Geometric Dynamics of Plasma in Jet Spaces with Berwald-Moór Metric

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Abstract: In this paper we construct the differential equations of the stream lines that characterize plasma regarded as a non-isotropic medium geometrized by a jet rheonomic time-invariant Berwald-Moór metric. Section 1 contains historical notes regarding the Plasma Physics and its geometrical description. Section 2 analyzes the generalized Lagrange geometrical approach of the non-isotropic plasma on 1-jet spaces. Section 3 studies the non-isotropic plasma as a medium geometrized by the jet rheonomic Berwald-Moór metric.

Key words and phrases: jet Finsler spaces, relativistic rheonomic Berwald-Moór metric, energy-stress-momentum d-tensor of non-isotropic plasma, conservation laws, DEs of stream lines.

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1 Introduction

During that so-called the radiation epoch, in which photons are strongly coupled with the matter, the interactions between the various constituents of the Universal Matter include radiation-plasma coupling, which is described by the plasma dynamics. Although it is not traditional to characterize the radiation epoch by the dominance of plasma interactions, however, it may be also called the plasma epoch (please, see [8]). This is because, in the plasma epoch, the electromagnetic interaction dominates all the four fundamental physical forces (electrical, magnetic, gravitational and nuclear).

In the present days, the Plasma Physics is an established field of Theoretical Physics, although the formulation of magnetohydrodynamics in a curved space-time is a relatively new development (please see, Punsly [19]). The MHD processes in an isotropic space-time are extensively studied by a lot of physicists. For example, the MHD equations in an expanding Universe are investigated by Kleidis, Kuiroukidis, Papadopoulos and Vlahos in [8]. Considering the interaction of the gravitational waves with the plasma in the presence of a weak magnetic field, Papadopoulos also investigates the relativistic hydromagnetic equations [17]. The electromagnetic-gravitational dynamics into plasmas with pressure and viscosity is studied by Das, DeBenedictis, Kloster and Tariq in the paper [4].

It is important to note that all preceding physical studies are done on an isotropic four-dimensional space-time, represented by a semi-(pseudo-) Riemannian space with the signature (+, +, +, −). Consequently, the Riemannian geometrical methods are used as a pattern over there.

From a geometrical point of view, using the Finslerian geometrical methods, the plasma dynamics was extended on non-isotropic space-times by Gîrtu and Ciubotariu in the paper [6]. More general, after the development of Lagrangian geometry of tangent bundle by Miron and Anastasiei [11], the generalized Lagrange geometrical objects describing the relativistic magnetized plasma were studied by M. Gîrtu, V. Gîrtu and Postolache in the paper [7].

According to Olver’s opinion [16], we appreciate that the 1-jet spaces are basic geometrical objects in the study of classical and quantum field theories. For such a reason, inspired by the geometrical methods of geometric dynamics developed by Udriște in [20]-[21], and using as a pattern the Miron-Anastasiei’s geometrical ideas from [11], Neagu recently developed in the book [14] that so-called "multitime Riemann-
Lagrange geometry” on 1-jet spaces, in the sense of distinguished d-connections, d-torsions, and d-curvatures. We would like to point out that the geometrical construction on 1-jet spaces exposed in the monograph [14] was initiated by Asanov in [2] and further developed by Neagu and Udriste. Under the influence of the Riemann-Lagrange geometrical ideas from [14], the paper [13] creates a multitime extension on 1-jet spaces of the geometrical objects that characterize plasma in semi-Riemannian and Lagrangian approaches.

On the other hand, more studies of Russian researchers (Asanov [1], Mikhailov [10], Garas’ko and Pavlov [5]) emphasize the importance in the study of physical fields of the Finsler geometry which is characterized by the total equality of all non-isotropic directions. For such a reason, Asanov, Pavlov and their co-workers [5], [18] offer some dimensional linear space (with the material reality, in the framework of the theory of space-time structure and gravitation, as well as in unified gauge field theories, by the Berwald-Moór metric)

\[ F : TM \rightarrow \mathbb{R}, \quad F(y) = (y^1 y^2 ... y^n)^{1/2}, \]

whose Finsler geometry is intensively studied by Matsumoto and Shimada in the paper [9]. Taking into account that our natural physical intuition distinguishes four dimensions in a natural correspondence with the material reality, in the framework of the 4-dimensional linear space \((n = 4)\) with Berwald-Moór metric, Pavlov and his co-workers [5], [18] offer some new physical approaches and geometrical interpretations for:

1. physical events = points in the 4-dimensional space;
2. straight lines = shortest curves;
3. intervals = distances between the points along of a straight line;
4. light pyramids \(\Leftrightarrow\) light cones in a pseudo-Euclidean space.

In such a physical perspective and because of all preceding geometrical and physical reasons, this paper is devoted to the development on the 1-jet space \(J^1(\mathbb{R}, M^4)\) of the geometric dynamics of plasma endowed with the relativistic rheonomic time-invariant Berwald-Moór metric

\[ \hat{F} : J^1(\mathbb{R}, M^4) \rightarrow \mathbb{R}, \]

\[ \hat{F}(t, y) = \sqrt{h_{11}(t)} \sqrt{y_1^2 y_1^3 y_1^3}, \]

where \(h_{11}(t)\) is a Riemannian metric on \(\mathbb{R}\) and \((t, x^1, x^2, x^3, y^1, y_1^2, y_1^3, y_1^4)\) are the coordinates on the 1-jet space \(J^1(\mathbb{R}, M^4)\).

We underline that the influence of the Riemann-Lagrange geometrical ideas from [14], the paper [13] creates a multitime extension on 1-jet spaces of the geometrical objects that characterize plasma in semi-Riemannian and Lagrangian approaches.

2 Generalized Lagrange geometrical approach of the non-isotropic plasma on 1-jet spaces

Let \((\mathbb{R}, h_{11}(t))\) be the set of real numbers endowed with a Riemannian structure, where the coordinate \(t\) plays the role of relativistic time. The Christoffel symbol of the Riemannian metric \(h_{11}(t)\) is

\[ \hat{\kappa}_{11} = \frac{h_{11}}{2} \frac{dt}{dt}, \quad h_{11} = \frac{1}{h_{11}} > 0. \]

Let us consider that \(M^n\) is a spatial real manifold of dimension \(n\), whose local coordinates are \((x^i)_{i=1}^n\). Notice that, in this Section, the latin indices run from 1 to \(n\). Moreover, the Einstein convention of summation is used throughout this work. Let \(J^1(\mathbb{R}, M)\) be the 1-jet space of dimension \(2n + 1\), whose local coordinates are \((t, x^i, y_1^i)\). These transform by the rules

\[ \tilde{t} = \tilde{t}(t), \quad \tilde{x}^i = \tilde{x}^i(x^i), \quad \tilde{y}_1^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{dt} y_1^j, \]

where \(\frac{dt}{dt} \neq 0\) and rank \((\partial \tilde{x}^i/\partial x^j) = n\).

Let \(\mathcal{RGM}C^n = (J^1(\mathbb{R}, M), G^{(1)(1)}_{ij} = h_{11} g_{ij})\) be a relativistic rheonomic generalized Lagrange space (for more details, please see Neagu [14], [15]), where \(g_{ij}(t, x^k, y_1^k)\) is a metrical d-tensor on \(J^1(\mathbb{R}, M)\), which is symmetrical, non-degenerate and of constant signature.

Let us consider that \(\mathcal{RGM}C^n\) is endowed with a nonlinear connection having the form [14], [15]

\[ \Gamma = \left( M^{(i)}_{(1)1} = -\hat{\kappa}_{11} y_1^i, \quad N^{(i)}_{(1)1} \right). \]

The nonlinear connection \(\Gamma\) produces on \(J^1(T, M)\) the following dual adapted bases of d-vectors and d-covectors:

\[ \left\{ \frac{\delta}{\delta t}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y_1^i} \right\} \subset \mathcal{X}(J^1(\mathbb{R}, M)), \]

\[ \left\{ dt, dx^i, \delta y_1^i \right\} \subset \mathcal{X}^*(J^1(\mathbb{R}, M)), \]

where

\[ \delta = \frac{\partial}{\partial t} + \hat{\kappa}_{11} y_1^m \frac{\partial}{\partial y_1^m}, \quad \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N^{(m)}_{(1)i} \frac{\partial}{\partial y_1^m}. \]
\[ \delta y_i^1 = dy_i^1 - \delta x^1 y_i^1 dt + N_{(1)_m}^i dx^m. \]

It is important to note that the d-tensors on the 1-jet space \( J^1(\mathbb{R}, M) \) behave like classical tensors. For example, on the 1-jet space \( J^1(\mathbb{R}, M) \) we have the global metrical d-tensor

\[ G = h_{11} dt \otimes dt + g_{ij} dx^i \otimes dx^j + h^{11} g_{ij} \delta y_i^1 \otimes \delta y_j^1, \]

which is endowed with the physical meaning of non-isotropic gravitational potential. Obviously, the d-tensor \( G \) has the adapted components

\[ G_{AB} = \begin{cases} 
  h_{11}, & \text{for } A = 1, \ B = 1 \\
  g_{ij}, & \text{for } A = i, \ B = j \\
  h^{11} g_{ij}, & \text{for } A = (1), \ B = (1) \\
  0, & \text{otherwise}.
\end{cases} \]

Following the geometrical ideas of Asanov and Neagu, the preceding geometrical ingredients lead us to the Cartan canonical 1-linear connection (given in adapted components)

\[ C\Gamma = \left( z_1^{11}, G_{jk}^1, L_{jk}, C_{j(k)}^{(1)} \right), \]

where

\[ G_{jk}^1 = g_{km}^{i} \frac{\delta g_{mjk}}{\delta t}, \]

\[ L_{jk}^{i} = \frac{g_{im}}{2} \left( \frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right), \]

\[ C_{j(k)}^{(1)} = \frac{g_{km}}{2} \left( \frac{\delta g_{jm}}{\delta y_i^1} + \frac{\delta g_{km}}{\delta y_i^1} - \frac{\delta g_{jk}}{\delta y_i^1} \right). \]

In the sequel, the Cartan linear connection \( C\Gamma \), given by (1), induces the \( \mathbb{R} \)-horizontal (\( h_\mathbb{R} \)) covariant derivative

\[ D^{(1)}_{i(j)(1)\ldots(i)} G_{1k(1)(l)\ldots(i)} = \frac{\delta D^{(1)}_{i(j)(1)\ldots(i)} }{\delta t} + D^{(1)}_{i(j)(1)\ldots(i)} z_1^{11} \]

\[ + D^{(1)}_{i(j)(1)\ldots(i)} G_{m1}^j + D^{(1)}_{i(j)(1)\ldots(i)} C_{mk1}^j \]

\[ + D^{(1)}_{i(j)(1)\ldots(i)} z_1^{11} + \]

\[ - D^{(1)}_{i(j)(1)\ldots(i)} L_{jk}^{i} - D^{(1)}_{i(j)(1)\ldots(i)} C_{jk}^{(1)} \]

\[ - D^{(1)}_{i(j)(1)\ldots(i)} L_{jk}^{i} - D^{(1)}_{i(j)(1)\ldots(i)} C_{jk}^{(1)} \]

the \( M \)-horizontal (\( h_M \)) covariant derivative

\[ D^{(1)}_{i(j)(1)\ldots(i)} G_{1k(1)(l)\ldots(i)} = \frac{\delta D^{(1)}_{i(j)(1)\ldots(i)} }{\delta x^k} + D^{(1)}_{i(j)(1)\ldots(i)} L_{mp}^i \]

\[ + D^{(1)}_{i(j)(1)\ldots(i)} J_{mp}^j + \]

\[ - D^{(1)}_{i(j)(1)\ldots(i)} L_{mp}^i - D^{(1)}_{i(j)(1)\ldots(i)} J_{mp}^j - \]

and the vertical (\( v \)) covariant derivative

\[ D_{1k(1)(l)\ldots(p)}^{(1)(j)(1)\ldots(i)} = \frac{\partial D_{1k(1)(l)\ldots(p)}^{(1)(j)(1)\ldots(i)}}{\partial y_i^1} + D_{1k(1)(l)\ldots(p)}^{(1)(j)(1)\ldots(i)} C_{m1}^{(1)} \]

\[ + D_{1k(1)(l)\ldots(p)}^{(1)(j)(1)\ldots(i)} C_{mk1}^{(1)} \]

\[ - D_{1k(1)(l)\ldots(p)}^{(1)(j)(1)\ldots(i)} C_{mk1}^{(1)} \]

\[ - D_{1k(1)(l)\ldots(p)}^{(1)(j)(1)\ldots(i)} C_{mk1}^{(1)} \]

where

\[ D = D_{1k(1)(l)\ldots(p)}^{(1)(j)(1)\ldots(i)} (t, x^r, y^1_r) \delta \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial y_i^1} \otimes \frac{\partial}{\partial y_1^{1}} \otimes \cdots \]

\[ \otimes dx^k \otimes \delta y_i^1 \otimes \cdots \]

is an arbitrary d-tensor on \( J^1(\mathbb{R}, M) \).

**Remark 1** The Cartan covariant derivatives produced by \( C\Gamma \) have the metrical properties

\[ h_{11/1} = h_{11}^{11} = 0, \quad h_{11|k} = h_{11}^{11|k} = 0, \]

\[ g_{ij/1} = g_{ij}^{ij} = 0, \quad g_{ij|k} = g_{ij}^{ij|k} = 0, \]

\[ g_{ij}^{(1)}(k) = g_{ij}^{(1)}(k) = 0. \]

For the study of the magnetized non-viscous plasma dynamics, in a relativistic generalized Lagrangian geometrical approach on 1-jet spaces, we use the following geometrical objects [21, 13]:

1. the unit relativistic time dependent velocity-d-field of a test particle, which is given by

\[ U = u_1^i (t, x^k, y_1^k) \frac{\partial}{\partial y_i^1}, \]

where, if we take \( \varepsilon^2 = h_{11} g_{ij} y_1^i y_1^j > 0 \), then we put \( u_1^i = y_1^i / \varepsilon \). Obviously, we have \( h_{11} u_{11} u_1^1 = 1 \), where \( u_{11} = g_{im} u_1^m \);

2. the distinguished relativistic time dependent 2-form of the (electric field)-(magnetic induction), which is given by

\[ H = H_{ij} (t, x^k, y_1^k) dx^i \wedge dx^j; \]

3. the distinguished relativistic time dependent 2-form of the (electric induction)-(magnetic field), which is given by

\[ G = G_{ij} (t, x^k, y_1^k) dx^i \wedge dx^j; \]
4. the relativistic time dependent Minkowski energy d-tensor of the electromagnetic field inside the non-isotropic plasma, which is given by

\[ E = E_{ij}(t, x^k, y^k) dx^i \otimes dx^j + h^{11} E_{ij}(t, x^k, y^k) \delta y^i_1 \otimes \delta y^j_1. \]

The adapted components of the relativistic time dependent Minkowski energy are defined by

\[ E_{ij} = \frac{1}{4} g_{ij} H^r G^{rs} + g^{rs} H_{ir} G_{js}, \]

where \( G^{rs} = g^{rp} g^{sq} G_{pq}. \) Moreover, we suppose that the adapted components of the relativistic time dependent Minkowski energy verify the non-isotropic Lorentz conditions [13]

\[ E_{ij}^{(1)}|_{(m)} u^i_1 = 0, \]

where \( E_{ij}^{(1)} = g^{mp} E_{pi}. \) Obviously, if we use the notations \( H^r_{ir} = g^{mp} H_{ir}^{(1)}; G^r_{ir} = g^{rs} G_{si}, \) we have

\[ E_{ij}^{(1)} = \frac{1}{4} \delta_{ij}^{(1)} H^r G^{rs} - H^r_{ir} G^r_{ij}, \]

where \( \delta_{ij}^{(1)} \) is the Kronecker symbol.

In our jet generalized Lagrangian geometrical approach, the relativistic time dependent non-isotropic plasma is characterized by an energy-stress-momentum d-tensor \( \mathcal{T} \), which is defined by [7], [13]

\[ \mathcal{T} = \mathcal{T}_{ij}(t, x^k, y^k) dx^i \otimes dx^j + h^{11} \mathcal{T}_{ij}(t, x^k, y^k) \delta y^i_1 \otimes \delta y^j_1, \]

where [4]

\[ \mathcal{T}_{ij} = \left( \rho + \frac{p}{c^2} \right) h^{11} u^i_1 u^j_1 + p g_{ij} + E_{ij}. \] (3)

The entities \( c = \) constant, \( p = p(t, x^k, y^k) \) and \( \rho = \rho(t, x^k, y^k) \) have the physical meanings of: - the speed of light, the non-isotropic hydrostatic pressure and the non-isotropic proper mass density of plasma. Notice that the adapted components of the energy-stress-momentum d-tensor \( \mathcal{T} \), that characterizes the non-isotropic plasma, are given by

\[ \mathcal{T}_{ij} = \begin{cases} \mathcal{T}_{ij}, & \text{for} \quad C = i, \quad F = j \\ h^{11} \mathcal{T}_{ij}, & \text{for} \quad C = \left( ^{(1)}_{(i)} \right), \quad F = \left( ^{(1)}_{(j)} \right) \\ 0, & \text{otherwise.} \end{cases} \]

(4)

In the jet generalized Lagrange framework for plasma, we postulate that the following non-isotropic conservation laws of the components (3) and (4) are true:

\[ \mathcal{T}_{A:M}^{(M)} = 0, \quad \forall A \in \left\{ 1, i, \left( ^{(1)}_{(i)} \right) \right\}, \]

(5)

where the capital latin letters \( A, M, \ldots \) are indices of kind 1, \( i \) or \( \left( ^{(1)} \right), \). \( "A: M" \) represents one of the local covariant derivatives \( h_{\mathcal{R}}, h_{\mathcal{M}}, \) or \( v- \) and

\[ \mathcal{T}_{A}^{M} = G^{MD} \mathcal{T}_{DA} = \begin{cases} \mathcal{T}_{i}^{m}, & \text{for} \quad A = i, \quad M = m \\ \mathcal{T}_{i}^{m}, & \text{for} \quad A = \left( ^{(1)}_{(i)} \right), \quad M = \left( ^{(1)}_{(m)} \right) \\ 0, & \text{otherwise.} \end{cases} \]

Note that the d-tensor \( \mathcal{T}_{i}^{m} \) is given by the formula

\[ \mathcal{T}_{i}^{m} = g^{mp} \mathcal{T}_{pi} = \left( \rho + \frac{p}{c^2} \right) h^{11} u^m_1 u^i_1 + p \delta_i^m + E_i^m. \]

It is easy to see that the jet non-isotropic conservation laws (5) reduce to the following local non-isotropic conservation equations:

\[ \mathcal{T}_{i}^{m}|_{(m)} = 0, \quad \mathcal{T}_{i}^{(1)}|_{(m)} = 0. \]

(6)

Moreover, by direct computations, we deduce that the non-isotropic conservation equations (6) become

\[ h^{11} \left[ \left( \rho + \frac{p}{c^2} \right) u^m_{(1)} \right]_{(m)} u^i_1 + \left( \rho + \frac{p}{c^2} \right) h^{11} u^1_{(m)} u^m_{(i)} = 0, \]

\[ + p_{,i} - g_{ir} h^{r}_{,i} = 0, \]

\[ h^{11} \left[ \left( \rho + \frac{p}{c^2} \right) u^1_{(1)} \right]_{(m)} u^i_1 + \left( \rho + \frac{p}{c^2} \right) h^{11} u^1_{(m)} u^1_{(i)} = 0, \]

\[ - u^1_{(m)}_{(i)} + p \left( ^{(1)}_{(i)} \right) - g_{ir} h^{r}_{,i} = 0, \]

(7)

where \( p_{,i} = \delta p/\delta x^i, p \left( ^{(1)}_{(i)} \right) = \partial p/\partial y^i_1 \) and

- \( h^{r} = - g^{rs} E_{s}^{m} \) is the non-isotropic horizontal Lorentz force;
- \( h^{r+1} = - g^{rs} E_{s}^{m} \left( ^{(1)}_{(m)} \right) \) is the non-isotropic vertical Lorentz d-tensor force.

Contracting now the non-isotropic conservation equations (7) with \( u^i_1 \) and taking into account the non-isotropic Lorentz conditions (2), we find the non-isotropic continuity equations of plasma, namely

\[ \left[ \left( \rho + \frac{p}{c^2} \right) u^m_{(1)} \right]_{(m)} + p_{,m} u^m_{(1)} = 0, \]

\[ \left[ \left( \rho + \frac{p}{c^2} \right) u^1_{(1)} \right]_{(m)} + p \left( ^{(1)}_{(m)} \right) = 0, \]

(8)
where we also used the equalities

0 = h^{11}_{i1} u^i_{1|m} u^m_{1} = \frac{1}{2} (h^{11}_{i1} u^i_{1})_{,m} = -h^{11}_{i1} u^i_{1|m} u^m_{1},
0 = h^{11}_{i1} u^i_{1(l)} \mid_{(m)} = \frac{1}{2} (h^{11}_{i1} u^i_{1})_{(l)} \mid_{(m)} = -h^{11}_{i1(l)} \mid_{(m)} u^i_{1},

the symbols ”,m” and ”(l) \mid_{(m)}” being the derivative operators $\delta/\delta x^m$ and $\partial/\partial y^m$.

Replacing the continuity laws (8) into the conservation equations (7), we find the non-isotropic relativistic Euler equations for plasma, namely

\[
\left( \rho + \frac{\mathbf{P}}{c^2} \right) h^{11}_{i1} u^i_{1|m} u^m_{1} - \rho_{,m} \left( h^{11}_{i1} u^i_{1} - \delta^i_1 \right) \quad - g_{im} \mathbf{F}^m = 0, \\
\left( \rho + \frac{\mathbf{P}}{c^2} \right) h^{11}_{i1} u^i_{1(l)} \mid_{(m)} - \rho_{(l)} \mid_{(m)} \left( h^{11}_{i1} u^i_{1} - \delta^i_1 \right) \quad - g_{im} \mathbf{F}^m = 0. 
\]

If we take now $y^m_{1} = dx^m_{1}/dt$, then we have

\[
u^1_0 = \frac{1}{\varepsilon_0} \frac{dx^m_{1}}{dt} = \frac{dx^m_{1}}{ds}, \quad \varepsilon_0^2 = h^{11}(t) g_{ij}(t, x, dx/dt) \frac{dx^i_{1} dx^j_{1}}{dt}. 
\]

where $s$ is a natural parameter of the curve $c = (x^k(t))$, having the property $ds/dt = \varepsilon_0$. Introducing this $u^m_0$ into the non-isotropic Euler equations (9), we obtain the equations of the non-isotropic stream lines for jet plasma, which are given by the following second order DE systems:

**horizontal non-isotropic stream line DEs:**

\[
\frac{d^2 x^k_{1}}{ds^2} + \left[ L^k_{rm} - \frac{c^2}{\mathbf{P} + \rho c^2} \delta^k_p \mathbf{P}^m_{p} \right] \frac{dx^r_{1} dx^m_{1}}{ds} ds = \frac{h_{11} c^2}{\mathbf{P} + \rho c^2} \left[ \frac{h}{\mathbf{F} - g_{km} \mathbf{P}^m_{p}} \frac{N^{(k)}_{(1)m} dx^m_{1}}{\varepsilon_0 ds} ds \right. \\
- \frac{h_{11} N^{(p)}_{(1)m} \partial g_{pq} dx^p dx^m_{1} dx^k_{1}}{\varepsilon_0} \\
- \frac{h_{11} N^{(r)}_{(1)m} \partial g_{pq} dx^p dx^m_{1} dx^k_{1}}{2 \partial y^r_{1}} \\
- \frac{h_{11} \mathbf{F}^k_{1}}{2 \partial y^r_{1}} \frac{dx^p dx^m_{1} dx^k_{1}}{ds} ds.
\]

**vertical non-isotropic stream line DEs:**

\[
\left[ C^k_{r(m)} - \frac{c^2}{\mathbf{P} + \rho c^2} \delta^k_p \mathbf{P}^m_{p} \right] \frac{dx^r_{1} dx^m_{1}}{ds} ds = \frac{h_{11} c^2}{\mathbf{P} + \rho c^2} \left[ \frac{h}{\mathbf{F} - g_{km} \mathbf{P}^m_{p}} \right].
\]

Remark 2 If the metrical d-tensor $g_{ij}(t, x, y)$ is Finslerian-like one, that is we have

\[
g_{ij}(t, x, y) = \frac{h_{11}}{2} \frac{\partial^2 F^2}{\partial y^i_1 \partial y^j_1}, 
\]

where $F : J^1(\mathbb{R}, M) \to \mathbb{R}_+$ is a jet Finslerian metric, then we use the canonical spatial nonlinear connection $N = \left( N^{(k)}_{(1)m} \right)$ of the jet Finsler space, whose general formula is given in [15]. Consequently, the DEs of the stream lines of plasma in non-isotropic jet Finsler spaces reduce to:

- **horizontal non-isotropic stream line DEs:**

\[
\frac{d^2 x^k_{1}}{ds^2} + \left[ L^k_{rm} - \frac{c^2}{\mathbf{P} + \rho c^2} \delta^k_p \mathbf{P}^m_{p} \right] \frac{dx^r_{1} dx^m_{1}}{ds} ds = \frac{h_{11} c^2}{\mathbf{P} + \rho c^2} \left[ \frac{h}{\mathbf{F} - g_{km} \mathbf{P}^m_{p}} \frac{N^{(k)}_{(1)m} dx^m_{1}}{\varepsilon_0 ds} ds \right. \\
- \frac{h_{11} N^{(p)}_{(1)m} \partial g_{pq} dx^p dx^m_{1} dx^k_{1}}{\varepsilon_0} \\
- \frac{h_{11} N^{(r)}_{(1)m} \partial g_{pq} dx^p dx^m_{1} dx^k_{1}}{2 \partial y^r_{1}} \\
- \frac{h_{11} \mathbf{F}^k_{1}}{2 \partial y^r_{1}} \frac{dx^p dx^m_{1} dx^k_{1}}{ds} ds.
\]

- **vertical non-isotropic stream line DEs:**

\[
\left[ C^k_{r(m)} - \frac{c^2}{\mathbf{P} + \rho c^2} \delta^k_p \mathbf{P}^m_{p} \right] \frac{dx^r_{1} dx^m_{1}}{ds} ds = \frac{h_{11} c^2}{\mathbf{P} + \rho c^2} \left[ \frac{h}{\mathbf{F} - g_{km} \mathbf{P}^m_{p}} \right].
\]
3 The non-isotropic plasma as a medium geometrized by the jet rheonomic Berwald-Moór metric

Let us note that in this Section we work with a fixed 4-dimensional spatial manifold \( M^4 \). Consequently, throughout this Section, the latin indices run only from 1 to 4. Also note that in this Section we will focus only on the jet relativistic rheonomic Berwald-Moór metric [12].

\[ \dot{F}(t, y) = \sqrt{h^{11}(t)} \frac{4}{\sqrt{y_1^1 y_1^2 y_1^3 y_1^4}}. \]  

(1)

Using the notation \( G_{1111} = y_1^1 y_1^2 y_1^3 y_1^4 \), the fundamental metrical d-tensor produced by the relativistic rheonomic Berwald-Moór metric (1) is given by (no sum by \( i \) or \( j \))

\[ g_{ij}(t, x, y) = \frac{h_{11}(t)}{2} \frac{\partial^2 \dot{F}^2}{\partial y_i^j \partial y_j^i} = \frac{1 - 2 \delta_{ij}}{G_{1111}} \frac{1}{y_i^j y_j^i}, \]  

(2)

where \( \delta_{ij} \) is the Kronecker symbol. The matrix \( g = (g_{ij}) \) admits the inverse \( g^{-1} = (g^{jk}) \), whose entries are (no sum by \( j \) or \( k \))

\[ g^{jk} = \frac{2(1 - 2 \delta_{jk})}{G_{1111}} y_i^j y_k^i. \]  

(3)

Using some general formulas from the paper [15], we find the following geometrical results [12]:

Theorem 3 For the jet relativistic rheonomic Berwald-Moór metric (1), the energy action functional

\[ \tilde{E}(x(\cdot)) = \int_a^b \sqrt{y_1^i y_1^j y_1^k y_1^l} h^{11} \sqrt{h_{11}} \ dt, \]

where \( y_1^i = dx^i/dt \), produces on the 1-jet space \( J^1(\mathbb{R}, M^4) \) the canonical time dependent spray

\[ \dot{S} = \left( H^{(i)}_{(1)1} = -\frac{\dot{x}_i^1}{2} y_1^i, \quad C^{(i)}_{(1)1} = -\frac{\dot{x}_i^1}{3} y_1^i \right). \]

Corollary 4 The canonical nonlinear connection on the jet space of first order \( J^1(\mathbb{R}, M^4) \), associated to the jet relativistic rheonomic Berwald-Moór metric (1), is given by

\[ \Gamma = \left\{ \begin{array}{ll} M^{(i)}_{(1)1} = 2h_{(1)1} = -\frac{\dot{x}_i^1}{2} y_1^i, & \\
N^{(i)}_{(1)j} = \frac{G^{(i)(1)}_{(1)1}}{\partial y_i^j} = -\frac{\dot{x}_i^1}{3} \delta_{ij}, & \end{array} \right. \]  

(4)

Remark 5 The nonlinear connection (4) produces the dual adapted bases of d-vector fields

\[ \left\{ \delta_t, \delta x^i, \delta y_i^j \right\} \subset \mathcal{X}(E), \]  

(5)

where,

\[ \delta_t = \frac{\partial}{\partial t} + \dot{x}_1^1 y_1^1 \frac{\partial}{\partial y_1^j}, \]  

\[ \delta x^i = \frac{\partial}{\partial x^i} + \frac{\dot{x}_i^1}{3} \frac{\partial}{\partial y_1^i}, \]

and d-covector fields

\[ \left\{ dt, \ dx^i, \ dy_i^j \right\} \subset \mathcal{X}^*(E), \]  

(6)

where

\[ \delta y_i^j = dy_i^j - \dot{x}_1^1 y_1^j dt - \frac{\dot{x}_i^1}{3} dx^i \]

and \( E = J^1(\mathbb{R}, M^4) \). Note that the distinguished geometrical elements of the adapted bases (5) and (6) transform like classical tensors.

On the 1-jet space \( J^1(\mathbb{R}, M^4) \), we will describe the Cartan canonical connection produced by the relativistic rheonomic Berwald-Moór metric (1) in local adapted components. Thus, using the formulas (1), (2) and (3), by direct computations, we obtain [12]:

Theorem 6 The Cartan canonical \( \tilde{\Gamma} \)-linear connection, produced by the rheomorphic Berwald-Moór metric (1), has the following adapted local components:

\[ C\tilde{\Gamma} = \left( \begin{array}{ccc} \dot{x}_1^1, & G^{k}_{j(1)}, & L^i_{jk} = -\frac{\dot{x}_i^1}{3} C^{(i)(1)}_{j(k)}, & C^{(i)(1)}_{j(k)} \end{array} \right), \]  

(7)

where, if we use the notation

\[ A^i_{jk} = \frac{2\delta_i^j + 2\delta_i^k + 2\delta_{jk} - 8\delta_j^i \delta_{jk} - 1}{8} \]

(no sum by \( i, j \) or \( k \)), then we have

\[ C^{(i)(1)}_{j(k)} = A^i_{jk} \cdot \frac{y_1^i y_1^k}{y_1^j y_1^j} \quad \text{(no sum by \( i, j \) or \( k \)).} \]

Remark 7 The below properties of the d-tensor \( C^{(i)(1)}_{j(k)} \) are true (sum by \( m \)):

\[ C^{(i)(1)}_{j(k)} = C^{(i)(1)}_{k(j)}, \quad C^{(i)(1)}_{j(m)} y_1^m = 0, \quad C^{(m)(1)}_{j(m)} = 0. \]

For more details, please see also the papers [3] and [9].
Remark 8 The coefficients $A^l_{ij}$ have the following values:

$$A^l_{ij} = \begin{cases} 
-\frac{1}{8}, & i \neq j \neq l \neq i \\
\frac{1}{8}, & i = j \neq l \text{ or } i = l \neq j \text{ or } j = l \neq i \\
-\frac{3}{8}, & i = j = l.
\end{cases}$$

The geometric dynamics of the non-isotropic plasma regarded as a medium geometrized by the jet rheonomic Berwald-Moór metric (1) is obtained using the DEs of stream lines (11) and (10) for the particular Berwald-Moór geometrical objects (2), (3), (5) and (7). Consequently, we find the following geometrical DEs for non-isotropic plasma:

- **the horizontal** Berwald-Moór non-isotropic stream line DEs are given by

$$\frac{d^2x^k}{ds^2} + \frac{c^2}{p + p c^2}P_{,m} \left( 1 - 4\delta^{km} \right) \frac{dx^m}{ds} \frac{dx^k}{ds} = \frac{h_{11}c^2}{p + p c^2} \mathcal{F}^k,$$

where $k \in \{1, 2, 3, 4\}$ is a fixed index and we do sum by $m$;

- **the vertical** Berwald-Moór non-isotropic stream line DEs are given by

$$p^{(1)}_{\#(m)} \left( 1 - 4\delta^{nk} \right) \frac{dx^m}{ds} \frac{dx^k}{ds} = h_{11} \mathcal{F}^k,$$

where $k \in \{1, 2, 3, 4\}$ is a fixed index and we do sum by $m$.

Remark 9 In the particular case when the hydrostatic pressure is dependent only by $t$ and $x$ (i.e. we have $p = p(t, x^k)$), the DEs of stream lines for non-isotropic plasma endowed with Berwald-Moór metric become:

- **horizontal** Berwald-Moór non-isotropic stream line DEs:

$$\frac{d^2x^k}{ds^2} + \frac{c^2}{p + p c^2}P_{,m} \left( 1 - 4\delta^{km} \right) \frac{dx^m}{ds} \frac{dx^k}{ds} = \frac{h_{11}c^2}{p + p c^2} \mathcal{F}^k,$$

where $p_{,m} = \partial p / \partial x^m$;

- **vertical** Berwald-Moór non-isotropic stream line DEs:

$$\mathcal{F}^{k1} = 0.$$

Open Problem. There exist real physical interpretations for the preceding jet Finsler-Berwald-Moór geometric dynamics of non-isotropic plasma?

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