Picard solution of Painlevé VI and related tau-functions

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Abstract

In this paper we obtain explicit expressions for tau-functions related to Picard type solutions of the Painlevé VI equation in terms of theta functions and their derivatives.

1 Introduction

In this paper we study a special case of the Painlevé VI equation [1,2]

\[ q''(t) = \frac{1}{2} \left( \frac{1}{q(t)} + \frac{1}{q(t) - 1} + \frac{1}{q(t) - t} \right) q'(t)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{q(t) - t} \right) q'(t) + \]
\[ \frac{q(t)(q(t) - 1)(q(t) - t)}{t^2(t - 1)^2} \left[ \alpha + \beta - \frac{t}{q(t)^2} + \gamma \frac{t - 1}{(q(t) - 1)^2} + \delta \frac{t(t - 1)}{(q(t) - t)^2} \right] \]  

(1)

when

\[ \alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = \frac{1}{2}. \]  

(2)

This case was originally considered by Picard [3]. Due to a special choice of parameters (2) a general solution of (1) is known

\[ q_0(t) = \wp(c_1 \omega_1 + c_2 \omega_2; \omega_1, \omega_2) + \frac{t + \frac{1}{3}}{3}, \]  

(3)

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where $\wp(u; \omega_1, \omega_2)$ is the Weierstrass elliptic function with half-periods $\omega_{1,2}$, $c_{1,2}$ are complex constants and $\omega_{1,2}(t)$ are two linearly independent solutions of the hypergeometric equations
\[ t(1-t)\omega''(t) + (1-2t)\omega'(t) - \frac{1}{4}\omega(t) = 0. \] (4)

The properties of the Picard solutions have been studied recently by M. Mazzocco [4]. In particular, she investigated its monodromy properties and algebraic solutions which correspond to $c_1$ and $c_2$ being rational numbers.

Algebraic solutions of Painlevé VI play an important role in many applications in theoretical physics (see, for example, [6, 7]). In such cases a calculation of related tau-functions can be simpler due to a presence of an algebraic relation between the solution $q(t)$ and the variable $t$.

The goal of this paper is different. We aim to present explicit expressions for tau-functions related to the Picard solutions and its images under birational canonical transformations [5] for generic values of complex parameters $c_1$ and $c_2$. To our knowledge this has not been done before.

More explicitly, we consider a sequence of tau-functions [5] obtained by a parallel shift $l_3$ from the Picard solution. We calculate the first two tau-functions and other members of the sequence can be obtained using the standard Toda-type second order relations.

2 Properties of the Painlevé VI equation

In this section we briefly review the main properties of the equation (1) which we denote as $P_{VI}(\alpha, \beta, \gamma, \delta)$.

Following [5] one can introduce two different parameterizations of parameters in the Painlevé VI equation: $(\kappa_0, \kappa_1, \kappa_\infty, \theta)$
\[ \alpha = \frac{1}{2}\kappa_\infty, \quad \beta = -\frac{1}{2}\kappa_0^2, \quad \gamma = \frac{1}{2}\kappa_1^2, \quad \delta = \frac{1}{2}(1 - \theta^2), \] (5)
and $(b_1, b_2, b_3, b_4)$
\[ \kappa_0 = b_1 + b_2, \quad \kappa_1 = b_1 - b_2, \quad \kappa_\infty = b_3 - b_4, \quad \theta = b_3 + b_4 + 1. \] (6)

This equation is equivalent to the Hamiltonian system $H_{VI}(t; q, p)$ described by the equations
\[ \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \] (7)
with the Hamiltonian function
\[ H_{VI}(t; q, p) = \frac{1}{t(t-1)}[q(q-1)(q-t)p^2 - \{\kappa_0(q-1)(q-t) + \kappa_1q(q-t)+(\theta-1)q(q-1)\}p + \kappa(q-t)], \] (8)
where $q \equiv q(t)$, $p \equiv p(t)$ and
\[ \kappa = \frac{1}{4}(\kappa_0 + \kappa_1 + \theta - 1)^2 - \frac{1}{4}\kappa_\infty^2. \] (9)
One can introduce an auxiliary Hamiltonian $h(t)$,

$$h(t) = (t-1)H(t) + e_2(b_1, b_3, b_4) t - \frac{1}{2} e_2(b_1, b_2, b_3, b_4),$$

where $e_i(x_1, \ldots, x_n)$ is the $i$-th elementary symmetric function in $n$ variables and a set of $x_i$'s can be a subset of $b_i$'s as in (10).

Okamoto [5] showed that for each pair $\{q(t), p(t)\}$ satisfying (7), the function $h(t)$ solves the $E_{VI}$ equation which is

$$h'(t) \left[ t(1-t)h''(t) \right]^2 + h'(t)[2h(t) - (2t-1)h'(t)] + b_1 b_2 b_3 b_4 = \prod_{k=1}^{4} \left( h'(t) + b_k^2 \right)$$

and $q(t)$ solves $P_{VI}(\alpha, \beta, \gamma, \delta)$.

Conversely, for each solution $h(t)$ of (11), such that $\frac{\partial^2}{\partial t^2} h(t) \neq 0$, there exists a solution $\{q(t), p(t)\}$ of (7), where $q(t)$ solves (1). An explicit correspondence between three sets $\{q(t), q'(t)\}$, $\{q(t), p(t)\}$ and $\{h(t), h'(t), h''(t)\}$ is given by birational transformations, which can be found in [5].

The group of Backlund transformations of $P_{VI}$ is isomorphic to the affine Weyl group of the type $F_4$: $W_a(F_4)$. It contains the following transformations of parameters (only five of them are independent)

$$w_1 : b_1 \leftrightarrow b_2, \quad w_2 : b_2 \leftrightarrow b_3, \quad w_3 : b_3 \leftrightarrow b_4, \quad w_4 : b_3 \rightarrow -b_3, \quad b_4 \rightarrow -b_4, \quad (12)$$

and the parallel transformation

$$l_3 : b \equiv (b_1, b_2, b_3, b_4) \rightarrow b^+ \equiv (b_1, b_2, b_3 + 1, b_4). \quad (14)$$

The auxiliary function $h_+(t)$ corresponding to parameters $b^+ = l_3(b)$ is given in [5]

$$h_+(t) = h(t) - q(q-1)p + (b_1 + b_4)q - \frac{1}{2}(b_1 + b_2 + b_4). \quad (15)$$

Following [5] one can calculate $h_+(t)$ in terms of $h(t)$ and its first and second derivatives

$$h_+(t) = \frac{t(t-1)h''(t) + 2h(t)[b_3(b_3 + 1) + h'(t)] + b_3(1-2t)h'(t) - b_1 b_2 b_4}{2(h'(t) + b_3^2)} \quad (16)$$

and vice versa

$$h(t) = \frac{t(t-1)h''(t) + 2h_+(t)[b_3(b_3 + 1) + h_+'(t)] - (b_1 + 1)(1-2t)h'_+(t) + b_1 b_2 b_4}{2(h'_+(t) + (b_3 + 1)^2)} \quad (17)$$

For each solution of the $P_{VI}$ equation one can introduce a corresponding tau-function via

$$H(t, q(t), p(t); b) = \frac{d}{dt} \log T(t, b). \quad (18)$$
Obviously tau-functions are defined up to an arbitrary normalization factor.

Following [5] let us introduce a family of tau-functions $T_m(t)$

$$T_m(t) = \exp \left\{ \int dt H(t, q(t), p(t); b_m) \right\},$$

(19)

where

$$b_m \equiv l^m_3(b) = (b_1, b_2, b_3 + m, b_4), \quad m \in \mathbb{Z}.$$  

(20)

As shown in [5] they satisfy the second order Toda-type equation

$$\frac{dt}{dt} \left[ t(t-1) \frac{d}{dt} \log T_m(t) \right] + (b_1 + b_3 + m)(b_3 + b_4 + m) = c(m) \frac{T_{m+1}(t)T_{m-1}(t)}{T_m^2(t)},$$

(21)

where $c(m)$ is a nonzero constant.

### 3 Elliptic functions and useful identities

In this section we list all definitions and properties of elliptic functions used later in the text. Following [8] we will use the standard theta functions $\theta_i(x|\tau)$ with quasi-periods $\pi$ and $\pi\tau$.

The elliptic modulus $k$ and its complement $k'$ are defined in a standard way by

$$k = \frac{\theta_2(0|\tau)^2}{\theta_3(0|\tau)^2}, \quad k' = \frac{\theta_4(0|\tau)^2}{\theta_3(0|\tau)^2}, \quad k^2 + k'^2 = 1.$$  

(22)

It will be more convenient to use the parameter $t = k^2$ as the second argument of elliptic functions and hereafter we will follow this notation (except for the theta-functions), i.e.

$$\tau = i \frac{K'(t)}{K(t)}, \quad q = e^{i\pi\tau}, \quad K(t) = \frac{\pi}{2} \theta_3^2(0|\tau), \quad K'(t) = K(1-t)$$

(23)

where $K(t)$ and $K'(t)$ are the complete elliptic integrals of the first kind of the parameters $t$ and $1-t$.

We introduce Jacobi elliptic functions

$$\text{sn}(u, t) = \frac{1}{k^{1/2}} \frac{\theta_1(v|\tau)}{\theta_4(v|\tau)}, \quad \text{cn}(u, t) = \frac{(k')^{1/2} \theta_2(v|\tau)}{k^{1/2} \theta_4(v|\tau)}, \quad \text{dn}(u, t) = (k')^{1/2} \frac{\theta_3(v|\tau)}{\theta_4(v|\tau)}.$$  

(24)

where

$$u = \frac{2K(t)}{\pi} v$$

(25)

and define the fundamental elliptic integral of the second kind [8] by

$$\mathcal{E}(u, t) = \int_0^u \text{dn}^2(x, t)dx.$$  

(26)

It satisfies

$$\mathcal{E}(K(t), t) = E(t),$$

(27)
where \( E(t) \) is the complete elliptic integral of the second kind.

Using (23) and identities for complete elliptic integrals of the first and second kind

\[
\rho_t K(t) = \frac{E(t) - K(t)}{2t}, \quad \rho_t E(t) = \frac{E(t) + K(t)}{2t}
\]

one can obtain

\[
\rho_t \tau(t) = \frac{i\pi}{4t(t-1)K^2(t)}.
\]

We also need the derivatives of Jacobi elliptic functions with respect to the parameter \( t \). Differentiating the formula

\[
u = \int_0^\infty \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}
\]

with respect to \( t \) and calculating the remaining integral we get

\[
\frac{d}{dt}\text{sn}(u,t) = -\frac{\text{sn}(u,t)\text{cn}^2(u,t)}{2(t-1)} + \frac{\text{cn}(u,t)\text{dn}(u,t)}{2t(t-1)}[u(t-1) + \mathcal{E}(u,t)].
\]

From (32) it is easy to obtain

\[
\frac{d}{dt}\text{cn}(u,t) = \frac{\text{sn}^2(u,t)\text{cn}(u,t)}{2(t-1)} - \frac{\text{sn}(u,t)\text{dn}(u,t)}{2t(t-1)}[u(t-1) + \mathcal{E}(u,t)]
\]

and

\[
\frac{d}{dt}\text{dn}(u,t) = \frac{\text{sn}^2(u,t)\text{dn}(u,t)}{2(t-1)} - \frac{\text{sn}(u,t)\text{cn}(u,t)}{2t(t-1)}[u(t-1) + \mathcal{E}(u,t)].
\]

Integrating the well known formula between a logarithmic derivative of \( \theta_4(x|\tau) \) and \( \mathcal{E}(u,t) \) we obtain

\[
\theta_4(x|\tau) = \theta_4(0|\tau) \exp \left\{ -\frac{2x^2}{\pi^2} E(t) K(t) + \int_0^{2xK(t)/\pi} \mathcal{E}(y,t) dy \right\}.
\]

We can use (35) for calculation of the derivatives \( \theta_4(x|\tau)' \) and \( \theta_4(x|\tau)'' \).

Finally, combining (35) with the differential equation satisfied by theta-functions

\[
\frac{4}{i\pi} \frac{\partial}{\partial \tau} \theta_i(u|\tau) + \frac{\partial^2}{\partial u^2} \theta_i(u|\tau) = 0, \quad i = 1, 2, 3, 4,
\]

one can calculate the derivative \( \rho_t \theta_4(x|\tau) \) in terms of theta-functions and \( \mathcal{E}(u,t) \).

We notice that formula (35) is convenient for expansion of \( \theta_4(x|\tau) \) in a series in \( x \) up to any required order. Say,

\[
\theta_4(x|\tau) = \theta_4(0|\tau) \exp \left\{ -\frac{2x^2}{\pi^2} E(t) K(t) \right\} \left[ 1 + 2\frac{x^2K^2(t)}{\pi^2} + \frac{2(3-2t)x^4K^4(t)}{3\pi^4} + O(x^6) \right].
\]
4 Picard solution and tau-functions

The restriction on parameters (2) for the Picard solution (3) can be rewritten in terms of parameters $b_i$ (6) as

$$b_1 = b_2 = 0, \quad b_3 = b_4 = -1/2.$$  \hfill (38)

It is convenient to fix a particular branch of the Picard solution (3) by choosing two linearly independent solutions of equation (4)

$$\omega_1(t) = \frac{\pi}{2} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right) = K(t), \quad \omega_2(t) = \frac{i\pi}{2} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - t\right) = iK'(t) = iK(1 - t),$$  \hfill (39)

where $K(t)$ and $K'(t)$ are the complete elliptic integrals of the first kind as defined in the previous section.

With a choice of half-periods (39) for the Weierstrass function the expressions for the invariants $e_1, e_2, e_3$ take the following form

$$e_1 = 1 - \frac{t + 1}{3}, \quad e_2 = t - \frac{t + 1}{3}, \quad e_3 = -\frac{t + 1}{3}.$$  \hfill (40)

After some simple calculations we can rewrite the Picard solution of $P_{VI}$ as

$$q_0(t) = \frac{1}{\text{sn}^2(c_1 K(t) + i c_2 K'(t), t)},$$  \hfill (41)

where $c_1$ and $c_2$ are the same parameters as in (3).

Let us make a change of variables

$$c_1 = \frac{2x}{\pi} + 1, \quad c_2 = \frac{2y}{\pi} + 1,$$  \hfill (42)

where $x, y$ are new parameters. A reason for this is that the resulting tau-functions look simpler in $x$ and $y$.

With a substitution (42) formula (41) takes the form

$$q_0(t) = t \frac{\text{cn}^2(z, t)}{\text{dn}^2(z, t)},$$  \hfill (43)

where we defined a new variable $z$

$$z = \frac{2K(t)}{\pi} (x + \tau y).$$  \hfill (44)

The hamiltonian $H_0(t)$ for the choice of parameters (38) can be calculated from (7-8) in terms of $q_0(t)$ and its first derivative

$$H_0(t) = \frac{t^2 + (1 - 2t)q_0(t)}{4t(t - 1)(q_0(t) - t)} + \frac{t(t - 1)q_0'(t)^2}{4q_0(t)(q_0(t) - 1)(q_0(t) - t)}.$$  \hfill (45)

Using formulas from section 3 one can explicitly calculate the derivative $q_0'(t)$
\[
\frac{d}{dt}q_0(t) = q'_0(t) = \frac{1}{dn^2(z,t)} + \frac{sn(z,t)cn(z,t)}{dn^3(z,t)} \left[ \frac{\pi}{2K(t)} \log \theta_2(x + \tau y|\tau) \right]'_x + \frac{iy}{K(t)}. \tag{46}
\]

Substituting (46) into (45) we produce the following expression for the function \(H_0(t)\):

\[
H_0(t) = -\frac{1}{4(t-1)} - \frac{cn^2(z,t)}{4t(t-1)sn^2(z,t)} + \frac{\mathcal{E}(x,y,t)^2}{4t(t-1)}, \tag{47}
\]

where

\[
\mathcal{E}(x,y,t) = \frac{\pi}{2K(t)} [\log \theta_1(x + \tau y|\tau)]'_x + \frac{iy}{K(t)} \tag{48}
\]

and \(z\) is defined by (44).

To calculate the tau-function \(T_0(t)\) for the Picard solution (43) we have to calculate the indefinite integral of (47) with respect to the variable \(t\) which looks like a hopeless problem.

Now we formulate the central result of this paper.

**Theorem 4.1** The tau-function for the Picard solution (43) is given by

\[
T_0(t) = \exp \left\{ \int H_0(t) dt \right\} = c_0(x,y) q^{y^2/\pi^2} t^{-1/4} \frac{\theta_1(x + \tau y|\tau)}{\theta_4(0|\tau)}, \tag{49}
\]

where \(c_0(x,y)\) is an integration constant.

**Proof:**

First we rewrite \(T_0(t)\) as

\[
T_0(t) = c_0(x,y) q^{y^2/\pi^2} \frac{sn(z,t)\theta_4(x + \tau y|\tau)}{\theta_4(0|\tau)}. \tag{50}
\]

The proof is straightforward and reduces to differentiations. We shall do this in a few steps. Taking logarithmic derivative of (50) we obtain

\[
\frac{\partial}{\partial t} \log T_0(t) = \frac{i y^2}{\pi} \partial_t \tau(t) + \frac{cn(z,t)dn(z,t)}{sn(z,t)} \left[ \frac{2(x + \tau y)}{\pi} \partial_t K(t) + \frac{2yK(t)}{\pi} \partial_t \tau(t) \right] + \frac{\partial_t \log \theta_4(u|\tau)}{\theta_4(0|\tau)} \left|_{u=x+\tau y} \right. - \frac{\partial_x \theta_4(0|\tau)}{\theta_4(0|\tau)} \partial_t \tau(t). \tag{51}
\]

Using (28-30) one can evaluate the first two terms in (51). The derivative \(\partial_t \log \theta_4(u|\tau)\) was calculated in (32). Differentiating (35) twice and using the equation (36) one can evaluate all derivatives in the last term of (51). Combining all contributions and using a simple formula

\[
\log \theta_1(x + \tau y|\tau)]'_x = \log \theta_4(x + \tau y|\tau)]'_x + \frac{2K(t) cn(z,t)dn(z,t)}{\pi} \frac{sn(z,t)}{sn^3(z,t)} \tag{52}
\]

we obtain after simplifications the expression (47) for \(H_0(t)\).
It is quite remarkable that the indefinite integral of the function \((47)\) gives such a simple answer \((49)\). We were able to produce this expression by expanding \((47)\) in a series in \(x\) at \(y = 0\) and integrating term by term. Analyzing the resulting series we compared it to the expansion of \(\theta_1(x|\tau)\) in \(x\) which is similar to the expansion \((37)\). This allowed us to arrive at the final answer \((49)\).

To solve the equation \((21)\) for \(T_m(t)\) we need to calculate the second tau-function \(T_1(t)\) corresponding to the solution with parameters

\[
b_1 = (0, 0, 1/2, -1/2).
\]

First we have from \((10)\)

\[
h_0(t) = t(t - 1)H_0(t) + t/4 - 1/8
\]

for the Picard solution with \(b_0 = (0, 0, -1/2, -1/2)\) and

\[
h_1(t) = t(t - 1)H_1(t) - t/4 + 1/8
\]

for the solution with parameters \((53)\).

Now \(h_1(t)\) is obtained using the birational canonical transformation \((16)\) with parameters \((38)\) and \(h(t)\) replaced with \(h_0(t)\). Using \((54-55)\) one can arrive at the following answer:

\[
H_1(t) = -\frac{\text{sn}^2(z, t)}{4\text{dn}^2(z, t)} + \frac{1}{4t(t - 1)} \left[ \frac{\pi}{2K(t)} \log \theta_4(x + \tau y|\tau) \right] + \frac{iy}{K(t)} - t \frac{\text{sn}(z, t)\text{cn}(z, t)}{\text{dn}(z, t)} \right]^2.
\]

In fact, the equation satisfied by \(h_0(t)\) and \(h_1(t)\) is the same and it is easy to check that

\[
h_1(t, x, y) = h_0(t, x + \frac{\pi}{2}, y + \frac{\pi}{2})
\]

and

\[
H_1(t, x, y) = H_0(t, x + \frac{\pi}{2}, y + \frac{\pi}{2}) + \frac{1}{4t} + \frac{1}{4(t - 1)},
\]

where we show explicitly a dependence on the fixed parameters \(x\) and \(y\).

Taking \((58)\) into account it is easy to integrate \(H_1(t)\) using formula \((49)\).

The answer is given by the following

**Theorem 4.2** The \(\tau\)-function \(T_1(t)\) is given by

\[
T_1(t) = \exp\int H_1(t)dt = c_1(x, y) q^{\sigma^2/\pi^2}(1 - t)^{1/4} \frac{\theta_3(x + \tau y|\tau)}{\theta_4(0|\tau)},
\]

where \(c_1(x, y)\) is an arbitrary integration constant.

If we define the sequence of tau-functions \(T_m(t)\) corresponding to Picard type solutions with parameters

\[
b_m = (0, 0, -1/2 + m, -1/2), \quad m \in \mathbb{Z}
\]

8
and two initial conditions

\[ T_0(t) = q^{\nu^2/\pi^2} t^{-1/4} \frac{\theta_1(x + \tau y|\tau)}{\theta_4(0|\tau)}; \quad T_1(t) = q^{\nu^2/\pi^2} (1 - t)^{1/4} \frac{\theta_3(x + \tau y|\tau)}{\theta_4(0|\tau)}, \]  

(61)

then other tau-functions \( T_m(t) \), for \( m > 1 \) or \( m < 0 \) can be calculated from the difference-differential equation

\[ \frac{d}{dt} \left[ t(t - 1) \frac{d}{dt} \log T_m(t) \right] + \left( m - \frac{1}{2} \right)^2 = c(m) \frac{T_{m+1}(t)T_{m-1}(t)}{T_m^2(t)}, \]  

(62)

where \( c(m) \) is determined by a normalization of tau-functions.

Note that the expressions for \( T_m(t) \), \( m \neq 0, 1 \) will be more complicated and involve explicitly the function \( \mathcal{E}(u, t) \) defined in (26). We will not calculate them here.

5 Conclusion

In this paper we constructed a sequence of tau-functions for the Picard type solutions of Painlevé VI equation with parameters (60). In fact, starting with the tau-function for the Picard solution (49) one can write many different birational canonical transformations and calculate corresponding sequences of tau-functions. We have successfully applied this approach to sum up some infinite form factor expansions for the 2D Ising model. All details will be given in forthcoming publications.

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