One Dimensional Asynchronous Cooperative Parrondo’s Games

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Abstract

An analytical result and an algorithm are derived for the probability distribution of the one-dimensional cooperative Parrondo’s games. We show that winning and the occurrence of the paradox depends on the number of players. Analytical results are compared to the results of the computer simulation and to the results based on the mean-field approach.

1 Introduction

Devised as a pedagogical illustration of the Brownian ratchet mechanism, Parrondo’s games are coin flipping games with an apparently paradoxical property that alternating plays of two losing games produce a winning game. In the original setup a player has some capital, which in game $A$ is increased by one with probability $p$ and decreased by one with probability $1 - p$ \cite{1,2,3}. Game $B$ is more complicated and the rules are that the probability of winning is $p_1$ if the capital is multiple of $M$ and if it is not, the probability of winning is $p_2$. A bias is introduced into probabilities such that two games have a tendency to lose (capital is a decreasing function of the number of runs). A third game, game $A + B$, may be constructed as a random combination of games $A$ and $B$, and this game is in the long run winning (the capital is an increasing function of the number of runs). New
types of Parrondo’s game have been recently introduced in which the feedback is introduced through spatial neighbor dependence [4], [5]. These games, termed cooperative Parrondo’s games, rely on the state of player’s neighbors depending whether a player has either lost or won the previous game. Each of $N$ players, arranged in a circle, owns a capital $C_i(t), i = 1, \ldots, N$, which evolves by combination of two games. Game A is the same as in the classical setup, namely the probability of winning and losing is $p^{(A)}$ and $1 - p^{(A)}$ respectively. Game B is different from its counterpart in the original setup since it depends on the state of the neighbors to the left and to the right of the player. It was shown that alternation of games A and B, which may be losing or fair when played individually, leads to a winning outcome. In [4], the evolution of probabilities in games B and C was studied using mean field type equations since it was assumed that the corresponding exact equations were too complicated to be obtained analytically. We derive here the exact probability evolution equations analytically and show excellent agreement with the results based on computer simulations. We also demonstrate that the existence of the paradox depends on the number of players, when a set of probabilities introduced in [4] is used. For arbitrary set of probabilities, we establish the constraints that prevent the occurrence of the paradox when games are played by three players. The same approach may be applied to arbitrary number of players however with considerable increase in complexity of expressions. The paper is organized as follows: Following a short presentation of essential rules of the games in section 2, we derive the probability transition matrix and the corresponding probability evolution equation in section 3. Stationary probability distribution and constraints in the form of inequalities for the paradox to occur are derived in section 4 and finally we compare analytical results with computer simulation results and the mean field approach of Toral [4] in section 5. We conclude with suggestions about possible new directions and applications of this game.

2 Features of the Game

Each player may be in one of two states: state 0 (“loser”) or state 1 (“winner”). The state of the whole ensemble of $N$ players may be represented as a binary string $s = (s_1, \ldots, s_N), s_i \in (0, 1)$ of length $N$, or equivalently as state $s$ in decimal notation. We also assume periodic boundary conditions, i.e. $s_{N+1} = s_1$. To each state corresponds a vector, equivalent to a basis vector.
\( |s\rangle \) in \( M = 2^N \) dimensional state space \( S_M \)

\[
S_M = \{ |s\rangle | s = 0, 1, ... M - 1 \}.
\]  

(1)

For example, state (011) is equivalent to state 3, and the corresponding vector is \( |3\rangle = [00010000]^T \), while state (111) is equivalent to state 7 and the corresponding vector is \( |7\rangle = [00000001]^T \). Game A is the same as in the classical setup, while probabilities of winning in game B depend on the present state of left and right neighbors, denoted as a pair \( (s_{k-1} s_{k+1}) \), and with player at position \( k \) are given by:

\[
\cdot p_{0}^{(B)} \text{ when } (s_{k-1} s_{k+1}) = (00), \text{ in decimal notation } 0,
\]

\[
\cdot p_{1}^{(B)} \text{ when } (s_{k-1} s_{k+1}) = (01), \text{ in decimal notation } 1,
\]

\[
\cdot p_{2}^{(B)} \text{ when } (s_{k-1} s_{k+1}) = (10), \text{ in decimal notation } 2,
\]

\[
\cdot p_{3}^{(B)} \text{ when } (s_{k-1} s_{k+1}) = (11), \text{ in decimal notation } 3.
\]

Winning or losing in any particular game leaves a player in state 1 ("winner") or 0 ("loser") respectively, until he gets a random chance to play again. Capital \( C(t) \) is a function of the ensemble that is incremented by 1 or decremented by 1 if one of the players wins or loses respectively. Following a play by one of the players, the state of the ensemble has changed from a state \( s(t) \) at time \( t \) to a state \( s(t+1) \) at time \( t+1 \). If the probability that an ensemble in state \( s(t) \) (or \( |s(t)\rangle \)) is \( \pi_s(t) \), then the probability distribution \( p(t) \) at time \( t \) is:

\[
|\pi(t)\rangle = \sum_{s=0}^{M-1} \pi_{s}(t) \ |s\rangle,
\]

(2)

while the corresponding probability distribution evolution equation is

\[
|\pi(t+1)\rangle = \mathcal{T} \ |\pi(t)\rangle
\]

(3)

It was reported in [4] that the combination of losing games A and B, leads to a winning result for the set of probabilities \((p_0 = 1, p_1 = p_2 = 0.16, p_3 = 0.7)\). Numerical simulations show that this set is not unique and in subsequent sections we discuss constraints that determine this paradoxical result, however in further exposition we will adhere exclusively to this set.

### 3 Probability Transition Matrix

The analysis is performed via discrete time Markov chains (DTMCs) and we first derive the probability transition matrix for game B. Since at each
moment of time only one player plays and therefore changes state, the change may be represented by a Hamming distance between the initial \((i)\) and the final \((f)\) state of player \(k\) is defined as:

\[
d_H = \sum_{k=1}^{N} |i_k - f_k|.
\] (4)

Clearly, player may either remain in the same state as before the play \((d_H^k = |i_k - f_k| = 0)\) or change his state \((d_H^k = |i_k - f_k| = 1)\). We will consider each case separately.

**Case 1: \(d_H = 0\)**

If the ensemble is in state \(s\), then the \(k\)-th player is in state \(s_k\). State \(s\) of the ensemble also defines the neighborhood of the \(k\)-th player, a pair \(\eta = (s_{k-1}, s_{k+1})\) which in turn determines the probability of winning. Since the ensemble initially in state \(i\) may switch to state \(f = i\) in one of \(N\) different ways as a result of one of the players switching from state \(i_k\) to state \(f_k = i_k\), the probability of transition is in this case equal to the sum of probabilities of independent events

\[
T_{fi} = w(i \rightarrow f) = \frac{1}{N} \sum_{k=1}^{N} w(i_k, f_k),
\] (5)

where probabilities \(w\) depend on whether state \(f_k\) is 1 (winning) or 0 (losing), and upon the probability of winning, i.e.

\[
w(i_k, f_k) = \begin{cases} 
1 - p_{(B)}^{(i_k, f_k)} & f_k = 0 \\
p_{(B)}^{(i_k, f_k)} & f_k = 1 
\end{cases}
\] (6)

**Case 1: \(d_H = 1\)**

In this case \(k\)-th player switches from state \(i_k\) to state \(f_k\), \((i_k \neq f_k)\), with probability

\[
T_{fi} = \frac{1}{N} w(i_k, f_k).
\] (7)

**3.1 Probability transition matrix for \(N = 3\)**

As an example we consider an ensemble of three players \((N = 3)\) playing game \(B\), and let us assume that the ensemble is in state \(s = (011) = 3\).
Possible transitions and the corresponding probabilities are:

\[
\begin{align*}
T_{33} &= w(011 \rightarrow 011) = w(3 \rightarrow 3) = \frac{1}{3}[(1 - p_3) + p_1 + p_2], \\
T_{31} &= w(011 \rightarrow 001) = w(3 \rightarrow 1) = \frac{1}{3}(1 - p_1), \\
T_{23} &= w(011 \rightarrow 010) = w(3 \rightarrow 2) = \frac{1}{3}(1 - p_2), \\
T_{73} &= w(011 \rightarrow 111) = w(3 \rightarrow 7) = \frac{1}{3}p_3.
\end{align*}
\] (8)

Explicitly, probability \(T_{13}\) represents the probability that second player, in state \(s_2 = 1\), switches to state \(f_2 = 0\). This probability is equal to the product of the probability that it is this player’s turn to play i.e. 1/3, and the probability that this player loses, i.e. switches to state \(f_2 = 0\). The neighborhood of the second player, \((s_1 s_3) = (0 1) = 1\), determines the choice \(p_1\). Similarly, transition \(T_{33}\) may take place when either one of the three players switches to a same state, hence the corresponding probability is the sum shown in the first expression of (8). Finally, using \(p_i^1 = 1 - p_i, \ i = 0, 1, 2, 3\)

\[
Q_1 = 1 - p_3 + p_1 + p_2 \quad \text{and} \quad Q_2 = 2 - p_2 - p_1 + p_0
\]

the transition matrix for game \(B\) \((N = 3)\) is:

\[
\mathbf{T}^{(B)} = \begin{pmatrix}
3p_0^1 & p_0^1 & p_0^1 & 0 & p_0^1 & 0 & 0 & 0 \\
p_0 & Q_2 & 0 & p_1^1 & 0 & p_1^1 & 0 & 0 \\
p_0 & 0 & Q_2 & p_2^1 & 0 & 0 & p_1^1 & 0 \\
0 & p_1 & p_3 & Q_1 & 0 & 0 & 0 & p_3^1 \\
p_0 & 0 & 0 & 0 & Q_2 & p_1^1 & p_2^1 & 0 \\
0 & p_2 & 0 & 0 & p_1 & Q_1 & 0 & p_3^1 \\
0 & 0 & p_1 & 0 & p_2 & 0 & Q_1 & p_3^1 \\
0 & 0 & 0 & p_3 & 0 & p_3 & p_3 & 3p_3
\end{pmatrix}
\] (9)

Corresponding matrix for game \(A\) may be easily obtained by performing the following replacement: \(p_0^{(B)} \rightarrow p^{(A)}\) for each \(p \in \{0, 1, 2, 3\}\).

Furthermore, we introduce a vector of the capital \(|C\rangle\) whose components represent the capital generated by each ensemble state. Elements \(C_s\) of this vector represent normalized capital generated by that specific state (equal to the sum of all winning and losing individual states in a given ensemble state). Naturally, player state 0 generates capital \(-1\), while state 1 generates capital \(+1\). Explicitly,

\[
C_s = \frac{1}{N} \sum_{i=1}^{N} (-1)^{s_i+1}, \quad \text{so that} \quad C_s \in [-1, 1].
\] (10)
In other words, elements of $|C\rangle$ are average values of the capital generated by each ensemble state separately. For example for $N = 3$, the vector of the capital is:

$$|C\rangle = (1/3) [-3 - 1 -1 1 1 1 1 3]^T.$$ (11)

In the above expression the third vector element corresponding to the state $|010\rangle = 2$ is equal to $C_2 = (1/3)((-1) + (+1) + (-1)) = (1/3)(-1)$. This also implies that the ensemble switching from a state $s(t)$ to a state $s(t+1) = (010) = |2\rangle$, under the assumption that such a transition is possible, generates average capital $\langle C(t+1)\rangle = \langle C | 2 \rangle = -1/3$. Furthermore, an ensemble remaining in state $|2\rangle$ throughout its temporal evolution would in the average generate capital $(-1/3)$ in each turn of the game. Hence the capital generated by the ensemble is

$$\langle C \rangle = \langle C | \pi \rangle,$$ (12)

where $\langle C \rangle$ denotes the average value of the generated capital. In order to evaluate the probability for one of the games, either $B$ or $A+B$ to be winning, it should be noted that:

\[
\begin{align*}
P_{\text{win}} + P_{\text{lose}} & = 1 \\
P_{\text{win}} - P_{\text{lose}} & = \langle C \rangle
\end{align*}
\]

(13)

where $P_{\text{win}}$ and $P_{\text{lose}}$ are probabilities of winning and losing in a certain game, so that

$$P_{\text{win}} = (1/2)(1 + \langle C \rangle).$$ (14)

This expression implies that condition $P_{\text{win}}^{(B)} < 1/2$, i.e. that game $B$ is losing, is equivalent to the condition $\langle C^{(B)} \rangle < 0$, and that $P_{\text{win}}^{(A+B)} > 1/2$, i.e. that game $A + B$ is winning, is equivalent to $\langle C^{(A+B)} \rangle > 0$.

### 4 Analysis of the Games

#### 4.1 Equilibrium distribution

The equilibrium (stationary) state occurs when the probability distribution remains invariant under the action of $\mathcal{T}$, that is, $|\pi(t+1)\rangle = \mathcal{T} |\pi(t)\rangle = |\pi\rangle$. In order to evaluate the probability distribution in this case, we need
to solve \((1 - T)\pi = 0\). For game \(A\), for which there is a probability \(p\) for a player to win (alternatively \((1 - p)\) to lose), the stationary distribution is easily obtained by setting \(p_0 = p_1 = p_2 = p_3 = p\) and reads

\[
\pi^{(A)} = [(1 - p)^3, (1 - p)^2p, (1 - p)^2p, (1 - p)p^2, (1 - p)p^2, (1 - p)p^2, p^3]^T
\] (15)

The probabilities in the above expression may also be readily obtained by associating to each ensemble state, from 0 to 7 in decimal notation, corresponding probabilities for each player. For game \(B\) the stationary distribution is

\[
\pi^{(B)} = \left[ \frac{(1 - p_0)[2 - (p_1 + p_2)](1 - p_3)\alpha}{p_0p_3(p_1 + p_2)}, \frac{[2 - (p_1 + p_2)](1 - p_3)\alpha}{p_0p_3(p_1 + p_2)}, \frac{\alpha(1 - p_3)}{p_3}, \frac{[2 - (p_1 + p_2)](1 - p_3)\alpha}{p_0p_3(p_1 + p_2)} \right]^T
\] (16)

where

\[
\alpha = \frac{1}{\frac{(1-p_0)+3p_3[2-(p_1+p_2)](1-p_3)}{p_0p_3(p_1+p_2)} - \frac{3(1-p_3)}{p_3} + 1}.
\] (18)

Substituting the set of probabilities given in [4], \(p_0 = 1, p_1 = p_2 = 0.16, p_3 = 0.7\), in the expression for the stationary distribution for game \(B\), we get

\[
\pi^{(B)} = [0, .24901, .24901, .04743, .24901, .04743, .04743, .11067]^T
\] (19)

For the randomized game \(A + B\), probabilities \(p_i (i = 0, 1, 2, 3)\) for game \(B\), are replaced with the corresponding probabilities \(q_i (i = 0, 1, 2, 3)\)given by:

\[
q_i = \gamma p + (1 - \gamma)p_i, \quad (i = 0, 1, 2, 3).
\] (20)

and where parameter \(\gamma\) represents the relative probability of playing game \(A\), where we have assumed the value of one half. The corresponding stationary distribution for the randomized game, with \(p = 1/2\), is

\[
\pi^{(A+B)} = [.06006, .18019, .18019, .08875, .18019, .08875, .08875, .13312]^T.
\] (21)

Expressions [19] and [21], although not leading to the paradoxical result illustrate the probabilities associated with each state in games \(B\) and \(A + B\) respectively.
4.2 Constraints of the games

The probability of winning using the stationary distribution is given by

\[ p_{\text{winning}} = \langle \pi \mid \rho \rangle = \sum_{s=0}^{M-1} \pi_s \rho_s, \]  
(22)

where \( \rho_s \) is the winning probability in state \( \pi_s \). Components \( \rho_s \) of vector \( |\rho\rangle \) may be defined using (14) or alternatively as,

\[ \rho_s = \sum_{k=1}^{N} \frac{s_k}{N}, \]  
(23)

so that the probability that the ensemble generates a winning outcome when switching to a state \( s = (s_1, \ldots, s_N) \), is determined by the fraction of winning players in the ensemble. The summation in the above expression should actually be over the indices corresponding to the winning players, however use of all indices is justified since states 0 do not contribute to the summation. Since for the paradox to occur we must have \( P_{\text{win}}^{(B)} < 1/2 \) and \( P_{\text{win}}^{(A+B)} > 1/2 \) simultaneously, a simple calculation yields the following inequality

\[ (p_1 + p_2)[1 + \frac{1}{2}(p_0 - p_3)] > 2[1 - \frac{1}{4}(p_0 + 3p_3)]. \]  
(24)

We may also assume that probabilities \( p_1 \) and \( p_2 \) should be equal, so that finally we get

\[ 1 > p_1 > \frac{1 - \frac{1}{4}(p_0 + 3p_3)}{1 + \frac{1}{2}(p_0 - p_3)}. \]  
(25)

If \( p_0 = 1 \), and \( p_3 = 0.7 \) are inserted in the above expression, the inequality is not satisfied and there is no paradox, justifying the conclusion based on numerical simulations that there is no paradox for games played by three players when probabilities are those given by Toral in [4]. However, for the same number of players and a set of probabilities, e.g. \( p_0 = 0.09, p_1 = p_2 = 0.52 \) and \( p_3 = 0.89 \), the paradox exists.

5 Results

All results in this section are based on probability values given in [4], namely \( (p_0 = 1, p_1 = p_2 = 0.16, p_3 = 0.7, P^{(A)} = 0.499) \). First, in Fig. 1 we present analytical results that show how paradox depends on the number of players when \( N \leq 12 \). Namely, in this range of \( N \) values paradox exists if \( N \) is
different from 3, 4, 7 or 8. Numerical simulations, shown in Fig. 2, indicate
that the paradox occurs if the number of players is greater than 8 and no
exceptions are noticed up to 1000 players. The mean-field equation for the
evolution of the common probability in game $B$

$$P^{(B)}(t + 1) = (1 - P^{(B)}(t))^2 p_0 + P^{(B)}(t)(1 - P^{(B)}(t))(p_1 + p_2) + P^{(B)}(t)^2 p_3,$$

(26)
in which each term reflects one of the four possibilities given in the definition
of the game, introduced in [4], naturally yields results irrespective of the
number of players.

As a comparative illustration of results based on all three approaches, we
show in Fig. 3 the average capital per turn as a function of the probability $p_3$
of game $B$ played by five players. Analytical and numerical simulation results
show perfect agreement while results based of the mean field equation show
very good agreement only in the (approximate) range $0.16 < p_3 < 0.83$.
For values of $p_3$ below 0.16 the capital based on mean field calculations
shows that there is no stable solution of the iterative mean field equation.
Namely, mean field equation in this area switches to one of the two solutions
at each successive step of calculations while it deviates considerably from
the analytical and numerical results for $p_3 > 0.83$. We emphasize that due
to the very nature of the mean-field approach this large discrepancy for low
and high values of $p_3$ remains for arbitrary number of players. However the
dependence of the average capital as function of other probabilities, $p_0$, $p_1$
and $p_2$ is considerably better, as presented in Fig. 4. for the case of $p_0$.

6 Conclusion

An algorithm for obtaining analytical expressions for the evolution of the probabili-ties and constraints of the games is illustrated in the case of three
players. These results may prove to be of important use in applications
to social and economic models where exact results may be indispensa-ble. Usefulness of these games to social and economic models, and possibly in
biological applications, may be expanded by analyzing games played by all
players simultaneously that we introduce in the follow up paper [6]. Also, the
approach presented here may be useful in analysis of games based on spin
models. Moreover, we show that the mean-field approach, although useful
when large number of players is involved, may be quite inaccurate in certain
range of probabilities that define spatial neighbor dependence.
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Fig. 1. Averaged capital per turn as a function of the number of players obtained using analytical expressions for the evolution of the probability distribution. For $1 \leq N \leq 6$ it is easy to distinguish winning and losing games, however for $N \geq 7$ game $B$ is barely losing and in order to accentuate that conclusion we use magnified scale for the capital. The spline fit is performed solely for visualization purposes.

Fig. 2. Numerically obtained averaged capital per turn (realization of the games) as a function of the number of players ($N$). Capital is averaged over 1000 runs of the game, and over $100N$ turns in each run after the system entered the stationary state.

Fig. 3. Mean-field approach results compared to the analytical and numerical results for the average capital per turn as a function of $p_{3}^{(B)}$ for $N = 5$, ($p_0 = 1, p_1 = p_2 = 0.16$). Analytical and numerical results show excellent agreement while mean-field results show considerable divergence for small and large values of $p_{3}^{(B)}$.

Fig. 4 Average capital per turn of the game as a function of $p_{0}^{(B)}$. Due to the large number of players only results based on the computer simulations are compared to the mean-field results. The other probabilities $p_{1} = p_{2} = 0.16, p_{3} = 0.7$ are kept fixed. The graph shows very good agreement.
The graph shows the average capital per turn, \( \langle C(t) \rangle \), as a function of the number of players on a lattice. The x-axis represents the number of players, ranging from 3 to 12. The y-axis represents the average capital per turn, ranging from -0.10 to 0.10.

Two curves are depicted:

- A solid black line labeled \( (B) \) indicates the capital per turn for a specific subset of players.
- A dashed gray line labeled \( (A+B) \) represents the combined capital per turn for all players.

The graph illustrates the dynamics of capital accumulation as the number of players increases, with peaks and troughs indicating changes in the average capital.
The graph represents the average capital per turn ($<C(t)>$) as a function of the number of players on a lattice. The graph includes two simulations:

- Simulation (B) represented by black dashed line.
- Simulation (A+B) represented by gray dashed line.

The x-axis represents the number of players on the lattice, ranging from 10 to 1000, and the y-axis represents the average capital per turn, ranging from -0.001 to 0.005.
The graph shows the average capital per turn, \( <C(t)> \), as a function of \( p_3^{(B)} \). The graph includes data points for simulation, analytical solution, and mean-field equation.
\[ \langle C(t) \rangle \]

average capital per turn

\[ p_0^{(B)} \]

\( p \)

simulation (N=100)

mean-field