GENERALIZED TEVELEV DEGREES OF $\mathbb{P}^1$

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ABSTRACT. Let $(C, p_1, \ldots, p_n)$ be a general curve. We consider the problem of enumerating covers of the projective line by $C$ subject to incidence conditions at the marked points. These counts have been obtained by the first named author with Pandharipande and Schmitt via intersection theory on Hurwitz spaces and by the second named author with Farkas via limit linear series. In this paper, we build on these two approaches to generalize these counts to the situation where the covers are constrained to have arbitrary ramification profiles: that is, additional ramification conditions are imposed at the marked points, and some collections of marked points are constrained to have equal image.

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1. INTRODUCTION

Throughout, we work over $\mathbb{C}$.

1.1. Tevelev degrees. Let $X$ be a smooth, projective variety. We are interested in the following question:

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**Question 1.** Let \((C, p_1, \ldots, p_n)\) be a general pointed curve of genus \(g\), let \(\beta \in H_2(X, \mathbb{Z})\) be a curve class on \(X\), and let \(q_1, \ldots, q_n \in X\) be general points. Then, how many morphisms \(f : C \to X\) are there in class \(\beta\) (that is, \(f_*([C]) = \beta\)) satisfying \(f(p_i) = q_i\) for \(i = 1, 2, \ldots, n\)?

Equivalently, we ask for the set-theoretic degree of the canonical morphism

\[
\tau : \mathcal{M}_{g,n}(X, \beta) \to \mathcal{M}_{g,n} \times X^n.
\]

We assume on the one hand that the expected dimension of \(\mathcal{M}_{g,n}(X, \beta)\) is equal to the dimension of \(\mathcal{M}_{g,n} \times X^n\), equivalently

\[
\int_{\beta} c_1(T_X) = \dim(X)(n + g - 1),
\]

and on the other hand that all dominating components of \(\mathcal{M}_{g,n}(X, \beta)\) are generically smooth of the expected dimension. In this case, the answer Tevelev degrees of \(X\). This question was considered by Tevelev in the case \(X = \mathbb{P}^1\), \(\beta = (g+1)[\mathbb{P}^1]\), \(n = g+3\) in [12].

Alternatively, one can formulate a virtual analogue of the question in Gromov-Witten theory in terms of the space of stable maps \(\overline{\mathcal{M}}_{g,n}(X, \beta)\), equipped with its virtual fundamental class. The resulting counts are referred to as virtual Tevelev degrees and can be expressed in terms of the quantum cohomology of \(X\), see [1] and [3]. It is expected that when \(X\) and \(\beta\) are sufficiently positive, the geometric and virtual Tevelev degrees agree, see [9] for partial results in this direction.

We will deal in this work exclusively with geometric Tevelev degrees, so henceforth drop the modifier “geometric.” These are, in general, more difficult to compute than virtual Tevelev degrees, and at present complete answers are only available for \(X = \mathbb{P}^1\), which we review in the next section.

1.2. **Tevelev Degrees of** \(\mathbb{P}^1\). We now specialize to the case \(X = \mathbb{P}^1\). We will generalize the situation slightly, imposing the condition that \(r \geq 1\) of the marked points on the source curve \(C\) map to the same marked point of the target.

Fix a genus \(g \geq 0\), an integer \(\ell \in \mathbb{Z}\) and a positive integer \(r \in \mathbb{Z}_{\geq 1}\). Call

\[
\begin{align*}
&d = g + 1 + \ell \quad \text{and} \quad n = g + 3 + 2\ell. \\
\end{align*}
\]

Assume

\[
\begin{align*}
&r \leq d \quad \text{and} \quad n - r + 1 \geq 3. \\
\end{align*}
\]

Let \(\overline{\mathcal{H}}_{g,d,r}\) be the Deligne-Mumford stack parametrizing degree \(d\) Harris-Mumford admissible covers

\[
\pi : (C, p_1, \ldots, p_n) \to (D, q_1, \ldots, q_{n+r+1})
\]

where \(C\) is a genus \(g\) nodal curve with \(n\) distinct marked points \(p_1, \ldots, p_n \in C^\text{sm}\), \(D\) is a genus 0 nodal curve with \(n - r + 1\) distinct marked points \(q_1, \ldots, q_{n+r+1} \in D^\text{sm}\), \(\pi\) sends \(p_i\) to \(q_i\) for \(i = 1, \ldots, n - r\) and \(p_i\) to \(q_{n+r+1}\) for \(i \geq n - r + 1\). We require \(\pi\) to have exactly \(2g + 2d - 2\) simple branch points \(z_1, \ldots, z_{2g+2d-2}\). In addition, we require that \([z_1, \ldots, z_{2g+2d-2}]\) \(\cap \{q_1, \ldots, q_{n+r+1}\} = 0\) and that \((D, q_1, \ldots, q_{n+r+1}, z_1, \ldots, z_{2g+2d-2})\) is a stable curve. See Notation [3] below for a more precise definition.

Define

\[
\tau_{g,d,r} : \overline{\mathcal{H}}_{g,d,r} \to \overline{\mathcal{M}}_{0,n-r+1}
\]
to be the morphism that remembers only the stabilized domain and the target curves with all of the marked points (while ramification and branch points are forgotten and the curves are stabilized). Note that by condition \(1\), the domain and the target of \(\tau_{g,\ell, r}\) have the same dimension.

Finally, define the Tevelev degree

\[
\text{Tev}_{g, \ell, r} = \frac{\deg(\tau_{g, \ell, r})}{(2g + 2\ell - 2)!}
\]

and set \(\text{Tev}_{g, \ell, r} = 0\) in all other cases. In the case \(r = 1\), it is straightforward to check that this recovers the definition of the previous section, that is,

\[
\text{Tev}_{g, \ell, 1} = \text{Tev}^{p^1}_{g, \ell, g + 3 + 2\ell} = \text{Tev}^{p^1}_{g, \ell, n}.
\]

The factor \((2g + 2\ell - 2)!\) removes the redundancy of the different possible labelings of the ramification points on the domain curve.

In \([4]\), explicit closed formulas for \(\text{Tev}_{g, \ell, r}\) are found using the following recursion, obtained via intersection theory on the space \(\overline{\mathcal{M}}_{g,d,n,r}\) after a nodal degeneration of \(C\):

**Proposition.** \([4]\) Proposition 7] Fix \(g, r \geq 1\) and \(\ell \in \mathbb{Z}\) satisfying the conditions in \([2]\). Then, we have the recursion

\[
\text{Tev}_{g, \ell, r} = \text{Tev}_{g-1, \ell, \max(1,r-1)} + \text{Tev}_{g-1, \ell+1, r+1}.
\]

The genus 0 case is instead established by hand and then the whole recursion is explicitly solved, giving the following:

**Theorem.** \([4]\) Theorem 12] Fix \(g \geq 0, \ell \in \mathbb{Z}\) and \(r \geq 1\) satisfying conditions \([2]\). Then, we have:

- for \(r = 1\),
  \[
  \text{Tev}_{g, \ell, r} = 2^g - 2 \sum_{i=0}^{\ell-2} \binom{g}{i} + (-\ell + 2) \binom{g}{-\ell - 1} + \ell \binom{g}{-\ell},
  \]

- for \(r > 1\),
  \[
  \text{Tev}_{g, \ell, r} = 2^g - 2 \sum_{i=0}^{\ell-2} \binom{g}{i} + (-\ell + r - 3) \binom{g}{-\ell - 1} + (\ell - 1) \binom{g}{-\ell} - \sum_{i=\ell+1}^{\ell-2} \binom{g}{i}.
  \]

When \(\ell \geq r - 1\), all of the binomial coefficients are interpreted to vanish, leaving simply \(\text{Tev}_{g, \ell, r} = 2^g\). This agrees with the virtual Tevelev degrees computed in \([1]\).

A different degeneration approach via the theory of limit linear series is given in \([7]\), resulting in the following formula:

**Theorem.** \([7]\) Theorem 1.3] Fix \(g \geq 0, \ell \in \mathbb{Z}\) and \(r \geq 1\) satisfying conditions \([1]\) and \([2]\). Then,

\[
\text{Tev}_{g, \ell, r} = \int_{\mathcal{G}(2,\ell+1)} \sigma_{g, \ell} \sigma_{r-1} \left[ \sum_{i+j=n-2-r} \sigma_{i} \sigma_{j} \right] - \int_{\mathcal{G}(2,\ell)} \sigma_{r} \sigma_{r-2} \left[ \sum_{i+j=n-3-r} \sigma_{i} \sigma_{j} \right]
\]

Here, the second term is interpreted to vanish when \(r = 1\). In the case \((r, n) = (1,3)\), the formulas of \([4]\) and \([7]\) recover the classical counts of Castelnuovo \([2]\).
1.3. **Generalized Tevelev degrees for $\mathbb{P}^1$**. In this paper, we consider a natural generalization of the previous problem, in which incidence conditions with arbitrary ramification profiles are imposed. Let $g \geq 0$ be a genus, $\ell \in \mathbb{Z}$ and fix $k \geq 0$ vectors of integers 
\[ \mu_h = (e_{h,1}, \ldots, e_{h,r_h}) \in \mathbb{Z}_{\geq 1}^{r_h} \]
with $r_h \geq 1$ for $h = 1, \ldots, k$. For each $h$, define
\[ |\mu_h| = \sum_{j=1}^{r_h} e_{h,j}. \]

Call
\[ (3) \quad d[g, \ell] = g + 1 + \ell \]
and assume
\[ (4) \quad g + 3 + 2\ell = k \sum_{h=1}^{k} |\mu_h| \]
and that for every $h = 1, \ldots, k$ we have
\[ (5) \quad |\mu_h| \leq d[g, \ell]. \]

Also call
\[ n[\mu_1, \ldots, \mu_k] = \sum_{h=1}^{k} r_h. \]
the total number of markings and
\[ b[g, \ell, \mu_1, \ldots, \mu_k] = 2g + 2d - 2k \sum_{h=1}^{k} |\mu_h| + n \]
the number of additional (simple) branch points.

**Notation 1.** Unless there is some ambiguity, we will simply write $d$, $n$ and $b$ in place of $d[g,1]$, $n[\mu_1, \ldots, \mu_k]$ and $b[g,1,\mu_1, \ldots, \mu_k]$.

**Definition 2.** Denote by $\mathcal{M}_{g,d,\mu_1,\ldots,\mu_k}$ the Deligne-Mumford stack of admissible covers (see also [6, ?]) whose objects over a scheme $S$ are commutative diagrams
\[
\begin{array}{ccc}
C & \xrightarrow{\pi} & D \\
\downarrow p_i & & \downarrow q_i \\
S & & Q_t
\end{array}
\]
where:
(i) $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, k\}$ and $t \in \{1, \ldots, b\}$;
(ii) $(C \to S, p_1, \ldots, p_n)$ is a proper flat family of reduced connected genus $g$ curves with at most nodal singularities and $p_1, \ldots, p_n : S \to C$ are sections lying in the smooth locus of $C/S$;
(iii) $(D \to S, q_1, \ldots, q_k, Q_1, \ldots, Q_b)$ is a stable $(k+b)$-pointed curve of genus 0;
(iv) $\pi \circ p_i = q_j$ for $1 \leq j \leq k$ and $n - \sum_{h=1}^{k} r_h + 1 \leq i \leq n - \sum_{h=1}^{k} r_h$ (see Figure 1 for an illustration);
(v) $\pi^{-1}(\text{Sing}(D/S)) = \text{Sing}(C/S)$ set-theoretically;
(vi) $\pi$ is finite and étale with the following exceptions:
Figure 1. Explanation of the indices.

(a) for each \(s \in S\) and each \(x \in C_s\) a nodal point of \(C_s\) (necessarily) sent to a nodal point \(y\) of \(D_s\) we require that locally \(C, D\) and \(\pi\) are described by:

\[
\begin{align*}
C : x y = a, & \ a \in \mathcal{O}_{S, x}, \ x, y \text{ generate } \mathfrak{m}_{C, x} \\
D : u v = a^p, & \ a \in \mathcal{O}_{S, y}, \ u, v \text{ generate } \mathfrak{m}_{D, y} \\
\pi : u = x^p, & \ v = y^p
\end{align*}
\]

for some \(p \in \mathbb{Z}_{\geq 1}\),

(b) there are unique smooth points \(P_i : S \to C\), one over each \(Q_i : S \to D\), where \(\pi\) has ramification index 2,

(c) for \(1 \leq h \leq k\) and \(1 \leq j \leq r_h\), \(\pi\) has ramification index \(e_{h,j}\) at \(p_{n-\sum_r r_r+j}\).

Note that, by Riemann-Hurwitz formula, \(\pi_s : C_s \to D_s\) is a degree \(d\) map for all \(s \in S\). The sections \(p_i\) for \(i = 1, \ldots, n\) and \(q_i\) for \(i = 1, \ldots, k\) are called *markings*, the sections \(P_i\) for \(i = 1, \ldots, b\) *ramification points* and the sections \(Q_i\) for \(i = 1, \ldots, b\) *branch points* of the cover \(\pi\).

Morphisms in \(\mathcal{G}_{g,d,\mu_1,\ldots,\mu_k}\) are cartesian diagrams

\[
\begin{array}{ccc}
C & \xrightarrow{\pi} & D \\
\downarrow^{f_D} \downarrow_{f_C} & & \downarrow^{f_D'} \downarrow_{f_C'} \\
S & \xrightarrow{f_S} & C'
\end{array}
\]

with \(f_C \circ p_i = f_S \circ p'_i\) for \(i = 1, \ldots, n\), \(f_D \circ q_j = f_D' \circ q'_j\) for \(j = 1, \ldots, k\) and \(f_D \circ Q_t = f_D' \circ Q'_t\) for \(t = 1, \ldots, b\).

Roughly speaking, these are Harris-Mumford admissible covers in [6 Paragraph 4] with some extra data.

**Notation 3.** When

\[
\mu_h = (1, \ldots, 1)
\]

underlined.
for all \( h = 1, \ldots, k \), we simply write:
\[
\overline{\cal H}_{g, d, r_1, \ldots, r_k} := \overline{\cal H}_{g, d, \mu_1, \ldots, \mu_k},
\]
\[
\tau_{g, \ell, r_1, \ldots, r_k} := \tau_{g, \ell, \mu_1, \ldots, \mu_k},
\]
\[
n[r_1, \ldots, r_k] := n[\mu_1, \ldots, \mu_k],
\]
\[
b[g, \ell, r_1, \ldots, r_k] := b[g, \ell, \mu_1, \ldots, \mu_k].
\]

Assume now that
\[
k \geq 3
\]
and let
\[
\tau_{g, \ell, \mu_1, \ldots, \mu_k} : \overline{\cal H}_{g, d, \mu_1, \ldots, \mu_k} \to \overline{\cal M}_{g, n} \times \overline{\cal M}_{0, k}
\]
be the morphism that remembers only the domain and the target curves with all the marked points (while ramification and branch points are forgotten and the curves are stabilized).

Note that, by Equations 3 and 4, the domain and the target of \( \tau_{g, \ell, \mu_1, \ldots, \mu_k} \) have both dimension
\[
dim [g, \ell, \mu_1, \ldots, \mu_k] = 4g + 2\ell + (k - 3) - \sum_{h=1}^{k} |\mu_h| + n = 3g - 3 + n + (k - 3)
\]

**Notation 4.** Unless there is ambiguity, we will write \( \dim \) instead of \( \dim [g, \ell, \mu_1, \ldots, \mu_k] \).

**Definition 5.** Define
\[
\text{Tev}_{g, \ell, \mu_1, \ldots, \mu_k} = \begin{cases} 
\deg \tau_{g, \ell, \mu_1, \ldots, \mu_k} \\
0
\end{cases}
\]
when conditions 4, 5 and 6 are satisfied,
\[
\text{Tev}_{g, \ell, \mu_1, \ldots, \mu_k} = \begin{cases} 
\deg \tau_{g, \ell, \mu_1, \ldots, \mu_k} \\
0
\end{cases}
\]
otherwise,
to be the **generalized Tevelev degrees of** \( \mathbb{P}^1 \).

The goal of this paper is to give explicit formulas for these Tevelev degrees following the two degeneration approaches of 3 (via excess intersections on \( \overline{\cal H}_{g, d, \mu_1, \ldots, \mu_k} \)) and 7 (via limit linear series).

1.4. **Results and overview of the paper.** In 2 we employ the strategy of 3 to compute the generalized Tevelev degrees of \( \mathbb{P}^1 \).

To begin, in 2 we reduce the general problem to the case in which \( e_{h, j} = 1 \) for all \( h, j \).

Namely, we prove the following equality:

**Proposition 6.** Let \( g \geq 0, \ell \in \mathbb{Z} \) and \( \mu_1, \ldots, \mu_k \) vectors with \( \mu_h \in \mathbb{Z}_{\geq 1}^{r_h} \) where \( r_h \geq 1 \) for all \( h = 1, \ldots, k \). Assume that the inequalities 4, 5 and 6 hold. Then we have:
\[
\text{Tev}_{g, \ell, \mu_1, \ldots, \mu_k} = \text{Tev}_{g, \ell, (\mu_1, \ldots, \mu_k)}.
\]

In 3 we use the same strategy as in 3 to find a recursion for the Tevelev degrees: we degenerate \( C \) to a 2-nodal curve obtained by attaching a rational bridge to a general smooth curve of genus \( g - 1 \), and reduce, via excess intersections on Hurwitz spaces, to the case of genus \( g = 0 \). We find:

**Proposition 7.** Let \( g \geq 1, \ell \in \mathbb{Z} \) an integer and \( r_1, \ldots, r_k \in \mathbb{Z}_{\geq 1} \) positive integers. Assume that the Equations 4, 5 and 6 hold with
\[
\mu_h = (1, \ldots, 1)
\]
for \( h = 1, \ldots, k \). Then, we have the recursion
\[
\text{Tev}_{g, \ell, r_1, \ldots, r_k} = \text{Tev}_{g-1, \ell, r_1, \ldots, r_k, r_k-1} + \text{Tev}_{g-1, \ell+1, r_1, \ldots, r_k, r_k+1, r_k+1}.
\]
where by definition we set

\[ \text{Tev}_{g-1,\ell,r_1,...,r_{k-1},0} = \text{Tev}_{g-1,\ell,r_1,...,r_{k-1}} \]

The genus 0 case is established by hand in §4.

**Proposition 8.** For \( \ell \geq 0 \) and \( r_1, ..., r_k \in \mathbb{Z}_{\geq 1} \) positive integers such that the Equations (4), (5) and (6) hold with

\[ \mu_h = (1,...,1) \]

for \( h = 1, ..., k \). Then we have

\[ \text{Tev}_{0,\ell,r_1,...,r_k} = 1. \]

Finally, in §5 we completely solve the recursion, obtaining the following closed formulas for the Tevelev degrees:

**Theorem 9.** Let \( g \geq 0, \ell \in \mathbb{Z} \) and \( \mu_1,...,\mu_k \) be vectors where \( \mu_i \in \mathbb{Z}_{\geq 1} \) with \( r_i \geq 1 \). Assume that the Equations (4), (5) and (6) hold. Then

\[ \text{Tev}_{g,\ell,\mu_1,...,\mu_k} = 2^g - 2 \sum_{i=0}^{\ell-2} \binom{g}{i} \left( (-\ell - k - 2 + \sum_{h=1}^{k} |\mu_h|) \binom{g}{-\ell - 1} \right) \]

\[ + \left( \ell - k + \sum_{h=1}^{k} \delta_{|\mu_h|,1} \right) \binom{g}{-\ell} - \sum_{h=1}^{k} \sum_{i=\ell+1}^{k-2} \binom{g}{i}. \]

Here \( \delta_{n,1} \) is the Kronecker delta.

In §6 we give a second approach to the generalized Tevelev degrees, expanding on that of §7. Namely, we consider a compact type degeneration of \( C \) and apply the theory of Eisenbud-Harris limit linear series [5]. Some care is needed to handle incidence conditions imposed at marked points lying on different components of the degeneration of \( C \). Our method essentially reduces the problem to a computation in genus 0, and yields the following:

**Theorem 10.** Let \( g \geq 0, \ell \in \mathbb{Z} \) and \( \mu_1,...,\mu_k \) be vectors where \( \mu_h \in \mathbb{Z}_{\geq 1}^{r_h} \) with \( r_h \geq 1 \). Suppose that equations (4), (5) and (6) hold. Then, the generalized Tevelev degree \( \text{Tev}_{g,\ell,\mu_1,...,\mu_k} \) is equal to

\[ \sum_{(I,J):I \cap J = \emptyset} (-1)^{\# J} \int_{\text{Gr}(2,d+1-J)} \left( \prod_{h \in I} \sigma_{|\mu_h|-1} \prod_{h \in J} \sigma_{|\mu_h|-2} \right) \cdot \sigma_1^g \]

\[ \cdot \sum_{\{(i,j):i+j=k-3-J\}} \sigma_i \sigma_j. \]

Here, we adopt the standard notation \( \sigma_i = s_i(\mathcal{V}) \in A^i(\text{Gr}(2,d+1)), \) where \( \mathcal{V} \) is the universal rank 2 subbundle on the Grassmannian \( \text{Gr}(2,d+1) \), so that \( \sigma_i = 0 \) if \( i < 0 \) or \( i > d-1 \).

In the case \( |\mu_1| = \cdots = |\mu_k| = 1 \), the formula recovers that of [7] Theorem 1.2.

We also show in §6.4 that the recursion of Proposition 7 and the genus 0 base case of Proposition 8 can be extracted from the formula of Theorem 10.

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2. Reduction to Unramified Marked Points

Let \( g \geq 0 \) be a genus, \( \ell \in \mathbb{Z} \) an integer and \( \mu_1, \ldots, \mu_k \) vectors with \( \mu_h \in \mathbb{Z}_{\geq 1} \) where \( r_h \geq 1 \) for all \( h = 1, \ldots, k \). Assume that the Equations (4), (5) and (6) hold.

In this section, we prove that the generalized Tevelev degrees depend only on the total incidence order over each marked point of the target. Namely, we find the following equality, which implies Proposition 6

\[
\text{Te}_v^{g, \ell, \mu_1, \ldots, \mu_k} = \text{Te}_v^{g, \ell, \mu_1, \ldots, \mu_{k-1}, |\mu_k|}
\]

where \((|\mu_k|)\) is a vector in \( \mathbb{R} \).

Note that we may assume \( r_k \geq 2 \). The strategy is simple: we show that, upon coalescing the points \( p_{n-r_k+1}, \ldots, p_n \) to into the single point \( p_{n-r_k+1} \), the incidence conditions at \( p_{n-r_k+1}, \ldots, p_n \) turn into a single incidence condition at \( p_{n-r_k+1} \) with ramification order \( |\mu_k| \).

Formally, let \( \Gamma_0 \) be the stable graph

\[
\Gamma_0 = \begin{array}{c}
p_1 \\
\vdots \\
p_{n-r_k} \\
p_n \\
p_{n-r_k+1} \\
\vdots \\
p_n
\end{array}
\]

where \( g \) is the usual gluing map.

Following the notation of Definition 2, let \( \overline{\mathcal{M}}_{g,n} \) be the Hurwitz space whose general point parametrizes covers \( \pi : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree \( |\mu_k| \) sending \( r_k \) marked points \( p_{n-r_k+1}, \ldots, p_n \) to a single point \( q_{n-k} \) with ramification indices \( e_{k,1}, \ldots, e_{k,r_k} \), respectively, with an additional marked point \( p \) of total ramification over the target. (We assume as usual that \( \pi \) is otherwise simply branched.)

Let

\[
\xi_{\Gamma_0} : \overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n-r_k+1} \times \overline{\mathcal{M}}_{0,r_k+1} \to \overline{\mathcal{M}}_{g,n}
\]

be the map remembering the (stabilized) marked source.

The space \( \overline{\mathcal{H}}_{0,|\mu_k|,|\mu_k|,|\mu_k|} \) also defines a boundary divisor

\[
\overline{\mathcal{H}}_{g,d,0,0, \mu_1, \ldots, \mu_k-1, |\mu_k|} \times \overline{\mathcal{H}}_{0,|\mu_k|,|\mu_k|,|\mu_k|} \to \overline{\mathcal{H}}_{g,d,0,0, \mu_1, \ldots, \mu_k}
\]

by gluing the source curves at the points of order \( |\mu_k| \) ramification and gluing \( d - |\mu_k| \) additional rational tails mapping isomorphically to the target, see also Lemma 11.

We will show that there exists a commutative diagram

\[
\begin{array}{cccc}
\overline{\mathcal{H}}_{g,d,0,0, \mu_1, \ldots, \mu_k-1, |\mu_k|} \times \overline{\mathcal{H}}_{0,|\mu_k|,|\mu_k|,|\mu_k|} & \xrightarrow{\tau_{g,d,0,0, \mu_1, \ldots, \mu_k}} & \overline{\mathcal{H}}_{g,d,0,0, \mu_1, \ldots, \mu_k} \\
\overline{\mathcal{M}}_{g,n-r_k+1} \times \overline{\mathcal{M}}_{0,r_k+1} & \xrightarrow{\xi_{\Gamma_0} \times 1} & \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,k}
\end{array}
\]
such that the induced map from the top left space to the cartesian product is birational onto the components of the fiber product that dominate $\mathcal{M}_{g,n-r_k+k} \times \hat{\mathcal{M}}_{0,n-r_k+1} \times \mathcal{M}_{0,k}$. Here, the disjoint union is over a certain set of combinatorial choices, as detailed in Lemma 11 and the two maps

$$
\tau_{g,\ell,\mu_1,\ldots,\mu_k} : \mathcal{H}_{g,\ell,\mu_1,\ldots,\mu_k} \to \mathcal{M}_{g,n} \times \mathcal{M}_{0,k}
$$

$$
\tau_{g,\ell,\mu_1,\ldots,\mu_k,1} : \mathcal{H}_{g,\ell,\mu_1,\ldots,\mu_k,1} \to \mathcal{M}_{g,n-r_k+1} \times \mathcal{M}_{0,k}
$$

are as in §1.3. We will conclude by comparing their degrees.

We now proceed to the proof. We have a commutative diagram

$$
\begin{array}{ccc}
\bigcup_{(\Gamma,\Gamma')} \mathcal{H}_{(\Gamma,\Gamma')} & \xrightarrow{\bigcup_{(\Gamma,\Gamma')} \xi_{\mu,\mu'}} & \mathcal{H}_{g,\ell,\mu_1,\ldots,\mu_k} \\
\downarrow & & \downarrow \\
\bigcup_{\Gamma_0} \mathcal{H}_{\Gamma_0} \times \mathcal{M}_{0,k} & \xrightarrow{\bigcup_{\Gamma_0} \xi_{\mu,\mu'} \times 1} & \mathcal{M}_{g,n} \times \mathcal{M}_{0,k} \\
\downarrow & & \downarrow \\
\mathcal{H}_{\Gamma_0} \times \mathcal{M}_{0,k} & \xrightarrow{\xi_{\mu,\mu'} \times 1} & \mathcal{M}_{g,n} \times \mathcal{M}_{0,k}
\end{array}
$$

(9)

In the middle row, we have added back the data of the $b$ simple ramification points, so that the composite vertical arrow on the right is again $\tau_{g,\ell,\mu_1,\ldots,\mu_k}$. The lower disjoint union on the left side of the diagram is over all stable graphs $\Gamma_0$ that can be obtained by distributing $b$ legs to the two vertices of $\Gamma_0$. The lower square in the diagram is then not Cartesian, but the disjoint union of the $\mathcal{H}_{\Gamma_0} \times \mathcal{M}_{0,k}$ maps properly and birationally to the corresponding fiber product.

The upper left entry $\bigcup_{(\Gamma,\Gamma')} \mathcal{H}_{(\Gamma,\Gamma')}$ is the disjoint union of boundary divisors of $\mathcal{H}_{g,\ell,\mu_1,\ldots,\mu_k}$, where the index set of the disjoint union consists of isomorphism classes of $\Gamma_0$-structures on $\Gamma$ (See [11], Definition 2.5) such that the composition

$$
E(\hat{\Gamma}_0) \subseteq E(\Gamma) \to E(\Gamma')
$$

is surjective. Then, by [8] Proposition 3.2, the upper square is Cartesian (a priori) on the level of closed points, and $\mathcal{H}_{(\Gamma,\Gamma')}$ is identified with the underlying reduced space of the fiber product. (Note: our diagram differs from that of [8] Proposition 3.2] in that we have extra factors of $\mathcal{M}_{0,k}$ parametrizing the marked target, on which the horizontal maps act by the identity; the Cartesianness of the upper square is preserved.)

Finally, we denote the composition

$$
\mathcal{H}_{(\Gamma,\Gamma')} \to \mathcal{H}_{\Gamma_0} \times \mathcal{M}_{0,k}
$$

by $\tau_{(\Gamma,\Gamma')}$. Because we are interested in degrees of the vertical maps, we need only consider components $\mathcal{H}_{(\Gamma,\Gamma')}$ mapping dominantly under $\tau_{(\Gamma,\Gamma')}$. We will see in Corollary 12 that, when restricted to such components, the upper part of diagram (9) is in fact Cartesian. The following lemma shows that, up to relabelling of the ramification and branch legs, there is only one possible $(\Gamma, \Gamma')$ for which $\tau_{(\Gamma,\Gamma')}$ is dominant.
Lemma 11. The pairs $(\Gamma, \Gamma')$ yielding dominating components in Diagram $\mathfrak{9}$ have the following form:

\[ \Gamma = \begin{array}{c}
p_1 \quad \cdots \quad p_{n-r_k+1} \\
\vdots \\
p_{n-r_k} \\
\end{array} \begin{array}{c}
g \\
\mu_k \\
\mu_k \\
0 \\
\end{array} \begin{array}{c}
b - r_k + 1 \\
r_k - 1 \\
\end{array} \]

\[ \Gamma' = \begin{array}{c}
q_1 \quad \cdots \\
\vdots \\
q_{k-1} \\
\end{array} \begin{array}{c}
0 \\
0 \\
\end{array} \begin{array}{c}
b - r_k + 1 \\
r_k - 1 \\
\end{array} \]

Here, the green legs correspond to ramification and branch points, the black legs correspond to markings, the blue numbers over a vertex of $\Gamma$ indicate the degree of the map when restricted to the corresponding vertex and the blue numbers over an edge indicate the ramification index at the correspond node. In particular, over the right vertex of $\Gamma'$, there are $d - |\mu_k|$ additional genus 0 vertices of $\Gamma$, each with degree 1, which are not drawn. Also, the picture does not depict the fact that our covers are subject to restrictions (iii) and (iv,c) appearing in the definition of $\mathcal{P}_g,d,\mu_1,\ldots,\mu_k$.

Proof. We start with some preliminary observations:

(i) In order for the map $\tau_{(\Gamma,\Gamma')} \to \Gamma'$ to be surjective, the graph obtained from $\Gamma$ forgetting all the legs corresponding to ramification points and then stabilizing is canonically identified with $\Gamma_0$. In particular, the set $V(\Gamma)$ contains canonically defined vertices $v_k$ and $v_0$ of genus $g$ and 0, respectively, corresponding to the vertices of $\Gamma_0$, which persist after the contraction. We also denote the degrees of $v_k$ and $v_0$ by $d_k$ and $d_0$, respectively.

(ii) From the fact that the graph $\Gamma$ comes endowed with a $\Gamma_0$ structure such that the composition $E(\Gamma_0) \subseteq E(\Gamma) \to E(\Gamma')$ is surjective, we have that $\Gamma'$ has exactly one edge and two vertices. Denote by $w_0$ and $w_1$ the vertices of $\Gamma'$.

(iii) Every vertex $v$ of $\Gamma$ different from $v_0$ and $v_k$ cannot contain more than one ramification or marked point. Indeed, if $v$ contains multiple marked points, then the image of $\tau_{(\Gamma,\Gamma')}$ would be supported inside

\[ \partial(\mathcal{M}_k) \times \mathcal{M}_{0,1} \]

where $\partial$ denotes the boundary. On the other hand, if $v$ contains a ramification point in addition to another ramification or marked point, then the fibers of $\tau_{(\Gamma,\Gamma')} \to \Gamma'$ must have positive dimension, because the positions of the ramification points of a cover do not influence its image under $\tau_{(\Gamma,\Gamma')}$. In either case, we have a contradiction of the surjectivity of $\tau_{(\Gamma,\Gamma')}$. 

(iv) In order for the map $\tau_{(\Gamma,\Gamma')}$ to be surjective, exactly one vertex of $\Gamma'$ contains at most one leg corresponding to a marked point of the target. Without loss of generality, take $w_1$ to be this vertex.

Note (iii) implies that all edges of $\Gamma$ different from $v_0$ and $v_k$ are leaves, incident to one of $v_0$ or $v_k$ and no other vertices. In particular, $v_k$ and $v_0$ are joined by an edge $e$ of $\Gamma$ corresponding to a node whose ramification index we denote $\alpha \in \mathbb{Z}_{\geq 1}$. Moreover, we have that $v_k$ and all of the leaves attached to $v_0$ map to one of $w_0, w_1$, and that $v_0$ and all of the leaves attached to $v_k$ map to the other vertex of $\Gamma'$.
We distinguish two cases.

Case I) $v_g$ maps to $w_0$.

Let $\overline{H}_{w_0}, \overline{H}_{w_1}$ be the factors of $\overline{H}(\Gamma, \Gamma')$ corresponding to the cover over $w_0, w_1$, respectively. Denoting by $n_0, n_1$ be the valences of $w_0, w_1$, respectively, we have
\[
\dim(\overline{H}_{w_0}) = n_0 - 3,
\]
\[
\dim(\overline{H}_{w_1}) = n_1 - 3.
\]

We will show via a parameter count that the only way for the map
\[
\tau(\Gamma, \Gamma') : \overline{H}(\Gamma, \Gamma') = \overline{H}_{w_0} \times \overline{H}_{w_1} \rightarrow (\overline{M}_{g,n-r_k+1} \times \overline{M}_{0,k}) \times \overline{M}_{0,r_k+1}
\]
to be surjective is for $(\Gamma, \Gamma')$ to have the claimed form. Observe that the composition of $\tau(\Gamma, \Gamma')$ with the projection to $\overline{M}_{g,n-r_k+1} \times \overline{M}_{0,k}$ does not depend on the factor $\overline{H}_{w_1}$. Similarly, the composition of $\tau(\Gamma, \Gamma')$ with the projection to $\overline{M}_{0,r_k+1}$ does not depend on the factor $\overline{H}_{w_0}$.

It follows that
\[
\dim(\overline{H}_{w_0}) \geq \dim(\overline{M}_{g,n-r_k+1} \times \overline{M}_{0,k})
\]
\[
\dim(\overline{H}_{w_1}) \geq \dim(\overline{M}_{0,r_k+1}),
\]
and then that both inequalities are in fact equalities because the source and target of $\tau(\Gamma, \Gamma')$ each have dimension $\dim - 1$. In particular, we have $n(w_1) = r_k + 1$.

Now, we claim that the legs corresponding to $p_{n-r_k+1}, \ldots, p_n$ are incident to the vertex $v_0$, and that the only edge attached to $v_0$ is $e$. Suppose instead that these legs lie on genus 0 vertices (distinct from $v_0$) attached to $v_0$ and mapping to $w_0$. Note that these vertices cannot contain legs corresponding ramification point, by (iii) above. Therefore, we have $r_k$ distinct vertices of genus 0 containing the legs corresponding to $p_{n-r_k+1}, \ldots, p_n$. Let also $s$ be the number of genus 0 vertices (also distinct from $v_0$) mapped to $w_0$ with degree 2, joined to $v_0$ by an edge with ramification index 2 and carrying exactly one leg of $\Gamma$ corresponding to a ramification point.

Then, the degree of $v_0$ would be at least
\[
d_0 = a + 2s + |\mu_k|
\]
and the number of legs on $v_0$ corresponding to ramification points would be at least
\[
(2d_0 - 2) - (a - 1) - (|\mu_k| - r_k) - s = a + 3s + |\mu_k| - 1 + r_k,
\]
which is strictly larger than the number $n(w_1) - 1 = r_k$ allowed. This proves the claim.

We deduce that $d_0 = d$ and $a = d_0 \geq |\mu_k|$. Finally, using again the fact that $n(w_1) = r_k + 1$, we find that in fact $a = d_0 = |\mu_k|$ and that there are no ramification points other than those attached to $v_0$ over $w_1$. It follows that $\Gamma \rightarrow \Gamma'$ must be of the claimed form.

Case II) $v_g$ maps to $w_1$.

We show that this is not possible. Let $b$ be the total number of additional simple ramification points as before. Arguing as in the previous case, we find that $\dim(\overline{H}_{w_1}) = n(w_1) - 3 = b - r_k + 1$. We claim that the maximal number of legs corresponding to ramification points and lying over $w_1$ is $b - r_k + 1$. This yields a contradiction, as in this case $\dim(\overline{H}_{w_1}) \leq b - r_k + 1 + 2 - 3 = b - r_k$.

To prove the claim, suppose that over $w_1$ there are exactly $s$ additional genus 0 vertices with degrees equal to 2 and containing each exactly 1 simple ramification point. Also let $\gamma$ be equal to
\[ |\mu_k| \text{ if the marked point on } w_1 \text{ is } q_k \text{ and } 0 \text{ otherwise. Then the maximum number of ramification points over } w_1 \text{ is at most} \]
\[
(2d_g + 2g - 2) - (a - 1) + s - \sum_{h=1}^{k-1} |\mu_h| + \sum_{h=1}^{k-1} r_h
\]
\[
\leq 2d + 2g - 2 - (a - 1) - 3s - 2\gamma - \sum_{h=1}^{k-1} |\mu_h| + \sum_{h=1}^{k-1} r_h
\]
\[
\leq 2d + 2g - 2 + 1 - s - \gamma - |\mu_k| - \sum_{h=1}^{k-1} |\mu_h| + \sum_{h=1}^{k-1} r_h
\]
\[
= b - r_k + 1 - s - \gamma,
\]
where in the first inequality we use the fact that \( d_g + 2s + \gamma \leq d \) and in the second inequality the fact that \( a + 2s + \gamma \geq |\mu_k| \).

\[ \square \]

**Corollary 12.** The upper square in diagram \((9)\) is Cartesian when restricted to the components \( \mathcal{X}_{(\Gamma, \Gamma')} \) of Lemma\(11\).

**Proof.** We already have that the square in question from \((9)\) is Cartesian on the level of reduced spaces. On the other hand, a computation on the level of complete local rings shows that the fiber product in the upper square is reduced when restricted to the components \( \mathcal{X}_{(\Gamma, \Gamma')} \) because the covers parametrized by \( \mathcal{X}_{(\Gamma, \Gamma')} \) have exactly one ramified node, see \([8] \S3.2\). \[ \square \]

**Lemma 13.** Fix \( \mu = (e_1, \ldots, e_r) \in \mathbb{Z}_{\geq 1}^r \) and let \( \epsilon_{\mu} : \mathcal{X}_{0, [\mu], r} \rightarrow \overline{M}_{0, r+1} \) be the morphism recalling the (stabilized) domain genus 0 curve with all the markings, as defined above. Then,
\[
\deg(\epsilon_{\mu}) = (r - 1)!.
\]

**Proof.** Up to scaling, there is a unique meromorphic function with zeroes and poles prescribed by the \( r + 1 \) marked points. Then, \((r - 1)!\) is the number of ways the ways to label the ramification points of the resulting cover. \[ \square \]

**Proof of Proposition\(8\)** We compare the Televy degrees arising from diagram \((9)\). We have
\[
\deg(\tau_{g, l, \mu_1, \ldots, \mu_r}) = \left( b[g, l, \mu_1, \ldots, \mu_r] \right) (r_k - 1)! \deg(\tau_{g, l, \mu_1, \ldots, \mu_{k-1}, [\mu_k]})
\]
\[
= b[g, l, \mu_1, \ldots, \mu_k] \scriptstyle{\text{Tev}}_{g, l, \mu_1, \ldots, \mu_{k-1}, [\mu_k]} \]
where the first factor counts the number of possible \(( \Gamma, \Gamma' )\) as in Lemma\(11\), and the second factor comes from Lemma\(13\). The multiplicities of the contributions from each \( \mathcal{X}_{(\Gamma, \Gamma')} \) are 1, owing to Corollary\(12\), Equation \((7)\) follows, so we are done. \[ \square \]

## 3. Genus recursion

In this section, we prove the recursion of Proposition\(7\). The idea is exactly the same as that used in \([4]\) Proof of Proposition \(7\), which we briefly recall here for the reader’s convenience.

Let \( g \geq 1 \) be a genus, \( \ell \in \mathbb{Z} \) an integer and \( r_1, \ldots, r_k \in \mathbb{Z}_{\geq 1} \) positive integers. Set
\[
\mu_h = \left( \frac{1, \ldots, 1}{r_h} \right)
\]
for \( h = 1, \ldots, k \), and assume that the Equations (4), (5) and (6) hold. Let \( \Gamma_0 \) be the graph

\[
\Gamma_0 = \begin{array}{c}
p_1 \\
\vdots \\
p_{n-1} \\
g-1 \\
0 \\
p_n
\end{array}
\]

Form the commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{H}}_{(\Gamma, \Gamma')} & \xrightarrow{\prod \xi_{(\Gamma, \Gamma')}} & \tilde{\mathcal{H}}_{g, d, r_1, \ldots, r_k} \\
\downarrow & & \downarrow \\
\tilde{\mathcal{H}}_{\Gamma_0 \times M_{0,k}} & \xrightarrow{\prod \xi_{(\Gamma_0, \Gamma_0')}} & \tilde{\mathcal{H}}_{g, n+b} \times M_{0,k} \\
\downarrow & & \downarrow \\
\mathcal{M}_{\Gamma_0 \times M_{0,k}} & \xrightarrow{\xi_{(\Gamma_0, \Gamma_0')}} & \mathcal{M}_{g, n} \times M_{0,k}
\end{array}
\]

and denote by \( \tau_{(\Gamma, \Gamma')} \) the composite morphism

\[
\tilde{\mathcal{H}}_{(\Gamma, \Gamma')} \rightarrow \mathcal{M}_{\Gamma_0 \times M_{0,k}}
\]
as in the previous section.

**Lemma 14.** The pairs \( (\Gamma, \Gamma') \) appearing in diagram (10) for which \( \tilde{\mathcal{H}}_{(\Gamma, \Gamma')} \) dominates \( \mathcal{M}_{\Gamma_0 \times M_{0,k}} \) are the following:

- **type I:**

\[
\Gamma = \begin{array}{c}
p_1 \\
\vdots \\
p_{n-1} \\
g-1 \\
0 \\
p_n
\end{array}
\]

\[
\Gamma' = \begin{array}{c}
q_1 \\
\vdots \\
q_{k-1} \\
0 \\
0 \\
q_k
\end{array}
\]
• **type II:**

\[
\Gamma =
\begin{array}{c}
\vdots \\
p_n \\
\hline
p_{n-2}
\end{array}
\begin{array}{c}
g-1 \\
\hline
1
\end{array}
\begin{array}{c}
d \\
\hline
1
\end{array}
\begin{array}{c}
p_1 \\
\hline
1
\end{array}
\begin{array}{c}
1 \\
\hline
2
\end{array}
\begin{array}{c}
1 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\]

\[
\Gamma' =
\begin{array}{c}
\vdots \\
q_{k-1} \\
\hline
q_k
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\]

- **type III:**

\[
\begin{array}{c}
p_{n-1} \\
\hline
p_n \\
\hline
1
\end{array}
\begin{array}{c}
g-1 \\
\hline
1
\end{array}
\begin{array}{c}
d-1 \\
\hline
1
\end{array}
\begin{array}{c}
p_1 \\
\hline
2
\end{array}
\begin{array}{c}
2 \\
\hline
2
\end{array}
\begin{array}{c}
1 \\
\hline
2
\end{array}
\begin{array}{c}
1 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\begin{array}{c}
0 \\
\hline
2
\end{array}
\]

Conventions here are the same as in Lemma 11.

Moreover, the upper square in diagram (10) is Cartesian when restricted to these components \(\mathcal{M}_{(\mathcal{G}, \Gamma')}.\)

**Proof.** The proof follows exactly the lines of the proof of [4, Proposition 7]. □

To compute the degree of \(\tau_{g, d, n, r_1, \ldots, r_k} : \mathcal{M}_{g, d, n, r_1, \ldots, r_k} \to \mathcal{M}_{g, n} \times \mathcal{M}_{0, k},\) the excess intersection formula gives

\[
\deg(\tau_{g, d, n, r_1, \ldots, r_k}) = \sum_{(\mathcal{G}, \Gamma')} \deg(\{ c(N_{(\mathcal{G}, \Gamma')}) \cap s(\mathcal{M}_{(\mathcal{G}, \Gamma'), \mathcal{M}_{g, d, n, r_1, \ldots, r_k}) \}_{\dim-2})
\]

where the sum is over all the pairs \((\mathcal{G}, \Gamma')\) appearing in diagram (10), the line bundle \(N_{(\mathcal{G}, \Gamma')} = \tau^*_{(\mathcal{G}, \Gamma')} N_{\mathcal{G}_{1, k}}\) is the pullback under \(\tau_{(\mathcal{G}, \Gamma')}\) of the normal bundle of \(\mathcal{G}_{1, k} \times 1\) and the terms in the sum are all in homological dimension \(\dim-2.\) On the right hand side, \(\deg\) denotes the degree of the given class given upon pushforward to \(\mathcal{M}_{\mathcal{G}} \times \mathcal{M}_{0, k},\) as a multiple of the fundamental class.
Following the computations done in [4] Proposition 7, we see that:

- from the $(\Gamma, \Gamma')$ of type I, we get the contribution
  $$-b[g, l, r_1, ..., r_k] \, \Tev_{g-1, l, r_1, ..., r_k+1},$$

- from the $(\Gamma, \Gamma')$ of type II, we get the contribution
  $$2b[g, l, r_1, ..., r_k] \, \Tev_{g-1, l, r_1, ..., r_k+1},$$

- and from the $(\Gamma, \Gamma')$ of type III, we get the contribution
  $$b[g, l, r_1, ..., r_k] \, \Tev_{g-1, l, r_1, ..., r_k-1}.$$

Proposition [4] follows by summing these three terms.

4. GENUS 0

Now, we consider the genus 0 case. The proof here differs slightly from that of the analogous [4] Proof of Proposition 8.

Let $g = 0$, $\ell \geq 0$ and $r_1, ..., r_k \in \mathbb{Z}_{\geq 1}$ positive integers such that the Equations (4), (5) and (6) hold with

$$\mu_h = (1, ..., 1)_{r_h}$$

for $h = 1, ..., k$.

Let us first explain the argument. A map $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d$ is given by a pair of polynomials $s_0, s_1$ of degree $d$ up to simultaneous scaling. We may parametrize the coefficients of the $f_i$ by the projective space $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(d))) \cong \mathbb{P}^{2d+1}$. Then, the $n = 2d + 1$ incidence conditions on $f$ are cut out by hyperplanes $\mathbb{P}^{2d+1}$, which are expected to intersect in a single point giving a unique map $\pi$ of degree $d$. The following shows that this is indeed the case when the $p_i, q_j$ are general.

Formally, consider the closed subscheme

$$V_{d, r_1, ..., r_k} \subseteq \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(d))) \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^k$$

defined by the condition that the $C$-point

$$[(s_0 : s_1, p_1, ..., p_n, q_1 = [q_1^0 : q_1^1], ..., q_k = [q_k^0 : q_k^1])$$

of $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d)) \times H^0(\mathbb{P}^1, \mathcal{O}(d))) \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^k$ belongs to $V_{d, r_1, ..., r_k}$ if and only if $q_j^1 s_0(p_i) - q_0^0 s_1(p_i) = 0$ for $1 \leq j \leq k$ and $n - \sum_{h=j+1}^{k} r_h + 1 \leq i \leq n - \sum_{h=1}^{k} r_h$.

Also, let

$$V^o_{d, r_1, ..., r_k} \subseteq V_{d, r_1, ..., r_k}$$

be the open subscheme consisting of points

$$(\pi = [s_0 : s_1, p_1, ..., p_n, q_1, ..., q_k])$$

where the morphism $\pi$ has exactly degree $d$, is simply branched (i.e., the ramification index of $\pi$ is $\leq 2$ at every point of $\mathbb{P}^1$ and different ramification points in the domain are mapped to different branch points in the target) and the branch points of $\pi$ are distinct from the $q_1, ..., q_k$.

Clearly,

$$V^o_{d, r_1, ..., r_k} \neq \emptyset.$$
We have the following commutative triangle:

\[
\begin{array}{ccc}
V_{d,r_1,...,r_k} & \longrightarrow & \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(d))) \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^k \\
\longrightarrow & p & \downarrow \text{projection} \\
& (\mathbb{P}^1)^n \times (\mathbb{P}^1)^k & 
\end{array}
\]

(11)

The next lemma implies that $\tau_{0,\ell,r_1,...,r_k}$ has degree 1.

**Lemma 15.** The restriction $p|_{V^\circ_{d,r_1,...,r_k}}$ of the map in Diagram(7) to $V^\circ_{d,r_1,...,r_k}$ is injective on closed points. Moreover, the set-theoretic image $p(V^\circ_{d,r_1,...,r_k})$ is dense in $(\mathbb{P}^1)^n \times (\mathbb{P}^1)^k$.

**Proof.** First, we prove injectivity. Let

\[
\pi_1, \pi_2 : \mathbb{P}^1 \to \mathbb{P}^1
\]

be two degree $d$ maps in the fiber under $p$ of $(p_1, p_2, q_1, ..., q_k)$. Then, the quotient map

\[
f = \frac{\pi_1}{\pi_2} : \mathbb{P}^1 \to \mathbb{P}^1
\]

has degree at most $2d = n - 1$ and the preimage of 1 under $f$ contains at least $n$ points $p_1, ..., p_n$. Therefore, $f$ is the constant map equal to 1 and $\pi_1 = \pi_2$. This shows that $p|_{V^\circ_{d,r_1,...,r_k}}$ is injective on closed points.

The second claim follows from the fact that

\[
\dim(V^\circ_{d,r_1,...,r_k}) \geq \dim((\mathbb{P}^1)^n \times (\mathbb{P}^1)^k)
\]

combined with the first claim. \qed

**Proof of Proposition 8.** Lemma 15 shows that a general choice of points on the source and target curve gives rise a unique cover $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ counted in $\text{Tev}_{0,\ell,r_1,...,r_k}$, so the conclusion is immediate. \qed

5. Explicit solution to the recursion

5.1. Results for $\ell \geq 0$. We start with case $\ell \geq 0$, in which our formulas are simpler.

**Proposition 16.** Suppose $g \geq 0$, $\ell \geq 0$ and $r_1, ..., r_k \in \mathbb{Z}_{\geq 1}$. Assume that the Equations (4), (5) and (6) hold with

\[
\mu_h = (1, ..., 1)
\]

for $h = 1, ..., k$. Then, we have

\[
\text{Tev}_{g,\ell,r_1,...,r_k} = \begin{cases} 
2^g - \sum_{h=1}^{k} \sum_{j=0}^{r_n-2} \binom{g}{j} & \text{for } \ell = 0, \\
\text{Tev}_{g,0,r_1-\ell,...,r_k-\ell} & \text{for } \ell < \max(r_1, ..., r_k), \\
2^g & \text{for } \ell \geq \max(r_1, ..., r_k).
\end{cases}
\]

Here, for $\ell < \max(r_1, ..., r_k)$, we set:

\[
\text{Tev}_{g,0,r_1-\ell,...,r_k-\ell} := \text{Tev}_{g,0,r_1-\ell,...,r_k-\ell,1,...,1}
\]

where the indices $h \in \{i_1, ..., i_t\}$ are those for which $r_h > \ell$ and at the end there are $g + 3 - \sum_{h=1}^{t} (r_h - \ell)$ numbers 1.
Proof. We start by noticing that for \( \ell = \max(r_1, \ldots, r_k) \), the two definitions of the right hand side coincide. The proof of the proposition is by induction on \( g \).

• Suppose that \( g = 0 \). Then, by Proposition \ref{prop:base_case}, the left hand side is equal to 1, while for the left hand side we have to distinguish three cases:

  - If \( \ell = 0 \), then \((d, n) = (1, 3)\) and thus \( k = 3 \) and \( r_h = 1 \) for all \( h \leq k \). The right hand side is then equal to 1.
  - If \( \ell < \max(r_1, \ldots, r_k) \), then there are at most 3 and at least 1 indices \( h \in \{1, \ldots, k\} \) for which \( r_h > \ell \). Moreover, for these \( h \), it must be \( r_h = \ell + 1 \). The right hand side is therefore equal to \( \text{Tev}_{0,0,1,1,1} \), which is equal to 1 by Proposition \ref{prop:base_case}.
  - If \( \ell \geq \max(r_1, \ldots, r_k) \), then we have \( 2^g = 1 \) on the right hand side.

• Suppose that \( g \geq 1 \). Then, we distinguish two cases:

  - Suppose \( \ell \geq \max(r_1, \ldots, r_k) \). Then, this inequality persists during the recursion, and we have

    \[
    \text{Tev}_{g, \ell, r_1, \ldots, r_k} = \text{Tev}_{g-1, \ell, r_1, \ldots, r_k} + \text{Tev}_{g-1, \ell+1, r_1, \ldots, r_k+1} = 2^{g-1} + 2^{g-1} = 2^g
    \]

    where in the second equality we have used the inductive hypothesis. We leave to the reader to check that the Equations (4), (5) and (6) also persist during the recursion.

  - Suppose instead that \( \ell < \max(r_1, \ldots, r_k) \). Then the non-strict inequality persists during the recursion and, for \( r_k < d \), we can write

    \[
    \text{Tev}_{g, \ell, r_1, \ldots, r_k} = \text{Tev}_{g-1, \ell, r_1, \ldots, r_k} + \text{Tev}_{g-1, \ell+1, r_1, \ldots, r_k+1} = 2^{g-1} - \sum_{h=1}^{k-1} \sum_{j=0}^{n-\ell-2} \binom{g-1}{j} - \sum_{j=0}^{n-\ell-2} \binom{g-1}{j} + 2^{g-1} - \sum_{h=0}^{k-1} \sum_{j=0}^{n-\ell-3} \binom{g-1}{j} - \sum_{j=0}^{n-\ell-3} \binom{g-1}{j}
    \]

    The case \( r_k = d \) requires special attention, because in this case \( d[g-1, \ell + 1] < r_k + 1 \) and

    \[
    \text{Tev}_{g-1, \ell, r_1, \ldots, r_{k-1}, r_k+1} = 0.
    \]

  - A priori, we cannot apply the inductive hypothesis in the second and third equalities above. However, the proof goes through also in this case because when \( r_k = d \), by condition \ref{cond:rh}, it must be that \( r_h \leq 1 + \ell \) for all \( h < k \) and thus also

    \[
    2^{g-1} - \sum_{h=0}^{k-1} \sum_{j=0}^{n-\ell-3} \binom{g-1}{j} - \sum_{j=0}^{n-\ell-2} \binom{g-1}{j} = 2^{g-1} - \sum_{j=0}^{g-1} \binom{g-1}{j} = 0.
    \]

\( \square \)
5.2. Results for general \( \ell \). Finally, the master result which also covers the \( \ell \geq 0 \) case takes the following form:

**Theorem 17.** Suppose \( g \geq 0, \ell \in \mathbb{Z} \) and \( r_1, \ldots, r_k \in \mathbb{Z}_{\geq 1} \). Assume that the inequalities (4), (5) and (6) hold with

\[
\mu_h = \left( \frac{1, \ldots, 1}{r_h} \right)
\]

for \( h = 1, \ldots, k \). Then,

\[
\text{Tev}_{g, \ell, r_1, \ldots, r_k} = 2^g - 2 \sum_{i=0}^{-\ell-2} \binom{g}{i} + (-\ell - k - 2 + n) \binom{g}{-\ell -1} + \left[ \ell - k + \sum_{h=1}^{k} \delta_{r_h, 1} \right] \binom{g}{-\ell} - \sum_{h=1}^{k} \sum_{i=-\ell+1}^{-\ell} \binom{g}{i}.
\]

Here \( \delta_{r_h, 1} \) is the Kronecker delta.

**Proof.** We prove also this statement by induction on \( g \).

- If \( g = 0 \), then the left hand side is equal to 1, and all terms of the right hand side except \( 2^g = 1 \) vanish when \( \ell > 0 \). The only other possibility is \( \ell = 0 \), in which case \( d = \ell + 1 = 1 \) and \( r_h = 1 \) for \( h = 1, 2, \ldots, k \), so the only remaining term

\[
\left( \ell - k + \sum_{h=1}^{k} \delta_{r_h, 1} \right)
\]

also vanishes.

- Suppose instead that \( g \geq 1 \). We may assume that \( r_1 \leq \ldots \leq r_k \). Also, since in the case \( r_1 = \ldots = r_{k-1} = 1 \) the formula reduces to [4] Theorem 8, we may assume that \( k > 1 \) and \( r_{k-1} > 1 \). We then write

\[
\text{Tev}_{g, \ell, r_1, \ldots, r_k} = \text{Tev}_{g-1, \ell, r_1, \ldots, r_{k-1}, r_k-1} + \text{Tev}_{g-1, \ell+1, r_1, \ldots, r_{k-1}, r_k+1}
\]

and, for \( r_k \neq d \), we can apply to inductive hypothesis to rewrite the right hand side as

\[
2^{g-1} - 2 \sum_{j=0}^{-\ell-2} \binom{g-1}{j} + (-\ell - k - 2 + \sum_{h=1}^{k-1} r_h + r_k - 1) \binom{g-1}{-\ell - 1} + \left( \ell - k + \sum_{h=1}^{k} \delta_{r_h, 1} + \delta_{r_{k-1}, 1} \right) \binom{g-1}{-\ell} - \sum_{h=1}^{k-1} \sum_{j=\ell+1}^{-\ell} \binom{g-1}{j} - \sum_{j=\ell+1}^{-\ell} \binom{g-1}{j}
\]

\[(12)\]

\[
+ 2^{g-1} - 2 \sum_{j=0}^{-\ell-3} \binom{g-1}{j} + (-\ell - 1 - k - 2 + \sum_{h=1}^{k-1} r_h + r_k + 1) \binom{g-1}{-\ell-2} + \left( \ell + 1 - k + \sum_{h=1}^{k-1} \delta_{r_h, 1} + \delta_{r_{k-1}, 1} \right) \binom{g-1}{-\ell-1} - \sum_{h=1}^{k-1} \sum_{j=\ell}^{-\ell-2} \binom{g-1}{j} - \sum_{j=\ell}^{-\ell-2} \binom{g-1}{j}.
\]

Now, transferring a term \( \binom{g-1}{-\ell-1} \) from the last row to the second row and combining the corresponding terms in rows 1 and 3 and in 2 two and 4, we get the desired formula.

When \( r_k = d \), we have \( d[g-1, \ell+1] < r_k+1 \), so the inductive hypothesis may not be applied directly. In this case, we have

\[
\text{Tev}_{g-1, \ell+1, r_1, \ldots, r_{k-1}, r_k+1} = 0.
\]
Moreover, by condition 3, it must be that $\ell > -2$ and $t_h < \ell + 2$ for all $h < k$. Also, since we are assuming $r_{k-1} > 1$, it must be that $\ell \geq 0$, and thus the sum of the terms in the third and fourth rows of Equation 12 is

$$2^{g-1} - \sum_{j=0}^{g-1} \binom{g-j}{j} = 0.$$ 

This concludes the proof. \[\square\]

**Proof of Theorem 5** Combine Theorem 17 and Proposition 6. \[\square\]

6. Schubert Calculus Formula via Limit Linear Series

In this section, we explain a second approach to the generalized Tevelev degrees and establish the alternate formula of Theorem 10.

The strategy is as follows. We wish to count degree $d$ morphisms $\pi : C \to \mathbb{P}^1$ satisfying certain incidence conditions. Such a morphism is determined by a line bundle $\mathcal{L}$ on $C$ of degree $d$, along with two linearly independent sections $f_0, f_1 \in H^0(C, \mathcal{L})$ spanning a rank 1 linear series $V = \langle f_0, f_1 \rangle \subset H^0(C, \mathcal{L})$. The condition that $f(p) = q$ for $p \in C$ and $q \in \mathbb{P}^1$ translates to a linear condition on the sections $f_0, f_1$ upon evaluation at $p$.

We then count $V, f_0, f_1$ on $C$ satisfying the desired properties by considering the limit of this data on a degeneration of $C$. Our approach is via the theory of limit linear series [5], with which we assume some familiarity.

6.1. Degeneration setup. For notational convenience, we will assume that henceforth $\mu_h = (1)^{[\mu_h]} = (1)^h$ for $h = 1, 2, \ldots, k$; by Proposition 6, we may reduce to this case, but in fact, the proof that follows works equally well in the general case.

We fix, for the entirety of the proof of Theorem 10 a 1-parameter family $\psi : \mathcal{C} \to B, p_1, \ldots, p_n : B \to \mathcal{C}$ of $n$-pointed genus $g$ curves. $B$ may be taken to be a complex analytic disk or the spectrum of a discrete valuation ring with residue field $C$. We assume that the total space $\mathcal{C}$ is smooth, the generic fiber $\mathcal{C}_g$ is a general curve over the residue field of $\eta \in B$ (or a 1-parameter family of general curves over a punctured disc), and the special fiber $C_0$ has the following form.

The nodal curve $C_0$ contains a rational component (spine) $C_{sp}$, to which general elliptic tails $E_1, \ldots, E_g$ are attached at general points $s_1, \ldots, s_g \in C_{sp}$, respectively, and rational tails $R_1, \ldots, R_k$ are attached at general points $t_1, \ldots, t_k \in C_{sp}$, respectively.

We assume that the $r_h$ pre-images of the target point $q_h \in \mathbb{P}^1$ are situated at general points of the rational tail $R_h$ for $h = 1, \ldots, k$ (see Figure 3). The family may be regarded as arising from the collision (in the limit) of these $r_h$ pre-images at the point $t_h$ for each $h$. Note that we allow $r_h = 1$, so the pointed curve $C_0$ may not be stable.

Let $G^1_{\mathcal{C} / B} \to B$ be the proper moduli space of (possibly crude) limit linear series of rank 1 and degree $d$ on the fibers of $\psi$. We recall some aspects of the construction, see e.g. [10], Proposition 3.2.5] for a modern treatment with further details.

Let $\text{Pic}_{d, sp}(\mathcal{C} / B)$ be the relative Picard scheme of line bundles of degree $d > 0$ on the fibers of $\psi$, where on the special fiber of $\mathcal{C}$ we impose the finer requirement that the line bundle have degree $d$ on $C_{sp}$ and 0 on all other components. Then, $\mathcal{C} \times_B \text{Pic}_{d, sp}(\mathcal{C} / B)$ is equipped with a Poincaré line bundle $\mathcal{L}_{sp}$, as well as twists $\mathcal{L}_i$ for all other components $C_i \subset C_0$, which have the property that the restriction of $\mathcal{L}_i$ to $C_0 \times \{\mathcal{L}_i\}$ has degree $d$ on $C_i$ and degree 0 on all other components.

From here, $G^1_{\mathcal{C} / B}$ may be constructed as a closed subscheme (with conditions defined by the vanishing of sections at the nodes of $C_0$) of a product of Grassmannian bundles $\text{Gr}(2, \mathcal{W}_{i})$ over $\text{Pic}_{d, sp}(\mathcal{C} / B)$. Here, $\mathcal{W}_i$ is the vector bundle on $\text{Pic}_{d, sp}(\mathcal{C} / B)$ whose fiber is given by the
space $H^0(C, \mathcal{L})$ at a point over the generic fiber of $B$ and $H^0(C_i, \mathcal{L}_i)$ over the special fiber. In particular, $G^1_d(\mathcal{E}/B)$ carries tautological rank 2 bundles $\mathcal{V}_i$ for each component $C_i \subset C_0$, which agree over the generic fiber of $B$, and are given by the aspect of the linear series on $C_i$ over the special fiber.

Now, let $\text{Coll}^1_d(\mathcal{E}/B) \to G^1_d(\mathcal{E}/B)$ the $\mathbb{P}^3$-bundle associated to $\mathcal{V}_i^{sp}$. A point of $\text{Coll}^1_d(\mathcal{E}/B)$ parametrizes a point of $G^1_d(\mathcal{E}/B)$ and a pair of sections $[f_0 : f_1]$, taken up to simultaneous scaling, of the underlying linear series $\mathcal{V}$ (on the generic fiber of $\mathcal{E}$) or of the $C_{\text{sp}}$-aspect $\mathcal{V}_{\text{sp}}$ (on the special fiber). Such a point may be regarded as a complete collineation $C^2 \to V$ or $C^2 \to V_{\text{sp}}$ in the sense of $[13]$. We will refer to the divisor on $\text{Coll}^1_d(\mathcal{E}/B)$ parametrizing complete collineations where $f_0, f_1$ are linearly dependent as the rank 1 locus.

Fix, for the rest of the proof of Theorem $[10]$ general points $q_1, \ldots, q_k \in \mathbb{P}^1$.

**Definition 18.** We define a point $(V, [f_0 : f_1])$ of the generic fiber of $\text{Coll}^1_d(\mathcal{E}/B) \to B$ to be a Tevelev point if $[f_0 : f_1]$ defines a degree $d$ cover $\pi : C \to \mathbb{P}^1$ satisfying the required incidence conditions of Definition $[13] (iv)$.

By definition, the number of Tevelev points on the generic fiber is equal to $\text{Tev}_{g, d, r_1, \ldots, r_k}$. By assumption, Tevelev points of $\text{Coll}^1_d(\mathcal{E}/B)$ define covers $f$ of degree $d$, so in particular are supported away from the rank 1 locus.

In order to enumerate Tevelev points, we compute their limits in the special fiber.

**Definition 19.** A point of $\text{Coll}^1_d(\mathcal{E}/B)_0$ is a limit Tevelev point if it lies in the closure of the locus of Tevelev points on the generic fiber of $\text{Coll}^1_d(\mathcal{E}/B)$

The strategy of proof of Theorem $[10]$ is as follows. In $[6.2]$ we first give set-theoretic restrictions on limit Tevelev points, then show that all points of $\text{Coll}^1_d(\mathcal{E}/B)_0$ satisfying these conditions are in fact limit Tevelev. Moreover, the scheme-theoretic closure of the locus of Tevelev points is étale over $B$, so it suffices to enumerative limit Tevelev points. This enumeration is then carried out in $[6.3]$.

### 6.2. Characterization of limit Tevelev points.

The following claims, Propositions $[20]$, $[22]$ and $[24]$ give constraints on a point $(V, [f_0 : f_1]) \in \text{Coll}^1_d(\mathcal{E}/B)_0$ to be limit Tevelev.

**Proposition 20.** Consider a limit Tevelev point $(V, [f_0 : f_1]) \in \text{Coll}^1_d(\mathcal{E}/B)_0$ and its aspect $V_{\text{sp}}$ on $C_{\text{sp}}$. Then, $(V_{\text{sp}}, [f_0 : f_1])$ has the following properties.

---

**Figure 2.** The curve $C_0$. [diagram of the curve $C_0$ with labels $C_{\text{sp}}$, $s_0$, $s_1$, $t_0$, $t_1$, $E_1$, $E_g$, $R_k$, $R_1$, $p_1$, $\ldots$, $p_n$, $p_{n-r_k+1}$, and points and lines representing the curve structure.]

---
We divide the proof into two steps.  

**Proof.** By Riemann-Hurwitz formula, the vanishing sequence of the aspect $V_{sp}$ is at most $(d-2,d)$ at $s_i$, so because $V$ is a (possibly crude) limit linear series, $V_{sp}$ must have vanishing sequence at least $(0,2)$, giving (1).

Regarding $t_h$ as the simultaneous limit of the $r_h$ points

$$p_{n-r_h-\ldots-r_{h,i+1},\ldots,-r_1}$$

on the generic fiber, the condition that

$$q_h^1 \cdot f_0 - q_h^0 \cdot f_1 \in H^0(\mathbb{P}^1, \mathcal{O}(d))$$

vanishes on the divisor given by the sum of these $r_h$ points has flat limit in $\text{Coll}^1(\mathcal{C} / B)$ given by the condition that it vanishes on the divisor $r_h \cdot t_h$. This yields (2).

Condition (2’) is immediate from (2), unless the section

$$q_h^1 \cdot f_0 - q_h^0 \cdot f_1 \in H^0(\mathbb{P}^1, \mathcal{O}(d))$$

is identically zero. However, the condition on the generic fiber that $V$ contains a 1-dimensional subspace of sections vanishing at

$$p_{n-r_h-\ldots-r_{h,i+1},\ldots,-r_1}$$

is closed, and limits to the desired one. ∎

**Proposition 22.** Suppose $(V, [f_0 : f_1]) \in \text{Coll}^1(\mathcal{C} / B)_0$ satisfies properties (1), (2), (2’) of Proposition 20. Then, $(V_{sp}, [f_0 : f_1])$ satisfies the following genericity properties.

- (3) $f_0, f_1$ share no common zeroes on $C_{sp}$. In particular, $f_0, f_1$ are linearly independent, and $V_{sp} = \langle f_0, f_1 \rangle$ has no base-points.
- (4) Properties (1) and (2) of Proposition 20 hold “exactly” that is, $V_{sp}$ is simply ramified at each $s_1, \ldots, s_g$, and $[f_0 : f_1]$ has ramification index exactly $r_h$ at $t_h$ for $h = 1, 2, \ldots, k$.

**Proof.** We divide the proof into two steps.

**Step 1** We first show that $f_0, f_1$ are linearly independent. Suppose instead that $[f_0 : f_1] = [\lambda_0 f : \lambda_1 f]$. Note that we require $f \in V_{sp}$ and that $V_{sp}$ is ramified at $s_1, \ldots, s_g$.

Define the integer $\alpha \geq 0$ as follows. We declare $\alpha = 0$ if $q_h \neq [\lambda_0 : \lambda_1]$ for all $h$, $\alpha = r_h$ if $q_h = [\lambda_0 : \lambda_1]$.

Then, for every $q_h$ for which $q_h \neq [\lambda_0 : \lambda_1]$, condition (2) implies that $f$ must vanish at $t_h$ to order $r_h$. In total, $f$ is constrained to vanish on a fixed divisor $D_a \subset C_{sp}$ of degree $n - \alpha$.

On the other hand, suppose that $q_h = [\lambda_0 : \lambda_1]$. Then, we have the additional condition from (2’) that $V_{sp}$ is ramified to order $r_h$ at $t_h$.

We now claim that linear series $V_{sp}$ with all of the above properties cannot exist. To see this, we consider a further degeneration of our marked $\mathbb{P}^1$ in which all of the points in the support of $D_a$ coalesce to a single point $p_a$, and consider the limit $\tilde{V}_{sp}$ of $V_{sp}$ on
this new marked rational curve. Then, we still have the conditions that \( \mathcal{V}_{sp} \) is ramified at \( s_1, \ldots, s_g \), and that it is ramified to order \( r_h \) at \( t_h \). The condition that we have a non-zero section \( f \) vanishing on \( D_h \) now becomes the condition that \( \mathcal{V}_{sp} \) is ramified to order \( n - \alpha - 1 \) at \( p_q \).

The total amount of ramification on \( \mathcal{V}_{sp} \) is therefore at least

\[
g + (n - \alpha - 1) + (\alpha - 1) = g + n - 2 = 2d - 1,
\]

which is more than the amount allowed by the Riemann-Hurwitz formula, a contradiction.

**Step 2** Now we prove that conditions (3) and (4) hold.

Let \( \mathcal{H} \) be the Hurwitz stack parametrizing maps \( \mathbb{P}^1 \to \mathbb{P}^1 \) of the same degree and with exactly the same ramification profiles as \( [f_0 : f_1] : \mathbb{P}^1 \to \mathbb{P}^1 \). In the domain curve, we mark all the points \( p_i, s_j, t_k \). On the target curve, we mark \( q_h \) for \( h = 1, \ldots, k \) unless both \( f_0 \) and \( f_1 \) vanish at \( t_h \) to order \( r_h \).

We have a forgetful map

\[
\tau : \mathcal{H} \to M_{0,g+k} \times M_{0,N}
\]

remembering the marked source and target. Because all of our points have been chosen to be general, \( \tau \) is dominant and thus \( \dim(\mathcal{H}) \geq \dim(M_{0,g+k} \times M_{0,N}) \). A parameter count shows that this is only possible when \( [f_0 : f_1] \) has degree \( d \) (i.e. \( f_0, f_1 \) share no common factor), \( [f_0 : f_1] \) has ramification index exactly \( r_h \) at \( t_h \) for \( h = 1, 2, \ldots, k \), and has ramification index exactly 2 at \( s_i \) for \( i = 1, \ldots, g \). The proof is complete.

\( \square \)

**Remark 23.** Given (3), condition (2') is superfluous, as the linear independence of \( f_0, f_1 \) implies that the section

\[
q^1_h \cdot f_0 - q^0_h \cdot f_1
\]

from (2) is not identically zero.

**Proposition 24.** Consider a limit Tevelev point \((V, [f_0 : f_1]) \in \text{Coll}^1_d(\mathcal{E} / \mathcal{B}^0) \). Then, the aspects of \( V \) on the tails of \( \mathcal{C}_{sp} \) have the following properties.

5. The aspect \( V_{E_j} \) is the unique linear series on \( E_j \) with vanishing sequence \((d - 2, d)\) at \( s_j \) for \( i = 1, 2, \ldots, g \).

6. The aspect \( V_{R_h} \) is the unique linear series on \( R_h \cong \mathbb{P}^1 \) spanned by a section \( f_0^h \) vanishing to order \( d \) at \( t_h \) and a section \( f_1^h \) vanishing to order \( d - r_h \) at \( t_h \) and to order 1 at each of

\[
P_{n - r_h - r_{h+1}} \cdots P_{n - r_{h+1}}
\]

In particular, \( V \) is a fine limit linear series, and the data of a limit Tevelev point is determined by \((V_{sp}, [f_0 : f_1])\).

**Proof.** By properties (1) and (4), the vanishing sequence of \( V_{E_j} \) at \( s_j \) is at least \((d - 2, d)\), which is the largest possible, and uniquely determines \( V_{E_j} \) (as the image of the complete linear system \([\mathcal{O}_{E_j}(2s_j)] \) in \([\mathcal{O}_{E_j}(d s_j)] \). This yields (5).

By properties (2) and (4), the vanishing sequence of \( V_{R_h} \) at \( t_h \) is at least \((d - r_h, d)\), so \( V_{R_h} \) contains a section vanishing to order \( d \) at \( t_h \). On the other hand, \( V_{R_h} \) contains a non-zero section vanishing at the \( r_h \) marked points

\[
P_{n - r_h - r_{h+1}} \cdots P_{n - r_{h+1}}
\]

because the same is true on the generic fiber. This section can then vanish to order at most \( d - r_h \) at \( t_h \), so we must have equality, yielding (6).
Proposition 25. Suppose that \((V, [f_0 : f_1]) \in \text{Coll}^1_d(\mathcal{C} / B)_0\) satisfies conditions (1)–(6) above. Then, 
\((V, [f_0 : f_1])\) is a limit Tevelev point.

Moreover, \((V, [f_0 : f_1])\) smooths transversally to the generic fiber, that is, the scheme-theoretic
closure of the locus of Tevelev points is étale over \(B\).

Proof. Let \(G^1_d(\mathcal{C} / B)\) be the moduli space of fine limit linear series on the fibers of \(\mathcal{C}\), as constructed in the proof of [5, Theorem 3.3]. By [5, Theorem 4.5], the special fiber of \(G^1_d(\mathcal{C} / B)\) is pure of the expected relative dimension \(2d - 2 - g\), so \(G^1_d(\mathcal{C} / B) \to B\) is flat of the same dimension. Indeed, \(G^1_d(\mathcal{C} / B)\) may be regarded as a locally closed subscheme inside a product of Grassmannian bundles over \(B\), which has the expected dimension over the special fiber, and thus must be flat of the expected dimension globally by semicontinuity.

As before, let \(\mathcal{V}_{sp}\) be the rank 2 vector bundle on \(G^1_d(\mathcal{C} / B)\) with fiber equal to the linear series \(V\) on the general fiber of \(\psi : \mathcal{C} \to B\), and fiber \(V_{sp}\) on special fiber. Let \(\text{Coll}^1_d(\mathcal{C} / B)\) be the \(\mathbb{P}^1\)-bundle over \(G^1_d(\mathcal{C} / B)\) associated to the rank 4 bundle \(\tau_{\mathcal{V} \oplus 2}\).

We now cut out the locus of Tevelev points \(\text{Coll}^1_d(\mathcal{C} / B)_{\text{Teve}} \subset \text{Coll}^1_d(\mathcal{C} / B)\) via the \(n\) linear conditions that \([f_0 : f_1]\) is incident to \(q_h\) at \(p_i\) when \(n - \sum_{h=j}^k r_h + 1 \leq i \leq n - \sum_{h=j+1}^k r_h\) and \(1 \leq j \leq k\). These conditions have flat limits given by condition (2) of Proposition 20.

We see now that the points on the special fiber of \(\text{Coll}^1_d(\mathcal{C} / B)_{\text{Teve}}\), satisfying properties (1)–(6) therefore lie in the closed subscheme cut out by these linear conditions. Moreover, in a neighborhood of these points, the dimension of this closed subscheme, when restricted to the special fiber, has the expected dimension of 0, by a dimension count as in the proof of Proposition 22.

Therefore, applying semicontinuity once more implies that the points satisfying (1)–(6) smooth in a flat family of relative dimension 0 to Tevelev points on the general fiber.

It remains to check the transversality. Again, we appeal to the dictionary between linear series and Hurwitz spaces: a non-zero relative tangent vector to \(\text{Coll}^1_d(\mathcal{C} / B)_{\text{Teve}}\) over \(B\) would give rise to a non-zero relative tangent vector of the map \(\tau : \mathcal{H} \to M_{0,g+k} \times M_{0,k}\) where \(\mathcal{H}\) is the Hurwitz space of covers \(C_{sp} = \mathbb{P}^1 \to \mathbb{P}^1\) defined by the points of \(\text{Coll}^1_d(\mathcal{C} / B)_{\text{Teve}}\) and \(\tau\) remembers the marked source and target. However, we have chosen a general point of \(\tau : \mathcal{H} \to M_{0,g+k} \times M_{0,k}\), over which \(\tau\) is unramified by generic smoothness, so we have reached a contradiction. This completes the proof.

6.3. Enumeration of limit Tevelev points. By the results of the previous section, we have reduced to the following problem: fix

\[(\mathbb{P}^1, s_1, \ldots, s_g, t_1, \ldots, t_k) \in M_{0,g+k}\]
general. Then, we wish to count \((V, [f_0 : f_1]) \in \text{Coll}^1_d(\mathbb{P}^1)\) satisfying the following properties (cf. Propositions 20, 22 and 25):

1. \(V\) is ramified at each \(s_j\) for \(j = 1, 2, \ldots, g\);
2. \([f_0 : f_1]\) is incident to order \(r_h\) at \(q_h\) when evaluated at \(t_h\) for \(h = 1, 2, \ldots, k\);
3. \(f_0, f_1\) are linearly independent, that is, \((V, [f_0 : f_1])\) lies away from the rank 1 locus.

Note that in this degenerate problem we have dropped the subscript \(sp\).

Remark 26. From the results of the previous section, given (1) and (2), asking for \(f_0\) and \(f_1\) to be linearly independent is the same as asking for \((2')\) and thus the same as asking for \(f_0\) and \(f_1\) to share no common zero.
The space $\text{Coll}_d^1(\mathbb{P}^1)$ is the $\mathbb{P}^3$-bundle over the Grassmannian $\text{Gr}(2, d+1)$ associated to the vector bundle $\mathcal{V} \oplus \mathcal{V}$, where $\mathcal{V}$ is the tautological rank 2 subbundle on $\text{Gr}(2, d+1)$. As before, we denote the projection map by $\varphi : \text{Coll}_d^1(\mathbb{P}^1) = \mathbb{P}(\mathcal{V} \oplus \mathcal{V}) \to \text{Gr}(2, d+1)$.

Conditions (1), (2), respectively, give rise to the following cycles on $\text{Coll}_d^1(\mathbb{P}^1)$:

(i) An intersection of $g$ Schubert cycles of class $\sigma_1 \in A^1(\text{Gr}(2, d+1))$, pulled back by $\varphi$;
(ii) an intersection of $\sum_{h=1}^k r_h = n$ relative hyperplanes of class $c_1(\mathcal{O}(\mathcal{V} \oplus \mathcal{V})(1)) \in A^1(\text{Coll}_d^1(\mathbb{P}^1))$.

In particular, the cycles (i), (ii) define an intersection in $\text{Coll}_d^1(\mathbb{P}^1)$ of dimension $(2(d-1)+3) - n - g = 0$.

However, we note that the set-theoretic intersection of these cycles typically includes excess components supported in the rank 1 locus, and thus fail (3). Indeed, one may check via parameter counts that there exist points in this intersection where $[\lambda_0 : \lambda_1] = q_h$, and furthermore that if $r_h > 2$, we have excess loci of positive dimension. In order to avoid this excess intersection, we replace (2) with the following combination of properties (2) and (2') appearing in Proposition 20:

(2) $V$ has a non-zero section vanishing to order at least $r_h$ at $t_h$ and $[f_0 : f_1]$ is incident at $q_h$ when evaluated at $t_h$ for $h = 1, 2, \ldots, k$.

For each $h$, property (2) defines a cycle (ii") on $\text{Coll}_d^1(\mathbb{P}^1)$, again of codimension $r_h - 1$, given by the intersection of (the pullback of) a Schubert cycle of class $\sigma_{r_h-1} \in A^{r_h-1}(\text{Gr}(2, d+1))$ and a relative hyperplane on $\text{Coll}_d^1(\mathbb{P}^1)$. The following now shows that the new intersection avoids the rank 1 locus, and the excess intersections are zero-dimensional and transverse.

**Proposition 27.** The intersection of the cycles (i), (ii"), is transverse and is supported away from the rank 1 locus.

Moreover, the only base-points of $(V, [f_0 : f_1])$ occur among the points $t_1, \ldots, t_k$. For each base-point $t_h$, we must have $r_h > 1$, and $V$ has vanishing sequence exactly $(1, r_h)$.

Otherwise, $(V, [f_0 : f_1])$ satisfies properties (1), (2) exactly.

**Proof.** The proof is similar to that of Proposition 22. We divide it into three steps.

**Step 1** We first show that if $(V, [f_0 : f_1])$ lies in the intersection of (i) and (ii"), then $f_0$ and $f_1$ must be linearly independent. Suppose instead $f_0$ and $f_1$ are linearly dependent, and write $[f_0 : f_1] = [\lambda f : \mu f]$. We have:

(a) $V$ satisfies condition (1),
(b) $V$ has a non-zero section $u_h$ vanishing to order $r_h$ at $t_h$ for each $h = 1, \ldots, k$, and
(c) $V$ has a non-zero section (namely, $f$) vanishing simultaneously at all or all but one of the points $t_1, \ldots, t_k$ (if $[\lambda : \mu] = q_h$, then $f$ need not vanish at the corresponding point of $C_{pq}$).

Let $J \subseteq \{1, \ldots, k\}$ be the set of indices $h$ such that the sections $u_h$ and $f$ are linearly dependent for a general choice of $s_j, t_h$. Consider a degeneration of our marked $\mathbb{P}^1$ in which all of the points $t_h$ for $h \in J$ coalesce to a single point $p$, and consider the limit $\tilde{V}$ of $V$ on the new marked curve. Then (a), (b) and (c) give:

(a) $\tilde{V}$ satisfies condition (1),
(b) $\tilde{V}$ has a non-zero section $\tilde{u}_h$ vanishing to order $r_h$ at $t_h$ for each $h \notin J$, and
(c) $\tilde{V}$ has a non-zero section $\tilde{f}$ vanishing at $p$ at least to order $-1 + \sum_{h \in J} r_h$ and vanishing at $t_h$ for $h \notin J$. 

Moreover, we may assume that \( \bar{u}_h \) and \( \bar{f} \) are linearly independent for each \( h \notin J \) (otherwise we also coalesce \( p \) and \( t_0 \) and keep repeating this process). Then, the total ramification of \( \overline{V} \) is at least
\[
\left( g + \sum_{h \notin J} r_h \right) + \left( -1 + \sum_{h \notin J} r_h \right) - 1 = g + n - 2
\]
while, by the Riemann-Hurwitz formula, the maximum allowed is \( g + n - 3 \). This is a contradiction.

**Step 2** Now, we prove the claim about base-points, again by a parameter count on Hurwitz spaces. As in Proposition 22 we find that demanding additional \( f_0, f_1 \) share a common factor imposes too many conditions on the cover \([f_0 : f_1]\), with the following exception. If \( f_0, f_1 \) are required to vanish to order 1 on some \( t_h \) for which \( r_h > 1 \), then we have added exactly one vanishing condition, because \( V \) was already constrained to be ramified at \( t_h \). On the other hand, we lose the incidence condition at \( t_h \), which is now automatic, so such \([f_0 : f_1]\) are still expected to exist. On the other hand, if \( r_h = 1 \), requiring that \( f_0, f_1 \) vanish at \( t_h \) adds 2 conditions, so such \([f_0 : f_1]\) are not expected to exist in this case.

More precisely, let \( \mathcal{H} \) be the Hurwitz stack parametrizing maps \( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) of the same degree and with exactly the same ramification profiles as \([f_0 : f_1]\), with \( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \). In the domain curve, we mark all the points \( s_j, t_h \) while, this time, on the target curve we mark \( q_h \) for \( h = 1, ..., k \) unless both \( f_0 \) and \( f_1 \) vanish at \( t_h \). For the corresponding forgetful map
\[
\tau : \mathcal{H} \rightarrow M_{0,g+k} \times M_{0,N}
\]
(remembering the marked source and target) to be dominant, a parameter count shows that the only possible common zeros of \( f_0 \) and \( f_1 \) are \( t_1, ..., t_k \), and that these zeroes must appear with multiplicity at most 1. This proves the second claim.

**Step 3** Transversality is obtained as in the proof of Proposition 25 a non-zero tangent vector in the intersection gives rise to a non-zero relative tangent vector of a generically étale forgetful morphism \( \tau : \mathcal{H} \rightarrow M_{0,n_i} \times M_{0,n_o} \) where \( \mathcal{H} \) is a Hurwitz space of covers \( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \), yielding a contradiction.

**Proof of Theorem** By Proposition 27 we need to count the points in the intersection of (i), (ii") where \( V \) is base point free.

First, the intersection number of the cycles (i), (ii") is given by
\[
\int_{\text{Coll}_1(\mathbb{P}^1)} \varphi^* \left( \sigma_1^g \cdot \prod_{h=1}^k \sigma_{n_h-1} \right) \cdot c_1(\mathcal{O}(1))^k
\]
\[
= \int_{\text{Gr}(2,d+1)} \sigma_1^g \cdot \left( \prod_{h=1}^k \sigma_{n_h-1} \right) \cdot \varphi_*(c_1(\mathcal{O}(1))^k)
\]
\[
= \int_{\text{Gr}(2,d+1)} \sigma_1^g \cdot \left( \prod_{h=1}^k \sigma_{n_h-1} \right) \cdot s_{k-3}(\mathcal{V} \oplus \mathcal{V})
\]
\[
= \int_{\text{Gr}(2,d+1)} \sigma_1^g \cdot \left( \prod_{h=1}^k \sigma_{n_h-1} \right) \cdot \left( \sum_{i+j=k-3} \sigma_i \sigma_j \right)
\]

Moreover, the intersection is transverse, and given set-theoretically by the points \( (V,[f_0 : f_1]) \) described in Proposition 27.
More generally, suppose \( J \subset \{1, 2, \ldots, k\} \) and consider the points \( \{V, [f_0 : f_1]\} \) in the intersection of (i),(ii) for which \( V \) has a simple base-point at \( t_h \) for all \( h \in J \). Then twisting down \( V \) by these \( r_h \) and dividing \( f_0, f_1 \) by the polynomials vanishing on the \( t_h \) gives a new point \( (V, [f_0 : f_1]) \in \text{Coll}^1_{d-\#J}(\mathbb{P}^3) \).

We have that \( (V, [f_0^I : f_1^I]) \) satisfies property (1) as before, and satisfies property (2”) at the points \( t_h \) with \( h \in I := \{1, 2, \ldots, k\} \setminus J \). Now, at points \( t_h \) with \( h \in J \), \( V \) is base-point free of ramification index (at least) \( r_h - 1 \) (with no incidence condition). Computing the number of such \( \{V, [f_0^I : f_1^I]\} \) similarly to before yields

\[
\int_{\text{Gr}(2,d+1-\#J)} \sigma^I_k \cdot \left( \prod_{h \in I} \sigma_{r_h-1} \right) \cdot \left( \prod_{h \in J} \sigma_{r_h-2} \right) \cdot \left( \sum_{i+j=k-3-\#J} \sigma_i \sigma_j \right).
\]

Note that this count includes \( \{V, [f_0^I : f_1^I]\} \) that may have simple base-points at \( t_h \) with \( h \in I \). Note that it is still the case that \( f_0^I, f_1^I \) are linearly independent and that the intersection is transverse. We require \( d - \#J > 0 \) and \( k - 3 - \#J \geq 0 \); if this is not the case, the integral is interpreted to be zero.

Now, the limit Tevelev points are exactly those where \( V \) is base-point free. Thus, varying over all possible \( J \subset \{1, 2, \ldots, k\} \) and applying inclusion-exclusion yields Theorem 10.

6.4. Comparison with Propositions 7 and 8. As a check, we now demonstrate how Theorem 10 recovers the formulas of before, namely Propositions 7 and 8. We continue to assume for notational convenience that \( \mu_h = (1)^h \).

Proposition 28. Let \( g \geq 1 \), \( \ell \in \mathbb{Z} \) an integer and \( r_1, \ldots, r_k \in \mathbb{Z}_{\geq 1} \) positive integers. Assume that the Equations (4), (5) and (6) hold with \( \mu_h = (1)^h \) for \( h = 1, \ldots, k \). Then, the formula for \( \text{Tevelev}_{g, \ell, r_1, \ldots, r_k} \) of Theorem 10 satisfies the recursion of Proposition 7.

Proof. We claim that the desired recursion already holds on each individual term of the formula of Theorem 10 indexed by a partition \( J \sqcup J = \{1, 2, \ldots, k\} \). Indeed, this is immediate from the Pieri rule

\[
\sigma_{r_h-1} \cdot \sigma_1 = \sigma_{r_h-2} \cdot \sigma_{11} + \sigma_{r_h}
\]

and the fact that

\[
\int_{\text{Gr}(2,N)} \beta \cdot \sigma_{11} = \int_{\text{Gr}(2,N-1)} \beta
\]

for any \( \beta \in A_0(\text{Gr}(2,N-1)) \).

Proposition 29. Let \( \ell \geq 0 \) and \( r_1, \ldots, r_k \in \mathbb{Z}_{\geq 1} \) positive integers such that the inequalities (5), (4) and (6) hold with \( \mu_h = (1)^h \) for \( h = 1, \ldots, k \). Then the formula for \( \text{Tevelev}_{0,\ell,r_1,\ldots,r_k} \) of Theorem 10 satisfies Proposition 8 that is, we have \( \text{Tevelev}_{0,\ell,r_1,\ldots,r_k} = 1 \).

Proof. Fix a non-negative integer \( m \), and define \( \text{Tevelev}^{m}_{0,\ell,r_1,\ldots,r_k} \) by

\[
\sum_{J \sqcup J = \{1,2,\ldots,k\}} (-1)^J \int_{\text{Gr}(2,d+1-\#J)} \left( \prod_{h \in I} \sigma_{r_h-1} \prod_{h \in J} \sigma_{r_h-2} \right) \cdot \left( \sum_{i+j=k-3-\#J} \sigma_i \sigma_j \right).
\]

where \( \text{Tevelev}^{m}_{0,\ell,r_1,\ldots,r_k} \) differs from \( \text{Tevelev}_{0,\ell,r_1,\ldots,r_k} \) in that we require \( i \geq m \) in the last term.
We prove the stronger statement that \( \text{Tev}^m\sigma_{0,d,r_1,...,r_k} = 1 \) whenever \( 0 \leq m \leq d-1 \) by induction on \( k \). Note that when \( m > d - 1 \), all of the terms vanish automatically, because \( \sigma_i = 0 \) for \( i > d - 1 \).

First, consider the base case \( k = 3 \). Then, the statement is simply that

\[
\int_{\text{Gr}(2,d+1)} \sigma_{r_1-1}\sigma_{r_2-1}\sigma_{r_3-1} = 1
\]

whenever \( r_1, r_2, r_3 \leq d \), which follows from the Pieri rule.

For the inductive step, fix a partition \( I' \coprod J' = \{1,2,\ldots,k-1\} \), and consider the summands of \( \text{Tev}^m\sigma_{0,d,r_1,...,r_k} \) indexed by \( (I' \cup \{k\}) \coprod J' \) and \( I' \coprod (J' \cup \{k\}) \).

Matching the summand indexed by \( (i + 1, j) \) from the partition \( (I' \cup \{k\}) \coprod J' \) with that indexed by \( (i, j) \) from the partition \( I' \coprod (J' \cup \{k\}) \) leaves one additional term from the first partition, coming from \( i = m, j = k - 4 - \# J' - m \). By the Pieri rule, observe further that

\[
\sigma_{r_1-1}\sigma_{r_2-1} - \sigma_{r_1-2}\sigma_{1} = \sigma_{r_1+1}
\]

Thus, we find the two partitions in question contribute

\[
(-1)^{\# J'} \int_{\text{Gr}(2,d+1-\# J')} \left( \prod_{h \in J'} \sigma_{r_1-1} \prod_{i \in J'} \sigma_{r_2-2} \cdot \sum_{i+j=k-4-\# J'} \sigma_{r_i+j} \sigma_j \right)
\]

\[
+(-1)^{\# J'} \int_{\text{Gr}(2,d+1-\# J')} \left( \prod_{h \in J'} \sigma_{r_1-1} \prod_{i \in J'} \sigma_{r_2-2} \cdot \sigma_{r_1-1} \cdot m \sigma_{k-3-\# J' - m} \right)
\]

Summing over all \( I', J' \) and reindexing the sum in the first term, we find that

\[
\text{Tev}^m\sigma_{0,d,r_1,...,r_k} = \text{Tev}^m\sigma_{0,d,r_1,...,r_{k-1}} + \sum_{I' \coprod J' = \{1,2,\ldots,k-1\}} (-1)^{\# J'} \int_{\text{Gr}(2,d+1-\# J')} \left( \prod_{h \in J'} \sigma_{r_1-1} \prod_{i \in J'} \sigma_{r_2-2} \cdot \sigma_{r_1-1} \cdot m \sigma_{k-3-\# J' - m} \right)
\]

By the inductive hypothesis, if \( m + r_k \leq d - 1 \), then \( \text{Tev}^m\sigma_{0,d,r_1,...,r_{k-1}} = 1 \). On the other hand, if \( m + r_k > d - 1 \), then every term in the sum defining \( \text{Tev}^m\sigma_{0,d,r_1,...,r_{k-1}} \) vanishes, so \( \text{Tev}^m\sigma_{0,d,r_1,...,r_{k-1}} = 0 \).

It therefore suffices to show that the second term is equal to 0 if \( m + r_k \leq d - 1 \) and 1 if \( m + r_k > d - 1 \). We claim that the sum is equal to simply

\[
\int_{\text{Gr}(2,d+1)} \sigma_{r_1-1}\sigma_m\sigma_{r_2-1 + \cdots + r_{k-1} - 2 - m} = \int_{\text{Gr}(2,d+1)} \sigma_{r_1-1}\sigma_m\sigma_{2d-1 - r_k - m}.
\]

To see this, we employ the same inductive strategy as before: we fix a partition \( I'' \coprod J'' = \{1,2,\ldots,k-2\} \) and match the terms of the sum where \( J' = J'' \) and \( I' = I'' \cup \{k-1\} \), using the fact that

\[
\sigma_{k-3-\# J'' - m} - \sigma_{k-2}(\# J'' + 1 - m)\sigma_{11} = \sigma_{k-4-\# J'' - m}.
\]

Continuing in this fashion yields the claim.

Again, by the Pieri rule, the final integral is equal to 1 whenever each of the three terms does not vanish. Because \( 2 \leq r_k \leq d \) and \( 0 \leq m \leq d - 1 \), this happens exactly when \( r_k + m \geq d \), which is exactly what we need. \( \square \)
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