Overlap and domain wall fermions: what is the price of chirality?

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In this contribution the costs of simulations employing domain wall and overlap fermions are estimated. In the discussion we will stay within the quenched approximation.

Chirally invariant formulations of lattice QCD are still relatively new. However, besides their conceptual advantages, a number of physics applications have already been performed. In these works it became clear that simulations with such formulations are expensive, even when they are restricted to the quenched approximation. Consequently, at the time of the conference no simulations have been performed with such dynamical quarks so far.

Our conclusions concerning the costs of simulations with dynamical overlap or domain wall fermions will therefore be rather indirect: we will estimate the overhead of using chiral invariant formulations to standard Wilson fermions in the quenched case and assume this overhead to be the same for an unquenched simulation. The discussion below will be organized in the form of four statements.

Statement 1

The 5-dimensional domain wall construction in the limit \( N_s \to \infty \) is completely equivalent to a 4-dimensional lattice formulation of overlap fermions.

This is a mathematical statement that can be proven rigorously. Denoting by \( s \) the extra 5th dimension, the 5-dimensional operator reads

\[
D_5 = \frac{1}{2} \{ \gamma_5 (\partial_s^* + \partial_s) - a_s \partial_s^* \partial_s \} + M
\]

with

\[
M = D_w - m_0.
\]

Here \( D_w \) is the standard Wilson-Dirac operator, \( a_s \) the lattice spacing in the extra dimension and \( m_0 \) a mass parameter.

Keeping \( a_s \) finite, in the limit \( N_s \to \infty \) we obtain a 4-dimensional operator

\[
aD_4 = 1 - A(A^\dagger A)^{-1/2}
\]

with

\[
A = -a_s M (2 + a_s M)^{-1}
\]

that satisfies the Ginsparg-Wilson relation. If also \( a_s \to 0 \), Neuberger’s overlap operator is recovered.

Statement 2

Domain wall fermions do not perform better than overlap fermions.

The mathematical equivalence of domain wall and overlap fermions gives rise to the suspicion that also in practical applications no clear preference for either formulation –as far as the expense of the simulation is concerned– can be given. We give one example below for the cost of computing the pion propagator \( \Gamma_\pi \) to a certain relative precision. In fig. 1 we plot the ratio \( R \),

\[
R = \left| \frac{\Gamma_\pi^{\text{exact}}(T/3) - \Gamma_\pi^{\text{approx}}(T/3)}{\Gamma_\pi^{\text{exact}}(T/3)} \right|
\]

Aiming at a, say, per mil precision we find for this particular example that domain wall fermions are a factor of two more expensive than overlap fermions. It is clear that for both formulations additional improvements might be implemented. However, it seems very unlikely that...
one particular formulation will give an order of magnitude better performance.

![Figure 1](image.png)

Figure 1. The ration $R$ of eq. (5) as a function of time needed (on a particular machine) to obtain a relativ precision for the pion propagator. Results are averaged over ten configurations on a $8^3 \times 24$ lattice at $\beta = 5.85$.

Statement 3

*Keeping $N_s$ finite: the residual mass is not all.*

The locality of chirality breaking effects is only guaranteed at distances $|x - y| \gg N_s a$. This is discussed in [3, 4]. First investigations of this question have been performed in [5]. I find it very important to study the effects of this observation further on a quantitative level.

Statement 4

*Whatever time estimate is found for Wilson fermions: multiply the effort by a factor of $O(100)$ for chiral symmetric actions.*

The factor referred to in this statement is really only an order of magnitude estimate. The reasons for this large factor are different for overlap and domain wall fermions.

In the case of overlap fermions it turns out that the polynomial required to approximate the square root has typically a degree of $O(100)$, since this polynomial has to be evaluated in every step of a linear solver, the cost of simulations with overlap fermions increase correspondingly. It is also noteworthy that so far no way of preconditioning the overlap operator has been found.

In the case of domain wall fermions there is some experience [5] that the number of conjugate gradient iterations is larger for the 5-dimensional problem. This in addition to the additional number of slices in the extra dimension gives again a large value of the factor compared to standard Wilson fermions.

Conclusion

If the estimate of about 100Tflops to solve most of the problems in lattice QCD as discussed in the panel contributions is indeed correct, then this would mean a demand of 10 Petaflops for simulations with chirally invariant formulations of lattice-QCD. We then would be close to the number [10] anticipated by K. Wilson in his 1989 Capri contribution.

When, as it is done at this conference, we emphasize that thinking about better algorithms is one of the most important things the lattice community should address, then this is even more true for chirally invariant formulations of lattice-QCD. And, finally, the hope is that putting effort into the development of better algorithms, it can be shown that statement number 4 is not correct.

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