Finite temperature hydrodynamic modes of trapped quantum gases

András Csordás
Research Group for Statistical Physics of the Hungarian Academy of Sciences
Pázmány Péter Sétány 1/A
1117 Budapest, Hungary

Robert Graham
Fachbereich Physik, Universität-GH,
45117 Essen, Germany

The hydrodynamic equations of an ideal fluid formed by a dilute quantum gas in a parabolic trapping potential are studied analytically and numerically. Due to the appearance of internal modes in the fluid stratified by the trapping potential, the spectrum of low-lying modes is found to be dense in the high-temperature limit, with an infinitely degenerate set of zero-frequency modes. The spectrum for Bose-fluids and Fermi-fluids is obtained and discussed.

I. INTRODUCTION

The successful trapping of dilute Bose- and Fermi-gases in magnetic traps and their subsequent cooling to temperatures below quantum-degeneracy has made the study of their hydrodynamics a subject of high current interest. The basic hydrodynamic equations of the fluid formed by such gases in local thermodynamic equilibrium are well known. In the limit where the fluid can be considered ideal they are the continuity equation for the mass-density $\rho$ and velocity field $\vec{u}$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0,$$  \hspace{1cm} (1.1)

the Euler equation for the velocity field

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla P + \vec{f}(\vec{x}),$$  \hspace{1cm} (1.2)

with the external force per unit mass $\vec{f} = -\frac{1}{m} \nabla V(\vec{x})$ and the pressure $P$ related to the internal energy density $\varepsilon$ by $P = \frac{2}{3} \varepsilon$. It satisfies

$$\frac{\partial P}{\partial t} + \vec{u} \cdot \nabla P = -\frac{5}{3} \left[ \nabla (\vec{u} P) - \rho \vec{u} \cdot \vec{f} \right]$$  \hspace{1cm} (1.3)

The thermodynamic equilibrium distributions of the density $\rho_0(\vec{x})$ and pressure $P_0(\vec{x})$ with $\nabla P_0 = \rho_0 \vec{f}$ are given by the ideal quantum-gas expressions at constant temperature $\beta = 1/k_B T$ and chemical potential $\mu$

$$\rho_0(\vec{x}) = m \int \frac{d^3k}{(2\pi)^3} f_+ (\vec{k}, \vec{x})$$ \hspace{1cm} (1.4)

$$P_0(\vec{x}) = \frac{2}{3} \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} f_+ (\vec{k}, \vec{x}),$$ \hspace{1cm} (1.5)

with the single-particle distribution

$$f_+ (\vec{k}, \vec{x}) = \frac{1}{e^{\beta \frac{\hbar^2 k^2}{2m} + V(\vec{x}) - \mu} + 1}$$ \hspace{1cm} (1.6)

of the Bose-Einstein (upper sign) or Fermi-Dirac (lower sign) form.

The derivation of these equations from the Boltzmann-equation is well-known, see [1]. In recent years a number of papers have already been devoted to the study of solutions of these equations linearized around the equilibrium state in parabolic traps. Griffin, Wu and Stringari derived a closed equation for the velocity fluctuations and gave explicit solutions for surface waves of a Bose gas in an isotropic trap and, for a classical gas, also in the axially symmetric anisotropic trap. In the latter case they gave also solutions for modes corresponding to irrotational flow. A
further studies of the hydrodynamic regime of a trapped bose-gases was presented in [6]. Fermi-gases were considered by Brunn and Clark [7]. Besides considering the low temperature limit for the degenerate Fermi gas, these authors gave an analytical solution for the mode spectrum in an isotropic trap in the high-temperature limit and identified one branch of the dispersion relation as ‘internal waves’ driven by the inhomogeneous trap potential. This is a point which we intend to examine further in the present paper. Amoruso et al. [8] also derived special solutions to the linearized hydrodynamic equations for the low-temperature limit of the degenerate Fermi gas. In a recent paper [9] we have also studied this low-temperature regime for Fermi gases and gave solutions for the completely anisotropic parabolic trap. The present paper will therefore concentrate on temperatures of the order of the degeneracy temperature or above. In a number of papers effects beyond the scope of eqs.(1.3) - (1.6) were also considered. Vichi and Stringari [10] considered the effects of mean-fields due to interactions on the collective oscillations of Fermi gases in a trap, while Pethick, Smith and collaborators [11], [12], [13], Vichi [14] and Guéry-Odelin et al. [15] discussed the collisional damping of collective modes in Bose gases and Fermi gases respectively. In [11][12] a simple interpolation formula was proposed between the mode-frequencies $\omega_c$ in the collisionless regime and the hydrodynamic regime, $\omega_h$, of the form

$$\omega^2 = \omega_c^2 + \frac{\omega_h^2 - \omega_c^2}{1 - i\omega\tau} \quad (1.7)$$

where

$$\tau^{-1} = \frac{8\pi a^2}{m} < \rho_0 v > \quad (1.8)$$

is the mean collision rate. This description was further examined in ref. [15]. Eq.(1.7) is based on general considerations of non-equilibrium thermodynamics [16]. Damping of the hydrodynamics in Bose gases was also studied by Griffin and collaborators in a series of papers, see [17] where further references can be found.

In the present paper we will not be concerned with damping effects. Instead our goal in the present paper is to study further the collision-dominated dissipation-less hydrodynamic regime in harmonic traps with arbitrary anisotropy. We do this by giving a systematic treatment of the linearized hydrodynamic equations based on eqs.(1.1) - (1.6) applicable (within our basic assumptions) in the whole temperature range from the high temperature domain, where the Boltzmann limit $f_\pm \simeq \exp(-\beta \left(\frac{h^2 k^2}{2m} + V(x) - \mu\right)$ applies, to the regime close to the degeneracy temperature for bosons and down to nearly vanishing temperature for fermions. We shall discuss a class of exact solutions of the dissipation-less equations applicable to the whole temperature-domain covered by the theory, generalizing results obtained previously for traps with axial symmetry. It can be shown that in the high-temperature limit of a classical Boltzmann gas the linearized hydrodynamic equations in a completely anisotropic trap are integrable and separable in elliptic coordinates, just as their low temperature counterparts [13][14]. However, at lower temperatures where effects of quantum statistics become important, the integrability and separability are lost, which manifests itself e.g. in effects of avoided level crossings.

Of special interest in the present paper, besides the common sound modes, will be the phenomenon of ‘internal waves’, which are characteristic of fluids whose equilibrium state is stratified by an external potential. Internal waves in trapped Fermi gases were mentioned in [9] but have not yet been investigated in detail for trapped quantum-gases. For the discussion of internal waves in classical contexts like waves in the atmosphere see [19].

II. LINEARIZED HYDRODYNAMIC EQUATIONS AND HILBERT-SPACE OF THEIR SOLUTIONS

In the present section we write down the five linearized hydrodynamic equations whose solution is the central theme of this paper. In previous work on these equations they were reduced to a set of three wave-equations for the velocity field, which we shall also write down for completeness, and some special solutions of this latter set of equations were given. However, the appropriate boundary conditions are hard to formulate for the velocity field and therefore it is not clear, so far, which function space is spanned by the solutions, and whether a scalar product can be placed on this function space, and if so what it is. This question is of particular practical and theoretical relevance for the present problem, because in general the solutions have to be constructed numerically by converting the differential operators to matrices using a basis and the scalar product in the solution-space. It is important to choose the correct scalar product because, as we shall see, the problem possesses a dense-lying discrete spectrum of low-lying states, and it is a priori far from clear, whether all these states are needed to span the complete space of solutions, and if not, how the correct states are to be distinguished. It is our aim here to devote particular attention to this open problem and to present an answer. The way to achieve this will be to deviate from the previous line of approach by deriving, instead of three coupled wave-equations for the components of the velocity field, two coupled wave-equations for the pressure and the density. For these we shall construct a scalar product in which the wave-operator is self-adjoint so that its eigenfunctions form a complete set in a well-defined Hilbert space.
A. Linearized hydrodynamic equations

Let us introduce small deviations $\delta \rho$, $\delta P$ of density and pressure from equilibrium
\[ \rho = \rho_0(\vec{x}) + \delta \rho(\vec{x}, t), \quad P = P_0(\vec{x}) + \delta P(\vec{x}, t) \] (2.1)
where $\rho_0(\vec{x})$ and $P_0(\vec{x})$ solve the time-independent hydrodynamic equations with vanishing velocity-field $\vec{u}_0 = 0$, namely $\nabla P_0(\vec{x}) = \rho_0(\vec{x})\vec{f}(\vec{x})$, which defines the mechanical equilibrium condition. In principle there are many equilibrium profiles $\rho_0(\vec{x}), P_0(\vec{x})$ satisfying this requirement. In our present context the physically relevant one is the thermodynamic equilibrium of maximum local entropy. The entropy-maximum is achieved by the special profiles $\rho_0(\vec{x}), P_0(\vec{x})$ corresponding to a state with uniform temperature $T$ and chemical potential $\mu$. Using eqs.(1.4), (1.5) $\rho_0(\vec{x}), P_0(\vec{x})$ can be written as
\[ \rho_0(\vec{x}) = A_{+} m \left( \frac{mk_BT}{2\pi\hbar^2} \right)^{3/2} F_{+} \left( \frac{3}{2}, V(\vec{x}) - \mu \right) \]
\[ P_0(\vec{x}) = A_{+} k_BT \left( \frac{mk_BT}{2\pi\hbar^2} \right)^{3/2} F_{+} \left( \frac{5}{2}, V(\vec{x}) - \mu \right) \] (2.2)
where the upper (lower) sign refers to bosons (fermions) and
\[ A_{+} = 1, \quad A_{+} = 2. \] (2.3)
Eqs.(2.2), but with space-dependent $T(\vec{x}), \mu(\vec{x})$ apply also to the states of local thermodynamic equilibrium, in which the system always is in the hydrodynamic limit. They can then be taken to define two of the four fields $P(\vec{x}), \rho(\vec{x}), T(\vec{x}), \mu(\vec{x})$ in terms of the other two. In the fermionic case we need to assume the presence of two equally populated hyperfine sub-states, the collisions between which can then ensure the local thermodynamic equilibrium. The Bose-Einstein integrals $F_{-}(s, z)$ and Fermi-Dirac integrals $F_{+}(s, z)$ are defined by
\[ F_{\mp}(s, z) = \int_0^\infty \frac{x^{s-1}}{e^{x+z} \mp 1} \, dx \] (2.4)
satisfying the familiar recursion relation.
\[ \frac{\partial F_{\mp}(s, z)}{\partial z} = -F_{\mp}(s-1, z) \] (2.5)
In the present case $z$, and therefore also $F_{\mp}(s, z)$, is space-dependent via $z = z(\vec{x}) = (V(\vec{x}) - \mu)/k_BT$. However, we shall usually suppress the $z$- and $\vec{x}$-dependence in our notation for simplicity and just write $F_{\mp}(s)$.

Let us see under which conditions this thermodynamic equilibrium state is stable against mechanical perturbations. Displacing a volume-element of fluid mechanically in the direction of increasing pressure, i.e. in the direction of $\vec{f}$, its volume is compressed adiabatically, so that its density is increased, per unit displacement, by $(\partial \rho_0/\partial P_0)_S \nabla P_0$, whereas the density in the ambient equilibrium-gas changes by $\nabla \rho_0$ in the same displacement. A restoring force per unit volume in the direction opposite to the displacement
\[ \vec{f} \cdot \left[ (\partial \rho_0/\partial P_0)_S \nabla P_0 - \nabla \rho_0 \right] < 0 \] (2.6)
must result for a mechanically stable state. Using the relation $(\partial \rho_0/\partial P_0)_S = 3\rho_0/5P_0$, valid for the ideal quantum-gases, and eqs.(2.2)-(2.4) we may rewrite (2.6) as
\[ \vec{f}^2 \frac{m^2}{(k_BT)^2} \left( \frac{3}{5} F_{+}^2 \left( \frac{4}{2} \right) - F_{+} \left( \frac{1}{2} \right) \right) < 0. \] (2.7)

A restoring force does not result if the whole fluid-layer on an equi-potential surface is displaced in the same way orthogonal to the equi-potential surface; because then no ambient fluid remains, which could give rise to the buoyancy force (2.6). Instead a new mechanical equilibrium is reached. This mechanism gives rise to the zero-frequency modes discussed later.
and its derivative $F'_z(z)$ with respect to its argument $z$ in terms of which we can write

$$\frac{P_0(\vec{x})}{\rho_0(\vec{x})} = \frac{k_B T}{m} F'_z((V(\vec{x}) - \mu)/k_B T)$$

(2.15)

$$\nabla \left( \frac{P_0(\vec{x})}{\rho_0(\vec{x})} \right) = -\vec{f}(\vec{x}) F'_z((V(\vec{x}) - \mu)/k_B T)$$

(2.16)

Suppressing in the following the subscript $\mp$ and also the argument $(V(\vec{x}) - \mu)/k_B T$ of $F$ and $F'$ for notational simplicity we can rewrite eqs. (2.12) - (2.13) as

$$\partial_t^2 \rho = \nabla^2 \delta P - \vec{f} \cdot \nabla \delta \rho - \left( \nabla \cdot \vec{f} \right) \delta \rho$$

(2.14)

$$\partial_t^2 \delta P = \frac{5 k_B T}{3 m} F \nabla^2 \delta P - \left( \frac{5}{3} F' + \frac{2}{3} \right) \vec{f} \cdot \nabla \delta P$$

(2.15)

$$- \frac{5 k_B T}{3 m} F \vec{f} \cdot \nabla \delta \rho + \left[ \left( \frac{5}{3} F' + \frac{2}{3} \right) \vec{f} \cdot \nabla f^2 - \frac{5 k_B T}{3 m} F \left( \nabla \cdot \vec{f} \right) \right] \delta \rho$$

(2.16)

So far we have gained in simplicity compared to eq. (2.11) because we have now only two coupled wave-equations instead of three. More important, however, is the fact that it is clear physically that the density and pressure perturbations must go to zero in the limit of large distances from the center of a confining trap. It should be noted that the same cannot be said for the velocity field. Indeed, it is clear from (2.8) that for $|\vec{x}| \to \infty$ where $\rho_0(\vec{x}) \to 0$ the velocity field $\vec{u}$ is not necessarily bounded by the hydrodynamic equations. However, in spite of the improvement
of the formulation of the linearized hydrodynamics we have achieved so far, the self-adjoint-ness of the wave-operator $H$ defined by writing eqs. (2.16), (2.17) in the form

$$\partial_t^2 \left( \frac{\delta P}{\delta \rho} \right) = -H \cdot \left( \frac{\delta P}{\delta \rho} \right)$$

(2.18)

remains to be clarified. Can a scalar product be found in which the operator $H$ is hermitian? This is the question to which we turn next.

B. Scalar product and hermiticity of the wave-operator

In order to find a useful scalar product on the space of solutions of eqs. (2.12) - (2.13) we proceed as follows. First we find a Lagrangian for eqs. (2.12) - (2.13), which must be a quadratic functional of $\delta \rho(\vec{x}, t), \delta P(\vec{x}, t)$. It can be found by making a general ansatz and comparing the coefficient-functions of the resulting Euler-Lagrange equations with those in eqs. (2.12) - (2.13). From the Lagrangian density we can pass to the associated ‘energy density’ $H$ whose space-integral

$$E = \int d^3x H$$

(2.19)

must be conserved by time-translation invariance. We shall then define the scalar product $\langle P_1 | P_2 \rangle$ in such a way that the conserved ‘energy’ takes the form

$$E = \langle P | H P \rangle$$

(2.20)

for vectors $P$ satisfying the time-independent wave-equation $H P = \omega^2 P$.

Thus we begin with the Lagrangian

$$L = \int d^3x \mathcal{L}$$

(2.21)

for whose density we find after some calculation

$$\mathcal{L} = \frac{\beta}{2} (\partial_t \delta P)^2 + \alpha (\partial_t \delta P) (\partial_t \delta \rho) - \frac{5}{3} \frac{k_B T F}{m} \alpha (\partial_t \delta \rho)^2$$

$$- \frac{1}{2} \left( \alpha + \frac{5}{3} \frac{k_B T F}{m} \beta \right) \left( \nabla \delta P \right)^2 + \bar{E} \cdot \left( \delta \rho \nabla \delta P - \delta P \nabla \delta \rho \right) + I \delta P \delta \rho + \frac{1}{2} J (\delta \rho)^2$$

(2.22)

where the coefficients $\alpha, \beta, \bar{E}, I, J$ are defined by

$$\alpha = -\frac{K}{(2 + 5F')F + \frac{1}{2} \frac{\nu}{k_B T}}$$

$$\beta = \frac{m}{k_B T} \frac{1 + F'}{F} \alpha$$

$$\bar{E} = -\frac{1}{6} \alpha (5F' + 2) \bar{f}$$

$$I = \frac{1}{2} \left( \alpha + \frac{5}{3} \frac{k_B T F}{m} \beta \right) \bar{\nabla} \cdot \bar{f} + \frac{\beta}{6} (5F' + 2) \bar{f}^2$$

$$J = \frac{\alpha}{3} (5F' + 2) \bar{f}^2$$

(2.23)

(2.24)

(2.25)

(2.26)

(2.27)

The coefficient $K$ in the relation for $\alpha$ is arbitrary, and can be used for normalization.

Next we pass to the associated density $\mathcal{H}$ defined by the Legendre transformation

$$\mathcal{H} = (\partial_t \delta \rho) \frac{\partial \mathcal{L}}{\partial (\partial_t \delta \rho)} + (\partial_t \delta P) \frac{\partial \mathcal{L}}{\partial (\partial_t \delta P)} - \mathcal{L},$$

(2.28)
with the conserved quantity (2.19). It is now useful to employ the vector-notation already defined in eq.(2.18) by defining

\[ \mathbf{P}(\mathbf{x}, t) = \left( \delta P(\mathbf{x}, t), \delta \rho(\mathbf{x}, t) \right). \] (2.29)

With the harmonic time-dependence

\[ \mathbf{P}(\mathbf{x}, t) = \mathbf{P}(\mathbf{x}) \cos(\omega t + \varphi) \] (2.30)

the conserved quantity \( E \) can be written as

\[ E = \omega^2 \sin^2(\omega t + \varphi) \int d^3x \left[ \frac{\beta}{2} \left( \delta P \right)^2 + \alpha \delta P \delta \rho - \frac{5}{3} \frac{k_B T}{m} \frac{F \alpha}{2} \left( \delta \rho \right)^2 \right] 
+ \cos^2(\omega t + \varphi) \int d^3x \left[ \frac{1}{2} \left( \alpha + \frac{5}{3} \frac{k_B T}{m} F \beta \right) \left( \nabla \delta P \right)^2 
- \vec{E} \cdot \left( \delta \rho \nabla \delta P - \delta P \nabla \delta \rho \right) - I \delta P \delta \rho - \frac{1}{2} J (\delta \rho)^2 \right] \] (2.31)

In order to meet our goal (2.20) we should define the scalar product \( \langle \mathbf{P}_1 | \mathbf{P}_2 \rangle \) in such a way that the coefficients of \( \sin^2 \) and \( \cos^2 \) in (2.31) become both equal to \( \omega^2 \langle \mathbf{P} | \mathbf{P} \rangle \). Therefore, we can conclude from the coefficient of \( \sin^2 \) that the norm becomes

\[ \langle \mathbf{P} | \mathbf{P} \rangle = \int d^3x \left[ \frac{\beta}{2} \delta P^2 + \alpha \delta P \delta \rho - \frac{5}{6} \frac{k_B T}{m} F \alpha (\delta \rho)^2 \right] \] (2.32)

Using the relations (2.23), (2.24) for \( \alpha \) and \( \beta \) it can be checked that the norm is positive, as required, if \( \alpha < 0 \), (which can always be achieved by the choice of the constant \( K \)), and \( \beta > 0 \), which requires the inequality \( 1 + F' > 0 \), and

\[ \frac{5}{3} \frac{k_B T}{m} F \alpha \beta - \alpha^2 > 0 \] which in turn requires the stronger inequality

\[ 5F'((V(\mathbf{x}) - \mu)/k_B T) + 2 > 0 \] (2.33)

Using the definition (2.14) of \( F = F_\mathbf{x} \) it is easy to check that (2.33) is equivalent to the stability condition (2.6) of the thermodynamic equilibrium state. The functions \( F(z), F'(z) \) are plotted in figs.1 for bosons and 2 for fermions. From eq.(2.32) we deduce that the scalar product on the complexified space of solutions can be defined as

\[ \langle \mathbf{P}_1 | \mathbf{P}_2 \rangle = \int d^3x (\delta P_1^*, \delta \rho_1^*) \mathbf{S} \left( \delta P_2, \delta \rho_2 \right) \] (2.34)

with the matrix

\[ \mathbf{S} = \begin{pmatrix} \frac{\beta}{2} & \frac{\alpha}{2} \\ \frac{\alpha}{2} & \frac{5}{3} \frac{k_B T}{m} F \alpha \end{pmatrix} \] (2.35)

The coefficient of \( \cos^2 \) in eq.(2.31) can now be checked to be of the form

\[ \langle \mathbf{P} | \mathbf{H} \mathbf{P} \rangle = \omega^2 \langle \mathbf{P} | \mathbf{P} \rangle \] (2.36)

for vectors \( \mathbf{P} \) satisfying \( \mathbf{H} | \mathbf{P} \rangle = \omega^2 | \mathbf{P} \rangle \). Indeed, restricting to a real space, for simplicity, because \( \mathbf{H} \) is real, we find by direct evaluation

\[ \langle \mathbf{P} | \mathbf{H} \mathbf{P} \rangle = \int d^3x (\delta P, \delta \rho) \mathbf{S} \cdot \mathbf{H} \left( \frac{\delta P}{\delta \rho} \right) \]
\[ = \int d^3x \left[ \frac{1}{2} \left( \alpha + \frac{5}{3} \frac{k_B T}{m} F \beta \right) \left( \nabla \delta P \right)^2 
- \vec{E} \cdot \left( \delta \rho \nabla \delta P - \delta P \nabla \delta \rho \right) - I \delta P \delta \rho - \frac{1}{2} J (\delta \rho)^2 \right] \] (2.37)

We can furthermore show by direct calculation that the wave-operator is hermitian in the scalar product (2.34), if suitable boundary conditions are imposed. We find after straightforward, but lengthy calculation
\[ \langle P_1|H P_2 \rangle - \langle P_2|H P_1 \rangle = \int d^3x \left( P_1 S \cdot H P_2 - P_2 S \cdot H P_1 \right) \]
\[ \quad = \frac{1}{2} \int d^3x \nabla \cdot \left\{ (\alpha + \frac{5k_B T}{3m} F) \left( \delta P_1 \nabla \delta P_2 - \delta P_2 \nabla \delta P_1 \right) - \tilde{f} (\delta P_1 \delta \rho_2 - \delta P_2 \delta \rho_1) \right\} \]  
\[ \quad = \frac{1}{2} \int d^3x \nabla \cdot \left\{ \left( \alpha + \frac{5k_B T}{3m} F \right) \left( \delta P_1 \nabla \delta P_2 - \delta P_2 \nabla \delta P_1 \right) - \tilde{f} (\delta \rho_1 \delta P_2 - \delta P_2 \delta \rho_1) \right\} \]  
\[ \quad = \frac{1}{2} \int d^3x \nabla \cdot \left\{ \left( \alpha + \frac{5k_B T}{3m} F \right) \left( \delta P_1 \nabla \delta P_2 - \delta P_2 \nabla \delta P_1 \right) - \tilde{f} \delta (\delta P_1 \delta \rho_2 - \delta P_2 \delta \rho_1) \right\} \]  

Thus, we must impose boundary conditions at infinity in such a way that the surface-integral we obtain from (2.38) by the application of Gauss’ law vanishes. Since the coefficient functions \( \alpha \) and \( \beta \) grow for \(|\vec{x}| \to \infty \) like \( \exp \left( \frac{V(\vec{x}) - F}{k_B T} \right) \), the fluctuations \( \delta P(\vec{x}) \) and \( \delta \rho(\vec{x}) \) for all solutions must vanish sufficiently rapidly for \( |\vec{x}| \to \infty \).

Finally, let us transform the scalar product (2.34) to the more symmetrical form
\[ \langle P_1|P_2 \rangle = \int d^3x (u_1 u_2 + v_1 v_2) \]  
by the linear transformation
\[ P = \begin{pmatrix} \delta P \\ \delta \rho \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix} \]  
with a matrix \( M \) in lower triangular form
\[ M = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \]  

diagonalizing and normalizing the kinetic term in the Lagrangian \( \mathcal{L} \) (and hence also in the ‘energy-density’ \( \mathcal{H} \)). The latter requirement yields
\[ a = \sqrt{\frac{5k_B T}{m}} \sqrt{-\frac{F}{\alpha (5F' + 2)}} \]
\[ b = \sqrt{\frac{9m}{5k_B T}} \frac{1}{\sqrt{-\alpha F (5F' + 2)}} \]  
\[ c = \sqrt{\frac{3m}{5k_B T}} \frac{1}{\sqrt{-\alpha F}} \]  

In the new variables \( u, v \) the wave-equation now reads
\[ -\frac{\partial^2}{\partial t^2} \begin{pmatrix} u \\ v \end{pmatrix} = \tilde{\mathcal{H}} \begin{pmatrix} u \\ v \end{pmatrix} \]  
with the manifestly hermitian wave-operator
\[ \tilde{\mathcal{H}} = \begin{pmatrix} \frac{5k_B T}{3m} \nabla \cdot \frac{\vec{F}}{\vec{F}^2} & \frac{\sqrt{3} \sqrt{\alpha F' + 2} \vec{f}}{\sqrt{\vec{F}^2}} + \frac{\sqrt{3} m}{3k_B T} \frac{\sqrt{5F' + 2} \vec{f}^2}{\vec{F}^2} \\ -\frac{\sqrt{3} \sqrt{\alpha F' + 2} \vec{f}}{\sqrt{\vec{F}^2}} & \frac{m}{5k_B T} \frac{(5F' + 2)^2 \vec{f}^2}{\vec{F}^2} \end{pmatrix} \]  

C. Zero-frequency modes and isothermal modes

For arbitrary temperature \( T \) and trap-potential \( V(\vec{x}) \) eqs. (2.12), (2.13) possess a class of exact time-independent solutions, which depend on an arbitrary function \( G(V(\vec{x})) \), and its derivative \( G' = dG/dV \), namely
\[ \delta \rho(\vec{x}) = -\varepsilon G'(V(\vec{x})) \]
\[ \delta P(\vec{x}) = \frac{\varepsilon}{m} G(V(\vec{x})). \]  

7
\[ \varepsilon \text{ is a parameter which is sufficiently small to make the linearized theory consistent. The norm } \langle \mathbf{P}_0 \rangle \text{ of these solutions is} \]
\[ \langle \mathbf{P}_0 | \mathbf{P}_0 \rangle = \varepsilon^2 \int d^3 x \frac{|\alpha|}{2m} \left[ 1 + \frac{F'}{k_B T F} G^2 + 2GG' + \frac{5}{3} k_B T F G' \right]. \]  
(2.46)

Since all functions under the integral depend on \( \vec{x} \) only via \( V(\vec{x}) \) the integration \( \int d^3 x \ldots \) can be replaced by \( \text{const} \int dV \sqrt{V} \ldots \), if \( V \) is a homogeneous function of second order of \( \vec{x} \), e.g. a parabolic potential. The scalar product exists and is positive under the condition (2.33), if \( G(V) \) vanishes sufficiently rapidly, e.g. like \( \exp(-V/k_B T) \), for \( V \to \infty \). Then these solutions belong to the Hilbert-space and have to be considered. Physically, they appear because of the coexistence of a continuum of mechanical equilibrium states and the thermodynamic equilibrium. As we shall see in the following section these states are not isolated from all the other states but occur, for any local wave-number, as the end-point of a spectral branch of states if the local wave-number is turned in the direction of \( \vec{f} = - \frac{1}{m} \nabla V \).

There are some further exact solutions of the wave-equations (2.16, 2.17) which hold for all temperatures in the fermionic -and all temperature \( T > T_c \) in the bosonic case. They are obtained by extending the ansatz (2.45) for the zero-frequency modes according to

\[ \begin{pmatrix} \delta P(\vec{x}, t) \\ \delta \rho(\vec{x}, t) \end{pmatrix} = \varepsilon \alpha y \beta z \gamma \left( \frac{1}{m} G(V(\vec{x})) \right) e^{-i \omega t} \]  
(2.47)

with \( \alpha, \beta, \gamma = 0 \) or 1. Inserting this ansatz in eq. (2.16), and using the property that for \( \alpha = \beta = \gamma = 0 \) eq. (2.47) is a zero-frequency mode, we find after a simple calculation

\[ (\omega^2 - \alpha \omega_1^2 - \beta \omega_2^2 - \gamma \omega_3^2)G'(V) = 0. \]  
(2.48)

Next we insert the ansatz also in eq. (2.17) and obtain by a similar calculation

\[ \left( \omega^2 + (\alpha \omega_1^2 + \beta \omega_2^2 + \gamma \omega_3^2) \left( \frac{5}{3} F' + \frac{2}{3} \right) \right) G(V) + \frac{5}{3} k_B T (\alpha \omega_1^2 + \beta \omega_2^2 + \gamma \omega_3^2) F G'(V) = 0. \]  
(2.49)

Eq. (2.48) determines the mode-frequencies as

\[ \omega_{\alpha \beta \gamma} = \sqrt{\alpha \omega_1^2 + \beta \omega_2^2 + \gamma \omega_3^2}, \]  
(2.50)

while (2.49), for \( \alpha, \beta, \gamma \) not all equal to zero, fixes the yet undetermined function \( G(V) \) in the ansatz (2.47) as

\[ G(V) = \text{const} F \left( \frac{3}{2}, \frac{V(\vec{x}) - \mu}{k_B T} \right). \]  
(2.51)

It follows with (2.47) that \( \delta P(\vec{x}, t) \) and \( \delta \rho(\vec{x}, t) \) for these modes are related by

\[ \delta P(\vec{x}, t) = \frac{k_B T}{m} \frac{F(3, \frac{V(\vec{x}) - \mu}{k_B T})}{F(1, \frac{k_B T}{k_B T})} \delta \rho(\vec{x}, t), \]  
(2.52)

which is the relation between changes of pressure and density implied by the local thermodynamic equilibrium (2.2) if the temperature is kept constant. These isothermal modes were already found in [3] for the special case of isotropic and axially symmetric parabolic trap-potentials.

The modes (2.50) contain as special cases the three Kohn-modes \( \omega_{100} = \omega_1, \omega_{010} = \omega_2, \omega_{001} = \omega_3 \), corresponding to oscillations of the center of mass of the trapped gas. It is interesting to note that collisionless Kohn-modes of the form

\[ \delta \rho(\vec{x}, t) = \varepsilon \frac{\partial}{\partial x_i} \rho_0(\vec{x}) e^{-i \omega t} \]  
(2.53)

with the same frequencies \( \omega_i \) also exist. It therefore follows from the phenomenological formula (1.7) that these modes are not damped by the relaxation mechanisms present in the system, in agreement with the general statement made by the Kohn-theorem.

For fermions the result (2.51) and (2.47) for the frequencies and mode-functions apply to all temperatures and can therefore also be extrapolated to \( T \to 0 \). Indeed, for \( T \to 0 \), modes with the frequencies (2.51) where
independent of $\vec{x}, t$ also that $\omega$
The amplitudes in these modes with a spatially constant amplitude proportional to $\sum_{i=1}^{3} A_i$ in the equation (2.8) for momentum conservation we arrive at the eigenvalue problem:

$$\delta \rho = -\frac{im}{\omega} \left( \frac{m k_B T}{2\pi h^2} \right)^{3/2} \left[ F \left( \frac{3}{2} \right) \sum_{i=1}^{3} A_i F \left( \frac{1}{2} \right) \frac{m}{k_B T} \sum_{i=1}^{3} A_i \omega_i^2 x_i^2 \right] e^{-i\omega t}. \quad (2.56)$$

$$\delta P(\vec{x}, t) = \frac{i k_B T}{\omega} \left( \frac{m k_B T}{2\pi h^2} \right)^{3/2} \left[ -\frac{5}{3} F \left( \frac{5}{2} \right) \sum_{i=1}^{3} A_i F \left( \frac{3}{2} \right) \frac{m}{k_B T} \sum_{i=1}^{3} A_i \omega_i^2 x_i^2 \right] e^{-i\omega t}. \quad (2.57)$$

A comparison of (2.56)-(2.57) with the local equilibrium relations (1.4), (1.4) reveals that the temperature oscillates in these modes with a spatially constant amplitude proportional to $\sum_{i=1}^{3} A_i$. Finally, using the results (2.56), (2.57) in the equation (2.8) for momentum conservation we arrive at the eigenvalue problem:

$$\omega^2 A_i = 2A_i \omega_i^2 + \frac{2}{3} \omega_i^2 \sum_{j=1}^{3} A_j \quad i = 1, 2, 3. \quad (2.58)$$

The eigenvector $\vec{A}$ and the eigenvalue $\omega^2$ are clearly temperature independent and follow from the cubic secular equation

$$(\omega^2)^3 - \frac{8}{3} (\omega_x^2 + \omega_y^2 + \omega_z^2) (\omega^2)^2 + \frac{20}{3} \left( \omega_x^2 \omega_y^2 + \omega_y^2 \omega_z^2 + \omega_z^2 \omega_x^2 \right) \omega^2 - 16 \omega_x^2 \omega_y^2 \omega_z^2 = 0. \quad (2.59)$$

In the special case of an axially symmetric trap the cubic equation can be reduced to a quadratic one and a result first obtained in [9] is recovered.

### III. SHORT WAVE-LENGTH SOLUTIONS

The two coupled wave-equations derived in the previous section in various forms are difficult to solve for arbitrary temperature in a system which is made spatially inhomogeneous by an external potential $V(\vec{x}) \neq 0$. An exception, however, are waves of wave-lengths, which are short on the spatial scale on which $V(\vec{x})$ and hence also $P_0(\vec{x}), \rho_0(\vec{x})$ vary. Such waves, in the representation with $u(\vec{x}, t), v(\vec{x}, t)$, can be written as

$$\begin{pmatrix} u(\vec{x}, t) \\ v(\vec{x}, t) \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} a_0(\vec{x}) \\ b_0(\vec{x}) \end{pmatrix} e^{iS(\vec{x}, t)} \quad (3.1)$$

The eikonal $S(\vec{x}, t)$ defines the local wave-vector by the relation

$$\vec{k}(\vec{x}, t) = \nabla S(\vec{x}, t) \quad (3.2)$$

The amplitudes $a_0, b_0$ and also $\vec{k}$ vary slowly in space, on the same scale as $V(\vec{x}), P_0(\vec{x}), \rho_0(\vec{x})$. The frequency $\omega$ is independent of $\vec{x}, t$. Inserting the ansatz in the equation (2.43) and neglecting derivatives of $a_0, b_0$ and $\vec{k}$, and assuming also that $\omega^2 \gg |\nabla \cdot \vec{f}|$, we obtain the secular equation as the vanishing of the determinant

$$\begin{vmatrix} \frac{5}{3} \frac{k_B T}{m} F \vec{k}^2 + \frac{1}{60} \frac{m}{k_B T} \vec{P}^2 & -\omega^2 \\ -\frac{i}{\sqrt{3}} \frac{\sqrt{5} F''}{\vec{f}} + \frac{5}{3} \frac{\sqrt{5} F''}{\vec{f}} + \frac{2}{5} \frac{m}{k_B T} \vec{f}^2 & \frac{5}{3} \frac{\sqrt{5} F''}{\vec{f}} + \frac{2}{5} \frac{m}{k_B T} \vec{f}^2 - \omega^2 \end{vmatrix} = 0 \quad (3.3)$$

9
We note that terms with \( \vec{f}^2 \) and \( \vec{k} \cdot \vec{f} \) are essential to keep in this approximation together with the \( k^2 \)-terms, because \( |\vec{f}| \) grows at large distances from the trap-center, at least for parabolic traps, and provides the physically crucial confining mechanism. On the other hand, \( a_0(\vec{x}), b_0(\vec{x}) \), and \( \vec{\nabla} \cdot \vec{f} \) do not grow in a similar way and are therefore consistently negligible.

From eq. (3.3) we deduce the local dispersion-law for waves of short-wavelength \( 2\pi/k \)

\[
\omega^2 = \omega^2_\pm(\vec{k}, \vec{x}) = \frac{1}{2} \left[ \frac{5 k_B T}{3 m} F k^2 + \frac{m}{k_B T} \left( \frac{1}{60 F} + \frac{5 F'}{5} \right) \vec{f}^2 \right] \\
\pm \sqrt{\frac{1}{4} \left[ \frac{5 k_B T}{3 m} F k^2 + \frac{m}{k_B T} \left( \frac{1}{60 F} + \frac{5 F'}{5} \right) \vec{f}^2 \right]^2 - \frac{1}{3} (5 F' + 2) (\vec{f} \times \vec{k})^2} 
\]

In the same level of approximation the pressure and density oscillations are related by

\[
\delta P(\vec{k}, x) = \frac{1}{k^2} \left[ \omega^2 - i \vec{f} \cdot \vec{k} \right] \delta \rho(\vec{k}, x)
\]

as follows from eq. (2.12).

The dispersion law (3.4) contains a lot of physics and will be discussed now. First we note that for \( \vec{f} \neq 0 \) there are two branches of the dispersion law, one of high frequency and another one of lower frequency, which are both physical. Thus, there are two different types of waves in these systems. Both branches correspond to frequencies \( \omega^2 \geq 0 \) for all \( \vec{k} \), i.e. to stable oscillation waves. Another simple observation is that the local dispersion-relation is anisotropic and depends on the angle between \( \vec{f} \) and \( \vec{k} \). The physical nature of the two branches is most easily seen by assuming that the angle between \( \vec{f} \) and \( \vec{k} \) is sufficiently small to permit the expansion of the square-root in (3.4) in the second term of its radicand. We obtain then to lowest non-vanishing order

\[
\omega^2_+ = \frac{5 k_B T}{3 m} F k^2 + \frac{m}{k_B T} \left( \frac{1}{60 F} + \frac{5 F'}{5} \right) \vec{f}^2 
\]

\[
\omega^2_- = \frac{5 F' + 2}{3 m} \vec{f} \times \vec{k}\]

The high-frequency branch is easily recognized in this limit as an adiabatic sound mode, in particular if the identity

\[
c^2_s = \frac{\partial P_0}{\partial \rho_0} \bigg|_s = \frac{5 P_0}{3 \rho_0} = \frac{5 k_B T}{3 m} F
\]

is used, which is valid for the ideal quantum gases, with the definition of \( F = F_\perp \) by eqs. (2.14), (2.15). The low-frequency branch reaches its lowest frequency \( \omega_- = 0 \), in the present approximation, for all waves traveling locally in the direction parallel to the force of the trap \( \vec{f} \), so that \( \vec{f} \times \vec{k} = 0 \). The existence of such zero-frequency modes has already been discussed in the preceding section. Looking at exact solutions at high-temperature in the next section we shall see them appear again.

The low-frequency branch for given \( |\vec{k}| \) achieves its highest frequency if the wave propagates locally in directions orthogonal to \( \vec{f} \). The maximum frequency for modes orthogonal to \( \vec{f} \) is then reached for short wave-lengths

\[
c^2_s k^2 \gg \frac{m}{k_B T} \left( \frac{1}{60 F} + \frac{F'}{5} \right) \vec{f}^2
\]

and given by

\[
\omega_{\max} = \sqrt{\frac{5 F' + 2 |\vec{f}|}{3 c_s}}
\]

Waves with the properties of the low-frequency branch solutions found here are typical for media which are stratified by an external force and are called ‘internal waves’. One of their surprising and counter-intuitive properties is that in regions where (3.9) applies the group-velocity \( \vec{\nabla} k \omega_-(\vec{k}) \) is orthogonal to the wave-vector \( \vec{k} \). For a text-book discussion of such waves see [10]. Indeed, the dispersion-relations (3.6), (3.7) can e.g. be directly compared with eqs.(53),(54) given there.
IV. SOLUTION IN THE HIGH-TEMPERATURE REGION

A. Hilbert-space of polynomial solutions

The high-temperature regime is defined by

$$ T \gg T_{\text{deg}} \quad (4.1) $$

where $T_{\text{deg}}$ is the degeneracy temperature at which the de Broglie wave-length becomes of the order of the mean particle distance. In this regime we can approximate

$$ f_{\mp}(\vec{p}, \vec{x}) = e^{-\left(\frac{\vec{p}^2}{2m} + V(\vec{x}) - \mu\right)/k_B T} \quad (4.2) $$

and $\rho_0(\vec{x}) = \rho_0(0)e^{-V(\vec{x})/k_B T}$, $P_0(\vec{x}) = \frac{\hbar^2 m}{2}\rho_0(\vec{x})$. It is useful to introduce new dimensionless variables $P_1(\vec{x}), \rho_1(\vec{x})$ via the definitions

$$ \delta P(\vec{x}, t) = \frac{k_B T}{m}\rho_0(\vec{x})P_1(\vec{x}, t) \quad (4.3) $$

$$ \delta \rho(\vec{x}, t) = \rho_0(\vec{x})\rho_1(\vec{x}, t) $$

and to rewrite the coupled wave-equations for $\delta P$, $\delta \rho$ as coupled equations for $P_1$ and

$$ P_1 - \rho_1 = T_1 = \frac{\delta T}{T} \quad (4.4) $$

where $\delta T(\vec{x}, t)$ is the deviation of the temperature from equilibrium. Separating the time-dependence $e^{-i\omega t}$ we arrive at

$$ \omega^2 \begin{pmatrix} P_1 \\ P_1 - \rho_1 \end{pmatrix} = \begin{pmatrix} \frac{-k_B T}{m} \nabla^2 & \vec{f} \cdot \nabla \\ \vec{f} \cdot \nabla & \frac{-k_B T}{m} \nabla^2 \end{pmatrix} + \frac{m}{k_B T} \begin{pmatrix} \vec{f} \\ \vec{f} \end{pmatrix} \begin{pmatrix} P_1 \\ P_1 - \rho_1 \end{pmatrix} \quad (4.5) $$

We specialize these equations to a harmonic potential

$$ V(\vec{x}) = \frac{m}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right) \quad (4.6) $$

It is then clear, via term by term inspection, that there are polynomial solutions of eq. (4.5), in which $P_1$ is polynomial in the Cartesian components of $\vec{x}$ of total order $n$ and $P_1 - \rho_1$ is a corresponding polynomial of total order $n - 2$. This is so because then each term on the right-hand side of (4.5) either decreases the total order of the polynomial on the left-hand side by 2 via the operation of $\nabla^2$ or keeps the same order of the polynomial. It follows that we can pick freely as the highest total power an integer $n$ for $P_1$ and determine the eigenvalues $\omega_n^2$ by comparing the coefficients of all terms with this highest total power, imposing the condition for nontrivial solvability. It is clear that we get polynomials of arbitrary total order in this way. Moreover these solutions lie in the Hilbert-space because via eq. (4.3) $\delta P$ and $\delta \rho$ fall off sufficiently rapidly for $|\vec{x}| \to \infty$, if $P_1$ and $\rho_1$ are polynomials in the Cartesian components of $\vec{x}$.

To see this explicitly we specialize the scalar product (4.34) for the present high-temperature case by replacing $F(3/2, (V(\vec{x}) - \mu)/k_B T) = \exp(-(V(\vec{x}) - \mu)/k_B T)$, $F_\pm = 1, F_\mp = 0$ which gives $\alpha = -(K/2) \exp((V(\vec{x}) - \mu)/k_B T), \beta = -((m/k_B T)\alpha$, and the scalar product

$$ \langle \mathbf{P} | \mathbf{P} \rangle = \frac{K}{4} \frac{k_B T}{m} (\rho_0(0))^2 e^{-\frac{m}{k_B T} \frac{\vec{p}^2}{2m}} \int d^3x e^{-\frac{V(\vec{x})}{k_B T}} \left[ P_1 \tilde{P}_1 - P_1 \tilde{P}_1 - \rho_1 \tilde{P}_1 + \rho_1 \tilde{P}_1 + \frac{5}{3} \rho_1 \tilde{P}_1 \right] \quad (4.7) $$

It is also clear now, that for harmonic potentials $V(\vec{x})$ the polynomial solutions for $P_1$ and $\rho_1$ of all orders are complete in a space with this scalar product, a fact which is very familiar from the quantum mechanics of the harmonic oscillator.

To be specific let us consider a polynomial of $x, y, z$ for $P_1$ of total order $n$. It has then terms of highest total order of the form

$$ P_1 = \sum_{n_1, n_2, n_3} A_{n_1 n_2 n_3} x^{n_1} y^{n_2} z^{n_3} + \text{lower order} \quad (4.8) $$
with \( n_1 + n_2 + n_3 = n \), and \((n+2)^2 = (n+2)(n+1)/2\) different coefficients \( A_{n_1 n_2 n_3} \). For the same mode \( P_1 - \rho_1 \) must be a polynomial of total order \( n - 2 \) with \((n) = n(n-1)/2\) terms \( x^{n_1} y^{n_2} z^{n_3} \) of highest total order \( n_1 + n_2 + n_3 = n - 2 \) with coefficients \( B_{n_1 n_2 n_3} \).

The solvability-condition for the linear homogeneous equations connecting all these coefficients then gives a secular equation for the eigenvalues \( \omega^2 \) of order

\[
\binom{n+2}{2} + \binom{n}{2} = n^2 + n + 1
\]

(4.9)

with just as many solutions. \((n+1)(n+2)/2\) of these modes can be considered as modes of sound waves, modified by the external potential, while the remaining \( n(n-1)/2 \) can be considered as modes of internal waves, modified by their coupling to sound waves. The second kind of modes therefore exists only for \( n \geq 2 \).

How many linearly independent zero-frequency modes appear at a given total order \( n \)? If \( n \) is odd, the polynomials for \( P_1 \) and \( \rho_1 \) have odd parity and cannot describe a zero-frequency mode, which must have even parity by eq. (2.43). If \( n \) is even, on the other hand, there is precisely one linearly independent zero-frequency mode associated with that integer, which may be written in the form

\[
\begin{pmatrix}
\delta P(\vec{x}) \\
\delta \rho(\vec{x})
\end{pmatrix}
= \text{const} \frac{\rho_0(\vec{x})}{m} \frac{V(\vec{x})}{k_B T} \left[ \frac{\rho(\vec{x})}{n/2} \right]
\]

(4.10)

The order of the secular equation and the number of frequencies with given value of \( n \) grows quadratically with \( n \). For \( n = 0 \) we have just one coefficient \( A_{000} \) and \( P_1 - \rho_1 = 0 \), and the frequency \( \omega^2 = 0 \). We shall see in section [IVD] that this mode must be excluded because of particle-number conservation. Thus, there is actually no physical mode with \( n = 0 \). For \( n = 1 \) we have already three coefficients \( A_{100}, A_{010}, A_{001} \) but no B-coefficient yet. The equations for the three A-coefficients are decoupled and we obtain the three frequencies

\[
\omega_{100} = \omega_x, \omega_{010} = \omega_y, \omega_{001} = \omega_z
\]

(4.11)

corresponding to a rigid center of mass motion of the atom-cloud in the trap. These are the high-temperature limits of the Kohn-modes already encountered in section [II]. Also the other temperature-independent modes discussed there appear again as follows: For \( n = 2 \) we have 6 A-coefficients and one B-coefficient and all together 7 solutions for \( \omega^2 \). Because of separate symmetry of the trapping potential under reflections of the \( x \)-, \( y \)-, and \( z \)-axis the three equations for \( A_{110}, A_{101}, A_{011} \) are decoupled from each other and the rest and give immediately

\[
\omega_{110} = \sqrt{\omega_x^2 + \omega_y^2}, \omega_{101} = \sqrt{\omega_x^2 + \omega_z^2}, \omega_{011} = \sqrt{\omega_y^2 + \omega_z^2}.
\]

(4.12)

Similarly, for \( n = 3 \) the coefficient \( A_{111} \) is decoupled from the rest, and the corresponding mode frequency is

\[
\omega_{111} = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}.
\]

(4.13)

All these decoupled modes are isothermal modes, corresponding to \( \delta T = 0 \). In fact, we have already seen in subsection [II] that these modes exist for all temperatures (for bosons for \( T > T_c \)). The remaining four coefficients for \( n = 2 \), \( A_{200}, A_{020}, A_{002} \) and \( B_{000} \) are coupled. Of the four resulting coupled modes one is a zero-frequency mode \( \omega_{200}^{(0)} = 0 \), for which

\[
\frac{1}{\omega_x^2} A_{200}^{(0)} + \frac{1}{\omega_y^2} A_{020}^{(0)} + \frac{1}{\omega_z^2} A_{002}^{(0)} = \frac{m}{2k_B T} B_{000}.
\]

(4.14)

This is the only internal-wave mode with \( n = 2 \). The remaining three modes with \( n = 2 \) are modified sound modes, for whose squared frequencies \( \omega_x^2 \) the cubic equation (2.53) is re-obtained as it must be, since the corresponding modes were found in section [II] to exist for all temperatures with temperature-independent mode-frequencies (but with mode-functions for pressure and density which depend on temperature and the quantum-statistics). The solutions of the cubic secular equation are simple if one trap frequency, say \( \omega_z \), is much smaller or much larger than the two others. Then one solution for \( \omega^2 \) is small,

\[
\omega \approx \sqrt{\frac{12}{5} \omega_z},
\]

(4.15)

or large.
\[ \omega \simeq \sqrt{\frac{8}{3} \omega_z}, \]  

(4.16)

respectively, while the other two are large and given by

\[ \omega_{\pm}^2 \simeq \frac{4}{3} (\omega_0^2 + \omega_z^2) \pm \frac{4}{3} \sqrt{(\omega_0^2 + \omega_z^2)^2 - \frac{15}{4} \omega_0^2 \omega_z^2}, \]

(4.17)

or small

\[ \omega_{\pm}^2 \simeq \frac{5}{4} (\omega_0^2 + \omega_z^2) \pm \frac{5}{4} \sqrt{(\omega_0^2 + \omega_z^2)^2 - \frac{96}{25} \omega_0^2 \omega_z^2}, \]

(4.18)

respectively. Other simple solutions are obtained in the isotropic case and the axially symmetric case, cf. the following sections. Quartic equations for \( \omega^2 \) similar to eq. (2.19) are obtained from the solvability conditions for the three quadruples of amplitudes of the \( n = 3 \) modes \((A_{02}, A_{12}, A_{20}, A_{10}), (A_{03}, A_{21}, A_{01}, A_{11}), \) and \((A_{01}, A_{20}, A_{03}, A_{10})\), which decouple from each other because they differ in parity. The solutions of these equations and of \( \{2.59\} \) for arbitrary trap-frequencies are tedious expressions and will not be given here. They and the eigenvalues \( \omega_n^2 \) for \( n \geq 3 \) can easily be determined numerically for specific ratios of the trap-frequencies when needed, where efficient use can be made of the already mentioned conservation of the \( x-, y-, \) and \( z\)-reflection parities.

B. Isotropic case

The solution for the isotropic case \( \omega_x = \omega_y = \omega_z = \omega_0 \) in the high-temperature regime has already been given by Bruun and Clark [7] based on the velocity equation (2.11). It is not clear, however, how the Hilbert space is defined in dimensionless frequency and space-coordinate via a square-integrable field. It is therefore worthwhile to check this case again, using our present description. Introducing dimensionless frequency and space-coordinate via

\[ \Omega^2 = \frac{\omega^2}{\omega_0^2}, \quad \tilde{r} = \sqrt{\frac{m \omega_0^2}{k_B T}} \]

(4.19)

and imposing a polynomial ansatz

\[
P_1(\tilde{r}) = r^l Y_m(\Theta, \varphi) Q_n(r^2) \]

\[
P_1(\tilde{r}) - \rho_1(\tilde{r}) = r^l Y_m(\Theta, \varphi) T_{n-1}(r^2), \]

(4.20)

with \( Q_n \) and \( T_n \) polynomials of order \( n \) and \( Y_m(\Theta, \varphi) \) the spherical harmonics, we obtain

\[
\left( \Omega^2 - l - r \frac{d}{dr} \right) Q_n(r^2) = \left[ \frac{5}{2} \Omega^2 - r^2 \right] T_{n-1}(r^2) \]

(4.21)

\[
-\frac{2}{3} \left( \frac{d^2}{dr^2} + \frac{2(l+1)}{r} \frac{d}{dr} \right) Q_n(r^2) = \left[ \Omega^2 - 2 - \frac{2l}{3} - \frac{2}{3} r \frac{d}{dr} \right] T_{n-1}(r^2). \]

(4.22)

We should mention that the quantum-number \( n \) defined by the polynomial order of \( Q_n, T_n \) differs from the quantum-number \( n \) defined in section [V A], which corresponds to \( 2n + l \) within the present definitions. As was mentioned already, the characteristic equation follows from the linear set of equations for the largest powers. Let us consider separately

(i) \( n = 0 \):

\[
Q_0(r^2) = A_0^{(0)}, \quad T_{-1}(r^2) = 0 \]

(4.23)

Eq. (4.22) is consistent with \( T_{-1} = 0 \) and gives no additional information. From eq. (1.21) we conclude

\[
(\Omega^2 - l) A_0^{(0)} = 0. \]

(4.24)

Thus we have the spectrum
\[\omega = \sqrt{\omega_0}\] 

(4.25)

for the modes

\[\delta P = \frac{k_B T}{m} \delta \rho = \frac{k_B T}{m} \rho_0(x) A_0^{(0)} Y_{lm}(\Theta, \varphi)\]

\[\delta T = 0.\] 

(4.26)

These isothermal modes were first found by Griffin, Wu and Stringari (ii) \(n \geq 1\):

\[Q_n(r^2) = \sum_{i=0}^{n} A_i^{(n)} r^{2i}\]

\[T_{n-1}(r^2) = \sum_{i=0}^{n-1} B_i^{(n)} r^{2i}\] 

(4.27)

From (4.21) we obtain for \(i = n\):

\[(\Omega^2 - l - 2n) A_n^{(n)} + B_{n-1}^{(n)} = 0\] 

(4.28)

\[i \neq n, \quad (\Omega^2 - l - 2i) A_i^{(n)} + B_{i-1}^{(n)} - \frac{5}{2} \Omega^2 B_i^{(n)} = 0\] 

(4.29)

\[i = 0, \quad (\Omega^2 - l) A_0^{(n)} - \frac{5}{2} \Omega^2 B_0^{(n)} = 0\] 

(4.30)

From (4.22) we get correspondingly

\[\frac{4}{3} i(2i + 2l + 1) A_i^{(n)} + \left[ \Omega^2 - \frac{2(l+1)}{3} - \frac{4i}{3} \right] B_{i-1}^{(n)} = 0\] 

(4.31)

Let us consider the highest power \(i = n\) first, which yields two linear homogeneous algebraic equations for \(A_n^{(n)}, B_{n-1}^{(n)}\) of the form

\[\mathbf{C}_{n-1} \begin{pmatrix} A_n^{(n)} \\ B_{n-1}^{(n)} \end{pmatrix} = 0\] 

(4.32)

\[\mathbf{C}_{n-1} = \begin{pmatrix} \frac{4}{3} i [2n + 2l + 1] & \Omega^2 - \frac{2(l+1)}{3} - \frac{4i}{3} \\ \Omega^2 - l - 2n & 1 \end{pmatrix}\] 

(4.33)

The solvability condition gives for \(n \geq 1\)

\[\Omega^4 - \frac{5}{3} \Omega^2 (l + 2n + \frac{2}{5}) + \frac{2}{3} l(l+1) = 0\] 

(4.34)

\[\Omega_{1,2}^2 = \frac{\omega_{1,2}}{\omega_0^2} = \frac{1}{2} \left[ \frac{5}{3} \left( l + 2n + \frac{2}{5} \right) \pm \sqrt{\frac{25}{9} \left( l + 2n + \frac{2}{5} \right)^2 - \frac{8}{3} l(l+1)} \right]\] 

(4.35)

This result was first obtained by Bruun and Clark. We differ from their result only for the case \(n = 0\), where we find that only one branch of the solution (4.33) exists, namely the branch which yields \(\Omega^2 = l\), as seen from (4.22), (4.24). In the case \(n \geq 1, l = 0\) one of the solutions (4.33) vanishes. It gives the zero-frequency solutions discussed in previous sections. The other solution at \(\Omega^2 = \frac{\omega_{1,2}^2}{\omega_0^2}\) behaves normally and describes a sound-mode, as can e.g. be seen from the factor \(5/3\) which is characteristic of the inverse of the adiabatic compressibility in ideal gases.

Let us turn to the mode-functions in the case \(n \geq 1\). Their coefficients can be obtained recursively starting with \(B_0^{(n)}\), which is fixed only by normalization, then solving (4.30)
\[ A_0^{(n)} = \frac{5\Omega^2}{2\Omega^2 - l} B_0^{(n)}, \]  

then rising to higher values of \( i \) by first solving for
\[ A_1^{(n)} = \frac{1}{4} \frac{3\Omega^2 - 2(l + 1) - 4i}{i(2i + 2l + 1)} B_i^{(n)}, \]

and then using (4.29) with (4.37) and (4.34) to obtain also \( B_i^{(n)} \) in terms of \( B_{i-1}^{(n)} \) as
\[ B_i^{(n)} = -\frac{n - i}{i(2i + 2l + 1)} B_{i-1}^{(n)}. \]

Solving the recursion-relation (4.38) we obtain
\[ B_i^{(n)} = \frac{(1 - n)(1 - n + 1) \cdots (1 - n + i - 1)}{(2l + 3) \cdot (2l + 4) \cdots (2l + 2i + 1)} \frac{1}{2^i} i! B_0^{(n)} \]

which identifies the polynomial \( T_{n-1}(r^2) \) as the confluent hypergeometric function
\[ T_{n-1}(r^2) = \sum_{i=0}^{n-1} B_i^{(n)} r^{2i} = B_0^{(n)} _1 F_1(1 - n, \frac{2l + 3}{2}, \frac{r^2}{2}) \]

which is proportional to the generalized Laguerre-polynomial \( L_n^{(l+\frac{1}{2})}(r^2) \). Substituting the result into eq. (4.37) we obtain also the polynomial \( Q_n(r^2) \) as the combination of hypergeometric functions
\[ Q_n(r^2) = \sum_{i=0}^{n} A_i^{(n)} r^{2i} = B_0^{(n)} \left\{ \frac{3}{4n} \left[ \Omega_{1,2}^2 + \frac{2l}{3} \right] _1 F_1 \left( -n, \frac{2l + 3}{2}, \frac{r^2}{2} \right) \right. \]
\[ \left. - \frac{2l + 1}{2n} _1 F_1 \left( -n, \frac{2l + 1}{2}, \frac{r^2}{2} \right) \right\} \]

which is proportional to \( (3\Omega_{1,2}^2 + 2l)L_n^{(l+\frac{1}{2})}(r^2) - 2(1 + 2l + 2n)L_n^{(l-\frac{1}{2})}(r^2) \). The physical modes are then
\[ \delta P(\vec{x}, t) = \frac{k_B T}{m} \rho_0(\vec{x}) r^l Y_{lm}(\Theta, \varphi) Q_n(r^2) \]
\[ \frac{\delta T(\vec{x}, t)}{T} = r^l Y_{lm}(\Theta, \varphi) T_{n-1}(r^2) \]

It is quite remarkable that the spatial perturbation of the temperature in the two physically very different branches of the spectrum, the sound-modes and the internal modes, is exactly the same, because \( T_{n-1}(r^2) \) is independent of \( \omega_z^2 \) and therefore the same for both branches. (We note, however, that this is strictly true only in the Boltzmann limit). The spatial distribution of the pressure and the density, on the other hand, is very different for both kinds of modes as one would expect. 

For \( l = 0 \) and arbitrary \( n \) we obtain the mode-functions of the zero-frequency modes. They form the bottom of a ladder of rotational modes for each value of \( n \). The mode-functions are given by combinations of Hermite-polynomials of the radial variable \( (m\omega_z^2\vec{x}^2/k_BT)^{1/2} \) and form a complete set in the Hilbert-space of radial functions defined by the scalar product (4.4). This simply means that an arbitrary radial, i.e. angle-independent, mode-function within our Hilbert-space is a zero-frequency mode.

**C. Anisotropic harmonic potential and conserved operator**

Let us now turn to the case of an anisotropic harmonic trapping potential. Then the generator of rotations \( \mathbf{L} = -i(\vec{x} \times \vec{\nabla}) \) no longer commutes with the wave-operator \( \mathbf{H} \) defined by the matrix-differential operator on the right-hand side of (4.3). On the other hand, for harmonic trapping potentials the polynomial solutions discussed in section IV A continue to exist also in this case. Therefore, one must strongly suspect that a complete set of operators commuting with \( \mathbf{H} \) exists also in the fully anisotropic case. For Bose-Einstein condensed bosons and for fermions at
temperature \( T = 0 \) a similar situation prevails (but not at temperatures between the high- and the low-temperature limits, cf. the discussion below), and two operators commuting with the corresponding wave-operator were constructed in our previous papers [9], [18]. One way to find these operators is to introduce elliptic coordinates and to separate the wave-equation in these coordinates. The separation-constants introduced by this procedure appear naturally as eigenvalues of certain differential operators, any combination of which can be identified with the searched for commuting operators. In [9], [18] this procedure was fully carried through in the low-temperature limit.

For our present high-temperature regime we shall not present the most general anisotropic case. Rather we restrict ourselves to axially symmetric traps with \( \omega_x = \omega_y = \omega_\perp \), where one of the conserved operators, namely \( \frac{1}{2}L_z \) with

\[
L_z = -i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right),
\]

(4.44)

still follows from symmetry and we only need to find a second one to demonstrate integrability.

One can find a conserving operator via the separation of the wave equation for \( P_1 \) in elliptical cylindric coordinates. This procedure leads to an operator \( B' \), which has the important property

\[
B' \left( \frac{P_1}{P_1 - \rho_1} \right) = B \left( \frac{P_1}{P_1 - \rho_1} \right),
\]

(4.45)

where the eigenvalue \( B \) is the separation constant. \( B' \) has the form

\[
B' = \begin{pmatrix} \hat{R} - \frac{4}{5} & 2 \\ 0 & \hat{R} + \frac{4}{5} \end{pmatrix},
\]

(4.46)

where (with \( \omega_x^2 = \omega_y^2 = \omega_\perp^2 \))

\[
\hat{R} = \vec{x} \cdot \vec{\nabla} - \frac{k_B T}{m} \left( \frac{1}{\omega_x^2} \frac{\partial^2}{\partial x^2} + \frac{1}{\omega_y^2} \frac{\partial^2}{\partial y^2} + \frac{1}{\omega_\perp^2} \frac{\partial^2}{\partial z^2} \right).
\]

(4.47)

Returning back to our original variables \( \delta P \) and \( \delta \rho \) one can prove via explicit calculation that the operator \( B \), which corresponds to \( B' \) in that representation, has vanishing commutator with \( H \):

\[
\left[ H, B \right] = 0
\]

(4.48)

Let us determine the spectrum of \( B \) in the Hilbert-space of polynomials discussed in section [IV A], where \( P_1 \) is a polynomial of total order \( n \) and \( P_1 - \rho_1 \) a polynomial of total order \( n - 2 \). We obtain directly from (4.45), (4.47) the eigenvalue

\[
B = n - \frac{4}{5}
\]

(4.49)

i.e. \( B \) is the conserved operator which introduces the principal quantum number \( n \) of section [IV A] which is the total number of nodal surfaces of the pressure and density oscillations in any given mode.

D. Restrictions by particle-number conservation

In the Hilbert-space defined by the scalar product (2.32) there may still be unphysical states, which do not satisfy the conservation of the total number of particles

\[
N = \frac{1}{m} \int d^3 x \left( \rho_0(\vec{x}) + \delta \rho(\vec{x}, t) \right)
\]

(4.50)

which applies to a closed system. In terms of the variable \( \rho_1 = \delta \rho/\rho_0 \) eq.(4.50), with \( N = \frac{1}{m} \int d^3 x \rho_0(\vec{x}) \) and \( \rho_0(\vec{x}) = \rho_0(0)e^{-V(\vec{x})/k_B T} \), implies the condition

\[
\int d^3 x e^{-V(\vec{x})/k_B T} \rho_1 = 0
\]

(4.51)

In order to see which modes satisfy (4.51) we use the existence of the conserved \( B \) with (4.46), from which, by elimination of \( P_1 \), the eigenvalue equation for \( \rho_1 \)
\[ B \rho_1 = \left( -\frac{4}{5} + \hat{R} \right) \rho_1 \]  

(4.52)

can be derived very easily. We can use this to insert \( \hat{R} \) in front of \( \rho_1 \) in eq. (4.51) to obtain

\[ \int d^3 x e^{-V(\vec{x})/k_B T} \hat{R} \rho_1 = \left( B + \frac{4}{5} \right) \int d^3 x e^{-V(\vec{x})/k_B T} \rho_1 \]  

(4.53)

Now we use the explicit form of \( \hat{R} \) and the quadratic form of the potential on the left-hand side of (4.53) and apply partial integration to let the derivatives act on the exponential factor. Boundary terms are not incurred by this operation because \( \rho_1 \) is polynomial and the exponential factor vanishes rapidly at infinity. We find that the left-hand side of (4.53) vanishes, and therefore

\[ \left( B + \frac{4}{5} \right) \int d^3 x e^{-V(\vec{x})/k_B T} \rho_1 = 0 \]  

(4.54)

Hence, all modes for which the eigenvalue \( B \) satisfies

\[ B \neq -\frac{4}{5} \]  

(4.55)
satisfy the restriction (4.51) imposed by particle-number conservation. Looking at the spectrum (4.49) of \( B \) we see that (4.55) is satisfied for all modes with the exception of the mode with \( n = 0 \), which is a special zero-frequency mode. For this mode \( \rho_1(\vec{x}) = \text{const} \) and (4.51) is obviously violated. This mode must therefore be excluded from the physical spectrum. All the other modes, and in particular all the other zero-frequency modes, satisfy particle-number conservation.

**E. Solutions for axially symmetric traps**

We introduce scaled cylinder coordinates \( r, z, \varphi \) via

\[ x_1 = \sqrt{\frac{k_B T}{M \omega_z^2}} r \cos \varphi, \quad x_2 = \sqrt{\frac{k_B T}{M \omega_z^2}} r \sin \varphi, \quad x_3 = \sqrt{\frac{k_B T}{M \omega_z^2}} z \]  

(4.56)

and the ansatz

\[ P_1 = e^{\imath m \varphi} z^\alpha Q(\rho^2, z^2), \quad P_1 - \rho_1 = e^{\imath m \varphi} z^\alpha T(\rho^2, z^2) \]  

(4.57)

where \( \alpha = 0, 1 \) determines the parity under inversion of the \( z \)-axis and \( m = -\infty, \cdots, 0, 1, 2, \cdots \infty \) is the quantum number of angular momentum around the \( z \)-axis. The equations to be solved are then, with \( \Omega^2 = \omega^2/\omega_z^2, \lambda = \omega^2/\omega_z^2 \)

\[ \Omega^2 T = -2 \left[ \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} + 2|m| + 1 \frac{\partial}{\rho} + 2\alpha \frac{\partial}{z} \right] Q \]

\[ + \frac{2}{3} \left[ m + 2 + \lambda (\alpha + 1) + \rho \frac{\partial}{\rho} + \lambda z \frac{\partial}{z} \right] T \]  

(4.58)

\[ \Omega^2 Q = \left[ \frac{5}{2} \Omega^2 - \rho^2 - \lambda^2 z^2 \right] T + \left[ |m| + \lambda \alpha + \rho \frac{\partial}{\rho} + \lambda z \frac{\partial}{z} \right] Q. \]

For \( Q \) and \( T \) we make a polynomial ansatz in \( \rho^2, z^2 \) of order \( n \) and \( n - 1 \), respectively. The eigenvalue of \( B' \) for these solutions is \( B = 2n + |m| + \alpha - \frac{4}{5} \). The terms of \( Q \) and \( T \) of highest order read

\[ Q = \sum_{l=0}^{n} A_l \rho^{2l} z^{2(n-l)} + \text{lower order} \]

\[ T = \sum_{l=0}^{n-1} B_l \rho^{2l} z^{2(n-l)} + \text{lower order} \]  

(4.59)
Inserting the ansatz and comparing coefficients of the highest order terms we obtain the two sets of equations

\[ \Omega^2 A_0 = |m| + \lambda(2n + \alpha) A_0 - \lambda^2 B_0 \]
\[ \Omega^2 A_1 = -B_0 + |m| + 2 + \lambda(2n - 2 + \alpha) A_1 - \lambda^2 B_1 \]
\[ \vdots \]
\[ \Omega^2 A_l = -B_{l-1} + |m| + 2l + \lambda(2n - 2l + \alpha) A_l - \lambda^2 B_l \]
\[ \vdots \]
\[ \Omega^2 A_n = -B_{n-1} + |m| + 2n + \lambda \alpha A_n \]

and

\[ \Omega^2 B_l = -\frac{4}{3}(n - l)(2n - 2l - 1 + 2\alpha)A_l \]
\[ + \frac{2}{3}|m| + 2l + \lambda(2n - 2l + \alpha - 1) B_l \]
\[ - \frac{8}{3}(l + 1)(|m| + l + 1)A_{l+1} \]

Let us examine simple special cases.

For \( n = 0 \) we have \( Q = A_0, T = 0 \) and obtain

\[ \Omega^2 = \frac{\omega_2^2}{\omega_1^2} = |m| + \lambda \alpha \quad (|m| = 0, 1, \ldots, \alpha = 0, 1) \] (4.62)

This result is a special case of the result (4.11) (for \( m = 0, \alpha = 1 \) and \( m = \pm 1, \alpha = 0 \)), and (4.12) (for \( m = \pm 1, \alpha = 1 \) and one mode with \( |m| = 2, \alpha = 0 \)), and (4.13) (for one mode with \( |m| = 2, \alpha = 1 \)), but the case \( |m| \geq 3 \) has no simple counterpart in the fully anisotropic case.

For \( n = 1 \) we get already the three coupled equations

\[ \lambda^2 B_0 + [\Omega^2 - |m| - (2 + \alpha)\lambda] A_0 = 0 \]
\[ B_0 + (\Omega^2 - |m| - 2 - \lambda \alpha) A_1 = 0 \] (4.63)
\[ \left[ \Omega^2 - \frac{2}{3}|m| + 2 + \lambda + \alpha \lambda \right] B_0 + \frac{4}{3}(1 + 2\alpha) A_0 + \frac{8}{3}(|m| + 1) A_1 = 0 \]

Specializing further to \( m = 0 \) and \( \alpha = 0 \) we find the exact solutions

\[ \omega_1^2 = 0 \quad \text{with} \quad A_0 = \lambda A_1 = \frac{\lambda}{2} B_0 \] (4.64)

and

\[ \omega_{2,3}^2 = \frac{\omega_1^2}{3} \left[ 4\lambda + 5 \pm \sqrt{16(\lambda - 1)^2 + 9} \right] . \] (4.65)

These mode-frequencies were first obtained in [1].

The zero-frequency mode is the axially symmetric counterpart of (4.14) while (4.65) corresponds to two of the three mode-frequencies which solve (2.59). The third solution of (2.59) in the axially symmetric case is \( \omega^2 = 2\omega_2^2 \) and is in the completely anisotropic case the counterpart of the second of the two modes described by (4.62) with \( |m| = 2, \alpha = 0 \).

In order to account for all modes in our comparison between the axially symmetric and the fully anisotropic case, we may finally note that the second of the two modes with \( |m| = 2, \alpha = 1 \) described by (4.62) has as a completely anisotropic counterpart a particular solution of the quartic secular-equation for \( \omega^2 \) for the amplitudes \( (A_{021}, A_{201}, A_{003}, B_{001}) \) appearing in the scheme of section [IV A] for \( n = 3 \).
V. NUMERICAL DETERMINATION OF THE TEMPERATURE-DEPENDENT MODE-SPECTRUM

At intermediate temperatures the coupled wave equations (2.10), (2.17) are generally not separable and the spectrum can only be found numerically. Best suited for numerical work are the wave equations in the manifestly hermitian form (2.44) with the scalar product in the simple form (2.39).

The numerical analysis will be performed for the axially symmetric case, choosing for the anisotropy parameter \( \lambda = \omega^2_2/\omega^2_1 = 8 \), partially for historical reasons as this was the geometry of the first TOP trap at JILA [20]. The number of particles is chosen as \( N = 10^6 \). First the chemical potential is determined for the given particle-number \( N \) as a function of temperature. This is done in the standard way, by integrating eq. (1.4) to obtain \( N(\mu, T) \) and solving for \( \mu(N, T) \). The results for our chosen set of parameters are displayed in fig. 8 both for bosons and for fermions, in the domain where \( \mu \leq 0 \), to which we shall restrict our attention, in the following. We may remark here that it follows from the form of the potential and (1.4) that \( \mu \) is a scaling function \( S \) of \( N, \bar{\omega} = (\omega_2\omega_y\omega_z)^{1/3} \), and \( T \) of the form \( \mu/\hbar^2 = S(k_BT/\hbar\bar{\omega}N^{1/3}) \).

In order to determine the spectrum, the wave-operator (2.44) is represented in the basis of the harmonic oscillator eigenfunctions with widths \( \sqrt{k_BT/m\omega^2_z} \) and \( \sqrt{k_BT/m\omega^2_1} \) in axial and radial direction, respectively. The basis is cut-off at a finite size of order 100 both for \( u \) and \( v \) and the resulting finite-dimensional Hermitian matrix is diagonalized. The size of the finite basis is varied in order to control that the eigenvalues obtained are converged numerically. The truncation of the basis to a finite size introduces some spurious eigenmodes and eigenfrequencies, which can be distinguished however, and subsequently eliminated, by the fact that they don’t converge but disappear and reappear somewhere else as the size of the basis is varied.

Some of the results are displayed in figs. 18 which we now discuss. Fig. 18 gives an overview, in the domain \( 94 < k_BT/\hbar\bar{\omega} < 195 \), of the spectrum of eigenvalues \( \omega^2 \), for a gas of bosons (but very similar results not shown here are also found for fermions). For clarity only modes with azimuthal quantum number \( m = 0 \) and even parity are shown. The basis used consisted of oscillator eigenfunctions of order \( 2n_z \) in \( z \)-direction and order \( 2n_\rho \) in radial direction, with integer \( n_\rho + n_z \leq 10 \). All frequencies obtained by the diagonalization of the matrix of \( H \) in this basis (except for the spurious ones introduced by the truncation of the basis) are shown in the figure. As can be seen from fig. 8 the chosen temperature domain extends from the BEC-temperature to the high-temperature region. In fact, we have checked that our numerical code applied to the case of an isotropic trap gives a similar result, which for \( k_BT/\hbar\bar{\omega} = 195 \) coincides with the analytically known eigenvalues (4.35) in the high-temperature limit with high precision. This fact offers us the possibility to assign the quantum numbers of the high-temperature domain to the whole corresponding temperature-dependent branch of frequencies.

An obvious feature of fig. 8 is the band-like structure of the spectrum, which is caused by the anisotropy of the trap: Because of the higher stiffness of the trap in axial direction for the assumed value of \( \lambda = 8 \), nodes of the mode-function in axial direction are more costly in energy \( \hbar\bar{\omega} \), for sound-like modes, than nodes in radial direction. Therefore we can assign to the ‘bands’ the quantum number \( n_z \) of nodes of sound-like modes in axial direction with \( n_z = 10 \) in the highest ‘band’ (which can consist of a single mode only in the subspace we consider in fig. 8) and \( n_z = 0 \) in the lowest. It should be noted that the two lowest bands are not split and form a single broad band. Within a given band sound-like modes differ only by the radial quantum number \( n_\rho \) counting the number of nodes in radial direction. By our choice of a finite-dimensional basis we are restricted to modes with quantum numbers \( n_\rho + n_z \leq 10 \). The presence of internal waves complicates the assignment of quantum numbers, because each pair of quantum numbers \( n_z, n_\rho \) appears twice, once at higher frequency for a sound-like mode and once at lower frequency for an internal wave mode. Thus internal modes with high values of \( n_z \) give also frequencies in the low lying bands of fig. 8. The number of eigenfrequencies in the ‘bands’ depends, of course, on the size of the basis.

By these considerations we arrive at the following assignment of quantum numbers to the frequencies shown in fig. 8. The mode with the largest frequency forms the ‘band’ on the top and must be a sound-mode with \( n_z = 10, n_\rho = 0 \). The next lower band must contain two sound-like modes with \( n_z = 9, n_\rho = 1 \) and \( n_z = 9, n_\rho = 0 \). We see from the figure that this exhausts already the number of modes (=2) in the second ‘band’ from the top, which can therefore not contain an internal wave mode. Similarly, in the 3rd, 4th, 5th, etc. ‘band’ from the top there are 3, 4, 5, etc. sound-like modes with \( n_z = 8, 7, 6 \), etc. and 3, 4, 5, etc. different values of \( n_\rho \), respectively. In the figure we can follow this counting of different frequencies down to the third ‘band’ from the bottom, thereby accounting for all eigenvalues in these ‘bands’. It follows that none of these ‘bands’ contains any internal wave modes.

The lowest two bands, on the other hand, in particular the lowest one, contain many more different frequencies than the 10 and 11 different sound-frequencies with \( n_z = 1, n_\rho = 0 \), respectively. These must therefore be considered as the internal waves. Among the internal waves are also the zero-frequency modes. For each value of \( n = n_z + n_\rho \geq 1 \) in our subspace there is precisely 1 zero-frequency mode, so the eigenvalue \( \omega^2 = 0 \) is 10-fold degenerate in our subspace. By their special nature the zero-frequency modes have quantum-numbers \( n_z = n_\rho \). Internal modes of high \( n = 2(n_z + n_\rho) \), for which \( n_\rho \) differs only slightly from \( n_z \) have frequencies close to zero.
The discussion of the spectrum we have given depends on our arbitrary restriction of the number of nodal surfaces to a finite and not very large number. If modes with an arbitrary number of nodal surfaces are permitted, then the mode-spectrum becomes dense, due to the existence of small internal wave frequencies from modes with arbitrarily high quantum-numbers. This can already be seen from the analytically determined spectrum $(1.37)$ for the isotropic case.

Within the subspace of Hilbert space we consider here there are many internal mode frequencies also below the geometric mean trap frequency, as can be seen with more clarity in fig.2. Only a small number of sound-modes can occur in this regime. The mode frequencies displayed in figs.3, 4 have a surprisingly weak temperature-dependence throughout the range considered. For large $T$ this can also be seen from the analytical results. For the sound-modes the velocity of sound increases with temperature roughly $\sim \sqrt{k_B T}$, but the wavelength of any given mode also increases with temperature $\sim \sqrt{k_B T}$ due to the expansion of the size of the thermal cloud, so that both temperature-dependences effectively cancel. However, for internal modes the compensation between the speed of sound and the wave-length does not work in the same way, as can e.g. be seen from eq.(3.7). In fig.5 we consider a magnification of a part of the frequency spectrum (in this instance for the isotropic case) where the increase of some frequencies with temperature can be seen, which freely cross other levels (which must have therefore different quantum numbers $n, l$), which are nearly temperature-independent and belong to sound-like modes. However, in the axially symmetric case avoided crossings can also be seen, as shown in fig.6 for two internal wave modes at small frequency below the geometric mean trap-frequency. This indicates that the conserved operator $B$ of section IV.C ceases to exist at intermediate temperatures. It would be futile, therefore, to look for analytical solutions of the spectrum in the intermediate temperature range, as the system appears to be non-integrable.

Generally, the differences between the results for fermions and bosons in the region above the degeneracy temperature are qualitatively not very big. One difference due to quantum statistics can be seen in fig.5, where the temperature dependence of internal wave modes and sound modes is displayed for a Fermi gas. The frequencies of the internal wave modes curve downwards, those of the sound modes curve upwards. In the Bose gas case the opposite tendency is found as shown in figures 2 and 4.

VI. CONCLUSIONS

In the present paper we have given a systematic analysis of the hydrodynamic modes of quantum gases in a harmonic trap with general anisotropy in the collision-dominated non-dissipative limit. Provided the hydrodynamic limit is applicable the analysis applies for Fermi-gases at all temperatures and for Bose-gases at temperatures above Bose-Einstein condensation. Our results extend previous works by allowing for traps with arbitrary anisotropy, and by treating bosons and fermions side by side on an equal footing. In addition to analytical solutions in certain special cases and limits we also present numerical solutions in the whole temperature-domain covered by the theory. Our analysis is based on the reduction of the five conservation-laws for the densities of mass, momentum and energy to two coupled wave-equations for the mass-density and the pressure. We have constructed a scalar product with a positive $L_2$-norm on the space of solutions, in which the two-component wave-operator is hermitian and, because of the stability of the hydrodynamic modes which we demonstrate, it is non-negative. However, a class of solutions with vanishing frequencies was found which is a consequence of the existence of mechanical equilibrium states in addition to the unique thermodynamic equilibrium which maximizes the entropy. A further class of exact solutions consisting of isothermal modes was identified among which are the center of mass modes required by the Kohn-theorem. These results generalize earlier results by Griffin et al. and Brunn and Clark to traps without axial symmetry.

We studied the two coupled wave equations for pressure and density in the short-wavelength limit. Two different branches of solutions could be identified in this way, the high-frequency branch being associated with pressure-driven sound waves, the low-frequency branch with potential-driven internal waves. The explicit dispersion relation of the lower branch found in the short wavelength limit and its characteristic properties like the existence of a maximal frequency, the anisotropy of the dispersion-relation and the orthogonality of the local group velocity and the local wave-vector makes the identification with internal waves manifest and unambiguous. We also examined the high-temperature limit of the two coupled wave equations and demonstrated the existence of polynomial solutions by exhibiting a conserved operator whose eigenvalues fix the respective polynomial order, or, equivalently, the number of nodal surfaces of the solutions. We constructed the solutions for density and pressure also explicitly.

The wave equations were finally also solved numerically. Surprisingly, the mode-spectrum found turned out to be quasi-continuous, which can be understood by the overlap of the spectrum of low-frequency internal waves of short wavelengths with the spectrum of sound waves with large wavelengths. Within a finite-dimensional subspace of the Hilbert space, defined by restricting from above the number of nodal surfaces, we found a band-like structure of the eigenfrequencies in a strongly anisotropic trap, where the bands are labeled by the number of nodal surfaces.
orthogonal to the more strongly confined direction while the modes within a given band differ by the number of nodes in the other directions. The number of internal wave modes in such a finite-dimensional subspace is also limited, and these modes are then found primarily in the lowest lying band. We note that the restriction to a finite subspace of Hilbert space is also physically motivated, since typical excitation-mechanisms like the modulation of the trapping potential, will also excite only modes in a certain subspace with appreciable amplitude.

The analysis we have presented is subject to some obvious limitations which we now discuss briefly. It is clear that only systems in the collision-dominated hydrodynamic limit have been considered here. For bosons a necessary requirement is therefore a large scattering length and a sufficiently large number-density to ensure a large cross-section for elastic collisions. For fermions this requires, besides a large number-density, the simultaneous trapping of several hyperfine states, in order to allow for the interaction of the different fermionic species by elastic collisions, which would be forbidden for a single species by the Pauli-principle. We have neglected throughout the spin-wave excitations, which can also occur in the latter systems, which is permitted because they decouple from the density-waves by symmetry as long as the external potential is the same for all components. The collision-rates in degenerate Fermi-gases are suppressed by a Fermi-blocking factor compared to the classical collision rates \( T^{2} \) and scale proportional to \( (T/T_F)^2 \). Therefore, at least in the low-temperature domain, it is necessary to use atomic species with particularly large positive or negative s-wave scattering-lengths. Another limitation of our analysis is the neglect of mean-field effects of the interaction in comparison to the pressure term. This seems to be a rather good approximation for the experimentally realized trapped quantum-gases, which behave like ideal quantum-gases to a good approximation. The most severe limitation of our calculations is certainly the neglect of dissipation. The reason for this restriction (which is discussed further in [3] for bosons, and in [5] for fermions) lies not so much in the negligibility of dissipative effects for the physically excited modes, but in the particular purpose we set out to achieve in this paper, namely to give an account of the mode-spectrum in the whole temperature-domain covered by the theory. This goal cannot be achieved so far with the inclusion of damping effects, but remains an interesting aim for future work. It seems clear that with the inclusion of damping the zero-frequency modes we have found will turn into purely over-damped modes. However, as was exemplified for the Kohn-modes, the phenomenological theory behind eq.(1.7) permits to obtain also a result for the damping of some modes (with the result that it is vanishing for the Kohn-modes) whose hydrodynamic frequencies are only known in the absence of dissipation [1][2]. The requirement is that the mode-frequencies without damping are also known in the collisionless limit. Then an estimate of the collision-time can be used in eq.(1.7) to obtain an interpolation between the collision-dominated and the collisionless regime including the damping due to the finite value of the collision-time [1][2]. In the past this estimate proved to be quite useful in the comparison of the experimental results for bosons [3], and it may be hoped that further results along such lines not only for bosons but also for the rapidly developing experiments on trapped Fermi-gases may be obtained in the near future.

ACKNOWLEDGMENTS

This work has been supported by a project of the Hungarian Academy of Sciences and the Deutsche Forschungsgemeinschaft under Grant No. 436 UNG 113/ 144. A. Cs. would like to acknowledge support by the Hungarian Academy of Sciences under Grant No. AKP 98-20 2.2 and the Hungarian National Scientific Research Foundation under Grant Nos. OTKA F020094, T029552, T025866. R. G. wishes to acknowledge support by the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 237 ”Unordnung und große Fluktuationen”.

[1] M.H. Anderson et al., Science 269, 198 (1995); K.B. Davis et al., Phys. Rev. Lett. 75, 3969 (1995); C.C. Bradley et al., ibid. 78, 985 (1997).
[2] B. DeMarco and D.S. Jin, Science 285, 1703 (1999)
[3] M.-O. Mewes, G. Ferrari, F. Schreck, A. Sinatra, and C. Salomon, physics/9909007 (September 6, 1999)
[4] L.P. Kadanoff and G. Baym, Quantum Statistical Mechanics (W.A. Benjamin, N.Y., 1962), Ch. 6.
[5] A. Griffin, W.-C. Wu, and S. Stringari, Phys. Rev. Lett. 78, 1838 (1997)
[6] Y. Kagan, E.L. Surkov and G. Shlyapnikov, Phys. Rev. Lett. 789, 2604 (1997)
[7] G. M. Bruun and Ch. W. Clark, cond-mat/9905263
[8] M. Amoruso, I. Meccoli, A. Minguzzi, and M.P. Tosi, Eur. Phys. J. D 7, 441 (1999)
[9] A. Csordás and R. Graham, cond-mat/0007049
[10] L. Vichi and S. Stringari, cond-mat/9905154
Fig. 1. The bosonic function $F_+(z) = f_+(\frac{z}{2}, z)/f_-(\frac{z}{2}, z)$ (full line) and its derivative $F'_+(z)$ (broken line) as a function of $z$. 

\[ F_+(z) = \frac{f_+(\frac{z}{2}, z)}{f_-(\frac{z}{2}, z)} \]
FIG. 2. The fermionic function $F_+(z) = F_+(\frac{\sqrt{2}}{2}, z)/F_+(\sqrt{2}, z)$ (full line) and its derivative $F'_+(z)$ (broken line) as a function of $z$.

FIG. 3. The dimensionless chemical potential $\mu/\hbar\omega$ in the range $\mu < 0$ as a function of the dimensionless temperature $t = k_B T/\hbar\omega$ with $\omega = (\omega_0^2 + \omega_\perp^2)^{1/3}$, for bosons (full curve) and for fermions (dashed curve); number of atoms $N = 10^6$. 
FIG. 4. Dimensionless squared hydrodynamic mode-frequencies \( \left( \frac{\omega}{\bar{\omega}} \right)^2 \) of a bosonic gas above the BEC-transition as function of the dimensionless temperature \( t \), for the \( m=0 \) modes of even parity with up to 10 nodal surfaces; anisotropy parameter \( \lambda = \left( \frac{\omega_z}{\omega_\perp} \right)^2 = 8 \); number of atoms \( N = 10^6 \); total size of basis 132.

FIG. 5. Squared internal mode frequencies of fig.4 below the geometric mean trap frequency as function of temperature, for the same parameters as in fig.4.
FIG. 6. Squared mode-frequencies increasing with temperature and level crossings for a bose-gas with the same parameters as in fig.4 for an isotropic trap.

FIG. 7. Squared mode-frequencies of internal waves decreasing with temperature, and an avoided level crossing for a bose-gas in an axially symmetric trap with the same parameters as in fig.4.
FIG. 8. Squared mode-frequencies for a fermionic gas above the Fermi-Temperature as a function of $t = k_B T / \hbar \bar{\omega}$ for sound modes (upper part) and for internal wave modes (lower part).