Reciprocity Laws on Algebraic Surfaces via Iterated Integrals

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(with an appendix by Matt Kerr)

Abstract

This paper presents a proof of reciprocity laws for the Parshin symbol and for two new local symbols, defined here, which we call 4-function local symbols. The reciprocity laws for the Parshin symbol are proven using a new method - via iterated integrals. The usefulness of this method is shown by two facts - first, by establishing new local symbols - the 4-function local symbols and their reciprocity laws and, second, by providing refinements of the Parshin symbol in terms of bi-local symbols, each of which satisfies a reciprocity law. The $K$-theoretic variant of the first 4-function local symbol is defined in the Appendix. It differs by a sign from the one defined via iterated integrals. Both the sign and the $K$-theoretic variant of the 4-function local symbol satisfy reciprocity laws.

Key words: reciprocity laws, complex algebraic surfaces, iterated integrals
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0 Introduction

This paper is the second one in a series of papers on reciprocity laws on varieties via iterated integrals (after [H1]). We construct and prove reciprocity laws for both classical and new symbols. Here, we present a new prove of the reciprocity laws for the Parshin symbol, in addition to well-known approaches such as [P1], [P2], [Ka], [Kh], [FV], [PR1]. Besides proofs of the Parshin reciprocity laws, the new method gives new symbols on algebraic surfaces and new reciprocity laws.

The present paper is a corrected and substantially improved version of the preprint “Refinement of the Parshin symbol for surfaces” [H3]. In an email to the author [D2], Deligne pointed out that the refinements of the Parshin symbols were not independent of choices of local uniformizers. After examining carefully the origin of the refinement - namely, iterated integrals of differential forms over membranes - we realized that the refinement becomes independent of local uniformizers by introducing bi-local symbols. A key property of the bi-local symbols is that they are almost the same as the tame symbol on a curve, however, they are defined over surfaces.

We call these symbols bi-local, since they depend on two points \( P \) and \( Q \). We fix two points \( P \) and \( Q \) on a curve \( C \) on a surface \( X \) and we localize using two uniformizers: one for the curve \( C \) on which the points \( P \) and \( Q \) lie, and one for the point \( P \). We represent the uniformizers by rational functions. Then we evaluate a certain rational function at the points \( P \) and \( Q \) and take the ratio of the two values. One can think of the point \( P \) as the point of interest and of the point \( Q \) as a base point. The points \( P \) and \( Q \) are in the union of the support of the divisors of the functions \( f_1, \ldots, f_4 \), such that \( P \) belongs to an intersection of two irreducible components of the divisors and \( Q \) belongs to only one component, serving as a base point of loops on the curve \( C \), where \( Q \) belongs.

We construct a refinement of the Parshin symbol in the sense that the latter is a product of bi-local symbols and all of them satisfy a reciprocity law. We also introduce 4-function local symbols, which have similar properties they can be factorized in simpler bi-local symbols that satisfy reciprocity laws. Moreover, such a presentation in terms of bi-local symbols provides proofs of the reciprocity laws for the local symbols.

Three of the six symbols that compose the Parshin symbol for a surface have only values \( \pm 1 \). The composition of the remaining three symbols, which gives the Parshin symbol up to a sign, will be used in a follow-up paper constructing a two-dimensional analogue of the Contou-Carrére symbol and its reciprocity laws.

Another reason for using bi-local symbols is that they are computationally effective.

Unlike the paper [H1], where we used iterated integrals over paths for reciprocity laws, here we define a higher dimensional analogue, which we call \textit{iterated integrals over membranes}. It took five or six years to complete the many details around these new ideas. An apology is due from the author to the mathematical community and to my former student for that delay.

The idea for \textit{iterated integrals over membranes} had its genesis in an attempt to generalize Manin’s non-commutative modular symbol [M] to a non-commutative Hilbert modular symbol [H2], which remains an ongoing project. However, I received an encouraging email from Manin [M2] about my work [H2].

Before exploring reciprocity laws on surfaces, one has to establish reciprocity laws on curves, for example, the Weil reciprocity law. In [H1], the proof of the Weil reciprocity is via iterated integrals. It uses only double iteration, in contrast to higher order iteration,
which is used in the formulas for a parallel transport with respect to a connection (see [H1].) Similarly, the Parshin symbol on a surface and the two 4-function local symbols use relatively simple iterated integrals over membranes. More complicated iterated integrals over membranes might also be considered for the purpose of reciprocity laws. However, in general, they will produce very complicated formulas. Simpler formulas occur only when we consider at most double iteration. For double iteration, that produces the Parshin symbol for surfaces and the two 4-function local symbols.

The sources of new symbols in our approach are iterated integrals. More precisely, every iterated integral leads to a reciprocity law. In [H1] Theorems 2.9 and 3.3, we use higher order iteration on a complex curve. Then the reciprocity laws are complicated. One can do the same for surfaces. However, we have chosen to consider at most double iterated integrals, which lead to relatively simple reciprocity laws. Over a surface there are three such (iterated) integrals:

(i) a 2-form leading to an analogue of “the sum of the residues is zero”;
(ii) iteration a 2-form with a 1-form - leading to the Parshin symbol;
(iii) an iteration of a 2-form with a 2-form which leads to both 4-function local symbols.

The algebraic varieties in this paper are defined over the complex numbers $\mathbb{C}$. However, all the constructions on a variety $X_K$ would work over any algebraically closed subfield $K \subset \mathbb{C}$, simply by considering the induced embedding $X_K \subset X_C$.

There are several interesting formulas that we would like to introduce to the attention of the reader. For explaining the formulas defining the reciprocity laws, it would be instructive to make a comparison with the Weil reciprocity law stated in terms of the tame symbol.

The divisor of non-zero rational functions $f$ on a complex smooth projective curve is formal sum

$$ (f) = \sum_i a_i P_i, $$

such that $P_i$’s are points where $f$ has zeros or poles and the coefficients $a_i \in \mathbb{Z}$ are the orders of vanishing of $f$ at the points $P_i$. Let also

$$ (g) = \sum_j b_j Q_j. $$

Following Weil, we define

$$ f((g)) = \prod_j f(Q_j)^{b_j} \quad \text{and} \quad g((f)) = \prod_i g(P_i)^{a_i}. $$

**Theorem 0.1. (Weil reciprocity law)** If the support of the divisor of $f$ and the support of the divisor of $g$ are disjoint then

$$ f((g)) = g((f)). $$

Weil reciprocity law can be expressed it terms of the tame symbol, in order to include the cases when the support of $f$ and $g$ have common points. The tame symbol on a curve $C$ is defined as

$$ \{f, g\}_P = (-1)^{ab} \left( \frac{f^b}{g^a} \right) (P), $$
where $a = \text{ord}_P(f)$ and $b = \text{ord}_P(g)$. If $Q$ is in the support of $g$ but not in the support of $f$ then
\[ \{f, g\}_Q = f(Q)^b, \]
where $b = \text{ord}_Q(g)$. As a consequence
\[ \prod_{Q \in \text{Support}(g)} \{f, g\}_Q = f((g)). \]

We can express the Weil reciprocity, using the tame symbol.

**Theorem 0.2.** (Weil reciprocity law in terms of the tame symbol) The tame symbol satisfies the following reciprocity law
\[ \prod_P \{f, g\}_P = 1, \]
where the product is taken over all points $P$ of the smooth projective curve $C$.

Note that the tame symbol is (possibly) different from 1 only when the point $P$ is in the union of the support of the divisors of $f$ and $g$.

Now let $X$ be a smooth complex projective surface, let $C$ be a smooth curve on the surface $X$ and let $P$ be a point on the curve $C$. For a non-zero rational function $f_k$ on the surface $X$, let
\[ a_k = \text{ord}_C(f_k) \]
be the order of vanishing of $f_k$ on the curve $C$. Let also $x$ be a rational function on the surface $X$, representing an uniformizer at the curve $C$, such that no pair of irreducible components of the support of the divisor of $x$ intersect at the point $P$. Let
\[ b_k = \text{ord}_P((x^{-a_k}f_k)|_C). \]

We recall the Parshin symbol
\[ \{f_1, f_2, f_3\}_{C,P} = (-1)^K \left( \prod_{i=1}^3 D_i \right)(P), \]
where
\[ D_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \]
and
\[ K = a_1a_2b_3 + a_2a_3b_1 + a_3a_1b_2 + b_1b_2a_3 + b_2b_3a_1 + b_3b_1a_2. \]

The Parshin symbol satisfies the following reciprocity laws.

**Theorem 0.3.** Let $f_1, f_2, f_3$ be non-zero rational functions on a smooth (complex) projective surface $X$, the following reciprocity laws hold:

(a) \[ \prod_P \{f_1, f_2, f_3\}_{C,P} = 1, \]
where the product is taken over all points $P$ over a fixed curve $C$. Here we assume that the union of the support of the divisors $\bigcup_{i=1}^3 |\text{div}(f_i)|$ in $X$ have normal crossing and no three components have a common point.
\[ \prod_{C} \{f_1, f_2, f_3\}_{C,P} = 1, \]

where the product is taken over all curves \( C \) passing through a fixed point \( P \). Here we assume that the union of the support of the divisors \( \bigcup_{i=1}^{3} |\text{div}(f_i)| \) in \( \tilde{X} \) have normal crossings and no two components have a common point with the exceptional curve \( E \) in \( \tilde{X} \) above the point \( P \). We denote by \( \tilde{X} \) the blow-up of \( X \) at the point \( P \).

We obtain both the Weil reciprocity law and the Parshin reciprocity law via iterated integrals. Using the techniques of iterated integrals over membranes (Subsection 1.4), we define two new 4-function local symbols.

**Definition 0.4.** With the above notation, we define two new 4-function local symbols:

\[ \{f_1, f_2, f_3, f_4\}^{(1)}_{C,P} = (-1)^L \frac{\left(\frac{a_2}{f_2}\right)^{a_2b_4-b_3a_4}}{\left(\frac{a_4}{f_4}\right)^{a_1b_2-b_1a_2}}(P), \]

and

\[ \{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = (-1)^L \frac{\left(\frac{a_2+b_2}{f_2}\right)^{-(a_3b_4-b_3a_4)}}{\left(\frac{a_4+b_4}{f_4}\right)^{-(a_1b_2-b_1a_2)}}(P), \]

where \( L = (a_1b_2 - b_1a_2)(a_3b_4 - b_3a_4) \).

For them, we have the following reciprocity laws.

**Theorem 0.5.** (Reciprocity laws for the new 4-function local symbols) Let \( f_1, f_2, f_3 \) be non-zero rational functions on a smooth (complex) projective surface \( X \), the following reciprocity laws hold:

(a) \[ \prod_{P} \{f_1, f_2, f_3, f_4\}^{(1)}_{C,P} = 1, \]

where the product is taken over all point \( P \) of a fixed curve \( C \). Here we assume that the union of the support of the divisors \( \bigcup_{i=1}^{4} |\text{div}(f_i)| \) in \( X \) have normal crossing and no three components have a common point.

(b) \[ \prod_{C} \{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = 1, \]

where the product is taken over all curves \( C \) passing through a fixed point \( P \). Here we assume that the union of the support of the divisors \( \bigcup_{i=1}^{4} |\text{div}(f_i)| \) in \( \tilde{X} \) have normal crossings and no two components have a common point with the exceptional curve \( E \) in \( \tilde{X} \) above the point \( P \). We denote by \( \tilde{X} \) the blow-up of \( X \) at the point \( P \).

Our approach is based on new types of symbols which we call bi-local symbols. They allow us to refine the local symbols that we study (the Parshin symbol, the 4-function
symbols) in the sense that the local symbols of interest are presented as products of the bi-local symbols and then reciprocity laws are proven for the latter.

For the reader interested in $K$-theoretic approach, we have included a second proof of the reciprocity laws for the new 4-function local symbols, based on Milnor $K$-theory. It can be found in Section 4 and the Appendix.

We learned from Pablos Romo that recently a third proof of the reciprocity laws for the 4-function local symbols, as well as new results about refinements of the Parshin symbol were obtained [PR2].

Let us relate the work in this paper to other results in this area. Brylinski and McLaughlin (see [BrMcL]) used gerbes to define the Parshin symbol. Here we give an alternative, more analytic approach, based on iterated integrals over membranes. We should mention a few other approaches to tame symbols and to the Parshin symbol, for example, [D1], [OZh], [PR1].

Structure of the paper

In Subsection 1.1, we recall basic properties of iterated integrals over paths. Then, in Subsection 1.2, we prove Weil reciprocity by establishing first a reciprocity law for a bi-local symbol, and then removing the dependence on the base point, we recover the Weil reciprocity for the tame symbol on a curve. Subsection 1.3 gives a construction of two foliations. They are needed for the definition of iterated integrals on membranes, presented in Subsection 1.4. Such integrals are the key technical ingredient in this paper.

Section 2, contains the first type of reciprocity laws for the Parshin symbol and for the first 4-function local symbol, where the product of the symbols is over all points $P$ of a fixed curve $C$ on a surface $X$. The proofs are based on the reciprocity laws for bi-local symbols expressed as iterated integrals on membranes. Certain products of bi-local symbols become local symbols such as the Pashin symbol or the first 4-function local symbol. We call such products a refinement of the Pashin symbol or a refinement of the first 4-function local symbol.

Section 3 is about the second type of reciprocity laws, where the product of the symbols is taken over all curves $C$ on $X$ passing though a fixed point $P$. We first establish a technical result about the Parshin symbol and first 4-function symbol under blow-up. We also define bi-local symbols suitable for the second type of reciprocity law. Then the corresponding reciprocity laws are proven for the bi-local symbols, the Parshin symbol, and the second 4-function local symbol. The bi-local symbols in Section 3 provide a second type of refinement of the Parshin symbol and for the second 4-function local symbol.

For convenience of the reader, we conclude with Section 4, by giving an alternative proof of the reciprocity laws of the 4-function local symbols based on Milnor $K$-theory.

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1 Geometric and analytic background

1.1 Iterated path integrals on complex curves

This Subsection contains a definition and properties of iterated integrals, which will be used for the definition of bi-local symbols and for another proof of the Weil reciprocity law in Subsection 1.2.

Let \( C \) be a smooth complex curve. Let \( f_1 \) and \( f_2 \) be two non-zero rational functions on \( C \). Let \( \gamma : [0, 1] \rightarrow C \) be a path, which is a continuous, piecewise differentiable function on the unit interval.

**Definition 1.1.** We define the following iterated integral

\[
\int_{\gamma} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{0 < t_1 < t_2 < 1} \gamma^* \left( \frac{df_1}{f_1} \right) (t_1) \wedge \gamma^* \left( \frac{df_2}{f_2} \right) (t_2).
\]

The two Lemmas below are due to K.-T. Chen [Ch].

**Lemma 1.2.** An iterated integral over a path \( \gamma \) on a smooth curve \( C \) is homotopy invariant with respect to a homotopy, fixing the end points of the path \( \gamma \).

**Lemma 1.3.** If \( \gamma = \gamma_1 \gamma_2 \) is a composition of two paths, where the end of the first path \( \gamma_1 \) is the beginning of the second path \( \gamma_2 \), then

\[
\int_{\gamma_1 \gamma_2} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{\gamma_1} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_{\gamma_1} \frac{df_1}{f_1} \int_{\gamma_2} \frac{df_2}{f_2} + \int_{\gamma_2} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}.
\]

Let \( \sigma \) be a simple loop around a point \( P \) on \( C \) with a base point \( Q \). Let \( \sigma = \gamma \sigma_0 \gamma^{-1} \), where \( \sigma_0 \) is a small loop around \( P \), with a base the point \( R \) and let \( \gamma \) be a path joining the points \( Q \) with \( R \).

The following Lemma is essential for the proof of the Weil reciprocity (see also [HI]).

**Lemma 1.4.** With the above notation, the following holds

\[
\int_{\sigma} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{\gamma} \frac{df_1}{f_1} \int_{\sigma_0} \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2}.
\]

**Proof.** First, we use Lemma 1.3 for the composition \( \gamma \sigma_0 \gamma^{-1} \). We obtain

\[
\int_{\sigma} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{\gamma} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_{\gamma} \frac{df_1}{f_1} \int_{\sigma_0} \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_{\gamma^{-1}} \frac{df_1}{f_1} \int_{\gamma} \frac{df_2}{f_2} + \int_{\gamma^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}.
\]
Then, we use the homotopy invariance of iterated integrals, Lemma 1.2, for the path \( \gamma \gamma^{-1} \). Thus,
\[
0 = \int_{\gamma \gamma^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}.
\]
Finally, we use Lemma 1.3 for the composition of paths \( \gamma \gamma^{-1} \). That gives
\[
0 = \int_{\gamma \gamma^{-1}} \frac{df_1}{f_2} \circ \frac{df_2}{f_2} = \int_{\gamma} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_{\gamma^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}.
\]
(1.2)
The Lemma 1.4 follows from Equations (1.1) and (1.2).

1.2 Weil reciprocity via iterated path integrals

Here, we present a proof of the Weil reciprocity law, based on iterated integrals and bi-local symbols. This method will be generalized in the later Subsections in order to prove reciprocity laws on complex surfaces. Similar ideas about the Weil reciprocity law are contained in [H1], however, without bi-local symbols.

Let \( x \) be a rational function on a curve \( C \), representing an uniformizer at \( P \). Let
\[
a_i = \text{ord}_P(f_i).
\]
and let
\[
g_i = x^{-a_i}f_i.
\]
Then
\[
\frac{df_i}{f_i} = a_i \frac{dx}{x} + \frac{dg_i}{g_i}.
\]
Let \( \sigma_0^\epsilon \) be a small loop around the point \( P \), whose points are at most at distance \( \epsilon \) from the point \( P \). One can take the metric inherited from the Fubini-Study metric on \( \mathbb{P}^k \).

Put \( \sigma_0^\epsilon = \sigma_0 \) in Lemma 1.4 then
\[
\int_{\gamma} \frac{df_1}{f_1} \int_{\sigma_0^\epsilon} \frac{df_2}{f_2} = 2\pi i a_2 \int_{\gamma} \frac{df_1}{f_1} = 2\pi i a_2 \left( a_1 \int_{\gamma} \frac{dx}{x} + \int_{\gamma} \frac{dg_1}{g_1} \right).
\]
Similarly,
\[
\int_{\sigma_0^\epsilon} \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2} = 2\pi i a_1 \left( -a_2 \int_{\gamma} \frac{dx}{x} - \int_{\gamma} \frac{dg_2}{g_2} \right).
\]
From [H1], we have that
\[
\lim_{\epsilon \to 0} \int_{\sigma_0^\epsilon} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \frac{(2\pi i)^2}{2} a_1 a_2.
\]
(1.3)
Using Lemma 1.4 we obtain
\[
\int_{\sigma} \frac{df_1}{f_2} \circ \frac{df_2}{f_2} = 2\pi i \left( a_2 \log(g_1) - a_1 \log(g_2) + \pi i a_1 a_2 \right) \mid_{\sigma}.
\]
After exponentiation, we obtain
Lemma 1.5. With the above notation the following holds
\[
\exp \left( \frac{1}{2\pi i} \int_{f_1} \frac{df_1}{f_2} \circ \frac{df_2}{f_2} \right) = (-1)^{a_1 a_2} \frac{f_1}{g_2} (P) \left( \frac{g_1}{f_2} \right)^{-1} = (-1)^{a_1 a_2} \frac{f_1}{f_2} (Q) \left( \frac{f_1}{f_2} \right)^{-1}
\]

Definition 1.6. (Bi-local symbol on a curve) With the above notation, we define a bi-local symbol
\[
\{ f_1, f_2 \}_P = (-1)^{a_1 a_2} \frac{f_1}{f_2} (P) \left( \frac{f_1}{f_2} \right)^{-1}. \tag{1.4}
\]

Let the curve \( C \) be of genus \( g \) and let \( P_1, \ldots, P_n \) be the points of the union of the support of the divisors of \( f_1 \) and \( f_2 \). Let \( \sigma_1, \ldots, \sigma_n \) be simple loops around the points \( P_1, \ldots, P_n \), respectively. Let also \( \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \) be the \( 2g \) loops on the curve \( C \) such that
\[
\pi_1(C, Q) = < \sigma_1, \ldots, \sigma_n, \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n > / \sim,
\]
where \( \delta \sim 1 \), for
\[
\delta = \prod_{i=1}^{n} \sigma_i \prod_{j=1}^{g} [\alpha_j, \beta_j].
\]
From Theorem 3.1 in [H1], we have

Lemma 1.7.
\[
\int_{\alpha \beta \alpha^{-1} \beta^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{\alpha} \frac{df_1}{f_1} \int_{\beta} \frac{df_2}{f_2} - \int_{\alpha} \frac{df_2}{f_2} \int_{\beta} \frac{df_1}{f_1}.
\]

Using the above result, we obtain that
\[
0 = \int_{\delta} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \in (2\pi i)^2 \mathbb{Z} + \sum_{i=1}^{n} \int_{\sigma_i} \frac{df_1}{f_1} \cdot \frac{df_2}{f_2},
\]
where the sum is over simple loops \( \sigma_i \) around each of the points \( P_i \). Then we obtain:

Theorem 1.8. (Reciprocity law for the bi-local symbol [1.4]) With the above notation, the following holds
\[
\prod_{P} \{ f_1, f_2 \}_P = 1.
\]

If we want to make the above reciprocity law into a reciprocity law for a local symbol we have to remove the dependency on the base point \( Q \). This can be achieved in the following way: In the reciprocity law for the bi-local symbol, the dependency on \( Q \) is
\[
\prod_{P} f_1(Q)^{a_2} f_2(Q)^{-a_1} = f_1(Q)^{(2\pi i)^{-1} \sum_{P} \text{Res}_P \frac{df_1}{f_2}} =
\]
\[
f_2(Q)^{-{(2\pi i)^{-1} \sum_{P} \text{Res}_P \frac{df_1}{f_2}}} = f_1(Q)^{0} f_2(Q)^{0} = 1.
\]
Thus, we recover Weil reciprocity:
Theorem 1.9. (Weil reciprocity) The local symbol

\[ \{f_1, f_2\}_P = (-1)^{a_1 a_2} \frac{f_2^{a_2}}{f_1^{a_1}}(P). \]

satisfies the following reciprocity law

\[ \prod_P \{f_1, f_2\}_P = 1, \]

where the product is over all points \( P \) in \( C \).

1.3 Two foliations on a surface

The goal of this Subsection is to construct two foliations on a complex projective algebraic surface \( X \) in \( \mathbb{P}^k \). This is an algebraic-geometric material, needed for the definition of iterated integrals on membranes, presented in Subsection 1.4.

Let \( f_1, f_2, f_3 \) and \( f_4 \) be four non-zero rational functions on the surface \( X \). Let

\[ C \cup C_1 \cup \cdots \cup C_n = \bigcup_{i=1}^4 |\text{div}(f_i)|, \]

where we fix one of the irreducible components \( C \). Let

\[ \{P_1, \ldots, P_N\} = C \cap (C_1 \cup \cdots \cup C_n). \]

We can assume that the curves \( C, C_1, \ldots, C_n \) are smooth and that the intersections are transversal (normal crossings) and no three of them intersect at a point, by allowing blow-ups on the surface \( X \).

The two foliations have to satisfy the following Conditions:

1. There exists a foliation \( F'_v \) such that

   (a) \( F'_v = (f - v)_0 \) are the level sets of a rational function

   \[ f : X \to \mathbb{P}^1, \]

   for small values of \( v \), (that is, for \( |v| < \epsilon \) for a chosen \( \epsilon \));

   (b) \( F'_v \) is smooth for all but finitely many values of \( v \);

   (c) \( F'_v \) has only nodal singularities;

   (d) \( \text{ord}_C(f) = 1 \);

   (e) \( R_i \notin C_j \), for \( i = 1, \ldots, M \) and \( j = 1, \ldots, n \), where

   \[ \{R_1, \ldots, R_M\} = C \cap (D_1 \cup \cdots \cup D_m) \]

   and

   \[ F'_0 = (f)_0 = C \cup D_1 \cup \cdots \cup D_m. \]
2. There exists a foliation $G_w$ such that

(a) $G_w = (g - w)0$ are the level sets of a rational function $g : X \to \mathbb{P}^1$;

(b) $G_w$ is smooth for all but finitely many values of $w$;

(c) $G_w$ has only nodal singularities;

(d) $g|_C$ is non constant.

3. Coherence between the two foliations $F'$ and $G$:

(a) All but finitely many leaves of the foliation $G$ are transversal to the curve $C$.

(b) $G_{g(P_i)}$ intersects the curve $C$ transversally, for $i = 1, \ldots, N$. (For definition of the points $P_i$ see the beginning of this Subsection.)

(c) $G_{g(R_i)}$ intersects the curve $C$ transversally, for $i = 1, \ldots, M$. (For definition of the points $R_i$ see condition 1(e).)

The existence of $f \in \mathbb{C}(X)^\times$ satisfying properties 1(a-d) is a direct consequence from the following result, which follows immediately from (a special case of) a result of Thomas [Th, Theorem 4.2].

**Theorem 1.10.** Consider a smooth curve $C$ in a smooth projective surface $X$, with hyperplane section $H_X$. There exists a large constant $N \in \mathbb{N}$ and a pencil in $|NH_X|$, given as the level sets $(f - x)0$ of some rational function $f$ such that $(f - x)0$ is smooth for all but finitely many values of $x$, at which it has only nodal singularities, and $C \subset (f)0$.

Moreover, a general choice of $g \in \mathbb{C}(X)^\times$ will satisfy 2(a-d) and 3(a-c). (For instance, the quotient of two generic linear forms on $\mathbb{P}^k$ restricted to $C$ will not have branch points in $\{P_i\} \cup \{R_j\}$.)

It remains to examine property 1(e). The proof of Theorem 4.2 in [op. cit.] contains the basic

**Observation:** The base locus of the linear system $H^0(I_C(N))$ is the smooth curve $C$ for $N >> 0$. So by Bertini’s theorem the general element of the linear system is smooth away from $C$.

Consider $C \subset X$. By the Observation, there exists $\mathcal{F} \in H^0(X, O(N))$ such that $\text{ord}_C(\mathcal{F}) = 1$ and $(\mathcal{F}) = C + D$, where $D$ is a second smooth curve on $X$, meeting $C$ transversally (if at all).

**Claim:** We may choose $\mathcal{F}$ so that condition 1(e) holds, that is, $R_i \notin C_j$ for each $i, j$, where $\{R_1, \ldots, R_M\} = C \cap D$. Equivalently, $C \cap D \cap C_j = \emptyset$.

**Proof.** Define $H^0(I_C(N))^{reg}$ to be the subset of $H^0(X, I_C(N))$ whose elements $\mathcal{F}$ satisfy $\text{ord}_C(\mathcal{F}) = 1$ and $(\mathcal{F}) = C + D$ as above. Assume that for every $N >> 0$ and $\mathcal{F} \in H^0(I_C(N))^{reg}$ we have $D \cap C \cap C_j \neq \emptyset$ for some particular $j$. If we obtain a contradiction (for some $N$) then the claim is proved, since this is a closed condition for each $j$. 

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According to our assumption, \((\mathcal{F})\) always has an ordinary double point at the intersection \(\Delta := C \cap C_j \neq \emptyset\). In the exact sequence
\[
0 \to H^0(X, I^2_C(N)) \to H^0(X, I_C(N)) \to H^0(C, \mathcal{N}^*_C/X(N)) \to H^1(X, I^2_C(N)),
\]
the last term vanishes by ([GH], Vanishing Theorem B) for \(N\) sufficiently large. Hence, every section over \(C\) of the twisted conormal sheaf \(\mathcal{N}^*_C/X(N)\) has a zero along \(\Delta = C \cap C_j\).

Next consider the exact sequence
\[
0 \to H^0(C, I_\Delta \otimes \mathcal{N}^*_C/X(N)) \to H^0(C, \mathcal{N}^*_C/X(N)) \to \mathbb{C} |_{\Delta} \to H^1(C, I_\Delta \otimes \mathcal{N}^*_C/X(N)).
\]
The last term vanishes again by [loc. cit.]. Denote the third arrow by \(\text{ev}_\Delta\). Then we can take a section of \(\mathcal{N}^*_C/X(N)\) not vanishing on \(\Delta\) simply by taking an element in the preimage of \(\text{ev}_\Delta(1, \ldots, 1)\). This produces the desired contradiction.

Consider a metric on the surface \(X\), which respects the complex structure. For example, we can take the metric inherited from the Fubini-Study metric on \(\mathbb{P}^k\) via the embedding \(X \to \mathbb{P}^k\). Let \(U_1', \ldots, U_M'\) be disks of radii \(\epsilon\) on \(C\), centered respectively at \(R_1, \ldots, R_M\). Let
\[
C_0 = C - \bigcup_{j=1}^M U_\epsilon - \{P_1, \ldots, P_N\}.
\]

**Definition 1.11.** With the above notation, let \(F_v\) be the connected component of
\[
F_v' - \left( \bigcup_{i=1}^M G_g(U'_i) \right) \cap F_v',
\]
containing \(C_0\), for \(|v| << \epsilon\), where
\[
G_g(U'_i) = \bigcup_{w \in U'_i} G_g(w)
\]

**Lemma 1.12.** With the above notation, for small values of \(|v|\), we have that each leaf \(F_v\) is a continuous deformation of \(F_0 = C_0\), preserving homotopy type.

**Proof.** From Property 3(c), it follows that \(C\) and \(D_i\) meet at \(R_j\) (if at all) at a non-zero angle. At the intersection \(R_i\), locally we can represent the curves by \(xy = 0\). The deformation leads to \(v - xy = 0\), which is a leaf of \(F'\), locally near \(R_i\). Consider a disk \(U\) of radius \(\epsilon_i\) at \((x, y) = (0, 0)\) in the \(xy\)-plane. Then for \(|v| << \epsilon_i\) we have that \(U\) separates \(F'_v\) into 2 components, one close to the \(x\)-axis and the other close to the \(y\)-axis. We do the same for each of the points \(R_1, \ldots, R_M\) and we take the minimum of the bounds \(\epsilon_i\). Then \(F_v\) will consist of points close to the curve \(C_0\). \(\square\)

### 1.4 Iterated integrals on a membrane. Definitions and properties

In this Subsection, we define types of iterated integrals over membranes, needed in most of this manuscript.
Let $\tau$ be a simple loop around $C_0$ in $X - C_0 - \left( \bigcup_{i=1}^{M} G_{g(\tau_i')} \right)$. Let $\sigma$ be a loop on the curve $C_0$. We define a membrane $m_\sigma$ associated to a loop $\sigma$ in $C^0$ by

$$m_\sigma : [0, 1]^2 \to X$$

and

$$m_\sigma(s, t) \in F_{f(\tau(t))} \cap G_{g(\sigma(s))}.$$

Note that for fixed values of $s$ and $t$, we have that

$$F_{f(\tau(t))} \cap G_{g(\sigma(s))}$$

consists of finitely many points, where $F$ and $G$ are foliations satisfying the Conditions in Subsection 1.3 and Lemma 1.12.

**Claim:** The image of $m_\sigma$ is a torus.

Indeed, consider a tubular neighborhood around a loop $\sigma$ on the curve $C_0$. One can take the following tubular neighborhood:

$$\bigcup_{|v|<\epsilon} F_v \cap G_{g(\sigma)}$$

of $F_v \cap G_{g(\sigma)}$. Its boundary is $F_{f(\tau)} \cap G_{g(\sigma)}$, where $\tau$ is a simple loop around $C_0$ on $X - \bigcup_{i=1}^{n} C_i - \bigcup_{j=1}^{m} D_j$ and $|f(\tau(t))| = \epsilon$.

We shall define the simplest type of iterated integrals over membranes. Also, we are going to construct local symbols in terms of iterated integrals $I_1, I_2, I_3, I_4$ on membranes, defined below.

We define the following differential forms

$$A(s, t) = m^* \left( \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \right)(s, t)$$

$$b(s, t) = m^* \left( \frac{df_3}{f_3} \right)(s, t)$$

and

$$B(s, t) = m^* \left( \frac{df_3}{f_3} \wedge \frac{df_4}{f_4} \right)(s, t).$$

The first diagram

$$\begin{array}{c}
  t \\
  \downarrow \quad A \\
  s
\end{array}$$

denotes

$$I_1 = \int_{0}^{1} \int_{0}^{1} A(s, t).$$

The second diagram
denotes
\[ I_2 = \int_0^1 \int_{0<t_1<t_2<1} A(s, t_1) \wedge b(s, t_2). \]

The third diagram
\[ t \quad \begin{array}{|c|c|}
  \hline
  A & b \\
  \hline
\end{array} \\
\begin{array}{c}
  s_1 \\
  s_2
\end{array} \]
denotes
\[ I_3 = \int \int_{0<s_1<s_2<1} \int_0^1 A(s_1, t) \wedge b(s_2, t). \]

And the fourth diagram
\[ t_2 \quad \begin{array}{|c|c|}
  \hline
  B & \\
  \hline
\end{array} \\
\begin{array}{c}
  A \\
  \hline
\end{array} \\
\begin{array}{c}
  s_1 \\
  s_2
\end{array} \]
denotes
\[ I_4 = \int \int_{0<s_1<s_2<1} \int \int_{0<t_1<t_2<1} A(s_1, t_1) \wedge B(s_2, t_2). \]

Local symbols will be defined via the above four types of iterated integrals. The integrals that we define below, used for defining bi-local symbols, are a technical tool for proving reciprocity laws for the local symbols. Bi-local symbols also satisfy reciprocity laws.

Consider the dependence of \( \log(f_i(m(s, t))) \) on the variables \( s \) and \( t \) via the parametrization of the membrane \( m \).

**Definition 1.13.** Let
\[ l_i(s, t) = \log(f_i(m(s, t))) \]

We have
\[ dl_i(s, t) = \frac{\partial l_i(s, t)}{\partial s} ds + \frac{\partial l_i(s, t)}{\partial t} dt. \]
\[ b(s, t) = dl_3(s, t) \]
\[ A(s, t) = \frac{\partial l_1(s, t)}{\partial s} \frac{\partial l_2(s, t)}{\partial t} ds \wedge dt - \frac{\partial l_1(s, t)}{\partial t} \frac{\partial l_2(s, t)}{\partial s} ds \wedge dt \] (1.5)
\[ B(s, t) = \frac{\partial l_3(s, t)}{\partial s} \frac{\partial l_4(s, t)}{\partial t} ds \wedge dt - \frac{\partial l_3(s, t)}{\partial t} \frac{\partial l_4(s, t)}{\partial s} ds \wedge dt \] (1.6)
The above equations express the differential forms $A, B$ and $b$ is terms of monomials in terms of first derivatives of $l_1, l_2, l_3, l_4$. We are going to define bi-local symbols associated to monomials in first derivatives of $l_1, l_2, l_3, l_4$, which occur in

$$A(s, t), \quad A(s, t_1) \wedge b(s, t_2), \quad A(s_1, t) \wedge b(s_2, t), \quad \text{and} \quad A(s_1, t_2) \wedge B(s_2, t_2)$$

**Definition 1.14.** (Iterated integrals on membranes) Let $f_1, \ldots, f_{k+l}$ be rational functions on $X$, where the pairs $(k, l)$ will be superscripts of the integrals. Let $m$ be a membrane as above. We define:

(a) $I^{(1,1)}(m; f_1, f_2) =$

$$= \int_0^1 \int_0^1 \left( \frac{\partial l_1(s, t)}{\partial s} ds \right) \wedge \left( \frac{\partial l_2(s, t)}{\partial t} dt \right)$$

(b) $I^{(1,2)}(m; f_1, f_2, f_3) =$

$$\begin{align*}
= \int & \int \int_{0 \leq s \leq 1; \ 0 \leq t_1 \leq t_2 \leq 1} \left( \frac{\partial l_1(s, t_1)}{\partial s} \frac{\partial l_2(s, t_1)}{\partial t_1} ds_1 dt_1 \right) \\
& \wedge \left( \frac{\partial l_3(s, t_2)}{\partial t_2} dt_2 \right)
\end{align*}$$

(c) $I^{(2,1)}(m; f_1, f_2, f_3) =$

$$\begin{align*}
= \int & \int \int_{0 \leq s_1 \leq s_2 \leq 1; \ 0 \leq t \leq 1} \left( \frac{\partial l_1(s_1, t)}{\partial s_1} \frac{\partial l_2(s_1, t)}{\partial t} ds_1 dt \right) \\
& \wedge \left( \frac{\partial l_3(s_2, t)}{\partial s_2} ds_2 \right)
\end{align*}$$

(d) $I^{(2,2)}(m; f_1, f_2, f_3, f_4) =$

$$\begin{align*}
= \int & \int \int \int_{0 \leq s_1 \leq s_2 \leq 1; \ 0 \leq t_1 \leq t_2 \leq 1} \left( \frac{\partial l_1(s_1, t_1)}{\partial s_1} \frac{\partial l_2(s_1, t_1)}{\partial t_1} ds_1 dt_1 \right) \\
& \wedge \left( \frac{\partial l_3(s_2, t_2)}{\partial s_2} \frac{\partial l_4(s_2, t_2)}{\partial t_2} ds_2 dt_2 \right)
\end{align*}$$

**Proposition 1.15.** (a) $I_1 = I^{(1,1)}(m; f_1, f_2) - I^{(1,1)}(m; f_2, f_1)$;

(b) $I_2 = I^{(1,2)}(m; f_1, f_2, f_3) - I^{(1,2)}(m; f_2, f_1, f_3)$;

(c) $I_3 = I^{(2,1)}(m; f_1, f_2, f_3) - I^{(2,1)}(m; f_2, f_1, f_3)$;

(d) $I_4 = I^{(2,2)}(m; f_1, f_2, f_3, f_4) - I^{(2,2)}(m; f_2, f_1, f_3, f_4) - I^{(2,2)}(m; f_1, f_2, f_4, f_3) + I^{(2,2)}(m; f_2, f_1, f_4, f_3)$

Consider a metric on the projective surface $X$ inherited from the Fubini-Study metric on $\mathbb{P}^k$. Let $\tau$ be a simple loop around the curve $C$ of distance at most $\epsilon$ from $C$. We are going to take the limit as $\epsilon \to 0$. Informally, the radius of the loop $\tau$ goes to zero. Then we have the following lemma.

**Lemma 1.16.** With the above notation the following holds:

(a) $\lim_{\epsilon \to 0} I^{(1,1)}(m_\sigma, f_1, f_2) = (2\pi i) \text{Res}_f \frac{df_2}{f_2} \int_\sigma \frac{df_1}{f_1}$
\[ \lim_{\epsilon \to 0} I^{(1,2)}(m_\sigma, f_1, f_2, f_3) = \frac{(2\pi i)^2}{2} \text{Res}_{f_2} \frac{df_2}{f_2} \text{Res}_{f_3} \frac{df_3}{f_3} \int_{\sigma} \frac{df_1}{f_1} \]

\[ \lim_{\epsilon \to 0} I^{(2,1)}(m_\sigma, f_1, f_2, f_3) = -(2\pi i) \text{Res}_{f_2} \frac{df_2}{f_2} \int_{\sigma} \frac{df_1}{f_1} \circ \frac{df_3}{f_3} \]

\[ \lim_{\epsilon \to 0} I^{(2,2)}(m_\sigma, f_1, f_2, f_3, f_4) = -\frac{(2\pi i)^2}{2} \text{Res}_{f_2} \frac{df_2}{f_2} \text{Res}_{f_4} \frac{df_4}{f_4} \int_{\sigma} \frac{df_1}{f_1} \circ \frac{df_3}{f_3} \]

**Proof.** First, we consider the integrals in parts (a) and (c), where there is integration with respect to the variable \( t \) in the definition of the membrane \( m \). Let \( m(s, \cdot) \) denote the loop obtained by fixing the first variable \( s \) and varying the second variable \( t \). Then, there is no iteration along the loop \( m(s, \cdot) \) around the curve \( C \), for fixed value of \( s \). Using Properties 1(d) and 1(e), the integration over the loop \( m(s, \cdot) \) gives us a single residue. This process is independent of the base point of the loop \( m(s, \cdot) \). That proves parts (a) and (c).

For parts (b) and (d), we have a double iteration along the loop \( m(s, \cdot) \) around the curve \( C \), where the value of \( s \) is fixed and the second argument varies. After taking the limit as \( \epsilon \) goes to 0, the integral along \( m(s, \cdot) \), with respect to \( t_1 \) and \( t_2 \), becomes a product of two residues (see Equation (1.3)), which are independent of a base point. That proves parts (b) and (d). \qed

2 First type of reciprocity laws

2.1 Reciprocity laws for bi-local symbols

In this Subsection, we define bi-local symbols and prove their reciprocity laws. Using them, in the following two Sections, we establish the first type of reciprocity laws for the Parshin symbol and for a new 4-function new symbol. By a first type of reciprocity law, we mean that the product of the local symbols is taken over all points \( P \) of a fixed curve \( C \) on the surface \( X \).

Consider the fundamental group of \( C_0 \). We recall that \( C_0 \) is essentially the curve \( C \) without several intersection points and without several open neighborhoods. More precisely,

\[ C_0 = C - \left( \bigcup_{j=1}^{m} G_{U_j} \right) \cap C - \left( \bigcup_{i=1}^{n} C_i \right) \cap C. \]

where \( U_j \) is a small neighborhood of \( R_j \) on the complex curve \( C \). We recall the notation for the intersection points

\[ \{P_1, \ldots, P_N\} = C \cap (C_1 \cup \cdots \cup C_n), \]

\[ \{R_1, \ldots, R_M\} = C \cap (D_1 \cup \cdots \cup D_m), \]

Let

\[ \pi_1(C_0, Q) = \langle \sigma_1, \ldots, \sigma_n, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \rangle / \sim \]

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be a presentation of the fundamental group, where
\[ \delta \sim 1, \]
for
\[ \delta = \prod_{i=1}^{n} \sigma_{i} \prod_{j=1}^{g} \{\alpha_{j}, \beta_{j}\}. \]

We are going to drop the indices \( i \) and \( j \). Thus, we are going to write \( P \) instead of \( P_{i} \) or \( R_{j} \) and \( \sigma \) instead of \( \sigma_{i} \). Consider the definition of a membrane \( m_{\sigma} \), associated to a loop \( \sigma \), given in the beginning of Subsection 1.4. Let \( m_{\sigma}(s, \cdot) \) be the loop obtained by fixing the variable \( s \) and letting the second argument vary. Similarly, \( m_{\sigma}(\cdot, t) \) denotes the loop obtained by fixing the variable \( t \) and letting the first argument vary.

**Definition 2.1.** Let \( a_{k} = \text{ord}_{C}(f_{k}) \) and \( b_{k} = \text{ord}_{P}((x-a_{k}f_{k})|_{C}) \), where \( x \) is a rational function, representing an uniformizer such that \( \text{ord}_{C}(x) = 1 \) and \( P \) is not an intersection of any two of the components of the divisor of \( x \).

It is straightforward to represent the order of vanishing as residues, given by the following:

**Lemma 2.2.** We have
\[ a_{k} = \frac{1}{2\pi i} \int_{m_{\sigma}(s, \cdot)} \frac{df_{k}}{f_{k}} \quad \text{and} \quad b_{k} = \frac{1}{2\pi i} \int_{m_{\sigma}(\cdot, t)} \frac{df_{k}}{f_{k}}. \]

Using properties 1(d) and 3(b), we should think of \( m_{\sigma}(\cdot, t) \) and \( m_{\sigma}(s, \cdot) \) as translates of \( \sigma \) and of \( \tau \), respectively. Then the above integrals are residues, which detect the order of vanishing. For example \( a_{k} \) is the order of vanishing of \( f_{k} \) along a generic point of \( C \).

Then the following theorem holds, whose proof is immediate from Lemmas 1.16 and 2.2.

**Theorem 2.3.** (a) \( (2\pi i)^{-2} \lim_{\epsilon \to 0} I^{(1,1)}(m_{\sigma}, f_{1}, f_{2}) = a_{2}b_{1} \),
(b) \( (2\pi i)^{-2} \lim_{\epsilon \to 0} I^{(1,2)}(m_{\sigma}, f_{1}, f_{2}, f_{3}) = (\pi i)a_{2}a_{3}b_{1} \),
(c) \[ \exp \left( (2\pi i)^{-2} \lim_{\epsilon \to 0} I^{(2,1)}(m_{\sigma}, f_{1}, f_{2}, f_{3}) \right) = \left( \{f_{2}, f_{3}\}_{P}^{Q} \right)^{-a_{1}}, \]
(d) \[ \exp \left( \frac{2}{(2\pi i)^{3}} \lim_{\epsilon \to 0} I^{(2,2)}(m_{\sigma}, f_{1}, f_{2}, f_{3}, f_{4}) \right) = \left( \{f_{1}, f_{3}\}_{P}^{Q} \right)^{-a_{2}a_{4}}. \]

Let us denote by \( \alpha \) the loop \( \alpha_{j} \) and by \( \beta \) the loop \( \beta_{j} \). Then the following lemma holds

**Lemma 2.4.** (a) \( (2\pi i)^{-2} \lim_{\epsilon \to 0} I^{(1,1)}(m_{(\alpha, \beta)}, f_{1}, f_{2}) = 0 \),
(b) \( (2\pi i)^{-2} \lim_{\epsilon \to 0} I^{(1,2)}(m_{(\alpha, \beta)}, f_{1}, f_{2}, f_{3}) = 0 \),
(c) \( (2\pi i)^{-2} \lim_{\epsilon \to 0} I^{(2,1)}(m_{(\alpha, \beta)}, f_{1}, f_{2}, f_{3}) = 0 \),
(d) \( (2\pi i)^{-2} \lim_{\epsilon \to 0} I^{(2,2)}(m_{(\alpha, \beta)}, f_{1}, f_{2}, f_{3}, f_{4}) = 0 \).
(c) \[ \exp \left( \frac{(2\pi i)^{-2}}{\epsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} I^{(2,1)}_{(m_{[\alpha, \beta]}, f_1, f_2, f_3)} \right) = 1, \]

(d) \[ \exp \left( \frac{2}{(2\pi i)^5} \lim_{\epsilon \rightarrow 0} I^{(2,2)}_{(m_{[\alpha, \beta]}, f_1, f_2, f_3, f_4)} \right) = 1. \]

**Proof.** It follows from Lemmas 1.16 and 1.4. A more modern proof follows from the well-definedness of the integral Beilinson regulator on $K_2$ on the level of homology (see [Ke]).

**Definition 2.5.** (Bi-local symbols on a surface) For a simple loop $\sigma$ around a point $P$ in $C_0$, based at $Q$, let

\[ \text{Log}^{(i,j)}[f_1, \ldots, f_{i+j}]^{(1), Q}_{C, P} = \lim_{\epsilon \rightarrow 0} I^{i,j}(m_{\sigma}, f_1, \ldots, f_{i+j}), \]

\[ ^{1,2}[f_1, f_2, f_3]^{(1), Q}_{C, P} = \exp \left( \frac{(2\pi i)^{-2}}{\epsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} I^{(1,2)}_{(m_{\sigma}, f_1, f_2, f_3)} \right), \]

\[ ^{2,1}[f_1, f_2, f_3]^{(1), Q}_{C, P} = \exp \left( \frac{(2\pi i)^{-2}}{\epsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} I^{(2,1)}_{(m_{\sigma}, f_1, f_2, f_3)} \right), \]

\[ ^{2,2}[f_1, f_2, f_3, f_4]^{(1), Q}_{C, P} = \exp \left( \frac{2}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} I^{(2,2)}_{(m_{\sigma}, f_1, f_2, f_3, f_4)} \right). \]

The following reciprocity laws hold for the above bi-local symbols.

**Theorem 2.6.** (a) $\sum_P \text{Log}^{1,1}[f_1, f_2]^{(1), Q}_{C, P} = 0$.

(b) $\prod_P ^{1,2}[f_1, f_2, f_3]^{(1), Q}_{C, P} = 1$.

(c) $\prod_P ^{2,1}[f_1, f_2, f_3]^{(1), Q}_{C, P} = 1$.

(d) $\prod_P ^{2,2}[f_1, f_2, f_3, f_4]^{(1), Q}_{C, P} = 1$.

**Proof.** Parts (b), (c) and (d) follow directly from Theorem 2.3 and from Weil reciprocity. Part (a) follows again from Theorem 2.3 and the theorem that the sum of the residues of a differential form on a curve is zero. \[ \square \]

### 2.2 Parshin symbol and its first reciprocity law.

In this Subsection, we construct a refinement of the Parshin symbol in terms of six bi-local symbols. Using this presentation of the Parshin symbol, Definition 2.7 and Theorem 2.8 we prove the first reciprocity of the Parshin symbol (Theorem 2.10).

**Definition 2.7.** We define the following bi-local symbol

\[ P^Q_{C, P} = \left( ^{1,2}[f_1, f_2, f_3]^{(1), Q}_{C, P} \right) \left( ^{1,2}[f_2, f_3, f_1]^{(1), Q}_{C, P} \right) \left( ^{1,2}[f_3, f_1, f_2]^{(1), Q}_{C, P} \right) \times \]

\[ \times \left( ^{2,1}[f_1, f_2, f_3]^{(1), Q}_{C, P} \right) \left( ^{2,1}[f_2, f_3, f_1]^{(1), Q}_{C, P} \right) \left( ^{2,1}[f_3, f_1, f_2]^{(1), Q}_{C, P} \right) \]

at the points $P = P_i \in C \cap (C_1 \cup \cdots \cup C_n)$ and a fixed point $Q$ in $C - C \cap (C_1 \cup \cdots \cup C_n)$.

Using Theorem 2.3 parts (b) and (c), we obtain:
**Theorem 2.8.** (Refinement of the Parshin symbol) We have the following explicit formula

\[ Pr^Q_{C,P} = (-1)^K \left( \frac{f_1^{D_1} f_2^{D_2} f_3^{D_3}}{f_1^{D_1} f_2^{D_2} f_3^{D_3}} \right) (P), \]

where

\[ D_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \]

and

\[ K = a_1a_2b_3 + a_2a_3b_1 + a_3a_1b_2 + b_1b_2a_3 + b_2b_3a_1 + b_3b_1a_2. \]

Note that \( Pr^Q_{C,P} \) is essentially the Parshin symbol, which can be defined in the following way

**Definition 2.9.** (The Parshin symbol)

\[ \{ f_1, f_2, f_3 \}_{C,P} = (-1)^K \left( \frac{f_1^{D_1} f_2^{D_2} f_3^{D_3}}{f_1^{D_1} f_2^{D_2} f_3^{D_3}} \right) (P). \]

The only difference between the two symbols is the constant factor in \( Pr^Q_{C,P} \), depending only on the base point \( Q \) (the denominator of \( Pr^Q_{C,P} \)). Rescaling by that constant leads to the Parshin symbol.

**Theorem 2.10.** (First reciprocity law for the Parshin symbol) For the Parshin symbol, the following reciprocity law holds

\[ \prod_P \{ f_1, f_2, f_3 \}_{C,P} = 1, \]

where the product is taken over points \( P \) in \( C \cap (C_1 \cup \cdots \cup C_n) \). (When \( P \) is another point of \( C \) then the symbol is trivial.) Here we assume that the union of the support of the divisors \( \bigcup_{i=1}^3 |\text{div}(f_i)| \) in \( X \) have normal crossing and no three components have a common point.

**Proof.** We are going to use the reciprocity laws for bi-local symbols stated in Theorem 2.6 parts (b) and (c). Then the reciprocity law for the bi-local symbol \( Pr^Q_P \) follows. There is relation between the Parshin symbol and \( Pr^Q_C \), namely,

\[ \{ f_1, f_2, f_3 \}_{C,P} = Pr^Q_{C,P} \left( \frac{f_1^{D_1} f_2^{D_2} f_3^{D_3}}{f_1^{D_1} f_2^{D_2} f_3^{D_3}} \right) (Q). \]

Now, we remove the dependence on the base point \( Q \). In order to do that, note that

\[ \prod_P f_1(Q)^{D_1} = g_1(Q)^{\sum_P D_1}. \]

Here \( g_1 = x^{-a_1} f_1 \), where \( x \) is a rational function on the surface \( X \), representing an uniformazer at the curve \( C \), such that the components of the divisor of \( x \) do not intersect at the points \( P \) or \( Q \). Moreover,

\[ D_1 = (2\pi i)^{-2} \left( \text{Log}^{1,1}[f_2, f_3]_P^{(1),Q} - \text{Log}^{1,1}[f_1, f_2]_P^{(1),Q} \right) \]
by Theorem 2.3 part (a) and Proposition 1.15 part (a). Using Theorem 2.6 part (a), for the above equality, we obtain
\[ \sum_P D_1 = 0. \]
Therefore,
\[ \prod_P g_1(Q)^{D_1} = 1. \]
Similarly,
\[ \prod_P g_2(Q)^{D_2} = 1 \quad \text{and} \quad \prod_P g_3(Q)^{D_3} = 1, \]
where \( g_k = x^{-a_k} f_k \).

### 2.3 New 4-function local symbol and its first reciprocity law

In this Subsection, we define a new 4-function local symbol on a surface. We also express the new 4-function local symbol as a product of bi-local symbols (Definition 2.11 and Proposition 2.12), which serves as a refinement similar to the refinement of the Parshin symbol in Subsection 2.2. Using the reciprocity laws for bi-local symbols established in Subsection 2.1, we obtain the first type of reciprocity law for the new 4-function local symbol (Theorem 2.14).

**Definition 2.11.** We define the following bi-local symbol, which will lead to the 4-function local symbol on a surface.

\[
PR_{C,P}^{Q} = \left( 2,2 \right)_{[f_1, f_2, f_3, f_4]_{P}}^{(1),Q} \left( 2,2 \right)_{[f_1, f_2, f_3, f_4]_{P}}^{(1),Q}^{-1} \times \left( 2,2 \right)_{[f_2, f_3, f_4]_{P}}^{(1),Q}^{-1} \left( 2,2 \right)_{[f_2, f_1, f_3, f_4]_{P}}^{(1),Q}^{-1}.
\]

Using Theorem 2.3 part (d), we obtain:

**Proposition 2.12.** Explicitly, the bi-local symbol \( PR_{C,P}^{Q} \) is given by

\[
PR_{C,P}^{Q} = (-1)^L \left( \frac{a_2}{f_2} \right)_{a_3 b_4 - b_3 a_4} \left( \frac{a_4}{f_4} \right)_{a_1 b_2 - b_1 a_2} (P) \cdot \left( \frac{a_2}{f_2} \right)_{a_3 b_4 - b_3 a_4} \left( \frac{a_4}{f_4} \right)_{a_1 b_2 - b_1 a_2} (Q),
\]

where

\[
L = (a_1 b_2 - a_2 b_1)(a_3 b_4 - a_4 b_3).
\]

**Definition 2.13.** (4-function local symbol) With the above notation, we define a 4-function local symbol

\[
\{ f_1, f_2, f_3, f_4 \}_{C,P}^{(1)} = (-1)^L \left( \frac{a_2}{f_2} \right)_{a_3 b_4 - b_3 a_4} \left( \frac{a_4}{f_4} \right)_{a_1 b_2 - b_1 a_2} (P).
\]
It is an easy exercise to check that the symbol \( \{f_1, f_2, f_3, f_4\}_{C,P}^{(1)} \) is independent of the choices of local uniformizers. See also the Appendix for \( K \)-theoretical approach for the 4-function local symbol. Note that the relation between the bi-local symbol \( PR_{C,P} \) and the local symbol \( \{f_1, f_2, f_3, f_4\}_{C,P}^{(1)} \) is only a constant factor depending on the base point \( Q \). There is a similar relation between the bi-local symbol \( PR_{C,P} \) and the Parshin symbol \( \{f_1, f_2, f_3\}_{C,P} \).

**Theorem 2.14.** (Reciprocity law for the 4-function local symbol) The following reciprocity law for the 4-function local symbol on a surface holds

\[
\prod_P \{f_1, f_2, f_3, f_4\}_{C,P}^{(1)} = 1,
\]

where the product is taken over points \( P \) on a fixed curve \( C \). Here we assume that the union of the support of the divisors \( \bigcup_{i=1}^{4} \text{div}(f_i) \) in \( X \) have normal crossing and no three components have a common point.

**Proof.** Using Theorem 2.6 part (d), we obtain that the bi-local symbol \( PR_{C,P} \) satisfies a reciprocity law, namely,

\[
\prod_P PR_{C,P} = 1,
\]

where the product is over all points \( P \) in \( C \cap (C_1 \cup \cdots \cup C_n) \). In order to complete the proof of Theorem 2.14, we proceed similarly to the proof of the first Parshin reciprocity law. Namely,

\[
\prod_P g_1(Q)^{a_2(a_3b_4-a_4b_3)} = g_1(Q)^{a_2} \sum_P a_3b_4-a_4b_3 = g(Q)^{b_2} = 1,
\]

where \( g_1 = x^{-a_1} f_1 \) and \( x \) is a rational function representing an uniformizer at the curve \( C \), such that the components of the divisor of \( x \) do not intersect at the points \( P \) or \( Q \). The last equality of (2.3) holds, because

\[
a_3b_4 - a_4b_3 = (2\pi i)^{-2} \left( \log^{1,1}[f_3, f_4]_{C,P}^{(1)} - \log^{1,1}[f_4, f_3]_{C,P}^{(1)} \right) = 0
\]

and

\[
\sum_P (2\pi i)^{-2} \left( \log^{1,1}[f_3, f_4]_{C,P}^{(1)} - \log^{1,1}[f_4, f_3]_{C,P}^{(1)} \right) = 0,
\]

by Theorem 2.3 (a) and Theorem 2.6 (a), respectively.

There is one more interesting relation for the 4-function symbol, whose is a direct consequence of the explicit formula of the symbol.

**Theorem 2.15.** Let

\[
R_{ijkl} = \{f_i, f_j, f_k, f_l\}_{C,P}.
\]

Then \( R_{ijkl} \) has the same symmetry as the symmetry of a Riemann curvature tensor with respect to permutations of the indices, namely

\[
R_{ijkl} = -R_{jikl} = -R_{ijlk} = -R_{klij}.
\]
3 Second type of reciprocity laws

3.1 Bi-local symbols revisited

In this Subsection, we define bi-local symbols, designed for proofs of the second type of reciprocity laws for local symbols. These bi-local symbols also satisfy reciprocity laws. Using them, in the following two sections, we establish the second type of reciprocity laws for the Parshin symbol and for a new 4-function new symbol. By a second type of reciprocity law, we mean that the product of the local symbols is taken over all curves $C$ on the surface $X$, passing through a fixed point $P$.

Let $C_1, \ldots, C_n$ be curves in $X$ intersecting at a point $P$. Assume that $C_1, \ldots, C_n$ are among the divisors of the rational functions $f_1, \ldots, f_4$. Let $\tilde{X}$ be the blow-up of $X$ at the point $P$. Assume that after the blow-up the curves above $C_1, \ldots, C_n$ meet transversally the exceptional curve $E$ and no two of them intersect at a point on the exceptional curve $E$.

Let $D$ be a curve on $\tilde{X}$ such that $D$ intersects $E$ in one point. Setting

$$\tilde{P}_k = E \cap \tilde{C}_k,$$

where $\tilde{C}_k$ is the curve above $C_k$ after the blow-up, and

$$Q = E \cap D,$$

**Definition 3.1.** We define the following bi-local symbols

$$i,j[f_1, \ldots, f_{i+j}]^{(2), D}_{C_k, P} := i,j[f_1, \ldots, f_{i+j}]^{(1), Q}_{E, \tilde{P}_k}.$$

**Theorem 3.2.** The following reciprocity laws for bi-local symbols hold:

(a)

$$\prod_{C_k}^{1,2} [f_1, f_2, f_3]^{(2), D}_{C_k, P} = 1,$$

(b)

$$\prod_{C_k}^{2,1} [f_1, f_2, f_3]^{(2), D}_{C_k, P} = 1,$$

(c)

$$\prod_{C_k}^{2,2} [f_1, f_2, f_3, f_4]^{(2), D}_{C_k, P} = 1,$$

where the product is over the curves $C$, among the divisors of at least one of the rational functions $f_1, \ldots, f_4$, which pass through the point $P$.

The proof is reformulation of Theorem 2.6 where the triple $(C_k, P, D)$ in the above Theorem correspond to the triple $(P, Q, C)$ with $P = C_k \cap E$ and $Q = D \cap E$ in Theorem 2.6 where the curve $C$ in Theorem 2.6 corresponds to the curve $E$. 

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3.2 Parshin symbol and its second reciprocity law.

In this Subsection, we present an alternative refinement of the Parshin symbol in terms of bi-local symbols (Definition 3.3). This implies the second reciprocity law for the Parshin symbol, since each of the bi-local symbols satisfy the second type of reciprocity laws (see Subsection 3.1).

**Definition 3.3.** We define the following bi-local symbol, useful for the proof of the second reciprocity law of the Parshin symbol

\[
P_{\tau_{C,E}}^D = \left(\{1,2 f_1, f_2, f_3\}_{C,E}^{(2)}\right) \left(\{1,2 f_2, f_3, f_1\}_{C,E}^{(2)}\right) \times \left(\{2,1 f_1, f_2, f_3\}_{C,E}^{(2)}\right) \times \left(\{2,1 f_2, f_3, f_1\}_{C,E}^{(2)}\right) \times \left(\{2,1 f_3, f_1, f_2\}_{C,E}^{(2)}\right),
\]

Let \(\tilde{P} = \tilde{C} \cap E\), \(Q = D \cap E\). Then

\[
P_{\tau_{C,E}}^D = P_{\tau_{E,\tilde{P}}}^Q.
\]

Similarly to the proof of Theorem 2.10, we can remove the dependence of the bi-local symbol \(P_{\tau_{E,\tilde{P}}}^Q\) on the base point \(Q\).

**Definition 3.4.** The second Parshin symbol \(\{f_1, f_2, f_3\}_{C,E}^{(2)}\) is the symbol, explicitly given by

\[
\{f_1, f_2, f_3\}_{C,E}^{(2)} = (-1)^K \left(\frac{f_1^{D_1} f_2^{D_2} f_3^{D_3}}{\tilde{P}}\right),
\]

where

\[
D_1 = \begin{vmatrix}
c_2 & c_3 \\
d_2 & d_3
\end{vmatrix}, \quad D_2 = \begin{vmatrix}
c_3 & c_1 \\
d_3 & d_1
\end{vmatrix}, \quad D_3 = \begin{vmatrix}
c_1 & c_2 \\
d_1 & d_2
\end{vmatrix}
\]

and

\[
K = c_1 c_2 d_3 + c_2 c_3 d_1 + c_3 c_1 d_2 + d_1 d_2 c_3 + d_2 d_3 c_1 + d_3 d_1 c_2,
\]

with \(c_k = \text{ord}_E(f_k)\) and \(d_i = \text{ord}_{\tilde{P}}((y^{-c_k} f_k)|_E)\). Here \(y\) is a rational function representing an uniformizer at \(E\) such that the components of the divisor of \(y\) do not intersect at the point \(\tilde{P}\).

**Proposition 3.5.** The second Parshin symbol is equal to the inverse of the Parshin symbol. More precisely,

\[
\{f_1, f_2, f_3\}_{C,E}^{(2)} = (\{f_1, f_2, f_3\}_{C,P})^{-1}
\]

Let

\[
a_i = \text{ord}_C(f_i)
\]

and

\[
b_i = \text{ord}_P((x^{-a_i} f_i)|_C),
\]

where \(x\) is a rational function representing a uniformizer at \(C\), whose support does not contain other components passing through the point \(P\).

**Lemma 3.6.** With the above notation, the following holds

\[
\text{ord}_E(f_i) = c_i = a_i + b_i.
\]
Proof. We still assume that after the blow-up the union of the support of the rational functions $f_1, f_2, f_3$ have normal crossings and no three curves intersect at a point. Before the blow-up, let $C_1, \ldots, C_n$ be all the components of the union of the support of the three rational functions that meet at the point $P$. And let $E$ be the exceptional curve above the point $P$. Then for $C = C_1$, we have

$$b_i = \sum_{j=2}^{n} \text{ord}_{C_j}(f_i)$$

and

$$\text{ord}_E(f_i) = \sum_{j=1}^{n} \text{ord}_{C_j}(f_i).$$

That proves the Lemma. $\square$

Proof. (of Proposition 3.5) Consider the pairs $(C, P)$ on the surface $X$ and $(E, \tilde{C})$ on on the blow-up $\tilde{X}$. Then by the above Lemma, we have

$$[c_i \ d_i] = \left[ \begin{array}{c} \text{ord}_E(f_i) \\ \text{ord}_{\tilde{C}}(f_i) \end{array} \right] = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \cdot \left[ \begin{array}{c} a_i \\ b_i \end{array} \right]$$

(3.1)

The Parshin symbol is invariant under change of variables given by $\left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$. Also the Parshin symbol is send to its reciprocal when we change the variables by a matrix $\left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$. That proves the Proposition. $\square$

**Theorem 3.7.** (Second reciprocity law for the Parshin symbol) We have

$$\prod_C \{f_1, f_2, f_3\}_{C,P} = 1,$$

where the product is over the curves $C$ from the support of the divisors of the rational functions $\bigcup_{i=1}^{3} |\text{div}(f_i)|$, which pass through the point $P$. (For all other choices of curves $C$, the Parshin symbol will be equal to 1.) Here we assume that the union of the support of the divisors $\bigcup_{i=1}^{3} |\text{div}(f_i)|$ in $\tilde{X}$ have normal crossings and no two components have a common point with the exceptional curve $E$ in $\tilde{X}$ above the point $P$. We denote by $\tilde{X}$ the blow-up of $X$ at the point $P$.

Proof. We can use Proposition 3.5 and the first reciprocity law for the Parshin symbol given in Theorem 2.10 Then Theorem 3.7 follows. $\square$

### 3.3 The second 4-function local symbol and its second reciprocity law

In this Subsection, We define a second type of 4-function local symbol (Definition 3.10), which satisfies the second type reciprocity laws. By a second reciprocity law, we mean that the product of the local symbols is taken over all curves $C$ on the surface $X$, which pass through a fixed point $P$. The 4-function local symbol has a refinement (see Definition 3.8) which provides a proof of the second reciprocity law (Theorem 3.11).
Definition 3.8. We define a bi-local symbol, useful for the second reciprocity law for a new 4-function local symbol. Let
\[ PR_{C,P}^D = \left[ f_1, f_2, f_3, f_4 \right]_{C,E}^{(2),D} \left[ f_1, f_2, f_3, f_4 \right]_{C,E}^{(2),D} \]
\[ \times \left( 2, 2 \right) \left[ f_2, f_1, f_3, f_4 \right]_{C,E}^{(2),D} \left( 2, 2 \right) \left[ f_2, f_1, f_3, f_4 \right]_{C,E}^{(2),D} \].

Let
\[ L = (c_1d_2 - c_2d_1)(c_3d_4 - c_4d_3), \]
where
\[ c_i = \text{ord}_E(f_i), \]
\[ d_i = \text{ord}_P((x-a_i) f_i|_E), \]
for a rational function \( x \), representing a uniformizer at \( E \), whose support does not contain other components passing through the point \( P = E \cap \bar{C} \).

Lemma 3.9.
\[ PR_{C,E}^D = (-1)^L \left( \frac{c_2}{f_2} \frac{c_3}{f_3} \frac{c_4}{f_4} \right) \left( \frac{c_2}{f_2} \frac{c_3}{f_3} \frac{c_4}{f_4} \right) \left( \frac{c_2}{f_2} \frac{c_3}{f_3} \frac{c_4}{f_4} \right). \]

where \( Q = D \cap E \).

It follows directly from Equation [2.1] and Lemma [3.6]

Definition 3.10. The second 4-function local symbol has the following explicit representation:
\[ \{ f_1, f_2, f_3, f_4 \}_{C,P}^{(2)} = (-1)^L \left( \frac{f_1^{a_1+b_1}}{f_2^{a_1+b_1}} \frac{f_3^{a_2+b_2}}{f_3^{a_2+b_2}} \frac{f_4^{a_3+b_3}}{f_4^{a_3+b_3}} \right) \left( \frac{f_1^{a_1+b_1}}{f_2^{a_1+b_1}} \frac{f_3^{a_2+b_2}}{f_3^{a_2+b_2}} \frac{f_4^{a_3+b_3}}{f_4^{a_3+b_3}} \right). \]

Theorem 3.11. (Reciprocity law for the second 4-function local symbol) We have the following reciprocity law
\[ \prod_C \{ f_1, f_2, f_3, f_4 \}_{C,P}^{(2)} = 1, \]
where the product is over the curves \( C \) from the support of the divisors of the rational functions \( \bigcup_{i=1}^4 |\text{div}(f_i)| \), which pass through the point \( P \). Here we assume that the union of the support of the divisors \( \bigcup_{i=1}^4 |\text{div}(f_i)| \) in \( \bar{X} \) have normal crossings and no two components have a common point with the exceptional curve \( E \) in \( \bar{X} \) above the point \( P \). We denote by \( \bar{X} \) the blow-up of \( X \) at the point \( P \).

Proof. Using Theorem [3.2] we obtain a reciprocity law for the bi-local symbol \( PR_{C,E}^{(2),D} \). Multiplying each symbol by the same constant, depending only on \( Q \), we can remove the dependence on \( Q \). Explicitly, the separation between the dependence on \( D \) and the second 4 function local symbol are given in Lemma [3.9] Then we can use Lemma [3.6] in order to express the coefficients \( c_i \) and \( d_i \) in terms of \( a_i \) and \( b_i \), which implies the reciprocity law stated in the Theorem [3.11].
4 An alternative proof of the reciprocity laws for the 4-function local symbols

In this Section, we give alternative proofs of the two reciprocity laws of the 4-function local symbol, based in Milnor $K$-theory. We will use the $K$-theoretic interpretation of the 4-function local symbol, presented in the Appendix by M. Kerr.

**Definition 4.1.** The quotient of the $K$-theoretic symbol $K[f_1, f_2, f_3, f_4]_{C,P}^{(1)}$, from the Appendix, and the 4-function local symbol $\{f_1, f_2, f_3, f_4\}_{C,P}^{(1)}$ from Definition 2.13 is given by

$$(f_1, f_2, f_3, f_4)_{C,P}^{(1)} = (-1)^{a_1a_2a_3b_4+a_2a_3a_4b_1+a_3a_4a_1b_2+a_4a_1a_2b_3}.$$

Here $a_k = \text{ord}_C(f_k)$ and $b_k = \text{ord}_P((x^{-a_k} f_k)|_C)$, where $x$ is a rational function representing an uniformizer at $C$ such that $P$ is not an intersection point of the irreducible components of the support of the divisor $(x)$.

For each point $P$ on a fixed curve $C$ the values $a_k$ remain the same. Therefore, we have the following interpretation in terms of integrals. Let

$$\omega_k = (-a_k \frac{dx}{x} + \frac{df_k}{f_k})|_C$$

be a differential form on the curve $C$. Then

$$b_k(P) = \frac{1}{2\pi i} \text{Res}_P(\omega_k)$$

**Proposition 4.2.** We can express the sign $(f_1, f_2, f_3, f_4)_{C,P}^{(1)}$ in terms of residues

$$(f_1, f_2, f_3, f_4)_{C,P}^{(1)} = \exp\left(\frac{1}{2} (a_1a_2a_3\text{Res}_P(\omega_4) + a_2a_3a_4\text{Res}_P(\omega_1) + a_3a_4a_1\text{Res}_P(\omega_2) + a_4a_1a_2\text{Res}_P(\omega_3))\right)$$

**Theorem 4.3.** The sign $(f_1, f_2, f_3, f_4)_{C,P}^{(1)}$ is also a symbol, satisfying the following reciprocity law:

$$\prod_P (f_1, f_2, f_3, f_4)_{C,P}^{(1)} = 1,$$

where the product is over all points $P$ of the curve $C$.

**Proof.** It follows from the fact that the sum of the residues on a curve is equal to zero and from the previous Proposition. \qed

**Theorem 4.4.** The $K$-theoretic symbol satisfies the following reciprocity law

$$\prod_C K[f_1, f_2, f_3, f_4]_{C,P}^{(1)} = 1.$$

where the product is over all points $P$ of the curve $C$.

The proof follows directly from the $K$-theoretic definition given in the Appendix.
Proof. (an alternative proof of Theorem 2.14) Using the reciprocity law for the \( K \)-theoretic symbol \( K[f_1, f_2, f_3, f_4]^{(2)}_{C, P} \) such as in the Appendix and the above Theorem, we obtain another proof of the reciprocity law for the 4-function local symbol.

Now, we proceed toward an alternative proof of the second type of reciprocity laws for the new 4-function local symbol.

Let \( E \) be the exceptional curve for the blowup of \( X \) at the point \( P \). Let \( \tilde{C} \) be the irreducible component sitting above the curve \( C \) in the blow-up. We define \( \tilde{P} = \tilde{C} \cap E \).

A direct observation leads to

\[
\{f_1, f_2, f_3, f_4\}^{(2)}_{C, P} = \left( \{f_1, f_2, f_3, f_4\}^{(1)}_{E, \tilde{P}} \right)^{-1}
\]

for the 4-function local symbols. Similarly we define

\[
(f_1, f_2, f_3, f_4)^{(2)}_{C, P} = \left( (f_1, f_2, f_3, f_4)^{(1)}_{E, \tilde{P}} \right)^{-1}
\] (4.1)

for the sign and

\[
K[f_1, f_2, f_3, f_4]^{(2)}_{C, P} = \left( K[f_1, f_2, f_3, f_4]^{(1)}_{E, \tilde{P}} \right)^{-1}
\] (4.2)

for the \( K \)-theoretic symbol.

Theorem 4.5. For the sign and the \( K \) theoertic symbol we have a second type of reciprocity laws.

\[
\prod_C (f_1, f_2, f_3, f_4)^{(2)}_{C, P} = 1
\]

and

\[
\prod_C K[f_1, f_2, f_3, f_4]^{(2)}_{C, P} = 1,
\]

where the product is taken over all curves \( C \), passing through the point \( P \). Here we assume that the union of the support of the divisors \( \bigcup_{i=1}^{4} \text{div}(f_i) \) in \( \tilde{X} \) have normal crossings and no two components have a common point with the exceptional curve \( E \) in \( \tilde{X} \) above the point \( P \). We denote by \( \tilde{X} \) the blow-up of \( X \) at the point \( P \).

Proof. For the \( K \)-theoretic symbol we have

\[
\prod_C K[f_1, f_2, f_3, f_4]^{(2)}_{C, P} = \left( \prod_{\tilde{P}} K[f_1, f_2, f_3, f_4]^{(1)}_{E, \tilde{P}} \right)^{-1} = 1.
\]

The first equality follows from the definition of \( K[f_1, f_2, f_3, f_4]^{(2)}_{C, P} \) and the second equality from Theorem 4.4.

Proof. (an alternative proof of Theorem 3.11) We have the following equalities

\[
\prod_C \{f_1, f_2, f_3, f_4\}^{(2)}_{C, P} = \prod_C K[f_1, f_2, f_3, f_4]^{(2)}_{C, P} \prod_C K[f_1, f_2, f_3, f_4]^{(2)}_{C, P} = 1.
\]

The first equality follows from Definition 4.1 and Equations 4.1 and 4.2. The second equality follows from Theorem 4.5.
Appendix A. By Matt Kerr

There is a well-known K-theoretic approach to the Parshin symbol, which we shall recall below. The purpose of this appendix is to provide (up to sign) a K-theoretic interpretation for the 4-function symbol.

To begin, note that if only two components \( C \) and \( C' \) of \( \bigcup_i |(f_i)| \) meet at a point \( P \in X \), then the Parshin symbol has the local symmetry property

\[
\{f_1, f_2, f_3\}_{C,P} = (\{f_1, f_2, f_3\}_{C',P})^{-1}
\]
as does the (second) 4-function symbol. This is just a special case of the second reciprocity law.

Now it is well known that the Parshin symbol may be computed by the composition

\[
\begin{aligned}
P_{C,P} : K_3^M(\mathbb{C}(X)) &\xrightarrow{Tame_C} K_2(\mathbb{C}(C)) \\
&\xrightarrow{Tame_P} \mathbb{C}^x.
\end{aligned}
\]

Since \( P_{C,P} \) is invariant under blow-up and satisfies the local symmetry property, this reduces checking the two reciprocity laws to Weil reciprocity.

One is tempted to believe that the 4-function symbol \( \{f_1, f_2, f_3, f_4\}_{C,P}^{(1)} \) (Definition 2.13) can be identified with the image of \( \{f_1, f_2\} \otimes \{f_3, f_4\} \) under the composition

\[
\begin{aligned}
Q_{C,P} : K_2(\mathbb{C}(X)) \otimes^2 &\xrightarrow{Tame_C^2} (\mathbb{C}(C))^x \otimes^2 \\
&\xrightarrow{Tame_P} \mathbb{C}^x
\end{aligned}
\]

and to argue in the same manner. Indeed, the first reciprocity law for \( Q_{C,P} \) again follows from Weil reciprocity, and it is also invariant under blow-up.

However, a short computation shows that \( Q_{C,P} \) and Definition 2.13 differ by the factor

\[
(-1)^{a_1b_2b_3a_4+b_1a_2b_4+b_1a_2a_3b_4+b_1b_2a_3a_4}.
\]

Indeed, we have

\[
\begin{aligned}
K[f_1, f_2, f_3, f_4]_{C,P} := \\
Tame_P\{Tame_C\{f_1, f_2\}, Tame_C\{f_3, f_4\}\} = \\
Tame_P\{Tame_C\{x^{a_1}y^{b_1}g_1, x^{a_2}y^{b_2}g_2\}, Tame_C\{x^{a_3}y^{b_3}g_3, x^{a_4}y^{b_4}g_4\}\} = \\
Tame_P\left\{(-1)^{a_1a_2}x^{a_1b_2-a_2b_1}g_1^{(a_2/g_2)^{a_1}}, (-1)^{a_3a_4}x^{a_3b_4-a_4b_3}g_3^{(a_4/g_4)^{a_3}}\right\} = \\
(-1)^{(a_1b_2-a_2b_1)(a_3b_4-a_4b_3)} \left\{(-1)^{a_1a_2}\left(\frac{g_1}{g_2}\right)^{a_1}ight\}^{a_3b_4-a_4b_3} = \\
(-1)^{(a_3a_4)} \left\{(-1)^{a_3a_4}\left(\frac{g_3}{g_4}\right)^{a_3}ight\}^{a_1b_2-a_2b_1} = \\
(-1)^{b_1a_2a_3a_4+b_1b_2a_3a_4+a_1a_2b_3a_4+a_1a_2a_3b_4}\{f_1, f_2, f_3, f_4\}_{C,P}^{(1)}.
\end{aligned}
\]

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