OPEN-MULTICOMMUTATIVITY OF THE PROBABILITY MEASURE FUNCTOR

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Abstract. For the functors acting in the category of compact Hausdorff spaces, we introduce the so-called open multi-commutativity property, which generalizes both bicommutativity and openness, and prove that this property is satisfied by the functor of probability measures.

1. Introduction

It is well-known that the construction of space of probability measures $P$ is functorial in the category Comp of compact Hausdorff spaces. The functor $P$ is normal in the sense of E.V. Shchepin [1]. It is well-known that the functor $P$ is open, i.e. it preserves the class of open surjective maps. This was first proved by Ditor and Eifler [2]. E.V. Shchepin [1] discovered tight relations between the properties of openness and bicommutativity. In particular, he proved that every open functor in Comp is bicommutative, i.e. preserves the class of the bicommutative diagrams in the sense of Kuratowski.

In this paper we introduce the so-called open multi-commutativity property and show that this property is satisfied by the functor $P$.

2. Preliminaries

2.1. Openness. We say that a functor in Comp is open if it preserves the class of open surjective maps.

2.2. Bicommutativity. A commutative diagram

\[
\begin{array}{c}
D = X \xrightarrow{f} Y \\
g \\
Z \xrightarrow{T}
\end{array}
\]

in Comp is said to be bicommutative if its characteristic map $\chi_D = (f, g) : X \rightarrow Y \times_T Z$ is onto. We say that a functor in Comp is bicommutative if it preserves the bicommutative diagrams (see [1]).

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2.3. **Open-multicommutativity.** Suppose that $O$ is a finite partially ordered set. We regard $O$ as a category and consider a functor $F: O \to \text{Comp}$ (i.e. a diagram in $\text{Comp}$ indexed by the objects of $O$). A cone over $F$ consists of a space $X \in \vert \text{Comp} \vert$ and a family of maps $\{X \to Fd\}_{d \in |O|}$ that satisfy obvious commutativity conditions. Given such a cone, $C = (\{X \to Fd\}_{d \in |O|})$, we denote by $\chi_C: X \to \lim F$ its characteristic map.

We say that the cone $C$ is **open-multicommutative** if its characteristic map is an open onto map.

We say that a functor in $\text{Comp}$ is **open-multicommutative** if it preserves the class of open-multicommutative diagrams.

Note that, if $|O|$ consists of one object then the open-multicommutativity reduces to openness.

3. **Main result**

In the sequel, we need a more detailed description of the diagrams under consideration. We denote the spaces in the diagram by $X_i$, where $i$ is a generic element of a partially ordered set. If $i \geq j$, then we denote by $\varphi_{ij}$ the map from $X_i$ to $X_j$. The limit, $\lim D$, of the diagram $D$ can be naturally embedded into the set

$$\prod \{X_i \mid i \text{ is a maximal element of the set of indices}\}.$$ 

We will denote this embedding by $h$.

One can naturally define the category of diagrams in $\text{Comp}$ with the same index set $O$. We denote this category by $\text{Comp}^O$.

**Lemma 3.1.** The operations of the limits of the inverse systems and the limits of the diagrams commute.

**Proof.** Straightforward. \hfill \Box

Let $D$ be a diagram in $\text{Comp}^O$. Without loss of generality, we may suppose that the set of maximal elements of $O$ is $\{1, \ldots, k\}$. Then $\lim D \subset X_1 \times \cdots \times X_k$.

Consider the diagram $P(D)$. The limit of this diagram is

$$\lim P(D) \subset P(X_1) \times \cdots \times P(X_k).$$

There exists the unique map $\chi: P(\lim D) \to \lim P(D)$.

**Theorem 3.2.** The map $\chi$ is open.

**Proof.** We will need the following lemmas.
Lemma 3.3. Let maps \( f_{X_i} : X_i \to X'_i \) be such that the diagrams

\[
\begin{array}{ccc}
X_i & \xrightarrow{\varphi_{ij}} & X_j \\
\downarrow f_{X_i} & & \downarrow f_{X_j} \\
X'_i & \xrightarrow{\varphi'_{ij}} & X'_j
\end{array}
\]

are commutative for every \( i \geq j \). In other words, \((f_{X_i})\) is a morphism in \( \text{Comp}^O \) of the diagram \( D \) to a diagram \( D' \) in which the spaces and maps are endowed with "prime". Then the diagram

\[
\begin{array}{ccc}
P(\lim D) & \xrightarrow{\chi} & \lim P(D) \\
\left\| \right\| & \downarrow & \left\| \right\|
\end{array}
\]

\[
\begin{array}{ccc}
P\left( \prod_{i=1}^{k} f_{X_i} \circ h^{-1} \right) & \xrightarrow{\chi'} & \lim P(D') \\
\downarrow & & \downarrow
\end{array}
\]

is bicommutative.

Proof. In order to prove the bicommutativity of this diagram we should, given \( \tau_0 \in P(D) \) and \( \mu = (\mu_1, \ldots, \mu_k) \in \lim P(D) \) such that

\[
\chi'(\tau_0) = \prod_{i=1}^{k} P(f_{X_i})(\mu) = (Pf_{X_1}(\mu_1), \ldots, Pf_{X_i}(\mu_k)),
\]

find a measure \( \tau \in P(D) \) for which \( \chi(\tau) = \mu \) and \( P\left( \prod_{i=1}^{k} f_{X_i} \right)(\tau) = \tau_0 \).

Let us define diagrams \( D_i, i \in \{1, \ldots, k\} \) in the following way. We replace every \( X_j, j = i + 1, \ldots, k \), by \( X'_j \) and every \( \varphi_{js} \) by \( \varphi_{js} \circ f_{X_j} \). Every other maps and spaces are determined by the following conditions.

1. If \( s \geq t \) and \( X'_s \in D_i \) then \( X'_s \in D_i \).
2. Only one of \( X_j \) or \( X'_j \) can be in \( D_i \) for every \( j \).
3. If \( s \geq t \) and \( X_s \in D_i \) then \( X_t \in D_i \).
4. For \( s \geq t \) if \( X_s, X_t \in D_i \) then \( \varphi_{st} \in D_i \), if \( X'_s, X'_t \in D_i \) then \( \varphi'_{st} \in D_i \), if \( X_s, X'_t \in D_i \) then \( \varphi'_{st} \circ f_{X_t} \in D_i \).

Now for every \( i \in \{1, \ldots, k\} \) denote by \( D^i \) the square diagram

\[
\begin{array}{ccc}
\lim D_i & \xrightarrow{\pi_i} & X_i \\
\downarrow & & \downarrow f_{X_i} \\
\lim D_{i-1} & \xrightarrow{\pi_i} & X'_i
\end{array}
\]
This diagram is well-defined. Indeed, let \((x_j) \in \lim D_i\) and

\[
H \left( (x_j)_j \right) = h \left( \prod_{j=1}^{i-1} x_j \times f_{x_i} \times \prod_{j=i+1}^k \left( h^{-1}(x_j)_j \right) \right) = (y_j)_j \notin \lim D_{i-1}.
\]

For every \(s \notin \{1, \ldots, k\}\) we consider arbitrary index \(t \geq s\). Without loss of generality we can assume that \(t \in \{1, \ldots, k\}\). There are five possibilities.

1. \(X_t, X_s \in D_i\) and \(X_t, X_s \in D_{i-1}\). Then \(H^i_t(x_t) = y_t = x_t \in X_t\) and \(H^i_t(x_s) = y_s = x_s \in X_s\). But \(\varphi_{ts}(y_t) = \varphi_{ts}(x_t) = x_s = y_s\) therefore \((y_j)_j \in \lim D_{i-1}\).

2. \(X'_t, X'_s \in D_i\) and \(X'_t, X'_s \in D_{i-1}\). Then \(H^i_t(x_t) = y_t = x_t \in X'_t\) and \(H^i_s(x_s) = y_s = x_s \in X'_s\). But \(\varphi_{ts}(y_t) = \varphi_{ts}(x_t) = x_s = y_s\) therefore \((y_j)_j \in \lim D_{i-1}\).

3. \(X_t, X'_s \in D_i\) and \(X_t, X'_s \in D_{i-1}\). Then \(H^i_t(x_t) = y_t = x_t \in X'_t\) and \(H^i_s(x_s) = y_s = x_s \in X'_s\). But

\[
(\varphi_{ts} \circ f_{X_t})(y_t) = (\varphi_{ts} \circ f_{X_t})(x_t) = x_s = y_s
\]

therefore \((y_j)_j \in \lim D_{i-1}\).

4. \(X_t, X'_s \in D_i\) and \(X'_t, X'_s \in D_{i-1}\). Then \(H^i_t(x_t) = y_t = f_{X_t}(x_t) \in X'_t\) and \(H^i_s(x_s) = y_s = f_{X_s}(x_s) \in X'_s\). Since diagrams \((1)\) are commutative

\[
\varphi'_{ts}(y_t) = \varphi'_{ts}(f_{X_t}(x_t)) = f_{X_s}(\varphi_{ts}(x_t)) = f_{X_s}(x_s) = y_s
\]

therefore \((y_j)_j \in \lim D_{i-1}\).

5. \(X_t, X_s \in D_i\) and \(X_t, X'_s \in D_{i-1}\). Then \(H^i_t(x_t) = y_t = x_t \in X_t\) and \(H^i_s(x_s) = y_s = f_{X_s}(x_s) \in X'_s\) and \(\varphi'_{ts} \circ f_{X_t} \in D_{i-1}\). The commutativity of diagrams \((1)\) implies

\[
(\varphi'_{ts} \circ f_{X_t})(y_t) = f_{X_s}(\varphi_{ts}(x_t)) = f_{X_s}(x_s) = y_s
\]

therefore \((y_j)_j \in \lim D_{i-1}\).

These are all possibilities which can happen and this implies that \((y_j)_j \in \lim D_{i-1}\). Thus, the diagram is well defined.

Every diagram \(\mathcal{D}^i\) is bicommutative. Since \(P \pi_1(\tau_0) = P f_{X_1}(\mu_1)\), applying the functor \(P\) to the diagram \(D^1\) one can find \(\tau_1 \in P(\lim D_1)\) such that

\[
P \pi_1(\tau_1) = \mu_1, \quad PH^1(\tau_1) = P \left( h \circ \left( \prod_{j=2}^k 1_{X_j} \times f_{X_i} \times \prod_{j=i+1}^k \left( h^{-1} \right) \right) \right)(\tau_1) = \tau_0.
\]

We assume that for every \(i \in \{1, \ldots, k\}\) we can define \(\tau_i \in P(\lim D_i)\) such that

\[
P \pi_i(\tau_i) = \mu_i, \quad PH^i(\tau_i) = P \left( h \circ \left( \prod_{j=1}^{i-1} 1_{X_j} \times f_{X_i} \times \prod_{j=i+1}^k \left( h^{-1} \right) \right) \right)(\tau_1) = \tau_{i-1}.
\]
This holds due to the fact that
\[ P f_{X_i}(\mu_i) = P \pi_i(\tau_0) = P \pi_i(H^1(\tau_1)) = P \pi_i(H^1(H^2(\tau_2))) = P \pi_i(H^1 \circ H^2 \circ \ldots \circ H^{i-1})(\tau_{i-1}) = P \pi_i(\tau_{i-1}). \]

Consider now the map \( P \left(h \circ \prod_{j=1}^{k} f_{X_j} \circ h^{-1}\right). \) Since
\[
\prod_{j=1}^{k} f_{X_j} = \left(f_{X_1} \times \prod_{j=2}^{k} 1_{X_j}\right) \circ \left(1_{X_1} \times f_{X_2} \times \prod_{j=3}^{k} 1_{X_j}\right) \circ \ldots \circ \left(1_{X_1} \times \prod_{j=s+1}^{k} 1_{X_j}\right) \circ \ldots \circ \left(\prod_{j=1}^{s-1} 1_{X_j} \times f_{X_s} \times \prod_{j=s+1}^{k} 1_{X_j}\right),
\]
we have
\[
P \left(h \circ \prod_{j=1}^{k} f_{X_j} \circ h^{-1}\right)(\tau_k) = P \left(H^1 \circ H^2 \circ \ldots \circ H^k\right)(\tau_k)
= P \left(H^1 \circ H^2 \circ \ldots \circ H^{k-1}\right)(\tau_{k-1}) = \ldots = P \left(H^1\right)(\tau_1) = \tau_0.
\]

As it has been proved before
\[ \chi(\tau_k) = (P \pi_1, \ldots, P \pi_k)(\tau_k) = (\mu_1, \ldots, \mu_k). \]
The measure \( \tau = \tau_1 \) is the measure we were looking for.

\[ \square \]

**Lemma 3.4.** The map \( \chi \) is surjective.

**Proof.** Denote by
\[
\Gamma_1 = \{ j \notin \{1, \ldots, k\} \mid 1 \geq j \}
\]
\[
\Gamma_2 = \{ j \notin \{1, \ldots, k\} \mid 2 \geq j \} \setminus \Gamma_1
\]
\[
\vdots
\]
\[
\Gamma_k = \{ j \notin \{1, \ldots, k\} \mid k \geq j \} \setminus (\Gamma_1 \cup \ldots \cup \Gamma_{k-1}).
\]
We denote the indices in \( \Gamma_i \) as \( j_1^l, \ldots, j_{m_l}^l \), where \( l \in \{1, \ldots, k\}. \) Let \( (\mu_1, \ldots, \mu_n) \in \lim P(D). \)

We have to find a measure \( \tau \in P(\lim D) \) such that \( \chi(\tau) = (\mu_1, \ldots, \mu_n). \) Denote by \( D_{11} \) the following bicommutative diagram
\[
\begin{array}{ccc}
X_1 \times X_{j_1} & \xrightarrow{\pi_1} & X_1 \\
\downarrow{\pi_1} & & \downarrow{\varphi_{1..j_1}} \\
X_1 & \xrightarrow{\varphi_{1..j_1}} & X_{j_1}
\end{array}
\]
Since \((\mu_1, \ldots, \mu_n) \in \lim_D \{P(X_1), \ldots, P(X_k)\}\), applying the functor \(P\) to this diagram we can find a measure \(\tau_{11}\) for which \(P\pi_1(\tau_{11}) = \mu_1\) and \(P\pi_2(\tau_{11}) = \mu_{j_1}\). Next we define the diagram \(D_{it}\), where \(i \in \{1, \ldots, k\}\) and \(j_i \in \Gamma_i\) as

\[
\begin{array}{c}
X_1 \times X_{j_1} \times \cdots \times X_{j_i} \times X_{i} \times \cdots X_{i} \\
\pi_{i12\ldots j_{i-1}} \\
X_1 \times X_{j_1} \times X_{j_1} \times X_{j_{i-1}} \times X_{i} \times X_k \\
\varphi_{i\ldots j_i} \circ \pi_{j_{i-1}12\ldots j_{i-1}}
\end{array}
\]

Applying in natural order for all these diagrams \(D_{it}\), where \(i\) runs from 1 to \(k\) and \(j_i \in \Gamma_i\) we find at the very end a measure \(\tau_n = \tau \in P(\lim D)\). Then for every \(i \in \{1, \ldots, k\}\) we have \(P\pi_i(\tau) = P\pi_i \circ P\pi_{i12\ldots i} \circ P\pi_{i12\ldots i+1} \circ \cdots \circ P\pi_{i12\ldots n-1}(\tau) = \mu_i\). For each \(j \notin \{1, \ldots, k\}\) there exists \(i \in \{1, \ldots, k\}\) such that \(j \in \Gamma_i\) and \(P\phi_{i\ldots j} \circ P\pi_i \circ P\pi_{i12\ldots i+1} \circ \cdots \circ P\pi_{i12\ldots n-1}(\tau) = P\phi_{i\ldots j} \circ P\pi_i(\tau) = \mu_j\). By the definition of the limit of a diagram we have that \(P\phi_{i\ldots j} \circ P\pi_i = P\pi_j\) for every \((i, j) \in \Gamma\) and therefore \(P\pi_j(\tau) = \mu_j\). This proves the lemma. \(\Box\)

1) First we restrict ourselves on the case that \(X_i\) are finite. Since \(\chi\) is an affine surjective map (see Lemma 3.3) of the compact convex polyhedron \(P(\lim D)\) onto the compact convex polyhedron \(\lim P(D)\), the map \(\chi\) is open (see [4]).

2) The case of zero-dimensional spaces \(X_i\) is treated similarly as in [4]. 3) Suppose now that \(X_i\) are arbitrary compact metrizable spaces. There exist zero-dimensional compact spaces \(X_i'\) and continuous surjective maps \(f_i : X_i' \to X_i\). Apply for the diagram

\[
P(\lim D') \xrightarrow{\chi'} \lim P(D')
\]

\[
P\left(h_0 \prod_{i=1}^k f_i \circ h^{-1}\right) \downarrow \quad \downarrow h_0 \prod_{i=1}^k P(f_i) \circ h^{-1}
\]

\[
P(\lim D) \xrightarrow{\chi} \lim P(D)
\]

the lemma 2 and the lemma 2.1 from [1] together with the fact that the characteristic map \(\chi'\) is open we see that the map \(\chi\) is open as well. \(\Box\)

Now we are going to prove the main result.

**Theorem 3.5.** The probability measure functor \(P\) is open-multicommutative.

**Proof.** Consider an arbitrary cone \(\{T, h_i, i = 1\ldots n\}\) of the diagram \(D\) such that the characteristic map

\[
\chi_{T, D} : T \to \lim D
\]

is surjective and open. It was proved before that the map

\[
\chi : P(\lim D) \to \lim P(D)
\]
is surjective and open. Since the functor $P$ preserves open and surjective maps, the map

$$P(\chi_{T,D}) : P(T) \to P(\lim D)$$

is also open and surjective. Consider the composition $\chi \circ P(\chi_{T,D})$. For every measure $\nu \in P(\lim D)$ we have $\chi(\nu) = (\nu_1, ..., \nu_n)$ where $\nu_i = P\pi_i(\nu) \in P(X_i)$. Let $\mu \in P(T)$ and $\varphi \in C(T)$. It holds

$$P\pi_i \circ P(\chi_{T,D})(\mu)(\varphi) = P(\chi_{T,D})(\mu)(\varphi \circ \pi_i)$$

$$= \mu(\varphi \circ \pi_i \circ \chi_{T,D}) = \mu(\varphi \circ \pi_i \circ h_i) = Ph_i(\mu)(\varphi).$$

This implies that $\chi \circ P(\chi_{T,D}) = \chi_{P(T),P(D)}$. Thus the characteristic map $\chi_{P(T),P(D)}$ of the cone $\{P(T), Ph_i, i = 1...n\}$ is open and surjective. Since the cone is arbitrarily chosen, this implies that the functor $P$ is open-multicommutative.

\[\square\]

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