Degree of the exceptional component of foliations of degree two and codimension one in $\mathbb{P}^3$

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Abstract. The purpose of this work is to obtain the degree of the exceptional component of the space of holomorphic foliations of degree two and codimension one in $\mathbb{P}^3$. This component is the closure of the orbit of the foliation defined by the differential form

$$\omega = (3fg - 2gf)/x_0,$$

where $f = x_0^2x_3 - x_0x_1x_2 + x_1^3/3$, $g = x_0x_2 - x_1^2/2$

under the natural action of the group of automorphisms of $\mathbb{P}^3$. Our first task is to unravel a geometric characterization of the pair $g, f$. This leads us to the construction of a parameter space as an explicit fiber bundle over the variety of complete flags. Using tools from equivariant intersection theory, especially Bott’s formula, the degree is expressed as an integral over our parameter space.

1. Introduction

A holomorphic foliation of codimension one and degree $d$ in the complex projective space $\mathbb{P}^n$ is given by a differential 1-form $\omega = A_0dx_0 + \cdots + A_ndx_n$ where $A_0, \ldots, A_n$ are homogeneous polynomials of degree $d+1$, satisfying the conditions (i) $A_0x_0 + \cdots + A_nx_n = 0$ and (ii) (integrability) $\omega \wedge d\omega = 0$.

These two conditions define a closed subscheme $\mathcal{F}(d,n)$ of the projective space of global sections of the twisted cotangent sheaf $\Omega^1_{\mathbb{P}^n}(d+2)$. Naturally, $\mathcal{F}(d,n)$ possesses a decomposition into irreducible components. This decomposition is completely known in cases $d = 0$ and $d = 1$, thanks to the pioneering book by [8]. For $n \geq 3$, $d = 2$ a full description can be found in the celebrated work by [3] where we learn that there are six components of $\mathcal{F}(2,3)$, among which is the exceptional component $E(3)$, our main object of interest here.

There are a few other known components for $d \geq 3$, such as the pullback, the rational and the logarithmic components, see [1], [5] and [1]. In [5] the authors managed to compute the degree of some rational components. In particular, the degrees of the rational components $\mathcal{R}(n,2,3)$ were computed for $n \leq 5$; in [9] this is extended to the rational components $\mathcal{R}(n,2,2r+1)$ for arbitrary $n, r$.

The exceptional component $E(3)$ is the closure of the orbit, under the natural action of $\text{Aut}(\mathbb{P}^3)$, of the foliation defined by the differential form

$$\omega_0 = \frac{3f_0dg_0 - 2g_0df_0}{x_0},$$

where $f_0 = x_0^2x_3 - x_0x_1x_2 + x_1^3/3$, $g_0 = x_0x_2 - x_1^2/2$. 

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Our goal is to describe the geometry of the family of pairs \(g, f\) in the orbit closure, enabling us to compute the degree of \(E(3)\). Our strategy starts with a complete flag in \(\mathbb{P}^3\)

\[ p \ (\text{point}) \in \ell \ (\text{line}) \subset v \ (\text{plane}), \]

over which we describe suitable cubic forms \(f\) and quadratic forms \(g\) that fulfill some special conditions. The polynomials \(f, g\) will give us the differential 1-form

\[ \omega = \frac{3fdg - 2gdf}{h}, \]

where \(h\) is an equation of the plane \(v\) of the given flag.

The main technical difficulty is to solve the indeterminacies of the rational map \((g, f) \mapsto \omega\). This is done by a careful analysis of the irreducible components of the indeterminacy locus, blown up one at a time. Macaulay2 ([10]) is extensively used. We get a description of a parameter space for the exceptional component, over which we can apply appropriate tools from equivariant intersection theory to compute the desired degree.

2. A parameter space for the Exceptional Component

The component \(E(3)\) is the orbit closure of the foliation defined by the 1-form \(\omega_0\) as in (1). The dimension of \(E(3)\) is equal to 13, see [2]. Explicitly, we have

\[ \omega_0 = (x_1x_2^2 - 2x_1^2x_3 + x_0x_2x_3)dx_0 + x_0(3x_1x_3 - 2x_2^2)dx_1 + x_0(x_1x_2 - 3x_0x_3)dx_2 + x_0(2x_0x_2 - x_1^2)dx_3. \]

The singular locus of the foliation (2) consists of a union of three curves:

1. the conic given by ideal \(\langle x_0, x_2^2 - 2x_1x_3 \rangle\) (it lies in the plane \(x_0 = 0\));
2. the line \(\ell_0\) defined by \(x_0 = x_1 = 0\);
3. the twisted cubic given by \(\langle 2x_2^2 - 3x_1x_3, x_1x_2 - 3x_0x_3, x_1^2 - 2x_0x_2 \rangle\).

These three components meet at the point \(p_0 := (0 : 0 : 0 : 1)\) in \(\mathbb{P}^3\). Let us examine the geometry of the surface defined by the cubic form \(f_0\). We see that it is an irreducible cubic, singular along the line \(\ell_0\). Moreover, given a point \(p_t = (0 : 0 : t : 1) \in \ell_0\), the tangent cone to this cubic surface at the point \(p_t\) has equation

\[ x_0^2 - tx_0x_1 = x_0(x_0 - tx_1). \]

Thus, the tangent cone is a pair of planes containing the double line \(\ell_0\), one of which is fixed and the other varies with the point \(p_t\).

We have also perceived that at the special point \(p_0 = (0 : 0 : 0 : 1)\) the tangent cone to the cubic is the double plane \(x_0^2 = 0\).

Therefore, the cubic \(f_0\) comes endowed with a companion complete flag

\[ \varphi_0 : p_0 = \{x_0 = x_1 = x_2 = 0\} \in \ell_0 = \{x_0 = x_1 = 0\} \subset v_0 = \{x_0 = 0\}. \]

As for the quadric

\[ g_0 = x_0x_2 - \frac{x_1^2}{2}, \]
we have a cone containing the line $\ell_0$ and vertex $p_0$. Moreover, the tangent plane to the cone at a smooth point on $\ell_0$ is precisely the same plane $v_0$.

2.1. Remark. The flag can move under the action of an automorphism of $\mathbb{P}^3$, but it can be recovered directly from the 1-form $\omega$ defining the exceptional foliation, just by looking at the singular locus. For any such $\omega$, the singular locus has three components - a conic living on the new plane, the new line and a twisted cubic that meets the other two components just at the new point; the line is tangent to the twisted cubic, the plane is the osculating plane and the conic is the osculating conic. By the way, the dimension 13 mentioned just above is the dimension of the family of pointed twisted cubics.

These considerations about the cubic/quadric pairs as in (1) lead us to consider the construction of a parameter space for the family of exceptional foliations as a fiber bundle over the variety $F$ of complete flags on $\mathbb{P}^3$,

$$p(\text{point}) \in \ell(\text{line}) \subset \pi(\text{plane}).$$

Given such a flag we take:

- A cubic surface $f$ with the properties:
  1. $f$ is singular along $\ell$;
  2. the tangent cone to $f$ at each point $q \in \ell$ is the union of two planes $\pi \cup \alpha_q$.
  3. $\alpha_p = \pi$, that is, at the point $p$ the tangent cone to $f$ is the double plane.

- A quadratic cone $g$ with the properties:
  1. the line $\ell$ is contained in the cone $g$;
  2. the plane $\pi$ is tangent to $g$ along the line $\ell$;
  3. the point $p$ is in the vertex of the cone.

An exceptional foliation is then given by the differential form

$$\omega = \frac{3fdg - 2gdf}{h},$$

where $h$ is an equation of the plane $\pi$.

2.2. Definition. For simplicity we will call here a cubic or a quadric that meet the requirements above as special.

We realize that after the construction of special pairs $(f,g)$, it is still necessary to impose the condition that the differential form $3fdg - 2gdf$ be divisible by the equation of the plane. We refer to this as the divisibility condition. In order to better understand this divisibility condition, let us fix a complete flag as in (1).

The requirements about special cubics/quadrics show that we have

$$\begin{cases}
  f = (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3) x_0^3 + a_4x_0x_1^2 + a_5x_0x_1x_2 + a_6x_1^3 \\
  g = b_0x_0^3 + b_1x_0x_1 + b_2x_0x_2 + b_3x_1^3.
\end{cases}$$

We register the following invariant description.
2.3. Lemma. The 7 monomials appearing in $f$
\[x_0^3, x_0^2 x_1, x_0^2 x_2, x_0^2 x_3, x_0 x_1^2, x_0 x_1 x_2, x_1^3\]
are the monomial generators of the subspace of cubics,
\[x_0^2(x_0, \ldots, x_3) + x_0(x_0, x_1)(x_0, x_1, x_2) + (x_0, x_1)^3 \subset S_3.\]
Likewise, the 4 monomials $x_0^2, x_0 x_1, x_0 x_2, x_1^2$ in the quadric $g$
generate the subspace
\[x_0(x_0, x_1, x_2) + (x_0, x_1)^2 \subset S_2.\]

Varying the flag, we obtain equivariant vector subbundles
\[(5) \quad \mathcal{A} \subset S_3 \quad \text{and} \quad \mathcal{B} \subset S_2,\]
with respective ranks 7 and 4. Projectively we obtain, at this stage, the variety
\[\mathbb{P}(\mathcal{A}) \times \mathbb{P}(\mathcal{B}),\]
a $\mathbb{P}^6 \times \mathbb{P}^3$ bundle of pairs $(f, g)$ over a fixed flag.

Continuing the discussion about the divisibility condition, we may write
\[3fg - 2gdf = x_0 \omega_1 + \left[(3a_6 b_1 - 2a_4 b_3)x_1^4 + (3a_6 b_2 - 2a_5 b_3)x_1^2 x_2\right] dx_0,\]
where $\omega_1$ is a 1-form. From this it follows that the divisibility condition is
given, on the fiber over our fixed flag $\varphi$, by the equations
\[\begin{cases}
3a_6 b_1 &= 2a_4 b_3, \\
3a_6 b_2 &= 2a_5 b_3
\end{cases}\quad \text{and} \quad \begin{cases}
a_6 &= b_3 = 0.
\end{cases}\]

This locus consists of two irreducible components inside $\mathbb{P}^6 \times \mathbb{P}^3$, both of
codimension two:
\[(6) \quad (\ast) \quad \begin{cases}
3a_6 b_1 &= 2a_4 b_3 \\
3a_6 b_2 &= 2a_5 b_3 \\
a_4 b_2 &= a_5 b_1
\end{cases}\quad \text{and} \quad (\ast\ast) \quad a_6 = b_3 = 0.
\]

For a pair $(g, f)$ satisfying $(\ast\ast)$, we actually get
\[\begin{cases}
f &= x_0 (a_0 x_0^2 + a_1 x_0 x_1 + a_2 x_0 x_2 + a_3 x_0 x_3 + a_4 x_1^2 + a_5 x_1 x_2), \\
g &= x_0 (b_0 x_0 + b_1 x_1 + b_2 x_2).
\end{cases}\]

It means that a general element in the second component $(\ast\ast)$ consists of a
cubic and a quadric both divisible by the equation of the plane, certainly
not interesting for our study of the exceptional component.

Henceforth, we refer to the divisibility condition as the equations $(\ast)$ in
\cite{B}.

Let $G = G(1, 3)$ be the Grassmann variety of lines in $\mathbb{P}^3$, with tautological sequence
\[0 \to S \to G \times \mathbb{C}^4 \to Q \to 0,\]
where rank $(S) = 2$. Denote by $Q^\vee$ the dual of $Q$. We also have the tautological sequence over $\mathbb{P}^3$,
\[0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathbb{P}^3 \times \mathbb{C}^4 \to \mathcal{P} \to 0\]
Then,
\[
\begin{align*}
\mathbb{P}(S) &= \{(p, \ell) \mid p \in \ell\} \subset \mathbb{P}^3 \times G \\
\mathbb{P}(Q^\vee) &= \{ (\ell, \pi) \mid \ell \subset \pi \} \subset G \times \mathbb{P}^3.
\end{align*}
\]
The variety of complete flags is just the fiber product
\[ \mathcal{F} = \mathbb{P}(S) \times_G \mathbb{P}(Q^\vee) = \{(p, \ell, \pi) \mid p \in \ell \subset \pi \} \subset \mathbb{P}^3 \times G \times \mathbb{P}^3. \]
The tautological bundles of these spaces lift to bundles over \( \mathcal{F} \) still denoted by the same letters. They fit together into the diagram (pullbacks omitted),
\[ \mathcal{O}_{Q^\vee}(-1) = \mathcal{O}_{\mathbb{P}^3}(-1) \hookrightarrow Q^\vee \twoheadrightarrow \mathcal{P}^\vee \twoheadrightarrow S_1 := (\mathbb{C}^4)^\vee \]
\( \varphi_0 : \langle x_0 \rangle \subset \langle x_0, x_1 \rangle \subset \langle x_0, x_1, x_2 \rangle \subset \langle x_0, \ldots, x_3 \rangle \)
where the bottom row indicates the corresponding fibers over the flag \( \mathcal{F} \).

In view of Lemma 2.3, we have the surjections
\[ (\mathcal{O}_{Q^\vee}(-2) \otimes S_1) \oplus (\mathcal{O}_{Q^\vee}(-1) \otimes Q^\vee \otimes \mathcal{P}^\vee) \oplus \text{Sym}_3 Q^\vee \to A \subset S_3 \]
\[ (\mathcal{O}_{Q^\vee}(-1) \otimes \mathcal{P}^\vee) \oplus \text{Sym}_2 Q^\vee \to B \subset S_2. \]

By construction, the vector bundles \( A \) and \( B \) fit into the exact sequences
\[ S_1 \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \hookrightarrow A \twoheadrightarrow \mathcal{A} := A/\left(S_1 \otimes \mathcal{O}_{\mathbb{P}^3}(-2)\right) \]
\[ \mathcal{O}_{\mathbb{P}^3}(-2) \hookrightarrow B \twoheadrightarrow \mathcal{B} := B/\left(\mathcal{O}_{\mathbb{P}^3}(-2)\right). \]

2.4. **Lemma.** \( \mathcal{A} \) is isomorphic to \( \left(Q^\vee/\mathcal{O}_{\mathbb{P}^3}(-1)\right) \otimes \mathcal{B} \).

**Proof.** Indeed, on the fiber over \( \varphi_0 \) we have
\[ \left(Q^\vee/\mathcal{O}_{\mathbb{P}^3}(-1)\right)_{\varphi_0} = \langle x_0, x_1 \rangle/\langle x_0 \rangle = \langle x_1 \rangle \]

hence
\[ \left(Q^\vee/\mathcal{O}_{\mathbb{P}^3}(-1)\right)_{\varphi_0} = \langle x_1 \rangle \otimes (x_0 x_1, x_0 x_2, x_1^3) = \mathcal{A}_{\varphi_0}, \]
where the \( = \) signs mean isomorphisms of representations of the stabilizer of the flag \( \varphi_0 \). \( \square \)

2.5. **Definition.** We define \( X = \mathbb{P}(B) \) the corresponding projective bundle of special quadrics. The fiber \( X_{\varphi_0} \) is the \( \mathbb{P}^3 \) of special quadrics over the flag \( \varphi_0 = (p_0, \ell_0, v_0) \) fixed in \( \mathcal{F} \).

The divisibility condition \( (*) \) \( (3) \) can be rewritten as a system of linear equations in variables \( (a) = (a_4 : a_5 : a_6) \) with coefficients \( (b) = (b_0 : b_1 : b_2 : b_3) \in \mathbb{P}^3 \),

\[ \begin{bmatrix} 2b_3 & 0 & -3b_1 \\ 0 & -2b_3 & 3b_2 \\ -b_2 & b_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \]

The matrix of the coefficients has determinant zero and generic rank two. This rank drops when \( b_1 = b_2 = b_3 = 0 \leftrightarrow x_0^2 \). Off the point \( x_0^2 \) the solution space is spanned by the vector product of two rows,
\[ (-3b_1 b_2 : -3b_2^2 : -2b_2 b_3) = (3b_1 : 3b_2 : 2b_3). \]

The idea now is to describe the locus of special cubic forms with the divisibility condition as a \( \mathbb{P}^4 \)-bundle over \( X \) (since it has codimension two in the \( \mathbb{P}^6 \) of special cubics). But to make it possible, we need to replace \( X \) by a new space, \( X' \), for which the rank of the coefficients matrix in (7) is two everywhere.
Precisely, think of the fiberwise solution space to (7) as defining a rational map $\psi : X \to \mathbb{P}(\mathcal{A})$, which on the fiber over the standard flag $\varphi_0$ reads

$$\psi : (b) \mapsto (a_4 : a_5 : a_6) = (3b_1 : 3b_2 : 2b_3).$$

Look at the closure $X'$ of the graph of $\psi$. On the fiber over $\varphi_0$, this is the blowup of $\mathbb{P}^3 = \mathbb{P}(\mathcal{B})$ at the point $x_0^2$. As a matter of fact, this turns out to be the restriction to the fiber over $\varphi_0$ of the blowup of $X$ along the section $\mathbb{P} \left( \mathcal{O}_{\mathbb{P}^3}(-2) \right) \hookrightarrow \mathbb{P}(\mathcal{B})$ over $F$. We have the diagram

$$\xymatrix{X' \ar[d]^\psi \ar[rd]^\psi' \ar@/_/[drr] & & \mathbb{P}(\mathcal{A}) \ar@{-->}[rr]^\varphi & & \mathbb{P}(\mathbb{P}(\mathcal{A})) \ar[d] \ar[l]^\varphi \ar@{-->}[ddr] \ar@/_/[r] & \mathbb{P}(\mathbb{P}(\mathcal{B})) \ar[ll]_{\mathbb{P}^1\text{-bundle}} \ar[d]\ar[dl]\mathbb{F} \ar[ul] \ar[l]_{\mathbb{P}^1\text{-bundle}}};$$

where the leftmost inclusion is defined by taking the square of the equation of the plane in the flag. The equality $\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{B})$ comes from Lemma 2.4 under this identification,

$$\mathcal{O}_{\mathcal{A}}(-1) = \mathcal{O}_{\mathcal{B}}(-1) \otimes (\mathcal{Q}/\mathcal{O}_{\mathbb{P}^3}(1)).$$

Pulling back the above tautological line subbundle via $\psi'$, we get the diagrams

$$\xymatrix{S_1 \otimes \mathcal{O}_{\mathcal{P}^3}(-2) \ar[d]|^\| \ar[r] & A' \ar[d]^\| & \mathcal{O}_{\mathcal{A}}(-1) \ar[d]^\| \ar[r]\ar@{-->}[urr] & \mathcal{O}_{\mathcal{A}}(-1) \ar[d]^\| \ar@{-->}[ddr] \ar@/_/[r] & \mathcal{O}_{\mathcal{P}^3}(-2) \ar[r] & A \ar[r]^\| & \mathcal{A}}$$

and

$$\xymatrix{\mathcal{O}_{\mathcal{P}^3}(-2) \ar[d]^\| \ar[r] & B' \ar[d]^\| & \mathcal{O}_{\mathcal{B}}(-1) \ar[d]^\| \ar[r]\ar@{-->}[urr] & \mathcal{O}_{\mathcal{B}}(-1) \ar[d]^\| \ar@{-->}[ddr] \ar@/_/[r] & \mathcal{O}_{\mathcal{P}^3}(-2) \ar[r] & B \ar[r]^\| & \mathcal{B}}$$

where rank $A' = 5$, rank $B' = 2$. We have

$$X' = \mathbb{P}(B'),$$

a $\mathbb{P}^1$ bundle over the $\mathbb{P}^2$ bundle $\mathbb{P}(\mathbb{P}(\mathcal{B}))$ over the flag variety $F$. Define

$$Y = \mathbb{P}(A'),$$

an equivariant $\mathbb{P}^4$–bundle over $X'$.

2.6. **Proposition.** A general point in $Y$ corresponds to a pair $(g, f)$ of special quadric, cubic satisfying the divisibility condition.

**Proof.** The assertion follows from the previous considerations. □
The next step is to solve the indeterminacies of the rational map
\[ Y \longrightarrow E(3) \subset \mathbb{P}(H^0(\Omega^1_{\mathcal{E}}(4))) \]
\[ (g, g', f) \longmapsto \omega = \frac{3fdg - 2gdf}{x_0}. \]
This will be accomplished by a sequence of 4 blowups,
\[ Y_4 \longrightarrow Y_3 \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow Y \longrightarrow E(3). \]

2.7. Remark. Since all constructions performed so far are equivariant, we drop the reference to the fiber over \( \varphi_0 \), and simplify notation writing
\[ X = X_{\varphi_0}, \ A = A_{\varphi_0}, \ldots, etc. \]
In view of Lemma 2.4, the rational map (8) can also be written as the rational linear projection map
\[ \psi: X \longrightarrow \mathbb{P}(A) \]
\[ (g, g') \longmapsto 3u_1x_0x_1 + 3u_2x_0x_1x_2 + 2u_3x_1^2. \]
(14) \[ X' = \{(g, g') = ((b_0 : b_1 : b_2, (u_1 : u_2 : u_3)) \mid b_iu_j = b_ju_i \}, \]
where (15) \[ g' := u_1x_0x_1 + u_2x_0x_2 + u_3x_1^2. \]
Here, the \( y \) are homogeneous coordinates in \( \mathbb{P}^2 = \) fiber of \( \mathbb{P}(B) \) over \( \varphi_0 \).

3. SOLVING THE INDETERMINACIES

Let us work over an affine cover of \( X' \) to deal with the indeterminacies of the map \( (g, g', f) \longmapsto \omega. \)
Notice that over a fixed flag, a point in \( Y \) has 7 affine coordinates.
Notation as in (14), consider the open dense set of \( X' \) given by \( b_0 = u_1 = 1. \)
The interesting equations are
\[ \begin{cases} b_2 = b_1u_2 \\ b_3 = b_1u_3 \end{cases} \quad \text{and} \quad \begin{cases} 3a_6 = 2a_4u_3 \\ a_5 = a_4u_2 \end{cases}, \]
where the former corresponds to the blowup of our $\mathbb{P}^3$ of special quadrics at $x_0^3$ \[\text{8 A. ROSSINI AND I. VAINSENCHER}\] and the latter to divisibility $(\ast)$ \[\text{10}.\] So,

\[
\begin{align*}
  f &= (a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3) x_0^2 + a_4 \left( x_0 x_1^2 + u_2 x_0 x_1 x_2 + \frac{2}{3} u_3 x_1^3 \right) \\
  g &= x_3^2 + b_1 x_0 x_1 + b_1 u_2 x_0 x_2 + b_1 u_3 x_1^2.
\end{align*}
\]

Computing $\omega$ as in \[\text{13},\] we find using Macaulay2

\[
\omega = [(3a_0 b_1 - 2a_1) x_0^2 x_1 + (6a_0 b_1 u_3 + a_1 b_1 - 4a_4) x_0 x_1^2 + \\
+ (3a_0 b_1 u_2 - 2a_2) x_0^2 x_2 - 2a_3 x_2^2 x_3 + \cdots] dx_0 + \cdots
\]

For $a_0 = 0$ the 1-form $\omega$ vanishes only if $a_0 = a_1 = a_2 = a_3 = a_4 = 0$, which is impossible. Thus, in order to study the indeterminacy locus we can take $a_0 = 1$.

Setting $a_0 = 1$ and collecting coefficients of $\omega$, we get a non-reduced and reducible scheme, given by an ideal $J$. Its radical $\text{rad}(J) := J_{\text{red}}$ presents two irreducible components:

(1) A component $C$, with ideal

\[\text{(17)}\]

\[J_C = \langle b_1 - 4u_3, a_1 - 6u_3, a_2, a_3, a_4 - 12u_3^2, u_2 \rangle\]

The ideal $J_C$ is generated by a regular sequence. The affine coordinates here are $a_1, a_2, a_3, a_4, b_1, u_2, u_3$. Therefore we have a locally complete intersection, which represents the curve over the flag $\varphi_0$ given by

\[
f = (x_0 + 2u_3 x_1)^3, \quad g = (x_0 + 2u_3 x_1)^2.
\]

(2) A component $E$ with ideal

\[\text{(18)}\]

\[J_E = \langle a_1, a_2, a_3, a_4, b_1 \rangle,
\]

which is the whole $\mathbb{P}^2$–fiber of the exceptional divisor of $X'$ over $g = x_0^3$, and $f = x_0^3$.

Notice that there is precisely one point in the intersection of these two components, since

\[J_C + J_E = \langle a_1, a_2, a_3, a_4, b_1, u_2, u_3 \rangle.
\]

This ideal represents the single point $(g, g', f) = (x_0^2, x_0 x_1, x_0^3) \in \mathbb{Y}$.

Let’s see what happens when we blowup $\mathbb{Y}$ first along $C$, followed by a blowup along $E'$, the strict transform of $E$.

Let $\mathbb{A}^7$ be the affine neighborhood defined by $b_0 = a_0 = u_1 = 1$. The blowup of this $\mathbb{A}^7$ along $C$ is

\[
\mathbb{Y}_1 \subset (\mathbb{A}^7 \times \mathbb{P}^5; \langle (s_0, s_1), \ldots, s_5 \rangle) \mid s_i \cdot e_j = s_j \cdot e_i \subset \mathbb{A}^7 \times \mathbb{P}^5,
\]

where $e_i$, $0 \leq i \leq 5$, are the equations of $J_C$, ordered as in \[\text{17},\].

There are six choices for the local equation of the first exceptional divisor. Choose, say, exc$_1 = u_2$, i.e, look to the affine chart $s_5 = 1$. Then, we make
the following substitutions

\[
\begin{align*}
    b_1 - 4w_3 &= s_0 u_2 \\
    a_1 - 6w_3 &= s_1 u_2 \\
    a_2 &= s_2 u_2 \\
    a_3 &= s_3 u_2 \\
    a_4 - 12u_2^2 &= s_4 u_2 
\end{align*}
\]

The new 7 affine coordinates on the blowup are

(19) \[ s_0, s_1, s_2, s_3, s_4, u_2, u_3. \]

After the blowup \( Y_1 \to Y \) of \( Y \) along \( C \), the strict transform of the radical \( J_{red} \) is given by

(20) \( J'_E = \langle s_3, s_2, 3s_0 - 2s_1, 6u_3 + s_1 u_2, 3s_4 + s_2^2 u_2 \rangle. \)

It coincides with the strict transform \( E' \) of \( E \) under the first blowup. Now, take \( E' \) as our new blowup center and denote by \( Y_2 \to Y_1 \) the blowup of \( Y_1 \) along \( E' \),

\[ Y_2 = \{ ((u, s, t), (t_0 : \ldots : t_4)) \mid t_i \cdot e_j = t_j \cdot e_i \} \subset \mathbb{A}^7 \times \mathbb{P}^4, \]

where \( e_i, \ 0 \leq i \leq 4, \) are the equations of \( J_{E'} \), ordered as in (20), and \( (u, s, t) \) as in (19).

Choose now the equation \( \text{exc}_2 = s_2 \) in (20) as the new local exceptional equation. Equations of \( Y_2 \) become

\[
\begin{align*}
    s_3 &= t_0 s_2 \\
    3s_0 - 2s_1 &= t_2 s_2 \\
    6u_3 + s_1 u_2 &= t_3 s_2 \\
    3s_4 + s_2^2 u_2 &= t_4 s_2 
\end{align*}
\]

The new 7 affine coordinates on \( Y_2 \) are

(21) \[ t_0, t_2, t_3, t_4, u_2, s_1, s_2. \]

The two blowups so far performed are not enough to solve the indeterminacies of our map \((g, f) \mapsto \omega\). However they do make the new scheme of indeterminacies to become reduced. Although reduced, the indeterminacy locus is still reducible, with two components. One of them is given by the ideal

(22) \( J_R = \langle u_2, t_3 - 1, t_2, t_0, 3s_1 - 2t_4 \rangle. \)

The ideal \( J_R \) represents a ruled surface \( R \) which is a \( \mathbb{P}^1 \)–subbundle of the (transform of the) exceptional divisor \( \mathbb{P}(\mathcal{N}_{C/Y}) \) – note the equation \( \text{exc}_1 = u_2 \) representing the curve \( C \) and the other four linear equations on the new affine variables.

Denote by \( Y_3 \to Y_2 \) the blowup of \( Y_2 \) along \( R \). This blowup, in the affine chart \( s_5 = 1, \ t_1 = 1, \) is

\[
Y_3 = \{ ((u, s, t, l), (v_0 : \ldots : v_4)) \mid v_i \cdot e_j = v_j \cdot e_i \} \subset \mathbb{A}^7 \times \mathbb{P}^4,
\]

where \( e_i, \ 0 \leq i \leq 4, \) are the equations of \( J_R \), ordered as in (22), and \( (u, s, t, l) \) as in (21).
Choose now the equation $\text{exc}_3 = u_2$ in (22) as the new local exceptional equation, i.e, take $v_0 = 1$. Equations of $\mathcal{Y}_3$ become

$$
\begin{align*}
t_3 - 1 &= v_1u_2 \\
t_2 &= v_2u_2 \\
t_0 &= v_3u_2 \\
3s_1 - 2t_4 &= v_4u_2
\end{align*}
$$

The new 7 affine coordinates on $\mathcal{Y}_3$ are

(23)

$v_1, v_2, v_3, v_4, u_2, s_2, t_4$

and the indeterminacy locus reduces to

(24)

$J_L = \langle s_2, v_1, v_2, v_3, v_4, t_4 \rangle$.

The component represented by the ideal $J_L$ (24) is easy to describe. Since the affine variables are as listed in (23), the six linear equations represent a line inside the (transform of) the second blowup center $E'$ (note the presence of the equation $\text{exc}_2 = s_2$ in $J_L$), a line parametrized by the variable $u_2$. We call $L$ the component described by the ideal $J_L$ (24). This is a reduced and irreducible local complete intersection. Since it is the full indeterminacy locus, a blowup along this subscheme will solve the indeterminacies over the neighborhood $[b_0 = 1, u_1 = 1], [s_5 = t_1 = v_0 = 1]$. That is, the map to $\omega$ becomes a morphisms, cf. [7, ex. 7.17.3, p. 168].

Denote by $\mathcal{Y}_4 \to \mathcal{Y}_3$ the blowup of $\mathcal{Y}_3$ along $L$,

$$\mathcal{Y}_4 = \{(u, s, t, v, z_0 : \ldots : z_5) \mid z_i \cdot e_j = z_j \cdot e_i\} \subset \mathbb{A}^7 \times \mathbb{P}^5,$$

where $e_i$, $0 \leq i \leq 5$, are the equations of $J_L$, ordered as in (24), and $(u, s, t, v)$ as in (23).

In $\mathcal{Y}_4$ the map is solved, at least in the affine chart $[s_5 = t_1 = v_0 = 1]$. The calculations in all other standard neighborhoods reveal that this sequence of four blowups, along $C, E, R$ and $L$, will solve the map $(g, f) \mapsto \omega$ over the neighborhood $[b_0 = 1, u_1 = 1]$.

There are other 5 standard neighborhoods to be checked to complete an affine cover of $\mathcal{X}'$, corresponding to $[b_3 = 1], [b_2 = 1], [b_1 = 1], [b_0 = 1 \text{ and } u_2 = 1], [b_0 = 1 \text{ and } u_3 = 1]$.

Similar calculations show that these four blowups described will solve the map over each of these neighborhoods.

The whole discussion of the indeterminacy loci was made over the fixed flag (3). Since the blowup centers as described were actually fibers of bundles over the variety of complete flags $\mathcal{F}$, we have at the end a bundle which constitutes the desired parameter space:

$$\begin{align*}
\mathcal{Y}_4 &\xrightarrow{\text{blowup } L} \mathcal{Y}_3 &\xrightarrow{\text{blowup } R} \mathcal{Y}_2 &\xrightarrow{\text{blowup } E} \mathcal{Y}_1 &\xrightarrow{\text{blowup } C} \mathcal{Y} \\
\mathcal{Y} &\xrightarrow{\mathbb{P}^4 \text{ bundle}} \mathcal{X}' &\xrightarrow{\text{blowup } \mathbb{P}(\mathcal{O}_{\mathbb{P}^3}(-2))} \mathcal{X} &\xrightarrow{\mathbb{P}^3 \text{ bundle}} \mathcal{F} = \text{ all flags}
\end{align*}$$

(25)

This can be summarized as follows:
3.1. Theorem. Let $\mathcal{Y}_4$ be the variety obtained by the four blowups as described above. Then $\mathcal{Y}_4$ is equipped with a morphism $\Phi$ onto the exceptional component $E(3)$. \hfill \Box

3.2. Proposition. The map $\Phi : \mathcal{Y}_4 \rightarrow E(3)$ is generically injective.

Proof. For a given $\omega \in E(3)$ (off the boundary) we already saw that the flag can be recovered (see Remark 2.1). Hence, we can look at a fiber over a fixed flag.

So fix the flag $\varphi_0 = (p_0, \ell_0, v_0)(\mathbb{K})$. The fiber of $\mathbb{K}$ at this flag is the $\mathbb{P}^3$ of special quadrics

$$g = b_0x_0^2 + b_1x_0x_1 + b_2x_0x_2 + b_3x_1^2.$$  

Now, we will pay attention to the dense open set where $b_0$ and the quadric $g$ can be recovered from $\omega$. By construction, one can see that there are no indeterminacies on this neighborhood $[b_0 = 1]$. Moreover,

$$2g\frac{\partial f}{\partial x_3} - 3f\frac{\partial g}{\partial x_3} = 2a_3gx_0^3,$$

and the quadric $g$ can be recovered from $\omega$ just by looking at the coefficient of $dx_3$ (at least on the dense open set $a_3 = 1$).

The coefficient of $dx_2$ in $\omega$ is

$$\frac{1}{x_0} \left( 2g\frac{\partial f}{\partial x_2} - 3f\frac{\partial g}{\partial x_2} \right) = (2a_2b_0 - 3a_0)x_0^3 + (2a_3b_0 + 2a_2b_1 - 3a_1)x_0^2x_1 + (-a_3b_1 + 2a_2b_3)x_0x_1^2 - a_2x_0^2x_2 - a_5x_0x_1x_2 - 3a_3x_0^2x_3.$$

Hence the coefficients $a_2$ and $a_5$ can be recovered from $x_0^2x_2dx_2$ and $x_0x_1x_2dx_2$, respectively. And since the coefficients $b_0$ and $b_1$ are also known we obtain the coefficients $a_0$ and $a_1$ from $x_0^3dx_2$ and $x_0^2x_1dx_2$ respectively. \hfill \Box

4. Calculation of Degree

By construction, $\mathcal{Y}_4$ is equipped with a line bundle $\mathcal{W}$, pullback of the line bundle $\mathcal{O}_{\mathbb{P}^4}(-1)$ over $\mathbb{P}^{44} = \mathbb{P}(H^0(\Omega^1_{\mathbb{P}^3}(4)))$. The fiber of $\mathcal{W}$ over a point in $\mathcal{Y}_4$ is the rank one space spanned by the computed 1-form $\omega$.

$$\mathcal{W} = \Phi^*(\mathcal{O}(-1)) \quad \downarrow \quad \mathcal{O}(-1) \quad \downarrow$$

$$\mathcal{Y}_4 \quad \xrightarrow{\Phi} \quad \mathbb{P}(H^0(\Omega^1_{\mathbb{P}^3}(4))) \supset E(3) = \Phi(\mathcal{Y}_4)$$
4.1. **Theorem.** The degree of the exceptional component of codimension one and degree two foliations in $\mathbb{P}^3$ is given by

$$\int_{\mathbb{Y}_4} -c_1^{13}(\mathcal{W}) \cap [\mathbb{Y}_4].$$

**Proof.** We have $\dim(\mathbb{Y}_4) = 13$. Since the map $\Phi$ is generically injective the required degree is (cf. definition of $\deg_{\tilde{f}} X$ in [6], page 83)

$$\int_{\mathbb{Y}_4} c_1(\Phi^* \mathcal{O}(1))^{13} \cap [\mathbb{Y}_4] = \int_{\mathbb{Y}_4} c_1(\Phi^* \mathcal{O}(-1))^{13} \cap [\mathbb{Y}_4] = \int_{\mathbb{Y}_4} c_1(\mathcal{W})^{13} \cap [\mathbb{Y}_4].$$

□

In order to obtain the value of the integral in Theorem 4.1 we apply Bott’s formula

$$\int_{\mathbb{Y}_4} -c_1^{13}(\mathcal{W}) \cap [\mathbb{Y}_4] = \sum_F -c_1^{T}(\mathcal{W})^{13} \cap [F]_T,$$

where the sum runs through all fixed components $F$ under a convenient action of the torus $\mathbb{C}^*$. The $N_{F|\mathbb{Y}_4}$ appearing in the denominator denotes the normal bundle of a fixed component $F$ in $\mathbb{Y}_4$. Fix an action

$$\mathbb{C}^* \times \mathbb{P}^3 \rightarrow \mathbb{P}^3 \quad (t, x_i) \rightarrow t^{w_i}x_i$$

of $\mathbb{C}^*$ in $\mathbb{P}^3$ with distinct weights $w_i$, $i \in \{0, 1, 2, 3\}$. The only fixed flags are the 24 standard ones

$$p_{ijk} = \{x_i = x_j = x_k = 0\} \in \ell_{ij} = \{x_i = x_j = 0\} \subset v_i = \{x_i = 0\}.$$

Over each one of these fixed flags, we can find 72 fixed isolated points and 5 fixed lines. So, there is a total of $72 \cdot 24 = 1728$ fixed points and $5 \cdot 24 = 120$ fixed lines.

A detailed exposition of the fixed points and the computations of their contributions on Bott’s formula (27), including scripts for Macaulay2 for this sum and for the resolution of singularities also in the other neighborhoods can be found in

https://sites.google.com/a/ifsudestemg.edu.br/nucleo-de-matematica/artur/trabalho

4.2. **Theorem.** The degree of the exceptional component of foliations of codimension one and degree two in $\mathbb{P}^3$ is $168208$.

In the next Table 1 we list the components of the space of foliations of degree two and codimension one in $\mathbb{P}^3$ as described in [3], and their respective degrees. Notice that the degree of the logarithmic components have not been found yet in the literature to the best of our knowledge.
5. A geometric interpretation of the degree

For a codimension one foliation in $\mathbb{P}^n$, given by the differential form

$$\omega = \sum_{i=0}^{n} A_i dx_i,$$

the hyperplane defined by the distribution at a point $p \in \mathbb{P}^n$ is

$$H = \sum_{i=0}^{n} A_i(p)x_i = 0.$$

Thus, a tangent direction $v = (v_0 : \ldots : v_n) \in \mathbb{P}(T_p\mathbb{P}^3)$ lies on this hyperplane $H$ if

$$\sum_{i=0}^{n} A_i(p) \cdot v_i = 0.$$

This can be thought of as a linear equation on the coefficients of the $A_i$. Hence, the point $(p, v) \in \mathbb{P}(T\mathbb{P}^3)$ defines a hyperplane in the projective space of distributions.

(29) $$\omega(p) \cdot v = 0.$$  

The equation (29) shows that the degree of a $m$–dimensional component of the space of codimension one foliations in $\mathbb{P}^n$ can be interpreted as the number of such foliations that are tangent to $m$ general directions in $\mathbb{P}^n$.

In particular, Theorem 1.2 means that there are 168208 exceptional foliations tangent to 13 general directions in $\mathbb{P}^3$.

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