Numerical methods for mean-field stochastic differential equations with jumps

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Abstract
In this paper, we are devoted to the numerical methods for mean-field stochastic differential equations with jumps (MSDEJs). By combining with the mean-field Itô formula (see Sun, Yang, and Zhao, Numer. Math. Theor. Meth. Appl., 10, pp. 798–828 (2017)), we first develop the Itô formula and further construct the Itô-Taylor expansion for MSDEJs. Then based on the Itô-Taylor expansions, we propose the strong order γ and the weak order η Itô-Taylor schemes for MSDEJs. We theoretically prove the strong convergence rate γ of the strong order γ Itô-Taylor scheme and the weak convergence rate η of the weak order η Itô-Taylor scheme, respectively. Some numerical tests are also presented to verify our theoretical conclusions.

Keywords Mean-field stochastic differential equations with jumps · Itô formula · Itô-Taylor expansion · Itô-Taylor schemes · Error estimates

Mathematics Subject Classification (2010) 60H35 · 65C20 · 60H10

1 Introduction

Let (Ω, 𝒟, ℙ, 𝒉) be a complete filtered probability space with ℙ = (𝒟ₜ)₀≤ₜ≤T being the filtration of the following two mutually independent stochastic processes:

- The 𝑚-dimensional Brownian motion: \( W = (W_t)_{0 ≤ t ≤ T} \);

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The Poisson random measure on $\mathbb{E} \times [0, T]$: $\{\mu(A \times [0, t]), A \in \mathcal{E}, 0 \leq t \leq T\}$, where $\mathbb{E} = \mathbb{R}^d \setminus \{0\}$ and $\mathcal{E}$ is its Borel field.

To be more precise, $\mathcal{F}_t$ is defined by

$$\mathcal{F}_t = \sigma(\mu([0, s] \times A) : 0 \leq s \leq t, A \in \mathcal{E}) \lor \sigma(W_s : 0 \leq s \leq t) \lor \mathcal{N},$$

where $\mathcal{N}$ contains all the sets of $\mathcal{P}$-measure zero.

Suppose that $\mu$ has the intensity measure $\nu(de, dt) = \lambda(de)dt$, which implies that $\mathbb{E} [\mu([0, t] \times A)] = \lambda(A)t$ for $A \in \mathcal{E}$ and $t \in [0, T]$. Here, $\lambda$ is a $\sigma$-finite measure on $(\mathbb{E}, \mathcal{E})$ satisfying $\int_{\mathbb{E}} (1 \wedge |e|^2) \lambda(de) < +\infty$. Then, we have the compensated Poisson random measure:

$$\tilde{\mu}(de, dt) = \mu(de, dt) - \lambda(de)dt,$$

such that $\{\tilde{\mu}(A \times [0, t]) = (\mu - \nu)(A \times [0, t])\}_{0 \leq t \leq T}$ is a martingale for any $A \in \mathcal{E}$ with $\lambda(A) < \infty$. Moreover, let $F$ be the distribution of the jump size, then it holds that

$$\lambda(de) = \lambda F(de),$$

where $\lambda = \lambda(\mathbb{E}) < \infty$ is the intensity of the Poisson process $N_t = \mu(\mathbb{E} \times [0, t])$, which counts the number of jumps of $\mu$ occurring in $[0, t]$. Then, the Poisson measure $\mu$ generates a sequence of pairs $\{(\tau_i, Y_i), i = 1, 2, \ldots, N_T\}$ with $\{\tau_i \in [0, T], i = 1, 2, \ldots, N_T\}$ representing the jump times of the Poisson process $N_t$ and $\{Y_i \in \mathbb{E}, i = 1, 2, \ldots, N_T\}$ the corresponding jump sizes satisfying $Y_i \overset{iid}{\sim} F$. For more details of the Poisson random measure or Lévy measure, the readers are referred to [8, 26].

We consider the following mean-field stochastic differential equations with jumps (MSDEJs) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$:

$$X^{t_0, \xi, \xi'}_t = x + \int_{t_0}^t \mathbb{E} \left[ \int_{X^{t_0, \xi, \xi'}_s} b(s, X^{t_0, \xi, \xi'}_s, x) \right]_{x=X^{t_0, \xi, \xi'}_s} ds + \int_{t_0}^t \mathbb{E} \left[ \int_{X^{t_0, \xi, \xi'}_s} \sigma(s, X^{t_0, \xi, \xi'}_s, x) \right]_{x=X^{t_0, \xi, \xi'}_s} dW_s + \int_{t_0}^t \mathbb{E} \left[ \int_{\mathbb{E}} c(s, X^{t_0, \xi, \xi'}_s, x, e) \right]_{x=X^{t_0, \xi, \xi'}_s} \mu(de, ds) \tag{1.1}$$

for $0 \leq t_0 \leq t \leq T$, where $t_0$ and $T$ are, respectively, the deterministic initial and terminal time; the initial condition $\xi'$ and $\xi$ are $\mathcal{F}_{t_0}$ measurable; $b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$, and $c : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{E} \rightarrow \mathbb{R}^d$ are the so-called drift, diffusion, and jump coefficients, respectively. We point out that in (1.1) the integrand in the last term needs to be an adapted left-continuous process and $X^{t_0, \xi, \xi'}_s (x = \xi' \text{ or } \xi)$ denote the almost sure left-hand limit of $X^{t_0, \xi, \xi'}_s$ at time $s$, i.e.,

$$X^{t_0, \xi, \xi'}_s = \lim_{r \uparrow s} X^{t_0, \xi, \xi'}_r, \quad a.s.$$

Notice that in the expectation operator in (1.1), we denote by $X^{t_0, \xi, \xi'}_t$ the solution of (1.1) with $\xi = \xi'$. In general, $\xi$ and $\xi'$ are of different values and can be independent.

Our aim is to solve $X^{t_0, \xi, \xi'}_t$ numerically. For this end, we need to solve (1.1) by two steps in succession. First, we solve the solution $X^{t_0, \xi, \xi'}_t$ of (1.1) with the same...
initial values $\xi = \xi'$. Then, we solve the solution $X^{t_0, \xi, \xi'}$ of (1.1) with different initial values by using $X^{t_0, \xi', \xi'}$.

The main motivation we study the numerical methods for the MSDEJ (1.1) with $\xi \neq \xi'$ is that we intend to solve mean-field backward stochastic differential equations with jumps driven by an MSDEJ which depends on two different initial values in general (see [18]). For this reason, we need to study the numerical methods for MSDEJs with two different initial values first.

Mean-field stochastic differential equations (MSDEs), also called McKean-Vlasov SDEs, were first studied by Kac [24, 25] in the 1950s. Since then, MSDEs have been encountered and intensively investigated in many areas such as kinetic gas theory [1, 21, 31], quantum mechanics [22], quantum chemistry [27], McKean-Vlasov type partial differential equations (PDEs) [3, 4, 15, 20], mean-field games [6, 7, 9, 16], and mean-field backward stochastic differential equations (MBSDEs) [2, 3, 5, 29, 30]. In the last decade, MSDEJs have also received much attention because of its wide applications in the research on nonlocal PDEs [12], MBSDEs with jumps [18, 19], economics and finance [10], and mean-field control and mean-field games with jumps [23, 32]. Therefore, it is important and necessary to study the numerical solutions of MSDEJs.

Compared with the well-developed theory of numerical methods for stochastic differential equations with jumps (SDEJs) (see [11, 17, 26] and references therein), little attention has been paid to the numerical methods for MSDEJs. In this work, we aim to propose the general Itô-Taylor schemes for solving MSDEJs. It is worth noting that Kloeden proposed an efficient Gauss-quadrature method for MSDEs in [14], and the authors studied the mean-field Itô formula and proposed the general Itô-Taylor schemes for MSDEs in [28]. By using the mean-field Itô formula in [28], we first develop the Itô formula for MSDEJs, then based on which, we construct the Itô-Taylor expansion for MSDEJs and further propose the Itô-Taylor schemes of strong order $\gamma$ and weak order $\eta$ for solving MSDEJs. Taking $\gamma = 0.5, 1.0$ and $\eta = 2.0$, we obtain the Euler scheme, the strong order 1.0 Taylor scheme, and the weak order 2.0 Taylor scheme, respectively. The rigorous error estimates indicate that the order of strong convergence of the strong order $\gamma$ Taylor scheme is $\gamma$ and the order of weak convergence of the weak order $\eta$ Taylor scheme is $\eta$. Some numerical tests are also carried out to show the efficiency and the accuracy of the proposed schemes for solving MSDEJs and to verify our theoretical conclusions. The numerical results are consistent with our theoretical ones and show that the efficiency of the proposed schemes depends on the level of the intensity of the Poisson random measure.

2 Preliminaries

In this section, we state a standard result on the existence and uniqueness of the strong solutions of the MSDEJ (1.1). For this end, we set the following assumptions on $b$, $\sigma$ and $c$.

(A1) $b(\cdot, x', x), \sigma(\cdot, x', x)$, and $c(\cdot, x', x, e)$ are deterministic continuous processes, for any fixed $(x', x, e) \in \mathbb{R}^d \times \mathbb{R}^d \times E$. 

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There exists a constant $L > 0$ such that for all $t \in [0, T]$ and $x, x', y, y' \in \mathbb{R}^d$

$$|b(t, x', x) - b(t, y', y)| + |\sigma(t, x', x) - \sigma(t, y', y)| \leq L(|x' - y'| + |x - y|).$$

There exists a function $\rho : \mathbb{E} \to \mathbb{R}^+$ satisfying $\int_\mathbb{E} \rho^2(e) \lambda(\mathbb{E}) \, de < +\infty$, such that for all $t \in [0, T], x, x', y, y' \in \mathbb{R}^d$ and $e \in \mathbb{E}$

$$|c(t, x', x, e) - c(t, y', y, e)| \leq \rho(e)(|x' - y'| + |x - y|),$$

$$|c(t, 0, 0, e)| \leq \rho(e).$$

There exists a constant $K > 0$ such that for all $t \in [0, T], x, x' \in \mathbb{R}^d$, and $e \in \mathbb{E}$

$$|b(t, x', x)| + |\sigma(t, x', x)| \leq K(1 + |x'| + |x|),$$

$$|c(t, x', x, e)| \leq \rho(e)(1 + |x'| + |x|).$$

Now we state the existence and uniqueness of the solution of the MSDEJ (1.1) and some useful properties in the following theorem (see Theorem 3.1 and Proposition 3.1 in [12]).

**Theorem 2.1** Under the assumptions (A1) − (A4), the MSDEJ (1.1) admits a unique strong solution $X_t^{\xi_0, \xi'_0}$ on $[t_0, T]$.

In addition, for any $p \geq 2$, there exists a constant $C_p > 0$ depending only on $L, K$ and $\rho(e)$ such that for any $\mathcal{F}_{t_0}$ measurable $\xi_1, \xi_2, \xi'_1, \xi'_2$ satisfying $\mathbb{E}[|\xi_i|^p + |\xi'_i|^p] < \infty$ with $i = 1, 2$, we have

$$\mathbb{E}\left[ \sup_{s \in [t_0, t]} |X_s^{\xi_0, \xi'_1, \xi_1}|^p \Big| \mathcal{F}_{t_0} \right] \leq C_p (1 + |\xi_1|^p),$$

$$\mathbb{E}\left[ \sup_{s \in [t_0, T]} |X_s^{\xi_0, \xi'_1, \xi_1} - X_s^{\xi_0, \xi'_2, \xi_1}|^p \Big| \mathcal{F}_{t_0} \right] \leq C_p \left( |\xi_1 - \xi_2|^p + \mathbb{E}[|\xi'_1 - \xi'_2|^p] \right),$$

$$\mathbb{E}\left[ \sup_{s \in [t_0, t_0 + \delta]} |X_s^{\xi_0, \xi'_1, \xi_1} - \xi_1|^p \Big| \mathcal{F}_{t_0} \right] \leq C_p (1 + |\xi_1|^p)\delta,$$

a.s. for all $\delta > 0$ with $t_0 + \delta \leq T$, where $X_s^{\xi_0, \xi'_1, \xi_1}$ and $X_s^{\xi_0, \xi'_2, \xi_2}$ are the solutions of (1.1) with initial conditions $(\xi_1, \xi'_1)$ and $(\xi_2, \xi'_2)$, respectively.

**3 The Itô formula and Itô-Taylor expansion**

In this section, we develop the Itô formula and Itô-Taylor expansion for MSDEJs, which are the foundation for proposing the Itô-Taylor schemes for MSDEJs.
3.1 Itô’s formula for MSDEJs

In this subsection, based on the mean-field Itô formula [28], we rigorously prove the Itô formula for MSDEJs. To proceed, we define \( f^\beta(t, x) \) and \( g^\beta(t, x, e) \) by

\[
    f^\beta(t, x) = \mathbb{E} \left[ f(t, \beta_t, x) \right], \quad g^\beta(t, x, e) = \mathbb{E} \left[ g(t, \beta_t, x, e) \right],
\]

for functions \( f(t, x', x) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) and \( g(t, x', x, e) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{E} \to \mathbb{R} \).

Let \( X_t \) be a \( d \)-dimensional Itô process with jumps satisfying the MSDEJ

\[
    dX_t = b^\beta(t, X_t)dt + \sigma^\beta(t, X_t)dW_t + \int_{\mathbb{E}} c^\beta(t, X_t, e)\mu(de, dt), \quad 0 \leq t \leq T
\]

with \( \beta_t \) a \( d \)-dimensional Itô process with jumps defined by

\[
    d\beta_t = \psi_t dt + \varphi_t dW_t + \int_{\mathbb{E}} h_{t-}(e)\mu(de, dt),
\]

where \( \psi_t \) and \( \varphi_t \) are two progressively measurable processes such that \( \int_0^T |\psi_t|dt < +\infty \) and \( \int_0^T \text{Tr}[\varphi_t \varphi_t^T]dt < +\infty \), and \( h_t \) is a progressively measurable process such that \( \int_0^T \int_{\mathbb{E}} |h_t(e)|^2\lambda(de)dt < +\infty \). Here, \( \text{Tr}[A] \) denotes the trace of a matrix \( A \) and

\[
    c^\beta(t, X_t, e) = \mathbb{E} \left[ c(t, \beta_t, x, e) \right].
\]

To proceed, we first define two differential operators \( L^0 \) and \( L^1 \) as

\[
    L^0 f^\beta(s, x) = \frac{\partial f^\beta}{\partial s}(s, x) + \nabla_x f^\beta(s, x)b^\beta(s, x)
\]

\[
    + \frac{1}{2} \text{Tr}[\nabla_{xx} f^\beta(s, x)(\sigma^\beta(s, x))\sigma^\beta(s, x)^T],
\]

\[
    \hat{L}^1 f^\beta(s, x) = \nabla_x f^\beta(s, x)\sigma^\beta(s, x) = (L^1 f^\beta(s, x), \ldots, L^m f^\beta(s, x))
\]

with

\[
    L^j f^\beta(s, x) = \sum_{k=1}^d \frac{\partial f^\beta}{\partial x^k}(s, x)\sigma^\beta_{kj}(s, x), \quad j = 1, \ldots, m,
\]

\[
    \frac{\partial f^\beta}{\partial s}(s, x) = \mathbb{E} \left[ \frac{\partial f}{\partial s}(s, \beta_s, x) + \nabla_x f(s, \beta_s, x)\psi_s + \frac{1}{2} \text{Tr}[f_{xx'}(s, \beta_s, x)\varphi_s\varphi_s^T] \right],
\]

and

\[
    \nabla_x f^\beta(s, x) = \mathbb{E}[\nabla_x f(s, \beta_s, x)], \quad f^\beta_{xx}(s, x) = \mathbb{E}[f_{xx}(s, \beta_s, x)],
\]

where \( \nabla_x f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}) \) is a \( d \)-dimensional row vector, and \( f_{xx} = (\frac{\partial^2 f}{\partial x_k \partial x_j})_{d \times d} \) is a \( d \times d \) matrix.

Based on Itô’s formula for MSDEs [28], we develop the following Itô’s formula for MSDEJs.
Theorem 3.1 (Mean-field Itô formula with jumps) Let \( X_t \) and \( \beta_t \) be two \( d \)-dimensional Itô processes with jumps defined by (3.1) and (3.2), respectively. Let \( f = f(t, x', x) \in C^{1,2,2} \), then \( f^\beta(t, X_t) \) is an Itô process with jumps and satisfies

\[
\begin{align*}
    f^\beta(t, X_t) &= f^\beta(0, X_0) + \int_0^t L^0 f^\beta(s, X_s)ds + \int_0^t \widetilde{L}^1 f^\beta(s, X_s)dW_s \\
    &\quad + \int_0^t \int_{\mathcal{E}} L^{-1}_e f^\beta(s, X_{s-})\mu(de, ds),
\end{align*}
\]

where \( L^0 \) and \( \widetilde{L}^1 \) are defined by (3.4) and

\[
L^{-1}_e f^\beta(s, x) = f^\beta(s, x + c^\beta(s-, x, e)) - f^\beta(s-, x).
\]

Proof For simplicity, we consider the case \( d = m = q = 1 \). The general case can be obtained similarly.

Assume that the Poisson random measure \( \mu \) generates a sequence of pairs \( \{(\tau_i, Y_i), i = 1, 2, \ldots, N_t\} \), where \( N_t = \mu(E \times [0, t]) \) represents the total number of jumps of \( \mu \) up to time \( t \), and \( (\tau_i, Y_i) \) are the \( i \)th jump time and jump size, respectively. Then we can write the MSDEJ (3.1) as

\[
X_t = X_0 + \int_0^t b^\beta(s, X_s)ds + \int_0^t \sigma^\beta(s, X_s)dW_s + \sum_{i=1}^{N_t} c^\beta(\tau_i, X_{\tau_i-}, Y_i),
\]

where \( c^\beta(\tau_i, X_{\tau_i-}, Y_i) = \mathbb{E}[c(\tau_i, \beta_{\tau_i-}, x, e)]|_{(x,e)=(X_{\tau_i-},Y_i)}. \)

Let \( \tau_0 = 0 \) and \( \tau_{N_t+1} = t \), and we have

\[
f^\beta(t, X_t) - f^\beta(0, X_0) = \sum_{i=0}^{N_t} \left( f^\beta(\tau_{i+1}, X_{\tau_{i+1}}) - f^\beta(\tau_i, X_{\tau_i}) \right)
\]

\[
= \sum_{i=0}^{N_t} \left( f^\beta(\tau_{i+1}, X_{\tau_{i+1}}) - f^\beta(\tau_{i+1-}, X_{\tau_{i+1}-}) \right) + \sum_{i=0}^{N_t} \left( f^\beta(\tau_{i+1-}, X_{\tau_{i+1}-}) - f^\beta(\tau_i, X_{\tau_i}) \right).
\]

Note that \( X_t \) is an MSDE on each time interval \([\tau_i, \tau_{i+1}]\) for \( i = 0, \ldots, N_t \), then by the mean-field Itô formula for MSDEs (see Theorem 2.3 in [28]), we obtain

\[
f^\beta(\tau_{i+1-}, X_{\tau_{i+1}-}) - f^\beta(\tau_i, X_{\tau_i}) = \int_{\tau_i}^{\tau_{i+1}-} L^0 f^\beta(s, X_s)ds + \int_{\tau_i}^{\tau_{i+1}-} \widetilde{L}^1 f^\beta(s, X_s)dW_s.
\]

At each jump time \( \tau_i, i = 1, \ldots, N_t \), \( f^\beta(t, X_t) \) has a jump

\[
f^\beta(\tau_i, X_{\tau_i}) - f^\beta(\tau_i-, X_{\tau_i-}) = f^\beta(\tau_i, X_{\tau_i-} + c^\beta(\tau_i-, X_{\tau_i-}, Y_i)) - f^\beta(\tau_i-, X_{\tau_i-}).
\]
Then by (3.8)–(3.10) we get

\[
\begin{align*}
 f^\beta(t, X_t) - f^\beta(0, X_0) &= \sum_{i=0}^{N_1} \left( \int_{t_{i-1}}^{t_i} L^0 f^\beta(s, X_s) ds + \int_{t_{i-1}}^{t_i} L^1 f^\beta(s, X_{\tau_i -}) dW_s \right) \\
 &\quad + \sum_{i=1}^{N_1} \left( f^\beta(t_i, X_{\tau_i -}) - f^\beta(t_i, X_{\tau_i -}) - c^\beta(t_i, X_{\tau_i -}, Y_i) \right) \\
 &= \int_0^t L^0 f^\beta(s, X_s) ds + \int_0^t L^1 f^\beta(s, X_{\tau_i -}) dW_s \\
 &\quad + \int_0^t \int_E L^{-1}_{e} f^\beta(s, X_{\tau_i -}) \mu(de, ds),
\end{align*}
\]

where \( L^{-1}_{e} \) is defined by (3.6). We complete the proof.

It is worth noting that the Itô formula (3.5) can be written equivalently as

\[
\begin{align*}
 f^\beta(t, X_t) &= f^\beta(0, X_0) + \int_0^t L^0 f^\beta(s, X_s) ds + \int_0^t L^1 f^\beta(s, X_{\tau_i -}) dW_s \\
 &\quad + \int_0^t \int_E L^{-1}_{e} f^\beta(s, X_{\tau_i -}) \mu(de, ds),
\end{align*}
\]

where

\[
L^0 f^\beta(s, x) = L^0 f^\beta(s, x) + \int_E L^{-1}_{e} f^\beta(s, x) \lambda(de).
\] (3.11)

Note that when \( f \) is independent of \( x_i \), the Itô formula (3.5) for MSDEJs reduces to the one for standard SDEJs [8, 26]. Hence, the Itô formula for MSDEJs can be seen as a generalization of the one for standard SDEJs.

### 3.2 Itô-Taylor expansion for MSDEJs

In this subsection, by utilizing Itô’s formula, we construct the Itô-Taylor expansion for MSDEJs. To proceed, we introduce multiple Itô integrals and coefficient functions as below.

#### 3.2.1 Multiple Itô integrals

In this subsection, we introduce the multiple stochastic integrals.

(A) Multi-indices

Let \( \alpha = (j_1, \cdots, j_l) \) be a multi-index with \( j_i \in \{-1, 0, 1, \cdots, m\} \), \( i = 1, \cdots, l \). Set \( l(\alpha) = l \) to be the length of \( \alpha \), and let \( \mathcal{M} \) be the set of all multi-indices, i.e.,

\[
\mathcal{M} = \left\{(j_1, j_2, \cdots, j_l) : j_i \in \{-1, 0, 1, \cdots, m\}, \right. \\
\left. i \in \{1, 2, \cdots, l\}, l \in \mathbb{N}_+ \right\} \cup \{v\},
\]
where $\nu$ is the multi-index of length zero, i.e., $l(\nu) = 0$. For a given $\alpha \in \mathcal{M}$ with $\ell(\alpha) \geq 1$, $-\alpha$ and $-\alpha^-$ are two multi-indices obtained by deleting the first and the last component of $\alpha$, respectively. We also denote by

$$
n(\alpha) : \text{the number of the components of } \alpha \text{ equal to } 0,
$$

$$
s(\alpha) : \text{the number of the components of } \alpha \text{ equal to } -1.
$$

Moreover, for a given $\alpha \in \mathcal{M}$, let $e = (e_1, \ldots, e_{s(\alpha)})$ denote a vector $e \in \mathbb{E}^{e(\alpha)}$.

(B) Multiple integrals

For a given $\alpha \in \mathcal{M}$, we define the multiple Itô integral operator $I_\alpha$ on the adapted right continuous processes $\{f = f(t, e_1, \ldots, e_{s(\alpha)}), t \geq 0\}$ with left limits by

$$I_\alpha[f(\cdot)]_{\rho, \tau} := \begin{cases} f(\tau), & l = 0, \\
\int_\rho^\tau I_{\alpha - [f(\cdot)]_{\rho, s}} ds, & l \geq 1 \text{ and } j_l = 0, \\
\int_\rho^\tau I_{\alpha - [f(\cdot)]_{\rho, s}} dW^h_s, & l \geq 1 \text{ and } j_l = 1, \\
\int_\rho^\tau \int_{\mathbb{E}} I_{\alpha - [f(\cdot)]_{\rho, s}} d\mu(de_{s(\alpha)}, ds), & l \geq 1 \text{ and } j_l = -1,
\end{cases}
$$

where $\rho$ and $\tau$ are two stopping times satisfying $0 \leq \rho \leq \tau \leq T$, a.s. and all the integrals exist. For instance,

$$I_f[\cdot]_{0, t} = f(t), \quad I_{(0)}[\cdot]_{0, t} = \int_0^t f(s) ds, \quad I_{(1)}[\cdot]_{0, t} = \int_0^t f(s) dW_s^1,
$$

$$I_{(-1, -1)}[\cdot]_{0, t} = \int_0^t \int_0^{s_2^-} \int_{\mathbb{E}} f(s_1 -, e_1, e_2) d\mu(de_1, ds_1) d\mu(de_2, ds_2).
$$

3.2.2 Coefficient functions

For a given function

$$f(t, x', x, e) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{E} \rightarrow \mathbb{R},$$

by (3.4) and (3.6), we have

$$L^0 f^\beta(t, x, e) = \frac{\partial f^\beta}{\partial t}(t, x, e) + \sum_{k=1}^d b_k^\beta(t, x) \frac{\partial f^\beta}{\partial x_k}(t, x, e)$$

$$+ \frac{1}{2} \sum_{i,k=1}^d \sum_{j=1}^d \sigma_{ij}^\beta(t, x) \sigma_{kj}^\beta(s, x) \frac{\partial^2 f^\beta}{\partial x_i \partial x_k}(t, x, e),$$

with

$$\frac{\partial f^\beta}{\partial t}(t, x, e) = \mathbb{E} \left[ \frac{\partial f}{\partial t}(t, \beta_t, x, e) + \nabla_{x'} f(t, \beta_t, x, e) \psi_t + \frac{1}{2} \text{Tr} \left[ f_{x' x'}(t, \beta_t, x, e) \varphi_t \varphi_t^\top \right] \right],$$

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and

\[ L^j f^\beta (t, x, e) = \sum_{k=1}^{d} \frac{\partial f^\beta}{\partial x^k}(t, x, e) \sigma^\beta_{kj}(t, x), \quad j = 1, \ldots, m, \]  
(3.14)

\[ L_{e_2}^{-1} f^\beta (t, x, e_1) = f^\beta (t, x + c^\beta (t, x, e_2), e_1) - f^\beta (t, x, e_1). \]

Based on (3.13) and (3.14), we present the following coefficient functions and hierarchical and remainder sets.

(C) Itô coefficient functions

For a given \( \alpha = (j_1, \ldots, j_l) \in \mathcal{M} \) and a smooth function \( f(t, x', x) \), we define the coefficient function \( f^\alpha_\beta \) by

\[
f^\alpha_\beta (t, x, e) := \begin{cases} f^\beta (t, x), & l = 0, \\ L^{j_l} f_{-\alpha}^\beta (t, x, e_{j_l}, \ldots, e_{\pi(-\alpha)}), & l \geq 1 \text{ and } j_l \geq 0, \\ L_{e_{j_l}(\alpha)}^{-1} f_{-\alpha}^\beta (t, x, e_{j_l}, \ldots, e_{\pi(-\alpha)}), & l \geq 1 \text{ and } j_l = -1, 
\end{cases}
\]  
(3.15)

where \( e = (e_1, \ldots, e_{\pi(\alpha)}) \in \mathbb{E}^\pi(\alpha) \). The dependence on \( e \) in (3.15) is introduced by the repeated application of the operator \( L_{e_2}^{-1} \) in (3.14). Take \( m = d = q = 1 \) and let \( f(t, x', x) = x \), and we can deduce the following examples:

\[
f^0_\beta (t, x) = b^\beta (t, x), \quad f^1_1 (t, x, e) = \sigma^\beta (t, x), \quad f^1_{-1} (t, x, e) = c^\beta (t, x, e),
\]

\[
f^1_{(1, -1)} (t, x, e) = L^1 c^\beta (t, x, e) = \mathbb{E} \left[ \frac{\partial c}{\partial x} (t, \beta_{t, -}, x, e) \right] \sigma^\beta (t, x).
\]

Here, we have assumed that the functions \( b, \sigma, c, \) and \( f \) satisfy all the smoothness and integrability conditions needed in the definitions of (3.15).

(D) Hierarchical and remainder sets

We call a subset \( A \subset \mathcal{M} \) a hierarchical set if it satisfies

\[ A \neq \emptyset, \quad \sup_{\alpha \in A} l(\alpha) < \infty, \quad \text{and} \quad -\alpha \in A \, \text{for each} \, \alpha \in \mathcal{M} \setminus \{\nu\}; \]

and its remainder set \( B(A) \) is defined by

\[ B(A) = \{\alpha \in \mathcal{M} \setminus A : -\alpha \notin A\}. \]

Take \( m = 1 \) for instance and we give a hierarchical set

\[ \mathcal{A}_0 = \{v, (-1), (0), (1)\}, \]

and its remainder set is

\[ B(\mathcal{A}_0) = \{(-1, -1), (0, -1), (1, -1), (-1, 0), (0, 0), (1, 0), (1, -1), (0, 1), (1, 1)\}. \]

3.2.3 The Itô-Taylor expansion

In this subsection, by using Itô’s formula (3.5) for MSDEJs, we present the Itô-Taylor expansions of

\[ f^\beta (t, X_t) = \mathbb{E} \left[ f(t, \beta_t, x) \right]_{x=X_t} \]
for the solution \( X_t \) of the MSDEJ (3.1) with \( \beta_t \) defined by (3.2) satisfying
\[
\mathbb{E} \left[ \int_0^T \int_{\mathcal{E}} |\beta_t|^2 \lambda(de)dt \right] < \infty.
\]

**Theorem 3.2 (Itô-Taylor expansion)** Let \( \rho \) and \( \tau \) be two stopping times with \( 0 \leq \rho \leq \tau \leq T \), a.s.. Then for a given hierarchical set \( \mathcal{A} \subset \mathcal{M} \) and a function \( f(t, x', x) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \), we have the Itô-Taylor expansion
\[
f^\beta(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}} I_{\alpha} \left[ f_{\alpha}^\beta(\rho, X_\rho) \right]_{\rho, \tau} + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} I_{\alpha} \left[ f_{\alpha}^\beta(\cdot, X_\cdot) \right]_{\rho, \tau},
\]
(3.16)
provided that all of the coefficient functions \( f_{\alpha}^\beta \) are well defined and all of the multiple Itô integrals exist.

**Proof** By an iterated application of the Itô formula (3.5), the proof of the above theorem is analogous to the ones of the Itô-Taylor expansions for standard SDEs [13] and SDEJs [26]. So, we omit it here. \( \square \)

For notational simplicity, we have suppressed the dependence on \( e \in \mathcal{E}(\alpha) \) in the coefficients \( f_{\alpha} \) in (3.16). Moreover, when \( f \) is independent of \( x' \), the Itô-Taylor expansions (3.16) for MSDEJs reduce to the ones for standard SDEJs [26]. Hence, the Itô-Taylor expansions for MSDEJs can be seen as a generalization of the ones for standard SDEJs.

## 4 Itô-Taylor schemes for MSDEJs

In this section, based on the Itô-Taylor expansions (3.16), we propose the general Itô-Taylor schemes for the MSDEJ (1.1).

Without loss of generality, we let \( t_0 = 0 \) and take \( (\xi', \xi) = (x_0, X_0) \). For simplicity, we write \( X_t^{0, x_0, X_0} = X_t^{x_0, X_0} \) to obtain
\[
X_t^{x_0, X_0} = X_0 + \int_0^t \mathbb{E} \left[ b(s, X_s^{x_0, X_0}, x) \right] dx_{s=x_0} \quad dW_s
+ \int_0^t \mathbb{E} \left[ \sigma(s, X_s^{x_0, X_0}, x) \right] dx_{s=x_0} \quad ds,
\]
(4.1)
where \( X_t^{x_0, x_0} = X_t^{x_0, x_0} x_{x=x_0} \).

By choosing different hierarchical sets \( \mathcal{A} \) in the Itô-Taylor expansion (3.16), we shall derive the two types of strong order \( \gamma \) and weak order \( \eta \) Itô-Taylor schemes for solving the MSDEJ (4.1). To this end, we take a uniform time partition on \([0, T]\
\)
\[
0 = t_0 < t_1 < \cdots < t_N = T,
\]
where \( t_{k+1} \) is \( \mathcal{F}_{t_k} \)-measurable for \( k = 0, 1, \ldots, N - 1 \).
Now, we let $X^x_k$ and $X^x_k$ be the approximation of the solution $X_t^{x_0,x_0}$ and $X_t^{x_0,x_0}$ at time $t = t_k$, respectively, and denote by

$$f_kX^x_k(t_k, X^x_k) = \mathbb{E}\left[f\left(t_k, X^x_k, x\right)\right]_{x=x^x_k},$$

$$g_kX^x_k(t_k, X^x_k, e) = \mathbb{E}\left[g\left(t_k, X^x_k, x, e\right)\right]_{x=X^x_k},$$

for $f(t, x', x) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $g(t, x', x, e) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{E} \to \mathbb{R}$.

### 4.1 Strong Itô-Taylor schemes

To construct the strong Itô-Taylor schemes for the MSDEJ (4.1), for $\gamma = 0.5, 1.0, 1.5, \cdots$, we define the hierarchical set $\mathcal{A}_\gamma$ by

$$\mathcal{A}_\gamma = \left\{ \alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \leq 2\gamma \text{ or } l(\alpha) = n(\alpha) = \gamma + \frac{1}{2} \right\}$$

and denote its remainder set by $\mathcal{B}(\mathcal{A}_\gamma)$. Take $f(t, x', x) = x$ and let $\beta_t = X_t^{x_0,x_0}$, then by Theorem 3.2, for $k = 0, 1, \ldots, N - 1$, we have the Itô-Taylor expansion

$$X_t^{x_0,x_0} = \sum_{\alpha \in \mathcal{A}_\gamma} \mathcal{I}_\alpha \left[ f_\alpha X_t^{x_0,x_0} \left( t_k, X_t^{x_0,x_0} \right) \right]_{t_k, t_{k+1}} + \sum_{\alpha \in \mathcal{B}(\mathcal{A}_\gamma)} \mathcal{I}_\alpha \left[ f_\alpha X_t^{x_0,x_0} \left( \cdot, X_t^{x_0,x_0} \right) \right]_{t_k, t_{k+1}}. \quad (4.2)$$

By removing the remainder term in (4.2), we propose the following general strong order $\gamma$ Itô-Taylor scheme for solving the MSDEJ (4.1).

**Scheme 4.1** (Strong order $\gamma$ Itô-Taylor scheme)

$$X_t^{x_0,x_0} = \sum_{\alpha \in \mathcal{A}_\gamma} \mathcal{I}_\alpha \left[ f_\alpha X_t^{x_0,x_0} \left( t_k, X_t^{x_0,x_0} \right) \right]_{t_k, t_{k+1}}.$$

Based on Scheme 4.1, by taking $\gamma = 0.5$ and 1.0, we will give two specific strong Taylor schemes for MSDEJs in the following subsections.

#### 4.1.1 The Euler scheme

Take $\gamma = 0.5$ in Scheme 4.1, and we have

$$\mathcal{A}_{0.5} = \{v, (-1), (0), (1)\}$$

and

$$\mathcal{B}(\mathcal{A}_{0.5}) = \{(-1, -1), (0, -1), (1, -1), (-1, 0), (0, 0), (1, 0), (-1, 1), (0, 1), (1, 1)\}.$$
Then by (4.1), we get the strong order 0.5 Itô-Taylor scheme for solving the MSDEJ (4.1)

\[
X_{k+1}^{X_0} = X_k^{X_0} + b_k^{X_0}(t_k, X_k^{X_0})\Delta t_k + \sigma_k^{X_0}(t_k, X_k^{X_0})\Delta W_k + \int_{t_k}^{t_{k+1}} \int_{\mathbb{E}} c_k^{X_0}(t_k, X_k^{X_0}, e) \mu(de, dt)
\]

which is the so-called Euler scheme. Here

- \(\Delta t_k = t_{k+1} - t_k\) and \(\Delta W_k = W_{t_{k+1}} - W_{t_k}\);
- \(N_i = \mu(\mathbb{E} \times [0, t])\) and \((\tau_i, Y_i)\) are the \(i\)th jump time and jump size.

### 4.1.2 The strong order 1.0 Itô-Taylor scheme

Take \(\gamma = 1\) in Scheme 4.1, and we have \(A_1 = \{(v, (-1), (0), (1), (1, 1), (1, -1), (-1, 1), (-1, -1))\}\) and

\[
\mathcal{B}(A_1) = \{(0, -1), (-1, 0), (0, 0), (1, 0), (0, 1), (-1, 1, 1), (0, 1, 1), (-1, 1, -1), (-1, -1, -1), (0, -1, 1), (1, -1, 1), (-1, -1, 1), (0, -1, -1), (1, -1, -1)\}.
\]

Then by (4.1), we get the strong order 1.0 Itô-Taylor scheme

\[
X_{k+1}^{X_0} = X_k^{X_0} + b_k^{X_0}(t_k, X_k^{X_0})\Delta t_k + \sigma_k^{X_0}(t_k, X_k^{X_0})\Delta W_k + \int_{t_k}^{t_{k+1}} \int_{\mathbb{E}} c_k^{X_0}(t_k, X_k^{X_0}, e) \mu(de, dt)
+ \int_{t_k}^{t_{k+1}} \int_{\mathbb{E}} L^1 \sigma_k^{X_0}(t_k, X_k^{X_0})dW_z dW_z
+ \int_{t_k}^{t_{k+1}} \int_{\mathbb{E}} L^{-1} \sigma_k^{X_0}(t_k, X_k^{X_0})\mu(de, dz)dW_z
+ \int_{t_k}^{t_{k+1}} \int_{\mathbb{E}} L^{-1} \sigma_k^{X_0}(t_k, X_k^{X_0})\mu(de_1, d_2)\mu(de_2, ds).
\]
Combining with the Itô formula (3.5) for MSDEJs and the properties of jump times, we can write the scheme (4.4) as

\[ X_{k+1}^{X_0} = X_k^0 + bX_k^{X_0}(t_k, X_k^0) \Delta t_k + \sigma X_k^{X_0}(t_k, X_k^0) \Delta W_k + \sum_{i=N_{k+1}}^{N_k} cX_k^{X_0}(t_k, X_k^0, Y_i) \]

which is readily applicable for scenario simulation.

4.2 Weak Itô-Taylor schemes

To construct the weak Itô-Taylor schemes for MSDEJs, for \( \eta = 1.0, 2.0, \ldots \), we define the hierarchical set \( \Gamma_\eta \) by

\[ \Gamma_\eta = \{ \alpha \in \mathcal{M} : l(\alpha) \leq \eta \} \tag{4.5} \]

and denote its remainder set by \( \mathcal{B}(\Gamma_\eta) \). Take \( f(t, x', x) = x \) and let \( \beta_t = X_t \), then by Theorem 3.2, for \( k = 0, 1, \ldots, N - 1 \), we have the Itô-Taylor expansion

\[ X_{k+1}^{X_0, X_0} = \sum_{\alpha \in \Gamma_\eta} I_\alpha [f^X_{\alpha}(t_k, X_{tk}^{X_0, X_0})]_{tk, tk+1} + \sum_{\alpha \in \mathcal{B}(\Gamma_\eta)} I_\alpha [f^X_{\alpha}(\cdot, X_{tk}^{X_0, X_0})]_{tk, tk+1}. \tag{4.6} \]

Remove the remainder term in (4.6), and we propose the following general weak order \( \eta \) Itô-Taylor schemes for solving the MSDEJ (4.1).

Scheme 4.2 (Weak order \( \eta \) Itô-Taylor scheme)

\[ X_{k+1}^{X_0} = \sum_{\alpha \in \Gamma_\eta} I_\alpha \left[ f^X_{\alpha}(t_k, X_k^0) \right]_{tk, tk+1}. \]

Based on Scheme 4.2, by taking \( \eta = 1.0 \) and 2.0, we will present two specific weak Taylor schemes for MSDEJs.
4.2.1 The Euler scheme

Taking $\eta = 1.0$ in (4.5) leads to

$$\Gamma_{1.0} = \{v, (-1), (0), (1)\},$$

$$\mathcal{B}(\Gamma_{1.0}) = \{(-1, -1), (0, -1), (1, -1), (-1, 0), (0, 0), (1, 0), (-1, 1), (0, 1), (1, 1)\}.$$  

Then by Scheme 4.2, we get the Euler scheme (4.3)

$$X_{k+1}^0 = X_k^0 + bX_k^0(t_k, X_k^0)\Delta t_k + \sigma X_k^0(t_k, X_k^0)\Delta W_k + \sum_{i=N_{k+1}}^{N_{k+1}} cX_k^0(t_k, X_k^0, Y_i),$$

which is also the weak order 1.0 Itô-Taylor scheme.

4.2.2 The weak order 2.0 Itô-Taylor scheme

Taking $\eta = 2.0$ in (4.5) gives

$$\Gamma_{2.0} = \{v, (-1), (0), (1), (1, 1), (1, -1), (-1, 1), (-1, -1), (0, 0), (1, 0), (-1, 0), (0, -1)\},$$

$$\mathcal{B}(\Gamma_{2.0}) = \{(-1, 1, 1), (0, 1, 1), (-1, 1, 1), (-1, 1, -1), (0, 1, -1), (1, 1, -1), (-1, -1, 1),$$

$$(-1, 1, 1), (1, 1, -1), (-1, -1, -1), (0, -1, -1), (1, -1, -1), (-1, 0, 0),$$

$$(-1, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 1, -1), (0, 0, 1),$$

$$(-1, 1, 1), (1, 0, 1), (0, -1, 0), (1, 1, 0), (0, 1, -1), (1, 0, -1)\}.$$  

Then by Scheme 4.2, we get the weak order 2.0 Itô-Taylor scheme

$$X_{k+1}^0 = X_k^0 + bX_k^0(t_k, X_k^0)\Delta t_k + \sigma X_k^0(t_k, X_k^0)\Delta W_k + \sum_{i=N_{k+1}}^{N_{k+1}} cX_k^0(t_k, X_k^0, Y_i)$$

$$+ \frac{1}{2} L^1 \sigma X_k^0(t_k, X_k^0)((\Delta W_k)^2 - \Delta t_k) + \sum_{i=N_{k+1}}^{N_{k+1}} L^1 cX_k^0(t_k, X_k^0, Y_i)(W_{t_i} - W_{t_k})$$

$$+ \sum_{i=N_{k+1}}^{N_{k+1}} \left( \sigma X_k^0(t_k, X_k^0) + cX_k^0(t_k, X_k^0, Y_i) \right) - \sigma X_k^0(t_k, X_k^0) \left( W_{t_{k+1}} - W_{t_k} \right)$$
where $\Delta Z_k$ is a random variable defined by

$$
\Delta Z_k = \int_{t_k}^{t_{k+1}} \int_{t_k}^s d W (s) d s = \Delta W_k \Delta t_k - \int_{t_k}^{t_{k+1}} \int_{t_k}^s d z d W_z .
$$

Remark 4.1 Note that to solve the MSDEJ (4.1) for $X_0 \neq x_0$, we need two steps in succession. We take the Euler scheme (4.3) for instance to illustrate this procedure

- Step 1: solve (4.1) with $X_0 = x_0$ to obtain $\{X_n^{X_0}\}_{n=0}^{N}$

$$
X_{n+1}^{X_0} = X_n^{X_0} + b(X_n^{X_0}) \Delta t_n + \sigma(X_n^{X_0}) \Delta W_n + \sum_{i=N_{n+1}}^{N_n + 1} c(X_n^{X_0}, Y_i);
$$

- Step 2: solve (4.1) with $X_0 \neq x_0$ to get $\{X_n^{X_0}\}_{n=0}^{N}$ after we got $\{X_n^{X_0}\}_{n=0}^{N}$

$$
X_{n+1}^{X_0} = X_n^{X_0} + b(X_n^{X_0}) \Delta t_n + \sigma(X_n^{X_0}) \Delta W_n + \sum_{i=N_{n+1}}^{N_n + 1} c(X_n^{X_0}, Y_i). \tag{4.7}
$$

Here $f_t^{X_0}(t_n, x, e)$ denotes $E[f_t^{X_0}(t_n, X_n^{X_0}, x, e)]$ for $f = b$, $\sigma$ and $c$.

5 Error estimates for strong Taylor schemes

In this section, based on the relationship between the local and global convergence rates, we shall prove the error estimates of the strong order $\gamma$ Itô-Taylor Scheme 4.1.
We shall perform the analysis in the case where $X_0 = x_0$ at first. For simplicity, we write the MSDEJ (4.1) with $X_0 = x_0$ as

$$X_t = X_0 + \int_0^t \mathbb{E}\left[b(s, X_s, x)\right]_{x=X_t} ds + \int_0^t \mathbb{E}\left[\sigma(s, X_s, x)\right]_{x=X_s} dW_s$$

$$\quad + \int_0^t \int_{\mathbb{E}} \mathbb{E}\left[c(s, x_{s-}, x, e)\right]_{x=X_{s-}} \mu(de, ds).$$

(5.1)

### 5.1 The general error estimate theorem

Let $\{X_t, Y(s)\}_{t \leq s \leq T}$ be the solution of the MSDEJ (5.1) starting from the point $(t, Y)$, that is

$$X_{t, Y(t + h)} = Y + \int_t^{t+h} b_{X_{t, Y}(s)} ds + \int_t^{t+h} \sigma_{X_{t, Y}(s)} dW_s$$

$$\quad + \int_t^{t+h} \int_{\mathbb{E}} c_{X_{t, Y}(s-)}(e) \mu(de, ds),$$

where $h \in [0, T - t]$. Let $\tilde{X}_{t, Y(t + h)}$ be the one-step approximation of $X_{t, Y(t + h)}$, and $\tilde{X}_{0, X_0}(t_k)$ is the corresponding solution of the one-step scheme

$$\tilde{X}_{0, X_0}(t_k) = \tilde{X}_{t_{k-1}, \tilde{X}_{0, X_0}(t_{k-1})}(t_k),$$

(5.3)

with $\tilde{X}_{0, X_0}(0) = X_0$. For simplicity, we denote $X_{0, X_0}(t_k)$ by $X(t_k)$ and $\tilde{X}_{0, X_0}(t_k)$ by $\tilde{X}_k$. Then, the one-step scheme (5.3) becomes

$$\tilde{X}_k = \tilde{X}_{t_{k-1}, \tilde{X}_{k-1}}(t_k).$$

(5.4)

To present the general error estimate theorem for the one-step scheme (5.3), we first give the following two lemmas.

**Lemma 5.1** Let $X_{t, Y}(s)$ and $X_{t, \tilde{Y}}(s)$ be the solutions of the MSDEJ (5.2) with initial conditions $X_{t, Y}(t) = Y$ and $X_{t, \tilde{Y}}(t) = \tilde{Y}$, respectively. Let $Z = X_{t, Y}(t + h) - X_{t, \tilde{Y}}(t + h) - (Y - \tilde{Y})$, then under the assumptions (A1) – (A4), we have

$$\mathbb{E}[|Z|^2] \leq C\mathbb{E}[|Y - \tilde{Y}|^2]h,$$

(5.5a)

$$\mathbb{E}[|X_{t, Y}(t + h) - X_{t, \tilde{Y}}(t + h)|^2] \leq (1 + Ch)\mathbb{E}[|Y - \tilde{Y}|^2],$$

(5.5b)

where $C$ is a positive constant depending on $\lambda(E)$, the Lipschitz constant $L$ and the function $\rho(e)$ in assumption (A3).
Proof For any $0 \leq \theta \leq h$, by applying the Itô formula (3.5) to $|X_{t,Y}(t+\theta) - X_{t,\tilde{Y}}(t+\theta)|^2$, we obtain

$$
|X_{t,Y}(t+\theta) - X_{t,\tilde{Y}}(t+\theta)|^2
= |Y - \tilde{Y}|^2 + \int_t^{t+\theta} \left| \sigma^{X_{t,Y}}(s, X_{t,Y}(s)) - \sigma^{X_{t,\tilde{Y}}}(s, X_{t,\tilde{Y}}(s)) \right|^2 ds
+ 2 \int_t^{t+\theta} (X_{t,Y}(s) - X_{t,\tilde{Y}}(s)) \left( b^{X_{t,Y}}(s, X_{t,Y}(s)) - b^{X_{t,\tilde{Y}}}(s, X_{t,\tilde{Y}}(s)) \right) ds
+ 2 \int_t^{t+\theta} (X_{t,Y}(s) - X_{t,\tilde{Y}}(s)) \left( \sigma^{X_{t,Y}}(s, X_{t,Y}(s)) - \sigma^{X_{t,\tilde{Y}}}(s, X_{t,\tilde{Y}}(s)) \right) dW_s
+ \int_t^{t+\theta} \int_E \left( |X_{t,Y}(s-)-X_{t,\tilde{Y}}(s-)| + c^{X_{t,Y}}(s-, X_{t,Y}(s-), e)
- c^{X_{t,\tilde{Y}}}(s-, X_{t,\tilde{Y}}(s-), e) \right)^2 ds
+ \int_t^{t+\theta} \int_E \left( |X_{t,Y}(s-)-X_{t,\tilde{Y}}(s-)|^2 \right) \mu(de, ds).
$$

Then taking the expectation $\mathbb{E}[-]$ on the above equation and noting the fact that

$$
\mathbb{E}\left[ \int_s^t \int_E g(r-, e) \mu(de, dr) \right] = \mathbb{E}\left[ \int_s^t \int_E g(r-, e) \lambda(de) dr \right]
$$

for some adapted function $g$ with $0 \leq s \leq t$, we deduce

$$
\mathbb{E}[|X_{t,Y}(t+\theta) - X_{t,\tilde{Y}}(t+\theta)|^2]
= \mathbb{E}[|Y - \tilde{Y}|^2] + \mathbb{E}\left[ \int_t^{t+\theta} \left| \sigma^{X_{t,Y}}(s, X_{t,Y}(s)) - \sigma^{X_{t,\tilde{Y}}}(s, X_{t,\tilde{Y}}(s)) \right|^2 ds \right]
+ 2 \mathbb{E}\left[ \int_t^{t+\theta} (X_{t,Y}(s) - X_{t,\tilde{Y}}(s)) \left( b^{X_{t,Y}}(s, X_{t,Y}(s)) - b^{X_{t,\tilde{Y}}}(s, X_{t,\tilde{Y}}(s)) \right) ds \right]
+ \mathbb{E}\left[ \int_t^{t+\theta} \int_E \left( c^{X_{t,Y}}(s, X_{t,Y}(s), e) - c^{X_{t,\tilde{Y}}}(s, X_{t,\tilde{Y}}(s), e) \right)^2 ds \right]
+ 2 \mathbb{E}\left[ \left| X_{t,Y}(s) - X_{t,\tilde{Y}}(s) \right| \left| c^{X_{t,Y}}(s, X_{t,Y}(s), e) - c^{X_{t,\tilde{Y}}}(s, X_{t,\tilde{Y}}(s), e) \right| \lambda(de) ds \right].
$$

Under the assumptions (A2) and (A3), we get

$$
\mathbb{E}\left[ |X_{t,Y}(t+\theta) - X_{t,\tilde{Y}}(t+\theta)|^2 \right]
\leq \mathbb{E}[|Y - \tilde{Y}|^2] + 4L^2 \int_t^{t+\theta} \mathbb{E}\left[ |X_{t,Y}(s) - X_{t,\tilde{Y}}(s)|^2 \right] ds
+ 4L \int_t^{t+\theta} \mathbb{E}\left[ |X_{t,Y}(s) - X_{t,\tilde{Y}}(s)|^2 \right] ds
+ 8K_1 \int_t^{t+\theta} \mathbb{E}\left[ |X_{t,Y}(s) - X_{t,\tilde{Y}}(s)|^2 \right] ds
\leq \mathbb{E}[|Y - \tilde{Y}|^2] + (4L^2 + 4L + 8K_1) \int_t^{t+\theta} \mathbb{E}\left[ |X_{t,Y}(s) - X_{t,\tilde{Y}}(s)|^2 \right] ds.
$$
where \( K_1 = \int_E \rho^2(s) \lambda(de) \lor \lambda(E) \). Then by the Gronwall lemma [13], we obtain
\[
\mathbb{E} \left[ (X_{t,Y}(t + \theta) - X_{t,Y}(t))^2 \right] \leq e^{4(L^2 + L^2 + 2K_1)h} \mathbb{E} \left[ (Y - \bar{Y})^2 \right],
\]
which leads to the inequality (5.5b).

By the definition of \( Z \), we have
\[
Z = \int_t^{t+h} \left( b_{X_t,Y}(s, X_{t,Y}(s)) - b_{X_{t,Y}}(s, X_{t,Y}(s)) \right) ds
+ \int_t^{t+h} \left( \sigma_{X_t,Y}(s, X_{t,Y}(s)) - \sigma_{X_{t,Y}}(s, X_{t,Y}(s)) \right) dW_s
+ \int_t^{t+h} \left( c_{X_t,Y}(s, X_{t,Y}(s), e) - c_{X_{t,Y}}(s, X_{t,Y}(s), e) \right) \mu(de, ds)
= \int_t^{t+h} \left( \bar{b}_{X_t,Y}(s, X_{t,Y}(s)) - \bar{b}_{X_{t,Y}}(s, X_{t,Y}(s)) \right) ds
+ \int_t^{t+h} \left( \bar{\sigma}_{X_t,Y}(s, X_{t,Y}(s)) - \bar{\sigma}_{X_{t,Y}}(s, X_{t,Y}(s)) \right) dW_s
+ \int_t^{t+h} \left( \bar{c}_{X_t,Y}(s, X_{t,Y}(s), e) - \bar{c}_{X_{t,Y}}(s, X_{t,Y}(s), e) \right) \bar{\mu}(de, ds),
\]
where the function \( \bar{b} \) is defined by
\[
\bar{b}(t, x', x) = b(t, x', x) + \int_E c(t, x', x, e) \lambda(de).
\]

By assumptions (A2) and (A3), it is easy to conclude that \( \bar{b} \) satisfies the Lipschitz condition. Then we take square on both sides of (5.7) and take \( \mathbb{E}[\cdot] \) on the derived equation to get
\[
\mathbb{E}[|Z|^2] \leq 3 \mathbb{E} \left[ \int_t^{t+h} \left( \bar{b}_{X_t,Y}(s, X_{t,Y}(s)) - \bar{b}_{X_{t,Y}}(s, X_{t,Y}(s)) \right) ds \right]^2
+ 3 \mathbb{E} \left[ \int_t^{t+h} \left| \sigma_{X_t,Y}(s, X_{t,Y}(s)) - \sigma_{X_{t,Y}}(s, X_{t,Y}(s)) \right|^2 ds \right]
+ 3 \mathbb{E} \left[ \int_t^{t+h} \int_E \left| c_{X_t,Y}(s, X_{t,Y}(s), e) - c_{X_{t,Y}}(s, X_{t,Y}(s), e) \right|^2 \lambda(de) ds \right].
\]

Then by using (5.6) and (5.8), we deduce
\[
\mathbb{E}[|Z|^2] \leq 12(L^2 + K_1^2)h \mathbb{E} \left[ \int_t^{t+h} \left( \mathbb{E}[|X_{t,Y}(s) - X_{t,Y}(s)|^2] + |X_{t,Y}(s) - X_{t,Y}(s)|^2 \right) ds \right]
+ 6L^2 \mathbb{E} \left[ \int_t^{t+h} \left( \mathbb{E}[|X_{t,Y}(s) - X_{t,Y}(s)|^2] + |X_{t,Y}(s) - X_{t,Y}(s)|^2 \right) ds \right]
+ 6K_1 \mathbb{E} \left[ \int_t^{t+h} \left( \mathbb{E}[|X_{t,Y}(s) - X_{t,Y}(s)|^2] + |X_{t,Y}(s) - X_{t,Y}(s)|^2 \right) ds \right].
\]
\[ \leq 12(2L^2 + 2K_1^2 + K_1)(1 + h) \left[ \int_t^{t+h} \mathbb{E}[|X_{t,Y}(s) - X_{t,Y}(s)|^2] ds \right] \]
\[ \leq 12(2L^2 + 2K_1^2 + K_1)(1 + h) \left[ \int_t^{t+h} \mathbb{E}[|Y - \bar{Y}|^2] \times e^{4(L^2 + L + 2K_1)h} ds \right] \]
\[ \leq 12(2L^2 + 2K_1^2 + K_1)(1 + h)e^{4(L^2 + L + 2K_1)h} \mathbb{E}[|Y - \bar{Y}|^2] h, \]
which proves (5.5a). The proof ends. \[ \square \]

**Lemma 5.2** Under the assumptions (A1) – (A4), for \( k = 0, \cdots, N - 1 \), we have
\[ \mathbb{E}\left[ \mathbb{E}^t \left[ (X_{t_k, \bar{X}_k}(t_{k+1}) - \bar{X}_k|F_{t_k}) \right] \right] \leq C (1 + \mathbb{E}[|Y|^2]) h^2, \tag{5.9} \]
where \( C \) is a positive constant depending on \( \lambda(E) \), the function \( \rho(e) \) in (A3) and the linear growth constant \( K \) in (A4).

**Proof** By using the properties of stochastic integrals w.r.t. the Poisson random measure, the proof of Lemma 5.2 is similar to the one of Lemma 4.2 in [28]. So we omit it here. \[ \square \]

Based on Lemmas 5.1 and 5.2, we give the general error estimate theorem for the one-step scheme (5.3).

**Theorem 5.1** Let \( X_{t,Y}(t + h) \) be defined as (5.2). If \( \bar{X}_{t,Y}(t + h) \) satisfies
\[ \mathbb{E}\left[ \mathbb{E}^t \left[ (X_{t,Y}(t + h) - \bar{X}_{t,Y}(t + h)|F_t) \right] \right] \leq C^* (1 + \mathbb{E}[|Y|^2] + |Y|^2) \frac{1}{2} h^{p_1}, \tag{5.10a} \]
\[ \left( \mathbb{E}\left[ X_{t,Y}(t+h) - \bar{X}_{t,Y}(t+h)|F_t \right] \right)^{\frac{1}{2}} \leq 1C^* \left( 1 + \mathbb{E}[|Y|^2] + |Y|^2 \right)^{\frac{1}{2}} h^{p_2}, \tag{5.10b} \]
where \( t \in [0, T - h] \), \( p_1 \) and \( p_2 \) are parameters satisfying \( p_2 \geq \frac{1}{4} \) and \( p_1 \geq p_2 + \frac{1}{2} \), and \( C^* > 0 \) is a constant independent of \( h \), \( X_{t,Y}(t + h) \) and \( \bar{X}_{t,Y}(t + h) \). Then for \( k = 1, \cdots, N \), it holds that
\[ \left( \mathbb{E}[|X(t_k) - \bar{X}_k|^2] \right)^{\frac{1}{2}} \leq C (1 + \mathbb{E}[|X_0|^2])^{\frac{1}{2}} h^{p_2 - \frac{1}{2}}, \tag{5.11} \]
where \( C \) is a constant independent of \( h \), \( X_{t,Y}(t + h) \) and \( \bar{X}_{t,Y}(t + h) \).

**Proof** By Theorem 2.1, Lemma 5.2, and Lemma 4.4 in the paper [28], it holds that for all \( k = 0, \cdots, N \),
\[ \mathbb{E}[|\bar{X}_k|^2] \leq C (1 + \mathbb{E}[|X_0|^2]) \tag{5.12} \]
Then based on Lemmas 5.1–5.2 and the inequality (5.12), the estimate (5.11) can be proved by using the similar proof procedure of Theorem 4.1 in [28], which provides the error estimate results for MSDEs only driven by the Brownian motion. The proof ends. \[ \square \]

**Remark 5.1** Theorem 5.1 implies that when the weak local error estimate of the one-step scheme (5.3) is of order \( p_1 \) and its strong local error estimate is of order \( p_2 \), then the global strong order of the scheme (5.3) is \( p_2 - \frac{1}{2} \). This result is consistent with the ones for SDEs and MSDEs.
5.2 The error estimates for strong Taylor schemes

In this subsection, utilizing Theorem 5.1, we prove the error estimates of Scheme 4.1 to reveal the orders of strong convergence of strong Taylor schemes.

Let \( W_t^n = t \) and \( \alpha = (i_1, i_2, \cdots, i_k) \in \mathcal{M} \) be a given multi-index. Then, we have the following two lemmas.

**Lemma 5.3** Let \( f^\beta \) and \( \beta \) be defined by (3.1) and (3.2), respectively. Assume that \( f^\alpha \) and \( I_\alpha[f^\beta(\cdot)]_{t,t+h} \) exist. Then for all \( f^\beta \) satisfying

\[
|f^\beta_\alpha(t, x)| \leq C(1 + \mathbb{E}[|\beta_t|^2] + |x|^2)^{1/2},
\]

we have the estimates:

\[
\mathbb{E} \left[ \left( I_\alpha[f^\alpha X_{t+h}^Y(\cdot, X_t, Y(\cdot))]_{t,t+h} \right)^2 \right] \leq CMh^{l(\alpha) + n(\alpha)},
\]

(5.13)

\[
\mathbb{E} \left[ I_\alpha[f^\alpha X_{t+h}^Y(\cdot, X_t, Y(\cdot))]_{t,t+h} \right] \leq C\sqrt{M}h^{l(\alpha)} \quad \text{if} \quad l(\alpha) = n(\alpha) + s(\alpha),
\]

(5.14)

where \( M = 1 + \mathbb{E}[|Y|^2] + |Y|^2 \).

**Proof** If \( \alpha = v \), i.e., \( l(\alpha) + n(\alpha) = 0 \), we get

\[
\mathbb{E} \left[ \left( I_\alpha[f^\alpha X_{t+h}^Y(\cdot, X_t, Y(\cdot))]_{t,t+h} \right)^2 \right] \leq C\mathbb{E}[1 + \mathbb{E}[|Y|^2] + |Y|^2].
\]

which leads to (5.13) with \( p(z) = l(\alpha) + n(\alpha) = 0 \).

Now we consider \( l(\alpha) > 0 \). If \( i_k = 0 \), then by the Holder’s inequality, we obtain

\[
\mathbb{E} \left[ \left( I_\alpha[f^\alpha X_{t+h}^Y(\cdot, X_t, Y(\cdot))]_{t,t+h} \right)^2 \right] \leq h \int_t^{t+h} \mathbb{E} \left[ \left( I_{\alpha_{0}}[f^\alpha X_{t+s}^Y(\cdot, X_t, Y(\cdot))]_{t,s} \right)^2 \right] ds.
\]

(5.15)

If \( i_k = 1 \), by Itô’s isometry formula, we have

\[
\mathbb{E} \left[ \left( I_\alpha[f^\alpha X_{t+h}^Y(\cdot, X_t, Y(\cdot))]_{t,t+h} \right)^2 \right] \leq \int_t^{t+h} \mathbb{E} \left[ \left( I_{\alpha_{1}}[f^\alpha X_{t,s}^Y(\cdot, X_t, Y(\cdot))]_{t,s} \right)^2 \right] ds.
\]

(5.16)
If \( i_k = -1 \), by Itô’s isometry formula, we deduce
\[
\mathbb{E}\left[ \left( \int_t^{t+h} \mathbb{E}\left[ f_{\alpha}^{X_t,Y}(\cdot, X_{t,Y}(\cdot)) \right]_{s,t+h} \right)^2 \bigg| \mathcal{F}_t \right] = \mathbb{E}\left[ \left( \int_t^{t+h} \mathbb{E}\left[ f_{\alpha}^{X_t,Y}(\cdot, X_{t,Y}(\cdot)) \right]_{s,t+h} \lambda(de, ds) \right)^2 \bigg| \mathcal{F}_t \right]
\]
(5.17)
Combining with (5.15)–(5.17), we deduce the recurrence relation
\[
\frac{p(\alpha)}{p(\alpha-1)} + (1 + \mathbb{I}_{\{i_k=0\}}) = \sum_{j=1}^{k} \left( 1 + \mathbb{I}_{\{i_j=0\}} \right) = l(\alpha) + n(\alpha),
\]
which implies that (5.13) holds true.

Similarly, we can prove (5.14). The proof ends.

Based on Theorem 5.1 and Lemma 5.3, we prove the error estimate of the strong order \( \gamma \) Taylor scheme in the following theorem.

**Theorem 5.2** Let \( X(t) \) and \( \tilde{X}_k \) be the solutions of the MSDEJ (5.1) and the strong order \( \gamma \) Taylor scheme 4.1, respectively. Let \( f(t, x', x) = x \) and
\[
|f_{\alpha}^{\beta}(t, x)| \leq C \left( 1 + \mathbb{E}[|\beta|^2] + |x|^2 \right)^{1/2}, \quad (t, x) \in [0, T] \times \mathbb{R}^d
\]
for all \( \alpha \in \mathcal{A}_\gamma \cup \mathcal{B}(A_{\gamma}) \) with \( \beta \) defined by (3.2). Then it holds that
\[
\max_{k \in \{1, 2, \ldots, N\}} \mathbb{E}[|X(t_k) - \tilde{X}_k|^2] \leq C \left( 1 + \mathbb{E}[|X_0|^2] \right) (\Delta t)^{2\gamma},
\]
where \( \Delta t = \Delta t_k = T/N \) for \( k = 0, 1, \ldots, N - 1 \).

**Proof** By the Itô-Taylor expansion (3.16), we have
\[
X_{t,Y}(t + h) = Y + \sum_{\alpha \in \mathcal{A}_{\gamma}, \nu} I_{\alpha} \left[ f_{\alpha}^{X_{t,Y}}(t, Y) \right]_{t,t+h} + R^\gamma,
\]
(5.18)
where
\[
R^\gamma = \sum_{\alpha \in \mathcal{B}(A_{\gamma})} I_{\alpha} \left[ f_{\alpha}^{X_{t,Y}}(\cdot, X_{t,Y}(\cdot)) \right]_{t,t+h}.
\]
Moreover, the strong order \( \gamma \) Taylor scheme 4.1 can be written as
\[
\tilde{X}_{t,Y}(t + h) = Y + \sum_{\alpha \in \mathcal{A}_{\gamma}, \nu} I_{\alpha} \left[ f_{\alpha}^{X_{t,Y}}(t, Y) \right]_{t,t+h}.
\]
(5.19)
Then, we subtract (5.19) from (5.18) and obtain
\[ R^\gamma = X_{t,Y}(t+h) - \tilde{X}_{t,Y}(t+h). \]  
\hspace{1cm} (5.20)

According to Lemma 5.3, we deduce
\[
E\left[ |R^\gamma|^2 |F_t \right] = E\left[ \left| \sum_{a \in B(A_\gamma)} I_a \left[ f_a^{X_{t,Y}} (\cdot, X_{t,Y}(\cdot)) \right]_{l,t+h} \right|^2 |F_t \right] 
\leq C \sum_{a \in B(A_\gamma)} E\left[ \left| I_a \left[ f_a^{X_{t,Y}} (\cdot, X_{t,Y}(\cdot)) \right]_{l,t+h} \right|^2 |F_t \right] 
\leq C \sum_{a \in B(A_\gamma)} (1 + E[|Y|^2] + |Y|^2) h^{l(\alpha)+n(\alpha)} 
\leq C (1 + E[|Y|^2] + |Y|^2) h^{2p_2},
\]
where \( 2p_2 = \min_{\alpha \in B(A_\gamma)} \{ l(\alpha) + n(\alpha) \} \). Since \( A_\gamma = \{ \alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \leq 2\gamma \} \), then we get
\[ p_2 = \frac{1}{2} \min_{\alpha \in B(A_\gamma)} \{ l(\alpha) + n(\alpha) \} = \gamma + \frac{1}{2}. \]

Now we prove \( p_1 \geq p_2 + \frac{1}{2} \). By Lemma 5.3, we can deduce
\[
E\left[ |R^\gamma|^2 |F_t \right] = E\left[ \left| \sum_{a \in B(A_\gamma)} I_a \left[ f_a^{X_{t,Y}} (\cdot, X_{t,Y}(\cdot)) \right]_{l,t+h} \right|^2 |F_t \right] 
\leq \sum_{a \in B(A_\gamma)} \left| E\left[ \left| I_a \left[ f_a^{X_{t,Y}} (\cdot, X_{t,Y}(\cdot)) \right]_{l,t+h} \right|^2 |F_t \right] \right| 
= \sum_{a \in B(A_\gamma)} \left| E\left[ \left| I_a \left[ f_a^{X_{t,Y}} (\cdot, X_{t,Y}(\cdot)) \right]_{l,t+h} \right|^2 |F_t \right] \right| 
\leq C \sum_{a \in B(A_\gamma)} (1 + E[|Y|^2] + |Y|^2)^{1/2} h^{l(\alpha)} 
\leq C (1 + E[|Y|^2] + |Y|^2)^{1/2} h^{p_1},
\]
where \( p_1 = \min_{\alpha \in B(A_\gamma)} \{ l(\alpha) : l(\alpha) = n(\alpha) + s(\alpha) \} \). Simple calculation yields
\[ p_1 = \min_{\alpha \in B(A_\gamma)} \{ l(\alpha) : l(\alpha) = n(\alpha) + s(\alpha) \} = \begin{cases} 
\gamma + 1, & \gamma = 1, 2, \ldots \\
\gamma + \frac{3}{2}, & \gamma = 0.5, 1.5, \ldots 
\end{cases},
\]
which implies that \( p_1 \geq p_2 + \frac{1}{2} \). Then by Theorem 5.1, we complete the proof. \qed

Based on Lemmas 5.1–5.3 and Theorem 5.1, by repeating the same procedures as in their proof, Lemmas 5.1–5.3 and Theorem 5.1 can be extended to general cases where \( X_0 \neq x_0 \). Then we can prove that Theorem 5.2 holds true for general cases in the same way.
Theorem 5.3 Let $X_t^{x_0, X_0}$ and $\tilde{X}_k^{X_0}$ be the solutions of the MSDEJ (4.1) and the strong order $\gamma$ Taylor scheme 4.1, respectively. Let $f(t, x', x) = x$ and

$$|f^{\beta}_a(t, x)| \leq C(1 + \mathbb{E}[|\beta_t|^2] + |x|^2)^{1/2}, \quad (t, x) \in [0, T] \times \mathbb{R}^d$$

for all $\alpha \in \mathcal{A}_\gamma \cup \mathcal{B}(\mathcal{A}_\gamma)$ with $\beta$ defined by (3.2). Then it holds that

$$\max_{k \in (1, 2, \ldots, N)} \mathbb{E}[|X_{t_k}^{x_0, X_0} - \tilde{X}_k^{X_0}|^2] \leq C(1 + \mathbb{E}[|x_0|^2 + |X_0|^2])(\Delta t)^{2\gamma},$$

where $\Delta t = \Delta t_k = T/N$ for $k = 0, 1, \ldots, N - 1$.

Remark 5.2 From Theorem 5.3, we come to the conclusion that the order of strong convergence of the strong order $\gamma$ Taylor scheme 4.1 is $\gamma$.

6 Error estimates for weak Taylor schemes

In this section, we focus on the error estimates of the weak order $\eta$ Itô-Taylor Scheme 4.2. For simplicity, we only perform the analysis for $X_0 = x_0$. And based on the analysis for $X_0 = x_0$, the error estimates for general cases can be obtained in a similar way.

Let $C_{p, 2k, 2k}^{k, 2k}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ be the set of functions $\varphi(t, x', x): [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that all their derivatives with respect to $t, x'$ and $x$ up to $k, 2k$ and $2k$, respectively, are continuous and of polynomial growth.

For a given $\eta \in \{1, 2, \ldots\}$ and function $g \in C_{p, \eta}^{2\eta}(\mathbb{R}^d; \mathbb{R})$, we define

$$u(s, y) = \mathbb{E}[g(X_t^{s, y})], \quad (s, y) \in [0, T] \times \mathbb{R}^d$$

where $(s, y) \in [0, T] \times \mathbb{R}^d$ and $X_t^{s, y}, s \leq t \leq T$, is the solution of the MSDEJ (5.1) starting from $(s, y)$. Then we get

$$u(0, X_0) = \mathbb{E}[g(X_T^{0, X_0})] = \mathbb{E}[g(X_T)].$$

Lemma 6.1 (Kolmogorov backward equation) Assume that the coefficients of the MSDEJ (5.1) have the components $b^k, \sigma^{k,j}, c^k \in C_{p, \eta}^{\eta, 2\eta}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ for $1 \leq k \leq d$ and $1 \leq j \leq m$ with uniformly bounded derivatives. Then the functional $u$ defined in (6.1) is the unique solution of the nonlocal Kolmogorov backward partial integral differential equations (PIDEs)

$$\begin{align*}
\mathcal{L}^0 u(s, y) &= 0, \quad (s, y) \in [0, T] \times \mathbb{R}^d, \\
u(T, y) &= g(y), \quad y \in \mathbb{R}^d,
\end{align*}$$

where $\mathcal{L}^0$ is defined by (3.11). Moreover, we have

$$u(s, \cdot) \in C_{p, \eta}^{2\eta}(\mathbb{R}^d; \mathbb{R}), \quad 0 \leq s \leq T.$$

Proof The details of the proof can be found in Lemma 7.1 and Theorem 7.3 in [12]. So, we omit it here. \qed
To proceed, we introduce the following two lemmas. For a given \( x \in \mathbb{R} \) and \( p \in \mathbb{N} \), we denote by \([x]\) the integer part of \( x \) and \( \mathcal{A}_p \) the set of multi-indices \( \alpha = (j_1, \ldots, j_l) \) of length \( l \leq p \) with components \( j_i \in \{-1, 0\}, \) for \( i \in \{1, \ldots, l\} \).

**Lemma 6.2** Let \( X_t \) be the solution of the MSDEJ (5.1), and \( \rho \) and \( \tau \) be two stopping times with \( \tau \) being \( \mathcal{F}_\rho \)-measurable and \( 0 \leq \rho \leq \tau \leq T \) a.s.. Given \( \alpha \in \mathcal{M} \), let \( p = l(\alpha) - \left[ \frac{l(\alpha) + n(\alpha)}{2} \right] \) and \( f(t, x) \in C^{p, 2p}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}) \) be a \( \mathcal{F}_t \)-adapted process such that for any \( \alpha \in \mathcal{A}_p \)

\[
\mathbb{E} \left[ (f_{\alpha}(t, X_t))^2 \bigg| \mathcal{F}_\rho \right] \leq K, \quad a.s., \quad t \in [\rho, \tau],
\]

for some constant \( K \). Moreover, for an adapted process \( g(\cdot) = g(\cdot, e) \) with \( e \in \mathbb{E}^{s(\alpha)} \), if \( \mathbb{E} \left[ (g(t, e))^2 \bigg| \mathcal{F}_\rho \right] < +\infty \) a.s. for \( t \in [\rho, \tau] \), then

\[
\mathbb{E} \left[ \left| f(\tau, X_\tau) I_\alpha[g(\cdot)]_{\rho, \tau} \bigg| \mathcal{F}_\rho \right| \right] \leq C_1(\tau - \rho)^{l(\alpha)}, \quad (6.5)
\]

where the positive constants \( C_1 \) and \( C_2 \) do not depend on \( (\tau - \rho) \).

**Proof** It is obvious that (6.5) holds for \( |\alpha| = 0 \). Suppose that (6.5) holds for all \( |\alpha| \leq l \). Now, we consider \( \alpha = (j_1, \ldots, j_{i+1}) \) with \( j_{i+1} = -1 \) and obtain

\[
I_{\alpha}[g(\cdot)]_{\rho, \tau} = \int_\rho^\tau \int_E I_{\alpha-}[g(\cdot)]_{\rho, s-} \mu(de, ds). \quad (6.6)
\]

Then, by the Itô formula (3.5), we get

\[
f(\tau, X_\tau) I_{\alpha}[g(\cdot)]_{\rho, \tau} = \int_\rho^\tau L^0 f(s, X_s) I_{\alpha}[g(\cdot)]_{\rho, s} ds \\
+ \int_\rho^\tau \mathcal{L}^1 f(s, X_s) I_{\alpha}[g(\cdot)]_{\rho, s} dW_s \\
+ \int_\rho^\tau \int_E \left( f(s, X_s) I_{\alpha}[g(\cdot)]_{\rho, s} \\
- f(s, X_s-) I_{\alpha-}[g(\cdot)]_{\rho, s-} \right) \mu(de, ds). \quad (6.7)
\]

When \( s \) is a jump time, we have

\[
f(s, X_s) = L_e^{-1} f(s, X_s-) + f(s, X_s-),
\]

\[
I_{\alpha}[g(\cdot)]_{\rho, s} = I_{\alpha}[g(\cdot)]_{\rho, s-} + I_{\alpha-}[g(\cdot)]_{\rho, s-}.
\]

By inserting the above equations into (6.7), we deduce

\[
f(\tau, X_\tau) I_{\alpha}[g(\cdot)]_{\rho, \tau} = \int_\rho^\tau L^0 f(s, X_s) I_{\alpha}[g(\cdot)]_{\rho, s} ds \\
+ \int_\rho^\tau \mathcal{L}^1 f(s, X_s) I_{\alpha}[g(\cdot)]_{\rho, s} dW_s \\
+ \int_\rho^\tau \int_E \left( f(s, X_s) I_{\alpha-}[g(\cdot)]_{\rho, s-} \\
+ L_e^{-1} f(s, X_s-) I_{\alpha}[g(\cdot)]_{\rho, s-} \right) \mu(de, ds). \quad (6.8)
\]
which leads to

\[
\mathbb{E}\left[ f(\tau, X_\tau) I_\alpha[g(\cdot)]_{\rho, \tau} \right] = \left| \int_\rho^\tau \int_\varepsilon \int_\mathbb{E} \left[ L^0 f(s, X_s) I_\alpha[g(\cdot)]_{\rho, s} \right] ds \right|
\]

\[
+ \left| \int_\rho^\tau \int_\varepsilon \int_\mathbb{E} \left[ f(s, X_s) I_\alpha[g(\cdot)]_{\rho, s} \right] \lambda(de) ds \right|
\]

\[
+ \left| \int_\rho^\tau \int_\varepsilon \int_\mathbb{E} \left[ L^{-1}_e f(s, X_s) I_\alpha[g(\cdot)]_{\rho, s} \right] \lambda(de) ds \right|
\]

\[
\leq C(\tau - \rho)^{l+1} + \int_\rho^\tau \int_\varepsilon \int_\mathbb{E} \left[ \left( L^0 f(s, X_s) I_\alpha[g(\cdot)]_{\rho, s} \right) \right] ds \lambda(de) ds.
\] (6.9)

Moreover, by the conditions of Lemma 6.2, we have

\[
L^0 f(t, x) \in C^{p-1,2(p-1)}, \quad L^{-1}_e f(t, x) \in C^{p,2p},
\]

\[
\mathbb{E}\left[ (L^0 f_\alpha(t, X_t))^2 \right] \leq K, \quad \mathbb{E}\left[ (L^{-1}_e f_\alpha(t, X_t))^2 \right] \leq K,
\]

\[
\mathbb{E}\left[ (I_{(\alpha(1))} g(\cdot))_{\rho, s}^2 \right] < +\infty, \quad \text{for} \quad t \in [\rho, \tau].
\]

Then, we can repeatedly apply (6.9) \( p \) times to get

\[
\mathbb{E}\left[ f(\tau, X_\tau) I_\alpha[g(\cdot)]_{\rho, \tau} \right] \leq C(s - \rho)^{(\alpha)+n(\alpha)},
\] (6.10)

Using the conditions of Lemma 6.2, for \( s \in [\rho, \tau] \), we deduce

\[
\mathbb{E}\left[ I_\alpha[g(\cdot)]_{\rho, s}^2 \right] \leq C(s - \rho)^{(\alpha)+n(\alpha)},
\]

which implies that

\[
\left| \mathbb{E}\left[ f_\beta(s, X_s) I_\alpha[g(\cdot)]_{\rho, s} \right] \right|^2 \leq \mathbb{E}\left[ f_\beta(s, X_s)^2 \right] \mathbb{E}\left[ I_\alpha[g(\cdot)]_{\rho, s}^2 \right] \leq C(s - \rho)^{(\alpha)+n(\alpha)}.
\] (6.11)

Then by (6.10) and (6.11), we obtain

\[
\mathbb{E}\left[ f(\tau, X_\tau) I_\alpha[g(\cdot)]_{\rho, \tau} \right] \leq C \int_\rho^\tau \int_\varepsilon \int_\mathbb{E} \left[ f_\beta(s, X_s)^2 \right] \lambda(de) ds \leq C(s - \rho)^{(\alpha)+n(\alpha)}.
\] (6.12)
Since \( p = l(\alpha) - \left[\frac{l(\alpha)+n(\alpha)}{2}\right] \), then by (6.12), we have
\[
\mathbb{E}\left[ f(\tau, X_\tau)I_\alpha[g(\cdot)]_{\rho,\tau} \big| \mathcal{F}_\rho \right] \leq C(\tau - \rho)^{l+1}.
\]

Similarly, we can prove (6.5) for \( \alpha = (j_1, \ldots, j_{i+1}) \) with \( j_{i+1} = 0 \) or 1. The proof ends.

\[\Box\]

**Lemma 6.3** Let \( \rho \) and \( \tau \) be two stopping times with \( \tau \) being \( \mathcal{F}_\rho \)-measurable and \( 0 \leq \rho \leq \tau \leq T \), a.s.. Given \( \alpha \in \mathcal{M} \) and \( \{g(t, e), t \in [\rho, \tau]\} \) with \( e \in \mathcal{E}(\alpha) \) is an adapted process. If \( g(t, e) \) is \( 2s(\alpha)+3 \) integrable for a given \( q \in \mathbb{N}^+ \), then for any square integrable adapted process \( \{h(t), t \in [\rho, \tau]\} \), it holds that
\[
\mathbb{E}\left[ h(\tau)I_\alpha[g(\cdot)]_{\rho,\tau} \big| \mathcal{F}_\rho \right] \leq C_1(\tau - \rho)^q(\ell(\alpha)+n(\alpha)-s(\alpha))^{s(\alpha)},
\]
where the positive constants \( C_1 \) and \( C_2 \) do not depend on \( (\tau - \rho) \).

**Proof** The proof of Lemma 6.3 can be found in Lemma 3.2 in [17] and Lemma 4.5.5 in [26]. We omit it here.

Based on Lemmas 6.1–6.3, we now prove the error estimates of the weak order \( \eta \) Itô-Taylor Scheme 4.2 in the following theorem.

**Theorem 6.1** Let \( X_t \) and \( X_k \) be the solutions of the MSDEJ (5.1) and the weak order \( \eta \) Itô-Taylor scheme 4.2, respectively. Assume that \( \mathbb{E}[|X_0|^q] < \infty \) for \( q \geq 1 \) and \( b^k, \sigma^k, c^k \in \mathcal{C}_{1,2}^{p+1,2}(0, T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R} \) are Lipschitz continuous for \( 1 \leq k \leq d \) and \( 1 \leq j \leq m \). Let the coefficients \( f_\alpha \) with \( f(t, x', x) = x \) satisfy
\[
|f_\alpha(t, x)| \leq K(1 + \mathbb{E}[|\beta_1|] + |x|), \quad (t, x) \in [0, T] \times \mathbb{R}^d
\]
for all \( \alpha \in \Gamma_{\eta} \cup \mathcal{B}(\Gamma_{\eta}) \) with \( K > 0 \) being a constant and \( \beta_1 \) defined by (3.2). Then for any function \( g \in \mathcal{C}^{2(\eta+1)}_p(\mathbb{R}^d; \mathbb{R}) \), it holds that
\[
\mathbb{E}\left[ g(X_T) - g(X_N) \right] \leq C(\Delta t)\eta,
\]
where \( C \) is a positive constant independent of \( \Delta t \).

**Proof** For simplicity, we consider \( d = m = 1 \). The proof of the general case is similar. According to (6.2) and (6.3), it holds that
\[
H = \mathbb{E}\left[ g(X_N) \right] - \mathbb{E}\left[ g(X_T) \right] = \mathbb{E}\left[ u(T, X_N) - u(0, X_0) \right].
\]

By the Itô-Taylor expansion (3.16) and (6.3), we deduce
\[
\mathbb{E}\left[ u(t, X_t^x, y) - u(s, y) \big| \mathcal{F}_s \right] = 0
\]
for any \( 0 \leq s \leq t \leq T \) and \( y \in \mathbb{R}^d \). Then by (6.4), (6.15) and (6.16), we get
\[
H = \mathbb{E}\left[ \sum_{k=1}^{N} (u(t_k, X_k) - u(t_{k-1}, X_{k-1})) \right]
\]
for any \( 0 \leq s \leq t \leq T \) and \( y \in \mathbb{R}^d \). Then by (6.4), (6.15) and (6.16), we get
Then it is easy to obtain $H \leq H_1 + H_2$ with

$$H_1 = \left\{ \mathbb{E} \left[ \sum_{k=1}^{N} \frac{\partial u}{\partial y} (t_k, X_{t_k}^{h-1}, X_{k-1}) (X_k - X_{t_k}^{h-1}, X_{k-1}) \right] \right\}$$

(6.18)

$$H_2 = \left\{ \mathbb{E} \left[ \sum_{k=1}^{N} \frac{1}{2} \frac{\partial^2 u}{\partial y^2} (t_k, X_{t_k}^{h-1}, X_{k-1} + \theta_k (X_k - X_{t_k}^{h-1}, X_{k-1})) \times (X_k - X_{t_k}^{h-1}, X_{k-1})^2 \right] \right\}$$

(6.19)

with the parameter $\theta_k \in (0, 1)$.

By Theorem 3.2 and Scheme 4.2, we have

$$X_{t_{k+1}}^{h-1}, X_{k-1} - X_k = \sum_{\alpha \in B(\Gamma, \alpha)} I_\alpha \left[ f_{\alpha}^X (\cdot, X_{t_k}^{h-1}, X_{k-1}) \right]_{l_{k-1}, l_k}.$$  

(6.20)

Utilizing the estimate (5.12) and the condition (6.13), one can prove that for every $p \in \{1, 2, \ldots\}$, there exist constants $C$ and $r$ such that for every $q \in \{1, \ldots, p\}$

$$\mathbb{E} \left[ \max_{0 \leq n \leq N} |X_n|^q \right] \leq C (1 + |X_0|^q).$$

(6.21)

Since $l(\alpha) = \eta + 1$ for $\alpha \in B(\eta)$, then we obtain $2\eta + 1 \geq 2p$ for $p = l(\alpha) - \left[ \frac{l(\alpha)}{2} \right]$. Then by (6.4) and (6.21), for $\alpha \in \mathcal{A}_p$ and $k = 1, \ldots, N$, we deduce

$$\mathbb{E} \left[ \left( U_\alpha (t_k, X_{t_k}) \right)^2 \left| \mathcal{F}_\rho \right] < +\infty, \ a.s. \right.$$  

(6.22)

$$\mathbb{E} \left[ \left( V(t_k, X_k) \right)^2 \left| \mathcal{F}_\rho \right] < +\infty, \ a.s. \right.$$  

(6.23)

where

$$U(t_k, X_{t_k}) = \frac{\partial u}{\partial y} (t_k, X_{t_k}^{t_{k+1}, X_{k-1}}).$$

$$V(t_k, X_{t_k}) = \frac{\partial^2 u}{\partial y^2} (t_k, X_{t_k}^{t_{k+1}, X_{k-1} + \theta_k (X_k - X_{t_k}^{t_{k+1}, X_{k-1}})}).$$

Moreover, by Theorem 2.1 and the condition (6.13), we get

$$\mathbb{E} \left[ \left| f_{\alpha}^X (z, X_{t_k}^{t_{k+1}, X_{k-1}}) \right|^2 \left| \mathcal{F}_{t_{k+1}} \right] \leq C (1 + \mathbb{E} [ |X_0|^2 ] + |X_0|^2).$$

(6.24)

Then based on (6.20), (6.22), and (6.24), by Lemma 6.2, we have

$$H_1 \leq \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{\alpha : l(\alpha) = \eta + 1} \left[ \mathbb{E} \left[ \frac{\partial u}{\partial y} (t_k, X_{t_k}^{t_{k+1}, X_{k-1}}) I_\alpha \left[ f_{\alpha}^X (\cdot, X_{t_k}^{t_{k+1}, X_{k-1}}) \right]_{l_{k-1}, l_k} \left| \mathcal{F}_{t_{k+1}} \right] \right] \right]$$

$$\leq C \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{\alpha : l(\alpha) = \eta + 1} \left( t_k - t_{k-1} \right)^{\eta + 1} \right] \leq C (\Delta t_k)^{\eta}.$$  

(6.25)
Similarly, based on (6.20), (6.23), and (6.24), by Lemma 6.3 with $q = 1$, we deduce
\[
H_2 \leq C\mathbb{E}\left[\sum_{k=1}^{N} \sum_{\{\alpha,j: (\alpha,j)=\eta+1\}} \mathbb{E}\left[\left(\frac{\partial^2 u}{\partial y^2}\right)\left(t_k, X_{tk-1}, X_{tk-1} + \theta_k (X_k - X^{tk-1}_{tk-1})\right)\right]ight]
\times \left[I_{\alpha}\left[f_\alpha (\cdot, X^{\eta+1}_{\eta+1})\right]\right]_{t_{k-1}}^{t_k}\left[F_{\eta+1}\right]
\leq C(\Delta t)^\eta.
\] (6.26)

Then by using (6.15), (6.25), and (6.26), we get (6.14). The proof ends.

Based on the above analysis, by repeating the same procedures as above, it is easy to deduce the error estimates of Scheme 4.2 for general cases where $X_0 \neq x_0$.

**Theorem 6.2** Let $X^{x_0,x_0}_t$ and $X^{x_0}_t$ be the solutions of the MSDEJ (4.1) and the weak order $\eta$ Itô-Taylor scheme 4.2, respectively. Assume that $\mathbb{E}[|x_0|^q + |X_0|^q] < \infty$ for $q \geq 1$ and $b^k, \sigma^k, c^k \in C^{n+1,2(\eta+1),2(\eta+1)}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ are Lipschitz continuous for $1 \leq k \leq d$ and $1 \leq j \leq m$. Let the coefficients $f_\alpha$ with $f(t, x', x) = x$ satisfy
\[
|f_\alpha^d (t, x)| \leq K \left(1 + \mathbb{E}[|\beta_t|] + |x|\right), \quad (t, x) \in [0, T] \times \mathbb{R}^d
\]
for all $\alpha \in \Gamma_\eta \cup B(\Gamma_\eta)$ with $K > 0$ being a constant and $\beta_t$ defined by (3.2). Then for any function $g \in C^{2(\eta+1)}(\mathbb{R}^d; \mathbb{R})$, it holds that
\[
\mathbb{E}[g(X_T^{x_0,x_0}) - g(X_T^{x_0})] \leq C(\Delta t)^\eta,
\]
where $C$ is a positive constant independent of $\Delta t$.

**Remark 6.1** Theorem 6.2 indicates that the order of weak convergence of the weak order $\eta$ Itô-Taylor scheme 4.2 is $\eta$.

### 7 Numerical examples

In this section, we carry out some numerical tests to verify our theoretical conclusions and to show the efficiency and the accuracy of the proposed schemes for solving MSDEJs. For each example, we shall test the Euler scheme (4.3), the strong order 1.0 Taylor scheme (4.4), and the weak order 2.0 Taylor scheme (4.7), respectively.

For simplicity, we adopt the uniform time partition, and the time partition number $N$ is given by $N = \frac{T}{\Delta t}$. We denote by $\mathbb{E}[|X_T - X_N|]$ and $\mathbb{E}[X_T - X_N]$ the strong errors and the weak errors between the exact solution $X_T$ of the MSDEJ (1.1) and the numerical solution $X_N$ of the proposed schemes. The Monte Carlo method is used to approximate the expectation $\mathbb{E}[\cdot]$ appeared in coefficients and errors with sample times $M$. The “exact” solutions of the MSDEJs are identified with the numerical one using a small step-size $\Delta t_{\text{exact}} = 2^{-12}$. Moreover, we will test the efficiency of our schemes w.r.t. the level of the intensity $\lambda$ of the Poisson measure $\mu$ by the magnitudes of the sample times $M$ and the running time (RT) for different values of $\lambda$. 

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In what follows, we denote by Euler, S-1.0 and W-2.0 the Euler scheme, the strong order 1.0 Taylor scheme, and the weak order 2.0 Taylor scheme, respectively. The convergence rate (CR) with respect to \( \Delta t \) is obtained by using linear least square fitting to the numerical errors. In all the tests, we set \( T = 1.0 \). The unit of RT is the second.

**Example 7.1** Consider the following MSDEJ with \( X_0 = x_0 \):

\[
\begin{align*}
  dX_t &= a\left(\mathbb{E}[X_t] + X_t\right)dt + bX_t dW_t + \int_0^t c e(\mathbb{E}[X_s] + X_{s-}) \mu(de, ds),
\end{align*}
\]

where \( a, b, \) and \( c \) are constants.

We set \( a = 1.25, b = 0.75, c = 0.25, \) and \( X_0 = 0.1 \). Assume that the jump sizes \( \{Y_i, i = 1, \ldots, N_T\} \) satisfy \( Y_i \sim U\left(-\frac{1}{2}, \frac{1}{2}\right) \), which is the uniform distribution on \( \left[-\frac{1}{2}, \frac{1}{2}\right] \). And we use the Euler scheme (4.3), the strong order 1.0 Taylor scheme (4.4) and the weak order 2.0 Taylor scheme (4.7) to solve (7.1), respectively. We have listed the strong and weak errors and convergence rates of the schemes for different intensity \( \lambda \) in Tables 1, 2, and 3, respectively.

The numerical results listed in Tables 1, 2, and 3 show that the Euler scheme (4.3), the strong order 1.0 Taylor scheme (4.4), and the weak order 2.0 Taylor scheme (4.7) are stable and accurate for solving the linear MSDEJ (7.1). Moreover, we can draw the following conclusions.

1. The orders of strong convergence of the Euler scheme, the strong order 1.0 Taylor scheme, and the weak order 2.0 Taylor scheme are 0.5, 1.0, and 1.0, respectively;

---

**Table 1** Errors and convergence rates of the Euler scheme

| \( \lambda \) | \( \mathbb{E}[X(T) - X_N] \) | \( \mathbb{E}[X(T) - X_N] \) |
|--------------|-------------------|-------------------|
| \( N \)      | 16                | 32                | 64                | 128               | 256               | CR | M | RT |
| 0.1          | 2.739E-01         | 1.405E-01         | 6.630E-02         | 4.517E-02         | 3.377E-02         | 0.768 | 45 | 2.94 |
| 0.5          | 1.839E-01         | 9.217E-02         | 4.957E-02         | 3.212E-02         | 2.063E-02         | 0.783 | 65 | 4.18 |
| 1.0          | 2.174E-01         | 1.236E-02         | 7.031E-02         | 4.236E-02         | 2.479E-02         | 0.781 | 75 | 4.87 |
| 2.0          | 2.106E-01         | 1.173E-01         | 5.759E-02         | 3.800E-02         | 2.610E-02         | 0.765 | 85 | 5.46 |
| 3.0          | 1.479E-01         | 8.452E-02         | 5.922E-02         | 3.605E-02         | 2.614E-02         | 0.623 | 100 | 6.14 |

---

**Table 2** Errors and convergence rates of the strong order 1.0 Taylor scheme

| \( \lambda \) | \( \mathbb{E}[X(T) - X_N] \) | \( \mathbb{E}[X(T) - X_N] \) |
|--------------|-------------------|-------------------|
| \( N \)      | 16                | 32                | 64                | 128               | 256               | CR | M | RT |
| 0.1          | 2.192E-01         | 1.222E-01         | 6.122E-02         | 2.696E-02         | 1.034E-02         | 1.099 | 100 | 6.07 |
| 0.5          | 2.987E-01         | 1.510E-01         | 7.575E-02         | 4.380E-02         | 1.907E-02         | 0.972 | 200 | 12.94 |
| 1.0          | 3.229E-01         | 1.699E-01         | 8.444E-02         | 4.397E-02         | 1.934E-02         | 1.007 | 300 | 24.51 |
| 2.0          | 3.401E-01         | 1.880E-01         | 9.122E-02         | 4.380E-02         | 2.007E-02         | 1.027 | 500 | 59.56 |
| 3.0          | 2.996E-01         | 1.640E-01         | 8.564E-02         | 4.028E-02         | 1.946E-02         | 0.992 | 900 | 107.35 |
2. The orders of weak convergence of the Euler scheme, the strong order 1.0 Taylor scheme, and the weak order 2.0 Taylor scheme are 1.0, 1.0, and 2.0, respectively;
3. The efficiency of the schemes depends on the level of the intensity \( \lambda \) of the Poisson measure \( \mu \). As the intensity \( \lambda \) increases, the sample times \( M \) and the running time RT increase.

All of the conclusions above are consistent with our theoretical results.

| Table 2 | Errors and convergence rates of the strong order 1.0 Taylor scheme |
|---------|---------------------------------------------------------------|
|         | \( N \) | 16 | 32 | 64 | 128 | 256 | CR | M | RT |
| \( \lambda \) | \( \mathbb{E}[|X(T) - X_N|] \) | 0.1 | 1.679E-01 | 8.939E-02 | 4.560E-02 | 2.292E-02 | 1.126E-02 | 0.976 | 100 | 6.76 |
|         |         | 0.5 | 2.027E-01 | 1.107E-01 | 5.738E-02 | 2.859E-02 | 1.408E-02 | 0.965 | 200 | 13.63 |
|         |         | 1.0 | 2.084E-01 | 1.138E-02 | 5.930E-02 | 2.962E-02 | 1.449E-02 | 0.963 | 400 | 41.21 |
|         |         | 2.0 | 2.138E-01 | 1.164E-01 | 6.046E-02 | 3.053E-02 | 1.477E-02 | 0.964 | 800 | 99.20 |
|         |         | 3.0 | 2.006E-01 | 1.092E-01 | 5.665E-02 | 2.877E-02 | 1.404E-02 | 0.960 | 1000 | 130.32 |
|         | \( \mathbb{E}[|X(T) - X_N|] \) | 0.1 | 1.679E-01 | 8.939E-02 | 4.560E-02 | 2.292E-02 | 1.126E-02 | 0.976 | 100 | 6.76 |
|         |         | 0.5 | 2.027E-01 | 1.107E-01 | 5.738E-02 | 2.859E-02 | 1.408E-02 | 0.965 | 200 | 13.63 |
|         |         | 1.0 | 2.084E-01 | 1.138E-02 | 5.930E-02 | 2.962E-02 | 1.449E-02 | 0.963 | 400 | 41.21 |
|         |         | 2.0 | 2.138E-01 | 1.164E-01 | 6.046E-02 | 3.053E-02 | 1.477E-02 | 0.964 | 800 | 99.20 |
|         |         | 3.0 | 2.006E-01 | 1.092E-01 | 5.665E-02 | 2.877E-02 | 1.404E-02 | 0.960 | 1000 | 130.32 |

| Table 3 | Errors and convergence rates of the weak order 2.0 Taylor scheme |
|---------|---------------------------------------------------------------|
|         | \( N \) | 8 | 16 | 32 | 64 | 128 | CR | M | RT |
| \( \lambda \) | \( \mathbb{E}[|X(T) - X_N|] \) | 0.1 | 3.686E-02 | 1.329E-02 | 6.709E-03 | 2.672E-03 | 1.271E-03 | 1.203 | 100 | 7.28 |
|         |         | 0.5 | 5.150E-02 | 1.722E-02 | 6.706E-03 | 3.670E-03 | 1.849E-03 | 1.183 | 200 | 16.15 |
|         |         | 1.0 | 4.686E-02 | 1.681E-02 | 6.918E-03 | 3.749E-03 | 2.165E-03 | 1.104 | 500 | 78.36 |
|         |         | 2.0 | 5.022E-02 | 1.974E-02 | 8.703E-03 | 4.663E-03 | 2.825E-03 | 1.039 | 800 | 114.51 |
|         |         | 3.0 | 4.808E-02 | 1.841E-02 | 9.511E-03 | 5.843E-03 | 3.511E-03 | 0.921 | 1500 | 228.86 |
|         | \( \mathbb{E}[|X(T) - X_N|] \) | 0.1 | 3.438E-02 | 1.013E-02 | 2.618E-03 | 5.500E-04 | 8.881E-05 | 2.140 | 2800 | 450.22 |
|         |         | 0.5 | 4.033E-02 | 1.157E-02 | 2.537E-03 | 7.168E-04 | 1.165E-04 | 2.089 | 3000 | 481.91 |
|         |         | 1.0 | 3.529E-02 | 9.424E-03 | 2.526E-03 | 5.719E-04 | 1.246E-04 | 2.033 | 3500 | 567.14 |
|         |         | 2.0 | 3.945E-02 | 1.101E-02 | 2.587E-03 | 3.550E-04 | 1.099E-04 | 2.193 | 4500 | 826.78 |
|         |         | 3.0 | 3.836E-02 | 1.057E-02 | 2.542E-03 | 6.279E-04 | 4.323E-05 | 2.366 | 5500 | 1201.71 |
Consider the following general nonlinear MSDEJs

\[ dX_{t,x_0}^0, X_0 = \left( \left( X_t^{0,x_0}, X_0 \right)^{5/3} + 2\lambda^2 \mathbb{E}\left[X_t^{0,x_0} \right] \right) dt + \frac{1}{2} \mathbb{E}\left[X_t^{0,x_0} \right] dW_t \]

\[ + \int \frac{e}{2(1 + \lambda^2)} \left( X_t^{0,x_0}, X_0 + \mathbb{E}\left[(X_t^{0,x_0}, X_0)^2 \right] \right) \mu(de, dt), \tag{7.2} \]

where \( x_0 \) and \( X_0 \) are initial values and \( \lambda \) is the intensity of the Poisson measure \( \mu \).

Let the jump sizes satisfy \( Y_i \overset{\text{iid}}{\sim} U(-\frac{1}{2}, \frac{1}{2}), i = 1, \ldots, N_T \). For simplicity, we set \( \lambda = 1.0 \). In Tables 4 and 5, we have listed the errors and convergence rates of the schemes for different initial values of \( x_0 \) and \( X_0 \). We have also plotted the errors and convergence rates in Figs. 1, 2, 3, and 4, respectively.

From the numerical results in Tables 4 and 5, we come to the conclusion that the Euler scheme (4.3), the strong order 1.0 Taylor scheme (4.4), and the weak order 2.0 Taylor scheme (4.7) are stable and accurate for solving the nonlinear MSDEJ (7.2) with different initial values of \( x_0 \) and \( X_0 \). Tables 4 and 5 also show that the orders
Fig. 1  Strong errors and convergence rates of the schemes with $X_0 = x_0 = 0.1$

Fig. 2  Weak errors and convergence rates of the schemes with $X_0 = x_0 = 0.1$
Fig. 3  Strong errors and convergence rates of the schemes with $X_0 = 0.05$, $x_0 = 0.15$

Fig. 4  Weak errors and convergence rates of the schemes with $X_0 = 0.05$, $x_0 = 0.15$
of strong convergence of the schemes (4.3), (4.4), and (4.7) are 0.5, 1.0, and 1.0, respectively, and the orders of weak convergence are 1.0, 1.0, and 2.0, respectively, which verify again our theoretical conclusions.

Note that the efficiency of the proposed schemes, especially the high-order ones, depends on the level of the intensity of the Poisson measure. This is mainly due to the existence of the double integrals involving the Poisson measure $I_{(-1,1)}$, $I_{(-1,-1)}$, $I_{(0,-1)}$, and $I_{(-1,0)}$, the computation complexity of which is dependent on the number of jumps. Hence, to construct more efficient high-order schemes for MSDEJs, we will focus on the jump-adapted methods in our future work, which avoid the integrals involving the Poisson measure.

8 Conclusions

In this paper, we developed the Itô formula and the Itô-Taylor expansion for MSDEJs, then based on which we proposed the strong order $\gamma$ and the weak order $\eta$ Itô-Taylor schemes for solving MSDEJs. We rigorously proved the error estimates of the proposed schemes, which show that the order of strong convergence of the strong order $\gamma$ Taylor scheme and the order of weak convergence of the weak order $\eta$ Taylor scheme are $\gamma$ and $\eta$, respectively. Numerical experiments verified our theoretical conclusions and indicated that the efficiency of the schemes depends on the level of the intensity of the Poisson measure. In the future work, we shall consider the jump-adapted methods for solving MSDEJs.

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References

1. Bossy, M., Talay, D.: A stochastic particle method for the McKean-Vlasov and the Burgers equation. Math. Comput. 66, 157–192 (1997)
2. Buckdahn, R., Djehiche, B., Li, J., Peng, S.: Mean-field backward stochastic differential equations: A limit approach. Ann. Probab. 37, 1524–1565 (2009)
3. Buckdahn, R., Li, J., Peng, S.: Mean-field backward stochastic differential equations and related partial differential equations. Stoch. Process Appl. 119, 3133–3154 (2009)
4. Buckdahn, R., Li, J., Peng, S.: Mean-field stochastic differential equations and associated PDEs. Ann. Probab. 45, 824–878 (2017)
5. Bensoussan, A., Yam, P., Zhang, Z.: Well-posedness of mean-field type forward-backward stochastic differential equations. Stoch. Process Appl. 125, 3327–3354 (2015)
6. Carmona, R., Delarue, F.: Probabilistic analysis of mean-field games. SIAM J. Control Optim. 51, 2705–2734 (2012)
7. Cardaliaguet, P., Delarue, F., Lasry, J.M., Lions, P.L.: The Master Equation and the Convergence Problem in Mean Field Games, Volume 201 of Annals Of Mathematics Studies. Princeton (2019)
8. Cont, R., Tankov, P.: Financial Modelling with Jump Processes. Chapman and Hall/CRC Press, London (2004)
9. Guéant, O., Lasry, J., Lions, P.: Mean-Field Games and Applications, Paris-Princeton Lectures on Mathematical Finance, vol. 2003, pp. 205–266. Springer, Berlin (2010)
10. Hafayed, M., Tabet, M., Boukaf, S.: Mean-field maximum principle for optimal control of forward-backward stochastic systems with jumps and its application to mean-variance portfolio problem. Commun. Math. Stat. 3, 163–186 (2015)
11. Higham, D., Kloeden, P.: Numerical methods for nonlinear stochastic differential equations with jumps. Numer. Math. 101, 101–119 (2005)
12. Hao, T., Li, J.: Mean-field SDEs with jumps and nonlocal integral-PDEs. Nonlinear Differ. Equa. Appl. 23, 1–51 (2016)
13. Kloeden, P., Platen, E.: Numerical Solution of Stochastic Differential Equations. Springer, Berlin (1992)
14. Kloeden, P., Shardlow, T.: Gauss-quadrature method for one-dimensional mean-field SDEs. SIAM J. Sci. Comput. 39, A2784–A2807 (2017)
15. Kotelenez, P.: A class of quasilinear stochastic partial differential equations of McKean-Vlasov type with mass conservation. Probab. Theory Related Fields 102, 159–188 (1995)
16. Lasry, J., Lions, P.: Mean-field games. Japan J. Math. 2, 229–260 (2007)
17. Liu, X., Li, W.: Weak approximations and extrapolations of stochastic differential equations with jumps. SIAM J. Numer. Anal. 37, 1747–1767 (2000)
18. Li, J.: Mean-field forward and backward SDEs with jumps and associated nonlocal quasi-linear integral-PDEs. Stoch. Process. Appl. 128, 3118–3180 (2018)
19. Li, J., Min, H.: Controlled mean-field backward stochastic differential equations with jumps involving the value function. J. Syst. Sci. Complex 29, 1238–1268 (2016)
20. McKean, H.: A class of markov processes associated with nonlinear parabolic equations. PNAS 55, 1907–1911 (1966)
21. Méléard, S.: Asymptotic Behaviour of Some Interacting Particle Systems; McKean-Vlasov and Boltzmann Models in Probabilistic Models for Nonlinear Partial Differential Equations, pp. 42–95. Springer, Berlin (1996)
22. Mendoza, M., Aguilar, M., Valle, F.: A mean-field approach that combines quantum mechanics and molecular dynamics simulation: the water molecule in liquid water. J. Mol. Struct. 426, 181–190 (1998)
23. Ni, Y., Li, X., Zhang, J.: Mean-field stochastic linear-quadratic optimal control with Markov jump parameters. Syst. Control Lett. 93, 69–76 (2016)
24. Kac, M.: Foundations of kinetic theory. Proceedings of the 3rd Berkeley Symposium on Mathematical Statistics and Probability 3, 171–197 (1956)
25. Kac, M.: Probability and Related Topics in the Physical Sciences. Interscience Publishers, New York (1958)
26. Platen, E., Bruti-Liberati, N.: Numerical Solution of Stochastic Differential Equations with Jumps in Finance, Stochastic Modelling and Applied Probability. Springer, Berlin (2010)
27. Stevenson, P., Stone, J., Strayer, M.: Hartree-Fock mean-field models using separable interactions. Office of Scientific & Technical Information Technical Reports 217, U8 (1999)
28. Sun, Y., Yang, J., Zhao, W.: Itô-Taylor schemes for solving mean-field stochastic differential equations. Numer. Math. Theor. Meth. Appl. 10, 798–828 (2017)
29. Sun, Y., Zhao, W.: An explicit second-order numerical scheme for solving mean-field backward stochastic differential equations. Numer. Algorithms 84, 253–283 (2020)
30. Sun, Y., Zhao, W., Zhou, T.: Explicit θ-scheme for solving mean-field backward stochastic differential equations. SIAM J. Numer. Anal. 56, 2672–2697 (2018)
31. Talay, D., Vaillant, O.: A stochastic particle method with random weights for the computation of statistical solutions of McKean-Vlasov equations. Ann. Appl. Probab. 13, 140–180 (2003)
32. Wang, B., Zhang, J.: Mean-field games for large population multi-agent systems with Markov jump parameters. SIAM J. Control Optim. 50, 2308–2334 (2012)

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