Testing quantum correlations with generalized quasi-probabilities under uncontrollable noises

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We introduce a method for testing quantum correlations in terms of generalized quasi-probability distributions when uncontrollable noises are involved in the measuring devices as well as in the system prepared. The order parameter of the generalized quasi-probability functions provides distributed matching values to the photon number statistics, as it modifies the bosonic operator ordering. Our formalism provides a noise adaptive quasi-probability function by a disproportional assignment of the values to the photon number distribution under the effect of noises. Remarkably, using the formalism, it is possible to observe quantum correlations of continuous variable systems in the presence of severe uncontrollable noises. Our scheme provides a useful tool to test quantum correlations in various protocols with near-term noisy quantum information processors with continuous variable systems.

I. INTRODUCTION

Verifying quantum correlation in continuous variable (CV) system is an important issue in the field of quantum technology. Quantum correlation is a crucial ingredient for a building block of quantum information processing such as quantum computation [1], quantum communications [2–5], and quantum cryptography [6]. One of the most important challenges towards building a large scale quantum network [7–11] is the establishment of quantum correlations between distant nodes. An efficient identification of quantum correlation, that lies outside of classical counter part, is thus essential in developing scalable quantum architectures, which is known to require special treatments with non-trivial process in its measurements especially when the CV system is exploited [12–14].

CV quantum states can be represented in a couple of different ways: the wave functional description has been conventionally used while the normalized positive operators, called density matrix, can also provide its corresponding representation. At the same time, in quantum optics, it has been well-known that a quasi-probability distribution can also be used as the equivalent description of density matrices [15–17]. The phase space representation with quasi-probabilities such as the Glauber-Sudarshan P-function [18, 19], Wigner function [20], and Husimi Q-function [21] can demonstrate the nonclassical features of quantum states, which are incompatible with classical counterparts, in a rather explicit manner. Quantum correlation in the multi-mode quantum systems can also be found when the non-classical character of the multi-mode distribution has been extended to the correlated systems [22]. In this framework, negativity [23, 24], non-locality [25, 26], contextuality [27, 28], entanglement [29, 30] and coherence [31] of quantum states have been studied, which provide useful resources for quantum information processing [32–37].

A direct measurement of quasi-probabilities from the state has been an important issue [38–43], while the probability function can be measured typically by employing the homodyne detection for the tomographic reconstruction [44–48]. If the field can be confined in a cavity, the quasi-probability function is directly measurable by the atom-field interaction as it is described in cavity quantum electrodynamic systems [42, 43]. Recent progress of the photon number resolving detectors [49], e.g., based on superconducting circuits [50–52] enhances the possibility to directly measure the quasi-probability of quantum states and their non-classical features.

However, all these measurement scheme are not quite perfect and very sensitive to the existing noise on the measured systems and the measuring components. When the imperfection of the measuring devices become significant, quantum characteristics of the system can disappear sharply. Coarse-graining measurement [53] has been regarded as a reason why one can not detect quantum effects at macroscopic scale. Conditions for this hypothesis were suggested to confirm its validity in macro-realismin [54], saying that coarse-graining measurement and classical Hamiltonian are responsible for the emergence of classicality out of quantum world. The possibility of detection of quantum correlation with extremely coarse-graining measurement was reported [55], but has been also understood in this hypothesis without contradiction because the local measurement used in Ref. [55] requires Kerr nonlinearity, implicating another requirement of precision measurements [56]. A recent observation [56] that micro-macro entanglement is hard to detect with coarse-graining measurement has strengthen the validity of hypothesis.

Here we introduce a method to detect quantum correlations with Bell-type inequality tests, in terms of generalized quasi-probability distributions when uncontroll-
lable noises are involved in the measuring devices as well as in the system prepared. Our formalism provides a noise adaptive quasi-probability function when the noise source is identified. In this framework, a violation of the proposed Bell-type inequalities formulated with quasi-probability functions is a direct indication of the existence of quantum correlation in the states. Remarkably, it allows us to observe quantum correlations of CV systems even in the presence of severe noises included in the systems or detectors. Our work provides a useful tool to test quantum correlations in various quantum information protocols in near-term noisy devices.

II. GENERALIZED QUASI-PROBABILITY DISTRIBUTIONS

We start with reviewing the generalized representation of the quasi-probability distributions. Note that, in the quantum version of phase space formalism, the whole space can be spanned with a complete basis called generalized parity operator [15–17]

\[ \hat{\Pi}(\alpha; s) = \frac{1}{1-s} \sum_{n=0}^{\infty} \left( \frac{s+1}{s-1} \right)^n |\alpha,n\rangle \langle \alpha,n| . \]  

(1)

Here, |\alpha,n\rangle is the displaced number basis produced by applying the Glauber displacement operator \( \hat{D}(\alpha) \) to the number basis |\alpha,n\rangle = \hat{D}(\alpha) |n\rangle with a complex variable \( \alpha \). Its eigenvalues \( (s+1)/(s-1))^n/(1-s) \) are defined with one real parameter \( s \). Then, the generalized quasi-probability distribution of a quantum state \( \hat{\rho} \) in phase space can be obtained by the expectation value of \( \hat{\Pi}(\alpha; s) \) as,

\[ W(\alpha; s) = \frac{2}{\pi} \text{Tr}[\hat{\rho} \hat{\Pi}(\alpha; s)] , \]

(2)

which is so called the \( s \)-parameterized quasi-probability distribution function. This is a unified form of quasi-probability distribution functions with different orders parameter \( s \): (i) If \( s \) tends to one from the left, it becomes the Glauber-Sudarshan P-function [18, 19],

\[ P(\alpha) = \lim_{s \to 1} - \frac{2}{\pi} \text{Tr}[\hat{\rho} \hat{\Pi}(\alpha; s)] . \]

(3)

Note that all the eigenvalues become infinity as \( s \to 1 \), representing the singularity of the P-function [15, 16].

(ii) If we set \( s = 0 \), \( \hat{\Pi}(\alpha; 0) = \sum_{n=0}^{\infty} (-1)^n |\alpha,n\rangle \langle \alpha,n| = \hat{D}(\alpha)(-1)^n \hat{D}(\alpha) \), and the Wigner function [20]

\[ W(\alpha) = \frac{2}{\pi} \text{Tr}[\hat{\rho} \hat{\Pi}(\alpha; 0)] = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \langle \alpha,n| \rho |\alpha,n\rangle \]

(4)

is obtained. (iii) If \( s = -1 \), \( \hat{\Pi}(\alpha; -1) = |\alpha\rangle \langle \alpha| \) so that the Eq. (2) becomes the Husimi Q-function [21] as

\[ Q(\alpha) = \frac{1}{\pi} \text{Tr}[\hat{\rho} \hat{\Pi}(\alpha; -1)] = \frac{1}{\pi} \langle \alpha| \rho |\alpha\rangle . \]

(5)

In general, a quasi-probability distribution \( W(\alpha; s') \) can be regarded as a smoothed quasi-probability distribution of \( W(\beta; s) \) with an order parameter \( s > s' \). This can be represented by the convolution of \( W(\beta; s) \) and a Gaussian weight [15, 16]

\[ W(\alpha; s') = \frac{2}{\pi(s-s')} \int d^2\beta \ W(\beta; s) \exp \left( -\frac{2|\alpha-\beta|^2}{s-s'} \right) . \]

(6)

As the effects of noises in phase space can be modeled by Gaussian smoothing [39–41, 44–48], the decreasing \( s' \) is often considered as a loss of non-classicality. For example, as \( s' \) decreases the negativity of \( W(\beta; s) \) is reduced, and the expectation value becomes non-negative everywhere in phase space when it becomes the Husimi Q-function \( s' = -1 \). This can be understood as the smoothing of \( W(\beta; s) \) over the area satisfying the Heisenberg minimum uncertainty, associated with the ideal simultaneous measurements of position and momentum. Due to the non-negativity of the expectation value in whole phase space as well as the possibility of the projection of quantum states on the coherent state basis \( |\alpha\rangle \), the Husimi Q-function can be often interpreted as a proper probability distribution. Note that the coherent states are overlapping each other and comprise the over-complete basis in phase space.

The eigenvalues of the generalized parity operator in Eq. (1) are bounded when \( s \leq 0 \), while they diverge when \( s > 0 \). For example, +1 and −1 are the degenerate eigenvalues when \( s = 0 \), and the change of eigenvalues with \( n \) is plotted in Fig. 1 when \( s = -0.1 \). We intend to use the generalized parity operator \( \hat{\Pi}(\alpha; s) \) as an observable to test quantum correlations so that we shall focus on the non-positive \( s \) region in the later part of this paper. Then, the \( s \)-parameterized quasi-probability distributions \( W(\alpha; s) \) as the expectation values of \( \hat{\Pi}(\alpha; s) \) cover from the Wigner function with \( s = 0 \) to the Husimi Q-function with \( s = -1 \).

III. THE EFFECT OF DETECTION NOISES

A noisy quantum measurement can be understood based on the notion of coarse-graining operation [57], which transforms a probability density in phase space
Note that the relation in Eq. (10) is generally valid for any reconstruction method of the quasi-probability distribution. For example, the result is consistent with the analysis of the noise effects when reconstructing the quasi-probability distribution by homodyne measurements [44–48].

IV. EFFECTS OF ENVIRONMENTAL NOISE

We start with the introduction of the convolution law of the quasi-probability distribution functions [58]. Let us consider a beam splitter with the transmissivity $t$ and reflectivity $r$ satisfying $r^2 + t^2 = 1$. The P-function $(s = 1)$ of one output mode (denoted by mode $d$) is the simple convolution of the two input modes (denoted by $a$ and $b$ modes) [15, 16] so that their characteristic functions are in the convolution relation as $\chi_d(\alpha; s = 1) = \chi_a(\alpha; s = 1) \chi_b(\alpha; s = 1)$. As the generalized characteristic function of a quantum state $\hat{\rho}$ with a parameter $s$ can be written by $\chi(\alpha; s) = \text{Tr}[\hat{\rho}\exp(\alpha a^\dagger - \alpha^* a)]\exp(s|\alpha|^2/2) = \chi(\alpha; 1)\exp((s - 1)|\alpha|^2/2)$, we can obtain the convolution law for the generalized characteristic functions,

$$\chi_d(\alpha; s) = \chi_d(\alpha; 1)\exp\left(\frac{s - 1}{2}|\alpha|^2\right) = \chi_a(\alpha; r; 1)\exp\left(\frac{s - 1}{2}|\alpha|^2\right) \times \chi_b(\alpha; t; 1)\exp\left(\frac{s - 1}{2}|\alpha|^2\right) = \chi_d(\alpha; r; s)\chi_b(\alpha; t; s).$$

Therefore, we can arrive at the convolution law for the generalized quasi-probability distributions as

$$W_d(\alpha; s) = \frac{1}{t^2} \int d^2\beta W_a(\beta; s)W_b\left(\alpha - \frac{r\beta}{t}; s\right).$$

Let us now consider the effects of environmental noises. Since dissipation induced by interactions with environment tends to smoothed quasi-probability distributions, we shall see that the state evolution under environmental noises can be effectively described as dynamical changes of the order parameter $s$ in this section. Note that such an attenuated dynamics can be also understood within the framework of noisy measurements [59], in agreement with the description in the previous section.

As an exemplary model, suppose that a quantum system encounters and interacts with thermal environment regarded as a reservoir. The effect of the reservoir can be modeled by mixture of the mode for the system and the mode of thermal fields by a beam splitter. The evolution of the quasi-probability distribution can be described by solving the Fokker-Planck equation [58],

$$\frac{\partial W(\alpha; s; \tau)}{\partial \tau} = \left[ \frac{\nu}{2} \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha^*} \alpha^* + \frac{\partial^2}{\partial \alpha \partial \alpha^*} \right] W(\alpha; s; \tau).$$

\hspace{1cm} (13)
We then obtain time evolution at time $\tau$ given by the convolution of the original field and thermal environment like Eq. (12) as

$$W(\alpha; s; \tau) = \frac{1}{t^2(\tau)} \int d^2\beta W^0(\beta; s) W\left(\alpha - r(\tau)\beta/\tau; s; 0\right),$$

where $r(\tau) = \sqrt{1 - e^{-\gamma \tau}}$ and $t(\tau) = e^{-\gamma \tau}$ are given in terms of the energy decay rate $\gamma$ and

$$W^0(\beta; s) = \frac{2}{\pi(1 + 2n - s)} \exp\left(-\frac{2|\beta|^2}{1 + 2n - s}\right)$$

is the generalized quasi-probability function for the thermal state with mean photon number $\bar{n}$. By rescaling with respect to $\beta' = r(\tau)\beta/\tau(\tau)$ and $\alpha' = \alpha/t(\tau)$, Eq. (14) can be recast into

$$\frac{2}{\pi(1 + 2n - s)r(\tau)^2} \int d^2\beta' W(\beta'; s) \exp\left(-\frac{2t(\tau)^2|\alpha' - \beta'|^2}{(1 + 2n - s)r(\tau)^2}\right).$$

From Eq. (6), the effect of the thermal environment can then be identified with temporal changes of the quasi-probability distribution as

$$W(\alpha; s; \tau) = \frac{1}{t^2(\tau)} W\left(\frac{\alpha}{t(\tau)}; s'; \tau\right),$$

with the parameter function,

$$s'(\tau) = \frac{s - r^2(\tau)(1 + 2\bar{n})}{t^2(\tau)}.$$

Therefore, the evolution of quasi-probability distribution under environmental noise can be effectively described by dynamical changes of the order parameter $s$.

V. BELLO INEQUALITIES WITH QUASI-PROBABILITY DISTRIBUTIONS

In the following we shall propose an efficient formalism to test quantum correlations in phase space under noises based on the results of the previous sections. Suppose that we have a two-mode system that we would like to test and its $s$-parameterized quasi-probability distribution can be reconstructed by measurements. Let us first introduce a Bell-type inequality [60] formulated by the generalized quasi-probability distribution functions.

From the fact that the generalized parity operator in Eq. (1) has the bounded eigenvalues with a non-positive $s$, we will use $\tilde{\Pi}(\alpha; s)$ to define the observable for testing quantum correlations. We define the effective observable operator in the form of

$$\hat{O}(\alpha; s) = X(s)\tilde{\Pi}(\alpha; s) + Y(s)\mathbb{I},$$

where $X(s)$ and $Y(s)$ are arbitrary functions of the parameter $s$ in the region $-1 \leq s \leq 0$, and $\mathbb{I}$ is the identity operator. In that circumstance, the eigenvalue spectrum of the measurement operator is given as

$$e_n(s) = \frac{X(s)(s + 1)^n}{1 - s} + Y(s),$$

which takes the clear maximum when $n = 0$ with the value $e_0(s) = X(s)/(1 - s) + Y(s)$. The minimum is given as $e_1(s) = -(s + 1)X(s)/(1 - s)^2 + Y(s)$ when $n = 1$ if $X(s) > 0$.

The general requirement on this operator is

$$|\langle \hat{O}(\alpha; s) \rangle| \leq 1.$$  

Therefore, the conditions we can take are $X(s)/(1 - s) + Y(s) = 1$ and $-(s + 1)X(s)/(1 - s)^2 + Y(s) = -1$ by which we arrive at a solution $X(s) = (1 - s)^2$ and $Y(s) = s$ so that the operator has the form

$$\hat{O}(\alpha; s) = (1 - s)^2\tilde{\Pi}(\alpha; s) + s\mathbb{I},$$

with eigenvalues

$$e_n(s) = (1 - s)\left(\frac{s + 1}{s - 1}\right)^n + s.$$  

Note that when $s = 0$ it becomes $\hat{O}(\alpha; 0) = \tilde{\Pi}(\alpha; 0) = \sum_{n=0}^{\infty}(-1)^n|\alpha, n\rangle\langle\alpha, n|$, and when $s = -1$, $\hat{O}(\alpha; -1) = 2|\alpha\rangle\langle\alpha| - \mathbb{I}$. We can thus recover the Wigner and Husimi Q functions for $s = 0$ and $-1$, respectively.

Now we formulate a Bell-type inequality in terms of the generalized quasi-probability distributions. Suppose that two local parties choose observables, $\hat{A}_i$ and $\hat{B}_i$, respectively, where $a, b \in \{1, 2\}$. The measurement operators of the local observables are defined here as

$$\hat{A}_a = \hat{O}(\alpha_a; s), \quad \hat{B}_a = \hat{O}(\beta_a; s),$$

where $-1 \leq s \leq 1$, and the Bell operator can be then constructed in a similar way with the CHSH-type [61] as

$$\mathcal{B} = \hat{A}_1 \otimes \hat{B}_1 + \hat{A}_1 \otimes \hat{B}_2 + \hat{A}_2 \otimes \hat{B}_1 - \hat{A}_2 \otimes \hat{B}_2.$$  

For any separable states $\rho = \sum_i P_i \rho_i \otimes \rho_i$ with $\sum_i P_i = 1$, $\langle \hat{A}_a \otimes \hat{B}_b \rangle = \text{Tr}[\hat{\rho}\hat{A}_a \otimes \hat{B}_b] = \sum_s P_s \text{Tr}[\hat{\rho}_s \hat{A}_a \otimes \hat{B}_b] \propto \sum_i P_i (\hat{A}_i \otimes \hat{B}_i) + (\hat{A}_1 \otimes \hat{B}_2 + \langle \hat{A}_2 \rangle) (\hat{B}_1 - (\hat{A}_2 \otimes \hat{B}_2)).$ Since, in our case, the expectation values of the local observables are bounded by $|\langle \hat{A}_a \rangle| < 1$ and $|\langle \hat{B}_b \rangle| \leq 1$ for any non-positive $s$, the expectation value of the Bell operator is bounded as $|\langle \mathcal{B} \rangle| \equiv |\mathcal{B}| \leq 2$ by separable states.

Finally, we can write the Bell-type inequality based on the generalized quasi-probability distributions as
for $-1 \leq s \leq 1$, where $W(\alpha, \beta; s) = (4/\pi^2)</s}\langle \Pi(\alpha; s \otimes \Pi(\beta; s)) = \text{is the two-mode quasi-probability distribution functions and } W(\alpha; s) \text{ and } W(\beta; s) \text{ are its marginal distributions. Therefore, the violation of the inequality (25) guarantees that the state is entangled. Note that this is not a test of quantum non-locality, which has a different criterion in terms of a local realistic theory [60–62], but a method for witnessing quantum correlations of CV systems in phase space.}

VI. TESTING QUANTUM CORRELATIONS UNDER NOISES

In this section we apply the Bell-type inequalities with quasi-probability distributions to test quantum correlations under noises. We shall show that efficient certification of quantum correlations is possible under severe detection or environmental noises.

As a representative example, we consider the two-mode squeezed vacuum state (TMSV)

$$|\text{TMSV} \rangle = \sum_{n=0}^{\infty} \frac{\tan^a \xi}{\cosh \xi} |n, n\rangle,$$

with a squeezing parameter $\xi > 0$, which can be generated, e.g., by non-degenerate optical parametric amplifiers [63]. TMSV has been often considered to be the normalized EPR states, i.e., the maximally entangled state associated with position and momentum [25]. For a non-positive parameter $s$, its generalized quasi-probability distribution function is given by

$$W(\alpha, \beta; s) = \frac{4}{\pi^2 R(s)} \exp \left( -\frac{2}{R(s)} S(s)(|\alpha|^2 + |\beta|^2) + \sinh 2\xi (\alpha^{*}\beta + \alpha \beta^{*}) \right),$$

where $R(s) = s^2 - 2s \cosh 2\xi + 1$ and $S(s) = \cosh 2\xi - s$, and its marginal single-mode distribution is $W(\alpha; s) = (2/\pi S(s)) \exp[-2|\alpha|^2/S(s)].$

A. Quantum correlations under detection noises

Let us first consider a test of quantum correlations under detection noises. Assuming that the detection efficiencies in two modes are the same as $\eta$, the reconstructed $s$-parameterized quasi-probability distributions under detection noises are given by Eq. (9) as

$$W_{\eta}(\alpha, \beta; s) = \frac{1}{\eta} W(\alpha, \beta; s^\prime)$$

$$W_{\eta}(\alpha; s) = \frac{1}{\eta} W(\alpha; s^\prime).$$

Therefore, the Bell function for the reconstructed quasi-probability distributions and the rescaled order parameter $s^\prime = s/\eta + (1 - 1/\eta)$ is given as

$$|\mathcal{B}(s^\prime)| = \left| \frac{\pi^2 (1 - s^\prime)^4}{4} [W(\alpha_1, \beta_1; s^\prime) + W(\alpha_1, \beta_2; s^\prime)$$

$$+ W(\alpha_2, \beta_1; s^\prime) - W(\alpha_2, \beta_2; s^\prime)] + \pi s^\prime (1 - s^\prime)^2$$

$$\times [W(\alpha_1; s^\prime) + W(\beta_1; s^\prime)] + 2s^2 \right|$$

$$= \left| \frac{\pi^2 (1 - s^\prime)^4 \eta^2}{4} [W_{\eta}(\alpha_1, \beta_1; s) + W_{\eta}(\alpha_1, \beta_2; s)$$

$$+ W_{\eta}(\alpha_2, \beta_1; s) - W_{\eta}(\alpha_2, \beta_2; s)] + \pi s^\prime (1 - s^\prime)^2 \eta$$

$$\times [W_{\eta}(\alpha_1; s) + W_{\eta}(\beta_1; s)] + 2s^2 \right| \leq 2,$$
cosh 2 as the measurement operator in Eq. (21) becomes the photon on-off detection when \( s' = -1.0 \). As increasing \( \xi \), a narrower peak of violations appears at the region \( s = 0 \) and \( \eta = 1 \), which is the detection of the entanglement between multiple photons of two modes. The measurement operator in Eq. (21) becomes the parity operator when \( s' = 0 \). Note that the parity measurement can detect the correlation between higher number of photons than the on-off measurement but is more fragile under detection noises.

**B. Dynamical quantum correlations under thermal environment**

Let us then investigate the dynamic behavior of quantum correlations of TMSV under thermal environmental noises. We assume that the effects of thermal noises are independent in two modes with the same energy decay rate \( \gamma \) and average thermal photon number \( \bar{n} \) and identified before testing. Therefore, using the relations of the rescaled quasi-probability distributions in Eq. (16) and the order parameters in Eq. (17), the evolution of the quasi-probability distributions of TMSV can be represented by

\[
W(\alpha, \beta; s, \tau) = \frac{1}{t^4(\tau)} W\left(\frac{\alpha}{t(\tau)}, \frac{\beta}{t(\tau)}; s'(\tau); 0\right),
\]

where

\[
W\left(\frac{\alpha}{t(\tau)}, \frac{\beta}{t(\tau)}; s'(\tau); 0\right) = \frac{1}{\pi t^2(\tau) s'(\tau)} \exp\left(-\frac{2|\alpha|^2}{t^2(\tau)} + 2|\alpha|^2 + |\beta|^2 + \sinh 2\xi \frac{\alpha^* \beta + \alpha \beta^*}{t^2(\tau)}\right).
\]

Here, \( R'(s, \tau) = s'(\tau)^2 - 2s'(\tau) \cosh 2\xi + 1 \) and \( S'(s, \tau) = \cosh 2\xi - s'(\tau) \) with the parameters \( s'(\tau) = (s - r^2(\tau))(1 + 2\bar{n})/t^2(\tau) \) and \( r(\tau) = \sqrt{1 - e^{-\gamma \tau}} \) and \( t(\tau) = \sqrt{e^{-\gamma \tau}} \).

Therefore, we can set the Bell inequality by rescaling with the dynamically changing parameters \( \alpha' = \alpha/t(\tau), \beta' = \beta/t(\tau), \) and \( s'(\tau) \) as

\[
|B(s'(\tau))| = \left| \frac{\pi(1 - s'(\tau))^2}{4} \left[ W(\alpha_1', \beta_1'; s'(\tau)) + W(\alpha_2', \beta_1'; s'(\tau)) + W(\alpha_1', \beta_2'; s'(\tau)) + W(\alpha_2', \beta_2'; s'(\tau)) + W(\alpha_1', \beta_1'; s'(\tau)) + W(\alpha_2', \beta_2'; s'(\tau)) \right] + \pi s'(\tau)(1 - s'(\tau))^2 W(\alpha_1', s'; \tau) + W(\beta_1', s'; \tau) + 2s'(\tau)^2 \right|
\]

\[
= \left| \frac{\pi(1 - s'(\tau))^2}{4} \left[ W(\alpha_1, \beta_1; s; \tau) + W(\alpha_2, \beta_1; s; \tau) + W(\alpha_1, \beta_2; s; \tau) + W(\alpha_2, \beta_2; s; \tau) + W(\alpha_1, \beta_1; s; \tau) + W(\alpha_2, \beta_2; s; \tau) \right] + \pi s'(\tau)(1 - s'(\tau))^2 W(\alpha_1; s; \tau) + W(\beta_1; s; \tau) + 2s'(\tau)^2 \right| \leq 2.
\]

In Fig. 4, we plot the dynamics of quantum correlations of TMSV detected by the Bell inequality in Eq. (34) under thermal environment. It is remarkable that our method successfully keeps detecting the quantum correlations of TMSV (\( \xi = 0.3 \)) for long time, for example, up to the dimensionless time \( r(\tau) \sim 0.8, 0.7, 0.5 \) for...
dimensional quantum correlations. The measurement of dimensionless binning with eigenvalues
the thermal environments with average photon number \( \bar{n} = 0, 0.5, 2 \), respectively. This is much longer than
the allowed time for observing entanglement of the same TMSV (\( \xi = 0.3 \)) before death by previous schemes, e.g.
\( r(\tau) \sim 0.35, 1.3, 0.6 \) for \( \bar{n} = 0, 0.5, 2 \), respectively, given in Ref. [64].

VII. REMARKS

We can generalize our method further for testing high-dimensional quantum correlations. The measurement of
\( \Pi(\alpha; s) \) for \( s \leq 0 \) can be associated with a number resolving detection under noises and subsequent dichotomic
(2-dimensional) binning with eigenvalues \( \pm 1 \). Similarly, we can map the number \( n \) into the discretized phases by
\( \omega = \exp(2\pi i/d) \) for the measurement with arbitrary \( d \) outcomes. The eigenvalues are assigned as complex variables
\( \omega^n \). When \( d = 2 \), \( \omega^n = (-1)^n \). Therefore, the generalized quasi-probability function with \( d \)-dimensional outcomes
\( W(\alpha; s_d) \) can be defined as

\[
W(\alpha; s_d) = \frac{2}{\pi(1 - s_d)} \sum_{n=0}^{\infty} \omega^n |\alpha, n\rangle \langle \alpha, n|.
\]

as the expectation value of the generalized parity operator

\[
\hat{\Pi}(\alpha; s_d) = \frac{1}{1 - s_d} \sum_{n=0}^{\infty} (s_d + 1 - s_d - 1)^n |\alpha, n\rangle \langle \alpha, n|.
\]

with a complex order parameter \( s_d = -i \cot(\pi/d) \). Note that Eq. (35) becomes equivalent with Eq. (2) for \( d = 2 \).

The measurement process observing the operator in Eq. (36) can be associated with the number resolving detection and subsequent binning into the complex eigenvalues \( \omega^n \) in order. By using Eq. (35), different type of Bell inequalities can be tested with arbitrary \( d \)-outcome local measurements [65–69]. For example, see the results in Ref. [70, 71]. We can rewrite the \( d \)-dimensional quasi-probability function under noises as,

\[
W_{\eta}(0; s_d) = \frac{2}{\pi(1 - s_d)} \sum_{n=0}^{\infty} (1 - \eta + \eta \omega)^n P(n) \equiv \frac{W(0; s'_d)}{\eta},
\]

by which it would be possible to observe high-dimensional quantum correlations under noises likewise the method proposed in this paper when \( d = 2 \) (we left the detailed analysis for a future study). Note that the relation in Eq. (10) is also valid here as

\[
1 - s'_d \equiv \frac{1 - s_d}{\eta}.
\]

It would be also valuable to extend this method for testing multi-mode quantum correlations of CV systems in phase space [72–74].

Our result implicates that quantum characteristics of the system do not sharply disappear by coarse-graining but may be hidden behind the effect of noises. Our work clearly shows that it is possible to directly observe quantum correlations with coarse-graining measurements by harnessing the knowledge on the imperfection of the measuring devices or the measured systems. This approach would help us to explore the border between classicality and quantumness at the macroscopic scale and provides us a way to circumvent the difficulty in observing quantum nature of complex systems.

In summary, we have introduced a method for detecting quantum correlations in terms of generalized quasi-probability distribution functions in the presence of uncontrollable noises involved in the measuring devices or the system prepared. The parameter for the different quasi-probabilities provides distributed matching values to the photon number statistics as it modifies the bosonic operator ordering. Our formalism provides the noise adaptive quasi-probability distribution functions. Using the formalism, it is possible to observe quantum correlations of CV systems in an adaptive manner when the different amount of noise exists in the systems as well as in the detectors. Remarkably, the proposed method allows us to detect quantum correlations under severe noises. It has been also shown that the critical point of the correlation is varied with respect to the noise added.
to the systems even when the order parameter for the correlation is adjusted.

Our scheme can be directly applicable to detect quantum correlations with inefficient detectors or in the systems under noises. An immediate experimental demonstration of our scheme is expected with current optical technologies. We believe that our work provides a useful tool to certify quantum correlations in various protocols in near-term noisy quantum information processors with CV systems.

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[1] M. Nielsen and I. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2000).
[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[3] A. Furusawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science 282, 702–706 (1998).
[4] S. Pirandola, J. Eisert, C. Weedbrook, A. Furusawa, and S. L. Braunstein, Nature Photonics 9, 641–652 (2015).
[5] B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau, and R. Hanson, Nature 526, 682 (2015).
[6] A. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[7] H. J. Kimble, Nature, 453, 1023 (2008).
[8] J. Yin, Y. Cao, Y. Li, J. Ren, S. Liao, L. Zhang, W. Cai, W. Liu, B. Li, H. Dai, M. Li, Y. Huang, L. Deng, L. Li, Q. Zhang, N. Liu, Y. Chen, C. Lu, R. Shu, C. Peng, J. Wang, and J. Pan, Phys. Rev. Lett. 119, 200501 (2017).
[9] S. Wehner, D. Elkouss, and R. Hanson, Science 362, 9288 (2018).
[10] S.-W. Lee, T. C. Ralph, and H. Jeong, Phys. Rev. A 100, 052303 (2019).
[11] S. M. Lee, S.-W. Lee, H. Jeong, and H. S. Park, Phys. Rev. Lett. 124, 060501 (2020).
[12] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. 77, 513 (2005).
[13] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. 84, 621 (2012).
[14] G. Adesso, S. Ragy, and A. R. Lee, Open Syst. Inf. Dyn. 21, 1440001 (2014).
[15] K. E. Cahill and R. J. Glauber, Phys. Rev. 177, 1857 (1969).
[16] K. E. Cahill and R. J. Glauber, Phys. Rev. 177, 1882 (1969).
[17] H. Moya-Cessa and P. L. Knight, Phys. Rev. A 48, 2479 (1993).
[18] R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[19] E. C. G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963).
[20] E. Wigner, Phys. Rev. 40, 749 (1932).
[21] K. Husimi, Proc. Phys. Math. Soc. Jpn. 22, 264 (1940).
[22] M. S. Kim, W. Son, V. Bužek, and P. L. Knight, Phys. Rev. A 65, 032323 (2002).
[23] J. P. Dahl, H. Mack, A. Wolf, and W. P. Schleich, Phys. Rev. A 74, 042323 (2006).
[24] M. Walschaers, C. Fabre, V. Parigi, and N. Treps, Phys. Rev. Lett. 119, 183601 (2017).
[25] K. Banaszek and K. Wódkiewicz, Phys. Rev. A 58, 4345 (1998); Phys. Rev. Lett. 82, 2009 (1999); Acta Phys. Slovaca 49, 491 (1999).
[26] S.-W. Lee, H. Jeong, and D. Jaksch, Phys. Rev. A 80, 022104 (2009).
[27] R. W. Spekkens, Phys. Rev. Lett. 101, 020401 (2008).
[28] A. Asadian, C. Budroni, F. E. S. Steinhoff, P. Rabl, and O. Gühne, Phys. Rev. Lett. 114, 250403 (2015).
[29] J. Sperling and W. Vogel, Phys. Rev. A 79, 042337 (2009).
[30] J. Sperling, E. Meyer-Scott, S. Barkhofen, B. Brecht, and C. Silberhorn, Phys. Rev. Lett. 122, 053602 (2019).
[31] J. Sperling and I. A. Walmsley, Phys. Rev. A 97, 062318 (2018).
[32] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, New J. Phys. 14, 113011 (2012).
[33] H. Pashayan, J. J. Wallman, and S. D. Bartlett, Phys. Rev. Lett. 115, 070501 (2015).
[34] S. Rahimi-Keshari, T. C. Ralph, and C. M. Caves, Phys. Rev. X 6, 021039 (2016).
[35] F. Shahandeh, A. P. Lund, and T. C. Ralph, Phys. Rev. Lett. 119, 120502 (2017).
[36] J. Sperling and W. Vogel, arXiv.1907.12427 (2019).
[37] J. Weimbub and D. K. Ferry, Applied Physics Reviews 5, 041104 (2018).

We can rewrite Eq. (8) with respect to the probability $P(n)$ by Eq. (7) as
$$
\sum_{m=0}^{\infty} \frac{(s+1)}{s-1} \sum_{n=m}^{\infty} \frac{(n)}{m} (1-\eta)^{n-m} \eta^m P(n)
\approx \sum_{n=0}^{\infty} (1-\eta)^n \frac{(s+1)\eta}{(s-1)(1-\eta)} \frac{(n)}{m} P(n)
\approx \sum_{n=0}^{\infty} \left(1 - \eta + \frac{(s+1)}{s-1} \eta\right)^n P(n),
$$
where we used $\sum_{m=0}^{\infty} \binom{n}{m} = \sum_{m=0}^{\infty} \binom{n}{m} = \cdots = \binom{0}{m} = 0$ and the relation $\sum_{m=0}^{\infty} \alpha^m (\eta) = (1+\alpha)^n$. 

Appendix
[38] S. Wallentowitz, and W. Vogel, Phys. Rev. A 53, 4528 (1996).
[39] K. Banaszek, and K. Wódkiewicz, Phys. Rev. Lett. 76, 4344 (1996).
[40] K. Banaszek, C. Radzewicz, K. Wódkiewicz, and J. S. Krasinski, Phys. Rev. A 60, 674 (1999).
[41] K. Banaszek, A. Dragan, K. Wódkiewicz, and C. Radzewicz, Phys. Rev. A 66, 043803 (2002).
[42] P. Bertet, A. Auffeves, P. Maioli, S. Osnaghi, T. Meunier, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 89, 200402 (2002).
[43] R. Juarez-Amaro and H. Moya-Cessa, Phys. Rev. A 68, 023802 (2003).
[44] R. Loudon, The quantum theory of light, 2nd edn. (Oxford University Press, Oxford, 1983).
[45] U. Leonhardt, and H. Paul, Phys. Rev. A 48, 4598 (1993).
[46] A. I. Lvovsky, H. Hansen, T. Aichele, O. Benson, J. Mlynek, and S. Schiller, Phys. Rev. Lett. 87, 050402 (2001).
[47] T. Richter and M. G. A. Paris, J. Opt. B: Quantum Semiclass. Opt. 3, S42–54 (2001).
[48] H.-W. Lee, Optics Communications 337, 62-65 (2015).
[49] M. D. Eisaman, J. Fan, A. Migdall, and S. V. Polyakov, Review of Scientific Instruments 82, 071101 (2011).
[50] D. I. Schuster, A. A. Houck, J. A. Schreier, A. Wallraff, J. M. Gambetta, A. Blais, L. Frunzio, J. Majer, B. Johnson, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Nature 445, 515–518 (2007).
[51] M. Hofheinz, H. Wang, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O’Connell, D. Sank, J. Wenner, J. M. Martinis, and A. N. Cleland, Nature 459, 546–549 (2009).
[52] F. Marsili, D. Bitauld, A. Gaggero, S. Jahanmirinejad, R. Leoni, F. Mattioli, and A. Fiore, New Journal of Physics 11, 045022 (2009).
[53] J. Kofler and Č. Brukner, Phys. Rev. Lett. 99, 180403 (2007).
[54] J. Kofler and Č. Brukner, Phys. Rev. Lett. 101, 090403 (2008).
[55] H. Jeong, M. Paternostro, and T. C. Ralph, Phys. Rev. Lett. 102, 060403 (2009).
[56] S. Raeisi, P. Sekatski, and C. Simon, Phys. Rev. Lett. 107, 250401 (2011).
[57] P. Ehrenfest, T. Ehrenfest-Afanasyeva, Mechanics Enzyklopädie der Mathematischen Wissenschaften, Vol. 4., Leipzig, 1911. (Reprinted in: Ehrenfest, P., Collected Scientific Papers, North Holland, Amsterdam, 1959, pp. 213–300.)