CONTINUOUS GALERKIN METHODS ON QUASI-GEOMETRIC MESHES FOR DELAY DIFFERENTIAL EQUATIONS OF PANTOGRAPH TYPE

QIUMEI HUANG* AND XIUXIU XU

Beijing Institute for Scientific and Engineering Computing
Beijing University of Technology
Beijing 100124, China

HERMANN BRUNNER

Department of Mathematics, Hong Kong Baptist University
Hong Kong, China

and

Department of Mathematics & Statistics, Memorial University of Newfoundland
St. John’s, NL, A1C 5S7, Canada

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Abstract. We analyze the optimal global and local convergence properties of continuous Galerkin (CG) solutions on quasi-geometric meshes for delay differential equations with proportional delay. It is shown that with this type of meshes the attainable order of nodal superconvergence of CG solutions is higher than of the one for uniform meshes. The theoretical results are illustrated by a broad range of numerical examples.

1. Introduction. The delay differential equation (DDE) with proportional delay,
\[ u'(t) + a(t)u(t) + b(t)u(qt) = f(t), \quad t \in J := [0, T], \quad 0 < q < 1, \]
\[ u(0) = u_0, \]
(1)
is generally known as the pantograph equation (although in its original form (Iserles [13]) the coefficients \( a \) and \( b \) are constants).

Its numerical solution, especially by Runge-Kutta methods, collocation methods, and discontinuous Galerkin methods – on uniform meshes as well as on quasi-geometric ones – has received considerable attention in the last twenty years. Of the numerous papers (most of which contain extensive lists of references) we mention Liu [16], Bellen, Guglielmi and Torelli [3], Koto [14], Bellen [1], Brunner and Liang [8] (Runge-Kutta methods); Takama, Muroya and Ishiwata [17], Brunner, Hu and Lin [6], and Bellen, Brunner, Maset and Torelli [2] (collocation methods); Brunner, Huang and Xie [7], and Huang, Xie and Brunner [12] (discontinuous Galerkin (DG)

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* Corresponding author: Hermann Brunner.
The monographs by Bellen and Zennaro [4] and Brunner [5] review various numerical methods and their convergence analysis of (1) (and related more general functional integral and integro-differential equations) with vanishing delays.

The optimal order of convergence of DG methods on uniform meshes for pantograph-type DDEs (1) have been studied in [7]. There it is shown the attainable order of the DG solution $U$ in the space $S_{m}^{(m-1)}(J_0)$ of (discontinuous) piecewise polynomial of degree $m \geq 0$ is $O(h^{m+1})$, and that the optimal order of superconvergence at the mesh points cannot exceed $O(h^{m+2})$ when $m \geq 1$. This is in sharp contrast to DG solutions for ODEs [10] and for DDEs of constant delay [15], where the optimal orders of superconvergence at the mesh points are $O(h^{2m+2})$ and $O(h^{2m+1})$, respectively.

For DDEs with proportional delay, Xu, Huang and Chen [19] located all the superconvergence points of CG solutions based on uniform meshes. Their analysis is based on the supercloseness between the CG solution and the interpolant of the true solution.

The fact that for $0 < q < 1$ the image of a subinterval $(t_{n-1}^{-}, t_{n}^{+})$ under the mapping $\theta(t) = qt$ will in general lie in the union $(t_{i-1}^{-}, t_{i}^{+}) \cup (t_{i}^{+}, t_{i+1}^{+})$ ($i < n-2$) of two adjacent previous subintervals makes the theoretical analysis and the computation of numerical solutions rather complex. In order to avoid this problem, different kinds of strategies have been proposed. A very prominent one consists in replacing the uniform meshes by a so-called “quasi-geometric meshes” (see for example [1, 2, 3, 6]).

Thus, there arises the question as to what the orders of global convergence and local superconvergence of CG solutions will be under such quasi-geometric meshes. It is the aim of this paper to show that when employing certain quasi-geometric meshes a higher order of convergence accuracy can be attained for CG solutions at the nodal points.

The outline of the paper is as follows. In Section 2, we introduce the CG method for (1) on quasi-geometric meshes and discuss the existence and uniqueness of the CG solution. The main results on the optimal order of global convergence and local superconvergence of the CG solution are stated in Section 3. Section 4 shows that form of the discretized CG method and the collocation method are identical for delay differential equations with proportional delay. In Section 5, we provide numerical experiments to illustrate our theory. Possible extensions and future research work are described in the final Section 6.

2. The continuous Galerkin method for delay differential equations with proportional delay. In this section, we introduce the CG method for DDE (1) under quasi-geometric mesh. Then we give the computational form, and the existence and uniqueness of the CG solution. We assume that the given functions $a$, $b$ and $f$ in (1) are continuous on $J$.

2.1. The CG method. Suppose that on a given (small) initial subinterval $J_0 = [0, t_0]$ of $[0, T]$, $t_0 = q^kT$ for a suitable value of $k$, the approximation $\phi(t)$ of the exact solution $u$ is known. For instance, $\phi(t)$ can be obtained by the CG method or by the truncation of the Taylor expansion of the exact solution $u(t)$. Subsequently, we solve the following equation:

$$
\begin{align*}
&u'(t) + a(t)u(t) + b(t)u(qt) = f(t), \quad t_0 \leq t \leq T, \\
&u(t) = \phi(t), \quad qt_0 \leq t \leq t_0.
\end{align*}
$$

(2)
On the interval \([t_0, T]\), we introduce macro-mesh \(\{\xi_n\}\) by setting
\[t_0 = \xi_0 < \xi_1 < \cdots < \xi_k = T, \quad \xi_n = q^{k-n}T \quad (0 \leq n \leq k),\]
with increasing size \(H_n := \xi_n - \xi_{n-1} = Tq^{k-n}(1 - q) \quad (\mu = 1, \cdots, k)\) denoting the macro-steps.

On the subinterval \(I_n := [\xi_{n-1}, \xi_n] \quad (\mu = 1, \cdots, k)\), we insert \(l + 1\) equally spaced nodes, with distance \(h_n := \frac{H_n}{l}\), so that the resulting quasi-geometric mesh \(J_h\)
\[t_0 = \xi_0 < t_1 < \cdots < t_l = \xi_1 < \cdots < t_{2l} = \xi_2 < \cdots < t_{kl} = \xi_k = T\]
has the property
\[qt_n = t_{n-1}, \quad n = l, \cdots, kl.\]
That is, the linearity of \(\theta(t) := qt \quad (0 < q < 1)\) implies that \(\theta\) maps current mesh points of the quasi-geometric mesh into previous ones. We will use the notation
\[N = kl, \quad J_n := [t_{n-1}, t_n], \quad h_n := \frac{t_n - t_{n-1}}{2}, \quad t_{n-1/2} := \frac{t_n + t_{n-1}}{2}, \quad h := \max_{1 \leq n \leq N} h_n \quad (1 \leq n \leq N).\]
The corresponding CG finite element space is defined by
\[S_m^{(0)}(J_n) = \{v \in C(J) : v|J_n \in P_m, 1 \leq n \leq N\},\]
where \(P_m\) denotes the space of (real) polynomials of degree not exceeding \(m\), with \(m \geq 1\).

In the CG method, we are looking for an approximation solution \(U \in P_m(J_n)\) for (2) such that
\[\int_{J_n} [U'(t) + a(t)U(t) + b(t)U(qt)]\eta(t)dt = \int_{J_n} f(t)\eta(t)dt, \quad \eta(t) \in P_{m-1}(J_n) \quad 1 \leq n \leq N. \quad (3)\]
Here, we let \(U(t) = \phi(t), \quad qt_0 \leq t \leq t_0\). Because of the continuity of \(U(t)\), we have \(U(t_{n-1}) = \lim_{t \to t_{n-1}^-} U(t) = \lim_{t \to t_{n-1}^+} U(t)\). \(U(t) \in J_n\) has only \(m\) degrees of freedom on each subinterval, so \(\eta(t) \in P_{m-1}(J_n)\).

It is obvious that the exact solution \(u\) also satisfies (3), that is
\[\int_{J_n} [u'(t) + a(t)u(t) + b(t)u(qt)]\eta(t)dt = \int_{J_n} f(t)\eta(t)dt, \quad \eta(t) \in P_{m-1}(J_n) \quad 1 \leq n \leq N. \quad (4)\]
If \(t \leq t_l\), then \(qt \leq qt_l = t_0\), \(r\) becomes
\[u'(t) + a(t)u(t) = f(t) - b(t)\phi(t) =: g(t), \quad t_0 \leq t \leq t_l.\]
(3), (4) can be split into
\[\int_{J_n} [U'(t) + a(t)U(t)]\eta(t)dt = \int_{J_n} g(t)\eta(t)dt, \quad \eta(t) \in P_{m-1}(J_n) \quad 1 \leq n \leq l, \quad (5)\]
\[\int_{J_n} [U'(t) + a(t)U(t) + b(t)U(qt)]\eta(t)dt = \int_{J_n} f(t)\eta(t)dt, \quad \eta(t) \in P_{m-1}(J_n) \quad l < n \leq N,\]
and
\[ \int_{J_n} [u'(t) + a(t)u(t)]\eta(t)dt = \int_{J_n} g(t)\eta(t)dt, \quad \eta(t) \in \mathcal{P}_{m-1}(J_n) \quad 1 \leq n \leq l, \]
\[ \int_{J_n} [u'(t) + a(t)u(t) + b(t)u(qt)]\eta(t)dt = \int_{J_n} f(t)\eta(t)dt, \quad \eta(t) \in \mathcal{P}_{m-1}(J_n) \quad l < n \leq N, \]
respectively. Let \( e(t) = u(t) - U(t) \), then subtracting (5) from (6) gives
\[ B_n(e, \eta) = \int_{J_n} [e'(t) + a(t)e(t)]\eta(t)dt = 0, \quad \eta(t) \in \mathcal{P}_{m-1}(J_n) \quad 1 \leq n \leq l, \]
\[ B_n(e, \eta) = \int_{J_n} [e'(t) + a(t)e(t) + b(t)e(qt)]\eta(t)dt = 0, \quad \eta(t) \in \mathcal{P}_{m-1}(J_n) \quad l < n \leq N. \]
In other words, the CG error \( e(t) \) has the orthogonality property in each \( J_n \).

Suppose that \( L_1(s), \ldots, L_{m+1}(s) \) are given basis functions on \([-1, 1]\). We denote by \( l_{n,1}(t), \ldots, l_{n,m+1}(t) \) the corresponding local basis functions on the subinterval \( J_n \), obtained by transforming \( L_1(s), \ldots, L_{m+1}(s) \) from \([-1, 1]\) to \( J_n \). \( \tilde{L}_1(s), \ldots, \tilde{L}_m(s) \) are given basis function on \([-1, 1]\]. Denote by \( \tilde{l}_{n,1}(t), \ldots, \tilde{l}_{n,m}(t) \) the corresponding basis functions on the subinterval \( J_n \), obtained by transforming \( \tilde{L}_1(s), \ldots, \tilde{L}_m(s) \) from \([-1, 1]\) to \( J_n \). On \( J_n \) \((n = 1, \ldots, N)\), the CG solution can be written as
\[ U_n(t) = \sum_{i=1}^{m+1} u_{n,i}l_{n,i}(t) = \sum_{i=1}^{m+1} u_{n,i}L_i\left(\frac{t - t_{n-1}/2}{h_n}\right) \]
where the unknown coefficients \( u_{n,i} \) are determined by the CG equation (5).

2.2. The computational form of the CG equation. After selecting the basis functions, the CG equation (5) can be rewritten as a system of linear algebraic equations for the vector
\[ U_n = (u_{n,1}, \ldots, u_{n,m+1})^T. \]
The structure of these \( N \) systems changes for each value of \( n \) as we pass the following phase 1 to phase 2. To make this more precise, we first define
\[ g_0 = (L_1(1), \ldots, L_{m+1}(1)), \]
\[ g_1 = (L_1(-1), \ldots, L_{m+1}(-1)), \]
\[ f_n = \left( \int_{-1}^{1} f(t_{n-1}/2 + sh_n)\tilde{L}_1(s)ds, \ldots, \int_{-1}^{1} f(t_{n-1}/2 + sh_n)\tilde{L}_m(s)ds \right)^T, \]
and introduce the matrices
\[ A = \left( \int_{-1}^{1} L_j'(s)\tilde{L}_i(s)ds \right)_{1 \leq j \leq m+1, 1 \leq i \leq m}, \]
\[ A_n = \left( \int_{-1}^{1} a(t_{n-1}/2 + sh_n)L_j(s)\tilde{L}_i(s)ds \right)_{1 \leq i \leq m, 1 \leq j \leq m+1}, \]
\[ \tilde{A} = \left( \begin{array}{c} g_1 \\ A \end{array} \right), \quad \tilde{A}_n = \left( \begin{array}{c} 0 \\ A_n \end{array} \right), \quad \tilde{f}_n = \left( \begin{array}{c} g_0U_{n-1} \\ h_nf_n \end{array} \right) (g_0U_n = \phi(t_0)). \]
reflecting contributions corresponding to the nondelay terms in CG equations. The contributions of the delay term will introduced below with two distinct phases.

- **Phase 1:** \(1 \leq n \leq l\). In this phase, for \(t \in J_n\), the images \(q_t < q_{t-l} = t_0\) lies in \([q_{t_0}, t_0]\). We let

\[
\mathbf{b}_n = \left( \int_{-1}^{1} b(t_{n-1/2} + sh_n)\phi(q(t_{n-1/2} + sh_n))\tilde{L}_i(s)ds \right)_{1 \leq i \leq m}^T, \\
\tilde{\mathbf{b}}_n = \begin{pmatrix} 0 \\ \mathbf{b}_n \end{pmatrix}.
\]

\(U_n\) is given by the solution of the linear algebraic system:

\[
(\hat{A} + h_n\hat{A}_n)U_n = \hat{f}_n - h_n\tilde{\mathbf{b}}_n. 
\] (8)

- **Phase 2:** \(l < n \leq N\). In this phase, for any \(t \in J_n\), the images \(q_t\) are in \(J_{n-l}\). We define

\[
\mathbf{B}_n = \left( \int_{-1}^{1} b(t_{n-1/2} + sh_n)L_j(s)\tilde{L}_i(s)ds \right)_{1 \leq i \leq m, 1 \leq j \leq m+1}, \\
\hat{\mathbf{B}}_n = \begin{pmatrix} 0 \\ \mathbf{B}_n \end{pmatrix}.
\]

\(U_n\) is given by the solution of the linear algebraic system:

\[
(\hat{A} + h_n\hat{A}_n)U_n = \hat{f}_n - h_n\hat{\mathbf{B}}_nU_{n-1}. 
\] (9)

### 2.3. Existence and uniqueness of the CG equation

We will briefly discuss the existence and uniqueness of the CG solution (defined by the solutions of the linear algebraic systems (8), (9)).

**Theorem 2.1.** Assume that the given functions \(a\), \(b\), and \(f\) are continuous on \([t_0, T]\). Then for any \(q \in (0, 1)\), there exists \(\tilde{h} > 0\) such that for all \(h \in (0, \tilde{h})\), each of the linear algebraic systems (8) and (9) possesses a unique solution \(U_n \in \mathbb{R}^{m+1}\).

**Proof.** Noting that the left-hand sides of the linear algebraic systems (8) and (9) are all

\[
\hat{A} + h_n\hat{A}_n,
\]

and the given functions \(a(t)\) and \(b(t)\) are in \(C(J)\), we just need to show that \(\hat{A}\) is nonsingular.

Suppose that the basis functions on \([-1, 1]\] are given by (see subsection 3.1 for details)

\[
L_j(s) = M_{j-1}(s), \quad j = 1, \cdots, m+1, \\
\tilde{L}_i(s) = M_{i-1}(s), \quad i = 1, \cdots, m.
\]

From the properties of Legendre polynomials, we have

\[
\hat{A} = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \frac{2}{3} & \cdots & 0 & 0 & 0 \\
0 & \frac{2}{3} & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{1}{2m-3} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{2}{2m-5} & 0 \\
0 & 0 & 0 & \cdots & (2m-3)(2m-5) & 0 & (2m-1)(2m-3)
\end{pmatrix}
\]
which means \( \tilde{A} \) is nonsingular. Therefore, for any \( q \in (0, 1) \), there exists a positive \( \tilde{h} \), so that for all \( h \in (0, \tilde{h}) \) and \( 1 \leq n \leq N \), (5) defines a unique CG solution \( U \in S_{m}^{(0)}(J_{h}) \) for DDE (2).

3. Global convergence and local superconvergence of the CG solution. In this section, we first list some preliminaries, then give the global convergence and local superconvergence of the CG solution, respectively.

3.1. Preliminaries. The follow preliminaries are given in this subsection which will play a key role in the proofs of the following Theorems 3.2 and 3.4.

**Lemma 3.1.** (Gronwall) Assume that \( \{k_{j}\}_{j \geq 0} \) is a given nonnegative sequence, the sequence \( \{\varepsilon_{n}\}_{n \geq 0} \) satisfies

\[
\varepsilon_{n} \leq \gamma_{n} + \sum_{j=0}^{n-1} k_{j} \varepsilon_{j} \quad (n \geq 1)
\]

for some \( \rho_{0} \geq 0, q_{j} \geq 0 (j \geq 0) \) with \( \gamma_{n} = \rho_{0} + \sum_{j=0}^{n-1} q_{j} \). Then

\[
\varepsilon_{n} \leq \gamma_{n} \exp\left(\sum_{j=0}^{n-1} k_{j}\right) \quad (n \geq 1).
\]

We then introduce the Legendre polynomials on the interval \([-1, 1]\). They are defined by

\[
P_{0}(s), P_{1}(s) = s, \cdots, P_{n}(s) = \frac{1}{2^{n}n!} \frac{d^{n}}{ds^{n}}(s^{2} - 1)^{n}, \quad n \geq 2.
\]

The Legendre polynomials satisfy the following orthogonality relation

\[
\int_{-1}^{1} P_{j}(s)P_{i}(s)ds = \begin{cases} 
0 & \text{if } i \neq j; \\
\frac{2}{2i+1} & \text{if } i = j.
\end{cases}
\]

Define \( M_{0}(s) = 1, M_{1}(s) = s \) and \( M_{k+1}(s) = \int_{-1}^{1} P_{k}(t)dt \) \((k = 1, 2, \cdots)\), we get another family of polynomials, which we call the M-type series:

\[
M_{0}(s) = 1, \quad M_{1}(s) = s, \quad M_{n+1}(s) = \frac{1}{2^{n}n!} \frac{d^{n-1}}{ds^{n-1}}(s^{2} - 1)^{n}, \quad n = 1, 2, \cdots.
\]

The M-type series has the following properties: \( (M_{i}, M_{j}) \neq 0 \) when \( i = j \) or \( i - j = \pm 2 \), otherwise \( (M_{i}, M_{j}) = 0 \). \( M_{j}(\pm 1) = 0 \) \((j \geq 2)\), and

\[
M_{k+1}(s) = \frac{1}{k(k+1)} (P_{k+1}(s) - P_{k-1}(s)) = \frac{1}{k(k+1)} (s^{2} - 1)P'_{k}(s) \quad (k \geq 1).
\]

The zeros of \( M_{n+1}(s) \) are called the Gauss-Lobatto points of degree \( n + 1 \).

3.2. Global error estimate of the CG solution.

**Theorem 3.2.** Assume the following.

1. The functions \( a(t), b(t), f(t), \phi(t) \) describing the DDE (2) are in \( C^{m}(J) \).
2. \( u(t) \) is the exact solution of the initial value problem for the pantograph DDE (2).
3. \( U \in S_{m}^{(0)}(J_{h}) \) is the CG approximation of \( u \) defined by (3).
4. \( J_{h} \) is a quasi-geometric mesh for \([t_{0}, T]\) and \( h \) is sufficiently small.
Then for $u \in W^{m+1,\infty}([t_0, T])$, the following optimal global convergence estimate holds:

$$||u - U||_{\infty} \leq Ch^{m+1}||u||_{m+1,\infty} \quad (m \geq 2).$$

(10)

**Proof.** For $t \in J_n$, we take

$$t = t_{n-1/2} + h_n s, \quad s \in [-1, 1].$$

For $u(t) = u(t_{n-1/2} + h_n s)$, expanding $u'(t_{n-1/2} + h_n s)$ as a Legendre series,

$$u'(t_{n-1/2} + h_n s) = \sum_{j=0}^{\infty} d_{n,j+1} P_j(s),$$

$$d_{n,j} = (j - 1/2) \int_{-1}^{1} u'(t_{n-1/2} + h_n s) P_{j-1}(s) ds \quad (j = 1, 2, \cdots).$$

(11)

When $j = 1$, $d_{n,1} = \frac{1}{2}(u(t_n) - u(t_{n-1}))$. Integration $u'(t_{n-1/2} + h_n s)$ from $-1$ to $s$, we get an $M$-type series as following

$$u(t_{n-1/2} + h_n s) = \sum_{j=0}^{\infty} d_{n,j} M_j(s),$$

with

$$d_{n,0} = \frac{1}{2}(u(t_{n-1}) + u(t_n)).$$

It is easy to see that

$$u(t_0) = \sum_{j=0}^{\infty} d_{1,j} M_j(-1) = d_{1,0} - d_{1,1} = \phi(t_0).$$

For (11), we use multi-integration by parts and have (see [9] for details)

$$d_{n,j} = O(h_n^j) ||u||_{j,\infty} \quad (j = 0, 1, 2, \cdots, m + 1).$$

(12)

The following idea is motivated by [9]. We construct a new $m$-degree polynomial interpolation $u_I$ of the exact solution $u$ in $J_n$,

$$u_I(t_{n-1/2} + h_n s) = \sum_{j=0}^{m} d_{n,j} M_j(s) - \sum_{j=2}^{m} d_{n,j}^* M_j(s) \quad (m \geq 2).$$

We let $u_I(t) = u(t)$, $qt_0 \leq t \leq t_0$, and $d_{n,j}^*$ is to be determined below. It is obvious that

$$||u - u_I||_{\infty} \leq Ch^{m+1} ||u||_{m+1,\infty}. \quad (13)$$

Then, the remainder term of $J_n$ is:

$$R(t_{n-1/2} + h_n s) = u(t_{n-1/2} + h_n s) - u_I(t_{n-1/2} + h_n s)$$

$$= \sum_{j=2}^{m} d_{n,j}^* M_j(s) + \sum_{j=m+1}^{\infty} d_{n,j} M_j(s).$$

(14)

From the formula (7) we see that

$$B_n(R, \eta) = \int_{J_n} (R'(t) + a(t) R(t)) \eta(t) dt, \quad \eta(t) \in \mathcal{P}_{m-1}(J_n), \quad 1 \leq n \leq l, \quad (15)$$

$$B_n(R, \eta) = \int_{J_n} (R'(t) + a(t) R(t) + b(t) R(qt)) \eta(t) dt,$$

$$\eta(t) \in \mathcal{P}_{m-1}(J_n), \quad l < n \leq N. \quad (16)$$
In (15) and (16), we let the test functions $\eta(t)$
\[ \eta(t_{n-1/2} + h_n s) = \sum_{i=0}^{m-1} \beta_i M_i(s) \in P_{m-1}(J_n). \]
and use the transformation $t = t_{n-1/2} + h_n s$ to get
\[
B_n(R, \eta) = \int_{-1}^{1} (R'(t_{n-1/2} + h_n s) + h_n a(t_{n-1/2} + h_n s)R(t_{n-1/2} + h_n s))\eta(t_{n-1/2} + h_n s)ds = \sum_{i=0}^{m-1} \beta_i(\sum_{j=2}^{m} d_{n,j}^* C_{ij} + \sum_{j=m+1}^{\infty} d_{n,j} C_{ij}), \quad 1 \leq n \leq l,
\]
\[
B_n(R, \eta) = \int_{-1}^{1} (R'(t_{n-1/2} + h_n s) + h_n a(t_{n-1/2} + h_n s)R(t_{n-1/2} + h_n s) + h_n b(t_{n-1/2} + h_n s)R(q(t_{n-1/2} + h_n s)))\eta(t_{n-1/2} + h_n s)ds = \sum_{i=0}^{m-1} \beta_i(\sum_{j=2}^{m} d_{n,j}^* C_{ij} + \sum_{j=m+1}^{\infty} d_{n,j} C_{ij} + \sum_{j=2}^{m} d_{n-k,j}^* C_{ij}' + \sum_{j=m+1}^{\infty} d_{n-k,j} C_{ij}'), \quad l \leq n \leq N,
\]
where
\[
C_{ij} = \int_{-1}^{1} (M_j'(s) + h_n a(t_{n-1/2} + h_n s)M_j(s))M_i(s)ds,
\]
\[
C_{ij}' = \int_{-1}^{1} h_n b(t_{n-1/2} + h_n s)M_j(s)M_i(s)ds.
\]

By the orthogonality properties of $P_i(s)$ ($i = 0, 1, \ldots, m$) and $M_{j+1}(s) = \frac{1}{2j+1} (P_{j+1}(s) - P_{j-1}(s))$ ($j \geq 1$), we have
\[
\int_{-1}^{1} M_j'(s)M_i(s)ds = \begin{cases} \frac{2}{(2i-1)(2j+3)} & \text{if } i - j = 1, \\ \frac{2}{(2i-3)(2j+1)} & \text{if } i - j = -1, \\ 0 & \text{otherwise}. \end{cases}
\]
\[
\int_{-1}^{1} M_j(s)M_i(s)ds = \begin{cases} O(1) & \text{if } i = j, \\ O(1) & \text{if } i - j = \pm 2, \\ 0 & \text{otherwise}. \end{cases}
\]

Thus, we obtain the following estimates:
\[
C_{ij} = O(h_n^{\vert i-j \vert - 1}), \quad C_{ij}' = O(h_n^{\vert i-j \vert - 1}).
\]

In order to determine the coefficients $d_{n,j}^*$ in (17) and (18), we require that $d_{n,j}^*$ satisfy the following equations:
\[
\sum_{j=2}^{m} d_{n,j}^* C_{ij} = - \sum_{j=m+1}^{\infty} d_{n,j} C_{ij}, \quad i = 1, \ldots, m-1, \quad 1 \leq n \leq l,
\]
We then show that
\[
\sum_{j=2}^{m} d_{n,j}^* C_{ij} = - \sum_{j=m+1}^{\infty} d_{n,j} C_{ij} - \sum_{j=2}^{m} d_{n-k,j}^* C_{ij} - \sum_{j=m+1}^{\infty} d_{n-k,j}^* C_{ij},
\]
\[i = 1, \cdots, m-1, \quad l < n \leq N. \tag{24}\]

To prove (25), we proceed as follows:

**Step 1.** We prove that
\[
\sum_{j=m+1}^{\infty} d_{n,j} C_{ij} = O(h_n^{m+1-i})\|u\|_{m+1, \infty}, \quad i = 1, \cdots, m-1, \quad 1 \leq n \leq N. \tag{26}\]

We combine (12) and (22) to obtain
\[
\sum_{j=m+1}^{\infty} d_{n,j} C_{ij} = d_{n,m+1} C_{i,m+1} + \sum_{j=m+2}^{\infty} d_{n,j} C_{ij} = O(h_n^i h_n^{m+1-i})\|u\|_{m+1, \infty} = O(h_n^{2m+1-i})\|u\|_{m+1, \infty}, \quad i = 1, \cdots, m-1, \quad 1 \leq n \leq N.
\]

Here, it is easy to see from (19), (20) and (21) that $C_{i,j} = 0$ when $j - i \geq 3$, then
\[\sum_{j=m+2}^{\infty} d_{n,j} C_{ij} = 0.
\]

**Step 2.** We show that
\[
d_{n,j}^* = O(h_n^{2m+2-j})\|u\|_{m+1, \infty}, \quad j = 2, \cdots, m, \quad 1 \leq n \leq l. \tag{27}\]

Combining (23) and (26), we have
\[
\sum_{j=2}^{m} d_{n,j}^* C_{ij} = O(h_n^{2m+1-i})\|u\|_{m+1, \infty}, \quad i = 1, \cdots, m-1, \quad 1 \leq n \leq l.
\]

This leads to
\[
\begin{pmatrix}
O(1) & O(h) & O(h^2) & \cdots & O(h^{m-2}) \\
O(h) & O(1) & O(h) & \cdots & O(h^{m-3}) \\
O(1) & O(h) & O(1) & \cdots & O(h^{m-4}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O(h^{m-4}) & O(h^{m-5}) & O(h^{m-6}) & \cdots & O(1)
\end{pmatrix}
\times
\begin{pmatrix}
d_{n,2}^* \\
d_{n,3}^* \\
d_{n,4}^* \\
\vdots \\
d_{n,m}^*
\end{pmatrix}
= \begin{pmatrix}
O(h_n^{2m}) \\
O(h_n^{2m-1}) \\
O(h_n^{2m-2}) \\
\vdots \\
O(h_n^{m+2})
\end{pmatrix}\|u\|_{m+1, \infty}.
\]

When $h_n$ is sufficiently small, the coefficient matrices of the systems (23) are diagonally dominant. Thus, (27) holds.

**Step 3.** We then use mathematical induction to prove that
\[
d_{n,j}^* = O(h_n^{2m+2-j})\|u\|_{m+1, \infty}, \quad j = 2, \cdots, m, \quad l < n \leq N. \tag{28}\]
For $n = l + 1$, it is easy to show that the right-hand side term of (24) satisfies
\[ \sum_{j=m+1}^{\infty} dt_{i+1,j} C_{ij} + \sum_{j=2}^{m} d_{i,j} C_{ij}^* + \sum_{j=m+1}^{\infty} d_{i,j}^* C_{ij}^* = O(h_{i+1}^{2m+1-i}) \| u \|_{m+1,\infty}, \]
\[ i = 1, \ldots, m - 1. \]
This, together with (24), implies that
\[ \sum_{j=2}^{m} d_{i+1,j}^* C_{ij} = O(h_{i+1}^{2m+1-i}) \| u \|_{m+1,\infty}, \quad i = 1, \ldots, m - 1. \]
Similarly to the proof of (27) we find that
\[ d_{i+1,j}^* = O(h_{i+1}^{2m+2-j}) \| u \|_{m+1,\infty}, \quad j = 2, \ldots, m. \]
Assume that (28) is valid for $n = l + 2, \ldots, N - 1$. Considering the case $n = N$, we have
\[ \sum_{j=m+1}^{\infty} d_{N,j} C_{ij} + \sum_{j=2}^{m} d_{N-l,j} C_{ij}^* + \sum_{j=m+1}^{\infty} d_{N-l,j}^* C_{ij}^* = O(h_N^{2m+1-i}) \| u \|_{m+1,\infty}, \]
\[ i = 1, \ldots, m - 1. \]
This leads to
\[ \sum_{j=2}^{m} d_{N,j}^* C_{ij} = O(h_N^{2m+1-i}) \| u \|_{m+1,\infty}, \quad i = 1, \ldots, m - 1. \]
Therefore, we have
\[ d_{N,j}^* = O(h_N^{2m+2-j}) \| u \|_{m+1,\infty}, \quad j = 2, \ldots, m. \]
So (25) holds.
The above arguments show that $B_n(R, \eta)$ can be expressed as
\[ B_n(R, \eta) = \beta_0 \sum_{j=2}^{m} d_{n,j}^* C_{0,j} + \sum_{j=m+1}^{\infty} d_{n,j} C_{0,j} \]
\[ = O(h_n^{2m+1}) \beta_0 \| u \|_{m+1,\infty}, \quad 1 \leq n \leq l, \quad (29) \]
\[ B_n(R, \eta) = \beta_0 \sum_{j=2}^{m} d_{n,j}^* C_{0,j} + \sum_{j=m+1}^{\infty} d_{n,j} C_{0,j} \]
\[ + \sum_{j=1}^{m} d_{n-k,j}^* C_{0,j} + \sum_{j=m+1}^{\infty} d_{n-k,j} C_{0,j} \]
\[ = O(h_n^{2m+1}) \beta_0 \| u \|_{m+1,\infty}, \quad l < n \leq N. \quad (30) \]
By the inverse estimate $|\beta_0| \leq C \int_{-1}^{1} |\eta(s)| ds$ (see [9] for details), (29) and (30), there exists a constant $C > 0$ such that
\[ B_n(R, \eta) \leq C h_n^{2m+1} \int_{-1}^{1} |\eta(s)| ds \| u \|_{m+1,\infty} = C h_n^{2m} \int_{J_n} |\eta(t)| dt \| u \|_{m+1,\infty} \]
\[ \leq C h_n^{2m} \int_{J_n} |\eta(t)| dt \| u \|_{m+1,\infty}, \quad 1 \leq n \leq N. \quad (31) \]
Now, we let
\[ \theta = u_I - U, \quad e = u - U = u - u_I + (u_I - U) = R + \theta. \quad (32) \]
From (7), we have
\[ B_i(\theta, \eta) = -B_i(R, \eta), \quad 1 \leq i \leq N. \]  
(33)

When \( t_0 \leq t \leq t_l \), i.e., \( 1 \leq i \leq l \), we have
\[ B_i(\theta, \eta) = \int_{J_i} (\theta'(t) + a(t)\theta(t))\eta(t)dt \leq Ch^{2m} \int_{J_i} |\eta(t)|dt\|u\|_{m+1, \infty}. \]

Summing from 1 to \( n \) for \( J_i \) and taking \( \eta(t) = \theta'(t) \), we obtain
\[
\int_{t_0}^{t_n} \theta^2(t)dt \\
\leq -\int_{t_0}^{t_n} a(t)\theta(t)\theta'(t)dt + Ch^{2m} \int_{t_0}^{t_n} |\theta'(t)|dt\|u\|_{m+1, \infty} \\
\leq \int_{t_0}^{t_n} (\varepsilon\theta^2(t) + \frac{a^2(t)}{4\varepsilon} \theta^2(t))dt + \int_{t_0}^{t_n} (\varepsilon\theta^2(t) + \frac{C^2}{4\varepsilon} h^{4m} \|u\|_{m+1, \infty}^2)dt \\
\leq 2\varepsilon \int_{t_0}^{t_n} \theta^2(t)dt + \frac{a^2}{4\varepsilon} \int_{t_0}^{t_n} \theta^2(t)dt + \frac{C^2}{4\varepsilon} h^{4m} (t_n - t_0)\|u\|_{m+1, \infty}^2, \quad 1 \leq n \leq l. \\
\]

Where we use the \( ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon} \) and \( \bar{a} := \max_{t \in [t_0, t_n]} a(t) \). Choose \( \varepsilon < \frac{1}{2} \) and omitting the first term of the right-hand side, there yields
\[
\int_{t_0}^{t_n} \theta^2(t)dt \leq C \int_{t_0}^{t_n} \theta^2(t)dt + Ch^{4m} \|u\|_{m+1, \infty}^2, \quad 1 \leq n \leq l. \\
\]

Here and in the following analysis, \( C \) denotes a generic constant that may assume different values at different places. When \( t \geq t_l \), i.e., \( l < i \leq N \)

\[ B_i(\theta, \eta) = \int_{J_i} (\theta'(t) + a(t)\theta(t) + b(t)\eta(qt))\eta(t)dt \\
\]

Summing from 1 to \( n \) for \( J_i \) and combining (31), (33), we have
\[
\int_{t_0}^{t_n} (\theta'(t) + a(t)\theta(t) + b(t)\eta(qt))\eta(t)dt \leq Ch^{2m} \int_{t_0}^{t_n} |\eta(t)|dt\|u\|_{m+1, \infty}. \\
\]

Selecting \( \eta(t) = \theta'(t) \), similarly, we can obtain
\[
\int_{t_0}^{t_n} \theta^2(t)dt \\
\leq -\int_{t_0}^{t_n} a(t)\theta(t)\theta'(t)dt - \int_{t_0}^{t_n} b(t)\theta(qt)\theta'(t)dt + \int_{t_0}^{t_n} Ch^{2m} \|u\|_{m+1, \infty}\|\theta'(t)\|dt \\
\leq \varepsilon \int_{t_0}^{t_n} \theta^2(t)dt + \frac{a^2}{4\varepsilon} \int_{t_0}^{t_n} \theta^2(t)dt + \varepsilon \int_{t_0}^{t_n} \theta^2(t)dt \\
+ \frac{\bar{b}^2}{4\varepsilon} \int_{t_0}^{t_n} \theta^2(qt)dt + \varepsilon \int_{t_0}^{t_n} \theta^2(t)dt + \frac{C^2}{4\varepsilon} h^{4m} (t_n - t_0)\|u\|_{m+1, \infty}^2 \\
\leq 3\varepsilon \int_{t_0}^{t_n} \theta^2(t)dt + C \int_{t_0}^{t_n} \theta^2(t)dt + Ch^{4m} \|u\|_{m+1, \infty}^2, \\
\]

where \( \bar{a} := \max_{t \in [t_0, t_n]} a(t) \), \( \bar{b} := \max_{t \in [t_0, t_n]} b(t) \), and

\[
\int_{t_0}^{t_n} \theta^2(qt)dt = \frac{1}{q} \int_{t_0}^{t_n} \theta^2(t)dt \leq \frac{1}{q} \int_{t_0}^{t_n} \theta^2(t)dt. \\
\]
We suppose \( \varepsilon \) is sufficiently small and get
\[
\int_{t_0}^{t_n} \theta^2(t) dt \leq C \int_{t_0}^{t_n} \theta^2(t) dt + \mathrm{Ch}^4m \| u \|_{m+1,\infty}^2. \tag{34}
\]
Since \( \theta(t_0) = 0 \), we have
\[
\theta(t) = \int_{t_0}^{t} \theta'(t) dt.
\]
Then \( \theta^2(t) \leq (t_n - t_0) \int_{t_0}^{t_n} \theta^2 dt \) by Schwarz inequality. Integrating \( \theta^2(t) \) in \( J_n \) and combining (34) yields
\[
\int_{J_n} \theta^2 dt \leq Ch_n \int_{t_0}^{t_n} \theta^2 dt \leq Ch \int_{t_0}^{t_n} \theta^2 dt + \mathrm{Ch}^{4m+1} \| u \|_{m+1,\infty}^2.
\]
Application of Gronwall’s inequality yields the supercloseness estimate
\[
\int_{J_n} \theta^2 dt \leq \mathrm{Ch}^{4m+1} \| u \|_{m+1,\infty}^2.
\]
By the inverse property \( \| \theta \|_{L_\infty(J_n)} \leq \mathrm{Ch}^{-\frac{1}{2}} \| \theta \|_{L_2(J_n)} \), we have
\[
\| \theta(t) \|_{I_n,\infty} \leq \mathrm{Ch}^{2m} \| u \|_{m+1,\infty}. \tag{35}
\]
From the interpolation error estimates and (13) and (32), we see that (10) is true. \( \square \)

Observe that the maximum stepsize of the partition \( J_h \) for \([t_0, T]\) is attained in the last interval \( I_k \), where the distance is
\[
h = \frac{H_k}{2l} = \frac{T(1 - q)}{2l}.
\]
If we want the method described above to converge on the original interval \([0, T]\), we must also consider how to choose \( t_0 \). It is suggestive that \( t_0 \) is chosen as to depend on \( l \), such that
\[
t_0 = q^k T \leq 2h = \frac{T(1 - q)}{l}, \tag{36}
\]
that is,
\[
\kappa = \left\lceil \log (1 - q) - \log l \right\rceil \frac{\log q}{\log \kappa}
\]
is the minimum integer for which (36) holds. Hence, the following convergence theorem holds.

**Theorem 3.3.** Assume the following:

1. The function \( a(t), b(t), f(t) \), describing the DDE (1) are in \( C^m(J) \).
2. \( u \in W^{m+1,\infty}([0, T]) \) is the exact solution of the pantograph DDE (1).
3. \( \phi \in C^m(J_0) \), and \( \| u - \phi \|_{J_0,\infty} \leq Ch_0^{m+1} \| u \|_{m+1,\infty} \), where \( h_0 = \frac{T}{2} \).
4. \( J_h \) is a quasi-geometric mesh for \([t_0, T]\) and \( h \) is sufficiently small.
5. \( U \in S_m(J_h) \) is the CG solution of \( u \) in \([t_0, T]\).
6. \( \bar{U} \) is the approximation of \( u \) in \( J \) (i.e., \( \bar{U} = \phi \) in \([0, t_0] \), \( \bar{U} = U \) in \([t_0, T]\)).

Then the optimal global convergence estimate
\[
\| u - \bar{U} \|_{\infty} \leq Ch_{m+1} \| u \|_{m+1,\infty} \quad (m \geq 2)
\]
is true.
3.3. Local superconvergence of the CG solution. In this subsection, we get the superconvergence at the mesh points and other character points under the quasi-geometric mesh.

The following theorem gives the result of the local superconvergence.

**Theorem 3.4.** For (2), under the same assumptions in Theorem 3.2, there follows the local nodal superconvergence

\[ |(u - U)(t_n)| \leq Ch^{2m} \| u \|_{m+1,\infty}, \ n = 1,\ldots, N \ (m \geq 2). \]

Assuming that \( u \in W^{m+2,\infty}(J) \), the optimal superconvergence order at the other Lobatto points \( t_{nr} \) is given by

\[ |(u - U)(t_{nr})| \leq Ch^{m+2} \| u \|_{m+2,\infty}, \ n = 1,\ldots, N, \ r = 1,\ldots, m - 1 \ (m \geq 2), \]

where \( t_{nr} = t_{n-1/2} + h_n s_r \) and \( s_r \) are the zeros (excluding \( \pm 1 \)) of the \((m+1)\)-degree polynomial \( M_{m+1}(s) \) in \([-1,1]\).

**Proof.** This proof is based on the results of the proof of Theorem 3.2. We have from (14), (35) that

\[ e(t) = (R + \theta)(t) \leq \sum_{j=2}^{m} d_{n,j} M_j(s) + \sum_{j=2}^{\infty} d_{n,j} M_j(s) + Ch^{2m} \| u \|_{m+1,\infty}. \]

Since \( M_j(\pm 1) = 0 \ (j \geq 2) \), we obtain the local nodal superconvergence

\[ e(t_n) \leq Ch^{2m} \| u \|_{m+1,\infty}, \ n = 1,\ldots, N \ (m \geq 2). \]

When \( m \geq 2 \), we have \( M_{m+1}(s_r) = 0 \ (r = 1,2,\ldots, m - 1) \), therefore,

\[ e(t_{nr}) \leq \sum_{j=2}^{m} d_{n,j}^* M_j(s_r) + \sum_{j=m+2}^{\infty} d_{n,j} M_j(s_r) + Ch^{2m} \| u \|_{m+1,\infty}. \]

Combining \( d_{n,m+2} = O(h_n^{m+2}) \| u \|_{m+2,\infty} \) and \( d_{n,j} = O(h_n^{2m+2-j}) \| u \|_{m+1,\infty} \), we are led to

\[ e(t_{nr}) \leq Ch^{m+2} \| u \|_{m+2,\infty}, \]

as asserted. □

**Theorem 3.5.** Let the assumptions 1, 2, 4, 5, 6 in Theorem 3.3 hold and suppose that \( \| u - \phi \|_{\infty, J_0} \leq Ch_0^{m} \| u \|_{2m,\infty, J_0} \). Then the optimal order of nodal superconvergence is given by

\[ |(u - U)(t_n)| \leq Ch^{2m} \| u \|_{2m,\infty}, \ n = 1,\ldots, N \ (m \geq 2). \]

If we have \( \| u - \phi \|_{\infty, J_0} \leq Ch_0^{m+2} \| u \|_{m+2,\infty, J_0} \), the attainable order of superconvergence at the other Lobatto points \( t_{nr} \) is

\[ |(u - U)(t_{nr})| \leq Ch^{m+2} \| u \|_{m+2,\infty}, \ n = 1,\ldots, N, \ r = 1,\ldots, m - 1 \ (m \geq 2), \]

where \( t_{nr} = t_{n-1/2} + h_n s_r \) and \( s_r \) are the zeros (excluding \( \pm 1 \)) of the \((m+1)\)-degree polynomial \( M_{m+1}(s) \) in \([-1,1]\).

**Remark 1.** We can also constant the auxiliary problem of the original problem to prove these results. In this framework, the superconvergence of the nodal points will be given and then the supercloseness between \( U \) and some interpolation \( \Pi_h u \) of \( u \) will be obtained, and the superconvergence points are located by combining the two above results. We leave this to another paper.

The optimal superconvergence analysis can also be extended to DDEs with nonlinear vanishing delays \( \theta(t) \).
3.4. Delay differential equations with nonlinear vanishing delay. We assume that the delay item \( \theta \) is no longer \( gt \), but more general delay function which is subject to the following conditions

1. \( \theta(0) = 0 \) and \( \theta(t) < t \) for \( t > 0 \),
2. \( \min_{i \in J} \theta^i(t) =: q_0 > 0 \).

The delay differential equation (1) becomes

\[
\dot{u}(t) + a(t)u(t) + b(t)u(\theta(t)) = f(t), \quad 0 \leq t \leq T.
\]

It is easy to verify that the same results are true for the CG solution in \( S_m(0)(J_h) \), where the quasi-geometric mesh is replaced by a quasi-graded mesh, where the \( \xi_\mu \) are now given by

\[
\xi_\mu = \theta^{k-\mu}(T) \quad (0 \leq \mu \leq k),
\]

and the nodes

\[
t_0 = \xi_0 < t_1 < \cdots < t_l = \xi_1 < \cdots < t_{2l} = \xi_2 < \cdots < t_{(k-1)l} = \xi_k - 1
\]

are defined recursively by

\[
t_{n-l} = \theta(t_n), \quad n = l, \ldots, k,
\]

with the last \( l + 1 \) nodes being assigned arbitrarily (see [2] for details). The computational form of CG solutions will require an obvious modification. We leave the details to the reader.

4. The discretized CG method. Let \( \bar{h}_n = 2h_n \). On \( J_n \) let \( U' \) be given by

\[
U'(t_{n-1} + s\bar{h}_n) = \sum_{j=1}^{m} \tilde{L}_j(s)\bar{u}_{n,j}, \quad t = t_{n-1} + s\bar{h}_n, \quad s \in (0, 1),
\]

where \( \bar{u}_{n,j} := U'(t_{n-1} + c_j\bar{h}_n) \) and

\[
\tilde{L}_j(s) := \prod_{k \neq j} \frac{s - c_k}{c_j - c_k}.
\]

Setting

\[
\beta_j(s) := \int_0^s \tilde{L}_j(s)ds \quad (j = 1, \ldots, m),
\]

we obtain

\[
U(t_{n-1} + s\bar{h}_n) = U_{n-1} + \bar{h}_n \sum_{j=1}^{m} \beta_j(s)\bar{u}_{n,j}, \quad t = t_{n-1} + s\bar{h}_n, \quad s \in [0, 1].
\]

Here, \( 0 < c_1 < c_2 < \cdots < c_m \leq 1 \) are given numbers and \( U_{n-1} := U(t_{n-1}) \). Letting \( \eta(t_{n-1} + s\bar{h}_n) = \tilde{L}_i(s) \in P_{m-1}([0, 1]) \) \((i = 1, \ldots, m)\) in the CG equation (5), the CG equation (5) can be rewritten as

\[
\int_0^1 [U'(t_{n-1} + s\bar{h}_n) + a(t_{n-1} + s\bar{h}_n)U(t_{n-1} + s\bar{h}_n)]\tilde{L}_i(s)ds
\]

\[
= \int_0^1 g(t_{n-1} + s\bar{h}_n)\tilde{L}_i(s)ds, \quad 1 \leq i \leq m, \quad 1 \leq n \leq l.
\]

\[
\int_0^1 [U'(t_{n-1} + s\bar{h}_n) + a(t_{n-1} + s\bar{h}_n)U(t_{n-1} + s\bar{h}_n) + b(t_{n-1} + s\bar{h}_n)U(g(t_{n-1} + s\bar{h}_n))]\tilde{L}_i(s)ds
\]
= \int_0^1 f(t_{n-1} + s \tilde{h}_n) L_i(s) ds, \quad 1 \leq i \leq m, \quad l < n \leq N.

(a) Discretized local CG time-stepping: We approximate the CG equation by using interpolatory $m$-point quadrature formulas with the abscissas $\{c_i\}$. Denoting the resulting “discretized” CG approximation by $\bar{U}$, with

$$ \bar{U}'(t_{n-1} + s \tilde{h}_n) = \sum_{j=1}^m \bar{L}_j(s) \bar{u}_{n,j}, \quad s \in (0, 1) \quad (\bar{u}_{n,j} = \bar{U}'(t_{n-1} + c_{n,j} \tilde{h}_n)),$$

$$ \bar{U}(t_{n-1} + s \tilde{h}_n) = \bar{U}_{n-1} + \tilde{h}_n \sum_{j=1}^m \beta_j(s) \bar{u}_{n,j}, \quad s \in (0, 1),$$

the discretized CG equation is given by

$$ \tilde{u}_{n,i} + a(t_{n,i})(\bar{U}_{n-1} + \tilde{h}_n \sum_{j=1}^m \beta_j(c_i) \bar{u}_{n,j}) = g(t_{n,i}), \quad 1 \leq i \leq m, \quad 1 \leq n \leq l. $$

$$ \tilde{u}_{n,i} + a(t_{n,i})(\bar{U}_{n-1} + \tilde{h}_n \sum_{j=1}^m \beta_j(c_i) \bar{u}_{n,j}) + b(t_{n,i})(\bar{U}_{n-1} + \tilde{h}_n \sum_{j=1}^m \beta_j(c_i) \bar{u}_{n-1,j}) $$

$$ = f(t_{n,i}), \quad 1 \leq i \leq m, \quad l < n \leq N.$$  

Here $t_{n,i} = t_{n-1} + c_n \tilde{h}_n$.

(b) Local collocation time-stepping: for given collocation points

$$ X_h := \{t_{n,i} + \bar{c}_i \tilde{h}_n : 0 < \bar{c}_1 < \bar{c}_2 < \cdots < \bar{c}_m \leq 1 \ (n = 1, \cdots, N)\}, $$

the collocation solution $u_h$ has to satisfy the collocation equation

$$ u_h'(t) + a(t)u_h(t) + b(t)u_h(qt) = f(t), \quad t \in X_h. $$

We set

$$ u_h'(t_{n-1} + s \tilde{h}_n) = \sum_{j=1}^m \tilde{L}_j(s) w_{n,j}, \quad s \in (0, 1), \quad w_{n,j} := u_h'(t_{n-1} + \bar{c}_j \tilde{h}_n) $$

where

$$ \tilde{L}_j(s) := \prod_{k \neq j} \frac{s - \bar{c}_k}{\bar{c}_j - \bar{c}_k}. $$

It follows that on $[t_{n-1}, t_n]$ we have

$$ u_h(t_{n-1} + s \tilde{h}_n) = u_{n-1} + \tilde{h}_n \sum_{j=1}^m \beta_j(s) w_{n,j}, \quad s \in [0, 1]. $$

Here, $u_{n-1} = u_h(t_{n-1})$. The collocation equation can be written as

$$ w_{n,i} + a(t_{n,i})(u_{n-1} + \tilde{h}_n \sum_{j=1}^m \beta_j(\bar{c}_i) w_{n,j}) = g(t_{n,i}), \quad 1 \leq i \leq m, \quad 1 \leq n \leq l. $$

$$ w_{n,i} + a(t_{n,i})(u_{n-1} + \tilde{h}_n \sum_{j=1}^m \beta_j(\bar{c}_i) w_{n,j}) + b(t_{n,i})(u_{n-1} + \tilde{h}_n \sum_{j=1}^m \beta_j(\bar{c}_i) w_{n-1,j}) $$

$$ = f(t_{n,i}), \quad 1 \leq i \leq m, \quad l < n \leq N, $$

with $\bar{t}_{n,i} := t_{n-1} + \bar{c}_n \tilde{h}_n$. 
If we choose \(c_n = \bar{c}_n\), the discretized CG equation is equivalent to the one describing the continuous collocation method for the DDE (2).

5. Numerical experiments. In this section, three examples are chosen to illustrate the convergence results established in the previous sections. In the following tables, we use the notations

\[
err := ||u - U||\infty, \quad R := \frac{\log(err_{N1}/err_{N2})}{\log(h_{N1}/h_{N2})},
\]

\[
erl := \max_{1 \leq n \leq N, 1 \leq r \leq m-1} |u(t_{nr}) - U(t_{nr})|, \quad R_l := \frac{\log(erl_{N1}/erl_{N2})}{\log(h_{N1}/h_{N2})},
\]

\[
er := \max_{1 \leq n \leq N} |u(t_n) - U(t_n)|, \quad R_n := \frac{\log(ern_{N1}/ern_{N2})}{\log(h_{N1}/h_{N2})},
\]

where \(t_{nr} (n = 1, \cdots, N, r = 1, \cdots, m - 1)\) denote the Lobatto points, and \(t_n (n = 1, \cdots, N)\) denote the nodal points.

Example 1. We first use the CG method to solve the following DDE with proportional delay:

\[
\begin{align*}
  u'(t) &= au(t) + bu(qt) + \cos(t) - a\sin(t) - b\sin(qt), \quad 0 < t \leq 1, \\
  u(0) &= 0.
\end{align*}
\]

Its exact solution is \(u(t) = \sin(t)\) for any \(0 < q < 1\).

In the initial subinterval \(J_0 = [0, t_0]\), we select \(t_0 = q^k\) with \(k = \kappa + 1\), and the approximation \(\phi(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\) to the exact solution \(u(t)\) by Taylor expansion.

We choose the quasi-geometric mesh \(J_h\) with the mesh size \(h = \frac{T(1-q)}{2l}, \tilde{N} = N + 1 = kl + 1\). The numerical results are obtained by piecewise quadratic CG solution \(m = 2\), and piecewise cubic CG solution \(m = 3\), respectively.

In Tables 1, 2 and 3 we present the errors and convergence orders of the piecewise quadratic CG solution for (37), with \(a = -1, b = 0.5, q = 0.1, 0.5, 0.9\).

| \(l\) | \(N\) | \(h\)  | \(err\)  | \(R\)   | \(erl\)  | \(R_l\)  | \(ern\)  | \(R_n\)  |
|-----|-----|------|--------|--------|--------|--------|--------|--------|
| 32  | 65  | 0.0141 | 1.7761e-07 | 1.3194e-09 | 2.2170e-08 | 3.0020 | 2.3098e-09 | 4.0196 |
| 64  | 129 | 0.0070 | 2.2170e-08 | 3.0020 | 8.2762e-11 | 2.5431e-11 | 3.9995 |
| 128 | 385 | 0.0035 | 2.7692e-09 | 3.0011 | 5.1979e-12 | 3.9971 | 1.5992e-12 | 3.9912 |
| 256 | 769 | 0.0018 | 3.4601e-10 | 3.0006 | 3.1937e-13 | 3.9967 | 1.1580e-13 | 3.7877 |

| \(l\) | \(N\) | \(h\)  | \(err\)  | \(R\)   | \(erl\)  | \(R_l\)  | \(ern\)  | \(R_n\)  |
|-----|-----|------|--------|--------|--------|--------|--------|--------|
| 8   | 41  | 0.0313 | 1.6970e-06 | 2.8765e-08 | 5.3983e-09 | 3.9967 | 3.9967 | 3.9967 |
| 16  | 97  | 0.0156 | 2.1326e-07 | 2.9923 | 1.8217e-09 | 3.9810 | 3.3729e-10 | 4.0005 |
| 32  | 225 | 0.0078 | 2.6722e-08 | 2.9965 | 1.1460e-10 | 3.9906 | 2.1087e-11 | 3.9996 |
| 64  | 513 | 0.0039 | 3.3440e-09 | 2.9984 | 7.1770e-12 | 3.9971 | 1.3352e-12 | 3.9813 |
Table 3. Errors of the piecewise quadratic CG solution for $q = 0.9$

| $l$ | $N$  | $h$    | $err$         | $R$  | $erl$ | $R_l$ | $ern$ | $R_n$ |
|-----|------|--------|---------------|------|-------|-------|-------|-------|
| 2   | 59   | 0.0250 | 6.0306e-07    | 6.8775e-09 | 4.7920e-10 |
| 4   | 145  | 0.0125 | 7.6555e-08    | 2.9777  | 4.3524e-10 | 3.9820 | 2.9951e-11 | 4.0000 |
| 8   | 337  | 0.0062 | 9.6414e-09    | 2.7359e-11 | 3.9917 | 1.8849e-12 | 3.9900 |
| 16  | 785  | 0.0031 | 1.2096e-09    | 2.9947  | 1.6924e-12 | 4.0149 | 1.4067e-13 | 3.7442 |

The above tables reveal that $\|u - U\|_\infty = O(h^3)$, $|(u - U)(t_{nr})| = O(h^4)$, $|(u - U)(t_n)| = O(h^4)$ ($m = 2$).

In Tables 4, 5 and 6 we present the errors and convergence orders of the piecewise cubic CG solution for (37), with $q = 0.1, 0.5, 0.9$.

Table 4. Errors of the piecewise cubic CG solution for $q = 0.1$

| $l$ | $N$  | $h$    | $err$         | $R$  | $erl$ | $R_l$ | $ern$ | $R_n$ |
|-----|------|--------|---------------|------|-------|-------|-------|-------|
| 2   | 3    | 0.2250 | 1.4841e-05    | 4.5365e-07 |        |       |       |       |
| 4   | 5    | 0.1125 | 1.0334e-06    | 3.8441  | 1.6909e-08 | 4.7457 |
| 8   | 9    | 0.0563 | 6.7528e-08    | 3.9358  | 5.6590e-10 | 4.9011 |
| 16  | 33   | 0.0281 | 4.3062e-09    | 3.9710  | 1.8229e-11 | 4.9562 |

Table 5. Errors of the piecewise cubic CG solution for $q = 0.9$

| $l$ | $N$  | $h$    | $err$         | $R$  | $erl$ | $R_l$ | $ern$ | $R_n$ |
|-----|------|--------|---------------|------|-------|-------|-------|-------|
| 1   | 23   | 0.0500 | 4.2293e-08    | 1.1384e-09 |       |       |       |       |
| 2   | 59   | 0.0250 | 2.6931e-09    | 3.9730  | 3.5219e-11 | 4.9011 |
| 3   | 100  | 0.0167 | 5.3511e-10    | 3.9856  | 4.6101e-12 | 5.0148 |

Table 6. Errors of the piecewise cubic CG solution for $q = 0.5$

| $l$ | $N$  | $h$    | $err$         | $R$  | $erl$ | $R_l$ | $ern$ | $R_n$ |
|-----|------|--------|---------------|------|-------|-------|-------|-------|
| 2   | 7    | 0.1250 | 1.5582e-06    | 2.7586e-08 | 6.3699e-10 |
| 4   | 17   | 0.0625 | 1.0244e-07    | 9.3970  | 9.3059e-10 | 4.8900 | 9.9871e-12 | 5.9951 |
| 8   | 41   | 0.0313 | 6.5491e-09    | 3.9674  | 3.0059e-11 | 4.8519 | 1.5665e-13 | 5.9944 |
| 16  | 97   | 0.0156 | 4.1370e-10    | 3.9846  | 9.5568e-13 | 4.9751 | 8.8818e-16 | 7.4625 |

The numerical results reveal that the induced errors behave like $\|u - U\|_\infty = O(h^3)$, $|(u - U)(t_{nr})| = O(h^4)$, $|(u - U)(t_n)| = O(h^4)$ ($m = 2$).

In Tables 4, 5 and 6 we present the errors and convergence orders of the piecewise cubic CG solution for (37), with $q = 0.1, 0.5, 0.9$.

Table 7. Errors of the piecewise cubic CG solution on quasi-geometric and uniform mesh

| quasi-geometric mesh $q=0.1$ | uniform mesh $q=0.1$ |
|-------------------------------|----------------------|
| $N$  | $h$    | $ern$ | $R_n$ | $N$  | $h$    | $ern$ | $R_n$ |
|-----|--------|-------|-------|-----|--------|-------|-------|
| 3   | 0.2250 | 2.6799e-08 | 3   | 0.1667 | 1.4741e-07 |
| 5   | 0.1125 | 4.2575e-10 | 5   | 0.1000 | 8.0524e-09 | 5.6912 |
| 9   | 0.0563 | 6.8141e-12 | 9   | 0.0556 | 3.8018e-10 | 5.1942 |
| 33  | 0.0281 | 1.0514e-13 | 33  | 0.0278 | 1.6836e-11 | 4.4970 |
Table 8. Errors of the piecewise cubic CG solution on quasi-geometric and uniform mesh

| N     | h     | err  | R_n | N     | h     | err  | R_n |
|-------|-------|------|-----|-------|-------|------|-----|
| 23    | 0.0500| 1.6728e-11 | 10 | 0.0500 | 2.3973e-10 | 59    | 0.0250 | 2.6135e-13 | 6.0002 | 20 | 0.0250 | 8.7779e-12 | 4.7714 |
| 100   | 0.0167 | 1.5321e-14 | 30 | 0.0167 | 1.2087e-12 | 4.8899 |

From Tables 7 and 8, we see that

\[ |(u - U)(t_n)| = O(h^6) \quad (m = 3), \]

on the quasi-geometric mesh and

\[ |(u - U)(t_n)| = O(h^5) \quad (m = 3), \]

on the uniform mesh.

**Example 2.** Here we solve a DDE with a nonlinear vanishing delay, namely

\[ u'(t) = au(t) + bu(\theta(t)) + f(t), \quad 0 < t \leq 1, \]

\[ u(0) = 1. \tag{38} \]

with \( \theta(t) = \arctan(t) \). We set \( f(t) \) to make the exact solution \( u(t) = e^t \). In initial subinterval \( J_0 = [0, t_0] \), the approximation \( \phi(t) \) of the exact solution \( u(t) \) are provided by Taylor expansion are \( \phi(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \) with \( m = 2 \) and \( \phi(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} \) with \( m = 3 \).

We chose a quasi-graded mesh \( J_h \) with last \( l + 1 \) nodes being assigned equally spaced, \( \tilde{N} = N + 1 = \kappa l + 1 \).

In Tables 9 and 10 we show the error behavior of the piecewise quadratic and cubic CG solutions for (38), with \( a = -2, \ b = 1 \).

Table 9. Errors of the piecewise quadratic CG solution

| l | N     | h     | err  | R | erl | R_t | ern | R_n |
|---|-------|-------|------|---|-----|-----|-----|-----|
| 2 | 259   | 0.0535 | 3.7028e-06 | 7.5049e-08 | 1.3945e-08 |
| 4 | 2081  | 0.0268 | 4.6613e-07 | 2.9969 | 8.1931e-10 | 4.0990 |
| 6 | 7033  | 0.0178 | 1.3843e-07 | 2.9949 | 9.4865e-10 | 3.9883 | 1.5923e-10 | 4.0408 |
| 8 | 16673 | 0.0134 | 5.8468e-08 | 2.9980 | 3.0239e-10 | 3.9769 | 4.8486e-11 | 4.1361 |

Table 10. Errors of the piecewise cubic CG solution

| l | N     | h     | err  | R | erl | R_t | ern | R_n |
|---|-------|-------|------|---|-----|-----|-----|-----|
| 1 | 32    | 0.1065 | 2.2019e-07 | 3.9806e-09 |
| 2 | 259   | 0.0535 | 1.4186e-07 | 3.9856 | 1.2883e-10 | 4.9990 |
| 3 | 877   | 0.0357 | 2.8306e-08 | 3.9863 | 1.7072e-11 | 4.9987 |

The results in Tables 9 and 10 reveal that the convergence orders of the CG solutions for (38) are, respectively,

\[ ||u - U||_\infty = O(h^3), \quad ||(u - U)(t_{nr})|| = O(h^4), \quad ||(u - U)(t_n)|| = O(h^4) \quad (m = 2), \]

and

\[ ||u - U||_\infty = O(h^4), \quad ||(u - U)(t_{nr})|| = O(h^5) \quad (m = 3). \]
Table 11. Errors of the piecewise cubic CG solution on quasi-graded and uniform mesh

| quasi-graded mesh | uniform mesh |
|------------------|--------------|
| $N$   | $h$  | $ern$ | $R_{st}$ | $N$   | $h$  | $ern$ | $R_{st}$ |
| 32    | 0.1065 | 1.078e-10 | 6   | 0.0833 | 4.438e-08 | 6   | 0.0833 | 4.438e-08 |
| 259   | 0.0535 | 7.798e-13 | 11  | 0.0455 | 1.792e-09 | 15  | 0.0333 | 3.967e-10 |
| 877   | 0.0357 | 8.038e-14 | 5.2942 | 5.6200 | 3.967e-10 | 4.8627 |

From Tables 11, we see that

$$|(u - U)(t_n)| = O(h^6) \quad (m = 3),$$
on the quasi-graded mesh and

$$|(u - U)(t_n)| = O(h^5) \quad (m = 3),$$
on the uniform mesh.

**Example 3.** We solve the DDE with nonlinear delay,

$$u'(t) = au(t) + bu(t^2) - a \cos(t) - \sin(t) - b \cos(t^2), \quad 0 < t \leq \frac{1}{2},$$

$$u(0) = 1,$$

with $\theta(t) = t^2$. The exact solution is $u(t) = \cos(t)$. In initial subinterval $J_0 = [0, t_0]$, the approximation of the exact solution $u(t)$ is provided by Taylor expansion are

$\phi(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!}$ with $m = 2$ and $\phi(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}$ with $m = 3$.

We choose quasi-graded meshes $J_h$ with last $l + 1$ nodes ($l = 2, 4, 8, 16, 32, 64$) being chosen equally spaced, $N = N + 1 = kl + 1 (k = \kappa + 1)$.

Figure 1 exhibits the behavior of the error of the piecewise quadratic CG solution for (39). We find that superconvergence occurs at the Lobatto points and the mesh points. In Tables 12 and 13, we illustrate the error behavior of the piecewise quadratic and cubic CG solutions for (39), with $a = -1, b = 0.5$.

![Figure 1. Errors for the CG solution of Example 3](image)

Table 12. Errors of the piecewise quadratic CG solution

| $l$ | $N$   | $h$   | $err$   | $R_{el}$ | $R_{st}$ |
|-----|-------|-------|---------|----------|----------|
| 8   | 25    | 0.0156| 1.142e-07| 1.659e-10| 1.305e-10|
| 16  | 49    | 0.0078| 1.445e-08| 2.9824    | 3.9916   |
| 32  | 97    | 0.0039| 1.817e-09| 2.9913    | 3.9605   |
| 64  | 257   | 0.0019| 2.279e-10| 2.9954    | 3.0334   | 3.197e-14| 3.9396   |
Table 13. Errors of the piecewise cubic CG solution

| l  | N  | h      | err N | R  | err R | err Rl | err Rn |
|----|----|--------|-------|----|-------|--------|--------|
| 1  | 3  | 0.1250 | 1.8848e-06 | 5.4469e-08 | 5.1176e-10 |
| 3  | 7  | 0.0417 | 2.4047e-08 | 3.9700 | 2.2334e-10 | 5.0033 | 7.8237e-13 | 5.9013 |
| 5  | 16 | 0.0250 | 3.1325e-09 | 3.9900 | 1.7334e-11 | 5.0037 | 4.2411e-14 | 5.7063 |
| 7  | 22 | 0.0179 | 8.1709e-10 | 3.9939 | 3.2190e-12 | 5.0037 | 5.5511e-15 | 6.0433 |

As the Tables 12 and 13 show, the convergence orders of CG solution for (39) are respectively given by

\[ ||u - U||_\infty = O(h^3), \quad |(u - U)(t_{nr})| = O(h^4), \quad |(u - U)(t_n)| = O(h^4) \quad (m = 2), \]

and

\[ ||u - U||_\infty = O(h^4), \quad |(u - U)(t_{nr})| = O(h^5), \quad |(u - U)(t_n)| = O(h^6) \quad (m = 3). \]

This illustrates the correctness of the theory.

6. Concluding remarks. The following three problems remain to be addressed in future research work.

- Analysis of the attainable order of local superconvergence of the DG method for pantograph-type DDEs under geometric meshes.
- Postprocessing of the CG solutions for pantograph-type DDEs under geometric meshes.
- Postprocessing of the DG solutions for pantograph-type DDEs under geometric meshes.

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REFERENCES

[1] A. Bellen, Preservation of superconvergence in the numerical integration of delay differential equations with proportional delay, *IMA J. Numer. Anal.*, 22 (2002), 529–536.

[2] A. Bellen, H. Brunner, S. Maset and L. Torelli, Superconvergence in collocation methods on quasi-graded meshes for functional differential equations with vanishing delays, *BIT Numer. Math.*, 46 (2006), 229–247.

[3] A. Bellen, N. Guglielmi and L. Torelli, Asymptotic stability properties of \( \theta \)-methods for the pantograph equation, *Appl. Numer. Math.*, 24 (1997), 279–293.

[4] A. Bellen and M. Zennaro, *Numerical Methods for Delay Differential Equations*, Oxford University Press, Oxford, 2003.

[5] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, Cambridge, 2004.

[6] H. Brunner, Q. Hu and Q. Lin, Geometric meshes in collocation methods for Volterra integral equations with proportional delays, *IMA J. Numer. Anal.*, 21 (2001), 783–798.

[7] H. Brunner, Q. Huang and H. Xie, Discontinuous Galerkin methods for delay differential equations of pantograph type, *SIAM. J. Numer. Anal.*, 48 (2010), 1944–1967.

[8] H. Brunner and H. Liang, Stability of collocation methods for delay differential equations with vanishing delays, *BIT Numer. Math.*, 50 (2010), 693–711.

[9] C. Chen, *Structure Theory of Superconvergence of Finite Elements*, Hunan Press of Science and Technology, Changsha, 2001 (in Chinese).

[10] M. Delfour, W. Hager and F. Trochu, Discontinuous Galerkin methods for ordinary differential equations, *Math. Comp.*, 36 (1981), 455–473.

[11] K. Deng, Z. Xiong and Y. Huang, The Galerkin continuous finite element method for delay-differential equation with a variable term, *Appl. Math. Comput.*, 186 (2007), 1488–1496.
[12] Q. Huang, H. Xie and H. Brunner, Superconvergence of discontinuous Galerkin solutions for delay differential equation of pantograph type, *SIAM. J. Sci. Comput.*, 33 (2011), 2664–2684.

[13] A. Iserles, On the generalized pantograph functional-differential equation, *European J. Appl. Math.*, 4 (1993), 1–38.

[14] T. Koto, Stability of Runge-Kutta methods for the generalized pantograph equation, *Numer. Math.*, 84 (1999), 233–247.

[15] D. Li and C. Zhang, Superconvergence of a discontinuous Galerkin Method for first-order linear delay differential equations, *J. Comput. Math.*, 29 (2011), 574–588.

[16] Y. Liu, On the \( \theta \)-method for delay differential equations with infinite lag, *J. Comput. Appl. Math.*, 71 (1996), 177–190.

[17] N. Takama, Y. Muroya and E. Ishiwata, On the attainable order of collocation methods for delay differential equations with proportional delay, *BIT Numer. Math.*, 40 (2000), 374–394.

[18] X. Xu and Q. Huang, Continuous Galerkin methods for delay differential equations of pantograph type, *Math. Pract. Theory*, 24 (2014), 280–288 (in Chinese).

[19] X. Xu, Q. Huang and H. Chen, Local superconvergence of continuous Galerkin solutions for delay differential equations of pantograph type, *J. Comput. Math.*, 34 (2016), 186-199.

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E-mail address: qmhuang@bjut.edu.cn
E-mail address: xuxiuxiu@emails.bjut.edu.cn
E-mail address: hbrunner@math.hkbu.edu.hk