The nonlinear ship rolling and safe basin erosion in stochastic beam seas

D Deleanu and C L Dumitrache
Constanta Maritime University, Department of General Engineering Sciences, Mircea cel Batran street, No. 104, 900663, Romania.
E-mail: dumitrudeleanu@yahoo.com

Abstract. A short number of steep breaking waves hitting a ship from the side may affect its dynamic behaviour and determine extreme roll angles and, eventually, capsizing. Even the capsise is a rare event, its consequences for the ship and crew are often fatal. Generally, the ship rolling in regular or stochastic beam seas could be described by a second-order non-linear differential equation with the roll angle as dependent variable. Non-linearity comes from the restoring and damping moments, which are usually represented by polynomials of roll angle or of its time derivative. In the paper, we used such a model equation to estimate the ship rolling and capsizing in a stochastic beam sea. The sea action was simulated by a harmonic function with random frequency and phase. The roll equation was solved using a simple, fast and accurate iterative scheme based on Taylor expansion which proved to be very competitive in terms of accuracy with more elaborate methods and which allowed a substantial reduction of the CPU time. Due to these properties of the scheme, we were able to conduct an extensive investigation on fractal erosion of safe basins and to represent the boundaries between capsizing and non-capsizing regions in wave frequency – wave amplitude plane and normalized integrity curves for different combinations of random wave parameters. The restoring and damping coefficients corresponded to a vehicle ferry model, considered to be either with or without bilge keels.

1. Introduction
In ship motion analysis, the study of large amplitude nonlinear rolling is crucial because it is closely linked with capsizes dynamics. For such roll analysis, linear approximation is no longer valid and, as a consequence, obtaining closed – form solutions becomes difficult or even impossible.

The nonlinear effects occur mainly due to nature of restoring, damping and external loadings’ moments. Various nonlinear models for ship rolling have been proposed in the last decades. In most of them, the restoring moment is approximated by quadratic, cubic or even higher – order polynomials of roll angle. The damping is usually inserted in the model equation by means of linear – quadratic or linear – cubic terms in the angular roll velocity. Finally, the sea loads could be set for regular or stochastic waves [1 – 3].

Capsizing or keeling over is that catastrophic situation in which the ship is turned on its side or is upside down. Both theoretical and experimental studies have identified several mechanisms for ship capsise, including pure resonant rolling, parametric excitation, broaching and loss of stability at a wave crest. It was emphasized the fact that more attention should be paid to capsize under transient, rather than steady – state conditions. This situation corresponds to a short train of regular/random waves hitting the ship from the side in otherwise calm whether conditions. Dangerous large – amplitude motions and,
finally, capsizing can appear when the ship is affected by no more than 8 – 10 sufficiently steep waves. Tools of nonlinear dynamics such as safe basins, integrity curves or Lyapunov exponents are involved in an attempt to understand the mechanisms behind ship capsizing [4 – 8].

If the rolling in beam waves is considered, it is possible to ignore the coupling of rolling and the other five degrees of freedom of ship motion. It results that for solving the rolling problem and thus the capsizing scenario, we can numerically integrate a second – order nonlinear differential equation with the roll angle as dependent variable. Numerical integrators incorporated in modern software packages, as ode45 in Matlab, encounter efficiency problems for the unbounded solutions associated with capsizing. This is why, in this paper, we have developed our own software that uses a simple and accurate iterative scheme based on Taylor expansion to solve a typical nonlinear roll equation [9]. The coefficients for damping and restoring moments correspond to an intact vehicle ferry model equipped or not with bilge keels.

2. The nonlinear roll equation

The differential equation of ship roll motion under the random beam seas is given by

\[(l_{44} + A_{44}) \ddot{\theta} + B_{44} \dot{\theta} + B_{44q} \theta^3 + \Delta \cdot GZ(\theta) = M_{sea}(t)\]  

where \(l_{44} + A_{44}\) represents the sum between ship’s moment of inertia in roll and added mass inertia, \(B_{44}\) and \(B_{44q}\) denote the linear and nonlinear damping coefficients, \(\Delta\) the ship displacement, \(GZ(\theta)\) the arm of static stability and \(M_{sea}(t)\) the torque caused by random beam waves. Additionally, \(\theta\) represents the roll angle and a dot over denotes the time differentiation.

Considering a cubic polynomial approximation for \(GZ(\theta)\), namely \(C_1 \theta + C_3 \theta^3\), and dividing by the total moment of inertia, one obtains the following equation:

\[\ddot{\theta} + d_1 \dot{\theta} + d_3 \dot{\theta}^3 + k_1 \theta + k_3 \theta^3 = \xi(t)\]

where

\[d_1 = \frac{B_{44}}{l_{44} + A_{44}}, d_3 = \frac{B_{44q}}{l_{44} + A_{44}}, k_1 = \frac{\Delta C_1}{l_{44} + A_{44}}, d_3 = \frac{\Delta C_3}{l_{44} + A_{44}}, \xi(t) = \frac{M_{sea}(t)}{l_{44} + A_{44}}\]

The excitation of the stochastic beam waves is shaped by a harmonic function with random frequency and phase:

\[\xi(t) = \mu \cos(\omega t + \psi), \psi = \sigma B(t) + \Gamma\]

where \(\omega\) is the center frequency, \(\mu\) and \(\sigma\) the amplitude and the intensity of the excitation, \(B(t)\) a Wiener process and \(\Gamma\) a random variable with the uniform distribution in \([0, 2\pi]\) (see [10]).

For numerical simulation, the damping and restoring coefficients we chosen for a real vehicle ferry ship [4]. If the ship was equipped with bilge keels then \(d_1 = 0.0476542 s^{-2}, d_3 = 3.765 s^{-1}\), otherwise \(d_1 = 0.01265913 s^{-2}, d_3 = 0.4954 s^{-1}\). For the restoring coefficients, a cubic spline interpolation of the static stability curve results in \(k_1 = 0.69199703 s^{-2}, k_3 = -0.5392039 s^{-2}\). The initial conditions are of the form \(\theta(0) = \theta_0\) and \(\dot{\theta}(0) = \dot{\theta}_0\).

3. The iterative scheme

In order to benefit from a better flexibility in the integration of equation (2) and to obtain a reasonably short CPU time, especially in the transition area from bounded to unbounded solutions associated with capsizing, we developed a software based on a simple and accurate scheme proposed in [11].

Adapted to our problem, the scheme reads as

\[\theta_n = \theta_{n-1} + \dot{\theta}_{n-1} \Delta t + 0.5 \dot{\theta}_{n-1}(\Delta t)^2\]

\[\theta_{n+1} = \theta_{n-1} + 2 \dot{\theta}_{n-1} \Delta t + 2 \dot{\theta}_{n-1}(\Delta t)^2\]

\[\dot{\theta}_{n} = (\theta_{n+1} + \theta_{n-1})/(2 \Delta t), n \geq 2\]

\[\dot{\theta}_{n} = -d_1 \dot{\theta}_{n} - d_3 \dot{\theta}_{n}^3 - k_1 \theta_{n} - k_3 \theta_{n}^3 + \xi((n-1)\Delta t)\]

The starting values are
\[ \theta_1 = \theta(0), \dot{\theta}_1 = \theta(0), \ddot{\theta}_1 = \mu - d_1\dot{\theta}(0) - d_3\ddot{\theta}(0) - k_1\theta(0) - k_3\theta^3(0) \]  

The scheme (4) is \(O(\Delta t^2)\) of accuracy.

4. Numerical simulations

In this section, with the help of scheme (4), we carried out a thorough numerical investigation on the solutions of roll equation (2) for different combinations \((\omega, \mu, \sigma)\) of wave parameters. The computations were performed with the Matlab software package.

To begin with, let us form an image of the influence exerted by the intensity \(\sigma\) of the random phase \(\psi\) on the shape of the wave moment. Figure 1 presents the graph of \(\xi(t)\) for \(\omega = 0.9, \mu = 0.3, \sigma \in \{0.2, 0.6\}, \Delta t = 0.0014, t \in [0, 4 \cdot 2\pi/\omega]\). The continuous dark line is associated with the periodic wave \(f = \mu \cos(\omega t)\), that is the function \(\xi(t)\) with \(\sigma = 0\). It is obvious that for small \(\sigma\) function \(\xi(t)\) is a narrow – band process, the difference between the regular and random waves being relatively insignificant. The shape of the waves’ moments separates more and more with the increase of \(\sigma\).

![Figure 1](image1)

**Figure 1.** The time evolution of function \(\xi(t)\) for \(\omega = 0.9, \mu = 0.3, \Delta t = 0.0014\) and \((a) \sigma = 0.2; (b) \sigma = 0.6\). \(f\) represents \(\xi(t)\) with \(\sigma = 0\).

The size of the step \(\Delta t\) influences to some extend the stochastic wave moment, as illustrated in figure 2. In order to keep the solution obtained with (4) as close as possible to the one provide by Matlab’s solver ode45, in the following computations the value \(\Delta t = 0.001\) was used.

![Figure 2](image2)

**Figure 2.** The same as in figure 1 but for \(\Delta t = 0.0028\).

As we mentioned in the introductory section, the probability of capsizing for given wave parameters is higher in the transient regime. If capsize does not occur within 8 – 10 cycles of forcing than it is unlikely to appear later. The cumulative conditions \(\theta > 1.5\, \text{rad} \) and \(t < 10 \cdot 2\pi/\omega\) were the basis of the representations in figure 3, where the dark (green) colour corresponds to the pairs \((\omega, \mu)\) for which
the solutions of equation (2) are bounded and the light one (yellow) to the pairs in which the ship is in a capsized state. The four panels, which describe the behaviour of the ship without bilge keels, indicate that, as the degree of stochasticity increases, lower and lower amplitudes $\mu$ are sufficient for the ship to capsize (for a given $\sigma$). A similar situation is encountered for the ship with bilge keels in operation.

![Figure 3](image)

**Figure 3.** Separation of the plane $(\omega, \mu)$ in regions with non-capsizing effect (dark area) or capsizing effect (light area) for the ship not equipped with bilge keels: 
(a) $\sigma = 0$; (b) $\sigma = 0.2$; (c) $\sigma = 0.4$; (d) $\sigma = 0.4$.

The initial conditions were $\theta(0) = \dot{\theta}(0) = 0$.

Safe basin concept has been introduced in 1990s for the study of nonlinear ship rolling and capsize. The safe basin denotes the set of initial conditions in the phase plane $(\theta, \dot{\theta})$ which define the separatrix between capsize and non-capsize areas and illustrate the high sensitivity of capsize to initial conditions. In fact, for a given set of parameters $(\omega, \mu)$, the safe basin of attraction is formed by all initial conditions $(\theta(0), \dot{\theta}(0))$ that do not lead to capsise. The size, shape and locations of safe basins provide valuable information about the engineering integrity of the ship. The numerical simulation showed that an increasing of forcing amplitude $\mu$ leads to a process of fractal erosion of the safe basin, finished with a sharp decrease of safe area. We will prove this in the following by examples for three $\omega$ values.

The initial conditions were selected from a vast set having $40,401 = 201 \times 201$ elements, obtained by dividing the rectangle $[-1.2, 1.2] \times [-1.0, 1.0]$ in equally spaced segments. The running time was chosen equal to $10T$, with $T = 2\pi/\omega$. Each point is tested against the escape criterion $\theta > 1.5 \text{ rad}$, and a colored small rectangle around the initial condition is used to indicate the time after which this criterion was satisfied. It turns out that the white region will correspond to the safe basin.

**Case 1:** $\omega = 0.9 \text{ rad/s} \quad (10T = 69.77 \text{ s})$
Figure 3 indicates that this is a relatively safe frequency, because the ship capsizing occurs for large forcing $\mu$. Keeping fixed initial conditions, one observes a rapid decrease in the amplitudes $\mu$ required for ship to capsize with the wave randomness increase, as illustrated in figure 4.

![Figure 4](image)

**Figure 4.** Separation of the plane $(\sigma, \mu)$ in regions with non-capsizing effect (dark area) or capsizing effect (light area) for the ship (a) without bilge keels; (b) with bilge keels.

The central frequency was $\omega = 0.9$ rad/s and $\theta(0) = \dot{\theta}(0) = 0$.

a) **Ship without bilge keels**

For a perfect sinusoidal excitation ($\sigma = 0$) and $\mu = 0$, the safe basin contains 66% from the tested initial conditions (see figure 5(a)). By increasing $\mu$ the basin erodes slightly from outside but the important loss occurs from the inside. Here, the start is given by a few thin “fingers” that begin to invade into the safe basin (see figure 5(b)). If $\mu$ continues to grow, these objects thicken and finally cover the entire safe area (figures 5(c) and (d)).

![Figure 5](image)

**Figure 5.** Safe basin’s fractal erosion for the ship without bilge keels if $\omega = 0.9$ and $\sigma = 0$.

(a) $\mu = 0$; (b) $\mu = 0.2$; (c) $\mu = 0.32$; (d) $\mu = 0.62$.

The colour bar shows the time required for the criterion $\theta < 1.5$ rad to be violated.
For $\sigma \neq 0$, this internal erosion occurs and develops for lower amplitudes $\mu$. The appearance of invasive formations in the safe basin may differ with the increase of $\sigma$, as shown in figure 6.

Figure 6. Safe basin’s fractal erosion for the ship without bilge keels if $\omega = 0.9$ and $\sigma = 0.4$. (a) $\mu = 0.2$; (b) $\mu = 0.22$.

b) Ship with bilge keels

The damping supplement introduced by bile keels has two immediate effects: the widening of the safe basin (which reaches 88% of the tested initial conditions for $\sigma = \mu = 0$) and doubling or even tripling the value of the amplitude $\mu$ required for overturning (depending on the initial conditions). Thus, figure 7 shows a basin in the shape of a curvilinear parallelogram that erodes only from outside, through stripes of initial conditions that pass into the capsizing area with the increase of $\mu$.

Figure 7. Safe basin’s fractal erosion for the ship with bilge keels if $\omega = 0.9$ and $\sigma = 0.0$. (a) $\mu = 0.0$; (b) $\mu = 1.1$; (c) $\mu = 1.2$; (d) $\mu = 1.6$. 
The increase of the degree of stochasticity of the waves causes the disappearance of the safe basin to occur at much lower values of amplitude $\mu$ (see figure 4(b)). The erosion mechanism may follow other pathways, as shown in figure 8. It is also worth noting that very small increases in amplitude $\mu$ have a tremendous effect on the safe basin’s size.

**Figure 8.** Safe basin’s fractal erosion for the ship with bilge keels if $\omega = 0.9$ and $\sigma = 0.6$. $(a) \mu = 0.3135$; $(b) \mu = 0.315$.

**Case 2:** $\omega = 0.6 \text{ rad/s (}107 = 104.7 \text{ s)}$

Figure 3 tells us that this is one of the most dangerous central wave frequencies for the ship, capsizing occurring for a minimal forcing amplitude $\mu$. On the other hand, the degree of stochasticity influences only to a small extent the transition to boundless solutions of the roll equation, as shown in figures 3 and 9.

**Figure 9.** Separation of the plane $(\sigma, \mu)$ in regions with non-capsizing effect (dark area) or capsizing effect (light area) for the ship $(a)$ without bilge keels; $(b)$ with bilge keels.

The central frequency was $\omega = 0.6 \text{ rad/s and } \dot{\theta}(0) = \theta(0) = 0$.

Which is really remarkable for $\sigma \neq 0$, no matter if the ship is equipped with bilge keels or not, is the unfailing and then the collapse of the safe basin when exceeding a threshold of the amplitude $\mu$. Figures 10 and 11 represent two examples in this sense, the erosion mechanism undergoing only small changes compared to the case $\omega = 0.9$.
Figure 10. Safe basin’s fractal erosion for the ship without bilge keels if $\omega = 0.6$ and $\sigma = 0.4$. 
(a) $\mu = 0.088$; (b) $\mu = 0.0885$.

Figure 11. Safe basin’s fractal erosion for the ship with bilge keels if $\omega = 0.6$ and $\sigma = 0.4$. 
(a) $\mu = 0.272$; (b) $\mu = 0.273$.

The integrity curves show the relative influence of wave excitation’s amplitude $\mu$ on capsize relative to vessel safety in the absence of incident waves. They could be generated by plotting the safe area, normalized to unity at $\mu = 0$. In order to have a common reporting base, this normalization was done relative to the case of the ship equipped with bilge keels. This situation corresponds to a maximum safe basin. Figures 12 and 13 present the integrity curves associated to the ship without/with bilge keels for $\omega \in \{0.3, 0.6, 0.9\}$ and $\sigma \in \{0.0, 0.2, 0.4, 0.6\}$. They were obtained by interpolating 10 to 20 safe basins’ sizes for different forcing amplitudes $\mu$.

Analysing the two figures we find that, as $\sigma$ grows, the values of amplitude $\mu$ for which the safe basin is completely empty are closer and closer for the different values of the frequency $\omega$. The sea stochasticity has a levelling effect on the influence of parameter $\omega$. At the same time, the additional damping due to the bilge keels considerably increases the area below the normalized integrity curves. As an example, for $\omega = 0.6$ and $\sigma = 0.4$, the ratio of these areas is four in favour of the ship equipped with bilge keels.
Figure 12. Normalized integrity curves for the ship without bilge keels.

Figure 13. Normalized integrity curves for the ship with bilge keels.
In order to highlight even more clearly the influence of sea stochasticity on the size of the safe basin as well as its sudden erosion when reaching a limit value of amplitude $\mu$, figure 14 shows the normalized integrity curves for $\omega = 0.6$ and $\sigma \in \{0.0, 0.2, 0.4, 0.6\}$. Once the profile of the waves’ disturbing moment has moved away from the purely sinusoidal one (regular sea), the integrity curves almost overlap and the surface below them is only slight different from that of a rectangle.

![Figure 14. Influence of sea stochasticity on the safe basin for a fixed central frequency $\omega$.](image)

### 5. Conclusions

In the present study, a simple, fat and flexible iterative scheme based on Taylor expansion was applied to a nonlinear equation for estimating the roll motion of a ship in a stochastic beam sea. The speediness of the scheme was a determining factor in revealing several mechanisms responsible for eroding the safe basin of the ship and in construction of the so-called normalized integrity curves.

The main conclusions of the study are as follows:

a) As the amplitude of the sea excitation increases, the safe basins erode both from the outside and from the inside. How this erosion occurs depends on the excitation central frequency and on the damping moment;

b) For a regular beam sea, there are frequencies for which the ship is more prone for capsizing. In a stochastic beam sea, as the degree of randomness increases, a leveling effect of the excitation frequency’s influence on the ship’s response occurs;

c) In a stochastic sea there is, for each central frequency, a threshold for the excitation amplitude at which the safe basin suddenly erodes. These limits tend to approach with the removal of the purely sinusoidal perturbation associated with the regular sea;

d) The usefulness of the bilge keels in preventing the ship capsizing was well evidenced by the obtained numerical results.

### 6. References

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