Asymptotic shape of isolated magnetic domains

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We investigate the energy of an isolated magnetized domain \(\Omega \subset \mathbb{R}^n\) for \(n = 2, 3\). In non-dimensionalized variables, the energy given by

\[
\mathcal{E}(\Omega) = \int_{\mathbb{R}^n} |\nabla \chi_\Omega|^2 \, dx + \int_{\mathbb{R}^n} |\nabla h_\Omega|^2 \, dx
\]

penalizes the interfacial area of the domain as well as the energy of the corresponding magnetostatic field. Here, the magnetostatic potential \(h_\Omega\) is determined by \(\Delta h_\Omega = \partial_1 \chi_\Omega\), corresponding to uniform magnetization within the domain. We consider the macroscopic regime \(|\Omega| \to \infty\), in which we derive compactness and \(\Gamma\)-limit which is formulated in terms of the cross-sectional area of the anisotropically rescaled configuration. We then give the solutions for the limit problems.

1. Introduction

Ferromagnetic materials exhibit the formation of magnetic domains, i.e. regions with almost uniform magnetization oriented along certain crystalline directions. In the prototypical case of a sample with initial uniform magnetization, the application of an external magnetic field can induce a phase transformation. The initial time of the phase transformation is characterized by the nucleation and growth of small isolated magnetic domains of the new phase (figure 1). The shape and energetics of these domains are essential to understand the phase transformation of the magnetic material and its related hysteresis [1–6]. An isolated magnetic domain can also be formed by a ferrofluidic droplet under the application of an external field. In this paper, we investigate the shape and energetics of isolated magnetic domains from the viewpoint of
Figure 1. Nucleation of a single magnetic domain during cooling with applied magnetic field, see ([17], fig. 1). The red arrow indicates the magnetization vector inside the domain. (Online version in colour.)

calculus of variations without relying on any specific ansatz function. We note that the shape and properties of magnetic domains in solids and fluids have been studied in the physical literature both experimentally and numerically [7–9]; for applications see e.g. [10–16]. However, in these works, the energy of the optimal domain shape is determined either numerically or within a certain class of ansatz functions.

We assume that the magnetization \( m_{\Omega} \in L^2(\mathbb{R}^n, \mathbb{R}^n) \) points in direction \( e_1 \) within the magnetic domain \( \Omega \subset \mathbb{R}^n \) and vanishes outside, i.e. \( m_{\Omega} = e_1 \chi_\Omega \) where \( \chi_\Omega \) is the characteristic function of \( \Omega \). By Maxwell’s equations, the induced magnetostatic field \( h_{\Omega} \in L^2(\mathbb{R}^n, \mathbb{R}^n) \) is then given by the unique (distributional) solution of

\[
h_{\Omega} := -\nabla(-\Delta_{\mathbb{R}^n})^{-1}(\text{div} m_{\Omega}).
\]

We note that (up to a factor 2) the same stray field is created if the magnetization is \( e_1 \) in \( \Omega \) and \(-e_1\) otherwise which would model the situation of a uniformly magnetized domain in an infinitely extended uniaxial magnetic material. Our choice of magnetization also describes the situation of a uniformly magnetized ferrofluidic droplet. We assume that an external field with relative strength \( \lambda > 0 \) is applied in the \( e_1 \)-direction. Also taking into consideration an energy related to the interface of the magnetic domain, the energy of the single magnetic domain in a non-dimensionalized setting is given by

\[
E(\Omega) = \int_{\mathbb{R}^n} |\nabla \chi_\Omega| \, dx + \int_{\mathbb{R}^n} |h_{\Omega}|^2 \, dx + \lambda \int_{\mathbb{R}^n} \chi_\Omega \, dx.
\]

The parameter \( \lambda \) describes the relative strength of the external magnetic field. We note that the last term in (1.2), the Zeeman term, is a trivial contribution to the energy by our assumption that the volume of the magnetic domain is given by

\[
\int_{\mathbb{R}^n} \chi_\Omega \, dx = \mu,
\]

for some fixed \( \mu > 0 \). We note that in the above model neither boundary effects, nor material irregularities which might lead to pinning of the domain, are included. We note that the physically relevant case corresponds to three-dimensional domains \( \Omega \). However, for the mathematical interest, we formulate the model and give some results in other dimensions as well. We refer to remark 1.1 for more details on the origin and derivation of the energy.

In [18], it has been shown that minimizers of our energy functional (1.2) with volume constraint (1.3) exist in all dimensions \( n \) and for all prescribed masses \( \mu \geq 0 \). For \( 2 \leq n \leq 7 \), for any local minimizer of (1.2), there is a regular representative \( \Omega \) for its positivity set which is open, bounded and has a smooth boundary. Furthermore, in this case, the sets \( \Omega, \mathbb{R}^n \setminus \Omega \) and \( \partial \Omega \) are connected. We note that the positivity set is only defined a.e. and the regular representative is obtained after modification by a zero set. Also, the scaling of the ground state energy and some qualitative
properties of the shape have been investigated in [18] for \( n = 3 \) and in [19] for \( n \geq 4 \). We note that for small volume \( \mu \), the energy (1.2) is a perturbation of the perimeter energy and minimizers have approximately the shape and energy of a ball with mass \( \mu \). We consider the opposite regime of large volume \( \mu \gg 1 \) where the impact of the magnetostatic energy is essential to determine the minimal energy and optimal shape of the domain. In this regime, minimizers of the energy (1.2) get more and more elongated and assume the shape of a thin needle. More precisely, the scaling of the ground state energy is given by \( \inf_{|\Omega| = \mu} \mathcal{E}(\Omega) \sim \mu^{2/(2n+1)} \) for \( n \geq 3 \) and \( \mu \geq 1 \). The scalingwise upper bound is obtained by ellipsoids with length of order \( L \sim \mu^{3/(2n+1)} \) and radius of order \( R \sim \mu^{2/(2n+1)} \) (cf. [19]); see figure 2. In this paper, we rescale the variables accordingly to maintain a bounded configuration. For the rescaled energy, we establish a compactness result (theorem 2.2) and a limit energy in the framework of \( \Gamma \)-convergence (theorem 2.3). The limit energy is formulated in terms of the horizontal cross-section area of the rescaled domain. The limit energy is local for \( n = 3 \) and can be solved algebraically for both dimensions \( n = 2, 3 \) (theorem 2.4). Interestingly, for \( n = 2 \), the solution of the rescaled limit model yields the cross-section area of an exact ellipse and is close to an ellipse for \( n = 3 \).

We give some remarks about the relation to the underlying physical models:

**Remark 1.1 (Relation to Landau–Lifshitz energy).** The energy of a ferromagnetic body \( U \subset \mathbb{R}^3 \) is given by the Landau–Lifshitz energy ([20], Ch. 3.2)

\[
\mathcal{E}_{LL}[m] = \int_{U} \left( A|\nabla m|^2 + K(1 - m_1^2)^2 - J_s H_{\text{ext}} \cdot m \right) dx + \frac{\mu_0}{2} \int_{\mathbb{R}^3} |H|^2 dx. \tag{1.4}
\]

The terms in (1.4) in order of appearance are as follows: the exchange energy favours alignment of neighbouring spins. The anisotropy energy favours an alignment with certain crystal lattice directions; in our case, we consider a uniaxial material where the directions \( \pm e_1 \) are preferred. The external field energy (also called Zeeman energy) describes the energy associated with an external magnetic field \( H_{\text{ext}} \). The constant \( J_s \) is the so-called saturation magnetization. Finally, the last integral on the right-hand side of (1.4) is the stray field energy (or magnetostatic energy) where \( H \) is the demagnetization or stray field energy and \( \mu_0 \) is the vacuum permeability. In bulk samples, one observes large magnetic domains with uniform magnetization, separated by sharp one-dimensional transition layers (so-called Bloch walls) of thickness of order \( \ell_{\text{trans}} := \sqrt{A/K} \) with rapid rotation of the magnetization ([20], Sec. 3.6.1).

From \( |m| = 1 \), we get the well-known estimate \( a|\nabla m|^2 + b(1 - m_3^2) \geq 2\sqrt{ab}|\nabla m_3| \). Hence,

\[
\mathcal{E}_{LL}[m] \geq \int_{U} \left( 2\sqrt{AK}|\nabla m_1| - J_s H_{\text{ext}} \cdot m \right) dx + \frac{\mu_0}{2} \int_{\mathbb{R}^3} |H|^2 dx. \tag{1.5}
\]
We consider an external field of the form $H_{\text{ext}} = -\lambda 8(AK)^{3/2} \mu_0^{-2} e_1$ and non-dimensionalize the model by rescaling length and energy in units of $\ell_{\text{res}} = 4\mu_0^{-1} \sqrt{AK}$ and $(AK)^{3/2} \mu_0^{-2}$. With the sharp interface assumption $n \in \{ \pm 1 \}$, we arrive at (1.2). This sharp interface approximation is valid if the width of the interfacial layers in (1.4) is much smaller than the domain size [21]. Note that we consider the renormalized energy where the infinite contribution of the Zeeman energy outside the domain is neglected. In view of (2.3), our results (for $n = 3$) are hence physically relevant for sufficiently large domains in the sense that $\ell_{\text{trans}} \ll (V/\ell_{\text{res}}^2)^{2/7}(\ln V/\ell_{\text{res}}^3)^{-1/7}$, where $V$ is the volume of the domain.

**Remark 1.2 (Relation to ferrofluidic droplets).** A ferrofluid is a colloidal liquid consisting of ferromagnetic particles suspended in a surrounding fluid [22]. The ferromagnetic particles are coated with a surfactant which inhibits agglomeration. These fluids are interesting for applications since they can be moved by the application of an external field, in which case fascinating droplet shape patterns may occur. Correspondingly, the associated magnetic energy is given by the exchange energy, Zeeman energy and the magnetostatic energy as in (1.4). Furthermore, for the total energy of the system, we assume an interfacial energy at the boundary of the ferrofluidic droplet. For our model to apply, we assume a constant density of the magnetic dipoles within the ferrofluidic droplet which then leads to the energy (1.2).

**Notation.** We write $X \lesssim Y$ if there exists a universal constant $C > 0$ such that $X \leq CY$. Analogously, we also use $\gtrsim$. If a constant depends on a parameter $p$, we write $C_p$. For $x \in \mathbb{R}^n$, we write $x = (x_1, x')$, $x' = (x_2, \ldots, x_n)$ and analogously $\mathbb{V} = (\partial_1, \mathbb{V}')$. The volume of the $n$-dimensional unit ball is denoted by $\omega_n$. The vector space of all functions of bounded variation is denoted $\text{BV}$. For the perimeter of a Lebesgue measurable set $E \subset \mathbb{R}^n$, we write $\mathcal{P}(E)$. The set $E$ is said to have finite perimeter if $\mathcal{P}(E) < \infty$ i.e. $\chi_E \in \text{BV}(\mathbb{R}_n, (0, 1))$, where $\chi_E$ is the characteristic function of $E$. The Fourier Transform and its inverse are given by

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i x \cdot \xi} \, dx, \quad f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i x \cdot \xi} \, d\xi.$$  

We recall that the homogeneous $H^s$-norm for functions $f: \mathbb{R} \to \mathbb{R}$ is given by

$$||f||^2_{H^s} = \int_\mathbb{R} |\xi|^s |\hat{f}(\xi)|^2 \, d\xi.$$  

**2. Setting and statement of results**

**(a) Setting and rescaled model**

In the limit of large $\mu$, we expect minimizers to have the approximate shape of elongated ellipsoids. It is hence convenient to work in anisotropically rescaled variables. For prescribed volume $\mu > 1$ (cf. (1.3)), we define

$$u(\tilde{x}) := \chi_{\Omega} \left( \frac{\tilde{x}_1}{R_\mu}, \frac{\tilde{x}'}{L_\mu} \right).$$  

The parameters $R_\mu, L_\mu > 0$ are given by

$$R_\mu = \mu^{1/3}, \quad L_\mu = \mu^{2/3} \text{ for } n = 2,$$

$$R_\mu = \mu^{2/7}(\ln \mu)^{-1/7}, \quad L_\mu = \mu^{3/7}(\ln \mu)^{2/7} \text{ for } n = 3$$  

and

$$R_\mu = \mu^{2/(2n+1)}, \quad L_\mu = \mu^{3/(2n+1)} \text{ for } n \geq 4.$$  

These length scales correspond to the optimal scaling of diameter $R_\mu$ and length $L_\mu$ for the magnetic domain which are obtained by minimizing the energy in the class of ellipsoidal
configurations with volume $\mu$. For the aspect ratio, we write

$$\epsilon := \frac{R\mu}{L\mu} < 1.$$  \hfill (2.5)

We correspondingly rescale our energy by $R^{n-2}_\mu L_\mu$, according to the energy of these ellipsoidal configurations. With the change of variables (2.1) and expressing the non-local part of the energy in terms of Fourier variables, the renormalized and rescaled energy takes the form

$$E^{(n)}_\epsilon[u] := \frac{E(\mu) - \lambda |\mu|}{R^{n-2}_\mu L_\mu} = P^{(n)}_\epsilon[u] + N^{(n)}_\epsilon[u],$$

where the parts of the energy related to perimeter and non-local interaction are

$$P^{(n)}_\epsilon[u] := \int_{\mathbb{R}^n} \sqrt{|\nabla' u|^2 + \epsilon^2 |\partial_1 u|^2} \, dx$$ \hfill (2.6)

and

$$N^{(n)}_\epsilon[u] := \gamma_n(\epsilon) \int_{\mathbb{R}^n} \frac{\xi_1^2}{\epsilon^2 \xi_1^2 + |\xi|^2} |\hat{u}(\xi)|^2 \, d\xi.$$ \hfill (2.7)

Here, the prefactor in front of the non-local energy in (2.7) is given by

$$\gamma_2(\epsilon) := \epsilon, \quad \gamma_3(\epsilon) := \frac{1}{7 |\ln \epsilon| - 3 \ln(\ln \mu)} \quad \text{and} \quad \gamma_n(\epsilon) := 1 \text{ for } n \geq 4.$$ \hfill (2.8)

We note that the limit $\epsilon \to 0$ is equivalent to $\mu \to \infty$ and that $\gamma_3(\epsilon) = \frac{1}{7 |\ln(\epsilon)|^{-1} + o(1)}$ as $\epsilon \to 0$. In view of the assumption $|\Omega| = \mu$ and since $R^{n-1}_\mu L_\mu$, the function $u$ has unit mass. Correspondingly, we consider the set of admissible functions

$$\mathcal{A} = \left\{ u \in \text{BV}(\mathbb{R}^n, [0, 1]) : \int_{\mathbb{R}^n} u(x) \, dx = 1 \right\},$$ \hfill (2.9)

and set $E^{(n)}_\epsilon[u] = \infty$ if $u \not\in \mathcal{A}$.

(b) Main results

In [18,19,23], existence of minimizers with prescribed volume has been established for the energy (1.2). We recall the scaling of the ground state energy (figure 2):

**Theorem 2.1 (Scaling of minimal energy [18,19,23]).** Let $n \geq 2$. Then we have

$$c_n \leq \inf_{u \in \mathcal{A}} E^{(n)}_\epsilon[u] \leq C_n \quad \text{for all } \epsilon \in (0, 1)$$ \hfill (2.10)

for some constants $c_n, C_n > 0$ which depend only on $n$.

The proof of this statement is given in [18,23] for dimension $n = 3$ and [19] for general $n \geq 3$ where the scaling laws (2.2)–(2.4) have been derived for suitable (integral) notions of diameter and length of the configuration. Theorem 2.1 shows that our choice of $R_\mu$ and $L_\mu$ provides us with a non-trivial limit as $\epsilon \to 0$. In dimensions 2 and 3, under the assumption of uniformly bounded support of $u$, the statement of theorem 2.1 also follows from theorem 2.3 which additionally yields the asymptotically leading order constant in the expansion of the energy. Indeed, it is easy to see that by continuity of $\epsilon \mapsto \inf_{u \in \mathcal{A}} E^{(n)}_\epsilon[u]$, the infimum of $E^{(n)}_\epsilon[u]$ on $\mathcal{A}$ is of order one for $\epsilon \to 0$. Taking an (almost) minimizing sequence $u_\epsilon$ for $\epsilon \to 0$, we can apply theorem 2.3 to obtain the required bounds. In the case $n \geq 4$, we do not know the $\Gamma$-limit of the energy $E^{(n)}_\epsilon$ but with the ideas presented in lemma A.2, we are able to prove the lower bound in theorem 2.1 with a different argument and we also obtain the upper bound in arbitrary dimension $n \geq 4$. The main difference between $n = 2, 3$ and $n \geq 4$ is the property that $\gamma_n(\epsilon) \to 0$ (for $\epsilon \to 0$) only for $n = 2, 3$. This translates to the fact that in dimension $n \geq 4$ the leading order term no longer consists of a single expression (cf. proposition 3.8 and lemma A.2).
The limit problem is formulated in terms of the cross-sectional area

\[ A[u] : \mathbb{R} \to [0, \infty), \quad A[u] := \int_{\mathbb{R}^{n-1}} u(., x') \, dx'. \tag{2.11} \]

In view of our unit mass assumption on \( u \) in (2.9), we then have for \( A := A[u] \) that

\[ \int_{\mathbb{R}} A(x) \, dx = 1. \tag{2.12} \]

For sequences with uniformly bounded energy, we have the following compactness result:

**Theorem 2.2 (Compactness).** Let \( n \in \{2, 3, 4\} \). Let \( \rho > 0 \) and let

\[ s \in \left[0, \frac{1}{14}\right] \quad \text{if} \quad n = 2, \quad s \in \left[0, \frac{1}{10}\right] \quad \text{if} \quad n = 3 \quad \text{and} \quad s \in \left[0, \frac{1}{22}\right] \quad \text{if} \quad n = 4. \tag{2.13} \]

Then for any sequence \( u_{\epsilon} \in A \) with \( E_{\epsilon}(u_{\epsilon}) \leq C \) and \( spt \, u_{\epsilon} \subset B_\rho(0) \), \( \forall \epsilon > 0 \), we have

\[ A[u_{\epsilon}] \to A \quad \text{in} \quad H^s(\mathbb{R}) \quad \text{as} \quad \epsilon \to 0 \tag{2.14} \]

(after selection of a subsequence) for some \( A \in \mathcal{A}_0^{(n)} \) where

\[ \mathcal{A}_0^{(n)} := \left\{ A \in X : A \text{ satisfies (2.12)} \right\}, \]

and where \( X = H^{1/2}(\mathbb{R}) \) if \( n = 2 \), \( X = H^1(\mathbb{R}) \) if \( n = 3 \) and \( X = \bigcap_{s \leq 7} H^s(\mathbb{R}) \) if \( n = 4 \).

In particular, we have \( A[u_{\epsilon}] \to A \) in \( L^2 \) for all \( 2 \leq n \leq 4 \), see ([24], Theorem 4.54). We note that the regularity of the limit space is better than the topology of the convergence in (2.14). To have compactness in the topology of the limit space would require us to rule out oscillations on the frequency scale \( \epsilon^{-1} \). However, uniform control over the energy does not seem to provide sufficient control in this regime. We also note that the topology (2.14) does not distinguish between two configurations \( u_{\epsilon,1} \) and \( u_{\epsilon,2} \) if they have the same cross-sectional area and in particular is not even Hausdorff.

In the macroscopic limit \( \epsilon \to 0 \) and for dimensions \( n \in \{2, 3\} \), our functionals \( \Gamma \) converge to limit functionals given by

\[ E_{\epsilon}^{(2)}[A] = 2|\{x \in \mathbb{R} : A(x) > 0\}| + \frac{1}{2} \int_{\mathbb{R}} |\partial_1|^{1/2} A(\xi_1)^2 \, d\xi_1 \tag{2.15} \]

and

\[ E_{\epsilon}^{(3)}[A] = 2\sqrt{\pi} \int_{\mathbb{R}} \sqrt{A(\xi_1)} \, d\xi_1 + \frac{1}{14\pi} \int_{\mathbb{R}} |\partial_1 A(\xi_1)|^2 \, d\xi_1, \tag{2.16} \]

for \( A \in \mathcal{A}_0^{(n)} \) and \( E_{\epsilon}^{(n)}[A] = +\infty \) else. More precisely, we have

**Theorem 2.3 (Gamma-limit).** Let \( n \in \{2, 3\} \). Then

\[ E_{\epsilon}^{(n)} \xrightarrow{\Gamma} E_0^{(n)} \quad \text{w.r.t to the topology (2.14)}. \tag{2.17} \]

In particular,

(i) For any sequence \( u_{\epsilon} \in A \) with uniformly bounded support such that (2.14) holds for some \( A \in \mathcal{A}_0^{(n)} \), we have

\[ \liminf_{\epsilon \to 0} E_{\epsilon}^{(n)}[u_{\epsilon}] \geq E_0^{(n)}[A]. \quad (\text{liminf estimate}) \tag{2.18} \]

(ii) For any \( A \in \mathcal{A}_0^{(n)} \), there is a sequence \( u_{\epsilon} \in A \) with uniformly bounded support such that (2.14) holds and

\[ \limsup_{\epsilon \to 0} E_{\epsilon}^{(n)}[u_{\epsilon}] \leq E_0^{(n)}[A]. \quad (\text{limsup estimate}) \tag{2.19} \]
The notion of \( \Gamma \)-convergence in particular ensures that minimizers (with uniformly bounded support) converge to minimizers of the limit functional. Indeed, by (2.19) any sequence of minimizers \( u_\epsilon \) of \( E^{(n)}_\epsilon \) verifies the hypothesis of theorem 2.2 and by lower semicontinuity (2.18) it converges (up to extracting a subsequence) to a minimizer \( A \) of the limit functional \( E^{(n)}_0 \). We note that we need the assumption of uniformly bounded support in theorem 2.3 and hence only characterize minimizing sequences with this property. In fact, we believe that minimizing sequences should have equibounded support since the minimal energy of the limit functional is sublinear as a function of the volume. However, a proof for this statement seems quite technical and is ongoing work. We note that ([18], Thm. 1.2) gives a characterization of an integral version of the length and radius of configuration.

The limit energies which determine the asymptotic shape of the rescaled needles can be solved explicitly (for \( n = 2 \)) or implicitly (for \( n = 3 \)). An illustration of these solutions is shown in figure 3.

**Theorem 2.4 (Solutions of limit problems).** Let \( n \in \{2, 3\} \). The unique (up to translations) minimizers with bounded support \( A^{(n)} \in A^{(n)}_0 \) of \( E^{(n)}_0 \) are given by

\[
A^{(n)}(x_1) = \frac{R^{(n)}(R^{(n)}(t))^n}{L^{(n)}}, \quad \text{where } t := \frac{|x_1|}{L^{(n)}},
\]

and where \( R^{(n)}_*, L^{(n)}_* > 0 \) and \( R^{(n)} : \mathbb{R} \to [0, 1] \) are given as follows:

(i) For \( n = 2 \), we have \( R^{(2)}_* = (1/2\pi^2)^{1/3} \) and \( L^{(2)}_* = (2/\pi)^{1/3} \) and

\[
R^{(2)}(t) = \chi_{(-1,1)}(t) \sqrt{1 - t^2}.
\]

(ii) For \( n = 3 \), \( R^{(3)} \) is given by (3.54) with parameters \( R^{(3)}_* \approx 1.511 \) and \( L^{(3)}_* \approx 0.202 \) solving the system (3.55)–(3.57).

We remark that for \( n = 2 \), the minimizer of \( E^{(2)}_0 \) is exactly given by the cross-section of an ellipse; for \( n = 3 \) this is only approximately the case. We believe that the optimal needle shape for the full model should be rotationally symmetric; however, to show this rigorously seems quite difficult and is outside of the scope of our analysis. We have the following physical intuition for the limit model: with increasing volume the aspect ratio \( R_* / L_* \) of the domain becomes smaller and the magnetostatic energy related to interaction between charges in horizontal direction becomes more important compared to interactions in other directions. In particular, the self-interaction of magnetic charges within a single layer \( \{x_1\} \times \mathbb{R}^{n-1} \) for \( x_1 \in \mathbb{R} \) becomes dominant in the limit. We note that \( \partial_1 A \) corresponds to the magnetic charge density per layer. A problem similar to the limit problem of theorem 2.4 for \( n = 3 \) has been solved in ([25], Sec. 5). Different to the problem considered in [25], we cannot assume an a priori bound on the support of our solution and we

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**Figure 3.** Plot for the solutions \( R^{(n)}(t) \) in theorem 2.4 for \( 0 \leq t \leq 1 \) with associated radially symmetric and rescaled needle domain \( \Omega = \{ |x'| \leq \rho(x_1) \} \). For the solution of the limit problem \( \Omega \) for \( n = 2 \) is precisely the unit disc, for \( n = 3 \) it approximates the unit disc (indicated by the dashed line). (Online version in colour.)
need to rule out the possibility of losing mass at infinity. Also, our problem includes a volume constraint which is not present in [25].

(c) Comparison to previous results

Non-local problems with an isoperimetric term and a competing non-local interaction term have been studied extensively in recent years [26,27], in particular for nuclear stability [28], elasticity and micromagnetics. Coulomb-type potentials with interfacial penalization and prescribed volume have been considered (e.g. [29–36]) since they naturally appear in physical models, for example in the modelling of diblock-copolymers, see [37–41]. The phase transformation in a ferromagnetic slab under the influence of an external field has been investigated in [25] where also the nucleation of ferromagnetic domains plays an essential role, see also [42]. Nucleation and related problems in shape memory alloys have been e.g. considered in [43–45]. There are various other related models from materials science where dilute/large volume regimes are relevant, such as epitaxial growth (e.g. [46]), dislocations (e.g. [47]) or superconductors (e.g. [48,49]).

In many papers related to our setting, a perfect ellipsoid shape is assumed [50–56]. This is a common simplification due to the fact that the self-demagnetization of ellipsoids can be treated analytically. From this perspective, our three-dimensional result which shows that asymptotic minimizers are not perfect ellipsoids is of particular interest. We note, however, that similar structures have been observed numerically, e.g. in [57]. There, the authors study a sessile ferrofluid drop for a specified range of surface tension, gravitational and magnetic forces. These ranges are characterized by the magnetic and gravitational Bond numbers which express the relative importance of gravitational and magnetic forces with respect to surface tension. Compared to our setting, gravitation (in the vertical direction) is considered as an additional effect which favours flat droplets. The authors deduce numerically that if the gravitational Bond number is below a certain threshold, then the height of the drop is increasing, seen as a function of the magnetic Bond number. If the magnetic Bond number is large enough, an inversion of the curvature is observed; compare figure 3 and ([57], fig. 4). Other examples of optimal shapes for ferromagnetic liquid droplets that are not ellipsoidal are given in [7]. The energy considered there includes an exchange and core energy in addition to surface and demagnetization energy in our paper. The authors constrain the set of possible shapes and textures by imposing a cylindrical symmetry. They then use numerics to calculate the energy minimizers in their restricted class of admissible shapes to be doughnut- or apple-shaped.

The competition between a non-local repulsive potential and an attractive confining term is found also in other problems, for example in models studying the interaction of dislocations [58,59] or [60–62]. The latter considered a non-local anisotropic potential energy with an isotropic and local confinement term, other [63] study problems with a non-local anisotropic perimeter. In our case, we combine a non-local anisotropic energy potential with an anisotropic but local perimeter. Another anisotropic and non-local repulsive energy that has been treated variationally using ansatz–free analysis is [60] (based on [61,62]). The model studied in [62] describes the interaction of positive dislocations in the plane. The energy and results have subsequently been generalized to arbitrary dimensions in [60] and consist of an isotropic quadratic confinement and an anisotropic, non-local and repulsive term with a fixed mass constraint. The energy density of the latter expression \( W_\alpha(x) \) is associated with the isotropic Coulomb potential together with an anisotropic part of the form \( \alpha x_2^n/|x|^n \). The authors prove that their energy admits a unique minimizer which is calculated to be an ellipsoid with half axis depending on \( \alpha \). Their approach relies on the strict convexity of the energy for \( \alpha \in (-1, n-2] \) and the Euler–Lagrange conditions. Borrowing and extending techniques from complex analysis, it is shown that the optimality conditions are satisfied by ellipsoids. Although similar in its statements, this result exhibits some major differences compared to our work. Firstly, we note that our (rescaled) model incorporates anisotropy in both parts of the energy. In particular, our perimeter bound degenerates for \( \epsilon \rightarrow 0 \), while the attractive potential in [60] is purely isotropic. Secondly, we point out that our repulsive energy potential is ‘more degenerate’ in the following sense: writing \( \mathcal{N}_1 \) in \( x \)-coordinates (see
lemma 2.1 in [18]) we find the convolution kernel to be \( \partial^2 \Gamma_n(x) \sim -1/|x|^n + n x^2/|x|^{n+2} \), where \( \Gamma_n \) is the usual Coulomb kernel in \( n \) dimensions. Since the relevant energy estimates take place close to 0, the kernel \( \partial^2 \Gamma_n(x) \) is more singular compared to \( W_n(x) \). Thirdly, we want to highlight the non-convexity of our problem. As we pointed out earlier, a crucial step in [60] is to use strict convexity for the existence of minimizers and the conclusion that a solution to the Euler–Lagrange conditions is indeed a global minimizer. Our energy is not accessible through those methods. Lastly, we also note that our result is not only concerned with minimizers of (1.2) but also with sequences bounded in energy. In particular, we are able to give highly non-trivial information about the convergence of those configurations for \( \epsilon \to 0 \).

(d) Assumption of bounded support

For both theorems 2.2 and 2.3, we assume that \( spt u_\epsilon \subset B_1(0) \), uniformly for some fixed \( B_1 \). The assumption of bounded support in theorem 2.2 is used in the proof of lemma 3.1 in order to ensure the boundedness of the second moment of \( u_\epsilon \) (in \( x' \)-direction) and in the proof of theorem 2.2 for the conclusion that no mass is lost in the limit process (in direction of large \( |x_1| \)). For theorem 2.2, the assumption seems to be necessary: indeed, consider two disjoint needles of mass 1/2, with optimal cross-section and with increasing distance in lateral direction as \( \epsilon \to 0 \). We then have the uniform bound \( E[u_\epsilon] \lesssim 1 \), however, we do not have compactness of the sequence \( u_\epsilon \) in any \( L^p \)-space. However, in the case of a sequence of minimizers, it might be possible to alleviate this assumption.

3. Proofs

(a) Preliminaries

The main idea behind the compactness and the liminf estimate in theorem 2.3 is to replace the rescaled characteristic function by the cross-sectional area \( A(x_1) := A[u](x_1) \) of the domain (cf. (2.11)) The Fourier transform of the cross-section also appears as the first constant term of the Taylor expansion of \( \hat{u}_\epsilon \) in transversal direction since

\[
\hat{u}(\xi_1,0) = \frac{1}{(2\pi)^{(n-1)/2}} \hat{A}(\xi_1).
\]

(3.1)

For the proof, we hence take into account only the low frequencies of our configuration, i.e. we expand \( \hat{u} \) around \( \xi' = 0 \) as

\[
\hat{u}(\xi_1,\xi') = \frac{1}{(2\pi)^{(n-1)/2}} \hat{A}(\xi) + \xi' \cdot \nabla_{\xi}\hat{u}(\xi_1,0) + \frac{1}{2} \xi' \cdot D^2\hat{u}(\xi_1,\xi')\xi'
\]

(3.2)

for some \( \xi' \in (0, \xi') \) and then perform a cut-off in transversal direction in the Fourier space. A corresponding lower bound for the energy is given in the next lemma:

**Lemma 3.1 (Lower bound on non-local energy).** Let \( n \geq 2 \) and \( u \in A \) and let \( A := A[u] \). Let \( spt u \subset \mathbb{R} \times B'_\rho(0) \) for some \( \rho > 0 \). Then, for all \( \eta > 0 \)

\[
N^{(0)}_\epsilon[u] \geq (1 - \rho^2 \eta^2) N_{\epsilon,\eta,1}[u] + N_{\epsilon,\eta,2}[u],
\]

(3.3)

where

\[
N_{\epsilon,\eta,1}[u] = \frac{\gamma_{n}(\epsilon)}{(2\pi)^{n-1}} \int_{\mathbb{R}^n} \frac{\xi_1^2}{\epsilon^2 \xi_1^2 + |\xi'|^2} \left| \hat{A}(\xi_1) \chi_{D'_\rho(\xi_1)}(\xi') \right|^2 d\xi
\]

(3.4)

and

\[
N_{\epsilon,\eta,2}[u] = \gamma_{n}(\epsilon) \int_{\mathbb{R}^n} \frac{\xi_1^2}{\epsilon^2 \xi_1^2 + |\xi'|^2} \left| \hat{u}(\xi) - \frac{1}{(2\pi)^{(n-1)/2}} \hat{A}(\xi_1) \chi_{D'_\rho(\xi_1)}(\xi') \right|^2 d\xi
\]

(3.5)

and where

\[
D'_\eta(\xi_1) := \left\{ \xi' \in \mathbb{R}^{n-1} : |\xi'| \leq \eta \hat{A}(\xi_1)^{1/2} \right\}.
\]

(3.6)
Proof. With the identity

\[ |\hat{u}|^2 = |\hat{u}(\xi_1, 0)|^2 \chi_{D^*_R(\xi_1)}(\xi') + |\hat{u} - \hat{u}(\xi_1, 0)\chi_{D^*_R(\xi_1)}(\xi')|^2 + 2\text{Re} \left[ (\hat{u} - \hat{u}(\xi_1, 0))\hat{u}(\xi_1, 0)\chi_{D^*_R(\xi_1)}(\xi') \right], \] (3.7)

and since \( \hat{u}(\xi_1, 0) = (2\pi)^{-(n-1)/2} \hat{A}(\xi_1) \), we have

\[ N_{e,n,1}[u] \geq N_{e,n,2}[u] - 2Ie[u], \] (3.8)

where

\[ Ie[u] := \frac{\gamma_n(\epsilon)}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}} \int_{D^*_R(\xi_1)} \frac{\xi_1^2}{\epsilon^2 \xi_1^2 + |\xi'|^2} \text{Re} \left[ (\hat{u}(\xi) - (2\pi)^{-(n-1)/2} \hat{A}(\xi_1)) \hat{A}(\xi) \right] d\xi' d\xi_1. \]

Using spherical coordinates \( \xi' = (r, \omega) \) in transversal direction, we estimate

\[ Ie[u] \leq \frac{\gamma_n(\epsilon)}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}} \int_0^{\eta \hat{A}(\xi_1)^{1/2}} \frac{\xi_1^2 \hat{A}(\xi_1)}{\epsilon^2 \xi_1^2 + r^2} \int_{r \xi_1 = r} (\hat{u}(\xi) - \hat{A}(\xi_1)) d\omega |r^{n-2} dr d\xi_1. \]

We calculate

\[ \hat{u}(\xi) - \hat{A}(\xi_1) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-i \xi' \cdot x} [\cos(\xi' \cdot x') + i \sin(\xi' \cdot x') - 1] dx' dx_1. \] (3.9)

The integral over the sine function vanishes by symmetry upon integration in \( |\xi'| = r \). Expanding the cosine function and since \( \text{spt } u \subset \mathbb{R} \times B_R(0) \), we hence get

\[ |\int_{|\xi'| = r} (\hat{u}(\xi) - \hat{A}(\xi_1)) d\omega| \leq \frac{r^2}{2(2\pi)^{n/2}} \int_{\mathbb{R} \times B_R(0)} |x'|^2 |u(x)| dx \leq \frac{\eta^2 \rho^2}{2\pi} N_{e,n,1}[u_\epsilon]. \] (3.10)

Inserting (3.10) into the expression for \( I \) and since \( r \leq \eta |\hat{A}(\xi_1)|^{1/2} \), we obtain

\[ 2Ie[u] \leq \frac{2\gamma_n(\epsilon)}{(2\pi)^{(n-1)/2}} \int_0^{\eta \hat{A}(\xi_1)^{1/2}} \frac{\xi_1^2 \hat{A}(\xi_1)}{\epsilon^2 \xi_1^2 + r^2} \left| \hat{A}(\xi_1) \right| \frac{r^2 \rho^2}{2} r^{n-2} dr d\xi_1 \leq \frac{\eta^2 \rho^2}{\sqrt{2\pi}} N_{e,n,1}[u_\epsilon]. \]

Since \( \hat{A} \) does not depend on \( \xi' \), in order to calculate the integral for \( N_{e,n,1}[u] \), it is useful to first integrate in \( \xi' \) and then in \( \xi_1 \). For this, the following integral formula is useful:

**Lemma 3.2 (Integral formula).** Let \( \beta \geq 0, k \in \mathbb{N} \) and let \( \xi' \in \mathbb{R}^{n-1} \). Then

\[ \int_{|\xi'| \leq \beta} \frac{\xi_1^2 |\xi'|^{2k}}{\epsilon^2 \xi_1^2 + |\xi'|^2} d\xi' = \frac{\omega_{n-2}}{n-1+2k} \epsilon^{-2} n^{-1+2k} F_2(1, \frac{n-1+2k}{2}, \frac{n+1+2k}{2}, -\left( \frac{\beta}{\epsilon |\xi_1|} \right)^2), \]

where \( F_2 \) is the hypergeometric function. In particular, for \( k = 0 \)

\[ \int_{|\xi'| \leq \beta} \frac{\xi_1^2}{\epsilon^2 \xi_1^2 + |\xi'|^2} d\xi' = \begin{cases} \frac{n}{2} \arctan \left( \frac{\beta}{\epsilon |\xi_1|} \right) & \text{for } n = 2, \\ \pi \xi_1^2 \ln \left( 1 + \left( \frac{\beta}{\epsilon |\xi_1|} \right)^2 \right) & \text{for } n = 3. \end{cases} \] (3.11)

Furthermore, if \( n + 2k \geq 4 \) and \( \beta/\epsilon |\xi_1| \to \infty \), we have the asymptotic expansion

\[ \int_{|\xi'| \leq \beta} \frac{\xi_1^2}{\epsilon^2 \xi_1^2 + |\xi'|^2} |\xi'|^{2k} d\xi' = \frac{\omega_{n-2}}{n-3+2k} \xi_1^2 \beta^{n-3+2k} + O(\epsilon \xi_1^3 \beta^{n-4+2k}). \] (3.12)
Proof. We write $N := n - 2 + 2k$. With the change of variables $s = |\xi'|/(e|\xi|)$, we get
\[
\int_{|\xi'| \leq \beta} \frac{\xi^2}{e^2 \xi^2 + |\xi'|^2} \xi'^2 \, d\xi' = \omega_{n-2} e^{-1} N\int_0^{\beta/(e|\xi|)} s^N \frac{1}{1 + s^2} \, ds.
\]
The last integral can be expressed as a hypergeometric function \([64], 15.6.1\)
\[
\int_0^{\beta/(e|\xi|)} s^N \frac{1}{1 + s^2} \, ds = \left( \frac{\beta}{e|\xi|} \right)^{N+1} \frac{1}{N+1} 2F_1 \left( 1, \frac{N+1}{2}, \frac{N+3}{2}, - \left( \frac{\beta}{e|\xi|} \right)^2 \right),
\]
which yields the assertion. Then (3.11) is directly followed by \([65], \text{Sec. 7.3.2 (148), p. 476}\) for $n = 3$ and by \((65), \text{Sec. 7.3.2 (83), p. 473}\) for $n = 2$ since $2F_1(1, \frac{1}{2}, \frac{3}{2}, z) = 2F_1(1, 1, 3, z)$ (see \([65], \text{Sec. 7.3.1 (4), p. 545}\)).

Now assume that $N \geq 2$. With the help of \((65), \text{Sec. 7.3.1 (6)}, \text{we also get the expansion}\)
\[
2F_1 \left( 1, \frac{N+1}{2}, \frac{N+3}{2}, -s^2 \right) = \left( \frac{N+1}{2} \right) \frac{O(1/s^2)}{\Gamma((N+1)/2)}.
\]
Together with $\Gamma((N-1)/2)/(\Gamma((N+1)/2) = 2/(N-1)$ (3.12) then follows.

(b) Proof of theorem 2.2

We consider a sequence of functions $u_\epsilon \in A$ with $\text{spt } u_\epsilon \subset B_\rho(0)$ (for some fixed $\rho > 0$) satisfying the uniform bound $L^{(n)}_\rho[u_\epsilon] \leq C$. The compactness result is based on the decomposition in lemma 3.1. This decomposition gives a lower bound on the non-local energy in terms of the cross-section area $A_\epsilon := A[u_\epsilon]$, i.e.

\[
N^{(n)}_\epsilon [u_\epsilon] \geq (1 - \rho^2 \eta^2) \frac{\gamma_\epsilon}{(2\pi)^{n-1}} \int_{|\xi'| \leq \eta} \frac{\xi^2}{\xi^2 + |\xi'|^2} |\tilde{A}_\epsilon(\xi')|^2 \, d\xi'
\]

(3.13)

noting that the prefactor is positive for $\eta$ sufficiently small. At the core of the proof is the next lemma which allows us to get uniform bounds on Sobolev norms for $A_\epsilon$:

Lemma 3.3 (Bound on cross-section $A$). Let $n = 2, 3$. Let $u \in A$ with $\text{spt } u \subset \mathbb{R} \times B_\rho(0)$ for some $\rho > 2$ and let $A := A[u]$. Then we have the bounds

(i) For any $s \in [0, \frac{1}{14}]$ if $n = 2$ and $s \in [0, \frac{1}{10}]$ if $n = 3$, we have
\[
\int_\mathbb{R} |\xi|^2 |\tilde{A}(\xi)|^2 \, d\xi \lesssim P^{(n)}_\epsilon |u|^2 + N^{(n)}_\epsilon |u| + C_{\rho, \sigma},
\]

(3.14)

(ii) Let $\eta \in (0, \frac{1}{2\rho})$. Then we have
\[
\int_{S_{\epsilon, \eta}} |\xi|^{n-1} |\tilde{A}(\xi)|^2 \, d\xi \lesssim N^{(n)}_\epsilon |u|,
\]

(3.15)

where
\[
S_{\epsilon, \eta} := \left\{ \xi \in \mathbb{R} : q_{\epsilon, \eta}(\xi) \geq e^{(\eta/2)^{1/2}} \right\},
\]

(3.16)

Proof. Let $\eta \in (0, 1/2\rho)$ to be fixed later. By lemmas 3.1 and 3.2, we have

\[
N^{(n)}_\epsilon |u| \gtrsim \gamma_\epsilon \int_\mathbb{R} \left( \int_{|\xi'| \leq \eta} \frac{\xi^2}{\xi^2 + |\xi'|^2} \, d\xi' \right) |\tilde{A}(\xi)|^2 \, d\xi
\]

(3.17)

\[
\gtrsim \int_\mathbb{R} \left[ \frac{|\xi|}{|\ln e|} \frac{\xi^2}{\xi^2} \ln(1 + q_{\epsilon, \eta}(\xi)^2) \right] |\tilde{A}(\xi)|^2 \, d\xi \quad \text{for } \begin{cases} n = 2, \\ n = 3. \end{cases}
\]

(3.18)
With the control of the anisotropic perimeter, we also have
\[ P_\epsilon^{(n)}[u] \geq \epsilon \int_{\mathbb{R}^n} |\nabla u(x)| \, dx \geq \epsilon |\nabla A(\xi)| = \epsilon |\xi_1||\widehat{A}(\xi_1)| \quad \forall \xi_1 \in \mathbb{R}. \tag{3.19} \]

The proofs for (i)–(ii) are based on the lower bounds in (3.18)–(3.19) and are given below:

(i) We use the decomposition \( \mathbb{R} = (R^{(n)}_{\epsilon,\eta}) \cup (R^{(n)}_{\epsilon,\eta}') \) in frequency space where
\[ R^{(n)}_{\epsilon,\eta} := \{ \xi_1 \in \mathbb{R} : \widetilde{q}_{\epsilon,\eta}(\xi_1) \geq \left[ \begin{array}{l} 1, \quad \text{for } n = 2 \\ \epsilon^{-\eta}, \quad \text{for } n = 3 \end{array} \right] \}, \tag{3.20} \]
using the notation (3.16). The factor \( \epsilon^{-n} \) for \( n = 3 \) ensures that the logarithm in (3.18) can be controlled. Indeed, for \( \xi_1 \in R^{(n)}_{\epsilon,\eta} \), we have \( \arctan(q_{\epsilon,\eta}(\xi_1)) \geq 1 \) and \( \ln(1 + q_{\epsilon,\eta}(\xi_1)^2) \geq |\ln(\epsilon)| \). Hence, restricting integration to \( R^{(n)}_{\epsilon,\eta} := R^{(n)}_{\epsilon,\eta}' \) from (3.18), we get
\[ \int_{R_{\epsilon}} |\xi_1|^{n-1}|\widehat{A}(\xi_1)|^2 \, d\xi_1 \lesssim N^{(n)}_{\epsilon}[u]. \tag{3.21} \]

In particular, for \( s < 1/2(n - 1) \), we get
\[ \int_{R_{\epsilon}} |\xi_1|^{2s}|\widehat{A}(\xi_1)|^2 \, d\xi_1 \leq 2|\widehat{A}|_{L^\infty} + \left( \sup_{\xi_1 \in R_{\epsilon}} |\xi_1|^{2s-(n-1)} \right) \int_{R_{\epsilon}} |\xi_1|^{n-1}|\widehat{A}(\xi_1)|^2 \, d\xi_1 \lesssim N^{(n)}_{\epsilon}[u] + 1. \tag{3.22} \]

The above calculations show that (i) holds with integration restricted to \( R^{(n)}_{\epsilon,\eta} \). To show a corresponding estimate on \( R^{(n)}_{\epsilon,\eta}' \), we use the further decomposition
\[ R^{(n)}_{\epsilon} = \bigcup_{k=0}^{N} I_k, \quad \text{where } I_k = \left\{ \xi_1 \in R^{(n)}_{\epsilon} : \left( \frac{1}{\epsilon} \right)^{\alpha_k} \leq |\xi_1| \leq \left( \frac{1}{\epsilon} \right)^{\alpha_{k+1}} \right\}, \]
\( I_0 := \{ \xi_1 \in R^{(n)}_{\epsilon} : 0 \leq |\xi_1| \leq \epsilon^{-\alpha_1} \} \) and \( I_N := \{ \xi_1 \in R^{(n)}_{\epsilon} : \epsilon^{-\alpha_N} \leq |\xi_1| \leq \infty \} \) and for a finite set of numbers \( 0 < \alpha_1 < \ldots < \alpha_N < \infty \) with \( N \in \mathbb{N} \) to be chosen in the sequel.

With the choice \( \alpha_N := 2/(1 - 2s) \) and by (3.19), we get
\[ \int_{I_N} |\xi_1|^{2s}|\widehat{A}(\xi_1)|^2 \, d\xi_1 \lesssim \left( \frac{P_\epsilon^{(n)}[u]}{\epsilon} \right)^2 \int_{\epsilon^{-\alpha_N}}^{\epsilon} |\xi_1|^{2s-2} \, d\xi_1 \lesssim \epsilon^{(1-2s)\alpha_N-2} \left( P_\epsilon^{(n)}[u] \right)^2. \tag{3.23} \]

It remains to show the corresponding estimates on \( I_k \) for \( k = 0, \ldots, N - 1 \). We give the proof separately for \( n = 2, 3 \) with a different choice of numbers \( \alpha_k \) for \( 2 \leq k \leq N - 1 \).

The case \( n = 2 \): in this case, we decompose \( R^{(n)}_{\epsilon,\eta} = I_0 \cup I_1 \), i.e. \( N = 1 \) and the estimate for \( I_1 = I_N \) has been given above. By construction, we have \( ||q_{\epsilon,\eta}||_{L^\infty(R^{(n)}_{\epsilon,\eta})} \leq 1 \). Since \( \arctan(q) \geq q \) for \( q \leq 1 \) from (3.18) and by definition of \( q_{\epsilon,\eta} \), we hence immediately get the estimate
\[ \int_{I_0} |\widehat{A}(\xi_1)|^{5/2} \, d\xi_1 \leq C_\eta \epsilon N^{(n)}_{\epsilon}[u]. \tag{3.24} \]

By Hölder’s inequality, it follows that
\[ \int_{I_0} |\xi_1|^{2s}|\widehat{A}(\xi_1)|^2 \, d\xi_1 \leq ||\xi_1||_{L^\infty(I_0)} |||\widehat{A}|^{5/2}|_{L^{4/5}(I_0)} ||I_0||_{1/3}^{1/3} \leq C_\epsilon \epsilon^{-2\alpha_1 - \frac{1}{2}\alpha_1 + \frac{3}{5}} (N^{(n)}_{\epsilon}[u])^{4/5}. \tag{3.25} \]

The right-hand side above is uniformly bounded in \( \epsilon \) if \( \alpha_1 := 4/(1 + 10s) \) and by Young’s inequality for \( \eta := 1/4\rho \) we get \( C_\eta (N^{(n)}_{\epsilon}[u])^{4/5} \leq C_\rho + CN^{(n)}_{\epsilon}[u] \). To close the estimate, we need \( \alpha_1 \geq \alpha_N \). That is, we need \( 4/(1 + 10s) \geq 2/(1 - 2s) \) or equivalently \( s \leq \frac{1}{14} \).
The case $n = 3$: for $\xi_1 \in R^c_{\epsilon, \eta}$, we have $|\hat{A}| \leq \eta^{-2} e^{2(1-\eta)}|\xi_1|^2$. With $\alpha_1 := (4(1 - \eta))/(5 + 6s)$, we hence get
\begin{equation}
\int_{I_1} |\xi_1|^2 |\hat{A}(\xi_1)|^2 \, d\xi_1 \lesssim \frac{e^{4(1-\eta)}}{\eta^4} \int_0^{\epsilon^{-1}} \xi_1^{4+2s} \, d\xi_1 \leq C_\eta e^{4(1-\eta)-(5+2s)\alpha_1} \leq C_\eta. \tag{3.25}
\end{equation}
It remains the estimates on $I_k$ for $k = 1, \ldots, N - 1$. Noting that $|I_k| \leq e^{-\alpha_{k+1}}$ and $e^{-\alpha_k} \leq |\xi_1| \leq e^{-\alpha_{k+1}}$ in $I_k$, estimate (3.18) implies
\begin{equation}
N^{(3)}_\epsilon[u] \gtrsim \frac{1}{|\ln \epsilon|} \int_{I_k} e^{-2\alpha_k} \ln(1 + \eta^2 e^{2\alpha_{k+1} - 2} |\hat{A}(\xi_1)|) |\hat{A}(\xi_1)|^2 \, d\xi_1 \\
\geq \frac{C_\eta}{|\ln \epsilon|} e^{2\alpha_{k+1} - 2 - 2\alpha_k} \int_{I_k} |\hat{A}(\xi_1)|^3 \, d\xi_1, \tag{3.26}
\end{equation}
using that $\ln(1 + y) \gtrsim y$ for $y \leq 1$. This holds if $e^{2\alpha_{k+1} - 2} |\hat{A}| \leq e^{2\alpha_{k+1} - 2} \leq 1$ for $\epsilon \leq 1$ if $\alpha_2 > 1$ and hence $\alpha_{k+1} > 1$ for $k \geq 1$ (this condition holds, see (3.28) below). Hence,
\begin{equation}
\int_{I_k} |\hat{A}(\xi_1)|^3 \, d\xi_1 \leq C_\eta |\ln \epsilon| e^{2(1+\alpha_k) - 2\alpha_{k+1}} N^{(3)}_\epsilon[u]. \tag{3.27}
\end{equation}
For $k = 1, \ldots, N - 1$, we hence get by Hölder’s inequality
\begin{align*}
\int_{I_k} |\xi_1|^2 |\hat{A}(\xi_1)|^2 \, d\xi_1 &\lesssim |\xi_1|^{2s} \|\hat{A}\|_{L_\infty(I_k)} |\xi_1|^{1/3} \|\hat{A}\|^2_{L^3(I_k)} \\
&\leq C_\eta |\ln \epsilon|^{2/3} e^{\frac{4}{3}(1+\alpha_k) - \frac{5}{3} + 2s\alpha_{k+1}} (N^{(3)}_\epsilon[u])^{2/3}.
\end{align*}
As in two dimensions, we then apply Young’s inequality to (3.27) and note that in order to be bounded, the exponent of $\epsilon$ has to be strictly positive. This motivates us to define
\begin{equation}
\alpha_{k+1} := \frac{4(\alpha_k + 1 - \eta)}{5 + 6s}. \tag{3.28}
\end{equation}
We note that for $\eta < \frac{15}{16} \left(\frac{1}{10} - s\right)$, this definition produces a monotonically increasing sequence of $\alpha_k$ such that $\alpha_2 > 1$ and $\alpha_{k+1} - \alpha_k \geq (15/(5 + 6s))(\frac{1}{10} - s) - (4/(5 + 6s))\eta > 0$. In particular, since $\alpha_2 > 1$ the Taylor approximation in (3.26) is justified. Furthermore, for $0 < s < \frac{1}{10}$ after finitely many iterations, we obtain $\alpha_{k+1} > \alpha_N$ which terminates the algorithm.

(ii) For $\xi_1 \in S_{\epsilon, \eta}$ and since $\eta < 1/4$, we have $q_{\epsilon, \eta}(\xi_1) \geq C_\eta \epsilon^{(\eta/2) - 1} \geq C_\eta$. This yields $\arctan(q_{\epsilon, \eta}(\xi_1)) \geq C_\eta$ and $\ln(1 + q_{\epsilon, \eta}(\xi_1)^2) \geq C_\eta |\ln \epsilon|$. Inserting these estimates into (3.18) yields assertion (ii).

We next give the proof of theorem 2.2 for $n = 2, 3$. Since the estimates have already been carried out in lemma 3.3, it remains to give the functional analytic argument based on weak compactness in $H^s$ together with an additional argument showing that the weak limit is in the (better) space $H^{(n-1)/2}$. The proof for $n = 4$ proceeds differently since the estimates on the set $R_{\epsilon, \eta}$ coming from the hypergeometric function obtained in lemma 3.2 no longer directly provide us with a control over a $H^s$-norm of $A$. Instead, an additional step must be carried out which we will detail in the appendix.

**Proposition 3.4 (Compactness for $n \in \{2, 3\}$).** Let $n = 2, 3$. For any sequence $u_\epsilon \in A$ with $E^{(n)}_\epsilon[u_\epsilon] \leq C$ and spt $u_\epsilon \subset B_\rho(0)$ for some $\rho > 0$ (up to a subsequence), we have
\begin{equation}
A[u_\epsilon] \rightharpoonup A \quad \text{in } H^s(\mathbb{R}) \text{ as } \epsilon \to 0, \tag{3.29}
\end{equation}
for any fixed $s \in [0, \frac{1}{10}]$ if $n = 2$ and $s \in [0, \frac{1}{16}]$ if $n = 3$ and for some $A \in A^{(n)}_0$.

**Proof.** Assume that $s$ satisfies the assumptions above. Since $u_\epsilon$ satisfies $\|u_\epsilon\|_{L_\infty(\mathbb{R})} \leq 1$ and $\|u_\epsilon\|_{L^\infty(\mathbb{R})} \leq 1$, we also have $\|u_\epsilon\|_{L^2} \leq 1$ and by the Banach–Alaoglu theorem, for a subsequence (still denoted by $u_\epsilon$) we have $u_\epsilon \rightharpoonup u$ in $L^2$ and $u_\epsilon \rightarrow u$ in $L^\infty$ for some $u \in L^\infty$. We write $A_\epsilon := A[u_\epsilon]$ and $A := A[u]$. By Fubini’s theorem, we then also get $A_\epsilon \rightarrow A$ in $L^2$ and $A_\epsilon \rightharpoonup A$ in $L^\infty$. From lemma
3.3(i), it follows that \( \|A_\varepsilon\|_{L^p} \leq C_{\rho, \delta} \) uniformly in \( \varepsilon \in (0, 1) \). After selection of a subsequence and by Banach–Alaoglu, we then have \( A_\varepsilon \rightharpoonup A \) in \( H^p \).

Since \( u_\varepsilon \subset B_\rho \) we have \( u \subset B_\rho \) and \( \int_{\mathbb{R}} u(x) \, dx = 1 \). It remains to show that \( A \in H^{(n-1)/2}(\mathbb{R}) \); we fix \( \eta := \frac{1}{4\rho} \). By lemma 3.3 (ii) and since \( \|A_\varepsilon\|_{L^2} \leq C_\rho \), we have the uniform estimate \( \|(\hat{A}_\varepsilon \chi_{S_\varepsilon, \eta})^\vee)\|_{H^{(n-1)/2}(\mathbb{R})} \leq C_\rho \). Since \( H^{(n-1)/2}(\mathbb{R}) \) is a reflexive space and by application of Banach–Alaoglu there exists \( B \in H^{(n-1)/2}(\mathbb{R}) \) such that, after selection of a subsequence

\[
(\hat{A}_\varepsilon \chi_{S_\varepsilon, \eta})^\vee \rightharpoonup B \quad \text{in} \quad H^{(n-1)/2}(\mathbb{R}) \quad \text{as} \quad \varepsilon \to 0.
\]

The global energy bound, together with the definition of \( S_\varepsilon, \eta \) in (3.16) yields, after extracting a subsequence, that \( |\hat{A}_\varepsilon \chi_{\mathbb{R} \setminus S_\varepsilon, \eta}| \to 0 \) a.e. as \( \varepsilon \to 0 \). Therefore, \( \chi_{\mathbb{R} \setminus S_\varepsilon, \eta} \to 0 \) a.e. as \( \varepsilon \to 0 \) by the Paley–Wiener theorem. Since \( ||\chi_{S_\varepsilon, \eta}||_{L^\infty} \leq 1 \) and \( A_\varepsilon \rightharpoonup A \), by lemma 3.5, we then have \( \hat{A}_\varepsilon \chi_{S_\varepsilon, \eta} \rightharpoonup \hat{A} \) in \( L^2 \) as \( \varepsilon \to 0 \). By (3.30) and the uniqueness of the weak limit, this implies \( A = B \in H^{(n-1)/2} \).

We have used the following auxiliary result in the proof of proposition 3.4:

**Lemma 3.5 (Auxiliary lemma).** Let \( g_n \in L^2(\mathbb{R}), f_n \in L^\infty(\mathbb{R}), n \in \mathbb{N}, \) be two sequences with \( ||f_n||_{L^\infty} \leq 1, f_n \to f \) pointwise a.e. and \( g_n \to g \) in \( L^2 \) for some \( g \in L^2(\mathbb{R}), f \in L^\infty(\mathbb{R}) \). Then

\[
f_n \cdot g_n \to f \cdot g \quad \text{in} \quad L^2(\mathbb{R}) \quad \text{as} \quad n \to \infty.
\]

**Proof.** For \( \varphi \in L^2(\mathbb{R}) \), we write

\[
|\langle \varphi, f_n g_n \rangle - \langle \varphi, f g \rangle| \leq ||g_n||_{L^2} |||\varphi f_n - f|||_{L^2} + ||\varphi f \cdot (g_n - g)|| =: I_1 + I_2.
\]

Since \( |\varphi(f_n - f)| \leq ||\varphi||_{L^2} \) we get \( I_1 \to 0 \) for \( n \to \infty \) by dominated convergence. Furthermore, since \( \varphi f \in L^2(\mathbb{R}) \) and by the weak convergence of \( g_n \) we have \( I_2 \to 0 \).

(c) **Proof of theorem 2.3**

The proof of theorem 2.3 follows from the liminf estimate in proposition 3.6 together with the construction of the recovery sequence in proposition 3.8. As in the proof of theorem 2.2, the liminf is based on the decomposition of the nonlinear energy from lemma 3.1. To get the optimal leading order constant for our estimate, we additionally take the limit \( \eta \to 0 \). For the upper bound, we use a constant recovery sequence in our rescaled setting (for sufficiently smooth limit function \( A \)).

**Proposition 3.6 (Liminf estimate).** Let \( n \in \{2, 3\} \). Then for any sequence \( u_\varepsilon \in A \) such that (2.14) holds, we have \( \liminf_{\varepsilon \to 0} E_\varepsilon^{(n)}[u_\varepsilon] \geq E_0^{(n)}[A] \).

**Proof.** To estimate the non-local part for both \( n = 2, 3 \), we will use (3.11) and restrict the integration in \( 4 \)-direction to the set \( S_{\varepsilon, \eta} \) defined in (3.16).

The case \( n = 3 \): Taking a subsequence, we can assume \( A_\varepsilon \to A \) a.e. By the assumption \( u_\varepsilon \subset B_\rho \), we also have \( A_\varepsilon \subset [-\rho, \rho] \). In view of theorem 2.2, we then get \( ||\sqrt{A_\varepsilon}||_{L^1} \to ||\sqrt{A}||_{L^1} \). By the isoperimetric inequality, this implies

\[
P_\varepsilon^{(n)}[u_\varepsilon] \geq \int_{\mathbb{R}^3} |
abla' u_\varepsilon(x)| \, dx \geq 2\sqrt{\pi} \int_{\mathbb{R}} \sqrt{A_\varepsilon(x_1)} \, dx_1 \to 2\sqrt{\pi} \int_{\mathbb{R}} \sqrt{A(x_1)} \, dx_1.
\]

For the estimate of the non-local term, we assume that \( \eta \in (0, \frac{1}{4}) \) is small enough such that \( 1 - \rho^2 \eta^2 > 0 \) in (3.3). Using (3.11) and since by lemma 3.3 (ii), it holds that \( (\hat{A}_\varepsilon \chi_{S_\varepsilon, \eta})^\vee \rightharpoonup A \) in \( H^1 \) for
\[ \epsilon \to 0 \text{ for fixed } \eta, \text{ we get} \]
\[
\liminf_{\epsilon \to 0} N^\epsilon_n[u_\epsilon] \geq \liminf_{\epsilon \to 0} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left( 1 - \rho^2 \eta^2 \right) e^{\int \frac{\xi^2}{2 \rho^2} \frac{\xi^2}{2} + \frac{\xi^2}{2} d\xi'} d\xi_1
\]

(3.3)

\[
\liminf_{\epsilon \to 0} \frac{1 - \rho^2 \eta^2}{2\pi} \int_{\mathbb{R}^n} \left( 1 - \rho^2 \eta^2 \right) \frac{\ln(1 + (\eta^2 - 2) \liminf_{\epsilon \to 0} \int_{S_{\eta,\rho}} \xi_1^2 |A_\epsilon(\xi_1)|^2 d\xi_1}
\]

(3.11)

\[
\geq \frac{1}{28\pi} (2 - \eta^2) \liminf_{\epsilon \to 0} \int_{S_{\eta,\rho}} \xi_1^2 |A_\epsilon(\xi_1)|^2 d\xi_1
\]

(3.34)

\[
\geq \frac{1}{28\pi} (2 - C_\rho \eta) \int_{\mathbb{R}^n} \xi_1^2 |A_\epsilon(\xi_1)|^2 d\xi_1,
\]

(3.35)

by the weak lower semicontinuity of the \(H^1\)-norm. Combining (3.33) with (3.35) yields the lower bound for \(E^\epsilon_n[u_\epsilon] \) for \( \eta \to 0 \).

The case \( n = 2 \): the perimeter of any measurable subset of \( \mathbb{R} \) with positive measure is estimated from below by 2. In particular, if \( A_\epsilon(x_1) > 0 \) this implies that \( \int_{\mathbb{R}^2} |\nabla u_\epsilon(x, x')| dx' \geq 2 \). By the lower semicontinuity of the measure of the positivity set, this implies

\[
\liminf_{\epsilon \to 0} \int_{\mathbb{R}^2} |\nabla u_\epsilon(x)| dx \geq 2 |\{ x \in \mathbb{R} : A(x) > 0 \}|
\]

(3.36)

As before we assume that \( \eta \in (0, \frac{1}{4}) \) is small enough such that \( 1 - \rho^2 \eta^2 > 0 \) in (3.3). With (3.11) and the definition of \( S_{\epsilon,\rho} \) in (3.16), we get

\[
\liminf_{\epsilon \to 0} N^\epsilon_n[u_\epsilon] \geq \liminf_{\epsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}^n} \left( 1 - \rho^2 \eta^2 \right) \frac{\xi_1^2}{2 |\xi| |A_\epsilon(\xi)|^2} d\xi_1
\]

(3.11)

\[
\geq \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{2} |\xi_1| |A_\epsilon(\xi)|^2 d\xi_1,
\]

(3.37)

since \((A_\epsilon \chi_{S_{\epsilon,\rho}})' \to A \) in \( H^{1/2} \) (by lemma 3.3 (ii)), by weak lower semicontinuity of the \(H^{1/2}\)-norm and since \( \arctan(\epsilon^{(n/2)-1}) \to \pi/2 \) for \( \epsilon \to 0 \). The result follows by adding (3.36) and (3.37) and taking the limit \( \eta \to 0 \).

Before we give the proof of the recovery sequence, we derive a series representation for the Fourier transform of a symmetric characteristic function. In our application, we will consider functions of the form \( A(x_1) = \omega_{n-1} \rho^{n-1}(x_1) \):

**Lemma 3.7 (Fourier transform for rotationally symmetric set).** Let \( n \in \mathbb{N} \). For \( u(x_1, x') := \chi_{(0,\rho(x_1))}(|x'|) \) for \( \rho \in C_c^\infty(\mathbb{R}, [0, \infty)) \), we have

\[
\widehat{\chi}(\xi_1, \xi') = \frac{\omega_{n-1}}{(2\pi)^{n/2}} \sum_{k=0}^{\infty} \frac{(-1)^k |\xi|^{2k}}{k! \Gamma(k + 1 + (n-1)/2)2^{2k+(n-1)/2}} \int_{\text{spt} \rho} e^{-i\xi_1 \cdot \rho(x_1)} 2^{k+n-1} dx_1
\]

(3.38)

and there exists a constant \( C < \infty \) such that

\[
|\widehat{\chi}(\xi_1, \xi')| \leq C |\xi|^{-n/2} \int_{\text{spt} \rho} \rho^{(n/2)-1}(x_1) dx_1.
\]

(3.39)

**Proof.** We apply the Hankel transform to the radially symmetric function \( u(\cdot, \xi') \) for fixed \( x_1 \in \mathbb{R} \) (cf. [66], B.5). This yields

\[
\widehat{\chi}(\xi_1, \xi') = \frac{\omega_{n-1}}{(2\pi)^{n/2}} \int_{\text{spt} \rho} e^{-i\xi_1 \cdot \rho(x_1)} \frac{1}{|\xi|^{n-3/2}} \int_{0}^{\infty} \sigma^{(n-1)/2} \chi_{(0,\rho(x_1))}(\sigma) J_{(n-3)/2}(\xi \sigma) d\sigma dx_1
\]

\[
= \frac{\omega_{n-1}}{(2\pi)^{n/2}} \frac{1}{|\xi|^{(n-1)/2}} \int_{\text{spt} \rho} e^{-i\xi_1 \cdot \rho(x_1)} 2^{k+n-1} dx_1.
\]

(3.40)
Here, $I_{\alpha} = \sum_{k=0}^{\infty} (-1)^k (k! \Gamma(k + 1 + \alpha))^{-1} (z/2)^{2k + \alpha}$ is the Bessel function of first kind of order $\alpha \in \mathbb{Q}$. With this series representation (3.38) follows. The bound (3.39) follows from (3.40) by applying the estimate $J_{\alpha}(t) \lesssim t^{-1/2}$ for $\alpha = (n - 1)/2 > 0$.

**Proposition 3.8 (Recovery sequence).** For any $A \in A_{0}^{(n)}$, $n \in \{2, 3\}$, there is a sequence $u_{\epsilon} \in A$ such that (2.14) holds and $\lim sup_{\epsilon \to 0} E_{\epsilon}^{(n)}[u_{\epsilon}] \leq E_{0}^{(n)}[A]$.

**Proof.** By approximation, we assume $A \in C_{c}^{\infty}(\mathbb{R})$, the result in the general case then follows by taking a diagonal sequence. We choose the constant recovery sequence $u_{\epsilon} := u \in A$ where $u(x) := \chi_{(0, \rho(x))}(|x|)$ with $\rho$ given by $A(x) = \omega_{n-1} \rho^{n-1}(x)$. The perimeter terms are easily estimated

$$P_{\epsilon}^{(n)}[u] = \int_{\mathbb{R}^2} \sqrt{(|\nabla u(x)|^2 + \epsilon^2 |\partial_t u(x)|^2)} \, dx \quad (3.41)$$

$$\leq \begin{cases} 2||A > 0|| + C\epsilon |A|_{W^{1,1}(\mathbb{R})} & \text{for } n = 2, \\ \sqrt{\pi} \int_{\mathbb{R}} \sqrt{A(x)} \, dx + C\epsilon & \text{for } n = 3. \end{cases} \quad (3.42)$$

For the estimate of the non-local term $\Lambda_{\epsilon}^{(n)}[u_{\epsilon}]$, we recall that according to lemma 3.7 the Fourier transform of $u$ is given by

$$\hat{u}(\xi) = \frac{\omega_{n-1}}{(2\pi)^{n/2}} \sum_{k=0}^{\infty} \frac{(-1)^k |\xi'|^{2k}}{k! \Gamma(k + 1 + (n - 1)/2)} S_{k}(\xi_{1}), \quad (3.43)$$

where $S_{k} := (A^{(2k/(n-1))+1}) \wedge$ are Schwartz functions. In particular, $S_{0} = \hat{A}$. We decompose the Fourier space into two domains, i.e. $\mathbb{R}^n = \mathcal{D} \cup \mathcal{D}^{c}$, where $\mathcal{D} := \{(\xi_{1}, \xi') \in \mathbb{R}^n : |\xi_{1}| \geq \sigma := 1 + |\xi_{1}|^2\}$ and we claim that the contribution of this set to the non-local energy is zero for $\epsilon \to 0$. From (3.39) in lemma 3.7, we know that $|\hat{u}(\xi)|^2 \lesssim 1/|\xi'|^n$ since $A$ is bounded. With $m_{\epsilon}(\xi) := \xi_{1}^2 / (\epsilon^2 \xi_{1}^2 + |\xi'|^2)$ this implies

$$\int_{\mathcal{D}} m_{\epsilon}(\xi) |\hat{u}(\xi)|^2 \, d\xi \lesssim \int_{\mathcal{D}} \int_{\sigma}^{\infty} \frac{\xi_{1}^2}{\epsilon^2 \xi_{1}^2 + |\xi'|^2} |\xi'|^n \, d\xi' \, d\xi_{1} \lesssim \int_{\mathcal{D}} \frac{\xi_{1}^2}{\sigma^3} \, d\xi_{1}. \quad (3.44)$$

Since we chose $\sigma = 1 + |\xi_{1}|^2$, the term $\xi_{1}^2 \sigma^{-3}$ is integrable and thus

$$\gamma_{n}(\epsilon) \int_{\mathcal{D}} m_{\epsilon}(\xi) |\hat{u}(\xi)|^2 \, d\xi \lesssim \gamma_{n}(\epsilon). \quad (3.44)$$

The second domain is the set $\mathbb{R}^n \setminus \mathcal{D}$. In this region, we can neglect higher order terms in $|\xi'|$. For $k \geq 1$, we have by (3.12) of lemma 3.2 for $\beta = \sigma$

$$\int_{|\xi'| \leq \sigma} m_{\epsilon}(\xi) |\xi'|^{2k} |\xi'| \, d\xi' \lesssim |\xi|^{-\sigma} \sigma^{2k+n-3} + \epsilon |\xi|^{-3} \sigma^{2k+n-4}. \quad (3.45)$$

With (3.43) and since $A$ is smooth, (3.43) implies

$$\gamma_{n}(\epsilon) \int_{\mathcal{D}^{c}} m_{\epsilon}(\xi) |\hat{u}(\xi)|^2 \, d\xi \leq \frac{\gamma_{n}(\epsilon)}{(2\pi)^{n-1}} \int_{\mathcal{D}^{c}} m_{\epsilon}(\xi) |\hat{A}(\xi)|^2 \, d\xi + \gamma_{n}(\epsilon)C_{A}. \quad (3.46)$$

Evoking lemma 3.2 once again with $\beta = 1 + \xi_{1}^2$ we get by (3.11) for $n = 2$, resp. $n = 3$

$$\gamma_{n}(\epsilon) \int_{\mathcal{D}^{c}} m_{\epsilon}(\xi) |\hat{A}(\xi)|^2 \, d\xi = \begin{cases} \frac{1}{\pi} \int_{\mathbb{R}} |\xi_{1}| \arctan \frac{1+\xi_{1}^2}{\pi |\xi_{1}|} |\hat{A}(\xi)|^2 \, d\xi_{1}, & \text{for } n = 2, \\ \frac{1}{28|\ln \epsilon|} \int_{\mathbb{R}} \xi_{1}^2 \ln \left(1 + \frac{1+\xi_{1}^2}{\pi |\xi_{1}|} \right) |\hat{A}(\xi)|^2 \, d\xi_{1}. & \text{for } n = 3. \end{cases} \quad (3.47)$$

Passing to the limit $\epsilon \to 0$ in the perimeter estimate (3.41) and the estimates for the non-local term (3.44), (3.46) and (3.47) yields the assertion for $A_{\epsilon} \in C_{c}^{\infty}(\mathbb{R})$. ■
(d) Proof of theorem 2.4

In this section, we calculate the solutions of the limit problems (figure 3), thus giving the proof of theorem 2.4. Both energies $E_n$, $n = 2, 3$, from (2.15) to (2.16) are invariant under translation. In the following, we separately consider the cases $n = 2, 3$ and thus omit the superscript $(n)$ for the sake of readability.

The case $n = 2$: from the energy $E_2$, it directly follows that the positivity set of any minimizing sequence must be bounded and connected. Indeed, a monotonicity argument shows that, otherwise, shifting the separate components together decreases the energy (see lemma A.3). Among functions with fixed positivity set, the energy is strictly convex and hence there exists a unique minimizer among this class of functions. We first consider the case when the positivity set is given by $I := (-1, 1)$. We get

$$0 = \pi \frac{d}{d\eta} \bigg|_{\eta = 0} ||A + \eta \psi||^2_{H^{1/2}} = \int_{\mathbb{R}} |\tau \hat{A}(\tau) \hat{\psi}(\tau)| \, d\tau = \int_{\mathbb{R}} H(A'(t)) \psi(t) \, dt$$

(3.48)

for all $\psi \in C^\infty_c(I)$ with $\int_I \psi(t) \, dt = 0$. Here, $H$ is the Hilbert transform given by

$$H(f)(s) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(t)}{s-t} \, dt.$$  

(3.49)

Identity (3.48) implies that $H(A') = C_1$ on $I$ for some constant $C_1 \in \mathbb{R}$ where $C_1$ is determined by the volume constraint. For fixed $I$, this uniquely determines $A$ ([67], eq.(45)), see also ([68], Thm. 1) for the complementary case that $f$ and $H(f)$ are known on disjoint intervals. We claim that

$$A(t) = \frac{2}{\pi} \sqrt{1-t^2} \chi_I(t).$$  

(3.50)

Obviously, condition (2.12) then holds. Furthermore, $A'(t) = -(2t/\pi) \phi(t)$ where $\phi(t) := (1/\sqrt{1-t^2}) \chi_I$. We recall that by ([69], (4.111)), we have

$$H(t \mapsto t \phi(t))(s) = -\frac{1}{\pi} \int_{\mathbb{R}} \phi(t) \, dt + s H(\phi)(s).$$  

(3.51)

Since $H(\phi)(s) = -(\text{sgn}(s)(1/\sqrt{s^2 - 1}) \chi_{\mathbb{R}\setminus I}(s)$ (cf. [70], (21)), (3.51) implies

$$H(A') = \frac{2}{\pi} \left(\frac{1}{\pi} \int_I \frac{1}{\sqrt{1-t^2}} \, dt - \frac{s \text{sgn}(s)}{\sqrt{s^2 - 1}} \chi_{\mathbb{R}\setminus I}(s)\right) = \frac{2}{\pi} \left(1 - \frac{s \text{sgn}(s)}{\sqrt{s^2 - 1}} \chi_{\mathbb{R}\setminus I}(s)\right),$$

which confirms (3.50) for prescribed positivity set $I$. By translation invariance, it is enough to consider intervals of the form $I_{L_0^n} := (-L_0^n, L_0^n)$ for some $L_0^n > 0$. With the same calculation as before, we see that the minimizer among the class of functions with positivity set $I_{L_0^n}$ is given by $A_{L_0^n}(L_0^n t) = (1/L_0^n) A(t)$ where $A$ is given by (3.50). We have $A_{L_0^n}(\tau) = \sqrt{\frac{2}{\pi L_0^n}} |\tau|^{1/2} \chi_{I_{L_0^n}}(\tau))$ ([70], p. 11 (8)), where $J_1$ is the Bessel function of the first kind. In particular, $||A_{L_0^n}||^2_{H^{1/2}} = 2/(\pi L_0^n)$ (cf. [71], 703) and hence $E_0^2[A_{L_0^n}] = 4L_0^n - 1/(\pi L_0^n)^2$. Minimizing in $L_0^n$ we obtain $L_0^n = (2/\pi)^{1/3}$ and hence $A_{L_0^n}(t) = (2/\pi)^{2/3} \sqrt{1 - |t/L_0^n|^2} \chi_I(\tau/L_0^n)$. Comparing this to (2.20) we find $R_0 = (1/(2\pi^2))^{1/3}$. The above calculation also shows that the interval length $L_0^n$ is uniquely determined by the minimization of the energy which—together with the fact that we have uniqueness among functions with fixed positivity set—with the previous arguments shows that the problem admits a unique minimizer (up to translation).

The case $n = 3$: we first consider the minimization problem for symmetrically decreasing functions with support on some fixed interval $[-L_0^n, L_0^n]$. By the direct method of the calculus of variations, there exists a minimizer $A$ in this class of functions. In terms of $R := R_0$, defined as in (2.20), the Euler–Lagrange equation takes the form

$$\frac{R^3}{L_0^n} (R^2 \psi''(t)) = \frac{7}{R(t)} - 2 \beta_0 \quad \text{in} \ [-1, 0],$$

(3.52)
for some Lagrange multiplier $\beta_0 > 0$ (the positivity follows from a simple rescaling argument) and $R_+ > 0$ satisfying $\pi R_+^2 = A(0)$. Multiplying (3.52) by $(R')^2 = 2RR'$, integrating and since $R' \geq 0$ in $[-1, 0]$, we get

$$R^2(R')^2 = \frac{L^2}{R_+^3} \left( 7R - \beta_0 R^2 + \frac{\alpha_0}{\beta_0} \right) \quad \text{for } t \in [-1, 0],$$  \hspace{1cm} (3.53)

for some $\alpha_0 \in \mathbb{R}$ with $(R')^2(-1) = (L^2/4R_+^3)(\alpha_0/\beta_0)$. Since $R(0) = 0$ and $R(0) = 1$ in (3.53), we get $\alpha_0 = \beta_0(\beta_0 - 7)$.

By explicit integration with $g(\rho) \coloneqq 7\beta_0(\rho - 1) - \beta_0^2(\rho^2 - 1)$, we get

$$t + 1 = \frac{R_+^{3/2}}{L_+\beta_0^{3/2}} \left[ \frac{7}{2} \left( \arctan \left( \frac{7}{2\sqrt{\alpha_0}} \right) - \arctan \left( \frac{7 - 2\beta_0 R}{2\sqrt{g(R)}} \right) \right) + \left( \sqrt{\alpha_0} - \sqrt{g(R)} \right) \right].$$  \hspace{1cm} (3.54)

It remains to determine the parameters $\beta_0, R_+, L_+$. Setting $t = 0$ in (3.54) and since $R(0) = 1$, we obtain

$$L_+ = \left( \frac{R_+}{\beta_0} \right)^{3/2} \left[ \frac{7}{2} \left( \arctan \left( \frac{7}{2\sqrt{\alpha_0}} \right) - \frac{\pi}{2} \right) + \sqrt{\alpha_0} \right].$$  \hspace{1cm} (3.55)

The condition $\pi R_+^2 L_+ \int_{\mathbb{R}} R^2 \, dt = 1$ reads

$$R_+^{-7/2} = 2\pi \int_0^1 \frac{R^3}{\sqrt{\alpha_0/\beta_0} + 7R - \beta_0 R^2} \, dR$$

$$= -\frac{\pi}{24\beta_0^{7/2}} \left[ -1470\sqrt{\alpha_0} + 32\alpha_0^{3/2} + (252\alpha_0 + 5145) \left( \arctan \left( \frac{7}{2\sqrt{\alpha_0}} \right) - \frac{\pi}{2} \right) \right].$$  \hspace{1cm} (3.56)

Since the minimizer is also optimal with respect to volume preserving rescalings of the form $R_+ = (1/\epsilon^{1/2}) R(t/\epsilon)$, we get the additional condition $0 = \pi R_+ L_+ \int R \, dt - (6\pi/7)(R_+^4/L_+) \int R^2(R')^2 dt$. Integrating (3.53), we can replace $\int R^2(R')^2 dt$ and an explicit integration of $\int R \, dt$ yields

$$0 = R_+^{7/2} \left( -\frac{105}{2} \sqrt{\alpha_0} + 5 \left( \frac{\alpha_0 + 147}{4} \right) \left( \frac{\pi}{2} - \arctan \left( \frac{7}{2\sqrt{\alpha_0}} \right) \right) \right) - \frac{12}{7} R_+^2 L_+ \alpha_0^{3/2} + \frac{6}{7\pi} \beta_0^{7/2}.$$

Inserting (3.55) and (3.56) into (3.57), we then obtain the unique solution $\alpha_0 \approx 104.332$ for $\alpha_0$. Substituting this into (3.55) and (3.56), we obtain $\beta_0 \approx 14.297$, $R_+ \approx 1.511$ and $L_+ \approx 0.202$. From (3.53), it follows that $R' \neq 0$ for $t \in [-1, 1]\setminus\{0\}$.

In view of ([72], ch. 3) and the Pólya–Szegö inequality ([73], Theorem 1.1), see also ([74], ch. III), symmetric rearrangement does not increase the energy. In fact, since our minimizer satisfies $|\{ x : A'(x) = 0 \} \cap \text{spt}(A) | = 0$, any configuration whose symmetric rearrangement is given by this minimizer has strictly larger energy ([75], Thm. 1). Hence, our arguments show that within the class of functions with bounded support, the unique minimizer, up to translation, is given by $A$. Since the energy of any function $u \in \mathcal{A}$ can be approximated by the energy of functions with bounded support, this shows that $A$ is a minimizer of the energy within the class of functions $\mathcal{A}$ and that this minimizer is unique, up to translation, within the class of functions with bounded support. Furthermore, any minimizer satisfies the Euler–Lagrange equation. In turn, a simple calculation shows that solutions of the Euler–Lagrange equation with finite mass necessarily have compact support. Indeed, this follows since the ordinary differential inequality $A' \leq -C\sqrt{A}$ does not allow for solutions with finite mass and unbounded support as a straightforward calculation shows.

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Appendix A

In the appendix, we give the proof of theorem 2.1 and theorem 2.2 for \( n = 4 \) as well as some auxiliary estimate for the \( H^{3/2} \)-norm used in the proof of theorem 2.4.

**Lemma A.1 (Compactness for \( n = 4 \)).** Let \( n = 4 \). Then for any sequence \( u_\epsilon \in A \) with \( \text{spt } u_\epsilon \subset B_\rho(0) \) for some \( \rho > 0 \), \( E^{(4)}_{\epsilon}[u_\epsilon] \leq 1 \) and \( 0 \leq s < \frac{1}{2} \), there is a subsequence (not relabelled) and a function \( A \in A^{(4)}_\rho \) such that

\[
A[u_\epsilon] \rightarrow A \quad \text{in } H^1(\mathbb{R}) \text{ as } \epsilon \rightarrow 0. \tag{A 1}
\]

Although the proof is only concerned with \( n = 4 \), we will carry out steps in full generality, i.e. \( n \geq 4 \) whenever possible since this gives a hint for the structure of the estimates and the loss of regularity for \( n \geq 4 \).

**Proof.** Using estimate (3.12) of lemma 3.2 for \( k = 0 \) and \( \beta = \eta|\widehat{A}_\epsilon(\xi)|^{1/2} \), we get for \( n \geq 4 \) and \( q_{\epsilon,\eta}(\xi_1) := \eta|\widehat{A}_\epsilon|^{1/2}/(\epsilon|\xi_1|) \geq 1 \)

\[
y_\eta(\epsilon) \int_{|\xi_1| \leq \eta|\widehat{A}_\epsilon(\xi)|^{1/2}} \frac{\xi_1^2}{\epsilon^{2} + |\xi'|^2} \, d\xi' = \frac{\eta n - 2}{n - 3} \xi_1^2 \eta|\widehat{A}_\epsilon(\xi_1)|^{(n-3)/2} + R,
\]

where \( R = \epsilon O(\eta^{-4}|\xi_1|^3|\widehat{A}_\epsilon(\xi_1)|^{(n-4)/2}) \). This implies that, by lemma 3.1

\[
N(\epsilon) \geq C_\eta \int_{\mathbb{R} \cap |q_{\epsilon,\eta}| \geq 1} \xi_1^2 |\widehat{A}_\epsilon(\xi_1)|^{(n+1)/2} \, d\xi_1. \tag{A 3}
\]

for \( \epsilon \) small enough. Our goal is to bound some \( H^s \)-norm of \( A_\epsilon \). Setting \( p = (n + 1)/4 \) and \( q = (n + 1)/(n - 3) \), we have \( p^{-1} + q^{-1} = 1 \) and for \( \alpha > (n - 3)/(n + 1) \), it follows by Hölder’s inequality

\[
\int_{(-1,1) \cap |q_{\epsilon,\eta}| \geq 1} \xi_1^{2s} |\widehat{A}_\epsilon(\xi_1)|^2 \, d\xi_1 
\leq \left( \int_{(-1,1) \cap |q_{\epsilon,\eta}| \geq 1} \xi_1^{(2s+\alpha)p} |\widehat{A}_\epsilon(\xi_1)|^{2p} \, d\xi_1 \right)^{1/p} \left( \int_{(-1,1) \cap |q_{\epsilon,\eta}| \geq 1} \xi_1^{-\alpha q} \, d\xi_1 \right)^{1/q}. \tag{A 4}
\]

By our choice of \( p \), we have \( 2p = (n + 1)/2 \) and since \( \alpha > (n - 3)/n + 1 \) it holds that \( \alpha q > 1 \), so the second integral converges. To apply our estimate (A 3), we require \( (2s + \alpha)p \leq 2 \), which is equivalent to \( 2s < 8/(n + 1) - ((n - 3)/(n + 1)) = (11 - n)/(n + 1) \). This yields compactness of \( (\widehat{A}_\epsilon(\xi)) \) for some \( s > 0 \) in all dimensions \( 4 \leq n \leq 10 \). For the complementary estimate, i.e. if \( q_{\epsilon,\eta}(\xi_1) \leq 1 \), we proceed as in the proof of lemma 3.3 for \( n = 2, 3 \) to find that \( \int_0^{\infty} |\xi_1|^{2s} |\widehat{A}_\epsilon| \, d\xi_1 \) is bounded, provided \( \gamma \geq \frac{1}{2} \).

Furthermore, since \( q_{\epsilon,\eta}(\xi_1) \leq 1 \), we have the lower bound

\[
y_\eta(\epsilon) \int_{|\xi_1| \leq \eta|\widehat{A}_\epsilon(\xi)|^{1/2}} \frac{\xi_1^2}{\epsilon^{2} + |\xi'|^2} \, d\xi' \geq \epsilon^{-3} |\xi_1|^{n-1} \frac{(\eta|\widehat{A}_\epsilon|^{1/2}/\epsilon|\xi_1|)^{n-1}}{\epsilon^2},
\]

from which it follows that

\[
\int_{R \cap |q_{\epsilon,\eta}| \geq 1} |\widehat{A}_\epsilon|^{(n+3)/2} \, d\xi_1 \leq C_\eta \epsilon^2.
\]
We can estimate by Hölder’s inequality
\[
\left\| \left( \mathcal{A}_\varepsilon \chi_{[0,\varepsilon^{-1}) \cap [q_\varepsilon t \pm 1]} \right) \right\|_{H^2}^2 \lesssim \left( \sup_{\xi \in [0,\varepsilon^{-1})} |\xi_1|^2 \right) |\mathcal{A}_\varepsilon \chi_{[q_\varepsilon t \pm 1]}|_{L^2}^2 \varepsilon^{-(n-1)/(n+3)}(A5)
\]
for \( \gamma \leq 8/(2s(n+3) + n - 1) \). Both inequalities for \( \gamma \) yield together that \( s < (5-n)/(2(n+7)) \), in particular \( s < \frac{1}{20} \) for \( n = 4 \). Note that this estimate does not provide us with control over an \( H^\varepsilon \) norm for the part \( q_\varepsilon \gamma \geq 1 \) with positive \( s \) for dimensions \( n \geq 5 \). The better regularity for the limit function \( \mathcal{A} \in \mathcal{A}_0^{(n)} \) follows as in Proposition 3.4 by using (A2) and lemma 3.5. Since \( \gamma_n(e) = 1 \) for \( n \geq 4 \), we can take \( S_\varepsilon = R_\varepsilon \) in the notation of lemma 3.3.

**Lemma A.2.** The lower bound in theorem 2.1 holds for \( n = 4 \), the upper bound holds for all \( n \geq 4 \).

**Proof.** Assume that the infimum in (2.10) for \( n = 4 \) is zero. By (A3) and (A4) (for \( s = 0 \)), this would imply that \( |\mathcal{A}_\varepsilon|_{L^2} \) converges to zero, i.e. \( |\mathcal{A}_\varepsilon|_{L^2} \to 0 \) and since the support is bounded, also \( |\mathcal{A}_\varepsilon|_{L^1} \to 0 \). But this would contradict the assumption that \( |\mathcal{A}_\varepsilon|_{L^1} = 1 \). Hence, the infimum is strictly positive.

For the upper bound in (2.10), it is enough to construct a sequence with uniformly bounded energy. So let \( n \geq 4 \) and \( \mathcal{A} \in C_0^\infty(\mathbb{R}) \) with \( |\mathcal{A}|_{L^\infty} \leq 1 \) and let \( \mathcal{A} \) be given as in lemma 3.7. Then obviously the perimeter \( P^{(n)}_\varepsilon \) is bounded. For the non-local energy, we can estimate as for (3.44) for \( \varepsilon := 1 + |\xi_1|^2 \)
\[
\int_{\mathbb{R}} \int_{|\xi'| \geq \varepsilon} \frac{\xi_1^2}{\xi_1^2 + |\xi'|^2} |\hat{u}(\xi)|^2 \, d\xi' \, d\xi_1 \lesssim 1.
\]

For the remaining part of the non-local energy, we note that by lemma 3.7
\[
|\hat{u}(\xi)|^2 = \frac{(2\pi)^n}{\omega_n} \sum_{k=0}^\infty \left( -1 \right)^k \sum_{\ell=0}^{k} \frac{\xi_1^2}{\xi_1^2 + |\xi'|^2} |\hat{u}(\xi)| \, d\xi' \sim \xi_1^2 |\hat{u}(\xi)| \, d\xi' \sim \xi_1^2 \sigma^{-2} |\hat{u}(\xi)| \, d\xi' \sim \xi_1^2 \sigma^{-2} \int_{\text{sp}(A)} e^{-ix_1 \xi} |A(x_1)|^{1/2} f_{(n-1)/2} (\sigma |A(x_1)|^{1/(n-1)}) \, dx_1 \, d\xi_1,
\]
where the last step is due to the calculation in lemma 3.7. We decompose \( \mathbb{R} \times \text{sp}(A) = \{ (\xi_1, x_1) : \sigma |A(x_1)|^{1/(n-1)} \geq 1 \} \cup \{ (\xi_1, x_1) : \sigma |A(x_1)|^{1/(n-1)} \leq 1 \} \) and since for \( \alpha \in \frac{1}{2} \text{N} \), we have \( |J_\alpha(t)| \lesssim t^{-1/2} \) for \( t \geq 1 \) and \( 1^\alpha \) for \( t \leq 1 \), we get
\[
\int_{\mathbb{R}} \int_{|\xi'| \leq s} \frac{\xi_1^2}{\xi_1^2 + |\xi'|^2} |\hat{u}(\xi)|^2 \, d\xi' \, d\xi_1 \sim \int_{\text{sp}(A)} \xi_1^2 \sigma^{-2} \int_{\text{sp}(A)} e^{-ix_1 \xi} |A(x_1)|^{1/2} f_{(n-1)/2} (\sigma |A(x_1)|^{1/(n-1)}) \, dx_1 \, d\xi_1 \lesssim |\text{sp}(A)| \int_{\mathbb{R}} \xi_1^2 \sigma^{-3} \, d\xi_1 + \int_{\mathbb{R}} \xi_1^2 \sigma^{-3} |\mathcal{A}(\xi_1)| \, d\xi_1.
\]
These integrals are finite since \( \xi_1^2 (1 + \xi_1^2)^{-3} \) is integrable and \( A \) is smooth.

\[\blacksquare\]
We note that the \( H^{1/2} \)-norm of \( A \) can also be expressed (up to a constant which only depends on \( s \) and the space dimension, see [76], Prop. 3.4) as

\[
||A||^2_{H^{1/2}(\mathbb{R}^n)} = C_{s,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |A(x) - A(y)|^2 |x - y|^{n+1} \, dx \, dy,
\]

With this identity, we prove the following auxiliary estimate that has been used in the proof of theorem 2.4:

**Lemma A.3.** Let \( \phi, \psi \in H^{1/2}(\mathbb{R}) \). Assume that \( A := \phi + \psi \) with \( \phi, \psi \geq 0 \) with \( \{ \phi > 0 \} \subset (-\infty, 0] \) and \( \{ \psi > 0 \} \subset [\tau, \infty) \) for some \( \tau \geq 0 \). Then

\[
||A||^2_{H^{1/2}(\mathbb{R}^n)} > ||A_t||^2_{H^{1/2}(\mathbb{R})},
\]

where \( A_t(x) := \phi(x) + \psi(x + t) \) for \( t \in (0, \tau] \).

**Proof.** A direct calculation shows that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |A_t(x) - A_t(y)|^2 |x - y|^2 \, dx \, dy = -4 \frac{d}{dt} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x) \psi(y + t) |x - y|^2 \, dx \, dy,
\]

using that \( \phi \) and \( \psi \) have disjoint support for \( t \leq \tau \). Without loss of generality, we can assume \( \tau = \text{dist}(\{ \phi > 0 \}, \{ \psi > 0 \}) \). With the change of variables \( z = y + t \), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x) \psi(y + t) |x - y|^2 \, dx \, dy = \frac{d}{dt} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x) \psi(z) |x - z|^2 |x - z + t|^2 \, dx \, dz
\]

\[
= -2 \int_{\text{spt}(\psi)} \int_{\text{spt}(\phi)} (x - z + t) \frac{\phi(x) \psi(z)}{|x - z + t|^4} \, dx \, dz.
\]

Since \( \phi, \psi \geq 0 \) and \( x - z \leq -\tau \), the above expression stays strictly positive as long as \( t \in (0, \tau) \). \( \blacksquare \)

### References

1. DeSimone A, Kohn RV, Müller S, Otto F. 2000 Magnetic microstructures - a paradigm of multiscale problems. *ICIAM* 99, 175–190.
2. Desimone A, Kohn RV, Müller S, Otto F. 2002 A reduced theory for thin-film micromagnetics. *Commun. Pure Appl. Math.* 55, 1408–1460. (doi:10.1002/cpa.3028)
3. Fabian K, Hubert A. 1999 Shape-induced pseudo-single-domain remanence. *Geophys. J. Int.* 138, 717–726. (doi:10.1046/j.1365-246x.1999.00916.x)
4. Geiß C, Heider F, Soffel H. 1996 Magnetic domain observations on magnetite and titanotrichmelliné grains (0.5–10 µm). *Geophys. J. Int.* 124, 75–88. (doi:10.1111/j.1365-246X.1996.tb06353.x)
5. Pokhail T, Moskowitz B. 1997 Magnetic domains and domain walls in pseudo-single-domain magnetite studied with magnetic force microscopy. *J. Geo. Res.* 102, 22 681–22 694. (doi:10.1029/97JB01856)
6. Strukov B.A., Strukov A.P. 1998 *Ferroelectric phenomena in crystals*. Berlin, Heidelberg: Springer.
7. Banerjee S, Widom M. 2001 Shapes and textures of ferromagnetic liquid droplets. *Braz. J. Phys.* 31, 360–365. (doi:10.1590/S0103-97332001000300005)
8. Clark NA. 2013 Ferromagnetic ferrofluids. *Nature* 504, 229–230. (doi:10.1038/504229a)
9. Roodan VA, Gómez-Pastora J, Karampelas IH, González-Fernández C, Bringas E, Ortiz I, Chalmers JJ, Furlani EP, Swihart MT. 2020 Formation and manipulation of ferrofluid droplets with magnetic fields in a microdevice: a numerical parametric study. *Soft Matter* 16, 9506–9518. (doi:10.1039/D0SM01426E)
10. Brusentsov NA, Gogosov V, Brusentsova T, Sergeev A, Jurchenko N, Kuznetsov AA, Kuznetsov OA, Shumakov L. 2001 Evaluation of ferromagnetic fluids and suspensions for the site-specific radiofrequency-induced hyperthermia of MX11 sarcoma cells in vitro. *J. Magn. Magn. Mater.* 225, 113–117. (doi:10.1016/S0304-8853(00)01238-5)
11. Hess AJ, Liu Q, Smalyukh II. 2015 Optical patterning of magnetic domains and defects in ferromagnetic liquid crystal colloids. *Appl. Phys. Lett.* 107, 071906. (doi:10.1063/1.4928552)
12. Li Q, Kartikowati CW, Horie S, Ogi T, Iwaki T, Okuyama K. 2017 Correlation between particle size/domain structure and magnetic properties of highly crystalline Fe₃O₄ nanoparticles. Sci. Rep. 7, 1–7. (doi:10.1038/s41598-016-0028-x)

13. Newbower R. 1973 Magnetic fluids in the blood. IEEE Trans. Mag. 9, 447–450. (doi:10.1109/TMAG.1973.1067671)

14. Ochoński W. 1989 Dynamic sealing with magnetic fluids. Wear 130, 261–268. (doi:10.1016/0043-1648(89)90238-X)

15. Szczech M, Horak W. 2015 Tightness testing of rotary ferromagnetic fluid seal working in water environment. Ind. Lubr. Tribol. 67, 455–459. (doi:10.1108/ILT-02-2015-0014)

16. Uhlmann E, Spur G, Bayat N, Patzwald R. 2002 Application of magnetic fluids in tribotechnical systems. J. Magn. Magn. Mater. 252, 336–340. (doi:10.1016/S0304-8853(02)00724-2)

17. Murakami Y, Kasai H, Kim JJ, Mamishin S, Shindo D, Mori S, Tonomura A. 2009 Ferromagnetic domain nucleation and growth in colossal magnetoresistive manganite. Nat. Nanotechnol. 5, 37–41. (doi:10.1038/nnano.2009.342)

18. Knüpfer H, Nolte F. 2018 Optimal shape of isolated ferromagnetic domains. SIAM J. Math. Anal. 50, 5857–5886. (doi:10.1137/18M1175719)

19. Stantejsky D. 2018 A model problem in micromagnetics: scaling, γ-limit and shape of minimizers. Master’s thesis, University of Heidelberg.

20. Hubert A, Schöfer R. 1998 Magnetic domains. Berlin, Heidelberg: Springer.

21. Anzellotti G, Baldo S, Visintin A. 1991 Asymptotic behavior of the Landau-Lifshitz model of ferromagnetism. Appl. Math. Optim. 23, 171–192. (doi:10.1007/BF01442396)

22. Rosensweig R. 1985 Ferrohydrodynamics, vol. 1. Cambridge, UK: Cambridge University Press.

23. Nolte F. 2018 Optimal scaling laws for domain patterns in thin ferromagnetic films with strong perpendicular anisotropy. PhD thesis, Ruprecht Karl University of Heidelberg, Heidelberg, Germany.

24. Demengel F, Demengel G. 2012 Functional spaces for the theory of elliptic partial differential equations. London, UK: Springer.

25. Knüpfer H, Muratov C. 2011 Domain structure of bulk ferromagnetic crystals in applied fields near saturation. J. Nonlinear Sci. 21, 921–962. (doi:10.1007/s00332-011-9105-2)

26. Candau-Tilh J, Goldman M. 2021 Existence and stability results for an isoperimetric problem with a non-local interaction of Wasserstein type. Preprint.

27. Pegon M. 2021 Large mass minimizers for an isoperimetric problem with a repulsive integrable potential. Nonlinear Anal. 211, 112395. (doi:10.1016/j.na.2021.112395)

28. Gamow G. 1930 Mass defect curve and nuclear constitution. Proc. R. Soc. A 126, 632–644.

29. Alama S, Bronsard L, Topaloglu I, Zuniga A. 2020 A nonlocal isoperimetric problem on Rᴺ. Calc. Var. Partial Differ. Equ. 60, 1–27. (doi:10.1007/s00526-020-01865-8)

30. Bonacini M, Cristina R. 2014 Local and global minimality results for a nonlocal isoperimetric problem on Rᴺ. SIAM J. Math. Anal. 46, 2310–2349. (doi:10.1137/130929898)

31. Bonacini M, Knüpfer H, Röger M. 2016 Optimal distribution of oppositely charged phases: perfect screening and other properties. SIAM J. Math. Anal. 48, 1128–1154. (doi:10.1137/15M1020927)

32. Frank R, Lieb E. 2015 A compactness lemma and its application to the existence of minimizers for the liquid drop model. SIAM J. Math. Anal. 47, 4436–4450. (doi:10.1137/15M1010658)

33. Julin V. 2014 Isoperimetric problem with a coulomb repulsive term. Indiana Univ. Math. J. 63, 77–89. (doi:10.1512/iumj.2014.63.5185)

34. Knüpfer H, Muratov C. 2013 On an isoperimetric problem with a competing nonlocal term I: the planar case. Commun. Pure Appl. Math. 66, 1129–1162. (doi:10.1002/cpa.21451)

35. Knüpfer H, Muratov C. 2014 On an isoperimetric problem with a competing nonlocal term II: the general case. Commun. Pure Appl. Math. 67, 1974–1994. (doi:10.1002/cpa.21479)

36. Lu J, Otto F. 2013 Nonexistence of a minimizer for Thomas-Fermi-Dirac-von Weizsäcker model. Commun. Pure Appl. Math. 67, 1605–1617. (doi:10.1002/cpa.21477)

37. Bahiana M, Oono Y. 1990 Cell dynamical system approach to block copolymers. Phys. Rev. A 41, 6763–6771. (doi:10.1103/PhysRevA.41.6763)

38. Choksi R, Ren X. 2003 On the derivation of a density functional theory for microphase separation of diblock copolymers. J. Stat. Phys. 113, 151–176. (doi:10.1023/A:1025722804873)

39. Leibler L. 1980 Theory of microphase separation in block copolymers. Macromolecules 13, 1602–1617. (doi:10.1021/ma60078a047)
40. Matsen MW, Schick M. 1994 Stable and unstable phases of a diblock copolymer melt. *Phys. Rev. Lett.* 72, 2660–2663. (doi:10.1103/PhysRevLett.72.2660)

41. Ohta T, Kawasaki K. 1986 Equilibrium morphology of block copolymer melts. *Macromolecules* 19, 2621–2632. (doi:10.1021/ma00164a028)

42. Otto F, Viehmann T. 2009 Domain branching in uniaxial ferromagnets: asymptotic behavior of the energy. *Calc. Var. Partial Differ. Equ.* 38, 135–181. (doi:10.1007/s00526-009-0281-y)

43. Conti S, Schweizer B. 2006 Rigidity and gamma convergence for solid-solid phase transitions with SO(2) invariance. *Commun. Pure Appl. Math.* 59, 830–868. (doi:10.1002/cpa.20115)

44. Knüpfer H, Kohn RV. 2010 Minimal energy for elastic inclusions. *Proc. R. Soc. A* 467, 695–717. (doi:10.1098/rspa.2010.0316)

45. Knüpfer H, Otto F. 2019 Nucleation barriers for the cubic-to-tetragonal phase transformation in the absence of self-accommodation. *ZAMM Z. Angew. Math. Mech.* 99, e201800179. (doi:10.1002/zamm.201800179)

46. Bella P, Goldman M, Zwicknagl B. 2015 Study of island formation in epitaxially strained films on unbounded domains. *Arch. Ration. Mech. Anal.* 218, 163–217. (doi:10.1007/s00205-015-0858-x)

47. Conti S, Garroni A, Ortiz M. 2015 The line-tension approximation as the dilute limit of linear-elastic dislocations. *Arch. Ration. Mech. Anal.* 218, 699–755. (doi:10.1007/s00205-015-0869-7)

48. Choksi R, Conti S, Kohn RV, Otto F. 2008 Ground state energy scaling laws during the onset and destruction of the intermediate state in a type I superconductor. *Commun. Pure Appl. Math.* 61, 595–626. (doi:10.1002/cpa.20206)

49. Conti S, Goldman M, Otto F, Serfaty S. 2018 A branched transport limit of the Ginzburg-Landau functional. *J. de l’École Polytechnique–Mathématiques* 5, 317–375. (doi:10.5802/jep.72)

50. Agarwal S, Carbou G, Labbé S, Prieur C. 2011 Control of a network of magnetic ellipsoidal samples. *Math. Control Relat. Fields* 1, 129–147. (doi:10.3934/mcrf.2011.1.129)

51. Aharoni A. 1988 Elongated single-domain ferromagnetic particles. *J. Appl. Phys.* 63, 5879–5882. (doi:10.1063/1.340280)

52. Alouges F, Beauchard K. 2008 Magnetization switching on small ferromagnetic ellipsoidal samples. *ESAIM: COCV* 15, 676–711. (doi:10.1051/cocv:2008047)

53. Berkovskiy B, Kalikmanov V. 1985 Topological instability of magnetic fluids. *J. de Phys. Lett.* 46, 483–491. (doi:10.1051/jphyslet:019850046011048300)

54. Brancher J, Zouaoui D. 1987 Equilibrium of a magnetic liquid drop. *J. Magn. Magn. Mater.* 65, 311–314. (doi:10.1016/0304-8853(87)90058-8)

55. Brown W. 1968 The fundamental theorem of fine-ferromagnetic-particle theory. *J. Appl. Phys.* 39, 993–994. (doi:10.1063/1.1656363)

56. Di Fratta G. 2016 The Newtonian potential and the demagnetizing factors of the general ellipsoid. *Proc. R. Soc. A* 472, 20160197. (doi:10.1098/rspa.2016.0197)

57. Séro-Guillaume OE, Zouaoui D, Bernardin D, Brancher JP. 1992 The shape of a magnetic liquid drop. *J. Fluid Mech.* 241, 215–232. (doi:10.1017/S0022112092002015)

58. Kimura M, van Meurs P. 2020 Regularity of the minimiser of one-dimensional interaction energies. *ESAIM: COCV* 26, 27. (doi:10.1051/cocv/2019043)

59. van Meurs P. 2021 Expansions for the linear-elastic contribution to the self-interaction force of dislocation curves. *arXiv*. (doi:10.1017/S0956792521000322)

60. Carrillo JA, Mateu J, Mora MG, Rondi L, Scardia L, Verdera J. 2020 The equilibrium measure for an anisotropic nonlocal energy. *Calc. Var. Partial Differ. Equ.* 60, 1–28. (doi:10.1007/s00526-021-01928-4)

61. Carrillo JA, Mateu J, Mora MG, Rondi L, Scardia L, Verdera J. 2019 The ellipse law: Kirchhoff meets dislocations. *Commun. Math. Phys.* 373, 507–524. (doi:10.1007/s00220-019-03368-w)

62. Mora G, Rondi L, Scardia L. 2018 The equilibrium measure for a nonlocal dislocation energy. *Commun. Pure Appl. Math.* 72, 136–158. (doi:10.1002/cpa.21762)

63. Eleuteri M, Lussardi L, Torricelli A. 2020 Limits of non-local anisotropic perimeters. *arXiv*.

64. Olver FWJ et al. 2021 NIST digital library of mathematical functions. *Ann. Math. Artif. Intell.* 38, 105–119.

65. Prudnikov AP. 1990 *Integrals series: more special functions*, vol. 3. Boca Raton, FL: CRC Press.

66. Grafakos L. 2009 *Classical fourier analysis*. New York, NY: Springer.

67. Söhngen H. 1939 Die Lösungen einer Integralgleichung und deren Anwendung in der Tragflügeltheorie. *Math. Z.* 45, 245–264. (doi:10.1007/BF01580284)

68. Rüland A. 2019 Quantitative invertibility and approximation for the truncated Hilbert and Riesz transforms. *Revista Matemática Iberoamericana* 35, 1997–2024. (doi:10.4171/rmi/1107)
69. King FW. 2009 Hilbert transforms, vol. 1. Cambridge, UK: Cambridge University Press.
70. Erdélyi A, Magnus W, Oberhettinger F, Tricomi FG. 1954 Tables of integral transforms, vol. 2. New York-Toronto-London: McGraw-Hill Book Company, Inc.
71. Gradshteyn I, Ryzhik I. 2000 Table of integrals, series, and products, 6th edn. San Diego, CA: Academic Press, Inc.
72. Lieb EH, Loss M. 2001 Analysis (Graduate Studies in Mathematics). Providence, RI: American Mathematical Society.
73. Brothers J, Ziemer W. 1988 Minimal rearrangements of Sobolev functions. J. für die reine und Angew. Math. (Crelles Journal) 1988, 153–179. (doi:10.1515/crll.1988.384.153)
74. Pólya G, Szegö G. 1951 Isoperimetric inequalities in mathematical physics, vol. 27. Princeton, NJ: Princeton University Press.
75. Ferone A, Volpicelli R. 2004 Convex rearrangement: equality cases in the Polya-Szegö inequality. Calc. Var. 21, 259–272. (doi:10.1007/s00526-003-0256-3)
76. Nezza ED, Palatucci G, Valdinoci E. 2012 Hitchhiker’s guide to the fractional Sobolev spaces. Bull. des Sci. Math. 136, 521–573. (doi:10.1016/j.bulsci.2011.12.004)