Some Exact Solutions of the Semilocal Popov Equations with Many Flavors

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Abstract

In 2+1 dimensional nonrelativistic Chern-Simons gauge theories on $S^2$ which has a global $SU(M)$ symmetry, the semilocal Popov vortex equations are obtained as Bogomolny equations by minimizing the energy in the presence of a uniform external magnetic field. We study the equations with many flavors and find several families of exact solutions. The equations are transformed to the semilocal Liouville equations for which some exact solutions are known. In this paper, we find new exact solutions of the semilocal Liouville equations. Using these solutions, we construct solutions to the semilocal Popov equations. The solutions are expressed in terms of one or more arbitrary rational functions on $S^2$. Some simple solutions reduce to $CP^{M-1}$ lump configurations.

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I. INTRODUCTION

The Popov equations are a set of vortex-type equations which involve a U(1) gauge potential and a single scalar field on $S^2$ [1]. They are a variant of the Bogomolny equations [2] for abelian Higgs vortices [3, 4] on $S^2$. They are obtained by dimensional reduction of the SU(1,1) Yang-Mills instanton equations on the four-manifold $S^2 \times H^2$, where $H^2$ is a hyperbolic plane. When the radii of $S^2$ and $H^2$ are equal, the equations are integrable. The explicit solutions were constructed in Ref. [5] in terms of rational functions on a sphere. They have even vortex numbers and can be obtained from the solutions of the Liouville equation [6].

The Popov equations also arise in 2+1 dimensional Chern-Simons systems with nonrelativistic matter fields on $S^2$ [7]. These systems on $\mathbb{R}^2$ have been considered in the context of the quantum Hall effect, superconductivity and other phenomena related with fractional statistics [8–10]. The Popov equations obtained in this way have a straightforward generalization to the semilocal version [11, 12] which involves two scalar fields with a global SU(2) symmetry. Though they are not integrable, we were able to construct two families of exact solutions to those equations [7], both of which involve rational functions on $S^2$ but with different vortex numbers. In particular, we showed that the solution with a unit vortex number and reflection symmetry is precisely given by the $CP^1$ lump configuration with unit size. The magnetic field of the solution is that of a Dirac monopole with unit magnetic charge on $S^2$.

In this paper, we consider the semilocal Popov equations with more than two scalar fields which arise in 2+1 dimensional Chern-Simons systems with more than two nonrelativistic matter fields on $S^2$. We will construct some exact solutions to those equations. First, we transform the equations into semilocal Liouville equations to which some exact solutions were found before for two matter fields [13, 14] and for more than two matter fields [15]. Here, we will find new exact solutions. From these solutions, we will obtain many families of exact solutions of the semilocal Popov equations. The solutions are expressed in terms of some number of arbitrary rational functions on $S^2$. Some simple solutions turn out to be precisely given by $CP^{M-1}$ lump configurations, where $M$ is the number of matter fields.

The rest of the paper is organized as follows: In Section II we consider nonrelativistic Chern-Simons matter systems with several scalar fields on $S^2$ and derive the semilocal Popov
II. SEMILOCAL POPOV EQUATIONS FROM NONRELATIVISTIC CHERN-SIMONS MATTER SYSTEMS

In this section, we derive the semilocal Popov equations with global SN($M$) symmetry by generalizing the result in Ref. [7]. Let us consider a 2 + 1 dimensional Chern-Simons gauge theory coupled to nonrelativistic matter fields on $S^2$ with the metric $ds^2 = \Omega dz d\bar{z}$, where

$$\Omega = \frac{8}{(1 + |z|^2)^2}. \quad (1)$$

Here, the radius of $S^2$ is fixed to be $\sqrt{2}$ for convenience. With $M$ matter fields, the action reads

$$S = \int dt \int_{S^2} \left\{ \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \sum_{k=1}^{M} \left[ \Omega (i \phi_k^* D_t \phi_k - V) - (|\tilde{D}_z \phi_k|^2 + |\tilde{D}_{\bar{z}} \phi_k|^2) \right] \right\} \frac{i}{2} dz d\bar{z}, \quad (2)$$

where $\kappa$ is the Chern-Simons coefficient, which is assumed to be negative in this paper, and $k = 1, \ldots, M$ denotes the flavor index of the matter fields. The covariant derivatives are defined by

$$D_t \phi_k = (\partial_t - ia_t) \phi_k, \quad \tilde{D}_z \phi_k = (\partial_z - ia_z - iA_{z}^{ex}) \phi_k. \quad (3)$$

Note that we have applied an external $U(1)$ gauge potential $A^{ex}_z$ given by

$$A^{ex}_z = \frac{i}{2} \frac{g z}{1 + |z|^2}, \quad (4)$$

which generates the uniform magnetic field $F^{ex}_{z\bar{z}}$ of a magnetic charge $g$ on $S^2$ given by

$$F^{ex}_{z\bar{z}} = \frac{ig}{(1 + |z|^2)^2} = \frac{i g}{8} \Omega. \quad (5)$$

The potential $V$ has the form

$$V = -\frac{g}{8} \sum_{k=1}^{M} |\phi_k|^2 - \frac{1}{2|\kappa|} \left( \sum_{k=1}^{M} |\phi_k|^2 \right)^2. \quad (6)$$
This action has a manifest SU($M$) global symmetry, as well as a U(1) local gauge symmetry. It has been studied on a plane to understand the quantum Hall effect and other related phenomena for multi-layer systems [10, 14].

The variation of the time component of the gauge potential $a_t$ gives the Gauss constraint

\[ F_{z\bar{z}} = -i \frac{\Omega}{2\kappa} \sum_{k=1}^{M} |\phi_k|^2, \quad (7) \]

which represents the characteristic nature of the Chern-Simons theory that the magnetic field is proportional to the charge density. From the action in Eq. (2), the energy is calculated as

\[ E = \int_{S^2} \left[ \sum_{k=1}^{M} (|\bar{D}_z \phi_k|^2 + |\bar{D}_\bar{z} \phi_k|^2) + \Omega V \right] \frac{i}{2} dz \wedge d\bar{z}, \quad (8) \]

which has no explicit contribution from the Chern-Simons term. With the help of the Gauss constraint in Eq. (7), we can rewrite the energy in a manifestly positive definite form as

\[ E = 2 \int_{S^2} \sum_{k=1}^{M} |\bar{D}_z \phi_k|^2 \frac{i}{2} dz \wedge d\bar{z}. \quad (9) \]

Then, the energy vanishes if

\[ \bar{D}_z \phi_k = 0, \quad (k = 1, \ldots, M). \quad (10) \]

This equation can be solved with respect to the gauge potentials away from the zeros of $\phi_k$:

\[ a_z + A_z^{ex} = -i \partial_z \ln \phi_k, \quad (k = 1, \ldots, M). \quad (11) \]

Then,

\[ \partial_z \ln \left( \frac{\phi_k}{\phi_1} \right) = 0; \quad (12) \]

hence the ratio

\[ w_k(z) \equiv \frac{\phi_k}{\phi_1} \quad (13) \]

is locally holomorphic with $w_1(z) \equiv 1$. Because the $\phi_k$’s have zeros at discrete points thanks to Eq. (10) [16], $w_k(z)$ should be rational functions of $z$. The field strength $F_{z\bar{z}} = \partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z$ is then given by

\[ F_{z\bar{z}} = -i \partial_z \partial_{\bar{z}} \ln |\phi_k|^2 - i \frac{g}{8} \Omega, \quad (14) \]

where we used Eq. (5). We can combine this equation with Eq. (7) by eliminating $a_{\bar{z}}$ to obtain

\[ \partial_z \partial_{\bar{z}} \ln |\phi_k|^2 = -\frac{\Omega}{2\kappa} \left( \frac{\kappa g}{4} - \sum_{k=1}^{M} |\phi_k|^2 \right) \quad (15) \]
away from the zeros of $\phi_k$. For a single scalar field $M = 1$, this becomes the Popov equation \[1\] with $\kappa$ being negative. Popov showed that it is integrable if $g = -2$. The exact solutions are written in terms of rational functions on a sphere \[3\]. They can be obtained from the solutions of the Liouville equation \[6\]. For $M = 2$, which is the simplest semilocal case, we were able to find some exact solutions that also involved rational functions on $S^2$. Here, we would like to consider the $M > 2$ case.

Before we try to find solutions to Eq. (15), a few remarks related to the role of the external field in Eq. (5) are in order. The original form of the Popov equations is \[1\]

\[
D_z \phi = 0,
\]

\[
F_{zz} = -i \frac{\Omega}{4} (C^2 - |\phi|^2),
\]

where $D_z = \partial_z - ia_z$ contains no external field. The constant $C$ is the ratio of the radii of $S^2$ and $H^2$ in the four-manifold $S^2 \times H^2$ on which the Popov equations are obtained as a dimensional reduction of the SU(1,1) Yang-Mills instanton equations. Comparing these equations with Eqs. (10), (7) and (14), we see that the external field provides the constant term in Eq. (17), which is missing from the Gauss constraint in Eq. (7). The external magnetic charge $g$ is identified as $g = -2C^2$. If we require $g$ to be quantized as an integer, then $C^2 = |g|/2$ becomes half-integer. We will see below that for each integer $g \leq -2$, the $M = 1$ Popov equations can be related to semilocal Popov equations with $M = |g| - 1$ flavors and $g = -2$ external magnetic charge.

The first Chern number $N$ is now defined as

\[
N = \frac{1}{2\pi} \int_{S^2} (F_{zz} + F_{zz}) dz \wedge d\bar{z} = g + \frac{1}{2\pi} \int_{S^2} F_{zz} dz \wedge d\bar{z},
\]

where the contribution from the external field is included. It is an integer and is the same as the vortex number, which counts the number of isolated zeros of $\phi_k$. Inserting Eq. (7) into Eq. (18), we obtain a Bradlow-type constraint \[17\] on $N$:

\[
N = g + \frac{1}{2\pi |\kappa|} \int_{S^2} \Omega \left( \sum_{k=1}^{M} |\phi_k|^2 \right) \frac{i}{2} dz \wedge d\bar{z} \geq g.
\]

Therefore, the vortex number should be equal to or greater than the external magnetic charge $g$. 

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III. EXACT SOLUTIONS

From now on, we consider only the case \( g = -2 \). As mentioned above, this is the value for which the Popov equation with a single scalar is integrable. Also, with a suitable rescaling of the scalars, we fix \( \kappa = -2 \) without loss of generality. Then, Eq. (15) becomes

\[
\partial_\z \partial_{\bar{z}} \ln |\phi_k|^2 = \frac{2}{(1 + |z|^2)^2} \left( 1 - \sum_{k=1}^{M} |\phi_k|^2 \right)
\]

away from the zeros of \( \phi_k \). Defining

\[
e^{u_k} = \frac{|\phi_k|^2}{(1 + |z|^2)^2},
\]

we can rewrite Eq. (20) as the semilocal Liouville equation

\[
\partial_\z \partial_{\bar{z}} u_k = -K_{kl} e^{u_l},
\]

where \( K_{kl} = 2 \) for all \( k \) and \( l \). Though this is not integrable because \( K \) is not one of the Cartan matrices of Lie algebras, some exact solutions have been constructed in Refs. [13–15]. Here, we would like to obtain new exact solutions. Then, through a transformation of Eq. (21), these provide exact solutions to the semilocal Popov equations on \( S^2 \).

Though Eq. (20) has no manifest global \( \text{SU}(M) \) symmetry on the left-hand side, the original equations, Eqs. (7) and (10), do. Also, they reduce to equations with fewer flavors if we put some of the \( \phi_k \)'s to be zero. Therefore, we can obtain solutions from those of the fewer-flavor case by trivial embedding and a suitable global \( \text{SU}(M) \) rotation. To be more specific, let \( \tilde{\phi}_k \) \( (k = 1, \ldots, m) \) be a solution of the Popov equations with \( m \) flavors. Then, the configuration transformed by a constant \( \text{SU}(M) \) matrix,

\[
\phi_k = \sum_{l=1}^{m} U_{kl} \tilde{\phi}_l, \quad (U \in \text{SU}(M)),
\]

should also satisfy the semilocal Popov equations with \( M \) flavors, provided \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) are solutions of the \( M = 2 \) equations. From Ref. [7], we can write the corresponding solutions as follows: For \( m = 1 \),

\[
\phi_k = U_{k1} \frac{(1 + |z|^2)(P(z)Q'(z) - Q'(z)P(z))}{\sqrt{1 + |z|^2(\sqrt{P(z)^2 + |Q(z)|^2})}},
\]

\[
a_{\bar{z}} = i \left[ \frac{P(z)P'(z) + Q(z)Q'(z)}{|P(z)|^2 + |Q(z)|^2} - \frac{z}{1 + |z|^2} \right] - A_{\bar{z}}^{ex},
\]

\[
= i \frac{P(z)P'(z) + Q(z)Q'(z)}{|P(z)|^2 + |Q(z)|^2},
\]

(24)
where $U_{k1}$'s are constant such that $\sum_k |U_{k1}|^2 = 1$. $P(z)$ and $Q(z)$ are arbitrary polynomials in $z$ on $S^2$ without common zeros. If the highest order of $P(z)$ and $Q(z)$ is $n$, the vortex number is $N = 2n - 2$, which is even. Note that all the scalar fields $\phi_k$ share the same vortex points.

For $m = 2$, solutions are
\[
\phi_k = \sqrt{\frac{3}{2}} \left( 1 + |z|^2 \right) \left( P(z) Q'(z) - Q(z) P'(z) \right) \left[ U_{k1} P(z) + U_{k2} Q(z) \right],
\]
\[
a_z = \frac{3i}{2} \frac{P(z) P'(z) + Q(z) Q'(z)}{|P(z)|^2 + |Q(z)|^2},
\] (25)

where $\sum_k U_{kl} U_{km} = \delta_{lm}$ ($l, m = 1, 2$). In this family of solutions, $\phi_k$'s share a part of the vortex points at the zeros of $PQ' - QP'$. If $U_{k1} P(z) + U_{k2} Q(z)$ are generically polynomials of order $n$, the vortex number is $N = 3n - 2$ [7]. In particular, we have solutions with unit vorticity $N = 1$ for $n = 1$. The simplest $N = 1$ solution is obtained with $P(z) = 1$ and $Q(z) = z$:
\[
\phi_k = \sqrt{\frac{3}{2}} \frac{U_{k1} + U_{k2} z}{\sqrt{1 + |z|^2}},
\]
\[
a_z = \frac{3i z}{2 \sqrt{1 + |z|^2}}.
\] (26)

Here, the conformal factor $1 + |z|^2$ in the numerator was cancelled by the same factor in the denominator. Equation (26) is the the $CP^{M-1}$ lump configuration with unit winding number and unit size [18]. The scalars satisfy
\[
\sum_{k=1}^{M} |\phi_k|^2 = \frac{3}{2},
\] (27)

which defines $S^{2M-1}$ fibered as a circle bundle over $CP^{M-1}$. Note that the radius $\sqrt{3/2}$ is the ratio of the radii $C$ as discussed in Section III. The Chern-Simons gauge field $a_z$ generates a uniform magnetic field of magnetic charge $g = 3$ on $S^2$, part of which is cancelled by the external magnetic field so that the vortex number is $N = 1$.

So far, we have discussed solutions that are simple embeddings of the $M = 2$ solutions found in Ref. [7] but more nontrivial solutions can also be obtained. To proceed, we rewrite Eq. (22) by using Eq. (13):
\[
\partial_z \partial_{\bar{z}} u_1 = -2 \sum_{k=1}^{M} |w_k(z)|^2 e^{u_1},
\] (28)
where \(w_1(z) = 1\) and the other \(w_k(z)\)'s are arbitrary rational functions on \(S^2\). A simple ansatz considered in Ref. [15] is

\[
w_k(z) = \sqrt{\frac{(M - 1)}{(k - 1)}} w^{k-1}(z),
\]

(29)

where \(w(z)\) is a rational function on \(S^2\). In other words, \(w_k\)'s are proportional to the \((k-1)\)-th power of \(w\). With this ansatz, Eq. (28) becomes

\[
\partial_z \partial_{\bar{z}} u_1 = -2(1 + |w(z)|^2)^{M-1} e^{u_1}.
\]

(30)

Now, we define \(v\) by

\[
ev = \frac{|w'|^2}{(1 + |w|^2)^{M+1}} e^v.
\]

(31)

Then, Eq. (30) can be written as

\[
\partial_w \partial_{\bar{w}} v = \frac{2}{(1 + |w|^2)^2} \left( \frac{M + 1}{2} - e^v \right),
\]

(32)

where we have changed the differentiation variable from \(z\) to \(w\). This is the Popov equation with \(C = \sqrt{\frac{M+1}{2}}\) in Eq. (17), for which the external magnetic charge is \(g = -(M + 1)\) as discussed in Section II. Thus, increasing the number of flavors corresponds to increasing the external magnetic charge. As in \(M = 2\) case, the constant solution \(e^v = \frac{M+1}{2}\) of Eq. (32) gives the solution to the semilocal Popov equation with \(M\) flavors:

\[
\phi_k = \sqrt{\frac{M + 1}{2}} \left( \frac{M - 1}{k - 1} \right) \frac{(1 + |z|^2) P^{M-k} Q^{k-1} (P Q' - Q P')}{|P|^2 + |Q|^2}^{\frac{M+1}{2}},
\]

(33)

\[
a_z = i \frac{M + 1}{2} \frac{P P' + Q Q'}{|P|^2 + |Q|^2},
\]

(33)

where \(P\) and \(Q\) are polynomials in \(z\) without common zeros defined by

\[
w(z) = \frac{Q(z)}{P(z)}.
\]

(34)

For the \(M = 2\) case, Eq. (33) reduces to Eq. (25) and may be considered as a generalization of Eq. (25) to more flavors. If \(w(z)\) is a generic rational function of degree \(n\), the vortex number is calculated by counting the number of zeros:

\[
N = n(M - 1) + 2n - 2 = n(M + 1) - 2.
\]

(35)
This result can also be confirmed by considering the behavior of $\phi_k$ around $z = \infty$:

$$\phi_k \sim c \left( \frac{z}{|z|} \right)^{n(M+1)-2},$$ (36)

which can be removed by using a gauge transformation of the winding number $n(M+1) - 2$ defined on an annulus on $S^2$ enclosing $z = \infty$. A simple solution with circular symmetry and vortex number $N = M - 1$ may be obtained by choosing $P(z) = 1$ and $Q(z) = z$:

$$\phi_k = \sqrt{\frac{M + 1}{2}} \left( \frac{M - 1}{k - 1} \right) \frac{z^{k-1}}{(1 + |z|^2)^{M/2}}, \quad \alpha = \frac{M + 1}{2} \frac{z}{1 + |z|^2}.$$ (37)

This is a $CP^{M-1}$ lump configuration with

$$\sum_{k=1}^{M} |\phi_k|^2 = \frac{M + 1}{2},$$ (38)

which defines an $S^{2M-1}$ of radius $\sqrt{\frac{M+1}{2}}$. Compared with Eq. (27), this lump solution has a larger winding number and a bigger radius.

The solution, Eq. (33), involves two polynomials in $z$, i.e., one rational function $w(z)$. More general solutions are also possible for a given $w(z)$. To construct such solutions, it is illuminating to notice that the solutions, Eq. (33), come from the identity

$$\partial_z \partial_{\bar{z}} \ln(1 + |w(z)|^2) = \frac{|w'(z)|^2}{(1 + |w(z)|^2)^2}.$$ (39)

To obtain Eq. (33), we rewrite the identity as

$$\partial_z \partial_{\bar{z}} \ln \frac{1}{(1 + |w(z)|^2)^{M+1}} = -2 \frac{M + 1}{2} \frac{|w'(z)|^2(1 + |w(z)|^2)^{M-1}}{(1 + |w(z)|^2)^{M+1}}.$$ (40)

Note that a factor $(1 + |w(z)|^2)^{M-1}$ has been multiplied both in the numerator and in the denominator. Then, the expansion of this factor immediately gives the relation in Eq. (29).

This kind of consideration suggests that we generalize Eq. (39) to

$$\partial_z \partial_{\bar{z}} \ln \prod_{k=1}^{n} \frac{1}{(c_k^2 + |w(z)|^2)^2} = -2 \sum_{k=1}^{n} \frac{|c_k^2| |w'(z)|^2}{(c_k^2 + |w(z)|^2)^2} \equiv -2 \prod_{k=1}^{n} \frac{|w'(z)|^2}{(c_k^2 + |w(z)|^2)^2} \sum_{k=0}^{2n-2} p_k |w(z)|^{2k},$$ (41)
where $c_k$’s and $p_k$’s are positive constants. Then, clearly,
\[
\exp^{u_k} = \frac{p_k|w'(z)|^2|w(z)|^{2k}}{\prod_{k=1}^{n}(c_k^2 + |w(z)|^2)^2} \tag{42}
\]
solves Eq. (28) with $M = 2n - 1$ and $w_k(z) = \sqrt{p_k} w^k(z)$. This is because the Laplacian of the logarithm of any (anti-)holomorphic function vanishes except at the zeros. Note that the number of flavors $M$ is determined by the number of terms in the last summation of Eq. (41) because the numerator in Eq. (42) should be the absolute value of a rational function. A further generalization similar to Eq. (40) is
\[
\partial_z \partial_{\bar{z}} \ln \prod_{k=1}^{n} \frac{1}{(c_k^2 + |w(z)|^2)^{n_k+1}} = -2 \sum_{k=1}^{n} \frac{n_k + 1}{2} \frac{c_k^2|w'(z)|^2(c_k^2 + |w(z)|^2)^{n_k-1}}{(c_k^2 + |w(z)|^2)^{n_k+1}}
\equiv -2 \sum_{k=0}^{M-1} \frac{|w'(z)|^2}{\prod_{k=1}^{n}(c_k^2 + |w(z)|^2)^{n_k+1}} \sum_{k=0}^{M-1} p_k |w(z)|^{2k}, \tag{43}
\]
where $M = \sum_k n_k + n - 1$. Then,
\[
\exp^{u_k} = \frac{p_k|w'(z)|^2|w(z)|^{2k}}{\prod_{k=1}^{n}(c_k^2 + |w(z)|^2)^{n_k+1}} \tag{44}
\]
solves the semilocal Liouville equations with $M$ flavors. Up to SU($M$) global transformations, this is the most general solutions we can construct with one rational function $w(z)$.

Equation (42) and its generalization, Eq. (44), are new families of solutions to the semilocal Liouville equation. Finally, multiplying the conformal factor by Eq. (21) gives solutions to the semilocal Popov equations, namely,
\[
|\phi_k|^2 = \frac{p_k(1 + |z|^2)^2|w'(z)|^2|w(z)|^{2k}}{\prod_{k=1}^{n}(c_k^2 + |w(z)|^2)^{n_k+1}}. \tag{45}
\]
Then, we can write the scalar fields and the gauge potential as
\[
\phi_k = \frac{\sqrt{p_k}(1 + |z|^2)|w'(z)| w^k(z)}{\prod_{k=1}^{n}(c_k^2 + |w(z)|^2)^{(n_k+1)/2}},
\]
\[
a_z = -i \frac{2}{\prod_{k=1}^{n}(c_k^2 + |w(z)|^2)^{n_k+1}} \sum_{k=1}^{n} \frac{n_k + 1}{2} \frac{w(z)w'(z)}{c_k^2 + |w(z)|^2}, \tag{46}
\]
where we chose a local gauge that is different from Eq. (33) for convenience.

So far, we obtained solutions that involve only one rational function on $S^2$. One can find solutions that depend on more than one rational functions, on which we briefly comment here. For this purpose, we need an identity which generalizes Eq. (39) [15, 19]:
\[
\partial_z \partial_{\bar{z}} \ln \left( \sum_{i=1}^{n} |f_i(z)|^2 \right) = \sum_{i<j} \frac{|f_{ij}(z)|^2}{(\sum_{i=1}^{n} |f_i(z)|^2)^2}, \tag{47}
\]
where \( f_i \)'s are arbitrary rational functions and \( f_{ij} = f_j f_j' - f_j f_i' \). Now, if we define \( u_{ij} \) as

\[
e^{u_{ij}} = \frac{|f_{ij}(z)|^2}{(\sum_{i=1}^n |f_i(z)|^2)^2},
\]

the \( u_{ij} \)'s satisfy

\[
\partial_z \partial_{\bar{z}} u_{ij} = -2 \sum_{i<j} e^{u_{ij}};
\]

hence, they provide solutions for \( M = n(n-1)/2 \). It is now obvious that a further generalization similar to Eqs. (40), (41) and (43) generates more solutions.

IV. CONCLUSION

In this paper, we have considered the semilocal Popov equations on \( S^2 \) with an arbitrary number of scalar fields. The equations naturally arise in certain Chern-Simons gauge theories on \( S^2 \). The gauge field couples to nonrelativistic matter fields having additional global symmetry, and an external uniform magnetic field is turned on. The energy of the system is minimized if the semilocal Popov equations are satisfied. As in our earlier work [7], we transformed the equations to semilocal Liouville equations. We found new exact solutions of those equations. From these and other known solutions, we constructed many families of exact solutions of the semilocal Popov equations with an arbitrary number of scalar fields. Among them, we identified \( CP^{M-1} \) lump configurations with winding numbers less than \( M \).

The solutions found here are not the most general ones. Also, there are many distinct families of solutions some of which have the same vortex numbers. It is, however, not clear whether they are smoothly connected to each other in the solution space. In this paper, we derived the semilocal Popov equations in the context of 2+1 dimensional Chern-Simons gauge theories. It would be interesting if we could find other relevant physical systems, such as the Yang-Mills instanton equations, which is the case for the Popov equations with a single scalar.

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