THE DECOMPOSABILITY PROBLEM FOR TORSION-FREE
ABELIAN GROUPS IS ANALYTIC-COMPLETE

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Abstract. We discuss the decomposability of torsion-free abelian groups. We show that among computable groups of finite rank this property is $\Sigma^0_3$-complete. However, when we consider computable groups of infinite rank, it becomes $\Sigma^1_1$-complete (and $\Sigma^1_1$-complete for groups of infinite rank in general), so it cannot be characterized by a first-order formula in the language of arithmetic.

1. Introduction

A group is computable if its domain can be enumerated effectively and the binary operation of the group is computable (via the enumeration). Such an enumeration is called a computable presentation. In other words, a computable group has a computable word problem. As with many other computable structures, there are a variety of questions we can ask about them. Perhaps the first question that comes to mind is what groups can be presented computably. This type of question is typically answered by looking at a specific class of groups and identifying which groups in that class have computable copies. Downey and others [2] studied the class of completely decomposable groups.

Definition 1.1: An abelian group $G$ is completely decomposable if it can be written

$$G = \bigoplus_i H_i$$

where each $H_i$ is a subgroup of $Q$.

In particular, they considered groups of the form

$$G_S = \bigoplus_{p \in S} Q_p,$$

where $S$ is a set of primes and $Q_p$ is the subgroup of $Q$ generated by the elements \(\{\frac{1}{p^k} : k \in \omega\}\). They found that there is a computable copy of $G_S$ iff $S$ is $\Sigma^0_3$.

Khisamiev [7] looked at (countable) reduced torsion groups, which are uniquely determined by their Ulm sequences. He was able to completely characterize the Ulm sequences of length $< \omega^2$ which can occur in a computably presented group. Ash, Knight, and Oates [1], working slightly later, independently duplicated his results.

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Another type of question that arises involves the notion of computable categoricity.

**Definition 1.2:** A computably presentable group is *computably categorical* if any two computable copies of the group have a computable isomorphism between them.

Additionally, a computably presentable group is $\Delta^0_n$-categorical if any two computable copies of the group have a $\Delta^0_n$-isomorphism between them.

Downey and Melnikov ([3],[4]) studied homogeneous completely decomposable groups, in which each summand has the same type (see Definition 2.2). They found that every completely decomposable group is $\Delta^0_5$-categorical, but homogeneous completely decomposable groups are actually $\Delta^0_3$-categorical.

Studying computable groups can also yield results telling us the difficulty of determining some of their most fundamental properties. For example, Downey and Montalban [5] studied the isomorphism problem for torsion-free abelian groups. They found that:

1) the set of pairs of computable indices for isomorphic torsion-free abelian groups is $\Sigma^1_1$-complete, and

2) the set of isomorphic pairs of torsion-free abelian groups is $\Sigma^1_1$-complete

In their research of completely decomposable groups, Downey and Melnikov [4] found that the index set of completely decomposable groups can be described by a $\Sigma^0_7$ formula (though it is not known if this is sharp). The property with which this paper is concerned is a similar one: decomposability.

**Definition 1.3:** An abelian group is *decomposable* if it can be written as the direct sum of two (or more) nontrivial subgroups. Otherwise, it is *indecomposable*.

Often the best way to study an abelian group is by writing it as a direct sum of its indecomposable subgroups, so determining whether a group is decomposable is a problem at the heart of abelian group theory.

It is known that the only indecomposable torsion groups are the cocyclic groups (groups of the form $\mathbb{Z}(p^k)$, with $k \in \omega + 1$), and that every mixed group is decomposable. Torsion-free groups of rank 1 are indecomposable, but beyond this no classification has been found. It has been conjectured by some (Kudinov, Melnikov) that this is in part because the class of torsion-free decomposable groups is not arithmetical. We shall see that this is indeed the case. In fact, the class is not even hyperarithmetical.

After a short discussion of some concepts from logic and algebra, we will construct an indecomposable group of rank 2. We will apply the ideas used in this example to show that the index set of computable decomposable groups of finite rank is $\Sigma^0_3$-complete. We will then shift our focus to groups of infinite rank and show that:

1) the index set of computable decomposable torsion-free abelian groups is $\Sigma^1_1$-complete, and
2) the set of decomposable torsion-free abelian groups is $\Sigma_1^1$-complete

2. SOME COMPUTATIONAL COMPLEXITY HIERARCHIES

The arithmetical hierarchy was developed to describe the complexity of properties based on their formulas in the language of arithmetic. A computable set (or relation) $S \subset \omega$ is said to be $\Sigma^0_n$ (or $\Pi^0_n$). A set $S_1$ is $\Sigma^0_{n+1}$ if it can be characterized by a formula of the form

$$x \in S_1 \Leftrightarrow (\exists y \in \omega) R_1(x, y)$$

where $R_1$ is a $\Pi^0_n$ relation. Likewise, a set $S_2$ is $\Pi^0_{n+1}$ if it can be characterized by a formula of the form

$$x \in S_2 \Leftrightarrow (\forall y \in \omega) R_2(x, y)$$

where $R_2$ is a $\Sigma^0_n$ relation.

In other words, the $n$ represents how many times the formula alternates quantifiers over $\omega$ (or some other infinite computable set), and a $\Sigma^0_n$ formula starts with an existential quantifier, while a $\Pi^0_n$ formula starts with a universal quantifier.

For example, given a computable group $G$, an element $g \in G$, and a fixed prime $p$, there is a $\Pi^0_2$ formula that says whether $p$ infinitely divides $g$

$$p |^\infty g \Leftrightarrow (\forall k \in \omega) (\exists h \in G) p^k h = g$$

Any set that is characterized by a $\Sigma^0_n$ or $\Pi^0_n$ formula for some $n$ is said to be arithmetical. If we allow quantifiers over functions from $\omega$ to $\omega$ (or between any two computable sets), then our formula will be analytic. We say that a formula is $\Sigma^1_1$ if it is of the form

$$(\exists f \in \omega^\omega) R(f)$$

where $R$ is any arithmetical formula. Similarly, a $\Pi^1_1$ formula is of the form

$$(\forall f \in \omega^\omega) R(f)$$

where $R$ is arithmetical.

For any complexity class $\Gamma$, we say that a set $A$ is $\Gamma$-complete if any other set $B \in \Gamma$ can be "coded" into $A$. That is to say, there is a computable function $f : \omega \rightarrow \omega$ such that $x \in B$ iff $f(x) \in A$.

The canonical example of a $\Sigma^1_1$-complete set is the index set of computable trees in $\omega^{<\omega}$ with an infinite path.

The Borel hierarchy is a complexity class structure for Polish spaces (like $\omega^\omega$) which defines $\Sigma^0_1$ sets to be open sets and $\Pi^0_1$ sets to be closed sets. In this hierarchy, a set is $\Sigma^1_1$ or analytic if it is the image of a Polish space under a continuous mapping. For example, the set of trees (not necessarily computable) in $\omega^{<\omega}$ with an infinite path is $\Sigma^1_1$-complete.

To summarize, this paper shows that

1) the index set of computable decomposable torsion-free abelian groups of finite rank is $\Sigma^0_3$-complete,

2) the index set of computable decomposable torsion-free abelian groups of infinite rank is $\Sigma^1_1$-complete, and
3) the set of decomposable torsion-free abelian groups of infinite rank is $\Sigma_1^1$-complete.

3. Algebra Background

In this paper, we will exclusively discuss torsion-free abelian groups. Also, the term basis will refer to a maximal linearly independent subset of a group. The rank of a group is the cardinality of any of its bases.

**Definition 3.1:** Given an abelian group $G$, an element $x \in G$, and a prime $p$, the height of $p$ at $x$ is given by

$$h_p(x) = \sup\{k : p^k| x\}$$

We call

$$\chi_G(x) = (h_2(x), h_3(x), h_5(x),...)$$

the characteristic of $x$ in $G$.

**Definition 3.2:** We define an equivalence relation on characteristics by saying that $\chi_G(x) \sim \chi_G(y)$ if

- for all $p$, $h_p(x) = \infty \iff h_p(y) = \infty$, and
- $h_p(x) = h_p(y)$ for all but finitely many $p$

We call the equivalence classes types. In other words, $x$ and $y$ have the same type iff there exist integers $m$ and $n$ such that $\chi_G(mx) = \chi_G(ny)$.

We can put a partial order on types by declaring for two types $\alpha, \beta$ that $\alpha \preceq \beta$ if, given any element $a$ of type $\alpha$ and any element $b$ of type $\beta$,  

- for all $p$, $h_p(a) = \infty \Rightarrow h_p(b) = \infty$ and
- $h_p(a) \leq h_p(b)$ for all but finitely many $p$

**Definition 3.3:** A nonzero element has strictly maximal type if no nonzero element linearly independent from it has a greater or equal type.

There is a class of subgroups which we must introduce before giving any proofs.

**Definition 3.4:** In an abelian group $G$, a subgroup $H$ is called pure if for every $x \in H$ and $m \in \omega$, if $m$ divides $x$ in $G$, $m$ also divides $x$ in $H$.

If $S$ is a set of elements in $G$, the pure subgroup generated by $S$ is the smallest pure subgroup of $G$ containing $S$.

For example, $Z$ is a pure subgroup of $Z^2$, but $Z$ is not a pure subgroup of $Q$. In fact, the only pure subgroups of $Q$ are $Q$ and the trivial subgroup.

4. An Example of an Indecomposable Group

The following example can be found in Fuchs [6]. Let $G_0$ be the free abelian group generated by two elements, $x_1$ and $x_2$. For every $k > 0$, we add elements of the form

$$\frac{x_1}{3^k} \text{ and } \frac{x_2}{5^k}$$
to $G_0$. We also add the element $\frac{x_1 + x_2}{2}$ to the group. We denote by $G$ the group generated by all these elements.

Note that $\{x_1, x_2\}$ is still a basis for this group, and that

$$\chi_G(x_1) = (0, \infty, 0, 0, 0, ...)$$

$$\chi_G(x_2) = (0, 0, \infty, 0, 0, ...)$$

Furthermore, any element of the form $q_1x_1 + q_2x_2$ with both coefficients nonzero has type $(0, 0, 0, ...)$. Thus, $x_1$ and $x_2$ both have strictly maximal type.

**Proposition 4.1:** In a decomposable group $G (= A \oplus B)$, if $x \in G$ decomposes as $x = a + b$ and an integer $m$ divides $x$, then $m$ divides $a$ and $b$ as well.

**Proof:** Let $y \in G$ be such that $my = x$, and suppose $y$ decomposes $y = a_1 + b_1$. Then we see that

$$ma_1 + mb_1 = my = x = a + b$$

$ma_1 \in A$ and $mb_1 \in B$, so $ma_1 = a$ and $mb_1 = b$. \(\square\)

**Corollary 4.2:** In a decomposable group $G$, every element of strictly maximal type must be contained in a direct summand.

**Proof:** Suppose that $x$ is an element of strictly maximal type that is not in $A$ or $B$. Then we can write $x = a + b$ with $a \in A$ and $b \in B$, and both $a$ and $b$ nonzero. Because $x$ has strictly maximal type, there is a prime $p$ and an integer $k$ such that $p^k$ divides $x$, but neither $a$ nor $b$. However, this contradicts the proposition. \(\square\)

Thus, we can assume $x_1 \in A$ and $x_2 \in B$. Now consider the decomposition of the element

$$\frac{x_1 + x_2}{2} = a + b$$

with $a \in A$ and $b \in B$. It is clear that $2a = x_1$ and $2b = x_2$, but there are no elements in $G$ which satisfy these equations. Thus, the group is indecomposable.

The proofs contained in this paper will mimic this technique of creating elements of strictly maximal type and then introducing elements which force them to be contained in the same direct summand. We call these elements *links*.

**Definition:** Let $x$ and $y$ be two elements of strictly maximal type in a torsion-free abelian group $G$. If there is a prime $p$ which divides the sum $x + y$ but neither $x$ nor $y$, then the element $\frac{x + y}{p}$ is a link connecting $x$ and $y$. We say that $x$ and $y$ are connected by a *chain* of links if there are elements $x_1, x_2, ..., x_n$ such that the sequence $\{x_0 = x, x_1, x_2, ..., x_n, x_{n+1} = y\}$ has the property that for $0 \leq i \leq n$, there is a link connecting $x_i$ and $x_{i+1}$.

The following proposition gives us a simple way to construct indecomposable groups.

**Proposition 4.3:** If a torsion-free group has a basis of elements of strictly maximal type, with each pair of them having a link or a chain of links connecting them,
then it is indecomposable.

Proof: Every element of strictly maximal type must be contained in a direct summand, and any two of these elements with a link between them must be in the same direct summand. Transitivity, this is also true of any two elements with a chain of links connecting them. Thus, the entire basis is contained in a single direct summand, so the group is indecomposable.

5. Groups of Finite Rank

Remark 5.1: Let $G$ be a group of finite rank, and assume $G = A \oplus B$. Then

(1) $\text{rank}(G) = \text{rank}(A) + \text{rank}(B)$

(2) If $\{a_1, \ldots, a_n\}$ is a basis for $A$ and $\{b_1, \ldots, b_m\}$ is a basis for $B$, then $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ is a basis for $G$ with the following property:

If there exists an element $g = \sum_{i=1}^{n} q_i a_i + \sum_{j=1}^{m} r_j b_j$, then there exist elements $g_A, g_B$ such that $g_A = \sum_{i=1}^{n} q_i a_i$ and $g_B = \sum_{j=1}^{m} r_j b_j$

Conversely, if $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ is a basis for $G$ with this property, then the pure subgroup generated by the $a_i$'s and the pure subgroup generated by the $b_j$'s give a decomposition of $G$. Thus, a group of finite rank is decomposable iff it has a basis with this property.

Let $[G]^{<\omega}$ denote the set of all finite sets in $G$. The following $\Pi^0_2$ formula describes the property of being a basis of $G$. For $\bar{x} \in [G]^{<\omega}$,

$$BASIS(\bar{x}) \Leftrightarrow [(\forall y \in G) (\exists q \in Q^{<\omega}) (|q| = |\bar{x}| \land y = \sum_i q_i x_i)$$

$$\land (\forall \bar{q} \in Q^{<\omega}) (|\bar{q}| = |\bar{x}| \land \sum_i q_i x_i = 0) \Rightarrow \bar{q} = \bar{0}]$$

If we take the conjunction of that formula with one describing the property in Remark 4.1, we have the following $\Sigma^0_3$ formula for decomposable groups of finite rank:

$$\exists \bar{a}, \bar{b} \in [G]^{<\omega}) \{BASIS(\bar{a} \cup \bar{b}) \land \bar{a} \neq \emptyset \land \bar{b} \neq \emptyset \land (\forall y \in G) (\forall \bar{q} \in Q^{<\omega})$$

$$\langle |\bar{q}| = |\bar{a}| + |\bar{b}| \land y = \sum_i q_i a_i + \sum_j q_j b_j \Rightarrow w = \sum_i q_i a_i \rangle$$

Theorem 5.2: The set of decomposable groups of finite rank is $\Sigma^0_3$-complete.

Proof: Recall that $\text{Cof} = \{n : W_n \text{ is cofinite}\}$ is $\Sigma^0_3$-complete. In order to prove our result, we construct a function from $\omega$ to groups of rank 2 such that $G_n$ is decomposable iff $W_n$ is cofinite.

Construction: We start with a group $G$ generated by the following elements:
$\langle g_1, g_2, \frac{g_1 + g_2}{2}, \frac{g_1}{3}, \frac{g_2}{5}, \frac{g_1}{7}, \frac{g_2}{11}, \ldots \rangle$

($g_1$ and $g_2$ are linearly independent).

$g_1$ is divisible by all odd-indexed primes, and $g_2$ is divisible by all even-indexed primes (except $p_0 = 2$), so they have incomparable (indeed, strictly maximal) types. Thus, like the example above, our initial group $G$ is indecomposable.

The group $G_n$ is generated by adding $\frac{g_2}{p_{2k+1}}$ for every $k$ such that $\Phi_n(k) \downarrow$.

**Verification:** If $W_n$ is coinfinite, then $g_1$ is still divisible by infinitely many primes that do not divide $g_2$. Thus, the types remain incomparable, and the group remains indecomposable.

If $W_n$ is cofinite, then the type of $g_2$ is strictly greater than the type of $g_1$. There are finitely many primes that divide $g_1$ but not $g_2$. Denote their product by $m$.

**Lemma 5.3:** $G_n = A \oplus B$, where $A$ is the pure subgroup generated by $a = \frac{g_1 + mg_2}{2}$ and $B$ the pure subgroup generated by $g_2$.

**Proof:** We observe that $\frac{m+g_2}{2} = a - \frac{m-1}{2}g_2$ (Note that $m$ is a product of odd primes).

Any element of the form $\frac{g_1}{p} \in G_n$ can be written

$$\frac{g_1}{p} = \frac{2}{p}a - \frac{m}{p}g_2$$

If $p \nmid m$, then $p \mid g_2$, so $\frac{m}{p}g_2 \in B$. Thus, every generating element of the group can be uniquely decomposed, so the group is decomposable. \(\square\)

$G_n$ is decomposable iff $W_n$ is cofinite, so the theorem is proved. \(\square\)

### 6. Groups of Infinite Rank

We can adapt the formula used for groups of finite rank to describe decomposable groups of infinite rank. However, this means the first existential quantifier is searching over infinite sets instead of finite sets, so the $\Sigma^0_3$ formula becomes a $\Sigma^1_1$ formula (here BASIS is a $\Pi^0_2$-formula on infinite sets):

$$\exists \bar{a}, \bar{b} \in [G]^{\leq \omega} \ [\text{BASIS}(\bar{a} \sqcup \bar{b}) \land \bar{a} \neq \emptyset \land \bar{b} \neq \emptyset \land (\forall y \in G)
(\forall q \in Q^{\omega})(\exists w \in G)(y = \sum_i q_i a_i + \sum_j q_j b_j) \Rightarrow w = \sum_i q_i a_i)]$$

**Theorem 6.1:** The set of decomposable groups of infinite rank is $\Sigma^1_1$-complete.

**Proof:** We will construct a function from trees in $\omega^{< \omega}$ to torsion-free abelian groups of infinite rank that takes a tree $T$ and gives a group $G_T$ that is decomposable iff $T$ has an infinite path. (Recall that the set of trees in $\omega^{< \omega}$ which have an
infinite path is $\mathbf{\Sigma}_1^1$-complete.)

**Construction of the group $G$:** We start with a countably infinite set of linearly independent elements: $x_1, x_2, \ldots$ and $\{x_\sigma\}_{\sigma \in \omega^{<\omega}}$ (which we denote as the $x$-elements), and $y_1, y_2, \ldots$ (the $y$-elements). These elements form a basis for our group. We will give them all strictly maximal type and introduce links connecting all the $x$-elements and separate links connecting all the $y$-elements.

The initial group $G_0$ is generated by the following elements:

- For $i, k > 0$ and $\sigma \in \omega^{<\omega}$,
  $$\frac{x_i}{p^k_{(0,i)}}, \frac{y_i}{p^k_{(1,i)}}, \text{ and } \frac{x_\sigma}{p^k_{(2,\sigma)}}$$

- For $0 < i < j$,
  $$\frac{x_i + x_j}{p^k_{(3,\langle i,j \rangle)}} \text{ and } \frac{y_i + y_j}{p^k_{(4,\langle i,j \rangle)}}$$

- For every $i \geq 0$ and $\sigma, \rho \in \omega^{<\omega}$,
  $$\frac{x_i + x_\sigma}{p^k_{(5,\langle i,\sigma \rangle)}} \text{ and } \frac{x_\sigma + x_\rho}{p^k_{(6,\langle \sigma,\rho \rangle)}}$$

- For $n > 1$,
  $$\frac{y_1 + y_2 + \ldots + y_n}{p^k_{(7,n)}}$$

All the $x$- and $y$-elements are elements of strictly maximal type, and due to the links, all the $x$-elements must be in the same direct summand of $G_0$ (as do the $y$-elements). Thus, $G_0$ can only be decomposed as $G_0 = A \oplus B$, where $A$ is the pure subgroup containing all the $x$-elements, and $B$ is the pure subgroup containing all the $y$-elements.

Now we add to $G_0$ links of the form

$$\frac{x_i + y_i}{p^k_{(8,i)}}$$

for $i \geq 0$, and denote by $G$ the group generated by these elements. Now every $x$- and $y$-element are connected by a chain of links, so $G$ is indecomposable.

**Construction of $G_T$:** Given a tree $T$ in $\omega^{<\omega}$, we will add elements to $G$ to form a group $G_T$ that will be decomposable if $T$ has an infinite path through it. The idea is that if there is an infinite path $\pi$, then $G_T = A_T \oplus B_\pi$, where $A_T$ is the pure subgroup containing the $x$-elements, and $B_\pi$ is the pure subgroup of $G_T$ containing all the elements of the form $y_i + x_\pi|_i$. Note that if there is more than one infinite path through $T$, there will be more than one way to decompose $G_T$.

Enumerate $T$ so that each string in $T$ is enumerated after all of its initial segments. When we see $\sigma \in T$ with $|\sigma| = n$, we do the following: (It’s worth noting that in each case, the introduction of the first element creates the second element. We list both simply to remind the reader that the second element also exists)
(1) For $i \leq n$, we add to the group the elements
\[
\frac{y_i + x_{\sigma|1}}{p_{(1,i)}} \quad \text{and} \quad \frac{x_{\sigma|1}}{p_{(1,i)}}
\]
(2) For $i < n$, we add to the group the elements
\[
\frac{(y_i + x_{\sigma|1}) + (y_n + x_\sigma)}{p_{(4,(i,n))}} \quad \text{and} \quad \frac{x_{\sigma|1} + x_\sigma}{p_{(4,(i,n))}}
\]
(3) We add to the group the elements
\[
\frac{y_n + x_\sigma}{p_{(8,n)}} \quad \text{and} \quad \frac{x_n - x_\sigma}{p_{(8,n)}}
\]
(4) Finally, we add the elements
\[
\frac{(y_1 + x_{\sigma|1}) + (y_2 + x_{\sigma|2}) + \ldots + (y_n + x_\sigma)}{p_{(7,n)}} \quad \text{and} \quad \frac{x_{\sigma|1} + x_{\sigma|2} + \ldots + x_\sigma}{p_{(7,n)}}
\]

Verification: If an infinite path $\pi$ does exist, we shall see that $G_T = A_T \oplus B_\pi$ (as described above).

Each $x_i$ and $x_\sigma$ is contained in $A_T$. We have $y_j = -x_{\pi|j} + (y_j + x_{\pi|j})$. Both of these elements are infinitely divisible by $p_{(1,j)}$ because $x_{\pi|j}$ went through step (1) infinitely often.

For $0 < i < j$,
\[
\frac{y_i + y_j}{p_{(4,(i,j))}} = \frac{(y_i + x_{\pi|1}) + (y_j + x_{\pi|j})}{p_{(4,(i,j))}} - \frac{x_{\pi|1} + x_{\pi|j}}{p_{(4,(i,j))}}
\]
These elements were created during step (2) of some stage.

For $i > 0$,
\[
\frac{x_i + y_i}{p_{(8,i)}} = \frac{x_i - x_{\pi|1}}{p_{(8,i)}} + \frac{y_i + x_{\pi|1}}{p_{(8,i)}}
\]
These elements were created during step (3) of some stage.

For $n > 1$,
\[
\frac{y_1 + y_2 + \ldots + y_n}{p_{(7,n)}} = \frac{-x_{\pi|1} + x_{\pi|2} + \ldots + x_{\pi|n}}{p_{(7,n)}} + \frac{(y_1 + x_{\pi|1}) + (y_2 + x_{\pi|2}) + \ldots + (y_n + x_{\pi|n})}{p_{(7,n)}}
\]
These elements were created during step (4) of some stage.

We see that all the generating elements of $G_T$ can be uniquely decomposed, so $G_T = A_T \oplus B_\pi$.

Now suppose $G_T$ is decomposable as $G_T = A' \oplus B'$. All the $x$-elements still have strictly maximal type, so they must be in the same direct summand ($A'$).

Each $y_j$ can be decomposed $y_j = a_j + b_j$, where $a_j \in A'$ and $b_j \in B'$. We know $a_j$ and $b_j$ are infinitely divisible by $p_{(1,j)}$ because $y_j$ is. The only other basis elements
that could be infinitely divisible by this prime are the elements \( x_\sigma \) with \(|\sigma| = j\).

**Lemma 6.2:** If there is any \( y_j \in A' \), then \( G_T = A' \) (and \( B' = 0 \)).

**Proof:** Suppose \( y_j \in A' \) \((y_j = a_j)\), and that another element \( y_i \notin A' \). Let \( q = p(4,i,j) \) (we can assume that \( j < i \)). Then \( q \) must divide \( y_i + y_j \), and thus, \( b_i + b_j \) (which is just \( b_i \)). There is some finite sum such that

\[
k_0 b_i = k_1 y_i + \sum_{|\sigma| = i} k_\sigma x_\sigma
\]

with each \( k_\sigma, k_0, k_1 \in \omega \). (Recall that the only other basis elements that could be infinitely divisible by \( p(1,i) \) are the elements \( x_\sigma \) with \(|\sigma| = i\).) We can also write

\[
k_0 a_i = (k_0 - k_1) y_i - \sum_{|\sigma| = i} k_\sigma x_\sigma
\]

Note that if \( k_0 \neq k_1 \), then \( y_i \in A' \). Thus, \( k_1 = k_0 \), so

\[
b_i = y_i + \frac{1}{k_0} \sum_{|\sigma| = i} k_\sigma x_\sigma
\]

and this must be divisible by \( q \). However, \( q \) does not divide \( y_i \), nor any nontrivial linear combination of \( y_i \) with \( x\)-elements (though \( q \) does divide \( y_i + y_j \)). Therefore, \( y_i \in A' \). This is true for every \( y_i \), so \( A' = G_T \). \( \square \)

There is no \( y_j \in B' \), either. This is because there are no elements

\[
\frac{x_j}{p^{<8,j>}} \quad \frac{y_j}{p^{<8,j>}}
\]

So we see that every \( y_j = a_j + b_j \), with both components being nonzero.

Now suppose \( y_1, y_2 \) decompose as

\[
y_1 = \sum_{|\sigma| = 1} k_\sigma x_\sigma + b_1 \quad \text{and} \quad y_2 = \sum_{|\rho| = 2} l_\rho x_\rho + b_2
\]

We shall denote \( p(7,2) \) by \( r \). \( r|(y_1 + y_2) \), so it must also divide

\[
a_1 + a_2 = \sum_{|\sigma| = 1} k_\sigma x_\sigma + \sum_{|\rho| = 2} l_\rho x_\rho
\]

Although \( r \) does not divide any \( x_\sigma \), from step (4) of the construction we see that \( r \) divides elements of the form \( x_\sigma + x_\sigma' \) where \(|\sigma| = 1 \) (and \( \sigma' \in T \)). Thus, \( r \) also divides elements of the form

\[
\sum_m l_m (x_\sigma + x_\sigma') = k x_\sigma + \sum_m l_m x_\sigma'
\]

where \( k = \sum_m l_m \) (and \( \sigma' \in T \)).

From this we see that

\[
r| \left( \sum_{|\sigma| = 1} k_\sigma x_\sigma + \sum_{|\rho| = 2} l_\rho x_\rho \right) \iff k_\sigma \equiv \sum_{\rho > \sigma} l_\rho (\text{mod } r)
\]
for each $\sigma$ with $|\sigma| = 1$.

Similarly, $p_{(7,3)} \mid (y_1 + y_2 + y_3)$, so for each $\tau \in T$ with $|\tau| = 3$, $p_{(7,3)} \mid (x_\sigma + x_\rho + x_\tau)$, where $\sigma \prec \rho \prec \tau$.

By the same reasoning, we see that if $y_3 = \sum_{|\tau| = 3} m_\tau x_\tau + b_3$, then for each $\sigma \in T$ with $|\sigma| = 1$,

$$k_\sigma \equiv \sum_{\rho \succ \sigma} l_\rho \equiv \sum_{\tau \succ \sigma} m_\tau \pmod{p_{(7,3)}}$$

There are infinitely many such equivalences, so we see that

$$k_\sigma = \sum_{\rho \succ \sigma} l_\rho = \sum_{\tau \succ \sigma} m_\tau = ...$$

It is also true that for each $\rho \in T$ with $|\rho| = 2$,

$$l_\rho \equiv \sum_{\tau \succ \rho} m_\tau \pmod{p_{(7,3)}}$$

Continuing this process, we see that the following also holds:

$$l_\rho = \sum_{\tau \succ \rho} m_\tau = ...$$

Thus, if we choose a $\sigma$ of length 1 such that $k_\sigma \neq 0$ (which we are guaranteed by the fact that $y_1 \notin B'$), there must be a $\rho$ of length 2 such that $\sigma \prec \rho$ and $l_\rho \neq 0$, and a $\tau$ of length 3 such that $\rho \prec \tau$ and $m_\tau \neq 0$. By repeating this process, we find an infinite path through $T$.

Thus, $G_T$ is decomposable iff $T$ has an infinite path. $\square$

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