Groups definable in partial differential fields with an automorphism

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Abstract

In this paper we study groups definable in existentially closed partial differential fields of characteristic 0 with an automorphism which commutes with the derivations. In particular, we study Zariski dense definable subgroups of simple algebraic groups, and show an analogue of P.J. Cassidy’s result for partial differential fields. We also show that these groups have a smallest definable subgroup of finite index.

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1 Introduction

Fields with operators appear everywhere in mathematics, and are particularly present in areas close to algebra. The development of differential and difference algebra dates back to J. Ritt (26) in the 1950’s, and was then further expanded by E. R. Kolchin (16, 17) and R. Cohn (11) in the 1960’s. The study of differential and difference fields has been important in mathematics since the 1940’s and has applications in many areas of mathematics.
One can also mix the operators, this gives the notion of differential-difference fields, i.e., a field equipped with commuting derivations and automorphisms. These fields were first studied from the point of view of algebra by Cohn in [12].

Model theorists have long been interested in fields with operators, until recently mainly on fields of characteristic 0 with one or several commuting derivations (ordinary or partial differential fields), and on fields with one automorphisms (difference fields). The first author also started in [5] the model-theoretic study of the existentially closed difference-differential fields of characteristic 0, around 2005 (one derivation, one automorphism). Then work of D. Pierce (23) and of O. León-Sánchez (19) brought back the model-theory of differential fields with several commuting derivations in the forefront of research in the area, as well as when a generic automorphism is added to these fields. However, unlike in the pure ordinary differential case and in the pure difference case, little is known on the possible interactions between definable subsets of existentially closed differential fields with several derivations, nor with an added automorphism.

In this paper we study groups definable in existentially closed differential-difference fields. We were motivated by the following result of P. J. Cassidy (Theorem 19 in [8]; we phrase it differently):

**Theorem.** Let $\mathcal{U}$ be a differentially closed field of characteristic 0 (with $m$ commuting derivations), let $H$ be a simple algebraic group defined and split over $\mathbb{Q}$, and $G \leq H(\mathcal{U})$ a $\Delta$-algebraic subgroup of $H(\mathcal{U})$ which is Zariski dense in $H$. Then $G$ is definably isomorphic to $H(L)$, where $L$ is the constant field of a set $\Delta'$ of commuting derivations. Furthermore, the isomorphism is given by conjugation by an element of $H(\mathcal{U})$.

She has similar results for Zariski dense $\Delta$-closed subgroups of semi-simple algebraic groups. A version of her result for (existentially closed) difference fields was also proved by Z. Chatzidakis, E. Hrushovski and Y. Peterzil (Proposition 7.10 of [10]):

**Theorem.** Let $(\mathcal{U}, \sigma)$ be a model of ACFA. Let $H$ be a simple algebraic group defined over $\mathcal{U}$, and let $G$ be a Zariski dense definable subgroup of $H(\mathcal{U})$. If $SU(G)$ is infinite then $G = H(\mathcal{U})$. If $SU(G)$ is finite, there are an isomorphism $f : H \to H'$ of algebraic groups, and integers $m > 0$ and $n$ such that some subgroup of $f(G)$ of finite index is conjugate to a subgroup of $H'(\text{Fix}(\sigma^m \text{Frob}^n))$. In particular, the generic types of $G$ are non-orthogonal to the formula $\sigma^m(x) = x^{p^n}$. If $H$ is defined over $\text{Fix}(\sigma)^{\text{alg}}$, then we may take $H = H'$ and $f$ to be conjugation by an element of $H(\mathcal{U})$.

In this paper, we generalise Cassidy’s results to the theory $\text{DCF}_{mA}$, the
model companion of the theory of fields of characteristic 0 with \( m \) derivations and an automorphism which commute, and one of our main results is:

**Theorem 4.1** Let \( \mathcal{U} \) be a model of \( \text{DCF}_m \mathbb{A} \), let \( H \) be a simple algebraic group defined and split over \( \mathbb{Q} \), and \( G \) a definable subgroup of \( H(\mathcal{U}) \) which is Zariski dense in \( H \).

Then \( G \) has a definable subgroup \( G_0 \) of finite index, the Kolchin closure of which is conjugate (say by an element \( g \in H(\mathcal{U}) \)) to \( H(L) \), where \( L \) is the field of constants of a set of definable commuting derivations. Furthermore, either \( G_0^g = H(L) \), or \( G_0^g \subseteq H(\text{Fix}(\sigma^{\ell})(L)) \) for some integer \( \ell \geq 1 \). In the latter case, if \( H \) is centerless, we are able to describe precisely the subgroup \( G_0^g \) as \( \{ g \in H(L) \mid \sigma^r(g) = \varphi(g) \} \) for some integer \( r \) and algebraic automorphism \( \varphi \) of \( H(L) \).

We have analogous results for Zariski dense definable subgroups of semi-simple centerless algebraic groups (Theorem 4.7). Using an isogeny result (Proposition 3.5), and introducing the appropriate notion of definably quasi-(semi-)simple definable group (see Definition 3.1), gives then slightly more general results in the definably quasi-simple case, see Corollary 4.2. The results in the (definably-quasi-) semi-simple case require a little more work and are less easy to state, see Theorem 4.7 for a precise statement.

Inspired by results of Hrushovski and A. Pillay on groups definable in pseudo-finite fields, we then endeavour to show that definable groups which are definably quasi-semi-simple have a smallest definable subgroup of finite index (this smallest definable subgroup is called the connected component). This is done in Corollary 5.10 and follows from several intermediate results. We first show the result for Zariski dense definable subgroups of a simply connected algebraic simple group \( H \), give a precise description of the connected component (Theorem 5.8), and show that every definable Zariski dense subgroup of \( H(\mathcal{U}) \) is quantifier-free definable. We then show the existence of a smallest definable subgroup of finite index for an arbitrary simple algebraic group \( H \) (Theorem 5.9), to finally reach the conclusion. Part of the study involves giving a description of definable subgroups of (arbitrary) algebraic groups and we obtain the following result, of independent interest:

**Theorem 5.1** Let \( H \) be an algebraic group, \( G \leq H(\mathcal{U}) \) a Zariski dense definable subgroup. Then there are an algebraic group \( H' \), a quantifier-free definable subgroup \( R \) of \( H'(\mathcal{U}) \), together with a quantifier-free definable \( f : R \to G \), with \( f(R) \) contained and of finite index in \( G \), and \( \text{Ker}(f) \) finite central in \( R \).

We also show how this result extends to existentially closed difference field of any characteristic, and state the appropriate generalization to \( \text{Fix}(\sigma^t) \) (Theorems 5.3 and 5.4).
The paper is organised as follows. Section 2 contains the algebraic and model-theoretic preliminaries. Section 3 introduces the notions of definably quasi-(semi-)simple groups and shows the isogeny result \( (3.5) \). Section 4 contains the main results of the paper: description of Zariski dense definable subgroups of simple and semi-simple algebraic groups \( (4.1, 4.2 \text{ and } 4.7) \). Section 5 proves the above mentioned Theorem \( 5.1 \) and shows that definably quasi-semi-simple definable groups have a definable connected component.

\section{Preliminaries}

This section is divided in five subsections: 2.1 - Differential and difference algebra; 2.2 - Model theory of differential and difference fields; 2.3 - The results of Cassidy; 2.4 - Quantifier-free canonical bases; 2.5 - Some well-known group-theoretic results.

Notation and conventions: All rings are commutative, all fields are commutative of characteristic 0.

If \( K \) is a field, then \( K^{\text{alg}} \) denotes an algebraic closure of \( K \) (in the sense of the theory of fields).

\subsection{Differential and difference algebra}

\textbf{Definition 2.1.} For more details, please see [17], [11] and [12].

(1) Recall that a \textit{derivation} on a ring \( R \) is a map \( \delta : R \to R \) which satisfies \( \delta(a + b) = \delta(a) + \delta(b) \) and \( \delta(ab) = a\delta(b) + \delta(a)b \) for all \( a, b \in R \).

(2) A \textit{differential ring}, or \( \Delta \)-ring, is a ring equipped with a set \( \Delta = \{ \delta_1, \ldots, \delta_m \} \) of commuting derivations. A \textit{differential field} is a differential ring which is a field.

(3) A \textit{difference ring} is a ring equipped with a distinguished automorphism, which we denote by \( \sigma \). (This differs from the usual definition which only requires \( \sigma \) to be an endomorphism.) A \textit{difference field} is a difference ring which is a field.

(4) A \textit{difference-differential ring} is a differential ring equipped with an automorphism \( \sigma \) (which commutes with the derivations). A \textit{difference-differential field} is a difference-differential ring which is a field.

\textbf{Notation 2.2.} (1) If \( K \) is a difference field, then \( \text{Fix}(\sigma)(K) \), or \( \text{Fix}(\sigma) \) if there is no ambiguity, denotes the \textit{fixed field} of \( K \), \( \{ a \in K \mid \sigma(a) = a \} \).
(2) Let $K \subset U$ be difference-differential fields, and $A \subset U$. Then $K(A)_\Delta$ denotes the differential field generated by $A$ over $K$, $K(A)_\sigma$ the difference field generated by $A$ over $K$, and $K(A)_{\sigma, \Delta}$ the difference-differential field generated by $A$ over $K$. (Note that we require $K(A)_\sigma$ and $K(A)_{\sigma, \Delta}$ to be closed under $\sigma^{-1}$.)

Polynomial rings and the corresponding ideals and topologies

**Definition 2.3.** Let $K$ be a difference-differential ring, $y = (y_1, \ldots, y_n)$ a tuple of indeterminates.

- Then $K\{y\}$ (or $K\{y\}_\Delta$) denotes the ring of polynomials in the variables $\delta_1^{i_1} \cdots \delta_m^{i_m} y_j$, where $1 \leq j \leq n$, and the superscripts $i_k$ are non-negative integers. It becomes naturally a differential ring, by setting $\delta_k(\delta_1^{i_1} \cdots \delta_m^{i_m} y_j) = \delta_1^{i_1} \cdots \delta_m^{i_m} y_{j_k}$, where $i_k = j_k$ if $\ell \neq k$, and $j_k = i_k + 1$. The elements of $K\{y\}$ are called differential polynomials, or $\Delta$-polynomials.

- $K[y]_\sigma$ denotes the ring of polynomials in the variables $\sigma^i(y_j)$, $1 \leq j \leq n$, $i \in \mathbb{Z}$, with the obvious action of $\sigma$; thus it is also a difference ring. They are called difference polynomials, or $\sigma$-polynomials.

- $K\{y\}_\sigma$ denotes the ring of polynomials in the variables $\sigma^i \delta_1^{i_1} \cdots \delta_m^{i_m} y_j$, with the obvious action of $\sigma$ and derivations. They are called difference-differential polynomials, or $\sigma$-$\Delta$-polynomials.

- A $\Delta$-ideal of a differential ring $R$ is an ideal which is closed under the derivations in $\Delta$ and it is called linear if it is generated by homogeneous linear $\Delta$-polynomials.

Similarly, a $\sigma$-ideal $I$ of a difference ring $R$ is an ideal closed under $\sigma$; if it is also closed under $\sigma^{-1}$, we will call it reflexive; if whenever $a \sigma^n(a) \in I$, then $a \in I$, it is perfect. Finally, a $\sigma$-$\Delta$-ideal is an ideal which is closed under $\sigma$ and $\Delta$.

**Remarks 2.4.** As with the Zariski topology, if $K$ is a difference-differential field, the set of zeroes of differential polynomials, $\sigma$-polynomials and $\sigma$-$\Delta$-polynomials in some $K^n$ are the basic closed sets of a Noetherian topology on $K^n$, see Corollary 1 of Theorem III in [12]. We will call these sets $\Delta$-closed (or Kolchin closed, or $\Delta$-algebraic), $\sigma$-closed/algorithmic and $\sigma$-$\Delta$-closed/algorithmic respectively. These topologies are called the Kolchin topology (or $\Delta$-topology), $\sigma$-topology and $\sigma$-$\Delta$-topology respectively. There are natural notions of closures and of irreducible components.
**Fact 2.5.** Recall that if $K$ is a differential subfield of $U$, then $K\Delta$ denote the $K$-vector space generated by the elements of $\Delta$.

1. ([17], Proposition 0.5.7) Every commuting linearly independent subset of $U\Delta$ is a subset of a commuting basis of $U\Delta$.

2. ([17], Corollary to Proposition 0.5.8) Let $\Delta' \subset U\Delta$ be a set of commuting derivations. Then the $\Delta'$-differential field $U$ is differentially closed.

3. ([17], 0.8.13) (Sit) Let $A$ be a perfect $\Delta$-ideal of the $\Delta$-algebra $U\{y\}$. A necessary and sufficient condition that the set of zeroes $Z(A)$ of $A$ be a subring of $U$, is that there exist a vector subspace and Lie subring $D$ of $U\Delta$ such that $A = [Dy]$ (the $\Delta$-ideal generated by all $Dy$, $D \in D$). When this is the case, there exists a commuting linearly independent subset $\Delta'$ of $U\Delta$ such that $Z(A)$ is the field of absolute constants of the $\Delta'$-field $U$.

Item (1) is valid under weaker hypotheses.

**Definition 2.6.** For short, we will call field of constants in $U$ a subfield of $U$, which is the field of constants of some subset $\Delta' \subset U\Delta$ of commuting derivations.

**Remarks 2.7.** Recall that we are in characteristic 0, this result is false in positive characteristic. We let $K$ be a differential subfield of the differentially closed field $U$. Consider the commutative monoid $\Theta$ (with 1) generated by $\delta_1, \ldots, \delta_m$, and let $K\Theta$ be the $K$-vector space with basis $\Theta$. It can be made into a ring, using the commutation rule $\delta_i \cdot a = a\delta_i + \delta_i(a)$, $i = 1, \ldots, m$.

Each element $f$ of $U\Theta$ defines a linear differential operator $L_f : \mathbb{G}_a \to \mathbb{G}_a$, defined by $a \mapsto f(a)$. One has $L_{f,g} = L_f \circ L_g$. Every $\Delta$-closed subgroup of $\mathbb{G}_a(U)$ is then defined as the set of zeroes of a finite set of linear differential operators in $U\Theta$, and for $n \geq 1$, every $\Delta$-closed subgroup of $\mathbb{G}_a^n(U)$ is defined by conjunctions of equations of the form $L_1(x_1) + \cdots + L_n(x_n) = 0$, with the $L_i$ in $U\Theta$, and with the $L_i$ in $K\Theta$ if the subgroup is defined over $K$, see e.g. Proposition 11 in [7], or Proposition 0.8.12 in [17]. Sets of elements of $K\Theta$ generate what is called a linear $\Delta$-ideal. The following result is certainly well-known, but we did not know of a reference.

**Lemma 2.8.** Let $S$ be a $K$-subspace of $K\Theta$, and assume that it is closed under left multiplication by the $\delta_i$, $i = 1, \ldots, m$, and that it does not contain 1. Then the differential ideal $I$ generated by the set

$$S(x) := \{f(x) \mid f \in S\} \subset K\{x\}$$

does not contain $x$ and is prime.
Proof. Note that \( I \) is simply the \( K\{x\}_\Delta \)-module generated by \( S(x) \), i.e., an element of \( I \) is a finite \( K\{x\}_\Delta \)-linear combination of elements of \( S(x) \). Every element \( f \) in \( K\{x\}_\Delta \) can be written uniquely as \( f = f_0 + f_1 + f_\geq 1 \), with \( f_0 \) the constant term, \( f_1 \) the sum of the linear terms, and \( f_\geq 1 \) the sum of terms of \( f \) of total degree \( \geq 2 \). Note that \((f + g)_i = f_i + g_i \) for \( i \in \{0, 1, \geq 1\} \). Moreover \((fg)_0 = f_0g_0, (fg)_1 = f_0g_1 + f_1g_0, (fg)_\geq 1 = f_\geq 1 g + fg_\geq 1 + f_1g_1\).

Since all elements of \( S(x) \) have degree \( \geq 1 \), it follows that all elements of \( I \) have constant term 0. This easily implies that if \( f \in I \), then \( f \in S(x) \): as \( f \) is a \( K\{x\}_\Delta \)-linear combination of elements of \( S(x) \), and as the elements of \( S(x) \) have degree 1, it follows that \( f_1 \) is a \( K \)-linear combination of elements of \( S(x) \) of degree 1, i.e., belongs to \( S(x) \). But \( 1 / x \notin S \) implies \( x / x \in S(x) \), and therefore \( x / x \notin I \).

The primeness of \( I \) follows from the fact that \( I \) is generated by linear differential polynomials, so that, as a ring, \( K\{x\}_\Delta / I \) is isomorphic to a polynomial ring (in maybe infinitely many indeterminates) over \( K \).

2.2 Model theory of differential and difference fields

Notation 2.9. We consider the language \( L \) of rings, let \( \Delta = \{\delta_1, \ldots, \delta_m\} \). We define \( L_{\Delta} = L \cup \Delta \), \( L_\sigma = L \cup \{\sigma\} \) and \( L_{\sigma, \Delta} = L_{\Delta} \cup \{\sigma\} \) where the \( \delta_i \) and \( \sigma \) are unary function symbols.

2.10. The theory \( \text{DCF}_m \)

The model theoretic study of differential fields (with one derivation, in characteristic 0) started with the work of A. Robinson (27) and of L. C. Blum (2). For several commuting derivations, T. McGrail showed in 21 that the \( L_\Delta \)-theory of differential fields of characteristic zero with \( m \) commuting derivations has a model companion, which we denote by \( \text{DCF}_m \). The \( L_\Delta \)-theory \( \text{DCF}_m \) is complete, \( \omega \)-stable and eliminates quantifiers and imaginaries. Its models are called \textit{differentially closed}. Differentially closed fields had appeared earlier in the work of Kolchin (16). From now on till 2.13 definable will mean \( L_\Delta \)-definable in \( U \), maybe with parameters.

An interesting corollary of Lemma 2.8 is the following:

Theorem 2.11. Let \( U \models \text{DCF}_m, L \leq U \) be a subfield of constants. Then there is no definable non-trivial automorphism of \( L \).

Proof. Suppose that \( \psi : L \to L \) is a definable isomorphism. The graph of \( \psi \) defines an additive subgroup \( S \) of \( L \times L \leq U \times U \).
By Remark 2.7 there are linear differential polynomials $F_i(x)$ and $G_i(y)$, $i = 1, \ldots, s$, such that

\[ S = \{(x, y) \in L \times L \mid F_1(x) = G_1(y), \ldots, F_s(x) = G_s(y)\}. \]

Let $H_1, \ldots, H_r$ be the linear combinations of the $\delta_i$ such that $H_1(x) = H_2(x) = \cdots = 0$ defines the field $L$. Then, because $S$ is the graph of an automorphism of $L$, we have

\[ \bigcap_{i=1}^s \ker(F_i) \cap L = \{0\} = \bigcap_{i=1}^s \ker(G_i) \cap L. \]

Hence, $x$ belongs to the differential ideal generated by the $F_i(x)$ and the $H_j(x)$, and this implies (see Lemma 2.8) that there are linear differential polynomials $L_1, \ldots, L_{s+r}$ such that $\sum_{i=1}^s L_i(F_i(x)) + \sum_{j=1}^r L_{s+j}(H_j(x)) = x$; letting $G(y) = \sum_{i=1}^s L_i(G_i(y)) + \sum_{j=1}^r L_{s+j}(H_j(y))$, we get $x = G(y)$. Since $\psi^{-1}$ is injective, $\ker(G)$ must be trivial, i.e. $G(y) = ky$ for some $k \in U$. However, since $\psi$ is a field automorphism we must have $k = 1$, i.e. $\psi = \text{id}$.

2.12. Definable and algebraic closure, independence. Let $(U, \Delta)$ be a differentially closed field. If $A \subset U$, then $dcl(A)$ and $acl(A)$ denote the definable and algebraic closure in the sense of the theory $DCF_m$. Then $dcl(A)$ is the smallest differential field containing $A$, and $acl(A)$ is the field-theoretic algebraic closure of $dcl(A)$. Independence in $U$ is given by independence in the sense of the theory ACF (of algebraically closed fields) of the algebraic closures, i.e., $A \downarrow C B$ iff $acl(CA)$ and $acl(CB)$ are linearly disjoint over $acl(C)$.

A result we will often use is the following:

**Theorem 2.13.** (Pillay, [24], Theorem 4.1 and Corollary 4.2) Let $G$ be a group definable in $U$, which is connected (i.e., with no definable subgroup of finite index). Then $G$ embeds definably into $H(U)$, for some algebraic group $H$.

This result is stated for one derivation, but the author remarks that it generalises easily to several commuting derivations.

2.14. The theories ACFA and DCF$_mA$

The $L_\sigma$-theory of difference fields has a model companion denoted ACFA ([20], see also [9] and [10]). León-Sánchez showed that the $L_{\sigma, \Delta}$-theory of difference-differential fields admits a model companion, DCF$_mA$, and he gave an explicit axiomatisation of this theory in [19]. (When $m = 1$, the theory
was extensively studied by the third author, in [6], see also [5]).

The theories ACFA and DCF\textsubscript{m}A have similar properties, they are model-complete, supersimple and eliminate imaginaries, but they are not complete and do not eliminate quantifiers. The completions of both theories are obtained by describing the isomorphism type of the difference subfield $\mathbb{Q}^{alg}$.

In what follows we will view ACFA as DCF\textsubscript{m}A with $m = 0$, and we fix a (sufficiently saturated) model $\mathcal{U}$ of DCF\textsubscript{m}A. Until subsection 2.3, definable will mean $L_{\sigma,\Delta}$-definable in $\mathcal{U}$, and a small $A$ will be one such that $\mathcal{U}$ is $\left(|A| + \aleph_0\right)^+$ saturated.

2.15. The fixed field

We let $\text{Fix}(\sigma) := \{x \in \mathcal{U} : \sigma(x) = x\}$ be the fixed field of $\mathcal{U}$; by Theorem 2.16(8) below, it is a pseudo-finite field and $\text{Fix}(\sigma^k)$ is the unique extension of $\text{Fix}(\sigma)$ of degree $k$.

**Theorem 2.16.** ([19], Propositions 3.1, 3.3 and 3.4). Let $a, b$ be tuples in $\mathcal{U}$ and let $A \subseteq \mathcal{U}$. We will denote by $\text{acl}(A)$ the model theoretic closure of $A$ in the $L_{\sigma,\Delta}$-structure $\mathcal{U}$, and by $\mathcal{C}$ the subfield of absolute constants of $\mathcal{U}$. Then:

1. $\text{acl}(A)$ is the (field-theoretic) algebraic closure of the difference-differential field generated by $A$.

2. If $A = \text{acl}(A)$, then the union of the quantifier-free diagramme of $A$ and of the theory DCF\textsubscript{m}A is a complete theory in the language $L_{\sigma,\Delta}(A)$.

3. $\text{tp}(a/A) = \text{tp}(b/A)$ if and only if there is an $L_{\sigma,\Delta}(A)$-isomorphism $\text{acl}(Aa) \to \text{acl}(Ab)$ sending $a$ to $b$.

4. Every $L_{\sigma,\Delta}$-formula $\varphi(x)$ is equivalent modulo DCF\textsubscript{m}A to a disjunction of formulas of the form $\exists y \varphi(x, y)$, where $\varphi$ is quantifier-free (positive), and such that for every tuples $a$ and $b$ (in a difference-differential field of characteristic 0), if $\psi(a, b)$ holds, then $b \in \text{acl}(a)$.

5. Every completion of DCF\textsubscript{m}A is supersimple (of SU-rank $\omega^m+1$). Independence is given by independence (in the sense of ACFA) of algebraically closed sets: $a$ and $b$ are independent over $C$ if and only if the fields $\text{acl}(Ca)$ and $\text{acl}(Cb)$ are linearly disjoint over $\text{acl}(C)$.

6. Every completion of DCF\textsubscript{m}A eliminates imaginaries.
(7) If \( k \geq 1 \), and \( \mathcal{U} \models DCF_m A \), then the difference-differential field \( \mathcal{U}[k] = (\mathcal{U}, +, \cdot, \Delta, \sigma^k) \) is also a model of \( DCF_m A \), and the algebraic closure of \( \text{Fix}(\sigma) \) is a model of \( DCF_m \).

(8) \( \text{Fix}(\sigma) \) and \( \text{Fix}(\sigma) \cap C \) are pseudo-finite fields.

Remarks 2.17. (a) Item (4) is stated in a slightly different way in [19]. Here we prefer to have our set defined positively, at the cost of it consisting of maybe several elements. This gives us that every definable subset of \( \mathcal{U}^n \) is the projection of a \( \sigma\Delta \)-algebraic set \( W \) by a projection with finite fibers.

(b) By item (5) independence is given by independence (in the sense of ACF) of algebraically closed sets. This shows in particular that \( DCF_m A \) is one-based over ACF (see [1]).

(c) As with ACFA, if \( G \) is a definable subgroup of \( H(\mathcal{U}) \) for some algebraic group \( H \), we define the prolongations

\[
p_n : H(\mathcal{U}) \to H(\mathcal{U}) \times \sigma(H(\mathcal{U})) \times \cdots \times \sigma^n(H(\mathcal{U})), \quad g \mapsto (g, \sigma(g), \ldots, \sigma^n(g)),
\]

and let \( G_{(n)} \) be the Kolchin closure of \( p_n(G) \). Then an element \( g \in G \) is a generic of \( G \) if and only if \( p_n(g) \) is a generic of the \( \Delta \)-closed subgroup \( G_{(n)} \) of \( H(\mathcal{U}) \times \sigma(H(\mathcal{U})) \times \cdots \times \sigma^n(H(\mathcal{U})) \) for each \( n \). This implies that \( G \) and its \( \sigma\Delta \)-closure have the same SU-rank, and therefore that \( G \) has finite index in its \( \sigma\Delta \)-closure.

(d) Let \( A \subset \mathcal{U} \) be a small difference-differential subfield, and let \( L \) be a small difference-differential field extending \( A \). Assume that \( L \cap A^{\text{alg}} = A \). Then there is an \( A \)-embedding of \( L \) into \( \mathcal{U} \). Indeed, our assumption implies that \( L \otimes A A^{\text{alg}} \) is an integral domain, whence the difference-differential structure of \( L \) and of \( A^{\text{alg}} \) extend uniquely to the field \( M \) of fractions of \( L \otimes A A^{\text{alg}} \). Because \( A^{\text{alg}} = \text{acl}(A) \), the conclusion now follows by item (2) and the saturation hypothesis on \( \mathcal{U} \): \( qftp(L/A) \) is realized in \( \mathcal{U} \).

(e) This has the following consequence, which we will use: Let \( q \) be a quantifier-free type over a difference-differential subfield \( A \) of \( \mathcal{U} \), and suppose that \( q \) is stationary, i.e., if \( a \) realizes \( q \), then \( A(a)_{\sigma,\Delta} \cap A^{\text{alg}} = A \). Let \( f : A \to A' \subset \mathcal{U} \) be an isomorphism; then \( f(q) \) is realised in \( \mathcal{U} \).

(f) When \( m = 0 \), all these results appear in [19]. When \( m = 1 \), they appear in [5], [6].
(g) Item (8) has a more general formulation in Proposition 3.4(iv) of [19] as it applies to any field of constants of a set \( \Delta' \subset \text{Fix}(\sigma)(C)\Delta \) of commuting derivations.

From item (b) above, we can deduce the following:

**Lemma 2.18.** Let \( G \) be a group definable in \( \text{DCF}_mA \). Then there are a definable subgroup \( G_1 \) of finite index in \( G \), and a definable homomorphism \( f : G_1 \to H(\mathcal{U}) \), where \( H \) is an algebraic group, and the kernel of \( f \) is finite.

**Proof.** By Remark 2.17(b), and Theorem 4.9, Remark 4.10 of [1], \( G \) has a definable subgroup \( G_0 \) of finite index, and there is a definable homomorphism \( g : G_0 \to H(\mathcal{U}) \), where \( H \) is an algebraic group and \( \ker(g) \) is finite. \( \Box \)

**2.19. The fixed field.** The fixed field plays an important role in the study of interactions between definable sets. For instance, any non-stationary type is non-orthogonal to the fixed field (as proved by the first author in [6]). The next few results of this subsection are not particularly difficult, but to our knowledge do not appear in print.

**Definition 2.20.** Let \( M \) be a \( \mathcal{L} \)-structure. A definable subset \( D \) of \( M^r \) is stably embedded if every \( M \)-definable subset of \( D^n \) is definable with parameters from \( D \), for any \( n \geq 1 \).

Let \( (\mathcal{U}, \sigma, \Delta) \) be a sufficiently saturated model of \( \text{DCF}_mA \). For \( \ell \geq 1 \), we consider the difference-differential field \( F_\ell := \text{Fix}(\sigma^\ell) \). It is the only Galois extension of \( F_1 := F \) of degree \( \ell \), and its Galois group over \( \text{Fix}(\sigma) \) is generated by \( \sigma \). Similarly, if \( m \geq 1 \), \( \text{Fix}(\sigma^{m\ell}) \) is the unique Galois extension of \( F \) of degree \( m\ell \). The lemma below will be useful in determining the induced structure.

**Lemma 2.21.** Let \( k, \ell \geq 1 \) be integers. Then the difference-differential field \( F_{k\ell} \) is interpretable in the difference-differential field \( F_k \). Furthermore, given an \( \mathcal{L}_{\sigma,\Delta} \)-formula \( \varphi(x) \) (\( x \) an \( n \)-tuple of variables), there is an \( \mathcal{L}_{\sigma,\Delta}(F_k) \)-formula \( \varphi^*(y) \), \( y \) a \( \ell n \)-tuple of variables, and an \( F_k \)-basis \( \mathcal{B} \) of \( F_{k\ell} \) such that for any \( n \)-tuple \( c \in F_{k\ell} \), if \( d \) denotes the coordinates of \( c \) with respect to \( \mathcal{B} \), then

\[
F_{k\ell} \models \varphi(c) \iff F_k \models \varphi^*(d).
\]

The formulas \( \varphi \) and \( \varphi^* \) have the same complexity.

**Proof.** We will first do the case \( k = 1 \). Let \( \alpha \) be such that \( F(\alpha) = F_\ell \), and let \( f(X) = X^\ell + \sum_{i=0}^{\ell-1} a_i X^i \in F[X] \) be its minimal polynomial over
\[ F. \text{ If } b_0, \ldots, b_{k-1} \in F \text{ are such that } \sigma(\alpha) = \sum_{i=0}^{k-1} b_i \alpha^i, \text{ then, identifying } F_{k} \text{ with } F \oplus F \alpha \oplus \cdots \oplus F \alpha^{k-1}, \text{ one can define on } F^{k} \text{ the structure of the difference-differential field } F_k: \text{ Addition is obvious, multiplication by } \alpha \text{ is given by a matrix involving the coefficients } \bar{a} \text{ of } f(X), \text{ and } \sigma \text{ is definable using } \sigma(\alpha) = \sum b_i \alpha^i. \text{ As to the derivations, note that for } 1 \leq j \leq m \text{ one has }
\[ \delta_j(\alpha) = -\frac{\sum_{i=0}^{k} \delta_i(a_i) \alpha^i}{\sum_{i=1}^{k} i a_i \alpha^{i-1}}. \]

This interpretation of } F_k \text{ in } F \text{ is done quantifier-free, so that from a formula } \varphi \text{ one easily gets the formula } \varphi^*, \text{ with the same complexity, when } k = 1.

For the general case, let } k > 1, \text{ and observe that } F \text{ is quantifier-free definable in the difference-differential field } F_k \text{ by the formula } \sigma(x) = x. \text{ If } \alpha \text{ is such that } F_{k\ell} = F(\alpha), \text{ then the procedure given above allows us to interpret } F_{k\ell} \text{ inside } F, \text{ by defining an } \mathcal{L}_{\sigma,\Delta}\text{-structure on } F^{k\ell}. \text{ In order to get the full result, it suffices to show that some embedding of the difference-differential field } F_k \text{ inside the copy } F^{k\ell} \text{ of } F_{k\ell} \text{ is definable: let } \beta \in F_k \text{ be such that } F_k = F(\beta), \text{ write } \beta = \sum e_j \alpha^j, \text{ and let } \iota: F_k \to F^{k\ell} \text{ be defined by sending } \beta \text{ to the element with coordinates } (e_j) \text{ with respect to the basis } \{1, \alpha, \ldots, \alpha^{k\ell-1}\}. \text{ Thus, } \iota \text{ is the restriction to } F_k \text{ of the isomorphism } F_{k\ell} \to F^{k\ell}. \text{ We are therefore able to interpret quantifier-free in } F \text{ and also in } F_k, \text{ the } \mathcal{L}_{\sigma,\Delta}\text{-structures } (F_{k\ell}, F_k, F, \Delta, \sigma). \text{ (This interpretation is uniform in the parameters } (a, b, e).) \text{ The result follows easily.}

**Proposition 2.22.** Fix } \ell \geq 1. \text{ Then } F_{\ell} \text{ is stably embedded, and its induced structure is that of the pure difference-differential field. If } \ell = 1, \text{ it is the pure differential field.}

**Proof.** The first part follows from elimination of imaginaries (Prop. 3.3 in [19]): if } c \text{ is a code for a definable subset } S \text{ of } F^n, \text{ then } \sigma^*(c) = c. \text{ So every definable subset of } F_{\ell}^n \text{ is definable using parameters from } F_{\ell}.

By Proposition 2.16(4), every } \mathcal{L}_{\sigma,\Delta}\text{-formula } \varphi(\bar{x}) \text{ is equivalent (modulo } \text{DCF}_{m,A}) \text{ to a formula of the form } \exists \bar{y} \psi(\bar{x}, \bar{y}), \text{ where } \psi(\bar{x}, \bar{y}) \text{ is quantifier-free, and whenever } (a, b) \text{ realises } \psi, \text{ then } b \in acl(a). \text{ Let } r \text{ be a bound on the degree of } b \text{ over the difference-differential field generated by } a, \text{ and } N \text{ the least common multiple of all integers } \leq r. \text{ By Lemma 2.21 applied to } F_{\ell N}, \text{ if one fixes an } F\text{-basis } B \text{ of } F_{\ell N} \text{ over } F, \text{ there is an } \mathcal{L}_{\Delta}(F_{\ell})\text{-formula } \exists \bar{z} \psi^*(\bar{t}, \bar{z}) \text{ such that for any } \bar{c} \in F_{\ell N}, \text{ if } \bar{d} \text{ denotes the tuple of coordinates of } \bar{c} \text{ with respect to } B, \text{ then }
\[ U \models \exists \bar{y} \psi(\bar{c}, \bar{y}) \iff F \models \exists \bar{z} \psi^*(\bar{d}, \bar{z}). \]

This finishes the proof. \qed
Corollary 2.23. We consider $F_\ell$ as a difference-differential field and we denote as $\text{acl}_{F_\ell}(A)$ the algebraic closure in the sense of model theory in $F_\ell$. If $A \subset F_\ell$, then $\text{acl}_{F_\ell}(A) = \text{acl}(A) \cap F_\ell$, and independence in $F_\ell$ is given by independence (in the sense of ACF) of algebraic closures.

Proof. Certainly, in $F_\ell$, $A$ and $B$ independent over $C$ implies that $\text{acl}_{F_\ell}(CA)$ and $\text{acl}_{F_\ell}(CB)$ are algebraically independent over $\text{acl}_{F_\ell}(C)$. Conversely, assume that $A = \text{acl}_{F_\ell}(CA)$ and $B = \text{acl}_{F_\ell}(CB)$ are algebraically independent over $C = \text{acl}_{F_\ell}(C)$, but that there is some formula $\psi(a, b, c) \in \text{tp}_{F_\ell}(A/B, C)$ ($a \in A$, $b \in B$ and $c \in C$) which forks over $C$. As $\text{Th}(F_\ell)$ is supersimple, this means that there is a sequence $(b_i)_{i \in \omega}$ of independent realisations (in the sense of $F_\ell$) of $\text{tp}_{F_\ell}(b/Ca)$ such that $\{\psi(a, b_i, c) \mid i \in \omega\}$ is inconsistent. By the other direction, we have that, letting $B_i = \text{acl}_{F_\ell}(Cb_i)$, the difference-differential fields $A, (B_i)_{i \in \omega}$ are algebraically independent over $C$, and therefore so are $\text{acl}(A) = A^{al}$, $\text{acl}(B_i) = B_i^{al}$ over $\text{acl}(C) = C^{al}$. Observe also that there is a formula $\psi'(a, b, c)$ such that

$$\mathcal{U} \models \psi'(a, b, c) \iff F_\ell \models \psi(a, b, c).$$

Thus, we would have that $\{\psi'(a, b_i, c) \mid i \in \omega\}$ is inconsistent, which gives the desired contradiction.

2.3 The results of Cassidy

2.24. Before stating Cassidy’s results, we need some definitions and discussion of the various notions. We work inside a (sufficiently saturated) differentially closed field $\mathcal{U}$ of characteristic 0, and in this subsection definable will mean definable in the differential field $\mathcal{U}$.

Definition 2.25. (See Chapter II in [7]) An (affine) $\Delta$-algebraic group, or differential algebraic group, is a $\Delta$-closed subset of affine space, whose group laws are locally given by everywhere defined differential rational maps.

This context was extended to the non-affine setting, see e.g. [17] chapter 1 §2. A $\Delta$-algebraic group is then definable in $\mathcal{U}$, but there are definable groups which are not $\Delta$-algebraic.

Definition 2.26. (1) A (definable) group is linear if it (definably) embeds into some $\text{GL}_n(\mathcal{U})$. If the algebraic group $G$ has no infinite abelian quotient, then $G$ is linear.
(2) Recall that an algebraic group $G$ is simple if it is non-abelian, and has no proper connected normal algebraic subgroup. It is semi-simple if it connected, linear, and has no non-trivial connected abelian normal algebraic subgroup. Note that a finite center is allowed. A semi-simple algebraic group is isogenous to a (finite) product of simple normal algebraic subgroups, called its simple components. See [14], Theorem 27.5.

(3) A $\Delta$-algebraic group $G$ is $\Delta$-simple if it is non-abelian and has no proper connected normal $\Delta$-closed subgroup. Again, a finite center is allowed.

(4) Similarly, a linear $\Delta$-algebraic group $G$ is $\Delta$-semi-simple if it has no non-trivial connected normal abelian $\Delta$-closed subgroup.

The following results were shown by Cassidy in [8]:

**Theorem 2.27.** (Cassidy, [8], Theorem 14) Let $G$ be a connected linear $\Delta$-semi-simple $\Delta$-algebraic group. Then $G$ is $\Delta$-isomorphic to a Zariski dense $\Delta$-algebraic subgroup of a connected semi-simple algebraic group $H$.

**Theorem 2.28.** (Cassidy, [8], Theorem 15) Let $G$ be a Zariski dense connected $\Delta$-closed subgroup of a semi-simple algebraic group $A \leq \text{GL}(n,U)$, with simple components $A_1, \ldots, A_t$. Then there exist connected nontrivial $\Delta$-simple normal $\Delta$-closed subgroups $G_1, \ldots, G_t$ of $G$ such that

1. If $i \neq j$, then $[G_i, G_j] = 1$.
2. The product morphism $G_1 \times \cdots \times G_t \to G$, $(g_1, \ldots, g_t) \mapsto g_1g_2\cdots g_t$, is onto, with finite kernel.
3. $G_i$ is the identity component of $G \cap A_i$, and is Zariski dense in $A_i$.
4. $G$ is $\Delta$-semi-simple.

**Remarks 2.29.** Note the following consequence: Let $G$ be a group definable in $U$, with no definable subgroup of finite index, and with no definable infinite abelian quotient. Assume that if $A$ is a definable infinite abelian subgroup, then $N_G(A)$ has infinite index in $G$. Then there is a definable homomorphism $\varphi : G \to H(U)$ for some semi-simple algebraic group $H$, with $\varphi(G)$ Zariski dense in $H$ and $\text{Ker}(\varphi)$ finite. Indeed, Pillay’s Theorem 2.13 gives the existence of a definable embedding $\varphi : G \to H(U)$ for some algebraic group $H$, and we may assume that $\varphi(G)$ is Zariski dense in $H$. Thus $H$ has no infinite definable abelian quotient, since $[G, G] = G$ implies $[H, H] = H$, so that $H$ is linear. Further, if $A$ is a normal abelian algebraic subgroup of
$H$, then $A(U) \cap \varphi(G)$ is normal in $\varphi(G)$, and therefore must be finite. Replace $H$ by $H/A$, and compose $\varphi$ with the projection $H \to H/A$. Applying if necessary the same procedure to $H/A$, we may assume that some quotient of $H$ is semi-simple, and that the map $\varphi : G \to H(U)$ has finite kernel.

By quantifier-elimination in $DCF_m$, we know that $\varphi(G)$ is a $\Delta$-algebraic subgroup of $H(U)$.

This is slightly stronger than Cassidy’s result, since our group $G$ is only “$L_\Delta$-definable”, i.e., is a Boolean combination of $\Delta$-closed sets.

**Theorem 2.30.** (Cassidy, [8], Theorem 19). Let $H$ be a simple algebraic group defined and split over $\mathbb{Q}$, and $G \leq H(U)$ be a $\Delta$-algebraic subgroup which is Zariski dense in $H$. Then $G$ is definably isomorphic to $H(L)$, where $L$ is the constant field of a set $\Delta'$ of commuting derivations. Furthermore, the isomorphism is given by conjugation by an element of $H(U)$.

**Remarks 2.31.** Cassidy’s Theorem 2.30 is stated in different terms. Instead of speaking of simple algebraic groups, defined and split over $\mathbb{Q}$ in [8], she speaks of simple Chevalley groups. In fact, many of her results are stated in terms of Chevalley groups, but we chose not to do that. Recall that any simple algebraic group is isomorphic to one which is defined and split over the prime field, $\mathbb{Q}$ in our case.

Note also that the field $L$ of Theorem 2.30 is algebraically closed. We will therefore be able to use Fact 2.32 below.

**Fact 2.32.** Let $G$ be a simple algebraic group, let $K$ be an algebraically closed field. Then

(a) The group $G(K)$ has no infinite normal subgroup;

(b) The field $K$ is definable in the pure group $G(K)$.

Both assertions are well-known, but we were not able to find an easy reference for the first assertion: it follows from the fact that if $g \in G(K) \setminus Z(G(K))$, then the infinite set $(g^{G(K)}g^{-1})$ is Zariski closed and irreducible, contains 1, is closed under conjugation, and therefore generates a Zariski closed normal subgroup of $G(K)$, which must equal $G(K)$ since $G$ has no proper infinite normal algebraic subgroup. The second is also well-known, see for instance Theorem 3.2 in [18].

### 2.4 Quantifier-free canonical bases

As $DCF_mA$ is supersimple there is a notion of canonical basis for complete types which is defined as a sort of amalgamation basis, and is not easy to
describe. In our case, we will focus on an easier concept: canonical bases of quantifier-free types. They are defined as follows:

We work in a model \((U, \sigma, \Delta)\) of \(DCF_m\). Let \(a\) be a finite tuple in \(U\), and \(K \subset U\) a difference-differential field. We define the quantifier-free canonical basis of \(tp(a/K)\), denoted by \(qf-Cb(a/K)\), as the smallest difference-differential subfield \(k\) of \(K\) such that \(k(a)_{\sigma, \Delta}\) and \(K\) are linearly disjoint over \(k\). Another way of viewing this field is as the smallest difference-differential subfield of \(K\) over which the smallest \(K\)-definable \(\sigma\)-\(\Delta\)-closed set containing \(a\) is defined (this set is called the \(\sigma\)-\(\Delta\)-locus of \(a\) over \(K\)). Analogous notions exist for \(DCF_m\) and ACFA. We were not able to find explicit statements of the following easy consequences of the Noetherianity of the \(\sigma\)-\(\Delta\)-topology, so we will indicate a proof.

**Lemma 2.33.** Let \(a, K \subset U\) be as above.

1. \(qf-Cb(a/K)\) exists and is unique; it is finitely generated as a difference-differential field.

2. Let \(K \subset M \subset K(a)_{\sigma, \Delta}\). Then \(M = K(b)_{\sigma, \Delta}\) for some finite tuple \(b\) in \(M\).

**Proof.** (1) Let \(n = |a|\), and write \(K\{y\}_{\sigma} = \bigcup_{r \in \mathbb{N}} K[r]\), where

\[
K[r] = K[\sigma^{i_1_i_1} \delta_2^{i_2} \cdots \delta_m^{i_m} y_j \mid 1 \leq j \leq n, |i| + \sum_j i_j \leq r].
\]

Then each \(K[r]\) is finitely generated over \(K\) as a ring, and is therefore Noetherian. For each \(r\), consider the ideal \(I[r] = \{f \in K[r] \mid f(a) = 0\}\), and the corresponding \(\sigma\)-\(\Delta\)-closed subset \(X[r]\) of \(U^n\) defined by \(I[r]\). Then the sets \(X[r]\) form a decreasing sequence of \(\sigma\)-\(\Delta\)-closed subsets of \(U^n\), which stabilises for some \(r\), which we now fix. Note that the ideal \(I[r]\) is a prime ideal (of the polynomial ring \(K[r]\)), and as such has a smallest field of definition, say \(k_0\), and that \(k_0\) is finitely generated as a field, and is unique. We now let \(k\) be the difference-differential field generated by \(k_0\).

**Claim 1.** \(k(a)_{\sigma, \Delta}\) and \(K\) are linearly disjoint over \(k\).

**Proof.** This follows from the fact that \(X[s] = X[r]\) for every \(s \geq r\).

(2) Consider \(B := qf-Cb(a/M)\). By (1), \(B\) is finitely generated as a difference-differential field.

**Claim 2.** \(KB = M\).
Proof. Indeed, by definition, \( B(a)_{\sigma, \Delta} \) and \( M \) are linearly disjoint over \( B \). Hence, \( KB(a)_{\sigma, \Delta} \) and \( M \) are linearly disjoint over \( KB \). But this is only possible if \( KB = M \). □

Remarks 2.34. Given fields \( K \subset L \) (of characteristic 0), the field \( L \) is a regular extension of \( L_0 := K^{alg} \cap L \). So, if \( L = K(a)_{\sigma, \Delta} \) for some (maybe infinite) tuple \( a \), then \( B := \text{qf-Cb}(a/K^{alg}) \) is contained in \( L_0 \), and we have \( KB = L_0 \): the inclusion is clear, and the reverse inclusion follows by noting that the linear disjointness of \( B(a)_{\sigma, \Delta} \) and \( K^{alg} \) over \( B \) implies the linearly disjointness of \( KB(a)_{\sigma, \Delta} \) and \( K^{alg} \) over \( KB \), whence \( KB \) must contain \( L_0 \).

In positive characteristic \( p \), if \( L \) is a separable extension of \( K \), then \( L \) and \( K^s \) are linearly disjoint over their intersection \( L_0 \): this is because \( K^s/K \) is Galois. We will use this remark later.

2.5 Some well-known group-theoretic results

Lemma 2.35. Let \( G_1 \) and \( G_2 \) be groups, and \( H \leq G_1 \times G_2 \) a proper subgroup, such that \( H \) projects onto \( G_1 \) and onto \( G_2 \) (via the two natural projections \( G_1 \times G_2 \to G_i \)). Let \( S_1 \) be defined by \( H \cap (G_1 \times (1)) = S_1 \times (1) \) and \( S_2 \) by \( H \cap ((1) \times G_2) = (1) \times S_2 \). Then \( S_i \) is a normal subgroup of \( G_i \), and \( H/(S_1 \times S_2) \) is the graph of a group isomorphism \( f : G_1/S_1 \to G_2/S_2 \).

Furthermore, if the \( G_i \) and \( H \) are definable, so are the \( S_i \) and the isomorphism \( f \).

Proof. Let \( g \in G_1 \); because \( H \) projects onto \( G_1 \), there is some \( g' \in G_2 \) such that \( (g, g') \in H \); thus \( (g, g')^{-1}(S_1 \times (1))(g, g') = g^{-1}S_1g \times (1) \) is contained in \( H \), which shows \( S_1 \leq G_1 \). Similarly, \( S_2 \leq G_2 \).

For the second assertion, we may assume that \( S_1 = S_2 = (1) \). I.e., given \( g \in G_1 \), there is a unique \( g' \in G_2 \) such that \((g, g') \in H \), and given \( g' \in G_2 \) there is a unique \( g \in G_1 \) such that \((g, g') \in H \): this exactly says that \( H \) is the graph of a bijection between \( G_1 \) and \( G_2 \), and a moment’s thought gives that it is an isomorphism.

The last statement is obvious. □

Corollary 2.36. Let \( H \leq G \times G_2 \) be as above, with \( G \) and \( H \) definable, centerless, and \( G_2 = \prod_{i=1}^s G'_i \), and where \( G \) and each \( G'_i \) is definable and definably simple. Then for some index \( i \), the image of \( H \) in \( G \times G'_i \), via the natural projection \( G \times G_2 \to G \times G'_i \), is the graph of a definable isomorphism \( f \) between \( G \) and \( G'_i \).

If moreover \( G = H(L) \) and the \( G'_i = H_i(L_i) \), where \( H \) and the \( H_i \) are simple algebraic groups defined over \( \mathbb{Q} \), and the fields \( L \) and \( L_i \) are fields of constants
of the differentially closed field \( U \), then \( f \) is the restriction of an algebraic isomorphism \( H \to H_i \) and \( L = L_i \).

**Proof.** In the notation of Lemma \[2.35\] \( S_1 = (1) \), and \( S_2 \) is a definable normal subgroup of \( \prod G'_j \), hence is a product of some of the factors, and in fact of \( s - 1 \) of them: \( H/(S_1 \times S_2) \) is the graph of a group isomorphism \( f \) between the definably simple group \( G \) and \( G_1/S_2 \), so \( G_1/S_2 \) must be (isomorphic via the natural projection to) one of the \( G'_i \).

The moreover part follows from results of Borel-Tits (see Theorem A in \[4\], or 2.7, 2.8 in \[30\], or Theorem 4.17 in \[25\]) which describe abstract isomorphisms between simple algebraic groups: there are an algebraic isomorphism \( \varphi : H \to H_i \), and field isomorphism \( \psi : L \to L_i \), such that \( f = \bar{\psi} \circ \varphi \), where \( \bar{\psi} \) is the isomorphism \( H_i(L) \to H_i(L_i) \) induced by \( \psi \). Since \( f \) and \( \varphi \) are definable, so is \( \bar{\psi} \). By Fact \[2.32\] (b), \( \psi \) is also definable. A result of S. Suer (Theorem 3.6 in \[32\]) then tells us that \( L = L_i \), and using Theorem \[2.11\] we obtain that \( \psi \) is the identity. Hence \( f = \varphi \). \( \Box \)

**Definition 2.37.** Let \( G_1 \leq H_i(U) \) be Zariski dense subgroups of the algebraic groups \( H_i \) for \( i = 1, 2 \), and let \( f : G_1 \to G_2 \) be an isomorphism. By abuse of language, we will say that \( f \) is an algebraic isomorphism if it is the restriction to \( G_1 \) of an algebraic isomorphism \( H_1 \to H_2 \).

### 3 The isogeny result

We work in a sufficiently saturated model \((U, \Delta, \sigma)\) of DCF\(_m\)A. We will often work in its reduct to \( L_\Delta \). Unless otherwise mentioned, definable will mean \( L_{\sigma,\Delta}\)-definable in \( U \) with parameters.

**Definition 3.1.** Let \( G_1, G_2, G \) be definable groups.

1. Recall that \( G_1 \) and \( G_2 \) are **definably isogenous** if there are definable subgroups \( H_i \) of \( G_i \) of finite index, finite normal subgroups \( S_i \) of \( H_i \) for \( i = 1, 2 \), and a definable isomorphism \( H_1/S_1 \to H_2/S_2 \).

2. We say that \( G \) is **definably quasi-simple** if \( G \) has no definable abelian subgroup of finite index, and if whenever \( H \) is a definable infinite subgroup of \( G \) of infinite index, then its normaliser \( N_G(H) \) has infinite index in \( G \).

3. We say that \( G \) is **definably quasi-semi-simple** if whenever \( H \) is a definable infinite abelian subgroup of \( G \), then its normaliser \( N_G(H) \) has infinite index in \( G \), and if \( G \) has no definable normal subgroup \( N \) such that \( G/N \) is definably isogenous to an infinite abelian group.
Remark 3.2. Conditions (2) and (3) will ensure that if there is a definable homomorphism \( f : G \to H(U) \) for some algebraic group \( U \), and with \( \text{Ker}(f) \) finite, then the Zariski closure of \( f(G) \) will be a linear algebraic group.

In our context (of a supersimple theory), a definable group will in general have infinitely many definable subgroups of finite index, so it will not have a smallest one. We devised these properties to take care of that problem, and we will show below that they are preserved by isogenies. We first need a lemma:

Lemma 3.3. Assume that \( G \) is an infinite definable group.

1. Assume that \( G \) is abelian, and let \( 0 \neq n \in \mathbb{N} \). Then \( [G : G^n] \) is finite.
2. Assume that \( G \) is definably isogenous to an abelian group. Then \( G \) has an infinite abelian subgroup \( A \) of finite index.

Proof. (1) By Lemma 2.18, we know that there are a subgroup \( A \) of finite index in \( G \), and a definable \( f : A \to H(U) \) for some algebraic group \( H \), with \( \text{Ker}(f) \) finite. Then it suffices to show the conclusion for \( f(A) \); because \( \text{Ker}(f) \) is finite, \( A^n \) has index at most \( |\text{Ker}(f)| \) in \( f^{-1}(f(A))^n \).

Hence we may assume that \( A \) is Zariski dense in \( H \). Then \( H \) is abelian as well, and we may assume it is connected.

Claim. The \( n \)-torsion subgroup \( Z \) of \( H(U) \) is finite.

The proof is by induction on \( \dim(H) \). Assume first that \( H \) has no proper algebraic subgroup. Then \( H \) is one of \( \mathbb{G}_a \), \( \mathbb{G}_m \) or a simple abelian variety. In all three cases, \( Z \) is finite (characteristic 0 implies that \( \mathbb{G}_a(U) \) is torsion-free).

Assume now that \( N \) is a minimal (proper, infinite) algebraic subgroup of \( H \); then by the first case, \( Z \cap N(U) \) is finite. The group \( Z N(U)/N(U) \) is contained in the \( n \)-torsion subgroup of \( (H/N)(U) \), which is finite by induction hypothesis. This proves the claim.

The claim implies that \( A \) and \( A^n \) have the same SU-rank, and therefore that \( A^n \) has finite index in \( A \). As \( [G : A] \) is finite and \( A^n \leq G^n \), we obtain the result.

(2) Let \( A \) be a definable subgroup of \( G \) of finite index, and \( Z \) a normal finite (of size \( n \)) subgroup of \( A \) such that \( A/Z \) is abelian and infinite.

Going to a definable subgroup of \( A \) of finite index \( (C_A(Z)) \), which has index bounded by the size of the largest abelian subgroup of \( \text{Sym}(n) \), we may assume that \( Z \) is central in \( A \). Thus the map

\[ A \times A \to Z, \ (g, h) \mapsto [g, h] \]

is a bilinear map. Then \( A^n \) is a definable subgroup of \( A \), which is abelian since every element of \( A^n \) is of the form \( g^n \) for some \( g \in A \). By (1), \( (A/Z)^n \)
Lemma 3.4. Let $G$ be a definable group, $G_0$ a definable subgroup of $G$ of finite index, and $Z$ a finite normal subgroup of $G$.

(1) Assume that $G/Z$ has a definable abelian subgroup $A'$, such that $N_{G/Z}(A')$ has finite index in $G/Z$. Then $G$ has a definable abelian subgroup $A$, with $N_G(A)$ of finite index. Furthermore, $[A' : ZA]$ is finite.

(2) $G$ is definably quasi-simple if and only if $G_0$ is definably quasi-simple.

(3) $G$ is definably quasi-simple if and only if $G/Z$ is definably quasi-simple.

(4) The assertions of items (2) and (3) hold with “quasi-semi-simple” in place of “quasi-simple”.

Proof. (1) Going to a definable subgroup of $G$ of finite index, we may assume that $Z$ is central in $G$. Let $A \leq G$ be the subgroup of $G$ which contains $Z$ and projects onto $A'$. Then $A$ is definable, and by Lemma 3.3, it has a definable abelian subgroup $A_0$ of finite index. The proof of Lemma 3.3 shows that in fact we may take $A_0 = A|Z|$. As $N_{G/Z}(A')$ has finite index in $G/Z$, so does $N_G(A)$ in $G$, because $Z$ is central in $G$, and so does $N_G(A_0)$ since $A_0$ is a characteristic subgroup of $A$.

(2) Suppose $G_0$ is definably quasi-simple, let $H$ be an infinite definable subgroup of $G$ of infinite index, and assume that $N_G(H)$ has finite index in $G$. Then $N_G(H) \cap N_G(G_0)$ has finite index in $G$, is contained in $N_G(H \cap G_0)$, and therefore $N_G(H \cap G_0) \cap G_0 = N_{G_0}(H \cap G_0)$ has finite index in $G_0$. However $[H : H \cap G_0]$ finite implies $H \cap G_0$ infinite; as $G_0$ is definably quasi-simple, $N_{G_0}(H \cap G_0)$ has infinite index in $G_0$, therefore in $G$, and we get the desired contradiction. That $G$ has no definable abelian subgroup of finite index is clear.

For the other direction, assume that $G$ is definably quasi-simple. Let $H$ be an infinite definable subgroup of $G_0$ of infinite index in $G_0$, and suppose that $N_{G_0}(H)$ has finite index in $G_0$; then $N_G(H)$, which contains $N_{G_0}(H)$, has finite index in $G$, which gives us the desired contradiction. Clearly if $A$ is a definable abelian subgroup of $G_0$ of finite index in $G_0$, then $A$ has finite index in $G$.

(3) By (2), going to a definable subgroup of $G$ of finite index, we may assume that $Z$ is central in $G$. We will first deal with the condition on definable abelian subgroups. If $A$ is a definable abelian subgroup of finite index in $G$, then so is $AZ/Z$ in $G/Z$. If $A$ is a definable abelian subgroup of $G/Z$
of finite index, use Lemma 3.3 to obtain a definable abelian subgroup $B$ of finite index in $G$.

Assume $G/Z$ is definably quasi-simple, and let $H$ be an infinite definable subgroup of $G$ of infinite index. Then $HZ/Z$ is infinite and has infinite index in $G/Z$, so its normalizer $N$ has infinite index in $G/Z$, and if $N' \geq Z$ is such that $N'/Z = N$, then $N'$ has infinite index in $G$ and normalizes $HZ$. As $Z$ is normal in $G$, we have $N_G(H)/Z \leq N$, which shows that $[G : N_G(H)]$ is infinite. The other direction is immediate because $Z$ is normal.

(4) To prove the equivalence of the first condition of definably quasi-semi-simple: reason as in (2) and (3), noting that if $Z$ is the center of $G$, $A' \leq G/Z$ is abelian definable and $N_{G/H}(A')$ has finite index in $G$, and if $A$ lifts $A'$, then $N_{G(A)}$ has also finite index in $G$, and normalises the characteristic abelian subgroup $A|Z$ of $A$ (see the proof of Lemma 3.3).

To prove the equivalence of the second condition: let $G_0$ be a definable subgroup of $G$ of finite index, and assume that it has a normal subgroup $N$ such that $G_0/N$ is isogenous to an abelian group (and therefore has a definable abelian subgroup of finite index, by Lemma 3.3). Without loss of generality, $G_0$ is normal in $G$, so that if $g \in G$, then $G_0/N^g$ is virtually abelian. As $N$ has finitely many conjugates, the map $G_0 \mapsto \prod_{g \in G/G_0} G_0/N^g$ has kernel $M := \bigcap N^g$, so that $G_0/M$ embeds into the virtually abelian group $\prod_{g \in G/G_0} G_0/N^g$. The other direction is clear: if $G/N$ is virtually abelian, then so is $G_0/N \cap G_0$.

Finally, let $Z$ be a finite normal subgroup of $G$, which we may assume to be central. A normal subgroup $N$ of $G/Z$ with $(G/Z)/N$ virtually abelian, pulls back to a normal subgroup $N'$ of $G$ with $G/N' \approx (G/Z)/N$; and if $N$ is a normal subgroup of $G$ with $G/N$ virtually abelian, then so is $GZ/NZ$.

**Proposition 3.5.** Let $G$ be a group definable in $U$, and assume that $G$ is definably quasi-simple (resp. definably quasi-semi-simple). Then there are a definable subgroup $G_0$ of finite index in $G$, a simple (resp. semi-simple) algebraic group $H$ defined and split over $\mathbb{Q}$, and a definable homomorphism $\phi : G_0 \to H(U)$, with finite kernel and Zariski dense image. Moreover, if $\bar{G}$ is the connected component of the Kolchin closure of $\phi(G_0)$, then $\bar{G}$ is $\Delta$-simple (resp. $\Delta$-semi-simple).

**Proof.** By Lemma 2.18 there is a definable subgroup $G_0$ of $G$ of finite index, and a definable homomorphism $\phi : G_0 \to H(U)$ where $H$ is a connected algebraic group, $\phi(G_0)$ is Zariski dense in $H$, and $\text{Ker}(\phi)$ is finite. Note that, as $G_0$ has no infinite definable quotient which is definably isogenous to an abelian group, the connected algebraic group $H$ has no abelian quotient, and therefore $H$ is linear. We will now show that we can assume that $H$ is...
simple (resp. semi-simple).

Assume first that $G$ is definably quasi-simple, and let $N$ be a maximal normal proper algebraic subgroup of $H$. Then $N(U) \cap \phi(G_0)$ is finite, and composing $\phi$ with the natural projection $H(U) \to (H/N)(U)$, replacing $H$ by $H/N$, we obtain that $H$ can be chosen to be simple.

Similarly, if $G$ is definably quasi-semi-simple, and because $H$ is linear, there is an algebraic connected normal subgroup $N$ of $H$, maximal among the connected normal solvable algebraic subgroups of $H$. Then $H/N$ has no proper connected normal abelian algebraic subgroup, and therefore is semi-simple. An easy induction on the class of solvability of $N$ shows that $N(U) \cap \phi(G_0)$ must be finite. Again, we may replace $H$ by $H/N$ to obtain that $H$ is semi-simple.

Let $\bar{G}$ be the Kolchin closure of $\phi(G_0)$ in $H$; replacing $G_0$ by a subgroup of finite index, we may assume that $\bar{G}$ is connected for the Kolchin topology. Then $\bar{G}$ is Zariski dense in $H$, and by Theorem 2.28 $\bar{G}$ is $\Delta$-semi-simple. Further, if $H$ is simple, then $\bar{G}$ is $\Delta$-simple. That $H$ can be taken defined and split over $\mathbb{Q}$ is because every semi-simple algebraic group is isomorphic to one such.

\section{Definable subgroups of semi-simple algebraic groups}

In this section we give a description of Zariski dense definable subgroups of simple and semi-simple algebraic groups. We work in a sufficiently saturated model $(\mathcal{U}, \Delta, \sigma)$ of DCF$_m$A. Unless otherwise mentioned, definable will mean $L_{\sigma, \Delta}$-definable with parameters.

\textbf{Theorem 4.1.} Let $H$ be a simple algebraic group defined and split over $\mathbb{Q}$, and $G$ a definable subgroup of $H(U)$ which is Zariski dense in $H$. Then $G$ has a definable subgroup $G_0$ of finite index, the Kolchin closure of which is conjugate to $H(L)$ (say by an element $g \in H(U)$), where $L$ is a field of constants in $\mathcal{U}$. Furthermore, either $G_0^\sigma = H(L)$, or $G_0^\sigma \subseteq H(\text{Fix}(\sigma^\ell)(L))$ for some integer $\ell \geq 1$. In the latter case, if $H$ is centerless, we are able to describe precisely the subgroup $G_0^\sigma$ as $\{h \in H(L) \mid \sigma^n(h) = \varphi(h)\}$ for some $n$ and algebraic automorphism $\varphi$ of $H(L)$.

\textbf{Proof.} Replacing $G$ by a subgroup of finite index, we may assume that the Kolchin closure $\bar{G}$ of $G$ is connected. Then $\bar{G}$ is also Zariski dense in $H$, and by Theorem 2.30 $\bar{G}$ is conjugate to $H(L)$, for some field of constants $L \leq \mathcal{U}$.
We may therefore assume that $G \leq H(L)$.

The strategy is the same as in the proof of Proposition 7.10 in [10]. Going to the $σ$-closure of $G$ within $H(L)$, and then to a subgroup of finite index, we may assume that $G$ is quantifier-free definable, and that it is connected for the $σ$-$Δ$-topology. If $G = H(L)$, then we are done. Assume that $G \neq H(L)$. We will first do the case when $H$ is centerless.

In the notation of Remark 2.17(c), let $n$ be the smallest integer such that the $Δ$-algebraic group $G_{(n)}$ is not equal to $H(L) \times σ(H(L)) \times \cdots \times σ^n(H(L))$ (Here we use that $G$ is quantifier-free definable and connected for the $σ$-$Δ$-topology). If $π$ is the projection on the last factor $σ^n(H(L))$, then $π(G_{(n)}) = σ^n(H(L))$: this is because if $h$ is a generic of $G$ for the $σ$-$Δ$-topology, then $h$, $σ^n(h)$ are generics of $H(L)$, $σ^n(H(L))$ respectively for the Kolchin topology. As $G_{(n)}$ projects onto $G_{(n-1)}$, it follows that $G_{(n)} \leq G_{(n-1)} \times σ^n(H(L))$ satisfy the hypotheses of Lemma 2.35 and its Corollary 2.36.

Using Corollary 2.36 and the fact that $H$ is defined over $\mathbb{Q}$, it follows that $σ^i(L) = L$ for some $i$, and that for some algebraic automorphism $ϕ$ of $H(L)$, the group $G_{(n)}/\prod_{0 \leq j \leq n,j \neq i} σ^j(H(L))$ defines the graph of the restriction of $ϕ$ to $H(L)$. By minimality of $n$, we have $i = n$, $σ^n(L) = L$.

Thus, the group $G$ will be defined by the equation $σ^n(g) = φ(g)$ within $H(L)$ (the left to right inclusion is clear; equality comes from the fact that both $G$ and $\{h \in H(L) \mid σ^n(h) = φ(h)\}$ are quantifier-free definable and connected for the $σ$-$Δ$-topology).

We now show, still with $H$ centerless, that if $G \neq H(L)$ is as above, then $G \leq H(\text{Fix}(σ^ℓ)(L))$ for some $ℓ$. By Proposition 14.9 of [13], the group $\text{Inn}(H)$ of inner automorphisms of $H(L)$ has finite index in the group $\text{Aut}(H)$ of algebraic automorphisms of $H(L)$. Moreover $σ^n$ induces a permutation of $\text{Aut}(H)/\text{Inn}(H)$, and hence there are some $r \in \mathbb{N}^*$ and $h \in H(L)$ such that

$$σ^{n(r-1)}(ϕ) \circ σ^{n(r-2)}(ϕ) \circ \cdots \circ ϕ = λ_h,$$

where $λ_h$ is conjugation by $h$. I.e., our group $G$ is contained in the subgroup $G'$ of $H(L)$ defined by $\{g \in H(L) \mid σ^nr(g) = λ_h(g)\}$.

By the existential closedness of $U$, there is some $u \in H(L)$ such that $σ^nr(u) = h^{-1}u$. So, if $g \in G'$, then

$$σ^nr(u^{-1}gu) = σ^nr(u^{-1})λ_h(g)σ^nr(u) = u^{-1}h(h^{-1}gh)(h^{-1}u) = u^{-1}gu.$$

I.e., $u^{-1}G'u \leq H(\text{Fix}(σ^nr) \cap L)$. 

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This does the case when $H$ is centerless. Assume now that the center $Z$ of $H$ is non-trivial. By the first part we know that there are $u \in H(U)$ and $\ell \geq 1$ such that $(u^{-1}GZu)/Z \leq (H/Z)(\text{Fix}(\sigma^\ell(L)))$. Since $Z$ is finite and characteristic, there is some $s \in \mathbb{N}$ such that for all $a \in Z$, we have $\prod_{i=0}^{s-1} \sigma^i(a) = 1$. If $g \in u^{-1}Gu$, then $\sigma^\ell(g)g^{-1} \in Z$; hence $\sigma^\ell(g)g^{-1} = 1$, and $u^{-1}Gu \leq H(\text{Fix}(\sigma^\ell))$.

**Corollary 4.2.** Let $G$ be an infinite group definable in $\mathcal{U}$, and suppose that $G$ is definably quasi-simple. Then there are a simple algebraic group $H$ defined and split over $\mathbb{Q}$, a definable subgroup $G_0$ of $G$ of finite index, and a definable group homomorphism $\phi : G_0 \to H(U)$, with the following properties:

1. $\text{Ker}(\phi)$ is finite.
2. The Kolchin closure of $\phi(G_0)$ is $H(L)$ for some field $L$ of constants in $\mathcal{U}$.
3. Either $\phi(G_0) = H(L)$, or for some integer $\ell$, $\phi(G_0)$ is a subgroup of $H(\text{Fix}(\sigma^\ell) \cap L)$.

**Proof.** By Proposition 3.5 we can reduce to the case where $G$ is a definable subgroup of a simple algebraic group $H$. Then apply Theorem 4.1 to conclude.

**Lemma 4.3.** Let $H$ be a simple algebraic group, defined and split over $\mathbb{Q}$, let $L \leq U$ be a field of constants, and let $\varphi$ be an algebraic automorphism of $H$. Let $\ell \geq 1$, and consider the subgroup $G \leq H(L)$ defined by $\sigma^\ell(g) = \varphi(g)$. Then $G$ is definably quasi-simple.

**Proof.** By Lemma 3.4 we may assume that $Z(H) = (1)$. Note that, as $\varphi$ is defined over $\mathbb{Q}^{alg}$ and therefore over $L$, the existential closedness of $\mathcal{U}$ gives that $G$ is Kolchin dense in $H(L)$, and in particular Zariski dense in $H$, so that no infinite abelian subgroup of $G$ can have finite index in $G$ (since its Zariski closure would be abelian). Note also that as $H(L)$ is connected (for the Kolchin topology), the group $G$ is connected for the $\sigma$-$\Delta$-topology.

Let $U$ be an infinite definable subgroup of $G$ of infinite index, and assume by way of contradiction that its normalizer $N$ has finite index in $G$.

Consider $p_\ell$ as defined in Remark 2.17(c), and $U(\ell) \leq G(\ell)$. Then

$$U(\ell) \trianglelefteq N(\ell) = G(\ell).$$

Indeed, we know that $p_\ell(U)$ and $p_\ell(N)$ are Kolchin dense in $U(\ell)$ and $N(\ell)$ respectively, with $p_\ell(U) \trianglelefteq p_\ell(N)$; hence any element of $p_\ell(N)$ normalizes the
Kolchin closure $U_{(t)}$ of $p_t(U)$, and since $N_{G_{(t)}}(U_{(t)})$ is Kolchin closed we get $U_{(t)} \leq N_{(t)}$; that $N_{(t)} = G_{(t)}$ is because $[G : N]$ is finite and the group $G_{(t)}$ is connected for the Kolchin topology. In particular, $U_{(0)} \leq G_{(0)} = H(L)$, and as the group $H(L)$ is simple (by Fact 2.32(a)), the Kolchin closure of $U$ must be $H(L)$.

Moreover, as every generic of $U$ is a generic of its $\sigma$-$\Delta$-closure $\tilde{U}$, it follows that $G$ normalizes $\tilde{U}$. So, we may replace $U$ by $\tilde{U}$; then $G$ also normalises the connected component of $U$ (for the $\sigma$-$\Delta$-topology), and so we may assume that $U$ is $\sigma$-$\Delta$-closed and connected. By Theorem 4.1, for some $r \leq \ell$ and algebraic automorphism $\psi$ of $H(L)$, the group $U$ is defined within $H(L)$ by the equation $\sigma^r(g) = \psi(g)$. We will show that this is impossible unless $r = \ell$ (and $\psi = \varphi$). Indeed, suppose that $r < \ell$, take a generic $(u, g)$ of $U \times G$ over all parameters necessary to define $G$, $\varphi$, $U$ and $\psi$. Consider now $(u, \sigma^r(u))$, and $(g, \sigma^r(g))$. The elements $u$, $g$ and $\sigma^r(g)$ are independent generics of the algebraic group $H$. Then

$$\sigma^r(g^{-1}ug) = \sigma^r(g^{-1})\psi(u)\sigma^r(g), \psi(g^{-1}ug) = \psi(g^{-1})\psi(u)\psi(g).$$

However, since $u \in U \leq G$, we have $\sigma(g^{-1}ug) = \psi(g^{-1}ug)$, i.e., $\sigma^r(g)\psi(g^{-1}) \in C_H(\psi(u))$. As $\psi$ is an automorphism of $H$, the elements $\sigma^r(g), \psi(g)$ and $\psi(u)$ are independent generics of $H$; this gives us the desired contradiction, as $\sigma^r(g)\psi(g)^{-1}$ and $\psi(u)$ are independent generics of the non-abelian algebraic group $H$. □

4.4. The semi-simple case needs some additional lemmas. Indeed, Zariski denseness, or even Kolchin denseness, and the previous results do not suffice to give a complete description. Here is a typical example: Let $H$ be a simple algebraic group defined and split over $\mathbb{Q}$, and consider the subgroup $G$ of $H(U)^2$ defined by

$$G = \{(g_1, g_2) \in H(U)^2 \mid \sigma(g_1) = g_2\}.$$ 

Then $G$ is Kolchin dense in $H(U)^2$, however $G$ is isomorphic to $H(U)$, via the projection on the first factor. We will now prove several lemmas which will allow us to bypass this difficulty.

**Lemma 4.5.** Let $G_1, \ldots, G_t$ be centerless $\Delta$-simple $\Delta$-algebraic groups, with $G_i$ Zariski dense in some simple algebraic group $H_i$, and $G \leq G_1 \times \cdots \times G_t$ a $\Delta$-algebraic connected (for the Kolchin topology) subgroup, which projects via the natural projections onto each $G_i$. Then there are a set $\Psi \subset \{1, \ldots, t\}^2$, and algebraic isomorphisms $\psi_{i,j} : H_i \to H_j$ whenever $(i, j) \in \Psi$, such that

$$G = \{(g_1, \ldots, g_t) \in \prod_{i=1}^t G_i \mid g_j = \psi_{i,j}(g_i), (i, j) \in \Psi\}.$$ 

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Moreover, if \((i, j), (j, k) \in \Psi\) with \(k \neq i\), then \((j, i), (i, k) \in \Psi\), \(\psi_{j,i} = \psi_{i,j}^{-1}\), and \(\psi_{i,k} = \psi_{j,k}\psi_{i,j}\).

Furthermore, \(G\) has no subgroup of finite index.

Proof. By Corollary 2.36 if \(G\) projects onto \(\prod_{i=2}^{t} G_i\) and onto \(G_1\), but \(G \neq \prod_{i=1}^{t} G_i\), then there is some index \(i \geq 2\), and an isomorphism \(\psi_{1,i} : G_1 \to G_i\) such that

\[
G = \{(g_1, \ldots, g_t) \in \prod G_i \mid g_i = \psi_{1,i}(g_1)\}.
\]

We define \(\Psi\) be the set of pairs \((i, j) \in \{1, \ldots, t\}^2\) such that the image \(G_{i,j}\) of \(G\) under the natural projection \(\prod_{\ell=1}^{t} H_\ell \to H_i \times H_j\) is a proper subgroup of \(G_i \times G_j\). By Corollary 2.36 again, if \((i, j) \in \Psi\), then \(G_{i,j}\) is the graph of an algebraic isomorphism \(G_i \to G_j\) (i.e., the restriction to \(G_i\) of an algebraic isomorphism \(H_i \to H_j\)). Then the set \((\Psi, \psi_{i,j})\) satisfies the moreover part of the conclusion, and we have

\[
G \leq \{(g_1, \ldots, g_t) \in \prod_{i=1}^{t} G_i \mid g_j = \psi_{i,j}(g_i), (i, j) \in \Psi\}.
\]

To prove equality, we let \(T \subset \{1, \ldots, t\}\) be maximal such that whenever \(i, j \in T\), then \((i, j) \notin \Psi\); then the natural projection \(\prod_{\ell \in T} H_\ell \to \prod_{\ell \in T} H_\ell\) defines an injection on \(G\), and sends \(G\) to a subgroup \(G'\) of \(\prod_{\ell \in T} G_\ell\), with the property that whenever \(k \neq \ell \notin T\), then \(G'\) projects onto \(G_k \times G_\ell\). By the first case and an easy induction, this implies that \(G' = \prod_{\ell \in T} G_\ell\). The last assertion follows from it being true for each \(G_i\) by Theorem 2.30 and Fact 2.32.

\[\tag{2.32}\]

**Lemma 4.6.** Let \(H_1, \ldots, H_r\) be simple centerless algebraic groups defined and split over \(\mathbb{Q}\), \(L_1, \ldots, L_r\) \(\Delta\)-closed subfields of \(\mathcal{U}\), and \(G \leq \prod_{i=1}^{r} H_i(L_i)\) a Kolchin dense quantifier-free definable subgroup, which is connected for the \(\sigma\)-\(\Delta\)-topology. Let \(\hat{G}_i \leq H_i(L_i)\) be the \(\sigma\)-\(\Delta\)-closure of the projection of \(G\) on the \(i\)-th factor \(H_i(L_i)\).

Then there is a partition of \(\{1, \ldots, r\}\) into subsets \(I_1, \ldots, I_s\), such that for each \(1 \leq k \leq s\), the following holds:

If \(i \neq j \in I_k\), then there are an integer \(n_{ij} \in \mathbb{Z}\) and an algebraic isomorphism \(\theta_{ij} : H_i(L_i) \to H_j(\sigma^{n_{ij}}(L_j))\) such that if \(\pi_{I_k}\) is the projection \(\prod_{j \in I_k} H_j(L_j) \to \prod_{j \in I_k} H_j(L_j)\), and \(i \in I_k\) is fixed, then

\[
\pi_{I_k}(G) = \{(g_j)_{j \in I_k} \in \prod_{j \in I_k} H_j(L_j) \mid \theta_{ij}(g_i) = \sigma^{n_{ij}}(g_j) \text{ if } j \neq i\}.
\]

Moreover, \(G \simeq \prod_{k=1}^{s} \pi_{I_k}(G)\), and \(G\) projects onto each \(\hat{G}_i\).

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Proof. We use the prolongations $p_n$ defined in \[\text{2.17}\] and choose $N$ large enough so that $G = \{g \in \prod_{i=1}^{r} H_i(L_i) \mid p_N(g) \in G(N)\}$. Then $G(N)$ is a $\Delta$-algebraic subgroup of

$$
\prod_{i=1}^{r} (\tilde{G}_i)_{(N)} \leq \prod_{1 \leq i \leq r, 0 \leq k \leq N} H_i(\sigma^k(L_i)).
$$

Let $\Psi \subset \{(1, \ldots, r) \times \{0, \ldots, N\}\}^2$ be the set of pairs given by Lemma 4.5 and $\psi_{(i,k),(j,\ell)}$, $((i, k), (j, \ell)) \in \Psi$, the corresponding set of algebraic isomorphisms

$$
\psi_{(i,k),(j,\ell)} : H_i(\sigma^k(L_i)) \to H_j(\sigma^\ell(L_j)).
$$

So, if $(g_1, \ldots, g_r) \in G$, then

$$
\psi_{(i,k),(j,\ell)}(\sigma^k(g_i)) = \sigma^\ell(g_j). \quad (1)
$$

Note the following, whenever $((i, k), (j, \ell)) \in \Psi$:

- If $k + 1, \ell + 1 \leq N$, then $((i, k + 1), (j, \ell + 1)) \in \Psi$, with $\psi_{(i,k+1),(j,\ell+1)} = \psi_{(i,k),(j,\ell)}^{\sigma}$ (here, $\psi_{(i,k),(j,\ell)}^{\sigma}$ denotes the isomorphism obtained by applying $\sigma$ to the coefficients of the isomorphism $\psi_{(i,k),(j,\ell)}$);
- If $k, \ell \geq 1$, then $((i, k - 1), (j, \ell - 1)) \in \Psi$, with $\psi_{(i,k-1),(j,\ell-1)} = \psi_{(i,k),(j,\ell)}^{\sigma^{-1}}$;
- If $k \leq \ell$, then applying $\sigma^{-k}$ to equation (1) gives

$$
((i, 0), (j, \ell - k)) \in \Psi, \text{ and } \psi_{(i,k),(j,\ell)} = \psi_{(i,0),(j,\ell-k)}^{\sigma^k}.
$$

- Finally, if $i = j$ and $k < \ell$, then $\tilde{G}_i$ is defined by an equation $\sigma^{n_i}(g) = \varphi_i(g)$ within $H_i(L_i)$ for some integer $n_i$ and algebraic automorphism $\varphi_i$ of $H(L)$, $((i, 0), (i, n_i)) \in \Psi$ with associated isomorphism $\psi_{(i,0),(i,n_i)} = \varphi_i$, and $\ell - k$ is a multiple of the integer $n_i$. This is because $G$ projects onto a subgroup of finite index of $\tilde{G}_i$, and therefore the $\Delta$-algebraic group $G(N)$ projects onto the $\Delta$-algebraic group $(\tilde{G}_i(N))$. However, from Fact 2.32 and the definition of $\tilde{G}_i$ within $H_i(L)$, we deduce that $\tilde{G}_i$ has no subgroup of finite index, and therefore that $G$ projects onto each $\tilde{G}_i$.

By Lemma 4.5, we know that the set $\Psi$ and the $\psi_{i,j}$ completely determine the $\tilde{G}_i$, and by the above observations, each condition $\sigma^k(g_i) = \psi_{(i,k),(j,\ell)}(\sigma^\ell(g_j))$ is implied by

$$
\sigma^{k-\ell}(g_i) = \psi_{(i,k),(j,\ell)}^{-\ell}(g_j). \quad (2)
$$
The set $\Psi$ defines a structure of graph on $\{1, \ldots, r\} \times \{0, \ldots, N\}$, which in turn induces a graph structure on $\{1, \ldots, r\}$, by $E(i, j)$ iff there are some $k, \ell$ such that $((i, k), (j, \ell)) \in \Psi$. If $E(i, j)$, then the isomorphism $\tilde{G}_i \to \tilde{G}_j$ is given by equation (2). Then $\{\{1, \ldots, r\}, E\}$ has finitely many connected components, say $I_1, \ldots, I_s$, and for each $r$, if $i \in I_r$, then $I_r = \{i\} \cup \{j \mid E(i, j)\}$. Lemma 4.5 shows that $G = \prod_{r=1}^s \pi_{I_r}(G)$, and gives the desired description of $\pi_{I_r}(G)$, with $\theta_{i,j} = \psi_{(i,k),(j,\ell)}^{i-\ell}$ and $n_{i,j} = k - \ell$, if $((i, k), (j, \ell)) \in \Psi$ and $i \in I_r$. 

**Theorem 4.7.** Let $G$ be a definable subgroup of $H(\mathcal{U})$, where $H$ is a semi-simple algebraic group defined and split over $\mathbb{Q}$, and with trivial center. Assume that $G$ is Zariski dense in $H$.

1. Assume that the $\sigma$-$\Delta$-closure of $G$ is connected for the $\sigma$-$\Delta$-topology. Then there are $s$ and simple normal algebraic subgroups $H_1, \ldots, H_s$ of $H$, a projection $\pi : H \to H_1 \times \cdots \times H_s$ which restricts to an injective map on $G$, fields of constants $L_i$ in $\mathcal{U}$, definable subgroups $G_i$ and $G'_i$ of $H_i(L_i)$ for $1 \leq i \leq s$, and $h \in \pi(H)(\mathcal{U})$, such that

$$G_1 \times \cdots \times G_s \leq h^{-1} \pi(G)h \leq G'_1 \times \cdots \times G'_s,$$

and each $G_i$ is a normal subgroup of finite index of $G'_i$.

2. Assumptions as in (1). If in addition $G$ is $\sigma$-$\Delta$-closed, then $h^{-1} \pi(G)h = G_1 \times \cdots \times G_s$, and for each $i$, either $G_i = H_i(L_i)$, or for some integer $\ell_i$ and automorphism $\varphi_i$ of $H_i(L_i)$, $G_i$ is defined within $H_i(L_i)$ by $\sigma^{\ell_i}(g) = \varphi_i(g)$.

**Proof.** By Theorem 2.28 if $H_1, \ldots, H_s$ are the simple algebraic components of $H$, and $\tilde{G}$ is the Kolchin closure of $G$, then $\tilde{G}$ is $\Delta$-semi-simple; if $G_i$ is the connected (for the $\Delta$-topology) component of $G \cap H_i(\mathcal{U})$, then the morphism $\rho : \tilde{G}_1 \times \cdots \times \tilde{G}_s \to \tilde{G}$ is an isogeny, and because $H$ is centerless, is an isomorphism.

By Theorem 2.28 we know that there are $\Delta$-definable subfields $L_i$ of $\mathcal{U}$, such that each $G_i$ is conjugate to $H_i(L_i)$ within $H_i(\mathcal{U})$. But as $[H_i, H_j] = 1$ for $i \neq j$, there is $h \in H(\mathcal{U})$ such that $h^{-1} \tilde{G}_i h \leq H_i(L_i)$ for all $i$. We will replace $G$ by $h^{-1}Gh$, so that $\tilde{G}_i = H_i(L_i)$ for every $i$.

(1) For each $i$, consider the projection $\pi_i$ on the $i$-th factor $H_i(L_i)$, and let $G'_i = \pi_i(G)$. Further, let $G_i = H_i(L_i) \cap G$. So, $G_1 \times \cdots \times G_r$ is a subgroup of $G$.

**Claim 1.** $G'_i$ is Kolchin dense in $H_i(L_i)$, for $i = 1, \ldots, r$. 

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Proof. Since $G$ is Kolchin dense in $\tilde{G}$, any generic $g := (g_1, \ldots, g_r)$ of $G$ is a generic of the $\Delta$-algebraic group $\tilde{G}$. Then $g_i$ is a generic of $H_i(L_i)$ for all $i$, and the claim is proved. □

Claim 2. For all $i \in \{1, \ldots, r\}$, $G_i \leq G'_i$.

Proof. Let $q : H \to H_2 \times \cdots \times H_r$ be the projection on the last $r - 1$ factors. Then $G \cap \text{Ker}(q)$ is normal in $H$, contained in $H_1(L_1) \times (1)^{r-1}$, and equals $G_1 \times (1)^{r-1}$. As $G$ projects onto $G'_1$, we get $G_1 \leq G'_1$. The proof for the other indices is similar. □

Claim 3. If $G_i \neq (1)$, then $[G'_i : G_i] < \infty$. If moreover $G$ is quantifier-free definable, then $G_i = G'_i$.

Proof. Both $G_i$ and $G'_i$ are definable subgroups of the simple $\Delta$-algebraic group $H_i(L_i)$ and $G'_i$ is Kolchin dense in $H_i(L_i)$.

If $G'_i = H_i(L_i)$, then $G_i = G'_i$ since $H_i(L_i)$ is a simple (abstract) group (by 2.32 and because $Z(H) = (1)$). If $G'_i \neq H_i(L_i)$, then by Theorem 4.1 Claim 1 and Lemma 4.3 $G'_i$ is definably quasi-simple. Moreover, since $G'_i$ is Zariski dense in the centerless algebraic group $H_i$ (by Claim 1), it follows that $G'_i$ has no finite normal subgroup, and so that any definable normal subgroup of $G'_i$ must have finite index in $G_i$. Together with Claim 2, this gives the result when $G$ is definable.

If $G$ is quantifier-free definable, so is every $G_i$, and therefore $G_i$ is closed in the $\sigma$-$\Delta$-topology. This implies that $G_i = G'_i$, because $G$, and therefore also $G'_i$, is connected for the $\sigma$-$\Delta$-topology. □

If all $G_i$ are non-trivial, we have shown that our group $G$ is squeezed between $G_1 \times \cdots \times G_r$ and $G'_1 \times \cdots \times G'_r$. And that if $G$ is quantifier-free definable, then $G = \prod_{i=1}^r G_i$.

Assume now that some $G_i$ are trivial. If $\tilde{G}$ denotes the $\sigma$-$\Delta$-closure of $G$, and $G_i$ the $\sigma$-$\Delta$-closure of $G'_i$, then these groups are connected for the $\sigma$-$\Delta$-topology, quantifier-free definable, and $\tilde{G}$ is a proper subgroup of $\prod_{i=1}^r G_i$. Hence Lemma 4.6 applies, and gives a subset $T$ of $\{1, \ldots, r\}$ such that the natural projection $\pi_T$ defines an isomorphism $\tilde{G} \to \prod_{i \in T} \tilde{G}_i$, which restricts to an embedding $G \to \prod_{i \in T} G'_i$ with $\prod_{i \in T} G'_i : \pi_T(G) < \infty$. Moreover, applying Claim 3 to $G''_i := \pi_T(G) \cap H_i(L_i)$, $i \in T$, we get

$$\prod_{i \in T} G''_i \leq \pi_T(G) \leq \prod_{i \in T} G'_i,$$

with $G''_i$ a normal subgroup of $G'_i$ of finite index. This finishes the proof of (1) (modulo a change of notation).

We showed that $\pi_T(G) = \prod_{i \in T} G_i$, which, together with Theorem 4.1 proves (2). □
Remarks 4.8. In the case of $Z(H) \neq (1)$, we can obtain a similar result in a particular case: let $H_i(L_i)$ are the subgroups of $G$ given by Theorem 2.28 and define $G_i = G \cap H_i(L_i)$ as above. Then if all $G_i$ are infinite or trivial, the same proof gives some subset $T$ of $\{1, \ldots, r\}$, and an isogeny $\prod_{i \in T} G_i$ onto a subgroup of finite index of $G$.

In the general case, however, we can only obtain such a representation of a proper quotient of $G$: the problem arises from the fact that the groups $G_i$ may be finite non-trivial, so that the projection $\pi_T$ defined in the proof will restrict to an isogeny on $G$. So, we might as well work with the image of $G$ in $H/Z(H)$.

5 Definable subgroups of finite index

We work in a sufficiently saturated model $(U, \sigma, \Delta)$ of DCF$_mA$. Unless otherwise mentioned, definable will mean $L_{\sigma, \Delta}$-definable with parameters.

The aim of this section is to show that a definably quasi-simple group definable in $U$ has a definable connected component. To do that, we investigate definable subgroups of algebraic groups which are not quantifier-free definable, and obtain a description similar to the one obtained by Hrushovski and Pillay in Proposition 3.3 of [13].

Theorem 5.1. Let $H$ be an algebraic group, $G \leq H(U)$ a Zariski dense definable subgroup. Then there are an algebraic group $H'$, a quantifier-free definable subgroup $R \leq H'(U)$, together with a quantifier-free definable homomorphism $f : R \to G$, with $f(R)$ contained and of finite index in $G$, and $\text{Ker}(f)$ finite central in $R$.

Proof. We follow closely the proof of Hrushovski-Pillay given in [13, Prop. 3.3].

Passing to a subgroup of $G$ of finite index, we may assume that its $\sigma$-$\Delta$-closure $\hat{G}$ is connected for the $\sigma$-$\Delta$-topology. We work over some small $F_0 \prec U$ over which $G$ is defined. By Theorem 2.16(4), using compactness, we know that there is some quantifier-free definable set $W$, and projection $\pi : W \to \hat{G}$ with finite fibers and such that $G = \pi(W)$; replacing if necessary $\pi$ by the transpose of its graph, we may assume that $\pi$ is the projection on the first $t$ coordinates (where $t$ is such that $H \leq \mathbb{A}^t$ or $\mathbb{P}^{t-1}$).

Let $b, c$ be independent generics of $G$, let $a \in G$ be such that $ab = c$, and let $\hat{b}, \hat{c}$ be tuples in $U$ such that $(b, \hat{b}), (c, \hat{c}) \in W$. So $\hat{b} \in \text{acl}(F_0b)$, and $\hat{c} \in \text{acl}(F_0c)$. We let $a_1 \in U$ be such that $\text{acl}(F_0a) \cap F_0(b, \hat{b}, c, \hat{c})_{\sigma, \Delta} = F_0(a, a_1)_{\sigma, \Delta}$. Note that because $a = cb^{-1}$ and $\text{acl}(F_0a)$ is Galois over $F_0(a)_{\sigma, \Delta}$, the field $F_0(b, \hat{b}, c, \hat{c})_{\sigma, \Delta}$ is a regular extension of $\text{acl}(F_0a) \cap F_0(b, \hat{b}, c, \hat{c})_{\sigma, \Delta}$, and
Lemma 2.33. Moreover, \( qftp(b, \hat{b}, c, \hat{c})/F_0(a, a_1)_{\sigma, \Delta} \) is stationary (see Remark 2.17(e)), and \( F_0(a, a_1)_{\sigma, \Delta} \) contains \( qf-Cb(b, \hat{b}, c, \hat{c}/acl(F_0a)) \) (the quantifier-free canonical basis, see subsection 2.4).

Observe that \( qftp(c, \hat{c}, a, a_1/F_0(b, \hat{b})_{\sigma, \Delta} \) is stationary: this is because \( qftp(c, \hat{c}/F_0(b, \hat{b})_{\sigma, \Delta} \) is stationary, and \( (a, a_1) \in F_0(b, \hat{b}, \hat{c}, \hat{c})_{\sigma, \Delta} \). We now define \( b_1 \) by

\[
\text{acl}(F_0b) \cap F_0(a, a_1, c, \hat{c})_{\sigma, \Delta} = F_0(b, b_1)_{\sigma, \Delta},
\]

and \( c_1 \) by

\[
\text{acl}(F_0c) \cap F_0(a, a_1, b, b_1)_{\sigma, \Delta} = F_0(c, c_1)_{\sigma, \Delta}.
\]

By Remark 2.34 we obtain:

- \( qf-Cb(a, a_1, c, \hat{c}/acl(F_0b)) \subseteq F_0(b, b_1)_{\sigma, \Delta} \)
- \( qf-Cb(a, a_1, b, b_1/acl(F_0c)) \subseteq F_0(c, c_1)_{\sigma, \Delta} \).

Since \( b_1 \in \text{acl}(F_0c)(a, a_1)_{\sigma, \Delta} \) and \( c_1 \in \text{acl}(F_0b)(a, a_1)_{\sigma, \Delta} \), we obtain that \( b_1 \in F_0(a, a_1, c, c_1)_{\sigma, \Delta} \) and \( a_1 \in F_0(b, b_1, c, c_1)_{\sigma, \Delta} \). I.e., we have

\[
F_0(a, a_1, c, c_1)_{\sigma, \Delta} = F_0(a, a_1, b, b_1)_{\sigma, \Delta} = F_0(b, b_1, c, c_1)_{\sigma, \Delta}. \tag{1}
\]

As in [133], \( (a, a_1) \) defines the germ of a generically defined and invertible \( \sigma-\Delta \)-rational map \( g_{a,a_1} \) from (the set of realisations of) \( q_1 = qftp(b, b_1/F_0) \) to \( q_2 = qftp(c, c_1/F_0) \). (In our setting, this means: there are definable sets \( U_1 \) and \( U_2 \), such that the \( \sigma-\Delta \)-closure of \( U_i \) equals the \( \sigma-\Delta \)-closure of the set of realisations of \( q_i \), and such that \( g_{a,a_1} \) defines a \( \sigma-\Delta \)-rational invertible map \( U_1 \rightarrow U_2 \). We may if necessary replace \( U_i \) by a smaller set.)

Choose \( (\tilde{a}, \tilde{a}_1) \in U \) realising \( qftp(a, a_1/F_0) \) and independent from \( (a, b, c) \) over \( F_0 \). Let \( F_0' \prec U \) contain \( F_0(\tilde{a}) \) and such that \( (a, b, c) \) is independent from \( F_0' \) over \( F_0 \).

Let \( (b', b'_1) \) be such that

\[
qftp(a, a_1, b, b_1, c, c_1/F_0) = qftp(\tilde{a}, \tilde{a}_1, b', b'_1, c, c_1/F_0).
\]

Then we have

\[
F_0(\tilde{a}, \tilde{a}_1, c, c_1)_{\sigma, \Delta} = F_0(\tilde{a}, \tilde{a}_1, b', b'_1)_{\sigma, \Delta} = F_0(b', b'_1, c, c_1)_{\sigma, \Delta}, \tag{1'}
\]

so that

\[
F_0'(b', b'_1)_{\sigma, \Delta} = F_0'(c, c_1)_{\sigma, \Delta} \tag{2}
\]
because \((b', b'_1) \in F'_0(\tilde{a}, \tilde{a}_1, c, c_1)_{\sigma, \Delta} \subset F'_0(c, c_1)_{\sigma, \Delta}\). In particular,

\[(b, b_1) \text{ and } (b', b'_1) \text{ are independent over } F'_0. \tag{3}\]

(because \(b\) and \(c\) are). We now let \(d = (\tilde{a})^{-1}a\), and \(r = qftp(a, a_1/F'_0)\) (the unique non-forking extension of \(qftp(a, a_1/F_0)\) to \(F'_0\)).

**Claim 1.** The set of realisations of \(qftp(a, a_1)\) depends on \((\alpha, \alpha_1)\) over \(F'_0\).

Proof. (iv) and (v) are clear from \((\tilde{a})^{-1}a\) and \(\tilde{a} \in F'_0\). Then (iv) and the fact that \(F'_0\) is independent from \((a, b, c)\) over \(F_0\) give (i). We get (ii) from the independence of \((b, b_1)\) and \((b', b'_1)\) over \(F'_0\), that \((b, b_1)\) realises \(q'_1\), and \(qftp(b', b'_1/F'_0) = qftp(b, b_1/F'_0)\). Then (iii) follows from the fact that \(g^{-1}_{\tilde{a}, \tilde{a}_1}\) is the germ of a function from \(q_2\) to \(q_1\). (c, \(c_1\)) \(\mapsto\) \((b', b'_1)\), and is defined over \(F'_0\). Hence \(h_{a, a_1} := g^{-1}_{a, a_1}g_{(a, a_1)}\) is defined over \(F'_0(a, a_1)_{\sigma, \Delta}\) and sends \((b, b_1)\) to \((b', b'_1)\). Finally (vi) follows from equations (1) and (1').

**Claim 2.** The set of realisations of \(r\) is closed under generic composition, and if \((\alpha, \alpha_1)\) and \((a', a'_1)\) are \(F'_0\)-independent realisations of \(r\), then \(F'_0(\alpha, \alpha_1, a', a'_1)_{\sigma, \Delta}\) contains a realisation \((a'', a''_1)\) of \(r\), which is independent from \((\alpha, \alpha_1)\) and from \((a', a'_1)\) over \(F'_0\).

Proof. Note that if \(A_1, A_2, A_3, A_4\) realise \(q'_1\), and are such that \(A_1\) and \(A_2\) are independent over \(F'_0\) and \(A_3\) and \(A_4\) are independent over \(F'_0\), then \(qftp(A_1, A_2/F'_0) = qftp(A_3, A_4/F'_0)\). Since the assertion of the claim only depends on \(qftp(\alpha, \alpha_1, a', a'_1/F'_0)\), we may assume that \((\alpha, \alpha_1) = (a, a_1)\), and assume that \((a', a'_1)\) is independent from \((a, b, \tilde{b}')\) over \(F'_0\), and this is what we will do.

If \((b'', b''_1) \in \mathcal{U}\) is such that:

\[qftp(a', a'_1, b', b'_1, b'', b''_1/F'_0) = qftp(a, a_1, b, b_1, b', b'_1/F'_0),\]

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then from the fact that
\[ F_0'(a, a_1, b, b_1)_{\sigma, \Delta} = F_0'(a, a_1, b', b_1')_{\sigma, \Delta} = F_0'(b, b_1, b', b_1')_{\sigma, \Delta} \]
(by equations (1) and (1')), applying them to the tuple \((a, a_1, b', b_1', b'', b''')\) and using (3) we obtain that \((b, b_1)\) and \((b'', b''')\) are independent over \(F_0'\), and realise \(q_{ftp}'\).

So, if \((a'', a''') \in F_0'(b, b_1, b'', b''')_{\sigma, \Delta}\) is such that \(q_{ftp}(a'', a''', b, b_1, b'', b''')/F_0' = q_{ftp}(a, a_1, b, b_1, b', b'_1/F_0')\), then \(q_{ftp}(a'', a'''_{1}/F_0') = r\). Because \(b, b', b''\) are independent over \(F_0'\), we get that \((a'', a''')\) is independent from \((a, a_1)\) and from \((a', a'_1)\) over \(F_0'\). It remains to show that \((a'', a''') \in F_0'(a, a_1, a', a'_1)_{\sigma, \Delta}\). From the definition of \(h_{a,a_1}\) and \(h_{a',a'_1}\), we get \((\tilde{a})^{-1}a b = b', (\tilde{a})^{-1}a'b' = b''\), and therefore \((\tilde{a})^{-1}a'\tilde{a}b = b'' = (\tilde{a})^{-1}a''b\), from which we deduce that \(a'(\tilde{a})^{-1}a = a''\), so that \(a'' \in F_0'(a,a')\). On the other hand, we know that \((b', b'_1) \in F_0'(a, a_1, b, b_1)_{\sigma, \Delta}\), and that \((b'', b''') \in F_0'(a', a'_1, b'_1, b''')_{\sigma, \Delta}\), so that
\[ (a'', a''') \in F_0'(a, a_1, a', a'_1, b, b_1)_{\sigma, \Delta}. \]

Hence
\[ (a', a''') \in acl(F_0', a, a') \cap F_0'(a, a_1, a', a'_1, b, b_1)_{\sigma, \Delta} = F_0'(a, a_1, a', a'_1)_{\sigma, \Delta} \]
because \((b, b_1)\) is independent from \((a, a_1, a', a'_1)\) over \(F_0'\), and the claim is proved.

**Claim 3.** \(F_0'(a, a_1, a', a'_1)_{\sigma, \Delta} = F_0'(a, a_1, a'', a''')_{\sigma, \Delta} = F_0'(a', a'_1, a'', a''')_{\sigma, \Delta}. \)

**Proof.** Let us consider the map \(((a, a_1), (a', a'_1)) \Rightarrow (a'', a''')\), which is defined at least when \((a, a_1)\) and \((a', a'_1)\) are independent over \(F_0'\). Observe now that \(q_{ftp}(b, b_1, b', b'_1/F_0') = q_{ftp}(b', b'_1, b, b_1/F_0')\), and so we get a realisation \((\tilde{a}, \tilde{a}_1)\) of \(r\) which represents the germ of the inverse of \((a, a_1); \) then \((\tilde{a})^{-1}\tilde{a} = a^{-1}\tilde{a}, \) and therefore using (1), \((\tilde{a}, \tilde{a}_1) \in acl(F_0'(a) \cap F_0'(a, a_1, b, b_1)_{\sigma, \Delta} = F_0'(a, a_1)_{\sigma, \Delta}\) (because \(b\) is independent from \(a\) over \(F_0')\). Reasoning as in the proof of Claim 2 and using \((\tilde{a})^{-1}a' = (\tilde{a})^{-1}a''a^{-1}\tilde{a}, \) and \(\tilde{a}^{-1}a = \tilde{a}(a')^{-1}(\tilde{a})^{-1}a''\), proves the claim.

Similarly, using the fact that the first part of the tuple lives in the algebraic group \(H\), one gets that the map that associates to \(((a, a_1), (a', a'_1))\) the tuple \((a'', a''')\) as above, is generically associative. Hence we are in presence of a normal group law in the sense of [34] (pages 356-357), involving however infinite tuples.

We will now reason as in [21] (Lemma 2.3 and Propositions 3.1 and 4.1 in [21]): as the \(\sigma, \Delta\)-topology is Noetherian, we can find an algebraic group \(H'\) and a quantifier-free definable and connected subgroup \(R\) of \(H'(U)\) such that
$r$ is the generic type of $R$. More precisely: as in Lemma 2.3 of [24], we work in the pure field $\mathcal{U}$, and replace $(a, a_1)$ by the infinite tuple obtained by closing $(a, a_1)$ under $\sigma, \sigma^{-1}$ and the $\delta_i$. This allows to represent the normal group law as a normal group law on some inverse limit of algebraic sets, together with a definable map from the set of realisations of $r$ to this inverse limit. (This map is the one which associates to a tuple $(g, g_1)$ realising $r$ the infinite tuple of coordinates $(g, g_1)_{\sigma, \Delta}$. We will denote it by $\nabla$). Then Proposition 3.1 of [24] shows how to replace this inverse limit by an inverse limit of algebraic groups. And finally, as in Theorem 4.1 of [24], the Noetherianity of the $\sigma$-$\Delta$-topology guarantees that the map $\nabla$ from the set of realizations of $r$ to this inverse limit of $\mathcal{U}$-points of algebraic groups must yield an injection at some finite stage, say to $H'(\mathcal{U})$. The algebraic relations between the coordinates then give rise to the $\sigma$-$\Delta$-equations satisfied by $(g, g_1)$, i.e., they quantifier-freely define the group $R \leq H'(\mathcal{U})$.

Let us now look at $p = qftp(a, a_1, d/F'_0)$, and recall (by (iv) of Claim 1) that $F'_0(a, a_1)_{\sigma, \Delta} = F'_0(d, d)_{\sigma, \Delta}$, and let $K$ be the subgroup of $(H' \times H)(\mathcal{U})$ generated by the realisations of $p$. Observe that it is definable by a quantifier-free $L_{\Delta}$-formula.

As in [13], it follows that $K$ is the graph of a group epimorphism $f : R \rightarrow \tilde{G}$ that is finite-to-one. Indeed, if $a$ is a generic of $G \leq H(\mathcal{U})$, then $f^{-1}(a) = (a, a_1)$ is algebraic over $F_0(a, a_1)_{\sigma, \Delta}$, so that $SU(a/F_0) = SU(a_1/F_0)$, i.e., $SU(R) = SU(\tilde{G})$; hence $f(R)$ must have finite index in $\tilde{G}$. As $f^{-1}(g)$ is finite for every realisation of $p$, Ker ($f$) must be finite. Because $R$ is connected for the $\sigma$-$\Delta$-topology, the kernel is central.

Claim 4. $f(R) \leq G$.

Proof. Let $(g, g_1)$ be a generic of $R$, i.e., a realisation of $r$. Then $g \in \tilde{G}$. We know that $qftp(b, b, c, \hat{c}/F'_0(a, a_1)_{\sigma, \Delta})$ is stationary, and therefore so is its image under any $F'_0$-automorphism of the difference-differential field $\mathcal{U}$ sending $(a, a_1)$ to $(g, g_1)$, so that by Remark 2.17(e), there are $(h, \hat{h}, u, \hat{u})$ in $\mathcal{U}$ such that

$$qftp(a, a_1, b, b, c, \hat{c}/F'_0(a, a_1)_{\sigma, \Delta}) = qftp(g, g_1, h, \hat{h}, u, \hat{u}/F'_0).$$

Thus $h, u \in G$, and so does $g = uh^{-1}$. □

Observe that $f(R)$ has finite index in $G$, because it has finite index in $\tilde{G} \geq G$. This finishes the proof of the theorem. □

Remark 5.2. In the notation of Theorem 5.1 consider $R(n)$ and $G(n)$, as well as the natural $L_{\Delta}$-map $f(n) : R(n) \rightarrow G(n)$. While the map $f$ is not onto $\tilde{G}$, the map $f(n)$ is surjective onto $G(n)$ for all $n \geq 0$ (in the differential field
This follows from quantifier-elimination in DCF\textsubscript{m}. Moreover, the image of R in ˜G is dense for the σ-∆-topology, i.e., this is the appropriate notion of a dominant map between difference-differential varieties.

**Theorem 5.3.** Let \( \mathcal{U} \) be a model of ACFA (possibly of positive characteristic \( p \)), let H be an algebraic group defined over \( \mathcal{U} \), and \( G \leq H(\mathcal{U}) \) a definable subgroup, with σ-closure \( \tilde{G} \). Then there is an algebraic group \( H' \), a quantifier-free definable subgroup \( R \) of \( H'(\mathcal{U}) \), and a quantifier-free definable homomorphism \( f : R \to \tilde{G} \), with finite kernel and such that \( [\tilde{G} : f(R)] \) is finite.

**Proof.** If the characteristic is 0, this follows from Theorem 5.1 with \( m = 0 \). Suppose now that the characteristic is \( p > 0 \), and let \( F_0 \prec \mathcal{U} \) be countable and contain the parameters needed to define \( H \) and \( G \).

**Claim.** There is a quantifier-free definable set \( W \), and a quantifier-free definable map \( f : W \to \tilde{G} \), such that \( f(W) = G \), and for every \( a \in G \), \( f^{-1}(a) \) is finite, and separably algebraic over \( F_0(a)_\sigma \).

**Proof.** We know this holds with \( f^{-1}(a) \) algebraic over \( F_0(a)_\sigma \) (by Theorem 2.16(4)), we need to show that we can find such an \( f \) with \( f^{-1}(a) \) separably algebraic over \( F_0(a)_\sigma \) when \( a \in G \). This follows from the following observation: \( tp(a/F_0) \) is uniquely determined by the isomorphism type over \( F_0 \) of the difference field \( F_0(a)^s_\sigma \), the separable closure of the field \( F_0(a)_\sigma \). Indeed, the automorphism \( \sigma \) of \( F_0(a)^s_\sigma \) extends uniquely to \( F_0(a)^{alg}_\sigma \). A compactness argument then shows the claim.

Thus the proof of Theorem 5.1 can be reproduced almost verbatim, using the following observation:

Given a difference subfield \( A \) of \( \mathcal{U} \), \( scl(A) := acl(A) \cap A^s \) is Galois over \( A \), so that if \( b \in \mathcal{U} \), then \( scl(A) \cap A(b)_\sigma \) contains qf-Cb\((b/acl(A))\), because \( A^s \) and \( A(b)_\sigma \) are linearly disjoint over their intersection. Thus, replacing everywhere in the proof of 5.1 acl by scl gives the result.

Another corollary worth mentioning is the following:

**Theorem 5.4.** Let \( \mathcal{U} \models DCF_{m} A, \ell \geq 1, \) let H be an algebraic group defined over \( F_\ell \), and \( G \leq H(F_\ell) \) a Zariski dense definable subgroup. Then there are an algebraic group \( H' \), a quantifier-free \( L_{\sigma,\Delta} \)-definable subgroup \( R \) of \( H'(F_\ell) \), and a quantifier-free definable homomorphism \( f : R \to G \), with finite kernel and such that \( [G : f(R)] \) is finite.

**Proof.** Reason as in the proof of Theorem 5.1 using Lemma 2.21 to get \( H' \) and \( R \) definable in \( F_\ell \).
Definition 5.5. Let $H$ be an algebraic group. It is *simply connected* if it is connected and whenever $f : H' \to H$ is an isogeny from a connected algebraic group $H'$ onto $H$, then $f$ is an isomorphism.

The *universal covering of the connected algebraic group* $H$ is a simply connected algebraic group $\hat{H}$, together with an isogeny $\pi : \hat{H} \to H$. It satisfies the following universal property (see 18.8 in [22]): if $\varphi : H' \to H$ is an isogeny of connected algebraic groups, then there is a unique algebraic homomorphism $\psi : \hat{H} \to \hat{H}'$ such that $\varphi \circ \psi = \pi$.

Remark 5.6. (1) The definition of simply connected in arbitrary characteristic is a little more complicated. The algebraic groups we will consider will be semi-simple algebraic groups, defined and split over $\mathbb{Q}$, and we will be considering their rational points in some algebraically closed field $K$.

(2) Every simple algebraic group has a universal covering and it is itself a simple algebraic group, see section 5 in [28] for properties, or Chapter 19 in [22].

(3) Note that if $H$ is a simple algebraic group and $K$ is algebraically closed, then $H(K)/Z(H(K))$ is simple as an abstract group.

(4) Moreover, since a semi-simple algebraic group is isogenous to the product of its simple factors, it follows that the universal covering of a semi-simple algebraic group is simply the product of the universal coverings of its simple factors.

Lemma 5.7. Let $H$ be a simple algebraic group defined over the algebraically closed field $L$ of characteristic 0, and $\pi : \hat{H} \to H$ its universal covering. Then any algebraic automorphism of $H(L)$ lifts to one of $\hat{H}(L)$.

Proof. Let $\varphi$ be an algebraic automorphism of $H(L)$, and consider the map $p : \hat{H}(L) \to H(L)$ defined by $\varphi \circ \pi$. Then there is a map $\psi : \hat{H}(L) \to \hat{H}(L)$ such that $\pi = \varphi \circ \pi \circ \psi$. It then follows easily that $\psi$ is an isomorphism: $\psi(\hat{H}(L))$ is a subgroup of $\hat{H}(L)$ which projects onto $H(L)$ via $\pi$, hence must equal $\hat{H}(L)$. So $\psi$ is onto, and because $\text{Ker}(\pi)$ is finite, it must be injective.

Theorem 5.8. Let $H$ be a simply connected simple algebraic group defined and split over $\mathbb{Q}$. Then every Zariski dense definable subgroup of $H(U)$ is quantifier-free definable. Equivalently, every Zariski dense definable subgroup $G$ of $H(U)$ has a smallest definable subgroup $G^0$ of finite index, and $G^0$ is quantifier-free definable.
Furthermore, if \( G \leq H(U) \) is definable, Zariski dense and connected for the \( \sigma\Delta \)-topology, then there is an \( L_\Delta \)-definable subfield \( L \) of \( U \), such that 
\[
h^{-1}G^0h \leq H(L) \text{ for some } h \in H(U), \text{ and either } h^{-1}G^0h = H(L), \text{ or } \]
for some integer \( n \) and algebraic automorphism \( \theta \) of \( H(L) \).

Proof. Let us first discuss the equivalence of the first two assertions. If any Zariski dense definable subgroup of \( H(U) \) is quantifier-free definable, then the Noetherianity of the \( \sigma\Delta \)-topology implies that there is a smallest one, \( G^0 \). Conversely, assume that any Zariski dense definable subgroup \( G \) of \( H(U) \) has a smallest definable subgroup of finite index, \( G^0 \), and that \( G^0 \) is quantifier-free definable. Then so \( G \), since it is a finite union of cosets of \( G^0 \).

So, we let \( G \leq H(U) \) be Zariski dense in \( H \), quantifier-free definable, and connected for the \( \sigma\Delta \)-topology. Then the Kolchin closure \( \bar{G} \) of \( G \) is connected for the Kolchin topology, and by Theorem 2.30, \( \bar{G} \) is conjugate to \( H(L) \), for some field \( L \) of constants in \( U \). Hence we may assume that \( G \leq H(L) \). We want to show that \( G \) has no definable subgroup of finite index.

First note that if \( G = H(L) \), then \( G \) has no definable subgroups of finite index (by Fact 2.32(a)), and the result is proved. Assume therefore that \( G \) is a proper subgroup of \( H(L) \). Let \( H' = H/Z(H) \). By Theorem 1.1 there are an integer \( n \geq 1 \) and an algebraic automorphism \( \theta' \) of \( H'(L) \) such that the group 
\[
G' = GZ(H)/Z(H) \text{ is defined by } \]
\[
G' = \{ g \in H'(L) \mid \sigma^n(g) = \theta'(g) \}.
\]

As \( H \) is simply connected, \( H \to H/Z(H) \) is the universal covering of \( H/Z(H) \). By Lemma 5.7 there is an algebraic automorphism \( \theta \) of \( H(L) \) which lifts \( \theta' \). 

Claim. \( G = \{ g \in H(L) \mid \sigma^n(g) = \theta(g) \} \).

Indeed, the group on the right-hand side is \( \sigma\Delta \)-closed and connected (for the \( \sigma\Delta \)-topology), and it contains \( G \); as \( G \) is a subgroup of finite index of \( GZ(H) \), and is \( \sigma\Delta \)-closed, the equality follows. \( \square \)

Assume by way of contradiction that \( G \) has a definable subgroup of finite index \( > 1 \). By Proposition 5.1 there are a quantifier-free definable group \( R \) (living in some algebraic group \( S \)) and a (quantifier-free) definable map \( f : R \to G \) with finite non-trivial central kernel, and image of finite index \( > 1 \) in \( G \).

For every \( r \geq 1 \), the map \( f \) induces a dominant \( \Delta \)-map \( f_{(r)} : R_{(r)} \to G_{(r)} \), and for \( r \geq n - 1 \), this map has finite central kernel, since for \( r \geq n - 1 \), the natural map \( G_{(r)} \to G_{(n-1)} \) has trivial kernel. Observe that \( f_{(r)} \) is surjective.
by Remark 5.2.

Fix \( r \geq n - 1 \), and consider the (epimorphism) \( f(r) : R(r) \to G(r) \cong H(L)^n \). Because \( H \) is simply connected, so is \( H^n \), and therefore the map \( f(r) \) has an algebraic homomorphic section \( g : G(r) \to R(r) \) (i.e., \( f(r)g = \text{id}_{G(r)} \)); since \( G(r) \) is connected, we obtain that \( R(r) \cong H(L)^n \times \text{Ker}(f(r)) \). Since \( H(L) \) equals its commutator subgroup, it follows that \([R(r), R(r)] \cong H(L)^n\) is a \( \Delta \)-definable normal subgroup of \( R(r) \) which projects via \( f(r) \) onto \( G(r) \cong H(L)^n \). As \( R \) is connected for the \( \sigma-\Delta \)-topology, \( R(r) \) is connected for the \( \Delta \)-topology, and we must therefore have \( \text{Ker}(f(r)) = \{1\} \).

So, we have shown that the definable group \( G \) is quantifier-free definable and has no proper definable subgroup of finite index. This finishes the proof.

**Theorem 5.9.** Let \( H \) be a simple algebraic group, \( G \leq H(U) \) be a definable subgroup which is Zariski dense in \( H \). Then \( G \) has a smallest definable subgroup \( G^0 \) of finite index. Let \( \pi : \hat{H} \to H \) be the universal covering of \( H \), and let \( \hat{G} \) be the connected component of the \( \sigma-\Delta \)-closure of \( \pi^{-1}(G) \). Then \( G^0 = \pi(\hat{G}) \).

**Proof.** By Lemma 5.8 \( \hat{G} \) has no definable subgroup of finite index. Hence, neither does \( \pi(\hat{G}) \). As \( \hat{G} \cap G \) has finite index in \( \pi(\hat{G}) \), we have that \( \pi(\hat{G}) \subseteq G \). We have shown that \( \pi(\hat{G}) \) is contained in any Zariski dense definable subgroup of \( H \), therefore \( \pi(\hat{G}) \) is the smallest definable subgroup of finite index of \( G \).

**Corollary 5.10.** Let \( G \) be a definably quasi-semi-simple definable group. Then \( G \) has a definable connected component.

**Proof.** Let \( P \) be the property “having a definable connected component”. The result follows easily from Proposition 3.5, Proposition 4.7, Theorem 5.9, and the following remarks:

(a) If \( G_0 \) is a definable subgroup of finite index of \( G \), then \( G_0 \) has \( P \) if and only if \( G \) has \( P \);

(b) If the group \( G \) is the direct product of its definable subgroups \( G_1, G_2 \), and \( G_1, G_2 \) have \( P \), then so does \( G \);

(c) Let \( f : G \to G_1 \) be a definable onto map, with \( \text{Ker}(f) \) finite. Then \( G_1 \) has \( P \) if and only if \( G \) has \( P \). One direction is clear: if \( G \) has \( P \), then so does \( G_1 \), because an infinite strictly decreasing of definable subgroups of \( G_1 \) of finite index would yield such a sequence for \( G \); for the other, assume that \( G_1 \) has \( P \), and let \( G_1^0 \) be its connected component; then
$f^{-1}(G^0_1)$ is definable and has finite index in $G$. So, we may assume that $G_1$ is connected.

If $G_0$ is a definable subgroup of finite index of $G$, then $f(G_0) = G_1$, so that $G_0\ker(f) = G$; hence $[G : G_0] \leq |\ker(f)|$. This being true for any definable subgroup $G_0$ of $G$ of finite index, implies that there is minimal one, and that its index in $G$ is bounded by $|\ker(f)|$.

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