On Gauss third-order Jacobsthal numbers and their applications

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Abstract. We define the Gauss third-order Jacobsthal numbers. Then we give a formula for the Gauss third-order Jacobsthal numbers by using the third-order Jacobsthal numbers. The Gauss modified third-order Jacobsthal numbers are described and the relation with modified third-order Jacobsthal numbers are explained. We show that there is a relation between the Gauss third-order Jacobsthal numbers and the third-order Jacobsthal numbers. Their Binet’s formulas are obtained. We also define the matrices of the Gauss third-order Jacobsthal numbers and the Gauss modified third-order Jacobsthal numbers. We examine properties of the matrices.

Keywords. Gauss third-order Jacobsthal numbers · modified third-order Jacobsthal numbers · third-order Jacobsthal numbers

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1 Introduction

The investigation of Gaussian numbers is a research topic of great interest. The set of these numbers is denoted by $\mathbb{Z}[i]$. Gaussian numbers were investigated by Gauss in 1832. In 1963, Horadam [12], introduced the concept of Gaussian Fibonacci numbers. And then Jordan [13] considered the two different sequences of Gaussian numbers and extended some relationships which are known about the common Fibonacci sequences.

In 2013, Asci and Gurel [1], introduced the concept of Gaussian Jacobsthal numbers. They also studied the complex Jacobsthal polynomials [2]. Polynomials that can be defined by Jacobsthal-like recursion relations are called Jacobsthal Polynomials and they were studied in 1997 by Horadam and in 2000 by Djordjević (see, [9,11]). More mathematicians were involved
in the study of Jacobsthal polynomials such as Djordjević and Srivastava [10], among others. Cook and Bacon [8] presented a natural manner of extension of the Jacobsthal numbers into the third-order Jacobsthal numbers. In addition to this, Cerda-Morales obtained some interesting identities for the classical third-order Jacobsthal numbers. And, he gave a closed form for modified third-order numbers. In 2017, Cerda-Morales gave the extension of Jacobsthal numbers into the quaternion algebra [3]. And, the author generalized the methods are given by Cook and Bacon. They showed that these formulas are similar to the Binet formulas for the classical Jacobsthal numbers.

For example, the third-order Jacobsthal numbers, the modified third-order numbers, the third-order Jacobsthal quaternions, the dual third-order Jacobsthal quaternions and their properties have been studied in [3–6].

2 Gauss third-order Jacobsthal numbers, Gauss modified third-order Jacobsthal numbers and their some properties

2.1 Gauss third-order Jacobsthal numbers

Now, we introduce Gauss third-order Jacobsthal numbers $J_{\text{G}}^{(3)}_n$ and present some of their basic properties.

**Definition 2.1** Gauss third-order Jacobsthal numbers $J_{\text{G}}^{(3)}_n$ are defined by

$$J_{\text{G}}^{(3)}_n = J_{\text{G}}^{(3)}_{n+2} + J_{\text{G}}^{(3)}_{n+1} + 2J_{\text{G}}^{(3)}_n, \quad n \geq 1,$$

with $J_{\text{G}}^{(3)}_1 = 1$, $J_{\text{G}}^{(3)}_2 = 1 + i$ and $J_{\text{G}}^{(3)}_3 = 2 + i$.

The first few terms are as follows:

$$J_{\text{G}}^{(3)}_4 = J_{\text{G}}^{(3)}_3 + J_{\text{G}}^{(3)}_2 + 2J_{\text{G}}^{(3)}_1 = 2 + i + 1 + i + 2(1) = 5 + 2i,$$

$$J_{\text{G}}^{(3)}_5 = J_{\text{G}}^{(3)}_4 + J_{\text{G}}^{(3)}_3 + 2J_{\text{G}}^{(3)}_2 = 5 + 2i + 2 + i + 2(1 + i) = 9 + 5i,$$

$$J_{\text{G}}^{(3)}_6 = J_{\text{G}}^{(3)}_5 + J_{\text{G}}^{(3)}_4 + 2J_{\text{G}}^{(3)}_3 = 9 + 5i + 5 + 2i + 2(1 + i) = 18 + 9i.$$

**Theorem 2.2** For $n \geq 4$, we have

$$J_{\text{G}}^{(3)}_n = J_n^{(3)} + iJ_{n-1}^{(3)},$$

where $J_n^{(3)}$ is the $n$-th third-order Jacobsthal number.
Proof. We can prove the theorem by the induction method on $n$. For $n = 4$, we have
$$JG_4^{(3)} = JG_3^{(3)} + JG_2^{(3)} + 2JG_1^{(3)} = 5 + 2i = J_4^{(3)} + iJ_3^{(3)}.$$ 
Now, assume that the theorem holds for $n \leq k$, that is
$$JG_k^{(3)} = J_k^{(3)} + iJ_{k-1}^{(3)}.$$ 
Then, for $n = k + 1$, we have
$$JG_{k+1}^{(3)} = JG_k^{(3)} + JG_{k-1}^{(3)} + 2JG_{k-2}^{(3)}$$
$$= J_k^{(3)} + iJ_{k-1}^{(3)} + J_{k-1}^{(3)} + iJ_{k-2}^{(3)} + 2 \left( J_{k-2}^{(3)} + iJ_{k-3}^{(3)} \right)$$
$$= J_k^{(3)} + J_{k-1}^{(3)} + 2J_{k-2}^{(3)} + i \left( J_{k-1}^{(3)} + J_{k-2}^{(3)} + 2J_{k-3}^{(3)} \right)$$
$$= J_{k+1}^{(3)} + iJ_k^{(3)}.$$ 
Thus, the result follows.

We defined Binet’s formulas for the Gauss third-order Jacobsthal numbers. Let $\omega_1$ and $\omega_2$ be the solutions of the quadratic equation $x^2 + x + 1 = 0$:
$$\omega_1 = \frac{-1 + i\sqrt{3}}{2}, \quad \omega_2 = \frac{-1 - i\sqrt{3}}{2}.$$ 
So, we obtain
$$JG_n^{(3)} = \frac{2^n}{7} (2 + i) - \frac{\omega_1^n (\omega_1 + i)}{(2 - \omega_1)(\omega_1 - \omega_2)} + \frac{\omega_2^n (\omega_2 + i)}{(2 - \omega_2)(\omega_1 - \omega_2)}.$$ 
For simplicity of notation, let
$$Z_n = \frac{1}{\omega_1 - \omega_2} \left[ (2 - \omega_2)(\omega_1 + i)\omega_1^{n+1} - (2 - \omega_1)(\omega_2 + i)\omega_2^{n+1} \right]$$
$$= \left\{ \begin{array}{ll}
2 + i & \text{if } n \equiv 0 \pmod{3} \\
-3 + 2i & \text{if } n \equiv 1 \pmod{3} \\
1 - 3i & \text{if } n \equiv 2 \pmod{3}
\end{array} \right. \quad (2.2)$$ 
Then, we can write
$$JG_n^{(3)} = \frac{1}{7} (2^n (2 + i) - Z_n), \quad (2.3)$$ 
where $Z_n + Z_{n+1} + Z_{n+2} = 0$, $Z_0 = 2 + i$ and $Z_1 = -3 + 2i$. Furthermore, we easily obtain the identities stated in the following result:
Theorem 2.3 For $n \geq 1$, we have
\[
\left( J_{G_n^{(3)}} \right)^2 + \left( J_{G_{n+1}^{(3)}} \right)^2 + \left( J_{G_{n+2}^{(3)}} \right)^2 = \frac{1}{7} \left( 3 \cdot 2^{n}(3 + 4i) - 2^{n+1}(2 + i)X_n - 2i \right),
\]
where $(X_n)_{n \geq 0}$ is the sequence defined by $X_{n+2} = -X_{n+1} - X_n$, $X_0 = -i$ and $X_1 = 1$.

Proof. Using Eq. (2.3), we have
\[
\begin{align*}
7 \left[ J_{G_n^{(3)}} \right] & = 2^n (2 + i) - Z_n + 2^{n+1} (2 + i) - Z_{n+1} + 2^{n+2} (2 + i) - Z_{n+2} \\
& = 7 \cdot 2^n (2 + i) - (Z_n + Z_{n+1} + Z_{n+2}) \\
& = 7 \cdot 2^n (2 + i). 
\end{align*}
\]
Then, $J_{G_n^{(3)}} + J_{G_{n+1}^{(3)}} + J_{G_{n+2}^{(3)}} = 2^n (2 + i)$.

Remark 2.1 For $n \geq 1$, we have
\[
7 \left[ J_{G_n^{(3)}} \right] = 2^n (2 + i) - Z_n + 2^{n+1} (2 + i) - Z_{n+1} + 2^{n+2} (2 + i) - Z_{n+2} \\
= 7 \cdot 2^n (2 + i) - (Z_n + Z_{n+1} + Z_{n+2}) \\
= 7 \cdot 2^n (2 + i).
\]

Definition 2.4 The Gauss third-order Jacobsthal number matrix $J_{g_n^{(3)}}$ is defined by
\[
J_{g_n^{(3)}} = \begin{bmatrix}
J_{G_{n+1}^{(3)}} & J_{G_{n}^{(3)}} + 2J_{G_{n-1}^{(3)}} & 2J_{G_{n-2}^{(3)}} \\
J_{G_{n}^{(3)}} & J_{G_{n-1}^{(3)}} + 2J_{G_{n-2}^{(3)}} & 2J_{G_{n-3}^{(3)}} \\
J_{G_{n-1}^{(3)}} & J_{G_{n-2}^{(3)}} + 2J_{G_{n-3}^{(3)}} & 2J_{G_{n-4}^{(3)}}
\end{bmatrix}, \quad n \geq 4.
\]
where $J_{g}^{(3)} = \begin{bmatrix} 5 + 2i & 4 + 3i & 4 + 2i \\ 2 + i & 3 + i & 2 + 2i \\ 1 + i & 1 & 2 \end{bmatrix}$ for convenience.

Note that the matrix $J_{g}^{(3)}$ has order $3 \times 3$ and its coefficients are gaussian integers. Now, we study important properties of this matrix, whose coefficients are Gauss third-order Jacobsthal numbers.

**Proposition 2.5** For $n \geq 3$, we have

\[
\det\left( J_{g}^{(3)} \right) = 2^{n-1}(1 - 2i).
\]

**Proof.** First, note that

\[
J_{g}^{(3)} = \begin{bmatrix} J_{n+1}^{(3)} & J_{n}^{(3)} & 2J_{n-1}^{(3)} & 2J_{n-2}^{(3)} \\ J_{n}^{(3)} & J_{n-1}^{(3)} & 2J_{n-2}^{(3)} & 2J_{n-3}^{(3)} \\ J_{n-1}^{(3)} & J_{n-2}^{(3)} & 2J_{n-3}^{(3)} & 2J_{n-4}^{(3)} \\ J_{n-2}^{(3)} & J_{n-3}^{(3)} & 2J_{n-4}^{(3)} & 2J_{n-5}^{(3)} \end{bmatrix} + i \begin{bmatrix} J_{n}^{(3)} & J_{n-1}^{(3)} & 2J_{n-2}^{(3)} & 2J_{n-3}^{(3)} \\ J_{n-1}^{(3)} & J_{n-2}^{(3)} & 2J_{n-3}^{(3)} & 2J_{n-4}^{(3)} \\ J_{n-2}^{(3)} & J_{n-3}^{(3)} & 2J_{n-4}^{(3)} & 2J_{n-5}^{(3)} \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 + i & 1 & 2 \\ 1 & i & 0 \\ 0 & 1 & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1}.
\]

So we use multiplicative property of the determinant and we get

\[
\det\left( J_{g}^{(3)} \right) = \begin{vmatrix} 1 + i & 1 & 2 \\ 1 & i & 0 \\ 0 & 1 & i \end{vmatrix}^{n-1} = 2^{n-1}(1 - 2i).
\]

Thus, the result follows.

\[\square\]

Using Proposition 2.5 and properties of the determinant we can deduce that.

**Proposition 2.6** For $n, m \geq 3$, we have

\[
\det\left( J_{g}^{(3)} \right) \det\left( J_{g}^{(3)} \right) + \det\left( J_{g}^{(3)} \right) \det\left( J_{g}^{(3)} \right) = -5 \cdot 2^{m+n-2}(3 + 4i).
\]
Proof. From definition of determinant, we have
\[
\det \left( Jg^{(3)}_n \right) \det \left( Jg^{(3)}_m \right) + \det \left( Jg^{(3)}_{n+1} \right) \det \left( Jg^{(3)}_{m+1} \right) \\
= 2^n(1-2i)2^n(1-2i) + 2^{m-1}(1-2i)2^{n-1}(1-2i) \\
= 5 \cdot 2^{m+m-2}(1-2i)^2.
\]
Then, the result is obtained.

Using \( n = m \) in Proposition 2.6, we get

Corollary 2.7 For \( n \geq 3 \), we have
\[
\det \left( \left( Jg^{(3)}_n \right)^2 \right) + \det \left( \left( Jg^{(3)}_{n+1} \right)^2 \right) = -5 \cdot 2^{2(n-1)}(3 + 4i).
\]

Proposition 2.8 For \( n, m \geq 3 \), we have
\[
\left( \frac{1}{2} - i \right) \det \left( Jg^{(3)}_{n+m} \right) = \det \left( Jg^{(3)}_n \right) \det \left( Jg^{(3)}_m \right).
\]

Proof. From definition of determinant, we have
\[
\det \left( Jg^{(3)}_n \right) \det \left( Jg^{(3)}_m \right) = (2^{n-1}(1-2i)) \left( 2^{m-1}(1-2i) \right) \\
= 2^{n+m-2}(1-2i)(1-2i) \\
= \frac{1}{2}(1-2i) \cdot 2^{n+m-1}(1-2i) \\
= \frac{1}{2}(1-2i) \det \left( Jg^{(3)}_{n+m} \right).
\]

Proposition 2.9 For \( n \geq 3 \), we have
\[
\det \left( \left( Jg^{(3)}_{n+1} \right)^n \right) - \det \left( \left( Jg^{(3)}_n \right)^n \right) = 2^{n^2-n}(1-2i)^n(2^n + 1).
\]

Proof.
\[
\det \left( \left( Jg^{(3)}_{n+1} \right)^n \right) - \det \left( \left( Jg^{(3)}_n \right)^n \right) = (2^n(1-2i))^n + (2^{n-1}(1-2i))^n \\
= 2^n(1-2i)^n + 2^{n(n-1)}(1-2i)^n \\
= 2^{n^2-n}(1-2i)^n(2^n + 1).
\]

Then, the result follows.
2.2 Gauss modified third-order Jacobsthal numbers

In this subsection, we introduce Gauss modified third-order Jacobsthal numbers \( \{KG_n^{(3)}\}_{n \geq 1} \) and present some of their basic properties.

**Definition 2.10** Gauss modified third-order Jacobsthal numbers \( KG_n^{(3)} \) are defined by

\[
KG_{n+3}^{(3)} = KG_{n+2}^{(3)} + KG_{n+1}^{(3)} + 2KG_n^{(3)}, \quad n \geq 1, \tag{2.4}
\]

with \( KG_1^{(3)} = 1 + 3i, \ KG_2^{(3)} = 3 + i \) and \( KG_3^{(3)} = 10 + 3i \).

The first few terms are as follows:

\[
\begin{align*}
KG_4^{(3)} &= KG_3^{(3)} + KG_2^{(3)} + 2KG_1^{(3)} \\
&= 10 + 3i + 3 + i + 2(1 + 3i) = 15 + 10i,
KG_5^{(3)} &= KG_4^{(3)} + KG_3^{(3)} + 2KG_2^{(3)} \\
&= 15 + 10i + 10 + 3i + 2(3 + i) = 31 + 15i,
KG_6^{(3)} &= KG_5^{(3)} + KG_4^{(3)} + 2KG_3^{(3)} \\
&= 31 + 15i + 15 + 10i + 2(10 + 3i) = 66 + 31i.
\end{align*}
\]

**Theorem 2.11** For \( n \geq 4 \), we have

\[
KG_n^{(3)} = K_n^{(3)} + iK_{n-1}^{(3)},
\]

where \( K_n^{(3)} \) is the \( n \)-th modified third-order Jacobsthal number.

**Proof.** We can prove the theorem by the induction method on \( n \). For \( n = 4 \), we have

\[
KG_4^{(3)} = K_4^{(3)} + iK_3^{(3)}.
\]

Now, assume that the theorem holds for \( n \leq k \), that is

\[
KG_k^{(3)} = K_k^{(3)} + iK_{k-1}^{(3)}.
\]

Then, for \( n = k + 1 \), we have

\[
KG_{k+1}^{(3)} = KG_k^{(3)} + KG_{k-1}^{(3)} + 2KG_{k-2}^{(3)} \\
= K_k^{(3)} + iK_{k-1}^{(3)} + K_{k-1}^{(3)} + iK_{k-2}^{(3)} + 2\left(K_{k-2}^{(3)} + iK_{k-3}^{(3)}\right) \\
= K_k^{(3)} + K_{k-1}^{(3)} + 2K_{k-2}^{(3)} + i\left(K_{k-1}^{(3)} + K_{k-2}^{(3)} + 2K_{k-3}^{(3)}\right) \\
= K_{k+1}^{(3)} + iK_k^{(3)}.
\]

Thus, the result follows. \( \square \)
We find Binet’s formulas for the Gauss modified third-order Jacobsthal numbers. Let \( \omega_1 \) and \( \omega_2 \) be the solutions of the quadratic equation \( x^2 + x + 1 = 0 \):

\[
\omega_1 = -1 + i\sqrt{3} \quad \text{and} \quad \omega_2 = -1 - i\sqrt{3}.
\]

So, we obtain

\[
KG^{(3)}_n = 2^{n-1}(2 + i) + \omega_1^{n-1}(\omega_1 + i) + \omega_2^{n-1}(\omega_2 + i).
\]

**Theorem 2.12** For \( n \geq 3 \), we have

\[
KG^{(3)}_n = JG^{(3)}_n + 2JG^{(3)}_{n-1} + 6JG^{(3)}_{n-2}.
\]

**Proof.** Using the relation

\[
KG^{(3)}_n = JG^{(3)}_n + 2JG^{(3)}_{n-1} + 6JG^{(3)}_{n-2},
\]

for the \( n \)-th modified third-order Jacobsthal proved in [7]. Then, we have

\[
KG^{(3)}_n = K^{(3)}_n + iK^{(3)}_{n-1}
\]

\[
= J^{(3)}_n + 2J^{(3)}_{n-1} + 6J^{(3)}_{n-2} + i \left( J^{(3)}_{n-1} + 2J^{(3)}_{n-2} + 6J^{(3)}_{n-3} \right)
\]

\[
= J^{(3)}_n + iJ^{(3)}_{n-1} + 2 \left( J^{(3)}_{n-1} + iJ^{(3)}_{n-2} \right) + 6 \left( J^{(3)}_{n-2} + iJ^{(3)}_{n-3} \right)
\]

\[
= JG^{(3)}_n + 2JG^{(3)}_{n-1} + 6JG^{(3)}_{n-2}.
\]

Thus, the result follows. \( \Box \)

**Theorem 2.13** For \( n \geq 3 \), we have

\[
13KG^{(3)}_n + 48KG^{(3)}_{n-1} + 20KG^{(3)}_{n-2} = 147JG^{(3)}_n.
\]

**Proof.** Using the relation

\[
13KG^{(3)}_n + 48KG^{(3)}_{n-1} + 20KG^{(3)}_{n-2} = 147J^{(3)}_n,
\]

for the \( n \)-th modified third-order Jacobsthal proved in [7]. Then, we obtain

\[
13KG^{(3)}_n + 48KG^{(3)}_{n-1} + 20KG^{(3)}_{n-2}
\]

\[
= 13 \left( K^{(3)}_n + iK^{(3)}_{n-1} \right) + 48 \left( K^{(3)}_{n-1} + iK^{(3)}_{n-2} \right) + 20 \left( K^{(3)}_{n-2} + iK^{(3)}_{n-3} \right)
\]

\[
= 13K^{(3)}_n + 48K^{(3)}_{n-1} + 20K^{(3)}_{n-2} + i \left( 13K^{(3)}_{n-1} + 48K^{(3)}_{n-2} + 20K^{(3)}_{n-3} \right)
\]

\[
= 147J^{(3)}_n + 147iJ^{(3)}_{n-1}
\]

\[
= 147 \left( J^{(3)}_n + iJ^{(3)}_{n-1} \right)
\]

\[
= 147JG^{(3)}_n.
\]

Then, the result follows. \( \Box \)
Theorem 2.14 For $n \geq 1$ and $m \geq 2$, we have
\[
J_{n+m}^{(3)} = J_{n+1}^{(3)} J_{m+1}^{(3)} + \left( J_n^{(3)} + 2 J_{n-1}^{(3)} \right) J_m^{(3)} + 2 J_n^{(3)} J_{m-1}^{(3)}.
\]

Proof.
\[
J_{n+1}^{(3)} J_{m+1}^{(3)} + \left( J_n^{(3)} + 2 J_{n-1}^{(3)} \right) J_m^{(3)} + 2 J_n^{(3)} J_{m-1}^{(3)}
\]
\[
= J_{n+1}^{(3)} \left( J_{m+1}^{(3)} + i J_m^{(3)} \right) + \left( J_n^{(3)} + 2 J_{n-1}^{(3)} \right) \left( J_m^{(3)} + i J_{m-1}^{(3)} \right)
\]
\[
+ 2 J_n^{(3)} \left( J_{m-1}^{(3)} + i J_{m-2}^{(3)} \right)
\]
\[
= J_{n+1}^{(3)} J_{m+1}^{(3)} + \left( J_n^{(3)} + 2 J_{n-1}^{(3)} \right) J_m^{(3)} + 2 J_n^{(3)} J_{m-1}^{(3)}
\]
\[
+ i \left( J_{n+1}^{(3)} J_m^{(3)} + \left( J_n^{(3)} + 2 J_{n-1}^{(3)} \right) J_{m-1}^{(3)} + 2 J_n^{(3)} J_{m-2}^{(3)} \right)
\]
\[
= J_{n+m}^{(3)} + i J_{n+m-1}^{(3)}
\]
\[
= J_{n+m}^{(3)}.
\]

Then, the result is obtained.

Definition 2.15 The Gauss modified third-order Jacobsthal number matrix $K_{n}^{(3)}$ is defined by
\[
K_{n}^{(3)} = \begin{bmatrix}
J_{n}^{(3)} & J_{n-1}^{(3)} & J_{n-2}^{(3)} & J_{n-3}^{(3)} \\
J_{n+1}^{(3)} & J_{n}^{(3)} & J_{n-1}^{(3)} & J_{n-2}^{(3)} \\
J_{n+2}^{(3)} & J_{n+1}^{(3)} & J_{n}^{(3)} & J_{n-1}^{(3)} \\
J_{n+3}^{(3)} & J_{n+2}^{(3)} & J_{n+1}^{(3)} & J_{n}^{(3)}
\end{bmatrix}, \ n \geq 4.
\]

where $K_{n}^{(3)} = \begin{bmatrix}
15 + 10i & 16 + 5i & 20 + 6i \\
10 + 3i & 5 + 7i & 6 + 2i \\
3 + i & 7 + 2i & 2 + 6i
\end{bmatrix}$ for convenience.

Proposition 2.16 For $n \geq 3$, we have
\[
\det \left( K_{n}^{(3)} \right) = 147 \cdot 2^{n-2} (1 - 2i).
\]

Proof. Note that the usual modified third-order Jacobsthal number $K_{n}^{(3)}$ satisfies the next relation
\[
\begin{bmatrix}
K_{n+1}^{(3)} & K_{n}^{(3)} & 2 K_{n-1}^{(3)} \\
K_{n+2}^{(3)} & K_{n+1}^{(3)} & 2 K_{n}^{(3)} \\
K_{n+3}^{(3)} & K_{n+2}^{(3)} & 2 K_{n+1}^{(3)}
\end{bmatrix} = \begin{bmatrix}
15 & 16 & 20 \\
10 & 5 & 6 \\
3 & 7 & 2
\end{bmatrix} \begin{bmatrix}
1 & 1 & 2 \end{bmatrix}^{n-3}.
\]
Then, we have

\[
Kg_{n}^{(3)} = \begin{bmatrix}
K_{n+1}^{(3)} & K_{n}^{(3)} + 2K_{n-1}^{(3)} & 2K_{n}^{(3)} \\
K_{n}^{(3)} & K_{n-1}^{(3)} + 2K_{n-2}^{(3)} & 2K_{n-1}^{(3)} \\
K_{n-1}^{(3)} & K_{n-2}^{(3)} + 2K_{n-3}^{(3)} & 2K_{n-2}^{(3)}
\end{bmatrix} + i \begin{bmatrix}
K_{n}^{(3)} & K_{n-1}^{(3)} + 2K_{n-2}^{(3)} & 2K_{n-1}^{(3)} \\
K_{n-1}^{(3)} & K_{n-2}^{(3)} + 2K_{n-3}^{(3)} & 2K_{n-2}^{(3)} \\
K_{n-2}^{(3)} & K_{n-3}^{(3)} + 2K_{n-4}^{(3)} & 2K_{n-3}^{(3)}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
15 & 16 & 20 \\
10 & 5 & 6 \\
3 & 7 & 2
\end{bmatrix} \begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}^{n-3} + i \begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}^{n-4}.
\]

So we use multiplicative property of the determinant and we get

\[
\det(Jg_{n}^{(3)}) = \begin{vmatrix}
15 & 16 & 20 & 1 + i & 1 & 2 \\
10 & 5 & 6 & 1 & i & 0 \\
3 & 7 & 2 & 0 & 1 & i
\end{vmatrix}^{n-4}
\]

\[
= 588 \cdot 2^{n-4}(1-2i)
\]

\[
= 147 \cdot 2^{n-2}(1-2i).
\]

Thus, the result follows.

\[
\Box
\]

**Proposition 2.17** For \(n \geq 3\), we have

\[
2 \cdot \det(Kg_{n}^{(3)}) = 147 \cdot \det(Jg_{n}^{(3)}).
\]

**Proof.** The proof is easily deduced from Propositions 2.5 and 2.16.

\[
\Box
\]

**3 Conclusion**

We defined new numbers by using definitions of Gauss sequence, the third-order Jacobsthal number and modified third-order Jacobsthal number are studied. The properties of those numbers were examined. Some theorems about these numbers were presented, their matrix representations are established and identities with their determinants are demonstrated.
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