Spin-flip quantum transition driven by the time-oscillating Rashba field

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Abstract

Focusing on the spin-flip quantum transition, we study the time-dependent phenomena by the oscillating Rashba spin–orbit interaction. An electron is confined by a harmonic potential surrounded by a cylindrical hard-wall in a two-dimensional (2D) quantum dot. The oscillating Rashba external field having a frequency $w$ is applied perpendicular to the 2D plane. The projection and discrete Fourier transform analyses reveal that the interstate transition causes the characteristic spin-flip quantum transition when the Rashba field has a resonant frequency. Particularly in the cylindrical hard-wall confinement, a typical Rabi oscillation results with a spin flipping. The perturbation approach up to the fourth-order terms satisfactorily explains the origin of the oscillating components found in the spin density.

1. Introduction

The spin–electric coupling via the spin–orbit interaction (SOI) is the typical quantum phenomenon. Over a half-century ago, Rashba [1] theoretically predicted that the SOI couples an electron spin with an electric field. However, it was not until the end of the last century that this coupling has experimentally verified to be manipulated [2]. After the discovery of the Rashba SOI, the remarkable progress in the spin and SOI related phenomena has been sparked such as a single electron coherent oscillations [3], spin resonance [4], a spin field-effect transistor proposed by Datta and Das [5], the Aharonov–Casher effect as a function of Rashba SOI splitting strength [6, 7], a spin filter and analyzer [8], the Rashba SOI coupling in the 2D magnetoeexcitons [9], and the spin current and relating transport phenomena [10–15]. Furthermore, with an extreme development in the nano-fabrication technique of quantum dots (QDs) [16], the spin states in these QDs are considered to be promising for physical realization of the quantum computation algorithm [17–19]. Thus, spintronics [5, 20] opens new avenues of exploration in both science and technology. Manchon et al have reviewed the new perspectives for the SOI coupling [21].

It is well known that an electron spin is a natural choice for creating a quantum bit (qubit) because the spin of a single electron forms a two-level system. Accordingly, the key element is an ability to induce transitions between the spin-up ($\alpha$) and spin-down ($\beta$) states and to prepare arbitrary superpositions of these two basis states [22]. However, comparing with studies of the above static SOI, few works have been carried out for the dynamics with the time-dependent (TD) SOI couplings. Nowack et al [22] have achieved the pioneering experiment, by which they observed coherent transitions (Rabi oscillations) between $\alpha$ and $\beta$ states and succeeded in the coherent control of the single electron spin in the QD using an oscillating electric field generated by a local gate. For the TD SOI coupling phenomena, Földi et al [23] have theoretically studied the similar Rabi flopping in the quantum ring system. They found that the spin components oscillate with Rabi frequency. Considering the future spin–based quantum information, van Berg et al [24] have demonstrated an electronically controlled spin–orbit qubit in an InSb nanowire. They have succeeded to observe the Rabi oscillation frequencies up to 104 MHz. Furthermore, Echeverría–Arrondo and Sherman [25] have studied the
coupled spin and orbital dynamics of a spin-1/2 particle in a harmonic potential subject to ultrastrong SOI and external magnetic field. Walls [26] has studied the spin dynamics of a single electron under parametric modulation of a lateral QD’s electrostatic potential in the presence of spin–orbit coupling.

In this work, we study TD phenomena caused by the spin–electric coupling via the Rashba SOI because the understanding of the TD quantum transitions between these spin $\alpha$ and $\beta$ states is fundamental but crucial in the spintronics. We study how the Rashba external field time-modulate by the frequency $\omega$ causes the quantum transition for an electron confined in the QD by solving the TD Schrödinger equation numerically. We then employ the projection and discrete Fourier transform (disFT) analyses, and study the interstate transitions by a comparison with the analytical results by the perturbation approach and investigate the possibility of the Rabi oscillation driven by the TD Rashba SOI. In order to realize the spintronics, the coherency of the quantum electronic spin states is crucial [27], but is influenced significantly by the spin-flips and/or spin-precession [28]. Therefore, the deep understanding of the spin–flip quantum transition driven by the time-modulate Rashba field is important not only in the elucidation of the spin dynamics but also in the design of the novel devices for the future quantum computation [22]. In section 2, we briefly summarize the influence on the electronic states by the static Rashba SOI. Then, we discuss the spin dynamics caused by the TD Rashba SOI in section 3. We mention how to solve the TD Schrödinger equation numerically (section 3.1) and show the resulting snapshots (section 3.2). We study the spin–flip quantum transitions by the projection analysis in section 3.3. The resulting state probabilities have been investigated by the perturbation approach of the TD expansion coefficients, taking into account up to the 4th order terms (section 3.4). In section 3.5, we discuss the difference in the spin–flip processes, in accordance with the cylindrical hard-wall and harmonic potentials and study the possibility of the Rabi oscillation caused by the TD Rashba SOI (section 3.6). In section 4, we further discuss the non-resonant Rashba SOI, and then conclude. We add the calculational details and others in Appendices.

2. Static system

2.1. Confinement by hybrid potential

We confine an electron by the 2D isotropic harmonic potential of the strength $\omega_0$. We further surround the harmonic potential by the cylindrical hard-wall to realize the practical 2D QD system. Figure 1 illustrates the potential profile of the present 2D QD fabricated by InSb [29]. The hybridized potential is produced on the $100 \times 100$ nm square substrate by the cylindrical potential having a radius $r_0 = 50$ nm. By employing the effective mass ($m^*$) approximation, the static and unperturbed Hamiltonian $\hat{H}_0(x, y)$ is then given by

$$\hat{H}_0(x, y) = \frac{\hat{p}^2}{2m^*} + \frac{1}{2}m^*\omega_0(x^2 + y^2) + V_{\text{cyl}} \equiv \frac{\hat{p}^2}{2m^*} + V_{\text{hyb}}(x, y),$$

where we represent the hybrid potential by $V_{\text{hyb}}(x, y)$. As such, by varying $\omega_0$, we can tune the eigenstate, in accordance with the change in the confinement from the cylindrical hard-wall ($\omega_0 = 0$) to the harmonic potential ($\omega_0 > 15$ effective atomic unit (a.u.), appendix A) through the hybrid form.

We have the corresponding static Schrödinger equation of

$$\hat{H}_0(x, y)\varphi_{nl}^{(0)} = E_{nl}^{(0)}\varphi_{nl}^{(0)},$$

where symbols $n$ and $l$ are a radial and angular quantum number, respectively. We solve this static Schrödinger equation (2) numerically by employing a finite difference technique where the wave function is discretized into square grid points of real space (i.e., numerically exact diagonalization). The second(2nd)-order differential in the kinetic energy is carried out by the central difference method. We employ a Cartesian $x$ and $y$ grid by dividing the real space into a mesh of $64 \times 64$ (1.5625 nm squares), and then carry out the numerical diagonalization in a twofold larger space due to the spin hybridization between the spin-up ($\alpha$) and spin-down ($\beta$) states. Consequently, the eigenstates result being degenerate doubly by $\alpha$ and $\beta$. Since the present 2D QD is supposed to be fabricated by InSb [29], an effective mass (ratio) $m^*/m_0 = 0.0138$ [30] and a dielectric constant (ratio) $\epsilon_0/\epsilon_0 = 17.88$ [31, 32] of InSb are employed in the numerical calculation. We have reported the calculational details in the present numerical diagonalization in our previous works [33–35].

We give the calculated eigenstates for the unperturbed Hamiltonian $\hat{H}_0$ in figure 1. The solid lines indicate those eigenstates for the present 2D hybrid QD whereas the broken lines are those for the ideal harmonic confinement. The yellow line indicates an intercept potential between the harmonic and cylindrical

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3 For the classification of the perturbed eigenstates in this work, we expeditiously use the notation $(n, l)$ of the unperturbed state.
confinements. According to figure 1, the confinement for the ground and lower excited states are well approximated by the harmonic potential when $\omega_0$ is larger than 15 a.u. (appendix A).

2.2. Static Rashba SOI
We here apply the Rashba external electric field $\mathbf{\Xi} = (\Xi_x, \Xi_y, \Xi_z)$ in this 2D system with a time-modulation. Before the discussion of the time-developing phenomena, we summarize the static Rashba phenomena in the 2D QD system. By employing the spin Pauli operator $\hat{\sigma}$, we have the Rashba SOI Hamiltonian $\hat{H}_R$ for an electron confined in the 2D QD by,

$$\hat{H}_R = \hat{\sigma} \cdot (\mathbf{\Xi} \times \hat{\mathbf{p}})$$

$$= \begin{pmatrix}
\Xi_x \hat{p}_y - \Xi_y \hat{p}_x & -\Xi_z (\hat{p}_y + i \hat{p}_z) \\
-\Xi_z (\hat{p}_y - i \hat{p}_z) & -\Xi_x \hat{p}_y - \Xi_y \hat{p}_z
\end{pmatrix}$$

$$= -\Xi_x (\hat{p}_y + i \hat{p}_z) \hat{\sigma}_z - \Xi_z (\hat{p}_y - i \hat{p}_z) \hat{\sigma}_z + (\Xi_x \hat{p}_y - \Xi_y \hat{p}_z) \hat{\sigma}_z. \quad (3)$$

In equation (3), we employ the spin raising (+) and lowering (−) operators $\hat{\sigma}_z$, defined by,

$$\hat{\sigma}_z \equiv \frac{1}{2} (\hat{\sigma}_x \pm i \hat{\sigma}_y),$$

where $\hat{\sigma}_x$ and $\hat{\sigma}_y$ are the spin Pauli matrices. Accordingly, we decompose the Rashba Hamiltonian of equation (3) into three parts by,

$$\hat{H}_R \equiv \hat{H}_R^+ + \hat{H}_R^- + \hat{H}_R^z.$$
with

\[ \hat{H}_R = -\Xi_0 (\hat{p}_x + i\hat{p}_y) \hat{\sigma}_x, \]
\[ \hat{H}_R^x = -\Xi_0 (\hat{p}_y - i\hat{p}_x) \hat{\sigma}_x, \]
\[ \hat{H}_R^z = (\Xi_0 \hat{p}_x - \Xi_0 \hat{p}_y) \hat{\sigma}_z. \]

(4)

Also, we represent the effective Rashba electric field by the polar coordinates;

\[ \Xi = (\Xi_x, \Xi_y, \Xi_z) = \Xi_0 (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta), \]

where \( \Xi_0 \) is the so-called Rashba coupling constant \([36–38]\).

We show how the Rashba SOI changes the unperturbed ground state \((00) = |\psi_0^{(00)}\rangle\) against the applied direction \(\theta\) in figure 2(a). The Rashba SOI causes the largest stabilization when the Rashba electric field is applied perpendicular to the 2D plane \((\theta = 0)\) and the smallest stabilization when \(\theta = \pi/2\). One should further note that the Rashba field of \(\theta = 0\) hybridizes the opposite spin most strongly, and the perturbed ground state \((00\alpha)\) results (figure 2(a)). Also in the first (1st) excited states \((01)\) and \((0\bar{1})\), the Rashba field applied perpendicular to the 2D plane \((\theta = 0)\) resolves the spin degeneration most largely whereas the Rashba field of \(\theta = \pi/2\) restores the degeneration. The completely complementary energetics is found both in the perturbed ground and 1st excited states so as to conserve the Kramers degeneracy between the two perturbed states; e.g., \((00\alpha)\) and \((00\beta)\), and \((01\alpha)\) and \((01\beta)\). These features are caused by the \(b\)-like components of the Rashba SOI \((\hat{H}_R^x\) and \(\hat{H}_R^z\)) in

\[ \text{Figure 2. Rashba SOI energetics of the ground (a) and 1st excited state (b) reproduced from our previous work [33–35]} \]

(Copyright 2014, 2016, 2017, The Japan Society of Applied Physics). We show the energy difference from the unperturbed ground \((n, l) = (0, 0)\) and 1st excited \((0, 1)\) state over varied \(\theta\). The energy difference is normalized by \(n^2 \Xi_0^2\). We also show the spin densities \(\rho^x\) (left) and \(\rho^z\) (right) for the Rashba SOI perturbed state at \(\theta = 0\). Hereafter, we expediently use the unperturbed notation \((n, l)\) for the classification of the perturbed eigenstates. In the present InSb QD, the confinement strength of \(1/\omega\) a.u. corresponds to 11.9 meV.

\[^4\text{We denote the eigenstate by the isotropic confinement of the harmonic potential by } (n, l) \text{ and that by the anisotropic case by } [n, \sigma]. \text{ The corresponding eigenfunction is also denoted by } \psi_n^\alpha \text{ (Laguerre polynomial) for the isotropic harmonic potential and by } \phi_{n\sigma} \text{ (Hermitian polynomial) for the elliptic harmonic potential.}\]
equation (3) and their contributions become most significant at $\theta = 0$. Accordingly, we here apply the time-modulated Rashba field perpendicular to the 2D plane.

3. Dynamical system

3.1. Suzuki–Trotter decomposition

We now study the dynamical properties of the confined electron with the application of the time-modulated Rashba electric field $\hat{F}_R(t)$ by solving the following TD Schrödinger equation:

$$i\hbar \frac{d}{dt}\Psi(x, y; t) = \hat{H}(x, y; t)\Psi(x, y; t).$$  

(5)

Here, the TD Hamiltonian $\hat{H}(x, y; t)$ is given by the sum of the static/unperturbed Hamiltonian $\hat{H}_0$ of equation (1) and the time-modulated Rashba Hamiltonian $\hat{H}_R(x, y; t)$ (appendix B):

$$\hat{H}(x, y; t) = \hat{H}_0(x, y) + \hat{H}_R(x, y; t),$$

(6)

where we tune the Rashba electric field periodically having a frequency $\omega$ by,

$$\hat{H}_R(x, y; t) = -\Xi_0(t) \cdot (\hat{p}_x \hat{\sigma}_x - \hat{p}_y \hat{\sigma}_y),$$

$$= -\Xi_0(t) \cdot [i(\hat{p}_x + i\hat{p}_y) \hat{\sigma}_x + (\hat{p}_y - i\hat{p}_x) \hat{\sigma}_y],$$

(7)

with

$$\Xi_0(t) = \Xi_0 \cdot \sin \omega t.$$  

(8)

We solve the TD Schrödinger equation (5) numerically by employing the real-space and real-time grid approach via Crank–Nicolson method. For this purpose, we rewrite the time-evolution propagator by separating the potential operator from the kinetic one based on Suzuki–Trotter exponential product theory [39–42] with taking into account the decomposition up to the second order of $\Delta t$. One should note that the time-modulated Rashba field also causes the oscillating scalar potential $V_s(t)$ by which the other type of the quantum transition is generated. We can here exclude this type of the quantum transition, as discussed in appendix B.

The general solution of this TD Schrödinger equation (5) is approximately given by,

$$\Psi(x, y; t + \Delta t) \approx \exp \left[ \frac{\Delta t}{\hbar} \hat{H}_R(t + \frac{\Delta t}{2}) \right] \Psi(x, y; t),$$  

(9)

where $\Delta t$ is a time-grid interval (appendix C). By employing the Suzuki–Trotter exponential decomposition approach, the time-development operator of equation (9) is further rewritten into,

$$\exp \left[ \frac{\Delta t}{\hbar} \hat{H}_R(t + \frac{\Delta t}{2}) \right]$$

$$= \exp \left[ -\frac{m}{\hbar} \Xi_0 \left( t + \frac{\Delta t}{4} \right) \hat{\sigma}_x \right] \exp \left[ -\frac{\Delta t}{\hbar} \frac{\hbar^2}{2m} \hat{\sigma}_x \right] \exp \left[ \frac{m}{\hbar} \Xi_0 \left( t + \frac{\Delta t}{4} \right) \hat{\sigma}_x \right]$$

$$\times \exp \left[ +\frac{m}{\hbar} \Xi_0 \left( t + \frac{\Delta t}{4} \right) \hat{\sigma}_y \right] \exp \left[ -\frac{\Delta t}{\hbar} \frac{\hbar^2}{2m} \hat{\sigma}_y \right] \exp \left[ -\frac{m}{\hbar} \Xi_0 \left( t + \frac{\Delta t}{4} \right) \hat{\sigma}_y \right]$$

$$\times \exp \left[ \frac{\Delta t}{\hbar} \left( \frac{1}{2} m\omega_0^2 (x^2 + y^2) + V_s(t) - m\Xi_0^2 \left( t + \frac{\Delta t}{2} \right) \right) \right]$$

$$\times \exp \left[ +\frac{m}{\hbar} \Xi_0 \left( t + \frac{\Delta t}{4} \right) \hat{\sigma}_y \right] \exp \left[ -\frac{\Delta t}{\hbar} \frac{\hbar^2}{2m} \hat{\sigma}_y \right] \exp \left[ -\frac{m}{\hbar} \Xi_0 \left( t + \frac{\Delta t}{4} \right) \hat{\sigma}_y \right]$$

$$\times \exp \left[ -\frac{m}{\hbar} \Xi_0 \left( t + \frac{\Delta t}{4} \right) \hat{\sigma}_x \right] \exp \left[ -\frac{\Delta t}{\hbar} \frac{\hbar^2}{2m} \hat{\sigma}_x \right] \exp \left[ +\frac{m}{\hbar} \Xi_0 \left( t + \frac{\Delta t}{4} \right) \hat{\sigma}_x \right].$$

(10)

Consequently, we can determine the TD wave function by changing the time-grid interval $\Delta t$ through equation (10). The exponential operators for the kinetic term (spatial gradient) are commutative between the $x$ and $y$ components when the Cartesian coordinate system is employed. These components are, then, rewritten using Cayley’s approach into the second order approximation and are further represented in terms of the Crank–Nicolson matrix form [43];
\[
\exp \left[ i \Delta t \frac{\hbar}{2m} \right] \approx 1 + \frac{i \Delta t \hbar^2}{2m \Delta \omega^2} \frac{1}{1 - \frac{i \Delta t \hbar^2}{2m \Delta \omega^2}}
\]

(11)

As such, a pair of simultaneous linear equations is produced, independently toward the directions of \(x\) and \(y\). Each simultaneous linear equation is easily solved because the matrix appearing on the left-hand side is straightforwardly decomposed into the LU form. Thus, this procedure enables us to solve the TD Schrödinger equation directly and quickly with precision [43].

The Suzuki–Trotter exponential approach decompose the present time-evolution operator (equation (10)) into thirteen exponential products and causes a slight complexity in the numerical calculation. However, this decomposition is effective for the CPU time with an order \(O(N)\) of the division number \(N\). In order to confirm the numerical accuracy in the Suzuki–Trotter exponential decomposition approach, we also solve the TD Schrödinger equation (5) directly by the real-time and real-space finite difference method with the Crank–Nicolson diagonal algorithm. This direct difference method is simple to solve the TD Schrödinger equation but consumes lots of CPU time with an order of \(O(N^3)\). We compare both results through the projection analysis of the calculated TD wave functions and confirm the numerical accuracy in the Suzuki–Trotter exponential decomposition approach (appendix D).

### 3.2. Snapshots

We show the snapshots of the \(\alpha\) and \(\beta\) spin densities against time in figure 3. An electron having an \(\alpha\) spin is initially in the ground state of the cylindrical hard-wall \(\omega_0 = 0\) (a) or in that of the harmonic potential \(\omega_0 = 15\) (b), noted by \((00\alpha)\) see footnote 3. The external Rashba field is applied perpendicular to the 2D plane with the resonant frequency \(w = \omega_R\), being equal to the energy difference between the ground and first-excited states; i.e., the minimum excitation \(5\) of \(\omega_R = \frac{\lambda}{2m} \left( \frac{z_1-\lambda_1}{\sqrt{z_1^2}} \right)\) for \(\omega_0 = 0\) and of \(\omega_R = 15\) for (b). Here symbols \(z_{0,1}\) and \(z_{0,1}\) are zero points of the Bessel function, mentioned in appendix A.

Figure 3(a) demonstrates that the \(\alpha\) spin decreases whereas the \(\beta\) spin increases \((t \sim 3\) a.u.). The density distribution of the induced \(\beta\) spin has a single node toward the radial direction. At \(t \sim 6\), the \(\alpha\) spin ‘fully’ disappears whereas the \(\beta\) spin becomes tangible. After that, the \(\beta\) spin decreases but the \(\alpha\) spin increases, and the system at \(t \sim 9\) provides the same spin–distributions found at \(t \sim 3\). The system then restores completely the initial spin distribution at \(t \sim 12\). These TD features repeat periodically.

Figure 3(b) is the snapshots of the spin densities \(\alpha\) and \(\beta\) confined by \(\omega_0 = 15\). The electron initially in the ground state \((00\alpha)\) reduces the \(\alpha\) spin but increases the \(\beta\) spin \((t \sim 3)\). The induced \(\beta\) spin shows the distribution with a single radial node, as found in the cylindrical hard-wall confinement. In the cylindrical hard-wall confinement, we also find the ‘full’ exchange from the \(\alpha\) spin to the \(\beta\) one at \(t \sim 6\). However, we cannot find such a full spin-exchange in the harmonic confinement of \(\omega_0 = 15\). At \(t \sim 4\), a characteristic distribution of the double radial nodes is found in the \(\alpha\) spin. This feature indicates that the excitation into higher states results. Furthermore, we cannot find the peculiar time at which the system completely restores the initial state within the short time-period. A comparable localization of both spin densities to the center is also characteristics of the eigenfunctions in the harmonic confinement (b).

### 3.3. State probability

We discuss the above TD features by employing the projection analysis. We project the numerically calculated TD wave function \(|\Psi(t)\rangle_S\) into the eigenstate \(|\psi_{nl}\rangle\) of the unperturbed Hamiltonian \(\hat{H}_0\) of equation (1), and calculate the state probability \(P^n_{\alpha}(t) = |\langle \psi_{nl} | \Psi(t; t) \rangle|^2\) against time by varying the eigenstate \((n\ell\sigma)\).

For the electron confined in the cylindrical hard-wall, we find the meaningful two state–probabilities \(P^n_{\alpha}(t)\) and \(P^n_{\beta}(t)\), and show them in figure 4(a). It elucidates that the electron initially in the ground state having the spin \(\alpha\), \((00\alpha)\), transits into the first excited state with changing its spin oppositely, \((01\beta)\). At \(t \sim 3\), the electron is stochastically equally in both the ground and 1st excited states having the opposite spin. Then, at \(t \sim 6\), the electron is resonantly excited into the 1st excited state by changing its spin completely. After that, the electron excited in the state \((01\beta)\) returns to the ground state with recovering the original \(\alpha\) spin and the system restores the initial state at \(t \sim 12\). Thus, the electron repeats the resonant interstate transition between the ground and 1st excited states periodically with the characteristic spin exchanging/flopping. Consequently, the electron confined in the cylindrical hard-wall results in the interstate (Rabi-like) oscillation. One can find the small amount of the state probability of the 2nd excited state \((10\alpha)\) around \(t \sim 6\) and also note the beating due to higher frequencies in the state probabilities of \((00\alpha)\) and \((01\beta)\).

\(^5\) The present cylindrical hard-wall confinement gives the minimum excitation frequency \(\omega_R = \frac{\lambda}{2m} \left( \frac{z_1-\lambda_1}{\sqrt{z_1^2}} \right)\) of an order of \(\text{GHz} \sim \text{THz}\).
We give the state probability for the harmonic confinement against time in figure 4(b). The Rashba field is also time-modulated by having the resonant frequency of $\omega_h = 15$ (b). Hereafter, the value of $\Xi_0 \approx 2 \times 10^{-1}$ a.u. is supposed for the Rashba constant $[37]$. Snapshot times are roughly estimated and indicated in figures in the effective atomic unit (a.u.); the present effective unit of an atomic time is $\frac{h}{2m} \beta = 4.947 \times 10^{-13}$ section with the effective Bohr radius $a_0^*$. The external Rashba field is applied perpendicular to the 2D plane having the resonant frequency $\omega_h$ equal to the minimum excitation energy between the ground and first-excited states; $\omega_h = \frac{1}{2m} \left( \frac{\Delta^2 - \Delta \gamma}{\Delta \gamma} \right)$ for (a) and $\omega_h = 15$ for (b).

![Figure 3](image-url) Snapsots of TD $\alpha$ and $\beta$ spin density profiles. An electron is confined by the cylindrical hard-wall $\omega_h = 0$ (a) or harmonic potential $\omega_h = 15$ (b). Hereafter, the value of $\Xi_0 \approx 2 \times 10^{-1}$ a.u. is supposed for the Rashba constant $[37]$. Snapshot times are roughly estimated and indicated in figures in the effective atomic unit (a.u.); the present effective unit of an atomic time is $\frac{h}{2m} \beta = 4.947 \times 10^{-13}$ section with the effective Bohr radius $a_0^*$. The external Rashba field is applied perpendicular to the 2D plane having the resonant frequency $\omega_h$ equal to the minimum excitation energy between the ground and first-excited states; $\omega_h = \frac{1}{2m} \left( \frac{\Delta^2 - \Delta \gamma}{\Delta \gamma} \right)$ for (a) and $\omega_h = 15$ for (b).

We give the state probability for the harmonic confinement against time in figure 4(b). The Rashba field is also time-modulated by having the resonant frequency of $w = 15 (= \omega_h)$, i.e., the minimum excitation is from the ground state to the 1st excited one. Similarly to the cylindrical hard-wall confinement, the electron is initially in the ground state $(00\alpha)$, and transits into the first excited state $(01\beta)$. However, different from the cylindrical hard-wall confinement, the state probability of $(01\beta)$ amounts at most to $\sim 62\%$ even in its maximum ($t \sim 3.8$). The remaining $38\%$ is shared by the higher excited states of $(10\alpha)$ and $(11\beta)$, so on. Around $t \sim 9.3$, most of the dissipated electron return to the original ground state. Nevertheless, the system does not recover its initial state completely. Parts of the electron still remains the other higher states. The Rashba SOI in the harmonic confinement causes the multiple resonant interstate transitions to the higher excited states.

### 3.4. Expansion coefficients

In order to deepen our understanding on the present TD features, we here study the oscillation components included in each projection $\langle \varphi_0^{(0)} | \mathcal{W}(t) | \varphi_\nu \rangle$. By employing the disFT analysis $[44]$, we extract the oscillation components for the harmonic confinement $\omega_h = 15$, and show them in figures 5(a)–(d). The present disFT analysis finds the oscillation components in the four states of the ground $(00\alpha)$ (a), 1st $(01\beta)$ (b), 2nd $(10\alpha)$ (c)
and 3rd \((11\beta)\) (d) excited states. The disFT amplitudes of the former three states are in the same order but that of the 3rd excited state is smaller than these three formers by an order of magnitude. The disFT analysis further demonstrates that each state has the multiple oscillation components and all those components are interestingly an integral multiplication of the resonant frequency \(\omega_0\). Figures 5 demonstrates that the ground (a) and 2nd (b)
excited states have the same oscillation components of $\pm \omega_0$, $-3 \omega_0$, and $-5 \omega_0$ whereas the 1st (b) and 3rd (d) excited states have those of $\pm 2 \omega_0$ and $-4 \omega_0$. That is, the former two states have the odd numbered multiplication of $\omega_0$ whereas the latter has the even numbered multiplication of $\omega_0$.

By employing the spinor representation, the TD wave function $|\Psi(t)\rangle_S$ of the TD Schrödinger equation (5) is decomposed into,

$$|\Psi(t)\rangle_S = (|\alpha\rangle, |\beta\rangle) \begin{pmatrix} \Phi_\alpha(t) \\ \Phi_\beta(t) \end{pmatrix}. \tag{12}$$

Each spinor function can be further expressed in the expansion form for the unperturbed eigenstates by,

$$|\Phi_\alpha(t)\rangle_S = \sum_{n\alpha} C_{n\alpha}(t) \exp \left( \frac{E_{n\alpha}^{(0)}}{\hbar} t \right) |\varphi_{n\alpha}^{(0)}\rangle,$$

$$|\Phi_\beta(t)\rangle_S = \sum_{n\beta} C_{n\beta}(t) \exp \left( \frac{E_{n\beta}^{(0)}}{\hbar} t \right) |\varphi_{n\beta}^{(0)}\rangle, \tag{13}$$

where $(n\alpha\beta)$ is the quantum numbers of the eigenstate of the static/unperturbed Hamiltonian (1), and $|\varphi_{n\alpha}^{(0)}\rangle$ and $E_{n\alpha}^{(0)}$ are the corresponding eigenfunction and eigenvalue, respectively. When the TD external field $\hat{V}(t)$ is small enough, the perturbation expansion approach further rewrites the TD expansion coefficient $C_{n\alpha\beta}(t)$ of equation (13) by,

$$C_{n\alpha\beta}(t) = C_{n\alpha\beta}(t_0) + \sum_{m\alpha'} \left( \frac{1}{\hbar} \right) \int_{t_0}^{t} dt' V_{n\alpha\beta,m\alpha'}(t') \cdot C_{m\alpha'}(t_0) + \sum_{m\alpha'} \sum_{k\alpha''} \left( \frac{1}{\hbar} \right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' V_{n\alpha\beta,k\alpha''}(t') V_{k\alpha'',m\alpha'}(t'') \cdot C_{m\alpha'}(t_0) + \ldots. \tag{14}$$

Here, the TD transition matrix element $V_{n\alpha\beta,m\alpha'}(t)$ is given by,

$$V_{n\alpha\beta,m\alpha'}(t) = \exp \left( \frac{E_{n\alpha}^{(0)} - E_{m\alpha'}^{(0)}}{\hbar} t \right) \langle \varphi_{n\alpha}^{(0)} | \hat{V}(t) | \varphi_{m\alpha'}^{(0)} \rangle. \tag{15}$$

Consequently, the projection $\langle \varphi_{n\alpha}^{(0)} | \Psi(t) \rangle_S$ is given by,

$$\langle \varphi_{n\alpha}^{(0)} | \Psi(t) \rangle_S = C_{n\alpha}(t) e^{-i\omega_{n\alpha}^{(0)} t}, \tag{16}$$

where $\omega_{n\alpha}^{(0)} = E_{n\alpha}^{(0)}/\hbar$.

Since the present Rashba external field is applied perpendicular to the 2D plane with the frequency $\omega$, the corresponding TD perturbation potential $\hat{V}(t)$ is given by

$$\hat{V}(t) = \hat{H}_R \cdot \sin \omega t = \frac{\hat{H}_R}{2\hbar} e^{-i\omega t} + \frac{\hat{H}_R}{2\hbar} e^{i\omega t} \equiv e^{-i\omega t} + e^{i\omega t}. \tag{17}$$

Here, the time-independent operator $\hat{F}$ in equation (17) is defined by

$$\hat{F} = -\frac{\hat{H}_R}{2\hbar} = \frac{\Xi}{2\hbar} (\hat{p}_x \hat{\sigma}_x - \hat{p}_y \hat{\sigma}_y). \tag{18}$$

Owing to the feature of the Rashba SOI, one should note the following relation;

$$\langle n|\hat{F}|m \rangle \equiv E_{nm} = -\frac{1}{2\hbar} \langle n|\hat{H}_R|m \rangle,$$

$$\langle n|\hat{F}^\dagger|m \rangle = \frac{1}{2\hbar} \langle n|\hat{H}_R|m \rangle = -E_{nm}. \tag{19}$$

Let us now estimate the expansion coefficients for the present case, where the electron is initially in the ground state $|\varphi_{00}^{(0)}\rangle$ having an $\alpha$ spin, $(00\alpha)$. Namely, the initial condition is given by

$$C_{n\alpha}(0) = 1, \quad C_{n\beta}(0) = 0 \quad (n \neq 00\alpha). \tag{20}$$
For example, we calculate the expansion coefficient $C_{00x}(t)$ for the ground state $(00\alpha)$ based on the approximation up to the fourth-order perturbation theory (appendix E):

$$C_{00x}(t) \approx C_{00x}(0) + c_{00x}^{(1)}(t) + c_{00x}^{(2)}(t) + c_{00x}^{(3)}(t) + c_{00x}^{(4)}(t).$$

(21)

These TD coefficients by the perturbation expansion up to the 4th order terms are explicitly given by,

\[
\begin{align*}
\text{1st order component } c_{00x}^{(1)}(t): \\
&= \frac{1}{i\hbar} \int_0^t dt' V_{00x,00x}(t') \cdot C_{m\sigma'}(0), \\
&= \frac{F_{00x,00x}}{i\hbar} \int_0^t dt' \{e^{-i(\omega_{00x} - \omega_{00x})t'} - e^{-i(\omega_{00x} + \omega_{00x})t'}\} \\
&= \frac{F_{00x,00x}}{\hbar w} \cdot 2(\cos wt - 1).
\end{align*}
\]

(23)

Accordingly, the TD coefficient $c_{00x}^{(1)}(t)$ is simplified by

$$c_{00x}^{(1)}(t) \propto F_{00x,00x} e^{\pm iwt}.$$  

(24)

However, the 1st order transition matrix element of the Rashba Hamiltonian between the same state gives $F_{00x,00x} = 0$. Consequently, the 1st order term $c_{00x}^{(1)}(t)$ has no oscillation component.

\[
\begin{align*}
\text{2nd order component } c_{00x}^{(2)}(t): \\
&= \frac{1}{i\hbar^2} \int_0^t \int_0^t dt' dt'' V_{00x,00x}(t') V_{00x,00x}(t'') \\
&= -\frac{F_{00x,013} F_{013,00x}}{\hbar^2} \int_0^t dt' \left[\frac{1}{i(\omega_{013} - \omega_{00x} - w)} e^{i2\omega_{013} t'} - e^{i(\omega_{00x} - \omega_{013} + w) t'}\right] \\
&\quad - \frac{1}{i(\omega_{013} - \omega_{00x} - w)} \left[1 - e^{i(\omega_{00x} - \omega_{013} + w) t'}\right] \\
&\quad - \frac{1}{i(\omega_{013} - \omega_{00x} + w)} \left[1 - e^{i(\omega_{013} - \omega_{00x} - w) t'}\right] \\
&\quad + \frac{1}{i(\omega_{013} - \omega_{00x} + w)} \{e^{i2\omega_{013} t'} - e^{i(\omega_{00x} - \omega_{013} + w) t'}\}.
\end{align*}
\]

(24)
\[
\frac{F_{00,01,\beta}F_{01,3,00,\alpha}}{\hbar^2} \left[ \frac{1}{i(\omega_{01,\beta} - \omega_{00,\alpha} + w)} + \frac{1}{i(\omega_{01,\beta} - \omega_{00,\alpha} - w)} \right] t \\
- \frac{1}{2w(\omega_{01,\beta} - \omega_{00,\alpha} - w)} e^{-2\omega t} \\
+ \frac{1}{2w(\omega_{01,\beta} - \omega_{00,\alpha} - w)} + \frac{1}{2w(\omega_{01,\beta} - \omega_{00,\alpha} - w)} e^{2\omega t} - \frac{1}{2w(\omega_{01,\beta} - \omega_{00,\alpha} + w)} \\
- w(\omega_{01,\beta} - \omega_{00,\alpha} - w) \left( \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} - \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} \right) \\
+ \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} \left( \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} - \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} \right) e^{i(\omega_{01,\beta} - \omega_{00,\alpha} - w)t} - 1) \\
- \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} - w} \left( \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} - \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} \right) e^{i(\omega_{01,\beta} - \omega_{00,\alpha} + w)t} - 1) \right]. 
\] (25)

Therefore, the TD coefficient \( c_{00,01}^{(2)}(t) \) is simplified by

\[
\begin{align*}
\frac{F_{00,01,\beta}F_{01,3,00,\alpha}}{\hbar^2} \left[ \frac{1}{2w(\omega_{01,\beta} - \omega_{00,\alpha} - w)} e^{-2\omega t} \\
+ \frac{1}{2w(\omega_{01,\beta} - \omega_{00,\alpha} - w)} + \frac{1}{2w(\omega_{01,\beta} - \omega_{00,\alpha} - w)} e^{2\omega t} - \frac{1}{2w(\omega_{01,\beta} - \omega_{00,\alpha} + w)} \\
- w(\omega_{01,\beta} - \omega_{00,\alpha} - w) \left( \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} - \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} \right) \\
+ \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} \left( \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} - \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} \right) e^{i(\omega_{01,\beta} - \omega_{00,\alpha} - w)t} - 1) \\
- \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} - w} \left( \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} - \frac{1}{\omega_{01,\beta} - \omega_{00,\alpha} + w} \right) e^{i(\omega_{01,\beta} - \omega_{00,\alpha} + w)t} - 1) \right]. 
\end{align*} 
\] (26)

One should note that the Rashba SOI results in the non-zero transition matrix elements of \( F_{00,01,\beta} \) and \( F_{01,3,00,\alpha} \). Accordingly, the 2nd order term multiply causes the oscillation components of \( w_{220} = \pm 1, 0 \) because of \( w = \omega_0 \) and the harmonicity of the confinement \( \omega_{01,\beta} - \omega_{00,\alpha} = \omega_0 \).

3rd order component \( c_{00,01}^{(3)}(t) \):

\[
\begin{align*}
\frac{1}{3} \sum \sum \sum \left( \frac{1}{\hbar} \right)^3 \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' V_{00,01,\beta}(t') V_{01,3,00,\alpha}(t'') V_{k_1,\sigma_1,k_2,\sigma_2}(t'''). 
\end{align*} 
\] (27)

We cannot find the appropriate states \( |k_1\rangle \) and \( |k_2\rangle \) which cause the non-zero 3rd order Rashba SOI matrix elements for the initial condition of equation (20). Consequently, \( c_{00,01}^{(3)}(t) = 0 \) results.

4th order component \( c_{00,01}^{(4)}(t) \):

\[
\begin{align*}
\frac{1}{4} \sum \sum \sum \left( \frac{1}{\hbar} \right)^4 \\
\times \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' V_{00,01,\beta}(t') V_{01,3,00,\alpha}(t'') V_{k_1,\sigma_1,k_2,\sigma_2}(t''') V_{k_3,\sigma_3,k_4,\sigma_4}(t'''). 
\end{align*} 
\] (28)

One should remember that the Rashba SOI in the 2D harmonic confinement system restricts the possible intermediate states to produce the non-zero transition matrix elements by \( |k_1\rangle = |01,\beta\rangle, |k_2\rangle = |10,\alpha\rangle, |k_3\rangle = |01,\beta\rangle \). Consequently, the remaining virtual time integration in equation (28),

\[
\int_0^t dt' V_{k_1,\sigma_1}(t'),
\]

produces the oscillation components of \( \omega_{k_1} = \omega_k \pm w \), and the 4th order term has the oscillation components of
We apply the above results for the present harmonic confinement \( \omega_0 = 15 \) with paying our attention to the equivalent energy difference in the neighboring eigenstates due to the harmonicity. We further subtract the eigenfrequency \( \omega_{\text{disFT}} \) for the direct comparison with the projection \( \langle \psi_{\text{disFT}}^{(0)} | \psi(t) \rangle \) (see equation (16)). As such, the Rashba external field having the resonant frequency \( \omega_r = \omega_0 \) causes the oscillation components of \( \mp \omega_0, \mp 3 \omega_0, -5 \omega_0 \) in the ground state projection. These peaks completely coincide with those found by the disFT analysis as shown in figure 5(a). The perturbation approach further elucidates that the components of \( \mp \omega_0 \) and \( -3 \omega_0 \) are caused by the second order perturbation term whereas the component \( -5 \omega_0 \) is caused by fourth order term.

\[
\begin{align*}
(\omega_{00\alpha} - \omega_{01\beta} \pm \omega) & = \omega_{00\alpha} - \omega_{00\alpha} \pm 2\omega, \\
(\omega_{00\alpha} - \omega_{10\beta} \pm \omega) & = \omega_{10\alpha} - \omega_{00\alpha} \pm 2\omega, \\
(\omega_{00\alpha} - \omega_{10\beta} \pm \omega) & = \omega_{00\alpha} - \omega_{10\alpha} \pm 2\omega, \quad \omega_{00\alpha} - \omega_{10\alpha} \pm 2\omega, \quad \omega_{00\alpha} - \omega_{10\alpha} \pm 2\omega, \quad \omega_{00\alpha} - \omega_{10\alpha} \pm 2\omega.
\end{align*}
\]

(29)

Eventually, one should note that the Rashba field applied perpendicular to the 2D plane polarizes the electron spin confined in the harmonic potential and the resonant oscillation of the Rashba field separates the polarized spin state into \( \alpha \) and \( \beta \) completely by the different frequencies. The \( \alpha \) spin state oscillates with the odd multiple harmonic frequencies whereas the \( \beta \) spin state does with the even ones when the electron is initially polarized by the \( \alpha \) state.

We also show the oscillation components for the cylindrical hard-wall confinement \( \omega_0 = 0 \) in figure 6. The disFT analysis reveals that the present time-modulated Rashba field causes the significant interstate transition among three states of \((00\alpha), (01\beta), \) and \((10\alpha)\). Figure 6 further demonstrates that each state has the multiple frequencies, similarly to the harmonic confinement. Although the lack of the equivalency in the energy differences among the neighboring eigenstates causes a complicatedness to represent the disFT peak explicitly in figure 6, the present fourth-order perturbation approach gives the consistent assignment in the disFT peaks of the cylindrical confinement (table E1 in appendix E).
3.5. Multiple transitions

We here discuss the selection rule for the Rashba SOI coupling for an electron confined by the ideal cylindrical hard-wall or harmonic potential, respectively. When the confinement is achieved by the 2D central force field, the interstate SOI transition basically occurs between those states having the different angular momenta (l and l') due to the in-plane momentum components. The present Rashba field is applied perpendicular to the 2D plane, and the ls-like SOI coupling result [33, 34]. Consequently, the conservation of the total angular momentum $j_z = l + s_z$ governs the present Rashba SOI selection rule. Eventually, the interstate Rashba SOI transition matrix element $(\langle nl's'\sigma'|\hat{H}_R|nl\sigma\rangle)$ is given by,

$$\langle nl's'\sigma'|\hat{H}_R|nl\sigma\rangle \propto \delta_{\alpha\alpha'} \delta_{l'+l+\sigma',\sigma},$$  \hspace{1cm} (30)

where $\hat{H}_R$ is the static part of the Rashba SOI Hamiltonian of equation (7) and is given by,

$$\hat{H}_R = -\Xi(\hat{p}_x \hat{\sigma}_y - \hat{p}_y \hat{\sigma}_x).$$ \hspace{1cm} (31)

Since the eigenstate in the cylindrical hard-wall confinement is expressed by the Bessel function, the selection rule is fully given by equation (30). For example, an electron having a spin $\alpha$ is initially in the ground state; i.e., $l = 0$ and $s_z = \frac{1}{2}$. Then, the selection rule for the angular momentum causes the restriction of $l' = l \pm 1$ for the transited state. The conservation of the total angular momentum $j_z = \frac{1}{2}$ of equation (30) further restricts the Rashba SOI transition into the state having $l' = 1$ and $s_z' = -\frac{1}{2}$. Consequently, via the Rashba SOI, the ground state (00$\alpha$) of the cylindrical hard-wall is coupled with those (01$\beta$), (11$\beta$), (21$\beta$), (31$\beta$), ... Furthermore, one should note that the transited state becomes the next ‘initial’ state, and such new interstate transitions occur based on equation (30). For example, the 1st excited state (01$\beta$) is the transited state from the ground state (00$\alpha$), but becomes the new ‘initial’ state having $l = 1$. Then, the selection rule for the angular momentum newly gives $l' = (l \pm 1) = 2$ or 0, but the conservation of the total angular momentum $j_z' = \frac{1}{2}$ restricts the interstate transitions to those having the quantum numbers of $l' = 0$ and $s_z' = \frac{1}{2}$. Consequently, the possible states are those having the different radial quantum number $n$ of (10$\alpha$), (20$\alpha$), (30$\alpha$), (40$\alpha$), ..., namely, in the cylindrical hard-wall confinement, the Rashba SOI causes the ‘multiple and simultaneous’ interstate transitions of (01$\beta$), (11$\beta$), (21$\beta$), (31$\beta$), ... and (10$\alpha$), (20$\alpha$), (30$\alpha$), (40$\alpha$), ..., as illustrated in figure 7. Note that the number of the possible transition states are infinite but all the energy differences among the eigenstates are not equivalent.

In the harmonic confinement, the eigenstates are represented by the Laguerre polynomial, and the condition for the radial quantum number $n' = n \pm 1$ is added to the selection rule. Accordingly, the interstate Rashba SOI coupling is given by,

$$\langle nl's'\sigma'|\hat{H}_R|nl\sigma\rangle \propto \delta_{\alpha\alpha'} \times \delta_{l'\pm l}(\alpha),$$ \hspace{1cm} (32)

Figure 7 illustrates the possible Rashba SOI interstate couplings in the harmonic confinement. The harmonic confinement causes the multiple but inter-neighboring-state couplings, and the conservation of $j_z$ changes the spin state alternately. Eventually, the ‘multiple but successive’ interstate couplings results with the spin flip-flopping, as illustrated in figure 7. The meaning of the ‘successive’ couplings is naturally symbolic and has no actual time-delays, as found in the cylindrical hard-wall confinement.

These different selection rules and energy eigenvalues between the cylindrical hard-wall and harmonic confinements produce the characteristic difference in the Rashba SOI interstate couplings. In the harmonic confinement of $\omega_0$, the Rashba external field having the oscillating frequency $w = \omega_0$ produces the infinite number of the interstate resonant couplings. Contrary, the cylindrical hard-wall confinement does not have the equivalency in the energy difference between the neighboring eigenstates. As such, the resonant interstate coupling is specified uniquely, in accordance with the Rashba excitation frequency $w$. One should, then, remember that the numerical calculations based on the Suzuki–Trotter method result in the not infinite but finite number of the four interstate transitions in the harmonic confinement (figure 4(b)). This inconsistency is caused by the incompleteness of the harmonic confinement in the present hybrid potential where the cylindrical hard-wall terminates the infinite expansion of the Rashba eigenfunctions and breaks the equivalency in the energy difference between the neighboring eigenstates.

In order to understand the multiple resonant-couplings in the ideal harmonic confinement, we rewrite the TD Schrödinger equation (5) into the simultaneous (rate) equations for the expansion coefficients $C_q(t)$, and solve them numerically but directly by assuming the ideal harmonic confinement. The corresponding simultaneous equations is given by
Here, $C_n(t)$ is the TD expansion coefficient defined in equation (13) and $\tilde{V}(t)$ is the TD perturbation potential. Symbols $\varphi_m^{(0)}$ and $E_m^{(0)}$ are the $m$th eigenfunction and eigenvalue of the unperturbed Hamiltonian of equation (1). In equation (33), we appropriately rewrite the quantum state $(k, l, \sigma)$ by a single index $m$ for a simplicity. In the following numerical calculation, we employ thirty eigenstates $\varphi_m^{(0)} (m = 1, 2, \ldots, 30)$ of the 2D harmonic confinement (appendix A), and solve the simultaneous equation equation (33) by the finite difference method. We also suppose the minimum excitation from the ground state to the 1st excited state of the 2D harmonic potential of $150 \omega_0 = \omega_0$. Accordingly, the Rashba field applied perpendicular to the 2D plane has the oscillation frequency $\omega_0$. In the calculation of equation (33), we further estimate the interstate Rashba SOI matrix elements analytically by using the Laguerre polynomial eigenfunctions. We tabulate several of those results in table A1 in appendix A.

In figure 8, we show the resulting state-probability obtained by equation (33) against time. Figure 7 and equation (32) predict that the TD Rashba SOI with the frequency $\omega_0 = 15$ causes the multiple and successive interstate couplings between the neighboring eigenstates with changing the spin alternately. The solution of the simultaneous equation (33) surely reproduces all the possible interstate couplings among the employed 30 eigenstates, as shown in figure 8. One should, however, note the monotonous reduction in each state-probability. This reduction prevents the restitution of the initial state, and causes no periodicity in the interstate oscillation.

3.6. Rabi oscillation

The Rabi oscillation is the typical dynamics found in the interstate resonant transition caused by the oscillating TD external field. We here discuss whether the TD Rashba field causes the Rabi oscillation or not. We focus on the hybrid confinement system of $\omega_0 = 0 \omega_0$ (a) and 15 (b), and the external Rashba field is periodically oscillating with the resonant frequency $\omega_R$ of the minimal excitation from the ground state to the 1st excited state. By

---

$\frac{i\hbar}{\lambda} \frac{d}{dt} C_n(t) = \sum_i \exp \left( - \frac{E_m^{(0)} - E_n^{(0)}}{\hbar} t \right) \langle \varphi_n^{(0)} | \tilde{V}(t) | \varphi_m^{(0)} \rangle \cdot C_m(t)$. (33)

---

Figure 7. Possible interstate transitions by the Rashba SOI in the cylindrical hard-wall or harmonic potential confinement. An electron is supposed to be initially in the ground state $(00)$ having an $\alpha$ spin.
varying the Rashba SOI coupling constant $\Xi_0$, we calculate the TD state-probability and estimate the time period of the interstate transitions; i.e., the interstate (Rabi) frequency $\Omega$. We give the resulting $\Omega^2$ against $\Xi_0$ by red squares in figures 9(a) and (b). We also show the linear dependence of $\Omega^2 - \Xi_0^2$ based on the least square method (LSM) by the red dots lines. Thus, in the cylindrical hard-wall system ($\omega_0 = 0$ (a)), a remarkable linearity in $\Omega \propto \Xi_0$ results. Contrary, the corresponding linearity decreases in the harmonic confinement system ($\omega_0 = 15$ (b)).

When the interstate transition occurs resonantly between the two states of the initial (ground state $|n\rangle$) and the final (excited state $|k\rangle$) only, the rotational wave approximation (RWA) determines the state probability at time $t$ by,

$$P_i(t) = |C_i(t)|^2 \propto \sin^2\left(\frac{\Omega t}{2}\right), \ (i = n \ or \ k).$$  \hspace{1cm} (34)

Here, $\Omega$ is the Rabi frequency of the inter-two-state resonant transition. By using the interstate transition matrix element $F_{kn}$ defined by equation (19), we have

$$\Omega^2 = \frac{4}{\hbar^2}|F_{kn}|^2.$$  \hspace{1cm} (35)

Thus, the interstate Rabi frequency $\Omega$ increases linearly against the strength of the interstate coupling. For the present Rashba SOI coupling of equation (18), this matrix element $F_{kn}$ is given by,
\[ F_{kn} = -\frac{1}{\hbar} \langle k | \hat{H}_B | n \rangle = -\frac{1}{\hbar} \varepsilon_0 \langle k | \hat{p}_x \sigma_x - \hat{p}_z \sigma_z | n \rangle. \]  
\r
Consequently, under the two-state transition approach with the RWA, the interstate Rabi frequency \( \Omega \) is proportional to the Rashba SOI coupling constant \( \varepsilon_0 \) and given by,

\[ \Omega^2 = \frac{1}{\hbar^2} | \langle 000 | \hat{p}_x \sigma_x - \hat{p}_z \sigma_z | 010 \rangle |^2 \cdot \varepsilon_0^2 \equiv D^2 \cdot \varepsilon_0^2. \]  
\r
We can estimate this constant \( D \) by the two approaches; the numerical estimation \( D_{\text{LSM}} \) by the LSM application, and the direct calculation \( D \) by

\[ D = \frac{|\langle 000 | \hat{p}_x \sigma_x - \hat{p}_z \sigma_z | 010 \rangle|}{\hbar}. \]  
\r
We then compare these two values \( D_{\text{LSM}} \) and \( D \).

We return the change of the state probability against time in the cylindrical hard-wall confinement. Figure 4(a) elucidates that the Rashba electric field having the minimum excitation frequency \( \omega = \frac{\hbar}{2m} \left( \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_f} \right) \) approximately causes the inter-two-state transition between \((000)\) and \((01\beta)\). Namely, the linear relationship \( \Omega \propto \varepsilon_0 \) found in figure 9(a) is consistently explained by the two-state transition with the RWA. The LSM application to the cylindrical hard-wall confinement gives \( D_{\text{LSM}} = 6.87 \) (red dots line in figure 9(a)). The value of \( D = 7.84 \) (green line in figure 9(a)) is also calculated by equation (38), where the interstate Rashba SOI transition matrix element is directly obtained by the Bessel eigenfunctions of the states \((000)\) and \((01\beta)\) (appendix A). We can find the consistent coincidence between these two constants \( D_{\text{LSM}} = 6.87 \) and \( D = 7.84 \). A slight discrepancy from the relation \( \Omega^2 \propto \varepsilon_0^2 \) is found when the Rashba coupling \( \varepsilon_0 \) increases (figure 9(a)). With an increase in \( \varepsilon_0 \), one should take into account the non-resonant transitions into higher states (figures 4 and 6). Consequently, the inter-two-state transition and RWA approach is inapplicable and overestimates the proportional constant \( D \), as discussed in section 4.

We also discuss the proportional constant \( D \) in the hybrid confinement of \( \omega = 15 \). The LSM application gives \( D_{\text{LSM}} = 6.66 \) (red dots line in figure 9(b)), whereas equation (38) gives \( D = 15.0 \) (green line) where we employ the Laguerre polynomial eigenfunctions to estimate the interstate Rashba SOI coupling (appendix A). The two-state transition with the RWA gives the inconsistently larger \( D \) value in the hybrid confinement. The confinement of \( \omega = 15 \) causes the considerable harmonicity as found in figure 1. Accordingly, the TD Rashba external field having the resonant frequency \( \omega = \omega_0 \) has a potential to produce the multiple resonant transitions by the Rashba SOI. Actually, figure 4(b) elucidates that the inter-three-state transitions occur multiply among \((000)\), \((01\beta)\) and \((11\alpha)\). We then obtain the value of \( D \) by solving the simultaneous equation (33) numerically with taking into account these three states. We illustrate the calculated \( \Omega^2 \propto \varepsilon_0^2 \) relation (purple dots line) in figure 9(b). The three-state approximation still causes the relation in \( \Omega^2 \propto \varepsilon_0^2 \) (appendix F) and reduces the proportional constant \( D = 7.47 \) from 15.0, significantly approaching to \( D_{\text{LSM}} = 6.66 \). Thus, in the present hybrid confinement of \( \omega = 15 \), the TD Rashba SOI couplings are well understood by the inter-three-state transitions.

With an increase in the number of the possible transition states, the slope of the relationship \( \Omega = \Delta \) decreases (figure 9(b)). The ideal harmonic confinement has the infinite number of the possible states by the resonant transition. Eventually in the static state, an infinitesimal state-probability is distributed in each state, and the interstate oscillation of \( \Omega \) is not generated. Namely, the finite number of the possible interstate transitions is crucial for the Rabi oscillation for the non-dissipated system. When the interstate transitions occur infinitely, the system cannot return its initial state.

4. Non-resonant Rashba SOI

In this section, we study the non-resonant TD phenomena when the time-modulated Rashba field has a detuning frequency \( \delta \) from the resonant one \( \omega_B \). We here set \( \delta \) by a quarter of \( \omega_B \). Accordingly, the practical detuning is \( \delta = \frac{\hbar}{2m} \left( \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_f} \right) / 4 \) for \( \omega_0 = 0 \) and \( \delta = \frac{15}{4} \) for \( \omega_0 = 15 \) because we consider the minimum excitation from the ground state to the 1st excited state for both confinements. We then calculate the state probability based on the ST-EDM approach, and obtain the frequency components of the projections by employing the disFT analysis, similarly to the resonant cases described above.

We show the resulting state probabilities against time in figures 10(a) and (b). The comparison with those in the resonant excitation (figures 4(a) and (b)) reveals that the simple ‘sinusoidal’ interstate oscillation results but the interstate exchanges are not complete in both systems of \( \omega_0 = 0 \) and 15. It is also characteristic that the interstate transitions to the (2nd and 3rd) higher excited states are not remarkable. In spite of the sinusoidal oscillation, the disFT analysis elucidates that the larger numbers of the frequency components are included in
the non-resonant interstate oscillation (figures 11 and 12). The perturbation approach reveals that these frequency components are produced by the detuning $\delta$ which splits the resonant peaks multiply. In the following, we discuss how the non-resonant Rashba SOI coupling splits the resonant peaks via the case of the harmonic confinement ($\omega_0 = 15$).
Figure 12. Distribution of the oscillation components in the projection confined by the cylindrical hard-wall (figure 10(a)) by the disFT analysis. The Rashba field is oscillated by the non-resonant frequency of \( \omega = \frac{\delta}{|r|} \left( 1 + \frac{1}{4} \right) \). The disFT analysis elucidates that the meaningful projected eigenstates are (00\(\alpha\)), (01\(\beta\)), and (10\(\alpha\)).

Table 2. Possible oscillation components for the resonant and non-resonant excitations in the harmonic confinement. The Rashba field is time-modulated by the frequency \( \omega \) and we consider the minimum excitation from the ground state to the 1st excited state. The resonant frequency is, then, \( \omega_0 = \omega_{\text{Rashba}} = 15 \).

| Term \( \text{Resonant} (\omega = \omega_0) \) | Non-resonant \( (\omega = \omega_0 + \delta) \) |
|--------------------------------------|--------------------------------|
| \( \chi_{10}^{(1)}(t) \) | \( -2\omega_0 \) \( -2\omega_0 - \delta, \delta \) |
| \( \chi_{10}^{(2)}(t) \) | \( -3\omega_0 \) \( -3\omega_0 - 2\delta, -3\omega_0 - \delta \) |
| \( \chi_{10}^{(3)}(t) \) | \( -3\omega_0 \) \( -\omega_0 + 2\delta, -\omega_0 + \delta \) |
| \( \chi_{10}^{(4)}(t) \) | \( -4\omega_0 \) \( -4\omega_0 - \delta \) \( -4\omega_0 - 2\delta, -4\omega_0 - \delta \) |
| \( \chi_{10}^{(5)}(t) \) \( \chi_{10}^{(6)}(t) \) | \( -2\omega_0 - 2\omega_0 \) \( 2\omega_0 + 3\delta, -2\omega_0 - \delta \) \( -\omega_0 - 2\omega_0 + \delta \) \( 2\delta, \delta \) |
| \( \chi_{10}^{(7)}(t) \) | \( -5\omega_0 \) \( -5\omega_0 - 4\delta, -5\omega_0 - 3\delta \) \( -5\omega_0 - 2\delta, -5\omega_0 - \delta \) |
| \( \chi_{10}^{(8)}(t) \) | \( 3\omega_0 - 3\omega_0 \) \( -3\omega_0 - 3\delta \) \( 3\omega_0 - 3\delta \) |
| \( \omega_0 - \omega_0 \) | \( \omega_0 + 3\delta, \omega_0 + 2\delta \) \( -\omega_0 + 2\delta, -\omega_0 + \delta \) |

The disFT analysis extracts the oscillation components in the non-resonant SOI couplings in figure 10(b). We show those of the ground (00\(\alpha\)), 1st (01\(\beta\)), 2nd (10\(\alpha\)) and 3rd (11\(\beta\)) excited states separately in figures 11(a)–(d), respectively. We should remember that the projection to the ground state includes the five harmonic sounds in the resonant excitation (fourth order perturbation treatment). They are the odd-number multiplied frequencies of \( \mp \omega_0, \mp 3\omega_0 \) and \( -5\omega_0 \) (figure 5(a)). The disFT analysis for the non-resonant excitation demonstrates that these four resonant peaks split multiply as shown in figure 11(a). Similar peak-splittings are further found in the other states of (01\(\beta\)), (10\(\alpha\)) and (11\(\beta\)) (figures 11(b)–(d)).

We calculate the non-resonant oscillation components in the harmonic confinement up to the 4th order perturbed terms, and summarize them in table 2 with the comparison of those resonant components. As mentioned in section 3.4, the 2nd order perturbation approach elucidates that the expansion coefficient \( C_{00\alpha}(t) \) for the ground state includes the two oscillation components of \( -\omega_0 \) and \( -3\omega_0 \) by the minimal resonant excitation. The 4th order perturbation approach further adds the other three components of \( \omega_0, 3\omega_0 \) and \( -5\omega_0 \). In the non-resonant excitation, the perturbation approach predicts that these frequency components are splitted multiply into \( -5\omega_0 \) \( -2\delta, -3\omega_0 - \delta, -\omega_0 + 2\delta, -\omega_0 + \delta \) and \( -5\omega_0 - 4\delta, -5\omega_0 - 3\delta, \cdots \), as shown in table 2. By changing the eigenstate index \( k \) in table 2, we can obtain the non-resonant oscillation component.
For the Rabi SOI coupling, we replace the transition matrix element under the resonant coupling with the RWA does not change the relation based on the three-state (purple dot line) by solving the system equation \((33)\) numerically.

We also give the disFT oscillation components in the cylindrical hard-wall confinement under the non-resonant excitation \(w = \omega_R + \delta\) in figure 12. The perturbation approach predicts accurately the characteristic peaks and explains the characteristic peak-splittings even in the non-resonant excitation.

Finally, we study how the interstate Rabi oscillation changes under the non-resonant Rashba TD field. In figures 13(a) and (b), we give the change in the interstate Rabi frequency \(\Omega\) by varying the Rashba coupling constant \(\Xi_0\). In the cylindrical confinement (\(\omega_R = 0\)), one can find the relationship of \(\Omega^2 \propto \Xi_0^3\) even in the non-resonant excitation (blue squares in figure 13(a)). We also show the corresponding relationship of \(\Omega^2 \propto \Xi_0^3 \) under the resonant coupling (red squares). The LSM procedure elucidates the proportional constant of \(D_{\text{LSM}}^\text{non-res} = 8.36\) (non-resonant) and \(D_{\text{LMS}}^\text{res} = 6.87\) (resonant). Namely, the non-resonant coupling hardly changes the proportional constant \(D\). The non-resonant coupling causes a parallel displacement in \(\Omega^2 \propto \Xi_0^3\) only. Contrary, in the harmonic confinement of \(\omega_R = 15\), less linear dependence results for the non-resonant excitation (blue squares in figure 13(b)). Consequently, the LSM approach gives a proportional constant of \(D_{\text{LSM}}^\text{non-res} = 4.68\) (non-resonant), significantly different from \(D_{\text{LMS}}^\text{res} = 6.66\) (resonant). However, a similar parallel displacement by \(\delta^2 = \left(\frac{\Xi_0^2}{\delta^2}\right)\) also appears between non-resonant (blue squares) and resonant (red ones) results.

We apply the two-state transition approach with the RWA to the non-resonant excitation \(w = \omega_R + \delta\), and have the interstate oscillation frequency \(\Omega\) by

\[
\Omega^2 = \frac{4}{\hbar^2} F_{kk} F_{kk} \delta^2. \tag{39}
\]

For the Rabi SOI coupling, we replace the transition matrix element \(F_{kk}\) by equation \((36)\) and have

\[
\Omega^2 = \left\langle \left| n | \hat{\sigma}_x - \hat{\sigma}_x | k \right| n \right\rangle \cdot \Xi_0^3 + \delta^2. \tag{40}
\]

Equation \((40)\) elucidates that the relation \(\Omega^2 \propto \Xi_0^3\) is conserved even in the non-resonant excitation. Only a parallel displacement by \(\delta^2\) results.

For the cylindrical hard-wall confinement, we have already calculated the interstate Rashba SOI matrix element \(\left\langle n | \hat{\sigma}_x - \hat{\sigma}_x | k \right| n \rangle\) analytically by using Bessel eigenfunction (appendix A). The calculated proportional constant of \(D = 7.84\) consistently explains the LSM value of \(D_{\text{LMS}} = 6.87\). According to equation \((40)\), the two-state transition with the RWA does not change the relation \(\Omega^2 \propto \Xi_0^3\) and conserves the proportional constant \(D\).
even in the non-resonant excitation. Indeed, the LSM value in the non-resonant excitation is \( D_{\text{LSM}} = 8.36 \) (blue dots line in figure 13(a)) and roughly coincides with \( D = 7.84 \) in the resonant excitation. Consequently, we can conclude that the two-state transition with the RWA reasonably explains the interstate non-resonant oscillation found in the cylindrical hard-wall confinement (figures 10(a) and 13(a)). Furthermore, the resulting parallel shift fully coincides with the theoretical prediction of \( \delta = \frac{\hbar}{2m} \left( \frac{\xi_{l+1}^2 - \xi_l^2}{\xi_l^2} \right) / 4 \).

In figure 13(b), we also plot \( \Omega^2 \) against \( \Xi_{l=0}^2 \) (blue squares) in the harmonic confinement (\( \omega_0 = 15 \)) under the non-resonant excitation. The applied Rashba field similarly has the non-resonant frequency of \( w = \omega_R + \delta \) where \( \omega_R \) is the minimal resonant frequency \( \omega_0 \) and \( \delta = \omega_0 / 4 \). Figure 10(b) elucidates that the inter-two-state coupling occurs between the ground and 1st excited states but the transition is non-resonant. Nevertheless, we can find the relationship \( \Omega^2 - \Xi_{l=0}^2 \) (blue squares) in figure 13(b). As such, the proportional constant estimated by the LSM is \( D_{\text{LSM}} = 4.68 \) (blue dots line), being significantly different from \( D = 15.0 \) predicted by the two-state approach with the RWA (green line). Thus, the two-state transition with the RWA (green line) hardly explains the non-resonant interstate oscillation in the harmonic confinement. For the resonant excitation in the harmonic confinement, one should remember that the three-state approach based on the simultaneous equation (33) consistently explains the interstate oscillation (figure 9(b)). However, even the three-state transition approach cannot explain the interstate non-resonant oscillation (purple dots line in figure 13(b)). This discrepancy is caused because the harmonicity of the confinement causes the complicated multiple transitions even in the non-resonant SOI coupling. Only the parallel displacement \( \delta^2 \) is the common feature in the non-resonant excitation, irrespective of the difference in the confinements.

5. Conclusion

We computationally study the TD phenomena caused by the oscillating Rashba SOI. When the Rashba field is applied to the cylindrical hard-wall confinement system with having the resonant frequency, the typical Rabi oscillation results with causing the spin flipping. The TD Rashba field applied to the harmonic confinement system causes a characteristic beating, and separates the polarized spin state into \( \alpha \) and \( \beta \) completely by the different frequencies. The perturbation approach up to the 4th order terms satisfactorily explains the characteristic beating found in the spin density.

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Appendix A. Eigen states and Rashba SOI interstate matrix element

For the two limited cases, the harmonic and cylindrical confinements, we have the exact eigenfunctions and eigenvalues analytically. For the 2D harmonic confinement, the eigenfunction \( \varphi_{nl}^{(0)}(r) \) is given by,

\[
\varphi_{nl}^{(0)}(r) = A_{nl} z^{l+1} \exp \left( -\frac{z^2}{2} \right) L_{l}^{(0)}(z^2) e^{i\phi},
\]

where the coefficients \( A_{nl} \) and \( \zeta \) of the Laguerre polynomial \( L_{l}^{(0)}(z^2) \) and the variable \( z \) are given by,

\[
A_{nl} = \zeta \left( \frac{n!}{\sqrt{\pi}} \right) \left( \frac{n!}{(n - l)!} \right)!
\]

\[
\zeta = \sqrt{\frac{m\omega_0}{\hbar}}
\]

\[
z = \zeta r.
\]

The corresponding eigenvalue is also given by

\[
E_{nl} = \hbar\omega_0(2n + |l| + 1).
\]

We have also the following eigenfunction and eigenvalue for the cylindrical confinement:

\[
\varphi_{nl}^{(0)}(r) = B_{nl} z^{l+1} \left( \frac{r}{b_0} \right) e^{i\phi},
\]

where the symbol \( J_l \) is the Bessel function and \( z_{l,n+1} \) is the \((n + 1)\)th zero point of the \( l\)th Bessel function. The coefficient \( B_{nl} \) and eigenvalue \( E_{nl} \) are, then, given by,
The Rashba external oscillating field of \( \Xi(t) = \Xi_0 \sin wt \) applied perpendicular to the 2D quantum plane induces the oscillating scalar potential \( V_{\Xi}(t) \).

\[
V_{\Xi}(t) = -e \left( \frac{\Xi_0}{A} \right) \cdot z \cdot \sin wt. \tag{B.1}
\]

Here, \( A \) is the transformation constant of the Rashba external field to the electric one.

The present system as given in figure 1 is the 2D (or quasi-2D) system. For the later discussion, we give the thickness perpendicular to the 2D plane (\( z \) direction) by \( L_z \). An electron is, then, confined toward \( z \) direction by the hard-wall potential. Accordingly, we can represent the eigenfunction \( \phi_{nlm}(x, y, z) \) in the form of the separation of variables; an in-plane part of \( \varphi_{nl}^{(0)}(x, y) \) and an out-of-plane part of \( Z_n(z) \).

\[
\phi_{nlm}(x, y, z) = \varphi_{nl}^{(0)}(x, y) \cdot Z_n(z). \tag{B.2}
\]

As mentioned in appendix A, the in-plane part \( \varphi_{nl}^{(0)}(x, y) \) is given by equation (A.1) for the 2D harmonic confinement and by equation (A.4) for the cylindrical hard-wall potential. The out-of-plane part \( Z_n(z) \) is given by

---

**Table A1.** Rashba SOI interstate matrix element for harmonic confinement. Interstate Rashba matrix elements are estimated by using the Bessel eigenstates for the cylindrical hard-wall confinement and the Laguerre polynomial eigenstates for the harmonic potential confinement.

Note that \( \langle \varphi' \varphi' | \mathcal{H}_R | \varphi \varphi \rangle = \langle \varphi \varphi | \mathcal{H}_R | \varphi' \varphi' \rangle \) because of \( \mathcal{H}_R^{1} = \mathcal{H}_R \) (equation (1)). All values are normalized by \( -\hbar^2 \Xi_0 \).

| Transition matrix element | Cylindrical hard-wall | Harmonic potential |
|---------------------------|-----------------------|-------------------|
| \((00n) | \mathcal{H}_R | (01,l)\) | 2.0710 | 0.3445 |
| \((01n) | \mathcal{H}_R | (00n)\) | 2.0710 | 0.3445 |
| \((01n) | \mathcal{H}_R | (10n)\) | 2.6792 | 0.3445 |
| \((10n) | \mathcal{H}_R | (01,l)\) | 2.6792 | 0.3445 |
| \((10n) | \mathcal{H}_R | (11,l)\) | 4.1315 | 0.4872 |
| \((11n) | \mathcal{H}_R | (10n)\) | 4.1315 | 0.4872 |
| \((11n) | \mathcal{H}_R | (20n)\) | 4.7304 | 0.4872 |
| \((20n) | \mathcal{H}_R | (11,l)\) | 4.7304 | 0.4872 |
| \((20n) | \mathcal{H}_R | (21,l)\) | 6.1539 | 0.5967 |
| \((21n) | \mathcal{H}_R | (20n)\) | 6.1539 | 0.5967 |
| \((21n) | \mathcal{H}_R | (30n)\) | 6.7506 | 0.5967 |
| \((30n) | \mathcal{H}_R | (21,l)\) | 6.7506 | 0.5967 |

\[
B_n^1 = \frac{1}{\sqrt{x_0 l+1}(x_0 l+1)} \tag{A.5}
\]

and

\[
P_n^1 = \frac{\hbar^2}{2m} \left( \frac{x_0 l+1}{x_0 l+1} \right)^2. \tag{A.6}
\]

Based on equations (A.3) and (A.6), we obtain the exact eigenvalues both for the harmonic and cylindrical confinements, and show them in the left (\( \omega_0 = 0 \)) and right (\( \omega_0 = 15 \)) sides of figure 1, respectively. Then, we can confirm the numerical accuracy of the present calculations in the numerical diagonalization by the comparison of the computed eigenstates with those analytical results. The numerically diagonalized eigenstates at \( \omega_0 = 0 \) fully coincide with the analytical results of equation (A.6). Although the surrounding of the harmonic potential by the cylindrical hard-wall causes the finite termination of the harmonic eigenfunctions, the numerically diagonalized eigenvalues (at least, up to 2nd excited states) almost coincide with the analytical results by equation (A.3) when \( \omega_0 < 15 \). In the present hybridized potential, the confinement potential changes from the harmonic to the cylindrical one at \( r = r_0 \). Accordingly, the eigenstate having an energy of \( E < E_0 (\Xi_0 \omega_0 r_0) \) is approximated by the harmonic state \( n \). We, then, introduce the critical index \( N_e = \frac{m^2 \omega_0^2 r_0^2}{2A} \) to judge the characteristics of the hybridized confinement potential. In the present case, \( N_e = 3.84 \) and the 1st and 2nd excited states of the hybridized confinement are approximated by the harmonic states.

By employing the analytical expression of the eigenfunction of equation (A.1) for the harmonic confinement and of equation (A.4) for the cylindrical hard-wall, we can estimate the interstate Rashba SOI couplings. We tabulate several of them in table A1.

**Appendix B. Transition by TD scalar potential**

The Rashba external oscillating field of \( \Xi(t) = \Xi_0 \sin wt \) applied perpendicular to the 2D quantum plane induces the oscillating scalar potential \( V_{\Xi}(t) \).
where we introduce the quantum number \( n_z \).

We study the quantum transition caused by the TD scalar potential \( V_z(t) \). For a convenience of the later discussion, we suppose the confinement is achieved by the ideal 2D harmonic potential and the electron is initially in the ground state \( |00\rangle \) with a spin \( \alpha \). Since \( V_z(t) \) does not include any spin operators, the electron conserves the spin state \( \alpha \) during the corresponding quantum transition. One should further note that the orthogonal character of the in-plane part of equation (B.2) should conserve the in-plane quantum state \( n_l 00 = (00) \) during the quantum transition. Therefore, the transition matrix integration for the in-plane component is unity. As such, the quantum transition by \( V_z(t) \) is simply estimated in terms of the transition matrix element for the out-of-plane component.

The matrix element of the perpendicular transition between the states \( n_z \) and \( m_z \) is given by,

\[
M_{n_z, m_z} \propto \left| \langle Z_{n_z} | Z_{m_z} \rangle \right|^2 = \left( \frac{4[(m_z + n_z) - 1] n_z L_z}{(m_z^2 - n_z^2)^{3/2}} \right)^2 \\
= \begin{cases} 
\frac{8m_z n_z L_z}{(m_z^2 - n_z^2)^{3/2}} & (m_z = n_z: \text{odd}) \\
0 & (m_z = n_z: \text{even})
\end{cases}.
\]  

(B.4)

Since the initial state is the ground state of \( n_z = 1 \) (odd parity), the parity of the eigenfunction toward the \( z \) axis restricts the possible lowest excited state of \( m_z = 2 \). Accordingly, the resulting transition matrix element of equation (B.4) is given by,

\[
\langle Z_{m_z=1} | Z_{m_z=2} \rangle = -\frac{16L_z}{9\pi^2}.
\]  

(B.5)

One should note the transition matrix element is proportional to the thickness \( L_z \). Here, we suppose the 2D and/or quasi-2D systems whose atomic layer thickness \( L_z \) is extremely small. Therefore, we can conclude that the TD transition caused by the TD scalar potential \( V_z(t) \) of equation (B.1) is negligible in the present TD phenomena.
Furthermore, one should consider the following evidence. The transition by the TD scalar potential \( V_{\text{sd}}(t) \) occurs between the two quantum states in the different subband groups having the different \( n_x \) values (figure B1). Accordingly, the corresponding energy difference is inversely proportional to \( L_z^2 \). The small value of \( L_z \) in the present 2D and/or quasi-2D system results in the energy difference, being extremely larger than the energy difference caused by the harmonic potential \( \hbar \omega_b \) as shown in figure B1. This evidence also strengthens the negligibility of the transition by the TD scalar potential \( V_{\text{sd}}(t) \).

**Appendix C. Time-propagator by Suzuki–Trotter decomposition**

A general solution of the TD Schrödinger equation (5) is given as,

\[
\Psi(x, y; t + \Delta t) = \hat{T} \exp \left[ \frac{1}{\hbar} \int_t^{t+\Delta t} \mathcal{H}(x, y; t') \, dt' \right] \Psi(x, y; t),
\]

(C.1)

where symbol \( \hat{T} \) is a time-ordering operator and \( \Delta t \) is a time-grid interval. Then, the time-propagator in equation (9) is further reduced into the following exponential operator;

\[
\hat{T} \exp \left[ \frac{1}{\hbar} \int_t^{t+\Delta t} ds \mathcal{H}(s) \right] = \exp \left[ \frac{\Delta t}{\hbar} \mathcal{H}(t) + \Delta t \mathcal{F} \right] = \exp \left[ \frac{\Delta t}{\hbar} \hat{H}(t) + \Delta t \mathcal{F} \right],
\]

(C.2)

where \( \mathcal{F} \equiv \frac{\partial}{\partial t} \) is the well-known operator defined by the following equation:

\[
A(t) e^{\Delta t \mathcal{F}} B(t) = A(t + \Delta t) B(t).
\]

(C.3)

Consequently, the Suzuki–Trotter decomposition approximates the time propagator in the second order of \( \Delta t^2 \),

\[
\hat{T} \exp \left[ \frac{1}{\hbar} \int_t^{t+\Delta t} ds \mathcal{H}(s) \right] \approx \exp \left[ \frac{\Delta t}{\hbar} \hat{H}(t) + \Delta t \mathcal{F} \right] = \exp \left[ \frac{\Delta t}{\hbar} \hat{H}(t) + \frac{\Delta t}{2} \right] + O(\Delta t^3).
\]

(C.4)

We further rewrite the kinetic and TD Rashba terms into the quadratic form,

\[
\hat{K}(t) + \mathcal{H}_{\text{R}}(t) = -\frac{\hbar^2}{2m} \left\{ \left( \frac{\partial}{\partial x} + \frac{i m}{\hbar} \varepsilon_x(t) \hat{\sigma}_y \right)^2 + \left( \frac{\partial}{\partial y} - \frac{i m}{\hbar} \varepsilon_x(t) \hat{\sigma}_x \right)^2 \right\} - m \varepsilon_z^2(t)
\]

\[
= \hat{K}_x(t) + \hat{K}_y(t) - m \varepsilon_z^2(t),
\]

(C.5)

where we define the renormalized TD kinetic component \( \hat{K}_x(t) \) and \( \hat{K}_y(t) \) by

\[
\hat{K}_x(t) = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial x} + \frac{i m}{\hbar} \varepsilon_x(t) \hat{\sigma}_y \right)^2,
\]

\[
\hat{K}_y(t) = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial y} - \frac{i m}{\hbar} \varepsilon_x(t) \hat{\sigma}_x \right)^2.
\]

(C.6)

We, then, include the last term \(-m \varepsilon_z^2(t)\) in equation (C.5) in to the confinement potential \( V_{\text{hyb}} \), and have the TD scalar potential \( V'(t) \) defined by,

\[
V'(t) = \frac{1}{2} m \omega_b^2 (x^2 + y^2) + V_{\text{eff}}(x, y) - m \varepsilon_z^2(t).
\]

(C.7)

Consequently, the Suzuki–Trotter exponential product decomposition rewrites the time developing operator (equation (C.2)) by,

\[
\exp \left[ \frac{\Delta t}{\hbar} \hat{H}(t) + \frac{\Delta t}{2} \right] = \exp \left[ \frac{1}{\hbar} \Delta t \hat{K}_x(t) + \frac{\Delta t}{4} \right] \exp \left[ \frac{1}{\hbar} \Delta t \hat{K}_y(t) + \frac{\Delta t}{4} \right] \times \exp \left[ \frac{\Delta t}{\hbar} V'(t) + \frac{\Delta t}{2} \right] \times \exp \left[ \frac{1}{\hbar} \Delta t \hat{K}_x(t) + \frac{\Delta t}{4} \right] \exp \left[ \frac{1}{\hbar} \Delta t \hat{K}_y(t) + \frac{\Delta t}{4} \right].
\]

(C.8)
The renormalized kinetic terms $\hat{K}_x$ and $\hat{K}_y$ are further decomposed into

$$
\left( \frac{\partial}{\partial x} + iX_x(x, y) \right)^2 = \exp\left[ -i\Theta(x, y) \right] \left( \frac{\partial}{\partial x} \right)^2 \exp[+i\Theta(x, y)],
$$

where

$$
\Theta(x, y) = \int_0^x dx' X_x(x', y)
$$

Accordingly, the exponential terms including the renormalized TD kinetic terms $\hat{K}_x(t + \frac{\Delta t}{4})$ or $\hat{K}_y(t + \frac{\Delta t}{4})$ in equation (C.8) are expressed by

$$
\exp\left[ \frac{1}{2} \frac{\Delta t}{i\hbar} \hat{K}_x \left( t + \frac{\Delta t}{4} \right) \right] = \exp\left[ \frac{\hbar^2}{2m} \left( \frac{\partial}{\partial x} + i\frac{m}{\hbar}z_x \left( t + \frac{\Delta t}{4} \right) \right)^2 \right] \\
= \exp\left[ -i\frac{m}{\hbar} z_x \left( t + \frac{\Delta t}{4} \right) \right] \exp\left[ -\frac{\Delta t}{i\hbar} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right] \\
\times \exp\left[ +i\frac{m}{\hbar} z_x \left( t + \frac{\Delta t}{4} \right) \right]
$$

$$
\exp\left[ \frac{1}{2} \frac{\Delta t}{i\hbar} \hat{K}_y \left( t + \frac{\Delta t}{4} \right) \right] = \exp\left[ \frac{\hbar^2}{2m} \left( \frac{\partial}{\partial y} - i\frac{m}{\hbar}z_y \left( t + \frac{\Delta t}{4} \right) \right)^2 \right] \\
= \exp\left[ +i\frac{m}{\hbar} z_y \left( t + \frac{\Delta t}{4} \right) \right] \exp\left[ -\frac{\Delta t}{i\hbar} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} \right] \\
\times \exp\left[ -i\frac{m}{\hbar} z_y \left( t + \frac{\Delta t}{4} \right) \right]
$$

Finally, we can rewrite the time development operator by equation

$$
\exp\left[ \frac{\Delta t}{i\hbar} \hat{H} \left( t + \frac{\Delta t}{2} \right) \right] \\
= \exp\left[ -i\frac{m}{\hbar} \left( t + \frac{\Delta t}{4} \right) \right] \exp\left[ -\frac{\Delta t}{i\hbar} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right] \exp\left[ +i\frac{m}{\hbar} \left( t + \frac{\Delta t}{4} \right) \right] \\
\times \exp\left[ +i\frac{m}{\hbar} \left( t + \frac{\Delta t}{4} \right) \right] \exp\left[ -\frac{\Delta t}{i\hbar} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} \right] \\
\times \exp\left[ -i\frac{m}{\hbar} \left( t + \frac{\Delta t}{4} \right) \right] \exp\left[ -\frac{\Delta t}{i\hbar} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right] \\
\times \exp\left[ +i\frac{m}{\hbar} \left( t + \frac{\Delta t}{4} \right) \right] \exp\left[ -\frac{\Delta t}{i\hbar} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} \right] \\
\times \exp\left[ -i\frac{m}{\hbar} \left( t + \frac{\Delta t}{4} \right) \right]
$$

Appendix D. Comparison of the disFT peaks obtained by the Suzuki–Trotter exponential decomposition approach with those by the finite difference method

We give the disFT oscillation components of the projection calculated by the finite difference approach via the Crank–Nicolson method in figure D1. The resulting distribution of the disFT frequencies completely coincide with those obtained by the Suzuki–Trotter exponential decomposition approach (figures 5 and 6).

Appendix E. Expansion coefficients for the general transitions

We here give the expansion coefficient $C_{kl}(t)$ approximately by employing the perturbation expansion up to the fourth-order terms;

$$
C_{kl}(t) \approx C_{kl}(0) + C_{kl}^{(1)}(t) + C_{kl}^{(2)}(t) + C_{kl}^{(3)}(t) + C_{kl}^{(4)}(t),
$$

where we represent the (spatial) quantum number of the unperturbed state simply by the single letter, i.e., $|\psi_{alb}\rangle = |\psi_{kl}\rangle$. We also introduce the initial condition by,
Then, we extract the oscillation components included in $C_{tk}(s(t))$ when the time-modulated Rashba field with the frequency $\omega$ (equation (7)) is applied perpendicular to the 2D plane as the TD perturbation $V(t)$. The expansion coefficients of equation (E.1) are given by the multiple virtual-time integrals by,

$$
\begin{align*}
C_{mob}(0) &= 1, \\
C_{mob}(0) &= 0 \quad (l \neq n, \sigma' \neq \eta).
\end{align*}
$$

(E.2)

A single execution of the virtual-time integration,

$$
\int_0^{t_f} dr V_{k,n}(t) \cdot C_{mob}(0),
$$

causes the oscillation components $\omega_k - \omega_n \pm \omega$, in addition to the static component 0. Consequently, the first-order expansion term $c_{kr}^{(1)}(s(t))$ of equation (E.1) has the oscillation components of $\omega_k - \omega_n \pm \omega$ and 0. The twice executions of the virtual-time integrations in $c_{kr}^{(2)}(s(t))$ causes the oscillation components of

$$
(\omega_k - \omega_n \pm \omega) + (\omega_k - \omega_n \pm \omega) = \omega_k - \omega_n \pm 2\omega \quad \text{or} \quad \omega_k - \omega_n \pm \omega
$$

$$
\omega_k - \omega_n \pm \omega, \quad \omega_k \pm \omega_k \pm \omega, \quad 0.
$$

Accordingly, $c_{kr}^{(2)}(s(t))$ has the following components:

$$
c_{kr}^{(2)}(s(t)) \Rightarrow \omega_k - \omega_n \pm 2\omega, \quad \omega_k - \omega_n, \quad \omega_k - \omega_k \pm \omega, \quad 0
$$

(E.4)

Analogously, the third- and fourth-order terms have the oscillation components of

$$
c_{kr}^{(3)}(s(t)) \Rightarrow \omega_k - \omega_n \pm 3\omega, \quad \omega_k - \omega_n \pm \omega, \quad \omega_k - \omega_k \pm \omega, \quad 0,
$$

$$
\omega_k \pm \omega_k \pm \omega,
$$

(E.5)

$$
c_{kr}^{(4)}(s(t)) \Rightarrow \omega_k - \omega_n \pm 4\omega, \quad \omega_k - \omega_n \pm 2\omega, \quad \omega_k - \omega_n, \quad \omega_k \pm \omega_k \pm 2\omega, \quad \omega_k - \omega_k \pm \omega, \quad \omega_k - \omega_k \pm 2\omega, \quad \omega_k - \omega_k \pm 3\omega, \quad \omega_k - \omega_k \pm \omega, \quad 0.
$$

(E.6)

We tabulate the resulting components included in the higher-order expansion coefficient $c_{kr}^{(i)}(s(t))$ and the corresponding components included in the projection $\langle \varphi_{\alpha \sigma}^{(0)} | \Psi(t) \rangle$ in table E1. In order to estimate the
oscillation components of the resonant Rashba SOI coupling, one should replace the Rashba oscillation frequency \( \omega \) by the resonant frequency \( \omega + \delta \). whereas one can estimate the oscillation components of the non-resonant Rashba SOI coupling by replacing by \( \omega + \delta \).

### Appendix F. Three-state approach

#### F.1. Rate equation

We suppose that an electron is confined by the ideal harmonic potential \( \omega_0 \) and the external field is applied having an oscillation frequency \( \omega = \omega_0 + \delta \) (\( \delta \) is a detuning from the resonant frequency \( \omega_0 \) and \( \delta \ll \omega_0 \)). We consider that the interstate transition occurs among the three states of \( |i\rangle, |m\rangle \) and \( |k\rangle \) in an energy order. Accordingly, we have the following three simultaneous differential equations,

\[
\begin{align*}
\frac{i\hbar}{\partial t} C_m(t) &= V_m(t) \cdot C_m(t) + V_m(t) \cdot C_n(t) + V_m(t) \cdot C_k(t), \\
\frac{i\hbar}{\partial t} C_n(t) &= V_m(t) \cdot C_n(t) + V_m(t) \cdot C_m(t) + V_m(t) \cdot C_k(t), \\
\frac{i\hbar}{\partial t} C_k(t) &= V_m(t) \cdot C_k(t) + V_m(t) \cdot C_m(t) + V_m(t) \cdot C_n(t).
\end{align*}
\]

(F.1)

When the excitation energy is given by \( \hbar \omega_0 + \delta \), the interstate matrix elements of equation (F.1) are given by,

\[
\begin{align*}
V_m(t) &= \exp[-i(\omega_0 + \delta)t] F_{mn} + \exp[i(\omega_0 + \delta)t] F^*_{mn}, \\
V_m(t) &= \exp[-i(2\omega_0 + \delta)t] F_{nn} + \exp[2i(\omega_0 + \delta)t] F^*_{nn}, \\
V_m(t) &= \exp[-i(\omega_0 + \delta)t] F_{mk} + \exp[i(\omega_0 + \delta)t] F^*_{mk}, \\
V_m(t) &= \exp[-i(2\omega_0 + \delta)t] F_{mm} + \exp[2i(\omega_0 + \delta)t] F^*_{mm}, \\
V_m(t) &= \exp[-i(\omega_0 + \delta)t] F_{nk} + \exp[i(\omega_0 + \delta)t] F^*_{nk}, \\
V_m(t) &= \exp[-i(\omega_0 + \delta)t] F_{mn} + \exp[i(3\omega_0 + \delta)t] F^*_{mn}, \\
V_m(t) &= \exp[-i(2\omega_0 + \delta)t] F_{mm} + \exp[2i(2\omega_0 + \delta)t] F^*_{mm}.
\end{align*}
\]

(F.2)

We then employ the RWA to neglect the highly oscillating terms. Eventually, the simultaneous equations of equation (F.1) is reduced to,
\[ i\hbar \frac{\partial}{\partial t} C_n(t) = e^{i\omega t} F^{n*} \cdot C_n(t), \]
\[ i\hbar \frac{\partial}{\partial t} C_m(t) = e^{-i\omega t} F_m \cdot C_m(t) + e^{i\omega t} F^{n*} \cdot C_k(t), \]
\[ i\hbar \frac{\partial}{\partial t} C_k(t) = e^{-i\omega t} F_{km} \cdot C_k(t). \]  

We define \( F \equiv \frac{F_{mn}}{\hbar} \) and \( f \equiv \frac{F_{kn}}{\hbar} \) and rewrite equation (F.3) by,
\[ i\hbar \frac{\partial}{\partial t} C_n(t) = e^{i\omega t} F^{*} \cdot C_n(t), \]
\[ i\hbar \frac{\partial}{\partial t} C_m(t) = e^{-i\omega t} F \cdot C_m(t) + e^{i\omega t} f^{*} \cdot C_k(t), \]
\[ i\hbar \frac{\partial}{\partial t} C_k(t) = e^{-i\omega t} f \cdot C_k(t). \]  

We employ the unitary transformation and rewrite these coefficients \( C_n(t), C_m(t), C_k(t) \) into \( b_n(t), b_m(t), b_k(t) \) by,
\[ C_n(t) = e^{i\delta t} b_n(t), \]
\[ C_m(t) = b_m(t), \]
\[ C_k(t) = e^{-i\delta t} b_k(t). \]  

Consequently, equation (F.4) is rewritten by
\[ i\hbar \frac{\partial}{\partial t} b_n(t) = \delta b_n(t) + F^{*} b_m(t), \]
\[ i\hbar \frac{\partial}{\partial t} b_m(t) = F b_n(t) + f^{*} b_k(t), \]
\[ i\hbar \frac{\partial}{\partial t} b_k(t) = f^{*} b_m(t) - \delta b_k(t). \]  

Here, we approximate that the interstate transitions occur periodically with having the unique frequency \( \Omega \). Consequently, the unitary transformed TD coefficient \( b_l(t) \) is expressed by \( b_l(t) \propto e^{i\omega t} \). The interstate frequency \( \Omega \) is feasibly determined by solving the following secular equation:
\[ \begin{pmatrix} \delta & F^{*} & 0 \\ F & -\Omega & f^{*} \\ 0 & f & -\delta \end{pmatrix}. \]  

F.2. Application to the initial state |00\alpha⟩
Since the present Rashba external field is applied perpendicular to the 2D plane with the frequency \( \omega \), the corresponding TD perturbation potential \( \hat{V}(t) \) is given by
\[ \hat{V}(t) = \hat{F}_{R} \cdot \sin \omega t = \frac{\hat{F}_{R}}{2i} e^{-i\omega t} + \frac{\hat{F}_{R}}{2i} e^{i\omega t} \equiv \hat{F} e^{-i\omega t} + \hat{F}^{*} e^{i\omega t}. \]  

Here, we defined the time-independent operator \( \hat{F} \) in equation (17) by
\[ \hat{F} = \frac{\hat{F}_{R}}{2i} = \frac{\omega}{2i} (\hat{x} \hat{\sigma}_x - \hat{y} \hat{\sigma}_y). \]  

The electron is first in the ground state |0⟩ = |00\alpha⟩ and transits into the states |m⟩ = |01\beta⟩ and |k⟩ = |10\alpha⟩. Accordingly, the interstate Rashba SOI matrix element \( F \) and \( f \) are given by
\[ F = \frac{1}{\hbar} F_{mn} = -\frac{1}{2i} \langle 01\beta | \hat{F}_{R} | 00\alpha \rangle, \]
\[ f = \frac{1}{\hbar} F_{km} = -\frac{1}{2i} \langle 10\alpha | \hat{F}_{R} | 01\beta \rangle. \]  

We estimate \( F \) and \( f \) directly by using the Laguerre polynomial eigenfunctions:
\[ \langle 01\beta | \hat{F}_{R} | 00\alpha \rangle = \langle 10\alpha | \hat{F}_{R} | 01\beta \rangle = 0.3445. \]  

Eventually, we have the relation of \( F \equiv f \) for the present harmonic three states, and the secular equation (F.7) gives the interstate frequency \( \Omega \) of,
\[ \Omega^2 = 2|F|^2 + \delta^2. \]
Equation (F.12) demonstrates that the relationship $\Omega^2 - \Xi^2$ is still conserve in the three-state approach with the RWA. Therefore, the three-state approximation with the RWA also results the linear relation $\Omega \propto \Xi$ by the resonant excitation of $\delta = 0$. It further elucidates that the detuning $\delta$ from the resonant frequency results in the parallel displacement in the relationship $\Omega^2 - \Xi^2$, as found in figures 13(a) and (b).

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**References**

[1] Rashba E I 1960 Sov. Phys. Solid state 2 1109
[2] Nitta J, Akazaki T, Takayangi H and Enoki T 1997 Phys. Rev. Lett. 78 1335
[3] Koppens F H L, Buizert C, Tielrooij K J, Vink I T, Nowack K C, Meunier T, Kouwenhoven L P and Wanderssyen L M K 2006 Nature 442 766
[4] Pioro-Ladrière M, Obata T, Tokura Y, Shin Y-S, Kubo T, Yoshihata K, Tanjyama T and Taruza S 2008 Nat. Phys. 4 776
[5] Datta S and Das B 1990 Appl. Phys. Lett. 56 665
[6] König M, Tschetschetkin A, Hankiewicz E M, Sinova J, Hock V, Daumer V, Schfer M, Becker C R, Buhmann H and Molenkamp L W 2006 Phys. Rev. Lett. 96 076804
[7] Bergsten T, Kobayashi T, Sekine Y and Nitta J 2006 Phys. Rev. Lett. 97 196803
[8] Aharony A, Tokura Y, Cohen G Z, Entin-Wohlman O and Katsumoto S 2011 Phys. Rev. B 84 035323
[9] Hakojiya T, Liberman M A, Moskalenko S A and Podlesny I V 2011 J. Phys. Condens. Matter 23 345405
[10] Mishchenko E G, Shytov A V and Halperin B I 2004 Phys. Rev. Lett. 93 226602
[11] Raimondi R and Forini C 2006 Phys. Rev. B 74 035340
[12] Niu Z 2012 Appl. Phys. Lett. 101 062601
[13] Konschelle F 2014 Eur. Phys. J. B 87 119
[14] Alidoust M and Hallerman K 2015 J. Phys. Condens. Matter 27 235501
[15] Bobkova I V, Bobkova A M, Zyznin A A and Alidoust M 2016 Phys. Rev. B 94 134506
[16] For a review, see e.g. Jakšič I and Wójcik A (ed) 1998 Quantum Dots (Berlin: Springer)
[17] Nielsen M A and Chuang I L 2011 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[18] Loss D and DiVincenzo D P 1998 Phys. Rev. A 57 120
[19] Awschalom D D, Loss D and Samarth N 2002 Semiconductor Spintronics and Quantum Computation (Berlin: Springer)
[20] Wolf S A, Awschalom D D, Buhrman R A, Daughton J M, von Molnár S, Roukes M L, Chtchelkanova A Y and Treger D M 2001 Science 294 1488
[21] Manchon A, Koo H C, Nitta H, Frolov S M and Duine R A 2015 Nat. Mater. 14 871
[22] Nowack K C, Koppens F H L, Nazarov Y V and Wanderssyen L M K 2007 Science 318 1430
[23] Ford P, Benedict M G, Kulkmin O and Peeters F M 2009 Phys. Rev. B 80 165303
[24] van den Berg J W G, Nadji-Perve G, Priebiag V S, Plissard S R, Bakkers E P A M, Frolov S M and Kouwenhoven L P 2013 Phys. Rev. Lett. 110 066806
[25] Echeverría-Arrondo C and Sherman E Y 2013 Phys. Rev. B 88 155328
[26] Walls D J 2007 Phys. Rev. B 76 195307
[27] Kikkawa J M and Awschalom D D 1998 Phys. Rev. Lett. 80 4513
[28] Wei J, Haug R J, Klitzing K V and Ploog K 1993 Phys. Rev. Lett. 71 4019
[29] Windler R 2000 Phys. Rev. B 62 4245
[30] Koteles E S and Datars W R 1974 Phys. Rev. B 9 568
[31] Young K F and Frederik H P R 1973 J. Phys. Chem. Ref. Data 2 329
[32] Adachi S 1986 Phys. Rev. B 35 7454
[33] Yokozuka T, Ido K, Clark R, Takeda K and Tokura Y 2014 Japan. J. Appl. Phys. 53 031801
[34] Yamagami T, Ido K, Takeda K and Tokura Y 2016 Japan. J. Appl. Phys. 55 045201
[35] Tojo T, Inui M, Takeda K and Tokura Y 2017 Japan. J. Appl. Phys. 56 075201
[36] Grundler D 2000 Phys. Rev. Lett. 84 6074
[37] Sato Y, Kita T, Gozu S and Yamada S 2001 J. Appl. Phys. 89 8017
[38] Arugu T and Hatta S 2009 J. Vac. Soc. Japan 52 577
[39] Trotter H F 1959 Proc. Amer. Math. Soc. 10 145
[40] Suzuki M 1976 Commun. Math. Phys. 51 183
[41] Suzuki M 1976 Prog. Theor. Phys. 56 1454
[42] Tamaki M and Yamamoto R (ed) 1988 Computational Physics and Chemistry (in Japanese) (Tokyo: Kaibundo)
[43] Watanabe N and Tsukada M 2000 Phys. Rev. B 62 2914
[44] Brigham E O 1988 The Fast Fourier Transform and Its Applications (Englewood Cliffs, NJ: Prentice-Hall)