DYNAMICS OF A 3D BRINKMAN-FORCHHEIMER EQUATION WITH INFINITE DELAY

WENJING LIU AND RONG YANG*

Faculty of Science, Beijing University of Technology, Ping Le Yuan 100, Chaoyang District
Beijing, 100124, China

XIN-GUANG YANG

Department of Mathematics and Information Science, Henan Normal University
Xinxiang, 453007, China

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Abstract. This paper is concerned with the pullback dynamics and asymptotic stability for a 3D Brinkman-Forchheimer equation with infinite delay. The well-posedness of weak solution to the 3D Brinkman-Forchheimer flow with infinite delay is investigated in the weighted space $C^\kappa(H)$ firstly, then the pullback attractors are presented for the process of weak solution. Moreover, the existence of global attractor and the exponential stability analysis of stationary solutions are shown, which is based on the estimate of corresponding steady state equation.

1. Introduction. The Brinkman-Forchheimer equation describes the motion of fluid flow in a saturated porous medium (see [20, 23]), which has received much attention on several issues over the last decades. Vafai and Kim [26] have obtained an exact solution to this problem using a Brinkman-Forchheimer system, i.e., the extended Darcy equation. Whitaker [29] has investigated the theoretical development of the Forchheimer equation. In literatures [17, 22, 27], the authors have discussed the continuous dependence on the Darcy and Forchheimer coefficients even though the variable viscosity, which explains the turbulence of fluid flow in the medium. The existence of attractors for Brinkman-Forchheimer model has also been studied (see [13, 21, 25, 28] and references therein), which is an interesting and important problem understanding the large time behavior of solutions for non-autonomous evolutionary equations. But all of them are related to the non-delayed situations.

The time delay effect and memory are originated from the boundary controllers in engineering, which means that the system is controlled by a force taking into account the history of the solutions. In fact, the hereditary features have been extensively investigated during the last few decades due to their importance in fluid
dynamics, see Barbu and Sritharan [1]. Very recently, the authors investigated
the well-posedness and dynamics for the 3D Brinkman-Forchheimer flow with finite
delay in [14, 15, 31]. While, to our knowledge, the infinite delay effects have not
yet been studied so far.

Inspired by [19], we consider the asymptotic behavior of weak solution to the fol-
lowing non-autonomous 3D Brinkman-Forchheimer equation with unbounded vari-
able delay, which is written as

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + \alpha u + |u| u + \gamma |u|^2 u + \nabla p &= f(t, u_t) + g(t, x), \quad t > \tau, x \in \Omega, \\
\nabla \cdot u &= 0, \quad t > \tau, x \in \Omega, \\
u u &= 0, \quad t > \tau, x \in \partial \Omega, \\
u |t = \tau &= u(x), \quad x \in \Omega, \\
u |t = \tau(\theta, x) &= u(\tau + \theta, x) = \phi(\theta, x), \quad \theta \in (-\infty, 0], \quad x \in \Omega,
\end{aligned}
\]

(1.1)

where \( \tau \in \mathbb{R}, \Omega \subset \mathbb{R}^3 \) is an open and bounded domain with a smooth boundary
\( \partial \Omega, \phi(0, x) = \varphi(x) \). \( u = (u_1, u_2, u_3) \) is the velocity. \( p \) is pressure. \( \nu > 0 \) and \( \alpha > 0 \)
denote the Brinkman kinematic viscosity and the Darcy coefficient respectively.
\( \beta > 0, \gamma > 0 \) are the Forchheimer coefficients. \( g \) is a non-delayed external force
field. \( u_t(\cdot) \) represents the history of the state from time \(-\infty \) up to the present time
\( t \), which denotes constant, variable and distributed delays, see Caraballo and Real
[4, 5]. \( f \) is the external force containing some hereditary characteristics.

The asymptotic behaviour of dynamical system can provide useful information
on the future evolution of the system, which will be the main aim of this article.
The main results of this article can be summarized as follows.

(I) Under appropriate assumptions, we first establish the existence, regularity
and stability of weak solution to (1.1) by the classic Galerkin approximation
and energy method, see Theorem 3.2. It is worth mentioning that the weak
solution exists in space \( C^\kappa(H) \) for any \( \kappa > 0 \), which generalized the result of
[19]. The difficulty is the convergence of the delay term, we will use a refined
version of Dini’s Theorem to deal with the delay effect.

(II) On one hand, we prove that there exists the pullback attractor of fixed
bounded sets for non-autonomous system, see Theorem 4.7. On the other
hand, we establish, under the same assumptions, the existence of \( D_{\sigma} \)-attractor,
i.e. a pullback attractor for a given class of families of time varying subsets of
the phase space, which will be denoted by \( A_{\sigma} \) in this case, see Theorem 4.10.
Comparing with [19], we establish the structure of pullback attractor in a
general framework featuring the notion of tempered universe, and the relation
between families of pullback attractors in various universes is also obtained.

(III) The long time asymptotic phenomenon is studied. Based on the above re-
results, we show the existence of global attractor in the non-autonomous case,
see Theorem 5.3. If \( f \) is independent of time and \( g \) converges to \( g_\infty \) in some
metric space as \( t \to \infty \), we establish the existence and uniqueness of stationary
solution of (1.1) by the Brouwer fixed point theorem. In the special external
force case, we are able to ensure the asymptotic convergence to the stationary
solutions with an exponential rate for the unbounded variable delay, see
Theorem 5.6.

The paper is organized as follows. A number of spaces and some lemmas for
preparing what follows are given in Section 2. The well-posedness of weak solution
to (1.1) is established in Section 3. The results of the existence of two kinds of
pullback attractors for this 3D non-autonomous dynamical system with infinite
delay and time-dependent external force are proved in Section 4. In Section 5, as
the long time behavior of the 3D Brinkman-Forchheimer equation, the existence of
global attractor and the exponential stability analysis of stationary solutions are
obtained.

2. Preliminary. We begin by introducing some spaces, lemmas and a refined ver-
sion of Dini’s Theorem.

• Some generic functional spaces
Denoting \( L^p(\Omega) = (L^p(\Omega))^3 \) and \( \| \cdot \|_p \) is the norm in \( L^p(\Omega) \). Let \( V := \{ u | u \in (C_0^\infty(\Omega))^3, \text{div} u = 0 \} \), \( H \) be the closure of \( V \) in \( L^2(\Omega) \) topology, \( \| \cdot \|_H \) and \((\cdot, \cdot)\) denote the norm and inner product in \( H \) respectively, where, for \( u, v \in H \),
\[
(u, v) = \sum_{i=1}^{3} \int_{\Omega} u_i(x)v_i(x)dx.
\]

\( V \) is the closure of the set \( V \) in \( (H^1_0(\Omega))^3 \) topology, \( \| \cdot \|_V \) and \((\cdot, \cdot)\) denote the norm and inner product in \( V \) respectively, where, for \( u, v \in V \),
\[
((u, v)) = \sum_{i,j=1}^{3} \int_{\Omega} \partial u_j \partial x_i \partial v_j \partial x_i dx.
\]

Clearly, \( V \rightarrow H \equiv H' \rightarrow V' \), \( H' \) and \( V' \) are dual spaces of \( H \) and \( V \) respectively, where the injection is dense and continuous. \( \| \cdot \|_* \) and \( \langle \cdot, \cdot \rangle \) denote the norm in \( V' \) and the dual product between \( V \) and \( V' \) respectively.

Let \( X \) be a Banach space with norm \( \| \cdot \|_X \). The Hausdorff semi-distance \( \text{dist}_X \) between two sets \( C_1, C_2 \subset X \) is defined by
\[
\text{dist}_X(C_1, C_2) = \sup_{x \in C_1} \inf_{y \in C_2} \| x - y \|_X,
\] (2.1)
which will be used in the definition of attractors.

We use \( P \) to denote the Helmhholz-Leray orthogonal projection in \( L^2(\Omega) \) onto the
space \( H \), \( A := -P\Delta \) is the Stokes operator with domain \( \mathcal{D}(A) = (H^2(\Omega))^3 \cap V \).
\( \{ \lambda_j \}_{j=1}^{\infty} \) are the eigenvalues of \( A \) with corresponding eigen-
functions \( \{ \omega_j \}_{j=1}^{\infty} \).

• The weighted spaces for delay term
Given \( T > \tau \) and \( u : (-\infty, T) \rightarrow H \), for any \( t \in (\tau, T) \), let \( u_t \) be the function
defined on \( (-\infty, 0] \) satisfying \( u_t(\theta) = u(t + \theta), \theta \in (-\infty, 0] \). There are several
phase spaces that have been used to deal with the infinite delays \[11, 12\]. We will use
the following phase space for our problem,
\[
C_\kappa(H) = \{ \varphi \in C((-\infty, 0]; H) : \lim_{s \rightarrow -\infty} e^{\kappa s} \varphi(s) \text{ exists in } H \}, \quad \text{for any } \kappa > 0,
\]
which is a Banach space with the norm
\[
\| \varphi \|_\kappa := \sup_{s \in (-\infty, 0]} e^{\kappa s} \| \varphi(s) \|_H.
\]
Thus,
\[
\| u_t(\cdot) \|_\kappa := \sup_{\theta \in (-\infty, 0]} e^{\kappa \theta} \| u(t + \theta) \|_H.
\]

• Some lemmas and a refined version of Dini’s Theorem
We directly cite the following two lemmas, which will be useful in the proof of the well-posedness of global weak solution.

**Lemma 2.1.** (See [16], Lemma 1.3) Assume that \( Q \in \mathbb{R}^n_x \times \mathbb{R}_t \) is a bounded domain, \( g_\mu \in L^q(Q), \ g \in L^q(Q), \ 1 < q < +\infty, \) and \( \|g_\mu\|_{L^q(Q)} \leq C, g_\mu \to g \) almost everywhere (a.e.) in \( Q, \) then we have
\[
g_\mu \rightharpoonup g \text{ in } L^q(Q),
\]
where “\( \rightharpoonup \)” means weak convergence.

**Lemma 2.2.** (See [9], Lemma 4.4) Assume that \( p \geq 2, \) then for any \( a \in \mathbb{R}^n, \ b \in \mathbb{R}^n, \)
\[
\langle |a|^{p-2}a - |b|^{p-2}b, a - b \rangle \geq \gamma_0 |a - b|^p,
\]
where \( \gamma_0 > 0 \) is a constant depending only on \( n \) and \( p. \)

Next, we supply a proof for a refined version of Dini’s Theorem, which is useful for proving the uniform convergence.

**Theorem 2.3.** Suppose a sequence of functions \( \{S_n(x)\}_{n=1}^\infty \) on the finite interval \( [a, b] \) satisfy that
\[
S_n(x) \to S(x), \text{ a.e. for } a \leq t \leq b, \text{ as } n \to \infty;
S_n(a) \to S(a) \text{ and } S_n(b) \to S(b), \text{ as } n \to \infty.
\]
If \( \{S_n(x)\}_{n=1}^\infty \) is monotonic on \( [a, b] \) for any \( n \in \mathbb{N}^+ \) and \( S(x) \) is continuous, then \( \{S_n(x)\}_{n=1}^\infty \) uniformly converges to \( S(x) \) on \([a, b].\)

**Proof.** Since \( S(x) \) is continuous on \([a, b], \) then it is uniformly continuous. Combining the fact \( S_n(x) \to S(x) \) a.e. on \([a, b], \) for any \( \varepsilon > 0, \) we can choose \( m + 1 \) points \( a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b \) such that
\[
|S(x) - S(y)| < \frac{\varepsilon}{2} \text{ for any } x, y \in [x_{i-1}, x_i], \ 1 \leq i \leq m; \tag{2.4}
\]
\[
S_n(x_i) \to S(x_i) \quad \forall \ 0 \leq i \leq m, \text{ as } n \to \infty. \tag{2.5}
\]
Hence, there exists \( N = \max\{N_0, N_1, \cdots, N_m\} \) such that
\[
|S_n(x_i) - S(x_i)| < \frac{\varepsilon}{2} \quad \forall \ 0 \leq i \leq m, \text{ when } n > N. \tag{2.6}
\]
Since \( \{S_n(x)\} \) is monotonic on \([a, b], \) then for any \( x \in [a, b], \) there exists \( 1 \leq i \leq m \) such that
\[
x \in [x_{i-1}, x_i], \quad |S_n(x) - S(x)| \leq \max \{|S_n(x_{i-1}) - S(x)|, |S_n(x_i) - S(x)|\}. \tag{2.7}
\]
Using (2.4) and (2.6), for any \( x \in [x_{i-1}, x_i], \) one has
\[
|S_n(x_{i-1}) - S(x)| \leq |S_n(x_{i-1}) - S(x_{i-1})| + |S(x_{i-1}) - S(x)| < \varepsilon, \text{ for } n > N; \tag{2.8}
\]
\[
|S_n(x_i) - S(x)| \leq |S_n(x_i) - S(x_i)| + |S(x_i) - S(x)| < \varepsilon, \quad \text{for } n > N. \tag{2.9}
\]
Collecting (2.7), (2.8) and (2.9) together, one has
\[
|S_n(x) - S(x)| < \varepsilon, \text{ when } n > N, \tag{2.10}
\]
which finishes the proof. \( \square \)
3. Global well-posedness for \((1.1)\). In this section we prove the existence, regularity and stability of weak solution to the system \((1.1)\). By the Helmholtz-Leray projection, the system \((1.1)\) can be transformed to the abstract equivalent form

\[
\begin{cases}
\frac{du}{dt} + \nu Au + P(\alpha u + \beta |u|u + \gamma |u|^2 u) = Pf(t, u_t) + Pg(t, x), & t > \tau, x \in \Omega, \\
u u|_{t=0} = 0, & t > \tau, \\
u u|_{t=\tau} = \varphi(x), & x \in \Omega, \\
u u(\theta, x) = u(\tau + \theta, x) = \phi(\theta, x), & \theta \in (-\infty, 0], x \in \Omega.
\end{cases}
\] (3.1)

In order to state the problem in the correct framework, we first establish the suitable assumption on the delay term \(f\).

\textbf{(H-1)} The function \(f : [\tau, T] \times C(\kappa(H)) \to L^2(\Omega)\) satisfies:

(I-1) For any \(\xi \in C(\kappa(H))\), the function \(f(\cdot, \xi)\) is measurable and \(f(\cdot, 0) \equiv 0\);

(I-2) There exists a \(L_f > 0\) such that

\[|f(t, \xi) - f(t, \eta)|_2 \leq L_f |\xi - \eta|_\kappa, \text{ for all } t \in [\tau, T], \; \xi, \eta \in C(\kappa(H)).\]

In fact, an example of forcing term with distributed delay satisfying the above assumption is given in [19], Example 2.

\textbf{Definition 3.1.} Let \(T > \tau\), assume that the initial data \((\varphi, \phi) \in H \times C(\kappa(H)), g \in L^2_{\text{loc}}(\mathbb{R}; V')\). We shall say that \(u(x, t) \in C([\tau, T]; H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; L^4(\Omega))\)

is a weak solution to \((1.1)\) with the initial data \((\varphi, \phi)\) if it satisfies:

\[
\begin{cases}
\left( \frac{\partial u}{\partial t}, v \right) + \nu (Au, v) + \alpha(u, v) + \beta(|u|u, v) + \gamma(|u|^2 u, v) = \left( f(t, u_t), v \right) + \langle g(t, x), v \rangle, \\
u u|_{t=0} = \varphi, \quad u|_{t=\tau} = \nu u(\theta, x) = u(\tau + \theta, x) = \phi(\theta, x) \text{ for } \theta \in (-\infty, 0], \\
\end{cases}
\]

where the first equation holds for all \(v \in V\) in the sense of \(D'(\tau, T)\). Furthermore, we call \(u(x, t)\) the strong solution of \((1.1)\), if it is a weak solution satisfying that \(u \in L^\infty(\tau, T; V) \cap L^2(\tau, T; D(A))\).

Based on the Definition 3.1, we will show the existence of weak solution to \((1.1)\) in the following.

\textbf{Theorem 3.2.} For any \(T > \tau\) and \(\kappa > 0\), suppose that the initial data \((\varphi, \phi) \in H \times C(\kappa(H)), g \in L^2_{\text{loc}}(\mathbb{R}; V')\), \(f\) satisfies hypothesis \((H-1)\), then problem \((1.1)\) possesses a weak solution \(u(t, x)\) satisfying

\[u \in C([\tau, T]; H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; L^4(\Omega)),\]

and the following estimates hold:

\[
\|u(t)\|_H^2 \leq \|u_0\|_\kappa^2 + e^{-(\eta - 2L_f)(t-\tau)} \|\varphi\|_\kappa^2 + \int_\tau^t e^{-(\eta - 2L_f)(t-s)} \frac{\|g(s)\|_2^2}{\nu} ds; \quad (3.3)
\]

\[
\|u(t)\|_H^2 + \nu \int_\tau^t \|u\|_V^2 ds + 2\gamma \int_\tau^t |u|^4 ds \leq \|\varphi\|_H^2 + e^{2L_f(t-\tau)} \|\varphi\|_\kappa^2 + 2 \int_\tau^t e^{2L_f(t-s)} \frac{\|g(s)\|_2^2}{\nu} ds, \quad (3.4)
\]

for all \(t \geq \tau, \kappa > 0\) and \(0 < \eta \leq \min\{2\alpha, 2\kappa\}\).

Furthermore, if \((\varphi, \phi) \in V \times C(\kappa(H)), g \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))\), then the weak solution becomes strong, i.e., \(u \in L^2(\tau, T; D(A)) \cap L^\infty(\tau, T; V)\) and \(\frac{\partial u}{\partial t} \in L^2(\tau, T; L^2(\Omega))\).

\textbf{Proof.} \textbf{Step 1:} Approximated solution of \((1.1)\).
Let \( \{w_j\}_{j=1}^\infty \in V \) be an orthonormal basis of \( H \), which also are the eigenfunctions for the Stokes operator \( A \). Let \( V_m = \text{span}\{w_1, \cdots, w_m\} \) be subspace of \( V \) and the projector \( P_m : H \rightarrow V_m \) is given by

\[
P_m u = \sum_{j=1}^m (u, w_j)w_j \quad \text{for} \quad u \in H.
\]

We construct the approximated solution \( u_m(t) = \sum_{j=1}^m h_{j,m}(t)w_j \) satisfying the following Cauchy problem

\[
\begin{cases}
\frac{d}{dt} u_m + \nu(A u_m, w_j) + \alpha(u_m, w_j) + \beta(|u_m|u_m, w_j) + \gamma(|u_m|^2u_m, w_j) \\
= (f(t, u_{m,t}), w_j) + (g(t, x), w_j), \quad \forall 1 \leq j \leq m; \\
\quad u_m(\tau, x) = P_m \varphi, \quad u_m(\theta, x) = P_m \phi(\theta, x) \quad \text{for} \quad \theta \in (-\infty, 0].
\end{cases}
\]

The problem (3.5) is an ordinary functional differential equation with respect to the unknown variables \( \{h_{j,m}(t)\}_{j=1}^m \), which has a unique local solution (in an interval \([\tau, t^*] \) with \( \tau < t^* \leq T \)) by the existence and uniqueness of a local solution of ordinary functional differential equations with infinite delay [12], Th. 1.1, p. 36, or see [11]. In fact, the global solution \( t^* = T \) can be deduced by the \textit{a priori} estimates below.

**Step 2: The \textit{a priori} estimates for \( \{u_m\} \).**

Multiplying the first equation in (3.5) by \( h_{j,m}(t) \) and summing in \( j \), we obtain that for a.e. \( t \in [\tau, T] \),

\[
\begin{align*}
\frac{d}{dt} \|u_m\|^2_H &+ 2\nu \|u_m\|^2_V + 2\alpha \|u_m\|^2 + 2\beta \|u_m\|^3 + 2\gamma \|u_m\|^4 \\
&= 2(f(t, u_m, t), u_m) + 2(g(t), u_m) \\
&\leq 2\|f(t, u_m, t)\|_H^2 + 2\|g(t)\|_V^2 \& \|u_m\|_V \quad (3.6)
\end{align*}
\]

the last inequality is obtained by using Young’s inequality.

Therefore, for any \( 0 < \eta \leq \min\{2\alpha, 2\kappa\} \), one has

\[
\begin{align*}
\frac{d}{dt} \|u_m\|^2_H &+ \nu \|u_m\|^2_V + \eta \|u_m\|^2_H + 2\beta \|u_m\|^3 + 2\gamma \|u_m\|^4 \\
&\leq 2L_f \|u_m, t\|^2 + \frac{1}{\nu} \|g(t)\|_V^2. \quad (3.7)
\end{align*}
\]

Multiplying by \( e^{\eta t} \) in both sides of (3.7) and integrating over \([\tau, t]\) with the time variable, we deduce that

\[
\begin{align*}
e^{\eta t} \|u_m(t)\|^2_H + \nu \int_\tau^t e^{\eta s} \|u_m\|^2_V ds + 2\int_\tau^t e^{\eta s}(\beta \|u_m\|^3 + \gamma \|u_m\|^4)ds \\
\leq e^{\eta t} \|\varphi\|^2_H + \int_\tau^t e^{\eta s}(2L_f \|u_m, s\|^2 + \frac{1}{\nu} \|g(s)\|^2_\nu)ds,
\end{align*}
\]

i.e.

\[
\begin{align*}
\|u_m(t)\|^2_H &+ \nu \int_\tau^t e^{-\eta(t-s)} \|u_m\|^2_V ds + 2\int_\tau^t e^{-\eta(t-s)}(\beta \|u_m\|^3 + \gamma \|u_m\|^4)ds \\
&\leq e^{-\eta(t-\tau)} \|\varphi\|^2_H + \int_\tau^t e^{-\eta(t-s)}(2L_f \|u_m, s\|^2 + \frac{1}{\nu} \|g(s)\|^2_\nu)ds.
\end{align*}
\]

(3.9)
Thus, using (3.9), one has
\[
\|u_{m,t}\|_\kappa^2 \leq \max\{ \sup_{\theta \in (-\infty, \tau-t]} e^{2\kappa \theta} \|\phi(\theta + t - \tau)\|_H^2, \sup_{\theta \in [\tau-t, 0]} e^{2\kappa \theta} e^{-\eta(t+\theta)} \|\varphi\|_H^2 \}
\]
\[
+ e^{2\kappa \theta} \int_{\tau}^{t+\theta} e^{-\eta(t+\theta-s)} (2L_f \|u_{m,s}\|_\kappa^2 + \frac{1}{\nu} \|g(s)\|_*^2) ds.
\]
(3.10)

On the one hand, for any \(0 < \eta \leq \min\{2\alpha, 2\kappa\}\), one has
\[
\sup_{\theta \in (-\infty, \tau-t]} e^{2\kappa \theta} \|\phi(\theta + t - \tau)\|_H^2 = \sup_{\theta \in (-\infty, 0]} e^{2\kappa \theta} \|\phi(\theta)\|_H^2
\]
\[
eq e^{-2\kappa(t-\tau)} \|\phi\|_\kappa^2 \leq e^{-\eta(t-\tau)} \|\phi\|_\kappa^2,
\]
(3.11)

On the other hand,
\[
\sup_{\theta \in [\tau-t, 0]} e^{2\kappa \theta} e^{-\eta(t+\theta)} \|\varphi\|_H^2 \leq e^{-\eta(t-\tau)} \|\varphi\|_H^2 \leq e^{-\eta(t-\tau)} \|\phi\|_\kappa^2,
\]
(3.12)

and
\[
\sup_{\theta \in [\tau-t, 0]} e^{2\kappa \theta} \int_{\tau}^{t+\theta} e^{-\eta(t+\theta-s)} (2L_f \|u_{m,s}\|_\kappa^2 + \frac{1}{\nu} \|g(s)\|_*^2) ds
\]
\[
\leq \int_{\tau}^{t} e^{-\eta(t-s)} (2L_f \|u_{m,s}\|_\kappa^2 + \frac{1}{\nu} \|g(s)\|_*^2) ds.
\]
(3.13)

Combining inequalities (3.10) - (3.13), we deduce that
\[
\|u_{m,t}\|_\kappa^2 \leq e^{-\eta(t-\tau)} \|\phi\|_\kappa^2 + \int_{\tau}^{t} e^{-\eta(t-s)} (2L_f \|u_{m,s}\|_\kappa^2 + \frac{1}{\nu} \|g(s)\|_*^2) ds,
\]
(3.14)
i.e.
\[
e^{\eta t} \|u_{m,t}\|_\kappa^2 \leq e^{\eta \tau} \|\phi\|_\kappa^2 + \int_{\tau}^{t} e^{\eta s} (2L_f \|u_{m,s}\|_\kappa^2 + \frac{1}{\nu} \|g(s)\|_*^2) ds.
\]
(3.15)

Thanks to the Gronwall Lemma, it yields
\[
e^{\eta t} \|u_{m,t}\|_\kappa^2 \leq e^{\eta \tau} \|\phi\|_\kappa^2 e^{2L_f(t-\tau)} + \int_{\tau}^{t} e^{2L_f(t-s)} e^{\eta s} \frac{\|g(s)\|_*^2}{\nu} ds,
\]
(3.16)
i.e.
\[
\|u_{m,t}\|_\kappa^2 \leq e^{-(\eta-2L_f)(t-\tau)} \|\phi\|_\kappa^2 + \int_{\tau}^{t} e^{-(\eta-2L_f)(t-s)} \frac{\|g(s)\|_*^2}{\nu} ds \leq C,
\]
(3.17)
where \(C\) is a constant depending on \(\nu, \eta, L_f, \tau\) and \(T\).
Thus, by (3.6) and (3.17), one has

\[
\|u_m(t)\|_{H^1}^2 + \nu \int_\tau^t \|u_m\|_{V'}^2 \, ds + 2 \int_\tau^t (\alpha |u_m|^2 + \beta |u_m|^3 + \gamma |u_m|^4) \, ds \\
\leq \|\phi\|_H^2 + 2L_f \int_\tau^t \|u_{m,s}\|_H^2 \, ds + \int_\tau^t \|g(s)\|_{\nu'}^2 \, ds \\
\leq \|\phi\|_H^2 + 2L_f \int_\tau^t (e^{-(\eta-2L_f)(s-\tau)}) \|\phi\|_H^2 + \int_\tau^s e^{-(\eta-2L_f)(s-r)} \frac{\|g(r)\|_{\nu'}^2}{\nu} \, dr \, ds \\
+ \int_\tau^t \|g(s)\|_{\nu'}^2 \, ds \\
\leq \|\phi\|_H^2 + e^{2L_f(t-\tau)} \|\phi\|_H^2 + 2L_f \int_\tau^t \int_\tau^s e^{-(\eta-2L_f)(s-r)} \frac{\|g(r)\|_{\nu'}^2}{\nu} \, ds \, dr \\
+ \int_\tau^t \|g(s)\|_{\nu'}^2 \, ds \\
\leq \|\phi\|_H^2 + e^{2L_f(t-\tau)} \|\phi\|_H^2 + \int_\tau^t e^{2L_f(t-s)} \frac{\|g(s)\|_{\nu'}^2}{\nu} \, ds + \int_\tau^t \|g(s)\|_{\nu'}^2 \, ds,
\]

which means that

\[
\{u_m\} \text{ is bounded in } L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; L^4(\Omega)). \tag{3.19}
\]

Recalling that

\[
\frac{\partial u_m}{\partial t} = -\nu A u_m - P(\alpha u_m + \beta |u_m| u_m + \gamma |u_m|^2 u_m) + Pf(t, u_m, t) + Pg(t, x).
\]

On the one hand, \( |u_m|^2 u_m \) is bounded in \( L^\frac{4}{3}(\tau, T; L^\frac{4}{3}(\Omega)) \rightarrow L^\frac{4}{3}(\tau, T; V') \). On the other hand, by hypothesis \( (H-1) \) and (3.17), one has

\[
\|f(t, u_m, t)\|_{L^2(\tau, T; L^2(\Omega))} \leq L_f^2 \int_\tau^T \|u_m, t\|_{L^2(\Omega)}^2 \, dt \leq C. \tag{3.20}
\]

Thus, we infer that the sequence

\[
\left\{ \frac{\partial u_m}{\partial t} \right\} \text{ is uniformly bounded in } L^2(\tau, T; V') + L^\frac{4}{3}(\tau, T; L^\frac{4}{3}(\Omega)) \rightarrow L^\frac{4}{3}(\tau, T; V').
\]

In particular, combining (3.19) and (3.21), by the Lions-Magenes Theorem, one has

\[
\left\{ u_m \right\} \text{ is uniformly bounded in } C([\tau, T]; H). \tag{3.22}
\]

**Step 3: Passing to limit for deriving the global solution of (1.1) by compact argument and energy method.**

By the Aubin-Lions Lemma, the following space

\[
W = \left\{ u \mid u \in L^2(\tau, T; V); \frac{\partial u}{\partial t} \in L^\frac{4}{3}(\tau, T; V') \right\}
\]

is compactly embedded in \( L^2(\tau, T; H) \). Combining the preceding uniform estimates (3.19)-(3.22), we can deduce that there exists a subsequence \( u_m \) (without relabeling)
such that, when $m \to \infty$,

\begin{align}
  &u_m \to u \text{ strongly in } L^2(\tau, T; H); \\
  &u_m \to u \text{ weakly * in } L^\infty(\tau, T; H); \\
  &u_m \to u \text{ weakly in } L^2(\tau, T; V);
\end{align}

(3.23)

\begin{align}
  |u_m| u_m &\to \chi \text{ weakly in } L^2(\tau, T; L^2(\Omega)); \\
  |u_m|^2 u_m &\to \zeta \text{ weakly in } L^{\frac{2}{3}}(\tau, T; L^{\frac{4}{3}}(\Omega)); \\
  f(t, u_{m,t}) &\to \xi \text{ weakly in } L^2(\tau, T; L^2(\Omega)); \\
  \frac{\partial u_m}{\partial t} &\to \frac{\partial u}{\partial t} \text{ weakly in } L^{\frac{3}{2}}(\tau, T; V'),
\end{align}

(3.26)

(3.27)

(3.28)

(3.29)

with $u \in C([\tau, T]; H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; L^4(\Omega))$.

Thus, using Lemma 2.1 and combining (3.23), (3.26), (3.27), we derive that

\[ \chi = |u|u, \quad \zeta = |u|^2u. \]

(3.30)

Furthermore, we prove the following limiting process (as $m \to \infty$) by the energy method,

\[ f(t, u_{m,t}) \to f(t, u_t) \text{ strongly in } L^2(\tau, T; L^2(\Omega)). \]

(3.31)

Notice that

\[ |f(t, u_{m,t}) - f(t, u_t)|_2 \leq Lf \|u_{m,t} - u_t\|_\kappa, \]

(3.32)

thus combining (3.20) and (3.28) together, we only need to show that, when $m \to \infty$,

\[ u_{m,t} \to u_t \text{ in } C_\kappa(H) \] for any $\tau \leq t \leq T$. 

(3.33)

Since for any $\tau \leq t \leq T$,

\[
\|u_{m,t} - u_t\|_\kappa^2 \leq \max\{ \sup_{\theta \in (-\infty, \tau-t]} e^{2\kappa\theta} \|P_m\theta(\theta + t - \tau) - \phi(\theta + t - \tau)\|_H^2, \\
\sup_{\theta \in [\tau-t, 0]} e^{2\kappa\theta} \|u_m(\theta + t) - u(\theta + t)\|_H^2 \}
\]

(3.34)

\[
\leq \max\{ \|P_m\phi - \phi\|_\kappa^2, \sup_{\theta \in [\tau,T]} \|u_m(\theta) - u(\theta)\|_H^2 \},
\]

then we split into three steps to prove (3.33).

\[ (1) \text{: } \|P_m\phi - \phi\|_\kappa \to 0 \text{ as } m \to \infty. \]

For any $s \leq 0$, one has

\[
\|P_m\phi - \phi\|_\kappa \leq \sup_{\theta \in (-\infty, s]} e^{\kappa\theta} \|P_m\phi(\theta) - \phi(\theta)\|_H + \sup_{\theta \in [s, 0]} e^{\kappa\theta} \|P_m\phi(\theta) - \phi(\theta)\|_H
\]

(3.35)

Defining \[ \lim_{\theta \to -\infty} e^{\kappa\theta} \phi(\theta) = \tilde{\phi} \] in $H$, i.e., for any $\varepsilon > 0$, there exists a large enough constant $M > 0$ such that

\[ \|e^{\kappa\theta} \phi(\theta) - \tilde{\phi}\|_H < \frac{\varepsilon}{6}, \quad \text{when } \theta \leq -M. \]

Then, taking $s \leq -M$, there exists $N_1 > 0$ such that

\[
\sup_{\theta \in (-\infty, s]} e^{\kappa\theta} \|P_m\phi(\theta) - \phi(\theta)\|_H \leq \sup_{\theta \in (-\infty, s]} \|P_m(e^{\kappa\theta} \phi(\theta)) - P_m\tilde{\phi}\|_H + \\
\|P_m\phi - \tilde{\phi}\|_H + \sup_{\theta \in (-\infty, s]} \|\tilde{\phi} - e^{\kappa\theta} \phi(\theta)\|_H
\]

(3.36)

\[ \leq \frac{\varepsilon}{2}, \quad \text{for any } m > N_1. \]
Combining (3.35)-(3.37), we finish the proof of convergence of initial datum in $C(\mathbb{R})$.

(II): $\| u_m(t) \|_H^2 \to \| u(t) \|_H^2$ uniformly on $[\tau, T]$ as $m \to \infty$ by the energy method.

From (3.6) and (3.17), the following energy inequality holds for all $u_m$:

\[ \| u_m(t) \|_H^2 + 2\nu \int_s^t \| u_m(r) \|_V^2 \, dr \leq \| u_m(s) \|_H^2 + \frac{1}{2\alpha} \int_s^t |f(r, u_m, r)|_2^2 \, dr + 2 \int_s^t \langle g(r), u_m(r) \rangle \, dr \]

\[ \leq \| u_m(s) \|_H^2 + \frac{L^2}{2\alpha} \int_s^t \| u_m, r \|_\gamma^2 \, dr + 2 \int_s^t \langle g(r), u_m(r) \rangle \, dr \]

\[ \leq \| u_m(s) \|_H^2 + 2 \int_s^t \langle g(r), u_m(r) \rangle \, dr + C(t-s), \]

for all $\tau \leq s \leq t \leq T$, where $C$ is a constant depending on $\alpha, \nu, \eta, L_f, \tau$ and $T$.

By (3.24)-(3.30), passing to the limit $m \to \infty$ in (3.5), the following holds

\[ \left( \frac{\partial u}{\partial \theta}, v \right) + \nu (Au, v) + \alpha (u, v) + \beta (|u|u, v) + \gamma (|u|^2 u, v) = (\xi, v) + \langle g, v \rangle, \]

for all $v \in V$, with the initial data $u|_{t=\tau} = \varphi(x)$ and $u_\tau(x, t) = u(\tau + \theta, x) = \phi(\theta, x)$.

Taking $v = u$ in (3.39), one obtains the following energy inequality:

\[ \| u(t) \|_H^2 + 2\nu \int_s^t \| u(r) \|_V^2 \, dr \leq \| u(s) \|_H^2 + \frac{1}{2\alpha} \int_s^t |\xi(r)|_2^2 \, dr \]

\[ + 2 \int_s^t \langle g(r), u(r) \rangle \, dr, \]

for all $\tau \leq s \leq t \leq T$. By (3.28), one has

\[ \int_s^t |\xi(r)|_2^2 \, dr \leq \liminf_{m \to \infty} \int_s^t |f(r, u_m, r)|_2^2 \, dr \quad \forall \tau \leq s \leq t \leq T. \]

Plugging (3.41) into (3.40), we arrive at

\[ \| u(t) \|_H^2 + 2\nu \int_s^t \| u(r) \|_V^2 \, dr \leq \| u(s) \|_H^2 + 2 \int_s^t \langle g(r), u(r) \rangle \, dr + C(t-s), \]

for all $\tau \leq s \leq t \leq T$, where $C$ is the same constant in (3.38).

Now, consider functions $J_m(t), J(t) : [\tau, T] \to \mathbb{R}$ defined by

\[ J_m(t) = \| u_m(t) \|_H^2 - 2 \int_\tau^t \langle g(r), u_m(r) \rangle \, dr - Ct; \]

\[ J(t) = \| u(t) \|_H^2 - 2 \int_\tau^t \langle g(r), u(r) \rangle \, dr - Ct, \]

where $C$ is the same constant in (3.38).
From (3.38) and (3.42), one knows that, for any \( m \in \mathbb{N}^+ \), \( J_m(t) \) and \( J(t) \) are non-increasing and continuous functions on \([\tau, T]\).

On the other hand, by using (3.23), we have
\[
\|u_m(t) - u(t)\|_H^2 \to 0 \quad \text{a.e. for } \tau \leq t \leq T \text{ as } m \to \infty, \tag{3.45}
\]
especially from the construction of initial data, one has
\[
\|u_m(\tau) - u(\tau)\|_H^2 \to 0 \text{ as } m \to \infty. \tag{3.46}
\]

By (3.25), one has, for any \( \tau \leq t \leq T \),
\[
\left| \int_\tau^t \langle g(r), u_m(r) \rangle \, dr - \int_\tau^t \langle g(r), u(r) \rangle \, dr \right| \leq \int_\tau^T \|g(r), u_m(r) - u(r)\| \, dr \tag{3.47}
\]
\[
\to 0, \text{ as } m \to \infty,
\]
i.e., \( \int_\tau^t \langle g(r), u_m(r) \rangle \, dr \) converges uniformly to \( \int_\tau^t \langle g(r), u(r) \rangle \, dr \) on \([\tau, T]\).

Combining (3.45), (3.46) and (3.47) together, one has
\[
J_m(t) \to J(t), \quad \text{a.e. for } \tau \leq t \leq T \text{ and } J_m(\tau) \to J(\tau), \text{ as } m \to \infty. \tag{3.48}
\]

Since \( T \) is arbitrary, using a refined version of Dini’s Theorem 2.3 (see Preliminary), we obtain that
\[
J_m(t) \to J(t), \quad \text{uniformly on } [\tau, T], \text{ as } m \to \infty. \tag{3.49}
\]

Reusing (3.47), one achieves that
\[
\sup_{t \in [\tau, T]} \left| \|u_m(t)\|_H^2 - \|u(t)\|_H^2 \right| \to 0, \text{ as } m \to \infty. \tag{3.50}
\]

(III): \( \|u_m - u\|_{C([\tau, T]; H)} \to 0 \text{ as } m \to \infty \) by a contradiction argument.

If this was not so, then there exists \( \varepsilon > 0 \) and two subsequences (without relabeling) \( \{u_m\} \) and \( \{t_m\} \in [\tau, T] \) such that \( \|u_m(t_m) - u(t_0)\|_H \geq 2\varepsilon \) for any \( m \in \mathbb{N}^+ \). Therefore, there exists \( t_0 \in [\tau, T] \) such that \( \lim_{m \to \infty} t_m = t_0 \). Notice that \( \lim_{m \to \infty} \|u(t_m) - u(t_0)\|_H = 0 \), so there exists \( M > 0 \) such that
\[
\|u_m(t_m) - u(t_0)\|_H \geq \|u_m(t_m) - u(t_m)\|_H - \|u(t_m) - u(t_0)\|_H \geq \varepsilon, \quad \forall m \geq M. \tag{3.51}
\]

However, on the one hand, by using (3.50),
\[
\|u_m(t_m)\|_H - \|u(t_0)\|_H \leq \left( \|u_m(t_m)\|_H - \|u(t_m)\|_H \right) + \left( \|u(t_m)\|_H - \|u(t_0)\|_H \right) \to 0, \text{ as } m \to \infty, \tag{3.52}
\]
i.e. \( \lim_{m \to \infty} \|u_m(t_m)\|_H = \|u(t_0)\|_H \).

On the other hand, using (3.21), one has
\[
\|u_m(t) - u_m(s)\|_H \leq \int_s^t \|u_m'(r)\|_H \, dr \leq (t - s)^{\frac{1}{2}} \|u_m'\|_{L^2(\tau, T; V')} \tag{3.53}
\]
which means that \( \{u_m\} \) is equicontinuous on \([\tau, T]\) in \( V' \). Combining the fact \( \|u_m\|_{L^\infty(\tau, T; V')} \leq C\|u_m\|_{L^\infty(\tau, T; H)} \leq C \) (\( C \) is independent of \( m \)), then by the Ascoli-Arzelà Theorem, we have
\[
u_m \to u, \quad \text{strongly in } C([\tau, T]; V'). \tag{3.54}
\]
Thus, combining the facts \( \|u_m(t_m)\|_H \leq \|u_m\|_{L^\infty(\tau, T; H)} \leq C \) and \( \lim_{m \to \infty} t_m = t_0 \), one has
\[
u_m(t_m) \to u(t_0), \quad \text{weakly in } H. \tag{3.55}
\]
Combining (3.52) and (3.55), by Radon’s Theorem, one has
\[ u_m(t_m) \rightarrow u(t_0), \quad \text{strongly in } H, \] (3.56)
which contradicts to (3.51), hence \( \|u_m - u\|_{C([\tau,T];H)} \rightarrow 0 \) as \( m \rightarrow \infty \).

Combining Step (I) and Step (III) together, we finish the proof of both (3.33) and (3.31).

Finally, combining (3.24)-(3.31) and passing to the limit of (3.5), we conclude that \( u \) is exactly a weak solution to (1.1). Collecting (3.17) and (3.18) together, one obtains the estimates of (3.3)-(3.4).

**Step 4: Regularity of weak solution.**

Let \( \{w_j\}_{j=1}^{\infty} \in V \) be an orthonormal basis of \( H \), which satisfies
\[ Aw_j = \lambda_j w_j, \quad \text{in } \Omega; \quad w_j = 0, \quad \text{on } \partial \Omega. \] (3.57)
Substituting \( w_j \) by \( \frac{Aw_j}{\lambda_j} \) in (3.5), one has
\[ \left( \frac{\partial u_m}{\partial t}, Aw_j \right) + \nu(Au_m, Aw_j) + \alpha(u_m, Aw_j) + \beta(|u_m|^{2d}u_m, Aw_j) \]
\[ + \gamma(|u_m|^2u_m, Aw_j) = (f(t, u_m), Aw_j) + (g, Aw_j). \] (3.58)
Multiplying (3.58) by \( h_{j,m}(t) \) and summing in \( j \), we obtain that
\[ \frac{d}{dt} \|u_m\|^2_V + 2\nu|Au_m|^2 + 2\alpha\|u_m\|^2 + 2\beta(\nabla(|u_m|u_m), \nabla u_m) \]
\[ + 2\gamma(\nabla(|u_m|^2u_m), \nabla u_m) = 2(f(t, u_m), Au_m) + 2(g, Au_m) \]
\leq C\nu|f(t, u_m)|^2 + \frac{\nu}{2}|Au_m|^2 + C\nu|g(t, x)|^2 + \frac{\nu}{2}|Au_m|^2. \] (3.59)
Noting that
\[ (\nabla(|u_m|u_m), \nabla u_m) = 2(|u_m|\nabla u_m, \nabla u_m) \geq 0, \]
\[ (\nabla(|u_m|^2u_m), \nabla u_m) = 3|u_m|^2\nabla u_m, \nabla u_m) \geq 0, \]
combining with (3.59), we have
\[ \frac{d}{dt} \|u_m\|^2_V + \nu|Au_m|^2 \leq C\nu|f(t, u_m)|^2 + \nu|g(t, x)|^2. \] (3.60)
Integrating (3.60) over \([\tau, t]\) with the time variable and using (3.20), we deduce that
\[ \|u_m(t)\|^2_V + \nu \int_{\tau}^{t} |Au_m|^2 ds \leq C, \]
i.e. we get that
\[ \{u_m\} \text{ is uniformly bounded in } L^\infty(\tau, T; V) \cap L^2(\tau, T; D(A)). \] (3.61)
Since \( V \) is continuously embedded in \( L^6(\Omega) \), then by \( \|u_m\|_{L^\infty(\tau, T; V)} \leq C \), one has
\[ |u_m|^2u_m \in L^2(\tau, T; L^2(\Omega)), \]
and we infer that the sequence
\[ \left\{ \frac{\partial u_m}{\partial t} \right\} \text{ is uniformly bounded in } L^2(\tau, T; L^2(\Omega)). \] (3.62)
Combining (3.61) and (3.62), we finish the proof. \( \square \)
Furthermore, we also obtain the following theorem about the stability of (1.1).

**Theorem 3.3.** For any \( T > \tau \) and given functions \((\varphi_i, \phi_i, g_i) \in H \times C_\kappa(H) \times L^2_0([\tau, T]; V')\) for \( i = 1, 2 \), if \( f \) satisfies hypothesis (H-1), then problem (1.1) possesses two weak solutions \( \{w_i\}_{i=1,2} \subset C([\tau, T]; H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; L^4(\Omega)) \) with respect to \( \{ (\varphi_i, \phi_i, g_i) \}_{i=1,2} \), and the following stability estimates hold:

\[
\max_{r \in [\tau, t]} \|u^1(r) - u^2(r)\|^2_H \leq 2e^{\frac{\nu}{4\alpha}(t-\tau)}(\|\varphi_1 - \varphi_2\|^2_H + \frac{L^2_2}{4\alpha\kappa}\|\phi_1 - \phi_2\|^2_\kappa) + \frac{1}{\nu} \int_{\tau}^{t} \|g_1 - g_2\|^2_2 ds; \tag{3.63}
\]

\[
\|u^1_t - u^2_t\|^2_\kappa \leq 2e^{\frac{\nu}{4\alpha}(t-\tau)}\left(\frac{L^2_2}{4\alpha\kappa} + 1\right)\|\phi_1 - \phi_2\|^2_\kappa + \frac{1}{\nu} \int_{\tau}^{t} \|g_1 - g_2\|^2_2 ds. \tag{3.64}
\]

**Proof.** Set \( w = u^1 - u^2 \), then

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2_H + \nu \langle Aw, w \rangle + \alpha(w, w) + \beta(|u^1|^2 u^1 - |u^2|^2 u^2, w) + \gamma(|u^1|^2 u^1 - |u^2|^2 u^2, w)
\]

\[= (f(t, u^1_t) - f(t, u^2_t), w) + (g_1 - g_2, w)
\]

\[\leq |f(t, u^1_t) - f(t, u^2_t)|_2 \|w\|^2_H + \|g_1 - g_2\|_V \|w\|_V.
\]

\[
\leq \frac{1}{4\alpha} |f(t, u^1_t) - f(t, u^2_t)|^2_2 + \alpha \|w\|^2_H + \frac{1}{2\nu} \|g_1 - g_2\|^2_2 + \frac{\nu}{2} \|w\|^2_V.
\]

By Lemma 2.2, we derive that

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2_H + \frac{\nu}{2} \|w\|^2_V \leq \frac{1}{4\alpha} |f(t, u^1_t) - f(t, u^2_t)|^2_2 + \frac{1}{2\nu} \|g_1 - g_2\|^2_2. \tag{3.65}
\]

Integrating the above inequality from \( \tau \) to \( t \) and by hypothesis (H-1), we get

\[
\|w(t)\|^2_H + \nu \int_{\tau}^{t} \|w\|^2_V ds
\]

\[\leq \|w(\tau)\|^2_H + \frac{1}{2\alpha} \int_{\tau}^{t} |f(s, u^1_s) - f(s, u^2_s)|^2_2 ds + \frac{1}{\nu} \int_{\tau}^{t} \|g_1 - g_2\|^2_2 ds \tag{3.66}
\]

\[\leq \|\varphi_1 - \varphi_2\|^2_H + \frac{L^2_2}{2\alpha} \int_{\tau}^{t} \|u^1_s\|^2_\kappa ds + \frac{1}{\nu} \int_{\tau}^{t} \|g_1 - g_2\|^2_2 ds.
\]

Similarly with (3.34), for any \( \tau \leq s \leq T \),

\[
\|w_s\|^2_\kappa \leq \max\{e^{2\alpha(\tau-s)}\|\phi_1 - \phi_2\|^2_\kappa, \max_{r \in [\tau, s]} \|w(r)\|^2_H\}. \tag{3.67}
\]

Plugging (3.67) into (3.66), one has

\[
\max_{r \in [\tau, t]} \|w(r)\|^2_H \leq \|\varphi_1 - \varphi_2\|^2_H + \frac{1}{\nu} \int_{\tau}^{t} \|g_1 - g_2\|^2_2 ds + \frac{L^2_2}{2\alpha} \left(\frac{1}{4\kappa}\|\phi_1 - \phi_2\|^2_\kappa + \int_{\tau}^{t} \max_{r \in [\tau, s]} \|w(r)\|^2_H ds\right). \tag{3.68}
\]

Thanks to the Gronwall Lemma, one obtains (3.63). Plugging this result into (3.67), we finish the proof. \(\square\)

4. **Pullback attractors for (1.1).** In this section, we shall show the existence of pullback attractor for fixed bounded sets and families of sets depending on time. In order to present our results on pullback dynamics of problem (1.1), we first recall some basic definitions of the theory.
4.1. Theory of pullback attractors. For the description of pullback attractors, the functional space \( X := H \times C_\kappa(H) \) is used as our phase space equipped with the norm \( \| (\varphi, \phi) \|_X = \| \varphi \|_H + \| \phi \|_\kappa \). From Theorem 3.2 and Theorem 3.3, we have the following result about a continuous process.

**Lemma 4.1.** Assume that \( g \in L^2_{\text{loc}}(\mathbb{R}; V') \), \( f \) satisfies hypothesis (H-1). Then, the unique weak solution of (1.1) generates a biparametric family of maps \( U(t, \tau) : X \to X \) given by

\[
U(t, \tau)(\varphi, \phi) = (u(t), u_\tau), \quad \forall -\infty < \tau \leq t < +\infty,
\]

which defines a semi-process on \( X \) with the following properties:

(i) \( U(t, \tau) \in C(X, X) \) for all \( t \geq \tau \),
(ii) \( U(\tau, \tau) = \text{Id} \) (the identity map) for all \( \tau \in \mathbb{R} \),
(iii) \( U(t, \tau) = U(t, r)U(r, \tau) \) for all \( -\infty < \tau \leq r \leq t < +\infty \).

**Definition 4.2.** A universe \( D \) defined in a metric space \( X \) is a class of families \( \hat{D} \) of the form \( \hat{D} = \{ D(t) | t \in \mathbb{R} \} \), where each \( D(t) \) is a nonempty bounded subset of \( X \).

Let \( \mathcal{P}(X) \) denote the family of all nonempty subsets of \( X \). For our problem, we first consider \( \mathcal{D}_F \), the universe of fixed nonempty bounded subsets of \( X \), which contains all families

\[
\hat{D} = \{ D(t) : D(t) = D, t \in \mathbb{R}, \text{ where } D \text{ is a bounded set of } X \} \subset \mathcal{P}(X). \quad (4.2)
\]

Then, we also consider the corresponding \( \sigma \)-indexed universe \( \mathcal{D}_\sigma \) of all families \( \hat{D}_\sigma = \{ D_\sigma(t) : t \in \mathbb{R} \} \subset \mathcal{P}(X) \) such that

\[
\lim_{\tau \to -\infty} \left( e^{\sigma \tau} \sup_{(\varphi, \phi) \in D_\sigma(\tau)} \| (\varphi, \phi) \|_X^2 \right) = 0. \quad (4.3)
\]

Notice that these universes are inclusion closed.

Our results are concerned with families of universes determined by the time-dependent force \( g = g_\sigma \). Let us put \( \sigma_0 = \eta - 2L_f \) and assume that

\[
\int_{-\infty}^{0} e^{\sigma s} \| g(s) \|_V^2 ds < +\infty, \quad \text{for some } \sigma \in (0, \sigma_0], \quad (4.4)
\]

which is equivalent to

\[
\int_{-\infty}^{t} e^{-\sigma(t-s)} \| g(s, x) \|_V^2 ds < +\infty \quad \text{if } g \in L^2_{\text{loc}}(\mathbb{R}; V'), \quad (4.5)
\]

for all \( t \in \mathbb{R} \) and \( 2L_f < \eta \).

**Definition 4.3.** Given a universe \( \mathcal{D} \) defined on \( X \), a family \( \mathcal{A} = \{ A(t) \}_{t \in \mathbb{R}} \) is called a pullback \( \mathcal{D} \)-attractor of a process \( U(t, \tau) : X \to X \), if the following properties hold:

(i) Compactness: \( A(t) \) is a nonempty compact set of \( X \), \( \forall t \in \mathbb{R} \),
(ii) Invariance: \( U(t, \tau)A(\tau) = A(t), \forall t \geq \tau \),
(iii) Pullback \( \mathcal{D} \)-attraction: \( \lim_{\tau \to -\infty} \text{dist}_X(U(t, \tau)D(\tau), A(t)) = 0, \forall t \in \mathbb{R} \) and \( \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D} \).

In addition, a \( \mathcal{D} \)-attractor \( A \) is said to be minimal, if whenever \( C = \{ C(t) \} \) is another \( \mathcal{D} \)-attracting family of closed sets, then \( A(t) \subset C(t) \) for all \( t \in \mathbb{R} \).
4.2. Pullback attractor for fixed bounded sets. In this subsection, we verify that there exists the pullback attractor of fixed bounded sets for non-autonomous system (1.1), which is usually denoted by $\mathcal{A}_{CDF}$ (the attractor of Cralue, Debussche and Flandoli), see [7]. We first construct the appropriate families of pullback absorbing sets. Then the asymptotic compactness for the process will be proceeded. Finally the pullback attractor can be constructed.

**Lemma 4.4.** Assume that $g \in L^2_{loc}(\mathbb{R}; V')$, $f$ satisfies hypothesis (H-1), $2L_f < \eta \leq \min\{2\alpha, 2\kappa\}$ and (4.4) holds. Then the family $\hat{B}_0 = \{B_0(t) : t \in \mathbb{R}\}$ with $B_0(t) = B_H(0, \rho(t)) \times B_{C_\kappa(H)}(0, \rho(t))$, where

$$
\rho^2(t) = 1 + \frac{1}{\nu} \int_{-\infty}^te^{-\sigma(t-s)}\|g(s)\|^2 ds,
$$

(4.6)
is pullback absorbing for bounded sets for the semi-process $U$ in Lemma 4.1.

**Proof.** From the estimate (3.3) in Theorem 3.2, one has

$$
\|u(t)\|_H^2 \leq \|u_0\|_H^2 e^{-\sigma_0(t-r)}\|\phi\|_\infty^2 + \int_{-\infty}^te^{-\sigma_0(t-s)}\\|g(s)\|_2^2 ds
\leq e^{-\sigma(t-r)}\|\phi\|_\infty^2 + \int_{-\infty}^te^{-\sigma(t-s)}\\|g(s)\|_2^2 ds,
$$

(4.7)
for all $t \geq \tau$, $0 < \eta \leq \min\{2\alpha, 2\kappa\}$ and $0 < \sigma \leq \sigma_0$.

Then for any bounded set $D$ of $X$ and any $t$, there exists a time $\tau(D,t)$ such that

$$
U(t, \tau)D \subset B_0(t), \quad \forall \ t > \tau,
$$

(4.8)
which finishes the proof. \qed

Next, we show the pullback asymptotic compactness.

**Definition 4.5.** For a semi-process $U$ defined on a metric space $(X,d)$, if there exists a family $\hat{B}_0 = \{B_0(t) : t \in \mathbb{R}\} \subset P(X)$ such that, for any $t$, any sequence $\{\tau_n\}$ with $\tau_n \leq t$, $\lim_{n \to +\infty} \tau_n = -\infty$ and $x_n \in B_0(\tau_n)$, the sequence $U(t, \tau_n)x_n$ is relatively compact in $X$, then it is said that $U$ is $\hat{B}_0$-pullback asymptotically compact.

**Lemma 4.6.** Assume that $g \in L^2_{loc}(\mathbb{R}; V')$, $f$ satisfies hypothesis (H-1), $2L_f < \eta \leq \min\{2\alpha, 2\kappa\}$ and (4.4) holds. Let $U$ be the semi-process defined in Lemma 4.1 and $\hat{B}_0$ in Lemma 4.4. Then $U$ is $\hat{B}_0$-pullback asymptotically compact.

**Proof.** For any $t_0 \in \mathbb{R}$ and any sequence $(\varphi_n, \phi_n) \in B_0(\tau_n) = B_H(0, \rho(\tau_n)) \times B_{C_\kappa(H)}(0, \rho(\tau_n))$ with $\lim_{n \to +\infty} \tau_n = -\infty$. Consider the sequence $U(t_0, \tau_n)(\varphi_n, \phi_n) = (u^n(t_0), u^n_\theta(\theta)) = (u^n(t_0), \xi^n(\theta))$ for $\theta \in (-\infty, 0]$, where $u^n(\cdot)$ is a sequence of solutions in their respective intervals $[\tau_n, t_0]$ with the initial data $(\varphi_n, \phi_n)$. Then we need to prove that the sequence $\{u^n(t_0)\}$ is relatively compact in $H$ and $\{\xi^n\}$ is relatively compact in $C_\kappa(H)$. The proof is analogous to the Step 3 in the proof of Theorem 3.2, which will be split into the following two steps.

**Step 1:** We prove that $\xi^n$ is relatively compact in $C([-T, 0]; H)$ for any $T > 0$.

Since $\xi^n(\theta) = u^n(t_0 + \theta)$, then it is equivalent to prove that $u^n(\cdot)$ is relatively compact in $C([t_0 - T, t_0]; H)$. Observe that $\phi_n \in B_{C_\kappa(H)}(0, \rho(\tau_n))$ and $\lim_{n \to +\infty} \tau_n = -\infty$. Then for any $T > 0$, there exists a $t_0$ such that $\tau_n \leq t_0 - T$ for all $n$. Then for any $t_0 - T < s < t_0$, we have

$$
\|u^n(s) - u^n(t_0)\|_{C([-T, 0]; H)} \leq \|u^n(s) - u^n(t_0 + \theta)\|_{C([-T, 0]; H)} \leq \|u^n(\cdot)\|_{C([-T, 0]; H)}
$$

for all $\theta$, and hence $\{u^n(\cdot)\}$ is relatively compact in $C([-T, 0]; H)$.

**Step 2:** We prove that $\varphi_n$ is relatively compact in $C([-T, 0]; H)$ for any $T > 0$.

Since $\varphi_n(\theta) = \varphi_n(t_0 + \theta)$, then it is equivalent to prove that $\varphi_n(\cdot)$ is relatively compact in $C([-T, 0]; H)$. Observe that $\varphi_n \in B_H(0, \rho(\tau_n))$ and $\lim_{n \to +\infty} \tau_n = -\infty$. Then for any $T > 0$, there exists a $t_0$ such that $\tau_n \leq t_0 - T$ for all $n$. Then for any $t_0 - T < s < t_0$, we have

$$
\|\varphi_n(s) - \varphi_n(t_0)\|_{C([-T, 0]; H)} \leq \|\varphi_n(s) - \varphi_n(t_0 + \theta)\|_{C([-T, 0]; H)} \leq \|\varphi_n(\cdot)\|_{C([-T, 0]; H)}
$$

for all $\theta$, and hence $\{\varphi_n(\cdot)\}$ is relatively compact in $C([-T, 0]; H)$.
where

$$
\left| \nabla \phi \right| \leq 1 + e^{-\sigma(t_0-T)} \int_{-\infty}^{t_0} e^{\sigma s} \left\| g(s, x) \right\|_\nu^2 \, ds =: R(t_0, \hat{T}), \forall t \in [t_0 - \hat{T}, t_0].
$$

(4.9)

Therefore,

$$
\left| u^n(t) \right|_H^2 \leq \left| u^n_t \right|_\kappa^2 \leq R(t_0, \hat{T}), \quad \forall t \in [t_0 - \hat{T}, t_0], \forall n \geq n_0(t_0, \hat{T});
$$

(4.10)

$$
\left| u^n_{t_0 - \hat{T}} \right|_\kappa^2 \leq R(t_0, \hat{T}), \quad \forall n \geq n_0(t_0, \hat{T}).
$$

(4.11)

Let $y^n(\cdot) = u^n(\cdot + t_0 - \hat{T})$. Then, for each $n \geq n_0(t_0, \hat{T})$, the function $y^n$ is a weak solution to the following equation:

$$
\frac{\partial y^n}{\partial t} - \nu \Delta y^n + \alpha y^n + \beta |y^n| y^n + \gamma |y^n|^2 y^n + \nabla \tilde{p} = \tilde{f}(t, y^n_t) + \tilde{g}(t), \quad \text{in} \ (0, \hat{T}) \times \Omega,
$$

$$
\nabla \cdot y^n = 0, \quad \text{in} \ (0, \hat{T}) \times \Omega,
$$

$$
y^n = 0, \quad \text{on} \ (0, \hat{T}) \times \partial \Omega,
$$

$$
y^n_0(\theta, x) = u^n_{t_0 - \hat{T}}(\theta, x), \quad y^n_0(\theta, x) = u^n_{t_0}(\theta, x) = \xi^n(\theta, x), \quad \theta \in (-\infty, 0), \ x \in \Omega,
$$

where $\tilde{p}(t) = p(t + t_0 - \hat{T})$, $\tilde{f}(t, y^n_t) = f(t + t_0 - \hat{T}, y^n_{t + t_0 - \hat{T}})$ and $\tilde{g}(t) = g(t + t_0 - \hat{T})$.

Combining (3.4), (4.10) and (4.11), for all $n \geq n_0(t_0, \hat{T})$, we have

$$
\left| u^n(t_0 - \hat{T}) \right|_H^2 + \left| u^n_t \right|_\kappa^2 + 2 \int_{0}^{\hat{T}} e^{2L\nu(T - s)} \left\| g(s) \right\|_\nu^2 \, ds \leq K(t_0, \hat{T}).
$$

(4.12)

Combining with (4.9), one has

$$
\left\| \tilde{f}(t, y^n_t) \right\|_{L^2(0, \hat{T}; L^2(\Omega))}^2 \leq L^2 \int_{0}^{\hat{T}} \left\| y^n_t \right\|_\kappa^2 \, dt = L^2 \int_{t_0 - \hat{T}}^{t_0} \left| u^n_t \right|_\kappa^2 \, dt \leq C,
$$

(4.13)

where $C$ is independent of $n$.

Hence, $\{(y^n)\}$ is bounded in $L^2(0, \hat{T}; V')$ similarly with (3.21). Thus, there exists a subsequence (without relabeling) and some function $y(\cdot)$ such that

$$
y^n \rightarrow y \text{ weakly * in } L^\infty(0, \hat{T}; H);
$$

(4.14)

$$
y^n \rightarrow y \text{ weakly in } L^2(0, \hat{T}; V);
$$

(4.15)

$$
y^n' \rightarrow y' \text{ weakly in } L^2(0, \hat{T}; V');
$$

(4.16)

$$
y^n \rightarrow y \text{ strongly in } L^2(0, \hat{T}; H);
$$

(4.17)

$$
\tilde{f}(t, y^n_t) \rightarrow \tilde{f}(t, y_t') \text{ weakly in } L^2(0, \hat{T}; L^2(\Omega)).
$$

(4.18)

Furthermore, by a standard compactness method, one can prove that $y$ belonging to $C([0, \hat{T}]; H) \cap L^2(0, \hat{T}; V) \cap L^4(0, \hat{T}; L^4(\Omega))$ is the unique weak solution to the following equation:

$$
\frac{\partial y}{\partial t} - \nu \Delta y + \alpha y + \beta |y| y + \gamma |y|^2 y + \nabla \tilde{p} = \tilde{f}(t) + \tilde{g}(t), \quad \text{in} \ (0, \hat{T}) \times \Omega,
$$

$$
\nabla \cdot y = 0, \quad \text{in} \ (0, \hat{T}) \times \Omega,
$$

$$
y = 0, \quad \text{on} \ (0, \hat{T}) \times \partial \Omega,
$$

(4.19)

with the initial data $y(0, x)$.
which means that \( u^n(\cdot) \) is relatively compact in \( C([t_0 - \hat{T}, \delta); H) \) and \( \xi^n(\cdot) \to y(\cdot + \hat{T}) \) strongly in \( C([-\hat{T} + \delta, 0); H) \).

(4.21)

Taking \( \delta = \hat{T} - T \), we finish the proof. Since \( u^n(t_0) = y^n(\hat{T}) \), then the sequence \( \{u^n(t_0)\} \) is relatively compact in \( H \).

Moreover, since for any fixed \( T > 0 \), there exists \( n_0(t_0, T) > 0 \) such that \( u^n(s + t_0) = y^n(s + T) \) with \( s \in [-T, 0] \) satisfies the estimate (4.10) for any \( n \geq n_0(t_0, T) \), then we also have

\[
\|\psi(s)\|^2_H \leq 1 + Me^{\sigma T}, \forall \ s \in [-T, 0], \forall \ T > 0,
\]

(4.22)

where \( \psi(s) = y(s + T), M = : e^{-\sigma t_0} \int_{-\infty}^t e^{\sigma s} \|u(s, x)\|_H^2 ds \).

**Step 2: We show that \( \xi^n \) converges to \( \psi \) in \( C_\epsilon(H) \).**

Our goal is to prove that for any \( \epsilon > 0 \), there exists \( n_\epsilon \) such that

\[
\sup_{s \in (-\infty, 0]} e^{2\kappa s} \|\xi^n(s) - \psi(s)\|^2_H \leq \epsilon, \forall \ n \geq n_\epsilon.
\]

(4.23)

Notice that, for any \( T > 0 \),

\[
\sup_{s \in (-\infty, 0]} e^{2\kappa s} \|\xi^n(s) - \psi(s)\|^2_H \leq \max\{ \sup_{s \in (-\infty, -T]} e^{2\kappa s} \|\xi^n(s) - \psi(s)\|^2_H, \sup_{s \in [-T, 0]} \|\xi^n(s) - \psi(s)\|^2_H \}.
\]

Fix large enough \( T_\epsilon > 0 \) such that \( \max\{e^{-2\kappa T_\epsilon}, Me^{\epsilon(\sigma - 2\kappa)T_\epsilon}\} \leq \frac{\epsilon}{2} \). From Step 1, one knows that there exists \( n_\epsilon > 0 \) (depending only on \( t_0 \) and \( T_\epsilon \)) such that

\[
\sup_{s \in [-T_\epsilon, 0]} \|\xi^n(s) - \psi(s)\|^2_H \leq \epsilon \text{ and } \tau_n \leq t_0 - T_\epsilon, \forall \ n \geq n_\epsilon.
\]

(4.24)

Hence, to prove (4.23), we just need to show that

\[
\sup_{s \in (-\infty, -T_\epsilon]} e^{2\kappa s} \|\xi^n(s) - \psi(s)\|^2_H \leq \epsilon, \forall \ n \geq n_\epsilon.
\]

(4.25)

By (4.22) and the choice of \( T_\epsilon \), for all \( k \in \mathbb{N} \cup \{0\} \) and all \( s \in [-\epsilon(T_\epsilon + k + 1), -(T_\epsilon + k)], \)

\[
e^{2\kappa T_\epsilon} \|\psi(s)\|^2_H \leq e^{-2\kappa(T_\epsilon + k)} (1 + Me^{\epsilon(\sigma - 2\kappa)T_\epsilon} e^{k(\sigma - 2\kappa)}
\]

\[
\leq e^{-2\kappa T_\epsilon} e^{-2\kappa k} + Me^{\epsilon(\sigma - 2\kappa)T_\epsilon} e^{k(\sigma - 2\kappa)} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},
\]

i.e. \( \sup_{s \in (-\infty, -T_\epsilon]} e^{2\kappa s} \|\psi(s)\|^2_H \leq \frac{\epsilon}{2} \). Therefore, in order to prove (4.25), we just need to show that

\[
\sup_{s \in (-\infty, -T_\epsilon]} e^{2\kappa s} \|\xi^n(s)\|^2_H \leq \frac{\epsilon}{2}, \forall \ n \geq n_\epsilon.
\]

(4.27)

Recalling that

\[
\xi^n(s) = \begin{cases} 
\phi^n(s + t_0 - \tau_n), & \text{if } s \in (-\infty, \tau_n - t_0), \\
u^n(s + t_0), & \text{if } s \in [\tau_n - t_0, 0].
\end{cases}
\]

(4.28)
Thus, we just need to prove that
\[
\max\left\{ \sup_{s \in (-\infty, \tau_n - t_0)} e^{2\kappa s} \| \phi^n(s + t_0 - \tau_n) \|^2_H, \sup_{s \in [\tau_n - t_0, -T]} e^{2\kappa s} \| u^n(s + t_0) \|^2_H \right\} \leq \epsilon \frac{\epsilon}{2}.
\] (4.29)

Using (4.24), the first term above can be estimated as follows:
\[
\sup_{s \in (-\infty, \tau_n - t_0)} e^{2\kappa s} \| \phi^n(s + t_0 - \tau_n) \|^2_H = \sup_{s \in (-\infty, \tau_n - t_0)} e^{2\kappa(s + t_0 - \tau_n)} e^{2\kappa(\tau_n - t_0)} \| \phi^n(s + t_0 - \tau_n) \|^2_H \leq e^{2\kappa(\tau_n - t_0)} \| \phi_n \|^2_H \leq e^{2\kappa(\tau_n - t_0)} \rho^2 \left( \tau_n \right) \leq e^{2\kappa(\tau_n - t_0)} + M e^{(\sigma - 2\kappa)T_\epsilon}, \quad \forall \ n \geq n_\epsilon \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\] (4.30)

For the second term, using (4.11) with \( \hat{T} = T_\epsilon \), one has
\[
\sup_{s \in [\tau_n - t_0, -T]} e^{2\kappa s} \| u^n(s + t_0) \|^2_H = \sup_{\theta \in [\tau_n - t_0 + T_\epsilon, 0]} e^{2\kappa(\theta - T_\epsilon)} \| u^n(t_0 - T_\epsilon + \theta) \|^2_H \leq e^{-2\kappa T_\epsilon} \| u^n_{t_0 - T_\epsilon} \|^2_H \leq e^{-2\kappa T_\epsilon} R(t_0, T_\epsilon) \leq e^{-2\kappa T_\epsilon} + M e^{(\sigma - 2\kappa)T_\epsilon} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},
\]
which finishes the proof of (4.29).

Based on Lemma 4.1, Lemma 4.4 and Lemma 4.6, the following theorem is a strict result of the existence theory of pullback attractor in [6, 10].

**Theorem 4.7.** Assume that \( g \in L^2_{loc}(\mathbb{R}; V') \), \( f \) satisfies hypothesis (H-1), \( 2L_f < \eta \leq \min\{2\alpha, 2\kappa\} \) and (4.4) holds. Then the semi-process \( U \) defined in Lemma 4.1 possesses the minimal pullback-\( D_F \) attractor \( \mathcal{A}_F = \{ A_F(t) : t \in \mathbb{R} \} \), which is given by
\[
\mathcal{A}_F(t) = \bigcup_{D \text{ bounded}} \Lambda(D, t),
\] (4.31)
where
\[
\Lambda(D, t) := \bigcap_{s \leq t} \bigcup_{\tau \leq s} U(t, \tau)D^x, \quad \forall \ t \in \mathbb{R}.
\] (4.32)

**Remark 1.** The pullback attractor defined by (4.31) is the minimal family of closed sets that attracts all bounded sets, however, uniqueness of the pullback attractor defined above does not hold in general, (see [2, 8]).
4.3. Pullback attractor for families of sets depending on time. In this subsection, under the same assumptions, we extend the previous result to the more recent framework of pullback attractors in a universe of families of time-dependent sets with a tempered growth condition, following the ideas of [3, 18, 19].

First, we briefly recall some concepts of dynamical systems when the universe of objects that can be attracted is not fixed but is composed of time-dependent families.

**Definition 4.8.** For $\mathcal{D} \subset \mathcal{P}(X)$, if for any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, any sequence $\lim_{n \to +\infty} \tau_n = -\infty$ and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in $X$, then the semi-process $U$ on $X$ is said to be pullback $\mathcal{D}$-asymptotically compact.

A family $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \in \mathcal{D}$ is said to be pullback $\mathcal{D}$-absorbing for the process $U$, if for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \hat{D}) \leq t$ such that

$$U(t, \tau)D(\tau) \subset D_0(t) \text{ for all } \tau \leq \tau_0(t, \hat{D}). \quad (4.33)$$

**Lemma 4.9.** Assume that $g \in L_{loc}^2(\mathbb{R}; V')$, $f$ satisfies hypothesis (H-1), $2L_f < \eta \leq \min\{2\alpha, 2\kappa\}$ and (4.4) holds. Then the family $\hat{B}_0$ defined in Lemma 4.4 belongs to the $\sigma$-indexed universe $\mathcal{D}_\sigma$ and is pullback $\mathcal{D}_\sigma$-absorbing for the semi-process $U$. Furthermore, the semi-process $U$ is pullback $\mathcal{D}_\sigma$-asymptotically compact.

**Proof.** Recalling the definition of $\hat{B}_0$, one has

$$\lim_{\tau \to -\infty} \sup_{(\varphi, \phi) \in B_0(\tau)} \|\varphi, \phi\|_X^2 = 2 \lim_{\tau \to -\infty} \int_{-\infty}^\tau e^{\sigma \tau} \frac{\|g(s)\|_V^2}{\nu} ds = 0, \quad (4.34)$$

which implies that $\hat{B}_0 \in \mathcal{D}_\sigma$.

For any $(\varphi, \phi) \in D_\sigma(\tau)$, let $U(t, \tau)(\varphi, \phi) = (u(t), u_\tau)$. From (4.7), we have

$$\|u(t)\|^2_{H^2} \leq \|u_t\|^2_{L^2} + e^{\sigma(\tau-t)} \|\phi\|_X^2 + \int_{-\infty}^t e^{-\sigma(t-s)} \|g(s)\|^2_{V'} ds. \quad (4.35)$$

Combining (4.3), one obtains that $\hat{B}_0$ is pullback $\mathcal{D}_\sigma$-absorbing.

Given any $\hat{D} \in \mathcal{D}_\sigma$, any sequence $\lim_{n \to +\infty} \tau_n = -\infty$ and any sequence $x_n \in D(\tau_n)$, since $\hat{B}_0$ is pullback $\mathcal{D}_\sigma$-absorbing, then one can choose a diagonal sequence of times $\{\tau_n(m)\}_m$ related to $\hat{D}$ to ensure that $x_n(m) \in B_0(\tilde{\tau}_m)$ with an appropriate time sequence $\lim_{m \to +\infty} \tilde{\tau}_m = -\infty$. Then, recalling the fact that $U$ is $\hat{B}_0$-pullback asymptotically compact from Lemma 4.6, one achieves that $U$ is pullback $\mathcal{D}_\sigma$-asymptotically compact.

Finally, based on Lemma 4.9, the following results are straightforward consequences of [[18], Theorem 18, Remark 19 and Corollary 20] or [[3], Theorem 7].

**Theorem 4.10.** Assume that $g \in L_{loc}^2(\mathbb{R}; V')$, $f$ satisfies hypothesis (H-1), $2L_f < \eta \leq \min\{2\alpha, 2\kappa\}$ and (4.4) holds. Then, the family $\mathcal{A}_\sigma = \{A_\sigma(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with

$$A_\sigma(t) = \bigcup_{\hat{D} \in \mathcal{D}_\sigma} \Lambda(D, t)^X, \quad (4.36)$$

is pullback $\mathcal{D}_\sigma$-asymptotically compact.
Assume that \( \text{Theorem 5.3}. \)

where

\[
\Lambda(D, t) := \bigcap_{s \leq t} \bigcup_{\tau \leq s} U(t, \tau) D(\tau) \quad t \in \mathbb{R},
\]

(4.37)

is the minimal pullback \( D_{\alpha} \)-attractor belongs to \( D_{\sigma} \) for the semi-process \( U \) on \( X \) associated to (1.1). Moreover, using the structures of pullback attractors, the following relationship holds:

\[
\mathcal{A}_F \subset \mathcal{A}_\sigma \subset \mathcal{A}_{\sigma_0}.
\]

(4.38)

5. Asymptotic behavior for long time. In this section, we will proceed in the following way. First, we establish existence of uniformly bounded absorbing families of sets for the semi-process \( U \). Then, taking into account the relationship between uniform pullback attraction and forward attraction, we show the existence of global attractor. Finally, we analyze the long time behavior of solutions in a neighborhood of a stationary solution for the 3D Brinkman-Forchheimer equation with infinite delay.

5.1. Global attractor for long time. First, we give the definition of global attractor and then we prove the existence of global attractor in this subsection.

Definition 5.1. A family of nonempty compact subsets \( \mathcal{A} = \{ \mathcal{A}(t) : t \in \mathbb{R} \} \) is said to be a global attractor for the semi-process \( U \) if the following hold:

(i) It is invariant, i.e.,

\[
U(t, \tau) \mathcal{A}(\tau) = \mathcal{A}(t), \quad \forall \, \tau \leq t.
\]

(5.1)

(ii) For all bounded subsets \( B \) of \( X \),

\[
\lim_{t \to +\infty} \text{dist}(U(t, \tau) B, \mathcal{A}(t)) = 0, \quad \forall \, \tau \in \mathbb{R}.
\]

(5.2)

Lemma 5.2. Assume that \( g \in L^2(\mathbb{R}^{+};V') \), \( f \) satisfies hypothesis (H-1), \( 2L_f < \eta \leq \min\{2\alpha, 2\kappa\} \) and (4.4) holds. Then the bounded set \( B_\rho = \{(\varphi, \phi) \in X, (\varphi, \phi) \in B_H(0, \rho) \times B_{C_k(H)}(0, \rho)\} \) is a bounded absorbing set for the semi-process \( U \) in Lemma 4.1, where

\[
\rho^2 = 1 + \frac{1}{\nu} \left( \int_{-\infty}^{0} e^{\sigma s} ||g(s)||_2^2 ds + \int_{0}^{+\infty} ||g(s)||_2^2 ds \right).
\]

(5.3)

Proof. From (4.7), for all \( t \geq \tau \) and \( 2L_f < \eta \leq \min\{2\alpha, 2\kappa\} \), one has

\[
\|u(t)\|^2_H \leq \|u_0\|^2_k \leq e^{-\sigma(t-\tau)} \|\phi\|_k^2 + e^{-\sigma t} \int_{-\infty}^{0} e^{\sigma s} \frac{||g(s)||_2^2}{\nu} ds + \int_{0}^{+\infty} \frac{||g(s)||_2^2}{\nu} ds.
\]

(5.4)

Then for any bounded set \( B \) of \( X \), there exists a time \( t_B \) such that

\[
U(t, \tau) B \subset B_\rho, \quad \forall \, t \geq t_B,
\]

(5.5)

i.e., \( B \) is absorbing in \( B_\rho \).

Theorem 5.3. Assume that \( g \in L^2(\mathbb{R}^{+};V') \), \( f \) satisfies hypothesis (H-1), \( 2L_f < \eta \leq \min\{2\alpha, 2\kappa\} \) and (4.4) holds. Then the semi-process \( U \) defined by (4.1) associated to (1.1) has a global attractor \( \mathcal{A} = \{ \mathcal{A}(t) : t \in \mathbb{R} \} \), which is given by

\[
\mathcal{A}(t) = \bigcup_{B \text{ bounded}} \Lambda(B, t)^X,
\]

(5.6)
where
\[ \land(B, t) := \bigcap_{s \geq t} \bigcup_{s \geq t} U(t, \tau)B, \quad \forall \ t \in \mathbb{R}. \] (5.7)

**Proof.** Based on Lemma 5.2, using the Theorem I.1.1 modified by Remark I.1.4 in [24], we just need to show that the semi-process \( U \) is asymptotically compact, i.e., for any bounded set \( B \) of \( X \), any \( x_n \in B \) and any sequence \( \lim_{n \to +\infty} t_n = +\infty \), the sequence \( U(t_n, \tau)x_n \) is relatively compact. Notice that, for any fixed \( \tau \in \mathbb{R} \), \( U(t + r, \tau + r)x = U(t, \tau)x \), for any \( r \in \mathbb{R}, \ x \in X, \ t \geq \tau \). Thus,
\[ U(t_n, \tau)x_n = U(0, \tau - t_n)x_n. \] (5.8)

By Lemma 4.9, one obtains that, \( \forall \ \tau \in \mathbb{R} \), the sequence \( U(0, \tau - t_n)x_n \) is relatively compact. \( \square \)

5.2. **Exponential stability of steady state solutions.** In this subsection, assuming that \( f \) is independent of time and \( g_\infty(x) = \lim_{t \to +\infty} g(t, x) \) exists in some metric space, then we can derive the generalized steady state elliptic equation of (1.1) as
\[
\begin{cases}
-\nu \triangle u^* + \alpha u^* + \beta |u^*|u^* + \gamma |u^*|^2 u^* + \nabla p = f(u^*) + g_\infty, & \text{in } \Omega, \\
\nabla \cdot u^* = 0, & \text{in } \Omega, \\
u^* = 0, & \text{on } \partial \Omega,
\end{cases}
\] (5.9)

where \( g_\infty \in V' \) is independent of \( t \).

Thus, using the Brouwer fixed point theorem, we first give a general result of the existence and uniqueness of solution for the corresponding elliptic equation when the viscosity is large. Then, a direct approach will be used to study the stability properties, i.e., every weak solution of (1.1) converges to the unique weak stationary solution exponentially fast as \( t \to +\infty \).

**Definition 5.4.** Assume that \( g_\infty \in V' \), we shall say that \( u^* \in V \) is a weak solution to (5.9) if it satisfies:
\[
\nu a(u^*, v) + \alpha (u^*, v) + \beta (|u^*|u^*, v) + \gamma (|u^*|^2 u^*, v) = (f(u^*), v) + (g_\infty, v), \] (5.10)

for all \( v \in V \).

**Theorem 5.5.** Suppose that \( f \) satisfies hypothesis (H-1) and \( \nu \lambda_1 > L_f - \alpha \), where \( \lambda_1 \) is the first eigenvalue of \( A \), then
1. For all \( g_\infty \in V' \), there exists a unique weak solution \( u^* \in V \) to (5.9);
2. If \( g_\infty \in L^2(\Omega) \), the solution \( u^* \) is exactly a strong solution, i.e. \( u^* \in D(A) \).

**Proof.** Notice that, if \( \varphi \) is independent of time, then
\[
\|\varphi\|_H := \sup_{s \in [-\infty, 0]} e^{\lambda s} \|\varphi\|_H = \|\varphi\|_H.
\]

Thus, the remaining proof is analogous to Theorem 4.3 in [30]. \( \square \)

**Theorem 5.6.** For any \( T > \tau \) and the initial data \( (\varphi, \phi) \in H \times C(\Omega) \). Suppose that \( f \) is independent of time and satisfies hypothesis (H-1), the parameters satisfy \( \nu \lambda_1 > L_f - \alpha \). \( g \in L^2(\mathbb{R}^+; V') \cap C([0, +\infty); V') \) and there exists \( g_\infty \in V' \) such that, for small enough \( \mu > 0 \),
\[
\lim_{t \to +\infty} ||g - g_\infty||_s = 0 \quad \text{and} \quad \int_0^\infty e^{\mu s} ||g - g_\infty||_s^2 ds < +\infty.
\]
Let \( u(t, x) \) be the unique weak solution to (1.1) with the initial data \((\varphi, \phi)\) and external force field \( g \), \( u_\infty \in V \) be the unique weak solution to (5.9) with \( g_\infty \). Then, there exist two positive constants \( C \) and small enough \( \lambda \in (0, \mu] \) (both of them depend only on \( \nu, \lambda_1, \alpha, L_f \) and \( \kappa \)), such that

\[
\|u(t) - u_\infty\|^2_H \leq C e^{-\lambda t} (\|\varphi - u_\infty\|^2_H + \|\phi - u_\infty\|^2_\kappa + \int_0^t e^{\lambda s} \|g - g_\infty\|^2_\ast ds), \tag{5.11}
\]

\[
\|u_t - u_\infty\|^2_\kappa \leq \max \left\{ e^{-2\alpha t} \|\phi - u_\infty\|^2_\kappa, C e^{-\lambda t} (\|\varphi - u_\infty\|^2_H + \|\phi - u_\infty\|^2_\kappa + \int_0^t e^{\lambda s} \|g - g_\infty\|^2_\ast ds) \right\}, \tag{5.12}
\]

for all \( t \geq 0 \).

**Proof.** Denoting \( w(t) = u(t) - u_\infty \), then \( w \) satisfies

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2_H + \nu \|w\|^2_V + \alpha \|w\|^2_\kappa + \beta (|u| |u| - |u_\infty| |u_\infty|, w) + \gamma (|u|^2 u - |u_\infty|^2 u_\infty, w) \\
= (f(u_t) - f(u_\infty), w) + (g - g_\infty, w).
\]

By Lemma 2.2, one has

\[
\beta (|u| |u| - |u_\infty| |u_\infty|, w) + \gamma (|u|^2 u - |u_\infty|^2 u_\infty, w) \geq 0. \tag{5.13}
\]

Therefore, we can derive that

\[
\frac{d}{dt} \|w\|^2_H \leq -2\nu \|w\|^2_V - 2\alpha \|w\|^2_\kappa + 2(f(u_t) - f(u_\infty), w) + 2(g - g_\infty, w) \\
\leq -2\nu \|w\|^2_V - 2\alpha \|w\|^2_\kappa + 2\|f(u_t) - f(u_\infty)\|_H \|w\|_H + 2 \|g - g_\infty\|_\ast \|w\|_V \\
\leq -2\nu \|w\|^2_V - 2\alpha \|w\|^2_\kappa + \varepsilon_1 \|w\|^2_H + \frac{1}{\varepsilon_1} \|f(u_t) - f(u_\infty)\|^2_H \\
+ \varepsilon_2 \|w\|^2_V + \frac{1}{\varepsilon_2} \|g - g_\infty\|^2_\ast \\
\leq (-2\nu + \varepsilon_2) \|w\|^2_V + (-2\alpha + \varepsilon_1) \|w\|^2_\kappa + \frac{L_f^2}{\varepsilon_1} \|w_t\|^2_\kappa + \frac{1}{\varepsilon_2} \|g - g_\infty\|^2_\ast \\
\leq ((-2\nu + \varepsilon_2) \lambda_1 - 2\alpha + \varepsilon_1) \|w\|^2_H + \frac{L_f^2}{\varepsilon_1} \|w_t\|^2_\kappa + \frac{1}{\varepsilon_2} \|g - g_\infty\|^2_\ast,
\]

holds for any \( \varepsilon_1 > 0 \) and \( 0 < \varepsilon_2 < 2\nu \), where the last inequality uses the poincaré inequality.

For any fixed positive \( \lambda \), we get

\[
\frac{d}{dt} (e^{\lambda t} \|w\|^2_H) = \lambda e^{\lambda t} \|w\|^2_H + e^{\lambda t} \frac{d}{dt} \|w\|^2_H \\
\leq \left( \lambda + (-2\nu + \varepsilon_2) \lambda_1 - 2\alpha + \varepsilon_1 \right) e^{\lambda t} \|w\|^2_H \\
+ \frac{L_f^2 e^{\lambda t}}{\varepsilon_1} \|w_t\|^2_\kappa + \frac{e^{\lambda t}}{\varepsilon_2} \|g - g_\infty\|^2_\ast. \tag{5.14}
\]

Simple computation shows that

\[
\|w_t\|^2_\kappa = \max \{ e^{-2\alpha t} \|\phi - u_\infty\|^2_\kappa, \sup_{\theta \in [-t, 0]} e^{2\alpha \theta} \|w(t + \theta)\|^2_H \}. \tag{5.15}
\]
Substituting (5.15) into (5.14) and integrating in time interval, one has

\[
e^{\lambda t} \|w(t)\|_H^2 \leq \|w(0)\|_H^2 + \left( \lambda + (-2\nu + \varepsilon_2)\lambda_1 - 2\alpha + \varepsilon_1 \right) \int_0^t e^{\lambda s} \|w(s)\|_H^2 ds
\]

\[
+ \frac{L_f^2}{\varepsilon_1} \|\varphi - u_\infty\|_\kappa^2 \int_0^t e^{(\lambda - 2\kappa) s} ds + \frac{1}{\varepsilon_2} \int_0^t e^{\lambda s} \|g - g_\infty\|_\kappa^2 ds
\]

\[
+ \frac{L_f^2}{\varepsilon_1} \int_0^t \sup_{\theta \in [0, s]} e^{(2\kappa - \lambda) \theta} e^{\lambda (s + \theta)} \|w(s + \theta)\|_H^2 ds.
\]

Taking \( \lambda < 2\kappa \), we obtain that

\[
e^{\lambda t} \|w(t)\|_H^2 \leq \|\varphi - u_\infty\|_H^2 + \frac{L_f^2}{(2\kappa - \lambda)\varepsilon_1} \|\varphi - u_\infty\|_\kappa^2 + \frac{1}{\varepsilon_2} \int_0^t e^{\lambda s} \|g - g_\infty\|_\kappa^2 ds
\]

\[
\left( \frac{L_f^2}{\varepsilon_1} + \lambda + (-2\nu + \varepsilon_2)\lambda_1 - 2\alpha + \varepsilon_1 \right) \int_0^t \sup_{\theta \in [0, s]} e^{\lambda \theta} \|w(\theta)\|_H^2 ds,
\]

for any \( \varepsilon_1 > 0 \) and \( 0 < \varepsilon_2 < 2\nu \).

Taking \( \varepsilon_1 = L_f \), small enough \( 0 < \lambda \leq \mu \) and \( \varepsilon_2 \) such that

\[
0 < \lambda + \varepsilon_2\lambda_1 < 2(\nu\lambda_1 - L_f + \alpha),
\]

then

\[
e^{\lambda t} \|w(t)\|_H^2 \leq \|\varphi - u_\infty\|_H^2 + \frac{L_f}{2\kappa - \lambda} \|\varphi - u_\infty\|_\kappa^2 + \frac{1}{\varepsilon_2} \int_0^t e^{\lambda s} \|g - g_\infty\|_\kappa^2 ds,
\]

and

\[
\|w_t\|_\kappa^2 \leq \max\{e^{-2\kappa t} \|\varphi - u_\infty\|_\kappa^2, \sup_{\theta \in [-t, 0]} e^{2\kappa \theta} C e^{-\lambda (t + \theta)} (\|\varphi - u_\infty\|_H^2 + \|\varphi - u_\infty\|_\kappa^2
\]

\[
+ \int_0^{t+\theta} e^{\lambda s} \|g - g_\infty\|_\kappa^2 ds \}
\]

\[
\leq \max\{e^{-2\kappa t} \|\varphi - u_\infty\|_\kappa^2, C e^{-\lambda t} (\|\varphi - u_\infty\|_H^2 + \|\varphi - u_\infty\|_\kappa^2
\]

\[
+ \int_0^t e^{\lambda s} \|g - g_\infty\|_\kappa^2 ds \},
\]

holds for \( 0 < \lambda < 2\kappa \), i.e., the estimates (5.11) and (5.12) can be proved. \( \square \)

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E-mail address: L18003383946@163.com
E-mail address: ysihan2010@163.com
E-mail address: yangxinguang@hotmail.com