Combinatorial Structure of Finite Dimensional Representations of Yangians: the Simply-Laced Case

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Abstract

We compute the decomposition of representations of Yangians into \( g \)-modules for simply-laced \( g \). The decomposition has an interesting combinatorial tree structure. Results depend on a conjecture of Kirillov and Reshetikhin.

0 Introduction

Let \( g \) be a complex semisimple Lie algebra of rank \( r \), and \( Y(g) \) its Yangian (\cite{4}), a Hopf algebra which contains the universal enveloping algebra \( U(g) \) of \( g \) as a Hopf subalgebra. Write \( \alpha_1, \ldots, \alpha_r \) for the fundamental roots and \( \omega_1, \ldots, \omega_r \) for the fundamental weights of \( g \). As defined in \cite{4}, denote by \( W_m(\ell) \) a particular irreducible \( Y(g) \) module all of whose \( g \)-weights \( \lambda \) satisfy \( \lambda \preceq m\omega_\ell \), where \( \alpha \preceq \beta \) means \( \beta - \alpha \) is a nonnegative integer linear combination of the roots \( \{\alpha_i\} \).

Specifically, \( W_m(\ell) \) decomposes into \( g \)-modules as

\[
W_m(\ell)|_g \simeq \bigoplus_{\lambda \preceq m\omega_\ell} V_\lambda^{\otimes n_\lambda}
\]

where \( V_\lambda \) is the irreducible \( g \)-module with highest weight \( \lambda \) and it occurs \( n_\lambda \) times in \( W_m(\ell) \). In particular, \( n_{m\omega_i} = 1 \).

There is a formula for the multiplicities \( n_\lambda \) in \( \text{(1)} \) based on the conjecture that every finite-dimensional representation of \( Y(g) \) can be obtained from one specific representation by means of the “reproduction scheme,” defined in \cite{7}. If this conjecture holds, then write \( \lambda = m\omega_\ell - \sum n_i\alpha_i \), and it is proved in \cite{7} that

\[
n_\lambda = Z(\ell, m|n_1, \ldots, n_r) = \sum_{\text{partitions}} \prod_{n \geq 1} \prod_{k=1}^r \left( P_n^{(k)}(\nu) + \nu_n^{(k)} \right)
\]

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The sum is taken over all ways of choosing partitions $\nu^{(1)}, \ldots, \nu^{(r)}$ such that $\nu^{(i)}$ is a partition of $n_i$ which has $\nu^{(i)}_n$ parts of size $n$ (so $n = \sum_{n \geq 1} n \nu^{(i)}_n$). The function $P$ is defined by

$$
P^{(k)}_n(\nu) = \min(n, m) \delta_{k, \ell} - 2 \sum_{h \geq 1} \min(n, h) \nu^{(k)}_h + \sum_{j \neq k, h \geq 1} \min(-c_{k,j} n, -c_{j,k} h) \nu^{(j)}_h
$$

where $C = (c_{i,j})$ is the Cartan matrix of $\mathfrak{g}$. We define $\binom{a}{b}$ to be 0 whenever $a < b$; since the values of $P$ can be negative, many of the binomial coefficients in (2) can be zero.

Yangians are closely related to quantum affine universal enveloping algebras $U_q(\hat{\mathfrak{g}})$ when $q$ is not a root of unity, and $U_q(\mathfrak{g})$ is a Hopf subalgebra of $U_q(\hat{\mathfrak{g}})$ in much the same way that $\mathfrak{g}$ is a Hopf subalgebra of $Y(\mathfrak{g})$. View $V_\lambda$ as a highest weight module over $U_q(\mathfrak{g})$; then an affinization of $V_\lambda$ is defined in [3] to be a $U_q(\hat{\mathfrak{g}})$-module all of whose weights as a $U_q(\mathfrak{g})$-module satisfy $\mu \preceq \lambda$ and $\lambda$ appears with multiplicity 1. In the case that $\lambda = m\omega_\ell$, $V_{m\omega_\ell}$ has a unique minimal affinization (3) with respect to a partial ordering defined in [2], and it is believed (3) that the decomposition of this minimal affinization into $U_q(\mathfrak{g})$-modules is the same as the decomposition for the Yangian module $W_m(\ell)$.

In Section 1, we view the values of $P^{(k)}_n(\nu)$ as the coordinates of certain strings of weights of $\mathfrak{g}$ which lie inside the Weyl chamber. This interpretation allows us to compute the values of $n_\lambda$ much more efficiently. Furthermore, the “initial substring” relation on the labelling by strings of weights imposes the structure of a rooted tree on the set of $\mathfrak{g}$-modules which make up $W_m(\ell)$, rooted at $V_{m\omega_\ell}$ and with the children of any $V_\lambda$ having highest weights $\mu = \lambda - \delta$ with $\delta$ in the positive root lattice.

In Section 2, we use this added structure to study the asymptotics of the dimension of $W_m(\ell)$ as $m$ gets large, based on the fact that the tree structure of $W_m(\ell)$ lifts to $W_{m+1}(\ell)$. We show that the conjecture implies that the dimension grows asymptotically to a polynomial in $m$, and compute the degree of this polynomial for every simply-laced $\mathfrak{g}$ and choice of $\omega_\ell$.

In Section 3 we give a list of the decompositions of $W_m(\ell)$ for all simply-laced $\mathfrak{g}$ and small values of $m$ as derived numerically from the conjecture, using the results of Section 1. For any choice of $\mathfrak{g}$, representations $W_1(\ell)$ are called fundamental representations, since every finite-dimensional representation of $Y(\mathfrak{g})$ appears as a quotient of a submodule of a tensor product of fundamental representations. In the context of $U_q(\hat{\mathfrak{g}})$-modules, the decompositions of most of the fundamental representations were calculated in [3] using completely different techniques, and those calculations agree with ours.

A similar idea can be used to give a combinatorial interpretation to the values in equations (2) and (3) when $\mathfrak{g}$ is not simply-laced. The resulting structure is not as regular as in the simply-laced case, but should yield similar results.
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1 Structure in the simply-laced case

Assume that our Lie algebra $g$ of rank $r$ is simply-laced. Then equation (3) becomes

$$P_n^{(k)}(\nu) = \min(n, m)\delta_{k,\ell} - \sum_{j=1}^{r} c_{j,k} \left( \sum_{h \geq 1} \min(n, h)\nu_h^{(j)} \right)$$

(4)

Fix a highest weight $m\omega_\ell$, and pick an arbitrary $\nu = (\nu^{(1)}, \ldots, \nu^{(r)})$, where each $\nu^{(i)}$ is a partition of some integer $n_i$. Then for any nonnegative integer $n$, the values $(P_n^{(1)}, \ldots, P_n^{(r)})$ can be thought of as the $\omega$-coordinates of some weight; define

$$\mu_n = \sum_{k=1}^{r} P_n^{(k)} \omega_k$$

A given $\nu$ contributes a nonzero term to the sum in (3) if and only if the corresponding weights $\mu_0 = 0, \mu_1, \mu_2, \ldots$ all lie in the dominant Weyl chamber. The motivation for seeing these as weights is that the sum in (4) can be naturally realized as subtracting some linear combination of roots; if we let

$$d_n = \sum_{k=1}^{r} \left( \sum_{h \geq 1} \min(n, h)\nu_h^{(k)} \right) \alpha_k$$

(5)

then $\mu_n = \min(n, m)\omega_\ell - d_n$.

Think of $\nu^{(1)}, \ldots, \nu^{(r)}$ as Young diagrams with $\nu^{(k)}$ having $\nu_h^{(k)}$ rows of length $h$. Then we can tell whether a sequence of vectors $d_0, d_1, d_2, \ldots$ can arise from $\nu^{(1)}, \ldots, \nu^{(r)}$ by looking at their successive differences $\delta_i = d_i - d_{i-1}$. If we write $\delta_n$ out as a linear combination of the roots $\{\alpha_i\}$, then the $\alpha_k$-coordinate is the number of boxes in the $n$th column of the Young diagram of $\nu^{(k)}$, since the sum $\sum_h \min(n, h)\nu_h^{(k)}$ in (5) is the number of boxes in the first $n$ columns. Thus a sequence arises from partitions if and only if the $\delta_i$ are nonincreasing; that is, $\forall i \geq 1 : \delta_i \geq \delta_{i+1}$.

If we let $s$ be the size of the largest part in any of the partitions in $\nu$, then $d_s = d_i$ for all $t > s$ (and $s$ is the smallest index for which this is true), and all the information we need to identify a particular summand of $W_m(\ell)$ is the (strictly increasing) chain of weights $d_0 = 0 < d_1 < \cdots < d_s$, which we define to have length $s$. Note that the chain of length 0 consisting of only $d_0 = 0$ is permissible, arises from empty partitions, and corresponds to the $V_{m\omega_\ell}$ component of $W_m(\ell)$.

In summary, we have proven the following:
Theorem 1 Let \( g \) be a simply-laced complex semisimple Lie algebra of rank \( r \) with fundamental roots \( \alpha_1, \ldots, \alpha_r \) and fundamental weights \( \omega_1, \ldots, \omega_r \), and assume the decomposition of \( W_m(\ell) \) into \( g \)-modules in (1) is given by the conjecture in (2) and (3). Then that decomposition can be refined into a direct sum of parts indexed by chains of weights \( d = d_0, \ldots, d_s \) with successive differences \( \delta_i = d_i - d_{i-1} \) (and \( \delta_{s+1} = 0 \)) such that

(i) \( d_0 = 0 \) and \( d_0 < d_1 < \cdots < d_s \),

(ii) \( \min(n, m)\omega_\ell - d_n \) lies in the positive Weyl chamber for \( 0 \leq n \leq s \), and

(iii) \( \delta_i \preceq \delta_{i+1} \) for all \( 1 \leq i \leq s \).

The summand with label \( d = d_0, \ldots, d_s \) consists of the \( g \)-module of highest weight \( m\omega_\ell - d_s \) with multiplicity

\[
\prod_{n \geq 1} \prod_{k=1}^r \left( P_n^{(k)}(d) + d_n^{(k)} \right) \frac{d_n^{(k)}}{d_n^{(k)}}
\]

where the values of \( P_n^{(k)}(d) \) and \( d_n^{(k)} \) are defined by the relations

\[
\min(m, n)\omega_\ell - d_n = \sum_{k=1}^r P_n^{(k)}(d)\omega_k
\]

\[
\delta_n - \delta_{n+1} = \sum_{k=1}^r d_n^{(k)}\alpha_k
\]

and all of the multiplicities are nonzero.

This decomposition is a refinement of the one in (1) since it is possible to find two different chains \( d_0, \ldots, d_s \) and \( d_0', \ldots, d_t' \) with \( d_s = d_t' \). This happens any time the sum in (4) has more than one nonzero term. One example of this occurs in \( W_2(4) \) for \( E_6 \); see Figure 1.

Corollary 2 If \( d_0, \ldots, d_s \) is a valid label then any initial segment \( d_0, \ldots, d_{s'} \) (for \( 0 \leq s' < s \)) is a valid label also. Conversely, given any label \( d_0, \ldots, d_s \), we can extend it to another valid label by appending any weight \( d_{s+1} \) which satisfies the conditions that \( \min(s+1, m)\omega_\ell - d_{s+1} \) is in the positive Weyl chamber and, if \( s > 0 \), that \( d_s < d_{s+1} \leq d_s + \delta_s \).

This follows immediately from conditions (i)--(iii). Since \( d_0 \) must be 0, this completely describes an effective algorithm for computing the conjectured decomposition of a given \( W_m(\ell) \). The computations in Section 3 were computed using this algorithm. The fact that an initial segment of a valid label is still a valid label is the key result which fails to hold true when \( g \) is not simply-laced.

Since truncating any label gives you another label, we can impose a tree structure on the parts of this decomposition, with a node of the tree corresponding to a summand in the decomposition from Theorem 1. The “children”
Figure 1: Tree-structure[1] of the decomposition of $W_2(4)$ for $E_6$.

of the node with label $d_0, \ldots, d_s$ are all the nodes indicated by Corollary 2; we can label the edges joining them to their parent with the various choices for the increment $\delta_{s+1}$. For each $n \geq 0$, the $n$th row of the tree consists of all the nodes with labels of length $n$.

As an example of this structure, the tree for the decomposition of $W_2(4)$ for $g = E_6$ is given in Figure 1. Scalars in front of modules, as in $2V_\omega + \omega$, indicate multiplicity. The label $(a_1, \ldots, a_6)$ corresponds to an increment $\delta = \sum a_i \alpha_i$, so condition (iii) says that the labels along any path down from $V_\omega$ will be nonincreasing in each coordinate. The labels on the edges are technically unnecessary, since they can be obtained by subtracting the highest weight of the child from the highest weight of the parent. However, as the next corollary shows, they do record useful information that is not apparent by looking directly at the highest weights.

**Corollary 3** If $d_0, \ldots, d_s$ is a valid label for $W_m(\ell)$, then it is also a valid label for $W_{m'}(\ell)$ for any $m' > m$, and for any $m' \geq s$.

Both parts are based on the fact that condition (ii) is the only one that depends on $m$. For $m' > m$, if $\min(n, m) \omega_\ell - d_n$ is a nonnegative linear combination of the $\{\omega_\ell\}$ then adding some nonnegative multiple of $\omega_\ell$ will not change that fact. And if $m' \geq s$ then the value of $m'$ is irrelevant; the weights we look at are just $n \omega_\ell - d_n$ for $0 \leq n \leq s$.

If we can lift labels from $W_m(\ell)$ to $W_{m+1}(\ell)$, we can also lift the entire tree structure. Specifically, the lifting of labels extends to a map from the tree of

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1Thanks to Paul Taylor’s diagrams package.
$W_m(\ell)$ to the tree of $W_{m+1}(\ell)$ which preserves the increment $\delta$ of each edge and lifts each $V_\lambda$ to $V_{\lambda+\omega_t}$. The $m' \geq s$ part of Corollary \[ \text{Corollary}\] tells us that this map is a bijection on rows $0, 1, \ldots, m$ of the trees, where the labels have length $s \leq m$. On this part of the tree, multiplicities are also preserved. This follows from the formula for multiplicities in Theorem \[ \text{Theorem}\]: the only values of $P_n^{(k)}(d)$ that change are for $n = m + 1$, but $d_n^{(k)} = 0$ when $n$ is greater than the length of the label, so the product of binomial coefficients is unchanged.

### 2 The Growth of Trees

In this section, we will prove that as $m$ gets large, the dimension of the representation $W_m(\ell)$ grows like a polynomial in $m$, and will give a method to compute the degree of the polynomial growth. All statements assume the conjectural formulas for multiplicities of $g$-modules. Roots and weights are numbered as in \[ \text{Theorem}\].

Since the tree decompositions for $W_m(\ell)$ for $m = 1, 2, 3, \ldots$ stabilize, we can define $T(\ell)$ to be the tree whose top $n$ rows coincide with those of $W_m(\ell)$ for all $m \geq n$. The highest weight associated with an individual node appearing in $T(\ell)$ is only well-defined up to addition of any multiple of $\omega_t$, but the difference $\delta$ between any node and its parent is well-defined. (These differences are the labels on the edges of the tree in Figure \[ \text{Figure}\].) We can characterize each node by the string of successive differences $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_s$ which label the $s$ edges in the path from the root of the tree to that node. The multiplicity of a node of $T(\ell)$ is well-defined, as already noted.

The tree of $W_m(\ell)$ matches $T(\ell)$ exactly in the top $m$ rows. The number of rows in the tree of $W_m(\ell)$ is bounded by the largest $\alpha$-coordinate of $m\omega_t$, since if $\delta_1, \ldots, \delta_s$ is a label of $W_m(\ell)$ then $m\omega_t - \sum_{i=1}^s \delta_i$ must be in the positive Weyl chamber, and whatever $\alpha$-coordinate is nonzero in $\delta_s$ must be nonzero in all of the $\delta_i$. Therefore to prove that the dimension of $W_m(\ell)$ grows as a polynomial in $m$, it suffices to prove that the dimension of the part of $W_m(\ell)$ which corresponds to the top $m$ rows of $T(\ell)$ does so.

Now we need to examine the structure of the tree $T(\ell)$. The path $\delta_1, \ldots, \delta_s$ to reach a vertex is a sequence of weights whose $\alpha$-coordinates are nonincreasing. Write this instead as $\Delta^{m_1} \ldots \Delta^{m_t}$ where the $\Delta_i$ are strictly decreasing and $m_i$ is the number of times $\Delta_i$ occurs among $\delta_1, \ldots, \delta_s$; we will say this path has path-type $\Delta_1 \ldots \Delta_t$. The number of path-types that can possibly appear in the tree $T(\ell)$ is finite, since each $\Delta_i$ is between $\omega_t$ and $0$ and has integer $\alpha$-coordinates.

We need to understand which path-types $\Delta_1 \ldots \Delta_t$ and which choices of exponents $m_i$ correspond to paths which actually appear in $T(\ell)$. Given a path $\delta_1, \ldots, \delta_s$, assume that $m > s$ and recall $\mu_n = m\omega_t - \delta_n = n\omega_t - \sum_{i=1}^n \delta_i$. Condition (ii) from Theorem \[ \text{Theorem}\] requires that $\mu_n$ is in the positive Weyl chamber for $1 \leq n \leq s$; that is, the $\omega$-coordinates of $\mu_n$ must always be nonnegative. (These coordinates are just the values of $P_n^{(k)}$ from Theorem \[ \text{Theorem}\].) Since $\mu_n = \mu_{n-1} + \omega_t - \delta_n$, we need to keep track of which $\omega$-coordinates of $\omega_t - \delta_n$ are positive and which are negative.
For a path-type $\Delta_1 \ldots \Delta_t$, we say that $\Delta_i$ provides $\omega_k$ if the $\omega_k$-coordinate of $\omega_k - \Delta_i$ is positive, and that it requires $\omega_k$ if the coordinate is negative. Geometrically, $\Delta_i$ providing $\omega_k$ means that each $\Delta_i$ in the path moves the sequence of $\mu$s away from the $\omega_k$-wall of the Weyl chamber, while requiring $\omega_k$ moves towards that wall. The terminology is justified by restating what condition (ii) implies about path-types in these terms:

**Lemma 4** The tree $T(t)$ contains paths of type $\Delta_1 \ldots \Delta_t$ if and only if, for every $\Delta_n$, $1 \leq n \leq t$, every $\omega_i$ required by $\Delta_n$ is provided by some $\Delta_k$ with $k < n$.

The “only if” part of the equivalence is immediate from the preceding discussion: the sequence $\mu_0, \mu_1, \ldots$ starts at $\mu_0 = 0$, and if it moves towards any wall of the Weyl chamber before first moving away from it, it will pass through the wall and some $\mu_i$ will be outside the chamber. Conversely, if $\Delta_1 \ldots \Delta_t$ is any path-type which satisfies the condition of the lemma, then $\Delta_1^{m_1} \ldots \Delta_t^{m_t}$ will definitely appear in the tree when $m_1 \gg m_2 \gg \cdots \gg m_t$. This ensures that the coordinates of the $\mu_i$ are always nonnegative, since the sequence of $\mu$s moves sufficiently far away from any wall of the Weyl chamber before the first time it moves back towards it. We could compute the exact conditions on the $m_i$ for a specific path; in general, they all require that $m_n$ be bounded by some linear combination of $m_1, \ldots, m_{n-1}$, and the first $m_i$ appearing with nonzero coefficient in that linear combination has positive coefficient.

Now we can show that the number of nodes of path-type $\Delta_1 \ldots \Delta_t$ appearing on the $m$th level of the tree grows as $m^{t-1}$. Consider the path $\Delta_1^{m_1} \ldots \Delta_t^{m_t}$ as a point $(m_1, \ldots, m_t)$ in $\mathbb{R}^t$. The path ends on row $m$ if $m = m_1 + \cdots + m_t$, so solutions lie on a plane of dimension $t - 1$; the number of solutions to that equality in nonnegative integers is ${m + t - 1 \choose t - 1}$, which certainly grows as $m^{t-1}$, as expected. The further linear inequalities on the $m_i$ which ensure that $\mu_1, \ldots, \mu_m$ remain in the Weyl chamber correspond to hyperplanes through the origin which our solutions must lie on one side of, but the resulting region still has full dimension $t - 1$ since the generic point with $m_1 \gg m_2 \gg \cdots \gg m_t$ satisfies all of the inequalities, as shown above.

The highest weight of the $\mathfrak{g}$-module at the node associated with the generic solution of the form $m_1 \gg m_2 \gg \cdots \gg m_t$ grows linearly in $m$. Its dimension, therefore, grows as a polynomial in $m$, and the degree of the polynomial is just the number of positive roots of the Lie algebra which are not orthogonal to the highest weight. The only positive roots perpendicular to this generic highest weight are those perpendicular to every highest weight which comes from a path of type $\Delta_1 \ldots \Delta_t$, and the number of such roots is the degree of polynomial growth of the dimensions of the representations of the $\mathfrak{g}$-module. This can also be expressed as $\frac{1}{2}\dim(O_\lambda)$, where $\lambda$ is a weight orthogonal to any given positive root if and only if all highest weights of type $\Delta_1 \ldots \Delta_t$ are.

We can figure out how the multiplicities of nodes with a specific path-type grow as well. Theorem 1 gives a formula for multiplicities as a product of binomial coefficients over $1 \leq k \leq r$ and $n \geq 1$. The only terms in the product which are not 1 correspond to nonzero values of $\delta_n - \delta_{n+1}$. In the path
\( \Delta_1^{m_1} \ldots \Delta_t^{m_t} \), these occur only when \( n = m_1 + \cdots + m_t \) for some \( 1 \leq i \leq t \), so that \( \delta_n - \delta_{n+1} = \Delta_i - \Delta_{i+1} \) (where \( \Delta_{i+1} \) is just 0). Following our previous notation, let \( \delta_n - \delta_{n+1} = \alpha_k \). If we take any \( k \) for which \( \alpha_k \) is nonzero, there are two possibilities for the contribution to the multiplicity from its binomial coefficient. If \( \omega_k \) has been provided by at least one of \( \Delta_1, \ldots, \Delta_i \), then the value \( P_n^{(k)} \) is a linear combination of \( m_1, \ldots, m_t \), which grows linearly as \( m \) gets large. In this case, the binomial coefficient grows as a polynomial in \( m \) of degree \( \alpha_k \). On the other hand, if \( \omega_k \) has not been provided, then the binomial coefficient is just 1.

For any \( k, 1 \leq k \leq r \), define \( f(k) \) to be the smallest \( i \) in our path-type such that \( \Delta_i \) provides \( \omega_k \); we say that \( \Delta_i \) provides \( \omega_k \) for the first time. Then the total contribution to the multiplicity from the coordinate \( k \) will be the product of the contributions when \( n = m_1 + \cdots + m_j \) for \( j = f(k), f(k) + 1, \ldots, t \). As \( m \) gets large, the product of these contributions grows as a polynomial of degree \( \sum_{j=f(k)}^{t} (m_1 + \cdots + m_j) \); that is, the sum of the decreases in the \( \alpha_k \)-coordinate of the \( \Delta_i \). But since \( \Delta_{i+1} \) is just 0, that sum is exactly the \( \alpha_k \)-coordinate of \( \Delta_{f(k)} \).

So given a path-type \( \Delta_1 \ldots \Delta_t \), which Lemma 4 says appears in \( T(\ell) \), the total of the multiplicities of the nodes of that path-type which appear in the top \( m \) rows of \( T(\ell) \) grows as a polynomial of degree

\[
 g(\Delta_1 \ldots \Delta_t) = t + \sum_{k=1}^{r} \text{\( \alpha_k \)-coordinate of \( \Delta_{f(k)} \)}
\]

(6)

where we take \( \Delta_{f(k)} \) to be 0 if \( \omega_k \) is not provided by any \( \Delta \) in the path-type. This value is just the sum of the degrees of the polynomial growths described above.

Finally, since there are only finitely many path-types, the growth of the entire tree \( T(\ell) \) is the same as the growth of the part corresponding to any path-type \( \Delta_1 \ldots \Delta_t \) which maximizes \( g(\Delta_1 \ldots \Delta_t) \). So we have proven the following, up to some calculation:

**Theorem 5** Let \( g \) be simply-laced with decompositions of \( W_m(\ell) \) given by Theorem 4. Then the dimension of the representation \( W_m(\ell) \) as \( m \) gets large is asymptotic to a polynomial in \( m \) of degree \( \frac{1}{2} \dim(\mathcal{O}_\lambda) + g(\Delta_1 \ldots \Delta_t) \), where the path-type \( \Delta_1 \ldots \Delta_t \) is one which maximizes the value of \( g \), and \( \mathcal{O}_\lambda \) is the adjoint orbit of a weight \( \lambda \) which is orthogonal to exactly those positive roots orthogonal to all highest weights of nodes with path-type \( \Delta_1 \ldots \Delta_t \).

1. If \( g \) is of type \( A_n \), then the maximum value of \( g(\Delta_1 \ldots \Delta_t) \) is 0, for all \( 1 \leq \ell \leq n \).

2. If \( g \) is of type \( D_n \), then the maximum value of \( g(\Delta_1 \ldots \Delta_t) \) is \( \lfloor \ell/2 \rfloor \), for \( 1 \leq \ell \leq n-2 \), and 0 for \( \ell = n-1, n \).

3. If \( g \) is of type \( E_6, E_7, \) or \( E_8 \), the maximum value of \( g(\Delta_1 \ldots \Delta_t) \) is
We will complete the proof by exhibiting the path-types which give the indicated values of \( g \) and proving they are maximal.

If \( \Delta_1 \ldots \Delta_t \) maximizes the value of \( g \), then it cannot be obtained from any other path-type by inserting an extra \( \Delta \), since any insertion would increase the length \( t \) and would not decrease the sum in the definition of \( g \). Therefore each \( \Delta_k \) in our desired path-type must be in the positive root lattice, allowable according to Lemma 4, and must be maximal (under \( \preceq \)) in meeting those requirements; we will call a path-type maximal if this is the case.

In particular, if \( \omega_{\ell} \) is in the root lattice then \( \Delta_1 \) will be \( \omega_{\ell} \), and a \( g \)-value of 0 corresponds exactly to an \( \omega_{\ell} \) which is not in the root lattice and is a minimal weight. Thus the 0s above can be verified by inspection; these are exactly the cases in which \( W_m(\ell) \) remains irreducible as a \( g \)-module. Similarly, if \( \omega_{\ell} \) is not in the root lattice but there is only one point in the lattice and in the Weyl chamber under \( \omega_{\ell} \), the path-type will consist just of that point. We can now limit ourselves to path-types of length greater than one.

If \( g \) is of type \( D_n \) then for each \( \omega_{\ell} \), \( 2 \leq \ell \leq n-2 \), there is a unique maximal path-type:

\[
\omega_{\ell} \succ \omega_{\ell} - \omega_2 \succ \omega_{\ell} - \omega_4 \succ \cdots \succ \omega_{\ell} - \omega_{\ell-2} \quad \text{when } \ell \text{ is even}
\]

\[
\omega_{\ell} - \omega_1 \succ \omega_{\ell} - \omega_3 \succ \cdots \succ \omega_{\ell} - \omega_{\ell-2} \quad \text{when } \ell \text{ is odd}
\]

In both cases, the only contribution to \( g \) comes from the length of the path, which is \( \lfloor \ell/2 \rfloor \). This also means that the nodes of the tree \( T(\ell) \) will all have multiplicity 1 in this case.

When \( g \) is of type \( E_6, E_7 \) or \( E_8 \), the following weights have a unique maximal path-type (of length \( > 1 \)), whose \( g \)-value is given in Theorem 5:

\[
E_6 \quad \ell = 4 \quad \omega_4 \succ \omega_4 - \omega_2 \succ \omega_4 - \omega_1 - \omega_6 \succ \omega_2 + \omega_4 - \omega_3 - \omega_7 \succ 2\omega_2 - \omega_4
\]

\[
E_7 \quad \ell = 3 \quad \omega_3 \succ \omega_3 - \omega_1 \succ \omega_3 - \omega_6 \succ \omega_1 + \omega_6 - \omega_4 \succ 2\omega_1 - \omega_3
\]

\[
E_8 \quad \ell = 6 \quad \omega_6 \succ \omega_6 - \omega_1
\]

\[
\ell = 1 \quad \omega_1 \succ \omega_1 - \omega_8
\]

\[
\ell = 7 \quad \omega_7 \succ \omega_7 - \omega_8 \succ \omega_7 - \omega_1 \succ \omega_7 + \omega_8 - \omega_6 \succ 2\omega_8 - \omega_7
\]

\[
\ell = 8 \quad \omega_8
\]

We will consider the remaining weights in \( E_8 \) next. Consider the incomplete path-type

\[
\omega_{\ell} \succ \omega_{\ell} - \omega_8 \succ \omega_{\ell} - \omega_1 \succ \omega_{\ell} - \omega_6 + \omega_8 \succ \omega_{\ell} + \omega_1 - \omega_4 + \omega_8 \succ \cdots
\]

where \( \omega_{\ell} \) is any fundamental weight which is in the root lattice and high enough that all of the weights in question lie in the Weyl chamber. The path so far provides \( \omega_8, \omega_1, \omega_6 \) and \( \omega_4 \); notice that for any \( \omega_i \) which has not been provided,
all of its neighbors in the Dynkin diagram have. Therefore we can extend this path four more steps by subtracting one of \(\alpha_2\), \(\alpha_3\), \(\alpha_5\) and \(\alpha_7\) at each step, to produce a path in which every \(\omega_i\) has been provided. This can be extended to a full path-type by subtracting any \(\alpha_i\) at each stage until we reach the walls of the Weyl chamber.

The resulting path-type is maximal, and is the unique maximal one up to a sequence of transformations of the form

\[\cdots \succ \Delta \succ \Delta - \lambda \succ \Delta - \lambda - \mu \succ \cdots \succ \Delta \succ \Delta - \Delta \succ \Delta - \Delta - \lambda - \mu \succ \cdots\]

which do not affect the rate of growth \(g\). All relevant weights are in the Weyl chamber if and only if \(\omega_\ell \succ \xi = (4,8,10,14,12,8,6,2)\); this turns out to be everything except \(\omega_1\), \(\omega_7\) and \(\omega_8\), whose path-types are given above. If the path-type could start at \(\xi\), it would have growth \(g = 8\), though this is not possible since the last weight in the path-type would be 0 in this case. But each increase of the starting point of the path by any \(\alpha_i\) increases \(g\) by 2 (1 from the length of the path and 1 from the multiplicity). So the growth for any \(\omega_\ell \succ \xi\) is a linear function of its height with coefficient 2; \(g = 2\text{ht}(\omega_\ell) - 120\).

The only remaining cases are \(\omega_4\) and \(\omega_5\) when \(g\) is of type \(E_7\). Both work like the general case for \(E_8\), beginning instead with the incomplete path-types

\[\ell = 4 \quad \omega_4 \succ \omega_4 - \omega_1 \succ \omega_4 - \omega_6 \succ \omega_4 \succ \cdots\]

\[\ell = 5 \quad \omega_5 - \omega_7 \succ \omega_5 - \omega_2 \succ \omega_5 + \omega_7 - \omega_1 - \omega_6 \succ \omega_5 \succ \cdots\]

This concludes the proof of Theorem 5.

The same argument used for \(E_8\) shows that for any choice of \(g\), all “sufficiently large” weights \(\omega_\ell\) in a particular translate of the root lattice will have growth given by \(2\text{ht}(\omega_\ell) - c\) for some fixed \(c\). A weight is sufficiently large if every \(\omega_i\) is provided in its maximal path. Thus we can easily check that \(\omega_4\) and \(\omega_5\) qualify for \(E_7\), and in both cases \(c = 63\). Similarly, \(\omega_4\) for \(E_6\), qualifies, and \(c = 36\). While there are no sufficiently large fundamental weights for \(A_n\) or \(D_n\), we can compute what the maximal path-type would be if one did exist, and in all cases, \(c\) is the number of positive roots. A uniform explanation of this fact would be nice, even though the exhaustive computation does provide a complete proof.

### 3 Computations

This section gives the decompositions of \(W_m(\ell)\) into \(g\)-modules predicted by the conjectural formulas in \(\text{[6]}\). We also give the tree structure defined in Section \(\text{[1]}\).

The representations \(W_m(\ell)\) when \(m = 1\) are called fundamental representations. In the setting of \(U_q(\mathfrak{g})\)-module decompositions of \(U_q(\hat{\mathfrak{g}})\) modules, the decompositions of the fundamental representations for all \(\mathfrak{g}\) and most choices of \(\omega_\ell\) appear in \(\text{[6]}\), calculated using techniques unrelated to the conjecture used in \(\text{[6]}\) to give formulas (2) and (3). Those computations agree with the ones given below. In particular, the choices of \(\omega_\ell\) not calculated in \(\text{[6]}\) are exactly those in which the maximal path-type (Theorem \(\text{[6]}\)) is not unique.
\( A_n \)

As already noted, when \( \mathfrak{g} \) is of type \( A_n \), the \( Y(\mathfrak{g}) \)-modules \( W_m(\ell) \) remain irreducible when viewed as \( \mathfrak{g} \)-modules.

\( D_n \)

Let \( \mathfrak{g} \) be of type \( D_n \). As already noted, the fundamental weights \( \omega_{n-1} \) and \( \omega_n \) are minimal with respect to \( \preceq \), so \( W_m(n-1) \) and \( W_m(n) \) remain irreducible as \( \mathfrak{g} \)-modules. Now suppose \( \ell \leq n-2 \). Then the structure of the weights in the Weyl chamber under \( \omega_\ell \) does not depend on \( n \), and so the decomposition of \( W_m(\ell) \) in \( D_n \) is the same for any \( n \geq \ell + 2 \).

As mentioned in the proof of Theorem 5, there is a unique maximal path-type for each \( \omega_\ell \), and there are no multiplicities greater than 1. The decomposition is therefore very simple: if \( \ell \leq n-2 \) is even, then

\[
W_m(\ell) \simeq \bigoplus_{k_2+k_4+\ldots+k_{\ell-2}+k_2=1+k\leq m} V_{k_2\omega_2+k_4\omega_4+\ldots+k_{\ell-2}\omega_{\ell-2}+(m-k)\omega_\ell}
\]

and if \( \ell \) is odd, then

\[
W_m(\ell) \simeq \bigoplus_{k_1+k_3+\ldots+k_{\ell-2}=k\leq m} V_{k_1\omega_1+k_3\omega_3+\ldots+k_{\ell-2}\omega_{\ell-2}+(m-k)\omega_\ell}
\]

where the minor difference is because \( \omega_\ell \) for \( \ell \) odd is not in the root lattice. The sum \( k \) is the level of the tree on which that module appears, and the parent of a module is obtained by subtracting 1 from the first of \( k_{\ell-2}, k_{\ell-4}, \ldots \) which is nonzero (or from \( k_\ell \) if nothing else is nonzero and \( \ell \) is even).

\( E_n \)

When \( \mathfrak{g} \) is of type \( E_n \) the tree structure is much more irregular: these are the only cases in which a \( \mathfrak{g} \)-module can appear in more that one place in the tree and in which a node on the tree can have multiplicity greater than one.

We indicate the tree structure as follows: we list every node in the tree, starting with the root and in depth-first order, and a node on level \( k \) of the tree is written as \( \oplus V_\lambda \). This is enough information to recover the entire tree, since the parent of that node is the most recent summand of the form \( k^{-1} V_\mu \). Comparing Figure 1 to its representation here should make the notation clear.

Due to space considerations, for \( E_6 \) we list calculations for \( m \leq 3 \), for \( E_7 \) we list \( m \leq 2 \), and for \( E_8 \) only \( m = 1 \). The tree decomposition for \( W_3(4) \) for \( E_7 \), for example, would have 836 components.

\( E_6 \)

\( W_m(1) \) remains irreducible for all \( m \).
$W_1(2) \simeq V_{\omega_2} \oplus V_0$

$W_2(2) \simeq V_{2\omega_2} \oplus V_{\omega_2} \oplus V_0$

$W_3(2) \simeq V_{3\omega_2} \oplus V_{2\omega_2} \oplus V_{\omega_2} \oplus V_0$

$W_1(3) \simeq V_{\omega_3} \oplus V_{\omega_0}$

$W_2(3) \simeq V_{2\omega_3} \oplus V_{\omega_3+\omega_6} \oplus V_{2\omega_6}$

$W_3(3) \simeq V_{3\omega_2} \oplus V_{2\omega_3+\omega_6} \oplus V_{\omega_3+2\omega_6} \oplus V_{3\omega_6}$

$W_1(4) \simeq V_{\omega_4} \oplus V_{\omega_1+\omega_6} \oplus 2V_{\omega_2} \oplus V_0$

$W_2(4) \simeq V_{2\omega_4} \oplus V_{\omega_1+\omega_4+\omega_6} \oplus V_{2\omega_1+2\omega_6} \oplus 2V_{\omega_2+\omega_4} \oplus V_{\omega_3+\omega_6} \oplus 2V_{\omega_1+\omega_2+\omega_6} \oplus 3V_{2\omega_2} \oplus V_{\omega_4} \oplus V_{\omega_1+\omega_6} \oplus 2V_{\omega_2} \oplus V_0$

$W_3(4) \simeq V_{3\omega_2} \oplus V_{\omega_1+2\omega_4+\omega_6} \oplus V_{2\omega_1+\omega_4+2\omega_6} \oplus 2V_{\omega_2+\omega_4} \oplus V_{\omega_3+3\omega_6} \oplus 2V_{\omega_2+2\omega_4} \oplus V_{\omega_3+\omega_6} \oplus 2V_{\omega_1+\omega_2+2\omega_6} \oplus 3V_{2\omega_2+\omega_4} \oplus V_{2\omega_4} \oplus 2V_{\omega_2+3\omega_3+\omega_6} \oplus 3V_{2\omega_2+2\omega_4+\omega_6} \oplus 4V_{3\omega_2+\omega_4} \oplus 3V_{2\omega_2+\omega_4} \oplus 4V_{2\omega_2+\omega_4} \oplus V_{3\omega_1+\omega_3+\omega_6} \oplus 4V_{3\omega_2+\omega_4} \oplus 2V_{\omega_3+\omega_6} \oplus 2V_{2\omega_2+\omega_4} \oplus V_{\omega_3+\omega_5} \oplus 2V_{\omega_1+\omega_2+\omega_6} \oplus 3V_{2\omega_2} \oplus V_{\omega_4} \oplus V_{\omega_1+\omega_6} \oplus 2V_{\omega_2} \oplus V_0$

$W_1(5) \simeq V_{\omega_5} \oplus V_{\omega_1}$

$W_2(5) \simeq V_{2\omega_5} \oplus V_{\omega_1+\omega_5} \oplus V_{2\omega_1}$

$W_3(5) \simeq V_{3\omega_5} \oplus V_{\omega_1+2\omega_5} \oplus V_{2\omega_1+\omega_5} \oplus V_{3\omega_1}$

$W_m(6)$ remains irreducible for all $m$.

$E_7$

$W_1(1) \simeq V_{\omega_1} \oplus V_0$

$W_2(1) \simeq V_{2\omega_1} \oplus V_{\omega_1} \oplus V_0$

$W_1(2) \simeq V_{\omega_2} \oplus V_{\omega_7}$

$W_2(2) \simeq V_{2\omega_2} \oplus V_{\omega_2+\omega_7} \oplus V_{2\omega_7}$

$W_1(3) \simeq V_{\omega_3} \oplus V_{\omega_6} \oplus 2V_{\omega_1} \oplus V_0$
\[ W_2(3) \simeq V_{\omega_2} \oplus V_{\omega_3} \oplus V_{\omega_4} \oplus V_{\omega_6} \oplus 2V_{\omega_1+\omega_5} \oplus 2V_{\omega_1+\omega_6} \oplus 2V_{\omega_1+\omega_7} \oplus 3V_{\omega_2} \oplus V_{\omega_4} \oplus V_{\omega_3} \oplus V_{\omega_6} \oplus 2V_{\omega_1} \oplus V_0 \]

\[ W_1(4) \simeq V_{\omega_1} \oplus V_{\omega_7} \oplus V_{\omega_5} \oplus V_{\omega_2} \oplus 3V_{\omega_3} \oplus V_{\omega_4} \oplus V_{\omega_6} \oplus V_{\omega_1} \oplus 3V_{\omega_1} \oplus V_0 \oplus V_0 \]

\[ W_2(4) \simeq V_{\omega_2} \oplus V_{\omega_1+\omega_6} \oplus V_{\omega_2+\omega_6} \oplus V_{\omega_1+\omega_7} \oplus 2V_{\omega_2+\omega_4} \oplus V_{\omega_3+\omega_5+\omega_7} \oplus 2V_{\omega_1+\omega_2+\omega_7} \oplus \]

\[ W_1(5) \simeq V_{\omega_5} \oplus V_{\omega_1+\omega_7} \oplus 2V_{\omega_2} \oplus 2V_{\omega_7} \]

\[ W_2(5) \simeq V_{\omega_2} \oplus V_{\omega_1+\omega_7} \oplus V_{\omega_2+\omega_6} \oplus V_{\omega_1+\omega_7} \oplus V_{\omega_2+\omega_5} \oplus V_{\omega_3+\omega_7} \oplus V_{\omega_1+\omega_5} \oplus V_{\omega_2} \oplus 3V_{\omega_3} \oplus V_{\omega_2} \oplus V_{\omega_4} \oplus V_{\omega_6} \oplus V_{\omega_1} \oplus V_6 \]

\[ W_1(6) \simeq V_{\omega_1} \oplus V_{\omega_1} \oplus V_0 \]

\[ W_2(6) \simeq V_{\omega_2} \oplus V_{\omega_1} \oplus V_{\omega_6} \oplus V_{\omega_1} \oplus V_0 \]

\[ W_m(7) \text{ remains irreducible for all } m. \]

\[ E_8 \]

\[ W_1(1) \simeq V_{\omega_1} \oplus V_{\omega_1} \oplus V_0 \]

\[ W_1(2) \simeq V_{\omega_2} \oplus V_{\omega_1} \oplus 2V_{\omega_1} \oplus 2V_{\omega_1} \oplus V_0 \]
\[ W_1(3) \simeq V_{\omega_1} \oplus V_{\omega_8} \oplus 2V_{\omega_1 + \omega_8} \oplus 3V_{\omega_2} \oplus V_{\omega_7} \oplus V_{2\omega_8} \oplus 3V_{\omega_7} \oplus V_{\omega_1} \oplus 4V_{\omega_1} \oplus 2V_{\omega_8} \oplus 3V_{\omega_7} \oplus V_0 \oplus V_0 \]

\[ W_1(4) \simeq V_{\omega_1} \oplus V_{\omega_1 + \omega_6} \oplus 2V_{\omega_2 + \omega_7} \oplus V_{2\omega_1 + \omega_6} \oplus 3V_{\omega_2 + \omega_8} \oplus V_{\omega_6 + \omega_8} \oplus V_{2\omega_7 + \omega_6} \oplus 3V_{\omega_7 + \omega_8} \oplus 2V_{\omega_1 + \omega_7} \oplus \]

\[ W_1(5) \simeq V_{\omega_1} \oplus V_{\omega_1 + \omega_7} \oplus 2V_{\omega_2 + \omega_8} \oplus 3V_{\omega_3} \oplus V_{\omega_6 + \omega_8} \oplus 2V_{\omega_7 + \omega_6} \oplus 3V_{\omega_7 + \omega_8} \oplus 2V_{\omega_1 + \omega_7} \oplus \]

\[ W_1(6) \simeq V_{\omega_1} \oplus V_{\omega_1 + \omega_8} \oplus 2V_{\omega_1} \oplus 2V_{\omega_2} \oplus 2V_{\omega_7} \oplus V_{2\omega_8} \oplus 2V_{\omega_7} \oplus 3V_{\omega_7} \oplus V_{\omega_1} \oplus 3V_{\omega_2} \oplus 3V_{\omega_8} \oplus 2V_{\omega_1 + \omega_7} \oplus \]

\[ W_1(7) \simeq V_{\omega_1} \oplus V_{\omega_1 + \omega_8} \oplus 2V_{\omega_1} \oplus 2V_{\omega_2} \oplus 2V_{\omega_7} \oplus V_{2\omega_8} \oplus 2V_{\omega_7} \oplus 3V_{\omega_7} \oplus V_{\omega_1} \oplus 3V_{\omega_2} \oplus 3V_{\omega_8} \oplus 2V_{\omega_1 + \omega_7} \oplus \]

\[ W_1(8) \simeq V_{\omega_1} \oplus V_{\omega_1 + \omega_8} \oplus 2V_{\omega_1} \oplus 2V_{\omega_2} \oplus 2V_{\omega_7} \oplus V_{2\omega_8} \oplus 2V_{\omega_7} \oplus 3V_{\omega_7} \oplus V_{\omega_1} \oplus 3V_{\omega_2} \oplus 3V_{\omega_8} \oplus 2V_{\omega_1 + \omega_7} \oplus \]

References

[1] Bourbaki, N. Groupes et algèbres de Lie, ch. 4, 5 et 6. Masson, Paris, 1981.

[2] Chari, V. Minimal quantizations of representations of affine Lie algebras: the rank 2 case. Publ. Res. Inst. Math. Sci. 31 (1995), no. 5, 873–911.

[3] Chari, V; Pressley, A. Quantum affine algebras and their representations. Representations of groups (Banff, AB, 1994), 59–78, CMS Conf. Proc. 16, Amer. Math. Soc., Providence, RI, 1995.

[4] Drinfel’d, V. G. Hopf algebras and the quantum Yang-Baxter equation, Soviet Math. Dokl. 32 (1985), 254–258.

[5] Drinfel’d, V. G. A new realization of Yangians and quantized affine algebras. Soviet Math. Dokl. 36 (1988), 212–216.

[6] Kirillov, A. N.; Reshetikhin, N. Yu. Representations of Yangians and multiplicities of occurrence of the irreducible components of the tensor product of representations of simple Lie algebras. J. Soviet Math. 52 (1990), 3156–3164.
[7] Kulish, P. P.; Reshetikhin, N. Yu.; Sklyanin, E. K. Yang-Baxter equations
and representation theory: I. Lett. Math. Phys. 5 (1981), no. 5, 393–403

[8] Reshetikhin, N. Yu. Private communication.

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