Symmetry Groups and Equivalence Transformations in the Nonlinear Donnell–Mushtari–Vlasov Theory for Shallow Shells

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Abstract—In the case of transversely only loaded shallow shells, the nonlinear Donnell–Mushtari–Vlasov theory for large deflection of isotropic thin elastic shells leads to a system of two coupled nonlinear forth-order partial differential equations known as Marguerre’s equations. This system involves two arbitrary elements – the curvature tensor of the shell middle-surface and the function of transversal load per unit surface area. In the present note, the point symmetry groups of Marguerre’s equations are established, the corresponding group classification problem being solved. It is shown that Marguerre’s equations are equivalent to the von Kármán equations for large deflection of plates in the time-independent case and in the time-dependent case as well. It is also observed that the same holds true in respect of the field equations for anisotropic shallow shells.

1 Introduction

Within the framework of the nonlinear Donnell–Mushtari–Vlasov (DMV) theory (see, e.g., [1]) the state of equilibrium of a transversely loaded thin isotropic elastic shell of uniform thickness is determined by the following system of two coupled nonlinear forth-order partial differential equations:

\[
D \Delta^2 w - \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} w_{\alpha\beta}\Phi_{\mu\nu} - \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} b_{\alpha\beta}\Phi_{\mu\nu} = p,
\]

\[
(1/Eh) \Delta^2 \Phi + (1/2) \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} w_{\alpha\beta}w_{\mu\nu} + \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} b_{\alpha\beta}w_{\mu\nu} = 0,
\]

in two independent variables, associated with the coordinates on the shell middle-surface \( F \), and two dependent variables – the transversal displacement function \( w \), and Airy’s stress function \( \Phi \). Here, \( \varepsilon^{\alpha\beta} \) is the alternating tensor of \( F \); \( b_{\alpha\beta} \) is the curvature tensor of \( F \); \( D \), \( E \) and \( h \) are the bending rigidity, Young’s modulus and thickness of the shell, respectively (i.e., \( D, E \) and \( h \) are given constants); \( p \) is the function of transversal load per unit surface area; a semicolon is used for covariant differentiation with respect to the metric tensor \( a_{\alpha\beta} \) of the surface \( F \); \( \Delta \) is the Laplace-Beltrami operator on \( F \). Here and throughout, Greek indices have the range 1, 2 and the usual summation convention over a repeated index (one subscript and one superscript) is employed.

The present note is concerned with a special case of the nonlinear DMV theory; the so-called shallow shells are considered. In fact, this is an approximation of the theory, which is usually introduced as follows (cf., e.g., [2]). Let \( (x^1, x^2, x^3) \) be a fixed right-handed rectangular Cartesian coordinate system in the 3-dimensional Euclidean space in which the middle-surface \( F \) of a shell is embedded, and let this surface be given by the equation

\[
x^3 = f(x^1, x^2), \quad (x^1, x^2) \in \Omega \subset \mathbb{R}^2,
\]

where \( f : \mathbb{R}^2 \to \mathbb{R} \) is assumed to be a single-valued and smooth function possessing as many derivatives as may be required on the domain \( \Omega \). Let us take \( x^1, x^2 \) to serve as coordinates on

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the surface \( F \). Then, relative to this coordinate system, the components of the fundamental tensors and the alternating tensor of \( F \) are given by the expressions:

\[
\begin{align*}
a_{\alpha\beta} &= \delta_{\alpha\beta} + f_{,\alpha} f_{,\beta}, \\
b_{\alpha\beta} &= a^{-1/2} f_{,\alpha\beta}, \\
\varepsilon_{\alpha\beta} &= a^{-1/2} e_{\alpha\beta},
\end{align*}
\]

(2)

where

\[
a = \det(a_{\alpha\beta}) = 1 + (f_{,1})^2 + (f_{,2})^2;
\]

\[
\delta_{\alpha\beta} = \delta_{\alpha\beta} \text{ is the Kronecker delta symbol; } e_{\alpha\beta} \text{ is the alternating symbol; here and in what follows, a comma is used for partial differentiation with respect to the coordinates on } F.
\]

A shell is said to be shallow on the domain \( \Omega_0 \subset \Omega \) when the inequalities

\[
|f_{,\alpha}| |f_{,\beta}| \leq \varepsilon^2 \ll 1, \quad \varepsilon = \text{const},
\]

hold for every point \((x^1, x^2) \in \Omega_0\). Hence, for shallow shells the quadratic terms in the right-hand sides of expressions (2) are small compared to unity and may be neglected. Thus, allowing for a relative error of order \( O(\varepsilon^2) \), one may regard the intrinsic geometry of the shell middle-surface \( F \) as Euclidean and \((x^1, x^2)\) may be thought of as an Euclidean coordinate system in which:

\[
\begin{align*}
a_{\alpha\beta} &= \delta_{\alpha\beta}, \\
b_{\alpha\beta} &= f_{,\alpha\beta}, \\
\varepsilon_{\alpha\beta} &= e_{\alpha\beta};
\end{align*}
\]

(3)

(4)

(5)

the mean curvature \( H \) of the surface \( F \) and its Gaussian curvature \( K \) (note that the latter is not necessarily equal to zero within the allowed relative error) take the form

\[
\begin{align*}
H &= (1/2)\delta_{\alpha\beta} f_{,\alpha\beta}, \\
K &= (1/2)e^{\alpha\mu} e^{\beta\nu} f_{,\alpha\beta} f_{,\mu\nu},
\end{align*}
\]

(6)

(7)

and system (8) reads

\[
\begin{align*}
D\delta^{\alpha\beta} \delta^{\mu\nu} w_{\alpha\beta\mu\nu} - e^{\alpha\mu} e^{\beta\nu} w_{\alpha\beta\mu\nu} \Phi_{\mu\nu} - e^{\alpha\mu} e^{\beta\nu} b_{\alpha\beta} \Phi_{\mu\nu} &= p, \\
(1/Eh)\delta^{\alpha\beta} \delta^{\mu\nu} \Phi_{,\alpha\beta\mu\nu} + (1/2)e^{\alpha\mu} e^{\beta\nu} w_{,\alpha\beta\mu\nu} + e^{\alpha\mu} e^{\beta\nu} b_{\alpha\beta} w_{,\mu\nu} &= 0,
\end{align*}
\]

(8)

Thus, we arrive at the equilibrium equations for shallow shells within the framework of the nonlinear DMV theory. Since equations (8) follow from Marguerre’s shell theory \( \text{as well, they are also known as Marguerre’s equations for large deflection of plates with small initial curvature (i.e., shallow shells). These equations are well accepted and play an important role in the shell theory (see, e.g., } [3], [4] \text{ and the references therein). They also include as a special case, with } b_{\alpha\beta} = 0, \text{ the well-known von Kármán equations for large deflection of plates } [4].\]

2 Symmetry Groups

The aim of the present work is to study, following [1], [8] and [9], the invariance properties of system (8) relative to local one-parameter Lie groups of local point transformations acting on open subsets of the 4-dimensional Euclidean space \( \mathbb{R}^4 \), with coordinates \((x^1, x^2, w, \Phi)\), representing the involved independent and dependent variables. For that purpose Lie infinitesimal technique is used and, as a rule, the results obtained are expressed in terms of the infinitesimal generators (operators) of the groups; in the present case, the latter are vector fields of the form

\[
X = \xi^\mu \frac{\partial}{\partial x^\mu} + \eta \frac{\partial}{\partial w} + \varphi \frac{\partial}{\partial \Phi},
\]

(9)

where \( \xi^\mu, \eta \) and \( \varphi \) are functions of the variables \( x^1, x^2, w \) and \( \Phi \). The system considered involves an arbitrary tensor field \(- b_{\alpha\beta}\) and an arbitrary function \(- p \). This gives rise to a group classification problem with respect to the arbitrary element \(- the set \{b_{\alpha\beta}, p\}. \)
The infinitesimal criterion of invariance leads to the following system of determining equations (DE system) for the components $\xi^1, \xi^2, \eta$ and $\varphi$ of the vector fields of form (8) generating point symmetry groups admitted by system (8):

\begin{align*}
\xi^1 &= C_1 x^1 + C_2 x^2 + C_3, \\
\xi^2 &= -C_2 x^1 + C_1 x^2 + C_4, \\
\eta &= \eta(x^1, x^2), \\
\varphi &= B_1 x^1 + B_2 x^2 + B_3,
\end{align*}

(10) \quad (11) \quad (12) \quad (13)

\begin{align*}
b_{\alpha\mu}\xi^{\mu}_{,\beta} + b_{\beta\mu}\xi^{\mu}_{,\alpha} + \xi^{\mu} b_{\alpha\beta,\mu} &= -\eta_{,\alpha\beta}, \\
e^{\alpha\mu}\xi^{\beta\nu} b_{\alpha\beta} \eta_{,\mu\nu} &= 0, \\
D\delta^{\alpha\beta}\xi^{\mu}_{,\alpha\beta\mu} &= 2p\xi^{\mu}_{,\mu} + \xi^{\mu} P_{,\mu}.
\end{align*}

(14) \quad (15) \quad (16)

where $C_1, \ldots, C_4, B_1, B_2,$ and $B_3$ are arbitrary real constants. This result is established in [10] through the standard computational procedure (see [7, Sec. 5] or [8, Sec. 2.4]), and therefore we can assert that the system (8) admits a vector field of form (9) if and only if (10) – (16) hold. In other words, all possible symmetries of the kind under consideration inherent to system (8) can be found via the solution of the DE system. So, the problem to solve consists in finding the solutions of this system. The main difficulty here comes out of the determining equations (14) – (16). They show that the space of solutions $L$ to the DE system, i.e., the Lie algebra of symmetries associated with system (8), depends on the choice of the arbitrary element $\{b_{\alpha\beta}, p\}$. At this juncture, we do face a group classification problem. That is to say, that we will have to determine all those specializations (special forms) of the arbitrary element for which system (8) admits vector fields of the form (9).

Proceeding to analyze this problem, we will first show that equations (12), (14) – (16) may be replaced by the following three equivalent ones:

\begin{align*}
\eta &= -\xi^{\mu} f_{,\mu} + A_1 x^1 + A_2 x^2 + A_3, \\
2P\xi^{\mu}_{,\mu} + \xi^{\mu} P_{,\mu} &= 0, \\
2K\xi^{\mu}_{,\mu} + \xi^{\mu} K_{,\mu} &= 0,
\end{align*}

(17) \quad (18) \quad (19)

where $A_1, A_2$ and $A_3$ are arbitrary real constants, and

\begin{align*}
P &= 2D\delta^{\mu\nu} H_{,\mu\nu} + p.
\end{align*}

(20)

Indeed, (17) represents the general solution of equations (14) of the form (12) when (2), (10) and (11) hold; under the same assumption, by substituting (17) into (15) and (16), and taking into account (8) and (9), after some algebra we obtain expressions (18) and (19). Using this result, hereafter we will assume that the DE system consists of equations (10), (11), (13), (17), (18) and (19).

Now, a 6-dimensional space of solutions to the DE system arises immediately. It corresponds to the 6-dimensional Lie algebra $L_0$ of the vector field whose components are given as follows

\begin{align*}
\xi^\mu &= 0, \\
\eta &= A_1 x^1 + A_2 x^2 + A_3, \\
\varphi &= B_1 x^1 + B_2 x^2 + B_3,
\end{align*}

(21)

where $A_1, A_2, A_3, B_1, B_2$ and $B_3$ are arbitrary real constants. Evidently, the above solutions of the DE system do not depend on the choice (specialization) of the arbitrary element.
Hence, for any specialization of the arbitrary element system (8) admits the 6-parameter Lie group $G_0$ generated by all linear combinations of the vector fields

$$\frac{\partial}{\partial w}, \ x^1 \frac{\partial}{\partial w}, \ x^2 \frac{\partial}{\partial w}, \ \frac{\partial}{\partial \Phi}, \ x^1 \frac{\partial}{\partial \Phi}, \ x^2 \frac{\partial}{\partial \Phi},$$

these vector fields constitute a basis of the associated Lie algebra $L_0$. Simultaneously, $G_0$ is the largest Lie group of point transformations admitted by system (8) for any choice of the arbitrary element. Indeed, by setting

$$f = \sin(x^1) \sin(x^2),$$

for instance, one can easily verify that for this particular choice of the arbitrary element the DE system has no other solutions besides those given by (21).

So far, we have obtained the so-called kernel of the full symmetry groups associated with system (8), that is the group $G_0$. The next step is to identify the cases in which the system under consideration possesses larger groups of point symmetries. In the light of all the above, it means to characterize in a suitable manner all those specializations of the arbitrary element for which system (8) admits vector fields of the form

$$Y = \xi^\mu \frac{\partial}{\partial x^\mu} - \xi^\mu f_{,\mu} \frac{\partial}{\partial w} \quad (\xi^\mu \neq 0),$$

where $\xi^\mu$ are given by the expressions (10) and (11), keeping in mind that (18) and (19) remain the only conditions (necessary and sufficient) for (8) to be invariant under a group generated by a vector field of form (22). Note that all determining equations are thus taken into account.

Taking into account (6), (7) and (20) we can see at once that for $f = p = 0$ the invariance conditions (13) and (14) are satisfied. Hence, system (8), with $f = p = 0$, possesses a larger group of point symmetries (in addition to $G_0$) and (10), (11) and (22) show that this is the complete 4-parameter group of homothetic motions of the Euclidean plane. In this special case, (8) coincide with the homogeneous von Kármán equations, so that we have arrived at the result obtained in [11] (see also [12]). This example gives us a good motivation for studying the general case.

We begin with the following observation. Given a vector field $Y$ of form (22), the function

$$\tilde{w} = w + f,$$  

is an invariant of the corresponding Lie group of transformations acting on $\mathbb{R}^4$. Therefore, we can introduce new coordinates $(x^1, x^2, \tilde{w}, \Phi)$ on $\mathbb{R}^4$, the new dependent variable $\tilde{w}$ being defined by (23), in which $Y$ takes the form of an infinitesimal operator of the Lie algebra associated with the group of homothetic motions of the Euclidean plane, namely

$$Y = \xi^\mu \frac{\partial}{\partial x^\mu},$$

(note that the components $\xi^1$ and $\xi^2$ still have the form (10) and (11), respectively, as before), and system (8) reads

$$D\delta^{\alpha\beta} \delta^{\mu\nu} \tilde{w}_{,\alpha\beta\mu\nu} - e^{\alpha\mu} e^{\beta\nu} \tilde{w}_{,\alpha\beta} \Phi_{,\mu\nu} = P,$$

$$(1/Eh)\delta^{\alpha\beta} \delta^{\mu\nu} \Phi_{,\alpha\beta\mu\nu} + (1/2)e^{\alpha\mu} e^{\beta\nu} \tilde{w}_{,\alpha\beta} \tilde{w}_{,\mu\nu} = K.$$

Thus, the problem of invariance of system (8) with respect to a vector field (22) converts into the problem of invariance of system (23) under a vector field (24) as a change of the variables does not affect the group properties of a system of differential equations. Now, taking into account the invariance conditions (13) and (14) (note that they remain unchanged under the above coordinate transformation), we can conclude that system (23) admits a one-parameter group $G$ of homothetic motions of the Euclidean plane generated by a vector field of form (24).
if and only if the corresponding arbitrary element \( \{ b_{\alpha \beta}, p \} \) is such that the functions \( P \) and \( K \), defined by formulae (4) and (24), are invariants of \( G \) (when \( C_1 = 0 \)) or eigenfunctions (when \( C_1 \neq 0 \)) of its generator, the latter being regarded as an operator acting on the smooth functions \( \zeta : M \to R, \ M \subset R^2 \). This result may be thought of as a general solution to the group classification problem under consideration in terms of the function \( p \) and the two characteristic invariants, \( H \) and \( K \), of the shell middle-surface \( F \).

### 3 Equivalence Transformations

Let us now briefly discuss, in the context of the shell theory, the meaning of the coordinate transformation \( \omega : R^4 \to R^4 \), \((x^1, x^2, w, \Phi) \to (x^1, x^2, w + f, \Phi)\) introduced in the previous Section. Omitting tilde’s in equations (25), we can say that systems (8) and (25) belong to the same class – they have the same differential structure, and differ from one another only in the form of the arbitrary element. Hence, according to [7, Definition 6.4], we may conclude that \( \omega : R^4 \to R^4 \) is an equivalence transformation for system(8). In addition, we see that (25) is nothing but a system of nonhomogeneous von Kármán equations with special right-hand sides. It is noteworthy that Marguerre’s equations [8], i.e., the equilibrium equations for shallow shells, turned out to be equivalent to the von Kármán equations, i.e., the equilibrium equations for plates, regardless of the invariance properties of system (8). To the best of the author’s knowledge, this fact has not been noticed before in the literature, though both the von Kármán equations and Marguerre’s equations have been studied and utilized for many years until now (see, e.g., [1], [2], [4], [5], [11], [12], [13] and the references therein). In our opinion, the established correspondence between these two systems of equations will certainly be of use in solving a wide range of problems arising in the shell theory. In particular, the results for the von Kármán equations obtained in [4] can be easily conveyed to the theory of shallow shells.

Several additional applications of the transformation \( \omega \) can be given too. Evidently, it maps:

- the time-dependent Marguerre’s equations

\[
D_{\alpha \beta \mu \nu} \partial^\omega \omega_{\alpha \beta \mu \nu} - e^{\alpha \mu e^{\beta \nu} w_{\alpha \beta \mu \nu} - e^{\alpha \mu e^{\beta \nu} b_{\alpha \beta \mu \nu} + \rho \ddot{w} = p,}
\]

\[
(1/Eh) \delta_{\alpha \beta \mu \nu} \Phi_{\alpha \beta \mu \nu} + (1/2)e^{\alpha \mu e^{\beta \nu} w_{\alpha \beta \mu \nu} + e^{\alpha \mu e^{\beta \nu} b_{\alpha \beta \mu \nu} = 0,}
\]

(\( \rho \) is the mass per unit area of the shell middle-surface, and a superposed dot is used to denote partial derivative with respect to the time \( t \)) into the time-dependent von Kármán equations

\[
D_{\alpha \beta \mu \nu} \partial^\omega \omega_{\alpha \beta \mu \nu} - e^{\alpha \mu e^{\beta \nu} w_{\alpha \beta \mu \nu} + \rho \ddot{w} = P,}
\]

\[
(1/Eh) \delta_{\alpha \beta \mu \nu} \Phi_{\alpha \beta \mu \nu} + (1/2)e^{\alpha \mu e^{\beta \nu} w_{\alpha \beta \mu \nu} + e^{\alpha \mu e^{\beta \nu} = K;}
\]

- the time-dependent equations for anisotropic shallow shells

\[
D_{\alpha \beta \mu \nu} w_{\alpha \beta \mu \nu} - e^{\alpha \mu e^{\beta \nu} w_{\alpha \beta \mu \nu} - e^{\alpha \mu e^{\beta \nu} b_{\alpha \beta \mu \nu} + \rho \ddot{w} = p,}
\]

\[
E_{\alpha \beta \mu \nu} \Phi_{\alpha \beta \mu \nu} + (1/2)e^{\alpha \mu e^{\beta \nu} w_{\alpha \beta \mu \nu} + e^{\alpha \mu e^{\beta \nu} = 0,}
\]

(where \( D_{\alpha \beta \mu \nu} \) and \( E_{\alpha \beta \mu \nu} \) denote the material constants) into the following system of von Kármán-type equations

\[
D_{\alpha \beta \mu \nu} w_{\alpha \beta \mu \nu} - e^{\alpha \mu e^{\beta \nu} w_{\alpha \beta \mu \nu} + \rho \ddot{w} = D_{\alpha \beta \mu \nu} f_{\alpha \beta \mu \nu} + p,}
\]

\[
E_{\alpha \beta \mu \nu} \Phi_{\alpha \beta \mu \nu} + (1/2)e^{\alpha \mu e^{\beta \nu} w_{\alpha \beta \mu \nu} = K.
\]

The above list could be continued.

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