Ising models on the Regularized Apollonian Network

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We investigate the critical properties of Ising models on a Regularized Apollonian Network (RAN), here defined as a kind of Apollonian Network (AN) in which the connectivity asymmetry associated to its corners is removed. Different choices for the coupling constants between nearest neighbors are considered, and two different order parameters are used to detect the critical behaviour. While ordinary ferromagnetic and anti-ferromagnetic models on RAN do not undergo a phase transition, some anti-ferrimagnetic models show an interesting infinite order transition. All results are obtained by an exact analytical approach based on iterative partial tracing of the Boltzmann factor as intermediate steps for the calculation of the partition function and the order parameters.

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Many real world networks exhibit complex topological properties as the small word effect, related to a very short minimal path between nodes, and the scale-free property, related to the power-law nature of the connectivity distribution. These properties have important implications in the real phenomena as virus spreading in computers, sharing of technological information and diffusion of epidemic diseases, to name just a few.

In this context, the Apollonian network\textsuperscript{11} is a particularly useful theoretical tool, since it is scale-free, displays small-world effect, can be embedded in a Euclidean lattice and shows space-filling as well as matching graph properties. Therefore, in spite of its deterministic nature, it shares the most relevant characteristics of real world networks.

Phase transitions has been detected for a number of different physical models on Apollonian Networks. For example, the ideal gas undergoes to Bose-Einstein condensation\textsuperscript{2–6} and epidemics exhibits a transition between an absorbing state and an active state\textsuperscript{7–8}. In particular, in\textsuperscript{6} it has been adopted an analytical strategy which has some similarities with that in this paper.

In this work we focus on the infinite order transition exhibited by some Ising models on the Apollonian network. Previous studies of similar Ising models\textsuperscript{9,10} and Potts model\textsuperscript{11} have not detected critical properties, due to the fact that infinite order transitions are elusive. On the contrary, a second order phase transition, as a function of the noise parameter, has been detected for a majority vote model\textsuperscript{12}.

Ising models on different hierarchical fractal have been studied\textsuperscript{13} and in the case of diamond fractal they have been exactly solved by exact renormalization\textsuperscript{14,15}. These models, differently to present model, show a second order phase transition. Indeed, the standard Ising behavior is second order transition in plane models\textsuperscript{16,17}, nevertheless, it may be very intricate, with many phases\textsuperscript{18}, when the interactions are more complicated.

We start by regularizing the standard Apollonian Network in order to remove the connectivity asymmetry associated to its corners which consistently simplify the analytical computation of the thermodynamics of the Ising models.

The Regularized Apollonian Network (RAN) is defined starting form a $g = 0$ generation network with 4 nodes all connected, forming a tetrahedral structure with 6 bonds. Each of the four triples of nodes individuates a different triangle. At generation $g = 1$ a new node is added inside each of the four triangles and it is connected with the surrounding three nodes, creating 12 new triangles. Then the procedure is iterated at any successive generation inserting new nodes in the last created triangles, and connecting each of them with the three surrounding nodes.

In RAN the connectivity of any of the already existing nodes (so-called old nodes) is doubled when generation is updated, while the connectivity of the newly created nodes (the new nodes) always equals 3, leading to the following relevant property: the connectivity at generation $g$ of a node only depends on its age. More explicitly, its connectivity is $3 \times 2^{g-g'}$ where $g'$ is the generation in which it was created. Besides, RAN has the following properties: (i) the total number of nodes is $N_g = (4 \times 3^g + 4)/2$; (ii) the number of new nodes created at generation $g \geq 1$ is $4 \times 3^{g-1} \approx (2/3) N_g$ (equality for large $g$); (iii) the average connectivity is $C_g = 2U_g/N_g \approx 6$ (for large $g$); (iv) the total number of bonds is $U_g = 2 \times 3^{g+1}$; (v) the number of new bonds created at generation $g \geq 1$ is $4 \times 3^g = (2/3)U_g$.

The number of nodes having coordination $k$ is $m(k,g)$ which equals $4 \times 3^{g-g'-1}$ if $k = 3 \times 2^{g'}$ with $g' = 0, ..., g - 1$; equals 4 if $k = 3 \times 2^{g}$; and equals 0 otherwise. Accordingly, the cumulative distribution $P(k) = \sum_{k' \geq k} m(k,g)/N_g$ exhibits, for large values of $g$, a power-law behavior i.e., $P(k) \propto 1/k^{\eta}$, with $\eta = \ln(3)/\ln(2) \approx 1.585$.

Analogously to AN, RAN is scale-free and displays the
small word effect. Furthermore, since RAN network can be decomposed in four AN networks cutting a finite number of couplings, the thermodynamics on the two models is the same.

Ising models are defined according to the following Hamiltonian:

$$H_g = -\sum_{i,j} J_{ij} \sigma_i \sigma_j - \sum_i h_i \sigma_i - q \sum_{i,j,k} \sigma_i \sigma_j \sigma_k,$$  \hspace{1cm} (1)

where the first sum goes on all $U_g/2$ connected pairs of nodes of RAN of generation $g$, the second sum goes on all $N_g$ nodes, and the third sum only goes on the $4 \times 3^g$ triangles of the last generation $g$ (one of the nodes $i,j$ or $k$ must be lastly generated). The constants $J_{ij}$ and $h_i$ may depend on the connectivities (on the age) of the involved nodes. The couplings $J_{ij}$ can be both positive (ferromagnetic) or negative (anti-ferromagnetic). The constant $q$ is introduced only for technical reasons, the relevant physics corresponding to $q=0$.

The partition function is

$$Z_g = \sum_\# \exp(-\beta H_g),$$  \hspace{1cm} (2)

where the sum goes on all $2^{N_g}$ configurations and $\beta$ is the inverse temperature, i.e., $\beta = 1/T$ (we consider an unitary Boltzmann’s constant $k_B$). Then, the thermodynamical variables can be obtained from

$$\Phi = \lim_{g\to\infty} \left(1/N_g\right) \log(Z_g).$$  \hspace{1cm} (3)

Our strategy consists in performing a partial sum in $\Phi$ with respect to the $4 \times 3^{g-1}$ spin variables over nodes created at the last generation $g$. This sum creates new effective interaction between all remaining spins and new magnetizations, and a new value for the parameter $q$. In other words, we exactly map the $g$ generation model in the same $g-1$ model with new parameters. This technique works for any possible choice of the parameters $J_{ij}$, $h_i$ and $q$, but we will consider here only some simple cases.

We stress that our approach is different when compared to the transfer matrix technique \cite{9,10} and it gives exact expressions for the thermodynamical variables in the $g \to \infty$ limit. While it confirms the absence of transition in the ordinary ferromagnetic and anti-ferromagnetic models, it detects an infinite order phase transition in a simple anti-ferromagnetic model occurring at a finite temperature, in contrast with what was found in \cite{9,10}, where no critical behavior at a finite temperature was identified for this kind of models.

In order to illustrate our strategy we start with the simplest case in which all interactions $J_{ij}$ are equal ($J_{ij} = J$), $q = 0$ and all $h_i = 0$. Without loss of generality, one can chose $J = 1$ (ferromagnetism) or $J = -1$ (anti-ferromagnetism).

Since a new node is only linked to three older surrounding nodes (it is created inside a triangle), the summation on the spin over a new node creates an extra-interaction among the three surrounding spins on the older nodes. Therefore, the partial sum on spins over new nodes in \cite{2} yields, after some lengthy but straightforward calculations, the following equality

$$\log[Z_g(\beta,J)] = \log[Z_{g-1}(\beta,J_1)] + 4 \times 3^{g-1} A(\beta),$$  \hspace{1cm} (4)

where

$$J_1 = J + (1/2\beta) \log[2 \cosh(2\beta) - 1].$$  \hspace{1cm} (5)

Also,

$$A(\beta) = (1/4) \log[2 \cosh(2\beta) - 1] + \log[2 \cosh(\beta)].$$  \hspace{1cm} (6)

In (4), $Z_{g-1}(\beta,J_1)$ is the partition function of the same model at generation $(g-1)$ with a different value $J_1$ ($J_1$ has the same sign of $J$). Note that in RAN (contrary to AN equality \cite{4}) is exact.

Performing the thermodynamical limit $g \to \infty$, one obtains from \cite{4}

$$\Phi(\beta,J) = (1/3) \Phi(\beta,J_1) + (2/3) A(\beta),$$  \hspace{1cm} (7)

where we have used $N_{g-1}/N_g \to 1/3$ and $4 \times 3^{g-1}/N_g \to 2/3$. We have thus re-expressed the thermodynamical function $\Phi(\beta,J)$ in terms of $\Phi(\beta,J_1)$, proving the absence of transition. In fact, since $\Phi(\beta,J)$ only depends on the product $\beta J$, and since the above equation can be iterated, a single non-analytical point would imply an infinite number of non-analytical points.

Iteration of (7) gives

$$\Phi(\beta,J) = \frac{2}{3} \sum_{k=0}^\infty \frac{1}{3^k} A(\beta J_k)$$  \hspace{1cm} (8)

where

$$J_k = J_{k-1} + (1/2\beta) \log[2 \cosh(2\beta J_{k-1}) - 1]$$  \hspace{1cm} (9)

with $J_0 = J$. If $J_0 = J = 1$ (ferromagnetism), the positive $J_k$ increase monotonically and diverge for large $k$. On the contrary, if $J_0 = J = -1$ (anti-ferromagnetism) the negative $J_k$ converge to $0$ for large $k$. In both cases it is easy to verify that the sum (8) converges.

Since we have proven that the both constant coupling cases (the ferromagnet and the anti-ferromagnet one) do not undergo a phase transition, we now extend our scope to consider a different model.

Let us now come back to the Hamiltonian \cite{10} and let us assume that the action of the external magnetic field $h_i$ is proportional to the connectivity of the node, i.e., $h_i = h z_i$ where $z_i$ is the connectivity of node $i$. Accordingly, we can define the following spontaneous magnetization:

$$M = \lim_{g \to \infty} \frac{\sum_i z_i \langle \sigma_i \rangle}{\sum_i z_i} = \frac{1}{6\beta} \left[ \frac{\partial \Phi}{\partial h} \right]_{h=q=0^+},$$  \hspace{1cm} (10)

where...
Specific heat
Entropy

\[ u \text{ of } k \text{ units of the Boltzmann’s constant} \]

\[ \Phi(\beta, h, q, u) = (1/3)\Phi(\beta, J_1, h_1, q_1) + (2/3)A(\beta, h, q) \]

where the sum only goes on the \( 4 \times 3^g \) triangles of last generation \( g \). Note that the factor \( 1/2 \) in the last term comes from the fact that in the limit \( g \to \infty \) one has \( 4 \times 3^g/N_g \to 2 \), accordingly \( |L| \leq 1 \).

Since we have proven that a constant value for the couplings \( J_{ij} \) leads to absence of transition, we will assume now, on the contrary, that they depend on the connectivity of the nodes \( i \) and \( j \) (in turn, the connectivity of a node depends only on its age). The simplest age dependence for an anti-ferrimagnetic model is \( J_{ij} = -u \) with \( u > 1 \) if the connectivity of at least one of the two nodes \( i \) or \( j \) is 3 and \( J_{ij} = -1 \) otherwise. This is the same of assuming that \( J_{ij} = -u \) for bonds involving nodes of last generation and \( J_{ij} = -1 \) otherwise. Then, following the same procedure, we take a partial sum with respect to the spins over last created nodes and we obtain the exact equality

\[ \Phi(\beta, h, q, u) = (1/3)\Phi(\beta, J_1, h_1, q_1) + (2/3)A(\beta, h, q) \]

We stress that, while the initial anti-ferromagnetic model (described by \( \Phi(\beta, h, q, u) \)) had two possible values for the couplings \((-u \text{ and } -1)\), the effective model after partial sum (described by \( \Phi(\beta, J_1, h_1, q_1) \)) has the single value \( J_1 \) for all of them. The new parameters \( J_1, h_1, q_1 \) and \( A \) can be again explicitly computed in terms of \( \beta, h, q, u \).

Assuming \( h = 0 \) and \( q = 0 \) one has that \( h_1 \) and \( q_1 \) are also equal to zero, and we obtain again \[ J_1 = -1 + (1/2\beta)\log[2 \cosh(2\beta u)] - 1, \]

while the remaining \( J_k \) for \( k \geq 2 \) are again obtained by \[ \Phi(\beta, h, q, u) = (1/3)\Phi(\beta, J_1, h_1, q_1) + (2/3)A(\beta, h, q) \]

The resulting entropy and specific heat, as a function of the temperature \( T \) (in units of the Boltzmann’s constant \( k_B \)) are depicted in Fig. 1 and Fig. 2 for various values of \( u \). Note that the case \( u = 1 \) is the regular anti-ferromagnet with non vanishing zero temperature entropy. Interestingly, whenever \( u > 1 \), as a consequence of the larger value of the coupling constants involving new nodes, frustration is removed and the zero temperature entropy drops to 0.

Let us call \( \beta_c = (1/2)\log[2 \cosh(2u\beta_c)] - 1 \) the (non vanishing) value of \( \beta \) for which \( J_1 \) vanishes, which only depends on \( u \), then, there are three cases: (i) \( \beta = \beta_c \): in this case \( J_1 = 0 \) and one immediately obtains \( \Phi = (1/3)\log(2) + (1/6)\log[2 \cosh(2u\beta_c)] + (2/3)\log[2 \cosh(u\beta_c)] \); (ii) \( \beta > \beta_c \): in this case \( J_1 > 0 \), i.e., the initial anti-ferrimagnetic model is mapped into a ferromagnetic model. The \( J_k \) increases monotonically and diverges for large \( k \), (iii) \( \beta < \beta_c \): in this case \( J_1 < 0 \),
i.e., the initial anti-ferromagnetic model is mapped into an anti-ferromagnetic model. The $J_k$ converge monotonically to 0 for large $k$.

Is $\beta_c$ a critical point? The answer to this question is not easy since all thermodynamical functions seem to behave regularly at $\beta_c$ (see Figs. 1 and 2). To address this question we have to compute the spontaneous magnetization $M$ and the coordination $L$.

Given $A(\beta, h, q)$ in (12), it turns out that $(\partial A/\partial h)$ and $\partial A/\partial q$, as well as $(\partial J_1/\partial h)$ and $(\partial J_1/\partial q)$, calculated at $h = q = 0$ vanish. Then, given (10) and (11), we obtain from (12):

$$L = T_{21} M(J_1) + T_{22} L(J_1)$$

where $M(J_1)$ and $L(J_1)$ are the magnetization and coordination of the model with couplings $J_1$. Also,

$$T_{11}(\beta J_0) = \frac{1}{3} \frac{\partial h_1}{\partial h} = \frac{2}{3} + (1/4)[\tanh(3\beta J_0) + \tanh(\beta J_0)],$$

$$T_{12}(\beta J_0) = \frac{1}{9} \frac{\partial h_1}{\partial h} = (1/12)[\tanh(3\beta J_0) - 3 \tanh(\beta J_0)],$$

$$T_{21}(\beta J_0) = \frac{\partial h_1}{\partial q} = (1/4)[3 \tanh(3\beta J_0) - \tanh(\beta J_0)],$$

$$T_{22}(\beta J_0) = \frac{1}{3} \frac{\partial h_1}{\partial q} = T_{11}(\beta J_0) - \frac{2}{3},$$

(14)

where all derivative are calculated at $h = q = 0$.

This relation can be iterated and one obtains

$$m = \prod_{k=0}^{\infty} T(\beta J_k)m(J_\infty),$$

(15)

where $m$ is the vector whose two elements are $M$ and $L$; $T(\beta J_k)$ are the $2 \times 2$ matrices with elements $T_{ij}(\beta J_k)$ and $m(J_\infty)$ corresponds to spontaneous magnetization and coordination of the model with couplings $J_\infty$.

If $\beta < \beta_c$, the sequence of $J_k$ converges to 0, and $\prod_{k=0}^{\infty} T(\beta J_k)$ is a vanishing matrix; therefore, $m = 0$ independently on $m(J_\infty)$. If $\beta > \beta_c$, on the contrary, one find that $J_\infty = \infty$, with both component of $m(\beta J_\infty)$ equal to 1 (ferromagnet with infinitely large couplings). Furthermore, in this case, $\prod_{k=0}^{\infty} T(\beta J_k)$ does not vanish, and therefore $m \neq 0$. We have thus proven the existence of a transition at $\beta_c$.

The magnetization $M$ and the coordination $L$ for the $u = 5$ model are depicted in the first panel of Fig. 3. Apparently, the transition occurs at a temperature around 26, but this is a wrong perception. The second panel shows, in fact, that the critical temperature is about $T_c = 49.16$ which is the correct transition temperature which we found analytically.

FIG. 3: (color on line) In the first panel are depicted the spontaneous magnetization $M$ (full) and the coordination $L$ (dashed) versus the temperature $T$ (in units of the Boltzmann’s constant $k_B$) for the $u = 5$ model. In the second panel are depicted $1/\log(M)$ (full) and $1/\log(L)$ (dashed) versus the temperature $T$ (in units of the Boltzmann constant $k_B$). In the first panel it seems that transition occurs at a temperature around 26, but this is a wrong perception. The second panel shows, in fact, that the critical temperature is about $T_c = 49.16$ which is the correct transition temperature which we found analytically.
Qualitatively identical results come out whenever \( u > 1 \), leading to: (i) \( \beta_c = (1/2) \log[2 \cosh(2u\beta_c) - 1] \) individualizes the transition temperature; (ii) the transition is of infinite order. In the limit \( u \to 1 \) one easily verify that \( T_c = 0 \), which confirms the absence of transition in the ordinary anti-ferromagnetic model.

Our minimal choice, \( J_{i,j} = -u \) for newly created bonds and \( J_{i,j} = -1 \) otherwise, is the simplest but it is not the only one which leads to paramagnetic/ferromagnetic infinite order phase transition. Indeed, there are many possible hierarchical choices for the \( J_{i,j} \) which lead to the same qualitative behavior.

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