A NOTE ON DIOPHANTINE APPROXIMATION WITH GAUSSIAN PRIMES

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1. Introduction

Let $\theta \in \mathbb{R}$ be an irrational number. Then the continued fraction expansion of $\theta$ yields infinitely many natural numbers $q$ such that

$$|\theta - \frac{a}{q}| \leq q^{-2},$$

where $(a, q) = 1$. In other words, for infinitely many $q \in \mathbb{N}$, we have

$$||q\theta|| \leq q^{-1},$$

where $||x||$ is the distance of $x$ to the nearest integer. The problem of approximating irrational numbers by rational numbers with prime denominator is considerably more difficult and has a long history. The question is for which $\gamma > 0$ one can prove the infinitude of primes $p$ such that

$$||p\theta|| \leq p^{-\gamma + \varepsilon}.$$  \hspace{1cm} (1)

The first results in this direction are due to Vinogradov [8] who showed that $\gamma = 1/5$ is admissible. Vaughan [7] improved this exponent to $\gamma = 1/4$ using his famous identity for the von Mangoldt function. It should be noted that using Vaughan’s method, an asymptotic result of the following form can be established.

Theorem 1.1. Let $\theta \in \mathbb{R}$ be irrational and $\varepsilon > 0$ be an arbitrary constant. There exists an infinite increasing sequence of natural numbers $(N_k)_{k \in \mathbb{N}}$ such that

$$\sum_{N_k/2 < p \leq N_k \atop ||p\theta|| \leq \delta_k} 1 \sim 2\delta_k \sum_{N_k/2 < p \leq N_k} 1 \quad \text{as } k \to \infty$$ \hspace{1cm} (2)

if

$$N_k^{-1/4 + \varepsilon} \leq \delta_k \leq 1/2.$$ \hspace{1cm} (3)

The next important step was Harman’s work [2] in which he used his sieve method to show that (1) holds for infinitely many primes $p$ if $\gamma = 3/10$. Harman’s method doesn’t imply the asymptotic (2) for $\delta_k = N_k^{-3/10 + \varepsilon}$ since it uses a lower bound sieve. However, Harman’s sieve can be employed to recover Vaughan’s result and hence (2) for the same $\delta_k$-range as in (3). We further mention the work of Heath-Brown and Jia [3] who used bounds for Kloosterman sums to obtain a further improvement of the exponent to $\gamma = 16/49 = 1/3 - 0.0068...$. Finally, the exponent $\gamma = 1/3$ was achieved in a landmark paper...
by Matomäki [6] who incorporated the Kuznetsov formula into the method to bound sums of Kloosterman sums. This exponent $\gamma = 1/3$ is considered to be the limit of currently available techniques.

The purpose of this note is to formulate the problem for Gaussian primes and establish a result corresponding to Theorem 1.1 in this context, thereby proving the infinitude of Gaussian primes satisfying an inequality corresponding to (1). To this end, we shall apply a version of Harman’s sieve for $\mathbb{Z}[i]$. Our method will require additional counting arguments, as compared to the classical method. Our results are stated in section 10.

As usual, throughout this paper, $\varepsilon$ denotes an arbitrarily small positive number which may change from line to line.

2. Setup

Throughout the sequel, $\theta$ is a complex number such that $\theta \notin \mathbb{Q}(i)$, and $x \geq 1$. We compare the quantities

$$S(x, \delta) := \sum_{x/2 < N(p) \leq x, ||p\theta|| \leq \delta} 1$$

and

$$S(x) := \sum_{x/2 < N(p) \leq x} 1.$$

Here the sums run over Gaussian primes $p$, $N(n)$ denotes the norm of $n \in \mathbb{Z}[i]$, $0 < \delta \leq 1/2$, and we define

$$||z|| := \max \{||\Re(z)||, ||\Im(z)||\},$$

where $\Re(z)$ is the real part and $\Im(z)$ is the imaginary part of $z \in \mathbb{C}$. Hence, $||z||$ measures the distance of $z$ to the nearest Gaussian integer with respect to the supremum norm. By the prime number theorem for Gaussian primes,

$$S(x) \sim 2 \cdot \frac{x}{\log x} \quad \text{as } x \to \infty. \quad (4)$$

Our goal is to construct an infinite increasing sequence $(N_k)_{k \in \mathbb{N}}$ of natural numbers such that

$$S(N_k, \delta_k) \sim 4\delta_k^2 S(N_k) \quad \text{as } k \to \infty$$

for $N_k^{-\gamma+\varepsilon} \leq \delta_k \leq 1/2$, where $\gamma$ is a suitable positive number. In this paper, we shall show that $\gamma = 1/24$ is admissible. The full analog of Theorem (1.1) would be achieved if $\gamma = 1/24$ could be replaced by $\gamma = 1/8$ (the exponent 1/4 in (3) needs to be halved because our setting is 2-dimensional).

3. Application of Harman’s sieve for $\mathbb{Z}[i]$}

In the following, let $A$ be a finite set of non-zero Gaussian integers, $P$ be a subset of the set $\mathbb{P}$ of Gaussian primes and $z$ be a positive parameter. By $\mathcal{S}(A, P, z)$ we denote the
number of elements of $A$ which are coprime to $P(z)$, the product of all Gaussian primes in $P$ with norm $\leq z$, i.e.

$$S(A, P, z) = \sharp\{ n \in A : p \nmid n \text{ for all } p \in P \text{ such that } N(p) \leq z \}.$$ 

The following is a version of Harman’s sieve for $\mathbb{Z}[i]$.

**Theorem 3.1 (Harman).** Let $A, B$ be finite sets of non-zero Gaussian integers with norm $\leq x$. Suppose for any sequences $(a_n)_{n \in \mathbb{Z}[i]}$ and $(b_n)_{n \in \mathbb{Z}[i]}$ of complex numbers satisfying $|a_n|, |b_n| \leq 1$ the following hold:

$$\sum_{N(m) \leq M, \, mn \in A} a_m = \lambda \sum_{N(m) \leq M, \, mn \in B} a_m + O(Y),$$  

$$\sum_{x^{\alpha} < N(m) \leq x^{\alpha+\beta}, \, mn \in A} a_m b_n = \lambda \sum_{x^{\alpha} < N(m) \leq x^{\alpha+\beta}, \, mn \in B} a_m b_n + O(Y)$$  

for some $\lambda, Y > 0$, $0 < \alpha < 1/2$, $\beta \leq 1/2$, and $M > x^{\alpha}$. Then we have

$$S(A, P, x^{\beta}) = \lambda S(B, P, x^{\beta}) + O(Y \log^3 x).$$  

**Proof.** The proof is parallel to that of Harman’s sieve for the classical case, Theorem 3.1, in [4] (Fundamental Theorem) with $R = 1$ and $c_r = 1$, making repeated use of the Buchstab identity in the setting of $\mathbb{Z}[i]$ (see Chapter 11 in [4]). Therefore, we omit the details. □

In the usual terminology, the sums in (5) are referred to as type I bilinear sums, and the sums in (6) as type II bilinear sums.

We shall apply Theorem 3.1 to the situation when

$$A := \{ n \in \mathbb{Z}[i] : x/2 < N(n) \leq x, \, ||n\theta|| \leq \delta \} \quad \text{with } 0 < \delta \leq 1/2,$$

$$B := \{ n \in \mathbb{Z}[i] : x/2 < N(n) \leq x \} \quad \text{and } \beta = \frac{1}{2}.$$ 

The parameters $\alpha$ and $M$ will later be chosen suitably. We note that

$$S(x, \delta) = \sum_{x/2 < N(p) \leq x, \, ||n\theta|| \leq \delta} 1 = S(A, P, x^{1/2})$$  

and

$$S(x) = \sum_{x/2 < N(p) \leq x} 1 = S(B, P, x^{1/2}).$$
4. DETECTING SMALL $||n\theta||$

We observe that

$$||n\theta|| \leq \delta \iff ([\delta - \Re(n\theta)] - [\delta - \Re(n\theta)]) ([\delta - \Im(n\theta)] - [\delta - \Im(n\theta)]) = 1.$$ 

Hence, the type I sum in question can be written in the form

$$\sum_{N(m) \leq M} a_m = \sum_{N(m) \leq M} a_m \cdot \sum_{x/(2N(m)) < N(n) \leq x/N(m)} ([\delta - \Re(mn\theta)] - [\delta - \Re(mn\theta)]) \times ([\delta - \Im(mn\theta)] - [\delta - \Im(mn\theta)]) = 1.$$ 

Further, using $[x] = x - \psi(x) - 1/2$, the inner sum over $n$ can be expressed in the form

$$\sum_{x/(2N(m)) < N(n) \leq x/N(m)} ([\delta - \Re(mn\theta)] - [\delta - \Re(mn\theta)]) \times ([\delta - \Im(mn\theta)] - [\delta - \Im(mn\theta)])$$

$$= 4\delta^2 \sum_{x/(2N(m)) < N(n) \leq x/N(m)} 1 - 2\delta \sum_{x/(2N(m)) < N(n) \leq x/N(m)} (\psi (\delta - \Im(mn\theta)) - \psi (-\delta - \Im(mn\theta))) \psi (\delta - \Re(mn\theta)) - \psi (-\delta - \Re(mn\theta))) +$$

$$= 4\delta^2 \sum_{x/(2N(m)) < N(n) \leq x/N(m)} 1 - 2\delta S_1 - 2\delta S_2 + S_3,$$

say. Next, we approximate the function $\psi(x)$ by a trigonometric polynomial using the following lemma due to Vaaler (see [1], Theorem A6).

**Lemma 4.1** (Vaaler). For $0 < |t| < 1$ let

$$W(t) = \pi t (1 - |t|) \cot \pi t + |t|.$$ 

Fix a natural number $J$. For $x \in \mathbb{R}$ define

$$\psi^*(x) := - \sum_{1 \leq |j| \leq J} (2\pi i)^{-1} W \left( \frac{j}{J + 1} \right) e(jx)$$

and

$$\sigma(x) := \frac{1}{2J + 2} \sum_{|j| \leq J} \left( 1 - \frac{|j|}{J + 1} \right) e(jx).$$
Then $\sigma(x)$ is non-negative, and we have

$$|\psi^*(x) - \psi(x)| \leq \sigma(x)$$

for all real numbers $x$.

Throughout the sequel, $J$ denotes a natural number such that $J \geq \delta^{-1}$ which will be fixed in section 9. From Lemma 4.1 we deduce that

$$S_1 \ll \frac{x/N(m)}{J} + \sum_{1 \leq |j| \leq J} \frac{1}{|j|} \times \left| \sum_{x/(2N(m)) < N(n) \leq x/N(m)} (e(j(\delta - \Im(mn\theta))) - e(j(-\delta - \Im(mn\theta)))) \right|$$

$$\ll \frac{x/N(m)}{J} + \sum_{1 \leq |j| \leq J} \frac{1}{|j|} \times \left| \sum_{x/(2N(m)) < N(n) \leq x/N(m)} (e(j\delta) - e(-j\delta)) \cdot e(-j\Im(mn\theta)) \right|$$

$$\ll \frac{x/N(m)}{J} + \sum_{1 \leq |j| \leq J} \min\{\delta, |j|^{-1}\} \cdot \sum_{x/(2N(m)) < N(n) \leq x/N(m)} e(j\Im(mn\theta)) \right| .$$

In a similar way, we obtain

$$S_2 \ll \frac{x/N(m)}{J} + \sum_{1 \leq |j| \leq J} \min\{\delta, |j|^{-1}\} \cdot \sum_{x/(2N(m)) < N(n) \leq x/N(m)} e(j\Re(mn\theta)) \right| .$$
and

\[ S_3 \ll \frac{x/N(m)}{J^2} + \]

\[ \frac{1}{J} \cdot \sum_{1 \leq |j_1| \leq J} \min\{\delta, |j_1|^{-1}\} \cdot \sum_{x/(2N(m)) < N(n) \leq x/N(m)} e\left(j_1 \Im(mn)\right) + \]

\[ \frac{1}{J} \cdot \sum_{1 \leq |j_2| \leq J} \min\{\delta, |j_2|^{-1}\} \cdot \sum_{x/(2N(m)) < N(n) \leq x/N(m)} e\left(j_2 \Re(mn)\right) + \]

\[ \sum_{1 \leq |j_1| \leq J} \min\{\delta, |j_1|^{-1}\} \cdot \min\{\delta, |j_2|^{-1}\} \times \]

\[ \sum_{x/(2N(m)) < N(n) \leq x/N(m)} e\left(j_1 \Im(mn) + j_2 \Re(mn)\right) \].

Summing over \( m \) and using \( |a_m| \leq 1 \) and \( J \geq \delta^{-1} \), we get

\[ \sum_{N(m) \leq M \atop mn \in A} a_m = 4\delta^2 \sum_{N(m) \leq M} a_m \sum_{x/(2N(m)) < N(n) \leq x/N(m)} 1 + O\left(\delta x^{1+\varepsilon} J^{-1} + \delta(E_1 + E_2) + E_3\right) \]

\[ = 4\delta^2 \sum_{N(m) \leq M} a_m + O\left(\delta x^{1+\varepsilon} J^{-1} + \delta(E_1 + E_2) + E_3\right), \] (10)

where

\[ E_1 = \sum_{1 \leq |j| \leq J} \min\{\delta, |j|^{-1}\} \cdot \sum_{N(m) \leq M} \sum_{x/(2N(m)) < N(n) \leq x/N(m)} e\left(j \Im(mn)\right); \]

\[ E_2 = \sum_{1 \leq |j| \leq J} \min\{\delta, |j|^{-1}\} \cdot \sum_{N(m) \leq M} \sum_{x/(2N(m)) < N(n) \leq x/N(m)} e\left(j \Re(mn)\right) \]

and

\[ E_3 = \sum_{1 \leq |j_1| \leq J} \sum_{1 \leq |j_2| \leq J} \min\{\delta, |j_1|^{-1}\} \cdot \min\{\delta, |j_2|^{-1}\} \times \]

\[ \sum_{N(m) \leq M} \sum_{x/(2N(m)) < N(n) \leq x/N(m)} e\left(j_1 \Im(mn) + j_2 \Re(mn)\right); \] (11)
In a similar way, using $|a_m|, |b_n| \leq 1$ and $J \geq \delta^{-1}$, we derive the asymptotic estimate

$$\sum_{x^n < N(m) \leq x^{n+\beta}} a_m b_n = 4\delta^2 \sum_{N(m) \leq M} a_m b_n + O\left(\delta x^{1+\varepsilon}J^{-1} + \delta(F_1 + F_2) + F_3\right),$$

where

$$F_1 = \sum_{1 \leq |j| \leq J} \min\{\delta, |j|^{-1}\} \cdot \sum_{x^n < N(m) \leq x^{n+\beta}} a_m b_n e(j\Im(mn\theta)),$$

$$F_2 = \sum_{1 \leq |j| \leq J} \min\{\delta, |j|^{-1}\} \cdot \sum_{x^n < N(m) \leq x^{n+\beta}} a_m b_n e(j\Re(mn\theta))$$

and

$$F_3 = \sum_{1 \leq |j_1| \leq H_1} \sum_{1 \leq |j_2| \leq J} \min\{\delta, |j_1|^{-1}\} \cdot \min\{\delta, |j_2|^{-1}\} \cdot \sum_{x^n < N(m) \leq x^{n+\beta}} a_m b_n e(j_1\Im(mn\theta) + j_2\Re(mn\theta)).$$

5. Transformations of the sums $E_i$ and $F_i$

We note that

$$E_1 = E_2.$$  

We further have, by breaking the $|j|$-range into $O(\log 2J)$ dyadic intervals,

$$E_1 \ll (\log 2J) \cdot \sup_{1 \leq H \leq J} \min\{\delta, H^{-1}\} \cdot E_1(H),$$

where

$$E_1(H) = \sum_{1 \leq |j| \leq H} \sum_{N(m) \leq M} \left| \sum_{x^n < N(m) \leq x^{n+\beta}} e(j\Im(mn\theta)) \right|.$$  

Similarly,

$$E_3 \ll (\log 2J)^2 \cdot \sup_{1 \leq |H_1| \leq J} \sup_{1 \leq |H_2| \leq J} \min\{\delta, H_1^{-1}\} \cdot \min\{\delta, H_2^{-1}\} \cdot E_3(H_1, H_2),$$
where
\[ E_3(H_1, H_2) = \sum_{|j_1| \leq H_1, |j_2| \leq H_2} \sum_{N(m) \leq M} \sum_{x/(2N(m)) < N(m) \leq x/N(m)} e\left(j_1 \Im(m\theta) + j_2 \Re(m\theta)\right) . \]

We note that
\[ E_3(H_1, H_2) = \sum_{j \neq 0} \sum_{N(m) \leq M} \sum_{x/(2N(m)) < N(m) \leq x/N(m)} e\left(\Im(jm\theta)\right) \]
and hence,
\[ E_1(H) = E_3(H, 1/2). \]
Thus, it suffices to estimate \( E_3(H_1, H_2) \) for \( H_1 \geq 1 \) and \( H_2 \geq 1/2 \) to bound \( E_1, E_2 \) and \( E_3 \).

Similarly,
\[ F_1 = F_2 \]
and
\[ F_1 \ll (\log 2J) \cdot \sup_{1 \leq H \leq J} \min\{\delta, H^{-1}\} \cdot F_1(H), \]
where
\[ F_1(H) = \sum_{1 \leq |j| \leq H} \sum_{x^{a} < N(m) \leq x^{a+\beta}} a_m b_n e\left(j \Im(m\theta)\right) , \]
and
\[ F_3 = (\log 2J)^2 \cdot \sup_{1 \leq |H_1| \leq J} \min\{\delta, H_1^{-1}\} \cdot \min\{\delta, H_2^{-1}\} \cdot F_3(H_1, H_2), \]
where
\[ F_3(H_1, H_2) = \sum_{|j_1| \leq H_1, |j_2| \leq H_2} \sum_{(j_1, j_2) \neq (0,0)} \sum_{x^{a} < N(m) \leq x^{a+\beta}} a_m b_n e\left(j_1 \Im(m\theta) + j_2 \Re(m\theta)\right) \]
\[ = \sum_{j \neq 0} \sum_{x^{a} < N(m) \leq x^{a+\beta}} a_m b_n e\left(\Im(jm\theta)\right) \]
and hence,
\[ F_1(H) = F_3(H, 1/2). \]
Thus, it suffices to estimate \( F_3(H_1, H_2) \) for \( H_1 \geq 1 \) and \( H_2 \geq 1/2 \) to bound \( F_1, F_2 \) and \( F_3 \).
So we have reduced the problem to bounding the type I sums $E_3(H_1, H_2)$ and the type II sums $F_3(H_1, H_2)$.

6. Treatment of Type II Sums

To treat the type II sums, we first reduce them to type I sums. We begin by splitting $F_3(H_1, H_2)$ into subsums of the form

$$F_3(H_1, H_2, K, K') := \sum_{|\Re(j)| \leq H_1, |\Im(j)| \leq H_2} \sum_{n, n' \in A} a_{mn} b_n e(\Re jm n)$$

where $K < K' \leq 2K$. Next, we apply the Cauchy-Schwarz inequality, getting

$$F_3(H_1, H_2, K, K')^2 \ll H_1 H_2 K \cdot \sum_{|\Re(j)| \leq H_1, |\Im(j)| \leq H_2} \left| \sum_{n, n' \in A} b_n e(\Re jm n) \right|^2,$$

where we use the bound $|a_m| \leq 1$. Expanding the square and re-arranging summation, we get

$$F_3(H_1, H_2, K, K')^2 \ll H_1 H_2 K \cdot \sum_{|\Re(j)| \leq H_1, |\Im(j)| \leq H_2} \sum_{x/(2K') < N(n), N(n') \leq x/K} b_n b_m$$

\[
\times \sum_{\max\{K, x/(2N(n)), x/(2N(n'))\} < \min\{K', x/N(n), x/N(n')\}} e(\Re jm (n - n')) \]

\[
\ll H_1 H_2 K x + H_1 H_2 K \cdot \sum_{j \neq 0, |\Re(j)| \leq H_1, |\Im(j)| \leq H_2} x/(2K') \sum_{N(n) < \min\{K', x/N(n), x/N(n')\}} e(\Re jm (n - n')) \]

\[
\ll H_1 H_2 K x + H_1 H_2 K \cdot \sum_{0 < N(n) \leq 4(H_1^2 + H_2^2) x/K} \sum_{j \neq 0} \sum_{n, n' \in A} b_n b_m e(\Re jm n) \]

\[
\times \sum_{\max\{K, x/(2N(n)), x/(2N(n'))\} < \min\{K', x/N(n), x/N(n')\}} e(\Re jm (n - n')) \]

Here the second line arrives by isolating the diagonal contribution of $n_1 = n_2$ and using the bound $|b_n| \leq 1$, and the third line arrives by writing $n = j(n_1 - n_2)$.
7. Estimating sums of linear exponential sums

Our next task is to bound linear exponential sums of the form

\[
\sum_{\hat{y} < N(m) \leq y} e(\Im(m\kappa)) = \sum_{(m_1, m_2) \in \mathbb{Z}^2 \atop \hat{y} < m_1^2 + m_2^2 \leq y} e(\Re(m_2) + \Im(m_1)) \\
= \sum_{(m_1, m_2) \in \mathbb{Z}^2 \atop m_1^2 + m_2^2 \leq y} e(\Re(m_2) + \Im(m_1)) - \sum_{(m_1, m_2) \in \mathbb{Z}^2 \atop m_1^2 + m_2^2 \leq \hat{y}} e(\Re(m_2) + \Im(m_1)),
\]

(29)

where \( \kappa \) is a complex number and \( 0 \leq \hat{y} < y \). Here we use the following simple slicing argument. We have

\[
\sum_{(m_1, m_2) \in \mathbb{Z}^2 \atop m_1^2 + m_2^2 \leq y} e(\Re(m_2) + \Im(m_1)) \\
\leq \sum_{-\sqrt{y} \leq m_1 \leq \sqrt{y}} \sum_{-\sqrt{y-m_1^2} \leq m_2 \leq \sqrt{y-m_1^2}} e(\Re(m_2) + \Im(m_1)) \\
\leq y^{1/2} \cdot \min \left\{ ||\Re(\kappa)||^{-1}, \sqrt{y} \right\},
\]

(30)

where we use the classical bound

\[
\sum_{a < m \leq b} e(mz) \ll \min \left\{ b - a + 1, ||z||^{-1} \right\}
\]

for linear exponential sums. Similarly, by interchanging the rules of \( m_1 \) and \( m_2 \), we get

\[
\sum_{(m_1, m_2) \in \mathbb{Z}^2 \atop m_1^2 + m_2^2 \leq y} e(\Re(m_2) + \Im(m_1)) \ll y^{1/2} \cdot \min \left\{ ||\Im(\kappa)||^{-1}, \sqrt{y} \right\}.
\]

Taking the geometric mean of these two estimates gives

\[
\sum_{(m_1, m_2) \in \mathbb{Z}^2 \atop m_1^2 + m_2^2 \leq y} e(\Re(m_2) + \Im(m_1)) \ll y^{1/2} \cdot \min \left\{ ||\Im(\kappa)||^{-1}, \sqrt{y} \right\}^{1/2} \cdot \min \left\{ ||\Re(\kappa)||^{-1}, \sqrt{y} \right\}^{1/2}.
\]

Using \((29)\), we deduce that

\[
\sum_{\hat{y} < N(m) \leq y} e(\Im(m\kappa)) \ll y^{1/2} \cdot \min \left\{ ||\Im(\kappa)||^{-1}, \sqrt{y} \right\}^{1/2} \cdot \min \left\{ ||\Re(\kappa)||^{-1}, \sqrt{y} \right\}^{1/2}.
\]

(31)
To bound the sums appearing in sections 5 and 6 we need to bound sums of linear sums of roughly the shape

$$\sum_{n \sim Z} \sum_{m \sim Y} e(\Re(mn\theta))$$.

Considering (31), we are left with bounding expressions of the form

$$G_\theta(y, z) := \sum_{0 < N(n) \leq z} \min \{||\Im(n\theta)||^{-1}, \sqrt{y}\}^{1/2} \cdot \min \{||\Re(n\theta)||^{-1}, \sqrt{y}\}^{1/2},$$  \hspace{1cm} (32)

where \(y, z \geq 1\). To this end, we break the above into partial sums

$$G_\theta(y, z, \Delta_1, \Delta'_1, \Delta_2, \Delta'_2)$$

:= \sum_{0 < N(n) \leq z} \min \{||\Im(n\theta)||^{-1}, \sqrt{y}\}^{1/2} \cdot \min \{||\Re(n\theta)||^{-1}, \sqrt{y}\}^{1/2}$$  \hspace{1cm} (33)

with \(0 \leq \Delta_1 < \Delta'_1 \leq 1/2\) and \(0 \leq \Delta_2 < \Delta'_2 \leq 1/2\) and bound them by

$$G_\theta(y, z, \Delta_1, \Delta'_1, \Delta_2, \Delta'_2) \ll \min \{\Delta_1^{-1}, \sqrt{y}\}^{1/2} \cdot \min \{\Delta'_2^{-1}, \sqrt{y}\}^{1/2} \cdot \Sigma_\theta(z, \Delta'_1, \Delta'_2),$$  \hspace{1cm} (34)

where

$$\Sigma_\theta(z, \Delta'_1, \Delta'_2) = \sum_{\substack{0 < N(n) \leq z \\|\Im(n\theta)||\leq \Delta'_1 \\|\Re(n\theta)||\leq \Delta'_2}} 1.$$  \hspace{1cm} (35)

In the next section, we shall prove that for infinitely many Gaussian integers \(q\), a bound of the form

$$\Sigma_\theta(z, \Delta'_1, \Delta'_2) \ll \left(1 + \frac{z}{|q|^2}\right) \cdot (1 + \Delta'_1|q|) (1 + \Delta'_2|q|)$$  \hspace{1cm} (36)

holds. We shall also see that for these \(q\), we have

$$\Sigma_\theta(z, \Delta'_1, \Delta'_2) = 0 \text{ if } \max\{\Delta'_1, \Delta'_2\} < 1/(\sqrt{8}|q|) \text{ and } z \leq |q|^2/8.$$  \hspace{1cm} (37)

Plugging (36) into (33) gives

$$G_\theta(y, z, \Delta_1, \Delta'_1, \Delta_2, \Delta'_2) \ll \left(1 + \frac{z}{|q|^2}\right) \cdot \min \{\Delta_1^{-1}, \sqrt{y}\}^{1/2} \cdot \min \{\Delta_2^{-1}, \sqrt{y}\}^{1/2} \times$$

$$\left(1 + \Delta'_1|q|\right) \cdot (1 + \Delta'_2|q|).$$  \hspace{1cm} (38)

Next, we write

$$G_\theta(y, z) = G_\theta(y, z, 0, 2^{-L-1}, 0, 2^{-L-1}) + \sum_{i=1}^{L} \sum_{j=1}^{L} G_\theta(y, z, 2^{-i-1}, 2^{-j-1}, 2^{-i-1}, 2^{-j-1}) +$$

$$+ \sum_{j=1}^{L} G_\theta(y, z, 0, 2^{-L-1}, 2^{-j-1}, 2^{-j}) + \sum_{i=1}^{L} G_\theta(y, z, 2^{-i-1}, 2^{-i}, 0, 2^{-L-1}),$$
where $L$ satisfies $1/(2\sqrt{y}) \leq 2^{-L-1} < 1/\sqrt{y}$. Using (38), we deduce that
\[ G_\theta(y, z) \ll \left(1 + \frac{z}{|q|^2}\right) \left(y^{1/2} + |q|^2\right) \cdot (\log 2y)^2. \] (39)

If $z \leq |q|^2/8$, then using (37), we have
\[
G_\theta(y, z) = \sum_{i=1}^{L} \sum_{j=1}^{L} G_\theta(y, z, 2^{-i-1}, 2^{-i}, 2^{-j-1}, 2^{-j}) + \sum_{j=1}^{L} G_\theta(y, z, 0, 2^{-L-1}, 2^{-j-1}, 2^{-j}) + \sum_{i=1}^{L} G_\theta(y, z, 2^{-i-1}, 2^{-i}, 0, 2^{-L-1}),
\]
where $1/(2\sqrt{8}|q|) \leq 2^{-L-1} < 1/(\sqrt{8}|q|)$. In this case, using (38), we deduce that
\[ G_\theta(y, z) \ll \left(|q|y^{1/4} + |q|^2\right) \cdot \log^2(2|q|). \] (40)

8. COUNTING

In this section, we prove (36) and (37). To bound the quantity $G_\theta(y, z)$, we need information about the spacing of the points $n\theta$ modulo 1, where $n \in \mathbb{Z}[i]$. We begin by using the Hurwitz continued fraction development of $\theta$ in $\mathbb{Z}[i]$ (see [5]) to approximate $\theta$ in the form
\[ \theta = \frac{a}{q} + \gamma, \]
where $a, q \in \mathbb{Z}[i]$, $(a, q) = 1$ and
\[ |\gamma| \leq |q|^{-2}. \]

As in the classical case, this continued fraction development yields a sequence of infinitely many $q \in \mathbb{Z}[i]$ satisfying the above. Now it follows that
\[
||n_1\theta - n_2\theta|| = \left\|\frac{(n_1 - n_2)a}{q} + (n_1 - n_2)\gamma\right\| \geq \left\|\frac{(n_1 - n_2)a}{q}\right\| - |n_1 - n_2| \cdot |\gamma| \geq \frac{1}{\sqrt{2}|q|} - \frac{|n_1 - n_2|}{|q|^2}
\]
if $n_1, n_2 \in \mathbb{Z}[i]$ such that $n_1 \not\equiv n_2 \mod q$. We cover the set
\[ \mathcal{N} := \{ n \in \mathbb{Z}[i] : 0 < \mathcal{N}(n) \leq z \} \]
by $O(1 + z/|q|^2)$ disjoint rectangles
\[ \mathcal{R} = \{ s \in \mathbb{C} : a_1 < \Re(s) \leq b_1, \ a_2 < \Im(s) \leq b_2 \}, \]
where $|b_1 - a_1| \leq |q|/4$, so that
\[ \mathcal{N} \subset \bigcup_{\mathcal{R}}. \]
Note that if \( n_1, n_2 \in \mathbb{Z}[i] \cap R \), then \( |n_1 - n_2| \leq |q|/(2\sqrt{2}) \) and hence, by (41), if \( n_1, n_2 \in \mathbb{Z}[i] \cap R \) and \( n_1 \neq n_2 \), then
\[
||n_1 \theta - n_2 \theta|| \geq \frac{1}{2\sqrt{2}|q|}. \tag{42}
\]

Now,
\[
\Sigma_\theta(z, \Delta', \Delta'_2) \leq \sum_R \Sigma_\theta(R, \Delta', \Delta'_2),
\]
where
\[
\Sigma_\theta(R, \Delta', \Delta'_2) := \sum_{n \in \mathbb{Z}[i] \cap R \atop \|\Im(n\theta)\| \leq \Delta'_1 \atop \|\Re(n\theta)\| \leq \Delta'_2} 1 + \sum_{n \in \mathbb{Z}[i] \cap R \atop \|\Im(n\theta)\| \leq 1 - \Delta'_1 \atop \|\Re(n\theta)\| \leq \Delta'_2} 1 + \sum_{n \in \mathbb{Z}[i] \cap R \atop \|\Im(n\theta)\| \leq \Delta'_1 \atop \|\Re(n\theta)\| \geq 1 - \Delta'_2} 1. \tag{43}
\]

If \( \{\Im(n\theta)\} \leq \Delta'_1 \leq 1/2 \) and \( \{\Re(n\theta)\} \leq \Delta'_2 \leq 1/2 \) for \( i = 1, 2 \), then
\[
|(|\{\Re(n_1\theta)\} + i\{\Im(n_1\theta)\}) - (\{\Re(n_2\theta)\} + i\{\Im(n_2\theta)\})| \geq ||n_1 \theta - n_2 \theta||,
\]
and hence, by (41), if \( n_1, n_2 \in \mathbb{Z}[i] \cap R \) and \( n_1 \neq n_2 \), then
\[
|(|\{\Re(n_1\theta)\} + i\{\Im(n_1\theta)\}) - (\{\Re(n_2\theta)\} + i\{\Im(n_2\theta)\})| \geq \frac{1}{2\sqrt{2}|q|}.
\]

It follows that
\[
\sum_{n \in \mathbb{Z}[i] \cap R \atop \|\Im(n\theta)\| \leq \Delta'_1 \atop \|\Re(n\theta)\| \leq \Delta'_2} 1 \ll V_{1/(2\sqrt{2}|q|)} (\Delta'_1, \Delta'_2),
\]
where \( V_D (\Delta'_1, \Delta'_2) \) is the maximal number of points of distance \( \geq D \) that can be put into a rectangle with dimensions \( \Delta'_1 \) and \( \Delta'_2 \). The remaining three sums in the last line of (43) can be estimated similarly. It follows that
\[
\Sigma_\theta(R, \Delta', \Delta'_2) \ll V_{1/(2\sqrt{2}|q|)} (\Delta'_1, \Delta'_2).
\]

Clearly,
\[
V_D (\Delta'_1, \Delta'_2) \ll \left(1 + \frac{\Delta'_1}{D}\right) \left(1 + \frac{\Delta'_2}{D}\right).
\]

Putting everything together, we obtain (36). Further, (37) holds because \( 0 < N(n) \leq |q|^2/8 \) implies
\[
||n\theta|| = \left|\left| \frac{na}{q} + n\gamma \right|\right| \geq \left|\left| \frac{na}{q} \right|\right| - |n| \cdot |\gamma| \geq \frac{1}{\sqrt{2|q|}} - \frac{\sqrt{|q|^2/8}}{|q|^2} = \frac{1}{\sqrt{8|q|}}. \tag{44}
\]
9. Final estimations of the sums $E_i$ and $F_i$

Recall the conditions $H_1 \geq 1$ and $H_2 \geq 1/2$. Combining (28), (31), (32) and (39), we get

\[
F_3(H_1, H_2, K, K')^2 \ll (H_1 H_2 x)^\varepsilon \cdot \left( H_1^2 H_2^2 x K + H_1 H_2 x K^{1/2} \cdot \left( 1 + \frac{(H_1^2 + H_2^2) x/K}{|q|^2} \right) (K^{1/2} + |q|^2) \right),
\]

where we use the facts that the number $\tau(n)$ of divisors $j$ of $n$ is $O(\mathcal{N}(n)\varepsilon)$ and that the number of solutions $(n_1, n_2)$ with $x/(2K') < \mathcal{N}(n_1), \mathcal{N}(n_2) \leq x/K$ of the equation $n/j = n_1 - n_2$ is $O(x/K)$. Multiplying out and taking square root yields

\[
F_3(H_1, H_2, K, K') \ll (H_1 H_2 x)^\varepsilon \cdot \left( H_1 H_2 x K^{1/2} + (H_1 + H_2) x K^{-1/4} + |q| x^{1/2} K^{1/4}) \right). \tag{45}
\]

Recall the definition of $F_3(H_1, H_2)$ in (25). From (45), we conclude that

\[
F_3(H_1, H_2) \ll (H_1 H_2 x)^\varepsilon \cdot \left( H_1 H_2 x^{(1+\alpha)/2} + (H_1 + H_2) x^{1-\alpha/4} + |q| x^{1/2+(\alpha+\beta)/4}) \right). \tag{46}
\]

by splitting the summation range of $\mathcal{N}(m)$ into $O(\log 2x)$ dyadic intervals $(K, K')]$. We also split $E_3(H_1, H_2)$, defined in (19), into $O(\log 2M)$ parts

\[
E_3(H_1, H_2, K, K') := \sum_{j \neq 0} \sum_{K < \mathcal{N}(m) \leq K'} \sum_{x/(2\mathcal{N}(m)) < \mathcal{N}(n) \leq x/\mathcal{N}(m)} e(\Im(jmn\theta)).
\]
with $1/2 \leq K < K' \leq 2K$, which, using (31), (32) and (33), we estimate by

$$E_3(H_1, H_2, K, K')$$

$$\ll (x/K)^{1/2} \cdot \sum_{j \neq 0} \sum_{K < N(m) \leq K'} \min \left\{ \left| 2 \delta, H \right|, \left| 3 \delta, H \right| \right\} \cdot \sum_{|\Re(j)| \leq H_1} \sum_{|\Im(j)| \leq H_2} \min \left\{ \left| \Re((j)m\theta) \right|^{-1}, \sqrt{x/K} \right\}^{1/2} \times$$

$$\ll x^{x} \cdot (x/K)^{1/2} \cdot \sum_{0 < N(l) \leq (H_1^2 + H_2^2)K'} \min \left\{ \left| \Im(l\theta) \right|^{-1}, \sqrt{x/K} \right\}^{1/2} \times$$

$$\ll x^{x} \cdot (x/K)^{1/2} \cdot \left( 1 + \frac{(H_1^2 + H_2^2)K}{|q|^2} \right) \left( (x/K)^{1/2} + |q|^2 \right)$$

$$\ll x^{x} \cdot (x/K)^{1/2} \cdot \left( xK^{-1} + (H_1^2 + H_2^2)x|q|^{-2} + (H_1^2 + H_2^2)x^{1/2}K^{-1/2} + |q|^2x^{1/2}K^{-1/2} \right).$$

If $(H_1^2 + H_2^2)K' \leq |q|^2/8$, then using (40) instead of (39), we obtain

$$E_3(H_1, H_2, K, K') \ll (x|q|)^{x} \left( |q|x^{3/4}K^{-3/4} + |q|^2x^{1/2}K^{-1/2} \right).$$

(48)

We deduce that for all $K \geq 1/2$,

$$E_3(H_1, H_2, K, K') \ll (x|q|)^{x} \times$$

$$\left( (H_1^2 + H_2^2)x|q|^{-2} + (H_1^2 + H_2^2)x^{1/2}K^{-1/2} + |q|x^{3/4}K^{-3/4} + |q|^2x^{1/2}K^{-1/2} \right).$$

(49)

which implies

$$E_3(H_1, H_2) \ll (x|q|)^{x} \times$$

$$\left( (H_1^2 + H_2^2)x|q|^{-2} + (H_1^2 + H_2^2)x^{1/2}M^{1/2} + |q|x^{3/4} + |q|^2x^{1/2} \right).$$

(50)

Now, from (17) and (50), we obtain

$$E_3 \ll (Jx|q|)^{x} \cdot \left( \delta^2 J^2x|q|^{-2} + \delta^2 J^2x^{1/2}M^{1/2} + \delta^2 |q|x^{3/4} + \delta^2 |q|^2x^{1/2} \right),$$

(51)

where we use the inequality

$$\min\{\delta, H_1^{-1}\} \cdot \min\{\delta, H_2^{-1}\} \leq \delta^2,$$

and from (24) and (16), we obtain

$$F_3 \ll (Jx)^{x} \left( x^{1+\alpha+\beta/2} + \delta Jx|q|^{-1} + \delta Jx^{1-\alpha/4} + \delta |q|x^{1/2+(\alpha+\beta)/4} \right),$$

(52)

where we use the inequalities

$$\min\{\delta, H_1^{-1}\} \cdot \min\{\delta, H_2^{-1}\} \leq (H_1 H_2)^{-1}$$

and

$$\min\{\delta, H_1^{-1}\} \cdot \min\{\delta, H_2^{-1}\} \leq \delta(H_1 H_2)^{-1/2}$$

(the first for the diagonal, the second for the non-diagonal contribution).
Further, from (14), (15), (20) and (50), we infer
\[
E_1, E_2 \ll (Jx|q|)^{c} \cdot \left( \delta J^{2}x|q|^{-2} + \delta J^{2}x^{1/2}M^{1/2} + \delta |q|x^{3/4} + \delta |q|x^{1/2} \right),
\] (53)
where we use the inequality
\[
\min\{\delta, H^{-1}\} \leq \delta,
\]
and from (21), (22), (26) and (52), we infer
\[
F_1, F_2 \ll (Jx|q|)^{c} \cdot \left( x^{(1+\alpha+\beta)/2} + \delta^{1/2}Jx|q|^{-1} + \delta^{1/2}Jx^{1-\alpha/4} + \delta^{1/2}|q|x^{1/2+(\alpha+\beta)/4} \right),
\] (54)
where we use the inequalities
\[
\min\{\delta, H^{-1}\} \leq H^{-1} \quad \text{and} \quad \min\{\delta, H^{-1}\} \leq \delta^{1/2}H^{-1/2}.
\]

Combing (10), (51) and (53), we obtain
\[
\sum_{\mathcal{N}(m)\leq M} a_{m} = 4\delta^{2} \sum_{\mathcal{N}(m)\leq M} a_{m} + O\left( (Jx|q|)^{c} \cdot \left( \delta J^{2}x|q|^{-2} + \delta J^{2}x^{1/2}M^{1/2} + \delta |q|x^{3/4} + \delta |q|x^{1/2} \right) \right),
\] (55)
and combining (12), (52) and (54), we obtain
\[
\sum_{x^{\alpha}<\mathcal{N}(m)\leq x^{\alpha+\beta}} a_{m}b_{n} = 4\delta^{2} \sum_{\mathcal{N}(m)\leq M} a_{m}b_{n} + O\left( \left( \delta xJ^{-1} + x^{(1+\alpha+\beta)/2} + \delta Jx|q|^{-1} + \delta Jx^{1-\alpha/4} + \delta |q|x^{1/2+(\alpha+\beta)/4} \right) \right).
\] (56)

Now we choose \( J := \lfloor \delta^{-1}x^{3/2} \rfloor, x := |q|^{12} \) (and hence \(|q| := x^{1/12}\)), \( M = x^{2/3}, \alpha := 1/3 \) and \( \beta := 1/2 \) so that
\[
\sum_{\mathcal{N}(m)\leq M} a_{m} = 4\delta^{2} \sum_{\mathcal{N}(m)\leq M} a_{m} + O\left( \delta^{2}x^{1-\epsilon} + x^{5/6+8\epsilon} \right)
\] (57)
and
\[
\sum_{x^{\alpha}<\mathcal{N}(m)\leq x^{\alpha+\beta}} a_{m}b_{n} = 4\delta^{2} \sum_{\mathcal{N}(m)\leq M} a_{m}b_{n} + O\left( \delta^{2}x^{1-\epsilon} + x^{11/12+8\epsilon} \right).
\] (58)

10. Conclusion

Having proved (57) and (58), we deduce that (5) and (6) hold with \( Y = \delta^{2}x^{1-\epsilon} \) if \( \delta \geq x^{-1/24+5\epsilon} \). Now using Theorem 3.1, (4), (8) and (9), it follows that
\[
\sum_{x/2<\mathcal{N}(p)\leq x \atop |np|<\delta} 1 = 4\delta^{2}(1+o(1)) \cdot \sum_{x/2<\mathcal{N}(p)\leq x} 1,
\]
provided that \( x = |q|^{12} \), where \( a/q \) is a Hurwitz continued fraction approximant of \( \theta \) and \( \delta \geq x^{-1/24+\epsilon} \) for any fixed \( \epsilon > 0 \). So by taking \( N_{k} = |q_{k}|^{12} \), where \( q_{k} \) is the \( k \)-th Hurwitz continued fraction denominator for \( \theta \), we have the following result.
Theorem 10.1. Let $\theta$ be a complex number such that $\theta \not\in \mathbb{Q}(i)$ and $\varepsilon > 0$ be an arbitrary constant. Then there exists an infinite increasing sequence of natural numbers $(N_k)_{k \in \mathbb{N}}$ such that

$$\sum_{N_k/2 < N(p) \leq N_k} 1 \sim 4\delta_k^2 \cdot \sum_{N_k/2 < N(p) \leq N_k} 1 \quad \text{as } k \to \infty$$

if $N_k^{-1/24+\varepsilon} \leq \delta_k \leq 1/2$.

From this, we deduce the following.

Corollary 10.2. Let $\theta$ be a complex number such that $\theta \not\in \mathbb{Q}(i)$ and $\varepsilon > 0$ be an arbitrary constant. Then there exist infinitely many Gaussian primes such that

$$||p\theta|| \leq N(p)^{-1/24+\varepsilon}.$$

11. Notes

(I) An improvement of the exponent $1/24$ in Theorem 10.1 and hence Corollary 10.2 would be possible if the bound (31) could be sharpened. We note that if the pair $(m_1, m_2)$ in (30) runs over a rectangle instead of a disk, then the variables $m_1$ and $m_2$ can be separated, and the sum is therefore a product of two linear exponential sums, which implies a much better bound than (31) in this situation. Covering the disk by rectangles and summing up their contributions trivially does not seem to lead to a substantial improvement over the bound (31), though.

(II) In general, it may be an interesting line of future research to improve the exponent $1/24$ in Corollary 10.2 possibly by refining our method, using lower bound sieves and employing Kloosterman sums.

(III) Another interesting line could be to investigate Diophantine approximation problems of this type in general number fields.

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