THE Y-PRODUCT

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Abstract. We present a topological construction that provides many examples of non-commutative Frobenius algebras that generalizes the well-known pair-of-pants. When applied to the solid torus, in conjunction with Crane-Yetter theory, we provide a topological proof of the Verlinde formula. We also apply the construction to a solid handlebody of higher genus, leading to a generalization of the Verlinde formula (not the higher genus Verlinde formula); in particular, we define a generalized $S$-matrix. Finally, we discuss the relation between our construction and Yetter’s construction of a handle as a Hopf algebra, and give a generalization.

1. Basic Constructions

Convention: Manifolds may be unorientable, in which case the opposite orientation $\overline{M}$ will just mean $M$ itself. We will work with smooth manifolds possibly with boundary and corners; the boundary will implicitly come with a collar neighborhood structure.

Let $Y$ be the cone over the discrete space of three points, or equivalently, three closed intervals with one endpoint from each identified:

\[ Y = \bigcup_{i=1}^{3} [0, 1] \cup [0, 1] \cup [0, 1] / 0 \sim 0 \]  

Definition 1.1. Let $M$ be a manifold with boundary $\partial M = N$. Let $Q$ be the double of $M$, i.e. $Q = M \cup_{N} \overline{M}$.

The $Y$-product on $Q$, denoted $Y_Q$, is the manifold constructed as follows. Thicken $Y$ into a surface with corners, denoted $\tilde{Y}$, as follows:

\[ \tilde{Y} := \bigcup_{i=1}^{3} \bigcup_{I_1, I_2} \bigcup_{I_3} \]  

Then glue three copies of $M \times [-1, 1]$ to $N \times \overline{Y}$ by identifying $N \times [-1, 1] \subset M \times [-1, 1]$ with $N \times I_1, N \times I_2, N \times I_3$, respectively. The resulting manifold with boundary (no corners) is the $Y$-product on $Q$.

It is clear that the six copies of $M$ at the ends of the three copies of $M \times [-1, 1]$ glue up to form three disjoint copies of $Q$. (See Figure 2 in Section 2.1)

Clearly, the usual pair of pants is obtained from $M = [0, 1], N = \partial M, Q = S^3$.

This is the basic construction of the $Y$-product; in our main application, we will consider variants where $N$ is not the entire boundary of $M$, so that $Q$ is a manifold with boundary. One may also consider embedded manifolds, say a knot $K$ in $S^3$, and take $Q = K \# \overline{K}$.

An intuitive description of $Y_Q$ is to hold $Q$ above still water, so that it and its reflection is $Q \cup Q$. Slowly lower $Q$ into the water, until the surface of the water cuts $Q$ at exactly $N$, so that we see just one $Q$. This intuitive picture is the key idea behind the half-handle decomposition of $Y_Q$ as described in Section 2.1 - each time the water passes a critical point, a corresponding handle is added to build $Y_Q$.

A $Y$-product is naturally a cobordism $Y_Q : Q \cup Q \to Q$. By taking the dual cobordism, we get a cobordism $\mathcal{K}_Q : Q \to Q \cup Q$, which we call the $Y$-coproduct.

They naturally come with (co)units: simply take $M \times [-1, 1]$ where we take $M \times \{1\} \cup N \times [-1, 1] \cup M \times \{1\}$ as the incoming or outgoing boundary. Alternatively, in fitting with the $Y$-product construction, we attach $M \times [-1, 1]$ to $N \times \overline{I}$ ($\overline{I}$ is defined in (1.2)) along $N \times [-1, 1]$. It is easy to see that attaching $\overline{I}$ to one of the upper arms of $\tilde{Y}$ results in the identity cobordism. (For more details, see discussions in Section 2.1 before and in the examples.) We denote the unit and counit by $I_Q : \emptyset \to Q$ and $I_Q : Q \to \emptyset$, respectively.
More generally, given a fat graph $\Gamma$ (a graph with a thickening to a surface like $\bar{Y}$), we can associate a construction that attaches $M \times I$’s to $N \times \Gamma$. Moreover, under certain modifications, e.g. resolving a 4-valent vertex into two 3-valent vertices, the resulting manifold will be unchanged; for example, the usual diagram for depicting the Frobenius algebra relations (shown below) would automatically make $Z(Q)$ a Frobenius algebra, for any appropriate TQFT $Z$ (here the graph thickenings are implicitly given by blackboard framing): 

![Diagram](attachment:image.png)

See Example 3.11 3.12

2. Handle Decompositions for Manifolds with Corners/ Relative Cobordisms

In order to get a handle on Y-products for manifolds $Q$ with boundary, we need a theory of handle decompositions on manifolds with corners, or more precisely, relative cobordisms.

It seems that such theory of handle decompositions/Morse theory is still not very well-known (at least, they were not known to the authors at the onset of this project). Thus, while there are already several works that lay out such a theory in full (see e.g. [BNR2016], [Lau2011]), we give a lightning tour through some of the theory, presenting only constructions and propositions (mostly without proof) that are relevant to our application. The only things new are some terminology.

Let us briefly recall the theory of handle decompositions; we will expand on this theory to include handle decompositions for manifolds with corners/relative cobordisms.

Let $D^k$ be the standard $k$-dimensional disk.

**Definition 2.1.** An abstract $n$-dimensional $k$-handle, is $H_k := D^k \times D^{n-k}$, with the following distinguished submanifolds:
- $\partial D^k \times D^{n-k}$: the attaching region;
- $D^k \times \{0\}$: the attaching sphere;
- $D^k \times \{0\}$: the core;
- $D^k \times \partial D^{n-k}$: the belt region;
- $\{0\} \times D^{n-k}$: the belt sphere;
- $\{0\} \times D^{n-k}$: the co-core.

**Definition 2.2.** Let $M$ be an $n$-manifold with (possibly empty) boundary $\partial M$. Let $\varphi : \partial D^k \times D^{n-k} \to \partial M$ be an embedding of the attaching region of an abstract $k$-handle into the boundary of $M$. The manifold $M' = M \cup D^k \times D^{n-k}/\varphi$ is said to be obtained from $M$ by attaching a $k$-handle. (We smooth corners at the boundary of the attaching region in a canonical manner.) We refer to the region corresponding to the abstract handle as the $(k)$-handle.

**Definition 2.3.** An elementary cobordism of index $k$ is a cobordism $M : \partial_* M \to \partial_* M$ that is obtained from the identity cobordism $\partial_* M \times [0,1]$ by attaching a $k$-handle to its outgoing boundary.

**Definition 2.4.** A handle decomposition of a cobordism $M : \partial_* M \to \partial_* M$ is a filtration of $M$

$$\partial_* M \times [0,1] =: M_0 \subset M_1 \subset \cdots \subset M_{l-1} \subset M_l := M$$

such that $M_{i+1}$ is obtained from $M_i$ by attaching a handle to the outgoing boundary of $M_i$.

By smoothing corners at the boundary of attaching regions, it is clear that a handle decomposition allows us to write any cobordism as a composition of elementary cobordisms.

**Definition 2.5.** Given a handle decomposition of a cobordism $M : \partial_* M \to \partial_* M$, the dual handle decomposition is the handle decomposition of $M : \partial_* M \to \partial_* M$ by simply treating each $k$-handle as a $(n-k)$-handle (where $n$ is the dimension of $M$).

Thus, a handle decomposition on $M : \emptyset \to N$ naturally endows a handle decomposition on its double $Q = M \cup_{N} \bar{M} : \emptyset \to N \to \emptyset$.

**Proposition 2.6.** Any cobordism admits a handle decomposition.
Now let us recall relative cobordisms, which are cobordisms between manifolds with boundary (see e.g. [BNR2016]). Note that in some texts, relative cobordisms are only defined between manifolds with boundary when the boundaries are the same, and the vertical wall (as defined below) is always the identity cobordism.

**Definition 2.7.** A relative cobordism from a manifold with boundary $M_{in}$ to another manifold with boundary $M_{out}$, denoted $(W, \partial W) : (M_{in}, \partial M_{in}) \rightarrow (M_{out}, \partial M_{out})$ is a manifold $W$ with corners $\partial^2 W, \partial_2^3 W$ which separate the boundary $\partial W$ into three submanifolds $\partial_1 W, \partial_2 W, \partial_3 W$, and identifications $M_{in} \cong \partial_3 W, M_{out} \cong \partial_1 W$: the three submanifolds satisfy:

- their union is $\partial W$,
- $\partial_2 W, \partial_3 W$ are disjoint,
- $\partial_2 W \cap \partial_3 W = \partial^2 W$,
- $\partial_1 W \cap \partial_3 W = \partial_3^2 W$.

We refer to $\partial_1 W, \partial_2 W,$ and $\partial_3 W$ as the incoming boundary, outgoing boundary, and vertical wall of $W$, respectively.

We call $\partial^2 W$ and $\partial_2^3 W$ the incoming corner and outgoing corner, respectively. Note that $\partial_2 W$ is a cobordism from the former to the latter.

If $W$ is oriented, we make the following choices for orientations:

- $\partial_1 W$ has the induced orientation,
- $\partial_2 W, \partial_3 W$ have the opposite induced orientation,
- $\partial^2 W, \partial_3^2 W$ have the induced orientation with respect to $\partial_1 W, \partial_2 W$, respectively.

See Figure 1.

**Figure 1.** Comparison of positive and negative type half-handles. The light gray regions are part of the vertical boundary, the dark gray regions are the attaching regions. Note that the vertical boundaries are the same.

We may sometimes implicitly identify $M_{in}$ with $\partial_2 W$ and $M_{out}$ with $\partial_3 W$.

**Definition 2.8.** Let $hD^k$ denote the half-disk, $hD^k = \{x = (x_1, \ldots, x_k) | |x| \leq 1, x_1 \geq 0\}$. Let $h\partial D^k$ denote the hemisphere portion of the boundary, $h\partial D^k = \{x = (x_1, \ldots, x_k) | |x| = 1, x_1 \geq 0\}$, and let $\partial_i hD^k$ denote the flat portion of the boundary, $\partial_i hD^k = \{x = (x_1, \ldots, x_k) | |x| \leq 1, x_1 = 0\}$.

Let $0 \leq k \leq n - 1$. An abstract $n$-dimensional $k^+$-half-handle is $hH^+_{k} := D^k \times hD^{n-k}$, with the following distinguished submanifolds:

- $\partial D^k \times hD^{n-k}$: the attaching region;
- $\partial D^k \times \{0\}$: the attaching sphere;
- $D^k \times \{0\}$: the core;
- $D^k \times h\partial D^{n-k}$: the belt region;
- $\{0\} \times h\partial D^{n-k}$: the belt disk;
- $\{0\} \times hD^{n-k}$: the co-core.

An abstract $n$-dimensional $k^-$-half-handle is $hH^-_{k} := hD^{k+1} \times D^{n-k-1}$, with the following distinguished submanifolds:

- $h\partial D^{k+1} \times D^{n-k-1}$: the attaching region;
- $h\partial D^{k+1} \times \{0\}$: the attaching disk;
Proof. Follows from the doubling argument and the argument for usual handle decompositions.

Definition 2.11. Given a handle decomposition of a relative cobordism \((W, \partial W) : (\partial W, \partial^2 W) \to (\partial W, \partial^2 W)\), the dual handle decomposition is the handle decomposition of \((W, \partial W) : (\partial W, \partial^2 W) \to (\partial W, \partial^2 W)\), by simply treating each \(k\)-handle as a \((n-k)\)-handle, and each \(k^*\)-half-handle as a \((n-1-k)^*\)-half-handle (where \(n\) is the dimension of \(W\)).

Proposition 2.12. Any relative cobordism admits a half-handle decomposition.

Proof. The idea is to double the relative cobordism along the vertical boundary, find a \(\mathbb{Z}/2\)-invariant Morse function, apply Morse theory to get a handle decomposition, and observe that we get a half-handle decomposition on the original relative cobordism. See [BNR2016, Section 2] for a complete proof.

Next, we consider the dual decomposition:

Proposition 2.13. Let \(M\) be a relative cobordism, and let \((M_i)_{i=0, \ldots, l}\) be a half-handle decomposition. Let \(M'\) be the relative cobordism with the same underlying manifold \(M\), but incoming and outgoing boundaries/corners are swapped.

Then there exists a half-handle decomposition \((M'_i)_{i=0, \ldots, l}\) of \(M'\) such that, if the \(i\)-th (half-)handle attachment for \(M\) is a \(k\)-handle/\(k^*\)-half-handle, then the \((l+1-i)\)-th (half-)handle attachment for \(M'\) is a \((n-k)\)-handle/\((n-1-k)^*\)-half-handle.

Proof. Follows from the doubling argument and the argument for usual handle decompositions.

\[\text{possibly with "trivial steps", which occur when a handle (not a half-handle) is attached.}\]
Before we move on to applications, we consider another variant of cobordisms, namely *cornered cobordisms*, (or *cobordism with corners* in [Yet199701]). These are also cobordisms between manifolds with boundary, but they are closer to the other notion of relative cobordism as mentioned above (when the boundary is fixed).

Cornered cobordisms were used in [Tha2021] to formulate the extended Crane-Yetter theory (see Definition 3.3 below). Note the use of a semicolon instead of a comma in (3.16), for the theory on cornered cobordisms (Definition 3.3) to define maps associated to relative cobordisms (see (3.16), Example 3.6, Figure 13).

Another application is to help formulate the following useful fact about negative type half-handles, which intuitively says that attaching negative type half-handles does not change the “bulk” of a relative cobordism:

**Lemma 2.15.** Let \( M \) be a relative cobordism obtained from attaching only negative type half-handles to an identity relative cobordism. Then, treating \( M \) as a cornered cobordism with corner \( \partial_2^c M \), \( M \) is an identity cornered cobordism.

**Proof:** This follows easily by dualizing (i.e. taking the dual decomposition of) the fact that attaching positive type half-handles does not change the diffeomorphism type of the relative cobordism (because, as stated before, the attaching region for positive type half-handles is a ball). One can also prove this directly by applying a deformation retract of a negative half-handle onto the vertical handle within it. \( \square \)

By considering dual handles, we see that attaching a positive half-handle can be interpreted as the identity cornered cobordism with \( \partial_2^c \) as corner.

### 2.1. Half-handle Decomposition for Y-product

Now let us consider the Y-product construction from the perspective of (half-)handle decompositions. Recall that the Y-product on \( Q = M \cup_N \overline{M} \), where \( M \) is a manifold with boundary, is obtained by gluing three copies of \( M \times [-1, 1] \) to \( N \times \overline{Y} \) (see (1.2)). Following that construction, if we only glue on two copies of \( M \times [-1, 1] \), say to \( N \times I_1 \) and \( N \times I_2 \), the resulting manifold has two boundary components, \( M \cup_N N \times I_3 \cup_N \overline{M} \) and \( M \cup_N \overline{M} \); it is clearly the identity relative cobordism on \( Q \). Thus, most of the interesting topology happens in \( M \times I_3 \), here treated as a relative cobordism from \( \overline{M} \cup M \) to \( N \times I \). See Figure 2 below.

Suppose we are given some half-handle decomposition of \( M \) as a relative cobordism from \( \emptyset \) to \( N \); let the handles, in order of attachment, be \( H_1, H_2, \ldots, H_l \). Lay out \( M \times I_3 \) like a “folding fan”: if \( M \) is embedded in some half-space \( \mathbb{R}_{x_0} \times \mathbb{R}^l \) away from the boundary, then sweeping the half-space through an extra dimension will trace out

\[
M \times I_3 \simeq \{ (\cos \theta \cdot x, y, \sin \theta \cdot x) | (x, y) \in M \} \subset \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}
\]

Then it is clear that \( H_1 \times I_3, H_2 \times I_3, \ldots, H_l \times I_3 \) give a half-handle decomposition of \( M \times I_3 \), where each \( H_i \times I_3 \) is exactly one index higher than \( H_i \).

A half-handle decomposition for the counit can be obtained by similar methods. This time, we think of \( M \) as a relative cobordism from \( N \) to \( \emptyset \), and properly embed \( M \) in \( \mathbb{R}_{x_0} \times \mathbb{R}^l \), with \( N \) in \( \{0\} \times \mathbb{R}^l \). By sweeping through an extra dimension as before, we will get the counit relative cobordism, and each (half-)handle sweeps out a (half-)handle.
2.2. Solid tori. We describe two examples of Y-products, both are Y-products on the solid torus; these will serve as the topological basis of our main result.

Before that, let us fix some notation/conventions - the large number of automorphisms of the solid torus makes it necessary to fix particular choices of curves/bases to avoid confusion; for example, the seemingly innocuous automorphism of “flipping a donut” actually implements the “charge conjugation” operation on the Verlinde algebra (see (3.17)).

Let $X$ be the solid torus $X = S^1 \times D^2$. Choose $\bar{l} := S^1 \times \{\ast\}$, $\bar{m} := \{\ast\} \times \partial D^2$ to be the longitude and meridian, respectively. Orientations are chosen so that $(\bar{n}, \bar{l}, \bar{m})$ is the orientation of $X$, where $\bar{n}$ is the outward normal vector at the boundary.

Given a diagram of another solid torus, we may represent an identification of that solid torus with $X$ by drawing and labeling two simple closed curves on that solid torus’s boundary, indicating which curve goes to $\bar{l}$ and $\bar{m}$.

**Example 2.16.** Define $X'$ as $X$ after applying surgery along the core circle with 0-framing (trivial with respect to the product structure of $X$, or equivalently, $\bar{l}$ provides the framing); by definition, there is a cornered cobordism $\mathcal{H}_2 : X \to X'$, corresponding to the attachment of a 2-handle to the core circle.

Now $X'$ is clearly another solid torus. We define $\Phi : X' \simeq X$ to be the diffeomorphism that sends $\bar{m}$ and $\bar{l}$ (as curves in $\partial X' = \partial X$) to $\bar{l}$ and $-\bar{m}$ respectively. Similarly, we define $\overline{\Phi} : X' \simeq X$ to be the diffeomorphism that sends $\bar{m}$ and $\bar{l}$ to $-\bar{l}$ and $\bar{m}$ respectively.

We denote the cornered cobordisms resulting from composing $\Phi, \overline{\Phi}$ with $\mathcal{H}_2$ by $\Psi := \Phi \circ \mathcal{H}_2 : X \to X$, $\overline{\Psi} := \overline{\Phi} \circ \mathcal{H}_2 : X \to X$ respectively.

Note that $\overline{\Psi}$ is the dual cobordism of $\Psi$, and vice versa (with appropriate choice of identification $\overline{X} \simeq X$).

If we identify the boundary $\partial X$ with $\mathbb{R}^2$, such that $\bar{l} = (1, 0)^T$ and $\bar{m} = (0, 1)^T$, then $\Psi|_{\partial X}$ can be identified with a clockwise $\pi/2$ rotation, which has two fixed points $(0,0)$ and $(1/2,1/2)$, and swaps $(1/2,0)$ and $(0,1/2)$. Similarly, $\overline{\Psi}|_{\partial X}$ can be identified with an anti-clockwise $\pi/2$ rotation. This will be useful for extending $\Psi$ to higher genus solid handlebodies (see Section 2.3). \hfill \triangle

Now let us describe the two Y-products on $X$.

**Example 2.17 (Solid torus I).** Consider $M = D^2 \times [0, 1]$ as a relative cobordism

$$(D^2 \times [0, 1], \partial D^2 \times [0, 1]) : (\varphi, \psi) \mapsto (D^2 \times \{0, 1\}, \partial D^2 \times \{0, 1\})$$

which is half-way to building $X$; we give it the following half-handle decomposition: start with the $0^\ast$-half-handle, followed by a $1^\ast$-half-handle, and finally a 2-handle (see Figure 3).

To be more specific, the $0^\ast$-half-handle starts at the $(1/4,0)$ point on $\partial X$. Then the $1^\ast$-half-handle grows out of the $0^\ast$-half-handle and closes up into a meridian. Finally, the 2-handle fills in the meridian.

\footnote{Note that the last two steps can in fact be replaced with a single $1^\ast$-half-handle; we choose this half-handle decomposition for use in computations later.}
This should define a Y-product on $X$, which we denote $Y^{(1)}_X$, with the corresponding half-handle decomposition on the Y-product on the solid torus $X$ as shown in Figure 4 below.

Here's a more wordy description of the half-handle decomposition of $Y^{(1)}_X$, in terms of a surgery picture, i.e., we keep track of just what happens to the 3-manifold as we add half-handles. Start with $X \sqcup X$. The $1^-$-half-handle cuts out two half 3-disks $hD^3$ from the boundaries of $X$’s (to be precise 3, we take out a
$hD^3$ around the $(1/4, 0)$ point from the left $X$, and a $hD^3$ around the $(3/4, 0)$ point from the right $X$, and attaches a solid cylinder. The boundary of this new 3-manifold $M_1$ is a genus two surface.

Next, the attaching sphere of the $2^-$-half-handle is a circle on the boundary of $M_1$ (the genus two surface) that travels along the meridians of the $X$’s and goes back and forth along the solid cylinder just attached. A neighborhood of the attaching sphere is removed, and replaced with a thickened 2-disk, obtaining $M$.

In terms of the boundary, $\partial M_2$ is now already a torus. Indeed, just focusing on what happens to the boundary surfaces, we see that we have performed a Y-product on the torus. However, there is one more handle, a 3-handle corresponding to the 2-handle of $M$. What is the attaching sphere of this 3-handle? Observe that the attaching sphere of the previous $2^-$-half-handle also bounds a 2-disk in $M_1$ (it goes through the solid cylinder). The attaching sphere of the 3-handle is the union of this 2-disk and the core of the $2^-$-half-handle.

Another way to view this Y-product is that it is simply the usual pair of pants times $D^2$. This is apparent from the construction as described in Section 1. In particular, we see that the vertical boundary is the usual pair of pants times $S^1$.

A half-handle decomposition for the counit for the corresponding Y-coproduct is given in Figure 5.

![Figure 5](image)

**Figure 5.** Half-handle decomposition for counit for $Y^{(1)}_X$. The $2^-$-half-handle corresponds to the $1^-$-half-handle in the bottom row in Figure 4 and the $3^-$-half-handle, here shown to be broken into a $3^+$-half-handle followed by a 4-handle, corresponds to the final $2^+$-half-handle in Figure 3.

**Example 2.18 (Solid torus II).** Now let us consider the other way to have the solid torus be the double of a manifold with corners. Slicing the solid torus like a bagel, we see that it is the double of $hD^2 \times S^1$ (with corner $\{-1, 1\} \times S^1$, where $\{-1, 1\}$ are the corners of $hD^2$), so that the two halves glue along an annulus.

This half-handle decomposition is simpler than the previous one. Indeed, it is clear from Figure 6 that just taking the $0^-$- and $1^-$-half-handle gives us a half-handle decomposition of $hD^2 \times S^1$. Again, to be precise, the $0^-$-half-handle starts at the $(0, 3/4)$ point on $\partial X$. Then the $1^-$-half-handle grows out of the $0^-$-half-handle and closes up into a longitude.

![Figure 6](image)

**Figure 6.** Half-handle decomposition for $X$, for $Y^{(2)}_X$. Note that in the third diagram, the attaching region of the $1^+$-half-handle is the shaded region, the outgoing boundary is made of the two half-disks facing each other, and the vertical boundary is the top curved surface.

The corresponding half-handle decomposition for $Y^{(2)}_X$ is shown in Figure 7. In terms of surgery on 3-manifolds, for the $1^+$-half-handle, we take out a $hD^3$ around the $(0, 3/4)$ point from the left $X$, and a $hD^3$ around the $(0, 1/4)$ point from the right $X$, and attaches a solid cylinder. Next, the attaching sphere of $\Psi$ and the Y-products; see Proposition 2.20. It becomes particularly important when discussing higher genus handlebodies in Section 2.3 as the Y-products are no longer symmetric.

\[\text{(Footnote 3)}\] This may seem overly pedantic, as the boundary of $X$ is so symmetric as to have no special points, but these choices matter in the interaction between $\Psi/\Psi$ and the Y-products; see Proposition 2.20. It becomes particularly important when discussing higher genus handlebodies in Section 2.3 as the Y-products are no longer symmetric.
the 2*-half-handle is a circle on the boundary that travels along the longitudes of the $X$’s and goes back and forth along the solid cylinder just attached. A neighborhood of the attaching sphere is removed, and replaced with a thickened 2-disk.

\[ \mathbf{Y}_X^{(2)} : X \sqcup X = 1^*\text{-hf-hdl} \rightarrow 2^*\text{-hf-hdl} \rightarrow X \]

**Figure 7.** $\mathbf{Y}_X^{(2)}$ half-handle decomposition

The resulting Y-product is different from the previous example. Indeed, we may view the solid torus as $I \times \text{Ann}$. Then this Y-product is $(Y$-product on $I$) times $\text{Ann}$. The Y-product on $I$ is homotopy equivalent to a point, and thus this Y-product $\mathbf{Y}_X^{(2)}$ is homotopy equivalent to a circle, whereas the previous Y-product $\mathbf{Y}_X^{(1)}$ is homotopy equivalent to a usual pair of pants.

Note that the vertical boundary of this Y-product is also diffeomorphic to the usual pair of pants times $S^1$.

A half-handle decomposition for the counit for the corresponding Y-coproduct is given in Figure 8 below.

\[ \xrightarrow{2^*\text{-hf-hdl}} D_3 \xrightarrow{3^*\text{-hf-hdl}} \emptyset \]

**Figure 8.** Half-handle decomposition for counit for $\mathbf{Y}_X^{(2)}$. The $2^*$- and $3^*$-half-handles correspond to the $1^+$- and $2^*$-half-handles in Figure 6.

**Remark** 2.19. We note that, as one might expect, the two half-handle decompositions for $X$ can be related by handle cancellations and sliding. As mentioned before, the $1^*$-half-handle and 2-handle in the first example can be merged/canceled to get a $1^*$-half-handle, so that the overall half-handle decomposition of the solid torus consists of half-handles of index $0^*, 1^*, 1^*, 2^*$ in that order. In the second example, the half-handle decomposition consists of half-handles of index $0^*, 1^*, 1^*, 2^*$ in that order. It is easy to check that the $1^*$- and $1^*$-half-handles can slide off each other and be swapped in the order of handle attachments. But this changes the resulting Y-products because the manifold obtain from the first half of the handles is changed.

**Proposition 2.20.** As relative cobordisms $X \sqcup X \rightarrow X$,

\[
\begin{align*}
\mathbf{Y}_X^{(1)} \circ (\Psi \sqcup \overline{\Psi}) &= \overline{\Psi} \circ \mathbf{Y}_X^{(2)} \\
\mathbf{Y}_X^{(1)} \circ (\Psi \sqcup \overline{\Psi}) &= \Psi \circ \mathbf{Y}_X^{(2)} \circ P
\end{align*}
\]

where $P$ swaps the two copies of $X$. 

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Proof. Figure 9 proves the first equality. They are equal as relative cobordisms by a simple 2-3 handle cancellation. We just note that in the left diagram, the attaching region for the 1− and 2−-half-handles of $Y^{(1)}_X$ looks like that of $Y^{(2)}_X$ because of $\Psi$.

The second equality needs an extra $P$ because $\Psi$ turns $\partial X$ the other way compared to $\overline{\Psi}$, so the diagram for $Y^{(1)}_X \circ (\Psi \cup \Psi)$ would look like the left diagram in Figure 9 but the second $X$ would be stacked on top of the first.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{$Y^{(1)}_X \circ (\overline{\Psi} \cup \overline{\Psi}) = \overline{\Psi} \circ Y^{(2)}_X$}
\end{figure}

Proposition 2.21. Let $K$ be the cornered cobordism $K : X \to X$ given by attaching two 2-handles as in Figure 10. Then as relative cobordisms $X \cup X \to X$,

\begin{align*}
Y^{(2)}_X \circ (\Psi \cup \Psi) &= \Psi \circ Y^{(1)}_X \circ (K \cup \mathrm{id}_X) = \Psi \circ Y^{(1)}_X \circ (\mathrm{id}_X \cup K) \\
Y^{(2)}_X \circ (\overline{\Psi} \cup \overline{\Psi}) \circ P &= \overline{\Psi} \circ Y^{(1)}_X \circ (K \cup \mathrm{id}_X) = \overline{\Psi} \circ Y^{(1)}_X \circ (\mathrm{id}_X \cup K)
\end{align*}

Proof. In Figures 11, 12 half-handles are only added in the second step (at $Y^{(1)}_X, Y^{(2)}_X$), and the same half-handles are added in both figures. Thus, from the final diagrams, it is clear that performing an extra $\Psi$ in Figure 11 results in Figure 12.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure10.png}
\caption{Cornered cobordism $K : X \to X$; as a 4-manifold, $K$ is $D^2 \times S^2$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=1\textwidth]{figure11.png}
\caption{$Y^{(1)}_X \circ (K \cup \mathrm{id}_X)$. In the penultimate equality, the 2- and 3-handles cancel.}
\end{figure}
Figure 12. $Y_X^{(2)} \circ (\Psi \cup \Psi)$. In the last equality, the 1-handle is canceled by one of the 2-handles.

We also consider similar results for the coproducts, which follow simply from dualizing the picture:

**Proposition 2.22.** As relative cobordisms $X \to X \cup X$,

$$(\Psi \cup \Psi) \circ \mathcal{A}^{(1)}_X = \mathcal{A}^{(2)}_X \circ \Psi$$

$$(\overline{\Psi} \cup \overline{\Psi}) \circ \mathcal{A}^{(1)}_X = P \circ \mathcal{A}^{(2)}_X \circ \overline{\Psi}$$

where $P$ swaps the two copies of $X$.

**Proposition 2.23.** Let $K$ be the cornered cobordism $K : X \to X$ given by attaching two 2-handles as in Figure 10. Then as relative cobordisms $X \cup X \to X$,

$$(\overline{\Psi} \cup \overline{\Psi}) \circ \mathcal{A}^{(2)}_X = (K \cup \text{id}_X) \circ \mathcal{A}^{(1)}_X \circ \overline{\Psi} = (\text{id}_X \cup K) \circ \mathcal{A}^{(1)}_X \circ \overline{\Psi}$$

where $P$ swaps the two copies of $X$.

**Remark 2.24.** Proposition 2.21 essentially formally follows from Proposition 2.22 (which in turn essentially formally follows from Proposition 2.20). Represent the first equation in Proposition 2.22 as follows:

(2.1) $\Psi \cup \Psi = \Psi$

Then, we have

(2.2) $\Psi \circ \Psi = \Psi$

**2.3. Solid handlebodies.** We can generalize the discussion above to handlebodies of higher genus. We fix a specific genus $g$ solid handlebody, denoted $X_g$, constructed from $g$ copies of $X$, with a 1-handle joining the $(1/2, 1/2)$ point on the $i$-th $\partial X$ to the $(0, 0)$ point on the $(i+1)$-st $\partial X$. This is a little awkward to draw (see Figure 17) but the point is that $\Psi$ naturally extends to $X_g$.

Indeed, applying $\Psi$ to the $i$-th $X$ is compatible with applying $\overline{\Psi}$ to the $(i+1)$-st $X$. We thus define $\Psi_g$ to be the cornered cobordism that results from alternately applying $\Psi$ and $\overline{\Psi}$ to the copies of $X$ in $X_g$ (starting with $\Psi$ on the first copy); we define $\overline{\Psi}_g$ to be the same except it starts with $\overline{\Psi}$ on the first copy of $X$.

For application in Example 3.17, we also consider slight variants $\Psi'_g$, $\overline{\Psi}'_g$ of $\Psi_g$, $\overline{\Psi}_g$. The only alteration made is near the $(0, 0)$ point of the boundary of the first solid torus, where the identifications $\Phi$, $\overline{\Phi} : X' \simeq X$ are twisted so that they also fix a neighborhood of $(0, 0)$. (In the diagram below, the segments labeled $\tilde{m}, \tilde{l}$ in the right diagram are the images of $\tilde{m}, \tilde{l}$ in the left diagram.)

(2.3) $\Phi' = \Phi$ except in a neighborhood of $(0, 0)$:

\[ \tilde{m}, \tilde{l} \rightarrow \]
Skeins, Witten-Reshetikhin-Turaev.

3.1. generalization, the HOMFLY-PT polynomial was constructed \[FYH\] explosion of activity on skeins, in particular, Kauffman found a skein formulation \[Kau1987\], and a common attention by Conway \[Con1970\]. After Jones’s discovery of his polynomial invariant \[Jon1985\], there was an itly, since Alexander introduced his polynomial invariant of knots and links \[Ale1928\], and brought to greater

2.3), in particular we derive a generalized Verlinde formula.

Follows easily from the solid torus case (Proposition 2.20).

Proof. Follows easily from the solid torus case (Proposition 2.20). \[\Box\]

3. VERLINDE FORMULA

The main application we have for this construction is a topological explanation, via Crane-Yetter theory, for the Verlinde formula \[Ver1988\], as stated in Proposition 5.3, in the context of premodular categories.

As is well known, the skein module of a solid torus is the Verlinde algebra. We prove that the Y-products considered in Section 2.2 give rise to the fusion and convolution products on the Verlinde algebra, and the cornered cobordism \(\Psi\) gives rise to the S-matrix. Thus, Proposition 2.20 gives a topological proof of the Verlinde formula.

We discuss implications of applying the same techniques to the higher genus solid handlebodies (Section 3.3), in particular we derive a generalized S-matrix.

See the discussion in \[mathoverflowVerlinde\] on the Verlinde formula.

3.1. Skeins, Witten-Reshetikhin-Turaev. Skeins, or skein relations, have been around, at least implicitly, since Alexander introduced his polynomial invariant of knots and links \[Ale1928\], and brought to greater attention by Conway \[Con1970\]. After Jones’s discovery of his polynomial invariant \[Jon1985\], there was an explosion of activity on skeins, in particular, Kauffman found a skein formulation \[Kau1987\], and a common generalization, the HOMFLY-PT polynomial was constructed \[FYH\] \[PT1988\]. Skeins are, in a sense, 1-chains with non-abelian coefficients. A skein in a 3-manifold \(M\) is an embedded graph with extra data, known as a “coloring”, attached to its edges and vertices. More generally, a skein is a linear combination of such graphs. There should be a well-defined notion of the evaluation of a skein in a ball; two skeins are considered equivalent if they agree outside a ball and have the same evaluation in the ball. In general, the graphs are allowed to intersect the boundary of \(M\) transversely at a set of points

\[
\Phi' = \Phi \text{ except in a neighborhood of } (0,0) :
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics{image}
\end{array}
\end{array}
\]

\[
(2.4)
\]
V, known as a “boundary value”, which inherit the coloring from the graph. The skein module of M with boundary value V, denoted Skein(M; V), is the space of skeins with a fixed boundary value V.

From here on, we will assume that the reader is familiar with the basics of skein theory. We give a quick summary of the notations and conventions about premodular categories, as presented in [Tha2020] that are relevant for our applications in this paper; see also [KJ2011].

We fix a semisimple premodular (i.e. ribbon fusion) category A over an algebraically closed field k. Here are some notation:

- 1: unit simple object;
- X*: the (left) dual of X; we suppress associativity, unit, and pivotal morphisms X \cong X**;
- X_i for i \in \text{Irr}(A): representative for each isomorphism class of simple object, such that X_{i^*} = X_i^*;
- d_i: categorical dimension of X_i;
- √d_i: choice of square root such that \sqrt{d_1} = 1 and \sqrt{d_i} = \sqrt{d_i^*};
- D = \sum_{i \in \text{Irr}(A)} d_i^2: the global dimension; D \neq 0 by [ENO2003].
- c_{X,Y}: braiding of X strand over Y strand:

\[ c_{X,Y} = \begin{array}{l}
\text{\includegraphics[width=0.1\textwidth]{braiding}}
\end{array} \]

- \theta_X: the twist operator, given by a left-hand twist:

\[ \theta_X = \begin{array}{l}
\text{\includegraphics[width=0.1\textwidth]{twist}}
\end{array} \]

- s_{ij} = \text{tr} c_{j,i^*} \circ c_{i^*,j^*}: the S-matrix; we also define \overline{s}_{ij} = \text{tr} c_{i^*,j^*} \circ c_{j,i^*} = s_{i^*j}, which we refer to as the \overline{S}-matrix; equivalently,

\[ i \begin{array}{l}
\text{\includegraphics[width=0.1\textwidth]{S-matrix}}
\end{array} = s_{ij} \cdot \text{id}_{X_j}; \quad j \begin{array}{l}
\text{\includegraphics[width=0.1\textwidth]{S-matrix}}
\end{array} = \overline{s}_{ij} \cdot \text{id}_{X_j} \]

(Note \ s_{ij} as defined here is denoted \ s_{ij} in [BK2001]).

Concatenation of objects will mean tensor product; this will always be the “standard” one, e.g. the tensor product in A or tensor product of vector spaces. Most sums over simple objects, typically indexed with a lower case Latin alphabet (usually i, j, k) will implicitly be over \text{Irr}(A).

We define the functor (\cdot): A^{\otimes n} \rightarrow \text{Vec} by

\[ \langle V_1, \ldots, V_n \rangle = \text{Hom}_A(1, V_1 \cdots V_n). \]

The pivotal structure gives functorial isomorphisms

\[ z: \langle V_1, \ldots, V_n \rangle \cong \langle V_n, V_1, \ldots, V_{n-1} \rangle \]

such that zn = 1d, so up to canonical isomorphism, \langle V_1, \ldots, V_n \rangle only depends on the cyclic order of V1, \ldots, Vn.

There is a non-degenerate pairing ev: \langle V_1, \ldots, V_n \rangle \otimes \langle V_1^*, \ldots, V_n^* \rangle \rightarrow k obtained by post-composing with evaluation maps. When two nodes in a graph are labeled by the same Greek letter, say α, it stands for a summation over a pair of dual bases:

\[ \varphi_\alpha \in \langle V_1, \ldots, V_n \rangle, \varphi^\alpha \in \langle V_1^*, \ldots, V_n^* \rangle \]

where \phi_\alpha \in \langle V_1, \ldots, V_n \rangle, \phi^\alpha \in \langle V_1^*, \ldots, V_n^* \rangle are dual bases with respect to the pairing ev.

A dashed line stands for the regular coloring, i.e. the sum of all colorings by simple objects i, each taken with coefficient d_i:

\[ \begin{array}{l}
\text{\includegraphics[width=0.2\textwidth]{regular_coloring}}
\end{array} = \sum_{i \in \text{Irr}(A)} d_i \cdot i \]

An oriented edge labeled X is the same as the oppositely oriented edge labeled X*.
Note that although we will be discussing various pivotal multifusion categories, the morphisms are described in terms of morphisms in \( \mathcal{A} \), in particular all morphisms depicted graphically are of morphisms in \( \mathcal{A} \), unless specified otherwise.

Finally, we record some facts and lemmas that are useful for computations. We refer readers to [KJ2011] for proofs.

\[
\begin{align*}
V_1 & \cdots V_n \\
\begin{array}{c}
\vdots \\
\end{array} & = V_1 \cdots V_n
\end{align*}
\]

(3.8)

\[
V_1 \cdots V_n = \frac{\text{dim}(\text{Hom}(X_k, V_1 \otimes \cdots \otimes V_n))}{d_k} X_k
\]

(3.9)

where the shaded region can contain anything.

(3.11) \[ \frac{1}{D} \left\{ \begin{array}{c} \gamma_i \\
\gamma_i \end{array} \right\} = \delta_{i,1} \text{id}_{X_i} \quad \text{(when } \mathcal{A} \text{ is modular)} \]

More generally, from [Müg2003], when \( \mathcal{A} \) is just premodular, if \( J \subset \text{Irr}(\mathcal{A}) \) is the subset of transparent simple objects, then

\[
\frac{1}{D} \left\{ \begin{array}{c} \gamma_i \\
\gamma_i \end{array} \right\} = \begin{cases} 
\text{id}_{X_i} & \text{if } i \in J \\
0 & \text{else}
\end{cases}
\]

(3.12)

The Verlinde algebra is the Grothendieck ring \( V = K(\mathcal{A}) \otimes \mathbb{Z} \). It has a natural basis \( x_i := [X_i] \). The multiplication is given by the fusion rules:

\[
x_i x_j = \sum_{k \in \text{Irr}(\mathcal{A})} N_{ij}^k x_k
\]

(3.13)

where \( N_{ij}^k \) is the multiplicity of \( X_k \) in the direct sum decomposition of \( X_i \otimes X_j \). We refer to this as the fusion product.

We can define another algebra structure \( * \) on \( V \), which we refer to as the convolution product, given by:

\[
x_i * x_j = (\delta_{ij}/d_i) \cdot x_i
\]

(3.14)

The S-matrix diagonalizes the fusion rules (see [Müg2003] Lemma 2.4 iii], [BK2001]):

**Proposition 3.1 (Verlinde Formula).** Let \( s: V \simeq V \) be the linear operator that, in the basis \( \{x_i\} \), is given by the S-matrix; that is,

\[
s(x_i) = \sum_j s_{ij} x_j
\]

Then

\[
s(xy) = s(x) * s(y)
\]
3.2. Crane-Yetter. In the second author’s PhD thesis [Tha2021], we developed an extended Crane-Yetter theory in terms of skeins.

**Definition 3.2.** The state spaces associated to a 3-manifold with boundary $M$, depending on choice of boundary value $V$ on $\partial M$, is defined to be the skein module $Z_{CY}(M; V) = \text{Skein}(M; V)$. \hfill $\triangle$

Before we give the value of Crane-Yetter on a cornered cobordism, let us briefly discuss the relation to WRT. For this discussion $\mathcal{A}$ is modular. We showed in [Tha2021] that, as widely expected (see e.g. [BFMGI2007]), WRT theory is a boundary theory of the Crane-Yetter theory, which we formulate as follows. Given a cobordism between closed 2-manifolds $W, M$, WRT assigns vector spaces $Z_{WRT}(W; M, \varphi)$, $Z_{WRT}(M; \varphi)$ and a linear map between them

\begin{align}
Z_{WRT}(M, \varphi) : Z_{WRT}(M; V_-, \varphi) & \to Z_{WRT}(M; V_+) \\
\end{align}

(3.15)

Now given a relative cobordism between 3-manifolds with boundary which restricts to $M : N_\rightarrow N_+$ on the vertical boundary, say $(W, M) : (M_\rightarrow N_\rightarrow (M_+, N_+)$, Crane-Yetter theory gives us a map

\begin{align}
Z_{CY}(W, M, \varphi) : Z_{CY}(M; V_-) & \to Z_{CY}(M; V_+) \\
\varphi_- & \to Z_{CY}(W; N_+)(\varphi_- \cup \varphi) \\
\end{align}

(3.16)

that is, we glue a skein $\varphi_-$ in $M_-$ to $\varphi$ along the boundary value $V_-$, treat it as a skein in $M_\cup N_-$, $M$, and apply the Crane-Yetter map associated to the 4-manifold $W$ with corner $N_+$ (see Figure 13 below).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13.png}
\caption{$Z_{WRT}$ in terms of $Z_{CY}$}
\end{figure}

We showed in [Tha2021] that Equations 3.15 and 3.16 agree up to a scalar depending only on $\mathcal{A}$ and the signature $\sigma$ and Euler characteristic $\chi$ of $W$: $Z_{CY}(W) = \kappa^{\sigma(W)}\chi(W)/2 Z_{WRT}(\partial W)$. \hfill $\square$

We define the value of $Z_{CY}$ on a cornered cobordism $(W, N)$ by first defining it on handles, or more precisely elementary (cornered) cobordisms, and then, by decomposing $(W; N)$ into elementary ones, we define $Z_{CY}(W; N)$ to be the composition; we then show that the result is independent of the choice of decomposition. We will only need to compute cobordisms with handles of index 2 and 3, but we the value of $Z_{CY}$ on handles of all indices for completeness:

**Definition 3.3.** Let $W : M \rightarrow N M'$ be an elementary cornered cobordism of index $k$, that is, an identity cornered cobordism on $M$ composed with a $k$-handle attachment $\mathcal{H}_k$. For a boundary value $V \in Z_{CY}(N)$, we define

\begin{align}
Z_{CY}(W; N) : Z_{CY}(M; V) & \to Z_{CY}(M'; V) \\
\end{align}

case-by-case: given a colored ribbon graph $\Gamma \in Z_{CY}(M; V)$, $Z_{CY}(W; N)(\Gamma)$ is constructed as follows:

- $k = 0$: $\mathcal{H}_0$ adds a new $S^3$ component to $M$; we define

\begin{align}
Z_{CY}(W; N)(\Gamma) := D \cdot \Gamma \cup \emptyset_{S^3} \\
\end{align}

where $\emptyset_{S^3}$ is the empty skein in $S^3$.

\footnote{Note here $Z_{CY}$ is $Z_{CY}^k$ in [Tha2021].}
• \(k = 4\): \(\mathcal{H}_4\) kills off an \(S^3\) component of \(M\); writing \(\Gamma = \Gamma' \cup \Gamma''\) with \(\Gamma'\) in the \(S^3\) component and \(\Gamma''\) in the other component of \(M\),

\[ Z_{\text{CY}}(W; \mathcal{N})(\Gamma) = Z_{\text{WRT}}(\Gamma') \cdot \Gamma'' \]

• \(k = 1\): the attaching region of \(\mathcal{H}_1\) is a pair of balls; by an isotopy, we may arrange that \(\Gamma\) is disjoint from the attaching region, and regard \(\Gamma\) as a graph in \(M'\), and define

\[ Z_{\text{CY}}(W; \mathcal{N})(\Gamma) := \mathcal{O}^{-1} \cdot \Gamma \]

• \(k = 2\): similar to the \(k = 1\) case, the underlying graph being \(X\) in \(\mathcal{H}_2\) (and its neighborhood) defines an embedding of the solid torus \(\mathcal{D}^2 \times \partial \mathcal{D}^2 \to M'\). Let \(\gamma = [-\varepsilon, \varepsilon] \times \partial \mathcal{D}^2 \subset \mathcal{D}^2 \times \partial \mathcal{D}^2\) be the core of the solid torus, trivially framed, and let \(\Gamma' = (\gamma, \text{regular})\) be the colored ribbon graph obtained by applying the regular coloring to \(\gamma\). Then, sending \(\Gamma'\) to \(M'\) under the embedding,

\[ Z_{\text{CY}}(W; \mathcal{N})(\Gamma) = \Gamma \cup \Gamma' \]

• \(k = 3\): first suppose that \(\Gamma\) intersects the co-core of \(\mathcal{H}_3\) transversely and in exactly one ribbon, and suppose the label of this ribbon is a simple object \(i\). If \(i = 1\), we may ignore this ribbon and isotope \(\Gamma\) to be disjoint from \(\mathcal{H}_3\); then we may define

\[ Z_{\text{CY}}(W; \mathcal{N})(\Gamma) = \delta_{i,1} \cdot \Gamma \]

In general, by isotopy, we may arrange \(\Gamma\) to be transverse to the co-core of \(\mathcal{H}_3\), and then apply (3.8) to get \(\Gamma = \sum_j \Gamma_j\), where each \(\Gamma_j\) satisfies the previous assumptions. Then we extend to this case by linearity.

\(\triangle\)

**Proposition 3.4** ([Tha2021 Prop. 5.13]). Let \(W : M \rightarrow \mathcal{N} M'\) be a cornered cobordism, and let \(W = W_1 \circ \cdots \circ W_1\) be a decomposition of \(W\) into a composition of elementary cornered cobordisms. We define

\[ Z_{\text{CY}}(W; \mathcal{N}) = Z_{\text{CY}}(W_1; \mathcal{N}) \circ \cdots \circ Z_{\text{CY}}(W_1; \mathcal{N}) \]

Then \(Z_{\text{CY}}(W; \mathcal{N})\) is independent of handle decomposition of \(W\).

The proof boils down to checking that modifying the decomposition (into elementary cobordisms) by handle slides and handle pair creation/annihilation does not affect \(Z_{\text{CY}}(W; \mathcal{N})\), and is a simple exercise in skein theory. Note that in [Tha2021] we work in PL topology, but since we work in low dimensions, it is the same.

### 3.3. Solid tori.
Recall that we let \(X = \mathcal{D}^2 \times S^1\) denote the solid torus, Let \(C\) denote the core circle \(\{0\} \times S^1\) in \(X\), framed trivially with respect to \(X\) as a product (i.e. the normal vector is a fixed vector \(v\) in every slice \(\mathcal{D}^2 \times \{\theta\}\)), and fixed the orientation on \(C\) agreeing with \(\hat{\ell}\).

The skein module of \(X\) with empty boundary condition has basis \(\{\varphi_i\}_{i \in \text{Irr}(\mathcal{A})}\), where \(\varphi_i\) is the skein with underlying graph being \(C\) and labeled by the simple object \(X_i\) (this is stated more generally in Lemma 3.13 see e.g. [Tha2021 Eqn. 9.15]). Then \(Z_{CY}(X)\) can be identified with the Verlinde algebra:

\[ (3.17) \quad Z_{CY}(X) \cong V, \quad \varphi_i \mapsto x_i \]

Then, as mentioned at the beginning of Section 2.2 the “flipping a donut” automorphism translate into charge conjugation \(i \mapsto i^*\) in the Verlinde algebra.

**Example 3.5.** The cornered cobordisms \(\Psi, \overline{\Psi}\) implement the \(S, \overline{S}\)-matrices on \(Z_{CY}(X)\), i.e.

\[ Z_{CY}(\Psi)(\varphi_j) = s(\varphi_j) = \sum_i s_{ij} \varphi_i \]

\[ Z_{CY}(\overline{\Psi})(\varphi_j) = \overline{s}(\varphi_j) = \sum_i \overline{s}_{ij} \varphi_i \]

This follows from a direct computation as in Figure 14 below. \(\triangle\)
$\varphi_j = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure14a.png}
\end{array}$

$X\xrightarrow{2\text{-}handle} X'$

$\Psi : m \mapsto l \mapsto -m$

$\varphi_j = \sum s_{ij} \varphi_i$

FIGURE 14. Crane-Yetter applied to $\Psi$ results in the S-matrix on $V$; computation for $\overline{S}$-matrix is similar.

**Example 3.6.** Consider the Y-product $Y_X^{(2)}$ on $X$ as described in Example 2.18 where we worked out a half-handle decomposition. This is a relative cobordism

$$(Y_X^{(2)}, Y_{\partial X}^{(2)} : (X \cup \partial X) \rightarrow (X, \partial X))$$

In order to apply Crane-Yetter theory, we interpret this relative cobordism as a cornered cobordisms by taking the outgoing corner $\partial^2 = \partial X$ as the corner:

$$(Y_X^{(2)}, \partial X) : Y_{\partial X}^{(2)} \cup (X \cup X) \rightarrow X$$

By Lemma 2.15 this cornered cobordism is in fact the identity cobordism, since only half-handles of the negative type are used to build $Y_X^{(2)}$.

Let us put the empty skein in the vertical boundary $Y_{\partial X}^{(2)}$, $\varphi_i$ in one $X$, and $\varphi_j$ in the other $X$. It is easy to see that under $Y_{\partial X}^{(2)} \cup (X \cup X) \simeq X$, the two skeins $\varphi_i, \varphi_j$ simply stack on top of each other (see Figure 15), as skeins, this is equal to the linear combination $\sum k N_{ij}^k \varphi_k$ (use (3.9)).

$$Z_{CY}(Y_X^{(2)})(\varphi_i \otimes \varphi_j) = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure15a.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure15b.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure15c.png}
\end{array} = \sum N_{ij}^k \varphi_k$$

FIGURE 15.

Thus, $Y_X^{(2)}$ gives the fusion product on the Verlinde algebra:

$$(3.18) \quad Z_{CY}(Y_X^{(2)})(\varphi_i \otimes \varphi_j) = \sum k N_{ij}^k \varphi_k$$

and so (3.17) is an isomorphism of algebras. $\triangle$

**Example 3.7.** Now consider the Y-product $Y_X^{(1)}$ from Example 2.17. The (half-)handles of $Y_X^{(1)}$ are of index $1^-, 2^-, 3$ in that order. Since the 3-handle does not affect the vertical boundary, we see that $Y_X^{(1)}$, treated again as a cornered cobordism with $\partial^2 = \partial X$ as corner, is the identity cornered cobordism (from the $1^-,-2^-$-half-handles) composed with a 3-handle:

$$(Y_X^{(1)}, \partial X) : Y_{\partial X}^{(1)} \cup (X \cup X) \rightarrow (X, \partial X)$$

$\varphi_i \otimes \varphi_j = \sum \delta_{ij} \varphi_i$

FIGURE 16.
From the computation in Figure 16, we see that \( Y_X^{(1)} \) gives the convolution product on the Verlinde algebra

\[
Z_{CY}(Y_X^{(1)})(\varphi_i \otimes \varphi_j) = 1/d_i \cdot \delta_{ij} \cdot \varphi_i
\]

Finally, we put everything together to give an alternative proof of the Verlinde formula based on Crane-Yetter theory:

**Theorem 3.8 (Verlinde Formula).** Under an identification of the Verlinde algebra \( V \) with the skein module \( Z_{CY}(X) \), the fusion product, convolution product, and \( S, \overline{S} \)-matrix on \( V \) are the results of evaluating the Crane-Yetter functor on the relative cobordisms \( Y_X^{(2)}, Y_X^{(1)} : X \cup X \to X \), and \( \Psi, \overline{\Psi} : X \to X \), respectively.

Thus, since the Y-products are related by \( \Psi \) (resp. \( \overline{\Psi} \)), the \( \overline{S} \)-matrix (resp. \( S \)-matrix) is an algebra morphism from \( V_g \) with the fusion product to \( V_g \) with the convolution product, i.e.

\[
s(x) \ast s(y) = s(xy) \\
\overline{s}(x) \ast \overline{s}(y) = \overline{s}(xy)
\]

for \( x, y \in V \).

**Proof.** Examples 3.7, 3.6 show that under the identification \( x_i \mapsto \varphi_i \), the Y-products \( Y_X^{(2)}, Y_X^{(1)} \) give the fusion and convolution products, respectively. Example 3.5 shows that \( \Psi \) gives the \( S \)-matrix (resp. \( \overline{S} \)-matrix). Similar to Theorem 3.8, this follows from Proposition 2.21 and computing \( Z_{CY}(K) \).

For the other way, that is, \( S, \overline{S} \)-matrices as algebra maps from convolution to fusion, we have:

**Theorem 3.9.** Let \( J \in \text{Irr}(A) \) be the subset of transparent simple objects, and let \( \varphi_J = \sum_{i \in J} d_i \varphi_i \). Then

\[
s((D \varphi_J \cdot x) \ast y) = s(x \ast (D \varphi_J \cdot y)) = s(x) \ast s(y)
\]

\[
\overline{s}((D \varphi_J \cdot x) \ast y) = \overline{s}(x \ast (D \varphi_J \cdot y)) = \overline{s}(x) \ast \overline{s}(y)
\]

**Proof.** Similar to Theorem 3.8, this follows from Proposition 2.21 and computing \( Z_{CY}(K) \).

It follows from Definition 3.3 that, given \( \varphi_i \), its image under \( Z_{CY}(K) \), is obtained by adding \( \Xi \) to \( \varphi_i \), where \( \Xi \) is the skein consisting of the two 2-handle attaching circles that define \( K \) (in Figure 10), both labeled with \( \Omega \). In other words, \( \Xi = s(\sum_{i \in \text{Irr}(A)} d_i \varphi_i) \), and \( Z_{CY}(K)(\varphi_i) = \varphi_i \cdot \Xi \).

By [3.11], when \( A \) is modular, we have \( \Xi = D \cdot \varphi_1 = D \cdot \varphi_X^b \); more generally, by [3.12], for \( A \) premodular, we have \( \Xi = D \cdot (\sum_{i \in J} d_i \varphi_i) = D \varphi_J \).

**Remark 3.10.** We may also prove Theorem 3.9 for modular \( A \) by appealing to the more general result that, as mentioned at the beginning of this section (in the discussion relating Crane-Yetter to Reshetikhin-Turaev), \( Z_{WRT} \) and \( Z_{CY} \) should only differ by a factor \( \kappa \overline{\Delta}^{1/2} \) which only depends on \( A \) and on the signature \( \sigma \) and Euler characteristic \( \chi \) of the 4-manifold in question. Here, as a 4-manifold, \( K \) is \( S^2 \times D^2 \), which has \( \sigma = 0, \chi = 2 \), while the identity cobordism on \( X \), as a 4-manifold, is \( S^1 \times D^2 \times D^2 \) and has \( \sigma = 0, \chi = 0 \). This contributes a factor of \( \Delta^{1/2} = \Delta \) to \( Z_{CY}(K) \) as compared to \( Z_{CY}(id_X) \), which agrees with out computation above.
3.4. Y-coproducts, units, counits.

**Example 3.11.** [Y-coproduct etc for $Y^{(1)}_X$, continuation of Example 3.7] To compute $\mathcal{X}^{(1)}_X$, we use the half-handle decomposition dual to Figure 4.

We start with $\varphi_i$. The dual to the 3-handle is a 1-handle, which essentially does not affect $\varphi_i$ except give a factor of $1/\mathcal{D}$. Next, the dual to the $2^–$-half-handle is a $1^–$-half-handle, which we present as a $1^–$-half-handle followed by a 2-handle. The $1^–$-half-handle only changes the boundary, and the 2-handle’s belt circle is an unknot that is contractible, so it gives a factor of $\mathcal{D}$. Finally, the dual to the $1^–$-half-handle is a $2^+$-half-handle, which we present as a $2^–$-half-handle followed by a 3-handle. Again, the $2^–$-half-handle only changes the boundary, and the 3-handle cuts $\varphi_i$ twice. The overall composition gives

$$Z_{\text{CY}}(\mathcal{X}^{(1)}_X)(\varphi_i) = \frac{1}{d_i} \cdot \varphi_i \otimes \varphi_i$$

The counit, as given in Figure 5 when evaluated under $Z_{\text{CY}}$, simply amounts to embedding $X$ in $S^3$ in a standard way ($\tilde{I}$ must give 0-framing), and evaluating the skein under $Z_{\text{WRT}}$. In particular, $\varphi_i \Rightarrow d_i$. It is clear that this is a counit for $Z_{\text{CY}}(\mathcal{X}^{(1)}_X)$, as expected.

By performing a similar analysis to the dual of the counit, we see that the main interesting step for the unit is the attachment of a 2-handle (part of the 1$^–$-half-handle that “punches out the donut hole”), which gives us the skein consisting of the core circle labeled by $\Omega$, i.e. $\sum d_i \varphi_i$, which is clearly the unit for the convolution product.

To summarize, we have the following structures, which form a Frobenius algebra (see discussion around (1.3), or check by direction computation):

\[
\begin{align*}
\varphi_i \otimes \varphi_j &\Rightarrow \frac{\delta_{ij}}{d_i} \varphi_i \\
1 &\Rightarrow \sum d_i \varphi_i \\
\varphi_i &\Rightarrow \frac{1}{d_i} \varphi_i \otimes \varphi_i \\
\varphi_i &\Rightarrow d_i
\end{align*}
\]

(3.23)

**Example 3.12.** Similar to Example 3.11, we use the half-handle decomposition dual to Figure 7.

We start with $\varphi_i$. The dual to the $2^+$-half-handle is a $1^+$-half-handle, which effectively punches out a hole in the side of the donut, while adding a $\Omega$-labeled circle around the hole. Enlarge the hole so that our 3-manifold resembles the second diagram in Figure 7. The dual to the $1^–$-half-handle is a $2^+$-half-handle, which effectively cuts the $1^–$-half-handle that connects the two solid tori. The result is $\sum_k \varphi_i \varphi_k \otimes \varphi_k^*$, or if we chose to push $\varphi_i$ to the bottom solid torus, $\sum_k \varphi_k \otimes \varphi_k^* \varphi_i$. (the $d_k$ factor from $\Omega$ is canceled by the $\alpha$ that arises from the $2^+$-half-handle, as we saw in Example 3.7).

The counit, as given in Figure 8, when evaluated under $Z_{\text{CY}}$, is given by $\varphi_i \Rightarrow \delta_{i0}$: the $2^+$-half-handle denies any $\varphi_i$ except for $i = 1$. It is clear from the dual to this half-handle decomposition that the unit is given by $\varphi_0$.

To summarize, we have the following structures, which form a Frobenius algebra (see discussion around (1.3), or check by direction computation):

\[
\begin{align*}
\varphi_i \otimes \varphi_j &\Rightarrow \sum_k N^k_{ij} \varphi_k \\
1 &\Rightarrow \varphi_0 \\
\varphi_i &\Rightarrow \sum_k \varphi_i \varphi_k \otimes \varphi_k^* = \sum_k \varphi_k \otimes \varphi_k^* \varphi_i \\
\varphi_i &\Rightarrow \delta_{i0}
\end{align*}
\]

(3.24)

\[\triangle\]
In [MS1989], they give a generalization of the Verlinde formula to the “n-point function characters at genus $g$” (here $S_{ij} = s_{ij}/O^{1/2}$):

$$\dim V(g, i_1, \ldots, i_n) = \sum_p \frac{S_{i_1 p} \cdots S_{i_n p}}{S_{0 p} \cdots S_{0 p}} \left( \frac{1}{S_{0 p}} \right)^{2g-2}$$

(3.25)

We may arrive at (3.25) as follows. Consider the following computation, each diagram representing a cobordism $X_{\psi^n} \to \emptyset$ (see (2.2)), here shown for $n = 3$ inputs and $g = 1$ loop (the unlabeled dots mean $\psi$, and the numbered dots refer to the Y-(co)product or counit):

The last “$\simeq$” is not an equality because $1_1 \circ \psi \neq 1_2$, they differ by an additional 2-handle. Under Crane-Yetter, the difference is only a factor of $\mathcal{D}$: the $\Omega$ loop around the attaching circle of $\psi$ is contractible because of $1_1$, thus contributes a factor of $\mathcal{D}$.

Putting $\varphi_{i_1}, \ldots, \varphi_{i_n}$ as input, the left most diagram gives

$$\varphi_{i_1} \circ \cdots \circ \varphi_{i_n} \mapsto \sum_p \frac{S_{i_1 p} \cdots S_{i_n p}}{d_{i_1 p} - 1} \cdots \varphi_p \mapsto \sum_p \frac{S_{i_1 p} \cdots S_{i_n p}}{d_{i_1 p} - 1} \cdots \varphi_{i_1} \circ \cdots \circ \varphi_{i_n} \mapsto \cdots$$

$$\cdots \mapsto \sum_p \mathcal{D} \cdot \frac{S_{i_1 p} \cdots S_{i_n p}}{d_{i_1 p} - 1 + 2g} \cdot \varphi_p \mapsto \sum_p \mathcal{D} \cdot \frac{S_{i_1 p} \cdots S_{i_n p}}{d_{i_1 p} - 1 + 2g - 2} \cdot \varphi \mapsto \cdots$$

(3.27)

which is equal to the right hand side of (3.25). The right most diagram gives

$$\varphi_{i_1} \circ \cdots \circ \varphi_{i_n} \mapsto \varphi_{i_1} \cdots \varphi_{i_n} \mapsto \sum_{k_1} \varphi_{k_1} \circ \varphi_{k_1}^* \cdots \varphi_{i_1} \cdots \varphi_{i_n} \mapsto \sum_{k_1} \varphi_{k_1} \circ \varphi_{k_1}^* \varphi_{i_1} \cdots \varphi_{i_n} \mapsto \cdots$$

$$\cdots \mapsto \sum_{k_1, \ldots, k_g} \varphi_{k_1} \varphi_{k_1}^* \cdots \varphi_{k_g} \varphi_{k_g}^* \varphi_{i_1} \cdots \varphi_{i_n} \mapsto \dim V(g, i_1, \ldots, i_n)$$

(3.28)

Note that the genus $g$ here has nothing to do with the genus in $X_g$ in the following section, and (3.25) is not the generalization of the Verlinde formula that we promised in the abstract.

3.5. Higher genus handlebodies. It is easy to see that skeins of the form shown in Figure 17 span $Z_{CV}(X_g; A)$ (use (3.8) and isotope to look like that, then use skein relations in a ball around $\varphi$).

(Figure 17. We depict for genus 4, but the general case is clear: At each $X$, going from left to right, one strand is “left behind”, alternately the leftmost and rightmost strand, so the label on the “left behind” strand would be $i_1, i_2, i_2 - 1, \ldots, i_{g/2} + 1$. Note the labels $(0, 0)$ and $(1/2, 1/2)$ indicate how $X$ is identified with that solid torus.

Define the genus $g$ Verlinde algebra to be $V_g = \sum_{i_1, \ldots, i_g} V(i_1, \ldots, i_g)$, where $V(i_1, \ldots, i_g) = \text{End}_A(X_{i_1} \otimes \cdots \otimes X_{i_g})$, and the direct sum is over all $g$-tuples of simple objects.

We define the internal genus $g$ Verlinde algebra (see [GJS2019]) to be the functor

$$V_g(-) : \mathcal{A} \to \text{Vec}$$

$$A \mapsto V_g(A) := \bigoplus_{i_1, \ldots, i_g} V^{(A)}(i_1, \ldots, i_g) := \bigoplus_{i_1, \ldots, i_g} \text{Hom}_A(AX_i \cdots X_{i_g}, X_{i_1} \cdots X_{i_g})$$

$$f : A \to B \mapsto - \circ (f \otimes \text{id})$$

(3.29)
By Figure 17, we have linear maps \( V_{(i_1, \ldots, i_g)}^{(A)} \to Z_{\text{CY}}(X_g; A) \). Note that the marked point on the boundary, labeled with object \( A \in \mathcal{A} \), is placed at a fixed point of \( \Psi' \) (and of \( \Psi'_g \)).

By Lemma 3.13 below, we may unambiguously denote skeins in \( Z_{\text{CY}}(X_g; A) \) by an element in \( V_g^{(A)} \).

**Lemma 3.13.** The linear maps \( V_{(i_1, \ldots, i_g)}^{(A)} \to Z_{\text{CY}}(X_g; A) \) are isomorphisms onto their images; moreover, their images give a direct sum decomposition of \( Z_{\text{CY}}(X_g; A) \). Thus, we have a natural isomorphism \( V_g^{(-)} \simeq Z_{\text{CY}}(X_g; -) : A^{\text{op}} \to \text{Vec} \).

**Proof.** Follows from standard skein-theoretic techniques.

We may refine \( V_{(i_1, \ldots, i_g)}^{(A)} \) by the simple object that the morphism passes through: we define \( V_{(i_1, \ldots, i_g)}^{(A)} = \text{span}(\text{Hom}(X_k, X_{i_1} \cdots X_{i_g}) \circ \text{Hom}(AX_{i_1} \cdots X_{i_g}, X_k)) \), and we have \( V_{(i_1, \ldots, i_g)}^{(A)} = \bigoplus_k V_{(i_1, \ldots, i_g)}^{(A)} \).

The generalization of the convolution product for the genus \( g \) Verlinde algebra is given by composition in each component \( V_{(i_1, \ldots, i_g)} \) (see Definition 3.15, Example 3.18), while the generalization of the fusion product is not so obvious (see Definition 3.16, Example 3.19).

The operations defined below should be thought of as natural transformations \( g_{V_g^{(-)} \to V_g^{(-)} \otimes V_g^{(-)} \to V_g^{(- \otimes \text{op})}} \); when restricted to \( 1 \), we get linear maps \( V_g \to V_g \) or \( V_g \otimes V_g \to V_g \). Again by Lemma 3.13, we may regard the operations defined below as natural transformations \( Z_{\text{CY}}(X_g; -) \to Z_{\text{CY}}(X_g; -) \) or \( Z_{\text{CY}}(X_g; -) \otimes Z_{\text{CY}}(X_g; -) \to Z_{\text{CY}}(X_g; - \otimes \text{op}) \).

**Definition 3.14.** The generalized \( S \)-matrix and generalized \( \overline{S} \)-matrix are given by, for \( \varphi \in V_{(i_1, \ldots, i_g)}^{(A)} \),

\[
\varphi = \sum_{k_1, \ldots, k_g} d_{k_1} \cdots d_{k_g} \in V_{(i_1, \ldots, i_g)}^{(A)}; \quad \varphi = \sum_{k_1, \ldots, k_g} d_{k_1} \cdots d_{k_g} \in V_{(i_1, \ldots, i_g)}^{(A)}.
\]

**Definition 3.15.** For \( \varphi \in V_{(i_1, \ldots, i_g)}^{(A)} \), \( \varphi' \in V_{(j_1, \ldots, j_g)}^{(B)} \), we define their generalized convolution product to be

\[
\varphi \ast \varphi' = \frac{\delta_{i_1,j_1}}{d_{i_1}} \cdots \frac{\delta_{i_g,j_g}}{d_{i_g}} \cdot \varphi' \circ (\text{id}_B \otimes \varphi) \in V_{(i_1, \ldots, i_g)}^{(BA)}.
\]

**Definition 3.16.** For \( \varphi \in V_{(i_1, \ldots, i_g)}^{(A)} \), \( \varphi' \in V_{(j_1, \ldots, j_g)}^{(B)} \), we define their generalized fusion product to be given

\[
\varphi \cdot \varphi' := \sum_{k_1, \ldots, k_g} d_{k_1} \cdots d_{k_g} \in V_{(i_1, \ldots, i_g)}^{(BA)}.
\]
Example 3.17. Recall that $Ψ'_g,Ψ'_g$ are designed to be the identity in a neighborhood of the marked point on the boundary, so $Z_{\text{CY}}(Ψ'_g),Z_{\text{CY}}(Ψ'_g)$ are well-defined as linear maps from $Z_{\text{CY}}(X_g;A)$ to itself. From Figure 18 we can see that, under Crane-Yetter, the cornered cobordisms $Ψ'_g,Ψ'_g$ realize the generalized $S,S$-matrices on $Z_{\text{CY}}(X_g;A)$: that is, $Z_{\text{CY}}(Ψ'_g)(\phi) = s(\phi), Z_{\text{CY}}(Ψ'_g)(\phi) = \pi(\phi)$.

In relation to $Ψ_g,Ψ_g$, recall that the marked point at $p := (0,0)$ should come with a framing, say tangent to $l$; then $Ψ_g$ defines a linear map from $Z_{\text{CY}}(X_g;(A,p,l))$ to $Z_{\text{CY}}(X_g;(A,p,-\tilde{m}))$, and $Ψ_g$ defines a linear map from $Z_{\text{CY}}(X_g;(A,p,l))$ to $Z_{\text{CY}}(X_g;(A,p,\tilde{m}))$. Then we may get back to $Z_{\text{CY}}(X_g;(A,p,\tilde{l}))$ by applying a diffeomorphism from $X_g$ to itself that untwists the framing on $p$. Alternatively, we may compose with a relative cobordism which is trivial topologically, but on the vertical boundary, we have a skein that untwists the marked point.

△

Example 3.18. Similar to Example 3.7, the half-handles that make up $Y^{(1)}_{X_g}$ are of negative type, so the most interesting part for skeins is the 3-handles. In the last copy of $X$ in $X_g$, the 3-handle cuts a pair of skeins labeled $i_k$ and $j_k$, where $k = [g/2] + 1$, which forces $i_k = j_k$ and gives a factor of $1/d_i$. Upon simplifying the skeins, we see that in the penultimate copy of $X$, the 3-handle again cuts only a pair of skeins, labeled $i_{k'}$ and $j_{k'}$ for some $k' (= k + 1$ or $k - 1$ depending on parity of $g$), which forces $i_{k'} = j_{k'}$ and gives a factor of $1/d_{i_{k'}}$. Repeat this procedure, we see that we end up with $Z_{\text{CY}}(Y^{(1)}_{X_g}(\varphi \otimes \varphi')) = \varphi \cdot \varphi'$. In other words, under Crane-Yetter, $Y^{(1)}_{X_g}$ realizes the generalized convolution product.

△

Example 3.19. Similar to Example 3.6, it is easy to see that $Y^{(2)}_{X_g}$ stacks one $X_g$ on top of another, and so $Z_{\text{CY}}(Y^{(2)}_{X_g}(\varphi \otimes \varphi'))$ simply stacks $\varphi$ on top of $\varphi'$. Using methods similar to the computation of $Z_{\text{CY}}(Y^{(2)}_{X_g})$ (see Figure 15), and adding an appropriate identity morphism in the vertical boundary to combine marked points into one, we see that, under Crane-Yetter, $Y^{(2)}_{X_g}$ realizes the generalized fusion product: $Z_{\text{CY}}(Y^{(2)}_{X_g}(\varphi \otimes \varphi')) = \varphi \cdot \varphi'$.

△

Theorem 3.20. Under the identification of the internal genus $g$ Verlinde algebra $V_{g}(-)$ with the skein module functor $Z_{\text{CY}}(X_g;-)$, the generalized fusion product, generalized convolution product, and generalized $S,S$-matrices on $V_{g}(-)$ are the results of evaluating the Crane-Yetter functor on the relative cobordisms $Y^{(2)}_{X_g}, Y^{(1)}_{X_g}$: $X_g \cup X_g \to X_g$, and $Ψ'_g,Ψ'_g : X_g \to X_g$, respectively.

Thus, since the $Y$-products are related by $Ψ'_g$ (resp. $Ψ'_g$), the generalized $S$-matrix (resp. $S$-matrix) is an algebra morphism from $V_g$ with the (resp. opposite) generalized fusion product to $V_g$ with the generalized convolution product.

Proof. Examples 3.19, 3.18, 3.17 prove the first part of the statement.

Proposition 2.29 shows that the $Y$-products are related by $Ψ_g,Ψ_g$; by evaluating under Crane-Yetter for empty boundary conditions (i.e. $A = B = 1$), we get

\[
Z_{\text{CY}}(Y^{(1)}_{X_g}) \circ (Z_{\text{CY}}(Ψ_g) \otimes Z_{\text{CY}}(Ψ_g)) = Z_{\text{CY}}(Ψ_g) \circ Z_{\text{CY}}(Y^{(2)}_{X_g})
\]

\[
Z_{\text{CY}}(Y^{(1)}_{X_g}) \circ (Z_{\text{CY}}(Ψ'_g) \otimes Z_{\text{CY}}(Ψ'_g)) = Z_{\text{CY}}(Ψ'_g) \circ Z_{\text{CY}}(Y^{(2)}_{X_g}) \circ P
\]

where here $P$ is the swapping of factors for tensor product of vector spaces; thus, for $x,y \in V_g$,

\[
s(x) \ast s(y) = s(x \cdot y)
\]

\[
\pi(x) \ast \pi(y) = \pi(y \cdot x).
\]

Now for non-empty boundary condition, we need to use $Ψ'_g,Ψ'_g$ instead of $Ψ_g,Ψ_g$. As discussed in Example 3.17, the former is obtain from the latter by untwisting the marked point on the boundary. It is clear by direct inspection that for $x,y \in V_g^{(A)}$, $y \in V_g^{(B)}$,

\[
s(x) \ast s(y) = s(x \cdot y)
\]

\[
\pi(x) \ast \pi(y) = c_{B,A} \circ \pi(y \cdot x).
\]
Figure 18. Computation of $Z_{CY}(\Psi)$ for genus $g = 4$; for odd genus the argument is essentially the same. The top row represents the skeins after adding the 2-handles. In the bottom row, we see that the skeins labeled with $i_*$'s criss-cross with the skeins labeled with $\Omega$. This is used inductively (we proceed from right to left) - for the torus to its left, we know that when we pull the skeins in, we get the criss-cross pattern, as we see in the second row.

4. Relation to Yetter's Handle as Hopf Algebra

We would like to say a few words about the relation of our work to Yetter's work on the handle as a Hopf algebra [Yet199701].
In [Yet199701], Yetter constructs a formal Hopf algebra structure on the torus with one boundary component (which we refer to as $H$ for the rest of this discussion). He considers the category $S^1$, where the objects are surfaces with boundary identified with $S^1$, and morphisms are cornered cobordisms. The tensor product of two surfaces $N_1 \otimes N_2$ is $(N_1 \cup N_2) \cup \text{POP}$ (the pair of pants). Then the multiplication and comultiplication on $H$ are given by cornered cobordisms $m : H \otimes H \to H$ and $\Delta : H \to H \otimes H$ (see Figure 19).

![Figure 19. Multiplication and comultiplication on $H$, from Yetter’s context.](image)

We would like to describe $H$ as a bialgebra in terms of the $Y$-product. In order to do so, we will need to modify the notion of $Y$-product slightly. Before we describe the variant, notice that the multiplication $m$ in Figure 19 is similar to $Y^{(1)}_X$. Indeed, it is simply $Y_H$ considered as a cornered cobordism with corner given by the outgoing corner. Likewise, the coproduct on $H$ looks similar to $Y^{(2)}_X$, and can be given by $Y_H$ considered as a cornered cobordism with incoming corner of $Y_H$ as corner.

Given $Q = M \cup N \overline{M}$, we define the reduced $Y$-product, denoted $\overline{Y}_Q$, to be $Y_Q$ as a cornered cobordism with corner given by the outgoing corner $\partial Q$. The incoming boundary of $\overline{Y}_Q$ is $Q \# Q$, where for $Q_1, Q_2$, with identifications of their boundaries with $\partial Q$, $Q_1 \# Q_2 := (Q_1 \cup Q_2) \cup Y_{\partial Q}$ (here $Y_{\partial Q}$ is the vertical boundary of $Y_Q$; $\#$ would be $\emptyset$ in Yetter’s context).

In terms of half-handle decomposition, suppose we are given a half-handle decomposition of $M$ as a relative cobordism $(M, N') : \emptyset \to (N, P = \partial N)$ such that all half-handles are of negative type and come before regular handles. Such half-handle decomposition always exists: any handle decomposition for the vertical boundary $N'$ as a cobordism $\emptyset \to P$ can be upgraded to a half-handle decomposition for the relative cobordism $(N' \times I, N') : \emptyset \to (N', P)$ by changing each $k$-handle to a $k'$-half-handle, then any handle decomposition for the cornered cobordism $(M; P) : N' \to N$ will complete the desired half-handle decomposition.

We can build, as in Section 2.1, a half-handle decomposition for $Q$, which we may breakdown to a sequence of cobordisms $\emptyset \to N' \to N \to N' \to \emptyset$. Obviously, $Q \cup Q$ can be given the half-handle decomposition which is the concatenation of two copies of that sequence. By attaching $Y_{\partial Q}$ to $Q \cup Q$, it is not hard to see that the middle $N' \to \emptyset \to N'$ is canceled out, leaving the sequence $\emptyset \to N' \to N \to N' \to N \to N' \to \emptyset$ as a half-handle decomposition for $\overline{Y}_Q$.

This operation of attaching $Y_{\partial Q}$ to $Q \cup Q$ is a relative cobordism $Q \cup Q \to \overline{Y}_Q$: it is clear from the construction of the full $Y$-product and our choice of half-handle decomposition for $M$ that it is contained in the full $Y$-product $Y_Q : Q \cup Q \to Q$. In other words, we may apply the same construction for $Y_Q$ to get a half-handle decomposition for $\overline{Y}_Q$, by simply restricting to adding handles for the latter half of $M : \emptyset \to N' \to N$.

As a concrete example, we can build $H$ from the sequence of (half-)handles $0^-, 1, 1, 1^*$. The resulting $H \# H$ has (half-)handles $0^-, 1, 1, 1, 1^*$. Then $\overline{Y}_H$ is given by one 2-handle that eliminates the middle two 1-handles of $H \# H$.

The reduced $Y$-coproduct, unit, and counit, denoted $\overline{\Delta}_Q$, $\overline{\varepsilon}_Q$, and $\overline{\eta}_Q$ respectively, are also given by similar considerations. Note that the unit object for $\#$ is not $\emptyset$, but $N' \times I$, which has boundary $N' \cup \partial N'$. The unit $\overline{\eta}_Q$ is a cornered cobordism from $N' \times I$ to $Q$.

The following is a generalization of the bialgebra structure on $H$, in particular, it applies to any connected surface with one boundary component:

**Proposition 4.1.** Let $Q$ be a connected $n$-manifold with boundary a sphere $S^{n-1}$, and suppose that $Q$ can be presented as the double of a manifold in two ways, say $Q = M_1 \cup N_1 \overline{M}_1 = M_2 \cup N_2 \overline{M}_2$, such that $N_1$ and $N_2$
intersect transversely. Furthermore, suppose that $M_0 := M_1 \cap M_2$ is diffeomorphic to a ball (after smoothing corners), and $M_0 \cap \partial Q, M_1 \cap \partial Q, M_2 \cap Q$ are $(n-1)$-balls (after smoothing corners). Then the $Y$-product from $M_1$, $Y_Q^{(1)}$, and $Y$-coproduct from $M_2$, $X_Q^{(2)}$, obey the bialgebra relation.

**Proof.** Denote by $N'_0 := M_1 \cap \partial Q, N'_2 := M_2 \cap \partial Q$, or equivalently they are the closures of $M_1 \setminus N_1, M_2 \setminus N_2$, and denote by $N'_0 := M_0 \cap \partial Q$; by hypothesis, they must be $(n-1)$-balls.

Denote $P := N_1 \cap N_2$; it separates $N_1$ into two pieces, each a mirror copy of the other, and we denote one by $N_{10}$. Similarly, $P$ separates $N_2$ into $N_{20}$ and $\overline{N}_{20}$.

We denote by $P_1, P_2$ the closures of $\partial N_{10} \setminus P, \partial N_{20} \setminus P$.

We represent $Q$ schematically as follows:

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.4\textwidth]{diagram1.png}
\end{array}
\end{array}
\end{equation}

We see how $N'_0$ is split into two $D^{n-1}$ along a $D^{n-2}$, which we denote by $P_0$. Note that these are schematic diagrams, $N_{10}$ and $N_{20}$ are not necessarily $(n-1)$-balls.

Since $M_0$ is an $n$-ball by hypothesis, it can be thought of as a trivial cobordism from $N_{20} \cup_P N_{10}$ to $D^{n-1}$ (technically, this would be a cobordism that is a mix between cornered cobordism and relative cobordism; the corner is $P_1$, and the vertical boundary is the $D^{n-1}$ between $P_1$ and $P_2$; if we double $M_0$ along the vertical boundary, then we get a trivial cornered cobordism). The rightmost diagram above shows $M_0$ for Yetter’s case, which is when $Q$ is a torus with one boundary component.

$Q^{(1)}$ and $Q^{(2)}$ are schematically represented as follows:

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.4\textwidth]{diagram2.png}
\end{array}
\end{array}
\end{equation}

In each middle piece $M_0 \cup M_0$, the union is taken over one of the $D^{n-1}$’s in $N'_0$.

Finally $(Q^{(1)}Q)^{(2)}(Q^{(1)}Q) = (Q^{(2)}Q)^{(1)}(Q^{(2)}Q)$ is schematically represented as follows:

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.4\textwidth]{diagram3.png}
\end{array}
\end{array}
\end{equation}

We will discuss the square in the middle later.

Consider the following diagrams, which represent $X_Q^{(2)} \circ Y_Q^{(1)}$ the same way that a fat graph represents a relative cobordism as described in Section 11, that is, the diagram forms a “spine” onto which copies of $M_0 \times I$ are glued (we show this for parts of $Y_Q^{(1)}$ in the right diagram below). Note that the fact that the piece labeled (*) is also $M_0 \times I$ relies on the fact that $N'_0$ is simply a disk.

\footnote{Here the superscript in $\#^{(1)}$ is used to indicate the $Y$-product from which the $\#$ operation is for.}

[1] Section 11.
We denote $\Xi_1 := \mathcal{X}_Q^{(2)} \circ Y_Q^{(1)}$ for brevity. The first diagram is the union of the two subsequent diagrams along the thinner lines. The last diagram (before the semicolon) is easily seen to represent the same cornered cobordism as $\Xi_1$. The top half represents $Y_Q^{(1)}$; the "++" at the top represents the incoming boundary $Q\#^{(1)}Q$ of $Y_Q^{(1)}$ as in (4.2), and the "+" in the middle represents $Q$ as in (4.1).

Now the other side of the bialgebra relation is $(Y_Q^{(1)} \#^{(2)} Y_Q^{(1)}) \circ (\mathcal{X}_Q^{(2)} \#^{(1)} \mathcal{X}_Q^{(2)})$, which we denote by $\Xi_2$ for brevity; we depict it by the same graphical representation as follows:

We want to show that $\Xi_1$ and $\Xi_2$ are in fact the same cornered cobordism. In the diagram below, we observe that each piece of $M_0 \times I$ in $\Xi_1$ has a corresponding part in $\Xi_2$:

There is only one piece in $\Xi_2$ that is missing from $\Xi_1$, which is the piece in the center; we refer to this piece as the heart.

The heart is schematically represented as follows (the second diagram shows the heart when $Q$ is the torus with one boundary component):

The dotted lines represent some fixed points under the symmetry associated with $\mathcal{X}_Q^{(2)}$. Concretely, the horizontal dotted line at the top is a copy of $N_{20} \cup P_2 \overline{N_{20}}$, the vertical dotted line on the left and right are copies of $N_{10}$, and the dotted arc at the bottom is a copy of $P_1 \times I$.

From the discussion on $M_0$ as a trivial cobordism after (4.1), it follows easily that the heart is a trivial cornered cobordism, where the corner is the dotted line.

Thus, we may apply the following operation which folds and flattens the heart, and $\Xi_2$ will remain unchanged as a cornered cobordism:
(We may also think of the operation above as occurring after we remove the heart from $\Xi_2$, and the diagrams depict how to identify the new boundary faces.)

Observe that shrinking the “fins” (see (4.8)) after the collapse does not affect $\Xi_2$, for the same reason that $\tilde{i}$ is a unit for $\tilde{Y}$ (see (1.2) and discussion on (co)unit after that). After a slight modification, it’s clear that we have $\Xi_1$.

\[(4.9)\]
\[\Xi_2 = \begin{array}{c}
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Diagram 3} \\
\text{Diagram 4}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Diagram 5} \\
\text{Diagram 6}
\end{array}
\end{array} = \Xi_1\]

\[\square\]

References

[Ale1928] James W Alexander, *Topological invariants of knots and links*, Transactions of the American Mathematical Society 30 (1928), no. 2, 275–306.

[BFMG2007] John W. Barrett, João Faria Martins, and J. Manuel García-Islas, *Observables in the turaev-viro and crane-yetter models*, Journal of Mathematical Physics 48 (2007), no. 9, 093508, available at [https://doi.org/10.1063/1.2759440](https://doi.org/10.1063/1.2759440)

[BK2001] Bojko Bakalov and Alexander A Kirillov, *Lectures on tensor categories and modular functors*, Vol. 21, American Mathematical Soc., 2001.

[BKR2016] Maciej Borodzik, András Némethi, and Andrew Ranicki, *Morse theory for manifolds with boundary*, Algebraic & Geometric Topology 16 (2016), no. 2, 971–1023.

[Con1970] John H Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational problems in abstract algebra, 1970, pp. 329–358.

[EnO2005] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik, *On fusion categories*, Annals of Mathematics (2005), 581–642.

[FYH+1985] Peter Freyd, David Yetter, Jim Hoste, WB Raymond Lickorish, Kenneth Millett, and Adrian Ocneanu, *A new polynomial invariant of knots and links*, Bulletin (new series) of the American mathematical society 12 (1985), no. 2, 239–246.

[GJS2019] Sam Gunningham, David Jordan, and Pavel Safronov, *The finiteness conjecture for skein modules*, arXiv preprint arXiv:1908.05233 (2019).

[Jon1985] Vaughan FR Jones, *A polynomial invariant for knots via von neumann algebras*, Bull. Amer. Math. Soc.(NS) 12 (1985), no. 1, 103–111.

[Kam1987] Louis H Kauffman, *State models and the jones polynomial*, Topology 26 (1987), no. 3, 385–407.

[KJ2011] Alexander Kirillov Jr, *String-net model of turaev-viro invariants*, arXiv preprint arXiv:1106.6033 (2011).

[Lau2011] François Laudenbach, *A morse complex on manifolds with boundary*, Geometriae Dedicata 153 (2011), no. 1, 47–57.

[mathoverflowVerlinde] URL:https://mathoverflow.net/q/151221 (version: 2013-12-08).

[Mil2016] John Milnor, *Morse theory.(am-51), volume 51*, Morse theory.(am-51), volume 51, 2016.

[MS1989] Gregory Moore and Nathan Seiberg, *Classical and quantum conformal field theory*, Communications in Mathematical Physics 123 (June 1989), no. 2, 177–254.

[Müg2003] Michael Müger, *On the structure of modular categories*, Proceedings of the London Mathematical Society 87 (2003), no. 2, 291–308.

[PT1988] JH PRZYTYCKI and Pawel Traczyk, *Invariants of links of conway type*, Kobe J. Math. 4 (1988), 115–139.

[Tha2020] Ying Hong Tham, *Reduced tensor product on the drinfeld center*, arXiv preprint arXiv:2004.09611 (2020).

[Tha2021] Ying Hong Tham, *On the category of boundary values in the extended crane-yetter tqft*, Ph.D. Thesis, 2021.

[Ver1988] Erik Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, Nuclear Physics B 300 (1988), 360–376.

[Yet199701] David Yetter, *Portrait of the handle as a hopf algebra*, Physics, Geometry 184 (199701).