A CRITERION FOR I-ADIC COMPLETENESS

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ABSTRACT. Let $I$ denote an ideal in a commutative Noetherian ring $R$. Let $M$ be an $R$-module.

The $I$-adic completion is defined by $\hat{M} = \varprojlim M/I^n M$. Then $M$ is called $I$-adic complete whenever the natural homomorphism $M \to \hat{M}$ is an isomorphism. Let $M$ be $I$-separated, i.e.

$\bigcap_n I^n M = 0$. In the main result of the paper it is shown that $M$ is $I$-adic complete if and only if $\text{Ext}^1_R(F,M) = 0$ for the flat test module $F = \bigoplus_{i=1}^r R x_i$, where $\{x_1, \ldots, x_r\}$ is a system of elements such that $\text{Rad} I = \text{Rad} R$. This result extends several known statements starting with C. U. Jensen’s result (see [9, Proposition 3]) that a finitely generated $R$-module $M$ over a local ring $R$ is complete if and only if $\text{Ext}^1_R(F,M) = 0$ for any flat $R$-module $F$.

1. INTRODUCTION

Let $R$ denote a commutative Noetherian ring. For an ideal $I \subset R$ we consider the $I$-adic completion. For an $R$-module $M$ it is defined by $\varprojlim M/I^n M = \hat{M}$. In the case of $(R,m)$ a local ring and a finitely generated $R$-module $M$ it was shown (see [13]) that the following conditions are equivalent:

(i) $M$ is $m$-adic complete.
(ii) $\text{Ext}^1_R(F,M) = 0$ for any flat $R$-module $F$.
(iii) $\text{Ext}^1_R(F,M) = 0$ for $F = \bigoplus_{i=1}^r R x_i$, where $x_1, \ldots, x_r \in m$ are elements that generate an $m$-primary ideal.

Here $R_x$ denotes the localization of $R$ with respect to $\{x^\alpha | \alpha \in \mathbb{N}_{\geq 0}\}$. This is an extension (in the local case) of a result of C. U. Jensen (see [9, Proposition 3]) who proved the equivalence of the first two conditions. The main result of the present paper is an extension to the case of $I$-adic completion. More precisely we prove the following result:

Theorem 1.1. Let $I$ be an ideal of a commutative Noetherian ring $R$. Let $M$ denote an arbitrary $R$-module that is $I$-separated, i.e.

$\bigcap_n I^n M = 0$. Then the following conditions are equivalent:

(i) $M$ is $I$-adic complete.
(ii) $\text{Ext}^1_R(F,M) = 0$ for all $i \geq 0$ and any flat $R$-module $F$ with $F \otimes_R R/I = 0$.
(iii) $\text{Ext}^1_R(R x, M) = 0$ for all elements $x \in I$.
(iv) $\text{Ext}^1_R(\bigoplus_{i=0}^r R x_i, M) = 0$ for a system of elements $\{x_1, \ldots, x_r\}$ such that $\text{Rad} x R = \text{Rad} I$.

Note that in his paper (see [9, Proposition 3]) C. U. Jensen proved the following: Let $R$ denote a semi local ring and $M$ a finitely generated $R$-module. Then $M$ is complete if and only if $\text{Ext}^1_R(F,M) = 0$ for all flat (resp. countably generated flat) $R$-modules $F$. A different proof for the vanishing of $\text{Ext}^1_R(F,M)$ for all $i \geq 1$, all flat $R$-modules and $M$ a complete $R$-module follows by results of R.-O. Buchweitz and H. Flenner (see [5, Theorem 2]).

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Moreover another criterion for $I$-adic completeness was shown by A. Frankild and S. Sather-Wagstaff. Let $I \subset R$ an ideal contained in the Jacobson radical of $R$. Let $M$ be a finitely generated $R$-module. In their paper (see [7]) they proved that $M$ is $I$-adically complete if and only if $\Ext^{i}_R(\hat{R}^I, M) = 0$ for all $1 \leq i \leq \dim_R M$, where $\hat{R}^I$ denotes the $I$-adic completion of $R$.

Whence, the main result of the present paper is the construction of a simple flat test module $F$ for the completeness of $M$ in the $I$-adic topology in terms of the vanishing of $\Ext^{i}_R(F, M)$. This generalizes the case of the maximal ideal proved in [13]. Instead of Matlis Duality used in [13] (not available in the case of $I$-adic topology) here we use some homological techniques.

2. Preliminary Results

In the following $R$ denotes a commutative Noetherian ring. For an ideal $I \subset R$ we consider the $I$-adic completion. For an $R$-module $M$ it is defined by $\lim_{\leftarrow} M/I^\alpha M = \hat{M}^I$. We say that $M$ is $I$-adically complete whenever the natural homomorphism $\tau = \tau^I_{M} : M \to \hat{M}^I$ (induced by the natural surjections $M \to M/I^\alpha M$) is an isomorphism. It implies that $M$ is $I$-separated, i.e. $\cap_{\alpha} I^\alpha M = 0$. We begin with a few preparatory results needed later on.

Proposition 2.1. Let $I \subset R$ denote an ideal. Let $M$ denote an $R$-module such that $M \otimes_R R/I = 0$. Then $M \otimes_R X / I = 0$ for any $R$-module $X$ with $\text{Supp}_R X \subseteq V(I)$.

Proof. The assumption $M \otimes_R R/I = 0$ implies that $M = I^\alpha M$ for all $\alpha \geq 1$, or in other words $M \otimes_R R/I = 0$. First of all let $X$ be a finitely generated $R$-module. Then $I^\alpha X = 0$ for a certain $\alpha \in \mathbb{N}$. Therefore

$$M \otimes_R X = M \otimes_R X / I = (M \otimes_R R/I^\alpha) \otimes_R X = 0.$$ 

If $X$ is not necessarily a finitely generated $R$-module, then $X \simeq \lim_{\leftarrow} X_\alpha$ for a direct system of finitely generated submodules $X_\alpha \subset X$ ordered by inclusions. Whence $\text{Supp}_R X_\alpha \subseteq V(I)$ and

$$M \otimes_R X = \lim_{\leftarrow} (M \otimes_R X_\alpha) = 0,$$

as required. \hfill \Box

Let $I \subset R$ denote an ideal. For an $R$-module $X$ the left derived functors $\Lambda_i^I(X), i \in \mathbb{Z}$, are defined by $H_i(\lim_{\leftarrow} (R/I^\alpha \otimes_R F))$, where $F$ denotes a flat resolution of $X$. The functors $\Lambda_i^I(\cdot)$ were first systematically studied by Greenlees and May (see [8]) and by A.-M. Simon (see [14]) and more recently in [1] and [11]. Note that there is a natural surjective homomorphism $\Lambda_0^I(X) \to \hat{X}^I$. Since $R$ is Noetherian it follows that $M/I^\alpha M \simeq \hat{M}^I/I^\alpha \hat{M}^I$ for all $\alpha \geq 1$ for any $R$-module $M$ (see [16, Section 2.2]). Whence the completion $\hat{M}^I$ is $I$-adically complete.

In the following we will discuss the assumption on the $R$-module $M$ in Proposition 2.1. It provides a certain kind of a Nakayama Lemma. Its proof is known (see [14, 5.1] and also [15, 1.3, Lemma (ii)]). Here we add these arguments.

Proposition 2.2. Let $I$ denote an ideal in a commutative Noetherian ring $R$. Let $M$ denote an arbitrary $R$-module. Then $M \otimes_R R/I = 0$ if and only if $\hat{M}^I = 0$ and if and only if $\Lambda_0^I(M) = 0$.

Proof. If $M \otimes_R R/I = 0$, then $M/I^\alpha M = 0$ for all $\alpha \geq 1$ and $\hat{M}^I = \lim_{\leftarrow} M/I^\alpha M = 0$. Conversely, suppose that $\hat{M}^I = 0$. Then $0 = \hat{M}^I/1\hat{M}^I \simeq M/IM$, as required. If $\Lambda_0^I(M) = 0$, then clearly $\hat{M}^I = 0$. Let $F_1 \to F_0 \to M \to 0$ denote a free presentation of $M$. If $\Lambda_0^I(M) = 0$, then $\hat{F}_1^I \to \hat{F}_0^I$ is onto. Therefore $F_1/IF_1 \to F_0/IF_0$ is onto too, i.e. $M/IM = 0$ as required. \hfill \Box
In the following let \( \mathbf{x} = \{x_1, \ldots, x_r\} \) denote a system of elements of \( R \) such that \( \text{Rad} x \mathbf{x} = \text{Rad} I \). Then we consider the Čech complex \( \mathcal{C}_n \) of the system \( \mathbf{x} \) (see e.g. [11, Section 3]). Note that this is the same as the complex \( K_{\infty}(\mathbf{x}) \) introduced in [1] as the direct limit of the Koszul complexes \( K_r(\mathbf{x}) \). There is a natural morphism \( \mathcal{C}_n \to R \).

In our context the importance of the Čech complex is the following result. It allows the expression of the right derived functors \( \Lambda_i(\cdot) \) in different terms.

**Theorem 2.3.** Let \( I \subset R \) denote an ideal of a commutative Noetherian ring \( R \). Let \( M \) denote an \( R \)-module. Then there are natural isomorphisms

\[
\Lambda_i(M) \simeq H_i(\text{Hom}_R(\mathcal{C}_n, E_R(M)))
\]

for all \( i \in \mathbb{Z} \), where \( E_R(M) \) denotes an injective resolution of \( M \).

**Proof.** This statement is one of the main results of [11, Section 4]. In the formulation here we do not use the technique of derived functors and derived categories. (For a more advanced exposition based on derived categories and derived functors the interested reader might also consult [1].)

Let us summarize a few basic results on completions used in the sequel. The following results are well-known.

**Proposition 2.4.** Let \( J \subset I \) denote ideals of a commutative Noetherian ring. Let \( \mathbf{x} = \{x_1, \ldots, x_r\} \) denote a system of elements of \( R \). Let \( M \) denote an \( R \)-module.

(a) Suppose that \( M \) is \( I \)-adic complete. Then it is also \( J \)-adic complete.

(b) Suppose that \( \text{Rad} I = \text{Rad} x \mathbf{x} \). Assume that \( M \) is complete in the \( x_i \text{R-adic topology for } i = 1, \ldots, r \) and \( M \) is \( I \)-separated. Then \( M \) is \( I \)-adic complete.

**Proof.** For the proof of (a) see [16, 2.2.6]. Let \( \hat{M}^I \) denote the \( I \)-adic completion of \( M \). Because the \( I \)-adic topology and the topology defined by \( \{(x_1^n, \ldots, x_r^n)R\}_{n \geq 0} \) are equivalent it follows \( \hat{M}^I \simeq \lim_{\leftarrow} M / (x_1^n, \ldots, x_r^n)M \). Therefore an element \( m \in \hat{M}^I \) has the form \( m = \sum_{i=0}^{r} \sum_{j=1}^{\infty} x_i^j m_{a,i} \) with \( m_{a,i} \in M \). Since \( M \) is \( x_i \text{R-adically complete} \) \( \hat{M}^I \simeq \lim_{\leftarrow} M / x_i^n M \) and \( m_i = \sum_{j=1}^{\infty} x_i^j m_{a,i} \) for some \( m_i \in M \). Then \( m' = \sum_{i=1}^{r} m_i \) maps to \( m \) under \( \tau : M \to \hat{M}^I \). Because \( M \) is \( I \)-separated it is \( I \)-adically complete. This proves (b).

The following result is helpful in order to compute projective limits of \( \text{Ext} \)-modules. Here \( \lim_{\leftarrow} \{M, x\} \) denotes the first right derived functor of the projective limit (see [17]).

Let \( M \) denote an \( R \)-module and \( x \in R \). We consider the following projective system \( \{M, x\} \), where \( M_\alpha = M \) for all \( \alpha \in \mathbb{N} \) and the transition map \( M_\alpha \to M_\alpha \) is the multiplication by \( x \).

**Lemma 2.5.** With the previous notation it follows that

\[
\lim_{\leftarrow} \{M, x\} \simeq \text{Ext}_R^i(R_x, M) \text{ for } i = 0, 1 \text{ and } \text{Ext}_R^i(R_x, M) = 0 \text{ for all } i \geq 2.
\]

If \( M \) is \( x \text{R-separated} \), then \( \lim_{\leftarrow} \{M, x\} = \text{Hom}_R(R_x, M) = 0 \).

**Proof.** Let \( \{R, x\} \) denote the direct system with \( R_\alpha = R \) for all \( \alpha \in \mathbb{N} \) and the transition map \( R_\alpha \to R_{\alpha+1} \) is the multiplication by \( x \). Then \( R_x \simeq \lim_{\leftarrow} \{R, x\} \) as it is well known. Therefore, \( \text{Hom}_R(\{R, x\}, M) \simeq \{M, x\} \). Now, by [12, Lemma 2.6] (see also Lemma 4.1) there are short exact sequences

\[
0 \to \lim_{\leftarrow} \{\text{Ext}_R^{i-1}(R, M), x\} \to \text{Ext}_R^i(R_x, M) \to \lim_{\leftarrow} \{\text{Ext}_R^i(R, M), x\} \to 0
\]

for all \( i \in \mathbb{Z} \). Whence the results of the first part follow. If \( M \) is \( x \text{R-separated} \), the vanishing of \( \text{Hom}_R(R_x, M) \) is easy to see.
The previous result is a slight modification of Lemma [13, Lemma 2.7].

**Remark 2.6.** A morphism of two complexes \( X \rightarrow Y \) of \( R \)-modules is called a quasi-isomorphism (or homology isomorphism) whenever the induced homomorphisms on the cohomology modules \( H^i(X') \rightarrow H^i(Y') \) are isomorphisms for all \( i \in \mathbb{Z} \). For the definitions of \( \text{Hom}_R(X', Y') \) and \( X' \otimes_R Y' \) we refer to [17] and [2]. For the details on homological algebra of complexes of modules we refer to [2].

3. On \( I \)-adic Completions

Let \( I \subset R \) denote an ideal of a commutative Noetherian ring \( R \). In the first main result we shall prove a vanishing result for a certain Ext-module for an \( I \)-adic complete \( R \)-module.

**Theorem 3.1.** Let \( M \) denote an arbitrary \( R \)-module. Let \( F \) denote a flat \( R \)-module satisfying \( F \otimes_R R/I = 0 \). Then \( \text{Ext}_R^i(F, \hat{M}^i) = 0 \) for all \( i \in \mathbb{Z} \).

**Proof.** For the proof we use the techniques summarized in Theorem 2.3. To this end we fix a few notions. Let \( \underline{x} = \{x_1, \ldots, x_r\} \) denote a system of elements of \( R \) such that \( \text{Rad} \underline{x}R = \text{Rad} I \). Let \( \hat{C}_\underline{x} \) denote the \( \hat{C} \)ech complex with respect to \( \underline{x} \) as defined in [11, Section 3]. It is a bounded complex of flat \( R \)-modules with a natural morphism \( \hat{C}_\underline{x} \rightarrow R \) of complexes. Let \( E \) denote an injective resolution of \( \hat{M}^i \) as an \( R \)-module. By applying \( \text{Hom}_R(\cdot, E) \) it induces a morphism of complexes \( E \rightarrow \text{Hom}_R(\hat{C}_\underline{x}, E) \).

Next we investigate the complex \( \text{Hom}_R(\hat{C}_\underline{x}, E) \). It is a left bounded complex of injective \( R \)-modules. Moreover, by virtue of Theorem 2.3 it follows that \( \Lambda^i(E) \simeq H_i(\text{Hom}_R(\hat{C}_\underline{x}, E)) \) for all \( i \in \mathbb{Z} \). Therefore \( H_i(\text{Hom}_R(\hat{C}_\underline{x}, E))) = 0 \) for all \( i \neq 0 \) and \( H_0(\text{Hom}_R(\hat{C}_\underline{x}, E)) ) \simeq \hat{M}^i \) since \( \hat{M}^i \) is \( I \)-adic complete and \( \Lambda^i(E) = 0 \) for all \( i > 0 \) (see [14, 5.2]). Therefore, the morphism \( E \rightarrow \text{Hom}_R(\hat{C}_\underline{x}, E) \) is a quasi-isomorphism.

Now we consider the complexes

\[
\text{Hom}_R(F, \text{Hom}_R(\hat{C}_\underline{x}, E)) \simeq \text{Hom}_R(F \otimes_R \hat{C}_\underline{x}, E).
\]

We claim that they are homologically trivial complexes. For that reason it will be enough to show that \( F \otimes_R \hat{C}_\underline{x} \) is homologically trivial since \( E \) is a bounded below complex of injective \( R \)-modules. Since \( F \) is a flat \( R \)-module it yields that \( H^i(F \otimes_R \hat{C}_\underline{x}) \simeq F \otimes_R H^i(\hat{C}_\underline{x}) = 0 \) for all \( i \in \mathbb{Z} \). For the vanishing apply Proposition 2.1 with \( H^i(\hat{C}_\underline{x}) \simeq H_i(\text{Hom}_R(F, \cdot)) \) for all \( i \in \mathbb{Z} \) and \( \text{Supp}_R H_i(R) \subseteq V(I) \).

As shown above \( E \rightarrow \text{Hom}_R(\hat{C}_\underline{x}, E) \) is a quasi-isomorphism of left bounded complexes of injective \( R \)-modules. Applying the functor \( \text{Hom}_R(F, \cdot) \) induces a quasi-isomorphism

\[
\text{Hom}_R(F, E) \rightarrow \text{Hom}_R(F, \text{Hom}_R(\hat{C}_\underline{x}, E)).
\]

The complex at the right is cohomologically trivial, while the cohomology of the complex at the left is \( \text{Ext}_R^i(F, \hat{M}^i) \). Therefore, the Ext-modules vanish, as required.

In the following result we prove another behaviour of certain Ext-modules in respect to the \( I \)-adic completion.

**Theorem 3.2.** Let \( I \subset R \) denote an ideal. Let \( M, X \) be arbitrary \( R \)-modules with \( \tau : M \rightarrow \hat{M}^i \) the natural map. Suppose that \( \text{Supp}_R X \subset V(I) \). Then

(a) \( \text{Ext}_R^i(X, \hat{M}^i / \tau(M)) = 0 \) and

(b) the natural homomorphism \( \text{Ext}_R^i(X, M) \rightarrow \text{Ext}_R^i(X, \hat{M}^i) \) is an isomorphism

for all \( i \in \mathbb{N} \).
Proof. Let $\mathfrak{x} = \{x_1, \ldots, x_r\}$ denote elements of $R$ such that $\text{Rad } xR = \text{Rad } I$. Let $\check{\mathcal{C}}_\mathfrak{x}$ denote the Čech complex with respect to $\mathfrak{x}$. Then there is a short exact sequence of complexes

$$0 \rightarrow D_\mathfrak{x}[-1] \rightarrow \check{\mathcal{C}}_\mathfrak{x} \rightarrow R \rightarrow 0,$$

where $D_\mathfrak{x}$ is the global Čech complex. That is $D_i^\mathfrak{x} = \check{\mathcal{C}}^{i+1}_\mathfrak{x}$ for $i \geq 0$ and $D_i^\mathfrak{x} = 0$ for $i < 0$. Now let $E'$ denote an injective resolution of $M$. By applying the functor $\text{Hom}_R(-, E')$ to the short exact sequence of complexes it provides a short exact sequence

$$0 \rightarrow E' \rightarrow \text{Hom}_R(\check{\mathcal{C}}_\mathfrak{x}, E') \rightarrow \text{Hom}_R(D_\mathfrak{x}[-1], E') \rightarrow 0$$

of left bounded complexes of injective $R$-modules. The complex in the middle is an injective resolution of $\hat{M}^I$ as follows by 2.3 and the fact that $\hat{M}^I$ is $I$-adic complete (see [14, 5.2]). Therefore the complex at the right is quasi-isomorphic to $\hat{M}^I/\tau(M)$ considered as a complex concentrated in homological degree zero. In order to prove the statement in (a) let $L$ denote a projective resolution of $X$. Then $\text{Ext}^i_R(X, \hat{M}^I/\tau(M)) \simeq H^i(\text{Hom}_R(L, \text{Hom}_R(D_\mathfrak{x}[-1], E'))))$ (see [2]) and it will be enough to show that the last modules vanish for all $i \in \mathbb{Z}$.

To this end consider the isomorphism of complexes

$$\text{Hom}_R(L, \text{Hom}_R(D_\mathfrak{x}[-1], E')) \simeq \text{Hom}_R(L \otimes_R D_\mathfrak{x}, E')[1].$$

Now $D_\mathfrak{x}$ is a bounded complex of flat $R$-modules. Therefore there is the quasi-isomorphism $L \otimes_R D_\mathfrak{x} \rightarrow X \otimes_R D_\mathfrak{x}$. It will be enough to show that the complex $X \otimes_R D_\mathfrak{x}$ is homologically trivial. But this is true since $X \otimes_R D_i^\mathfrak{x} = 0$ for all $i \in \mathbb{Z}$ because of $\text{Supp}_R X \subseteq V(I)$. This proves the statement in (a).

By view of the above investigations there is a quasi-isomorphism $E' \rightarrow \text{Hom}_R(\check{\mathcal{C}}_\mathfrak{x}, E')$ where the second complex is an injective resolution of $\hat{M}^I$. This proves the statement in (b). \hfill \Box

We will continue with a result on the vanishing of certain Ext-modules. As a step towards to our main result we shall prove a partial result in order to characterize the completion.

**Theorem 3.3.** Let $R$ denote a commutative Noetherian ring. Let $M$ denote an arbitrary $R$-module. Let $x \in R$ be an element such that $0 :_M x^\alpha = 0 :_M x^\beta$ for all $\alpha \geq \beta$. Then the following conditions are equivalent:

(i) $M$ is $xR$-adic complete.

(ii) $\text{Ext}^i_R(R_x, M) = 0$ for $i = 0, 1$.

**Proof.** Let $N = \cup_{\alpha \geq 1} :_M x^\alpha$. Then $N = :_M x^\beta$ as follows by the definition of $\beta$. That is, in both of the statements we may replace $x$ by $x^\beta$ without loss of generality. That is, we may assume that $xn = 0$. Then there is a commutative diagram with exact rows:

$$\begin{array}{cccccc}
0 & \rightarrow & M/N & \xrightarrow{x^n} & M & \rightarrow & M/x^nM & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M/N & \xrightarrow{x^n} & M & \rightarrow & M/x^nM & \rightarrow & 0.
\end{array}$$

By passing to the inverse limit it provides an exact sequence

$$0 \rightarrow \lim_{\leftarrow} \{M/N, x\} \rightarrow M \rightarrow \hat{M}^x \rightarrow \lim_{\leftarrow} \{M/N, x\} \rightarrow 0.$$ 

By view of Lemma 2.4 it yields that $\lim_{\leftarrow} \{M/N, x\} \simeq \text{Ext}^i_R(R_x, M/N)$ for $i = 0, 1$. Furthermore, the short exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ induces an isomorphism $\text{Ext}^i_R(R_x, M) \simeq \text{Ext}^i_R(R_x, M/N)$. This follows by the long exact cohomology sequence and $\text{Ext}^i_R(R_x, N) = 0$ for
all \( i \in \mathbb{Z} \). For this vanishing note that the multiplication by \( x \) acts on \( R_x \) as an isomorphism and on \( N \) as the zero map.

That is, the homomorphism \( M \to \tilde{M}^x \) is an isomorphism, if and only if \( \text{Ext}^i_R(R_x, M) = 0 \) for \( i = 0, 1 \) which proves (i) \( \iff \) (ii).

**Proof of Theorem 1.1.** The implication (i) \( \implies \) (ii) is a consequence of Theorem 3.1. Moreover, the implications (ii) \( \implies \) (iii) is trivial and (iii) \( \implies \) (iv) is easy to see. In order to prove (iv) \( \implies \) (i) we first note that \( \text{Ext}^1_R(R_x, R) = 0 \) for all \( i = 1, \ldots, r \). By Theorem 3.3 this implies that \( M \) is \( x_i \)-radically complete for \( i = 1, \ldots, r \). Since \( x = \{ x_1, \ldots, x_r \} \) generates \( I \) up to the radical it follows that \( \tilde{M} \) is \( I \)-adic complete by Proposition 2.4 since \( M \) is \( I \)-adic separated. This completes the proof. \( \square \)

4. An alternative proof of Theorem 3.1

In this section there is an alternative proof of Theorem 3.1 based on some results on inverse limits. The results of the following two lemmas might be also of some independent interest. These statements are particular cases of certain spectral sequences on inverse limits (see [10]). Here we give an elementary proof based on the description of \( \lim \limits_{\leftarrow}^1 \) as discussed in [17].

**Lemma 4.1.** Let \( R \) denote a commutative ring. Let \( \{ M_n \} \in \mathbb{N} \) be a direct system of \( R \)-modules. Let \( N \) denote an arbitrary \( R \)-module. Then there is a short exact sequence

\[
0 \to \lim \limits_{\leftarrow}^1 \text{Ext}^{i-1}_R(M_n, N) \to \text{Ext}^i_R(\lim \limits_{\leftarrow} M_n, N) \to \lim \lim \text{Ext}^i_R(M_n, N) \to 0
\]

for all \( i \in \mathbb{Z} \).

**Proof.** This result was proved in ([12, Lemma 2.6]) for a Noetherian ring. The same arguments work in the general case. \( \square \)

In the following we need a certain dual statement of Lemma 4.1. In a certain sense it will be the key argument for the second proof of Theorem 3.1.

**Lemma 4.2.** Let \( R \) denote a commutative ring. Let \( M \) denote an arbitrary \( R \)-module. Let \( \{ N_n \} \in \mathbb{N} \) be an inverse system of \( R \)-modules with \( \lim \limits_{\leftarrow}^1 N_n = 0 \). Then there is a short exact sequence

\[
0 \to \lim \limits_{\leftarrow}^1 \text{Ext}^{i-1}_R(M, N_n) \to \text{Ext}^i_R(M, \lim \limits_{\leftarrow} N_n) \to \lim \lim \text{Ext}^i_R(M, N_n) \to 0
\]

for all \( i \in \mathbb{Z} \).

**Proof.** Because of \( \lim \limits_{\leftarrow}^1 N_n = 0 \) there is a short exact sequence

\[
0 \to \lim \limits_{\leftarrow} N_n \to \prod N_n \to \prod N_n \to 0,
\]

where the third homomorphism is the transition map (see [17]). It induces a long exact cohomology sequence

\[
\cdots \to \prod \text{Ext}^{i-1}_R(M, N_n) \xrightarrow{f} \prod \text{Ext}^{i-1}_R(M, N_n) \to \text{Ext}^i_R(M, \lim \limits_{\leftarrow} N_n) \\
\to \prod \text{Ext}^i_R(M, N_n) \xrightarrow{g} \prod \text{Ext}^i_R(M, N_n) \to \cdots.
\]

To this end recall that \( \text{Ext} \) transforms direct products into direct products in the second variable and cohomology commutes with direct products (see e.g. [6]). Now it is known (see e.g. [17]) that

\[
\text{Coker} f \simeq \lim \limits_{\leftarrow} \text{Ext}^{i-1}_R(M, N_n) \text{ and } \text{Ker} g = \lim \lim \text{Ext}^i_R(N, N_n).
\]

This completes the proof. \( \square \)
Remark 4.3. The assumption that $\lim_{i}^1 N_{ik} = 0$ is fulfilled whenever the projective system $\{N_{ik}\}$ satisfies the Mittag-Leffler condition. That is, for instance when the transition map $N_{ik+1} \rightarrow N_{ik}$ is surjective for all $k \geq 1$.

Note that the proof of the following Theorem is motivated by some arguments done by Buchweitz and Flenner (see [5]).

Theorem 4.4. Let $R$ denote a commutative ring. Let $M$ denote an $R$-module. Let $F$ denote a flat $R$-module satisfying $F \otimes R I = 0$. Then $\text{Ext}_R^i(F, \hat{M}^1) = 0$ for all $i \in \mathbb{Z}$.

Proof. By definition we have $\hat{M}^1 = \lim_{\rightarrow} M/I^n M$. By Lemma 4.2 there is a short exact sequence

$$0 \rightarrow \lim_{\rightarrow}^1 \text{Ext}_R^{i-1}(F, M/I^n M) \rightarrow \text{Ext}_R^i(F, \hat{M}^1) \rightarrow \lim_{\rightarrow} \text{Ext}_R^i(F, M/I^n M) \rightarrow 0.$$  

In order to show the vanishing of $\text{Ext}_R^i(F, \hat{M}^1)$ for all $i \in \mathbb{Z}$ it will be enough to show the vanishing of $\text{Ext}_R^i(F, M/I^n M)$ for all $i \in \mathbb{Z}$ and all $\alpha \geq 1$. We claim that

$$\text{Ext}_R^i(F, M/I^n M) \cong \text{Ext}_R^{i/n}(F/I^n F, M/I^n M)$$

for all $i \in \mathbb{Z}$ and all $\alpha \geq 1$. In order to show these isomorphisms let $L$ be a projective resolution of $F$ as an $R$-module. As $F$ is flat as an $R$-module $\text{Tor}_R^i(F, R/I^n) = 0$ for all $i > 0$ and $L \otimes R I^n$ is a projective resolution of $F/I^n F$ as an $R/I^n$-module. By adjunction there are isomorphisms of complexes

$$\text{Hom}_{R/I^n}(L \otimes R I^n, M/I^n M) \cong \text{Hom}_R(L, M/I^n M).$$

By taking cohomology it proves the above claim. The vanishing follows now because of $F/I^n F = 0$ as a consequence of the assumption $F \otimes R I = 0$. □

Remark 4.5. In fact Theorem 4.4 is a slight sharpening of Theorem 3.1. To this end recall that for an $R$-module $M$ and an ideal $I \subset R$ the I-adic completion $\hat{M}^1$ is not necessarily I-adic complete (as used in the proof of Theorem 3.1). For an explicit example see J. Bartijn’s Thesis [3, I, §3, page 19]. Note that in this example the ring is not Noetherian. It grows out of [4, III, §2, Exerc. 12]. See also A. Yekutieli (see [18, Example 1.8]).

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