1. Introduction

In a paper written in 1983, V.N. Grishin proposed to complement the product, left and right division operations of Lambek’s syntactic calculus with a dual set of operations: coproduct, and the subtraction operations of right and left difference. In its most elementary form, the resulting categorial type logic, which we’ll refer to as the Lambek-Grishin calculus (LG), is given by the preorder axioms for the derivability arrow →, together with the invertible rules of inference below, characterizing the operations ⊗/,\ as a residuated triple, and ⊕, ⊙, □ as a dual residuated triple.

\[
\begin{align*}
A \to C/B & \iff A \otimes B \to C \iff B \to A\setminus C \\
B \odot C & \to A \iff C \to B \oplus A \iff C \odot A \to B
\end{align*}
\]

From this basis, extended versions can be obtained in terms of linear distributivity principles. These allow for interaction between the ⊗ and ⊕ families while preserving their individual (non-commutative, non-associative) characteristics.

LG exhibits two kinds of symmetry, given by the translation tables below. We write \(\cdot\) for the left-right symmetry of the original syntactic calculus; it preserves derivability: \(A \to B\) iff \(A' \to B'\). The \(\cdot\) symmetry relates the operations of the \(\otimes\) family to their duals. This symmetry is arrow-reversing: \(A \to B\) iff \(B'' \to A''\).

From this basis, extended versions can be obtained in terms of linear distributivity principles. These allow for interaction between the \(\otimes\) and \(\oplus\) families while preserving their individual (non-commutative, non-associative) characteristics.

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Abbreviating a long list of definitional equations \((C/D)'\) = \(D'/C'\), \((D\setminus C)'\) = \(C'/D'\), . . . For atoms, \(p'' = p = p''\).
With his 1993 paper, Jim Lambek was among the first to bring Grishin’s work to the attention of a wider audience; also he had the paper translated by his student Ćubrić so as to make it accessible for researchers lacking fluency in Russian. Linguistic exploration is of a more recent date. In Moortgat (2009), I give a survey of results obtained so far. Semantically, LG derivations are associated with terms of the linear lambda calculus, as is the case for the original categorial grammars. But because LG logically is a multiple-conclusion system, the target terms are obtained via a continuation-passing-style translation into multiplicative intuitionistic linear logic. The translation introduces a distinction between values and contexts of evaluation; the context is explicitly included into the meaning composition process. Recent work in formal semantics (e.g. (De Groote 2001; Barker and Shan 2006)) has forcefully argued for this view on the syntax-semantics interface. Symmetric LG provides a solid prooftheoretic basis for a continuation semantics, and for the different evaluation strategies that go with it. Syntactically, Grishin’s distributivity principles make it possible to interleave the composition of phrases out of their constituent parts with the composition of evaluation contexts for the semantic values associated with these phrases. This creates new possibilities for handling discontinuous dependencies that arise when syntactic and semantic composition are out of tune.

My aim in this paper is to complement the symmetry between (dual) residuated type-forming operations with an orthogonal opposition that contrasts residuated and Galois connected operations. Whereas the (dual) residuated operations are monotone, the Galois connected operations (and their duals) are antitone. The paper is organized as follows. In §2 the vocabulary is extended with a Galois connected pair and a dual Galois connected pair, and the algebraic properties of these operations is discussed. In §3 the distributivity principles for the $\otimes$ and $\oplus$ families are generalized to include the four negative operations. In §4 the (dual) Galois connected operations are given a continuation-passing-style translation. Linguistic applications of the new vocabulary are discussed in §5. We conclude with some directions for further research.

The translation introduces little puzzles of its own. In the references, the author of a well-known study on partially ordered algebraic systems appears in disguise as L. Fooks — the English transliteration of the Russian transliteration doesn’t quite disclose the identity of the Hungarian mathematician with the German name.
2. Residuation and Galois connections

Let us recall some key concepts from Dunn (1991); Galatos et al. (2007). Consider two posets \((X, \leq), (Y, \leq')\) with mappings \(f : X \rightarrow Y, g : Y \rightarrow X\). The pair \((f, g)\) is called a residuated pair \((rp)\), a dual residuated pair \((drp)\), a Galois connection \((gc)\), a dual Galois connection \((dgc)\) depending on which of the following biconditionals holds:

\[
\begin{align*}
(rp) & \quad f x \leq' y \iff x \leq gy \\
(drp) & \quad y \leq' f x \iff gy \leq x \\
(gc) & \quad y \leq' f x \iff x \leq gy \\
(dgc) & \quad f x \leq' y \iff gy \leq x
\end{align*}
\]

Instead of the above biconditionals, one can use an alternative characterization in terms of the tonicity properties and the properties of the compositions of the operations involved:

\[
\begin{align*}
(rp) & \quad f, g : \text{isotone}, \quad x \leq g f x, \quad f g y \leq' y \\
(drp) & \quad f, g : \text{isotone}, \quad g f x \leq x, \quad y \leq' f g x \\
(gc) & \quad f, g : \text{antitone}, \quad x \leq g f x, \quad y \leq' f g y \\
(dgc) & \quad f, g : \text{antitone}, \quad f g x \leq x, \quad g f y \leq' y
\end{align*}
\]

In the context of categorial type logic, we speak about types and derivability between types, i.e. we consider just one inequality. For the residuated operators of Lambek’s syntactic calculus, one can read \(f\) as the operation of multiplying to the right with some fixed type; \(g\) then is right division by that type. The composition law \(f g y \leq' y\) takes the form of the familiar rightward application schema \((A/B) \otimes B \rightarrow A\). By \(\cdot^\circ\) symmetry, multiplication to the left and left division similarly form a residuated pair. By arrow reversal under \(\cdot^\circ\), we obtain the dual residuated pairs.

In addition to these binary operations, residuated with respect to each of their operands, one can also introduce the unary case of residuated pairs in the categorial type language, although neither Lambek nor Grishin have done so. The defining biconditional is

\[
(rp) \quad \Diamond A \rightarrow B \iff A \rightarrow \Box B
\]

\(^3\)Completeness with respect to relational semantics is discussed in Areces et al. (2004) for the Galois connected operations, and in Kurtonina and Moortgat (2010) for the \(\otimes/\oplus\) families.
The use of such a pair has been advocated in [Morrill (1994)] to impose *island constraints* in order to block overgeneration resulting from the structural rule of associativity. In [Kurtonina and Moortgat (1997)], the residuated unary operators are used to establish embedding results, showing that in moving from associative/commutative LP to the non-associative/non-commutative base logic NL no expressivity is lost: associativity and/or commutativity can be recovered in a *controlled* form. On another festive occasion ([Lambek 2007]), the recipient of this Festschrift has spoken stern words about the infatuation with diamonds and boxes that one finds in certain categorial circles, so I will say no more about them in this paper.

Let us rather turn to monotone *decreasing* type-forming operations. Such operations are already familiar from the binary vocabulary, where the (co)implications among themselves form (dual) Galois connected pairs satisfying $A \rightarrow C/B \iff B \rightarrow A\backslash C$ and $B \odot C \rightarrow A \iff C \odot A \rightarrow B$, as we saw. If the language also contains multiplicative units for $\otimes$ and $\oplus$, one obtains four negations defined in terms of (co)implication with respect to these units: $1 \odot A, A \backslash 0$, and the $\leftrightarrow$-symmetric pair. This is the way the negations are introduced in [Grishin (1983)]. A multiplicative unit for product is not unproblematic for the linguistic applications: it allows for typing of the empty string or structure which easily leads to overgeneration. A simple way of avoiding such problems is to keep the language unit-free and to introduce the antitone operations as unary connectives in their own right. For a Galois connected pair $^0,^0$ this was done in [Areces et al. (2004)]. Here we add a $\infty$-symmetric dual Galois connected pair $^1,^1$. The Galois principles for these operations manifest themselves in the following form.

\begin{align*}
(gc) & \quad B \rightarrow A^0 \iff A \rightarrow ^0 B ; \\
(dgc) & \quad 1B \rightarrow A \iff A^1 \rightarrow B
\end{align*}

The compositions of $^0,^0$ (in either order), and similarly of $^1,^1$, are isotone and idempotent. For the Galois connected operations, the compositions are expanding; for the dual Galois operations, they are contracting, i.e. we have the arrows below. Together with monotonicity and idempotence, this means composing the Galois connected negations yields a *closure* operation; dually, from the composition of $^1,^1$ one obtains an *interior* operation.

\begin{align*}
A \rightarrow ^0(A^0) , & \quad A \rightarrow (^0A)^0 ; \\
A \rightarrow (A^1)^1 & \quad (A^1)^1 \rightarrow A , \quad A \rightarrow (A^1)^1 ; \quad A\rightarrow ^0(A^0)
\end{align*}
3. Distributivity principles

The properties discussed above depend exclusively on the (dual) Galois principles. The next natural step is to investigate possible forms of interaction between the negative operations and the rest of the vocabulary. Our aim here is to keep the four negations distinct, rather than to opt for collapse into one pair of a cancelling pre- and postnegation (Abrusci, 2002; Lambek, 1993), or a single involutive negation (de Groote and Lamarche, 2002).

For communication between the $\otimes$ and $\oplus$ families, Grishin proposes two groups of interaction principles. We present them in the rule format of Moot (2007) and Moortgat (2009). One group consists of the rules in Figure 1, which we will collectively refer to as \((\text{distr})\). The other group, \((\text{distr})^{-1}\), is obtained by taking the converses of the inference rules of Figure 1 with premise and conclusion changing place.

From the principles in \((\text{distr})\), using the (dual) residuation principles, one easily derives the type transitions below. They change the dominance relation between the product and the difference operation; whereas the difference operation is dominated by the product on the left of the arrow, on the right the difference operation is the main connective.

\[
\begin{align*}
(A \otimes B) \otimes C &\rightarrow A \otimes (B \otimes C) \\
C \otimes (A \otimes B) &\rightarrow A \otimes (C \otimes B) \\
C \otimes (B \otimes A) &\rightarrow (C \otimes B) \otimes A \\
(B \otimes A) \otimes C &\rightarrow (B \otimes C) \otimes A
\end{align*}
\]

From the \((\text{distr})^{-1}\) principles, one derives the type transitions below. For the interaction between product and difference operations, these are the converses of the above.

\[
\begin{align*}
A \otimes (B \otimes C) &\rightarrow (A \otimes B) \otimes C \\
A \otimes (C \otimes B) &\rightarrow C \otimes (A \otimes B) \\
(C \otimes B) \otimes A &\rightarrow C \otimes (B \otimes A) \\
(B \otimes C) \otimes A &\rightarrow (B \otimes A) \otimes C
\end{align*}
\]
\[
\begin{align*}
A & \rightarrow B, & A & \rightarrow B, & A & \rightarrow B, & A & \rightarrow B, \\
1B & \rightarrow A^0, & 1B & \rightarrow 0A, & 1B & \rightarrow 0A, & B^1 & \rightarrow A^0 \\
A & \rightarrow B \oplus C, & A & \rightarrow B \oplus C, & A & \rightarrow B \oplus C, & A & \rightarrow B \oplus C \\
B^1 & \rightarrow A/C, & B^1 & \rightarrow C/A, & 1C & \rightarrow A/B, & 1C & \rightarrow B/A \\
A & \otimes B \rightarrow C, & A & \otimes B \rightarrow C, & A & \otimes B \rightarrow C, & A & \otimes B \rightarrow C \\
C & \oplus A \rightarrow 0B, & A & \otimes C \rightarrow 0B, & C & \oplus B \rightarrow A^0, & B & \oplus C \rightarrow A^0
\end{align*}
\]

Figure 2: Generalization of (\textit{distr}) for $1$, $1$, $0$, $0$.

For interaction between $\otimes$ and $\oplus$, the $(\textit{distr})^{-1}$ principles have the following effect.

\[
\begin{align*}
(A \oplus B) \otimes C & \rightarrow A \oplus (B \otimes C) \\
C \otimes (A \oplus B) & \rightarrow (C \otimes B) \oplus A \\
(B \otimes A) \otimes C & \rightarrow (B \otimes C) \oplus A
\end{align*}
\]

How can we generalize the distributivity principles to include the (dual) Galois connected operations? In the case where these operations are defined in terms of multiplicative units this is straightforward: in $(\textit{distr})$ or $(\textit{distr})^{-1}$, one replaces a subformula of the $\otimes$ term by $1$ and/or of the $\oplus$ term by $0$. We can extrapolate from the patterns involving the multiplicative units to obtain versions appropriate for our unit-free setting. We illustrate with the $(\textit{distr})$ principles. Compare the derivations below for a case of interaction among the (dual) Galois connected operators:

\[
\begin{align*}
\frac{A \vdash B}{1 \otimes A \vdash B \oplus 0} \quad \text{(distr)} & \quad \sim \quad \frac{A \vdash B}{B^1 \vdash 0A}
\end{align*}
\]

Interaction between Galois connected and residuated families takes the following form:

\[
\begin{align*}
\frac{A \vdash B \oplus C}{1 \otimes A \vdash B \oplus C} \quad \text{(distr)} & \quad \sim \quad \frac{A \vdash B \oplus C}{B \otimes 1 \vdash C/A}
\end{align*}
\]

Taking into account the $\rhd$ and $\bowtie$ symmetries, we obtain the generalized $(\textit{distr})$ principles of Figure 2.
Characteristic theorems depending on the principles of Figure 2 are the laws of the excluded middle below. They follow from the first row of inferences with the premise instantiated as the identity arrow. Compare the version with multiplicative units, where these become $1 \odot A \to A \setminus 0$ iff $1 \to (A \setminus 0) \oplus A$ (not- $A$ or $A$) etc.

$$1_A \to A^0 , \quad 1_A \to ^0 A , \quad A^1 \to ^0 A , \quad A^1 \to A^0$$

As long as one makes a choice for either the $(\text{distr})$ or the $(\text{distr})^{-1}$ group of distributivity principles, the four negations remain distinct operations. Grishin himself follows a different route: to the mixed-associativity laws of $(\text{distr})^{-1}$, he adds the corresponding excluded middle laws as extra axioms, leading to the identifications $1 \odot A \leftrightarrow A \setminus 0$ and $0/A \leftrightarrow A \odot 1$. The mixed-associativity laws of $(\text{distr})$ then become derivable, i.e. the distributivity rules become invertible. For the linguistic applications we have in mind, invertibility of the distributivity rules is not an option: we need the full group of distributivities (mixed associativity and mixed commutativity laws); invertible distributivity rules in that situation mean that the non-associativity/non-commutativity of the $\otimes$ and $\oplus$ operations is no longer preserved, as shown in [Bastenhof (2010)].

With respect to the de Morgan laws and the expressibility of the (co)implications in terms of (co)product and negation, the choice between the $(\text{distr})$ or $(\text{distr})^{-1}$ principles again leads to one-way arrows rather than equalities. For the de Morgan laws, from the $(\text{distr})$ principles one derives the inequalities below (and variants with $\Leftrightarrow$-symmetric formulas on the left and/or on the right of the arrow).

$$(A \otimes B)^1 \to ^0 B \oplus ^0 A \quad A^1 \otimes B^1 \to ^0 (B \oplus A)$$

$$(A \otimes B)^1 \to ^0 A \oplus ^0 B \quad B^1 \otimes A^1 \to ^0 (B \oplus A)$$

Inequalities of the following type then express the relation between (co)implication and (co)product plus negation.

$$A \setminus B \to A^0 \oplus B \quad B \otimes ^1 A \to B \otimes A$$

$$A \setminus B \to B \oplus A^0 \quad ^1 A \otimes B \to B \otimes A$$

In Figure 3 we give the neighbours of $A \setminus B$ and $B/A$ in terms of the (dual) Galois negations, given $(\text{distr})$. A vertical $\Leftrightarrow$ symmetry axis runs through the middle of the picture. For $B \otimes A$ and $A \otimes B$, the dual
situation obtains: take the $\infty$-symmetric image of the formulas, and turn around the arrows.

With a choice for $(distr)^{-1}$, the arrows in the above inequalities are turned around. We don’t elaborate on this option, because the illustrations we’ll discuss in §5 only make use of the $(distr)$ principles. Before turning to these illustrations, we extend the Curry-Howard interpretation to the (dual) Galois connected operations.

4. Proofs and terms

As argued in §1, the computational semantics of LG takes the form of a continuation-passing-style (CPS) translation associating the derivations of our multiple-conclusion source logic with derivations of single-conclusion LP. The latter are Curry-Howard isomorphic with terms of the linear lambda calculus. Our purpose in this section is to extend the call-by-value CPS translation for (the (co)implication fragment of) LG of Bernardi and Moortgat (2010) to the (dual) Galois negations. To this end, we present LG in the format of a Display Logic, and we define a mapping $[-]$ acting on its types and derivations:

$$[-] : LG^\beta \rightarrow LP^{\alpha;\beta[r]}$$

**Types** The target calculus has the same atoms as the source, plus a distinguished atom $r$, the response type. The source calculus connectives are all interpreted in terms of linear implicative types with
head type $r$. We write $A^\perp$ for $A \rightarrow r$. For source types $A$, the target language distinguishes values $\lceil A \rceil$, continuations $\lceil A \rceil^\perp$, and computations $\lceil A \rceil^{\perp \perp}$. Because the target logic is non-directional, the translation identifies left-right symmetric source types: $\lceil A \rceil = \lceil A^{\perp \perp} \rceil$. For atoms $p \in \mathcal{A}$, $\lceil p \rceil = p$. For complex types, we have the mapping below.

\[
\lceil A \backslash B \rceil = \lceil B \rceil^\perp \rightarrow \lceil A \rceil^\perp ; \lceil A \otimes B \rceil = \lceil A \rceil^\perp \rightarrow \lceil B \rceil^\perp ; \lceil A^0 \rceil = \lceil 1^A \rceil = \lceil A \rceil^\perp
\]

**Proofs and terms** The presentation of LG as a display sequent calculus in [Moortgat (2009)] essentially follows [Gore (1997)] but adds a mechanism to make a distinction between active and passive formulas. Sequent structures are built out of labeled formulas, considered passive: input formulas (hypotheses) are labeled with variables $x, y, z, \ldots$, output formulas (conclusions) with covariables $\alpha, \beta, \gamma, \ldots$. A characteristic feature of the Display Logic format is that for every logical connective (not just for product and coproduct) there is a matching structural connective. We opt for clarity rather than economy of notation, and use the same symbols for logical and structural operations, marking off the latter by means of center dots.

Input structures $I$ and output structures $O$ are then built according to the grammar below.

\[
I ::= \; x : A \mid I \cdot \otimes \cdot I \mid I \cdot \otimes \cdot O \mid O \cdot \otimes \cdot I \mid \perp I \mid \perp O
\]

\[
O ::= \; \alpha : A \mid O \cdot \oplus \cdot O \mid I \cdot \backslash \cdot O \mid O \cdot \slash \cdot I \mid I \mid O
\]

The (dual) residuation and (dual) Galois principles can now be formulated at the structural level. We don’t repeat these rules: simply replace the formula variables $A, B, \ldots$ of the arrow presentation by structure variables $X, Y, \ldots$ (with input or output interpretation depending on the context) and the logical connectives by their structural counterpart. For example,

\[
\frac{A \rightarrow C / B}{A \otimes B \rightarrow C} \quad \sim \quad \frac{X + Z \cdot / Y}{X \cdot \otimes Y + Z}
\]

These rules are invertible; they allow you to display any formula making up a structure as the single occupant of the sequent an-

\footnote{The relation between display calculus and the Gentzen-style categorial sequent calculi is discussed in [Areces and Bernardi (2004)].}
tecedent or succedent, depending on its input/output polarity. Se-
quents related by the (dual) residuation or Galois rules we call
*display equivalent*. The distributivity principles, likewise, take
the form of structural rules in the Display Logic presentation. For ex-
ample,

\[
\frac{A \otimes B \rightarrow C \oplus D}{C \otimes A \rightarrow D/B} \quad \frac{X \cdot \otimes \cdot Y \vdash Z \cdot \oplus \cdot W}{Z \cdot \otimes \cdot X \vdash W \cdot \bot \cdot Y}
\]

As said, we make a distinction between active and passive for-
mulas. A sequent can have at most one active formula, which is
unlabeled and displayed as the sole antecedent or succedent for-
mula. In all, then, this gives us three kinds of sequent: \(X \vdash Y\) (all
formulas are passive), \(X \vdash A\) (active output formula), \(A \vdash Y\) (ac-
tive input formula). As will become clear below, there are explicit
inference rules to activate a passive formula, on the input or on the
output side.

In Bernardi and Moortgat (2010) proofs of the source calculus
are coded by their own term language, a suitably adapted version of
the \(\tilde{\lambda} \mu\tilde{\mu}\) calculus of Curien and Herbelin (2000). Here we define the
CPS translation directly on the proofs of the source. The target cal-
culus consists of natural deduction proofs in correspondence with a
fragment of the linear lambda calculus. The translation respects the
following invariants:

- target judgements are of the form \(\Gamma \vdash M : B\), where \(\Gamma\), the
typing environment for the target terms, is a multiset of type
declarations \(\bar{x} : [A]\) (resp. \(\bar{\alpha} : [A]^{\bot}\)) for the passive input
(resp. output) formulas making up the structures appearing in
the source proofs;

- source sequents \(X \vdash Y\) are mapped to target terms of type \(r\);
structural rules rewriting \(X \vdash Y\) to \(X' \vdash Y'\) leave the associated
term unaffected;

- source sequents \(X \vdash A\) are mapped to terms of type \([A]^{\bot \bot}\)
(computations);

- source sequents \(A \vdash Y\) are mapped to terms of type \([A]^{\bot}\) (con-
tinuations).

Below we present the rules of the source calculus, followed by
their \([\cdot]\) translation. First the identity group ((Co)Axiom, Cut) and
the rules for activating a displayed passive formula.
The logical rules of the source calculus introduce an active input or output formula in the conclusion. Rules with a passive premise simply replace a structural connective by the corresponding logical one. Rules with active premise(s) compose the active formula of the conclusion out of the active subformula(e) of the premise(s).

Below the rules for the (dual) Galois negations and their translations. In the case of $(\alpha \cdot L)$, we can have the identity transformation, because $[A]^{\perp \perp} = [0A]^\perp$: the term coding the premise, a computation of type $A$, can also be interpreted as a continuation of type $0A$, as required for the term coding the conclusion.

Finally, the rules for the (co)implications. We give the rules for formulas $A \setminus B$ and $A \odot B$ (rather than $B \odot A$, which is the dual of $A \setminus B$) in order to highlight the correspondence between the interpretation
of implication and co-implication.

\[
\begin{align*}
X \vdash (x : A) \cdot \beta : (B : B) & \quad \rightarrow & \quad (x : A) \cdot (\beta : B) \vdash X & \quad \rightarrow & \quad A \vdash B \vdash X & \quad \rightarrow & \quad X \vdash A \vdash B \vdash Y & \quad \rightarrow & \quad X \vdash A \vdash Y \vdash A \vdash B & \quad \rightarrow & \quad X \vdash A \vdash B \vdash R (x : A) & \quad \rightarrow & \quad X \vdash A \vdash B \vdash R \end{align*}
\]

\[\begin{align*}
\left[\setminus R\right] &= \left[\setminus L\right] = \lambda h.(h \Lambda \tilde{x} . S^r) : [A \setminus B]^{\perp} = [A \ominus B]^{\perp} \\
\left[\setminus L\right] &= \lambda u.(M^A)^{\perp} (u K^B) : [A \setminus B]^{\perp} \\
\left[\setminus R\right] &= \lambda k.(k \left[\setminus (L)\right]) : [A \ominus B]^{\perp}
\end{align*}\]

For the binary vocabulary, I have shown in [Moortgat, 2009] that LG enjoys Cut elimination. Extending this result to the unary negative operations presents no problems. Below the transformation for a principal cut on \(^0A\) in the source calculus together with the image (normalization/\(\beta\) conversion) under the \(\left[\cdot\right]\) translation. The remaining cases are obtained from the \(\Rightarrow\) and \(\Rightarrow\) symmetries.

\[
\begin{align*}
X \vdash (x : A) & \quad \rightarrow & \quad X \vdash 0 (x : A) & \quad \rightarrow & \quad X \vdash 0 A \vdash 0 R & \quad \rightarrow & \quad X \vdash 0 Y & \quad \rightarrow & \quad X \vdash 0 \vdash 0 L & \quad \rightarrow & \quad X \vdash 0 Y & \quad \rightarrow & \quad (\lambda k \Lambda \tilde{x} . S^r) \left[M^A\right]^{\perp} & \quad \rightarrow & \quad \left(M^A\right)^{\perp} \Lambda \tilde{x} . S^r
\end{align*}
\]

5. Illustrations

Let us turn to the possible uses of the negative operations in combination with the rest of the vocabulary. We give examples of new expressive facilities that rely exclusively on the residuation and Galois principles, and examples involving also the distributivity principles (\(\text{distr}\)). To accommodate the lexical recipes of a simple extensional Montague-style interpretation, we compose the derivational semantics given by the CPS translation with a mapping \(\left[\cdot\right]\).

\[
\begin{align*}
\mathcal{L}_G^{\alpha} \rightarrow \mathcal{L}_P^{\alpha} \rightarrow \mathcal{L}_L^{\beta} \rightarrow \mathcal{L}_L^{\beta} \rightarrow \mathcal{L}_L^{\beta}
\end{align*}
\]
On the type level, $|\cdot|$ associates the atomic syntactic types in $A$ and the response type $r$ with target semantic types built from the atomic semantic types $e, t$. For atomic syntactic types in $A$, $|\cdot|$ coincides with the mapping from syntactic to semantic types of a direct (non-continuized) interpretation, with $|np| = e$, $|s| = t$, $|n| = e \rightarrow t$, for example. For the continuation response type, let us assume $|r| = t$. As a result of the identification $|r| = |s|$, the interpretation of a sentence computation, $|[s]^{\perp\perp}|$, will be given by a term of type $(t \rightarrow t) \rightarrow t$. If this sentence stands on its own, i.e. if there is no bigger context of which it forms a part, we can evaluate it to a truth-value denoting expression by providing the trivial continuation — the identity function of type $t \rightarrow t$.

On the level of proofs/terms, source constants of type $A$ are associated with closed target terms of type $|A|$. These lexical recipes are not required to be linear. But on complex source types and terms, $|\cdot|$ acts homomorphically, so that, apart from possible non-linear contributions of the lexical items, the linearity of the source terms is reflected in the translation.

\[ |(M \ N)| = (|M| \ |N|) \quad ; \quad |\lambda x. M| = \lambda \tilde{x}. |M| \]

**Scope** Our first example illustrates the use of the interior operation, i.e. the composition of the dual Galois connected operations $^1(\cdot)^1$. This example makes no use of the distributivity postulates. Suppose we assign type $^1(np^1)$ to quantifier phrases (‘everyone’, ‘some student’, …). The type contracts to $np$, accounting for the fact that such phrases syntactically behave as simple noun phrases. In the case where a sentence contains multiple quantifier phrases, there is a derivational ambiguity as to the points in the derivation where the $(^1R)$ rules apply. These choice points lead to the different scope construals for such a sentence.

Below, we give two derivations, using the compact format introduced in [Moortgat (2009)](2009): the display equivalences and the formula (de)activation steps leading from one active formula to the next are compiled away; for legibility, only the (co)axiom formulas and the input values of the endsequent are explicitly labeled.

Tracing the steps in backward chaining fashion, the two derivations have the same initial moves: the focus is shifted from the goal formula $s$ first to the subject, then to the direct object; the main connective in each case is rewritten to its structural counterpart by the
(1·L) rules. At that point, the derivations diverge. In the case of (†), (1·R) introduces the conegation on the direct object np.

\[
\frac{np \vdash \beta : np}{np^{-1} \vdash np^1} \quad (1\cdot R) \\
\frac{1(np^1) \vdash np}{s \vdash \alpha : s} \quad (1\cdot L)
\]

\[
\frac{np \vdash \beta : np \quad np^{-1} \vdash np^1}{1(np^1) \vdash np} \quad \gamma : np \vdash np \quad (1\cdot L)
\]

\[
\frac{(np^1) \vdash (np \vdash \cdot (\gamma : np) \cdot \cdot s)}{1(np^1) \vdash (np \vdash \cdot (\gamma : np) \cdot \cdot s)} \quad 1\cdot L
\]

The stepwise construction of the [−] translation below shows that this derivation is mapped to an interpretation where the direct object outscopes the subject.

\[
\frac{\lambda k.(\tilde{\beta}) : [np^1]^{1\perp}}{\Rightarrow \quad \lambda \tilde{\beta}.(\gamma : np) = \tilde{\gamma} : [np^1] = [np]^{1\perp}} \quad (1\cdot L)
\]

\[
\frac{\lambda u.(\gamma : np) : [np]^{1\perp}}{= \quad \lambda \tilde{\gamma}.(tv \lambda u.(\gamma : np)) = \tilde{\gamma} : [np^{-1}]^{1\perp}}
\]

\[
\frac{\lambda k.(\gamma : np) : [np^{-1}]^{1\perp}}{1\cdot L \quad \lambda \tilde{\gamma}.(tv \lambda u.(\gamma : np)) = \tilde{\gamma} : [np^{-1}]^{1\perp}}
\]

\[
\frac{\lambda \tilde{\gamma}.(do \lambda \tilde{\gamma}.(tv \lambda u.(\gamma : np))) = \tilde{\gamma} : [s]^{1\perp}}{1\cdot L \quad \lambda \tilde{\gamma}.(do \lambda \tilde{\gamma}.(tv \lambda u.(\gamma : np))) = \tilde{\gamma} : [s]^{1\perp}}
\]

Some comments on the steps. The focus shifting rules ⇒, ⇐, ⇔, are shorthand for a sequence of steps: first the deactivation of the active formula of the premise, then display equivalences to bring a new formula in focus, and finally a μ or ˜μ step activating that
new formula. Deactivation of the premise active formula is achieved by means of a cut against a (co)axiom; these cuts introduce the (co)variables \( \bar{\gamma}, \bar{tv}, \) and \( \bar{su} \). The \( \mu \) or \( \bar{\mu} \) steps then build a computation or continuation term for the conclusion by binding a (co)variable of the appropriate type, \( \bar{\beta}, \bar{\gamma}, \bar{\alpha} \) in the case at hand. The conclusion of the \( (1, L) \) rules, similarly, is obtained from an implicit cut on a (co)axiom, introducing the (co)variables \( \bar{k} \) and \( \alpha \) do of type \( [np^1]^\perp \) and \( [1(np^1)] \) respectively.

So far the direct object wide scope interpretation. The alternative derivation, shown below, proceeds with \( (\dagger) \) where we had \( (\ddagger) \) before. In the case of \( (\ddagger) \), the \( (1R) \) rule introduces the conegation on the subject \( np \).

\[
\frac{\begin{array}{c}
x : np \vdash np \\
\vdash s : \alpha \vdash s
\end{array}}{\begin{array}{c}
\vdash np \vdash np \setminus s \vdash np \cdot \setminus s
\end{array}} \quad \Leftrightarrow \quad \frac{\begin{array}{c}
\vdash (np \setminus s) \vdash np \cdot \setminus (np^1)
\end{array}}{\begin{array}{c}
\vdash np \vdash s \cdot \setminus (np \setminus s) \vdash \setminus (np^1)
\end{array}}
\]

We compute the \([\cdot] \) translation. The abbreviated right branch is mapped to a term \( \bar{k} \) of type \( [np^1]^\perp = [np]^\perp \), which this time takes the direct object role. This derivation results in an interpretation where the subject outscapes the direct object.

\[
\begin{align*}
\Leftrightarrow & \lambda \overline{\alpha} . (u \overline{\alpha} \bar{\alpha}) \quad : \quad [np \setminus s]^\perp \\
\Leftrightarrow & \lambda u' . (k \bar{\alpha} (u' \lambda u . (u \overline{\alpha} \bar{\alpha}))) \quad : \quad [(np \setminus s)/np]^\perp \\
\Leftrightarrow & \lambda \bar{\alpha} . (k \bar{\alpha} (tv \lambda u . (u \overline{\alpha} \bar{\alpha}))) \quad : \quad [np]^\perp \\
\Leftrightarrow & \lambda k . (k \bar{\alpha} (\bar{k} (tv \lambda u . (u \overline{\alpha} \bar{\alpha})))) \quad : \quad [np^1]^\perp \\
\Leftrightarrow & \lambda \bar{\alpha} \bar{\gamma} \lambda \bar{\alpha} \bar{\gamma} (do (tv \lambda u . (u \overline{\alpha} \bar{\alpha}))) \quad : \quad [1(np^1)]^\perp \\
\Leftrightarrow & \lambda \bar{\alpha} \bar{\gamma} (\bar{\gamma} \lambda \bar{\alpha} (do (tv \lambda u . (u \overline{\alpha} \bar{\alpha})))) \quad : \quad [1(np^1)]^\perp \\
\Leftrightarrow & \lambda \bar{\alpha} \bar{\gamma} (\bar{\alpha} (do (tv \lambda u . (u \overline{\alpha} \bar{\alpha})))) \quad : \quad [s]^\perp
\end{align*}
\]
The table below gives the $\cdot \mid$ translation of the constants, for a sample sentence ‘everyone likes someone’, assuming a non-logical target constant ‘like’ of type $e \rightarrow e \rightarrow t$, and the logical constants $\exists, \forall$ (ignoring the person/thing distinction).

| source       | $\cdot \mid$ translation                                                                 |
|--------------|------------------------------------------------------------------------------------------|
| everyone : $[np]\downarrow$ | $\forall : (e \rightarrow t) \rightarrow t$                                                |
| someone : $[np]\downarrow$    | $\exists : (e \rightarrow t) \rightarrow t$                                                |
| likes : $([s] \rightarrow [np] \rightarrow [np]) \rightarrow [np]$ | $\forall \lambda y . (\forall \lambda x . (c ((:\text{like } y \ x) x)))$ $\rightarrow \lambda v . (\forall \lambda x . (c ((:\text{like } y \ x) x)))$ $\rightarrow \lambda c . \lambda y . (\forall \lambda x . (c ((:\text{like } y \ x) x)))$ |

The familiar Montague-style interpretations result from the composition of the $\cdot \mid$ and $\cdot \downarrow$ translations, and a final evaluation step, providing the identity function $\lambda p.p$ for the abstraction over the parameter $c$ of type $t \rightarrow t$.

\[
|\lambda \alpha . (\text{do } \lambda y . (\text{tv} (\lambda u . (u \alpha x))) y))| = \\
\lambda c . (\forall \lambda y . (\forall \lambda x . (c ((:\text{like } y \ x) x)))
\]

Comparing this analysis of scope-taking with the available alternatives, we notice that the generalized quantifier type $(e \rightarrow t) \rightarrow t$ arises as the $\cdot \mid$ image of the syntactic source type $np$, and the response type of the continuation semantics: there is no syntactic type $s$ involved. This is in contrast with the usual type assignments to quantifier phrases, such as $s/((np \downarrow s)$ in the standard Lambek calculus, or the ‘wrapping’ alternative $(s \uparrow np) \downarrow s$ of Morrill and Valentin (this volume), or again the $(s \odot s) \odot np$ assignment of Bernardi and Moortgat (2007).

The identification of the scope domain with syntactic type $s$ has been criticized in [Dalrymple et al. (1997)] on the basis of readings where a quantifier phrase takes scope at a non-sentential level. Examples would be in situ interpretations of quantifier phrases within nominal modifiers, or as complements of relational noun constructions (‘a solution for every problem’, ‘every picture of a star’). [Carpenter (1997)] obtains $n$-internal local readings by assigning the modifier head or relational noun a lifted type based on the syntactic category $s$: $(n\downarrow n)/(s \uparrow np) \downarrow s$ for the preposition ‘for’, $n/(s \uparrow np) \downarrow s$ for ‘picture of’; the lexical semantics for these items is then given in terms of lower-order constants.
With the negative operations, we can create noun phrase internal scope possibilities without introducing an artificial syntactic category. This time, we use the expanding composition of Galois connected operations \(0_{\cdot}0\). With a typing \(n/0(np^0)\) for ‘picture of’, the double negation on the argument produces a lifted semantic type \((e \to t) \to t\) under the combined \([\cdot]\) and \(|\cdot|\) translations. Such doubly negated arguments would be appropriate also for higher-order transitive verbs (‘seeks’, ‘needs’: \((np/s)^0(np^0)\)) allowing for a de dicto versus a de re interpretation of the direct object, and for complement-taking verbs (‘claims’, ‘thinks’: \((np/s)^0(s^0)\)) where both the main clause and the embedded clause need their own \(s\) continuation.

In the table below, we give the CPS translation of these syntactic source types, together with their image under \(|\cdot|\) and terms expressing lexical semantics. At the target end, ‘pic’ is a non-logical constant of type \(e \to e \to t\). The target non-logical constants ‘seek’ and ‘claim’ are of type \(((e \to t) \to t) \to e \to t\) and \(((t \to t) \to t) \to e \to t\) respectively. Note that the \(|\cdot| \circ [\cdot]\) image of \(np/n\) for the determiners is of the appropriate semantic type for the standard Montagovian lexical recipes.

| source      | \(|\cdot|\) translation                          |
|-------------|-------------------------------------------------|
| picture of  | \(\lambda k, \lambda q. (k \lambda x. q \lambda y. (\text{pic } y x))\) |
| \([n]^+ \to [np]^+\) | \(((e \to t) \to t) \to ((e \to t) \to t) \to t\) |
| seeks       | \(\lambda v, \lambda c. (\text{seek } q x))\) |
| \((\text{s}^+\to [np]^+)\to [np]^+\) | \(((t \to t) \to e \to t) \to ((e \to t) \to t) \to t\) |
| claims      | \(\lambda c. (\text{claim } q x))\) |
| \((\text{s}^+\to [np]^+)\to [s]^+\) | \(((t \to t) \to e \to t) \to ((t \to t) \to t) \to t\) |
| every       | \(\lambda Q, \lambda P. (\forall \lambda x. ((P x) \Rightarrow (Q x)))\) |
| \([np]^+ \to [n]^+\) | \((e \to t) \to (e \to t) \to t\) |
| some        | \(\lambda Q, \lambda P. (\exists \lambda x. ((P x) \wedge (Q x)))\) |
| \([np]^+ \to [n]^+\) | \((e \to t) \to (e \to t) \to t\) |

The lexical entries are put to work to compute some scope ambiguities below. We give the CPS translation of the derivations, and the result of the \(|\cdot|\) translation of the constants. We emphasize again that the interpretations we have discussed so far are obtained on the basis of the pure logic of residuated and Galois connected operations: they do not rely on interaction principles.
every picture of some teacher $\vdash np$

$$\lambda \bar{a}. ((\text{pictureof}) (\text{every} (\bar{a})) \ldots) \lambda \bar{x}. ((\text{some} (\bar{x})) (\text{teacher}))$$

$$= \lambda \bar{a}. (\forall \lambda x. (\exists \lambda y. (\text{teacher} y) \land (\text{pic} y x) \Rightarrow (\bar{x} x)))$$

$$\lambda \bar{a}. ((\exists \lambda y. (\text{teacher} y) \land (\forall \lambda x. (\text{pic} y x) \Rightarrow (\bar{x} x)))))$$

Alice claims some unicorn left $\vdash s$

$$\lambda \bar{a}. ((\text{claims} \ldots (\exists \lambda k (\bar{a} [a.])) \ldots) \lambda \bar{x}. ((\text{some} (\bar{x})) (\text{uni}))))$$

$$= \lambda c. (\exists \lambda x. ((\text{unicom} x) \land (\exists \lambda y. (\text{left} (\bar{y})))) (\text{alice})))$$

Let us turn now to some examples where the distributivity principles do come into play. For the relation between the binary implication and coimplication the crucial observation is that from the same premises $X \vdash B$ and $C \vdash Y$, we can derive an input implication $B \cdot C$ or an output coimplication $B \odot C$; compare

$$X \cdot \otimes \cdot B \cdot C \vdash Y \quad \text{versus} \quad X \vdash B \odot C \cdot \oplus \cdot Y$$

Semantically, we have seen that implication and coimplication combine the same pieces of information: the latter is interpreted as $\lambda k. (k \ M^{[B]})$, i.e. the lifted form of the interpretation of the former. From a syntactic point of view, there is a difference. The implication $B \cdot C$ must concatenate externally with its argument $X$. But in the case where $X$ is a product structure, the conditions for the application of the $(distr)$ interaction principles are met, and the coimplication can infix itself within $X$ and associate with any of its leafs $A$ into a formula $(B \odot C) \odot A$.

In [Bernardi and Moortgat (2007)] we have shown that this property of nested coimplications allows us to syntactically model the type schema for in situ binding $q(A, B, C)$ from [Moortgat (1996)] with a type $(B \odot C) \odot A$ (the type $A \otimes (C \odot B)$ would do as well). An expression with such a type behaves locally as an $A$ within a domain of type $B$ which is mapped into $C$. See the derivation below for a ‘compiled’ sequent rule ($qL$). The notation $X[Y]$ for an input structure singles out a substructure $Y$ of $X$ reachable via a path of
structural products. For output structures, we write \( X[Y] \) to pick out a substructure \( Y \) reachable along a path of structural implications. With \( \tilde{Y} \) we mean the image of the input product context \( Y \) under the residuation inferences.

\[
\begin{align*}
X[A] \vdash B & \quad C \vdash \tilde{Y}[D] \quad \text{rp} \\
X[A] \cdot \odot \tilde{Y}[D] & \vdash B \odot C \quad \odot R \\
X[A] \vdash (B \odot C) \cdot \odot \tilde{Y}[D] & \quad \text{drp} \\
X[ (B \odot C) \cdot \odot \cdot A ] & \vdash \tilde{Y}[D] \quad \text{distr'} \\
Y[ X[ (B \odot C) \odot A ] & \vdash \tilde{Y}[D] \quad \odot L \\
Y[ X[ q(A, B, C) ] & \Rightarrow D \quad \text{rp} \\
Y[ X[ q(A, B, C) ] & \Rightarrow D \quad qL
\end{align*}
\]

Semantically, there is a difference as to how the types \( q(A, B, C) \) and \( (B \odot C) \odot A \) package the meaning contributions of the subformulae \( A, B \) and \( C \). Under the direct interpretation, \( q(A, B, C) \), the semantic type corresponding to \( q(A, B, C) \), is \( (A' \rightarrow B') \rightarrow C' \). Contrast this with the CPS interpretation for \( (B \odot C) \odot A \),

\[
[(B \odot C) \odot A] = ((B \setminus C)^+ \rightarrow [A]^+)^+
\]

which consists essentially of a pair of an \( A \) value and a lifted \( B \setminus C \) value (i.e. a \( B \odot C \) continuation). Because our target language is restricted to the simply typed linear lambda calculus, the pair is expressed as a curried higher-order function.

Given this CPS interpretation, the \( | \cdot | \) translation of an expression of type \( (B \odot C) \odot A \) can have the schematic form below

\[
\lambda h.((h \lambda u.(u \mid M^{B,C} \mid |N^A|)) \mid N^A)
\]

with \( |M| \) and \( |N| \) the lexical contributions of the \( B \setminus C \) value and \( A \) value respectively. We illustrate with an example from inflectional morphology. Take a past tense transitive verb ‘tease+ed’. Suppose we see the tense morpheme as a function taking a subjectless, non-tensed form of the verb (type \( i \), with interpretation \( |i| = e \rightarrow t \)) to a tensed verb phrase with external subject argument (type \( np\setminus s \)). Tense combines as an affix with the tenseless verbal head, allowing it to combine with whatever internal arguments (and modifiers)
it may have. For transitive ‘tease+ed’, the lexicon then will contain the following information, assuming at the target side constants ‘tease’ and ‘past’ of type $e \rightarrow e \rightarrow t$ and $t \rightarrow t$ respectively.

\[
\text{tease+ed} : (i/np) \odot ((np\backslash s) \odot i)
\]

\[
|\text{tease+ed}| = \lambda h.((h \lambda u.((u |\text{ed}((\text{np}\backslash s)i)))) (|\text{tease}[^i/np]|))
\]

\[
|\text{tease}[^i/np]| = \lambda Q \lambda y.(Q (\text{tease}^{e-s} y))
\]

\[
|\text{ed}((\text{np}\backslash s)i)| = \lambda V \lambda P.((V \lambda c.((c (\text{past}^{e-s} (P x)))))
\]

A derivation for ‘Molly teased Leopold’ is given below together with its $[\cdot]$ and $|\cdot|$ translations.

\[
\begin{align*}
\vdash & \cdot np \sqcup np \sim i \sqcup i \cdot \\
\vdash & i/np \sqcup i \sqcup np \backslash i \end{align*}
\]

\[
\begin{array}{c}
\hline
\text{L} \\
\hline
\end{array}
\begin{array}{c}
\vdash & \cdot np \sqcup np \sim s \sqcup s \cdot \\
\vdash & np\backslash s \sqcup np \backslash s \cdot i \\
\hline
\text{R} \\
\hline
\end{array}
\begin{array}{c}
\vdash & (np\backslash s) \cdot \odot ((i/np) \cdot \odot np) \sqcup (np\backslash s) \sqcup i \\
\vdash & (i/np) \odot ((np\backslash s) \odot i) \sqcup (np \cdot \backslash s) \cdot / np \\
\hline
\text{L} \\
\hline
\end{array}
\begin{array}{c}
\vdash & np \cdot \odot ((i/np) \odot ((np\backslash s) \odot i) \cdot \odot np) \sqcup s \\
\hline
\text{su} \\
\hline
\end{array}
\begin{array}{c}
\vdash & \text{verb+tense} \\
\hline
\text{do} \\
\hline
\end{array}
\]

\[
\lambda \tilde{\alpha}.(\text{verb+tense} | \tilde{\alpha} \tilde{\beta}.(\tilde{\beta} \lambda h.((h \lambda u.((u \tilde{\alpha}) [su])) | [do]))))
\]

\[
= \lambda c.((c (\text{past} \circ (\text{tease} \circ \text{leopold} \circ \text{molly})))
\]

A type assignment of the form $(B \odot C) \odot A$ is appropriate for an infix functor that associates with a particular host $A$, as in the verb+tense combination. We can use the composition $\lambda_1(\cdot)^8$ for infixes that have no such host requirements, and can be placed freely within their domain of application. Examples that come to mind are parenthetical adverbs. A lexical type assignment $s/s$ to an adverb such as ‘hopefully’ only allows it to occur in sentence-initial position, as in ‘Hopefully, John left’. With a doubly-negated type assignment $\lambda_1((s/s)^8)$, the sentence-initial position is still available, because of the contraction $\lambda_1((s/s)^8) \sqcup s/s$, but in addition the word can occupy any sentence-internal position, as in ‘John, hopefully, left’. ‘John left, hopefully’. In the table below, one finds the continuized interpretation for ‘adv’ with the simple $s/s$ assignment, using a non-logical constant ‘hpfy’ of type $t \rightarrow t$ at the target side,
and for ‘adv’ with the doubly-negated type \(^1((s/s)^0)\). The interpretation for the latter is simply the lifted form of the interpretation of the former.

| source | \(\cdot\) | translation |
|--------|-------|-------------|
| adv : \([s]^+ \rightarrow [s]^+\) | \(\lambda c\lambda p.(c (\text{hpfy} \ p)) : (t \rightarrow t) \rightarrow t\) |
| adv' : \((\lfloor s \rfloor) \rightarrow (\lfloor s \rfloor)\)^+ \(\perp\) | \(\lambda k.(k |\text{adv}|) : ((t \rightarrow t) \rightarrow t) \rightarrow t\) |

These examples must suffice to give the reader an idea of the possible uses of the (dual) Galois connected operations in syntax and semantics.

6. Conclusions, further directions

Where do we go from here? In this paper we have looked at (dual) Galois connected unary type-forming operations. As with the (dual) residuated (co)product family, the concept of Galois connected families generalizes to operations of greater arity. Below, using \textit{ad hoc} notation, the binary case, with a Galois connected triple \(\rightarrow, \&\), and a dual Galois connected triple \(\leftarrow, \oplus, \!\!\!\downarrow\). The (dual) residuated triples are added for comparison: mind the direction of the arrows! The new connectives are downward monotonic in all positions. So far, no linguistic applications have been proposed.

\((rp)\) \(A \rightarrow C/B \iff A \otimes B \rightarrow C \iff B \rightarrow A\!\!\!/C\)
\((drp)\) \(A \leftarrow B \otimes C \iff B \oplus A \leftarrow C \iff B \leftarrow C \sqcap A\)
\((gc)\) \(A \rightarrow C\!\!\!\uparrow B \iff A \!\!\!\downarrow B \leftarrow C \iff B \rightarrow A\!\!\!/\!\!\!\downarrow\!
\((dgc)\) \(A \leftarrow C \!\!\!\downarrow B \iff A \!\!\!\uparrow B \rightarrow C \iff B \leftarrow A \!\!\!\uparrow\!

A second theme for further research concerns the distributivity principles \((distr)\) and \((distr)^{-1}\). The analysis of infixation phenomena in this paper relies on the \((distr)\) interactions. In [Bastenhof (2010)], however, one finds an analysis of relativization on the basis of a type assignment \(((n\!\!\!/n) / (s \otimes^0 np))\) to the relative pronoun. For extraction of the gap, this analysis uses the \((distr)^{-1}\) interactions between \(\oplus\) and \(\otimes\); these are combined with the \((distr)\) principles of Fig 2 for the interaction between \(\!\!\!\uparrow\) and \(\otimes\). As we saw above, the \((distr)\) and \((distr)^{-1}\) principles cannot be combined in their full generality.
without spoiling the non-associative and non-commutative character of $\otimes/\oplus$. The mixture of Bastenhof (2010) is one way of avoiding overgeneration. The general picture of a controlled combination of the $(\text{distr})$ and $(\text{distr})^{-1}$ principles is a topic for further research.

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