BOWEN ENTROPY FOR FIXED-POINT FREE FLOWS

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Abstract. In this paper, we devote to the study of the Bowen’s entropy for fixed-point free flows and show that the Bowen entropy of the whole compact space is equal to the topological entropy. To obtain this result, we establish the Brin-Katok’s local entropy formula for fixed-point free flows in ergodic case.

1. Introduction. Let $(X,d)$ be a compact metric space. A pair $(X,\phi)$ is called a flow, if $\phi : X \times \mathbb{R} \to X$ is a continuous map satisfying $\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s \in \mathbb{R}$, where $\phi_t(\cdot) = \phi(\cdot, t)$ is a homeomorphism on $X$. A Borel probability measure $\mu$ on $X$ is called $\phi$-invariant if for any Borel set $B$, it holds $\mu(\phi_t(B)) = \mu(B)$ for all $t \in \mathbb{R}$. It is called ergodic if any $\phi$-invariant Borel set has measure 0 or 1. The set of all Borel probability measures, all $\phi$-invariant Borel probability measures and all ergodic $\phi$-invariant Borel probability measures on $X$ are denoted respectively by $\mathcal{M}(X)$, $\mathcal{M}_\phi(X)$ and $\mathcal{E}_\phi(X)$. We shall assume throughout the paper that $\phi$ is a continuous real flow on a compact metric space $X$ without fixed points.

Measure theoretic and topological entropy play crucial roles in the study of behavior of dynamical systems, which measure the rate of increase in dynamical complexity as the system evolves with time. Topological entropy for one parameter flows on compact metric spaces is defined by Bowen [2, 3]. For $x \in X$, $\varepsilon > 0$, $t \geq 0$, the usual Bowen ball is defined by

$$B_t(x, \varepsilon, \phi) = \{ y \in X : d(\phi_s x, \phi_s y) < \varepsilon, \text{ for all } 0 \leq s \leq t \}.$$ 

For $\delta > 0$, we call a subset $E$ of $X$ a $(t, \delta)$-spanning set if for each $x \in X$, there exists $e \in E$ such that $x \in B_t(e, \delta, \phi)$. Let $r_t(\delta, \phi)$ denote the minimum cardinality of $(t, \delta)$-spanning sets. Then topological entropy of $(X, \phi)$ is defined by

$$\tilde{h}_{top}(\phi) = \lim_{\delta \to 0} \lim_{t \to \infty} \frac{1}{t} \log r_t(\delta, \phi).$$

To investigate the topological entropies of mutually conjugate expansive flows, Thomas [15] introduced a concept of topological entropy for flows which takes into consideration all possible reparametrizations of the flow. Later on, in [16] he showed

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this definition of entropy is equivalent to Bowen’s definition for any continuous real flow without fixed points. Let $I$ be a closed interval of real numbers containing the origin, a continuous map $\alpha : I \rightarrow \mathbb{R}$ is called a reparametrization on $I$ if it is a homeomorphism onto its image and $\alpha(0) = 0$. Let $\text{Rep}(I)$ denote the set of all such reparametrization on $I$. For $x \in X$, $\varepsilon > 0$, $t \geq 0$, set

$$B(x, t, \varepsilon, \phi) = \{y \in X : d(\phi_\alpha(x), \phi_s y) < \varepsilon, \forall 0 \leq s \leq t, \text{ for some } \alpha \in \text{Rep}([0, t])\},$$

and call such set a $(t, \varepsilon, \phi)$-ball or a reparametrization ball in $X$. Clearly, all the reparametrization balls are open sets.

Traditionally, time-1 map $\phi_1$ is considered when we investigate measure theoretic entropy of flow. The sets of all $\phi_1$-invariant Borel probability measures and ergodic $\phi_1$ invariant Borel probability measures are denoted by $\mathcal{M}_{\phi_1}(X)$ and $\mathcal{E}_{\phi_1}(X)$ respectively. Abramov [1] proved that $\hat{h}_{\mu}(\phi_1) = \|h_t\mu(\phi_1)\|$ for all $t \in \mathbb{R}$ which is well known as the Abramov entropy formula. Bowen and Ruelle [5] showed that

$$\hat{h}_{\text{top}}(\phi) = \sup\{h_{\mu}(\phi_1) : \mu \in \mathcal{M}_{\phi_1}(X)\}. \quad (1)$$

However, an invariant measure for $\phi_1$ is not, in general, $\phi$-invariant and similarly, a $\phi$-ergodic measure is not necessarily ergodic for $\phi_1$. On the other hand, the entropy of time-1 map of two measure theoretic equivalent flows may not equal. There are many successful attempts to describe measure theoretic entropy of flows, such as [13, 14, 12], which focus on the whole flow itself. Dou etc. [8] introduced Bowen entropy on subsets and local measure theoretic entropy for compact metric flows through reparametrization balls inspired by [4] and proved a variational principle for compact metric flows without fixed points using Feng and Huang’s method in [9].

For convenience, we recall some of the standard facts on Bowen topological entropy for compact metric flows (see [8]). Let $(X, \phi)$ be a flow, $Z \subset X$, $s \geq 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$, set

$$\mathcal{M}_{N,\varepsilon}^s(\phi, Z) = \inf \sum_i \exp(-s t_i),$$

where the infimum is taken over all finite or countable families of reparametrization balls $\{B(x_i, t_i, \varepsilon, \phi)\}$, $x_i \in X$ and $t_i \geq N$ such that $Z \subset \bigcup B(x_i, t_i, \varepsilon, \phi)$. Then the following limits exist:

$$\mathcal{M}_s^s(\phi, Z) = \lim_{N \to \infty} \mathcal{M}_{N,\varepsilon}^s(\phi, Z),$$

$$\mathcal{M}_s^s(\phi, Z) = \lim_{\varepsilon \to 0} \mathcal{M}_{\varepsilon}^s(\phi, Z).$$

The Bowen topological entropy on $Z$, $h_{\text{top}}^B(\phi, Z)$, is defined as the critical value of the parameters, where $\mathcal{M}_s^s(\phi, Z)$ jumps from $\infty$ to $0$, i.e.,

$$h_{\text{top}}^B(\phi, Z) = \inf\{s : \mathcal{M}_s^s(\phi, Z) = 0\} = \sup\{s : \mathcal{M}_s^s(\phi, Z) = \infty\}.$$

For $Z = X$, simplify $h_{\text{top}}^B(\phi, X)$ as $h_{\text{top}}^B(\phi)$. Replacing the reparametrization balls by usual Bowen balls $B_t(x, \varepsilon, \phi)$, we can have the definition of the usual Bowen topological entropy on a subset $Z$ for flow $(X, \phi)$, denoted by $\hat{h}_{\text{top}}^B(\phi, Z)$. If we replace the reparametrization balls by the Bowen balls $B(x, n, \varepsilon, \phi_1) = \{y \in X : d(\phi_i x, \phi_i y) < \varepsilon, i = 0, 1, \ldots, n-1\}$, we have the definition of usual Bowen topological entropy on a subset $Z$ for time-1 map, denoted by $h_{\text{top}}^B(\phi_1, Z)$. Simplify $h_{\text{top}}^B(\phi, X),$
\( h^{B}_{\text{top}}(\phi_1, X) \) as \( \tilde{h}^{B}_{\text{top}}(\phi), h^{B}_{\text{top}}(\phi_1) \), respectively. Moreover, Dou etc. defined lower and upper measure theoretic entropy for any Borel probability measure \( \mu \),

\[
\underline{h}_\mu(\phi) = \int \underline{h}_\mu(\phi, x) d\mu \quad \text{and} \quad \bar{h}_\mu(\phi) = \int \bar{h}_\mu(\phi, x) d\mu,
\]

where

\[
\underline{h}_\mu(\phi, x) = \lim_{\varepsilon \to 0} \liminf_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon, \phi))
\]

and

\[
\bar{h}_\mu(\phi, x) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon, \phi)).
\]

Similarly, by replacing \( B(x, t, \varepsilon, \phi) \) into \( B(x, n, \varepsilon, \phi_1) \), we can have the definitions of the measure theoretic lower and upper local entropies for the time-1 map, denoted by \( \underline{h}_\mu(\phi_1) \) and \( \bar{h}_\mu(\phi_1) \) respectively. By Brin-Katok’s formula (see \[7\]), \( \underline{h}_\mu(\phi_1) = \bar{h}_\mu(\phi_1) = h_\mu(\phi_1) \) for \( \mu \in \mathcal{M}_{\phi_1}(X) \), and \( \underline{h}_\mu(\phi_1, x) = \bar{h}_\mu(\phi_1, x) = h_\mu(\phi_1) \) for \( \mu \)-a.e. \( x \in X \) when \( \mu \in \mathcal{E}_{\phi}(X) \).

Dou etc. \[8\] posed a question whether \( \underline{h}_\mu(\phi) = \underline{h}_\mu(\phi_1) \) and \( \bar{h}_\mu(\phi) = \bar{h}_\mu(\phi_1) \) hold for every \( \mu \in \mathcal{M}(X) \). In this paper, inspired by Brin and Katok \[7\], we establish the Brin-Katok’s formula for compact metric flows without fixed points in ergodic case (Theorem 1.1), which partially give a positive answer to the above question. We mention here that in \[14\], Sun and Vargas proved Katok’s formula for fixed-point free flows, and they apply their idea to the proof of Theorem 1.1. Furthermore, following the idea of Zheng and Chen \[18\], we prove that the Bowen topological entropy of whole space coincides with the classical topological entropy for compact metric flows without fixed points, which relates topological dynamics and dimension theory. The following theorems present the main results of this paper.

**Theorem 1.1.** Let \( (X, \phi) \) be a compact metric flow without fixed points, and \( \mu \in \mathcal{E}_{\phi}(X) \). Then for \( \mu \)-a.e. \( x \in X \),

\[
\underline{h}_\mu(\phi, x) = \bar{h}_\mu(\phi, x) = h_\mu(\phi_1).
\]

In particular, \( \underline{h}_\mu(\phi) = \bar{h}_\mu(\phi) = h_\mu(\phi_1) \).

**Theorem 1.2.** Let \( (X, \phi) \) be a compact metric flow without fixed points. Then

\[
h^{B}_{\text{top}}(\phi) = \tilde{h}_{\text{top}}(\phi)
\]

2. **Brin-Katok’s formula for flows.** In this section, we will give the proof of Theorem 1.1, which can be obtained from the following Proposition 1 and 2.

**Proposition 1.** Let \( (X, \phi) \) be a compact metric flow, \( \mu \in \mathcal{E}_{\phi}(X) \), then for \( \mu \)-a.e. \( x \in X \),

\[
\bar{h}_\mu(\phi, x) \leq \frac{1}{|\tau|} h_\mu(\phi_\tau),
\]

for all \( \tau \in \mathbb{R} \setminus \{0\} \).

**Proof.** Case 1. Consider \( \tau > 0 \) and \( t = n\tau, n \in \mathbb{N} \).

For any \( \varepsilon > 0 \), choose \( \eta > 0 \) such that \( d(\phi_s x, \phi_s y) < \varepsilon, \forall s \in [0, \tau] \) if \( d(x, y) < \eta \).

For \( x \in X \), write \( B(x, n, \eta, \phi_\tau) = \{ y \in X : d(\phi_{i\tau} x, \phi_{i\tau} y) < \eta, i = 0, 1, \ldots, n - 1 \} \).

Then

\[
B(x, n, \eta, \phi_\tau) \subset B_t(x, \varepsilon, \phi) \subset B(x, t, \varepsilon, \phi).
\]
Choose a finite measurable partition $\xi$ of $X$ with $\text{diam}(\xi) < \frac{\epsilon}{2}$. Let $\xi_n := \xi \vee \phi_{-1}^{n-1} \xi \vee \cdots \vee \phi_{-1}^{(n-1)} \xi$ and $\xi_n(x)$ be the element of $\xi_n$ containing $x$. According to Shannon-McMillan-Breiman theorem (see [6, 10, 11]), the limit $\lim_{n \to \infty} \frac{1}{n} \log \mu(\xi_n(x))$ exists for $x$ in a full $\mu$-probability set. Let $h_\mu(\phi, \xi, x) := -\lim_{n \to \infty} \frac{1}{n} \log \mu(\xi_n(x))$, since $\mu$ is ergodic and $h_\mu(\phi, \xi, x) = h_\mu(\phi, \xi, \phi_\tau x)$, it follows that for $\mu$-a.e. $x \in X$,
\[
\lim_{n \to \infty} \frac{\log \mu(\xi_n(x))}{n} = h_\mu(\phi, \xi).
\] (2)

Since $\xi_n(x) \subset B(x, n, \eta, \phi)$, we have $h_\mu(\phi, \xi, x) \leq h_\mu(\phi, \xi)$. Therefore, for $\mu$-a.e. $x \in X$,
\[
\lim_{\epsilon \to 0} \limsup_{t \to \infty} -\frac{\log \mu(\xi_n(x))}{t} \leq \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{\log \mu(\xi_n(x))}{t} = h_\mu(\phi, \xi). 
\]

Thus, we obtain that
\[
\bar{T}_\mu(\phi, x) \leq \frac{1}{\tau} h_\mu(\phi_\tau)
\]
holds for $\mu$-a.e. $x \in X$.

**Case 2.** Consider $\tau > 0$ and $t > 0$.

For $t > 0$, choose $n_t \in \mathbb{N}$ such that $n_t \tau < t < (n_t + 1) \tau$, then
\[
B(x, (n_t + 1) \tau, \varepsilon, \phi) \subset B(x, t, \varepsilon, \phi) \subset B(x, n_t \tau, \varepsilon, \phi).
\]
Therefore, for $\mu$-a.e. $x \in X$,
\[
\lim_{\epsilon \to 0} \limsup_{t \to \infty} -\frac{\log \mu(\xi_n(x))}{t} \leq \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{\log \mu(B(x, (n_t + 1) \tau, \varepsilon, \phi))}{t} \leq \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{\log \mu(B(x, n_t \tau, \varepsilon, \phi))}{n_t \tau} \leq \frac{1}{\tau} h_\mu(\phi_\tau).
\]

(by Case 1.)

**Case 3.** Consider $\tau < 0$.

Then we have $-\tau > 0$, similar argument like in Case 2, we have for $\mu$-a.e. $x \in X$
\[
\bar{T}_\mu(\phi, x) \leq -\frac{1}{\tau} h_\mu(\phi_{-\tau}) = -\frac{1}{\tau} h_\mu(\phi_\tau) = \frac{1}{|\tau|} h_\mu(\phi_\tau).
\]

\[
\square
\]

**Lemma 2.1.** [15] Let $(X, \phi)$ be a compact metric flow without fixed points. For any $\varepsilon > 0$, there exists $\varepsilon > 0$ such that for any $x, y \in X$, and any closed interval $I$ containing the origin, and any reparametrization $\alpha \in \text{Rep}(I)$, if $d(\phi_\alpha(x), \phi_s(y)) < \varepsilon$ for all $s \in I$, then it holds that
\[
|\alpha(s) - s| < \begin{cases} \varepsilon_1 |s|, & \text{if } |s| > 1; \\ \varepsilon_1, & \text{if } |s| \leq 1. \end{cases}
\]

**Proposition 2.** Let $(X, \phi)$ be a compact metric flow without fixed points, $\mu \in \mathcal{E}_\phi(X)$, then for $\mu$-a.e. $x \in X$,
\[
h_\mu(\phi, x) \geq \frac{1}{|\tau|} h_\mu(\phi_\tau),
\]
for all $\tau \in \mathbb{R} \setminus \{0\}$. 
Proof. Fix $\tau > 0$, without loss of generality we may assume that $h_\mu(\phi_\tau) > 0$. For sufficiently small $r > 0$, choose $L \in \mathbb{N}$ such that $L \geq \frac{2 \log 6 + r}{r^\tau}$ and $h_\mu(\phi_{LT}) = Lh_\mu(\phi_\tau) > 2(\log 6 + r)$.

Case 1. Consider $t = nLT$, $n \in \mathbb{N}$.

Let $\xi = \{A_1, \cdots, A_m, A_{m+1}\}$ be a finite measurable partition satisfying that
- $A_1, \cdots, A_m$ are piecewise disjoint compact sets;
- $A_{m+1} = X \setminus \bigcup_{i=1}^m A_i$,

and

$$h_\mu(\phi_{LT}, \xi) \geq h_\mu(\phi_{LT}) - r.$$ 

For $b \in (0, r)$, by a similar argument with (2), for sufficiently large $N$, when $n \geq N$, $\mu(E) > 1 - b$, where

$$E = \{x \in X : \mu(\xi_{n', LT}(x)) < \exp[-n'(h_\mu(\phi_{LT}, \xi) - b)] \text{ for all } n' \geq n\},$$

where $\xi_{n', LT}(x)$ denotes the element in $\bigvee_{i=0}^{n'-1} \phi_{LT}^i \xi$ containing $x$.

Let $\eta_0 = \min \{d(A_i, A_j) : 1 \leq i \neq j \leq m\}$. Fix $\eta \in (0, \eta_0)$, choose $\theta > 0$ such that

$$d(\phi_\tau(z), z) < \frac{\eta}{3}, |s| \leq \theta$$

for all $z \in X$. We also choose $\varepsilon \in (0, \frac{\eta}{3})$ corresponding to $\varepsilon_1 = \frac{\theta}{4LT}$ in Lemma 2.1.

For any $x \in X$ and $n \geq N$, define

$$W_n := \left\{ A \in \bigvee_{i=0}^{n-1} \phi_{LT}^{-i} \xi : A \cap B(x, \varepsilon, \phi) \neq \emptyset \right\}.$$

Then

$$B(x, \varepsilon, \phi) \subset H_n := \bigcup_{A \in W_n} A.$$

Claim. $\sharp W_n \leq 6^n$.

In fact, if $A \cap B(x, \varepsilon, \phi) \neq \emptyset$, $A \in \bigvee_{i=0}^{n-1} \phi_{LT}^{-i} \xi$, take $y \in A \cap B(x, \varepsilon, \phi)$, there exists $\alpha \in \text{Rep}([0, \ell])$ such that $d(\phi_{\alpha(x)} y, \phi_{\alpha(y)} y) < \varepsilon$, $0 \leq s \leq t$. Fix $s_1 \in [0, \ell]$, set $u := s - s_1$, $\gamma(u) := \alpha(s) - \alpha(s_1)$, then $\gamma \in \text{Rep}([-s_1, t-s_1])$ satisfying

$$d(\phi_{\gamma(u)} \phi_{\alpha(s_1)} y, \phi_{\alpha(s)} y) < \varepsilon,$$

$-s_1 \leq u \leq t - s_1$.

By Lemma 2.1, one obtains

$$|\gamma(u) - u| < \begin{cases} \varepsilon_1|u| = \frac{\theta}{4LT}|u|, & \text{if } |u| > 1; \\ \varepsilon_1 = \frac{\theta}{4LT} \leq \frac{\theta}{4}, & \text{if } |u| \leq 1. \end{cases}$$

For $s_2 \in [-s_1, t - s_1]$ satisfying $|s_1 - s_2| \leq LT$, we have

$$|(\alpha(s_1) - s_1) - (\alpha(s_2) - s_2)| \leq \frac{\theta}{4}.$$

Associate each $A \in W_n$, consider the following sequence:

$$S_{\alpha} = \left\{ \left\lfloor \frac{\alpha(kLT) - kLT}{\theta/4} \right\rfloor \right\}, \; k = 0, 1, \cdots, n - 1,$$

where $|z|$ denotes the largest integer less or equal $z$. Note that the first term of $S_{\alpha}$ is zero and two consecutive terms of it differ at most by 1. Therefore there exist at
most $3^{n-1}$ such sequences. If for another $\tilde{A} \in W_n$, there exists $z \in \tilde{A} \cap B(x,t,\varepsilon,\phi)$, we can choose $\beta \in \text{Re}(0,\ell]$ such that $d(\phi_{\beta(s)}x,\phi_\varepsilon z) < \varepsilon$, $0 \leq s \leq t$. Similarly, we can define the sequence $S_\beta$. If $S_n = S_\beta$, then for any $s \in [0,t]$, 
\[
|\alpha(s) - \beta(s)| \leq \left| (\alpha(s) - s) - (\alpha\left(\frac{s}{L_T}\right) - \frac{s}{L_T}) \right| \\
+ \left| (\alpha\left(\frac{s}{L_T}\right) - \frac{s}{L_T}) - (\beta\left(\frac{s}{L_T}\right) - \frac{s}{L_T}) \right| \\
+ \left| (\beta(s) - s) - (\beta\left(\frac{s}{L_T}\right) - \frac{s}{L_T}) \right| \\
\leq \theta + \frac{\theta}{4} \left| (\alpha\left(\frac{s}{L_T}\right) - \frac{s}{L_T}) \right| + \frac{\theta}{4} \left| (\beta\left(\frac{s}{L_T}\right) - \frac{s}{L_T}) \right| \\
\leq \theta.
\]
Moreover, $d(\phi_{\alpha(s)}x,\phi_{\beta(s)}x) < \frac{\eta}{3}$, $\forall s \in [0,t]$. Therefore, 
\[
d(\phi_s y,\phi_\varepsilon z) \leq d(\phi_s y,\phi_{\alpha(s)}x) + d(\phi_{\alpha(s)}x,\phi_{\beta(s)}x) + d(\phi_{\beta(s)}x,\phi_s z) \\
\leq \varepsilon + \frac{\eta}{3} + \varepsilon < \eta, \forall s \in [0,t].
\]
In particular, $d(\phi_{L_T}y,\phi_{L_T}z) \leq \eta, i = 0, 1, \cdots, n - 1$.

For any $A \in \bigvee_{i=0}^{n-1} \phi_{L_T}^{-1}\xi$, there exist $i_0, i_1, \cdots, i_{n-1} \in\{1, 2, \cdots, m+1\}$ such that
\[
A = A_{i_0} \cap \phi_{L_T}^{-1}(A_{i_1}) \cap \cdots \phi_{L_T}^{-1}(A_{i_{n-1}}).
\]
By the choice of $\eta_0$, for the fixed $S_\alpha$, there exist at most $2^n$ many $A$’s such that $A \cap B(x,t,\varepsilon,\phi) \neq \emptyset$. Hence, $\#W_n \leq 6^n$. We finish the proof of claim.

Set 
\[
D_n = \{x \in E : \mu(B(x,t,\varepsilon,\phi)) > 6^{2n}\exp(-(\mu(\phi_{L_T},\xi) - r)n)\}.
\]
If we can prove $\sum_{n=\infty}^\infty \mu(D_n) < \infty$, then apply the Borel-Cantelli Lemma, for $\mu$-a.e. $x \in E$, 
\[
\liminf_{n \to \infty} \frac{-\log \mu(B(x,nL_T,\varepsilon,\phi))}{nL_T} + \frac{r + \log 6}{L_T} \geq \frac{1}{L_T}\mu(\phi_{L_T},\xi).
\]
Since $L \geq \frac{r + \log 6}{L_T}$, we obtain that 
\[
r + \frac{\mu(\phi,x)}{L_T} \geq \frac{1}{L_T}\mu(\phi_{L_T},\xi) \geq \frac{1}{L_T}\mu(\phi_{L_T}) - \frac{r}{L_T} = \frac{1}{L_T}\mu(\phi_{L_T}) - \frac{r}{L_T},
\]
hence, 
\[
2r + \frac{\mu(\phi,x)}{L_T} \geq \frac{1}{L_T}\mu(\phi_{L_T}).
\]
As $r > 0$ was chosen arbitrary we conclude that for $\mu$-a.e. $x \in X$, 
\[
\frac{\mu(\phi,x)}{L_T} \geq \frac{1}{L_T}\mu(\phi_{L_T}).
\]
For any $x \in E$, in those $\#W_n$-many elements in $\bigvee_{i=0}^{n-1} \phi_{L_T}^{-1}\xi$ which intersect with $B(x,t,\varepsilon,\phi)$, there exists at least one whose measure is greater than $6^n\exp[-(\mu(\phi_{L_T},\xi) - r)n]$. The total number of such atoms will not exceed $6^{-n}\exp[(\mu(\phi_{L_T},\xi) - r)n]$. Hence $Q_n$, the total number of elements in $\bigvee_{i=0}^{n-1} \phi_{L_T}^{-1}\xi$ that intersect $D_n$, satisfies 
\[
Q_n \leq \#W_n 6^{-n}\exp[(\mu(\phi_{L_T},\xi) - r)n] \leq \exp[(\mu(\phi_{L_T},\xi) - r)n].
\]
Let $S_n$ denote the total measure of such $Q_n$ elements of $\vee_{i=0}^{n-1} \phi_{L^i}^{-1} \xi$ whose intersections with $E$ have positive measure. By the definition of $E$, 
\[ S_n \leq Q_n \exp \left[-n(h_\mu(\phi_{L^t}, \xi) - b)\right] \leq \exp \left([b - r]n\right). \]
From which it follows that $\mu(D_n) \leq S_n \leq \exp \left([b - r]n\right)$. Since $b < r$, we have $\sum_{n=1}^\infty \mu(D_n) < \infty$ for sufficiently large $N$.

**Case 2.** Consider $t > 0$.
Choose $n_t \in \mathbb{N}$ such that $n_t L_t \leq t < (n_t + 1)L_t$. Then we have
\[ B(x, (n_t + 1)L_t, \varepsilon, \phi) \subseteq B(x, t, \varepsilon, \phi) \subseteq B(x, n_t L_t, \varepsilon, \phi). \]
Therefore, for $\mu$-a.e. $x \in X$
\[ \lim_{t \to \infty, \varepsilon \to 0} \frac{\log \mu(B(x, t, \varepsilon, \phi))}{t} \geq \lim_{t \to \infty, \varepsilon \to 0} \frac{\log \mu(B(x, n_t L_t, \varepsilon, \phi))}{t} \]
\[ \geq \lim_{t \to \infty, \varepsilon \to 0} \frac{\log \mu(B(x, (n_t + 1)L_t, \varepsilon, \phi))}{(n_t + 1)L_t} \]
\[ = \lim_{t \to \infty, \varepsilon \to 0} \frac{\log \mu(B(x, n_t L_t, \varepsilon, \phi))}{n_t L_t} \]
\[ \geq \frac{1}{\tau} h_\mu(\phi_\tau). \quad \text{(by Case 1.)} \]

**Case 3.** Consider $\tau < 0$.
Then $-\tau > 0$, similar argument in Case 2 we have for $\mu$-a.e. $x \in X$
\[ h_\mu(\phi, x) \geq -\frac{1}{\tau} h_\mu(\phi_{-\tau}) = -\frac{1}{\tau} h_\mu(\phi_\tau) = \frac{1}{|\tau|} h_\mu(\phi_\tau). \]

3. **Proof of Theorem 1.2.** We need the following variational principle which reveals the basic relationship between Bowen topological entropy and measure theoretic entropy for compact metric flows without fixed-points.

**Theorem 3.1.** [8] Let $(X, \phi)$ be a compact metric flow without fixed points. If $K$ is a non-empty compact subset of $X$, then
\[ h^B_{\text{top}}(\phi, K) = \sup \{ h_\mu(\phi) : \mu \in \mathcal{M}(X), \ \mu(K) = 1 \}. \quad (3) \]

**Proof of Theorem 1.2.** For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\phi_s x, \phi_s y) < \varepsilon$ for any $0 \leq s \leq 1$ and $x, y \in X$ whenever $d(x, y) < \delta$. Then it is easy to see that
\[ B(x, [t], \delta, \phi_1) \subseteq B_t(x, \varepsilon, \phi) \subseteq B(x, [t], \varepsilon, \phi_1), \]
hence $h^B_{\text{top}}(\phi_1) = \hat{h}^B_{\text{top}}(\phi)$. Moreover, since the reparametrization ball $B(x, t, \varepsilon, \phi)$ always contains the usual Bowen ball $B_t(x, \varepsilon, \phi)$, we have $h^B_{\text{top}}(\phi) \leq \hat{h}_{\text{top}}(\phi)$. Therefore,
\[ h^B_{\text{top}}(\phi) \leq h^B_{\text{top}}(\phi_1). \]
From [2] and [4], $h^B_{\text{top}}(\phi_1) = h_{\text{top}}(\phi_1) = \hat{h}_{\text{top}}(\phi)$. Thus,
\[ h^B_{\text{top}}(\phi) \leq \hat{h}_{\text{top}}(\phi). \]
On the other hand, for each $\mu \in \mathcal{E}_{\phi_1}(X)$ and any $t \in \mathbb{R}$, set $\mu_t(B) := \mu(\phi_t(B))$ and $m(B) := \int_0^1 \mu_t(B)dt$ for every Borel set $B$, then we check at once that $\mu_t \in \mathcal{E}_{\phi_1}(X)$
and \( m \in \mathcal{E}_\phi(X) \). Since \((X, \mu, \phi_1)\) and \((X, \mu, \phi)\) are isomorphic (for a definition see \cite{17}), we have \( h_{\mu_t}(\phi_1) = h_{\mu}(\phi_1) \). The concavity of the function \(-x \log x\) implies

\[
h_m(\phi_1) \geq \int_0^1 h_{\mu_t}(\phi_1) dt = h_{\mu}(\phi_1).
\]

Thus \( \sup \{ h_{\mu}(\phi_1) : \mu \in \mathcal{E}_\phi(X) \} \leq \sup \{ h_m(\phi_1) : m \in \mathcal{E}_\phi(X) \} \). By (1) and the classical variational principle for \((X, \phi_1)\) (see \cite{17}),

\[
\sup \{ h_m(\phi_1) : m \in \mathcal{E}_\phi(X) \} = \hat{h}_{\text{top}}(\phi),
\]

\[
\sup \{ h_{\mu}(\phi_1) : \mu \in \mathcal{E}_\phi(X) \} = h_{\text{top}}(\phi) = \hat{h}_{\text{top}}(\phi).
\]

Actually, we obtain

\[
\hat{h}_{\text{top}}(\phi) = \sup \{ h_m(\phi_1) : m \in \mathcal{E}_\phi(X) \}
= \sup \{ h_m(\phi_1) : m \in \mathcal{M}_\phi(X) \}
= \sup \{ h_{\mu}(\phi_1) : \mu \in \mathcal{E}_\phi(X) \}.
\]

According to Theorem 1.1 and (3), we have

\[
\hat{h}_{\text{top}}^B(\phi) = \sup \{ h_{\mu}(\phi) : \mu \in \mathcal{M}(X) \}
\geq \sup \{ h_{\mu}(\phi) : \mu \in \mathcal{E}_\phi(X) \}
= \sup \{ h_{\mu}(\phi_1) : \mu \in \mathcal{E}_\phi(X) \}
= \hat{h}_{\text{top}}(\phi),
\]

which completes the proof. \qed

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