Anti-symplectic involutions for Lagrangian spheres in a symplectic quadric surface

Joontae Kim and Jiyeon Moon

Abstract

We show that the space of anti-symplectic involutions of a monotone $S^2 \times S^2$ whose fixed points set is a Lagrangian sphere is connected. This follows from a stronger result, namely that any two anti-symplectic involutions in that space are Hamiltonian isotopic.

1 Introduction and main results

A symplectic manifold $(M, \omega)$ is an even dimensional smooth manifold $M$ equipped with a closed non-degenerate 2-form $\omega$. Recall that an anti-symplectic involution of a symplectic manifold $(M, \omega)$ is a diffeomorphism $R \in \Diff(M)$ satisfying $R^2 = \id_M$ and $R^*\omega = -\omega$. Its fixed point set $\Fix(R) = \{x \in M \mid R(x) = x\}$ is a Lagrangian submanifold, that is, $\dim \Fix(R) = \frac{1}{2}\dim M$ and $\omega$ vanishes on the tangent bundle of $\Fix(R)$, if it is non-empty. In this case, $\Fix(R)$ is called a real Lagrangian. We abbreviate by $\mathcal{A}(M, \omega)$ the space of anti-symplectic involutions of $(M, \omega)$. The space of smooth involutions of a smooth manifold $M$ is denoted by $\I(M)$. Throughout the paper, $\Diff(M)$ and all of its subspaces are endowed with the $C^\infty$-topology. It is a prominent question to investigate the injectivity of the natural forgetful map $\pi_0\mathcal{A}(M, \omega) \to \pi_0\I(M)$ in a symplectic ruled surface, the total space of an $S^2$-fibration over a closed oriented surface with a symplectic form that is non-degenerate on the fibres. This formulates a version of the symplectic isotopy conjecture \cite[Problem 14 in Section 14.2]{8} about the symplectomorphism group of ruled surfaces. In particular, if the map is injective, then any two elements in $\mathcal{A}(M, \omega)$ are isotopic if and only if they are smoothly isotopic.

In this paper, we explore the question when $(M, \omega)$ is the monotone symplectic quadric surface $(Q, \omega) = (S^2 \times S^2, \omega_0 \oplus \omega_0)$, where $\omega_0$ is a Euclidean area form normalized by $\int_{S^2} \omega_0 = 1$, and when the fixed point set of $R \in \mathcal{A}(Q, \omega)$ is a Lagrangian sphere. Based on foliations by $J$-holomorphic curves, Gromov \cite{2} showed that the symplectomorphism group $\Symp(Q, \omega)$ deformation retracts onto the isometry group $(\SO(3) \times \SO(3)) \times \Z/2$, where $\Z/2$ acts by interchanging two $S^2$-factors, and hence the natural map $\pi_0\Symp(Q, \omega) \to \pi_0\Diff(Q)$ is injective, whence the symplectic isotopy conjecture holds in this case. Hind \cite{3} proved that every Lagrangian sphere in $Q$ is Hamiltonian isotopic to the antidiagonal $\Sigma = \{(x, -x) \mid x \in S^2\}$. Therefore, any two real Lagrangian spheres $L_0 = \Fix(R_0)$ and $L_1 = \Fix(R_1)$ in $Q$ are Hamiltonian isotopic, and it is natural to ask whether the corresponding involutions $R_0$ and $R_1$ are Hamiltonian isotopic. We say that $R_0, R_1 \in \mathcal{A}(M, \omega)$ are Hamiltonian isotopic if there

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exists $\phi \in \text{Ham}(M, \omega)$ such that $\phi^* R_1 := \phi^{-1} \circ R_1 \circ \phi = R_0$, where $\text{Ham}(M, \omega)$ denotes the group of Hamiltonian diffeomorphisms of $M$. The classification of Hamiltonian isotopy classes of anti-symplectic involutions is a priori finer than the one of real Lagrangians, see Figure 1. We write $R_{\Sigma}(x, y) = (-y, -x)$ for the anti-symplectic involution of $Q$ with $\text{Fix}(R_{\Sigma}) = \Sigma$.

The main result of the paper is rather modest.

**Main Theorem.** Every anti-symplectic involution of $(S^2 \times S^2, \omega_0 \oplus \omega_0)$ whose fixed point set is a Lagrangian sphere is Hamiltonian isotopic to $R_{\Sigma}$.

In particular, any two anti-symplectic involutions of $Q$ whose fixed point set is a Lagrangian sphere are Hamiltonian isotopic. We abbreviate by $\mathcal{A}(Q, \omega, S^2) = \{ R \in \mathcal{A}(Q, \omega) \mid \text{Fix}(R) \cong S^2 \}$ and $\mathcal{I}(Q, S^2) = \{ \sigma \in \mathcal{I}(Q) \mid \text{Fix}(\sigma) \cong S^2 \}$. As an immediate corollary, we obtain a sort of a symplectic phenomenon.

**Theorem A.** $\pi_0 \mathcal{A}(Q, \omega, S^2) = 0$, while $\pi_0 \mathcal{I}(Q, S^2) \neq 0$.

That $\pi_0 \mathcal{I}(Q, S^2) \neq 0$ easily follows since the actions of $R_{\Sigma}$ and $\tau(x, y) = (y, x)$ on homology are different, that is, $(R_{\Sigma})_* \neq \tau_*$ on $H_2(Q)$, and hence they form different components in $\mathcal{I}(Q, S^2)$. In consequence, the natural map $\pi_0 \mathcal{A}(Q, \omega, S^2) \to \pi_0 \mathcal{I}(Q, S^2)$ is injective, but not surjective. It is known that $\mathcal{A}(Q, \omega)$ has at least four components for topological reasons; two of them have no fixed points (but the corresponding quotient manifolds are not diffeomorphic), and the fixed point sets of the remaining two cases are $S^2$ and $T^2$, see [1, 6.11.7 and 6.11.8] for details. By [6], every real Lagrangian $T^2$ in $Q$ is Hamiltonian isotopic to the Clifford torus, defined as the product of the equators in each $S^2$-factor. Hence, there is no obvious reason to find non-isotopic anti-symplectic involutions of $Q$ having $T^2$ as the fixed point set.

**Remark 1.1.** We refer to an intriguing work by Kharlamov–Shevchishin [5] about anti-symplectic involutions in rational symplectic 4-manifolds. The scheme of their proof can be adapted to obtain our main theorem. It can be used to show the connectedness of the space of anti-symplectic involutions of $Q$ corresponding to other topological types. Nevertheless, our approach is still interesting in its own right since this is more explicit and direct.

**An outline of the proof.** We first recall the proof of the result by Hind [3]. A crucial ingredient is the following non-trivial statement.

**Theorem 1.2** (Hind). Let $L$ be a Lagrangian sphere in $Q$. Then there exists a tame almost complex structure $J$ on $Q$ such that each leaf of the Gromov’s foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ associated to $J$ intersects $L$ transversely at a single point.

See Section 2.1 for the definition of Gromov foliation theorem. If $L = \Sigma$, then the Gromov’s foliations associated to the standard complex structure $J_{\text{std}} = j_{\mathbb{C}P^1} \oplus j_{\mathbb{C}P^1}$ satisfies the desired properties in Theorem 1.2. Since any Lagrangian sphere $L$ in a symplectic 4-manifold has self-intersection number $-2$, the homology class $[L]$ is equal to one of $\pm (A_1 - A_2) \in H_2(Q)$, where $A_1 = [S^2 \times \{ \text{pt} \}]$ and $A_2 = [\{ \text{pt} \} \times S^2]$ are generators. Hence, the intersection number of $L$ and any Gromov’s leaf is equal to $\pm 1$, and they intersect minimally and transversely under the situation of Theorem 1.2. With this understood, any orientation preserving diffeomorphism from $L$ to $\Sigma$ extends uniquely to a self-diffeomorphism of $Q$ by sending the leaves of the Gromov’s foliations corresponding to $J$ to the ones from $J_{\text{std}}$. Then
Moser isotopy furnishes a symplectomorphism acting trivially on homology, which turns out to be a Hamiltonian diffeomorphism by the celebrated result of Gromov. Therefore, any Lagrangian sphere \( L \) is mapped to \( \overline{\Delta} \) by a Hamiltonian diffeomorphism.

When \( L = \text{Fix}(R) \) is a real Lagrangian sphere with \( R \in \mathcal{A}(Q, \omega) \), we can prove Theorem 1.2 using \( R \)-anti-invariant tame almost complex structures, see Proposition 2.3. This allows us to construct a diffeomorphism of \( Q \) which is equivariant with respect to the involutions \( R \) and \( R_{\overline{\Delta}} \). Finally, one can still apply the Moser trick in an equivariant way to obtain \( \Phi \in \text{Ham}(Q, \omega) \) satisfying \( R = \Phi^*R_{\overline{\Delta}} \). In contrast to the original work by Hind, we do not need a sophisticated method, namely SFT, to obtain Theorem 1.2. Instead, the classical methods in \( J \)-holomorphic curves theory are totally enough.

| \( R \) | Hamiltonian isotopic | \( \Rightarrow \) | isotopic | \( \Rightarrow \) | smoothly isotopic |
|--------|----------------------|---------------|---------|---------------|------------------|
| \( \downarrow \) | \( \downarrow \) | \( \downarrow \) |
| \( \text{Fix}(R) \) | Hamiltonian isotopic | \( \Rightarrow \) | Lagrangian isotopic | \( \Rightarrow \) | smoothly isotopic |

Figure 1: Relations between isotopies of \( R \) and \( \text{Fix}(R) \)

2 Proof of the Main Theorem

2.1 Recollection of Gromov’s foliation theorem

Let \( \mathcal{J} \) be the space of tame almost complex structures on the monotone symplectic quadric \( Q \), which is non-empty and contractible [8, Theorem 2.6.4]. Fix \( J \in \mathcal{J} \) and consider the moduli space of \( J \)-holomorphic spheres in the homology class \( A_i \),

\[
\mathcal{M}(A_i, J) = \{ u : \mathbb{C}P^1 \to Q \mid \partial_J(u) = 0, \, u_*[\mathbb{C}P^1] = A_i \}/\text{PSL}(2, \mathbb{C}),
\]

where \( \partial_J(u) = \frac{1}{2}(du + J \circ du \circ j_{\mathcal{C}(P^1)}) \in \Omega^{0,1}(\mathbb{C}P^1, u^*TQ) \) denotes the complex anti-linear part of \( du \). Since every non-constant \( J \)-holomorphic sphere in \( Q \) has positive Chern number and \( A_i \) are primitive classes, every \( u \in \mathcal{M}(A_i, J) \) is not multiply covered, and hence is simple, see [7, Lemma 9.4.6]. From the adjunction formula [7, Theorem 2.6.4 or Corollary E.1.7],

\[
A_i \cdot A_i - \langle c_1(Q), A_i \rangle + \chi(\mathbb{C}P^1) = 0,
\]

which is a numerical criterion for simple \( J \)-holomorphic curves to be embedded, every \( u \in \mathcal{M}(A_i, J) \) is embedded. It follows from the automatic transversality, see [4, Theorem 1] or [7, Lemma 3.3.3], that for every \( J \in \mathcal{J} \) the moduli space \( \mathcal{M}(A_i, J) \) is a smooth manifold of dimension

\[
\dim_{\mathbb{C}} Q \cdot \chi(\mathbb{C}P^1) + 2\langle c_1(Q), A_i \rangle - \dim_{\mathbb{R}} \text{PSL}(2, \mathbb{C}) = 2.
\]

Since every \( J \)-holomorphic sphere in the homology class \( A_i \) has the minimal possible positive energy, Gromov’s compactness theorem asserts that \( \mathcal{M}(A_i, J) \) is compact. By positivity of intersections [7, Section 2.6], any two \( J \)-holomorphic spheres in \( \mathcal{M}(A_1, J) \) and \( \mathcal{M}(A_2, J) \) intersect transversely at a single point as \( A_1 \cdot A_2 = 1 \). Let \( \widehat{\mathcal{M}}(A_i, J) \) be the space of parametrized \( J \)-holomorphic spheres in \( Q \) representing \( A_i \). Since the evaluation map \( \text{ev}_i : \mathcal{M}_1(A_i, J) := \widehat{\mathcal{M}}(A_i, J) \times_{\text{PSL}(2, \mathbb{C})} \mathbb{C}P^1 \to Q, \quad \text{ev} ([u, z]) = u(z) \) (2.1)
is a diffeomorphism [7, Proposition 9.4.4], for each \(i = 1, 2\) there exists a unique curve \(\Sigma_i(x)\) in \(\mathcal{M}(A_i, J)\) passing through any given point \(x \in Q\). Therefore, \(\mathcal{F}_1 = \mathcal{M}(A_1, J)\) and \(\mathcal{F}_2 = \mathcal{M}(A_2, J)\) determine two transversal foliations of \(Q\) whose leaves are embedded \(J\)-holomorphic spheres in the homology classes \(A_1\) and \(A_2\), respectively. We call \(\mathcal{F}_1\) and \(\mathcal{F}_2\) the Gromov’s foliations associated to \(J\). We let \(\mathcal{F}^{\text{std}}_i\) be the Gromov’s foliations associated to the standard complex structure \(J^{\text{std}} = j_{\mathbb{C}P^1} \oplus j_{\mathbb{C}P^1}\). Analogously, their leaves passing through \(x \in Q\) are denoted by \(\Sigma^{\text{std}}_i(x) \in \mathcal{F}^{\text{std}}_i\).

### 2.2 A real analogue of Theorem 1.2

We begin with the following homological result, see [6, Section 2.1] for the proof.

**Lemma 2.1.** For any \(R \in \mathcal{A}(Q, \omega)\) the map \(R_*\) induced in \(H_2(Q) \cong \mathbb{Z}^2\) satisfies either

\[
C_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{or} \quad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Moreover, the following holds:

(i) \(R_* = C_1\) if and only if \(\text{Fix}(R)\) is diffeomorphic to \(S^2\).

(ii) \(R_* = C_2\) if and only if \(\text{Fix}(R)\) is either empty or diffeomorphic to \(T^2\).

For \(R \in \mathcal{A}(Q, \omega)\) we abbreviate by \(\mathcal{J}_R = \{ J \in \mathcal{J} \mid J = -R^*J \}\) the space of tame almost complex structures on \(Q\) which are \(R\)-anti-invariant. Note that \(\mathcal{J}_R\) is non-empty and contractible by Sévennec’s arguments [9, Proposition 1.1]. One can directly check that the anti-symplectic involution \(R^{\Sigma}_2\) sends a leaf in \(\mathcal{F}^{\text{std}}_1\) to a leaf in \(\mathcal{F}^{\text{std}}_2\), but it reverses the complex orientation on the leaves. More generally, we observe the following.

**Lemma 2.2.** Let \(R \in \mathcal{A}(Q, \omega)\) such that \(\text{Fix}(R)\) is a Lagrangian sphere and let \(J \in \mathcal{J}_R\). If \(\Sigma_1 \in \mathcal{F}_1\), then \(R(\Sigma_1) \in \mathcal{F}_2\). Similarly, if \(\Sigma_2 \in \mathcal{F}_2\), then \(R(\Sigma_2) \in \mathcal{F}_1\).

**Proof.** Let \(u : \mathbb{C}P^1 \to Q\) be a \(J\)-holomorphic sphere in the homology class \(A_1\), which parametrizes \(\Sigma_1 \in \mathcal{F}_1\). Pick any anti-holomorphic involution \(\rho\) of \(\mathbb{C}P^1\) which is necessarily orientation-reversing. We then verify that

\[
\hat{\partial}_J(R \circ u \circ \rho) = dR \circ \hat{\partial}_J(u) \circ \rho = 0
\]

\[
(R \circ u \circ \rho)_*[\mathbb{C}P^1] = R_*(-u_*[\mathbb{C}P^1]) = R_*(-A_1) = A_2.
\]

Here the last equality follows from Lemma 2.1. We have seen that \(R \circ u \circ \rho\) is a \(J\)-holomorphic sphere representing the class \(A_2\), that is, \(R(\Sigma_1) \in \mathcal{F}_2\). The same argument also works for a leaf \(\Sigma_2 \in \mathcal{F}_2\). \(\square\)

Now we prove Theorem 1.2 using \(R\)-anti-invariant tame almost complex structures. The proof is quite elementary.

**Proposition 2.3.** Let \(L = \text{Fix}(R)\) be a real Lagrangian sphere in \(Q\) and \(J \in \mathcal{J}_R\). Then each leaf of the Gromov’s foliations \(\mathcal{F}_1\) and \(\mathcal{F}_2\) associated to \(J\) intersects \(L\) transversely at a single point.
Proof. Without loss of generality, we consider the case when a leaf \( \Sigma_1 \in \mathcal{F}_1 \) is given. Since the algebraic intersection number of \( L \) and \( \Sigma_1 \) is non-zero, more precisely, \([L] \cdot [\Sigma_1] = \pm (A_1 - A_2) \cdot A_1 = \mp 1\), it follows that \( L \) and \( \Sigma_1 \) must have an intersection point. To see that \( \Sigma_1 \) and \( L \) has a unique intersection point, let \( x, y \in \Sigma_1 \cap L \) be given. By Lemma 2.2, we can consider another leaf \( \Sigma_2 := R(\Sigma_1) \) of \( \mathcal{F}_2 \). Since \( x, y \) are fixed points of \( R \), we observe that \( x, y \in \Sigma_2 \cap L \), and hence \( x, y \in \Sigma_1 \cap \Sigma_2 \). This implies that \( x = y \) as claimed. It remains to check that \( \Sigma_1 \) intersects \( L \) transversely at a single point \( x \). Let \( X \in T_x \Sigma_1 \cap T_x L \) be given. Since \( T_x L = \{ X \in T_x Q \mid dR(X) = X \} \) and \( dR(T_x \Sigma_1) = T_x \Sigma_2 \), we deduce that \( X = dR(X) \in T_x \Sigma_2 \cap T_x L \), which implies \( X \in T_x \Sigma_1 \cap T_x \Sigma_2 \). Since \( \Sigma_1 \) and \( \Sigma_2 \) intersect transversely, we must have \( X = 0 \). Therefore, \( \Sigma_1 \) and \( L \) intersect transversely for dimensional reasons.

\[\square\]

2.3 A diffeomorphism of \( Q \) induced by transversal foliations

Let \( R \in \mathcal{A}(Q, \omega) \) such that \( L = \text{Fix}(R) \) is a Lagrangian sphere. We shall construct an explicit diffeomorphism \( \Phi \) of \( Q \) which is equivariant with respect to the anti-symplectic involutions \( R \) and \( R_{\Sigma} \).

Choose any diffeomorphism \( \phi : L \to \Sigma \) which is orientation-preserving. Pick \( J \in \mathcal{J}_R \) and write \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) for the corresponding Gromov’s foliations. It follows from Proposition 2.3 that we can extend \( \phi \) to a unique diffeomorphism \( \Phi : Q \to Q \) determined by sending the leaves of the foliations \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) to the leaves of the standard foliations \( \mathcal{F}^{\text{std}}_1 \) and \( \mathcal{F}^{\text{std}}_2 \). To express \( \Phi \) explicitly, for \( i = 1, 2 \) we define the map

\[ \pi_i : Q \to L, \quad \pi_i(x) = \Sigma_i(x) \cap L, \]

which sends a leaf in \( \mathcal{F}_i \) to its unique intersection point with \( L \). Since every leaf in \( \mathcal{F}_i \) and \( L \) intersect transversely, \( \pi_i \) is smooth, see Remark 2.4. The diffeomorphism \( \Phi \) of \( Q \) is then defined by

\[ \Phi(x) := \Sigma_1^{\text{std}}(\phi \pi_1(x)) \cap \Sigma_2^{\text{std}}(\phi \pi_2(x)). \tag{2.2} \]

whose inverse is given by

\[ \Phi^{-1}(x) = \Sigma_1(\phi^{-1} \pi_1^{\text{std}}(x)) \cap \Sigma_2(\phi^{-1} \pi_2^{\text{std}}(x)). \]

Here \( \pi_i^{\text{std}} \) are defined analogously using \( \mathcal{J}^{\text{std}}_i \). Since \( \Phi \) preserves the leaves of \( \mathcal{F}_i \) and \( \mathcal{F}_i^{\text{std}} \) and \( \phi \) is orientation preserving, \( \Phi \) acts trivially on homology, that is, the induced map \( \Phi_* \) on \( H_*(Q) \) is the identity.

Remark 2.4. The discussion below is somewhat technical and implicit in [3], but for the sake of completeness, we explain that \( \Phi \) and \( \Phi^{-1} \) are smooth following the arguments in the proof of [7, Theorem 9.4.7]. In that proof, it is shown that for any \( J \in \mathcal{J} \) the maps

\[ \Lambda : Q \to Q, \quad \Lambda(x) = \Sigma_1(x) \cap \Sigma_2(x) \]

\[ \sigma_i : Q \to \mathcal{M}(A_i, J), \quad \sigma_i(x) = \Sigma_i(x) \]

are smooth. It thus suffices to show that \( \pi_i \) are smooth since then \( \Phi = \Lambda \circ (\phi \times \phi) \circ (\pi_1 \times \pi_2) \) is smooth as well. Consider the map \( \eta_i : \mathcal{M}(A_i, J) \to L \) defined by \( \eta_i(\Sigma_i) = \Sigma_i \cap L \). By Proposition 2.3, it is well-defined and bijective. From the expression \( \pi_i = \eta_i \circ \sigma_i \), it remains to check that \( \eta_i \) is smooth. Let \( \text{pr}_i : \mathcal{M}_1(A_i, J) \to \mathcal{M}(A_i, J) \) denote the projection, where \( \mathcal{M}_1(A_i, J) \) is defined in (2.1). Since the evaluation map \( \text{ev}_i \) in (2.1) is a diffeomorphism,
η\textsuperscript{-1} = \text{pr}_i \circ \text{ev} \big|_{L}^{-1} \) is smooth. From the splitting \( T_x L \oplus T_x \Sigma_i(x) = T_x Q \) for each \( x \in L \), we deduce that the differential of \( \eta\textsuperscript{-1} \) at every point of \( L \) is bijective. Therefore, \( \eta \) is a diffeomorphism by the inverse function theorem. This shows that \( \Phi \) is smooth, and the same argument also yields that \( \Phi\textsuperscript{-1} \) is smooth.

By virtue of the choice of \( J \) in \( J_R \), the following observation is straightforward.

**Lemma 2.5.** The diffeomorphism \( \Phi: Q \to Q \) defined in (2.2) satisfies \( \Phi \circ R = R_{\Sigma} \circ \Phi \).

**Proof.** Using Lemma 2.2, for all \( x \in Q \) we have

\[
\begin{align*}
\Sigma_1(x) &= R(\overline{\Sigma}_2(R(x))), \\
\Sigma_1^{\text{std}}(x) &= R_{\Sigma}(\overline{\Sigma}_2^{\text{std}}(R_{\Sigma}(x))), \\
\pi_1(x) &= \pi_2(R(x)).
\end{align*}
\]

Here the last item follows from

\[
\pi_1(x) = \Sigma_1(x) \cap L = R(\overline{\Sigma}_1(x) \cap L) = R(\overline{\Sigma}_1(x)) \cap L = \overline{\Sigma}_2(R(x)) \cap L = \pi_2(R(x)).
\]

We readily see that

\[
\Phi(R(x)) = \Sigma_1^{\text{std}}(\phi \pi_1(R(x))) \cap \Sigma_2^{\text{std}}(\phi \pi_2(R(x)))
\]

\[
= \Sigma_1^{\text{std}}(\phi \pi_2(x)) \cap \Sigma_2^{\text{std}}(\phi \pi_1(x))
\]

\[
= R_{\Sigma}(\overline{\Sigma}_2^{\text{std}}(R_{\Sigma}(\phi \pi_2(x)))) \cap R_{\Sigma}(\overline{\Sigma}_1^{\text{std}}(R_{\Sigma}(\phi \pi_1(x))))
\]

\[
= R_{\Sigma}(\overline{\Sigma}_2^{\text{std}}(\phi \pi_2(x))) \cap R_{\Sigma}(\overline{\Sigma}_1^{\text{std}}(\phi \pi_1(x)))
\]

\[
= R_{\Sigma}[\overline{\Sigma}_2^{\text{std}}(\phi \pi_2(x)) \cap \overline{\Sigma}_1^{\text{std}}(\phi \pi_1(x))]
\]

\[
= R_{\Sigma}(\Phi(x)).
\]

This completes the lemma.

### 2.4 Equivariant Moser trick

The diffeomorphism \( \Phi \) does not necessarily preserve the symplectic form \( \omega \), so we employ the Moser’s trick to adjust the pull-back \( (\Phi^{-1})^* \omega \) while keeping its compatibility with the involutions.

**Lemma 2.6.** There exists a diffeomorphism \( \Psi: Q \to Q \) such that

(i) \( \Psi^* \omega = (\Phi^{-1})^* \omega \).

(ii) \( \Psi \circ R_{\Sigma} = R_{\Sigma} \circ \Psi \).

(iii) \( \Psi \) acts trivially on homology.

Consider a smooth family of closed 2-forms on \( Q \) given by

\[
\omega_t := (1 - t)(\Phi^{-1})^* \omega + t \omega, \quad t \in [0, 1].
\]
Lemma 2.7. For each $t \in [0, 1]$ we have

(i) $\omega_t$ is a symplectic form.

(ii) $[\omega_t] = [\omega]$ in $H^2_{dR}(Q)$.

(iii) $R^*_\Delta \omega_t = -\omega_t$.

Proof. To show (i), we claim that $\omega_t$ is compatible with $(\Phi^{-1})^*J$, that is, $\omega_t(X, (\Phi^{-1})^*JX) > 0$ for all non-zero $X \in TQ$. By naturality, we have $(\Phi^{-1})^*\omega(X, (\Phi^{-1})^*JX) > 0$. Since the Gromov’s foliations associated to $(\Phi^{-1})^*J$ coincide with the standard foliations $\mathcal{F}_{\text{std}}^i$ (in particular, the tangent bundle $TQ$ admits a decomposition into two transversal $(\Phi^{-1})^*J$-invariant subbundles which are $\omega$-orthogonal), we deduce $\omega(X, (\Phi^{-1})^*JX) > 0$. Moreover, $(\Phi^{-1})^*\omega$ evaluates to 1 over a leaf of $\mathcal{F}_{\text{std}}^i$ for $i = 1, 2$, so $(\Phi^{-1})^*[\omega] = [\omega] \in H^2_{dR}(Q)$. This implies (ii). By Lemma 2.5, (iii) follows. \hfill \Box

We are in position to apply the Moser trick, see [8, Theorem 3.2.4]. Since $[\omega_1] = [\omega_0]$, there exists a 1-form $\beta$ on $Q$ such that $\omega_1 - \omega_0 = d\beta$. Then $\omega_t = (\Phi^{-1})^*\omega + td\beta$. Since $R^*_\Delta d\beta = -d\beta$, the anti-averaged 1-form $\tilde{\beta} := \frac{1}{2}(\beta - R^*_\Delta \beta)$ satisfies $\omega_t = (\Phi^{-1})^*\omega + td\beta$ and $R^*_\Delta \tilde{\beta} = -\tilde{\beta}$. Let $X_t$ be the Moser vector field associated to $\tilde{\beta}$, that is, the unique solution of the equation $\omega_t(X_t, \cdot) + \tilde{\beta} = 0$. The flow $\Psi_t$ of $X_t$ satisfies $\Psi_t^* \omega_t = \omega_0$ and $\Psi_t \circ R^*_\Delta = R^*_\Delta \circ \Psi_t$ for all $t \in [0, 1]$. Noting that $R^*_\Delta X_t = X_t$, the equation

$$\frac{d}{dt}(R^*_\Delta \circ \Psi_t \circ R^*_\Delta) = dR^*_\Delta \left(\frac{d}{dt}(\Psi_t \circ R^*_\Delta)\right) = dR^*_\Delta(X_t(\Psi_t \circ R^*_\Delta)) = X_t(R^*_\Delta \circ \Psi_t \circ R^*_\Delta),$$

and the uniqueness of the solution show that $\Psi_t$ and $R^*_\Delta$ commute. The time-1 flow $\Psi := \Psi_1$ satisfies the properties (i)–(iii). The condition (iii) follows since $\Psi$ is isotopic to the identity. This completes the proof of Lemma 2.6.

Proof of the Main Theorem. Note that $\Gamma = \Psi \circ \Phi \in \text{Symp}(Q, \omega)$ acts trivially on homology and that $\Gamma \circ R = R^*_\Delta \circ \Gamma$. By Gromov’s theorem, $\Gamma$ is symplectically isotopic to the identity, and hence $\Gamma \in \text{Ham}(Q, \omega)$ as $H^1_{\text{dR}}(Q) = 0$. This shows that $R \in A(Q, \omega)$ is Hamiltonian isotopic to $R^*_\Delta$ as sought. \hfill \Box

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Korea Institute for Advanced Study (KIAS), 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea

*E-mail address:* joontae@kias.re.kr

Department of Mathematics, Ajou University, 206 Worldcup-ro, Suwon 16499, Republic of Korea

*E-mail address:* j9746@ajou.ac.kr