Abstract. Level \( m \)-stratifications on PEL Shimura varieties are defined and studied by Wedhorn using BT-\( m \)s with PEL structure, and then by Vasiu for general Hodge type Shimura varieties using Shimura \( F \)-crystals. The theory of foliations is established by Oort for Siegel modular varieties, and by Mantovan for PEL Shimura varieties. It plays an important role in Hamacher’s work to compute the dimension of Newton strata of PEL Shimura varieties.

We study level \( m \) stratifications on good reductions at \( p > 2 \) of Shimura varieties of Hodge type by constructing certain torsors together with equivariant morphisms, and relating them to truncated displays. We then use the results obtained to extend the theory of foliations to these reductions. As a consequence, combined with results of Nie and Zhu, we get a dimension formula for Newton strata.

Contents

0. Introduction 2
1. Preliminaries 3
   1.1. Dieudonné crystals and Dieudonné modules 3
   1.2. Integral canonical models 4
   1.3. Definitions of some stratifications 5
   1.4. Dilatations and some group theoretic settings 7
2. Level \( m \) stratifications 10
   2.1. Constructing torsors over \( S \) and \( S_0 \) 10
   2.2. The morphism \( \zeta_m \) 12
   2.3. Independence of symplectic embeddings 18
3. Truncated displays with additional structure 19
   3.1. Truncated displays 19
   3.2. Truncated displays with additional structure 20
   3.3. More properties about level \( m \) stratifications 23
4. Geometry of Newton strata 25
   4.1. Group theoretic preparations 25
   4.2. Central leaves 27
   4.3. Foliations 34
References 39

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0. Introduction

The theory of foliations is established by Oort in [29] for Siegel modular varieties, and by Mantovan in [21] for good reductions of PEL shimura varieties. The theory roughly says that a Newton stratum is an “almost product” of a central leaf with an isogeny leaf. The goal of this paper is to extend such a theory to good reductions at char $p \geq 3$ of Shimura varieties of Hodge type. As a consequence, thanks to results and ideas in [10], [28], [42] and [48], we get a dimension formula for Newton strata.

The first step is to understand level $m$-stratifications on these reductions. They are defined and studied by Wedhorn in [43] for those of PEL type using BT-$m$-s with PEL structure, and by Vasiu in [39] for general Hodge type Shimura varieties using Shimura $F$-crystals.

We introduce a new way to understand level $m$-stratifications. It relies on explicit constructions of some torsors on the reductions, and it is closely related to truncated displays in [19]. Our construction uses group schemes defined in [39], and is greatly influenced by ideas and methods there. We refer to section 2 and 3 for details.

We also consider classical level $m$-stratifications defined by geometric isomorphism types of BT-$m$s. We summarize the main results as follows.

Theorem 1. Let $\mathcal{X}_0$ be the good reduction of a Shimura variety of Hodge type.

1. Each level $m$ stratum $\mathcal{X}_0^s$ is a smooth locally closed subscheme of $\mathcal{X}_0$. The closure $\overline{\mathcal{X}_0^s}$ is a union of level $m$ strata, and $\mathcal{X}_0^s \rightarrow \overline{\mathcal{X}_0^s}$ is an affine immersion.
2. There is a quasi-finite fppf cover $\mathcal{T} = \text{Spec} A \rightarrow \mathcal{X}_0$, such that the BT-$m$ with additional structure is constant.
3. Each level $m$ stratum is open and closed in its classical level $m$ stratum.

By [38], for $m$ big enough, all level $m$ strata are central leaves. Using the above theorem, as well results in [10], [16], [28], [29] and [40], we can prove the followings.

Theorem 2. Let $\mathcal{X}_0$ be as above. Then

1. Each central leaf $\mathcal{X}_0^c$ is a smooth locally closed subscheme of $\mathcal{X}_0$. The closure $\overline{\mathcal{X}_0^c}$ is a union of central leaves, and $\mathcal{X}_0^c \rightarrow \overline{\mathcal{X}_0^c}$ is an affine immersion.
2. Each central leaf is open and closed in its central leaf, and it is also closed in its Newton stratum.
3. Central leaves in a Newton stratum $\mathcal{X}_0^b$ are of the same dimension $\langle \rho, \nu_G(b) \rangle$.

We define Igusa towers in 4.2, and give the almost product morphism in 4.3. The ideals and technics are from [12], [16], [21], and [29]. With results in [48], we deduce that

Theorem 3. The Newton stratum $\mathcal{X}_0^b$ is of dimension $\langle \rho, \mu + \nu(b) \rangle - \frac{1}{2} \text{def}(b)$. 
1. Preliminaries

1.1. Dieudonné crystals and Dieudonné modules. We will first recall results related to crystals of $p$-divisible groups that will be used in this paper. Our main references are [1], [4] and [14].

Let $\kappa$ be a perfect field of characteristic $p > 0$ and $T$ be a $W(\kappa)$-scheme such that $p$ is locally nilpotent. We refer to [4] for the definition of crystalline site and crystals. Let $(T/W(\kappa))_{cris}$ be the crystalline site of $T$ over $W(\kappa)$. There is a contravariant functor $G \to \mathbb{D}(G)$ from the category of $p$-divisible groups over $T$ to the category of crystals (of quasi-coherent sheaves) over $(T/W(\kappa))_{cris}$. This functor is defined using the Lie algebra of the universal vector extension of the dual $p$-divisible group $G^\vee$. The formation of $\mathbb{D}$ is compatible with base change. In particular, if $p = 0$ on $T$, then the absolute Frobenius $\sigma$ on $T$ induces the relative Frobenius on $G$, and hence a morphism of crystals $\sigma^*(\mathbb{D}(G)) \to \mathbb{D}(G)$.

Suppose now that $T_0$ is a $\kappa$-scheme, and $G_0$ is a $p$-divisible group over $T_0$. Let $T_0 \to T$ be an object of $(T_0/W(\kappa))_{cris}$ on which $p$ is locally nilpotent, and $G$ be a lifting of $G_0$ to $T$. By construction of $\mathbb{D}$, we have an isomorphism $\mathbb{D}(G_0)(T) \cong \mathbb{D}(G)(T)$. Moreover, the $O_T$-module $\mathbb{D}(G)(T)$ sits in an exact sequence

$$0 \to (\text{Lie}G)^\vee \to \mathbb{D}(G)(T) \to \text{Lie}G^\vee \to 0.$$

**Definition 1.1.1.** Let $T$ be a $\kappa$-scheme. A Dieudonné crystal over $T$ is triple $(\mathcal{E}, \varphi, \nu)$ where

1. $\mathcal{E}$ is a crystal of finite locally free modules over $(T/W(\kappa))_{cris}$,
2. $\varphi : \sigma^*\mathcal{E} \to \mathcal{E}$ and $\nu : \mathcal{E} \to \sigma^*\mathcal{E}$ are homomorphisms of $O_T,cris$-modules such that $\varphi \circ \nu = p \cdot \text{id}_{\mathcal{E}}$ and $\nu \circ \varphi = p \cdot \text{id}_{\sigma^*\mathcal{E}}$.

It is well known that $\mathbb{D}(G)$ is a Dieudonné crystal, for a $p$-divisible group $G/T$.

Let $A_0$ be a formally smooth $\kappa$-algebra, by a lifting of $A_0$, we mean a $p$-adically complete flat $W(\kappa)$-algebra $A$ such that $A \otimes \kappa \cong A_0$. Such a lifting always exists by [4] Lemma 1.1.2 and Lemma 1.2.2, it is unique by [4] Remark 1.2.3 (b). Moreover, $A$ is formally smooth over $W(\kappa)$ (with respect to the $p$-adic topology), and the Frobenius $\sigma : A_0 \to A_0$ lifts to $A$ (but NOT necessarily unique).

If $A_0$ is formally finitely generated (see [4] Lemma 1.3.1 for the definition), then $A$ is necessarily regular. This is because $W(\kappa) \to A$ is flat with geometrically regular special fiber, while $A$ is a quotient of $W(\kappa)[[x_1, \ldots, x_r]][y_1, \ldots, y_s]$, the $p$-adic completion of $W(\kappa)[[x_1, \ldots, x_r]][y_1, \ldots, y_s]$, and hence all maximal ideals contain $p$. If $A = W(\kappa)[[x_1, \ldots, x_n]]$, the homomorphism given by Frobenius on $W(\kappa)$ and $p$-th power on indeterminants is a lift of the Frobenius on Let $A_0 := A \otimes \kappa$.

**Definition 1.1.2.** Fixing the pair $(A, \sigma)$, a Dieudonné module over $A_0$ is quadruple $(M, \varphi, \nu, \nabla)$ where

1. $M$ is a locally free $A$-module,
2. $\nabla : M \to M \otimes \Omega^1_{A/W(\kappa)}$ is an integrable, topologically quasi-nilpotent connection,
3. $\varphi : \sigma^*M \to M$ and $\nu : M \to \sigma^*M$ are horizontal homomorphisms of $A$-modules such that $\varphi \circ \nu = p \cdot \text{id}_M$ and $\nu \circ \varphi = p \cdot \text{id}_{\sigma^*M}$.

This definition is a special case of [4] Definition 2.3.4, as $\ker(A \to A_0) = (p)$ is equipped with the natural PD-structure, and $\mathcal{D}$ there is our $A$. We refer to [4] Remark 2.2.4 c) for the definition of a topologically quasi-nilpotent connection.
Proposition 1.1.3. The category of Dieudonné crystals over Spec $A_0$ is equivalent to the category of Dieudonné modules over $A_0$.

Proof. This is a direct consequence of [3] Proposition 2.2.2 and Remark 2.2.4 b) and h).

We will use some constructions that are needed for the proof.

Construction 1.1.4. The Dieudonné module attached to a Dieudonné crystal $E$ is as follows. The $A$-module $M$ is $\varprojlim_n E(A_n)$. Let $A(2)$ be the $p$-adic completion of the PD-envelop of $A \otimes A$ with respect to $\ker(A \otimes A \to A_0)$. There are homomorphisms $i_1 : A \to A(2)$, $a \to a \otimes 1$ and $i_2 : A \to A(2)$, $a \to 1 \otimes a$. The crystal property of $E$ gives an isomorphism $\varepsilon : i_2^*M \to i_1^*M$. Let $\theta : M \to i_2^*M$ be $m \mapsto 1 \otimes m$, then $\nabla = \theta - \text{id}_M \otimes 1$, $m \mapsto \varepsilon(1 \otimes m) - m \otimes 1 \in M \otimes (K/K[2]) = M \otimes \Omega_{A/W(A)}$. Here $K$ is the kernel of $A(2) \to A$. The maps $F$ and $V$ follows from [3] Remark 2.2.4 h).

Construction 1.1.5. Notations as above. Let $A'$ be a $p$-adically complete and $p$-torsion free $W(\kappa)$-algebra, equipped with a list of Frobenius $\sigma'$. Let $i : A \to A'$ be a homomorphism of $W(\kappa)$-algebras. Then $i^*(E)(A')$ is again an $A'$-module with connection and Frobenius. The module is just $M \otimes A'$ and the connection is just $i^*\nabla$. Since $i \circ \sigma$ and $\sigma' \circ i$ has the same reduction modulo $p$, the crystal property of $E$ induces a canonical isomorphism $\varepsilon' : \sigma'^\ast i^* (M) \to i^* \sigma^* (M)$ as follows. Let $\varepsilon'^{[2]} : (A(2)/K[2]) \otimes M \to M \otimes (A(2)/K[2])$ be the reduction modulo $K[2]$ of $\varepsilon$, which is the isomorphism given by $\nabla + iM \otimes 1$, then $\varepsilon' = (\sigma' \circ i \circ \sigma)^\ast (\varepsilon'^{[2]})$.

The Frobenius on $M \otimes A'$ is given by

$$\sigma'^\ast (M_{A'}) = \sigma'^\ast i^* (M) \xrightarrow{\varepsilon'} i^* \sigma^* (M) \to i^* M = M_{A'}.$$

Explicitly, $\varepsilon'$ has the following description: let $a_1, \cdots, a_n \in A$ be such that $da_1, \cdots, da_n$ form a basis of $\Omega^n_{A/W(A)}$. Note that they exist Zariski locally. Then

$$\varepsilon'(m \otimes 1) = \sum_i \nabla (\partial_i^\bullet (m \otimes 1)) \otimes \frac{z_i}{\partial_i^\bullet},$$

Here $\partial_i^\bullet = (i_1, \cdots, i_n)$ is a multi-index, $\nabla (\partial_i^\bullet) = \nabla (\frac{\partial_{i_1}}{\partial_a})^{i_1} \cdots \nabla (\frac{\partial_{i_n}}{\partial_a})^{i_n}$, $z_i = z_i^1 \cdots z_i^n$ where $z_i = \sigma' \circ i (a_i) - i \circ \sigma (a_i)$. The map $\varepsilon'$ is well defined and independent of choices of $a_1, \cdots, a_n$, and hence we always have $\varepsilon'$ by Zariski gluings.

1.2. Integral canonical models. We recall Kisin’s construction of good reductions of Shimura varieties of Hodge type.

Let $(G, X)$ be a Shimura datum of Hodge type. Let $p$ be a prime. Assume that $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension of $\mathbb{Q}_p$. Then $G_{\mathbb{Q}_p}$ extends to a reductive group scheme $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$. Let $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, for any compact open subgroup $K^p \subseteq G(\mathbb{A}_f^p)$ that is small enough, Kisin proves in [15] that the Shimura variety $\text{Sh}_{K,K^p}(G, X)$ has an integral canonical model provided that $p > 2$.

We recall the constructions in [15]. Let $i : (G, X) \hookrightarrow (\text{GSp}(V, \psi), X')$ be a symplectic embedding. By [15] Lemma 2.3.1, there exists a $\mathbb{Z}_p$-lattice $V_{\mathbb{Z}_p} \subseteq V_{\mathbb{Q}_p}$, such that $i_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \to \text{GL}(V_{\mathbb{Q}_p})$ extends uniquely to a closed embedding $G_{\mathbb{Z}_p} \hookrightarrow \text{GL}(V_{\mathbb{Z}_p})$. So there is a $\mathbb{Z}$-lattice $V_\mathbb{Z} \subseteq V$ such that $G_{\mathbb{Z}_p}$, the Zariski closure of $G$ in $\text{GL}(V_{\mathbb{Z}_p})$, is reductive, as the base change to $\mathbb{Z}_p$ of $G_{\mathbb{Z}_p}$ is $G_{\mathbb{Z}_p}$. Moreover, we can assume $V_{\mathbb{Z}_p}$ is such that $V_{\mathbb{Z}_p}[2] \cong V_{\mathbb{Z}}$. Let $d = [V_{\mathbb{Z}_p}/V_{\mathbb{Z}}]$, $g = \frac{1}{2} \text{dim}(V)$, $K = K_p K^p$, $E$ be the reflex field of $(G, X)$ and $v$ be a place of $E$ over $p$, then the integral
canonical model $\mathcal{S}_K(G, X)$ of $\mathbb{H}_K(G, X)$ is constructed as follows. We can choose $K' \subseteq \text{GSp}(V, \psi')(k_f)$ small enough such that $K' \supseteq K$, that $\mathbb{H}_K'(\text{GSp}(V, \psi), X)$ affords a moduli interpretation, and that the natural morphism
$$f : \mathbb{H}_K(G, X) \to \mathbb{H}_K'(\text{GSp}(V, \psi), X)_E$$
is a closed embedding. Let $\mathcal{A}_{g,d,K'/\mathbb{Z}(p)}$ be the moduli scheme of abelian schemes over $\mathbb{Z}(p)$-schemes with a polarization of degree $d$ and level $K'$ structure. Then $\mathcal{A}_{g,d,K'/\mathbb{Z}(p)} \otimes \mathbb{Q} = \mathbb{H}_K'(\text{GSp}(V, \psi), X)$. The integral canonical model $\mathcal{S}_K(G, X)$ is the normalization of the Zariski closure of $\mathbb{H}_K(G, X)$ in $\mathcal{A}_{g,d,K'/\mathbb{Z}(p)} \otimes O_E(v)$.

**Theorem 1.2.1.** The $O_{E,(v)}$-scheme $\mathcal{S}_K(G, X)$ is smooth, and morphisms in the inverse system $\varprojlim_{K'} \mathcal{S}_K(G, X)$ are étale.

**Proof.** This is [15] Theorem 2.3.8. □

**Remark 1.2.2.** The morphism $\mathcal{S}_K(G, X) \to \mathcal{A}_{g,d,K'/O_{E,(v)}}$ is finite, as $\mathcal{A}_{g,d,K'/O_{E,(v)}}$ is Nagata.

Moreover, the scheme $\mathcal{S}_K(G, X)$ is uniquely determined by the Shimura datum and the group $K$ in the sense that $\varprojlim_{K'} \mathcal{S}_K(G, X)$ satisfies a certain extension property (see [15] 2.3.7 for a precise statement) and there is an $G(\hat{\mathbb{A}}_f)$-action on $\varprojlim_{K'} \mathcal{S}_K(G, X)$ extending the one on $\varprojlim_{K'} \mathbb{H}_K(G, X)$.

Let $\mathcal{A} \to \mathcal{S}_K(G, X)$ be the pull back to $\mathcal{S}_K(G, X)$ of the universal abelian scheme on $\mathcal{A}_{g,d,K'/\mathbb{Z}(p)}$, and $V$ be $H^1_{dR}(\mathcal{A}/\mathcal{S}_K(G, K))$. Kisin also constructed in [15] Proposition 1.3.2, there is a tensor $s \in V^\otimes_{\mathbb{Z}(p)}$ defining $G_{\mathbb{Z}(p)} \subseteq \text{GL}(V_{\mathbb{Z}(p)})$. This tensor gives a section $s_{dR}/E$ of $V^\otimes_{\mathbb{H}_K(G,X)}$, which is actually defined over $O_{E,(v)}$. More precisely, we have the following result.

**Proposition 1.2.3.** The section $s_{dR}/E$ of $V^\otimes_{\mathbb{H}_K(G,X)}$ extends to a section $s_{dR}$ of $V^\otimes$.

**Proof.** This is [15] Corollary 2.3.9. □

The extended section $s_{dR}$ has a lot of good properties. One sees easily that it is a locally direct summand of $V^\otimes$. Let $x$ be a closed point in the special fiber of $\mathcal{S}_K(G, X)$, $\overline{x}$ a lifting to a $W(k(x))$-point of $x$, then $s_{dR,\overline{x}} \in V^\otimes_{\overline{x}}$ is invariant under the Frobenius on $V^\otimes_{\overline{x}}$ (see [15] Corollary 1.4.3). Here we use the canonical isomorphism $V_{\overline{x}} \cong H^1_{\text{crys}}(A_{\overline{x}}/k(x))$. Let $\mathcal{R}$ be the completion of the stalk at $x$ of $\mathcal{S}_K(G, X)$ with respect to the maximal ideal, and $D$ be the contravariant Dieudonné functor, then $s_{dR}$ is parallel with respect to the connection on $D(A_{\mathcal{R}}(\mathcal{R}))$.

1.3. **Definitions of some stratifications.** We define various stratifications on reductions of Shimura varieties of Hodge type and varieties over them. Note that it is technically necessary to define stratifications on varieties over reductions of Shimura varieties of Hodge type.

Let $\mathcal{S}_0$ be the special fiber of $\mathcal{S}_K(G, X)$. It is smooth over $\kappa = O_{E,(v)}/(v)$. Using the morphism $\mathcal{S}_0 \to \mathcal{S}_{g,d,K'/\kappa}$, one gets classical Newton stratification (resp. classical level $m$ stratification, resp. classical central leaves) on an $\mathcal{S}_0$-scheme $X$ by putting together points of $X_\kappa$ whose attached $p$-divisible groups (resp. BT-ms,
resp. $p$-divisible groups) are geometrically isogenous (resp. geometrically isomorphic, resp. geometrically isomorphic). Let $i$ be a geometrically isogenous (resp. isomorphism, resp. isomorphism) class of $p$-divisible groups (resp. BT-ms, resp. $p$-divisible groups), we will write $X^{cN,i}$ (resp. $X^{cL,i}$, resp. $X^{cC,i}$) for the corresponding stratum. The following statement should be well known.

**Proposition 1.3.1.** Notations as above, we have

1. Each $X^{cN,i}$ (resp. $X^{cL,i}$, resp. $X^{cC,i}$) is locally closed in $X$.  
   
2. Each classical central leaf is closed in the classical Newton stratum containing it.

**Proof.** The proof of [43] Proposition 1.8 implies that the stack of BT-ms (of given height) is an algebraic stack. So $X^{cL,i}$ is locally closed. The locally closedness for $X^{cN,i}$ is in [29] 2.1. Statement (2) is [29] Theorem 2.2. This implies that $X^{cC,i}$ is locally closed.

We are interested in refinements of the above stratifications. Let $x \in X$ be a point (not necessarily closed) and $\tau$ be the geometric point of $X$ induced by embedding $k(x)$ to an algebraically closed field $k(\tau)$. Then we have a Frobenius invariant tensor $s_{\text{cris},\tau} \in \mathcal{Y}_{W(k(\tau))}^0 = H^1_{\text{cris}}(\mathcal{A}_\tau, k(\tau))^\otimes$. Let $W_m$ be the ring of Witt vectors of length $m$, then by passing to $W_m(k(\tau))$, we have a tensor $s_{\text{cris},\tau} \in \mathcal{Y}_{W_m(k(\tau))}^0 = \mathcal{D}(\mathcal{A}_\tau[p^m])^\otimes$.

**Definition 1.3.2.** The Newton stratification on $X$ is a decomposition of the topological space $X = \coprod_{x \in C} X^N.x$ such that $x, y \in X$ are in the same subset $X^N.x$ if and only if there is an algebraically closed field $k$ with embeddings $k(x) \hookrightarrow k$ and $k(y) \hookrightarrow k$ and induced geometric points $\mathcal{F}$ and $\mathcal{G}$, such that there is an isomorphism $H^1_{\text{cris}}(\mathcal{A}_\tau, k(\mathcal{F})) \otimes B(k) \to H^1_{\text{cris}}(\mathcal{A}_\tau, k(\mathcal{G})) \otimes B(k)$ mapping $s_{\text{cris},\tau}$ to $s_{\text{cris},\tau}$. Each $X^N.x$ is called a Newton stratum.

We can also define level $m$-stratifications.

**Definition 1.3.3.** The level $m$ stratification on $X$ is a decomposition of the topological space $X = \coprod_{j \in J} X^L.j$ such that $x, y \in X$ are in the same subset $X^L.j$ if and only if there is an algebraically closed field $k$ with embeddings $k(x) \hookrightarrow k$ and $k(y) \hookrightarrow k$ and induced geometric points $\mathcal{F}$ and $\mathcal{G}$, such that there is an isomorphism $\mathcal{D}(\mathcal{A}_\tau[p^m]) \to \mathcal{D}(\mathcal{A}_\tau[p^m])$ mapping $s_{\text{cris},\tau}$ to $s_{\text{cris},\tau}$. Each $X^L.j$ is called a level $m$ stratum.

**Definition 1.3.4.** Let $x \in \mathcal{J}_0(\tau)$ be a point. The central leaf $C_x$ of $\mathcal{J}_0$ crossing $x$ is the subset $y \in \mathcal{J}_0.\tau$ such that there exists an algebraically closed field $k$ with embeddings $k(x) \hookrightarrow k$ and $k(y) \hookrightarrow k$ and induced geometric points $\mathcal{F}$ and $\mathcal{G}$, and an isomorphism $H^1_{\text{cris}}(\mathcal{A}_\tau, k(\mathcal{F})) \to H^1_{\text{cris}}(\mathcal{A}_\tau, k(\mathcal{G}))$ of Dieudonné modules mapping $s_{\text{cris},\tau}$ to $s_{\text{cris},\tau}$.

For simplicity, we will skip the superscript $cN$, $cL$, $cC$, $N$ and $L$ when there is no risk of confusion.

The level 1 stratification is precisely the Ekedahl-Oort stratification defined and studied in [46]. Each classical Newton stratum (resp. classical level $m$ stratum, resp. classical central leaf) is a union of Newton strata (resp. level $m$ strata, resp. central leaves). Moreover, we have the following result of Vasiu.
Proposition 1.3.5. Each Newton stratum is a union of connected components of the classical Newton stratum containing it. In particular, Newton strata are locally closed in $X_\mathbf{v}$.

Proof. This is [40] Theorem 5.3.1 (b). \qed

1.4. Dilatations and some group theoretic settings. The Shimura datum $(G, X)$ determine a cocharacter $\mu : \mathbb{G}_m, W(\kappa) \to G_{\mathbb{Z}_p} \otimes W(\kappa)$ which is unique up to $G_{\mathbb{Z}_p}(W(\kappa))$-conjugacy. We introduce in this subsection some group theoretic settings which are essentially from [39] section 4. For simplicity, we write $G_R$ for $G_{\mathbb{Z}_p} \otimes R$, for a $\mathbb{Z}_p$-algebra $R$. Let $P_+ \subseteq G_{W(\kappa)}$ (resp. $L \subseteq G_{W(\kappa)}$, $P_- \subseteq G_{W(\kappa)}$) be the subgroup whose Lie algebra is the submodule of Lie($G_{W(\kappa)}$) of non-negative weights (resp. of weight 0, of non-positive weights) with respect to $\mu$ composed with the adjoint action of $G_{W(\kappa)}$ on Lie($G_{W(\kappa)}$). Then $P_+$ and $P_-$ are parabolic subgroups of $G_{W(\kappa)}$ in opposite position, and $L$ is the common Levi subgroup of $P_+$ and $P_-$. To construct what we need, we need to introduce dilatations. Let $R$ be a D.V.R. with uniformizer $t$ and residue field $k$, $X$ be a smooth scheme over $R$ and $Y_k \subseteq X_k$ be a closed subscheme which is smooth over $k$. Let $I$ be the ideal defining $Y_k \subseteq X$, $\tilde{X}$ be the blow up of $Y_k$ on $X$ and $X' \subseteq \tilde{X}$ be the open subscheme such that $IO_{\tilde{X}}$ is generated by $t$. Following [2], $X'$ is called the dilatation of $Y_k$ on $X$.

Proposition 1.4.1.

(1) The $R$-scheme $X'$ is smooth.

(2) The natural $R$-morphism $u : X' \to X$ whose generic fiber is an isomorphism is universal in the following sense. For any flat $R$-scheme $Z$ and any $R$-morphism $v : Z \to X$ such that $v_k$ factors through $Y_k$, there is a unique $R$-morphism $v' : Z \to X'$ satisfying $v = u \circ v'$.

(3) Dilatations commute with products: Let $X_i, i = 1, 2$ be smooth $R$-schemes, and $Y_k \subseteq X_1 \otimes k$ be closed smooth subvarieties, then $(X_1 \times_R X_2)'$, the dilatation of $Y_1 \times_k Y_2$ on $X_1 \times_R X_2$, is canonically isomorphic to $X'_1 \times_R X'_2$. In particular, if $X$ is an $R$-group scheme and $Y_k \subseteq X_k$ is a subgroup scheme, then $X'$ is a group scheme over $R$, and the natural morphism $u : X' \to X$ is a group homomorphism.

Proof. The first statement is [2] 3.2 Proposition 3, the second statement is [2] 3.2 Proposition 1, and the last statement is [2] 3.2 Proposition 2 (d). \qed

Let $P_{+,0}$ be the special fiber of $P_+$, and $G'$ be the dilatation of $P_{+,0}$ on $G_{W(\kappa)}$ with natural homomorphism $u : G' \to G$. Let $U_+$ (resp. $U_$), and $H$ be $U_+ \times L \times U_-$, $f : H \to G_{W(\kappa)}$ be the natural morphism induced by product, and $f_- : H \to H$ be given by $(a, b, c) \to (a, b, c^\varphi)$. Let $f_{0-} : H \to G_{W(\kappa)}$ be the composition $f \circ f_-$. Then $f_{0-}$ is generically an open embedding. Note that on the special fiber, $f_{0-}$ factors through $P_{+,0}$, so by Proposition 1.4.1 (2), $f_{0-}$ factors through $u : G' \to G$. The morphism $H \to G'$ is denoted by $u'$.

Note that $u'$ is also generically an open embedding, so we have sequences of injections

$$H(W(\overline{\kappa})) \to G'(W(\overline{\kappa})) \to G_{W(\kappa)}(W(\overline{\kappa})) \quad \text{and} \quad H(W(\overline{\kappa})[\varepsilon]/(\varepsilon^2)) \to G'(W(\overline{\kappa})[\varepsilon]/(\varepsilon^2)) \to G_{W(\kappa)}(W(\overline{\kappa})[\varepsilon]/(\varepsilon^2)).$$

By Proposition 1.4.1 (2), $G'(W(\overline{\kappa})) = \{ g \in G_{W(\kappa)}(W(\overline{\kappa})) \mid g \mod p \in P_{+,0}(\overline{\kappa}) \}$ and $G'(W(\overline{\kappa})[\varepsilon]/(\varepsilon^2)) = \{ g \in G_{W(\kappa)}(W(\overline{\kappa})[\varepsilon]/(\varepsilon^2)) \mid g \mod p \in P_{+,0}(\overline{\kappa}[\varepsilon]/(\varepsilon^2)) \}$. One
sees easily that the image of $H(W(\pi))$ (resp. $H(W(\pi)[\varepsilon]/(\varepsilon^2))$) in $G_{W(\pi)}(W(\pi))$ (resp. $G_{W(\pi)}(W(\pi)[\varepsilon]/(\varepsilon^2))$) equals to $G'(W(\pi))$ (resp. $G'(W(\pi)[\varepsilon]/(\varepsilon^2))$). So the natural morphism of smooth affine $\kappa$-schemes (resp. $W_m(\kappa)$-schemes) $u'_1: H_\kappa \to G'_\kappa$ (resp. $u'_m : H_{W_m(\kappa)} \to G'_{W_m(\kappa)}$) is an isomorphism.

From now on, we will identify $H_\kappa$ (resp. $H_{W_m(\kappa)}$) via $u'_1$ (resp. $u'_m$), and view $H_\kappa$ (resp. $H_{W_m(\kappa)}$) as a group scheme via the identification. We will write $H_m$ (resp. $G'_m, U_{+,m}, U_{-,m}, L_m, G_m$) for $H_{W_m(\kappa)}$ (resp. $G'_{W_m(\kappa)}, U_{+,W_m(\kappa)}, U_{-,W_m(\kappa)}, L_{W_m(\kappa)}, G_{W_m(\kappa)}$). We need a precise description of the group structure on $H_m$. We start with $u': H \to G'$. To describe $u'$, it suffices to consider the map $H(H) \to G'(H)$, but $H(H) \subseteq H_Q(H_Q)$ as well as $G'(H) \subseteq G'_Q(H_Q)$, so for $(h_1, h_2, h_3) \in H(H)$, $u'((h_1, h_2, h_3)) = h_1 h_2 h_3$. This means that for $(h_1, h_2, h_3)$ and $(g_1, g_2, g_3)$ in $H(H)$, the product $u'((h_1, h_2, h_3)) \cdot u'((g_1, g_2, g_3)) = h_1 h_2 h_3 g_1 g_2 g_3$. When we reduce $u'$ by $p^m$, we get an isomorphism, and the group structure on $H_m$ is such that $(h_1, h_2, h_3) \cdot (g_1, g_2, g_3) = (f_1, f_2, f_3)$, where $f_1 f_2 f_3 = h_1 h_2 h_3 g_1 g_2 g_3$.

There is a homomorphism $p_m : G'_m \to P_{+,m}$ given by $(h_1, h_2, h_3) \mapsto (h_1, h_2)$. It is direct that $p_m \circ u'_m|_{P_{-,m}}$ is the identity. We remark that the construction of $p_m$ relies on the fact that $u'_m$ is an isomorphism, and such a homomorphism does NOT exist over $W(\kappa)$.

1.4.2. The functor $W_m$. Let $X/W(\kappa)$ be of finite type. Let $W_m(X)$ be the presheaf on the category of affine $\kappa$-schemes such that $W_m(X)(R) = X(W_m(R))$, for all $\kappa$-algebra $R$. Clearly, a morphism $f : X \to Y$ of $W(\kappa)$-schemes induces a natural transformation $W(f) : W_m(X) \to W_m(Y)$. The functor $W_m(-)$ has the following properties.

**Theorem 1.4.3.**

(1) The functor $W_m(X)$ is represented by a $\kappa$-scheme, $W_m(-)$ is compatible with fiber products.

(2) If $X$ is affine (resp. separated), then $W_m(X)$ is also affine (resp. separated).

(3) If $f : X \to Y$ is an immersion (resp. open immersion, resp. closed immersion), then $W(f) : W_m(X) \to W_m(Y)$ is also an immersion (resp. open immersion, resp. closed immersion). Moreover, if $\{X_i\}_{i \in I}$ is an open covering of $X$, then $\{W(X_i)\}_{i \in I}$ is an open covering of $W_m(X)$.

(4) Let $X_0$ be the special fiber of $X$, and $F : X_0 \to X_0$ be the absolute Frobenius. If $X/W(\kappa)$ is smooth, then the natural morphism $W_m(X) \to W_m(X)$ is a $F^m \cdot \Omega^1_{X_0/\kappa}$-torsor. In particular, $W_m(X)$ is smooth.

**Proof.** The first three statements are in [8] page 643, and the last statement follows from [8] Proposition 2 by using Case 2 on page 263.

**Remark 1.4.4.** For $m = 1$, we have $W_1(X) = X_\kappa$. If $X/W(\kappa)$ is smooth (resp. a group scheme), then $W_m(X)$ is of dimension $m \cdot \dim(X_\kappa)$ (resp. a group scheme over $\kappa$).

**Remark 1.4.5.** If $f : X \to S$ is a smooth morphism of $W(\kappa)$-schemes that are of finite type, then the induced morphism $W_m(X) \to W_m(S)$ is also smooth, as one can check that nilpotent immersions lifts locally.

Now we come back to notations introduced before 1.4.2. Let $H_m$ be $W_m(H)$ which is the same as $W_m(G')$ and $G_m$ be $W_m(G_W(\kappa))$. Let $\sigma : H_m \to H_m$ be the morphism which maps $h \in H_m(R) = H(W_m(R))$ to $W_m(R) \xrightarrow{\text{Frob}} W_m(R) \xrightarrow{h} G'$.
for any \( \kappa \)-algebra \( R \). Then there is an action \( T_m : H_m \times_{\kappa} G_m \to G_m \) given by \( (h,g) \to h\sigma(h)^{-1} \), where the multiplication on the right-hand-side is induced by the composition \( H_{W_m(\kappa)} \xrightarrow{u_m} G'_{W_m(\kappa)} \xrightarrow{u} G_{W_m(\kappa)} \). We remark that our notations here conflicts with previous notations where \( H_m \) is defined to be the reduction mod \( p^m \) of \( H \). BUT we INTEND to do this, as these two schemes are related by the functor \( W_m \), and one should be viewed as the realization of the other in the different world.

By the smoothness of \( H_m \) and \( G_m \), we have the following corollary.

**Corollary 1.4.6.** The quotient \([H_m \backslash G_m]\) induced by \( T_m \) is a smooth Artin stack over \( \kappa \).

For \( m' \geq m \), we have natural morphisms \( H_{m'} \to H_m \) and \( G_{m'} \to G_m \) inducing a commutative diagram

\[
\begin{array}{ccc}
H_{m'} \times G_{m'} & \xrightarrow{T_{m'}} & G_{m'} \\
\downarrow & & \downarrow \\
H_m \times G_m & \xrightarrow{T_m} & G_m.
\end{array}
\]

Hence we have a natural morphism of algebraic stacks \([H_{m'} \backslash G_{m'}] \to [H_m \backslash G_m]\).

**Theorem 1.4.7.** There exists an integer \( N > 0 \), such that for any \( m' \geq m \geq N \), the natural morphism \([H_{m'} \backslash G_{m'}] \to [H_m \backslash G_m]\) induces a homeomorphism on topological spaces.

**Proof.** This is Vasiu’s [38] MAIN THEOREM A. \( \square \)
2. Level m stratifications

2.1. Constructing torsors over $\mathcal{S}$ and $\mathcal{S}_0$. We expect a smooth morphism $\mathcal{S}_0 \to [H_m \setminus G_m]$ whose fibers are level $m$ strata. But we can only do this over a Zariski cover. Fortunately, this is enough for the study of level $m$ stratifications. The construction here is a generalization of our construction in [49]. For simplicity, we will write $\mathcal{S}$ for $\mathcal{S}_K(G, X) \otimes W(\kappa)$.

We use notations as in 1.1. Let $\mathcal{I} = \text{Isom}((V_{\mathcal{I}}^\vee, s) \otimes O_{\mathcal{S}}, (V, s_{\text{dR}}) \otimes O_{\mathcal{S}})$, it is a $G_{W(\kappa)}$-torsor over $\mathcal{S}$ (see e.g. [46]). Let $L^1 \subseteq L^\vee_{\mathcal{I}}$ be the submodule of weight 1 with respect to $\mu$, let $V^1 \subseteq V$ be the Hodge filtration. Let $\mathcal{I}_+ \subseteq \mathcal{I}$ be the closed subscheme given by $\mathcal{I}_+ = \text{Isom}((V_{\mathcal{I}}^{1 \vee} \supseteq L^1, s) \otimes O_{\mathcal{S}}, (V \supseteq V^1, s_{\text{dR}}) \otimes O_{\mathcal{S}})$. Then $\mathcal{I}_+$ is a $P_m$-torsor over $\mathcal{S}$ (see [46]).

Let $\mathcal{I}_{+0}$ be the special fiber of $\mathcal{I}_+$ viewed as a $W(\kappa)$-scheme. Then $\mathcal{I}_{+0}$ is smooth over $\mathcal{S}_0$ and hence smooth over $\kappa$. Let $I'$ be the dilatation of $\mathcal{I}_{+0}$ on $\mathcal{I}$. Then by Proposition [14.1] $I'$ is smooth over $W(\kappa)$.

**Proposition 2.1.1.** The $W(\kappa)$-scheme $I'$ is naturally a $G'$-torsor over $\mathcal{S}$. Moreover, we have a commutative diagram of $\mathcal{S}$-morphisms

$$
\begin{array}{ccc}
G' \times I' & \to & I' \\
\downarrow \quad u_1 \times u & & \downarrow u \\
G_{W(\kappa)} \times I & \to & I,
\end{array}
$$

where the horizontal morphisms are actions, and the vertical ones are those induced by dilatations.

**Proof.** The natural composition $\mathcal{I}' \xrightarrow{\pi} \mathcal{I} \xrightarrow{\kappa} \mathcal{S}$ makes $\mathcal{I}'$ an $\mathcal{S}$-scheme. We first prove that $\mathcal{I}'$ is faithfully flat over $\mathcal{S}$.

We claim that $\mathcal{I}'$ is smooth over $\mathcal{S}$. We only need to check this at points on $\mathcal{S}_0$, as over Sh$K(G, X)$, the morphism $\mathcal{I}' \to I$ induces a canonical isomorphism $\mathcal{I}'|_{\text{Sh}_K(G, X)} \to I|_{\text{Sh}_K(G, X)}$. Let $\mathcal{S}$ be the ideal sheaf of $O_{\mathcal{I}}$ defining $\mathcal{I}_+$, then $O_{\mathcal{I}} = O_{\mathcal{S}}[\mathcal{I}]$. Let $\mathcal{I}' \subseteq \mathcal{I}_0$ be a point, and $x = \pi \circ u(x')$ be its image in $\mathcal{S}_0$, we will find open affine neighbourhoods $x' \in U' \subseteq \mathcal{I}'$ and $x \in V \subseteq \mathcal{S}$, such that $\pi \circ u(U') \subseteq V$ and that $U' \to V$ is smooth. Still write $x'$ for its image in $\mathcal{I}$, then by the $\mathcal{S}$-smoothness of $\mathcal{I}$ and $\mathcal{I}_+$, there is an open affine neighbourhood $U \subseteq \mathcal{I}$ of $x'$ and an open affine neighbourhood $V \subseteq \mathcal{S}$ of $x$, such that

1. there are elements $a_1, \ldots, a_i \in \mathcal{I}_U$ such that $(a_1, \ldots, a_i) = \mathcal{I}_U$
2. there are elements $b_1, \ldots, b_i \in O_U$ such that $da_1, \ldots, da_i, db_1, \ldots, db_i$ form a basis of $\mathcal{I}_{O_U}/O_U$.

The homomorphism $R[x_1, \ldots, x_i, y_1, \ldots, y_i] \to O_U$ given by $x_j \to a_j$, $1 \leq j \leq i$ and $y_s \to b_s$, $1 \leq s \leq t$ induces a decomposition $U \xrightarrow{\psi} H_{V}^1 \to V$ with $v$ étale. Now take $U' \subseteq \mathcal{I}'$ to be $u^{-1}(U)$, then $O_{U'} = O_U[\mathcal{I}_{U}] = O_{\mathcal{S}} \otimes O_{\mathcal{S}} \otimes O_{\mathcal{S}}[\mathcal{I}_{U}]$. $O_{\mathcal{S}}[\mathcal{I}_{U}]$ is smooth over $O_{\mathcal{S}}[\mathcal{I}_{U}]$, which is a polynomial algebra over $O_V$, and hence $U'$ is smooth over $V$.

Now we prove that $\mathcal{I}' \to \mathcal{S}$ is surjective on topological spaces.

Let $x$ be a geometric point of $\mathcal{S}_0$ with residue field $k = \mathcal{K}$, and $\bar{x} \in \mathcal{S}(W(\kappa))$ be a lifting of $x$. The $\mathcal{S}$-scheme $\mathcal{I}_+$ is flat, so $\bar{x}$ induces an exact sequence
Let \( (\mathbb{I}_x)' \) be the dilatation of the special fiber of \( \mathbb{I}_{x,x} \) on \( \mathbb{I}_x \), then \( O_{\mathbb{I}_x} = O_{\mathbb{I}_x} [\mathcal{F} \otimes \mathcal{O}_x W(k)]_p \). Note that there is a natural surjection \( O_{\mathbb{I}_x} \otimes \mathcal{O}_x W(k) \to O_{\mathbb{I}_x} [\mathcal{F} \otimes \mathcal{O}_x W(k)] \), so \( \mathbb{I}_x \) contains the special fiber of \( (\mathbb{I}_x)' \) which is non-empty. Note that the closed embedding \( (\mathbb{I}_x)' \hookrightarrow \mathbb{I}_x^r \) is actually an isomorphism, as they are both smooth affine over \( W(k) \) of the same dimension.

To prove that \( \mathbb{I}' \) is a \( G' \)-torsor over \( \mathcal{F} \), we still need to construct an action \( \rho' : G'_x \times \mathcal{T} \mathbb{I}' \to \mathbb{I}' \) and show that the morphism \( G'_x \times \mathcal{T} \mathbb{I}' \xrightarrow{(\rho', pr_2)} \mathbb{I} \times \mathcal{T} \mathbb{I}' \) is an isomorphism.

To construct \( \rho' \), consider the diagram of \( \mathcal{F} \)-morphism

\[
\begin{array}{ccc}
G'_x \times \mathcal{T} \mathbb{I}' & \xrightarrow{u_1 \times u} & G_W(k) \times \mathcal{T} \mathbb{I}' \\
\downarrow & & \downarrow u \\
G_{W(k)} \times \mathcal{T} \mathbb{I}' & \xrightarrow{\rho} & \mathbb{I}'
\end{array}
\]

By Proposition 1.4.13, the morphism \( u_1 \times u \) is the dilatation of \( P_{x,0} \times \mathcal{T} \mathbb{I}_+ \) on \( G_{W(k)} \times \mathcal{T} \mathbb{I}' \), and hence \( (G'_x \times \mathcal{T} \mathbb{I}')_\kappa \) is mapped to \( P_{x,0} \times \mathcal{T} \mathbb{I}_+ \), which is mapped to \( \mathbb{I}_+ \) via \( \rho \). So by the universality of \( \mathbb{I}' \xrightarrow{\rho} \mathbb{I} \), there is an unique \( \rho' : G'_x \times \mathcal{T} \mathbb{I}' \to \mathbb{I}' \) making the above diagram commutative. It is clearly an \( \mathcal{F} \)-morphism. That \( \rho' \) satisfies the association law follows similarly from Proposition 1.4.13.

Now we prove that the morphism \( G'_x \times \mathcal{T} \mathbb{I}' \xrightarrow{(\rho', pr_2)} \mathbb{I} \times \mathcal{T} \mathbb{I}' \) is an isomorphism. We consider the commutative diagram

\[
\begin{array}{ccc}
G'_x \xrightarrow{\rho} \mathbb{I}' \times \mathcal{T} \mathbb{I}' & \xrightarrow{(\rho', pr_2)} & \mathbb{I} \times \mathcal{T} \mathbb{I}' \\
\downarrow u \times u & & \downarrow u \times u \\
G_W(k) \times \mathcal{T} \mathbb{I}' & \xrightarrow{(\rho, pr_2)} & \mathbb{I} \times \mathcal{T} \mathbb{I}.
\end{array}
\]

The morphism \( \rho \times pr_2 \) is an isomorphism, and hence has an inverse \( i \). Note that \( \mathbb{I}'_0 \times \mathcal{T} \mathbb{I}'_0 \) is mapped to \( \mathbb{I}_+ \times \mathcal{T} \mathbb{I}_+ \), which is mapped to \( P_{x,0} \times \mathcal{T} \mathbb{I}_+ \) via \( i \), so by the universality of \( G'_x \times \mathcal{T} \mathbb{I}' \xrightarrow{\rho} G_W(k) \times \mathcal{T} \mathbb{I} \), there is a unique morphism \( i' : \mathbb{I}' \times \mathcal{T} \mathbb{I}' \to G'_x \times \mathcal{T} \mathbb{I}' \) such that \( (u_1 \times u) \circ i' = i \circ (u \times u) \). The universality also implies that \( i' \circ ((\rho', pr_2)) = id \).

For an \( \mathcal{F} \)-scheme \( T \) which is flat over \( W(k) \), and a morphism \( (t_1, t_2) : T \to \mathbb{I} \times \mathcal{T} \mathbb{I} \), such that \( t_i(T \times \mathcal{F}_0) \subseteq \mathbb{I}_+, \ i = 1, 2 \), the universality of \( u : \mathbb{I} \to T \) implies that there is a unique morphism \( (t_1', t_2') : T \to \mathbb{I} \times \mathcal{T} \mathbb{I}' \) such that \( (t_1, t_2) = (u \times u) \circ (t_1', t_2') \). One applies this to \( T = \mathbb{I}' \times \mathbb{I} \) and finds immediately \( ((\rho', pr_2)) \circ i' = id \).

Let \( \psi : \mathbb{I}_{x} \to \mathbb{I}' \) be the natural morphism which is equivariant with respect to their torsor structures. Let \( \psi_m : \mathbb{I}_{x,m} \to \mathbb{I}'_m \) be the reduction mod \( p^m \).

Applying the functor \( W_m(-) \) to the commutative diagram in the above proposition, by Theorem 1.4.3 and its remarks, we have that \( W_m(\mathbb{I}') = W_m(\mathcal{T}) \) is a
torsor under $W_m(G') = G'_m$, and a commutative diagram of $W_m(\mathcal{S})$-morphisms

$$
\begin{array}{ccc}
G'_m \times_{W_m(\kappa)} W_m(\mathcal{I}) & \longrightarrow & W_m(\mathcal{I}) \\
\downarrow_{W_m(u_1) \times W_m(u)} & & \downarrow_{W_m(u)} \\
G_{W(\kappa),m} \times_{W_m(\kappa)} W_m(\mathcal{I}) & \longrightarrow & W_m(\mathcal{I}).
\end{array}
$$

2.1.2. Constructing certain torsors over $\mathcal{S}_0$. To do our job, we have to construct torsors over a Zariski cover of $\mathcal{S}_0$, rather than over $W_m(\mathcal{S})$. We write $\mathcal{G}$ for the group $G_{W(\kappa)}$ (resp. $G'$, resp. $P_+$) and $\mathcal{T}^\circ$ for the $\mathcal{G}$-torsor $\mathcal{I}$ (resp. $\mathcal{I}'$, resp. $\mathcal{I}'_+$) over $\mathcal{S}_0$. By applying $W_m(-)$, we get a $W_m(\mathcal{G})$-torsor $W_m(\mathcal{T}^\circ)$ over $W_m(\mathcal{S})$.

Let $U^i = \text{Spec } R^i 1 \leq i \leq r$ be an open covering of $\mathcal{S}_0$. Let $U^i_m = U^i \times W_m(\kappa)$, $R^i_m = R^i \otimes_{W_m(\kappa)} W_m(\kappa)$, then by smoothness of $U^i_m$, there is a homomorphism of $W_m(\kappa)$-algebras $w^i_m : R^i_m \to W_m(R^i_0)$ such that $w^i_m$ induces the identity map on $R^i_0$. We will fix once and for all a system of $w^i_m$s such that $w^i_m$ is induced by $w^i_m$ mod $p^{m-1}$. Note that $w^i_m$ is the same as a morphism $U^i_0 \to W_m(U^i) \subseteq W_m(\mathcal{S}_0)$, which will also be denoted by $w^i_m$. By pulling back the $W_m(\mathcal{G})$-torsor $W_m(\mathcal{T}^\circ)$ along $w_i$, we get a $W_m(\mathcal{G})$-torsor $T^i_m$ over $U^i_0$. Let $U_0 = \prod_{i=1}^r U^i_0$, then we get a $W_m(\mathcal{G})$-torsor $T^i_m$ over $U_0$ by putting together $T^i_m$s.

The scheme $T^i_m/U_0$ should be viewed as a kind of “relative $W_m$ functor”. More precisely, for an affine scheme $T = \text{Spec } A$ over $U_0$, $T_m(T)$ is the subset of elements $t \in (\prod_{i=1}^r U^i_m)(W_m(A))$ such that the diagram

$$
\begin{array}{ccc}
\text{Spec } W_m(A) & \longrightarrow & \prod_{i=1}^r T^i_m|U^i_m \\
\downarrow & & \downarrow \\
\text{Spec } W_m(\prod_{i=1}^r R_0) & \longrightarrow & \prod_{i=1}^r U^i_m.
\end{array}
$$

is commutative, where the lower horizontal morphism is induced by the $w^i_m$s. This implies that $T_{m+1} \to T_m$ is a (trivial) torsor under $F^{m*}(\Omega^{1\mathcal{S}_0}_{\mathcal{S}_0}/\mathcal{S}_0) \otimes (\prod_{i=1}^r R_0)$

We apply the above constructions to $T^\circ = \mathcal{I}_+$ (resp. $\mathcal{I}'_+$) and get a $G'_m$-torsor (resp. $P_{+,m}$-torsor) over $U_0$. By abusing notations, the $G'_m$-torsor (resp. $P_{+,m}$-torsor) will be denoted by $\mathcal{I}_{+,m}$ (resp. $\mathcal{I}'_{+,m}$).

2.2. The morphism $\zeta_m$. We want a $G'_m$-equivariant morphism $\zeta_m : \mathcal{I}'_m \to G_m$ which induces the level $m$ stratification. The $\kappa$-scheme $\mathcal{I}'_m$ is smooth and affine, so to define the morphism $\zeta_m : \mathcal{I}'_m \to G_m$, it suffices to work with $\mathcal{I}'_m$-points. This means that we only need to determine how to attach an element in $G_m(T)$ to an element in $\mathcal{I}'_m(T)$ with $T$ affine and smooth over $U_0$. Let $T = \text{Spec } A$ and $t : T \to \mathcal{I}'_m$ be a morphism. Then $t$ corresponds to a morphism $W_m(T) \to \mathcal{I}'$ such that the diagram

$$
\begin{array}{ccc}
W_m(T) & \longrightarrow & \mathcal{I}' \\
\downarrow & & \downarrow \\
W_m(U_0) & \longrightarrow & \mathcal{S}_0.
\end{array}
$$
is commutative. Here the lower horizontal morphism is induced by \( w^t \), \( s \) at the beginning of this section. This diagram lifts to a commutative diagram

\[
\begin{array}{ccc}
W(T) & \xrightarrow{\overline{t}} & W'(V) \\
\downarrow & & \downarrow \\
W(U_0) & \longrightarrow & \mathcal{F}.
\end{array}
\]

We remark that \( W(A) \) is \( p \)-adically complete and \( p \)-torsion free.

We fix some notations first. Let \( S_0 \) be an \( \mathbb{F}_p \)-scheme and \( S \) be a \( \mathbb{Z}_p \)-scheme equipped with an automorphism \( \sigma : S \to S \) lifting the absolute Frobenius on \( S_0 \). For an \( S \)-scheme \( T \), we write \( \sigma^* T \) or \( T^{(\sigma)} \) for the pull back of \( T \) via \( \sigma \). For a coherent \( O_S \)-module \( N \) and an \( O_S \)-linear homomorphism \( f \), we write \( \sigma^* f \) or \( f^{(\sigma)} \) for the pull back of \( f \) via \( \sigma \). If \( N = M \otimes O_S \) for some finitely generated \( \mathbb{Z}_p \)-module \( M \), then there is an \( O_S \)-linear isomorphism \( \xi : N \to \sigma^* N \) given by \( m \otimes s \to m \otimes 1 \otimes s \). The inverse of \( \xi \) is given by \( m \otimes t \otimes s \to m \otimes \sigma(t)s \). It is easy to check that \( \sigma(f) = \xi^{-1} f^{(\sigma)} \xi \).

We now follow [14] Appendix A and [18]. Let \( \sigma \) be the Frobenius on \( W(A) \), and \( \nu \) be the Verschiebung. Note that \( \nu \) is injective. We write \( \nu^{-1} : \nu(W(A)) \to W(A) \) for the map by taking preimages. The tuple \((W(A), \nu(W(A)), A, \sigma, \nu^{-1})\) is a lifting frame (see [18] page 12 for the definition, and 2.2 for this statement).

Let \( \mathcal{D}(A_T) \) be the Dieudonné crystal attached to \( A_T[p^\infty] \). There is a canonical isomorphism \( \mathcal{D}(A_T)(W(A)) \cong \mathcal{V} \otimes W(A) \). The Frobenius on \( \mathcal{D}(A_T)(W(A)) \) induces via the isomorphism a \( \sigma \)-linear map \( \varphi : \mathcal{V} \otimes W(A) \to \mathcal{V} \otimes W(A) \), and a linear map \( v : \mathcal{V} \otimes W(A) \to (\mathcal{V} \otimes (W(A))^{(\sigma)} \). We also have \( \varphi^{\text{lin}} \circ v = p, v \circ \varphi^{\text{lin}} = p \). Let \( F^1 \subseteq \mathcal{V} \otimes W(A) \) be the preimage of \( \text{Lie}(A_T) \subseteq \mathcal{D}(A_T)(A) = \mathcal{V} \otimes A \), then by the proof of [18] Proposition 3.15, \( \tilde{\varphi} \) is well defined on \( F^1 \), and induces a surjection \( (\tilde{\varphi})^{\text{lin}} : \sigma^* F^1 \to \mathcal{V} \otimes W(A) \). Note that \( \nu^\sigma \) induces the Frobenius \( \nu^\sigma \varphi \) on \( \mathcal{V} \otimes W(A) \).

One defines the Frobenius on \( (\mathcal{V} \otimes W(A))^{(\sigma)} \) to be the one induced by \( \varphi \) on \( \mathcal{V} \otimes W(A) \) and \( \varphi^{\text{lin}} \) on \( (\mathcal{V} \otimes W(A))^{(\sigma)} \). Let \( \varphi^{\text{lin}} : (\mathcal{V} \otimes W(A))^{(\sigma)} \to W(A) \) be the linearization of \( \varphi \). Then for a normal decomposition \( \mathcal{V} \otimes W(A) = L \oplus M \) (see [18] page 12 for the definition) where \( L \) is a direct summand of \( \mathcal{V} \otimes W(A) \) lifting \( \text{Lie}(A_T) \), the map \( (\tilde{\varphi})^{\text{lin}} \otimes \varphi^{\text{lin}} : (L \oplus M)^{(\sigma)} \to \mathcal{V} \otimes W(A) \) is an isomorphism.

Let \( \tilde{t} \in \mathcal{V}(W(T)) \) be as before. Let us still write \( \tilde{t} \) for its image in \( L \). Then \( \tilde{t}(L^1 \otimes W(A)) \) is a lifting of \( \mathcal{V} \otimes A \), and \( \tilde{t}(L^1 \otimes W(A)) \oplus \tilde{t}(L^0 \otimes W(A)) \) is a normal decomposition of \( \mathcal{V} \otimes W(A) \). Let \( g_T \) be the composition of

\[
L \otimes W(A) \xrightarrow{\xi} (L \otimes W(A))^{(\sigma)} \xrightarrow{\tilde{\varphi}^{(\sigma)}} (\mathcal{V} \otimes W(A))^{(\sigma)} \\
(\mathcal{V} \otimes W(A))^{(\sigma)} = \tilde{t}^{(\sigma)}((L^1 \otimes W(A))^{(\sigma)}) \oplus \tilde{t}^{(\sigma)}((L^0 \otimes W(A))^{(\sigma)}) \xrightarrow{(\tilde{\varphi})^{\text{lin}} \otimes \varphi^{\text{lin}}} \mathcal{V} \otimes W(A)
\]

and

\[
\mathcal{V} \otimes W(A) \xrightarrow{\tilde{t}^{-1}} L \otimes W(A).
\]

**Proposition 2.2.1.** The element \( g_T \in \text{GL}(L)(W(A)) \) is in \( G_{W(\kappa)}(W(A)) \).

**Remark 2.2.2.** Let \( g_t \) be the image of \( g_T \) in \( \text{GL}(L)(W_m(A)) \). Then \( g_t \) depends only on \( t \) and not on the choice of the lifting \( \tilde{t} \). Let \( (\tilde{\varphi})^{\text{lin}} \) (resp. \( \varphi^{\text{lin}} \)) be the reduction
mod $V_m(A)$ of $(\varphi_p)_{\text{lin}}$ (resp. $\varphi_{\text{lin}}$). Then $g_t$ is the composition of

\[\begin{array}{c}
L \otimes W_m(A) \xrightarrow{\xi} (L \otimes W_m(A))(\sigma) \xrightarrow{t(\sigma)} (V \otimes W_m(A))(\sigma)
\end{array}\]

\[\begin{array}{c}
(V_{W_m(A)})(\sigma) = t(\sigma)((L_{W_m(A)})(\sigma)) \oplus t(\sigma)((L^0_{W_m(A)})(\sigma)) \xrightarrow{(\varphi_p)_{\text{lin}} \otimes \varphi_{\text{lin}}} \mathcal{V}_{W_m(A)}
\end{array}\]

and $\mathcal{V}_{W_m(A)} \xrightarrow{t^{-1}} L_{W_m(A)}$

We need preparations to prove this statement. Let $R_i$ be as before, $R$ be $\Pi_{i=1}^{r} R_i$ and $\hat{R}$ be the $p$-adic completion of $R$. By smoothness of $R$, there is a homomorphism $\sigma': \hat{R} \to \hat{R}$ lifting the Frobenius $R_0$. The tuple $(\hat{R}, (p), R_0, \sigma', \varphi'_{\text{lin}})$ is lifting frame, one can transfer statements for $\mathcal{W}(A)$ without any problem. Among there statements, I would like to mention that there is a Frobenius $\varphi'_{\text{lin}}$ of $\text{Aut}_{\hat{R}}(\mathcal{W})$ which is still denoted by $\varphi'_{\text{lin}}$. Moreover, by Proposition 1.3 there is a topologically quasi-nilpotent integrable connection $\nabla': \mathcal{W} \to \mathcal{W} \otimes \Omega^1_{\mathcal{W}}$ such that $\sigma'\mathcal{W} \to \mathcal{W}$ is $\nabla'$-parallel.

We need to work with normal decompositions. Let $G$ be the closed subscheme of $\text{Aut}_{\hat{R}}(\mathcal{W} \otimes \hat{R})$ that respects $s_{\text{dr}}$. Then $G$ is a reductive group scheme over $\hat{R}$. Let $P'$ be the stabilizer in $G$ of $\mathcal{W} \otimes \hat{R} \subseteq \mathcal{W} \otimes \hat{R}$, and $U'$ be the subgroup of $G$ acting trivially on $\mathcal{W} \otimes \hat{R}$. Then $P'$ is a parabolic subgroup of $G$ with unipotent radical $U$. Using terminologies in [15] 1.1.2, we have the following result.

**Lemma 2.2.3.** The filtration $\mathcal{V}^1 \otimes \hat{R} \subseteq \mathcal{W} \otimes \hat{R}$ is $G$-split.

**Proof.** Let $P'$ be the stabilizer in $\text{Aut}_{\hat{R}}(\mathcal{W} \otimes \hat{R})$ of $\mathcal{V}^1 \otimes \hat{R} \subseteq \mathcal{W} \otimes \hat{R}$, and $U'$ be its unipotent radical. Let $\chi: \mathbb{G}_m \to P'/U'$ be the cocharacter inducing the grading $\mathcal{V}^1_{\hat{R}} \oplus (\mathcal{V}^1_{\hat{R}}/\mathcal{V}^1_{\hat{R}})$, then by [15] Lemma 1.1.1, we only need to prove that $\chi$ factors through $P'/U$. There exists an fppf $\hat{R}$-algebra $A$ such that $\mathbb{I}_+(A) \neq \emptyset$. Take a $t \in \mathbb{I}_+(A)$, then $\mathcal{V}^1_{\hat{R}} \otimes t(L^0 \otimes A)$ is a $G_A$-splitting. So by [15] Lemma 1.1.1 again, the base change to $A$ of $\chi$ factors through $P'/U_A$, but this implies that $\chi$ factors through $P'/U$.

Let $\mathcal{V} \otimes \hat{R} = F^1 \oplus F^0$ be normal decomposition induced by a cocharacter $\mu'$ of $G$ (one can take, for example, $F^1 = \mathcal{V}^1 \otimes \hat{R}$ and $F^0$ a complement of it induced by a cocharacter lifting $\chi$), and $\mathcal{V}^1 \otimes \hat{R} = (F^0)^{\vee} \oplus (F^1)^{\vee}$ be the induced normal decomposition. Let $(\mathcal{V} \otimes \hat{R})^0$ be the submodule generated by elements in $(F^1)^{a} \otimes (F^0)^{b} \otimes (F^0)^{c} \otimes (F^1)^{d}$ with $a = d$. Note that $(\mathcal{V} \otimes \hat{R})^0$ is the submodule of weight 0 in $\mathcal{V} \otimes \hat{R}$ with respect to $\mu'$.

**Lemma 2.2.4.** The map $\varphi'$ takes integral values on $(\mathcal{V} \otimes \hat{R})^0$. In other words, $\varphi'$ induces a map $(\mathcal{V} \otimes \hat{R})^0 \to \mathcal{V} \otimes \hat{R}$. The section $s_{\text{dr}} \in \mathcal{V} \otimes \hat{R}$ lies in $(\mathcal{V} \otimes \hat{R})^0$, it is $\varphi'$-invariant and annihilated by $\nabla'$.

**Proof.** The map $\varphi'$ acts on $F^1$ and $F^0$ as $\varphi'$, and on $F^{1\vee}$ and $F^{0\vee}$ as $\varphi'_{\text{inv}}$. We know that $\varphi'_{\text{inv}}$ (resp. $\varphi'_{\text{lin}}$) takes integral values on $F^0$ (resp. $F^{0\vee}$, $F^{1\vee}$), and that $\varphi'(F^1) \subset p(\mathcal{V} \otimes \hat{R})$, so $\varphi'$ takes integral values on $(\mathcal{V} \otimes \hat{R})^0$, as $a = d$. 


To see that $s_{\text{DR}} \in (\mathcal{V} \otimes \hat{R})^0$, one only needs to notice that by definition of $\mathbb{I}$, $G$ acts trivially on $s_{\text{DR}}$, and hence $s_{\text{DR}}$ is of weight 0 with respect to $\mu'$, which means that $s_{\text{DR}} \in (\mathcal{V} \otimes \hat{R})^0$.

To prove the last statement, we need to use Faltings’s deformation theory. Let $x$ be a closed point in $\text{Spec} \, \hat{R}$, and $\hat{R}_x$ be the completion with respect to the maximal ideal defining $x$. Note that $x$ corresponds to a closed point of $U_0 = \text{Spec} \, R_0$. Let $k$ be the residue field of $x$, and $R_G$ be the completion at identity of $U_{-, W(k)}$. Then $\mathcal{A}[p^\infty]_{\hat{R}_x}$ is a deformation of $\mathcal{A}[p^\infty]_x$, and induces an isomorphism $\hat{R}_x \cong R_G$ by [15]

Proposition 2.3.5. Moreover, $s_{\text{DR}} = s_{\text{cirs}, x} \otimes 1$ over $\hat{R}_x$.

Let $\tilde{x}$ be a lifting of $x$, then $\mathcal{V}_x^w \subseteq \mathcal{V}_x$ gives the Hodge filtration. Let $\sigma'' : R_G \to R_G$ be Frobenius on $W(k)$ and $p$-th power in indeterminants. The Dieudonné module of $\mathcal{A}[p^\infty]_{\hat{R}_x}$ is the tuple $(M, F, \varphi'', \nabla'')$, where $M = \mathcal{V}_x \otimes R_G = \mathcal{V}_{R_G}$, $F = \mathcal{V}_x^w \otimes R_G \subseteq M$ is the Hodge filtration, $\varphi'' = u \circ \varphi_x$ with $u$ the universal element of $U_-$ and $\varphi_x$ the Frobenius on $\mathcal{V}_x$, and $\nabla''$ the connection. We have $\nabla''(s_{\text{DR}}) = 0$ by [15] 1.5.4. We also have $s_{\text{DR}} \in (\mathcal{V} \otimes R_G)^0$ and is $\varphi''$-invariant. Here the $\varphi''$-action on $\mathcal{V} \otimes R_G[p]$ and $(\mathcal{V} \otimes R_G)^0$ are constructed similarly. The only difference is that when defining $(\mathcal{V} \otimes R_G)^0$, we fix an isomorphism $t : L \otimes R_G \to \mathcal{V} \otimes R_G$ which maps $s \otimes 1$ to $s_{\text{DR}}$ and $L^1 \otimes R_G$ to $\mathcal{V}^1 \otimes R_G$. The normal decomposition in this case is the one induced by $t$.

Let $\iota : \hat{R} \to \hat{R}_x = R_G$ be the injection. Then by Construction [1.1.4] $\nabla'' = \iota^* \nabla'$, and hence $\nabla'(s_{\text{DR}}) = 0$. Since $\iota \circ \sigma'$ and $\sigma'' \circ \iota$ has the same reduction modulo $p$, by Construction [1.1.5] there is a canonical isomorphism $\varepsilon : \sigma'' \iota^*(\mathcal{V}_{\hat{R}}) \to \iota^* \sigma''(\mathcal{V}_x)$. The linearization of $\varphi''$ is given by

$$\sigma''(\mathcal{V}_{R_G}) = \sigma'' \iota^*(\mathcal{V}_{\hat{R}}) \xrightarrow{\varepsilon} \iota^* \sigma''(\mathcal{V}_x) \xrightarrow{\varphi''_{\text{lin}}} \mathcal{V}_x.$$ 

The description of $\varepsilon$ in Construction [1.1.6] shows that it respects $s_{\text{DR}}$, as $\nabla'(s_{\text{DR}}) = 0$. But then $\varphi'$ respects $s_{\text{DR}}$, as both $\varphi''$ and $\varepsilon$ respect $s_{\text{DR}}$, and $x$ is an arbitrary point. \qed

**Proof.** (of Proposition [2.2.4]) We use notations as before Proposition [2.2.1]. We will show that $g_t \in \text{GL}(L)(W(A))$ fixes $s_{\text{DR}}$. The homomorphism $\hat{R} \to W(A)$ induces $R_0 \to W(A)/(p) \to A$, and hence canonical isomorphisms

$$\mathcal{V}_{W(A)} \cong \mathbb{D}(\mathcal{A}_{W(A)/(p)})(W(A)) \cong \mathbb{D}(\mathcal{A}_A)(W(A)).$$

Let $F^i = t(L^i \otimes W(A))$, then $\mathcal{V}_{W(A)} = F^1 \oplus F^0$ is a normal decomposition of $\mathcal{V}_{W(A)}$. Moreover, the splitting is induced by a cocharacter of $G_{W(A)}$. Let $(\mathcal{V}^0_{W(A)})^0$ be the submodule of weight 0 in $\mathcal{V}^\infty_{W(A)}$. Then by the same argument as in the proof of Lemma [2.2.4] $\varphi$ takes integral value on $(\mathcal{V}^0_{W(A)})^0$, and that $\varphi_{\text{lin}} : (\mathcal{V}^0_{W(A)})^0(\sigma) \to \mathcal{V}^\infty_{W(A)}$ is the same as the restriction to $(\mathcal{V}^0_{W(A)})^0(\sigma)$ of $(\bar{\varphi}^\infty_{\text{lin}}) = \bar{\varphi}^\infty_{\text{lin}} \subseteq \mathcal{V}^\infty_{\text{lin}}$.

So we reduce to check that $s_{\text{DR}}$ is $\varphi$-invariant. Let $\iota : \hat{R} \to W(A)$ be the composition of $w : \hat{R} \to W(R_0)$ induced by the systems of morphisms $w_{\text{DR}}$'s at the beginning of [2.1.2] and the natural homomorphism $W(R_0) \to W(A)$. Noting that $W(A)$ is $p$-adically complete and $p$-torsion free, and that $\iota \circ \sigma' = \sigma \circ \iota$ mod $p$, the statement follows from the same argument as in the second half of the proof of Lemma [2.2.4] by using that $s_{\text{DR}}$ is $\varphi'$-invariant and annihilated by $\nabla'$. \qed
Applying Proposition 2.2.4 and the remark after it to $A$ such that $\text{Spec } A = \mathbb{V}_m$, we get a morphism $\zeta_m : \mathbb{V}_m \rightarrow G_m$, given by mapping $t \in \mathbb{V}_m(\mathbb{V}_m)\!)$ to $g_t$.

**Proposition 2.2.5.**

1. The morphism $\zeta_m$ is $G'_m$-equivariant considering left $G'_m$-actions, and hence induces a morphism $\zeta_m,\# : U_0 \rightarrow [G'_m\setminus G_m]$.

2. The level $m$ stratification on $U_0,\pi$ is induced by fibers of $\zeta_m,\# \otimes \pi$, and hence each stratum on $U_0,\pi$ is locally closed.

**Proof.** For 1, we have a morphism $\mathbb{V}_m \rightarrow \mathbb{V}_m$, which is equivariant with respect to $G'_m \rightarrow G_m$. Here the left action of $h$ in $G'_m$ (resp. $G_m$) on $\mathbb{V}_m$ (resp. $\mathbb{V}_m$) is given by $h \cdot t = t \circ h^{-1}$. We use notations as in Remark 2.2.2. We have $(h \cdot t)(\xi) = t(\xi) \circ h^{-1}(\xi) \circ \xi = (t(\xi) \circ \xi)(\xi^{-1} \circ h^{-1}(\sigma) \circ \xi) = (t(\sigma) \circ \xi) \circ \sigma(h)^{-1}$. Then the correspondence $t \mapsto t \circ h^{-1}$ induces the correspondence $g_t \mapsto h g_t \sigma(h)^{-1}$. This proves 1.

For 2, by definition of level $m$-stratification on $U_0$, we work with algebraically closed fields. Let $k$ be such a field, and $x,y$ be two $k$-points of $U_0$. Then $x$ and $y$ are in the same stratum if there is an isomorphism of Dieudonné modules $\rho : \mathbb{D}(A_x[p^\infty])(W_m(k)) \cong \mathbb{D}(A_y[p^\infty])(W_m(k))$ mapping $s_{\text{cris},x}$ to $s_{\text{cris},y}$. Let $t_x$ (resp. $t_y$) be a section of $\mathbb{V}_m,x$ (resp. $\mathbb{V}_m,y$). The constructions as in Proposition 2.2.4 give $g_t$ (resp. $g_{t_y}$) in $G_m(k)$. Let $\varphi_x$ (resp. $\varphi_y$) be the Frobenius, then $\varphi_y = \rho \circ \varphi_x(\rho^{-1})$. There is an $h \in G'_m(k)$ such that $t_y = h \circ t_x = \rho \circ t_x \circ h^{-1}$.

The formulas for $\varphi_y$ and $t_y$ imply immediately that $g_{t_y} = h g_t \sigma(h)^{-1}$.

Conversely, if for $x,y,t_x,t_y$ as above, there exists an $h \in G'_m(k)$ such that $g_{t_y} = h g_t \sigma(h)^{-1}$, then $\rho = (h^{-1} \circ t_y) \circ t_x^{-1}$ is an isomorphism of Dieudonné modules $\mathbb{D}(A_x[p^\infty])(W_m(k)) \cong \mathbb{D}(A_y[p^\infty])(W_m(k))$ mapping $s_{\text{cris},x}$ to $s_{\text{cris},y}$.

Let $o$ be a point in the topological space of $[G'_m\setminus G_m]$. Noting that $o$ is locally closed in $[G'_m\setminus G_m]$ and that it admits a representative over $\pi$, we know that $\zeta_m,\#(o) \subseteq U_0,\pi$ is locally closed. This proves 2.

**Theorem 2.2.6.** The morphism $\zeta_m,\#$ is smooth.

**Proof.** By faithfully flat descent, it is equivalent to that $\zeta_m$ is smooth. By II III Proposition 10.4, we only need to show that for any $\pi$-point $t$ of $\mathbb{V}_m$, the induced map on tangent spaces $T_{\zeta_m} : T_{\mathbb{V}_m} \rightarrow T_{\zeta_m(t)}G_m$ is surjective. The element $t$ (resp. $\zeta_m(t)$) corresponds to an element in $\Gamma(W_m(\pi))$ (resp. $G(W_m(\pi))$) which is still denoted by $t$ (resp. $\zeta_m(t)$). Note that $t$ lifts to an element $\tilde{t} \in \Gamma(W(\pi))$ with image $g_{\tilde{t}} \in G(W(\pi))$ which is a lift of $\zeta_m(t)$.

Let $\pi[e]$ be such that $e^2 = 0$. We will prove the following statement which implies the subjectiveness of $T_{\zeta_m}$.

(*) for any $g[e] \in G(W(\pi[e]))$ whose image in $G(W(\pi))$ deforms $g_{\tilde{t}}$, there is a $\tilde{t}[e] \in \Gamma(W(\pi[e]))$ deforming $t$, such that $g_{\tilde{t}[e]} = g[e]$.

Let $\pi : \mathbb{V}_m \rightarrow U_0$ be the projection. We fixed a homomorphism $O_U \rightarrow W(O_{U_0})$, it induces a homomorphism $\iota : R_G \cong O_{U,\pi(t)} \rightarrow W(O_{U,\pi(t)}) \cong W(R_G,0)$. Here $(\cdot)^{\iota}$ means the completion with respect to the ideal defining $\pi(t)$. Viewing $\tilde{t}$ as an $R_G$-point, we get a trivialization $G \times W(\pi)R_G \xrightarrow{\sim} \mathbb{V}_U \times U G$ of $\mathbb{V}_U$, $h \mapsto h \cdot t$. Using notations as in the proof of Lemma 2.2.4 by Faltings's deformation theory, the construction in 2.2.4 gives $ug_{\tilde{t}} \in G(R_G)$ when applied to $\tilde{t}$ and $\nabla_R$, and $h u g_{\tilde{t}} (\sigma(h)^{-1}$ when applied to $h \cdot t$. 

Let $\sigma$ be the Frobenius on $W(R_{G,0})$. Let $R_G(2)$ be $p$-adic completion of the PD-envelope of $R_G \otimes R_G$ with respect to the $\ker(R_G \otimes R_G \to R_{G,0})$, and $K$ be $\ker(R_G \otimes R_G \to R_G)$. The Dieudonn\'e crystal structure of $\mathbb{D}(A_{R_{G,0}})$ induces an isomorphism

$$\theta : (R_G(2)/K^{[2]}) \otimes_{R_G} V_{R_G} \to V_{R_G} \otimes_{R_G} (R_G(2)/K^{[2]}).$$

There are also canonical isomorphisms

$$V_{R_G} \otimes_{R_G} R_G(2) \xrightarrow{\alpha} V_{W(\mathbb{F})} \otimes_{W(\mathbb{F})} R_G(2) \xrightarrow{\beta} R_G(2) \otimes_{R_G} V_{R_G}.$$ 

By [13] 1.5.4, the composition $\theta_\mathbf{t} = \overline{t}^{-1} \circ \alpha \circ \theta \circ \beta \circ \overline{t}$ is an element in $U_-(R_G(1))$ via the identification $i$.

Now by Construction 1.1.5, the morphism $\zeta$ on $W(R_{G,0})$-points is given by

$$h \mapsto h u g \xi^{-1}((\sigma t \cdot \sigma m)\theta_\mathbf{t}) \xi \sigma(h)^{-1}, \ h \in G'(W(R_{G,0})).$$

Let $\overline{t}[c]$ be a deformation of $\overline{t}$. Then its image gives a deformation $\pi(\overline{t}[c])$ of $\pi(\overline{t})$. For $h \in \ker(G'(W(\mathbb{F}[c]))) \to G'(W(\mathbb{F}))),$ we have $\sigma(h) = \text{id}$, as $\sigma(W(\mathbb{F}[c])) \subseteq W(\mathbb{F})$. Similarly, we have $(\xi^{-1}((\sigma t \cdot \sigma m)\theta_\mathbf{t}) \xi \sigma(h)^{-1}) \pi(\overline{t}[c]) = \text{id}$ as $((\sigma t \cdot \sigma m)\theta_\mathbf{t}) \pi(\overline{t}) = \text{id}$. The multiplication morphism $P_+ \times U_- \to G$ is étale at $(1,1) \in P_+ \times U_-$, so by the description of $G'$, the morphism $G' \times R_G \to G,$ $(h, u) \mapsto hu$ is smooth at $(1,1)$. This implies $(\ast)$ and hence the theorem.

We will use the morphism $\zeta_{m,\#}$ to study level $m$ stratifications. Before stating the main theorem, we introduce the following terminology.

**Definition 2.2.7.** An immersion of schemes $S \to T$ is said to be pure if it is an affine morphism.

**Theorem 2.2.8.**

1. Each level $m$ stratum is locally closed in $\mathcal{S}_0,\pi$. Moreover, (given the reduced induced scheme structure) it is smooth and equi-dimensional.

2. The Zariski closure of a level $m$ stratum is a union of level $m$ strata.

3. Each level $m$ stratum is and pure in $\mathcal{S}_0,\pi$.

**Proof.** For (1), let $p : U_0,\pi \to \mathcal{S}_0,\pi$ be the natural morphism, then two geometric points of $U_0,\pi$ lie in the same level $m+1$ stratum if and only if their images in $\mathcal{S}_0,\pi$ are in the same stratum. This implies that a stratum of $\mathcal{S}_0,\pi$ is locally closed with respect to an open covering, and hence locally closed in $\mathcal{S}_0,\pi$. Let $\mathcal{S}_0,\pi \subseteq \mathcal{S}_0,\pi$ be a level $m+1$-stratum, then $p^{-1}(\mathcal{S}_0,\pi)$ is smooth by the previous theorem. This implies that $\mathcal{S}_0,\pi$ is smooth.

For (2), recall that $p : U_0,\pi \to \mathcal{S}_0,\pi$ is the natural morphism $\bigsqcup U_0,\pi \to \mathcal{S}_0,\pi$, where $\bigsqcup U_0,\pi$ is a finite open covering of $\mathcal{S}_0,\pi$. So for a subset $S \subseteq \mathcal{S}_0,\pi$, we have $p^{-1}(S) = \bigsqcup (S \cap U_0,\pi) = p^{-1}(S)$. We only need to show that the closure of a stratum in $U_0,\pi$ is a union of strata, as $p$ is surjective. Let $c'$ be a point in the topological space of $[G'_m \setminus G_m] \otimes \pi$. Let $C \subseteq [G'_m \setminus G_m] \otimes \pi$ be the subset of points that generalize to $c$, then the universally openness of $\zeta_{\#}$ implies that $U_{\mathcal{S}_0,\pi} = \bigcup_{c' \in C} U_{0,\pi}^{c'}$.

For (3), we only need to show that the immersion $U_0,\pi \to U_{0,\pi}$ is affine. It suffices to show that $O_{G'_m}$, the $G'_m,\pi$-orbit in $G_m,\pi$, is an affine scheme. Let’s write $c$ for its image in $[G'_0 \setminus G_0] \otimes \pi$ and $O_0$ for the corresponding $G'_0,\pi$-orbit. Then by
There is a canonical isomorphism $i : G_\mu \cong G_\nu$. The proof is extracted from [17].

2.3. Independence of symplectic embeddings. In this subsection, we prove that level $m$ stratifications are determined by Shimura data. The method here is a variation of the one in [17]. The definition of level $m$ stratifications is independent of choices of open covers, so we only need to show that the morphism $\zeta_m, \# : U_0 \to [G'_m \backslash G_m]$ is determined by its Shimura datum.

We first check that the stack $[G'_m \backslash G_m]$ is determined by its Shimura datum. The pair $(G, X)$ determines a unique $GSp(W(\kappa))$-conjugacy class of cocharacters that $\mu$ is one of them. The construction of $G_m$ depends only on $\mu$, and we use $G'_m$ to indicate the cocharacter.

**Lemma 2.3.1.** Let $\nu : G_{m,W(\kappa)} \to G_{W(\kappa)}$ be a cocharacter conjugate to $\mu$, then there is a canonical isomorphism $i^\mu_\nu : [G'_m \backslash G_m] \to [G'_m \backslash G_m]$.

**Proof.** Let $g^0 \in G_{W(\kappa)}(W(\kappa))$ be such that $g^0 \mu := g^0 \mu g^0 \nu^{-1} = \nu$. Then $g^0 \nu_+ = \nu_+$.

The composition $G^\mu \to G_{W(\kappa)}(W(\kappa))$ is such that its special fiber factors through $P^\mu_{+}$. So by the universality of dilatation, we have a unique homomorphism $G^\mu \to G^\nu$, which is necessarily an isomorphism, making the diagram

$$
\begin{array}{ccc}
G^\mu & \longrightarrow & G^\nu \\
\downarrow & & \downarrow \\
G_{W(\kappa)} & \longrightarrow & G_{W(\kappa)}
\end{array}
$$

commutative.

Let $i^\mu_\nu : G_{W(\kappa)} \to G_{W(\kappa)}$ be the morphism such that $g \mapsto g^0 g \sigma(g^0) \nu$. Consider the $G^\mu$-action (resp. $G^\nu$-action) on the source (resp. target) of $i^\mu_\nu$, given by the natural morphism $G^\mu \to G_{W(\kappa)}$ (resp. $G^\nu \to G_{W(\kappa)}$). Direct computation shows that $i^\mu_\nu$ is equivariant with respect to $G^\mu \to G^\nu$, and hence induces an isomorphism of stacks $i^\mu_\nu : [G'_m \backslash G_m] \to [G'_m \backslash G_m]$ after applying the Greenberg functor.

The morphism $i^\mu_\nu$ is independent of choices of $g^0$. As, if $g^1 \in G_{W(\kappa)}(W(\kappa))$ is another element such that $g^1 \mu = \nu$, then there is a unique $l \in L^\mu(W(\kappa))$, such that $g^1 = g^0 l$. But $i^\mu_\nu(g) = g^1 \sigma(g^1)^{-1} = g^0 l \sigma(l)^{-1} \sigma(g^0)^{-1} = i^\mu_\nu(l \cdot g)$, so they induce the same morphism on stacks. 

**Remark 2.3.2.** By canonical, we mean that for three cocharacters $\mu, \nu, \lambda$ conjugating to each other, there is the identity $i^\mu_\lambda \mu = i^\mu_\nu \lambda \circ i^\mu_\nu \mu$.

**Proposition 2.3.3.** The morphism $\zeta_m, \# : U_0 \to [G'_m \backslash G_m]$ depends only on the Shimura datum, not on the choice of symplectic embedding, the lattice and the Hodge-Tate tensor.

**Proof.** The proof is extracted from [17]. Let $i_1 : (G, X) \leftrightarrow (GSp(V_1, \psi_1))$ and $i_2 : (G, X) \leftrightarrow (GSp(V_2, \psi_2))$ be two symplectic embeddings. There is a symplectic embedding $i : (G, X) \leftrightarrow (GSp(V, \psi))$ with $V = V_1 \oplus V_2$. Let $L_1 \subseteq V_1$ and $L_2 \subseteq V_2$ be $Z_{\psi}$-lattices, $s_1 \in L_1^\psi$ and $s_2 \in L_2^\psi$ be tensors defining $GZ_{\psi}$. Let $L$ be $L_1 \oplus L_2$, and $s \in L^\psi$ be a tensor defining $GZ_{\psi} \subseteq GL(L)$. Let $A_1$ be the universal abelian scheme restricted to $\mathcal{S}$, and $A$ be the one induced by $i$. Then $A = A_1 \times A_2$. One
gets $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2$ by taking de Rham cohomology, with Hodge filtrations denoted by $\text{Fil}^1$.

Let $I := \text{Isom}(L^1, s \otimes O, (\mathcal{V}, s_{\text{dR}}))$, $I_1 := \text{Isom}(L^1, s_1 \otimes O, (\mathcal{V}_1, s_{1,\text{dR}}))$, and $J := \text{Isom}(L^1, L^1, s_1, s \otimes O, (\mathcal{V}_1, s_1, s_{1,\text{dR}}))$, the scheme of isomorphisms that maps $L^1$ to $\mathcal{V}_1$ and respects the Hodge-Tate tensors. Then the natural morphisms $I \leftarrow J \rightarrow I_1$ are $G_{W(p)}$-equivariant isomorphisms. Similarly, we have a natural isomorphism of $P_1$-torsors $I_1 \rightarrow I_1$. By the universality of dilatations, we get an isomorphism $\mathcal{I} \rightarrow \mathcal{I}$. By applying the Greenberg functor and pulling back to $U_0$, we get an isomorphism of $G_m$-torsors $\mathcal{I} \rightarrow \mathcal{I}_{1,m}$.

To deduce the proposition, one only needs to check that the diagram

$$
\begin{array}{ccc}
\mathcal{I}_m & \xrightarrow{\zeta_m} & \mathcal{I}_{1,m} \\
\downarrow & & \downarrow \\
G_m & & G_m
\end{array}
$$

is commutative. But this follow from that $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ as Dieudonné modules, and that the $\zeta_m$s are constructed using $F$-modules.

\[\square\]

Remark 2.3.4. As a direct consequence, we see level $m$ stratifications are independent of choices of symplectic embeddings.

3. Truncated displays with additional structure

3.1. Truncated displays. Let $R$ be a commutative ring of characteristic $p$, $W(R)$ be the ring of Witt vectors and $I_{m+1}(R)$ be $\ker(W_{m+1}(R) \rightarrow R)$. The inverse of Verschiebung induces a $\sigma$-linear bijection $v^{-1} : I_{m+1}(R) \rightarrow W_m(R)$. Let $J_{m+1}(R)$ be $\ker(W_{m+1}(R) \rightarrow W_m(R))$.

Definition 3.1.1. A truncated pair of level $m$ over $R$ is a tuple $(P, Q, \iota, \epsilon)$. Here $P$ is a projective $W_m(R)$-module of finite rank, $Q$ is a finitely generated $W_m(R)$-module, $\iota : Q \rightarrow P$ and $\epsilon : I_{m+1}(R) \otimes_{W_m(R)} P \rightarrow Q$ are homomorphisms. The following conditions are required:

1. The compositions $\epsilon \iota$ and $\epsilon(\epsilon \otimes 1)$ are the multiplication maps.
2. $\ker(\iota)$ is a finite projective $R$-module.
3. $\epsilon$ induces an exact sequence

$$
0 \longrightarrow J_{m+1}(R) \otimes \text{coker}(\iota) \longrightarrow Q \xrightarrow{\iota} P \longrightarrow \text{coker}(\iota) \longrightarrow 0.
$$

Definition 3.1.2. A normal decomposition of a truncated pair consists of projective $W_m$-modules $L \subseteq Q$ and $T \subseteq P$ such that we have isomorphisms $L \oplus T \xrightarrow{\iota \oplus \text{id}} P$ and $L \otimes (I_{m+1}(R) \otimes_{W_m(R)} T) \xrightarrow{id \otimes \epsilon} Q$.

Lemma 3.1.3. Every truncated pair admits a normal decomposition.

Proof. This is [19] Lemma 3.3. \[\square\]

Definition 3.1.4. A truncated display of level $m$ over $R$ is a tuple $(P, Q, \iota, \epsilon, F, V^{-1})$. Here $(P, Q, \iota, \epsilon)$ is a truncated pair, $F : P \rightarrow P$ and $V^{-1} : Q \rightarrow P$ are $\sigma$-linear maps such that $V^{-1} \epsilon = v^{-1} \otimes F$ and that the image of $V^{-1}$ generates $P$ as a $W_m(R)$-module.
For a truncated pair $P, Q, t, s$ with normal decomposition $(L, T)$, the set of pairs $(F, V^{-1})$ is in bijection with the set of $\sigma$-linear isomorphism $\Psi : L \oplus T \to P$ such that $\Psi|_{F} = V^{-1}|_{F}$ and $\Psi|_{T} = F|_{T}$. If $L$ and $T$ are free $W_{m}(R)$-modules, $\Psi$ is described by an invertible matrix over $W_{m}(R)$. The triple $(L, T, \Psi)$ is called a normal representation of $(P, Q, t, s, F, V^{-1})$.

We recall the main of truncated displays in [19]. We will fix a positive integer $h$. Let $\mathcal{B}_{m}$ be the category of BT-$m$ of height $h$ fibered over the category of affine $\mathbb{F}_{p}$-schemes, and $\mathcal{D}_{isp}$ be the category of truncated displays of level $m$ and rank $h$ fibered over the category of affine $\mathbb{F}_{p}$-schemes. Taking Dieudonné display of a $p$-divisible group induces a morphism $\Phi_{m} : \mathcal{B}_{m} \to \mathcal{D}_{isp}$.

Let $R$ be an $\mathbb{F}_{p}$-algebra, and $X$ be a BT-$m$ over $R$. We write $\text{Aut}(X)$ (resp. $\text{Aut}(\Phi_{m}(X))$, and $\text{Aut}^{\sigma}(X)$ for $\ker(\text{Aut}(X) \to \text{Aut}(\Phi_{m}(X)))$).

**Theorem 3.1.5.**

1. The fiber categories $\mathcal{B}_{m}$ and $\mathcal{D}_{isp}$ are both smooth algebraic stacks of dimension $0$ over $\mathbb{F}_{p}$.

2. The morphism $\Phi_{m}$ is smooth.

3. The group scheme $\text{Aut}^{\sigma}(X)$ is commutative infinitesimal finite flat over $R$. The natural morphism $\text{Isom}(X, Y) \to \text{Isom}(\Phi_{m}(X), \Phi_{m}(Y))$ is a torsor under $\text{Aut}^{\sigma}(X)$.

**Proof.** In 1, the statement for $\mathcal{B}_{m}$ is proved in [43], that for $\mathcal{D}_{isp}$ is [19] Proposition 3.15. The second statement is [19] Theorem 4.5, and the third statement is [19] Theorem 4.7. \qed

We recall Lau’s construction realizing $\mathcal{D}_{isp}$ as a disjoint union of quotient stacks. For an integer $d$ with $0 \leq d \leq h$, let $\mathcal{D}_{isp}^{d}$ be the substack such that $\text{coker}(\cdot)$ has rank $d$. Let $G_{m}$ be the presheaf on affine $\mathbb{F}_{p}$-schemes such that $G_{m}(R)$ is the set of $h \times h$ invertible $W_{m}(R)$-matrices. Then $G_{m}$ is represented by an affine smooth $\mathbb{F}_{p}$-scheme. There is a morphism $\tau_{m,d} : G_{m} \to \mathcal{D}_{isp}^{d}_{m}$ such that $\tau_{m,d}(g)$ is given by the normal decomposition $(L, T, \Psi)$ where $L = W_{m}(R)^{h-d}$, $T = W_{m}(R)^{d}$ and $\Psi$ has matrix representation $g$. Let $G_{m,d}$ be the sheaf of groups such that $G_{m,d}(R)$ is the set of matrices $(\begin{array}{cc} A & B \\ C & D \end{array})$ with $A \in \text{Aut}(L)$, $B \in \text{Hom}(T, L)$, $C \in \text{Hom}(L, L_{m+1}(R) \otimes T)$ and $D \in \text{Aut}(T)$. Then $G_{m,d}$ is also an affine smooth $\mathbb{F}_{p}$-scheme. Moreover, $[G_{m,d}]_{m} \cong \mathcal{D}_{isp}^{d}_{m}$.

### 3.2. Truncated displays with additional structure

We start with the datum $(G_{W}(\kappa), \mu)$. For the embedding $G_{W}(\kappa) \subseteq \text{GL}(L_{W}(\kappa))$, let $L^{1} \subseteq L^{0}_{W}(\kappa)$ (resp. $L^{0} \subseteq L^{0}_{W}(\kappa)$) be the sub-module of weight $1$ (resp. $0$). For a $\kappa$-algebra $R$, let $X_{m}(R)$ be the set of $\sigma$-linear isomorphisms $\Psi : (L^{0} \oplus L^{1}) \otimes W_{m}(R)$, such that

1. $(L^{1}_{W_{m+1}(R)}, L^{0}_{W_{m}(R)}), \Psi)$ is a normal representation of a truncated display structures on $L^{0}_{W_{m}(R)} \otimes W_{m}(R)$.

2. $s_{\text{str}} \otimes 1 \in L^{0}_{W(\kappa)} \otimes W_{m}(R)$ is $F$-invariant.

Our arguments before imply that $X_{m}(R) \cong G_{m}(R)$. We say two elements in $G_{m}(R)$ are equivalent if and only if there is an isomorphism between their corresponding truncated displays respecting the tensor $s_{\text{str}} \otimes 1$.

Let $KU_{-m}$ be $\ker(U_{-m+1} \to U_{-m})$. It represents the functor which associates to a $\kappa$-algebra $R$ the set $1 + L_{m+1}(R) \otimes \text{Lie}(U_{-})$. Let $K'_{m}$ be $U_{+m} \times L_{m+1} \times KU_{-m}$ whose group structure is such that $(h_{1}, h_{2}, h_{3}) = (h'_{1}, h'_{2}, h'_{3})$. If and only
if $h_1 h_3 h_2 = h' h'_1 h'_3 h'_2$. The group $KG'_m$ acts on $G'_m$ via $\sigma$-conjugation, and two elements in $G'_m(R)$ are equivalent if and only if they are in the same $KG'_m(R)$-orbit. We remark that if $G = GL_2$, and $\mu$ a cocharacter defined over $\mathbb{Z}_p$ whose subspace of weight 1 (resp. 0) is of rank $g$, then our $KG'_m$ here is precisely $G_{m,g}$ in the previous subsection.

There is a natural homomorphism $G'_m \to KG'_m$ which is identity on $U_+, m$ and $L_m$, and the morphism induced by “multiplication by $p$”

$$p : W_m(R) \to I_{m+1}(R), \quad (r_1, r_2, \cdots, r_m) \mapsto (0, r_1^p, r_2^p, \cdots, r_m^p)$$
on $U_-, m$. This homomorphism is faithfully flat, bijective on geometric points, and has finite kernel. Moreover, this homomorphism is compatible with their actions on $G_m$, i.e. the action of $g' \in G'_m$ on $g \in G_m$ is the same as that of its image in $KG'_m$ on $g$. This induces a morphism of stacks $[G'_m \setminus G_m] \to [KG'_m \setminus G_m]$ which is smooth and bijective on geometric points.

Let $\mathcal{S}, U^1 = Spec \mathbb{R}^1, 1 \leq i \leq r, U_i, \mathcal{I}'_m/U_0$ and $\zeta_m : \mathcal{I}'_m \to G_m$ be as before. We will construct a morphism $\mathcal{S}_0 \to [KG'_m \setminus G_m]$ whose geometric fibers are level $m$ strata. In fact, we will show that $\mathcal{S}'_0$ will construct a morphism $\mathcal{S}_0 \to [KG'_m \setminus G_m]$ whose geometric fibers are level $m$ strata. In fact, we will show that $\mathcal{S}'_0$ extends to a $KG'_m$-torsor $\mathcal{K}'_m$ over $U_0$ which descents to $\mathcal{S}_0$, and the morphism $\zeta_m$ extends to a $KG'_m$-equivariant morphism $K \zeta_m : K \mathcal{I}'_m \to G_m$ which also descents to $\mathcal{S}_0$.

The constructions of $K \mathcal{I}'_m$ and $K \zeta_m$ are straightforward: one takes $K \mathcal{I}'_m$ to be $\mathcal{I}'_m \times_{G_m} KG'_m$ and $K \zeta_m$ to be $\zeta_m \times_{G_m} \text{id}_{KG'_m}$. Here we use the easy fact the the morphism $G_m = G_m \times_{G_m} C'_m \to G_m \times_{G_m} KG'_m$ is an isomorphism. We need a better description of $\mathcal{I}'_m \to K \mathcal{I}'_m$. We describe $\mathcal{I}'_m$ first. It is regular, so it suffices to work with $\mathcal{I}'_m(A)$ with $A/U_0$ regular. Noting that $W(A)$ is $p$-torsion free, an element $t \in \mathcal{I}'_m(A)$ lifts to a $W(A)$-point $\hat{t}$ of $\mathcal{I}'$, where $\hat{t}$ is an isomorphism $L'_m \otimes W(A) \to V \otimes W(A)$ mapping $s$ to $s_{dR}$, such that $\hat{t} \otimes W(A)/(p)$ maps $L^1 \otimes (W(A)/(p))$ to $V^1 \otimes (W(A)/(p))$. Then $t$ is described as follows. Let $G$ be $\text{Aut}(V, s_{dR})$. Then $V^1 \otimes (W(A)/(p)) \subseteq V \otimes (W(A)/(p))$ is induced by a cocharacter of $\mathcal{G} \otimes (W(A)/(p))$ which lifts to a cocharacter of $\mathcal{G} \otimes W(A)$. So we have a splitting $V_{W(A)} = \hat{V}^1 \oplus \hat{V}^0$ such that $\hat{V}^1 \otimes (W(A)/(p)) = V^1 \otimes (W(A)/(p))$. The isomorphism $\hat{t}$ is then of the form $(a_{m} b_{m})$, with $a \in \text{Isom}(L^1_{W(A)}, \hat{V}^1), b \in \text{Hom}(L^0_{W(A)}, \hat{V}^0)$, $c \in p \cdot \text{Hom}(L^1_{W(A)}, \hat{V}^0) = (p) \cdot \text{Hom}(L^0_{W(A)}, \hat{V}^0)$, and $d \in \text{Isom}(L^0_{W(A)}, \hat{V}^0)$. Now $t$ is given by base-change to $W_m(A)$ of the corresponding elements in Hom or Isom, denoted by $(a_{m} b_{m})$. We remark that $c_m$ is of the form $p \otimes c_m$, where $c_m$ is an element in $(\text{Hom}(L^0_{W(A)}, \hat{V}^0) \otimes W_m(A))$. The element $t$ is independent of choices of splittings, although we need a splitting to write down the matrix.

The morphism $I'_m \to K \mathcal{I}'_m$ is then given by mapping $(a_{m} b_{m})$ to $(p \cdot c_{m} d_{m})$. Here $p \cdot c_{m} \in (\text{Hom}(L^0_{W(A)}, \hat{V}^0)) \otimes I_{m+1}(A)$ is induced by the multiplication by $p$ map $W_m(A) \to I_{m+1}(A)$.

Now we explain why $K \mathcal{I}'_m$ and $K \zeta_m$ descents to $\mathcal{S}_0$. Let $U^{ij}$ be $U^i \cap U^j$ and $R^{ij}$ be its affine coordinate ring. Let $K \mathcal{I}'_m$ be the restriction of $K \mathcal{I}'_m$ to $U^i_0$, and $K \zeta_m : K \mathcal{I}'_m \to G_m$ be the restriction of $K \zeta_m$ to $U^i_0$. There is a canonical isomorphism $c^{ij} : V_{U^i \cap U^j}(R^{ij}) \to V_{U^i \cap U^j}(R^{ij})$ given by the composition of canonical isomorphisms $V_{U^i \cap U^j}(R^{ij}) \to \mathbb{D}(A_{\mathcal{S}_0})(W(R^{ij})) \to V_{U^i \cap U^j}(R^{ij})$. The isomorphism $c^{ij}$ respects $s_{dR}$ and the $F - V$ module structures. The universal element
\[ K_{\mathbb{Y}}^\prime |_{U_0^{ij}} \xrightarrow{ij} \widetilde{K}_{\mathbb{Y}}^\prime |_{U_0^{ij}} \] is such that it lifts to a section of \( \mathbb{Y}_m \) over an affine regular fppf cover (namely, \( \mathbb{Y}_m |_{U_0^{ij}} \)), so to define a morphism, it suffices to work with Spec \( A \)-points with Spec \( A/U_0^{ij} \) regular such that there is a regular fppf affine cover Spec \( A' \rightarrow \text{Spec } A \) with a commutative diagram

\[
\begin{array}{ccc}
\text{Spec } A' & \xrightarrow{t'} & \mathbb{Y}_m |_{U_0^{ij}} \\
\downarrow & & \downarrow \pi \\
\text{Spec } A & \xrightarrow{t} & \widetilde{K}_{\mathbb{Y}}^\prime |_{U_0^{ij}}.
\end{array}
\]

Viewed as an \( A' \)-point, \( t = \pi \circ t' \) is induced by a \( W(A') \)-point \( \tilde{t} \) of \( \mathbb{Y} \) described as before. Let \( A'' = A' \otimes_A A' \) and \( i_1 : A' \rightarrow A'' \), \( i_2 : A' \rightarrow A'' \) be the two embeddings, then \( i_1^*t = i_2^*t \). Composing \( c^{ij} \) induces a well defined \( B \)-point of \( K_{\mathbb{Y}}^\prime |_{U_0^{ij}} \), denoted by \( c^{ij} \circ t \). Noting that \( i_1^*(c^{ij} \circ t) = i_2^*(c^{ij} \circ t) = i_2^*c^{ij} \circ i_2^*t = i_1^*(c^{ij} \circ t) \), \( c^{ij} \circ t \) descents to an \( A \)-point of \( K_{\mathbb{Y}}^\prime |_{U_0^{ij}} \) by faithfully flat descent. It is independent of the choices of \( A' \), as one could work with the universal morphism \( \mathbb{Y}_m |_{U_0^{ij}} \rightarrow K_{\mathbb{Y}}^\prime |_{U_0^{ij}} \) and then base-change to \( A \). So \( c^{ij} \) induces an isomorphism \( K_{\mathbb{Y}}^\prime |_{U_0^{ij}} \rightarrow K_{\mathbb{Y}}^\prime |_{U_0^{ij}} \), denoted by \( c^{ij} \), which is clearly \( KG' \)-equivariant. But the \( c^{ij} \)'s satisfy the cocycle condition as they are canonical, so one glues \( K_{\mathbb{Y}}^\prime |_{U_0^{ij}} \)s to a \( KG' \)-torsor \( \mathcal{S}_0 \). By construction of \( \zeta_m \) in [1.3] which uses only the \( F - V \) module structure on \( D(A_\mathcal{S}_0)(W(R_0^{ij})) \), we see that \( K_{\mathbb{Y}}^\prime |_{U_0^{ij}} = K_{\mathbb{Y}}^\prime |_{U_0^{ij}} \circ c^{ij} \). So the \( K_{\mathbb{Y}}^\prime |_{U_0^{ij}} \)s also glue.

One has to fix a finite open affine cover \( \{ U_1^i \}_{1 \leq i \leq r} \) of \( \mathcal{S} \) as well as a collection of homomorphisms \( f^i : R^i = O_{U_i} \rightarrow W(R_0^i) \) to construct \( K_{\mathbb{Y}}^\prime \) as well as \( K_{\mathbb{Y}} \). It turns out that the glued scheme and morphism are unique.

**Lemma 3.2.1.** The glued torsor and equivariant morphism are independent of choices of \( \{ (U^i, f^i) \} \).

**Proof.** It suffices to show that for any collection \( \{ U^i, g^i \}_{i,k} \), the glued torsor and equivariant morphism are the same as those given by \( \{ (U^i, f^i) \} \). Here \( \{ U^i \} \) is a finite open affine cover of \( U^i \), and \( g^i \) is a homomorphism \( R^k = O_{U^i} \rightarrow W(R_0^i) \). Note that we do not assume any compatibility in \( f^i \) and \( g^i \).

For \( U^i \subseteq U^j \) and \( U^j \subseteq U^l \), we write \( R^{i,j} \) for the coordinate ring of \( U^i \cap U^j \). We write \( \mathcal{V}_{U^i} |_{W(R^{k,l})} \) for \( \mathcal{V}_{U^i} \otimes_{W(R^i)} W(R^{k,j}) \), here we write \( f^i \) for the composition \( R^l \xrightarrow{f^i} W(R_0^i) \rightarrow W(R^{k,j}) \) by abusing notations. We have canonical isomorphisms

\[
\begin{array}{c}
\mathcal{V}_{U^1} |_{W(R^{k,l})} \\
\downarrow d^i \\
\mathcal{V}_{U^2} |_{W(R^{k,l})} \\
\downarrow d^i \\
\mathcal{V}_{U^3} |_{W(R^{k,l})}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{D}(A_{\mathcal{S}_0}(W(R^{k,j}))) \\
\downarrow (d^i)^{-1} \\
\mathcal{D}(A_{\mathcal{S}_0}(W(R^{k,l}))) \\
\downarrow (d^i)^{-1} \\
\mathcal{D}(A_{\mathcal{S}_0}(W(R^{k,l})))
\end{array}
\]
The composition of the horizontal $m$ strata and morphisms is $c^{ji}$. So we have a commutative diagram

\[
\begin{array}{ccc}
K^\nu_{m}|_{U^\nu_{m,k,l}} & \xrightarrow{c^{ji}} & K^\nu_{m}|_{U^\nu_{m,k,l}} \\
\downarrow{(d^{i})^{-1}od^i} & & \downarrow{(d^{ii})^{-1}od^i} \\
G_m & \xrightarrow{e^{i}k_{m}} & K^\nu_{m}|_{U^\nu_{m,k,l}}
\end{array}
\]

which means that the glued torsors and morphisms are unique. □

The glued torsor and morphism over $\mathcal{S}_0$ will still be denoted by $K^\nu_{m}$ and $K_\zeta$ respectively. The induced morphism $\mathcal{S}_0 \rightarrow [KG'_m \backslash KG_m]$ will be denoted by $K_\zeta$. Fibers of $K_\zeta$ are level $m$ strata.

**Corollary 3.2.2.** The morphism $K_\zeta$ is smooth, and it is determined by the Shimura datum.

**Proof.** The smoothness follows from Theorem 2.2.0 and that the natural morphism $\mathbb{V}_m \rightarrow K^\nu_{m} \times_{\mathcal{S}_0} U_0$ is faithfully flat. The morphism $\zeta$ is uniquely determined by the Shimura datum together with a collection $\{(U^i, f^i)\}_{i}$, and so is $K_\zeta$. But by the previous lemma, $K_\zeta$ is independent of choices of $K_\zeta$, so it is uniquely determined by the Shimura datum. □

3.3. **More properties about level $m$ stratifications.** We will start with an analog of [29] Theorem 1.3 which plays an essential role there.

**Proposition 3.3.1.** Let $\mathcal{S}_0$ be a level $m$ stratum, $x \in \mathcal{S}_0$ be a point. Then there is a quasi-finite fppf cover $T = \text{Spec} A$ of $\mathcal{S}_0$, such that there exists an isomorphism $(A[x[p^m]])_T \rightarrow A_T[p^m]$ whose induced isomorphism on Dieudonné modules (truncated displays) respects the Hodge-Tate tensor.

**Proof.** We work with $U_0$ and consider the commutative diagram

\[
\begin{array}{ccc}
\mathbb{V}_m & \xrightarrow{\zeta_m} & G_m \\
\downarrow{\zeta_m} & & \downarrow{\zeta_m} \\
\mathbb{B} \mathcal{S}_m \otimes \kappa & \xrightarrow{\Phi_m} & \mathbb{D} \text{isp}_m \otimes \kappa \\
\downarrow{} & & \downarrow{} \\
[KG'_m \backslash G_m] & \xrightarrow{[KG'_m \backslash G_m]} & [KGL'_m \backslash GL_m] \otimes \kappa.
\end{array}
\]

Let $G_m^{\circ}$ be the $G_m^{\circ}$-orbit in $G_m^{\circ}$ corresponding to $U_0^{\circ}$. Let $g \in G_m^{\circ}$ be the image under $\zeta$ of a section of $\mathbb{V}_m^{\circ}$. By [9] Corollary 17.16.3, there is a quasi-finite étale cover $T = \text{Spec} A$ of $U_0^{\circ}$ such that $\mathbb{V}_m(T) \neq \emptyset$. The composition $g : T \rightarrow \mathbb{V}_m \rightarrow G_m$ factors through $G_m^{\circ}$.

We claim that there is a finite flat cover $T'$ of $T$, such that there exists $h \in G_m^{\circ}(T')$ with $g_{T'} = h \cdot g_{T'}$. It suffices to show that, over a finite flat cover of $G_m^{\circ}$, the orbit morphism $G_m^{\circ} \rightarrow G_m^{\circ}$, $h \rightarrow h \cdot g$ admits a section. Let $H_g$ be the stabilizer of $g$ in $G_m^{\circ}$, and $(H_g)^{\text{red}}$ be the reduced identity component. Then $G_m^{\circ}/(H_g)^{\text{red}}$ is a finite flat cover of $G_m^{\circ}$. We only need to show that $G_m^{\circ} \rightarrow G_m^{\circ}/(H_g)^{\text{red}}$ admits a section.
Let \( g_0 \) be the image of \( g \) in \( G_0 := G_1 \), and \( H_{g_0} \) be the stabilizer of \( g_0 \) in \( G'_0 := G'_1 \). Then by Theorem 8.1, \( (H_{g_0})^0_{\text{red}} \) is unipotent. Let \( H_{g_0}^m \) be the inverse image of \( (H_{g_0})^0_{\text{red}} \) in \( G_{m,\alpha} \), then \( H_{g_0}^m \) is unipotent and contains \( (H_{g_0})^0_{\text{red}} \). This implies that \( (H_{g_0})^0_{\text{red}} \) is unipotent, and hence \( G_{m,\alpha}/(H_{g_0})^0_{\text{red}} \) admits a section.

We still write \( H \) for its image in \( KG_{m}'(T') \) and \( KGL_{m}'(T') \). Then \( H \) gives an isomorphism between truncated displays attached to \( (\mathcal{A}_z[m])_{T'} \) and \( (\mathcal{A}_{U_0}'[m])_{T'} \). But by Theorem 3.1.5 (3), there is a finite flat cover \( T'' \) of \( T' \), such that \( h_{T''} \) comes from an isomorphism of \( \text{BT-}m\text{-s} \).

Fiber of \( \mathcal{S}_0 \to [KG_{m}'(\mathcal{G}_m)] \to [KGL_{m}'(\mathcal{G}_m)] \otimes \kappa \) are precisely classical level \( m \) strata (see 1.3). For \( s \in [\{KGL_{m}'(\mathcal{G}_m)] \), we write \( \mathcal{S}_0^{cL,s} \) for the corresponding stratum. It is a locally closed subscheme of \( \mathcal{S}_0_{\alpha,\kappa} \), and, as a set\(^1\) it is a union of level \( m \) strata.

**Lemma 3.3.2.** There is a quasi-finite fpfp cover \( T = \text{Spec } A \) of \( \mathcal{S}_0^{cL,s} \) such that \( \mathcal{A}_T[p^m] \) is constant.

**Proof.** The proof is the same as the previous proposition, but we should consider the following commutative diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s} \\
\downarrow & & \downarrow \\
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s}
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s} \\
\downarrow & & \downarrow \\
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s} \\
\downarrow & & \downarrow \\
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s} \\
\downarrow & & \downarrow \\
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s} \\
\downarrow & & \downarrow \\
\mathcal{S}_0^{cL,s} & \to & \mathcal{S}_0^{cL,s}
\end{array}
\end{array}
\]

**Proposition 3.3.3.** Each \( \mathcal{S}_0^{cL,s} \) is a union of connected components of \( \mathcal{S}_0^{cL,s} \). In particular, it is open and closed in \( \mathcal{S}_0^{cL,s} \).

**Proof.** The proof is similar to Theorem 3.3, but we prefer to write down the details. The proposition follows from the following statement.

Let \( y \) be the generic point of an irreducible component of \( \mathcal{S}_0^{cL,s} \), then \( \{y\} \subseteq \mathcal{S}_0^{cL,s} \) if \( y \in \mathcal{S}_0^{cL,s} \).

This is because any irreducible component intersecting \( \{y\} \) lies in \( \mathcal{S}_0^{cL,s} \), while in one connected component of \( \mathcal{S}_0^{cL,s} \), one can joint two irreducible components which do not intersect by other irreducible components.

Now we prove \((*)\). Let \( x \in \mathcal{S}_0'(\overline{\kappa}) \). By passing to an affine cover, we can assume that \( T := \{y\} = \text{Spec}(A) \) with \( A \) an integral domain of finite type over \( \overline{\kappa} \). Let \( \text{Aut}_\overline{\kappa}(\mathcal{A}_x[p^m]) \) be the sheaf of automorphism of \( \mathcal{A}_x[p^m] \). By Lemma 2.4, it is represented by a \( \overline{\kappa} \)-scheme of finite type. Let \( \overline{K} \) be an algebraically closed field containing \( A \) and all residue fields \( k(x), x \in \text{Aut}_\overline{\kappa}(\mathcal{A}_x[p^m]) \). The assumption of \((*)\) means that there is an isomorphism \( f : \mathcal{A}_x[p^m]_{\overline{\kappa}} \to \mathcal{A}_T[p^m] \otimes_A \overline{\kappa} \), whose induced isomorphism on truncated displays respects the Hodge-Tate tensors. By Lemma 3.3.2 there is a quasi-finite and surjective morphism \( T' = \text{Spec}(B) \to T \), such that \( \mathcal{A}_T[p^m] \times_T T' \). By taking the reduced induced structure and passing to affine covers of irreducible components, we can assume that \( B \) is an integral domain of finite type over \( \overline{\kappa} \).

\footnote{One can replace “set” by “scheme”, see Remark 6.3.3}
Let’s write $- \otimes -$ for tensor product over $\mathcal{P}$. We have $A \hookrightarrow B \hookrightarrow \overline{\mathcal{P}}$, and that $\overline{\mathcal{P}} \otimes B$ is an integral domain with fraction field $L$. The isomorphism $f_L$ respects the Hodge-Tate tensors. By constancy of $\mathcal{A}_{T}[p^m]$, we also have an isomorphism $\alpha : \mathcal{A}_{T}[p^m] \times_A (B \otimes \overline{\mathcal{P}}) \to \mathcal{A}_{x}[p^m] \otimes B \otimes \overline{\mathcal{P}}$. The composition $\alpha_L \circ f_L$ is an automorphism of $\mathcal{A}_{x}[p^m] \otimes L$, so it is given by a $\mathcal{P}$-point $g$ of $\text{Aut}_{\mathcal{P}}(\mathcal{A}_{x}[p^m])$. Now the composition $\alpha^{-1} \circ g_{B \otimes \mathcal{P}}$ is such that its induced isomorphism on truncated displays respects the Hodge-Tate tensors after tensoring $W_m(L)$. But then it has to respect the tensors, as $W_m(B \otimes \overline{\mathcal{P}}) \subseteq W_m(L)$. In particular, $(*)$ holds.

**Remark 3.3.4.** The previous proposition implies that $\mathcal{S}_0^{cL,s}$ is a finite disjoint union of level $m$ strata as schemes, and hence smooth.

### 4. Geometry of Newton strata

Let $T = \text{Spec}(R)$ be an affine scheme over $\mathcal{S}$ such that $R$ is $p$-adically complete, $p$-torsion free and equipped with a lift of Frobenius $\delta$. Let $M$ be $\mathbb{D}(A)(R)$ and $\varphi$ be the Frobenius. If $\mathcal{W}_s(T) \neq \emptyset$, then $\varphi$ determines a $G_{\mathcal{Z}_p}(R)$-conjugacy class of $G_{\mathcal{Z}_p}(R)^{\mu^p}(p)\delta(G_{\mathcal{Z}_p}(R))$. More precisely, for $t \in \mathcal{W}_s(T)$ and $M^t := t(L_R^t), i = 1, 2$, we have a commutative diagram

$$
\begin{array}{c}
\xymatrix{
L_R^t \ar[r]^-{\xi} \ar[d]^-{\delta^*(L_R^t \oplus L_R^0)} & \delta^*(L_R^t \oplus L_R^0) \ar[r]^-{\varphi \otimes \text{id}} \ar[d]^-{\xi^{-1}} & \delta^*(M^1 \oplus M^0) \ar[r]^-{\varphi_{\mu^p}(\xi^{-1})} & M^t \ar[r]^-{\iota_{L_R^t}} & L_R^t.
}
\end{array}
$$

Noting that the composition of horizontal morphisms is $\varphi_{\mu^p}$ and that $g_1 \in G_{\mathcal{Z}_p}(R)$, we have $\varphi_{\mu^p} = g_1\mu^p(p)$. So $\varphi_{\mu^p}$ gives $\{h\mu^p(p)\delta(h)^{-1} \mid h \in G_{\mathcal{Z}_p}(R)\}$ by considering all elements in $\mathcal{W}_s(R)$.

Let $F$ be the fraction field of $W := W(\mathcal{P})$. Let $C(G_{\mathcal{Z}_p})$ (resp. $B(G)$) be the set of $G_{\mathcal{Z}_p}(W)$-\(\sigma\)-conjugacy (resp. $G(F)$-\(\sigma\)-conjugacy) classes in $G(F)$. Let $C(G_{\mathcal{Z}_p}, \mu^p)$ be the set of $G_{\mathcal{Z}_p}(W)$-\(\sigma\)-conjugacy classes in $G_{\mathcal{Z}_p}(W)\mu^p(p)G_{\mathcal{Z}_p}(W)$, and $B(G_{\mathcal{Z}_p}, \mu^p)$ be the image of $C(G_{\mathcal{Z}_p}, \mu^p) \to C(G_{\mathcal{Z}_p}) \to B(G)$. For a $\mathcal{P}$-point $x$ of $\mathcal{S}_0$, it lifts to a $W$-point $\tilde{x}$ of $\mathcal{S}$, and $\mathcal{W}_{\tilde{x}}(W) \neq \emptyset$. Applying the above construction to $R = W$, we get an element in $C(G_{\mathcal{Z}_p}, \mu^p)$. This indues maps $C_t : \mathcal{S}_0(\mathcal{P}) \to C(G_{\mathcal{Z}_p}, \mu^p)$ and $N_t : \mathcal{S}_0(\mathcal{P}) \to B(G_{\mathcal{Z}_p}, \mu^p)$. The pre-image under $C_t$ (resp. $N_t$) of an element in $C(G_{\mathcal{Z}_p}, \mu^p)$ (resp. $B(G_{\mathcal{Z}_p}, \mu^p)$) is the set of $\mathcal{P}$-points of a central leaf (resp. Newton stratum).

#### 4.1. Group theoretic preparations

Let $B \subseteq G_{\mathcal{Z}_p}$ be a Borel subgroup and $T \subseteq B$ be a maximal torus. Then $T$ splits over an unramified extension of $\mathbb{Q}_p$. Let $\Gamma$ be $\text{Gal}(\mathbb{P} / \mathbb{F}_p)$, then $X^*(T)$ (resp. $X_*(T)$), the group of characters (resp. cocharacters) is a $\Gamma$-module. Let $\pi_1(G)$ be the quotient of $X_*(T)$ by the coroot lattice. Let $W_G$ be the Weyl group of $G$, $\widetilde{W}_G := \text{Norm}_G(T)(T) / T(W) \cong W_G \rtimes X_*(T)$ be the extended affine Weyl group and $W_a$ be the affine Weyl group. We have a canonical exact sequence

$$
0 \longrightarrow W_a \longrightarrow \widetilde{W}_G \longrightarrow W_G \longrightarrow 0.
$$
Let $\Omega \subseteq \tilde{W}_G$ be the stabilizer of the alcove corresponding to the Iwahoric subgroup of $G(F)$ given by the preimage of $B(\mathbf{F})$. We define the length function on $\tilde{W}_G$ by
\begin{equation}
 l(wr) = l(w), \text{ for } w \in W_a, r \in \pi_1(G).
\end{equation}

4.1.2. Results about $B(G)$. To a $G(F)$-σ-conjugacy class $[b]$, Kottwitz defines two functorial invariants $\nu_G(b) \in (X_*(T)_G/W_G)^F \cong X_*(T)_{G,\text{dom}}^\sigma$ and $\kappa_G(b) \in \pi_1(G)_\Gamma$ in $[17]$. These two invariants determines $[b]$ uniquely.

We consider the partial order $\leq$ on $X_*(T)_G$ given by $\chi' \leq \chi$ if and only if $\chi - \chi'$ is a linear combination of positive roots with positive rational coefficients. We write $\overline{\mu^\sigma}$ the average of its $\Gamma$-orbit.

**Proposition 4.1.3.** For $b \in G_{Z_p}(W)\mu^\sigma(p)G_{Z_p}(W)$, we have $\nu_G(b) \leq \overline{\mu^\sigma}$ and $\kappa_G(b) = \mu^*_\sigma$. Here $\mu^*_\sigma$ is the image of $\mu^\sigma$ in $\pi_1(G)_\Gamma$.

**Proof.** This is $[35]$ Theorem 4.2. □

By works of Gashi, Kottwitz, Lucarelli, Rapoport and Richartz, we have
\[ B(G_{Z_p},\mu^\sigma) = \{ [b] \in B(G) \mid \nu_G(b) \leq \overline{\mu^\sigma} \text{ and } \kappa_G(b) = \mu^*_\sigma \}. \] (See $[22]$ 8.6).

To each $G(K)$-σ-conjugacy class $[b]$, one defines $M_b$ to be the centralizer in $G$ of $\nu_G(b)$, and $J_b$ be the group scheme representing
\[ J_b(R) = \{ g \in G(R \otimes_{\mathbb{Q}_p} K) \mid gb = b\sigma(g) \}. \]
The group $J_b$ is a inner form of $M_b$ which does not depend on the choices of representatives in $[b]$ (see $[17]$ 5.2).

**Definition 4.1.4.** For $[b] \in B(G)$, the defect of $[b]$ is defined by
\[ \text{def}_G(b) = \text{rank}_{\mathbb{Q}_p} G - \text{rank}_{\mathbb{Q}_p} J_b. \]

Hamacher gives a formula for $\text{def}_G(b)$ using root data.

**Proposition 4.1.5.** Let $w_1, \ldots, w_l$ be the sums over all elements in a Galois orbit of absolute fundamental weights of $G$. For $[b] \in B(G)$, we have
\[ \text{def}_G(b) = 2 \cdot \sum_{i=1}^l \{ \nu_G(b), w_i \}, \]
where $\{ \cdot \}$ means the fractional part of a rational number.

**Proof.** This is $[10]$ Proposition 3.8. □

4.1.6. Minimal points and fundamental elements. Let us write $K$ for $G_W(W)$, $K_1$ for $\ker(K \to G_W(\mathbf{F}))$ and $\mathcal{I}$ for the Iwahoric subgroup attached to $B$.

**Definition 4.1.7.** A element $x \in G(F)$ is called minimal if for any $y \in K_1 g K_1$, there is a $g \in K$ such that $y = gx \sigma(g)^{-1}$.

By $[42]$ Remark 9.1, if $x$ is minimal, then any element in the $K$-σ-orbit of $x$ is again minimal.

**Definition 4.1.8.** An element $x \in \tilde{W}$ is fundamental if $\mathcal{I} x \mathcal{I}$ lies in a single $\mathcal{I}$-σ-orbit.

If $x$ is a representative of a fundamental element, then it is minimal. By $[28]$ Theorem 1.3, it is also straight.
Theorem 4.1.9. Each orbit in $B(G_{z_r}, \mu^\sigma)$ contains a fundamental element in $W_G\mu^\sigma(p)W_G$.

Proof. This is [28] Proposition 1.5. □

4.1.10. Truncations of level 1 and Ekedahl-Oort strata. For a dominant cocharacter $\chi \in X_*(T)$, we write $W_\chi$ for the Weyl group of the centralizer of $\chi$, and $\chi W$ for the set of elements $w$ which are the shortest representatives of their cosets $W_\chi w$. Let $x_\chi = w_0 w_{0, \chi}$ denote the longest element of $W_G$ and where $w_{0, \chi}$ is the longest element of $W_\chi$. Let $\tau_\chi = x_\chi\chi(p)$. Then $\tau_\chi$ is the shortest element of $W_G\chi(p)W_G$.

Theorem 4.1.11. Let $\mathcal{T} = \{(w, \chi) \in W_G \times X_*(T)_{\text{dom}} \mid w \in \chi W\}$. Then the map assigning to $(w, \chi)$ the $K$-$\sigma$-conjugacy class of $K_1 w \tau_\chi K_1$ is a bijection between $\mathcal{T}$ and the set of $K$-$\sigma$-conjugacy classes in $K_1 \backslash G(F)/K_1$.

Proof. This is [41] Theorem 1.1 (1). □

Remark 4.1.12. Let $(w, \mu^\sigma) \in \mathcal{T}$ be corresponding to the $K$-$\sigma$-conjugacy classes in $K_1 \backslash G(F)/K_1$ attached to a fundamental element in $[b] \in B(G_{z_r}, \mu^\sigma)$. Then by the group theoretic arguments in [10] 7.3, $l(w) = \langle 2\rho, \nu_G(b) \rangle$, where $\rho$ is the half-sum of positive roots of $G$. This is the dimension of the Ekedahl-Oort stratum corresponding to $w$ by [46].

4.2. Central leaves. By Theorem 4.1.7, all central leaves are given by level $m$ stratum for $m$ big enough. So we can apply results about level $m$ strata to central leaves.

Corollary 4.2.1. Each central leaf is a smooth, equi-dimensional locally closed subscheme of $\mathcal{X}_{0, \pi}$. It is open and closed in the classical central leaf containing it, and closed in the Newton stratum containing it.

Proof. We only need to explain why it is closed in the Newton stratum, as other statements following from Theorem 2.2.8 and Proposition 3.3.3. But by [29] Theorem 2.2, a classical central leaf is closed in the classical Newton stratum, so a central leaf is closed in the classical Newton stratum. In particular, it is closed in the Newton stratum. □

Let $x, y \in \mathcal{X}_{0, \pi}$ be two points, and $X, Y$ be their attached $p$-divisible groups. For any $n \in \mathbb{Z}_{>0}$, we write $X_n$ (resp. $Y_n$) for its $p^n$-kernel. Let $I_n^{X,Y}$ (resp. $I_n^{X,Y}$) be the subset of elements $f \in \text{Isom}(X,Y)$ (resp. $f \in \text{Isom}(X,Y_n)$) such that $D(f): D(Y) \rightarrow D(X)$ (resp. $D(f): D(Y_n) \rightarrow D(X_n)$) respects the Hodge-Tate tensors. For $N \in \mathbb{Z}$ such that $N \geq n$, we write $\Phi_n: \text{Isom}(X,Y) \rightarrow \text{Isom}(X_n,Y_n)$ resp. $\Phi_n^N : \text{Isom}(X_n,Y_n) \rightarrow \text{Isom}(X_n,Y_n)$ for the natural maps induced by restricting to $p^n$-kernels.

Lemma 4.2.2. For each $n \in \mathbb{Z}$, there is an integer $N(X,Y,n)$, such that for any $N \geq N(X,Y,n)$, we have $\Phi_n(I_n^{X,Y}) = \Phi_n^N(I_n^{X,Y})$.

Proof. This is essentially the proof of [29] Lemma 1.5. More precisely, by the proof there, $\Phi_n(\text{Isom}(X,Y))$ is finite, and there is $N_1$ such that $\forall N \geq N_1$, we have $\Phi_n(\text{Isom}(X,Y)) = \Phi_n^N(\text{Isom}(X_n,Y_n))$. In particular, $\forall N \geq N_1$, $\Phi_n(I_n^{X,Y})$, $\Phi_n^N(I_n^{X,Y})$ and $\Phi_n^N(\text{Isom}(X_n,Y_n))$ are finite. But

$$\Phi_n(I_n^{X,Y}) \subseteq \Phi_n^{N+1}(I_{N+1}^{X,Y}) \subseteq \Phi_n^N(I_n^{X,Y}) \text{ and } \Phi_n(I_n^{X,Y}) = \bigcap N \Phi_n^N(I_n^{X,Y}).$$
so there is $N_0 \geq N_1$ such that $\Phi_n(I_{X,Y}) = \Phi_n^{N_0}(I_{X,Y}^{N_0})$ which proves the claim. \(\square\)

**Remark 4.2.3.** In [29], Lemma 1.5. shows that there is $N(X, Y, n)$, such that for any $N \geq N(X, Y, n)$, $\Phi_n(\text{Hom}(X, Y)) = \Phi_n^{N}(\text{Hom}(X_N, Y_N))$. One could ask for similar results for $\text{Hom}^0$, homomorphisms respecting Hodge-Tate tensors. However, the phase homomorphisms from $X_n$ to $Y_n$ respecting Hodge-Tate tensors does not quite make sense. As $\mathcal{D}(X_n)^\circ$ is constructed from $\mathcal{D}(X_n)$ by taking tensors, duals, and symmetric/exterior powers, and there are no obvious maps between $\mathcal{D}(X_n)^\circ$ and $\mathcal{D}(Y_n)^\circ$ induced by $f \in \text{Hom}(X_n, Y_n)$ unless $f$ is an isomorphism.

For $x$ and $X$ as above, we denote by $G^X$ (resp. $G_n^X$) the group of elements in $\text{Aut}(X)$ (resp. $\text{Aut}(X_n)$) whose induced map on Dieudonné module respects Hodge-Tate tensors. If $y \in \mathcal{S}_0(\pi)$ is in the central leaf crossing $x$, then $I_{X,Y}$ (resp. $I_{X,Y}^n$) is a trivial $G^X$-torsor (resp. $G_n^X$-torsor).

**Corollary 4.2.4.** Let $C \subseteq \mathcal{S}_0(\pi)$ be a central leaf. For each $n \in \mathbb{Z}_{>0}$, there is an integer $N(n, C)$, such that for any $N \geq N(n, C)$, we have $\Phi_n(I_{X,Y}^N) = \Phi_n(I_{X,Y}^N)$ for any $x, y \in C(\pi)$.

**Proof.** Fix $x \in C(\pi)$ with attached $p$-divisible group $X$, by the previous lemma, for any $n \in \mathbb{Z}_{>0}$, there is an integer $N(X, n)$, such that for $N \geq N(X, n)$, we have $\Phi_n^{N}(G_{X}^N) = \Phi_n(G_{X}^N)$. Then for any $y \in C(\pi)$ with attached $p$-divisible group $Y$, we have $\Phi_n^{N}(I_{X,Y}^N) = \Phi_n(I_{X,Y}^N)$ and $\Phi_n^{N}(G_{X}^N) = \Phi_n(G_{Y}^N)$, as $I_{X,Y}$ (resp. $I_{X,Y}^N$) is a trivial $G^X$-torsor (resp. $G_n^X$-torsor). One takes $N(n, C) = N(X, n)$ and finishes the proof. \(\square\)

**Proposition 4.2.5.** Let $[b] \in B(G_{\mathbb{Z}_p}, \mu^p)$ be a conjugacy class, and $\mathcal{S}_0^b$ be the corresponding Newton stratum. Then any central leaf in $\mathcal{S}_0^b$ is of dimension $(2p, n_G(b))$.

**Proof.** We only need to prove that central leaves in $\mathcal{S}_0^b$ are of the same dimension. The dimension formula then follows from Remark 4.1.12. Let $x \in \mathcal{S}_0^b(\pi)$ be a point, and $\mathcal{S}_0^b_x$ be the connected component of the central leaf containing $x$. The $G_n^X$-torsor $\mathbb{L}_x$ is trivial, we will fix such a section as well as its lifting $t_x$ in $\mathbb{L}(W)^\mathbb{A}$. The image of $t_x$ in $\mathbb{L}(W)$ will be denoted by the same notation. By [10] Proposition 1.4.4, an element $g \in G(K)$ such that $g^{-1}b(a(g) \in G_{\mathbb{Z}_p}(W)\mu^p(p)G_{\mathbb{Z}_p}(W)$ gives a point $gx \in \mathcal{S}_0^b(\pi)$, denoted by $y$, by considering the the Dieudonné submodule $t_x(gV_W)$. Multiplying $g$ by some $p^r$ if necessary, we can assume that there is an isogeny $f : A_x \to A_y$ of degree $p^{n_1}$ whose induced morphism on Dieudonné modules respects the Hodge-Tate tensors. We write $\mathcal{S}_0^b$ for the connected of the central leaf containing $y$. Note that for each central leaf, there is a point of the form $gx$.

We will construct a connected regular affine $\pi$-scheme $T$, with dominant and quasi-finite morphisms $\mathcal{S}_0^x \leftarrow T \to \mathcal{S}_0^y$. We assume that the symplectic embedding we choose is of good reduction\(^2\). Let $\mathcal{S}_0^x$ be the irreducible component of the central leaf containing (the image of) $x$. Let $n_2$ be such that all level $n_2$ strata (in $\mathcal{S}_0^x$) are central leaves, and $n = n_1 + n_2$. Let $N$ be $n + N(n, \mathcal{S}_0^x)$ with $N(n, \mathcal{S}_0^x)$ be as in the previous corollary. By Proposition 4.3.4, there is a quasi-finite fpf cover $s : T = \text{Spec} R \to \mathcal{S}_0^x$, such that there exists an isomorphism $i : (A_T[p^N])_T \to A_T[p^N]$ whose induced map on Dieudonné modules respects Hodge-Tate tensors. We write $i$ for the restriction of $i$ to $p^n$-kernels.

\(^2\)One could also do this by choosing a lifting $\overline{x}$ of $x$, and considering a section of $\mathbb{L}_{\overline{x},\overline{\pi}}$.

\(^3\)This is always possible, for example, by Zarhin’s trick, one can take $(V_{\mathbb{Z}_p} \oplus V'_{\mathbb{Z}_p})^4$. 
By passing to an irreducible component of $T$ containing a closed point $z$ mapping to $x$, we can assume that $T$ is integral, and that $\tau : T \to \mathcal{S}_0^*$ is dominant, quasi-finite and of finite type. Modifying $x$ (and hence $y = gx$) by a prime to $p$ isogeny if necessary, we can assume that $z$ is in the smooth locus of $T$. We could then pass to a smooth affine neighborhood of $z$, which will still be denoted by $T$.

Let $H$ be $\ker(f)$ and $\mathcal{H}$ be $i(H \times T)$ in $\mathcal{A}_T[p^n]$. We consider the abelian scheme $\mathcal{A}_T/\mathcal{H}$ over $T$.

1. The polarization and level structure on $\mathcal{A}_T$ descend to $\mathcal{A}_T/\mathcal{H}$, which gives a quasi-finite morphism $\tau' : T \to \mathcal{S}_{0,K'}$. The polarization descents as $\mathcal{H}$ is isotropic for the form defined by the polarization, while $i$ respects these forms. The quasi-finiteness follows from the arguments in [29] page 279, which shows that for $u \in \mathcal{S}_{0,K'}(\overline{\tau})$, $\tau(\tau^{-1}(u))$ is finite.

2. For $u \in T(\overline{\tau})$, $\tau'(u)$ factors through $\mathcal{S}_0$. By our choice of $i$, the isomorphism $i_u : \mathcal{A}_v[p^n] \to \mathcal{A}_u[p^n]$ lifts to an isomorphism $i_0 : \mathcal{A}_u[p^n] \to \mathcal{A}_u[p^n]$ respecting Hodge-Tate tensors. Choosing a $t_u \in \mathcal{I}(W)$ as at the beginning of the proof, and using the commutative diagram

\[
\begin{array}{ccc}
\mathcal{A}_v[p^n] & \xrightarrow{i_u} & \mathcal{A}_u[p^n] \\
\downarrow f & & \downarrow \ \\
\mathcal{A}_v[p^n] & \xrightarrow{i_0} & \mathcal{A}_u[p^n]/i_0(H),
\end{array}
\]

we can identify $\mathcal{D}(\mathcal{A}_u[p^n]/i_0(H)) \to \mathcal{D}(\mathcal{A}_u[p^n])$ with $t_u(gV_W)$ for some $g \in G(K)$ such that $g^{-1}b\sigma(g) \in G_{\mathbb{Z}_p}(W)\mu^\sigma(p)G_{\mathbb{Z}_p}(W)$. By [16] Proposition 1.4.4, this gives a point $gu \in \mathcal{S}^0_{0}(\overline{\tau})$.

Let $\mathcal{R}$ be the lifting of $R$ which is in particular regular. Let $\mathcal{M}$ (resp. $\mathcal{M}'$) be $\mathcal{D}(\mathcal{A}_T)(\mathcal{R})$ (resp. $\mathcal{D}(\mathcal{A}_T/\mathcal{H})(\mathcal{R})$). The isogeny $\mathcal{A}_T \to \mathcal{A}_T/\mathcal{H}$ gives $\mathcal{M}' \to \mathcal{M}$ which induces an isomorphism $\mathcal{M}'[\frac{1}{p}] \to \mathcal{M}[\frac{1}{p}]$. One could then translate the section $s_{\mathcal{M}} \in \mathcal{M}^\otimes$ to a section of $\mathcal{M}'[\frac{1}{p}]^\otimes$, denoted by $s'_{\mathcal{M}}[\frac{1}{p}]$.

We claim the followings.

3. $s'_{\mathcal{M}}[\frac{1}{p}]$ extends to a section $s'_{\mathcal{M}}$ over $\mathcal{R}$, and $\mathcal{I} := \text{Isom}_{\mathcal{R}}((W_v, s)_{\mathcal{R}}, (\mathcal{M}', s'_{\mathcal{M}}))$ is a $G_W$-torsor.

We will show how to finish the proof with (3), and then give a proof of it.

4. By (3), passing to a quasi-finite affine scheme which is etale over $T$, we could assume that $\mathcal{I}(\mathcal{R})$ and $\mathcal{I}(\mathcal{R})$ are non-empty. One could identify $\mathcal{M}' \to \mathcal{M}$ as an element $g \in G(\mathcal{R}[\frac{1}{p}])$ by choosing an element in $\mathcal{I}(\mathcal{R})$ resp. $\mathcal{I}(\mathcal{R})$. One could then identify

\[
\begin{array}{ccc}
\mathcal{M}' & \xrightarrow{\rho} & \mathcal{M} \\
\downarrow & & \downarrow \ \\
\mathcal{M}' & \xrightarrow{\rho} & \mathcal{M}
\end{array}
\]

The Hodge filtration on $\mathcal{M}_T$ is induced by a cocharacter of $G_T$ which lifts to a cocharacter of $G_{\overline{T}}$ that is conjugate to $\mu$. Let $\mathcal{M} = \text{Fil}^1(\mathcal{M}) \oplus \text{Fil}^0(\mathcal{M})$ be the splitting induced by this cocharacter. Here $\text{Fil}^1(\mathcal{M})$ is a lift of the Hodge filtration. Let $\text{Fil}_0(\mathcal{M})$ be the submodule of $\mathcal{M}$ generated by $\varphi(\text{Fil}^0(\mathcal{M}))$. Then $\varphi$ induces a $\alpha$-isomorphism $\text{Fil}^0(\mathcal{M}) \to \text{Fil}_0(\mathcal{M})$. By the above identification, we have $\mathcal{M}' = g\text{Fil}^1(\mathcal{M}) \oplus g\text{Fil}^0(\mathcal{M})$, and $g\text{Fil}_0(\mathcal{M}) \subseteq \mathcal{M}'$ a direct summand. The
commutative diagram

\[
\begin{array}{c}
\sigma^*(\Fil^0(\mathcal{M})) \xrightarrow{\imath^\text{im}} \Fil^0(\mathcal{M}) \\
\downarrow \sigma^* \downarrow \quad \downarrow \quad \downarrow \\
\sigma^*(g\Fil^0(\mathcal{M})) \xrightarrow{\imath^\text{im}} g\Fil^0(\mathcal{M})
\end{array}
\]

is such that all but the lower horizontal map are isomorphisms, so has to be an isomorphism. But then \(g\Fil^1(\mathcal{M})\) will be a lift of the Hodge filtration on \(\mathcal{M}'\).

Now one verifies the conditions in \[16\] 1.4.7, and applies \[16\] Proposition 1.4.9. This implies that we have \(\mathcal{I}_0 \overset{\tau}{\leftarrow} \Spec R \rightarrow \mathcal{I}_0^y\) with \(\tau\) quasi-finite and dominant, and \(\tau'\) quasi-finite. So we have \(\dim \mathcal{I}_0^x \leq \dim \mathcal{I}_0^y\). But we can get \(\dim \mathcal{I}_0^x \geq \dim \mathcal{I}_0^y\) by exactly the same proof. So we have \(\dim \mathcal{I}_0^x = \dim \mathcal{I}_0^y\) and that \(\tau'\) is also dominant.

Proof of (3). By the construction of \[15\] Proposition 1.3.2, there is a line \(L\) in some \(G_{Z_p}\)-representation \(V_{Z_p}^\otimes\) constructed from \(V_{Z_p}\) (by taking sums, tensor products, duals, symmetric/exterior products), such that \(G_{Z_p}\) is the stabilizer of \(L\), and that \(V_{Z_p}^\otimes = V_{Z_p}^\otimes \otimes (V_{Z_p}^\otimes)^\vee\). The line bundle \(\mathcal{E} := L \times_{G_{Z_p}} \mathbb{P}_R\) is a direct summand of \(\mathcal{M}^\otimes\), and \(\mathcal{E}[1/p] \subseteq \mathcal{M}[1/p]^{\otimes} \cong \mathcal{M}'[1/p]^{\otimes}\) extends to a direct summand of rank one of \(\mathcal{M}_U^\otimes\), denoted by \(\mathcal{E}\). Here \(U \subseteq \Spec R\) is an open affine subscheme of form \(\Spec R_f\) with \(f \notin (p)\).

(3.i) \(\mathcal{J}_1 := \Isom_U((V_W, \mathcal{L} \mathcal{W}, U, (\mathcal{M}', \mathcal{E})))\), isomorphism mapping \(L_U\) to \(\mathcal{E}'\), is a \(G_W\)-torsor. We only need to check the faithfully flatness. It is already a \(G_W\)-torsor over \(\mathcal{R}[1/p]\), so we only need to check this at points in the special fiber. Let \(t\) be a closed point of \(U_{\mathfrak{p}}\), and \(s\) be a closed point of \(\mathcal{J}_{1,t}\). By (2), \(s\) is in the closure of \(\mathcal{J}_1 \times_U U[1/p]\) in \(\Isom_U(V_U, \mathcal{M}_U^\otimes)\), and \(\mathcal{J}_{1,t}\) is a \(G_{\kappa}\)-torsor. In particular, we have \(O_{\mathcal{J}_{1,s}}\) is flat over \(W\), and

\[
\dim(O_{\mathcal{J}_{1,s}}) - \dim(O_{\mathcal{J}_{1,t}}) = (\dim(O_{\mathcal{J}_{1,s}}[1/p]) + 1) - (\dim(O_{\mathcal{J}_{1,t}}[1/p]) + 1) = \dim(G) = \dim(\mathcal{J}_{1,t}).
\]

This implies that \(\mathcal{J}_1 \to U\) is faithfully flat, and hence a \(G_W\)-torsor.

(3.ii) Noting that \(\mathcal{R}_{(p)} \cong O_U((p))\) is a DVR and \(G_W\) is quasi-split, we see that \(\mathcal{J}_{1, \pi_{(p)}}\) is a trivial torsor by \[27\] Theorem 7.1. The existence of \(\mathcal{R}_{(p)}\)-sections implies that the rational map \(s'_{dR}[1/p] : \mathcal{R} \dashrightarrow V^{\otimes}\) is defined at all points of codimension 1, and hence extends to \(\mathcal{R}\), as \(V^{\otimes}\) is a group and \(\mathcal{R}\) is normal. This extended section will be denoted by \(s'_{dR}\), and it is necessarily the closure of \(s'_{dR}[1/p]\) in \(V^{\otimes}\).

(3.iii) By the same argument as in (3.i), \(\mathcal{J} = \Isom_{\mathcal{R}}((V_W, s)_\mathcal{R}, (\mathcal{M}', s'_{dR}))\) is a \(G_W\)-torsor. \(\square\)

4.2.6. Slope filtrations and completely slope divisible \(p\)-divisible groups. We refer to \[32\] Definition 1.1 for the definition of slope filtrations and Definition 1.2 for that of completely slope divisible \(p\)-divisible groups.

Let \(\mathcal{G}\) be a \(p\)-divisible group over \(S\), and \(\mathcal{G} : 0 = \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}\) be its slope filtration. Then \(\mathcal{G}' := \mathcal{G}_i/\mathcal{G}_{i-1}\) is again a \(p\)-divisible group over \(S\). Assume that (contravariant) Dieudonné modules exists for \(p\)-divisible groups over \(S\), and write \(\mathcal{M}\) for the Dieudonné module of \(\mathcal{G}\). The slope filtration induces surjections of \(p\)-divisible groups \(\mathcal{G} \to \mathcal{G}/\mathcal{G}_1 \to \mathcal{G}/\mathcal{G}_2 \to \cdots \to \mathcal{G}/\mathcal{G}_{n-1}\). Taking Dieudonné modules,
we get $\mathcal{M}^* : 0 \subseteq \mathcal{M}^1 \subseteq \mathcal{M}^2 \subseteq \cdots \subseteq \mathcal{M}^n = \mathcal{M}$, with $\mathcal{M}^i$ the Dieudonné module of $G/g_{n-i}$. Clearly, $\mathcal{M}_i := \mathcal{M}^i/\mathcal{M}^{i+1}$ is the Dieudonné module of $G^{n-i}$. We will call $\mathcal{M}^*$ the slope filtration of $\mathcal{M}$, it is such that $\mathcal{M}^i \subseteq \mathcal{M}^{i+1}$ is a locally direct summand.

For $x \in \mathcal{I}_0(k)$, with $k$ algebraically closed, the slope filtration on $\mathcal{A}_x[p^\infty]$ induces a filtration on $M := \mathcal{D}(\mathcal{A}_x[p^\infty])(W(k))$, denoted by $M^*$. Fixing an isomorphism $t : Z_{\mathbb{Z}_p} \otimes W(k) \to M$ respecting the Hodge-Tate tensors, we could view $G_{W(k)}$ as a subgroup of $GL(M)$, and one can talk about $G_{W(k)}$-split filtrations (see [15] 1.1.2 for the definition) of $M$.

**Lemma 4.2.7.** The slope filtration on $M$ is $G_{W(k)}$-split.

**Proof.** Let $K$ be the fraction field of $W(k)$. By [17], $M_K^*$ is $G_K$-split, so by Proposition 1.1.4, $M^*$ is $G_{W(k)}$-split. \hfill $\square$

**Lemma 4.2.8.** Let $\mathcal{I}_0^h$ be a Newton stratum of $\mathcal{I}_0$. Then there exists a point $x \in \mathcal{I}_0^h(\mathbb{F}_p)$ such that $\mathcal{A}_x[p^\infty]$ is completely slope divisible.

**Proof.** The proof here is given by putting together [17] 4.3 and [15] Proposition 1.4.4. Let $x \in \mathcal{I}_0^h(\mathbb{F}_p)$ be a point and $M$ be the Dieudonné module of $\mathcal{A}_x[p^\infty]$. Fixing an isomorphism $t : W_{\mathbb{Z}_p} \otimes W(k) \to M$ respecting the Hodge-Tate tensors, we identify the Frobenius of $M$ with an element in $g \in G(K)$ where $K = W(\mathbb{F}_p)[1/p]$, and hence a $\sigma$-$K$-space structure on $V_K$ (see [17] 3, with $L = K$). Let $n$ be such that $\nu(g) \in Hom_K(G_{m,K}, G_K)$, and $E \subseteq K$ be the fixed field of $\sigma^n$.

We have two fiber functors from $\text{Rep}(G)$ to the category of finite dimensional $E$-vector spaces. Namely, $w_1((V', \rho)) = V'_E$, and $w_2((V', \rho)) = (V'_K)^{\rho(g)}$, where $(V'_K)^{\rho(g)}$ is defined at the end of [17] 3. $\text{Iso}(w_1, w_2)$ is a $G_E$-torsor, so there is a finite extension $E'$ of $E$ in $K$, such that $w_1$ and $w_2$ become isomorphic over $E'$. By replacing $n$ by $n[E' : E]$, we can assume that $w_1$ and $w_2$ are isomorphic over $E$. Moreover, the natural injection $(V'_K)^{\rho(g)} \to V'_K$ induces an isomorphism $(V'_K)^{\rho(g)} \otimes_E K \to V'_K$.

Let $\alpha : w_1 \to w_2$ be such an isomorphism over $E$, and $\alpha_K$ be its base-change to $K$. Let $\beta : w_2 \to w_1$ be the isomorphism given by $(V'_K)^{\rho(g)} \otimes_E K \to V'_K = V'_E \otimes_E K$. Then $\beta \circ \alpha_K$ gives an automorphism of $w_1$ over $K$, and hence an element $c \in G(K)$. Noting that $\alpha$ and $c$ determines each other uniquely, we could assume that $c$ is such that $c \sigma(c)^{-1} \in G_{Z_p}(W(\mathbb{F}_p))^\mu_2(p)G_{Z_p}(W(\mathbb{F}_p))$, by enlarging $E$ and changing $\alpha$ if necessary. Then by [16] 1.4.2 and Proposition 1.4.4, $c^{-1} \cdot V_{W(\mathbb{F}_p)}$ with Frobenius induced by $g$ is a Dieudonné module, and it comes from a point $c^{-1}x \in \mathcal{I}_0^h(\mathbb{F}_p)$. The slope filtration on $\mathbb{D}((G_{c^{-1}}[p^\infty])) = c \cdot V_{W(\mathbb{F}_p)}$ is induced by the cocharacter $c^{-1}\nu(g)$. The equality 4.3.3 of [17] means that

$$c \sigma(c)^{-1} \cdot \sigma(c \sigma(c)^{-1}) \cdot \sigma^2(c \sigma(c)^{-1}) \cdots \sigma^{n-1}(c \sigma(c)^{-1}) = c^{-1}\nu(g),$$

which implies that the Dieudonné module structure on $c^{-1} \cdot V_{W(\mathbb{F}_p)}$ is completely slope divisible. \hfill $\square$

Let $\mathcal{I}_0^h \subseteq \mathcal{I}_0(\mathbb{F}_p)$ be a Newton stratum, and $x \in \mathcal{I}_0^h(\mathbb{F}_p)$ be a point such that $\mathcal{A}_x[p^\infty]$ is completely slope divisible. Let $C_x \subseteq \mathcal{I}_0^h(\mathbb{F}_p)$ be the central leaf containing $x$. By Lemma [12] 4.2, $V_{W(\mathbb{F}_p)} := t^{-1}(M^*)$ is a $G_{W(\mathbb{F}_p)}$-split filtration of $V_{W(\mathbb{F}_p)}$. We write $P$ for $\text{stab}_{G_{W(\mathbb{F}_p)}}(V_{W(\mathbb{F}_p)}^*)$. Let $T = \text{Spec} R$ be a $C_x$-scheme which is smooth over $\mathbb{F}_p$. Let $\mathcal{R}$ be a lifting of $T$ which is smooth over $W(\mathbb{F}_p)$. Then by [32] Proposition 2.3,
\( \mathcal{A}_C[p^\infty] \) (and hence \( \mathcal{A}_T[p^\infty] \)) is completely slope divisible. In particular, \( \mathcal{A}_T[p^\infty] \), and hence \( \mathcal{M} := \mathbb{D}(\mathcal{A}_T)(\mathcal{R}) \) admits a slope filtration. We write \( \mathcal{M}^* \) for the slope filtration on \( \mathcal{M} \).

**Corollary 4.2.9.** The scheme \( I := \text{Isom}_R((V_{W(R)}^\vee, V_{W(R)}^\bullet, s), (\mathcal{M}, \mathcal{M}^*, s_{dR})) \) of isomorphisms respecting the filtrations as well as the Hodge-Tate tensors is a \( P \)-torsor over \( \mathcal{R} \).

**Proof.** There exists a morphism \( \mathcal{R} \to \mathcal{J} \) lifting \( R \to C_x \), so we have

\[
J := \text{Isom}_R((V_{W(R)}^\vee, V_{W(R)}^\bullet, s), (\mathcal{M}, \mathcal{M}^*, s_{dR})) = \text{Isom}_R((V_{W(R)}^\vee, s), (V_{W(R)}, s_{dR}))
\]

is a \( G_{W(R)} \)-torsor. By passing to a connected component of an fpf étale affine cover, we can assume that \( \mathcal{R} \) is an integral domain and that \( J(\mathcal{R}) \neq \emptyset \). Fixing a \( t \in J(\mathcal{R}) \), we could view \( \mathcal{M}^* \) as a filtration on \( V_{W(R)}^\vee \otimes \mathcal{R} \). Applying Lemma 4.2.7 to the generic point of \( T \), we see that \( \mathcal{M}^* \) is induced by a cocharacter of \( G \) which is geometrically conjugate to \( \nu \). Here \( \nu \) is a cocharacter (of \( G_{W(R)} \)) inducing \( V_{W(R)}^\vee \).

Let \( G_{W(R)}-\text{Fil} \) (resp. Fil) be the \( G_{W(R)} \)-split (resp. usual) filtrations on \( V_{R} \). By Proposition 1.1.5, \( G_{W(R)}-\text{Fil} \) is a proper smooth \( \mathcal{R} \)-scheme, and hence the morphism \( G_{W(R)}-\text{Fil} \to \text{Fil} \) is a closed immersion. The \( \mathcal{R} \)-point of Fil induced by \( \mathcal{M}^* \) is such that the generic point lies in \( G_{W(R)}-\text{Fil} \), so \( \mathcal{M}^* \) is \( G_{W(R)} \)-split.

Let \( \mathcal{P} \) be \( \text{stab}_{G_{W(R)}}(\mathcal{M}^*) \). It is a parabolic subgroup of \( G_{R} \) by Lemma 1.1.1. Then \( \text{trans}_{G_{W(R)}}(P_s, \mathcal{P}) \cong I_s \). But \( \text{trans}_{G_{W(R)}}(P_s, \mathcal{P}) \) is a \( P \)-torsor as \( P \) and \( \mathcal{P} \) are of the same type (because they are of the same type at all points in \( T \)), so \( I_s \) is a \( P \)-torsor.

**4.2.10. Igusa towers.** Let \( x \in \mathcal{J}_0^\bullet(R) \) be such that \( \mathcal{A}_C[p^\infty] \) is completely slope divisible and \( C_x \) be the central leaf containing \( x \). Let \( \mathcal{G} \) be \( \mathcal{A}_C[p^\infty] \). We write \( \mathcal{G}_x^* \) (resp. \( \mathcal{G}_x \)) for the slope filtration, and \( \mathcal{G}_x^i \) (resp. \( \mathcal{G}_x \)) for the central leaf containing \( x \).

Let \( M \) be the Dieudonné module of \( \mathcal{G}_x \), and \( \mathcal{M}^* \) be the slope filtration. We fix, once an for all in this subsection, an isomorphism \( t : V_{W(R)}^\vee \to M \) respecting the Hodge-Tate tensors and hence identify \( V_{W(R)}^\vee \) and \( M \).

The canonical slope decomposition \( \mathcal{G}_x^i \) gives a decomposition of Dieudonné modules \( M = \oplus M_i \), and hence a cocharacter \( \nu \) of \( G_{W(R)} \). We write \( P \) (resp. \( P' \)) for the stabilizer of \( M^* \) in \( G_{W(R)} \) (resp. \( \text{GL}(M) \)), \( U \) (resp. \( U' \)) for the unipotent radical, and \( L \) (resp. \( L' \)) for the centralizer of \( \nu \).

Let \( \text{Spec} \mathcal{R} \) be an affine scheme over \( C_x \). By smoothness of \( \mathcal{J}_W \), there is a morphism \( j : \text{Spec}(W(R)) \to \mathcal{J}_W \). We write \( W \) for \( j^*V \cong \mathbb{D}(\mathcal{G}_R)(W(R)) \), and \( W^* \) for the slope filtration induced by that on \( \mathbb{D}(\mathcal{G}_R)(W(R)) \).

For \( m \in \mathbb{Z}_+ \), let \( J'_m \) be the presheaf such that for a \( R \)-algebra \( A \), \( J'_m(A) \) is the set of \( W_m(A) \)-linear isomorphisms \( M \otimes W_m(A) \to W \otimes_{W(R)} W_m(A) \) mapping \( \mathcal{M}^*|_{W_m(A)} \) to \( W^*|_{W_m(A)} \). We define \( j_m \) to be the sub presheaf of \( J'_m \) such that the \( W_m(A) \)-linear isomorphisms \( M \otimes W_m(A) \to W \otimes_{W(R)} W_m(A) \) respect Hodge-Tate tensors.

**Lemma 4.2.11.**

1. Both \( j_m \) and \( J'_m \) are represented by smooth affine \( R \)-schemes. Moreover, \( j_m \) is a closed subscheme of \( J'_m \).

2. Both \( j_m \) and \( J'_m \) are independent of the choice of \( j \).

**Proof.** For (1), we can assume that \( j \) factors through an affine subscheme of \( \mathcal{J}_W \), whose \( p \)-adic completion is denoted by \( \mathcal{R} \). The homomorphism \( \mathcal{R} \to W(R) \) induced
by $j$ will still be denoted by $j$. We can choose a Frobenius on $R$ such that $j$ is compatible with Frobeni. Let $M$ be $\mathbb{D}(A[p^{\infty}])(R)$ and $M^\bullet$ be the slope filtration. Let $I'$ be $\text{Isom}_R((M, M^\bullet)_R, (M, M^{\bullet*}))$, and $I$ be $\text{Isom}_R((M, M^{\bullet*}, s)_R, (M, M^\bullet, s_{\text{an}}))$. Then $I'$ is a $P'$-torsor and by Corollary 4.2.9 $I$ is a $P$-torsor. Now $J'_m$ (resp $J_m$) is given by first applying the Greenberg functor to $I'$ to Spec $R$ (resp. $I$ to Spec $R$), and then pulling back via Spec $R \rightarrow W_m(R)$ given by $R \xrightarrow{j} W(R) \rightarrow W_m(R)$. In particular, (1) holds.

For (2), let $j_1$ and $j_2$ be two homorphisms $R \rightarrow W(R)$. Then the canonical isomorphism $i_{12}: j_1^*V \xrightarrow{\sim} (A_R)(W(R)) \rightarrow j_2^*V$ respects the Hodge-Tate tensors as well as the slope filtrations. This gives an isomorphism between the torsors induced by $j_1$ and $j_2$. It also satisfies the cocycle condition. \qed

**Definition 4.2.12.** The Igusa tower $J_{b,m}$ is the sheaf over $C_x$ which attaches to a $C_x$-scheme $T$ the set of isomorphisms $t : \oplus G_x^i[p^m] \rightarrow \oplus G_x^j[p^m]$, such that

1. $t$ extends étale locally to any $m' \geq m$;
2. for any affine scheme Spec $R$ over $T$, the element in $J'_m/U_m(R)$ induced by $t_R$ lies in $J_m/U_m(R)$.

Let $\Gamma_b$ be $L(W) \cap \text{Aut}(G_x)$ (here the intersection is via the Dieudonné functor which is an equivalence), and $\Gamma_{b,m}$ is the quotient of $\Gamma_b$ by the subgroup which is identity on $G_x[p^m]$. By Corollary 4.2.11 there is an integer $N$, such that for any $y \in C_x(\overline{\mathbb{Q}})$ and any $m \geq N$, $J_{b,m,y}(\overline{\mathbb{Q}})$ is a $\Gamma_{b,m}$-torsor. Here we use the canonical identifications $G_x = \oplus G_x^i[p^m]$ and $G_y = \oplus G_y^i[p^m]$.

**Proposition 4.2.13.** $J_{b,m}$ is representable. Moreover, for $m \geq N$, $J_{b,m} \rightarrow C_x$ is finite étale and Galois, with Galois group $\Gamma_{b,m}$.

**Proof.** As in [21] section 4, the sheaf $J'_{b,m}/C_x$ such that $J'_{b,m}(T)$ is the set of isomorphisms $t : \oplus G_x^i[p^m] \rightarrow \oplus G_x^j[p^m]$ which extends étale locally to any $m' \geq m$ is representable. For any open affine subscheme $T = \text{Spec } R$ of $J'_{b,m}$, the pull back to $T$ of the universal isomorphism on $J'_{b,m}$ gives a morphism $T \rightarrow J'_m/U_m$. By Lemma 4.2.11 (1), to make it factor through $J_m/U_m$ gives a closed condition, and by Lemma 4.2.11 (2), the closed subschemes obtained by using different $T$s glue to a closed subscheme of $J'_m$. This is precisely $J_{b,m}$.

For the rest statements, as in [21] Proposition 4, it suffices to prove that for each closed point $y \in C_x$, there is an isomorphism $t : \oplus G_x^i \otimes_{\overline{\mathbb{Q}}} R \rightarrow \oplus G_y^i \otimes_{\overline{\mathbb{Q}}} R$ whose induced map $R \rightarrow I'/U'$ factors through $I/U$. Here we write $C$ for $C_x$, and $R$ for $O_{\overline{\mathbb{Q}}}$ for simplicity, and $\mathcal{R}$, $I$ and $I'$ are as in the previous lemma. By [22] Lemma 3.4, there is an isomorphism $t_0 : \oplus G_x^i \otimes_{\overline{\mathbb{Q}}} R \rightarrow \oplus G_y^i \otimes_{\overline{\mathbb{Q}}} R$.

Let $K$ be a algebraically closed field containing $R$. Then there is an isomorphism $t_1 : G_x \otimes_{\overline{\mathbb{Q}}} K \rightarrow G_y \otimes_{\overline{\mathbb{Q}}} K$ respecting both the slope filtrations and the Hodge-Tate tensors. Its induced map $\oplus G_x^i \otimes_{\overline{\mathbb{Q}}} K \rightarrow \oplus G_y^i \otimes_{\overline{\mathbb{Q}}} K$ will still be denoted by $t_1$. Then $g := t_1^{-1} \circ t_{0,K}$ is a automorphism of $\oplus G_x^i \otimes_{\overline{\mathbb{Q}}} K$, and hence it is defined over $\overline{\mathbb{Q}}$ and still denoted by $g$. Now $t_0 \circ g^{-1}$ is precisely what we need. \qed

**From now on, we always assume** $m \geq N$ when working with $J_{b,m}$s. For $m' \geq m$, there is a natural projection $g : J_{b,m'} \rightarrow J_{b,m}$ induced by restricting to the $p^m$-torsion. This morphism is finite étale. Let $J_b = \lim_{\rightarrow m} J_{b,m}$, it is equipped with the action of $\Gamma_b$. Let $T_b$ be the group of self quasi-isogenies of $G_x$ respecting the the
tensors. We will show, as in [21], that the $\Gamma_b$-action on $J_b$ extends to a sub-monoid $S_b \subseteq T_b$.

For $\rho \in T_b$, we write $\rho = \oplus_i \rho^i$ for the decomposition to isoclinic factors. If $\rho^{-1}$ is an isogeny, we write $e_i(\rho) \geq f_i(\rho)$ respectively for the minimal and maximal integer such that $\ker(\rho^{e_i}) \subseteq \ker(\rho^{f_i}) \subseteq \ker(\rho^{\infty})$. We define

$$S_b = \{ \rho \in T_b | \rho^{-1} \text{ is an isogeny, } f_{i-1}(\rho) \geq e_i(\rho), \forall \ i \geq 2 \}.$$ 

It is not hard to see $S_b$ is a monoid, and that $p^{-1}$ and $fr^{-B} := \oplus_i p^{-\lambda_i B}$ are in $S_b$. Moreover, the proof of [22] Lemma 2.11 also works here, and hence $T_b = \langle S_b, p, fr^B \rangle$.

**Proposition 4.2.14.** Let $m, \rho$ be as before, and $e = e_1(\rho)$. There is a unique finite flat group scheme $\mathcal{H} \subseteq \mathcal{G}_{b,m}[p^r]$, such that the corresponding subgroups in $\mathcal{G}_{b,m}$ are $t(\ker(\rho^{-1}))$. The abelian scheme $\mathcal{A}/\mathcal{H}$, together with the polarization and level structure, induces a morphism $\rho_1 : J_{b,m} \rightarrow \mathfrak{A}$ which factors through $C_x$. Moreover, it induces a morphism $\rho : J_{b,m} \rightarrow J_{b,m-e}$ of Igusa towers.

**Proof.** The existence of $\mathcal{H}$ is proved in [22] Lemma 3.6. For $z \in J_{b,m}(\mathfrak{F})$, it is a pair $(y, j)$ with $y \in C_x(\mathfrak{F})$ and $j : \mathcal{G}_x[p^m] \rightarrow \mathcal{G}_y[p^m]$ an isomorphism whose induced map on Dieudonné modules respects the Hodge-Tate tensors. Here we use the canonical identifications $\mathcal{G}_x = \oplus_i \mathcal{G}_x^{i}[p^m]$ and $\mathcal{G}_y = \oplus_i \mathcal{G}_y^{i}[p^m]$. By our assumption, $j$ lifts to an isomorphism $j' : \mathcal{G}_x \rightarrow \mathcal{G}_y$ whose induced map on Dieudonné modules respects Hodge-Tate tensors. The isogeny $\rho^{-1} : \mathcal{G}_x \rightarrow \mathcal{G}_x$ gives a element $g \in L(K)$ such that $g^{-1}b\sigma(g) \in G_W(W)\mu^\infty G_W(W)$. It gives, via $j'$, a point $gy \in \mathcal{S}(\mathfrak{F})$ by [10]. The isomorphism $\mathcal{G}_x \rightarrow \mathcal{G}_{gy}$ which is the push out of $j'$ via $\rho^{-1}$ is an isomorphism respecting Hodge-Tate tensors. So $gy \in C_x(\mathfrak{F})$. By the proof of Proposition 4.2.5, for each open affine subscheme Spec($R$) $\subseteq J_{b,m}$, the composition $Spec(R) \rightarrow J_{b,m} \xrightarrow{\rho_1} \mathfrak{A}$ factors through $\mathcal{A}_0$. This factorization is necessarily unique by [16] Proposition 1.4.9, and hence glue to a morphism $i : J_{b,m} \rightarrow \mathcal{A}_0$ which necessarily factors through $C_x$.

Let $H$ be $\ker(\rho^{-1})$. Using the identification $\mathcal{G}_{J_{b,m}}/\mathcal{H} \cong \iota_* \mathfrak{G}$, the isomorphism

$$\oplus_i \mathcal{G}_x^{i}[p^{m-e}] \xrightarrow{\rho^{-1}} \oplus_i (\mathcal{G}_x^{i}/H^i)[p^{m-e}] \xrightarrow{i} \oplus_i (\mathcal{G}_y^{i}/H^i)[p^{m-e}] \cong \oplus_i \mathcal{G}_y^{i}[p^{m-e}]$$

gives the morphism $J_{b,m} \rightarrow J_{b,m-e}$. \hfill \Box

**Remark 4.2.15.** As in [21], by the same proof above, we can also define $\sigma$-semi-liner action of $F : \mathcal{G}_x \rightarrow \mathcal{G}_x^{(p)}$ on Igusa towers. More precisely, $Frob : J_{b,m} \rightarrow J_{b,m-1}$ is induced by the abelian scheme $\mathcal{A}^{(p)} = \mathcal{A}/\mathcal{A}[F]$. Here by $\sigma$-semi-liner, we mean the following diagram

$$\begin{array}{ccc}
J_{b,m} & \xrightarrow{\text{Frob}} & J_{b,m-1} \\
\downarrow & & \downarrow \\
C_x & \xrightarrow{\sigma} & C_x
\end{array}$$

is commutative. Moreover, $Frob = q \circ \sigma_{J_{b,m}}$. Here we write $\sigma$ for the absolute Frobenius.

4.3. Foliations.
4.3.1. Rapoport-Zink formal schemes of Hodge type. Rapoport-Zink formal schemes of Hodge type are first defined and constructed by Wansu Kim in [13]. Howard and Pappas give in [12] a more direct construction relying on the existence of the integral model. We will follow [12] in this paper.

Let’s fix some notations as in [12] 2.1.1. We write $\text{Nil}_W$ for the category of $W$-schemes $S$ such that $p$ is Zariski locally nilpotent in $O_S$. We write $\text{ANil}_W \subseteq \text{Nil}_W^{op}$ for the full subcategory of Noetherian $W$-algebras in which $p$ is nilpotent, and $\text{ANil}_W^{fr}$ for the category of Noetherian adic $W$-algebras in which $p$ is (topologically) nilpotent, and embed $\text{ANil}_W \subseteq \text{ANil}_W^{fr}$ as a full subcategory by endowing any $W$-algebra in $\text{ANil}_W$ with its $p$-adic topology. We say that an adic $W$-algebra $A$ is formally finitely generated if $A$ is Noetherian, and if $A/I$ is a finitely generated $W$-algebra for some ideal of definition $I \subseteq A$. Thus $\text{Spf}(A)$ is a formal scheme which is formally of finite type over $\text{Spf}(W)$. If, in addition, $p$ is nilpotent in $A$, then $A$ is a quotient of $W/(p^n)[[x_1, \cdots, x_r]][y_1, \cdots, y_s]$ for some $n, r,$ and $s$. We will denote by $\text{ANil}_W^{fr} \subseteq \text{ANil}_W^{fr}$ the full subcategory whose objects are $W$-algebras that are formally finitely generated and formally smooth over $W/(p^n)$, for some $n \geq 1$.

We start with classical Rapoport-Zink spaces. Let $X_0$ be a $p$-divisible group over $k = \mathbb{F}_p$. The Rapoport-Zink space $\mathcal{RZ}(X_0)$ of deformations of $X_0$ up to quasi-isogeny is the functor assigning to each scheme $S$ in $\text{Nil}_W$ the set of isomorphism classes of pairs $(X, \rho)$, where $X$ is a $p$-divisible group over $S$, and $\rho : X \times_k S \rightarrow X \times_S S$ is a quasi-isogeny. Here $S := S \times_k W$. As in [35], $\mathcal{RZ}(X_0)$ is represented by a formal scheme $\mathcal{RZ}(X_0)$ over $\text{Spf}(W)$ that is formally smooth and locally formally of finite type over $W$. If $(X_0, \lambda_0)$ is a principal polarized $p$-divisible group, one can also define $\mathcal{RZ}(X_0, \lambda_0)$ (see [12] 2.3.1 or [36]). It is represented by a closed sub formal scheme of $\mathcal{RZ}(X_0)$, denoted by $\mathcal{RZ}(X_0, \lambda_0)$, which is again formally smooth and locally formally of finite type over $W$.

Now we consider the morphism $\mathcal{A}_W \rightarrow \mathcal{A}_W$ as before, which is induced by an embedding of Shimura data $(G, X) \rightarrow (\text{GSp}(V, \psi), X')$ which are both of good reduction at $p$. For $x \in \mathcal{A}_W(W)$, we write $X_0$ for $X_0 \otimes \mathbb{F}_p$. The $G_1$-torsor $\mathcal{T}_x$ is trivial, and its sections lifts to elements in $\mathcal{U}(V)$. We fix such a lifting as before, and use it to translate the Dieudonné module structure on $\mathbb{D}(X_0)(W)$ to $V$. Let $V_W = F^1 \oplus F^0$ be the splitting on $V_0$ induced by $\mu$, then under the above identification, $F^1$ gives the Hodge filtration on $V_W$. Moreover, as at the beginning of this section, the Frobenius of $X_0$ gives an element $b \in B(G_1, \mu^\sigma)$.

**Definition 4.3.2.** We define the functor $\mathcal{RZ}_{\text{nil}}^i_{G_W} : \text{ANil}_W \rightarrow \text{Sets}$ as follows. For any $R \in \text{ANil}_W$, $\mathcal{RZ}_{\text{nil}}^i_{G_W}(R)$ is the set of isomorphism classes of triples $(X, \rho, t)$, with $(X, \rho) \in \mathcal{RZ}(X_0)(R)$, and $t : 1 \rightarrow \mathbb{D}(X)^\otimes$ be such that $t(1/p)$ is Frobenius equivariant, satisfying the following conditions.

1. For some nilpotent ideal $J \subseteq R$ with $p \in J$, the pull-back of $t$ to $\text{Spec}(R/J)$ is identified with $s$ under the isomorphism of isocrystals

\[
\mathbb{D}(\rho) : \mathbb{D}(X_{R/J})^\otimes[1/p] \rightarrow \mathbb{D}(X_0 \times_{\mathbb{F}_p} R/J)^\otimes[1/p]
\]

induced by the quasi-isogeny $\rho$.

2. $\text{Isom}(\mathbb{D}(X)(t), (V_W, s)_R)$, the sheaf of isomorphisms of crystals on $\text{Spec}(R)$ respecting the tensors, is a crystal of $G_W$-torsors over the (big fppf) crystalline site $\text{CRIS}(\text{Spec}(R)/W)$.

3. $\text{Isom}_R(\mathbb{D}(X)(t), (\text{Fil}_1, t), (V_W, F^1, s)_R)$ is a $P_+\text{-torsor}$. Here $\text{Fil}_1 \subseteq \mathbb{D}(X)(R)$ is the Hodge filtration.
An isomorphism $(X, \rho, t) \to (X', \rho', t')$ is an isomorphism of $p$-divisible groups $X \to X'$ compatible with additional structures in the obvious way.

**Definition 4.3.3.** The functor $\mathcal{RZ}_{G_W}^{\text{fsm}}$ on $\text{ANil}_{W}^{\text{fsm}}$ is defined by setting

$$
\mathcal{RZ}_{G_W}^{\text{fsm}}(A) = \varprojlim_n \mathcal{RZ}_{G_W}^{\text{nil}}(A/I^n),
$$

where $I$ is an ideal of definition of $A$.

We have the following results by [12] or [13] (see [12] Theorem 3.2.1).

**Theorem 4.3.4.** The functor $\mathcal{RZ}_{G_W}^{\text{fsm}}$ is represented by a closed sub formal scheme $\mathcal{RZ}_{G_W}(X_0)$ of $\mathcal{RZ}(X_0)$ which is formally smooth and formally locally of finite type over $\text{Spf}(W')$.

Now we briefly discuss the construction of $\mathcal{RZ}_{G_W}(X_0)$ in [12]. Let $i : \mathcal{A}_W^F \to \mathcal{A}_W^{F,s}$ be the (finite) morphism of smooth formal schemes obtained by $p$-adic completions. Let $\pi : X \to \mathcal{A}_W^F$ be the formal $p$-divisible obtained from $\mathcal{A}$. We use the same notation for the pull back to $\mathcal{A}_W^F$ of $\pi$. By [15], we have an $O_{\mathcal{A}_W^F}$-morphism of crystals $s_{\text{cris}} : 1 \to \mathcal{D}(X)^\otimes$ which gives $s$ when pulling back to $x$. Moreover, $s_{\text{cris}}[1/p]$ is Frobenius equivariant.

Let $\mathcal{RZ}(X_0, \lambda_0)$ be the symplectic Rapoport-Zink space attached to $x$, viewed as a point of $\mathcal{A}_W$, and $\Theta_G : \mathcal{RZ}_{G_W}^0(X_0) \to \mathcal{A}_W$ be the pull back via $i$ of the natural map $\Theta : \mathcal{RZ}(X_0, \lambda_0) \to \mathcal{A}_W$. Then $i' : \mathcal{RZ}_{G_W}^0(X_0) \to \mathcal{RZ}(X_0, \lambda_0)$ is a finite morphism of formal schemes and $\mathcal{RZ}_{G_W}^0(X_0)$ is formally smooth and locally formally of finite type ([12] Proposition 3.2.5). Let $\mathcal{RZ}_{G_W}(X_0)$ be presheaf on $\text{Nil}_W$ which attaches to $S \in \text{Nil}_W$ the set of elements $(X, \rho) \in \mathcal{RZ}_{G_W}^0(X_0)(S)$ such that for any field extension $k'/k$ and any $y \in S(k')$, the isomorphism

$$
\mathcal{D}(\rho) : \mathcal{D}(X_y)^{\otimes}(W')[1/p] \to V_{W'}[1/p]
$$

maps $y^*(s_{\text{cris}})(W')$ to $s$. Here $W'$ is the Cohen ring of $k'$. By [12] Proposition 3.2.7 $\mathcal{RZ}_{G_W}(X_0)$ is represented by smooth formal scheme which is open and closed in $\mathcal{RZ}_{G_W}^0(X_0)$, and by [12] Proposition 3.2.9, $\mathcal{RZ}_{G_W}(X_0)$ represents $\mathcal{RZ}_{G_W}^{\text{fsm}}$. The composition $\mathcal{RZ}_{G_W}(X_0) \to \mathcal{RZ}_{G_W}^0(X_0) \hookrightarrow \mathcal{RZ}(X_0, \lambda_0)$ is actually a closed immersion ([12] Proposition 3.2.11).

4.3.5. *The almost product structure of Newton strata.* Now we assume, in addition, that $x \in \mathcal{A}_0(\mathcal{F})$ is such that $X_0$ is completely slope divisible. Let $\mathcal{G}/C_x$ be as as in the previous subsection and $\mathcal{G}_i$ be its the slope filtration. Then by [21] Lemma 8, for any pair of integers $d \geq 0$ and $r \geq d/\delta f$, there exists a canonical isomorphism $\alpha : \mathcal{G}(q^r)[p^d] \cong \oplus_i \mathcal{G}_i(q^r)[p^d]$. Here $q = |\iota| = p^f$, and $\delta = \min_{i=1, \ldots, k-1}(\lambda_i - \lambda_{i+1})$ with $\lambda_i$ the slope of $\mathcal{G}_i$. As in [21] Lemma 8, $\alpha$ is compatible with additional structures, but in the following sense.

**Lemma 4.3.6.** Let $\text{Spec} \, R/C_x$ be an affine scheme and $\text{Spec} \, W_d(R) \to \mathcal{A}_W$ be a lifting. Let $N'$ be $\mathcal{D}(\mathcal{G}(q^r))(W(R))$, and $N'$ be the slope filtration. We write $-\delta$ for the reduction to $W_d$. Let $M = \oplus_i M_i$ be as at the beginning of the previous subsection, and $I = \text{Isom}_{W_d(R)}((M, M', s), (N, N', s_{\text{cris}}))$. Then $\alpha$ induces a section of the natural projection $I \to I/U$ via the identification $M = \oplus_i M_i$.

**Proof.** It suffices to assume that $\text{Spec} \, R$ is open affine in $C_x$. By passing to an étale algebra, we can assume that the torus constructed in Corollary 4.2.9 is trivial. A
section $t$ there gives a $t_d \in I(W_d(R))$. To prove the lemma, we only need to check that the composition

$$M_d = \oplus_i M_{i,d} \xrightarrow{t_d} \oplus_i N_{i,d} \xrightarrow{\alpha} N_d \xrightarrow{t_d^{-1}} M_d$$

lies in $G_W(W_d(R))$, where $N_i = D(G^{i,(q^r)})(W(R))$, and $t_d : \oplus_i M_{i,d} \rightarrow \oplus_i N_{i,d}$ is the composition $\oplus_i M_{i,d} = \oplus_i M_d/M_d^{-1} \rightarrow \oplus_i N_d/N_d^{-1} = \oplus_i N_i$.

By the construction of $\alpha$ as in [22] Lemma 4.1, we see that the composition above is base-change to $W_d(R)$ of

$$M = \oplus_i M_i \xrightarrow{k} \oplus_i N_i \xrightarrow{\alpha'} N \xrightarrow{t^{-1}} M$$

which is of the form $p^\nu \varphi$ for some pro-cocharacter $\nu$ of $G_W$. The construction implies that it is an isomorphism and hence respects Hodge-Tate tensors. \qed

Let $RZ_{G_W}^{n,d}(X_0) \subseteq RZ_{G_W}(X_0)$ be the closed sub formal scheme classifying quasi-isogenies $\rho : X_0 \times \overline{S} \rightarrow X \times_S \overline{S}$ such that $p^n \rho$ and $p^{d-n} \rho^{-1}$ are both isogenies (or equivalently, $p^n \rho$ is an isogeny whose kernel is killed by $p^d$). It is the pull back via $RZ_{G_W}(X_0) \hookrightarrow RZ(X_0)$ of $RZ_{G_W}^{n,d}(X_0)$. The universal isomorphism $\rho$.

Let $RZ_{G_W}^{n,d}(X_0)$s for a direct system as $n, d$ vary, and the direct limit is $RZ_{G_W}(X_0)$. For $\rho \in T_b$, action by $\rho$ induces a morphism $RZ_{G_W}^{n,d}(X_0) \rightarrow RZ_{G_W}^{n+n(\rho),d+d(\rho)}(X_0)$, with $n(\rho)$ resp. $d(\rho)$ the smallest integer such that $p^n(\rho) \rho$ and $p^{d(\rho)-n(\rho)} \rho$ are isogenies. Similarly, the $\sigma$-semi-linear action of $F : X_0 \rightarrow X_0^{(p)}$, $Frob : RZ_{G_W}(X_0) \rightarrow RZ_{G_W}(X_0)$, given by $(H, \rho) \rightarrow (H, \rho \circ F^{-1})$, induces a semi-linear morphism $RZ_{G_\mathcal{W}}^{n,d}(X_0) \rightarrow RZ_{G_W}^{n+1,d+1}(X_0)$.

Let $RZ_{G_W}^{n,d,\ast}(X_0)$ be the reduced fiber of $RZ_{G_W}^{n,d,\ast}(X_0)$. To simplify notations, we write $\mathcal{M}$ for $RZ_{G_W}(X_0)$ and $\overline{\mathcal{M}}^{n,d}$ for $RZ_{G_W}^{n,d,\ast}(X_0)$. For $m \geq d$, we define a morphism $\pi : J_{b,m} \times \overline{\mathcal{M}}^{n,d} \rightarrow \mathcal{X}_b$ as follows. Let $t : \oplus_i X_0^{[p^n]} \rightarrow \oplus_i \mathcal{G}_{j_{b,m}}^{i,(q^r)}[p^n]$ be the universal isomorphism. By the previous lemma, the isomorphism

$$X_0^{[q^r]}[p^n] = \oplus_i X_0^{[q^r]}[p^n] \rightarrow \oplus_i \mathcal{G}_{j_{b,m}}^{i,(q^r)}[p^n] \xrightarrow{\alpha^{-1}} \mathcal{G}_{j_{b,m}}^{i,(q^r)}[p^n]$$

is compatible with additional structures. We still write $t$ for its restriction to $p^d$-kernels. Let $(X, \rho, s)$ be the universal object on $\overline{\mathcal{M}}^{n,d}$, then $p^n \rho$ is an isogeny with $\ker(p^n \rho) \subseteq X_0[p^d]$. So $\ker(p^n \rho(p^r)) \subseteq X_0^{[p^r]}[p^d]$. The polarized abelian scheme $p^r A/p^r t(p^r \ker(p^n \rho(p^r)))$ (with level structure) gives $\pi$. Here $A$ is the abelian scheme on $J_{b,m}$, and $p_t$ is the projection of $J_{b,m} \times \overline{\mathcal{M}}^{n,d}$ to the $i$-th factor.

**Lemma 4.3.7.** The morphism $\pi$ is finite.

**Proof.** It follows essentially from the Siegel cases. More precisely, let $C_x^{G_{Sp}}$ be the central leaf crossing (the image of $\chi$ in $\mathcal{X}_b$, and $\mathfrak{p}$ be the Newton polygon of $b$. Let $C_x^{G_{Sp}}(b_{m,n})$ be the Igusa cover of $C_x^{G_{Sp}}$. We have a commutative diagram

$$\begin{array}{ccc}
J_{b,m} & \xrightarrow{\pi} & J_{b,m}^{G_{Sp}} \\
| & & | \\
C_x & \xrightarrow{\pi} & C_x^{G_{Sp}}
\end{array}$$

|
induced by the universal isomorphism on \( J_{b,m} \). We also have a closed embedding \( \overline{M}^n,d \hookrightarrow \overline{M}^n,d_{\GSp} \) such that the universal quasi-isogeny on \( \overline{M}^n,d \) is the pull back of the one on \( \overline{M}^n,d_{\GSp} \). Here \( \overline{M}^n,d_{\GSp} \) is \( RZ(X_0,\lambda_0)^{n,d} \). By doing the construction of \( \pi \) to \( J_{b,m}^{\GSp} \times \overline{M}^n,d_{\GSp} \), we get \( \pi' : J_{b,m}^{\GSp} \times \overline{M}^n,d_{\GSp} \to \mathcal{A}^b \), such that the composition

\[
J_{b,m} \times \overline{M}^n,d \quad \overset{i}{\longrightarrow} \quad J_{b,m}^{\GSp} \times \overline{M}^n,d_{\GSp} \quad \overset{\pi'}{\longrightarrow} \quad \mathcal{A}^b
\]

is \( \pi \). But \( \pi' \) is finite By [21] Proposition 10, and \( i \) is finite by construction, so \( \pi \) is finite.

Note that \( \overline{M}^n,d \) could be singular, but each of its connected components has an open dense smooth locus, we have the following “weak foliation”.

**Proposition 4.3.8.** Let \( U \) be the smooth locus of an irreducible component of \( \overline{M}^n,d \). Then the morphism \( \pi|_{J_{b,m} \times U} \) factors through \( \mathcal{A}^b \).

**Proof.** As in the first half of the proof of Proposition 4.2.14, each \( \pi \)-point of \( J_{b,m} \times U \) factors through \( \mathcal{A}^b \). But by the proof of Proposition 4.2.5, the statement follows from [16] Proposition 1.4.9.

The above result is weak, but it is enough to compute the dimension of Newton strata.

**Corollary 4.3.9.** The Newton stratum \( \mathcal{A}^b \) is of dimension \( \langle \rho, \mu + \nu(b) \rangle - \frac{1}{2} \dim(b) \).

**Proof.** Notations as above, we have \( \pi|_{J_{b,m} \times U} \) factors through \( \mathcal{A}^b \), and it is quasi-finite by Lemma 4.3.7. So we have \( \dim(\mathcal{A}^b) \geq \dim(J_{b,m}) + \dim(\overline{M}^n,d) \), for all \( m, d, n, r \). But \( \pi \) induces a finite surjection \( J_{b,m}(\overline{\pi}) \times \overline{M}^n,d(\overline{\pi}) \to \mathcal{A}^b(\overline{\pi}) \) when restricting to \( \overline{\pi} \)-points for \( d \) big enough. So we have \( \dim(\mathcal{A}^b) = \dim(C_x) + \dim(\overline{M}) \). By [48] Corollary 3.13, \( \dim(\overline{M}) = \langle \rho, \mu - \nu(b) \rangle - \frac{1}{2} \dim(b) \), and by Theorem 4.2.5 \( \dim(C_x) = (2\rho, \nu(b)) \). One deduces the formula immediately.

If one is willing to use a bigger \( r \), one has the following “strong foliation”

**Proposition 4.3.10.** For \( r \) big enough, the morphism \( \pi \) factors through \( \mathcal{A}^b \), and induces a finite morphism to it. Moreover, notations as before, we have the followings.

1. \( \pi = (f \circ f^\dagger \times 1) \circ \pi \);
2. \( \pi \circ q = \pi \), with \( q : J_{b,m} \to J_{b,m'} \) the natural projection;
3. \( \pi \circ i = \pi \), with \( i : \overline{M}^n,d' \to \overline{M}^n,d \) the natural immersion;
4. \( \pi \circ (\rho \times \rho) = \pi \), for \( \rho \in S_b \), \( m \geq d + d(\rho) + e(\rho) \), and \( r \geq (d + d(\rho))/\delta f \);
5. \( \pi \circ (\text{Forb}^f \times \text{Frob}^f) = (1 \times \sigma(f))\pi \), for \( m \geq d + 1 \), and \( r \geq (d + 1)/\delta f \).

**Proof.** As in the proof of [21] Proposition 9, (2) and (3) are direct consequences of the construction. Let \( U = \text{Spf}(A, I) \) be an open affine connected sub formal scheme of \( M_0 := M \times \text{Spec} \overline{\pi} \), with \( I = \sqrt{I} \). By [12] Remark 2.3.5 (c), the universal family on \( \text{Spf}(A, I) \) gives family on \( \text{Spec} A \). Let \( \rho \) be the quasi-isogeny on \( \text{Spec} A \), and \( n' \geq n, d' \geq d \) be such that \( p^{n'} \rho \) is an isogeny whose kernel is killed by \( p^{d'} \). Then for \( r \geq d'/\delta f \) and \( m \geq d', n' \geq \max\{d_i\} \), it factors through \( \mathcal{A} \).

Now for \( \overline{M}^n,d \), we can take finitely many \( U_i \)'s as above such that \( U_i \cap \overline{M}^n,d \) form an open cover of \( \overline{M}^n,d \). For \( m, n', d, r \) with \( m \geq d', d \geq \max\{d_i\}, n' \geq \max\{n_i\} \), form

\[
\pi_{m, n', d, r} : \overline{M}^n,d_{\GSp} \longrightarrow \mathcal{A}^b
\]
and $r \geq d'/\delta f$ such that $p^{d'} \rho$ is an isogeny whose kernel is killed by $p^{d'}$, each attached morphism $\pi_i : J_{b,m} \times \text{Spec} A_i \to \mathcal{A}_i$ factors through $\mathcal{A}_i$. In particular, they induce a morphism $\pi' : J_{b,m} \times (\bigcup_i U_i) \to \mathcal{A}_i$. By (3), we have

$$\pi = \pi'|_{J_{b,m} \times (\bigcup_i (\mathcal{M}^{m,r}_{d'}(U_i)))}$$

here $\pi$ is the morphism attached to $m, n, d, r$, with $m, r$ as at the beginning of this paragraph. The finiteness follows from Lemma 4.3.7. The equalities (1), (4) and (5) follow as in [21].

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