SOME NONLINEAR FUNCTIONS OF BERNOULLI AND EULER UMBRÆ

CHRISTOPHE VIGNAT, UNIVERSITÉ D’ORSAY

Abstract. In a recent paper [5], Yi-Ping Yu has given some interesting nonlinear moments of the Bernoulli umbra; the aim of this paper is to show the probabilistic counterpart of these results and to extend them to Bernoulli polynomials.

1. Introduction

In a recent rich contribution, Yi-Ping Yu gives several nonlinear moments of the Bernoulli umbra \( \mathfrak{B} \) defined by its generating function

\[
\exp (z \mathfrak{B}) = \frac{z}{\exp (z) - 1}, \quad |z| < 2\pi.
\]

This umbra is related to the Bernoulli numbers as

\[ \mathfrak{B}^n = B_n; \]

for example

\[ B_0 = 1; \quad B_1 = -\frac{1}{2}; \quad B_2 = \frac{1}{6}; \quad B_3 = 0; \quad B_4 = -\frac{1}{30}, \]

and all odd orders Bernoulli numbers except \( B_1 \) equal 0.

Similarly, the Euler umbra \( \mathfrak{E} \) is defined by the generating function

\[
\exp (z \mathfrak{E}) = \operatorname{sech} (z)
\]

We generalize here these umbræ and define the Bernoulli umbra \( \mathfrak{B} (x) \) as

\[
\exp (z \mathfrak{B} (x)) = \frac{ze^{zx}}{e^z - 1}
\]

and the Euler umbra \( \mathfrak{E} (x) \) as

\[
\exp (z \mathfrak{E} (x)) = \frac{2e^{zx}}{e^z + 1}.
\]

As a result,

\[
\mathfrak{B}^n (x) = B_n (x)
\]

and

\[
\mathfrak{E}^n (x) = E_n (x),
\]

respectively the Bernoulli and Euler polynomials of degree \( n \).

The aim of this paper is to compute some nonlinear functions of these umbræ as probabilistic nonlinear moments. In the following, we denote the expectation operator

\[
E h (X) = \int h (x) f_X (x) \, dx
\]

where \( f_X \) is the probability density function of the random variable \( X \). We will use the following characterization of the Bernoulli and Euler umbræ.

Theorem 1. The Bernoulli umbra \( \mathfrak{B} (x) \) satisfies, for all admissible function \( h \),

\[
h (\mathfrak{B} (x)) = Eh \left( x - \frac{1}{2} + iL_B \right)
\]

where \( L_B \) is the Lagrange interpolating polynomial of degree \( n \) for the Bernoulli numbers.

where the random variable $L_B$ follows a logistic distribution, with density
\[ f_{L_B}(x) = \frac{\pi}{2} \text{sech}^2(\pi x), \quad x \in \mathbb{R}. \]
Accordingly, the Euler umbra $\mathcal{E}(x)$ satisfies, for all admissible function $h$,
\[ h(\mathcal{E}(x)) = Eh \left(x - \frac{1}{2} + iL_E\right) \]
where the random variable $L_E$ follows the hyperbolic secant distribution
\[ f_{L_E}(x) = \text{sech}(\pi x). \]

Proof. Since
\[ \exp(it \mathfrak{B}(x)) = E \exp\left(it \left(x - \frac{1}{2} + iL_B\right)\right), \]
by identification with (1.1), the random variable $L_B$ has characteristic function
\[ E(e^{itL_B}) = \frac{t}{\sinh\left(\frac{t}{2}\right)}. \]
But from [6, 1.9.2]
\[ \int_0^{+\infty} \text{sech}^2(ax) \cos(xt) \, dx = \frac{\pi t}{2a^2} \text{csch}\left(\frac{\pi t}{2a}\right) \]
so that, with $a = \pi$, the density of $L_B$ is
\[ f_{L_B}(x) = \frac{\pi}{2} \text{sech}^2(\pi x), \]
which is a logistic density.
Accordingly, the characteristic function of the random variable $L_E$ is
\[ Ee^{itL_E} = \text{sech}\left(\frac{t}{2}\right). \]
From [6, 1.9.1],
\[ \int_0^{+\infty} \text{sech}(ax) \cos(xt) \, dx = \frac{\pi}{2a} \text{sech}\left(\frac{\pi t}{2at}\right) \]
so that, with $a = \pi$, the density of $L_E$ is
\[ f_{L_E}(x) = \text{sech}(\pi x). \]
Thus $\pi L_0$ follows an hyperbolic secant distribution. \hfill \Box

As a consequence, the Bernoulli polynomials read
\[ B_n(x) = \mathfrak{B}(x)^n = E \left(x - \frac{1}{2} + iL_B\right)^n \]
and the Bernoulli numbers
\[ B_n = \mathfrak{B}^n = \mathfrak{B}(0)^n = E \left(-\frac{1}{2} + iL_B\right)^n, \quad n \geq 0. \]
Similarly, the Euler polynomials read
\[ E_n(x) = \mathcal{E}(x)^n = E \left(x - \frac{1}{2} + iL_E\right)^n \]
and the Euler numbers
\[ E_n = 2^n \mathcal{E}\left(\frac{1}{2}\right)^n = 2^n E(iL_E)^n. \]

We note from [7, p. 471] that the random variable $L_B$ can also be obtained as
\[ L_B = \frac{1}{2\pi} \log \frac{U}{1-U} = \frac{1}{2\pi} \log \frac{E_1}{E_2}. \]
where $U$ is uniformly distributed on $[-1, +1]$, $E_1$ and $E_2$ are independent with exponential distribution $f_E(x) = \exp(-x)$, $x \in [0, +\infty]$ and equality is in the sense of distributions. As for the random variable $L_E$, from [7], it can be obtained as

$$L_0 = \frac{1}{\pi} \log |C| = \frac{1}{\pi} (\log |N_1| - \log |N_2|)$$

where $C$ is Cauchy distributed and $N_1$ and $N_2$ are two independent standard Gaussian random variables.

2. THE MOMENT $\log \mathfrak{B}(x)$

We compute

$$\log \mathfrak{B}(x) = E \log \left(x - \frac{1}{2} + iL_B\right)$$

which, by symmetry, is equal to

$$\frac{1}{2} E \log \left(\left(x - \frac{1}{2}\right)^2 + L_B^2\right) = \log \left|x - \frac{1}{2}\right| + \frac{1}{2} E \log \left(1 + \frac{L_B^2}{(x - \frac{1}{2})^2}\right)$$

but from [3, 2.6.30.2]

$$\int_0^{+\infty} \frac{\log (1 + bz^2)}{\sinh^2 c z} dz = h(b, c) = \frac{2}{c} \left(\log \frac{c}{\pi \sqrt{b}} - \psi \left(\frac{c}{\pi \sqrt{b}}\right)\right).$$

Thus, by bisection of the angle $2\pi z$,

$$\int_0^{+\infty} \frac{\log (1 + bz^2)}{\sinh^2 2\pi z} dz = \frac{1}{4} \int_0^{+\infty} \frac{\log (1 + bz^2)}{\sinh^2 \pi z \cos^2 \pi z} dz = \frac{1}{4} \int_0^{+\infty} \frac{\log (1 + bz^2)}{\cos^2 \pi z} \left(\frac{\cosh^2 \pi z}{\sinh^2 \pi z} - 1\right) dz$$

so that

$$\frac{\pi}{2} \int_{-\infty}^{+\infty} \frac{\log (1 + bz^2)}{\cosh^2 \pi z} dz = \pi \left(h(b, \pi) - 4h(b, 2\pi)\right).$$

We deduce, with $b = (x - \frac{1}{2})^{-2}$,

$$\frac{1}{2} E \log \left(1 + \frac{L_B^2}{(x - \frac{1}{2})^2}\right) = \frac{\pi}{2} \left(h(b, \pi) - 4h(b, 2\pi)\right) = \frac{\pi}{2} \left(\log \left(\frac{1}{\sqrt{b}}\right) - \psi \left(\frac{1}{\sqrt{b}}\right)\right) - \frac{4}{\pi} \left(\log \left(\frac{2}{\sqrt{b}}\right) - \psi \left(\frac{2}{\sqrt{b}}\right)\right)$$

$$= \log \left(\frac{1}{\sqrt{b}}\right) - 2 \log \left(\frac{2}{\sqrt{b}}\right) - \psi \left(\frac{1}{\sqrt{b}}\right) + 2 \psi \left(\frac{2}{\sqrt{b}}\right)$$

and, using the identity

$$\psi(2z) = \frac{1}{2} \psi(z) + \frac{1}{2} \psi \left(z + \frac{1}{2}\right) + \log 2,$$

we obtain after simplification

$$E \log \left(iL_B + x - \frac{1}{2}\right) = \log \left(\frac{1}{\sqrt{b}}\right) + E \log (1 + bL_B^2) = \psi \left(\frac{1}{2} + |x - \frac{1}{2}|\right).$$

For $x = 1$, we recover the result by Y.-P. Yu, namely

$$E \log \left(\frac{1}{2} + iL_B\right) = \psi(1) = -\gamma.$$
3. The moment \( \log E(x) \)

This moment can be obtained according to the same approach, namely, again with \( b = (x - \frac{1}{2})^{-2} \),

\[
\log E(x) = \log \frac{1}{\sqrt{b}} + \frac{1}{2} E \log \left( 1 + bL_B^2 \right)
\]

where the latter expectation is now computed using [3, 2.6.30] as

\[
\int_0^{+\infty} \frac{\log (1 + bz^2)}{\cosh(\pi z)} dz = 2 \log \frac{\Gamma\left(\frac{3}{4} + \frac{1}{2\sqrt{b}}\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{2\sqrt{b}}\right)} - \log \frac{1}{2\sqrt{b}}
\]

so that

\[
\log E(x) = \log \frac{2}{\Gamma^2\left(\frac{3}{4} + \frac{1}{2}|x - \frac{1}{2}|\right)} \frac{\Gamma\left(\frac{3}{4} + \frac{1}{2}|x - \frac{1}{2}|\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{2}|x - \frac{1}{2}|\right)}
\]

4. The moments \( B^{-k}(x) \) and \( E^{-k}(x) \)

By derivation of the preceding results, we deduce

\[
B^{-1}(x) = E\left(x - \frac{1}{2} + iL_B\right)^{-1} = \frac{d}{dx} \log \mathfrak{B}(x)
\]

so that we have

\[
B^{-1}(x) = \begin{cases} 
\psi'(x), & x > \frac{1}{2} \\
-\psi'(-x + 1), & x < \frac{1}{2} \\
0 & x = \frac{1}{2}
\end{cases}
\]

and we remark that \( B^{-1}(x) \) is not continuous in \( x = \frac{1}{2} \). Since moreover for any integer \( k \geq 1 \)

\[
E\left(x - \frac{1}{2} + iL_B\right)^{-k} = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} E\left(x - \frac{1}{2} + iL_B\right)^{-1}
\]

we deduce

\[
B^{-k}(x) = E\left(x - \frac{1}{2} + iL_B\right)^{-k} = \left\{ \begin{array}{ll}
\frac{(-1)^{k-1}}{(k-1)!} \psi^{(k)}(x), & x > \frac{1}{2} \\
\frac{1}{(k-1)!} \psi^{(k)}(-x + 1), & x < \frac{1}{2}
\end{array} \right.
\]

and in a particular case \( x = 1 \), since \( \psi^{(k)}(1) = (-1)^{k+1} k! \zeta(k + 1) \),

\[
B^{-k}(1) = E\left(\frac{1}{2} + iL_B\right)^{-k} = k! \zeta(k + 1).
\]

In the Euler case, we have

\[
E^{-1}(x) = \frac{d}{dx} \log E(x) = \begin{cases} 
\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right), & x > \frac{1}{2} \\
\psi\left(\frac{1-x}{2}\right) - \psi\left(1 - \frac{x}{2}\right), & x < \frac{1}{2} \\
0 & x = \frac{1}{2}
\end{cases}
\]

and \( E^{-1}(x) \) is not continuous in \( x = \frac{1}{2} \).

More generally, for any integer \( k \geq 1 \),

\[
E^{-k}(x) = \left\{ \begin{array}{ll}
\frac{(-1)^{k-1}}{(k-1)!} \left( \psi^{(k-1)}\left(\frac{x+1}{2}\right) - \psi^{(k-1)}\left(\frac{x}{2}\right) \right), & x > \frac{1}{2} \\
\frac{1}{(k-1)!} \left( \psi^{(k-1)}\left(\frac{1-x}{2}\right) - \psi^{(k-1)}\left(1 - \frac{x}{2}\right) \right), & x < \frac{1}{2}
\end{array} \right.
\]
5. THE MOMENT $\log \sin \frac{\pi B}{2}$

This moment can be easily computed from the moment representation as follows

$$\log \sin \frac{\pi B}{2} = E \log \sin \left( \frac{\pi}{2} + iL_B \right) = E \log \sin \left( \frac{-\pi}{4} - i \frac{\pi L_B}{2} \right) = E \log \sin \left( \frac{-\pi}{4} + i \frac{\pi L_B}{2} \right)$$

and expanding the product of sines we obtain

$$\frac{1}{2} E \log \left( \frac{1}{2} \cos \left( \frac{-\pi}{2} \right) + \frac{1}{2} \cos (i\pi L_B) \right) = -\frac{1}{2} \log 2 + \frac{1}{2} E \log \cosh (\pi L_B).$$

But since

$$\pi L_B = \frac{1}{2} \log \frac{U}{1-U},$$

we deduce

$$\cosh (\pi L_B) = \frac{1}{2} \sqrt{U (1-U)}$$

so that, with $E \log U = -1$, we deduce

$$E \log \cosh (\pi L_B) = -\log 2 + 1$$

and the result

$$\log \sin \frac{\pi B}{2} = \frac{1}{2} - \log 2$$

follows.

6. THE POCHHAMMER ($\mathcal{B}(x)$)$_n$

The Pochhammer symbol

$$(\mathcal{B} + 1)_n = \frac{\Gamma (\mathcal{B} + n + 1)}{\Gamma (\mathcal{B} + 1)}$$

has been evaluated in [1, p.149] as

$$(\mathcal{B} + 1)_n = \frac{n!}{(n+1)}.$$

We use the “intuitive argument” suggested by Carlitz [2] to compute its polynomial version ($\mathcal{B}(x)$)$_n$ as follows: a generating function of ($\mathcal{B}(x)$)$_n$ is

$$\varphi (x, t) = \sum_{n=0}^{+\infty} (\mathcal{B}(x))_n \frac{t^n}{n!} = E \exp \left( - \left( x - \frac{1}{2} + iL_B \right) \log (1-t) \right)$$

$$= (1-t)^{-\left( x-\frac{1}{2} \right)} E \exp (-iL_B \log (1-t))$$

with the characteristic function for the logistic density

$$E \exp (iL_B u) = \frac{\frac{1}{2}}{\sinh \left( \frac{u}{2} \right)}$$

so that

$$\varphi (x, t) = (1-t)^{-\left( x-\frac{1}{2} \right)} \frac{\frac{1}{2} \log (1-t)}{\sinh \left( \frac{\log(1-t)}{2} \right)} = - (1-t)^{-\left( x-1 \right)} \frac{\log (1-t)}{t}$$

This term is identified as the derivative

$$\frac{d}{dx} (1-t)^{-\left( x-1 \right)} = \frac{d}{dx} \sum_{n=0}^{+\infty} \frac{t^n}{n!} (x-1)_n = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \frac{d}{dx} (x-1)_n$$

with

$$\frac{d}{dx} (x-1)_n = (x-1)_n (\psi (x+n-1) - \psi (x-1))$$
so that the coefficient of \( \frac{t^n}{n!} \) in \( \varphi(x, t) \) is
\[
(\mathfrak{B}(x))_n = \frac{(x - 1)_{n+1}}{n+1}(\psi(x+n) - \psi(x-1)).
\]
We recover the result by Nörlund by taking the limit case \( x \to 1 \) which is \( \frac{n!}{n+1} \).

7. **The Pochhammer** \((\mathfrak{E}(x))_n\)

We use the same approach to compute the Pochhammer symbol of the Euler polynomial umbra; the generating function reads
\[
\varphi(x, t) = \sum_{n=0}^{\infty} (\mathfrak{E}(x))_n \frac{t^n}{n!} = E \exp\left(-\left(x - \frac{1}{2} + iL_E\right) \log(1 - t)\right)
\]
\[
= (1 - t)^{-(x - \frac{1}{2})} E \exp(-iL_E \log(1 - t))
\]
with the characteristic function of the hyperbolic secant distribution
\[
E^{iL_E t} = \sech\left(\frac{t}{2}\right)
\]
so that
\[
E \exp(iL_E \log(1 - t)) = \sech\left(\frac{1}{2} \log(1 - t)\right) = \frac{\sqrt{1 - t}}{1 - \frac{t}{2}}
\]
and
\[
\varphi(x, t) = \frac{1}{(1 - t)^{x-1}(1 - \frac{t}{2})} = \sum_{n=0}^{\infty} \frac{t^n n!}{2^n} \sum_{k=0}^{n} \frac{(x - 1)_k n!}{k!} 2^k
\]
so that
\[
(\mathfrak{E}(x))_n = \frac{n!}{2^n} \sum_{k=0}^{n} \frac{(x - 1)_k n!}{k!} 2^k.
\]

**References**

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