Z_c(4430), Z_c(4200), Z_1(4050), and Z_2(4250) as triangle singularities

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Abstract. Z_c(4430) discovered in B^0 \to \psi(2S)K^-\pi^+, Z_c(4200) found in B^0 \to J/\psi K^-\pi^+, and Z_1(4050) and Z_2(4250) observed in B^0 \to \chi_{c1} K^-\pi^+ are candidates of charged charmonium-like states. All surviving theoretical models interpreted these candidates as four-quark states, until we recently identified a compelling alternative. We discuss that kinematical singularities in triangle loop diagrams induce a resonance-like behavior that can consistently explain the properties (such as spin-parity, mass, width, and Argand plot) of Z_c(4430), Z_c(4200), Z_1(4050) and Z_2(4250) from experiments. In terms of the triangle singularities, we can also naturally understand interesting experimental findings such as the appearance (absence) of Z_c(4200)/(Z_c(4430))-like contribution in \Lambda_b^0 \to J/\psi p\pi^+, and the highly asymmetric shape of the spectrum bump for Z_1(4050); the other theoretical models have not successfully addressed these points. Although Pakhlov et al. proposed another triangle diagram to generate a Z_c(4430)-like bump, we argue that this scenario is very unlikely.

Introduction

A current trend of the hadron spectroscopy is to establish the existence and the internal structure of exotic hadrons which are not accommodated by the conventional q\bar{q} and qq\bar{q} structures. Such exotic hadrons could be tetraquark, pentaquark, hadron molecule, or hybrid states. Possible experimental signatures of the exotic states are: (i) the mass does not fit a quark-model prediction; (ii) the state matches a state for which Lattice QCD predicts a high gluon content; (iii) the state has a peculiar decay pattern; and so on. But the listed signatures may seem model-dependent criteria, and one may wonder if there is a more unambiguous signature.

The discoveries of charged quarkonium-like state candidates, Z_c and Z_0, are encouraging. For example, Z_c(4430) was discovered in \psi(2S)\pi^+ invariant mass distribution of B^0 \to \psi(2S)K^-\pi^+ [1, 2, 3], while Z_c(4200) in the J/\psi\pi^+ distribution of B^0 \to J/\psi K^-\pi^+ [4]. Z_1(4050) and Z_2(4250) are also reported in the analysis of B^0 \to \chi_{c1} K^-\pi^+ [5]. If these spectrum bumps are really associated with the existence of resonances, the quark content of these states minimally need four quarks, a clear signature of exotics. Among the charged charmonium-like states, Z_c(4430) has been an outstanding exotic candidate [6], and all the surviving theoretical interpretations of Z_c(4430) considered it to be a genuine four-quark state (including hadron-molecule interpretations) until the present work. Theoretical interpretations of Z_c(4200), Z_1(4050) and Z_2(4250) are similar, although these exotic candidates are reported only by the Belle experiment, and yet to be confirmed by an independent experiment.

We propose a completely different scenario based on the triangle singularity (TS) to interpret these exotic candidates [7, 8]. The TS arise in triangle diagrams as shown in Fig. 1 when the processes are kinematically allowed to occur even at the classical level. The TS is a kinematical effect and its existence and location are completely determined by the particle masses involved in the process in the zero-width limit of unstable particles. Although the unstable particles have finite widths in reality and thus the TS are somewhat relaxed, the TS still significantly enhance the amplitudes. In what follows, we demonstrate that the TS arising from Fig. 1 induce a resonance-like behavior, with which we can consistently understand the experimentally determined properties of Z_c(4430), Z_c(4200), Z_1(4050) and Z_2(4250) such as spin-parity, mass, width, and Argand plot.
by \(Z\) the Breit-Wigner form is used to model the \(Z\) phase-space distributions are also shown by the black dotted curves for comparison. Because of the presence of the triangle diagrams of Figs. 1(a) and 1(b) generate the red solid curves in Figs. 2(a) and 2(b), respectively. The in Fig. 2(b)).

\[\psi\text{-}(4260)\text{-like bump in the }\alpha J/\psi\pi\text{ distribution by the TS from the diagram (a) [(b,c)], while a }Z_1(4050)\text{-like bump in the }\chi_{c1}\pi^+\text{ distribution by the TS from the diagram (d) [(e)]. Figures taken from Refs. [7, 8]. Copyright (2019) APS.\]

\section*{Model}

In our model, the amplitude for a triangle diagram in Fig. 1 is given by

\[ T_{abc,H} = \int dp_1 \frac{\bar{V}_{abc23}(p_1, p_2; p_3)}{E - E_2(p_2) - E_3(p_3) - E_4(p_4)} \Gamma_{3,1}(p_3, p_4; p_1) \frac{1}{E - E_1(p_1) - E_2(p_2)} \Gamma_{12,ab}(p_1, p_2, p_4), \]  

where we have used the particle labels and their momenta defined in Fig. 1(f). The total energy in the center-of-mass (CM) frame is denoted by \(E\), while the energy of a particle \(x\) is \(E_x(p_x) = \sqrt{p_x^2 + m_x^2}\) with the mass \(m_x\) and momentum \(p_x\). For unstable intermediate particles 1 and 2, we use \(E_j(p_j) = \sqrt{p_j^2 + m_j^2} - \nu_j/2\) \((j = 1, 2)\) where \(\nu_j\) is the width. We use the mass and width values from Ref. [9]. In Eq. (1), the decay of an unstable particle \(R\) to lighter particle-pair \(i-j\) is described by a vertex \(\Gamma_{i,j,R}\) and the \(23 \to ab\) rescattering by \(\bar{V}_{abc23}\). We use an s-wave interaction of \(\bar{V}_{abc23}\) for Fig. 1(a-d) to be consistent with the experimentally determined spin-parity of \(Z_1(4430)\) and \(Z_2(4200)\): \(J^P = 1^-\). The spin-parity of \(Z_1(4050)\) has not been experimentally determined, and our model predict it to be \(J^P = 1^-\). For Fig. 1(e) where the intrinsic parity is different between the 23 and ab pairs, we use two types of \(\bar{V}_{abc23}\) from the s-wave pair to the p-wave pair \((J^P = 1^-\text{ for }Z_2(4050))\), and vice versa \((J^P = 1^-)\).

\section*{Results for \(Z_1(4430)\) and \(Z_2(4200)\)}

The \(\psi(2S)\pi [J/\psi\pi]\) invariant mass distribution for \(\bar{B}^0 \to \psi(2S)K^-\pi^+\) \([\bar{B}^0 \to J/\psi K^-\pi^+]\) is shown in Fig. 2(a) [2(b)]. The triangle diagrams of Figs. 1(a) and 1(b) generate the red solid curves in Figs. 2(a) and 2(b), respectively. The phase-space distributions are also shown by the black dotted curves for comparison. Because of the presence of the TS in the triangle diagram, a resonance-like peak clearly shows up at \(m_{\psi(2S)\pi} \approx 4.45\text{ GeV}\) in Fig. 2(a) \((m_{J/\psi}\rho \approx 4.2\text{ GeV}\) in Fig. 2(b)).

We simulate the peaks from the TS in terms of \(Z_c^+\) excitations. We use a model that goes as \(\bar{B}^0 \to Z_c^+ K^-\) followed by \(Z_c^+ \to J/\psi\pi^+\) to fit the Dalitz plot distributions generated by the triangle diagrams of Figs. 1(a) and 1(b). The Breit-Wigner form is used to model the \(Z_c\) propagation. The kinematical region included in the fit have the Dalitz plot distribution larger than 10% of the peak height. As shown by the blue dash-dotted curves in Figs. 2(a) and 2(b), the Breit-Wigner form can fit the peaks very well. The resulting Breit-Wigner parameters are shown in Table 1 along

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Triangle diagrams contributing to \(\bar{B}^0 \to \psi J/\psi\pi\) (a,b), \(A_0^\pm \to J/\psi\pi\) (c), and \(\bar{B}^0 \to \chi_{c1}\pi^+\) (d,e); \(\psi = J/\psi, \psi(2S)\). Particle labels and their momenta used in Eq. (1) are defined in (f). An isospin 1/2 nucleon resonances of 1400–1800 MeV is denoted by \(N^+\) in (c). A \(Z_1(4430)\) \([Z_2(4200)\)-like bump in the \(\psi\pi\) mass distribution is generated by TS from the diagram (a) [(b,c)], while a \(Z_1(4050)\) \([Z_2(4250)\)-like bump in the \(\chi_{c1}\pi^+\) distribution by the TS from the diagram (d) [(e)]. Figures taken from Refs. [7, 8]. Copyright (2019) APS.}
\end{figure}
with the experimentally determined ones. The comparison shows a remarkable agreement.

Now we show the triangle amplitude for $Z_c(4430)$ in the form of the Argand plot so that we can compare it with the counterpart from the LHCb [3]. The angle-independent part of the amplitude ($A$) supplemented with a complex constant background ($c_{bg}$) is given by

$$A(m_{ab}^2) = c_{bg} + c_{norm} \frac{1}{\Delta} \int_{m_{ab}^2(i)+\Delta/2}^{m_{ab}^2(i)-\Delta/2} d m_{ab}^2 \ A(m_{ab}^2) \ dm_{ab}^2,$$

(3)

where the $ab$ invariant mass is denoted by $m_{ab}$, and the $z$-component of the $Z_c$ spin by $\hat{z}_{ab}^z$; $Y_{lm}$ is the spherical harmonics. The quantity $M_{abc,H}$ is the invariant amplitude related to $T_{abc,H}$ of Eq. (1) through Eq. (B3) of Ref. [10]. We adjust complex constants $c_{norm}$ and $c_{bg}$ to fit the LHCb's Argand plot. Each point of the LHCb’s Argand plot has been fitted to dataset in the bin covering from $m_{\psi(2S)\pi} - \Delta/2$ to $m_{\psi(2S)\pi} + \Delta/2$. Thus we also average our amplitude as:

$$\tilde{A}(m_{ab}^2(i)) = \frac{1}{\Delta} \int_{m_{ab}^2(i)-\Delta/2}^{m_{ab}^2(i)+\Delta/2} d m_{ab}^2 \ A(m_{ab}^2) \ dm_{ab}^2,$$

(3)

where the central value of an $i$-th bin is $m_{ab}^2(i)$. As seen in Fig. 3, the LHCb’s $Z_c(4430)$ Argand plot is consistently described by the triangle diagram of Fig. 1(a) giving $\tilde{A}(m_{ab}^2(i))$. Thus the counterclockwise behavior of the LHCb’s $Z_c(4430)$ Argand plot [3] is not necessarily pointing to the existence of a resonance.

We now discuss $\Lambda_b^0 \to J/\psi p p^{-}$. According to the LHCb analysis [11], a significantly better description of the $\Lambda_b^0 \to J/\psi p p^{-}$ data is obtained by including the $Z_c(4200)$ amplitude. To this process, the triangle diagram of Fig. 1(c)
can contribute with a TS in the $Z_c(4200)$-region. The triangle diagram includes an isospin $1/2$ nucleon resonance ($N^*$), and several $N^*$ in the mass range of 1400–1800 MeV can be relevant to the TS. We show in Fig. 2(c) the $J/\psi\pi^-$ spectrum generated by the triangle diagram of Fig. 1(c) including $N^* = N(1440)\, 1/2^+,$ $N(1520)\, 3/2^-$, and $N(1680)\, 5/2^+.$ The triangle diagrams with different $N^*$ create different bumps in the region of $Z_c(4200).$ In a realistic situation, a single broad bump from the coherent sum of these bumps may show up. Other charmoniums ($\psi(2S)$, $X(3872)$, etc.) in the mass range of 3650-3900 MeV, which have a coupling to $J/\psi\pi\pi,$ might replace $\psi(3770)$ in Fig. 2(c), and also generate TS bumps in the $Z_c(4200)$-region. Because the $\Lambda_b^0 \to J/\psi p\pi^-$ data is statistically limited, the $Z_c(4200)$ amplitude in the LHCb analysis is assumed to have the same mass and width as the Belle analysis of $\bar{B}^0 \to \psi(2S)K^-\pi^+$ [4]. Thus, although some of the spectrum bumps of Fig. 2(c) seem to be in the lower end of the $Z_c(4200)$-region, they are still consistent with the LHCb’s observation.

The LHCb analysis [11] also found that their description of the $\Lambda_b^0 \to J/\psi p\pi^-$ data is hardly improved by including a $Z_c(4430)$ contribution. This interesting observation can be understood if $Z_c(4430)$ appears in $\bar{B}^0 \to \psi(2S)K^-\pi^+$ due to the TS. This is because, within experimentally observed hadrons, there is no triangle diagram like Fig. 1(c) available to cause a TS at the $Z_c(4430)$ position for the case of $\Lambda_b^0 \to J/\psi p\pi^-.$

**Comment on Pakhlov et al.’s triangle diagram**

Pakhlov et al. claimed that a triangle diagram, which includes an experimentally unobserved hadron, can generate a $Z_c(4430)$-like spectrum bump [12, 13] due to a kinematical effect. We however point out that the proposed mechanism is kinematically forbidden at the classical level. The Coleman-Norton theorem [14] dictates that such a diagram does not include a TS. Appropriately substituting the masses, widths, and vertex forms into our model discussed in the previous section, we find that Pakhlov et al.’s triangle diagram does not generate a $Z_c(4430)$-like bump, as expected from the Coleman-Norton theorem. The authors presented a clockwise Argand plot from the triangle diagram [13]. This result has been ruled out by the counter-clockwise Argand plot from the LHCb [3]. All these points strongly indicate that Pakhlov et al.’s scenario is very unlikely to explain $Z_c(4430).$

**Results for $Z_1(4050)$ and $Z_2(4250)$**

The $\chi_{c1}\pi^+$ invariant mass distributions for $\bar{B}^0 \to \chi_{c1}K^-\pi^+$ are shown in Fig. 4. The triangle diagram of Fig. 1(d) gives the red solid curve in Fig. 4(d). The diagram Fig. 1(e) generates the blue solid curve in Fig. 4(e-1) for the final $\chi_{c1}\pi^+$
FIGURE 4. Distributions of the \( \chi_{c1}\pi^+ \) invariant mass for \( B^0 \rightarrow \chi_{c1}K^-\pi^+ \). The red solid curve in the panel (d) is generated by the triangle diagram of Fig. 1(d). The blue [magenta] solid curve in the panel (e-1) [(e-2)] is from the diagram Fig. 1(e) with the final \( \chi_{c1}\pi^+ \) pair of \( J^P = 1^+ \) \( [J^P = 1^-] \). Breit-Wigner amplitudes, fitted to the solid curves, are given with the green dash-dotted curves. The phase-space distributions are the dotted curves. The normalization of the curves has been set in the same way as Fig. 2. The \( \chi \) triangle diagram of Fig. 1(d). The blue [magenta] solid curve in the panel (e-1) [(e-2)] is from the diagram Fig. 1(e) with the final \( \chi_{c1}\pi^+ \) pair of \( J^P = 1^+ \) \( [J^P = 1^-] \). Breit-Wigner amplitudes, fitted to the solid curves, are given with the green dash-dotted curves. The phase-space distributions are the dotted curves. The normalization of the curves has been set in the same way as Fig. 2. The scale has been doubled for the panels (e-1) and (e-2). Figure taken from Ref. [8]. Copyright (2019) APS.

pair with \( J^P = 1^+ \), while the magenta solid curve in Fig. 4(e-2) is the case with \( J^P = 1^- \). Clear resonance-like peaks are induced by the triangle singularities at \( m_{\chi_{c1} \pi} \sim 4.025 \) GeV in Fig. 4(d) and \( m_{\chi_{c1} \pi} \sim 4.22 \) GeV in Figs. 4(e-1) and 4(e-2). A characteristic feature of the bump in Fig. 4(d) is that it has a significantly asymmetric shape.

We can again simulate the TS-induced bumps with the fake \( Z^- \) excitation mechanisms. The \( B^0 \rightarrow \chi_{c1}K^-\pi^+ \) Dalitz plot distribution from the triangle diagram of Fig. 1(d) [1(e)] is fitted with the mechanism of \( B^0 \rightarrow ZK^- \) followed by \( Z \rightarrow \chi_{c1}\pi^+ \) by adjusting their Breit-Wigner mass and width. We include the kinematical region where the Dalitz plot distribution is larger than 10% of the peak height. The green dash-dotted curves in Fig. 2 are showing the quality of the fits. The Breit-Wigner form cannot fit well the red solid curve with the asymmetric bump in Fig. 4(d). Meanwhile, the bumps in Figs. 4(e-1) and 4(e-2) are reasonably well fitted. The resulting Breit-Wigner parameters are given in Table 1, along with the Belle analysis [5] on \( Z_1(4050) \) and \( Z_2(4250) \). A quite good agreement is seen for \( Z_1(4050) \). Also, our result from the triangle diagram of Fig. 1(e) easily agrees with the \( Z_2(4250) \) mass and width from the Belle analysis because they have rather large errors. The \( J^P = 1^- \) assignment to \( Z_2(4250) \) cannot be eliminated by this comparison alone.

The asymmetry of the bump shape generated by the triangle diagram Fig. 1(d) seems important to reproduce the Belle data in the \( Z_1(4050) \)-region. To make this statement clear, as in Fig. 5(left), we superimpose the spectra from the triangle diagrams of Figs. 1(d) and 1(e) on the Belle data (Fig. 14 of Ref. [5]). Although this is a qualitative comparison where any interferences among different mechanisms are not taken into account, the spectrum bumps from the triangle diagrams fit the data very well. Particularly, the data has a very sharp rise and a moderate fall-off at the \( Z_1(4050) \)-region, and the asymmetric bump shape from the triangle diagram of Fig. 1(d) reproduces it well. The Belle analysis [5] was not able to fit well this sharp peak of the data as seen in Fig. 14 of the reference, probably

| \( Z_1(4050) \)     | \( Z_2(4250) \)    |
|---------------------|---------------------|
| \( J^P \)           | \( J^P \)           |
| \( 1^- \)           | \( 1^- \)           |
| \( ?^7 \)           | \( ?^7 \)           |
| \( M_{BW} \) (MeV)  | \( 4041 \pm 1 \)    |
| \( 4051 \pm 14^{+20}_{-11} \) | \( 4247 \pm 53 \) |
| \( 4309 \pm 116 \)  | \( 4248^{+44+180}_{-29-35} \) |
| \( \Gamma_{BW} \) (MeV) | \( 115 \pm 17 \)    |
| \( 83^{+21+47}_{-17-22} \) | \( 345 \pm 67 \) |
| \( 468 \pm 90 \)    | \( 177^{+54+316}_{-39-61} \) |
FIGURE 5. Distributions of the $\chi_{c1}\pi^+$ invariant mass for $\bar{B}_0 \to \chi_{c1}K^-\pi^+$. (Left) The red, blue, and magenta solid curves in Figs. 4(d,e-1,e-2) are superimposed on the Belle data (Fig. 14 of Ref. [5]); in order to fit the data, a constant factor is multiplied to each of the curves and an incoherent constant background is added. (Right) The different spectra are generated by the triangle diagram of Fig. 1(e) with different masses for $X(3872)$ and $K^*(892)$ (see the text). We obtain the red solid, blue dashed, green dotted, and magenta dash-dotted curves using the $X(3872)\pi^+$ threshold energy smaller than the PDG value by 0, 50, 100, and 150 MeV, respectively. All these curves have the same peak height after being scaled. We turn off the on-shell $X(3872)\pi^+$ contribution in the red solid curve to obtain the black dash-two-dotted curve. Figures taken from Ref. [8]. Copyright (2019) APS.

because they used the Breit-Wigner form to simulate this bump. The data seem to disfavor the Breit-Wigner shape. As seen in Fig. 4(d), the triangle diagram of Fig. 1(d) generates the spectrum bump, the shape of which is significantly different from the Breit-Wigner.

Having seen that the asymmetric shape is crucial to explain the Belle data, one may wonder how the triangle diagram can create this peculiar shape. In Fig. 4(d), we can find that the spectrum has an abrupt bend at $m_{\chi_{c1}\pi} \sim 4.01$ GeV, where the $X(3872)\pi^+$ channel opens, and the sharp rise of the spectrum starts from this point. This is more clearly seen in an enlarged one shown by the red solid curve in Fig. 5(right). This seems to indicate that the sharp rise is partially due to the opening of the $X(3872)\pi^+$ channel. To confirm this speculation, we turn off the on-shell $X(3872)\pi^+$ contribution, which arises from $+ie$ in the denominator of Eq. (1), and show the resulting spectrum by the black dash-two-dotted curve in Fig. 5(right). Indeed, the asymmetry of the bump shape is essentially from the on-shell $X(3872)\pi^+$ contribution.

The large asymmetry seems to be also due to the proximity of the $X(3872)\pi^+$ threshold to the TS energy ($\sim 4.025$ GeV). We can examine this point by lowering the $X(3872)\pi^+$ threshold. We use $X(3872)$ and $K^*(892)$ masses of, in unit of MeV, $(m_{X(3872)}, m_{K^*(892)}) = (3822, 1084), (3772, 1218)$, and $(3722, 1330)$. In this way, we can lower the threshold by 50, 100, and 150 MeV, respectively, while the peak position of the spectrum is kept almost at the same place. Figure 5(right) indicates that, as the $X(3872)\pi^+$ threshold is lowered, the rise of the bump becomes significantly more moderate. Through the above analysis, we now understand the asymmetric shape of the $Z_1(4050)$ bump observed in the Belle data with well-founded physics: TS and the channel opening near the TS energy. The triangle diagram of Fig. 1(d) includes these physical contents.

The asymmetric bump shape associated with $Z_1(4050)$ is interesting because it could sensitively discriminate different theoretical interpretations of $Z_1(4050)$. A successful model should explain this characteristic spectrum shape of $Z_1(4050)$, in addition to the mass, width, and $J^P$. So far, this question has been successfully addressed by our model only. Higher statistics data is also highly hoped to establish the spectrum shape because the error bars are still rather large in the Belle data.
Summary

The identity of the charged charmonium-like state ($Z_c$) candidates is a hot problem in the field of the hadron spectroscopy, and this work is along this trend. We showed that the experimentally determined properties of $Z_c(4430)$, $Z_c(4200)$, $Z_1(4050)$ and $Z_2(4250)$ such as spin-parity, mass, width, and Argand plot are all explained well by the triangle loop diagrams we identified and the kinematical singularities involved. This scenario is completely different from the previous (and surviving) theoretical interpretations based on the four-quark picture (including hadron molecule), and is so far the only one giving a natural explanation for: (i) the appearance (absence) of $Z_c(4200)(Z_c(4430))$-like contribution in $\Lambda_b^0 \rightarrow J/\psi p\pi^-$; (ii) the highly asymmetric shape of the $Z_1(4050)$ bump.

ACKNOWLEDGMENTS

The author thanks K. Tsushima for collaboration. This work is in part supported by National Natural Science Foundation of China (NSFC) under contracts 11625523.

REFERENCES

[1] S.K. Choi et al. (Belle Collaboration), Phys. Rev. Lett. 100, 142001 (2008).
[2] K. Chilikin et al. (Belle Collaboration), Phys. Rev. D 88, 074026 (2013).
[3] R. Aaij et al. (LHCb Collaboration), Phys. Rev. Lett. 112, 222002 (2014).
[4] K. Chilikin et al. (Belle Collaboration), Phys. Rev. D 90, 112009 (2014).
[5] R. Mizuk et al. (Belle Collaboration), Phys. Rev. D 78, 072004 (2008).
[6] https://physics.aps.org/synopsis-for/10.1103/PhysRevLett.112.222002; https://home.cern/news/news/experiments/lhc-b-confirms-existence-exotic-hadrons
[7] S.X. Nakamura and K. Tsushima, Phys. Rev. D 100, 051502(R) (2019); arXiv:1901.07385.
[8] S.X. Nakamura, Phys.Rev. D 100, 011504(R) (2019).
[9] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018).
[10] H. Kamano, S.X. Nakamura, T.-S.H. Lee, and T. Sato, Phys. Rev. D 84, 114019 (2011).
[11] R. Aaij et al. (LHCb Collaboration), Phys. Rev. Lett. 117, 082003 (2016).
[12] P. Pakhlov, Phys. Lett. B702, 139 (2011).
[13] P. Pakhlov and T. Uglov, Phys. Lett. B748, 183 (2015).
[14] S. Coleman and R.E. Norton, Nuovo Cim. 38, 438 (1965).