Towards the Generalized Riemann Hypothesis using only zeros of the Riemann zeta function

William Banks

Abstract

For any real $\beta_0 \in \left[\frac{1}{2}, 1\right)$, let $\text{GRH}[\beta_0]$ be the assertion that for every Dirichlet character $\chi$ and all zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$, one has $\beta \leq \beta_0$ (in particular, $\text{GRH}[\frac{1}{2}]$ is the Generalized Riemann Hypothesis). In this paper, we show that the validity of $\text{GRH}[\frac{9}{10}]$ depends only on certain distributional properties of the zeros of the Riemann zeta function $\zeta(s)$.

Keywords: Riemann zeta function, Riemann hypothesis, Stationary phase, Zeros

Mathematics Subject Classification: Primary: 11M06, 11M26; Secondary: 11M20

Contents

1 Introduction and statement of results ................................................................. 1
2 Approach .............................................................................................................. 3
3 Integral bounds ..................................................................................................... 3
4 Twisting the von Mangoldt function .................................................................. 8
5 Proofs of Theorems 1.1 and 1.2 ........................................................................ 9
References ............................................................................................................. 12

1 Introduction and statement of results

The Riemann zeta function is a central object of study in analytic number theory. In terms of the complex parameter $s = \sigma + it$, the zeta function is defined in the half-plane $\sigma > 1$ by two equivalent expressions:

$$\zeta(s) := \sum_{n \geq 1} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$ 

Riemann [6] showed that $\zeta(s)$ extends analytically to a meromorphic function in the whole complex plane, its only singularity being a simple pole at $s = 1$. Moreover, the zeta function satisfies a functional equation relating its values at $s$ and $1 - s$.

The Riemann Hypothesis (RH) asserts that if $\rho = \beta + i\gamma$ is a zero of $\zeta(s)$ with real part $\beta > 0$, then $\beta = \frac{1}{2}$.

1There are many excellent accounts of the theory of the Riemann zeta function; we refer the reader to Titchmarsh [7] and to Borwein et al [1] for essential background.
More generally, for a Dirichlet character $\chi$ mod $q$, the Dirichlet $L$-function $L(s, \chi)$ is defined for $\sigma > 1$ by:

$$L(s, \chi) := \sum_{n \geq 1} \chi(n)n^{-s} = \prod_{p \text{ prime}} \left(1 - \chi(p)p^{-s}\right)^{-1}.$$ 

The function $L(s, \chi)$ extends to a meromorphic function (which is entire if $\chi$ is nonprincipal), and when $\chi$ is primitive it satisfies a simple functional equation relating its values at $s$ and $1 - s$; see, e.g., Bump [2, Chapter 1]. The Generalized Riemann Hypothesis (GRH), which was perhaps first formulated by Piltz in 1884 (see Davenport [3]), asserts that if $\rho = \beta + i\gamma$ is a zero of $L(s, \chi)$ with $\beta > 0$, then $\beta = \frac{1}{2}$.

For any principal character $\chi_0$ mod $q$ one has

$$L(s, \chi_0) = \zeta(s) \prod_{p \mid q} (1 - p^{-s}),$$

hence RH is equivalent to GRH for $L(s, \chi_0)$. On the other hand, for nonprincipal characters $\chi$, no direct relationship between RH and GRH for $L(s, \chi)$ has been previously established. To establish such a connection, we study the following weak form of the GRH for Dirichlet $L$-functions.

Hypothesis GRH$[\beta_0]$: Given $\beta_0 \in \left[\frac{1}{2}, 1\right)$, the inequality $\beta \leq \beta_0$ holds for all zeros $\rho = \beta + i\gamma$ of an arbitrary Dirichlet $L$-function $L(s, \chi)$.

Note that GRH$[\frac{1}{2}]$ is equivalent to the assertion that GRH holds for all Dirichlet $L$-functions. In the present paper, we show that hypothesis GRH$[\frac{9}{10}]$ can be reformulated entirely in terms of certain distributional properties of the zeros of the Riemann zeta function.

To state our results, we introduce some notation. If $T > 0$ is not the ordinate of a zero of the zeta function, then $N(T)$ is used to denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $0 < \beta < 1, 0 < \gamma < T$, and we define

$$S(T) := \frac{1}{\pi} \arg \zeta(1/2 + iT).$$

If $\gamma > 0$ is the ordinate of a zero, then we set

$$N(\gamma) := \frac{1}{2} (N(\gamma^+) + N(\gamma^-)), \quad S(\gamma) := \frac{1}{2} (S(\gamma^+) + S(\gamma^-)).$$

Using an explicit form of the well known relation (see, e.g., Montgomery and Vaughan [5, Corollary 14.2])

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}) \quad (T > 0), \quad (1.1)$$

one sees that the difference $S(\gamma^+) - S(\gamma^-)$ is an integer for every zero $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\gamma > 0$. Extending the definition of $S(T)$ appropriately, this also holds for complex zeros with $\gamma < 0$. Therefore, writing

$$e(\alpha) := e^{2\pi i\alpha} \quad (\alpha \in \mathbb{R}),$$

we can unambiguously define

$$Z(\rho) := \lim_{T \to \gamma} \overline{e(S(T))} \quad (1.2)$$

(the common value of the left and right limits) for every complex zero $\rho$ of $\zeta(s)$.

Throughout the paper, $C^\infty_c(\mathbb{R}^+)$ is the space of smooth functions $f : \mathbb{R}^+ \to \mathbb{C}$ with compact support in $\mathbb{R}^+$. Let $\mu$ and $\phi$ denote the Möbius and Euler functions, respectively.
Theorem 1.1 Assume RH. Suppose that for every function \( B \in C^\infty_c(\mathbb{R}^+) \) and every rational number \( \xi := m/q \) with \( 0 < m < q \) and \( (m, q) = 1 \), the bound
\[
\sum_{\rho = \frac{1}{2} + it} \xi^{-\frac{1}{2}} \frac{Z(\rho)B(\gamma)}{2\pi X} + \frac{\mu(q)}{\phi(q)} \sum_{n \geq 1} \Lambda(n)B(n/X) \ll X^{9/10+\varepsilon}
\]
holds for any \( \varepsilon > 0 \), where the sum on the left side runs over all complex zeros \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \), the quantity \( Z(\rho) \) is given by (1.2), and the implied constant depends only on \( \xi, B, \) and \( \varepsilon \). Then, the hypothesis GRH\([9, 10]\) is true.

We reiterate that the hypotheses of Theorem 1.1 involve only properties of the zeros of the Riemann zeta function. No conditions are imposed on the zeros of nonprincipal Dirichlet L-functions.

We also have the following converse of Theorem 1.1.

Theorem 1.2 Assume RH. If GRH\([9, 10]\) is true, then for every \( B \in C^\infty_c(\mathbb{R}^+) \) and every rational number \( \xi := m/q \) with \( 0 < m < q \) and \( (m, q) = 1 \), the bound (1.3) holds for any \( \varepsilon > 0 \).

Throughout the paper, implied constants in the symbols \( \ll, O, \) etc., often depend on various parameters (e.g., \( \xi, B, \varepsilon \)) as indicated by the notation (e.g., see (1.3) above); these constants are independent of all other parameters.

2 Approach

Using the explicit formula and assuming RH, we show that a sum of the form
\[
\sum_{n \geq 1} \Lambda(n)e(-n\xi)B(n/X)
\]
with \( \xi \in \mathbb{R}^+ \) and \( B \in C^\infty_c(\mathbb{R}^+) \) is equal to
\[
-\sum_{\rho = \frac{1}{2} + it} \xi^{-1/2} e^{-it \xi} Z(\rho)B(\frac{\gamma}{2\pi \xi X})
\]
up to an error of at most \( O_{\xi, B}(X^{9/10}) \); see Theorem 4.1. To handle the integrals that arise in our use of the explicit formula, we apply the method of the stationary phase (however, some care is needed to obtain adequate and explicit estimates for the error terms); see Lemma 3.4, which is the main workhorse for the proof of Theorem 4.1.

Given the close relationship (under RH) between the sums (2.1) and (2.2), the estimate (1.3) for rational numbers \( \xi := m/q \) allows us to deduce an equally strong bound
\[
\sum_{n \geq 1} \Lambda(n)\chi(n)B(n/X) \ll X^{9/10+\varepsilon}
\]
for any primitive Dirichlet character \( \chi \) modulo \( q > 1 \); see the proof of Theorem 1.1 in Sect. 5. Since \( B \in C^\infty_c(\mathbb{R}^+) \) is arbitrary, GRH\([9, 10]\) holds for the function \( L(s, \chi) \). Conversely, under GRH\([9, 10]\), it follows that (2.3) holds for every primitive character \( \chi \) mod \( q \). This leads to similar estimates for the sums (2.1) and (2.2); see the proof of Theorem 1.2 in Sect. 5.

3 Integral bounds

Our aim in this section is to establish certain integral bounds that are needed in the proof of Theorem 4.1 (see Sect. 4).
For each $B \in C_c^\infty(\mathbb{R}^+)$, let $X_B \geq 10$ be a number large enough so that

$$1 \notin X \supp(B) := \{ Xu : u \in \supp(B) \} \quad (X > X_B).$$

The value of $X_B$ is held fixed throughout the paper.

**Lemma 3.1** Let $\xi \in \mathbb{R}^+$, $B \in C_c^\infty(\mathbb{R}^+)$, and $X > X_B$. Then

$$\int_{\mathbb{R}^+} \left(1 - \frac{1}{u^3} - u\right)e(-u\xi) B(u/X) \, du \ll X^{-1}.$$

**Proof** Put

$$g(u) := \left(1 - \frac{1}{u^3} - u\right) B(u/X) \quad (u \in \mathbb{R}^+).$$

Then $g$ and all of its derivatives are smooth and supported on the set $X \supp(B)$ (note that $B(u/X) = 0$ for $u \in \{0, \pm 1\}$ since $B \in C_c^\infty(\mathbb{R}^+)$ and $X > X_B$). In particular, $g''(u) = 0$ unless $u \sim X$, in which case one has $g''(u) \ll X^{-2}$. Integrating by parts twice, we have (since $g$ and $g'$ vanish at 0 and $\infty$)

$$\int_{\mathbb{R}^+} e(-u\xi) g(u) \, du = \int_{\mathbb{R}^+} e(-u\xi) g''(u) \, du \ll e^{-\xi^2} X^{-2} \meas(X \supp(B)),$$

and the lemma follows. \qed

The next two technical lemmas are needed in the proof of Lemma 3.4 below.

**Lemma 3.2** Fix $u_0 \in \mathbb{R}^+$, and let $L, D : C_c^\infty(\mathbb{R}^+) \to C_c^\infty(\mathbb{R}^+)$ be the linear operators defined by

$$LF(u) := \frac{uF(u)}{(u_0 - u)} \quad \text{and} \quad DF(u) := F'(u) \quad (u \in \mathbb{R}^+).$$

For any integer $k \geq 0$, let $(DL)^k$ and $L(DL)^k$ be the linear operators given by

$$[DL]^k := D \circ L \circ \cdots \circ D \circ L \quad \text{and} \quad L[DL]^k := L \circ [DL]^k.$$

Then for every $F \in C_c^\infty(\mathbb{R}^+)$ and every integer $k \geq 0$ we have

$$[DL]^k F(u) \ll \max \left\{ \left| u - u_0 \right|^{-k}, \left| u - u_0 \right|^{-2k} \right\},$$

$$L[DL]^k F(u) \ll \max \left\{ \left| u - u_0 \right|^{-k}, \left| u - u_0 \right|^{-2k+1} \right\},$$

for all $u \in \mathbb{R}^+$, $u \neq u_0$.

**Proof** Let $F \in C_c^\infty(\mathbb{R}^+)$ be fixed in what follows. For any integers $A, B \geq 0$ and a real number $C > 0$, let $V(A, B, C)$ denote the set of functions $G \in C_c^\infty(\mathbb{R}^+)$ of the form

$$G(u) = \sum_{0 \leq i \leq A} \sum_{0 \leq j \leq B} \sum_{h, i, j \geq 0} c_{h, i, j} u^h F^{(i)}(u) u_0^{-i},$$

where the sum runs over nonnegative integers $h, i, j$, and the coefficients $c_{h, i, j}$ are complex numbers satisfying $|c_{h, i, j}| \leq C$. If $F$ is supported on the interval $[a, b]$, where $0 < a < b < \infty$, then the trivial bound

$$|G(u)| \leq C_{A, B} C \max \{b^A, 1\} \max_{0 \leq i \leq A+B} \left| F^{(i)}(u) \right| \max_{A \leq j \leq A+B} \left| u - u_0 \right|^{-j}$$

(3.1)
holds for all $G \in \mathcal{V}(A, B, C)$, where
\[
C_{A,B}^j := |\{(h, i, j) : h, i, j \geq 0, h \leq A \leq j, i + j = h + B\}| \leq A \sum_{h=0}^{A+B} j = 1 \ll 1.
\]

Next, noting that
\[
LG(u) = \sum_{h, i, j \geq 0 \atop h \leq A \leq j \atop i+j=h+B} c_{h,i,j} u^{h+1} F^{(i)}(u) \frac{u^{h+i} F^{(i)}(u)}{(u_0 - u)^{h+i+1}} = \sum_{h, i, j \geq 0 \atop h \leq A \leq j \atop i+j=h+B} c_{h-1,i,j-1} u^{h} F^{(i)}(u) \frac{u^{h+i} F^{(i)}(u)}{(u_0 - u)^{h+i+1}},
\]
it follows that
\[
L : \mathcal{V}(A, B, C) \to \mathcal{V}(A + 1, B, C).
\] (3.2)

Similarly, we write $DG(u) = G_1(u) + G_2(u) + G_3(u)$, where the functions $G_j$ are defined as follows. First, we have
\[
G_1(u) := \sum_{h, i, j \geq 0 \atop h \leq A \leq j \atop i+j=h+B} c_{h,i,j} u^{h+1} F^{(i)}(u) \frac{u^{h+i} F^{(i)}(u)}{(u_0 - u)^{h+i+1}} = \sum_{h, i, j \geq 0 \atop h \leq A \leq j \atop i+j=h+B} (h+1) c_{h+1,i,j} u^{h} F^{(i)}(u) \frac{u^{h+i} F^{(i)}(u)}{(u_0 - u)^{h+i+1}}.
\]

Since the coefficients $(h+1)c_{h+1,i,j}$ do not exceed $AC$ in absolute value, it is clear that $G_1 \in \mathcal{V}(A, B + 1, AC)$. Next, we have
\[
G_2(u) := \sum_{h, i, j \geq 0 \atop h \leq A \leq j \atop i+j=h+B} c_{h,i,j} u^{h} F^{(i+1)}(u) \frac{u^{h+i} F^{(i)}(u)}{(u_0 - u)^{h+i+1}} = \sum_{h, i, j \geq 0 \atop h \leq A \leq j \atop i+j=h+B} c_{h,i,j+1} u^{h} F^{(i)}(u) \frac{u^{h+i} F^{(i)}(u)}{(u_0 - u)^{h+i+1}}
\]
and clearly $G_2 \in \mathcal{V}(A, B + 1, C)$. Finally, we have
\[
G_3(u) := \sum_{h, i, j \geq 0 \atop h \leq A \leq j \atop i+j=h+B} c_{h,i,j} u^{j} F^{(i)}(u) \frac{u^{h+i} F^{(i)}(u)}{(u_0 - u)^{h+i+1}} = \sum_{h, i, j \geq 0 \atop h \leq A \leq j \atop i+j=h+B} (j-1) c_{h,i,j} u^{h} F^{(i)}(u) \frac{u^{h+i} F^{(i)}(u)}{(u_0 - u)^{h+i+1}}.
\]

As the coefficients $(j-1)c_{h,i,j}$ do not exceed $(A + B)C$ in absolute value, we have $G_3 \in \mathcal{V}(A, B + 1, (A + B)C)$. Consequently, $DG \in \mathcal{V}(A, B + 1, (2A + B + 1)C)$, and therefore
\[
D : \mathcal{V}(A, B, C) \to \mathcal{V}(A, B + 1, (2A + B + 1)C).
\] (3.3)

Observe that $F$ itself lies in $\mathcal{V}(0, 0, 1)$. Therefore, an inductive argument using (3.2) and (3.3) shows that for every integer $k \geq 0$ we have
\[
[D^k L]^k F \in \mathcal{V}(k, k, k \cdot 3^k) \quad \text{and} \quad L[D^k L]^k F \in \mathcal{V}(k + 1, k, k \cdot 3^k).
\]

Using (3.1), the result follows. $\square$

**Lemma 3.3** For any positive number $\lambda$, we have
\[
\int_{\mathbb{R}} e^{-iu^2} du = \frac{e^{-i\lambda^2}}{i\lambda} + i \int_{\mathbb{R}} e^{-iu^2} u^{-2} du \ll \lambda^{-1}.
\]

The next result is our primary tool; its proof is based on the well known stationary phase method.
Lemma 3.4 Let \( \xi \in \mathbb{R}^+ \), \( B \in C_c^\infty(\mathbb{R}^+) \), and \( X > X_B \). Suppose \( B \) is supported on the interval \([a, b] \), where \( 0 < a < b < \infty \). For any real number \( \gamma \neq 0 \), consider the integral \( I(\gamma) \) defined by

\[
I(\gamma) := \int_{\mathbb{R}^+} e^{(-u\xi)B(u/X)u^{-1/2+iy}} \, du.
\]

Put

\[
a_* := \frac{1 + 2a - \sqrt{1 + 4a}}{2}, \quad b_* := \frac{1 + 2b + \sqrt{1 + 4b}}{2}, \quad \gamma_* := \frac{\gamma}{2\pi \xi X}.
\]

(3.4)

Then, for \( \gamma_* \notin [a_*, b_*] \) the bound

\[
I(\gamma) \ll X^{1/2} \max \{ X^{-2} |\gamma|^{-2}, |\gamma|^{-4} \}
\]

holds, whereas for \( \gamma_* \in [a_*, b_*] \) we have the estimate

\[
I(\gamma) = \xi^{-1/2-iy} e^{\left( \frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e} + \frac{7}{8} \right) B(\gamma_*)} + O_{\xi,B}(X^{-1/10}).
\]

Proof Making the change of variables \( u \mapsto Xu \), we have

\[
I(\gamma) = X^{1/2+iy} \int_{\mathbb{R}^+} e^{iXf(u)} g(u) \, du = X^{1/2+iy} J \quad \text{(say)},
\]

where

\[
f(u) := -2\pi \xi u + \gamma \log u, \quad g(u) := \frac{B(u)}{u^{1/2}}.
\]

Note that \( \gamma_* = \gamma/(2\pi \xi X) \) is the only real number for which \( f'(\gamma_*) = 0 \).

For any given \( \Delta > 0 \), we write \( J = J_\infty + J_* \) with

\[
J_\infty := \int_{\mathbb{R}^+} e^{iXf(u)} g(u) \, du, \quad J_* := \int_{\mathbb{R}^+} e^{iXf(u)} g(u) \, du.
\]

We study \( J_\infty \) first. Let \( \mathcal{L} \) and \( \mathcal{D} \) be the operators defined in Lemma 3.2 with \( u_0 := \gamma_* \), and put

\[
g_k := [D\mathcal{L}]^k g \quad \text{and} \quad \tilde{g}_k := \mathcal{L}[D\mathcal{L}]^k g \quad (k \geq 0).
\]

According to Lemma 3.2,

\[
g_k(u) \ll_{B,k} \max \{ |u - \gamma_*|^{-k}, |u - \gamma_*|^{-2k} \} \quad (u \neq \gamma_*).
\]

(3.6)

Taking into account that \( f'(u) = 2\pi \xi u^{-1}(\gamma_* - u) \), we have

\[
J_\infty = \frac{1}{2\pi \xi} \int_{\{ |u| > \gamma_* \} \setminus \{ |u - \gamma_*| > \Delta \}} e^{iXf(u)} g_0(u) \, du = -\frac{1}{2\pi i \xi X} \int_{\{ |u| > \gamma_* \} \setminus \{ |u - \gamma_*| > \Delta \}} e^{iXf(u)} g_1(u) \, du,
\]

where we have used integration by parts in the last step. Similarly, by induction on \( k \), we see that

\[
J_\infty = \frac{1}{(-2\pi i \xi X)^k} \int_{\{ |u| > \gamma_* \} \setminus \{ |u - \gamma_*| > \Delta \}} e^{iXf(u)} g_k(u) \, du.
\]

(3.7)

The numbers \( a_* \) and \( b_* \) defined in (3.4) have the property that if \( \gamma_* \notin [a_*, b_*] \), then \( |u - \gamma_*| > |\gamma_*|^{1/2} \) for all \( u \in [a, b] \). Hence, choosing \( \Delta := |\gamma_*|^{1/2} \) in the case that \( \gamma_* \notin [a_*, b_*] \),
[a_*, b_*], it follows that J_\infty = J and J_* = 0. Setting k := 4 and combining (3.6) and (3.7), we derive the bound
\[ J \ll (\xi X)^{-4} \max_{\mathbb{B}} \left\{ |\gamma_*|^{-2}, |\gamma_*|^{-4} \right\} = \max_{\mathbb{B}} \left\{ (\xi X)^{-2}|\gamma|^{-2}, |\gamma|^{-4} \right\} \]

In view of (3.5), we obtain the first statement of the lemma.

From now on, we assume that \( \gamma_* \in [a_*, b_*] \). We further assume that \( X \) is large enough (depending on \( \mathbb{B} \)) so that
\[ \Delta := X^{-2/5} < \min\{1, 2^{-1/2}a_*\}. \] (3.8)
Write \( J = J_\infty + J_* \) as before with this \( \Delta \). Setting \( k := 5 \) (say) and combining (3.6) and (3.7), we derive the bound
\[ J_\infty \ll \frac{X}{\xi \mathbb{B}}. \] (3.9)

Turning to the estimate of \( J_* \), observe that the upper bound (3.8) implies that the interval \( \Omega_\Delta := [\gamma_* - \Delta, \gamma_* + \Delta] \) lies entirely inside \( \mathbb{R}^+ \); in particular,
\[ J_* := \int_{\Omega_\Delta} e^{iXf(\gamma)} g(\gamma) \, du. \]

Since \( \gamma_* \in [a_*, b_*] \) (and thus, \( \gamma_* \gg_{\mathbb{B}} 1 \)), the estimate
\[ f''(\gamma) = \frac{4\pi i\xi \gamma_*}{u^3} \ll_{\mathbb{B}} \xi \]
holds uniformly for all \( \gamma \in \Omega_\Delta \), hence by Taylor’s approximation we have
\[ e^{iXf(\gamma)} = e^{iXf(\gamma_*) + \int_{\gamma_*}^{\gamma} f''(\gamma)(u-\gamma)^2 \, du} (1 + r_1(\gamma)) \]
for some complex function \( r_1 \) such that
\[ r_1(\gamma) \ll_{\mathbb{B}} X|u - \gamma_*|^3 \quad (u \in \Omega_\Delta). \]

We can also write
\[ g(\gamma) = g(\gamma_*) + r_2(\gamma), \]
where \( r_2 \) satisfies the bound
\[ r_2(\gamma) \ll_{\mathbb{B}} |u - \gamma_*| \quad (u \in \Omega_\Delta). \]

Therefore, since
\[ \int_{\Omega_\Delta} |r_1(\gamma)| \, du \ll_{\mathbb{B}} X \int_{\gamma_* - \Delta}^{\gamma_* + \Delta} |u - \gamma_*|^3 \, du \ll \Delta^4, \]
\[ \int_{\Omega_\Delta} |r_2(\gamma)| \, du \ll_{\mathbb{B}} \int_{\gamma_* - \Delta}^{\gamma_* + \Delta} |u - \gamma_*| \, du \ll \Delta^2, \]
and \( \Delta^2 \ll \Delta^4 = X^{-3/5} \) by (3.8), we derive the estimate
\[ J_* = \int_{\Omega_\Delta} e^{iXf(\gamma_*) + \int_{\gamma_*}^{\gamma} f''(\gamma)(u-\gamma)^2 \, du} (1 + r_1(\gamma_*)) (g(\gamma_*) + r_2(\gamma)) \, du 
= e^{-i\gamma_*^{-1/2}B(\gamma_*)} \int_{\Omega_\Delta} e^{-\pi \xi X\gamma_*^{-1}(u-\gamma_*)^2} \, du + O_{\mathbb{B}}(X^{-3/5}), \]
where in the last step we used the identities
\[ Xf(\gamma) = -\gamma + \gamma \log \gamma, \quad f''(\gamma) = -2\pi \xi \gamma_*^{-1}, \quad g(\gamma) := \gamma_*^{-1/2}B(\gamma_*). \]
Next, we extend the range of integration in the preceding integral to all of $\mathbb{R}$ (with an acceptable error). Consider the integral
\[
\mathcal{K} := \int_{\mathbb{R}} e^{-\pi i \xi \psi^{-1}(u-\gamma)^2} \, du.
\]

Making the change of variables $u \mapsto c \, u + \gamma$, where $c := \sqrt{\psi}/(\pi \xi X)$, and applying Lemma 3.3, we have
\[
\mathcal{K} = c \int_{\mathbb{R}} e^{-iu^2} \, du \ll c^2 \Delta^{-1} \ll \frac{\xi}{\mathcal{B}} X^{-3/5}.
\]

Using this bound together with (3.10), it follows that
\[
\mathcal{J} = e^{-i\gamma \psi^{-1/2} + i\gamma} \mathcal{B}(\gamma) \int_{\mathbb{R}} e^{-\pi i \xi \psi^{-1}(u-\gamma)^2} \, du + O_{\xi, \mathcal{B}}(X^{-3/5}).
\]

Combining the previous bound with (3.9), we have
\[
\mathcal{J} = e^{-i\gamma \psi^{-1/2} + i\gamma} \mathcal{B}(\gamma) \int_{\mathbb{R}} e^{-\pi i \xi \psi^{-1}(u-\gamma)^2} \, du + O_{\xi, \mathcal{B}}(X^{-3/5}).
\]

The integral here can be explicitly evaluated:
\[
\int_{\mathbb{R}} e^{-\pi i \xi \psi^{-1}(u-\gamma)^2} \, du = e^{-i\gamma/4} \sqrt{\frac{\xi}{\psi}}.
\]

Inserting this result into (3.11) and recalling (3.5), after some simplification we find that
\[
\mathcal{I}(\gamma) = \xi^{-1/2} e^{\left(\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e} - \frac{1}{8}\right)} \mathcal{B}(\gamma) + O_{\xi, \mathcal{B}}(X^{-1/10}),
\]
and the proof is complete.

\[\square\]

4 Twisting the von Mangoldt function

**Theorem 4.1** Assume RH. Let $\xi \in \mathbb{R}^+$, $\mathcal{B} \in C^\infty_c(\mathbb{R}^+)$, and $X > X_\mathcal{B}$. Then
\[
\sum_{n \geq 1} \Lambda(n)e(-n\xi)\mathcal{B}\left(\frac{n}{X}\right) = -\sum_{\rho = \frac{1}{2} + i\gamma} \xi^{-1/2 - i\gamma} \zeta(\rho) \mathcal{B}\left(\frac{\gamma}{2\pi \xi X}\right) + O_{\xi, \mathcal{B}}(X^{9/10}).
\]

**Proof** Our goal is to estimate
\[
\sum_{n \geq 1} \Lambda(n)e(-n\xi)\mathcal{B}\left(\frac{n}{X}\right) = \sum_{n \geq 1} \Lambda(n)\varphi(n),
\]

where $\varphi(u) := e(-u\xi)\mathcal{B}(u/X)$. By the explicit formula (see, e.g., Iwaniec and Kowalski [4, Exercise 5, p. 109]) we have
\[
\sum_{n \geq 1} \Lambda(n)\varphi(n) = \int_{\mathbb{R}^+} \left(1 - \frac{1}{(u - 1)u(u + 1)}\right)\varphi(u) \, du - \sum_{\rho} \hat{\varphi}(\rho),
\]

where $\hat{\varphi}$ is the Mellin transform of $\varphi$ given by
\[
\hat{\varphi}(s) := \int_{\mathbb{R}^+} \varphi(u)u^{s-1} \, du.
\]

Using Lemma 3.1 to bound the integral in (4.1), we get that
\[
\sum_{n \geq 1} \Lambda(n)\varphi(n) = -\sum_{\rho} \hat{\varphi}(\rho) + O_{\xi, \mathcal{B}}(X^{-1}).
\]

(4.2)
Next, for any complex zero \( \rho = \frac{1}{2} + i\gamma \) of \( \xi(s) \) we have
\[
\psi(\rho) := \int_{\mathbb{R}^+} e(-u\xi)B(u/X)u^{-1/2+i\gamma} \, du,
\]
which is the integral \( \mathcal{I}(\gamma) \) considered in Lemma 3.4. Defining \( a, a_*, a, b_*, \gamma_* \) as in Lemma 3.4, it follows that
\[
\sum_{\rho=\frac{1}{2}+i\gamma} \psi(\rho) = \Sigma_1 + O_{\mathcal{B}}(X^{9/10}), \tag{4.3}
\]
where
\[
\Sigma_1 := \sum_{\substack{\rho=\frac{1}{2}+i\gamma \\
\gamma_* \in [a_* b_*]}} \xi^{-1/2-i\gamma} e(\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e} + \frac{7}{8})B(\gamma_*).
\]
Note that the error term in (4.3) is a consequence of the following bounds on the sums that arise naturally in our application of Lemma 3.4:
\[
\sum_{\substack{\rho=\frac{1}{2}+i\gamma \\
\gamma_* \notin [a_* b_*]}} X^{1/2} \max \{|X^{-2}| |\gamma_*^{-2} | |\gamma_*^{-4} \} \ll X^{1/2}, \quad \sum_{\substack{\rho=\frac{1}{2}+i\gamma \\
\gamma_* \in [a_* b_*]}} X^{-1/10} \ll X^{9/10}.
\]
The condition \( \gamma_* \notin [a_* b_*] \) in the above definition of \( \Sigma_1 \) is redundant (indeed, we have \( a_* < a < b < b_* \) by (3.4), hence \( B(\gamma_*) = 0 \) if \( \gamma_* \notin [a_* b_*] \)), and so we can simply write
\[
\Sigma_1 = \sum_{\rho=\frac{1}{2}+i\gamma} \xi^{-1/2-i\gamma} e(\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e} + \frac{7}{8})B\left(\frac{\gamma}{2\pi X}\right).
\]
Next, observe that (1.1) implies
\[
e\left(\frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8}\right) = e(N(T) - S(T) + O(T^{-1})) = \overline{e(S(T))} + O(T^{-1})
\]
provided that \( T > 0 \) is not the ordinate of a zero of \( \xi(s) \) (since \( N(T) \in \mathbb{Z} \)). Taking the limit as \( T \to \gamma \), we get that
\[
e\left(\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e} + \frac{7}{8}\right) = \zeta(\rho) + O(\gamma^{-1}),
\]
where (as in Sect. 1)
\[
\zeta(\rho) := \lim_{T \to \gamma} \overline{e(S(T))}.
\]
Thus, up to an acceptable error, we can replace \( \Sigma_1 \) in (4.3) with the quantity
\[
\Sigma_1 := \sum_{\rho=\frac{1}{2}+i\gamma} \xi^{-1/2-i\gamma} \zeta(\rho)B\left(\frac{\gamma}{2\pi X}\right).
\]
The theorem now follows by combining (4.2) and (4.3). \( \square \)

5 Proofs of Theorems 1.1 and 1.2

We begin by considering the GRH for individual Dirichlet \( L \)-functions. For convenience, we formulate the following hypothesis.

Hypothesis GRH[\( \beta_0 \), \( \chi \)]: Given \( \beta_0 \in \left[\frac{1}{2}, 1\right) \) and a Dirichlet character \( \chi \), the inequality \( \beta \leq \beta_0 \) holds for all zeros \( \rho = \beta + i\gamma \) of the \( L \)-function \( L(s, \chi) \).

Note that GRH[\( \beta_0 \)] is true if and only if GRH[\( \beta_0 \chi \)] holds for all characters \( \chi \).

**Lemma 5.1** Fix \( \beta_0 \in \left[\frac{1}{2}, 1\right) \), and let \( \chi \) be a nonprincipal character of modulus \( q \). Then the following are equivalent:

- - -
The contribution from the principal character $\chi$ nonprincipal characters is at most $O_{q}$, $\chi \star$ character.

Proof Consider the intermediate bound

$$\sum_{n \leq X} \Lambda(n) \chi(n) \ll X^{\beta_{0} + \epsilon}.$$  \hspace{1cm} (5.1)

The equivalence $(i) \iff (5.1)$ is standard and well known, and the implication $(5.1) \implies (ii)$ is immediate using partial summation. To prove $(ii) \implies (5.1)$ one uses a fixed test function $\mathcal{B} \in C_{c}^{\infty}(\mathbb{R}^{+})$ such that $0 \leq \mathcal{B}(u) \leq 1$ for all $u \in \mathbb{R}$, and $\mathcal{B}(u) = 1$ for all $u \in [\frac{1}{2}, 1]$. By $(ii)$ we have

$$\sum_{X/2 < n \leq X} \Lambda(n) \chi(n) = \sum_{X/2 < n \leq X} \Lambda(n) \chi(n) \mathcal{B}(n/X) \leq \sum_{n \geq 1} \Lambda(n) \chi(n) \mathcal{B}(n/X) \ll X^{\beta_{0} + \epsilon},$$

and then $(5.1)$ follows by a standard dyadic argument.

Lemma 5.2 Fix $\beta_{0} \in [\frac{1}{2}, 1)$. Let $\xi := m/q$ with $0 < m < q$ and $(m, q) = 1$. If GRH$[\beta_{0}, \chi]$ holds for all nonprincipal characters $\chi$ mod $q$, then for any $\epsilon > 0$ we have

$$\sum_{n \geq 1} \Lambda(n) e(-mn/q) \mathcal{B}(n/X) = \frac{\mu(q)}{\phi(q)} \sum_{n \geq 1} \Lambda(n) \mathcal{B}(n/X) + O_{\xi, \mathcal{B}, \epsilon}(X^{\beta_{0} + \epsilon})$$ \hspace{1cm} (5.2)

Proof Introducing an error of at most $O_{\mathcal{B}}(X^{1/2})$, the sum on the left side of $(5.2)$ can be replaced by

$$\sum_{(n,q)=1} \Lambda(n) e(-mn/q) \mathcal{B}(n/X) = \sum_{(a,q)=1} e(-am/q) \sum_{n \equiv a \mod{q}} \Lambda(n) \mathcal{B}(n/X)$$

$$= \frac{1}{\phi(q)} \sum_{(a,q)=1} e(-am/q) \sum_{\chi(1)} \sum_{n \equiv a \mod{q}} \Lambda(n) \chi(n) \mathcal{B}(n/X).$$

The contribution from the principal character $\chi_{0}$ is

$$\frac{1}{\phi(q)} \sum_{(a,q)=1} e(-am/q) \sum_{n \geq 1} \Lambda(n) \mathcal{B}(n/X) = \frac{\mu(q)}{\phi(q)} \sum_{n \geq 1} \Lambda(n) \mathcal{B}(n/X)$$

(the first sum is a Ramanujan sum), and by Lemma 5.1 $(ii)$ the total contribution from all nonprincipal characters is at most $O_{\xi, \mathcal{B}, \epsilon}(X^{\beta_{0} + \epsilon})$.

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1 Assume RH. Since RH is equivalent to GRH in the case of principal characters, we can assume $\chi$ is nonprincipal. Moreover, if $\chi$ is induced from a primitive character $\chi_{*}$ of conductor $q_{*}$, then $L(s, \chi)$ and $L(s, \chi_{*})$ have the same zeros in the critical strip since

$$L(s, \chi) = L(s, \chi_{*}) \prod_{p \mid q} (1 - \chi_{*}(p)p^{-s}).$$

Hence, to prove Theorem 1.1 it suffices to show that GRH$[\frac{\beta_{0}}{10}, \chi]$ holds for any Dirichlet $L$-function attached to a primitive Dirichlet character $\chi$ of modulus $q > 1$. In this situation,
the following identity is well known:
\[
\chi(n) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \mod q} \chi(m)e(-nm/q) \quad (n \in \mathbb{Z}),
\] (5.3)
where the sum runs over any complete set of residue classes \(m \mod q\), and \(\tau(\chi)\) is the Gauss sum given by
\[
\tau(\chi) := \sum_{n \mod q} \chi(n)e(n/q);
\]
see, e.g., Bump [2, Chapter 1]. In particular, for every \(B \in \mathcal{C}^\infty_c(\mathbb{R}^+)\) we have
\[
\sum_{n \geq 1} \Lambda(n)\chi(n)B(n/X) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{0 < m < q} \chi(m) \sum_{n \geq 1} \Lambda(n)e(-nm/q)B(n/X). \quad (5.4)
\]
Combining (1.3) and Theorem 4.1, the estimate
\[
\sum_{n \geq 1} \Lambda(n)e(-nm/q)B(n/X) = -F_{q,B}(X) + O_{q,B,\epsilon}(X^{9/10+\epsilon}) \quad (5.5)
\]
holds uniformly for \(0 < m < q\) with \((m, q) = 1\), where
\[
F_{q,B}(X) := \frac{\mu(q)}{\phi(q)} \sum_{n \geq 1} \Lambda(n)B(n/X).
\]
We insert (5.5) into (5.4). Since \(F_{q,B}(X)\) is independent of \(m\), and
\[
\sum_{0 < m < q} \chi(m) = 0
\]
for the nonprincipal character \(\chi\), we derive the bound
\[
\sum_{n \geq 1} \Lambda(n)\chi(n)B(n/X) \ll \frac{q}{q,B,\epsilon} X^{9/10+\epsilon}.
\]
Since \(B \in \mathcal{C}^\infty_c(\mathbb{R}^+)\) is arbitrary, Lemma 5.1 shows that GRH\([9/10, \chi]\) is true, and we are done. \(\square\)

**Proof of Theorem 1.2** Assume RH and GRH\([9/10]\). For any \(B \in \mathcal{C}^\infty_c(\mathbb{R}^+)\) and \(\xi := m/q\) with \(0 < m < q\) and \((m, q) = 1\), we have by Theorem 4.1:
\[
\sum_{n \geq 1} \Lambda(n)e(-n\xi)B\left(\frac{n}{X}\right) = -\sum_{\rho = \frac{1}{2} + iy} \xi^{-1/2-iy} \zeta(\rho)B\left(\frac{\gamma}{2\pi \xi X}\right) + O_{q,B,\epsilon}(X^{9/10}),
\]
and by Lemma 5.2:
\[
\sum_{n \geq 1} \Lambda(n)e(-n\xi)B(n/X) = \frac{\mu(q)}{\phi(q)} \sum_{n \geq 1} \Lambda(n)B(n/X) + O_{q,B,\epsilon}(X^{9/10+\epsilon}).
\]
Combining these results, we obtain (1.3). \(\square\)

**Acknowledgements**
The author thanks Henryk Iwaniec for some stimulating conversations. Many thanks also to the referees for their helpful comments and for pointing out flaws in the original version of the manuscript.

**Data Availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study

Received: 10 May 2022  Accepted: 30 January 2023  Published online: 25 February 2023
References
1. Borwein, P., Choi, S., Rooney, B., Weirathmueller A. (eds.): The Riemann Hypothesis. A Resource for the Afficionado and Virtuoso Alike. CMS Books in Mathematics. Springer, New York (2008)
2. Bump, D.: Automorphic Forms and Representations. Cambridge Studies in Advanced Mathematics, vol. 55. Cambridge University Press, Cambridge (1997)
3. Davenport, H.: Multiplicative Number Theory, 3rd edn. Revised and with a preface by H. L. Montgomery. Graduate Texts in Mathematics, vol. 74. Springer, New York (2000)
4. Iwaniec, H., Kowalski, E.: Analytic Number Theory. American Mathematical Society Colloquium Publications, vol. 53. American Mathematical Society, Providence, RI (2004)
5. Montgomery, H.L., Vaughan, R.C.: Multiplicative Number Theory. I. Classical Theory. Cambridge Studies in Advanced Mathematics, vol. 97. Cambridge University Press, Cambridge (2007)
6. Riemann, B.: Ueber die Anzahl der Primzahlen unter einer gegebenen Größe. Monatsberichte der Berliner Akademie (1859)
7. Titchmarsh, E.C.: The Theory of the Riemann Zeta-Function, 2nd edn. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York (1986)

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.