Thin Film Breakup and Rivulet Evolution Modeling

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Abstract.
The present paper is aimed at the modeling of a continuous film breakup into individual rivulets, leading to the formation of dry patches on the substrate surface. Following an approach already successfully applied to the prediction of still/moving droplet configuration, we attempt to model the details of a single possible film breakup and its evolution over a two-dimensional domain via a phenomenological model. Based on the momentum, energy and mass flow balance of the capillary ridge on the border of the dry patch, the proposed model is validated against both numerical prediction and experimental results from the open literature. Such a detailed prediction may not be practical for the simulation of complex geometrical configuration (which may include, as an example, multiple breakups on the surface of a whole aircraft subject to icing condition), but can be used to look for statistically significant parameters that can be used to provide proper boundary conditions for fully 3D CFD computations.

1. Introduction
A number of applications of engineering interest are related to the presence and evolution of a thin layer of liquid over a surface, often in presence of a phase change. Such liquid layer may be a collection of tiny droplets, either still moving, driven by gravity or shear stresses, as in fogging and defogging simulations, or can be made up by an ensemble of rivulets, or can be a continuous film, as in several heat transfer devices including de-humidifier or condenser, or may be any combination of these forms.

In most cases, the goal of prediction and simulations are just the evaluation of global average parameters such as percentage of wet surface, or the check of proper critical thresholds such as dry-out limits in a cooling device. Sometimes, however, more details are required. As an example, in-flight icing phenomena on aircraft surfaces are driven by the evolution of the surface water layer fed by a stream of super-cooled droplets from the clouds. Under different conditions of cloud water content, temperature and flow field we may have to deal with all of the above mentioned possible water layer, and we need to know in details such topology, since it will affect the water flow over the surface and, thus, the extent of the area subjected to possible icing condition to be protected.

In previous works [1, 2], the authors addressed the Modeling of the transition between still droplet and moving ones. Here, we present a model for the breakup of a continuous film and the prediction of the evolution of the consequent dry patch.

A first paper on separation of a laminar film flowing vertically under gravity was published by Hartley and Murgatroyd [3], who proposed a critical breakup threshold condition based on a force balance at the apex of the dry patch. Starting from a velocity profile driven by gravity
and viscous dissipation only, they defined critical values for both flow rate and film thickness, under which the film may separate into two rivulets, forming a stable dry patch. In the force balance, the pressure increase induced by film inertia compensates the liquid surface tension \( \sigma \).

Podgorski et al. [4], starting from experimental data on a laminar film flowing down an inclined plane, derived an approximate model for the prediction of the shape of a stable dry patch downstream of a film breakup. The presence of a capillary ridge near the contact line of the dry patch suggested a force balance along the contact line, in which the weight of capillary ridge is balanced by surface tension forces. Practically, Podgorski’s model is designed to handle the case where the film inertia is very low and the film thickness is smaller than the critical one defined by Hartley and Murgatroyd.

Also Rio and Limat [5] observed experimentally the presence of this capillary ridge and Wilson et al. [6] obtained its exact shape, solving analytically the governing lubrication equation.

The aim of this paper is to formulate a more general model, based mostly on the basic conservation equation, suited to the prediction of the shape of a stable dry patch, when a laminar liquid film flowing on an inclined plane separates into two rivulets. To enhance the flexibility of the model, both film inertia and capillary ridge weight are taken into account. Furthermore, since we start from the most general form of the balance equations, the shear stress driven case could be easily taken into consideration.

2. Problem Formulation and Model

Consider a thin film of a newtonian liquid with constant and uniform density \( \rho \) and viscosity \( \mu \) flowing down a plane inclined at an angle \( \alpha \) to the horizontal. Let \( h \) and \( u_\infty \) be the film thickness and the film velocity and assume that the external pressure is equal to the atmospheric one, i.e. relative pressure \( p_\infty = 0 \). Suppose that the film separates into two rivulets forming a stable dry patch and let \( \theta \) be the static contact angle of the liquid. Following Podgorski [4], it is assumed that the deviated film mass flow is progressively stored into a capillary ridge along the dry patch contact line. Thus, it is possible to derive the differential balances of mass, momentum and energy for an infinitesimal element of liquid inside the capillary ridge.

2.1. Mass, Momentum and Energy Balances

We will solve the continuity, momentum and energy equations in the capillary ridge along a curvilinear coordinate \( l \) aligned with the ridge itself. The elementary control volume at a position \( l \) is shown in Fig. 1(b): it includes a section of the ridge between \( l \) and \( l + dl \), and it is bounded by the dry patch contact line and the interface \( i \) between the ridge and the film. The flow direction \( l \) is inclined at an angle \( \beta \) with respect to the main flow direction \( y \) of the undisturbed upstream film. Above the interface \( i \) the liquid flow has the same characteristics of the undisturbed film far from the dry patch (\( u_\infty \) and \( p_\infty \)), while below \( i \), in the capillary ridge, the flow is assumed parallel to the contact line (its velocity and pressure become \( u \) and \( p \)). To let this be true, it is also assumed that the flow deviates instantly and enters the capillary ridge in \( l \) direction. The surface \( S \) represents a general section of the capillary ridge, see Fig. 1(a). It is not necessary to know its exact shape but only its extension.

From the continuity equation, applied to the control volume of Fig. 1(b), the exit mass flow \( (\rho uS)_{l+dl} \) must be equal to the sum of the incoming flow \( (\rho uS)_l \) from the upstream section of the capillary ridge and the mass flow \( \delta m_i \) entering through the interface \( i \) between the film and the ridge; thus

\[
\delta m_i + \rho(uS)_l = \rho(uS)_{l+dl}
\]

Rearranging Eq. (1) and neglecting the higher order terms we have

\[
\delta m_i = \rho(udS + Sdu) = d(\rho uS)
\]
Figure 1. (a) Dry Patch and capillary ridge section. (b) External forces on the control volume

The term $\delta m_i$ will be defined in section 2.2 as a function of the angle $\beta$. The subscript $i$ refers to a property of the liquid crossing the interface $i$ between the ridge and the film.

The momentum equation correlates the momentum fluxes to the external forces acting on the capillary ridge elementary volume. Referring to Fig. 1(b), these external forces are pressure forces on the faces of the volume, the surface tension $\sigma$ applied along $dl$, the gravity force $dF_g$ due to the weight of the ridge projected along the inclined plane,

$$dF_g = \rho g \sin \alpha \ S \ dl$$

and the surface shear stress $dF_r$ along the substrate,

$$dF_r = \tau_p \ w \ dl$$

where $w \ dl$ is the base area of the ridge element, where $\tau_p$ is applied. Equilibrium along the ridge streamwise coordinate $l$ and along its normal direction $n$ yield respectively

$$\left[ (\rho u^2 + p)_{l+dl} S_{l+dl} - (\rho u^2 + p)_l S_l \right] \cos \left( \frac{d\beta}{2} \right) - \left( \rho u^2 + p_i \right) S_{n,i} = dF_g \sin \beta - dF_r \tag{3}$$

$$dF_g \cos \beta + p_i h \ dl = \sigma (1 - \cos \theta) \ dl + \left[ (\rho u^2 + p)_{l+dl} S_{l+dl} + (\rho u^2 + p)_{l+dl} S_{l+dl} \right] \sin \left( \frac{d\beta}{2} \right) \tag{4}$$

where $S_{n,i}$ is the projection of the interface surface $i$ on the plane normal to the streamwise direction.

$$S_{n,i} = \frac{\delta m_i}{\rho u_i}$$
Rearranging Eqs. (3) and (4) and neglecting the higher order terms yields

\[
d\left[(\rho u^2 + p) S\right] - \left(\delta m_i u_i + p_i \frac{\delta m_i}{\rho u_i}\right) = dF_g \sin \beta - dF_y
\]

\[
dF_g \cos \beta + p_i h \, dl = \sigma(1 - \cos \theta) \, dl + (\rho u^2 + p) \, S \, d\beta
\]  

(5)  

(6)  

Finally, energy balance can be written as

\[
\rho(uS)_l \left(\frac{1}{2}u^2 + \frac{p}{\rho} + gH\right)_i + \delta m_i \, e_i = \rho(uS)_{l+dl} \left(\frac{1}{2}u^2 + \frac{p}{\rho} + gH\right)_{l+dl} + \rho uS \frac{dR}{\rho}
\]

(7)  

where \(H\) is the geodetic height of the corresponding face, \(dR\) represents the shear stress losses

\[
dR = \tau_p \frac{w \, dl}{S}
\]

and \(e_i\) is the specific energy of the liquid entering the capillary ridge through its interface \(i\).

Assuming energy conservation through the interface:

\[
e_i = \frac{1}{2} u^2_\infty + gH_i
\]

Applying continuity equation and neglecting the higher order terms, Eq. (7) offers

\[
d \left[\rho uS \left(\frac{1}{2}u^2 + \frac{p}{\rho}\right)\right] + \rho uS \left(gdH + \frac{dR}{\rho}\right) = \delta m_i \left(\frac{1}{2}u^2_\infty + g\Delta H\right)
\]

(8)  

where the right hand side defines the net energy input from the incoming film mass flow and \(dH\) is the geodetic height variation between sections \(l\) and \(l + dl\) of the elementary volume.

Finally, \(\Delta H\) is the height difference between the interface \(i\) and the ridge centroid

\[
\Delta H = \frac{1}{2} w \sin \alpha \sin \beta = H_i - H_l
\]

We can recast the problem in non-dimensional form assuming, as reference quantities, the incoming film height \(h\) and velocity \(u_\infty\). This allows to define the dimensionless lengths \(h^* = 1\), \(dl^*\), \(w^*\), \(\Delta H^*\) velocity \(u^*\) and surface \(S^*\). Furthermore, dimensionless pressures \(p^*\) are obtained normalizing via the inlet film kinetic energy \(\frac{1}{2}\rho u^2_\infty\). Finally, the friction factor and the dimensionless incoming mass flow are defined as:

\[
f = \frac{\tau_p}{\frac{1}{2} \rho u^2_\infty} ; \quad \dot{m}_i^* = \frac{1}{\rho u^2_\infty} \frac{\delta m_i}{dl}
\]

This normalization lead to the definition of the proper Weber Number and Bond Number

\[
We = \frac{1}{2} \rho u^2_\infty \frac{h}{\sigma} ; \quad Bo = \frac{\rho g h^2}{\sigma}
\]

Normalizing Eqs. (2), (5), (6) and (8) a system of four differential equations can be obtained

\[
\dot{m}_i^* = \frac{d}{dl^*} (u^* S^*)
\]

\[
\frac{d\beta}{dl^*} = -\frac{\left(1 - \cos \theta\right) - We \, p_i^* - Bo \sin \alpha \, S^* \cos \beta}{We \left(2u^2 + p^*\right) S^*}
\]

(9)  

(10)  

\[
\left(\frac{Bo}{We \sin \alpha \sin \beta - \frac{w^*}{S^*} f} \right) S^* + \dot{m}_i^* \left(2u_i^* + \frac{p_i^*}{u_i^*}\right) = \frac{d}{dl^*} \left[\left(2u^2 + p^*\right) S^*\right]
\]

(11)  

\[
\left(\frac{Bo}{We \sin \alpha \sin \beta - \frac{w^*}{S^*} f} \right) u^* S^* + \dot{m}_i^* \left(1 + \frac{Bo}{We} \Delta H^*\right) = \frac{d}{dl^*} \left[\left(u^* + p^*\right) u^* S^*\right]
\]

(12)
2.2. Cauchy Problem

Some more hypothesis are needed to solve Eqs. (9), (10), (11) and (12). In particular, from the assumption that the potential energy reduction balances fiction losses, we get

\[ gdH + \frac{dR}{\rho} = 0 \Rightarrow \frac{Bo}{We} \sin \alpha \sin \beta - \frac{w^*}{S^*} f = 0 \]

This is the same condition which is fulfilled in the film flow far from the dry patch. Furthermore, if the film flow is undisturbed up to interface \( i \), the mass flow \( \delta m_i \) entering the control volume through the interface \( i \) can be defined as the mass coming from an horizontal section \( dx = dl \cos \beta \)

\[ \delta m_i = \rho u_\infty h \frac{S}{w} \Rightarrow \dot{m}_i^* = \cos \beta \]

The velocity \( u_i \) and the pressure \( p_i \), that represent the entering condition of the liquid at the interface \( i \), are required to solve Eqs. (10) and (11). Following [4], it is assumed that the relative pressure \( p_i \) at the interface is zero, and that the energy transported by incoming mass flow \( \delta m_i \), see Eq. (8), is fully converted into kinetic energy

\[ \frac{1}{2} u_i^2 = \frac{1}{2} u_\infty^2 + g \Delta H \Rightarrow u_i = \sqrt{u_\infty^2 + 2g \Delta H} \]

\[ u_i^* = \sqrt{1 + (Bo/We) \Delta H^*} \]  

Finally, assuming that the distance between the centroid and the entering section of the control volume is a linear function of the length \( S/h \), the dimensionless height \( \Delta H^* \) is estimated as

\[ \Delta H^* = \xi \frac{S}{h} \sin \alpha \cos \beta \]

where \( \xi \) is a constant depending on the shape of capillary ridge section.

Note that from geometrical consideration on Fig.1(a) we can obtain that \( \xi = \frac{1}{2} \frac{h}{S^*/w} \), i.e the average height \( h \) of the ridge is \( 1/2 \xi \) time the inflow film height, \( h = \frac{1}{2} \frac{S}{\xi} \). Thus, the maximum possible value for \( \xi \) is 1/2, when the ridge does not exceed the film height and thus has the shape of a simple rectangle of height equal to \( h \), while the minimum value should be limited by the requirement of equilibrium between surface tension and ridge hydrostatic pressure. A reasonable value \( \xi = 0.3 \) was chosen in the following computations. Although a proper evaluation of \( \xi \) would require a 3D formulation, results will show that its actual value doesn’t affect considerably the results.

Under these assumptions, we obtain the following Cauchy Problem from Eqs. (9, 10, 11, 12)

\[ \dot{u}^* = \frac{d u^*}{dt^*} = \frac{\cos \beta}{S^*} \frac{\sqrt{1 + (Bo/We) \Delta H^* - u^*}}{p^*} \]

\[ \dot{S}^* = \frac{d S^*}{dt^*} = \frac{\cos \beta}{u^*} \frac{p^* - \sqrt{1 + (Bo/We) \Delta H^* - u^*}}{p^*} \]

\[ \dot{p}^* = \frac{d p^*}{dt^*} = \frac{\cos \beta}{u^* S^*} \left[ 1 + (Bo/We) \Delta H^* - (u^{*2} + p^*) \right] p^* - 2u^{*2} \frac{\sqrt{1 + (Bo/We) \Delta H^* - u^*}}{p^*} \]

\[ \dot{\beta} = \frac{d \beta}{dt^*} = - \frac{(1 - \cos \theta) - Bo \sin \alpha \frac{S^* \cos \beta}{We (2u^{*2} + p^*) S^*}} \]

\[ \beta \]
with the initial conditions
\[ u_0^* = 0 \quad ; \quad p_0^* = 1 + \frac{Bo}{We} \Delta H_0^* \quad ; \quad S_0^* = \frac{1 - \cos \theta}{Bo \sin \alpha - (We + Bo \Delta H_0^*) / r_0^*} \quad ; \quad \beta_0 = 0 \]

where \( r_0^* = r_0 / h = 1 / \beta_0 \) is the dimensionless radius of curvature at the apex of the dry patch.

Differential Eqs. (15), (16), (17) and (18) was solved and the functions \( u^*(l^*) \), \( S^*(l^*) \), \( p^*(l^*) \) and \( \beta(l^*) \) obtained. Knowing \( \beta(l^*) \) and introducing the dimensionless Cartesian Coordinates \( x^* = x / h \) and \( y^* = y / h \), see figure 1(a), it’s possible to calculate the dry patch shape \( y^*(x^*) \)
\[ y^* = \int_0^{l^*} \sin \beta \, dl^* \quad ; \quad x^* = \int_0^{l^*} \cos \beta \, dl^* \]

3. Results
We will compare the model described above with both the theoretical results of Podgorski et al. [4] and the experimental data reported in the same paper and in [5]. Podgorski model is still commonly used as reference benchmark for more sophisticated simulation, such as in [7]. In his model Podgorski et al. use the following equations to describe the dry patch shape
\[ x = r_0 \frac{\sin \beta}{\cos^2 \beta} \quad y = -\frac{r_0}{3} \frac{1 - 3 \sin^2 \beta}{\cos^3 \beta} - 1 \]

The radius of curvature at the apex of the dry patch is given by an empirical relation [4]
\[ r_0 = m \frac{(1 - \cos \theta)^4}{\theta - \cos \theta \sin \theta} \frac{h}{Ca Bo \sin \alpha} \]

where \( Ca = \mu u_\infty / \sigma \) is the Capillary number and \( m \) is a dimensionless constant depending on the shape of the capillary ridge.

Both Rio and Limat [5] and Podgorski et al. [4] made experimental tests on silicon oil 47V20 sold by Rhodorsil (\( \rho = 950 K g / m^3 \), \( \mu = 0.019 Pa \cdot s \), \( \sigma = 0.021 N / m \)). The film flow rate per unit length and the tilt angle of the inclined plane was chosen as input parameters. The advancing contact angle \( \theta_a \) of the liquid was experimentally measured both by Podgorski et al. and Rio and Limat. The main case parameters are given in Table 1.

Other quantities need to be defined to run our model. The initial condition on \( S^* \) is estimated considering the curvature radius \( r_0 \) from Eq. (20) with \( m = 0.17 \) (as in [5]) or \( m = 0.33 \) (as in [4]) depending on the case, while the mean velocity and the thickness of the film (required to define \( We, Bo \) and \( Ca \)) are computed assuming that gravity is balanced by viscous shear stress
\[ h = \sqrt[3]{\frac{3 \mu \dot{V}}{\rho g \sin \alpha}} \quad ; \quad u_\infty = \frac{\dot{V}}{h} \]

Fig. 2(a) compares our model with the experimental setup of Rio and Limat [5]: predicted dry patch shape shows a fully satisfying agreement with both experimental data and Podgorski model results. The profile of liquid velocity inside the capillary ridge \( u(x) \) and its section \( S(x) \)

| Ref. | \( V = u_\infty h \) | \( \alpha \) | \( \theta_a \) | \( m \) |
|------|----------------|-------|--------|-----|
| Rio and Limat [5] | 0.01cm²/s | 80° | 48° | 0.17 |
| Podgorski et al. [4] | 0.01cm²/s | 15° | 30° | 0.33 |

Table 1. Test case set up
Figure 2. Model validation: $\dot{V} = 0.01 \text{ cm}^2/\text{s}$, $\alpha = 80^\circ$, $\theta = 48^\circ$. (a) Dry patch shapes. (b) Flow velocity along the ridge. (c) Capillary ridge cross section. (d) Pressure inside the ridge.

Figure 3. Model validation: $\dot{V} = 0.01 \text{ cm}^2/\text{s}$, $\alpha = 15^\circ$, $\theta = 30^\circ$. Dry patch shapes.
are shown in Figs. 2(b) and 2(c): these profiles replicates fairly well the one predicted by Podgorski et al. model [4], too. It is not possible to compare the liquid pressure inside the capillary ridge because Podgorski et al. didn’t take it into account: anyway it can be seen in Fig. 2(d) that \( p(x) \) reaches its maximum value of \( 1.26 \times 10^{-3} \text{ N/cm}^2 \) near the apex of the dry patch \( (x \approx 0.335 \text{ cm}) \) and than it starts to decrease for increasing value of \( x \).

The maximum is due to the initial increase of the net potential energy \( g \Delta H \), i.e. of the energy that enter the ridge through the interface \( i \).

Fig. 3 shows, again, a fully satisfying matching between our model, Podgorski’s model and experimental data [4], for a different set of parameters (see table 1). This confirms the model accuracy and flexibility. The trends of \( u \), \( S \) and \( p \) are not shown because appears quite similar to the ones of Figs. 2.

It is worth to notice that the details of exact shape of capillary ridge section are not such important: the influence of \( \xi \) is showed in Fig. 4(a): the higher is its value, the less it influences the predicted dry patch shape; only at very low values of \( \xi \) the predicted dry patch becomes significantly narrower: let remember that \( \xi = 0.1 \) corresponds to an unrealistic height of the ridge five times taller then the incoming film, while when \( \xi = 0 \) the centroid of capillary ridge coincides with the ridge interface one, i.e. the capillary ridge has zero width (and thus infinite height).

Also the influence of the empirical parameter \( m \), needed by Eq. (20) to define the radius of curvature at the apex of the dry patch, is analyzed, see Fig. 4(b). The reason why \( m \) doesn’t affect significantly the shape of our predicted dry patch is that \( r_0 \) affects only the initial condition \( S_0 \), whilst it scales the dry patch shape obtained by Podgorski, see Eqs. (19). Moreover, being both \( m \) and \( \xi \) only connected with the ridge shape, their meaning must be similar and it would be desirable to correlate them.

Figure 4. Dry patch shapes: \( \dot{V} = 0.01 \text{ cm}^2/\text{s}, \alpha = 80^\circ, \theta = 48^\circ \): Influence of \( \xi \) (a) and \( m \) (b).

Fig. 5 illustrates the influence of both film flow rate \( \dot{V} \) and tilt angle of the inclined plane \( \alpha \) (the two main input parameter defining the general problem) on the dry patch shape. Silicon oil is again considered and the imposed liquid contact angle is \( \theta = 48^\circ \). The flow rate ranges from 0.01 to 0.04 cm²/s, while we consider a tilt angle in the range 15° \( \pm \) 90°, where \( \alpha = 90^\circ \) is the case of liquid film sliding on a vertical wall. Values of \( \dot{V} \) higher than 0.04 cm²/s are not considered because Rio and Limat observed experimentally that it exists a critical flow rate \( \dot{V}_c = 0.03 \div 0.04 \text{ cm}^2/\text{s} \) (not depending on \( \alpha \)) beyond which the dry patch closes [5].
Figure 5. Dry patch shapes: (a) $\theta = 48^\circ$, $\alpha = 80^\circ$, different flow rates: $V = 0.01, 0.02, 0.03, 0.04 \text{ cm}^2/\text{s}$; (b) $\theta = 48^\circ$, $V = 0.01 \text{ cm}^2/\text{s}$, different tilt angles: $\alpha = 15^\circ, 30^\circ, 60^\circ, 90^\circ$.

Fig. 5(a) shows a progressive reduction of the dry patch width when the film flow rate increases, for a fixed $\alpha$. This trend is due to the fact that the film inertia (i.e. the Weber number) and the film energy increase with the flow rate; hence the surface tension forces find it harder to widen the dry patch gap. Narrower dry patches are also obtained increasing the tilt angle $\alpha$ at a fixed $V$, although this reduction is less noticeable for high value of $\alpha$, where the curves progressively collapse asymptotically on the limiting case of $\alpha = 90^\circ$, see Fig. 5(b).

It is interesting to notice that the dry patch always tends to a quadratic shape for high values of $x$, see Fig. 6. This confirms what Wilson et al. [6] obtained from the lubrication approximation. It can be also found that when $x$ is high enough the mean velocity tends to a constant value, the capillary ridge section increases linearly and the liquid pressure decreases proportionally to $x^{-1/2}$:

$$\lim_{x \to \infty} y(x) \propto x^2; \quad \lim_{x \to \infty} u(x) \propto 1; \quad \lim_{x \to \infty} S(x) \propto x; \quad \lim_{x \to \infty} p(x) \propto x^{-\frac{1}{2}}$$

Such behavior appears reasonable: if $x$ is large enough the dry patch angle $\beta$ gets close to $\pi/2$, i.e. the problem reduces to an undisturbed flowing film. In terms of asymptotically solutions, this means that the velocity $u$ of the liquid inside the ridge reaches a constant value ($u_\infty$ should be the expected value), the pressure $p$ is equal to zero and the ridge collapses to a section of the flowing film itself ($S = h x$). From our model, the velocity $u$ tends to a value that is higher than $u_\infty$ but comparable. The only reason is probably due to an inaccurate definition of the height $\Delta H$, see Eq. (14), that is connected with the incoming net potential energy and doesn’t reach zero for high values of $x$ (it just tends to a constant value that is lower than the initial one). Podgorski model [4], otherwise, predicts a velocity $u$ inside the ridge proportional to $\sqrt{x}$: this, for $x \to \infty$, yields a non physical infinite velocity, notwithstanding the dry patch angle $\beta$ tends to $\pi/2$. The expression of $u(x)$ predicted by Podgorski can be obtained rearranging the governing equations of his model [4] and extracting the dry patch angle $\beta(x)$ from Eq. (19).

4. Conclusion

A model based on balance equations was developed with the aim to describe the shape of a stable dry patch on an inclined plane. The existence of a capillary ridge near the dry patch
Figure 6. Ratio between dry patch shape $y(x)$ and its parabolic approximation, $\dot{V} = 0.04 \text{cm}^2/\text{s}$, $\alpha = 80^\circ$, $\theta = 48^\circ$.

contact line, as proposed by other authors, was assumed. Excellent agreement with the available experimental data was found, although validation was limited to a high viscosity liquid (silicon oil) and the dominant effect was the capillary ridge weight. It would be helpful to validate the model when the liquid inertia is higher (e.g. water), but no experimental data is yet available.

Some empirical information are needed: in particular, in the definition of the height $\Delta H$, function of the ridge shape via the parameter $\xi$, and in the evaluation of the initial curvature of the dry patch, via a tunable parameter $m$.

However in the present model the effect of variation of either $\Delta H$ and $m$ is limited, and thus reasonable dry patch development predictions are possible without knowing the exact shape of the capillary ridge. Thus, the model seems more flexible than previous literature ones, such as Podgorski’s. Furthermore, the proposed model could be able to predict the dry patch shape in a wider range of cases, i.e. when the Weber number is higher, since the film inertia is taken into account.

In perspective, this approach could be extended to the break-up of a shear driven film, adding the shear stress to sum of the external force acting on the capillary ridge, or integrated in more complex problems involving heat and mass transfer.

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