Mixed estimator of spline truncated, Fourier series, and kernel in biresponse semiparametric regression model

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Abstract. Regression analysis is a method of analysis to determine the relationship between the response and the predictor variables. There are three approaches in regression analysis, namely the parametric, nonparametric, and semiparametric approaches. Biresponse Semiparametric regression model is a regression model that uses a combination approach between parametric and nonparametric components, where two response variables are correlated with each other. For data cases with several predictor variables, different estimation technique approaches can be used for each variable. In this study, the parametric component is assumed to be linear. At the same time, the nonparametric part is approached using a mixture of three estimation techniques, namely, spline truncated, Fourier series, and the kernel. The unknown data pattern is assumed to follow the criteria of each of these estimation techniques. The spline is used when the data pattern tends to change at certain time intervals, the Fourier series is used when the data pattern tends to repeat itself, and the kernel is used when the data does not have a specific way. This study aims to obtain parameter estimates for the mixed semiparametric regression model of spline truncated, Fourier series, and the kernel on the biresponse data using the Weighted Least Square (WLS) method. The formed model depends on the selection of knot points, oscillation parameters, and optimal bandwidth. The best model is based on the smallest Generalized Cross Validation (GCV).

Keywords: Biresponse, Fourier, Kernel, Semiparametric Regression, Spline Truncated

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1. Introduction

Regression analysis is a statistical method to determine the closeness of the relationship between the response variable and one or more predictor variables. This analysis aims to estimate the population average or the average value of the response variable based on the value of the independent variable. Three approaches are often used in regression analysis, namely the parametric, nonparametric, and semiparametric approaches. If the data pattern is known as linear, quadratic, or cubic, then the regression method used is the parametric regression method [1]. However, in reality, it is found that not all data follow certain patterns. If the relationship between the predictor variables and the response is unknown, then the nonparametric regression method is suitable for modeling the relationship between these variables [2]. Meanwhile, if the data pattern consists of parametric and nonparametric components, a semiparametric regression approach is used.
The use of semiparametric regression each year is increasingly in demand in data modeling because of its flexible and objective nature. The nonparametric component of semiparametric regression can be approached by various estimation techniques such as spline, Fourier series, and kernel [3]. Several previous studies regarding spline estimators have been conducted by other researchers [2, 4, 5], and [6]. Several studies for the Fourier series estimator were conducted [7-11]. Likewise, studies for the kernel estimators have been carried out [12-14].

Spline approach has the advantage of very good visual and statistical interpretation and has high flexibility, so it is good at handling smooth data/functions [3]. One type of basic function in spline that is often used is spline truncated. This approach can describe changes in the behavior of different curves at different intervals. The results in a better level of flexibility can adjust more effectively to local characteristics in data. Meanwhile, Fourier series is a series in the sinusoidal form (sine and cosine) used to represent periodic functions in general. One of the advantages of the nonparametric regression approach using the Fourier Series is that it can handle data that has repetitive patterns, namely the repetition of the response variable values for different predictor variables. Furthermore, kernel regression is also more flexible, easy mathematical form, and can achieve a relatively fast convergence rate [15]. The selection of smoothing parameters (bandwidth) is considered more important than selecting the kernel function.

In general, studies on semiparametric and nonparametric regression models only use the same estimation method for some or even all of the predictor variables. However, sometimes, it is found that data have different data patterns for each predictor variable. Therefore, to overcome these different data patterns, some researchers have developed mixed methods to estimate them [16-18]. Previous research on semiparametric regression on biresponse data has been carried out with the Fourier series estimator approach [19] and using the mixed estimator approach Fourier series and Spline truncated [20].

Previous studies only used a mixture of two estimators in semiparametric regression. However, there may be three or more data patterns that different estimators can approach. Therefore, this study will use a semiparametric regression model using the spline truncated estimator approach, the Fourier series, and the kernel, which will be applied to data with two responses (biresponse). Based on the formulation of the problem, this study examines the semiparametric regression estimator form of a mixture of Spline Truncated, Fourier Series, and Kernel Regression.

2. Materials and Methods

2.1 Semiparametric Regression Model

Semiparametric regression is a combination of parametric components and nonparametric components. Suppose paired data \((x_i, t_i, y_i)\) and the relationship between \(x_i, y_i\) and \(t_i\) are assumed to follow a semiparametric regression model.

\[
y_i = f(x_i) + g(t_i) + \epsilon_i, \quad i = 1, 2, ..., n. \tag{1}
\]

where, \(y_i\) is the response variable, \(x_i\) and \(t_i\) are predictor variables, and \(\epsilon_i\) is an independent random error and is normally distributed with mean zero and variance \(\sigma^2\). \(f(x_i)\) is a regression function a known pattern form (parametric component). In contrast, \(g(t_i)\) is a regression function with no known pattern (nonparametric component). There are several nonparametric component approaches such as spline truncated, Fourier series, and kernels. This spline method is very good at modeling data whose patterns change at certain subintervals [3]. Fourier series is a trigonometric polynomial function with a degree of flexibility in handling data that has repeating patterns [21]. In contrast, the kernel estimator is used for data that does not follow a specific way [22]. Some of the advantages of kernel estimators are flexibility, easy mathematical form, and a relatively fast convergence rate [2].

If the regression curve \(g(t_i)\) in equation (1) is approximated by spline truncated function of degree \(p\) with the points of knots \(K_1, K_2, ... , K_m (K_1 \leq K_2 \leq \cdots \leq K_m)\) given by the equation :
\( g(t_i) = \sum_{j=0}^{p} \xi_j t_i^j + \sum_{k=1}^{m} \Phi_k (t_i - K_k)^p_k \) \tag{2}

The truncated function is given by:
\[
(t_i - K_k)^p_+ = \begin{cases} (t_i - K_k)^p, & t \geq K_k \\ 0, & t < K_k \end{cases}
\]

Where \( \xi_0, \xi_1, \ldots, \xi_p, \Phi_1, \ldots, \Phi_m \) are unknown parameters, \( i = 1, 2, \ldots, n \) denotes the amount of data and \( K_k \) with \( k = 1, 2, \ldots, m \) is denotes k-knots.

The nonparametric component approach using Fourier series is to use the function:
\[
h(z_i) = b z_i + \frac{\alpha_0}{2} + \sum_{l=1}^{L} (\alpha_l \cos lz_i)
\]

Where, \( \alpha_0, \alpha_l, l = 1, 2, \ldots, L \) are model parameters.

Furthermore, if the kernel regression curve approaches the nonparametric component, the Nadaraya-Watson kernel estimator is used, namely:
\[
\hat{q}_\phi(v) = n^{-1} \sum_{i=1}^{n} K \left( \frac{v - v_i}{\phi} \right) y_i,
\]
\[
= n^{-1} \sum_{i=1}^{n} R_{\phi i}(v) y_i
\]

Where \( \hat{q}_\phi(v) \) is the kernel regression estimation function, \( \phi \) is bandwidth and function \( R_{\phi i}(v) \) is the weighting function. One of the K kernel functions that can be used is the following Gaussian Kernel:
\[
K(u) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} u^2 \right), -\infty < u < \infty
\]

2.2 Biresponse Semiparametric Regression

Biresponse semiparametric regression is a regression model that aims to obtain a relationship between two response variables and their predictor variables. Given a biresponse semiparametric regression model for \( i = 1, 2, \ldots, n \) as follows:
\[
y_{1i} = \beta_{01} + \beta_{11} x_{1i} + \cdots + \beta_{p1} x_{pi} + g_1(t_i) + \epsilon_{1i}
\]
\[
y_{2i} = \beta_{02} + \beta_{12} x_{1i} + \cdots + \beta_{p2} x_{pi} + g_2(t_i) + \epsilon_{2i}
\]

\( y_{1i} \) and \( y_{2i} \) are the response variables, \( g_1(t_i) \) and \( g_2(t_i) \) are nonparametric regression functions on response one and response two, respectively.

The errors in the two regression models in equation (6), namely \( \epsilon_{1i} \) and \( \epsilon_{2i} \) are assumed to fulfill the following conditions:
\[
E(\epsilon_{1i}) = E(\epsilon_{2i}) = 0, \quad \text{Cov}(\epsilon_{1i}, \epsilon_{1i'}) = \begin{cases} \sigma_{11}, & i = i' \\
0, & i \neq i' \end{cases}
\]
\[
\text{Cov}(\epsilon_{2i}, \epsilon_{2i'}) = \begin{cases} \sigma_{12}, & i = i' \\
0, & i \neq i' \end{cases}
\]

2.3 Methods
To get an estimation of the biresponse semiparametric regression model for the mixture of Spline Truncated, Fourier Series, and Kernel, use the following steps.

1. Given the response variable $Y_1$ and $Y_2$ with parametric component variable $x$, and nonparametric component variables $t, z, v$. Based on equation (6), it can be written as:

$$ y_{ji} = f_j(x_i) + g_j(t_i) + h_j(z_i) + q_j(v_i) + \varepsilon_{ji} $$

where $i = 1, 2, \ldots, n$ and $j = 1, 2$.

2. Approaching the function $f_j(x_i)$ with a linear regression function

3. Approaching the function $g_j(t_i)$ with the spline truncated with $M$ knots

4. Approaching the function $h_j(z_i)$ with the Fourier series function

5. Approaching the function $q_j(v_i)$ with the kernel function

6. Expressing the $f(x), g(t), h(z)$, and $q(v)$ curves in the form of a matrix where $\tilde{\beta}$ is a parameter for linear regression, $\tilde{\xi}$ spline truncated regression parameter, and $\tilde{\alpha}$ fourier series regression parameter.

7. Converting mixed of spline truncated, Fourier series, and kernel in biresponse semiparametric regression model in the form of a matrix.

8. Solving optimization using Weighted Least Square method

9. Equating the partial derivative with zero.

10. Getting the mix estimation

3. Results and Discussion

3.1 Mixed of Spline Truncated, Fourier Series, And Kernel in Biresponse Semiparametric Regression Model

Given data with response variable $Y_1$ and $Y_2$ and predictor variable consisting of a parametric component and a nonparametric component. Parametric component is $x$ and nonparametric component variables which are approached with spline truncated, fourier series and kernel, namely $t, z, v$. The biresponse semiparametric regression model containing these variables can be stated by equation (7). While $i$ states the amount of data, and $j$ is the number of responses with $i = 1, 2, \ldots, n$ and $j = 1, 2$. Meanwhile, $\varepsilon_{ji}$ is assumed to be independent, identical, and normally distributed with mean zero and variance $\sigma_{ji}^2$.

Based on equation (7), it can be seen that this equation is a biresponse semiparametric function consisting of four constituent components. The first component, the function $f$, is a parametric component that is assumed to follow a linear pattern. Thus, the function is approximated by using linear regression, namely

$$ f_j(x_i) = \beta_{0j} + \beta_j x_i $$

Meanwhile, the second component, the function $g$, is a nonparametric component approximated by linear spline truncated function. The relationship between the response variable $y_j$ and the predictor variables ($t_i$) is assumed to fluctuate in certain sub-intervals. The linear spline truncated regression curve can be defined as follows:

$$ g_j(t_i) = \xi_j t_i + \sum_{k=1}^{M} \Phi_{jk}(t_i - K_{jk})_+ $$

The truncated function is given by:

$$ (t_i - K_{jk})_+ = \begin{cases} (t_i - K_{jk})_+, & t_i \geq K_{jk} \\ 0, & t_i < K_{jk} \end{cases} $$
Furthermore, the third component, the function $h$, is a nonparametric component that is approximated by the Fourier series function. It is assumed that the relationship between the response variable $y_j$ and the predictor variables $(z_i)$ has a recurring pattern or forms a seasonality. The Fourier series regression curve can be defined as follows:

$$h_j(z_i) = b_{0j} + \frac{\alpha_{0j}}{2} + \sum_{l=1}^{L} \alpha_{lj} \cos lz_i$$  \hspace{1cm} (10)

Then the fourth component, namely the $q$ function, is a nonparametric component approached by the Kernel function. The pattern of the relationship between the response variable $y_j$ and the predictor variables $(v_i)$ is assumed to have no known way. In this study, the Nadaraya Watson kernel function is used, which is defined as follows:

$$q_j(v_i) = n^{-1} \sum_{i=1}^{n} R_{\phi_i}(v) y_{ji}$$

Then the fourth component, namely the $q$ function, is a nonparametric component approached by the Kernel function. The pattern of the relationship between the response variable $y_j$ and the predictor variables $(v_i)$ is assumed to have no known way. In this study, the Nadaraya Watson kernel function is used, which is defined as follows:

$$q_j(v_i) = n^{-1} \sum_{i=1}^{n} R_{\phi_i}(v) y_{ji}$$

$q_j(v_i)$ is an estimator of the Nadaraya-Watson kernel regression curve, and parameter $\phi$ represents the bandwidth parameter, and the $K$ function is a kernel function. As for this study, the Gaussian kernel function is used as in equation (5) so that:

$$K \left( \frac{v_j - v_i}{\phi_j} \right) = \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{(v_j - v_i)^2}{2\phi_j^2} \right)$$

### 3.2 Mixed Estimator Of Spline Truncated, Fourier Series, And Kernel In Biresponse Semiparametric Regression Model

Before estimating the model, equation (7) components are first expressed in the form of a matrix. The first part is for the parametric component in equation (8), for response $j = 1,2$ and $i = 1,2,\ldots,n$ model can be expressed in the form of a matrix to be:

$$
\begin{pmatrix}
    f_1(x_1) \\
    f_1(x_2) \\
    \vdots \\
    f_1(x_n) \\
    f_2(x_1) \\
    f_2(x_2) \\
    \vdots \\
    f_2(x_n)
\end{pmatrix}
= 
\begin{pmatrix}
    1 & x_1 & 0 & 0 \\
    1 & x_2 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots \\
    1 & x_n & 0 & 0 \\
    0 & 0 & 1 & x_1 \\
    0 & 0 & 1 & x_2 \\
    \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 1 & x_n
\end{pmatrix}
\begin{pmatrix}
    \beta_{01} \\
    \beta_{1} \\
    \beta_{02} \\
    \beta_{2}
\end{pmatrix}
$$

Furthermore

$$
\begin{pmatrix}
    \tilde{f}_1(x) \\
    \tilde{f}_2(x)
\end{pmatrix}
= 
\begin{pmatrix}
    X_1 & 0 \\
    0 & X_2
\end{pmatrix}
\begin{pmatrix}
    \tilde{\beta}_1 \\
    \tilde{\beta}_2
\end{pmatrix}
$$

in matrix notation can be written as:
\[ \hat{f}(x) = X\hat{\beta} \]  

With vector \( \hat{f}(x) \) is sized \( 2n \times 1 \), Matrix \( X \) is sized \( 2n \times 4 \) and vector \( \hat{\beta} \) is sized \( 4 \times 1 \).

Furthermore, the nonparametric component which is approached by Spline Truncated in equation (9), for response \( j = 1, 2 \) and \( i = 1, 2, ..., n \) model can be expressed in the form of a matrix to be:

\[
\begin{pmatrix}
g_1(t_1) \\
g_1(t_2) \\
\vdots \\
g_1(t_n) \\
g_2(t_1) \\
g_2(t_2) \\
\vdots \\
g_2(t_n)
\end{pmatrix}
= \begin{pmatrix}
t_1 (t_1 - K_{11})_+ & \cdots & (t_1 - K_{1M})_+ & 0 & 0 & \cdots & 0 \\
t_2 (t_2 - K_{11})_+ & \cdots & (t_2 - K_{1M})_+ & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
t_n (t_n - K_{11})_+ & \cdots & (t_n - K_{1M})_+ & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & t_1 (t_1 - K_{21})_+ & \cdots & (t_1 - K_{2M})_+ \\
0 & 0 & \cdots & 0 & t_2 (t_2 - K_{21})_+ & \cdots & (t_2 - K_{2M})_+ \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & t_n (t_n - K_{21})_+ & \cdots & (t_n - K_{2M})_+
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_M
\end{pmatrix}
\]

Furthermore, it can be written as:

\[
\begin{pmatrix}
g_1(t) \\
g_2(t)
\end{pmatrix}
= \begin{pmatrix}
G_1 & 0 \\
0 & G_2
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix}
\]  

in matrix notation can be written as:

\[ \hat{g}(t) = G\hat{\xi} \]  

With vector \( \hat{g}(t) \) is sized \( 2n \times 1 \), Matrix \( G \) is sized \( 2n \times (2M + 2) \), vector \( \hat{\xi} \) is sized \( (2M + 2) \times 1 \), and \( M \) is denoted number of knots.

Furthermore, the non-parametric component that is approached by the Fourier Series in equation (10), for response \( j = 1, 2 \) and \( i = 1, 2, ..., n \) model can be expressed in the form of a matrix to be:

\[
\begin{pmatrix}
h_1(z_1) \\
h_1(z_2) \\
\vdots \\
h_1(z_n) \\
h_2(z_1) \\
h_2(z_2) \\
\vdots \\
h_2(z_n)
\end{pmatrix}
= \begin{pmatrix}
z_1 \frac{1}{2} \cos \frac{z_1}{2} & \cdots & \cos \frac{Lz_1}{2} \\
z_2 \frac{1}{2} \cos \frac{z_2}{2} & \cdots & \cos \frac{Lz_2}{2} \\
\vdots & \vdots & \vdots \\
z_n \frac{1}{2} \cos \frac{z_n}{2} & \cdots & \cos \frac{Lz_n}{2} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\frac{b_1}{b_1} \\
\frac{a_0}{a_0} \\
\frac{a_1}{a_1} \\
\frac{a_2}{a_2} \\
\frac{a_3}{a_3}
\end{pmatrix}
\]

Furthermore

\[
\begin{pmatrix}
\bar{h}_1(z) \\
\bar{h}_2(z)
\end{pmatrix}
= \begin{pmatrix}
D_1 & 0 \\
0 & D_2
\end{pmatrix}
\begin{pmatrix}
\bar{a}_1 \\
\bar{a}_2
\end{pmatrix}
\]  

in matrix notation can be written as:

\[ \bar{h}(z) = D\bar{a} \]  

With vector \( \bar{h}(z) \) is sized \( 2n \times 1 \), Matrix \( D \) is sized \( 2n \times (2L + 4) \), vector \( \bar{a} \) is sized \( (2L + 4) \times 1 \), and \( L \) is denote number of oscillations.

The fourth component is a nonparametric component which the Kernel approaches. Based on equation (11), for response \( j = 1, 2 \) and \( i = 1, 2, ..., n \) model can be expressed in the form of a matrix to be:
\[
\begin{pmatrix}
q_1(v_1) \\
q_1(v_2) \\
\vdots \\
q_1(v_n)
\end{pmatrix}
= \begin{pmatrix}
n^{-1}R_{\phi 1}(v_1) & \ldots & n^{-1}R_{\phi n}(v_1) \\
n^{-1}R_{\phi 1}(v_2) & \ldots & n^{-1}R_{\phi n}(v_2) \\
\vdots \\
n^{-1}R_{\phi 1}(v_n) & \ldots & n^{-1}R_{\phi n}(v_n)
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{pmatrix}
\]

Furthermore

\[
\begin{pmatrix}
\bar{q}_1(v) \\
\bar{q}_2(v)
\end{pmatrix} = \begin{pmatrix} V(\phi)_1 & 0 \\
0 & V(\phi)_2 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\
\bar{y}_2 \end{pmatrix}
\]

in matrix notation can be written as:

\[
\bar{q}(v) = V(\phi)\bar{y}
\]

With vector \(\bar{q}(v)\) is sized \(2n \times 1\), Matrix \(V(\phi)\) is sized \(2n \times 2n\) and \(\bar{y}\) is sized \(2n \times 1\).

Therefore, equation (7) which is a biresponse semiparametric regression model with a truncated spline mixture estimator, Fourier series and kernel is expressed in matrix notation:

\[
\begin{align*}
\bar{y} &= \hat{f}(x) + \hat{g}(t) + \hat{h}(x) + \bar{q}(v) + \bar{\varepsilon} \\
&= X\hat{\beta} + G\hat{\xi} + D\hat{\alpha} + V(\phi)\bar{y} + \bar{\varepsilon}
\end{align*}
\]

Or it can also be stated as:

\[
\bar{y}^* = X\hat{\beta} + G\hat{\xi} + D\hat{\alpha} + \bar{\varepsilon}
\]  \hspace{1cm} (16)

Where \(\bar{y}^* = (I - V(\phi))\bar{y}\)

Then in equation (16), the parameter estimation is carried out using the optimization method Weighted Least Square (WLS) to be:

\[
\begin{align*}
\min_{\hat{\beta}, \hat{\xi}, \hat{\alpha}} Q(\hat{\beta}, \hat{\xi}, \hat{\alpha}) &= \min_{\hat{\beta}, \hat{\xi}, \hat{\alpha}} \left\{ (\bar{y}^* - X\hat{\beta} - G\hat{\xi} - D\hat{\alpha})^T W (\bar{y}^* - X\hat{\beta} - G\hat{\xi} - D\hat{\alpha}) \right\}
\end{align*}
\]  \hspace{1cm} (17)

\(W\) is the covariance variance matrix between the two responses. Meanwhile, a simplification is made in equation (17) below:

\[
\begin{align*}
Q &= (\bar{y}^* - X\hat{\beta} - G\hat{\xi} - D\hat{\alpha})^T W (\bar{y}^* - X\hat{\beta} - G\hat{\xi} - D\hat{\alpha}) \\
&= \bar{y}^{*T} W \bar{y}^* - 2\bar{y}^{*T} W X\hat{\beta} - 2\bar{y}^{*T} W G\hat{\xi} - 2\bar{y}^{*T} W D\hat{\alpha} + \hat{\beta}^T X^T W X\hat{\beta} + 2\hat{\beta}^T X^T W G\hat{\xi} \\
&\quad + 2\hat{\beta}^T X^T W D\hat{\alpha} + \hat{\xi}^T G^T W G\hat{\xi} + 2\hat{\xi}^T G^T W D\hat{\alpha} + \hat{\alpha}^T D^T W D\hat{\alpha}
\end{align*}
\]  \hspace{1cm} (18)

Then proceed to look for the parameter \(\hat{\beta}\), which is looking for the derivative of the function \(Q\) against \(\hat{\beta}\) obtained:

\[
\frac{\partial Q(\beta, \xi, \alpha)}{\partial \beta} = 0
\]

\[
-2\bar{y}^{*T} W X + 2X^T W X\hat{\beta} + 2X^T W G\hat{\xi} + 2X^T W D\hat{\alpha} = 0
\]

\[
X^T W X\hat{\beta} = \bar{y}^{*T} W X - X^T W G\hat{\xi} - X^T W D\hat{\alpha}
\]
\[
\hat{\beta} = (X^T WX)^{-1} X^T W \hat{y}^* - (X^T WX)^{-1} X^T W G \hat{\xi} - (X^T WX)^{-1} X^T WD \hat{\alpha}
\]

By assuming \( H = (X^T WX)^{-1} X^T W \), then equation (19) can be simplified to form \( \hat{\beta} \) to be:

\[
\hat{\beta} = H \hat{y}^* - HG \hat{\xi} - HD \hat{\alpha}
\]

Using the same method as before, derive the \( Q \) function in equation (18) against \( \hat{\xi} \) from getting the parameter \( \hat{\xi} \), namely:

\[
\frac{\partial Q(\hat{\beta}, \hat{\xi}, \hat{\alpha})}{\partial \hat{\xi}} = 0
\]

\[
-2\hat{y}^* W G + 2\hat{\beta}^T X^T W G + 2G^T W G \hat{\xi} + 2G^T W D \hat{\alpha} = 0
\]

\[
G^T W \hat{\xi} = \hat{y}^* W G - \hat{\beta}^T X^T W G - G^T W D \hat{\alpha}
\]

\[
\hat{\xi} = (G^T W G)^{-1} G^T W \hat{y}^* - (G^T W G)^{-1} G^T WX \hat{\beta} - (G^T W G)^{-1} G^T WD \hat{\alpha}
\]

By assuming \( L = (G^T W G)^{-1} G^T W \), then equation (21) can be simplified to form \( \hat{\xi} \) to be:

\[
\hat{\xi} = L \hat{y}^* - LX \hat{\beta} - LD \hat{\alpha}
\]

Then proceed to look for the \( \hat{\alpha} \) parameter by deriving the \( Q \) function in equation (18) against \( \hat{\alpha} \).

\[
\frac{\partial Q(\hat{\beta}, \hat{\xi}, \hat{\alpha})}{\partial \hat{\alpha}} = 0
\]

\[
-2\hat{y}^* W D + 2\hat{\beta}^T X^T W D + 2\hat{\xi}^T G^T W D + 2D^T W D \hat{\alpha} = 0
\]

\[
D^T W D \hat{\alpha} = \hat{y}^* W D - \hat{\beta}^T X^T W D - \hat{\xi}^T G^T W D
\]

\[
\hat{\alpha} = (D^T W D)^{-1} D^T W \hat{y}^* - (D^T W D)^{-1} D^T WX \hat{\beta} - (D^T W D)^{-1} D^T W G \hat{\xi}
\]

By assuming \( E = (D^T W D)^{-1} D^T W \), then equation (23) can be simplified to form \( \hat{\alpha} \) to be:

\[
\hat{\alpha} = E \hat{y}^* - EX \hat{\beta} - EG \hat{\xi}
\]

Therefore, parameters obtained still contain other parameters, then do a process of elimination - substitution to get the parameters without loading other parameters. It can be seen in equations (20), (22), and (24) that the equation contains three parameters. Therefore, we will simplify the equation, which has only two parameters. The first step, namely the elimination process for equations (20) and (22), is obtained:

\[
(I - HGLX) \hat{\beta} + (HD - HGLD) \hat{\alpha} = (H - HGL) \hat{y}^*
\]

Furthermore, the elimination process for equations (20) and (24) is also carried out:

\[
(I - HDEX) \hat{\beta} + (HG - HDEG) \hat{\xi} = (H - HDE) \hat{y}^*
\]

Just like before, made elimination back to equation (22) and (24) were obtained:

\[
(LX - LDEX) \hat{\beta} + (I - LDEG) \hat{\xi} = (L - LDE) \hat{y}^*
\]

After obtaining an equation containing two parameters, namely equation (25), (26), and (27), the process of elimination continues again to get parameters that do not contain other parameters. The elimination process is done in equation (26) and (27) to obtain \( \beta \) parameters is:

\[
\hat{\beta} = [(I - HDEX) - (HG - HDEG)(I - LDEG)^{-1}(LX - LDEX)]^{-1}(H - HDE)
\]

\[
- (HG - HDEG)(I - LDEG)^{-1}(L - LDE)[(I - V(\phi)) \hat{y}]
\]
\[ \hat{\beta} = B(K, l, \phi) \hat{y} \] (28)

Where \( B(K, l, \phi) = [(I - HDEX) - (HG - HDEG)(I - LDEG)^{-1}(LX - LDEX)]^{-1}[(H - HDE) - (HG - HDEG)(I - LDEG)^{-1}(L - LDE)](I - V(\phi)) \)

Furthermore, the substitution of parameter \( \hat{\beta} \) to equation (26) is obtained:

\[ (I - HDEX)B(K, l, \phi)\hat{y} + (HG - HDEG)\hat{\xi} = (H - HDE)\hat{y}^* \]

\[ (I - HDEX)B(K, l, \phi)\hat{y} + (HG - HDEG)\hat{\xi} = (H - HDE)(I - V(\phi))\hat{y} \]

\[ \hat{\xi} = (HG - HDEG)^{-1}[(H - HDE)(I - V(\phi)) - (I - HDEX)B(K, l, \phi)]\hat{y} \]

\[ \hat{\xi} = C(K, l, \phi) \hat{y} \] (29)

Substitute \( \hat{\beta} \) and \( \hat{\xi} \) to equation (24) to obtain the following \( \hat{\alpha} \) parameters:

\[ \hat{\alpha} = E\hat{y}^* - EXB(K, l, \phi)\hat{y} - EGC(K, l, \phi)\hat{y} \]

\[ = [E(I - V(\phi)) - EXB(K, l, \phi) - EGC(K, l, \phi)]\hat{y} \]

\[ \hat{\alpha} = F(K, l, \phi) \hat{y} \] (30)

Where \( F(K, l, \phi) = (I - V(\phi)) - EXB(K, l, \phi) - EGC(K, l, \phi) \)

Based on equations (28), (29) and (30), then mixed estimator of spline truncated, fourier series, and kernel in biresponse semiparametric regression model can be written as follows:

\[ \hat{\gamma} = X\hat{\beta} + G\hat{\xi} + D\hat{\alpha} + V(\phi)\hat{y} \]

\[ = (XB(K, l, \phi) + GC(K, l, \phi) + DF(K, l, \phi) + V(\phi))\hat{y} \]

\[ \hat{\gamma} = T(K, l, \phi) \hat{y} \] (31)

Where \( T(K, l, \phi) = XB(K, l, \phi) + GC(K, l, \phi) + DF(K, l, \phi) + V(\phi) \)

The model in equation (31) obtained depends on the best mix of knot point \( K \), the oscillation parameter \( l \), and the bandwidth \( \phi \) using the smallest Generalized Cross Validation (GCV), namely:

\[ GCV(K, l, \phi) = \frac{MSE(K, l, \phi)}{(N^{-1}trace(I - T(K, l, \phi)))^2} \]

Where \( MSE(K, l, \phi) = N^{-1}\hat{\gamma}^T(I - T(K, l, \phi))^T(I - T(K, l, \phi))\hat{\gamma} \) dan \( N = n_1 + n_2 \).

4. Conclusion

Based on the analysis conducted previously, it can be concluded that mixed estimator of spline truncated, fourier series, and kernel in biresponse semiparametric regression model can be written as follows:

\[ \hat{\gamma} = X\hat{\beta} + G\hat{\xi} + D\hat{\alpha} + V(\phi)\hat{y} \]

\[ = XB(K, l, \phi)\hat{y} + GC(K, l, \phi)\hat{y} + DF(K, l, \phi)\hat{y} + V(\phi)\hat{y} \]

Where

\[ B(K, l, \phi) = [(I - HDEX) - (HG - HDEG)(I - LDEG)^{-1}(LX - LDEX)]^{-1}[(H - HDE) - (HG - HDEG)(I - LDEG)^{-1}(L - LDE)](I - V(\phi)) \]
\[ C(K, l, \phi) = (HG - HDEG)^{-1}[(H - D)(I - V(\phi)) - (I - HDEX)B(K, l, \phi)] \]

\[ F(K, l, \phi) = (I - V(\phi)) - EXB(K, l, \phi) - EGC(K, l, \phi) \]

And the smallest GCV based on optimum knot point \( K \), the oscillation parameter \( l \), and the bandwidth \( \phi \) can be determined by the formula

\[
GCV(K, l, \phi) = \frac{MSE(K, l, \phi)}{(N^{-1}\text{trace}(I - T(K, l, \phi)))^2}
\]

Where \( MSE(K, l, \phi) = N^{-1}\hat{y}^T(I - T(K, l, \phi))\hat{y} \) dan \( N = n_1 + n_2 \).

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