = 1, 2$)

$$\left| \int_{-\infty}^{-M} \mathcal{H}(t + ih_i)r^{-t-ih_i} dt \right| \leq \int_{-\infty}^{-M} |\mathcal{H}(t + ih_i)r^{-t-ih_i}| dt$$

$$\leq \int_{M}^{\infty} \left( \frac{e}{t} \right)^{\omega t} \left( \frac{r}{\eta} \right)^{-t} t^\nu dt < \infty,$$  

(3.16)

and

$$\left| \int_{L_2} \mathcal{H}(z)r^{-z} dz \right| = \left| \int_{h_1}^{h_2} \mathcal{H}(-M + it)r^{M-it} dt \right|$$

$$\leq \int_{h_1}^{h_2} |\mathcal{H}(-M + it)r^{M-it}| dt$$

$$\leq \left( \frac{e}{M} \right)^{\omega M} \left( \frac{r}{\eta} \right)^{-M} M^\nu (h_2 - h_1) < \infty.$$  

(3.17)
By (3.15)-(3.17), integral (3.10) absolutely converges and makes sense with \( L = L_{H_0} \).

We next show (3.12). Note that (3.16) and (3.17) converge to 0 as \( M \to \infty \) and by the theory of residues,

\[
\frac{1}{2\pi i} \int_{C_M} \mathcal{H}(z)r^{-z}dz = \sum_{k=0}^{p} \text{Res}_{z=\delta_k} [\mathcal{H}(z)r^{-z}].
\]

Thus our claim follows immediately if we prove the convergence of the residue expansion. Let \( q \in \mathbb{N} \) \((q > p)\) be arbitrarily given. Then we can choose a real number \( N = N(q) > 0 \) satisfying (3.14) where \( p \) and \( M \) are replaced by \( q \) and \( N \) respectively. Set

\[
C_M' := L_1' \cup L_2 \cup L_3' \cup L_4'
\]

where

\[
L_1' := \{ z \in \mathbb{C} : \Re[z] = -N, h_1 \leq \Im[z] \leq h_2 \},
\]

\[
L_3' := \{ z \in \mathbb{C} : -N \leq \Re[z] \leq -M, \Im[z] = h_1 \},
\]

\[
L_4' := \{ z \in \mathbb{C} : -N \leq \Re[z] \leq -M, \Im[z] = h_2 \}.
\]

Observe that

\[
\frac{1}{2\pi i} \int_{C_M'} \mathcal{H}(z)r^{-z}dz = \sum_{k=p+1}^{q} \text{Res}_{z=\delta_k} [\mathcal{H}(z)r^{-z}]. \tag{3.18}
\]

By replacing \( M \) by \( N \) in (3.10) and (3.17),

\[
\lim_{N \to \infty} \left| \int_{L_1'} \mathcal{H}(z)r^{-z}dz \right| = \lim_{N,M \to \infty} \left| \int_{L_1' \cup L_4'} \mathcal{H}(z)r^{-z}dz \right| = 0,
\]

which implies

\[
\lim_{p,q \to \infty} \sum_{k=p+1}^{q} \text{Res}_{z=\delta_k} [\mathcal{H}(z)r^{-z}] = \lim_{N,M \to \infty} \frac{1}{2\pi i} \int_{C_M'} \mathcal{H}(z)r^{-z}dz = 0.
\]

Thus (3.12) is proved.

(ii) The case \( \omega < 0 \) is an analogue of the case \( \omega > 0 \). By (3.12), for sufficiently large \( M > 0 \) \((i = 1, 2)\)

\[
\left| \int_{L_1} \mathcal{H}(t + ih_i)r^{-t-ih_i}dt \right| \leq \int_{M}^{\infty} |\mathcal{H}(t + ih_i)r^{-t-ih_i}| dt \\
\leq \int_{M}^{\infty} \left( \frac{e}{t} \right)^{-\omega t} \left( \frac{\eta}{r} \right)^{M} dt < \infty, \tag{3.19}
\]

and

\[
\left| \int_{L_2} \mathcal{H}(z)r^{-z}dz \right| = \left| \int_{h_1}^{h_2} \mathcal{H}(M + it) r^{M-it}dt \right| \\
\leq \int_{h_1}^{h_2} |\mathcal{H}(M + it)r^{M-it}| dt \\
\leq \left( \frac{e}{M} \right)^{-\omega M} \left( \frac{\eta}{r} \right)^{M} M^{\omega}(h_2 - h_1) < \infty. \tag{3.20}
\]

Note that both (3.19) and (3.20) converge to 0 as \( M \to \infty \). Then we obtain our desired result by replacing \( \delta_k \) and \( M \) in the proof of the case \( \omega > 0 \) by \( e_k \) and \(-M\) respectively.
(iii) Finally we consider $\omega = 0$. Note that (3.16) and (3.17) hold if $r < \eta$, consequently (3.12) follows. If $r > \eta$, then we have (3.19) and (3.20) which give (3.13) immediately. The proposition is proved. \hfill \Box

In the remainder of this section, we assume $\alpha^* > 0$. (3.21)

Under (3.21), we can choose a contour $L = L_{Br}$ which is a vertical contour (or Bromwich contour) starting at the point $\ell - i\infty$ and terminating at the point $\ell + i\infty$ where $\ell$ satisfies (3.11). Actually, $H_{mn}^{\mu\nu}(r)$ does not depend on the choice of $L$ due to the following proposition and it is an analytic function of $r \in (0, \infty)$ (it is a holomorphic function of $r \in \mathbb{C}$ in the sector $|\arg r| < \frac{\omega \pi}{2}$. See [14, Theorem 1.2.(iii)]).

**Proposition 3.2.** Under (3.21), Mellin-Barnes integral (3.10) makes sense with $L = L_{Br}$. Furthermore, for $r \in (0, \infty)$

$$\frac{1}{2\pi i} \int_{L_{Br}} H(z) r^{-z} dz = \frac{1}{2\pi i} \int_{L_{Ha}^-} H(z) r^{-z} dz$$

(3.22)

if $\omega > 0$,

$$\frac{1}{2\pi i} \int_{L_{Br}} H(z) r^{-z} dz = \frac{1}{2\pi i} \int_{L_{Ha}^+} H(z) r^{-z} dz$$

(3.23)

if $\omega < 0$. If $\omega = 0$ then (3.22) and (3.23) hold for $r < \eta$ and $r > \eta$ respectively.

**Proof.** We only prove the case $\omega \geq 0$. The proof of the other case is almost same. Fix $r \in (0, \infty)$ and $L = L_{Br} = \{ z \in \mathbb{C} : \Re[z] = \ell \}$, where $\ell$ satisfies (3.11). Note that the relation

$$|H(\ell + it)^{-r-\ell}| \sim e^{-\omega \eta r} r^{-\ell} |t|^{\omega \ell + \Lambda} \exp \left\{ - \frac{|t| \alpha^*}{2} \right\}$$

holds as $|t| \to \infty$ uniformly in $\ell$ by (3.9). Hence the contour integral

$$\frac{1}{2\pi i} \int_{L_{Br}} H(z) r^{-z} dz$$

absolutely converges and makes sense. Due to Proposition 3.1 we only need to show that

$$\frac{1}{2\pi i} \int_{L_{Br}} H(z) r^{-z} dz = \sum_{k=0}^{\infty} \text{Res}_{z=\hat{a}_k} [H(z) r^{-z}] .$$

(3.24)

Given (sufficiently large) $p \in \mathbb{Z}_+$, we take $M$ as in the proof of Proposition 3.1. Define a closed rectangular contour $C^M$ which can be decomposed into four lines

$$C^M = L_v^M \cup L_v^{-M} \cup L_h^{-M} \cup L_h^M$$

where

$$L_v^M := \{ z \in \mathbb{C} : \Re[z] = \ell, |\Im[z]| \leq M \},$$

$$L_v^{-M} := \{ z \in \mathbb{C} : \Re[z] = -M, |\Im[z]| \leq M \},$$

$$L_h^M := \{ z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = M \},$$

$$L_h^{-M} := \{ z \in \mathbb{C} : -M \leq \Re[z] \leq \ell, \Im[z] = -M \}.$$
Then by the theorem of residues,

\[ \frac{1}{2\pi i} \int_{C_M} \mathcal{H}(z) r^{-z} \, dz = \sum_{k=0}^{P} \text{Res}_{z=\delta_k} [\mathcal{H}(z) r^{-z}] . \] (3.25)

By (3.8), for any \( M \)

\[|\mathcal{H}(t + ih) r^{-t-ih}| \sim \left( \frac{e}{|t|} \right)^{\omega|t|} \left( \frac{r}{\eta} \right)^{|t|^\Lambda}\]

as \( t \to -\infty \). Thus

\[ \lim_{M \to \infty} \int_{L^{-M}} |\mathcal{H}(z) r^{-z}| \, |dz| = \lim_{M \to \infty} \int_{-M}^{M} |\mathcal{H}(-M + ih) r^{-M-ih}| \, dh \]

\[ \leq \lim_{M \to \infty} \int_{-M}^{M} \left( \frac{e}{M} \right)^{\omega M} \left( \frac{r}{\eta} \right)^{M^\Lambda} \, dt \]

\[ \leq \lim_{M \to \infty} 2 \left( \frac{e}{M} \right)^{\omega M} \left( \frac{r}{\eta} \right)^{M^\Lambda+1} = 0. \]

On the other hand, by (3.9).

\[ \lim_{M \to \infty} \int_{L^M} |\mathcal{H}(z) r^{-z}| \, |dz| = \lim_{M \to \infty} \int_{-\ell}^{\ell} |\mathcal{H}(t + iM) r^{-t-iM}| \, dt \]

\[ \leq \lim_{M \to \infty} \int_{-\ell}^{\ell} e^{-\omega t} \left( \frac{\eta}{r} \right)^t M^\omega t^\Lambda \exp \left\{ -\alpha^* M \pi \frac{t}{2} \right\} \, dt \]

\[ = \lim_{M \to \infty} \exp \left\{ -\alpha^* M \pi \frac{t}{2} \right\} M^\Lambda \int_{-\ell}^{\ell} \left( \frac{\eta \omega}{r e^{\omega t}} \right)^t \, dt \]

\[ = \lim_{M \to \infty} \exp \left\{ -\alpha^* M \pi \frac{t}{2} \right\} M^\Lambda \cdot \frac{\left( \eta \omega M \right)^{-M} - \left( \eta \omega M \right)^{-\ell}}{\ln \eta - \ln r + \omega(\ln M - 1)} = 0 \]

since \( \alpha^* > 0, \omega \geq 0, \) and \( r < \eta \). Similarly,

\[ \lim_{M \to \infty} \int_{L^M} |\mathcal{H}(z) r^{-z}| \, |dz| = 0. \]

Thus, by taking \( p \to \infty \) in (3.25),

\[ \sum_{k=0}^{\infty} \text{Res}_{z=\delta_k} [\mathcal{H}(z) r^{-z}] = \lim_{M \to \infty} \frac{1}{2\pi i} \int_{C_M} \mathcal{H}(z) r^{-z} \, dz \]

\[ = \lim_{M \to \infty} \frac{1}{2\pi i} \left( \int_{L^M} + \int_{L^{-M}} + \int_{L_M^H} + \int_{L_M} \right) \]

\[ = \lim_{M \to \infty} \frac{1}{2\pi i} \int_{L^{Br}} \mathcal{H}(z) r^{-z} \, dz = \frac{1}{2\pi i} \int_{L^{Br}} \mathcal{H}(z) r^{-z} \, dz. \]

Therefore (3.21) holds, and the proposition is proved. \( \square \)

**Proposition 3.3** ([11 (2.2.2)]).

\[ \frac{d}{dr} \left\{ \mathcal{H}_{mn}^{(1)} \begin{bmatrix} r \mid \{c, \gamma\} \end{bmatrix} \right\} = -r^{-1} \mathcal{H}_{n+1}^{m+1} \begin{bmatrix} r \mid \{c, \gamma\} \end{bmatrix} \begin{bmatrix} (0, 1) \mid \{b, \delta\} \end{bmatrix}. \]
Proof. Observe that $\omega$ and $\alpha^*$ of $H_{\nu \mu}^{m,n}(r)$ and of

$$H_{\nu \mu}^{m+1,n+1} \quad r \quad [\alpha, \gamma] \quad (0, 1) \quad (1, 1) \quad [\delta, \delta]$$

are same, respectively. Hence $H_{\nu \mu}^{m+1,n+1}(r)$ is well-defined with the same contour used for $H_{\nu \mu}^{m,n}(r)$. Therefore the proposition easily follows from (3.1), (3.7)-(3.9), and the definition of the Fox $H$ function.

3.2. Algebraic asymptotic expansions of $H_{\nu \mu}^{m,n}(r)$ near zero and at infinity.

Proposition 3.1 gives explicit power and power-logarithmic expansion of $H_{\nu \mu}^{m,n}(r)$.

**Proposition 3.4.** Suppose (3.21) holds. Then for sufficiently large $p \in \mathbb{Z}_+$, if $\omega > 0$

$$H_{\nu \mu}^{m,n}(r) = \sum_{k=0}^{p} \text{Res}_{z=\hat{c}_k} \mathcal{H}(r) r^{-z} + O(r^{-M}), \quad \hat{c}_p < M < \hat{c}_{p+1}$$

as $r \to \infty$, and if $\omega < 0$

$$H_{\nu \mu}^{m,n}(r) = -\sum_{k=0}^{p} \text{Res}_{z=\hat{\delta}_k} \mathcal{H}(r) r^{-z} + O(r^M), \quad -\hat{\delta}_p < M < -\hat{\delta}_{p+1}$$

as $r \to 0$.

**Proof.** Braaksma [1] proved (3.26) for $\omega \geq 0$. We follow Braaksma’s method to prove (3.27) for the case $\omega < 0$.

Let $\omega < 0$. Given (sufficiently large) $p \in \mathbb{Z}_+$, take a constant $M > 0$ satisfying (3.14). Fix $h > 0$ and let $L^+_H$ be a right Hankel contour surrounding $L^+_H$. Precisely, $L^+_H$ is a right loop situated in a horizontal strip runs from $\infty - ih$ to $-M - ih$ and then to $-M + ih$ and finally terminating at the point $\infty + ih$. Let us denote by $C$ a closed rectangular contour which encircles $\hat{\delta}_0, \ldots, \hat{\delta}_p$ and satisfies

$$\int_{L^+_H} \mathcal{H}(z) r^{-z} dz + \int_C \mathcal{H}(z) r^{-z} dz = \int_{L^-_H} \mathcal{H}(z) r^{-z} dz.$$

Then

$$H_{\nu \mu}^{m,n}(r) = \frac{1}{2\pi i} \int_{L^+_H} \mathcal{H}(z) r^{-z} dz$$

$$= \frac{1}{2\pi i} \left( -\int_C \mathcal{H}(z) r^{-z} dz + \int_{L^-_H} \mathcal{H}(z) r^{-z} dz \right)$$

$$= -\sum_{k=0}^{p} \text{Res}_{z=\hat{\delta}_k} \mathcal{H}(z) r^{-z} + \frac{1}{2\pi i} \int_{L^-_H} \mathcal{H}(z) r^{-z} dz. \quad (3.28)$$

Using (3.7), (3.8), and (3.21), and modifying the proof of Proposition 3.2,

$$\frac{1}{2\pi i} \int_{L^-_H} \mathcal{H}(z) r^{-z} dz = \sum_{k=0}^{p} \text{Res}_{z=\hat{c}_k} \mathcal{H}(z) r^{-z} + \sum_{k=0}^{\infty} \text{Res}_{z=\hat{c}_k} \mathcal{H}(z) r^{-z}$$

$$= \frac{1}{2\pi i} \int_{L^-_H} \mathcal{H}(z) r^{-z} dz,$$
where \( L_{-M}^{M} \) is a Bromwich contour starting at the point \(-M - i\infty\) and terminating at the point \(-M + i\infty\).

We estimate the upper bound of contour integral along \( L_{-M}^{M} \). Recall (3.9):

\[
|H(-M + it)r^{M-it}| \sim e^{\omega M \eta^{-M} r^{M}|t|^{-\omega M + \Lambda} \exp \left\{ -|t|\alpha^* \frac{2}{2\pi} \right\}}
\]
as \( t \to \infty \). Thus, for \( r \leq 1 \),

\[
\left| \frac{1}{2\pi i} \int_{L_{-M}^{M}} H(z) r^{-z} dz \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(-M + it)r^{M-it}| dt \leq r^{M} \left( 1 + \frac{\omega M}{\eta^{M}} \int_{|t| \geq 1} |t|^{-\omega M + \Lambda} \exp \left\{ -|t|\alpha^* \frac{2}{2\pi} \right\}} dt \right) \leq r^{M}.
\]
The proposition is proved. \( \Box \)

If \( \omega > 0 \) and \( \gamma = 0 \) (i.e.) \( P_2 = \emptyset \), we have the following exponentially asymptotic behavior of \( H^{0\mu}_{\nu \mu}(r) \) (see [14, (1.7.13)]). For the proof we refer the reader to [4, Theorem 4].

**Proposition 3.5.** Assume (3.21) and \( \omega > 0 \). Then

\[
H^{0\mu}_{\nu \mu}(r) = O \left( r^{(\Lambda + \frac{1}{2})/\omega} \exp \left\{ \cos \left( \frac{\alpha^* + \sum_{j=m+1}^{\mu} \delta_j}{\omega} \pi \right) \omega \left( \frac{r}{\eta} \right)^{1/\omega} \right\}} \right)
\]
as \( r \to \infty \).

4. **Asymptotic estimates of the Fox H function**

Throughout this section we fix

\( d \in \mathbb{N}, \quad \alpha \in (0, 2), \quad \beta \in (0, \infty) \).

For \( \gamma \in [0, \infty), \sigma \in \mathbb{R} \) and \( z \in \mathbb{C} \) define

\[
H_{\sigma, \gamma}(z) := \frac{\Gamma \left( \frac{d}{2} + \gamma + \beta z \right) \Gamma(1 + z) \Gamma(-z)}{\Gamma(-\gamma - \beta z) \Gamma(1 - \sigma + \alpha z)}.
\]
Observe that

\[
\alpha^* = 2 - \alpha, \quad \Lambda = \frac{d}{2} + 2\gamma + \sigma - \frac{1}{2}, \quad \omega = 2\beta - \alpha, \quad \eta = \alpha^{-\alpha} \beta^{2\beta}.
\]

For each \( k \in \mathbb{Z}_+ \) we write

\[
\epsilon_{1,k} := k, \quad \delta_{1,k} := -\frac{d}{2} + \gamma + \frac{k}{\beta}, \quad \delta_{2,k} := -1 - k.
\]

Obviously, (3.5) holds, and thus \( \delta_{1,k}, \delta_{2,k} \) and \( \epsilon_{1,k} \) constitute \( P_1 \) and \( P_2 \) respectively.

**Remark 4.1.** (i) Note that \( H_{\sigma, \gamma}(z) \) has removable singularities at \( z = 0 \) and at \( z = -1 \) if \( \gamma = 0 \) and \( \gamma = \beta \), respectively. Indeed, by (3.1),

\[
H_{\sigma, 0}(z) = \frac{\beta \Gamma \left( \frac{d}{2} + \beta z \right) \Gamma(1 + z) \Gamma(1 - z)}{\Gamma(-\beta z) \Gamma(1 - \sigma + \alpha z)}.
\]
Therefore, $H_{\sigma,0}(z)$ has a removable singularity at $z = 0$. Similarly,
\[
H_{\sigma,\beta}(z) = -\frac{\beta \Gamma\left(\frac{d}{2} + \beta + \beta z\right) \Gamma(2 + z) \Gamma(-z)}{\Gamma(1 - \beta - \beta z) \Gamma(1 - \sigma + \alpha z)}.
\]
Thus, $H_{\sigma,\beta}(z)$ has a removable singularity at $z = 0$.

(ii) Assume that $\beta \in \mathbb{N}$ and $\gamma = 0$, then by (3.2),
\[
H_{\sigma,0}(z) = (2\pi)^{\frac{d}{2}} \frac{\beta - 1}{\prod_{k=1}^{\beta-1} \Gamma\left(k \beta - z\right) \Gamma(1 - \sigma + \alpha z)}.
\]
Hence $P_2 = \emptyset$.

(iii) If $\alpha = 1$ and $\sigma = 0$, then $H_{0,\gamma}(z) := \Gamma\left(\frac{d}{2} + \gamma + \beta z\right) \Gamma(-z) \Gamma(1 - \gamma - \beta z)$.

Thus, $d_{2,k} = 0$ for all $k \in \mathbb{Z}_+$.

For $r \in (0, \infty)$, we define
\[
\mathbb{H}_{\sigma,\gamma}(r) := \mathbb{H}_{23}^{21} \left[ \begin{array}{ccc} r & (1, 1) & (1 - \sigma, \alpha) \\ \frac{d}{2} + \gamma + \beta & (1, 1) & (1 + \gamma, \beta) \end{array} \right] = \frac{1}{2\pi i} \int_L H_{\sigma,\gamma}(z) r^{-z} dz.
\]
Here
\[
L = L_{B_r} = \{ z \in \mathbb{C} : \Re[z] = \ell_0 \}
\]
and $\ell_0$ is chosen to satisfy (4.1):
\[
\max\left(-1, -\frac{\gamma}{\beta} - \frac{d}{2\beta}\right) < \ell_0 < 0
\]
if $\gamma \notin \{0, \beta\}$,
\[
\max\left(-1, -\frac{d}{2\beta}\right) < \ell_0 < 1
\]
if $\gamma = 0$,
\[
\max\left(-2, -1 - \frac{d}{2\beta}\right) < \ell_0 < 0
\]
if $\gamma = \beta$. If $\alpha = 1$ and $\sigma = 0$, then we take $\ell_0$ such that
\[
-\frac{\gamma}{\beta} - \frac{d}{2\beta} < \ell_0 < 0.
\]
Since (3.21) holds (i.e. $\alpha < 2$), by Propositions 3.1 and 3.2 the value of $\mathbb{H}_{\sigma,\gamma}(r)$ is independent of the choice of $\ell_0$ as long as it is chosen as above.

By Proposition 3.4 we obtain the asymptotic behaviors of $\mathbb{H}_{\sigma,\gamma}(r)$ at infinity.

**Lemma 4.2.** It holds that
\[
\mathbb{H}_{\sigma,\gamma}(r) = -\frac{\Gamma\left(\frac{d}{2} + \gamma\right)}{\Gamma(-\gamma) \Gamma(1 - \sigma)} + O(r^{-1})
\]
for $r \geq 1$. In particular, if $\gamma \in \mathbb{Z}_+$ or $\sigma \in \mathbb{N}$, then
\[
\mathbb{H}_{\sigma,\gamma}(r) = \frac{\Gamma\left(\frac{d}{2} + \gamma + \beta\right)}{\Gamma(-\gamma - \beta) \Gamma(1 - \sigma + \alpha)} r^{-1} + O(r^{-2})
\]
for $r \geq 1$. If $\beta \in \mathbb{N}$ and $\gamma = 0$, then there exists a constant $c = c(d, \alpha, \beta, \sigma) > 0$ such that
\[
\mathbb{H}_{\sigma, 0}(r) = O \left( \exp \left\{ -cr^{1/(2\beta - \alpha)} \right\} \right)
\]
as $r \to \infty$.

Proof. Observe that $\mathcal{H}_{\sigma, \gamma}(z)$ has simple poles at $z = c_{1,k} = k$ for all $k \in \mathbb{Z}_{+}$. By (3.20), for sufficiently large $p \in \mathbb{Z}_{+}$,
\[
\mathbb{H}_{\sigma, \gamma}(r) = \sum_{k=0}^{p} \text{Res}_{z=c_{1,k}} \left[ \mathcal{H}_{\sigma, \gamma}(z) r^{-z} \right] + \sum_{k=0}^{p} \text{Res}_{z=c_{1,k}} \left[ \mathcal{H}_{\sigma, \gamma}(z) r^{-z} \right] + O(r^{-\epsilon})
\]
for $r \geq 1$. Additionally, if $\gamma \in \mathbb{Z}_{+}$ or $\sigma \in \mathbb{N}$, then
\[
\text{Res}_{z=0} \left[ \mathcal{H}_{\sigma, \gamma}(z) \right] = \lim_{z \to 0} \left( \frac{z \Gamma \left( \frac{d}{2} + \gamma + \beta z \right) \Gamma(1 + z) \Gamma(-z)}{\Gamma(-\gamma - \beta z) \Gamma(1 - \sigma + \alpha z)} \right)
\]
\[
= \frac{\Gamma \left( \frac{d}{2} + \gamma + \beta \right) \Gamma(1 + z) \Gamma(-z)}{\Gamma(-\gamma - \beta z) \Gamma(1 - \sigma + \alpha z)} = 0.
\]
Hence
\[
\mathbb{H}_{\sigma, \gamma}(r) = \text{Res}_{z=1} [\mathcal{H}_{\sigma, \gamma}(z)] r^{-1} + O(r^{-2}) = \frac{\Gamma \left( \frac{d}{2} + \gamma + \beta \right) \Gamma(1 + z) \Gamma(-z)}{\Gamma(-\gamma - \beta z) \Gamma(1 - \sigma + \alpha z)} r^{-1} + O(r^{-2}).
\]
Finally, assume $\beta \in \mathbb{N}$ and $\gamma = 0$. By (4.1),
\[
\mathbb{H}_{\sigma, 0}(r) = \mathbb{H}_{23}^{21} \left[ \frac{1}{2} \left( 1, 1 \right) \frac{1}{2} \left( 1 - \sigma, \alpha \right) \frac{1}{2} \left( 1, 1 \right) \frac{1}{2} \left( 1, \beta \right) \frac{1}{2} \left( 1, \beta \right) \right]
\]
\[
= (2\pi)^{\frac{d-1}{2}} \beta^{1/2} \frac{1}{2\pi} \int_{L} \prod_{k=1}^{d-1} \Gamma \left( \frac{k}{d} - z \right) \Gamma(1 - \sigma + \alpha z) \left( \frac{r}{\beta^{1/2}} \right)^{-z} \frac{dz}{z}
\]
\[
= (2\pi)^{\frac{d-1}{2}} \beta^{1/2} \mathbb{H}_{1, \beta+1}^{20} \left[ \frac{1}{2} \left( 1, \sigma, \alpha \right) \frac{1}{2} \left( \frac{d}{2}, \beta \right) \frac{1}{2} \left( 1, 1 \right) \frac{1}{2} \left( 1, 1 \right) \frac{1}{2} \left( 1 - \sigma, \alpha \right) \frac{1}{2} \left( 1, \beta \right) \frac{1}{2} \left( 1 - \sigma, \alpha \right) \frac{1}{2} \left( 1, \beta \right) \frac{1}{2} \left( 1, \beta \right) \frac{1}{2} \left( 1, \beta \right) \right].
\]
For the above Fox H function, we have
\[
\alpha^{*} = 2 - \alpha, \quad \Lambda = \frac{d}{2} + \sigma + 2\beta - \frac{3}{2}, \quad \omega = 2\beta - \alpha, \quad \eta = \alpha^{*} \beta^{2}.
\]
Note that
\[
\frac{1}{2} < \frac{\beta + 1 - \alpha}{2\beta - \alpha} \leq 1, \quad 2\beta - \alpha \geq 2 - \alpha > 0.
\]
Hence by Proposition 3.5
\[
\mathbb{H}_{1, \beta+1}^{20} \left( \frac{r}{\beta^{1/2}} \right) = O \left( \left( \frac{r}{\beta^{1/2}} \right)^{\left( \Lambda + \frac{1}{2} \right)/(2\beta - \alpha)} \exp \left\{ \cos \left( \frac{\beta + 1 - \alpha}{2\beta - \alpha} \pi \right) \left( 2\beta - \alpha \right) \left( \frac{r}{\eta \beta^{1/2}} \right)^{1/(2\beta - \alpha)} \right\} \right)
\]
\[
= O \left( \exp \left\{ -cr^{1/(2\beta - \alpha)} \right\} \right).
\]
The lemma is proved. \qed
Next we consider the asymptotic behavior of \( H_{\sigma, \gamma}(r) \) at zero.

**Lemma 4.3.** It holds that

\[
H_{\sigma, \gamma}(r) \sim \begin{cases} 
    r^{\frac{d + \gamma}{2}} : \gamma < \beta - \frac{d}{2} \\
    r |\ln r| : \gamma = \beta - \frac{d}{2} \\
    r : \gamma > \beta - \frac{d}{2}
\end{cases}
\]

as \( r \to 0 \). Additionally, if \( \gamma - \beta \in \mathbb{Z}_+ \) or \( \sigma + \alpha \in \mathbb{N} \), then

\[
H_{\sigma, \gamma}(r) \sim \begin{cases} 
    r^{\frac{d + \gamma}{2}} : \gamma < 2\beta - \frac{d}{2} \\
    r^2 |\ln r| : \gamma = 2\beta - \frac{d}{2} \\
    r^2 : \gamma > 2\beta - \frac{d}{2}
\end{cases}
\]

as \( r \to 0 \). If \( \alpha = 1 \) and \( \sigma = 0 \), then

\[ H_{0, \gamma}(r) \sim r^{\frac{d + \gamma}{2}} \]

as \( r \to 0 \).

**Proof.** Due to (3.27), it is sufficient to compare the order of residues among \( z = \vartheta_{1,0}, \vartheta_{1,1}, \vartheta_{2,0}, \) and \( \vartheta_{2,1} \).

First, let \( \gamma \neq \beta - \frac{d}{2} \). Then \( H_{\sigma, \gamma}(z) \) has a simple pole at \( z = \max\{\vartheta_{1,0}, \vartheta_{2,0}\} \). If \( \gamma > \beta - \frac{d}{2} \), then \( \vartheta_{1,0} < \vartheta_{2,0} \) so by (3.27)

\[
H_{\sigma, \gamma}(r) = \sum_{k=0}^{1} \text{Res}_{z=\vartheta_{2,k}} [H_{\sigma, \gamma}(z)r^{-z}] + O(r^{-\vartheta_{1,1}}) \sim r
\]

as \( r \to 0 \). Similarly, if \( \gamma < \beta - \frac{d}{2} \), then

\[
H_{\sigma, \gamma}(r) = \text{Res}_{z=\vartheta_{1,0}} [H_{\sigma, \gamma}(z)r^{-z}] + O(r^{-\vartheta_{1,0}}) \sim r^{\frac{d + \gamma}{2}}
\]

as \( r \to 0 \).

Next, assume \( \gamma = \beta - \frac{d}{2} \) (i.e. \( \vartheta_{1,0} = \vartheta_{2,0} \)). Then \( H_{\sigma, \gamma}(z) \) has a pole of order 2 at \( z = \vartheta_{0} = \vartheta_{1,0} = \vartheta_{2,0} \) so that

\[
\text{Res}_{z=\vartheta_{0}} [H_{\sigma, \gamma}(z)r^{-z}] = \lim_{z \to \vartheta_{0}} \frac{d}{dz} ((z + 1)^2 H_{\sigma, \gamma}(z)r^{-z})
\]

\[
= \left( \text{Res}_{z=-1} [H_{\sigma, \gamma}(z)] + \frac{|\ln r|}{\Gamma(\frac{d}{2})(1 - \sigma - \alpha)} \right) r.
\]

Then by (3.27), we obtain the first desired result.

Now we assume \( \gamma - \beta \in \mathbb{Z}_+ \) or \( \sigma + \alpha \in \mathbb{N} \). Then one can easily see that \( \text{Res}_{z=\vartheta_{2,0}} [H_{\sigma, \gamma}(z)r^{-z}] = 0 \). Hence it remains to compare the order of residues at \( z = \vartheta_{1,0}, \vartheta_{1,1}, \) and \( \vartheta_{2,1} \). Following the same argument of the above, we obtain the additional result.

Finally, we assume \( \alpha = 1 \) and \( \sigma = 0 \). Recall (4.2), and note \( \vartheta_{2,k} = 0 \) for all \( k \in \mathbb{Z}_+ \) which implies

\[
H_{0, \gamma}(r) = \sum_{k=0}^{\infty} \text{Res}_{z=\vartheta_{1,k}} [H_{0, \gamma}(z)r^{-z}] \sim r^{\frac{d + \gamma}{2}}
\]

as \( r \to 0 \). The lemma is proved. \( \square \)
For each $q \in \mathbb{Z}_+$ we define

$$\mathcal{H}_{\sigma, \gamma}^{(q)}(z) := \mathcal{H}_{\sigma, \gamma}(z) \left\{ \frac{\Gamma(1 + z)}{\Gamma(z)} \right\}^q$$

and

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) := \frac{1}{2\pi i} \int_L \mathcal{H}_{\sigma, \gamma}^{(q)}(z) r^{-z} dz. \quad (4.4)$$

Note that $\mathbb{H}_{\sigma, \gamma}^{(0)}(r) = \mathbb{H}_{\sigma, \gamma}(r)$ and by Proposition 3.3

$$\frac{d}{dr} \mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \mathbb{H}_{\sigma, \gamma}^{(q+1)}(r)$$

holds and (4.4) is well-defined for each $q \in \mathbb{Z}_+$. By 3.1

$$\mathcal{H}_{\sigma, \gamma}^{(q)}(z) = -\frac{\Gamma\left(\frac{d}{2} + \gamma + \beta z\right) \Gamma(1 - z) \Gamma(1 + z)}{\Gamma(-\gamma - \beta z) \Gamma(1 - \sigma + \alpha z)} z^{-q-1},$$

and thus $\mathcal{H}_{\sigma, \gamma}^{(q)}(z)$ does not have a pole at $z = 0$ if $q \geq 1$. Furthermore, $\mathcal{H}_{\sigma, \gamma}^{(q)}$ has a pole at $z = -k - 1$ of order at most 2 for each $k \in \mathbb{Z}_+$. Let us denote by

$$c_{1,k} := k + 1, \quad \nu_{1,k} := -\frac{d}{\beta} + \gamma + k, \quad \nu_{2,k} := -1 - k$$

the new elements of $P_1$ and $P_2$ for $\mathbb{H}_{\sigma, \gamma}^{(q)}(r)$. Then due to 3.20 again, we obtain an analogue of Lemma 4.2

**Lemma 4.4.** Let $q \in \mathbb{N}$. It holds that

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) \sim r^{-1}$$

as $r \to \infty$. Additionally, if $\beta \in \mathbb{N}$ and $\gamma = 0$, then there exists a constant $c = c(d, \alpha, \beta, \sigma, |q|)$ such that

$$\mathbb{H}_{\sigma, 0}^{(q)}(r) = O\left(\exp\left\{-cr^{1/(2\beta - \alpha)}\right\}\right)$$

as $r \to \infty$.

**Proof.** The proof is similar to the one of Lemma 4.2. The only difference is $\tilde{c}_k = c_{1,k} = k + 1$ which implies

$$\text{Res}_{z=\tilde{c}_k} \left[ \mathcal{H}_{\sigma, \gamma}^{(q)}(z) r^{-z} \right] = \frac{(-1)^k \cdot \Gamma\left(\frac{d}{2} + \gamma + \beta k\right)}{\Gamma(-\gamma - \beta k) \Gamma(1 - \sigma + \alpha k)} (k + 1)^q r^{-k-1}$$

for each $k \in \mathbb{Z}_+$. The lemma is proved. \hfill \Box

Lastly, we present another result which is necessary to obtain the upper estimates of classical derivatives of $p(t,x)$. Recall $\omega = 2\beta - \alpha$. Let $\kappa_1 = \kappa_1(d, \alpha, \beta, \gamma, \sigma)$ and $\kappa_2 = \kappa_2(d, \alpha, \beta, \sigma)$ denote constants

$$\kappa_1 := \frac{\Gamma\left(\frac{d+2\gamma}{2\beta}\right) \Gamma(1 - \sigma - \frac{d+2\gamma}{2\beta})}{\Gamma\left(\frac{d}{2}\right) \Gamma(1 - \sigma - \frac{d+2\gamma}{2\beta})}, \quad \kappa_2 := \lim_{z \to -1} \frac{(z+1) \Gamma\left(\frac{d}{2} + \gamma + \beta z\right)}{\Gamma(-\gamma + \beta) \Gamma(1 - \sigma - \alpha k)}.$$

**Lemma 4.5.** Let $q \in \mathbb{Z}_+$ and $\gamma \in [0, \infty)$.

(i) If $\gamma \notin \left\{ -\frac{d}{2}, -\frac{d}{2} - 1 \right\}$, then

$$\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left(-\frac{d + 2\gamma}{2\beta}\right)^q \cdot \kappa_1 r^{\frac{d+2\gamma}{2\beta}} + O(r^{\min\left(1, \frac{d+2\gamma}{2\beta}\right)}), \quad \left(0, 0 := 1\right)$$
as \( r \to 0 \). Additionally, if \( \gamma = \beta \) or \( \sigma + \alpha \in \mathbb{N} \), then

\[
\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left( -\frac{d + 2\gamma}{2\beta} \right)^q \cdot \kappa_1 r^{\frac{d + 2\gamma}{2\beta}} + O(r^{\min(2, \frac{d + 2\gamma}{2\beta})}) \quad \text{as} \quad r \to 0
\]

If \( \alpha = 1 \) and \( \sigma = 0 \), then

\[
\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left( -\frac{d + 2\gamma}{2\beta} \right)^q \cdot \kappa_2 r \ln r + O(r)
\]

\( \text{as} \quad r \to 0 \).

(ii) If \( \gamma = \beta - \frac{d}{2} \), then

\[
\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \left( -\frac{d + 2\gamma}{2\beta} \right)^q \cdot \kappa_2 r \ln r + O(r)
\]

\( \text{as} \quad r \to 0 \).

(iii) If \( \gamma = \beta - \frac{d}{2} - 1 \), then

\[
\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = \frac{\omega}{|\omega|} \cdot \left( -\frac{d + 2\gamma}{2\beta} \right)^q \cdot \kappa_2 r \ln r + O(r \ln r)
\]

\( \text{as} \quad r \to 0 \).

Proof. Due to Proposition 3.1 one can easily see our assertions hold if \( \omega \geq 0 \). Hence, we assume \( \omega < 0 \). Recall (3.27) and take \( M \geq 2 \) satisfying (3.11). If \( \gamma \neq \beta - \frac{d}{2} \) and \( \gamma \neq \beta - \frac{d}{2} - 1 \), then \( \mathfrak{d}_{1,0} \neq \mathfrak{d}_{2,0} \) and \( \mathfrak{d}_{1,1} \neq \mathfrak{d}_{2,1} \) and thus \( \mathcal{H}_{\sigma, \gamma}^{(q)}(z) \) has a simple pole at \( z = \mathfrak{d}_{2,0} = -1 \).

Hence

\[
\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = -\sum_{j=1}^{2} \operatorname{Res}_{z = \mathfrak{d}_{j,0}} \left[ \mathcal{H}_{\sigma, \gamma}^{(q)}(z) r^{-z} \right] - \operatorname{Res}_{z = \mathfrak{d}_{2,1}} \left[ \mathcal{H}_{\sigma, \gamma}^{(q)}(z) r^{-z} \right] + O(r^M)
\]

\[
= -\left( \operatorname{Res}_{z = -\frac{4\beta + 2\gamma}{2\beta}} \cdots + \operatorname{Res}_{z = -1} \cdots + \operatorname{Res}_{z = -2} \cdots \right) + O(r^{\min(2, \frac{d + 2\gamma}{2\beta})})
\]

\[
= -\left( -\frac{d + 2\gamma}{2\beta} \right)^q \kappa_1 r^{\frac{d + 2\gamma}{2\beta}} \cdot \frac{1}{\Gamma(-\gamma + \beta) \Gamma(1 - \sigma - \alpha)} r - \operatorname{Res}_{z = -2} \left[ \mathcal{H}_{\sigma, \gamma}^{(q)}(z) r^{-z} \right] + O(r^{\min(2, \frac{d + 2\gamma}{2\beta})})
\]

\( \text{as} \quad r \to 0. \) Note that the second term of (4.5) vanishes if \( \gamma = \beta \) or \( \sigma + \alpha \in \mathbb{N} \). Furthermore, if \( \alpha = 1 \) and \( \sigma = 0 \), then \( \mathfrak{d}_{2,k} = 0 \) for all \( k \in \mathbb{Z}_+ \). Therefore (i) is proved.

Suppose \( \gamma = \beta - \frac{d}{2} \). Then \( \mathfrak{d}_0 = \mathfrak{d}_{1,0} = \mathfrak{d}_{2,0} \) and \( \mathcal{H}_{\sigma, \gamma}^{(q)}(z) \) has a pole at

\[
z = \mathfrak{d}_0 = -\frac{d}{2} = -1
\]

of order 2 and

\[
\operatorname{Res}_{z = -1} \left[ \mathcal{H}_{\sigma, \gamma}^{(q)}(z) r^{-z} \right] = (-1)^q \cdot \kappa_2 r \ln r + \operatorname{Res}_{z = -1} \left[ \mathcal{H}_{\sigma, \gamma}^{(q)}(z) \right] r.
\]

(4.6)

Therefore,

\[
\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = -\operatorname{Res}_{z = -1} \left[ \mathcal{H}_{\sigma, \gamma}^{(q)}(z) r^{-z} \right] + O(r)
\]

\[
= (-1)^q \cdot \kappa_2 r \ln r + O(r)
\]
as \( r \to 0 \). Therefore (ii) is proved.

Finally, we let \( \gamma = \beta - \frac{d}{2} - 1 \). Then \( \partial_{1,1} = \partial_{2,0} \) and \( H_{\sigma, \gamma}^{(q)}(z) \) has a pole at \( z = \partial_{1,1} = - \frac{d + 2\gamma + 2}{2\beta} = -1 \) of order 2, and \( (4.6) \) holds. Note that \( H_{\sigma, \gamma}^{(q)}(z) \) has a simple pole at \( z = \partial_{1,0} = - \frac{d + 2\gamma + 2}{2\beta} \). Therefore,

\[
\mathbb{H}_{\sigma, \gamma}^{(q)}(r) = -\text{Res}_{z=-\frac{d+2\gamma}{2\beta}} \left[ H_{\sigma, \gamma}^{(q)}(z)r^{-z} \right] - \text{Res}_{z=-1} \left[ H_{\sigma, \gamma}^{(q)}(z)r^{-z} \right] + O(r)
\]

\[
= - \left( -\frac{d + 2\gamma}{2\beta} \right) ^q \kappa_1 r^{\frac{d+2\gamma}{2\beta}} - (-1)^q \cdot \kappa_2 r \ln r + O(r)
\]
as \( r \to 0 \). The lemma is proved. \( \square \)

5. Explicit representation of \( p(t, x) \) and its derivatives

Take the function \( \mathbb{H}_{\sigma, \gamma}(r) \) from (4.3), and define

\[
p_{\sigma, \gamma}(t, x) := \frac{2^{2\gamma}}{\pi^{d/2}} |x|^{-d - 2\gamma} t^{-\sigma} \mathbb{H}_{\sigma, \gamma} \left( \frac{|x|^{2\beta}}{2\beta} t^{-\alpha} \right), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d.
\]

Write

\[
p(t, x) := p_{0,0}(t, x) = \frac{|x|^{-d}}{\pi^{d/2}} \mathbb{H}_{0,0} \left( \frac{|x|^{2\beta}}{2\beta} t^{-\alpha} \right). \quad (5.1)
\]

Note that \( \mathbb{H}_{\sigma, \gamma}(r) \) is a bounded function of \( r \in (0, \infty) \). As a corollary of Lemmas (4.2) and (4.3) we obtain the following upper estimates of \( p_{\sigma, \gamma}(t, x) \) for \( \gamma \in [0, \infty) \) and \( \sigma \in \mathbb{R} \).

**Theorem 5.1.** Let \( \alpha \in (0, 2), \beta \in (0, \infty), \gamma \in [0, \infty), \) and \( \sigma \in \mathbb{R} \). Then for \( |x|^{2\beta} t^{-\alpha} \geq 1 \)

\[
|p_{\sigma, \gamma}(t, x)| \lesssim |x|^{-d - 2\gamma} t^{-\sigma}
\]

and for \( |x|^{2\beta} t^{-\alpha} \leq 1 \)

\[
|p_{\sigma, \gamma}(t, x)| \lesssim \begin{cases} t^{-\sigma - \frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma < \beta - \frac{d}{2} \\ |x|^{-d - 2\gamma + 2\beta} t^{-\sigma - \alpha} (1 + |\ln (|x|^{2\beta} t^{-\alpha})|) & : \gamma = \beta - \frac{d}{2} \\ |x|^{-d - 2\gamma + 2\beta} t^{-\sigma - \alpha} & : \gamma > \beta - \frac{d}{2} \end{cases}
\]

Furthermore,

(i) If \( \sigma + \alpha \in \mathbb{N} \), then for \( |x|^{2\beta} t^{-\alpha} \leq 1 \)

\[
|p_{\sigma, \gamma}(t, x)| \lesssim \begin{cases} t^{-\sigma - \frac{\alpha(d+2\gamma)}{2\beta}} & : \gamma < 2\beta - \frac{d}{2} \\ |x|^{-d - 2\gamma + 4\beta} t^{-\sigma - 2\alpha} (1 + |\ln (|x|^{2\beta} t^{-\alpha})|) & : \gamma = 2\beta - \frac{d}{2} \\ |x|^{-d - 2\gamma + 4\beta} t^{-\sigma - 2\alpha} & : \gamma > 2\beta - \frac{d}{2} \end{cases}
\]

(ii) If \( \alpha = 1 \) and \( \sigma = 0 \), then for \( |x|^{2\beta} t^{-\alpha} \leq 1 \)

\[
|p_{\sigma, \gamma}(t, x)| \lesssim t^{-\frac{d+2\gamma}{2\beta}}.
\]

(iii) If \( \gamma \in \mathbb{Z}_+ \), then for \( |x|^{2\beta} t^{-\alpha} \geq 1 \)

\[
|p_{\sigma, \gamma}(t, x)| \lesssim |x|^{-d - 2\gamma - 2\beta} t^{-\sigma + \alpha}.
\]
(iv) If $\beta \in \mathbb{N}$ and $\gamma = 0$, then there exists a constant $c = c(\alpha, \beta, \sigma) > 0$ such that for $|x|^{2\beta} t^{-\alpha} \geq 1$,
\[
|p_{\sigma,0}(t,x)| \leq |x|^{-d-\sigma} \exp \left\{ -|t^{-\alpha}| |x|^{2\beta} \right\}.
\]

(v) If $\gamma = \beta$, then for $|x|^{2\beta} t^{-\alpha} \geq 1$
\[
|p_{\sigma,\beta}(t,x)| \leq \begin{cases} 
|x|^{-d+4\beta} t^{-\sigma+\alpha} & : \sigma \in \mathbb{N} \\
|x|^{-d+2\beta} t^{-\sigma-2\alpha} & : \sigma \in \mathbb{R} \setminus \mathbb{N}
\end{cases}
\]
and for $|x|^{2\beta} t^{-\alpha} \leq 1$
\[
|p_{\sigma,\beta}(t,x)| \leq \begin{cases} 
|t^{-\sigma-\alpha-\frac{d\beta}{\sigma}}| & : \frac{d}{2} < \beta \\
|x|^{-d+2\beta} t^{-\sigma-2\alpha} \left( 1 + |\ln (|x|^{2\beta} t^{-\alpha})| \right) & : \frac{d}{2} = \beta \\
|x|^{-d+2\beta} t^{-\sigma-2\alpha} & : \frac{d}{2} > \beta.
\end{cases}
\]

**Remark 5.2.** (i) By Theorem 5.1, $p_{\sigma,\gamma}(t,\cdot) \in L_1(\mathbb{R}^d)$ for all $\gamma \in [0, \beta]$ and $\sigma \in \mathbb{R}$. Furthermore, if $\alpha = 1$ and $\sigma = 0$, then $p_{0,\gamma}(t,\cdot) \in L_1(\mathbb{R}^d)$ for all $\gamma \in [0, \infty)$.
(ii) Observe that
\[
p_{\sigma,\gamma}(t,x) = t^{-\sigma-\frac{\alpha(d+2\gamma)}{\alpha d}} p_{\sigma,\gamma}(1,t^{-\frac{\alpha d}{\sigma}} x) 
\] (5.2)
which implies
\[
\int_0^T \int_{\mathbb{R}^d} |p_{\sigma,\gamma}(t,x)| \, dx \, dt = \int_0^T t^{-\sigma-\frac{\alpha d}{\sigma}} \left( \int_{\mathbb{R}^d} |p_{\sigma,\gamma}(1,x)| \, dx \right) \, dt < \infty
\]
if $\sigma + \frac{\alpha \gamma}{\sigma} < 1$. Thus, under this condition, one can consider Riemann-Liouville fractional integral and the Fourier-Laplace transform of $p_{\sigma,\gamma}(t,x)$. However we do not use such transforms in this article.

The following theorem handles the interchangeability of $\Delta^\gamma$ and $D_t^\eta$.

**Theorem 5.3.** Let $\gamma \in [0, \infty)$. For any $\sigma \in \mathbb{R}$, $m \in \mathbb{N}$, and $\eta \in (-\infty, 1)$,
\[
D_t^m p_{\sigma,\gamma}(t,x) = p_{\sigma+m,\gamma}(t,x),
\]
and
\[
D_t^\eta p_{\sigma,\gamma}(t,x) = p_{\sigma+\eta,\gamma}(t,x).
\]

**Proof.** First we show
\[
\frac{\partial}{\partial t} p_{\sigma,\gamma}(t,x) = p_{\sigma+1,\gamma}(t,x) 
\] (5.3)
for any $\sigma \in \mathbb{R}$. By (3.1),
\[
\frac{\alpha z}{\Gamma(1-\sigma+\alpha z)} - \frac{1}{\Gamma(-\sigma+\alpha z)} + \frac{\sigma}{\Gamma(1-\sigma+\alpha z)}
\]
which implies
\[
\mathcal{H}_{\sigma,\gamma}(z) \alpha z = \mathcal{H}_{\sigma+1,\gamma}(z) + \sigma \mathcal{H}_{\sigma,\gamma}(z).
\]
Then by Proposition 3.3,

\[
\frac{\partial}{\partial t} \mathbb{H}_{\sigma, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) = \frac{t^{-1}}{2\pi i} \int_L \mathcal{H}_{\sigma, \gamma}(z) \alpha z \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right)^{-z} dz
\]

\[
= \frac{t^{-1}}{2\pi i} \int_L (\mathcal{H}_{1+\sigma, \gamma}(z) + \sigma \mathcal{H}_{\sigma, \gamma}(z)) \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right)^{-z} dz
\]

\[
= t^{-1} \left[ \mathbb{H}_{\sigma+1, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) + \sigma \mathbb{H}_{\sigma, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) \right].
\]

Therefore,

\[
\frac{\partial}{\partial t} p_{\sigma, \gamma}(t, x) = \frac{2^{2\gamma}}{\pi^{d/2}} |x|^{-d-2\gamma} t^{-\sigma} \left( \frac{\partial}{\partial t} \mathbb{H}_{\sigma, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) - \sigma t^{-1} \mathbb{H}_{\sigma, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right) \right)
\]

\[
= \frac{2^{2\gamma}}{\pi^{d/2}} |x|^{-d-2\gamma} t^{-\sigma-1} \mathbb{H}_{\sigma+1, \gamma} \left( \frac{|x|^{2\beta}}{2^{2\beta}} t^{-\alpha} \right)
\]

\[
= p_{\sigma+1, \gamma}(t, x),
\]

and (5.3) is proved. Thus to complete the proof of the theorem, due to (2.2) and (5.3), it is sufficient to show that for \(\sigma < 0\) and \(\eta < 1\)

\[
I_t^{|\sigma|} p_{\eta, \gamma}(t, x) = p_{\eta+\sigma, \gamma}(t, x).
\]

Take \(\ell_0 > -\frac{1-\eta}{\alpha}\) satisfying (3.11). For \(\Re[z] > -\frac{1-\eta}{\alpha}\), it holds that

\[
\frac{1}{\Gamma(-\sigma)} \int_0^t (t-s)^{-\sigma-1} s^{-\eta+\alpha z} ds = \frac{\Gamma(1-\eta + \alpha z)}{\Gamma(1-\sigma - \eta + \alpha z)} t^{-\sigma-\eta+\alpha z}.
\]

Observe that

\[
\mathcal{H}_{\eta, \gamma}(z) \Gamma(1-\eta + \alpha z) \Gamma(1-\sigma - \eta + \alpha z) = \mathcal{H}_{\eta+\sigma, \gamma}(z).
\]

By (3.3) and the Fubini theorem,

\[
I_t^{|\sigma|} p_{\eta, \gamma}(t, x) = \int_0^t \frac{(t-s)^{-\sigma-1}}{\Gamma(-\sigma)} p_{\eta, \gamma}(s, x) ds
\]

\[
= \frac{2^{2\gamma} \pi^{-d/2}}{2\pi i} \int_0^t \left( \frac{(t-s)^{-\sigma-1}}{\Gamma(-\sigma)} \mathcal{H}_{\eta, \gamma}(z) s^{-\eta} \left( \frac{|x|^{2\beta} s^{-\alpha}}{2^{2\beta}} \right)^{-z} dz ds
\]

\[
= \frac{2^{2\gamma} \pi^{-d/2}}{2\pi i} \int_0^t \mathcal{H}_{\eta, \gamma}(z) \left( \frac{|x|^{2\beta} s^{-\alpha}}{2^{2\beta}} \right)^{-z} \left[ \int_0^t (t-s)^{-\sigma-1} s^{-\eta+\alpha z} ds \right] dz
\]

\[
= \frac{2^{2\gamma} \pi^{-d/2}}{2\pi i} t^{-\sigma} \int_0^t \mathcal{H}_{\eta+\sigma, \gamma}(z) \left( \frac{|x|^{2\beta} s^{-\alpha}}{2^{2\beta}} \right)^{-z} dz
\]

\[
= \frac{2^{2\gamma} \pi^{-d/2}}{2\pi i} \int_0^t \mathcal{H}_{\eta+\sigma, \gamma}(z) \left( \frac{|x|^{2\beta} s^{-\alpha}}{2^{2\beta}} \right)^{-z} dz
\]

\[
= p_{\eta+\sigma, \gamma}(t, x).
\]

The theorem is proved. \(\square\)

Remark 5.4. Let \(0 < \varepsilon < T\). Then by Theorems 5.1 and 5.3, \(p(t, x)\) and \(\frac{\partial}{\partial t} p(t, x)\) are integrable in \(x \in \mathbb{R}^d\) uniformly in \(t \in [\varepsilon, T]\). Thus

\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} e^{-i(x, \xi)} p(t, x) dx = \int_{\mathbb{R}^d} e^{-i(x, \xi)} \frac{\partial}{\partial t} (t, x) dx.
\]
Theorem 5.5. Let \( \alpha, \beta, \gamma, \sigma \) be as in Theorem 5.1 and \( n \in \mathbb{N} \). Then for \( |x|^{2\beta t^{-\alpha}} \geq 1 \)

\[
|D_x^n p_{\alpha, \gamma}(t, x)| \leq \begin{cases} 
|x|^{-d-n t^{-\alpha}} \exp \left\{ -((|x|^{2\beta t^{-\alpha}})^{\frac{1}{2\beta}} + 1) \right\} & : \beta \in \mathbb{N}, \gamma = 0 \\
|x|^{-d-2\gamma-2\beta-n t^{-\sigma}+\alpha} & : \gamma \in \mathbb{Z}_+ \text{ or } \sigma \in \mathbb{N} \\
|x|^{-d-2\gamma-n t^{-\sigma}} & : \text{otherwise},
\end{cases}
\]

and for \( |x|^{2\beta t^{-\alpha}} \leq 1 \)

\[
|D_x^n p_{\alpha, \gamma}(t, x)| \leq \begin{cases} 
|x|^{2-n t^{-\sigma}} \exp \left\{ -((|x|^{2\beta t^{-\alpha}})^{\frac{1}{2\beta}} + 1) \right\} & : \gamma < \beta - \frac{d}{2} - 1 \\
|x|^{2-n t^{-\sigma} - 2\alpha} (1 + \ln |x|^{2\beta t^{-\alpha}}) & : \gamma = \beta - \frac{d}{2} - 1 \\
|x|^{-d-2\gamma+2\beta-n t^{-\sigma}+2\alpha} & : \gamma > \beta - \frac{d}{2} - 1.
\end{cases}
\]

Additionally, for \( |x|^{2\beta t^{-\alpha}} \leq 1 \), the followings hold:

(i) If \( \alpha = 1 \) and \( \sigma = 0 \),

\[
|D_x^n p_{0, \gamma}(t, x)| \leq |x|^{2-n t^{-d+2\beta+2}}.
\]

(ii) If \( \sigma + \alpha \in \mathbb{N} \),

\[
|D_x^n p_{\alpha, \gamma}(t, x)| \leq \begin{cases} 
|x|^{2-n t^{-\sigma} - \frac{\alpha(2d+2\gamma+2)}{2\beta}} & : \gamma < \beta - \frac{d}{2} - 1 \\
|x|^{2-n t^{-\sigma} - 2\alpha} (1 + \ln |x|^{2\beta t^{-\alpha}}) & : \gamma = \beta - \frac{d}{2} - 1 \\
|x|^{-d-2\gamma+2\beta-n t^{-\sigma}+2\alpha} & : \gamma > \beta - \frac{d}{2} - 1.
\end{cases}
\]

(iii) If \( \gamma = \beta \),

\[
|D_x^n p_{\sigma, \gamma}(t, x)| \leq \begin{cases} 
|x|^{2-n t^{-\sigma} - \frac{\alpha(2d+2)}{2\beta}} & : \gamma = \beta - \frac{d}{2} + 1 < \beta \\
|x|^{2-n t^{-\sigma} - 2\alpha} (1 + \ln |x|^{2\beta t^{-\alpha}}) & : \gamma = \beta - \frac{d}{2} + 1 = \beta \\
|x|^{-d+2\gamma-n t^{-\sigma}+2\alpha} & : \gamma = \beta - \frac{d}{2} + 1 > \beta.
\end{cases}
\]

Proof. Write \( R = R(t, x) := 2^{-2\beta}|x|^{2\beta t^{-\alpha}} \) and recall \( A^{(q)}_{\sigma, \gamma} \) from (4.4). By Proposition 3.3 and (3.1),

\[
D_x A^{(q)}_{\sigma, \gamma}(R) = -2\beta \frac{x^i}{|x|^{2-\sigma \gamma}} (R)
\]

for each \( q \in \mathbb{Z}_+ \) and \( i = 1, \ldots, d \). Hence

\[
|D_x^n p_{\sigma, \gamma}(t, x)|
= \left| -\frac{2^{2\gamma}}{\pi^{d/2}} x^i |x|^{-d-2\gamma-2 t^{-\sigma}} \left( (d+2\gamma) A_{\sigma, \gamma}(R) + 2\beta A^{(1)}_{\sigma, \gamma}(R) \right) \right|
\leq C |x|^{-d-2\gamma-1} t^{-\sigma} \left( (d+2\gamma) A_{\sigma, \gamma}(R) + 2\beta A^{(1)}_{\sigma, \gamma}(R) \right)
\]
and

\[ |D_{x^j} D_{x^i} p_{\sigma, \gamma}(t, x)| = |D_{x^j} \left( \frac{-2^{2\gamma}}{\pi^{d/2}} x^i |x|^{-d-2\gamma-2-t-\sigma} \left( (d+2\gamma) \mathbb{H}_{\sigma, \gamma}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(1)}(R) \right) \right) | \]

\[ = \frac{2^{2\gamma}}{\pi^{d/2}} t^{-\sigma} - \delta x^i |x|^{-d-2\gamma-2} \left\{ (d+2\gamma) \mathbb{H}_{\sigma, \gamma}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(1)}(R) \right\} \]

\[ + (d+2\gamma+2)x^i x^j |x|^{-d-2\gamma-4} \left\{ (d+2\gamma) \mathbb{H}_{\sigma, \gamma}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(1)}(R) \right\} \]

\[ + 2\beta x^i x^j |x|^{-d-2\gamma-4} \left\{ (d+2\gamma) \mathbb{H}_{\sigma, \gamma}^{(1)}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(2)}(R) \right\} \]

\[ \leq C |x|^{-d-2\gamma-2-t-\sigma} \sum_{q=1}^{2} \left( (d+2\gamma) \mathbb{H}_{\sigma, \gamma}^{(q-1)}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(q)}(R) \right). \]

Inductively, for any \( n \in \mathbb{N} \),

\[ |D^n_{x^i} p_{\sigma, \gamma}(t, x)| \leq C |x|^{-d-2\gamma-n-t-\sigma} \sum_{q=1}^{n} \left( (d+2\gamma) \mathbb{H}_{\sigma, \gamma}^{(q-1)}(R) + 2\beta \mathbb{H}_{\sigma, \gamma}^{(q)}(R) \right). \] (5.4)

By Lemma 4.2 and 4.3 and (5.4),

\[ |D^n_{x^i} p_{\sigma, \gamma}(t, x)| \leq \begin{cases} \frac{t^\alpha}{(2\pi)^\alpha} \exp \left\{ -c(|x|^{2\beta t-\alpha})^{\frac{1}{2\beta}} \right\} & : \gamma = 0, \beta \in \mathbb{N} \\ \left| x \right|^{-d-2\gamma-2\beta-n-t-\sigma+\alpha} & : \gamma \in \mathbb{Z}_+ \ 	ext{or} \ \sigma \in \mathbb{N} \\ \left| x \right|^{-d-2\gamma-n-t-\sigma} & : \text{otherwise} \end{cases} \]

for \( |x|^{2\beta t-\alpha} \geq 1 \). To estimate the upper bounds for \( |x|^{2\beta t-\alpha} \leq 1 \), observe that

\[ (d+2\gamma) \left( -\frac{d+2\gamma}{2\beta} \right)^{q-1} + 2\beta \left( -\frac{d+2\gamma}{2\beta} \right)^q = 0. \]

Thus, by Lemma 4.5 and (5.4),

\[ |D^n_{x^i} p_{\sigma, \gamma}(t, x)| \leq \begin{cases} \left| x \right|^{2-n-\sigma} t^{-\alpha} \frac{a(d+2\gamma+2)}{2\beta} & : \gamma < \beta - \frac{d}{2} - 1 \\ \left| x \right|^{2-n-\sigma} (1 + |\ln| x |^{2\beta t-\alpha}) & : \gamma = \beta - \frac{d}{2} - 1 \\ \left| x \right|^{-d-2\gamma+2\beta-n-\sigma} & : \gamma > \beta - \frac{d}{2} - 1 \end{cases} \]

for \( R \leq 1 \).

Additionally, if \( \sigma + \alpha \in \mathbb{N} \), by Lemma 4.6

\[ |D^n_{x^i} p_{\sigma, \gamma}(t, x)| \leq \begin{cases} \left| x \right|^{2-n-\sigma} t^{-\alpha} \frac{a(d+2\gamma+2)}{2\beta} & : \gamma < 2\beta - \frac{d}{2} - 1 \\ \left| x \right|^{2-n-\sigma} (1 + |\ln| x |^{2\beta t-\alpha}) & : \gamma = 2\beta - \frac{d}{2} - 1 \\ \left| x \right|^{-d-2\gamma+4\beta-n-\sigma} & : \gamma > 2\beta - \frac{d}{2} - 1 \end{cases} \]

for \( R \leq 1 \). Also, if \( \gamma = \beta \),

\[ |D^n_{x^i} p_{\sigma, \beta}(t, x)| \leq \begin{cases} \left| x \right|^{2-n-\sigma} t^{-\alpha} \frac{a(d+2)}{2\beta} & : \frac{d}{2} + 1 < \beta \\ \left| x \right|^{2-n-\sigma} (1 + |\ln| x |^{2\beta t-\alpha}) & : \frac{d}{2} + 1 = \beta \\ \left| x \right|^{-d+2\beta-n-\sigma} & : \frac{d}{2} + 1 > \beta \end{cases} \]

for \( R \leq 1 \). The theorem is proved. \( \square \)
6. Proofs of Theorems 2.1, 2.3 and 2.4

By Remark 5.2, \( p_{\sigma, \gamma}(t, \cdot) \in L_1(\mathbb{R}^d) \) if \( \gamma \in [0, \beta] \) and \( \sigma \in \mathbb{R} \), and \( p_{0, \gamma}(t, \cdot) \in L_1(\mathbb{R}^d) \) if \( \alpha = 1, \gamma \in [0, \infty) \), and \( \sigma = 0 \). Due to Lemmas 4.2, 4.3, Theorems 5.1, 5.3, and 5.5, to prove our desired results, it is enough to show

\[
\mathcal{F}\{p_{\sigma, \gamma}(t, \cdot)\} = |\xi|^{2\gamma} t^{-\sigma} E_{\alpha, 1-\sigma}(-t^\alpha |\xi|^{2\beta}), \tag{6.1}
\]

which is equivalent to

\[
p_{\sigma, \gamma}(t, x) = \Delta^\gamma p_{\sigma, 0}(t, x) = \Delta^\gamma \mathbb{D}^\sigma_1 p(t, x).
\]

We divide the proof into the cases \( d \geq 2 \) and \( d = 1 \).

Case 1: \( d \geq 2 \). We choose \( \ell_0 \in \mathbb{R} \) in (4.3) such that

\[
\max(-2, -1 - \frac{d-1}{4\beta}) < \ell_0 < -1 \tag{6.2}
\]

if \( \gamma = \beta \),

\[
\max(-1, -\frac{\gamma}{\beta} - \frac{d-1}{4\beta}) < \ell_0 < -\frac{\gamma}{\beta} \tag{6.3}
\]

if \( \gamma \in (0, \beta) \),

\[
\max(-1, -\frac{1}{\alpha} - \frac{d-1}{4\beta}) < \ell_0 < 0 \tag{6.4}
\]

if \( \gamma = 0 \). If \( \alpha = 1, \gamma \neq 0 \), and \( \sigma = 0 \), then we take \( \ell_0 \) such that

\[
-\frac{\gamma}{\beta} - \frac{d-1}{4\beta} < \ell_0 < -\frac{\gamma}{\beta}. \tag{6.5}
\]

Under the above restriction on \( \ell_0 \), (4.3) is well-defined and the value of \( p_{\sigma, \gamma}(t, x) \) is independent of the choice of \( \ell_0 \).

Due to the Fourier transform for radial function (see [34, Theorem IV.3.3])

\[
\mathcal{F}\{p_{\sigma, \gamma}(t, \cdot)\} \{\xi\} = \frac{2^{\frac{d+2\gamma}{2}}-2\gamma}{\xi^\frac{d-2\gamma}{2}} t^{-\sigma} \int_{0}^{\infty} \rho^{\frac{d-2\gamma}{2}-\sigma+1} \mathcal{H}_{\sigma, \gamma} \left( \frac{\rho^{2\beta^2}}{2^{2\beta}} t^{-\alpha} \right) J_{\frac{d}{2}-1}(\xi |\rho|) d\rho
\]

where \( J_{\frac{d}{2}-1} \) is the Bessel function of the first kind of order \( \frac{d}{2}-1 \), (i.e.) for \( r \in [0, \infty) \),

\[
J_{\frac{d}{2}-1}(r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \frac{d}{2})} \left( \frac{r}{2} \right)^{2k-1+\frac{d}{2}}.
\]

It is well-known if \( m > -1 \) then

\[
J_m(t) = \begin{cases} 
O(t^m) & : t \to 0+ \\
O(t^{-1/2}) & : t \to \infty.
\end{cases}
\]

Due to (6.2)-(6.5),

\[
\int_{0}^{\infty} \rho^{\frac{d}{2}-2\gamma-2\beta \ell_0} J_{\frac{d}{2}-1}(\xi |\rho|) d\rho \leq \int_{0}^{1} \rho^{-2\beta \ell_0 - 2\gamma - 1} d\rho + \int_{1}^{\infty} \rho^{\frac{d}{2}-2\gamma-2\beta \ell_0 - \frac{d}{2}} d\rho < \infty.
\]

Recall \( \Re[z] = \ell_0 \) along \( L \). By the above and (5.9),

\[
\int_{0}^{\infty} \int_{L} \rho^{\frac{d}{2}} \mathcal{H}_{\sigma, \gamma}(z) \left( \frac{\rho^{2\beta^2}}{2^{2\beta}} \right)^{-z} J_{\frac{d}{2}-1}(\xi |\rho|) |dz| d\rho < \infty.
\]
Therefore, by the Fubini theorem
\[
\int_0^{\infty} \rho^{-\frac{d}{2}+2\gamma} H_{\sigma,\gamma} \left( \frac{\rho^{2\beta}}{2\pi^{2\beta}} \right) J_{\frac{d}{2}-1}(|\xi|\rho) d\rho
\]
\[
= \frac{1}{2\pi i} \int_0^{\infty} \rho^{-\frac{d}{2}+2\gamma} \left[ \int_{L} H_{\sigma,\gamma}(z) \left( \frac{\rho^{2\beta} t^{-\alpha}}{2\pi^{2\beta}} \right)^{-z} dz \right] J_{\frac{d}{2}-1}(|\xi|\rho) d\rho
\]
\[
= \frac{1}{2\pi i} \int_{L} \left[ \int_0^{\infty} \rho^{-\frac{d}{2}+2\gamma} J_{\frac{d}{2}-1}(|\xi|\rho) \rho \right] H_{\sigma,\gamma}(z) \left( \frac{\rho^{2\beta} t^{-\alpha}}{2\pi^{2\beta}} \right)^{-z} dz.
\]
By using the formula (11.4.16),
\[
\int_0^{\infty} \rho^{-\frac{d}{2}+2\gamma} J_{\frac{d}{2}-1}(|\xi|\rho) d\rho = 2^{-\frac{d}{2}+2\gamma-2\beta z} |\xi|^{\frac{d}{2}+2\gamma+2\beta z-1} \frac{\Gamma(-\gamma-\beta z)}{\Gamma(\frac{d}{2}+\gamma+\beta z)}
\]
we have
\[
H_{\sigma,\gamma}(z) \frac{\Gamma(-\gamma-\beta z)}{\Gamma(\frac{d}{2}+\gamma+\beta z)} = \frac{\Gamma(z+1)\Gamma(-\gamma-\beta z)}{\Gamma(1-\sigma+\alpha z)}.
\]
Hence
\[
\frac{2^{\frac{d}{2}+2\gamma}}{2\pi i} \int_{L} \left[ \int_0^{\infty} \rho^{-\frac{d}{2}+2\gamma} J_{\frac{d}{2}-1}(|\xi|\rho) \rho \right] H_{\sigma,\gamma}(z) \left( \frac{\rho^{2\beta} t^{-\alpha}}{2\pi^{2\beta}} \right)^{-z} dz
\]
\[
= \frac{2^{\frac{d}{2}+2\gamma}}{2\pi i} \left[ \int_{L} \Gamma(z+1)\Gamma(-\gamma-\beta z) \left( \frac{\rho^{2\beta} t^{-\alpha}}{2\pi^{2\beta}} \right)^{-z} dz \right]
\]
\[
= \frac{\xi^{2\gamma} t^{-\sigma}}{2\pi i} \left[ \int_{L} \frac{\Gamma(1-z)\Gamma(z)}{\Gamma(1-\sigma+\alpha z)} \left( \frac{\xi^{2\beta} t^{-\alpha}}{2\pi^{2\beta}} \right)^{-z} dz \right]
\]
\[
= [\xi^{2\gamma} t^{-\sigma}] H_{12}^1 \left[ \left( \frac{\xi^{2\beta} t^{-\alpha}}{2\pi^{2\beta}} \right)^{-z} \right] \left( \frac{0,1}{0,1} (\sigma, \alpha) \right)
\]
\[
= [\xi^{2\gamma} t^{-\sigma}] E_{\alpha,1-\sigma}(-|\xi|^{2\beta} t^{\alpha}).
\]
Therefore,
\[
\mathcal{F}\{p_{\sigma,\gamma}(t, \cdot)\} = [\xi^{2\gamma} t^{-\sigma}] E_{\alpha,1-\sigma}(-|\xi|^{2\beta} t^{\alpha}).
\]
**Case 2:** \(d = 1, \gamma \in (0, \beta)\). We choose \(\ell_0\) such that
\[
\max(-1, -\frac{\gamma}{\beta} - \frac{1}{2\beta}) < \ell_0 < 0.
\]
Since \(p_{\sigma,\gamma}(t, x)\) is an even function,
\[
\mathcal{F}\{p_{\sigma,\gamma}(t, \cdot)\} = \int_{-\infty}^{\infty} e^{-i k x} p_{\sigma,\gamma}(t, x) dx
\]
\[
= 2 \int_0^{\infty} p_{\sigma,\gamma}(t, x) \cos(x) dx
\]
\[
= 2 \left( \int_0^{e^{\alpha/2\beta}} p_{\sigma,\gamma}(t, x) \cos(x) dx + \int_{e^{\alpha/2\beta}}^{\infty} p_{\sigma,\gamma}(t, x) \cos(x) dx \right)
\]
\[
= 2t^{-\sigma-\frac{\alpha}{\beta}} \left( \int_0^1 p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{\beta}} x) dx + \int_1^{\infty} p_{\sigma,\gamma}(1, x) \cos(t^{\frac{\alpha}{\beta}} x) dx \right).
\]
The last equality holds due to (5.2). Set Bromwich contours
\[
L_{<c} := \{ z \in \mathbb{C} : \Re[z] = \ell_{<c} \}, \quad L_{>c} := \{ z \in \mathbb{C} : \Re[z] = \ell_{>c} \}
\]
where

\[
\max(-1, -\frac{\gamma}{\beta} - \frac{1}{2\beta}) < \ell_{<c} < -\frac{\gamma}{\beta}, \quad -\frac{\gamma}{\beta} < \ell_{>c} < 0.
\]

By (3.9),

\[
\begin{align*}
\int_0^1 \int_{L_{<c}} & |2^{2z} \mathcal{H}_{\sigma, \gamma}(z)x^{-2\gamma - 2\beta z - 1}| \, |dz| \, dx \\
+ \int_1^\infty \int_{L_{>c}} & |2^{2z} \mathcal{H}_{\sigma, \gamma}(z)x^{-2\gamma - 2\beta z - 1}| \, |dz| \, dx < \infty.
\end{align*}
\]

Therefore, by the Fubini theorem,

\[
\begin{align*}
\int_0^1 p_{\sigma, \gamma}(1, x) \cos(t \frac{2\pi}{\ell} \xi x) \, dx &= \frac{2^{2\gamma} \pi^{-1/2}}{2\pi i} \int_0^1 \int_{L_{<c}} 2^{2z} \mathcal{H}_{\sigma, \gamma}(z)x^{-2\gamma - 2\beta z - 1} \cos(t \frac{2\pi}{\ell} \xi x) \, dz \, dx \\
&= \frac{2^{2\gamma} \pi^{-1/2}}{2\pi i} \int_{L_{<c}} 2^{2z} \mathcal{H}_{\sigma, \gamma}(z) \left[ \int_0^1 x^{-2\gamma - 2\beta z - 1} \cos(t \frac{2\pi}{\ell} \xi x) \, dx \right] \, dz.
\end{align*}
\]

Similarly,

\[
\begin{align*}
\int_1^\infty p_{\sigma, \gamma}(1, x) \cos(t \frac{2\pi}{\ell} \xi x) \, dx &= \frac{2^{2\gamma} \pi^{-1/2}}{2\pi i} \int_{L_{>c}} 2^{2z} \mathcal{H}_{\sigma, \gamma}(z) \left[ \int_1^\infty x^{-2\gamma - 2\beta z - 1} \cos(t \frac{2\pi}{\ell} \xi x) \, dx \right] \, dz.
\end{align*}
\]

For \( \eta \in \mathbb{R} \), it holds that (see [30] (1.8.1.1), (1.8.1.2))

\[
\int_0^1 x^{-2\lambda - 1} \cos(\eta x) \, dx = -\frac{1}{2\lambda} \frac{1}{\Gamma(-\lambda; \frac{1}{2}, 1 - \lambda; -\frac{\eta^2}{4})}, \quad \Re[\lambda] < 0
\]

and

\[
\int_1^\infty x^{-2\lambda - 1} \cos(\eta x) \, dx = \frac{\Gamma(-2\lambda) \cos \lambda \pi}{|\eta|^{-2\lambda}} + \frac{1}{2\lambda} \frac{1}{\Gamma(-\lambda; \frac{1}{2}, 1 - \lambda; -\frac{\eta^2}{4})}, \quad \Re[\lambda] > -\frac{1}{2},
\]

where \( 1_F(1 - \lambda; \frac{1}{2}, 1 - \lambda; -\frac{\eta^2}{4}) \) denotes the general hypergeometric function, i.e.

\[
1_F(\lambda; \eta; \frac{1}{2}, 1 - \lambda; -\frac{\eta^2}{4}) = \frac{\Gamma(\lambda) \Gamma(\eta)}{\Gamma(\lambda + \eta)}} \sum_{k=0}^{\infty} \frac{\Gamma(\lambda + k)}{\Gamma(\eta + k) \Gamma(\lambda + \eta + k)} \frac{z^k}{k!}, \quad p, q, \in \mathbb{C}, \quad \Re(\lambda) \notin \{-1, -2, \ldots\}.
Observe that for \( \gamma + \beta z \in \mathbb{C} \setminus \mathbb{N}, \)

\[
1_F(\left\{\begin{array}{c}
-\gamma - \beta z; \frac{1}{2}, 1 - \gamma - \beta z; -t^2 |\xi|^2 / 4
\end{array}\right.)
\]

\[
= \sqrt{\pi} \Gamma(1 - \gamma - \beta z) \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma - \beta z + k)}{\Gamma(\frac{1}{2} + k) \Gamma(1 - \gamma - \beta z + k)} \left(-t^2 |\xi|^2 / 4\right)^k/k!
\]

\[
= -\sum_{k=0}^{\infty} \frac{\sqrt{\pi} (\gamma + \beta z)}{(2k - 1) 2k(-\gamma - \beta z + k)} \left(-t^2 |\xi|^2 / 4\right)^k/k!
\]

\[
= 1 - \sum_{k=1}^{\infty} \frac{\gamma + \beta z}{(2k - 1) 2k(-\gamma - \beta z + k)} \left(t^2 |\xi|^2 / 4\right)^k/k!
\]

Since the series in \( F_{t, \xi}(z) \) converges absolutely for fixed \( t, \xi \), one can easily see that \( F_{t, \xi}(z) \) is holomorphic in \( \{ z \in \mathbb{C} : -\frac{2z - 1}{2\beta} < \Re[z] < -\frac{2z + 1}{2\beta}\} \) by Morera’s theorem. So we obtain

\[
\int_{0}^{1} p_{\sigma, \gamma}(1, x) \cos(t^2 \pi \xi x) dx = -\frac{2^{2\gamma - 1/2}}{2\pi i} \int_{L_{<c}} 2^{2\beta z} \frac{H_{\sigma, \gamma}(z)}{2(\gamma + \beta z)} F_{t, \xi}(z) dz,
\]

and

\[
\int_{1}^{\infty} p_{\sigma, \gamma}(1, x) \cos(t^2 \pi \xi x) dx
\]

\[
= \frac{2^{2\gamma - 1/2}}{2\pi i} \left( \int_{L_{<c}} 2^{2\beta z} H_{\sigma, \gamma}(z) \Gamma(-2\gamma - 2\beta z) \cos((\gamma + \beta z)\pi) |t^2 \pi \xi|^{2\gamma + 2\beta z} dz
\]

\[
+ \int_{L_{>c}} 2^{2\beta z} \frac{H_{\sigma, \gamma}(z)}{2(\gamma + \beta z)} F_{t, \xi}(z) dz \right).
\]

If \( \gamma \neq \beta \), by (3.1),

\[
\frac{H_{\sigma, \gamma}(z)}{2(\gamma + \beta z)} = -\frac{\Gamma(\frac{1}{2} + \gamma + \beta z) \Gamma(1 + z)}{2\Gamma(1 - \gamma - \beta z) \Gamma(1 - \sigma + \alpha z)}.
\]

Thus \( z = -\frac{2}{3} \) is a removable singularity. These facts lead to

\[
\operatorname{Res}_{z=-\frac{2}{3}} \left[ 2^{2\beta z} \frac{H_{\sigma, \gamma}(z)}{2(\gamma + \beta z)} F_{t, \xi}(z) \right] = 0
\]

and

\[
\frac{1}{2\pi i} \int_{L_{>c}} 2^{2\beta z} H_{\sigma, \gamma}(z) \Gamma(-2\gamma - 2\beta z) \cos((\gamma + \beta z)\pi) |t^2 |\xi|^2 \Gamma(\frac{1}{2} + \gamma + \beta z) \Gamma(1 + z) \Gamma(1 - \gamma - \beta z) \Gamma(1 - \sigma + \alpha z) dz
\]

\[
= \frac{1}{2\pi i} \int_{L_{<c}} 2^{2\beta z} H_{\sigma, \gamma}(z) \Gamma(-2\gamma - 2\beta z) \cos((\gamma + \beta z)\pi) (t^2 |\xi|^2)^z dz.
\]

Therefore,

\[
\int_{0}^{\infty} p_{\sigma, \gamma}(1, x) \cos(t^2 \pi \xi x) dx = \frac{2^{2\gamma - 1/2}}{2\pi i} \int_{L_{>c}} 2^{2\beta z} H_{\sigma, \gamma}(z) \Gamma(-2\gamma - 2\beta z) \cos((\gamma + \beta z)\pi) (t^2 |\xi|^2)^z dz.
\]
By (3.2),
\[ H_{\sigma,\gamma}(z) \Gamma(-2\gamma - 2\beta z) \cos((\gamma + \beta z)\pi) = \frac{\pi^{1/2}2^{-2\gamma - 2\beta z-1} \Gamma(1 + z) \Gamma(-z)}{\Gamma(1 - \sigma + \alpha z)}. \]
Hence
\[ F \{ p_{\sigma,\gamma}(t, \cdot) \} (\xi) = \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}} \int_{0}^{\infty} p_{\sigma,\gamma}(1, x) \cos(t \frac{\xi}{\sqrt{t^2 - \xi^2}} x) dx = \frac{|\xi|^{2\gamma} t^{-\gamma} \Gamma(1 + z) \Gamma(-z)}{2\pi i} \int_{L} H_{\sigma,\gamma}(z) \frac{\Gamma(1 + z) \Gamma(-z)}{\Gamma(1 - \sigma + \alpha z)} (t^\alpha |\xi|^{2\beta})^z dz \]
\[ = |\xi|^{2\gamma} t^{-\gamma} E_{\alpha,1-\sigma} (-|\xi|^{2\beta} t^\alpha). \]

**Case 3:** \( d = 1, \gamma = 0 \). Let
\[ \max(-1, -\frac{1}{\alpha}, -\frac{1}{2\beta}) < \ell_0 < 1. \]
Again, we follow the argument in Case 2. Note that
\[ \frac{H_{\sigma,0}(z)}{2\beta z} = -\frac{\Gamma(\frac{1}{2} + \beta z) \Gamma(1 + z) \Gamma(-z)}{2\pi i} \Gamma(1 - \sigma + \alpha z) \]
has a removable singularity at \( z = 0 \) if and only if \( \sigma \in \mathbb{N} \). Thus (6.6) holds if \( \sigma = 1 \) and we immediately obtain
\[ F \{ p_{1,0}(t, \cdot) \} = t^{-1} E_{\alpha,0}(-|\xi|^{2\beta} t^\alpha) = \frac{\partial}{\partial t} E_{\alpha}(-|\xi|^{2\beta} t^\alpha). \]
By Remark 5.4 and Theorem 5.3
\[ \frac{\partial}{\partial t} F \{ p(t, \cdot) \} = F \{ \frac{\partial p}{\partial t}(t, \cdot) \} = F \{ p_{1,0}(t, \cdot) \}. \]
Thus,
\[ F \{ p(t, \cdot) \} (\xi) = E_{\alpha}(-|\xi|^{2\beta} t^\alpha) + R(\xi). \]
By (5.2),
\[ F \{ p(t, \cdot) \} (\xi) = \int_{\mathbb{R}} e^{-ix\xi} p(t, x) dx = \int_{\mathbb{R}} e^{-ix\xi} t^{-\frac{\xi^2}{2}} p(1, t^{-\frac{\xi^2}{2}} x) dx \]
\[ = \int_{\mathbb{R}} e^{-it^{\alpha/2} x\xi} p(1, x) dx = F \{ p(1, \cdot) \} (t^{\frac{\xi^2}{2}} \xi). \]
Then by the Riemann-Lebesgue lemma, \( F \{ p(t, \cdot) \} (\xi) \) converges to 0 as \( t \to \infty \). This implies \( R(\xi) \equiv 0 \) and we obtain the desired result.

**Case 4:** \( d = 1, \gamma = \beta \). Now we additionally assume \( \sigma + \alpha \in \mathbb{N} \) and take \( \ell_0 \) so that
\[ \max(-2, -1 - \frac{1}{2\beta}) < \ell_0 < 0. \]
Note that every argument in Case 2 holds except that
\[ \frac{H_{\sigma,\beta}(z)}{2\beta (1 + z)} = -\frac{\Gamma(\frac{1}{2} + \beta + \beta z) \Gamma(2 + z) \Gamma(-z)}{2\pi i} \frac{\Gamma(1 - \beta - \beta z) \Gamma(1 - \sigma + \alpha z)(z + 1)}{\Gamma(1 - \sigma + \alpha z)(z + 1)} \]
has a removable singularity at \( z = -1 \) if \( \sigma + \alpha \in \mathbb{N} \). Thus (6.6) holds and we obtain (6.1). The theorems are proved.
References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Number 55. Courier Dover Publications, 1972.

[2] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71. Cambridge University Press, 1999.

[3] V. V. Anh and N. N. Leonenko. Spectral analysis of fractional kinetic equations with random data. *Journal of Statistical Physics*, 104(5-6):1349–1387, 2001.

[4] B. L. J. Braaksma. Asymptotic expansions and analytic continuations for a class of barnes-integrals. *Compositio Mathematica*, 15:239–341, 1964.

[5] Z.-Q. Chen, K.-H. Kim, and P. Kim. Fractional time stochastic partial differential equations. *Stochastic Processes and their Applications*, 125(4):1470–1499, 2015.

[6] Z.-Q. Chen, M. M. Meerschaert, and E. Nane. Space-time fractional diffusion on bounded domains. *Journal of Mathematical Analysis and Applications*, 393(2):479–488, 2012.

[7] P. Clément, G. Grispenberg, and S.-O. Londen. Schauder estimates for equations with fractional derivatives. *Transactions of the American Mathematical Society*, 352(5):2239–2260, 2000.

[8] P. Clément, S.-O. Londen, and G. Simonett. Quasilinear evolutionary equations and continuous interpolation spaces. *Journal of Differential Equations*, 196(2):418–447, 2004.

[9] S. D. Eidelman, S. D. Ivasyshen, and A. N. Kochubei. Analytic methods in the theory of differential and pseudo-differential equations of parabolic type, volume 152. Springer, 2004.

[10] S. D. Eidelman and A. N. Kochubei. Cauchy problem for fractional diffusion equations. *Journal of Differential Equations*, 199(2):211–255, 2004.

[11] R. Gorenflo, A. Iskenderov, and Y. Luchko. Mapping between solutions of fractional diffusion-wave equations. *Fractional Calculus and Applied Analysis*, 3(1):75–86, 2000.

[12] A. Hanyga. Multidimensional solutions of space-time-fractional diffusion equations. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 458(2018):429–450, 2002.

[13] S. Jo and M. Yang. Precise asymptotic approximations for kernels corresponding to Lévy processes. *Potential Analysis*, 40(3):203–230, 2014.

[14] A. A. Kilbas. *H-transforms: Theory and Applications*. CRC Press, 2004.

[15] I. Kim and K.-H. Kim. A generalization of the Littlewood-Paley inequality for the fractional Laplacian $(-\Delta)^{\alpha/2}$. *Journal of Mathematical Analysis and Applications*, 388(1):175–190, 2012.

[16] I. Kim, K.-H. Kim, and P. Kim. Parabolic Littlewood-Paley inequality for $\phi (-\Delta)$-type operators and applications to stochastic integro-differential equations. *Advances in Mathematics*, 249:161–203, 2013.

[17] I. Kim, K.-H. Kim, and S. Lim. An $L_q(L_p)$-theory for the time fractional evolution equations with variable coefficients. *Submitted*, 2014.

[18] I. Kim, K.-H. Kim, and S. Lim. An $L_q(L_p)$-theory for parabolic pseudo-differential equations: Calderón-Zygmund approach. *arXiv preprint arXiv:1503.04521*, 2015.

[19] I. Kim, K.-H. Kim, and S. Lim. Parabolic BMO estimates for pseudo-differential operators of arbitrary order. *Journal of Mathematical Analysis and Applications*, 427(2):557–580, 2015.

[20] A. N. Kochubei. Asymptotic properties of solutions of the fractional diffusion-wave equation. *Fractional Calculus and Applied Analysis*, 17(3):881–896, 2014.

[21] N. V. Krylov. On the Foundation of the $L_p$-theory of Stochastic Partial Differential Equations. *Stochastic Partial Differential Equations and Applications-VII*, pages 179–191, 2006.

[22] Z. Li, Y. Luchko, and M. Yamamoto. Asymptotic estimates of solutions to initial-boundary-value problems for distributed order time-fractional diffusion equations. *Fractional Calculus and Applied Analysis*, 17(4):1114–1136, 2014.

[23] F. Mainardi. Fractional diffusive waves in viscoelastic solids. *Nonlinear Waves in Solids, Fairfield*, pages 93–97, 1995.

[24] M. M. Meerschaert, R. L. Schilling, and A. Sikorskii. Stochastic solutions for fractional wave equations. *Nonlinear Dynamics*, pages 1–11, 2014.

[25] R. Metzler, E. Barkai, and J. Klafter. Anomalous diffusion and relaxation close to thermal equilibrium: a fractional fokker-planck equation approach. *Physical Review Letters*, 82(18):3563, 1999.
[26] R. Metzler and J. Klafter. Boundary value problems for fractional diffusion equations. *Physica A: Statistical Mechanics and its Applications*, 278(1):107–125, 2000.

[27] R. Metzler and J. Klafter. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339(1):1–77, 2000.

[28] R. Metzler and J. Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *Journal of Physics A: Mathematical and General*, 37(31):R161, 2004.

[29] I. Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, volume 198. Academic press, 1998.

[30] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. *Integrals and series: special functions*, volume 2. CRC Press, 1998.

[31] K. Sakamoto and M. Yamamoto. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *Journal of Mathematical Analysis and Applications*, 382(1):426–447, 2011.

[32] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional integrals and derivatives*, 1993.

[33] W. Schneider and W. Wyss. Fractional diffusion and wave equations. *Journal of Mathematical Physics*, 30(1):134–144, 1989.

[34] E. M. Stein and G. L. Weiss. *Introduction to Fourier analysis on Euclidean spaces*, Princeton university press, 1971.

[35] J. van Neerven, M. Veraar, L. Weis, et al. Stochastic maximal $L_p$-regularity. *The Annals of Probability*, 40(2):788–812, 2012.

[36] R. Zacher. Maximal regularity of type $L_p$ for abstract parabolic Volterra equations. *Journal of Evolution Equations*, 5(1):79–103, 2005.

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