On the Rate of Convergence to a Gamma Distribution on Wiener Space

E. Azmoodeh∗, P. Eichelsbacher† and L. Knichel‡

June 12, 2018

Abstract

In [NP09a], Nourdin and Peccati established a neat characterization of Gamma approximation on a fixed Wiener chaos in terms of convergence of only the third and fourth cumulants. In this paper, we investigate the rate of convergence in Gamma approximation on Wiener chaos in terms of the iterated Gamma operators of Malliavin Calculus. On the second Wiener chaos, our upper bound can be further extended to an exact rate of convergence in a suitable probability metric $d_2$ in terms of the maximum of the third and fourth cumulants, analogous to that of normal approximation in [NP15] under one extra mild condition. We end the paper with some novel Gamma characterization within the second Wiener chaos as well as Gamma approximation in Kolmogorov distance relying on the classical Berry–Esseen type inequality.

Keywords: Gamma approximation, Wiener chaos, Cumulants/Moments, Weak convergence, Malliavin Calculus, Berry–Esseen bounds, Stein’s method, Wasserstein distances

MSC 2010: 60F05, 60G50, 60H07

1 Introduction

Given a separable Hilbert space $\mathcal{H}$, we consider an isonormal Gaussian process $X = \{X(h) : h \in \mathcal{H}\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Our object of interest is a sequence $(F_n)_{n \geq 1}$ living inside a fixed Wiener chaos of order $q$ with fixed variance, e.g. $\mathbb{E}[F_n^2] = 1$. In recent years, these objects have been studied extensively, with one of the most famous results being the so-called fourth moment theorem, which first appeared in [NP05]. It states that

∗Ruhr University Bochum, Faculty of Mathematics, N-Süd EG/07, 44780 Bochum, Germany. E-mail: ehsan.azmoodeh@rub.de
†Ruhr University Bochum, Faculty of Mathematics, NA 3/66, 44780 Bochum, Germany. E-mail: peter.eichelsbacher@rub.de
‡Ruhr University Bochum, Faculty of Mathematics, N-Süd EG/07, 44780 Bochum, Germany. E-mail: lukas.knichel@rub.de. Lukas Knichel has been supported by the German Research Foundation (DFG) via Research Training Group RTG 2131 High dimensional phenomena in probability – fluctuations and discontinuity
\(F_n \overset{D}{\to} N\), where \(N \sim \mathcal{N}(0, 1)\) is a standard normal random variable, if and only if \(\mathbb{E}[F_n^4] \to 3\). In 2009, the authors of \([\text{NP09b}]\) proved a quantitative version of the fourth moment theorem combining Stein’s method for normal approximation with Malliavin calculus on the Wiener space. In this paper, they provide explicit bounds for the total variation distance between \(F_n\) and \(N\) in terms of the fourth cumulant of \(F_n\), namely

\[
d_{TV}(F_n, N) \leq 2\sqrt{\frac{q - 1}{3q} \kappa_4(F_n)}.
\]

Recall that for two random variables \(X\) and \(Y\), the total variation distance is

\[
d_{TV}(X, Y) := \sup_{A \in \mathcal{B}(\mathbb{R})} |P(X \in A) - P(Y \in A)|,
\]

where \(\mathcal{B}(\mathbb{R})\) is the set of all bounded Borel sets. In \([\text{NP15}]\), the optimal rate of convergence in the fourth moment theorem has been found. More precisely, if \(F_n\) converges in law to \(N\), then there exist constants \(0 < c < C\) (not depending on \(n\)), such that

\[
c \times \max\{\kappa_3(F_n), \kappa_4(F_n)\} \leq d_{TV}(F_n, N) \leq C \times \max\{\kappa_3(F_n), \kappa_4(F_n)\}.
\]

Note that the square root from the previous results has been removed and that the third cumulant comes into play.

Limit theorems for a Gamma target distribution have been considered e.g. in \([\text{NP09a}, \text{NP09b}]\) and \([\text{NPR10}]\). We consider the target \(G(\nu) \sim \text{CenteredGamma}(\nu)\). This means that \(G(\nu) = 2\hat{G}(\nu/2) - \nu\), where \(\hat{G}(\nu/2)\) is a standard Gamma random variable with density \(\hat{g}(x) = x^{\nu/2 - 1} e^{-x} \Gamma(\nu/2) x^{-1} \mathbf{1}_{(0,\infty)}(x)\). From now on, we still assume \(F\) to be inside a fixed Wiener Chaos of order \(q \geq 2\), but fix our variance to be \(\text{Var}(F) = \text{Var}(G(\nu)) = 2\nu\).

In \([\text{DPT18}]\), the authors used a Stein equation suitable for proving Stein-Malliavin upper bounds in 1-Wasserstein distance for the convergence of \(F_n\) to \(G(\nu)\). In Theorem 1.7, they showed that

\[
d_1(F, G(\nu)) \leq \max\left(1, \frac{2}{\nu}\right) \mathbb{E}\left[(2F + 2
nu - \Gamma_1(F))^2\right]^{1/2},
\]

where \(d_1\) denotes the 1-Wasserstein distance and \(\Gamma_1\) is the Gamma operator defined in the next section.

From \([\text{NPR10}]\) Theorem 3.6, for any random variable \(F\) in the \(q\)-th Wiener chaos with \(\mathbb{E}[F^2] = 2\nu\), we have the estimate

\[
\mathbb{E}\left[(2F + 2\nu - \Gamma_1(F))^2\right] \leq \frac{q - 1}{3q} |\kappa_4(F) - \kappa_4(G(\nu)) - 12\kappa_3(F) + 12\kappa_3(G(\nu))| \leq \text{const.} \times \max\{\kappa_3(F) - \kappa_3(G(\nu)), |\kappa_4(F) - \kappa_4(G(\nu))|\}.\tag{1}
\]

Combining these two results, we obtain an upper bound similar to the one in the fourth moment theorem, but worse by a whole square root, namely

\[
d_1(F, G(\nu)) \leq \text{const.} \times \max\{\kappa_3(F) - \kappa_3(G(\nu)), |\kappa_4(F) - \kappa_4(G(\nu))|\}^{1/2}.\tag{2}
\]

A natural question, which we will deal with in this paper, is if this square root can be removed using techniques similar to the ones in \([\text{NP15}]\).
As a generalization of the Wasserstein-1 distance $d_1$, we also define the following probability metrics. For $k \geq 1$, let

$$\mathcal{H}_k := \{ h \in C^{k-1}(\mathbb{R}) : h^{(k-1)} \in \text{Lip}(\mathbb{R}) \text{ and } \|h^{(1)}\|_{\infty} \leq 1, \ldots, \|h^{(k)}\|_{\infty} \leq 1 \}.$$ 

Furthermore define the corresponding distance between two random variables $X$ and $Y$ as

$$d_k(X,Y) := \sup_{h \in \mathcal{H}_k} \left| \mathbb{E}[h(X)] - \mathbb{E}[h(Y)] \right|.$$  \hspace{1cm} (3)

The outline of our paper is as follows: In section 2, we give a brief introduction to Malliavin calculus on the Wiener space and specify the notation used in the paper. The third section contains the main theoretical finding of this paper – an upper bound for the $d_2$ distance between a general element $F$ living in a finite sum of Wiener chaoses and our target distribution $G(\nu)$ in terms of iterated Gamma operators. In Section 4, we shift our focus to the case of a random variable $F = I^2(f)$ in the second Wiener chaos to establish an optimality result similar to the main result in \cite{NP15} by removing the square root in (2). Section 5 provides several new characterizations of the centered Gamma distribution $G(\nu)$ within the second Wiener chaos in terms of iterated Gamma operators. The final section deals with a different collection of techniques, mainly based on a classical Berry-Esseen lemma, to present several Gamma approximation results in the Kolmogorov distance.

## 2 Preliminaries: Gaussian Analysis and Malliavin Calculus

In this section, we provide a very brief introduction to Malliavin calculus and define some of the operators used in this framework. For a more detailed introduction and proofs, see for example the textbooks \cite{NP12} and \cite{Nua06}.

### 2.1 Isometric Gaussian Processes and Wiener Chaos

Let $\mathfrak{H}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$, and $X = \{X(h) : h \in \mathfrak{H}\}$ be an isometric Gaussian process, defined on some probability space $(\Omega, \mathcal{F}, P)$. This means that $X$ is a family of centered, jointly Gaussian random variables with covariance structure $\mathbb{E}[X(g)X(h)] = \langle g, h \rangle_{\mathfrak{H}}$. We assume that $\mathcal{F}$ is the $\sigma$-algebra generated by $X$. For an integer $q \geq 1$, we will write $\mathfrak{H}^\otimes q$ or $\mathfrak{H}^{\oslash q}$ to denote the $q$-th tensor product of $\mathfrak{H}$, or its symmetric $q$-th tensor product, respectively. If $H_q(x) = ((-1)^q e^{x^2/2} \frac{d^q}{dx^q} e^{-x^2/2}$ is the $q$-th Hermite polynomial, then the closed linear subspace of $L^2(\Omega)$ generated by the family $\{H_q(X(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$ is called the $q$-th Wiener chaos of $X$ and will be denoted by $\mathcal{H}_q$. For $f \in \mathfrak{H}^{\oslash q}$, let $I_q(f)$ be the $q$-th multiple Wiener-Itô integral of $f$ (see \cite{NP12} Definition 2.7.1)). An important observation is that for any $f \in \mathfrak{H}$ with $\|f\|_{\mathfrak{H}} = 1$ we have that $H_q(X(f)) = I_q(f^{\oslash q})$. As a consequence $I_q$ provides an isometry from $\mathfrak{H}^{\oslash q}$ onto the $q$-th Wiener chaos $\mathcal{H}_q$ of $X$. It is a well-known fact, called the Wiener-Itô chaotic decomposition, that any element $F \in L^2(\Omega)$ admits the expansion

$$F = \sum_{q=0}^{\infty} I_q(f_q),$$  \hspace{1cm} (4)
where \( f_0 = \mathbb{E}[F] \) and the \( f_q \in \mathcal{F}^{\otimes q}, q \geq 1 \) are uniquely determined. An important result is the following isometry property of multiple integrals. Let \( f \in \mathcal{F}^{\otimes p} \) and \( g \in \mathcal{F}^{\otimes q} \), where \( 1 \leq q \leq p \). Then
\[
\mathbb{E}[I_p(f)I_q(g)] = \begin{cases} 
  p! \langle f, g \rangle_{\mathcal{F}^{\otimes p}} & \text{if } p = q \\
  0 & \text{otherwise.} 
\end{cases}
\] (5)

### 2.2 The Malliavin Operators

We denote by \( \mathcal{S} \) the set of smooth random variables, i.e. all random variables of the form \( F = g(X(\phi_1), \ldots, X(\phi_n)) \), where \( n \geq 1 \), \( \phi_1, \ldots, \phi_n \in \mathcal{F} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) is a \( C^\infty \)-function, whose partial derivatives have at most polynomial growth. For these random variables, we define the Malliavin derivative of \( F \) with respect to \( X \) as the \( \mathcal{F} \)-valued random element \( DF \in L^2(\Omega, \mathcal{F}) \) defined as
\[
DF = \sum_{i=1}^\infty \frac{\partial g}{\partial x_i}(X(\phi_1), \ldots, X(\phi_n)) \phi_i.
\]
The set \( \mathcal{S} \) is dense in \( L^2(\Omega) \) and using a closure argument, we can extend the domain of \( D \) to \( \mathbb{D}^{1,2} \), which is the closure of \( \mathcal{S} \) in \( L^2(\Omega) \) with respect to the norm \( \| F \|_{\mathbb{D}^{1,2}} := \mathbb{E}|F|^2 + \mathbb{E}\|DF\|_{\mathcal{F}}^2 \). See [NP12] for a more general definition of higher order Malliavin derivatives and spaces \( \mathbb{D}^{p,q} \).

The Malliavin derivative satisfies the following chain-rule. If \( \varphi : \mathbb{R}^m \to \mathbb{R} \) is a continuously differentiable function with bounded partial derivatives and \( F = (F_1, \ldots, F_m) \) is a vector of elements of \( \mathbb{D}^{1,q} \) for some \( q \), then \( \varphi(F) \in \mathbb{D}^{1,q} \) and
\[
D\varphi(F) = \sum_{i=1}^m \frac{\partial \varphi}{\partial x_i}(F) DF_i.
\] (6)

Note that the conditions on \( \varphi \) are not optimal and can be weakened. For \( F \in L^2(\Omega) \), with chaotic expansion as in (4), we define the pseudo-inverse of the infinitesimal generator of the Ornstein-Uhlenbeck semigroup as
\[
L^{-1}F = -\sum_{p=1}^\infty \frac{1}{p} I_p(f_p).
\]
The following integration by parts formula is one of the main ingredients to proving the main theorem of section 3. Let \( F, G \in \mathbb{D}^{1,2} \). Then
\[
\mathbb{E}[FG] = \mathbb{E}[F]\mathbb{E}[G] + \mathbb{E}[\langle DG, -DL^{-1}F \rangle_{\mathcal{F}}].
\] (7)

### 2.3 Gamma Operators and Cumulants

Let \( F \) be a random variable with finite moments of all order. Define its cumulant generating function \( K_F(t) \) as the logarithm of the moment-generating function, that is \( K_F(t) = \log \mathbb{E}[e^{tX}] \). The \( j \)-th cumulant of \( F \), denoted by \( \kappa_j(F) \), is then defined as the \( j \)-th derivative of \( K_F \) evaluated at 0, i.e.
\[
\kappa_j(F) = \frac{d^j}{dt^j} K_F(t) \bigg|_{t=0}.
\]
Let $F$ be a random variable with a finite chaos expansion. We define the operators $\Gamma_i$, $i \in \mathbb{N}_0$ via

$$\Gamma_0(F) := F$$

and

$$\Gamma_{i+1}(F) := \langle D\Gamma_i(F), -DL^{-1}F \rangle_{\mathcal{H}}, \quad \text{for } i \geq 0.$$  

(8)

This is the Gamma operator used in the proof of the main theorem in [NP15], although it is defined differently there. Note that there is also an alternative definition, which can be found in most other papers in this framework, see for example Definition 8.4.1 in [NP12] or Definition 3.6 in [BBNP12]. For the sake of completeness, we also mention the alternative Gamma operators here, which we shall denote by $\Gamma_{alt}$. These are defined via

$$\Gamma_{alt,0}(F) := F \quad \text{and} \quad \Gamma_{alt,i+1}(F) := \langle DF, -DL^{-1}\Gamma_{alt,i}(F) \rangle_{\mathcal{H}}, \quad \text{for } i \geq 0.$$  

(9)

Note that these Gamma operators are related to the cumulants of $F$ by the following identity (from [NP10]): For all $j \geq 0$, we have

$$E[\Gamma_j(F)] = E[\Gamma_{alt,j}(F)] = \frac{1}{j!} \kappa_{j+1}(F).$$

Also note that we always have $\Gamma_1(F) = \Gamma_{alt,1}(F)$. Furthermore, on second Wiener chaos, we have $\Gamma_j(F) = \Gamma_{alt,j}(F)$ for all $j \in \mathbb{N}_0$.

3 The Stein-Malliavin Upper Bound

In the following, we will use centered versions of the Gamma-operators, i.e.

$$\overline{\Gamma}_j(F) := \Gamma_j(F) - E[\Gamma_j(F)] = \Gamma_j(F) - \frac{1}{j!} \kappa_{j+1}(F).$$

Theorem 3.1. Let $F$ be a centered random variable admitting a finite chaos expansion with $\text{Var}(F) = 2\nu$. Let $G(\nu) \sim \text{CenteredGamma}(\nu)$. Then there exists a constant $C > 0$ (only depending on $\nu$), such that

$$d_2(F, G(\nu)) \leq C \left\{ E \left[ (2F - \overline{\Gamma}_1(F))^2 \right] + E \left[ \left( \overline{\Gamma}_2(F) - 2\overline{\Gamma}_1(F) \right)^2 \right]^{1/2} \right. + \left. E \left[ \left( \overline{\Gamma}_3(F) - 2\overline{\Gamma}_2(F) \right)^2 \right]^{1/2} \right. + \left. |\kappa_3(F) - \kappa_3(G(\nu))| + |\kappa_4(F) - \kappa_4(G(\nu))| \right\}.$$  

(10)

To simplify computations, we begin with the following Lemma.

Lemma 3.2. Let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, where $g$ and $g'$ are bounded by a constant only depending on $\nu > 0$. Consider the solution $\varphi$ to the Stein equation $g(x) - E[g(G(\nu))] = 2(x + \nu)\varphi'(x) - x\varphi(x)$. Then
(a) $\varphi$ is again a Lipschitz function, where $\varphi$ and $\varphi'$ are bounded by a constant only depending on $\nu$.

(b) If $F \in \mathbb{D}^\infty$, then for any $r \in \mathbb{N}$:
\[
\mathbb{E}\left[g(F) \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right) \right] = -\mathbb{E}\left[\varphi'(F) \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right) \left( \Gamma_1(F) - 2F \right) \right] \\
- \mathbb{E}\left[\varphi(F) \left( \Gamma_{r+1}(F) - 2\Gamma_r(F) \right) \right].
\]

Proof. Part (a) is a consequence of [DP18, Theorem 2.3 (a)]. For part (b), note that $2\nu = \mathbb{E}[\Gamma_1(F)]$. Thus
\[
\mathbb{E}\left[g(F) \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right) \right] = \mathbb{E}\left[\left( g(F) - \mathbb{E}[g(\nu)] \right) \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right) \right] \\
= \mathbb{E}\left[\left( 2(F + \nu)\varphi'(F) - F\varphi(F) \right) \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right) \right] \\
= 2\mathbb{E}[F\varphi'(F)\Gamma_r(F)] + \mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_r(F)] - \mathbb{E}[F\varphi(F)\Gamma_r(F)] \\
- 4\mathbb{E}[F\varphi'(F)\Gamma_{r-1}(F)] - 2\mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_{r-1}(F)] + 2\mathbb{E}[F\varphi(F)\Gamma_{r-1}(F)] \\
=: \sum_{i=1}^{6} T_i.
\]

Now, we use the integration-by-parts formula (7) in combination with the chain rule (6) to obtain
\[
T_3 + T_2 = -\mathbb{E}[F\varphi(F)\Gamma_r(F)] + \mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_r(F)] \\
\quad = -\mathbb{E}[\Gamma_1(F)\Gamma_r(F)\varphi'(F)] - \mathbb{E}[\varphi(F)\Gamma_{r+1}(F)] + \mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_r(F)] \\
\quad = -\mathbb{E}[\Gamma_1(F)\Gamma_r(F)\varphi'(F)] - \mathbb{E}[\varphi(F)\Gamma_{r+1}(F)],
\]
and similarly
\[
T_6 + T_5 = 2\mathbb{E}[\Gamma_1(F)\Gamma_{r-1}(F)\varphi'(F)] + 2\mathbb{E}[\varphi(F)\Gamma_r(F)].
\]

Hence, putting everything together, the result follows. \qed

Proof of Theorem 3.7. As a starting point, we use the Stein equation (2.7) from [DP18]. Let $h \in \mathcal{H}_2$ be a test function, then by using the integration by parts formula (7), we get
\[
|\mathbb{E}[h(F)] - \mathbb{E}[h(G(\nu))]| = |\mathbb{E}\left[2(F + \nu) f'(F) - f(F) \right] | \\
\quad = |\mathbb{E}\left[2(F + \nu) f'(F) - f'(F) DF, -DL^{-1}F \right] | \\
\quad = |\mathbb{E}\left[f'(F) \left( 2F - \Gamma_1(F) \right) \right] |.
\]

Now set $g := f'$. Then $g$ is a bounded Lipschitz function whose derivative $g'$ is bounded by a constant only depending on $\nu$, see [DP18, Theorem 2.3 (b)]. Denote by $\varphi$ the solution to the Gamma Stein equation $g(x) - \mathbb{E}[g(G(\nu))] = 2(x + \nu)\varphi'(x) - x\varphi(x)$, and by $\psi$ the solution to $\varphi(x) - \mathbb{E}[\varphi(G(\nu))] = 2(x + \nu)\psi'(x) - x\psi(x)$. By Lemma 3.2 (a), both $\varphi$ and $\psi$ are Lipschitz,
where the functions themselves, as well as their derivatives are bounded by a constant only depending on \( \nu \). Now apply Lemma 3.2 (b) twice, to get

\[
\begin{align*}
\mathbb{E}\left[g(F)\left(2F - \Gamma_1(F)\right)\right] &= \mathbb{E}\left[\varphi'(F)\left(\Gamma_1(F) - 2F\right)^2\right] + \mathbb{E}\left[\varphi(F)\left(\Gamma_2(F) - 2\Gamma_1(F)\right)\right] \\
&= \mathbb{E}\left[\varphi'(F)\left(\Gamma_1(F) - 2F\right)^2\right] - \mathbb{E}[\varphi(F)]\left(\frac{1}{2} \kappa_3(F) - 2\kappa_2(F)\right) \\
&\quad + \mathbb{E}\left[\varphi(F)\left(\Gamma_2(F) - 2\Gamma_1(F)\right)\right] \\
&= \mathbb{E}\left[\varphi'(F)\left(\Gamma_1(F) - 2F\right)^2\right] - \mathbb{E}[\varphi(F)]\left(\frac{1}{2} \kappa_3(F) - 2\kappa_2(F)\right) \\
&\quad - \mathbb{E}\left[\psi'(F)\left(\Gamma_2(F) - 2\Gamma_1(F)\right)\left(\Gamma_1(F) - 2F\right)\right] - \mathbb{E}\left[\psi(F)\left(\Gamma_3(F) - 2\Gamma_2(F)\right)\right] \\
&= \mathbb{E}\left[\varphi'(F)\left(\Gamma_1(F) - 2F\right)^2\right] - \mathbb{E}[\varphi'(F)]\left(\Gamma_2(F) - 2\Gamma_1(F)\right)\left(\Gamma_1(F) - 2F\right) \\
&\quad - \mathbb{E}[\psi(F)]\left(\frac{1}{2} \kappa_3(F) - 2\kappa_2(F)\right) \\
&\quad + \mathbb{E}[\psi(F)]\left(\frac{1}{6} \kappa_4(F) - \kappa_3(F)\right).
\end{align*}
\]

The result follows by applying Cauchy-Schwarz inequality, as well as using the fact that \( \frac{1}{2} \kappa_3(G(\nu)) = 2\kappa_2(G(\nu)) = 2\kappa_2(F) = 4\nu \) and \( \frac{1}{6} \kappa_4(G(\nu)) = \kappa_3(G(\nu)) = 8\nu \), see (14).

**Remark 3.3.**

(i) The argument based on iterating the Stein equation, instead of applying Cauchy-Schwarz inequality after using the Malliavin integration by parts formula once, implemented in the proof of Theorem 3.1 is completely analogous to the main result from [NP15, p. 3129]. The backbone of this line of arguments is the fact that when applying higher order Gamma operators on chaotic random variables, the resulting random variables often become smaller (in terms of variance).

(ii) A natural framework in which to apply our main Theorem 3.1 is when the candidate random variable \( F \) is chaotic, meaning that \( F = I_q(f) \) for some \( q \geq 2 \), and kernel \( f \in \mathcal{F}_q \). In this framework, it is well-known (e.g. [NPR10]) that the first summand in the RHS of estimate (14) can be further controlled by using the third and fourth cumulants, namely that

\[
\mathbb{E}\left[\left(2F - \Gamma_1(F)\right)^2\right] = \text{Var} (\Gamma_1(F) - 2F) \leq \frac{q - 1}{3q}\left\{\kappa_4(F) - 12\kappa_3(F) + 48\nu\right\}.
\]

We emphasize that, when \( q \geq 4 \) and \( F \) is chaotic, the linear combination of the cumulants \( \kappa_4(F) - 12\kappa_3(F) + 48\nu \) is positive, see [NP09a, Corollary 4.4].

(iii) In order to interpret our upper bound in Theorem 3.1 in the language of the cumulants, analogous to that achieved in [NP15], for a chaotic random variable \( F = I_q(f) \) with \( q \geq 2 \), we need cumulant-type inequalities comparable to Proposition 4.3 in [BBNP12] for the remaining terms in the RHS of (14);

\[
\text{Var} (\Gamma_2(F) - 2\Gamma_1(F)) \quad \text{and} \quad \text{Var} (\Gamma_3(F) - 2\Gamma_2(F)).
\]
This is a deep problem and for the time being, such inequalities are difficult to tackle in full generality using the available techniques such as contraction operators or the recent machinery of Markov triplets \cite{Led12,ACP14,AMMP16}. For example, a suitable cumulant counterpart for studying the variance of the iterated Gamma quantity $\Gamma_3(F)$ is the 8th cumulant $\kappa_8(F)$. There exists an explicit representation (see \cite{BBNP12,Proposition 3.9}) of the quantity $\Gamma_3(F)$ in terms of appropriate contractions, involving the kernel function $f$. However, due to several zero-contractions appearing in the cumulant side $\kappa_8(F)$ which do not show up in the Wiener chaotic representation of $\Gamma_3(F)$, such comparison is impossible. The second major obstacle is that one needs to control the variance of an explicit linear structure of Gamma operators in terms of an “efficient” linear combination of cumulants. Here with “efficient” we mean that when plugging in the target random variable $G(\nu)$, the introduced cumulant combination vanishes. For instance, one has to note that $\kappa_4(G(\nu)) - 12\kappa_3(G(\nu)) + 48\nu = 0$. Thus, in Section 4 in order to analyze these variance quantities, we will focus on the case of $F$ belonging to the second Wiener chaos, where we have explicit representations in terms of the eigenvalues of the corresponding Hilbert-Schmidt operator.

As we will see later, when focussing on second Wiener chaos, the most critical term to analyze is $\text{Var} (\Gamma_3(F) - 2\Gamma_2(F))$. If we choose our test function $h$ to be smoother, we can deduce an upper bound in the smoother probability metric $d_3$ (see \cite{BBNP12} for definition) without the need of iterating Stein’s method.

**Proposition 3.4.** Let $F$ be a centered random variable admitting a finite chaos expansion with $\text{Var}(F) = 2\nu$. Let $G(\nu) \sim \text{CenteredGamma}(\nu)$. Then there exists a constant $C > 0$ (only depending on $\nu$), such that

$$d_3(F,G(\nu)) \leq C \left\{ \sqrt{\text{Var}(\Gamma_{alt,3}(F) - 2\Gamma_{alt,2}(F))} + |\kappa_3(F) - \kappa_3(G(\nu))| + |\kappa_4(F) - \kappa_4(G(\nu))| \right\}. \tag{11}$$

**Proof.** Let $h \in \mathcal{H}_3$ be a test function and denote by $f$ the solution to the Stein equation $h(x) - E[h(G(\nu))] = 2(x + \nu)f'(x) - xf(x)$. Now we use the Malliavin integration by parts
formula (7) a total number of three times to get
\[
\mathbb{E}[h(F)] - \mathbb{E}[h(G(\nu))] = \mathbb{E}[2(F + \nu)f'(F) - Ff(F)]
\]
\[
= \mathbb{E}[f'(F)\left(2F - \Gamma_{alt,1}(F)\right)]
\]
\[
= \mathbb{E}[f''(F)\left(2\Gamma_{alt,1}(F) - \Gamma_{alt,2}(F)\right)]
\]
\[
= \mathbb{E}[f''(F)\left(2\Gamma_{alt,1}(F) - \Gamma_{alt,2}(F)\right)] + \mathbb{E}[f''(F)\left(2\mathbb{E}[\Gamma_{alt,1}(F)] - \mathbb{E}[\Gamma_{alt,2}(F)]\right)]
\]
\[
= \mathbb{E}[f''(F)\left(2\Gamma_{alt,2}(F) - \Gamma_{alt,3}(F)\right)] + \mathbb{E}[f''(F)\left(1/2\kappa_3(F) - 4\nu\right)]
\]
\[
+ \mathbb{E}[f''(F)\left(1/2\kappa_3(F) - 4\nu\right)]
\]
\[
= \mathbb{E}[f''(F)\left(2\Gamma_{alt,2}(F) - \Gamma_{alt,3}(F)\right)] + \mathbb{E}[f''(F)\left(8\nu - 1/6\kappa_4(F)\right)] + \mathbb{E}[f''(F)\left(1/2\kappa_3(F) - 4\nu\right)].
\]

We know that \(\mathbb{E}[\kappa_3(G(\nu))] = 8\nu\) and \(\mathbb{E}[\kappa_4(G(\nu))] = 48\nu\). Combining this with the boundedness of \(f''\), \(f'''\) and Cauchy-Schwarz inequality, we obtain
\[
\|\mathbb{E}[h(F)] - \mathbb{E}[h(G(\nu))]\|
\]
\[
\leq C \left\{ \mathbb{E}\left[\left(\Gamma_{alt,3}(F) - 2\Gamma_{alt,2}(F)\right)^2\right] + |\kappa_3(F) - \kappa_3(G(\nu))| + |\kappa_4(F) - \kappa_4(G(\nu))|\right\}
\]
\[
\leq C \left\{ \sqrt{\text{Var}(\Gamma_{alt,3}(F) - 2\Gamma_{alt,2}(F))} + |\kappa_3(F) - \kappa_3(G(\nu))| + |\kappa_4(F) - \kappa_4(G(\nu))|\right\}.
\]

**Remark 3.5.** Here, we have used the traditional Gamma operators as defined in [3]. This way, we get a simple proof for the upper bound in a smoother integral probability metric. In the next section, we will focus only on random elements \(F\) belonging to the second Wiener chaos, and it can be checked that in this setup the two definitions of Gamma operators coincide.

### 4 The Case of Second Wiener Chaos

Throughout this section we assume that \(F = I_2(f)\), for some \(f \in \mathcal{H}\), belongs to the second Wiener chaos. It is a classical result (see [NP12, Section 2.7.4]) that these kind of random variables can be analyzed through the associated **Hilbert-Schmidt operator**

\[
A_f : \mathcal{H} \to \mathcal{H}, \quad g \mapsto f \otimes_1 g.
\]

Denote by \(\{c_{f,i} : i \in \mathbb{N}\}\) the set of eigenvalues of \(A_f\). We also introduce the following sequence of auxiliary kernels

\[
\left\{f \otimes_1^{(p)} f : p \geq 1\right\} \subset \mathcal{H},
\]
defined recursively as $f \otimes_1^{(1)} f = f$, and, for $p \geq 2$,
\[ f \otimes_1^{(p)} f = \left( f \otimes_1^{(p-1)} f \right) \otimes_1 f. \]

**Proposition 4.1.** (see e.g. [NP12, p. 43])

1. The random element $F$ admits the representation

\[ F = \sum_{i=1}^{\infty} c_{f,i} \left( N_i^2 - 1 \right), \tag{12} \]

where the $N_i$ are i.i.d. $\mathcal{N}(0, 1)$ and the series converges in $L^2(\Omega)$ and almost surely.

2. For every $p \geq 2$

\[ \kappa_p(F) = 2^{p-1}(p-1)! \sum_{i=1}^{\infty} c_{f,i}^p = 2^{p-1}(p-1)! \langle f, f \otimes_1^{(p-1)} f \rangle \]
\[ = 2^{p-1}(p-1)! \text{Tr} \left( A_f^p \right) \tag{13} \]

where $\text{Tr}(A_f^p)$ stands for the trace of the $p$th power of operator $A_f$.

When $\nu$ is an integer i.e. $G(\nu)$ is a centered $\chi^2$ random variable with $\nu$ degrees of freedom, then [12] shows us that $G(\nu)$ is itself an element of the second Wiener chaos, where $\nu$-many of the eigenvalues are 1 and the remaining ones are 0. Hence, in this case, we deduce from (13) that $\kappa_p(G(\nu)) = 2^{p-1}(p-1)! \nu$. Perhaps not surprisingly, this is also the case, when $\nu$ is any positive real number.

**Lemma 4.2.** Let $\nu > 0$ and $G(\nu) \sim \text{CenteredGamma}(\nu)$. Then

\[ \kappa_p(G(\nu)) = \begin{cases} 0, & p = 1; \\ 2^{p-1}(p-1)! \nu, & p \geq 2. \end{cases} \tag{14} \]

**Proof.** Since the cumulant generating function of a Gamma random variable is well-known, we can easily compute that of $G(\nu)$ to be

\[ K(t) = \frac{\nu}{2} \log \left( \frac{1}{1-2t} \right) - \nu t. \]

By simple induction over $p$, we obtain

\[ \frac{d^p}{dt^p} K(t) = \begin{cases} -\nu + \frac{\nu}{1-2t}, & p = 1; \\ \nu \frac{2^p(p-1)!}{2(1-2t)^{p+1}}, & p \geq 2. \end{cases} \]

The result now follows by letting $t = 0$. 

\[ \square \]
**Lemma 4.3.** Let $F = I_2(f)$, for some $f \in S^\otimes 2$, and denote by $A_f$ the corresponding Hilbert-Schmidt operator with eigenvalues $\{c_{f,i} : i \geq 1\}$. Then for $r \geq 1$

\[
\mathbb{E} \left[ \left( \Gamma_r(F) - 2 \Gamma_{r-1}(F) \right)^2 \right] = \frac{1}{(2r+1)!} \sum_{i=1}^{\infty} c_{f,i}^{2r+2} - \frac{4}{(2r)!} \sum_{i=1}^{\infty} c_{f,i}^{2r+1} + \frac{4}{(2r-1)!} \sum_{i=1}^{\infty} c_{f,i}^{2r},
\]

Proof. From [APP15] equation (24), which follows by induction on $r$, we have the representation

\[
\Gamma_r(F) = 2^r I_2 \left( f \otimes_1^{(r+1)} f \right).
\]

Using the isometry property [5], we obtain

\[
\text{Var} \left( \Gamma_r(F) - 2 \Gamma_{r-1}(F) \right) = 2^{2r+1} \| f \otimes_1^{(r+1)} f - f \otimes_1^{(r)} f \|_{S^\otimes 2}^2
\]

\[
= 2^{2r+1} \left( \langle f, f \otimes_1^{(2r+1)} f \rangle_{S^\otimes 2} - 2 \langle f, f \otimes_1^{(2r)} f \rangle_{S^\otimes 2} + \langle f, f \otimes_1^{(2r-1)} f \rangle_{S^\otimes 2} \right)
\]

\[
= 2^{2r+1} \text{Tr} \left( A_f^{2r+2} - 2 A_f^{2r+1} + A_f^{2r} \right).
\]

The result now follows with (13).

\[\square\]

### 4.1 Motivating Examples

Let $\nu > 0$. Assume that $\{F_n\}_{n \geq 1}$ is a sequence of random elements in the second Wiener chaos such that $\mathbb{E}(F_n^2) = 2\nu$ for all $n \geq 1$. The main Theorem 3.1 reads that there exists a general constant $C$, such that

\[
d_2(F_n, G(\nu)) \leq C \left\{ \text{Var} (\Gamma_1(F_n) - 2F_n) 
\right.
\]

\[
+ \sqrt{\text{Var} (\Gamma_2(F_n) - 2\Gamma_1(F_n))} \times \sqrt{\text{Var} (\Gamma_1(F_n) - 2F_n)} 
\]

\[
+ \sqrt{\text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n))} + |\kappa_3(F_n) - \kappa_3(G(\nu))| + |\kappa_4(F_n) - \kappa_4(G(\nu))| \right\}.
\]

As a consequence, in order for the square root in the upper bound in (2) to be removed, it is sufficient to verify the following statement. There exists a constant $C$ (independent of $n$, but may possibly depending on the sequence $\{F_n\}_{n \geq 1}$), such that the following variance-estimates hold:

\[
\text{Var} (\Gamma_2(F_n) - 2\Gamma_1(F_n)) \leq C \text{Var} (\Gamma_1(F_n) - 2F_n), \quad (16)
\]

\[
\text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) \leq C \text{Var}^2 (\Gamma_1(F_n) - 2F_n) \quad (17)
\]

Our major aim in the present section is to show that
(i) The variance-estimate \( \text{(16)} \) is universal in the sense that it holds for any random variable \( F \) in the second Wiener chaos having second moment \( \mathbb{E}(F^2) = 2\nu \). In particular it is not a matter of the fact whether the sequence \( F_n \) converges in distribution towards a centered Gamma target \( G(\nu) \).

(ii) The second variance-estimate \( \text{(17)} \) has a completely different flavor, and that occasionally holds too, meaning that it can be seen as a Gamma characterization estimate. By this we mean that the central assumption that the sequence \( F_n \) converges in distribution towards the Gamma target distribution \( G(\nu) \) is heavily used to establish the estimate.

In order to classify the convergence rate of a sequence, we introduce the following notation:

When \( (a_n)_{n \geq 1} \) and \( (b_n)_{n \geq 1} \) are two non-negative real number sequences, we write \( a_n \approx_C b_n \) if \( \lim_{n \to \infty} \frac{a_n}{b_n} = C \), for some constant \( C > 0 \).

**Example 4.4.** Let \( \alpha_n, \beta_n \) be two sequences of positive real numbers converging to zero as \( n \to \infty \) and assume that \( (1 - \alpha_n)^2 + \beta_n^2 = 1 \) for all \( n \geq 1 \). Consider the following sequence in the second Wiener chaos

\[
F_n = c_{1,n}(N_1^2 - 1) + c_{2,n}(N_2^2 - 1) := (1 - \alpha_n)(N_1^2 - 1) - \beta_n(N_2^2 - 1)
\]

\[
\overset{D}{\to} G(1), \quad \text{as } n \to \infty.
\]

Note that the second moment assumption \( \mathbb{E}(F_n^2) = 2(1 - \alpha_n)^2 + 2\beta_n^2 = 2 \) implies that \( \beta_n \approx_C \sqrt{\alpha_n} \). Hence, after some straightforward computations, we arrive in the asymptotic relations

\[
\begin{aligned}
\text{Var} \left( \frac{1}{N}(F_n - 2\frac{1}{N}(F_n)) \right) &\approx_C \left( \alpha_n - \alpha_n \right)^2, \\
\text{Var} \left( \frac{1}{N}(F_n - 2F_n) \right) &\approx_C \left( \alpha_n - \sqrt{\alpha_n} \right)^2.
\end{aligned}
\]

Also

\[
\text{Var} \left( \frac{1}{N}(F_n - 2\frac{1}{N}(F_n)) \right) = 2^5 \sum_{i=1}^{2} c_{i,n}(c_{i,n}^2 - c_{i,n})^2
\]

\[
\leq 4 \left( 2^3 \sum_{i=1}^{2} (c_{i,n}^2 - c_{i,n})^2 \right) = 4 \text{Var} \left( \frac{1}{N}(F_n - 2F_n) \right).
\]

Therefore, for some constant \( C \) (independent of \( n \)), both estimates \( \text{(16)}, \text{(17)} \) take place. Therefore, our main theorem \( \text{3.1} \) yields that

\[
d_2(F_n, G(1)) \leq_C \max \left\{ \left| \kappa_3(F_n) - \kappa_3(G(2)) \right|, \left| \kappa_4(F_n) - \kappa_4(G(2)) \right| \right\}.
\]

**Example 4.5.** In this example, instead, we consider the following sequence

\[
F_n = c_{1,n}(N_1^2 - 1) + c_{2,n}(N_2^2 - 1) := (1 - \alpha_n)(N_1^2 - 1) + \beta_n(N_2^2 - 1)
\]

\[
\overset{\text{law}}{\to} G(1), \quad \text{as } n \to \infty.
\]

Similar computations as in Example \( \text{4.3} \) yield that estimate \( \text{(16)} \) is in order. It is noteworthy that, as an alternative to the second estimate \( \text{(17)} \), the estimate

\[
\text{Var} \left( \frac{1}{N}(F_n - 2\frac{1}{N}(F_n)) \right) \leq_C \left( \kappa_4(F_n) - 6\kappa_3(F_n) \right)^2
\]
is also valid, which is enough for our purposes. Note that for the target random variable \( G(1) \), we have \( \kappa_4(G(1)) - 6\kappa_3(G(1)) = 0 \). Later on in Section 4.4, we will study this phenomenon in detail. Once again the square root in (2) can be improved.

4.2 Iterated Gamma Operators: Variance Estimates

4.2.1 Variance Estimate: \( \text{Var} (\Gamma_2(F_n) - 2\Gamma_1(F_n)) \leq C \text{Var} (\Gamma_1(F_n) - 2F_n) \)

We start with variance–estimate (16). We make use of a recent discovery in [APP15] that the second Wiener chaos is stable under the Gamma operators, meaning that for any element \( F \) in the second Wiener chaos, the resulting random variable \( \Gamma_r(F) \) remains inside the second Wiener chaos up to a constant for any \( r \geq 0 \).

Lemma 4.6. Let \( \nu > 0 \), and \( F = I_2(f) \) in the second Wiener chaos such that \( \mathbb{E}[F^2] = 2\nu \). Then, for every \( r \geq 1 \), with constant \( C = 4\nu \), we have

\[
\text{Var} (\Gamma_{r+1}(F) - 2\Gamma_r(F)) \leq C \text{Var} (\Gamma_r(F) - 2\Gamma_{r-1}(F)).
\]

(18)

In particular

\[
\text{Var} (\Gamma_2(F) - 2\Gamma_1(F)) \leq (4\nu) \text{Var} (\Gamma_1(F) - 2F).
\]

Also, for every \( r \geq 1 \), and with constant \( C = (4\nu)^r \), we have the following variance-estimate

\[
\text{Var} (\Gamma_{r+1}(F) - 2\Gamma_r(F)) \leq C \text{Var} (\Gamma_1(F) - 2F).
\]

(19)

Proof. Let’s first prove estimate (18). Then estimate (19) could be proven by iteration using similar arguments. Let \( r \geq 1 \). Denote by \( A_f \) the associated Hilbert-Schmidt operator. As in the proof of Lemma 4.3, the variance of the random quantity \( \Gamma_{r+1}(F) - 2\Gamma_r(F) \) can be rewritten as

\[
\text{Var} (\Gamma_{r+1}(F) - 2\Gamma_r(F)) = 2^{2r+3} \text{Tr} \left((A_f^{r+2} - A_f^{r+1})^2\right)
\]

\[
= 2^{2r+3} \text{Tr} \left(A_f^2(A_f^{r+1} - A_f^r)^2\right)
\]

\[
\leq 2^{2r+3} \text{Tr}(A_f^2) \times \text{Tr} \left((A_f^{r+1} - A_f^r)^2\right)
\]

\[
= 2(2 \text{Tr}(A_f^2)) \times \text{Tr} \left((A_f^{r+1} - A_f^r)^2\right)
\]

\[
= 2\kappa_2(F) \times \text{Tr} \left((A_f^{r+1} - A_f^r)^2\right)
\]

\[
= 4\nu \text{Var} (\Gamma_r(F) - 2\Gamma_{r-1}(F)),
\]

where in the third step, we have used the following trace inequality for non-negative operators (see [Liu07]),

\[
\text{Tr}(AB) \leq \text{Tr}(A) \text{Tr}(B) \quad \text{for } A, B \geq 0.
\]

\[ \square \]
Remark 4.7. A direct consequence of Lemma 4.6 is, that for a random element $F$ in the second Wiener chaos with $\text{Var}(\Gamma_1(F) - 2F) = 0$ (and therefore $F = G(\nu)$ in distribution), we necessarily obtain for $r \geq 2$,

$$0 = \text{Var}(\Gamma_{r+1}(F) - 2\Gamma_r(F)) = \frac{1}{(2r+3)!}\kappa_{2r+4}(F) - \frac{4}{(2r+2)!}\kappa_{2r+3}(F) + \frac{4}{(2r+1)!}\kappa_{2r+2}(F).$$  \hspace{1cm} (20)

Later on in Section 5, we will show that astonishingly the converse is also true. Precisely, for the random element $F$ in the second Wiener chaos with $\mathbb{E}(F^2) = 2\nu$, the sole assumption $\text{Var}(\Gamma_{r+1}(F) - 2\Gamma_r(F)) = 0$ for some $r \geq 2$, implies that $F$ necessarily is Gamma distributed.

4.2.2 Variance Estimate: $\text{Var}(\Gamma_3(F_n) - 2\Gamma_2(F_n)) \leq C \text{Var}^2(\Gamma_1(F_n) - 2F_n)$

We begin with the following important observation, namely that a sequence in the second Wiener chaos can only converge to a centered chi-squared distribution $\chi^2$, not to any other centered Gamma distribution.

Proposition 4.8. Let $(F_n = I_2(f_n))_{n \geq 1}$ be a sequence of random variables in the second Wiener chaos that converges in distribution to $G(\nu) \sim \text{CenteredGamma}(\nu)$. Denote by $c_{j,n}$ the $j$-th eigenvalue of the Hilbert-Schmidt operator $A_{f_n}$ associated with $F_n$. Then

(a) $\nu$ is an integer.

(b) There exists a set $I \subset \mathbb{N}$ with $\#I = \nu$, such that, $c_{j,n} \xrightarrow{n \to \infty} 1$ for $j \in I$, and $c_{j,n} \xrightarrow{n \to \infty} 0$ for $j \notin I$.

Proof. Since $F_n \xrightarrow{D} G(\nu)$ implies convergence of all cumulants, (13) and (14) imply that

$$2^{p-1}(p-1)! \sum_{k=1}^{\infty} c_{k,n}^p \to 2^{p-1}(p-1)!\nu \quad \text{as } n \to \infty$$

$$\Leftrightarrow \sum_{k=1}^{\infty} c_{k,n}^p \to \nu \quad \text{as } n \to \infty,$$

for all $p \geq 2$. The result then follows from the following lemma. \hfill $\square$

We drop the dependence on $n$. The corresponding result can be retrieved from this one by adding limits everywhere in the proof and exchanging summation and limits.

Lemma 4.9. Let $\nu > 0$ be a real number and let $\alpha_k$ be a sequence of real numbers such that

$$\sum_{k=1}^{\infty} \alpha_k^r = \nu, \quad \text{for all integers } r \geq 2.$$  \hspace{1cm} (21)

Then $\nu$ is an integer. Furthermore, there exist indices $k_1, \ldots, k_\nu$ such that $\alpha_{k_i} = 1$ for $i = 1, \ldots, \nu$ and $\alpha_k = 0$ for $k \notin \{k_1, \ldots, k_\nu\}$. 14
Proof. First note that we necessarily have $|a_k| \leq 1$ for all $k$, because if e.g. $|\alpha_1| > 1$ then $\sum_{k=1}^{\infty} \alpha_k^r \geq \alpha_1^r > \nu$ for sufficiently large even number $r$. Furthermore, we can deduce that $\alpha_k$ is positive for all $k$, because if e.g. $\alpha_1 < 0$, then

$$\sum_{k=2}^{\infty} \alpha_k^2 \geq \sum_{k=2}^{\infty} |\alpha_k|^3 \geq \sum_{k=2}^{\infty} \alpha_k^3 = \nu - \alpha_1^3 > \nu,$$

which implies $\sum_{k=1}^{\infty} \alpha_k^2 > \nu$ (contradicts (21)). Because of (21), there exists at least one $k \in \mathbb{N}$ such that $\alpha_k \neq 0$. W.l.o.g. let $\alpha_1 \neq 0$. We write $\alpha_1 = 1 - \varepsilon$ for some $\varepsilon \in [0, 1)$. Then

$$\sum_{k=1}^{\infty} \alpha_k^2 = \nu - \alpha_1^2 = \nu - 1 + \varepsilon(2 - \varepsilon),$$

and

$$\sum_{k=2}^{\infty} \alpha_k^3 = \nu - \alpha_1^3 = \nu - 1 + \varepsilon(\varepsilon^2 - 3\varepsilon + 3).$$

Since $2 - \varepsilon \leq \varepsilon^2 - 3\varepsilon + 3$, we obtain $\sum_{k=2}^{\infty} \alpha_k^2 \leq \sum_{k=2}^{\infty} \alpha_k^3$ on the one hand, and because $\alpha_k^2 \geq \alpha_k^3$, we have $\sum_{k=2}^{\infty} \alpha_k^2 \geq \sum_{k=2}^{\infty} \alpha_k^3$ on the other hand. Thus equality holds, and solving for $\varepsilon$ yields $\varepsilon = 0$. Now that we know that $\alpha_1 = 1$, we can write

$$\sum_{k=2}^{\infty} \alpha_k^r = \nu - 1,$$

for all integers $r \geq 2$.

Obviously, the right hand side cannot be negative, so it is either zero (in which case we conclude that all the remaining $\alpha_k$ are zero and we are done), or we continue inductively as before. Hence we find that $\alpha_1 = \ldots = \alpha_{\nu} = 1$, and then from $\sum_{k=\nu+1}^{\infty} \alpha_k^2 = 0$ we deduce that all remaining $\alpha_k$ are zero. \hfill \Box

Because of Proposition 4.8, from now on, we will only focus on cases where $\nu$ is an integer. Also recall that on second Wiener chaos $\Gamma_j = \Gamma_{alt,j}$ for all $j$, so we will always use the notation without the additional subscript. Unlike the variance estimate (16), in order to keep transparency in analyzing the validity of the second variance estimate (17), we discuss the following different cases separately.

Proposition 4.10. (The case of finitely many eigenvalues) Let $\nu > 0$ and $M \geq 2$. Assume $\sum_{i=1}^{M} c^2_{i,n} = \nu$ for all $n \geq 1$, and that as $n \to \infty$,

$$F_n := c_{1,n}(N_1^2 - 1) + c_{2,n}(N_2^2 - 1) + \cdots + c_{M,n}(N_M^2 - 1) \quad \overset{D}{\to} G(\nu),$$

where $G(\nu)$ is a centered Gamma random variable, and $\{N_i\}_{1 \leq i \leq M}$ is a family of independent $\mathcal{N}(0, 1)$ random variables. The $\nu \in \{1, 2, \ldots, M\}$ is an integer, and therefore the target $G(\nu)$ is a centered $\chi^2$ random variable with $\nu$ degrees of freedom. Set

$$\omega(n) := \max\{|1 - c_{i,n}| : i \in I\}, \quad \text{and} \quad \vartheta(n) := \sum_{i \in I^c} c^2_{i,n}.$$ (22)
Hence, we assume that \( \nu < M \).

\[
\text{Proof.} \quad \text{The first part follows immediately from Proposition \[4.8\]. Now, the second moment assumption } \mathbb{E}(F_i^2) = 2\sum_{1 \leq i \leq M} c_{i,n}^2 = 2\nu \text{ implies that (note that } \#I = \nu \text{ where } I \text{ is the set defined in item (b) of Proposition \[4.8\] ),}
\]

\[
\sum_{i \in I}(1 - c_{i,n})^2 = \sum_{i \in I}(1 + c_{i,n}^2 - 2c_{i,n}) = \sum_{i \in I}(1 + c_{i,n}^2) + \sum_{i \in I^c} c_{i,n}^2 - 2\sum_{i \in I} c_{i,n} - \sum_{i \in I} c_{i,n}^2
\]

\[
= 2\sum_{i \in I}(1 - c_{i,n}) - \sum_{i \in I^c} c_{i,n}^2.
\]

Therefore,

\[
\mathbb{E}(F_n - G(\nu))^2 = \sum_{i \in I}(1 - c_{i,n})^2 + \sum_{i \in I^c} c_{i,n}^2 = 2\sum_{i \in I}(1 - c_{i,n}) \to 0. \quad (23)
\]

Proof of \((a)\) : when \( \nu = M \), then for all \( 1 \leq i \leq M \), the coefficients \( c_{i,n} \to 1 \), as \( n \to \infty \).

Hence,

\[
\text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) = 2^7 \sum_{i=1}^{M} c_{i,n}^6(1 - c_{i,n})^2 \approx \left\{ \max\{|1 - c_{i,n}| : i = 1, \ldots, M\} \right\}^2,
\]

\[
\text{Var} (\Gamma_1(F_n) - 2F_n) = 2^3 \sum_{i=1}^{M} c_{i,n}^2(1 - c_{i,n})^2 \approx \left\{ \max\{|1 - c_{i,n}| : i = 1, \ldots, M\} \right\}^2.
\]

Therefore

\[
\text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) \approx_C \text{Var} (\Gamma_1(F_n) - 2F_n).
\]

Hence, we assume that \( \nu < M \). Then there exists at least one index \( 1 \leq j \leq M \), such that \( c_{j,n} \to 0 \), as \( n \to \infty \). Note that the complement set \( I^c \) contains exactly those indices. Since \( \#I^c \) is finite, and \( \vartheta(n) \leq 2\nu \omega(n) \), we have \( \sum_{i \in I^c} c_{i,n}^6 = o(\omega(n)^2) \). Hence

\[
\text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) = 2^7 \sum_{i=1}^{M} c_{i,n}^6(1 - c_{i,n})^2
\]

\[
= 2^7 \left\{ \sum_{i \in I} c_{i,n}^6(1 - c_{i,n})^2 + \sum_{i \in I^c} c_{i,n}^6(1 - c_{i,n})^2 \right\} \approx_C \max \left\{ \omega(n)^2, o(\omega(n)^2) \right\} \approx_C \omega(n)^2.
\]
Also,
\[
\text{Var} \left( \Gamma_1(F_n) - 2F_n \right) = 2^3 \left\{ \sum_{i \in I} c_{i,n}^2 (1 - c_{i,n})^2 + \sum_{i \in I^c} c_{i,n}^2 (1 - c_{i,n})^2 \right\} \approx_C \max \{ \omega(n)^2, \vartheta(n) \}. \tag{24}
\]

Hence,
\[
\text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) \approx_C \text{Var}^2 \left( \Gamma_1(F_n) - 2F_n \right) \quad \text{if and only if} \quad \vartheta(n) \approx C \omega(n).
\]

In addition, assumption \( \omega(n) \approx C \vartheta(n) \) is equivalent to
\[
\mathbb{E}(F_n - G(\nu))^2 = 2 \sum_{i \in I} (1 - c_{i,n}) \approx_C \omega(n) \approx_C \sqrt{\sum_{i \in I} (1 - c_{i,n})^2}. \tag{25}
\]

Therefore, when the degree of freedom \( \nu = 1 \), the cardinality of the set \( \# I = 1 \), and so \( \vartheta(n) \approx C \omega(n) \).

Proof of (b): It can be discussed in a similar way.

Remark 4.11. In the light of relation (23), always \( \vartheta(n) \leq 2 \nu \omega(n) \). Taking this into account together with
\[
\text{Var} \left( \Gamma_2(F_n) - 2\Gamma_1(F_n) \right) = 2^5 \sum_{i=1}^{M} c_{i,n}^4 (1 - c_{i,n})^2 = 2^5 \left\{ \sum_{i \in I} c_{i,n}^4 (1 - c_{i,n})^2 + \sum_{i \in I^c} c_{i,n}^4 (1 - c_{i,n})^2 \right\} \approx_C \max \{ \omega(n)^2, \omega(n)^2 \} \approx_C \omega(n)^2,
\]

one can conclude that the asymptotic estimate
\[
\text{Var} \left( \Gamma_2(F_n) - 2\Gamma_1(F_n) \right) \approx_C \text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right)
\]
takes place as soon as the sequence \( F_n \) in the second Wiener chaos converges in distribution towards the centered Gamma distribution \( G(\nu) \) without any further assumptions.

Example 4.12. The following simple example shows that, in general, many things can happen. Let \( \delta \in [0, 1] \), and consider the sequence \( F_n = \sum_{i=1}^{5} c_{i,n}(N_i^2 - 1) \) in the second Wiener chaos, where the coefficients \( c_{i,n} \) are given as
\[
c_{1,n} = \sqrt{1 + \frac{1}{n}}, \quad c_{2,n} = \sqrt{1 - \frac{1}{n}}, \quad c_{3,n} = \sqrt{1 - \frac{1}{n+\delta}}, \quad c_{4,n} = \sqrt{\frac{1}{2n+\delta}}, \quad \text{and} \quad c_{5,n} = \sqrt{\frac{1}{2n1+\delta}}.
\]

Then,
\[
c_{4,n}^2 \approx_C \frac{1}{n^2}, \quad c_{5,n}^2 \approx_C \frac{1}{n^2}, \quad \text{and} \quad \vartheta(n) \approx C \omega(n)^{1+\delta}.
\]

Therefore, when \( \delta = 0 \), then our favorite estimate
\[
\text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) \approx_C \text{Var}^2 \left( \Gamma_1(F_n) - 2F_n \right)
\]
takes place, and when $\delta = 1$, then

$$\text{Var} (\Gamma_2(F_n) - 2\Gamma_1(F_n)) \approx C \text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) \approx C \text{Var} (\Gamma_1(F_n) - 2F_n).$$

In general

$$\text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) \approx C \left( \text{Var} (\Gamma_1(F_n) - 2F_n) \right)^{\frac{2}{1+\delta}}.$$

One can also consider more involved intermediate rates such as $\vartheta(n) \approx C \omega(n)^{1+\delta} \log^\gamma(\omega(n))$ for some $\delta, \gamma \geq 0$.

**Corollary 4.13.** Let $M \geq 2$ and $\nu > 0$. Consider a sequence $(F_n)_{n \geq 1}$ of random elements in the second Wiener chaos such that $\mathbb{E}(F_n^2) = 2\nu$ for all $n \geq 1$, possessing the following representation

$$F_n = \sum_{1 \leq i \leq M} c_{i,n}(N_i^2 - 1), \quad n \geq 1.$$

Also, we assume that $F_n$ converges in distribution towards a centered Gamma distribution with parameter $\nu > 0$. Then there exist two constants $0 < C_1 < C_2$ (may depend on sequence $F_n$, but independent of $n$), such that for all $n \geq 1$,

(i) if $\nu = 1$, or $\vartheta(n) \approx C \omega(n)$, then

$$C_1 \text{Var} (\Gamma_1(F_n) - 2F_n) \leq \text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) \leq C_2 \text{Var} (\Gamma_1(F_n) - 2F_n).$$

(ii) if $\nu = M$, or $\vartheta(n) \approx C \omega(n)^2$, or $\vartheta(n) = o(\omega(n)^2)$, then

$$C_1 \text{Var} (\Gamma_1(F_n) - 2F_n) \leq \text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) \leq C_2 \text{Var} (\Gamma_1(F_n) - 2F_n).$$

**Remark 4.14.**  (Case $\nu = M$) Let $M \geq 2$ and $\nu > 0$. Assume that $\mathbb{E}(F_n^2) = 2\nu$ for all $n \geq 1$ where

$$F_n = \sum_{1 \leq i \leq M} c_{i,n}(N_i^2 - 1) \Rightarrow G(\nu = M), \quad \text{as } n \to \infty.$$  

The second moment assumption implies that $\sum_{i=1}^M (1 - c_{i,n})^2 = 2 \sum_{i=1}^M (1 - c_{i,n}) \geq 0$. On the
other hand (up to some constants),

\[
\left| \kappa_3(F_n) - \kappa_3(G(\nu)) \right| = \left| \sum_{i=1}^{M} (c_{i,n}^3 - 1) \right| = \left| \sum_{i=1}^{M} (c_{i,n} - 1)(c_{i,n}^2 + c_{i,n} + 1) \right| \\
= \left| \sum_{i=1}^{M} (c_{i,n} - 1) \left( (c_{i,n} - 1)^2 + 3c_{i,n} \right) \right| \\
= \left| \sum_{i=1}^{M} (c_{i,n} - 1) \left( (c_{i,n} - 1)^2 + 3(c_{i,n} - 1) + 3 \right) \right| \\
= \left| 3 \sum_{i=1}^{M} (c_{i,n} - 1) + 3 \sum_{i=1}^{M} (c_{i,n} - 1)^2 + \sum_{i=1}^{M} (c_{i,n} - 1)^3 \right| \\
= \left| 3 \sum_{i=1}^{M} (c_{i,n} - 1) + 3 \sum_{i=1}^{M} (c_{i,n} - 1)^2 + \sum_{i=1}^{M} (c_{i,n} - 1)^3 \right| \\
\approx C \left| \sum_{i=1}^{M} (c_{i,n} - 1) \right|, \\
\]

which in general is less than the rate \( \max \{ |1 - c_{i,n}| : i = 1, \ldots, M \} \). Similarly,

\[
\left| \kappa_4(F_n) - \kappa_4(G(\nu)) \right| \approx C \left| \sum_{i=1}^{M} (c_{i,n} - 1) \right|. \\
\]

Hence, the following remarks of independent interest are in order.

(i) Observations (26), and (27) reveal that either of the sole moment convergences \( \mathbb{E}(F_n^3) \to \mathbb{E}(G(\nu)^3) \) or \( \mathbb{E}(F_n^4) \to \mathbb{E}(G(\nu)^4) \) implies convergence in distribution of the sequence \( F_n \) towards the target distribution \( G(\nu) \). In other words, the third moment criterion implies the fourth moment criterion and vice versa. Such phenomenon has been already observed in the case of normal approximation, see [NV16].

(ii) It is worth mentioning that if \( M = \nu \geq 5 \), then [Zin13, Theorem 1.2] yields that in fact, in the stronger distance \( d_{TV} \), there exists a constant \( C \) (may depends on sequence \( F_n \), but independent of \( n \)) such that for all \( n \geq 1 \),

\[
d_{TV}(F_n, G(\nu)) \leq C \max \{ |1 - c_{i,n}| : i = 1, \ldots, M \}. \\
\]

Hence,

\[
d_{TV}(F_n, G(\nu)) \leq C \sqrt{\max \{ |\kappa_3(F_n) - \kappa_3(G(\nu))|, |\kappa_4(F_n) - \kappa_4(G(\nu))| \}}. \\
\]

We conjecture that in this setting, the estimate (28) continues to hold when removing the assumption \( \nu \geq 5 \). See also Proposition 4.22 in Section 4.5., and Conjecture 6.9.
Proposition 4.15. (The case of ultimately infinitely many non-zero eigenvalues) Let $\nu > 0$, and $(M_n)_{n \geq 1} \subset \mathbb{N} \cup \{+\infty\}$ be a sequence such that $M_n \uparrow \infty$. Consider a sequence $(F_n)_{n \geq 1}$ of random elements in the second Wiener chaos such that $\mathbb{E} [F_n^2] = 2\nu$ for all $n \geq 1$, possessing the following representation

$$F_n = \sum_{1 \leq i \leq M_n} c_{i,n}(N_i^2 - 1), \quad n \geq 1.$$ 

Also, we assume that $F_n$ converges in distribution towards a centered Gamma distribution $G(\nu)$ with parameter $\nu > 0$. Then, the asymptotic relation

$$\text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) \approx_C \text{Var}^2 (\Gamma_1(F_n) - 2F_n)$$

holds if and only if $\vartheta(n) \approx_C \omega(n)$. Consequently, whenever the aforementioned asymptotic condition takes place, there exist two constants $0 < C_1 < C_2$ (may depend on sequence $F_n$, but independent of $n$) such that for all $n \geq 1$,

$$C_1 \text{Var}^2 (\Gamma_1(F_n) - 2F_n) \leq \text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) \leq C_2 \text{Var}^2 (\Gamma_1(F_n) - 2F_n).$$

Proof. First note that since $M_n \uparrow \infty$, we have $M_n > \nu$ for large enough values of $n$. So without loss of generality, we assume $M_n = \infty$ for all $n \geq 1$. Using Proposition 4.14, we deduce that $\nu$ is an integer, and there exists a set $I \subset \mathbb{N}$ (independent of $n$) with $I = \{i : c_{i,n} \rightarrow 1 \text{ as } n \rightarrow \infty\}$, and also $\#I = \nu$. Then relation (24) yields that

$$\text{Var} (\Gamma_1(F_n) - 2F_n) \approx_C \omega(n)$$

if and only if $\vartheta(n) \approx_C \omega(n)$, where as before $\omega(n) = \max\{|1 - c_{i,n}| : i \in I\}$. Note that $\#I^c = \infty$. We claim that

$$\sum_{i \in I^c} c_{i,n}^6 = o(\omega(n)^2).$$

To this end, take a nested sequence $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_m \subseteq A_{m+1} \subseteq \ldots$ such that $A_m \rightarrow I^c$ as $m \rightarrow \infty$, and $\#A_m < \infty$ for all $m \geq 1$. Define

$$x_{m,r}(n) := \sum_{i \in A_m} c_{i,n}^r.$$ 

Then for each $m \in \mathbb{N}$, the estimate $x_{m,2}(n) \leq \sum_{i \in I^c} c_{i,n}^2 = \vartheta(n) \leq 2\nu \omega(n)$ holds. So the above analysis, together with the fact that $\#A_m$ is finite for $m \geq 1$, tells us that

$$x_{m,6}(n) = o(\omega(n)^2), \quad \forall m \geq 1.$$ 

Now, taking into account that $x_{m,6} \rightarrow x_{\infty,6}(n) := \sum_{i \in I^c} c_{i,n}^6$, as $m \rightarrow \infty$, and each $x_{m,6}(n) = o(\omega(n)^2)$, a direct application of monotone convergence theorem implies that $x_{\infty,6} = o(\omega(n)^2)$. Therefore,

$$\text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) = 2^7 \sum_{i=1}^{\infty} c_{i,n}^6 (1 - c_{i,n})^2$$

$$= 2^7 \left\{ \sum_{i \in I} c_{i,n}^6 (1 - c_{i,n})^2 + \sum_{i \in I^c} c_{i,n}^6 (1 - c_{i,n})^2 \right\} \approx_C \max\{|1 - c_{i,n}|^2 : i \in I\} \approx_C \omega(n)^2.$$

Hence the claim follows. $\square$
4.3 An Optimal Theorem

Now we are ready to state our main theorem providing an optimal rate of convergence in terms of the third and the fourth cumulants. The following result provides an analogous counterpart to the same phenomenon in the case of normal approximation on arbitrary Wiener chaos, see [NP15, Theorem 1.2].

**Theorem 4.16.** Let $\nu > 0$. Assume that

$$(F_n)_{n \geq 1} = \left( \sum_{i \geq 1} c_{i,n} (N_i^2 - 1) \right)_{n \geq 1}$$

is a sequence of elements in the second Wiener chaos such that $E(F_n^2) = 2 \sum_{i \geq 1} c_{i,n}^2 = 2\nu$ for all $n \geq 1$. Assume, in addition, as $n \to \infty$, that

$$\text{Var} (\Gamma_1(F_n) - 2F_n) \to 0. \quad (29)$$

Then $F_n$ converges in distribution towards a centered Gamma distribution $G(\nu)$ with parameter $\nu$. Furthermore, when $\vartheta(n) \approx_C \omega(n)$, where $\vartheta(n)$ and $\omega(n)$ are as in (22), then there exist two constants $0 < C_1 < C_2$ (possibly depending on the sequence $F_n$, but independent of $n$) such that for all $n \geq 1$,

$$C_1 M(F_n) \leq d_2(F_n, G(\nu)) \leq C_2 M(F_n), \quad (30)$$

where as before

$$M(F_n) := \max \left\{ \left| \kappa_3(F_n) - \kappa_3(G(\nu)) \right|, \left| \kappa_4(F_n) - \kappa_4(G(\nu)) \right| \right\}.$$

**Proof.** The asymptotic relation (29) implies that $F_n$ converges in distribution towards a centered Gamma distribution $G(\nu)$, which is a well known fact, see for example [NP09a].

(upper bound): This is a direct application of Theorem 3.1, Corollary 4.13, and Proposition 4.15. (lower bound): Fix a real number $\rho > 0$ whose range of values will be determined later on. Taking into account the second moment assumptions, it is a classical result (see [Luk70, Chapter 7]) that the characteristic functions $\varphi_{F_n}$ and $\varphi_{G(\nu)}$ are analytic inside the strip $\Delta_\nu := \{ z \in \mathbb{C} : |\text{Im} z| < \frac{1}{2\sqrt{\nu}} \}$. Moreover, in the strip of regularity $\Delta_\nu$, they follow the integral representations

$$\varphi_{F_n}(z) = \int_{\mathbb{R}} e^{izx} \mu_n(dx) \quad \text{and} \quad \varphi_{G(\nu)}(z) = \int_{\mathbb{R}} e^{izx} \mu_\nu(dx),$$

where $\mu_n$ and $\mu_\nu$ stand for the probability measures of $F_n$ and $G(\nu)$ respectively. Recall that all elements in the second Wiener chaos have exponential moments, see [NP12, Proposition 2.7.13, item (iii)]. Denote by $\Omega_{\rho,\nu}$ the domain

$$\Omega_{\rho,\nu} := \left\{ z = t + iy \in \mathbb{C} : |\text{Re} z| < \rho, |\text{Im} z| < \min\{ \frac{1}{2\sqrt{\nu}}, e^{-1} \} \right\}.$$
Then for any \( z \in \Omega_{\rho, \nu} \), together with a Fubini’s argument, we have that
\[
|\varphi_{F_n}(z) - \varphi_{G(\nu)}(z)| = \left| \int_{\mathbb{R}} e^{itx-yz} (\mu_n - \mu_\nu)(dx) \right| = \left| \sum_{k \geq 0} \frac{(-y)^k}{k!} \int_{\mathbb{R}} x^k e^{itx} (\mu_n - \mu_\nu)(dx) \right|
\leq \sum_{k \geq 0} \frac{e^{-k}}{k!} \left| \varphi^{(k)}(t) - \varphi^{(k)}(G(\nu)) \right| \leq \sum_{k \geq 0} \frac{e^{-k}}{k!} \rho^{k+1} d_2(F_n, G(\nu))
= \rho e^{\rho e^{-1}} d_2(F_n, G(\nu)).
\]
Hence \( |\varphi_{F_n}(z) - \varphi_{G(\nu)}(z)| \leq C_\rho \ d_2(F_n, G(\nu)) \) for every \( z \in \Omega_{\rho, \nu} \). Let \( R > 0 \) such that the disk \( D_R \subset \mathbb{C} \) with the origin as center and radius \( R \) is contained in the domain \( \Omega_{\rho, \nu} \) (note that \( R \) depends only on \( \nu \), since \( \rho \) is a free parameter. For example, one can choose \( \min\{2(\sqrt{\nu})^{-1}, e^{-1}\} < \rho < 2 \min\{2(\sqrt{\nu})^{-1}, e^{-1}\} \). Now for any \( z \in D_R \), and using the fact that
\[
\frac{1}{\varphi_{G(\nu)}^2(z)} = \left( e^{2iz} (1 - 2iz) \right)^\nu,
\]
one can readily conclude that the function \( \varphi_{G(\nu)}(z) \) is bounded away from 0 on the disk \( D_R \). Also, for any \( r \geq 2 \),
\[
|\kappa_r(F_n)| \leq 2^{r-1}(r-1)! \sum_{i \geq 1} |c_{i,n}|^r \leq 2^{r-1}(r-1)! \max_i |c_{i,n}|^{r-2} \sum_{i \geq 1} |c_{i,n}|^2 \quad \text{(31)}
\]
Therefore, for any \( z \in D_R \),
\[
\frac{1}{\varphi_{F_n}(z)} \leq \exp \left\{ \sum_{r \geq 2} \frac{|\kappa_r(F_n)|}{r!} |z|^r \right\} \leq \exp \left\{ \sum_{r \geq 2} \frac{2^{r-2}(r-1)! \sqrt{\nu}^r}{r!} |z|^r \right\}
\leq \exp \left\{ \sum_{r \geq 2} \frac{2^{r-2}(r-1)! \sqrt{\nu}^r}{r!} R^r \right\} := C_{R, \nu} < \infty.
\]
Hence the function \( \varphi_{F_n}(z) \) is also bounded away from 0 on the disk \( D_R \). Also, relation (31) implies that the following power series (complex variable) converge to some analytic function as soon as \( |z| < R \);
\[
\sum_{r \geq 1} \frac{\kappa_r(F_n)(iz)^r}{r!}, \quad \sum_{r \geq 1} \frac{\kappa_r(G(\nu))(iz)^r}{r!}
\]
Thus we come to the conclusion that the functions \( \varphi_{G(\nu)}(z) \) and \( \varphi_{F_n}(z) \) are analytic on the disk \( D_R \). Moreover, there exists a constant \( c > 0 \) such that \( |\varphi_{G(\nu)}(z)|, |\varphi_{F_n}(z)| \geq c > 0 \) for every \( z \in D_R \). This implies that on the disk \( D_R \) there exist two analytic functions \( g_n \) and \( g_\nu \) such that
\[
\varphi_{F_n}(z) = e^{g_n(z)}, \quad \varphi_{G(\nu)}(z) = e^{g_\nu(z)},
\]
i.e. \( g_n(z) = \log(\varphi_{F_n}(z)) \) and \( g_\nu(z) = \log(\varphi_{G(\nu)}(z)) \), for \( z \in D_R \). In fact, the functions \( g_n \) and \( g_\nu \) are given by the power series (32). Since the derivative of the analytic branch of the
complex logarithm is \((\log z)' = \frac{1}{z}\) (see Con95 Corollary 2.21), one can infer that for some constant \(C\), whose value may differ from line to line, and for every \(z \in D_R\), we have
\[
\left| \sum_{r \geq 2} \frac{\kappa_r(F_n) - \kappa_r(G(\nu))}{r!} (iz)^r \right| = \left| \log(\varphi_{F_n}(z)) - \log(\varphi_{G(\nu)}(z)) \right| \\
\leq C \left| \varphi_{F_n}(z) - \varphi_{G(\nu)}(z) \right| \leq C d_2(F_n, G(\nu)).
\]

Now, using Cauchy’s estimate for the coefficients of analytic functions, for any \(r \geq 3\), we obtain that
\[
\left| \kappa_r(F_n) - \kappa_r(G(\nu)) \right| \leq r! R^r \sup_{|z| \leq R} \left| \log \varphi_{F_n}(z) - \log \varphi_{G(\nu)}(z) \right|.
\]

Therefore,
\[
\max \left\{ \left| \kappa_3(F_n) - \kappa_3(G(\nu)) \right|, \left| \kappa_4(F_n) - \kappa_4(G(\nu)) \right| \right\} \leq C d_2(F_n, G(\nu)).
\]
\[
\square
\]

To demonstrate the power of Theorem 4.16, we consider a second order U-statistic with degeneracy order 1. The following example is taken from [AAP817 Section 3.1].

**Example 4.17.** Let \(\{h_i\}_{i \geq 1}\) be an orthonormal basis of \(\mathcal{H}\) and for \(i \geq 1\) set \(Z_i := I_1(h_i)\). For \(a \neq 0\) consider
\[
U_n = \frac{2a}{n(n-1)} \sum_{1 \leq i \neq j \leq n} Z_i Z_j = I_2 \left( \frac{2a}{n(n-1)} \sum_{1 \leq i \neq j \leq n} h_i \otimes h_j \right).
\]

Then \(nU_n \overset{D}{\to} a(Z_1^2 - 1)\) as \(n \to \infty\). Since the target is only distributed according to a centered Gamma distribution if \(a = 1\), we will restrict ourselves to this case and write \(G(1)\) for the target. Furthermore, in our setting, we need to fix the variance of our sequence to 2. Hence we consider
\[
W_n := \sqrt{\frac{n-1}{n}} nU_n = I_2 \left( \frac{2}{\sqrt{n(n-1)}} \sum_{1 \leq i \neq j \leq n} h_i \otimes h_j \right)
\]
\[
= I_2 \left( \frac{1}{\sqrt{n(n-1)}} \sum_{1 \leq i, j \leq n, i \neq j} h_i \otimes h_j \right) =: I_2(f_n)
\]

We consider the associated Hilbert-Schmidt operator \(A_{f_n} g = f_n \otimes 1 g\). Using the fact that \(\langle h_i \otimes h_j \rangle \otimes 1 h_k = \langle h_i, h_k \rangle_{\mathcal{H}} h_j\) we can explicitly compute the non-zero eigenvalues \(c_{1,n}, \ldots, c_{n,n}\) of \(A_{f_n}\). They are
\[
c_{1,n} = \sqrt{\frac{n-1}{n}}, \text{ and } c_{2,n} = \ldots = c_{n,n} = \frac{-1}{\sqrt{n(n-1)}}.
\]

23
Since our target has 1 degree of freedom, the assumptions of Theorem 4.16 are in order (see Proposition 4.10(a)) and thus the optimality result holds for $W_n$. Also, with the eigenvalues given above and Lemma 4.3, one may verify manually that $\text{Var}(\Gamma_3(W_n) - 2\Gamma_2(W_n)) \approx C\text{Var}^2(\Gamma_1(W_n) - 2W_n) \approx C.\frac{1}{n^2}$. As a consequence

$$d_2(W_n, G(1)) \approx C \left| \kappa_3(W_n) - \kappa_3(G(1)) \right| \approx C \left| \kappa_4(W_n) - \kappa_4(G(1)) \right| \approx C \frac{1}{n}.\left| \kappa_4(W_n) - \kappa_4(G(1)) \right|$$

4.4 Trace Class Operators

Lemma 4.18. Let $F = I_2(f)$ be a random element in the second Wiener chaos such that $A_f^3 - A_f^4 \geq 0$ (or $\leq 0$) is a non-negative (or non-positive) operator, where $A_f$ is the associated Hilbert-Schmidt operator. Then

$$\text{Var}(\Gamma_3(F) - 2\Gamma_2(F)) \leq 2 \times 3!^2 \left( \kappa_4(F) - 6\kappa_3(F) \right)^2.$$

Proof. Using relation (20), and the main result of [Liu07], one can write

$$\text{Var}(\Gamma_3(F) - 2\Gamma_2(F)) = \frac{1}{7!}\kappa_3(F) - \frac{4}{6!}\kappa_7(F) + \frac{4}{5!}\kappa_6(F)$$

$$= 2^7 \text{Tr}(A_f^3) - 2^6 \text{Tr}(A_f^7) + 2^7 \text{Tr}(A_f^7) = 2^7 \text{Tr}(A_f^3 - 2A_f^7 + A_f^6)$$

$$= 2^7 \text{Tr}\left((A_f^3 - A_f^4)^2\right) \leq 2^7 \left(\text{Tr}(A_f^4 - A_f^3)\right)^2 = 2 \times 3!^2 \left( \kappa_4(F) - 6\kappa_3(F) \right)^2.$$

Now, we can state the following non asymptotic version of the optimal rate of convergence towards the centered Gamma distribution $G(\nu)$.

Proposition 4.19. Let $\nu > 0$. Assume that $F = I_2(f)$ is a random element in the second Wiener chaos such that $E(F^2) = 2\nu$. Moreover, assume that $A_f^3 - A_f^4 \geq 0$ (or $\leq 0$). Then there exist two constants $0 < C_1 < C_2$, such that

$$C_1 M(F) \leq d_2(F, G(\nu)) \leq C_2 M(F),$$

where, as before, $M(F) := \max\left\{\left| \kappa_3(F) - \kappa_3(G(\nu)) \right|, \left| \kappa_4(F) - \kappa_4(G(\nu)) \right|\right\}$.

Proof. For the upper bound, combine Theorem 3.4 together with Lemma 4.6 and Lemma 4.18. The lower bound is derived from Theorem 4.16.

We close this section with two lemmas of independent interests. The first lemma gathers some non-asymptotic variance-estimates and will be used in the proof of Lemma 5.1 in Section 5. The second lemma displays that differences of all higher cumulants can be controlled from above by the quantity $M(F)$.
Lemma 4.20. Let \( F = I_2(f) \) be a general element in the second Wiener chaos. Then, for \( r \geq 1 \), the following estimates hold.

\[
\Var^2(\Gamma_{r+1}(F) - 2\Gamma_r(F)) \leq C \Var(\Gamma_r(F) - 2\Gamma_{r-1}(F)) \times \Var(\Gamma_{r+2}(F) - 2\Gamma_{r+1}(F)), \tag{34}
\]

\[
\Var^{2r}(\Gamma_2(F) - 2\Gamma_1(F)) \leq C \Var^{2r-1}(\Gamma_1(F) - 2F) \times \Var(\Gamma_{2r+1}(F) - 2\Gamma_{2r}(F)), \tag{35}
\]

where the general constant \( C \) is independent of \( F \). In particular,

\[
\Var^2(\Gamma_2(F) - 2\Gamma_1(F)) \leq C \Var(\Gamma_1(F) - 2F) \times \Var(\Gamma_3(F) - 2\Gamma_2(F)) .
\]

Moreover,

\[
\Var\left( (\Gamma_{2r+1}(F) - 2\Gamma_{2r}(F)) - 2(\Gamma_{2r}(F) - 2\Gamma_{2r-1}(F)) \right) \leq C \Var^2(\Gamma_r(F) - 2\Gamma_{r-1}(F))
\]

\[
\leq C \Var(\Gamma_{r-1}(F) - 2\Gamma_{r-2}(F)) \times \Var(\Gamma_{r+1}(F) - 2\Gamma_{r}(F)) .
\]

Proof. This is a direct application of \cite[Corollary 1]{Dra16b} with \( P = (A_j^{r+1} - A_j^r)^2, C = A_j^2 \), and the fact that, for \( r \geq 0 \), we have

\[
\Var(\Gamma_{r+1}(F) - 2\Gamma_r(F)) = 2^{2r+3} \Tr\left((A_j^{r+2} - A_j^{r+1})^2\right) .
\]

The estimate \cite{Dra16b} is also an application of \cite[Corollary 1]{Dra16a} with \( P = (A_j^2 - A_j)^2 \), and the convex function \( f(x) = x^{2r} \).

Lemma 4.21. Let \( \nu > 0 \), and \( F = I_2(f) \) in the second Wiener chaos so that \( \EE[F^2] = 2\nu \). Then, for every \( r \geq 1 \), there exists a constant \( C \) (depending only on \( \nu \), and \( r \)) such that

\[
\left| \EE[\Gamma_{r+1}(F)] - 2\EE[\Gamma_r(F)] \right| \leq C \, M(F), \tag{36}
\]

and also,

\[
\left| \kappa_r(F) - \kappa_r(G(\nu)) \right| \leq C \, M(F). \tag{37}
\]

Proof. We proof estimate \cite{Dra16a} by induction on \( r \). The estimate \cite{Dra16b} can be derived in a similar way. Obviously \cite{Dra16a} holds for \( r = 1, 2 \), so we assume that \( r \geq 3 \). Note that

\[
\left| \EE[\Gamma_{r+2}(F)] - 2\EE[\Gamma_{r+1}(F)] \right| = \left| \frac{\kappa_{r+3}(F)}{(r+2)!} - 2 \frac{\kappa_{r+2}(F)}{(r+1)!} \right|
\]

\[
\leq C \left| \frac{\kappa_{r+3}(F)}{(r+2)!} - 4 \frac{\kappa_{r+2}(F)}{(r+1)!} + 4 \frac{\kappa_{r+1}(F)}{r!} \right| + \left| \frac{\kappa_{r+2}(F)}{(r+1)!} - 2 \frac{\kappa_{r+1}(F)}{r!} \right|
\]

\[
= C \left| \frac{\kappa_{r+3}(F)}{(r+2)!} - 4 \frac{\kappa_{r+2}(F)}{(r+1)!} + 4 \frac{\kappa_{r+1}(F)}{r!} \right| + \left| \EE[\Gamma_{r+1}(F)] - 2\EE[\Gamma_r(F)] \right|.
\]

The second summand on the right hand side can be handled with the induction hypothesis. For the first summand, we have two possibilities. If \( r = 2s+1 \), for some \( s \geq 1 \), then

\[
\frac{\kappa_{r+3}(F)}{(r+2)!} - 4 \frac{\kappa_{r+2}(F)}{(r+1)!} + 4 \frac{\kappa_{r+1}(F)}{r!} \leq \Var(\Gamma_{s+1}(F) - 2\Gamma_s(F)) \leq C \Var(\Gamma_1(F) - 2F),
\]

25
and so we are done. Otherwise, \( r = 2s \) for some \( s \geq 2 \). Hence, using Cauchy–Schwarz inequality, we obtain that

\[
\left| \frac{\kappa_{r+3}(F)}{(r+2)!} - 4\frac{\kappa_{r+2}(F)}{(r+1)!} + 4\frac{\kappa_{r+1}(F)}{r!} \right| = \left| \frac{\kappa_{r+3}(F)}{(r+2)!} - 4\frac{\kappa_{r+2}(F)}{(2s+2)!} + 4\frac{\kappa_{r+1}(F)}{(2s+1)!} \right|
\]

\[
\leq C \left( \Var(\Gamma_{r+1}(F) - 2\Gamma_s(F)) \right) \times \left( \Var(\Gamma_s(F) - 2\Gamma_{s-1}(F)) \right)
\]

\[
\leq C \Var(\Gamma_1(F) - 2F).
\]

For the estimate (37), note that \( \kappa_{r+1}(G(\nu)) = 2r\kappa_r(G(\nu)) \). Therefore,

\[
\left| \kappa_{r+1}(F) - \kappa_{r+1}(G(\nu)) \right| \leq \left| \kappa_{r+1}(F) - 2r\kappa_r(F) \right| + \left| 2r\kappa_r(F) - \kappa_{r+1}(G(\nu)) \right|
\]

\[
\leq C \left( \Var(\Gamma_r(F)) - 2\Var(\Gamma_{r-1}(F)) \right) + \left| \kappa_r(F) - \kappa_r(G(\nu)) \right|.
\]

\[
\square
\]

4.5 A Further Example: Optimal Rate in Total Variation Distance

In this section we introduce a concrete example of a sequence within the second Wiener chaos. The corresponding Hilbert–Schmidt will only have two non-zero eigenvalues, both of which are converging to 1. A crucial observation is that although the presented example lies out of the favorable regimes discussed in Section 4.2.2, the optimal rate \( M(F_n) \) insists to hold in total variation distance.

**Proposition 4.22.** Consider the sequence \( \{F_n = c_{1,n}(N_1^2 - 1) + c_{2,n}(N_2^2 - 1) \}_{n \geq 1} \) in the second Wiener chaos where \( c_{1,n} = \sqrt{1 + \frac{1}{n}} \) and \( c_{2,n} = \sqrt{1 - \frac{1}{n}} \). Then

\[
d_{TV}(F_n, G(2)) \approx_C \max \left\{ |\kappa_3(F_n) - \kappa_3(G(2))|, |\kappa_4(F_n) - \kappa_4(G(2))| \right\} \approx_C \frac{1}{n^2}.
\]

**Proof.** First note that

\[
\kappa_4(F_n) - \kappa_4(G(2)) = 48 \sum_{j=1}^{2} \left( e_{j,n} - 1 \right) = 48 \frac{2}{n^2} \approx_C \frac{1}{n^2}.
\]

Similarly \( \kappa_3(F_n) - \kappa_3(G(2)) = 8 \sum_{j=1}^{2} \left( e_{j,n}^2 - 1 \right) \approx_C \frac{1}{n^2} \). To shorten notation, we write \( c_1 \) and \( c_2 \) instead of \( c_{1,n} \) and \( c_{2,n} \). We start by computing \( \varphi_n \), the probability density function of \( F_n \). The density of \( F_n \) is given by

\[
\varphi_n(x) = \frac{1}{2\pi \sqrt{c_1 c_2}} \int_{-c_1}^{x+c_2} e^{-\frac{1}{2} \left( \frac{t^2}{c_1} + \frac{x-t+c_2}{c_2} \right)} dt \cdot 1_{\{x > -c_1-c_2\}}(x)
\]

\[
= \frac{1}{2\pi \sqrt{c_1 c_2}} \int_{-c_1}^{x+c_2} e^{-\frac{1}{2} \left( \frac{t^2}{c_1} + \frac{x-t+c_2}{c_2} \right)} \cdot 1_{\{x > -c_1-c_2\}}(x) dt.
\]

26
Substituting \( t = (c_1 + c_2 + e)u - c_1 \), we get
\[
\varphi_n(x) = \frac{1}{2\pi \sqrt{c_1c_2}} e^{-\frac{x^2}{2c_1c_2}} \int_0^1 \frac{e^{(c_1-c_2)(c_1+c_2+x)u} - c_1-c_2}{2c_1c_2} (x + c_1 + c_2) \, du \mathbb{1}_{\{x > -c_1-c_2\}}(x)
\]
\[
= \frac{1}{2\sqrt{c_1c_2}} e^{-\frac{x+c_1}{2c_2}-\frac{1}{4}} \times \frac{1}{\pi} \int_0^1 \frac{e^{(c_1-c_2)(c_1+c_2+x)}u}{\sqrt{1-u}} \frac{1}{\sqrt{1-u}} \, du \mathbb{1}_{\{x > -c_1-c_2\}}(x)
\]
\[
= \frac{1}{2\sqrt{c_1c_2}} e^{-\frac{x+c_1}{2c_2}-\frac{1}{4}} \times \, _1F_1 \left( \frac{1}{2}, 1, \frac{c_1-c_2}{2c_1c_2} (c_1 + c_2 + x) \right) \times \mathbb{1}_{\{x > -c_1-c_2\}}(x).
\]
Here, \(_1F_1\) is the confluent hypergeometric function, which can be represented as
\[
_1F_1(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zu} u^{a-1} (1-u)^{b-a-1} \, du
\]
for \( \Re(b) > \Re(a) > 0 \). Note that if \( a = 1/2 \) and \( b = 1 \), we get
\[
_1F_1 \left( \frac{1}{2}, 1, z \right) = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} \int_0^1 e^{zu} \frac{1}{\sqrt{u}} \frac{1}{\sqrt{1-u}} \, du = \frac{1}{\pi} \int_0^1 e^{zu} \frac{1}{\sqrt{u}} \frac{1}{\sqrt{1-u}} \, du.
\]
Also note that the roles of \( c_1 \) and \( c_2 \) are completely interchangeable. It is just a matter of how we write down the convolution. Thus we can also write
\[
\varphi_n(x) = \frac{1}{2\sqrt{c_1c_2}} e^{-\frac{x+c_1}{2c_1c_2}} \times _1F_1 \left( \frac{1}{2}, 1, -\frac{c_1-c_2}{2c_1c_2} (c_1 + c_2 + x) \right) \times \mathbb{1}_{\{x > -c_1-c_2\}}(x).
\]
Also recall that the density of the target \( G(2) \) is given by
\[
\psi(x) = \frac{1}{2} e^{-\frac{x}{2}} \mathbb{1}_{\{x > -2\}}(x).
\]
The next step is to explicitly write down the total variation distance in terms of the density functions:
\[
d_{TV}(F_n, G(2)) = \frac{1}{2} \int_{-\infty}^{\infty} |\varphi_n(x) - \psi(x)| \, dx
\]
\[
= \frac{1}{2} \int_{-c_1-c_2}^{\infty} \psi(x) \, dx + \frac{1}{2} \int_{-c_1-c_2}^{\infty} \varphi_n(x) - \psi(x) \, dx
\]
\[
= \frac{1}{2} \left( 1 - e^{-\frac{c_1+c_2}{2}+1} \right) + \frac{1}{2} \int_{-c_1-c_2}^{\infty} \varphi_n(x) - \psi(x) \, dx
\]
\[
= : \frac{1}{2} (\alpha_1(n) + \alpha_2(n)).
\]
One can readily check that \( \alpha_1(n) \approx C \frac{1}{n^2} \). To examine the asymptotic behaviour of \( \alpha_2(n) \), we write
\[
\varphi_n(x) - \psi(x)
\]
\[
= \frac{1}{2} e^{-\frac{x}{2}} \left[ \frac{1}{\sqrt{c_1c_2}} e^{\frac{c_1+c_2}{2c_1}} e^{\frac{x}{2} - \frac{1}{2c_1}} _1F_1 \left( \frac{1}{2}, 1, -\frac{c_1-c_2}{2c_1c_2} (c_1 + c_2 + x) \right) - e^{-1} \right],
\]

27
and find a series expansion for the term inside the square brackets. Expanding \( \, _1F_1 \) as a series (see e.g. [AS72, p. 504]), we get

\[
_1F_1 \left( \frac{1}{2}, 1, -\frac{c_1 - c_2}{2c_1c_2} (c_1 + c_2 + x) \right) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma \left( \frac{1}{2} + k \right)}{\Gamma (1 + k)} \left[ -\frac{c_1 - c_2}{2c_1c_2} (c_1 + c_2 + x) \right]^k.
\]

On the other hand, we can expand the exponential around \(-c_1 - c_2\) as

\[
x \left( \frac{1}{2} - \frac{c_1}{c_2} \right) = e \left( -c_1 - c_2 \right) \left( \frac{1}{2} - \frac{\sqrt{x}}{c_1} \right) \left( x + c_1 + c_2 \right)^k.
\]

Thus, we obtain the following series expansion

\[
x \left( \frac{1}{2} - \frac{c_1}{c_2} \right) \times _1F_1 \left( \frac{1}{2}, 1, -\frac{c_1 - c_2}{2c_1c_2} (c_1 + c_2 + x) \right)
= \frac{e \left( -c_1 - c_2 \right) \left( \frac{1}{2} - \frac{\sqrt{x}}{c_1} \right)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{\Gamma \left( \frac{1}{2} + \ell \right)}{\Gamma (1 + \ell)} \left( -\frac{c_1 - c_2}{2c_1c_2} (c_1 + c_2 + x) \right)^{\ell} \left( \frac{1}{2} - \frac{\sqrt{x}}{c_1} \right)^{k-\ell} \left( x + c_1 + c_2 \right)^{k-\ell}.
\]

Now

\[
2 \alpha_2(n) = \int_{-\frac{\pi}{c_1 - c_2}}^{\infty} e^{-\frac{x}{2c_1c_2}} \left[ \frac{1}{\sqrt{c_1c_2}} e^{-\frac{c_1 + c_2}{2c_1c_2}} \left( \sum_{k=0}^{\infty} A_k(c_1, c_2)(x + c_1 + c_2)^k \right) - e^{-1} \right] \ dx
= \left( \frac{e^{-\frac{\pi}{2}(c_1+c_2)}}{\sqrt{c_1c_2}} - e^{-1} \right) \int_{-\frac{\pi}{c_1 - c_2}}^{\infty} e^{-\frac{x}{2c_1c_2}} \ dx 
+ \sum_{k=1}^{\infty} \frac{1}{\sqrt{c_1c_2}} e^{-\frac{c_1 + c_2}{2c_1c_2}} A_k(c_1, c_2) \int_{-\frac{\pi}{c_1 - c_2}}^{\infty} e^{-\frac{x}{2c_1c_2}} (x + c_1 + c_2)^k \ dx.
\]

Using the fact that \( \int_{-\frac{\pi}{c_1 - c_2}}^{\infty} \exp \left( -\frac{x}{2c_1c_2} \right) (x + c_1 + c_2)^k = k! 2^{k+1} \exp \left( \frac{c_1 + c_2}{2c_1c_2} \right) \) for all \( k \in \mathbb{N}_0 \), and setting

\[
B_1(c_1, c_2) := 2 \sqrt{c_1c_2} \quad \text{and} \quad B_2(c_1, c_2, k) := \frac{1}{\sqrt{c_1c_2}} e^{-\frac{c_1 + c_2}{2c_1c_2}} \times k! 2^{k+1} \exp \left( \frac{c_1 + c_2}{2c_1c_2} \right) \times \frac{\left( -c_1 - c_2 \right) \left( \frac{1}{2} - \frac{\sqrt{x}}{c_1} \right)}{\sqrt{\pi} 2^k} = \frac{2 k!}{\sqrt{c_1c_2\sqrt{\pi}}}.
\]

\[
28
\]
we get
\[
2 \alpha_2(n) = B_1(c_1, c_2) \left( \frac{e^{-\frac{1}{2}(c_1+c_2)}}{\sqrt{c_1c_2}} - e^{-1} \right) + \sum_{k=1}^{\infty} B_2(c_1, c_2, k) \left( \sum_{\ell=0}^{k} \frac{\Gamma(\frac{1}{2} + \ell) \left( -\frac{c_1-c_2}{c_1c_2} \right)^\ell \left( 1 + \frac{1}{c_1} \right)^{k-\ell}}{\Gamma(1+\ell) \ell! (k-\ell)!} \right),
\]
where \( B_1(c_1, c_2) \) and \( B_2(c_1, c_2, k) \) converge (for fixed \( k \)) to a constant as \( n \to \infty \), and thus do not contribute to the rate of convergence. One can easily check that \( \frac{e^{-\frac{1}{2}(c_1+c_2)}}{\sqrt{c_1c_2}} - e^{-1} \approx C \frac{1}{n^2} \) as \( n \to \infty \). All the other terms are of the form “something that converges to a constant” \( \times \) “a polynomial in \( c_1 \) and \( c_2 \)”. However, the terms for \( k = 1 \) and \( k = 2 \) also have the same rate of convergence, whereas the terms for \( k \geq 3 \) converge to zero at a faster rate. More precisely, we have
\[
2 \alpha_2(n) = 2 \left( \frac{1}{\sqrt{c_1c_2}} - e^{-\frac{1}{2}(c_1+c_2)} \right) + \frac{1}{\sqrt{c_1c_2c_1c_2}} (2c_1c_2 - c_1 - c_2)
\]
\[
+ \sum_{k=3}^{\infty} \frac{1}{4\sqrt{c_1c_2c_1c_2}} (8c_1^2c_2^2 - 8c_1c_2^2 + 3c_1^2 - 8c_1c_2^2 + 2c_1c_2 + 3c_2^2)
\]
\[
+ \sum_{k=3}^{\infty} B_2(c_1, c_2, k) \left( \sum_{\ell=0}^{k} \frac{\Gamma(\frac{1}{2} + \ell) \left( -\frac{c_1-c_2}{c_1c_2} \right)^\ell \left( 1 + \frac{1}{c_1} \right)^{k-\ell}}{\Gamma(1+\ell) \ell! (k-\ell)!} \right)
\]
\[
=: C(c_1, c_2) + \sum_{k=3}^{\infty} B_2(c_1, c_2, k) \left( \sum_{\ell=0}^{k} \frac{\Gamma(\frac{1}{2} + \ell) \left( -\frac{c_1-c_2}{c_1c_2} \right)^\ell \left( 1 + \frac{1}{c_1} \right)^{k-\ell}}{\Gamma(1+\ell) \ell! (k-\ell)!} \right).
\]
After some computations, we see that, as \( n \to \infty \), \( \frac{C(c_1, c_2)}{1/n^2} \to 1 \), whereas the remaining terms converge faster.

\[
\square
\]

5 Gamma Characterisation Within the Second Wiener Chaos

Let \( \nu > 0 \) and \( G(\nu) \) be a centered Gamma distributed random variable. Assume that \( F \) is a random element in the second Wiener chaos such that \( \mathbb{E}[F^2] = 2\nu \). The proof of Proposition 3.4 reveals that
\[
d_2(F, G(\nu)) \leq C \left( |\kappa_3(F) - \kappa_3(G(\nu))| + \sqrt{\text{Var}(\Gamma_2(F) - 2\Gamma_1(F))} \right).
\]
From observation 3.8, it appears that the sole condition \( \text{Var}(\Gamma_2(F) - 2\Gamma_1(F)) = 0 \) may not be enough to conclude that the random variable \( F \) is distributed like \( G(\nu) \), and in addition, one needs to match the third cumulants \( \kappa_3(F) = \kappa_3(G(\nu)) \). A simple example outside the second Wiener chaos is \( F \sim \mathcal{N}(0, 1) \). Then, obviously, \( \text{Var}(\Gamma_2(F) - 2\Gamma_1(F)) = 0 \), but \( F \)
is not Gamma distributed. Note that \( \kappa_3(F) = 0 \), whereas \( \kappa_3(G(\nu)) \neq 0 \). The next lemma clarifies that this is not the case.

**Lemma 5.1.** Let \( r \geq 0 \) be an integer and suppose that \( F = I_2(f) \) belongs to the second Wiener chaos, such that \( \mathbb{E}[F^2] = 2\nu \), where \( \nu > 0 \). Then

\[
F \overset{\text{law}}{=} G(\nu) \quad \text{if and only if} \quad \Delta_r(F) := \text{Var}(\Gamma_{r+1}(F) - 2\Gamma_r(F)) = 0, \quad \text{for some} \quad r \geq 0.
\]

Moreover

\[
\cdots \leq C \Delta_{r+1}(F) \leq C \Delta_r(F) \leq C \Delta_{r-1}(F) \leq \cdots \leq C \Delta_0(F) = \text{Var}(\Gamma_1(F) - 2F).
\]

**Proof.** The chain of estimates in (39) follows from Lemma 4.6. Also, it is well known that if \( F \sim \text{CenteredGamma}(\nu) \), then \( \Delta_0(F) = 0 \), and therefore \( \Delta_r(F) = 0 \) for all \( r \geq 1 \). For the other direction let

\[
F = \sum_{i \geq 1} c_i (N_i^2 - 1), \quad \text{and} \quad \mathbb{E}[F^2] = 2 \sum_{i \geq 1} c_i^2 = 2\nu.
\]

Assume that

\[
0 = \Delta_1(F) = \text{Var}(\Gamma_2(F) - 2\Gamma_1(F)) = \frac{\kappa_5(F)}{5!} - \frac{4}{4!}\kappa_4(F) + \frac{4}{3!}\kappa_3(F)
\]

\[
= 2^5 \sum c_i^6 - 4 \times 2^4 \sum c_i^5 + 4 \times 2^3 \sum c_i^4
\]

\[
= 2 \times \sum \left( 2^5 c_i^6 - 4 \times 2^4 c_i^5 + 4 \times 2^3 c_i^4 \right)
\]

\[
= \sum \left( 2^3 c_i^3 - 2^2 c_i^2 \right)^2.
\]

Hence, we either have \( c_i = 0 \) for all \( i \geq 1 \), which is impossible, or \( c_i = 1 \) for all \( i \geq 1 \). Now together with the condition \( 2 \sum_{i \geq 1} c_i^2 = 2\nu \), we can deduce that there are only finitely many non-zero coefficients \( c_i \), and moreover that \( \nu \) is an integer. Hence

\[
F = \sum_{i = 1}^{\nu} (N_i^2 - 1).
\]

The general case \( r \geq 2 \) follows from Lemma 4.20.

For \( r, \lambda \in (0, \infty) \), in what follows, we denote by \( \Gamma(r, \lambda) \), the Gamma distribution with shape parameter \( r \), and rate \( \lambda \), which has the following probability density function

\[
p_{r,\lambda}(x) = \begin{cases} 
\frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, & \text{if } x \geq 0, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \Gamma(r) \) stands for Gamma function. The side goal of the next lemma is to provide other sufficient conditions for the validity of variance-estimate (17).
Lemma 5.2. Let \( \nu > 0 \). Assume that \( F = I_2(f) \) is in the second Wiener chaos such that \( \mathbb{E}[F^2] = 2\nu \), and moreover \( \text{Cov}(\Gamma_3(F), 2\Gamma_2(F), \Gamma_2(F) - 2\Gamma_1(F)) \geq 0 \) (see Remark 5.3 item (iv)). For \( \beta \in \mathbb{R} \), define the biquadratic function

\[
\Phi_F(\beta) := \text{Var}\left( (\Gamma_3(F) - 2\Gamma_2(F)) - 2\beta^2(\Gamma_2(F) - 2\Gamma_1(F)) \right).
\]

Then, function \( \Phi_F \) attains its local maximum at \( \beta_{\text{max}} = 0 \), and the global minimum at the points

\[
\beta_{\text{min}}^\pm = \pm \sqrt{\frac{\text{E}\left[ (\Gamma_3(F) - 2\Gamma_2(F)) (\Gamma_2(F) - 2\Gamma_1(F)) \right]}{2 \text{Var}(\Gamma_2(F) - 2\Gamma_1(F))}}.
\]

Also, the following statements are equivalent.

(a) \( F \overset{D}{=} G(\nu) \).

(b) \( \Phi_F(\beta) = 0 \), for some \( \beta \neq \beta_{\text{min}}^\pm \).

In particular, \( \Phi_F(1) = 0 \) implies that \( F \overset{D}{=} G(\nu) \), and

\[
0 \leq \Phi_F(1) = \frac{\kappa_3(F)}{7!} - 8\frac{\kappa_7(F)}{6!} + 24\frac{\kappa_8(F)}{5!} - 32\frac{\kappa_5(F)}{4!} + 16\frac{\kappa_4(F)}{3!} \leq 2 \text{Var}^2(\Gamma_1(F) - 2F). \tag{40}
\]

Let

\[
\beta_0 := \pm \sqrt{\frac{\text{E}\left[ (\Gamma_3(F) - 2\Gamma_2(F)) (\Gamma_2(F) - 2\Gamma_1(F)) \right]}{\text{Var}(\Gamma_2(F) - 2\Gamma_1(F))}}. \tag{41}
\]

(I) In the case \( \beta_0 \leq 1 \),

\[
\text{Var}(\Gamma_3(F) - 2\Gamma_2(F)) \leq 2 \text{Var}^2(\Gamma_1(F) - 2F).
\]

(II) In the case \( \beta_0 \geq 1 \),

\[
\Phi_F(\beta_{\text{min}}^-) = \Phi_F(\beta_{\text{min}}^+) = \text{Var}(\Gamma_3(F) - 2\Gamma_2(F))
\]

\[
- \left( \frac{\text{E}\left[ (\Gamma_3(F) - 2\Gamma_2(F)) (\Gamma_2(F) - 2\Gamma_1(F)) \right]}{\text{Var}(\Gamma_2(F) - 2\Gamma_1(F))} \right)^2
\]

\[
\leq \text{Var}\left( (\Gamma_3(F) - 2\Gamma_2(F)) - 2(\Gamma_2(F) - 2\Gamma_1(F)) \right) \leq \text{Var}(\Gamma_3(F) - 2\Gamma_2(F)).
\]

Lastly, \( \Phi_F(\beta_{\text{min}}^\pm) = 0 \) implies that \( F \sim \left[ 2\Gamma\left( \frac{\ell_1}{2}, \frac{\ell_2}{2} \right) - \frac{k}{2k} \right] * \left[ 2\Gamma\left( \frac{\ell_1}{2}, 1 \right) - \ell_2 \right] \), where \( k = \beta_0^2 \), \( \beta_0 \) is given by (II), and the operation * stands for convolution. \( (\ell_1, \ell_2) \in \mathbb{N}_0^2 \) are such that

\[
\ell_1 \frac{k^2}{4} + \ell_2 = \nu, \text{ i.e.}
\]

\[
F \overset{\text{law}}{=} \sum_{i=1}^{\ell_1} \frac{k}{2} (N_i^2 - 1) + \sum_{j=1}^{\ell_2} (N_j^2 - 1), \tag{42}
\]

where \( \{N_i, N_j : 1 \leq i \leq \ell_1, 1 \leq j \leq \ell_2\} \) are i.i.d. \( \mathcal{N}(0, 1) \) variables with the convention that, when \( \ell_1 = 1 \) or \( \ell_2 = 0 \), the corresponding sum is understood as 0.
Proof. It is straightforward to deduce that the function $\Phi_F$ attains its local maximum at $\beta_{\max} = 0$, and the global minimum at points $\beta_{\min}^\pm$. Also

$$
\Phi_F(\beta_{\min}^\pm) = \text{Var}(\Gamma_3(F) - 2\Gamma_2(F)) - \left( \frac{\mathbb{E}[\left(\Gamma_3(F) - 2\Gamma_2(F)\right) \left(\Gamma_2(F) - 2\Gamma_1(F)\right)]}{\text{Var}(\Gamma_2(F) - 2\Gamma_1(F))} \right)^2.
$$

(43)

(a) $\Rightarrow$ (b): It is obvious that when $F \overset{D}{=} G(\nu)$, then $\Phi_F(0) = 0$. In fact $\Phi_F(\beta) = 0$ for all $\beta \in \mathbb{R}$, because

$$
0 \leq \Phi_F(\beta) \leq 2 \text{Var}(\Gamma_3(F) - 2\Gamma_2(F)) + 8\beta^4 \text{Var}(\Gamma_2(F) - 2\Gamma_1(F)).
$$

(b) $\Rightarrow$ (a): Assume that $\Phi_F(\beta') = 0$ for some $\beta' \neq \beta_{\min}^\pm$. Since $\Phi_F \geq 0$, this implies that $\Phi_F(\beta_{\min}^\pm) = 0$. Hence, relation (43) yields that the equality case happens in the Cauchy–Swartz inequality. Therefore, for some constant $k$ (in fact $k = 2(\beta_{\min}^\pm)^2 = \beta_0^2$), we have

$$
\Gamma_3 - 2\Gamma_2 \overset{a.s.}{=} k(\Gamma_2 - 2\Gamma_1).
$$

(44)

Hence,

$$
\Phi_F(\beta) = (k - 2\beta^2)^2 \text{Var}(\Gamma_2(F) - 2\Gamma_1(F)) = \left(2(\beta_{\min}^\pm)^2 - 2\beta^2\right)^2 \text{Var}(\Gamma_2(F) - 2\Gamma_1(F)).
$$

Now, assumption $\Phi_F(\beta') = 0$ tells us that $\text{Var}(\Gamma_2(F) - 2\Gamma_1(F)) = 0$, and so $F$ is distributed like $G(\nu)$. To continue the rest of the proof, let $F = \sum_i c_i(N_i^2 - 1)$ for some sequence of real numbers $(c_i)_{i \geq 1}$ such that $\sum_i c_i^2 < \infty$. Then we know that

$$
2 \text{Var}(\Gamma_1(F) - 2F) = 2\frac{\kappa_4(F)}{3!} - 4\frac{\kappa_3(F)}{2!} + 4\kappa_2(F) = \sum_i \left(2c_i^2 - 2c_i^2\right)^2.
$$

Also

$$
2 \text{Var}(\Gamma_3(F) - 2\Gamma_2(F)) = 2\frac{\kappa_8(F)}{7!} - 4\frac{\kappa_7(F)}{6!} + 4\frac{\kappa_6(F)}{5!} = \sum_i \left(2^4c_i^4 - 2^4c_i^4\right)^2.
$$

Hence,

$$
\left(2 \text{Var}(\Gamma_1(F) - 2F)\right)^2 = \left(\sum_i \left(2^2c_i^2 - 2^2c_i^2\right)^2\right)^2 = \sum_i \left(2^2c_i^2 - 2^2c_i^2\right)^4 + \sum_{i \neq j} \left(2^2c_i^2 - 2^2c_i^2\right)^2 \left(2^2c_j^2 - 2^2c_j^2\right)^2 =: A + B.
$$

Note that $B \geq 0$. Now

$$
A = \sum_i \left(2^4c_i^4 + 2^4c_i^2 - 2^5c_i^3\right)^2 = \sum_i \left[ \left(2^4c_i^4 - 2^4c_i^3\right) - \left(2^4c_i^3 - 2^4c_i^2\right) \right]^2
$$

$$
= \sum_i \left(2^4c_i^4 - 2^4c_i^3\right)^2 + \sum_i \left(2^4c_i^3 - 2^4c_i^2\right)^2 - 2\sum_i \left(2^4c_i^4 - 2^4c_i^3\right) \left(2^4c_i^3 - 2^4c_i^2\right) =: A_1 + A_2 - 2A_3.
$$

32
Note that $A_1 = 2 \text{Var}(\Gamma_3(F) - 2\Gamma_2(F))$. Moreover,

$$2 \text{Var}(\Gamma_2(F) - 2\Gamma_1(F)) = 2 \left\{ \frac{\kappa_6(F)}{5!} - 4 \frac{\kappa_5(F)}{4!} + 4 \frac{\kappa_4(F)}{3!} \right\} = \sum_i \left( 2^3 c_i^3 - 2^3 c_i^2 \right)^2.$$ 

Therefore $A_2 = 8 \text{Var}(\Gamma_2(F) - 2\Gamma_1(F))$. Also,

$$A_3 = \sum_i \left( 2^4 c_i^4 - 2^4 c_i^3 \right) \left( 2^2 c_i^3 - 2^2 c_i^2 \right) = 4 \left\{ \frac{\kappa_7(F)}{6!} - 4 \frac{\kappa_6(F)}{5!} + 4 \frac{\kappa_5(F)}{4!} \right\}$$

$$= 4 \mathbb{E} \left[ \left( \Gamma_3(F) - 2 \Gamma_2(F) \right) \left( \Gamma_2(F) - 2 \Gamma_1(F) \right) \right].$$

Finally, we arrive in

$$2 \text{Var}^2(\Gamma_1(F) - 2F) = \text{Var}(\Gamma_3(F) - 2\Gamma_2(F)) - 4 \mathbb{E} \left[ \left( \Gamma_3(F) - 2 \Gamma_2(F) \right) \left( \Gamma_2(F) - 2 \Gamma_1(F) \right) \right]$$

$$= \text{Var} \left( \left( \Gamma_3(F) - 2 \Gamma_2(F) \right) - 2 \left( \Gamma_2(F) - 2 \Gamma_1(F) \right) \right) + \frac{B}{2}$$

$$= \Phi_F(1) + \frac{B}{2}. \quad (45)$$

As a result, $\Phi_F(1) \leq 2 \text{Var}^2(\Gamma_1(F) - 2F)$. Also, $\Phi_F(1) = \frac{4}{2} = 2^7 \sum_i (c_i^2 - c_i^4)$. Hence, $\Phi_F(1) = 0$ implies that either $c_i = 0$ for all $i \geq 1$, which is impossible, or $c_i = 1$ for all $i \geq 1$.

Now, together with the condition $2 \sum_{i \geq 1} c_i^2 = 2\nu$, we can deduce that there are only finitely many non-zero coefficients $c_i$, and moreover that $\nu$ is an integer. Hence $F \overset{D}{=} G(\nu)$ is centered Gamma distributed.

Case (I): Note that $\Phi_F(0) = \Phi_F(\beta_0) = \text{Var}(\Gamma_3(F) - 2\Gamma_2(F))$, and also function $\Phi_F$ is increasing on $(-\infty, \beta_{\text{min}}^-) \cup (\beta_{\text{min}}^+, +\infty)$. Since $\beta_{\text{min}}^+ \leq \beta_0$, this implies that, if $\beta_0 \leq 1$,

$$\text{Var}(\Gamma_3(F) - 2\Gamma_2(F)) = \Phi_F(0) = \Phi_F(\beta_0)$$

$$\leq \Phi_F(1) = \text{Var} \left( \left( \Gamma_3(F) - 2 \Gamma_2(F) \right) - 2 \left( \Gamma_2(F) - 2 \Gamma_1(F) \right) \right)$$

$$\leq 2 \text{Var}^2(\Gamma_1(F) - 2F).$$

Case (II) can be discussed in a similar matter. Finally, if $\Phi_F(\beta_{\text{min}}^\pm) = 0$, then relation (44) implies that $\Gamma_{r+1}(F) - 2\Gamma_r(F) \overset{a.s.}{=} k^{r-1} (\Gamma_2(F) - 2\Gamma_1(F))$ for all $r \geq 2$. Note that we assume that $k \neq 0$, otherwise, one can immediately deduce that $F \overset{D}{=} G(\nu)$. Therefore,

$$\sum_i \left( \frac{2c_i}{k} \right)^{r-1} (c_i^3 - c_i^2) = \sum_i (c_i^3 - c_i^2), \quad \forall r \geq 2.$$

Hence, in general, the non-trivial possibility is that $c_i = k/2$ for some index $1 \leq i \leq \ell_1$, and $c_j = 1$ for some other index $1 \leq j \leq \ell_2$. Now taking into account the second moment assumption, we obtain that $\ell_1 \frac{k^2}{4} + \ell_2 = \nu$, i.e.

$$F = \sum_{i=1}^{\ell_1} \frac{k}{2} \left( N_i^2 - 1 \right) + \sum_{j=1}^{\ell_2} (N_j^2 - 1). \quad (46)$$
Remark 5.3.  

(i) Plainly, when \( k = 2 \), the random variable \( F \) appearing in (46) is in fact distributed like \( G(\nu) \). This is consistent with the fact that when \( k = 2 \), relation (44) turns into \( \Gamma_3(F) - 4\Gamma_2(F) + 4\Gamma_1(F) = 0 \). The latter means that \( \Phi_F(1) = 0 \), and therefore \( F \overset{\mathcal{D}}{=} G(\nu) \). Also note that \( k^2 \leq 4\nu \), and so \( \ell_1 \leq 16 \) and \( \ell_2 \leq \nu \).

(ii) By setting \( u = 2f \otimes_1 f - 2f \) and \( v = 2^2 f \otimes_1 (3) f - 2^2 f \otimes_1 f \), one can verify that

\[
\begin{align*}
    u \otimes_1 u &= (2f \otimes_1 f - 2f) \otimes_1 (2f \otimes_1 f - 2f) \\
                                      &= 2^2 f \otimes_1 (4) f - 2^3 f \otimes_1 (3) f + 2^2 f \otimes_1 f \\
                                      &= \{2^2 f \otimes_1 (4) f - 2^2 f \otimes_1 (3) f\} - \{2^2 f \otimes_1 (3) f - 2^2 f \otimes_1 f\} \\
                                      &= \{2^2 f \otimes_1 (4) f - 2^2 f \otimes_1 (3) f\} - v.
\end{align*}
\]

Therefore,

\[
\Phi_F(1) = \text{Var} \left( \Gamma_3(F) - 2\Gamma_2(F) - 2(\Gamma_2(F) - 2\Gamma_1(F)) \right) \\
       = 8\|u \otimes_1 u\|^2 \\
       \leq 8\|u\|^4 = 2\text{Var}^2(\Gamma_1(F) - 2F)
\]

Hence, relying on relation (45), the quantity \( B/2 \) is the exact amount one loses when applying the classical estimate \( \|f \otimes_1 g\| \leq \|f\| \|g\| \).

(iii) Let \( A_f \) stand for the associated Hilbert-Schmidt operator. Condition \( \beta_0 \leq 1 \) in item (I) of Lemma 5.2 is equivalent to the following trace inequality

\[
\text{Tr} \left( A_f(A_f^3 - A_f^2)^2 \right) \leq \frac{1}{2} \text{Tr}(A_f^3 - A_f^2)^2.
\]

(iv) We also point out that the condition \( \text{Cov} \left( \Gamma_3(F) - 2\Gamma_2(F), \Gamma_2(F) - 2\Gamma_1(F) \right) < 0 \) yields that

\[
\text{Var} \left( \Gamma_3(F) - 2\Gamma_2(F) \right) \leq \Phi_F(1) \leq 2\text{Var}^2(\Gamma_1(F) - 2F).
\]

(v) Let \( (F_n)_{n \geq 1} = (\sum_i c_i,n(N_i^2 - 1))_{n \geq 1} \) be a sequence in the second Wiener chaos such that \( \mathbb{E}(F_n^2) = 2\nu \) for all \( n \geq 1 \), and that \( F_n \) converges in distribution towards a centered Gamma random variable \( G(\nu) \). In addition, assume that either one of the following conditions

\[
\begin{align*}
    (1) \quad & \limsup_{n \to \infty} \frac{\sum_i c_i,n(c_i,n^3 - c_i,n^2)^2}{\sum_i (c_i,n^3 - c_i,n^2)^2} \leq \frac{1}{2}, \\
(2) \quad & \text{the numerical sequence } \chi(n) := \sum_i c_i,n(c_i,n^3 - c_i,n^2)^2 \text{ converges to 0 from below};
\end{align*}
\]

34
are fulfilled. Then there exists a constant $C$, independent of $n$, such that
\[ \text{Var} (\Gamma_3(F_n) - 2\Gamma_2(F_n)) \leq C \text{Var}^2 (\Gamma_1(F_n) - 2F_n). \]

(vi) In general, one has to note that the condition
\[ \text{Cov} (\Gamma_3(F) - 2\Gamma_2(F), \Gamma_2(F) - 2\Gamma_1(F)) = 0 \]
does not guarantee that $F \overset{D}{=} G(\nu)$. A simple counterexample is given by $F = c_1(N_1^2 - 1) + c_2(N_2^2 - 1)$ where (up to numerical error) $c_1 = 1.27$ and $c_2 = -0.62$. In fact, $(c_1, c_2)$ is one of the intersections of two curves $x^2 + y^2 = 2$, and $x^5(x-1)^2 + y^5(y-1)^2 = 0$.

The forthcoming results aim to provide neat characterizations of centered Gamma distribution $G(\nu)$, $\nu > 0$ inside the second Wiener chaos by recruiting the theory of real quadratic forms.

**Lemma 5.4.** Let $F = I_2(f)$ be in the second Wiener chaos such that $E[F^2] = 2\nu$. For $\beta_1, \beta_2 \in \mathbb{R}$ define the following non-negative definite binary quadratic form
\[ \Psi_2(\beta_1, \beta_2) := \text{Var} \left( \beta_1 (\Gamma_3(F) - 2\Gamma_2(F)) - 2\beta_2 (\Gamma_2(F) - 2\Gamma_1(F)) \right). \tag{48} \]

Moreover, assume that $\Delta = \text{discriminant} (\Psi_2) \neq 0$. Then $F \overset{D}{=} G(\nu)$ if and only if the quadratic form $\Psi_2$ is isotropic, i.e. $\Psi_2(\beta_1, \beta_2) = 0$ for some $(\beta_1, \beta_2) \neq (0,0)$. In fact, if $\Psi_2(\beta_1, \beta_2) = 0$ for some $(\beta_1, \beta_2) \neq (0,0)$, then $\Psi_2 = 0$ everywhere, and the random variable $F$ is distributed according to a centered Gamma distribution with parameter $\nu$.

**Remark 5.5.** One has to note that, by the Cauchy-Schwarz inequality, $\Delta \geq 0$. Also, the requirement $\Delta \neq 0$ is equivalent to saying that the form $\Psi_2$ is positive definite, i.e. $\det(A(\Psi_2)) > 0$, where $A(\Psi_2)$ is the associated symmetric matrix. Also, when $\Delta = 0$, then $\Gamma_3(F) - 2\Gamma_2(F) \overset{a.s.}{=} k (\Gamma_2(F) - 2\Gamma_1(F))$, for some constant $k$, and therefore the binary quadratic form $\Psi_2$ as in (48) reduces to $\Psi_2(\beta_1, \beta_2) = (k\beta_1 - 2\beta_2)^2 \text{Var} (\Gamma_2(F) - 2\Gamma_1(F))$. In this case, the sole requirement $\Psi_2(\beta_1, \beta_2) = 0$ for some $(\beta_1, \beta_2) \neq (0,0)$ implies that, in general, $F$ is of the form given in (42).

**Proof of Lemma 5.4.** Obviously, if $F \overset{D}{=} G(\nu)$, then $\Psi_2(\beta_1, \beta_2) = 0$. Now assume that for some $(\beta_1, \beta_2) \neq (0,0)$, we have that $\Psi_2(\beta_1, \beta_2) = 0$. If $\beta_1 = 0$, then, again, one can readily deduce that $F \overset{D}{=} G(\nu)$. Otherwise
\[ \Psi_2(\beta_1, \beta_2) = \beta_1^2 \text{Var} \left( (\Gamma_3(F) - 2\Gamma_2(F)) - 2\frac{\beta_2}{\beta_1} (\Gamma_2(F) - 2\Gamma_1(F)) \right) = 0, \]
which immediately implies that $\text{Var}(\Gamma_3(F) - 2\Gamma_2(F)) = 0$, and hence $F \overset{D}{=} G(\nu)$. \hfill \qed

In general, for $s \geq 1$, put
\[ D_s := \left\{ \Psi_s(\beta_1, \cdots, \beta_s) := \text{Var} \left( \sum_{r \in A} \beta_r (\Gamma_r(F) - 2\Gamma_{r-1}(F)) \right) : A \subseteq \mathbb{N} \text{ and } \#A = s \right\}, \tag{49} \]
and
\[ D = \bigcup_{s \geq 1} D_s. \]
Lemma 5.6. Let $F = I_2(f)$ be in the second Wiener chaos such that $\mathbb{E}[F^2] = 2\nu$. Then the set $\mathcal{D}$, containing non-negative quadratic forms of the form (49), characterizes the centered Gamma distribution in the sense that if for $\Psi_s \in \mathcal{D}_s \subseteq \mathcal{D}$, and $s \geq 1$,

$$\Psi_s(\beta_1, \ldots, \beta_s) = 0$$

for some $(\beta_1, \ldots, \beta_s) \neq (0, \ldots, 0)$, and if $\det(A(\Psi_s)) \neq 0$, then $F \sim \text{CenteredGamma}(\nu)$.

Proof. Assume that $\Psi_s(\beta_1, \ldots, \beta_s) = 0$ for some $(\beta_1, \ldots, \beta_s) \neq (0, \ldots, 0)$. Note that $\Psi_s \geq 0$, and hence by Sylvester’s law of inertia, together with the condition $\det(A(\Psi_s)) \neq 0$, one can deduce that $\Psi_s = 0$. Therefore $\text{Var} \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right) = 0$ for any $r \in A$, where the set $A$ is same as (49), which implies that $F$ is distributed like $G(\nu)$.

Remark 5.7. Let $A \subset \mathbb{N}$ with $\#A = d < \infty$. The characterization of random elements $F$ in the second Wiener chaos such that for some $(\beta_1, \ldots, \beta_d) \in \mathbb{R}^d \neq 0$,

$$\sum_{r \in A} \beta_r \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right) \overset{a.s.}{=} 0$$

is an interesting problem. The solution relates to the real roots of polynomial equations and the well-known Abel–Ruffini theorem (also known as Abel’s impossibility theorem) [Ruf99], and we leave it for future investigation. For instance, when $\#A = 3$, the problem can be reduced to the real solutions of the trinomial equation $x^n + ax + b = 0$ for some $n \in \mathbb{N}$, and hence the Glasser’s derivation method can be useful [Gla94].

6 A New Proof for a Bound in Kolmogorov Distance

In this section, we use techniques that date back to Tikhomirov from 1981 [Tik81], who used Stein’s equation on the level of characteristic functions in order to present a result for Gamma approximation in terms of the Kolmogorov distance. Similar lines of arguments have been recently employed in more generality in [AMPS17].

The starting point is the following classical Berry-Esseen lemma as stated in [Pet75, p. 104]. For a more general version of the lemma, the reader is referred to Zolotarev [Zol65].

Lemma 6.1. Let $F$ and $G$ be two cumulative distribution functions with corresponding characteristic functions $\varphi_F$, and $\varphi_G$. Then for every positive number $T > 0$, and every $b > 1/2\pi$, the estimate

$$d_{Kol}(F, G) := \sup_{x \in \mathbb{R}} |F(x) - G(x)| \leq b \int_{-T}^{T} \left| \frac{\varphi_F(t) - \varphi_G(t)}{t} \right| dt + bT \sup_{x} \int_{|y| \leq \frac{c(b)}{b}} |G(x + y) - G(x)| dy,$$

(50)

takes place, where $c(b)$ is a constant depending only on $b$, and it is given by the root of the following equation

$$\int_{0}^{\frac{c(b)}{b}} \frac{\sin^2 u}{u^2} du = \frac{\pi}{4} + \frac{1}{8b}. $$

36
In particular, if $\sup_x |G'(x)| \leq K$, then

\[ d_{Kol}(F, G) := \sup_{x \in \mathbb{R}} |F(x) - G(x)| \leq b \int_{-T}^{T} \left| \frac{\varphi_F(t) - \varphi_G(t)}{t} \right| dt + c(b)\frac{K}{T}. \tag{51} \]

In order to prove a Kolmogorov bound, we need an estimate on the difference of the characteristic functions and the distribution functions. The first is done in the following Lemma:

**Lemma 6.2.** Let $\nu > 0$ be an integer and let $F$ be a random variable admitting a finite chaos expansion with variance $\mathbb{E}[F^2] = 2\nu$. Let $G(\nu) \sim \text{CenteredGamma}(\nu)$. Define

\[ D(t) := \varphi_F(t) - \varphi_{G(\nu)}(t) = \mathbb{E}[e^{itF}] - \mathbb{E}[e^{itG(\nu)}], \quad t \in \mathbb{R}. \]

Then the following estimates take place:

\[ |D(t)| \leq \frac{1}{2} |t| \mathbb{E}[2(F + \nu) - \Gamma_1(F)] \leq \frac{1}{2} |t| \sqrt{\text{Var}(\Gamma_1(F) - 2F)}. \tag{52} \]

**Proof.** We consider the Stein operator associated to a centered Gamma random variable $G(\nu)$ (see [DPT8], equation 2.7):

\[ \mathcal{L}f(x) = 2(x + \nu)f'(x) - xf(x). \]

Using the integration by parts formula, we get for all $f \in C^1$ with bounded derivative

\[ \mathbb{E}[\mathcal{L}f(F)] = \mathbb{E}\left[f'(F)\left\{2(F + \nu) - \Gamma_1(F)\right\}\right]. \tag{53} \]

Also, for all $C^1$ functions $f : \mathbb{R} \to \mathbb{R}$, such that the expectation exists (e.g. if $f$ is polynomially bounded), we have

\[ \mathbb{E}[\mathcal{L}f(G(\nu))] = 0. \tag{54} \]

By considering real and imaginary part separately and using linearity, we can extend (53) and (54) to complex valued functions $f : \mathbb{R} \to \mathbb{C}$. Thus letting $f(x) = e^{itx}$ for $t \in \mathbb{R}$, we obtain

\[ \mathbb{E}[\mathcal{L}f(F)] = it \mathbb{E}[e^{itF} \{2(F + \nu) - \Gamma_1(F)\}]. \]

Therefore

\[ it \mathbb{E}[e^{itF} \{2(F + \nu) - \Gamma_1(F)\}] = \mathbb{E}[\mathcal{L}f(F)] = \mathbb{E}[\mathcal{L}f(F)] - 0 = \mathbb{E}[\mathcal{L}f(F)] - \mathbb{E}[\mathcal{L}f(G(\nu))] \]
\[ = it \times 2\nu \left( \mathbb{E}[e^{itF}] - \mathbb{E}[e^{itG(\nu)}] \right) - (1 - 2it) \left( \mathbb{E}[F e^{itF}] - \mathbb{E}[G(\nu) e^{itG(\nu)}] \right) \]
\[ = it \times 2\nu D(t) + (2t + i)D'(t). \]

So $D$ satisfies the differential equation

\[ (1 - 2it)D'(t) + 2\nu tD(t) = e(t), \quad \text{where } e(t) := t \mathbb{E}[e^{itF} \{2(F + \nu) - \Gamma_1(F)\}]. \tag{55} \]

Using the fact that $D(-t) = D(t)$ and $|D(t)| = |D(t)|$, we focus only on $t \geq 0$. The solution of the ordinary differential equation (55) with initial condition $D(0) = 0$ is given by

\[ D(t) = e^{-a(t)} \int_0^t \frac{e(s)}{1 - 2si} e^{a(s)} ds, \]

37
where
\[ a(t) = \int \frac{2\nu}{1 - 2ti} dt = \nu \log(4t^2 + 1) + i \left( t\nu - \frac{\nu}{2} \arctan(2t) \right). \]

Note that
\[ |e^{a(t)}| = (4t^2 + 1)^{\frac{\nu}{4}} \quad \text{and} \quad |e^{-a(t)}| = (4t^2 + 1)^{-\frac{\nu}{4}}. \]

Thus we can estimate
\[
|D(t)| \leq |e^{-a(t)}| \int_0^t \left| \frac{1}{1 - 2si} |e(s)| |e^{a(s)}| ds
\leq (4t^2 + 1)^{-\frac{\nu}{4}} \int_0^t \frac{(4s^2 + 1)^{\nu/4}}{\sqrt{4s^2 + 1}} |e(s)| ds
\leq \mathbb{E}[[2(F + \nu) - \Gamma_1(F)] (4t^2 + 1)^{-\frac{\nu}{4}} \int_0^t s (4s^2 + 1)^{\frac{\nu}{4} - \frac{1}{2}} ds]
= \mathbb{E}[[2(F + \nu) - \Gamma_1(F)] (4t^2 + 1)^{-\frac{\nu}{4}} \left( \frac{1}{2(\nu + 2)} \left( (4t^2 + 1)^{\frac{\nu}{4} + \frac{1}{2}} - 1 \right) \right)]
= \mathbb{E}[[2(F + \nu) - \Gamma_1(F)] \frac{1}{2(\nu + 2)} (\sqrt{4t^2 + 1} - (4t^2 + 1)^{-\nu/4})]
\leq \frac{1}{2} t \mathbb{E}[[2(F + \nu) - \Gamma_1(F)].
\]

The last estimate is due to Lemma 6.3 below. The second inequality in (52) is just Cauchy Schwarz.

Lemma 6.3. For any \( \nu > 0 \) and \( t \geq 0 \), we have that
\[
\sqrt{4t^2 + 1} - (4t^2 + 1)^{-\nu/4} \leq (2 + \nu) \times t.
\]

Proof. We make use of the following well-known inequalities:
\[
\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}, \quad \text{for all } x, y \geq 0; \tag{56}
1 - e^{-x} \leq x, \quad \text{for all } x \geq -1; \tag{57}
\log(x) \leq 2(\sqrt{x} - 1), \quad \text{for all } x > 0. \tag{58}
\]

With this we get
\[
\sqrt{4t^2 + 1} - (4t^2 + 1)^{-\nu/4} \leq 2t + 1 - e^{-\frac{\nu}{4} \log(4t^2 + 1)} \leq 2t + \frac{\nu}{4} \log(4t^2 + 1) \leq 2t + \frac{\nu}{2} (\sqrt{4t^2 + 1} - 1) \leq (2 + \nu) \times t.
\]
In order to estimate the second term in the Esseen-Lemma (50), we need to study the cumulative distribution function (CDF) of a centered Gamma random variable $G(\nu)$. This is done in the following Lemma.

**Lemma 6.4.** Let $\nu > 0$ be an integer and $G(\nu) \sim \text{CenteredGamma}(\nu)$. Denote by $G_\nu$ its CDF and by $g_\nu$ its probability density function (PDF). Then there exists a constant $K > 0$, such that for all $a, b \in \mathbb{R}$ we have

$$|G_\nu(a) - G_\nu(b)| \leq K|a - b|, \quad \text{if } \nu \geq 2,$$

and

$$|G_\nu(a) - G_\nu(b)| \leq K|a - b|^{1/2}, \quad \text{if } \nu = 1.$$  

**Proof.** The PDF of $G(\nu)$ is given by

$$g_\nu(x) = 2^{-\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)^{-1} (x + \nu)^{\frac{\nu}{2} - 1} e^{-\frac{x}{2}} \text{I}_{\{x > -\nu\}}(x).$$

If $\nu \geq 3$, then $g_\nu$ is continuous and hence $G_\nu$ is differentiable on the whole real line. One can readily verify that $g_\nu$ is bounded with $K := \sup_{x \in \mathbb{R}} |g_\nu(x)| = g_\nu(-2)$. So (59) is just an application of the mean value theorem.

When $\nu = 2$, then

$$g_\nu(x) = \frac{1}{2} e^{-\frac{x}{2} - 1} \text{I}_{\{x > -2\}}(x).$$

In this case $G_\nu$ is not differentiable in $x = -2$. However, because of the monotonicity, $g_\nu$ is bounded by $K := \sup_{x \in \mathbb{R}} |g_\nu(x)| = \lim_{x \downarrow -2} g_\nu(x) = 1/2$. Therefore (59) holds for all $a, b \in (-\infty, -2)$ and all $a, b \in (-2, \infty)$. Using the continuity of $G_\nu$, we can easily show that (59) extends to the whole real line.

When $\nu = 1$, the PDF has the form

$$g_\nu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} \text{I}_{\{x > -1\}}(x).$$

First note that $g_\nu$ is not bounded, in fact $\lim_{x \downarrow -1} g_\nu(x) = \infty$. Without loss of generality assume that $b > a$. We split the proof into three cases:

**Case 1 ($a < b \leq -1$):** Here (60) holds, since $G_\nu(a) = G_\nu(b) = 0$.

**Case 2 ($-1 < a < b$):** Define $C := \frac{1}{\sqrt{2\pi}}$. Then we have

$$g_\nu(x) \leq C \times \frac{1}{\sqrt{x + 1}} \quad \forall x > -1.$$

We compute (note that $G_\nu$ is increasing):

$$G_\nu(b) - G_\nu(a) = \int_a^b g_\nu(t) \, dt \leq C \int_a^b \frac{dt}{\sqrt{t + 1}} = 2C (\sqrt{b + 1} - \sqrt{a + 1}) \leq 2C \sqrt{b - a}.$$
Case 3 \((a \leq -1 < b)\): Using the continuity of \(G_\nu\) we get:

\[
G_\nu(b) - G_\nu(a) = G_\nu(b) - G_\nu(-1) = \lim_{\varepsilon \downarrow -1} G_\nu(b) - G_\nu(-1 + \varepsilon)
\]

\[
\leq 2 \lim_{\varepsilon \downarrow -1} 2C \sqrt{b + 1 - \varepsilon} = 2C \sqrt{b + 1} \leq 2C \sqrt{b - a}.
\]

\[\square\]

**Remark 6.5.** Now let \(a = -1\) and \(b \in (-1, 0)\). With similar arguments as above, this time using the upper bound \(g_\nu(t) \geq e^{-1/2} \sqrt{2\pi t + 1}\), we can show that

\[
G_\nu(b) - G_\nu(-1) \geq \sqrt{2} e^{-1/2} \sqrt{\pi} \times \sqrt{b - (-1)}.
\]

Thus in a vicinity of \(-1\), estimate (60) is actually the best we can do when \(\nu = 1\).

Now we have all the ingredients to show the following theorem.

**Theorem 6.6.** Let \(\nu > 0\) be an integer and let \(F\) be a random variable admitting a finite chaos expansion, such that \(\mathbb{E}[F^2] = 2\nu\). Let \(G(\nu) \sim \text{CenteredGamma}(\nu)\). Then

\[
d_{Kol}(F, G(\nu)) \leq \begin{cases} 
C \times \text{Var} \left( \Gamma_1(F) - 2F \right)^{1/4}, & \text{if } \nu \geq 2 \\
C \times \text{Var} \left( \Gamma_1(F) - 2F \right)^{1/6}, & \text{if } \nu = 1,
\end{cases}
\]

where \(C > 0\) is a constant only depending on \(\nu\).

**Proof.** If \(\nu \geq 2\) then putting the bounds from Lemma 6.2 and Lemma 6.4 into the Berry-Esseen lemma (50), we get for every \(T > 0\)

\[
d_{Kol}(F, G(\nu)) \leq b T \sup_{x \in \mathbb{R}} \int_{|y| \leq \varepsilon(b)} |G_\nu(x + y) - G_\nu(x)| dy
\]

\[
\leq b T \sqrt{\text{Var} \left( \Gamma_1(F) - 2F \right)} + b K \frac{c(b)^2}{T}
\]

\[
=: c_1 T \sqrt{\text{Var} \left( \Gamma_1(F) - 2F \right)} + \frac{c_2}{T}
\]

The minimum is achieved at

\[
T_{\min} = \left( \frac{c_2}{c_1} \right)^{1/4} \text{Var} \left( \Gamma_1(F) - 2F \right)^{-1/4},
\]

and is given by

\[
2 \sqrt{c_1 c_2} \text{Var} \left( \Gamma_1(F) - 2F \right)^{1/4}.
\]
If $\nu = 1$, then instead we get
\[
d_{Kol}(F, G(\nu)) \leq b \int_{-T}^{T} \left| \frac{\varphi_F(t) - \varphi_{G(\nu)}(t)}{t} \right| \, dt + bT \sup_{x \in \mathbb{R}} \int_{|y| \leq \epsilon(b)} |G_\nu(x + y) - G_\nu(x)| \, dy \\
\leq bT \sqrt{\text{Var}(\Gamma_1(F) - 2F)} + \frac{4}{3} bK \frac{c(b)^{3/2}}{T^{1/2}}
\]
\[=: \tilde{c}_1 T \sqrt{\text{Var}(\Gamma_1(F) - 2F)} + \frac{\tilde{c}_2}{T^{1/2}}.
\]

Again, minimising over $T > 0$ yields
\[T_{\min} = 2^{-2/3} \left( \frac{\tilde{c}_2}{\tilde{c}_1} \right)^{2/3} \text{Var}(\Gamma_1(F) - 2F)^{-1/3}
\]
and thus the minimum is
\[3 \times 2^{-2/3} \times \tilde{c}_1^{1/3} \tilde{c}_2^{2/3} \text{Var}(\Gamma_1(F) - 2F)^{1/6}.
\]

**Remark 6.7.** Most parts of this result are not new, we merely present an original proof to illustrate the power of other techniques that are mostly not relying on Stein’s method. In fact, using Theorem 1.7 from [DP18], as well as the fact that
\[d_{Kol}(F, G) \leq C \sqrt{d_1(F, G)},
\]
whenever the density of $G$ is bounded, we immediately retrieve the case $\nu \geq 2$. To our best knowledge, when $\nu = 1$, our result is new, as in this case the corresponding density $g_1$ is not bounded.

Since we were mainly interested in $F$ belonging to the second Wiener chaos, we have only focused on integer valued $\nu$. However, the proofs can easily be adapted to cover any $\nu > 0$, which leads to the following generalization.

**Theorem 6.8.** Let $\nu > 0$ be any positive real number and let $F$ be a random variable admitting a finite chaos expansion, such that $\mathbb{E}[F^2] = 2\nu$. Let $G(\nu) \sim \text{CenteredGamma}(\nu)$. Then
\[d_{Kol}(F, G(\nu)) \leq \begin{cases} 
C \times \text{Var}(\Gamma_1(F) - 2F)^{1/4}, & \text{if } \nu \geq 2 \\
C \times \text{Var}(\Gamma_1(F) - 2F)^{1/3}, & \text{if } \nu \in (0, 2),
\end{cases}
\]
where $C > 0$ is a constant only depending on $\nu$.

Under the light of the result presented in Section [4.5], we end the paper with the following conjecture.

**Conjecture 6.9.** Let $\nu > 0$, and $F = I_2(f)$ belonging to the second Wiener chaos so that $\mathbb{E}[F^2] = 2\nu$. Let $G(\nu) \sim \text{CenteredGamma}(\nu)$. Then there exist two general constants $0 < C_1 < C_2$ such that
\[C_1 M(F) \leq d_{TV}(F, G(\nu)) \leq C_2 M(F).
\] (61)
References

[AAPS17] B. Arras, E. Azmoodeh, G. Poly, and Y. Swan. A bound on the 2-Wasserstein distance between linear combinations of independent random variables. 2017, arXiv:1704.01376v2. To appear in Stochastic processes and their Applications.

[ACP14] E. Azmoodeh, S. Campese, and G. Poly. Fourth Moment Theorems for Markov diffusion generators. J. Funct. Anal., 266(4):2341–2359, 2014.

[AMMP16] E. Azmoodeh, D. Malicet, G. Mijoule, and G. Poly. Generalization of the Nualart-Peccati criterion. Ann. Probab., 44(2):924–954, 2016.

[AMPS17] B. Arras, G. Mijoule, G. Poly, and Y. Swan. A new approach to the Stein-Tikhomirov method: with applications to the second Wiener chaos and Dickman convergence, 2017, arXiv:1605.06819v2.

[APP15] E. Azmoodeh, G. Peccati, and G. Poly. Convergence towards linear combinations of chi-squared random variables: a Malliavin-based approach. In In memoriam Marc Yor—Séminaire de Probabilités XLVII, volume 2137 of Lecture Notes in Math., pages 339–367. Springer, Cham, 2015.

[AS72] M. Abramowitz and I. A. Stegun, editors. Handbook of mathematical functions with formulas, graphs, and mathematical tables, volume 55 of National Bureau of Standards applied mathematics series. U.S. Gov. Print. Off, Washington, DC, 10. print., dec. 1972, with corr edition, 1972.

[BBNP12] H. Biermé, A. Bonami, I. Nourdin, and G. Peccati. Optimal Berry-Esseen rates on the Wiener space: the barrier of third and fourth cumulants. ALEA Lat. Am. J. Probab. Math. Stat., 9(2):473–500, 2012.

[Con95] J. B. Conway. Functions of one complex variable. II, volume 159 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[DP18] C. Döbler and G. Peccati. The Gamma Stein equation and noncentral de Jong theorems. Bernoulli, 24(4B):3384–3421, 2018.

[Dra16a] S. S. Dragomir. Some Grüss' type inequalities for trace of operators in Hilbert spaces. Oper. Matrices, 10(4):923–943, 2016.

[Dra16b] S. S. Dragomir. Some trace inequalities for convex functions of selfadjoint operators in Hilbert spaces. Korean J. Math., 24(2):273–296, 2016.

[Gla94] M. L. Glasser. The quadratic formula made hard: A less radical approach to solving equations, 1994, arXiv:math/9411224.

[Led12] M. Ledoux. Chaos of a Markov operator and the fourth moment condition. Ann. Probab., 40(6):2439–2459, 2012.
[Liu07] L. Liu. A trace class operator inequality. *J. Math. Anal. Appl.*, 328(2):1484–1486, 2007.

[Luk70] E. Lukacs. *Characteristic functions*. Hafner Publishing Co., New York, 1970. Second edition, revised and enlarged.

[NP05] D. Nualart and G. Peccati. Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.*, 33(1):177–193, 2005.

[NP09a] I. Nourdin and G. Peccati. Noncentral convergence of multiple integrals. *Ann. Probab.*, 37(4):1412–1426, 2009.

[NP09b] I. Nourdin and G. Peccati. Stein’s method on Wiener chaos. *Probab. Theory Related Fields*, 145(1-2):75–118, 2009.

[NP10] I. Nourdin and G. Peccati. Cumulants on the Wiener space. *J. Funct. Anal.*, 258(11):3775–3791, 2010.

[NP12] I. Nourdin and G. Peccati. *Normal Approximations with Malliavin Calculus: From Stein’s Method toUniversality*. Cambridge Tracts in Mathematics. Cambridge University Press, 2012.

[NP15] I. Nourdin and G. Peccati. The optimal fourth moment theorem. *Proc. Amer. Math. Soc.*, 143(7):3123–3133, 2015.

[NPR10] I. Nourdin, G. Peccati, and G. Reinert. Invariance principles for homogeneous sums: universality of Gaussian Wiener chaos. *Ann. Probab.*, 38(5):1947–1985, 2010.

[Nua06] D. Nualart. *The Malliavin calculus and related topics*. Probability and its applications. Springer, Berlin and Heidelberg and New York, 2. ed. edition, 2006.

[NV16] L. Neufcourt and F. G. Viens. A third-moment theorem and precise asymptotics for variations of stationary Gaussian sequences. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13(1):239–264, 2016.

[Pet75] V. V. Petrov. *Sums of independent random variables*. Springer-Verlag, New York-Heidelberg, 1975. Translated from the Russian by A. A. Brown, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 82.

[Ruf99] P. Ruffini. *Teoria generale delle equazioni - in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto*. Nella stamperia di S. Tommaso d’Aquino, 1799.

[Tik81] A. N. Tikhomirov. On the convergence rate in the central limit theorem for weakly dependent random variables. *Theory of Probability & Its Applications*, 25(4):790–809, 1981.
[Zin13] R. Zintout. The total variation distance between two double Wiener-Itô integrals. *Statist. Probab. Lett.*, 83(10):2160–2167, 2013.

[Zol65] V. M. Zolotarev. On the closeness of the distributions of two sums of independent random variables. *Teor. Verojatnost. i Primenen.*, 10:519–526, 1965.